SOME IDENTITIES OF CARLITZ DEGENERATE BERNOULLI NUMBERS AND POLYNOMIALS

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ABSTRACT. In this paper, we study the Carlitz’s degenerate Bernoulli numbers and polynomials and give some formulae and identities related to those numbers and polynomials.

1. Introduction

As is well known, the ordinary Bernoulli polynomials are defined by the generating function

\[ (\frac{t}{e^t - 1}) e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \text{(see [1-20])} \].

When \( x = 0 \), \( B_n = B_n(0) \) are called the Bernoulli numbers.

From (1.1), we note that

\[ B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l} = d^{n-1} \sum_{a=0}^{d-1} B_n \left( \frac{a + x}{d} \right), \quad (n \geq 0, d \in \mathbb{N}) \].

It is easy to show that

\[ \frac{t}{e^t - 1} e^{t} - \frac{t}{e^t - 1} = t. \]

Thus, by (1.1) and (1.3), we get

\[ \sum_{n=0}^{\infty} (B_n(1) - B_n) \frac{t^n}{n!} = t. \]

By comparing the coefficients on the both sides, we have

\[ B_0 = 1, \quad B_n(1) - B_n = \delta_{n,1}, \quad \text{(see [21])}, \]

where \( \delta_{n,k} \) is the Kronecker’s symbol.

Let \( \chi \) be a Dirichlet character with conductor \( d \in \mathbb{N} \). Then, the generalized Bernoulli numbers attached to \( \chi \) are defined by the generating function

\[ \frac{t}{e^{\chi a} - 1} \sum_{a=0}^{d-1} \chi(a) e^{at} = \sum_{n=0}^{\infty} B_n,\chi \frac{t^n}{n!}, \quad \text{(see [12, 18, 20])}. \]

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Thus, by (1.6), we get
\[ B_{n,\chi} = d^{n-1} \sum_{a=0}^{d-1} \chi(a) B_n \left( \frac{a}{d} \right). \]

For \( \lambda \in \mathbb{C} \), L. Carlitz defined the degenerate Bernoulli polynomials as follows:
\[
(1.7) \quad \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x|\lambda) \frac{t^n}{n!}, \quad (\text{see [3, 4]}) .
\]

When \( x = 0 \), \( \beta_n(\lambda) = \beta_n(0|\lambda) \) are called the degenerate Bernoulli numbers.

By (1.7), we easily get
\[
(1.8) \quad \lim_{\lambda \to 0} \beta_n(x|\lambda) = B_n(x), \quad (n \geq 0).
\]

In this paper, we study the properties of degenerate Bernoulli numbers and polynomials and give some formulae and identities related to those numbers and polynomials.

2. Degenerate Bernoulli numbers and polynomials

We easily see that
\[
(2.1) \quad \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{1}{\lambda}} - \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} = t.
\]

From (1.7) and (2.1), we have
\[
(2.2) \quad \sum_{n=0}^{\infty} \left\{ \beta_n(1|\lambda) - \beta_n(\lambda) \right\} \frac{t^n}{n!} = t.
\]

By comparing the coefficients on the both sides of (2.2), we get
\[
(2.3) \quad \beta_n(1|\lambda) - \beta_n(\lambda) = \delta_{1,n}, \quad \beta_0(\lambda) = 1, \quad (n \in \mathbb{N}).
\]

Note that equation (2.3) is the \( \lambda \)-analogue of (1.5).

From (1.7), we can derive the following equation:
\[
(2.4) \quad t (1 + \lambda t)^{\frac{1}{\lambda}} = \left( (1 + \lambda t)^{\frac{1}{\lambda}} - 1 \right) \sum_{m=0}^{\infty} \beta_m(x|\lambda) \frac{t^m}{m!}
\]
\[= \left( \sum_{l=1}^{\infty} (1|\lambda) \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} \beta_m(x|\lambda) \frac{t^m}{m!} \right),\]
where
\[(x | \lambda)_n = x (x - \lambda) \cdots (x - \lambda (n - 1)) = \lambda^n \sum_{l=0}^{n} S_1 (n, l) \lambda^{-l} x^l.\]

