PBWD BASES AND SHUFFLE ALGEBRA REALIZATIONS FOR
\( U_v(L\mathfrak{sl}(n)), U_{v_1,v_2}(L\mathfrak{sl}(n)), U_v(L\mathfrak{sl}(m|n)) \) AND THEIR INTEGRAL FORMS

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Abstract. We construct a family of PBWD (Poincaré-Birkhoff-Witt-Drinfeld) bases for the quantum loop algebras \( U_v(L\mathfrak{sl}(n)), U_{v_1,v_2}(L\mathfrak{sl}(n)), U_v(L\mathfrak{sl}(m|n)) \) and their integral forms. This proves conjectures of \([HRZ, Z1]\) and generalizes the corresponding result of \([Ne]\). The key ingredient in our proofs is the interplay between these quantum affine algebras and the corresponding shuffle algebras, which are trigonometric counterparts of the elliptic shuffle algebras of \([FO1]–[FO3]\). Our approach is similar to that of \([E]\) in the formal setting, but the key novelty is an explicit shuffle algebra realization of the corresponding algebras, which is of independent interest. We use the latter to introduce certain integral forms of these quantum affine algebras and construct PBWD bases for them, which is crucially used in \([FT2]\) to study integral forms of type \(A\) shifted quantum affine algebras. The rational counterparts provide shuffle algebra realizations of the type \(A\) (super) Yangians. Finally, we also establish the shuffle algebra realizations of the integral forms of \([Gr, CP]\).

1. Introduction

1.1. Summary.

The quantum loop algebras \( U_v(L\mathfrak{g}) \) (aka quantum affine algebras with trivial central charge) admit two well-known presentations: the standard Drinfeld-Jimbo and the new Drinfeld realizations. The PBW property for the former has been established 25 years ago in the works of J. Beck, see \([B1, B2]\). In the latter case of the new Drinfeld realization, such results seem to be missing in the literature to our surprise. The only case we found addressed is the type \(A\) quantum loop algebras and their two-parameter generalizations, see \([HRZ, Theorem 3.11]\). However, the proof of that theorem is missing in \(loc.cit\). This gap has been also noticed in \([Z1, Z2]\), where a much weaker version has been established for the quantum loop superalgebra \( U_v(L\mathfrak{sl}(m|n)) \) of \([Y]\) by straightforward tedious arguments.

The goal of this paper is to fill in this gap by constructing a family of PBWD bases for the aforementioned quantum algebras and their integral forms. This is accomplished by establishing the shuffle realization of their positive halves (following the ideas of \([FO1]–[FO3]\)), which constitutes another main result of the paper. We note that the corresponding shuffle realization of \( U^>\mathfrak{g}(L\mathfrak{sl}(n)) \) can be implicitly deduced from \([Ne]\), but we provide an alternative simpler proof which also works for the other two cases of \( U^>\mathfrak{g}(L\mathfrak{sl}(n)), U^>\mathfrak{g}(L\mathfrak{sl}(m|n)) \) as well as for the integral forms \( \mathfrak{U}^>\mathfrak{g}(L\mathfrak{sl}(n)), \mathfrak{U}^>\mathfrak{g}_{v_1,v_2}(L\mathfrak{sl}(n)), \mathfrak{U}^>\mathfrak{g}(L\mathfrak{sl}(m|n)) \).

Let us point out right away both the similarities and the differences between the current work and a much older paper \([E]\) of B. Enriquez. In \([E]\), the author has established similar results for the quantum affine algebras in the formal setting, that is, when working over \(\mathbb{C}[[\hbar]]\) rather than over \(\mathbb{C}(v)\). In particular, the PBW theorem has been proved in \([E, Theorem 1.3]\), using an embedding of \( U^>\mathfrak{g}(L\mathfrak{g}) \) into the corresponding type \(\mathfrak{g}\) shuffle algebra \( S^<(\mathfrak{g}) \) \([E, Corollary 1.4]\), with the image \( S^<(\mathfrak{g}) \subset S^<(\mathfrak{g}) \) being the subalgebra generated by degree 1 components. In type \(A\), this coincides with our Proposition 3.4. However, the heart of our shuffle algebra...
isomorphism is the proof of the equality $\tilde{S}(\varphi) = S'_{\varphi}$, at least, for $\mathfrak{g} = \mathfrak{sl}_n$ (and similarly for $\mathfrak{g} = \mathfrak{sl}(m|n)$). This proves the (corrected) conjectural description of $S'_{\varphi}$ of [E, Remark 3.16].

We expect that similar arguments should provide PBWD bases of $U_\varphi(L\mathfrak{g})$ as well as establish their shuffle realizations, at least, for simply-laced simple $\mathfrak{g}$, which shall be discussed elsewhere. In contrast, the PBWD theorem for the Yangian $Y(\mathfrak{g})$ of any semisimple Lie algebra $\mathfrak{g}$ has been established long time ago in [Le].

A particular PBWD basis of the integral form $U_\varphi(L\mathfrak{g}_n)$ (closely related to the RTT integral form of $U_\varphi(L\mathfrak{g}_n)$) is used in [FT2] to define an integral form of type $A$ shifted quantum affine algebras of [FT1], see Remarks 2.12, 2.24. An important family of elements of that integral form, which is crucially used in [FT2], appear manifestly via their shuffle realizations (3.39).

As another particular case, viewing $U_\varphi(L\mathfrak{sl}_n)$ as a “vertical” subalgebra of the quantum toroidal algebra $U_\varphi(L\mathfrak{g}_n)$, we recover the PBWD basis of $U_\varphi(L\mathfrak{sl}_n)$ from [Ne], see Remark 3.24. Finally, let us make a few general comments about the PBWD bases constructed in this paper. As was pointed out to us by P. Etingof, the linear independence of the ordered monomials of (2.14) which is established in Section 3.2.2 can be immediately deduced by using the PBW property of $U(\mathfrak{g}_n[t, t^{-1}])$ as well as flatness of the deformation, cf. [E, Theorem 1.3]. However, we provide technical details as they are needed both for Section 3.2.3 and for the generalizations to the two-parameter and super-cases. At that point, we should note that while the two-parameter quantum affine algebras have been extensively studied since the original work [HRZ], see [JL, JZ1, JZ2] for a partial list of references (see also [Ta, BW1, BW2, JMY] for the case of two-parameter quantum finite groups), not many results have been established for them. In particular, it is still an open question whether these are flat deformations of the corresponding universal enveloping algebras. In [JZ2], an isomorphism between the Drinfeld-Jimbo and the new Drinfeld realizations of these algebras has been established (generalizing [HRZ, Theorem 3.12] for type $A$), see Remark 4.8, but it is not known how the construction of the PBW basis of [B1, B2] can be generalized to the former realization.

1.2. Outline of the paper.

- In Section 2.1, we recall the new Drinfeld realization of the quantum loop algebra $U_\varphi(L\mathfrak{sl}_n)$ and the triangular decomposition for it, see Proposition 2.9.

- In Section 2.2, we introduce the PBWD basis elements $e_\beta(r), f_\beta(r)$ of (2.11) and use them to construct the ordered PBWD monomials $e_h, f_h$ of (2.14). We construct the PBWD bases for $U_\varphi(L\mathfrak{sl}_n), U_\varphi(L\mathfrak{g}_n)$ and $U_\varphi(L\mathfrak{sl}_n)$ in Theorems 2.15 and 2.17.

- In Section 3.2, we introduce integral forms $\mathcal{U}_\varphi^\circ(L\mathfrak{sl}_n), \mathcal{U}_\varphi^\circ(L\mathfrak{sl}_n)$ as the $\mathbb{C}[v, v^{-1}]$-subalgebras generated by $\bar{e}_\beta(r), \bar{f}_\beta(r)$ of (2.18). We construct the PBWD bases for $\mathcal{U}_\varphi^\circ(L\mathfrak{sl}_n), \mathcal{U}_\varphi^\circ(L\mathfrak{g}_n)$ as well as prove their independence of any choices in Theorem 2.19. Following Remark 2.24, the integral form $\mathcal{U}_\varphi(L\mathfrak{sl}_n)$ of the entire $U_\varphi(L\mathfrak{sl}_n)$ introduced in Definition 2.20 is identified with the RTT integral form $\mathcal{U}_\varphi^{\text{rtt}}(L\mathfrak{sl}_n)$, which is used in [FT2] to establish Theorem 2.22. The latter implies the triangular decomposition for $\mathcal{U}_\varphi^{\text{rtt}}(L\mathfrak{sl}_n)$ of Corollary 2.23.

- In Section 3.1, we introduce the shuffle algebra $S^{(n)}$, which may be viewed as a trigonometric degeneration of the elliptic shuffle algebra of Feigin-Odesskii, see [FO1]–[FO3]. An embedding $\Psi: U_\varphi^\circ(L\mathfrak{sl}_n) \hookrightarrow S^{(n)}$ of Proposition 3.4 is a “simple version” of the shuffle realization of $U_\varphi^\circ(L\mathfrak{sl}_n)$ (which was used in [E] in the formal setting). The “hard version” of the shuffle realization, Theorem 3.5, establishes that $\Psi$ is an algebra isomorphism.

- In Section 3.2, we prove simultaneously Theorems 2.15 and 3.5. The key tool is the family of the specialization maps $\phi_{\varphi}$ of (3.12), while the main technical computations using $\phi_{\varphi}$ and the combinatorics of the shuffle algebra are presented in Lemmas 3.14, 3.15, 3.19.
In Section 3.3, we introduce a certain integral form $S(n)$ of the shuffle algebra $S(n)$, see Definition 3.31. While that construction looks rather cumbersome, it exactly matches the integral form $\U_{v_1,v_2}(L\mathfrak{s}\mathfrak{l}_n)$ under the isomorphism $\Psi$, see Theorem 3.34. In Proposition 3.36, we summarize the two key properties of $S(n)$ which play the crucial role in [FT2]

In Section 3.4, we prove simultaneously Theorems 2.19 and 3.34 following the arguments of Section 3.2. The proofs essentially boil down to the $n=2$ case, see Lemmas 3.40, 3.41.

- In Section 4, we generalize the key results of Sections 2, 3 to the two-parameter quantum loop algebras $U_{v_1,v_2}^>(L\mathfrak{s}\mathfrak{l}_n)$ of [HRZ]. We construct the PBWD bases for $U_{v_1,v_2}^>(L\mathfrak{s}\mathfrak{l}_n)$ in Theorem 4.3, thus, proving the conjecture of [HRZ], see Remark 4.4. We construct the PBWD bases for the integral form $U_{v_1,v_2}^>(L\mathfrak{s}\mathfrak{l}_n)$ in Theorem 4.7. Finally, we also establish the shuffle realization of $U_{v_1,v_2}^>(L\mathfrak{s}\mathfrak{l}_n)$ in Theorem 4.10.

- In Section 5, we generalize the key results of Sections 2, 3 to the quantum loop superalgebra $U_{v}^>(L\mathfrak{s}\mathfrak{l}(m|n))$ of [Y]. We construct the PBWD bases for $U_{v}^>(L\mathfrak{s}\mathfrak{l}(m|n))$ in Theorem 5.6, thus, proving the conjecture of [Z1], see Remark 5.7. We construct the PBWD bases for the integral form $U_{v}^>(L\mathfrak{s}\mathfrak{l}(m|n))$ in Theorem 5.10.

In Section 5.4, we introduce the corresponding shuffle algebra $S(m|n)$ and establish the isomorphism $U_{v}^>(L\mathfrak{s}\mathfrak{l}(m|n)) \sim S(m|n)$ in Theorem 5.17. The key new features of $S(m|n)$ are:

- its elements are now symmetric and skew-symmetric in different families of variables,
- additional wheel conditions appear.

- In Section 6.1, we recall the Yangian $Y_{h}^>(\mathfrak{s}\mathfrak{l}_n)$ and its Drinfeld-Gavarini subalgebra $\Y_{h}^>(\mathfrak{s}\mathfrak{l}_n)$, as well as the PBWD bases for those, see Theorems 6.5, 6.8.

In Section 6.2, we introduce a rational counterpart $\W^{(n)}$ of the shuffle algebra $S(n)$, equipped with an embedding $\Psi: Y_{h}^>(\mathfrak{s}\mathfrak{l}_n) \hookrightarrow \W^{(n)}$. However, it is no longer an isomorphism, and we establish explicit descriptions of the images of $Y_{h}^>(\mathfrak{s}\mathfrak{l}_n)$, $\Y_{h}^>(\mathfrak{s}\mathfrak{l}_n)$ in Theorems 6.20, 6.27, see also Definitions 6.17, 6.25.

- In Section 7.1, we recall the super Yangian $Y_{h}^>(\mathfrak{s}\mathfrak{l}(m|n))$ and its Drinfeld-Gavarini subalgebra $\Y_{h}^>(\mathfrak{s}\mathfrak{l}(m|n))$, as well as the PBWD bases for those, see Theorems 7.7, 7.8.

In Section 7.2, we introduce the rational counterpart $\W^{(m|n)}$ of the shuffle algebra $S(m|n)$, equipped with an embedding $Y_{h}^>(\mathfrak{s}\mathfrak{l}(m|n)) \hookrightarrow \W^{(m|n)}$. The latter is not an isomorphism, and we establish explicit descriptions of the images of $Y_{h}^>(\mathfrak{s}\mathfrak{l}(m|n))$, $\Y_{h}^>(\mathfrak{s}\mathfrak{l}(m|n))$ in Theorems 7.15, 7.16, see also Definition 7.14.

- In Section 8.1, we recall another integral form $U_{v}^>(L\mathfrak{s}\mathfrak{l}_n)$ of [Gr] and establish the PBWD basis for it as well as a shuffle realization, see Theorems 8.5, 8.8. We also recall the integral form $U_{v}(L\mathfrak{s}\mathfrak{l}_n)$ of [CP] and establish the PBWD property for it.

In Section 8.2, we discuss the generalizations of Sections 5, 7 to the case of nonisomorphic Dynkin diagrams of the Lie superalgebras $A(m,n)$. That shall be discussed further in [Ts].

1.3. Acknowledgments.
I am indebted to Pavel Etingof, Boris Feigin, Michael Finkelberg, and Andrei Neguţ for extremely stimulating discussions on the subject; to Naihuan Jing for a useful correspondence on two-parameter quantum algebras; to Luan Bezerra and Evgeny Mukhin for a useful correspondence on the quantum affine superalgebras. I am also grateful to MPIM (Bonn, Germany), IPMU (Kashiwa, Japan), and RIMS (Kyoto, Japan) for the hospitality and wonderful working conditions in the summer 2018 when this project was performed. I would like to thank Tomoyuki Arakawa and Todor Milanov for their invitations to RIMS and IPMU, respectively.

This work is partially supported by the NSF Grant DMS–1821185.
2. Quantum loop algebra $U_v(L\mathfrak{sl}_n)$ and its integral form $\mathfrak{U}_v(L\mathfrak{sl}_n)$

2.1. Quantum loop algebra $U_v(L\mathfrak{sl}_n)$.

Let $I = \{1, \ldots, n-1\}$, $(c_{ij})_{i,j \in I}$ be the Cartan matrix of $\mathfrak{sl}_n$, and $v$ be a formal variable. Following [D1], define the quantum loop algebra of $\mathfrak{sl}_n$ (in the new Drinfeld presentation), denoted by $U_v(L\mathfrak{sl}_n)$, to be the associative $\mathbb{C}(v)$-algebra generated by $\{e_{i,r}, f_{i,r}, \psi_{i,s}^\pm\}_{i \in I}^{r \in \mathbb{Z}, s \in \mathbb{N}}$ with the following defining relations:

$$[\psi_i^+(z), \psi_i^-(w)] = 0, \quad \psi_{i,0}^\pm = 1,$$

$$e_i(z)e_j(w) = e_j(w)e_i(z) \text{ if } c_{ij} = 0,$$

$$[e_i(z_1), [e_i(z_2), e_j(w)]_{v^{-1}}]_v + [e_i(z_2), [e_i(z_1), e_j(w)]_{v^{-1}}]_v = 0 \text{ if } c_{ij} = -1,$$

$$f_i(z)f_j(w) = f_j(w)f_i(z) \text{ if } c_{ij} = 0,$$

$$[f_i(z_1), [f_i(z_2), f_j(w)]_{v^{-1}}]_v + [f_i(z_2), [f_i(z_1), f_j(w)]_{v^{-1}}]_v = 0 \text{ if } c_{ij} = -1,$$

where $[a, b]_x := ab - xa \cdot ba$ and the generating series are defined as follows:

$$e_i(z) := \sum_{r \in \mathbb{Z}} e_{i,r}z^{-r}, \quad f_i(z) := \sum_{r \in \mathbb{Z}} f_{i,r}z^{-r}, \quad \psi_i^-(z) := \sum_{s \geq 0} \psi_{i,s}^-z^s, \quad \delta(z) := \sum_{r \in \mathbb{Z}} z^r.$$

Let $U_v^<(L\mathfrak{sl}_n)$, $U_v^>(L\mathfrak{sl}_n)$, and $U_v^0(L\mathfrak{sl}_n)$ be the $\mathbb{C}(v)$-subalgebras of $U_v(L\mathfrak{sl}_n)$ generated by $\{e_{i,r}\}_{i \in I}^{r \in \mathbb{Z}}, \{e_{i,r}\}_{i \in I}^{r \in \mathbb{Z}}$, and $\{\psi_{i,s}^\pm\}_{i \in I}^{s \in \mathbb{N}}$, respectively. The following is standard:

**Proposition 2.9** ([He]). (a) (Triangular decomposition of $U_v(L\mathfrak{sl}_n)$) The multiplication map $m : U_v^<(L\mathfrak{sl}_n) \otimes_{\mathbb{C}(v)} U_v^0(L\mathfrak{sl}_n) \otimes_{\mathbb{C}(v)} U_v^>(L\mathfrak{sl}_n) \to U_v(L\mathfrak{sl}_n)$ is an isomorphism of $\mathbb{C}(v)$-vector spaces.

(b) The algebra $U_v^<(L\mathfrak{sl}_n)$ (resp. $U_v^>(L\mathfrak{sl}_n)$ and $U_v^0(L\mathfrak{sl}_n)$) is isomorphic to the $\mathbb{C}(v)$-algebra generated by $\{e_{i,r}\}_{i \in I}^{r \in \mathbb{Z}}$ (resp. $\{f_{i,r}\}_{i \in I}^{r \in \mathbb{Z}}$ and $\{\psi_{i,s}^\pm\}_{i \in I}^{s \in \mathbb{N}}$) with the defining relations (2.2, 2.7) (resp. (2.3, 2.8) and (2.1)).

2.2. PBWD bases of $U_v(L\mathfrak{sl}_n)$.

Let $\{\alpha_i\}_{i=1}^{n-1}$ be the standard simple positive roots of $\mathfrak{sl}_n$, and $\Delta^+$ denote the set of positive roots: $\Delta^+ = \{\alpha_j + \alpha_{j+1} + \ldots + \alpha_i\}_{1 \leq j \leq i \leq n-1}$. Consider the following total ordering on $\Delta^+$:

$$\alpha_j + \alpha_{j+1} + \ldots + \alpha_i \leq \alpha_{j'} + \alpha_{j'+1} + \ldots + \alpha_{i'} \text{ iff } j < j' \text{ or } j = j', i \leq i'.$$

For every $\beta \in \Delta^+$, pick a total ordering $\leq_{\beta}$ on $\mathbb{Z}$. This gives rise to the total ordering on $\Delta^+ \times \mathbb{Z}$: $(\beta, r) \leq (\beta', r')$ if $\beta = \beta', r \leq_{\beta} r'$. For every pair $(\beta, k) \in \Delta^+ \times \mathbb{Z}$, we make the following three choices:

(1) a decomposition $\beta = \alpha_{i_1} + \ldots + \alpha_{i_p}$ such that $[\ldots [e_{\alpha_{i_1}}, e_{\alpha_{i_2}}], \ldots, e_{\alpha_{i_p}}]$ is a non-zero root vector of $\mathfrak{sl}_n$ (here $e_{\alpha_i}$ denotes the standard Chevalley generator of $\mathfrak{sl}_n$);

(2) a decomposition $\beta = r_1 + \ldots + r_p$ with $r_i \in \mathbb{Z}$;

(3) a sequence $(\lambda_1, \ldots, \lambda_{p-1}) \in \{v, v^{-1}\}^{p-1}$. 


