Lie algebra type noncommutative phase spaces are Hopf algebroids

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Abstract. For a noncommutative configuration space whose coordinate algebra is the universal enveloping algebra of a finite dimensional Lie algebra, it is known how to introduce an extension playing the role of the corresponding noncommutative phase space, namely by adding the commuting deformed derivatives in a consistent and nontrivial way, therefore obtaining certain deformed Heisenberg algebra. This algebra has been studied in physical contexts, mainly in the case of the kappa-Minkowski space-time. Here we equip the entire phase space algebra with a coproduct, so that it becomes an instance of a completed variant of a Hopf algebroid over a noncommutative base, where the base is the enveloping algebra.

Keywords: universal enveloping algebra, noncommutative phase space, deformed derivative, Hopf algebroid, completed tensor product

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1. Introduction

Recently, a number of physical models has been proposed [1, 9, 15], where the background geometry is described by a noncommutative configuration space of Lie algebra type. Descriptively, its coordinate algebra is the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. So-called $\kappa$-Minkowski space is the most explored example [15, 16, 17, 20]. That space has been used to build a model featuring the double special relativity, a framework modifying special relativity, proposed to explain some phenomena observed in the high energy gamma ray bursts.

The noncommutative phase space of the Lie algebra $\mathfrak{g}$ is introduced by enlarging $U(\mathfrak{g})$ with additional associative algebra generators, the deformed derivatives, which act on $U(\mathfrak{g})$ via an action ★ satisfying deformed Leibniz rules [19, 22]. The subalgebra generated by the deformed derivatives is commutative. In fact, this commutative algebra is a topological Hopf algebra isomorphic to the full algebraic dual $U(\mathfrak{g})^*$ of the enveloping algebra. In this article, we extend the coproduct of the topological Hopf algebra $U(\mathfrak{g})^*$ of deformed derivatives to a coproduct $\Delta : H \to H \hat{\otimes}_{U(\mathfrak{g})} H$ on the whole phase space $H$; this coproduct is moreover a part of a Hopf algebroid structure on $H$ over the noncommutative
base algebra \( U(\mathfrak{g}) \). Roughly, this means that the coproduct does not take value in a tensor product \( H \otimes H \) over the ground field, like for Hopf algebras, but in a tensor product of \( U(\mathfrak{g}) \)-bimodules, where \( H \) is a bimodule with the help of so called source and target maps. This tensor product \( \hat{\otimes}U(\mathfrak{g}) \) is understood in a completed sense, using a cofiltration on \( H \). In other words, we construct a Hopf algebroid internal \(^3\) to a tensor category of complete cofiltered vector spaces \(^{[18]}\).

The noncommutative phase space of Lie type is nontrivially isomorphic to a topological Heisenberg double of \( U(\mathfrak{g}) \) \(^{[22]}\). Heisenberg doubles of finite dimensional Hopf algebras are known to carry a Hopf algebroid structure \(^{[6]}\) \(^{[14]}\). However, our starting Hopf algebra \( U(\mathfrak{g}) \) is infinite-dimensional, though filtered by finite-dimensional pieces. While the generalities on such filtered algebras can be used to obtain the Hopf algebroid structure \(^{[18]}\), we here use the specific features of \( U(\mathfrak{g}) \) instead, and in particular the matrix \( \mathcal{O} \), introduced in the Section \(^2\).

From a geometric viewpoint, where \( U(\mathfrak{g}) \) is viewed as the algebra of left invariant differential operators on a Lie group, the matrix \( \mathcal{O} \) is interpreted as a transition matrix between a basis of left invariant and a basis of right invariant vector fields. Then our phase space appears as the algebra of formal differential operators around the unit of the Lie group. A different variant of the Hopf algebroid structure has been outlined in \(^{[12]}\) \(^{[13]}\), for the special case when the Lie algebra is the \( \kappa \)-Minkowski space, at a physical level of rigor.

We assume familiarity with bimodules, coalgebras, comodules, bialgebras, Hopf algebras, Hopf pairings and the Sweedler notation for comultiplications (coproducts) \( \Delta(h) = \sum h_{(1)} \otimes h_{(2)} \), and right coactions \( \rho(v) = \sum v_{(0)} \otimes v_{(1)} \) (with or without the explicit summation sign). We do not assume previous familiarity with Hopf algebroids. In noncommutative geometry, one interprets Hopf algebroids \(^{[2]}\) \(^{[4]}\) \(^{[6]}\) \(^{[14]}\) as formal duals to quantum groupoids.

The generic symbols for the multiplication map, comultiplication, counit and antipode will be \( m, \Delta, \epsilon, SS \) with various subscripts. All algebras are over a fixed ground field \( k \) of characteristic zero (in physical applications \( \mathbb{R} \) or \( \mathbb{C} \)). The opposite algebra of an associative algebra \( A \) is denoted \( A^{\text{op}} \), and the coopposite coalgebra to \( C = (C, \Delta) \) is \( C^{\co} = (C, \Delta^{\text{op}}) \). Given vector space \( V \), denote its algebraic dual by \( V^* = \text{Hom}(V, k) \), and the corresponding symmetric algebras \( S(V) \) and \( S(V^*) \). In our convention, if an algebra \( A \) is graded, its graded (homogeneous) components have upper indices, \( A = \oplus_{i=0}^{\infty} A^i \), \( A^i \subset A^{i+j} \), and, if \( B \) is filtered, its filtered components \( B_0 \subset B_1 \subset B_2 \subset \ldots \) have lower indices, \( B_i \), \( B_j \subset B_{i+j} \) and \( B = \bigcup_{i=0} B_i \). The Einstein summation convention on repeated Greek indices is assumed throughout the article.
2. Deformed phase space

Throughout the article \( \mathfrak{g} \) is a fixed Lie algebra over \( \mathbb{k} \) of some finite dimension \( n \). In a basis \( \hat{x}_1, \ldots, \hat{x}_n \) of \( \mathfrak{g} \),

\[
[x_{\mu}, x_{\nu}] = C^\lambda_{\mu\nu} \hat{x}_\lambda.
\] (1)

The generators of the universal enveloping algebra \( U(\mathfrak{g}) \) are also denoted by \( \hat{x}_k \), unlike the generators of the symmetric algebra \( S(\mathfrak{g}) \) which are denoted by \( x_1, \ldots, x_n \) (without hat symbol) instead. The dual basis of \( \mathfrak{g}^* \) and the corresponding generators of \( S(\mathfrak{g}^*) \) are denoted \( \partial^1, \ldots, \partial^n \).

\( S(\mathfrak{g}) = \bigoplus_{i=0}^{\infty} S^i(\mathfrak{g}) = \bigcup_{i=0}^{\infty} S_i(\mathfrak{g}) \) carries a graded and \( U(\mathfrak{g}) = \bigcup_i U_i(\mathfrak{g}) \) a filtered Hopf algebra structure. Both structures are induced along quotient maps from the tensor bialgebra \( T(\mathfrak{g}) \). By the PBW theorem, the linear map \( \xi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \) sending \( x_{i_1} \cdots x_{i_r} \mapsto \sum_{\sigma \in S_{r+1}} \hat{x}_{i_{\sigma(1)}} \cdots \hat{x}_{i_{\sigma(r)}} \) for any \( i_1, \ldots, i_r \), is an isomorphism of filtered coalgebras whose inverse may be identified with the projection to the associated graded ring [3]. When applied to spaces, we use the hat symbol for completions. Our main example is \( \hat{S}(\mathfrak{g}^*) = \lim_i S_i(\mathfrak{g}^*) \cong \prod_i S^i(\mathfrak{g}^*) \) which the completion of \( S(\mathfrak{g}^*) \) with respect to the degree of polynomial; it may be identified with the formal power series ring \( \mathbb{k}[[\partial^1, \ldots, \partial^n]] \) in \( n \) variables. For our purposes, it is useful to regard this ring as well as the algebraic duals \( U(\mathfrak{g})^* \) and \( S(\mathfrak{g}^*) \) either as topological or as cofiltered algebras (see Appendix A.2). For a multiindex \( K = (k_1, \ldots, k_n) \in \mathbb{N}_0^n \), denote \( |K| = k_1 + \ldots + k_n \), \( x_K := x_1^{k_1} \cdots x_n^{k_n} \) and \( \hat{x}_K := \hat{x}_1^{k_1} \cdots \hat{x}_n^{k_n} \). The multiindices add up componentwise. If \( J, K \) are multiindices the rule

\[
(\partial^J, x_K) := |J|! \delta^K_J \quad \text{('evaluation of a partial differential operator at a polynomial in zero')},
\]

continuously and linearly extends to a unique map \( \langle, \rangle : \hat{S}(\mathfrak{g}^*) \otimes S(\mathfrak{g}) \rightarrow \mathbb{k} \), which is a nondegenerate pairing, hence it identifies \( S(\mathfrak{g})^* \cong \hat{S}(\mathfrak{g}^*) \). The map \( \xi^T : U(\mathfrak{g})^* \rightarrow S(\mathfrak{g})^* \cong \hat{S}(\mathfrak{g}^*) \) dual to \( \xi \) is an isomorphism of cofiltered algebras. Introduce the opposite Lie algebra \( \mathfrak{g}^R \) generated by \( \hat{y}_\mu \), where

\[
[y_{\mu}, y_{\nu}] = -C^\lambda_{\mu\nu} \hat{y}_\lambda.
\] (2)

The Lie algebra \( \mathfrak{g}^R \) is isomorphic to \( \mathfrak{g} \) via the isomorphism \( \hat{y}_i \mapsto -\hat{x}_i \), and antisomorphic via \( \hat{y}_i \mapsto \hat{x}_i \). The latter induces the isomorphism

\[
U(\mathfrak{g}^L)^{op} \cong U(\mathfrak{g}^R).
\]

However, we consider spaces \( \mathfrak{g}^L := \mathfrak{g} \) and \( \mathfrak{g}^R \) distinct.

The \( n \)-th Weyl algebra \( A_n \) is the associative algebra generated by \( x_1, \ldots, x_n, \partial^1, \ldots, \partial^n \) subject to relations \([x_{\alpha}, x_{\beta}] = [\partial^\alpha, \partial^\beta] = 0 \) and \([\partial^\alpha, x_{\beta}] = \delta^\alpha_{\beta} \). The (semi)completed Weyl algebra \( \hat{A}_n \) is a completion of \( A_n \) by the "degree of a differential operator", hence allowing the formal power series in \( \partial^\alpha \)-s. \( A_n \) has a faithful representation, called Fock space,
on the polynomial algebra in $x_1, \ldots, x_n$, in which each $x_\mu$ acts as a multiplication operator and $\partial^\nu$ as a partial derivative; the action of $A_n$ extends continuously to a unique action of $\hat{A}_n$. We construct certain analogues of $A_n$ containing $U(\mathfrak{g}^*)$ or $U(\mathfrak{g}^R)$. They have a structure of a Hopf-algebraic smash product.

**DEFINITION 1.** Let $A$ be an algebra and $B$ a bialgebra.

A left action $\triangleright : B \otimes A \to A$ (right action $\lhd : A \otimes B \to B$), is a left (right) **Hopf action** if $\triangleright (aa') = \sum (b(1) \triangleright a) \cdot (b(2) \triangleright a')$ or, respectively, $(b \otimes a)(b' \otimes a') = \sum b(b'_1) \otimes (a \lhd b'_2) a'$, for all $a, a' \in A$ and $b, b' \in B$. We also say that $A$ is a left (right) $B$-module algebra. Given a left (right) Hopf action, the **smash product** $A_B \otimes (B_A)$ is an associative algebra which is a tensor product vector space $A \otimes B$ ($B \otimes A$) with the multiplication bilinearly extending the formulas

$$
(a\triangleright b)(a'\triangleright b') = \sum a(a'_{(1)} \triangleright b) a'_{(2)} b', \quad a, a' \in A, b, b' \in B,
$$

$$
(b\lhd a)(b'\lhd a') = \sum b(b'_{(1)} \lhd a) b'_{(2)} a', \quad a, a' \in A, b, b' \in B;
$$

where, for the emphasis, one writes $\triangleright a := a \otimes b$. 

