ON THE DONALDSON-UHLENBECK COMPACTIFICATION OF
INSTANTON MODULI SPACES ON CLASS VII SURFACES

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ABSTRACT. We study the following question: Let \((X, g)\) be a compact Gauduchon surface, \(E\) be a differentiable rank \(r\) vector bundle on \(X\) and \(D\) be a fixed holomorphic structure on \(D := \det(E)\). Does the complex space structure on \(\mathcal{M}^{\text{ASD}}(E)\) induced by the Kobayashi-Hitchin correspondence extend to a complex space structure on the Donaldson compactification \(\overline{\mathcal{M}}^{\text{ASD}}(E)\)? Our results answer this question in detail for the moduli spaces of SU(2)-instantons with \(c_2 = 1\) on general (possibly unknown) class VII surfaces.

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1. Introduction

1.1. The moduli problem for vector bundles on compact complex manifolds. An old, classical problem in complex geometry concerns the classification of the holomorphic structures on a fixed differentiable vector bundle on a complex manifold, up to isomorphism. The moduli problem for holomorphic vector bundles is devoted to studying the corresponding set of isomorphism classes and the geometric structures (topologies, complex space structures, Hermitian metrics, etc) one can naturally put on this set.

More precisely, let $X$ be a compact complex manifold of dimension $n$ and $E$ be a differentiable vector bundle of rank $r$ on $X$. A semi-connection on $E$ is a first order operator $\delta : A^0(E) \to A^{0,1}(E)$ satisfying the $\bar\partial$-Leibniz rule: $\delta(fs) = \bar\partial fs + f \delta(s)$ for any $f \in C^{\infty}(X, \mathbb{C})$ and $s \in A^0(E)$. The natural de Rham-type extension $A^{0,p}(E) \to A^{0,p+1}(E)$ will be denoted by the same symbol. A semi-connection $\delta$ is called integrable if $F_\delta = 0$, where $F_\delta \in A^{0,2}(\text{End}(E))$ is the endomorphism-valued $(0,2)$-form defined by the composition $\delta \circ \delta : A^0(E) \to A^{0,2}(E)$.

Denote by $\mathcal{Hol}(E)$ the set of holomorphic structures on $E$, and by $A^{0,1}(E)$ ($A^{0,1}(E)^{\text{int}}$) the space of (integrable) semi-connections on $E$. The Newlander-Nirenberg theorem gives a bijection $A^{0,1}(E)^{\text{int}} \to \mathcal{Hol}(E)$. We will denote by $\mathcal{E}_\delta$ the holomorphic structure on $E$ associated with an integrable semi-connection $\delta$. For an open set $U \subset X$ the space $\mathcal{E}_\delta(U)$ of holomorphic sections of $\mathcal{E}_\delta|_U$ coincides with the kernel of the first order operator $\delta_U : A^0(U, E) \to A^{0,1}(U, E)$.

Using this bijection and the natural $C^{\infty}$-topology on $A^{0,1}(E)^{\text{int}}$ we obtain a natural topology on the space $\mathcal{Hol}(E)$ of holomorphic structures on $E$. The moduli space of holomorphic structures on $E$ is the topological space obtained as the quotient

$$\mathcal{Hol}(E)/\text{Aut}(E) = A^{0,1}(E)^{\text{int}}/\text{Aut}(E)$$

of $\mathcal{Hol}(E)$ by the group $\text{Aut}(E) := \Gamma(X, \text{GL}(E))$ of differentiable automorphisms of $E$, which acts naturally on this space.

Since $\text{GL}(r)$ (the structure group of a vector bundle) is not a simple group, one considers a natural refinement of our moduli problem, which has a simple “symmetry group”. Let $\mathcal{D}$ be a holomorphic structure on the determinant line bundle $D := \text{det}(E)$, and $\lambda \in A^{0,1}(D)^{\text{int}}$ be the corresponding integrable semi-connection. A $\mathcal{D}$-oriented holomorphic structure on $E$ is a holomorphic structure $\mathcal{E}$ on $E$ with $\text{det}(\mathcal{E}) = \mathcal{D}$.

We recall that an $\text{SL}(r, \mathbb{C})$-vector bundle on $X$ is a rank $r$ vector bundle $E$ endowed with a trivialization of its determinant line bundle. Therefore, in this case, $D$ comes with a tautological trivial holomorphic structure $\Theta$: an $\text{SL}(r, \mathbb{C})$-holomorphic structure on $E$ is just a $\Theta$-oriented holomorphic structure in $E$ in our sense. This shows that classifying $\mathcal{D}$-oriented holomorphic structures on a fixed differentiable vector bundle gives the natural generalization of the classification problem for $\text{SL}(r, \mathbb{C})$-holomorphic structures on an $\text{SL}(r, \mathbb{C})$-vector bundle.

The set $\mathcal{Hol}_E(D)$ of $\mathcal{D}$-oriented holomorphic structures on $E$ can be identified with the subspace $A^{0,1}_{\lambda}(E)^{\text{int}} \subset A^{0,1}(E)^{\text{int}}$ defined by the condition $\text{det}(\lambda) = \lambda$. The gauge group $\mathcal{G}_E^\Theta := \Gamma(X, \text{SL}(E))$ acts naturally on this subspace. We will focus on the moduli space of $\mathcal{D}$-oriented holomorphic structures on $E$, which, by definition,
is the quotient space

\[ M_D(E) := \mathcal{H} ol_D(E)/G^C_E = A^{0,1}_\lambda(E)^{\text{int}}/G^C_E. \]

In general this quotient is highly non-Hausdorff and cannot be endowed with a natural complex space structure. It can be identified with the topological space associated with a holomorphic stack, but up till now it is not clear if, in our non-algebraic complex geometric framework, this approach leads to effective new results. On the other hand the classical point of view (studying moduli spaces whose points correspond to equivalence classes of holomorphic structures) has been used with effective results, for instance in making progress towards the classification of class VII surfaces [Te2, Te4].

We recall that a holomorphic structure \( E \) on \( E \) is called simple if \( H^0(\mathcal{E}nd(E)) = \mathcal{C}ld_E \). In contrast with \( M_D(E) \), the moduli space \( M_{D}^{\text{si}}(E) \) of simple \( D \)-oriented holomorphic structures has a natural, in general non-Hausdorff, complex space structure. This structure can be obtained in two different ways, which, a posteriori turn out to be equivalent. The first approach [FK] uses classical deformation theory. The second [LO] uses complex gauge theory. The equivalence of the two points of view has been established by Miyajima [Miy].

As in algebraic geometry, in order to define Hausdorff moduli spaces of holomorphic structures, one needs a stability condition, which depends on the choice of an Hermitian metric \( h \).

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As in algebraic geometry, in order to define Hausdorff moduli spaces of holomorphic structures, one needs a stability condition, which depends on the choice of an additional structure on \( X \). Whereas in algebraic geometry this additional structure is a polarisation on \( X \) (i.e. the choice of an ample line bundle on \( X \)), in our general complex geometric framework we need a Gauduchon metric on \( X \), i.e. a Hermitian metric \( g \) on \( X \) whose associated \( \omega \) satisfies the Gauduchon condition \( dd^c(\omega^{n-1}) = 0 \) [Gau]. Such a metric defines a Lie group morphism

\[ \deg_g : \text{Pic}(X) \to \mathbb{R}, \quad \deg_g(\mathcal{L}) = \int_X c_1(\mathcal{L}, h) \wedge \omega^{n-1}, \]

where \( c_1(\mathcal{L}, h) \) denotes the Chern form of the Chern connection associated with a Hermitian metric \( h \) on \( \mathcal{L} \). Using this degree map one can introduce a slope stability condition in the same way as in the algebraic framework:

For a coherent sheaf \( \mathcal{F} \) on \( X \) we put \( \deg_g(\mathcal{F}) := \deg_g(\det(\mathcal{F})) \). A non-zero torsion-free sheaf \( \mathcal{F} \) on \( X \) is called (semi-)stable (with respect to \( g \)) if for every subsheaf \( \mathcal{H} \subset \mathcal{F} \) with \( 0 < \text{rk}(\mathcal{H}) < \text{rk}(\mathcal{F}) \) one has

\[ \frac{\deg_g(\mathcal{H})}{\text{rk}(\mathcal{H})} < \frac{\deg_g(\mathcal{F})}{\text{rk}(\mathcal{F})} \quad \text{respectively} \quad \frac{\deg_g(\mathcal{H})}{\text{rk}(\mathcal{H})} \leq \frac{\deg_g(\mathcal{F})}{\text{rk}(\mathcal{F})}. \]

\( \mathcal{F} \) is called polystable (with respect to \( g \)) if it splits as a direct sum \( \mathcal{F} = \oplus_{i=1}^k \mathcal{F}_i \) of non-zero stable subsheaves \( \mathcal{F}_i \) such that \( \frac{\deg_g(\mathcal{F}_i)}{\text{rk}(\mathcal{F}_i)} = \frac{\deg_g(\mathcal{F})}{\text{rk}(\mathcal{F})} \) for \( 1 \leq i \leq k \).

We denote by \( M_D^{\text{si}}(E) \), \( M_D^{\text{st}}(E) \), \( M_D^{\text{pol}}(E) \) the moduli space of simple, stable, respectively polystable \( D \)-oriented holomorphic structures on \( E \). Any stable bundle is simple and polystable, so \( M_D^{\text{si}}(E) \subset M_D^{\text{st}}(E) \subset M_D^{\text{pol}}(E) \). Stability is an open condition with respect to the classical topology [LT], hence \( M_D^{\text{si}}(E) \) is open in both \( M_D^{\text{st}}(E) \), and \( M_D^{\text{pol}}(E) \). In particular \( M_D^{\text{si}}(E) \) comes with a natural complex space structure induced from \( M_D^{\text{si}}(E) \).

Note that in general, in the non-Kählerian framework, semi-stability is not an open condition (even with respect to the classical topology).
By choosing a Hermitian metric $h$ on $E$, $\mathcal{M}_{D}^{\text{rest}}(E)$ can be identified with the moduli space of projectively Hermitian-Einstein connections on $E$, defined as follows. Denote by $\mathcal{A}(E)$ the space of Hermitian connections on $(E, h)$. For a fixed Hermitian connection $a$ on the Hermitian line bundle $(D, \det(h))$ we put

$$\mathcal{A}_a(E) := \{ A \in \mathcal{A}(E) | \det(A) = a \}.$$ 

An element of $\mathcal{A}_a(E)$ will be called an $a$-oriented connection on $(E, h)$. We define the space and the moduli space of projectively Hermitian-Einstein $a$-oriented connections on $(E, h)$ by

$$\mathcal{A}^{\text{HE}}_a(E) := \{ A \in \mathcal{A}_a(E) | \Lambda_q F_A^0 = 0, \ (F_A^0)^{0.2} = 0 \} \ , \ \mathcal{M}^{\text{HE}}_a(E) := \mathcal{A}^{\text{HE}}_a(E)/\mathcal{G}_E,$n

where $F_A^0$ stands for the trace-free part of the curvature $F_A$ of $A$, and $\mathcal{G}_E := \Gamma(X, \text{SU}(E))$ is the SU(r)-gauge group of $(E, h)$. If $n = 2$, the conditions $\Lambda_q F_A^0 = 0, \ (F_A^0)^{0.2} = 0$ are equivalent to the anti-selfduality condition $(F_A^0)^+ = 0$, so $\mathcal{M}^{\text{HE}}_a(E)$ is just the moduli space $\mathcal{M}_a^{\text{ASD}}(E)$ of projectively ASD, $a$-oriented connections on $(E, h)$.

The Kobayashi-Hitchin correspondence [Bu1], [LT], [LY] generalising Donaldson’s fundamental work [D1] states that

**Theorem (The Kobayashi-Hitchin correspondence).** Let $D$ be a fixed holomorphic structure on $D = \det(E)$ and let $a$ be the Chern connection of the pair $(D, \det(h))$. The map $A \mapsto \mathcal{E}_A^h$ induces a homeomorphism

$$\mathcal{M}^{\text{HE}}_a(E) \xrightarrow{\simeq} \mathcal{M}^{\text{rest}}_D(E),$$

which restricts to a homeomorphism $\mathcal{M}^{\text{HE}}_a(E)^* \xrightarrow{\simeq} \mathcal{M}^{\text{rest}}_D(E)^*$ between the moduli space of irreducible projectively Hermitian-Einstein, $a$-oriented connections on $E$, and the moduli space of stable $D$-oriented holomorphic structures on $E$.

**Remark 1.1.** In the special case when $E$ is an SL(r, C)-bundle we can choose $h$ such that the distinguished trivialisation of $\det(E)$ is unitary. With this choice $(E, h)$ becomes an SU(r)-vector bundle, $a$ coincides with the trivial connection associated with this trivialisation, $\mathcal{A}_a(E)$ is the space of SU(r)-connections on $(E, h)$, and $\mathcal{M}^{\text{HE}}_a(E)$ is the moduli space of Hermitian-Einstein SU(r)-connections on $(E, h)$. In the case of an SL(r, C)-bundle $E$ (or of an SU(r)-bundle $(E, h)$) we will omit the subscripts $D, a$ in our notation, hence we will write $\mathcal{M}^{\text{rest}}(E), \mathcal{M}^{\text{HE}}(E)$ for the moduli spaces of polystable (stable) SL(r, C)-structures on $E$, and $\mathcal{M}^{\text{HE}}(E), \mathcal{M}^{\text{HE}}(E)^*$ for the moduli spaces of irreducible Hermitian-Einstein SU(r)-connections on $(E, h)$.

The Kobayashi-Hitchin correspondence has important consequences: using standard gauge-theoretical techniques one can prove that the quotient topology of $\mathcal{M}^{\text{HE}}_a(E)$ is Hausdorff. The point is that, for this moduli space, the gauge group $\mathcal{G}_E$ is the group of sections of a locally trivial bundle with compact standard fibre. Therefore $\mathcal{M}^{\text{rest}}_D(E), \mathcal{M}^{\text{HE}}_D(E)$ are Hausdorff spaces, and in particular $\mathcal{M}^{\text{HE}}_D(E)$ is a Hausdorff complex subspace of the (possibly non-Hausdorff) moduli space $\mathcal{M}^{\text{HE}}_D(E)$.

1.2. _Extending the complex structure to a compactification of $\mathcal{M}^{\text{HE}}_D(E)$. The fundamental questions._ The first natural question related to the Kobayashi-Hitchin correspondence is: _Does $\mathcal{M}^{\text{rest}}_D(E)$ have a natural complex space structure extending the canonical complex space structure of $\mathcal{M}^{\text{HE}}_D(E)$?_ The explicit examples described in [Te2], [Te3] and the general results proved in [Te4] show that in general the answer is negative. For instance, for a class VII surface $X$ with $b_2(X) = 1$, a
bundle $E$ with $c_2(E) = 0$, $c_1(E) = c_1(K_X)$ and a suitable Gauduchon metric $g$ on $X$, the moduli space $\mathcal{M}^\text{p}st_D(E)$ can be identified with a compact disk, whose interior corresponds to $\mathcal{M}^\text{p}st_D(E)$. Moreover, on a class VII surface $X$ with $b_2(X) = 2$ one obtains in a similar way a moduli space $\mathcal{M}^\text{p}st_D(E)$ which can be identified with $S^4$, and in this case $\mathcal{M}^\text{p}st_D(E)$ corresponds to the complement of the union of two circles in $S^4$. These examples show that, in the general (possibly non-Kählerian) framework the complex space structure on $\mathcal{M}^\text{p}st_D(E)$ does not extend to a complex space structure on $\mathcal{M}^\text{p}st_D(E)$. On the other hand, in general $\mathcal{M}^\text{p}st_D(E)$ is not compact, and when this is the case, even if the complex structure of $\mathcal{M}^\text{p}st_D(E)$ does extend to $\mathcal{M}^\text{p}st_D(E)$, the result is not satisfactory. This motivates the following:

**Question 1.** Let $(X, g)$ be a compact Gauduchon manifold, $E$ be a differentiable rank $r$ vector bundle on $X$ and $\mathcal{D}$ be a fixed holomorphic structure on $\mathcal{D} := \det(E)$. Does the complex space structure of $\mathcal{M}^\text{p}st_D(E)$ extend to a complex space structure on a natural compactification of it, which contains the space $\mathcal{M}^\text{p}st_D(E)$?

Note that, in the case $n = 2$, one has a good candidate for “a natural compactification of $\mathcal{M}^\text{p}st_D(E)$ which contains $\mathcal{M}^\text{p}st_D(E)$”: we identify $\mathcal{M}^\text{p}st_D(E)$ with $\mathcal{M}^\text{ASD}_a(E)^*$ via the Kobayashi-Hitchin isomorphism, and we embed the latter space in the Donaldson compactification of $\mathcal{M}^\text{ASD}_a(E)$. Therefore, in the case $n = 2$, one can ask a more precise version of Question 1:

**Question 2.** Let $(X, g)$ be a compact Gauduchon surface, $E$ be a differentiable rank $r$ vector bundle on $X$ and $\mathcal{D}$ be a fixed holomorphic structure on $\mathcal{D} := \det(E)$. Does the complex space structure on $\mathcal{M}^\text{ASD}_a(E)^*$ induced by the Kobayashi-Hitchin correspondence extend to a complex space structure on the Donaldson compactification $\mathcal{M}^\text{ASD}_a(E)$?

As the examples above show, in our general (possibly non-Kählerian) framework both questions have negative answer. Indeed, in our examples the space $\mathcal{M}^\text{ASD}_a(E)$ is already compact, and admits no complex space structure at all.

On the other hand we believe that both questions have positive answer when $(X, g)$ is Kähler. In other words

**Conjecture 1.** Let $(X, g)$ be a compact Kähler manifold, $E$ be a differentiable rank $r$ vector bundle on $X$ and $\mathcal{D}$ be a fixed holomorphic structure on $\mathcal{D} := \det(E)$. Then the natural complex space structure of $\mathcal{M}^\text{p}st_D(E)$ (induced from $\mathcal{M}^\text{p}st_D(E)$) extends to a natural compactification of it which contains $\mathcal{M}^\text{p}st_D(E)$. For $n = 2$ the natural complex space structure of $\mathcal{M}^\text{p}st_D(E)$ extends to the Donaldson compactification $\mathcal{M}^\text{ASD}_a(E)$ of $\mathcal{M}^\text{ASD}_a(E)$.

This conjecture is known to hold in the projective algebraic framework (when $\omega_g$ is the Chern form of an ample line bundle $\mathcal{H}$ on $X$) for $\text{SL}(2, \mathbb{C})$-bundles. In this case, for $n = 2$ one proves [Li] that $\mathcal{M}^\text{ASD}_a(E)$ can be identified with the image in a projective space of a regular map defined on a Zariski closed subset of the Gieseker moduli space associated with the data $c_1 = 0$, $c_2 = c_2(E)$, $r = 2$. This Gieseker moduli space is a projective variety. Taking into account Li’s result, and the fact that any compact Kähler surface admits arbitrary small deformations which are projective ([Koi1], [Bu3], [Bu4]), Conjecture 1 becomes very natural. The recent results of [GT] concern the higher dimensional projective case, and give further...
evidence for this conjecture.

In this article we will study in detail Question 2 in an interesting special case: the moduli space of SL(2, ℂ)-structures on an SL(2, ℂ)-bundle \(E\) with \(c_2(E) = 1\) on a class VII surface \(X\). One has

\[
\mathcal{M}^{\text{ASD}}(E) = \mathcal{M}^{\text{ASD}}(E) \cup (\mathcal{M}_0 \times X) = \mathcal{M}^{\text{ASD}}(E)^* \cup R \cup (\mathcal{M}^*_0 \times X) \cup (R_0 \times X),
\]

(1)

where \(R \subset \mathcal{M}^{\text{ASD}}(E)\) is the subspace of reducible SU(2)-instantons with \(c_2 = 1\), and \(\mathcal{M}_0 (\mathcal{M}^*_0, R_0)\) stands for the moduli space of flat, respectively flat irreducible, flat reducible, SU(2)-instantons. The last three terms in the decomposition (1) are compact. Therefore in our case, Question 2 reduces to a set of three more specific questions:

**Does the complex space structure of** \(\mathcal{M}^{\text{ASD}}(E)^* = \mathcal{M}^{\text{st}}(E)\) **extend across the compact strata**

(a) \(R_0 \times X\),

(b) \(\mathcal{M}^*_0 \times X\),

(c) \(R\)?

The factor \(R_0\) of the third summand in (1) can be further decomposed as a disjoint union as follows. Denote by \(C(X)\) the group of characters \(\text{Hom}(H_1(X, \mathbb{Z}), S^1)\). One has a natural identification

\[
\mathfrak{C}(X) := C(X)/\langle j \rangle \rightarrow R_0,
\]

where \(j\) is the involution induced by the conjugation \(S^1 \to S^1\). The group \(C(X)\) fits into the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & C^0(X) & \cong & S^1 & \longrightarrow & C(X) & \longrightarrow & \text{Tors}(H^2(X, \mathbb{Z})) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Pic}^0(X) & \cong & \mathbb{C}^* & \longrightarrow & \text{Pic}^T(X) & \overset{\varphi_1}{\longrightarrow} & \text{Tors}(H^2(X, \mathbb{Z})) & \longrightarrow & 0,
\end{array}
\]

where

\[
C^0(X) := \text{Hom}(H_1(X, \mathbb{Z})/\text{Tors}, S^1) \cong S^1,
\]

\[
\text{Pic}^T(X) := \{[\mathcal{L}] \in \text{Pic}(X) | c_1(\mathcal{L}) \in \text{Tors}(H^2(X, \mathbb{Z}))\}.
\]

The central vertical monomorphism maps a character \(\chi \in C(X)\) to the isomorphism class of the associated holomorphic Hermitian line bundle \(\mathcal{L}_\chi\), and identifies \(C(X)\) with \(\ker(\text{deg}_g|_{\text{Pic}^T(X)})\), which is independent of the Gauduchon metric \(g\). More precisely the congruence class \(\mathcal{C}^c(X) \in C(X)/C^0(X)\) associated with a torsion class \(c \in \text{Tors}(H^2(X, \mathbb{Z}))\) is identified with the vanishing circle of \(\text{deg}_g\) on the congruence class \(\text{Pic}^c(X) \in \text{Pic}(X)/\text{Pic}^0(X)\).

The involution \(j : C(X) \to C(X)\) maps \(C^c(X)\) diffeomorphically onto \(C^{-c}(X)\), in particular leaves invariant any circle \(C_c(X)\) associated with a class \(c\) belonging to the \(\mathbb{Z}_2\)-vector space

\[
\text{Tors}_2(H^2(X, \mathbb{Z})) := \ker (H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})).
\]

Let \(\mu_2\) be the multiplicative group \(\{\pm 1\}\). For \(c \in \text{Tors}_2(H^2(X, \mathbb{Z}))\) the set \(\rho^c(X)\) of fixed points of the induced involution \(C^c(X) \overset{j}{\longrightarrow} C^{-c}(X)\) is a \(\mu_2\)-torsor (in particular has two points), and the quotient

\[
\mathfrak{C}^c(X) := C^c(X)/\langle \rho_c \rangle
\]
Let $M$ be the quotient of $\text{Tors}(H^2(X,\mathbb{Z}))$ by the involution $c \mapsto -c$, and denote by $\mathfrak{T}_0(X)$, $\mathfrak{T}_1(X)$ the subsets of $\mathfrak{T}(X)$ which correspond respectively to $\text{Tors}_2(H^2(X,\mathbb{Z}))$ and $\text{Tors}(H^2(X,\mathbb{Z}))\backslash\text{Tors}_2(H^2(X,\mathbb{Z}))$.

For a class $c = [c] \in \mathfrak{T}(X)$ denote by $\mathfrak{C}(X)$ the quotient of $\mathfrak{C}(X) \cup \mathfrak{C}^{-c}(X)$ by the involution induced by $j$ on this union. Note that for $c = [c] \in \mathfrak{T}_0(X)$ one has $\mathfrak{C}(X) = \mathfrak{C}(X)$ whereas, for $c \in \mathfrak{T}_1(X)$, the choice of a representative $c \in c$ gives an identification $\mathfrak{C}(X) \cong \mathfrak{C}(X)$. Therefore the space $\mathcal{R}_0$ of reducible flat instantons on $X$ decomposes as a finite disjoint union of segments and circles:

$$\mathcal{R}_0 \cong \mathfrak{C}(X) = \bigcup_{c \in \mathfrak{T}_0(X)} \mathfrak{C}(X) = \left( \bigcup_{c \in \mathfrak{T}_0(X)} \mathfrak{C}(X) \right) \bigcup \left( \bigcup_{c \in \mathfrak{T}_1(X)} \mathfrak{C}(X) \right).$$

The set of all vertices appearing in the first term of (2) can be identified with the set of fixed points of $j$, which coincides with

$$\rho(x) := H^1(X,\mu_2) = \text{Hom}(H_1(X,\mathbb{Z}),\mu_2) = \bigcup_{c \in \text{Tors}_2(H^2(X,\mathbb{Z}))} \rho_c(X),$$

and has an important gauge theoretical interpretation: it corresponds to the moduli space of reducible flat SU(2)-instantons with non-abelian stabiliser $S^1$. Therefore, from a gauge theoretical point of view, the first term in the decomposition (2) (which is always non-empty) is substantially more complex than the second, because a segment $\mathfrak{C}(X)$ contains both abelian and non-abelian reductions. Our main result states

**Theorem 1.2.** Let $X$ be a class VII surface, and let $E$ be an $SL(2,\mathbb{C})$-bundle on $X$ with $c_2(E) = 1$. The complex space structure of $\mathcal{M}^{\text{ASD}}(E)^* = \mathcal{M}^{\text{st}}(E)$ extends across $\mathcal{R}_0 \times X$, and $\overline{\mathcal{M}}^{\text{ASD}}(E)$ is a smooth complex 4-fold at any reducible virtual point $([A], x) \in \mathcal{R}_0 \times X$.

