Construction of surfaces with large systolic ratio

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Abstract

Let \((M, g)\) be a closed, oriented, Riemannian manifold of dimension \(m\). We call a systole a shortest non-contractible loop in \((M, g)\) and denote by \(\text{sys}(M, g)\) its length. Let \(\text{SR}(M, g) = \frac{\text{sys}(M, g)^m}{\text{vol}(M, g)}\) be the systolic ratio of \((M, g)\). Denote by \(\text{SR}(k)\) the supremum of \(\text{SR}(S, g)\) among the surfaces of fixed genus \(k \neq 0\). In Section 2 we construct surfaces with large systolic ratio from surfaces with systolic ratio close to the optimal value \(\text{SR}(k)\) using cutting and pasting techniques. For all \(k_i \geq 1\), this enables us to prove:

\[
\frac{1}{\text{SR}(k_1 + k_2)} \leq \frac{1}{\text{SR}(k_1)} + \frac{1}{\text{SR}(k_2)}.
\]

We furthermore derive the equivalent intersystolic inequality for \(\text{SR}_h(k)\), the supremum of the homological systolic ratio. As a consequence we greatly enlarge the number of genera \(k\) for which the bound \(\text{SR}_h(k) \geq \text{SR}(k) \geq \frac{4 \log(k^2)}{\pi^2 k}\) is valid and show that \(\text{SR}_h(k) \leq \frac{(\log(195 k^2)+8)^2}{\pi(k-1)^2}\) for all \(k \geq 76\). In Section 3 we expand on this idea. There we construct product manifolds with large systolic ratio from lower dimensional manifolds.

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1 Introduction

In the present article we denote by a manifold a closed, oriented, Riemannian manifold \((M, g)\) of dimension \(m \geq 2\). We denote by \((S, g)\) a Riemannian surface.

A systole of \((M, g)\) is a shortest non-contractible loop. We denote by \(\text{sys}(M, g)\) its length. Normalizing by the volume of \((M, g)\) we obtain

\[
\text{SR}(M, g) = \frac{\text{sys}(M, g)^m}{\text{vol}(M, g)},
\]

the systolic ratio of \((M, g)\), which is invariant under scaling of \((M, g)\). Let

\[
\text{SR}(k) = \sup \{\text{SR}(S, g) \mid (S, g) \text{ Riemannian surface of genus } k \neq 0\}
\]

be the optimal systolic ratio in genus \(k\).

The exact value of \(\text{SR}(k)\) is only known for \(k = 1\). It was proven by Loewner (see [Pu], p. 71) that \(\text{SR}(1) = \frac{2}{\sqrt{3}}\). For large \(k\) it is known that (see [KS2])

\[
K_1 \frac{\log(k)^2}{k} \leq \text{SR}(k) \leq K_2 \frac{\log(k)^2}{k},
\]

(1)
where \( K_1 \) and \( K_2 \) are universal, but unknown constants. The best known upper bound is stated in [KS2], Theorem 2.2:

\[
\text{SR}(k) \leq \frac{1}{\pi} \frac{\log(k)^2}{k} (1 + o(1)), \text{ when } k \to \infty.
\]

(2)

It was furthermore shown in [BS] that there exists an infinite sequence of genera \((k_i)_i\), such that

\[
\frac{4}{9\pi} \frac{\log(k_i)^2 - c_0}{k_i} \leq \text{SR}(k_i),
\]

(3)

where \( c_0 \) is a fixed constant. This result comes from the study of hyperbolic surfaces, i.e. of constant curvature \(-1\). More families of hyperbolic surfaces satisfying the above inequality can be found in [KSV1], [KSV2] and [AM].

In the case of a surface \((S, g)\) one can also define the homological systole, which is a shortest homologically non-trivial loop in \((S, g)\). This is a shortest non-contractible loop that does not separate \((S, g)\) into two parts. We denote by \(\text{sys}_h(S, g)\) its length and define \(\text{SR}_h(S, g) = \frac{\text{sys}_h(S, g)^2}{\text{vol}(S, g)}\) as the homological systolic ratio. Let

\[
\text{SR}_h(k) = \sup\{\text{SR}_h(S, g) \mid (S, g) \text{ Riemannian surface of genus } k \neq 0\}
\]

be the optimal homological systolic ratio in genus \(k\).

It follows immediately that for any surface \((S, g)\)

\[
\text{sys}(S, g) \leq \text{sys}_h(S, g), \text{ hence } \text{SR}(k) \leq \text{SR}_h(k).
\]

Hence \(\text{SR}_h(k)\) has the same lower bound as \(\text{SR}(k)\) and it follows from [Gr2], Theorem 2.C that \(\text{SR}_h(k)\) satisfies an upper bound of order \(\frac{\log(k)^2}{k}\). In this article we show that \(\text{SR}(k)\) is smaller than \(\frac{\log(100k^2)+8}{\pi(k-1)}\) (see Theorem 1.3-3).

An open question is, whether \(\text{SR}(\cdot)\) and \(\text{SR}_h(\cdot)\) are monotonically decreasing functions with respect to the genus. Though we can not prove or disprove this result, we can at least show the following intersystolic inequalities:

**Theorem 1.1.** Let \(\text{SR}(k)\) and \(\text{SR}_h(k)\) be the supremum of the systolic ratio and the homological systolic ratio among all closed, oriented, Riemannian surfaces of genus \(k \geq 1\). Then for all \(k_i \geq 1\)

1. \(\frac{1}{\text{SR}(k_1+k_2)} \leq \frac{1}{\text{SR}(k_1)} + \frac{1}{\text{SR}(k_2)} \) or equally \(\text{SR}(k_1+k_2) \geq \left(\frac{1}{\text{SR}(k_1)} + \frac{1}{\text{SR}(k_2)}\right)^{-1}\).

2. \(\frac{1}{\text{SR}_h(k_1+k_2)} \leq \frac{1}{\text{SR}_h(k_1)} + \frac{1}{\text{SR}_h(k_2)} \) or equally \(\text{SR}_h(k_1+k_2) \geq \left(\frac{1}{\text{SR}_h(k_1)} + \frac{1}{\text{SR}_h(k_2)}\right)^{-1}\).

In [BB], p. 159, Babenko and Balacheff provide the equivalent inequality of Theorem 1.1.1 for connected sums of manifolds of dimension \(m \geq 3\). The above inequalities imply that \(\text{SR}(k)\) and \(\text{SR}_h(k)\) are at least of order \(\frac{1}{k}\).

Specializing on metrics \(\text{hyp}\) of constant curvature \(-1\), we obtain the optimal systolic ratio for compact hyperbolic surfaces:

\[
\text{SR}(k, \text{hyp}) = \sup\{\text{SR}(S, \text{hyp}) \mid (S, \text{hyp}) \text{ hyperbolic surface of genus } k \neq 0\},
\]

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and define in an analogous manner the optimal homological systolic ratio for compact hyperbolic surfaces. In this case the supremum is attained (see [Ma]) and much more is known about the corresponding maximal surfaces than in the general case (see [Ak], [AM], [Ba], [Ge], [Sc1], [Sc2] and [Sc3]). Especially

\[ \text{SR}(k, \text{hyp}) = \text{SR}_h(k, \text{hyp}) \] (see [Pa], Theorem 1.1).

This equation enables us to show the first statement of the following theorem:

**Theorem 1.2.** Let \( \text{SR}(k) \) and \( \text{SR}_h(k) \) be the supremum of the systolic ratio and the homological systolic ratio among all closed, oriented, Riemannian surfaces of genus \( k \geq 1 \) and \( \text{SR}(k, \text{hyp}) \) the supremum of the systolic ratio among compact hyperbolic surfaces of genus \( k \geq 2 \). We have:

1. \( \frac{1}{\text{SR}(k+1)} - \frac{\sqrt{3}}{2} \leq \frac{1}{\text{SR}(k)} \leq \frac{1}{\text{SR}(k+1, \text{hyp})} + \frac{1}{\pi} \).
2. \( \frac{1}{\text{SR}_h(k+1)} - \frac{\sqrt{3}}{2} \leq \frac{1}{\text{SR}_h(k)} \leq \frac{1}{\text{SR}_h(k+1)} + \frac{1}{\pi} \).
3. \( \frac{\text{SR}(k+1)}{\text{SR}(k)} \leq 1 + \frac{\text{SR}_h(k+1)}{\pi} \).

