Maxwell–Chern-Simons gauged non-relativistic $O(3)$ model with self-dual vortices

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Abstract

A non-relativistic version of the 2+1 dimensional gauged Chern-Simons $O(3)$ sigma model, augmented by a Maxwell term, is presented and shown to support topologically stable static self-dual vortices. Exactly like their counterparts of the ungauged model, these vortices are shown to exhibit Hall behaviour in their dynamics.

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I Introduction

Non-relativistic field theories in 2+1 dimensions supporting vortex solutions are important in planar physics. Vortices feature in the $O(3)$ ferromagnet Landau-Lifshitz model [1], the Ginzburg-Landau model of superconductivity [2], and in the charged quantum fluid [3]. The last two theories are $U(1)$ gauged theories while the former is ungauged. It is therefore interesting to consider the $U(1)$ gauged version of the non-relativistic $O(3)$ $\sigma$ model, the Landau-Lifshitz model of ferromagnetism, which is presented below.

The $U(1)$ gauging we perform employs both Maxwell and Chern-Simons dynamics. The inclusion of the Chern-Simons term enables the description of anyonic dynamics, but in addition to this, its presence leads to two very interesting features of the model. The first of these is that in the limit of vanishing gauge coupling constant the resulting $U(1)$ gauged $O(3)$ $\sigma$ model reduces to the Landau-Lifshitz model of ferromagnetic materials [1]. The second of these is that the resulting system supports self-dual vortex solutions, which is a very useful mathematical feature. Concerning the inclusion of the Maxwell term, this turns out to be necessary for the topological stability of the vortices.

Non-relativistic Chern-Simons vortices in 2+1 dimensions were first introduced by Jackiw and Pi [4] in the context of a $U(1)$ gauged non-linear Schrödinger equation, which apart from the self-interaction potential of the matter field resembles the Ginsburg-Landau equation. The theory supported self-dual solutions in the static limit, which were characterised by the large distance asymptotic property $\lim_{r \to \infty} |\Psi| = 0$ of the complex valued matter field $\Psi$, according to which these vortices do not have a non-trivial vacuum condensate. One consequence of this is, that the energy of their solutions is not bounded from below by a topological charge and hence is not guaranteed to be stable. To change this situation, namely to allow for vortices exhibiting a non-trivial condensate, Barashenkov and Harin [5] modified the model of Refs. [4]. One of the more remarkable features of the modified model of Refs. [5], which features both Maxwell and Chern-Simons dynamics, is that the static vortices are topologically stable and self-dual, and the resulting model agrees with the non-relativistic Ginzburg-Landau theory, with the La-
The Lagrangian formulation of Ref. [5]. Moreover in Ref. [6] the chiral aspect of this system was emphasised, resulting in the absence of negative (positive) vorticities in the model (conjugate-model). We shall have occasion to discuss this point in our case too. In the present paper, where we tackle the non-relativistic \( U(1) \) gauged \( O(3) \) \( \sigma \) model, analogous to the non-relativistic Ginzburg-Landau model, we shall follow the Lagrangian formulation of Ref. [5].

In a more general context independently of the presence of the Chern-Simons term, we discuss the dynamics of these vortices in the framework of the description given in Refs. [1, 2]. The purpose of this discussion is to highlight the common features in the non-relativistic dynamics of the vortices of all these 2+1 dimensional models. Most important amongst these features is the dependence of the correct definition of the momentum on the definition of the gauge invariant topological charge density. The prescription for doing this is a non-trivial matter and the fact that it turns out to be model-independent is, in our opinion, an important demonstration of the universality of this prescription given in Refs. [1, 2].

We present our model, and implicitly give the vortex solutions, below in Section 2. The dynamics of our vortices is given in Section 3, and a brief summary of our results is given in Section 4.