This result is surprising for two reasons: first, $\mathcal{R}_0$ is a union of segments and circles, which are not complex geometric objects; second, since the vertices of the segments $\mathfrak{C}(X)$ ($c \in \mathfrak{T}_0(X)$) are isolated non-abelian reductions, one expects essential singularities at these points.

Our second result concerns the extensibility of the complex space structure across $\mathcal{M}_0^* \times X$:

**Theorem 1.3.** Let $X$ be a class VII surface, and let $E$ be an $SL(2,\mathbb{C})$-bundle on $X$ with $c_2(E) = 1$. Then $\mathcal{M}_0^*$ consists of finitely many simple points, and the complex space structure of $\mathcal{M}^{\text{ASD}}(E)^* = \mathcal{M}^{\text{st}}(E)$ extends across $\mathcal{M}_0^* \times X$. For every $[A] \in \mathcal{M}_0^*$ the surface $([A]) \times X$ has an open neighbourhood in $\overline{\mathcal{M}}^{\text{ASD}}(E)$ which is a normal complex space whose singular locus is $([A]) \times X$, and the normal cone of this singular locus can be identified with the cone bundle of degenerate elements in $\mathcal{K}_X \otimes S^2(E)$, where $E$ is the holomorphic bundle associated with $A$.

Here we denoted by $S^2(E)$ the second symmetric power of $E$; an element $\eta \otimes \sigma \in \mathcal{K}_X(x)^{\gamma} \otimes S^2(E(x))$ is degenerate if the associated linear map $E(x) \to E(x)^{\gamma} \otimes \mathcal{K}_X(x)^{\gamma}$
has non-trivial kernel. Since \(\text{rk}(E) = 2\), this is equivalent to the condition that \(\sigma\) belongs to the image of the squaring map \(E(x) \to S^2(E(x))\).

Finally, the extensibility of the complex space structure across the subspace \(R \subset \mathcal{M}^{\text{ASD}}(E)\) has been studied in detail in a more general framework in \([Te5]\). In our special case the result is the following

**Theorem 1.4.** \([Te5]\) Let \(X\) be a class VII surface endowed with a Gauduchon metric \(g\) with \(\text{deg}(K_X) < 0\), and let \(E\) be an \(\text{SL}(2, \mathbb{C})\)-bundle on \(X\) with \(c_2(E) = 1\). Then \(\mathcal{M}^{\text{ASD}}(E)^* = \mathcal{M}^{\text{st}}(E)\) is a smooth 4-fold, and the reduction locus \(R \subset \mathcal{M}^{\text{ASD}}(E)\) is a union of \(b_2(X)|\text{Tors}(H^2(X, \mathbb{Z}))|\) circles. Any such circle has a neighbourhood which can be identified with a neighbourhood of the singular circle in a flip passage; in particular the holomorphic structure of \(\mathcal{M}^{\text{ASD}}(E)^*\) does not extend across any of these circles.

In other words, for class VII surfaces with \(b_2 > 0\), the holomorphic structure does not extend across the circles of reductions in the moduli space, but (supposing \(\text{deg}(K_X) < 0\)) the structure of the moduli space around such a circle is perfectly understood. Note that the condition \(\text{deg}(K_X) < 0\) is not restrictive. Indeed, using the classification of class VII surfaces with \(b_2 = 0\) \([Te1]\) and the results of \([Bu2]\) (see also \([Te4]\) Lemma 2.3), it follows that:

**Remark 1.5.** Any class VII surface whose minimal model is not an Inoue surface admits Gauduchon metrics \(g\) such that \(\text{deg}(K_X) < 0\).

A surprising corollary of our results is:

**Corollary 1.6.** Let \((X, g)\) be a primary Hopf surface endowed with a Gauduchon metric, and \(E\) be an \(\text{SL}(2, \mathbb{C})\)-bundle on \(X\) with \(c_2(E) = 1\). Then the natural complex structure on \(\mathcal{M}^{\text{st}}(E)\) is smooth and extends to a complex structure on \(\mathcal{M}^{\text{ASD}}(E)\), which becomes a 4-dimensional compact complex manifold.

This study of moduli spaces of \(\text{SU}(2)\)-instantons on class VII surfaces has several motivations. First, in recent articles the second author showed that \(\text{PU}(2)\)-instanton moduli spaces can be used to make progress on the classification of class VII surfaces, more precisely to prove the existence of curves on such surfaces \([Te2]\), \([Te4]\). A natural question is: can one obtain similar (or even stronger) results using moduli spaces of \(\text{SU}(2)\)-instantons? In order to follow this strategy one needs a thorough understanding of compactified such moduli spaces.

A second motivation is related to Corollary \([1.6]\) according to this result, the assignment \((X, g) \mapsto \mathcal{M}^{\text{ASD}}(E)\) defines a functor from the category of Gauduchon primary Hopf surfaces to the class of 4-dimensional smooth compact complex manifolds. Moreover, it is known that \(\mathcal{M}^{\text{ASD}}(E)^* = \mathcal{M}^{\text{st}}(E)\) is endowed with a canonical Hermitian metric \(g\) which is strongly \(\text{KT}\), i.e., satisfies \(\ddbar\omega_g = 0\) \([LT]\). The class of compact strongly \(\text{KT}\) Hermitian manifolds has been intensively studied in recent years. This class of manifolds intervenes in modern physical theories (II string theory, 2-dimensional supersymmetric \(\sigma\)-models) and also in Hitchin's theory of generalised Kähler geometry. Therefore, it is natural to ask

**Question 3.** In the conditions of Corollary \([1.6]\) does the canonical strongly \(\text{KT}\) metric on \(\mathcal{M}^{\text{ASD}}(E)^* = \mathcal{M}^{\text{st}}(E)\) extend to a smooth Hermitian metric on the complex 4-fold \(\mathcal{M}^{\text{ASD}}(E)\)?
If this question has a positive answer, the resulting metric on $\overline{M}^{\text{ASD}}(E)$ will be strongly KT, giving an interesting functor from the category of Gauduchon primary Hopf surfaces to the category of compact strongly KT 4-dimensional manifolds. This functor would yield a large class of examples of 4-dimensional strongly KT compact manifolds. We will come back to Question 3 in a future article.

A third motivation: the novelty of the methods used in the proofs, which emphasize surprising difficulties which occur in the non-Kählerian framework. We shall explain the ideas of proofs and these difficulties in the next subsections, and in the course of the subsequent proofs themselves, it will be apparent that our methods will be applicable in many other situations.

1.3. The idea of proof of Theorem 1.2. In the first part of the proof we study $\overline{M}^{\text{ASD}}(E)$ from the topological point of view. We show that any reducible flat $\text{SU}_2$-instanton $A$ on a class VII surface is regular, i.e. one has $H^2(A) = 0$. Using the local model theorem for virtual instantons [DK, Theorem 8.2.4], we will prove that $\overline{M}^{\text{ASD}}(E)$ is a topological 8-manifold around any virtual point $[(A), x] \in R_0 \times X$. Denoting by $\mathcal{M}^{\text{ASD}}(E)_{\text{reg}} \subset \overline{M}^{\text{ASD}}(E)$ the open subspace of regular irreducible instantons, we see that the subspace

$$\mathfrak{M} := \mathcal{M}^{\text{ASD}}(E)_{\text{reg}} \cup (R_0 \times X) \subset \overline{M}^{\text{ASD}}(E)$$

is a topological 8-manifold.

In a second step we will construct a complex manifold structure on the topological manifold $\mathfrak{M}$ using a gluing construction based on the following simple result proved in the appendix:

**Lemma 6.16.** Let $X$ be a topological $2n$-dimensional manifold, and $Y \subset X$ be an open subset endowed with a complex manifold structure. Let $U$ be an $n$-dimensional complex manifold, and $f : U \to X$ be a continuous, injective map with the properties:

- $X \setminus Y \subset \text{im}(f)$,
- The restriction $f|_{f^{-1}(Y)} : f^{-1}(Y) \to Y$ is holomorphic with respect to the holomorphic structure induced by the open embedding $f^{-1}(Y) \subset U$.

Then

1. $\text{im}(f)$ is an open neighbourhood of $X \setminus Y$ in $X$,
2. $f$ induces a homeomorphism $U \to \text{im}(f)$,
3. $f$ induces a biholomorphism $f^{-1}(Y) \to \text{im}(f) \cap Y$ with respect to the holomorphic structures induced by the open embeddings $f^{-1}(Y) \subset U$, $\text{im}(f) \cap Y \subset Y$,
4. There exists a unique complex manifold structure on $X$ which extends the fixed complex structure on $Y$, and such that $f$ becomes biholomorphic on its image.

The hard part of the proof of Theorem 1.2 is the construction of a pair $(U, f : U \to \mathfrak{M})$ such that, taking $Y := \mathcal{M}^{\text{ASD}}(E)_{\text{reg}}$ with the complex structure induced from $\mathcal{M}^{\text{ASD}}(E)_{\text{reg}}$, the hypothesis of Lemma 6.16 is fulfilled.

Note first that, surprisingly, the Gieseker (semi)stability condition for torsion-free sheaves, can be naturally extended to arbitrary Gauduchon surfaces. The point is that, for surfaces, in these conditions only the degree of the sheaf and its Euler-Poincaré characteristic intervene [Fr, p. 97]. We agree to call the torsion-free sheaves on $X$, which are not locally free, singular sheaves.
In our situation we obtain a natural homeomorphism $\varphi : S \to \mathcal{R}_0 \times X$ between the moduli space $S$ of singular rank 2 Gieseker stable sheaves $\mathcal{F}$ on $X$ with the properties

- $\det(\mathcal{F}) \cong \mathcal{O}_X$, $c_2(\mathcal{F}) = 1$,
- the double dual $\mathcal{F}^{**}$ is properly semi-stable.

Taking into account this identification, a natural choice would be to take for $\mathcal{U}$ an open neighbourhood of $S$ in the moduli space of all (locally free and singular) rank 2 Gieseker stable sheaves (with trivial determinant and $c_2 = 1$). Unfortunately in our non-Kählerian framework an unexpected difficulty arises: *Gieseker stability is not an open condition*. More precisely, in our case, any neighbourhood (in the moduli space of simple rank 2 sheaves with trivial determinant) of a point $[\mathcal{F}] \in S$ contains points corresponding to sheaves which are not even slope semi-stable.

Taking into account this difficulty, we will take $\mathcal{U}$ to be a sufficiently small open neighbourhood of $S$ in the moduli space $\mathcal{M}^{\text{si}}$ of simple rank 2 sheaves [KO] with trivial determinant. We will define a map

$$f : \mathcal{U} \to \mathcal{M}$$

whose restriction to $S$ coincides with $\varphi$ and whose restriction to $\mathcal{U} \cap \mathcal{M}^{\text{si}}(E)_{\text{reg}}$ coincides with the identity of this set. For a point $[\mathcal{F}] \in \mathcal{U}$ with $\mathcal{F}$ not semi-stable we put $f([\mathcal{F}]) := [\mathcal{E}_X]$, where $\mathcal{E}_X$ is a slope stable locally free sheaf which is determined up to isomorphy by the conditions

$$H^0(\text{Hom}(\mathcal{F}, \mathcal{E}_X)) \neq 0, \quad H^0(\text{Hom}(\mathcal{E}_X, \mathcal{F})) \neq 0.$$

The proof will be completed by showing that these properties determine a well-defined map $f : \mathcal{U} \to \mathcal{M}$ satisfying the assumptions of Lemma 6.16.

### 1.4. The idea of proof of Theorem 1.3

According to the main result of [Pl] the moduli space $\mathcal{M}_n^G$ consists of finitely many simple points. Let $[A] \in \mathcal{M}_n^G$ be a flat irreducible SU(2)-instanton, $\mathcal{E}$ be the associated stable holomorphic bundle, and $\pi : \mathbb{P}(\mathcal{E}) \to X$ its projectivisation. For a point $x \in X$ and a line $y \in \mathbb{P}(\mathcal{E}(x))$ we denote by $\eta_y : \mathcal{E}(x) \to q_y := \mathcal{E}(x)/y$ the corresponding epimorphism, and we put

$$F_y := \ker \left( \mathcal{E} \to \mathcal{E}_{|x} \xrightarrow{\eta_y} q_y \otimes \mathcal{O}_{|x} \right).$$

We will construct a torsion-free sheaf $\mathcal{F}$ on $\mathbb{P}(\mathcal{E}) \times X$, flat over $\mathbb{P}(\mathcal{E})$, such that for any point $y \in \mathbb{P}(\mathcal{E})$ the restriction $\mathcal{F}_{|[y] \times X}$, regarded as a sheaf on $X$, is isomorphic with $F_y$. The sheaf $\mathcal{F}$ defines an embedding $\mathbb{P}(\mathcal{E}) \to \mathcal{M}^{\text{si}}$ in the moduli space of simple, torsion-free sheaves with trivial determinant and $c_2 = 1$. The normal line bundle of the image $\mathcal{P}$ of this embedding can be computed explicitly. We will prove that $\mathcal{P}$ has an open neighbourhood $\mathcal{U}$ such that $\mathcal{U} \cap \mathcal{P} \subset \mathcal{M}^{\text{si}}(E)_{\text{reg}}$. On the other hand, using Fujiki’s contractibility criterion [Fuj], it follows that, there exists a modification $\mathcal{U} \to \mathcal{V}$ on a singular complex space $\mathcal{V}$, which contracts the projective fibres of $\mathcal{P}$. Using the continuity theorem [BTT] we obtain a continuous map $\mathcal{U} \to \mathcal{M}^{\text{ASD}}(E)$ which induces a homeomorphism between the complex space $\mathcal{V}$ and an open neighbourhood of $\{[A]\} \times X$ in $\mathcal{M}^{\text{ASD}}(E)$.

The article is organised as follows: Section 2 contains results on the topology $\mathcal{M}^{\text{ASD}}(E)$ around the virtual locus. These results are obtained using gauge theoretical methods. Using the constructions given in Section 3 we construct in Section 4 a holomorphic embedding $V_e : \mathfrak{A}_e \times X \to \mathcal{M}^{\text{si}}$, where $\mathfrak{A}_e$ is an open neighbourhood
of $\mathfrak{C}(X)$ in the quotient of $\text{Pic}^T(X)$ by the involution $l \mapsto l'$. This embedding plays a crucial role in the proof of Theorem 1.2. The proofs of Theorem 1.2 and Theorem 1.3 are given in Section 5. The appendix groups together general results needed in the proofs; many of these results are new, and are useful in many other situations.

## 2. The topological structure of $\overline{\mathcal{M}}_{\text{ASD}}(E)$ at the virtual points

### 2.1. Local models at the reducible virtual points

We start with the following regularity result.

**Proposition 2.1.** Let $X$ be a class VII surface endowed with a Gauduchon metric $g$, and $E$ be a holomorphic, split polystable $\text{SL}(2, \mathbb{C})$-bundle on $X$ with $c_2(E) = 0$. Then $H^2(\mathcal{E}nd_0(E)) = 0$.

**Proof.** The hypothesis implies $\mathcal{E} = \mathcal{L} \oplus \mathcal{L}'$, where $\mathcal{L}$ is a holomorphic line bundle on $X$ with $\text{deg}_g(\mathcal{L}) = 0$ and $c_1(\mathcal{L})^2 = 0$. Since the intersection form $q_X$ is negative definite the latter condition implies $c_1(\mathcal{L}) \in \text{Tors}(H^2(X, \mathbb{Z}))$ so, using the notation introduced in Section 1.2, $[\mathcal{L}] \in C^c(X)$ for a class $c \in \text{Tors}(H^2(X, \mathbb{Z}))$. Note that $\mathcal{E}nd_0(\mathcal{E}) \cong \mathcal{O}_X \oplus \mathcal{L}^\otimes 2 \oplus \mathcal{L}'^\otimes 2$ and, since $X$ is a class VII surface, $h^2(\mathcal{O}_X) = h^0(K_X) = 0$. The result follows from Lemma 2.2 below.

**Lemma 2.2.** Let $X$ be a class VII surface, and $[\mathcal{M}] \in C(X)$. Then $H^2(\mathcal{M}) = 0$.

**Proof.** By Serre duality we have $h^2(\mathcal{M}) = h^0(K_X \otimes \mathcal{M}^*)$. On the other hand, since $\mathcal{M}$ is associated with a character $\chi : H_1(X, \mathbb{Z}) \to S^1$, it follows that its Chern class $c_1(\mathcal{M})$ in Bott-Chern cohomology vanishes. Therefore it suffices to prove that, on a class VII surface $X$, the Bott-Chern class $c_1^{BC}(K_X) \in H^{1,1}_{BC}(X, \mathbb{R})$ is not represented by an effective divisor. Let $\pi : X \to X_{\text{min}}$ be the projection of $X$ on its minimal model. If a Bott-Chern class $\epsilon \in H^{1,1}_{BC}(X, \mathbb{R})$ is represented by an effective divisor $D$, then $\pi_*(\epsilon)$ will be represented by the effective divisor $\pi_*(D)$, hence it suffices to prove that the class

$$\pi_*(c_1^{BC}(K_X)) = c_1^{BC}(K_{X_{\text{min}}})$$

is not represented by an effective divisor on $X_{\text{min}}$.

**Case 1.** $X_{\text{min}}$ is an Inoue surface. An Inoue surface has no curve so, if $c_1^{BC}(K_{X_{\text{min}}})$ were represented by an effective divisor, this class would vanish. But $c_1^{BC}(K_{X_{\text{min}}})$ is non-trivial and pseudo-effective (see [Te3, Remark 4.2]).

**Case 2.** $X_{\text{min}}$ is a Hopf surface. Any primary Hopf surface $H$ contains a non-trivial anti-canonical effective divisor [Kod2, p. 696]. In other words one has $K_H \cong \mathcal{O}_H(-D)$ where $D > 0$. It follows that for any Hopf surface $H$ the class $-c_1^{BC}(K_{X_{\text{min}}})$ is non-trivial and pseudo-effective, so $c_1^{BC}(K_{X_{\text{min}}})$ cannot be represented by an effective divisor.

**Case 3.** $X_{\text{min}}$ is a minimal class VII surface with $b_2(X_{\text{min}}) > 0$. In this case, using [Na, Lemma 1.1.3] we see that even the de Rham class $c_1^{DR}(K_{X_{\text{min}}})$ is not represented by an effective divisor.

**Corollary 2.3.** Let $A$ be a reducible flat $\text{SU}(2)$-instanton on a class VII surface. Then $H^2_A = 0$. 
Proof. In general, if $A$ is a projectively ASD connection on a Gauduchon surface, and $\mathcal{E}$ is the associated polystable bundle $\mathcal{E}$ (see section 1.1), the cohomology spaces of the deformation elliptic complexes associated with $\mathcal{E}$ and $A$ can be compared explicitly [Te2 Section 1.4.4]. By [Te2 Corollary 1.21], if $b_1(X)$ is odd, the second cohomology spaces of the two complexes coincide. Note that that on Kähler surfaces the vanishing of $H^2(\mathcal{E}u_0(\mathcal{E}))$ does not imply the vanishing of $\mathbb{H}^2_A$. Indeed, if $g$ is Kähler, the harmonic space $\mathbb{H}^2_A$ contains $\mathbb{H}^1_A\omega_u$, hence it cannot vanish when $\mathcal{E}$ is a split polystable bundle. The result follows from Proposition 2.1.

The following proposition describes the local structure around a regular, reducible virtual instanton on any Riemannian 4-manifold with $b_1(X) = 1$ and $b_+(X) = 0$:

**Proposition 2.4.** Let $X$ be a connected, oriented, compact Riemannian 4-manifold with $b_1(X) = 1$ and $b_+(X) = 0$. Let $E$ be an $\text{SU}(2)$-bundle with $c_2(E) = 1$ on $X$. Let $A$ be a flat, reducible instanton on $X$ with $\mathbb{H}^2_A = 0$. The Donaldson compactification $\overline{M}^{\text{ASD}}(E)$ is an 8-dimensional topological manifold at $([A], x)$.

Proof. Let $E^0$ be the trivial $\text{SU}(2)$-bundle on $X$. For a point $u \in X$ denote by $\text{Gl}_u$ the space of gluing data at $u$, which is just the space $\text{Isom}^+(\Lambda^+_u, su(E^0_u)) \cong SO(3)$ of orientation preserving linear isometries $\Lambda^+_u \to su(E^0_u)$. Let $C^e_u$ be the $\varepsilon$-cone over $\text{Gl}_u$ in the vector space $\text{Hom}(\Lambda^+_u, su(E^0_u))$. In other words

$$C^e_u := \{ t\gamma \mid t \in [0, \varepsilon], \gamma \in \text{Gl}_u \} \cong [0, \varepsilon) \times \text{Gl}_u/\{0\} \times \text{Gl}_u.$$ 

The unions

$$\text{Gl}_X := \bigcup_{u \in X} \text{Gl}_u, \ C^e_X := \bigcup_{u \in X} C^e_u$$

have natural structures of a locally trivial fibre bundles over $X$ with standard fibres $SO(3)$, respectively the $\varepsilon$-cone $C^e := \{ t\gamma \mid t \in [0, \varepsilon], \gamma \in SO(3) \}$ over $SO(3)$ in $M_{3,3}(\mathbb{R})$. For an open set $U \subset X$ we denote by $\text{Gl}_U$, $C^e_U$ the restrictions of these bundles to $U$. The gauge group $G_{E^0} := \Gamma(X, SU(E^0))$ acts naturally on $\text{Gl}_X$ (hence also on $C^e_X$) via the morphisms

$$G_{E^0} \xrightarrow{\text{ev}} SU(E^0_u) \xrightarrow{\text{Ad}} SO(su(E^0_u)) \xrightarrow{\text{composition}} \text{Aut}(\text{Isom}^+(\Lambda^+_u, su(E^0_u))),$$

where $\text{Ad}$ is just the adjoint representation of $SU(E^0_u)$ on its Lie algebra $su(E^0_u)$ and the composition morphism on the right maps $\varphi \in SO(su(E^0_u))$ to the automorphism $\gamma \mapsto \varphi \circ \gamma$ of $\text{Gl}_u$. For an open set $U \subset X$ the bundles $\text{Gl}_U$, $C^e_U$ are obviously gauge invariant.

Let $A$ be a reducible flat instanton on $X$ with $\mathbb{H}^2_A = 0$, and let $x \in X$. In this case [DK Proposition 8.2.4] gives an open neighbourhood $U$ of $x$ in $X$, a positive number $\varepsilon > 0$, an open neighbourhood $W$ of $([A], x)$ in $\overline{M}^{\text{ASD}}(E)$ and a homeomorphism

$$B_{\varepsilon} \times C^e_U/G_A \to W,$$

where $B_{\varepsilon}$ is the radius $\varepsilon$ ball of $\mathbb{H}^1_A$, and $G_A$ the stabiliser of $A$.

1. Suppose first the $G_A \cong S^1$. In this case one has $\mathbb{H}^0_A = \mathbb{R}\sigma$, $\mathbb{H}^1_A = \mathbb{H}^1(X)\sigma$, where $\sigma \in \Gamma(X, su(E^0))$ is an $A$-parallel unit section. The stabiliser $G_A$ acts trivially on $\mathbb{H}^1_A$. Let $\Lambda^+_U$, $S(\Lambda^+_U)$, $B_{\varepsilon}(\Lambda^+_U)$ be the restriction of the bundle $\Lambda^+$ to $U$, in the vector space $\text{Hom}(\Lambda^+_U, su(E^0_U))$.
the corresponding unit sphere bundle, respectively the corresponding radius \( \varepsilon \) ball bundle. The maps \( a : \text{GL}_U \to S(\Lambda^+_U) \), \( b : C^U \to B_2(\Lambda^+_U) \) given by
\[
a(\gamma) = \gamma^{-1}(\sigma_u), \quad b(t\gamma) = t\gamma^{-1}(\sigma_u) \quad \forall u \in U \quad \forall \gamma \in \text{GL}_U \quad \forall t \in [0, \varepsilon)
\]
are surjective, and their fibres coincide with the \( G_A \)-orbits in \( \text{GL}_U \), \( C^U \) respectively. Therefore in this case one has a homeomorphism
\[
B_\varepsilon \times C^U / G_A \simeq B_\varepsilon \times B_2(\Lambda^+_U),
\]
which is obviously an 8-dimensional topological manifold.