Note that $1_B \otimes A$ and $B \otimes 1$ are subalgebras in $B_A \otimes A$, canonically isomorphic to $A$ and $B$. We may trade between the actions $\triangleright$ and homomorphisms of algebras $\psi : B \to \text{End} A$, $\psi(b)(a) := b \triangleright a$. If $B$ is a Hopf algebra with an antipode $SS$, we may replace $\psi$ with an antihomomorphism $\psi \circ SS$, or a right action $\lhd : a \otimes b \mapsto a \lhd b := b \triangleright SS(a)$. If $SS^2 = \text{id}$ (for instance, if $B$ is commutative or cocommutative, e.g. $B = U(\mathfrak{g})$), then there is an isomorphism $A_B \otimes B_A \cong B \otimes A$ of algebras, $a\triangleright b \mapsto \sum b(1) \triangleright (a \lhd b(2))$, with inverse $b\lhd a \mapsto \sum (b(1) \triangleright a) b(2)$.

Given $\mathfrak{g}^*$ as above, let $\mathcal{C}$ denote the $n \times n$-matrix with entries

$$
\mathcal{C}_\alpha^\beta := C_\alpha^\beta \partial^\gamma.
$$

In this notation,

$$
\phi := \frac{-\mathcal{C}}{e^{-\mathcal{C}} - 1} = \sum_{N=0}^\infty \frac{(-1)^N B_N}{N!} C_N
$$

is a power series in matrix $\mathcal{C}$, which in turn contains $\mathcal{D}^\beta$-s; in particular $\phi$ is itself a matrix whose entries $\phi_\beta^\gamma \in \hat{S}(\mathfrak{g}^*)$, $\alpha, \beta = 1, \ldots, n$, are formal power series in $\partial^\gamma$. The constants $B_N$ are the Bernoulli numbers. The formula $\phi(\partial^\alpha) = \phi_\alpha^\gamma$ defines a linear map $\phi(\hat{\mathcal{D}}_a) : \mathfrak{g} \to \hat{S}(\mathfrak{g}^*)$, which by the chain rule and continuity extends to a unique derivation $\phi(\hat{\mathcal{D}}_a) \in \text{Der}(\hat{S}(\mathfrak{g}^*))$. A crucial property of $\phi$ is that the corresponding
map $\phi: g^L \to \text{Der}(\hat{S}(g^*))$ is a Lie algebra homomorphism; it follows that it automatically extends to a unique Hopf action also denoted

$$\phi: U(g^L) \to \text{End}(\hat{S}(g^*)) .$$

(5)

Using the right action $\phi \circ SSU(g^*)$ we introduce $H^L := U(g^L)\sharp \hat{S}(g^*)$ and interpret it as the 'noncommutative phase space of Lie type'. We commonly identify $\hat{S}(g^*)$ with the subalgebra $1\sharp \hat{S}(g^*)$ and $U(g^L)\sharp 1$. It follows that in $H^L$

$$[\partial^\mu, x_\nu] = \left(\frac{-C}{e^C - 1}\right)^\mu_\nu. \quad (6)$$

This identity justifies interpretation of $\partial^\mu$ as deformed partial derivatives. Though we do not use this fact below, notice that $H^L$ may be considered as a free product of $U(g^L)$ and $\hat{S}(g^*)$ modulo the completion of the ideal of relations generated by $[\partial^\mu, x_\nu]$. The universal formula $[\partial^\mu, x_\nu]$ for $\phi$ is, in this context, derived in \cite{14}. The universal formula $[\partial^\mu, x_\nu]$ for $\phi$ is, in this context, derived in \cite{14}. There is an algebra isomorphism $A_n$ of algebras which we call the $\phi$-realization of $U(g)$ (by formal differential operators) given by $x_\nu \mapsto \hat{x}_\nu := x_\nu\phi^\nu_\nu$ on the generators and, if complemented by the formulas $\partial^\mu \mapsto \hat{\partial}^\mu$, it defines a unique continuous isomorphism of algebras $U(g^L)\sharp \hat{S}(g^*) \cong A_n (\phi$-realization of $H^L$).

Regarding that $g^R$ is a Lie algebra with structure constants $-C_{\beta\gamma}^\alpha$, the formula $[\partial^\mu, x_\nu] = \left(\frac{-C}{e^C - 1}\right)^\mu_\nu$ can be applied to it. This yields a matrix $\hat{\phi} := \left(\frac{C}{e^C - 1}\right)$ and a left Hopf action $\hat{\phi}: U(g^R) \to \text{End}(\hat{S}(g^*))$ hence a smash product $H^R := \hat{S}(g^*)\sharp U(g^R)$. Note that the tensor factors in this smash product are ordered differently than in the definition of $H^L$. The generators of $H^R$ are $\hat{y}_\mu, \hat{\partial}^\mu, \mu = 1, \ldots, n$. In addition to the relations in $U(g^R)$ and $\hat{S}(g^*)$ one also has

$$[\partial^\mu, \hat{y}_\nu] = \left(\frac{C}{e^C - 1}\right)^\mu_\nu. \quad (7)$$

The corresponding realization, $\hat{y}_\nu \mapsto x_\lambda \hat{\phi}^{\lambda}_\nu, \hat{\partial}^\mu \mapsto \hat{\partial}^\mu$, defines an isomorphism $H^R \cong A_n$.

**THEOREM 1.** There is an algebra isomorphism from $H^L := U(g^L)\sharp \hat{S}(g^*)$ to $H^R = \hat{S}(g^*)\sharp U(g^R)$ which fixes the commutative subalgebra $\hat{S}(g^*)$ (i.e. identifies $1\sharp \hat{S}(g^*)$ with $\hat{S}(g^*)\sharp 1$, and in particular $\partial^\mu \mapsto \hat{\partial}^\mu$), and which maps $\hat{x}_\nu \mapsto \hat{y}_\nu \hat{O}_\nu^\mu$, where $\hat{O} := e^C$ is an invertible $n \times n$-matrix with entries $\hat{O}_\nu^\mu \in \hat{S}(g^*)$ and inverse $\hat{O}^{-1} = e^{-C}$. After the identification, $[\hat{x}_\mu, \hat{y}_\nu] = 0$. Consequently, the images of $U(g^L) \hookrightarrow H^L$ and $U(g^R) \hookrightarrow H^R$ mutually commute. The following identities hold

$$[\hat{O}^\nu_\mu, \hat{y}_\nu] = C^\nu_\rho\hat{O}_\mu^\rho \quad (7)$$
\[ [O^\lambda_{\mu}, \hat{x}_\nu] = C^\rho_{\mu\nu} O^\lambda_\rho \]
\[ [(O^{-1})^\lambda_{\mu}, \hat{x}_\nu] = -C^\rho_{\mu\rho} (O^{-1})^\lambda_\mu \]
\[ [(O^{-1})^\lambda_{\mu}, \hat{y}_\nu] = -C^\rho_{\mu\rho} (O^{-1})^\lambda_\mu \]
\[ C^\tau_{\mu\nu} O^\lambda = C^\rho_{\mu\rho} O^\lambda_\rho O^\sigma_{\nu}, \quad C^\tau_{\mu\nu} (O^{-1})^\lambda = C^\rho_{\mu\rho} (O^{-1})^\lambda_\mu (O^{-1})^\sigma_\nu. \]

**Proof.** The above isomorphisms of algebras \( H^L \cong \hat{A}_n \) and \( H^R \cong \hat{A}_n \) combine into an isomorphism \( H^L \cong H^R \). If we express \( \hat{x}_\mu \) and \( \hat{y}_\nu \) within \( \hat{A}_n \) as \( x_\mu \phi_\mu \) and \( x_\sigma \phi_\sigma \) respectively, the commutation relation \([\hat{x}_\mu, \hat{y}_\nu] = 0\) becomes \([x_\mu \phi_\mu, x_\sigma \phi_\sigma] = 0\), which is the Proposition 5 (Appendix A.1).

Comparing the formulas for \( \phi \) and \( \hat{\phi} \), note that
\[ \hat{\phi} = \phi e^{-C}, \quad \hat{x}_\nu = \hat{y}_\mu (e^C)_\mu^\nu = \hat{y}_\mu O^\mu_\nu. \]

Rewrite \([\hat{x}_\mu, \hat{y}_\nu]\) now as
\[ [\hat{y}_\mu O^\rho_{\mu\nu}, \hat{y}_\nu] = [\hat{y}_\mu, \hat{y}_\nu] O^\rho_{\mu\nu} + \hat{y}_\lambda [O^\lambda_{\mu\nu}, \hat{y}_\nu] = \hat{y}_\lambda (-C^\rho_{\mu\rho} O^\rho_{\mu\nu} + [O^\lambda_{\mu\nu}, \hat{y}_\nu]). \]

Using induction and \([\partial^\gamma, \hat{y}_\nu]\) is \( \hat{S}(g^*) \) shows \([\hat{S}(g^*), \hat{y}_\nu] \subset \hat{S}(g^*) \). Thus, \((-C^\rho_{\mu\rho} O^\rho_{\mu\nu} + [O^\lambda_{\mu\nu}, \hat{y}_\nu]) \) implies \((7)\). Similarly, in \([\hat{x}_\mu, \hat{y}_\nu]\) we replace \( \hat{y}_\nu \) with \( \hat{x}_\lambda (O^{-1})^\lambda_\nu \) to prove \((9)\). To show \((8)\), calculate \( C^\lambda_{\mu\nu} \hat{y}_\mu O^\nu_{\lambda\delta} = C^\lambda_{\mu\nu} \hat{x}_\lambda = [\hat{x}_\mu, \hat{x}_\nu] = [\hat{y}_\mu O^\mu_{\nu\lambda}, \hat{y}_\nu] = \hat{y}_\mu O^\mu_{\nu\lambda}. \)

If in \((7)\) and \((9)\) we replace \( \hat{y}_\nu \) (resp. \( \hat{x}_\nu \)) on the left by \( \hat{y}_\nu (O^{-1})^\nu_{\mu} \) (resp. \( \hat{x}_\nu O^\nu_\mu \)), we get a quadratic (in \( O \) or \( O^{-1} \)) expression on the right, which are then compared with \((8)\) and \((10)\) to obtain \((11)\).

