Biquandle Module Invariants of Oriented Surface-Links

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Abstract

We define invariants of oriented surface-links by enhancing the biquandle counting invariant using biquandle modules, algebraic structures defined in terms of biquandle actions on commutative rings analogous to Alexander biquandles. We show that bead colorings of marked graph diagrams are preserved by Yoshikawa moves and hence define enhancements of the biquandle counting invariant for surface links. We provide examples illustrating the computation of the invariant and demonstrate that these invariants are not determined by the first and second Alexander elementary ideals and characteristic polynomials.

Keywords: Biquandle modules, counting invariants, surface-links, marked graph diagrams

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1 Introduction

In [1] the notion of quandle modules was introduced and used in [4] to generalize quandle cocycle invariants of oriented classical knots and links using dynamical cocycles. In later papers [2, 3, 5, 6, 7, 10] the second listed author and collaborators adapted the dynamical cocycle idea to various settings in classical, virtual and twisted virtual knot theory, defining enhancements of the quandle, rack and biquandle counting invariants. Quandle module colorings of knots can be understood as secondary colorings of quandle colored knots using “beads” which obey a kind of customized Alexander quandle coloring rule with coefficients depending on the base quandle coloring. In particular, quandle modules can be understood as generalized Alexander quandles for quandle-colored knots and links.

In [8] the first listed author and coauthors considered Alexander biquandle colorings of oriented surface-links represented by marked graph diagrams, also known as marked vertex diagrams or ch-diagrams. In particular, the methods of [8] distinguished most of the oriented surface-links of small ch-index as identified in [11] but did not distinguish the surface-links $6_{1}^{0,1}$ and $8_{1}^{1,1}$. In this paper we consider biquandle module invariants in the setting of orientable surface-links. We show that biquandle module colorings are preserved by a generating set of Yoshikawa moves, and hence define invariants of oriented surface-links. As an application, we exhibit a biquandle module invariant which does distinguish $6_{1}^{0,1}$ and $8_{1}^{1,1}$, showing in particular that biquandle module invariants are not determined by the first and second Alexander elementary ideals and characteristic polynomials. We provides explicit examples to illustrate the computation of the invariant and report the results of computer calculation of the invariant for some choices of biquandle modules for the oriented links of small ch-index.

The paper is organized as follows. In Section 2 we review the basics of marked graph diagrams and surface-link theory. In Section 3 we review biquandles and biquandle colorings of marked graph diagrams. In Section 4 we recall the definition of biquandle modules (updated with current notation) and show that biquandle module “bead” colorings are preserved by oriented Yoshikawa moves. We define biquandle module enhancement invariants for oriented surface-links and compute some examples. We end in Section 5 with some questions for future work.

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2 Marked Graph Diagrams

In this section, we review (oriented) marked graph diagrams representing surface-links.

**Definition 1.** A marked graph diagram, also called a marked vertex diagram or ch-diagram, is a planar 4-regular graph with vertices decorated as classical crossings and saddle crossings as depicted.

\[
\begin{array}{c}
\text{Classical} \\
\begin{array}{c}
\text{Saddle}
\end{array}
\end{array}
\]

A marked graph diagram is **orientable** if each edge in the graph can be directed such that the classical crossings receive “pass-through” orientations and saddle crossings receive “source-sink” orientations.

A marked graph diagram is **admissible** if the two diagrams resulting from smoothing all saddle points with the bars and against the bars are unlinks.

A marked graph diagram represents an orientable surface-link in $\mathbb{R}^4$ if the diagram is admissible; the surface is obtained by replacing each saddle crossing with a saddle as depicted and capping off the resulting unlinks above and below.

Non-admissible marked graph diagrams represent cobordisms between the links obtained through smoothing. Non-orientable admissible diagrams represent non-orientable surface-links, and non-admissible non-orientable diagrams represent non-orientable cobordisms. In particular, we may identify a classical link $L$ with the surface-link given by the trivial cobordism (i.e., $L \times [0, 1]$).