**Theorem 1.1** and **1.2** are obtained by constructing surfaces with large systolic ratio from surfaces with systolic ratio close to the optimal value \( \text{SR}(k) \), \( \text{SR}_h(k) \) or \( \text{SR}(k, \text{hyp}) \) using cutting and pasting techniques. As a result, we obtain the following statement: If \( (S, g) \) is a surface of genus \( k \), such that \( \text{SR}(S, g) \geq \frac{4}{9\pi} \log^2 \frac{k}{c_0} \), then

\[ \text{SR}(k + j_1) \geq \frac{4}{9\pi} \log^2 \left( \frac{k + j_1}{j_1} \right)^2 - c_1 \quad \text{and} \quad \text{SR}(j_2 \cdot k) \geq \frac{4}{9\pi} \log^2 \left( \frac{j_2 \cdot k}{j_2} \right)^2 - c_2 \quad \text{for all} \ \ j_1, j_2 \ll k. \]

Here the second inequality follows from **Theorem 1.1-1** by induction. This suggests that the bound \( \text{SR}_h(k) \geq \text{SR}(k) \gtrsim \frac{4}{9\pi} \log^2 \frac{k}{c_0} \) is valid for a large number of genera.

Furthermore, **Theorem 1.1** and **1.2** allow us to provide new lower bounds for \( \text{SR}(k) \) for small genera \( k \). In **Table 1** we give a summary of Riemannian surfaces of genus \( 1 \leq k \leq 25 \) with maximal known systolic ratio. Most of these are constructed from the examples presented in [Cas], [CK], [KSV1], [Sc1] and [Sc3] using **Theorem 1.1-1**. As the proof is constructive, the lower bound for \( \text{SR}(k) \) is attained in the thus constructed surfaces. The best known upper bounds for \( \text{SR}(k) \) in **Table 1** are due to the following sources (see also [Ka], Chapter 11 for a summary):

- genus 2: \( \text{SR}(2) \leq \frac{2}{\sqrt{3}}, \ [KST], \ \text{Theorem 1.3} \)
- genus 3-16: \( \text{SR}(k) \leq \frac{4}{3} \) for \( k \neq 1, \ [Gr1], \ \text{Corollary 5.2.B} \)
- genus 17-25:
  \[ \text{for all} \quad r \in \left( 0, \frac{1}{8} \right), \quad \frac{\log(2r^2 \text{SR}(k))^2}{4\pi \text{SR}(k) \left( \frac{1}{2} - 4r \right)^2 (k - 1)} \geq 1, \quad \ [KS2], \ [Ka], \ \text{inequality (11.4.1)}. \]

(4)
Revisiting the ideas of the proof of [Gr2], Theorem 2.C, we also show that

**Theorem 1.3.** Let $SR(k)$ and $SR_h(k)$ be the supremum of the systolic ratio and the homological systolic ratio among all closed, oriented, Riemannian surfaces of genus $k \geq 1$. Then

1. $SR_h(k) \leq \frac{4}{3}$ for all $k \geq 1$ and $SR_h(k) \leq \frac{2}{\sqrt{3}}$ for all $k \geq 20$.

2. $SR(2) = SR_h(2)$ and $SR(3) \geq SR_h(3) - 0.03$.

3. $SR_h(k) \leq \frac{(\log(195k)+8)^2}{\pi(k-1)}$ for all $k \geq 76$.

In fact using the same arguments as in the proof of **Theorem 1.3**, it can be shown that for $1 \leq k \leq 25$ $SR_h(k)$ satisfies the same upper bound as $SR(k)$ in Table 1.

| genus $k$ | surface (name and/or constructed from) | lower bound for $SR(k)$ | upper bound for $SR(k)$ and $SR_h(k)$ | reference for the lower bound |
|-----------|--------------------------------------|--------------------------|----------------------------------------|-----------------------------|
| 1         | $T_2^2$ hex                          | 1.15                     | 1.15                                   | Pu                         |
| 2         | $R_2$                                | 0.80                     | 1.15                                   | [CK], Fig. 2.1             |
| 3         | $R_3$                                | 0.66                     | 1.33                                   | Ca                         |
| 4         | $R_4$                                | 0.60                     | 1.33                                   | [CK], Fig. 2.1             |
| 5         | $S_5$                                | 0.48                     | 1.33                                   | Sc3                        |
| 6         | $I_6$                                | 0.42                     | 1.33                                   | Cas                        |
| 7         | $H_7$                                | 0.45                     | 1.33                                   | [KSV1]                     |
| 8         | $R_8$ (via $T_2^2$ hex, $H_7$)       | 0.32                     | 1.33                                   | Th. 1.1-1                  |
| 9         | via $R_2, H_7$                       | 0.29                     | 1.33                                   | Th. 1.1-1                  |
| 10        | via $R_3, H_7$                       | 0.27                     | 1.33                                   | Th. 1.1-1                  |
| 11        | $I(x|z)$                             | 0.28                     | 1.33                                   | Sc1                        |
| 12        | via $S_5, H_7$                       | 0.23                     | 1.33                                   | Th. 1.1-1                  |
| 13        | via $H_{14}$                         | 0.23                     | 1.33                                   | Th. 1.2-1                  |
| 14        | $H_{14}$                             | 0.25                     | 1.33                                   | [KSV1]                     |
| 15        | via $T_2^2$ hex, $H_{14}$            | 0.21                     | 1.33                                   | Th. 1.1-1                  |
| 16        | via $H_{17}$                         | 0.26                     | 1.33                                   | Th. 1.2-1                  |
| 17        | $H_{17}$                             | 0.29                     | 1.27                                   | [KSV1]                     |
| 18        | via $T_2^2$ hex, $H_{17}$            | 0.23                     | 1.22                                   | Th. 1.1-1                  |
| 19        | via $R_2, H_{17}$                    | 0.21                     | 1.16                                   | Th. 1.1-1                  |
| 20        | via $R_3, H_{17}$                    | 0.20                     | 1.12                                   | Th. 1.1-1                  |
| 21        | via $R_4, H_{17}$                    | 0.19                     | 1.08                                   | Th. 1.1-1                  |
| 22        | via $S_5, H_{17}$                    | 0.18                     | 1.04                                   | Th. 1.1-1                  |
| 23        | via $I_6, H_{17}$                    | 0.17                     | 1.00                                   | Th. 1.1-1                  |
| 24        | via $H_7, H_{17}$                    | 0.17                     | 0.97                                   | Th. 1.1-1                  |
| 25        | via $R_8, H_{17}$                    | 0.15                     | 0.94                                   | Th. 1.1-1                  |

Table 1: Upper and lower bounds for $SR(k)$ and $SR_h(k)$ in genus $1 \leq k \leq 25$.

The idea of Section 2 is to construct new surfaces with large systolic ratio from extremal surfaces. In Section 3 we expand on this idea. If $(M, g)$ and $(N, h)$ are two manifolds of dimension $m$ and
n, respectively, then
\[ \text{sys}(M \times N, g \times h) = \min(\text{sys}(M, g), \text{sys}(N, h)). \]

This enables us to construct manifolds with large systolic ratio from lower dimensional manifolds with large systolic ratio. We illustrate the consequences of this equation by two examples. First we construct \( n \)-dimensional Euclidean and general product-tori with large systolic ratio from lower dimensional ones, then we construct product manifolds of surfaces and tori. This enables us to prove Theorem 3.3

**Theorem 1.4.** Let \( \gamma_n \) be Hermite’s constant for flat tori in dimension \( n \), then
1. \( \gamma_{m+n} \geq \gamma(m) \cdot \gamma(n) \).
2. \( \gamma_{2n} \geq \gamma_n \).
3. \( \gamma(n) \geq \gamma_{n+1} \geq \gamma(n+1) \).

More refined methods can be applied to find lower bounds for Hermite’s constant for flat tori (see [Ma], p. 92) based on similar ideas. These lead to the laminated lattices, which provide the best known lower bounds for \( \gamma_n \) in certain dimensions (see [Ma], Table 14.4.1). However, Theorem 3.3 provides practical a priori bounds. Notably, the lower bound in Theorem 3.3 completes the known upper bound, which is Mordell’s inequality. The same inequalities hold for manifolds homeomorphic to Euclidean tori. These are stated in Theorem 3.4.

