A COVARIANCE EQUATION

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ABSTRACT. Let $S$ be a commutative semigroup with identity $e$ and let $\Gamma$ be a compact subset in the pointwise convergence topology of the space $S'$ of all non-zero multiplicative functions on $S$. Given a continuous function $F : \Gamma \to \mathbb{C}$ and a complex regular Borel measure $\mu$ on $\Gamma$ such that $\mu(\Gamma) \neq 0$. It is shown that

$$\mu(\Gamma) \int_{\Gamma} \overline{\varphi(t)} |F(\varphi)|^2 \, d\mu(\varphi) = \int_{\Gamma} \varphi(s) F(\varphi) \, d\mu(\varphi) \int_{\Gamma} \overline{\varphi(t)} F(\varphi) \, d\mu(\varphi)$$

for all $(s, t) \in S \times S$ if and only if for some $\gamma \in \Gamma$, the support of $\mu$ is contained in $\{F = 0\} \cup \{\gamma\}$. Several applications of this characterization are derived. In particular, the reduction of our theorem to the semigroup of non-negative integers $(\mathbb{N}_0, +)$ solves a problem posed by El Fallah, Klaja, Kellay, Mashregui and Ransford in a more general context. More consequences of our results are given, some of them illustrate the probabilistic flavor behind the problem studied herein and others establish extremal properties of analytic kernels.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let $S$ be a commutative semigroup with identity $e$ and let $S'$ be the completely regular space consisting of all non-zero multiplicative complex valued functions on $S$ furnished with the pointwise convergence topology. In this paper, we characterize the compactly supported complex regular Borel measures $\mu$ on $S'$ that satisfy the following equation

$$\mu(S') \int_{S'} \varphi(s) \overline{\varphi(t)} |F(\varphi)|^2 \, d\mu(\varphi) = \int_{S'} \varphi(s) F(\varphi) \, d\mu(\varphi) \int_{S'} \overline{\varphi(t)} F(\varphi) \, d\mu(\varphi)$$

for all $s, t \in S$, where $F$ is a fixed continuous function on $S'$. If $\mu(S') = 1$, then setting $\chi_s(\varphi) := \varphi(s)$, the latter equality can be written in the covariance equation form

$$\int_{S'} \left( F \chi_s - \int_{S'} F \chi_s \, d\mu \right) \left( \overline{F \chi_t} - \int_{S'} \overline{F \chi_t} \, d\mu \right) \, d\mu = 0$$

for all $s, t \in S$.

To handle this problem we first use the action of the algebra of shift operators on functions on $S \times S$. Then we appeal to Luecking's theorem on finite rank Toeplitz operators [4]. Several applications of our solution will be given in Sections 5 and 6. They all offer new results. The reduction of our characterization to the additive semigroup $\mathbb{N}_0$ of all non-negative integers solves a problem which was left open in recent paper by El Fallah, Kallaj, Kellay, Mashregui and

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In memory of Peter Maserick.
Ransford in a more general context. Indeed, we provide a solution to this question in the multi-dimensional setting.

Let $S$ denote a multiplicative commutative semigroup with identity $e$. A function $\rho : S \to \mathbb{C}$ is said to be multiplicative if its satisfies $\rho(st) = \rho(s)\rho(t)$ for all $(s, t) \in S^2$. It is clear, that if $\rho$ is non-zero multiplicative on $S$, then $\rho(e) = 1$. We denote by $S'$ the set of all non-zero multiplicative complex valued functions on $S$ and equip it with pointwise convergence topology under which it is a completely regular topological space.

Let $\mu$ be a compactly supported complex regular Borel measure $\mu$ on $S'$. It is known (see Rudin [9]) that $\mu$ has a polar decomposition $\mu = |\mu|\mu^*$, where $|\mu|$ is a nonnegative measure, the total variation of $\mu$, on $S^*$ and $h = h_\mu : S' \to \mathbb{C}$ a Borel measurable function with $|h| \equiv 1$.

The generalized Laplace transform of $\mu$ is the function defined on $S \times S$ by

\[
\widehat{\mu}(s, t) := \int_{S'} \rho(s)\overline{\rho(t)} \, d\mu(\rho), \quad (s, t) \in S \times S.
\]

and the variation of $\widehat{\mu}$ is the function

\[
|\mu|(s, t) := \int_{S'} \rho(s)\overline{\rho(t)} \, d|\mu|(\rho), \quad (s, t) \in S \times S.
\]

The above generalized Lapalace transforms can be viewed as $BV$-functions in the sense of Maserick [6] on the semigroup on $S \times S$ with appropriate involution.

If $F$ is a continuous function on $S'$, we denote by $F\mu$ the measure with density $F$ with respect to $\mu$.

**Definition 1.** Given complex Borel measure $\mu$ on a compact subset $\Gamma$ of $S'$ and a continuous function $F : \Gamma \to \mathbb{C}$, we shall say that the measure $\mu$ satisfies the covariance equation with respect to $F$ if

\[
\mu(\Gamma)|\widehat{F}|^2\mu(s, t) = \widehat{F}\mu(s, e)\widehat{F}\mu(e, t)
\]

for all $s, t \in S$.

