ON ISOMETRIC CORRESPONDENCE OF LEAVES

ION I. DINCĂ

Abstract. We prove that for a generic 3-dimensional integrable rolling distribution of contact elements (excluding developable seed and isotropic developable leaves) isometric correspondence of leaves of a general nature (independent of the shape of the seed) requires the Bäcklund transformation.

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1. Introduction

We shall consider the complexification

$$(\mathbb{C}^3, \langle \cdot, \cdot \rangle), \quad \langle x, y \rangle := x^T y, \quad |x|^2 := x^T x, \quad x, y \in \mathbb{C}^3$$

of the real 3-dimensional Euclidean space; in this setting surfaces are 2-dimensional objects of $\mathbb{C}^3$ depending on two real or complex parameters.

Isotropic (null) vectors are those vectors of length 0; since most vectors are not isotropic we call a vector simply vector and we shall emphasize isotropic for isotropic vectors. The same denomination will apply in other settings: for example we call quadric a non-degenerate quadric (a quadric projectively equivalent to the complex unit sphere).

Consider Lie’s viewpoint: one can replace a surface $x \subset \mathbb{C}^3$ with a 2-dimensional distribution of contact elements (pairs of points and planes passing through those points; the classical geometers call them facets): the collection of its tangent planes (with the points of tangency highlighted); thus a contact element is the infinitesimal version of a surface (the integral element $(x, dx)|_{pt}$ of the surface). Conversely, a 2-dimensional distribution of contact elements is not always the collection of the tangent planes of a surface (with the points of tangency highlighted), but the condition that a 2-dimensional distribution of contact elements is integrable (that is it is the collection of the tangent planes of a leaf (sub-manifold)) does not distinguish between the cases when this sub-manifold is a surface, curve or point, thus allowing the collapsing of the leaf.

A 3-dimensional distribution of contact elements is integrable if it is the collection of the tangent planes of an 1-dimensional family of leaves.

Tworollable (applicable or isometric) surfaces can be rolled (applied) one onto the other such that at any instant they meet tangentially and with same differential at the tangency point.

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Definition 1.1. The rolling of two isometric surfaces $x_0, x \subset \mathbb{C}^3$ (that is $|dx_0|^2 = |dx|^2$) is the surface, curve or point $(R, t) \subset O_3(\mathbb{C}) \times \mathbb{C}^3$ such that $(x, dx) = (R, t)(x_0, dx_0) := (Rx_0 + t, Rdx_0)$.

The rolling introduces the flat connection form (it encodes the difference of the second fundamental forms of $x_0, x$ and it being flat encodes the difference of the Gauß-Codazzi-Mainardi-Peterson equations of $x_0, x$).

Definition 1.2. Consider an integrable 3-dimensional distribution of contact elements $F = (p, m)$ centered at $p = p(u, v, w)$, with normal fields $m = m(u, v, w)$ and distributed along the surface $x_0 = x_0(u, v)$. If we roll $x_0$ on an isometric surface $x$ (that is $(x, dx) = (R, t)(x_0, dx_0) := (Rx_0 + t, Rdx_0)$), then the rolled distribution of contact elements is $(Rp + t, Rm)$ and is distributed along $x$; if it remains integrable for any rolling, then the distribution is called 3-dimensional integrable rolling distribution of contact elements with seed $x$ and leaf $Rp + t$.

Bianchi considered the most general form of a Bäcklund transformation as the focal surfaces (one transform of the other) of a Weingarten congruence (congruence upon whose two focal surfaces the asymptotic directions correspond; equivalently the second fundamental forms are proportional). Note that although the correspondence provided by the Weingarten congruence does not give the applicability (isometric) correspondence, the Bäcklund transformation is the tool best suited to attack the isometric deformation problem via geometric transformation, since it provides correspondence of the characteristics of the isometric deformation problem (according to Darboux these are the asymptotic directions), it is directly linked to the infinitesimal isometric deformation problem (Darboux proved that infinitesimal isometric deformations generate Weingarten congruences and Guichard proved the converse: there is an infinitesimal isometric deformation of a focal surface of a Weingarten congruence in the direction normal to the other focal surface; see Darboux (II, § 883-§ 924)) and it admits a version of the Bianchi Permutability Theorem for its second iteration.