II The model

In this Section we introduce our model and establish the self-duality equations minimising the energy. Since the vortices of our model will turn out to be solutions to the self-duality equations to the \( U(1) \) gauged \( O(3) \) \( \sigma \) model [7], our vortex solutions coincide with the latter and hence are implicitly given in this Section.
The model we introduce is described in terms of the $U(1)$ gauge field $A_\mu$ and the scalar field $\phi^a$ subject to the condition $\phi^a \phi^a = 1$, with $\mu = 0, i; \ i = 1, 2$ labeling the coordinates of the 2+1 dimensional space, and $a = \alpha, 3; \ \alpha = 1, 2$. A crucial role will be played by the gauging prescription we employ, which is the one used by Schroers \cite{7} in the construction of the relativistic Maxwell–$O(3)$ vortices, and subsequently in the construction of the relativistic Chern-Simons–$O(3)$ vortices by Ghosh et al \cite{8}, Kimm et al \cite{9} and ourselves \cite{10}. The prescription is characterised by the definition of the covariant derivative $D_\mu \phi^a$ as

$$D_\mu \phi^\alpha = \partial_\mu \phi^\alpha + A_\mu \varepsilon^{\alpha\beta} \phi^\beta, \quad D_\mu \phi^3 = \partial_\mu \phi^3. \quad (1)$$

This prescription of gauging was given employed earlier in Ref. \cite{11}. The proposed Lagrangian is

$$\mathcal{L} = -\frac{\mu^2}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{2} \varepsilon^{\mu\nu\lambda} F_{\mu\nu} A_\lambda$$

$$+ g(\phi^3) \varepsilon^{\alpha\beta} \phi^\alpha D_0 \phi^\beta - \frac{1}{2m} |D_i \phi^a|^2 - U(\phi^3) + \lambda (1 - |\phi^a|^2), \quad (2)$$

where $\lambda$ is a Lagrange multiplier. Both the function $g(\phi^3)$ and the $O(3)$ breaking potential $U(\phi^3)$ will be fixed by the criteria of topological stability. The choice for $g(\phi^3)$ can be made at this stage by requiring its correspondence with the Landau-Lifshitz theory in the ungauged limit, but we shall instead derive it below by requiring the existence of static self-dual vortices.

The Gauss Law constraint for (2) is

$$\mu^2 \partial_i E_i + \kappa B - g(\phi^3) |\phi^a|^2 = 0 \quad (3)$$

where $E_i = -F_{i0}$, and $B = \frac{1}{2} \varepsilon_{ij} F_{ij}$. The Hamiltonian is

$$\mathcal{H} = \frac{\mu^2}{2} (E_i^2 + B^2) + \frac{1}{2m} |D_i \phi^a|^2 + U(\phi^3). \quad (4)$$

In the static limit (3) and (4) reduce, respectively, to

$$- \mu^2 \Delta A_0 + \kappa B = g(\phi^3) |\phi^a|^2$$

$$\mathcal{H}_{static} = \frac{\mu^2}{2} (\partial_i A_0)^2 + \mathcal{H}_0 \quad (5)$$

$$\mathcal{H}_0$$
with $\mathcal{H}_0$ given by

$$\mathcal{H}_0 = \frac{\mu^2}{2} B^2 + \frac{1}{2m} |D_i \phi^a|^2 + U(\phi^3)$$

(7)

In the temporal gauge $A_0 = 0$, the static Hamiltonian (6) becomes equal to the density $\mathcal{H}_0$ given by (7), and the constraint (5) reduces to

$$\kappa B = g(\phi^3) |\phi^a|^2 = g(\phi^3)(1 - (\phi^3)^2).$$

(8)

