ON A THEOREM OF SCHOLZE-WEINSTEIN

VLADIMIR DRINFELD

Dedicated to the memory of Galim Mustafin

Abstract. Let $G$ be the Tate module of a $p$-divisible group $H$ over a perfect field $k$ of characteristic $p$. A theorem of Scholze-Weinstein describes $G$ (and therefore $H$ itself) in terms of the Dieudonné module of $H$; more precisely, it describes $G(C)$ for “good” semiperfect $k$-algebras $C$ (which is enough to reconstruct $G$).

In these notes we give a self-contained proof of this theorem and explain the relation with the classical descriptions of the Dieudonné functor from Dieudonné modules to $p$-divisible groups.

1. Introduction

Fix a prime $p$. Recall that an $\mathbb{F}_p$-algebra is said to be perfect (resp. semiperfect) if the Frobenius homomorphism $\text{Fr}: C \to C$ is bijective (resp. surjective).

The goal of these notes is to give a self-contained proof of Theorem 3.3.2. This theorem is due to P. Scholze and J. Weinstein (it is a very special case of the theory developed in §4 of [SW]). If $H$ is a $p$-divisible group over a perfect field $k$ of characteristic $p$ and $G$ is its Tate module, the theorem describes $G$ (and therefore $H$ itself) in terms of the Dieudonné module of $H$; more precisely, it describes $G(C)$ for “good” semiperfect $k$-algebras $C$ (which is enough to reconstruct $G$). The description is in terms of Fontaine’s ring $A_{\text{cris}}(C)$.

§3.4 and §3.6 are influenced by Fontaine’s book [F77]; they explain the relation between Theorem 3.3.2 and the classical descriptions of the Dieudonné equivalence [Dem, F77]. The idea of §3.6 is to switch from $A_{\text{cris}}(C)$ to a more manageable $W(k)$-module $\overline{M}(C)$, which was introduced by Fontaine [F77] (under a different name); the definition of $\overline{M}(C)$ is given in §3.6.2. Probably this idea is somewhat similar to [FF, §4.2]. Our §3.4 is a “baby version” of §3.6; instead of $\overline{M}(C)$ we work there with a certain submodule $M(C) \subset \overline{M}(C)$.

§2.5 and formula (3.11) reflect some exercises, which I had to do in order to understand [SW].

I thank P. Scholze and J. Weinstein for valuable advice and references. The author’s research was partially supported by NSF grant DMS-1303100.

2. Recollections on Fontaine’s functor $A_{\text{cris}}$

In §2.1-2.6 we follow [F94, §2.2] and [SW, §4], but the proof of the important Proposition 2.6.1 is different from the one given in [SW]. The material of §2.7 is influenced by Fontaine’s book [F77]; it is used in the proof of Proposition 3.4.1.

2010 Mathematics Subject Classification. Primary 14L05.

Key words and phrases. $p$-divisible group, Dieudonné equivalence, semiperfect ring.
We will use the following notation: we write $W(R)$ for the ring of $p$-typical Witt vectors of a ring $R$, and for $a \in R$ the Teichmüller element $(a, 0, 0, \ldots) \in W(R)$ is denoted by $[a]$.

2.1. The definition of $A_{\text{cris}}$. By the Fontainization of an $\mathbb{F}_p$-algebra $C$ we mean the perfect topological $\mathbb{F}_p$-algebra

$$C^\flat := \lim_{\leftarrow}(\ldots \to \Fr C \to \Fr C \to \Fr C).$$

Thus an element of $C^\flat$ is a sequence $(c_0, c_1, \ldots)$ of elements of $C$ such that $c_{n+1} = c_n$ for all $n$. Define $\nu_n : C^\flat \to C$ by $\nu(c_0, c_1, \ldots) := c_n$; then $\nu_n = \nu \circ \Fr c_n$.

Now suppose that $C$ is semiperfect. Then the homomorphisms $\nu_n : C^\flat \to C$ are surjective. Fontaine defined $A_{\text{cris}}(C)$ to be the $p$-adic completion of the PD hull of the epimorphism $W(C^\flat) \to C$ induced by $\nu_0 : C^\flat \to C$. Despite the fact that the definitions of $C^\flat$ and $W$ involve projective limits, one has the following

**Proposition 2.1.1.** For any $n \in \mathbb{N}$, the functor $C \mapsto A_{\text{cris}}(C)/p^n A_{\text{cris}}(C)$ commutes with filtered inductive limits.

**Proof.** The canonical homomorphism $f : W(C^\flat) \to A_{\text{cris}}(C)/p^n A_{\text{cris}}(C)$ factors through $W_n(C^\flat)$. Moreover, if $u \in \Ker(C^\flat \to C)$ then

$$f(V^i[w^n]) = f(p^i[w^{n-i}]) = p^i \cdot (p^{n-i})! \cdot \gamma_{p^{n-i}}([u]) = 0 \quad \text{for } 0 \leq i < n.$$ 

Therefore $f$ factors through $W_n(C^\flat)/\Ker \nu_n$. So $A_{\text{cris}}(C)/p^n A_{\text{cris}}(C)$ is the PD hull of the epimorphism

$$W_n(C^\flat)/\Ker \nu_n \to C \quad (2.1)$$

induced by $\nu_0 : C^\flat \to C$. The functor $C \mapsto W_n(C^\flat)/\Ker \nu_n$ commutes with filtered inductive limits. \qed

**Remark 2.1.2.** The isomorphism $C^\flat/\Ker \nu_n \to C$ induced by $\nu_n : C^\flat \to C$ transforms the map $(2.1)$ into the composed map $W_n(C) \to C \to C$.

2.1.3. The canonical epimorphism $A_{\text{cris}}(C) \to W(C)$. The ideal $\Ker(W(C) \to C)$ has a canonical PD structure, namely $\gamma_n(Va) := \frac{p^n - 1}{n!} \cdot V(a^n)$ for $a \in W(C)$. So the canonical epimorphism $W(C^\flat) \to W(C)$ uniquely extends to a PD homomorphism

$$\beta : A_{\text{cris}}(C) \to W(C).$$

2.2. The universal property. Let $C$ be a semiperfect $\mathbb{F}_p$-algebra and $n \in \mathbb{N}$. Let $\mathcal{A}_n$ be the category of PD thickenings $\tilde{C} \to C$ such that $p^n = 0$ in $\tilde{C}$.

**Proposition 2.2.1.** $A_{\text{cris}}(C)/p^n A_{\text{cris}}(C)$ is an initial object of $\mathcal{A}_n$.

The proof uses the following

**Lemma 2.2.2.** Let $\pi : \tilde{C} \to C$ be an object of $\mathcal{A}_n$ and $I := \Ker \pi$. Then

(i) for every $x \in I$ one has $x^p \in pI$;

(ii) if $a \in \tilde{C}$ and $x \in I$ then $(a + x)^{p^j} - a^{p^j} \in p^j I$ for all $j$; in particular, $(a + x)^{p^n} = a^{p^n}$;

---

1The definitions of PD thickening and PD hull include the condition $\gamma_m(p) = p^m/m!$. 

(iii) the ring homomorphism \( w_n : W_n(\tilde{C}) \to \tilde{C} \) defined by \( w_n(a_0, \ldots, a_n) := \sum_{j=0}^{n} p^j a_j^{p^{n-j}} \) factors as

\[
W_n(\tilde{C}) \xrightarrow{w_n} W_n(C) \xrightarrow{\tilde{w}_n} \tilde{C}.
\]

Proof. Since \( I \) has a PD structure we have (i). Statements (ii) and (iii) follow from (i). \( \Box \)

Proof of Proposition 2.2.1. It is clear that \( A_{\text{cris}}(C)/p^nA_{\text{cris}}(C) \) is an object of \( A_n \). To prove that this object is initial, one has to show that if \( \pi : \tilde{C} \to C \) is an object of \( A_n \) then there is a unique homomorphism \( f : W(C^\flat) \to \tilde{C} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
W(C^\flat) & \xrightarrow{f} & \tilde{C} \\
\downarrow & & \downarrow \pi \\
C^\flat & \xrightarrow{\nu} & C
\end{array}
\]

Define \( f \) to be the composition

\[
W(C^\flat) \xrightarrow{\alpha_n} W(C) \xrightarrow{\tilde{w}_n} \tilde{C},
\]

where \( \alpha_n \) is induced by \( \nu_0 \circ \operatorname{Fr}^{-n} : C^\flat \to C \) and \( \tilde{w}_n \) is as in (2.2). Then \( f \) has the required property.

