Cylindrical Invisibility Cloak with Simplified Material Parameters is Inherently Visible

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Abstract

It was proposed that perfect invisibility cloaks can be constructed for hiding objects from electromagnetic illumination (Pendry et al., Science 312, p. 1780). The cylindrical cloaks experimentally demonstrated (Schurig et al., Science 314, p. 997) and theoretically proposed (Cai et al., Nat. Photon. 1, p. 224) have however simplified material parameters in order to facilitate easier realization as well as to avoid infinities in optical constants. Here we show that the cylindrical cloaks with simplified material parameters inherently allow the zero$^{th}$-order cylindrical wave to pass through the cloak as if the cloak is made of a homogeneous isotropic medium, and thus visible. To all high-order cylindrical waves, our numerical simulation suggests that the simplified cloak inherits some properties of the ideal cloak, but finite scatterings exist.

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Recently there has been an increase of interest in realizing invisibility cloaks \[1, 2, 3, 4, 5, 6, 7, 8\]. In particular, the coordinate transformation method proposed in \[4\] is noticed to be especially powerful for designing invisibility cloaks that can in principle completely shield enclosed objects from electromagnetic (EM) illumination and at the same time cause zero disturbance to the foreign EM field. In 2D case, the cylindrical cloak obtained through such technique has anisotropic, spatially varying optical constants. In addition, some of the material parameters have infinite values at the interior surface of the cloak. To facilitate experimental realization at microwave frequencies, Schurig et al. have simplified the material properties such that only one material parameter is gradient and the requirement on infinite material constant is lifted \[9\]. The authors claimed that, in comparison to the ideal cloak, the simplified cloak maintains the power-flow bending with the penalty of nonzero reflectance at the outer interface. Similar simplification on a cylindrical cloak at optical frequencies is also employed in Ref. \[10\]. In this paper we provide a systematic theoretical study on simplified cylindrical cloaks. It is found that the bare device constructed using the simplified medium fails to be invisible. In addition, the device does not possess a spatial region that is completely in isolation from the outside world electromagnetically. Hence, perfect hiding with such a simplified cloak is not possible.

Consider a cylindrical cloak with its inner and outer boundaries positioned at \(r = a\) and \(r = b\) respectively. Both domains inside and outside the cloak are air. The structure is in general a three-layered cylindrical scatterer. We refer to the layers from inside to outside as layer 1, 2 and 3, respectively. Our analysis, similar to all previous works, is on normal incidence. That is, the \(k\) vector is perpendicular to the cloak cylinder axis. The EM wave is assumed to have TE polarization (i.e. electric field only exists in \(z\) direction). The TM polarization case can be derived by making \(E \rightarrow -H\), \(\varepsilon \rightarrow \mu\), and \(\mu \rightarrow \varepsilon\) substitutions. By default, we choose the cloak’s cylindrical coordinate as the global coordinate. In a homogeneous material region (i.e. cloak interior and exterior) the general solution is expressible in Bessel functions. Within the cloak medium, the general wave equation that governs the \(E_z\) field can be written as

\[
\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \frac{\partial E_z}{\partial r} \right) \right] + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\mu_r} \frac{\partial E_z}{\partial \theta} \right) + k_0^2 \varepsilon_r E_z = 0, \tag{1}
\]

where \(k_0\) is the free-space wave number, \(\mu_0\), \(\mu_r\), and \(\varepsilon_r\) are polarization-dependent permeability/permitivity profiles of the cloak. The time dependence \(\exp(i \omega t)\) has been used for
deriving Eq. 1.

Ideally, the cloak is designed to compress all fields within a cylindrical air region $r < b$ into the cylindrical annular region $a < r < b$. A corresponding coordinate transformation leads to a set of anisotropic and spatially variant material constants in the cloak shell, as described in [9]. Such an annular cylinder can indeed provide perfect invisibility cloaking [11], but it requires infinite values of optical constants at the cloak’s inner boundary. To circumvent the fabrication difficulty, the simplified parameters were used [9]. They are in the form of

$$
\mu_r = \left(\frac{r-a}{r}\right)^2, \\
\mu_\theta = 1, \\
\epsilon_z = \left(\frac{b}{b-a}\right)^2.
$$

To achieve cloaking for TM waves, the same set of material parameters, but for $\epsilon_r$, $\epsilon_\theta$ and $\mu_z$, is used [10]. However, we notice that the procedure for simplification of the material parameters adopted in [9] is questionable, as the derivation has assumed beforehand that $\mu_\theta$ is a constant. It is obvious that the invariant $\mu_\theta$ can be taken out of the differential operator in Eq. 1. Therefore wave behavior within the cloak shell is altered as compared to that in an ideal cloak.

