Solving Linear Programs with $\widetilde{O}(\sqrt{\text{rank}})$ Linear System Solves

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Abstract

We present an algorithm that given a linear program with $n$ variables, $m$ constraints, and constraint matrix $A$, computes an $\epsilon$-approximate solution in $\widetilde{O}(\sqrt{\text{rank}(A)} \log(1/\epsilon))$ iterations with high probability. Each iteration of our method consists of solving $\widetilde{O}(1)$ linear systems and additional nearly linear time computation, improving by a factor of $\Omega((m/\text{rank}(A))^{1/2})$ over the previous fastest method with this iteration cost due to Renegar (1988) [51]. Further, we provide a deterministic polynomial time computable $\widetilde{O}(\text{rank}(A))$-self-concordant barrier function for the polytope, resolving an open question of Nesterov and Nemirovski (1994) [47] on the theory of “universal barriers” for interior point methods.

Applying our techniques to the linear program formulation of maximum flow yields an $\widetilde{O}(|E|\sqrt{|V|} \log(U))$ time algorithm for solving the maximum flow problem on directed graphs with $|E|$ edges, $|V|$ vertices, and integer capacities of size at most $U$. This improves upon the previous fastest polynomial running time of $O(|E| \min\{|E|^{1/2}, |V|^{2/3}\} \log(|V|^2/|E|) \log(U))$ achieved by Goldberg and Rao (1998) [18]. In the special case of solving dense directed unit capacity graphs our algorithm improves upon the previous fastest running times of $O(|E| \min\{|E|^{1/2}, |V|^{2/3}\})$ achieved by Even and Tarjan (1975) [16] and Karzanov (1973) [22] and of $\widetilde{O}(|E|^{10/7})$ achieved more recently by Mądry (2013) [39].

1This paper is a journal version of the paper, “Path-Finding Methods for Linear Programming : Solving Linear Programs in $\widetilde{O}(\sqrt{\text{rank}})$ Iterations and Faster Algorithms for Maximum Flow” [34] and arXiv submissions [32, 33]. This paper contains several new results beyond these prior submissions. This paper provides the first proof of a $\widetilde{O}(r)$-self-concordant barrier for all polytopes $\{x \in \mathbb{R}^n : Ax \geq b\}$ with $r = \text{rank}(A)$ that is polynomial time computable (as opposed to the pseudo-polynomial time computability of the universal barrier of [47]). Further, this paper provides a conceptually simpler weight function than that in our prior works and a unified analysis by establishing new connections between the algorithm, the barrier, and $\ell_p$ Lewis weights [37, 9, 8]. Several components of [34, 32, 33] were not included in this journal version. Techniques, for leveraging this paper to solve linear programs exactly are deferred to [32] and techniques for analyzing the error induced by approximate linear system solves are deferred to [33]. These techniques are fairly standard and general and omitted from this paper for brevity. Further, techniques for reducing the cost of the linear systems found in [32] are also not included and have been improved in a sequence of recent work [35, 8, 2] and techniques solving generalized minimum cost flow as opposed to more restricted minimum cost flow problem considered in this paper are deferred to [33].
1 Introduction

Given a matrix, $A \in \mathbb{R}^{m \times n}$, and vectors, $b \in \mathbb{R}^{m}$ and $c \in \mathbb{R}^{n}$, solving a linear program

$$\min_{x \in \mathbb{R}^{n} : Ax \geq b} c^{T}x$$  \hspace{1cm} (1.1)$$
is a core algorithmic task for the theory and practice of computer science and operations research.

Since Karmarkar’s breakthrough result in 1984 [21], proving that interior point methods can solve linear programs in polynomial time for a relatively small polynomial, interior point methods have been an incredibly active area of research. Currently, the fastest asymptotic running times for solving (1.1) in many regimes are interior point methods. Previously, state-of-the-art interior point methods for solving (1.1) compute an $\epsilon$-approximate solution in either $\tilde{O}(\sqrt{m}\log(1/\epsilon))$ iterations of solving linear systems [51] or $\tilde{O}((m \cdot \text{rank}(A))^{1/4}\log(1/\epsilon))$ iterations of a more complicated but still polynomial time operation [56, 59, 61, 3].

However, in a breakthrough result of Nesterov and Nemirovski in 1994, they showed that there exists a universal barrier function that if computable would allow (1.1) to be solved in $O(\sqrt{\text{rank}(A)} \log(1/\epsilon))$ iterations [48]. Unfortunately, this barrier is more difficult to compute than solutions to (1.1) and despite this result, in many regimes the fastest interior point algorithms are still based on the $\tilde{O}(\sqrt{m}\log(1/\epsilon))$ iteration algorithm of Renegar from 1988.

In this paper we present a new interior point method that solves general linear programs in $\tilde{O}(\sqrt{\text{rank}(A)} \log(1/\epsilon))$ iterations thereby matching the theoretical limit proved by Nesterov and Nemirovski up to polylogarithmic factors. Further, we show how to achieve this convergence rate while only solving $O(1)$ linear systems and performing additional $\tilde{O}(\text{nnz}(A))$ work in each iteration.

Our algorithm is easily parallelizable and in the standard PRAM model of computation we achieve the first $\tilde{O}(\sqrt{\text{rank}(A)} \log(1/\epsilon))$-depth polynomial-work method for solving linear programs. Using state-of-the-art regression algorithms in [43, 38], our linear programming algorithm has a running time of $\tilde{O}((\text{nnz}(A) + (\text{rank}(A))^\omega)\sqrt{\text{rank}(A)} \log(1/\epsilon))$ where $\omega < 2.3729$ is the matrix multiplication constant [63]. Further, leveraging advances in solving sequences of linear systems this running time is improvable to $\tilde{O}((\text{nnz}(A) + \text{rank}(A)^2)\sqrt{\text{rank}(A)} \log(1/\epsilon))$ [35].

We achieve our results through an extension of standard path following techniques for linear programming [51, 19] that we call weighted path finding. We study the weighted central path, i.e. a weighted variant of the standard logarithmic barrier function [55, 17, 41] that was used implicitly by Mądry to achieve a breakthrough improvement to the running time for solving unit-capacity maximum flow problem [39]. We provide a general analysis of the weighted central path, discuss tools for manipulating points along the path and changing the path, and leverage this to produce an efficiently computable path that converges in $\tilde{O}(\sqrt{\text{rank}(A)} \log(1/\epsilon))$ iterations.

Ultimately, we show approximately following the central path re-weighted by $\ell_p$ Lewis weights, a fundamental concept in Banach space theory that has recently found applications for solving $\ell_p$ regression, yields our desired running times. We provide further intuition regarding this weighted central path, and show that it is in the central path induced by a barrier function with $\tilde{O}(\text{rank}(A))$-self-concordance. Further, we show that the value, gradient, and Hessian of this barrier are all computable deterministically in polynomial time. This Lewis weight barrier constitutes the first

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2Here and throughout the paper we use $\tilde{O}()$ to hide factors polylogarithmic in $m, n, U, |V|, |E|,$ and $M$.

3All approximate linear programming algorithms discussed in this paper can be leveraged to obtain exact solutions in weakly polynomial time through standard straightforward reductions (see e.g. [51]). This transformation replaces each $\log(1/\epsilon)$ factor in running times with $L$, a parameter that is at most the number of bits needed to represent (1.1) but in many cases can be much smaller.

4We assume that $A$ has no rows or columns that are all zero as these can be remedied by trivially removing constraints or variables respectively or immediately solving the linear program. Therefore $\text{nnz}(A) \geq \min\{m, n\}$.
barrier for polytopes whose self-concordance nearly matches that of the universal \[49, 36\] and entropic [5] barriers; neither of which are known to be either deterministically or polynomial time computable. Previous methods for computing such barriers required random sampling and run in pseudo-polynomial time, i.e. have running times which depend polynomially (as opposed to polylogarithmically) on the desired accuracy [1].

To further demonstrate the efficacy of our proposed interior point method, we show that it yields provably faster algorithms for solving the maximum flow problem, one of the most well studied problems in combinatorial optimization [52]. By applying our interior point method to a linear program formulation of maximum flow and applying state-of-the-art solvers for symmetric diagonally dominant linear systems [54, 26, 27, 23, 31, 6, 30, 29], to implement the iterations we achieve an algorithm on \(|V|\) node, \(|E|\) edge graphs with integer capacities in the range 0 to \(U\) in time \(O(|E|\sqrt{|V|}\log^{O(1)}(|V|)\log(U))\) with high probability. This improves upon the previous fastest polynomial running time of \(O(|E|\min\{|E|^{1/2}, |V|^{2/3}\} \log(|V|^2/|E|)\log(U))\) achieved in 1998 by Goldberg and Rao [18] for dense graphs. In the special case of solving dense unit capacity graphs our algorithm improves upon the previous fastest running times of \(O(|E|\min\{|E|^{1/2}, |V|^{2/3}\})\) achieved by Even and Tarjan in 1975 [16] and Karzanov in 1973 [22] and of \(O(|E|^{10/7})\) achieved by Mądry [39] more recently. Further, our algorithm is easily parallelizable and using [50, 30, 28], in the PRAM model we obtain a \(\tilde{O}(|E|\sqrt{|V|}\log(U))\)-work \(\tilde{O}(\sqrt{|V|})\)-depth algorithm. Using the same technique, we also solve the minimum cost flow problem in time \(\tilde{O}(|E|\sqrt{|V|}\log(M))\) with high probability where \(M\) is an upper bound on the absolute value of integer costs and capacities, improving upon the previous fastest algorithm of \(\tilde{O}(|E|^{1.5}\log(M))\) due to Daitch and Spielman [11].

1.1 Previous Work

Linear programming is an extremely well studied problem with a long history. There are numerous algorithmic frameworks for solving linear programming problems, e.g. simplex methods [12], ellipsoid methods [24], and interior point methods [21]. Each method has a rich history and an impressive body of work analyzing the practical and theoretical guarantees of the methods. Here we only present the major improvements on the number of iterations required to solve (1.1) and discuss the asymptotic running times of these methods. For a more comprehensive history linear programming and interior point methods we refer the reader to one of the many excellent references on the subject, e.g. [49, 65].

In 1984 Karmarkar [21] provided the first proof of an interior point method running in polynomial time. This method required \(O(m \log(1/\epsilon))\) iterations where the running time of each iteration was dominated by the time needed to solve a linear system of the form \(A^\top DAx = y\) for some diagonal matrix \(D \in \mathbb{R}^{m \times m}_{\geq 0}\) and some \(y \in \mathbb{R}^n\). Using low rank matrix updates and preconditioning, Karmarkar achieved a running time of \(O(m^{3.5} \log(1/\epsilon))\) for solving (1.1) inspiring a long line of research into interior point methods.

In 1988 Renegar provided an improved \(O(\sqrt{m} \log(1/\epsilon))\) iteration interior point method for solving (1.1). His method was based on type of interior point methods known as path following methods which solve (1.1) by incrementally minimizing a \(f_t(x) \overset{\text{def}}{=} t \cdot c^\top x + \phi(x)\) where \(\phi : \mathbb{R}^n \rightarrow \mathbb{R}\) is a barrier function such that \(\phi(x) \rightarrow \infty\) as \(x\) tends to boundary of the polytope and \(t\) is a parameter changed during the algorithm. Renegar provided a method based on using the log barrier \(\phi_t(x) \overset{\text{def}}{=} -\sum_{i \in [m]} \log((Ax - b)_i)\) which serves as the foundation for many modern interior point methods. As with Karmarkar’s result the running time of each iteration of this method was dominated by the time needed to solve a linear system of the form \(A^\top DAx = y\). Using a combination of techniques involving low rank updates, preconditioning and fast matrix multiplication, the amortized complexity of each iteration was improved \(58, 19, 49\) yielding the previous best known running time of \(O(m^{1.5}n \log(1/\epsilon))\) [57].
In seminal work of Nesterov and Nemirovski in 1994 [49], they generalized this approach and showed that path-following methods can be applied to minimize any linear cost function over any convex set if given a suitable barrier function. They introduced a measure of complexity of a barrier known as self-concordance and showed that given any \( \nu \)-self-concordant barrier for the set, an \( \tilde{O}(\sqrt{\nu} \log(1/\epsilon)) \) iteration method could be achieved. Further, they showed that for any convex set in \( \mathbb{R}^n \), there exists an \( O(n) \)-self-concordant barrier, called the universal barrier function. Therefore, in theory any such \( n \)-dimensional convex optimization problem can be solved in \( O(\sqrt{n} \log(1/\epsilon)) \) iterations. However, this result is traditionally considered to be primarily of theoretical interest as the universal barrier function is difficult to compute. Given the possible algorithmic implications of faster interior point methods, e.g. the flow problems of this paper, obtaining a barrier with near-optimal self-concordance that is easy to minimize is a fundamental open problem.

In 1989, Vaidya [61] made an important breakthrough in this direction. He proposed two barrier functions related to the volume of certain ellipsoids and obtained \( O((m \cdot \text{rank}(A))^{1/4} \log(1/\epsilon)) \) and \( O(\text{rank}(A) \log(1/\epsilon)) \) iteration linear programming algorithms [59, 61, 56]. Unfortunately, each iteration of these methods required computing the projection matrix \( D^{1/2} A (A^\top D A)^{-1} A^\top D^{1/2} \) for a positive diagonal matrix \( D \in \mathbb{R}^{m \times m} \). This was slightly improved by Anstreicher [3] who showed it sufficed to compute the diagonal of this projection matrix. Unfortunately, neither of these methods yield faster running times than [57] unless \( m \gg n \) and neither are immediately amenable to take full advantage of improvements in solving structured linear system solvers and thereby improve the running time for solving the maximum flow problem.

| Year   | Author           | Number of Iterations                         | Nature of iterations            |
|-------|------------------|---------------------------------------------|---------------------------------|
| 1984  | Karmarkar [21]   | \( O(m \log(1/\epsilon)) \)               | Linear system solve             |
| 1986  | Renegar [51]     | \( O(\sqrt{m} \log(1/\epsilon)) \)       | Linear system solve             |
| 1989  | Vaidya [60]      | \( O((m \cdot \text{rank}(A))^{1/4} \log(1/\epsilon)) \) | Matrix Inversion                |
| 1994  | Nesterov and Nemirovskii [49] | \( O(\sqrt{\text{rank}(A)} \log(1/\epsilon)) \) | Volume computation             |
|       | This paper       | \( O(\sqrt{\text{rank}(A)} \log(1/\epsilon)) \) | \( O(1) \) Linear system solves |

These results suggest that you can solve linear programs closer to the \( \tilde{O}(\sqrt{\text{rank}(A)} \log(1/\epsilon)) \) bound achieved by the universal barrier only if you pay more in each iteration. In this paper, we show that this is not the case. We provide a method that up to polylogarithmic factors matches the convergence rate of the universal barrier function while only having iterations of cost comparable to that of Karmarkar’s [21] and Renegar’s [51] algorithms.

1.2 Our Results

Our main result is provably faster algorithms which given \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^n, c \in \mathbb{R}^m, l_i \in \mathbb{R} \cup \{-\infty\}, \) and \( u_i \in \mathbb{R} \cup \{+\infty\} \) for all \( i \in [m] \) solve linear programs in the following form\(^5\)

\[
\text{OPT} \overset{\text{def}}{=} \min_{x \in \mathbb{R}^m} \min_{x : A^\top x = b} c^\top x, \quad \forall i \in [m] : l_i \leq x_i \leq u_i
\]

We assume throughout that \( A \) is non-degenerate, which we define as full column rank and no rows that are all zero. Further, we assume that for all \( i \in [m] \) the set \( \text{dom}(x_i) \overset{\text{def}}{=} \{x : l_i < x < u_i\} \), is neither the empty set or the entire real line, i.e. \( l_i < u_i \) and either \( l_i \neq -\infty \) or \( u_i \neq +\infty \) and we assume that the interior of the polytope, \( \Omega^\circ \overset{\text{def}}{=} \{x \in \mathbb{R}^m : A^\top x = b, l_i < x_i < u_i\} \), is non-empty.

\(^{5}\)Typically (1.2) is written as \( A x = b \) rather than \( A^\top x = b \). We chose this formulation to be consistent with the derivation of the self-concordant barrier in Section 5, and the standard use of \( n \) to denote the number of vertices and \( m \) to denote the number of edges in the linear program formulation of flow problems.
The problem of solving (1.2) without these assumptions is reducible to an instance where these assumptions hold, without increasing the running times the methods of this paper by more than polylogarithmic factors (see e.g., Appendix E of Part I [32]). Our main result is the following.

**Theorem 1** (Linear Programming). Given interior point $x_0 \in \Omega^\circ$ for linear program (1.2), the algorithm LPSolve (Algorithm 3) outputs $x \in \Omega^\circ$ with $c^\top x \leq \text{OPT} + \epsilon$ with constant probability in

$$O(\sqrt{n} \log^{13} m \cdot \log(mU/\epsilon) \cdot T_w)-\text{work and } O(\sqrt{n} \log^{13} m \cdot \log(mU/\epsilon) \cdot T_d)-\text{depth}$$

where $U = \max\{\|((u-l)/(u-x_0))\|_{\infty}, \|(u-l)/(x_0-l)\|_{\infty}, \|u-l\|_{\infty}, \|c\|_{\infty}\}$ and $T_w$ and $T_d$ are the work and depth needed to compute $(A^\top DA)^{-1}q$ for input positive diagonal matrix $D$ and vector $q$.

Note that (1.2) is the dual of (1.1) in the special case when $u_i = \infty$ for all $i \in [m]$. Consequently, in obtaining this result we solve (1.1) with the desired complexity (see Theorem 43). We consider this formulation with two-sided constraints, (1.2), as it directly encompasses the formulation of maximum flow and minimum cost flow as a linear program [11]. Interestingly, while it is well known that all linear programs, including (1.2), can be written in standard form, all known transformations to put (1.2) in standard form would increase the rank of $A$ causing an $\tilde{O}(\sqrt{\text{rank}(A)})$ iteration algorithms to be too slow to improve the running time for solving the maximum flow problem.

Using lower bounds results of Nesterov and Nemirovski, it is not hard to see that any general barrier for (1.2) must have self-concordance $\Omega(m)$. In particular, Proposition 2.3.6 of [49] shows that if any vertex of an $m$-dimensional polytope belongs to $k$ linearly independent $(m-1)$-dimensional facets, then the self-concordance of any barrier on $\Omega$ is at least $k$. Consequently, Theorem 1 corresponds to a method which converges at a rate faster than what would be predicted by standard interior point theory. That we solve (1.2) in $O(\sqrt{m} \log(1/\epsilon))$ is critical for achieving our faster maximum flow results. Leveraging Theorem 1 we show the following:

**Theorem 2** (Maximum Flow). Given a directed graph $G = (V,E)$ with integral costs $q \in \mathbb{Z}^E$ and capacities $c \in \mathbb{Z}_{\geq 0}^E$ with $\|q\|_{\infty} \leq M$ and $\|c\|_{\infty} \leq M$, we can compute a minimum cost maximum flow with constant probability with $O(|E| \sqrt{|V|} \log^{18} |E| \log M)$ work and $O(\sqrt{|V|} \log^{20} |E| \log M)$ depth.

We complement these results by designing a new barrier whose self-concordance nearly matches that of the universal barrier. We show that specialized to (1.1) an idealized version of our algorithm corresponds to following a path following scheme on a natural barrier induced by Lewis weights [37]. Formally, when $A$ is non-degenerate we provide an $O(\text{rank}(A) \log^5(m))$-self-concordant-barrier such that its gradient and Hessian are all polynomial time computable.

**Theorem 3** (Nearly Universal Barrier). Let $\Omega^\circ \triangleq \{x : Ax > b\}$ be non-empty for non-degenerate $A \in \mathbb{R}^{m \times n}$. There is an $O(n \log^5 m)$-self concordant barrier $\psi$ for $\Omega^\circ$ such that for all $\epsilon > 0$ and $x \in \Omega^\circ$ in $O(nm^{\omega-1} \cdot \log m \cdot \log(m/\epsilon))-\text{work and } O(\log^2 m \cdot \log(m/\epsilon))-\text{depth}$ it is possible to compute $g \in \mathbb{R}^n$ and $H \in \mathbb{R}^{n \times n}$ with $\|g - \nabla^2 \psi(x)\|_{\nabla^2 \psi(x)^{-1}} \leq \epsilon$, and $(1-\epsilon)\nabla^2 \psi(x) \preceq H \preceq (1+\epsilon)\nabla^2 \psi(x)$.

To obtain these results we provide several additional tools of possible independent interest. In Section 4 we provide several algebraic facts regarding Lewis weights and in Section B we provide several algorithms for computing Lewis weights in different contexts. Further, in Section C we provide results for a natural online learning problem which we leverage to handle approximation errors in our path finding schemes.

Ultimately, we hope the varied results of this paper will open the door towards developing even faster algorithms for convex programming more broadly. While the analysis in the paper is quite technical, ultimately the algorithms and heuristics they suggest, i.e. locally re-weighting the central path by Lewis weights (in the case of maximum flow, effective resistance), are straightforward and we hope may be used more broadly.
1.3 Geometric Motivation

To motivate our approach, consider the slightly simplified problem of designing an $\tilde{O}(\sqrt{n} \log(1/\epsilon)) = O(\sqrt{\text{rank}(A)} \log(1/\epsilon))$ iteration algorithm for solving (1.1) for non-degenerate $A$ where the running time of each iteration is dominated by the time needed to solve a linear system $A^T D A x = y$ for diagonal $D \in \mathbb{R}^{n \times n}$. The classic self-concordance theory for analyzing interior point methods established in [49] shows that it suffices to produce a simple enough $O(n)$-self-concordant barrier for the set $\Omega^o_{\nu} = \{ x \in \mathbb{R}^n : A x > b \}$. This seminal work of Nesterov and Nemirovski showed that given any $\nu$-self-concordant barrier for an open convex set $K$ there is an $O(\sqrt{\nu} \log(1/\epsilon))$ iteration interior point method, based on a technique known as path following, for minimizing linear functions over $K$. Further, the running time of each iteration is dominated by the time needed to compute a gradient of the barrier and approximately solve a linear system in its Hessian.

**Definition 4** (Self-concordance). A convex, thrice continuously differentiable function $\phi : K \to \mathbb{R}^n$ is a $\nu$-self-concordant barrier function for open convex set $K \subset \mathbb{R}^n$ if the following conditions hold:

- $\lim_{i \to \infty} \phi(x_i) \to \infty$ for all sequences $x_i \in K$ converging to boundary of $K$.
- $|D^3 \phi(x)[h,h,h]| \leq 2|D^2 \phi(x)[h,h]|^{3/2}$ for all $x \in K$ and $h \in \mathbb{R}^n$,
- $|D \phi(x)[h]| \leq \sqrt{\nu}|D^2 \phi(x)[h,h]|^{1/2}$ for all $x \in K$ and $h \in \mathbb{R}^n$.

To achieve our goals, ideally we would produce a $\tilde{O}(n)$-self-concordant barrier function for the feasible region such that the resulting path following scheme would have sufficiently low iteration costs. Unfortunately, as we have discussed no such barrier is known to exist, all previous $O(n)$-self-concordant barriers are more difficult to evaluate than linear programming, and it would be unclear how to generalize such an approach to solving (1.2). Deferring this last issue to Section 1.4, here we describe how to overcome the first two issues and derive a deterministic polynomial-time computable barrier functions with self-concordance $\tilde{O}(n)$.

Our barrier function can be derived from the following intuition regarding interior point methods. At a high level, interior point methods address the key difficulty of linear programming, making progress in the presence of non-differentiable inequality constraints, by leveraging a barrier function, $\phi$, which provides a local smooth approximation. These methods solve the linear program by performing Newtons method, i.e. solving a sequence of linear systems, which trade off the utility of minimizing cost, $c^T x$, and staying away from the constraints, i.e. minimizing $\phi$. Since these Newton steps correspond to minimizing linear functions over ellipsoids and these ellipsoids come from the second-order approximations of the barrier functions, interior point methods essentially approximate polytopes by a sequence of ellipsoids. Self-concordance can be viewed as a geometric condition that relates how well these ellipsoids approximate the domain. In particular, the following lemma shows that the second-order approximation of the barrier function at the minimum point well-approximates the domain.

**Theorem 5** (Dikin Ellipsoid Rounding [46, Thm 4.2.6]). Given a $\nu$-self-concordant barrier function $\phi$ for convex set $K \subset \mathbb{R}^n$, let $x_\phi$ be the minimizer of $\phi$ and $E \overset{\text{def}}{=} \{ x \in \mathbb{R}^n : (x-x_\phi)^T \nabla^2 \phi(x_\phi)(x-x_\phi) \leq 1 \}$ be the Dikin ellipsoid. $E$ is a $\nu + 2\sqrt{\nu}$-rounding of $K$, i.e. $E \subseteq K \subseteq (\nu + 2\sqrt{\nu})E$.

Consequently, to obtain a $\tilde{O}(n)$-self-concordant barriers it is necessary to obtain ellipsoids that are $\tilde{O}(n)$-roundings. The maximum volume contained ellipsoid or John ellipsoid has this property.

**Lemma 6** (John Ellipsoid Rounding [20]). For convex $K \subseteq \mathbb{R}^n$ and John ellipsoid, $J(K)$, i.e. the largest volume ellipsoid contained inside $K$, we have that $J(K) \subseteq K \subseteq nJ(K)$.
In contrast to other ellipsoids that yield approximation guarantees, e.g. the covariance matrix of the uniform distribution on the body [62], the John ellipsoid has the desirable property of being defined by a convex optimization problem and therefore can be computed in weakly polynomial time. There are multiple ways to express the John ellipsoid as the solution to a convex problem. Our barrier function is motivated by the following formulation, called \( D\-optimal\) design.

**Lemma 7 (Convex Formulation of John Ellipsoid [25]).** For any \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, \) and polytope interior \( \Omega^0 = \{ x \in \mathbb{R}^n : Ax > b \} \) the John ellipsoid equals \( \{ y \in \mathbb{R}^n : (y - x)^\top A^\top WA(y - x) \leq 1 \} \) where \( \{ x \in \Omega^0, w \in \mathbb{R}^m_\geq \} \) is the saddle point of the following convex concave problem

\[
\min_{x \in \Omega^0} \phi_\infty(x) \quad \text{where} \quad \phi_\infty(x) = \max_{\sum_{w_i = n, w_i \geq 0}} \ln \det \left( A^\top S_x^{-1} W S_x^{-1} A \right)
\]

where \( S_x \) and \( W \) are diagonal \( m \times m \) matrices with \( (S_x)_{ii} = a_i^\top x - b_i \) and \( W_{ii} \) are diagonal.

Motivated by Theorem 5 a natural approach towards obtaining a polynomial time computable \( \tilde{O}(n)\)-self-concordant barrier would simply be to pick a barrier function for \( \Omega^0 \) whose minimizer is the center of John ellipsoid. The function \( \phi_\infty(x) \) of (1.3) is such a function, but unfortunately, simply inducing a Dikin ellipse that approximates the feasible region is insufficient to be a self-concordant barrier. A self-concordant barrier also needs to not change two quickly; however \( \phi_\infty(x) \) is not even continuously differentiable. To see this, let \( J(\Omega, x) \) be the maximum volume ellipsoid inside \( K \) and centered at \( x \) and note that \( \phi_\infty(x) = c \log(\text{vol}(J(\Omega, x))) \) for a universal constant \( c \). Consequently, for \( \Omega = [-1, 1] \) we have \( \phi_\infty(x) = c \log(2(1 - |x|)) \), i.e. it is only affected by one constraint at each point, except at 0, where it is non-differentiable.

To make \( \phi_\infty(x) \) smooth, we could apply a standard approach of adding a strongly concave term, i.e. a regularizer, to the objective function \( \ln \det(A^\top S_x^{-1} W S_x^{-1} A) \). In general, if smooth \( f(x, y) \) is strongly concave in \( y \), then \( \max_y f(x, y) \) is smooth in \( x \). In fact, there are multiple ways to apply this approach to obtain a polynomial time computable universal barrier function. For example, it can be shown that that the following is an \( \tilde{O}(n) \) self-concordant barrier function

\[
\phi_p(x) \overset{\text{def}}{=} \max_{\sum_{i \in [m]} w_i = n, w_i \geq 0} \ln \det \left( A^\top S_x^{-1} W S_x^{-1} A \right) - \frac{n}{m} \sum_{i \in [m]} w_i \ln w_i - \frac{n}{m} \sum_{i \in [m]} \ln[S_x]_{ii}.
\]

**Lewis Weight Barrier:** In this paper, we provide a more elegant barrier that we believe further elucidates the geometric structure of the problem. In Section 5 for all \( p > 0 \) we consider the function

\[
\phi_p(x) \overset{\text{def}}{=} \begin{cases} \max_{w \in \mathbb{R}^m_+, w \geq 0} \frac{1}{2} f_p(x, w) & \text{if } p \geq 2 \\ \min_{w \in \mathbb{R}^m_+, w \geq 0} \frac{1}{2} f_p(x, w) & \text{if } p \leq 2 \end{cases}
\]

where

\[
f_p(x, w) \overset{\text{def}}{=} \ln \det \left( A^\top S_x^{-1} W^{1-\frac{2}{p}} S_x^{-1} A \right) - \left( 1 - \frac{2}{p} \right) \sum_{i \in [m]} w_i.
\]

We show that the maximizing \( (q > 2) \) or minimizing \( (q < 2) \) weights, \( w \in \mathbb{R}^m_\geq \), for \( \phi_p \) are the \( \ell_p\)-**Lewis weights** for the matrix \( S^{-1} A \) [37] and hence we call \( \phi_p \) the **Lewis weight barrier**.

Lewis weights are fundamental in the theory of Banach spaces and a key tool for approximating a matrix in \( \ell_p \)-norms. They generalize a fundamental \( \ell_2 \) measure of row importance known as **leverage scores** which are defined for \( A \in \mathbb{R}^{m \times n} \) as \( \sigma(A) = \text{diag}(A(A^\top A)^{-1} A^\top) \), i.e. the diagonals of the orthogonal projection matrix onto the image of \( A \). For all \( p > 0 \) the \( \ell_p \)-Lewis weights of \( A \) are the unique vector \( w_p(A) \) which is the leverage scores of \( W^{(1/2)-(1/p)} \) \( A \) for \( W_p = \text{Diag}(w_p) \). Intuitively,
the \( \ell_p \)-Lewis weight of a row \( i \), \( w_p(A)_i \), denotes the importance of the \( i^{th} \) row under \( \ell_p \) norm and it is known that sampling \( \tilde{O}(n^{\max(p,2/1)}) \) rows of \( A \) with probability proportional to \( \ell_p \) Lewis weight and reweighting yields a matrix \( B \) such that with high probability \( \|Bx\|_p \approx \|Ax\|_p \) multiplicatively for all \( x \) [4]. Recently, Cohen and Peng [9] studied Lewis weights in the context of solving \( \ell_p \)-regression, showed that Lewis weights computation can be written as a convex optimization problem for \( p \geq 2 \), and provided a nearly constant iteration algorithm for computing Lewis weights for \( p \in (0, 4) \).

In this paper we provide several complementary results regarding Lewis weights, including formulating their computation as a convex optimization problem for all \( p > 0 \) (Section 4) and providing additional algorithms for computing them (Section B). Further, we study the stability of Lewis weight barrier for \( \Omega \) (Section 3), we study the Lewis weight barrier for \( p \) and prove Theorem 3. The barrier \( \phi_p \) is essentially the limit of \( \phi_p \) for \( p \to \infty \) and consequently our analysis shows that the \( \ell_\Theta(\log m) \) generalization of the John ellipse yields a nearly universal barrier.

### 1.4 Path Finding

Though the explanation of the previous section suffices to prove Theorem 3, it is unclear how to leverage this analysis to prove Theorem 1.1. As discussed, there is no \( O(n) \)-self-concordant barrier for the feasible region of (1.2) and even if this issue could be overcome, naively implementing such a method would require the expensive operation of computing Lewis weights. However, computing these weights to high precision (or even certifying their properties) necessitates computing leverage scores which naively yields iteration costs comparable to that of Vaidya and Anstreicher’s interior point methods [59, 61, 56, 3], i.e. slower then solving \( O(1) \)-linear systems.

To overcome these issues we develop a scheme for dynamically re-weighting self-concordant barriers for \( \text{dom}(x_i) \) in (1.2). We provide 1-self-concordant barriers \( \phi_i \) for each \( \text{dom}(x_i) \) (see Section 3.1) and study the central path they induce, i.e. \( x_t \) for \( t > 0 \), where

\[
    x_t \overset{\text{def}}{=} \arg \min_{A^\top x = b} f_t(x) \quad \text{where} \quad f_t(x) \overset{\text{def}}{=} t \cdot c^\top x + \sum_{i \in [m]} \phi_i(x_i).
\]

(1.4)

Self-concordance theory yields that \( \sum_{i \in [m]} \phi_i(x_i) \) is a \( m \)-self-concordant barrier and therefore this yields an \( \tilde{O}(\sqrt{m}) \) iteration method; we directly attempt to improve this bound.

To motivate our improvement, note that the performance of this method is highly dependent on the representation of (1.1). Duplicating a constraint, i.e. a row of \( A \) and the corresponding entry in \( b_i \), \( \ell_i \) and \( u_i \), corresponds to doubling the contribution of some \( \phi_i \). Repeating a constraint many times can actually slow down the convergence of standard path following methods and in a series of papers [13, 14, 44, 45, 42], it was shown that by carefully duplicating constraints on Klee-Minty cubes standard interior point methods for the dual can take \( \Omega(\sqrt{m}) \) iterations.