Now let \( f : W(C^\flat) \to \tilde{C} \) be any homomorphism with the required property. If \( u \in C^\flat \) and \([u] \in W(C^\flat)\) is the corresponding Teichmüller element then Lemma 2.2.2(ii) implies that

\[
f([u]) = f([u^{p^n}]) = a^{p^n},
\]

where \( a \) is any element of \( \tilde{C} \) such that \( \pi(a) = \nu_0(u^{p^n}) \). Since

\[
\ker(W(\tilde{C})) \to W_n(\tilde{C})) = p^nW(C^\flat) \subset \ker f
\]

and \( W_n(\tilde{C}) \) is generated by the Teichmüller elements, we see that \( f \) is unique. \( \Box \)

### 2.3. Commutation of \( A_{\text{cris}} \) with a certain type of base change.

**Lemma 2.3.1.** Let \( B \) be a semiperfect \( \mathbb{F}_p \)-algebra. Let \( b_j, b'_j \in B \), where \( j \) runs through some set \( J \). Let \( I \subseteq B \) be the ideal generated by the elements \( b_j - b'_j \), \( j \in J \). Then for every \( n \in \mathbb{N} \) the ideal \( \ker(W_n(B) \to W_n(B/I)) \) is generated by \( V^m([b_j] - [b'_j]) \), where \( j \in J \), \( 0 \leq m < n \).

Proof. By induction, it suffices to check that the ideal \( \ker(W_n(B) \to W_{n-1}(B) \times W_n(B/I)) \) is generated by \( V^{n-1}([b_j] - [b'_j]) \), where \( j \in J \). This follows from semiperfectness and the identity \( x \cdot V^{n-1}y = V^{n-1}(F^{n-1}x \cdot y) \) in \( W(B) \). \( \Box \)

**Lemma 2.3.2.** Let \( C_1 \) and \( C_2 \) be semiperfect \( C \)-algebras. Then for every \( n \in \mathbb{N} \) the map \( W_n(C_1) \otimes W_n(C_2) \to W_n(C_1 \otimes C_2) \) is an isomorphism.

Proof. Write \( C_j = B_j/I_j \), where \( B_j \) is perfect (e.g., one can take \( B_j = C_j^\flat \)). It is easy to see that \( W_n(B_1 \otimes B_2) = W_n(B_1) \otimes W_n(B_2) \). The ideal \( \ker(B_1 \otimes B_2 \to C_1 \otimes C_2) \) is generated by elements of the form \( b_1 \otimes 1 - 1 \otimes b_2 \), where \( b_j \in B_j \) are such that the image of \( \{b_1, b_2\} \) in \( C_1 \times C_2 \) is contained in \( \operatorname{Im}(C \to C_1 \times C_2) \). So by Lemma 2.3.1, \( W_n(C_1 \otimes C_2) \) is the quotient of \( W_n(B_1) \otimes W_n(B_2) \) by the ideal generated by elements of the form \( V^m[b_1] \otimes 1 - 1 \otimes V^m[b_2] \), where \( b_1, b_2 \) are as above and \( 0 \leq m < n \). These elements have zero image in \( W_n(C_1) \otimes W_n(C_2) \).

\( \Box \)
Lemma 2.3.3. The formation of PD hulls commutes with flat base change.

Proof. This was proved by Berthelot [B74, Prop. I.2.7.1]. □

Proposition 2.3.4. Let $B$ be an $\mathbb{F}_p$-algebra and $B'$ a flat $B$-algebra. Suppose that $B$ and $B'$ are perfect. Let $C$ be a semiperfect $B$-algebra. Then the canonical map

$$A_{\text{cris}}(C) \hat{\otimes}_{W(B)} W(B') \rightarrow A_{\text{cris}}(C \otimes_B B')$$

is an isomorphism.

Proof. It suffices to show that for each $n \in \mathbb{N}$ the map

$$(2.3) \quad (A_{\text{cris}}(C)/p^n A_{\text{cris}}(C)) \otimes_{W_n(B)} W_n(B') \rightarrow A_{\text{cris}}(C \otimes_B B')/p^n A_{\text{cris}}(C \otimes_B B')$$

is an isomorphism. In the proof of Proposition 2.1.1 it was shown that $A_{\text{cris}}(C)/p^n A_{\text{cris}}(C)$ is the PD hull of the epimorphism

$$(2.4) \quad W_n(C^o/ \text{Ker } \nu_n) \rightarrow C$$

induced by $\nu_0 : C^o \rightarrow C$. One also has a similar description of $A_{\text{cris}}(C \otimes_B B')/p^n A_{\text{cris}}(C \otimes_B B')$; combining it with Lemma 2.3.2 we see that $A_{\text{cris}}(C \otimes_B B')/p^n A_{\text{cris}}(C \otimes_B B')$ is the PD hull of the epimorphism

$$W_n(C^o/ \text{Ker } \nu_n) \otimes_{W_n(B)} W_n(B') \rightarrow C \otimes_{W_n(B)} W_n(B') = C \otimes_B B'$$

obtained from (2.4) by base change via $f : W_n(B) \rightarrow W_n(B')$. It is easy to see that $f$ is flat, so the map (2.3) is an isomorphism by Lemma 2.3.3. □

2.4. The functor $A_{\text{cris}}$ for schemes.

Proposition 2.4.1. For any $n \in \mathbb{N}$, the functor $C \mapsto A_{\text{cris}}(C)/p^n A_{\text{cris}}(C)$ commutes with etale localization.

Proof. Follows from Proposition 2.2.1 combined with Lemma 2.3.3. □

Let $X$ be a semiperfect $\mathbb{F}_p$-scheme. Then one defines a $p$-adic formal scheme $A_{\text{cris}}(X)$ as follows: its underlying topological space is that of $X$, and its structure sheaf is the $p$-adic completion of the PD hull of the surjection $W(O_X) \rightarrow O_X$. By Proposition 2.4.1, $A_{\text{cris}}(X)$ is a scheme, and the functor $X \mapsto A_{\text{cris}}(X)$ commutes with etale localization.

2.5. The key example.

2.5.1. The ring $C$. Let $B$ be a perfect $\mathbb{F}_p$-algebra. Let $B[x_1^{p^{-\infty}}, \ldots, x_n^{p^{-\infty}}]$ denote the perfection of $B[x_1, \ldots, x_n]$; in other words, an element of $B[x_1^{p^{-\infty}}, \ldots, x_n^{p^{-\infty}}]$ is a finite sum

$$(2.5) \quad \sum_{\alpha \in \mathbb{Z}_+[1/p]^n} b_\alpha x^\alpha, \quad b_\alpha \in B,$$

where $\mathbb{Z}_+[1/p] := \{\alpha \in \mathbb{Z}[1/p] \mid \alpha \geq 0\}$ and for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+[1/p]^n$ we set

$$x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}.$$

Now let $C := B[x_1^{p^{-\infty}}, \ldots, x_n^{p^{-\infty}}]/(x_1, \ldots, x_n)$. Then $C$ is semiperfect. In [2.5.3, 2.5.4] we will describe $C^o$, $W(C^o)$, $W(C)$, and $C_{\text{cris}}$ for such $C$. This class of semiperfect $\mathbb{F}_p$-algebras is important because of Proposition 3.2.1 and the following remark.
Remark 2.5.2. Let $X$ be an affine scheme étale over $A^n_k$, where $k$ is a perfect field of characteristic $p$. Let $X_{\text{perf}}$ be the perfection of $X$. Then for each $m \in \mathbb{N}$ the fiber product (over $X$) of $m$ copies of $X_{\text{perf}}$ is isomorphic to $\text{Spec} C$, where $C$ is as in [2.5.1]. Let us note that fiber products of this type appear in [BMS].