With $d := \partial_u \cdot du + \partial_v \cdot dv + \partial_w \cdot dw = \partial_0 + \partial_u \cdot dw$, $V := p - x_0$ if the 3-dimensional distribution of contact elements is integrable and the rolled distribution remains integrable if we roll $x_0$ on an isometric surface $x$, $(x, dx) = (R, t)(x_0, dx_0)$ (that is we replace $x_0, V, m$ with $x, RV, Rm$), then along the leaves we have

$$0 = (Rm)^T \tilde{d}(RV + x) = m^T (\omega \times V + d(V + x_0) + \partial_w V dw),$$

$$\omega := N_0 \times R^{-1}dRN_0, N_0 := \frac{\partial_u x_0 \times \partial_v x_0}{|\partial_u x_0 \times \partial_v x_0|}.$$

Since we shall not need the integrability condition of this general integrable rolling distribution of contact elements, we shall not derive it.

Definition 1.3. A 3-dimensional integrable rolling distribution of contact elements is called generic if $m^T \partial_w V (m \times V) \times N_0 \neq 0$.

In [3] we proved that for a generic 3-dimensional integrable rolling distribution of contact elements (excluding developable seed and isotropic developable leaves) and with the symmetry of the tangency configuration (contact elements are centered on tangent planes of the surface $x_0$ and further pass through the origin of the tangent planes) the seed and any leaf are the focal surfaces of a Weingarten congruence (and thus we get Bäcklund transformation according to Bianchi’s definition) and for a generic 3-dimensional integrable rolling distribution of contact elements (excluding developable seed and isotropic developable leaves) isometric correspondence of leaves of a general nature (independent of the shape of the seed) requires the tangency configuration (contact elements are centered on tangent planes of the surface $x_0$). Further by applying, if necessary, a change of
variables \( w = w(\tilde{w}, u, v) \) we have \( N_0^T[d(V + x_0) \times \wedge d(V + x_0)] \neq 0 \) and we get the condition
\[
d(V + x_0)^T(I_3 - \frac{m N_0^T}{m^T N_0}) \odot [d(\frac{N_0 m^T}{m^T N_0}) \partial_w V - \partial_w(\frac{N_0 m^T}{m^T N_0}) d(V + x_0) + \frac{d(\frac{N_0 m^T}{m^T N_0}) \wedge d(V + x_0)}{N_0^T d(V + x_0) \times \wedge d(V + x_0)}] = 0, \quad du \wedge dv \wedge dw \neq 0.
\]

According to Bianchi (referring to 3-dimensional integrable rolling distributions of contact elements) "But, in view of the eventual applications to problems of deformations, it is opportune to limit the problem much more, and to suppose that every facet \( f \) and each of its associated facets \( f' \) has the center of one in the plane of the other".

We have now the main Theorem of this paper:

**Theorem 1.4.** For a generic 3-dimensional integrable rolling distribution of contact elements (excluding developable seed and isotropic developable leaves) isometric correspondence of leaves of a general nature (independent of the shape of the seed) requires the Bäcklund transformation.

The remaining part of the paper is organized as follows: in Section 2 we recall the rolling problem for surfaces and in Section 3 we provide the proof of Theorem 1.4.

2. THE ROLLING PROBLEM FOR SURFACES

Let \((u, v) \in D\) with \(D\) domain of \(\mathbb{R}^2\) or \(\mathbb{C}^2\) and \(x : D \rightarrow \mathbb{C}^3\) be a surface.

For \(\omega_1, \omega_2 \in \mathbb{C}^3\)-valued 1-forms on \(D\) and \(a, b \in \mathbb{C}^3\) we have
\[
a^T \omega_1 \wedge b^T \omega_2 = ((a \times b) \times \omega_1 + b^T \omega_1 a)^T \wedge \omega_2 = (a \times b)^T (\omega_1 \times \wedge \omega_2) + b^T \omega_1 \wedge a^T \omega_2;
\]
(2.1) in particular \(a^T \omega \wedge b^T \omega = \frac{1}{2} (a \times b)^T (\omega \times \wedge \omega)\).

Since both \(\times\) and \(\wedge\) are skew-symmetric, we have \(2 \omega_1 \times \wedge \omega_2 = \omega_1 \times \omega_2 + \omega_2 \times \omega_1 = 2 \omega_2 \times \wedge \omega_1\).

