Height inequality of algebraic points on curves over functional fields

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Introduction

In this paper, we shall give a linear and effective height inequality for algebraic points on curves over functional fields.

Let \( f: S \to C \) be a fibration of a smooth complex projective surface \( S \) over a curve \( C \), and denote by \( g \) the genus of a general fiber of \( f \). We assume that \( g \geq 2 \) and \( S \) is relatively minimal with respect to \( f \), i.e., \( S \) has no \((-1)\)-curves contained in a fiber of \( f \). Let \( k \) be the functional field of \( C \), and \( \bar{k} \) its algebraic closure. For an algebraic point \( P \in S(\bar{k}) \), we let \( E_P \) be the corresponding horizontal curve on \( S \). The geometric canonical height \( h_K(P) \) and the geometric logarithmic discriminant \( d(P) \) are defined as follows.

\[
h_K(P) = \frac{K_{S/C}E_P}{[k(P) : k]}, \quad d(P) = \frac{2g(\tilde{E}_P) - 2}{[k(P) : k]},
\]

where \( \tilde{E}_P \) is the normalization of \( E_P \), and \([k(P) : k] = FE_P \) is the degree of \( P \). It is a fundamental problem to give an effective bound of height by the geometric discriminant. Up to now, many height inequalities have been obtained.

**Szpiro**, \( h_K(P) \leq 8 \cdot 3^{3g+1}(g - 1)^2(d(P)/3^g + s + 1 + 1/3^3g) \),

**Vojta**, \( h_K(P) \leq (8g - 6)/3 \, d(P) + O(1) \),

**Parshin**, \( h_K(P) \leq (20g - 15)/6 \, d(P) + O(1) \),

**Esnault-Viehweg**, \( h_K(P) < 2(2g - 1)^2 \, (d(P) + s) \),

**Vojta**, \( h_K(P) \leq (2 + \varepsilon) \, d(P) + O(1) \),

**Moriwaki**, \( h_K(P) \leq (2g - 1) \, d(P) + O(1) \).

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where $s$ is the number of singular fibers of $f$. These inequalities can be found respectively in [Sz], [Vo1], [Pa], [EV], [Vo2] and [Mo]. It is a problem to get an inequality linear in $g$ with explicit $O(1)$. (cf. Lang’s comments on this problem, [La], p.153). The purpose of this paper is to give such an inequality.

**Theorem A.** Let $f : S \rightarrow C$ be a non-trivial fibration of genus $g \geq 2$ with $s$ singular fibers, and $P \in S(\bar{k})$ an algebraic point. If $f$ is semistable, then $$h_K(P) \leq (2g - 1)(d(P) + s) - K_{S/C}^2,$$
and the equality holds only if $f$ is smooth, i.e., $s = 0$.

If $f$ is non-semistable, then $$h_K(P) < (2g - 1)(d(P) + 3s) - K_{S/C}^2.$$

When we compare it with the canonical class inequality, the term $3s$ in the second inequality seems natural. Vojta obtains a canonical class inequality for semistable fibrations:

$$K_{S/C}^2 \leq (2g - 2)(2g(C) - 2 + s).$$

Furthermore, we have shown that if the equality holds, then $f$ is smooth (cf. [Ta2], Lemma 3.1). In [Ta1], in a quite natural way, we generalized Vojta’s inequality to the non-semistable case:

$$K_{S/C}^2 < (2g - 2)(2g(C) - 2 + 3s).$$

The first step of the proof of Theorem A is to obtain the first inequality for rational points by using Miyaoka-Yau inequality. The idea is motivated by Xiao’s proof of Manin’s Theorem (i.e., Mordell conjecture over functional fields), (cf. [Xi], Corollary to Theorem 6.2.7). In the semistable case, the height inequality for algebraic points can be obtained easily through base changes. The second step is based on the detailed study of the invariants of semistable reductions [Ta1]. Kodaira-Parshin’s trick plays an important role in this step.

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1. Preliminaries

Let $f : S \rightarrow C$ be a fibration of genus $g \geq 2$, let $F_1, \cdots, F_s$ be the singular fibers of $f$, and let $B = \sum_{i=1}^{s} F_i$. First of all, we consider the embedded resolution of the singularities of $B_{\text{red}}$. We denote by $K_{S/C}^2$, $\chi_f = \deg f_* \omega_{S/C}$ and $e_f = \sum_F (\chi_{\text{top}}(F) - (2 - 2g))$ the standard relative invariants of $f$. 
**Definition 1.1.** The embedded resolution of the singularities of $B$ is a sequence

$$(S, B) = (S_0, B_0) \xrightarrow{\sigma_1} (S_1, B_1) \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_r} (S_r, B_r) = (S', B')$$

satisfying the following conditions.