It is obvious that if $\mu(\Gamma) = 0$, then $\mu$ satisfies (1.2) if and only if either

\[
\int_{\Gamma} \rho(s)F(\rho)\, d\mu(\rho) = 0 \quad \forall s \in S
\]

or

\[
\int_{\Gamma} \rho(s)\overline{F(\rho)}\, d\mu(\rho) = 0, \quad \forall s \in S.
\]

Our goal herein is to describe the measures $\mu$ which satisfy (1.2). The main result is the following:

**Main Theorem.** Suppose that $\mu$ is a complex regular Borel measure with compact support $\Gamma \subset S'$ such that $\mu(\Gamma) \neq 0$ and let $F : \Gamma \to \mathbb{C}$ be a continuous function such that $F\mu \neq 0$. Then the following are equivalent

1. The measure $\mu$ satisfies the covariance equation with respect to $F$. 

(2) The measure $|\mu|$ satisfies the covariance equation with respect to $F$.

(3) There are a constant $c \in \mathbb{C} \setminus \{0\}$ and an element $\gamma \in \Gamma$ such that $F(\gamma) \neq 0$ and

$$\mu := c\delta_\gamma,$$

where $\delta_\gamma$ is the Dirac measure at $\gamma$.

Indeed, the constant $c$ and the multiplicative function $\gamma$ will be given explicitly in terms of $F$ and $\mu$ by the equalities

$$c := \mu(\Gamma) \quad \text{and} \quad \gamma(s) := \frac{\hat{\mu}(s,e)}{\mu(\Gamma)} + \frac{\hat{\mu}(s,e)}{\mu(\Gamma)} s \in \mathbb{S}.$$ 

As a corollary of this, we obtain the following:

**Corollary 1.** Let $\mu$ be a compactly supported complex regular Borel measure on $\mathbb{S}'$ such that $\mu(\mathbb{S}') \neq 0$, and let

$$\gamma(s) := \frac{\hat{\mu}(s,e)}{\mu(\mathbb{S}')}, \quad s \in \mathbb{S}.$$ 

Then $\mu$ satisfies the equality

$$\mu(\mathbb{S}') \int_{\mathbb{S}'^2} \varphi(s)\overline{\varphi(t)} d\mu(\varphi) = \int_{\mathbb{S}'^2} \varphi(s) d\mu(\varphi) \int_{\mathbb{S}'^2} \overline{\varphi(t)} d\mu(\varphi)$$

for all $s, t \in \mathbb{S}$ if and only if $\gamma \in \mathbb{S}'$ and $\mu = \mu(\mathbb{S}')\delta_\gamma$.

As a consequence of the latter result we shall derive the following:

**Corollary 2.** Let $Y$ be an integrable complexe random variable such that the expectation $\mathbb{E}[Y]$ is in $\mathbb{C} \setminus \{0\}$. Then a bounded complexe random vector $X = (X_1, \ldots, X_d)$ satisfies the equation

$$\mathbb{E}[Y]\mathbb{E}[X^m\overline{X^n}Y] = \mathbb{E}[X^mY]\mathbb{E}[\overline{X^n}Y]$$

for all multi-indices $m, n \in \mathbb{N}_0^d$ if and only $X$ is constant almost surely. Here, it is understood that

$$X^m := X_1^{m_1} \cdots X_d^{m_d}$$

for all $m = (m_1, \ldots, m_d) \in \mathbb{N}_0^d$.

The semigroup behind the latter corollary is $\mathbb{S} = (\mathbb{N}_0^d, +)$ introduced in Section 5. Variants of this corollary can be stated and proved for other semigroups, but we will omit to do it.

### 2. Maserick’s Approach to $BV$-functions

Throughout the sequel, $(\mathbb{S}, e, \ast)$ will denote a multiplicative commutative semigroup with identity $e$ and involution $\ast$ and consider the associated product semigroup $\mathbb{S} \times \mathbb{S}$ and furnish it with the involution denoted again by $\ast$ and given by

$$(s, t)^* := (t^*, s^*), \quad \text{for all } (s, t) \in \mathbb{S} \times \mathbb{S}.$$ 

For each $(a, b) \in \mathbb{S} \times \mathbb{S}$, we define the shift operator $E_{(a,b)}$ by

$$(E_{(a,b)}f)(s, t) = f(as, bt)$$
for all \((s, t) \in S \times S\) and \(f \in C^{S \times S}\). The complex span \(A(S)\) of all such operators is a commutative algebra with identity \(E_{(e, e)} = I\) and involution
\[
\left( \sum_{(a, b)} \alpha(a, b)E_{(a, b)} \right)^* = \sum_{(a, b)} \overline{\alpha(a, b)}E_{(b, a)}
\]
\((\alpha(a, b) \in \mathbb{C}, (a, b) \in S \times S)\). The semigroup \(S \times S\) is embedded in \(A(S)\) as a Hamel basis.

The algebra constructed in this way is isomorphic to the \(L^1\)-algebra (or semigroup algebra) constructed by Hewitt-Zuckermann [9]. The function space \(C(S \times S)\) can be algebraically identified with the dual \(A(S)^*\) of \(A(S)\) via \(T \mapsto (Tf)(e, e)\) \((f \in C^{S \times S}\) and \(T \in A(S))\). This map topologically identifies \(C^{S \times S}\) equipped with the topology of simple convergence and \(A(S)^*\) equipped with the weak*-topology. Following Berg et al. [2], \((S \times S)^*\) we will denote the set of all semicharacters on \(S \times S\). That is, \((S \times S)^*\) consists of the set of all members \(\eta\) of \(C^{S \times S}\) such that \(\eta(e, e) = 1\) and
\[
\eta(ss', tt') = \eta(s, t)\overline{\eta(s', t')}
\]
for all \((s, t), (s', t') \in S \times S\).