Since the weighting of \( \phi_i \) can affect convergence, we provide algorithms which dynamically re-weight the \( \phi_i \). We show that this can improve the convergence rate from \( \Omega(\sqrt{m}) \) to \( \tilde{O}(m) \). In Section 3, we study the weighted barrier function, \( \phi(x) = \sum_{i \in [m]} g_i(x) \phi_i(x) \) where \( g : \mathbb{R}^m \to \mathbb{R}^m \) is a weight function of the current point, and the weighted central path they induce, i.e.

\[
    x^g_t \overset{\text{def}}{=} \arg \min_{A^\top x = b} f_t(x) \quad \text{where} \quad f_t(x) \overset{\text{def}}{=} t \cdot c^\top x + \sum_{i \in [m]} g_i(x) \phi_i(x_i) \quad \text{for all} \ t \geq 0.
\]

To obtain our improved running times we investigate what properties of \( g(x) \) yield faster convergence. Standard analysis suggests that \( g \) should have low total size, i.e. \( \|g(x)\|_1 = O(n) \), and
induce Newton steps that do not change the Hessian much. Directly optimizing the weights for these conditions suggest that \( g(x) \) should be the \( \ell_1 \)-Lewis weights for the local re-weighting of the constraint matrix! In the special case where all \( \ell_i = 0 \) and \( u_i = +\infty \) this recovers the motivation that inspired considering \( \phi_\infty \) ! Here, we run into the same issues discussed in Section 1.3, e.g. that the John ellipse or \( \ell_1 \)-Lewis weights are unstable. Consequently, we consider the dual analog of the approach of the previous section and let \( g \) be the \( \ell_p \)-Lewis weights for \( p = 1 - 1 / \log m \) and show that this has the desired properties. Interestingly, in the case that the \( \ell_i = 0 \) and \( u_i = +\infty \) the \( x_t^q \) are dual to the central path induced by the \( \ell_q \)-Lewis weight barrier discussed in Section 1.3.

This reasoning yields dual algorithm to path following methods with the Lewis weight barrier: Newton step \( x \) for fixed \( g(x) \), update \( g(x) \), update \( t \), and repeat. All that remains is the issue of computing \( \ell_p \)-Lewis weights. To overcome this issue, we exploit that leverage scores and consequently \( \ell_p \)-Lewis weights for small \( p \) can be efficiently approximated for small \( p \). This was shown in the aforementioned exciting result Cohen and Peng [9]. In Section B we provide additional Lewis weight computation algorithms for all \( p \) which we leverage to compute multiplicative approximations to \( \ell_p \)-Lewis weights in our methods. Unfortunately, this error is still too much for path following to handle directly, as such large weights changes can decrease measures of centrality greatly.

To overcome this final issue, rather then using the weighted barrier \( \phi(x) = \sum_{i \in [m]} g_i(x) \phi_i(x) \) where the weights \( g(x) \) depends on the \( x \) directly, we instead maintain separate weights \( w \in \mathbb{R}_{>0}^m \) and current point \( x \) and use the barrier \( \phi(x, w) = \sum_{i \in [m]} w_i \phi_i(x_i) \). We then design a method where we maintain the invariants that \( x \) is close to the minimum of \( \phi(x, w) \) over \( A^T x = b \) and \( w \) is multiplicatively close to \( g(x) \). Since, each fixed \( w \in \mathbb{R}_{>0}^m \) induces a particular weighted central path, i.e. the minimizers of \( t \cdot e^T x + \phi(x, w) \), our method can be viewed as alternating between advancing along a weighted central path and changing the path. We call this technique, path finding.

We design this path-finding method in two steps. First, we show that we can take a Newton step on \( x \) and then update \( w \) while improving centrality and not changing \( w \) too much. This requires care, as with the weighted barrier it is difficult to certify that Newton steps are stable and do not change points multiplicatively. To overcome this, we explicitly measure the centrality of our points by the size of the Newton step in a mixed norm of the form \( \| \cdot \| = \| \cdot \|_\infty + C_{\text{norm}} \| \cdot \|_W \) to keep track of both the standard measure of centrality and this multiplicative change. Second, we show that given a multiplicative approximation to \( g(x) \) and bounds on the change of \( g(x) \), we can maintain the invariant that \( g(x) \) is close to \( w \) multiplicatively without moving \( w \) too much. We formulate this as a general two player game and provide an efficiently computable solution (See Section C).

By combining these insights and formulating minimum cost flow as a linear program, we prove Theorem 1 and Theorem 2. Measuring Newton step sizes with respect to the mixed norm helps explain how our method outperforms the self-concordance of the best barrier for (1.2). Self-concordance is based on \( \ell_2 \) analysis and the lower bounds for self-concordance are precisely the failure of \( \ell_2 \) to approximate \( \ell_\infty \). While ideally our methods might optimize over \( \ell_\infty \) directly, \( \ell_\infty \) is rife with degeneracies impairing this analysis. However, unconstrained minimization over a box is simple and by working with this mixed norm and carefully choosing weights we are taking advantage of the simplicity of minimizing \( \ell_\infty \) over most of the domain and only paying for the \( \tilde{O}(n) \)-self-concordance of a barrier for the subspace induced by the \( A^T x = b \) constraint.

1.5 Paper Organization

After providing notation in Section 2, in Section 3 we provide our analysis of weighted path finding, in Section 4 we provide our analysis of Lewis weights, and in Section 5 we prove the self-concordance of the Lewis weight barrier. The proofs of Theorems 1, 2, and 3 are then given in Section B. Algorithms for computing Lewis weights and many technical details are deferred to the appendix. Note that throughout we made only limited attempts to reduce polylogarithmic factors.
2 Notation

Vector Operations: We frequently apply scalar operations to vectors with the interpretation that these operations should be applied coordinate-wise, e.g. for $x, y \in \mathbb{R}^n$ we let $x/y \in \mathbb{R}^n$ with $[x/y]_i \overset{\text{def}}{=} (x_i/y_i)$, $xy \in \mathbb{R}^n$ with $[xy]_i = x_i y_i$, and $\log(x) \in \mathbb{R}^n$ with $[\log(x)]_i = \log(x_i)$ for all $i \in [n]$.

Matrices: We call a matrix $A$ non-degenerate if it has full column-rank and no zero rows. We call symmetric matrix $B \in \mathbb{R}^{n \times n}$ positive semidefinite (PSD) if $x^T B x \geq 0$ for all $x \in \mathbb{R}^n$ and positive definite (PD) if $x^T B x > 0$ for all $x \in \mathbb{R}^n$.

Matrix Operations: For symmetric matrices $A, B \in \mathbb{R}^{n \times n}$ we write $A \preceq B$ to indicate that $x^T A x \leq x^T B x$ for all $x \in \mathbb{R}^n$ and define $\prec$, $\preceq$, and $\succeq$ analogously. For $A, B \in \mathbb{R}^{n \times n}$, we let $A \circ B$ denote the Schur product, i.e. $[A \circ B]_{ij} \overset{\text{def}}{=} A_{ij} \cdot B_{ij}$ for all $i \in [n]$ and $j \in [m]$, and we let $A^{(2)} \overset{\text{def}}{=} A \circ A$. We use $\text{nnz}(A)$ to denote the number of nonzero entries in $A$.

Diagonals: For $A \in \mathbb{R}^{n \times n}$ we define $\text{diag}(A) \in \mathbb{R}^n$ with $\text{diag}(A)_i = A_{ii}$ for all $i \in [n]$ and for $x \in \mathbb{R}^n$ we define $\text{Diag}(x) \in \mathbb{R}^{n \times n}$ as the diagonal matrix with $\text{diag}(\text{Diag}(x)) = x$. We often use upper case to denote a vectors associated with diagonal matrices, e.g. $X \overset{\text{def}}{=} \text{Diag}(x)$ and $S = \text{Diag}(s)$.

Fundamental Matrices: For non-degenerate $A$ we let $P(A) \overset{\text{def}}{=} A(A^T A)^{-1} A^T$ denote the orthogonal projection matrix onto $A$’s image and $\sigma(A) \overset{\text{def}}{=} \text{diag}(P(A))$ denote $A$’s leverage scores. We let $\Sigma(A) \overset{\text{def}}{=} \text{Diag}(\sigma(A))$, $P^{(2)}(A) \overset{\text{def}}{=} P(A) \circ P(A)$, $A^{(2)} \overset{\text{def}}{=} \Sigma(A) - P^{(2)}(A)$, and $\bar{A}(A) \overset{\text{def}}{=} \Sigma(A)^{-1/2} A \Sigma(A)^{-1/2}$. $\Lambda(A)$ is a Laplacian matrix and $\bar{A}(A)$ is a normalized Laplacian matrix.

Norms: For PD $A \in \mathbb{R}^{n \times n}$ we let $\| \cdot \|_A$ denote the norm where $\|x\|_A^2 \overset{\text{def}}{=} x^T A x$ for all $x \in \mathbb{R}^n$. For positive $w \in \mathbb{R}^n_{\geq 0}$ we let $\| \cdot \|_w$ denote the norm where $\|x\|_w^2 \overset{\text{def}}{=} \sum_{i \in [n]} w_i x_i^2$ for all $x \in \mathbb{R}^n$. For any norm $\| \cdot \|$ and matrix $M$, its induced operator norm of $M$ is defined by $\|M\| = \sup_{\|x\|=1} \|Mx\|$.

Calculus: For a function of two vectors, i.e. $g(x, y) \in \mathbb{R}$ for all $x \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$, we let $\nabla x g(a, b) \in \mathbb{R}^{n_1}$ denote the gradient of $g$ as a function of $x$ for fixed $y$ at $(a, b) \in \mathbb{R}^{n_1 \times n_2}$, i.e. $[\nabla x g(a, b)]_i = \frac{\partial}{\partial x_i} g(a, b)$, and define $\nabla y$, $\nabla^2 x$, and $\nabla^2 y$ analogously. For $h : \mathbb{R}^n \to \mathbb{R}^m$ and $x \in \mathbb{R}^n$ we let $J_h(x) \in \mathbb{R}^{m \times n}$ denote the Jacobian of $h$ at $x$, i.e. $[J_h(x)]_{ij} \overset{\text{def}}{=} \frac{\partial}{\partial x_j} h_i(x)$ for all $i \in [m]$ and $j \in [n]$. For $f : \mathbb{R}^n \to \mathbb{R}$ and $x, h \in \mathbb{R}^n$ we let $Df(x)[h]$ denote the directional derivative of $f$ in direction $h$ at $x$, i.e. $Df(x)[h] \overset{\text{def}}{=} \lim_{t \to 0} [f(x + th) - f(x)]/t$.

Convex Sets: We call $U \subseteq \mathbb{R}^k$ convex if $t \cdot x + (1 - t) \cdot y \in U$ for all $x, y \in U$ and $t \in [0, 1]$ and symmetric if $x \in \mathbb{R}^k \Leftrightarrow -x \in \mathbb{R}^k$. For all $\alpha > 0$ and $U \subseteq \mathbb{R}^k$ we let $\alpha U \overset{\text{def}}{=} \{ x \in \mathbb{R}^k | x = \alpha^{-1} x, x \in U \}$. For all $p \in [1, \infty]$ and $r > 0$ we call the symmetric convex set $\{ x \in \mathbb{R}^k | \|x\|_p \leq r \}$ the $\ell_p$ ball of radius $r$.

Misc: For $z \in \mathbb{Z}$ we let $[z] \overset{\text{def}}{=} \{1, 2, \ldots, z\}$. We let $1_i$ denote the vector that has value 1 in coordinate $i$ and 0 elsewhere. We use $\tilde{O}$ to hide factors polylogarithmic in $m$, $n$, $U$, $|V|$, $|E|$, and $M$.

3 Weighted Path Finding

Here we introduce our weighted path finding scheme for solving (1.2). First we introduce the weighted central path (Section 3.2) and provide key properties of the path (Section 3.3) and weight functions (Section 3.4). Assuming a weight function (shown to exist in Section 4.4) we then provide the main lemmas we need for an $\tilde{O}(\sqrt{\text{rank}(A)} \log(U/e))$ iteration weighted path following algorithm for (1.2). In Section 3.5, 3.6 and 3.7 we study the effect of changing the path parameter, the point, and the weights, and in Section 3.8 we give our main subroutine for following the path.
3.1 Preliminaries

Recall that our goal is to efficiently solve (1.2) repeated below

$$\min_{x \in \mathbb{R}^m} c^T x.$$  
$$\forall i \in [m]: l_i \leq x_i \leq u_i$$

Here $$A \in \mathbb{R}^{m \times n}$$, $$b \in \mathbb{R}^n$$, $$c \in \mathbb{R}^m$$, $$l_i \in \mathbb{R} \cup \{-\infty\}$$, and $$u_i \in \mathbb{R} \cup \{+\infty\}$$ and we assume that $$A$$ is non-degenerate, that dom($$x_i$$) $$\defeq \{ x : l_i < x < u_i \}$$ is neither the empty set or the entire real line for all $$i \in [m]$$, and the interior of the polytope, $$\Omega^i \defeq \{ x \in \mathbb{R}^m : A^T x = b, l_i < x < u_i \}$$ is non-empty.

Rather than working directly with the different domains of the $$x_i$$ we take a slightly more general approach and let $$\phi_i : \text{dom}(x_i) \rightarrow \mathbb{R}$$ for all $$i \in [m]$$ denote a 1-self-concordant barrier function for dom($$x_i$$) (See Definition 4). In the remainder of the paper we will simply leverage that each $$\phi_i$$ is a 1-self-concordant barrier for each of the dom($$\phi_i$$) and not use any further structure about the barriers or the domains. It is easy to show that such $$\phi_i$$ exist and for completeness, below we provide an explicit 1-self-concordant barrier function for each possible dom($$x_i$$):

- **Case (1):** $$l_i$$ finite and $$u_i = +\infty$$: We use a log barrier defined as $$\phi_i(x) \defeq -\log(x - l_i)$$. Here
  \begin{align*}
  \phi_i'(x) &= -\frac{1}{x - l_i} , \\
  \phi_i''(x) &= \frac{1}{(x - l_i)^2} , \\
  \phi_i'''(x) &= -\frac{2}{(x - l_i)^3}
  \end{align*}

  and therefore clearly $$|\phi_i'''(x)| = 2(\phi_i''(x))^{3/2}$$, $$|\phi_i'(x)| = \sqrt{\phi_i''(x)}$$, and $$\lim_{x \rightarrow l_i^+} \phi_i(x) = +\infty$$.

- **Case (2):** $$l_i = -\infty$$ and $$u_i$$ finite: We use a log barrier defined as $$\phi_i(x) \defeq -\log(u_i - x)$$. Here
  \begin{align*}
  \phi_i'(x) &= \frac{1}{u_i - x} , \\
  \phi_i''(x) &= \frac{1}{(u_i - x)^2} , \\
  \phi_i'''(x) &= -\frac{2}{(u_i - x)^3}
  \end{align*}

  and therefore clearly $$|\phi_i'''(x)| = 2(\phi_i''(x))^{3/2}$$, $$|\phi_i'(x)| = \sqrt{\phi_i''(x)}$$, and $$\lim_{x \rightarrow u_i^-} \phi_i(x) = +\infty$$.

- **Case (3):** $$l_i$$ finite and $$u_i$$ finite: We use a trigonometric barrier defined as $$\phi_i(x) \defeq -\log \cos(a_i x + b_i)$$ for $$a_i = \frac{\pi}{u_i - l_i}$$ and $$b_i = -\frac{u_i + l_i}{2}$$. As $$x \rightarrow u_i^-$$ we have $$a_i x + b_i \rightarrow \frac{\pi}{2}$$ and as $$x \rightarrow l_i^+$$ we have $$a_i x + b_i \rightarrow -\frac{\pi}{2}$$ and therefore, in both cases $$\phi_i(x) \rightarrow +\infty$$. Further,
  \begin{align*}
  \phi_i'(x) &= a_i \tan(a_i x + b_i) , \\
  \phi_i''(x) &= \frac{a_i^2}{\cos^2(a_i x + b_i)} , \\
  \phi_i'''(x) &= \frac{2a_i^3 \sin(a_i x + b_i)}{\cos^3(a_i x + b_i)}
  \end{align*}

  Therefore, $$|\phi_i'(x)| \leq a_i / |\cos(a_i x + b_i)| = \sqrt{\phi_i''(x)}$$ and we have
  \begin{align*}
  |\phi_i'''(x)| &= \frac{2a_i^3 \sin(a_i x + b_i)}{\cos^3(a_i x + b_i)} \leq \frac{2a_i^3}{|\cos(a_i x + b_i)|^3} = 2(\phi_i''(x))^{3/2}.
  \end{align*}

While there is rich theory regarding self-concordance we will primarily use only following two lemmas regarding $$\phi_i$$. Lemma 8 bounds the change in the Hessian of $$\phi_i$$ Lemma 9 bounds the gradient of $$\phi_i$$.

**Lemma 8** ([46, Theorem 4.1.6]). If $$s \in \text{dom}(\phi_i)$$ for $$i \in [m]$$, and $$r \defeq \sqrt{\phi_i''(s)} |s - t| < 1$$ then $$t \in \text{dom}(\phi_i)$$ and $$(1 - r)\sqrt{\phi_i''(s)} \leq \sqrt{\phi_i''(t)} \leq (1 - r)^{-1} \sqrt{\phi_i''(s)}$$. Therefore $$\sqrt{\phi_i''(s)} \geq 1/U$$ where $$U$$ is the diameter of dom($$\phi_i$$).

**Lemma 9** ([46, Theorem 4.2.4]). $$\phi_i'(x) \cdot (y - x) \leq 1$$ for all $$x, y \in \text{dom}(\phi_i)$$ and $$i \in [m]$$.
3.2 The Weighted Central Path

Our path-finding algorithm maintains a feasible point \( x \in \Omega \), weights \( w \in \mathbb{R}^m_{>0} \), and minimizes the following penalized objective function for increasing \( t \) and small \( w \)

\[
\min_{A^\top x = b} f_t(x, w) \quad \text{where} \quad f_t(x, w) \overset{\text{def}}{=} t \cdot c^\top x + \sum_{i \in [m]} w_i \phi_i(x_i).
\] (3.1)

For every fixed set of weights, \( w \in \mathbb{R}^m_{>0} \) the set of points \( x_w(t) = \arg\min_{x \in \Omega} f_t(x, w) \) for \( t \in [0, \infty) \) form a path through the interior of the polytope that we call the weighted central path. We call \( x_w(0) \) a weighted center of \( \Omega \) and note that \( \lim_{t \to \infty} x_w(t) \) is a solution to (1.2) (Lemma 41).

While all weighted central paths converge to a solution of the linear program, different paths may have different algebraic properties which either improve or impair the convergence of a path following scheme. Consequently, our algorithm alternates between advancing down a central path (i.e. increasing \( t \)), moving closer to the weighted central path (i.e. updating \( x \)), and picking a better path (i.e. updating the weights \( w \)). More formally, we assume we have a feasible point \( \{x, w\} \in \{\Omega \times \mathbb{R}^m_{>0}\} \) and a weight function \( g(x) : \Omega \to \mathbb{R}^m_{>0} \), such that for any point \( x \in \mathbb{R}^m_{>0} \) the function \( g(x) \) returns a good set of weights that suggest a possibly better weighted path. Our algorithm then repeats the following: (1) if \( x \) close to \( \arg\min_{y \in \Omega} f_t(y, w) \), then increase \( t \) (2) otherwise, use projected Newton step to update \( x \) and move \( w \) closer to \( g(x) \).

In the remainder of this section we present how we measure both the quality of a current feasible point \( \{x, w\} \in \{\Omega \times \mathbb{R}^m_{>0}\} \), the quality of the weight function, and with a weight function control centrality. In Section 3.3 we derive and present both how close \( \{x, w\} \) is to the weighted central path and the step we take to improve this centrality and in Section 3.4 we present how we measure the quality of a weight function, i.e. how good the weighted paths it finds are. The remaining subsection analyze controlling centrality under changes to \( x, w, \) and \( t \).

3.3 Measuring Centrality

Here we explain how we measure the distance from \( x \) to the minimum of \( f_t(x, w) \) for fixed \( w \), denoted \( \delta_t(x, w) \). As \( \delta_t(x, w) \) measures the proximity of \( x \) to the weighted central path, we call it a centrality, measure of \( x \) and \( w \). To motivate \( \delta_t(x, w) \) we first compute a projected Newton step for \( x \). For all \( x \in \Omega \), we define \( \phi(x) \in \mathbb{R}^m \) by \( \phi(x)_i = \phi_i(x_i) \) for \( i \in [m] \), define \( \phi'(x) \), \( \phi''(x) \), and \( \phi'''(x) \) analogously, and let \( \Phi', \Phi'', \Phi''' \) denote their associated diagonal matrices. This yields

\[
\nabla_x f_t(x, w) = t \cdot c + w \phi'(x) \quad \text{and} \quad \nabla^2_{xx} f_t(x, w) = W \Phi''(x).
\]

Lemma 51 (proved in the appendix) shows that a Newton step for \( x \) is given by

\[
h_t(x, w) = -\left( I - \left(W \Phi''(x)\right)^{-1} A (A^\top (W \Phi''(x))^{-1} A)^{-1} A^\top \right) \left(W \Phi''(x)\right)^{-1} \nabla_x f_t(x, w)
\]

\[
= -\Phi''(x)^{-1/2} P_{x,w} W^{-1} \Phi''(x)^{-1/2} \nabla_x f_t(x, w)
\] (3.2)

where

\[
P_{x,w} \overset{\text{def}}{=} I - W^{-1} A_x \left(A_x^\top W^{-1} A_x\right)^{-1} A_x^\top \quad \text{for} \quad A_x \overset{\text{def}}{=} \Phi''(x)^{-1/2} A.
\] (3.3)

As with standard convergence analysis of interior point methods, we wish to keep the Newton step size in the Hessian norm, i.e. \( \|h_t(x, w)\|_{\phi''(x)} = \|\sqrt{\phi''(x)} h_t(x, w)\|_{\phi''(x)} \) small and the multiplicative change in the Hessian, \( \|\sqrt{\phi''(x)} h_t(x, w)\|_{\infty} \), small. While in standard logarithmic barrier analysis, i.e. \( w_i = 1 \) for all \( i \), we can bound the multiplicative change by the change in the hessian norm (since \( \|\cdot\|_{\infty} \leq \|\cdot\|_{2} \)), here we would like to use small weights and this comparison would be insufficient.
To track both these quantities simultaneously, we define the mixed norm for all \( y \in \mathbb{R}^m \) by
\[
\| y \|_{w+\infty} \overset{\text{def}}{=} \| y \|_{\infty} + C_{\text{norm}} \| y \|_{w}
\]
for \( C_{\text{norm}} > 0 \) defined in Definition 12. Note that \( \| \cdot \|_{w+\infty} \) is indeed a norm for \( w \in \mathbb{R}_{>0}^m \) as in this case both \( \| \cdot \|_{\infty} \) and \( \| \cdot \|_{w} \) are norms. However, rather than measuring centrality by the quantity
\[
\left\| \frac{\sqrt{\phi''(x)} h_t(x, w)}{w} \right\|_{w+\infty} = \left\| \frac{\nabla_x f_t(x, w)}{w} \right\|_{w+\infty}
\]
we instead find it more convenient to use the following idealized form
\[
\delta_t(x, w) \overset{\text{def}}{=} \min_{\eta \in \mathbb{R}^n} \left\| \frac{\nabla_x f_t(x, w) - A\eta}{\sqrt{\phi''(x)}} \right\|_{w+\infty}.
\]
This definition is justified by the following lemma which shows that these two quantities differ by at most a multiplicative factor of \( \| P_{x,w} \|_{w+\infty} \).

**Lemma 10.** For any norm \( \| \cdot \| \) and \( \| y \|_Q = \min_{\eta \in \mathbb{R}^n} \left\| y - \frac{A\eta}{w \sqrt{\phi''(x)}} \right\|_Q \), we have
\[
\| y \|_Q \leq \| P_{x,w} \| \| y \|_Q \leq \| P_{x,w} \| \cdot \| y \|_Q
\]
and therefore for all \( \{ x, w \} \in \{ \Omega^0 \times \mathbb{R}^m_{>0} \} \) we have
\[
\delta_t(x, w) \leq \| \frac{\sqrt{\phi''(x)} h_t(x, w)}{w} \|_{w+\infty} \leq \| P_{x,w} \|_{w+\infty} \cdot \delta_t(x, w).
\]

**Proof.** By definition \( P_{x,w} y = y - \frac{A\eta}{w \sqrt{\phi''(x)}} \) for some \( \eta \in \mathbb{R}^n \). Consequently,
\[
\| y \|_Q = \min_{\eta \in \mathbb{R}^n} \left\| y - \frac{A\eta}{w \sqrt{\phi''(x)}} \right\| \leq \| P_{x,w} \| \cdot \| y \|_Q.
\]
Further, letting \( \eta_q \) be such that \( \| y \|_Q = \left\| y - \frac{A\eta_q}{w \sqrt{\phi''(x)}} \right\| \) and noting \( P_{x,w} W^{-1}(\Phi'')^{-1/2} A = 0 \) yields
\[
\| P_{x,w} \| = \left\| P_{x,w} \left( y - \frac{A\eta_q}{w \sqrt{\phi''(x)}} \right) \right\| \leq \| P_{x,w} \| \cdot \| y - \frac{A\eta_q}{w \sqrt{\phi''(x)}} \| = \| P_{x,w} \| \cdot \| y \|_Q.
\]

We summarize this section with the following definition.

**Definition 11 (Centrality Measure).** For \( \{ x, w \} \in \{ \Omega^0 \times \mathbb{R}^m_{>0} \} \) and \( t \geq 0 \), we let \( h_t(x, w) \) denote the projected newton step for \( x \) on the penalized objective \( f_t \) given by
\[
h_t(x, w) \overset{\text{def}}{=} -\frac{1}{\sqrt{\phi''(x)}} P_{x,w} \left( \frac{\nabla_x f_t(x, w)}{w \sqrt{\phi''(x)}} \right)
\]

[12]
where $P_{x,w}$ is defined in (3.3). We measure the centrality of $\{x, w\}$ by

$$
\delta_t(x, w) \overset{\text{def}}{=} \min_{\eta \in \mathbb{R}^n} \left\| \nabla_x f_t(x, w) - A\eta \right\|_{w+\infty}
$$

(3.6)

where for all $y \in \mathbb{R}^m$ we let $\|y\|_{w+\infty} \overset{\text{def}}{=} \|y\|_{\infty} + C_{\text{norm}}\|y\|_w$ for $C_{\text{norm}} > 0$ defined in Definition 12.

### 3.4 The Weight Function

With the Newton step and centrality conditions defined, the specification of our algorithm becomes clearer. Our algorithm simply repeatedly (1) increases $t$ provided $\delta_t(x, w)$ is small and (2) decreases $\delta_t(x, w)$ by setting $x^{(\text{new})} \leftarrow x + h_t(x, w)$ and (3) moving $w^{(\text{new})}$ towards $g(x^{(\text{new})})$ for some weight function $g(x) : \Omega^0 \to \mathbb{R}^m$. To prove this algorithm converges, we need to show what happens to $\delta_t(x, w)$ when we change $t, x, w$. At the heart of this paper is understanding what conditions we need to impose on the weight function $g$ so that we can bound this change in $\delta_t(x, w)$ and hence achieve a fast convergent rate. In Lemma 14 we show that the effect of changing $t$ on $\delta_t$ is bounded by $C_{\text{norm}}$ and $\|g(x)\|_1$, in Lemma 15 we show that the effect that a Newton Step on $x$ has on $\delta_t$ is bounded by $\|P_{x,g(x)}g(x)\|_{g(x)+\infty}$, and in Lemma 16 and 17 we show the change of $w$ as $g(x)$ changes is bounded by $\|G(x)^{-1}J_g(x)(\Phi''(x))^{-1/2}\|_{g(x)+\infty}$.

For the remainder of the paper we assume we have a weight function $g(x) : \Omega^0 \to \mathbb{R}^m$ and make the following assumptions regarding our weight function. In Section 4.4 we prove that one exists.

**Definition 12 (Weight Function).** Differentiable $g : \Omega^0 \to \mathbb{R}^m$ is a $(c_1, c_s, c_k)$ -weight function if the following hold for all $x \in \Omega^0$ and $i \in [m]$:

- The size, $c_1$, satisfies $c_1 \geq \max\{1, \|g(x)\|_1\}$. This bounds how quickly centrality changes as $t$ changes.
- The sensitivity, $c_s$, satisfies $c_s \geq e_i^\top G(x)^{-1}A_x \left(A_x^\top G(x)^{-1}A_x\right)^{-1} A_x^\top G(x)^{-1}e_i$. This bounds how quickly Hessian change as $x$ changes.
- The consistency, $c_k$, satisfies $\|G(x)^{-1}J_g(x)(\Phi''(x))^{-1/2}\|_{g(x)+\infty} < 1 - c_k^{-1} < 1$. This bounds how much the weights change as $x$ changes, thereby governing how consistent the weights are with changes to $x$ along the weighted central path.

Throughout we assume we have such a weight function and define $C_{\text{norm}} \overset{\text{def}}{=} 24\sqrt{c_sc_k}$.

To motivate slack sensitivity, we show that it bounds $\|P_{x,w}\|_{w+\infty}$. This is used in Lemma 15.

**Lemma 13.** For any $w$ such that $\frac{1}{2}g(x) \leq w \leq \frac{5}{4}g(x)$, we have that

$$
\|P_{x,w}\|_{w+\infty} \leq c_\gamma \text{ where } c_\gamma \overset{\text{def}}{=} 1 + \frac{\sqrt{2c_s}}{C_{\text{norm}}} \leq 1 + \frac{1}{16c_k}.
$$

Proof. Letting $\|I - P_{x,w}\|_{w+\infty} \overset{\text{def}}{=} \max_{\|z\|_w=1} \| (I - P_{x,w})z\|_\infty$ we see that for any $y \in \mathbb{R}^m$, we have

$$
\|P_{x,y}\|_{w+\infty} = \|P_{x,w}\|_{w} + C_{\text{norm}} \|P_{x,w}y\|_w \leq \|y\|_\infty + \| (I - P_{x,w})y\|_\infty + C_{\text{norm}} \|y\|_w \\
\leq \|y\|_\infty + (\|I - P_{x,w}\|_{w+\infty} + C_{\text{norm}}) \|y\|_w \leq \left(1 + \frac{\|I - P_{x,w}\|_{w+\infty} + C_{\text{norm}}}{C_{\text{norm}}} \right) \|y\|_{w+\infty}.
$$

(3.7)
where we used the fact that \( \|P_{x,w}y\|_w \leq \|y\|_w \) for all \( y \) in the first inequality. Further, note that

\[
\|I - P_{x,w}\|_{w \to \infty}^2 = \max_{i \in [m]} \max_{\|y\|_w \leq 1} (e_i^\top (I - P_{x,w})y)^2 \leq \max_{i \in [m]} \|e_i^\top (I - P_{x,w})W^{-\frac{1}{2}}\|^2 \\
= \max_{i \in [m]} e_i^\top W^{-1}A_x \left(A_x^\top W^{-1}A_x\right)^{-1} A_x^\top W^{-1}e_i.
\]

Since \( \frac{4}{5}g(x) \leq w \leq \frac{5}{4}g(x) \), we have that

\[
\|I - P_{x,w}\|_{w \to \infty}^2 \leq 2 \max_{i \in [m]} e_i^\top G(x)^{-1}A_x \left(A_x^\top G(x)^{-1}A_x\right)^{-1} A_x^\top G(x)^{-1}e_i \leq 2c_w. \tag{3.8}
\]

Combining (3.7) and (3.8) and using \( C_{\text{norm}} = 24\sqrt{c_wk} \) yields the claims.

\[\square\]

### 3.5 Changing \( t \)

Here we bound how much centrality increases as we increase \( t \). We show that this rate of increase is governed by \( C_{\text{norm}} \) and \( \|w\|_1 \).

**Lemma 14.** For all \( \{x, w\} \in \{\Omega^\circ \times \mathbb{R}^m_{>0}\} \), \( t > 0 \) and \( \alpha \geq 0 \), we have

\[
\delta_{(1+\alpha)t}(x, w) \leq (1 + \alpha) \delta_t(x, w) + \alpha(1 + C_{\text{norm}}\sqrt{\|w\|_1}).
\]

**Proof.** Let \( \eta_t \in \mathbb{R}^m \) be such that

\[
\delta_t(x, w) = \left\| \nabla_x f_t(x, w) - A\eta_t \right\|_{w+\infty} = \left\| \frac{t \cdot c + w \phi'(x) - A\eta_t}{w \sqrt{\phi''(x)}} \right\|_{w+\infty}.
\]

Applying this to the definition of \( \delta_{(1+\alpha)t} \) and using that \( \|\cdot\|_{w+\infty} \) is a norm then yields

\[
\delta_{(1+\alpha)t}(x, w) = \min_{\eta \in \mathbb{R}^m} \left\| \frac{(1 + \alpha)t \cdot c + w \phi'(x) - A\eta}{w \sqrt{\phi''(x)}} \right\|_{w+\infty} \leq \left\| (1 + \alpha)\frac{t \cdot c + w \phi'(x)}{w \sqrt{\phi''(x)}} - (1 + \alpha)A\eta \right\|_{w+\infty} \leq (1 + \alpha) \left\| \frac{t \cdot c + w \phi'(x)}{w \sqrt{\phi''(x)}} \right\|_{w+\infty} + \alpha \left\| \frac{\phi'(x)}{\sqrt{\phi''(x)}} \right\|_{w+\infty} \leq (1 + \alpha) \delta_t(x, w) + \alpha \left\| \frac{\phi'(x)}{\sqrt{\phi''(x)}} \right\|_{w+\infty} + C_{\text{norm}} \left\| \frac{\phi'(x)}{\sqrt{\phi''(x)}} \right\|_{w+\infty}.
\]

The result follows from the fact that \( |\phi_i'(x)| \leq \sqrt{\phi_{ii}''(x)} \) for all \( i \in [m] \) and \( x \in \mathbb{R}^m \) by Definition 4. \[\square\]

### 3.6 Changing \( x \)

Here we analyze the effect of a Newton step of \( x \) on centrality. We show for sufficiently central \( \{x, w\} \in \{\Omega^\circ \times \mathbb{R}^m_{>0}\} \) and \( w \) sufficiently close to \( g(x) \) Newton steps converge quadratically.