Consider the scalar product \((\cdot, \cdot)\) on \(\mathfrak{M}_3(\mathbb{C}) : \langle X, Y \rangle := \frac{1}{2} \text{tr}(X^T Y)\). We have the isometry
\[
\alpha : \mathbb{C}^3 \rightarrow \mathfrak{o}_3(\mathbb{C}), \quad \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \quad x^T y = \langle \alpha(x), \alpha(y) \rangle = \frac{1}{2} \text{tr}(\alpha(x)^T \alpha(y)),
\]
\[
\alpha(x \times y) = [\alpha(x), \alpha(y)] = \alpha(\alpha(x)y) = yx^T - xy^T, \quad \alpha(Rx) = R \alpha(x) R^{-1}, \quad x, y \in \mathbb{C}^3, \quad R \in \mathfrak{o}_3(\mathbb{C}).
\]

Let \(x \subset \mathbb{C}^3\) be a surface applicable (isometric) to a surface \(x_0 \subset \mathbb{C}^3\).

(2.2) \((x, dx) = (R(t)(x_0), dx_0) := (Rx_0 + t, Rdx_0)\),

where \((R, t)\) is a sub-manifold in \(\mathfrak{o}_3(\mathbb{C}) \ltimes \mathbb{C}^3\) (in general surface, but it is a curve if \(x_0\), \(x\) are ruled and the rulings correspond under isometry or a point if \(x_0\), \(x\) differ by a rigid motion). The sub-manifold \(R\) gives the rolling of \(x_0\) on \(x\), that is if we rigidly roll \(x_0\) on \(x\) such that points corresponding under the isometry will have the same differentials, \(R\) will dictate the rotation of \(x_0\); the translation \(t\) will satisfy \(dt = -dRx_0\).

For \((u, v)\) parametrization on \(x_0\), \(x\) and outside the locus of isotropic (degenerate) induced metric of \(x_0\), \(x\) we have \(N_0 := \frac{\partial_x x_0 \times \partial_{x_0} x_0}{\partial_{x_0} x_0 \times \partial_{x_0} x_0}, N := \frac{\partial_x x \times \partial_x x}{\partial_{x} x \times \partial_{x} x}\) respectively positively oriented unit normal fields of \(x_0, x\) and \(R\) is determined by \(R = [\partial_x x \times \partial_x x \times N] [\partial_x x_0 \times \partial_x x_0 \times \det(R) N_0]^{-1}\); we take \(R\) with \(\det(R) = 1\); thus the rotation of the rolling with the other face of \(x_0\) (or on the other face of \(x\)) is \(R' := R(I_3 - 2N_0 N_0^T) = (I_3 - 2NN^T) R, \ \det(R') = -1\).

Therefore \(\mathfrak{o}_3(\mathbb{C}) \ltimes \mathbb{C}^3\) acts on 2-dimensional integrable distributions of contact elements \((x_0, dx_0)\) in \(T^*(\mathbb{C}^3)\) as: \((R, t)(x_0, dx_0) = (Rx_0 + t, Rdx_0)\); a rolling is a sub-manifold \((R, t) \subset \mathfrak{o}_3(\mathbb{C}) \ltimes \mathbb{C}^3\) such that \((R, t)(x_0, dx_0)\) is still integrable.
We have:

\[(3.1) \quad R^{-1}dRN_0 = R^{-1}dN - dN_0.\]

Applying the compatibility condition \(d\) to \[(2.2)\] we get:

\[(3.2) \quad R^{-1}dR \wedge dx_0 = 0, \quad dRR^{-1} \wedge dx = 0.\]

Since \(R^{-1}dR\) is skew-symmetric and using \[(2.4)\] we have

\[(3.3) \quad dx_0^T R^{-1}dRdx_0 = 0.\]

From \[(2.5)\] for \(a \in \mathbb{C}^3\) we get

\[R^{-1}dRa = R^{-1}dR(a^\perp + a^\top) = a^T N_0 R^{-1}dRN_0 - a^T R^{-1}dRN_0 N_0 = \omega \times a, \quad \omega := N_0 \times R^{-1}dRN_0 \]

\[\text{(det} R)R^{-1}(N \times dN) - N_0 \times dN_0 = R^{-1}(N \times dN) - N_0 \times dN_0.\]