2. Suppose now that \( G_A \simeq \text{SU}(2) \). In this case the bundle \( \text{su}(E^0) \) is trivial, and choosing a trivialisation of this bundle, one obtains identifications \( \mathbb{H}^0_A = \text{su}(2) \), \( \mathbb{H}^1_A = \mathbb{H}^1(X) \otimes \text{su}(2) \). Therefore in this case the quotient \( B_\varepsilon \times C^U / G_A \) is a locally trivial fibre bundle over \( U \) with fibre
\[
F_\varepsilon := B_\varepsilon(\text{su}(2)) \times C^\varepsilon / \text{SU}(2) \simeq B^3 \times C^\varepsilon / \text{SO}(3)
\]
Identifying \( C^\varepsilon \) with the topological cone
\[
[0, \varepsilon) \times \text{SO}(3) / \{0\} \times \text{SO}(3)
\]
we obtain a proper surjective map
\[
\pi_\varepsilon : B^3 \times [0, \varepsilon) \times \text{SO}(3) / \{0\} \times \text{SO}(3) \to F_\varepsilon.
\]
The quotient on the left can be identified with \( B^3 \times [0, \varepsilon) \), hence \( F_\varepsilon \) is just the quotient \( B^3 \times [0, \varepsilon) \) collapsing the fibres of the norm map \( B^3 \times \{0\} \to [0, \varepsilon) \). In other words \( F_\varepsilon \) is the quotient space of \( B^3 \times [0, \varepsilon) \) by the equivalence relation \( \sim_\varepsilon \) generated by
\[
\{( (v, 0), (w, 0) ) \in (B^3 \times [0, \varepsilon)) \times (B^3 \times [0, \varepsilon)) | \|v\| = \|w\| \}.
\]
The claim follows now by Lemma 2.5 below, taking into account that \( B^3 \times [0, \varepsilon) \) is a \( \sim \) saturated open neighbourhood of \((0, 0) \) in \( \mathbb{R}^3 \times [0, \infty) \) and that \( \sim_\varepsilon \) is just the equivalence relation induced by \( \sim \) on this neighbourhood.

Lemma 2.5. Put \( \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times [0, \infty) \), \( S^n_+ := S^n \cap \mathbb{R}^{n+1}_+ \), \( \eta := (0, 1) \in S^n_+ \), and let \( p : S^n_+ \to S^n \) be the surjective map defined by the conditions
- \( (1) \) the points \( \eta, x \) and \( p(x) \) belong to a geodesic of \( S^n \),
- \( (2) \) \( d(p(x), \eta) = 2d(x, \eta) \), where distances are computed using the standard Riemannian metric on \( S^n \).

The map \( \pi : \mathbb{R}^{n+1}_+ \to \mathbb{R}^{n+1}_+ \) defined by
\[
\pi(pw) = pw \quad \forall w \in S^n_+ \quad \forall p \in [0, \infty)
\]
is proper and identifies \( \mathbb{R}^{n+1}_+ \) with the topological quotient of \( \mathbb{R}^{n+1}_+ \) by the equivalence relation \( \sim \) generated by
\[
\{( (v, 0), (w, 0) ) \in \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+ | \|v\| = \|w\| \}.
\]
The map \( p : S^n_+ \to S^n \) is just the (maximal extension of the) homothety of coefficient 2 and center \( \eta \) on the sphere \( S^n \). Figure illustrates Lemma 2.5 in the case \( n = 2 \).

Combining Corollary 2.3 with Proposition 2.4 we obtain
Figure 1. The map $\pi$

**Proposition 2.6.** Let $X$ be a class VII surface, and $E$ be an $\text{SU}(2)$-bundle with $c_2(E) = 1$ on $X$. Then $\overline{\mathcal{M}}^{\text{ASD}}(E)$ is a topological 8-manifold around $R_0 \times X$. The union $\mathfrak{M} := \mathcal{M}^{\text{ASD}}(E)^* \cup (R_0 \times X)$ is open in $\overline{\mathcal{M}}^{\text{ASD}}(E)$ and is a topological 8-manifold.

2.2. The structure of $\overline{\mathcal{M}}^{\text{ASD}}(E)$ around the strata of irreducible virtual points. Let $X$ be a class VII surface. By [Pl, Theorem 2.1] it follows that for any stable holomorphic bundle $\mathcal{E}$ of rank 2 with $\det(\mathcal{E}) \simeq \mathcal{O}_X$ and $c_2(\mathcal{E}) = 0$ one has $H^2(\text{End}_0(\mathcal{E})) = 0$. Using the same argument as in the proof of Corollary 2.3 we obtain

**Proposition 2.7.** Let $A$ be an irreducible flat $\text{SU}(2)$-instanton on a class VII surface. Then $H^2_A = 0$.

On the other hand the method used in the proof of Proposition 2.4 gives:

**Proposition 2.8.** Let $X$ be a connected, oriented, compact Riemannian 4-manifold with $b_1(X) = 1$ and $b_+(X) = 0$. Let $E$ be an $\text{SU}(2)$-bundle with $c_2(E) = 1$ on $X$. Let $A$ be a flat, irreducible instanton on $X$ with $H^2_A = 0$. For a sufficiently small $\varepsilon > 0$ the subspace $\{[A]\} \times X$ has an open neighbourhood in $\overline{\mathcal{M}}^{\text{ASD}}(E)$ which can be identified with the cone bundle $C^\varepsilon_X$.

The following remark shows that an $\text{SL}(2, \mathbb{C})$-holomorphic structure $\mathcal{E}$ on $E^0$ defines a normal complex space structure on the cone bundle $C^\varepsilon_X$.

**Remark 2.9.** Let $(X, g)$ be Hermitian surface. The cone bundle $C^\varepsilon_X$ can be identified with a neighbourhood of the zero section in the cone bundle of degenerate elements in $\Lambda^{02}_X \otimes S^2(E^0)$. Therefore any holomorphic $\text{SL}(2, \mathbb{C})$-structure $\mathcal{E}$ on $E^0$ identifies $C^\varepsilon_X$ with a neighbourhood of the zero section in the cone bundle of degenerate elements in $K_X \otimes S^2(\mathcal{E})$, which is a normal complex space over $X$.

**Proof.** The standard fibre of $C^\varepsilon_X$ is a cone over $\text{SO}(3)$, hence it can be identified with a neighbourhood of the singular point in the quotient $\mathbb{R}^4/\mu_2$. If we fix a Spin$^c(4)$-structure $\sigma$ on $X$, then the $\text{SO}(3)$-bundle $\text{Gl}_X$ can be identified
with the quotient $U(\Sigma^+, E^0)/S^1$, where $\Sigma^+$ is the positive spinor bundle of $\sigma$. This identification is obtained using the standard isomorphism $\Lambda^+(X) \cong \text{su}(\Sigma^+)$. If $(X, g)$ is a Hermitian surface and $\sigma$ is its canonical Spin$^c$-structure, then $\Sigma^+ = \Lambda^{00} \oplus \Lambda^{02}$. Denoting by $S(E)$ the sphere bundle of $E^0$ we obtain a bundle isomorphism $U(\Sigma^+, E^0) \cong S(E^0) \times_X S(\Lambda^{02})$ given fibrewise by the formula

$$U(\Sigma^+, E^0) \ni a \mapsto (a^{-1}(j \cdot a(1_x), a(1_x))) \in S(\Lambda^{02}) \times_X S(E^0)$$

where $1_x$ is the standard generator of $\Lambda^{00}_x$, and $j \cdot$ stands for the multiplication with the quaternionic unit $j$ with respect to the quaternionic structure of $E$ given by the standard identification $\text{SU}(2) \cong \text{Sp}(1)$. The $S^1$-action on $S(\Lambda^{02}) \times_X S(E^0)$ induced by the standard action of $S^1$ on $U(\Sigma^+, E^0)$ is given by $\zeta \cdot (u, v) = (\zeta^{-2}u, \zeta v)$. This shows that the bundle $G_1 = U(\Sigma^+, E^0)/S^1$ can be identified with the image $\mathcal{S}$ of $S(\Lambda^{02}) \times_X S(E^0)$ in $\Lambda^{02} \otimes S^2(E^0)$ via the $S^1$-invariant map $(e, \nu) \mapsto (e \vee e) \otimes \nu$. It suffices to note that $\{t \in [0, \varepsilon], s \in \mathcal{S}\}$ is a neighbourhood of the zero section in the cone bundle of degenerate elements in $\Lambda^{02}_X \otimes S^2(E^0)$.

3. SINGULAR SHEAVES WITH TRIVIAL DETERMINANT AND $c_2 = 1$

3.1. A FAMILY OF $\text{SL}(2, \mathbb{C})$-BUNDLES WITH $c_2 = 0$. Let $X$ be a class VII surface. For such a surface one has $\chi(\mathcal{O}_X) = 0$, $H^1(\mathcal{O}_X) \cong \mathbb{C}$, and $\text{Pic}^0(X) \cong \mathbb{C}^*$. Fix an isomorphism $\theta : H^1(\mathcal{O}_X) \cong \mathbb{C}$ which maps $H^1(X, \mathbb{Z})$ onto $\mathbb{Z}$, and let $\varpi$ be the induced isomorphism $\text{Pic}^0(X) \cong \mathbb{C}^*$.

Since $X$ is fixed, we will omit it in the notations $\text{Pic}^T(X), \text{Pic}^c(X), C(X), \mathcal{C}(X), C^c(X), \mathcal{C}^c(X), \mathcal{C}^{-c}(X), \rho^c(X), \rho(X), \mathcal{T}(X), \mathcal{T}_0(X), \mathcal{T}_1(X)$ introduced in section 1.2.

Let $\mathcal{L}$ be a Poincaré line bundle on $\text{Pic}^T \times X$ normalised at a point $x_0 \in X$. For $l \in \text{Pic}^T$ we put $\mathcal{L}_l := \mathcal{L}_{l|} \times_X$, regarded as a line bundle on $X$. The universal property of the Poincaré line bundle shows that, for any $l \in \text{Pic}^T$, the isomorphism type of $\mathcal{L}_l$ is $l$. For a point $l \in \text{Pic}^T$ and a tangent vector $v \in T_l(\text{Pic}^T) = H^1(X, \mathcal{O}_X)$, the infinitesimal deformation $\epsilon_v(\mathcal{L}) \in \text{Ext}^1(\mathcal{L}_l, \mathcal{L}_l) = H^1(X, \mathcal{O}_X)$ associated with the pair $(\mathcal{L}, v)$ is given by

$$\epsilon_v(\mathcal{L}) = v.$$  

(3)

The quotient $\mathfrak{P}$ of $\text{Pic}^T$ by the involution $l \mapsto l^\ast$ decomposes as a disjoint union $\mathfrak{P} = \bigsqcup_{\varepsilon \in \mathbb{Z}} \mathfrak{P}^\varepsilon$ where, for a class $\varepsilon = [\varepsilon] \in \mathbb{Z}$, $\mathfrak{P}^\varepsilon$ stands for the quotient of $\text{Pic}^c \cdot \text{Pic}^{-c}$ by $j$. $\mathfrak{P}^\varepsilon$ is biholomorphic to $\mathbb{C}$ when $\varepsilon \in \mathcal{T}_0$, and can be identified with $\text{Pic}^c \cong \mathbb{C}^*$ when $\varepsilon \in \mathcal{T}_1$. The projection $\pi : \text{Pic}^T \rightarrow \mathfrak{P}$ is a branched double covering whose ramification locus is the set

$$\rho = H^1(X, \mu_2) \subset \bigcup_{\varepsilon \in \text{Tors}(H^2(X, \mathbb{Z}))} \text{Pic}^\varepsilon,$$

and whose branch locus is $\beta := \pi(\rho) \subset \bigcup_{\varepsilon \in \mathcal{T}_0} \mathfrak{P}^\varepsilon$. Since $\text{Pic}^0$ is an injective $\mathbb{Z}$-module, the short exact sequence $0 \rightarrow \text{Pic}^0 \rightarrow \text{Pic}^T \rightarrow \text{Tors}(H^2(X, \mathbb{Z})) \rightarrow 0$ splits. Fix a left splitting $\sigma : \text{Pic}^T \rightarrow \text{Pic}^0$, and put

$$\xi := \frac{1}{2}(\varpi \circ \sigma + \varpi \circ \sigma \circ j), \; \zeta := \frac{1}{2}(\varpi \circ \sigma - \varpi \circ \sigma \circ j) \in \mathcal{O}(\text{Pic}^T).$$

The zero locus of $\zeta$ is the ramification locus $\rho$. With these notations one has

$$\pi_*(\mathcal{O}(\text{Pic}^T)) = \mathcal{O}_X \xi \oplus \mathcal{O}_X \zeta.$$  

(4)
The push-forward $\mathcal{E} := (\pi \times \text{id}_X)_*(\mathcal{L})$ is a rank 2 bundle on $\mathfrak{P} \times X$. For $p \in \mathfrak{P}$ we put $\mathcal{E}_p := \mathcal{E}_{\{p\} \times X}$ regarded as a bundle on $X$. Taking into account the definition of $(\pi \times \text{id}_X)_*$, one obtains canonical isomorphisms

$$\mathcal{E}_\pi(l) = \mathcal{L}_{X_l + X_{l'}},$$

(5)

where $X_l$ stands for the smooth divisor $\{l\} \times X \subset \text{Pic}^T \times X$, where $X_l + X_{l'}$ is regarded as an effective divisor in $\text{Pic}^T \times X$, and where the right hand sheaf is regarded as an $\mathcal{O}_X$-module via the obvious projection $X_l + X_{l'} \to X$. For a ramification point $l \in \rho$, one obtains

$$\mathcal{E}_\pi(l) = \mathcal{L}_{2X_l} \text{ (viewed as an } \mathcal{O}_X \text{-module),}$$

(6)

where $2X_l$ is regarded as a non-reduced complex subspace of $\text{Pic}^T \times X$. Tensoring by $\mathcal{L}$ the short exact sequence

$$0 \to \mathcal{O}_{X_l} \to \mathcal{O}_{2X_l} \to \mathcal{O}_{X_l} \to 0$$

(7)

and regarding the central term as an $\mathcal{O}_X$-module, we obtain a canonical exact sequence

$$0 \to \mathcal{L}_l \to \mathcal{E}_\pi(l) \to \mathcal{L}_l \to 0,$$

(8)

whose extension class is precisely $\epsilon_{\mathfrak{P}_0}(\mathcal{L})$, where $\mathfrak{P}_0 \in \mathfrak{T}(\text{Pic}^T)$ is defined by the condition $d\zeta(\mathfrak{P}_0) = 1$ (see Proposition 6.8 in the Appendix). Taking into account formula (5) we get:

**Lemma 3.1.** In the conditions and with the notation above, let $p = \pi(l) \in \mathfrak{P}$. Then

1. If $l \notin \rho$ one has canonical identifications

$$\mathcal{E}_p = \mathcal{L}_l \oplus \mathcal{L}_{l'} = \mathcal{L}_l \oplus \mathcal{L}_{l'},$$

2. If $l \in \rho$ one has a canonical short exact sequence

$$0 \to \mathcal{L}_l \xrightarrow{j_l} \mathcal{E}_p \xrightarrow{\tau_l} \mathcal{L}_l \to 0,$$

(9)

whose extension class is the generator of $H^1(\mathcal{O}_X)$ induced by the fixed isomorphism $\partial : H^1(\mathcal{O}_X) \to \mathbb{C}$.

Using the identification (5) we see that:

**Remark 3.2.** Every representative $l \in p$ yields a canonical short exact sequence

$$0 \to \mathcal{L}_l \xrightarrow{j_l} \mathcal{E}_p \xrightarrow{\tau_l} \mathcal{L}_l \to 0.$$  

(10)

If $l' \neq l$ is a different representative, then $r_l \circ j_{l'} = \text{id}_{\mathcal{L}_l}$, $r_{l'} \circ j_l = \text{id}_{\mathcal{L}_{l'}}$.

**Definition 3.3.** Let $p = \pi(l) \in \mathfrak{P}$ and $x \in X$. If $l \notin \rho$ the two summands $\mathcal{L}_l$, $\mathcal{L}_{l'}$ of $\mathcal{E}_p$ will be called the canonical line sub-bundles of $\mathcal{E}_p$ and $\mathcal{L}_l(x)$, $\mathcal{L}_{l'}(x)$ the canonical lines of $\mathcal{E}_p(x)$. If $l \in \rho$ the image of $\mathcal{L}_l$ in $\mathcal{E}_p$ (via the monomorphism $j_l$ given by Lemma 3.1) will be called the canonical line sub-bundle of $\mathcal{E}_p$, and $\mathcal{L}_l(x)$ the canonical line of $\mathcal{E}_p(x)$.

**Proposition 3.4.** Fix a Gauduchon metric $g$ on $X$, let $l \in \text{Pic}^T$ with $\text{deg}_g(l) \geq 0$, and put $p := \pi(l)$. Then
Corollary 3.7. Let $l \notin \rho$. Then
\[
\operatorname{End}(\mathcal{E}_p) = \text{Cicd}_{\mathcal{L}_t} \oplus \text{Cicd}_{\mathcal{L}_t} \oplus H^0(\mathcal{L}_t^{\otimes 2}), \quad \text{Aut}(\mathcal{E}_p) = \mathbb{C}^* \text{id}_{\mathcal{L}_t} \times \mathbb{C}^* \text{id}_{\mathcal{L}_t} \times H^0(\mathcal{L}_t^{\otimes 2}).
\]

The multiplication on $\text{Cicd}_{\mathcal{L}_t} \oplus \text{Cicd}_{\mathcal{L}_t} \oplus H^0(\mathcal{L}_t^{\otimes 2})$ induced by the composition of endomorphisms is given by
\[
(z, \zeta, \lambda)(z', \zeta', \lambda') = (z' + z\lambda + \zeta', z\zeta + \zeta').
\]

(2) When $l \in \rho$ then
\[
\operatorname{End}(\mathcal{E}_p) = \text{Cicd}_{\mathcal{E}_p} \oplus \mathbb{C}(j_l \circ r_l), \quad \text{Aut}(\mathcal{E}_p) = \mathbb{C}^* \text{id}_{\mathcal{E}_p} \times \mathbb{C}(j_l \circ r_l).
\]

The multiplication on $\text{Cicd}_{\mathcal{E}_p} \oplus \mathbb{C}(j_l \circ r_l)$ induced by the composition of endomorphisms is given by
\[
(z, \zeta(j_l \circ r_l))(z', \zeta'(j_l \circ r_l)) = (z' + z\zeta + \zeta').
\]

Proof. (1) follows from Lemma 3.1 taking into account that, since $\deg_g(\mathcal{L}_t^{\otimes 2}) \leq 0$ and $\mathcal{L}_t^{\otimes 2} \neq \mathcal{O}_X$, one has $H^0(\mathcal{L}_t^{\otimes 2}) = 0$. Note that in our non-Kählerian framework $X$ might contain non-empty effective divisors with torsion fundamental class, so $H^0(\mathcal{L}_t^{\otimes 2})$ might be non-zero even if $l \notin \rho$.

For (2), let $\varphi \in \operatorname{End}(\mathcal{E}_p)$. The composition $r_l \circ \varphi \circ j_l$ must vanish because, if not, it would be an isomorphism, hence $\varphi \circ j_l$ would define a splitting of the extension $\mathcal{O}_X$. Therefore $\varphi$ maps $j_l(\mathcal{L}_t)$ into itself, giving an endomorphism $\varphi_0$ of $\mathcal{L}_t$, which can be written as $\varphi_0 = z \text{id}_{\mathcal{L}_t}$. The endomorphism $\psi := \varphi - z \text{id}_{\mathcal{E}_p}$ will vanish on $j_l(\mathcal{L}_t)$, so it can be written as $\psi = \chi \circ r_l$, for a morphism $\chi : \mathcal{L}_t \to \mathcal{E}_p$. Since, by the same argument as above one has $r_l \circ \chi = 0$, it follows that $\chi = j_l \circ u$ for a morphism $u : \mathcal{L}_t \to \mathcal{L}_t$. Writing $u = \zeta \text{id}_{\mathcal{L}_t}$, we obtain $\varphi = z \text{id}_{\mathcal{E}_p} + \zeta(j_l \circ r_l)$, which proves the claim.

Definition 3.5. Let $\operatorname{Div}(X)^{>0}$ be the set of non-empty, effective divisors on $X$.

Put
\[
\operatorname{Div}(X)^{>0} := \{ D \in \operatorname{Div}(X)^{>0} \mid c_1(\mathcal{O}(D)) \in \text{Tors}(H^2(X, \mathbb{Z})) \};
\]
\[
\nu(g) = \inf \{ \nu_0(D) \mid D \in \operatorname{Div}(X)^{>0}, \deg_g(D) < 0 \}, \quad \nu_0(g) = \inf \{ \nu_0(D) \mid D \in \operatorname{Div}(X)^{>0}, \deg_g(D) < 0 \}.
\]

One has obviously $\nu_0(g) \geq \nu(g)$, and using Bishop’s compactness theorem (see for instance [Ch]), it follows that $\nu(g) > 0$.

Remark 3.6. Let $\mathcal{L}$ be a non-trivial holomorphic line bundle on $X$. Suppose that $\deg_g(\mathcal{L}) < \nu(g)$ or $c_1^R(\mathcal{L}) = 0$ and $\deg_g(\mathcal{L}) < \nu_0(g)$. Then $H^0(\mathcal{L}) = 0$.

With these notations Proposition 3.4 gives

Corollary 3.7. Let $p = \pi(l) \in \mathcal{P} \setminus \beta$ with $0 \leq \deg_g(l) < \frac{1}{2} \nu_0(g)$. Then
\[
\operatorname{End}(\mathcal{E}_p) = \text{Cicd}_{\mathcal{L}_t} \oplus \text{Cicd}_{\mathcal{L}_t}^*, \quad \operatorname{Aut}(\mathcal{E}_p) = \mathbb{C}^* \text{id}_{\mathcal{L}_t} \times \mathbb{C}^* \text{id}_{\mathcal{L}_t}^*.
\]
Let \( A_\varepsilon \) be the tubular neighbourhood of \( C = \text{Hom}(H_1(X, \mathbb{Z}), S^1) \) in \( \text{Pic}^T \) given by
\[
A_\varepsilon := \{ l \in \text{Pic}^T \mid \deg_l(l) \in (-\varepsilon, \varepsilon) \},
\]
and let \( \mathfrak{A}_\varepsilon \) be the quotient \( \mathfrak{A}_\varepsilon := A_\varepsilon/\langle \iota \rangle \). \( A_\varepsilon \) is a disjoint union of annuli, and \( \mathfrak{A}_\varepsilon \) is a disjoint union of annuli and disks. Denoting by \( |\deg_l| : \mathfrak{P} \to [0, \infty) \) the map given by \( |\deg_l|([\pi_0(l)]) := |\deg_l(l)| \), we see that \( \mathcal{E} \) is just the zero set of \( |\deg_l| \), and \( \mathfrak{A}_\varepsilon \) is the open subset of \( \mathfrak{P} \) defined by the inequality \( |\deg_l|(p) < \varepsilon \).

**Corollary 3.8.** Let \((X, g)\) be a class VII surface endowed with a Gauduchon metric. For sufficiently small \( \varepsilon > 0 \) the following holds:

1. \( h^0(\mathcal{E} \text{nd}_0(\mathcal{E}_p)) = 1 \), \( h^2(\mathcal{E} \text{nd}_0(\mathcal{E}_p)) = h^2(\mathcal{E} \text{nd}_0(\mathcal{E}_p)) = 0 \) for any \( p \in \mathfrak{A}_\varepsilon \),
2. the family \( \mathcal{E} \) is versal at any point \( p \in \mathfrak{A}_\varepsilon \).

**Proof.** (1) By Corollary 3.7 it follows that \( h^0(\mathcal{E} \text{nd}_0(\mathcal{E}_p)) = 1 \) if \( |\deg_l|(p) \) is sufficiently small. Write \( p = \pi(l) \) with \( l \in A_\varepsilon \). By Lemma 2.2 we know that \( h^0(K_X \otimes \mathcal{L}^{j}) = h^0(K_X \otimes \mathcal{L}^{j}) = 0 \) if \( l \in C \). Using Remark 10 and the known property \( h^1(K_X) = 0 \), we see that \( h^0(K_X \otimes \mathcal{E} \text{nd}(\mathcal{E}_p)) = h^0(K_X \otimes \mathcal{E} \text{nd}_0(\mathcal{E}_p)) = 0 \) when \( p \in C \). By Grauer’s semicontinuity theorem it follows that, for sufficiently small \( \varepsilon > 0 \), one has \( h^2(\mathcal{E} \text{nd}(\mathcal{E}_p)) = h^2(\mathcal{E} \text{nd}_0(\mathcal{E}_p)) = 0 \) for any \( p \in \mathfrak{A}_\varepsilon \).