3. **Actions** and ** and some identities for them

There is a map \( \varepsilon_S : \hat{S}(g^*) \to k \), taking a formal power series to its constant term (‘evaluation at 0’). We introduce the “black action” of \( H^L \) on \( U(g^L) \) as the multiplication map, is the unique action for which \( \partial^\mu \uparrow 1 = 0 \) for all \( \mu \) and \( \hat{f} \uparrow 1 = \hat{f} \) for all \( \hat{f} \in U(g^L) \). It follows that \( O^\mu_{\nu} \uparrow 1 = \delta^\mu_{\nu} 1 = (O^{-1})^\mu_{\nu} \uparrow 1 \) and \( \hat{y}_\nu \uparrow 1 = \hat{x}_\mu (O^{-1})^\mu_{\nu} \uparrow 1 = \delta^\mu_{\nu} \hat{x}_\mu = \hat{x}_\nu \). Similarly, the right black action of \( H^R \) on \( U(g^R) \) is the composition
\[ U(g^R) \otimes H^R \hookrightarrow H^R \otimes H^R \overset{m}{\longrightarrow} H^L \cong \hat{S}(g^*) \overset{\epsilon_{\hat{S}(g^*)}}{\longrightarrow} U(g^L), \]
characterized by \( 1 \uparrow \partial^\mu = 0 \), and \( 1 \uparrow \hat{u} = \hat{u} \), for all \( \hat{u} \in U(g^R) \).
THEOREM 2. For any \( \hat{f}, \hat{g} \in U(\mathfrak{g}^L) \) the following identities hold

\[
\hat{x}_\alpha \hat{f} = \left( \mathcal{O}_\alpha^\beta \mapsto \hat{f} \right) \hat{x}_\beta \tag{14}
\]

\[
\mathcal{O}_\alpha^\gamma \mapsto (\hat{g} \hat{f}) = \left( \mathcal{O}_\alpha^\beta \mapsto \hat{g} \left( \mathcal{O}_\beta^\gamma \mapsto \hat{f} \right) \right) \tag{15}
\]

\[
(\mathcal{O}^{-1})_\alpha^\gamma \mapsto (\hat{g} \hat{f}) = \left( \left( \mathcal{O}^{-1} \right)_\beta^\gamma \mapsto \hat{g} \right) \left( \left( \mathcal{O}^{-1} \right)_\alpha^\beta \mapsto \hat{f} \right) \tag{16}
\]

\[
\hat{y}_\alpha \mapsto \hat{f} = \hat{f} \hat{x}_\alpha \tag{17}
\]

\[
(\hat{x}_\alpha \mapsto \hat{f}) \hat{g} = \left( \mathcal{O}_\alpha^\beta \mapsto \hat{f} \right) (\hat{x}_\beta \mapsto \hat{g}) \tag{18}
\]

Proof. We show (14) for monomials \( \hat{f} \) by induction on the degree of monomial; by linearity this is sufficient. For the base of induction, it is sufficient to note \( \mathcal{O}_\alpha^\beta \mapsto 1 = \delta_\alpha^\beta \). For the step of induction, let \( \hat{f} \) be of degree \( k \), and calculate

\[
\mathcal{O}_\alpha^\gamma \mapsto (\hat{x}_\nu \hat{f}) = \left( \mathcal{O}_\alpha^\beta \mapsto \hat{x}_\nu \hat{f} \right) \mathcal{O}_\gamma^\alpha \mapsto 1 + \hat{x}_\nu \hat{f} \mathcal{O}_\gamma^\alpha \mapsto 1
\]

\[
= [\mathcal{O}_\alpha^\beta, \hat{x}_\nu] \mapsto \hat{f} + \hat{x}_\nu [\mathcal{O}_\alpha^\gamma, \hat{f}] \mapsto 1 + \hat{x}_\nu \hat{f} \mathcal{O}_\delta^\alpha \mathcal{O}_\gamma^\beta \mapsto \hat{f}
\]

\[
= (\mathcal{O}_\alpha^\gamma + \delta_\alpha^\gamma \hat{x}_\nu) (\mathcal{O}_\gamma^\beta \mapsto \hat{f})
\]

\[
= (\mathcal{O}_\alpha^\beta \mapsto \hat{x}_\nu) (\mathcal{O}_\gamma^\beta \mapsto \hat{f})
\]

and use this in the following:

\[
\hat{x}_\alpha \hat{x}_\nu \hat{f} = \left( \mathcal{O}_\alpha^\beta \mapsto \hat{x}_\nu \right) \hat{x}_\beta \hat{f}
\]

\[
= (\mathcal{O}_\alpha^\beta \mapsto \hat{x}_\nu) (\mathcal{O}_\gamma^\beta \mapsto \hat{f}) \hat{x}_\gamma
\]

\[
= (\mathcal{O}_\alpha^\beta \mapsto \hat{x}_\nu \hat{f}) \hat{x}_\gamma
\]

Thus (14) holds for \( \hat{f} \)-s of degree \( k+1 \), hence, by induction, for all. Along the way, we have also shown (15) for \( \hat{g} \) of degree 1 and \( \hat{f} \) arbitrary. Now we do induction on the degree of \( \hat{g} \): replace \( \hat{g} \) with \( \hat{x}_\mu \hat{g} \) and calculate

\[
\mathcal{O}_\alpha^\beta \mapsto (\hat{x}_\mu \hat{g} \hat{f}) = \left( \mathcal{O}_\alpha^\beta \mapsto \hat{x}_\mu \right) (\mathcal{O}_\beta^\alpha \mapsto (\hat{g} \hat{f}))
\]

\[
= (\mathcal{O}_\alpha^\beta \mapsto \hat{x}_\mu) (\mathcal{O}_\beta^\alpha \mapsto \hat{g}) (\mathcal{O}_\gamma^\beta \mapsto \hat{f})
\]

\[
= (\mathcal{O}_\alpha^\beta \mapsto (\hat{x}_\mu \hat{g})) (\mathcal{O}_\gamma^\beta \mapsto \hat{f})
\]

The proof of (16) is similar to (14) and left to the reader. To show (17), we use (14) and expression \( \hat{y}_\alpha = \hat{x}_\beta (\mathcal{O}^{-1})_\alpha^\beta \):

\[
\hat{x}_\beta (\mathcal{O}^{-1})_\alpha^\beta \mapsto \hat{f} = \hat{x}_\beta \mapsto ((\mathcal{O}^{-1})_\alpha^\beta \mapsto \hat{f}) = \left( \mathcal{O}_\beta^\gamma \mapsto ((\mathcal{O}^{-1})_\alpha^\beta \mapsto \hat{f}) \right) \hat{x}_\gamma
\]

\[
= ((\mathcal{O}_\beta^\gamma (\mathcal{O}^{-1})_\alpha^\beta \mapsto \hat{f}) \mapsto \hat{x}_\gamma = \delta_\alpha^\gamma \hat{f} \hat{x}_\gamma = \hat{f} \hat{x}_\alpha
\]

Finally, (18) follows from (14) by multiplying from the right with \( \hat{g} \) and noticing that \( \hat{x}_\beta \mapsto \hat{g} = \hat{x}_\beta \hat{g} \) and the same for \( \hat{f} \) in place of \( \hat{g} \).

Now we state an analogue of the Theorem 2 for \( \triangleright \).
THEOREM 3. For any \( \hat{f}, \hat{g} \in U(\mathfrak{g}^R) \) the following identities hold

\[
\hat{f}(\hat{g}) = \hat{y}_\beta(\hat{f} \triangleright (O^{-1})^\beta_\alpha), \tag{19}
\]

\[
(\hat{g} \hat{f}) = (\hat{g} \triangleright (O^{-1})^\gamma_\alpha)(\hat{f} \triangleright (O^{-1})^\beta_\alpha), \tag{20}
\]

\[
(\hat{g} \hat{f}) = (\hat{g} \triangleright (O^{-1})^\gamma_\alpha)(\hat{f} \triangleright (O^{-1})^\beta_\alpha)
\]

\[
\hat{f} \triangleright \hat{g}_\alpha = \hat{y}_\alpha \hat{f}, \tag{22}
\]

\[
\hat{g}(\hat{f} \triangleright \hat{g}_\alpha) = (\hat{g} \triangleright \hat{y}_\alpha)(\hat{f} \triangleright (O^{-1})^\beta_\alpha), \tag{23}
\]

where

\[
\hat{z}_\alpha := O^\alpha_\beta \hat{y}_\beta = \hat{O}^\alpha_\beta \hat{x}_\rho (O^{-1})^\rho_\mu \in H. \tag{24}
\]

To emphasize the special role of the matrix \( O \), we make the following remark. The topological Hopf algebra \( \hat{S}(\mathfrak{g}^{L^*}) \) coacts on \( U(\mathfrak{g}^L) \) by a unique right coaction \( \rho_{\text{YD}} : U(\mathfrak{g}^L) \to U(\mathfrak{g}^L) \otimes \hat{S}(\mathfrak{g}^{L^*}) \), which is an algebra antihomomorphism and which is on generators \( \hat{x}_\mu \in \mathfrak{g}^L \) given by

\[
\rho_{\text{YD}} : \hat{x}_\mu \mapsto \hat{x}_\rho \otimes (O^{-1})^\rho_\mu.
\]

Postcomposing this coaction with the inclusion \( U(\mathfrak{g}^L) \otimes \hat{S}(\mathfrak{g}^{L^*}) \hookrightarrow U(\mathfrak{g}^L) \otimes \hat{S}(\mathfrak{g}^{L^*}) \) makes \( U(\mathfrak{g}^L) \) into a braided-commutative algebra in the category of Yetter-Drinfeld modules over \( \hat{S}(\mathfrak{g}^{L^*}) \), internally in a monoidal category of complete cofiltered vector spaces (see Appendix A.2 and [18], which places our Hopf algebroid below into a version of the framework of the scalar extension Hopf algebroid \([2,6]\).

4. Completed tensor product and bimodules

In this section we discuss the completed tensor products needed for the coproducts \( (\Delta_{\hat{S}(\mathfrak{g}^*)} \) in this and \( \Delta^L \) and \( \Delta^R \) in the next section), and introduce the maps \( \alpha^L, \beta^L, \alpha^R, \beta^R \) and use them to define \( U(\mathfrak{g}^L)\)-bimodule structure on \( H^L \) and \( U(\mathfrak{g}^R)\)-bimodule structure on \( H^R \).

The inclusions of filtered components \( U_k(\mathfrak{g}) \subset U_{k+1}(\mathfrak{g}) \subset U(\mathfrak{g}) \) induce epimorphisms of dual vector spaces \( U(\mathfrak{g})^* \rightarrow U_{k+1}(\mathfrak{g})^* \rightarrow U_k(\mathfrak{g})^* \), hence a complete cofiltration on \( U(\mathfrak{g})^* = \lim_k U_k(\mathfrak{g})^* \) (see Appendix A.2). At each finite level \( k \), \( U_k(\mathfrak{g}) \) is finite dimensional, hence \( (U_k(\mathfrak{g}) \otimes U_k(\mathfrak{g}))^* \cong U_k(\mathfrak{g})^* \otimes U_k(\mathfrak{g})^* \). Thus the multiplication \( U_k(\mathfrak{g}) \otimes U_l(\mathfrak{g}) \rightarrow U_{k+l}(\mathfrak{g}) \subset U_{k+l}(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \) for \( k = l \) dualizes to \( \Delta_k : U(\mathfrak{g})^* \rightarrow U_k(\mathfrak{g})^* \otimes U_k(\mathfrak{g})^* \) and the inverse limit is the coproduct \( \Delta_{U(\mathfrak{g})^*} := \lim_k \Delta_k : U(\mathfrak{g})^* \rightarrow \lim_k U_k(\mathfrak{g})^* \otimes U_k(\mathfrak{g})^* \cong \lim_k \lim_l U_{\mu}(\mathfrak{g})^* \otimes U_{\nu}(\mathfrak{g})^*. \) The right-hand side is by definition the completed tensor product \( U(\mathfrak{g})^* \otimes U(\mathfrak{g})^* \).
(For completed tensoring of elements and maps we below often use simplified notation, \(\otimes\).) Coproduct \(\Delta_{U(g)^*}\) transfers, along the dual isomorphism \(\xi^T: U(g)^* \xrightarrow{\cong} S(g)^*\) of cofiltered algebras, to the topological coproduct on the completed symmetric algebra \(\hat{S}(g)^* \cong S(g)^*\) (cf. [19]),

\[
\Delta_{\hat{S}(g)^*}: \hat{S}(g)^* \to \hat{S}(g)^* \hat{S}(g)^*.
\]

This construction can be performed both for \(g^L\) and for \(g^R\). The canonical isomorphism of Hopf algebras \(U(g^R) \cong U(g^L)^{\text{op}}\) induces the isomorphism of dual cofiltered Hopf algebras \(U(g^R)^* \cong (U(g^L)^*)^{\text{co}}\), commuting with \(\xi^T\), hence inducing an isomorphism of Hopf algebras \(\hat{S}(g^{Re}) \cong \hat{S}(g^{Le})^{\text{co}}\) fixing the underlying algebra \(\hat{S}(g)^*\). Thus, the coproduct on \(\hat{S}(g^{Re})\) is \(\Delta_{\hat{S}(g^{Re})}^{\text{op}}\), hence we just write \(\hat{S}(g)^*\) and use the algebra identification, with the (co)opposite signs \(\hat{S}(g)^*\) or \(\Delta_{\hat{S}(g)^*}^{\text{op}}\) whenever needed.

