Charmonium Spectrum from Quenched Lattice QCD with Tadpole Improvement Action

LIU Da-Qing(刘大庆)*
Institute of High Energy Physics, Chinese Academy of Sciences, PO Box 918-4, Beijing 100039

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We report our lattice simulation on the charmonium spectra in the quenched approximation. Because the complete adjustment on all the nonperturbative parameters needs much calculation time, we only adjust two of them, but with some rescaling for mass splitting. After the rescaling, the calculated masses of meson are 3.030 GeV ($\eta_c$), 3.080 GeV ($J/\psi$), 3.546 GeV ($h_c$) and 3.412 GeV ($\chi_{c0}$) respectively, which is in agreement with the experimental results.

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The charmonium spectra supply a touchstone for any theory on heavy quark systems. There are many studies on them using lattice quantum chromodynamics (QCD) with the non-relativistic quantum chromodynamics (NRQCD) approach[1] or the Wilson action in clover improvement.[2–6]

We report here our lattice results on the charmonium spectra in an anisotropic lattice. Unlike Refs.[3–6], in which $\xi$ was set to be 2 or 3 or even $\xi = 1$, we set here $\xi = 6$, i.e. $a_s \approx a_t$. We use this choice to reduce the computation time by working on a coarser lattice. We also take the Sheikholeslami–Wohlert action[7,8] (SW action) as well as with the tadpole improvement.

As we know, in a continuum the kinetic mass $M_2$ and the static one $M_1$ for quark are the same, but on a finite lattice they are generally different. Since the scaling of whole mass is mainly determined by $M_1$ while the scaling of mass splitting is mainly determined by $M_2$, this implies the different scaling. If this difference is not large, it is reasonable to suppose that some rescaling schedule will compensate for the difference.

To obtain the correct charmonium spectra on the lattice, we should adjust clover coefficients nonperturbatively around the tree-level values, which will need much computation time. To reduce this time, we just take those coefficients as the tree-level one. To remedy this deviation, one also needs a rescaling schedule.

To perform the simulation, we adopt the tadpole improvement action for gluon as follows:

$$S_G = \beta \left\{ \frac{5}{3} P_{sp} + \frac{4}{3} \frac{\xi P_{sp}}{u_s^4} - \frac{P_{sx}}{12} \frac{\xi P_{sx}}{u_s^4} \right\},$$

where $P_{sp}$ ($P_{st}$) is the sum of all spatial (temporal) plaquettes, $P_{sx}$ represents the sum of 2×1 spatial rectangular Wilson loops, and $P_{sx}$ represents the sum of the Wilson loops with two spatial and one temporal link. Here $\xi$ is the aspect ratio:

$$\xi = \frac{a_s}{a_t}.$$ (2)

The accuracy of this action, as many authors have pointed out,[9] is $O(a_s^4, a_t^2)$ for an on-shell quantity, while the accuracy of the Wilson gluon action is $O(a^2).$

As suggested by Lüscher and Weisz,[10] it is better to take the on-shell improvement scheme for quarks. Since we consider heavy quarks here, we adopt the approach proposed in Ref.[11], which illuminates how to improve the on-shell quantity for heavy quarks in Hamilton form as well as in action form.

We take the action as

$$S_q = \sum_x \tilde{\psi}(x) \left\{ m_0 a_s + \frac{1}{2} \xi \gamma_4 \nabla_4 \psi + \sum_i \gamma_i \nabla_i \left( 1 - \frac{1}{6} \Delta_i \right) - \xi_1 \Delta_4 \psi - \frac{1}{\xi_3} \sum_i \Delta_i - \frac{1}{12} \Delta_i^2 \right\} \psi(x),$$

$$\nabla_\mu \psi(x) = u_\mu(x) \psi(x + \hat{\mu}) - u_\mu^\dagger(x - \hat{\mu}) \psi(x - \hat{\mu}),$$

$$\Delta_\mu \psi(x) = u_\mu(x) \psi(x + \hat{\mu}) - u_\mu^\dagger(x - \hat{\mu}) \psi(x - \hat{\mu}) - 2 \psi(x),$$ (3)

where $m_0$ is the bare mass of the quark, and $u$ are the link variables. The following is the determination of the parameters $\xi_1, \xi_3, C_E$ and $C_B$ in Eq.(3).

