THE $T$-STRUCTURES GENERATED BY OBJECTS

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Abstract. Let $T$ be a well generated triangulated category, and let $S \subseteq T$ be a set of objects. We prove that there is a $t$-structure on $T$ with $T^{\leq 0} = \langle S \rangle^{-\infty,0}$.

This article is an improvement on the main result of Alonso, Jeremías and Souto [1], in which the theorem was proved under the assumption that $T$ has a nice enough model. It should be mentioned that the result in [1] has been influential—it turns out to be interesting to study all of these $t$-structures.

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0. Introduction

The main theorem of Alonso, Jeremías and Souto [1] is about constructing $t$-structures. We remind the reader.

Let $T$ be a triangulated category with coproducts, and assume $(T^{\leq 0}, T^{\geq 0})$ is a $t$-structure on $T$. Then it’s easy and classical that $T^{\leq 0}$ is closed under extensions, direct summands, coproducts and (positive) suspensions. One can ask the reverse question. Suppose we manage to somehow concoct a full subcategory $S \subseteq T$, closed under all these operations—in Keller and Vossieck [3, 4] and Alonso, Jeremías and Souto [1] such an $S$ is called a “cocomplete pre-aisle”. Given a cocomplete pre-aisle $S \subseteq T$, one can wonder if there is a $t$-structure with $S = T^{\leq 0}$.

Keller and Vossieck [4, Section 1] showed that a (cocomplete) pre-aisle $S$ is equal to $T^{\leq 0}$ if and only if the inclusion $S \hookrightarrow T$ has a right adjoint. And Alonso, Jeremías and Souto [1] construct such a right adjoint if $S$ is the cocomplete pre-aisle “generated” by a set of objects in $T$, and if $T$ has a sufficiently nice model—a locally presentable,

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cofibrantly generated model category will do. In this article we prove an improvement, we don’t assume $\mathcal{T}$ has any model. Our main theorem says

**Theorem 2.3.** Let $\mathcal{T}$ be a well generated triangulated category and let $S \subset \mathcal{T}$ be a set of objects. Then there is a $t$–structure on $\mathcal{T}$ with $\mathcal{T}^{\leq 0} = \langle S \rangle^{(-\infty,0]}$, where $\langle S \rangle^{(-\infty,0]}$ is our notation for the smallest cocomplete pre-aisle in $\mathcal{T}$ containing $S$.

Until this article the only model-free version of such a theorem assumed that the objects in $S$ are all compact in $\mathcal{T}$, see [1, Theorem A.1].

The proof of the more general theorem in [1], where the objects in $S$ aren’t restricted to be compact but the category $\mathcal{T}$ is assumed to have a nice model, hinges on a small-object argument in Quillen’s sense. We should say something about our proof which, needless to say, is totally different.

Suppose the pre-aisle $S$ is an aisle, meaning (in the terminology of Keller and Vossieck [3, 4]) that there is a $t$–structure with $S = \mathcal{T}^{\leq 0}$. Then for every object $t \in \mathcal{T}$ there must be an object $t^{\leq 0} \in S$, yielding a homological functor $H(-) = \text{Hom}(-, t^{\leq 0})$. The idea of this paper is to construct the functor $H$ directly, and then use Brown representability to exhibit it as $\text{Hom}(-, t^{\leq 0})$. It turns out that, as long as we’re willing to disregard set-theoretic issues, the definition of $H$ is simple enough—the reader can find it at the very beginning of Section 1. The construction makes sense in great generality, and it is straightforward to show that $H : \mathcal{T}^{\text{op}} \rightarrow \text{AB}$ is a homological functor respecting products. Here $\text{AB}$ stands for large abelian groups, the collection of elements might not be a set—it needn’t belong to our universe. The only subtle part is the set-theoretic problem: it is a little tricky to show that $H(x)$ is a small set for every $x \in \mathcal{T}$.

Finally we should mention one more result we prove in this article. But first the historical context: assuming all the objects in the set $S \subset \mathcal{T}$ are compact, Keller and Nicolás [2, Theorem A.9] give a refinement of Alonso Jeremías and Souto [1, Theorem A.1]. Not only do they show that the category $\langle S \rangle^{(-\infty,0]}$ is the aisle of a $t$–structure, they prove further that every object $x \in \langle S \rangle^{(-\infty,0]}$ can be expressed as the homotopy colimit of a countable sequence $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \cdots$, where each $x_\ell$ is an $\ell$–fold extension of coproducts of positive suspensions of objects in $S$. We give an analog of this which holds when the objects in $S$ aren’t assumed compact—it requires some notation to state our result precisely, the reader is referred to Proposition 2.5.

Assuming the existence of a nice model, as in the proof of the main theorem of Alonso Jeremías and Souto [1], does not offer an alternative approach to Proposition 2.5. The small object argument of [1] constructs sequences which decidedly aren’t countable.

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THE $T$-STRUCTURES GENERATED BY OBJECTS

1. $t$–STRUCTURES VIA REPRESENTABILITY

**Definition 1.1.** Let $T$ be a triangulated category, and let $S \subset T$ be a full, additive subcategory. Assume that $S$ is a pre-aisle, meaning

(i) $\Sigma S \subset S$.
(ii) $S \ast S = S$, meaning if $x \to y \to z$ is a triangle with $x, z \in S$ then $y \in S$.
(iii) $\text{smd}(S) = S$, in other words $S$ contains all direct summands of its objects.

For every pair of objects $t, t' \in T$ we make the following definitions:

(iv) The class $\tilde{H}_S(t, t')$ has for its elements the pairs of composable morphisms $t \to s \to t'$ with $s \in S$.

And now we define a relation on $\tilde{H}_S(t, t')$. Given two objects $t, t' \in T$, then $R(t, t') \subset \tilde{H}_S(t, t') \times \tilde{H}_S(t, t')$ is as follows

(v) Suppose we are given $(h, h') \in \tilde{H}_S(t, t') \times \tilde{H}_S(t, t')$, where

$$h = \{t \xrightarrow{f} s \xrightarrow{g} t'\} \quad \text{and} \quad h' = \{t \xrightarrow{f'} s' \xrightarrow{g'} t'\}.$$ 

The pair $(h, h')$ belongs to $R(t, t')$ if there is in $T$ a commutative diagram

![Diagram](image)

with $s'' \in S$.

