INTRODUCTION TO
EQUILIBRIUM THERMAL FIELD THEORY *

P V Landshoff
DAMTP, University of Cambridge
pvl@damtp.cam.ac.uk

Abstract

Within the next few years experiments at RHIC and the LHC will seek to create in the laboratory a quark-gluon plasma, the phase of matter through which the Universe passed very early in its life. It is believed that the plasma will survive long enough to reach thermal equilibrium. I give an introduction to the formalism of thermal field theory, the combination of statistical mechanics and quantum field theory needed to describe the plasma in thermal equilibrium, in a way that tries to keep close to the physics it describes.

Introduction

Thermal field theory is a combination of quantum field theory and statistical mechanics. This means that it is both difficult and interesting. The reason that we study it is that we want to describe the quark-gluon plasma, the phase that matter is believed to take above some critical temperature $T_c$. Lattice calculations suggest $T_c$ is about 100 MeV, or $10^{12}$K. In the plasma phase the quarks and gluons are deconfined; they can move rather freely through the whole plasma. This is the phase to which the universe evolved soon after the big bang, and before the end of the century experiments at the new collider RHIC will try to re-create it in the laboratory, by making gold nuclei collide together head-on and dump their kinetic energy into a small volume. Similar experiments, at much higher energy, are planned later for the LHC at CERN.

There is an obvious question: if a plasma is indeed produced, how will we know it? As yet there is no simple answer. There are estimates, necessarily based on very crude non-equilibrium theory, that suggest that the plasma will survive for a time long enough that it reaches thermal equilibrium before it eventually decays back into ordinary matter. So far, it is only equilibrium thermal field theory that is well formulated, and my lectures concentrate on this. For more information, the book by Le Bellac nowadays is the standard text, though the older book by Kapusta is still valuable, as is the classic Physics Report by Landsman and van Weert. As I want my description to stay as close as possible to physics I will develop the theory using operators rather than path integrals, and mostly I will use the so-called real-time formalism. In thermal field theory it is very important to pay attention to the physics: some of the clever mathematical formalism that has been developed is very powerful for calculating certain quantities, but it is all too easy to misuse it if one does not keep a close eye on the physics it is supposed to describe.

Because in relativistic theory particles are continually being created and destroyed, it is appropriate to use the grand ensemble formalism of statistical mechanics. The grand ensemble consists of a very large number of copies of the system under study. The energy and other conserved quantum numbers, such as total lepton and baryon number, vary from system to system in the ensemble, but their totals over the whole ensemble are fixed, and therefore their average values per system are fixed. If the system is large enough, fluctuations about the average values are very small, and so the grand

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* Lectures given in August 1998 at the University of Regensburg, based on a course given in February 1997 at the IX Jorge André Świeca Summer School, Campos do Jordão, Brazil
ensemble formalism does effectively achieve the desired physical situation in which the conserved quantum numbers are fixed for each system. In practice we do not specify the ensemble in terms of their values; rather we use related quantities, the temperature and various chemical potentials. These arise as Lagrange multipliers when we maximise the appropriate quantity in order to calculate the most probable configuration of the ensemble. This mathematics leads to the grand partition function

\[ Z = \sum_i \langle i | e^{-\beta (H - \mu N)} | i \rangle \]  

(1)

from which the macroscopic properties of the system in thermal equilibrium are calculated. Here \( \beta \) is the inverse temperature, \( \beta = 1/k_B T \), and usually we use units in which Boltzmann’s constant \( k_B = 1 \). The system’s Hamiltonian is \( H \) and \( N \) is a conserved quantum number, such as baryon number, with \( \mu \) the corresponding chemical potential. In the case of several conserved quantum numbers, \( \mu N \) is replaced with \( \sum_\alpha \mu_\alpha N_\alpha \).

The states \( |i\rangle \) are a complete orthonormal set of physical states of the system. In scalar field theory all states are physical and so

\[ Z = \text{tr} e^{-\beta (H - \mu N)} \]  

(2)

which is invariant under changes in the choice of orthonormal basis of states. In the case of gauge theories there are unphysical states, for example longitudinally-polarised photons or gluons, which must be excluded from the summation in (1). So then

\[ Z = \text{tr} \mathbb{P} e^{-\beta (H - \mu N)} \]  

(3)

where \( \mathbb{P} \) is a projection operator that removes unphysical states. The presence of \( \mathbb{P} \) can make things more complicated, and so to begin with I will consider scalar field theory, where it is not needed.

All the macroscopic properties of the system in thermal equilibrium may be calculated from \( Z \). In particular, for a system that is so large that its surface energy is negligible compared with its volume energy, the equation of state is

\[ PV = T \log Z \]  

(4)

Because we do not have sufficient knowledge of each system of the ensemble to specify which system state \( |i\rangle \) it is in, we use a density matrix \( \rho \) to describe the system: in thermal equilibrium

\[ \rho = Z^{-1} e^{-\beta (H - \mu N)} \]  

(5a)

The “thermal average” of an observable corresponding to an operator \( Q \), that is its average over all the systems of the ensemble, is

\[ < Q > = \text{tr} Q \rho = Z^{-1} \text{tr} Q e^{-\beta (H - \mu N)} \]  

(5b)

Notice that, throughout, all operators and all states are familiar zero-temperature ones. The temperature enters only in the exponential, which defines how the zero-temperature states are combined together to define the statistical-mechanical ensemble used to calculate the thermal averages of the zero-temperature operators.