We define the PBWD basis elements \( e_\beta(r) \in U^>_v(\mathfrak{sl}_n) \) and \( f_\beta(r) \in U^<_v(\mathfrak{sl}_n) \) via
\[
e_\beta(r) := [\cdots [e_{i_1, r_1}, e_{i_2, r_2}]_{\lambda_1}, e_{i_3, r_3}]_{\lambda_2}, \ldots, e_{i_p, r_p}]_{\lambda_{p-1}}, \]
\[
f_\beta(r) := [\cdots [f_{i_1, r_1}, f_{i_2, r_2}]_{\lambda_1}, f_{i_3, r_3}]_{\lambda_2}, \ldots, f_{i_p, r_p}]_{\lambda_{p-1}}.
\] (2.11)

In particular, \( e_{\alpha_i}(r) = e_{i,r} \) and \( f_{\alpha_i}(r) = f_{i,r} \). Note that \( e_\beta(r) \) and \( f_\beta(r) \) degenerate to the corresponding root generators \( e_\beta \otimes t^r \) and \( f_\beta \otimes t^r \) of \( \mathfrak{sl}_n[t,t^{-1}] = \mathfrak{sl}_n \otimes_\mathbb{C} \mathbb{C}[t,t^{-1}] \) as \( v \to 1 \), hence, the terminology.

**Remark 2.12.** The following particular choice features manifestly in [FT2] (cf. Remark 4.4):
\[
e_{\alpha_j+\alpha_{j+1}+\ldots+\alpha_i}(r) := [\cdots [e_{j,r}, e_{j+1,0}]_v, e_{j+2,0}]_v, \ldots, e_{i,0}]_v,
\]
\[
f_{\alpha_j+\alpha_{j+1}+\ldots+\alpha_i}(r) := [\cdots [f_{j,r}, f_{j+1,0}]_v, f_{j+2,0}]_v, \ldots, f_{i,0}]_v.
\] (2.13)

Let \( H \) denote the set of all functions \( h: \Delta^+ \times \mathbb{Z} \to \mathbb{N} \) with finite support. The monomials of the form
\[
e_h := \prod_{(\beta, r) \in \Delta^+ \times \mathbb{Z}} e_\beta(r)^{h(\beta,r)} \quad \text{and} \quad f_h := \prod_{(\beta, r) \in \Delta^+ \times \mathbb{Z}} f_\beta(r)^{h(\beta,r)} \quad \text{with} \quad h \in H
\] (2.14)
will be called the ordered PBWD monomials of \( U^>_v(\mathfrak{sl}_n) \) and \( U^<_v(\mathfrak{sl}_n) \), respectively.

Our first main result establishes the PBWD property of \( H \) and \( \mathfrak{sl}_n \) (cf. [Le]):

**Theorem 2.15.** (a) The ordered PBWD monomials \( \{e_h\}_{h \in H} \) form a \( \mathbb{C}(v) \)-basis of \( U^>_v(\mathfrak{sl}_n) \).
(b) The ordered PBWD monomials \( \{f_h\}_{h \in H} \) form a \( \mathbb{C}(v) \)-basis of \( U^<_v(\mathfrak{sl}_n) \).

The proof of Theorem 2.15 is presented in Section 3.2 and is based on the shuffle approach.

Let us relabel the Cartan generators via \( \psi_i := \begin{cases} \psi_i^+, & \text{if } r \geq 0 \\ \psi_i^-, & \text{if } r < 0 \end{cases} \) and pick a total ordering on \( \{\psi_i\}_{i \in I} \). Let \( H_0 \) denote the set of all functions \( g: I \times \mathbb{Z} \to \mathbb{Z} \) with finite support and such that \( g(i,r) \geq 0 \) for \( r \neq 0 \). The monomials of the form
\[
\psi_g := \prod_{(i,r) \in I \times \mathbb{Z}} \psi_i^{g(i,r)} \quad \text{with} \quad g \in H_0
\] (2.16)
will be called the ordered PBWD monomials of \( U^0_v(\mathfrak{sl}_n) \).

According to Proposition 2.9(b), the ordered PBWD monomials \( \{\psi_g\}_{g \in H_0} \) form a \( \mathbb{C}(v) \)-basis of \( U^0_v(\mathfrak{sl}_n) \). Combining this with Theorem 2.15 and Proposition 2.9(a), we finally get:

**Theorem 2.17.** The elements
\[
\{f_{h_-} \cdot \psi_{h_0} \cdot e_{h_+} \mid h_-, h_+ \in H, h_0 \in H_0\}
\]
form a \( \mathbb{C}(v) \)-basis of the quantum loop algebra \( U_v(\mathfrak{sl}_n) \).

### 2.3. Integral form \( \mathfrak{u}_v(\mathfrak{sl}_n) \) and its PBWD bases.

Following the above notations, define \( \widetilde{e}_\beta(r) \in U^>_v(\mathfrak{sl}_n) \) and \( \widetilde{f}_\beta(r) \in U^<_v(\mathfrak{sl}_n) \) via
\[
\widetilde{e}_\beta(r) := (v - v^{-1}) e_\beta(r), \quad \widetilde{f}_\beta(r) := (v - v^{-1}) f_\beta(r).
\] (2.18)

We also define \( \widetilde{e}_h, \widetilde{f}_h \) via the formula (2.14) but using \( \widetilde{e}_\beta(r), \widetilde{f}_\beta(r) \) instead of \( e_\beta(r), f_\beta(r) \). Define integral forms \( \mathfrak{u}_v^>(\mathfrak{sl}_n) \) and \( \mathfrak{u}_v^>(\mathfrak{sl}_n) \) as the \( \mathbb{C}[v,v^{-1}] \)-subalgebras of \( U^>_v(\mathfrak{sl}_n) \) and \( U^<_v(\mathfrak{sl}_n) \) generated by \( \{\widetilde{e}_\beta(r)\}_{\beta \in \Delta^+} \) and \( \{\widetilde{f}_\beta(r)\}_{\beta \in \Delta^+} \), respectively.
Theorem 2.19. (a) The subalgebras $\mathfrak{U}_v^0(\mathfrak{Ls}_n)$, $\mathfrak{U}_v^\infty(\mathfrak{Ls}_n)$ are independent of all our choices.

(b) The ordered PBWD monomials $\{\tilde{e}_h\}_{h \in H}$ form a basis of a free $\mathbb{C}[v, v^{-1}]$-module $\mathfrak{U}_v^0(\mathfrak{Ls}_n)$.

(c) The ordered PBWD monomials $\{f_h\}_{h \in H}$ form a basis of a free $\mathbb{C}[v, v^{-1}]$-module $\mathfrak{U}_v^\infty(\mathfrak{Ls}_n)$.

The proof of Theorem 2.19 is presented in Section 3.4 and is based on the shuffle approach.

Let $\mathfrak{U}_v^0(\mathfrak{Ls}_n)$ be the $\mathbb{C}[v, v^{-1}]$-subalgebra of $U_v^0(\mathfrak{Ls}_n)$ generated by $\{\psi^{\pm}_{i, \pm \delta}\}_{i \in I}$. Due to Proposition 2.9(b), the ordered PBWD monomials $\{\psi^g_{i \in H_0}\}_{(2.16)}$ form a basis of a free $\mathbb{C}[v, v^{-1}]$-module $\mathfrak{U}_v^0(\mathfrak{Ls}_n)$.

**Definition 2.20.** Define the integral form $\mathfrak{U}_v(\mathfrak{Ls}_n)$ as the $\mathbb{C}[v, v^{-1}]$-subalgebra of $U_v(\mathfrak{Ls}_n)$ generated by $\{\tilde{e}_\beta(r), f_\beta(r)\}_{\beta \in \Delta^+, \tilde{e}_i, \tilde{h}_i \in I}$.

**Remark 2.21.** Due to Theorem 2.19(a), the definition of $\mathfrak{U}_v(\mathfrak{Ls}_n)$ is independent of any choices.

The following result is proved in [FT2, Theorem 3.54] (cf. Theorem 2.17):

**Theorem 2.22 ([FT2]).** The elements

$$\left\{ \tilde{f}_{h-} \cdot \psi_{h_0}, \tilde{e}_{h+} \mid h-, h_0 \in H, h_0 \in H_0 \right\}$$

form a basis of a free $\mathbb{C}[v, v^{-1}]$-module $\mathfrak{U}_v(\mathfrak{Ls}_n)$.

In view of Theorem 2.19, this gives rise to the **triangular decomposition** of $\mathfrak{U}_v(\mathfrak{Ls}_n)$:

**Corollary 2.23.** The multiplication map

$$m: \mathfrak{U}_v^\infty(\mathfrak{Ls}_n) \otimes_{\mathbb{C}[v, v^{-1}]} \mathfrak{U}_v^0(\mathfrak{Ls}_n) \otimes_{\mathbb{C}[v, v^{-1}]} \mathfrak{U}_v^\infty(\mathfrak{Ls}_n) \rightarrow \mathfrak{U}_v(\mathfrak{Ls}_n)$$

is an isomorphism of $\mathbb{C}[v, v^{-1}]$-modules.

We conclude this section with the remark regarding the $\mathfrak{gl}_n$-counterpart:

**Remark 2.24.** (a) It is often more convenient to work with the quantum loop algebra of $\mathfrak{gl}_n$, denoted by $U_v(\mathfrak{Lgl}_n)$, which is roughly speaking obtained by adding one more Cartan current. Its integral form $\mathfrak{U}_v(\mathfrak{Lgl}_n)$ is defined analogously to $\mathfrak{U}_v(\mathfrak{Ls}_n)$ of Definition 2.20, and also admits a triangular decomposition $\mathfrak{U}_v(\mathfrak{Lgl}_n) \simeq \mathfrak{U}_v^\infty(\mathfrak{Lgl}_n) \otimes_{\mathbb{C}[v, v^{-1}]} \mathfrak{U}_v^0(\mathfrak{Lgl}_n) \otimes_{\mathbb{C}[v, v^{-1}]} \mathfrak{U}_v^\infty(\mathfrak{Lgl}_n)$. Here $\mathfrak{U}_v^\infty(\mathfrak{Lgl}_n) \simeq \mathfrak{U}_v^\infty(\mathfrak{Ls}_n)$, $\mathfrak{U}_v^0(\mathfrak{Lgl}_n) \simeq \mathfrak{U}_v^0(\mathfrak{Ls}_n)$, $\mathfrak{U}_v^\infty(\mathfrak{Lgl}_n) \simeq \mathfrak{U}_v^\infty(\mathfrak{Ls}_n)$.

(b) The proof of Theorem 2.22 presented in [FT2] crucially utilizes the identification of $\mathfrak{U}_v(\mathfrak{Lgl}_n)$ and the RTT integral form $\mathfrak{U}_v^{\text{RTT}}(\mathfrak{Lgl}_n)$ of [FRT] (see [FT2, 3(ii), Proposition 3.42]) under the $\mathbb{C}(v)$-algebra isomorphism $U_v^\text{RTT}(\mathfrak{Lgl}_n) \simeq \mathfrak{U}_v^{\text{RTT}}(\mathfrak{Lgl}_n) \otimes_{\mathbb{C}[v, v^{-1}]} \mathbb{C}(v)$ of [DF].

(c) Let us point out that the integral form $\mathfrak{U}_v(\mathfrak{Lgl}_n)$ provides a quantization of the thick slice $\overset{\dagger}{W}_0$ of [FT1, 4(viii)], see [FT2, Remark 3.61]. More precisely, we have

$$\mathfrak{U}_v(\mathfrak{Lgl}_n)/(v-1) \simeq \mathbb{C} \left[ \frac{t^\pm_{ji}}{t^0_{jk}} \right]_{r \geq 0, 1 \leq j < i \leq n} / \left( (t^+_{ji}, j \leq i \leq n) \right)_{1 \leq k \leq n}.$$  

(2.25)
3. Shuffle algebra realizations of $U_v^>(L\mathfrak{sl}_n)$ and $\Sigma_v^>(L\mathfrak{sl}_n)$

In this section, we establish the shuffle algebra realizations of $U_v^>(L\mathfrak{sl}_n)$ and $\Sigma_v^>(L\mathfrak{sl}_n)$ (hence, the independence of the latter of all choices made, Theorem 2.19(a)), and use those to prove Theorems 2.15(a), 2.19(b). As the assignment $e_{i,r} \mapsto f_{i,r}$ $(i \in I, r \in \mathbb{Z})$ gives rise to a $\mathbb{C}(v)$-algebra antiisomorphism $U_v^>(L\mathfrak{sl}_n) \to U_v^<(L\mathfrak{sl}_n)$ (resp. a $\mathbb{C}[v, v^{-1}]$-algebra antiisomorphism $\Sigma_v^>(L\mathfrak{sl}_n) \to \Sigma_v^<(L\mathfrak{sl}_n)$) mapping ordered PBWD monomials of the source to non-zero multiples of ordered PBWD monomials of the target, Theorems 2.15(b), 2.19(c) follow as well.

3.1. Shuffle algebra $S^{(n)}$.  

We follow the notations of [FT1, Appendix I(ii)] (cf. [Nc]).\(^1\) Let $\Sigma_k$ denote the symmetric group in $k$ elements, and set $\Sigma_{(k_1, \ldots, k_{n-1})} := \Sigma_{k_1} \times \cdots \times \Sigma_{k_{n-1}}$ for $k_1, \ldots, k_{n-1} \in \mathbb{N}$. Consider an $\mathbb{N}^I$-graded $\mathbb{C}(v)$-vector space $S^{(n)} = \bigoplus_{k=(k_1, \ldots, k_{n-1}) \in \mathbb{N}^I} S^{(n)}_k$, where $S^{(n)}_k$ consists of $\Sigma_k$-symmetric rational functions in the variables $\{x_{i,r}\}_{i \in I}$. We fix an $I \times I$ matrix of rational functions $(\zeta_{i,j}(z))_{i,j \in I} \in \text{Mat}_{I \times I}(\mathbb{C}(v)(z))$ via $\zeta_{i,j}(z) := \frac{z-v^{-c_i c_j}}{z-1}$.  

We now introduce the bilinear shuffle product $\star$ on $S^{(n)}$: given $F \in S^{(n)}_k$ and $G \in S^{(n)}_l$, define $F \star G \in S^{(n)}_{k+l}$ via

$$
(F \star G)(x_{1,1}, \ldots, x_{1,k_1+l_1}; \ldots; x_{n-1,1}, \ldots, x_{n-1,k_{n-1}+l_{n-1}}) := \frac{k! \cdot l!}{\prod_{i \in I} k_i!} \cdot \text{Sym}_{\Sigma_{k+l}} \left( F \left( \{x_{i,r}\}_{i \in I} \right)^{1 \leq r \leq k_i} \right) G \left( \{x_{i',r'}\}_{i' \in I}^{k_{i'+1} \leq r' \leq k_{i'+1}+l_i} \right) \cdot \prod_{i \in I} \prod_{r \leq k_i} \zeta_{i,i'}(x_{i,r}/x_{i',r'})
$$

(3.1)

Here $k! := \prod_{i \in I} k_i!$, while for $f \in \mathbb{C}(\{x_{i,1}, \ldots, x_{i,m_i}\}_{i \in I})$ we define its symmetrization via

$$
\text{Sym}_{\Sigma_m}(f)(\{x_{i,1}, \ldots, x_{i,m_i}\}_{i \in I}) := \frac{1}{m!} \cdot \sum_{(\sigma_1, \ldots, \sigma_{n-1}) \in \Sigma_m} f(\{x_{i,\sigma(1)}, \ldots, x_{i,\sigma(m)}\}_{i \in I})
$$

This endows $S^{(n)}$ with a structure of an associative unital algebra with the unit $1 \in S^{(n)}_{(0,\ldots,0)}$.

We will be interested only in the subspace of $S^{(n)}$ defined by the pole and wheel conditions:

- We say that $F \in S^{(n)}_k$ satisfies the pole conditions if

$$
F = \frac{f(x_{1,1}, \ldots, x_{n-1,k_{n-1}})}{\prod_{i=1}^{n} \prod_{r \leq k_i} \prod_{r' \leq k_{i+1}} (x_{i,r} - x_{i+1,r'})}, \text{ where } f \in (\mathbb{C}(v)[\{x_{i,r}\}_{i \in I}^{1 \leq r \leq k_i}])^{\Sigma_k}.
$$

(3.2)

- We say that $F \in S^{(n)}_k$ satisfies the wheel conditions\(^2\) if

$$
F(x_{i,r}) = 0 \text{ once } x_{i,r_1} = v_{\epsilon} x_{i,r_2}, v^s x_{i,r_2} \text{ for some } \epsilon, i, r_1, r_2, s,
$$

(3.3)

where $\epsilon \in \{\pm 1\}$, $i, i+\epsilon \in I$, $1 \leq r_1, r_2 \leq k_i$, $1 \leq s \leq k_{i+\epsilon}$.

Let $S^{(n)}_k \subset S^{(n)}_k$ denote the subspace of all elements $F$ satisfying these two conditions and set $S^{(n)} := \bigoplus_{k \in \mathbb{N}^I} S^{(n)}_k$. It is straightforward to check that the subspace $S^{(n)} \subset S^{(n)}$ is $\star$-closed.

The shuffle algebra $(S^{(n)}, \star)$ is related to $U_v^>(L\mathfrak{sl}_n)$ via the following construction:\(^3\)

\(^1\)These are trigonometric counterparts of the elliptic shuffle algebras of Feigin-Odesskii [FO1]–[FO3].

\(^2\)Following [FO1]–[FO3], the role of wheel conditions is exactly to replace rather complicated Serre relations.

\(^3\)In the formal setting (when working over $\mathbb{C}[\hbar]$ rather than over $\mathbb{C}(v)$) this goes back to [E, Corollary 1.4].
**Proposition 3.4.** The assignment $e_{i,r} \mapsto x_{i,1}^r$ ($i \in I, r \in \mathbb{Z}$) gives rise to an injective $\mathbb{C}(v)$-algebra homomorphism $\Psi: U_v^>(L\mathfrak{sl}_n) \to S^{(n)}$.

**Proof.** The assignment $e_{i,r} \mapsto x_{i,1}^r$ is compatible with relations (2.2, 2.7), hence, it gives rise to a $\mathbb{C}(v)$-algebra homomorphism $\Psi: U_v^>(L\mathfrak{sl}_n) \to S^{(n)}$, due to Proposition 2.9(b).

The injectivity of $\Psi$ follows from the general arguments based on the existence of a non-degenerate pairing on the source and a pairing on the target compatible with the former one via $\Psi$. This is explained in details in [Ne, Lemma 2.20, Proposition 2.30, Proposition 3.8]. □

The following result follows from its much harder counterpart [Ne, Theorem 1.1], but we will derive an alternative simpler proof as a corollary of our proof of Theorem 2.15, see Remark 3.23:

**Theorem 3.5.** $\Psi: U_v^>(L\mathfrak{sl}_n) \to S^{(n)}$ is a $\mathbb{C}(v)$-algebra isomorphism.

One of the benefits of our proof of Theorem 3.5 is that it will be directly generalized to establish the isomorphisms of Theorems 4.10 and 5.17 below, for which we cannot refer to [Ne].

### 3.2. Proof of Theorem 2.15(a).

Our proof of Theorem 2.15(a) will proceed in two steps: first, we shall establish the linear independence of the ordered PBWD monomials in Section 3.2.2, and then we will verify that they linearly span the entire algebra in Section 3.2.3 (note that usually the order of these two steps is opposite in the proof of PBW-type theorems). But before proceeding to the general case, we will first establish the result for $n = 2$ in Section 3.2.1.

#### 3.2.1. Proof of Theorem 2.15(a) for $n = 2$.

For $k \in \mathbb{N}$, set $[k]_v := \frac{v^k - v^{-k}}{v - v^{-1}}$ and $[k]_v! := [1]_v \cdots [k]_v$. We start from the following simple computation in the shuffle algebra $S^{(2)}$ (we shall denote variables $x_{i,r}$ simply by $x_r$):

**Lemma 3.6.** For any $k \geq 1$ and $r \in \mathbb{Z}$, the $k$-th power of $x^r \in S^{(2)}$ equals

$$x^r \star \cdots \star x^r = v^{-k(k-1)/2}[k]_v!/k! \cdot (x_1 \cdots x_k)^r.$$  

(3.7)

**Proof.** The proof is by induction in $k$. The case $k = 1$ is obvious. Applying the induction assumption to the $(k - 1)$-st power of $x^r$, the proof of (3.7) boils down to the verification of

$$\sum_{i=1}^k \prod_{1 \leq j \leq k, j \neq i} \frac{x_j - v^{-1}x_i}{x_j - x_i} = 1 + v^{-2} + v^{-4} + \ldots + v^{-2(k-1)}.$$  

(3.8)

The left-hand side is a rational function in $\{x_i\}_{i=1}^k$ of degree 0 and without poles, hence, a constant. To evaluate this constant, let $x_k \to \infty$: the last term tends to $v^{-2(k-1)}$, while the sum of the first $k - 1$ terms tends to $1 + v^{-2} + \ldots + v^{-2(k-2)}$ by the induction assumption, which results in the total constant $1 + v^{-2} + v^{-4} + \ldots + v^{-2(k-1)} = v^{1-k}[k]_v$. □

Theorem 2.15(a) for $n = 2$ is equivalent to the following result:

**Lemma 3.9.** For any total ordering $\preceq$ on $\mathbb{Z}$, the ordered monomials $\{e_{r_1}e_{r_2} \cdots e_{r_k}\}_{k \in \mathbb{N}}$ form a $\mathbb{C}(v)$-basis of $U_v^>(L\mathfrak{sl}_2)$.