Thus \(R^{-1}dR = \alpha(\omega)\) and \(\omega\) is flat connection form in \(T^*x_0:\)

\[(3.6) \quad d\omega + \frac{1}{2} \omega \times \omega = 0, \quad \omega \times \wedge dx_0 = 0, \quad (\omega)^\perp = 0.\]

With \(s := N_0^T(\omega \times dx_0) = s_{11}du^2 + s_{12}dudv + s_{21}dvdu + s_{22}d\omega^2\) the difference of the second fundamental forms of \(x, x_0\) we have

\[(3.7) \quad \omega = \frac{s_{12}\partial_u x_0 - s_{11}\partial_v x_0}{\partial_u x_0 \times \partial_v x_0} du + \frac{s_{22}\partial_u x_0 - s_{21}\partial_v x_0}{\partial_u x_0 \times \partial_v x_0} dv;\]

\((\omega \times \wedge dx_0 = 0\) is equivalent to \(s_{12} = s_{21}; \quad (d\omega)^\perp + \frac{1}{2} \omega \times \omega = 0, \quad (d\omega)^\top = 0\) respectively encode the difference of the Gauß-Codazzi-Mainardi-Peterson equations of \(x_0\) and \(x).\)

Using \(\frac{1}{2}dN_0 \times dN_0 = K|\partial_u x_0 \times \partial_v x_0|N_0du \wedge dv, \quad K\) being the Gauß curvature we get

\[(3.8) \quad dN_0 \times dN_0 = R^{-1}(dN \times dN) - \frac{1}{2} \omega \times (\omega \times dN_0) = dN_0 \times dN_0 + 2(\omega \times N_0) \times dN_0 + (\omega \times \omega); \quad \text{thus}\]

\[\frac{1}{2} \omega \times \wedge = dN_0^T \wedge \omega N_0.\]

Note also

\[(3.9) \quad \omega' = N_0 \times R^{-1}dR'N_0 = -\omega - 2N_0 \times dN_0\]

and

\[(3.10) \quad a^T \wedge \omega = 0, \quad \forall \omega \text{ satisfying } \text{[(2.6)]} \quad \Rightarrow \quad a^T \wedge dx_0 := \frac{a^T dx_0 + dx_0^T a}{2} = 0.\]

Note that the converse \(a^T \wedge dx_0 = 0, \quad \forall \omega \text{ satisfying } \text{[(2.6)]}\) is also true.

### 3. Proof of Theorem 1.4

#### 3.1. The case \(m = V \times N_0 + mN_0 + nV\)

We have \(0 = (Rm)^T d(RV + x) = m^T(\omega \times V + d(V + x_0) + \partial_w V dw),\) or, assuming \(m^T \partial_w V \neq 0,\)

\[dw = \frac{N_0^T[V \times d(V + x_0)] + mV^T(\omega \times N_0 + dN_0) - nV^T d(V + x_0)}{N_0^T(\partial_w V \times V) + nV^T \partial_w V} = 0.\]

Applying the compatibility condition \(d\) to this equation and using the equation itself we get

\[0 = -\partial_w[N_0^T[V \times d(V + x_0)] + mV^T(\omega \times N_0 + dN_0) - nV^T d(V + x_0)] \wedge \]

\[\frac{N_0^T[V \times d(V + x_0)] + mV^T(\omega \times N_0 + dN_0) - nV^T d(V + x_0)}{N_0^T(\partial_w V \times V) + nV^T \partial_w V} + \]

\[d - \frac{N_0^T[V \times d(V + x_0)]}{N_0^T(\partial_w V \times V) + nV^T \partial_w V} + d - \frac{mV^T}{nV^T} \wedge (\omega \times N_0 + dN_0) - \]

\[\frac{nV^T}{nV^T} \wedge d(V + x_0)\]
From (2.10) we get from the second relation of (3.2) \( (3.2) \):

\[
d N_0^T (\partial_w V \times V) + n V^T \partial_w V - d N_0^T (\partial_w V \times V)\]

\[
\frac{m^2 N_0^T (\partial_w V \times V) N_0^T (d N_0 \times d N_0)}{2 [N_0^T (\partial_w V \times V) + n V^T \partial_w V]^2} + \frac{m (d V \times V) \times d N_0}{[N_0^T (\partial_w V \times V) + n V^T \partial_w V]^2} \]

\[
V^T (\omega \times N_0 + d N_0) + \frac{m n N_0^T [\partial_w V \times d(V + x_0)]}{[N_0^T (\partial_w V \times V) + n V^T \partial_w V]^2} \wedge (N_0 \times V)^T (\omega \times N_0 + d N_0).
\]