As discussed in [19] [22], the coproduct is equivalently characterized by

\[
P \triangleright (\hat{f} \hat{g}) = m(\Delta_{\hat{S}(g)^*})(P \otimes \triangleright)(\hat{f} \otimes \hat{g}),
\]

for all \(P \in \hat{S}(g)^*\) (for instance, \(P = \partial^\mu\)) and all \(\hat{f}, \hat{g} \in U(g)\). Using the action \(\triangleright\) we assumed that we embedded \(\hat{S}(g)^* \to H^R \cong A_n\). The right hand version of (25) is that for all \(\hat{u}, \hat{v} \in U(g^R)\) and \(Q \in \hat{S}(g)^*\),

\[
(\hat{u} \hat{v}) \triangleright Q = m((\hat{u} \otimes \hat{v})(\triangleright \otimes \triangleright)\Delta_{\hat{S}(g)^*}^{\text{op}}(Q)).
\]

**DEFINITION 2.** The homomorphism \(\alpha^L: U(g^L) \to H^L\) is the inclusion \(U(g^L) \to U(g^L) \otimes 1 \to U(g^L) \otimes \hat{S}(g)^* = H^L\) and \(\alpha^R: U(g^R) \to H^R\) is the inclusion \(U(g^R) \to 1 \otimes U(g^R) \to \hat{S}(g)^* \otimes U(g^R) = H^R\). Thus, in our writing conventions, \(\alpha^L(\hat{f}) = \hat{f}\) and \(\alpha^R(\hat{u}) = \hat{u}\). Likewise, \(\beta^L: (A^L)^{\text{op}} \to H^L\) and \(\beta^R: (A^R)^{\text{op}} \to H^R\) are the unique antihomomorphisms of algebras extending the formulas (cf. [24])

\[
\beta^L(\hat{x}_\mu) = \hat{x}_\rho(\text{O}^{-1})^{\rho}_{\mu} = \hat{y}_\mu \in H^L, \quad \beta^R(\hat{y}_\alpha) := \text{O}^{\alpha}_{\sigma} \hat{y}_\sigma = \text{O}^{\alpha}_{\sigma} \hat{x}_\sigma(\text{O}^{-1})^{\rho}_{\mu} = \hat{z}_\alpha \in H^R.
\]

The extension \(\beta^L\) exists, because the extension of the map \(\hat{x}_\mu \mapsto \hat{y}_\mu\) on \(g\) to the antihomomorphism \(\beta^L_{\hat{f}}: T(g) \to H^L\) from the tensor algebra maps \([\hat{x}_\alpha, \hat{x}_\beta] = C^{\gamma}_{\alpha\beta} \hat{x}_\gamma\) to \([\hat{y}_\beta, \hat{y}_\alpha] - C^{\gamma}_{\alpha\beta} \hat{y}_\gamma\) = 0; similarly for \(\beta^R\).

**PROPOSITION 1.** (i) \(H^L\) is a \(U(g^L)\)-bimodule via the formula \(a.h.b := \alpha^L(a)\alpha^L(b)h\), for all \(a, b \in U(g^L), h \in H^L\). Likewise, \(H^R\) is a \(U(g^R)\)-bimodule via \(a.h.b := h\beta^R(a)\alpha^R(b)\), for all \(a, b \in U(g^R), h \in H^R\). From now on these bimodule structures are assumed.
(ii) For any \( \hat{f}, \hat{g} \in U(\mathfrak{g}^L) \) and any \( \hat{u}, \hat{v} \in U(\mathfrak{g}^R) \),

\[
\beta^L(\hat{g}) \triangleright \hat{f} \hat{g}, \quad \hat{u} \triangleright \beta^R(\hat{v}) = \hat{v}\hat{u}.
\] (28)

Proof. (i) The bimodule property of commuting of the left and the right \( U(\mathfrak{g}^L) \)-actions is ensured by \( \hat{x}_\mu, \hat{y}_\nu = 0 \). For the \( U(\mathfrak{g}^R) \)-actions it boils down to \( \hat{y}_\mu, \mathcal{O}_\nu \tilde{x}_\sigma (\mathcal{O}^{-1})^\rho_\nu = 0 \), what follows from the Theorem [12].

(ii) follows from (17) and (22), by induction on the filtered degree of \( \hat{g} \) (respectively, of \( \hat{v} \)).

PROPOSITION 2. Let \( \hat{H}^L := U(\mathfrak{g}^L) \# S(\mathfrak{g}^*) \) and \( \hat{H}^R := S(\mathfrak{g}^*) \circ U(\mathfrak{g}^R) \) be the completed smash products (Appendix A.2) which are cofiltered algebras. (i) The inclusions \( H^L \otimes H^L \rightarrow \hat{H}^L \otimes H^L \), \( H^R \otimes H^R \rightarrow \hat{H}^R \otimes \hat{H}^R \),

\[ H^L \otimes_{U(\mathfrak{g}^L)} H^L \rightarrow \hat{H}^L \otimes_{U(\mathfrak{g}^L)} \hat{H}^L \],
\[ H^R \otimes_{U(\mathfrak{g}^R)} H^R \rightarrow \hat{H}^R \otimes_{U(\mathfrak{g}^R)} \hat{H}^R \]

are isomorphisms; (ii) The actions \( \triangleright \) and \( \triangleright \triangleright \) extend to unique completed actions \( \triangleright : \hat{H}^L \otimes U(\mathfrak{g}^L) \rightarrow U(\mathfrak{g}^L) \) and \( \triangleright : U(\mathfrak{g}^R) \otimes \hat{H}^R \rightarrow U(\mathfrak{g}^R) \).

Proof. (i) The formal sums on the right have representatives at each cofiltered degree satisfying the finiteness needed to have representatives on the left hand side. For (ii) extend the recipe from (13).

DEFINITION 3. The right ideal \( I \subset H^L \otimes H^L \) is the right ideal generated by the set of all elements of the form \( \beta^L(\hat{f}) \otimes 1 - 1 \otimes \alpha^L(\hat{f}) \) where \( \hat{f} \in H^L \). In other words, \( I \) is the kernel of the map \( H^L \otimes H^L \rightarrow H^L \otimes_{U(\mathfrak{g}^L)} H^L \cong H^L \otimes H^L/I \).

The right ideal \( I' \subset H^L \otimes H^L \) is the set of all \( \sum_i h_i \otimes h'_i \in H^L \otimes H^L \) such that

\[
\sum_{i,j} (h_i \triangleright \hat{f}_j)(h'_i \triangleright \hat{g}_j) = 0, \quad \text{for all} \quad \sum_j \hat{f}_j \otimes \hat{g}_j \in U(\mathfrak{g}^L) \otimes U(\mathfrak{g}^L).
\]

Similarly, \( \hat{I} := \text{Ker} \ H^R \otimes H^R \rightarrow H^R \otimes_{U(\mathfrak{g}^R)} H^R \) and \( \hat{I}' \) is the left ideal in \( H^R \otimes H^R \) generated by elements of the form \( \alpha^R(\hat{g}_\mu) \otimes 1 - 1 \otimes \beta^R(\hat{g}_\mu) \). The completions of ideals (Appendix A.2) \( I, I', \hat{I}, \hat{I}' \) are \( \hat{I}, \hat{I}' \subset H^L \otimes H^L \cong \hat{H}^L \otimes \hat{H}^L \) and \( \hat{I}, \hat{I}' \subset H^R \otimes H^R \cong \hat{H}^R \otimes \hat{H}^R \), respectively.

More generally, for \( r \geq 2 \), let \( I^{(r)} \) be the kernel of the canonical projection \( (H^L)^{\otimes r} := H^L \otimes H^L \otimes \ldots \otimes H^L \) (r factors) to the tensor product of \( U(\mathfrak{g}^L) \)-bimodules \( H^L \otimes_{U(\mathfrak{g}^L)} H^L \otimes_{U(\mathfrak{g}^L)} \ldots \otimes_{U(\mathfrak{g}^L)} H^L \). \( I^{(r)} \) coincides with the smallest right ideal in the tensor product algebra \( (H^L)^{\otimes r} \) which contains \( 1 \otimes I \otimes 1^{(r-k)} \) for \( k = 0, \ldots, r-2 \). Let \( I^{(r)}_R \) be the set of all elements \( \sum_i h_{1i} \otimes h_{2i} \otimes \ldots \otimes h_{ri} \in (H^L)^{\otimes r} \) such that, for any \( \sum_j u_{1j} \otimes u_{2j} \otimes \ldots \otimes u_{rj} \in U(\mathfrak{g}^L)^{\otimes r} \),

\[
\sum_{i,j} (h_{1i} \triangleright u_{1j})(h_{2i} \triangleright u_{2j}) \cdots (h_{ri} \triangleright u_{rj}) = 0.
\]
THEOREM 4. (i) The restriction of $\triangleright: \mathcal{H}^L \otimes U(\mathfrak{g}^L) \to U(\mathfrak{g}^L)$ to $\hat{S}(\mathfrak{g}^*) \otimes U(\mathfrak{g}^*) \to U(\mathfrak{g}^L)$ turns $U(\mathfrak{g}^L)$ into a **faithful** left $\hat{S}(\mathfrak{g}^*)$-module. 
(ii) The right ideals $I = I'$ agree and the left ideals $\hat{I} = \hat{I}'$ agree. 
(iii) More generally, $I^{(r)} = \hat{I}^{(r)}$, $\hat{I}^{(r)} = \hat{I}^{(r)}$ for $r \geq 2$. 
(iv) Statements (ii) and (iii) hold also for the completed ideals.

**Proof.** We show part (ii) for the right ideals; the method of the proof easily extends to left ideals, and to (i), (iii) and (iv).

Let $\sum_i \hat{f}_i \otimes \hat{g}_i \in I$ and $v = \hat{x}_{i_1} \cdots \hat{x}_{i_k}$ a monomial in $U(\mathfrak{g}^L)$. Then

$$(\beta^L(v) \triangleright \hat{f}_i)\hat{g}_i - \hat{u}_i a^L(v) \triangleright \hat{g}_i = (\hat{y}_{i_k} \cdots \hat{y}_{i_1} \triangleright \hat{f}_i)\hat{g}_j - \hat{f}_i \hat{x}_{i_1} \cdots \hat{x}_i \triangleleft \hat{g}_j$$

what is zero by Eq. (17) and induction on $k$. Thus, by linearity, $I \subset I'$.