The propagators for quark are

$$\langle \psi(t', p') \bar{\psi}(t, p) \rangle = (2\pi)^3 \frac{\delta(p' - p)}{C(t' - t, p)} C(t, p),$$ (4)

where

$$C(t, p) = \frac{e^{-E|t|}}{\sinh E}$$ (5)

* Email: liudq@mail.ihep.ac.cn
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with the application of the residue theorem. Because the interaction terms only change vertices, it is not needed to consider the effect of interaction terms when we study propagators. We notice that $Z(p)$ is the time-independent function in Eq. (5). The energy $E$ of the quark with momentum $p$ is determined by

$$G^2 + w^2 + \xi_1 = 2G\xi_1 \cosh E\alpha_t,$$

$$G = m_0 + \xi_1 + \frac{1}{\xi_3} \sum_i \left(1 - \cos p_i + \frac{1}{6}(1 - \cos p_i)^2\right),$$

$$w^2 = \frac{1}{9} \sum_i \left(\frac{1}{6}(1 - \cos p_i)^2\right)^2.$$

To reach the above equation, we set $a_s = 1$. One may first expand $G$ and $w^2$ according to the power of $p_i$:

$$G = m_0 + \xi_1 + \frac{p_i^2}{2\xi_3} + O(p^4),$$

$$w^2 = \sum p_i^2 + O(p^6).$$

At $m_0 = 0$, one has

$$(\xi_1 E\alpha_t)^2 + O(a_t^4) = \sum p_i^2 + O(p_t^4),$$

where $a_t = 1/\xi_1$. Because of Eq. (8), we choose $\xi_1 = \xi$. For each particle, for instance, $J/\psi$, we can define the effective velocity of photon (EVP) $c_{\text{eff}}$ of the particle by fitting dispersion relation

$$E^2 \left(\sum p_i^2\right) = m^2 + c_{\text{eff}}^2 \sum p_i^2,$$

where $E$ is the energy and $p$ is the momentum of the particle. We then tune parameter $\xi_3$ to make

$$c_{\text{eff}} \approx 1$$

for particle $J/\psi$.

If we take the two-parameter on-shell transformation, we shall certainly have two free parameters. One of them may be chosen to obtain the coefficient of spin-orbit interaction $c_{so} = 0$ (see more detail in Ref. [11]) conventionally. For the others, such as $C_E$, there are a few choices. For instance, one may choose $C_E = 1$ or $C_E = 0^{[12]}$ in NRQCD.

If we follow the method developed in Refs. [4,11], we can obtain the tree-level parameters as

$$C_B = 1,$$

$$\xi_3 = \xi \frac{1}{1 + m_t} \left(\ln(1 + m_t) - \frac{2}{m_t(2 + m_t)}\right)^{-1},$$

$$C_E = \frac{\xi}{\xi_3(1 + m_t)} + \left(\frac{\xi}{\xi_3(1 + m_t)}\right)^2 \frac{m_t(2 + m_t)}{4},$$

where $m_t = m_0 a_s / \xi$. To obtain nonperturbative parameters we should adjust these parameters with the same physical principle which leads to Eq. (11). For instance, to obtain the nonperturbative $\xi_3$ we adjust $\xi_3$ to make Eq. (10) be satisfied.

We select $C_B = 1$ in our simulation. Noticing that $C_E = \xi_3 / \xi_3$ at $m_t = 0$ we just take a mass-independent setting for $C_E$ for simplicity:

$$C_E' = \frac{\xi}{\xi_3}.$$  (12)

The difference between Eqs. (11) and (12) is not large in the simulation. For example, at $m_0 a_s = 2.4$ we should have $C_E / C_E' = 0.87$ if $\xi_3$ is determined by Eq. (11) or $C_E / C_E' = 0.94$ if $\xi_3 = 3.3$ (see Table 3). The choices in Eq. (11) and/or Eq. (12) are at tree level and $C_E$ shall deviate from its nonperturbative value. The contribution of dynamical fermion is in fact neglected, and we shall discuss it in more detail in the future. Here we suppose that a mass splitting rescaling will compensate for the deviation.