This can be written for short

\[
(3.1) A + B \wedge V^T (\omega \times N_0 + d N_0) + C \wedge (N_0 \times V)^T (\omega \times N_0 + d N_0) = 0, \ \forall \omega \text{ satisfying } (2.6),
\]

where \( A \) is a scalar 2-form not depending on \( \omega \) and \( B, C \) are scalar 1-forms not depending on \( \omega \).

In order for \( w \) to be determined by \( \omega \) from (1.4), we need \((m \times V) \times N_0 \neq 0 \) and \( w \) cannot be linked to \( \omega \) by any other relation, either functional (as (3.1) a-priori is) or differential, thus in (3.1) \( \omega \) cancels independently of \( w \) and outside \( w \) we can replace \( \omega \) with any other solution of (2.6).

Replacing \( \omega \) respectively with \( 0, -2 N_0 \times d N_0, \omega - 2 N_0 \times d N_0 \) we get \( A = 0 \) and

\[
(2.1) \quad (B V + C N_0 \times V)^T \wedge d N_0 = 0, \quad (B N_0 \times V - C V)^T \wedge \omega = 0, \ \forall \omega \text{ satisfying } (2.6).
\]

From (2.10) we get from the second relation of (3.2) \((B N_0 \times V - C V)^T \odot d x_0 = 0\); with \( B = B_u d u + B_v d v \), \( C = C_u d u + C_v d v \) this becomes \( B_u (N_0 \times V)^T \partial_w x_0 - C_u V^T \partial_w x_0 = B_v (N_0 \times V)^T \partial_w x_0 - C_v V^T \partial_w x_0 = B_u (N_0 \times V)^T \partial_w x_0 + B_v (N_0 \times V)^T \partial_v x_0 - C_v V^T \partial_v x_0 = 0 \), so \( B_u = C_u V^T \partial_v x_0, \ B_v = C_v (N_0 \times V)^T \partial_v x_0 \) and \( (N_0 \times V)^T (C_u \partial_v x_0 - C_v \partial_w x_0) = 0 \).

We thus get

\[
\frac{m n N_0^T [\partial_w V \times d(V + x_0)]}{[N_0^T (\partial_w V \times V) + n V^T \partial_w V]^2} = \frac{b d x_0^T (N_0 \times V)}{|N_0 \times V|^2}
\]

and since \( m \neq 0 \) we get
\( n N_0^T [\partial_w V \times d(V + x_0)] \wedge dx_0^T (N_0 \times V) = 0 \)

and

\[
\begin{align*}
 b &= \frac{m n [N_0 \times V]^2}{[N_0^T (\partial_w V \times V) + n V^T \partial_w V]^2} \frac{N_0^T [\partial_w V \times \partial_u (V + x_0)]}{\partial_u x_0^T (N_0 \times V)} \\
&= \frac{m n [N_0 \times V]^2}{[N_0^T (\partial_w V \times V) + n V^T \partial_w V]^2} \frac{N_0^T [\partial_w V \times \partial_u (V + x_0)]}{\partial_u x_0^T (N_0 \times V)}.
\end{align*}
\]