**Observation 1.2.** In passing we note that, given $(h, h') \in \tilde{H}_S(t, t') \times \tilde{H}_S(t, t')$, where

$$h = \{t \xrightarrow{f} s \xrightarrow{g} t'\} \quad \text{and} \quad h' = \{t \xrightarrow{f'} s' \xrightarrow{g'} t'\},$$

then the diagram of Definition 1.1(v) forces the equality $gf = g'f'$.

**Example 1.3.** Suppose we are given composable morphisms $t \xrightarrow{e} s \xrightarrow{f} s' \xrightarrow{g} t'$ with $s, s' \in S$. Let

$$h = \{t \xrightarrow{e} s \xrightarrow{gf} t'\} \quad \text{and} \quad h' = \{t \xrightarrow{fe} s' \xrightarrow{g} t'\}.$$ 

The commutative diagram

![Diagram](image)

shows that $(h, h') \in R(t, t')$. 
Lemma 1.4. The $R(t, t')$ of Definition 1.1(v) is an equivalence relation.

Proof. Since $R(t, t')$ is obviously reflexive and symmetric, what needs proof is transitivity. Suppose therefore that we are given three elements $h, h', h'' \in \tilde{H}_{\mathcal{S}}(t, t')$ with $(h, h')$ and $(h', h'')$ in $R(t, t')$. If

$$h = \{ t \xrightarrow{f} s \xrightarrow{g} t' \}, \quad h' = \{ t \xrightarrow{f'} s' \xrightarrow{g'} t' \}, \quad h'' = \{ t \xrightarrow{f''} s'' \xrightarrow{g''} t' \}$$

then the diagrams exhibiting the fact that $(h, h')$ and $(h', h'')$ belong to $R(t, t')$ assemble to a commutative diagram

The commutative square

$$\begin{array}{ccc}
  s' & \xrightarrow{f} & s' \\
  \downarrow \vphantom{f} & & \downarrow \vphantom{f} \\
  \tilde{s} & \xrightarrow{g} & t' \\
\end{array}$$

may be factored through the homotopy pushout

$$\begin{array}{ccc}
  s' & \xrightarrow{f} & \tilde{s} \\
  \downarrow \vphantom{f} & \downarrow \vphantom{f} & \downarrow \vphantom{f} \\
  \tilde{s'} & \xrightarrow{g'} & \tilde{s'} \\
  \downarrow \vphantom{f} & \downarrow \vphantom{f} & \downarrow \vphantom{f} \\
  \tilde{s} & \xrightarrow{g} & t' \\
\end{array}$$

We remind the reader: by the definition of homotopy pushouts the square in the last diagram is commutative, and may be “folded” to a triangle $s' \rightarrow \tilde{s} \oplus \tilde{s'} \rightarrow \tilde{s''} \rightarrow \Sigma s'$. This makes $\tilde{s''}$ an object in $\mathcal{S} \ast \Sigma \mathcal{S} \subset \mathcal{S} \ast \mathcal{S} = \mathcal{S}$. The commutative diagram

now establishes that $(h, h'')$ belongs to $R(t, t')$. $\square$
**Definition 1.5.** With the notation of Definition 1.4 (iv) and (v), we define $H_S(t, t')$ to be the quotient of $\tilde{H}_S(t, t')$ by the equivalence relation $R(t, t').$

**Example 1.6.** The example to keep in mind is the following. Suppose $T$ is a triangulated category with a $t$–structure, and put $S = T^{\leq 0}$. Then an element of $\tilde{H}_S(t, t')$ is a pair of composable morphisms $t \to s \to t'$ with $s \in S = T^{\leq 0}$, and this composable string may factored further, uniquely, as $t \to s \to (t')^{\leq 0} \to t'$ where the map $(t')^{\leq 0} \to t'$ is the canonical map from the $t$–structure truncation. Example 1.3 applies, showing that every element of $\tilde{H}_S(t, t')$ is equivalent to an element $t \to s \to (t')^{\leq 0} \to t'$.

Now suppose the elements $t \to f \to (t')^{\leq 0} \to t'$ and $t \to f' \to (t')^{\leq 0} \to t'$ are equivalent to each other. Then there must exist in $T$ a commutative diagram

\[
\begin{array}{ccc}
 t & \xrightarrow{f} & (t')^{\leq 0} \\
 \downarrow{f'} & & \downarrow{t'} \\
 (t')^{\leq 0} & \xrightarrow{s} & t'
\end{array}
\]

with $s \in S$. The map $s \to t'$ must also factor as $s \to (t')^{\leq 0} \to t'$, allowing us to replace the above by the commutative diagram

\[
\begin{array}{ccc}
 t & \xrightarrow{f} & (t')^{\leq 0} \\
 \downarrow{f'} & & \downarrow{t'} \\
 (t')^{\leq 0} & \xrightarrow{s} & t'
\end{array}
\]

But now in the two commutative triangles

\[
\begin{array}{ccc}
 (t')^{\leq 0} & \xrightarrow{} & t' \\
 \downarrow{\alpha} & & \downarrow{} \\
 (t')^{\leq 0} & \xrightarrow{} & t'
\end{array}
\]

the maps $\alpha$ have to be identities, forcing $f = f'$.

Thus in the special case, where $S = T^{\leq 0}$ for some $t$–structure, the class $H_S(t, t')$ is canonically isomorphic to $\text{Hom}[t, (t')^{\leq 0}]$.