2.5.3. $C^p$, $W(C^p)$, and $W(C)$. Clearly $C^p$ is the ring of formal series [2.5] such that the set

$$\{ \alpha \in \mathbb{Z}_+[1/p]^n \mid b_\alpha \neq 0 \}$$

is discrete in $\mathbb{R}^n$. The Witt ring $W(C^p)$ identifies with the ring of formal series

$$\sum_{\alpha \in \mathbb{Z}_+[1/p]^n} a_\alpha x^\alpha, \quad a_\alpha \in W(B)$$

such that $a_\alpha \to 0$ when $\alpha$ runs through any bounded subset of $\mathbb{Z}_+[1/p]^n$ (boundedness in the sense of $\mathbb{R}^n$). Let is note that in formula (2.6) and similar formulas below $x^\alpha$ really means $X_1^{\alpha_1} \cdot \ldots \cdot X_n^{\alpha_n}$, where $X_i = \lfloor x_i \rfloor$ is the Teichmüller representative.

By Lemma 2.3.1, $W(C)$ is the quotient of $W(C^p)$ by the ideal topologically generated by $p^m x_i^{p^m}$, $m \geq 0$, $i = 1, \ldots, n$.

2.5.4. The ring $A_{\text{cris}}(C)$. For a real number $y \geq 0$ let $(y!)_p$ be the maximal power of $p$ dividing $\lfloor y \rfloor !$, where $\lfloor y \rfloor$ is the integral part of $y$; equivalently,

$$(y!)_p = p^{s(y)}, \quad \text{where } s(y) := \sum_{j=1}^{\infty} \lfloor y/p^j \rfloor.$$

For $\alpha \in \mathbb{Z}_+[1/p]^n$ let

$$\frac{(\alpha !)_p}{(\alpha !)_p} := \prod_{i=1}^{n} (\alpha_i !)_p.$$

Then $A_{\text{cris}}(C)$ identifies with the ring of formal series

$$\sum_{\alpha \in \mathbb{Z}_+[1/p]^n} a_\alpha x^\alpha (\alpha !)_p, \quad a_\alpha \in W(B), \quad a_\alpha \to 0.$$

2.6. The homomorphism $F : A_{\text{cris}}(C) \to A_{\text{cris}}(C)$. Let $C$ be a semiperfect $\mathbb{F}_p$-algebra. By functoriality, the endomorphism $F_{\text{cris}} \in \text{End} C$ induces endomorphisms $F \in \text{End} W(C^p)$ and $F \in \text{End} A_{\text{cris}}(C)$. There is no map $V : A_{\text{cris}}(C) \to A_{\text{cris}}(C)$; instead, the following proposition gives a partially defined map $V^{-1} = p^{-1}F$.

**Proposition 2.6.1.** Let $I_{\text{cris}}(C) \subset A_{\text{cris}}(C)$ be the kernel of the canonical epimorphism $A_{\text{cris}}(C) \twoheadrightarrow C$. Then there exists a unique $F$-linear map $F' : I_{\text{cris}}(C) \to A_{\text{cris}}(C)$ such that $F(b) = F'(pb)$ for $b \in A_{\text{cris}}(C)$,

$$F'(b) = (p-1)! \cdot \gamma_p(b) + \delta(b) \quad \text{for } b \in \text{Ker}(W(C^p) \to C), \quad \gamma_p \in \text{Spec} C,$$

where $\delta : W(C^p) \to W(C^p)$ is defined by

$$\delta(b) := \frac{F(b) - b^p}{p}.$$
This is essentially [SW, Lemma 4.1.8], but the proof given in §2.6.4 below is different from the one in [SW].

2.6.2. A general setup. Let \( \mathbb{Z}_p := \mathbb{Z}_p \cap \mathbb{Q} \). Let \( B \) be a \( \mathbb{Z}_p \)-algebra and \( I \subset B \) an ideal such that \( p \in I \). Let \( R \) be a ring equipped with an additive map \( \psi : I \to R \) such that

\[
\psi(b_1b_2) = \psi(pb_1)\psi(b_2) \quad \text{for} \ b_1 \in B, \ b_2 \in I, \tag{2.9}
\]

\[
\psi(p) = 1. \tag{2.10}
\]

Then the map \( \varphi : B \to R \) defined by

\[
\varphi(b) = \psi(pb)
\]

is a unital ring homomorphism. Moreover, \( \psi \) is a \( B \)-module homomorphism if the \( B \)-module structure on \( R \) is defined using \( \varphi \).

Let \((B', I')\) be the PD hull of \((B, I)\). If \( R \) is \( \mathbb{Z}_p \)-flat then \( pR \) is a PD ideal in \( R \), so \( \varphi \) extends uniquely to a PD morphism \( \varphi' : (B', I') \to (R, pR) \). Here is a statement in this spirit without assuming \( R \) to be \( \mathbb{Z}_p \)-flat.

**Lemma 2.6.3.** (i) Equip \( R \) with multiplication \( x \ast y := pxy \) and divided power operations \( \gamma_n(x) := \frac{x^n}{n!} \cdot x^n \). Then \( R \) is a (non-unital) PD algebra over \( B \).

(ii) The map \( \psi : I \to (R, \ast) \) is a homomorphism of \( B \)-algebras.

(iii) There exists a unique homomorphism \( \psi' : I' \to (R, \ast) \) of PD algebras over \( B \) extending \( \psi : I \to R \).

(iv) One has \( \psi'(b_1b_2) = \psi'(pb_1)\psi'(b_2) \) for \( b_1 \in B', \ b_2 \in I' \).

(v) The map \( \varphi' : B' \to R \) defined by \( \varphi'(b) := \psi'(pb) \) is a ring homomorphism. Moreover, \( \psi' : I' \to B' \) is a \( B' \)-module homomorphism if the \( B' \)-module structure on \( R \) is defined using \( \varphi' \).

**Proof.** Checking (i)-(ii) is straightforward. Statement (iii) follows from (ii) and the construction of \((B', I')\) in the proof of Theorem I.2.3.1 of [B74].

Since the map \( B/I \to B'/I' \) is an isomorphism, it suffices to check (iv) if \( b_1 \in I' \) and if \( b_1 \) belongs to the image of \( B \). In these cases (iv) follows from (iii).

Statement (v) follows from (iv). \( \square \)

2.6.4. Proof of Proposition 2.6.1. Uniqueness is clear. To prove existence, we apply Lemma 2.6.3 as follows.

Let \( B := W(C^o) \). Let \( I \subset B \) be the kernel of the composed map \( B \to C^o \xrightarrow{\nu_0} C \). Let \((B', I')\) be the PD hull of \((B, I)\). Let \( R := B' \). For \( b \in I \) set \( \psi(b) := (p-1)! \cdot \gamma_p(b) + \delta(b) \), where \( \delta(b) \in B \) is given by (2.8).

One checks that \( \psi(pb) = F(b) \) for \( b \in B = W(C^o) \). One also checks that \( \psi \) is additive and satisfies (2.9)-(2.10). Applying Lemma 2.6.3 we get \( \psi' : I' \to B' \). Passing to \( p \)-adic completions, one gets the desired map \( F' : I_{\text{cris}}(C) \to A_{\text{cris}}(C) \). \( \square \)

Let us note that for \( x \in W(C^o) \) one has

\[
F'Vx = F'(pF^{-1}x) = F(F^{-1}x) = x. \tag{2.11}
\]

\(^2\)The definitions of PD thickening and PD hull include the condition \( \gamma_m(p) = p^m/m! \).