---

As pointed out in the introduction, the linear independence can be deduced from the general arguments based on the flatness of the deformation and the PBW property of $U(\mathfrak{sl}_n[t, t^{-1}])$. However, the specialization maps of Section 3.2.2 and formulas (3.16, 3.17) are needed to prove that $\{e_h\}_{h \in \mathfrak{h}}$ span $U_v^>(L\mathfrak{sl}_n)$. We will use the same approach for two-parameter quantum loop algebra, for which the general argument does not apply.
Proof. For \( r_1 = \cdots = r_{k_1} < r_{k_1 + 1} = \cdots = r_{k_1 + k_2} < \cdots < r_{k_1 + \cdots + k_{l-1} + 1} = \cdots = r_{k_1 + \cdots + k_l} \), set \( k := k_1 + \cdots + k_l \) and choose \( \sigma \in \Sigma_k \) so that \( r_{\sigma(1)} \leq \cdots \leq r_{\sigma(k)} \). Then \( x_1^{r_1} \cdots x_k^{r_k} \) is a symmetric Laurent polynomial of the form \( \nu_{\bullet} m_{(r_{\sigma(1)}, \ldots, r_{\sigma(k)})}(x_1, \ldots, x_k) + \sum \nu_{\bullet} m_{v}(x_1, \ldots, x_k) \). Here \( m_{\bullet} \) are the symmetric monomials, the sum is over \( r' \) satisfying \( r_{\sigma(1)} \leq r'_{\sigma(1)} \leq \cdots \leq r'_{\sigma(k)} \), \( \nu_{\bullet} \in \mathbb{C}[v, v^{-1}] \), and \( \nu_{\bullet} \) is given by \( \nu_{\nu} = \frac{1}{k!} \sum_{i=1}^{k} (v^{-k_i(k_i-1)/2} / [k_i]_v!) \), due to Lemma 3.6. Recall that \( \{ m(s_1, \ldots, s_k)(x_1, \ldots, x_k) \}_{s_1 \leq \cdots \leq s_k} \) form a \( C(v) \)-basis of \( C[\{ x_i^{\pm 1} \}^{k}_{i=1}] \). Since \( S_{(2)}^{(2)} \simeq C[\{ x_i^{\pm 1} \}^{k}_{i=1}]^{\Sigma_k} \) as vector spaces, this implies that \( \{ x^{r_1} \ast x^{r_2} \cdots \ast x^{r_k} \}_{k \leq \mathbb{N}} \) form a \( C(v) \)-basis of \( S^{(2)} \). Combining this with the injectivity of \( \Psi \), we get the result. \( \square \)

3.2.2. Linear independence of \( e_h \).

For an ordered PBWD monomial \( e_h \) of \((2.14)\), define its degree \( \deg(e_h) = \deg(h) \in \mathbb{N} \) as a collection of \( d_{\beta} := \sum_{r \in \mathbb{Z}} h(\beta, r) \in \mathbb{N} (\beta \in \Delta^+) \) ordered with respect to the total ordering \((2.10)\) on \( \Delta^+ \). We consider the lexicographical ordering on \( \mathbb{N}^{n(n-1)/2} \):

\[
\{ d_{\beta} \}_{\beta \in \Delta^+} < \{ d_{\beta} \}_{\beta \in \Delta^+} \text{ if and only if } \exists \gamma \in \Delta^+ \text{ such that } d_\gamma > d'_\gamma \text{ and } d_\beta = d'_\beta \text{ for all } \beta < \gamma.
\]

Define the degree of a linear combination \( \sum_{h \in H} c_h e_h \) with only finitely many coefficients \( c_h \in \mathbb{C} \) being non-zero as \( \max\{ \{ \deg(e_h) \} | c_h \neq 0 \} \).

Assuming that \( \{ e_h \}_{h \in H} \) are not linearly independent, pick a nontrivial linear combination \( \sum_{h \in H} c_h e_h \) which is zero and is of the minimal possible degree, denoted \( d = \{ d_{\beta} \}_{\beta \in \Delta^+} \). Applying \( \Psi \) of Proposition 3.4, we get \( \sum_{h \in H} c_h \Psi(e_h) = 0 \). Note that each element \( \Psi(e_h) \) is homogeneous with respect to the \( \mathbb{N}^I \)-grading. Hence, without loss of generality, we may assume that all elements \( \{ \Psi(e_h) | c_h \neq 0 \} \) are of the same \( \mathbb{N}^I \)-degree, denoted \( k \).

In what follows, we shall need an explicit formula for \( \Psi(e_{\beta}(r)) \):

Lemma 3.10. For \( 1 \leq j < i < n \) and \( r \in \mathbb{Z} \), we have

\[
\Psi(e_{\alpha_j + \alpha_{j+1} + \cdots + \alpha_i}(r)) = (1 - v^2)^{i-j} p(x_{j,1}, \ldots, x_{i,1}) \frac{p(x_{j,1}, \ldots, x_{i,1})}{(x_{j,1} - x_{j+1,1}) \cdots (x_{i-1,1} - x_{i,1})},
\]

where \( p(x_{j,1}, \ldots, x_{i,1}) \) is a degree \( r + i - j \) monomial, up to a sign and an integer power of \( v \).

Proof. Straightforward computation. \( \square \)

Example 3.11. We have \( p(x_{j,1}, \ldots, x_{i,1}) = x_{j,1}^{r+1} x_{j+1,1} \cdots x_{i-1,1} \) for the choice of \((2.13)\).

For \( \beta = \alpha_j + \alpha_{j+1} + \cdots + \alpha_i \), define \( j(\beta) := j, i(\beta) := i \). We will use \( [\beta] \) to denote the integer interval \( [j(\beta); i(\beta)] \), while the length of \( \beta \) is defined as \( i(\beta) - j(\beta) + 1 \). Consider a collection of the intervals \( \{ [\beta] \}_{\beta \in \Delta^+} \) each taken with a multiplicity \( d_{\beta} \in \mathbb{N} \) and ordered with respect to the total ordering \((2.20)\) on \( \Delta^+ \) (the order inside each group is irrelevant), denoted by \( \cup_{\beta \in \Delta^+} [\beta]^{d_{\beta}} \). Define \( l \in \mathbb{N}^I \) via \( l := \sum_{\beta \in \Delta^+} d_{\beta}[\beta] \). Let us now define the specialization map

\[
\phi_{d}: S_{l}^{(n)} \rightarrow C(v)[\{ y_{\beta,s} \}^{1 \leq s \leq d_{\beta}}] \quad (3.12)
\]

Split the variables \( \{ x_{i,r} \}_{1 \leq i \leq l} \) into \( \sum_{\beta} d_{\beta} \) groups corresponding to the above intervals, and specialize those in the \( s \)-th copy of \( [\beta] \) to \( v^{-i(\beta)} \cdot y_{\beta,s} \), \( v^{-j(\beta)} \cdot y_{\beta,s} \) in the natural order (the variable \( x_{k, \bullet} \) gets specialized to \( v^{-k} y_{\beta,s} \)). For \( F = \frac{f(x_{1,1}, \ldots, x_{n-1,1}, v_{n-1})}{\Pi_{i=1}^{n-2} \Pi_{l \leq r \leq l+1} (x_{i,1} - x_{i+1,1})} \in S_{l}^{(n)} \), define \( \phi_{d}(F) \) as the corresponding specialization of \( f \). Note that \( \phi_{d}(F) \) is independent of our splitting of the variables \( \{ x_{i,r} \}_{1 \leq i \leq l} \) into groups and is symmetric in \( \{ y_{\beta,s} \}_{s=1}^{d_{\beta}} \) for any \( \beta \in \Delta^+ \).
Example 3.13. Let \( L = (1, 1, 0, \ldots, 0) \in \mathbb{N}^l \) and \( F = \frac{x_{1,1}x_{2,1}x_{1,1}^2}{(x_{1,1}-x_{2,1})(x_{1,1}-x_{3,1})} \in S_n^L \). Let us compute the images of \( F \) under all possible specialization maps:

(a) If the specialization corresponds to a single positive root \( \beta = \alpha_1 + \alpha_2 + \alpha_3 \), then \( \phi_d(F) \) is a function of a single variable \( y_{\beta,1} \) and equals \( (v^{-1}y_{\beta,1})^a(v^{-2}y_{\beta,1})^b(v^{-3}y_{\beta,1})^c \).

(b) If the specialization corresponds to two positive roots \( \beta_1 = \alpha_1, \beta_2 = \alpha_2 + \alpha_3 \), then \( \phi_d(F) \) is a function of two variables \( y_{\beta_1,1}, y_{\beta_2,1} \) and equals \( (v^{-1}y_{\beta_1,1})^a(v^{-2}y_{\beta_2,1})^b(v^{-3}y_{\beta_2,1})^c \).

(c) If the specialization corresponds to two positive roots \( \beta_1 = \alpha_1 + \alpha_2, \beta_2 = \alpha_3 \), then \( \phi_d(F) \) is a function of two variables \( y_{\beta_1,1}, y_{\beta_2,1} \) and equals \( (v^{-1}y_{\beta_1,1})^a(v^{-2}y_{\beta_2,1})^b(v^{-3}y_{\beta_2,1})^c \).

(d) If the specialization corresponds to \( \beta_1 = \alpha_1, \beta_2 = \alpha_2, \beta_3 = \alpha_3 \), then \( \phi_d(F) \) is a function of three variables \( y_{\beta_1,1}, y_{\beta_2,1}, y_{\beta_3,1} \) and equals \( (v^{-1}y_{\beta_1,1})^a(v^{-2}y_{\beta_2,1})^b(v^{-3}y_{\beta_3,1})^c \).

The key properties of the specialization maps \( \phi_d \) are summarized in the next two lemmas.

Lemma 3.14. If \( \deg(h) < d \), then \( \phi_d(\Psi(e_h)) = 0 \).

Proof. The above condition guarantees that \( \phi_d \)-specialization of any summand of the symmetrization appearing in \( \Psi(e_h) \) contains among all the \( \zeta \)-factors at least one factor of the form \( \zeta_{i, i+1}(v) = 0 \), hence, it is zero. The result follows.

Lemma 3.15. The specializations \( \{ \phi_d(\Psi(e_h)) \}_{d \in H} \) are linearly independent over \( \mathbb{C}(v) \).

Proof. Consider the image of \( e_h = \prod_{(\beta, r) \in \Delta^+ \times \mathbb{Z}} e_\beta(r)^{h(\beta, r)} \) under \( \Psi \). It is a sum of \( (\sum_{\beta \in \Delta^+} d_\beta)! \) terms, and as in the proof of Lemma 3.14 most of them specialize to zero under \( \phi_d, d := \deg(h) \). The summands which do not specialize to zero are parametrized by \( \Sigma_d := \prod_{\beta \in \Delta^+} \sum_{d_\beta} \). More precisely, given \( (\sigma_\beta)_{\beta \in \Delta^+} \in \Sigma_d \), the associated summand corresponds to the case when for all \( \beta \in \Delta^+ \) and \( 1 \leq s \leq d_\beta \), the \( \sum_{\beta < \beta'} d_{\beta'} + s \)-th factor of the corresponding term of \( \Psi(e_h) \) is evaluated at \( v^{-j(\beta)}y_{\beta, \sigma_\beta(s)}, \ldots, v^{-i(\beta)}y_{\beta, \sigma_\beta(s)} \). The image of this summand under \( \phi_d \) equals

\[
\prod_{\beta < \beta'} G_{\beta, \beta'} \prod_{1 \leq s' \leq d_{\beta'}} \left( \prod_{1 \leq s \leq d_\beta} \left( y_{\beta, s} - v^{-2}y_{\beta', s'} \right) \right) \prod_{1 \leq j' \leq d_{\beta'}} \left( y_{\beta, j'} - v^{-2}y_{\beta', j'} \right) \times \prod_{1 \leq j \leq d_{\beta'}} (y_{\beta, j} - y_{\beta', j})^{\delta_{j(\beta')} > j(\beta)} \delta_{1 \leq j'} \delta_{1 \leq j} \delta_{1 \leq j}^{\delta_{1 \leq j'}} \delta_{1 \leq j'}^{\delta_{1 \leq j}}. \tag{3.16}
\]

Here the collection \( \{ r_{\beta}(h, 1), \ldots, r_{\beta}(h, d_{\beta}) \} \) is obtained by listing every \( r \in \mathbb{Z} \) with multiplicity \( h(\beta, r) > 0 \) with respect to the total ordering \( \leq \) on \( \mathbb{Z} \), see Section 2.2. We also use the standard delta function notation: \( \delta_{\text{condition}} = \begin{cases} 1, & \text{if condition holds} \\ 0, & \text{if condition fails} \end{cases} \).
Note that the factors \( \{G_{\beta,\beta'}\}_{\beta<\beta'} \cup \{G_{\beta}\}_{\beta} \) in (3.16) are independent of \((\sigma_{\beta})_{\beta \in \Delta^+} \in \Sigma_d\). Therefore, the specialization \( \phi_d(\Psi(e_h)) \) has the following form:

\[
\phi_d(\Psi(e_h)) = c \cdot \prod_{\beta,\beta' \in \Delta^+} G_{\beta,\beta'} \cdot \prod_{\beta \in \Delta^+} G_{\beta} \cdot \prod_{\beta \in \Delta^+} \left( \sum_{\sigma_{\beta} \in \Sigma_d, \beta} G^{(\sigma_{\beta})}_{\beta} \right), \quad c \in \mathbb{C}^\times \cdot v^\mathbb{Z}. \tag{3.17}
\]

For \( \beta \in \Delta^+ \), we note that the sum \( \sum_{\sigma_{\beta} \in \Sigma_d} G^{(\sigma_{\beta})}_{\beta} \) coincides (up to a non-zero factor of \( \mathbb{C}^\times \)) with the value of the shuffle element \( x^{r_{\beta}(h,1)} \cdots x^{r_{\beta}(h,d_{\beta})} \in S_{d_{\beta}}^2 \) (in the shuffle algebra \( S(2)! \)) evaluated at \( \{y_{\beta,s}\}_{s=1}^{d_{\beta}} \). The latter elements are linearly independent, due to Lemma 3.9.

Thus, (3.17) together with the above observation completes our proof of Lemma 3.15. \( \square \)

Assuming the linear dependence of \( \{e_h\}_{h \in H} \), we have picked a non-trivial linear combination \( \sum_{h \in H} c_h e_h \) which is zero, and whose degree is \( d = \{d_{\beta}\}_{\beta \in \Delta^+} \). Applying the specialization map \( \phi_d \), we get \( \sum_{h \in H} \sigma_{\deg(h) = d} c_h \phi_d(\Psi(e_h)) = 0 \) by Lemma 3.14. Furthermore, we get \( c_h = 0 \) for all \( h \in H \) of degree \( \deg(h) = d \), due to Lemma 3.15. This contradicts the definition of \( d \).

This completes our proof of the linear independence of the ordered PBWD monomials \( e_h \).

Remark 3.18. The machinery of the specialization maps \( \phi_d \) that was used in the above proof is of its own interest (cf. [FHHSY, (1.4)]) and [Ne, (4.24)]).

3.2.3. Spanning property of \( e_h \).

We will actually show that any shuffle element \( F \in S_{k,n}^l \) belongs to the subspace \( M \subset S_{k,n}^l \), where \( M \subset S(n) \) denotes the \( \mathbb{C}(v) \)-subspace spanned by \( \{\Psi(e_h)\}_{h \in H} \). Let \( T_k \) denote a finite set consisting of all degree vectors \( d = \{d_{\beta}\}_{\beta \in \Delta^+} \in \mathbb{N}^{\frac{n(n-1)}{2}} \) such that \( \sum_{\beta \in \Delta^+} d_{\beta}[\beta] = k \). We order \( T_k \) with respect to the lexicographical ordering on \( \mathbb{N}^{\frac{n(n-1)}{2}} \). In particular, the minimal element \( d_{\text{min}} = \{d_{\beta}\}_{\beta \in \Delta^+} \in T_k \) is characterized by \( d_{\beta} = 0 \) for all non-simple roots \( \beta \in \Delta^+ \).

Lemma 3.19. If \( \phi_d(F) = 0 \) for all \( d' \in T_k \) such that \( d' > d \), then there exists an element \( F_d \in M \) such that \( \phi_d(F) = \phi_d(F_d) \) and \( \phi_d(F_d) = 0 \) for all \( d' > d \).

Proof. Consider the following total ordering on the set \( \{((\beta,s))_{\beta \in \Delta^+}\}^{1 \leq s \leq d_{\beta}} \):

\[
(\beta,s) \leq (\beta',s') \text{ iff } \beta < \beta' \text{ or } \beta = \beta', s \leq s'. \tag{3.20}
\]

First, we note that the wheel conditions (3.3) for \( F \) guarantee that \( \phi_d(F) \) (which is a Laurent polynomial in \( \{y_{s}\}_{s} \)) vanishes up to appropriate orders under the following specializations:

(i) \( y_{\beta,s} = v^{-2}y_{\beta',s'} \) for \( \beta, s < (\beta', s') \),
(ii) \( y_{\beta,s} = v^2y_{\beta',s'} \) for \( \beta, s < (\beta', s') \).

The orders of vanishing are computed similarly to [FHHSY, Ne]. Explicitly, let us view the specialization appearing in the definition of \( \phi_d \) as a step-by-step specialization in each interval \( [\beta] \), ordered first in the non-increasing length order, while the intervals of the same length are ordered in the non-decreasing order of \( \deg(\beta) \). As we specialize the variables in the \( s \)-th interval \( (1 \leq s \leq \sum_{\beta \in \Delta^+} d_{\beta}) \), we count only those wheel conditions that arise from the non-specialized yet variables. A straightforward case-by-case verification\(^5\) shows that the corresponding orders

\(^{5}\)This can be checked by treating each of the following cases separately: \( j = j' = i = i', j = j' = i < i' \), \( j < j' \).
of vanishing under the specializations (i) and (ii) equal \#\{(j, j') \in [\beta] \times [\beta']| j = j'\} - \delta_{\beta=\beta'} and \#\{(j, j') \in [\beta] \times [\beta']| j = j' + 1\}, respectively.

Second, we claim that \(\phi_d(F)\) vanishes under the following specializations:

(iii) \(y_{\beta,s} = y_{\beta',s'}\) for \((\beta, s) < (\beta', s')\) such that \(j(\beta) < j(\beta')\) and \(i(\beta) + 1 \in [\beta']\).

Indeed, if \(j(\beta) < j(\beta')\) and \(i(\beta) + 1 \in [\beta']\), there are positive roots \(\gamma, \gamma' \in \Delta^+\) such that \(j(\gamma) = j(\beta), i(\gamma) = i(\beta'), j(\gamma') = j(\beta'), i(\gamma') = i(\beta)\). Consider the degree vector \(d' = d + \delta_{\alpha, \gamma} - \delta_{\alpha, \gamma'}\). Then, \(d' > d\) and thus \(\phi_d(F) = 0\). The result follows.

Combining the above vanishing conditions for \(\phi_d(F)\), we see that it is divisible by the product \(\prod_{\beta < \beta'} G_{\beta, \beta'} \cdot \prod_{\beta \in \Delta^+} G_{\beta} \cdot G\) of (3.16). Therefore, we have

\[
\phi_d(F) = \prod_{\beta < \beta'} G_{\beta, \beta'} \cdot \prod_{\beta \in \Delta^+} G_{\beta} \cdot G
\] (3.21)

for some Laurent polynomial

\[
G \in \mathbb{C}(v)[\{y_{\beta,s}^{\pm 1}\}_{\beta \in \Delta^+}]_{d} \simeq \bigotimes_{\beta \in \Delta^+} \mathbb{C}(v)[\{y_{\beta,s}^{\pm 1}\}_{s=1}^{|\Sigma_{\beta}|}].
\] (3.22)

Combining this observation with formula (3.17) and the discussion after it, we see that there is a linear combination \(F_d = \sum_{h \in H} c_h e_h\) such that \(\phi_d(F) = \phi_d(F_d)\), due to Lemma 3.9. The equality \(\phi_d(F_d) = 0\) for \(d' > d\) is due to Lemma 3.14.