Equipped with the topology of pointwise convergence, \((S \times S)^*\) is completely regular. It is clear that if \(\varrho \in S'\), then
\[
\eta_{\varrho}(s, t) := \varrho(s)\overline{\varrho(t)}, \ (s, t) \in S \times S
\]
is a semicharacter on \(S \times S\). Moreover, the mapping \(\varrho \mapsto \eta_\varrho\) is a homeomorphism from \(S'\) onto \((S \times S)^*\). This identifies Radon measures on \(S'\) with those on \((S \times S)^*\).

**Definition 2.** Let \(\tau\) be a subset of \(A(S)\) and let \(P_\tau\) be convex cone in \(A(S)\) spanned by positive linear sums of finite products of elements of \(\tau\). The subset \(\tau\) is called admissible the sense of Maserick if

1. \(T^* = T\) for each \(T \in \tau\).
2. \(I - T \in P_\tau\) for each \(T \in \tau\).
3. \(A(S)\) is spanned by linear sums of finite products of members of \(\tau\).

An example of an admissible set is \(\tau := \{T_{a, \sigma}, a, \sigma \in S \times \mathbb{S}, \sigma \in \{-1, 1, -i, i\}\}\) where
\[
T_{a, \sigma} := \frac{1}{4} \left( I + \frac{\sigma}{2} E_a + \frac{\sigma}{2} E_{a^*} \right).
\]

Consider a function \(f : S \times S \to \mathbb{C}\) and let \(\Omega\) be a collection of subsets of \(P_\tau\) that satisfies
\[
\sum_{T \in \Lambda} T = I, \ \text{for all} \ \Lambda \in \Omega,
\]
\[
\{TT' : (T, T') \in \Lambda \times \Lambda\} \in \Omega \ \text{for all} \ \Lambda \in \Omega
\]
and each element of \(\tau\) belongs to some \(\Lambda \in P_\tau\). We impose the ordering on \(P_\tau\) given by
\[
\Lambda_1 \leq \Lambda_2 \iff \Lambda_2 = \Lambda_3 \Lambda_2
\]
for some \(\Lambda_3 \in P_\tau\). Then define
\[
\|f\|_\Lambda := \sum_{T \in \Lambda} |Tf(e, e)|.
\]
It is clear that the function $\Lambda \mapsto \|f\|_\Lambda$ is nondecreasing. The function $f$ is said to be of bounded variation (or just a $BV$-function) in the sense of Maserick if

$$\|f\| := \lim_{\Lambda} \|f\|_\Lambda < +\infty.$$  

We point out that this definition was given by Marsérick for arbitrary commutative semigroups with involution. Moreover, in a series of articles [5], [6], [7], he proved that $BV$-functions are precisely generalized Laplace transforms of compactly supported complex regular Borel measures on the space of semi-characters.

To give another way of describing generalized Laplace transforms we recall [2] that a function $f : \mathbb{S} \times \mathbb{S} \to \mathbb{C}$ is said to be positive definite if

$$\sum_{j,k=1}^{n} c_j \overline{c_k} f(a_j + a_k^*) \geq 0$$

for $n \in \mathbb{N}$ and finite sequences $(c_j) \in \mathbb{C}^n$, and $(a_j) \in (\mathbb{S} \times \mathbb{S})^n$. It was shown [5], [6], [7] that the space of bounded $BV$-functions is precisely the complex span of bounded positive definite functions. More generally, it was proved there that $BV$-functions complex linear sums of positive definite functions which are bounded in a certain sense (exponential boundedness).

Finally, using the above, Corollary 1 can be rephrased as

**Corollary 1’**. Let $f : \mathbb{S} \times \mathbb{S} \to \mathbb{C}$ be a $BV$-function in the sense of Maserick such that $f(e, e) \neq 0$. Then $f$ satisfies the equality

$$f(e, e)f(s, t) = f(s, e)f(e, t), \quad \text{for all } (s, t) \in \mathbb{S} \times \mathbb{S}$$

if and only if there are a constant $c \in \mathbb{C} \setminus \{0\}$ and a multiplicative function $\gamma \in \mathbb{S}'$ such that

$$f(s, t) = c\gamma(s)\gamma(t), \quad \text{for all } (s, t) \in \mathbb{S} \times \mathbb{S}.$$

We point out that in the case of the semigroup $\mathbb{S} := [0, 1], \wedge$ where $s \wedge t := \min(s, t)$, with the choice of $\tau$ as in (2.1) then $BV$-function on $[0, 1] \times [0, 1]$ in the sense of Maserick coincide with $BV$-functions in the classical sense. This can be deduced from [6].

3. Finite rank Toeplitz operators on the unit disc

We recall the basic scheme of Luecking’s method [4] about finite rank Toeplitz operators. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc of $\mathbb{C}$ and let $A^2(\mathbb{D})$ be classical Bergman space of $\mathbb{D}$. This is the space of all analytic functions on $\mathbb{D}$ which are square integrable with respect to the Lebesgue measure $dA$ on $\mathbb{D}$. The corresponding Bergman kernel is given by

$$K_\mathbb{D}(z, w) := \frac{1}{\pi(1 - z\overline{w})^2}, \quad z, w \in \mathbb{D}.$$  

Given a finite complex Borel measure $\nu$ on $\mathbb{D}$, the Toeplitz operator with symbol $\nu$ is defined on the space $\mathbb{C}[z]$ of all analytic polynomials by

$$T_{\nu}f(z) := \int_{\mathbb{D}} f(w)K_\mathbb{D}(z, w)d\nu(w), \quad z \in \mathbb{D}, \ f \in \mathbb{C}[z].$$
The measure $\mu$ induces a sesquilinear form defined $\mathbb{C}[z] \times \mathbb{C}[z]$, by

$$B_\mu(P, Q) := \int_{\mathbb{C}} P(z)\overline{Q(z)}d\nu(z)$$

for all $P, Q \in \mathbb{C}[z]$. The reproducing formula for the Bergman kernel shows that

$$B_\mu(P, Q) = \int_{\mathbb{D}} (T_\nu P)(z)\overline{Q(z)}dA(z)$$

for all analytic polynomials $P$ and $Q$. We are interested in those bounded Toeplitz operator which on $A^2(\mathbb{D})$ which are of finite rank. The characterization of those measures which induce such operators is given by the following result due to Luecking [4].