(2) Using the Riemann-Roch theorem we get \( h^1(\mathcal{E} \text{nd}_0(\mathcal{E}_p)) = 1 \) for any \( p \in \mathfrak{A}_\varepsilon \). It suffices to note that the infinitesimal deformation \( \epsilon_l(\mathcal{E}) \) is non-trivial for any non-trivial tangent vector \( v \). For \( p \neq \beta \) the claim follows from the similar property of the Poincaré line bundle \( \mathcal{L} \). For \( p \in \beta \) this follows from Proposition 6.8 [2].

### 3.2. Singular torsion-free sheaves with trivial determinant and \( c_2 = 1 \)

We start with the following simple classification result:

**Proposition 3.9.** Let \( \mathcal{F} \) be a singular torsion-free rank 2 sheaf on a non-algebraic complex surface \( X \) with \( \det(\mathcal{F}) \simeq \mathcal{O}_X \) and \( c_2(\mathcal{F}) = 1 \). Then \( \mathcal{F} \) fits in an exact sequence
\[
0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O}_{(x)} \to 0,
\]
where \( \mathcal{E} = \mathcal{F}'' \) is a rank 2-bundle on \( X \) with \( \det(\mathcal{E}) \simeq \mathcal{O}_X \), \( c_2(\mathcal{E}) = 0 \), and \( \mathcal{O}_{(x)} := \mathcal{O}/\mathcal{I}_x \).

**Proof.** Since \( \mathcal{F} \) is torsion-free on a surface, its singularity set \( S \subset X \) is 0-dimensional, \( \mathcal{F} \) embeds injectively in its bidual \( \mathcal{E} := \mathcal{F}'' \) (which is locally free), and the support of the quotient sheaf \( \mathcal{Q} := \mathcal{E}/\mathcal{F} \) is \( S \). Using the exact sequence
\[
0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{Q} \to 0
\]
we get \( \det(\mathcal{E}) \simeq \det(\mathcal{F}) \simeq \mathcal{O}_X \), and \( c_2(\mathcal{E}) = c_2(\mathcal{F}) = -h^0(\mathcal{Q}) \). On the other hand, since \( X \) is non-algebraic, we get [BLT] Théorème 0.3 [4] \( 4c_2(\mathcal{E}) - c_1(\mathcal{E})^2 \geq 0 \), hence, in our case, \( c_2(\mathcal{E}) \geq 0 \). This gives \( h^0(\mathcal{Q}) \leq c_2(\mathcal{F}) = 1 \). Since we suppose that \( \mathcal{F} \) is singular, \( \mathcal{Q} \) cannot vanish, hence \( h^0(\mathcal{Q}) = 1 \), which shows that \( \mathcal{Q} \) is just the structure sheaf of a simple point.

Let \( \mathcal{E} \) be rank 2 locally free sheaf on a surface \( X \), \( x \in X \) and \( \eta : \mathcal{E}(x) \to \mathbb{C} \) be a linear epimorphism. We will denote by \( \mathcal{F}(\mathcal{E}, x, \eta) \) the torsion-free sheaf \( \ker(\tilde{\eta}) \), where \( \tilde{\eta} : \mathcal{E} \to \mathcal{O}_{(x)} \) is the epimorphism induced by \( \eta \).

**Proposition 3.10.** If \( \mathcal{F}(\mathcal{E}, x, \eta) \cong \mathcal{F}(\mathcal{E}', x', \eta') \) then \( x = x' \) and any isomorphism \( f : \mathcal{F}(\mathcal{E}, x, \eta) \to \mathcal{F}(\mathcal{E}', x, \eta') \) is induced by a bundle isomorphism \( \varphi : \mathcal{E} \to \mathcal{E}' \) such that \( \eta' \circ \varphi(x) \in \mathbb{C}^* \eta \).
Therefore we will need families of admissible epimorphisms parameterised by products between the respective biduals which makes the diagram

\[
\begin{array}{c}
\mathcal{F}(\mathcal{E}, x, \eta) \\
\varphi \\
\mathcal{F}(\mathcal{E}', x, \eta')
\end{array}
\xrightarrow{f} \begin{array}{c}
\mathcal{F}(\mathcal{E}', x, \eta') \\
\varphi \\
\mathcal{E}'
\end{array}
\]

commutative. This shows that \( \varphi \) maps \( \mathcal{F}(\mathcal{E}, x, \eta) \) onto \( \mathcal{F}(\mathcal{E}', x, \eta') \) and \( f \) is just the restriction of \( \varphi \) to \( \mathcal{F}(\mathcal{E}, x, \eta) \). The former property is equivalent to \( \varphi(\ker(\eta)) = \ker(\eta') \), which, since \( O_{\{x\}} \) is a skyscraper sheaf with 1-dimensional stalk, is obviously equivalent to \( \eta' \circ \varphi(x) \in \mathbb{C}^* \eta \). But this is equivalent to \( \eta' \circ \varphi(x) = \eta \).

\[\blacksquare\]

**Corollary 3.11.** Let \( \mathcal{E} \) be rank 2 locally free sheaf on a surface \( X \), \( x, x' \in X \), and \( \eta, \eta' : \mathcal{E}(x) \to \mathbb{C} \) be epimorphisms. The following conditions are equivalent:

1. \( \mathcal{F}(\mathcal{E}, x, \eta) \cong \mathcal{F}(\mathcal{E}', x, \eta') \),
2. \( x = x' \) and there exists an automorphism \( \varphi \in \text{Aut}(\mathcal{E}) \) such that \( \eta' \circ \varphi(x) = \eta \).

We return to class VII surfaces and the objects constructed in the previous sections.

**Definition 3.12.** Let \( X \) be a class VII surface, let \( p \in \Psi \) and \( x \in X \). An epimorphism \( \eta : \mathcal{E}_p(x) \to \mathbb{C} \) will be called admissible if its kernel does not coincide with a canonical line of \( \mathcal{E}_p(x) \) (see Definition \( 3.3 \)).

Using Proposition \( 3.4 \) and Corollary \( 3.7 \) we get

**Corollary 3.13.** Let \( p = \pi(l) \in \Psi \) with \( 0 \leq |\deg(l)| < \frac{1}{2} \varphi_0(g) \).

1. For any admissible epimorphism \( \eta : \mathcal{E}_p(x) \to \mathbb{C} \), the sheaf \( \mathcal{F}(\mathcal{E}_p, x, \eta) \) is simple,
2. If \( \eta, \eta' : \mathcal{E}_p(x) \to \mathbb{C} \) are admissible epimorphisms, then \( \mathcal{F}(\mathcal{E}_p, x, \eta) \cong \mathcal{F}(\mathcal{E}_p, x, \eta') \).

The sheaves of the form \( \mathcal{F}(\mathcal{E}_p, x, \eta) \) will play a crucial role in the proof of our main result. Using Remark \( 3.2 \) we see that

**Remark 3.14.** Let \( \eta : \mathcal{E}_p(x) \to \mathbb{C} \) be an admissible epimorphism. A representative \( l \in p \) gives a short exact sequence

\[
0 \to L_i' \otimes I_x \overset{ji,x,\eta}{\longrightarrow} \mathcal{F}(\mathcal{E}_p, x, \eta) \overset{\tau_l,x,\eta}{\longrightarrow} L_i \to 0,
\]

where \( ji,x,\eta, \tau_l,x,\eta \) are induced by \( ji, \tau_l \).

4. **The map \( V_\varepsilon \) and its properties**

We will need holomorphic families of singular sheaves of the form \( \mathcal{F}(\mathcal{E}_p, x, \eta) \) parameterised by products \( \Psi \times U \), for sufficiently small open subsets \( U \subset X \). Therefore we will need families of admissible epimorphisms

\[
(\eta_{p,u} : \mathcal{E}_p(u) \to \mathbb{C})_{(p,u) \in \Psi \times U}
\]
defined for sufficiently small open subsets \( U \subset X \). Let \( U \subset X \) be a contractible, Stein open set. The restriction \( \mathcal{L}_U := \mathcal{L}|_{\text{Pic}^r \times U} \) is trivial. Choose a trivialisation morphism \( \sigma : \mathcal{L}_U \xrightarrow{\simeq} \mathcal{O} \text{Pic}^r \times U \). Taking push-forward via the double cover

\[
\pi \times \text{id}_U : \text{Pic}^r \times U \rightarrow \mathfrak{P} \times U,
\]

and using (4), we get a bundle isomorphism

\[
(\pi \times \text{id}_U)(\sigma) : \mathcal{E}_U \xrightarrow{\simeq} (\pi \times \text{id}_U)_*(\mathcal{O} \text{Pic}^r \times U) = \mathcal{O}_{\mathfrak{P} \times U} \oplus \mathcal{O}_{\mathfrak{P} \times U} \xi,
\]

where \( \mathcal{E}_U := \mathcal{O}_{\mathfrak{P} \times U} \). The projection on the second summand gives an epimorphism \( \eta^\sigma : \mathcal{E}_U \rightarrow \mathcal{O}_{\mathfrak{P} \times U} \).

Lemma 4.1. For any \( (p, u) \in \mathfrak{P} \times U \), \( \eta^\sigma_{p, u} : \mathcal{E}_p(u) \rightarrow \mathbb{C} \) is an admissible epimorphism.

Proof. Via the identifications

\[
\mathcal{E}_U(\mathfrak{P} \times U) = \mathcal{L}_U(\text{Pic}^r \times U) \xrightarrow{\simeq} \mathcal{O}(\text{Pic}^r \times U),
\]

the morphism \( \eta^\sigma \) is given by \( \eta^\sigma(\alpha) = \frac{1}{\xi} (\alpha - \alpha \circ (j \times \text{id}_U)) \). If \( l \neq l' \) this shows that \( \ker(\eta^\sigma_{p, u}) \) is identified with the diagonal line of the sum \( \mathcal{L}_l(u) \oplus \mathcal{L}_{l'}(u) \). If \( l = l' \) the same formula shows that \( \eta^\sigma \) defines a splitting of the restriction of the exact sequence (8) to \( U \), so \( \ker(\eta^\sigma_{p, u}) \) is a complement of \( \mathcal{L}_l(u) \) in \( \mathcal{E}_p(u) \).

4.1. The embedding \( V_\varepsilon \). Let \( \mathcal{M}^{\text{si}} \) be the moduli space of simple torsion-free sheaves on \( X \) with trivial determinantal line bundle and \( c_2 = 1 \), and let

\[
\mathcal{M}^{\text{si}}_{\text{reg}} := \{ [\mathcal{F}] \in \mathcal{M}^{\text{si}} | \text{Ext}^2_\xi(\mathcal{F}, \mathcal{F}) = 0 = \{ [\mathcal{F}] \in \mathcal{M}^{\text{si}} | \text{Ext}^2(\mathcal{F}, \mathcal{F}) = 0 \}
\]

be its regular part.

Proposition 4.2. Let \((X, g)\) be a class VII surface endowed with a Gauduchon metric. For sufficiently small \( \varepsilon > 0 \) the following hold:

1. \([\mathcal{F}(\mathcal{E}_p, x, \eta)] \in \mathcal{M}^{\text{si}}_{\text{reg}} \) for any \( p \in \mathfrak{A}_\varepsilon \), \( x \in X \) and admissible epimorphism \( \eta : \mathcal{E}_p(x) \rightarrow \mathbb{C} \).
2. The map \( V_\varepsilon : \mathfrak{A}_\varepsilon \times X \rightarrow \mathcal{M}^{\text{si}}_{\text{reg}} \) given by

\[
V(p, x) := [\mathcal{F}(\mathcal{E}_p, x, \eta)]
\]

(for an admissible epimorphism \( \eta : \mathcal{E}_p(x) \rightarrow \mathbb{C} \)) is well-defined, injective, holomorphic, immersive, and homeomorphic on its image.

In other words, for sufficiently small \( \varepsilon > 0 \), \( V_\varepsilon \) is a holomorphic embedding.

Proof. (1) Put \( \mathcal{F} := \mathcal{F}(\mathcal{E}_p, x, \eta) \), \( \mathcal{E} := \mathcal{E}_p \) to save on notation. The fact that \( \mathcal{F} \) is simple for sufficiently small \( |\deg|\)(p) follows from Corollary 3.13.

To prove that \( \text{Ext}^2(\mathcal{F}, \mathcal{F}) = 0 \), use the local-global spectral sequence. Since the singularity set of \( \mathcal{F} \) is 0-dimensional, one has

\[
H^1(\mathcal{Ext}^1(\mathcal{F}, \mathcal{F})) = 0,
\]

hence it suffices to show that, when \( |\deg|\)(p) is sufficiently small, one has

(a) \( H^2(\text{Hom}(\mathcal{F}, \mathcal{F})) = 0 \),
(b) \( \mathcal{Ext}^2(\mathcal{F}, \mathcal{F}) = 0 \).
To prove (a) note that the sheaf \( \text{Hom}(\mathcal{F}, \mathcal{F}) \) fits in a short exact sequence
\[
0 \to \text{Hom}(\mathcal{F}, \mathcal{F}) \to \text{Hom}(\mathcal{F}, \mathcal{E}) = \text{Hom}(\mathcal{E}, \mathcal{E}) \xrightarrow{\psi} \text{Hom}((\mathcal{F}/\mathcal{I}_x \mathcal{E}, \mathcal{O}_{(x)}) \to 0,
\]
where \( \psi \) is the composition
\[
\text{Hom}(\mathcal{E}, \mathcal{E}) \to \text{Hom}((\mathcal{F}/\mathcal{I}_x \mathcal{E}, \mathcal{E}/\mathcal{I}_x \mathcal{E}) \to \text{Hom}((\mathcal{F}/\mathcal{I}_x \mathcal{E}, \mathcal{E}/\mathcal{I}_x \mathcal{E}) \xrightarrow{\eta} \text{Hom}((\mathcal{F}/\mathcal{I}_x \mathcal{E}, \mathcal{O}_{(x)})).
\]
Since \( \text{Hom}((\mathcal{F}/\mathcal{I}_x \mathcal{E}, \mathcal{O}_{(x)}) \) is a torsion sheaf supported at \( \{x\} \), we have
\[
H^2(\text{Hom}(\mathcal{F}, \mathcal{F})) \cong H^2(\text{Hom}(\mathcal{E}, \mathcal{E}))
\]
which vanishes if \( \varepsilon \) is sufficiently small by Corollary 3.13.

For (b) note that the stalk \( \mathcal{F}_y \) is a free \( \mathcal{O}_y \)-module for any point \( y \neq \{x\} \), whereas
\[
\mathcal{F}_x \cong \mathcal{O}_x \oplus \{\mathcal{I}_x\}_x
\]
as \( \mathcal{O}_x \)-modules. Therefore we have to prove that
\[
\text{Ext}^2_{\mathcal{O}_x}((\mathcal{I}_x)_x, \mathcal{O}_x) = \text{Ext}^2_{\mathcal{O}_x}((\mathcal{I}_x)_x, \{\mathcal{I}_x\}_x) = 0.
\]
Using the exact sequence
\[
0 \to \mathcal{O}_x \to \mathcal{O}_x^{\mathbb{G}_m} \to \{\mathcal{I}_x\}_x \to 0
\]
for \( i \geq 2 \) [Ko section V.3, formula 3.20]. Using (16) again, we obtain an exact sequence
\[
\cdots \to \text{Ext}^2_{\mathcal{O}_x}((\mathcal{I}_x)_x, \mathcal{O}_x^{\mathbb{G}_m}) \to \text{Ext}^2_{\mathcal{O}_x}((\mathcal{I}_x)_x, \{\mathcal{I}_x\}_x) \to \text{Ext}^3_{\mathcal{O}_x}((\mathcal{I}_x)_x, \mathcal{O}_x) \to \cdots
\]
which shows that \( \text{Ext}^2_{\mathcal{O}_x}((\mathcal{I}_x)_x, \{\mathcal{I}_x\}_x) = 0 \), too.

(2) Corollary 3.13 shows that \( \mathcal{V}_\varepsilon \) is well-defined. The injectivity of \( \mathcal{V}_\varepsilon \) follows from Proposition 3.10, taking into account that \( \mathcal{E}_p, \mathcal{E}_{p'} \) are non-isomorphic when \( p \neq p' \) and \( |\text{deg}_p(p)|, |\text{deg}_p(p')| \) are sufficiently small.

We prove that \( \mathcal{V}_\varepsilon \) is holomorphic. Let \( U \subset X \) be a contractible, Stein open subset of \( X \), \( \sigma \) be a trivialization of \( \mathcal{L}_U := \mathcal{L}|_{\mathbb{P} \times U} \), and let \( \eta^\sigma : \mathcal{E}_U \to \mathcal{O}_{\mathbb{P} \times U} \) be the associated epimorphism. On the product \( \mathbb{P} \times U \times X \) we consider the following sheaves:
\[
\begin{align*}
\mathcal{U}^\sigma & := \mathcal{p}_{\mathbb{P} \times X}^*(\mathcal{E}), \\
\mathcal{U}_{\mathbb{P} \times U}^\sigma & := \mathcal{p}_{\mathbb{P} \times U}^*(\mathcal{E}_U).
\end{align*}
\]
Denoting by \( \Delta_{\mathbb{P} \times U} \approx U \) the graph of the inclusion map \( U \to X \), we get an obvious identification
\[
\mathcal{U}^\sigma|_{\mathbb{P} \times \Delta_{\mathbb{P} \times U}} = \mathcal{U}^\sigma|_{\mathbb{P} \times \Delta_{\mathbb{P} \times U}},
\]
so an obvious epimorphism
\[
\mathcal{w}^\sigma : \mathcal{U}^\sigma|_{\mathbb{P} \times \Delta_{\mathbb{P} \times U}} \to \mathcal{O}_{\mathbb{P} \times \Delta_{\mathbb{P} \times U}}
\]
which corresponds to \( \eta^\sigma \) via the canonical identifications. Put
\[
\mathcal{F}^\sigma := \ker \left( \mathcal{U}^\sigma \to \mathcal{U}^\sigma|_{\mathbb{P} \times \Delta_{\mathbb{P} \times U}} \xrightarrow{\mathcal{w}^\sigma} \mathcal{O}_{\mathbb{P} \times \Delta_{\mathbb{P} \times U}} \right).
\]
(17)
Since $\leftarrow \sigma$ and $O_{0} \times \Delta_{0}$ are flat over $\mathcal{P} \times U$, it follows by Lemma 4.4 proved below that $\mathcal{F}^{\sigma}$ is also flat over $\mathcal{P} \times U$, and that the restriction $\mathcal{F}^{\sigma}_{p,u}$ of $\mathcal{F}^{\sigma}$ to a fibre \{(p, u)\} \times X (regarded as a sheaf on X) is just the kernel of the composition

$\mathcal{E}_{p} \rightarrow \mathcal{E}_{p}(u) \rightarrow \mathcal{C}$.

Therefore one has

$[\mathcal{F}^{\sigma}_{p,u}] = V_{c}(p, u) \forall (p, u) \in \mathfrak{A}_{x} \times U$.

Since $\mathcal{F}^{\sigma}$ is flat over $\mathcal{P} \times U$, this proves that $V_{c}$ is holomorphic on $\mathfrak{A}_{x} \times U$. Using a covering of $X$ by Stein, contractible open subsets, we see that $V_{c}$ is holomorphic on $\mathfrak{A}_{x} \times X$.

We will show now that $V_{c}$ is an immersion. The tangent space $T_{[\mathcal{F}]} \mathcal{M}^{si}$ can be identified with the kernel $\text{Ext}_{0}^{1}(\mathcal{F}, \mathcal{F})$ of the trace map $\text{Ext}^{1}(\mathcal{F}, \mathcal{F}) \rightarrow H^{1}(X, O_{X})$ [10, section 3]. The local-global spectral sequence gives the exact sequence

$0 \rightarrow H^{1}(\text{Hom}_{0}(\mathcal{F}, \mathcal{F})) \rightarrow \text{Ext}_{0}^{1}(\mathcal{F}, \mathcal{F}) \rightarrow H^{0}(\text{Ext}^{1}(\mathcal{F}, \mathcal{F})) \rightarrow 0$,

where, on the right, we took into account that $H^{2}(\text{Hom}(\mathcal{F}, \mathcal{F})) = 0$. The image of $J$ is the subspace of $\text{Ext}_{0}^{1}(\mathcal{F}, \mathcal{F})$ consisting of trace-free extension classes of $\mathcal{F}$ by $\mathcal{F}$, which are locally split. The morphism $R$ has a simple geometric interpretation: If

$0 \rightarrow \mathcal{F} \xrightarrow{a} \mathcal{F}' \xrightarrow{b} \mathcal{F} \rightarrow 0$

represents an extension class $\varepsilon \in \text{Ext}_{0}^{1}(\mathcal{F}, \mathcal{F})$ then, for any $x \in X$, $R(\varepsilon)(x) \in \text{Ext}_{0}^{1}(\mathcal{F}_{X}, \mathcal{F}_{X})$ is the extension class of the $O_{x}$-module exact sequence

$0 \rightarrow \mathcal{F}_{x} \xrightarrow{a_{x}} \mathcal{F}'_{x} \xrightarrow{b_{x}} \mathcal{F}_{x} \rightarrow 0$.

The injectivity of the tangent map $D_{p,x}V_{c} : T_{(p, x)}(\mathcal{P} \times X) \rightarrow T_{[\mathcal{F}]} \mathcal{M}^{si} = \text{Ext}_{0}^{1}(\mathcal{F}, \mathcal{F})$ follows from Lemma 4.3 below.

Finally we prove that, for sufficiently small $\varepsilon > 0$ the map $V_{c}$ is homeomorphic on its image. Put

$\Delta_{c} := \{l \in \text{Pic}^{T_{c}} | \text{deg}_{g}(l) \in [-\varepsilon, \varepsilon]\}$, $\mathfrak{A}_{x} := \Delta_{c}/l \rightarrow l'$, and let $V_{c} : \mathfrak{A}_{x} \times X \rightarrow \mathcal{M}^{si}$ be the map defined again by

$V_{c}(p, x) := [\mathcal{F}(\mathcal{E}_{p}, x, \eta)]$,

for an admissible epimorphism $\eta : \mathcal{E}_{p}(x) \rightarrow \mathcal{C}$. It is easy to see that, for sufficiently small $\varepsilon > 0$, the map $V_{c}$ is continuous, and injective. Moreover, its image $V_{c}(\mathfrak{A}_{x} \times X)$ is Hausdorff. This follows using the non-separability criterion explained in section 6.4 and noting that, for two distinct pairs $(p, x) \neq (p', x')$ with $p, p' \in \mathfrak{A}_{x}$, one has $\text{Hom}(\mathcal{F}(\mathcal{E}_{p}, x, \eta), \mathcal{F}(\mathcal{E}_{p'}, x', \eta')) = 0$. Therefore $V_{c}$ induces a continuous bijection $\mathfrak{A}_{x} \times X \rightarrow V_{c}(\mathfrak{A}_{x} \times X)$ from a compact space to a Hausdorff space. By a well-known result in topology, this bijection is a homeomorphism. It suffices to note that $V_{c}$ is the restriction of $V_{c}$ to $\mathfrak{A}_{x} \times X$.

Lemma 4.3. (1) The partial derivative $\frac{\partial V_{c}}{\partial p} : T_{p, \mathcal{P}} \rightarrow \text{Ext}_{0}^{1}(\mathcal{F}, \mathcal{F})$ factorises as

$\frac{\partial V_{c}}{\partial p} = J \circ h$, where $h : T_{p, \mathcal{P}} \rightarrow H^{1}(\text{Hom}_{0}(\mathcal{F}, \mathcal{F}))$ is an isomorphism.

(2) The composition $R \circ \frac{\partial V_{c}}{\partial x} : T_{x}X \rightarrow H^{0}(\text{Ext}^{1}(\mathcal{F}, \mathcal{F}))$ is injective.
Proof. The definition of $\mathcal{F}^\sigma$ \((17)\) gives the exact sequence
\[
0 \to \mathcal{F}^\sigma \to \mathcal{O}_{\mathfrak{P} \times \Delta_{UX}} \to 0,
\]
where $w_{\sigma}$ stands for the composition $U \times \mathcal{O}_E \to U \times \mathcal{O}_{\mathfrak{P} \times \Delta_{UX}} \xrightarrow{w_{\sigma}} \mathcal{O}_{\mathfrak{P} \times \Delta_{UX}}$. The direct sum decomposition \((19)\) shows that
\[
\mathcal{F}^\sigma|_{\mathfrak{P} \times U \times U} = \mathcal{O}_{\mathfrak{P} \times U} \oplus \mathcal{I}_{\mathfrak{P} \times \Delta_{UX}},
\]
where $\Delta_{UX}$ is the graph of $\text{id}_U$.