This completes our proof of Lemma 3.19. \(\square\)

Let \(d_{\text{max}}\) and \(d_{\text{min}}\) denote the maximal and the minimal elements of \(T_k\), respectively. The condition of Lemma 3.19 is vacuous for \(d = d_{\text{max}}\). Therefore, Lemma 3.19 applies. Applying it iteratively, we will eventually find an element \(\tilde{F} \in M\) such that \(\phi_d(F) = \phi_d(\tilde{F})\) for all \(d \in T_k\). In the particular case of \(d = d_{\text{min}}\), this yields \(F = \tilde{F}\) (as the specialization map \(\phi_{d_{\text{min}}}\) essentially does not change the function, see Example 3.13(d)). Hence, \(F \in M\).

This completes our proof of Theorem 2.15(a). As explained in the beginning of Section 3, the result of Theorem 2.15(b) follows as well.

Remark 3.23. The above argument actually implies the surjectivity of \(\Psi\). Together with its injectivity established in Proposition 3.4, we obtain a new proof of Theorem 3.5.

Remark 3.24. In \([\text{Ne}]\), the shuffle realization of the quantum toroidal algebra \(U_{v, \bar{v}}(\hat{gl}_n)\) (which is an associative \(C(v, \bar{v})\)-algebra with \(v, \bar{v}\) being two independent formal variables) was established by crucially studying the slope \(\leq \mu\) subalgebras. In particular, combining the proofs of Proposition 3.9 and Lemma 3.14 of loc.cit., one obtains the PBWD basis of \(U_{v, \bar{v}}(\hat{gl}_n)\) with the PBWD basis elements given explicitly in the shuffle realization, see elements \(E^\mu_{(j,i)}\) of \([\text{Ne}, (3.46)]\). This gives rise to the PBWD basis of \(U_v(L\mathfrak{sl}_n)\) by viewing the latter as a “vertical” subalgebra of \(U_{v, \bar{v}}(\hat{gl}_n)\). The corresponding PBWD basis elements are given by \(\Psi^{-1}((1 - v^2)^{j-i} p(x_{j,1}, \ldots, x_{j,1}) |_{x_{j,1} = x_{j,1}})\), where \(p(x_{j,1}, \ldots, x_{j,1}) = \prod_{a=0}^{\mu} x_{a,1}^{[\mu(a-j) + 1]} - [\mu(a-j)]\) with \(\mu \in \frac{1}{j} \mathbb{Z}\). Note that as \(\mu\) varies over \(\frac{1}{j} \mathbb{Z}\), the degree of \(p\) varies over \(\mathbb{Z}\) multiplicity-free. Comparing this to Lemma 3.10, it is easy to see that the corresponding PBWD basis of \(U_v^>(L\mathfrak{sl}_n)\) is a particular case of our general construction from Theorem 2.15(a).
3.3. Integral form $\mathcal{G}^{(n)}$.

For $k \in \mathbb{N}$, set $|k| := \sum_{i=1}^{n-1} k_i$. Consider a $\mathbb{C}[v, v^{-1}]$-submodule $\mathcal{G}^{(n)}_k \subset S^{(n)}_k$ consisting of $F \in S^{(n)}_k$ of the form (3.2) with $f \in (v - v^{-1})^{|k|}(\mathbb{C}[v, v^{-1}][\{x_{i,r}^\pm \}_{1 \leq i \leq n, 1 \leq r \leq k_i}])^\Sigma_k$. Then $\mathcal{G}^{(n)} := \bigoplus_{k \in \mathbb{N}} \mathcal{G}^{(n)}_k$ is a $\mathbb{C}[v, v^{-1}]$-subalgebra of $S^{(n)}$. Due to Lemma 3.10, we have $\Psi(\Omega^\nu(L\mathfrak{sl}_n)) \subset \mathcal{G}^{(n)}$.

Describing the image of that embedding occupies the rest of this section.

Remark 3.25. For $r \in \mathbb{Z}$, set $F(x_1, x_2) := (v - v^{-1})^2(x_1 x_2) \in \mathcal{G}_2^{(2)}$. Following our proof of Lemma 3.9, it is easy to see that $[2]_v \cdot F \in \Psi(\Omega^\nu(L\mathfrak{sl}_2))$ but $F \notin \Psi(\Omega^\nu(L\mathfrak{sl}_2))$, cf. Lemma 3.40.

Pick $F \in \mathcal{G}^{(n)}$. For any degree vector $d = \{d_\beta\}_{\beta \in \Delta^+} \in \mathbb{N}^{n(n-1)}$ such that $\sum_{\beta \in \Delta^+} d_\beta[\beta] = k$, consider $\phi_2^d(F) \in \mathbb{C}[v, v^{-1}][\{y^\pm_{\beta, s}\}_{\beta \in \Delta^+} \}_{\beta \in \Delta^+}$ of (3.12). First, we note that $\phi_2^d(F)$ is divisible by $A := (v - v^{-1})^{|k|}$. (3.26)

Second, following the first part of the proof of Lemma 3.19, $\phi_2^d(F)$ is also divisible by $B := \prod_{(\beta, s) < (\beta', s')} (y_{\beta, s} - v^{-2}y_{\beta', s'})$ due to wheel conditions (3.3), where we use the total ordering (3.20) on the set $\{(\beta, s)\}_{\beta \in \Delta^+}$.

Combining these observations, define the reduced specialization map

$$\varphi_2^d: \mathcal{G}^{(n)}_k \longrightarrow \mathbb{C}[v, v^{-1}][\{y^\pm_{\beta, s}\}_{\beta \in \Delta^+} \}_{\beta \in \Delta^+}$$

via $\varphi_2^d(F) := \phi_2^d(F)/(AB)$. (3.28)

Let us now introduce another type of specialization maps. Consider a collection of positive integers $l = \{l_{\beta, i}\}_{\beta \in \Delta^+}$ (with all $l_{\beta} \in \mathbb{N}$). Define a degree vector $d = \{d_\beta\}_{\beta \in \Delta^+} \in \mathbb{N}^{n(n-1)}$ via $d_{\beta} := \sum_{i=1}^{l_{\beta}} l_{\beta, i}$. Let us define the specialization map

$$\varpi_2^l: \mathbb{C}[v, v^{-1}][\{y^\pm_{\beta, s}\}_{\beta \in \Delta^+} \}_{\beta \in \Delta^+} \longrightarrow \mathbb{C}[v, v^{-1}][\{z^\pm_{\beta, i}\}_{\beta \in \Delta^+} \}_{\beta \in \Delta^+}.$$ (3.29)

For each $\beta \in \Delta^+$, split the variables $\{y_{\beta, s}\}_{s=1}^{d_{\beta}}$ into $l_{\beta}$ groups of size $t_{\beta, i}$ (1 $\leq i \leq l_{\beta}$) each and specialize the variables in the $i$-th group to $v^{-2z_{\beta, i}}, v^{-4z_{\beta, i}}, v^{-6z_{\beta, i}}, \ldots, v^{-2l_{\beta}-1z_{\beta, i}}$. For $K \in \mathbb{C}[v, v^{-1}][\{z^\pm_{\beta, i}\}_{\beta \in \Delta^+} \}_{\beta \in \Delta^+}$, define $\varpi_2^l(K)$ as the corresponding specialization of $K$. Note that $\varpi_2^l(K)$ is independent of our splitting of the variables $\{y_{\beta, s}\}_{s=1}^{d_{\beta}}$ into groups.

Finally, for any $d = \{d_\beta\}_{\beta \in \Delta^+} \in \mathbb{N}^{n(n-1)}$ such that $\sum_{\beta \in \Delta^+} d_\beta[\beta] = k$ and any collection of positive integers $l = \{l_{\beta, i}\}_{\beta \in \Delta^+}$ such that $d_{\beta} = \sum_{i=1}^{l_{\beta}} l_{\beta, i}$, define the cross specialization map

$$\Upsilon_2^d: \mathcal{G}^{(n)}_k \longrightarrow \mathbb{C}[v, v^{-1}][\{z^\pm_{\beta, i}\}_{\beta \in \Delta^+} \}_{\beta \in \Delta^+}$$

via $\Upsilon_2^d(F) := \varpi_2^l(\varphi_2^d(F))$. (3.30)

Definition 3.31. $F \in S^{(n)}_k$ is integral if $F \in \mathcal{G}^{(n)}_k$ and $\Upsilon_2^d(F)$ is divisible by $\prod_{1 \leq i \leq l_{\beta, i}} t_{\beta, i}^\nu$ (the product of $v$-factorials) for all possible $d$ and $l$.

Example 3.32. In the simplest case $n = 2$, the symmetric Laurent polynomial $F \in S^{(2)}_k$ is integral iff it can be written in the form $F = (v - v^{-1})^k \cdot \bar{F}$ with $\bar{F} \in \mathbb{C}[v, v^{-1}][\{x_{i, r}^\pm \}_{1 \leq i \leq 2, 1 \leq r \leq k_i}])^\Sigma_k$ satisfying the following divisibility condition (for any splitting $k = k_1 + \ldots + k_l$ with $k_i \geq 1$): $\bar{F}(v^{-2z_1}, v^{-4z_1}, \ldots, v^{-2k_1z_1}, \ldots, v^{-2z_l}, \ldots, v^{-2k_lz_l})$ is divisible by $[k_1]_v! \cdots [k_l]_v!$. (3.33)
Set \( \mathcal{G}^{(n)} := \bigoplus_{k \in \mathbb{N}} \mathcal{G}_{k}^{(n)} \), where \( \mathcal{G}_{k}^{(n)} \subset \mathcal{G}_{k}^{(n)} \) denotes the \( \mathbb{C}[v, v^{-1}] \)-submodule of all integral elements. The following is the key result of this section:

**Theorem 3.34.** The \( \mathbb{C}(v) \)-algebra isomorphism \( \Psi : U_{v}^{\gamma}(L \mathfrak{s} \mathfrak{l}_{n}) \overset{\sim}{\longrightarrow} S^{(n)} \) of Theorem 3.5 gives rise to a \( \mathbb{C}[v, v^{-1}] \)-algebra isomorphism \( \Psi : \mathcal{U}_{v}^{\gamma}(L \mathfrak{s} \mathfrak{l}_{n}) \overset{\sim}{\longrightarrow} \mathcal{G}^{(n)} \).

The proof of Theorem 3.34 is presented in Section 3.4.

**Corollary 3.35.** (a) \( \mathcal{G}^{(n)} \) is a \( \mathbb{C}[v, v^{-1}] \)-subalgebra of \( S^{(n)} \).

(b) Theorem 2.19(a) holds, that is, the subalgebra \( \mathcal{U}_{v}^{\gamma}(L \mathfrak{s} \mathfrak{l}_{n}) \) is independent of all choices.

In [FT2], we crucially use the following two properties of the integral form \( \mathcal{G}^{(n)} \):

**Proposition 3.36.** (a) For any \( 1 \leq l < n \), consider the linear map \( t_{l}^{(n)} : S^{(n)} \rightarrow S^{(n)} \) given by

\[
  t_{l}^{(n)}(F)(\{x_{i,r}\}_{i \in I}) := \prod_{r=1}^{k_{l}}(1-x_{i,r}) \cdot F(\{x_{i,r}\}_{i \in I}) \text{ for } F \in S^{(n)}_{k}, k \in \mathbb{N}^{l}. \tag{3.37}
\]

Then

\[
  F \in \mathcal{G}^{(n)} \iff t_{l}^{(n)}(F) \in \mathcal{G}^{(n)}. \tag{3.38}
\]

(b) For any \( k \in \mathbb{N}^{l} \) and a collection \( g_{i}(\{x_{i,r}\}_{r=1}^{k_{i}}) \in \mathbb{C}[v, v^{-1}][\{x_{i,r}\}_{r=1}^{k_{i}}]^{\Sigma_{i}} \) \((i \in I)\), define

\[
  F := (v - v^{-1})^{[k]} \cdot \frac{\prod_{i=1}^{l} \prod_{1 \leq r \neq r' \leq k_{i}}(x_{i,r} - x_{i,r'}) \cdot \prod_{i=1}^{l} g_{i}(\{x_{i,r}\}_{r=1}^{k_{i}})}{\prod_{i=1}^{l} \prod_{1 \leq r \leq k_{i}}(x_{i,r} - x_{i+1,r})}. \tag{3.39}
\]

Then \( F \in \mathcal{G}^{(n)}_{k} \).

**Proof.** (a) Obvious from the above definition of the integral form \( \mathcal{G}^{(n)} \).

(b) The presence of the factor \( \prod_{i=1}^{l} \prod_{1 \leq r \neq r' \leq k_{i}}(x_{i,r} - x_{i,r'}) \) in \( F \) guarantees that for any degree vector \( d = \{d_{\beta}\}_{\beta \in \Delta^{+}} \) satisfying \( \sum_{\beta \in \Delta^{+}} d_{\beta} = k \), the reduced specialization \( \varphi_{d}(F) \) is still divisible by \( \prod_{\beta \in \Delta^{+}} \prod_{1 \leq s \neq s' \leq d_{\beta}}(y_{\beta,s} - y_{\beta,s'}) \). Thus, if at least one element in the collection \( h = \{t_{\beta,i}\} \) is larger than 1, the further specialization \( \Upsilon_{d_{h}}(F) = \varphi_{d}(\varphi_{d}(F)) \) vanishes. Meanwhile, the divisibility condition of Definition 3.31 is vacuous if all \( t_{\beta,i} = 1 \). Thus \( F \in \mathcal{G}^{(n)}_{k} \). \( \square \)

### 3.4. Proofs of Theorem 2.19(b) and Theorem 3.34.

The results of Theorem 2.19(b) and Theorem 3.34 follow from the following two statements:

(I) For any \( k \geq 1, \{\beta_{k}\}_{k=1}^{k} \subset \Delta^{+}, \{r_{k}\}_{k=1}^{k} \subset \mathbb{Z} \), we have \( \Psi(\bar{e}_{\beta_{1}}(r_{1}) \cdots \bar{e}_{\beta_{k}}(r_{k})) \in \mathcal{G}^{(n)} \).

(II) Any element \( F \in \mathcal{G}^{(n)} \) may be written as a \( \mathbb{C}[v, v^{-1}] \)-linear combination of \( \{\Psi(\bar{e}_{h})\}_{h \in H} \).

The first result (I) follows easily from our definition of \( \mathcal{G}^{(n)} \), while the proof of (II) will closely follow our proof of Lemma 3.19 as well as the validity of (II) for \( n = 2 \), see Lemma 3.41.

We start by establishing both (I) and (II) for \( n = 2 \).

#### 3.4.1. \( n = 2 \) case.

For \( n = 2 \), the description of the integral form \( \mathcal{G}^{(n)} \subset S^{(n)} \) is the simplest, see Example 3.32. Set \( \bar{e}_{r} := (v - v^{-1})e_{r} \in U_{v}^{\gamma}(L \mathfrak{s} \mathfrak{l}_{2}) \). The following result establishes (I) for \( n = 2 \):

**Lemma 3.40.** For any \( k \geq 1 \) and \( r_{1}, \ldots, r_{k} \in \mathbb{Z} \), we have \( \Psi(\bar{e}_{r_{1}} \cdots \bar{e}_{r_{k}}) \in \mathcal{G}_{k}^{(2)} \).
Proof. Pick any splitting \( k = k_1 + \ldots + k_l \) with all \( k_i \geq 1 \). We claim that as we specialize the variables \( x_1, \ldots, x_k \) to \( \{ v^{-2} z_i \}_{1 \leq i \leq k_i} \), the image of any summand of the symmetrization appearing in \( \Psi(e_{r_1} \cdots e_{r_k}) \in S_k^{(2)} \), is divisible by the product \( \prod_{i=1}^l [k_i]! \) of \( v \)-factorials.

To prove the latter, we fix \( 1 \leq i \leq l \) and consider the relative position of the variables \( v^{-2} z_{i_1}, v^{-4} z_{i_2}, \ldots, v^{-2k_i} z_{i_l} \). If there is an index \( 1 \leq r < k_i \) such that \( v^{-2r} z_{i_r} \) is placed to the left of \( v^{-2r} z_{i_r} \), then the specialization of the corresponding \( \zeta \)-factor equals \( \frac{v^{-2r} z_{i_r} - v^{-2r} z_{i_r}}{v^{-2r} z_{i_r} - v^{-2r} z_{i_r}} = 0 \). However, if \( r \leq r < k_i \) then the specialization of the corresponding \( \zeta \)-factors equals

\[
\prod_{1 \leq r < r' \leq k_i} \frac{v^{-2r} z_{i_r} - v^{-2r'} z_{i_r}}{v^{-2r} z_{i_r} - v^{-2r} z_{i_r}} = v^{-k_i(k_i-1)/2}[k_i]!.
\]

Combining this over all \( 1 \leq i \leq l \), we see that \( \prod_{i=1}^l [k_i]! \) indeed divides the above specialization of \( \Psi(e_{r_1} \cdots e_{r_k}) \). This completes our proof of Lemma 3.40.

For simplicity of the exposition, we will assume that the order \( \preceq \) on \( \mathbb{Z} \) is the usual one \( \leq \). The following result implies (II) for \( n = 2 \):

**Lemma 3.41.** Any symmetric Laurent polynomial \( F \in \mathbb{C}[v,v^{-1}][x_i^{\pm 1}]_{i=1}^k \Sigma_k \) satisfying the divisibility condition (3.33) may be written as a \( \mathbb{C}[v,v^{-1}]- \)linear combination of \( \{ \Psi(e_h) \}_{h \in H} \).

Proof. We may assume that \( F \) is homogeneous of the total degree \( N \). Let \( V_N \) denote the set of all ordered \( k \)-tuples of integers \( \underline{r} = (r_1, r_2, \ldots, r_k) \), \( r_1 \leq \cdots \leq r_k \), such that \( r_1 + \cdots + r_k = N \). This set is totally ordered with respect to the lexicographical ordering:

\( \underline{r} < \underline{r}' \) iff there is \( 1 \leq i \leq k \) such that \( r_i < r_i' \) and \( r_j = r_j' \) for all \( j > i \).

Let us present \( F \) as a linear combination of the monomial symmetric polynomials:

\[
F(x_1, \ldots, x_k) = \sum_{\underline{r} \in V_N} \mu_{\underline{r}} m_\underline{r}(x_1, \ldots, x_k) \quad \text{with} \quad \mu_{\underline{r}} \in \mathbb{C}[v,v^{-1}].
\]

Pick the maximal element \( \underline{r}_{\max} = (r_1, \ldots, r_k) \) of the finite set \( V_N(\hat{F}) := \{ \underline{r} \in V_N | \mu_{\underline{r}} \neq 0 \} \) and consider a splitting \( k = k_1 + \ldots + k_l \) such that

\[
r_1 = \cdots = r_{k_1} < r_{k_1+1} = \cdots = r_{k_1+k_2} < \cdots < r_{k_1+\cdots+k_{l-1}+1} = \cdots = r_k.
\]

Evaluating \( F \) at the corresponding specialization \( \{ v^{-2r} z_i \}_{1 \leq i \leq k_i} \), we see that the coefficient of the lexicographically largest monomial in the variables \( \{ z_i \}_{i=1}^l \) equals \( \mu_{\underline{r}_{\max}} \). Therefore, the divisibility condition (3.33) implies that \( \prod_{i=1}^l [k_i]! \) divides \( \mu_{\underline{r}_{\max}} \), that is, \( \frac{\mu_{\underline{r}_{\max}}}{\prod_{i=1}^l [k_i]!} \in \mathbb{C}[v,v^{-1}] \).

Set \( F^{(0)} := F \) and define \( F^{(1)} \in \mathbb{C}[v,v^{-1}][x_i^{\pm 1}]_{i=1}^k \Sigma_k \) via

\[
F^{(1)} := F^{(0)} - k! \prod_{i=1}^l [k_i]! \frac{\mu_{\underline{r}_{\max}}}{\prod_{1 \leq i \leq l} [k_i]!} \Psi(e_{r_1} \cdots e_{r_k}).
\]

Due to Lemma 3.40, \( F^{(1)} \) also satisfies the divisibility condition (3.33). Applying the same argument again to \( F^{(1)} \), we obtain the element \( F^{(2)} \) also satisfying (3.33). Proceeding further, we thus construct a sequence of symmetric Laurent polynomials \( \{ F^{(s)} \}_{s \in \mathbb{N}} \) satisfying (3.33).