(i) $\sigma_i$ is the blowing-up of $S_{i-1}$ at a singular point $(B_{i-1,\text{red}}, p_{i-1})$, which is not an ordinary double point.

(ii) $B_{r,\text{red}}$ has at worst ordinary double points as its singularities.

(iii) $B_i$ is the total transformation of $B_{i-1}$.

It is well-known that embedded resolution exists uniquely. We use the notations $m_i$ and $\bar{m}_i$ to denote respectively the multiplicities of $(B_{i,\text{red}}, p_i)$ and $(\bar{B}_{i,\text{red}}, p_i)$, where $\bar{B}_{i,\text{red}}$ is the strict transform of $B_{i,\text{red}}$ in $S_i$. Then it is obvious that

\[(1) \quad \bar{m}_i \geq m_i - 2.\]

Now we let $\pi : \tilde{C} \to C$ be a base change of degree $d$. Let $S_1$ be the normalization of $S \times_C \tilde{C}$. We can resolve the singularities of $S_1$ by using embedded resolution of $B$. It goes as follows.

\[\begin{array}{ccc}
S_2 & \xrightarrow{\eta} & S'_1 \\
\uparrow & \downarrow {\tau} & \downarrow {\sigma} \\
S_2 & \xrightarrow{\rho_2} & S_1 \\
\end{array} \quad \xrightarrow{\pi_r} \quad \begin{array}{ccc}
& & \\
& & \\
S' & \xrightarrow{\sigma} & S \\
\end{array}\]

where $S'_1$ is the normalization of $S_1 \times_S S'$ (hence it is also the normalization of $S' \times_C \tilde{C}$), and $S_2$ is the minimal resolution of the singularities of $S'_1$. All of the morphisms are induced naturally. So $S_2$ is also a resolution of $S_1$. We shall call such a $\rho_2$ the embedded resolution of the singularities of $S_1$.

Let $f_2 : S_2 \to \tilde{C}$ be the induced fibration, $\tilde{\rho} : S_2 \to \tilde{S}$ the contraction of the $(-1)$-curves contained in the fibers of $f_2$. Then we have an induced fibration $\tilde{f} : \tilde{S} \to \tilde{C}$, which is a relatively minimal fibration determined uniquely by $f$ and $\pi$. We shall call $\tilde{f}$ the pullback fibration of $f$ under the base change $\pi$.

\[\begin{array}{ccc}
\tilde{S} & \xleftarrow{\tilde{\rho}} & S_2 \\
\downarrow {\tilde{f}} & & \downarrow {f_2} \\
\tilde{C} & = & \tilde{C} \\
\end{array} \quad \xrightarrow{f_1} \quad \begin{array}{ccc}
S' & \xrightarrow{\sigma} & S \\
\downarrow {f_1} & & \downarrow {f} \\
\tilde{C} & = & \tilde{C} \\
\end{array} \quad \xrightarrow{\pi} \quad \begin{array}{ccc}
& & \\
& & \\
C & \to & C \\
\end{array}\]

Let

\[(2) \quad \Pi_2 = \rho_1 \circ \rho_2 : S_2 \to S.\]
If $\tilde{f}$ is semistable, then we say that $\pi$ is a *semistable reduction* of $f$. By using Kodaira-Parshin’s construction, we shall construct some special semistable reductions $\pi$.

Note that if $f$ is semistable, then $\tilde{S} = S_2$, $\tilde{\rho}$ is the identity endomorphism, and $\Pi_2$ is a morphism of $\tilde{S}$ to $S$ such that

\[ K_{\tilde{S}/\tilde{C}} = \Pi_2^* K_{S/C}, \quad K_{\tilde{S}/\tilde{C}}^2 = dK_{S/C}^2. \]

**Lemma 1.2.** There exists a semistable reduction $\pi : \tilde{C} \to C$ of $f$ such that

(i) $\pi$ is ramified uniformly over the $s$ critical points of $f$, and the ramification index of $\pi$ at any ramified point is exactly $e$.

(ii) $e$ is divided by all the multiplicities of the components of $\sigma^* B$, and it can be arbitrarily large.

In fact, a base change satisfying the above two conditions must be a semistable reduction. If $b = g(C) > 0$, then the existence follows from Kodaira-Parshin’s construction. Now we consider the case $b = 0$. If $s \geq 3$, then we can construct a base change totally ramified over the $s$ points. Then the existence is reduced to the case $b > 0$. If $s = 2$, the existence is obvious. Since $f$ is non-trivial, we claim that $s$ is at least 2. So the desired semistable reductions always exist.