(1) Let $(p, x) \in \mathfrak{P} \times U$, and put $\mathcal{E} := \mathcal{E}_p$, $\mathcal{F} := \mathcal{F}^\sigma_{(p, x)} = \mathcal{F}(\mathcal{E}_p, x, \eta_p, p, x)$. The direct sum decomposition \((19)\) shows that
\[
\mathcal{F}^\sigma|_{\mathfrak{P} \times \{x\} \times U} = \rho_p^\mathcal{P} (\mathcal{O}_U \oplus \mathcal{I}_x),
\]
hence this restriction can be regarded as a constant family of sheaves on $U$ parameterized by $\mathfrak{P} \times \{x\}$. Taking into account the geometric interpretation of the morphism $R$, this shows that, for every $p \in \mathfrak{P}$ and for every $\xi \in T_p(\mathfrak{P})$ one has
\[
R\left(\frac{\partial V_{\xi}}{\partial p}(\xi, 0)\right) = R_{\xi}(\mathcal{F}^\sigma) = 0,
\]
so $\frac{\partial V_{\xi}}{\partial p}$ factorizes as $\frac{\partial V_{\xi}}{\partial p} = J \circ h$, for a morphism $h : T_p \mathfrak{P} \to H^1(\text{Hom}_0(\mathcal{F}, \mathcal{F}))$. We will prove that $h$ is an isomorphism. Note first that the obvious morphisms
\[
\theta : \text{Ext}^1(\mathcal{F}, \mathcal{F}) \to \text{Ext}^1(\mathcal{F}, \mathcal{E}), \quad \tau : \text{Ext}^1(\mathcal{E}, \mathcal{E}) \to H^1(\text{Hom}(\mathcal{E}, \mathcal{E})) \to \text{Ext}^1(\mathcal{F}, \mathcal{E})
\]
associated with the embedding $\mathcal{F} \subset \mathcal{E}$ induce morphisms
\[
\theta_0 : H^1(\text{Hom}(\mathcal{F}, \mathcal{F})) \to H^1(\text{Hom}(\mathcal{F}, \mathcal{E})), \quad \tau_0 : H^1(\text{Hom}(\mathcal{E}, \mathcal{E})) \to H^1(\text{Hom}(\mathcal{F}, \mathcal{E})),
\]
which (taking into account the exact sequence \((13)\) are isomorphisms. Let now $v \in T_p(\mathfrak{P})$, and $w = (v, 0) \in T_{(p, x)}(\mathfrak{P} \times X)$. By Lemma \(4.4\) below, we obtain
\[
\theta_0(h(v)) = \tau_0(\epsilon_{(v, 0)}(U Y)) = \tau_0(\epsilon_{\mathcal{E}}(h)).
\]
On the other hand, by Proposition \(6.8\) in the appendix, it follows that $\epsilon_{\mathcal{E}}(h) \neq 0$ if $v \neq 0$. This proves that $h$ is injective. Taking into account that $\text{dim}(T_p(\mathfrak{P})) = \text{dim}(H^1(\text{Hom}_0(\mathcal{F}, \mathcal{F}))) = 1$, the statement (1) follows.

(2) Using \((19)\) we obtain an identification $\mathcal{F}^\sigma|_{\{p\} \times U \times U} = \mathcal{O}_{\{p\} \times U \times U} \oplus \mathcal{I}_{\{p\} \times \Delta_{UX}}$. By Proposition \(6.9\) proved in the appendix we obtain for a tangent vector $\eta \in T_x X$
\[
R\left(\frac{\partial V_{\eta}}{\partial x}(\eta)\right) = R(\epsilon_{(0, \eta)}(\mathcal{F}^\sigma)) = \epsilon_{\partial}(\mathcal{F}^\sigma) = \eta,
\]
which proves (2).

\section*{Lemma 4.4.}
Let $S, X$ be complex manifolds with $X$ compact, and let
\[
0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{Q} \to 0
\]
be a short exact sequence of coherent sheaves on $S \times X$, where $\mathcal{E}$ and $\mathcal{Q}$ are flat over $S$. We denote by $\mathcal{E}_s, \mathcal{F}_s$ the sheaves on $X$ given by $\mathcal{E}|_{\{s\} \times X}, \mathcal{F}|_{\{s\} \times X}$. Then
\begin{enumerate}
\item $\mathcal{F}$ is flat over $S$, and for any $s \in S$ the morphism $j_s : \mathcal{F}_s \to \mathcal{E}_s$ induced by $j$ is injective,
\end{enumerate}
(2) For any \( s \in S \) and any tangent vector \( w \in T_xS \) one has
\[
\theta(\epsilon_w(\mathcal{F})) = \tau(\epsilon_w(\mathcal{E})),
\]
where \( \theta : \text{Ext}^1(\mathcal{F}_s, \mathcal{F}) \to \text{Ext}^1(\mathcal{F}_x, \mathcal{E}_x) \), \( \tau : \text{Ext}^1(\mathcal{E}_s, \mathcal{E}_x) \to \text{Ext}^1(\mathcal{F}_s, \mathcal{E}_s) \)
are the morphisms induced by the monomorphism \( j_s : \mathcal{F}_s \to \mathcal{E}_s \).

**Proof.** (1) Using the short exact sequence (20) and the flatness of \( \mathcal{E} ', \mathcal{E} \) over \( S \), it follows that for any \( (s, x) \in S \times X \) and any \( \mathcal{O}_S \)-module \( \mathcal{A} \), one has \( \text{Tor}_k(\mathcal{F}(s, x), \mathcal{A}) = 0 \) for \( k > 0 \). Therefore \( \mathcal{F} \) is flat over \( S \). The injectivity of \( j_s \) follows from a well-known flatness criterion [Mat, p. 150].

(2) We may suppose that \( S \subset \mathbb{C} \) is a disk centered at \( 0 \), \( s = 0 \) and \( w = \frac{d}{dx} \). The fibre \( X_0 \) is a divisor in \( Y := S \times X \). Tensoring the short exact sequence
\[
0 \to \mathcal{O}_{X_0} \xrightarrow{z'} \mathcal{O}_{2X_0} \to \mathcal{O}_{X_0} \to 0
\]
with the \( \mathcal{O}_S \)-flat sheaves \( \mathcal{F}, \mathcal{E}, \mathcal{L} \) we obtain the following the commutative diagram with exact rows and columns
\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{F}_{X_0} & \xrightarrow{z'} & \mathcal{F}_{2X_0} & \xrightarrow{p'} & \mathcal{F}_{X_0} & \to & 0 \\
& & \downarrow{j_0} & & \downarrow{j_0} & & \downarrow{j_0} & & \\
0 & \to & \mathcal{E}_{X_0} & \xrightarrow{z''} & \mathcal{E}_{2X_0} & \xrightarrow{p''} & \mathcal{E}_{X_0} & \to & 0 \\
& & \downarrow{q_0} & & \downarrow{q_0} & & \downarrow{q_0} & & \\
0 & \to & \mathcal{L}_{X_0} & \xrightarrow{\tilde{z}} & \mathcal{L}_{2X_0} & \xrightarrow{\tilde{p}} & \mathcal{L}_{X_0} & \to & 0 \\
& & 0 & & 0 & & 0 & & \\
\end{array}
\]
The morphisms \( z', z'', \tilde{z} \) are induced by multiplication with \( z \), and \( p', p'', \tilde{p} \) are restriction morphisms. The infinitesimal deformations \( \epsilon_w(\mathcal{F}), \epsilon_w(\mathcal{E}) \) are the extension classes of the first two horizontal rows. By definition, \( \tau(\epsilon_w(\mathcal{E})) \) is the extension class of
\[
0 \to \mathcal{E}_{X_0} \xrightarrow{\epsilon''} p''^{-1}(\mathcal{F}_{X_0}) \xrightarrow{p''} \mathcal{F}_{X_0} \to 0,
\]
and \( \theta(\epsilon_w(\mathcal{F})) \) is the extension class of
\[
0 \to \mathcal{E}_{X_0} \to \mathcal{F}_{2X_0} \xrightarrow{j_0} \mathcal{E}_{X_0} \xrightarrow{p' \circ p''} \mathcal{F}_{X_0} \to 0.
\]
Let \( k : \mathcal{F}_{2X_0} \oplus \mathcal{E}_{X_0} \to \mathcal{E}_{2X_0} \) be the morphism given by \( k(x, y) = J_0(x) - z''(y) \). A simple diagram chasing shows that \( \ker(k) = (z' \oplus j_0)(\mathcal{F}_{X_0}) \) and \( \im(k) = p''^{-1}(\mathcal{F}_{X_0}) \). Therefore \( k \) induces an isomorphism
\[
\mathcal{F}_{2X_0} \oplus \mathcal{E}_{X_0} \xrightarrow{k} \mathcal{E}_{2X_0} \xrightarrow{p''^{-1}(\mathcal{F}_{X_0})}.
\]
It also defines an isomorphism between the extensions (21), (22).

### 4.2. Topological properties of the map \( V_2 \).

We will need the following notations:
\[
\mathcal{M}_{\text{sing}}^\text{si} := \{ [\mathcal{F}] \in \mathcal{M}^\text{si} | \mathcal{F} \text{ is singular} \},
\mathcal{M}_{\text{lf}}^\text{si} := \{ [\mathcal{F}] \in \mathcal{M}^\text{si} | \mathcal{F} \text{ is locally free} \},
\mathcal{M}_{\text{st}}^\text{si} = \{ [\mathcal{F}] \in \mathcal{M}^\text{si} | \mathcal{F} \text{ is stable} \},
\mathcal{M}_{\text{reg}}^\text{lf} = \{ [\mathcal{E}] \in \mathcal{M}_{\text{lf}}^\text{si} | H^2(\mathcal{E} \otimes \mathcal{O}(\mathcal{E})) = 0 \}.
\]
Note that $\mathcal{M}_{\text{sing}}^s$ is an analytic set of $\mathcal{M}^s$, $\mathcal{M}_{\text{reg}}^s$ is its (Zariski open) complement, and $\mathcal{M}_{\text{reg}}^s (\mathcal{M}_{\text{reg}}^s)$ is a Hausdorff open subset of $\mathcal{M}_{\text{reg}}^s$ which can be identified with the moduli space $\mathcal{M}^q(E)$ (respectively $\mathcal{M}^q(E)_{\text{reg}}$ introduced in the section [1].

The first result of this section shows that the map $V_\varepsilon$ has an important geometric interpretation: for a point $p_0 \in \mathfrak{A}_c$ (with $\varepsilon$ sufficiently small) any small singular simple deformation of $F(p_0, x_0, \eta_0)$ can be obtained by deforming the triple $(p_0, x_0, \eta_0)$, so any such deformation remains in the image of $V_\varepsilon$.

**Proposition 4.5.** Let $(X, g)$ be a class VII surface endowed with a Gauduchon metric. For sufficiently small $\varepsilon > 0$ the map $\mathfrak{A}_c \times X \to \mathcal{M}_{\text{sing}}^s$ induced by $V_\varepsilon$ is open.

**Proof.** Let $(p_0, x_0) \in \mathfrak{A}_c \times X$, $\eta_0 : \mathcal{E}_s(x_0) \to \mathbb{C}$ be an admissible epimorphism, and let $\mathcal{H}$ be a sheaf on $B \times X$, flat over $B$ (where $B \subset \mathbb{C}^4$ is the standard ball), with an identification $\mathcal{H}_0 = F(p_0, x_0, \eta_0)$, which is a universal deformation of $F(p_0, x_0, \eta_0)$ in the category of simple, torsion-free sheaves with trivial determinant and $c_2 = 1$. Note that $\mathcal{M}^s$ is smooth of dimension 4 at $[F(p_0, x_0, \eta_0)]$ by regularity and the Riemann-Roch theorem [10]. Put

\[ S := \{ s \in B | \mathcal{H}_s \text{ is singular} \}, \]

let $\mathscr{F}$ be the restriction of $\mathcal{H}$ to $S \times X$ and, for $s \in S$, denote by $\mathcal{S}_s$ the restriction of $\mathscr{F}$ to $\{s\} \times X$ (which coincides with $\mathcal{H}_s$). It suffices to prove that there exists an open neighbourhood $U$ of 0 in $S$, and a holomorphic map $f : U \to \mathfrak{A}_c \times X$ such that $f(0) = (p_0, x_0)$, and for any $s \in U$ the isomorphism type of $\mathcal{S}_s$ coincides with $V_\varepsilon(f(s))$. By Proposition [3.9] and [HL] Lemma 9.6.1 it follows that $\mathscr{F}_s$ is locally free, and for any $s \in S$ the induced morphism $\mathscr{F}_s'|_{\{s\} \times X} \to \mathscr{F}'|_{\{s\} \times X}$ coincides with the canonical embedding $\mathcal{S}_s \to \mathcal{S}_s'$, in particular it is a monomorphism. Using the short exact sequence

\[ 0 \to \mathscr{F} \to \mathscr{F}' \to \mathscr{F}'|_{\mathscr{F}} \to 0 \]

and a well-known flatness criterion [Mat, p. 150], we see that $\mathcal{D} := \mathcal{F}'|_{\mathcal{F}}$ is flat over $S$, and for any $s \in S$, the sequence

\[ 0 \to \mathcal{S}_s \to \mathcal{S}_s' \to \mathcal{S}_s' \to 0 \]

is exact. Using again Proposition [3.9] it follows that for any $s \in S$

1. there exists a unique point $\psi(s) \in X$ such that $\mathcal{Q}_s \simeq \mathcal{O}(\psi(s))$,
2. $\mathcal{F}_s'$ is a flat family of locally free sheaves on $X$ with trivial determinant and $c_2 = 0$.

But the map $x \mapsto \mathcal{O}(x)$ defines a biholomorphism between $X$ and the Douady moduli space of length 1 quotient sheaves of $\mathcal{O}_X$. Therefore the flatness of $\mathcal{D}$ over $S$ shows that the map $\psi : S \to X$ is holomorphic. On the other hand (supposing $\varepsilon$ is sufficiently small) the family $\mathcal{F}$ is versal at $p_0$ by Corollary [3.8] it follows that there exists an open neighbourhood $U$ of 0 in $S$, a holomorphic map $\varphi : U \to \mathfrak{A}_c$, and an isomorphism

\[ \mathcal{F}_s'|_{U \times X} \simeq (\varphi \times \text{id}_X)^*(\mathfrak{D}). \]

Therefore, with these notations, we see that for any $s \in S$, the sheaf $\mathcal{S}_s$ fits in an exact sequence

\[ 0 \to \mathcal{S}_s \to \mathcal{E}_s \xrightarrow{r_s} \mathcal{O}(\psi(s)) \to 0. \]

Since $\mathcal{S}_s$ is simple, it follows easily that $r_s$ is induced by an admissible epimorphism $\mathcal{E}_s(\psi(s)) \to \mathbb{C}$. Therefore the isomorphism type of $\mathcal{S}_s$ is $V(\varphi(s), \psi(s))$. 

\[ \square \]
Proposition 4.5 can be reformulated as follows:

**Corollary 4.6.** In the conditions of Proposition 4.5 the following holds: For any pair $(p_0, x_0) \in \mathfrak{A}_c \times X$, and for any neighbourhood $N$ of $(p_0, x_0)$, the point $V_c(p_0, x_0)$ has a neighbourhood $U$ such that $U \setminus V_c(N) \subset M^\infty$.

We will need the following stronger version of this statement:

**Proposition 4.7.** In the conditions of Proposition 4.5 the following holds: For any pair $(p_0, x_0) \in \mathfrak{A}_c \times X$, and for any neighbourhood $N$ of $(p_0, x_0)$, the point $V_c(p_0, x_0)$ has a neighbourhood $U$ such that $U \setminus V_c(N) \subset M^\infty$.

For the proof we will need the following simple general results concerning filtrable $SL_p^2, C_q$-bundles with $c_2 = 1$ on class VII surfaces. Let $X$ be a class VII surface, and put $b := b_2(X)$. The intersection form $q_X$ of $X$ is negative definite. Using the first Donaldson theorem [D2] and the fact that $c_1(K_X)$ is a characteristic element for $q_X$, one obtains (see [Te4], [Te5]) a basis $e_1, \ldots, e_b$ of $H^2(X, \mathbb{Z})/\text{Tors}(H^2(X, \mathbb{Z}))$ such that $q_X(e_i, e_j) = -\delta_{ij}$, $c_1(K_X) + \text{Tors} = \sum_{i=1}^b c_i$.

Using these classes, the filtrable $SL(2, C)$-bundles with $c_2 = 1$ on $X$ can be easily classified as follows.

**Lemma 4.8.** Let $E$ be a filtrable $SL(2, C)$-bundle with $c_2 = 1$ on $X$, and $q : E \to M$ be an epimorphism on a torsion-free coherent sheaf of rank 1. Then one of the following holds

1. $M$ is locally free, there exists $i \in \{1, \ldots, b\}$ such that $c_1(M) + \text{Tors} \in \{\pm e_i\}$, and $\text{ker}(q) \cong M'$. Therefore in this case $E$ fits in a short exact sequence

   $$0 \to M' \to E \to M \to 0.$$

   with $c_1(M) + \text{Tors} \in \{\pm e_i\}$.

2. There exists a line bundle $L$ on $X$ with torsion Chern class, and a point $x \in X$ such that

   (a) The pair $(x, L \otimes K_X)$ satisfies the Cayley-Bacharach condition: any section of $L \otimes K_X$ vanishes at $x$.

   (b) $M \cong L \otimes \mathcal{I}_x$, and $\ker(q) \cong L^\ast$.

   Therefore in this case $E$ fits in a short exact sequence

   $$0 \to L^\ast \to E \to L \otimes \mathcal{I}_x \to 0.$$

Recall that any filtrable locally free sheaf of rank 2 on a complex surface admits an epimorphism onto a torsion-free coherent sheaf of rank 1, hence Lemma 4.8 classifies all filtrable $SL(2, C)$-bundles with $c_2 = 1$ on class VII surfaces. Lemma 4.8 shows that

**Lemma 4.9.** Let $X$ be a class VII surface. The set

$$\{c_1(M^\infty) | M \text{ is a rank 1 torsion-free quotient of an } SL(2, C)-\text{bundle } E \text{ with } c_2 = 1 \text{ on } X\}$$

is finite.
Proof: (of Proposition 4.10) As in the proof of Proposition 4.9 let \((p_0, x_0) \in \mathbb{A}_x \times X, \eta_0 : \mathcal{E}_p(x_0) \to \mathbb{C}\) be an admissible epimorphism, and let \(\mathcal{E}\) be a sheaf on \(B \times X\), flat over \(B\) (where \(B \subset \mathbb{C}^4\) is the standard ball), with an identification \(\mathcal{H}_0 = \mathcal{F}(p_0, x_0, \eta_0)\), which is a universal deformation of \(\mathcal{F}(p_0, x_0, \eta_0)\) in the category of simple, torsion-free sheaves with trivial determinant and \(c_2 = 1\). Taking into account Corollary 4.6 the conclusion of Proposition 4.7 reduces to the

Claim: For sufficiently small \(\alpha > 0\) one has \(\{|[H_x]| z \in B_\alpha|\} \subset M_{\text{st}}^\alpha\).

Suppose by reductio ad absurdum that this does not hold. Then there would exist a sequence \((z_n)\) in \(B\) such that \(\lim_{n \to \infty} z_n = 0\), and for any \(n \in \mathbb{N}\) one has

1. \([H_{z_n}] \notin V_\epsilon(N),\)
2. \([H_n] is not stable.

By Corollary 4.6 the sheaf \(H_{z_n}\) will be locally free for any sufficiently large \(n\). We may assume that this is the case for any \(n\). Since \(H_{z_n}\) is not stable, there exists a line bundle \(L_n\) with \(\deg(L_n) \leq 0\) and a non-trivial morphism \(u_n : H_{z_n} \to L_n\). We may of course assume that \(L_n\) is “minimal”, i.e. it coincides with the bidual \(\text{im}(u_n)^{\alpha}\).

By Lemma 4.10 below it follows that the sequence \((\deg_g(L_n))\) is also bounded from below. Using Lemma 4.9 (which applies, because \(H_{z_n}\) are locally free) it follows that \((L_n)\) is a sequence of a compact subset of \(\text{Pic}(X)\), hence, passing again to a subsequence if necessary, we may assume that this sequence converges to a point \([L_X] \in \text{Pic}(X)\). We will have \(\deg_g(L_X) \leq 0\), and, using Flenner’s semi-continuity theorem (see Proposition 6.10), it follows \(\text{Hom}(H_0, L_X) \neq 0\). Since \(H_0 = \mathcal{F}(p_0, x_0, \eta_0)\), choosing a representative \(l_0 \in p_0\) and using Remark 3.14 we obtain \(\text{Hom}(L_{l_0}, L_X) \neq 0\), or \(\text{Hom}(L_{l_0}, L_X) \neq 0\). But \(\deg_g(L_{l_0} \otimes L_X) \leq \varepsilon\), \(\deg_g(L_{l_0} \otimes L_X) \leq \varepsilon\). Choosing \(\varepsilon < \varepsilon(g)\) it will follow by Remark 3.9 that \(L_X \simeq L_{l_0}\), or \(L_X \simeq L_{l_0}\), so in particular \([L_X] \in \text{Pic}^\alpha\). But this implies that for any sufficiently large \(n \in \mathbb{N}\) one has \([L_n] \in \text{Pic}^\alpha\). Since \(H_n\) is locally free, Lemma 4.8 shows that for any (sufficiently large \(n\)) there exists \(x_n \in X\) such that \(\text{Hom}(H_n, L_{l_0} \otimes L_{x_n}) \neq 0\). Using the compactness of \(X\), we get a point \(x_\infty \in X\) such that \(\text{Hom}(H_0, L_{x_\infty} \otimes L_{x_\infty}) \neq 0\). But it is easy to see that (supposing \(|\deg_g(l_0)| is sufficiently small) one has

\[\text{Hom}(\mathcal{F}(p_0, x_0, \eta_0), L_{l_0} \otimes L_x) = \text{Hom}(\mathcal{F}(p_0, x_0, \eta_0), L_{l_0} \otimes L_x) = 0 \forall x \in X.\]

Therefore we obtained a contradiction, which completes the proof.

Lemma 4.10. Let \((X, g)\) be a compact Gauduchon surface, \(M\) be a complex manifold and \(\mathcal{E}\) be a sheaf on \(M \times X\), flat over \(M\). For any compact subset \(K \subset M\) the set

\[\{|\deg_g(L)| \exists u \in K\text{ such that }\text{Hom}(\mathcal{E}_u, L) \neq 0\}\]

is bounded from below.

Proof. Using [10 Proposition 3.1] it follows that there exists a sheaf epimorphism \(q : \mathcal{F} \to \mathcal{E}\), where \(\mathcal{F}\) is a locally free sheaf on \(M \times X\). Therefore, for any \(u \in M\) we get an epimorphism \(q_u : \mathcal{F}_u \to \mathcal{E}_u\), where \(\mathcal{F}_u\) is locally free. Let \(h\) be a Hermitian metric on \(\mathcal{F}\), \(h_u\) the induced metric on \(\mathcal{F}_u\), and for any line bundle \(L\) on \(X\), let \(h_L\) be a Hermitian-Einstein metric on \(L\). We denote by \(A_u, A_L\) the Chern connections associated with \(h_u, h_L\). One has

\[F_{A_u \otimes A_L} = F_{A_u} \otimes \text{id}_L + \text{id}_{F_u} \otimes F_{A_L} = F_{A_L} \otimes \text{id}_L + F_{A_u} \text{id}_{F_u} \otimes L\]
Since \( K \) is compact, and \( i\Delta F_{\Lambda} = c_{\Lambda} \), where the Einstein constant \( c_{\Lambda} \) is proportional to \( \deg_{\mathcal{L}}(\mathcal{L}) \), it follows that there exists \( d_K \in \mathbb{R} \) such that \( i\Delta_g(F_{\Lambda \otimes \Lambda}) \) is negative definite for any \( u \in K \) and any line bundle \( \mathcal{L} \) with \( \deg(\mathcal{L}) < d_K \). Using a well-known vanishing theorem \([30]\), we obtain \( \text{Hom}(\mathcal{F}_u, \mathcal{L}) = 0 \) for any \( u \in K \) and line bundle \( \mathcal{L} \) with \( \deg(\mathcal{L}) < d_K \). It suffices to note that \( \text{Hom}(\mathcal{F}_u, \mathcal{L}) = 0 \) implies \( \text{Hom}(\mathcal{E}_u, \mathcal{L}) = 0 \).

Put \( \mathcal{V}_\varepsilon := \mathcal{V}_\varepsilon(\mathfrak{A}_\varepsilon \times X) \).

**Corollary 4.11.** Let \((X, g)\) be a class VII surface endowed with a Gauduchon metric. For sufficiently small \( \varepsilon > 0 \) there exists a Hausdorff open neighborhood \( U \) of \( \mathcal{V}_\varepsilon \) in \( \mathcal{M}^i \) such that

1. \( \mathcal{V}_\varepsilon \) is closed in \( \mathcal{U} \),
2. \( \mathcal{U} \setminus \mathcal{V}_\varepsilon \subset \{ \mathcal{M}^i_{\text{reg}} \} \).