\(^3\)This means that \( R \) is a PD ideal of the \( B \)-algebra \( B \oplus R \) obtained from \( R \) by formally adding the unit.
Proposition 2.6.5. Let $\beta : A_{\text{cris}}(C) \to W(C)$ be the canonical epimorphism constructed in §2.1.3. Then the following diagram commutes:

$$
\begin{array}{ccc}
I_{\text{cris}}(C) & \xrightarrow{F'} & A_{\text{cris}}(C) \\
\beta \downarrow & & \downarrow \beta \\
V(W(C)) & \xrightarrow{V^{-1}} & W(C)
\end{array}
$$

Proof. By (2.11), the identity $\beta(F'(b)) = V^{-1}(\beta(b))$ holds if $b \in \text{Ker}(W(C^\flat) \to C^\flat)$; it also holds if $b$ is the Teichmüller representative of an element of $\text{Ker}(C^\flat \to C)$. So it holds for any element $b \in \text{Ker}(W(C^\flat) \to C^\flat \to C)$. This implies that

$$
\beta(F'(\gamma_n(b))) = V^{-1}(\beta(\gamma_n(b))) \text{ for all } n \in \mathbb{N}, b \in \text{Ker}(W(C^\flat) \to C^\flat \to C).
$$

This is enough because the ideal $I_{\text{cris}}(C) \subset A_{\text{cris}}(C)$ is topologically generated by $\gamma_n(b)$, where $b$ and $n$ are as above. \qed

Corollary 2.6.6. Let $\beta_n : A_{\text{cris}}(C) \to W_n(C)$ be the map induced by $\beta : A_{\text{cris}}(C) \to W(C)$. Then for every $n \in \mathbb{N}$ the operator $F' : I_{\text{cris}}(C) \to A_{\text{cris}}(C)$ maps $\text{Ker} \beta_n$ to $\text{Ker} \beta_{n-1}$. So we have the map $(F')^n : \text{Ker} \beta_n \to A_{\text{cris}}(C)$. \qed

Remark 2.6.7. Using (2.11), we see that the restriction of $(F')^n$ to $\text{Ker}(W(C^\flat) \to W_n(C))$ is very simple: namely, $(F')^n V^n u = u$ for $u \in W(C^\flat)$ and $(F')^n V^i [c] = (p^{n-i} - 1)! \cdot \gamma_{p^{n-i}}([c])$ for $c \in \text{Ker}(C^\flat \to C)$.

2.7. The module $M(C)$ and the map $M(C) \to A_{\text{cris}}(C)$. In this section we define and study a $W(C^\flat)[F, V]$-module $M(C)$ and a canonical $W(C^\flat)[F]$-morphism $M(C) \to A_{\text{cris}}(C)$. This material is used in the proof of Proposition 3.4.1.

Let us note that $M(C)$ is a submodule of the module $\overline{M}(C)$ introduced in §6.2 below; the latter goes back to Fontaine’s book [F77].

2.7.1. Definition of $M(C)$. For any $\mathbb{F}_p$-algebra $B$ set

$$
W(B)[V^{-1}] := \lim_{\longrightarrow} (W(B) \xrightarrow{V} W(B) \xrightarrow{V} \ldots) = \bigcup_n V^{-n} W(B).
$$

We equip $W(B)$ and $W(B^\flat)$ with their natural topologies (they come from the presentation of $W(B)$ as a projective limit of $W_m(B)$ and the presentation of $W(B^\flat)$ as a projective limit of $W(B^\flat)/\text{Ker} \nu_n$). We equip $W(B)[V^{-1}]$ with the inductive limit topology. Then $W(B)[V^{-1}]$ is a complete topological $W(B^\flat)$-module equipped with operators $F, V : W(B)[V^{-1}] \to W(B)[V^{-1}]$ satisfying the usual identities. Each element of $W(B)[V^{-1}]$ has a unique expansion

$$
\sum_{m=-\infty}^{\infty} V^m [x_m],
$$

where $x_m \in B$ and $x_m = 0$ for sufficiently negative $m$.

Recall that $C^\flat$ is the projective limit of $C^\flat/\text{Ker} \nu_n$, where $\nu_n$ is as in §2.1. We equip $C^\flat$ with the projective limit topology.

\footnote{An element $u \in W(B^\flat)$ acts on $W(B)[V^{-1}]$ by $V^{-n} x \mapsto V^{-n} ((F^{-n} u) \cdot x)$.}
The projective limit of the topological modules $W(C^\flat/\ker\nu_n)[V^{-1}]$, $n \in \mathbb{N}$, is a topological $W(C^\flat)$-module equipped with operators $F, V$ satisfying the usual identities. Let $M(C)$ be the preimage of $W(C) \subset W(C)[V^{-1}]$ in this projective limit. Again, $M(C)$ is a topological $W(C^\flat)$-module equipped with operators $F, V : M(C) \to M(C)$. The map $\nu_0 : C^\flat \to C$ induces a canonical epimorphism $M(C) \twoheadrightarrow W(C)$.

**Proposition 2.7.2.** (i) $M(C)$ is complete with respect to the above topology.
(ii) Each element of $M(C)$ has a unique expansion
\[
\sum_{n=-\infty}^{\infty} V^n[x_n], \text{ where } x_n \in C^\flat, \ x_n \in \ker(C^\flat \to C) \text{ for } n < 0, \lim_{n \to -\infty} x_n = 0.
\]
(iii) $\ker(M(C) \xrightarrow{p} M(C)) = 0$.
(iv) An element \( (2.12) \) belongs to $p^rM(C)$ if and only if $x_n \in \ker(C^\flat \nu_r \to C)$ for $n < 0$ and $x_n \in \ker(C^\flat \nu_{r-n} \to C)$ for $0 \leq n < r$.
(v) The topology on $M(C)$ defined above is equal to the $p$-adic topology of $M(C)$.
(vi) The $W(C^\flat)$-module $M(C)/pM(C)$ is canonically isomorphic to the associated graded of the filtration
\[
C^\flat \supset \ker\nu_1 \supset \ker\nu_2 \supset \ldots.
\]
The isomorphism is as follows: the map $C^\flat/\ker\nu_1 \to M(C)/pM(C)$ is induced by the map
\[
C^\flat \to M(C), \ c \mapsto [c] \in W(C^\flat) \subset M(C),
\]
and for $r > 0$ the map $\ker\nu_r/\ker\nu_{r+1} \to M(C)/pM(C)$ is induced by the map
\[
\ker\nu_r \to M(C), \ c \mapsto p^{-r}[c] = V^{-r}[c^{p^{-r}}] \in M(C).
\]

The proof of the proposition is straightforward and left to the reader.

2.7.3. The submodule $M_0(C) \subset M(C)$. Let $M_0(C)$ be the set of all $x \in M(C)$ such that $V^n x \in W(C^\flat)$ (or equivalently, $p^n x \in W(C^\flat)$) for some $n \geq 0$. Clearly $M_0(C)$ is a $W(C^\flat)[F, V]$-submodule of $M(C)$. One can also think of $M_0(C)$ as a subset of $W(C^\flat)[V^{-1}] = W(C^\flat)[p^{-1}]$; an element $x \in W(C^\flat)[p^{-1}]$ belongs to $M_0(C)$ if and only if for some (or all) $n \geq 0$ such that $V^n x \in W(C^\flat)$ one has $V^n x \in \ker\beta_n$, where $\beta_n : W(C^\flat) \to W_n(C)$ is as in Corollary 2.6.6.

**Proposition 2.7.4.** $M(C)$ is the $p$-adic completion of $M_0(C)$.

Proof. One has $p^nM(C) \cap M_0(C) = p^nM_0(C)$. It remains to use Proposition 2.7.2(i,v) and density of $M_0(C)$ in $M(C)$.

2.7.5. The map $f : M(C) \to A_{\text{cris}}(C)$. Define $f_0 : M_0(C) \to A_{\text{cris}}(C)$ as follows: $f_0(x) := (F^n)^{n}V^n x$, where $n \geq 0$ is so big that $V^n x \in W(C^\flat)$. The map $f_0$ is well-defined: indeed, $(F^n)^{n}V^n x$ is defined by Corollary 2.6.6 (because $V^n x \in \ker\beta_n$) and moreover, $(F^n)^{n}V^n x$ does not depend on $n$ by (2.11). The map $f_0$ is a $W(C^\flat)[F]$-module homomorphism. By Proposition 2.7.2(v), it uniquely extends to a $W(C^\flat)[F]$-module homomorphism $f : M(C) \to A_{\text{cris}}(C)$.

Recall that each element of $M(C)$ has a unique expansion
\[
\sum_{n=-\infty}^{\infty} V^n[x_n], \text{ where } x_n \in C^\flat, \ x_n \in \ker(C^\flat \nu_0 \to C) \text{ for } n < 0, \lim_{n \to -\infty} x_n = 0.
\]
Proposition 2.7.6. (i) The homomorphism \( f : M(C) \to A_{\text{cris}}(C) \) takes \([2.13]\) to
\[
\sum_{m=0}^{\infty} V^m [x_m] + \sum_{l=1}^{\infty} (p^l - 1)! \cdot \gamma_{p^l}([x_{-l}]).
\]
(ii) The composite map \( M(C) \xrightarrow{f} A_{\text{cris}}(C) \xrightarrow{\beta} W(C) \) is equal to the canonical epimorphism \( M(C) \to W(C) \).