**Lemma 15.** Let \( \{x_0, w\} \in \{\Omega^\circ \times \mathbb{R}^m_{>0}\} \) such that \( \delta_t(x_0, w) \leq \frac{1}{10} \) and \( \frac{4}{5}g(x) \leq w \leq \frac{5}{4}g(x) \) and consider a Newton step \( x_1 = x_0 + h_t(x_0, w) \). Then, \( \delta_t(x_1, w) \leq 4(\delta_t(x_0, w))^2 \).

**Proof.** Let \( \phi_0 \overset{\text{def}}{=} \phi(x_0) \) and let \( \phi_1 \overset{\text{def}}{=} \phi(x_1) \). By the definition of \( h_t(x_0, w) \) and the formula of \( P_{x_0,w} \),
we know that there is some \( \eta_0 \in \mathbb{R}^n \) such that

\[
-\sqrt{\phi''_0}h_t(x_0, w) = \frac{t \cdot c + w\phi'_0 - A_{\eta_0}}{w \sqrt{\phi_0}}.
\]

Therefore, \( A_{\eta_0} = t \cdot c + w\phi'_0 + w\phi''_0h_t(x_0, w) \). Recalling the definition of \( \delta_t \) this implies that

\[
\delta_t(x_1, w) = \min_{\eta \in \mathbb{R}^n} \left\| \frac{t \cdot c + w\phi'_0 - A_{\eta}}{w \sqrt{\phi_1}} \right\|_{w+\infty} \leq \left\| \frac{t \cdot c + w\phi'_0 - A_{\eta_0}}{w \sqrt{\phi_0}} \right\|_{w+\infty}.
\]

By mean value theorem \( \phi'_1 - \phi'_0 = \phi''(\theta)h_t(x_0, w) \) for \( \theta \) between \( x_0 \) and \( x_1 \) coordinate-wise. Hence,

\[
\delta_t(x_1, w) \leq \left\| \frac{\phi''(\theta)h_t(x_0, w) - \phi''_0h_t(x_0, w)}{\sqrt{\phi_1}} \right\|_{w+\infty} = \left\| \frac{\phi''(\theta) - \phi''_0}{\sqrt{\phi_1}} \sqrt{\phi_0}h_t(x_0, w) \right\|_{w+\infty}.
\]

To bound the first term, we use Lemma 8 as follows

\[
\left\| \frac{\phi''(\theta) - \phi''_0}{\sqrt{\phi_1}} \sqrt{\phi_0}h_t(x_0, w) \right\|_{w+\infty} \leq \left\| \frac{\phi''(\theta) - \phi''_0}{\sqrt{\phi_0}} \right\|_{\infty} \left\| \sqrt{\phi_0}h_t(x_0, w) \right\|_{w+\infty} \leq \left( 1 - \left\| \sqrt{\phi_0}h_t(x_0, w) \right\|_{w+\infty} \right)^{-2} - 1 \cdot \left( 1 - \left\| \sqrt{\phi_0}h_t(x_0, w) \right\|_{w+\infty} \right)^{-1}.
\]

Using (3.5), i.e. Lemma 10, the bound \( c_\gamma \leq 2 \) (Lemma 13), and that \( \delta_t(x_0, w) \leq \frac{1}{10} \) yields

\[
\left\| \sqrt{\phi_0}h_t(x_0, w) \right\|_{w+\infty} \leq \left\| \sqrt{\phi_0}h_t(x_0, w) \right\|_{w+\infty} \leq c_\gamma \cdot \delta_t(x_0, w) \leq \frac{1}{5}.
\]

Using \((1 - t)^{-2} - 1\cdot(1 - t)^{-1} \leq 4t\) for \( t \leq 1/5 \), we have

\[
\left\| \frac{\phi''(\theta) - \phi''_0}{\sqrt{\phi_1}} \right\|_{w+\infty} \leq 4 \left\| \sqrt{\phi_0}h_t(x_0, w) \right\|_{w+\infty}
\]

Combining the formulas (3.9) and (3.10) yields that \( \delta_t(x_1, w) \leq 4(\delta_t(x_0, w))^2 \) as desired.

\[
3.7 \text{ Changing } w
\]

In Section 3.6 we used the assumption that the weights, \( w \), were multiplicatively close to \( g(x) \), for the current point \( x \in \Omega^\circ \). To maintain this invariant when we change \( x \) we will need to change \( w \) to move it closer to \( g(x) \). Here we bound how much \( g(x) \) can move as we move \( x \) (Lemma 16) and we bound how much changing \( w \) can decrease centrality (Lemma 17). Together these lemmas will allow us to show that we can keep \( w \) close to \( g(x) \) while still improving centrality (Section 3.8).

\textbf{Lemma 16.} For all \( t \in [0, 1] \), let \( x_t \overset{\text{def}}{=} x_0 + t\Delta \) for \( \Delta \in \mathbb{R}^m \) and \( g_t = g(x_t) \) such that \( x_t \in \Omega^\circ \). Then
for \( \epsilon = \| \sqrt{\phi''_t} \Delta \|_{g_0+\infty} \leq \frac{1}{10} \) we have \( \| \log(g_1) - \log(g_0) \|_{g_0+\infty} \leq (1 - c_k(g)^{-1} + 4\epsilon) \leq \frac{1}{9} \) and for all \( s, t \in [0, 1] \) and for all \( y \in \mathbb{R}^m \) we have \( \| y \|_{g_s+\infty} \leq (1 + 2\epsilon) \| y \|_{g_t+\infty} \).

**Proof.** Let \( q : [0, 1] \to \mathbb{R}^m \) be given by \( q(t) \overset{\text{def}}{=} \log(g_t) \) for all \( t \in [0, 1] \). Then, \( q'(t) = G_t^{-1}J_g(x_t)\Delta_x \). Letting \( Q(t) \overset{\text{def}}{=} \| q(t) - q(0) \|_{g_0+\infty} \) and using Jensen’s inequality yields that for all \( u \in [0, 1] \),

\[
Q(u) \leq Q(0) = \int_0^u \left\| G_t^{-1}J_g(x_t)(\Phi''_t)^{-1/2} \right\|_{g_0+\infty} \left\| \sqrt{\phi''_t} \Delta \right\|_{g_0+\infty} dt.
\]

Using Lemma 8 and \( \epsilon \leq \frac{1}{10} \), we have for all \( t \in [0, 1] \),

\[
\left\| \sqrt{\phi''_t} \Delta \right\|_{g_0+\infty} \leq \frac{\epsilon}{1 - \epsilon} \int_0^u \left\| G_t^{-1}J_g(x_t)(\Phi''_t)^{-1/2} \right\|_{g_0+\infty} dt.
\]

(3.11)

Note that \( Q \) is monotonically increasing. Let \( \theta = \sup_{u \in [0, 1]} \{ Q(u) \leq (1 - c_k^{-1} + 4\epsilon) \} \). Since \( \| q(t) - q(0) \|_{\infty} \leq \| Q(\theta) \|_{\infty} \leq \frac{1}{2} \) and \( q(t) = \log(g_t) \), we know that for all \( s, t \in [0, \theta] \), we have

\[
\left\| g_s - g_t \right\|_{\infty} \leq \| q(s) - q(t) \|_{\infty} + \| q(s) - q(t) \|_{2}\infty
\]

and therefore \( \| g_s / g_t \|_{\infty} \leq (1 + \| q(s) - q(t) \|_{\infty})^2 \leq (1 + (1 - c_k^{-1} + 4\epsilon)^2). \) Consequently,

\[
\| y \|_{g_s+\infty} \leq (1 + (1 - c_k^{-1} + 4\epsilon) \| y \|_{g_t+\infty} \leq (1 + 2\epsilon) \| y \|_{g_t+\infty}.
\]

Using (3.11), we have for all \( u \in [0, \theta] \),

\[
Q(u) \leq Q(0) = \int_0^u \left\| G_t^{-1}J_g(x_t)(\Phi''_t)^{-1/2} \right\|_{g_0+\infty} dt
\]

\[
\leq \frac{\epsilon}{1 - \epsilon} \int_0^u \left( 1 + 2\epsilon \right) \left\| G_t^{-1}J_g(x_t)(\Phi''_t)^{-1/2} \right\|_{g_t+\infty} dt
\]

\[
\leq \frac{\epsilon}{1 - \epsilon} (1 + 2\epsilon)(1 - c_k^{-1}) \theta < (1 - c_k^{-1} + 4\epsilon)\epsilon.
\]

Consequently, \( \theta = 1 \) and we have the desired result by the above bound on \( Q(1) \).

**Lemma 17.** Let \( v, w \in \mathbb{R}^m_{\geq 0} \) such that \( \epsilon = \| \log(w) - \log(v) \|_{w+\infty} \leq \frac{1}{10} \). Then for \( x \in \Omega^c \) we have

\[
\delta_t(x, v) \leq (1 + 4\epsilon)(\delta_t(x, w) + \epsilon).
\]

**Proof.** Let \( \eta_w \) be such that

\[
\delta_t(x, w) = \left\| c + w\phi'(x) - A\eta_w \right\|_{w+\infty}.
\]

(3.12)
The assumptions imply that \((1 + \epsilon)^{-2} w_i \leq v_i \leq (1 + \epsilon)^2 w_i\) for all \(i\) and consequently

\[
\delta_t(x, v) = \min_{\eta} \left\| \frac{c + v \phi'(x) - A \eta}{v \sqrt{\phi''(x)}} \right\|_{t+\infty} \leq \left(1 + \epsilon \right) \left\| \frac{c + v \phi'(x) - A \eta}{v \sqrt{\phi''(x)}} \right\|_{t+\infty} \leq (1 + \epsilon) \left\| \frac{c + v \phi'(x) - A \eta}{v \sqrt{\phi''(x)}} \right\|_{t+\infty}
\]

\[
\leq (1 + \epsilon) \left( \left\| \frac{c + w \phi'(x) - A \eta}{v \sqrt{\phi''(x)}} \right\|_{t+\infty} + \left\| \frac{(v - w) \phi'(x)}{v \sqrt{\phi''(x)}} \right\|_{t+\infty} \right)
\]

\[
\leq (1 + \epsilon)^{3} \delta_t(x, w) + (1 + \epsilon) \cdot \left\| \frac{\phi'(x)}{\sqrt{\phi''(x)}} \right\|_{t+\infty} \cdot \left\| \frac{(v - w)}{v} \right\|_{t+\infty}.
\]

The result follows from the fact that \(|\phi'_i(x)| \leq \sqrt{|\phi''_i(x)|}\) for all \(i \in [m]\) and \(x \in \mathbb{R}^m\) by Definition 4. \(\square\)

3.8 Centering

The results of the previous sections imply an efficient linear programming algorithm provided the weight function can be computed efficiently to high-precision. Unfortunately such a weight computation algorithm is unknown and instead only efficient approximate weight computation algorithms are presently available (See Appendix B). Consequently, here we show how to improve centrality even when the weight function is only computed approximately. Our algorithm is based on a solution to the \(\ell_\infty\) Chasing Game” summarized in Theorem 18 and proved in Appendix C.

**Theorem 18 (\(\ell_\infty\) Chasing Game).** For \(x^{(0)}, y^{(0)} \in \mathbb{R}^m\) and \(\epsilon \in (0, 1/5)\), consider the two player game consisting of repeating the following for \(k = 1, 2, \ldots\)

1. The adversary chooses \(U^{(k)} \subseteq \mathbb{R}^m\), \(u^{(k)} \in U^{(k)}\), and sets \(y^{(k)} = y^{(k-1)} + u^{(k)}\).

2. The adversary chooses \(z^{(k)}\) with \(\|z^{(k)} - y^{(k)}\|_\infty \leq R\) and reveals \(z^{(k)}\) and \(U^{(k)}\) to the player.

3. The player chooses \(\Delta^{(k)} \in (1 + \epsilon)U^{(k)}\) and sets \(x^{(k)} = x^{(k-1)} + \Delta^{(k)}\).

Suppose that each \(U^{(k)}\) is a symmetric convex set that contains an \(\ell_\infty\) ball of radius \(r_k\) and is contained in a \(\ell_\infty\) ball of radius \(R_k \leq R\) and the player plays the strategy

\[
\Delta^{(k)} = \arg \min_{\Delta \in (1 + \epsilon)U^{(k)}} \left\langle \nabla \Phi_{\mu}(x^{(k-1)} - z^{(k)}), \Delta \right\rangle \text{ where } \Phi_{\mu}(x) \overset{\text{def}}{=} \sum_{i \in [m]} (e^{\mu x_i} + e^{-\mu x_i}) \text{ and } \mu \overset{\text{def}}{=} \frac{\epsilon}{12 R}.
\]

If \(\Phi_{\mu}(x^{(0)} - y^{(0)}) \leq \frac{12 m \tau}{\epsilon} \) for \(\tau = \max_k \frac{R_k}{r_k}\) then this strategy guarantees that for all \(k\) we have

\[
\Phi_{\mu}(x^{(k)} - y^{(k)}) \leq \frac{12 m \tau}{\epsilon} \quad \text{and} \quad \|x^{(k)} - y^{(k)}\|_\infty \leq \frac{12 R}{\epsilon} \log \left( \frac{12 m \tau}{\epsilon} \right).
\]

We can think of the problem of maintaining weights as playing this game; we want to keep \(w\) is close to \(g(x)\) while the adversary controls the change in \(g(x)\) and the noise in approximating \(g(x)\). Theorem 18 shows that we control the error \(\ell_\infty\) if we can approximate \(g(x)\) in \(\ell_\infty\). Since we wish to maintain multiplicative approximations to \(g(x)\) we play this game with the log of \(w\) and \(g(x)\). Formally, our goal is to not move \(w\) too much in \(\|\cdot\|_{w+\infty}\) while keeping \(\|\log(g(x)) - \log(w)\|_\infty \leq K\)
Therefore, we know that for the Newton step, we have

\[ U_{\text{new}} \]

and therefore

\[ \delta \]

from our assumption on the symmetric convex set given by

\[ H \]

Hence, we have

\[ \text{for some error } K \] just small enough to not impair our ability to decrease \( \delta_t \) and approximate \( g \).

**Algorithm 1:** \((x^{(\text{new})}, w^{(\text{new})}) = \text{centeringInexact}(x, w, K)\)

\[
R = \frac{K}{48c_k \log(36c_1c_sc_km)}, \quad \delta = \delta_t(x, w) \quad \text{and} \quad \epsilon = \frac{1}{2c_k}.
\]

\[
x^{(\text{new})} = x - \frac{1}{\sqrt{\sigma'(x)}} P_{x,w} \left( \frac{tc - w \sigma'(x)}{\epsilon \sqrt{\sigma'(x)}} \right).
\]

Let \( U = \{ x \in \mathbb{R}^m \mid \| x \|_{w+\infty} \leq (1 - \frac{6}{7c_k}) \delta \} \).

Find \( z \) such that \( \| z - \log(g(x^{(\text{new})})) \|_{w+\infty} \leq R \).

\[
w^{(\text{new})} = \exp \left( \log(w) + \arg \min_{u \in (1+\epsilon)U} \langle \nabla \Phi_{\frac{1}{12R}}(z - \log(w)), u \rangle \right).
\]

In the following Theorem 19 we show that the above algorithm, \text{centeringInexact}, achieves precisely these goals. The algorithm consists primarily of taking a projected Newton step, which corresponds to solving a linear system, and a projection onto \( U \) (step 5), which in Section D we show can be done polylogarithmic depth and nearly linear work. Consequently, Theorem 19 is our primary subroutine for designing efficient linear programming algorithms in Section 6.

**Theorem 19.** Assume that \( K \leq \frac{1}{16c_k} \). Suppose that

\[ \delta \overset{\text{def}}{=} \delta_t(x, w) \leq R \quad \text{and} \quad \Phi_\mu(\log(g(x)) - \log(w)) \leq 36c_1c_sc_km \]

where \( \mu = \frac{\epsilon}{12R} \) and \( R = \frac{K}{48c_k \log(36c_1c_sc_km)} \). Let \((x^{(\text{new})}, w^{(\text{new})}) = \text{centeringInexact}(x, w, K)\), then

\[ \delta_t(x^{(\text{new})}, w^{(\text{new})}) \leq \left( 1 - \frac{1}{4c_k} \right) \delta \quad \text{and} \quad \Phi_\mu(\log(g(x^{(\text{new})})) - \log(w^{(\text{new})})) \leq 36c_1c_sc_km. \]

**Proof.** By Lemma 16, inequality (3.5), \( c_\gamma(g) \leq 1 + (1/16c_k(g)) \) (Lemma 13) and \( \delta \leq \frac{1}{16c_k} \), we have

\[
\| \log(g(x^{(\text{new})})) - \log(g(x)) \|_{g(x)+\infty} \leq (1 - \frac{1}{c_k} + 4c_\gamma \delta) \cdot c_\delta \leq (1 - \frac{1}{c_k}) \cdot c_\gamma \delta + 5\delta^2 \leq \left( 1 - \frac{13}{14c_k} \right) \delta. \tag{3.13}
\]

Using \( \Phi_\mu(\log(g(x)) - \log(w)) \leq 36c_1c_sc_km \), the definition of \( \mu \) and \( R \), and \( K \leq \frac{1}{16c_k} \), we have

\[
\| w - g(x) \| \| g(x) \|_{w+\infty} \leq \| \log(w) - \log(g(x)) \|_{w+\infty} + \| \log(w) - \log(g(x)) \|_{w+\infty} \leq \frac{17}{16} K \leq \frac{1}{14c_k}.
\]

Hence, we have

\[
\| \log(g(x^{(\text{new})})) - \log(g(x)) \|_{w+\infty} \leq \left( 1 + \frac{1}{14c_k} \right) \left( 1 - \frac{13}{14c_k} \right) \delta \leq \left( 1 - \frac{6}{7c_k} \right) \delta.
\]

Therefore, we know that for the Newton step, we have \( \log(g(x^{(\text{new})})) - \log(g(x)) \in U \) where \( U \) is the symmetric convex set given by \( U \overset{\text{def}}{=} \{ x \in \mathbb{R}^m \mid \| x \|_{w+\infty} \leq C \} \) where \( C \overset{\text{def}}{=} (1 - \frac{6}{7c_k}) \delta \). Note that from our assumption on \( \delta \), we have

\[
C \leq \delta \leq \frac{K}{48c_k \log(36c_1c_sc_km)} = R.
\]

and therefore \( U \) is contained in a \( \ell_\infty \) ball of radius \( R \). Therefore, we can play the \( \ell_\infty \) chasing
game on \( \log(g(x)) \) attempting to maintain the invariant that \( \| \log(w) - \log(g(x)) \|_\infty \leq K \) without taking steps that are more than \( 1 + \epsilon \) times the size of \( U \) where we pick \( \epsilon = \frac{1}{2c_\delta} \) so to not interfere with our ability to decrease \( \delta_t \) linearly. To apply Theorem 18 we need to ensure that \( R \) satisfies \( \frac{12R}{\epsilon} \log \left( \frac{12m \tau}{\epsilon} \right) \leq K \) where \( \tau \) is as defined in Theorem 18.

To bound \( \tau \), we need to lower bound the radius of \( \ell_\infty \) ball it contains. Since by assumption \( \| g(x) \|_1 \leq c_1(g) \) and \( \| g(x^{(\text{new})}) \) \( - \log(w) \|_\infty \leq \frac{1}{4} \), we have that \( \| u \|_1 \leq 2c_1(g) \). Hence, we have \( \| u \|_\infty^2 \geq \frac{1}{2c_1(g)} \| u \|_w^2 \) for all \( u \in \mathbb{R}^m \) and consequently, if \( \| u \|_\infty \leq \frac{\delta}{4c_{\|\cdot\|_1}} \), then \( u \in U \). Therefore, \( U \) contains a box of radius \( \frac{\delta}{4c_{\|\cdot\|_1}} \) and since \( U \) is contained in a box of radius \( \delta \), we have that

\[
\tau \leq 4c_{\|\cdot\|_1} \leq 96c_{s}c_{k}.
\]

where we used the fact that \( c_{\|\cdot\|_1} = 24c_{s}c_{k} \). Using that \( \epsilon = \frac{1}{2c_\delta} \), we have that

\[
\frac{12R}{\epsilon} \log \left( \frac{12m \tau}{\epsilon} \right) \leq 48Rc_k \log(36c_1c_sc_km) = K \quad \text{and} \quad \frac{12m \tau}{\epsilon} \leq 36c_1c_sc_km.
\]

This proves that we meet the conditions of Theorem 18. Consequently, \( \| \log(g(x^{(\text{new})})) - \log(w^{(\text{new})}) \|_\infty \leq K \) and \( \Phi_{\mu} \leq 36c_1c_sc_km \).

Since \( K \leq \frac{1}{4} \), Lemma 15 and \( \delta \leq \frac{1}{160c_\delta} \) shows that

\[
\delta_t(x^{(\text{new})}, w) \leq 4\delta_t(x, w)^2 \leq \frac{\delta}{40c_k}.
\] (3.14)

Step 5 shows that

\[
\| \log(w) - \log(w^{(\text{new})}) \|_{w, \infty} \leq \left( 1 + \frac{1}{2c_k} \right) \left( 1 - \frac{6}{7c_k} \right) \delta \leq \left( 1 - \frac{5}{14c_k} \right) \delta.
\]

Using this and (3.14), Lemma 17 shows that

\[
\delta_t(x^{(\text{new})}, w^{(\text{new})}) \leq \left( 1 + 4 \left( 1 - \frac{5}{14c_k} \right) \delta \right) \left( \delta_t(x^{(\text{new})}, w) + \left( 1 - \frac{5}{14c_k} \right) \delta \right)
\]

\[
\leq \left( 1 + \frac{1}{40c_k} \right) \left( \frac{\delta}{40c_k} + \left( 1 - \frac{5}{14c_k} \right) \delta \right)
\]

\[
\leq \left( 1 + \frac{1}{40c_k} - \left( \frac{5}{14c_k} - \frac{1}{40c_k} \right) \delta \leq \left( 1 - \frac{1}{4c_k} \right) \delta.
\]

\[
\square
\]

4 Lewis Weights

As discussed in Section 1.3 and Section 1.4, Lewis weights play a key role in designing a weight function needed to obtain our fastest linear programming algorithms and designing an efficiently computable self-concordant barrier. Formally, Lewis weights are defined as follows.

**Definition 20** (Lewis Weight). For all \( p > 0 \) and non-degenerate\(^6\) \( A \in \mathbb{R}^{m \times n} \) we define the \( \ell_p \) Lewis weight \( w^{(p)}(A) \) as the unique vector \( w \in \mathbb{R}^m \) such that \( w = \sigma(W^\frac{1}{2} - \frac{1}{p} A) \) where \( W = \text{Diag}(w) \).

\(^6\)The non-degeneracy assumptions on \( A \) are mild and made primarily for notational convenience. If \( A \) has a zero row its corresponding Lewis weight is defined to be 0 and if \( A \) is not full rank, much of the definitions and analysis still apply where inverses and determinants are replaced with pseudoinverse and pseudodeterminants respectively.
In this section we provide facts about Lewis weights that we use throughout the paper. First in Section 4.1 we show how Lewis weights can be written as the minimizer of a convex problem for all $p > 0$. This convex formulation is critical for our self-concordant barrier construction. Then in Section 4.1 we provide facts about the stability of Lewis weights which are critical for analyzing the performance of this barrier. Further, in Section 4.3 we show that Lewis weights yield ellipsoids that are provably good approximations to the polytope $\Omega = \{ x \in \mathbb{R}^n \mid \|Ax\|_\infty \leq 1 \}$. Finally, in Section 4.4 we use this analysis to show that Lewis weights yield a weight function in the context of Definition 12. In Appendix E we shed further light on Lewis weights and the algorithms we build with them showing that they interpolate between the natural uniform distribution over the rows of the matrix (i.e. $p \to 0$) and the weights that yield a John ellipse of $\Omega$ (i.e. $p \to \infty$).

### 4.1 Convex Formulation of Lewis Weights

Here we show that Lewis weights are the result of solving a particular convex optimization problem for $p > 0$ with $p \neq 2$. This convex formulation relies on the following potential.

**Definition 21** (Volumetric Potential). For non-degenerate $A \in \mathbb{R}^{m \times n}$ and $p > 0$ with $p \neq 2$ we define the volumetric potential as

$$V^A_p (w) \overset{\text{def}}{=} -\frac{1}{1 - \frac{2}{p}} \log \det \left( A^T W^{1 - \frac{2}{p}} A \right).$$

We will omit $A$ and $p$ when they are clear from context.

The main result of this section is the following lemma, claiming that Lewis weights are the unique solution to $\min_{w \succeq 0} V^A_p (w) + \sum_{i \in [m]} w_i$, or equivalently, $\min_{w_i \geq 0, \sum_{i \in [m]} w_i = n} V^A_p (w)$.

**Lemma 22.** For all non-degenerate $A \in \mathbb{R}^{m \times n}$ its $\ell_p$ Lewis weights exist and are unique for $p > 0$. For $p \neq 2$ the weights $w_p (A)$ are the unique minimizer of the following equivalent convex problems:

$$\min_{w \in \mathbb{R}^m_{\geq 0}} V^A_p (w) + \sum_{i \in [m]} w_i \quad \text{and} \quad \min_{w \in \mathbb{R}^m_{\geq 0}, \sum_{i \in [m]} w_i = n} V^A_p (w).$$

To prove Lemma 22 we first compute and bound the gradient and Hessian of $V^A_p (w)$.

**Lemma 23** (Gradient and Hessian of Volumetric Potential). For all non-degenerate $A \in \mathbb{R}^{m \times n}$, $w \in \mathbb{R}^m_{> 0}$, and $p > 0$ with $p \neq 2$ we have

$$\nabla V^A_p (w) = -W^{-1} \sigma_w \quad \text{and} \quad \nabla^2 V^A_p (w) = W^{-1} \left( \Sigma_w - \left( 1 - \frac{2}{p} \right) \Lambda_w \right) W^{-1}$$

where $W \overset{\text{def}}{=} \text{Diag} (w)$, $\sigma_w \overset{\text{def}}{=} \sigma (W^{\frac{1}{2} - \frac{1}{p}} A)$, $\Sigma_w \overset{\text{def}}{=} \Sigma (W^{\frac{1}{2} - \frac{1}{p}} A)$, and $\Lambda_w \overset{\text{def}}{=} P (W^{\frac{1}{2} - \frac{1}{p}} A)$. Consequently, $V^A_p$ is convex in $w$ and

$$\frac{2}{\max \{p, 2 \}} \cdot W^{-1} \Sigma_w W^{-1} \preceq \nabla^2 V^A_p (w) \preceq \frac{2}{\min \{p, 2 \}} \cdot W^{-1} \Sigma_w W^{-1}. \quad (4.1)$$

**Proof.** The formulas for $\nabla V^A_p (w)$ and $\nabla^2 V^A_p (w)$ follow immediately from a more general Lemma 50. Now, $0 \preceq \Lambda_w \preceq \Sigma_w$ by Lemma 47; consequently $W^{-1} \Sigma_w W^{-1} \preceq \nabla^2 V^A_p (w) \preceq \frac{2}{p} W^{-1} \Sigma_w W^{-1}$ when $p < 2$ and $\frac{2}{p} W^{-1} \Sigma_w W^{-1} \preceq \nabla^2 V^A_p (w) \preceq W^{-1} \Sigma_w W^{-1}$ when $p > 2$. In either case (4.1) holds. □

Using Lemma 23 we prove Lemma 22, our main result of this section.
Proof of Lemma 22. For all \( w \in \mathbb{R}_{>0}^m \) let \( f(w) = \psi_A(w) + \sum_{i=1}^m w_i \) and \( \sigma_w \overset{\text{def}}{=} \sigma(W^{\frac{1}{2}-\frac{1}{p}}A) \) where \( W = \text{Diag}(w) \). Lemma 23 and \( [\sigma_w]_i \leq 1 \) (Lemma 47) yields that for all \( w \in \mathbb{R}_{>0}^m \) if \( w_i > 1 \) then

\[
\frac{\partial f(w)}{\partial w_i} = 1 - \frac{[\sigma_w]_i}{w_i} \geq 1 - \frac{1}{w_i} > 0.
\]

Hence, we have \( \inf_{w_i > 0} f(w) = \inf_{1 > w_i > 0} f(w) \).

Now, if \( p > 2 \) and \( w_i \in [0,1] \) for all \( i \in [m] \) then since \( 1 - \frac{2}{p} > 0 \)

\[
[\sigma_w]_i = \sigma \left( W^{\frac{1}{2}-\frac{1}{p}}A \right)_i = w_i^{1-\frac{2}{p}} \left[ A(A^\top W^{1-\frac{2}{p}}A)^{-1}A^\top \right]_{ii} \geq w_i^{1-\frac{2}{p}} \left[ A(A^\top A)^{-1}A^\top \right]_{ii} = w_i^{1-\frac{2}{p}} \sigma(A)_i.
\]

Since \( A \) is non-degenerate, \( \sigma(A)_i \in (0,1] \) for all \( i \). Therefore for any \( j \) with \( w_j < \sigma(A)_j^{1/2} \), we have

\[
\frac{\partial f(w)}{\partial w_j} = 1 - \frac{[\sigma_w]_j}{w_j} \leq 1 - w_j^{\frac{2}{p}} \sigma(A)_j < 0
\]

and consequently, \( \inf_{w_i > 0} f(w) = \inf_{1 > w_i > \sigma_i^{1/2}} f(w) \).

Similarly, if \( p < 2 \), \( w_i \in [0,1] \) for all \( i \in [m] \), and \( w_{\min} = \min_{i \in [m]} w_i \) then since \( 1 - \frac{2}{p} < 0 \) we have \( W^{1-\frac{2}{p}} \succeq w_{\min}^{1-\frac{2}{p}} \mathbf{I} \). Consequently, by analogous derivation to (4.2) we have \( [\sigma_w]_i \geq (w_i/w_{\min})^{1-(2/p)} \sigma(A)_i \). Consequently, if \( j \in \arg \min_{i \in [m]} w_i \) this implies that \( [\sigma_w]_j \geq \sigma(A)_j \) and therefore if \( w_j < \sigma(A)_j \) we have \( \frac{\partial f(w)}{\partial w_j} < 0 \). Therefore, if we let \( \sigma_{\min} = \min_{i \in [m]} \sigma_i > 0 \) (since \( A \) is non-degenerate) we have \( \inf_{w_i > 0} f(w) = \inf_{1 > w_i > \sigma(A)_i} f(w) \).

In either case, since \( f \) is continuous the above reasoning argues that \( f \) achieves its minimum on the interior of the domain and therefore we have that the minimizer of \( w \) of \( f(w) \) satisfies \( \nabla f(w^*) = 0 \), i.e. \( [w^*_i] = [\sigma_w]_i \) for all \( i \in [n] \). This proves that the minimizer \( f(w) \) on \( w \in \mathbb{R}_{>0}^m \) exists and are Lewis weights. Further, Lemma 23 shows that

\[
\nabla^2 f(w) \preceq \frac{2}{\max(p,2)} \cdot W^{-1} \Sigma_w W^{-1} \succ 0
\]

for all \( w > 0 \) and therefore \( f \) is strictly convex where \( 1 \geq w_i \geq \min(\sigma, \sigma_i^{1/2}) \) for all \( i \). Consequently, the minimizer of \( f \) is unique and it is the unique point satisfying \( \nabla f(w) = 0 \) for \( w \in \mathbb{R}_{>0}^m \). Further, since \( \sum_{i=1}^m [\sigma_w]_{ii} = n \) by Lemma 47 we have \( \sum_{i=1}^m w п(A)_i = n \) and we have the desired equivalence of the two given objective functions.

\[
4.2 \quad \text{Stability of Lewis Weight Under Rescaling}
\]

Here we study the sensitivity of Lewis weight under rescaling \( A \). In Lemma 24 we compute the Jacobian of Lewis weights with respect to rescaling and in Lemma 25 we bound it.

**Lemma 24.** For all non-degenerate \( A \in \mathbb{R}^{m \times n} \), \( p > 0 \) with \( p \neq 2 \), and \( v \in \mathbb{R}^m \), let \( w(v) \overset{\text{def}}{=} w_p(VA) \) where \( V = \text{Diag}(v) \). Then, for \( A_v \overset{\text{def}}{=} A \left( W^{\frac{1}{2}-\frac{1}{p}} V A \right) \) and \( W_v \overset{\text{def}}{=} \text{Diag}(w(v)) \) we have

\[
J_w(v) = 2W_v \left( W_v - \left( 1 - \frac{2}{p} \right) A_v \right)^{-1} A_v V^{-1}.
\]
Lemma 23 and Lemma 50 yield that
\[ w(v) = \arg \min_{w \in \mathbb{R}^n} f(v, w) \]
and that the optimal is in the interior. Hence, the optimality conditions yield \( \nabla_w f(v, w(v)) = 0 \).
Taking derivative with respect to \( v \) on both sides, we have that
\[ \nabla_{ww} f(v, w(v)) + \nabla_w^2 f(v, w(v)) J_w(v) = 0. \]
Therefore, we have that
\[ J_w(v) = -(\nabla_{ww} f(v, w(v)))^{-1} \nabla_w^2 f(v, w(v)). \] (4.3)
Lemma 23 and Lemma 50 yield that
\[ \nabla_{ww}^2 f(v, w) = W^{-1}(\Sigma_w - \left(1 - \frac{2}{p}\right) \Lambda_w) W^{-1}. \] (4.4)
For \( \nabla_{ww}^2 f(v, w(v)) \), we note that \( \nabla_w f(v, w) = -W^{-1}\sigma(W^{\frac{1}{2}} V A) \). Taking derivative with respect to \( v \) and using Lemma 49 gives that
\[ \left[\nabla_{ww}^2 f(v, w)\right]_{ij} = -\frac{2}{w_i} \Lambda W^{\frac{1}{2}} V A]_{ij} \cdot v_j^{-1}. \] (4.5)
Combining (4.3), (4.4), and (4.5), we have
\[ J_w(v) = W_v \left(\Sigma_w - \left(1 - \frac{2}{p}\right) \Lambda_w\right)^{-1} W_v \cdot 2W_v^{-1} \Lambda_v V^{-1}. \]
The result follows from \( w(v) = \sigma(W^{\frac{1}{2}} V A) \) by Lemma 22.