**Noninteracting scalar bosons**

For some systems of bosons there is no conserved quantum number \( N \) and therefore no corresponding Lagrange multiplier, that is there is no chemical potential \( \mu \). For example, in the case of a heat bath of photons there is no constraint on their total number. Then in the scalar-field-theory case \( Z \) is just \( \text{tr} e^{-\beta H} \).
Consider first the case where interactions among the particles are so small that, once the system has thermalised, they can be neglected. This is a reasonable approximation for a weakly-interacting system, and is also a necessary preparation for setting up thermal perturbation theory. In the absence of interactions, the energies of the separate particles are good quantum numbers. To begin with, quantise the system in a finite volume $V$, so that the single-boson energies $\epsilon_r$ are discrete. We can choose as the basis states $|i\rangle$ of the system those labelled by the single-particle occupation numbers $n_r$:

$$|n_1 n_2 n_3 \ldots\rangle$$

and the eigenvalues of the noninteracting Hamiltonian $H_0$ are

$$n_1\epsilon_1 + n_2\epsilon_2 + n_3\epsilon_3 + \ldots$$

So the noninteracting grand partition function is

$$Z_0 = \sum_{\{n_r\}} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \ldots)}$$

$$= \prod_r \left( \sum_{n_r} e^{-\beta n_r \epsilon_r} \right) = \prod_r \frac{1}{1 - e^{-\beta \epsilon_r}}$$

and

$$\log Z_0 = -\sum_r \log \left( 1 - e^{-\beta \epsilon_r} \right)$$

In the continuum limit

$$\sum_r \to V \int \frac{d^3k}{(2\pi)^3}$$

and so the noninteracting equation of state is

$$P = \frac{T}{V} \log Z_0 = -T \int \frac{d^3k}{(2\pi)^3} \log(1 - e^{-\beta k^0})$$

where $k^0 = \sqrt{k^2 + m^2}$. If the bosons have non zero spin, there is an additional factor $g_s$ corresponding to the spin degeneracy of each single-boson state.

In the continuum limit we usually work with fields

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^0} a(k)e^{-ik \cdot x} + \text{h.c.}$$

(9a)

with

$$[a(k), a^{\dagger}(k')] = (2\pi)^3 2k^0 \delta^{(3)}(k - k')$$

(10a)

In the discrete case, we usually define

$$[a_r, a_s^{\dagger}] = \delta_{rs}$$

(10b)

If we sum this over $r$, the result is 1. But if we apply $V \int d^3k/(2\pi)^3$ to (10a), the result is rather $2k^0 V$. That is (10a) and (10b) have definitions of the operators $a$ differing by a factor $\sqrt{2k^0 V}$. We correct for this by defining the field in the discrete case to be

$$\phi(x) = \sum_r \frac{1}{\sqrt{2\epsilon_r V}} a_r e^{-i\epsilon_r t} e^{i\mathbf{k}_r \cdot \mathbf{x}} + \text{h.c.}$$

(9b)
The average number of particles in the single-particle level \( \epsilon_s \) is

\[
\langle a_s^\dagger a_s \rangle = Z_0^{-1} \sum_{\{n_r\}} \langle n_1 n_2 n_3 \ldots | e^{-\beta H_0} a_s^\dagger a_s | n_1 n_2 n_3 \ldots \rangle
\]

\[
= \left( \frac{1}{1 - e^{-\beta \epsilon_s}} \right)^{-1} \sum_{n_s} n_s e^{-n_s \beta \epsilon_s} = f(\epsilon_s) \tag{11a}
\]

where \( f \) is the Bose distribution

\[
f(\epsilon) = \frac{1}{e^{\beta \epsilon} - 1} \tag{11b}
\]

It is interesting to calculate the average of the square of the number of particles in the single-particle level \( \epsilon_s \):

\[
\langle (a_s^\dagger a_s)^2 \rangle = \left( \frac{1}{1 - e^{-\beta \epsilon_s}} \right)^{-1} \sum_{n_s} n_s^2 e^{-n_s \beta \epsilon_s} = (f(\epsilon_s))^2 + f(\epsilon_s)(f(\epsilon_s) + 1) \tag{11c}
\]

The average of the square is not equal to the square of the average, because the number of particles in the level \( \epsilon_s \) fluctuates from system to system in the ensemble.

We can now calculate a noninteracting thermal Green’s function, defined as the thermal average

\[
\langle T\phi(x)\phi(0) \rangle_0 = Z_0^{-1} \sum_i \langle i | e^{-\beta H_0} T\phi(x)\phi(0) | i \rangle \tag{12}
\]

Observe first that there is an operator identity

\[
T\phi(x)\phi(0) = \langle 0 | T\phi(x)\phi(0) | 0 \rangle + : \phi(x)\phi(0) : \tag{13}
\]

where : : denotes the usual normal product. The first term contributes to (12) just the usual zero-temperature Feynman propagator. To evaluate the contribution from the second, use the discrete case (9b) and so obtain double sums \( \sum_{r,s} \) of terms \( a_r^\dagger a_s, a_r a_s, a_r a_s^\dagger \). When we take the necessary expectation values, only the first survives, and then only for \( r = s \). So, using (11) we find that

\[
\langle : \phi(x)\phi(0) : \rangle_0 = \frac{1}{V} \sum_r \frac{1}{2\epsilon_r} f(\epsilon_r) e^{i\epsilon_r t - i k_r \cdot x} + \text{h.c.} \tag{14}
\]

Going to the continuum limit, we may write

\[
\langle T\phi(x)\phi(0) \rangle_0 = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} D_T(k) \tag{15}
\]

\[
D_T(k) = \frac{i}{k^2 - m^2 + i\epsilon} + 2\pi\delta(k^2 - m^2) n(k^0)
\]

where \( n(k^0) = f(|k^0|) \). The second term is the contribution from the heat bath; it contains the \( \delta \)-function because so far the heat-bath particles do not interact and so they are on shell.