Proof. Statement (i) follows from Remark \([2.6.7]\). Statement (ii) follows from (i) or from Proposition \([2.6.5]\) \( \Box \)

2.7.7. The image of \( f : M(C) \to A_{\text{cris}}(C) \). Let
\[
A'_{\text{cris}}(C) := \{ u \in A_{\text{cris}}(C) \mid p^n u \in F^n A_{\text{cris}}(C) \text{ for all } n \in \mathbb{N} \}
\]
(Roughly, \( A'_{\text{cris}}(C) \) is the set of all \( u \in A_{\text{cris}}(C) \) for which \( V^n u \) is defined for all \( n \).) Let \( A''_{\text{cris}}(C) \) be the set of all \( u \in A_{\text{cris}}(C) \) such that \( p^n u = F^n v_n \) for some sequence \( v_n \in A_{\text{cris}}(C) \) converging to 0. Clearly \( A''_{\text{cris}}(C) \subset A'_{\text{cris}}(C) \).

Lemma 2.7.8. \( \text{Im}(M(C) \xrightarrow{f} A_{\text{cris}}(C)) \subset A''_{\text{cris}}(C) \).

Proof. Let \( y \in M(C) \). Then \( p^n f(y) = f(F^n V^n y) = F^n f(V^n y) \). Moreover, \( f(V^n y) \to 0 \) because \( V^n y \to 0 \). \( \Box \)

Proposition 2.7.9. Suppose that \( C \) is as in \([2.5.1]\). Then \( f : M(C) \to A''_{\text{cris}}(C) \) is an isomorphism.

Proof. We will work with the explicit description of \( A_{\text{cris}}(C) \) from \([2.5.4]\). Since \( F(x^\alpha) = x^{p\alpha} \), one gets the following description of the subsets \( A''_{\text{cris}}(C) \subset A'_{\text{cris}}(C) \subset A_{\text{cris}}(C) \).

For \( \alpha \in \mathbb{Z}_+[1/p]^n \) let
\[
m(\alpha) := \max(0, \lfloor \log_p \alpha_1 \rfloor, \ldots, \lfloor \log_p \alpha_n \rfloor).
\]
Then \( A''_{\text{cris}}(C) \) (resp. \( A'_{\text{cris}}(C) \)) identifies with the ring of formal series
\[
\sum_{\alpha \in \mathbb{Z}_+[1/p]^n} a_\alpha x^\alpha / p^{m(\alpha)}, \quad a_\alpha \in W(B)
\]
such that \( a_\alpha \to 0 \) (resp. \( a_\alpha \to 0 \) when \( m(\alpha) \) is bounded).

Since \( M(C) \) and \( A''_{\text{cris}}(C) \) are \( \mathbb{Z}_p \)-flat topologically free \( \mathbb{Z}_p \)-modules, it suffices to check that the map \( \tilde{f} : M(C)/pM(C) \to A''_{\text{cris}}(C)/pA''_{\text{cris}}(C) \) induced by \( f \) is an isomorphism. By \([2.15]\), \( A''_{\text{cris}}(C)/pA''_{\text{cris}}(C) \) is a free \( B \)-module with basis \( y_\alpha, \alpha \in \mathbb{Z}_+[1/p]^n \), where \( y_\alpha \in A''_{\text{cris}}(C)/pA''_{\text{cris}}(C) \) is the image of \( x^\alpha / p^{m(\alpha)} \in A''_{\text{cris}}(C) \). On the other hand, by Proposition \([2.7.2]\) (vi), \( M(C)/pM(C) \) identifies with \( \text{gr } C^o \), i.e., the associated graded of the decreasing filtration on \( C^o \) whose \( i \)-th term equals \( \text{Ker } v_i \) if \( n \geq 1 \) and \( C^o \) if \( i \leq 0 \). It is clear that \( \text{gr } i C^o \) is a free \( B \)-module with basis \( x^\alpha \), where \( \alpha \in \mathbb{Z}_+[1/p]^n \) is such that the number \([2.14]\) equals \( i \). It is straightforward to check that \( \tilde{f}(x^\alpha) = y_\alpha \), so \( \tilde{f} \) is an isomorphism. \( \Box \)

3. The Dieudonné functor according to Scholze-Weinstein and Fontaine

We fix a perfect field \( k \) of characteristic \( p \).
3.1. Tate $k$-groups.

**Definition 3.1.1.** A Tate $k$-group is a group scheme $G$ over $k$ such that $\text{Ker}(G \rightarrow pG) = 0$, $\text{Coker}(G \rightarrow pG)$ is finite, and the map $G \rightarrow \lim_{\leftarrow n} G/p^nG$ is an isomorphism. The category of all (resp. connected) Tate $k$-groups will be denoted by $\text{Tate}_k$ (resp. $\text{Tate}_k^{\text{con}}$).

**Remark 3.1.2.** Any Tate $k$-group $G$ is affine because it is isomorphic to the projective limit of the finite group schemes $G/p^nG$.

3.1.3. **Relation to $p$-divisible groups.** If $H$ is a $p$-divisible group over $k$ then its Tate module $\text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, H) = \text{Hom}(\mathbb{Z}[p^{-1}]/\mathbb{Z}, H) = \lim_{\leftarrow n} \text{Hom}(p^{-n}\mathbb{Z}/\mathbb{Z}, H)$ is a Tate $k$-group. Thus we get an equivalence between the category of $p$-divisible groups and $\text{Tate}_k$. The inverse equivalence takes $G \in \text{Tate}_k$ to $\lim_{\rightarrow}(G/pG \rightarrow G/p^2G \rightarrow G/p^3G \rightarrow \ldots)$.

It identifies the full subcategory $\text{Tate}_k^{\text{con}} \subset \text{Tate}_k$ with the category of connected $p$-divisible groups.

**Remark 3.1.4.** If $H$ is a connected $p$-divisible $k$-group then its Tate module $\text{Hom}(\mathbb{Z}[p^{-1}]/\mathbb{Z}, H)$ can also be described as $\text{Ker}(H_{\text{perf}} \rightarrow H)$, where $H_{\text{perf}} = \lim_{\leftarrow n} (\cdots \xrightarrow{\text{Fr}} H \xrightarrow{\text{Fr}} H \xrightarrow{\text{Fr}} H)$. Indeed, in the connected case $H_{\text{perf}} = \lim_{\leftarrow n} \text{Hom}(p^{-n}\mathbb{Z}, H) = \text{Hom}(\mathbb{Z}[p^{-1}], H)$, so $\text{Ker}(H_{\text{perf}} \rightarrow H) = \text{Ker}(\text{Hom}(\mathbb{Z}[p^{-1}], H) \rightarrow \text{Hom}(\mathbb{Z}, H)) = \text{Hom}(\mathbb{Z}[p^{-1}]/\mathbb{Z}, H)$.

**Remark 3.1.5.** By §3.1.3, any Tate $k$-group can be uniquely represented as a direct product of a connected Tate group and a reduced one. Moreover, any reduced Tate $k$-group is a projective limit of etale group schemes.

3.2. **Tate $k$-groups as schemes.** Any Tate $k$-group $G$ is a semiperfect scheme. Indeed, $\text{Ker}(G \xrightarrow{\text{Fr}} G) \subset \text{Ker}(G \xrightarrow{p} G) = 0$, so $\text{Fr} : G \rightarrow G$ is a closed embedding.

**Proposition 3.2.1.** The underlying scheme of any connected Tate $k$-group is isomorphic to $\text{Spec} k[x_1^{p^{-\infty}}, \ldots, x_n^{p^{-\infty}}]/(x_1, \ldots, x_n)$ for some $n$.

**Proof.** Let $H$ be a connected $p$-divisible $k$-group. As an ind-scheme, $H$ is isomorphic to $\text{Spf} k[[x_1, \ldots, x_n]]$ for some $n$. So the proposition follows by Remark 3.1.4. \qed
3.3. The Dieudonné functor according to Scholze-Weinstein. Let $D_k$ be the Dieudonné ring of $k$ (i.e., the ring generated by $W(k)$ and elements $F, V$ subject to the usual relations). Let $W(k)[F] \subset D_k$ be the subring generated by $W(k)$ and $F$.