**Proof.** Using Propositions 4.2 and 4.7 every point \( e \in \mathcal{V}_\varepsilon \) has a neighbourhood \( \mathcal{U}_e \) in \( \mathcal{M}^i_{\text{reg}} \) such that \( \mathcal{V}_\varepsilon \cap \mathcal{U}_e \) is closed in \( \mathcal{U}_e \) and \( \mathcal{U}_e \setminus \mathcal{V}_\varepsilon \subset \mathcal{M}^i_{\text{reg}} \). It suffices to put

\[
\mathcal{U} := \left( \bigcup_{e \in \mathcal{V}_\varepsilon} \mathcal{U}_e \right) \cap \mathcal{N},
\]

where \( \mathcal{N} \) is a Hausdorff neighbourhood of \( \mathcal{V}_\varepsilon(\mathfrak{A}_\varepsilon \times X) \) (see Proposition 6.11).

5. **Extending the complex structure**

Let \((X, g)\) be a class VII surface endowed with a Gauduchon metric. We have seen that, for a pair \((p, x) \in (\mathfrak{P} \setminus \mathfrak{E}) \times X\) and an admissible epimorphism \( \eta : E(x) \to \mathbb{C} \), the sheaf \( \mathcal{F}(p, x, \eta) \) is not even slope semistable, because it is destabilised by its subsheaf \( \mathcal{L} \otimes L_x \), where \( l = [L] \) is the unique representative of \( p \) with negative degree. We will extend the holomorphic structure of \( \mathcal{M}^i(E) \) to its Donaldson compactification using the gluing Lemma 6.16. The parametrisation \( f \) intervening in this theorem will be defined on an open neighbourhood of \( \mathcal{V}_\varepsilon \) in \( \mathcal{M}^i \), and will map an element \([\mathcal{F}(p, x, \eta)] \in \mathcal{V}_\varepsilon \) with \( |\text{deg}_g(p)| > 0 \) to the isomorphism class of a stable, locally free Serre extension. This isomorphism class is non-separable (in the Hausdorff sense) from \([\mathcal{F}(p, x, \eta)] \) in \( \mathcal{M}^i(\mathfrak{A}) \).

5.1. **Families of Serre extensions.** The family of Serre extensions we need is defined as follows:

Let \( U \) be an open set of \( X \). Denote by \( \mathcal{L}_x, \mathcal{L}_{xU} \) the restriction of the Poincaré line bundle \( \mathcal{L} \) to \( A_x \times X \), respectively \( A_x \times U \). On the product \( \Pi^U_x := A_x \times U \times X \) consider the line bundle

\[
\mathcal{M} := p_{A_x X}^*(\mathcal{L}_x) \otimes p_{A_x U}^*(\mathcal{L}_{xU}),
\]

Put \( Z := A_x \times \Delta_{U \times X} \), where \( \Delta_{U \times X} \) is the graph of the embedding \( U \hookrightarrow X \). We have canonical isomorphisms

\[
\text{Ext}^1(\mathcal{L}_Z \otimes \mathcal{M}, p_{U}^*(\mathcal{K}_U) \otimes \mathcal{M}) = \text{Ext}^1(\mathcal{L}_Z, p_{U}^*(\mathcal{K}_U) \otimes \mathcal{M}^\otimes 2) = \omega_{U, \mathcal{M}} \otimes \mathcal{O}_Z \otimes (p_{U}^*(\mathcal{K}_U) \otimes \mathcal{M}^\otimes 2) \simeq \mathcal{O}_Z.
\]

For the second isomorphism we used formula (45) proved in the appendix; for the third isomorphism we used the obvious isomorphism \( \mathcal{M}|_Z = \mathcal{O}_Z \), and the obvious
identification between the normal bundle of $Z$ and the pull-back of the tangent bundle $T_U$. The local-global spectral sequence gives an exact sequence

$$0 \to H^1(p_U^*(K_U) \otimes \mathcal{M}^{\otimes 2}) \to \text{Ext}^1(I_Z \otimes \mathcal{M}^*, p_U^*(K_U) \otimes \mathcal{M}) \xrightarrow{\text{det}} H^0(\mathcal{E}xt^1(I_Z \otimes \mathcal{M}, p_U^*(K_U) \otimes \mathcal{M})) \quad (23)$$

The cohomology spaces $H^k(p_U^*(K_U) \otimes \mathcal{M}^{\otimes 2})$ can be computed using the Leray spectral sequence associated with the projection $p_{A,U} : \Pi^U \to A_c \times U$. Using the projection formula and denoting by $q_{A_c} : A_c \times U \to A_c$, $q_U : A_c \times U \to U$, $\pi_{A_c} : A_c \times X \to A_c$ the obvious projections, we get

$$R^k(p_{A,U*})(p_U^*(K_U) \otimes \mathcal{M}^{\otimes 2}) = \mathcal{E}xt^2_q(U^*(K_U) \otimes q_A^*(R^k\pi_{A,c*}(\mathcal{L}_e^{\otimes 2}))). \quad (24)$$

Since $H^0(\mathcal{L}_e^{\otimes 2}) = 0$ for generic $l$, formula (24) shows that $p_{A,U*}(p_U^*(K_U) \otimes \mathcal{M}^{\otimes 2}) = 0$. Let now $\varepsilon > 0$ be sufficiently small such that

1. $H^2(\mathcal{L}_e^{\otimes 2}) = 0$ for any $l \in A_c$ (see Lemma 2.2).
2. $H^0(\mathcal{L}_e^{\otimes 2}) = 0$ for any $l \in A_c \setminus \rho$ (see Remark 3.6).

The first condition implies $R^2(p_{A,U*})(p_U^*(K_U) \otimes \mathcal{M}^{\otimes 2}) = 0$, and the second implies that the sheaf $R^1(p_{A,U*})(p_U^*(K_U) \otimes \mathcal{M}^{\otimes 2})$ is supported on $\rho \times U$. Therefore, if $U$ is Stein, one obtains $H^i(R^k(p_{A,U*})(p_U^*(K_U) \otimes \mathcal{M}^{\otimes 2})) = 0$ for $i + k = 2$, so $R^2(p_{U*}(K_U) \otimes \mathcal{M}^{\otimes 2}) = 0$, implying that the morphism $a_U$ in (23) is surjective. For any $\nu \in a_U^{-1}(1)$ we obtain a short exact sequence

$$0 \to p_U^*(K_U) \otimes \mathcal{M} \to \mathcal{S}^{\nu} \to I_Z \otimes \mathcal{M}^* \to 0$$
on $\Pi^U$ with locally free central term (by Serre’s Lemma, see [OSS, Lemma 5.1.2]).

The 2-bundle $S'_{(l,u)}$ on $X$ defined by the restriction of $\mathcal{S}^{\nu}$ to $\{(l,u)\} \times X$ fits in a short exact sequence

$$0 \to L_l \to S'_{(l,u)} \to I_u \otimes L_l' \to 0,$$

so it is a Serre extension of $I_u \otimes L_l'$ by $L_l$. On the other hand

**Proposition 5.1.** Let $\varepsilon > 0$ be sufficiently small such that conditions (C1), (C2) above hold. Then

1. For every $(l,x) \in (A_c \setminus \rho) \times X$ one has $\dim(\text{Ext}^1(L_l' \otimes I_x, L_l)) = 1$, and there exists a locally free sheaf $S(l,x)$, unique up to isomorphism, which fits in an exact sequence of the form

$$0 \to L_l \to S(l,x) \to I_x \otimes L_l \to 0.$$

2. Suppose that $\varepsilon < \nu(g)$. Then $S(l,x)$ is (semi)stable if and only of $\deg_S(g) < 0$ (or $\deg_S(g) \leq 0$).

**Proof.** (1) The first statement follows easily using the exact sequence

$$0 \to H^1(\mathcal{L}_e^{\otimes 2}) \to \text{Ext}^1(L_l' \otimes I_x, L_l) \to H^0(\mathcal{E}xt^1(L_l' \otimes I_x, L_l)) \to H^2(\mathcal{L}_e^{\otimes 2}) \to 0$$

and the conditions (C1), (C2).

(2) If $\deg_S(g) \geq 0$ (or $\deg_S(g) > 0$) then $S(l,x)$ is obviously non-stable (non semi-stable). Conversely, suppose that $\deg_S(g) < 0$ (or $\deg_S(g) \leq 0$). If $L_l \to S(l,x)$ is a destabilising locally free subsheaf of rank 1 of $S(l,x)$, then $H^0(L_l' \otimes L_l' \otimes I_x) \neq 0$ and $\deg_S(L_l) \geq 0$ (or respectively $\deg_S(L_l) > 0$). This implies that the line bundle
\( \mathcal{L}^* \otimes \mathcal{L}_i \) has a non-trivial section whose zero divisor \( D \) contains \( x \), in particular is non-empty. We get
\[
\text{vol}_g(D) = |\deg_g(l)| - \deg_g(\mathcal{L}) \leq |\deg_g(l)|,
\]
which contradicts the assumption \( \varepsilon < o(g) \) (see Definition 3.5).
\[ \square \]

For \( l \in \rho \) one has \( \mathcal{L}_i^* \simeq \mathcal{L}_i \), hence \( H^1(\mathcal{L}_i^* \otimes 2) \simeq H^1(\mathcal{O}_X) \simeq \mathbb{C} \). Therefore in this case the situation is different:

**Remark 5.2.** For any line bundle \( \mathcal{L} \) on \( X \) one has \( \dim(\text{Ext}^1(\mathcal{L} \otimes \mathcal{I}_x, \mathcal{L})) = 2 \), and the set of isomorphism classes of locally free sheaves \( \mathcal{S} \) which fit into an exact sequence of the form
\[
0 \to \mathcal{L} \to \mathcal{S} \to \mathcal{L} \otimes \mathcal{I}_x \to 0.
\]
can be identified with the complement of a singleton in \( \mathbb{P}(\text{Ext}^1(\mathcal{L} \otimes \mathcal{I}_x, \mathcal{L})) \).

**Corollary 5.3.** Let \( \varepsilon \) be sufficiently small such that (C1), (C2) hold, and \((l, x) \in (A_\varepsilon \rho) \times X \). Then \( S'_{(l, x)} \simeq S(l, x) \) for any Stein open set \( U \ni x \) and any \( \nu \in \mathfrak{a}_U^{-1}(1) \).

**5.2. The proof of Theorem 1.2.** For a point \( p = (l, l') \in \mathfrak{C} \), we denote by \( a_p \), the gauge class of the flat SU(2) connection \( a_l \oplus a_{l'} \), where, for \( l \in \mathfrak{C} \), \( a_l \) stands for the flat \( U(1) \)-connection associated with \( l \). For \( p \in \mathfrak{R} \times \mathfrak{C} \) we put \( S(p, x) := S(l, x) \), where \( l \) is the representative of \( p \) of negative degree. Let \( \varepsilon > 0 \) be sufficiently small such that the properties of Proposition 5.1 hold, and let \( \mathcal{U} \) be an open neighbourhood of \( \mathcal{V}_\varepsilon \) in \( \mathcal{M}_a \) satisfying the properties of Corollary 4.1. Put
\[
\mathcal{V}_0 := \{ ([\mathcal{F}] \in \mathfrak{C}, x \in X, \eta : \mathcal{E}_p(x) \xrightarrow{\text{adm. epim.}} \mathbb{C}) \}
\]

Recall that, by Proposition 2.6, the union \( \mathfrak{M} := \mathcal{M}_{\text{ASD}}(E)^* \cup (\mathcal{R}_0 \times X) \) is open in \( \mathcal{M}_{\text{ASD}}(E) \) and is a topological 8-manifold. Define \( f : \mathcal{U} \to \mathfrak{M} \) by
\[
f([\mathcal{F}]) := \begin{cases} 
[\mathcal{F}] & \text{if } [\mathcal{F}] \in \mathcal{U} \setminus \mathcal{V}_\varepsilon, \\
[S(p, x)] & \text{if } [\mathcal{F}] = V_\varepsilon(p, x) \text{ with } (p, x) \in \mathcal{V}_\varepsilon \setminus \mathcal{V}_0, \\
(a_p, x) & \text{if } [\mathcal{F}] = V_\varepsilon(p, x) \text{ with } (p, x) \in \mathcal{V}_0.
\end{cases}
\]

In the first two lines of the above formula we have used the Kobayashi-Hitchin identification \( \mathcal{M}_a^{\text{ASD}}(E)^* \simeq \mathcal{M}_a^{\text{ASD}}(E) = \mathcal{M}_a^{\text{ASD}}(E) \).

**Proposition 5.4.** Let \( (X, g) \) be a class VII surface endowed with a Gauduchon metric. If \( \varepsilon \) and \( \mathcal{U} \) are sufficiently small, the following holds:

(1) \( f \) is injective.

(2) \( f^{-1}(\mathcal{M}_a^{\text{ASD}}(E)_{\text{reg}}) = \mathcal{U} \setminus \mathcal{V}_0 \), and the map \( \mathcal{U} \setminus \mathcal{V}_0 \to \mathcal{M}_a^{\text{ASD}}(E)_{\text{reg}} \) induced by \( f \) is holomorphic.

(3) \( f \) is continuous.

**Proof.** (1) We prove that (assuming \( \varepsilon \) and \( \mathcal{U} \) sufficiently small) \( f \) is injective. The restrictions \( f|_{\mathcal{U} \setminus \mathcal{V}_\varepsilon}, f|_{\mathcal{V}_\varepsilon} \) are obviously injective, so it suffices to prove that one has \( \text{im}(f|_{\mathcal{U} \setminus \mathcal{V}_\varepsilon}) \cap \text{im}(f|_{\mathcal{V}_\varepsilon}) = \emptyset \). Since \( f|_{\mathcal{U} \setminus \mathcal{V}_\varepsilon} \subset \mathcal{U} \) it suffices to prove that (assuming \( \varepsilon \) and \( \mathcal{U} \) sufficiently small) one has
\[
\mathcal{U} \cap f(\mathcal{V}_\varepsilon) = \emptyset.
\]
The set
\[
Y := \{ (l, [\mathcal{F}]) \in \text{Pic}^T \times \mathcal{U} \mid H^0(\mathcal{L}_l \otimes \mathcal{F}) \neq 0 \}
\]
is closed in $\text{Pic}^T \times \mathcal{U}$ by Grauert’s semicontinuity theorem. Let $\text{Pic}^T_{\{0,\varepsilon\}} \subset \text{Pic}^T$ be the union of compact annuli defined by the condition $0 \leq \deg_g(l) \leq \varepsilon$. The projection

$$Z := \mathcal{P}_u(Y \cap (\text{Pic}^T_{\{0,\varepsilon\}} \times \mathcal{U}))$$

is a compact subset of $\mathcal{U}$ which is disjoint from $\mathcal{V}_\varepsilon$, because (for sufficiently small $\varepsilon > 0$) one has

$$H^0(\mathcal{L}_l \otimes \mathcal{F}(p_0, x, \eta)) = 0$$

for any $(l, p) \in \text{Pic}^T_{\{0,\varepsilon\}} \times \mathfrak{A}$ and any admissible epimorphism $\eta : E_p(x) \to \mathbb{C}$. On the other hand one has obviously $f(\mathcal{V}_\varepsilon) \subset Z$, hence, replacing $\mathcal{U}$ by $\mathcal{U}\setminus Z$ if necessary, formula (26) will hold.

(2) The equality $f^{-1}(\mathcal{M}^\delta(E)_{\text{reg}}) = \mathcal{U}\setminus \mathcal{V}_0$ is obvious taking into account the definition of $f$ and the way in which $\varepsilon$ and $\mathcal{U}$ have been chosen. We prove that the map $\mathcal{U}\setminus \mathcal{V}_0 \to \mathcal{M}^\delta(E)_{\text{reg}}$ induced by $f$ is holomorphic. The holomorphy on $\mathcal{U}\setminus \mathcal{V}_\varepsilon$ is obvious. It remains to show that $f$ is holomorphic at any point $[\mathcal{F}(p_0, x_0, \eta_0)] \in \mathcal{V}_\varepsilon \setminus \mathcal{V}_0$. Using Proposition 4.2 we can find an open neighbourhood $N$ of $(p_0, x_0)$ in $(\mathfrak{A}, \mathcal{C}) \times X$ and an open embedding $\theta : N \times D \to \mathcal{U}$ such that $\theta(p, x, 0) = V_\varepsilon(p, x)$ for any $(p, x) \in N$, where $D$ is the unit disk. It suffices to prove that $f \circ \theta$ is holomorphic on $N \times D$ or, equivalently, that $f \circ \theta$ is separately holomorphic.

The holomorphy in the $N$ direction is obvious. For the holomorphy in the $D$-direction we proceed as follows. Fix $(p, x) \in N$. The map $\theta(p, x, \cdot) : D \to \mathcal{U}$ is defined by a sheaf $\mathcal{F}$ on $D \times X$, flat over $D$ with an identification $\mathcal{F}_0 = \mathcal{F}(p, x, \eta)$. Let $l$ be the representative of $p$ of negative degree, and

$$r_{l, x, \eta} : \mathcal{F}(p, x, \eta) = \mathcal{F}_0 \to \mathcal{L}_l$$

the corresponding destabilising epimorphism. The elementary transformation of the pair $(\mathcal{F}, r_{l, x, \eta})$ is a sheaf $\mathcal{F}'$ on $D \times X$, flat over $D$, with $\mathcal{F}'_z = \mathcal{F}_z$ for $z \neq 0$ and whose restriction to $\{0\} \times X$ fits into a short exact sequence

$$0 \to \mathcal{L}_l \to \mathcal{F}'_0 \to \mathcal{L}'_l \otimes \mathcal{I}_x \to 0.$$ (see Section 6.1.2). Using Corollary 6.7 it follows that the extension class associated to this exact sequence is non-trivial, hence $\mathcal{F}'_0 \simeq \mathcal{S}(p, x)$. On the other hand, since $\mathcal{F}'$ is flat over $D$, this proves that $(f \circ \theta)(p, x, \cdot) : D \to \mathcal{U}$ is holomorphic on $D$.

(3) Using (2) it follows that $f$ is continuous on $\mathcal{U}\setminus \mathcal{V}_0$. The critical continuity at the points of $\mathcal{V}_0$ follows using the continuity theorem [BTT] as follows. Let $([\mathcal{F}_n])_n$ be a sequence in $\mathcal{U}$ with $\lim_{n \to \infty} [\mathcal{F}_n] = [\mathcal{F}_\infty] = V_\varepsilon(p, x) \in \mathcal{V}_0$. We have to prove that

$$\lim_{n \to \infty} f([\mathcal{F}_n]) = ([\mathcal{F}_{p, x}])_0.$$ (27)

Using a standard argument, we can reduce the problem to three separate cases:

1. $([\mathcal{F}_n])_n$ is a sequence of $\mathcal{V}_0$.
2. $([\mathcal{F}_n])_n$ is a sequence of $\mathcal{V}_\varepsilon \setminus \mathcal{V}_0$.
3. $([\mathcal{F}_n])_n$ is a sequence of $\mathcal{U}\setminus \mathcal{V}_\varepsilon$.

In first case the claim (27) follows easily using the fact the map $V_\varepsilon$ is homeomorphism on its image (see Proposition 4.2). In the second case one applies the continuity theorem to a family of Serre extensions $\mathcal{R}'$ associated with a Stein open set $U \ni x$ and a lift $\nu \in \mathfrak{a}^{-1}_U(1)$ (see Section 5.1). In the third case one applies the
continuity theorem \[\text{[BTT]}\] to a versal deformation of the simple, torsion-free sheaf \(F(x, x, \eta)\), where \(\eta : \mathcal{E}_{px} \to \mathbb{C}\) is an admissible epimorphism.

Theorem 1.2 stated in Section 1.2 follows now directly from the gluing Lemma 6.16.

5.3. The proof of Theorem 1.3: Let \(X\) be a complex surface, \(\mathcal{E}\) be a holomorphic bundle on \(X\), \(\pi : \mathbb{P}(\mathcal{E}) \to X\) be the projectisation of \(\mathcal{E}\), \(\mathcal{L} \subset \pi^*(\mathcal{E})\) be the tautological line bundle on \(\mathbb{P}(\mathcal{E})\), and \(\mathcal{M}\) the quotient \(\pi^*(\mathcal{E})/\mathcal{L}\). On the product \(\mathbb{P}(\mathcal{E}) \times X\) consider the bundles

\[
\mathcal{E} := p_2^*(\mathcal{E}), \quad \mathcal{L} := p_1^*(\mathcal{L}), \quad \mathcal{M} := p_1^*(\mathcal{M}).
\]

Let \(Z \subset \mathbb{P}(\mathcal{E}) \times X\) be the graph of \(\pi\), and note that, via the obvious isomorphism \(Z = \mathbb{P}(\mathcal{E})\) the restrictions \(\mathcal{E}|_Z, \mathcal{L}|_Z, \mathcal{M}|_Z\) are identified with \(\pi^*(\mathcal{E}), \mathcal{L}\) and \(\mathcal{M}\) respectively. Therefore one obtains a short exact sequence

\[
0 \to \mathcal{L}|_Z \xrightarrow{\iota} \mathcal{E}|_Z \xrightarrow{\mu} \mathcal{M}|_Z \to 0.
\]

Let \(\tilde{\mu}\) be the composition \(\mathcal{E} \to \mathcal{E}|_Z \xrightarrow{\mu} \mathcal{M}|_Z\), and put \(\mathcal{F} := \ker(\tilde{\mu})\). For a point \(x \in X\) and a line \(y \in \mathbb{P}(\mathcal{E}(x))\) put \(q_y := \mathcal{E}(x)/y\), and let \(\eta_y : \mathcal{E}(x) \to q_y\) be the canonical epimorphism. The sheaf \(F_y\) on \(X\) given by the restriction \(\mathcal{F}|_{(y,x)}\) fits into the exact sequence

\[
0 \to F_y \to E \xrightarrow{\tilde{\eta}_y} q_y \otimes O(x) \to 0,
\]

where \(\tilde{\eta}_y\) is induced by \(\eta_y\). As in the proof of Proposition 4.2 we obtain for any \(y \in \mathbb{P}(\mathcal{E})\) a canonical exact sequence

\[
0 \to \text{Hom}(F_y, F_y) \to \text{Hom}(E, E) \xrightarrow{\psi_y} (y \otimes q_y) \otimes O(x) \to 0.
\]

Lemma 5.5. For any \(x \in X\) and \(y \in \mathbb{P}(\mathcal{E}(x))\) one has

1. \(\text{Ext}^k(F_y, F_y) = 0\) for \(k \geq 2\).
2. A canonical isomorphism \(\text{Ext}^1(F_y, F_y) \xrightarrow{\delta_y} \text{Ext}^2(q_y \otimes O(x), F_y)\).
3. Canonical isomorphisms

\[
H^2(\text{End}(F_y)) \cong H^2(\text{End}(E)) , \quad H^2(\text{End}_0(F_y)) \cong H^2(\text{End}_0(E)).
\]
4. A canonical short exact sequence

\[
0 \to T(x) \to H^0(\text{Ext}^1(F_y, F_y)) \xrightarrow{\delta_y} \Omega^2(x)^{\cdot} \otimes q_y \otimes y \to 0.
\]

Proof. The first claim follows as in the proof of Proposition 4.2; (2) follows directly from (28) taking into account that \(\mathcal{E}\) is locally free, and (3) follows from (29) taking into account that \(H^k(O(x)) = 0\) for \(k > 0\). For (4) use (2), the exact sequence

\[
0 \to \text{Ext}^1(q_y \otimes O(x), q_y \otimes O(x)) \to \text{Ext}^2(q_y \otimes O(x), F_y) \to \text{Ext}^2(q_y \otimes O(x), E) \xrightarrow{\delta_y} \text{Ext}^2(q_y \otimes O(x), q_y \otimes O(x)) \to 0,
\]

and the canonical isomorphisms (see Section 6.3):

\[
\text{Ext}^1(q_y \otimes O(x), q_y \otimes O(x)) = \text{Ext}^1(O(x), O(x)) = \text{Hom}(I_x, O(x)) = T(x),
\]

\[
\text{Ext}^2(q_y \otimes O(x), E) = \text{Ext}^1(I_x, q_y \otimes E) = \Omega^2(x)^{\cdot} \otimes q_y \otimes E(x),
\]

\[
\text{Ext}^2(q_y \otimes O(x), q_y \otimes O(x)) = \text{Ext}^1(I_x, O(x)) = \Omega^2(x)^{\cdot}.
\]

The morphism \(\Omega^2(x)^{\cdot} \otimes q_y \otimes E \to \Omega^2(x)^{\cdot}\) which corresponds to \(a\) via the latter identifications is \(\text{id}_{\Omega^2(x)^{\cdot}} \otimes \text{id}_{q_y} \otimes \eta_y\), so \(\ker(a) = \Omega^2(x)^{\cdot} \otimes q_y \otimes y\).
**Lemma 5.6.** If $\mathcal{E}$ is simple, then for any $y \in \mathbb{P}(\mathcal{E})$ the sheaf $\mathcal{F}_y$ is simple, and one has a canonical short exact sequence

$$0 \to y^* \otimes q_y \to H^1(\text{Hom}_0(\mathcal{F}_y, \mathcal{F}_y)) \to H^1(\text{Hom}_0(\mathcal{E}, \mathcal{E})) \to 0.$$  

If $H^2(\text{End}_0(\mathcal{E})) = 0$, then $H^2(\text{End}_0(\mathcal{F})) = 0$, and the local-global spectral sequence yields a canonical short exact sequence

$$0 \to H^1(\text{Hom}_0(\mathcal{F}_y, \mathcal{F}_y)) \to \text{Ext}^1_y(\mathcal{F}_y, \mathcal{F}_y) \to H^0(\text{Ext}^1_y(\mathcal{F}_y, \mathcal{F}_y)) \to 0. \tag{31}$$

Suppose now that $\mathcal{E}$ is simple and $H^i(\text{End}_0(\mathcal{E})) = 0$ for $i \in \{1, 2\}$. Combining (31), (30) with Lemma 5.6, and putting $\Theta_y := m^{-1}(T(x))$, we obtain two short exact sequences

$$0 \to \Theta_y \to \text{Ext}^1_y(\mathcal{F}_y, \mathcal{F}_y) \xrightarrow{\text{hom}} \Omega^2(x)^* \otimes q_y \otimes y \to 0,$$

$$0 \to y^* \otimes q_y \to \Theta_y \xrightarrow{m|\Theta_y} T(x) \to 0. \tag{32}$$

The sheaf $\mathcal{F}$ is flat over $\mathbb{P}(\mathcal{E})$, so it defines a holomorphic map $\Phi^\mathcal{E} : \mathbb{P}(\mathcal{E}) \to \mathcal{M}$. Denote by $T^\mathcal{E}_y$ the vertical tangent subbundle of the tangent bundle $T_{\mathbb{P}(\mathcal{E})}$. It is easy to see that

**Lemma 5.7.** Suppose that $\mathcal{E}$ is simple and $H^i(\text{End}_0(\mathcal{E})) = 0$ for $i \in \{1, 2\}$. Then $\Phi^\mathcal{E}$ takes values in $\mathcal{M}^\text{reg}_{\mathcal{E}}$, and for any $y \in \mathbb{P}(\mathcal{E})$ the differential $\Phi^\mathcal{E}_y : T_{\mathbb{P}(\mathcal{E})}(y) \to \text{Ext}^1_y(\mathcal{F}_y, \mathcal{F}_y)$ at $y$ has the following properties:

1. It maps isomorphically $T_{\mathbb{P}(\mathcal{E})}(y)$ on $\Theta_y$.
2. It maps $y^* \otimes q_y$ isomorphically onto the vertical tangent line $T^\mathcal{E}_y(y)$, and induces the canonical isomorphism $y^* \otimes q_y \xrightarrow{\text{iso}} T_y(\mathbb{P}(\mathcal{E}))(x)$.