Thus we have only two parameters $\xi_3$ and $m_0$ to be determined in the simulation: one is adjusted to give the correct energy-momentum relation for $J/\psi$ and the other will be verified to determine the lattice spacing $a_s (a_t)$.

In this study, we calculate five meson masses of charmonium quarkonia, i.e. $\eta_c(1S_0)$, $J/\psi(3S_1)$, $h_c(1P_1)$, $\chi_{c0}(3P_0)$ and $\chi_{c1}(3P_1)$ and their excited states. The operators for definite meson are chosen to be the standard ones.

To determine $\xi_3$, we calculate the energy for four low-lying momenta $\frac{L^2}{2p} p a_s = (0,0,0), (0,0,1), (0,1,1)$ and $(1,1,1)$ of the particle $J/\psi$ and fit them by

$$(E(|p|) a_t = (E(0) a_t)^2 + \frac{c_{\text{eff}}^2}{\xi_3} (|p| a_s)^2).$$

We tune $\xi_3$ to make the relation $c_0 = 1$ be satisfied to accuracy 1%.

There are several types of the schemes to determine $m_0$, or equivalent $a_t$. Here we may take the ratio of the masses of $S$-state meson:

$$\frac{m(\psi')}m(\eta_c) = 0.987.$$  (14)

Then, we extract $a_t$ by

$$a_t = \frac{m(1S)}{m(1S)_{\text{exp}}},$$  (15)

where $m(1S) = (m(1^1S) + 3m(1^3S))/4$ is the measured average mass of $1^1S$ and $1^3S$ state on the lattice, and $m(1S)_{\text{exp}}$ is taken as the experimental result of 3.0676 GeV.

Here we iterate $m_0$ and $\xi_3$ to make both Eqs. (13) and (14) hold at the same time.
Because of our computational limitation, our simulation is carried out on a $L^3 \times T = 6^3 \times 36$ lattice with $\xi = 6.0$. The parameters are listed in Table 1.

Table 1. Parameters in the simulation with the scale set by Eq. (14).

| $\beta$ | $u_0^4$ | sweep/config | configs. |
|---------|---------|--------------|---------|
| 2.5     | 0.423   | 40           | 100     |
| 2.8     | 0.463   | 40           | 100     |
| 3.0     | 0.486   | 40           | 100     |

For a definite meson operator we measure its correlation function as

$$C_{\text{state}}(t, \mathbf{p}) = \sum_{x} \langle \bar{\psi}(x, t) \Gamma \psi(x, t) \bar{\psi}(0, 0) \rangle e^{-i \mathbf{p} \cdot \mathbf{x}}.$$  \hfill (16)

The energy can be extracted by $n$-hyperbolic cosine ansatz ($n$ is the number of the extracted masses and $T$ is the period in the time direction):

$$C_{\text{state}}(t) = \sum_{j=0}^{n-1} a_j \left( e^{-m_j t} + e^{-m_j (T-t)} \right),$$ \hfill (17)

which means that we should find parameters $a(j)$ and $m(j)$ to minimize the function

$$\sum_{i=l_i}^{l_f} W_i \left( C(t_i) - \sum_{j=0}^{n-1} a_j \left( e^{-m_j t_i} + e^{-m_j (T-t_i)} \right) \right)^2 / C(t_i)^2,$$ \hfill (18)

where $W_i \propto C(t_i)^2 / \Delta C(t_i)$ is the weight of the correlation $C(t_i)$.

We choose $n = 2$ for all the measured particles in this work. In this choice, the calculated masses of the first excited states should be contaminated by the higher states. Therefore, these calculated masses are larger than the experimental ones.

