Open KdV hierarchy and minimal gravity on disk

Aditya Bawane, Hisayoshi Muraki and Chaiho Rim
Department of Physics, Sogang University, Seoul 04107, Korea

Abstract

We show that the minimal gravity of Lee-Yang series on disk is a solution to the open KdV hierarchy proposed for the intersection theory on the moduli space of Riemann surfaces with boundary.

1 Introduction

The generating function of the intersection theory on the moduli space of Riemann surfaces had been conjectured to satisfy the KdV hierarchy together with the string equation [1]. It was shown in [2, 3] that Witten’s conjecture is equivalent to the Virasoro constraints. On the other hand, minimal gravity is 2-dimensional quantum gravity coupled with minimal conformal matter so that the resulting theory still remains conformal and topological \((c = 0)\) [4, 5, 6], and also obeys the KdV hierarchy [7, 8]. As a result, KdV hierarchy plays an important role in understanding intersection theory as well as topological gravity.

Recently, the generating function for intersection theory on the moduli space of Riemann surfaces with boundary has been conjectured to satisfy the so-called open KdV hierarchy which contains the KdV flow parameters as well as new flow parameters related with the boundary [9]. It has also been checked that the open KdV hierarchy can be represented in modified form of Virasoro constraints [10]. It is naturally expected that the open KdV hierarchy also describes minimal gravity on a disk. We check this expectation in this letter.

In section 2, we summarize minimal gravity on sphere and its connection with KdV hierarchy and in section 3, open KdV hierarchy is explicitly checked for the free energy of minimal gravity on a disk. Section 4 is the conclusion.

2 Minimal gravity on sphere and KdV hierarchy

Minimal gravity \(M(2, 2p + 1)\) of Lee-Yang series is represented in terms of one-matrix model and its free energy on a sphere \(F_{\text{sphere}}\) is given as [11]

\[
F_{\text{sphere}} = \frac{1}{2} \int_0^{u_0} du |\mathcal{P}(u)|^2; \quad \mathcal{P}(u) = \sum_{m=0} t_m \frac{u^m}{m!}.
\] (1)

The polynomial equation \(\mathcal{P}(u) = 0\) is called the string equation. \(u_0\) is one of the solutions of the string equation and is related to the free energy as \(u_0 = \partial^2 F_{\text{sphere}} / \partial t_0^2\).
The free energy contains the set of parameters \( \{ t_m \} \) and becomes the generating function because derivatives with respect to \( \{ t_m \} \)’s evaluated on-shell provide correlations of the corresponding operators. By on-shell we mean that \( t_i = 0 \) for all \( i \), except \( t_{p−1} \) and \( t_{p+1} \). We normalize \( t_{p+1} \) to 1, and \( t_{p−1} \) is proportional to the cosmological constant \( \mu \). Multi-correlation on sphere is given as

\[
\left< \prod_{i=1}^{n} O_{a_i} \right>_{\text{sphere}} = \frac{\partial^n F_{\text{sphere}}}{\partial t_{a_1} \cdots \partial t_{a_n}}. \tag{2}
\]

Computing the above quantity on-shell gives us the physical correlation. For example, the two-point correlation

\[
\left< O_0 O_0 \right>_{\text{sphere}} = u^* \text{ is the desired result and further results in } \left< O_0 O_{n−1} \right>_{\text{sphere}} = u^*_n/n!, \tag{3}
\]

where the symbol * stands for on-shell value. One can also see that the free energy in (1) satisfies the KdV hierarchy on the sphere

\[
\frac{\partial^3 F_{\text{sphere}}}{\partial t_n \partial t_0^2} = \frac{\partial u}{\partial t_n} = \frac{u^n \partial u}{n! \partial t_0}. \tag{4}
\]

The KdV hierarchy in general is given by

\[
\frac{1}{\lambda^2} \frac{2n + 1}{2} \frac{\partial^3 F_c}{\partial t_0^2 \partial t_n} = \frac{\partial^2 F_c}{\partial t_0^2} \frac{\partial^2 F_c}{\partial t_{n−1}^2} + \frac{1}{2} \frac{\partial^2 F_c}{\partial t_0^2} \frac{\partial^2 F_c}{\partial t_0 \partial t_{n−1}} + \frac{1}{8} \frac{\partial^2 F_c}{\partial t_0 \partial t_{n−1}}. \tag{5}
\]

where \( n \geq 1 \). The string equation is

\[
\frac{\partial F_c}{\partial t_0} = \sum_{n \geq 0} t_{n+1} \frac{\partial F_c}{\partial t_n} + \frac{t_0^2}{2 \lambda^2}. \tag{6}
\]

The free energy has the genus expansion

\[
F_c = \sum_{g=0}^{\infty} \lambda^{2g−2} F_{(g)}, \tag{7}
\]

where \( F_{(0)} = F_{\text{sphere}} \) and \( \lambda \) is a formal expansion parameter. The sphere KdV and string equations can be deduced by considering the dominant part of the general equations. (The equation \( P(u) = 0 \) is obtained by a combination of (3) and (6) for \( g = 0 \).)