Indeed, if $b = 0$, $s = 1$, then $f$ is isotrivial (cf. [Be]). Let $p$ be the critical point. Then $f$ is locally trivial over $\mathbb{P}^1 - p = \mathbb{C}$. Because $\mathbb{C}$ is simply connected, $f$ must be trivial over $\mathbb{P}^1 - p$. Thus $F \times \mathbb{P}^1$ is birationally isomorphic to $S$ over $\mathbb{P}^1$. By the uniqueness of the relatively minimal model (since $g > 0$), we know that $f$ is trivial.

Let $\Sigma$ be a normal surface, and let $\rho : M \to \Sigma$ be any resolution of the singularities. If $\Gamma_1, \ldots, \Gamma_t$ are the exceptional curves of $\rho$, then the rational canonical divisor $K_\rho = \sum_{i=1}^t \alpha_i \Gamma_i$ of $\rho$ is defined by the adjunction formulas

\[ K_\rho \Gamma_i + \Gamma_i^2 = 2p_a(\Gamma_i) - 2, \quad \text{for} \quad i = 1, \ldots, t. \]

In particular, we have $K_{\rho_2}$ and $K_\eta$.

In Definition 1.1, we denote by $E_i$ the total inverse image of the exceptional curve of $\sigma_i$ in $S'$.

**Lemma 1.3.** Let $\pi$ be the semistable reduction constructed in Lemma 1.2. Then we have

\[ \tilde{\rho}^* K_{\tilde{S}/\tilde{C}} = \Pi_2^* K_{S/C} - \Pi_2^* \left( \sum_{i=1}^s (F_i - F_i, \text{red}) \right) + K_{\rho_2} - D'', \]

where $D'' = K_{S_2/C} - \tilde{\rho}^* K_{\tilde{S}/\tilde{C}}$ is an effective divisor supported on the exceptional set of $\tilde{\rho}$, and

\[ -K_{\rho_2} = \eta^* \pi_\tau^* \left( \sum_{i=1}^r (m_i - 2)E_i \right). \]
We refer to ([Ta1], §2.1 and §5) for the proof of this lemma. Note that in this case, \( \eta \) is the minimal resolution of rational double points of type \( A_n \), so \( K_\eta = 0 \).

In [Ta1], for each (singular) fiber \( F \) of \( f \), we associate to it three nonnegative rational numbers \( c_1^2(F) \), \( c_2(F) \) and \( \chi_F \).

**Definition 1.4.** Let \( \pi : \tilde{C} \longrightarrow C \) be a base change of degree \( d \) ramified over \( f(F) \) and some non-critical points. If the fibers of \( \tilde{f} \) over \( \pi^{-1}(f(F)) \) are semistable, then we define

\[
\begin{align*}
c_1^2(F) &= K_{S/C}^2 - \frac{1}{d}K_{\tilde{S}/\tilde{C}}^2, \\
c_2(F) &= e_f - \frac{1}{d}e_{\tilde{f}}, \\
\chi_F &= \chi_f - \frac{1}{d}\chi_{\tilde{f}}.
\end{align*}
\]

These invariants are independent of the choice of \( \pi \), and can be computed by embedded resolution of \( F \). One of them is zero iff \( F \) is semistable. Let \( I_{K}(f) = K_{S/C}^2 - \sum_{F} c_1^2(F) \), \( I_{\chi}(f) = \chi_f - \sum_{F} \chi_F \), \( I_{e}(f) = e_f - \sum_{F} c_2(F) \), where \( F \) runs over the singular fibers of \( f \). Then \( I_{K}(f) \), \( I_{\chi}(f) \) and \( I_{e}(f) \) are nonnegative invariants of \( f \), and one of the first two invariants vanishes if and only if \( f \) is isotrivial, i.e., all the nonsingular fibers are isomorphic. Note that if \( f \) is semistable, then these global invariants are nothing but the standard relative invariants of \( f \).