Furthermore in Theorem 3.5, we prove:

**Theorem 1.5.** Let \( b(M, g) = \sum_{i=0}^{m} b_i(M, g) \) be the sum of the Betti numbers of a manifold \((M, g)\) of dimension \( m \). Then in each dimension \( m \geq 3 \) there exist manifolds \( R_k^m = (S \times \mathbb{T}^{m-2}, g \times g_E) \), that are product manifolds of a surface \((S, g)\) of genus \( k \gg 2^m \) and an Euclidean torus \((\mathbb{T}^{m-2}, g_E)\), such that
\[
C_{1,m} \cdot \frac{\log(b(R_k^m))}{b(R_k^m)} \leq \text{SR}(R_k^m) \leq C_{2,m} \cdot \frac{\exp(C_{3,m} \sqrt{\log(b(R_k^m))})}{b(R_k^m)}.
\]

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2 Construction of surfaces with large systolic ratio

**proof of Theorem 1.1**

1. \( \text{SR}(k_1 + k_2) \geq \left( \frac{1}{\text{SR}(k_1)} + \frac{1}{\text{SR}(k_2)} \right)^{-1} \).

Let \( \epsilon_1 > 0 \) be a positive real number. For \( i \in \{1, 2\} \), let \((S_i, g_i)\) be a surface of genus \( k_i \geq 1 \), which satisfies
\[
\text{SR}(S_i, g_i) = \text{SR}(k_i) - \epsilon_1 \quad \text{and} \quad \text{sys}(S_i, g_i) = 1, \quad \text{hence} \quad \text{vol}(S_i, g_i) = \frac{1}{\text{SR}(S_i, g_i)}. \tag{5}
\]
To prove our theorem we construct a new surface $(S_c, g_c)$ of genus $k_1 + k_2$ from the surfaces $(S_1, g_1)$ and $(S_2, g_2)$ such that
\[
SR(S_c, g_c) = \left( \frac{1}{SR(S_1, g_1)} + \frac{1}{SR(S_2, g_2)} \right)^{-1}.
\]
As $(S_c, g_c)$ has genus $k_1 + k_2$, we obtain the inequality of Theorem 1.1-1 from the fact that $\epsilon_1$ can be chosen arbitrarily small.

We first construct $(S_c, g_c)$. For fixed $i$ let $a_i$ be a systole of $(S_i, g_i)$. We first divide each $a_i$ into two arcs, $b_i$ and $c_i$ of equal length. Then we cut $a_i$ along $c_i$ and call the surface obtained this way $(S_i^0, g_i)$. We denote by $\alpha_i$ the boundary curve of $(S_i^0, g_i)$. Let $\alpha_i^1$ and $\alpha_i^2$ be the two parts of $\alpha_i$ with common endpoints on $b_i$. We identify the boundary components of $(S_1^0, g)$ and $(S_2^0, g)$ in the following way
\[
\alpha_1^1 \sim \alpha_2^2 \quad \text{and} \quad \alpha_1^2 \sim \alpha_2^1
\]
to obtain a closed surface.
We denote the surface of genus $k_1 + k_2$ obtained according to this pasting scheme as $S_c = S_1^0 + S_2^0 \mod (6)$. (see Fig. 1)

We denote by $\alpha_c$ the curve, which is the image of $\alpha_1$ in $S_c$. As the metric on $S_c$, we take the metric of the parts to obtain a surface $(S_c, g_c)$ with the singularity, which is the curve $\alpha_c$.

![Figure 1: The surfaces $S_1^0$ and $S_2^0$ with identified boundaries.](image)

We now show that the length $\ell(\eta)$ of a non-contractible loop $\eta$ in $(S_c, g_c)$ satisfies
\[
\ell(\eta) \geq \text{sys}(S_1, g_1) = \text{sys}(S_2, g_2) = 1.
\]
As it is well-known that every non-contractible loop contains a simple non-contractible sub-loop, i.e. a non-contractible loop without self-intersection, of equal or shorter length, we assume that $\eta$ is a simple closed curve.
To prove that $\ell(\eta) \geq 1$ we distinguish two cases: either $\eta$ is contained in either $S_1^0 \subset S_c$ or $S_2^0 \subset S_c$ or not. Consider the first case.

**Case 1: $\eta$ is either contained in $S_1^0$ or contained in $S_2^0$**

We assume without loss of generality that $\eta$ is contained in $S_1^0$. We have to prove that $\ell(\eta) \geq 1$. Now if $\eta$ is non-contractible both in $S_1$ and in $S_1^0$, then $\eta$ is a non-contractible loop in $S_1$ and hence

$$\ell(\eta) \geq \text{sys}(S_1, g_1) = 1$$

and there is nothing to prove. Therefore it remains to prove the case, where $\eta$ is contractible in $S_1$, but non-contractible in $S_1^0$. It follows from surface topology that if $\delta$ is a closed curve that satisfies this condition, then

$$[\delta] = [\alpha_1]^l \in \pi_1(S_1^0), \text{ for some } l \in \mathbb{Z}.$$ 

As by assumption $\eta$ is additionally a simple loop it follows that $[\eta] = [\alpha_1]^{\pm 1}$. We assume without loss of generality that $[\eta] = [\alpha_1]$. We recall that $b_1 \subset S_1^0$ is the part of the systole $a_1 \subset S_1$, that is not cut in $S_1^0$. As $\eta$ runs around the cut in $S_1$, whose boundary is $\alpha_1$, it follows that there are two intersection points, $p_1$ and $p_2$ on $b_1$ (see Fig. 2), such that

- $p_1$ and $p_2$ divide $\eta$ into two parts, $\eta_1$ and $\eta_2$, such that
- $\eta_1$ is homotopic with fixed endpoints $p_1$ and $p_2$ to an arc $r_1\alpha_1^1 r_2$, where
- $r_1$ is the shorter arc on $b_1$ connecting $p_1$ and an endpoint of $\alpha_1^1$ and $r_2$ is the shorter arc on $b_1$ connecting $p_2$ and an endpoint of $\alpha_1^2$
- $\eta_2$ is homotopic with fixed endpoints to $r_1\alpha_1^2 r_2$.

![Figure 2: The surface $S_1^0$ with a curve $\eta$, such that $[\eta] = [\alpha_1]$ in $\pi_1(S_1^0)$](image)

We now show that

$$\ell(\eta_i) \geq \frac{1}{2} \quad \text{for} \quad i \in \{1, 2\}, \quad \text{hence} \quad \ell(\eta) \geq 1.$$

Let $b'$ be the arc of $b_1$ connecting $p_1$ and $p_2$. We have that $\ell(b') \leq \frac{1}{2}$. 

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Furthermore for \( i \in \{1, 2\} \)
\[
\ell(\eta_ib') = \ell(\eta_i) + \ell(b') \geq \ell(a_1) = 1.
\]
Because otherwise we could in \( S_1 \) replace \( a_1 \) by \( \eta_ib' \) to obtain a non-contractible loop shorter than \( a_1 \). But \( a_1 \) is the systole of \( S_1 \). A contradiction. Now as \( \ell(b') \leq \frac{1}{2} \), it follows that \( \ell(\eta_i) \geq \frac{1}{2} \) and hence our statement in (7). This concludes the proof in Case 1.

**Case 2:** \( \eta \) is not contained in either \( S_1^0 \) or \( S_2^0 \)

For fixed \( i \), we call a loop in \( S_c \) retractable into \( S_i^0 \subset S_c \) if and only if it is freely homotopic to a loop contained in \( S_i^0 \). We distinguish two subcases: either \( \eta \) is retractable into \( S_1^0 \) or \( S_2^0 \) or not.