Let $\mathfrak{D}_k$ be the category of those $D_k$-modules that are free and finitely generated over $W(k)$. Classical Dieudonné theory [Dem Ch. III] provides an equivalence

$$\mathfrak{D}_k^{op} \stackrel{\sim}{\rightarrow} \text{Tate}_k, \quad N \mapsto G_N.$$

**Definition 3.3.1.** $\text{Good}_k$ is the category of $k$-algebras isomorphic to

$$\text{Spec } B[x_1^{p^{-\infty}}, \ldots, x_n^{p^{-\infty}}]/(x_1, \ldots, x_n)$$

for some perfect $k$-algebra $B$.

The following theorem is due to P. Scholze and J. Weinstein.

**Theorem 3.3.2.** There is a canonical isomorphism of functors

$$(3.1) \quad G_N(C) \stackrel{\sim}{\rightarrow} \text{Hom}_{W(k)[F]}(N, A_{\text{cris}}(C)), \quad C \in \text{Good}_k, N \in \mathfrak{D}_k.$$ 

**Remark 3.3.3.** By Proposition 3.2.1 and Remark 3.1.5 the underlying scheme of any connected Tate $k$-group is isomorphic to $\text{Spec } C$, where $C \in \text{Good}_k$. So $G_N$ is completely determined by the functor $C \mapsto G_N(C), C \in \text{Good}_k$.

**Remark 3.3.4.** In §2.7.4 we defined $W(C)[F]$-submodules $A''_{\text{cris}}(C) \subset A'_{\text{cris}}(C) \subset A_{\text{cris}}(C))$. Using the map $V : N \to N$, one sees that

$$\text{Hom}_{W(k)[F]}(N, A_{\text{cris}}(C)) = \text{Hom}_{W(k)[F]}(N, A'_{\text{cris}}(C));$$

moreover, if $V : N \to N$ is topologically nilpotent then

$$\text{Hom}_{W(k)[F]}(N, A_{\text{cris}}(C)) = \text{Hom}_{W(k)[F]}(N, A''_{\text{cris}}(C)).$$

Note that $A'_{\text{cris}}(C)$ is much smaller than $A_{\text{cris}}(C)$; compare the denominators in (2.7) and (2.15).

Theorem 3.3.2 is a very special case of the theory developed in [SW] §4. More precisely, it is deduced from [SW Cor. 4.1.12] as follows. Let $H$ be the $p$-divisible group corresponding to $G_N$; then $G_N(C) = \text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p)C, H_C)$, where $(\mathbb{Q}_p/\mathbb{Z}_p)C$ and $H_C$ are the constant $p$-divisible groups over $\text{Spec } C$ with fibers $\mathbb{Q}_p/\mathbb{Z}_p$ and $H$. To get (3.1), it suffices to compute $\text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p)C, H_C)$ using [SW Cor. 4.1.12].

In what follows we give a self-contained construction of (3.1).

Recall that $\mathfrak{D}_k = \mathfrak{D}_k' \oplus \mathfrak{D}_k''$, where $\mathfrak{D}_k'$ (resp. $\mathfrak{D}_k''$) is the full subcategory of objects $N \in \mathfrak{D}_k$ such that the operator $V : N \to N$ is topologically nilpotent (resp. invertible). In §3.4 and §3.5 we will construct the isomorphism (3.1) for $N \in \mathfrak{D}_k'$ and $N \in \mathfrak{D}_k''$, respectively. Let us note that in §3.5 we just paraphrase the relevant part of [SW] §4. In §3.6 we sketch a conceptually better proof of Theorem 3.3.2 which treats $\mathfrak{D}_k'$ and $\mathfrak{D}_k''$ simultaneously.

**Remark 3.3.5.** The category $\text{Good}_k$ from Definition 3.3.1 is contained in the following category $\text{Good}_k'$ introduced in [SW] §4: a $k$-algebra $C$ is in $\text{Good}_k'$ if $C$ can be represented as a quotient of a perfect ring by an ideal generated by a regular sequence. In [SW] §4 the isomorphism (3.1) is established for $C \in \text{Good}_k'$. I do not know if the construction of (3.1) given below works for $C \in \text{Good}_k'$ (I did not check whether Propositions 2.7.9 and 3.6.7 hold in the more general setting).

---

5The theory from [SW] §4 was refined in [Lau].
3.4. The isomorphism (3.1) for $N \in \mathcal{D}_k'$. If $N \in \mathcal{D}_k'$ then $G_N := \text{Hom}_{D_k}(N, W)$, where $W$ is the Witt group scheme. This means that

$$G_N(C) = \text{Hom}_{D_k}(N, W(C))$$

for $N \in \mathcal{D}_k'$ and any $k$-algebra $C$.

**Proposition 3.4.1.** Let $N \in \mathcal{D}_k'$ and $C \in \text{Good}_k$. Then the canonical epimorphism $A_{\text{cris}}(C) \to W(C)$ (see §2.1.3) induces an isomorphism

$$\text{Hom}_{W(k)[F]}(N, A_{\text{cris}}(C)) \cong \text{Hom}_{D_k}(N, W(C)) \subset \text{Hom}_{W(k)[F]}(N, W(C)).$$

**Proof.** In §2.7.1 we defined a $W(C)[F]$-submodule $A_{\text{cris}}''(C) \subset A_{\text{cris}}(C)$. We have

$$\text{Hom}_{W(k)[F]}(N, A_{\text{cris}}(C)) = \text{Hom}_{W(k)[F]}(N, A_{\text{cris}}''(C))$$

by Remark 3.3.4 and the assumption $N \in \mathcal{D}_k'$.

In §2.7.1 and §2.7.5 we defined a $W(C)[F, V]$-module $M(C)$ and an $W(C)[F]$-morphism $f : M(C) \to A_{\text{cris}}(C)$. By Proposition 2.7.9 and the assumption $C \in \text{Good}_k$, this morphism induces an isomorphism $M(C) \cong A_{\text{cris}}''(C)$. So

$$\text{Hom}_{W(k)[F]}(N, A_{\text{cris}}(C)) = \text{Hom}_{W(k)[F]}(N, M(C)).$$

By Proposition 2.7.2(iii), $M(C)$ is $Z_p$-flat, so

$$\text{Hom}_{W(k)[F]}(N, A_{\text{cris}}(C)) = \text{Hom}_{D_k}(N, M(C)).$$

It remains to show that the map

(3.2) $$\text{Hom}_{D_k}(N, M(C)) \to \text{Hom}_{D_k}(N, W(C))$$

induced by the canonical epimorphism $M(C) \to W(C)$ is bijective. In fact, we will prove this without assuming that $C \in \text{Good}_k$.

Let us describe the preimage of $\alpha \in \text{Hom}_{D_k}(N, W(C))$ under the map (3.2). First note that for each $m, n \geq 0$ we have a map

$$V^{-m}F^n : W(C) \to W(C^\circ / \text{Ker } \nu_n)[V^{-1}],$$

where $\nu_n : C^\circ \to C$ is as in §2.1. If $m$ is so big that $V^m N \subset F^n N$ then we get a map

(3.3) $$N \to W(C^\circ / \text{Ker } \nu_n)[V^{-1}], \quad x \mapsto V^{-m}F^n \alpha(F^{-n}V^m x).$$

It is easy to check that this map does not depend on $m$; moreover, the maps (3.3) corresponding to different $n$ agree with each other and therefore define a map

$$N \to M(C) \subset \lim_{\leftarrow n} W(C^\circ / \text{Ker } \nu_n)[V^{-1}].$$

It is easy to check that this is the unique preimage of $\alpha$ under the map (3.2).
3.5. The isomorphism \((3.1)\) for \(N \in \mathcal{D}_{k'}\). In this subsection we follow [SW] §4.

Let \(H \in \text{Tate}_k\) be the projective limit of the group schemes \(\mu_{p^m}, m \in \mathbb{N}\); then
\[
H(C) = \text{Ker}((C^p)^x \to C^x)
\]
for any \(k\)-algebra \(C\).