**Lemma 25.** Under the setting of Lemma 24 for any \( v \in \mathbb{R}^m_{>0} \) and \( h \in \mathbb{R}^m \), we have that
\[ \|W_v^{-1} J_w(v) h\|_w(v) \leq p \cdot \|V^{-1} h\|_w(v), \] (4.6)
and that
\[ \|W_v^{-1} J_w(v) - pV^{-1} h\|_\infty \leq p \cdot \max \left\{ \frac{P}{2}, 1 \right\} \cdot \|V^{-1} h\|_w(v). \] (4.7)

**Proof.** Fix an arbitrary \( v \in \mathbb{R}^m_{>0} \) and \( h \in \mathbb{R}^m \) and let \( w \overset{\text{def}}{=} w(v), \Sigma \overset{\text{def}}{=} \Sigma(W^{\frac{1}{2}} V A), \Lambda \overset{\text{def}}{=} \Lambda(W^{\frac{1}{2}} V A), \tilde{\Lambda} \overset{\text{def}}{=} \tilde{\Lambda}(W^{\frac{1}{2}} V A), \) and \( P(2) = P(2)(W^{\frac{1}{2}} V A) \). By Lemma 24 and the fact that \( w \overset{\text{def}}{=} w_p(V A) = \sigma(W^{\frac{1}{2}} V A) \in \mathbb{R}^m_{>0} \), we have that
\[ J_w(v) h = 2\Sigma \left(\Sigma - \left(1 - \frac{2}{p}\right) \Lambda\right)^{-1} \Lambda V^{-1} h = 2\Sigma^{1/2} \tilde{\Lambda} \left(I - \left(1 - \frac{2}{p}\right) \bar{\Lambda}\right)^{-1} W^{1/2} V^{-1} h \] (4.8)
Consequently (4.6) follows from
\[ \|W_v^{-1} J_w(v) h\|_w = \|\Sigma^{-1/2} z\|_2 = 2 \left\| \tilde{\Lambda} \left(I - \left(1 - \frac{2}{p}\right) \bar{\Lambda}\right)^{-1} W^{1/2} V^{-1} h \right\|_2 \leq p \|W^{1/2} V^{-1} h\|_2 \]
where in the last step we used that $\overline{\Lambda}(I - (1 - \frac{2}{p})\overline{\Lambda})^{-1}$ is a symmetric matrix whose eigenvalues are of the form $\lambda/(1 - (1 - \frac{2}{p})\lambda)$ for each eigenvalue of $\lambda$ of $\overline{\Lambda}$ and since $0 \leq \overline{\Lambda} \leq I$

$$
\left\| \overline{\Lambda} \left( I - \left(1 - \frac{2}{p}\right) \overline{\Lambda} \right)^{-1} \right\|_2 \leq \max_{0 \leq \lambda \leq 1} \frac{\lambda}{1 - (1 - \frac{2}{p})\lambda} = \frac{p}{2}.
$$

(4.9)

Next, note that $\Lambda = \Sigma - P^{(2)}$ and therefore $I - \Lambda = W^{-1/2}P^{(2)}W^{-1/2}$. Combining this with (4.8) and that $I - \left(1 - \frac{2}{p}\right)\overline{\Lambda}$ is invertible yields

$$
[W^{-1}J_w(v) - pV^{-1}]_h = W^{-1/2} \left[ 2\overline{\Lambda} - p \left( I - \left(1 - \frac{2}{p}\right)\overline{\Lambda} \right) \right] \left( I - \left(1 - \frac{2}{p}\right)\overline{\Lambda} \right)^{-1} W^{1/2}V^{-1}h
$$

$$
= pW^{-1}P^{(2)}W^{-1/2} \left( I - \left(1 - \frac{2}{p}\right)\overline{\Lambda} \right)^{-1} W^{1/2}V^{-1}h.
$$

However, by Lemma 47 we know that $\|\Sigma^{-1}P^{(2)}x\|_\infty \leq \|x\|\Sigma = \|\Sigma^{1/2}x\|_2$ for all $x$ and therefore (4.7) follows from

$$
\| [W^{-1}J_w(v) - pV^{-1}]_h \|_\infty \leq p \left\| \left( I - \left(1 - \frac{2}{p}\right)\overline{\Lambda} \right)^{-1} W^{1/2}V^{-1}h \right\|_2 \leq p \cdot \max \left\{ 1, \frac{p}{2} \right\} \cdot \|V^{-1}h\|_w.
$$

To prove the inequality in the last step above, note that $(I - \left(1 - \frac{2}{p}\right)\overline{\Lambda})^{-1}$ is a symmetric matrix whose eigenvalues are of the form $1/(1 - (1 - \frac{2}{p})\lambda)$ for each eigenvalue of $\lambda$ of $\overline{\Lambda}$ and since $0 \leq \overline{\Lambda} \leq I$

$$
\left\| \left( I - \left(1 - \frac{2}{p}\right)\overline{\Lambda} \right)^{-1} \right\|_2 \leq \max_{0 \leq \lambda \leq 1} \frac{1}{1 - (1 - \frac{2}{p})\lambda} = \max \left\{ 1, \frac{p}{2} \right\}.
$$

(4.10)

4.3 Lewis Weight Rounding Properties

Here we show that the Lewis weights for a matrix $A \in \mathbb{R}^{m \times n}$ provide ellipses that provably approximate the polytope $\Omega = \{ x \in \mathbb{R}^n \|Ax\|_\infty \leq 1 \}$. This bound is critical to analyzing both the self-concordance of the Lewis weight barrier and the efficacy as lewis weights for the weighted central path. The main result of this section is the more general Lemma 26 which relates every $\ell_p$ Lewis weight to $\ell_r$ Lewis weight for $r \geq p$. In particular, it bounds how well $\ell_p$ lewis weights $w$ satisfy the optimality conditions of being a $\ell_r$ Lewis weight, i.e. the size of $\sigma(W_2^{\frac{1}{2}}A)w_i^{-1}$.

**Lemma 26.** For all non-degenerate $A \in \mathbb{R}^{m \times n}$ and $w \overset{\text{def}}{=} w_p(A)$ for $0 < p < r$ we have

$$
\sigma(W^{\frac{1}{2}}A)w_i^{-1} \leq c_{p,r,m} \overset{\text{def}}{=} (1 + \alpha)^{\frac{1}{1 + \alpha}} \left( \left( 1 + \frac{1}{\alpha} \right) m \right)^{\frac{\alpha}{1 + \alpha}} \leq 2m^{\frac{\alpha}{1 + \alpha}} \text{ where } \alpha = \frac{2}{p} - \frac{2}{r}.
$$

Lemma 26 shows that $\ell_p$ Lewis weights for large $p$ yield an ellipse that well approximates $\Omega$ though the following simple lemma.

**Lemma 27.** For non-degenerate $A \in \mathbb{R}^{m \times n}$, $w \overset{\text{def}}{=} w_p(A)$ for $p > 0$, and $W \overset{\text{def}}{=} \text{Diag}(w)$, define

$$
E \overset{\text{def}}{=} \{ x \in \mathbb{R}^n : x^\top A^\top WAx \leq 1 \} \text{ and } K \overset{\text{def}}{=} \{ x \in \mathbb{R}^n : \|Ax\|_\infty \leq 1 \}.
$$

23
Then $R \overset{\text{def}}{=} \max_{i \in [m]} \sigma(W^{\frac{1}{2}}A)x_iw_i^{-1}$ is the smallest value such that $E \subseteq \sqrt{R}K$. Further, $K$ is a $\sqrt{c_{p,\infty,m}}$-rounding (See Lemma 26) as $K \subset \sqrt{nE}$.

Proof. Note that $\sigma(W^{\frac{1}{2}}A)x_iw_i^{-1} = e_i^TA(A^TWA)^{-1}A^Te_i$ and

$$\max_{x \in E} \|Ax\|_\infty^2 = \max_{i \in [m],x \in E} \left( e_i^T A(A^TWA)^{-1}\frac{1}{2}(A^TWA)^{\frac{1}{2}}x \right)^2 = \max_{i \in [m]} e_i^T A(A^TWA)^{-1}A^Te_i = R.$$ Consequently, $R$ is as desired. Further, for any $x \in K$ we have $x^TA^TWAx \leq \sum_{i \in [m]} w_i \leq n$ and therefore $K \subset \sqrt{nE}$. \qed

Note that in Lemma 26 we have $\lim_{p \to \infty} c_{p,\infty,m} = 1$. This suggest that that as $p \to \infty$ the ellipse $E$ behaves like a John ellipse for $\Omega$. Indeed, in Appendix E.2 we show that $E$ converges to the John ellipse of $\Omega$.

To prove Lemma 26 we first provide the following helper lemma analyzing the effect of changing the power to which we might raise $W$ in $A^TWA$.

**Lemma 28.** Let $A \in \mathbb{R}^{m \times n}$, $p > 0$ and $w = w_p(A)$. Then, for any $r \geq p$, we have that

$$A^TWA^{1-\frac{2}{r}} \preceq A^TWA^{1-\frac{2}{p}} \preceq (1+\alpha)\left(\left(1 + \frac{1}{\alpha}\right)m\right)^\alpha A^TWA^{1-\frac{2}{r}}$$

where $\alpha = \frac{2}{p} - \frac{2}{r}$.

Proof. Since $r \geq p$ and $w_i \in (0,1]$ for all $i \in [m]$ we have that $w_i^{1-\frac{2}{p}} \leq w_i^{1-\frac{2}{r}}$ for all $i \in [m]$ and therefore $A^TWA^{1-\frac{2}{r}} \preceq A^TWA^{1-\frac{2}{p}}$.

To prove the other direction, let $\epsilon > 0$ be arbitrary and let $I_{w > \frac{2}{m}} \in \mathbb{R}^{m \times m}$ be the diagonal matrix where $[I_{w > \frac{2}{m}}]_{ii} = 1$ if $w_i > \frac{2}{m}$ and $[I_{w > \frac{2}{m}}]_{ii} = 0$ otherwise and let $I_{w \leq \frac{2}{m}} = I - I_{w > \frac{2}{m}}$. Note that

$$\text{tr}\left[(A^TWA^{1-\frac{2}{r}})^{-1}A^TWA^{1-\frac{2}{p}}I_{w \leq \frac{2}{m}}A\right] = \sum_{i \in [m]: w_i \leq \frac{2}{m}} w_i \leq m \cdot \frac{\epsilon}{m} = \epsilon$$

where we used that $w = \sigma(W^{\frac{1}{2}}A)$. Therefore, $A^TWA^{1-\frac{2}{r}}I_{w \leq \frac{2}{m}}A \preceq \epsilon \cdot A^TWA^{1-\frac{2}{p}}A$ and hence

$$A^TWA^{1-\frac{2}{r}}A \preceq \frac{1}{1 - \epsilon} A^TWA^{1-\frac{2}{p}}I_{w > \frac{2}{m}}A. \quad (4.11)$$

Now, we note that

$$A^TWA^{1-\frac{2}{p}}I_{w > \frac{2}{m}}A \preceq \left(\frac{m}{\epsilon}\right)^\frac{2}{p-2} A^TWA^{1-\frac{2}{r}}I_{w > \frac{2}{m}}A. \quad (4.12)$$

Combining (4.11) and (4.12), recalling $\alpha = \frac{2}{p} - \frac{2}{r}$ and choosing the minimizing $\epsilon = \frac{\alpha}{1+\alpha}$ yields

$$A^TWA^{1-\frac{2}{p}}A \preceq \frac{1}{1 - \epsilon} \left(\frac{m}{\epsilon}\right)^\alpha A^TWA^{1-\frac{2}{r}}A = \frac{(1+\alpha)^{1+\alpha}}{\alpha^\alpha} m^\alpha A^TWA^{1-\frac{2}{r}}A. \quad (4.13)$$

\qed

Using Lemma 28 we prove Lemma 26, the main result of this section.

**Proof of Lemma 26.** Note that

$$\sigma(W^{\frac{1}{2}}A)x_iw_i^{-1} = e_i^TA(A^TWA^{-\frac{2}{r}}A)^{-1}A^Te_iw_i^{-\frac{2}{r}}.$$ (4.13)
Applying Lemma 28 yields that
\[
A^\top W^{1-\frac{2}{p}} A \succeq \frac{1}{(1 + \alpha)(1 + \frac{1}{\alpha})m^\alpha} A^\top W^{1-\frac{2}{p}} A \text{ where } \alpha = \frac{2}{p} - \frac{2}{r}.
\]

Consequently, for all \(i \in [m]\) it follows that
\[
e_i^\top A(A^\top W^{1-\frac{2}{p}} A)^{-1} A^\top e_i w_i^{-\frac{2}{p}} \leq (1 + \alpha) ((1 + \alpha) m) \alpha e_i^\top A(A^\top W^{1-\frac{2}{p}} A)^{-1} A^\top e_i w_i^{-\frac{2}{p}}. \tag{4.14}
\]

Further, since \(w = \sigma(W^{\frac{1}{2}-\frac{1}{p}} A)\) we have that
\[
e_i^\top A(A^\top W^{1-\frac{2}{p}} A)^{-1} A^\top e_i w_i^{-\frac{2}{p}} = w_i^{\frac{2}{p} - 1} \sigma_i(W^{\frac{1}{2}-\frac{1}{p}} A) w_i^{-\frac{2}{p}} = w_i^\alpha \tag{4.15}
\]

Additionally, since \(w_i^{1-\frac{2}{p}} A^\top e_i A \preceq A^\top W^{1-\frac{2}{p}} A\) we have \(w_i^{1-\frac{2}{p}} A^\top e_i A(A^\top W^{1-\frac{2}{p}} A)^{-1} A^\top e_i \leq 1\) and
\[
e_i^\top A(A^\top W^{1-\frac{2}{p}} A)^{-1} A^\top e_i w_i^{-\frac{2}{p}} \leq w_i^{-1}. \tag{4.16}
\]

Combining (4.13), (4.14), (4.15), and (4.16) yields
\[
\sigma(W^{1-\frac{2}{p}} A)_i w_i^{-1} \leq \min \left\{ (1 + \alpha) \left( \left( 1 + \frac{1}{\alpha} \right) m \right)^\alpha w_i^\alpha, w_i^{-1} \right\} \leq (1 + \alpha)^{\frac{1}{1+\alpha}} \left( \left( 1 + \frac{1}{\alpha} \right) m \right)^\frac{\alpha}{1+\alpha}.
\]

where we used that \(\min\{ax^b, cx^c\} \leq ax^{\frac{b+c}{c}}\) for \(a \geq 1, b \geq c,\) and \(x \in [0, 1]\). The final inequality follows from the fact that if we let \(f(\alpha) \overset{\text{def}}{=} (1 + \alpha)^\frac{1}{1+\alpha}(1 + \frac{1}{\alpha})^{\frac{\alpha}{1+\alpha}}\) then \(f(\alpha) \leq 2\) for all \(\alpha \geq 0\) as the concavity of log shows that
\[
\log f(\alpha) = \frac{\log(1 + \alpha)}{1 + \alpha} + \frac{\alpha \cdot \log(1 + (1/\alpha))}{1 + \alpha} \leq \log \left( \frac{(1 + \alpha) + \alpha \cdot 1 + (1/\alpha)}{1 + \alpha} \right) = \log 2.
\]

\[\square\]

4.4 Weight Function

Here, we show that the Lewis weights for suitable \(p\) are a valid weight function (Definition 12). The main result of this section is the following theorem which bounds the weight function parameters. This theorem follows almost immediately from the calculations in Section 4. To obtain our fastest algorithms for linear programming, we choose \(p = 1 - 1/\log m\).

**Theorem 29.** For any \(p \in (0, 1)\) the weight function \(g : \Omega^o \to \mathbb{R}^m_{\geq 0}\) defined for all \(x \in \mathbb{R}^m_{\geq 0}\) as
\[
g(x) \overset{\text{def}}{=} w_p(A_x) \quad \text{where} \quad A_x \overset{\text{def}}{=} (\Phi''(x))^{-1/2} A. \tag{4.17}
\]

is a weight function in the context of Definition 12 and satisfies \(c_1(g) \leq n, c_s(g) \leq 2m^{1-p},\) and \(c_k(g) \leq 2^{\frac{2}{1-p}}.\) Further, for \(p = 1 - \frac{1}{\log m},\) we have \(c_1(g) \leq n, c_s(g) \leq 4,\) and \(c_k(g) \leq 2 \log m.\)

**Proof.** To bound the size, \(c_1(g),\) recall that \(w_p(A_x) = \sigma(W^{\frac{1}{2}-\frac{1}{p}} A_x)\) and therefore Lemma 47 implies \(\sum_{i \in [m]} w_p(A_x)_i = n.\) To bound the sensitivity, \(c_s(g),\) note that Lemma 26 and that \(p \leq 1\) yield
\[
e_i^\top G(x)^{-1} A_x \left( A_x^\top G(x)^{-1} A_x \right)^{-1} A_x^\top G(x)^{-1} e_i = g(x)_i^{-1} \sigma(G(x)^{\frac{1}{2}-\frac{1}{p}} A_x)_i \leq 2m^{\frac{\alpha}{1+\alpha}}
\]
where \( \alpha = \frac{2}{p} - \frac{2}{1} = \frac{2}{p}(1 - p) \). As \( \frac{\alpha}{1 + \alpha} = \frac{2-2p}{2} \leq 1 - p \), the bound on \( c_k(g) \) follows.

To bound the consistency, \( c_k(g) \), note that for arbitrary \( h \in \mathbb{R}^m \) and \( w(v) = w_p(\mathbf{V} \mathbf{A}) \) we have

\[
\mathbf{G}(x)^{-1} \mathbf{J}_g(x) \left( \Phi''(x) \right)^{-\frac{1}{2}} h = \mathbf{G}(x)^{-1} \mathbf{J}_w((\Phi''(x))^{-\frac{1}{2}})z
\]

where \( z = -\frac{1}{2}(\Phi''(x))^{-2} \Phi'''(x)h \). By Lemma 25, we have that

\[
\left\| \mathbf{G}(x)^{-1} \mathbf{J}_w((\Phi''(x))^{-\frac{1}{2}})z \right\|_{g(x)} \leq p \left\| (\Phi''(x))^{\frac{1}{2}} z \right\|_{g(x)}
\]

and

\[
\left\| \mathbf{G}(x)^{-1} \mathbf{J}_w((\Phi''(x))^{-\frac{1}{2}})z \right\|_{g(x)} \leq p \left\| (\Phi''(x))^{\frac{1}{2}} z \right\|_{g(x)} + p \left\| (\Phi''(x))^{\frac{1}{2}} z \right\|_{g(x)}.
\]

Combining (4.18), (4.19), (4.20) and using the definition \( \| \cdot \|_{g(x)+\infty} \overset{\text{def}}{=} \| \cdot \|_{\infty} + C_{\text{norm}} \| \cdot \|_{g(x)} \) yields

\[
\left\| \mathbf{G}(x)^{-1} \mathbf{J}_g(x) \left( \Phi''(x) \right)^{-\frac{1}{2}} h \right\|_{g(x)+\infty} \leq p \left\| (\Phi''(x))^{\frac{1}{2}} z \right\|_{\infty} + p(1 + C_{\text{norm}}) \left\| (\Phi''(x))^{\frac{1}{2}} z \right\|_{g(x)}.
\]

Note that \( \left\| \Phi''(x)^{\frac{1}{2}} z \right\|_{i} = \frac{1}{2} \left| \Phi''(x)^{-\frac{1}{2}} \Phi'''(x)h \right|_{i} \leq |h_i| \) by the self-concordance of \( \Phi \). Therefore,

\[
\left\| \mathbf{G}(x)^{-1} \mathbf{J}_g(x) \left( \Phi''(x) \right)^{-\frac{1}{2}} h \right\|_{g(x)+\infty} \leq p \| h \|_{\infty} + p(1 + C_{\text{norm}}) \| h \|_{g(x)} \leq p \left( 1 + \frac{1}{C_{\text{norm}}} \right) \| h \|_{g(x)+\infty}.
\]

Recalling that \( C_{\text{norm}} = 24 \sqrt{c_s(g) c_k(g)} \) and using \( c_s(g) \geq 1 \), the bound of \( c_k(g) = \frac{2}{1-p} \) follows from

\[
p \left( 1 + \frac{1}{C_{\text{norm}}} \right) \leq p + \frac{1}{24 c_k(g)} = 1 - \frac{2}{c_k(g)} + \frac{1}{24 c_k(g)} \leq 1 - \frac{1}{c_k(g)}.
\]

\[ \square \]

5 A Nearly Linear Self-concordant Lewis Weight Barrier

In this section, we construct an \( \tilde{O}(n) \)-self-concordant barrier for the set \( \Omega^o \overset{\text{def}}{=} \{ x \in \mathbb{R}^n \mid \mathbf{A} x > b \} \) for non-degenerate \( \mathbf{A} \in \mathbb{R}^{m \times n} \) and vector \( b \in \mathbb{R}^m \) using \( \ell_q \) Lewis weights.\(^7\) Interestingly, the central path for this barrier is the points \( x^{(t)} \in \mathbb{R}^n \), \( \lambda^{(t)} \in \mathbb{R}_{>0}^m \), and \( s^{(t)} \in \mathbb{R}_{>0}^m \) for \( t > 0 \) satisfying

\[
\lambda^{(t)}_i \cdot s^{(t)}_i = t \cdot w_q(\mathbf{A} x^{(t)})_i \quad \text{for all } i \in [m] \\
\mathbf{A}^\top \lambda^{(t)} = c, \\
\mathbf{A} x^{(t)} + s^{(t)} = b.
\]

where throughout this section we let \( \mathbf{A}_x \overset{\text{def}}{=} \mathbf{S}_x^{-1} \mathbf{A} \) and \( \mathbf{S}_x = \text{Diag}(\mathbf{A} x - b) \). For all \( x \in \Omega^o \) and \( w \in \mathbb{R}_{>0}^m \) we let

\[
f(x, w) \overset{\text{def}}{=} \ln \det \left( \mathbf{A}_x^\top \mathbf{W}^{\frac{2}{q}} \mathbf{A}_x \right) - \left( 1 - \frac{2}{q} \right) \sum_{i=1}^{m} w_i
\]

\(^7\)We use \( q \) throughout rather than \( p \) as in Section 4 to clearly distinguish between the different (but closely related) functions considered in each section.
and define the barrier as
\[ \psi(x) \defeq \begin{cases} \max_{w \in \mathbb{R}^m : w \geq 0} \frac{1}{2} f(x, w) & \text{if } q \geq 2 \\ \min_{w \in \mathbb{R}^m : w \geq 0} \frac{1}{2} f(x, w) & \text{if } q \leq 2 \end{cases}. \] (5.1)

Note with respect to \( w \) the function \( f(x, w) \) is just a scaling of the function we used for defining Lewis weights. Here we need these two cases to maintain that \( f \) is a convex function in \( x \).

Further, note that as \( q \to 0 \), Lemma 63 shows that \( w_q(A)_i = 1 \) (as long as \( A \) is in general position), and in this case \( \psi \) is the log barrier function.

We call this the **Lewis weight barrier function** as by Lemma 22 we can equivalently write
\[ \psi(x) = \ln \det \left( A_x^\top W_x^{1-\frac{2}{q}} A_x \right) \quad \text{where } \mathbf{W}_x = \Diag(w_q(A_x)). \]

The main result of this section is the following theorem which shows that the Lewis weight barrier is a self-concordant barrier. In particular this theorem shows that for \( q = \Theta(\log(m)) \) the Lewis-weight barrier is a \( O(n \log^5 m) \)-self concordant. Further, when \( q = 2 \) this theorem recovers the fact that the volumetric barrier function is a \( O(\sqrt{mn}) \)-self concordant [47, 3].

**Theorem 30.** Let \( \Omega^0 = \{ x : A x > b \} \) denote the interior of non-empty polytope for non-degenerate \( A \). For any \( q > 0 \), \( \psi : \Omega^0 \to \mathbb{R} \) defined in (5.1) is a barrier function such that for all \( x \in \Omega^0 \) and \( h \in \mathbb{R}^n \), we have

1. \( \nabla \psi(x)^\top \nabla^2 \psi(x)^{-1} \nabla \psi(x) \leq n \),

2. \( D^3 \psi[h, h, h] \leq 2v_q \| h \|^3 |\nabla^2 \psi(x)|^{3/2} \) for \( v_q = (q + 2)^{3/2} m^{1/2} + 4 \max\{ q, 2 \}^{2.5} \).

Consequently, \( v_q^3 \psi \) is a \( nv^2 \)-self-concordant barrier function for \( \Omega^0 \).

In the remainder of this section we prove Theorem 30. Leveraging the analysis of Section 4, this is a straightforward but tedious calculus exercise. We split the proof into parts. In Section 5.1, we compute the gradient of the Hessian of \( f \) and \( \psi \) and prove the first item in Theorem 30 (Lemma 32). In Section 5.2, we then prove the second item of Theorem 30 (Lemma 38) by bounding the stability of each component of the Hessian.

**5.1 Notation and Basic Properties of Lewis Weight Barrier**

For brevity, throughout the remainder of this section, we let \( w_x = \arg \max_{w \in \mathbb{R}^m : w \geq 0} \frac{1}{2} f(x, w) \) when \( q \geq 2 \), \( w_x = \arg \min_{w \in \mathbb{R}^m : w \geq 0} \frac{1}{2} f(x, w) \) when \( q \leq 2 \), and \( w_x = \sigma(A_x) \) when \( q = 2 \). We will show in Lemma 31 that \( w_x = w_q(A_x) \) for all \( q \). Further, for all \( x \in \Omega^0 \), we let

\[ P_x \defeq P(W_x^{\frac{1}{2} - \frac{1}{q}} A_x), \quad P_x^{(2)} \defeq P^{(2)}(W_x^{\frac{1}{2} - \frac{1}{q}} A_x), \quad \bar{A}_x \defeq \bar{A}(W_x^{\frac{1}{2} - \frac{1}{q}} A_x), \quad \text{and } \sigma_x \defeq \sigma(W_x^{\frac{1}{2} - \frac{1}{q}} A_x). \]

Leveraging this notation we compute and bound the gradient and Hessian of \( \psi \).

**Lemma 31.** For all \( x \in \Omega^0 \), \( \sigma_x = w_x = w_q(A_x) \) and for \( N_x \defeq 2\bar{A}_x(\mathbf{I} - (1 - \frac{2}{q})\bar{A}_x)^{-1} \), we have
\[ \nabla \psi(x) = -A_x^\top \sigma_x \quad \text{and } \nabla^2 \psi(x) = A_x^\top \Sigma_x^{1/2}(\mathbf{I} + N_x)\Sigma_x^{1/2} A_x \] (5.2)
Further, $N_x$ is a symmetric matrix with $0 \preceq N_x \preceq qI$ and therefore

$$A_x^\top \Sigma_x A_x \preceq \nabla^2 \psi(x) \preceq (1 + q) A_x^\top \Sigma_x A_x.$$  

(5.3)

Proof. By Lemma 50, deferred to the appendix, recalling that $c_q \overset{\text{def}}{=} 1 - \frac{2}{q}$ we have

$$\nabla_x f(x, w) = -2A_x^\top \sigma_{x,w},$$

$$\nabla_w f(x, w) = c_q W^{-1} \sigma_{x,w} - c_q,$$

$$\nabla^2_{xx} f(x, w) = A_x^\top (2 \Sigma_{x,w} + 4A_{x,w}) A_x,$$

$$\nabla^2_{ww} f(x, w) = -c_q W^{-1} (\Sigma_{x,w} - c_q A_{x,w}) W^{-1},$$

$$\nabla^2_{xw} f(x, w) = -2 c_q A_x^\top A_{x,w} W^{-1}.$$  

where $P_{x,w} \overset{\text{def}}{=} P(W^{\frac{1}{2} - \frac{1}{q}} A_x)$, $A_{x,w} \overset{\text{def}}{=} \Lambda(W^{\frac{1}{2} - \frac{1}{q}} A_x)$, and $\sigma_{x,w} \overset{\text{def}}{=} \sigma(W^{\frac{1}{2} - \frac{1}{q}} A_x)$.

Consequently, $f(x, w)$ is concave in $w$ when $q > 2$, convex in $w$ when $q < 2$ and each case the optimizer is in the interior of the set $\{ w_i \geq 0 \}$ by Lemma 22. Further, whenever $q \neq 2$ the optimality conditions imply that $\nabla_w f(x, w_x) = 0$ and considering the $q = 2$ case directly we see that in all cases $\sigma_x = w_x = w_q(A_x)$.

For $q \neq 2$ taking the derivative of $\nabla_w f(x, w_x) = 0$ with respect to $x$ yields that for $w(x) \overset{\text{def}}{=} w_x$,

$$\nabla^2_{ww} f(x, w_x) + \nabla^2_{ww} f(x, w_x) J_w(x) = 0.$$

Since $\nabla^2_{ww} f(x, w_x)$ is invertible, we have that in this case

$$J_w(x) = - (\nabla^2_{ww} f(x, w_x))^{-1} \nabla^2_{ww} f(x, w_w).$$

Using that $\psi(x) = \frac{1}{2} f(x, w_x)$ and taking the derivative of $x$ on both sides, we have

$$2 \nabla \psi(x) = \nabla_x f(x, w_x) + J_w(x)^\top \nabla_w f(x, w_x) = \nabla_x f(x, w_x) = -2A_x^\top \sigma_x$$

(5.4)

where we used that $\nabla_w f(x, w_x) = 0$ by optimality. Next, taking the derivative again yields that

$$2 \nabla^2 \psi(x) = \nabla^2_{xx} f(x, w_x) + \nabla^2_{xw} f(x, w_x) J_w(x)$$

$$= \nabla^2_{xx} f(x, w_x) - \nabla^2_{xw} f(x, w_x) \left( \nabla^2_{ww} f(x, w_x) \right)^{-1} \left( \nabla^2_{ww} f(x, w_x) \right)$$

Substituting in the computed values for $\nabla^2_{xx} f(x, w_x)$, $\nabla^2_{xw} f(x, w_x)$, $\nabla^2_{ww} f(x, w_x)$ and using that $\Sigma_x = W_x$ then yields that

$$\nabla^2 \psi(x) = A_x^\top (\Sigma_x + 2A_x) A_x + 2 c_q A_x^\top A_x (\Sigma_x - c_q A_x) A_x \Sigma_x^{-1} A_x A_x$$

(5.5)

Further, since when $q = 2$ we have $\psi(x) = \frac{1}{2} f(x, w)$ for any $w \in \mathbb{R}^n_{\geq 0}$ and $c_q = 0$ we see that (5.4) and (5.5) are correct for all $q > 0$. Rearranging, scaling, and leveraging that $\Sigma_x$ is PD (i.e. all leverage scores are positive) yields

$$\nabla^2 \psi(x) = A_x^\top \Sigma_x^{1/2} \left( I + 2A_x^\top + 2 c_q A_x (I - c_q A_x)^{-1} A_x \right) \Sigma_x^{1/2} A_x.$$  

Now, note that $0 \preceq A \preceq I$ and $c_q \in (-\infty, 1)$ for $q \in (0, \infty)$ and therefore no eigenvalue of $A$ has value $1/c_q$. Since, $x + c_q x^2 (1-c_q x)^{-1} = x (1-c_q x)^{-1}$ for $x \neq 1/c_q$ and $A$ and $I$ trivially commute we have that $\nabla^2 \psi(x) = A_x^\top \Sigma_x^{1/2} (I + N_x) \Sigma_x^{1/2} A_x$ as desired. Further, this implies that $N_x$ is symmetric with all eigenvalues in the range $[0, q]$ (see e.g. (4.9)), proving (5.3).  

□
Using Lemma 31 we can immediately bound $\nabla \psi(x)^{\top} \nabla^2 \psi(x)^{-1} \nabla \psi(x)$.

**Lemma 32.** For all $x \in \Omega^n$, we have $\nabla \psi(x)^{\top} \nabla^2 \psi(x)^{-1} \nabla \psi(x) \leq n$.

**Proof.** Since $\nabla \psi(x) = -A_x^{\top} \sigma_x$ and $\nabla^2 \psi(x) \succeq A_x^{\top} \Sigma_x A_x$ by Lemma 31 we have.

$$\nabla \psi(x)^{\top} \nabla^2 \psi(x)^{-1} \nabla \psi(x) \leq \sigma_x^{\top} A_x \left( A_x^{\top} \Sigma_x A_x \right)^{-1} A_x^{\top} \sigma_x = 1^{\top} \Sigma_x^{1/2} P \Sigma_x^{1/2} 1$$

where $P \overset{\text{def}}{=} \Sigma_x^{1/2} A_x \left( A_x^{\top} \Sigma_x A_x \right)^{-1} A_x^{\top} \Sigma_x^{1/2}$. Since $P$ is a projection matrix, $P \preceq I$ (Lemma 47) and

$$\nabla \psi(x)^{\top} \nabla^2 \psi(x)^{-1} \nabla \psi(x) \leq 1^{\top} \Sigma_x 1 = \sum_{i \in [m]} [\sigma_x]_i = n.$$

\[\square\]

### 5.2 Stability of Lewis Weight Barrier

Here we bound the directional derivatives of the barrier and show that they are not too large. Lemma 38 proved in this section, combined with Lemma 32 of the previous section immediately prove Theorem 30, bounding the self-concordance of $\psi$.