**Case 2.a): \( \eta \) is retractable into \( S_1^0 \) or \( S_2^0 \)**

Assume without loss of generality that \( \eta \) is retractable into \( S_1^0 \subset S_c \). We first prove the following lemma:

**Lemma 2.1.** For fixed \( i \in \{1, 2\} \), let \( \delta_i \in S_i^0 \) be an arc in \( S_c \) with endpoints \( p_1 \) and \( p_2 \) on \( \alpha_c \), such that \( \delta_i \) is homotopic with fixed endpoints to a geodesic arc \( d_i \subset \alpha_c \) of length \( \ell(d_i) < \ell(\alpha_c) = 1 \). Then
\[
\ell(\delta_i) \geq \ell(d_i).
\]

**proof of Lemma 2.1** Fix \( i \) and let us work in \( S_i^0 \). It follows from Case 1 that \( \alpha_i \) is a systole of \( S_i^0 \). Let \( b' \) be the remaining arc of \( \alpha_i \) connecting \( p_1 \) and \( p_2 \), such that \( \delta_ib' \subset S_i^0 \) is a closed curve. Then \( \delta_ib' \) is contained in \( S_i^0 \) and freely homotopic to the systole \( \alpha_i \) of \( S_i^0 \). It follows that
\[
\ell(\delta_i) + \ell(b') = \ell(\delta_ib') \geq \ell(\alpha_i) = \ell(dib') = \ell(d_i) + \ell(b'), \quad \text{hence} \quad \ell(\delta_i) \geq \ell(d_i).
\]

This proves our lemma. \( \square \)

Now if \( \eta \subset S_c \) is a loop that is retractable into \( S_i^0 \) then, using **Lemma 2.1** we can find a comparison curve \( \eta' \) for \( \eta \), such that
\[
\ell(\eta) \geq \ell(\eta'), [\eta] = [\eta'] \in \pi_1(S_c) \quad \text{and} \quad \eta' \subset S_i^0.
\]

We obtain \( \eta' \) from \( \eta \) by replacing any arc \( v \) of \( \eta \) that is contained in either \( S_1^0 \) or \( S_2^0 \) and that is homotopic with fixed endpoints to an arc \( d \subset \alpha_c \) by the boundary arc \( d \).

That **Lemma 2.1** can indeed be applied can be seen in the following way: We note that \( \eta \) has no self-intersection. Hence \( v \) has no self-intersection. Now if the length \( \ell(d) \) of \( d \) was bigger than one then this would imply that \( d \) and hence \( v \) has a self-intersection. A contradiction.

As \( \eta \) is retractable into \( S_i^0 \) and \( \eta' \) is freely homotopic to \( \eta \), due to our procedure \( \eta' \) is contained in \( S_i^0 \). By deforming \( \eta' \) slightly, we may assume that \( \eta' \) is contained in the interior of \( S_i^0 \). Therefore it follows from Case 1 that \( \ell(\eta') \geq 1 \). Hence
\[
\ell(\eta) \geq \ell(\eta') \geq 1.
\]

**Case 2.b): \( \eta \) is not retractable into either \( S_1^0 \) or \( S_2^0 \)**

\( \eta \) is not retractable into \( S_1^0 \) and not retractable into \( S_2^0 \). Now due to this property, \( \eta \) contains two subarcs \( \eta'_1 \) and \( \eta'_2 \) such that
- $\eta'_i$ has endpoints on $\alpha_c$ and is contained in $S_i^0$
- there is an arc $b^i$ of $\alpha_c$ connecting the endpoints of $\eta'_i$ on $\alpha_c$ such that
- $\eta'_1 b^1$ is not retractable into $S_2^0$ and $\eta'_2 b^2$ is not retractable into $S_1^0$.

It follows from these properties that

$$[\eta'_1 b^1] \neq 0 \in \pi_1(S_c) \quad \text{and} \quad [\eta'_2 b^2] \neq 0 \in \pi_1(S_c).$$

We now show that for fixed $i \in \{1, 2\}$:

$$\ell(\eta'_i) \geq \frac{1}{2}.$$ 

Consider without loss of generality $\eta'_1$. Again we distinguish two subcases: the arc $b^1$ is smaller or equal to $\frac{1}{2}$ or not.

**Case i):** $\ell(b^1) \leq \frac{1}{2}$

As $[\eta'_1 b^1] \neq 0 \in \pi_1(S_c)$ and $\eta'_1 b^1 \subset S_1^0$, $\eta'_1 b^1$ fulfills the conditions of Case 1 and therefore $\ell(\eta'_1 b^1) \geq 1$. As $\ell(b^1) \leq \frac{1}{2}$ it follows that

$$\ell(\eta'_1) \geq \frac{1}{2}.$$ 

This settles our claim in Case i.

**Case ii):** $\ell(b^1) > \frac{1}{2}$

In this case we close $S_1^0$ along $\alpha_1$ to obtain $S_1$. We denote all curves from $S_1^0$ by the same name in $S_1$. Let $d \subset S_1$ be the shortest geodesic arc on the systole $a_1$ of $S_1$ connecting the endpoints of $b^1$. As $\ell(b^1) > \frac{1}{2}$, we have that

$$\ell(d) < \frac{1}{2}.$$ 

Furthermore $[b^1 d^{-1}] = 0 \in \pi_1(S_1^0)$. Hence $\eta'_1 d$ is a closed curve in $S_1$ that, in $S_1$, is in the same free homotopy class as $\eta'_1 b^1$. Now

$$[\eta'_1 d] = [\eta'_1 b^1] \neq 0 \in \pi_1(S_1),$$

because otherwise it would follow that $[\eta'_1 b^1] = [a_1]^{\pm 1} \in \pi_1(S_1^0)$ (see Case 1). But then $\eta'_1 b^1$ would be a curve that is retractable into $S_2^0$, a contradiction.

Hence $\eta'_1 d$ is a non-contractible loop in $S_1$. Its length is bigger or equal to the length of the systole $a_1$ of $S_1$. It follows that

$$\ell(\eta'_1 d) = \ell(\eta'_1) + \ell(d) \geq 1 \quad \text{and} \quad \ell(d) \leq \frac{1}{2}, \quad \text{hence} \quad \ell(\eta'_1) \geq \frac{1}{2}.$$ 

This settles our claim in Case ii.

As the same arguments in Case i and Case ii for $\eta'_1$ apply to $\eta'_2$, we conclude that

$$\ell(\eta'_i) \geq \frac{1}{2} \quad \text{for} \quad i \in \{1, 2\}, \quad \text{hence} \quad \ell(\eta) \geq \ell(\eta'_1) + \ell(\eta'_2) \geq 1.$$
In total we obtain in both Case 1 and Case 2 that $\ell(\eta) \geq 1$.

As any non-contractible loop in $(S_c, g_c)$ has length greater than or equal to one, we have shown that

$$\text{sys}(S_c, g_c) = 1.$$ 

Due to Equation (5), we have that $\text{SR}(k_i) - \epsilon_1 = \text{SR}(S_i, g_i) = \text{vol}(S_i, g_i)^{-1}$, hence

$$\text{SR}(S_c, g_c) = \frac{1}{\text{vol}(S_1, g_1) + \text{vol}(S_2, g_2)} = \left(\frac{1}{\text{SR}(k_1) - \epsilon_1} + \frac{1}{\text{SR}(k_2) - \epsilon_1}\right)^{-1}.$$ 

Now for every $\epsilon_2 > 0$, we can approximate our non-smooth surface $(S_c, g_c)$ with a smooth surface $(S_{c_2}, g_{c_2})$ such that the distance function and the area of $(S_{c_2}, g_{c_2})$ is $\epsilon_2$-close to that of $(S_c, g_c)$. Letting $\epsilon_1$ and $\epsilon_2$ tend to zero we obtain:

$$\text{SR}(k_1 + k_2) \geq \text{SR}(S_c, g_c) \geq \left(\frac{1}{\text{SR}(k_1)} + \frac{1}{\text{SR}(k_2)}\right)^{-1}.$$ 

This concludes the proof of Theorem 1.1.1. \hfill \Box

2. $\text{SR}_h(k_1 + k_2) \geq \left(\frac{1}{\text{SR}_h(k_1)} + \frac{1}{\text{SR}_h(k_2)}\right)^{-1}$.

Let $\epsilon_1 > 0$ be a positive real number. For $i \in \{1, 2\}$, let $(S^h_i, g_i)$ be a surface of genus $k_i \geq 1$, which satisfies

$$\text{SR}_h(S^h_i, g_i) = \text{SR}_h(k_i) - \epsilon_1 \quad \text{and} \quad \text{sys}_h(S^h_i, g_i) = 1, \quad \text{hence} \quad \text{vol}(S^h_i, g_i) = \frac{1}{\text{SR}_h(S^h_i, g_i)}. \quad (8)$$

To prove Theorem 1.1.2 we construct a new surface $(S^h_c, g_c)$ of genus $k_1 + k_2$ from the surfaces $(S^h_1, g_1)$ and $(S^h_2, g_2)$ such that

$$\text{SR}_h(S^h_c, g_c) = \left(\frac{1}{\text{SR}_h(S^h_1, g_1)} + \frac{1}{\text{SR}_h(S^h_2, g_2)} + \epsilon_2\right)^{-1}, \quad (9)$$

where $\epsilon_2$ is a positive real number that can be chosen arbitrarily small. As $(S^h_c, g_c)$ has genus $k_1 + k_2$, we obtain the inequality of Theorem 1.1.2 from the fact that $\epsilon_1$ and $\epsilon_2$ can be chosen arbitrarily small.