It’s natural to wonder how much of the structure, which $\text{Hom}[t, (t')^{\leq 0}]$ always has, can be constructed on $H_S(t, t')$ without knowing that $S = T^{\leq 0}$ for some $t$–structure. This leads us to the next few results.
Definition 1.7. Let $\mathcal{T}$ be a triangulated category, let $S \subset \mathcal{T}$ be as in Definition 1.1 (i), (ii) and (iii), and let $H_S(t,t')$ be as in Definition 1.5. We define the following:

(i) If $h, h'$ are two elements of $H_S(t,t')$, then $h + h'$ is constructed as follows. First choose representatives for $h, h'$ in $\tilde{H}_S(t,t')$, that is choose factorizations

$$t \xrightarrow{f} s \xrightarrow{g} t', \quad t \xrightarrow{f'} s' \xrightarrow{g'} t'$$

which represent the equivalence classes $h, h'$. Then $h + h'$ is the equivalence class of

$$\begin{pmatrix} f \\ f' \end{pmatrix} s \oplus s' \xrightarrow{(g, g')} t$$

(ii) The element $0 \in H_S(t,t')$ is defined to be the equivalence class of $t \rightarrow 0 \rightarrow t'$.

The next lemma is straightforward, the proof is left to the reader.

Lemma 1.8. The operation $+$ of Definition 1.7(i) is well-defined, meaning the element $h + h' \in H_S(t,t')$ does not depend on the choice of representatives. The operation $+$ is commutative and associative. The element $0 \in H_S(t,t')$ of Definition 1.7(ii) satisfies $0 + h = h = h + 0$. Finally: if $h$ is represented by $t \xrightarrow{f} s \xrightarrow{g} t'$ and $h'$ is represented by $t \xrightarrow{f'} s' \xrightarrow{g'} t'$ then $h + h' = 0$.

Summarizing: Definition 1.7 gives $H_S(t,t')$ the structure of an abelian group.

Construction 1.9. Given a morphism $e : x \rightarrow y$, we define $\tilde{H}_S(e,t) : \tilde{H}_S(y,t) \rightarrow \tilde{H}_S(x,t)$ to be the map which precomposes with $e$. That is: $y \xrightarrow{f} s \xrightarrow{g} t$ goes to $x \xrightarrow{f \circ e} s \xrightarrow{g} t$. This map obviously sends equivalent elements of $\tilde{H}_S(y,t)$ to equivalent elements of $\tilde{H}_S(x,t)$, hence descends to a map which we will denote

$$H_S(e,t) : H_S(y,t) \rightarrow H_S(x,t)$$

The next lemma is another obvious one.

Lemma 1.10. Let $t \in \mathcal{T}$ be an object. With the notation of Construction 1.9, we have that $H_S(\cdot, t)$ is an additive functor from $\mathcal{T}$ to abelian groups. Here $H_S(x,t)$ is an abelian group with the structure given in Lemma 1.8.

Slightly more subtle is

Lemma 1.11. The functor $H_S(\cdot, t)$ of Lemma 1.10 is homological.

Proof. Suppose we are given an object $t \in \mathcal{T}$, and let $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ be a triangle in $\mathcal{T}$. We need to show that $H_S(\cdot, t)$ takes it to an exact sequence.

Suppose therefore that we are given an element $h \in H_S(y,t)$ which maps to zero in $H_S(x,t)$. Choose a representative for $h$, meaning composable morphisms $y \xrightarrow{f} s \xrightarrow{g} t$ with $s \in S$. We are given that the image of $h$ in $H_S(x,t)$ vanishes, meaning the composite
The equivalence means that there must be a commutative diagram

\[
\begin{array}{c}
 x \xrightarrow{f_\alpha} s \xrightarrow{g'} s' \xrightarrow{g''} t
\end{array}
\]

with \( s' \in S \). The composable morphisms \( y \xrightarrow{f} s \xrightarrow{g'} s' \xrightarrow{g''} t \) and Example 1.3 tell us that \( y \xrightarrow{f} s \xrightarrow{g'} s' \xrightarrow{g''} t \) is equivalent to \( y \xrightarrow{g'f} s' \xrightarrow{g''} t \). In other words \( h \in H_S(y,t) \) is also represented by \( y \xrightarrow{g'f} s' \xrightarrow{g''} t \). But now the vanishing of the composite \( x \xrightarrow{\alpha} y \xrightarrow{g'f} s' \) says that \( g'f : y \to s' \) must factor as \( y \xrightarrow{\beta} z \xrightarrow{\gamma} s' \). But then \( h \in H_S(y,t) \) must be the image of \( z \xrightarrow{\gamma} s' \xrightarrow{g''} t \) under the map \( H_S(\beta, t) : H_S(z, t) \to H_S(y, t) \).

And finally we look at the case where \( T \) has coproducts.

**Lemma 1.12.** Let \( T \) be a triangulated category with coproducts, let \( S \subseteq T \) be a full subcategory closed in \( T \) under coproducts, and assume that the hypotheses of Definition 1.1 (i), (ii) and (iii) are satisfied.

Then the functor \( H_S(\cdot, t) \) of Lemma 1.10 respects products. To expand: we view \( H_S(\cdot, t) \) as a functor \( T^{op} \to \text{Ab} \), and the products in \( T^{op} \) are the coproducts in \( T \). Suppose \( \{ x_\lambda, \lambda \in \Lambda \} \) is a set of objects in \( T \), there is always a natural map

\[
H_S \left( \prod_{\lambda \in \Lambda} x_\lambda , t \right) \xrightarrow{\rho} \prod_{\lambda \in \Lambda} H_S(x_\lambda, t)
\]

and we assert that this map is an isomorphism.

**Proof.** The inverse of the canonical map \( \rho \) is simple enough. Given an element \( \prod_{\lambda \in \Lambda} h_\lambda \in \prod_{\lambda \in \Lambda} H_S(x_\lambda, t) \) choose representatives, meaning for each \( \lambda \) choose for \( h_\lambda \) a representative \( x_\lambda \to s_\lambda \to t \). Form the composite

\[
\prod_{\lambda \in \Lambda} x_\lambda \xrightarrow{\rho} \prod_{\lambda \in \Lambda} s_\lambda \xrightarrow{\gamma} \prod_{\lambda \in \Lambda} t
\]

The reader can check that this construction gives a well-defined map

\[
\prod_{\lambda \in \Lambda} H_S(x_\lambda, t) \xrightarrow{\sigma} H_S \left( \prod_{\lambda \in \Lambda} x_\lambda , t \right)
\]

meaning the resulting element of \( H_S \left( \prod_{\lambda \in \Lambda} x_\lambda , t \right) \) is independent of the choice of representatives.
And now it is an exercise to check that \( \sigma \rho \) and \( \rho \sigma \) are both identities. \( \square \)

**Summary 1.13.** We have learned that, for any \( S \subset T \) as in Definition [1.1] and any object \( t \in T \), the functor \( H_S(\cdot, t) \) of Lemma [1.10] is homological. If \( T \) has coproducts and \( S \) is closed in \( T \) under coproducts then \( H_S(\cdot, t) \) also respects products.