Two marked graph diagrams represent ambient isotopic surface-links in $\mathbb{R}^4$ if and only if they are related by the Yoshikawa moves. In [9] the first author and coauthors identified the generating set of oriented Yoshikawa moves pictured here.

\[
\begin{align*}
\Gamma_1 : & \quad \xymatrix{ \circ \ar@{<->}[rr] & & \circ } & \quad \xymatrix{ \circ \ar@{<->}[rr] & & \circ } \\
\Gamma'_1 : & \quad \xymatrix{ \circ \ar@{<->}[rr] & & \circ } & \quad \xymatrix{ \circ \ar@{<->}[rr] & & \circ } \\
\Gamma_2 : & \quad \xymatrix{ \circ \ar@{<->}[rr] & & \circ } & \quad \xymatrix{ \circ \ar@{<->}[rr] & & \circ } \\
\Gamma_3 : & \quad \xymatrix{ \circ \ar@{<->}[rr] & & \circ } & \quad \xymatrix{ \circ \ar@{<->}[rr] & & \circ }
\end{align*}
\]
Marked graph diagrams and Yoshikawa moves provide a very convenient calculus for computing invariants of closed surface-links as well as cobordisms between knots and links. In particular, a classical knot or link diagram $L$ considered as a marked graph diagram can be pictured as a product $L \times [0,1]$. See [8, 9] for more.

3 Biquandles and Colorings

In this section we recall biquandles and the biquandle counting invariant for marked graph diagrams.

Definition 2. Let $X$ be a set. A biquandle structure on $X$ consists of two binary operations $\sqcup, \sqcap$ on $X$ satisfying for all $x, y, z \in X$

(i) $x \sqcup x = x \sqcap x$,

(ii) The maps $\alpha_x, \beta_x : X \to X$ and $S : X \times X \to X \times X$ defined by $\alpha_x(y) = y \sqcup x$, $\beta_x(y) = y \sqcap x$ and $S(x, y) = (y \sqcup x, x \sqcap y)$ are invertible, and

(iii) the exchange laws are satisfied:

\[
\begin{align*}
(x \sqcup y) \sqcap (z \sqcap y) &= (x \sqcup z) \sqcap (y \sqcup z), \\
(x \sqcap y) \sqcup (z \sqcup y) &= (x \sqcap z) \sqcup (y \sqcap z), \\
(x \sqcap y) \sqcap (z \sqcup y) &= (x \sqcup z) \sqcap (y \sqcup z).
\end{align*}
\]

Example 1. For any set $X$ and bijection $\sigma : X \to X$ the operations $x \sqcup y = x \sqcap y = \sigma(x)$ define a biquandle structure called a constant action biquandle. If $\sigma$ is the identity then we have a trivial biquandle.

Example 2. Let $X$ be any module over $\mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$. Then $X$ is a biquandle with operations

\[
x \sqcup y = tx + (s - t)y, \quad x \sqcap y = sx
\]
known as an *Alexander biquandle*. Equivalently, any abelian group $X$ with automorphisms $t, s : X \rightarrow X$ is an Alexander biquandle with

$$x \triangleright y = t(x) + s(y) - t(y), \quad x \triangleright y = s(x).$$

**Example 3.** Given an oriented marked surface diagram $L$, let $\{x_1, \ldots, x_n\}$ be a set of generators associates to the semi-arc in the diagram. A *biquandle word* is either a generator or obtained recursively from the generators as $a \triangleright b$ or $a \triangleright b$ where $a, b$ are biquandle words. Then the *fundamental biquandle* of $L$, $B(L)$, is the set of equivalence classes of biquandle words under the equivalence relation generated by the biquandle axioms and the *crossing relations*:

The biquandle axioms are chosen so that given a biquandle coloring of a diagram on one side of a move (Reidemeister move in the case of classical links, Yoshikawa move in the case of surface-links) there is a unique biquandle coloring of the diagram on the other side of the move. Hence by construction we have the following standard result:

**Theorem 1.** Let $X$ be a finite biquandle and $L, L'$ marked graph diagrams of ambient isotopic surface-links. Then the number of $X$-colorings of $L$ and the number of $X$-colorings of $L'$ are equal.