**Perturbation theory**

Suppose now that we introduce an interaction and again calculate \( \langle T\phi(x)\phi(0) \rangle \). \( \phi(x) \) is now the interacting Heisenberg-picture field: it is the familiar operator of zero-temperature field theory. We can develop a perturbation theory along the same lines as for zero-temperature scattering theory, by introducing an interaction picture that coincides with the Heisenberg picture at some time \( t_0 \):

\[
\phi_I(t, x) = \Lambda(t) \phi(t, x) \Lambda^{-1}(t)
\]

\[
\Lambda(t) = e^{i(t-t_0)H_0} e^{-i(t-t_0)H}
\]
where $H_{0I}$ is the free-field Hamiltonian in the interaction picture. Define as usual

$$\Lambda(t_1)\Lambda^{-1}(t_2) = U(t_1, t_2)$$

so that $\Lambda(t) = U(t, t_0)$. Because $U$ is constructed from ordinary zero-temperature operators, as usual we have

$$U(t_1, t_2) = T \exp \left( -i \int_{t_2}^{t_1} dt \ H_{t}^{\text{INT}}(t) \right)$$

We need $U(t_1, t_2)$ for complex $t_1$ and $t_2$, so that we integrate $t$ along some contour $C$ running from $t_2$ to $t_1$ in the complex plane and generalise the time ordering $T$ to an ordering $T_C$ along $C$: the operator whose argument is nearest to $t_1$ along the contour comes first.

Now, from the definition of $U$,

$$e^{-\beta H} = e^{-\beta H_{0I}} U(t_0 - i\beta, t_0)$$

so that

$$Z^{-1} \text{tr} e^{-\beta H} \phi(x)\phi(0) = Z^{-1} \text{tr} e^{-\beta H_{0I}} U(t_0 - i\beta, x^0) \phi_I(x) U(x^0, 0) \phi_I(0) U(0, t_0)
= Z_0 Z^{-1} \langle U(t_0 - i\beta, x^0) \phi_I(x) U(x^0, 0) \phi_I(0) U(0, t_0) \rangle_0$$

where I have used $U(t_1, t_2) U(t_2, t_3) = U(t_1, t_3)$.

Compare with what one has at zero temperature:

$$\langle 0| \phi(x)\phi(0)|0 \rangle = \langle 0| U(\infty, x^0) \phi_I(x) U(x^0, 0) \phi_I(0) U(0, -\infty)|0 \rangle$$

In (18) we have a non-interacting thermal average instead of a vacuum expectation value, the first argument of the first $U$ is $(t_0 - i\beta)$ instead of $\infty$, and the second argument of the last $U$ is $t_0$ instead of $-\infty$. In fact there is a close similarity between thermal perturbation theory and the usual zero-temperature Feynman perturbation theory. One draws familiar-looking Feynman graphs, now called thermal graphs, and the main difference is that instead of the internal lines in the ordinary Feynman graphs representing vacuum expectation values, in the thermal graphs they are non-interacting thermal averages.

In order to derive this, one needs to establish Wick’s theorem. It is a remarkable fact that indeed, for example,

$$\langle T_C \phi_I(x_1) \phi_I(x_2) \phi_I(x_3) \phi_I(x_4) \rangle_0 = \sum \langle T \phi_I \phi_I \rangle_0 \langle T \phi_I \phi_I \rangle_0$$

where the sum is over the possible pairings of the fields. For almost every ensemble other than one in thermal equilibrium there would be correction terms to (19a). To understand this, go back to the discrete case. One of the relations needed to establish (19a) is

$$\langle a_r^\dagger a_r^\dagger a_t a_u \rangle = \langle a_r^\dagger a_t \rangle \langle a_r^\dagger a_u \rangle + \langle a_r^\dagger a_u \rangle \langle a_r^\dagger a_t \rangle$$

This relation is straightforward except when the indices $r, s, t, u$ are all equal. Then, from (11), the right-hand side is just $2(f(e_r))^2$, while with (10b) the left-hand side is

$$\langle (a_r^\dagger a_r)^2 - (a_r^\dagger a_r) \rangle$$

which we calculate using (11c). We get the same result as for the right-hand side because of the fluctuation term $f(f + 1)$ in (11c); for most other ensembles this term would be different and so Wick’s theorem would not be valid.

One needs to choose a value for $t_0$. A common choice is $t_0 = 0$, with the contour $C$ for the $t$ integrations running from 0 to $-i\beta$ along the imaginary axis. This is the imaginary-time formalism. Alternatively,
$t_0 \to -\infty$, which with suitable contour choice gives the real-time formalism. Although under certain circumstances the imaginary-time formalism is a powerful calculational tool, the real-time formalism stays much closer to the physics and is more versatile.