Fix an algebraic closure \(\bar{k}\). For \(N \in \mathcal{D}_{k'}\) one has
\[
G_N(C) := \text{Hom}_{\text{Gal}(\bar{k}/k)}(N_0, H(C \otimes_k \bar{k})),
\]
where \(N_0\) is the following \(\mathbb{Z}_p\)-module with an action of \(\text{Gal}(\bar{k}/k)\):
\[
N_0 := (N \otimes_{W(k)} W(\bar{k}))^{V=1} := \text{Ker}(N \otimes_{W(k)} W(\bar{k}) \xrightarrow{\sim} N \otimes_{W(k)} W(\bar{k})).
\]

**Proposition 3.5.1.** Let \(C \in \text{Good}_k\). Then the map
\[
H(C) = \text{Ker}((C^p)^x \to C^x) \to A_{\text{cris}}(C), \quad c \mapsto \log[c] := -\sum_{n=1}^{\infty} (n-1)! \cdot \gamma_n(1 - [c])
\]
induces an isomorphism
\[
H(C) \xrightarrow{\sim} A_{\text{cris}}(C)^{F=p} := \text{Ker}(A_{\text{cris}}(C) \xrightarrow{F=p} A_{\text{cris}}(C)).
\]

In §3.5.2 we will give a proof of the proposition following [SW] §4 (where a more general statement is proved). Using Proposition 3.5.1, one gets the isomorphism \((3.1)\) as follows. Combining \((3.4)\) and \((3.5)\), one gets a canonical isomorphism
\[
G_N(C) = \text{Hom}_{\text{Gal}(\bar{k}/k)}(N_0, A_{\text{cris}}(C \otimes_k \bar{k})^{F=p}).
\]
We also have the canonical isomorphisms
\[
N_0 \otimes_{\mathbb{Z}_p} W(\bar{k}) \xrightarrow{\sim} N \otimes_{W(k)} W(\bar{k}), \quad A_{\text{cris}}(C) \otimes_{W(k)} W(\bar{k}) \xrightarrow{\sim} A_{\text{cris}}(C \otimes_k \bar{k})
\]
(the second one by Proposition 2.3.1). Combining \((3.3)\) and \((3.7)\), we see that an element of \(G_N(C)\) is the same as a \(\text{Gal}(\bar{k}/k)\)-equivariant \(D_k\)-homomorphism \(N \otimes_{W(k)} W(\bar{k}) \to A_{\text{cris}}(C) \otimes_{W(k)} W(\bar{k})\), which is the same as a \(D_k\)-homomorphism \(N \to A_{\text{cris}}(C)\).

3.5.2. Proof of Proposition 3.5.1. If \(c \in \text{Ker}((C^p)^x \to C^x)\) then
\[
F(\log[c]) = \log F([c]) = \log([c]^p) = p \log[c].
\]

Suppose that \(c \in \text{Ker}((C^p)^x \to C^x)\) and \(\log[c] = 0\). Note that for \(u \in \text{Ker}(W(C^p) \to C)\) one has \((1+u)^p - 1 \in pA_{\text{cris}}(C)\). In particular, \([c]^p \in 1 + pA_{\text{cris}}(C)\), so \([c]^p \in 1 + p^2A_{\text{cris}}(C)\). Since \(\log([c]^p) = 0\) and \([c]^p \in 1 + p^2A_{\text{cris}}(C)\), a standard argument\(^6\) shows that \([c]^p\) equals 1 in \(A_{\text{cris}}(C)\). But our assumption on \(C\) implies that the map \(W(C^p) \to A_{\text{cris}}(C)\) is injective, so \([c]^p\) equals 1 in \(W(C^p)\). Therefore \(c^p = 1\). By perfectness of \(C\), this implies that \(c = 1\). This proves injectivity of the map \((3.5)\).

The proof of surjectivity from [SW] §4.2 uses the Artin-Hasse exponential
\[
E_p(y) := \exp(y + \frac{y^p}{p} + \frac{y^{p^2}}{p^2} + \ldots) \in \mathbb{Z}_p[[y]].
\]

---

\(^6\)One uses that the formal series \(x^{-1} \cdot \log(1 + p^2x)\) belongs to \((\mathbb{Z}_p[[x]])^x\).
In [SW §4.2] it is proved that

\[(3.8) \quad \log[E_p(c)] = \sum_{n=-\infty}^{\infty} \frac{[c^p^n]}{p^n} := \sum_{m=0}^{\infty} p^m \cdot [c^{p^{-m}}] + \sum_{n=1}^{\infty} (p^n - 1)! \cdot \gamma_p([c]), \quad c \in \text{Ker}(C^\flat \to C).\]

(a way to think about the r.h.s is explained in [3.6.9 below). Formula (3.8) is an immediate consequence of the formula

\[(3.9) \quad [E_p(c)] = E_p([c]) \cdot \exp \sum_{m=1}^{\infty} p^m [c]^{p^{-m}}, \quad c \in \text{Ker}(C^\flat \to C).\]

which is proved in [SW §4.2] as follows. First, note that the exponent in the r.h.s of (3.9) has the form \(\exp(py)\), where \(y \in W(C^o)\) is topologically nilpotent; so the exponent makes sense as an element of \(W(C^o)\). Let \(z \in W(C^o)\) be the r.h.s. of (3.9), then it is easy to check that \(F(z) = z^p\) and the image of \(z\) in \(C^o\) equals \(E_p(c)\). These properties of \(z\) mean that \(z = [E_p(c)]\).

Let us now prove surjectivity. Let \(u \in A_{\text{cris}}(C)\) be such that \(F u = pu\); we want to represent \(u\) as \(\log[c], c \in \text{Ker}((C^o)^\times \to C^\times)\). Fix an isomorphism

\[C \sim \to B[x_1^{p^{-\infty}}, \ldots, x_n^{p^{-\infty}}]/(x_1, \ldots, x_n),\]

where \(B\) is a perfect \(k\)-algebra. Using the realization of \(A_{\text{cris}}(C)\) from [2.5.4 write

\[(3.10) \quad u = \sum_{\alpha \in \mathbb{Z}+(1/p)^n} a_\alpha \frac{x^\alpha}{(\alpha!)^p}, \quad a_\alpha \in W(B), \quad a_\alpha \to 0,\]

where \(x^\alpha\) really means \([x_1^{\alpha_1}] \cdot \ldots \cdot [x_n^{\alpha_n}]\). Since \(F u = pu\), it is clear that \(u\) is an infinite \(W(B)\)-linear combination of elements as in the r.h.s. of (3.8). So using (3.8), it is straightforward to find \(c \in \text{Ker}((C^o)^\times \to C^\times)\) such that \(u = \log[c]\). To formulate the answer, we will use the standard monomorphism

\[W(B) \hookrightarrow B[[t]]^\times, \quad a \mapsto f^a;\]

recall that if \(a = \sum_{i=0}^{\infty} V^i[b_i], b_i \in B\), then \(f^a(t) := \prod_{i=0}^{\infty} E_p(b_i t^i)\). Here is the formula for \(c\), which follows from (3.8):

\[(3.11) \quad c = \prod_{\alpha \in S} f^{a_\alpha}(x^\alpha) \in \text{Ker}((C^o)^\times \to C^\times), \quad \text{where } S := [0,p)^n \setminus [0,1)^n \subset \mathbb{Z}_+[1/p]^n.\]

The product converges because \(a_\alpha \to 0\).

3.6. **Another approach to the proof of Theorem 3.3.2.** In §3.4-3.5 we treated \(\mathfrak{D}'_k\) and \(\mathfrak{D}_k''\) separately, which is not good philosophically; this is related to the fact that we used the definition of the functor \(N \mapsto G_N\) from [Dem], which has a similar drawback. Here is a sketch of a conceptually better proof of Theorem 3.3.2, which uses the description of \(G_N\) given by Fontaine [F77]. This description is recalled in [3.6.5 below.}
3.6.1. Witt covectors and bivectors. For any discrete \( \mathbb{F}_p \)-algebra \( R \) we introduced in \([2.7.1]\) the \( W(R^o) \)-module \( W(R)[V^{-1}] \). Each element of \( W(R)[V^{-1}] \) has a unique expansion

\[
(3.12) \quad \sum_{m=-\infty}^{\infty} V^m [x_m], \quad x_m \in R,
\]

where \( x_m = 0 \) for sufficiently negative \( m \).