For each pair of $l_i$ and $l_f$ in Eq. (18), there are corresponding parameters $m_j$ and $a_j$. Then, one may choose a pair of $l_i$ and $l_f$ and the corresponding parameters $m_j$ and $a_j$ to make the function

$$\sum_{j} \left( \frac{\Delta^2_{m_j}}{m_j^2} + \frac{\Delta^2_{a_j}}{a_j^2} \right)$$

reach its minimal points for all the sets. Since nearly all the minimal points for a definite operator are approached at $l_i = 1$ and $l_f = T - 1$, we always set $l_i = 1$ and $l_f = T - 1$.

In Table 2, we list the results of energy vs momentum of the typical particles and their dispersion relations using Eq. (13) at $\beta = 3.0$ and $m_0 a_s = 1.06$.

From the simulation we find that the difference between EVPs for the particle $\eta_c$ and $J/\psi$ is always not larger than 2%. This is obvious since their mass splitting is mainly due to the spin splitting. A similar result has been obtained by other authors.\(^{[3-5]}\)

| $\left( \frac{L}{2\pi} p_a \right)^2$ | 0 | 1 | 2 | 3 |
|-------------------------------------|---|---|---|---|
| $\eta_c$                           | 0.276(3) | 0.309(2) | 0.340(2) | 0.369(2) |
| $J/\psi$                           | 0.286(1) | 0.317(1) | 0.349(1) | 0.3775(13) |
| $h_c$                              | 0.409(13) | 0.400(10) | 0.416(8) | 0.432(7) |
| $X_{c1}$                           | 0.401(9) | 0.414(8) | 0.432(7) | 0.448(7) |
| $X_{c0}$                           | 0.380(11) | 0.405(11) | 0.428(12) | 0.450(12) |

From the data we see that the EVPs for all the particles except mesons $\eta_c$ and $J/\psi$ are usually smaller than unity, which means that the kinetic mass $M_2$ is larger than the static mass $M_1$. To obtain the correct $\chi_f$ mass splitting, one should take a mass-dependence schedule of $C_E$, which leads to a programme of adjustment of three parameters: $m_0 a_s$, $\xi_3$ and $C_E$.

To combine the scale of $M_1$, $M_2$ and $M_E$ (mass reflects the spin–orbit interaction) and decrease the number of adjusted parameters, we adopt such progress: we adjust only two parameters $m_0 a_s$ and $\xi_3$ with the mass splitting rescaling. We take Eq. (14) to determine $m_0 a_s$ and measure the mass of particle $\eta_c$ and $J/\psi$. For other particles, since we should consider the spin–orbit splitting, we redefine its mass as follows.

We rescale the mass splitting between $1^3P_1$ and averaged $1S$ state as

$$m(\chi_{c1}) - m(1S) = 0.443 \text{ GeV.}$$ \hfill (19)

Therefore, the modified mass for a definite meson is

$$m_{\text{mod}} = 0.443 \frac{m_0 a_s - m(1S) a_t}{m(\chi_{c1}) a_t - m(1S) a_t} + m(1S) \text{ (GeV)},$$ \hfill (20)

where $m_0 a_s$ is the dimensionless measured mass on the lattice and $a_t$ is extracted by Eq. (15).

The simulation parameters are listed in Table 3. Our results for scale are not inconsistent with the result obtained from the Sommer scale $\sigma_0$ in the calculation of the gluon spectra\(^{[14]}\) although they are obtained in a very different aspect.

We list our modified charmonium spectra in Table 4, which are, as well as in other references, in
agreement with the experiment. From the data we find that at finite lattice spacing, the ratio \( \frac{m(J/\psi)}{m(\eta_c)} \) is smaller than the experimental one. To obtain the correct ratio one may decrease \( m_0 \) and therefore achieve a decreased lattice spacing \( a_s \). An estimation for the decreased \( m_0 \) shows that it approximately leads to the same result as in Ref. [14]. To decrease the calculation, we take the scale determined by Eq. (14) and notice that the ratios of \( a_s \) obtained by us to that obtained byMorningstar and Peadon are nearly the same at different \( \beta \), i.e. the ratio is \( 1.6 \) at \( \beta = 3.0 \) and \( 1.7 \) at \( \beta = 2.5 \).