According to our proof of Lemma 3.9 (see the formula for \( \nu_{\underline{r}} \)), the sequence \( \nu_{\underline{r}} \Sigma_k^{(i)} \in V_N \) of the maximal elements of \( V_N(F^{(i)}) \) strictly decreases. Meanwhile, the sequence of the minimal powers of any variable in \( F^{(s)} \) is a non-decreasing sequence. Hence, \( F^{(s)} = 0 \) for some \( s \in \mathbb{N} \). This completes our proof of Lemma 3.41. \( \square \)
3.4.2. General case.

Let us now generalize the arguments of Section 3.4.1 to prove (I) and (II) for any \( n > 2 \). The proof of the former is quite similar (though is more elaborate) to that of Lemma 3.40:

**Lemma 3.43.** \( \Psi(\bar{e}_\beta_1(r_1) \cdots \bar{e}_\beta_l(r_l)) \in \mathcal{G}^{(n)} \) for any \( l \geq 1, \{\beta_i\}_{i=1}^l \subset \Delta^+, \{r_i\}_{i=1}^l \subset \mathbb{Z} \).

**Proof.** Define \( k := \sum \beta_i \in \mathbb{N}^l \) via \( k := \sum_{i=1}^l [\beta_i] \), so that \( F := \Psi(\bar{e}_\beta_1(r_1) \cdots \bar{e}_\beta_l(r_l)) \) belongs to \( \mathcal{S}_k^{(n)} \).

First, we note that \( F \) is divisible by \( (\nu - \nu^{-1})^{k/2} \), due to Lemma 3.10. Thus \( F \in \mathcal{S}_k^{(n)} \).

It remains to show that \( \Upsilon_{d,I}(F) \) also satisfies the divisibility condition of Definition 3.31 for any degree vector \( d = \{d_{\beta}\}_{\beta \in \Delta^+} \in \mathbb{N}^{n(n-1)/2} \) and a collection of positive integers \( t = \{t_{\beta,i}\}_{1 \leq i \leq t_\beta} \) such that \( k = \sum_{\beta \in \Delta^+} d_{\beta} / t_\beta \in \mathbb{N}^{n(n-1)/2} \).

1. Without loss of generality, we may assume that \( \{x_{l,r}\}_{j(\beta) \leq i(\beta), 1 \leq r < t_\beta} \).
2. We may also assume that \( x_{l,r} \) was specialized to \( (\nu - \nu^{-1})^{i(\beta)} \), under the first specialization (3.12), while \( y_{\beta,r} \) was specialized to \( \nu^{-2r} \) under the second specialization (3.29), for any \( j(\beta) \leq \lambda(\beta) \).

For each \( j(\beta) \leq \lambda(\beta) \), consider the relative position of the variables \( x_{l,r}, x_{l,r'}, x_{l+1,r'}, x_{l+1,r'+1} \). As \( x_{l,r}, x_{l,r'} \) cannot enter the same function \( \Psi(\bar{e}_{\beta}(\bullet)) \), \( x_{l,r} \) is placed either to the left of \( x_{l,r'} \) or to the right. In the former case, we gain the factor \( \zeta_{\beta}(x_{l,r}/x_{l,r'}) \), which upon the specialization \( \phi_{d,I}(F) \) contributes a factor \( (y_{\beta,r} - \nu^{-2r} y_{\beta,r'}) \). Likewise, if \( x_{l+1,r'} \) is placed to the left of \( x_{l,r} \), we gain the factor \( \zeta_{\beta}(x_{l+1,r'}/x_{l,r}) \), which upon the specialization \( \phi_{d,I}(F) \) contributes a factor \( (y_{\beta,r} - \nu^{-2r} y_{\beta,r'}) \) as well. In the remaining case when \( x_{l,r'} \) is the to the left of \( x_{l+1,r} \), we will not gain the factor \( \zeta_{\beta}(x_{l,r'}/x_{l+1,r'}) \), which upon the specialization \( \phi_{d,I}(F) \) specializes to 0. As \( I \) ranges from \( j(\beta) < i(\beta) - 1 \), we thus gain the \( (i(\beta) - j(\beta)) \)-th power of \( (y_{\beta,r} - \nu^{-2r} y_{\beta,r'}) \).

Note that this power exactly coincides with the power of \( (y_{\beta,r} - \nu^{-2r} y_{\beta,r'}) \) in \( B \) of (3.27), by which we divide \( \phi_{d,I}(F) \) to define the reduced specialization \( \phi_{d,I}(F) \) of (3.28).

However, we have not used above \( \zeta \)-factors \( \Omega_{\beta}(i(\beta), j(\beta), x_{i(\beta), r}/x_{i(\beta), r'} \) for \( i(\beta), r \) to the left of \( x_{j(\beta), r'} \). If there is \( 1 \leq r < t_\beta \), we have that \( x_{i(\beta), r+1} \) placed to the left of \( x_{i(\beta), r} \), then \( \zeta_{\beta}(i(\beta), j(\beta), x_{i(\beta), r+1}/x_{i(\beta), r}) \) specializes to zero upon (3.29). In the remaining case when each \( x_{i(\beta), r} \) is placed to the left of \( x_{i(\beta), r+1} \), the total contribution of the \( \zeta \)-factors equals \( \nu^{-t_\beta(\beta, l(\beta, l'' = 1)^2[t_\beta,i]_2!} \) as in the above proof of Lemma 3.40.

This completes our proof of Lemma 3.43.

Let \( \tilde{M} := \mathcal{S}_k^{(n)} \subset \mathcal{G}_k^{(n)} \) denote the \( \mathbb{C} [\nu, \nu^{-1}] \)-submodule spanned by \( \{\Psi(e_h)\}_{h \in H} \). Recalling our proof of Theorem 3.5 in Section 3.2.3, it suffices to establish the following result (cf. Lemma 3.19):

**Lemma 3.44.** Let \( F \in \mathcal{G}_k^{(n)} \). If \( \phi_{d}(F) = 0 \) for all \( d \in \mathbb{N}^l \) such that \( d' > d \), then there exists an element \( F_d \in \tilde{M} \) such that \( \phi_{d}(F_d) = \phi_{d}(F_d) \) and \( \phi_{d}(F_d) = 0 \) for all \( d' > d \).

**Proof.** The proof follows from the formulas (3.21, 3.22) from our proof of Lemma 3.19, formula (3.17) together with the discussion after it, and finally Lemma 3.41.

This completes our proofs of Theorem 3.34 and Theorem 2.19(b). As explained in the beginning of Section 3, the result of Theorem 2.19(c) follows as well.
4. Generalizations to $U_{v_1,v_2}(L\mathfrak{s}_n)$

The two-parameter quantum loop algebra $U_{v_1,v_2}(L\mathfrak{s}_n)$ was introduced in [HRZ]$^6$ as a generalization of $U_q(L\mathfrak{s}_n)$ (one recovers the latter from the former by setting $v_1 = v, v_2 = v^{-1}$ and identifying some Cartan elements, see [HRZ, Remark 3.3(4)]). The key two results of [HRZ]:

1) The Drinfeld-Jimbo type realization of $U_{v_1,v_2}(L\mathfrak{s}_n)$, see [HRZ, Theorem 3.12];
2) The PBW basis of its subalgebras $U_{v_1,v_2}(L\mathfrak{s}_n)$, see [HRZ, Theorem 3.11].

However, the latter result ([HRZ, Theorem 3.11]) is stated without any glimpse of a proof.

The primary goal of this section is to generalize Theorem 2.15 to the case of $U_{v_1,v_2}(L\mathfrak{s}_n)$, thus proving [HRZ, Theorem 3.11]. At the same time, we also generalize Theorem 3.5 to establish the shuffle realization of $U_{v_1,v_2}(L\mathfrak{s}_n)$, which is of independent interest. The latter is used to establish the PBW bases of the integral form of $U_{v_1,v_2}(L\mathfrak{s}_n)$, generalizing Theorem 2.19.

4.1. Two-parameter quantum loop algebra $U_{v_1,v_2}(L\mathfrak{s}_n)$.

For the purpose of this section, it suffices to work only with the subalgebra $U_{v_1,v_2}(L\mathfrak{s}_n)$ of $U_{v_1,v_2}(L\mathfrak{s}_n)$. Let $v_1, v_2$ be two independent formal variables and set $\mathbb{K} := \mathbb{C}[v_1^{1/2}, v_2^{1/2}]$. Following [HRZ, Definition 3.1], define $U_{v_1,v_2}(L\mathfrak{s}_n)$ to be the associative $\mathbb{K}$-algebra generated by $\{e_{i,r}\}_{i \in I, r \in \mathbb{Z}}$ with the following defining relations:

\[
(z - (\langle j, i \rangle \langle i, j \rangle)^{1/2}w)e_i(z)e_j(w) = (\langle j, i \rangle z - (\langle j, i \rangle \langle i, j \rangle)^{-1})^{1/2}w)e_j(w)e_i(z),
\]

(4.1)

\[
e_{i}(z)e_{j}(w) = e_{j}(w)e_{i}(z) \text{ if } c_{ij} = 0,
\]

\[
[e_{i}(z_{1}), e_{i}(z_{2}), e_{i+1}(w)]_{v_{2}}v_{1} + [e_{i}(z_{2}), e_{i}(z_{1}), e_{i+1}(w)]_{v_{2}}v_{1} = 0,
\]

(4.2)

\[
[e_{i}(z_{1}), e_{i}(z_{2}), e_{i-1}(w)]_{v_{2}^{-1}}v_{1}^{-1} + [e_{i}(z_{2}), e_{i}(z_{1}), e_{i-1}(w)]_{v_{2}^{-1}}v_{1}^{-1} = 0,
\]

where we set $\langle i, j \rangle := v_{1}^{\delta_{ij}}v_{2}^{\delta_{i+1,j+1}-\delta_{ij}}$ and $e_{i}(z) = \sum_{r \in \mathbb{Z}} e_{i,r}z^{-r}$ as before.

4.2. PBWD bases of $U_{v_1,v_2}(L\mathfrak{s}_n)$.

We shall follow the notations of Section 2.2, except that now $(\lambda_{1}, \ldots, \lambda_{p-1}) \in \{v_1, v_2\}^{p-1}$. Similarly to (2.11), define the PBWD basis elements $e_{\beta}(r) \in U_{v_1,v_2}(L\mathfrak{s}_n)$ via

\[
e_{\beta}(r) := \prod_{(\beta, r) \in \Delta^{+} \times \mathbb{Z}} e_{\beta}(r)^{h(\beta, r)} (h \in H)
\]

will be called the ordered PBWD monomials of $U_{v_1,v_2}(L\mathfrak{s}_n)$.

Our first main result establishes the PBWD property of $U_{v_1,v_2}(L\mathfrak{s}_n)$:

**Theorem 4.3.** The ordered PBWD monomials $\{e_{h}\}_{h \in H}$ form a $\mathbb{K}$-basis of $U_{v_1,v_2}(L\mathfrak{s}_n)$.

The proof of Theorem 4.3 is outlined in Section 4.5 and is based on the shuffle approach.

**Remark 4.4.** In [HRZ, (3.14)], the PBWD basis elements are chosen as follows:

\[
e_{\alpha_{j}+\alpha_{j+1}+\ldots+\alpha_{i}}(r) := \prod_{\beta \in \Delta^{+}} e_{\beta}(r)^{h(\beta, r)} (e_{j,r}v_{1}, e_{j+2,0}v_{1}, \ldots, e_{i,0}v_{1}).
\]

(4.5)

In this particular case, Theorem 4.3 recovers the conjectured result [HRZ, Theorem 3.11].

**Remark 4.6.** The entire two-parameter quantum loop algebra $U_{v_1,v_2}(L\mathfrak{s}_n)$ admits a triangular decomposition as in Proposition 2.9. Hence, an analogue of Theorem 2.17 holds for $U_{v_1,v_2}(L\mathfrak{s}_n)$ as well, providing PBWD $\mathbb{K}$-bases of $U_{v_1,v_2}(L\mathfrak{s}_n)$.

---

$^6$To be more precise, this recovers the algebra of loc.cit. with the trivial central charges.
4.3. Integral form $\mathcal{U}_{v_1,v_2}(Ls_l n)$ and its PBWD bases.

Following (2.18), define $\tilde{e}_\beta(r) \in U_{v_1,v_2}(Ls_l n)$ via $\tilde{e}_\beta(r) := (v_1^{1/2}v_2^{-1/2} - v_2^{-1/2}v_1^{1/2})e_\beta(r)$. We also define $\tilde{e}_h$ via (2.14) but using $\tilde{e}_\beta(r)$ instead of $e_\beta(r)$. Define an integral form $\mathcal{U}_{v_1,v_2}(Ls_l n)$ as the $\mathbb{C}[v_1^{1/2}, v_2^{1/2}, v_1^{-1/2}, v_2^{-1/2}]$-subalgebra of $U_{v_1,v_2}(Ls_l n)$ generated by $\{\tilde{e}_\beta(r)\}_{r \in \Delta^+}$.

The following counterpart of Theorem 2.19 provides a much stronger version of Theorem 4.3:

**Theorem 4.7.** (a) The subalgebra $\mathcal{U}_{v_1,v_2}(Ls_l n)$ is independent of all our choices. (b) Elements $\{\tilde{e}_h\}_{h \in H}$ form a basis of a free $\mathbb{C}[v_1^{1/2}, v_2^{1/2}, v_1^{-1/2}, v_2^{-1/2}]$-module $\mathcal{U}_{v_1,v_2}(Ls_l n)$.

The proof of Theorem 4.7 follows easily from the one of Theorem 4.3 presented below in the same way as we deduced the proof of Theorem 2.19 in Section 3.4 from that of Theorem 2.15.

**Remark 4.8.** Similarly to Remark 2.24(a), it is often more convenient to work with the two-parameter quantum loop algebra of $gl_n$, denoted by $U_{v_1,v_2}(Lg_l n)$. Its integral form $\mathcal{U}_{v_1,v_2}(Lg_l n)$ is defined analogously to $\mathcal{U}_{v_1,v_2}(Ls_l n)$. Then, the argument of [FT2, Proposition 3.42], cf. Remark 2.24(b), identifies $U_{v_1,v_2}(Lg_l n)$ with the RTT integral form $\mathcal{U}_{v_1,v_2}(Lg_l n)$ under the $\mathbb{K}$-algebra isomorphism $U_{v_1,v_2}(Lg_l n) \simeq \mathcal{U}_{v_1,v_2}(Lg_l n) \otimes \mathbb{C}[v_1^{1/2}, v_2^{1/2}, v_1^{-1/2}, v_2^{-1/2}]$ of [JL]. The analogue of [FT2, Theorem 3.54] provides PBWD bases of $U_{v_1,v_2}(Lg_l n)$, cf. Theorem 2.22.

4.4. Shuffle algebra $\widetilde{S}^{(n)}$.

Define the shuffle algebra $(\widetilde{S}^{(n)}, \star)$ analogously to $(S^{(n)}, \star)$ with the following modifications:

(1) All vector spaces are defined over $\mathbb{K}$;
(2) The choice of $(\zeta_{i,j}(z))_{i,j \in I} \in \text{Mat}_{I \times I}(\mathbb{K}(z))$ is modified as follows:

$$\zeta_{i,j}(z) = \left( \frac{z - v_1 v_2}{z - 1} \right)^{\delta_{j,i-1}} \left( \frac{z - v_1 v_2}{z - 1} \right)^{\delta_{j,i}} \left( \frac{v_1 v_2}{z - 1} \right)^{\delta_{j,i+1}};$$

(3) The wheel conditions (3.3) for $F$ are modified as follows:

$$F(\{x_i, r\}) = 0 \text{ once } x_{i,r_1} = v_1^{1/2}v_2^{-1/2}x_{i+e,s} = v_1 v_2^{-1}x_{i,r_2} \text{ for some } e \in \{\pm 1\}, i, r_1, r_2, s.$$ 

Note that this recovers the shuffle algebra $S^{(n)}$ for $v_1 = v_2 = 1$.

The following result is completely analogous to Proposition 3.4:

**Proposition 4.9.** The assignment $e_{i,r} \mapsto x_{i,r}^r$ $(i \in I, r \in \mathbb{Z})$ gives rise to an injective $\mathbb{K}$-algebra homomorphism $\Psi : U_{v_1,v_2}^\otimes(Ls_l n) \rightarrow \widetilde{S}^{(n)}$.

Our proof of Theorem 4.3 also implies the counterpart of Theorem 3.5 (see Remark 4.11):

**Theorem 4.10.** $\Psi : U_{v_1,v_2}^\otimes(Ls_l n) \rightarrow \widetilde{S}^{(n)}$ is a $\mathbb{K}$-algebra isomorphism.

4.5. Proof of Theorem 4.3.

The proof of Theorem 4.3 is completely analogous to our proof of Theorem 2.15(a) and is based on the embedding into the shuffle algebra $\widetilde{S}^{(n)}$ of Proposition 4.9. Indeed, the linear independence of $\{e_h\}_{h \in H}$ is deduced as in Section 3.2.2 with the only modification of the specialization maps $\phi_l^\otimes : \widetilde{S}^{(n)}_l \rightarrow \mathbb{K}\{y_{\pm 1}^{1 \leq s \leq d_{l,n}}\}$, where $v^{-k}$ is replaced by $(v_1^{1/2}v_2^{-1/2})^{-k}$.

Then, the results of Lemmas 3.14 and 3.15 still hold, thus proving the linear independence of $\{e_h\}_{h \in H}$. The proof of the fact that $\{e_h\}_{h \in H}$ span $U_{v_1,v_2}^\otimes(Ls_l n)$ follows as in Section 3.2.3. To be more precise, Lemma 3.19 still holds and its iterative application immediately implies that any shuffle element $F \in \widetilde{S}^{(n)}$ belongs to the $\mathbb{K}$-subspace spanned by $\{\Psi(e_h)\}_{h \in H}$.

**Remark 4.11.** The last statement together with Proposition 4.9 implies Theorem 4.10.
5. Generalizations to \( U_v(L\mathfrak{sl}(m|n)) \)

The quantum loop superalgebra \( U_v(L\mathfrak{sl}(m|n)) \) was introduced in [Y], both in the Drinfeld-Jimbo and the new Drinfeld realizations, see [Y, Theorem 8.5.1] for an identification of those. The representation theory of these algebras was partially studied in [Z1] crucially using a weak version of the PBW theorem for \( U_v^>(L\mathfrak{sl}(m|n)) \), see [Z1, Theorem 3.12]. Motivated by [HRZ], the author also conjectured the PBW theorem for \( U_v^>(L\mathfrak{sl}(m|n)) \), see [Z1, Remark 3.13(2)].

The primary goal of this section is to generalize Theorem 2.15 to the case of \( U_v^>(L\mathfrak{sl}(m|n)) \), thus proving the conjecture of [Z1]. Simultaneously, we also generalize Theorem 3.5 to establish the shuffle realization of \( U_v^>(L\mathfrak{sl}(m|n)) \), which is of independent interest. The latter is used to establish the PBWD bases of the integral form of \( U_v^>(L\mathfrak{sl}(m|n)) \), generalizing Theorem 2.19.

The shuffle algebras associated to quantum loop superalgebras involve both symmetric and skew-symmetric functions (“bosons” and “fermions”) and seem to be new in the literature.

5.1. Quantum loop superalgebra \( U_v^>(L\mathfrak{sl}(m|n)) \).

For the purpose of this section, it suffices to work only with the subalgebra \( U_v^>(L\mathfrak{sl}(m|n)) \) of \( U_v(L\mathfrak{sl}(m|n)) \). Let \( I = \{1, \ldots, m+n-1\} \) from now on. Consider a free \( \mathbb{Z} \)-module \( \oplus_{i=1}^{m+n} \mathbb{Z} \epsilon_i \) with the bilinear form \( \langle \cdot, \cdot \rangle \) determined by \( \langle \epsilon_i, \epsilon_j \rangle = (-1)^{i<j} \delta_{ij} \). Let \( \nu \) be a formal variable and define \( \{v_i\}_{i \in I} \subset \{\nu, \nu^{-1}\} \) via \( v_i := \nu^{\epsilon_i} \). For \( i, j \in I \), set \( \bar{c}_{ij} := (\epsilon_i - \epsilon_{i+1}, \epsilon_j - \epsilon_{j+1}) \).