**Lemma 1.5.** ([Ta1], Theorem A’) If \( \tilde{f} \) is the pullback fibration of \( f \) under a base change of degree \( d \), then we have

\[
\begin{align*}
I_{K}(\tilde{f}) &= dI_{K}(f), \\
I_{\chi}(\tilde{f}) &= dI_{\chi}(f), \\
I_{e}(\tilde{f}) &= dI_{e}(f).
\end{align*}
\]

In what follows, we consider the computation of \( c_1^2(F) \). For this, we have to introduce an invariant \( c_{-1}(F) \) of \( F \). In fact, if \( \pi \) is the semistable reduction in Lemma 1.2, then \( c_{-1}(F) \) can be defined as

\[
c_{-1}(F) = \frac{1}{\deg \pi} \# \{ \text{curves over } F \text{ contracted by } \tilde{\rho} \}.
\]

Then we have (cf. (4) or [Ta1], Theorem 3.1)

\[
c_1^2(F) = 4(g - p_a(F_{\text{red}})) + F_{\text{red}}^2 + \sum_{p \in F} \alpha_p - c_{-1}(F),
\]

where \( \alpha_p = \sum_i (m_i - 2)^2 \), and the \( m_i \)'s come from the embedded resolution of the singular point \((F, p)\). In fact, we have proved that

\[
\sum_{p \in F} \alpha_p \leq 2p_a(F_{\text{red}}),
\]

with equality if and only if \( p_a(F_{\text{red}}) = 0 \), i.e., \( F \) is a tree of nonsingular rational curves. (cf. [Ta1], Lemma 3.2). Hence we have
Lemma 1.6. If $F$ is a singular fiber of $f$, then
\[ c_1^2(F) + c_{-1}(F) \leq 4g - 3, \]
and if $p_a(F_{\text{red}}) > 0$, then
\[ c_1^2(F) + c_{-1}(F) \leq 4g - 4. \]

Finally, we refer to [Hi] for the details of the following Miyaoka’s inequality.

Lemma 1.7. ([Mi], Corollary 1.3) If $S$ is a smooth surface such that the canonical divisor $K_S$ is nef (numerically effective), and $E_1, \ldots, E_n$ are disjoint ADE curves on $S$, then for any (reduced) effective normal crossing divisor $D$ disjoint to the $E_i$’s, we have
\[
\sum_{i=1}^{n} m(E_i) + 3\chi_{\text{top}}(D) \leq 3c_2(S) - (K_S + D)^2,
\]
where $m(E)$ is defined as follows,
\[
m(A_r) = 3(r + 1) - \frac{3}{r + 1},
m(D_r) = 3(r + 1) - \frac{3}{4(r - 2)}, \quad \text{for } r \geq 4,
m(E_6) = 21 - \frac{1}{8},
m(E_7) = 24 - \frac{1}{16},
m(E_8) = 27 - \frac{1}{40}.
\]

2. The proof of Theorem A for semistable curves

First of all, we give some notations. Let $f : S \rightarrow C$ be a relatively minimal semistable fibration. We denote by $f^\# : S^\# \rightarrow C$ the corresponding stable model of $f$, and by $q$ a singular point of the singular fibers of $f^\#$. Then $q$ is a rational double point of type $A_n$ or a nonsingular point of $S^\#$. Denote by $E_q$ the inverse image of $q$ in $S$, and let $\mu_q$ be the Milnor number of $(S^\#, q)$, i.e., the number of $(-2)$-curves in $E_q$. If $\mu_q = 0$, i.e., $q$ is a nonsingular point of $S^\#$, then $(S^\#, q)$ can be thought of as a “singular” point of type $A_0$, thus we have $m(E_q) = 0$ (cf. Lemma 1.7).

Theorem 2.1. If $f : S \rightarrow C$ is non-trivial and semistable, and $P \in S(\bar{k})$ is an algebraic point, then
\[ h_K(P) \leq (2g - 1)(d(P) + s) - K_S^2/C, \]
and if the equality holds, then $f$ is smooth.

Proof. Case I. $P$ is a $k$ rational point. Let $E$ be the corresponding section of $f$, and $E^\#$ the image of $E$ in $S^\#$.

If $b = g(C) > 0$, then we know that

$$K_S \sim K_{S/C} + (2b - 2)F$$

is nef. Now we want to use Miyaoka’s inequality. If $q \in E^\#$, then $q$ can not be a nonsingular point of $S^\#$. Let $E_0^q$ be the $(-2)$-curve in $E_q$ intersecting with $E$. Then

$$E_q - E_0^q = E_q' + E_q''.$$ 

In this case, we replace $q$ by $q'$ and $q''$. Note that $m(E_q) = 3(\mu_q + 1) - 3/\mu_q + 1$, and $\mu_q = \mu_q' + \mu_q'' + 1$, $(\mu_q'$ and $\mu_q''$ may be zero). Hence

$$\varepsilon_q := m(E_q) - m(E_q') - m(E_q'') = \frac{3}{\mu_q' + 1} + \frac{3}{\mu_q'' + 1} - \frac{3}{\mu_q + 1}.$$ 