It remains to show the converse inclusion $I' \subset I$. Suppose on the contrary that there is an element $\sum_i h_i \otimes h'_i \in I'$, but not in $I$; then by adding any element in $I'$ the sum is still in $I'$ and not in $I$. Observe that $\hat{x}_j \partial^L \otimes \hat{x}_{j'} \partial^{K'} = \hat{x}_j \partial^K \otimes \partial^{L'} (\hat{x}_{j'}) \partial^{K'} = \partial^L (\hat{x}_{j'}) \hat{x}_j \partial^K \otimes \partial^{K'} \mod I'$. The tensor factor $\beta(\hat{x}_{j'}) \hat{x}_j \partial^K$ belongs to $H^L$, hence it is also a linear combination of elements of the form $\hat{x}_{j'} \partial^{K'}$. Therefore, without loss of generality, we can assume

$$\sum_i h_i \otimes h'_i = \sum_{J,K,L} a_{J,K,L} \hat{x}_j \partial^K \otimes \partial^L. \quad (29)$$

By one of the results in [22],

$$P \triangleright \hat{f} = \sum (P, \hat{f}(2)) \phi \hat{f}(1) \quad (30)$$

where $P \in \hat{S}(\mathfrak{g}^*)$, $\hat{f} \in U(\mathfrak{g}^L)$ and $\langle \cdot, \cdot \rangle_{\phi} : \hat{S}(\mathfrak{g}^*) \otimes U(\mathfrak{g}^L) \to k$ is a nondegenerate topological Hopf algebra pairing defined with the help of $\phi$ (see also [14]). By the finite dimensionality of each cofiltered component $\hat{S}(\mathfrak{g}^*)$, replace the basis $\{ \partial^K \}_{K} \subset \hat{S}(\mathfrak{g}^*)$ with the unique family $\{ \partial^K \} \subset \hat{S}(\mathfrak{g}^*)$ satisfying $\langle \partial^K, \hat{x}_j \rangle = [K] \delta^K_J$. Thus we effectively replace $\partial^K$ and $\partial^L$ with $\partial^K$ and $\partial(L)$ in (29), with new coefficients $b_{J,K,L}$,

$$\sum_i h_i \otimes h'_i = \sum_{J,K,L} b_{J,K,L} \hat{x}_j \partial^K \otimes \partial(L).$$

This sum is typically infinite even if the sum over $a_{J,K,L}$ is finite. But this does not matter as the vanishing reasoning below is term by term. Choose $K$ and $J$ such that $(|K|, |L|)$ is a minimal bidegree for which $b_{J,K,L} \neq 0$ for at least some $J$. The condition $\sum_i h_i \otimes h'_i \in I'$ implies

$$\sum (h_i \triangleright \hat{x})(h'_i \triangleright \hat{x}_L) = 0.$$
Recall that $\Delta_{U(\mathfrak{g}')}\hat{x}_K = \sum_{K_1+K_2=K} \hat{x}_{K_1} \otimes \hat{x}_{K_2}$; in particular $|K_1| \leq |K|$, $|K'_2| \leq |K'|$. Therefore, $[30]$ gives

$$\partial^M \triangleright \hat{x}_K = \sum_{K_1} (\partial^M \triangleright \hat{x}_{K_2}) \Phi \hat{x}_{K_1}$$

$\partial^M \triangleright \hat{x}_K$ is zero for $|M| > |K|$; as $|K|$ is minimal only $M = K$ term survives and likewise for $L$. Therefore, only the terms with this fixed $K$ and $L$ survive in $\sum_i (h_i \triangleright \hat{x}_K)(h'_i \triangleright \hat{x}_L)$, and in that case where the black action is simply the pairing (times unit element) what implies $\sum J b_{J,K,L}x_J = 0$, in contradiction with the minimality of $(K,L)$.

5. Bialgebroid structures

Let us now use the shorter notation $\mathcal{A}^L := U(\mathfrak{g}^L)$, $\mathcal{A}^R := U(\mathfrak{g}^R)$. A suggestive symbol $\mathcal{A}$ denotes an abstract algebra in the axioms where either $\mathcal{A}^L$ or $\mathcal{A}^R$ (or both) may substitute in here intended examples. In this section, we equip the isomorphic associative algebras $H^L$ and $H^R$ with different structures: $H^L$ is a left $\mathcal{A}^L$-bialgebroid and $H^R$ a right $\mathcal{A}^R$-bialgebroid. We start by exhibiting the coring structures of these bialgebroids; an $\mathcal{A}$-coring is an analogue of a coalgebra where the ground field is replaced by a noncommutative algebra $\mathcal{A}$.

**DEFINITION 4.** [2, 7] Let $\mathcal{A}$ be a unital algebra and $C$ an $\mathcal{A}$-bimodule with left action $(a,c) \mapsto a.c$ and right action $(c,a) \mapsto c.a$. A triple $(C, \Delta, \epsilon)$ is an $\mathcal{A}$-coring if

(i) $\Delta : C \to C \otimes_\mathcal{A} C$ and $\epsilon : C \to \mathcal{A}$ are $\mathcal{A}$-bimodule maps called coproduct (comultiplication) and counit;

(ii) $\Delta$ is coassociative: $(\Delta \otimes_\mathcal{A} \text{id}) \circ \Delta = (\text{id} \otimes_\mathcal{A} \Delta) \circ \Delta$, where in the codomain the associativity isomorphism $(C \otimes_\mathcal{A} C) \otimes_\mathcal{A} C \cong C \otimes_\mathcal{A} (C \otimes_\mathcal{A} C)$ for the $\mathcal{A}$-bimodule tensor product is understood;

(iii) The counit axioms $(\epsilon \otimes_\mathcal{A} \text{id}) \circ \Delta \cong \text{id} \cong (\text{id} \otimes_\mathcal{A} \epsilon) \circ \Delta$ hold, where the identifications of $\mathcal{A}$-bimodules $C \otimes_\mathcal{A} C \cong C$, $c \otimes a \mapsto c.a$ and $\mathcal{A} \otimes_\mathcal{A} C \cong C$, $a \otimes d \mapsto a.d$ are understood.

**PROPOSITION 3.** (i) $\exists!$ linear maps $\Delta^L : H^L \to H^L \otimes_{\mathcal{A}^L} H^L$ and $\Delta^R : H^R \to H^R \otimes_{\mathcal{A}^R} H^R$ such that $\Delta^L$ and $\Delta^R$ respectively satisfy

$$P \triangleright (\hat{f} \hat{g}) = m(\Delta^L(P)(\triangleright \otimes \triangleright)(\hat{f} \otimes \hat{g})), \quad \hat{f}, \hat{g} \in \mathcal{A}^L, \quad P \in H^L, \quad (31)$$

$$\hat{u} \triangleright Q = m((\hat{u} \otimes \hat{v})(\triangleright \otimes \triangleright)\Delta^R(Q)), \quad \hat{u}, \hat{v} \in \mathcal{A}^R, \quad Q \in H^R. \quad (32)$$
(ii) $\Delta^L$ is a unique left $A^L$-module map extending $\Delta_{\hat{S}(g^*)}$ from $\hat{S}(g^*)$ to $H^L$ and $\Delta^R$ a unique right $A^R$-module map extending $\Delta^R_{\hat{S}(g^*)}$ from $\hat{S}(g^*)$ to $H^R$. Equivalently,
\[
\Delta^L(\hat{f} \delta P) = \hat{f} \Delta_{\hat{S}(g^*)}(P), \quad \Delta^R(Q \delta \hat{v}) = \Delta^R_{\hat{S}(g^*)}(Q) \delta \hat{v},
\]
for all $P,Q \in \hat{S}(g^*)$, $\hat{f} \in A^L$ and $\delta \in A^R$. In particular, $\Delta^L(\hat{x}_\mu) = \hat{x}_\mu \otimes 1$ and $\Delta^R\hat{y}_\mu = 1 \otimes \hat{y}_\mu$.

(iii) $\Delta^L\mathcal{O}^\mu = \mathcal{O}^\mu_\gamma \otimes \mathcal{O}^\mu_\lambda$, $\Delta^R\mathcal{O}^\mu_\gamma = \mathcal{O}^\mu_\gamma \otimes \mathcal{O}^\mu_\nu,
\Delta^L(\mathcal{O}^{-1})^\mu_\gamma = (\mathcal{O}^{-1})^\mu_\gamma \otimes (\mathcal{O}^{-1})^\mu_\lambda$, $\Delta^R(\mathcal{O}^{-1})^\mu_\gamma = (\mathcal{O}^{-1})^\mu_\gamma \otimes (\mathcal{O}^{-1})^\mu_\nu,
\Delta^L(\hat{y}_\mu) = \Delta^L(\hat{x}_\mu(\mathcal{O}^{-1})^\mu_\gamma) = \hat{x}_\mu(\mathcal{O}^{-1})^\mu_\gamma \otimes (\mathcal{O}^{-1})^\mu_\nu = 1 \otimes \hat{y}_\mu,
\Delta^R(\hat{y}_\mu) = \Delta^R(\hat{y}_\mu(\mathcal{O}^{-1})^\mu_\nu) = 1 \otimes \hat{y}_\nu.

(iv) $(H^L, \Delta^L, \epsilon^L)$ and $(H^R, \Delta^R, \epsilon^R)$ satisfy all the axioms for $A^L$-coring and $A^R$-coring respectively, provided we replace the tensor product of bimodules by the completed tensor of (cofiltered) bimodules.

Taking into account our bimodule structures, the counit axioms, Definition 4 (iii), read
\[
\sum \alpha(\epsilon^L(h_{(1)}))h_{(2)} = h = \sum \beta(\epsilon^L(h_{(2)}))h_{(1)}, \quad h \in H^L
\]
\[
\sum h_{(2)} \beta^R(\epsilon^R(h_{(1)})) = h = \sum h_{(1)} \alpha^R(\epsilon^R(h_{(2)})), \quad h \in H^R.
\]

(v) The coring structures from (iv) canonically extend to of an internal $A^L$-coring $(H^L, \Delta^L, \epsilon^L)$ and an internal $A^R$-coring $(H^R, \Delta^R, \epsilon^R)$ (see [3]) in the category of complete cofiltered vector spaces with $\otimes$-tensor product (see Proposition 3 and Appendix A.2). Bimodule structures on $H^L, H^R$, involve homomorphisms $\alpha^L := j_L \circ \alpha_L, \epsilon^R := j_R \circ \epsilon_R$, and antihomomorphisms $\beta^L := j_L \circ \beta_L, \beta^R := j_R \circ \beta_R$, where $j_L : H^L \hookrightarrow H^L$ and $j_R : H^R \hookrightarrow H^R$ are the inclusions.

Proof. The equivalence of the two statements in (ii) is evident. By Theorem 4 (ii), the satisfaction of the formulas (31), (32) uniquely determines $\Delta^L(P)$ and $\Delta^R(Q)$, showing the uniqueness. To show the existence, we set the values of $\Delta^L$ and $\Delta^R$ by (33) and check that (31) and (32) hold. We already know this for $P,Q \in \hat{S}(g^*)$ by (25) and (26). Using the action axiom for $\circ$, observe that
\[
\hat{x}_\mu \circ (P \circ (\hat{f} \delta\hat{g})) = \hat{x}_\mu \cdot m(\Delta^L(P)(\circ \otimes \circ)(\hat{f} \otimes \hat{g}))
\]
\[
= m(\hat{x}_\mu \Delta^L(P)(\circ \otimes \circ)(\hat{f} \otimes \hat{g}))
\]
(34)
\[
= m(\Delta^L(\hat{x}_\mu P)(\circ \otimes \circ)(\hat{f} \otimes \hat{g}))
\]
for all $\hat{f}, \hat{g} \in A^L$, hence (31) holds for all $P \in H^L$. Likewise check (32) for all $Q \in H^R$. Thus, (i). The statement in (ii) that $\Delta^L, \Delta^R$ then extend $\Delta_{\hat{S}(g^*)}, \Delta^R_{\hat{S}(g^*)}$ is the statement that (31), (32) specialize to (25), (26) when $P,Q \in \hat{S}(g^*)$. The rest of (ii) follows from uniqueness in (i).
(iii) By Theorem 4 (ii), the first 4 formulas follow from (15), (20), (16), (21). The formulas for $\Delta L(\hat{y}_\alpha)$ and $\Delta R(\hat{x}_\alpha)$ are straightforward.