Following [Y] (cf. [Z1, Theorem 3.3]), define \( U_v^>(L\mathfrak{sl}(m|n)) \) to be the associative \( \mathbb{C}(\nu) \)-superalgebra generated by \( \{e_{i,r}\}_{i \in I} \), with the \( \mathbb{Z}_2 \)-grading \( [e_{m,r}] = 1, [e_{i,r}] = 0 \) \( i \neq m, r \in \mathbb{Z} \), and with the following defining relations:

\[
(z - \nu^{\bar{c}_{ij}} w)e_i(z)e_j(w) = (\nu^{\bar{c}_{ij}} z - w)e_j(w)e_i(z) \quad \text{if} \quad \bar{c}_{ij} \neq 0,
\]

\[
[e_i(z), e_j(w)] = 0 \quad \text{if} \quad \bar{c}_{ij} = 0,
\]

\[
[e_i(z_1), [e_i(z_2), e_j(w)]_{\nu^{-1}}] + [e_i(z_1), [e_i(z_2), e_j(w)]_{\nu^{-1}}] = 0 \quad \text{if} \quad \bar{c}_{ij} = \pm 1, i \neq m,
\]

\[
[[e_{m-1}(w), e_m(z_1)]_{\nu^{-1}}, e_{m+1}(w)]_{\nu} e_m(z_2)] + [[[e_{m-1}(w), e_m(z_2)]_{\nu^{-1}}, e_{m+1}(w)]_{\nu} e_m(z_1)] = 0,
\]

where \( e_i(z) = \sum_{r \in \mathbb{Z}} e_{i,r} z^{-r} \) as before, and we use the super-bracket notations:

\[
[a, b] := [a, b]_1, \quad [a, b]_x := ab - (-1)^{|a||b|} x : ba \quad \text{for homogeneous} \quad a, b
\]

we set \( |a| = 0 \) if \( |a| = 0 \), and \( |a| = 1 \) if \( |a| = 1 \).

5.2. PBWD bases of \( U_v^>(L\mathfrak{sl}(m|n)) \).

Let \( \Delta^+ = \{\alpha_j + \alpha_{j+1} + \ldots + \alpha_1\}_{1 \leq j \leq m+n-1} \). For \( \beta \in \Delta^+ \), define its parity \( p(\beta) \in \mathbb{Z}_2 \) via

\[
p(\beta) = \begin{cases} 1, & \text{if} \quad m \in [\beta] \\ 0, & \text{if} \quad m \notin [\beta]. \end{cases}
\]

We shall follow the notations of Section 2.2. In particular, define the PBWD basis elements \( e_\beta(r) \in U_v^>(L\mathfrak{sl}(m|n)) \) via (2.11).

Let \( \bar{H} \) denote the set of all functions \( h: \Delta^+ \times \mathbb{Z} \to \mathbb{N} \) with finite support and such that \( h(\beta, r) \leq 1 \) if \( p(\beta) = 1 \). The monomials of the form

\[
e_h := \prod_{(\beta, r) \in \Delta^+ \times \mathbb{Z}} e_\beta(r)^{h(\beta, r)} \quad \text{with} \quad h \in \bar{H}
\]

will be called the ordered PBWD monomials of \( U_v^>(L\mathfrak{sl}(m|n)) \).

Our first main result establishes the PBWD property of \( U_v^>(L\mathfrak{sl}(m|n)) \):
Theorem 5.6. The ordered PBWD monomials \( \{e_h\}_{h \in H} \) form a \( \mathbb{C}(v) \)-basis of \( U_v^0(L\mathfrak{sl}(m|n)) \).

The proof of Theorem 5.6 is presented in Section 5.5 and is based on the shuffle approach.

Remark 5.7. In [Z1, (3.12)], the PBWD basis elements are chosen as follows:

\[
e_{a_1+a_2+\ldots+a_i}(r) := \left[ e_{j_1}, e_{j_2}v_{j_2}, e_{j_3}v_{j_3}v_{j_3}, \ldots, e_{i_0}v_i \right] .
\]

(5.8)

In this particular case, Theorem 5.6 recovers the conjecture of [Z1, Remark 3.13(2)].

Remark 5.9. The entire quantum loop superalgebra \( U_v(L\mathfrak{sl}(m|n)) \) admits a triangular decomposition as in Proposition 2.9, see [Z1, Theorem 3.3]. Hence, an analogue of Theorem 2.17 holds for \( U_v(L\mathfrak{sl}(m|n)) \) as well, providing PBWD \( \mathbb{C}(v) \)-bases of \( U_v(L\mathfrak{sl}(m|n)) \).

5.3. Integral form \( \mathcal{U}_v^\infty(L\mathfrak{sl}(m|n)) \) and its PBWD bases.

Following (2.18), define \( \bar{e}_\beta(r) \in U_v^0(L\mathfrak{sl}(m|n)) \) via \( \bar{e}_\beta(r) := (v-v^{-1})e_\beta(r) \). We also define \( \bar{e}_\alpha \) via (2.14) but using \( \bar{e}_\beta(r) \) instead of \( e_\beta(r) \). Define an integral form \( \mathcal{U}_v^\infty(L\mathfrak{sl}(m|n)) \) as the \( \mathbb{C}[v,v^{-1}] \)-subalgebra of \( U_v^0(L\mathfrak{sl}(m|n)) \) generated by \( \{\bar{e}_\beta(r)\}^{r \in \Delta^+} \).

The following counterpart of Theorem 2.19 provides a much stronger version of Theorem 5.6:

Theorem 5.10. (a) The subalgebra \( \mathcal{U}_v^\infty(L\mathfrak{sl}(m|n)) \) is independent of all our choices.
(b) Elements \( \{\bar{e}_h\}_{h \in H} \) form a basis of a free \( \mathbb{C}[v,v^{-1}] \)-module \( \mathcal{U}_v^\infty(L\mathfrak{sl}(m|n)) \).

The proof of Theorem 5.10 follows easily from the one of Theorem 5.6 presented below in the same way as we deduced the proof of Theorem 2.19 in Section 3.4 from that of Theorem 2.15.

Remark 5.11. Similarly to Remark 2.24(a), it is often more convenient to work with the quantum loop superalgebra \( U_v(L\mathfrak{gl}(m|n)) \). Its integral form \( \mathcal{U}_v(L\mathfrak{gl}(m|n)) \) is defined analogously to \( \mathcal{U}_v(L\mathfrak{gl}_n) \). Then, the argument of [FT2, Proposition 3.42], cf. Remark 2.24(b), identifies \( \mathcal{U}_v(L\mathfrak{gl}(m|n)) \) with the RTT integral form \( \mathcal{U}_v^{\text{RTT}}(L\mathfrak{gl}(m|n)) \) (see [Z3, Definition 3.1]) under the \( \mathbb{C}(v) \)-algebra isomorphism \( U_v(L\mathfrak{gl}(m|n)) \simeq \mathcal{U}_v^{\text{RTT}}(L\mathfrak{gl}(m|n)) \otimes_{\mathbb{C}[v,v^{-1}]} \mathbb{C}(v) \), cf. [DF]. Hence, the analogue of [FT2, Theorem 3.54] provides PBWD bases of \( \mathcal{U}_v^{\text{RTT}}(L\mathfrak{gl}_{n_1,n_2}) \), cf. Theorem 2.22.

5.4. Shuffle algebra \( S^{(m|n)} \).

Consider an \( \mathbb{N}^I \)-graded \( \mathbb{C}(v) \)-vector space \( S^{(m|n)} = \bigoplus_{k \in \mathbb{N}^I} S_{k}^{(m|n)} \), where \( S_{k}^{(m|n)} \) consists of rational functions in the variables \( x_{i,r} \leq r \leq k_i \), which are:

1) symmetric in \( x_{i,r} \) for every \( i \neq m \);
2) skew-symmetric in \( x_{m,r} \).

We fix an \( I \times I \) matrix of rational functions \( (\zeta_{i,j}(z))_{i,j \in I} \in \text{Mat}_{I \times I}(\mathbb{C}(v)(z)) \) defined via

\[
\zeta_{i,j}(z) = \begin{cases} 
\frac{z-v}{z-v^{-1}} & \text{if } j = i \\
\frac{z-v^{-1}}{z-v} & \text{if } j = i + 1 \\
\frac{z-v^{-1}}{z-v} & \text{if } j = i - 1 \\
1 & \text{otherwise}
\end{cases}.
\]

(5.12)

This allows us to endow \( S^{(m|n)} \) with a structure of an associative unital algebra with the shuffle product defined via (3.1), where a symmetrization along the variables \( \{x_{m,\bullet}\} \) is replaced by an anti-symmetrization. As before, we will be interested only in the subspace of \( S^{(m|n)} \) defined by the pole and wheel conditions (but now there are two kinds of the latter one):
We say that \( F \in S_{\Lambda}^{(m|n)} \) satisfies the pole conditions if
\[
F = \frac{f(x_{1,1}, \ldots, x_{m+n-1,k_{m+n-1}})}{\prod_{i=1}^{m+n-2} \prod_{r \leq k_i} (x_{i,r} - x_{i+1,r}),}
\]
where \( f \in (\mathbb{C}(v)\{\{x_{i,r}^{\pm 1}\}_{i \in I}^{1 \leq r \leq k_i}\}) \) is a Laurent polynomial which is symmetric in \( \{x_{i,r}^{k_i}\}_{r=1}^{k_m} \) for every \( i \neq m \) and is skew-symmetric in \( \{x_{m,r}^{k_m}\}_{r=1}^{k_m} \).

We say that \( F \in S_{\Lambda}^{(m|n)} \) satisfies the first kind wheel conditions if
\[
F(\{x_{i,r}\}) = 0 \text{ once } x_{i,r_1} = v_i x_{i+\epsilon,s} = v_i^2 x_{i+\epsilon,s} \text{ for some } \epsilon, i \neq m, r_1, r_2, s,
\]
where \( \epsilon \in \{\pm 1\}, i \in I \setminus \{m\}, i + \epsilon \in I, 1 \leq r_1, r_2 \leq k_i, 1 \leq s \leq k_{i+\epsilon} \).

We say that \( F \in S_{\Lambda}^{(m|n)} \) satisfies the second kind wheel conditions if
\[
F(\{x_{i,r}\}) = 0 \text{ once } x_{m-1,s} = v x_{m,r_1} = x_{m+1,s'} = v^{-1} x_{m+1,s'} \text{ for some } r_1, r_2, s, s',
\]
where \( 1 \leq r_1, r_2 \leq k_m, 1 \leq s \leq k_{m-1}, 1 \leq s' \leq k_{m+1} \).

Let \( S_{\Lambda}^{(m|n)} \subset S_{\Lambda}^{(m|n)} \) denote the subspace of all elements \( F \) satisfying these three conditions and set \( S^{(m|n)} := \bigoplus_{k \in \mathbb{N}^I} S_{\Lambda}^{(m|n)} \). It is straightforward to check that \( S^{(m|n)} \subset S_{\Lambda}^{(m|n)} \) is \( \ast \)-closed.

Similar to Proposition 3.4, the shuffle algebra \( (S^{(m|n)}, \ast) \) is related to \( U_v^\infty (L\mathfrak{s}l(m|n)) \) via:

**Proposition 5.16.** The assignment \( e_{i,r} \mapsto x_{i,r}^k \) \((i \in I, r \in \mathbb{Z})\) gives rise to an injective \( \mathbb{C}(v) \)-algebra homomorphism \( \Psi : U_v^\infty (L\mathfrak{s}l(m|n)) \to S^{(m|n)} \).

Our proof of Theorem 5.6 also implies the counterpart of Theorem 3.5 (see Remark 5.25):

**Theorem 5.17.** \( \Psi : U_v^\infty (L\mathfrak{s}l(m|n)) \cong S^{(m|n)} \) is a \( \mathbb{C}(v) \)-algebra isomorphism.

5.5. **Proof of Theorem 5.6.**

The proof of Theorem 5.6 is similar to our proof of Theorem 2.15(a) and is based on the embedding into the shuffle algebra \( S^{(m|n)} \) of Proposition 5.16. Therefore, we outline the proof, highlighting the key changes.

But before proceeding to the general case, let us first establish the result in the simplest case \( m = n = 1 \):

**Lemma 5.18.** For any total ordering \( \preceq \) on \( \mathbb{Z} \), the ordered monomials \( \{e_{r_1} e_{r_2} \cdots e_{r_k} \}_{r_1 \preceq \cdots \preceq r_k} \) form a \( \mathbb{C}(v) \)-basis of \( U_v^\infty (L\mathfrak{s}l(1|1)) \).

**Proof.** This immediately follows from the \( \mathbb{C}(v) \)-algebra isomorphism \( S^{(1|1)} \cong \bigoplus_{k \in \mathbb{N}} \Lambda_k \), where \( \Lambda_k \) denotes the vector space of skew-symmetric Laurent polynomials in \( k \) variables, while the algebra structure on the direct sum arises via the standard skew-symmetrization maps \( \Lambda_k \otimes \Lambda_1 \to \Lambda_{k+1} \).

Following Section 3.2.2, for any degree vector \( d = \{d_\beta\}_{\beta \in \Delta^+} \), define the specialization map \( \phi_d : S_{\Lambda}^{(m|n)} \to \mathbb{C}(v) \{y_{\beta,s}^{1 \leq s \leq d_\beta} \} \) (here \( l = \sum_{\beta \in \Delta^+} d_\beta \beta \in \mathbb{N}^I \)) via (3.12) with the only change that the variable \( x_{k,\ast} \) of the \( s \)-th copy of the interval \( [\beta] \) gets specialized to \( v^{-k} y_{\beta,s} \) if \( k \leq m \) and to \( v^{k-2m} y_{\beta,s} \) if \( k > m \). Note that any element in the image of \( \phi_d \) is a Laurent polynomial which is symmetric in \( \{y_{\beta,s}\}_{s=1}^{d_\beta} \) if \( p(\beta) = 0 \) and is skew-symmetric in \( \{y_{\beta,s}\}_{s=1}^{d_\beta} \) if \( p(\beta) = \bar{1} \).
The resulting specialization maps \( \phi_d \) still satisfy Lemmas 3.14 and 3.15 (\( H \) should be replaced by \( \bar{H} \) in the formulation of the latter). Moreover, for any \( h \in \bar{H} \) with \( \deg(h) = d \), we have the following generalization of the key formulas (3.16, 3.17):

\[
\phi_d(\Psi(e_h)) = c \cdot \prod_{\beta < \beta'} \tilde{G}_{\beta, \beta'} \cdot \prod_{\beta \in \Delta^+} \bar{G}_\beta \cdot \prod_{\beta_\sigma \in \Sigma_{d_\beta}} \left( \sum_{\sigma_\beta} \bar{G}_{\beta}^{(\sigma_\beta)} \right), \quad c \in \mathbb{C}^\times \cdot v^z, \tag{5.19}
\]

where

\[
\tilde{G}_{\beta, \beta'} = \prod_{1 \leq s \leq d_\beta} \left( y_{\beta, s} - v^{-2} y_{\beta', s'} \right)^{\nu^-(\beta, \beta')}, \quad \tilde{G}_\beta = \prod_{1 \leq s \leq d_\beta} \left( y_{\beta, s} - v^{2} y_{\beta, s'} \right)^{\nu^+(\beta, \beta')},
\]

\[
\tilde{G}_{\beta}^{(\sigma_\beta)} = \prod_{s=1}^{d_\beta} y_{\beta, \sigma_\beta(s)}^{r_{\beta}(h, s)}, \quad \left\{ \begin{array}{ll}
\prod_{s < s'} \frac{y_{\beta, \sigma_\beta(s)} - v^{-2} y_{\beta, \sigma_\beta(s')}}{y_{\beta, \sigma_\beta(s)} - v^{2} y_{\beta, \sigma_\beta(s')}} & \text{if } m > i(\beta), \\
\prod_{s < s'} \frac{y_{\beta, \sigma_\beta(s)} - v^{-2} y_{\beta, \sigma_\beta(s')}}{y_{\beta, \sigma_\beta(s)} - v^{2} y_{\beta, \sigma_\beta(s')}} & \text{if } m < j(\beta), \\
(-1)^{\sigma_\beta} & \text{if } m \in \beta.
\end{array} \right.
\]

Here the collection \( \{r_{\beta}(h, 1), \ldots, r_{\beta}(h, d_\beta)\} \) is defined as after (3.16) (that is, listing every \( r \in \mathbb{Z} \) with multiplicity \( h(\beta, r) > 0 \) with respect to the total ordering \( \preceq_\beta \) on \( \mathbb{Z} \)), while the powers \( \nu^{\pm}(\beta, \beta') \) are given by the following explicit formulas:

\[
\nu^-(\beta, \beta') = \# \{ (j, j') \in [\beta] \times [\beta'] | j = j' < m \} + \# \{ (j, j') \in [\beta] \times [\beta'] | j = j' + 1 > m \}, \tag{5.21}
\]

\[
\nu^+(\beta, \beta') = \# \{ (j, j') \in [\beta] \times [\beta'] | j = j' > m \} + \# \{ (j, j') \in [\beta] \times [\beta'] | j = j' + 1 < m \}. \tag{5.22}
\]

For \( \beta \in \Delta^+ \), we note that the sum \( \sum_{\sigma_\beta \in \Sigma_{d_\beta}} \bar{G}_{\beta}^{(\sigma_\beta)} \) coincides (up to a non-zero factor of \( \mathbb{C}^\times \)) with the value of the shuffle element \( x^{r_{\beta}(h, 1)} \cdots x^{r_{\beta}(h, d_\beta)} \) viewed either as

1) an element of \( \mathcal{S}^{(2)} \) if \( m > i(\beta) \),
2) an element of \( \mathcal{S}^{(0)} \) if \( m < j(\beta) \),
3) an element of \( \mathcal{S}^{(1)} \) if \( m \in \beta \),

evaluated at \( \{y_{\beta, s}\}_{s=1}^{d_\beta} \). The latter elements are linearly independent, due to Lemmas 3.9, 5.18. Combining this with (5.19), we get the linear independence of \( \{e_h\}_{h \in \bar{H}} \) as in Section 3.2.2.

Remark 5.23. This reduction to the rank 1 cases together with Lemma 5.18 explains why \( H \) had to be replaced by \( \bar{H} \) in the current setting.

The fact that the ordered PBWD monomials \( \{e_h\}_{h \in \bar{H}} \) span \( U^\infty_{\mathfrak{g}}(\mathfrak{sl}(m|n)) \) follows from the validity of Lemma 3.19 in the current setting. Let us now prove the latter using the same ideas and notations as before.

First, we note that the wheel conditions (5.14, 5.15) for \( F \) guarantee that \( \phi_d(F) \) (which is a Laurent polynomial in \( \{y_{\beta, s}\} \)) vanishes up to appropriate orders under the following specializations:
(i) \( y_{\beta,s} = v^{-2} y_{\beta',s'} \) for \((\beta, s) < (\beta', s')\),
(ii) \( y_{\beta,s} = v^2 y_{\beta',s'} \) for \((\beta, s) < (\beta', s')\).

A straightforward case-by-case verification shows that these orders of vanishing equal the corresponding powers of \( y_{\beta,s} - v^{-2} y_{\beta',s'} \) and \( y_{\beta,s} - v^2 y_{\beta',s'} \) appearing in \( \bar{G}_{\beta,\beta'} \) (if \( \beta < \beta' \)) or \( \bar{G}_{\beta} \) (if \( \beta = \beta' \)) of (5.20). In the former case, those are explicitly given by (5.21, 5.22).

**Remark 5.24.** We should point out right away that the computation of the corresponding orders requires an extra argument in the case \( \beta = \beta', m \in [\beta] \). Recall that the way we counted these orders in the proof of Lemma 3.19 was by realizing the specialization \( \phi_d \) as a step-by-step specialization in each interval in the specified order. A priori, we can choose another order of the intervals or even another way to perform this specialization. Let us now illustrate how our argument should be modified in the particular case \( \beta = \beta', m \in [\beta] \). Note that if we first specialize the variables in the interval \([\beta]\) to the corresponding \(v\)-multiples of \( y_{\beta,s} \), then the wheel conditions contribute \( i(\beta) - j(\beta) \) to the order of vanishing at \( y_{\beta,s} = v^2 y_{\beta,s'} \) and \( i(\beta) - j(\beta) - 1 \) to the order of vanishing at \( y_{\beta,s} = v^{-2} y_{\beta,s'} \). If instead we first specialize the variables in the interval \([\beta]\) to the corresponding \(v\)-multiples of \( y_{\beta,s'} \), then the wheel conditions contribute \( i(\beta) - j(\beta) - 1 \) to the order of vanishing at \( y_{\beta,s} = v^2 y_{\beta,s'} \) and \( i(\beta) - j(\beta) \) to the order of vanishing at \( y_{\beta,s} = v^{-2} y_{\beta,s'} \). Thus, none of these two specializations provides the desired orders of vanishing simultaneously for \( y_{\beta,s} = v^2 y_{\beta,s'} \) and \( y_{\beta,s} = v^{-2} y_{\beta,s'} \). However, picking the maximal of the orders separately for \( y_{\beta,s} = v^2 y_{\beta,s'} \) and \( y_{\beta,s} = v^{-2} y_{\beta,s'} \), we recover \( i(\beta) - j(\beta) \) for both of them, so that they equal the corresponding powers of \( y_{\beta,s} - v^2 y_{\beta,s'} \) and \( y_{\beta,s} - v^{-2} y_{\beta,s'} \) appearing in \( \bar{G}_{\beta} \) of (5.20).