Thus, by (2.4), we get
\[(2.5) \quad \sum_{n=0}^{\infty} (x | \lambda)_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \frac{(1 | \lambda)_{l+1}}{l+1} \left( \begin{array}{c} n \\ l \end{array} \right) \beta_{n-l} (x | \lambda) \right) \frac{t^n}{n!}.\]

By comparing the coefficients on both sides of (2.5), we get
\[(2.6) \quad (x | \lambda)_n = \sum_{l=0}^{n} \frac{(1 | \lambda)_{l+1}}{l+1} \left( \begin{array}{c} n \\ l \end{array} \right) \beta_{n-l} (x | \lambda), \quad (n \geq 0).\]

Note that
\[x^n = \lim_{\lambda \to 0} (x | \lambda)_n = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) \frac{B_{n-l} (x)}{l+1}.\]

On the other hand,
\[(2.7) \quad \sum_{n=0}^{\infty} \beta_n (x | \lambda) \frac{t^n}{n!} = \left( \frac{t}{(1 + \lambda t)^{\frac{n}{l}} - 1} \right) (1 + \lambda t)^{\frac{n}{l}}\]
\[= \left( \sum_{l=0}^{\infty} \beta_l (\lambda) \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} (x | \lambda)_m \frac{t^m}{m!} \right)\]
\[= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \beta_l (\lambda) \left( \begin{array}{c} n \\ l \end{array} \right) (x | \lambda)_{n-l} \right) \frac{t^n}{n!}.\]

By comparing the coefficients on both sides of (2.7), we have
\[(2.8) \quad \beta_n (x | \lambda) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) \beta_l (\lambda) (x | \lambda)_{n-l}, \quad (n \geq 0).\]

Therefore, by (2.3), (2.6) and (2.8), we obtain the following theorem.

**Theorem 2.1.** For \(n \geq 0\), we have
\[\beta_n (x | \lambda) = \sum_{l=0}^{n} \left( \begin{array}{c} n \\ l \end{array} \right) \beta_l (\lambda) (x | \lambda)_{n-l},\]
\[(x | \lambda)_n = \sum_{l=0}^{n} \frac{(1 | \lambda)_{l+1}}{l+1} \left( \begin{array}{c} n \\ l \end{array} \right) \beta_{n-l} (x | \lambda),\]
and
\[\beta_0 (\lambda) = 1, \quad \beta_n (1 | \lambda) - \beta_n (\lambda) = \delta_{1,n}.\]

From (1.7), we can derive the following equation:
\[(2.9) \quad \frac{t}{(1 + \lambda t)^{\frac{n}{l}} - 1} = \frac{t}{(1 + \lambda t)^{d/\lambda} - 1} \sum_{a=0}^{d-1} (1 + \lambda t)^{\frac{a+1}{n}}\]
\[
= \frac{1}{d} \left( \frac{dt}{(1 + \frac{1}{d}dt)^{\frac{1}{d}} - 1} \right) \sum_{a=0}^{d-1} \left( 1 + \frac{\lambda d}{d^{a+1}} \right) \frac{t^n}{n!} \\
= \frac{1}{d} \sum_{a=0}^{d-1} \left( \sum_{n=0}^{\infty} d^n \beta_n \left( \frac{a + x}{d} \right) \frac{\lambda^n}{n!} \right) \frac{t^n}{n!}, \quad (d \in \mathbb{N}).
\]

Therefore, by (1.7) and (2.9), we obtain the following theorem.

**Theorem 2.2.** For \( n \geq 0, d \in \mathbb{N} \), we have
\[
\beta_n (x | \lambda) = d^n \sum_{a=0}^{d-1} \beta_n \left( \frac{a + x}{d} \right) \frac{\lambda^n}{n!}.
\]

**Remark.** Theorem (2.2) is the \( \lambda \)-analogue of (1.2). That is,
\[
B_n (x) = \lim_{\lambda \to 0} \beta_n (x | \lambda) = d^n \sum_{a=0}^{d-1} B_n \left( \frac{a + x}{d} \right), \quad (d \in \mathbb{N}).
\]