**Luecking’s Theorem.** Suppose that $\nu$ is a finite complex Borel measure on $\mathbb{D}$. Then for following are equivalent:

1. $T_\nu$ has finite rank
2. $\nu$ has finite support.

Moreover, in the affirmative case, the rank of $T_\nu$ is equal to the number of points in the support of $\nu$.

The latter theorem will be of particular use in the proof of our main theorem.

4. **Proof of the main results**

Throughout this section, let $\Gamma$ be a compact subset of $S'$ and let $\mu$ be complex Borel measure on $\Gamma$ such that $\mu(\Gamma) = 1$. Then for each $s \in S$ the number

$$|s|_\Gamma := \sup_{\omega \in \Gamma} |\omega(s)|$$

is well-defined and finite due to the continuity of the function $\omega \rightarrow \omega(s)$.

Finally, we point out that if $\mu$ satisfies the covariance equation with respect to $F$, then in view of (1.2) we have that

$$(4.1) \quad \int_{\mathbb{D}} P(\varrho(s))Q(\varrho(t))|F(\varrho)|^2d\mu(\varrho) = \int_{\mathbb{D}} P(\varrho(s))F(\varrho)d\mu(\varrho) \int_{\mathbb{D}} Q(\varrho(t))F(\varrho)d\mu(\varrho)$$

for all $s, t \in S$ and all analytic polynomials $P, Q$.

Now, fix $s \in S$ and consider the complex Borel measure $\nu_s$ on $\mathbb{D}$ defined by

$$(4.2) \quad \int_{\mathbb{D}} f(z)d\nu_s(z) := \int_{\Gamma} f \left( \frac{\varrho(s)}{2(1+|s|_\Gamma)} \right) |F(\varrho)|^2d\mu(\varrho)$$

for all compactly supported continuous functions $f$ on $\mathbb{D}$.

It’s obvious that the measure $\nu_s$ is supported in the closed disc of $\mathbb{C}$ centered at the origin with radius $1/2$.

**Lemma 4.1.** If $\mu$ satisfies the covariance equation with respect to $F$, then the Toeplitz operator $T_{\nu_s}$ induced by $\nu_s$ is of rank at most 1.
Due to (4.1), it follows that for all $P, Q \in \mathbb{C}[z]$

\[
\int_\Gamma P(\varrho(s))|F(\varrho)|^2d\mu(\varrho) = \int_\Gamma P(\varrho(s))F(\varrho)d\mu(\varrho) \int_\Gamma \overline{F(\varrho)}d\mu(\varrho)
\]

\[
\int_\Gamma Q(\varrho(s))|F(\varrho)|^2d\mu(\varrho) = \int_\Gamma F(\varrho)d\mu(\varrho) \int_\Gamma Q(\varrho(s)\overline{F(\varrho)})d\mu(\varrho).
\]

Using this, a little computing shows that for all bounded analytic functions $f$ on $\mathbb{D}$ we have

\[
(T_{\nu_s}f)(z) = \int_\mathbb{D} f(w)K_{\mathbb{D}}(z, w)d\nu_s(w)
\]

\[
= \frac{1}{\pi} \sum_{j=0}^{+\infty} (j + 1) z^j \int_\mathbb{D} f(w)w^jd\nu_s(w)
\]

\[
= \frac{1}{\pi} \sum_{j=0}^{+\infty} (j + 1) z^j \int_\Gamma f \left( \frac{\varrho(s)}{2 + |s|r} \right) \left( \frac{\varrho(s)}{2(1 + |s|r)} \right)^j |F(\varrho)|^2d\mu(\varrho)
\]

\[
= \frac{1}{\pi} \int_\Gamma f \left( \frac{\varrho(s)}{2(1 + |s|r)} \right) F(\varrho)d\mu(\varrho) \sum_{j=0}^{+\infty} (j + 1) \int_\Gamma \left( \frac{z\varrho(s)}{2(1 + |s|r)} \right)^j \overline{F(\varrho)}d\mu(\varrho)
\]

\[
= \int_\Gamma f \left( \frac{\varrho(s)}{2(1 + |s|r)} \right) F(\varrho)d\mu(\varrho) \int_\Gamma K_{\mathbb{D}} \left( z, \frac{\varrho(s)}{2(1 + |s|r)} \right) \overline{F(\varrho)}d\mu(\varrho).
\]

Hence $T_{\nu_s}f$ belongs to the linear span of the element

\[
z \mapsto \int_\Gamma K_{\mathbb{D}} \left( z, \frac{\varrho(s)}{2(1 + |s|r)} \right) \overline{F(\varrho)}d\mu(\varrho)
\]

of $A^2(\mathbb{D})$. In particular, if $\nu_s(\mathbb{D}) \neq 0$, then

\[
(T_{\nu_s}f)(z) = \int_\Gamma f \left( \frac{\varrho(s)}{2(1 + |s|r)} \right) F(\varrho)d\mu(\varrho) \int_\Gamma K_{\mathbb{D}} \left( z, \frac{\varrho(s)}{2(1 + |s|r)} \right) \overline{F(\varrho)}d\mu(\varrho)
\]

\[
= \int_\Gamma f \left( \frac{\varrho(s)}{2(1 + |s|r)} \right) F(\varrho)d\mu(\varrho) \int_\mathbb{D} K_{\mathbb{D}}(z, w)d\nu_s(w)
\]

\[
= \frac{\int_\mathbb{D} f(w)d\nu_s(w)}{\nu_s(\mathbb{D})} \int_\mathbb{D} K_{\mathbb{D}}(z, w)d\nu_s(w)
\]

This completes the proof.

\[\square\]

**Lemma 4.2.** Suppose that $\mu$ satisfies the covariance equation with respect to $F$. Then the $\mu$ follows are equivalent.