Second, we claim that \( \phi_d(F) \) vanishes under the following specializations:

(iii) \( y_{\beta,s} = y_{\beta',s'} \) for \((\beta, s) < (\beta', s')\) such that \( j(\beta) < j(\beta') \) and \( i(\beta) + 1 \in [\beta'] \).

Indeed, if \( j(\beta) < j(\beta') \) and \( i(\beta) + 1 \in [\beta'] \), there are positive roots \( \gamma, \gamma' \in \Delta^+ \) such that \( j(\gamma) = j(\beta), i(\gamma) = i(\beta'), j(\gamma') = j(\beta'), i(\gamma') = i(\beta) \). Consider the degree vector \( d' = (d, d) \in T_k \) given by \( d'_\alpha = d_\alpha + \delta_{\alpha,\gamma} + \delta_{\alpha,\gamma'} - \delta_{\alpha,\beta} - \delta_{\alpha,\beta'} \). Then, \( d' > d \) and thus \( \phi_d(F) = 0 \). The result follows.

Finally, we also note that the skew-symmetry of the elements of \( S^{(m,n)} \) with respect to the variables \( \{x_{m,\bullet}\} \) implies that \( \phi_d(F) \) vanishes under the following specializations:

(iv) \( y_{\beta,s} = y_{\beta',s'} \) for all \( \beta < \beta' \) (and any \( s, s' \)) such that \([\beta] \ni m \in [\beta'] \).

Combining the above vanishing conditions for \( \phi_d(F) \), we see that it is divisible by the product \( \prod_{\beta < \beta'} \bar{G}_{\beta,\beta'} \cdot \prod_{\beta} \bar{G}_{\beta} \) of (5.20). Therefore, we have

\[
\phi_d(F) = \prod_{\beta < \beta', \beta, \beta' \in \Delta^+} \bar{G}_{\beta,\beta'} \cdot \prod_{\beta \in \Delta^+} \bar{G}_{\beta} \cdot \bar{G},
\]

where \( \bar{G} \) is a Laurent polynomial in \( \{y_{\beta,s}\}_{1 \leq s \leq d_\beta} \) which is symmetric in \( \{y_{\beta,s}\}_{s=1}^{d_\beta} \) if \( p(\beta) = 0 \) and is skew-symmetric in \( \{y_{\beta,s}\}_{s=1}^{d_\beta} \) if \( p(\beta) = 1 \).

Combining this observation with Lemmas 3.9, 5.18 and formulas (5.19, 5.20) implies the validity of Lemma 3.19 in the current setting. Hence, \( \{\Psi(\epsilon_h)\}_{h \in H} \) linearly span \( S^{(m,n)} \).

This completes our proof of Theorem 5.6.

**Remark 5.25.** The last statement together with Proposition 5.16 implies Theorem 5.17.
6. Generalizations to the Yangian $Y_h(\mathfrak{sl}_n)$

The PBWD bases for the Yangian $Y_h(\mathfrak{g})$ of any semisimple Lie algebra $\mathfrak{g}$ has been constructed 25 years ago in [Le]. While the Yangian deforms the universal enveloping of the loop algebra $\mathfrak{g}[t]$, that is $Y_h(\mathfrak{g})(t)/h \simeq U(\mathfrak{g}[t])$, there is a canonical construction of the Drinfeld-Gavarini dual (Hopf) subalgebra $Y'_h(\mathfrak{g}) \subset Y_h(\mathfrak{g})$ such that $Y'_h(\mathfrak{g})(/h)$ is a commutative $C$-algebra, see [FT2, Appendix A] and the original references [D2, Ga]. Following [Ga], we have established the PBWD bases for the Drinfeld-Gavarini dual $Y_h(\mathfrak{g})$ in [FT2, Theorem A.21].

As just mentioned, the PBWD results (cf. Theorems 2.15, 2.17, 2.19, 2.22) are known both for $Y_h(\mathfrak{g})$ and $Y'_h(\mathfrak{g})$ for an arbitrary semisimple $\mathfrak{g}$. Thus the main goal of this section is to establish the shuffle realizations of $Y_h(\mathfrak{sl}_n)$ and $Y'_h(\mathfrak{sl}_n)$ similar to those of Theorems 3.5, 3.34. For the latter purpose, it suffices to consider only the subalgebras $Y_{h}(\mathfrak{sl}_n), Y'_{h}(\mathfrak{sl}_n) \simeq Y_{h}(\mathfrak{sl}_n)$.

6.1. Algebras $Y_{h}(\mathfrak{sl}_n)$ and $Y'_{h}(\mathfrak{sl}_n)$.

Let $I = \{1, \ldots, n - 1\}$ and $(c_{ij})_{i,j \in I}$ be as in Section 2.1, and let $h$ be a formal variable. Following [D1], define $Y_{h}(\mathfrak{sl}_n)$ to be the associative $\mathbb{C}[h]$-algebra generated by $\{e_{ij}\}_{i,j \in I}$ with the following defining relations:

$$[e_{i,r+1}, e_{j,s+1}] - [e_{i,r}, e_{j,s+1}] = \frac{c_{ij}h}{2} (e_{i,r}e_{j,s} + e_{j,s}e_{i,r}), \quad (6.1)$$

$$[e_{i,r}, e_{j,s}] = 0 \text{ if } c_{ij} = 0,$$

$$[e_{i,r_1}, [e_{i,r_2}, e_{j,s}]] + [e_{i,r_2}, [e_{i,r_1}, e_{j,s}]] = 0 \text{ if } c_{ij} = -1. \quad (6.2)$$

Let $\{\alpha_i\}_{i=1}^{n-1}, \Delta^+$ be as in Section 2.2. For every $(\beta, r) \in \Delta^+ \times \mathbb{N}$, make the following choices:

1. a decomposition $\beta = \alpha_{i_1} + \ldots + \alpha_{i_p}$ such that $[\ldots [e_{\alpha_{i_1}}, e_{\alpha_{i_2}}, \ldots, e_{\alpha_{i_p}}]$ is a non-zero root vector $e_\beta$ of $\mathfrak{sl}_n$ (here $e_{\alpha_i}$ denotes the standard Chevalley generator of $\mathfrak{sl}_n$);

2. a decomposition $r = r_1 + \ldots + r_p$ with $r_i \in \mathbb{N}$.

We define the \textit{PBWD basis elements} $e_\beta(r) \in Y_{h}(\mathfrak{sl}_n)$ via

$$e_\beta(r) := [\ldots [e_{i_1,r_1}, e_{i_2,r_2}, e_{i_3,r_3}], \ldots, e_{i_p,r_p}]. \quad (6.3)$$

Let $H^+$ denote the set of all functions $h: \Delta^+ \times \mathbb{N} \to \mathbb{N}$ with finite support. The monomials

$$e_h := \prod_{(\beta, r) \in \Delta^+ \times \mathbb{N}} e_\beta(r)^{h(\beta, r)} \quad \text{with } h \in H^+ \quad (6.4)$$

will be called the \textit{ordered PBWD monomials} of $Y_{h}(\mathfrak{sl}_n)$.

The following is due to [Le] (cf. [FT2, Theorem B.10]):

\textbf{Theorem 6.5 ([Le])}. Elements $\{e_h\}_{h \in H^+}$ form a basis of a free $\mathbb{C}[h]$-module $Y_{h}(\mathfrak{sl}_n)$.

\textbf{Remark 6.6}. According to [Le], this result holds for any total ordering on $\Delta^+ \times \mathbb{N}$ used in (6.4).

Define $\overline{e}_\beta(r) \in Y_{h}(\mathfrak{sl}_n)$ via

$$\overline{e}_\beta(r) := h \cdot e_\beta(r). \quad (6.7)$$

We also define $\overline{e}_h$ via the formula (6.4) but using $\overline{e}_\beta(r)$ instead of $e_\beta(r)$. Define an integral form $Y'_{h}(\mathfrak{sl}_n)$ as the $\mathbb{C}[h]$-subalgebra of $Y_{h}(\mathfrak{sl}_n)$ generated by $\{\overline{e}_\beta(r)\}_{\beta \in \Delta^+}$.

The following result is proved in [FT2, Theorem A.21]:

\textbf{Theorem 6.8 ([FT2])}. (a) The subalgebra $Y'_{h}(\mathfrak{sl}_n)$ is independent of all our choices. 
(b) The \textit{ordered PBWD monomials} $\{\overline{e}_h\}_{h \in H^+}$ form a basis of a free $\mathbb{C}[h]$-module $Y'_{h}(\mathfrak{sl}_n)$.

\textsuperscript{7}See [FT2, Appendix B] for a correction of a gap in the proof of [Le].
6.2. Rational shuffle algebra $W^{(n)}$ and its integral form $\mathfrak{W}^{(n)}$.

Define the shuffle algebra $(\bar{W}^{(n)}, \star)$ analogously to the shuffle algebra $(S^{(n)}, \star)$ of Section 3.1 with the following modifications:

1. All rational functions $F \in \bar{W}^{(n)}$ are defined over $\mathbb{C}[\hbar]$;
2. The matrix $(\zeta_{i,j}(z))_{i,j \in I} \in \text{Mat}_{I \times I}(\mathbb{C}[\hbar](z))$ is defined via $\zeta_{i,j}(z) = \frac{z + \overline{c_{i,j}}}{z - 1}$;
3. The pole conditions (3.2) for $F \in \bar{W}^{(n)}_k$ are modified as follows:
   \[
   F = \frac{f(x_{1,1}, \ldots, x_{n-1,k_n-1})}{\prod_{i=1}^{n-2} \prod_{r \leq k_i} (x_{i,r} - x_{i+1,r'})}, \quad \text{where } f \in (\mathbb{C}[\hbar][(x_{i,r})_{i \in I \leq k_i}])_{\Sigma}^{\pm};
   \] (6.9)
4. The wheel conditions (3.3) for $F \in \bar{W}^{(n)}$ are modified as follows:
   \[
   F(\{x_{i,r}\}) = 0 \text{ once } x_{i,r} = x_{i+s,r} + \frac{\hbar}{2} = x_{i,r} + \hbar \text{ for some } s \in \{\pm 1\}, i, r, s.
   \] (6.10)

The shuffle algebra $(\bar{W}^{(n)}, \star)$ is related to $Y_h^>(\mathfrak{g}_n)$ via the following construction:

**Proposition 6.11.** The assignment $e_{i,r} \mapsto x_{i,1}^r$ ($i \in I, r \in \mathbb{N}$) gives rise to a $\mathbb{C}[\hbar]$-algebra homomorphism $\Psi: Y_h^>(\mathfrak{g}_n) \to \bar{W}^{(n)}$.

The following is the Yangian counterpart of Lemma 3.10:

**Lemma 6.12.** For $1 \leq j < i < n$ and $r \in \mathbb{N}$, we have

\[
\Psi(e_{\alpha_j + \alpha_{j+1} + \cdots + \alpha_i}(r)) = \hbar^{i-j} p(x_{j,1}, \ldots, x_{i,1}) \frac{p(x_{j,1}, \ldots, x_{i,1})}{(x_{j,1} - x_{j+1,1}) \cdots (x_{i-1,1} - x_{i,1})},
\]

where $p(x_{j,1}, \ldots, x_{i,1}) \in \mathbb{C}[\hbar][x_{j,1}, \ldots, x_{i,1}]$ is a degree $r$ monomial, up to a sign.

**Example 6.13.** For the particular choice $e_{\alpha_j + \alpha_{j+1} + \cdots + \alpha_i}(r) = \cdots [e_{j,r}, e_{j+1,0}, \cdots, e_{i,0}]$ (used in [FT2, Section 2]), we have $p(x_{j,1}, \ldots, x_{i,1}) = (-1)^{i-j} x_{j,1}^r$.

For $\underline{d} = \{d_\beta\}_{\beta \in \Delta^+} \in \mathbb{N}_+^{n(n-1)/2}$ and $\underline{l} = \sum_{\beta \in \Delta^+} d_\beta[\beta] \in \mathbb{N}_l$, define the specialization map

\[
\phi_{\underline{d}}: \bar{W}^{(n)}_k \longrightarrow \mathbb{C}[\hbar][\{y_{\beta,s}\}_{\beta \in \Delta^+, 1 \leq s \leq d_\beta}]^{\Sigma \underline{d}}
\] (6.14)

as in (3.12) with the only change that the variable $x_{k,\bullet}$ of the $s$-th copy of the interval $[\beta]$ gets specialized to $y_{\beta,s} - \frac{k \hbar}{2}$. Then, arguing exactly as in Section 3.2.2, we get:

**Proposition 6.15.** The elements $\{\Psi(e_{\hbar})\}_{h \in H^+}$ are linearly independent.

Combining this with Theorem 6.5, we obtain:

**Proposition 6.16.** $\Psi: Y_h^>(\mathfrak{g}_n) \to \bar{W}^{(n)}$ is an injective $\mathbb{C}[\hbar]$-algebra homomorphism.

In contrast to Theorem 3.5, the embedding $\Psi: Y_h^>(\mathfrak{g}_n) \hookrightarrow \bar{W}^{(n)}$ is not an isomorphism. The description of the image is similar to Theorem 3.34, but is significantly simpler.

**Definition 6.17.** $F \in \bar{W}^{(n)}_k$ is good if $\phi_{\underline{d}}(F)$ is divisible by $h^{\sum_{\beta \in \Delta^+} d_\beta[i(\beta) - j(\beta)]}$ for any degree vector $\underline{d} = \{d_\beta\}_{\beta \in \Delta^+}$ such that $k = \sum_{\beta \in \Delta^+} d_\beta[\beta]$.

**Example 6.18.** In the simplest case $n = 2$, any element $F \in \bar{W}^{(n)}_k$ ($k \in \mathbb{N}_l$) is good.

Set $W^{(n)} := \bigoplus_{k \in \mathbb{N}_l} W^{(n)}_k$ with $W^{(n)}_k \subset \bar{W}^{(n)}_k$ denoting the $\mathbb{C}[\hbar]$-submodule of all good elements.
Lemma 6.19. $\Psi(Y_h^n(\mathfrak{sl}_n)) \subseteq W(n)$.

Proof. Let $F = \Psi(e_{i_1,r_1} \cdots e_{i_N,r_N}) \in W_k(n)$ and $k = \sum_{\beta \in \Delta^+} d_\beta[\beta]$. For $\beta \in \Delta^+, 1 \leq s \leq d_\beta$, consider $\zeta$-factors between pairs of $x_{r_i}$-variables that are specialized to $(y_{\beta,s} - \frac{t h}{2}, y_{\beta,s} - \frac{(t+1) h}{2})$ with $j(\beta) \leq l \leq i(\beta) - 1$. Each of them contributes a multiple of $h$ into $\phi_2(\Psi(F))$. Since there are exactly $\sum_{\beta \in \Delta^+} d_\beta(i(\beta) - j(\beta))$ of such pairs, we get $F \in W_k(n)$. \hfill \Box

The following is the key result of this section:

Theorem 6.20. The $\mathbb{C}[h]$-algebra embedding $\Psi: Y_h^n(\mathfrak{sl}_n) \hookrightarrow \bar{W}(n)$ of Proposition 6.16 gives rise to a $\mathbb{C}[h]$-algebra isomorphism $\Psi: Y_h^n(\mathfrak{sl}_n) \overset{\sim}{\to} W(n)$.

In view of Example 6.18, Theorem 6.20 for $n = 2$ is equivalent to the following result:

Lemma 6.21. Any symmetric polynomial $F \in \mathbb{C}[h][\{x_i\}_{i=1}^k]^{\Sigma_k}$ may be written as a $\mathbb{C}[h]$-linear combination of $\{\Psi(e_i)\}_{i \in H^+}$.

The proof of Lemma 6.21 is completely analogous to that of Lemma 3.41, and relies on the following simple computation (cf. Lemma 3.6):

Lemma 6.22. For any $k \geq 1$ and $r \in \mathbb{N}$, the $k$-th power of $x^r \in \bar{W}_1^{(2)}$ equals

$$x^r \ast \cdots \ast x^r = k \cdot (x_1 \cdots x_k)^r.$$  \hspace{1cm} (6.23)

Proof. The proof is by induction in $k$ and boils down to the verification of

$$\sum_{i=1}^k \prod_{1 \leq j \leq k, \ j \neq i} \frac{x_j - x_i + h}{x_j - x_i} = k,$$  \hspace{1cm} (6.24)

which is proved similarly to (3.8). \hfill \Box

The proof of Theorem 6.20 for $n = 2$ is completely analogous to those of Theorems 2.15, 2.19 and crucially utilizes the $n = 2$ case of Lemma 6.21. We leave details to the interested reader.

Definition 6.25. $F \in \bar{W}_k(n)$ is integral if $F$ is divisible by $h^{|\mathfrak{k}|}$.

Remark 6.26. Any integral $F \in \bar{W}_k(n)$ is good as $|\sum_{\beta \in \Delta^+} d_\beta[\beta]| = \sum_{\beta \in \Delta^+} d_\beta(i(\beta) - j(\beta) + 1)$.

Set $W(n) := \bigoplus_{k \in \mathbb{N}^+} W_k(n)$, where $W_k(n) \hookrightarrow W(n)$ denotes the $\mathbb{C}[h]$-submodule of all integral elements. The following is our second key result of this section:

Theorem 6.27. The $\mathbb{C}[h]$-algebra isomorphism $\Psi: Y_h^n(\mathfrak{sl}_n) \overset{\sim}{\to} W(n)$ of Theorem 6.20 gives rise to a $\mathbb{C}[h]$-algebra isomorphism $\Psi: Y_h^n(\mathfrak{sl}_n) \overset{\sim}{\to} W(n)$.

The proof of Theorem 6.27 is completely analogous to that of Theorem 3.34, but is much simpler. In particular, adapting Lemma 3.43 to the current setting, the key combinatorial computation from its proof is not needed, while Lemma 3.44 is adapted without any changes.

Remark 6.28. Let us note right away that the key simplification in the proof of Theorem 6.27 (comparing to that of Theorem 3.34) as well as in the definition of integral elements of Definition 6.25 (comparing to those of Definition 3.31) is due to the following rank 1 computations:

1. $h^k(x_1 \cdots x_k)^r \in \Psi(Y_h^n(\mathfrak{sl}_2))$ for any $k, r \in \mathbb{N}$, due to Lemma 6.22;
2. $(v - v^{-1})^k[v]_v!(x_1 \cdots x_k)^r \in \Psi(\mathfrak{U}_v(L\mathfrak{sl}_2))$ for any $k \in \mathbb{N}, r \in \mathbb{Z}$, due to Lemma 3.6;
3. $(v - v^{-1})^k(x_1 \cdots x_k)^r \not\in \Psi(\mathfrak{U}_v(L\mathfrak{sl}_2))$ for any $k > 1, r \in \mathbb{Z}$, due to Lemma 3.40.
7. Generalizations to the super Yangian $Y_h(\mathfrak{sl}(m|n))$

The super Yangian $Y_h(\mathfrak{gl}(m|n))$ was first introduced in [Na], following the RTT formalism of [FRT]. Its finite-dimensional representations were classified in [Z4]. Around the same time, the super Yangians $Y_h(A(m, n))$ of Lie superalgebras of type $A(m, n)$ were introduced in [S] in the new Drinfeld presentation, where it was shown that basic general results from the theory of usual Yangians (including PBW bases) still hold. The explicit relation between these two super Yangians was established in [Go].