Example [1.6] showed us that, in the special case where \( S = T^{\leq 0} \) for some \( t \)-structure, the functor \( H_S(\cdot, t) \) is representable—it is naturally isomorphic to \( \text{Hom}(\cdot, t^{\leq 0}) \). The previous paragraph amounts to saying that, for a general \( S \), the functor \( H_S(\cdot, t) \) satisfies the obvious necessary conditions for representability.

Now we come to

**Proposition 1.14.** Suppose \( S \subset T \) are as in Definition [1.1] and let \( t \in T \) be an object.

Assume the functor \( H_S(\cdot, t) \) of Lemma [1.10] is representable, more explicitly assume we are given an isomorphism \( \text{Hom}(\cdot, t) \to H_S(\cdot, t) \) which we fix. Then

(i) The object \( x \) belongs to \( S \subset T \).

(ii) There is a unique morphism \( \varepsilon : x \to t \), so that the image of \( \text{id} : x \to x \) under the map \( \text{Hom}(x, x) \to H_S(x, t) \) is represented by \( x \xrightarrow{\text{id}} x \xrightarrow{\varepsilon} t \).

(iii) Any map \( s \to t \), with \( s \in S \), factors in \( T \) uniquely as \( s \xrightarrow{\exists} x \xrightarrow{\varepsilon} t \).

Finally: the subcategory \( S \) is equal to \( T^{\leq 0} \), for some \( t \)-structure on \( T \), if and only if \( H_S(\cdot, t) \) is representable for every \( t \in T \).

**Proof.** We begin by proving (i), starting with the given isomorphism \( \text{Hom}(\cdot, x) \to H_S(\cdot, t) \). The identity map \( \text{id} \in \text{Hom}(x, x) \) must go under the isomorphism \( \text{Hom}(x, x) \to H_S(x, t) \) to an element of \( H_S(x, t) \); choose a representative \( x \xrightarrow{\text{id}} x \xrightarrow{\varepsilon} t \) with \( s \in S \). The composable morphisms \( s \xrightarrow{\text{id}} s \xrightarrow{h} t \) may be viewed as representing an element of \( H_S(s, t) \), which must be the image of an \( f \in \text{Hom}(s, x) \) under the isomorphism \( \text{Hom}(s, x) \to H_S(s, t) \). We know the image \( x \xrightarrow{g} s \xrightarrow{h} t \) of \( \text{id} \in \text{Hom}(x, x) \) under the isomorphism \( \text{Hom}(x, x) \to H_S(x, t) \), and Yoneda tells us that \( f \) has to go to \( s \xrightarrow{gf} s \xrightarrow{h} t \). But \( f \) was chosen to have image \( s \xrightarrow{\text{id}} s \xrightarrow{h} t \), and therefore \( s \xrightarrow{gf} s \xrightarrow{h} t \) and \( s \xrightarrow{\text{id}} s \xrightarrow{h} t \) must be equivalent. Precomposing with \( g : x \to s \) we have that \( x \xrightarrow{gf} s \xrightarrow{h} t \) are also equivalent. The natural isomorphism \( \text{Hom}(x, x) \to H_S(x, t) \) therefore takes the elements \( fg, \text{id} \in \text{Hom}(x, x) \) to the same image, and we deduce that \( fg = \text{id} \). Thus \( x \) is a direct summand of \( s \in S \), and Definition [1.1][iii] guarantees that \( x \) belongs to \( S \). This proves (i).

To prove (ii) recall that the image of \( \text{id} : x \to x \), under the isomorphism \( \text{Hom}(x, x) \to H_S(x, t) \), is represented by the composable pair \( x \xrightarrow{f} s \xrightarrow{g} t \) which we chose at the beginning of the proof. But now we know that \( x \in S \) and Example [1.3] applied to the
composable morphisms $x \xrightarrow{id} x \xrightarrow{f} s \xrightarrow{g} t$, tells us that $x \xrightarrow{f} s \xrightarrow{g} t$ is equivalent to $x \xrightarrow{id} x \xrightarrow{gf} t$. Furthermore this representative is unique: if $x \xrightarrow{id} x \xrightarrow{h} t$ is equivalent to $x \xrightarrow{id} x \xrightarrow{h'} t$ then Observation [1.2] teaches us that $h = h'$. As in the statement of the current Proposition this preferred representative, of the image of $id \in \text{Hom}(x, x)$ under the isomorphism $\text{Hom}(x, x) \rightarrow H_S(x, t)$, will be denoted $x \xrightarrow{id} x \xrightarrow{\varepsilon} t$.

Now for the proof of (iii). Suppose we are given a map $s \xrightarrow{f} t$ with $s \in S$. Then $s \xrightarrow{id} s \xrightarrow{f} t$, viewed as representing an element in $H_S(s, t)$, corresponds under the isomorphism $H_S(s, t) \cong \text{Hom}(s, x)$ to a morphism $\alpha : s \rightarrow x$. The isomorphism $\text{Hom}(s, x) \rightarrow H_S(s, t)$ takes $\alpha$ to the equivalence class of $s \xrightarrow{\alpha} x \xrightarrow{\varepsilon} t$. The fact that $s \xrightarrow{id} s \xrightarrow{f} t$ is equivalent to $s \xrightarrow{\alpha} x \xrightarrow{\varepsilon} t$, coupled with Observation [1.2] tells us that $f = \varepsilon \alpha$.