Now equation (3.2) becomes:

\[
\begin{align*}
\frac{d(V + x_0)^T \odot dN_0}{m} &- \frac{1}{m} \frac{d(V + x_0)^T (V \times N_0 + n V) \odot [-\frac{m n}{m^2} [N_0^T (\partial_w V \times V) \\
+ n V^T \partial_w V] + \frac{1}{m} (dV \times N_0 + m dN_0 + dV + n dV)^T \partial_w V + \frac{1}{m} \frac{N_0^T [\partial_w V \times d(V + x_0)]}{N_0^T [\partial_w V \times \partial_u (V + x_0)]} \odot N_0^T [\partial_w V \times d(V + x_0)]}{m} \\
+ \frac{1}{m} (dV \times N_0 + m N_0 \times n V^T \partial_c (V + x_0) - \frac{d(V + x_0)^T (V \times N_0 + n V) [\partial_m N_0 \times m \partial_u N_0 + m \partial_u N_0 + \partial_u n V + n \partial_u V)^T \partial_u (V + x_0)]}{m} \\
+ \frac{1}{m} \frac{(dV \times N_0 + n V)^T \partial_u (V + x_0) + \frac{1}{m} (\partial_m V \times N_0 + V \times \partial_u N_0 + m \partial_u N_0 + \partial_u n V + n \partial_u V)^T \partial_u (V + x_0)}{m} \\
- \frac{1}{m} (\partial_u V \times N_0 + V \times \partial_u N_0 + m \partial_u N_0 + \partial_u n V + n \partial_u V)^T \partial_u (V + x_0)) = 0.
\end{align*}
\]

Taking \( d \mathbf{m} \) from \( B = \frac{bdx_0^2 V}{|V|^3} \) and using \( A = 0 \) the derivatives of \( \mathbf{m} \) and \( \mathbf{n} \) in the above equation disappear and we get only:

\[
\begin{align*}
&n \left\{ n^2 \left[ \frac{N_0^T [\partial_u V \times d(V + x_0)]}{N_0^T [\partial_u (V + x_0) \times \partial_c (V + x_0)]} \odot \frac{d(V + x_0)^T V}{m} \frac{N_0^T \partial_u (V + x_0) \times \partial_c (V + x_0)}{\partial_u x_0^T (N_0 \times V)} - \left[ (\partial_u V \times V) \odot \partial_c (V + x_0) + [(\partial_u V \times V) \odot \partial_c (V + x_0)] \odot (\partial_u V \times V) \right] \right\} \\
&+ \frac{N_0^T [\partial_u V \times d(V + x_0)]}{N_0^T [\partial_u (V + x_0) \times \partial_c (V + x_0)]} \odot \frac{d(V + x_0)^T V}{m} \left[ \frac{\partial_u x_0^T (N_0 \times V)}{\partial_u N_0^T \partial_u V \times \partial_u (V + x_0)} - \partial_u V^T \partial_u (V + x_0) + \partial_u V^T \partial_u (V + x_0) \right] \right\} \\
&\odot \left[ \frac{d(V + x_0)^T V}{m} \frac{N_0^T [\partial_u V \times \partial_u (V + x_0)]}{\partial_u x_0^T (N_0 \times V)} - \partial_u V^T \partial_u (V + x_0) \right] \\
&\odot \left[ \frac{d(V + x_0)^T V}{m} \frac{N_0^T [\partial_u V \times \partial_u (V + x_0)]}{\partial_u x_0^T (N_0 \times V)} - \partial_u V^T \partial_u (V + x_0) \right]
\end{align*}
\]
\[
\begin{aligned}
&\frac{1}{m}(\partial_u V \times N_0 + m \partial_u N_0)^T \partial_u (V + x_0) N_0^T (\partial_u V \times V) + \frac{|V|^2}{m} N_0^T (\partial_u N_0 \times \partial_v N_0) N_0^T (\partial_v V \times V) - \\
&\frac{1}{m} d(V + x_0)^T (V \times N_0) \left[ - \frac{1}{m} \partial_u x_0^T N_0^T [\partial_u V \times \partial_v (V + x_0)] + \frac{1}{m} \partial_v V^T \partial_v (V + x_0) - \right. \\
&\left. \frac{1}{m} \partial_v V^T \partial_v (V \times N_0)^T \partial_v (V + x_0) - \frac{1}{m} N_0^T (\partial_u V \times \partial_v V) V^T \partial_v (V + x_0) + \\
&\left( \frac{1}{m} \partial_v x_0^T N_0^T [\partial_u V \times \partial_v (V + x_0)] \right) - \frac{1}{m} \partial_u V^T \partial_v (V + x_0) + \\
&\frac{1}{m} \partial_v V^T \partial_v (V \times N_0)^T \partial_v (V + x_0) + \frac{1}{m} N_0^T (\partial_u V \times \partial_v V) \partial_v (V + x_0) + \frac{1}{m} \partial_v V^T \partial_v (V + x_0) N_0^T (\partial_u V \times V) + \\
&\frac{1}{m} (\partial_u V \times N_0 + m \partial_u N_0)^T \partial_v (V + x_0) V^T \partial_v V - \frac{1}{m} \partial_v V^T \partial_v (V + x_0) N_0^T (\partial_u V \times V) - \\
&\frac{1}{m} (\partial_v V \times N_0 + m \partial_v N_0)^T \partial_v (V + x_0) V^T \partial_u V + \frac{1}{m} \partial_u V^T \partial_u x_0 [V^T \partial_v (V + x_0)] - \frac{1}{m} \partial_v x_0^T \partial_v x_0 (N_0 \times V) - \frac{1}{m} \partial_u V^T \partial_u x_0 \\
&V^T \partial_u (V + x_0) \right] + m \left[ \frac{N_0^T [\partial_u V \times d(V + x_0)]}{N_0^2 [\partial_u (V + x_0) \times \partial_v (V + x_0)]} \odot d(V + x_0)^T V \right] \frac{N_0^T (\partial_u V \times N_0) N_0^T (\partial_u (V + x_0) \times \partial_v (V + x_0))}{m^2} - \\
&\frac{1}{m} \partial_v V^T \partial_v [N_0^T (\partial_u V \times \partial_v (V + x_0)) - N_0^T (\partial_u V \times \partial_v (V + x_0))] + \partial_v V^T \partial_v (N_0 \times V) \partial_v (V + x_0) - \\
&\partial_v V^T \partial_v (N_0 \times V)^T \partial_v (V + x_0) - \partial_v V^T \partial_v (N_0 \times N_0) \partial_v (V + x_0) + \partial_v V^T \partial_v (N_0 \times N_0) \partial_v (V + x_0) + \\
&\frac{N_0^T [\partial_u V \times d(V + x_0)]}{N_0^2 [\partial_u (V + x_0) \times \partial_v (V + x_0)]} \odot d(V + x_0)^T (V \times N_0) \frac{N_0^T (\partial_u V \times \partial_v (V + x_0))}{m^2} - \\
&\partial_u V^T \partial_v V^T \partial_v (V + x_0) + \partial_v V^T \partial_v V^T \partial_v (V + x_0) + \partial_v V^T \partial_v (\partial_v V^T \partial_v x_0 - \partial_v V^T \partial_v x_0) + \\
&\frac{1}{m^2} (d(V + x_0)^T V \odot d x_0^T V \frac{N_0^T [\partial_u V \times \partial_v (V + x_0)]}{\partial u x_0 (N_0 \times V)} \frac{N_0^T (\partial_u V \times V)}{N_0^2 [\partial_u (V + x_0) \times \partial_v (V + x_0)]} - \\
&\frac{1}{m^2} (d(V + x_0)^T V \odot (\partial_u V \times N_0)^T \partial_v (V \times N_0) \frac{N_0^T (\partial_u V \times d(V + x_0)]}{N_0^2 [\partial_u (V + x_0) \times \partial_v (V + x_0)]} \odot [ - \\
&\frac{1}{m} (d(V + x_0)^T (V \times N_0)\frac{- 1}{m} \partial_v x_0^T N_0^T [\partial_u V \times \partial_v (V + x_0)] + \frac{1}{m} \partial_v V^T \partial_v x_0 \partial_v (V + x_0) + \\
&\left( \frac{1}{m} \partial_v x_0^T N_0^T [\partial_u V \times \partial_v (V + x_0)] \right) - \frac{1}{m} \partial_u V^T \partial_v x_0 \partial_v (V + x_0) + \
\end{aligned}
\]
\[
\frac{1}{m} d(V + x_0)^T V \left[\left(-\frac{1}{m} \partial_{\omega} x_0 \left(V_{\omega} - \partial_{\omega} (V + x_0)\right) + \frac{1}{m} \partial_{\omega} V^T \partial_{\omega} x_0 (V \times N_0)^T \partial_{\omega} (V + x_0) + \frac{1}{m} \partial_{\omega} V^T \partial_{\omega} x_0 (N_0 \times V)\right)\right] \\
+ \frac{1}{m} V^T \partial_{\omega} V \left(\partial_{\omega} V^T \partial_{\omega} x_0 - \partial_{\omega} V^T \partial_{\omega} x_0\right) \right] \\
+ \frac{1}{m} N_0^T \left[\partial_{\omega} V \times (V + x_0)\right] + \frac{1}{m} \partial_{\omega} V^T \partial_{\omega} x_0 (V \times N_0)^T \partial_{\omega} (V + x_0) - \\
\frac{1}{m} \partial_{\omega} x_0 \left(\frac{N_0^T \left[\partial_{\omega} V \times (V + x_0)\right]}{\partial_{\omega} x_0 (N_0 \times V)} - \frac{1}{m} \partial_{\omega} V^T \partial_{\omega} x_0 (V \times N_0)^T \partial_{\omega} (V + x_0)\right) \\
+ \frac{1}{m} N_0^T \left(\partial_{\omega} V \times \partial_{\omega} V\right) \partial_{\omega} \partial_{\omega} x_0 (V + x_0) + \frac{1}{m} N_0^T \left(\partial_{\omega} V \times \partial_{\omega} V\right) \partial_{\omega} (V + x_0) - \\
\frac{1}{m} \partial_{\omega} V^T \partial_{\omega} x_0 (V \times N_0)^T \partial_{\omega} (V + x_0) \right] \\
+ \frac{1}{m} \partial_{\omega} V \left(\partial_{\omega} V^T \partial_{\omega} x_0 - \partial_{\omega} V^T \partial_{\omega} x_0\right) + \frac{1}{m} \left(\partial_{\omega} V \times N_0\right)^T \partial_{\omega} (V + x_0) V^T \partial_{\omega} V - \\
\frac{1}{m} \left(\partial_{\omega} V \times \partial_{\omega} V\right) \times \left(\frac{1}{m} N_0^T \left(\partial_{\omega} V \times \partial_{\omega} V\right) \partial_{\omega} \partial_{\omega} x_0 (N_0 \times V)\right) - \\
\frac{1}{m} N_0^T \left(\partial_{\omega} V \times (V + x_0)\right) \partial_{\omega} V + \left|V\right|^2 \frac{1}{m} N_0^T \left(\partial_{\omega} N_0 \times \partial_{\omega} N_0\right) \partial_{\omega} x_0 (V + x_0) \right]\]
\]
3.2. The case $m = V + mN_0$.
We have $0 = (Rm)^T d(RV + x) = m^T (\omega \times V + d(V + x_0) + \partial_w V dw)$, or, assuming $m^T \partial_w V \neq 0,$