**Real-time formalism**

One is interested in equilibrium properties of the plasma at finite times. Presumably these are independent of how it reached thermal equilibrium. So, as in familiar scattering theory, we are free to imagine that the interaction slowly switches off as we go into the remote past, and then when we take $t_0 \to -\infty$ the interaction-picture fields become the usual noninteracting $in$ fields. The corresponding $in$ states are direct products of non-interacting single-particle states. We need to choose how the contour $C$ runs from $-\infty$ to $(-\infty - i\beta)$, and the choice that keeps the formalism in the most direct contact with the physics is the so-called Keldysh one: along the real axis from $-\infty$ to $\infty$, back to $-\infty$, then straight down to $(-\infty - i\beta)$:

For most applications (though not all[6]) it turns out that the vertical part of the contour may be omitted. Then we may write the propagator that corresponds to a line of a thermal graph as a $2 \times 2$ matrix:

$$D(x_1, x_2) = \begin{bmatrix}
\langle T \phi_{in}(x_1) \phi_{in}(x_2) \rangle_0 & \langle \phi_{in}(x_2) \phi_{in}(x_1) \rangle_0 \\
\langle \phi_{in}(x_1) \phi_{in}(x_2) \rangle_0 & \langle T \phi_{in}(x_1) \phi_{in}(x_2) \rangle_0
\end{bmatrix}$$

(20)

When both $x_1$ and $x_2$ are on the $-\infty$ to $\infty$ part of $C$, the ordering $T_c$ along the contour is ordinary time ordering $T$; this corresponds to the element $D_{11}$ of $D$. When both are on the $\infty$ to $-\infty$ part of $C$, $T_c$ is anti-time-ordering $\bar{T}$; this corresponds to $D_{22}$. The off-diagonal elements correspond to $x_1$ being on one part of $C$ and $x_2$ on the other. There is translation invariance: the elements of $D$ depend only on the difference between $x_1$ and $x_2$.

I have already shown how to calculate $D_{11}$; the result is given in (15). The other elements of $D$ may be calculated in the same way, and its Fourier transform is

$$D(k) = \begin{bmatrix}
\frac{i}{2\pi \delta+(k^2-m^2)} & 2\pi \delta-(k^2-m^2) \\
 \frac{-i}{2\pi \delta-(k^2-m^2)} & \frac{2\pi \delta+(k^2-m^2)}{k^2-m^2+i\epsilon} + 2\pi \delta(k^2-m^2) n(k^0) \langle 1 \ 1 \rangle
\end{bmatrix}$$

(21a)

It may also be written in the form

$$D(k) = M(k^0) \bar{D}(k) M(k^0)$$

(21b)

with

$$M(k^0) = \sqrt{n(k^0)} \begin{bmatrix}
e^{\frac{1}{2}\beta|k^0|} & e^{-\frac{1}{2}\beta k^0} \\
e^{\frac{1}{2}\beta k^0} & e^{\frac{1}{2}\beta|k^0|}
\end{bmatrix}$$

$$\bar{D}(k) = \begin{bmatrix}
D(k) & 0 \\
0 & D^*(k)
\end{bmatrix}$$

$$D(k) = \frac{i}{k^2-m^2+i\epsilon}$$

(21c)

For the case of a fermion field, there is a rather similar matrix propagator, but with the Fermi-Dirac distribution replacing the Bose distribution.
Matrix structure

The elements of the matrix propagator (20) are not independent. For example,

\[
D_{21}(x) = \langle \phi_{\text{in}}(x) \phi_{\text{in}}(0) \rangle_0 = Z_0^{-1} \text{tr } e^{-\beta H_{\text{in}}} \phi_{\text{in}}(x) \phi_{\text{in}}(0) \\
= Z_0^{-1} \text{tr } e^{-\beta H_{\text{in}}} \phi_{\text{in}}(0) e^{\beta H_{\text{in}}} \phi_{\text{in}}(x) e^{\beta H_{\text{in}}} \\
= Z_0^{-1} \text{tr } e^{-\beta H_{\text{in}}} \phi(0) \phi(x^0 - i\beta, x) \\
= D_{12}(x^0 - i\beta, x) \tag{22a}
\]

Here, I have used a general property of traces, that \(\text{tr}(AB) = \text{tr}(BA)\), and the fact that \(H_{\text{in}}\) is the time-translation operator for the noninteracting field \(\phi_{\text{in}}\). The Fourier transform of (22a) is

\[
D_{12}(k) = e^{-\beta k^0} D_{21}(k) \tag{22b}
\]

Also, from their definitions (20), one can see that \(D_{11}\) and \(D_{22}\) may be expressed in terms of \(D_{12}\) and \(D_{21}\). It is this, together with (22b), which is responsible for the matrix structure (21b).