We first construct $(S^h_c, g_c)$. For fixed $i$ let $a_i$ be a homological systole of $(S^h_i, g_i)$. We first divide each $a_i$ into two arcs, $b_i$ and $c_i$, where $c_i$ has length

$$\ell(c_i) = \epsilon_2 < \frac{1}{4}.$$ 

Then we cut each $a_i$ along the length of $c_i$ and call the surface obtained this way $(S^0_i, g_i)$. We denote by $\gamma_i$ the boundary curve of $(S^0_i, g_i)$. We now take an Euclidean cylinder $C$, such that

- $C$ has height 1 and $\ell(\partial_1 C) = \ell(\partial_2 C) = \epsilon_2$, hence $\text{vol}(C) = \epsilon_2$
- $\gamma$ is the simple closed geodesic in $C$ of length $\ell(\gamma) = \epsilon_2$, which is freely homotopic to the boundary curve $\partial_1 C$ and such that $\text{dist}(\gamma, \partial_1 C) = \frac{1}{2}$. 

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We connect the boundary components of \((S_1^o, g_1)\) and \((S_2^o, g_2)\) by connecting them with the cylinder \(C\) in the following way
\[
\gamma_1 \sim \partial_1 C \quad \text{and} \quad \gamma_2 \sim \partial_2 C
\]
to obtain a closed surface. We denote the surface of genus \(k_1 + k_2\) obtained according to this pasting scheme as
\[
S_c^h = S_1^o + S_2^o + C \mod (10).
\]
We denote the boundary curves of the embedded cylinder by the same name as in the cylinders itself.

We now show that any non-separating loop in \((S_c^h, g_c)\) has length bigger than or equal to one. Let \(\eta\) be such a loop. To simplify our proof, we assume that \(\eta\) has no self-intersection and that the arcs of \(\eta\) contained in \(C\) are geodesic, i.e. straight lines. To prove our statement we distinguish two cases: either \(\eta\) intersects \(\gamma\) transversally or not.

**Case 1.** \(\eta\) intersects \(\gamma\) transversally

Due to our assumption, the subarc \(\eta' \subset C\) of \(\eta\) intersecting \(\gamma\) is a straight line. Hence \(\eta'\) traverses \(C\). It follows that its length is bigger than the height of the cylinder \(C\), which is 1. In this case we have that
\[
\ell(\eta) \geq \ell(\eta') \geq 1
\]
This settles our proof in **Case 1**.

**Case 2.** \(\eta\) does not intersect \(\gamma\) transversally

In the second case, \(\eta\) is a non-contractible loop that does not intersect \(\gamma\) transversally. Again we distinguish two subcases: \(\eta\) is contained in \(C\) or not. If \(\eta\) is contained in \(C\), then \(\eta\) is a separating loop, a contradiction to our assumption.

If \(\eta\) is not contained in \(C\) and does not intersect \(\gamma\) transversally, then \(\eta\) is freely homotopic to a loop \(\eta''\), which is contained in the interior of one of the \((S_i^o)_{i=1,2}\), say \(S_1^o\) and such that \(\ell(\eta) \geq \ell(\eta'')\). In this case we have that
\[
\ell(\eta) \geq \ell(\eta'') \geq \text{sys}_h(S_1^o, g_1) \geq 1.
\]
Here the second inequality follows from the fact that any non-separating simple loop in \(S_1^o\) is also a non-separating simple loop in \(S_1^h\). This settles our proof in **Case 2**. In total we conclude that \(\text{sys}_h(S_c^h, g_c) \geq 1\).

From the homological systolic ratio \(\text{SR}_h(S_c^h, g_c)\) of \((S_c^h, g_c)\) we obtain inequality (9). As in (9) \(\epsilon_1\) and \(\epsilon_2\) can be chosen arbitrarily small this yields
\[
\text{SR}_h(k_1 + k_2) \geq \text{SR}_h(S_c^h, g_c) \geq \left(\frac{1}{\text{SR}_h(k_1)} + \frac{1}{\text{SR}_h(k_2)}\right)^{-1}.
\]
Here the first inequality follows from the fact that \((S_c^h, g_c)\) can be approximated by a smooth surface. This concludes the proof of **Theorem 1.1**. 

□
proof of Theorem [1.2] The first inequalities in Theorem 1.2-1 and 1.2-2 are a simple consequence of Theorem 1.1-1 and 1.1-2, respectively. Here we set \( k_1 = 1 \) and \( k_2 = k \) and use the fact that \( \text{SR}(1) = \text{SR}_h(1) = \frac{2}{\sqrt{3}} \).

We now prove the second inequality in Theorem 1.2-1 and then show how to obtain the remaining inequalities in a similar fashion. We have to show that

\[
1. \quad \frac{1}{\text{SR}(k)} \leq \frac{1}{\text{SR}(k+1, \text{hyp})} + \frac{1}{\pi}.
\]

Let \( \epsilon_1 > 0 \) be a positive real number. Let \((S, \text{hyp})\) be a hyperbolic surface of genus \( k + 1 \geq 2 \), which satisfies

\[
\text{SR}(S, \text{hyp}) = \text{SR}(k+1, \text{hyp}).
\]

As \( \text{SR}(k+1, \text{hyp}) = \text{SR}_h(k+1, \text{hyp}) \), we may assume that \( S \) has a systole \( \alpha \) which is a non-separating simple closed curve. We can rescale \((S, \text{hyp})\) to obtain a surface \((S_{\max}, g_{\text{sc}})\) satisfying

\[
\text{SR}(S_{\max}, g_{\text{sc}}) = \text{SR}(k+1, \text{hyp}) \quad \text{and} \quad \text{sys}(S_{\max}, g_{\text{sc}}) = 1, \quad \text{hence} \quad \text{vol}(S_{\max}, g_{\text{sc}}) = \frac{1}{\text{SR}(k+1, \text{hyp})}.
\]

Let \((S^c, g_{sc})\) be the surface which we obtain by cutting open \((S_{\max}, g_{sc})\) along \( \alpha \). As \( \alpha \) is non-separating \((S^c, g_{sc})\) has signature \((k, 2)\). Let \( \alpha_1 \) and \( \alpha_2 \) be the boundary geodesics of \((S^c, g_{sc})\).

Let \( D \) be a sphere of constant curvature, whose great circles have length \( \ell(\alpha) = 1 \). Let \( D_1 \) and \( D_2 \) be the hemispheres which we obtain by cutting \( D \) along a great circle. It follows from the geometry of the sphere that

\[
\text{vol}(D_1) = \text{vol}(D_2) = \frac{\ell(\alpha)^2}{2\pi} = \frac{1}{2\pi}.
\]

For fixed \( i \in \{1, 2\} \), let \( \delta \subset D_i \) be a curve connecting two boundary points, \( p_1 \) and \( p_2 \) of \( D_i \). It follows from the geometry of \( D_i \), that there is a comparison boundary arc \( \delta' \) of \( D_i \), connecting \( p_1 \) and \( p_2 \) that is shorter than or of equal length as \( \delta \) and such that \( \delta(\delta')^{-1} \) is a contractible loop.

\[
\ell(\delta) \geq \ell(\delta') \quad \text{and} \quad [\delta(\delta')^{-1}] = 0 \in \pi_1(D_i), \quad \text{where} \quad \delta' \subset \partial D_i.
\]

To prove our statement, we construct a surface \((S', g')\) of genus \( k \) by pasting \( D_1 \) and \( D_2 \) along the boundary geodesics of \((S^c, g_{sc})\). Let \( \eta \) be a non-contractible simple loop in \((S', g')\). We first show that

\[
\ell(\eta) \geq 1, \quad \text{hence} \quad \text{sys}(S', g') \geq 1.
\]

If \( \eta \subset S' \) is contained in \( S^c \subset S_{\max} \), then \( \ell(\eta) \geq \text{sys}(S_{\max}, g_{sc}) = 1 \) and there is nothing to prove. If \( \eta \) is not contained in \( S^c \), then \( \eta \) intersects \( \alpha_1 \) or \( \alpha_2 \) transversally. Then it follows from the comparison statement [12] that there is a non-contractible comparison loop \( \eta' \subset S^c \subset S_{\max} \), such that

\[
\ell(\eta) \geq \ell(\eta').
\]

Hence for any non-contractible simple loop in \( S' \) there is a non-contractible loop \( \eta' \) in \( S_{\max} \), whose length is smaller or equal to the length of \( \eta \). Therefore

\[
\ell(\eta) \geq \ell(\eta') \geq \ell(\alpha) = 1, \quad \text{hence} \quad \text{sys}(S', g') \geq 1.
\]
As \( \text{sys}(S', g') \geq 1 \) and \( \text{vol}(S_c, g_{sc}) = \text{SR}(k + 1, \text{hyp})^{-1} \) (see (11)), we obtain for the systolic ratio of \((S', g')\)

\[
\text{SR}(k) \geq \text{SR}(S', g') \geq \frac{1}{\text{vol}(S_c, g_{sc}) + \text{vol}(D)} = \left( \frac{1}{\text{SR}(k + 1, \text{hyp}) + \frac{1}{\pi}} \right)^{-1}.
\]

Here the first inequality follows from the fact that \((S', g')\) can be approximated by a smooth surface. The above inequality implies the second inequality in Theorem 1.2-1. This concludes the proof of Theorem 1.2-1.