The uniqueness of $\alpha$ is proved as follows. Suppose we have elements $\alpha, \alpha'$ in $\text{Hom}(s, x)$ with $\varepsilon \alpha = \varepsilon \alpha'$. Consider the triangle $s \xrightarrow{\alpha-\alpha'} x \xrightarrow{\beta} s' \rightarrow \Sigma s$. As $x, \Sigma s$ both lie in $S$ so does $s'$, while $\varepsilon : x \rightarrow t$ has to factor $x \xrightarrow{\beta} s' \xrightarrow{\gamma} t$. The diagram

now exhibits that the pairs

$h = \{s \xrightarrow{\alpha} x \xrightarrow{\varepsilon} t\}, \quad h' = \{s \xrightarrow{\alpha'} x \xrightarrow{\varepsilon} t\}$

are equivalent as in Definition [1.4 (v)]. But this means that the isomorphism $\text{Hom}(s, x) \cong H_S(s, t)$ takes $\alpha, \alpha' \in \text{Hom}(s, x)$ to the same image, hence $\alpha = \alpha'$.

Now for the last part of the Proposition. Example [1.6] teaches us that, if $S = T^{\leq 0}$ for some $t$-structure, then $H_S(-, t)$ is representable for every $t \in T$. What needs proof is the converse. Assume therefore that every $H_S(-, t)$ is representable. By (i), (ii) and (iii) we deduce that, for every $t \in T$, there exists a morphism $\varepsilon : x \rightarrow t$ such that

(iv) $x \in S$

(v) Every morphism $s \rightarrow t$, with $s \in S$, factors uniquely through $\varepsilon$.

But this exactly says that the inclusion $S \rightarrow T$ has a right adjoint, and $\varepsilon : x \rightarrow t$ is the counit of adjunction. From [4 Section 1] it now follows that $S = T^{\leq 0}$ for a (unique) $t$-structure on $T$. □

**Discussion 1.15.** Let $T$ be a triangulated category with coproducts and let $S \subset T$ be a set of objects. Form the category $S = \langle S \rangle^{[-\infty, 0]}$, the smallest full subcategory of $T$ satisfying the hypotheses of Definition [1.11 (i), (ii) and (iii) and closed in $T$ under coproducts. Summary [1.13] allows us to deduce that the functor $H_S(-, t)$ is homological...
and respects products—still in the gorgeous generality of any triangulated category $\mathcal{I}$ with coproducts.

Now assume $\mathcal{I}$ is well generated. Then Brown representability holds, see [6] Theorem 8.3.3—by this theorem showing that $H_S(-, t)$ is representable only requires solving the set-theoretic problem, we need to prove that $H_S(x, t)$ is a small set for every $x, t \in \mathcal{I}$.

The proof of Theorem [2.3] will show how to solve this set theoretic problem. And then Proposition [1.14] will come to our aid—since we will know that $H_S(-, t)$ is representable for every $t \in \mathcal{I}$, Proposition [1.14] will allow us to deduce that $\langle S \rangle (-\infty, 0] = T^{\leq 0}$ for some $t$–structure.

2. Main theorem

**Construction 2.1.** Let $\mathcal{I}$ be a well-generated triangulated category, and let $S \subset \mathcal{I}$ be a small set of objects. Choose a large enough regular cardinal $\alpha$ so that

(i) The category $\mathcal{I}$ is $\alpha$–compactly generated.

(ii) The set $S$ is contained in $T^{\alpha}$.

Now we proceed by transfinite induction, on the ordinal $i \leq \alpha$, to build up full subcategories $S(i) \subset T^{\alpha} \cap \langle S \rangle (-\infty, 0]$ as follows:

(iii) The objects of $S(1)$ are the coproducts of $< \alpha$ objects in $\cup_{j=0}^{\infty} \Sigma^j S$.

(iv) If $i$ is any ordinal $< \alpha$ and $i + 1$ is its successor, then the objects of $S(i + 1)$ are all the coproducts of $< \alpha$ objects in $S(i) * (S(i))$.

(v) If $i' \leq \alpha$ is limit ordinal, then $S(i') = \cup_{i < i'} S(i)$.

By induction we see that each $S(i)$ satisfies $\Sigma S(i) \subset S(i)$, and for any limit ordinal $i$ we have that $S(i) * S(i) = S(i)$. By the above and the fact that $\alpha$ is a regular cardinal we have that $S(\alpha)$ satisfies

(vi) $S \subset S(\alpha) \subset T^{\alpha} \cap \langle S \rangle (-\infty, 0]$.

(vii) $\Sigma S(\alpha) \subset S(\alpha)$.

(viii) $S(\alpha) * S(\alpha) = S(\alpha)$.

(ix) Any coproduct of $< \alpha$ objects in $S(\alpha)$ lies in $S(\alpha)$.

Now we assert:

**Lemma 2.2.** Let the notation be as in Construction 2.1. Any morphism $t \to s'$, with $t \in T^{\alpha}$ and $s' \in \langle S \rangle (-\infty, 0]$, can be factored as $t \to s \to s'$ with $s \in S(\alpha)$.

**Proof.** Let $\mathcal{R} \subset \mathcal{I}$ be the full subcategory defined by the formula

$$\text{Ob}(\mathcal{R}) = \left\{ r \in \mathcal{I} \mid \begin{array}{l} \text{Every morphism } t \to r, \text{ with } t \in T^{\alpha}, \\ \text{can be factored as } t \to s \to r \text{ with } s \in S(\alpha) \end{array} \right\}$$

and we observe the following

(i) $S \subset \mathcal{R}$.

This is obvious: any map $t \to s$ can be factored as $t \to s \xrightarrow{id} s$. 

(ii) \( \Sigma \mathcal{R} \subset \mathcal{R} \).