1. For all $s \in \mathbb{S}$, the Toeplitz operator $T_{\nu_s}$ induced by $\nu_s$ is of rank 1.
2. There exists $s \in \mathbb{S}$ such that Toeplitz operator $T_{\nu_s}$ induced by $\nu_s$ is of rank 1.
3. $F\mu \neq 0$.
4. $\int_\Gamma F(\varrho)d\mu(\varrho) \neq 0$. 

\[\end{document}\]
\( \int_{\Gamma} F(\varrho) d\mu(\varrho) \neq 0. \)

(6) \( \int_{\Gamma} F(\varrho)d\mu(\varrho) \neq 0 \) and \( \int_{\Gamma} F(\varrho)d\mu(\varrho) \neq 0. \)

(7) \( \int_{\Gamma} |F(\varrho)|^2 d\mu(\varrho) \neq 0. \)

(8) For all \( s \in S \), we have \( \int_{\Gamma} |F(\varrho)|^2 d\mu(\varrho) \neq 0 \), and there is a unique complex number \( a_s \in \mathbb{D} \) such that

\[
\nu_s := \int_{\Gamma} |F(\varrho)|^2 d\mu(\varrho) \delta_{a_s}
\]

where \( \delta_{a_s} \) is the Dirac mass at \( a_s \).

**Proof.** The facts that (1) \( \Rightarrow \) (2), (6) \( \Rightarrow \) (4) \( \Rightarrow \) (3), (6) \( \Rightarrow \) (5) \( \Rightarrow \) (3) and (8) \( \Rightarrow \) (7) are obvious. That (7) \( \Rightarrow \) (6) follows from (4.1). The proof of Lemma 4.1 shows that (2) \( \Rightarrow \) (3). Finally, assume that (1) holds. By Luecking’s theorem there are complex numbers \( m_s \in \mathbb{C} \setminus \{0\} \) and \( a_s \in \mathbb{D} \) such that

\[
\nu_s = m_s \delta_{a_s}.
\]

Therefore,

\[
\int_{\Gamma} |F(\varrho)|^2 d\mu(\varrho) = \nu_s(\mathbb{D}) = m_s \neq 0
\]

showing that (8) holds. This completes the proof of the lemma. \( \square \)

We are ready now to prove the Main Theorem.

**Proof of Main Theorem.** First we prove that (1) \( \Rightarrow \) (3). Assume that \( \mu \) and \( F \) satisfy the hypothesis (1) of the Main Theorem. Without loss of generality we may assume that \( \mu(\Gamma) = 1 \). If \( \mu \) satisfies the covariance equation with respect to \( F \), then by Lemma 4.2 we see that

\[
\int_{\mathbb{D}} f(z) d\nu_s(z) = f(a_s) \int_{\Gamma} |F(\varrho)|^2 d\mu(\varrho),
\]

for all \( s \in S \) and continuous functions on \( \mathbb{D} \). Setting

\[
\gamma(s) := 2(1 + |s|_\Gamma) a_s, \quad s \in S,
\]

this yields

\[
\int_{\Gamma} f(\varrho(s)) |F(\varrho)|^2 d\mu(\varrho) = f(\gamma(s)) \int_{\Gamma} |F(\varrho)|^2 d\mu(\varrho),
\]

for all \( s \in S \) and continuous functions on \( \mathbb{C} \). Taking \( f(z) = z \) and applying (4.1) yields

\[
\gamma(s) = \frac{\int_{\Gamma} \varrho(s)|F(\varrho)|^2 d\mu(\varrho)}{\int_{\Gamma} |F(\varrho)|^2 d\mu(\varrho)}
\]

\[
= \frac{\int_{\Gamma} \varrho(s) F(\varrho) d\mu(z)}{\int_{\Gamma} F(\varrho) d\mu(\varrho)}
\]

for all \( s \in S \). Taking \( f(z) = \bar{z} \) and applying (4.1) yields

\[
\gamma(s) = \frac{\int_{\Gamma} \varrho(s)|F(\varrho)|^2 d\mu(\varrho)}{\int_{\Gamma} |F(\varrho)|^2 d\mu(\varrho)}
\]
A COVARIANCE EQUATION

\[ \gamma(s) = \frac{\int_{\Gamma} g(s)|F|^2(\varrho)d\mu(\varrho)}{\int_{\Gamma}|F(\varrho)|^2d\mu(\varrho)} = \frac{\int_{\Gamma} g(s)F(\varrho)d\mu(z)}{\int_{\Gamma}F(\varrho)d\mu(\varrho)} \]

for all \( s \in \mathcal{S} \). Using this and applying again (4.1) a little computing shows that for all polynomials \( P, Q \in \mathbb{C}[z] \) and pairs \((s, t) \in \mathcal{S} \times \mathcal{S}\), we have

\[ \int_{\Gamma} P(\varrho(s))Q(\varrho(t))|F|^2(\varrho)d\mu(\varrho) = P(\gamma(s))Q(\gamma(t)) \int_{\Gamma}|F|^2d\mu(\varrho), \]