We obtain the second inequality in Theorem 1.2-2 by replacing the surface \((S_{\text{max}}, g_{sc})\) of genus \(k + 1\) in the previous proof by a surface \((S_{\text{hom}}, g)\) of genus \(k + 1\) satisfying

\[
\text{SR}_h(k + 1) = \text{SR}_h(S_{\text{hom}}, g) - \epsilon_1 \quad \text{and} \quad \text{sys}_h(S_{\text{hom}}, g) = 1.
\]

where \(\epsilon_1 > 0\) is a positive real number that can be arbitrarily close to zero. We cut a surface \((S_{\text{hom}}, g)\) along a homological systole \(\beta\). Then we paste two hemispheres of boundary length \(\ell(\beta)\) along the boundary curves of the open surface to obtain a surface \((S'_{\text{hom}}, g_h)\) of genus \(k\). Then we apply similar arguments as in the case of the surface \((S', g')\) to obtain

\[
\text{sys}_h(S'_{\text{hom}}, g_h) \geq 1.
\]

The inequality then follows by from a calculation of the homological systolic ratio \(\text{SR}_h(S'_{\text{hom}}, g_h)\) of \((S'_{\text{hom}}, g_h)\).

We obtain inequality in Theorem 1.2-3 from the following construction. We cut a surface \((S, g)\) of genus \(k + 1\), whose systolic ratio is close to the optimal value \(\text{SR}(k + 1)\) along its homological systole \(\beta'\). Then we paste two hemispheres of boundary length \(\ell(\beta')\) along the boundary curves of the open surface to obtain a surface \((S''_{\text{hom}}, g_h)\) of genus \(k\). The inequality then follows from a calculation of the systolic ratio \(\text{SR}(S'', g'')\) of \((S'', g'')\). Here we use the fact that

\[
\frac{\ell(\beta')^2}{\text{vol}(S, g)} = \frac{\text{sys}_h(S, g)^2}{\text{vol}(S, g)} \leq \text{SR}_h(k).
\]

This concludes the proof of Theorem 1.2-3 and hence of Theorem 1.2.

proof of Theorem 1.3 This proof is very similar to the proof of Gr2, Theorem 2.C. However our statement is different. We first show how to obtain the first inequality in Theorem 1.3-1. Then we show how to obtain the second inequality and Theorem 1.3-2 by a simple modification of the proof. We show

For all \(\epsilon_1 \in (0, 10^{-5}]\), \(\text{SR}_h(k) - \epsilon_1 \leq \frac{4}{3} \) for all \(k \neq 0\)

We prove our statement by induction: As \(\text{SR}(1) = \text{SR}_h(1) = \frac{2}{\sqrt{3}}\), the statement is true for \(k = 1\). We assume that for all \(1 \leq k' < k\)

\[
\text{SR}_h(k') - \epsilon_1 \leq \frac{4}{3} \quad \text{thus} \quad \frac{1}{\text{SR}_h(k') - \epsilon_1} \geq \frac{3}{4}.
\]
Let \((S_{\text{hom}}, g)\) be a surface of genus \(k > 1\), which satisfies
\[
\text{vol}(S_{\text{hom}}, g) = 1 \quad \text{and} \quad \text{sys}_h(S_{\text{hom}}, g)^2 = SR_h(k) - \epsilon_1, \quad \text{hence} \quad SR_h(k) - \epsilon_1 = \text{sys}_h(S_{\text{hom}}, g)^2.
\]

Let \(\alpha\) be a systole of \((S_{\text{hom}}, g)\). Two cases can occur. Either \(\alpha\) is separating or \(\alpha\) is non-separating:

**Case 1. \(\alpha\) is non-separating**

In this case it follows with Equation (13) and as \(\text{sys}_h(S_{\text{hom}}, g)^2 \geq \text{sys}(S_{\text{hom}}, g)^2\) that
\[
SR_h(k) - \epsilon_1 = \text{sys}_h(S_{\text{hom}}, g)^2 = \ell(\alpha)^2 = \text{sys}(S_{\text{hom}}, g)^2 \leq SR(k) \leq \frac{4}{3},
\]

**Case 2. \(\alpha\) is separating**

Let \((S^1, g)\) and \((S^2, g)\) be the surfaces of signature \((k_1, 1)\) and \((k_2, 1)\), which we obtain by cutting open \((S_{\text{hom}}, g)\) along the systole \(\alpha\). Let \(\alpha_1\) be the boundary geodesics of \((S^1, g)\) and \(\alpha_2\) be the boundary geodesics of \((S^2, g)\).

Let \(D\) be a sphere of constant curvature, whose great circles have length \(\ell(\alpha)\). Let \(D_1\) and \(D_2\) be the hemispheres which we obtain by cutting \(D\) along a great circle. It follows from the geometry of the sphere that
\[
\text{vol}(D_1) = \text{vol}(D_2) = \frac{\ell(\alpha)^2}{2\pi}.
\]

To prove our statement, we construct two surfaces \((S^{1p}, g_1)\) of genus \(k_1\) and \((S^{2p}, g_2)\) of genus \(k_2\) by pasting \(D_1\) and \(D_2\) along the boundary geodesics of \((S^1, g)\) and \((S^2, g)\), respectively. It follows from the similar arguments as in the proof of **Theorem 1.2-1**, that
\[
\text{sys}_h(S^{1p}, g_1)^2 = \text{sys}_h(S^{2p}, g_2)^2 = \text{sys}_h(S_{\text{hom}}, g)^2 = SR_h(k) - \epsilon_1.
\]

It follows from the above equation that for all \(i \in \{1, 2\}:
\[
SR_h(k_i) \geq SR_h(S^{ip}, g_i) = \frac{\text{sys}_h(S_{\text{hom}}, g)^2}{\text{vol}(S^{ip}, g_i)} \quad \text{thus} \quad \frac{1}{SR_h(k_i)} \leq \frac{\text{vol}(S^{ip}, g_i)}{\text{sys}_h(S_{\text{hom}}, g)^2}.
\]

Combining the above two inequalities and using the fact that \(\text{vol}(S^{1p}, g_1) + \text{vol}(S^{2p}, g_2) = 1 + \frac{\ell(\alpha)^2}{\pi}\), we have that
\[
\frac{1}{SR_h(k_1)} + \frac{1}{SR_h(k_2)} \leq \frac{1 + \frac{\ell(\alpha)^2}{\pi}}{SR_h(k) - \epsilon_1},
\]

Now \(\ell(\alpha)^2 = \text{sys}(S_{\text{hom}}, g)^2 = SR(S_{\text{hom}}, g) \leq SR(k)\). This yields
\[
\frac{1}{SR_h(k_1)} + \frac{1}{SR_h(k_2)} \leq \frac{1 + \frac{SR(k)}{\pi}}{SR_h(k) - \epsilon_1}.
\]

Applying the induction hypothesis and using the fact that \(SR(k) \leq \frac{4}{3}\), we obtain:
\[
\frac{3}{2} \leq \frac{1}{SR_h(k_1)} + \frac{1}{SR_h(k_2)} \leq \frac{1 + \frac{4}{3\pi}}{SR_h(k) - 10^{-5}}, \quad \text{hence} \quad 0.96 \geq SR_h(k).
\]
But this proves our hypothesis. This settles our claim in Case 2 and therefore concludes the proof of the first part of Theorem 1.3-1. Letting $\epsilon_1$ go to zero, we obtain the first part of Theorem 1.3.