This shows that

**Proposition 5.8.** Suppose that $\mathcal{E}$ is simple and $H^i(\text{End}_0(\mathcal{E})) = 0$ for $i \in \{1, 2\}$. The map $\Phi^\mathcal{E} : \mathbb{P}(\mathcal{E}) \to \mathcal{M}$ is a codimension one holomorphic embedding whose normal line bundle $\mathcal{N}$ is isomorphic to $\pi^*(\mathcal{K} \otimes \mathcal{L} \otimes 1) \otimes \mathcal{E} \otimes (\mathcal{O}_\mathcal{E}(2))$.

We can now prove Theorem 1.3 stated in Section 1.2

**Proof.** (of Theorem 1.3) Let $\mathcal{E}$ be the holomorphic $\text{SL}(2, \mathbb{C})$-bundle associated with the irreducible, flat $\text{SU}(2)$-connection $A$. Identify $\mathbb{P}(\mathcal{E})$ with its image in $\mathcal{M}$ via $\Phi^\mathcal{E}$. Using the same methods as in the proof of Proposition 4.7 and Corollary 4.11 we obtain an open, Hausdorff neighbourhood $\mathcal{U}$ of $\mathbb{P}(\mathcal{E})$ in $\mathcal{M}^{\text{reg}}_{\mathcal{E}}$ such that $\mathcal{U} \cap \mathbb{P}(\mathcal{E}) \subset (\mathcal{M}^{\text{reg}}_{\mathcal{E}})_{\text{reg}}$. The continuity theorem [BTT] for flat families gives a continuous map $\tau : \mathcal{U} \to \mathcal{M}^{\text{ASD}}(\mathcal{E})$ which agrees with the Kobayashi-Hitchin correspondance on $\mathcal{U} \cap \mathbb{P}(\mathcal{E})$ and with $\pi$ on $\mathbb{P}(\mathcal{E})$. Choose $\mathcal{U} \subset \mathcal{U}$ such that $\mathcal{U}$ is a compact tubular neighbourhood of $\mathbb{P}(\mathcal{E})$ in $\mathcal{U}$ with smooth boundary $\mathcal{B}$. By Proposition 2.8 we may identify an open neighbourhood of $\{A\} \times X$ with the cone bundle $C^{\text{cone}}_X$. Using the continuity of $\tau$ we may suppose (taking $\mathcal{U}$ sufficiently small) that $\tau(\mathcal{U}) \subset C^{\text{cone}}_X$. By Lemma 5.9 proved below (which is a special case of [H] Lemma 4.4), the image $\mathcal{W} := \tau(\mathcal{U})$ is open in $C^{\text{cone}}_X$. Since $\tau(\mathcal{U})$ is compact, it is closed, so the obvious inclusion $\tau(\mathcal{U}) \subset \tau(\mathcal{U}) = \mathcal{W}$ is an equality.

By Proposition 5.8 the pair $(\mathcal{U}, \pi : \mathbb{P}(\mathcal{E}) \to X)$ satisfies the hypothesis of [F] Theorem 2', so there exists a normal complex space $\mathfrak{V}$ with an embedding $X \hookrightarrow \mathfrak{V}$,
and a modification $c : \mathcal{U} \to \mathcal{W}$, such that the pair $(\mathcal{W}, c)$ is the blowing down of $\mathcal{U}$ along $\pi$. Moreover, by [Tom] Theorem 3.7 it follows that $c$ coincides with the monoidal transformation of $\mathcal{W}$ with centre $c(\mathcal{P}(E)) = X$.

Put $\mathcal{V} := c(\mathcal{U})$. Since $\tau$ is constant on the fibres of $c$, it induces a continuous map $\overline{\nu} = c(\overline{\mathcal{U}}) \to \overline{\mathcal{W}}$, which is obviously bijective. But $\overline{\nu}$ is compact and $\overline{\mathcal{W}}$ is Hausdorff, so this map is a homeomorphism. Endowing $\mathcal{V}$, which is an open neighbourhood of $\{(A) \times X \in \mathcal{M}^{\text{ASD}}(E)$, with the complex space structure induced from $\mathcal{V}$ via the homeomorphism $\mathcal{V} \to \mathcal{W}$, we obtain the desired normal complex space structure around $\{(A) \times X \}$, and this structure obviously extends the natural complex space structure of $\mathcal{M}^{\text{ASD}}(E)^*$. 

The identification between the normal cone of $X$ in $\mathcal{V}$ and the cone of degenerate elements in $\mathcal{K}_X \otimes S^2(E)$ follows from the isomorphism $\mathcal{N} \cong \pi^*(\mathcal{K}_X) \otimes O_E(-2)$ given by Proposition 5.8 using [Ful, Section B.6].

The following result used in the proof is a special case of [Ta, Lemma 4.4]. We give a short proof for completeness.

**Lemma 5.9.** With the notation introduced in the proof of Theorem 1.3, the set $\mathcal{W} := \pi(\mathcal{U})$ is open in $C_X^\mathcal{U}$.

**Proof.** The set $\mathcal{W} \setminus X = \tau(\mathcal{I}(\mathcal{P}(E) \cup B))$ is obviously contained in $C_X^\mathcal{U} \setminus (X \cup \tau(B))$. Since the restriction of $\tau$ to $\mathcal{U} \setminus \mathcal{P}(E)$ is an open embedding, $\mathcal{W} \setminus X$ is open in the complement $C_X^\mathcal{U} \setminus (X \cup \tau(B))$; it also closed in this complement, because it can be written as $\tau(\mathcal{I}) \cap (C_X^\mathcal{U} \setminus (X \cup \tau(B)))$, and $\tau(\mathcal{U})$ is compact. Since $\mathcal{W} \setminus X$ is connected, it follows that it coincides with a connected component of $C_X^\mathcal{U} \setminus (X \cup \tau(B))$.

Let $\eta \in (0, \varepsilon)$ be such that $C_X^\mathcal{U} \setminus \tau(B) = \emptyset$. The set $C_X^\mathcal{U} \setminus X$ is contained in $C_X^\mathcal{U} \setminus (X \cup \tau(B))$, is connected, and intersects $\mathcal{W} \setminus X$, so it is contained in this connected component of $C_X^\mathcal{U} \setminus (X \cup \tau(B))$. Therefore $C_X^\mathcal{U} \subset \mathcal{W}$, so $\mathcal{W}$ contains a neighbourhood of $X$ in $C_X^\mathcal{U}$. Recalling that $\tau$ is an open embedding on $\mathcal{U} \setminus \mathcal{P}(E)$, it follows that $\mathcal{W}$ is open.

**6. Appendix**

**6.1. Elementary transformations of sheaves and sheaf deformations.**

**6.1.1. Elementary transformations of coherent sheaves.** Let $Y$ be a complex manifold, and $\xi : X \to Y$ be the embedding map of an effective divisor $X$ of $Y$. For a coherent sheaf $\mathcal{S}$ on $Y$ we denote by $\mathcal{S}|_X$ the restriction $\xi^*(\mathcal{S})$, which is a coherent sheaf on $X$, and by $\mathcal{S}_X$ the sheaf $\xi_*\xi^*(\mathcal{S}) = \mathcal{S}/\mathcal{I}_X\mathcal{S}$, which is a torsion coherent sheaf on $Y$.

Let $p : \mathcal{F} \to \mathcal{H}$ be an epimorphism of coherent sheaves on $Y$, where $\mathcal{H}$ has the property

$$\mathcal{I}_X\mathcal{H} = 0,$$

i.e., $\mathcal{H}$ is isomorphic to the direct image of a coherent sheaf on $X$. Let $p_X : \mathcal{F}_X \to \mathcal{H}$ be the morphism induced by $p$, and put

$$\mathcal{F}' := \ker(p), \quad \mathcal{H}' := \ker(p_X).$$

The exact sequence

$$0 \to \mathcal{F}' \hookrightarrow \mathcal{F} \xrightarrow{p} \mathcal{H} \to 0$$
Therefore the kernel of the canonical epimorphism
\[ \mathcal{F}_X \rightarrow \mathcal{F}_X \xrightarrow{p_X} \mathcal{H} \rightarrow 0, \]
which proves that the image of \( \mathcal{F}'_X \) in \( \mathcal{F}_X \) is \( \mathcal{H}' \). Thus \( \mathcal{F}' \) comes with an epimorphism \( p' : \mathcal{F}' \rightarrow \mathcal{H}' \), where \( \mathcal{H}' \) again has the property \( \mathcal{I}_X \mathcal{H}' = 0 \).

**Definition 6.1.** Let \( p : \mathcal{F} \rightarrow \mathcal{H} \) be an epimorphism of coherent sheaves on \( Y \) where \( \mathcal{I}_X \mathcal{H} = 0 \). The elementary transformation of the pair \((\mathcal{F}, p)\) is the pair \((\mathcal{F}', p')\), where \( \mathcal{F}' := \ker(p) \), \( \mathcal{H}' = \ker(p_{\mathcal{H}}) \) and \( p' \) is induced by the composition \( \mathcal{F}' \rightarrow \mathcal{F}'_X \rightarrow \mathcal{F}_X \) (whose image is \( \mathcal{H}' \)).

Note that
\[ \mathcal{H}'' := \ker(p_{\mathcal{H}}) : \mathcal{F}'_X \rightarrow \mathcal{H}'_X ) = \ker(\mathcal{F}'_X \rightarrow \mathcal{F}_X ) = \mathcal{I}_X \mathcal{F}' / \mathcal{I}_X \mathcal{F}. \quad (34) \]
Denoting by \((\mathcal{F}'', p'')\) the elementary transformation of \((\mathcal{F}', p')\) one has
\[ \mathcal{F}'' = \ker(p') = \ker(\mathcal{F}' \rightarrow \mathcal{F}_X ) = \ker(\mathcal{F}' \rightarrow \mathcal{F}_X ) = \mathcal{F}' \cap (\mathcal{I}_X \mathcal{F}) = \mathcal{I}_X \mathcal{F}, \]
where for the last equality we have used the obvious inclusion \( \mathcal{I}_X \mathcal{F} \subset \mathcal{F}' \). The epimorphism \( p'' : \mathcal{F}'' \rightarrow \mathcal{H}'' \) is just the canonical epimorphism \( \mathcal{I}_X \mathcal{F} \rightarrow \mathcal{I}_X \mathcal{F} / \mathcal{I}_X \mathcal{F}' \).

Using our assumption that \( X \) is a divisor in \( Y \) we obtain:

**Remark 6.2.** Regarding 0 \( \rightarrow \mathcal{O}(-X) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_X \rightarrow 0 \) as a resolution of \( \mathcal{O}_X \) by locally free \( \mathcal{O}_Y \)-modules, we get for any coherent sheaf \( \mathcal{A} \) on \( Y \)
\[ \mathcal{O} \rightarrow \mathcal{O}_X = \ker(\mathcal{A}(X) \rightarrow \mathcal{A}), \quad \mathcal{O} \rightarrow \mathcal{O}_X = \mathcal{O} \rightarrow \mathcal{O}_X = 0 \quad \text{for} \quad k \geq 2. \]
Therefore the kernel of the canonical epimorphism \( \mathcal{A}(-X) \rightarrow \mathcal{I}_X \mathcal{A} \) is \( \mathcal{T} \mathcal{O}_{1}(\mathcal{A}, \mathcal{O}_X) \).

The exact sequence (33) can be continued to the left as follows
\[ 0 \rightarrow \mathcal{T} \mathcal{O}_{1}(\mathcal{F}', \mathcal{O}_X) \rightarrow \mathcal{T} \mathcal{O}_{1}(\mathcal{F}, \mathcal{O}_X) \rightarrow \mathcal{H}(-X) \rightarrow \mathcal{F}'_X \rightarrow \mathcal{F}_X \rightarrow \mathcal{H} \rightarrow 0. \]
We thus obtain

**Proposition 6.3.** With the notations and the assumptions above, suppose that \( \mathcal{T} \mathcal{O}_{1}(\mathcal{F}, \mathcal{O}_X) = 0 \), and let \((\mathcal{F}', p')\) be the elementary transformation of \((\mathcal{F}, p)\). Then
\begin{itemize}
  \item[(1)] \( \mathcal{T} \mathcal{O}_{1}(\mathcal{F'}, \mathcal{O}_X) = 0 \),
  \item[(2)] One has a short exact sequence
    \[ 0 \rightarrow \mathcal{H}(-X) \xrightarrow{\rho} \mathcal{F}'_X \xrightarrow{p'_{\mathcal{H}}} \mathcal{H}' \rightarrow 0, \quad (35) \]
    where \( \rho \) identifies \( \mathcal{H}(-X) \) with \( \ker(p'_{\mathcal{H}}) = \mathcal{I}_X \mathcal{F} / \mathcal{I}_X \mathcal{F}' \) via the identifications \( \mathcal{I}_X \mathcal{F} = \mathcal{F}(-X), \mathcal{I}_X \mathcal{F}' = \mathcal{F}'(-X), \mathcal{F} / \mathcal{F}' = \mathcal{H} \).
  \item[(3)] The second elementary transformation \((\mathcal{F}'', p'') : \mathcal{F}'' \rightarrow \mathcal{H}'' \) can be identified with the pair \((\mathcal{F}(-X), p \otimes \text{id}) : \mathcal{F}(-X) \rightarrow \mathcal{H}(-X) \).
\end{itemize}
Supposing again \( \mathcal{T} \mathcal{O}_{1}(\mathcal{F}, \mathcal{O}_X) = 0 \), the short exact sequence of \( \mathcal{O}_{2X} \)-modules
\[ 0 \rightarrow \mathcal{O}(X)(-X) \xrightarrow{i_{2X}} \mathcal{O}_{2X} \xrightarrow{\pi_{2X}} \mathcal{O}_X \rightarrow 0 \quad (36) \]
associated with the decomposition \( 2X = X + X \) gives a short exact sequence
\[ 0 \rightarrow \mathcal{F}(X)(-X) \xrightarrow{i_{X}} \mathcal{F}_{2X} \xrightarrow{\pi_{X}} \mathcal{F}_X \rightarrow 0 \quad (37) \]
of $\mathcal{O}_X$-modules. Let $\varepsilon_X(\mathcal{F}) \in \text{Ext}^1_{\mathcal{O}_X}(\mathcal{F}_X, \mathcal{F}_X(-X))$ be the extension class of $(37)$, and let $\varepsilon_X(\mathcal{F}, \mathcal{F}) \in \text{Ext}^1_{\mathcal{O}_X}(\mathcal{H}', \mathcal{H}(-X))$ be the extension class of $(35)$. The two extensions can be compared using the morphisms

$$j : \mathcal{H}' \hookrightarrow \mathcal{F}_X, \; p_X \otimes \text{id} : \mathcal{F}_X(-X) \to \mathcal{H}(-X).$$

Recall that the functors $\text{Ext}$ are contravariant with respect to the first, and covariant with respect to the second argument. For an element $u \in \text{Ext}_A(M, N)$ and morphisms $\mu : M' \to M, \nu : N \to N'$ we denote by $\nu u \mu$ the corresponding element of $\text{Ext}_A(M', N')$.

**Theorem 6.4.** Let $p : \mathcal{F} \to \mathcal{H}$ be an epimorphism of coherent sheaves on $Y$ where $\mathcal{I}_X \mathcal{H} = 0$, and $\text{Tor}_1(\mathcal{F}, \mathcal{O}_X) = 0$. With the notations above we have

$$A_{\pi_X}(\varepsilon_X(\mathcal{F}, \mathcal{F})) = (p_X \otimes \text{id})(\varepsilon_X(\mathcal{F})) j,$$

where

$$\text{Coh}(X) \times \text{Coh}(X) = \text{Coh}(X) \times \text{Coh}(X)$$

is the natural transformation induced by the epimorphism $\pi_X : \mathcal{O}_X \to \mathcal{O}_X$.

**Proof.** Put $r := r_X, q := \pi^X$ to simplify the notation. Taking into account that $j$ is a monomorphism, and $p_X \otimes \text{id}$ is an epimorphism, it follows that $(p_X \otimes \text{id})(\varepsilon(\mathcal{F})) j$ is the isomorphism class of the extension corresponding to the upper line in the diagram

$$
\begin{array}{c}
0 \\
\mathcal{I}_X \mathcal{H}(-X) \xrightarrow{r} \mathcal{H}' \to 0 \\
0 \\
\mathcal{F}_X(-X) \xrightarrow{q} \mathcal{F}_X \\
0 \\
\mathcal{F}_X'(-X) \xrightarrow{i_X \otimes \text{id}} \mathcal{F}_X(-X) \xrightarrow{p_X \otimes \text{id}} \mathcal{F}_X(-X)
\end{array}
$$

where the morphisms $\bar{r}, \bar{q}$ are induced by $r$ and $q$ respectively. The morphism $i_X : \mathcal{F}_X' \to \mathcal{F}_X$ is induced by the inclusion $i : \mathcal{F}' \to \mathcal{F}$. The right-hand vertical exact sequence in the diagram and the exact sequence

$$\mathcal{F}_2' \to \mathcal{F}_2 \xrightarrow{p_X \otimes \text{id}}, \mathcal{H} \to 0$$

show that

$$q^{-1}(\mathcal{H}') = \ker(p_X \circ q) = \mathcal{F}'/\mathcal{I}_X \mathcal{F}.$$  

(39)

On the other hand, the definition of $r$ and the left-hand vertical exact sequence give

$$r(\mathcal{F}_X(-X)) = \mathcal{I}_X \mathcal{F}'/\mathcal{I}_X \mathcal{F}, \; r(\ker(p_X \otimes \text{id})) = r(\im(i_X \otimes \text{id})) = \mathcal{I}_X \mathcal{F}'/\mathcal{I}_X \mathcal{F}. \; (40)$$
Using \((39), (40)\) we get an obvious isomorphism of \(\mathcal{O}_X\)-modules.

\[
q^{-1}(\mathcal{H}')/r(\ker(p_X \otimes \text{id})) \cong \mathcal{F}_X.
\]

It is easy to check that \(\rho, \rho'_{\mathcal{X}}\) correspond to \(\bar{r}, \bar{q}\) via this isomorphism. \(\blacksquare\)

Denote by \(\mathfrak{o}\) the double origin in \(\mathbb{C}\), regarded as a non-reduced complex space.

**Remark 6.5.** The extension \((36)\) has a right splitting which is multiplicative (makes \(\mathcal{O}_X\) a sheaf of \(\mathcal{O}_X\)-algebras) if and only if the embedding \(X \hookrightarrow \mathcal{X}\) of \(X\) in its second order infinitesimal neighbourhood \(\mathcal{X}\) has a left inverse. Equivalently, this means that \(\mathcal{X}\) has the structure of an \(\mathfrak{o}\)-fibre bundle over \(X\) such that \(X \hookrightarrow \mathcal{X}\) becomes a section in this bundle.

**Corollary 6.6.** Let \(p : \mathcal{F} \to \mathcal{H}\) be an epimorphism of coherent sheaves on \(Y\) where \(\mathcal{I}_X \mathcal{H} = 0\), and \(\mathcal{T}or_1(\mathcal{F}, \mathcal{O}_X) = 0\). Let \(\sigma : \mathcal{O}_X \to \mathcal{O}_X\) be a multiplicative right splitting of \((36)\), and \(\varepsilon_X^\sigma(\mathcal{F}) \in \text{Ext}^1_{\mathcal{O}_X}(\mathcal{F}_X, \mathcal{F}_X(-X))\) be the extension class of \((36)\), regarded as an extension of \(\mathcal{O}_X\)-modules via \(\sigma\). Then

\[
\varepsilon_X(\mathcal{F}, p) = (p_X \otimes \text{id})(\varepsilon_X^\sigma(\mathcal{F}))j.
\]

**Proof.** We apply the natural transformation \(A_\sigma\) in \((38)\) taking into account that \(A_\sigma \circ \sigma_{p_X} = \text{id}, \varepsilon_X^\sigma(\mathcal{F}) = A_\sigma(\varepsilon_X(\mathcal{F}))\), and that \(p_X \otimes \text{id}\), \(j\) are morphisms of \(\mathcal{O}_X\)-modules. \(\blacksquare\)

6.1.2. **Elementary transformations of sheaf deformations.** Suppose now that \(Y = D \times X\), where \(X\) is a compact complex manifold, and \(D \subset \mathbb{C}\) is the standard disk. We will identify \(X\) with the fibre \(X_0 = \{0\} \times X\) over the origin \(0 \in D\). Let \(\mathcal{V}\) be a coherent sheaf on \(X\), and \(\mathcal{F}\) be a coherent sheaf on \(Y\), flat over \(D\), endowed with a fixed isomorphism \(\mathcal{F}|_X = \mathcal{V}\). In other words \(\mathcal{F}\) is a deformation of \(\mathcal{V}\) parameterised by \(D\). The corresponding infinitesimal deformation is an element \(\varepsilon_{\mathcal{V}}(\mathcal{F}) \in \text{Ext}^1(\mathcal{V}, \mathcal{V})\). On \(X\) we fix a short exact sequence

\[
0 \to \mathcal{T}' \xrightarrow{j_0} \mathcal{V} \xrightarrow{p_0} \mathcal{T} \to 0
\]

where \(\mathcal{T}' := \ker(p_0)\) and \(j_0 : \mathcal{T}' \hookrightarrow \mathcal{V}\) is the inclusion morphism. We denote by \(\mathcal{T}'^\mathcal{Y}, \mathcal{T}^\mathcal{Y}\) the direct images of \(\mathcal{T}, \mathcal{T}\) to \(Y\), and by \(p : \mathcal{F} \to \mathcal{T}^\mathcal{Y}\) the epimorphism induced by \(p_0\). Taking into account that in this special situation \(\mathcal{O}_X(-X)\) is trivial on \(X\), Proposition 6.3 shows that the elementary transformation of the pair \((\mathcal{F}, p)\) gives a subsheaf \(\mathcal{F}' \subset \mathcal{F}\), which comes with a short exact sequence

\[
0 \to \mathcal{T}'^\mathcal{Y} \to \mathcal{F}'_X \to \mathcal{T}'^\mathcal{Y} \to 0.
\]

Restricting to \(X\) we obtain a short exact sequence

\[
0 \to \mathcal{T} \xrightarrow{\beta} \mathcal{V}' \xrightarrow{p'_{0}} \mathcal{T}' \to 0,
\]

whose extension class is an element \(\varepsilon(\mathcal{F}, p_0) \in \text{Ext}^1(\mathcal{T}', \mathcal{T})\). The following result shows that the extension class \(\varepsilon(\mathcal{F}, p_0)\) can be computed explicitly in terms of the infinitesimal deformation \(\varepsilon_{\mathcal{V}}(\mathcal{F})\). Put \(D^* := D \setminus \{0\}\).