Throughout this section, to simplify the notation, we fix an arbitrary point $x \in \Omega^n$ and a direction $h \in \mathbb{R}^n$ and define $x_t \overset{\text{def}}{=} x + th$, $s_t = A x_t - b$, and $A_t = A x_t$ and further define $w_t$, $W_t$, $\Sigma_t$, $P_t^{(2)}$, $A_t$, $\Lambda_t$, and $N_t$ (Lemma 31) analogously.

First, we bound the derivatives of the slacks and weights in the following Lemma 33 and 34.

**Lemma 33.** For all $x \in \Omega^n$ and $h \in \mathbb{R}^n$ we have

$$\left\| S_t^{-1} \frac{d}{dt} s_t \right\|_{W_t, A_t} \leq \|h\|_{A_t^{\top} W_t A_t} \text{ and } \left\| S_t^{-1} \frac{d}{dt} s_t \right\|_{\infty} \leq \sqrt{2m^{1/2}} \|h\|_{A_t^{\top} W_t A_t}.$$

**Proof.** Since $S_t^{-1} \frac{d}{dt} s_t = A_t h$ we have $\left\| S_t^{-1} \frac{d}{dt} s_t \right\|_{W_t} = \|h\|_{A_t^{\top} W_t A_t}$. For the second inequality note that by Cauchy Schwarz,

$$\left\| S_t^{-1} \frac{d}{dt} s_t \right\|_{\infty} = \|A_t h\|_{\infty} = \max_{i \in [m]} \langle e_i, A_t h \rangle = \max_{i \in [m]} \left| \langle (A_x^{\top} W_x A_x)^{-1/2} A_x^{\top} A_x (A_x^{\top} W_x A_x)^{-1/2} h \rangle \right|$$

$$\leq \sqrt{\max_{i \in [m]} [A_x (A_x^{\top} W_x A_x)^{-1} A_x^{\top}]_{ii} \|h\|_{A_x^{\top} W_x A_x}}.$$  

The result follows that Lemma 26 shows

$$\max_{i \in [m]} [A_x (A_x^{\top} W_x A_x)^{-1} A_x^{\top}]_{ii} = \max_{i \in [m]} \sigma(W_x^{1/2} A_x)_{i} [W_x]_{i}^{-1} \leq 2m^{1/2}.$$

\[\square\]

**Lemma 34.** For all $x \in \Omega^n$ and $h \in \mathbb{R}^n$ we have

$$\left\| W_t^{-1} \frac{d}{dt} w_t \right\|_{W_t} \leq q \|h\|_{A_t^{\top} W_t A_t} \text{ and } \left\| W_t^{-1} \frac{d}{dt} w_t \right\|_{\infty} \leq q \left( \sqrt{2} \cdot m^{1/2} + \max \left\{ \frac{q}{2}, 1 \right\} \right) \|h\|_{A_t^{\top} W_t A_t}.$$

\[29\]
Proof. Since that $w_t = w_q(S_t^{-1}A_t)$, chain rule, Lemma 25, and Lemma 33 shows that the function $p(x) \equiv w_q(S_x^{-1}A_t)$ satisfies

$$\|J_p(x_t)h\|_{W_t^{-1}} \leq q \| (S_t^{-1})^{-1} S_t^{-2} \frac{d}{dt} s_t \|_{W_t^{-1}} = q \| S_t^{-1} \frac{d}{dt} s_t \|_{W_t^{-1}} = q \| h \|_{A_t^T W_t A_t}.$$ 

The same tools also show that

$$\| W_t^{-1} J_p(x_t)h \|_{\infty} \leq q \| (S_t^{-1})^{-1} S_t^{-2} \frac{d}{dt} s_t \|_{\infty} + q \cdot \max \left\{ \frac{q}{2}, 1 \right\} \cdot \left( \| S_t^{-1} \|_{W_t^{-1}} - \| S_t^{-2} \|_{W_t^{-1}} \right).$$

Since $\frac{d}{dt} w_t = J_p(x_t)h$ the result follows.

Now, recall that by Lemma 31 we have $\nabla^2 \psi(x) = A_t^T \Sigma_t^{1/2} (I + N_t) \Sigma_t^{1/2} A_t$ where $N_t \equiv 2A_t(I - c_q \hat{A}_t)^{-1}$ and $c_q = 1 - (2/q)$. Since we have already bounded the stability of $A_t$ and $\Sigma_t$ all that remains is to bound the stability of $\Lambda_t$ and leverage this to bound the stability $N_t$ and $\Sigma_t^2(x_t)$.

To simplify these calculation for all $t > 0$ and $\alpha \in \mathbb{R}$ we define $z_{t, \alpha} \in \mathbb{R}^n$ be defined for all $i \in [n]$ by $[z_{t, \alpha}]_i = \frac{d}{dt} \ln ([w_t]_i / [s_t]_i)$ and $Z_{t, \alpha} \equiv \text{Diag}(z_{t, \alpha})$. We will repeatedly use the fact

$$\frac{d}{dt} W_t^\alpha S_t^{-1} = W_t^\alpha S_t^{-1} \frac{d}{dt} \ln(W_t^\alpha S_t^{-1}) = W_t^\alpha S_t^{-1} Z_{t, \alpha}.$$ \hspace{1cm} (5.6)

In the following lemma we bound $z_{t, \alpha}$ and use this to simplify these derivative bounds.

**Lemma 35.** For all $x \in \Omega^\circ$ and $h \in \mathbb{R}^n$ we have and $z_{t, \alpha} \in \mathbb{R}^n$ defined for all $i \in [n]$ by $[z_{t, \alpha}]_i = \frac{d}{dt} \ln ([w_t]_i / [s_t]_i)$ we have that

$$\| z_t \| \Sigma \leq (\| \alpha \| q + 1) \| h \| \nabla^2 \psi(x_t) \text{ and } \| z_t \| \Sigma \leq \left( (\| \alpha \| q + 1) \sqrt{2m} \frac{1}{\sqrt{2}} + q |\alpha| \max \left\{ \frac{q}{2}, 1 \right\} \right) \| h \| \nabla^2 \psi(x_t)$$

**Proof.** Note that $[z_{t, \alpha}]_i = \alpha \cdot \left( \frac{d}{dt} [w_t]_i / [s_t]_i \right) - \frac{d}{dt} [s_t]_i / [s_t]_i$ and consequently the result therefore follows from triangle inequality, Lemma 33, Lemma 34, and $A_t^T W_t A_t \leq \nabla^2 \psi(x_t)$.

Using this we can bound the stability of $A_t$

**Lemma 36.** For all $x \in \Omega^\circ$ and $h \in \mathbb{R}^n$ we have

$$\left\| \frac{d}{dt} A_t \right\|_2 \leq \max\{3q, 16\} \| h \| \nabla^2 \psi(x_t).$$

**Proof.** Let $Q_t \equiv W_t^{-1/4} P_t W_t^{-1/4}$. Since $W_t = \Sigma_t$ this implies that

$$\Lambda_t = I - \Sigma_t^{-1/2} P_t^{(2)} \Sigma_t^{-1/2} = I - Q_t^{(2)}.$$ 

Now, let, $z_{t, \alpha}$ be defined as in Lemma 35 and let $Z_{t, \alpha} \equiv \text{Diag}(z_{t, \alpha})$. Since

$$\frac{d}{dt} A_t^T W_t^{-1/2} A_t^{-1} = -A_t^T W_t^{-1/2} A_t^{-1} \frac{d}{dt} \left[ A_t^T W_t^{-1/2} A_t \right] \left( A_t^T W_t^{-1/2} A_t \right)^{-1}$$
and
\[
\frac{d}{dt} S_{t}^{-2} W_{t}^{1-\frac{q}{2}} = \frac{d}{dt} \left[ \left( S_{t}^{-1} W_{t}^{1-\frac{q}{2}} \right)^{2} \right] = 2S_{t}^{-2} W_{t}^{1-\frac{q}{2}} z_{t, \frac{1}{2} - \frac{1}{q}}
\]
(\text{using (5.6)}), we have that
\[
\frac{d}{dt} [Q]_{ij} = \frac{d}{dt} e_{i}^{\top} W_{t}^{1-\frac{1}{q}} A_{t} \left( A_{t}^{\top} W_{t}^{1-\frac{q}{2}} A_{t} \right)^{-1} A_{t}^{\top} W_{t}^{1-\frac{1}{q}} e_{j} - [\bar{z}_{t, \frac{1}{2} - \frac{1}{q}}]_{ij} + Q_{ij} [\bar{z}_{t, \frac{1}{2} - \frac{1}{q}}]_{ij} - 2 \left[ Q_{ij} W_{t}^{1/4} z_{t, \frac{1}{2} - \frac{1}{q}} W_{t}^{1/4} Q_{ij} \right].
\]
Consequently,
\[
\frac{d}{dt} Q_{t} = Z_{t, \frac{1}{2} - \frac{1}{q}} Q_{t} + Q_{t} Z_{t, \frac{1}{2} - \frac{1}{q}} - 2Q_{t} W_{t}^{1/4} z_{t, \frac{1}{2} - \frac{1}{q}} W_{t}^{1/4} Q_{t}
\]
and by chain rule we have that
\[
\frac{d}{dt} Q_{t}^{(2)} = 2Q_{t} \circ \left[ \frac{d}{dt} Q_{t} \right] = 2Z_{t, \frac{1}{2} - \frac{1}{q}} Q_{t}^{(2)} + 2Q_{t}^{(2)} Z_{t, \frac{1}{2} - \frac{1}{q}} - 4 \left[ Q_{t} \circ Q_{t} W_{t}^{1/4} z_{t, \frac{1}{2} - \frac{1}{q}} W_{t}^{1/4} Q_{t} \right].
\]
Therefore, for all $y \in \mathbb{R}^{n}$ we have
\[
y^{\top} \left[ \frac{d}{dt} Q_{t}^{(2)} \right] y = 4y^{\top} \Sigma_{t}^{1/2} Z_{t, \frac{1}{2} - \frac{1}{q}} P_{t}^{\top} \Sigma_{t}^{-1/2} y - 4y^{\top} \left[ P_{t} \circ \Sigma_{t}^{-1/2} P_{t} Z_{t, \frac{1}{2} - \frac{1}{q}} P_{t} \Sigma_{t}^{-1/2} \right] y
\]
Applying Lemma 47 yields
\[
\left| y^{\top} \left[ \frac{d}{dt} Q_{t}^{(2)} \right] y \right| \leq 4\left\| \Sigma_{t}^{-1/2} y \right\| \left\| z_{t, \frac{1}{2} - \frac{1}{q}} \left\| \Sigma_{t} + 4\left\| \Sigma_{t}^{-1/2} y \right\| ^{2} \left\| z_{t, \frac{1}{2} - \frac{1}{q}} \left\| \Sigma_{t} \right\|.\right.
\]
Since, $\left\| \Sigma_{t}^{-1/2} y \right\| \Sigma_{t} = \left\| y \right\|_{2}$ applying Lemma 35 then yields that
\[
\left\| \frac{d}{dt} Q_{t}^{(2)} \right\| _{2} \leq 4 \left( \left( \frac{1}{4} - \frac{1}{q} \right) + 1 \right) + \left( \left( \frac{1}{2} - \frac{1}{q} \right) + 1 \right) \left\| h \right\| \nabla^{2} \psi(x_{t}) \right.
\]
Since $|q-4| + |2q-4| + 8 \leq \max \{ 3q, 16 \}$ and $\frac{d}{dt} Q_{t}^{(2)} = \frac{d}{dt} \bar{A}_{t}$ the result follows. \hfill \Box

Using this we can now bound the stability of $N_{t}$.

\textbf{Lemma 37.} For all $x \in \Omega^{o}$ and $h \in \mathbb{R}^{n}$ we have
\[
\left\| \left( I + N_{t} \right)^{-\frac{1}{2}} \frac{d}{dt} N_{t} \left( I + N_{t} \right)^{-\frac{1}{2}} \right\| _{2} \leq 2 \max \left\{ 1, \frac{q}{2} \right\} \left\| \frac{d}{dt} \bar{A}_{t} \right\| _{2} \leq 4 \max \{ q, 2 \} \left\| h \right\| \nabla^{2} \psi(x_{t}) \right.
\]
\textbf{Proof.} Recall that $N_{t} = 2\bar{A}_{t} \left( I - c_{q} \bar{A}_{t} \right)^{-1}$ for $c_{q} = 1 - \frac{2}{q}$. Direct calculation yields
\[
\frac{d}{dt} N_{t} = 2 \left[ \frac{d}{dt} \bar{A}_{t} \right] \left( I - c_{q} \bar{A}_{t} \right)^{-1} - 2 \bar{A}_{t} \left( I - c_{q} \bar{A}_{t} \right)^{-1} \left( -c_{q} \frac{d}{dt} \bar{A}_{t} \right) \left( I - c_{q} \bar{A}_{t} \right)^{-1}
\]
\[
= 2 \left[ I + c_{q} \bar{A}_{t} \left( I - c_{q} \bar{A}_{t} \right)^{-1} \right] \cdot \left[ \frac{d}{dt} \bar{A}_{t} \right] \cdot \left[ I - c_{q} \bar{A}_{t} \right]^{-1}
\]
\[
= 2 \cdot \left[ I - c_{q} \bar{A}_{t} \right]^{-1} \cdot \left[ \frac{d}{dt} \bar{A}_{t} \right] \cdot \left[ I - c_{q} \bar{A}_{t} \right]^{-1}.
\]
Now, since \( c_q \in (-\infty, 1) \) and \( 0 \leq \tilde{A} \leq I \) we have

\[
I + N_t = (I - c_q A_t)^{-1}(I + (2 - c_q) \tilde{A}) \succeq (I - c_q A_t)^{-1}
\]

and therefore \( \|(I + N_t)^{-1/2}(I - c_q A_t)^{-1/2}\|_2 \leq 1 \). Further as \( \|(I - c_q A_t)^{-1}\|_2 \leq \max\{1, \frac{q}{2}\} \) (see, e.g. (4.10)), the result follows from Lemma 36.

Now we can combine everything to prove the desired result

**Lemma 38.** For all \( x \in \Omega^c \) and \( h, y \in \mathbb{R}^n \) we have

\[
|y^T \left[ \frac{d}{dt} \nabla^2 \psi(x_t) \right] y| \leq \left( (q + 2)^{3/2} \sqrt{2m \frac{1}{\tau_t^2}} + 6 \max\{q, 2\} \right) \|h\| \|\nabla^2 \psi(x_t)\| \left[ y^T \nabla^2 \psi(x_t) y \right].
\]

**Proof.** Since \( \nabla^2 \psi(x) = A_t^\top \Sigma_t^{1/2}(I + N_t) \Sigma_t^{1/2} A_x \) by Lemma 31 we have

\[
y^T \left[ \frac{d}{dt} \nabla^2 \psi(x_t) \right] y = 2y^T A_t^\top W_t^{1/2} z_{t,1/2} [I + N_t] W_t^{1/2} A_t y + y^T A_t^\top W_t^{1/2} \left[ \frac{d}{dt} N_t \right] W_t^{1/2} A_t y.
\]

Cauchy Schwarz, and the facts that \( I + N_t \) is PSD and \( \nabla^2 \psi(x) = A_t^\top \Sigma_t^{1/2}(I + N_t) \Sigma_t^{1/2} A_x \) yield

\[
|y^T A_t^\top W_t^{1/2} z_{t,1/2} [I + N_t] W_t^{1/2} A_t y| \leq \|W_t^{1/2} A_t y\|_2 \|z_{t,1/2} [I + N_t] W_t^{1/2} A_t y\|_2 \\
\leq \|z_{t,1/2}\|_2 \sqrt{\|I + N_t\|_2} \|y\|_2 \|\nabla^2 \psi(x_t)\|.
\]

and

\[
|y^T A_t^\top W_t^{1/2} \left[ \frac{d}{dt} N_t \right] W_t^{1/2} A_t y| \leq \|y\|^2 \|\nabla^2 \psi(x_t)\| \left( I + N_t \right)^{-\frac{1}{2}} \frac{d}{dt} N_t \left( I + N_t \right)^{-\frac{1}{2}} \|h\| \|\nabla^2 \psi(x_t)\| \left[ y^T \nabla^2 \psi(x_t) y \right]
\]

Now, by Lemma 35

\[
\|z_{t,1/2}\|_2 = \|z_{t,1/2}\|_\infty \leq \left( (q + 1)^2 \sqrt{2m \frac{1}{\tau_t^2}} + \frac{q}{2} \max\{q, 2\} \right) \|h\| \|\nabla^2 \psi(x_t)\|.
\]

Further, since \( \|I + N_t\|_2 \leq 1 + q \) combining and applying Lemma 37 yields that \( |y^T \left[ \frac{d}{dt} \nabla^2 \psi(x_t) \right] y| \) is bounded by

\[
(2 \sqrt{1 + q} \left( (q + 1)^2 \sqrt{2m \frac{1}{\tau_t^2}} + \frac{q}{2} \max\{q, 2\} \right) + 4 \max\{q, 2\}^2 \|h\| \|\nabla^2 \psi(x_t)\| \left[ y^T \nabla^2 \psi(x_t) y \right]
\]

and the result follows by basic calculations. \( \square \)

### 6 Efficient Algorithms

In this section we show how to leverage the results of the previous sections to obtain efficient algorithms and derive the main results of this paper. In Section 6.1 we prove Theorem 1 and Theorem 43, our main results on linear programming, in Section 6.2 we prove Theorem 2, our main result on minimum cost maximum flow, and in Section 6.3 we prove Theorem 3 and a more general Theorem 46, our main results on a polynomial time computable nearly-universal self-concordant barrier. The algorithms in this section make critical use of algorithms for computing Lewis weights provided an analyzed in Appendix B and are stated as needed.
6.1 Linear Programming Algorithm

Here we show how to combine the results of the preceding sections to obtain our efficient linear programming algorithm and prove Theorem 1 and Theorem 43. Our algorithm uses the following result regarding approximately computing Lewis weights proved in Appendix B.

**Theorem 39** (Approximate Weight Computation). Let \( A \in \mathbb{R}^{m \times n} \) be non-degenerate and let \( T_w \) and \( T_d \) denote the work and depth needed to compute \((A^TDA)^{-1}z\) for arbitrary positive diagonal matrix \( D \) and vector \( z \). For all \( \epsilon \in (0, 1) \), \( p \in (0, 4) \), \( w^{(0)} \in \mathbb{R}_{>0}^m \) with \( \|w^{(0)}\|_\infty \leq 2^{-20}p^2(4-p) \), the algorithm \( \text{computeApxWeight}(x, w^{(0)}, \epsilon) \) can be implemented to return \( w \) that with high probability in \( n \) \( \|w_p(A)^{-1}(w_p(A) - w)\|_\infty \leq \epsilon \) in \( O(p^{-1}(4-p)^{-2}\epsilon^2 \log^2(n/(p\epsilon))) \) steps each of which can be implemented in \( O(\text{nnz}(A) + T_w) \) work and \( O(T_d) \) depth.

Without \( w^{(0)} \) the algorithm \( \text{computeInitialWeight}(A, p, \epsilon) \) (Algorithm 7) can be implemented to have the same guarantee with \( O(\sqrt{n}(4-p)^{-3}p^{-3}) \log \frac{m}{n} \log^2(n/(p\epsilon)) \) steps of the same cost.

Leveraging this result in Algorithm 2 we give the procedure, \( \text{pathFollowing} \), for approximately following the weighted central path induced by Lewis weights. In Theorem 40 provide its guarantees.

**Algorithm 2:** \((x^{(final)}, w^{(final)}) = \text{pathFollowing}(x, w, t_{start}, t_{end}, \epsilon)\)

\[
t = t_{start}, K = \frac{1}{16\epsilon^2}, \alpha = \frac{R}{1600\sqrt{n}\log^3 m} \quad \text{where} \quad R \quad \text{is defined in Theorem 19.}
\]

```
repeat
\( (x^{(new)}, w^{(new)}) = \text{centeringInexact}(x, w, K) \) where \( \text{computeApxWeight} \) to approximate \( g(x) \) (defined in Section 4.4).
\( t \leftarrow \text{median}(1 - \alpha \cdot t, t_{end}, 1 + \alpha \cdot t). \)
\( x \leftarrow x^{(new)}, w \leftarrow w^{(new)}. \)
until \( t = t_{end}; \)
for \( i = 1, \ldots, 4c_k \log \left( \frac{1}{\epsilon} \right) \) do
\( (x, w) = \text{centeringInexact}(x, w, K) \) where it use the function \( \text{computeApxWeight} \) to approximate \( g(x) \).
end
Output: \((x, w)\).
```

**Theorem 40.** Suppose that

\[
\delta_{t_{start}}(x, w) \leq \frac{1}{2^{16}\log^3 m} \quad \text{and} \quad \Phi_{\mu}(\log g(x) - \log w) \leq 36c_1c_sc_k m.
\]

where \( \mu \) is defined in Theorem 19. Let \((x^{(final)}, w^{(new)}) = \text{pathFollowing}(x, w, t_{start}, t_{end})\), then with high probability in \( n \),

\[
\delta_{t_{end}}(x^{(final)}, w^{(final)}) \leq \epsilon \quad \text{and} \quad \Phi_{\mu}(\log g(x^{(final)}) - \log w^{(final)}) \leq 36c_1c_sc_k m.
\]

Furthermore, \( \text{pathFollowing}(x, w, t_{start}, t_{end}) \) can be implemented

\[
O\left(\sqrt{n}\log^{13} m \cdot \kappa \cdot T_w\right) \quad \text{work and} \quad O\left(\sqrt{n}\log^{13} m \cdot \kappa \cdot T_d\right) \quad \text{depth}
\]

where \( \kappa = \left|\log \frac{t_{end}}{t_{start}}\right| + \log \frac{1}{\epsilon} \) and \( T_w \) and \( T_d \) are the work and depth needed to compute \((A^TDA)^{-1}q\) for input positive diagonal matrix \( D \) and vector \( q \).
Proof. We first note that
\[ R = \frac{1}{768c_k^2 \log(36c_1c_km)} = \frac{1}{3072 \log^2 m \log(288nm \log m)} \geq \frac{1}{216 \log^3 m}. \]

This algorithm maintains the invariant that \( \delta_t(x,w) \leq R \) and \( \Phi_\mu(\log g(x) - \log w) \leq 36c_1c_km \) on each iteration. By the definition of \( \Phi_\mu \), \( \mu \) and \( K \), we have \( \| \log g(x) - \log w \|_\infty \leq R \) and by (3.13), we have \( \| \log g(x) - \log g(x^{\text{new}}) \|_\infty \leq R \). Therefore, we have
\[ \| \log g(x^{\text{new}}) - \log w \| \leq 2R \leq \frac{1}{80(\frac{p}{2} + \frac{2}{\epsilon})} \] (6.1)
where we used the formula of \( R \) and \( p \) at the end. Thus, the weight \( w \) satisfies the condition of Theorem 58 and the algorithm centeringInexact can use the function computeApxWeight to find the approximation of \( g(x^{\text{new}}) \). Consequently,
\[ \delta_t(x^{\text{new}},w^{\text{new}}) \leq \left( 1 - \frac{1}{4c_k} \right) \delta \quad \text{and} \quad \Phi_\mu(\log g(x^{\text{new}}) - \log w^{\text{new}}) \leq 36c_1c_km. \]

Using Lemma 14, (6.1) and Theorem 29, we have
\[ \delta_t^{\text{(new)}}(x^{\text{new}},w^{\text{new}}) \leq (1 + \alpha) \left( 1 - \frac{1}{4c_k} \right) \delta + \alpha \left( 1 + C_{\text{norm}} \sqrt{\|w\|_1} \right) \leq \delta - \frac{\delta}{8c_k} + 100\alpha \sqrt{\|w\|} \log m \leq \delta. \]

Hence, we proved that the invariant. The \( \delta_t < \epsilon \) bounds follows from the last loop.

For the runtime, note that \( R = \Omega(\log^{-3} m) \) and hence \( \alpha = \Omega(n^{-1/2} \log^{-5} m) \). Therefore, the total number of step is \( O(\sqrt{n} \log^5 m \cdot [\|t_{\text{end}}/t_{\text{start}}\| + \log(1/\epsilon)]) \). Finally, each step involves computing projection to the mixed ball and computing Lewis weights. Theorem 62 shows that the projection can be formed in \( O(m \log m) \) time and \( O(\log m) \) depth. Theorem 18 shows that we need to compute Lewis weight with \( 1 \pm R = 1 \pm \Theta(\log^{-3} m) \) multiplicative approximation. Theorem 39 shows that we can compute the Lewis weight using \( O((1/R^2) \log^2 (m/R)) = O(\log^8 m) \) linear systems.

To leverage this result we first provide Lemma 41, which bounds how large a \( \delta_t \) is needed guarantee an approximately optimal solution. Further, in Lemma 42, we show how much the approximate centrality hurts our guarantee. Using these lemmas and the previous section, we conclude by describing our linear programming algorithm, LPSolve, and prove Theorem 1 and Theorem 43.

Lemma 41 ([46, Theorem 4.2.7]). Let \( x^* \in \mathbb{R}^m \) denote an optimal solution to (1.2) and \( x_t = \text{arg min} f_t(x,w) \) for some \( t > 0 \) and \( w \in \mathbb{R}^m_{>0} \). Then the following holds
\[ c^\top x_t(w) < c^\top x^* \leq \frac{\|w\|_1}{t}. \]

Proof. By the optimality conditions of (1.2) we know that \( \nabla_x f_t(x_t(w)) = t \cdot c + w \phi'(x_t(w)) \) is orthogonal to the kernel of \( A^\top \). Furthermore since \( x_t(w) - x^* \in \text{ker}(A^\top) \) we have
\[ (t \cdot c + w \phi'(x_t(w)))^\top (x_t(w) - x^*) = 0. \]
Using that \( \phi'_i(x_t(w)_i) \cdot (x_t^* - x_i(w)_i) \leq 1 \) by Lemma 9 then yields
\[
c^\top (x_t(w) - x^*) = \frac{1}{t} \sum_{i \in [m]} w_i \cdot \phi'_i(x_t(w)_i) \cdot (x_t^* - x_i(w)_i) \leq \frac{\|w\|_1}{t}.
\]
\[
\]

Lemma 42. For \( x \) such that \( \delta_t(x, g(x)) \leq \frac{1}{2^{16} \log^3 m} \) and \( x_t \overset{\text{def}}{=} \arg \min f_t(x, w) \) we have
\[
\left\| \sqrt{\phi''(x_t)} (x - x_t) \right\|_{\infty} \leq 8 \delta_t(x, g(x)).
\]

Proof. We prove this statement via our centering algorithm. We use Theorem 19 with exact weight computation and start with \( x^{(1)} = x \) and \( w^{(1)} = g(x^{(1)}) \). In each iteration, \( \delta_t \) is decreased by a factor of \( 1 - (4c_k)^{-1} \). (3.5) shows that
\[
\| \log (\phi''(x_t)) - \log (\phi''(x^{(k+1)})) \|_{\infty} \leq \left( 1 - 4 \delta_t(x^{(k)}, w^{(k)}) \right)^{-1} \leq e^{8 \delta_t(x^{(k)}, w^{(k)})}.
\]
Therefore, for any \( k \), we have
\[
\| \log (\phi''(x^{(k)})) - \log (\phi''(x_t)) \|_{\infty} \leq e^{8 \sum_{i=1}^{k} \delta_t(x^{(i)}, w^{(i)})} \leq e^{32c_k \delta_t(x^{(1)}, g(x^{(1)}))} \leq 2
\]
where we used that \( \delta_t \) is decreased by a factor of \( 1 - \frac{1}{4c_k} \), \( c_k \leq 2 \log m \) and that \( \delta_t \leq \frac{1}{2^{16} \log^3 m} \).

Using this on (6.2), we have
\[
\left\| \sqrt{\phi''(x_t)} (x^{(1)} - x_t) \right\|_{\infty} \leq 4 \sum_{i=1}^{k} \delta_t(x^{(i)}, w^{(i)}) \leq 8 \delta_t(x^{(1)}, w^{(1)}).
\]

Algorithm 3: \( x^{(\text{final})} = \text{LPSolve}(x_0, \epsilon) \)

\begin{itemize}
\item \textbf{Input:} an initial point \( x_0 \) such that \( A^\top x_0 = b \).
\item \( w = \text{computeInitialWeight}(x_0, \frac{1}{2^{16} \log^3 m}) \), \( d = -w_i \phi'_i(x_0) \).
\item \( t_1 = (2^{26} m^{3/2} U^2 \log^4 m)^{-1} \), \( t_2 = \frac{2m}{\epsilon} \), \( \epsilon_1 = \frac{\epsilon}{2^{16} \log^3 m} \), \( \epsilon_2 = \frac{\epsilon}{8U^2} \).
\item \( (x^{(\text{new})}, w^{(\text{new})}) = \text{pathFollowing}(x_0, w, 1, t_1, \epsilon_1) \) with cost vector \( d \).
\item \( (x^{(\text{final})}, w^{(\text{final})}) = \text{pathFollowing}(x^{(\text{new})}, w^{(\text{new})}, t_1, t_2, \epsilon_2) \) with cost vector \( c \).
\item \textbf{Output:} \( x^{(\text{final})} \).
\end{itemize}

Proof of Theorem 1. By Theorem 39, we know \text{computeInitialWeight} gives an weight
\[
\| G(x)^{-1} (g(x_0) - w) \|_{\infty} \leq \frac{1}{2^{16} \log^3 m} \leq R.
\]
By the definition of \( R \), we have that \( \Phi_\mu (\log g(x_0) - \log w) \leq 36 c_1 c_2 c_k m \) and that \( x_0 \) is the minimum
of
\[
\min_x d^\top x - \sum_i w_i \phi_i(x) \text{ given } A^\top x = b.
\]

Therefore, \((x, w)\) satisfies the assumption of theorem 40 because \(\delta_t = 0\) and \(\Phi_\mu\) is small enough. Hence, we have
\[
\delta^d_{t_1}(x^{(\text{new})}, w^{(\text{new})}) \leq \frac{1}{218 \log^3 \frac{1}{m}} \quad \text{and} \quad \Phi_\mu(\log g(x^{(\text{new})}) - w^{(\text{new})}) \leq 36c_1c_sc_km
\]
where we used the superscript \(d\) to indicate \(\delta\) is defined using the cost vector \(d\). Using this notation and (3.5), we have
\[
\delta^c_{t_1}(x^{(\text{new})}, w^{(\text{new})}) \leq \left\| P_{x^{(\text{new})}, w^{(\text{new})}} \left( \frac{t_1c + w^{(\text{new})}\phi'(x^{(\text{new})})}{\sqrt{\phi''(x^{(\text{new})})}} \right) \right\|_{w + \infty}
\]
\[
\leq \left\| P_{x^{(\text{new})}, w^{(\text{new})}} \left( \frac{t_1d + w^{(\text{new})}\phi'(x^{(\text{new})})}{\sqrt{\phi''(x^{(\text{new})})}} \right) \right\|_{w + \infty} + t_1 \left\| P_{x^{(\text{new})}, w^{(\text{new})}} \left( \frac{c - d}{\sqrt{\phi''(x^{(\text{new})})}} \right) \right\|_{w + \infty}
\]
\[
\leq 2 \cdot \delta^d_{t_1}(x^{(\text{new})}, w^{(\text{new})}) + 100\sqrt{n} \log m \cdot t_1 \cdot \left\| P_{x^{(\text{new})}, w^{(\text{new})}} \left( \frac{c - d}{\sqrt{\phi''(x^{(\text{new})})}} \right) \right\|_{\infty}
\]
(6.3)
where we used Lemma 13 at the end.

Next, we note that for any \(x\) and \(w\), let \(Q = W^{-1}A_x (A_x^\top W^{-1}A_x)^{-1} A_x^\top W^{-1}\), then (3.3) and sensitivity \(c_s \leq 4\) shows that
\[
\left\| P_{x, w} W^{-1} \right\|_{\infty \to \infty} \leq \left( \min_{i \in [m]} w_i \right)^{-1} + \left\| Q \right\|_{\infty \to \infty} \leq m + m \max_{i \in [m]} Q_{ii} \leq 5m.
\]
Putting it into (6.3) gives
\[
\delta^c_{t_1}(x^{(\text{new})}, w^{(\text{new})}) \leq \frac{1}{217 \log^3 \frac{1}{m}} + 500m^3/2 \log m \cdot t_1 \cdot \left\| \frac{c - d}{\sqrt{\phi''(x^{(\text{new})})}} \right\|_{\infty} \leq \frac{1}{216 \log^3 \frac{1}{m}}
\]
where we used that \(\|c - d\|_\infty \leq \|c\|_\infty + \|\phi'(x_0)\|_\infty \leq 2U\) (Lemma 9), \(\min_y \sqrt{\phi''(y)} \geq \frac{1}{\mathcal{B}}\) (Lemma 8) and that we have chosen \(t_1\) small enough.