Now by the Stone-Weierstrass theorem this implies that

\[ \int_{\Gamma} |F|^2(\varrho)d\mu(\varrho) = \int_{\Gamma} \delta_{\gamma} \]

for all continuous functions \( f \) on \( \Gamma \cup \{\gamma\} \). Hence \( \gamma \in \Gamma \) and

\[ |F|^2 \mu = \left( \int_{\Gamma} |F|^2(\varrho)d\mu(\varrho) \right) \delta_{\gamma}. \]

This proves that part (3) of the theorem holds. The proof of the converse is straightforward. Finally, it is obvious to see that the fact (3) \( \Rightarrow \) (2) \( \Rightarrow \) (1) is true, which completes the proof.

Next we prove Corollary 1.

**Proof.** Follows by taking the constant function \( F \equiv 1 \) in the main theorem. This completes the proof. \( \square \)

The proof of Corollary 2 will be shifted to the next section.

5. Applications to Classical Semigroups

Although the theory works for general semigroups, we limit ourselves concrete semigroups to illustrate the results. There are many interesting examples of such semigroups for which the multiplicative functions are explicit [2].

5.1. The semigroup \((\mathbb{N}_0^d, +)\). For a positive integer \( d \) the semi-group \((\mathbb{N}_0^d, +)\) is the set of all multi-indices \( m = (m_1, \cdots, m_d) \), where the entries are nonnegative integers, equipped the standard addition. The case \( d = 1 \) corresponds to the setting of the question posed in the paper [3] asking for the description of those signed measures \( \mu \) on the closed unit disc \( \overline{\mathbb{D}} \) that satisfy the equality

\[ \mu(\overline{\mathbb{D}}) \int_{\overline{\mathbb{D}}} z^n \bar{z}^m d\mu(z) = \int_{\overline{\mathbb{D}}} z^n d\mu(z) \int_{\overline{\mathbb{D}}} \bar{z}^m d\mu(z) \]

for all \( m, n \) in the set \( \mathbb{N}_0 \) of all non-negative integers. A direct application of our main result solves this problem even in a more general abstract setting. Indeed,
**Proposition 1.** Let $d$ be a positive integer. Suppose that $F$ is a continuous function on $\mathbb{C}^d$ and $\mu$ is a compactly supported complex Borel measure on $\mathbb{C}^d$ such that $\mu(\mathbb{C}^d) \neq 0$ and $F \mu \neq 0$, then the following are equivalent

1. $\mu$ satisfies the covariance equation with respect to $F$.

   $$(5.2) \quad \mu(\mathbb{C}^d) \int_{\mathbb{C}^d} z^n \bar{z}^m |F(z)|^2 d\mu(z) = \int_{\mathbb{C}^d} z^n F(z) d\mu(z) \int_{\mathbb{C}^d} \bar{z}^m F(z) d\mu(z)$$

   for all multi-indices $m, n \in \mathbb{N}_0^d$.

2. There is a constant $c \in \mathbb{C} \setminus \{0\}$ and a vector $\zeta \in \mathbb{C}^d$ such that $F(\zeta) \neq 0$ such that $\mu = c \delta_{\zeta}$.

**Proof.** As mentioned before, we consider the finitely generated abelian semigroup $S = (\mathbb{N}_0^d, +)$ with neutral element 0. It is not hard to see the corresponding semigroup $S'$ can be identified algebraically and topologically with $\mathbb{C}^d$ following the correspondence

$g_z(m) = z^m, \ m \in \mathbb{N}_0^d, \ z \in \mathbb{C}^d$

with the understanding that $0^0 = 1$. To complete the proof, it suffices to apply Corollary 1 to this semigroup. □

**Remark 1.** Under the assumptions of Proposition 1, if $F \equiv 1$ and $\mu(\mathbb{C}^d) \neq 0$, then $\mu$ satisfies (5.2) if and only if there is a nonzero constant $c \in \mathbb{C}$ such that $\mu = c \delta_{\zeta}$ where

$$\zeta := \frac{1}{\mu(\mathbb{C}^d)} \left( \int_{\mathbb{C}^d} z_1 d\mu(z), \ldots, \int_{\mathbb{C}^d} z_d d\mu(z) \right).$$

Hence the reduction of this result to the case $d = 1$ and $F \equiv 1$ solves the problem (5.1) posed by [3].

Now we prove Corollary 2.

**Proof of Corollary 2.** Suppose that $Y$ is an integrable random variable with expectation $\mathbb{E}[Y] \neq 0$. Let $X$ be a bounded complex random vector and which satisfies (1.7). We may assume that the complex random vector $X$ takes its values inside a compact $\Gamma$ of $\mathbb{C}^d$. Consider the compactly supported complex Borel $\mu$ on $\mathbb{C}^d$ defined by

$$(5.3) \quad \int_{\mathbb{C}^d} f(z) d\mu(z) = \mathbb{E}[f(X)Y]$$

for all continuous function on $\mathbb{C}^d$ supported in $\Gamma$. Then $\mu(\mathbb{C}^d) = \mathbb{E}[Y] \neq 0$ and $\mu$ satisfies (5.2) with $F \equiv 1$ so that by Proposition 1 we see that for some vector $\zeta \in \mathbb{C}^d$ and a nonzero constant $c$ we have $\mu = c \delta_{\zeta}$. Hence

$$\mathbb{E}[f(X)Y] = cf(\zeta)$$

for all bounded Borel measurable functions $f$ on $\mathbb{C}^d$. Denote by $\mathbb{E}[Y|X = x]$ the conditional expectation of $Y$ given $X = z$. Then the latter equality implies that

$$\int_{\mathbb{C}^d} \mathbb{E}(Y|X = z) f(x) dP_X(z) = cf(\zeta)$$
for all bounded Borel measurable functions $f$ on $\mathbb{C}^d$ showing that
\[ dP_X = \delta_\zeta \text{ and } \mathbb{E}(Y|X = \zeta) = c = \mathbb{E}(Y). \]

The converse is obvious, completing the proof.