Witten’s conjecture for intersection theory was proved by Kontsevich [12] using the one-matrix model. The generating function for minimal gravity on \( g = 0, 1, 2 \) has also been constructed using the KdV hierarchy [13, 14], and the resulting correlations (but off-shell, i.e., with arbitrary \( t_k \) parameters) have been shown to obey the recursion relations of topological gravity, as suggested by Witten. (In the string equation, \( t_1 \) is to be shifted by 1 for the comparison of the two cases.)

### 3 Minimal gravity on a disk and KdV hierarchy

A similar KdV hierarchy (“open KdV hierarchy”) has been proposed for intersection theory on the moduli space of Riemann surfaces with boundary, using an additional flow parameter \( s \). The flow along \( t_n \) is given as

\[
\frac{2n + 1}{2} \frac{\partial F^o}{\partial t_n} = \lambda \frac{\partial F^o}{\partial s} \frac{\partial F^o}{\partial t_{n−1}} + \lambda \frac{\partial^2 F^o}{\partial s^2 \partial t_{n−1}} + \frac{\lambda^2}{2} \frac{\partial F^o}{\partial t_0} \frac{\partial^2 F^o}{\partial t_0 \partial t_{n−1}} − \frac{\lambda^2}{4} \frac{\partial^3 F^o}{\partial t_0^3}, \quad n \geq 1. \tag{8}
\]
The open string equation is given by
\[
\frac{\partial F^o}{\partial t_0} = \sum_{n \geq 0} t_{n+1} \frac{\partial F^o}{\partial t_n} + s \frac{\partial F^o}{\partial t_0} + \lambda.
\] (8)

The open KdV together with the string equation is shown to be equivalent to the Virasoro constraints [10].

In this section, we explicitly check that the open KdV equations are satisfied by the free energy on a disk. The free energy with a (1, 1) boundary is given by [15, 16]
\[
F_{\text{disk}} = \sqrt{\pi} \int_0^\infty \frac{dl}{l^{3/2}} e^{-l\mu_B} \int_0^\infty dx e^{-lu},
\] (9)
where \(\mu_B\) is the boundary cosmological constant. In (9), \(u\) satisfies the string equation and KdV on a sphere and is, therefore, \(u = u(x, t_{n>0})\) a function of \(x\) and \(t_n\)'s \((n > 0)\) but independent of \(t_0\).

The genus expansion of the free energy with a boundary
\[
F^o = \sum_{g=0}^\infty \lambda^{g-1} F^o_{(g)},
\] (10)
shows that \(F^o_{(0)}\) satisfies the open KdV at this order:
\[
\frac{2n+1}{2} \frac{\partial F^o_{(0)}}{\partial t_n} = \frac{\partial F^o_{(0)}}{\partial s} \frac{\partial F^o_{(0)}}{\partial t_{n-1}} + \frac{1}{2} \frac{\partial F^o_{(0)}}{\partial t_0} \frac{\partial^2 F^o_{(0)}}{\partial t_0 \partial t_{n-1}}.
\] (11)

The one-point correlation on a disk is non-trivial. The correlation is found from (9) by differentiating with respect to \(t_n\), using the KdV on sphere (but staying off-shell):
\[
\langle O_n \rangle_{\text{disk}} = \frac{\partial F_{\text{disk}}}{\partial t_n} = -\sqrt{\pi} \int_0^\infty \frac{dl}{l} e^{-l\mu_B} \int_0^\infty du \frac{e^{-lu}}{\sqrt{l}} u^n,
\] (12)
where \(u_0 = u(x = t_0, t_{n>0})\) and is the same as the one in [11]. Since \(u\) has a gravitational dimension, one may set \(u = u_0 \xi\) so that \(u_0\) is dimensionful while \(\xi\) is dimensionless. The integration over \(u\) results in the incomplete gamma function. However, it is more convenient to rewrite the monomial \(\xi^n\) as a linear combination of Legendre polynomials \(P_k\):
\[
\xi^n = \sum_{k=n,n-2,\ldots \geq 0} (2k+1) n! a_{n,k} P_k(\xi)
\] (13)
where
\[
a_{n,k} = \frac{1}{2^{(n-k)/2}((n-k)/2)! (n+k+1)!!}.
\] (14)