Suppose we are given a morphism \( f : t \rightarrow \Sigma r \) with \( t \in \mathcal{T}^\alpha \) and \( r \in \mathcal{R} \). Then \( \Sigma^{-1}f : \Sigma^{-1}t \rightarrow r \) is a morphism from \( \Sigma^{-1}t \in \mathcal{T}^\alpha \) to \( r \in \mathcal{R} \), and may be factored as \( \Sigma^{-1}t \rightarrow s \rightarrow r \) with \( s \in S(\alpha) \). Thus \( f \) has a factorization as \( t \rightarrow \Sigma s \rightarrow \Sigma r \) with \( \Sigma s \in \Sigma S(\alpha) \subset S(\alpha) \).

(iii) The subcategory \( \mathcal{R} \) is closed in \( \mathcal{T} \) under coproducts.

To see this let \( \{ r_\lambda, \lambda \in \Lambda \} \) be any set of objects in \( \mathcal{R} \), let \( t \) be an object in \( \mathcal{T}^\alpha \), and suppose we are given a morphism \( t \xrightarrow{f} \bigoplus_{\lambda \in \Lambda} r_\lambda \).

The fact that \( t \) belongs to \( \mathcal{T}^\alpha \) permits us to find a subset \( \Lambda' \subset \Lambda \) of cardinality \( < \alpha \), and a factorization \( t \xrightarrow{\bigoplus_{\lambda \in \Lambda'} t_\lambda} \bigoplus_{\lambda \in \Lambda'} r_\lambda \xrightarrow{\bigoplus_{\lambda \in \Lambda} f_\lambda} \bigoplus_{\lambda \in \Lambda} r_\lambda \) with each \( t_\lambda \in \mathcal{T}^\alpha \). But then for each \( \lambda \in \Lambda' \) we can factor \( f_\lambda : t_\lambda \rightarrow r_\lambda \) as \( t_\lambda \rightarrow s_\lambda \rightarrow r_\lambda \) with \( s_\lambda \in S(\alpha) \), giving a factorization of \( f \) as \( t \xrightarrow{\bigoplus_{\lambda \in \Lambda'} s_\lambda} \bigoplus_{\lambda \in \Lambda} r_\lambda \).

But since the cardinality of \( \Lambda' \subset \Lambda \) is \( < \alpha \) and each \( s_\lambda \) belongs to \( S(\alpha) \), Construction 2.1(ix) gives that \( \bigoplus_{\lambda \in \Lambda'} s_\lambda \) must belong to \( S(\alpha) \).

(iv) The subcategory \( \mathcal{R} \) satisfies \( \mathcal{R} \ast \mathcal{R} \subset \mathcal{R} \).

To prove (iv) apply [7, Lemma 1.5], with \( \mathcal{A} = \mathcal{C} = S(\alpha) \) and \( \mathcal{X} = \mathcal{Z} = \mathcal{R} \); we have that any pair of morphisms \( t \rightarrow x \) and \( t \rightarrow z \), with \( t \in \mathcal{T}^\alpha \) and with \( x \in \mathcal{X} \) and \( z \in \mathcal{Z} \), factor (respectively) as \( t \rightarrow a \rightarrow x \) and \( t \rightarrow c \rightarrow z \) with \( a \in \mathcal{A} \) and \( c \in \mathcal{C} \). Since \( S(\alpha) = \mathcal{C} \subset \mathcal{T}^\alpha \) and \( \mathcal{T}^\alpha \) is triangulated [7, Remark 1.6] applies. We conclude that any map \( t \rightarrow y \), with \( t \in \mathcal{T}^\alpha \) and \( y \in \mathcal{X} * \mathcal{Z} = \mathcal{R} * \mathcal{R} \), must factor as \( t \rightarrow b \rightarrow y \) with \( b \in \mathcal{A} * \mathcal{C} = S(\alpha) * S(\alpha) \subset S(\alpha) \).

Since \( \langle S \rangle^{(-\infty,0]} \) is the smallest subcategory of \( \mathcal{T} \) satisfying (i), (ii), (iii) and (iv) it must be contained in \( \mathcal{R} \), proving the lemma.

**Theorem 2.3.** Let \( \mathcal{T} \) be a well generated triangulated category, and let \( \mathcal{S} \subset \mathcal{T} \) be a set of objects. Then there is a \( t \)-structure on \( \mathcal{T} \) with \( \mathcal{T}^\leq0 = \langle \mathcal{S} \rangle^{(-\infty,0]} \).

**Proof.** Put \( \mathcal{S} = \langle \mathcal{S} \rangle^{(-\infty,0]} \). By Discussion 1.15 it suffices to show that, for every pair of objects \( x, t \in \mathcal{T} \), the collection \( H_\mathcal{S}(x,t) \) is a set.
For this choose a regular cardinal $\alpha$ large enough so that $T$ is $\alpha$–compactly generated, and $S \cup \{x\} \subset T^\alpha$. Let $S(\alpha)$ be as in Construction 2.1. Suppose we are given a representative $x \overset{f}{\rightarrow} s \overset{g}{\rightarrow} t$ for an element $h \in H_S(x, t)$. Then $f : x \rightarrow s$ is a morphism from $x \in T^\alpha$ to $s \in S = \langle S \rangle ^{(-\infty, 0]}$, and Lemma 2.2 allows us to factor $f$ as $x \overset{f'}{\rightarrow} s' \overset{f''}{\rightarrow} s$ with $s' \in S(\alpha)$. The string of composable morphisms $x \overset{f'}{\rightarrow} s' \overset{f''}{\rightarrow} s \overset{g}{\rightarrow} t$, coupled with Example 1.3, yields an equivalence between $x \overset{f}{\rightarrow} s \overset{g}{\rightarrow} t$ and $x \overset{f'}{\rightarrow} s' \overset{g'}{\rightarrow} t$, that is $x \overset{f}{\rightarrow} s \overset{g}{\rightarrow} t$ is equivalent to $x \overset{f'}{\rightarrow} s' \overset{g'}{\rightarrow} t$. Thus every $h \in H_S(x, t)$ may be represented as $x \rightarrow s' \rightarrow t$ with $s' \in S(\alpha)$, and there is only a set of these.