Define a dressed thermal propagator matrix \(D'(x_1, x_2)\) analogous to \(D(x_1, x_2)\) in (20), but with the interacting Heisenberg field instead of \(\phi_{\text{in}}\). For example,

\[
D'_{12}(x_1, x_2) = \langle \phi(x_2) \phi(x_1) \rangle \tag{22c}
\]

Then, because \(H\) is the time-translation operator for \(\phi\), we can again derive

\[
D'_{12}(k) = e^{-\beta k^0} D'_{21}(k) \tag{22d}
\]

and so deduce that \(D'\) has a matrix structure similar to that of \(D\):

\[
D'(k) = M \begin{pmatrix} D'(k) & 0 \\ 0 & D'^*(k) \end{pmatrix} M \tag{23}
\]

where \(M(k^0)\) is the same matrix as in (21c). Define the thermal self-energy matrix \(\Pi\) by

\[
-i\Pi = D^{-1} - D'^{-1} \tag{24a}
\]

Then it follows that \(\Pi\) has the structure

\[
-i\Pi = M^{-1} \begin{pmatrix} -i\Pi(k, T) & 0 \\ 0 & [ -i\Pi(k, T) ]^* \end{pmatrix} M^{-1} \tag{24b}
\]

If we then solve (24a) for \(D'\), we find

\[
D' = M \begin{pmatrix} \frac{k^2 - m^2 - \Pi}{-i} & 0 \\ 0 & \frac{k^2 - m^2 - \Pi}{-i} \end{pmatrix} M \tag{25}
\]

So it is natural to interpret \(\text{Re } \Pi\) as a temperature-dependent shift to the mass \(m^2\). \(\Pi\) also has an imaginary part, so the propagation of the field through the heat bath decays with time.

In scalar field theory,

\[
-i\Pi = \bullet + \bullet + \bullet + \cdots
\]

To calculate the contribution to \(\Pi_{12}\) from the second term, for example, one needs
where each line $i \overline{j}$ represents $D_{ij}(k)$ and each vertex 2 is the same as the normal vertex 1, but opposite in sign.

**Imaginary-time formalism**

The real-time formalism stays close to the physics, but has the calculational complication that the propagator is a matrix. In the imaginary-time formalism there is not this complication, though except for a few simple cases there is the need to perform an analytic continuation from imaginary to real time at the end of the calculation. In the imaginary-time formalism the $t$ integration runs along the imaginary axis, so ordinary time-ordering is replaced with ordering in imaginary time:

$$
\bar{D}(x_1, x_2) = \theta(-\text{Im } t)D_{21}(x_1, x_2) + \theta(\text{Im } t)D_{12}(x_1, x_2)
$$

(26)

where $t = x_1^0 - x_2^0$. Because both $x_1^0$ and $x_2^0$ are integrated from 0 to $i\beta$, we need $\bar{D}(x_1, x_2)$ for values of $\text{Im } t$ in the range $-\beta$ to $+\beta$. In this finite interval it has a Fourier-series expansion:

$$
\bar{D}(t, x) = \frac{i}{\beta} \sum_{n=-\infty}^{\infty} D_n(x) e^{\omega_n t}
$$

(27a)

where $\omega_n = n\pi/\beta$. However, the relation (22a) implies that $\bar{D}(t, x) = \bar{D}(t + i\beta, x)$, so that only even values of $n$ contribute to the sum. In the case of fermions, the anticommutativity of the fields results in a minus sign appearing in the corresponding relation (22a), and so then only odd values of $n$ contribute.

If we apply a 3-dimensional Fourier transformation to (27a) and invert the Fourier summation over $n$, we find

$$
D_n(k) = \int_0^{i\beta} dt \ e^{-\omega_n t} D_{21}(t, k)
$$

(27b)

which turns out to be just the ordinary Feynman propagator with $k^0 = i\omega_n$. So the Feynman rules are just like the zero-temperature ones, except that the energy-conserving $\delta$-function at each vertex is replaced with a Konecker delta which imposes conservation of the discrete energy, and round each loop of a thermal graph

$$
\int \frac{d^4k}{(2\pi)^4} \rightarrow \frac{i}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3}
$$

(28)

**Gauge theories**

For gauge theories there is the complication that the grand partition function has to include the projection operator $\mathbb{P}$ onto physical states: see (3). The manipulations that set up the scalar-field perturbation theory include the use of the basic property of traces, trace $AB = \text{trace } BA$. To follow exactly the same route for a gauge theory one would need to use trace $AB\mathbb{P} = \text{trace } B\mathbb{P}$, but this is not a valid identity. It turns out[7] that there are two formalisms for perturbation theory for a thermal gauge theory:

1 Only the two physical degrees of freedom of the gauge field (the transverse polarisations) acquire the additional thermal propagator; the other components of the gauge field, and the ghosts, remain frozen at zero temperature. (This is for the bare propagators; self-energy insertions in the unphysical bare propagators do depend on the temperature.)

2 All components of the gauge field, and the ghosts, become heated to temperature $T$.

In the zero-temperature field theory, the ghosts are introduced in order to cancel unwanted contributions from the unphysical components of the gauge field[8], and the two formalisms of the thermal perturbation theory lead to the same answers for calculations of physical quantities for that reason. Often, using formalism 1 makes calculations simpler. It also makes them stay closer to the physics.
Photon or dilepton emission from a plasma

As an application, consider the emission of a real or virtual photon of momentum $q$ from a quark-gluon plasma. This is supposed to be an important diagnostic test of whether a plasma has been created in an experiment and has reached thermal equilibrium, and is then a way to measure its temperature.