Applying Miyaoka’s inequality to $D = E$ and

$$\{E_q \mid q \notin E^\#\} \cup \{E_q', E_q'' \mid q \in E^\#\},$$

we have

$$\sum_q m(E_q) + 3\chi_{\text{top}}(E) \leq 3c_2(S) - (K_S + E)^2 + \varepsilon$$

where $\varepsilon = \sum_{q \in E^\#} \varepsilon_q$. Note that $f$ is semistable, so $e_f$ is the number of singular points of the singular fibers of $f$, hence we have $\sum_q (\mu_q + 1) = e_f$. Since $h_K(P) = -E^2$, (6) implies that

$$h_K(P) \leq \sum_q \frac{3}{\mu_q + 1} + (2g - 1)(2b - 2) - K_{S/C}^2 + \varepsilon.$$ 

Now we consider the base change $\pi : \tilde{C} \to C$ of degree $d$ constructed in Lemma 1.2. Let $\tilde{f} : \tilde{S} \to \tilde{C}$ be the pullback fibration of $f$ under $\pi$, $\tilde{P}$ the corresponding rational point of $\tilde{f}$. We use the notation $\tilde{\cdot}$ to denote the corresponding objects of $\tilde{f}$. Then the following equalities can be verified easily.

$$K_{\tilde{S}/\tilde{C}}^2 = dK_{S/C}^2, \quad s = \frac{d}{e}s, \quad \mu_{\tilde{\cdot}} + 1 = e(\mu_q + 1), \quad \varepsilon = \frac{d}{e^2}\varepsilon,$$

$$2g(\tilde{C}) - 2 = d(2b - 2) + d \left(1 - \frac{1}{e}\right)s, \quad h_K(\tilde{P}) = dh_K(P).$$

Applying (7) to $\tilde{f}$, we have

$$dh_K(P) \leq \frac{d}{e^2} \sum_q \frac{3}{\mu_q + 1} + (2g - 1) \left((2b - 2)d + d \left(1 - \frac{1}{e}\right)s\right) - dK_{S/C}^2 + \frac{d}{e^2}\varepsilon,$$
i.e.,

\[ h_K(P) - (2g - 1)(d(P) + s) + K_{S/C}^2 \leq -\frac{(2g - 1)s}{e} + \frac{1}{e^2} \left( \sum_q \frac{3}{\mu_q + 1} + \varepsilon \right). \]

Let \( e \) be large enough we can see that the left hand side \( \leq 0 \), or \(<0\) if \( s > 0 \).

Now we consider the case \( b = 0 \). Since \( f \) is non-trivial, we have \( s \geq 5 \) [Ta2]. Then we consider also the base change given in Lemma 1.2. Since \( g(\tilde{C}) > 0 \), the height inequality for \( \tilde{P} \) holds, which implies the inequality for \( P \).

**Case II.** \( P \) is an algebraic point of degree \( d \). Let \( E_P \) be the corresponding reduced and irreducible horizontal curve on \( S \), \( \tilde{C} \) the normalization of \( E_P \), and \( \pi: \tilde{C} \rightarrow C \) the morphism induced by \( f \). Let \( \tilde{f}: \tilde{S} \rightarrow \tilde{C} \) be the pullback of \( f \) under \( \pi \). Since \( f \) is semistable, from (2) and (3) we know that \( \Pi_2 \) is a morphism of \( \tilde{S} \) to \( S \) such that

\[ K_{\tilde{S}/\tilde{C}} = \Pi_2^*(K_{S/C}), \quad K_{\tilde{S}/\tilde{C}}^2 = d_P K_{S/C}^2. \]

By the construction of \( \tilde{f}: \tilde{S} \rightarrow \tilde{C} \), there is a section \( \tilde{E} \) of \( \tilde{f} \) such that the induced map of \( \tilde{E} \) to \( E_P \) is a birational morphism, hence \( \Pi_2(\tilde{E}) = E_P \). By projection formula we have

\[
\begin{align*}
    h_K(P) &= \frac{1}{d_P} E_P K_{S/C} \\
    &= \frac{1}{d_P} \tilde{E} \cdot \Pi_2^*(K_{S/C}) \\
    &= \frac{1}{d_P} \tilde{E} K_{\tilde{S}/\tilde{C}} \\
    &\leq (2g - 1) \left( \frac{2g(\tilde{C}) - 2}{d_P} + \frac{s}{d_P} \right) - \frac{1}{d_P} K_{\tilde{S}/\tilde{C}}^2 \\
    &\leq (2g - 1)(d(P) + s) - K_{S/C}^2.
\end{align*}
\]

If \( s > 0 \), then the inequality holds strictly. Q.E.D.