(iv) To show that $\Delta L$ is an $A_L$-bimodule map note that by (ii) it commutes with the left $A_L$-action. It remains to show that $\Delta L$ commutes also with the right $A_L$-action. This is sufficient to check on the generators $\hat{x}_\mu$ of $A_L$ and arbitrary $P \in \hat{S}(g^*)$:

$$\Delta L(P, \hat{x}_\mu) = \sum P(1) \otimes A L \beta(\hat{x}_\mu) P(2) = \sum P(1) \otimes A L \alpha(\hat{x}_\mu) (O^{-1})_P P(2) = \sum P(1) \otimes A L (O^{-1})_P (O^{-1})_P P(2) = \Delta L(\hat{x}_\mu) (O^{-1})_P P(2) = \Delta L(\beta(\hat{x}_\mu)(P))$$

By Theorem 4 (iii) for $r = 3$, the action axiom for $\triangleright$ and associativity in $H L$ implies the coassociativity of $\Delta L$.

We exhibit the counits $\epsilon L$ and $\epsilon R$ (and $\tilde{\epsilon}_L$, $\tilde{\epsilon}_R$) by the actions on 1, $\epsilon L(h) := h \triangleright 1_{A_L}, \epsilon R(h) := 1_{A_R} \triangleright h.$ (35)

The counit axioms (34) for $\epsilon L$ are checked on the generators $\hat{x}_\mu$:

$$\sum \alpha(\epsilon L(\hat{x}_\mu(1))) \hat{x}_\mu(2) = \alpha(\epsilon L(\hat{x}_\mu)) 1 = \hat{x}_\mu,$$

$$\sum \beta(\epsilon L(\hat{x}_\mu(2))) \hat{x}_\mu(1) = \beta(\epsilon L(1)) \hat{x}_\mu = \hat{x}_\mu.$$

Similarly, one checks the counit identities for $\epsilon R$.

Using formal expressions in the completions, (v) is straightforward.

**DEFINITION 5.** [2, 4, 6] Given an algebra $A$, a **left $A$-bialgebroid** $(H, m, \alpha, \beta, \Delta, \epsilon)$ is a unital algebra $(H, m)$ which is an $A$-bimodule via the formula $a.h.a' := \alpha(a)\beta(a')h$, where the homomorphisms of algebra $\alpha : A \to H$ and $\beta : A^{op} \to H$ are fixed and have commuting images; $\Delta : A \to H \otimes A H$ is $A$-bimodule map, which is coassociative and with counit $\epsilon : H \to A$. It is required that

(i) $\epsilon$ is a **left character** on the $A$-ring $(H, m, \alpha)$ in the sense that the formula $h \otimes f \mapsto \epsilon(h \alpha(f))$ defines an action $H \otimes A \to A$ extending the left regular action $A \otimes A \to A$;

(ii) the coproduct $\Delta : H \to H \otimes_A H$ corestricts to the **Takeuchi product** [25]

$$H \times_A H \subset H \otimes_A H$$

which is by definition the $A$-subbimodule

$$\{ \sum b_i \otimes b'_i \in H \otimes_A H \mid \sum b_i \otimes b'_i \alpha(a) = \sum b_i \beta(a) \otimes b'_i, \forall a \in A \}$$
of $H \otimes_A H$ and is an algebra \cite{2, 23} with factorwise multiplication.

(iii) The corestriction $\Delta\mid : H \to H \times_A H$ is an algebra map.

Notice that, because $I \subset H \otimes H$ is just a right ideal in general, the tensor product $H \otimes_A H = H \otimes H/I$ does not carry a well-defined multiplication induced from $H \otimes H$, unlike $H \times_A H$ which does.

Interchanging the left and right sides in all modules and binary tensor products in the definition, we get a right $A$-bialgebroid \cite{2}. The $A$-bimodule structure on $H$ is then given by $a . h . b := h a (b) \beta (a)$. In short, $(H, m, \alpha, \beta, \Delta, \epsilon)$ is a right $A$-bialgebroid iff $(H, m, \beta, \alpha, \Delta^\text{op}, \epsilon)$ is a left $A^\text{op}$-bialgebroid. $H^L$ is not quite an $A^L$-bialgebroid: $\Delta^L$ takes value in the completion $H^L \hat{\otimes}_{A^L} H^L$. Moreover, to check the compatibility of $\Delta^L$ with the multiplication $m$, we should be able to extend $m$ to $H^L \otimes H^L \to H^L$ which is impossible. But there is an extension $H^L \otimes H^L \cong \hat{H}^L \hat{\otimes} H^L \cong \hat{H}^L$. Thus, in addition to $\hat{\otimes}$ for $\Delta^L$, we also need to replace $H^L$ by the completed smash product $\hat{H}^L$ from Proposition \cite{2}

**Proposition 4.** $(\hat{H}^L, \hat{m}, \hat{\alpha}^L, \hat{\beta}^L, \hat{\Delta}^L, \hat{\epsilon}^L)$ has a structure of left $A^L$-bialgebroid and $(\hat{H}^R, \hat{m}, \hat{\alpha}^L, \hat{\beta}^L, \hat{\Delta}^L, \hat{\epsilon}^L)$ of a right $A^R$-bialgebroid, provided $\hat{\otimes}$ is replaced by $\hat{\circ}$ everywhere in the axioms.

*Proof.* The internal coring axioms are checked in Proposition \cite{3}

To check that $\sum h_\lambda \otimes f_\lambda \mapsto \sum \epsilon^L (h_\lambda \alpha (f_\lambda))$ is an action and (i) holds for $\epsilon^L$, observe from the definition \cite{23} that $\epsilon^L (h \alpha (f)) = (h \alpha (f) \mapsto 1 = h \mapsto \hat{f}$, for all $\hat{f} \in A^L$, $h \in H^L$. Analogously check (i) for $\epsilon^R, \epsilon^L, \epsilon^R$.

To show that $\hat{\Delta}^L$ corestricts to the completed Takeuchi product $\hat{H}^L \hat{\otimes}_{A^L} \hat{H}^L$, calculate for $P \in H^L$ and $f, g, h \in A^L$,

\[
((P_1) \hat{\beta}^L (g) \mapsto \hat{f}) \cdot (P_2) \mapsto \hat{h}) = (P_1) \mapsto (\hat{\beta}^L (\hat{g}) \mapsto \hat{f}) \cdot (P_2) \mapsto \hat{h}) \stackrel{\quad(28)}{=} (P_1) \mapsto ((\hat{f} \hat{g}) \hat{h}) = (P_1) \mapsto \hat{f} \cdot ((P_2) \hat{\alpha} (\hat{g}) \mapsto \hat{h}),
\]

thus, by Theorem \cite{4} (ii), (iv), $P_1 \hat{\beta}^L (\hat{g}) \otimes_{A^L} P_2 = P_1 \otimes_{A^L} P_2 \hat{\alpha} (\hat{g})$, hence $\hat{\Delta}^L (P) \in H^L \hat{\otimes}_{A^L} H^L$.

We now check directly that the corestriction $\hat{\Delta}^L : \hat{H}^L \to \hat{H}^L \hat{\otimes}_{A^L} \hat{H}^L$ is a homomorphism of algebras,

$\hat{\Delta}^L (h_1 h_2) = \hat{\Delta}^L (h_1) \hat{\Delta}^L (h_2)$, for all $h_1, h_2 \in H^L$.

To this aim, recall that $\Delta \hat{S}(g^*) : \hat{S}(g^*) \to \hat{S}(g^*) \hat{\otimes} \hat{S}(g^*)$ is a homomorphism, and that by Proposition \cite{3} (ii), $\hat{\Delta}^L|_{1_\hat{S}(g^*)}$ is the composition

$1_{\hat{S}(g^*)} \cong \hat{S}(g^*) \xrightarrow{\Delta \hat{S}(g^*)} \hat{S}(g^*) \hat{\otimes} \hat{S}(g^*) \hookrightarrow \hat{H}^L \hat{\otimes}_{A^L} \hat{H}^L,$
hence homomorphism as well (the inclusion is a homomorphism, because the product is factorwise). We use this when applying to the tensor factor $P(2)Q$ in the calculation

\[
\hat{\Delta}^L((u\sharp P)(v\sharp Q)) = \hat{\Delta}^L(u(P(1) \triangleright v)\sharp P(2)Q) \\
= (u(P(1) \triangleright v)\sharp P(2)Q(1)) \otimes (1\sharp P(3)Q(2)) \\
= [(u\sharp P(1))(v\sharp Q(1))] \otimes (1\sharp P(3)Q(2)) \\
= (u(P(1) \triangleright v)\sharp P(2)Q(1)) \otimes (1\sharp P(3)Q(2)) \\
= [(u\sharp P(1)) \otimes (1\sharp P(2))][(v\sharp Q(1)) \otimes (1\sharp Q(2))] \\
= \Delta^L(u\sharp P)\Delta^L(v\sharp Q).
\]

6. The antipode and Hopf algebroid

A Hopf algebroid is roughly a bialgebroid with an antipode. In the literature, there are several nonequivalent versions. In the framework of G. Böhm [2], there are two variants which are equivalent if the antipode is bijective (as it is here the case): nonsymmetric and symmetric. The nonsymmetric involves one-sided bialgebroid with an antipode map satisfying axioms which involve both the antipode map and its inverse. The “symmetric” version involves two bialgebroids and axioms neither involve nor require the inverse of the antipode. We choose this version here, because we naturally constructed two actions, $\triangleright$ and $\blacktriangleleft$, which lead to the two coproducts, $\Delta^L$ and $\Delta^R$, as we have shown in Section 5.

DEFINITION 6. Given two algebras $A^L$ and $A^R$ with fixed isomorphism $(A^L)^{op} \cong A^R$, a symmetric Hopf algebroid is a pair of a left $A^L$-bialgebroid $H^L$ and a right $A^R$-bialgebroid $H^R$, isomorphic and identified as algebras $H \cong H^L \cong H^R$, such that the compatibilities

\[
\alpha^L \circ \epsilon^L \circ \beta^R = \beta^R, \quad \beta^L \circ \epsilon^L \circ \alpha^R = \alpha^R, \\
\alpha^R \circ \epsilon^R \circ \beta^L = \beta^L, \quad \beta^R \circ \epsilon^R \circ \alpha^L = \alpha^L.
\]

(36)

hold between the source and target maps $\alpha^L, \alpha^R, \beta^L, \beta^R$, and the counits $\epsilon^L, \epsilon^R$; the comultiplications $\Delta^L$ and $\Delta^R$ satisfy the compatibility relations

\[
(\Delta^R \otimes_{A^L} id) \circ \Delta^L = (id \otimes_{A^R} \Delta^L) \circ \Delta^R
\]

(37)

and there is a map $SS : H \to H$, called the antipode which is an antihomomorphism of algebras and satisfies

\[
SS \circ \beta^L = \alpha^L, \quad SS \circ \beta^R = \alpha^R \\
m \circ (SS \otimes id) \circ \Delta^L = \alpha^R \circ \epsilon^R \\
m \circ (id \otimes SS) \circ \Delta^R = \alpha^L \circ \epsilon^L
\]

(39)
THEOREM 5. For $A^L = U(g^L)$, $A^R = U(g^R)$, $\hat{H}^L := U(g^L)\hat{\Delta}(g^2)$, $\hat{H}^R := \hat{S}(g^*)\hat{\Delta}(g^R)$, $\epsilon^L, \epsilon^R, \alpha^L, \beta^L, \alpha^R, \beta^R$ from Section 4, and $\hat{\Delta}^L$, $\hat{\Delta}^R$, defined in Section 5, the axioms for Hopf algebroid $H$ are satisfied provided the tensor product in axioms is replaced by the completed tensor product. The antipode map $SS : H \to H$ is a unique continuous (in the sense of formal power series) antihomomorphisms of algebras such that

$$SS(\partial^\mu) = -\partial^\mu,$$

(hence, by continuity, $SS(O) = SS(e^C) = e^{-C} = O^{-1}$, and

$$SS(\hat{y}_\mu) = \hat{x}_\mu.$$  \hfill (40)

The antipode $SS$ is bijective. For general $g$, $SS^2 \neq id$. More precisely,

$$SS(\hat{y}_\mu) = \hat{x}_\mu - C_{\mu\lambda} \lambda, \quad SS^{-2}(\hat{x}_\mu) = SS^{-1}(\hat{y}_\mu) = \hat{x}_\mu - C_{\mu\lambda} \lambda$$  \hfill (41)

$$SS^2(\hat{x}_\mu) = \hat{x}_\mu + C_{\mu\lambda} \lambda, \quad SS^{-2}(\hat{y}_\mu) = \hat{y}_\mu + C_{\mu\lambda} \lambda.$$  \hfill (42)

Proof. In this proof, we simply write $\epsilon^L, \Delta^L$ etc. without hat symbol, as it is not essential for the arguments below.