Then the integration over \(\xi\) is given as the modified Bessel function \(K_n\) of the second kind:
\[
\int_1^\infty d\xi e^{-u_0 \xi} P_k(\xi) = \sqrt{\frac{2}{\pi u_0 l}} K_{1/2+k}(u_0 l).
\] (15)

Further, its integration over \(l\) is performed (after analytic continuation if necessary) to give
\[
\int_0^\infty \frac{dl}{l} e^{-l\mu_B} K_{1/2+k}(u_0 l) = \frac{2\pi (-1)^{k+1}}{2k+1} \cosh((1/2 + k)\tau),
\] (16)
where we put $\mu_B/u_0 = \cosh(\tau)$. Therefore, the correlation number is given as follows in terms of the Chebyshev polynomial $T_n(\cosh(x)) = \cosh(nx)$:

$$\langle O_n \rangle_{\text{disk}} = 2\pi \sqrt{2} u_0^{n+1/2} (-1)^{n+1} \sum_{k=n,n-2,\ldots \geq 0} a_{n,k} T_{2k+1}(\cosh(\tau/2)).$$  \hfill (17)

If one uses the identity of Chebyshev polynomials $T_{n+2} = 2T_2 T_n - T_{|n-2|}$ and the two-point correlation on sphere $\langle O_0 O_{n-1} \rangle_{\text{sphere}} = u_0^n/n!$, one has the following recursive relation

$$\frac{2n+1}{2} \langle O_n \rangle_{\text{disk}} = -u_0 T_2(\cosh(\tau/2)) \langle O_{n-1} \rangle_{\text{disk}} + \frac{1}{2} \langle O_0 \rangle_{\text{disk}} \langle O_0 O_{n-1} \rangle_{\text{sphere}}.$$  \hfill (18)

Comparing the recursion in (18) with the open KdV on disk in (11), one concludes that the free energy on a disk follows the open KdV and the flow along $s$ reads:

$$\frac{\partial F_{(0)}^o}{\partial s} = -u_0 T_2(\cosh(\tau/2)) = u_0 \cosh(\tau) = -\mu_B.$$  \hfill (19)

Considering (11), (18) and (19) we arrive at the conclusion that $F_{(0)}^o$ and $F_{\text{disk}}$ are related by the Legendre transformation

$$F_{(0)}^o(s) = F_{\text{disk}}(\mu_B) - \mu_B s.$$  \hfill (20)

We note that the above calculations are done off-shell and therefore, (19) holds off-shell also. Using this result, one can prove that the open KdV on the disk holds for $F_{\text{disk}}$ (and multi-correlations) using just the integral representation of $F_{\text{disk}}$ in (9) and the fact $-\mu_B e^{-l\mu_B} = \partial e^{-l\mu_B}/\partial l$. It is worth pointing out that the result of [9, 10] for $g = 0$

$$\frac{\partial F_{(0)}^c}{\partial s} = \frac{1}{2} \left( \frac{\partial F_{(0)}^c}{\partial t_0} \right)^2 + \frac{\partial^2 F_{(0)}^c}{\partial t_0^2}$$  \hfill (21)

still holds\footnote{The boundary parameter $\mu_B$ is independent of KdV parameters but $\tau$ depends on KdV parameters due to $u_0$.} essentially because of the Chebyshev polynomial identity $T_2 = 2T_1^2 - 1$.

### 4 Conclusion

We demonstrate that the open KdV hierarchy holds for the matrix models of the Lee-Yang series of the minimal gravity on disk. In order to prove this, the fact that the $s$ flow of the generating function on a disk is governed by the boundary cosmological constant $\mu_B$ is essential. This result can be compared with [13, 18], where intersection theory with boundary is investigated using a Penner-type matrix model, but has no boundary parameter. One could investigate more general boundary conditions and boundary correlations on the disk, and the corresponding KdV hierarchies. It might be interesting to investigate if the $s_n$ flows of [10, 17, 19] have an interpretation in the Lee-Yang series, which would be related to the other boundary condition than the simple (1,1) boundary condition considered in the text.

The minimal gravity $M(2, 2p + 1)$ is described by one variable $u$ which is the coordinate of $A_1$ Frobenius manifold [20]. It is known that its dual $A_{2p}$ Frobenius manifold (which has $2p$ number of coordinates $\{u^a\}$) can describe the same correlation on sphere. However, the correlation on disk is not fully understood in

\footnote{To make it match exactly, we may rescale $F_{\text{disk}} \rightarrow c F_{\text{disk}}$ with $c^2 = -1/(2\pi^2)$.}
terms of the dual manifold due to the non-canonical nature of the integral representation of the free energy [21, 22]. The open KdV hierarchy can be a guide to define the free energy on disk whose details will be considered in a separate paper.

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