### 3. The proof of Theorem A for non-semistable curves

Let \( f: S \rightarrow C \) be a non-semistable fibration with \( s \) singular fibers \( F_1, \ldots, F_s \). Let \( P \) be an algebraic point of degree \( d_P \), and \( E_P \) the corresponding horizontal curve on \( S \). We shall prove in this section that

\[
(8) \quad h_K(P) < (2g - 1)(d(P) + 3s) - K_{S/C}^2.
\]

We let \( \pi: \tilde{C} \rightarrow C \) be the semistable reduction of \( f \) constructed in Lemma 1.2. For convenience, we recall the construction of the pullback fibration \( \tilde{f} \) under \( \pi \) given in Sect. 1.
Note that \( \Pi_2 \) is defined as \( \rho_1 \circ \rho_2 \).

Let \( C_P \) be the normalization of \( E_P \), \( \pi_P : C_P \to C \) the morphism induced by \( f \), and \( f_P : S_P \to C_P \) the pullback fibration of \( f \) under \( \pi_P \). By the construction of \( f_P \), there is a section of \( f_P \) which maps birationally onto \( E_P \).

Now considering the normalization of one component of the fiber product \( \tilde{C} \times_{C_P} C_P \), we obtain a curve \( \hat{C} \) such that the following diagram commutes.

\[
\begin{array}{ccccccc}
\tilde{C} & \xrightarrow{\psi} & C_P \\
\phi \downarrow & & \downarrow \pi_P \\
\hat{C} & \xrightarrow{\pi} & C
\end{array}
\]

Among all the components of \( \tilde{C} \times_{C_P} C_P \), we can choose \( \hat{C} \) such that the induced morphism \( \phi \) is of the least degree.

Let \( \hat{f} : \hat{S} \to \hat{C} \) be the pullback fibration of \( \tilde{f} \) under \( \phi \). By the uniqueness of the relatively minimal model (since \( g > 0 \)) and the universal property of fiber product, we know that \( \hat{f} \) is nothing but the pullback of \( f_P \) under \( \psi \).

\[
\begin{array}{cccc}
\hat{S} & \xrightarrow{\phi} & S_P \\
\Phi_2 \downarrow & & \downarrow \pi_P \\
\tilde{S} & \xleftarrow{\rho} & S_2 & \xrightarrow{\Pi_2} & S
\end{array}
\]

Hence \( \hat{f} \) has a section \( \hat{E} \) induced by the section of \( f_P \) mentioned above. Now we know that the induced rational map of \( \hat{E} \) to \( E_P \) is of degree \( \deg \psi \). Since \( \tilde{f} \) is semistable, the induced map \( \Phi_2 : \hat{S} \to \tilde{S} \) is a morphism. Denote by \( \hat{E} \) the image of \( \hat{E} \) in \( \tilde{S} \), and let \( E_2 \) be the strict transform of \( \tilde{E} \) in \( S_2 \).

We claim that the restriction map \( \Phi_2|_{\hat{E}} : \hat{E} \to \tilde{E} \) is birational.

\[
\begin{array}{cccc}
\hat{E} & \xrightarrow{\Phi_2|_{\hat{E}}} & \hat{S} & \xrightarrow{\tilde{f}} & \hat{C} \\
\Phi_2 \downarrow & & \downarrow \phi & & \downarrow \phi \\
\tilde{E} & \xrightarrow{\Phi_2} & \tilde{S} & \xrightarrow{\tilde{f}} & \tilde{C}
\end{array}
\]
Indeed, if $\Phi_2|_E$ is not birational, then the degree of the morphism $\tilde{f}|_{\tilde{E}} : \tilde{E} \rightarrow \tilde{C}$ is less than that of $\phi$. On the other hand, we can see that the map of $\tilde{E}$ to $E_P$ can be lifted to $C_P$, hence $\tilde{f}|_{\tilde{E}}$ is factorized through $\tilde{C} \times_C C_P$. Hence there is a component of $\tilde{C} \times_C C_P$ such that degree of the induced morphism to $\tilde{C}$ is less than that of $\phi$, which contradicts the choice of $\tilde{C}$. This proves the claim.