Hence, \((x^{(\text{new})}, w^{(\text{new})})\) satisfy the assumption of Theorem 40 for the original cost function \(c\). Now, we only need to bound how large \(t_2\) should be and how small \(\epsilon_2\) should be in order to get \(x\) such that \(c^\top x \leq \text{OPT} + \epsilon\). By Lemma 41 and \(\|w^{(\text{final})}\| \leq 2m\), we have
\[
c^\top x t_2 \leq \text{OPT} + \frac{2m}{t_2}.
\]
Also, Lemma 42 shows that we have
\[
\left\| \sqrt{\phi''(x_{t_2})} \left( x^{(\text{final})} - x_{t_2} \right) \right\|_{\infty} \leq 8\epsilon_2.
\]
Using \(\min_y \sqrt{\phi''(y)} \geq \frac{1}{\mathcal{B}}\), we have \(\|x^{(\text{final})} - x_{t_2}\|_{\infty} \leq 8\epsilon_2 U\) and hence our choice of \(t_2\) and \(\epsilon_2\) yields
\[
c^\top x^{(\text{final})} \leq \text{OPT} + \frac{2m}{t_2} + 8\epsilon_2 U^2 \leq \text{OPT} + \epsilon.
\]
Theorem 43. Let \( x_0 \in \Omega \) defined as \( \{ A^T x = b, x \geq 0 \} \) for non-degenerate \( A \in \mathbb{R}^{m \times n} \). There is an algorithm that finds \( y \in \mathbb{R}^n \) with \( A y \leq c \) and \( b^T y \geq \max_{A y \leq c} b^T y - \epsilon \) with constant probability in

\[
O \left( \sqrt{n} \log^{13} m \cdot \log \left( \frac{mU}{\epsilon} \right) \cdot \mathcal{T}_w \right) \quad \text{work and} \quad O \left( \sqrt{n} \log^{13} m \cdot \log \left( \frac{mU}{\epsilon} \right) \cdot \mathcal{T}_d \right) \quad \text{depth}
\]

where \( U \) defined as \( \max \{ \text{diam}(\Omega), \| c \|_\infty, \| 1/ x_0 \|_\infty \} \), \( \text{diam}(\Omega) \) is the diameter of \( \Omega \), and \( \mathcal{T}_w \) and \( \mathcal{T}_d \) is the work and depth needed to compute \( (A^T D A)^{-1} q \) for input positive diagonal matrix \( D \) and vector \( q \).

Proof. Use Algorithm LPSolve to solve the linear program \( \min_{A^T x = b, x \geq 0} c^T x \). Following the proof of Theorem 1 and using \( \phi_i(x_i) = -\log x_i \) for all \( i \), we can find \( x \) and \( w \) such that \( \delta_t(x, w) \leq \frac{1}{t} \) with \( t = (\frac{\Omega}{t})^{O(1)} \) in the time same work and depth as Theorem 1. Further, (3.5) shows that \( \eta = (A_x^T W^{-1} A_x)^{-1} A_x^T \frac{\nabla f_t(x, w)}{w \sqrt{\phi''(x)}} \) satisfies

\[
\left\| \frac{\nabla f_t(x, w)}{w \sqrt{\phi''(x)}} \right\|_\infty \leq 1.
\]

We will prove that \( y = \frac{q}{t} \) has the desired properties. Since \( \phi''(x) = x_i^{-2} \), we have that

\[
\left\| W^{-1} X \left( t c - \frac{w}{x} - A \eta \right) \right\|_\infty \leq 1.
\]

In particular, we have \( (A y)_i \leq c_i - \frac{w_i}{t x_i} + \frac{w_i}{t x_i} \leq c_i \) for all \( i \in [n] \). Similarly, we have that \( (A y)_i \geq c_i - \frac{w_i}{t x_i} - \frac{w_i}{t x_i} \). Hence, we have

\[
b^T y = x^T A y \geq c^T x - \frac{2}{t} \sum_{i \in [m]} w_i \geq c^T x - \frac{3n}{t}
\]

and picking \( t = \frac{3n}{\epsilon} \) gives the result. \( \Box \)

6.2 Minimum Cost Maximum Flow

Here we show how to use the interior point method of the previous Section 6.1 to solve the maximum flow problem and the minimum cost flow problem and thereby prove Theorem 2. Formally, the maximum flow and minimum cost flow problems [11] is as follows. Let \( G = (V, E) \) be a connected directed graph where each edge \( e \in E \) has capacity \( c_e > 0 \). We call \( x \in \mathbb{R}^E \) a \( s-t \) flow for \( s, t \in V \) if \( x_e \in [0, c_e] \) for all \( e \) in \( E \) and for each vertex \( v \notin \{ s, t \} \) the amount of flow entering \( v \), i.e. \( \sum_{e=(a,v) \in E} f_e \) equals the amount of flow leaving \( v \), i.e. \( \sum_{e=(v,b) \in E} f_e \). The value of \( s-t \) flow is the amount of flow leaving \( s \) (or equivalently, entering \( t \)). The maximum flow problem is to compute a \( s-t \) flow of maximum value. In the minimum cost maximum flow problem there are costs \( q_e \in \mathbb{R} \) on each edge \( e \in E \) and the goal is to compute a maximum \( s-t \) flow of minimum cost, \( \sum_{e \in E} q_e f_e = q^T f \).

Since the minimum cost flow problem includes the maximum flow problem, we focus on this general formulation. The problem can be written as the following linear program

\[
\min_{0 \leq x \leq c} q^T x \text{ such that } Ax = Fe_t
\]

where \( F \) is the maximum flow value, \( e_t \in \mathbb{R}^{|V|-1} \) is an indicator vector of size \(|V| - 1\) that is non-zero at vertices \( t \) and \( A \) is a \(|V\setminus \{ s\}| \times |E| \) matrix such that for each edge \( e \), we have \( A_{e, \text{head}, e} = 1 \).
and \( A_{e_{tail}} = -1 \). In order words, the constraint \( Ax = Fe_t \) requires the flow to satisfies the flow conversation at all vertices except \( s \) and \( t \) and requires it flows \( F \) unit of flow into \( t \) (and therefore \( F \) out of \( s \)). We assume \( c_e \) and \( q_e \) are integer and \( M \) be the maximum absolute value of \( c_e \) and \( q_e \).

Note that \( \text{rank}(A) = |V| - 1 \) because the graph is connected and hence our algorithm takes only \( \tilde{O}(\sqrt{|V|} \log(U/\epsilon)) \) iterations to compute an \( \epsilon \)-approximate solution to this linear program. However, to solve minimum cost maximum flow with this we need to bound \( U/\epsilon \), compute \( F \), and turn the approximate solution into an exact minimum cost maximum flow. While there are many ways to deal with this issue we consider a different linear program formulation below related to [11].

**Lemma 44.** Given a directed graph \( G = (V, E) \) with integral costs \( q \in \mathbb{Z}^E \) and capacities \( c \in \mathbb{Z}^E_{\geq 0} \) with \( \|q\|_\infty \leq M \) and \( \|c\|_\infty \leq M \) in linear time we can find a new integral cost vector \( \tilde{q} \in \mathbb{Z}^E \) with \( \|\tilde{q}\|_\infty \leq \tilde{M} \) defined \( |E|^2 M^3 \) such that the following modified linear program

\[
\begin{align*}
\text{min} & \quad \tilde{q}^T x + \lambda (1^T y + 1^T z) - 2n\tilde{M}F \\
\text{subject to} & \quad Ax + y - z = Fe_t \\
& \quad 0 \leq x_i \leq c_i, \\
& \quad 0 \leq y_i \leq 4|V|M, \\
& \quad 0 \leq z_i \leq 4|V|M, \\
& \quad 0 \leq F \leq 2|V|M
\end{align*}
\]

with \( \lambda = 440|E|^4 \tilde{M}^2 M^3 \) satisfies the following conditions with constant probability:

1. \( F = |V|M, \ x = \frac{c}{2}, \ y = 2|V|M1 - (A \frac{c}{2})^- + Fe_t, \ z = 2|V|M1 + (A \frac{c}{2})^+ \) is an interior point of the linear program.

2. Given any feasible \( (x, y, z) \) with cost value within \( \frac{1}{16M} \) of the optimum. Then, one can find an exact minimum cost maximum \( s-t \) flow for graph \( G \) with costs \( q \) and capacities \( c \) in \( O(|E|) \) work and \( O(1) \) depth.

3. The linear system of the linear program can be solve in nearly linear time, i.e. for any positive diagonal matrix \( S \) and vector \( b \), it takes

\[
O \left( |E| \log^4 |V| \log(|V|/\eta) \right) \text{ work and } O \left( \log^6 |V| \log(|V|/\eta) \right) \text{ depth}
\]

to find \( x \) such that

\[
\|x - L^{-1}b\|_L \leq \eta \|x\|_L \quad (6.4)
\]

where \( L = [ A \mid I \mid - I \mid - e_t \]S[ A \mid I \mid - I \mid - e_t ]^T \).

**Proof.** By Lemma [11, Lemma 3.13], if we add the cost of every edge by a number uniformly at random from \( \left\{ \frac{1}{4|E|^2 M^2}, \frac{2}{3|E|^2 M^2}, \ldots, \frac{2|E|M}{4|E|^2 M^2} \right\} \). Then with probability at least \( 1/2 \), the new problem has an unique solution and this solution is a solution for the original problem. Applying this reduction and scaling the problem back to integral, we obtain the new cost vector such that the solution is unique.

For 1) Note that \( |V|M \leq y_i \leq 3|V|M \) and \( |V|M \leq z_i \leq 3|V|M \). So, \( (x, y, z) \) is an interior point.

For 2) Let \( \text{Val} = \tilde{q}^T x - 2|V|M\tilde{F} + \lambda (1^T y + 1^T z) \) be the objective value of \( (x, y, z) \). Let \( \text{Opt} \) be the objective value given by the minimum cost maximum flow.
First, we prove the the total excess demand $1^\top y + 1^\top z$ is small. Since $0 \leq F \leq 2|V|M$ and $|q^\top x| \leq |E|\bar{M}M$, we have that
\[ |q^\top x - 2|V|\bar{M}F| \leq 5|E|^2\bar{M}M \] (6.5)
for both the algorithm and for the optimum flow. By assumption, we know that $\text{Val} \leq \text{Opt} + \frac{1}{12M}$, we have that
\[ \lambda(1^\top y + 1^\top z) \leq 11|E|^2\bar{M}M. \]

Using $\lambda = 440|E|^4\bar{M}^2M^3$, we have that the total excess demand $1^\top y + 1^\top z \leq \epsilon = \frac{1}{40|E|^2\bar{M}M^2}$.

To route back the excess demand, we first scale the vector $x, y, z$ and $F$ by a $1 - \epsilon$ factor. Then, we create a spanning tree at $s$. At every vertex $v$, we route the excess demand from $v$ back to the source $s$ in the tree. To route one unit of excess demand at $v$, we pay at most $\sum_{i \in P_v} |\tilde{q}_i|$ where $P_v$ is the path from $s$ to $v$ on the tree. Since $\sum_{i \in P_v} |\tilde{q}_i| \leq |V|\bar{M}$, the cost we pay for routing is at most the potential decrease in the term $\lambda(1^\top y + 1^\top z)$. So the objective value of this new flow is at most
\[ (1 - \epsilon)\text{Val} \leq \text{Val} + 5\epsilon|E|^2\bar{M}M \leq \text{Val} + \frac{1}{12M} \leq \text{Opt} + \frac{1}{6M}, \]
where we used $\text{Val} \leq 5|E|^2\bar{M}M$ due to (6.5). Since we scale the vectors by $1 - \epsilon$ factor, the flow is feasible.

Due to the routing above, we can assume the flow $x$ has no excess demand with $\text{Val} \leq \text{Opt} + \frac{1}{6M}$. However, the flow $x$ may not be the maximum flow. Imagine now, we send the extra flow from $s$ to $t$ to make $x$ maximum. For every unit we send, we decrease the objective by at least $|V|\bar{M}$ due to the term $-2|V|\bar{M}F$ and the fact that the cost of that unit of flow is at most $|V|\bar{M}$. Since $\text{Val} \leq \text{Opt} + \frac{1}{6M}$, we can send at most $\frac{1}{6M\bar{M}}$ amount of extra flow. We call this new $x$ as $\tilde{x}$ and we let $\text{Val}$ be the objective value for this $\tilde{x}$. Note that the procedure above only decrease the objective value. So, we have again $\text{Val} \leq \text{Opt} + \frac{1}{6M}$. Finally, we note that $\tilde{x}$ is a weighted combination of maximum flow from $s$ to $t$. Since the minimum cost solution is unique and the cost are integral, the combined weight contributed by non-minimum-cost flow is at most $\frac{1}{6M}$. Since the flow is bounded by $M$, we know $\tilde{x}$ is at most $\frac{1}{6}$ far from the minimum cost solution for all edges.

For the total runtime, note that both the step $\tilde{x}$ and the step of routing excess demand cannot be omitted because it does not change the flow for every edge by more than $1/6$. So, we can simply round every number to nearest integer, which takes linear work and constant depth.

For Part 3, $L$ is symmetric diagonally dominant. The result follows from [30, Theorem 9.2]. □

Using the reduction mentioned above, one can obtain the promised minimum cost flow algorithm. (Further, using techniques from [11] this can be generalized to solving lossy flow problems.)

**Theorem 2 (Maximum Flow).** Given a directed graph $G = (V,E)$ with integral costs $q \in \mathbb{Z}^E$ and capacities $c \in \mathbb{Z}_{\geq 0}$ with $\|q\|_\infty \leq M$ and $\|c\|_\infty \leq \bar{M}$, we can compute a minimum cost maximum flow with constant probability with $O(|E|\sqrt{|V|}\log^{18}|E|\log M)$ work and $O(\sqrt{|V|}\log^{20}|E|\log M)$ depth.

**Proof.** Using the reduction (Lemma 44) and Theorem 1, we get an algorithm of minimum cost flow by solving

\[ O \left( \sqrt{|V|}\log^{13}|E| \cdot \log(|V|M) \cdot \mathcal{T}_w \right) \text{ work and } O \left( \sqrt{|V|}\log^{13}|E| \cdot \log(|V|M) \cdot \mathcal{T}_d \right) \text{ depth} \]

where $\mathcal{T}_w$ and $\mathcal{T}_d$ are the work and the depth of solving linear systems. It is known that for interior point methods, we only need to solve linear system with accuracy $\eta = \frac{1}{m^{O(1)}}$ ($\eta$ defined in (6.4))
because each step of interior point method only need to decrease the centrality $\delta_t$ by a constant factor. Hence, Lemma 44 shows that each linear system takes

$$\mathcal{T}_w = O\left(|E| \log^4 |V| \log(|V|)\right) \text{ work and } \mathcal{T}_d = O\left(\log^6 |V| \log(|V|)\right) \text{ depth.}$$

Hence, we have the result.

6.3 Computable Nearly Universal Barrier

Here we show how to combine the results of the preceding sections to obtain our main results on a polynomial time computable nearly-universal self-concordant barrier. We first provide and prove Theorem 3, a generalization of Theorem 3, and then show Theorem 3 as a special case. The results of this section use following result regarding computing Lewis weights proved in Appendix B.

**Theorem 45** (Exact Weight Computation). Let $A \in \mathbb{R}^{m \times n}$ be non-degenerate matrix and let $\epsilon \in (0, 1)$ and $p \in (0, \infty)$. For all $w^{(0)} \in \mathbb{R}^m$ with $\|w^{(0)} - w\|_\infty \leq \frac{p}{20(p+2)}$, the algorithm $\text{computeExactWeight}(A, p, w^{(0)}, \epsilon)$ (Algorithm 4) can be implemented to return $w$ such that $\|w(A)^{-1}(w(A) - w)\|_\infty \leq \epsilon$ in $O(mn^\omega - 1(p + p^{-1}) \log(n(1 + \frac{1}{p})\epsilon^{-1}))$ work and $O((p + p^{-1})^2 \log(m) \log(n(1 + \frac{1}{p})\epsilon^{-1}))$ depth.

Without $w^{(0)}$, the algorithm $\text{computeInitialWeight}(A, p, \epsilon)$ (Algorithm 7) can be implemented to achieve the same guarantee with $O(mn^\omega - 1(p + p^{-1}) \log(n) \log(n))$ work and $O((p + p^{-1})^2 \log(m) \log(n))$ depth.

**Theorem 46.** Let $\Omega^p = \{x : Ax > b\}$ denote the interior of non-empty polytope for non-degenerate $A \in \mathbb{R}^{m \times n}$. There is an $O(n \log^5 m)$-self concordant barrier $\psi$ defined using $\ell_q$ Lewis weight with $q = \Theta(\log m)$ (See (5.1)) satisfying

$$A_x^T W_x A_x^T \preceq \nabla^2 \psi(x) \preceq (q + 1)A_x^T W_x A_x^T$$

where $A_x = \text{Diag}(A x - b)$ and $w_x$ is the $\ell_q$ Lewis weight of the matrix $A_x$. Furthermore, we can compute or update the $w_x$, $\nabla \psi(x)$ and $\nabla^2 \psi(x)$ as follows:

- **Initial Weight:** For any $x \in \mathbb{R}^m$, we can compute a vector $\tilde{w}_x$ such that $(1 - \epsilon)w_x \preceq \tilde{w}_x \preceq (1 + \epsilon)w_x$ in $O(mn^\omega - 1(p + p^{-1}) \log^3 m \cdot \log(m/\epsilon))$-work and $O(\sqrt{n} \cdot \log^4 m \cdot \log(m/\epsilon))$-depth.

- **Update Weight and Compute Gradient/Hessian:** Given a vector $\tilde{w}_x$ such that $\tilde{w}_x = (1 \pm \frac{1}{100})w_x$, for any $y$ with $\|x - y\|_A^2 \leq \frac{c}{\log m}$ with some constant $c > 0$, we can compute $\tilde{w}_y$, $v$ and $H$ such that $\tilde{w}_y = (1 \pm \epsilon)w_y$,

$$\|v - \nabla \psi(x)\|_{\nabla^2 \psi(x)}^{-1} \leq \epsilon \text{ and } (1 - \epsilon)\nabla^2 \psi(x) \preceq H \preceq (1 + \epsilon)\nabla^2 \psi(x)$$

in $O(mn^\omega - 1 \cdot \log m \cdot \log(m/\epsilon))$-work and $O(\log^2 m \cdot \log(m/\epsilon))$-depth.

**Proof.** Theorem 30 with $q = \log m$ shows that there is such a barrier function $\psi$ that is $O(n \log^5 m)$. Lemma 31 shows that

$$\nabla \psi(x) = -A_x^T \sigma_x \text{ and } \nabla^2 \psi(x) = A_x^T \Sigma_x^{1/2}(I + N_x) \Sigma_x^{1/2} A_x.$$

Note that $\sigma_x = w_q(A_x)$ and we can compute $\tilde{\sigma}$ such that $\tilde{\sigma} \in (1 \pm \frac{c}{\sqrt{n}})\sigma_x$ in $O\left(\frac{mn^\omega - \frac{1}{2}}{\log^2 m} \cdot \log^3 m \cdot \log^2 m \right)$ work and $O\left(n^{1/2} \cdot \log^4 m \cdot \log \frac{\log m}{\epsilon}\right)$ depth using Theorem 45.
For the update version, let $w_x$ and $w_y$ be the Lewis weight corresponding to $x$ and $y$. Picking $c$ to be small enough constant, Lemma 34 shows that $\|w_y^{-1}(w_x - w_y)\|_\infty \leq O(c) \leq \frac{q}{20(q+2)}$. Hence, Theorem 45 shows that we can compute $w_x$ with $O(mn^{\omega-1} \cdot \log m \cdot \log(m/\epsilon))$-work and $O(\log^3 m \cdot \log(m/\epsilon))$-depth in this case.

For the gradient, with the approximate Lewis weight, Lemma 32 shows that

$$\|\nabla \psi(x) + A_x^\top \tilde{\sigma}\|_{\nabla^2 \psi(x)^{-1}} \leq \frac{\epsilon}{\sqrt{n}} \|\nabla \psi(x)\|_{\nabla^2 \psi(x)^{-1}} \leq \epsilon.$$

For the Hessian, we recall that $N_x = 2\bar{\Lambda}_x(I - (1 - \frac{2}{q})\bar{\Lambda}_x)^{-1}$ and $\bar{\Lambda}_x \overset{\text{def}}{=} \bar{\Lambda}(\Sigma_x^{\frac{1}{2}} - \frac{1}{q}A_x)$. Following calculations in Lemma 37 and Lemma 38, one can check that replacing $\sigma_x$ by $(1 \pm \epsilon)\sigma_x$ in the formula of $\nabla^2 \psi(x)$ (via $N_x$ and $\bar{\Lambda}_x$) only changes the matrix $\nabla^2 \psi(x)$ multiplicatively by $\pm \epsilon \log O(1) m$. Hence, we can compute it again in the same work and depth.

Leveraging this, we prove Theorem 3.

**Proof of Theorem 3.** This theorem is a specialization of Theorem 46. \hfill \qed

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A Projection Matrices, Leverages Scores, and log det

In this section, we prove various properties of projection matrices, leverage scores, and the logarithm of the determinant that we use throughout the paper.

First we provide the following theorem which gives various properties of projection matrices and leverage scores.

**Lemma 47** (Projection Matrices). Let $P \in \mathbb{R}^{m \times m}$ be an arbitrary orthogonal projection matrix and let $\Sigma = \text{Diag}(P)$. For all $i, j \in [m]$, $x, y \in \mathbb{R}^m$, and $X = \text{Diag}(x)$ we have

\begin{align*}
(1) \quad \Sigma_{ii} &= \sum_{j \in [m]} P_{ij}^{(2)} \\
(2) \quad 0 &\preceq P^{(2)} \preceq \Sigma \preceq \mathbf{I}, \text{(in particular, } 0 \leq \Sigma_{ii} \leq 1) \\
(3) \quad P_{ij}^{(2)} &\leq \Sigma_{ii} \Sigma_{jj} \\
(4) \quad \|\Sigma^{-1} P^{(2)} x\|_{\infty} &\leq \|x\|_{\infty} \\
(5) \quad \|\Sigma^{-1} P^{(2)} x\|_{\infty} &\leq \|x\|_{\infty} \\
(6) \quad \sum_{i \in [m]} \Sigma_{ii} &= \text{rank}(P) \\
(7) \quad \|y^{\top} XP^{(2)} y\| &\leq \|y\|_{\Sigma} \cdot \|x\|_{\Sigma} \\
(8) \quad \|y^{\top} (P \circ P XP)^2 y\| &\leq \|y\|_{\Sigma} \cdot \|x\|_{\Sigma}.
\end{align*}

**Proof.** To prove (1), we simply note that by definition of a projection matrix $P = PP$ and therefore

$$
\Sigma_{ii} = \sum_{j \in [m]} P_{ij}^{(2)} = \sum_{j \in [m]} P_{ij}^{2} = \sum_{j \in [m]} P_{ij}^{(2)}.
$$
To prove (2), we observe that since $P$ is a projection matrix, all its eigenvalues are either 0 or 1. Therefore, $\Sigma \preceq I$ and by (1) $\Sigma - P^{(2)}$ is diagonally dominant. Consequently, $\Sigma - P^{(2)} \succeq 0$. Rearranging terms and using the well known fact that the Shur product of two positive semi-definite matrices is positive semi-definite yields (2).

To prove (3), we use $P = PP$, Cauchy-Schwarz, and (1) to derive

$$P_{ij} = \sum_{k \in [m]} P_{ik} P_{kj} \leq \sqrt{\left( \sum_{k \in [m]} P_{ik}^2 \right) \left( \sum_{k \in [m]} P_{kj}^2 \right)} = \sqrt{\Sigma_{ii} \Sigma_{jj}}.$$  

Squaring then yields (3).

To prove (4), we note that by the definition of $P^{(2)}$ and Cauchy-Schwarz, we have

$$\left| e_i^T P^{(2)} x \right| = \left| \sum_{j \in [m]} P_{ij}^{(2)} x_j \right| \leq \sqrt{\left( \sum_{j \in [m]} \Sigma_{jj} x_j^2 \right) \left( \sum_{j \in [m]} \frac{P_{ij}^4}{\Sigma_{jj}} \right)}.$$  

Now, by (1) and (3), we know that

$$\sum_{j \in [m]} P_{ij}^4 \Sigma_{jj} \leq \sum_{j \in [m]} \frac{P_{ij}^2 \Sigma_{ii} \Sigma_{jj}}{\Sigma_{jj}} = \Sigma_{ii} \sum_{j \in [m]} P_{ij}^2 = \Sigma_{ii}^2.$$  

Since $\|x\|_\Sigma \overset{\text{def}}{=} \sqrt{\sum_{j \in [m]} \Sigma_{jj} x_j^2}$, combining (A.1) and (A.2) yields $\left| e_i^T P^{(2)} x \right| \leq \Sigma_{ii} \|x\|_\Sigma$ as desired.

To prove (5), we note that

$$\left| e_i^T P^{(2)} x \right| = \left| \sum_{j \in [m]} P_{ij}^{(2)} x_j \right| \leq \sum_{j \in [m]} P_{ij}^{(2)} |x_j| = \Sigma_{ii} \|x\|_\infty$$

To prove (6), we note that all the eigenvalues of $P$ are either 0 or 1 and $\sum_{i \in [m]} \Sigma_{ii} = \text{tr}(P)$.

To prove (7), we apply (4) and Cauchy-Schwarz to show

$$\left| y^T X P^{(2)} y \right| = \left| \sum_{i \in [m]} x_i \cdot y_i 1^T P^{(2)} y \right| \leq \sum_{i \in [m]} |x_i| \cdot |y_i| \cdot \Sigma_{ii} \cdot \|y\|_\Sigma \leq \|x\|_\Sigma \|y\|_\Sigma \|\Sigma\|_\Sigma \cdot$$

To prove (8), we note that by Cauchy Schwarz

$$\left| y^T (P \circ P X P) y \right| = \left| \sum_{i,j \in [m]} y_i y_j P_{ij} \left( \sum_{k \in [m]} P_{ik} P_{jk} x_k \right) \right|$$

$$\leq \sqrt{\left( \sum_{i,j \in [m]} |y_i| \cdot |y_j| \cdot P_{ij}^2 \right) \left( \sum_{i,j \in [m]} |y_i| \cdot |y_j| \cdot \left( \sum_{k \in [m]} P_{ik} P_{jk} x_k \right)^2 \right).}$$

Letting $|x|$ and $|y|$ be the vectors whose entries are the absolute values of the entries of $x$ and $y$ we
respectively, see that by (2) we have
\[
\sum_{i,j\in[m]} |y_i| \cdot |y_j| \cdot \mathbf{P}_{ij}^2 = \|y\|^2_{\mathbf{P}(2)} \leq \|y\|^2_{\mathbf{S}} = \|y\|^2_{\mathbf{S}}
\]
and
\[
\sum_{i,j\in[m]} |y_i| \cdot |y_j| \cdot \left( \sum_{k\in[m]} \mathbf{P}_{ik} \mathbf{P}_{jk} x_k \right)^2 = \sum_{i,j\in[m]} \left( \sum_{k\in[m]} \mathbf{P}_{ik} \sqrt{|y_i| |x_k|} \right) \left( \mathbf{P}_{jk} \sqrt{|y_j| |x_k|} \right)^2.
\]
Applying Cauchy Schwarz twice then yields that
\[
\sum_{i,j\in[m]} |y_i| \cdot |y_j| \cdot \left( \sum_{k\in[m]} \mathbf{P}_{ik} \mathbf{P}_{jk} x_k \right)^2 \leq \left( \sum_{i,k\in[m]} |y_i| \mathbf{P}_{ik}^2 |x_k| \right)^2 = \left( |y|^{\top} \mathbf{P}(2) |x| \right)^2 \leq \|y\|^2_{\mathbf{P}(2)} \|x\|^2 \leq \|y\|^2_{\mathbf{S}} \|x\|^2_{\mathbf{S}} = \|y\|^2_{\mathbf{S}} \|x\|^2_{\mathbf{S}}.
\]
Combining these inequalities than yields the desired bound on $|y^{\top} (\mathbf{P} \circ \mathbf{P} \mathbf{X} \mathbf{P}) y|$.

Next, we derive various matrix calculus formulas relating the projection matrix with the log determinant. We start by computing the derivative of the volumetric barrier function, $f(w) \overset{\text{def}}{=} \log \det(\mathbf{A}^{\top} \mathbf{W} \mathbf{A})$.

**Lemma 48 (Derivative of Volumetric Barrier).** For full rank matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$ let $f : \mathbb{R}^m_{>0} \to \mathbb{R}$ be given by $f(w) \overset{\text{def}}{=} \log \det(\mathbf{A}^{\top} \mathbf{W} \mathbf{A})$. For any $w \in \mathbb{R}^m_{>0}$, we have $\nabla f(w) = \mathbf{W}^{-1}\sigma(\mathbf{W}^{1/2} \mathbf{A})$.

**Proof.** Using the derivative of log det, we have that for all $i \in [m]$
\[
\frac{\partial f(w)}{\partial w_i} = \text{tr} \left[ (\mathbf{A}^{\top} \mathbf{W} \mathbf{A})^{-1} \frac{\partial}{\partial w_i} (\mathbf{A}^{\top} \mathbf{W} \mathbf{A}) \right] = \text{tr} \left[ (\mathbf{A}^{\top} \mathbf{W} \mathbf{A})^{-1} \mathbf{A}^{\top} e_i e_i^{\top} \mathbf{A} \right] = w_i^{-1}\sigma \left( \mathbf{W}^{1/2} \mathbf{A} \right)_i.
\]

Next we bound the rate of change of entries of the projection matrix.

**Lemma 49 (Derivative of Projection Matrix).** Given full rank $\mathbf{A} \in \mathbb{R}^{n \times m}$ and $w \in \mathbb{R}^m_{>0}$ we have
\[
D_w \mathbf{P}(\mathbf{W} \mathbf{A})[h] = \Delta \mathbf{P}(\mathbf{W} \mathbf{A}) + \mathbf{P}(\mathbf{W} \mathbf{A}) \Delta - 2 \mathbf{P}(\mathbf{W} \mathbf{A}) \Delta \mathbf{P}(\mathbf{W} \mathbf{A})
\]
where $\mathbf{W} = \text{Diag}(w)$ and $\Delta = \text{Diag}(h/w)$. In particular, we have that
\[
D_w \sigma(\mathbf{W} \mathbf{A})[h] = 2 \mathbf{A}(\mathbf{W} \mathbf{A}) \mathbf{W}^{-1} h.
\]

**Proof.** Note that
\[
\mathbf{P}(\mathbf{W} \mathbf{A}) = \mathbf{W} \mathbf{A} (\mathbf{A}^{\top} \mathbf{W}^2 \mathbf{A})^{-1} \mathbf{A}^{\top} \mathbf{W}.
\]
Using the derivative of matrix inverse, we have that
\[
D_w \mathbf{P}(\mathbf{W} \mathbf{A})[h] = \mathbf{H} \mathbf{A} (\mathbf{A}^{\top} \mathbf{W}^2 \mathbf{A})^{-1} \mathbf{A}^{\top} \mathbf{W} + \mathbf{W} (\mathbf{A}^{\top} \mathbf{W} \mathbf{A})^{-1} \mathbf{A}^{\top} \mathbf{H}
\]
\[
- 2 \mathbf{W} (\mathbf{A}^{\top} \mathbf{W}^2 \mathbf{A})^{-1} \mathbf{A}^{\top} \mathbf{H} \mathbf{W} (\mathbf{A}^{\top} \mathbf{W}^2 \mathbf{A})^{-1} \mathbf{A}^{\top} \mathbf{W}
\]
\[
= \Delta \mathbf{P} + \mathbf{P} \Delta - 2 \mathbf{P} \Delta \mathbf{P}.
\]
Therefore,
\[
D_w \sigma(WA)_i[h] = [D_w P(WA)[h]]_{ii} = 2\Delta_{ii}P_{ii} - 2(P\Delta P)_{ii}
\]
\[
= 2\sigma_i h_i - 2 \sum_{j \in [m]} P_{ij}^2 \Delta_{jj} = 2 \left( \left( \Sigma - P^{(2)} \right) \left( h/w \right) \right)_i = 2(\Lambda(h/w))_i.
\]

\[
\Box
\]

Consequently,
\[
\nabla \ln \det(A_x^\top W^{1 - \frac{2}{q}} A_x) = -2A_x^\top \sigma_{x,w},
\]
\[
\nabla_w \ln \det(A_x^\top W^{1 - \frac{2}{q}} A_x) = c_q W^{-1} \sigma_{x,w},
\]
\[
\nabla_{xx} \ln \det(A_x^\top W^{1 - \frac{2}{q}} A_x) = -2c_q A_x^\top A_{x,w} W^{-1}
\]

where, \( c_q \) is a constant defined in the paper.

**Lemma 50 (Potential Function Derivative).** For non-degenerate \( A \in \mathbb{R}^{m \times n} \) and \( q > 0 \) with \( q \neq 2 \), let \( p(x, w) \) be defined as \( \ln \det(A_x^\top W^{1 - \frac{2}{q}} A_x) \) for all \( x \in \mathbb{R}^n \) with \( Ax > b \) and all \( w \in \mathbb{R}^m_{>0} \) where \( A_x \) is the diagonal of \( A \), \( S_x = \text{Diag}(Ax - b) \), and \( w \in \text{Diag}(w) \). Then, the following hold
\[
\nabla_x p(x, w) = -2A_x^\top \sigma_{x,w},
\]
\[
\nabla_w p(x, w) = c_q W^{-1} \sigma_{x,w},
\]
\[
\nabla_{xx} p(x, w) = -2c_q A_x^\top A_{x,w} W^{-1}
\]

where, \( c_q = 1 - \frac{1}{q} \), \( \sigma_{x,w} \) is a constant defined in the paper.

**Proof.** To simplify the calculations, throughout this proof we overload notation and let \( p(s, w) \) be defined as \( \ln \det(A^\top S^{-1} W^{1 - \frac{2}{q}} S^{-1} A) \) where \( S = \text{Diag}(s) \) and let \( s_x \) be linear transformation of \( x \) this implies that the derivatives with respect to \( x \) to follow immediately from the derivatives with respect to \( s \).