5.2. **The multiplicative semi-group** $\mathbb{N}, \cdot)$. This is the set of all positive integers equipped the natural product with neutral element 1. Let $(p_j)_{j \in \mathbb{N}}$ be the sequence of primes. The fundamental theorem of arithmetic ensures that each $n \in \mathbb{N}$ admits a unique representation $n = p_1^{n_1} \cdots p_l^{n_l}$, where $l \in \mathbb{N}$ and $n_1 \cdots n_l \in \mathbb{N}_0$. We set $\kappa(n) := (n_1, \cdots, n_l)$ and for each sequence $z = (z_j)_{j \in \mathbb{N}}$, of complex numbers, we write
\[ z^{\kappa(n)} := \Pi_{j=1}^l z_j^{n_j}. \]

It is not hard to see that the topological semigroup $(\mathbb{N}, \cdot)'$ of multiplicative functions on $(\mathbb{N}, \cdot)$ can be identified algebraically and topologically with $\mathbb{C}^\infty := \mathbb{C}^\mathbb{N}$ be furnished with product topology following the correspondence
\[ \varrho_z(n) = z^{\kappa(n)}, \ n \in \mathbb{N}, \ z \in \mathbb{C}^\infty. \]

The reduction of our main result to this semigroup gives the following.

**Proposition 2.** Let $F : \mathbb{C}^\infty \to \mathbb{C}$ be a continuous function and let $\mu$ be a compactly supported complex regular Borel measure on $\mathbb{C}^\infty$ such that $\mu(\mathbb{C}^\infty) \neq 0$ and $F \mu \neq 0$. Then $\mu$ satisfies the covariance equation
\begin{align*}
(5.4) \quad \mu(\mathbb{C}^\infty) \int_{\mathbb{C}^\infty} z^{\kappa(m)} F(z) d\mu(z) = \int_{\mathbb{C}^\infty} F(z) z^{\kappa(n)} d\mu(z) \int_{\mathbb{C}^\infty} z^{\kappa(m)} F(z) d\mu(z)
\end{align*}

for all multi-indices $m, n \in \mathbb{N}$ if and only if there are a nonzero constant $c$ and an element $\zeta$ of $\mathbb{C}^\infty$ such that $F(\zeta) \neq 0$ and $\mu = c \delta_{\zeta}$.

**Remark**. Under the assumptions of Proposition 2, if $F$ is the constant function 1 and $\mu(\mathbb{C}^\infty) \neq 0$, then then $\mu$ satisfies (5.4) if and only if there is a constant $c \in \mathbb{C}$ such that $\mu = c \delta_{\zeta}$ where
\[ \zeta := \frac{1}{\mu(\mathbb{C}^\infty)} \left( \int_{\mathbb{C}^\infty} z_1 d\mu(z), \cdots, \int_{\mathbb{C}^\infty} z_d d\mu(z) \right). \]

5.3. **The semigroup** $\mathcal{S} = ([0, +\infty), +)$. The continuous multiplicative on this semigroup are form $\varrho_z(s) = e^{sz}, z \in \mathbb{C}$. Consider the subset of $\mathcal{S}'$ given by
\[ \Gamma := \{ s \mapsto e^{-sz}, z \in \mathbb{C}, \ \text{Re} \ z \geq 0 \} \cup \{ \infty \} \]
with the understanding that $\varrho_{\infty}(s) = 1_{\{0\}}(s), s \geq 0$. Then $\Gamma$ is a compact subset of $\mathcal{S}'$. Furthermore, complex regular Borel measures on $\Gamma \setminus \{\infty\}$ are in a bijective correspondence with finite complex regular measure on the right halfplane $\mathbb{H} = \{ z \in \mathbb{C}, \ \text{Re} \ z \geq 0 \}$. Hence every finite complex regular measure $\mu$ on $\mathbb{H}$ induces a compactly supported finite complex regular Borel measure $\nu$ on $\Gamma$ defined by its generalized Laplace transform
\[ \widehat{\nu}(s, t) = \int_{\Gamma \setminus \{\infty\}} \varrho(s) \varrho(t) d\nu(\varrho) = \int_{\mathbb{H}} e^{-sz - tz} d\mu(z), \ (s, t) \mathbb{S} \times \mathbb{S}. \]

Applying the main theorem to this semigroup yields the following
Proposition 3. Let $F : \mathbb{H} \to \mathbb{C}$ be a continuous function and let $\mu$ be a finite complex Borel measure on $\mathbb{H}$ such that $\mu(\mathbb{H}) \neq 0$ and $F \mu \neq 0$. Then $\mu$ satisfies the covariance equation with respect to $F$

\begin{equation}
\mu(\mathbb{H}) \int_{\mathbb{H}} e^{-sz-t\overline{z}} |F(z)|^2 d\mu(z) = \int_{\mathbb{H}} e^{-sz} F(z) d\mu(z) \int_{\mathbb{H}} e^{-t\overline{z}} \overline{F(z)} d\mu(z)
\end{equation}

for all $s, t \in [0, +\infty)$ if and only if there are a nonzero constant $c$ and an element $\zeta$ of $\mathbb{H}$ such that $F(\zeta) \neq 0$ and $\mu = c\delta_{\zeta}$.