We observe that
\[
(2.10) \quad \frac{t}{(1 + \lambda t)^{\frac{1}{d}} - 1} - \frac{t}{(1 + \lambda t)^{\frac{1}{d}} - 1} = \frac{t}{(1 + \lambda t)^{\frac{1}{d}} - 1} \left( (1 + \lambda t)^{\frac{1}{d}} - 1 \right) = t \sum_{l=0}^{n-1} (1 + \lambda t)^{\frac{1}{d}} = t \sum_{m=0}^{\infty} \left( \frac{n-1}{l | \lambda} \right) \frac{t^m}{m!}, \quad (n \in \mathbb{N}).
\]

On the other hand,
\[
(2.11) \quad \frac{t}{(1 + \lambda t)^{\frac{1}{d}} - 1} - \frac{t}{(1 + \lambda t)^{\frac{1}{d}} - 1} = \sum_{m=0}^{\infty} \left\{ \beta_m \left( n \mid \lambda \right) - \beta_m \left( \lambda \right) \right\} \frac{t^m}{m!} = t \sum_{m=0}^{\infty} \left\{ \beta_{m+1} \left( n \mid \lambda \right) - \beta_{m+1} \left( \lambda \right) \right\} \frac{t^m}{m!}.
\]

By (2.10) and (2.11), we get
\[
(2.12) \quad \sum_{m=0}^{\infty} \left( \frac{n-1}{l \mid \lambda} \right) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left( \frac{\beta_{m+1} \left( n \mid \lambda \right) - \beta_{m+1} \left( \lambda \right)}{m+1} \right) \frac{t^m}{m!}, \quad (n \in \mathbb{N}).
\]

Therefore, by comparing the coefficients on both sides of (2.12), we obtain the following theorem.
Theorem 2.3. For \( n \in \mathbb{N} \), \( m \geq 0 \), we have
\[
\sum_{l=0}^{n-1} \left( \frac{l}{\lambda} \right)_m = \frac{1}{m+1} \left( \beta_{m+1} (n \mid \lambda) - \beta_{m+1} (\lambda) \right).
\]

By replacing \( t \) by \( \frac{1}{\lambda} \log (1 + \lambda t) \) in (1.1), we get
\[
\log (1 + \lambda t) \frac{t}{\lambda} \left( \frac{1}{\lambda} \log (1 + \lambda t) \right)^{\frac{1}{\lambda} - 1} - 1 = \sum \limits_{n=0}^{\infty} B_n (x) \lambda^{-n} \frac{1}{n!} (\log (1 + \lambda t))^n
\]
\[
= \sum \limits_{n=0}^{\infty} \left( \sum \limits_{m=0}^{n} B_m (x) \lambda^{n-m} S_1 (n, m) \right) \frac{t^n}{n!}.
\]

On the other hand,
\[
\log (1 + \lambda t) \frac{t}{\lambda} \left( \frac{1}{\lambda} \log (1 + \lambda t) \right)^{\frac{1}{\lambda} - 1} - 1 = \left( \frac{\log (1 + \lambda t)}{\lambda t} \right) \frac{t}{\lambda} \left( \frac{1}{\lambda} \log (1 + \lambda t) \right)^{\frac{1}{\lambda} - 1} - 1
\]
\[
= \left( \sum \limits_{l=0}^{\infty} \frac{(-1)^l}{l+1} \lambda^l t^l \right) \left( \sum \limits_{m=0}^{\infty} \beta_m (x \mid \lambda) \frac{t^m}{m!} \right)
\]
\[
= \sum \limits_{n=0}^{\infty} \left( \sum \limits_{l=0}^{n} \frac{(-1)^l \lambda^l}{(n-l)!} \beta_{n-l} (x \mid \lambda) \frac{t^n}{n!} \right) \frac{t^n}{n!}
\]
\[
= \sum \limits_{n=0}^{\infty} \left( \sum \limits_{l=0}^{n} \frac{l!}{l+1} (-1)^l \lambda^l \frac{n}{l} \beta_{n-l} (x \mid \lambda) \frac{t^n}{n!} \right) \frac{t^n}{n!}.
\]

Therefore, by (2.13) and (2.14), we obtain the following theorem.