To prove the second part, we use the same arguments. Here we use the fact that we already know that $\text{SR}(k) \leq \frac{2}{\sqrt{3}}$ for $k \geq 20$. In fact using the same arguments, it can be shown that $\text{SR}_h(k)$ satisfies the same upper bound as $\text{SR}(k)$ in Table 4.

To prove the first part of Theorem 1.3-2, we also follow the above proof. However in this case, we obtain a contradiction in inequality (14) of Case 2 from the known value of $\text{SR}_h(1)$ and the upper and lower bound for $\text{SR}(2)$ and $\text{SR}_h(2)$ (see Table 2). Hence this case leads to a contradiction. It remains Case 1 from which follows that

$$\text{SR}(2) = \text{SR}_h(2).$$

We prove the second part,

$$\text{SR}(3) \geq \text{SR}_h(3) - 0.03$$

in a similar fashion. If Case 2 holds, then we use iteratively the second inequality in (14) to show that in this case $\text{SR}_h(3) \leq 0.69$, from which follows our statement. This settles the proof of the first and second part of Theorem 1.3.

**proof of Theorem 1.3-3** To prove the third part of the theorem we first derive a good upper bound for $\text{SR}(k)$ from inequality (11):

For all $r \in (0, \frac{1}{8})$, $f(r, \text{SR}(k)) := \frac{\log(2r^2 \text{SR}(k))^2}{4\pi \text{SR}(k) \left(\frac{1}{2} - 4r\right)^2 (k - 1)} \geq 1$.

Set (for $k \in \mathbb{R}_+$)

$$\text{SR}^u(k) := \begin{cases} 4/3 \left(\frac{\log(50k) + 1.4)^2}{\pi(k-1)} \right) & \text{for } k < 17 \\ k \geq 17. \end{cases}$$

That $\text{SR}^u(k) \geq \text{SR}(k)$ for $k \geq 17$ can be seen in the following way. For fixed $k \geq 17$ we have

- The function $f(r, \cdot)$ is a monotonically decreasing function in the interval $(0, 2r^2]$.

- Choosing $r' = \frac{4}{4\log(25k+25)-2} < 1/8$, it follows from (11) that $\text{SR}(r', k) \geq \text{SR}(k)$, where $\text{SR}(r', k)$ is given by $f(r', \text{SR}(r', k)) = 1$.

- It can be shown that for fixed $k \geq 17$, $f(r', \text{SR}^u(k)) < 1$ and $\lim_{k \to \infty} f(r', \text{SR}^u(k)) = 1$.

It follows that $\text{SR}^u(k) \geq \text{SR}(r', k) \geq \text{SR}(k)$ for $k \geq 17$.

Set (for $k \in \mathbb{R}_+$)

$$\text{SR}^u_h(k) := \begin{cases} 4/3 \left(\frac{\log(195k) + 8)^2}{\pi(k-1)} \right) & \text{for } k \leq 75 \\ k > 75. \end{cases}$$

We now prove by induction that

$$\text{SR}_h(k) \leq \text{SR}^u_h(k) \quad \text{for all } k \in \mathbb{N}\setminus\{0\}.$$

Therefore we use the same arguments as in the proof of the first part of the theorem. For fixed $k \in \mathbb{N}\setminus\{0\}$, we assume that our statement is proven for all $k' < k$. It is easy to see that the crucial point is inequality (14) in Case 2, which states that
\[
\frac{1}{\text{SR}_h(k_1)} + \frac{1}{\text{SR}_h(k_2)} \leq \frac{1 + \frac{\text{SR}(k)}{\pi}}{\text{SR}_h(k)}, \quad \text{where} \quad k_1 + k_2 = k.
\]

Applying the induction hypothesis and the upper bound for \(\text{SR}(\cdot)\) and \(\text{SR}_h(\cdot)\), we obtain from the above inequality:

\[
\frac{1}{\text{SR}_h(k_1)} + \frac{1}{\text{SR}_h(k_2)} \leq \frac{1}{\text{SR}_h(k_1)} + \frac{1}{\text{SR}_h(k_2)} \leq \frac{1 + \frac{\text{SR}_h(k)}{\pi}}{\text{SR}_h(k)}.
\] (15)

For \(k \leq 1000\) it can be verified by calculating \(\text{SR}_h(k_1)\) and \(\text{SR}_h(k_2)\) explicitly that

\[
\frac{1}{\text{SR}_h(k)} \leq \frac{1}{\text{SR}_h(k_1)} + \frac{1}{\text{SR}_h(k_2)} \leq \frac{1 + \frac{\text{SR}_h(k)}{\pi}}{\text{SR}_h(k)},
\]

hence \(\text{SR}_h(k) \geq \text{SR}_h(k)\), which implies our hypothesis for \(k \leq 1000\).

For \(k > 1000\) we note that the first term in (15) is minimal for \(k_1 = 75\) and \(k_2 = k - 75\). Hence:

\[
\frac{3}{4} + \frac{1}{\text{SR}_h(k_1)} + \frac{1}{\text{SR}_h(k_2)} = \frac{1}{\text{SR}_h(k_1)} + \frac{1}{\text{SR}_h(k_2)} \leq \frac{1}{\text{SR}_h(k)}.
\]

It can be verified that

\[
\frac{1}{\text{SR}_h(k)} \leq \frac{3}{4} + \frac{1}{\text{SR}_h(k_1)} + \frac{1}{\text{SR}_h(k_2)} \leq \frac{1}{\text{SR}_h(k)},
\]

which implies that \(\text{SR}_h(k) \geq \text{SR}_h(k)\).

But this proves our hypothesis. This settles our claim in Case 2 and therefore concludes the proof of Theorem 1.3.3.

In total we have proven Theorem 1.3.3. \(\square\)

### 3 Construction of manifolds with large systolic ratio

In this section we construct manifold with large systolic ratio from lower dimensional manifolds with large systolic ratio. To this end we first prove the following lemma:

**Lemma 3.1.** Let \((M, g)\) and \((N, h)\) be two closed, oriented Riemannian manifolds of dimension \(m\) and \(n\), respectively. If the product manifold \((M \times N, g \times h)\) has a systole, we have:

\[\text{sys}(M \times N, g \times h) = \min(\text{sys}(M, g), \text{sys}(N, h)).\]

**proof of Lemma 3.1** Let \(\eta\) be a systole of \((M \times N, g \times h)\) of length

\[\ell(\eta) = \text{sys}(M \times N, g \times h).
\]

Let \(p_M\) and \(p_N\) be the canonical projection from \((M \times N, g \times h)\) to \((M, g)\) and \((N, h)\), respectively. Consider the curves \(p_M(\eta)\) and \(p_N(\eta)\), respectively. We have:

\[\ell(\eta) \geq \ell(p_M(\eta)) \quad \text{and} \quad \ell(\eta) \geq \ell(p_N(\eta)).\]
Let $[\eta] \in \pi_1(M \times N)$ be the homotopy class of $\eta$. As $\eta$ is non-contractible we have that $[\eta] \neq 0$. As
\[
\pi_1(M \times N) = \pi_1(M) \times \pi_1(N) = p_M^*(\pi_1(M \times N)) \times p_N^*(\pi_1(M \times N)),
\]
either $[p_M(\eta)] \neq 0$ or $[p_M(\eta)] \neq 0$. Assume without loss of generality that $[p_M(\eta)] \neq 0$. It follows that
\[
sys(M \times N, g \times h) = \ell(\eta) \geq \ell(p_M(\eta)) \geq sys(M, g) \geq \min(sys(M, g), sys(N, h)).
\]
This proves our lemma. □

As a corollary we obtain:

**Corollary 3.2.** Let $(M, g)$ and $(N, h)$ be two closed, oriented Riemannian manifolds of dimension $m$ and $n$, respectively, such that $sys(M, g) = sys(N, h)$. Then for the product manifold $(M \times N, g \times h)$, we have:
\[
SR(M \times N, g \times h) = SR(M, g) \cdot SR(N, h).
\]

**proof of Corollary 3.2** This follows immediately from Lemma 3.1 and the fact that $vol(M \times N, g \times h) = vol(M, g) \cdot vol(N, h)$. □

Note that we can always scale one of the two manifolds in the product manifold to meet the conditions of the corollary. In the following we apply the corollary to two examples. First we construct $n$-dimensional Euclidean product-tori with large systolic ratio from lower dimensional ones, then we construct product manifolds of surfaces and tori.