One checks the relations (36) on generators, for which $\alpha^R(\hat{y}_\mu) = \hat{y}_\mu, \beta^R(\hat{y}_\mu) = O^\rho_\mu \hat{y}_\rho = O^\rho_\mu \hat{y}_\rho (O^{-1})^\rho_\mu, \alpha^L(\hat{x}_\mu) = \hat{x}_\mu, \beta^L(\hat{x}_\mu) = \hat{y}_\mu$.

Regarding that $\Delta^L$ and $\Delta^R$ restricted to $\hat{S}(g^*)$ coincide with $\hat{\Delta}_S(g^*)$, (37) and (38) restricted to $\hat{S}(g^*)$ reduce to the coassociativity. Algebra $H^L$ is generated by $\hat{S}(g^*)$ and $g^L$, so it is enough to check (37), (38) also on $\hat{y}_\mu = \hat{x}_\nu (O^{-1})^\nu_\mu$. This follows from the identities (in matrix form)

$$\Delta^L(\hat{x}O^{-1}) = \hat{x}O^{-1} \otimes_{A^L} O^{-1} = O^{-1} \otimes_{A^L} \hat{x}O^{-1} = 1 \otimes_{A^L} \hat{x}O^{-1},$$

$$\Delta^R(\hat{y}) = 1 \otimes_{A^R} \hat{y} = \hat{y} \otimes_{A^R} O^{-1} = \hat{x}O^{-1} \otimes_{A^R} O^{-1}.$$  \hfill (43)

Formula $SS(\partial^\mu) = -\partial^\mu$ clearly extends to a unique continuous antihomomorphism of algebras on the formal power series ring $\hat{S}(g^*)$. Similarly, by functoriality of $g \mapsto U(g)$, the antihomomorphism of Lie algebras, $SS : g^R \to g^R, \hat{y}_\mu \mapsto \hat{x}_\mu$, extends to a unique antihomomorphism $U(g^R) \to U(g^L)$. Regarding that $U(g^R)$ and $\hat{S}(g^*)$ generate $H^R$, it is sufficient to check that $SS$ is compatible with the additional relations in the smash product, namely $[\partial^\mu, \hat{y}_\nu] = \left(\begin{array}{c} C \mu \\ \nu \\ \end{array} \right)$.

Then $SS([\partial^\mu, \hat{x}_\nu]) = SS\left(\begin{array}{c} e^{-C} \\ \mu \\ \nu \\ \end{array} \right) = \left(\begin{array}{c} C \mu \\ \nu \\ \end{array} \right) - \left(\begin{array}{c} -C \mu \\ \nu \\ \end{array} \right)$, which equals $(e^{-C})^\mu_\nu [\hat{x}_\mu, -\partial^\mu] = [SS(\hat{y}_\mu O^\rho_\nu), -\partial^\mu] = [SS(\hat{y}_\mu), SS(\partial^\mu)]$.

To exhibit the inverse $SS^{-1}$, we similarly check that the obvious formulas $SS^{-1}(\hat{x}_\mu) = \hat{y}_\mu$, $SS^{-1}(\partial^\mu) = \partial^\mu$ define a unique continuous antihomomorphism $SS^{-1} : H \to H$. 

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For (41) calculate \( SS(\hat{x}_\mu) = SS(\hat{y}_\mu \mathcal{O}_\mu^\rho) = SS(\mathcal{O}_\mu^\rho) SS(\hat{y}_\rho) = (O^{-1})^\rho_\mu \hat{x}_\mu = (O^{-1})^\rho_\mu \hat{y}_\rho \mathcal{O}_\rho^\rho \) and use \([\mathcal{O}_\mu^\rho, \hat{y}_\sigma] = -C_{\tau\sigma}^\rho \mathcal{O}_\mu^\tau\) in the last step. Similarly, we get \( SS^{-1}(\hat{y}_\mu) = \mathcal{O}_\mu^\rho \hat{x}_\rho (O^{-1})^\rho_\mu \) and use \([\mathcal{O}_\mu^\rho, \hat{x}_\sigma] = -C_{\tau\sigma}^\rho \mathcal{O}_\mu^\tau\) for the second formula in (41). Notice that \( SS^{-1}(\hat{y}_\mu) = \hat{z}_\mu \) from Theorem 3, formula (22). For (42) similarly use the identities (in matrix form) \( SS^2(\hat{x}) = SS(O^{-1} \hat{y} \mathcal{O}) = O^{-1} \hat{x} \mathcal{O}, \ SS^{-2}(\hat{y}) = \mathcal{O} \hat{y} O^{-1}. \)

The formula \( SS(\beta L(\hat{x}_\mu)) = SS(\hat{y}_\mu) = \hat{x}_\mu = \alpha L(\hat{x}_\mu) \) shows \( SS \circ \beta L = \alpha L \) for the generators of \( \mathcal{A}^L \). Likewise for the rest of identities (39).

7. Conclusion and perspectives.

We have equipped the noncommutative phase spaces of Lie algebra type with the structure of a version of a Hopf algebroid over \( U(\mathfrak{g}) \). That means that we have found a left \( U(\mathfrak{g}) \)-bialgebroid \( \hat{H}^L \), and a right \( U(\mathfrak{g})^{op} \)-bialgebroid \( \hat{H}^R \), which are canonically isomorphic as associative algebras \( \hat{H}^L \cong \hat{H}^R \), and an antipode map \( SS \) satisfying a number of axioms involving a completed tensor product \( \hat{\otimes} \).

Hopf algebroids allow a version of Drinfeld’s twisting cocycles studied earlier in the context of deformation quantization [24], and are a promising tool for extending many constructions to the noncommutative case, and a planned direction for our future work. One can compare the Hopf algebroid corresponding to the abelian Lie algebra (thus, a completion of the usual Weyl algebra) with the Hopf algebroid corresponding to a nontrivial Lie algebra of the same dimension. It is a nontrivial fact that the latter could be obtained as a Drinfeld twist (in the Hopf algebroid sense) of the former. This enables twisting many geometric structures, including differential forms, from the undeformed to the deformed case. This has earlier been studied in the case of \( \kappa \)-spaces (e.g. in [12]), while the work for general finite-dimensional Lie algebras is in progress.

Appendix

A.1 Commutation \([\hat{x}_\alpha, \hat{y}_\beta] = 0\)

PROPOSITION 5. The identity \([\hat{x}_\mu, \hat{y}_\nu] = 0\) holds in the realization \( \hat{x}_\mu = x_\sigma \hat{\phi}_\mu^\sigma = x_\sigma \left( \frac{-C_{\sigma \gamma}}{C_{\sigma \gamma}^\rho} \right)^\rho_\mu \), \( \hat{y}_\mu = x_\rho \hat{\phi}_\mu^\rho = x_\rho \left( \frac{-C_{\rho \gamma}}{C_{\rho \gamma}^\rho} \right)^\rho_\mu \), where \( C_{\alpha \beta}^\nu = C_{\gamma \delta}^\nu \) (cf. the equations (6,3,4)).
Proof. For any formal series $P = P(\partial)$ in $\partial$-s, $\{P, \hat{x}_\mu\} = \frac{\partial P}{\partial (\hat{x}_\mu)} =: \delta_\mu P$. In particular (cf. [8]), from $[\hat{x}_\mu, \hat{x}_\nu] = C^\lambda_{\mu\nu} \hat{x}_\lambda$, one obtains a formal differential equation for $\phi^\sigma_\mu$:

\[
(\delta_\rho \phi^\gamma_\mu) \phi^\rho_\nu - (\delta_\rho \phi^\gamma_\nu) \phi^\rho_\mu = C^\sigma_{\mu\nu} \phi^\gamma_\rho.
\]  

(43)

By symmetry $C^\gamma_{jk} \mapsto -C^\gamma_{kj}$ (or as directly shown in [8]), the same equation holds with $(-\hat{\phi}) = \frac{\partial}{\partial e \cdot C}$ in the place of $\phi$. Similarly, the equation $[\hat{x}_\mu, \hat{y}_\nu] = 0$, i.e. $[x_\mu \phi^\gamma_\mu, x_\beta \phi^\gamma_\beta] = 0$, is equivalent to

\[
(\delta_\rho \phi^\gamma_\mu) \phi^\rho_\nu - (\delta_\rho \phi^\gamma_\nu) \phi^\rho_\mu = 0
\]  

(44)

Recall that $\phi = \frac{\partial}{\partial e}$ = $\sum_{N=0}^\infty (-1)^N \frac{B_N}{N!} (C^N)^i$, where $B_N$ are the Bernoulli numbers, which are zero unless $N$ is either even or $N = 1$. Hence, \( \hat{\phi} = \frac{\partial}{\partial e^{-1}} = \sum_{N=0}^\infty \frac{B_N}{N!} (C^N)^i = \frac{B_1}{2} \partial + \sum_{N=0}^\infty \text{even} \frac{B_N}{N!} (C^N)^i \), and $\phi - \hat{\phi} = -\frac{B_1}{2} \partial = C$. Notice that $\frac{\partial B_1}{\partial \phi} = C^2 \partial$. Therefore, subtracting (43) from (44) gives the condition

\[
(\delta_\rho \phi^\gamma_\mu) \phi^\rho_\nu - (\delta_\rho \phi^\gamma_\nu) \phi^\rho_\mu = C^\sigma_{\mu\nu} \phi^\gamma_\rho.
\]  

$C$ is homogeneous of degree 1 in $\partial\nu$-s, so we can split this condition into the parts of homogeneity degree $N$:

\[
[\delta_\rho (C^N)^{ij}] \phi^\rho_\nu - (\delta_\rho (C^N)^{ij}) \phi^\rho_\mu = C^\sigma_{\mu\nu} (C^N)^{i\sigma},
\]  

(45)

where the overall factor of $(-1)^N B_N/N!$ has been taken out. Hence the proof is reduced to the following lemma:

**LEMMA 1.** *The identities* (44) *hold for $N = 0, 1, 2, \ldots$.*

**Proof.** For $N = 0$, (44) reads $C^\gamma_{\mu\nu} = C^\gamma_{\nu\mu}$, which is the antisymmetry of the bracket. For $N = 1$ it follows from the Jacobi identity:

\[
(C^\gamma_{\mu\rho} C^\rho_{\nu\tau} - C^\gamma_{\nu\rho} C^\rho_{\mu\tau}) \partial^\tau = C^\rho_{\mu\nu} C^\gamma_{\rho\tau} \partial^\tau.
\]