The existence of self-dual vortices in the model (7), with a specific choice of the $O(3)$ breaking potential

$$U(\phi^3) = U_0(\phi^3) = \frac{\mu^2}{2} (1 - \phi^3)^2,$$

(9)

has been established in Refs. [7, 12]. The self-duality equations which minimise the energy of the Hamiltonian $\mathcal{H}_0$ given by (7) and (9), are

$$\varepsilon_{ij} D_i \phi^a = \varepsilon^{abc} D_j \phi^b \phi^c$$

(10)

$$B = \mu (1 - \phi^3),$$

(11)

which for the vortex field configurations in question are satisfied, in addition to the Gauss Law constraint (8). As the number of equations to be satisfied, (8), (10) and (11) exceeds the number of fields ($A_i, \phi^a$), it appears on first sight that the system is overdetermined. Fortunately however we have not yet specified the function $g(\phi^3)$, so we do this such that equations (8) and (11) become identical, thus reducing the number of the equations to the Bogomol’nyi equations (10)-(11). This choice is

$$g(\phi^3) = \kappa \mu (1 + \phi^3)^{-1},$$

leading to the final form of the proposed model

$$\mathcal{L} = -\frac{\mu^2}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\kappa}{2} \varepsilon^{\mu\nu\lambda} F_{\mu\nu} A_\lambda$$

$$+ \frac{\kappa \mu}{1 + \phi^3} \varepsilon^{\alpha\beta} \phi^a D_0 \phi^\beta - \frac{1}{2m} |D_i \phi^a|^2 - \frac{\mu^2}{2} (1 - \phi^3)^2.$$
The self-dual vortices supported by the static Hamiltonian (7) and (9), are well known from the work of Ref. [7], where it was shown that vortex configurations of vorticities \( N \geq 2 \) satisfy the Bogomol’nyi equations (10)-(11).

It is in order at this point to remark that we can achieve the topological lower bound by saturating the self-duality equations (10)-(11), but unlike in the case of the relativistic \( U(1) \) gauged \( O(3) \) sigma models [7, 8, 9, 10], it is not possible to achieve this by imposing instead the anti–self-duality equations with the opposite signs. The anti–self-duality equations saturate the energy of another model, namely that defined with the opposite sign of \( \kappa \) in (12). This is exactly what happens in the case of the Higgs, or Ginzburg-Landau model, which clearly explained in Ref. [6].

Departing from this limiting case of saturated Bogomol’nyi bounds, we can construct models which would support non–self-dual vortices of arbitrary \( N \), provided that the \( O(3) \) breaking potential \( U > U_0 = \mu^2 (1 - \phi^3)^2 \) everywhere. In that case we would have to relax the temporal gauge and treat the component \( A_0 \) of \( A_\mu \) in the static field configuration as a dynamical coordinate together with \((A_i, \phi^a)\) in the Euler-Lagrange equations, as in [5]. We do not elaborate on such details in the present Letter and suffice by noting the main features of these vortex solutions.

The most remarkable feature of these vortices is that the lower bound on their energy is given by a topological charge that is unrelated to the magnetic flux but is in fact the usual degree of the map for the corresponding ungauged \( O(3) \) sigma model:

\[
\varrho_0 = \frac{1}{8\pi} \varepsilon_{ijk} \varepsilon^{abc} \partial_i \phi^a \partial_j \phi^b \phi^c
\]  

(13)

This is shown in Ref. [7] and demonstrated in details in Ref. [10]. As a consequence, the value of the magnetic flux of these vortices is not quantised, inspite of the solution being topologically stable. The topological charge density (13) is not gauge invariant and hence strictly speaking inadequate to supply a lower bound to the energy density which is a gauge invariant quantity. As explained in Ref. [10], it is possible to define the gauge invariant topological charge density by adding a suitable total divergence term with vanishing surface integral, which renders it gauge invariant. Since we will need this expression of the gauge invariant topological charge density \( \varrho \) in
our dynamical considerations later, we quote it here
\[ \varrho = \varrho_0 + \frac{1}{4\pi} \varepsilon_{ij} \partial_i [ (\phi^3 - 1) A_j ] = \frac{1}{8\pi} \varepsilon_{ij} \varepsilon^{abc} D_i \phi^a D_j \phi^b \phi^c + \frac{1}{4\pi} B (\phi^3 - 1). \]

(14)

Topological charge densities like (14), whose volume integrals yield the degree of the map rather than the Chern-Pontryagin index associated with the gauge group, occur in $SO(d)$ gauged $O(d + 1)$ sigma models in $d$ dimensions, for all $d$.