Now we have

$$
\Pi_2^*E_2 = \deg \psi E_P, \quad \Phi_2^*\tilde{E} = \tilde{E},
$$

$$
\tilde{\rho}_*E_2 = \tilde{E}, \quad K_{\tilde{S}/\tilde{C}} = \Phi_2^*K_{\tilde{S}/\tilde{C}}.
$$

From Lemma 1.3,

$$
\tilde{\rho}^*K_{\tilde{S}/\tilde{C}} = \Pi_2^*K_{S/C} - D_\pi,
$$

where

$$
D_\pi = \Pi_2^* \left( \sum_{i=1}^{s} (F_i - F_i, \text{red}) \right) - K_{\rho_2} + D''
$$

is an effective divisor. By (9) and projection formula we have

$$
h_K(\hat{P}) = K_{\tilde{S}/\tilde{C}} \tilde{E} = \Phi_2^*K_{\tilde{S}/\tilde{C}} \tilde{E}
$$

$$
= K_{\tilde{S}/\tilde{C}} \tilde{E} = \tilde{\rho}^*K_{\tilde{S}/\tilde{C}} E_2
$$

$$
= (\Pi_2^*K_{S/C} - D_\pi)E_2
$$

$$
= \deg \psi K_{S/C} E_P - D_\pi E_2
$$

$$
= d_P \deg \psi h_K(P) - D_\pi E_2,
$$

thus

$$
h_K(P) = \frac{1}{d_P \deg \psi} h_K(\hat{P}) + \frac{1}{d_P \deg \psi} D_\pi E_2.
$$

Note that

$$
d \deg \phi = d_P \deg \psi.
$$

where $d$ is the degree of $\pi$.

**Lemma 3.1.**

$$
\frac{1}{d_P \deg \psi} h_K(\hat{P}) \leq (2g - 1)(d(P) + s) - I_K(f).
$$

*Proof.* First of all, we consider the case when $\hat{C} = \mathbb{P}^1$ and $\hat{f}$ is trivial, i.e., $f$ is isotrivial, so $I_K(f) = 0$. It is easy to see that $h_K(\hat{P}) = 0$. Since $f$ is non-trivial, $s \geq 2$, and the desired inequality holds.

In what follows, we assume that $\hat{f}$ is non-trivial if $\hat{C} = \mathbb{P}^1$. 
Since \( \hat{f} \) is semistable, by Theorem 2.1, we have

\[
(12) \quad h_K(\hat{P}) \leq (2g - 1) \left( 2g(\hat{C}) - 2 + \hat{s} \right) - K_{\hat{S}/\hat{C}}^2,
\]

where \( \hat{s} \) is the number of singular fibers of \( \hat{f} \). (Note that if \( \hat{f} \) is trivial, then we have a stronger inequality \( h_K(\hat{P}) \leq 2g(\hat{C}) - 2 \), because \( g(\hat{C}) > 0 \).) It is obvious that

\[
(13) \quad \hat{s} \leq \frac{d s}{e} \deg \phi = \frac{s}{e} d_P \deg \psi.
\]

From Lemma 1.5, we have

\[
(14) \quad K_{\hat{S}/\hat{C}}^2 = d_P \deg \psi I_K(f).
\]

By Hurwitz formula,

\[
2g(\hat{C}) - 2 = \deg \psi (2g(C_P) - 2) + r_{\psi}.
\]

Note that the ramification index of \( \pi \) at any ramified point is \( e \), by the construction of \( \psi \) we can see that the index of \( \psi \) at any ramified point is at most \( e \). Let \( x \) be a branch point of \( \psi \). Then we know that the contribution of \( \psi^{-1}(x) \) to \( r_{\psi} \) is at most \((1 - 1/e)\deg \psi\). On the other hand, \( \psi \) has at most \( d_P s \) branch points, thus

\[
r_{\psi} \leq \left( 1 - \frac{1}{e} \right) s d_P \deg \psi,
\]

it implies that

\[
(15) \quad 2g(\hat{C}) - 2 \leq \left( d(P) + \left( 1 - \frac{1}{e} \right) s \right) d_P \deg \psi.
\]

Combining (12)--(15), we have

\[
h_K(\hat{P}) \leq ((2g - 1)(d(P) + s) - I_K(f)) d_P \deg \psi.
\]

This completes the proof. \( \text{Q.E.D.} \)

Now we shall find the upper bound of \( D_\pi E_2/d_P \deg \psi \). So we only need to consider the first diagram at the beginning of this section. From (9),

\[
\Pi_2 \ast E_2 = \deg \psi \ E_P, \quad E_P(F_i - F_{i,\text{red}}) < d_P,
\]

by projection formula we have

**Lemma 3.2.**

\[
\frac{1}{d_P \deg \psi} \Pi_2^* \left( \sum_{i=1}^s (F_i - F_{i,\text{red}}) \right) E_2 < s.
\]
Lemma 3.3.

$$\frac{-K_{\rho_2}E_2}{d_P \deg \psi} \leq s - \# \{ F_i \mid p_a(F_{i,\text{red}}) = 0 \}.$$  

Proof. Note first that $p_a(F_{i,\text{red}}) = 0$ implies that $F_i$ is a tree of non-singular rational curves, so $F_i$ has no effect on $-K_{\rho_2}$ and $-K_{\rho_2}E_2$. For simplicity, we assume that $p_a(F_{i,\text{red}}) \neq 0$ for all $i$.