**Example 1 - tori** A lattice $\Lambda$ of dimension $n$ is a discrete subgroup of $\mathbb{R}^n$ that spans $\mathbb{R}^n$. An $n$-dimensional flat torus $\mathbb{T}^n = \mathbb{R}^n / \Lambda$ is the quotient of $\mathbb{R}^n$ and a lattice $\Lambda$. The shortest non-zero lattice vector of $\Lambda$ is the systole of $\mathbb{T}^n$. It’s length $sys(\mathbb{T}^n)$ is
\[
sys(\mathbb{T}^n) = \min_{\lambda \in \Lambda \setminus \{0\}} \|\lambda\|_2,
\]
where $\| \cdot \|_2$ denotes the Euclidean norm. If $A$ is a matrix representation of a basis of $\Lambda$, then $det(\Lambda)$, the determinant of $\Lambda$ is equal to $det(A)$ and
\[
vol(\mathbb{T}^n) = |det(\Lambda)| = |det(A)|.
\]

*Hermite’s constant or invariant* $\gamma_n$ is given by
\[
\gamma_n = \max\{SR(\mathbb{T}^n)\frac{2}{\pi} \mid \mathbb{T}^n \text{ flat torus of dimension } n\}
\]
It follows from this definition that it is the maximal value that the squared norm of the shortest non-zero lattice vector can attain among all lattices of determinant 1. Let $B^n$ be a $n$-dimensional Euclidean ball of radius 1. It was proven by Hlawka [Hl] and Minkowski [Mi] that
\[
\frac{n}{2\pi e} \leq \left( \frac{vol(B^n)}{2} \right)^{-\frac{2}{n}} \leq \gamma_n \leq 4 \cdot (vol(B^n))^{-\frac{2}{n}} \leq \frac{2n}{\pi e}.
\]
(16)

Here the approximations of the bounds apply for large $n$. From Corollary 3.2 we obtain:
Theorem 3.3. Let $\gamma_n$ be Hermite’s constant for flat tori in dimension $n$, then

1. $\gamma_{m+n} \geq \gamma_m \cdot \gamma_n$.
2. $\gamma_{2n} \geq \gamma_n$.
3. $\gamma_n \left( \frac{n}{n+1} \right) \geq \gamma_{n+1} \geq \gamma_n \left( \frac{n}{n+1} \right)$.

Let furthermore $(T^n, g)$ be a manifold homeomorphic to an Euclidean torus of dimension $n$ or shortly a torus. We define Hermite’s constant for general tori $\delta_n$ by

$$\delta_n = \sup \{ \text{SR}(T^n, g)^{\frac{1}{2n}} \mid (T^n, g) \text{ torus of dimension } n \}.$$

As $\gamma_n \leq \delta_n$ the same lower bound as in inequality (16) applies. An upper bound of order $n^2$ was conjectured in [Gr3]. The best known upper bound is of order $n^2$ and is stated in [Na, Theorem 4.2]. This implies for large $n$ that

$$\frac{n}{2\pi e} \leq \delta_n \leq \left( \frac{n}{e} \right)^2.$$

Using the same methods as in the Euclidean case we show that

Theorem 3.4. Let $\delta_n$ be Hermite’s constant for general tori of dimension $n$, then

1. $\delta_{m+n} \geq \delta_m \cdot \delta_n$.
2. $\delta_{2n} \geq \delta_n$.

The idea of the proof is to construct tori with large systoles from lower dimensional extremal ones.

proof of Theorem 3.3 Let $T^m$ and $T^n$ be two flat tori of dimension $m$ and $n$, respectively, such that

$$\text{SR}(T^m)^{\frac{1}{m}} = \gamma_m \quad \text{and} \quad \text{SR}(T^n)^{\frac{1}{n}} = \gamma_n.$$

We obtain Theorem 3.3-1 by scaling $T^m$ to obtain a torus $T_1^m$ and $T^n$ to obtain a torus $T_2^n$, such that

$$\text{sys}(T_1^m) = \text{sys}(T_2^n) = 1.$$

Let $T_1^m \times T_2^n$ be the product torus of dimension $m+n$. Applying Corollary 3.2 to $T_1^m \times T_2^n$ we obtain

$$\gamma_{m+n} \geq \text{SR}(T_1^m \times T_2^n) = \text{SR}(T_1^m) \cdot \text{SR}(T_2^n) = \gamma_m \cdot \gamma_n.$$

This inequality is equivalent to Theorem 3.3-1. The second inequality of the theorem follows by setting $n = m$ in the first inequality. The first inequality in Theorem 3.3 is Mordell’s inequality (see [Ma] or [Mo]). The second inequality follows by setting $m = 1$ in Theorem 3.3-1 and using the fact that $\gamma_1 = 1$.

This concludes the proof of Theorem 3.3.

Theorem 3.4 follows by the same arguments replacing $T^n$ by $(T^n, g)$ and will not be shown here.

Example 2 - products of tori and surfaces Denote from here on by $g_E$ be the Euclidean metric tensor. In this example we construct product manifolds with large systolic ratio from surfaces and tori with large systolic ratio. This enables us to prove:
Theorem 3.5. Let $b(M, g) = \sum_{i=0}^{m} b_i(M, g)$ be the sum of the Betti numbers of a manifold $(M, g)$ of dimension $m$. Then in each dimension $m \geq 3$ there exist manifolds $R_k^m = (S \times \mathbb{T}^{m-2}, g \times g_E)$, that are product manifolds of a surface $(S, g)$ of genus $k \gg 2^m$ and an Euclidean torus $(\mathbb{T}^{m-2}, g_E)$, such that

$$C_{1,m} \cdot \frac{\log(b(R_k^m))^2}{b(R_k^m)} \leq SR(R_k^m) \leq C_{2,m} \cdot \frac{\exp(C_{3,m}\sqrt{\log(b(R_k^m))})}{b(R_k^m)}.$$ 

Here the upper bound is the universal upper bound stated in [Sa], Theorem 1.2, inequality (1.5).

proof of Theorem 3.5 Let $(M, g)$ be an $m$-dimensional manifold and let

$$P_M(x) := \sum_{i=0}^{m} b_i(M, g) \cdot x^i$$

be the Poincaré polynomial. It is well-known that if $S = (S, g)$ is a surface of genus $k$ and $T = (\mathbb{T}^{m-2}, g_E)$ is an Euclidean torus of dimension $m - 2$, then

$$P_S(x) = 1 + 2k \cdot x + x^2 \quad \text{and} \quad P_T(x) = \sum_{i=0}^{m-2} \binom{m-2}{i} x^i, \quad \text{where} \quad \sum_{i=0}^{m-2} \binom{m-2}{i} = 2^{m-2}.$$ 

It follows furthermore from the Künneth theorem, that for $S \times T = (S \times \mathbb{T}^{m-2}, g \times g_E)$, we have

$$P_{S\times T}(x) = P_S(x) \cdot P_T(x).$$

It is easy to deduce from this formula, that

$$b(S \times T) = 4k + o(1), \quad \text{if} \quad k \gg 2^m. \quad (18)$$

We now choose our product manifold $R_k^m = (S \times \mathbb{T}^{m-2}, g \times g_E)$ in the following way. Let $\epsilon > 0$ be a positive real number and let in $R_k^m$, $(S, g)$ be a surface of genus $k \gg 2^m$, such that

$$\text{sys}(S, g) = 1 \quad \text{and} \quad \text{SR}(S, g) = \text{SR}(k) - \epsilon.$$ 

In $R_k^m$ let furthermore $\mathbb{T}^{m-2}$ be a Euclidean torus of dimension $m - 2$, such that

$$\text{sys}(\mathbb{T}^{m-2}) = 1 \quad \text{and} \quad \text{SR}(\mathbb{T}^{m-2}) = \gamma_{m-2}.$$ 

It follows from Corollary 3.2 and the inequalities (11) and (16) that

$$\text{SR}(S \times \mathbb{T}^{m-2}, g \times g_E) = \text{SR}(S, g) \cdot \text{SR}(\mathbb{T}^{m-2}, g_E) \geq K \frac{\log(k)^2}{k} \cdot (m - 2)^{\frac{m-2}{2}}.$$ 

Then the lower bound in Theorem 3.5 follows by applying Equation (18) to the above inequality. \qed
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