Note that the equivalence can also be checked without going to very large objects. If $x \overset{f}{\rightarrow} s \overset{g}{\rightarrow} t$ and $x \overset{f'}{\rightarrow} s' \overset{g'}{\rightarrow} t$ are equivalent, and $s$ and $s'$ both lie in $S(\alpha)$, then the definition of equivalence says there exists a commutative diagram

![Diagram](https://example.com/diagram.jpg)

with $\tilde{s} \in S = \langle S \rangle ^{(-\infty, 0]}$. The commutative square

![Commutative Square](https://example.com/commutative_square.jpg)

may be factored through the homotopy pushout, obtaining

![Factored Diagram](https://example.com/factored_diagram.jpg)

The fact that we have a homotopy pushout means there is a triangle $x \rightarrow s \oplus s' \rightarrow y \rightarrow \Sigma x$, and as $x$ and $s \oplus s'$ belong to $T^\alpha$ so does $y$. The map $y \rightarrow \tilde{s}$ is a morphism from $y \in T^\alpha$ to $\tilde{s} \in \langle S \rangle ^{(-\infty, 0]}$, and Lemma 2.2 permits us to factorize it as $y \rightarrow s'' \rightarrow \tilde{s}$.
with \( s'' \in S(\alpha) \). We obtain a commutative diagram

\[
\begin{array}{ccc}
  x & \xrightarrow{f} & s \\
  \downarrow{f'} & & \downarrow{s'} \\
  s' & \xrightarrow{g'} & t
\end{array}
\]

where \( s, s', s'' \) all lie in \( S(\alpha) \), and the objects \( x \) and \( t \) are given and fixed. If we delete \( \bar{s} \) the diagram still exhibits the equivalence of \( x \xrightarrow{f} s \xrightarrow{g} t \) and \( x \xrightarrow{f'} s' \xrightarrow{g'} t \), but now there is only a set of possible such diagrams to consider.

Summarizing: we have proved that \( \mathcal{H}_S(x, t) \) is a set, and the theorem follows. \( \square \)

**Reminder 2.4.** Before the next result we should remind the reader of basic notation.

Let \( \mathcal{T} \) be a triangulated category with coproducts and let \( T \subset \mathcal{T} \) be a set of objects. Then, inductively on the integer \( \ell > 0 \), we construct in \( \mathcal{T} \) full subcategories \( \operatorname{Coprod}_\ell(T) \) as follows:

(i) For \( \ell = 1 \) we declare the category \( \operatorname{Coprod}_1(T) \) to have for objects all possible coproducts of the objects belonging to \( T \).

(ii) Assume \( \operatorname{Coprod}_\ell(T) \) has been constructed. Then the category \( \operatorname{Coprod}_{\ell+1}(T) \) is defined by the formula

\[
\operatorname{Coprod}_{\ell+1}(T) = \operatorname{Coprod}_\ell(T) \ast \operatorname{Coprod}_1(T)
\]

This formula means that an object \( y \in \mathcal{T} \) belongs to \( \operatorname{Coprod}_{\ell+1}(T) \) if there exists in \( \mathcal{T} \) a triangle \( x \to y \to z \) with \( x \in \operatorname{Coprod}_\ell(T) \) and \( z \in \operatorname{Coprod}_1(T) \).

And now we are ready for the next result, which is an analog of Keller and Nicolás [2, Theorem A.9].

**Proposition 2.5.** Let \( \mathcal{T} \) be a well-generated triangulated category and assume \( S \subset \mathcal{T} \) is a set of objects. Pick a regular cardinal \( \alpha \) so that

(i) \( \mathcal{T} \) is \( \alpha \)-compactly generated.

(ii) \( S \subset \mathcal{T}^\alpha \).

And now let \( S(\alpha) \) be as in Construction 2.1.

Given any object \( x \in \overline{S}^{(-\infty,0]} \), there exists in \( \mathcal{T} \) a countable sequence \( x_1 \to x_2 \to x_3 \to \cdots \) such that

(iii) \( x_\ell \) belongs to \( \operatorname{Coprod}_\ell[S(\alpha)] \).

(iv) \( x \cong \operatorname{Hocolim} x_\ell \).

**Proof.** Fix for the proof an object \( x \in \overline{S}^{(-\infty,0]} \). We begin by constructing the sequence \( x_1 \to x_2 \to x_3 \to \cdots \) and a map \( \varphi : \operatorname{Hocolim} x_\ell \to x \), and then we will prove that \( \varphi \) is an isomorphism. The construction of the sequence is as follows.
(v) Let $\Lambda_1$ be the set of all maps $f_\lambda : s_\lambda \to x$, with $s_\lambda \in S(\alpha)$. Define

$$x_1 = \prod_{\lambda \in \Lambda_1} s_\lambda.$$ 

The formula makes it clear that $x_1$ belongs to $\text{Coproduct}_1[S(\alpha)]$. We let the map $\varphi_1 : x_1 \to x$ be the obvious.

(vi) Assume $x_\ell \in \text{Coproduct}_\ell[S(\alpha)]$ and the map $\varphi_\ell : x_\ell \to x$ have been defined. Complete $\varphi_\ell$ to a triangle $y \to x_\ell \xrightarrow{\varphi_\ell} x$, and let $\Lambda_\ell$ be the set of all maps $\Sigma^{-1}s_\lambda \to y$ with $s_\lambda \in S(\alpha)$. The composite $\coprod_{\Lambda_\ell} \Sigma^{-1}s_\lambda \to x_\ell \xrightarrow{\varphi_\ell} x$ factors through the composite $y \to x_\ell \xrightarrow{\varphi_\ell} x$ and therefore vanishes, allowing us to construct the commutative diagram

$$\coprod_{\Lambda_\ell} \Sigma^{-1}s_\lambda \quad \xrightarrow{g_\ell} \quad x_\ell \quad \xrightarrow{\varphi_\ell} \quad x_{\ell+1} \quad \coprod_{\Lambda_\ell} s_\lambda$$

where the row is a triangle. In other words we factor $\varphi_\ell : x_\ell \to x$ as a composite $x_\ell \xrightarrow{g_\ell} x_{\ell+1} \xrightarrow{\varphi_{\ell+1}} x$, and the triangle in the top row exhibits $x_{\ell+1}$ as belonging to $\text{Coproduct}_\ell[S(\alpha)] \ast \text{Coproduct}_1[S(\alpha)] = \text{Coproduct}_{\ell+1}[S(\alpha)]$.