Since the material parameters in Eq. 2 are azimuthally invariant (which is also true for the ideal parameter set), we can use the variable separation $E_z = \Psi(r)\Theta(\theta)$. Eq. 1 can then be decomposed into

$$
d^2\Theta d\theta^2 + m^2\Theta = 0, \\
d\left(\frac{r}{\mu_\theta} \frac{d\Psi}{dr}\right) + k_0^2 r \epsilon_z \Psi - m^2 \frac{1}{r \mu_r} \Psi = 0,
$$

where $m$ is an integer. The solution to Eq. 3 is $\exp(\text{im}\theta)$. Equation 4 is a second-order homogeneous differential equation. Two independent solutions are expected. At this moment, we assume the solution to Eq. 4, for a fixed $m$, can be written in general as $A_m Q_m + B_m R_m$, where $A_m$ and $B_m$ are constants. $Q_m$ and $R_m$ are functions of $r$. Now valid field solutions
in three layers (denoted by superscripts) can be described as

\[ E_1^z = \sum_m \mathcal{A}_m J_m(k_0 r) \exp(im\theta), \quad (5) \]

\[ E_2^z = \sum_m \{\mathcal{A}_m^2 Q_m + \mathcal{B}_m R_m\} \exp(im\theta), \quad (6) \]

\[ E_3^z = \sum_m \{\mathcal{A}_m^3 J_m(k_0 r) + \mathcal{B}_m^3 H_m^{(2)}(k_0 r)\} \exp(im\theta). \quad (7) \]

\(H_m^{(2)}\) is the Hankel function of the second kind, which represents outward-travelling cylindrical wave. The \(J_m\) and \(H_m^{(2)}\) terms in the 3rd layer are physically in correspondence to the incident and scattered waves, respectively. Hence, the scattering problem becomes to solve for, most importantly, \(\mathcal{A}_m^1\) (transmitted field) and \(\mathcal{B}_m^3\) (scattered field) subject to a given incidence \(\mathcal{A}_m^3\) [12]. Comparatively \(\mathcal{A}_m^2\) and \(\mathcal{B}_m^2\) are physically less interesting. The coefficients are solved by matching the tangential fields \((E_z, H_\theta)\) at the layer interfaces. Due to the orthogonality of the function \(\exp(im\theta)\), the cylindrical waves in different orders decouple. We hence can examine the transmission and scattering of the cloak for each individual order number \(m\).

By substituting the simplified material parameters into Eq. 4 we obtain

\[ (r-a)^2 \frac{d^2 \Psi}{dr^2} + \frac{(r-a)^2}{r} \frac{d\Psi}{dr} + \left[(r-a)^2 \left(\frac{b}{b-a}\right)^2 k_0^2 - m^2\right] \Psi = 0. \quad (8) \]

Equation 8 has two non-essential singularities at \(r = 0\) and \(r = a\) for \(m \neq 0\). It is worthwhile to mention that, with the ideal parameters, Eq. 4 can be written instead as

\[ (r-a)^2 \frac{d^2 \Psi}{dr^2} + (r-a) \frac{d\Psi}{dr} + \left[(r-a)^2 \left(\frac{b}{b-a}\right)^2 k_0^2 - m^2\right] \Psi = 0. \quad (9) \]

When \(m = 0\), Eq 8 can be further simplified to

\[ r^2 \frac{d^2 \Psi}{dr^2} + r \frac{d\Psi}{dr} + r^2 k_0^2 \left(\frac{b}{b-a}\right)^2 \Psi = 0. \quad (10) \]