We let

$$\sigma^*E_P = E_P + \sum_{i=1}^r a_{i-1}E_i,$$

where $\tilde{E}_P$ is the strict transform of $E_P$ and $a_i \geq 0$ is the multiplicity of the strict transform of $E_P$ at $p_i$. We have known that $(\pi_r \circ \eta)_*E_2 = \deg \psi \tilde{E}_P$, and

$$-K_{\rho_2} = (\pi_r \circ \eta)^* \left( \sum_{i=1}^r (m_{i-1} - 2)E_i \right),$$

hence

$$-K_{\rho_2}E_2 = (\pi_r \circ \eta)^* \left( \sum_{i=1}^r (m_{i-1} - 2)E_i \right)E_2$$

$$= \deg \psi \sum_{i=1}^r (m_{i-1} - 2)E_i \tilde{E}_P$$

$$= \deg \psi \sum_{i=1}^r (m_{i-1} - 2)a_{i-1}. \tag{16}$$

On the other hand,

$$\sigma^* \left( \sum_{i=1}^s F_{i,\text{red}} \right) = \sum_{i=1}^s F_{i,\text{red}} + \sum_{i=1}^r \tilde{m}_{i-1}E_i,$$

where $\tilde{F}_{i,\text{red}}$ is the strict transform of $F_{i,\text{red}}$, and $\tilde{m}_{i-1}$ is the multiplicity of the strict transform of $\sum_{i=1}^s F_{i,\text{red}}$ at $p_{i-1}$. From $\sum_{i=1}^s \tilde{F}_{i,\text{red}} \tilde{E}_P \geq 0$, we have

$$\sum_{i=1}^r a_{i-1}\tilde{m}_{i-1} \leq \sum_{i=1}^s F_{i,\text{red}}E_P \leq sd_P. \tag{17}$$

Combining (16) and (17) with (1), we have

$$-K_{\rho_2}E_2 \leq sd_P \deg \psi.$$

This completes the proof of the lemma. Q.E.D.
Lemma 3.4. 

\[ \frac{D''E_2}{d_P \deg \psi} \leq \sum_{i=1}^{s} c_{-1}(F_i). \]

Proof. We let \( \tilde{\rho} = \rho_k \circ \rho_{k-1} \circ \cdots \circ \rho_1 \) be the decomposition of \( \rho \) into \( k \) blowing-downs of \((-1)\)-curves. If we denote by \( \Delta_i \subset S_2 \) the total transform of the exceptional curve of \( \tilde{\rho}_i \), then

\[ D'' = K_{S_2/C} - \tilde{\rho}^* K_{S/C} = \sum_{i=1}^{k} \Delta_i. \]

It is easy to see that \( E_2 \Delta_i \) is the multiplicity of \( \tilde{\rho}_i \circ \cdots \circ \rho_1 (E_2) \) at \( \tilde{\rho}_i \circ \cdots \circ \rho_1 (\Delta_i) \), so

\[ E_2 \Delta_i \leq E_2 F = \frac{\deg \psi}{d} d_P, \]

where \( F \) is a fiber of \( f_2 \). On the other hand, the number of curves contracted by \( \tilde{\rho} \) is \( k = d \sum_{i=1}^{s} c_{-1}(F_i) \). Hence we have the desired inequality. \( \text{Q.E.D.} \)

Proof of (8). From (10) and the above lemmas, we have

\[ h_K(P) < (2g-1)(d(P) + s) - K_{S/C}^2 + \sum_{i=1}^{s} (c_1^2(F_i) + c_{-1}(F_i)) - \# \{ F_i \mid p_a(F_i, \text{red}) = 0 \} + 2s. \]

By Lemma 1.6,

\[ \sum_{i=1}^{s} (c_1^2(F_i) + c_{-1}(F_i)) \leq (4g-4)s + \# \{ F_i \mid p_a(F_i, \text{red}) = 0 \}. \]

Hence we have

\[ h_K(P) < (2g-1)(d(P) + 3s) - K_{S/C}^2. \]

\( \text{Q.E.D.} \)

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