Remark 3. The semigroup $\mathbb{S} = ([0, +\infty), +)$ is a sample example of the so-called conelike semigroups. These are subsets $\mathbb{S}$ of finite dimensional vector spaces that are stable under addition, contain the origin $0$ and satisfy: for all $x \in \mathbb{S}$, there is a real number $r_x \geq 0$ such that $rx \in \mathbb{S}$ for all $r \geq r_x$. They were introduced by Ressel [8] in connection with Bochner’s type theorem for topological semigroups. Using the work [10], these semigroups were shown to be of interest to holomorphic functions in tube domains [11]. Finally, our main theorem can be applied to derive results in the same spirit as Proposition 3 in the case of these semigroups.

6. Analytic Kernels

This section offers further applications of our results. We will establish a kind of extremal property for analytic kernels. For simplicity we consider the ball setting. Consider and open $\mathbb{B}_d$ in $\mathbb{C}^d$ and a closed ball $\overline{\mathbb{B}}_p$ in $\mathbb{C}^p$ both centered at $0$ with positive radius. Assume that $K : \mathbb{B}_d \times \overline{\mathbb{B}}_p \to \mathbb{C}$ be kernel of the form

\begin{equation}
K(z, w) = \sum_{m, n \in \mathbb{N}_0^d} a_{m,n} z^m w^n,
\end{equation}

with nonzero complexe coefficients $a_{m,n}$, where the series is uniformly convergent in the variable $w$ for all fixed $z$. We will prove the following

Theorem 1. Suppose that $K$ is a kernel of the form (6.1) and let $f : \mathbb{B}_d \to \mathbb{C}$ be holomorphic function. Then a finite complex Borel measure $\mu$ on $\overline{\mathbb{B}}_p$ such that $\mu(\overline{\mathbb{B}}_p) = 1$ satisfies the equality

\begin{equation}
|f(z)|^2 = \int_{\overline{\mathbb{B}}_p} |K(z, \overline{w})|^2 d\mu(w),
\end{equation}

in a neighborhood of $0$ if and only if there is are complex constant $c$ and an element $\zeta$ of $\mathbb{B}_p$ such that

$\mu = c\delta_{\zeta}$ and $f = cK(., \zeta)$

Proof. We expand $f$ in terms of a power series $f(z) = \sum_{m \in \mathbb{N}_0^d} b_m z^m$ in a neighborhood of $0$. If $\mu$ satisfies (6.2), then by integration, uniqueness of the power series expansion and (6.1) we see that

\begin{equation}
b_p \overline{b_q} = a_{m,n} \overline{a_{p,q}} \int_{\overline{\mathbb{B}}_p} w^n \overline{w}^m d\mu(w),
\end{equation}

for all $m, n, p, q \in \mathbb{N}_0^d$. The proof now follows from Proposition 1. \qed
7. CONCLUDING REMARKS

**Remark 4.** In view of Lemma 4.1 in [10] we see that generalized Laplace transforms of non-negative measure are preserved under surjective morphisms of semigroups. This, combined with Marserick’s work [6], shows that generalized Laplace transforms of compactly supported complex regular Borel measures remain preserved under these mappings. Below, we list a couple examples:

Since the mapping \( x \mapsto e^x \) is an isomorphism from the additive \( S = ([0, +\infty), +) \) onto the multiplicative semigroup \( T = ([1, +\infty), \cdot) \) The analog of Proposition 3 on \( T \) can be derived by a simple change of variable.

Every finitely generated abelian semigroup \( T \) with unit is surjective image of \( (\mathbb{N}_0^d, +, 0) \) for some \( d \in \mathbb{N} \), hence a similar observation shows an analog of Proposition 1 on \( T \) can be deduced by change of variable.

**REFERENCES**

[1] C. Berg, P. H. Maserick, *Exponentially bounded positive definite functions*, Illinois J. Math. 28 (1984), 162-179.
[2] C. Berg, J. P. R. Christensen, P. Ressel, *Harmonic Analysis on Semigroups*, Graduate Texts in Math., Vol. 100, Springer-Verlag, Berlin/New York, 1984.
[3] O. El-Fallah, K. Kellay, H. Klaja, J. Mashreghi, T. Ransford, *Dirichlet spaces with superharmonic weights and de Branges-Rovnyak spaces*, Complex Anal. Oper. Theory 10 (2016), no. 1, 97-107.
[4] D. Luecking, *Finite rank Toeplitz operators on the Bergman space* Proc. AMS 136 (2008), 1717–1723.
[5] P. H. Maserick, *Moments of measures on convex bodies*, Pacific J. Math. 68 (1977), 135-152.
[6] P. H. Maserick, *BV-functions, positive-definite functions and moment problems*, Trans. Amer. Math. Soc. 214 (1975), 137-152.
[7] P. H. Maserick, *Moment and BV-functions on commutative semigroups*, Trans. Amer. Math. Soc. 181 (1973), 61-75.
[8] P. Ressel, *Bochner’s theorem for finite-dimensional conelike semigroups*, Math. Ann. 296 (1993), no. 3, 431-440.
[9] W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1966.
[10] E. H. Youssfi, *Pull-back properties of moment functions and a generalization of Bochner-Weill’s theorem*, Math. Ann. 300 (1994), 435-450.
[11] E. H. Youssfi, *Harmonic analysis on conelike bodies and holomorphic functions on tube domains*, Journal of Functional analysis. 155 (1998), 381-435.

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