$$dw = \frac{mV^T(\omega \times N_0 + dN_0) - V^T d(V + x_0)}{V^T \partial_w V}.$$ 

Applying the compatibility condition $d$ to this equation and using the equation itself we get

$$0 = \frac{\partial_w[mV^T(\omega \times N_0 + dN_0) - V^T d(V + x_0)]}{V^T \partial_w V} \land \frac{mV^T(\omega \times N_0 + dN_0) - V^T d(V + x_0)}{V^T \partial_w V} +$$
$$\frac{\partial_w[V^T d(V + x_0)]}{V^T \partial_w V} \land \frac{V^T d(V + x_0)}{V^T \partial_w V} - \frac{d}{V^T \partial_w V} \land \frac{V^T d(V + x_0)}{V^T \partial_w V} - \frac{m^2 N_0^T(\partial_w V \times V)N_0^T (dN_0 \times dN_0)}{2(V^T \partial_w V)^2}$$
$$+ \frac{\partial_w[mV^T d(V + x_0)]}{(V^T \partial_w V)^2} + \frac{m \partial_w[V^T d(V + x_0)]}{(V^T \partial_w V)^2} + \frac{d}{V^T \partial_w V} \land \frac{m}{(V^T \partial_w V)^2} \land (N_0 \times V)^T (\omega \times N_0 + dN_0).$$

As in the previous case we get

$$\frac{mN_0^T [\partial_w V \times d(V + x_0)]}{(V^T \partial_w V)^2} = \frac{bdx_0^T (N_0 \times V)}{|N_0 \times V|^2}$$

with $b \subset \mathbb{C}$.

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