The model (12) supports self-dual solutions in the static limit, but the discussion of the dynamical properties of these vortices given below applies equally well to non–self-dual solutions of models departing from (12) by relaxing the restrictive choice of the $O(3)$ breaking potential $U_0(\phi^3)$ in (12) given by (9), in the manner specified above. That way we can avail of the topological inequalities established in Refs. [7, 10], which in these cases would not be saturated. The solutions of the Euler-Lagrange equations will then be subject also to the Gauss Law constraint (5), and involve an additional function parametrising the field $A_0$, in exactly the same way as in the second item of Ref. [5]. As we will not perform any numerical integrations in the present Letter, we do not pursue this direction here.

In what we have done above, constructing a non-relativistic $U(1)$ gauged $O(3)$ model for which we can state a topological lower bound on the energy of the static field configuration, the most important step is the determination of the function $g(\phi^3)$. The particular choice of potential $U_0$ given by (4) is however not obligatory and serves only to enable the saturation of this bound.

III Dynamics

Our final consideration is the description of the dynamics of our vortices based on the formalism developed in Ref. [1] for the vortices of the ungauged $O(3)$ sigma model, and in Ref. [2] for the analogous solitons of a
non-relativistic dynamical Ginzburg-Landau model. In both these examples, the definition of the momentum field density depended crucially on the gauge-invariant topological charge density that stabilises the vortex. In the present example, this involves employing the topological charge density defined by (14).

Our purpose here is to verify that the prescriptions introduced in Refs. [1, 2], namely that of employing the gauge-invariant topological charge density in the definition of the momentum field density hold also for the present model and are in that sense model independent. It should be pointed out that the conclusions of this section are independent of the form of the potential, and, the Chern-Simons term or the Maxwell term might be totally absent or be replaced by Lorentz non-invariant expressions.

For the presentation below, it is convenient to use the angular representation \((\Theta, \Phi)\) of the unit magnitude field \(\phi^a\)

\[
\phi^1 = \sin \Theta \cos \Phi, \quad \phi^2 = \sin \Theta \sin \Phi, \quad \phi^3 = \cos \Theta
\]

(15) used in Ref. [1].

Using formally the standard Noether procedure for field configurations a) approaching their asymptotic values fast enough for various surface terms arising in the variation of the action to vanish and b) such that all second spatial derivatives commute, one obtains the following expression for the field momentum density of the system:

\[
p_i^N = \kappa \mu (1 - \cos \Theta) (\partial_i \Phi - A_i) + \mu^2 \varepsilon_{ij} E_j B
\]

(16)

It is the sum of the momentum density \(\kappa \mu (1 - \cos \Theta) (\partial_i \Phi - A_i)\) of the global model properly covariantized to become gauge invariant and of the familiar Poynting contribution \(\varepsilon_{ij} E_j B\) of the pure Maxwell theory. Its volume integral \(P_i^N = \int d^2 x p_i^N\) yields the momentum.

\[1\] In fact, since the rest of the Lagrangian is not Lorentz invariant such terms will be induced by quantum effects.

\[2\] It is clear that the Chern-Simons term being metric-free does not contribute to the energy-momentum tensor.
In the presence of a vortex the second spatial derivatives do not commute. For a vortex with topological charge \( N \) and located at \( \mathbf{x}_0 \) one obtains \( \epsilon_{ij} \partial_i \partial_j \Phi(\mathbf{x}) = 2\pi N \delta(\mathbf{x} - \mathbf{x}_0) \). As a consequence \( P_i^N \) is not conserved. \( P_i^N \) is the correct expression for the momentum only in the absence of topological solitons. We follow the steps described in Refs. \[1\] and \[2\] and write for the momentum of the model valid in all topological sectors the formula:

\[
P_i = \int d^2 x [4\pi \kappa \mu \epsilon_{ij} x_j \varrho(\mathbf{x}) + \mu^2 \epsilon_{ij} E_j B]
\]

where \( \varrho \) is given by \([14]\). It is conserved even in the presence of an arbitrary number of vortices and antivortices.