We have now defined the sequence $x_1 \xrightarrow{g_1} x_2 \xrightarrow{g_2} x_3 \xrightarrow{g_3} \cdots$ as well as, for each $\ell > 0$, a map $\varphi_\ell : x_\ell \to x$. Since the $\varphi_\ell$'s are compatible with the $g_\ell$'s, we may factor the map from the sequence through some $\varphi : \text{Hocolim} x_\ell \to x$. It remains to prove that any such $\varphi$ is an isomorphism. To this end we prove

(vii) Let $t$ be any object in $T^\alpha$. Then the functor $\text{Hom}(t, -)$ takes the map $\varphi_1 : x_1 \to x$ to an epimorphism.

To prove (vii) let $f : t \to x$ be any morphism. Then $f$ is a map from $t \in T^\alpha$ to $x \in \langle S \rangle^{(-\infty, 0]}$, and Lemma 2.2 permits us to factor $f$ as $t \to s \to x$ with $s \in S(\alpha)$. But then $f$ factors through the coproduct of all $s \to x$, which in (v) was defined to be $\varphi_1 : x_1 \to x$.

Next we assert

(viii) Let $t$ be any object in $T^\alpha$. Applying the functor $\text{Hom}(t, -)$ to the displayed diagram in (vi) we obtain

$$\text{Hom}(t, x_\ell) \xrightarrow{\text{Hom}(t, g_\ell)} \text{Hom}(t, x_{\ell+1})$$

and we assert that the horizontal map and the slanted map have the same kernel.
To prove the assertion note first that the kernel of the horizontal map is obviously contained in the kernel of the slanted one, what needs proof is the reverse inclusion.

Choose therefore any map \( f : t \to x_\ell \) such that the composite \( t \xrightarrow{f} x_\ell \xrightarrow{\varphi_\ell} x \) vanishes. Recalling the triangle \( y \to x_\ell \xrightarrow{\varphi_\ell} x \) of (vi), the map \( f \) must factor as \( t \to y \to x_\ell \).

But the triangle \( \Sigma^{-1}x \to y \to x_\ell \) exhibits \( y \) as an object of \( \Sigma^{-1}(S)^{-\infty,0} \ast (S)^{-\infty,0} \subset \Sigma^{-1}(S)^{-\infty,0} \), making the morphism \( t \to y \) a map from \( t \in \mathcal{T}_\alpha \) to \( y \in \Sigma^{-1}(S)^{-\infty,0} \).

Lemma \( 2.2 \) applied to the suspension of this map, allows us to factor \( t \to y \) as a composite \( t \to \Sigma^{-1}s \to y \) with \( s \in S(\alpha) \). Hence \( t \to y \) factors through the coproduct of all \( \Sigma^{-1}s \to y \), and the triangle in the top row of the display diagram in (vi) shows that the composite \( t \xrightarrow{f} x_\ell \xrightarrow{g_\ell} x_{\ell+1} \) must vanish. We have proved (viii).

It remains to deduce the Proposition from (vii) and (viii), but this is standard. We remind the reader.

Let \( \mathcal{E}x(\mathcal{T}_\alpha, Ab) \) be the abelian category, whose objects are the additive functors \( (\mathcal{T}_\alpha)^{op} \to Ab \) which respect products of \( < \alpha \) objects. This category is developed extensively in \([6]\), see also the clever alternative description in Krause \([5]\). Let \( \mathcal{Y} : \mathcal{T} \to \mathcal{E}x(\mathcal{T}_\alpha, Ab) \) be the functor taking the object \( t \in \mathcal{T} \) to the object \( \mathcal{Y}(t) = \text{Hom}_\mathcal{T}(-, t)|_{\mathcal{T}_\alpha} \) in the category \( \mathcal{E}x(\mathcal{T}_\alpha, Ab) \). It’s obvious that \( \mathcal{Y} \) is a homological functor, and a theorem that it respects coproducts—see \([6]\) Proposition 6.2.6).

Now apply the functor \( \mathcal{Y} \) to everything in sight. By (vii) the map \( \mathcal{Y}(\varphi_1) : \mathcal{Y}(x_1) \to \mathcal{Y}(x) \) is an epimorphism and, since this epimorphism factors as \( \mathcal{Y}(x_1) \to \mathcal{Y}(x_\ell \to \mathcal{Y}(x) \) for every integer \( \ell > 0 \), the maps \( \mathcal{Y}(\varphi_\ell) : \mathcal{Y}(x_\ell) \to \mathcal{Y}(x) \) must all be epimorphisms. Now by (viii) the map \( \mathcal{Y}(g_\ell) : \mathcal{Y}(x_\ell) \to \mathcal{Y}(x_{\ell+1}) \) factors canonically as \( \mathcal{Y}(x_\ell \to \mathcal{Y}(x) \to \mathcal{Y}(x_{\ell+1}) \). Therefore the sequence

\[
\mathcal{Y}(x_1) \xrightarrow{\mathcal{Y}(g_1)} \mathcal{Y}(x_2) \xrightarrow{\mathcal{Y}(g_2)} \mathcal{Y}(x_3) \xrightarrow{\mathcal{Y}(g_3)} \cdots
\]

is ind-isomorphic to the sequence

\[
\mathcal{Y}(x) \xrightarrow{id} \mathcal{Y}(x) \xrightarrow{id} \mathcal{Y}(x) \xrightarrow{id} \cdots
\]

allowing us to compute, as (for example) in the proof of \([6]\) Theorem 8.3.3], that the map \( \mathcal{Y}(\varphi) : \mathcal{Y}(\text{Hocolim } x_\ell) \to \mathcal{Y}(x) \) must be an isomorphism.

But \( \alpha \) was chosen large enough so that \( \mathcal{T}_\alpha \) generates, hence the map \( \varphi : \text{Hocolim } x_\ell \to x \) must also be an isomorphism. \( \square \)

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