For \( \nabla_x p(x, w) \), Lemma 48 shows that
\[
\frac{\partial}{\partial s_i} p(s, w) = s_i^{-1} w_i^{1 + \frac{2}{q}} \sigma_{x,w} + 2 s_i^{-2} w_i^{1 - \frac{2}{q}} = 2 \frac{[\sigma_{x,w}]_i}{s_i}.
\]

Therefore \( \nabla_x p(x, w) = -2A_x^\top \sigma_{x,w} \) by chain rule.

For \( \nabla_w p(x, w) \), Lemma 48 and chain rule shows that
\[
\frac{\partial}{\partial w_i} p(x, w) = s_i^{-2} w_i^{1 + \frac{2}{q}} \sigma_{x,w} - 2 s_i^{-2} \left( 1 - \frac{2}{q} \right) w_i^{-\frac{2}{q}} = c_q \frac{[\sigma_{x,w}]_i}{w_i}.
\]

Therefore \( \nabla_w p(x, w) = c_q W^{-1} \sigma_{x,w} \).

For \( \nabla_{xx} p(x, w) \), the formula for \( \frac{\partial}{\partial s_i} p(s, w) \) given by (A.3) and Lemma 49 yields
\[
\frac{\partial^2}{\partial s_i \partial s_j} p(s, w) = 2 \frac{[\sigma_{x,w}]_i}{s_i^2} \delta_{i=j} - 4 \frac{[\Lambda_{x,w}]_{ij}}{s_i} \cdot s_j w_j^{-\frac{1}{2} + \frac{1}{q}} \cdot (-s_j^{-2} w_j^{1 - \frac{1}{q}})
\]
\[
= 2 \frac{[\sigma_{x,w}]_i}{s_i^2} \delta_{i=j} + 4 \frac{[\Lambda_{x,w}]_{ij}}{s_i s_j}.
\]

Therefore, \( \nabla_{xx} p(x, w) = A_x^\top (2\Sigma_{x,w} + 4A_{x,w})A_x \).
For $\nabla_{ww} p(x, w)$, the formula for $\frac{\partial^2}{\partial w_i \partial w_j} p(x, w)$ given by (A.4) and Lemma 49 yields

$$\frac{\partial^2}{\partial w_i \partial w_j} p(x, w) = -c_q \frac{[\sigma_{x, w}]_{ij}}{w_i^2} + 2c_q \frac{[\Lambda_{x, w}]_{ij}}{w_i} \cdot s_j w_j^{-\frac{1}{2} + \frac{1}{p}} \cdot \left( \frac{1}{2} - \frac{1}{p} \right) \left( s_j^{-1} w_j^{\frac{1}{2} - \frac{1}{p}} \right)$$

$$= -c_q \frac{[\sigma_{x, w}]_{ij}}{w_i^2} + 2c_q \frac{[\Lambda_{x, w}]_{ij}}{w_i} \cdot s_j w_j^{-\frac{1}{2} + \frac{1}{p}} \cdot \left( \frac{1}{2} - \frac{1}{p} \right) \left( s_j^{-1} w_j^{\frac{1}{2} - \frac{1}{p}} \right)$$

Therefore $\nabla_{ww} p(x, w) = -c_q W^{-1} (\Sigma_{x, w} - c_q \Lambda_{x, w}) W^{-1}$.

For $\nabla_{xw} p(x, w)$, the formula for $\frac{\partial}{\partial s_i} p(s, w)$ given by (A.3) and Lemma 49 yield

$$\frac{\partial}{\partial s_i} p(s, w) = -4 \frac{[\Lambda_{x, w}]_{ij}}{s_i} \cdot s_j w_j^{-\frac{1}{2} + \frac{1}{p}} \cdot \left( \frac{1}{2} - \frac{1}{p} \right) \left( s_j^{-1} w_j^{\frac{1}{2} - \frac{1}{p}} \right) = -2c_q \frac{[\Lambda_{x, w}]_{ij}}{s_i w_j}$$

Therefore, $-2c_q A_x^\top \Lambda_{x, w} W^{-1}$ by chain rule.

**Lemma 51.** For any vector $v$, any positive vector $w$ and matrix $A$, we have that

$$\arg \min_{A^\top x = 0} v^\top x + \frac{1}{2} \|x\|_w^2 = x_* \overset{\text{def}}{=} -W^{-1} v + W^{-1} A (A^\top W^{-1} A)^{-1} A^\top W^{-1} v.$$

**Proof.** Let $f(x) \overset{\text{def}}{=} v^\top x + \frac{1}{2} \|x\|_w^2$. Note that $\nabla f(x) = v + Wx$ and consequently, $x \in \ker(A^\top)$ is optimal if and only if $v + Wx \perp \ker(A^\top)$, i.e. $v + Wx \in \im(A)$, and $A^\top x = 0$. Since $A^\top x_* = 0$ and $w + Wx = A(A^\top W^{-1} A)^{-1} A^\top W^{-1} v \in \im(A^\top)$ the result follows.

**B Lewis Weight Computation**

Here, we describe how to efficiently compute approximations to Lewis weights and ultimately prove Theorem 39 and Theorem 45 (the Lewis weight computation results claimed and used in Section 6). We achieve our results by a combination of a number of technical tools, including projected gradient descent (for computing Lewis weights exactly in Section B.1 given a good initial weight), the Johnson-Lindenstrauss lemma (for computing Lewis weights approximately in Section B.2 given a good initial weight), and homotopy methods (for computing initial weights and completing the proofs of the main theorems in Section B.3).

Throughout the remainder of this section we let $A \in \mathbb{R}^{m \times n}$ denote an arbitrary non-degenerate matrix and $p \in (0, \infty)$ with $p \neq 2$. Further we let $w_p \overset{\text{def}}{=} w_p(A)$ and $W_p \overset{\text{def}}{=} \text{Diag}(w_p)$.

**B.1 Exact Computation**

Since Lewis weight can be found by the minimizer of a convex optimization problem (Lemma 22), we can use the gradient descent method directly to minimize $\mathcal{V}^A_p(w)$. Indeed, in this section we show how applying the gradient descent method in a carefully scaled space allows us to compute the weight to good accuracy in $O(\text{poly}(p))$ iterations. This results makes two assumptions to compute the weight: (1) we compute the gradient of $\mathcal{V}^A_p(w)$ exactly and (2) we are given a weight that is not too far from the true weight. In the remaining subsection we show how to address these issues.

First we state the following theorem regarding gradient descent method we use in our analysis. This theorem shows that if we take repeated projected gradient steps then we can achieve linear convergence up to bounds on how much the Hessian of the function changes over the domain of interest.

**Theorem 52** (Simple Constrained Minimization for Twice Differentiable Function). Let $H$ be a positive definite matrix and $Q \subseteq \mathbb{R}^m$ be a convex set. Let $f : Q \to \mathbb{R}$ be a twice differentiable function.
Suppose that there are constants $0 \leq \mu \leq L$ such that for all $x \in Q$ we have $\mu \cdot H \preceq \nabla^2 f(x) \preceq L \cdot H$. For any $x^{(0)} \in Q$ and any $k \geq 0$ if we apply the update rule

$$x^{(k+1)} = \arg \min_{x \in Q} f(x^{(k)}) + \nabla f(x^{(k)})^\top (x - x^{(k)}) + \frac{L}{2} \| x - x^{(k)} \|^2_H$$

then it follows that

$$\| x^{(k)} - x^* \|^2_H \leq \left( 1 - \frac{L}{2\mu} \right)^k \| x^{(0)} - x^* \|^2_H.$$
Proof. Let \( \hat{w} \) be the solution to the optimization problem given in Theorem 52.

Since the objective function and the constraints are axis-aligned, we can compute \( \hat{w} \) coordinate-wise and we see that this is the same as in the statement of this lemma.

To apply Theorem 52, we note that \( \| W^{-1}(w_p - w(0)) \|_\infty \leq \frac{p}{8(p+2)} \) implies that any \( w \in Q \) satisfies \( \| W^{-1}(w_p - w) \|_\infty \leq \frac{p}{8(p+2)} \) and hence Lemma 53 shows that

\[
\min \{ \frac{1}{4}, \frac{1}{2p} \} \| W^{-1} \|_{\infty} \leq \min \{ \frac{1}{4}, \frac{1}{2p} \} \| W^{-1} \|_{\infty} \leq \nabla^2 \mathcal{V}(w) \leq \max \{ 2, \frac{4}{p} \} W^{-1} \leq \max \{ 4, \frac{8}{p} \} W^{-1}.
\]

Hence, Theorem 52 and inequality (B.2) shows that

\[
\| w^{(k)} - w_p \|_{W^{-1}}^2 \leq \left( 1 - \frac{\min(\frac{1}{4}, \frac{1}{2p})}{\max(4, \frac{8}{p})} \right)^k \| w(0) - w_p \|_{W^{-1}}^2 \leq \left( 1 - \frac{1}{16(\frac{p}{7} + \frac{2}{p})} \right)^k \| w(0) - w_p \|_{W^{-1}}^2.
\]

The result follows as \( w(0) \) is close to \( w_p \) multiplicatively and therefore

\[
\| w(0) - w_p \|_{W^{-1}}^2 \leq \frac{3}{2} \sum_{i \in [n]} w_i(0) \cdot \| W^{-1}(w_p - w(0)) \|_\infty \leq 2n \| W^{-1}(w_p - w(0)) \|_\infty.
\]

\[\blacksquare\]

Note that the lemma does not show that \( w^{(k)} \) is a multiplicative approximation of \( w_p \). The following lemma shows that we can use \( w^{(k)} \) to get a multiplicative approximation.

Lemma 55. Given \( w \) such that \( \| W^{-1}(w_p - w) \|_\infty \leq \frac{p}{8(p+2)} \) and that \( \| w - w_p \|_{W_p^{-1}} \leq \frac{1}{4(1+\frac{2}{p})^2 \sqrt{n}} \).

Let \( \hat{w} = \text{diag}(A^T W_{p}^{1-\frac{2}{p}} A^{-1} A^T) \). Then, we have that

\[
\| W^{-1}(\hat{w} - w_p) \|_\infty \leq 4 \left( 1 + \frac{2}{p} \right)^2 \sqrt{n} \cdot \| w - w_p \|_{W_p^{-1}}.
\]

Proof. The definition of \( \hat{w} \) is motivated from the equality \( w_p = \) (\( \text{diag}(A^T W_{p}^{1-\frac{2}{p}} A^{-1} A^T) \)). To show \( \hat{w} \) is multiplicative close to \( w_p \), it therefore suffices to prove that \( A^T W_{p}^{1-\frac{2}{p}} A \) is multiplicatively close to \( A^T W_{p}^{1-\frac{2}{p}} \). Note that \( (1 - \alpha)A^T W_{p}^{1-\frac{2}{p}} A \leq A^T W_{p}^{1-\frac{2}{p}} \leq (1 + \alpha)A^T W_{p}^{1-\frac{2}{p}} A \) with

\[
\alpha \leq \text{tr} \left[ (A^T W_{p}^{1-\frac{2}{p}} A)^{-1}(A^T W_{p}^{1-\frac{2}{p}} - W_{p}^{1-\frac{2}{p}})A \right] = \sum_{i \in [n]} \frac{\sigma_i(W_{p}^{1-\frac{2}{p}} A)}{[w_{p,i}]^{2/p}} \left| w_{p,i}^{1-2/p} - [w_{p,i}]^{1-2/p} \right|.
\]
Since \( \|W_p^{-1}(w_p - w)\|_\infty \leq \frac{p}{8(p+2)} \) we have that for all \( i \in [n] \)
\[
|w_i^{1-2/p} - [w_p]_i^{1-2/p}| \leq 2 \left| 1 - \frac{2}{p} \right| |\frac{w_i - [w_p]_i}{[w_p]_i^{2/p}}|
\]
and therefore, by Cauchy Schwarz and that \( \sum_{i \in [n]} \sigma_i(W_p^{\frac{2}{p}-\frac{2}{p}}A)^2 = \sum_{i \in [n]} \sigma_i(W_p^{\frac{2}{p}-\frac{2}{p}}A) = n \) we have
\[
\alpha \leq 2 \left| 1 - \frac{2}{p} \right| \left\| \sum_{i \in [n]} \frac{\sigma_i(W_p^{\frac{2}{p}-\frac{2}{p}}A)}{[w_p]_i} |w_i - [w_p]_i| \right\| \leq 2 \left| 1 - \frac{2}{p} \right| \sqrt{n} \cdot \delta.
\]

The result follows from \( w_p = (\text{diag}(A(A^TW_p^{1-\frac{2}{p}}A^{-1}A^T))^2) \), that \( (1 - 2|1 - (2/p)|\delta \sqrt{n})^{-2/p} \geq 1 - 4(1 + \frac{2}{p})^2\sqrt{n} \delta \), and that \( (1 + 2|1 - (2/p)|\delta \sqrt{n})^{-2/p} \leq 1 + 4(1 + \frac{2}{p})^2\sqrt{n} \delta \).

Combining Lemma 54 and Lemma 55 yields the following Theorem 56, the main result of this section on weight computation.

**Algorithm 4: \( w = \text{computeExactWeight}(A, p, w^{(0)}, \epsilon) \)**

1. Let \( T = \left[ 32(\frac{p}{2} + \frac{3}{p}) \log(8n(1 + \frac{3}{p})\epsilon^{-1}) \right] \), \( r = \frac{p}{20(p+2)} \), and \( L = \max\{4, \frac{8}{p}\} \)
2. for \( k = 1, \cdots, T - 1 \) do
   - \( w^{(k+1)} = \text{median}\left( (1 - r) w^{(0)}, w^{(k)} - \frac{1}{L} \left( w^{(0)} - \frac{w^{(0)}}{w^{(0)}\sigma\left( W_p^{\frac{1}{p}-\frac{1}{p}}A \right)} \right), (1 + r) w^{(0)} \right) \)
3. **Output:** \( (\text{diag}(A(A^TW_p^{1-\frac{2}{p}}A^{-1}A^T)))^2 \) for \( W_T = \text{Diag}(w^{(T)}) \)

**Theorem 56 (Exact Weight Updates).** For all \( \epsilon \in (0, 1) \) and \( w^{(0)} \in \mathbb{R}_{>0}^m \) with \( \|w^{(0)}_p - w^{(0)}\|_\infty \leq \frac{p}{20(p+2)} \) the algorithm \( \text{computeExactWeight}(A, p, w^{(0)}, \epsilon) \) (Algorithm 4) outputs \( w \in \mathbb{R}_{>0}^m \)
with \( \|w_p(A)^{-1}(w_p(A) - w^{(0)})\|_\infty \leq \epsilon \) in \( O((p + \frac{1}{p})\log(n(1 + \frac{1}{p})\epsilon^{-1})) \) iterations, where each iteration involves computing \( \sigma(VA) \) for diagonal matrix \( V \) and extra linear time work and \( O(1) \) depth.

**Proof.** This result follows immediately from Lemma 54 and Lemma 55.

**B.2 Approximate Computation**

Here we show how to modify the algorithm and analysis of the previous subsection to use approximate leverage scores instead of exact leverage score in computing gradient. Further, we show how to use the Johnson-Lindenstrauss lemma to compute approximate leverage scores efficiently using a linear system solver. Together, these results give us efficient algorithms for improving the approximation quality of Lewis weights.

To analyze our algorithm, \( \text{computeApxWeight} \) (Algorithm 5) given below, we first give a helper
lemma showing that the optimality condition \( \sigma(W^\frac{1}{2} - \frac{1}{p} A) \)/w_i is stable under changes to w.

**Algorithm 5:** \( w = \text{computeApxWeight}(A, p, w(0), \epsilon) \)

\[
L = \max\{4, \frac{8}{p}\}, \quad r = \frac{p(4-p)}{2\sigma} \quad \text{and} \quad \delta = \frac{(4-p)\epsilon}{2^{256}}.
\]

Let the number of iterations \( T = \lceil 80(\frac{2}{p} + \frac{2}{r}) \log \left( \frac{8m}{32\epsilon} \right) \rceil. \]

for \( j = 1, \ldots, T - 1 \) do

\[
\text{Compute } \sigma^{(j)} \in \mathbb{R}^n \text{ such that } e^{-\delta} \sigma(W^\frac{1}{2} - \frac{1}{p} A) \leq \sigma^{(j)} \leq e^{\delta} \sigma(W^\frac{1}{2} - \frac{1}{p} A) \text{ for all } i \in [n].
\]

\[
w^{(j+1)} = \text{median} \left( (1 - r) w^{(0)}, w^{(j)} - \frac{1}{L} \left( w^{(0)} - w^{(0)}(0) \sigma^{(j)} \right), (1 + r) w^{(0)} \right).
\]

Output: \( (\text{diag}(A(A^\top W^{1-\frac{2}{p}} - A^{-1} A^\top)))^{\frac{2}{p}} \).

**Lemma 57.** Let \( w, v \in \mathbb{R}^m_{\geq 0} \) with \( w_i = e^\delta v_i \) for \( |\delta_i| \leq \delta \) for all \( i \in [n] \). Then, for all \( i \in [n] \)

\[
e^{\frac{2}{p} \delta_i - |1 - \frac{2}{p}| \delta} \cdot \frac{\sigma(W^\frac{1}{2} - \frac{1}{p} A)_i}{w_i} \leq \frac{\sigma(V^\frac{1}{2} - \frac{1}{p} A)_i}{v_i} \leq e^{\frac{2}{p} \delta_i + |1 - \frac{2}{p}| \delta} \cdot \frac{\sigma(W^\frac{1}{2} - \frac{1}{p} A)_i}{w_i}.
\]

**Proof.** Note that \( v_i^{-1} \sigma(V^\frac{1}{2} - \frac{1}{p} A)_i = v_i^{-\frac{2}{p}} a_i^\top (A^\top V^{-\frac{2}{p}} A)^{-1} a_i \) where \( a_i \) is the \( i \)-th row of \( A \). By the assumptions on \( w, v, \in \mathbb{R}^m_{\geq 0} \) we have

\[
v_i^{-1} \sigma(V^\frac{1}{2} - \frac{1}{p} A)_i = e^{\frac{2}{p} \delta_i - |1 - \frac{2}{p}| \delta} \cdot w_i^{-\frac{2}{p}} a_i^\top (A^\top V^{-\frac{2}{p}} A)^{-1} a_i \leq e^{\frac{2}{p} \delta_i + |1 - \frac{2}{p}| \delta} \cdot w_i^{-\frac{2}{p}} a_i^\top (A^\top W^{-\frac{2}{p}} A)^{-1} a_i.
\]

and the lower bound on \( \sigma(V^\frac{1}{2} - \frac{1}{p} A)_i \) follows similarly. \( \square \)

**Theorem 58 (Approximate Weight Computation).** If \( p \in (0, 4) \) and \( w(0) \in \mathbb{R}^m_{\geq 0} \) satisfies \( \|w(0)(w_p(A) - w(0))\|_\infty \leq r \) where \( r = \frac{p^2(4-p)}{2\sigma} \). For \( 0 < \epsilon < \frac{2}{p} - |1 - \frac{2}{p}| \), the algorithm \( \text{computeApxWeight}(x, w(0), \epsilon) \) returns \( w \) such that \( \|w_p(A)(w_p(A) - w)\|_\infty \leq \epsilon \) in \( O(p^{-1} \log(np^{-1} \epsilon^{-1})) \) steps. Each step involves computing \( \sigma \) up to \( \pm O((4 - p) \cdot \epsilon) \) multiplicative error with some extra linear time work.

**Proof.** Consider an execution of \( \text{computeApxWeight}(x, w(0), \epsilon) \) where there is no error in computing leverages scores, i.e. \( \sigma^{(j)} = \sigma(W^\frac{1}{2} - \frac{1}{p} A) \), and let \( v^{(j)} \) denote the \( w \) computed during this idealized execution of \( \text{computeApxWeight} \). We will show that \( w^{(j)} \) and \( v^{(j)} \) are multiplicatively close.

Suppose that \( w^{(j)} = e^{\delta^{(j)}} v^{(j)} \) with \( |\delta^{(j)}| \leq \delta \) for some \( \delta \geq 0 \). Define \( v^{(j+1)}, w^{(j+1)} \in \mathbb{R}^m_{\geq 0} \) to be \( v^{(j+1)} \) and \( w^{(j+1)} \) before taking the median, i.e.

\[
v^{(j+1)} = v^{(j)} - \frac{1}{L} \left( w^{(0)} - w^{(0)}(0) \sigma(W^\frac{1}{2} - \frac{1}{p} A) \right) \quad \text{and} \quad w^{(j+1)} = w^{(j)} - \frac{1}{L} \left( w^{(0)} - w^{(0)}(0) \sigma^{(j)} \right).
\]

Using \( \pm \delta \) to denote a real value with magnitude at most \( \delta \) and applying Lemma 57 with \( v = v^{(j)} \)
and \( w = w^{(j)} \), we have
\[
\bar{w}_i^{(j+1)} - v_i^{(j+1)} = w^{(j)} - v^{(j)} + \frac{w^{(0)}}{L} \left( \sum_{\delta} e^{\pm \delta} \sigma \left( \frac{1}{2} \frac{w^{(j)}}{w^{(j)}} - \frac{r}{v^{(j)}} \right) \right) \\
= (e^{\delta_i^{(j)}} - 1)v_i^{(j)} + \frac{w^{(0)}}{L} \left( e^{- \frac{\delta_i^{(j)}}{2} |1 - \frac{2}{p} | \delta^{(j)}|1 + \delta} - 1 \right) \cdot \frac{\sigma \left( \frac{1}{2} \frac{r}{v^{(j)}} \right)}{v^{(j)}}. \tag{B.3}
\]
Since \( \| W^{-1}_0(w^{(0)} - v^{(j)}) \|_\infty \leq r \) and that \( \| W^{-1}_0(w^{(0)} - w_p(A)) \|_\infty \leq r \), we have that \( w_p(A) = e^{\pm 3r v^{(j)}} \). Lemma 57 shows that for \( w = w_p(A) \) we have
\[
e^{-3(\frac{2}{p} + 1 - \frac{1}{p})} \cdot w_i^{(j)} \cdot \frac{\sigma \left( \frac{1}{2} \frac{r}{v^{(j)}} \right)}{v^{(j)}} \leq (v_i^{(j)})^{-1} \sigma \left( \frac{1}{2} \frac{r}{v^{(j)}} \right) \leq e^{3(\frac{2}{p} + 1 - \frac{1}{p})} \cdot w_i^{(j)} \cdot \frac{\sigma \left( \frac{1}{2} \frac{r}{v^{(j)}} \right)}{v^{(j)}}. \tag{B.4}
\]
where Using that \( w_i = \sigma \left( \frac{1}{2} \frac{r}{v^{(j)}} \right) \) \( \), \( (B.4) \), and \( (B.3) \), we have that
\[
\bar{w}_i^{(j+1)} - v_i^{(j+1)} = (e^{\delta_i^{(j)}} - 1)v_i^{(j)} + \frac{w^{(0)}}{L} \left( e^{- \frac{\delta_i^{(j)}}{2} |1 - \frac{2}{p} | \delta^{(j)}|1 + \delta} - 1 \right) e^{3(\frac{1}{p} + \frac{1}{2})r}.
\]
Since \( w^{(j+1)} \) and \( v^{(j+1)} \) are just truncation of \( \bar{w}^{(j+1)} \) and \( v^{(j+1)} \), we have the same bound for \( w_i^{(j+1)} - v_i^{(j+1)} \). Using \( u_i^{(j+1)} = e^{\delta_i^{(j+1)} - v_i^{(j+1)}} \), we get that
\[
(e^{\delta_i^{(j+1)}} - 1)v_i^{(j+1)} = (e^{\delta_i^{(j)}} - 1)v_i^{(j)} + \frac{w^{(0)}}{L} \left( e^{- \frac{\delta_i^{(j)}}{2} |1 - \frac{2}{p} | \delta^{(j)}|1 + \delta} - 1 \right) e^{3(\frac{1}{p} + \frac{1}{2})r}.
\]
Finally, we note that \( v^{(j+1)} = e^{\pm 2r v^{(0)}} \) and hence
\[
\frac{e^{\delta_i^{(j+1)}} - 1}{e^{\pm 2r}} (e^{\delta_i^{(j)}} - 1) + \frac{1}{L} \left( e^{- \frac{\delta_i^{(j)}}{2} |1 - \frac{2}{p} | \delta^{(j)}|1 + \delta} - 1 \right) e^{3(\frac{1}{p} + \frac{1}{2})r}.
\]
Using that \( L = \max \{ 4, \frac{8}{p} \} \), \( r = \frac{p^2 (4 - p)}{2^{2n}} \), \( |\delta^{(j)}| \leq \delta \leq \frac{1}{32} \) and \( p \leq 4 \) we obtain
\[
\frac{e^{\delta_i^{(j+1)}} - 1}{e^{\pm 2r}} (e^{\delta_i^{(j)}} - 1) + \frac{1}{L} \left( e^{- \frac{\delta_i^{(j)}}{2} |1 - \frac{2}{p} | \delta^{(j)}|1 + \delta} - 1 \right) e^{3(\frac{1}{p} + \frac{1}{2})r}.
\]
where we used \( e^x = 1 + \frac{4}{3} x \) for \( |x| \leq \frac{1}{2} \) in the first equality, we used \( e^x = 1 + x \pm x^2 \) for \( |x| \leq \frac{1}{2} \) in
the second equality.
For the first two terms, we have that
\[
\left| \left( 1 - \frac{2}{pL} \right) \delta_i^{(j)} + \frac{1}{L} \left| 1 - \frac{2}{p} \right| \delta^{(j)} \right| \leq \left( 1 - \frac{2}{pL} + \frac{1}{L} \left| 1 - \frac{2}{p} \right| \right) \delta^{(j)}.
\]
Using this, \(1 + x \leq e^x\) and (B.5) and \(|\delta_i^{(j)}| \leq \delta^{(j)} \leq 2r\), we have
\[
\delta^{(j+1)} \leq \left( 1 - \frac{2}{pL} + \frac{1}{L} \left| 1 - \frac{2}{p} \right| + 40 \left( 1 + \frac{4}{p} \right) r \right) \delta^{(j)} + \frac{3\delta}{L}.
\]
Using our choice of \(r\), we have \(40(1 + \frac{4}{p}) r \leq \frac{1}{2L} (\frac{2}{p} - \left| 1 - \frac{2}{p} \right|)\) and hence
\[
\delta^{(j+1)} \leq \left( 1 - \frac{2}{2L} \left( \frac{2}{p} - \left| 1 - \frac{2}{p} \right| \right) \right) \delta^{(j)} + \frac{3\delta}{L}
\]
and hence for all \(j \in [m]\)
\[
\delta^{(j)} \leq \frac{1}{\frac{1}{2L} \left( \frac{2}{p} - \left| 1 - \frac{2}{p} \right| \right)} \cdot \frac{3\delta}{L} \leq \frac{\frac{8\delta}{L}}{\frac{2}{p} - \left| 1 - \frac{2}{p} \right|} \leq \frac{\epsilon}{4}.
\]
Applying Lemma 54 and Lemma 55, and recalling that \(k = \left[ 80(\frac{p}{2} + \frac{2}{p}) \log \left( \frac{pn}{2\epsilon} \right) \right]\), we have
\[
\left\| W_p^{-1}(w_p - w^{(k)}) \right\|_\infty \leq \left\| W_p^{-1}(w_p - v^{(k)}) \right\|_\infty + \left\| W_p^{-1}(v^{(k)} - w^{(k)}) \right\|_\infty \leq 4 \left( 1 + \frac{2}{p} \right)^2 \sqrt{n} \cdot \|w - w_p\| W_p^{-1} + 2\delta^{(k)} \leq 4 \left( 1 + \frac{2}{p} \right)^2 \sqrt{n} \cdot 2\sqrt{n} \cdot \left( 1 - \frac{1}{16(\frac{p}{2} + \frac{2}{p})} \right) \frac{k}{160} + 2\delta^{(k)} \leq \epsilon.
\]
Unfortunately, we cannot use the previous lemma directly as computing \(\sigma\) exactly is too expensive for our purposes. However, in [53, 15] they showed that we can compute leverage scores, \(\sigma\), approximately by solving only polylogarithmically many regression problems (See [40, 38, 64, 7] for more details). These results use the fact that the leverage scores of the the \(i^{th}\) constraint, i.e. \(\sigma(A)_i\) is the \(\ell_2\) length of vector \(A(A^T A)^{-1} A^T 1_i\) and that by the Johnson-Lindenstrauss Lemma these lengths are persevered up to multiplicative error if we project these vectors onto certain random low dimensional subspace. Consequently, to approximate the \(\sigma\) we first compute the projected vectors and then use it to approximate \(\sigma\) and hence only need to solve \(O(1)\) regression problems. For completeness, we provide an algorithm and theorem statement below most closely resembling the one from [53].

\begin{algorithm}
  \textbf{Algorithm 6:} \(\sigma^{(\text{apx})} = \text{computeLeverageScores}(A, \epsilon)\)
  \begin{algorithmic}
    \State Let \(q^{(j)}\) be \(k\) random \(\pm 1/\sqrt{k}\) vectors of length \(m\) with \(k = O(\log(m)/\epsilon^2)\).
    \State Compute \(l^{(j)} = (A^T A)^{-1} A^T q^{(j)}\) and \(p^{(j)} = A l^{(i)}\).
    \State Output: \(\sum_{j=1}^k \left( \frac{p^{(j)}}{p^{(j)}} \right)^2\).
  \end{algorithmic}
\end{algorithm}
Lemma 59. For \( \epsilon \in (0, 1) \) with probability at least \( 1 - \frac{1}{m^{O(1)}} \) the algorithm \( \text{computeLeverageScores} \) returns \( \sigma^{(\text{apx})} \) such that for all \( i \in [m] \), \( (1 - \epsilon) \sigma_i(A) \leq \sigma_i^{(\text{apx})} \leq (1 + \epsilon) \sigma_i(A) \), by solving only \( O(\epsilon^{-2} \cdot \log m) \) linear systems.

In the next section we show how to combine these results to obtain our main result on approximate weight computation, Theorem 39.

B.3 Initial Weight and Final Theorems

Here, we show how to compute an initial weight without having an approximate weight to help the computation. While we can use the results of the previous section during the iterations of our linear programming algorithms (as we have shown that the Lewis weights do not change too quickly) we still need to design a routine to compute the initial weights. Here we show that the algorithm \( \text{computeInitialWeight} \) (Algorithm 7) simply calls the weight computation algorithms of the previous sections \( O(\sqrt{m}) \) times by first computing Lewis weights for \( p = 2 \), i.e. leverage scores, and then gradually decreasing \( p \) can achieve this goal.

**Algorithm 7:** \( w = \text{computeInitialWeight}(A, p_{\text{target}}, \epsilon) \)

\[
p = 2
\]
\[
\text{while } p \neq p_{\text{target}} \text{ do}
\]
\[
\text{Let } r \text{ be defined as in } \text{computeApxWeight} \text{ or computeExactWeight}
\]
\[
h = \frac{\min(2, p)}{\sqrt{n} \log \frac{m}{n}} \cdot r
\]
\[
p^{(\text{new})} = \text{median}(p - h, p_{\text{target}}, p + h).
\]
\[
w = \text{computeApxWeight}(p^{(\text{new})}, w^{(\text{new})}, \frac{r}{r}), \text{or computeExactWeight}(p^{(\text{new})}, w^{(\text{new})}, \frac{r}{r})
\]
\[
p = p^{(\text{new})}.
\]
\[
\text{end}
\]
\[
\text{Output: } \text{computeApxWeight}(p_{\text{target}}, w, \epsilon).
\]

The correctness of the above algorithm directly follows from the following lemma:

Lemma 60. For all \( q > 0 \) let \( \tilde{w}_q \in \mathbb{R}^m_{>0} \) denote the vector with \( \tilde{w}_q[i] = [w_p(A)]_{i}^{q/p} \) for all \( i \in [m] \).

If \( |p - q| \leq \frac{\min(2, p)}{\sqrt{n} \log(m^n)} \) then

\[
\left\| \log \left( \frac{w_p(A)}{w_q} \right) \right\|_{\infty} \leq \max \left\{ \frac{1}{2}, \frac{1}{p} \right\} \sqrt{n} \log \left( \frac{me^2}{n} \right) \cdot |p - q|.
\]

\[
(B.6)
\]

Proof. For notational convenience let \( w = w_p(A), W \stackrel{\text{def}}{=} \text{Diag}(w) \) and \( \Lambda \stackrel{\text{def}}{=} \Lambda(W_p^{-1/2}^{-1/2} A) \). Taking derivative with respect to \( p \) on both sides and using Lemma 49 yields

\[
\frac{dw_p(A)}{dp} = 2 \Lambda \left[ \left( \frac{1}{2} - \frac{1}{p} \right) w^{-1/2} \frac{1 - \frac{1}{p}}{p^2} \log w + w^{1/2} \frac{1}{p} \right]
\]

\[
= \Lambda \left[ \left( 1 - \frac{2}{p} \right) W^{-1} \frac{dw}{dp} + 2 \frac{1}{p^2} \log w \right].
\]

Hence, we have that

\[
\frac{dw_p(A)}{dp} = 2W \left( W - \left( 1 - \frac{2}{p} \right) \Lambda \right)^{-1} \Lambda \cdot \frac{1}{p} \log w.
\]

\[
(B.7)
\]

\(^8\)This is the only place our algorithm uses randomness for general linear programs. Since we can verify the centrality of central path by computing leverage score exactly (instead of using this theorem) every \( m^{O(1)} \) iterations of interior point method, \( 1 - \frac{1}{m^{O(1)}} \) probability is high enough even for the case \( \epsilon \) is doubly exponentially small.
Lemma 24 and 25 shows that for all \( h \in \mathbb{R}^m \)

\[
\left\| 2 \left( \mathbf{W} - (1 - \frac{2}{p})\mathbf{A} \right)^{-1} \mathbf{Ah} - ph \right\|_{\infty} \leq p \cdot \max \left\{ \frac{p}{2}, 1 \right\} \cdot \|h\|_{\mathbf{W}}.
\]

Setting \( h = p^{-2} \log w \) and using (B.7) we have

\[
\left\| \mathbf{W}^{-1} \frac{dw_p(A)}{dp} - \frac{\ln w_p}{p} \right\|_{\infty} \leq p \cdot \max \left\{ \frac{p}{2}, 1 \right\} \cdot \|p^{-2} \log w\|_{\mathbf{W}} \leq \max \left\{ \frac{1}{2}, \frac{1}{p} \right\} \|\log w\|_{\mathbf{W}}.
\]

Finally, we note that

\[
\|\ln w\|_{\mathbf{W}} = \sum_{i \in [m]} w_i \log^2 w_i \leq \sum_{w_i \leq \frac{1}{2}} w_i \log^2 w_i + \sum_{w_i \in (\frac{1}{2}, 1]} w_i \leq n \log^2 \frac{m}{n} + n \leq n \log^2 \frac{me}{n}
\]

where we used that \( \log^2 w \) is concave on \([0, \frac{1}{e}]\) and \( \sum_{i \in [n]} w_i \leq n \).