Theorem 2.4. For \( n \geq 0 \), we have
\[
\sum \limits_{n=0}^{n} B_m (x) \lambda^{n-m} S_1 (n, m) = \sum \limits_{m=0}^{\infty} \frac{l!}{l+1} \left( \frac{n}{l} \right) (-1)^l \lambda^l \beta_{n-l} (x \mid \lambda),
\]
where \( S_1 (n, m) \) is the Stirling number of the first kind.

By replacing \( t \) by \( \frac{1}{\lambda} \left( e^{\lambda t} - 1 \right) \) in (1.7), we get
\[
\frac{1}{\lambda} \left( e^{\lambda t} - 1 \right) e^{xt}
\]
\[
= \sum \limits_{m=0}^{\infty} \beta_m (x \mid \lambda) \frac{1}{m!} \lambda^{-m} \left( e^{\lambda t} - 1 \right)^m
\]
\[
= \sum \limits_{m=0}^{\infty} \beta_m (x \mid \lambda) \lambda^{-m} \sum \limits_{n=m}^{\infty} S_2 (n, m) \lambda^n \frac{n^n}{n!}
\]
\[
= \sum \limits_{n=0}^{\infty} \left( \sum \limits_{m=0}^{n} \lambda^{n-m} S_2 (n, m) \beta_m (x \mid \lambda) \frac{t^n}{n!} \right).
where $S_2(n, m)$ is the Stirling number of the second kind.

On the other hand,

$$
\frac{1}{\lambda} \left( \frac{e^{\lambda t} - 1}{e^t - 1} \right) e^{xt} = \frac{1}{\lambda^t} \left( \frac{t}{e^t - 1} \right) \left( e^{(x+\lambda)t} - e^{xt} \right) = \frac{1}{\lambda^t} \sum_{n=0}^{\infty} (B_n(x + \lambda) - B_n(x)) \frac{t^n}{n!} = \frac{1}{\lambda^t} \sum_{n=0}^{\infty} \left\{ \frac{B_{n+1}(x + \lambda) - B_{n+1}(x)}{n+1} \right\} \frac{t^n}{n!}.
$$

From (2.15) and (2.16), we have

$$
(2.17) \quad \frac{B_{n+1}(x + \lambda) - B_{n+1}(x)}{n+1} = \sum_{m=0}^{n} S_2(n, m) \frac{\lambda^{n-m+1} \beta_m(x|\lambda)}{n}. \tag{2.17}
$$

Therefore, by (2.17), we obtain the following theorem.

**Theorem 2.5.** For $n \geq 0$, we have

$$
\frac{B_{n+1}(x + \lambda) - B_{n+1}(x)}{n+1} = \sum_{m=0}^{n} S_2(n, m) \frac{\lambda^{n-m+1} \beta_m(x|\lambda)}{n}. \tag{2.17}
$$

**Remark.** From Theorem 2.3, we note that

$$
\lim_{\lambda \to 0} \frac{\lambda^{n-m+1} \beta_m(x|\lambda)}{n+1} = \frac{B_{m+1}(n) - B_{m+1}(\lambda)}{m+1}, \quad (m \geq 0, n \in \mathbb{N}).
$$

For $s \in \mathbb{C} \setminus \{1\}$, we define the degenerate Riemann zeta function as follows:

$$
(2.18) \quad \zeta(s, x|\lambda) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{(1+\lambda t)^{-s}}{1-(1+\lambda t)^{-s}} t^{s-1} dt,
$$

where $x \neq 0, -1, -2, \ldots$.

From (2.18), we note that

$$
\lim_{\lambda \to 0} \zeta(s, x|\lambda) = \zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s},
$$

where $x \neq 0, -1, -2, \ldots$.

By Laurent series and (2.18), we obtain the following theorem.

**Theorem 2.6.** For $n \in \mathbb{N}$, we have

$$
\zeta(1-n, x|\lambda) = -\frac{\beta_n(x|\lambda)}{n}.
$$

For $d \in \mathbb{N}$, let $\chi$ be a Dirichlet character with conductor $d$. Then, we define the generalized degenerate Bernoulli numbers attached to $\chi$ as follows:

$$
(2.19) \quad \frac{t}{(1+\lambda t)^d/\lambda - 1} \sum_{a=0}^{d-1} \chi(a)(1+\lambda t)^{\frac{a}{d}} = \sum_{n=0}^{\infty} \beta_{n, \chi}(\lambda) \frac{t^n}{n!}.
$$
Then, by (2.19), we get

\[
\sum_{n=0}^{\infty} \beta_{n, \chi} (\lambda) t^n = \frac{1}{d} \sum_{a=0}^{d-1} \chi(a) \frac{dt}{(1 + \lambda t)^{d/\lambda} - 1} (1 + \lambda t)^{\frac{d}{d-1}}
\]

Therefore, by (2.20), we obtain the following theorem.