Table 3. Parameters for the simulation with the scale set by Eq. (14).

| \( \beta \) | \( \xi_3 \) | \( m_4 \alpha_s \) | \( a_s \) (fm) | \( L_0 \) (fm) | EVP \( (J/\psi) \) | EVP \( (\eta_c) \) |
|----------|--------|---------|-------------|-----------|----------------|----------------|
| 2.5      | 3.93   | 0.322   | 1.93        | 1.000 (10)| 1.22 (21)     |                 |
| 2.8      | 3.92   | 0.258   | 1.55        | 1.006 (7) | 1.018 (15)    |                 |
| 3.0      | 6.38   | 0.208   | 1.25        | 1.005 (8) | 1.003 (17)    |                 |

Table 4. Results of the quarkonium mass \( M \) and the mass splitting \( \Delta M \) in units of GeV at \( \xi = 6.0 \) with the scale set by Eq. (14).

| state \( \beta \) | \( \beta = 2.5 \) | \( \beta = 2.8 \) | \( \beta = 3.0 \) | \( a_s \rightarrow 0 \) | Exp. (GeV) |
|-----------------|-----------------|-----------------|-----------------|-----------------|----------|
| \( \eta_c \)    | 3.033 (2)       | 3.032 (2)       | 3.030 (2)       | 3.030 (3)       | 2.980    |
| \( J/\psi \)    | 3.080 (7)       | 3.0795 (7)      | 3.080 (1)       | 3.080 (2)       | 3.097    |
| \( h_c \)       | 3.533 (20)      | 3.500 (10)      | 3.539 (6)       | 3.546 (17)      | 3.526    |
| \( \chi_{3s} \)| 3.511 (14)      | 3.511 (7)       | 3.511 (5)       |                 | 3.511    |
| \( \chi_{6s} \)| 3.471 (20)      | 3.447 (11)      | 3.436 (7)       | 3.412 (17)      | 3.415    |
| \( \pi_c \)     | 4.3094 (4)      | 4.3473 (3)      | 3.982 (2)       | 3.766 (5)       | 3.594    |
| \( \psi' \)     | 4.3162 (2)      | 4.3591 (1)      | 3.993 (1)       | 3.795 (2)       | 3.686    |
| \( h_c' \)      | 4.7484 (15)     | 4.964 (10)      | 4.633 (8)       | 4.564 (18)      |         |
| \( \chi_{3s} \)| 4.8893 (13)     | 5.079 (8)       | 4.689 (6)       | 4.534 (15)      |         |
| \( \chi_{4s} \)| 4.980 (21)      | 5.145 (13)      | 4.721 (10)      | 4.510 (23)      |         |

Comparing our results with those in Ref. [6], we find that our result for hyperfine splitting is smaller than that in Ref. [6]. Probably, this is caused by too large a lattice spacing and/or the contamination of hyperfine splitting by the spin-orbit terms in the determination of \( m_0 a_s \) in Eq. (14).

For the known excited states, our results, as those obtained in Ref. [4], are larger than the experimental ones. For unknown states, we need further exploration to explain why our results are also larger than those in Ref. [4].

Following Ref. [11], one should do this rescaling between \( \chi_J \) states. From the results, we find that our rescaling schedule also obtains the correct \( \chi_J \) states.

Since mass splitting \( \Delta m \) is not mainly dictated by the static mass \( M_1 \), Ref. [11] suggested the following strategy: forget about \( M_1 \) and adjust the bare mass so that the kinetic mass \( M_2 \) takes the physical value. Here we find that one may adjust parameters or rescale mass splitting to make mass splitting and the whole mass at the same scale. This rescaling combines the scale of the whole mass \( M \) and the mass splitting \( \Delta M \). After the rescaling, we obtain the results, which are in agreement with the experimental ones with the adjustment of only two parameters.

It is easy to see that the calculation time of the adjustment of three parameters is at least 3 \~{} 4 times larger than that of the adjustment of only two parameters. Therefore, this rescaling scheme is efficient.

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