This is the zero\(^{th}\)-order Bessel differential equation. Its non-essential singularity remains at \(r = 0\). Equation 10 suggests that an incoming zero\(^{th}\)-order cylindrical wave would effectively see the simplified cloak as a homogeneous isotropic medium whose effective refractive index is \(n_{\text{eff}} = \frac{b}{b-a}\). Its transmission through the cloak shell is therefore determined by the etalon effect of the finite medium.
When $m \neq 0$, the wave solution is governed by Eq. 8. By comparing Eqs. 9 and 8 we see that at radial positions $r >> a$, Eq. 8 asymptotically resembles Eq. 9 due to $r - a \approx r$. This indicates that all high-order cylindrical waves tend to behave similarly in both media at $r >> a$ positions, and hence the importance of parameter $b$, at which the cloak medium is truncated. In the following, we will derive the scattering coefficient $s_m$ subject to individual cylindrical wave incidences. The scattering coefficient in each cylindrical order is defined as $s_m = |B_m^3/A_m^3|$. Analytic derivation of $s_m$ can be done if two solutions to Eq. 8 (i.e. $Q_m$ and $R_m$) are known in closed form. However, despite the analoguences to Eq. 9, Eq. 8 fails the analytic Frobenius method [13]. Here we tackle the problem through the finite-element method. The field outside the cloak is computed numerically, and then is used for deriving coefficients $A_m^3$ and $B_m^3$ through a fitting procedure. $s_m$ is known in turn. Solutions with different azimuthal orders are obtained by varying the azimuthal dependence of a circular current source outside the cloak. The scattering problem is numerically manageable since functionals $\varepsilon_z\mu_\theta$ and $\mu_\theta/\mu_r$ in Eq. 4, unlike in the ideal cloak case, are both finite and do not possess any removable singularity for the simplified medium. The commercial software COMSOL is deployed to carry out calculations. For our case study, we fix $a = 0.1$m and operating frequency $f = 2$GHz. The performance of the cloak is examined as $b$ is increased from 0.2m. Similar parameters are also found in [14].

FIG. 1: Variation of the scattering coefficients, examined in each cylindrical wave order, as a function of $b$. For $m = 1, 2$, the curves are fitted using Savitzky-Golay smoothing filter according to numerically derived data points (dots).
The scattering coefficients in different azimuthal orders as a function of $b$ are shown in Fig. 1. As expected, the zero$^{th}$-order scattering coefficient is quite distinct from others due to the different governing wave equation. In fact, analytic solution exists when $m = 0$, as field in the cloak medium are Bessel functions \[11\]. The excellent agreement between the numerical and analytic results for zero$^{th}$-order scattering coefficient confirms the validity and accuracy of our approach. Using the analytic technique, the zero$^{th}$-order scattering coefficient is found to converge to 0.867 with respect to $b$. Despite that the effective index of the cloak approaches to 1 as $b$ increases, the phase variation of the zero$^{th}$-order wave within the cloak medium is increasing, as $k_0 \frac{b}{b-a}(b-a) = k_0 b$. This explains why the scattering coefficient converges to a value other than 0. The existence of zero$^{th}$-order scattering coefficient effectively disqualifies the cloak to be completely invisible.

Compared to the zero$^{th}$-order scattering coefficient, the high-order scattering coefficients (only those for $m = 1, 2$ are shown in Fig. 1) are noticed to be in a similar oscillatory fashion, and in general much smaller. The scattering coefficient tends to converge to a value closer to zero when the order number $m$ increases. Over certain ranges of $b$ value (e.g. around $b = 0.225m$ for $m = 1$), the computed $\mathcal{R}^2_m$ changes sign and hence the resulted scattering coefficient seems to be flipped from a negative value. We should attribute the relatively small high-order scattering coefficients to the cloak’s partial inherence of the ideal cloak based on coordinate transformation \[15\]. However, our numerical result shows that the high-order scattering coefficients do not converge to zero even when the cloak wall is very thick.

Besides the requirement of zero scattering (invisibility), a device also need to possess a spatial region which is in complete EM isolation from the exterior world in order to be an invisibility cloak. Therefore it is meaningful to know how much field penetrates into the simplified cloak subject to a foreign EM illumination. Again, the problem is studied by examining the individual cylindrical wave components separately. The transmission coefficient, defined as $t_m = |\mathcal{A}^1_m/\mathcal{A}^3_m|$, is used to characterize the field transmission. When $m = 0$, the amount of field transmitted into the cloak interior can be analytically derived, which is shown in Fig. 2 as a function of $b$. The transmission is noticed to be oscillatory, and converging to 1 as $b$ increases. Numerical calculation is also superimposed for validation. When $m \neq 0$, the FEM calculations show that the field inside the cloak is almost zero. The corresponding transmission coefficients are exclusively smaller than 0.005, hence are
not plotted in Fig. 2. This indicates that the contour \( r = a \) provides an insulation between its enclosed domain and the exterior domain, but only for all high-order cylindrical waves. Therefore, any objects placed inside the cloak are exposed to the zero\(^{th}\)-order cylindrical wave component. Reversely, the zero\(^{th}\)-order wave component of an EM source placed within the cloak (or scattered wave by objects inside the cloak) will transmit out. As a result, objects enclosed by a simplified cloak is sensible by a foreign detection unit.