Before a photon is emitted, the plasma is described by the density matrix

$$\rho = Z^{-1} \sum_i |i\text{ in}\rangle\langle i\text{ in}|e^{-\beta H}$$

(29)

The emission probability is calculated from squared matrix elements of the Heisenberg-picture electromagnetic current\(^9\):

$$W^\mu\nu(q) = Z^{-1} \sum_f \int d^4x e^{iq\cdot x} \langle f\text{ out}|J^\mu(x) \left( \sum_i |i\text{ in}\rangle\langle i\text{ in}|e^{-\beta H} \right) J^\nu(0)|f\text{ out}\rangle$$

(30a)

where I have introduced also a complete set of out states for the plasma. These satisfy the completeness relation

$$\sum_f |f\text{ out}\rangle\langle f\text{ out}| = 1$$

and so

$$W^\mu\nu(q) = Z^{-1} \sum_i \int d^4x e^{iq\cdot x} |i\text{ in}\rangle\langle i\text{ in}|e^{-\beta H} J^\nu(0)J^\mu(x)|i\text{ in}\rangle$$

(30b)

If we introduce a matrix $G^{\mu\nu}(q)$ analagous to $D_1^\mu$, but with thermal averages of products of electromagnetic currents instead of fields, $W^{\mu\nu}(q)$ is just $G^{\mu\nu}_{12}(q)$. So we draw thermal graphs where current $q$ enters at a 1 vertex and leaves at a 2 vertex, and distribute the labels 1 and 2 in all possible ways on the other vertices. For example,

![Thermal graphs](image)

where the solid lines are quarks and the dashed lines are gluons. The emission rate is calculated from a sum of matrix elements times their complex conjugates; the vertices labelled 1 correspond to a contribution to the matrix element and those labelled 2 to its complex conjugate. The sets of 1-vertices and of 2-vertices are joined by (12)-lines which, according to (21a) are on shell and represent particles in the heat bath. According to (21a),

$$D_{12}(k) = \begin{cases} 
2\pi \delta(k^2 - m^2) n(k^0) & (k^0 > 0) \\
2\pi \delta(k^2 - m^2) (1 + n(k^0)) & (k^0 < 0) 
\end{cases}$$

(31)

So, because the density of particles with momentum $k$ in the heat bath is $n(k^0)$, when $k^0 > 0$ a (12)-line represents a particle with momentum entering the photon-production reaction from the heat bath, while when $k^0 < 0$ it represents instead stimulated emission of a particle with momentum $-k$ into the heat bath. In the loop integrations, the energy on each line extends over both positive and negative values, so the thermal graphs sum together many physical processes. Consider the first graph, for example. Its right-hand part, with the 2-vertex, represents the processes

![Ordinary Feynman graphs](image)
depending on the signs of the relevant internal energies. In fact, energy-momentum conservation allows only the first one to be non-zero. I have not drawn in the other heat-bath particles, but remember that they are there as spectators. The other part of the thermal graph contains 1-vertices and (11)-lines, and according to (21a) the (11)-propagator is the ordinary Feynman zero-temperature propagator, plus a thermal part. If I use only the zero-temperature part of $D_{11}$ in each of these lines, I obtain the amplitude

\begin{equation}
\begin{array}{c}
\text{(ordinary Feynman graph)}
\end{array}
\end{equation}

(plus other terms which again vanish for kinematic reasons) and so part of the thermal graph represents the interference between this and $\Rightarrow$. If instead I use the thermal part $n(k^0)2\pi\delta(k^2)$ of the (11) gluon propagator, I obtain the amplitudes

\begin{equation}
\begin{array}{c}
\text{(ordinary Feynman graphs)}
\end{array}
\end{equation}

In each case, the incoming and outgoing gluon lines must have the same momentum $k$, so that these amplitudes again interfere with $\Rightarrow$, but now with the gluon $k$ being one of the spectator particles in the heat bath:

\begin{equation}
\begin{array}{c}
\text{(ordinary Feynman graph)}
\end{array}
\end{equation}

Similarly, I can identify physical processes that involve the thermal parts of the (11) quark propagators. Even a simple-looking thermal graph corresponds to a large number of physical processes\cite{10}, each of which can be rather complicated. An example is

\begin{equation}
\begin{array}{c}
\text{(thermal graph)}
\end{array}
\end{equation}

for which just one of the physical processes is the interference between

\begin{equation}
\begin{array}{c}
\text{(ordinary Feynman graphs)}
\end{array}
\end{equation}

A disconnected Feynman graph occurs because the thermal graph has “islands” of groups of 1-vertices entirely surrounded by 2-vertices.
Resummation and discontinuities

The matrix amplitude $G^\mu\nu(q,T)$ has structure similar to that of $D'$ in (23):

$$G(q,T) = M \begin{pmatrix} G(q,T) & 0 \\ 0 & G^*(q,T) \end{pmatrix} M$$

(32)

From the explicit form (21c) of $M$ we find that

$$G_{12}(q,T) = e^{\frac{1}{2} \beta |q^0|} n(q^0) \left(G(q,T) + G^*(q,T)\right)$$

$$G_{11}(q,T) = n(q^0) \left(e^{\beta q^0} G(q,T) + G^*(q,T)\right)$$

(33)

So, for positive $q^0$,

$$G_{12}(q,T) = 2n(q^0) \text{Re} G(q,T) = \frac{2}{e^{\beta q^0} + 1} \text{Re} G_{11}(q,T)$$

(34a)

With a small change of notation, $G(q,T) = iF(q,T)$, this becomes

$$-iF_{12}(q,T) = 2n(q^0) \text{Im} F(q,T) = \frac{2}{e^{\beta q^0} + 1} \text{Im} F_{11}(q,T)$$

(34b)

This relation between the (12) and (11) elements of $F(q,T)$ is reminiscent of the zero-temperature unitarity relation, which expresses the imaginary part of the forward elastic scattering amplitude (which is the Fourier transform of a matrix element of a time-ordered product of fields) to the sum of integrated squared amplitudes needed to calculate reaction rates\textsuperscript{[11]}. So here the imaginary part of $F_{11}(q,T)$ (which is the Fourier transform of a thermal average of a time-ordered product of currents) is related to $F_{12}(q,T)$, which up to a factor of $i$ is the sum (30a) of integrated squared amplitudes needed to calculate a thermal reaction rate.