The primary goal of this section is to generalize Theorem 6.20 to the case of $Y_h^\text{\lowercase{r}}(\mathfrak{sl}(m|n))$ (we note that $Y_h^\text{\lowercase{r}}(\mathfrak{sl}(m|n)) \simeq Y_h^\text{\lowercase{r}}(\mathfrak{gl}(m|n)) \simeq Y_h^\text{\lowercase{r}}(A(m, n))$). The resulting shuffle algebra $W^{(m|n)}$ is a mixture of the shuffle algebra $S^{(m|n)}$ from Section 5.4 and the rational shuffle algebra $W^{(n)}$ from Section 6.2. We also generalize Theorem 6.27 to the case of $Y_h^\text{\lowercase{r}}(\mathfrak{sl}(m|n))$.

### 7.1. Algebras $Y_h^\text{\lowercase{r}}(\mathfrak{sl}(m|n))$ and $Y_h^\text{\lowercase{r}}(\mathfrak{sl}(m|n))$

Let $I = \{1, \ldots, m+n-1\}$ and $(c_{ij})_{i,j \in I}$ be as in Section 5.1, and let $h$ be a formal variable. Following [S, Go] (see Remark 7.4 for a correction of the defining relations in [S, Definition 2]), define $Y_h^\text{\lowercase{r}}(\mathfrak{sl}(m|n))$ to be the associative $\mathbb{C}[h]$-superalgebra generated by $\{e_{i,r}\}_{i \in \mathbb{N}}$, with the $\mathbb{Z}_2$-grading $[e_{m,r}] = 1$, $[e_{i,r}] = 0$ $(i \neq m, r \in \mathbb{N})$, and with the following defining relations:

$$[e_{i,r+1}, e_{j,s}] - [e_{i,r}, e_{j,s+1}] = \frac{c_{ij}h}{2} (e_{i,r}e_{j,s} + e_{j,s}e_{i,r}) \quad \text{if} \quad c_{ij} \neq 0,$$

$$[e_{i,r}, e_{j,s}] = 0 \quad \text{if} \quad c_{ij} = 0,$$

$$[e_{i,r}, [e_{i,r_1}, e_{j,s}]] + [e_{i,r_2}, [e_{i,r_1}, e_{j,s}]] = 0 \quad \text{if} \quad c_{ij} = \pm 1, i \neq m,$$

$$[[e_{m-1,s}, e_{m,0}], [e_{m+1,s'}, e_{m,0}]] = 0,$$

where as before $[a, b] = ab - (-1)^{|a||b|} \cdot ba$.

**Remark 7.4.**
(a) The first relation of (7.2) implies the validity of the second one for $i = m = j \pm 1$, which is also listed among the defining relations of [S, Definition 2].
(b) Given relations (7.1, 7.2), the relation (7.3) is equivalent to:

$$[[e_{m-1,s}, e_{m,r_1}], [e_{m+1,s'}, e_{m,r_2}]] + [[e_{m-1,s}, e_{m,r_2}], [e_{m+1,s'}, e_{m,r_1}]] = 0.$$  

The latter should be used instead of a wrong relation $[[e_{m-1,s}, e_{m,r_1}], [e_{m+1,s'}, e_{m,r_2}]] = 0$ in [S].

Let $\{\alpha_i\}_{i=1}^{m+n-1}$, $\Delta^+$ be as in Section 5.2, and define the parity $p(\beta) \in \mathbb{Z}_2$ ($\beta \in \Delta^+$) via (5.4). Define the PBWD basis elements $e_\beta(r) \in Y_h^\text{\lowercase{r}}(\mathfrak{sl}(m|n))$ via (6.3). Let $\tilde{H}^+$ denote the set of all functions $h: \Delta^+ \times \mathbb{N} \rightarrow \mathbb{N}$ with finite support and such that $h(\beta, r) \leq 1$ if $p(\beta) = 1$. The monomials of the form

$$e_h := \prod_{(\beta, r) \in \Delta^+ \times \mathbb{N}} e_\beta(r)^{h(\beta, r)} \quad \text{with} \quad h \in \tilde{H}^+$$

will be called the ordered PBWD monomials of $Y_h^\text{\lowercase{r}}(\mathfrak{sl}(m|n))$. Analogously to [Le], we have:

**Theorem 7.7** ([S]). Elements $\{e_h\}_{h \in \tilde{H}^+}$ form a basis of a free $\mathbb{C}[h]$-module $Y_h^\text{\lowercase{r}}(\mathfrak{sl}(m|n))$.

Define $\tilde{e}_\beta(r) \in Y_h^\text{\lowercase{r}}(\mathfrak{sl}(m|n))$ via $\tilde{e}_\beta(r) := h \cdot e_\beta(r)$. We also define $\{\tilde{e}_h\}_{h \in \tilde{H}^+}$ via (7.6) but using $\tilde{e}_\beta(r)$ instead of $e_\beta(r)$. Define an integral form $Y_h^\text{\lowercase{s}}(\mathfrak{sl}(m|n))$ as the $\mathbb{C}[h]$-subalgebra of $Y_h^\text{\lowercase{r}}(\mathfrak{sl}(m|n))$ generated by $\{\tilde{e}_\beta(r)\}_{\beta \in \Delta^+}^{\in \mathbb{N}}$. The following is analogous to [FT2, Theorem A.21]:

**Theorem 7.8** ([FT2]).
(a) The subalgebra $Y_h^\text{\lowercase{r}}(\mathfrak{sl}(m|n))$ is independent of all our choices.
(b) The ordered PBWD monomials $\{\tilde{e}_h\}_{h \in \tilde{H}^+}$ form a basis of a free $\mathbb{C}[h]$-module $Y_h^\text{\lowercase{r}}(\mathfrak{sl}(m|n))$. 

7.2. Rational shuffle algebra $W^{(m|n)}$ and its integral form $\mathfrak{W}^{(m|n)}$.

Define the shuffle algebra $(W^{(m|n)}, \star)$ analogously to the shuffle algebra $(S^{(m|n)}, \star)$ of Section 5.4 with the following modifications:

1. All rational functions $F \in W^{(m|n)}$ are defined over $\mathbb{C}[\hbar]$;
2. The matrix $(\zeta_{i,j}(z))_{i,j \in I} \in \text{Mat}_{I \times I}(\mathbb{C}[\hbar](z))$ is defined via

$$
\zeta_{i,j}(z) = \begin{cases} 
\left(\frac{z+i}{z-1}\right)^{\delta_{i,m}} \left(\frac{z+i}{z-1}\right)^{\delta_{i,m}}, & \text{if } j = i \\
\left(\frac{z+i}{z-1}\right)^{\delta_{i,m}} \left(\frac{z+i}{z-1}\right)^{\delta_{i,m}}, & \text{if } j = i + 1 \\
\left(\frac{z+i}{z-1}\right)^{\delta_{i,m}} \left(\frac{z+i}{z-1}\right)^{\delta_{i,m}}, & \text{if } j = i - 1 \\
1, & \text{otherwise}
\end{cases}
$$

(7.9)

3. The pole conditions (5.13) for $F \in \tilde{W}^{(m|n)}_k$ are modified as follows:

$$
F = \frac{f \left( x_{1,1}, \ldots, x_{n-1,k_{n-1}} \right)}{\prod_{i=1}^{n-2} \prod_{r \leq k_i} (x_{i,r} - x_{i+1,r+1})} \quad f \in \mathbb{C}[\hbar]\{x_{i,r}\}_{i \in I},
$$

(7.10)

where polynomial $f$ is symmetric in $\{x_{i,r}\}_{i \in I}$ for $i \neq m$ and skew-symmetric in $\{x_{m,r}\}_{r=1}^{k_m}$;

4. The first kind wheel conditions (5.14) are modified as follows:

$$
F(\{x_{i,r}\}) = 0 \quad \text{once } x_{i,r} = x_{i+\epsilon,s} + \hbar/2 = x_{i,r} + \hbar, \quad \text{for some } \epsilon \in \{\pm 1\}, i \neq m, r_1, r_2, s;
$$

(7.11)

5. The second kind wheel conditions (5.15) are modified as follows:

$$
F(\{x_{i,r}\}) = 0 \quad \text{once } x_{m-1,s} = x_{m,r_1} + \hbar/2 = x_{m,s} + \hbar/2, \quad \text{for some } r_1, r_2, s, s'.
$$

(7.12)

In view of Theorem 7.7, the shuffle algebra $(W^{(m|n)}, \star)$ is related to $Y^\hbar_\mathfrak{g}(\mathfrak{sl}(m|n))$ via the following construction (cf. Propositions 6.11, 6.16):

**Proposition 7.13.** The assignment $e_{i,r} \mapsto x^r_{i,1}$ $(i \in I, r \in \mathbb{N})$ gives rise to a $\mathbb{C}[\hbar]$-algebra embedding $\Psi: Y^\hbar_\mathfrak{g}(\mathfrak{sl}(m|n)) \hookrightarrow W^{(m|n)}$.

For $d = \{d_\beta\}_{\beta \in \Delta^+}$, $l = \sum_{\beta \in \Delta^+} d_\beta$, define $\phi_d: \tilde{W}^{(m|n)}_k \rightarrow \mathbb{C}[\hbar]\{y_{l,s}\}_{\beta \in \Delta^+, 1 \leq s \leq d_\beta}$ via (6.14), but specializing $x^k_{\beta, \star}$ of the $s$-th copy of $[\beta]$ to $y_{l,s} - \frac{k\hbar}{2}$ if $k \leq m$ and to $y_{l,s} + \frac{(k-2m)\hbar}{2}$ if $k > m$.

**Definition 7.14.** (a) $F \in \tilde{W}^{(m|n)}_k$ is good if $\phi_d(F)$ is divisible by $\hbar^{\sum_{\beta \in \Delta^+} d_\beta \cdot (\beta - j(\beta))}$ for any degree vector $d = \{d_\beta\}_{\beta \in \Delta^+}$ such that $k = \sum_{\beta \in \Delta^+} d_\beta$.

(b) $F \in \tilde{W}^{(m|n)}_k$ is integral if $F$ is divisible by $\hbar^k$.

Set $W^{(m|n)} := \bigoplus_{k \in \mathbb{N}^t} \tilde{W}^{(m|n)}_k$ and $\mathfrak{W}^{(m|n)} := \bigoplus_{k \in \mathbb{N}^t} \mathfrak{W}^{(m|n)}_k$, where $W^{(m|n)}_k \subset \tilde{W}^{(m|n)}_k$ (resp. $\mathfrak{W}^{(m|n)}_k \subset \mathfrak{W}^{(m|n)}_k$) denotes the $\mathbb{C}[\hbar]$-submodule of all good (resp. integral) elements.

The following are the key results of this section:

**Theorem 7.15.** The $\mathbb{C}[\hbar]$-algebra embedding $\Psi: Y^\hbar_\mathfrak{g}(\mathfrak{sl}(m|n)) \hookrightarrow \tilde{W}^{(m|n)}$ of Proposition 7.13 gives rise to a $\mathbb{C}[\hbar]$-algebra isomorphism $\Psi: Y^\hbar_\mathfrak{g}(\mathfrak{sl}(m|n)) \xrightarrow{\sim} W^{(m|n)}$.

**Theorem 7.16.** The $\mathbb{C}[\hbar]$-algebra isomorphism $\Psi: Y^\hbar_\mathfrak{g}(\mathfrak{sl}(m|n)) \xrightarrow{\sim} W^{(m|n)}$ of Theorem 7.15 gives rise to a $\mathbb{C}[\hbar]$-algebra isomorphism $\Psi: \mathfrak{Y}^\hbar_\mathfrak{g}(\mathfrak{sl}(m|n)) \xrightarrow{\sim} \mathfrak{W}^{(m|n)}$.

Both Theorems 7.15, 7.16 are proved completely analogously to Theorems 5.17, 6.20, 6.27.
8. Further directions

In this section, we briefly outline some of the related results that will be addressed elsewhere.

8.1. Integral forms of Grojnowski and Chari-Pressley and their PBWD bases.

We follow the notations of Section 2. For \( i, r \in \mathbb{Z}, k \in \mathbb{N} \), define the divided power
\[
ed_{i,r}^{(k)} := e_{i,r}^k / [k]_v
\]
(8.1)
Following [Gr, Section 7.8], define the Grojnowski integral form \( U^>_v(Lsl_n) \) as the \( \mathbb{C}[v, v^{-1}] \)-subalgebra of \( U^>_v(Lsl_n) \) generated by all the divided powers \( \{e_{i,r}^{(k)}\}_{r \in \mathbb{Z}, k \in \mathbb{N}} \).

Following Remark 2.12, consider the following choice of PBWD basis elements \( \{e_{\beta}(r)\}_{\beta \in \Delta^+} \):
\[
e_{\alpha_j + \alpha_{j+1} + \ldots + \alpha_i}(r) := \prod_{i \neq j} e_{j,i}^r / [k]_v
\]
(8.2)
Set \( e_{\beta}(r)^{(k)} := e_{\beta}(r)^k / [k]_v \). Due to the computation of [Lu, Section 5.5], we have:

**Lemma 8.3.** \( e_{\beta}(r)^{(k)} \in U^>_v(Lsl_n) \) for any \( \beta \in \Delta^+, r \in \mathbb{Z}, k \in \mathbb{N} \).

The monomials of the form
\[
ed_h := \prod_{(\beta, r) \in \Delta^+ \times \mathbb{Z}} e_{\beta}(r)^{(h(\beta, r))} \quad \text{with} \quad h \in H
\]
(8.4)
will be called the ordered PBWD monomials of \( U^>_v(Lsl_n) \). Our first key result is:

**Theorem 8.5.** \( \{e_h\}_{h \in H} \) form a basis of a free \( \mathbb{C}[v, v^{-1}] \)-module \( U^>_v(Lsl_n) \).

The proof of Theorem 8.5 is completely analogous to those of Theorems 2.15, 2.19 and is based on the shuffle realization of \( U_v(Lsl_n) \) of Theorem 8.8. Following Definition 6.17, define:

**Definition 8.6.** \( F \in S_\mathbb{k}^{(n)} \) is good if it is of the form (3.2) with \( f \in \mathbb{C}[v, v^{-1}][\{x_i^{\pm 1}\}_{i \in I \leq k}] \), and \( \phi_d(F) \) is divisible by \( (v - v^{-1}) \sum_{\beta \in \Delta^+} d(\beta(\beta) - j(\beta)) \) for any degree vector \( d = \{d_\beta\}_{\beta \in \Delta^+} \) such that \( \mathbb{k} = \sum_{\beta \in \Delta^+} d_\beta[\beta] \).

Set \( S^{(n)} := \bigoplus_{\mathbb{k} \in \mathbb{N}^d} S_\mathbb{k}^{(n)} \) with \( S_\mathbb{k}^{(n)} \subseteq S_\mathbb{k}^{(n)} \) denoting the \( \mathbb{C}[v, v^{-1}] \)-submodule of all good elements.

**Lemma 8.7.** \( \Psi(U^>_v(Lsl_n)) \subseteq S^{(n)} \).

**Proof.** This follows from Lemma 3.6 and our proof of Lemma 6.19.

The following is the second key result of this section:

**Theorem 8.8.** The \( \mathbb{C}[v] \)-algebra isomorphism \( \Psi : U^>_v(Lsl_n) \cong S^{(n)} \) of Theorem 3.5 gives rise to a \( \mathbb{C}[v, v^{-1}] \)-algebra isomorphism \( \Psi : U^>_v(Lsl_n) \cong S^{(n)} \).

Let \( U_v(Lsl_n) \) be the \( \mathbb{C}[v, v^{-1}] \)-subalgebra of \( U^>_v(Lsl_n) \) generated by all divided powers \( f_{i,r}^{(k)} := f_{i,r}^k / [k]_v \! \). Define the \( \mathbb{C}[v, v^{-1}] \)-subalgebra \( U^>_0(Lsl_n) \) of \( U^>_v(Lsl_n) \) as in [CP, Section 3]. The Chari-Pressley integral form \( U_v(Lsl_n) \) is the \( \mathbb{C}[v, v^{-1}] \)-subalgebra of \( U^>_v(Lsl_n) \) generated by \( U^>_0(Lsl_n), U^>_0(Lsl_n), U^>_0(Lsl_n) \). Due to [CP, Proposition 6.1], we have:

**Theorem 8.9 ([CP]).** The multiplication map induces an isomorphism of \( \mathbb{C}[v, v^{-1}] \)-modules
\[
m : U^>_0(Lsl_n) \otimes_{\mathbb{C}[v, v^{-1}]} U^>_0(Lsl_n) \otimes_{\mathbb{C}[v, v^{-1}]} U^>_0(Lsl_n) \cong U_v(Lsl_n)
\]
Thus, Theorem 8.5 yields the PBWD property of \( U_v(Lsl_n) \) and its freeness over \( \mathbb{C}[v, v^{-1}] \).
8.2. Generalizations to all Dynkin diagrams associated with \( \mathfrak{sl}(m|n) \).

As Kac-Moody superalgebras admit many nonisomorphic Dynkin diagrams (in contrast to the ordinary Kac-Moody algebras), it is natural to study presentations of quantum groups associated to any of those. While the entire algebras are pairwise isomorphic (that is, they do not depend on the choice of Dynkin diagrams), this is no longer true for their positive subalgebras (denoted in this paper with the superscript \( > \)) generated by the \( e_\pm \)-generators. In the forthcoming note [Ts], the author will generalize the results of Sections 5, 7 to the positive subalgebras of the quantum affine superalgebra and super Yangian associated with the Lie superalgebra \( A(m,n) \). This features more general type \( A \) boson-fermion shuffle algebras.

More precisely, starting from a superspace \( V = V_0 \oplus V_1 \) endowed with a \( \mathbb{C} \)-basis \( v_1, \ldots, v_n \) such that each \( v_i \) is either even (\( v_i \in V_0 \)) or odd (\( v_i \in V_1 \)), one may define the quantum affine superalgebras \( U_\varnothing(L\mathfrak{g}(V)), U_\varnothing(L\mathfrak{s}(V)) \) as well as the super Yangians \( Y_\varnothing(\mathfrak{g}(V)), Y_\varnothing(L\mathfrak{s}(V)) \) (we note that for \( m = n \), the notation \( \mathfrak{sl}(V) \) is misleading and should be replaced by the associated Dynkin diagram of the Lie superalgebra \( A(n,n) \)). We note that the definitions for \( \mathfrak{g}(V) \) are based on the RTT approach of [FRT], while the definitions for \( \mathfrak{sl}(V) \) are based on the new Drinfeld presentation of [D1]. Applying the approach of [DF], we shall relate the above definitions arising via the RTT and new Drinfeld presentations. The resulting positive subalgebras \( U_\varnothing^{>}(L\mathfrak{g}(V)) \simeq U_\varnothing^{>}(L\mathfrak{sl}(V)) \) and \( Y_\varnothing^{>}(\mathfrak{g}(V)) \simeq Y_\varnothing^{>}(\mathfrak{sl}(V)) \) are generated by \( \{e_{i,r}\}_{r \in \mathbb{Z}} \) and \( \{e_{i,r}\}_{r \in \mathbb{N}} \), respectively. Here \( I, \Delta^+, \{\alpha_i\}_{i=1}^{n-1} \) are defined as in Section 2, but the parity \( p(\beta) \in \mathbb{Z}_2 \) is defined via \( p(\alpha_j + \alpha_{j+1} + \ldots + \alpha_i) = p(\alpha_j) + p(\alpha_{j+1}) + \ldots + p(\alpha_i) \) with

\[
p(\alpha_i) = \begin{cases} 0, & \text{if } v_i \text{ and } v_{i+1} \text{ have the same parity} \\ 1, & \text{otherwise} \end{cases}
\]

The construction of the PBWD bases for \( U_\varnothing^{>}(L\mathfrak{sl}(V)), Y_\varnothing^{>}(\mathfrak{sl}(V)) \) and their integral forms \( \hat{U}_\varnothing^{>}(L\mathfrak{sl}(V)), \hat{Y}_\varnothing^{>}(\mathfrak{sl}(V)) \) is similar to Theorems 5.6, 5.10, 7.7, 7.8. The corresponding ordered PBWD monomials are defined analogously to (5.5, 7.6) with the indexing sets \( H, \hat{H}^+ \) defined via the same conditions (but resulting in different sets, due to a new choice of \( p: \Delta^+ \to \mathbb{Z}_2 \)).

The associated shuffle algebras \( S(V), W(V) \) and their integral forms \( \mathfrak{S}(V), \mathfrak{W}(V) \) are defined similar to \( S^{(m|n)}, W^{(m|n)}, \mathfrak{S}^{(m|n)}, \mathfrak{W}^{(m|n)} \). Their elements are symmetric in the variables \( \{x_i\} \) if \( p(\alpha_i) = 0 \) and skew-symmetric if \( p(\alpha_i) = 1 \), hence, the name boson-fermion shuffle algebras.

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