Suppose now (45) holds for given $N = K \geq 1$. Then

\[
C^\gamma_{\nu\rho} (C^K)^\rho_{\mu\rho} C^\gamma_{\rho\sigma} = [\delta_\rho (C^K)^{\rho\mu}] (C^\sigma)^{\sigma\nu} C^\gamma_{\mu\rho} C^\rho_{\nu\tau} - C^\sigma_{\nu\sigma} (C^K)^{\rho\nu} C^\gamma_{\rho\sigma}.
\]

By the usual Leibniz rule for $\delta_\rho$, this yields

\[
C^\gamma_{\mu\nu} (C^K)^\rho_{\mu\rho} C^\gamma_{\rho\sigma} = \delta_\rho (C^K)^{\rho+1}_{\mu\nu} C^\sigma_{\mu\rho} C^\gamma_{\mu\rho} C^\rho_{\nu\tau} - C^\sigma_{\nu\sigma} (C^K)^{\rho+1}_{\mu\nu} C^\gamma_{\rho\sigma}.
\]

The identity (45) follows for $N = K + 1$ if the second and third summand on the right hand side add up to $-C^\gamma_{\nu\sigma} (C^K)^{\rho+1}_{\mu\nu}$. After renaming the indices, one brings the sum of these two to the form

\[
(C^K)^\rho_{\mu\nu} (C^\sigma_{\nu\lambda} C^\gamma_{\rho\sigma} + C^\sigma_{\nu\rho} C^\gamma_{\sigma\lambda}) \partial^\lambda = -(C^K)^\rho_{\mu\nu} C^\gamma_{\rho\lambda} \partial^\lambda C^\sigma_{\nu\sigma} = -(C^K)^{\rho+1}_{\mu\nu} C^\gamma_{\rho\sigma}
\]
as required. The Jacobi identity is used for the equality on the left.

A.2 Cofiltered algebras and completions

For our purposes, we may view the algebraic duals $U(g)^*$ and $S(g)^*$ as topological algebras, or as cofiltered algebras; the same for the completed tensor products. For the formal (adic) topology, the basis of neighborhoods of 0 in $U(g)^*$ and $S(g)^*$ is given by the ideals $\text{Ann } U_i(g)$, $\text{Ann } S_i(g)$ consisting of functionals vanishing on $i$-th filtered component. A complete cofiltration on a vector space $A$ is the (inverse) sequence of epimorphisms of its quotient spaces $\ldots \to A_{i+1} \to A_i \to A_{i-1} \to \ldots \to A_0$ such that $A = \lim_i A_i$; the quotient maps are $\pi_i : A \to A_i$, $\pi_{i,k} : A_{i+k} \to A_i$ and the identities $\pi_i = \pi_{i,k} \circ \pi_{i+k}$, $\pi_{i,k+l} = \pi_{i,k} \circ \pi_{i+k+l}$ hold. Strict morphisms of complete cofiltered vector spaces are the linear maps which induce the maps on the quotients. (This makes our category more rigid than the corresponding subcategory of pro-vector spaces.) The equality $A = \lim_i A_i$ means that the elements of $A$ may be viewed as sequences $(a_r)_{r \in \mathbb{N}_0} \in \prod_r A_r$ of compatible elements $a_r = \pi_{r+k}(a_{r+k})$. We say that $a = (a_r)_r$ has minimal cofiltered degree $\geq N$ if $\pi_N(a) = 0$ or $a_r = 0$ for $r < N$. In our main example, $U_i(g)^* := (U(g)^*)_i := U(g)^*/\text{Ann } U_i(g) \cong (U_i(g))^*$ and similarly for $S(g)^* \cong \hat{S}(g)^*$. Notice that we use the lower indices both for filtrations and for cofiltrations (but upper for gradations!).

The tensor product $A \hat{\otimes} B$ of complete cofiltered vector spaces is a cofiltered vector space with $N$-th cofiltered component (see [LS])

$$ (A \otimes B)_N = \frac{A \otimes B}{\cap_{p+q=N} \ker \pi_p \otimes \ker \pi_q}, $$

$(A \hat{\otimes} B)_N$ may be viewed as an abelian group of finite sums of the form $\sum \lambda a_\lambda \otimes b_\lambda$, where $a_\lambda = (a_\lambda)_s \in \prod_r A_r$, $b_\lambda = (b_\lambda)_s \in \prod_r B_r$ modulo the additive relation $\sim$ of equivalence for which $\sum a_\mu \otimes b_\mu \sim 0$ iff $\pi_p(a_\mu) \otimes \pi_q(b_\mu) = 0$ holds in $A_p \otimes B_q$ for all $p$ and $q$ such that $p+q = N$. In particular, the tails $(a_{r\lambda})_{r>2N}$, $(b_{r\lambda})_{r>2N}$ are irrelevant. The element in $A \hat{\otimes} B$ is any formal sum $\sum \lambda a_\lambda \otimes b_\lambda$ such that for any $p$ and $q$ $\pi^A_p(a_\lambda) \otimes \pi^B_q(b_\lambda) = 0$, for all but finitely many $\lambda$. A cofiltered algebra $A$ is a monoid internal to the $k$-linear category of (complete) cofiltered vector spaces, strict morphisms and with tensor product $\otimes$ [LS]. The bilinear associative unital multiplication map $\hat{m} : A \hat{\otimes} A \to A$ sends a formal sum $\sum a_\lambda \otimes a'_\lambda \in A \hat{\otimes} A$ to an element $\sum a_\lambda \cdot a'_\lambda \in A$ such that $(\sum a'_\lambda \cdot a_\lambda)_N$ is an equivalence class in $A_N$ of $\hat{m}(\sum a_\lambda \otimes a'_\lambda)$ where $\sum'$ denotes the finite sum of all elements such that $\pi_p(a_\lambda) \otimes \pi_q(b_\lambda) \neq 0$ for at least one pair $(p,q)$, $p+q = N$. 

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A vector subspace $W$ of a cofiltered vector space $V$ is cofiltered by $W_p := V_p \cap W$ with a canonical linear map $\varinjlim W_p \to V$, whose image is a complete cofiltered subspace $\bar{W} \subset V$, the completion of $W$. This is compatible with many additional structures, so defining the completions of algebras in cofiltered algebras, their modules and ideals (thus $\hat{I}, \hat{P}, \hat{I}(\nu), \hat{I}(\nu), \hat{I}(\nu)$ in Sections 4 and 5).

We want to define the completed smash product $H = U(g)\hat{\otimes} \hat{S}(g^*)$. Its underlying space is $U(g)\hat{\otimes} \hat{S}(g^*)$, where we treat $\hat{S}(g^*)$ as cofiltered (or as a topological ring), and the (filtered) ring $U(g)$ has the trivial cofiltration $U(g)$, in which every cofiltered component is the whole $U(g)$ (and carries the antidiscrete topology). The elements in $U(g)\hat{\otimes} \hat{S}(g^*)$ are therefore the formal sums $\sum u_\lambda \otimes a_\lambda$ such that $\pi_N(a_\lambda) = 0$ for all but finitely many $\lambda$. The basis of neighborhoods of 0 in $U(g)\hat{\otimes} \hat{S}(g^*)$ consists of the subspaces $V_i = U(g) \otimes \prod_{\nu \geq i} S^\nu(g^*)$ for all $i$. The elements in $V_i$ are therefore finite sums $\sum u_\lambda \otimes a_\lambda$, but $a_\lambda$ are in general infinite sums $\sum_{r=i+1} a_{\lambda, r}$, where $a_{\lambda, r} \in S^r(g^*)$.

The multiplication in $U(g)\hat{\otimes} \hat{S}(g^*)$ extends to a unique cofiltered algebra structure on $U(g)\hat{\otimes} \hat{S}(g^*)$ (the Hopf action admits a completed smash product). The formula $\sum_{\lambda, \mu}(u_\lambda \otimes a_\lambda)(u_\mu \otimes a_\mu) = \sum_{\lambda, \mu} u_\lambda u_\mu^r \otimes (a_\lambda \triangleleft u_\mu^r) a_\mu^r$ is indeed well defined if the number of pairs $(\mu, \lambda)$ such that $u_\lambda u_\mu^r \otimes \pi_N((a_\lambda \triangleleft u_\mu^r) a_\mu^r) \neq 0$ (only Sweedler summation) is a finite sum for all $N$. For fixed $\mu$, the minimal cofiltered degree (m.c.d.) of $(a_\lambda \triangleleft u_\mu^r) a_\mu^r$ is bigger than m.c.d. of $a_\mu^r$, at least by m.c.d. of $a_\lambda \triangleleft u_\mu^r(2)$. With summands $u_\mu^r(2)$ fixed, the m.c.d. of $a_\lambda \triangleleft u_\mu^r(2)$ can be made as large as wanted by choosing $a_\lambda$ of sufficiently high m.c.d. (that is, except for finitely many $\lambda$). Indeed, the m.c.d. of $a_\lambda \triangleleft u_\mu^r(2)$ is smaller than the m.c.d. of $a_\lambda$ only by a difference bounded above by the filtered degree of $u_\mu^r(2)$. This follows from the definition of the Hopf action $\triangledown = \triangleleft \circ SS_U(g)$, relation $[\partial^\mu, \hat{x}_\nu] = \delta^\nu_\mu$, higher order in $\partial^\nu$-s, and the induction on the m.c.d. of $a_\lambda(\partial^\nu)$. More details on this construction will be exhibited in [18].

References

1. AMELINO-CAMELLA G., ARZANO M.: Coproduct and star product in field theories on Lie-algebra non-commutative space-times, Phys. Rev. D65:084044 (2002) hep-th/0105120
2. BÖHM G.: Hopf algebroids, in Handbook of Algebra, 44 pages (2008), arXiv:0805.3806
3. BÖHM G.: Internal bialgebroids, entwining structures and corings, AMS Contemp. Math. 376 (2005) 207-226, math.QA/0311244
4. Böhm G, Szlachányi K.: Hopf algebroids with bijective antipodes: axioms, integrals and duals, Comm. Alg. 32 (11) (2004) 4433–4464 (math.QA/0305136)
5. Bourbaki N.: Lie groups and algebras, Ch. I-III, Hermann, Paris, 1971 (Ch.I), 1972 (Ch.II-III) (French); Springer 1975, 1989 (Ch. I-III, English).
6. Brzeziński T., Militaru G.: Bialgebroids, ×_A-bialgebras and duality, J. Alg. 251: 279–294 (2002) math.QA/0012164.
7. Brzeziński T., Wisbauer R., Corings and comodules. London Math. Soc. Lect. Note Ser., 309. Cambridge Univ. Press 2003.
8. Durov N., Meljanac S., Samsarov A., Škoda Z.: A universal formula for representing Lie algebra generators as formal power series with coefficients in the Weyl algebra, J. Alg. 309, n. 1, 318–359 (2007) math.QA/0604096.
9. Halliday S., Szabo R. J.: Noncommutative field theory on homogeneous gravitational waves, J. Phys. A39 (2006) 5189–5226, arXiv:hep-th/0602036.
10. Kontsevich M.: Deformation quantization of Poisson manifolds, Lett. Math. Phys. 66 (2003), no. 3, 157–216. 72; q-alg/9709040.
11. Meljanac S., Krešić-Jurić S., Stojić Marko: Covariant realizations of kappa-deformed space, Eur. Phys. J. C51 (2007) 229–240, hep-th/0702215.
12. Meljanac S., Škoda Z.: Heisenberg double versus deformed derivatives, Int. J. Mod. Phys. A 26, Nos. 27 & 28 (2011) 4845–4854, arXiv:1006.0478.