At zero temperature the amplitude is real for $q^2 < 4m^2$, and for $q^2 > 4m^2$ twice its imaginary part is the discontinuity across the set of branch cuts that run along the positive real axis from $q^2 = 4m^2$ to infinity. Such a relationship holds for each Feynman graph that contributes to the amplitude; the Feynman-graph relationship is known as the cutting rule, and is named after Cutkosky who first discovered it\textsuperscript{[11]}.

There are also cutting rules at nonzero temperature\textsuperscript{[3]}. The simplest graph that contributes to $F(q,T)$ is

(thermal graph)

This graph has a branch cut that runs from $q^2 = 4m^2$ to $\infty$, like at zero temperature, and another from $q^2 = 0$ to $-\infty$. It is real between these cuts. The discontinuities across each of these are proportional to the imaginary part of $F_{11}(q,T)$. However, this fact is not very useful, because for higher-order graphs there is no gap between the branch cuts and no region where $F_{11}(q,T)$ is real: it cannot be analytically continued from the upper side of the cuts to the lower side, and so the notion of a discontinuity is not useful. The relation (34b) between $F_{12}(q,T)$, from which the reaction rate is calculated, and the imaginary part of $F_{11}(q,T)$ is still valid, but it is no longer the discontinuity across a cut.

When we calculate $F_{11}(q,T)$, the internal lines of the graph correspond to thermal propagators $D_{11}(k)$ and $D_{11}(q-k)$. From the explicit form (21a) of $D_{11}$, we see that these internal propagators are singular at $k^2 = m^2$ and $(q-k)^2 = m^2$. When we perform the necessary integration over the internal loop momentum $k$, these singularities generate\textsuperscript{[11]} the branch points at $q^2 = 0$ and $4m^2$. However, Dyson resummation replaces the bare propagators $D_{11}$ with dressed propagators $D'_{11}$,

\[
\begin{array}{cccccccc}
- & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - \\
\end{array}
\]
so that the graph becomes

The resummation removes the singularity of $D_{11}(k)$ at $k^2 = m^2$ and replaces it with a singularity whose position, according to (25), is given by

$$k^2 = m^2 + \Pi(k, T) \quad \text{or} \quad k^2 = m^2 + \Pi^*(k, T)$$

(35)

The consequence is that the branch point of $F_{11}$ that was at $q^2 = 4m^2$ before the resummation has now moved off the real $q^2$ axis. But along the real axis the relation (34b) between $F_{12}(q, T)$ and the imaginary part of $F_{11}(q, T)$ is still valid.

**Infrared divergences**

The infrared divergences of zero-temperature become much worse at finite temperature: the Bose distribution diverges at zero energy and causes the usual logarithmic divergences to become power divergences. We know that the infrared divergences must cancel if the theory is to make sense, and in practice they always do, but there is no general theory yet to show this. To some extent, the situation can be rescued by including thermal self-energy insertions in the propagators, so that they acquire a mass proportional to the temperature. But, in the case of photons or gluons, not all the degrees of freedom have a mass, according to perturbation theory. If the so-called magnetic mass is non zero, it is nonperturbative.

Consider, for example, the effect on the decay rate $\pi^0 \to e^+e^-$ of the microwave background, which is a heat bath consisting only of photons. Let $\Gamma$ be the decay rate in vacuum. The heat bath will change it partly because it gives the electrons an additional temperature-dependent mass $\delta m_e^2 \propto e^2T^2$. This causes a change $\Delta \Gamma = \delta m_e^2 \partial \Gamma / \partial m_e^2$, which is associated with thermal graphs of the form

There is also the thermal graph

One finds that

$$\frac{\Delta \Gamma}{\Gamma} = \frac{\delta m_e^2}{\Gamma} \frac{\partial \Gamma}{\partial m_e^2} + \frac{m_e \alpha_{EM}}{\pi^3 Q} \int \frac{d^4k \delta(k^2)n(k^0)}{k^2} \int d^4p_1 d^4p_2 \delta^{(+)}(p_1^2 - m_e^2)\delta^{(+)}(p_2^2 - m_e^2)$$

$$\left(\frac{p_1}{p_1 \cdot k} - \frac{p}{p_2 \cdot k}\right)^2 \{\delta^{(4)}(p_1 + p_2 - P) - \delta^{(4)}(p_1 + p_2 + k - P)\}$$

(36)

where $Q^2 = (\frac{1}{4}m_e^2 - m_e^2)$. In the integral, the first $\delta^{(4)}$-function corresponds to the contribution from the internal vertices in the thermal graphs being both 1 or both 2 and the second to 12 and 21. Each term separately is infrared divergent, like $\int \frac{dk}{k^2}$, but the divergences cancel.