The consequences of \([17]\) for the dynamics of the vortices are quite surprising. Notice that the momentum of a static axially symmetric vortex located at \( \mathbf{a} \) and carrying topological charge \( N \) is \( 4\pi \kappa \mu N \epsilon_{ij} a_j \) and characterizes the position of the vortex and not its motion. Correspondingly, for a vortex moving in formation with velocity \( \mathbf{v} \) one obtains \( P_i = 4\pi \kappa \mu N \epsilon_{ij} (a_j^0 + v_j t) \). Conservation of momentum implies \( v_i = 0 \). Isolated vortices in the absence of external forces are \textit{spontaneously pinned}.

It is then natural to define the guiding centre \( \mathbf{R} \) of an isolated vortex of winding \( N \)

\[
R_i = -\frac{1}{4\pi \kappa \mu N} \epsilon_{ij} P_j
\]

Under a global displacement of the system by \( \mathbf{c} \) it changes to \( \mathbf{R} + \mathbf{c} \) and for an axially symmetric vortex it coincides with its geometric center. The above qualitative argument and the numerical study of the equations of motion in similar systems \([14]\) leads to the conclusion that \( \mathbf{R} \) is a faithful representation of the mean position of the vortex. In the presence of an external force \( F_j \) Newton’s law \( dP_j / dt = F_j \) translates into the vortex equation of mean motion

\[
\frac{d}{dt} R_i = -\frac{1}{4\pi \kappa \mu N} \epsilon_{ij} F_j
\]

Generically, up to a fine cyclotron motion in its details \([14]\), the vortex moves as a whole with a speed proportional to the external force and in a direction perpendicular to it. It exhibits the so called Hall behaviour.
As a final important comment we would like to mention that it is possible to develop the canonical structure of the model and show that the momentum defined in (17) is indeed the generator of spatial displacements satisfying the Poisson brackets \( \{ P_k, F \} = \partial_k F \) with any field \( F \). From this it follows that the two components of the momentum do not commute. Instead, they satisfy

\[
\{ P_1, P_2 \} = 4\pi \kappa \mu N
\]  

(20)

It is analogous to the Poisson bracket satisfied by the momentum of a planar charged particle under the influence of a homogeneous perpendicular magnetic field. Combined with the well-known fact that a central extension of the linear momentum algebra is possible only for the Euclidean algebra \( E(2) \) in two dimensions and for the translational algebra \( T(D) \) in any dimension \( D \), one concludes that the above Hall motion of the solitons is only possible in two-dimensional systems with \( E(2) \) symmetry or in general \( D \)-dimensional systems with just translational invariance.

IV Summary

We have presented a non-relativistic \( U(1) \) gauged \( O(3) \) sigma model with Maxwell and Chern-Simons dynamics which supports self-dual winding number \( N \geq 2 \) vortices. This model reduces to the Landau-Lifshitz theory of ferromagnetic materials in the ungauged limit.

It is possible to extend this model by making a different choice for the \( O(3) \) breaking potential, everywhere greater than the potential in (12), resulting in non–self-dual vortices with arbitrary vorticity as in Ref. [15]. This generalisation is deferred to some future work.

We have shown that the dynamics of our vortices exhibits the Hall behaviour familiar in the motion of planar charges in a perpendicular magnetic field [2], as well as in the motion of solitons in the global \( O(3) \) model and in the non-relativistic Maxwell gauged Ginsburg-Landau model [1]. In showing this we have employed the prescription used in Refs. [1, 2], namely that of defining the momentum field density of the soliton in terms of the gauge
invariant topological charge density (14), in our definition of the momentum field density (17). We conclude therefore that this prescription is model independent.

Our model exhibits nearly all the qualitative features of the Ginzburg-Landau model, except that in the limit of vanishing Maxwell term it is not possible to find zero energy self-dual solitons as in the analogous non-linear Schrödinger, or non-relativistic Ginzburg-Landau, system [4].
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