Next we numerically demonstrate (in COMSOL) scattering by the simplified cloaks with a plane-wave incidence. The incident plane wave travels from left to right and has the amplitude of 1. When \( b = 0.2 \)m, the \( E_z \) snapshot, \( E_z \) norm, and the scattered \( E_z \) snapshot are plotted in Fig. 3(a1)-(a3), respectively. It is noticed that the amplitude of the scattered field is about one half of the incident field. From the scattered field distribution, high-order Bessel terms constitute a significant portion. Scattering aside, the field inside the cloak shell is seen to have only zero\(^{th}\)-order Bessel term. We then increase \( b \) to 0.5m, and the corresponding numerical results are plotted in Fig. 3(b1)-(b3). Compared to the previous case, scattered field is reduced roughly by half in amplitude, and is now dominated by the zero\(^{th}\)-order Bessel term. The \( E_z \) norm is closer to be uniform outside the cloak, indicating better invisibility. The overall smaller scattering as well as the dominance of the zero\(^{th}\)-order scattering for the second cloak agree well with our derivation of the scattering coefficients in Fig. 1. When \( b \) is changed from 0.2m to 0.5m, the transmitted \( E_z \) field at the center of the cloak increases from 0.6032 to 0.8855 in norm, also in agreement with Fig. 2.

![FIG. 2: The zero\(^{th}\)-order transmission coefficient.](image-url)
FIG. 3: (a1)-(a3) $E_z$ snapshot, $E_z$ norm, and scattered $E_z$ field, respectively, for a simplified cloak with $a = 0.1\text{m}$ and $b = 0.2\text{m}$. (b1)-(b3) Same fields but for a simplified cloak with $a = 0.1\text{m}$ and $b = 0.5\text{m}$. (c1)-(c3) Same fields but for a near-to-ideal cloak with $a = 0.101\text{m}$ and $b = 0.5\text{m}$. For completeness, we also show the scattering by a near-to-ideal cloak in Fig. 3(c1)-(c3). The material parameters of the cloak are defined by $\mu_r = \frac{r-a}{r}$, $\mu_\theta = \frac{r}{r-a}$, and $\varepsilon_z = \left(\frac{b}{b-a}\right)^2 \frac{r-a}{r}$, with $a = 0.1\text{m}$ and $b = 0.5\text{m}$. To avoid the $r = 0.1\text{m}$ critical contour, we let the inner boundary of the cloak be positioned at $r = 0.101\text{m}$. In [11] we have confirmed analytically that the performance of an ideal cloak is extremely sensitive to the position of the cloak’s inner surface. In particular, the zero$^{th}$-order cylindrical wave will experience considerable reflection and transmission as it meets the cloak shell. This is confirmed in Fig. 3(c1)-(c3). The scattered field is almost purely the zero$^{th}$-order cylindrical wave. Even at such a small mis-location of the inner boundary, the amount of field leaking into the cloak is handsome, valued at $0.5466$ in norm. By comparing panels (c1)-(c3) and (b1)-(b3) in Fig. 3 we see that the improved cloak made from simplified medium inherits some merits of the near-to-ideal cloak, but with overall higher scattering, especially in the monopole wave component.

In conclusion, we have theoretically studied the cylindrical cloaks with simplified material
parameters. Such a simplified cloak is shown to inherit some properties of an ideal cloak based on coordinate transformation of Maxwell equations. However the penalty of using the simplified cloak is more than just nonzero reflectance at the cloak boundary. The monopole component of the incoming wave will always experience relatively high scattering. High-order cylindrical waves also experience finite (although smaller) scattering even when the cloak’s wall is kept very thick, as suggested by our numerical simulation. Besides that the device itself is visible, it cannot completely shield EM field, due to penetration of the monopole field component. Hence cloaking of objects will not be perfect. Lastly, considering that the monopole field treats a simplified cloak as a conventional glass tube, detection of any object placed within the cloak can be greatly enhanced by using a EM source with the maximum energy on its monopole component.

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