In fact there is more cancellation than just that of the infrared divergences. For $T \ll Q$ one can expand (36) in powers of $T^2/Q^2$. One finds that the first term, of order $\alpha_{EM}T^2/Q^2$, exactly cancels the electron-mass-shift contribution $\delta \Gamma$, so that the net change in the decay rate is of order $\alpha_{EM} T^4$.  

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The lesson is that, in thermal field theory, it is of importance to calculate all terms; gauge theories at finite temperature are rife with cancellations.

Another example where there is an apparent infrared problem is that of the calculation of the equation of state for the quark-gluon plasma. For instance, in the purely gluonic case thermal graphs of the form

\[
\text{become more and more divergent as more vertical lines are added. However, the problem goes away if one sums over all thermal graphs}^{[14]}.
\]

Consider scalar field theory, where the Hamiltonian \( H \) contains a term \( \int d^3x (-\frac{1}{2}m^2 \phi^2) \). Insert this into \( Z = \text{tr} e^{-H/T} \) and differentiate with respect to \( m^2 \):

\[
T \frac{\partial}{\partial m^2} \log Z = -\frac{1}{2} \int d^3x \langle (\phi(x))^2 \rangle = \langle (\phi(0))^2 \rangle V
\]  

(37a)

Here I have used translation invariance: \( \langle (\phi(x))^2 \rangle \) is independent of \( x \). With the relation \( PV = T \log Z \) from (4) and the definition (22c) of \( D'_{12}(x) \), this gives

\[
\frac{\partial P}{\partial m^2} = -\frac{1}{2} D'_{12}(0) = -\frac{1}{2} \int \frac{d^4q}{(2\pi)^4} D'_{12}(q)
\]  

(37b)

Insert the form of \( D'_{12}(q) \) obtained from the matrix product \( \mathbf{D}' = \mathbf{M} \tilde{\mathbf{D}}' \mathbf{M} \) in (24) and use the fact that the pressure must vanish when the particles are infinitely heavy:

\[
P = -\int_0^{\infty} dm^2 \int \frac{d^4q}{(2\pi)^4} \frac{e^{(q_0)}}{e^{\beta q^0} - 1} \frac{1}{q^2 - m^2 - \Pi(q, T, m)}
\]  

(37c)

where \( \Pi \) is the self energy defined in (24b) and \( e^{(q_0)} = \pm 1 \) according to whether \( q_0 \) is positive or negative. As any divergence of \( \Pi \) now appears in the denominator, the summation has made it harmless. One has to worry about the numerator possibly vanishing when \( q = 0 \) and \( m = 0 \), but this will be rendered harmless by the \( q^3 \) appearing in \( d^4q = q^3 dq d\Omega \). The quantities in (37) are all unrenormalised; there is still some work needed, involving the theory of composite operators, to cast the formula into one that involves only finite renormalised quantities\([15] \).

Linear response theory

Suppose that the thermal equilibrium of a plasma is disturbed by the switching on at \( t = 0 \) of an external electrostatic potential \( A'_0(x) \). Then the system’s Hamiltonian acquires an extra term

\[
H'(t) = \theta(t) \int d^3x J^0(x) A'_0(x)
\]  

(38)

where \( J^0 \) is the charge density. This will cause \( J^0 \) to change. As it is a Heisenberg-picture operator, its equation of motion is

\[
\frac{\partial J^0}{\partial t} = i [H + H', J^0]
\]  

(39a)

where \( H \) is the original Hamiltonian. When we take the thermal average of this equation, the contribution from \( H \) will disappear because originally there was thermal equilibrium. So the integrated change in \( \langle J^0(x) \rangle \) at very large time is

\[
\delta \langle J^0(x) \rangle = \int d^4x' G_R(x - x') A'_0(x')
\]  

(39b)
where
\[ G_R(x - x') = \theta(t - t')\langle [J^0(x), J^0(x')] \rangle \] (39c)

Taking the Fourier transform,
\[ \delta(J^0(k)) = G_R(k) A^0_{\text{ext}}(k) \] (39d)

We may express the retarded Green’s function \( G_R \) in terms of elements of the matrix Green’s function \( G_{\mu\nu} \)
\[ G_R = \frac{1}{2}(G^{00}_{11} - G^{00}_{22} + G^{00}_{21} - G^{00}_{12}) \] (40)

Each of the terms here may be calculated from perturbation theory. However, it is simpler to express \( G_R \) in terms of the function \( G^{00}(k) \) that appears in the diagonal matrix associated with \( G^{00} \) (see (23)):
\[ G_R = n(k^0)e^{\frac{1}{2}\beta k^0}|\{ (G^{00} + G^{00*}) \sinh \frac{1}{2} \beta |k^0| + (G^{00} - G^{00*}) \sinh \frac{1}{2} \beta k^0 \} | \] (41a)

On the other hand
\[ G_{11} = n(k^0)e^{\frac{1}{2}\beta |k^0|}\{ (G^{00} + G^{00*}) \sinh \frac{1}{2} \beta |k^0| + (G^{00} - G^{00*}) \cosh \frac{1}{2} \beta k^0 \} \] (41b)

So it is sufficient to calculate \( G^{00}_{11} \) and change its imaginary part by multiplying it by \( \tanh \frac{1}{2} \beta k^0 \).

This work was supported by the EC Programme “Training and Mobility of Researchers”, Network “Hadronic Physics with High Energy Electromagnetic Probes”, contract ERB FMRX-CT96-0008, and by PPARC

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