**Theorem 2.7.** For \( n \geq 0, d \in \mathbb{N} \), we have

\[
\beta_{n, \chi}(\lambda) = d^{n-1} \sum_{a=0}^{d-1} \chi(a) \beta_{n} \left( \frac{a}{d} \right) .
\]

### 3. Further Remark

Let \( p \) be a fixed prime number. Throughout this section, \( \mathbb{Z}_p \), \( \mathbb{Q}_p \) and \( \mathbb{C}_p \) will denote the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers and the completion of the algebraic closure of \( \mathbb{Q}_p \). The \( p \)-adic norm is normalized as \( |p|_p = \frac{1}{p} \). For \( \lambda, t \in \mathbb{C}_p \) with \( |\lambda t|_p < p^{-\frac{1}{p-1}} \), the degenerate Bernoulli polynomials are given by the generating function to be

\[
\frac{t}{(1 + \lambda t)^{\frac{d}{d-1}} - 1} (1 + \lambda t)^{\frac{d}{d-1}} = \sum_{n=0}^{\infty} \beta_n(x|\lambda) \frac{t^n}{n!}.
\]

Let \( d \) be a positive integer. Then, we define

\[
X = \lim_{N \to \infty} \left( \mathbb{Z}/dp^N \mathbb{Z} \right),
\]

\[
a + dp^N \mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \pmod{dp^N} \right\};
\]

\[
X^* = \bigcup_{0< a < dp} a + dp \mathbb{Z}_p.
\]

We shall usually take \( 0 \leq a < dp^N \) when we write \( a + dp^N \mathbb{Z}_p \). Now, we will use Theorem 2.2 to prove a \( p \)-adic distribution result.

**Theorem 3.1.** For \( k \geq 0 \), let \( \mu_{k, \beta} \) be defined by

\[
(3.1) \quad \mu_{k, \beta} \left( a + dp^N \mathbb{Z}_p \right) = \left( dp^N \right)^{k-1} \beta_{k} \left( a \right) \left| \frac{\lambda}{dp^N} \right|^\frac{1}{d}.
\]

Then \( \mu_{k, \beta} \) extends to a \( \mathbb{C}_p \)-valued distribution on the compact open sets \( U \subset X \).

**Proof.** It is enough to show that

\[
\sum_{i=0}^{p-1} \mu_{k, \beta} \left( a + idp^N + dp^{N+1} \mathbb{Z}_p \right) = \mu_{k, \beta} \left( a + dp^N \mathbb{Z}_p \right).
\]
Indeed, by (3.1), we get

\[
\sum_{i=0}^{p-1} \mu_{k,\beta} \left( a + idp^N + dp^{N+1}Z_p \right) = (dp^{N+1})^{k-1} \sum_{i=0}^{p-1} \beta_k \left( \frac{a + idp^N}{p} \bigg| \frac{\lambda}{dp^{N+1}} \right)
\]

\[
= (dp^{N})^{k-1} \sum_{i=0}^{p-1} \beta_k \left( \frac{a}{dp^N} + i \frac{\lambda}{dp^N} \bigg| \frac{\lambda}{dp^N} \right)
\]

\[
= (dp^{N})^{k-1} \beta_k \left( \frac{a}{dp^N} \bigg| \frac{\lambda}{dp^N} \right)
\]

\[
= \mu_{k,\beta} \left( a + dp^NZ_p \right).
\]

\[\square\]

The locally constant function \( \chi \) can be integrated against the distribution \( \mu_{k,\beta} \) defined by (3.1), and the result is

\[
\int_X \chi(x) \, d\mu_{k,\beta}(x) = \beta_{k,\chi}(\lambda), \quad (k \geq 0).
\]

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