**Corollary 6.7.** With the notations and under the assumptions above one has

1. The elementary transformation \(\mathcal{F}' := \ker(p)\) is flat over \(D\), hence it is a deformation of \(\mathcal{V}'\) parameterised by \(D\), which coincides with \(\mathcal{F}\) on \(D^* \times X\).
(2) \[ \varepsilon(F, p_0) = p_0(\varepsilon_{\pi^*}(F))j_0. \] (42)

Proof. For (1) note that the stalks \( O_{D,z} \) are principal ideal domains, hence flatness over \( D \) is equivalent to torsion-freeness as sheaf of \( O_D \)-modules. It suffices to note that \( F' \) is a subsheaf of \( F \).

(2) follows directly from Corollary \( \text{6.6} \) taking into account that, by definition, \( \varepsilon_{\pi^*}(F) \) is just the isomorphism class of the canonical extension

\[ 0 \to F_X(-X) = F_X \to F_{2X} \to F_X \to 0, \]

where \( F_{2X} \) is regarded as a sheaf of \( O_X \)-modules via the obvious multiplicative splitting \( \sigma : O_X \to O_{2X} \) induced by the composition \( 2X \subset Y = D \times X \to X \).

6.2. The push-forward of a family under a branched double cover. Let \( \pi : B \to \mathfrak{B} \) be a branched double covering of Riemann surfaces, \( b \in B \) be a ramification point, \( b = \pi(b) \) be the corresponding branch point. Let \( z \), \( \hat{z} \) be local coordinates of \( B \), \( \mathfrak{B} \) around \( b \), \( b \) such that \( z(b) = \hat{z}(b) = 0 \), \( \hat{z} \circ \pi = z^2 \) and let \( v \in T_b(B), v \in T_b(\mathfrak{B}) \) be tangent vectors such that \( dz(v) = d\hat{z}(v) \).

Proposition 6.8. Let \( X \) be a compact complex manifold, \( \mathcal{V} \) be a sheaf on \( B \times X \), flat over \( B \), and put \( \mathcal{E} := (\pi \times \text{id}_X)_*(\mathcal{V}) \). Let \( \epsilon_v(\mathcal{V}) \in \text{Ext}^1(\mathcal{V}_b, \mathcal{V}_b), \epsilon_v(\mathcal{E}) \in \text{Ext}^1(\mathcal{E}_b, \mathcal{E}_b) \) be the infinitesimal deformations of \( \mathcal{V}_b := (\pi_b \times X) \), \( \mathcal{E}_b := (\mathcal{E}_b)_b \times X \) (regarded as sheaves on \( X \)) corresponding to \( (v, \mathcal{V}), (v, \mathcal{E}) \) respectively. Then

(1) \( \mathcal{E}_b \) fits in an exact sequence

\[ 0 \to \mathcal{V}_b \xrightarrow{j_z} \mathcal{E}_b \xrightarrow{R} \mathcal{V}_b \to 0 \]

whose extension class is \( \epsilon_v(\mathcal{V}) \).

(2) One has \( \epsilon_v(\mathcal{V}) = R(\epsilon_v(\mathcal{E}))j_z \).

Proof. (1) Denote by \( b \subset B, X \subset B \times X \) the non-reduced complex subspaces associated with the effective divisors \( 2\{b\} \), \( 2(\{b\} \times X) \) respectively. One has obviously

\[ X = b \times X = (\pi \times \text{id}_X)^{-1}(\{b\} \times X). \]

Identify \( X \) with the divisor \( \{b\} \times X \) of \( B \times X \) to save on notations. Multiplication with \( z \) defines a sheaf monomorphism \( j_z : O_X \to O_X \), and a short exact sequence

\[ 0 \to O_X \xrightarrow{j_z} O_X \xrightarrow{R} O_X \to 0 \]

on \( B \times X \). Taking tensor product with \( \mathcal{V} \) and taking into account the flatness of \( \mathcal{V} \) over \( B \), we get a short exact sequence

\[ 0 \to \mathcal{V}_X \xrightarrow{j_z} \mathcal{V}_X \xrightarrow{R} \mathcal{V}_X \to 0. \] (43)

By definition, \( \epsilon_v(\mathcal{V}) \) is the extension class of \( (43) \) when \( \mathcal{V}_X \) is regarded as an \( O_X \)-module via the projection \( Q : X = b \times X \to X \).

On the other hand, since \( \pi \times \text{id}_X \) is a finite map, \( (\pi \times \text{id}_X)_*(\mathcal{V}) \) commutes with base change, hence

\[ \mathcal{E}_b = \{((\pi \times \text{id}_X)_*(\mathcal{V}))\}_{(b) \times X} = Q_*(\mathcal{V}_b). \]

It suffices to note that the sheaf \( Q_*\mathcal{V}_b \) coincides with \( \mathcal{V}_b \) itself regarded as an \( O_X \)-module, via the ring sheaf morphism \( O_X \to O_X \) induced by \( Q \).
Denote by $X$ the complex subspace of $B \times X$ corresponding to the effective divisor $4X$. The infinitesimal deformation $\epsilon_p(\mathcal{E})$ is the extension class of the lower line in the diagram below, where $\mathcal{Y}_X$, $\mathcal{Y}_X$ are regarded as $\mathcal{O}_X$-modules via the obvious projections $\mathcal{X} \to X$, $X \to X$. As in the proof of Theorem 6.4 we see that $\mathcal{R}(\epsilon_p(\mathcal{E}))[a] \in \text{Ext}^1(\mathcal{Y}_b, \mathcal{Y}_b)$ is the extension class of the upper line in the diagram below, where $\mathcal{J}_{z, p}, \mathcal{R}$ are induced by $\mathcal{J}_{z, p}$, $\mathcal{R}$ respectively. On the other hand one has

Using again the flatness of $\mathcal{V}$ over $B$ it follows that the morphism

$$F_z : \mathcal{Y}_X = \mathcal{Y}/\mathcal{I}_X \mathcal{Y} \to \mathcal{Y}/\mathcal{I}_X \mathcal{Y} = \mathcal{R}^{-1}(\mathcal{J}_z(\mathcal{Y}_X))/\mathcal{J}_z(\mathcal{Y}_X)$$

given by multiplication with $z$ is an isomorphism. It suffices to note that the upper triangles are commutative.

6.3. Extensions and families of ideal sheaves. Let $A$ be a commutative ring, and $M$, $N$ be $A$-modules. A free resolution

$$\cdots \to F_2 \overset{\delta_2}{\to} F_1 \overset{\delta_1}{\to} F_0 \overset{\varepsilon}{\to} N \to 0$$

of $N$ gives an isomorphism

$$\text{Ext}^1(N, M) \cong \ker(\text{Hom}(F_1, M) \to \text{Hom}(F_2, M))/\text{im}(\text{Hom}(F_0, M) \to \text{Hom}(F_1, M)).$$

Let

$$0 \to M \overset{a}{\to} E \overset{b}{\to} N \to 0$$

be an extension of $N$ by $M$. The element

$$\varepsilon(a, b) \in \ker(\text{Hom}(F_1, M) \to \text{Hom}(F_2, M))/\text{im}(\text{Hom}(F_0, M) \to \text{Hom}(F_1, M))$$

corresponding to the extension $(a, b)$ is given by the formula

$$\varepsilon(a, b) = [a^{-1} \circ p_1 \circ s \circ \delta_1],$$

where $s : F_0 \to E \times_N F_0$ is a right splitting of the pull-back extension

$$0 \to M \overset{(a, 0)}{\to} E \times_N F_0 \overset{p_2}{\to} F_0 \to 0.$$
Let $X$ be a complex manifold, $S \subset X$ be a codimension 2 locally complete intersection, and $\mathcal{I}_S$ be the ideal sheaf of $S$. Fix $x \in S$ and a system $(\xi_1, \xi_2) \in \mathcal{O}_x^{\mathbb{B}^2}$ of local equations of $S$. The Koszul resolution of $\mathcal{I}_{S,x}$ associated with $(\xi_0, \xi_1)$ reads

$$0 \to \mathcal{O}_x \xrightarrow{\delta_1} \mathcal{O}_x^{\mathbb{B}^2} \xrightarrow{q} \mathcal{I}_{S,x} \to 0,$$

where the morphisms $\delta_1, q$ are defined by

$$\delta_1(\varphi) = (-\xi_2 \varphi, \xi_1 \varphi), \quad q(\varphi_1, \varphi_2) = \xi_1 \varphi_1 + \xi_2 \varphi_2.$$

Using this resolution we get, for any coherent sheaf $M$ on $X$, an isomorphism

$$a^{M}_{\xi_1,\xi_2} : \text{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_{S,x}, M) \to M_x/\mathcal{I}_S M_x = M_{S,x}.$$

The isomorphisms $a^{M}_{\xi_1,\xi_2}$ associated with local equation systems $(\xi_1, \xi_2)$ give a canonical, global identification [Ha, Proposition 7.2, p. 179]

$$a^{M}_{\xi_1,\xi_2} : \text{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_S, M) \cong \omega_{S/X} \otimes \mathcal{O}_X M = \text{Hom}_{\mathcal{O}_S}(\mathcal{N}^2(\mathcal{I}_S/\mathcal{I}_S^2), \mathcal{M}_S)$$

$$\langle a^{M}_{\xi_1,\xi_2}(\varepsilon), d\xi_1 \wedge d\xi_2 \rangle = a^{M}_{\xi_1,\xi_2},$$

where $\omega_{S/X} := \text{Hom}_{\mathcal{O}_S}(\mathcal{N}^2(\mathcal{I}_S/\mathcal{I}_S^2), \mathcal{O}_S)$ [Ha, p. 141].

Let now

$$0 \to \mathcal{M} \xrightarrow{a} \mathcal{E} \xrightarrow{b} \mathcal{I}_S \to 0$$

be an extension of $M$ by $\mathcal{I}_S$, $\varepsilon(a, b) \in \text{Ext}^1(\mathcal{I}_S, M)$ be the corresponding extension class, and $\mathcal{E}(a, b) \in H^0(\text{Ext}^1(\mathcal{I}_S, M)) = H^0(\omega_{S/X} \otimes \mathcal{O}_X M)$ be its image via the canonical morphism. Formula (44) gives for a local equation system $(\xi_1, \xi_2)$ of $S$ at $x \in X$:

$$\langle \mathcal{E}(a, b), d\xi_1 \wedge d\xi_2 \rangle = a^{-1}_x(-\xi_2 \eta_1 + \xi_1 \eta_2) + \mathcal{I}_{S,x} M_x,$$

where $\eta_1, \eta_2 \in \mathcal{E}_x$ are lifts of $\xi_1, \xi_2$ via $b$.

**Proposition 6.9.** Let $X$ be a complex surface, $T$ be its tangent sheaf, and $x_0 \in X$.

Then

1. One has natural isomorphisms

$$\text{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_{x_0}, \mathcal{I}_{x_0}) \cong \mathcal{T}/\mathcal{I}_{x_0} \mathcal{T}, \quad H^0(\text{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_{x_0}, \mathcal{I}_{x_0})) = \mathcal{T}(x_0).$$

2. Let $D \subset \mathbb{C}$ be the standard disk, $\varphi : D \to X$ be a holomorphic map with $\varphi(0) = x_0$, and $\Phi \subset D \times X$ be its graph. Then

(a) $\mathcal{I}_\Phi$ is flat over $D$.

(b) The image $\mathcal{E}_\Phi(0)(\mathcal{I}_\Phi)$ of $\mathcal{E}_\Phi(0)(\mathcal{I}_\Phi) \in \text{Ext}^1(\mathcal{I}_{x_0}, \mathcal{I}_{x_0})$ via the canonical map $\text{Ext}^1(\mathcal{I}_{x_0}, \mathcal{I}_{x_0}) \to H^0(\text{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_{x_0}, \mathcal{I}_{x_0}))$ is given by

$$\mathcal{E}_\Phi(0)(\mathcal{I}_\Phi) = \varphi(0).$$

**Proof.** (1) Using (45) we get a canonical isomorphism

$$\text{Ext}^1(\mathcal{I}_{x_0}, \mathcal{I}_{x_0}) \to \mathcal{O}^2(x_0)^* \otimes \mathcal{O}_X \mathcal{I}_{x_0} = \Omega^2(x_0)^* \otimes \mathcal{O}_X (\mathcal{I}_{x_0}/\mathcal{I}_{x_0}^2)$$

$$= \Omega^2(x_0)^* \otimes \Omega^1(x_0) \simeq \mathcal{T}(x_0),$$

where the isomorphism $\mathcal{T}(x_0) \to \text{Hom}(\Omega^1(x_0), \Omega^1(x_0))$ is given by

$$\langle \frac{\partial}{\partial v}, \alpha_1 \wedge \alpha_2 \rangle = \alpha_1(v) \alpha_2 - \alpha_2(v) \alpha_1.$$
Compact subsets of the moduli space of simple sheaves.

6.4. Compact subsets of the moduli space of simple sheaves. Let \( f : \mathcal{X} \to S \) be a proper morphism of complex spaces, and \( \mathcal{A}, \mathcal{B} \) be coherent sheaves on \( \mathcal{X} \) such that \( \mathcal{A} \) is flat over \( S \). We denote by \( \mathfrak{An}_S \) the category of complex spaces over \( S \), and by \( H := \text{Hom}(\mathcal{A}, \mathcal{B}) \) the functor \( \mathfrak{An}_S \to \text{sets} \) given by

\[
H(T) := \text{Hom}_{\mathfrak{X}_T}(\mathcal{A}_T, \mathcal{B}_T),
\]

where \( \mathfrak{X}_T := T \times_S \mathcal{X} \), and \( \mathcal{A}_T, \mathcal{B}_T \) are the inverse image of \( \mathcal{A}, \mathcal{B} \) via the projection \( \mathfrak{X}_T \to \mathcal{X} \). A fundamental result of Flenner [Fl, Section 3.2] states that \( H \) is represented by a linear space over \( S \). In other words, there exists a coherent sheaf \( \mathcal{H} \) on \( S \) and, for any complex space \( T \to S \) over \( S \), a functorial bijection between \( \text{Hom}_{\mathfrak{X}_T}(\mathcal{A}_T, \mathcal{B}_T) \) and the set of holomorphic maps \( T \to \mathcal{V}(\mathcal{H}) \) over \( S \), where \( \mathcal{V}(\mathcal{H}) \) denotes the linear space associated with \( \mathcal{H} \) [Fl, Section 1.6]. Using this result and the isomorphisms \( \mathcal{V}(\mathcal{H})_s \simeq \mathcal{H}(s)^* \) [Fl, Section 1.8], one obtains

\[
\{ s \in S \mid \text{Hom}(\mathcal{A}_s, \mathcal{B}_s) \neq 0 \} = \text{supp}(\mathcal{V}(\mathcal{H})) = \text{supp}(\mathcal{H}),
\]

(49)

where \( \mathcal{A}_s, \mathcal{B}_s \) denote the restrictions of \( \mathcal{A}, \mathcal{B} \) to the fibre \( X_s := f^{-1}(s) \). This proves the following semi-continuity result:

**Proposition 6.10.** [Fl] The set \( \{ s \in S \mid \text{Hom}(\mathcal{A}_s, \mathcal{B}_s) \neq 0 \} \) is Zariski closed in \( S \).

Note that this statement is not a consequence of Grauert’s semi-continuity theorem. Let \( X \) be a compact complex space, and \( \mathcal{M}^{\text{se}} \) be the moduli space of simple sheaves on \( X \). Recall that \( \mathcal{M}^{\text{se}} \) is a (possibly non-Hausdorff) complex space. For the following corollary see also [KO]:

**Corollary 6.11.** Let \( ([\mathcal{F}_1], [\mathcal{F}_2]) \in \mathcal{M}^{\text{se}} \times \mathcal{M}^{\text{se}} \) be a non-separable pair. Then

\[
\text{Hom}(\mathcal{F}_1, \mathcal{F}_2) \neq 0, \quad \text{Hom}(\mathcal{F}_2, \mathcal{F}_1) \neq 0.
\]

**Proof.** For \( i \in \{1, 2\} \) let \( (U_i, p_i) \) be pointed Hausdorff complex spaces and \( \mathcal{F}_i \) a sheaf on \( U_i \times X \), flat over \( U_i \), such that \( \mathcal{F}_i|_{(U_i)\times X} \simeq \mathcal{F}_i \) and the map \( U_i \to \mathcal{M}^{\text{se}} \) induced by \( \mathcal{F}_i \) is an open embedding. Taking pull-backs we obtain sheaves \( \mathcal{A}_i \) over \( (U_1 \times U_2) \times X \) which are flat over \( U_1 \times U_2 \). Since \( ([\mathcal{F}_1], [\mathcal{F}_2]) \) is a non-separable pair, for any open neighbourhood \( W \) of \( (p_1, p_2) \) in \( U_1 \times U_2 \), there exists \( (u_1, u_2) \in W \) such that \( \mathcal{F}_i|_{(u_1)\times X} \simeq \mathcal{F}_2|_{(u_2)\times X} \) (regarded as sheaves on \( X \)). Since \( \mathcal{A}_i|_{(u_1, u_2)\times X} = \mathcal{F}_i|_{(u_1)\times X} \), the claim follows now from Proposition 6.10.

■
Proposition 6.12. Let $N \subset M$ be a compact, locally closed subset, such that for any pair $([F_1], [F_2]) \in N \times N$ with $[F_1] \neq [F_2]$ one has $\text{Hom}(F_1, F_2) = 0$, or $\text{Hom}(F_2, F_1) = 0$. Then $N$ has an open Hausdorff neighbourhood.

This follows from Corollary 6.11 and Proposition 6.15 explained below, which is a general existence criterion for Hausdorff open neighbourhoods of compact subspaces in locally compact spaces. Let $X$ be a topological space. A subset $A \subset X$ will be called separated in $X$ if any two distinct points of $A$ can be separated by disjoint neighbourhoods in $X$. For a point $x \in X$ we will denote by $\mathcal{V}_x$ the set of open neighbourhoods of $x$, and we put

$$M_x := \{ y \in X \mid \forall U \in \mathcal{V}_x \forall V \in \mathcal{V}_y, U \cap V \neq \emptyset \} \setminus \{ x \},$$

and, for a set $L \subset X$, we put $M_L := \bigcup_{x \in L} M_x$. We adopt Bourbaki’s definition of compactness, so compactness requires separateness.

Lemma 6.13. Let $X$ be locally Hausdorff topological space, and $L \subset X$ be a compact subspace, which is separated in $X$. Then

1. $L \cap M_L = \emptyset$.
2. $M_L$ is closed in $X$.

Proof. (1) Let $y \in L$ and $V$ be a separated open neighbourhood of $y$ in $X$. Since $L$ is separated in $X$, the point $y$ can be separated in $X$ from any point of $L \setminus V$. Using the compactness of $L \setminus V$, it follows that there exist open neighbourhoods $W$ of $y$ and $U$ of $L \setminus V$ with $W \cap U = \emptyset$. We may assume $W \subset V$. If, by reductio ad absurdum, $y$ belonged to $\bar{M}_L$, there would exist $x \in L$ such that $W \cap M_x \neq \emptyset$. The point $x$ cannot belong to $V$, because $V$ is separated and $W \subset V$; it cannot belong to $L \setminus V$ either, because in this case, for any point $z \in W$ the pair $(z, x)$ will be separated by the pair of open sets $(W, U)$.

(2) Suppose, by reductio ad absurdum, that there exists $y \in \bar{M}_L \setminus M_L$. Using (1) we obtain $y \notin L \cup M_L$, so for every $x \in L$ there exists $V_x \in \mathcal{V}_x$ and $V_y \in \mathcal{V}_y$ such that $V_x \cap V_y^x = \emptyset$. Let $\{x_1, \ldots, x_k\}$ be a finite subset of $L$ such that $L \subset \bigcup_{i=1}^k V_{x_i}$, and put $V_y := \bigcap_{i=1}^k V_{x_i}$. Thus $V_y \cap V_x = \emptyset$ for $1 \leq i \leq k$. Since $y \in \bar{M}_L$, there exists $x \in L$ and $p \in V_y \cap M_x$. Choosing $i \in \{1, \ldots, k\}$ such that $x \in V_{x_i}$, we see that the pair $(x, p)$ is separated in $X$ by the pair of open sets $(V_{x_i}, V_y)$, which contradicts $p \in M_x$.

Lemma 6.13 shows that

Corollary 6.14. In the conditions of Lemma 6.13, the set $S_L := X \setminus M_L$ is an open neighbourhood of $L$, and any pair $(x, y) \in L \times S_L$ with $x \neq y$ can be separated in $X$ by open sets.

With this preparation we can prove

Proposition 6.15. Let $X$ be a topological space with the property that any point has a fundamental system of compact neighbourhoods. Let $L \subset X$ be a compact subspace, which is separated in $X$. Then $L$ admits a Hausdorff open neighbourhood in $X$.

Proof. For each $x \in L$ let $C_x$ be a compact (hence Hausdorff) neighbourhood of $x$, and $K_x$ be a compact neighbourhood of $x$ which is contained in $S_L \cap C_x$. Note that, for every $x \in L$, the union $L \cup K_x$ is compact and separated in $X$, so Corollary 6.14
applies to this set. Let \( \{x_1, \ldots, x_k\} \) be a finite subset of \( L \) such that \( L \subset \bigcup_{i=1}^{k} \tilde{K}_{x_i} \). The set

\[
V := \left( \bigcup_{i=1}^{k} \tilde{K}_{x_i} \right) \cap \left( \bigcap_{i=1}^{k} S_{L \cup K_{x_i}} \right)
\]

is an open neighbourhood of \( L \). Moreover one has \( V \times V \subset \bigcup_{i=1}^{k} (K_{x_i} \times S_{L \cup K_{x_i}}) \), so any pair \((u, v) \in V \times V\) can be separated in \( X \) by open sets.

\[\Box\]

6.5. **The gluing lemma.** In many interesting gauge theoretical problems one obtains a moduli space which is a topological manifold, and contains a distinguished open set naturally endowed with a holomorphic structure. A natural question asks if this holomorphic structure extends to the whole moduli space. The following result gives a useful tool for dealing with this question:

**Lemma 6.16.** Let \( \mathcal{X} \) be a topological 2n-dimensional manifold, and \( \mathcal{Y} \subset \mathcal{X} \) be an open subset endowed with a complex manifold structure. Let \( \mathcal{U} \) be an n-dimensional complex manifold, and let \( f : \mathcal{U} \to \mathcal{X} \) be a map with the properties:

1. \( f \) is continuous and injective,
2. \( \mathcal{X} \setminus \mathcal{Y} \subset \text{im}(f) \),
3. The restriction \( f|_{f^{-1}(\mathcal{Y})} : f^{-1}(\mathcal{Y}) \to \mathcal{Y} \) is holomorphic.

Then

1. \( \text{im}(f) \) is an open neighbourhood of \( \mathcal{X} \setminus \mathcal{Y} \) in \( \mathcal{X} \), and \( f \) induces a homeomorphism \( \mathcal{U} \to \text{im}(f) \).
2. \( f \) induces a biholomorphism \( f|_{f^{-1}(\mathcal{Y})} : f^{-1}(\mathcal{Y}) \to \text{im}(f) \cap \mathcal{Y} \) with respect to the holomorphic structures induced by the open embeddings \( f^{-1}(\mathcal{Y}) \subset \mathcal{U} \), \( \text{im}(f) \cap \mathcal{Y} \subset \mathcal{Y} \).
3. There exists a unique complex manifold structure on \( \mathcal{X} \) which extends the fixed complex structure on \( \mathcal{Y} \), and such that \( f \) becomes biholomorphic on its image.

**Proof.** (1) follows from the invariance of domain theorem. For (2) note that \( f|_{f^{-1}(\mathcal{Y})} : f^{-1}(\mathcal{Y}) \to \mathcal{Y} \) is an injective holomorphic map between smooth complex manifolds of the same dimension, hence it induces a biholomorphism

\[
f^{-1}(\mathcal{Y}) \to f(f^{-1}(\mathcal{Y})) .
\]

But \( f(f^{-1}(\mathcal{Y})) = \text{im}(f) \cap \mathcal{Y} \). The existence statement in (3) is proved as follows. Let \( \mathcal{A} \) be a holomorphic atlas of \( \mathcal{Y} \) and \( \mathcal{B} \) a holomorphic atlas of \( \mathcal{U} \). Using (2) it is easy to see that the union

\[
\tilde{\mathcal{A}} := \mathcal{A} \cup \{ h \circ f^{-1} : f(U_h) \to V_h | h : U_h \to V_h \in \mathcal{B} \}
\]

is a holomorphic atlas on \( \mathcal{X} \). The unicity in (3) follows noting that any chart \( \chi \in \tilde{\mathcal{A}} \) is holomorphic with a holomorphic structure on \( \mathcal{X} \) satisfying the two conditions in (3).

\[\Box\]

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