In \([F77, \S V.1.3]\) Fontaine defines the group of Witt bivectors \( BW(R) \), which contains \( W(R)[V^{-1}] \) as a subgroup. Elements of \( BW(R) \) are formal expressions \((3.12)\) such that for some \( N < 0 \) the ideal generated by \( x_N, x_{N-1}, x_{N-2}, \ldots \) is nilpotent. The group \( BW(R)/W(R) \) is denoted by \( CW(R) \) and called the group of Witt covectors.

In fact, the definition of \( BW(R) \) from \([F77]\) p. 228 relies on that of \( CW(R) \). The latter is given in \([F77]\) Ch. II.1 and is based on Proposition 1.1 and Lemma 1.2 of \([F77]\) Ch. II.

If \( R \) is an algebra over a perfect \( \mathbb{F}_p \)-algebra \( k \) then \( BW(R) \) and \( CW(R) \) are \( W(k) \)-modules by an argument similar to Proposition II.2.2 of \([F77]\) (in which there is an extra assumption that \( k \) is a field). In particular, \( BW(R) \) and \( CW(R) \) are \( W(k^o) \)-modules.

3.6.2. The module \( \overline{M}(C) \). Let \( C \) be a semiperfect \( \mathbb{F}_p \)-algebra. Recall that \( C^o \) is the projective limit of \( C^o/\ker \nu_n \), where \( \nu_n \) is as in \([2.1]\). We equip \( C^o \) with the projective limit topology. Define \( BW(C^o) \) to be the projective limit of \( BW(C^o/\ker \nu_n) \). Following p. 229 of Fontaine’s book \([F77]\), consider the preimage of \( W(C) \subset BW(C) = BW(C^o/\ker \nu_0) \) in \( BW(C^o) \); we denote this preimage by \( \overline{M}(C) \). Thus elements of \( \overline{M}(C) \) are formal expressions

\[
(3.13) \quad \sum_{m=-\infty}^{\infty} V^m [x_m], \quad x_m \in C^o, \quad x_m \in \ker(C^o \xrightarrow{\nu_0} C) \text{ for } m < 0
\]

such that the ideal in \( C^o \) generated by \( x_{-1}, x_{-2}, \ldots \) is topologically nilpotent\(^7\). The topological nilpotence condition is automatic if \( C \) is as in the following lemma.

**Lemma 3.6.3.** The following properties of \( C \) are equivalent:

(i) the ideal \( \ker(C \xrightarrow{F} C) \) is nilpotent;

(ii) the ideal \( \ker(C \xrightarrow{F^o} C) \) is nilpotent.

**Proof.** Suppose that the product of any \( m \) elements of \( \ker(C \xrightarrow{F} C) \) is zero. Then the products of any \( m \) elements of \( \ker(C \xrightarrow{F^o} C) \) belongs to \( \ker(C \xrightarrow{F^{n-1}} C) \). So the product of any \( m^n \) elements of \( \ker(C \xrightarrow{F^o} C) \) is zero. \( \square \)

Clearly \( \overline{M}(C) \) is a \( W(C^o) \)-module equipped with maps \( F, V : \overline{M}(C) \to \overline{M}(C) \) satisfying the usual identities. Note that \( \overline{M}(C) \supset M(C) \), where \( M(C) \) is as in \([2.7.1]\).

Similarly to Proposition \([2.7.2]\) one proves the following

**Lemma 3.6.4.** (i) \( \overline{M}(C) \) is \( p \)-adically complete.

(ii) \( \ker(\overline{M}(C) \xrightarrow{p} \overline{M}(C)) = 0. \)

\(^7\)Fontaine’s notation for \( \overline{M}(C) \) is \( BW_0(\kappa(C)) \). Here \( \kappa(C) \) is our \( C^o \).

\(^8\)Here we are using that \( x_{-i} \) is topologically nilpotent for each \( i > 0 \).
(iii) If $C$ has the properties from Lemma 3.6.3 then one has a canonical $W(C^\flat)$-module isomorphism

$$M(C)/pM(C) \sim \prod_{i=0}^{\infty} \text{gr}^i C^\flat,$$

where $\text{gr}^i C^\flat$ corresponds to the filtration $C^\flat \supset \text{Ker} \nu_1 \supset \text{Ker} \nu_2 \supset \ldots$. □

3.6.5. Fontaine’s description of $G_N$. Let $k$ be a perfect field and $N \in \mathcal{D}_k$. Proposition V.1.2 of [F77] tells us that for any semiperfect $k$-algebra $C$ one has

$$(3.14)\quad G_N(C) = \text{Hom}_{D_k}(N, M(C)).$$

Note that by Lemma 3.6.4(ii), one has

$$(3.15)\quad \text{Hom}_{D_k}(N, M(C)) = \text{Hom}_{W(k)[F]}(N, M(C)).$$

3.6.6. The homomorphism $\bar{f} : \overline{M}(C) \to A'_{\text{cris}}(C)$. The homomorphism $f : M(C) \to A_{\text{cris}}(C)$ from §2.7.3 and Proposition 2.7.6 canonically extends to a homomorphism of $W(C^\flat)[F]$-modules $\bar{f} : \overline{M}(C) \to A_{\text{cris}}(C)$; namely, $\bar{f}$ takes an element $3.13)$ to

$$\sum_{m=0}^{\infty} V^m[x_m] + \sum_{l=1}^{\infty} (p^l - 1)! \cdot \gamma_{p^l}(x_{-l}) \in A_{\text{cris}}(C).$$

Similarly to Lemma 2.7.8 one shows that $\bar{f}(\overline{M}(C)) \subset A'(C)$, where $A'(C)$ is defined in §2.7.7.

Similarly to Proposition 2.7.9 one deduces from Lemma 3.6.4 the following

**Proposition 3.6.7.** If $C$ is as in §2.5.1 then the map $\bar{f} : \overline{M}(C) \to A'_{\text{cris}}(C)$ is an isomorphism. □

3.6.8. Proof of Theorem 3.3.2. By Remark 3.3.4 we have

$$\text{Hom}_{W(k)[F]}(N, A_{\text{cris}}(C)) = \text{Hom}_{W(k)[F]}(N, A'_{\text{cris}}(C)).$$

Combining this with (3.14)-(3.15) and Proposition 3.6.7 we get the desired isomorphism

$$G_N(C) \sim \text{Hom}_{W(k)[F]}(N, A_{\text{cris}}(C)), \quad C \in \mathcal{G}_{\text{odk}}, N \in \mathcal{D}_k.$$

3.6.9. A remark on the proof of Proposition 3.5.1. The r.h.s. of formula (3.8) equals $\bar{f}(w)$, where $w \in \overline{M}(C)^{V=1}$ is defined by $w := \sum_{n=-\infty}^{\infty} V^n[c]$ and $\bar{f}$ is as in 3.6.6.

**References**

[B74] P. Berthelot, Cohomologie cristalline des schémas de caractèreistique $p > 0$. Lecture Notes in Mathematics, 407, Springer-Verlag, Berlin-New York, 1974.

[BMS] B. Bhatt, M. Morrow, and P. Scholze, *Topological Hochschild homology and integral $p$-adic Hodge theory*, arXiv:1802.03261.

[Dem] M. Demazure, *Lectures on $p$-divisible groups*, Lecture Notes in Math. 302, Springer-Verlag, Berlin-New York, 1972.

[F77] J.-M. Fontaine, *Groupes $p$-divisibles sur les corps locaux*, Astérisque 47-48, Soc. Math. France, Paris, 1977.

[F94] J.-M. Fontaine, *Le corps des périodes $p$-adiques*. In: Périodes $p$-adiques, 59–111, Astérisque 223, Soc. Math. France, Paris, 1994.

[FF] L. Fargues and J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge $p$-adique*, 2017, https://webusers.imj-prg.fr/~laurent.fargues/Courbe_fichier_principal.pdf.
[Lau] E. Lau, *Dieudonné theory over semiperfect rings and perfectoid rings*, arXiv:1603.07831.

[SW] P. Scholze and J. Weinstein, *Moduli of p-divisible groups*, Camb. J. Math. 1 (2013), no. 2, 145–237.

University of Chicago, Department of Mathematics, Chicago, IL 60637