Combining these bounds yields that for all \( q, w_q \overset{\text{def}}{=} w_q(A), \) and \( \mathbf{W}_q \overset{\text{def}}{=} \text{Diag}(w_q) \)

\[
\left\| \frac{d}{dq} \ln(w_q/\tilde{w}_q) \right\|_{\infty} = \left\| \mathbf{W}_q^{-1} \frac{d}{dq} w_q - \log(\tilde{w}_q) \left\|_{\infty} \leq \max \left\{ \frac{1}{2}, \frac{1}{p} \right\} \sqrt{n} \log \left( \frac{me}{n} \right) + \frac{1}{p} \|\ln(w_q/\tilde{w}_q)\|_{\infty}.
\]

Now let \( \delta \) be the largest number for which \( q \) satisfying \( |p - q| \leq \delta \) implies that \( \|\ln(w_q/\tilde{w}_q)\|_{\infty} \leq 1 \).

Since for all such \( q \) we have

\[
\left\| \frac{d}{dq} \ln(w_q/\tilde{w}_q) \right\|_{\infty} \leq \max \left\{ \frac{1}{2}, \frac{1}{p} \right\} \sqrt{n} \log \left( \frac{me^2}{n} \right)
\]

and \( \ln(w_p/w_p) = 0 \), integration yields that (B.6) holds for all such \( q \). Therefore, it must be the case that \( \delta \leq \left[ \max \left\{ \frac{1}{2}, \frac{1}{p} \right\} \sqrt{n} \log \left( \frac{me^2}{n} \right) \right]^{-1} \) and the result follows.

We now have everything we need to prove our main theorems regarding exact and approximate Lewis weight computation. First we prove the result on exact weight computation (Theorem 45) and then we prove the result on approximate weight computation (Theorem 39).

**Proof of Theorem 45.** From Lemma 60, we see that each step of \( p \), we lies within the requirement of Theorem 56. Furthermore, Lemma 60 shows that it takes \( O(\sqrt{n} \cdot (p + \frac{1}{p}) \cdot \log \frac{w_p}{w}) \) steps in the `computeInitialWeight`. Each call of `computeExactWeight` involves \( O((p + \frac{1}{p}) \log(nw^{-1}(1 + \frac{1}{p})) \) iterations and each iteration involves computing leverage score, which takes \( O(mn^2) \) work and \( O(\log m) \) depth.

**Proof of Theorem 39.** From Lemma 60, we see that each step of \( p \), we lies within the requirement of Theorem 58. Furthermore, Lemma 60 shows that it takes \( O(\sqrt{n} \cdot ((4 - p)^{-1} + p^{-2}) \cdot \log \frac{w_p}{w}) \) steps in the `computeInitialWeight`. Each call of `computeApxWeight` involves \( O(p^{-1} \log(n/(pe))) \) iterations and each iteration involves computing leverage score up to accuracy \( \frac{\epsilon}{32(\frac{3}{4} - 1 - \frac{2}{p})} = \Theta((4 - p) \cdot \epsilon) \).

Finally, 59 shows this involves solving solving \( O((4 - p)^{-2} \epsilon^{-2} \log m) \) many linear systems.

### C Chasing Game

The goal of this section is to prove the following theorem:
Theorem 18 ($\ell_\infty$ Chasing Game). For $x^{(0)}, y^{(0)} \in \mathbb{R}^m$ and $\epsilon \in (0, 1/5)$, consider the two player game consisting of repeating the following for $k = 1, 2, \ldots$

1. The adversary chooses $U^{(k)} \subseteq \mathbb{R}^m$, $u^{(k)} \in U^{(k)}$, and sets $y^{(k)} = y^{(k-1)} + u^{(k)}$.

2. The adversary chooses $z^{(k)}$ with $\|z^{(k)} - y^{(k)}\|_\infty \leq R$ and reveals $z^{(k)}$ and $U^{(k)}$ to the player.

3. The player chooses $\Delta^{(k)} \in (1 + \epsilon)U^{(k)}$ and sets $x^{(k)} = x^{(k-1)} + \Delta^{(k)}$.

Suppose that each $U^{(k)}$ is a symmetric convex set that contains an $\ell_\infty$ ball of radius $r_k$ and is contained in a $\ell_\infty$ ball of radius $R_k \leq R$ and the player plays the strategy

$$\Delta^{(k)} = \underset{\Delta \in (1+\epsilon)U^{(k)}}{\arg \min} \left\langle \nabla \Phi_{\mu}(x^{(k-1)} - z^{(k)}), \Delta \right\rangle$$

where $\Phi_{\mu}(x) \overset{\text{def}}{=} \sum_{i \in [m]} (e^{\mu x_i} + e^{-\mu x_i})$ and $\mu \overset{\text{def}}{=} \frac{\epsilon}{12R}$.

If $\Phi_{\mu}(x^{(0)} - y^{(0)}) \leq \frac{12m\tau}{\epsilon}$ for $\tau = \max_k \frac{R_k}{r_k}$ then this strategy guarantees that for all $k$ we have

$$\Phi_{\mu}(x^{(k)} - y^{(k)}) \leq \frac{12m\tau}{\epsilon} \quad \text{and} \quad \|x^{(k)} - y^{(k)}\|_\infty \leq \frac{12R}{\epsilon} \log \left( \frac{12m\tau}{\epsilon} \right).$$

This theorem says that taking “projected gradient steps” using the potential function $\Phi_{\mu}(x)$, suffices to maintain a point $x^{(k)}$ sufficiently close to $y^{(k)}$ with respect to $\ell_\infty$ provided that $y^{(k)}$ are updated by a direction in $U^{(k)}$, noisy $z^{(k)}$ measurements to the $y^{(k)}$ are available, and slightly large movements to the $x^{(k)}$ (i.e. by $(1 + \epsilon)U^{(k)}$) are allowed. Formally, this theorem analyze the strategy of updating $x^{(k)}$ by setting the change, $\Delta^{(k)}$, to be the vector in $(1 + \epsilon)U^{(k)}$ that best minimizes the potential function of the observed position difference, i.e. $\Phi_{\mu}(x^{(k)} - z^{(k)})$ for careful choice of $\mu$.

To prove Theorem 18, we first show the following properties of the potential function $\Phi_{\mu}$.

Lemma 61. For all $x \in \mathbb{R}^m$ and $\mu > 0$, we have

$$e^{\mu \|x\|_\infty} \leq \Phi_{\mu}(x) \leq 2me^{\mu \|x\|_\infty} \quad \text{and} \quad \mu \Phi_{\mu}(x) - 2\mu m \leq \|\nabla \Phi_{\mu}(x)\|_1$$  \hspace{1cm} (C.1)

Furthermore, for any symmetric convex set $U \subseteq \mathbb{R}^m$ and any $x \in \mathbb{R}^m$, let $x^\ast \overset{\text{def}}{=} \arg \max_{y \in U} \langle x, y \rangle$ and $\|x\|_U \overset{\text{def}}{=} \max_{y \in U} \|x - y\|_\infty \leq \delta \leq \frac{1}{5\mu}$ we have

$$e^{-\mu \delta} \|\nabla \Phi_{\mu}(y)\|_U - \mu \|\nabla \Phi_{\mu}(y)^\ast\|_1 \leq \left\langle \nabla \Phi_{\mu}(x), \nabla \Phi_{\mu}(y)^\ast \right\rangle \leq e^{\mu \delta} \|\nabla \Phi_{\mu}(y)\|_U + \mu e^{\mu \delta} \|\nabla \Phi_{\mu}(y)^\ast\|_1.$$  \hspace{1cm} (C.2)

If additionally $U$ is contained in a $\ell_\infty$ ball of radius $R$ then

$$e^{-\mu \delta} \|\nabla \Phi_{\mu}(y)\|_U - \mu m R \leq \|\nabla \Phi_{\mu}(x)\|_U \leq e^{\mu \delta} \|\nabla \Phi_{\mu}(y)\|_U + \mu e^{\mu \delta} m R.$$  \hspace{1cm} (C.3)

Proof. For, notational convenience let $p_u(x) \overset{\text{def}}{=} e^{\mu x} + e^{-\mu x}$ for all $x \in \mathbb{R}$ so that $\Phi_{\mu}(x) = \sum_{i \in [m]} p_u(x_i)$. Equation (C.1) follows from the fact that for all $x \in \mathbb{R}$,

$$e^{\mu \|x\|} \leq p_u(x) \leq 2e^{\mu \|x\|} \quad \text{and} \quad p_{\mu}'(x) = \mu \text{sign}(x) \left( e^{\mu \|x\|} - e^{-\mu \|x\|} \right).$$

Next, let $x, y \in \mathbb{R}$ with $|x - y| \leq \delta$. Note that $|p_{\mu}'(x)| = |p_{\mu}'(|x|)| = \mu (e^{\mu \|x\|} - e^{-\mu \|x\|})$ and $|x - y| \leq \delta$.
implies that $|x| = |y| + z$ for some $z \in [-\delta, \delta]$. Using that $p'(|x|)$ is monotonic in $|x|$ we then have

$$|p'_\mu(x)| = p'_\mu(|x|) = p'_\mu(|y| + z) \leq p'_\mu(|y| + \delta) = \mu \left( e^{\mu|y|+\mu\delta} - e^{-\mu|y| - \mu\delta} \right)$$

$$= e^{\mu\delta} p'(|y|) + \mu \left( e^{\mu\delta - \mu|y|} - e^{-\mu|y| - \mu\delta} \right) \leq e^{\mu\delta} |p'(y)| + \mu e^{\mu\delta}. \quad (C.4)$$

By symmetry (i.e. replacing $x$ and $y$) this implies that

$$|p'_\mu(x)| \geq e^{-\mu\delta} |p'(y)| - \mu \quad (C.5)$$

Since $U$ is symmetric this implies that for all $i \in [m]$ we have $\operatorname{sign}(\nabla \Phi_\mu(y)^i) = \operatorname{sign}(\nabla \Phi_\mu(y)^i) = \operatorname{sign}(y_i)$. Therefore, if for all $i \in [n]$ we have $\operatorname{sign}(x_i) = \operatorname{sign}(y_i)$, by (C.4), we see that

$$\left\langle \nabla \Phi_\mu(x), \nabla \Phi_\mu(y)^b \right\rangle = \sum_{i \in [m]} p'_\mu(x_i) \nabla \Phi_\mu(y)^i \leq \sum_{i \in [m]} \left( e^{\mu\delta} p'_\mu(y_i) + \mu e^{\mu\delta} \right) \nabla \Phi_\mu(y)^i$$

$$\leq e^{\mu\delta} \left\langle \nabla \Phi_\mu(y), \nabla \Phi_\mu(y)^b \right\rangle + \mu e^{\mu\delta} \|\nabla \Phi_\mu(y)^b\|_1$$

$$= e^{\mu\delta} \|\nabla \Phi_\mu(y)\| U + \mu e^{\mu\delta} \|\nabla \Phi_\mu(y)^b\|_1.$$ 

Similarly, using (C.5), we have $e^{-\mu\delta} \|\nabla \Phi_\mu(y)\| U - \mu \|\nabla \Phi_\mu(y)^b\|_1 \leq \left\langle \nabla \Phi_\mu(x), \nabla \Phi_\mu(y)^b \right\rangle$ and hence (C.2) holds. On the other hand if $\operatorname{sign}(x_i) \neq \operatorname{sign}(y_i)$ then we know that $|x_i| \leq \delta$ and consequently $|p'_\mu(x_i)| \leq \mu (e^{\mu\delta} - e^{-\mu\delta}) \leq \frac{\mu}{2}$ since $\delta \leq \frac{1}{2\mu}$. Thus, we have

$$e^{-\mu\delta} |p'_\mu(y_i)| - \mu \leq -\frac{\mu}{2} \leq \operatorname{sign}(y_i) p'_\mu(x_i) \leq 0 \leq e^{\mu\delta} |p'_\mu(y_i)| + \mu e^{\mu\delta}.$$

Taking inner product on both sides with $\nabla \Phi_\mu(y)^b$ and using definition of $\| \cdot \|_U$ and $^b$, we get (C.2). Thus, (C.2) holds in general.

Finally we note that since $U$ is contained in a $\ell_\infty$ ball of radius $R$, we have $\|y^b\|_1 \leq mR$ for all $y$. Using this fact, (C.2), and the definition of $\| \cdot \|_U$, we obtain

$$e^{-\mu\delta} \|\nabla \Phi_\mu(y)\| U - \mu mR \leq \left\langle \nabla \Phi_\mu(x), \nabla \Phi_\mu(y)^b \right\rangle \leq \|\nabla \Phi_\mu(x)\| U$$

where the last inequality additionally uses $\nabla \Phi_\mu(y)^b \in U$. By symmetry (C.3) follows. \hfill \Box

**Proof of Theorem 18.** For the remainder of the proof, let $\|x\|_{U^{(k)}} = \max_{y \in U^{(k)}} \langle x, y \rangle$ and $x^{b(k)} = \arg\max_{y \in U^{(k)}} \langle x, y \rangle$. Since $U^{(k)}$ is symmetric, we know that $\Delta^{(k)} = -(1+\epsilon) \left( \nabla \Phi_\mu(x^{(k-1)} - y^{(k)}) \right)^{b(k)}$ and therefore by applying the mean value theorem twice we have that

$$\Phi^{(k)}_\mu(x^{(k)} - y^{(k)}) = \Phi^{(k-1)}_\mu(x^{(k-1)} - y^{(k)}) + \left\langle \nabla \Phi^{(k)}_\mu(\zeta_1), x^{(k)} - x^{(k-1)} \right\rangle$$

$$= \Phi^{(k)}_\mu(x^{(k-1)} - y^{(k)}) + \left\langle \nabla \Phi^{(k)}_\mu(\zeta_2), y^{(k)} - y^{(k-1)} \right\rangle + \left\langle \nabla \Phi^{(k)}_\mu(\zeta_1), x^{(k-1)} - x^{(k-1)} \right\rangle$$

for some $\zeta_1$ between $x^{(k)} - y^{(k)}$ and $x^{(k-1)} - y^{(k)}$ and some $\zeta_2$ between $x^{(k-1)} - y^{(k)}$ and $x^{(k-1)} - y^{(k-1)}$. 

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59
Now, using that $y(k) - y(k-1) \in U(k)$ and that $x(k) - x(k-1) = \Delta(k)$ we have
\[
\Phi_\mu(x(k) - y(k)) \leq \Phi_\mu(x(k-1) - y(k-1)) + \|\nabla \Phi_\mu(\zeta_2)\|_{U(k)} - (1 + \epsilon) \left( \nabla \Phi_\mu(\zeta_1), \left( \nabla \Phi_\mu(x(k-1) - z(k)) \right)^{b(k)} \right).
\] (C.6)

Since $U^k$ is contained within the $\ell_\infty$ ball of radius $R_k$, Lemma 61 shows that
\[
\|\nabla \Phi_\mu(\zeta_2)\|_{U(k)} \leq e^{\mu R_k} \|\nabla \Phi_\mu(x(k-1) - y(k-1))\|_{U(k)} + m \mu R_k e^{\mu R_k}.
\] (C.7)

Furthermore, since $\epsilon < \frac{1}{5}$ and $R_k \leq R$, by triangle inequality we have $\|\zeta_1 - (x(k-1) - z(k))\|_\infty \leq (1 + \epsilon)R_k + R \leq 3R$ and $\|z(k) - y(k-1)\|_\infty \leq 2R$. Therefore, applying Lemma 61 twice yields that
\[
\left( \nabla \Phi_\mu(\zeta_1), \left( \nabla \Phi_\mu(x(k-1) - z(k)) \right)^{b(k)} \right) \geq e^{-3\mu R} \|\nabla \Phi_\mu(x(k-1) - z(k))\|_{U(k)} - \mu m R_k
\] \[
\geq e^{-5\mu R} \|\nabla \Phi_\mu(x(k-1) - y(k-1))\|_{U(k)} - 2\mu m R_k.
\] (C.8)

Combining (C.6), (C.7), and (C.8) then yields that
\[
\Phi_\mu(x(k) - y(k)) \leq \Phi_\mu(x(k-1) - y(k-1)) - ((1 + \epsilon)e^{-5\mu R} - e^{\mu R}) \|\nabla \Phi_\mu(x(k-1) - y(k-1))\|_{U(k)}
\] \[
+ m \mu R_k e^{\mu R} + 2(1 + \epsilon) m \mu R_k.
\]

Since we chose $\mu = \frac{\epsilon}{12R}$ and $\epsilon \in (0, 1/5)$ we have $(1 + \epsilon)e^{-5\mu R} - e^{\mu R} \geq \frac{2\epsilon}{5}$ and
\[
m \mu R_k e^{\mu R} + 2(1 + \epsilon) m \mu R_k \leq \epsilon m \frac{7R_k}{24R}.
\]

Thus, we have
\[
\Phi_\mu(x(k) - y(k)) \leq \Phi_\mu(x(k-1) - y(k-1)) - \frac{2\epsilon}{5} \|\nabla \Phi_\mu(x(k-1) - y(k-1))\|_{U(k)} + \epsilon m \frac{7R_k}{24R}.
\]

Using Lemma 61 and the fact that $U_k$ contains a $\ell_\infty$ ball of radius $r_k$, we have
\[
\|\nabla \Phi_\mu(x(k-1) - y(k-1))\|_{U(k)} \geq r_k \|\nabla \Phi_\mu(x(k-1) - y(k-1))\|_1 \geq \frac{e r_k}{12R} \left( \Phi_\mu(x(k-1) - y(k-1)) - 2m \right).
\]

Therefore, we have that
\[
\Phi_\mu(x(k) - y(k)) \leq \left( 1 - \frac{e^2 r_k}{30R} \right) \Phi_\mu(x(k-1) - y(k-1)) + \frac{e^2 r_k}{15R} m + \epsilon m \frac{7R_k}{24R}
\] \[
\leq \left( 1 - \frac{e^2 r_k}{30R} \right) \Phi_\mu(x(k-1) - y(k-1)) + \epsilon m \frac{R_k}{3R}.
\]

Hence, if $\Phi_\mu(x(k-1) - y(k-1)) \leq \frac{12m \epsilon}{\epsilon}$, we have $\Phi_\mu(x(k) - y(k)) \leq \frac{12m \epsilon}{\epsilon}$. Since $\Phi_\mu(x(0) - y(0)) \leq \frac{12m \epsilon}{\epsilon}$ by assumption we have by induction that $\Phi_\mu(x(k) - y(k)) \leq \frac{12m \epsilon}{\epsilon}$ for all $k$. The necessary bound on $\|x(k) - y(k)\|_\infty$ then follows immediately from Lemma 61. \qed

D Appendix: Projection on Mixed Norm Ball

Here we give an algorithm to solve the following problem
\[
\max_{\|x\|_2 + \|t^{-1}x\|_\infty \leq 1} \langle a, x \rangle
\] (D.1)
for some given vector $l$ and $a$. This is used in Section 6 to compute weights. Note that

\[
\begin{align*}
\max_{\|x\|_2 + \|t^{-1}x\|_\infty \leq 1} \langle a, x \rangle &= \max_{0 \leq t \leq 1} \left[ \max_{\|x\|_2 \leq 1 - t \text{ and } -t_i \leq x_i \leq t_i} \langle a, x \rangle \right] \\
&= \max_{0 \leq t \leq 1} (1 - t) \left[ \max_{\|x\|_2 \leq 1} \max_{-\frac{t}{1-t} t_i \leq x_i \leq \frac{t}{1-t} t_i} \langle a, x \rangle \right] \\
&= \max_{0 \leq t \leq 1} (1 - t) f(t) \text{ where } f(t) \overset{\text{def}}{=} \max_{\|x\|_2 \leq 1} \max_{-\frac{t}{1-t} t_i \leq x_i \leq \frac{t}{1-t} t_i} \langle a, x \rangle. \tag{D.2}
\end{align*}
\]

After sorting the coordinates so that $|a_i|/l_i$ monotonically decrease with $i \in [n]$, and considering the maximization problem in $f(t)$ with only the $\|x\|_2$ or $-\frac{t}{1-t} t_i \leq x_i \leq \frac{t}{1-t} t_i$ constraints, it can be shown that the maximizing $x$ in the definition of $f$ is $x^{it}$ where for all $j \in [n]$

\[
x^{it}_j = \begin{cases} 
\frac{t}{1-t} \text{sign}(a_j) l_j \\ 
\sqrt{\frac{1 - \left(\frac{t}{1-t}\right)^2 \sum_{k \in [j]} l_k^2}{\|a\|_2^2 - \sum_{k \in [i]} a_k^2}} a_j 
\end{cases} \quad \text{if } j \in [i_t] \\
\sqrt{\frac{1 - \left(\frac{t}{1-t}\right)^2 \sum_{k \in [j]} l_k^2}{\|a\|_2^2 - \sum_{k \in [i]} a_k^2}} a_j \quad \text{otherwise}.
\tag{D.3}
\]

and $i_t$ is the first coordinate $i \in [n]$ such that

\[
\frac{1 - \left(\frac{t}{1-t}\right)^2 \sum_{k \in [j]} l_k^2}{\|a\|_2^2 - \sum_{k \in [i]} a_k^2} \leq \frac{\left(\frac{t}{1-t}\right)^2 l_i^2}{a_i^2}.
\]

Note that $i_t \geq i_s$ if $t \leq s$. Therefore, the set of $t$ such that $i_t = j$ is simply an interval given by\(^9\)

\[
\sqrt{l_j^2 \left(\|a\|_2^2 - \sum_{k \in [j]} a_k^2\right) + a_j^2 \sum_{k \in [j]} l_k^2} \leq \frac{t}{1-t} < \sqrt{l_{j-1}^2 \left(\|a\|_2^2 - \sum_{k \in [j-1]} a_k^2\right) + a_{j-1}^2 \sum_{k \in [j-1]} l_k^2}.
\tag{D.4}
\]

Therefore, we know that

\[
f(t) = \langle a, x^{(i_t)} \rangle = \frac{t}{1-t} \sum_{j \in [i_t]} |a_j| |l_j| + \sqrt{1 - \left(\frac{t}{1-t}\right)^2 \sum_{k \in [i_t]} l_k^2} \sqrt{\|a\|_2^2 - \sum_{k \in [i_t]} a_k^2}.
\]

Substituting this into D.2, we have that

\[
\max_{\|x\|_2 + \|t^{-1}x\|_\infty \leq 1} \langle a, x \rangle = \max_{0 \leq t \leq 1} g(t) \overset{\text{def}}{=} t \sum_{j \in [i_t]} |a_j| |l_j| + \sqrt{(1-t)^2 - t^2 \sum_{k \in [i_t]} l_k^2} \sqrt{\|a\|_2^2 - \sum_{k \in [i_t]} a_k^2}.
\]

\(^9\)There are some boundary cases we ignored for simplicity.
Note that
\[
g'(t) = \sum_{j \in [i]} |a_j| |l_j| + \frac{((1 - \sum_{k \in [i]} t_k^2) t - 1) \sqrt{\|a\|_2^2 - \sum_{k \in [i]} a_k^2}}{\sqrt{(1 - t)^2 - t^2 \sum_{k \in [i]} t_k^2}},
\]
\[
g''(t) = -\frac{\left(\sum_{k \in [i]} t_k^2\right) \cdot \sqrt{\|a\|_2^2 - \sum_{k \in [i]} a_k^2}}{\left((1 - t)^2 - t^2 \sum_{k \in [i]} t_k^2\right)^{3/2}}.
\]

Hence, \( g(t) \) is concave and its maximizer has a closed form via the quadratic formula. Therefore, one can compute the maximum value for each interval of \( t \) \((D.4)\) and find which is the best. This yields the following algorithm.

**Algorithm 8:** \( x = \text{projectMixedBall}(a, l) \)

1. Sort the coordinate such that \( |a_i| / l_i \) is in descending order.
2. Precompute \( \sum_{k=0}^{i} t_k^2, \sum_{k=0}^{i} a_k^2 \) and \( \sum_{j=1}^{i} |a_j| |l_j| \) for all \( i \).
3. Let \( g_i(t) = t \sum_{j \in [i]} |a_j| |l_j| + \sqrt{(1 - t)^2 - t^2 \sum_{k=0}^{i} t_k^2} \cdot \sqrt{\|a\|_2^2 - \sum_{k=0}^{i} a_k^2} \).
4. For each \( j \in \{1, \cdots, n\} \), find \( t_j = \arg \max_{t_{i=j}} g_j(t) \) using \((D.4)\).
5. Find \( i = \arg \max_j g_j(t_i) \).
6. **Output:** \( (1 - t_i) x^{(i)} \) defined by \((D.3)\).

The discussion above leads to the following theorem.

**Theorem 62.** For any \( a \in \mathbb{R}^n \) and \( l \in \mathbb{R}_{>0}^n \), the algorithm \( \text{projectMixedBall}(a, l) \) outputs a solution to \((D.1)\) in total work \( O(n \log n) \) and depth \( O(\log n) \) (in EREW model).

**Proof.** The correctness follows from the discussion above. For the runtime, it is known that sorting can be done in \( O(n \log n) \) work and \( O(\log n) \) depth in EREW model \([10]\) and that prefix sum can be done in \( O(n) \) work and \( O(\log n) \) depth in EREW model. The rest is easy. \(\square\)

### E Extreme Lewis Weights and Barrier

In this section we discuss the limits of Lewis weights and the Lewis weight barrier when \( p \to 0 \) and \( p \to \infty \). In Section E.1 we show that as \( p \to 0 \) Lewis weights converge to the uniform distribution over rows of a matrix under mild assumptions. This shows that under mild assumptions on the structure of a polytope, the Lewis weight barrier considered in Section 5 converges to the standard logarithmic barrier. In Section E.2 we consider the opposite extreme when \( p \to \infty \). In this case we show that \( \ell_\infty \) Lewis weights of the matrix \( A \) are precisely the weights that induce a John ellipses of the polytope \( \{ x \in \mathbb{R}^n : \|Ax\|_\infty \leq 1 \} \). This justifies the intuition given in the introduction regarding our barrier and path finding scheme as following a path induced by regularized John ellipses.

#### E.1 \( p \) Tends to 0

Here we show that \( \ell_p \) Lewis weights for a matrix \( A \in \mathbb{R}^{m \times n} \) in general position, i.e. any \( n \) rows are linearly independent, converge to uniform as \( p \to 0 \). Note that the assumption of general position is stronger than that of non-degeneracy and required for the statement to be true. For example, if there is a row that is perpendicular to all other rows, then it is not difficult to show that this row must have Lewis weight 1 for any \( p > 0 \).

**Lemma 63.** Given a matrix \( A \in \mathbb{R}^{m \times n} \) in general position, i.e. any \( n \) rows of \( A \) are linearly independent, then
\[
\lim_{p \to 0^+} w_p(A)_i = \frac{n}{m} \text{ for all } i.
\]
Proof. For $p \in (0, 2)$, Lemma 22 shows that the Lewis weight is given by
\[
w_p(A) = \arg\min_{w \in \mathbb{R}^n_{\geq 0}, \sum_{i \in [m]} w_i = n} \det(A^T W^{1-p} A).
\]
Considering $w \in \mathbb{R}^n$ with $w_i = \frac{n}{m}$ for all $i \in [m]$ we see that
\[
\min_{w \geq 0, \sum_{i \in [m]} w_i = n} \det(A^T W^{1-p} A) \leq \left(\frac{n}{m}\right)^{n(1-\frac{2}{p})} \det(A^T A).
\]
On the other hand the Cauchy–Binet formula shows that
\[
\det(A^T W^{1-p} A) = \sum_{S \subseteq \{m\}} \det(A_S)^2 \det(W_{S}^{1-p})
\]
where $A_S \subset \mathbb{R}^{n \times n}$ are the rows of $A$ at indices from $S$, $W_S \subset \mathbb{R}^{n \times n}$ is diagonal with the diagonals of $W$ at indices from $S$ and the summation is over all subsets of size $n$. Since $A$ is in general position, we have that $\det A_S \neq 0$ for all $S$. Therefore, for all subsets $S \subset [m]$ of size $n$ and all $W \succeq 0$.
\[
\det(W_{S}^{1-p}) \leq \frac{\det(A^T W^{1-p} A)}{\min_{S \subseteq \{m\}} \det(A_S)^2}.
\]
Now, let $W_p = \text{Diag}(w_p(A))$ be the diagonal matrix formed by the $\ell_p$ Lewis weight of $A$. Combining (E.1) and (E.2), we have that
\[
\det([W_p^{1-\frac{2}{p}}]_S) \leq c \left(\frac{n}{m}\right)^{n(1-\frac{2}{p})} \text{ where } c = \frac{\det A^T A}{\min_{S \subseteq \{m\}} (\det A_S)^2}.
\]
Hence, we have that $\det([W_p^{1-\frac{2}{p}}]_S) \geq c \cdot \left(\frac{n}{m}\right)^n$. Let $W_* = \lim_{p \to 0^+} W_p$. Taking limit $p \to 0^+$ on both sides, we have that $\det(W_*^S) \geq (n/m)^n$ for all subsets $S$ of size $n$. Since this holds for all subsets and since $\sum_{i \in [m]} w_i^n = n$, we have that $w_i^n = \frac{n}{m}$ for all $i$. Since
\[
\limsup_{p \to 0^+} \sum_{i \in [m]} w_p(A)_i = n = \liminf_{p \to 0^+} \sum_{i \in [m]} w_p(A)_i
\]
this shows that $\lim w_p$ exists and it converges to $\frac{n}{m}$.

E.2 $p$ Tends to Infinity

Here we show that as $p \to \infty$ the ellipse $E = \{x \in \mathbb{R}^n : x^T M x \leq 1\}$ for $M \overset{\text{def}}{=} \lim_{p \to +\infty} A^T W_p A$ where $W_p = w_p(A)$ is the John ellipse of the polytope $K = \{x \in \mathbb{R}^n : \|Ax\|_\infty \leq 1\}$, i.e. the ellipsoid of maximum volume contained inside $K$. To prove this we use the following lemma proved in [25] characterizing the John Ellipse.

Lemma 64. Given a polytope $\Omega = \{x \in \mathbb{R}^n : \|Ax\|_\infty \leq 1\}$ with $A \in \mathbb{R}^{m \times n}$. Let $E$ be the John ellipsoid of $\Omega$, namely, $E$ is the maximum volume ellipsoid contained inside $\Omega$. Then, we have that $E = \{x^T A^T W A x \leq 1\}$ with the diagonal matrix $W$ given by the vector maximizing
\[
\min_{w_i \geq 0, \sum_{i \in [m]} w_i = n} \log \det A^T W A.
\]
Using this we prove our desired result regarding the limits of Lewis weights as \( p \to \infty \).

**Lemma 65.** For non-degenerate \( A \in \mathbb{R}^{m \times n} \) let \( M \overset{\text{def}}{=} \lim_{p \to +\infty} A^\top W_p A \) where \( W_p = \text{Diag}(w_p(A)) \). Then \( E = \{ x \in \mathbb{R}^n : x^\top M x \leq 1 \} \) is the John ellipsoid of \( K = \{ x \in \mathbb{R}^n : \|Ax\|_\infty \leq 1 \} \).

**Proof.** Let \( c_{p,m} \) be the constant defined in Lemma 27. Further, for all \( p > 2 \) let \( M_p = c_{p,m}^2 A^\top W_p A \) and \( E_p = \{ x \in \mathbb{R}^n : x^\top M_p x \leq 1 \} \). Lemma 27 shows that \( E_p \subseteq K \). Further, letting \( s_n \) is the volume of the unit sphere we have that

\[
\text{vol}(E_p) = s_n \left( c_{p,m}^{2n} \det(A^\top W_p A) \right)^{-\frac{1}{2}} \geq s_n \left( c_{p,m}^{2n} \det(A^\top W_p^{1 - \frac{2}{p}} A) \right)^{-\frac{1}{2}}
\]

\[
= s_n \left( c_{p,m}^{2n} \cdot \min_{w_i \geq 0, \sum_{i=1}^n w_i = n} \log \det A^\top W^{1 - \frac{2}{p}} A \right)^{-\frac{1}{2}}
\]

where in the last step we used Lemma 22. Note that \( w^{1 - \frac{2}{p}} \to w \) as \( p \to \infty \) for all \( w > 0 \) and \( c_{p,m} \to 1 \) as \( p \to \infty \). Hence,

\[
\lim_{p \to +\infty} \text{vol}(E_p) \geq s_n \cdot \left( \min_{w_i \geq 0, \sum_{i=1}^n w_i = n} \log \det A^\top W A \right)^{-\frac{1}{2}}
\]

On the other hand, it is known that the John ellipsoid \( E^* \) of \( K \) is unique and its volume is given by the right hand side (Lemma 64). This implies that \( E_p \) converges to the John ellipsoid of \( K \). \( \square \)