**Definition 3.** Let $X$ be a finite biquandle. The number of $X$-colorings of a surface-link represented by a marked graph diagram $L$ is called the *biquandle counting invariant* of $L$ with respect to $X$, denoted $\Phi_X^Z(L)$.

**Remark 1.** We can also define $\Phi_X^Z(L)$ as the cardinality of the set of biquandle homomorphisms $f : B(L) \rightarrow X$, i.e. maps satisfying

$$f(x \triangleright y) = f(x) \triangleright f(y) \quad \text{and} \quad f(x \triangleright y) = f(x) \triangleright f(y)$$

for all $x, y \in B(L)$. 4
4 Biquandle Module Enhancements

We will now adapt an idea from previous work (see for example [2]) to enhance the biquandle counting invariant for surface-links.

**Definition 4.** Let $X$ be a finite biquandle and let $R$ be a commutative ring with identity. A **biquandle module** over $X$ with coefficients in $R$ is an assignment of units $t_{x,y}, r_{x,y} \in R^\times$ and elements $s_{x,y} \in R$ satisfying for all $x, y, z \in X$

\[
\begin{align*}
    t_{x,x} + s_{x,x} &= r_{x,x} & (i.i) \\
    r_{y \triangledown x \triangledown z} r_{x \triangledown y} &= r_{x \triangledown y \triangledown z} & (iii.i) \\
    r_{x \triangledown y \triangledown z} t_{y, z} &= t_{y \triangledown x \triangledown z} & (iii.ii) \\
    t_{x \triangledown y \triangledown z} s_{y,z} &= s_{y \triangledown x \triangledown z} & (iii.iii) \\
    t_{x \triangledown y \triangledown z} t_{y, z} &= t_{x \triangledown x \triangledown z} & (iii.iv) \\
    s_{x \triangledown y \triangledown z} t_{y, z} &= s_{x \triangledown x \triangledown z} & (iii.v) \\
    t_{x \triangledown y \triangledown z} s_{x,z} + s_{x \triangledown y \triangledown z} s_{y,z} &= s_{x \triangledown x \triangledown z} & (iii.vi)
\end{align*}
\]

We can specify an $X$-module with a triple $[t,s,r]$ of matrices $t,s,r \in M_n(R)$ whose row $x$ column $y$ entries are $t_{x,y}, s_{x,y}$ and $r_{x,y}$ respectively.

**Example 4.** Let $X$ be the biquandle structure on the set $X = \{1, 2\}$ specified by the operation tables

\[
\begin{array}{c|cc}
    \triangledown & 1 & 2 \\
    \hline
    1 & 2 & 2 \\
    2 & 1 & 1
\end{array}
\quad
\begin{array}{c|cc}
    \triangledown & 1 & 2 \\
    \hline
    1 & 2 & 2 \\
    2 & 1 & 1
\end{array}
\]

Then our python computations reveal $X$-module structures over $\mathbb{Z}_5$ including

\[
\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 4 & 4 \\ 3 & 2 \end{bmatrix}.
\]

Then for instance we have $t_{1,1} = 2$, $s_{1,1} = 2$ and $r_{1,1} = 4$, and for $x = 1, y = 2, z = 2$ we have for axiom (iii.vi)

\[
t_{1 \triangledown 2, 2 \triangledown 2} s_{1,2} + s_{1 \triangledown 2, 2 \triangledown 2} s_{2,2} = t_{1 \triangledown 2, 1 \triangledown 2} + s_{2,1} s_{2,2} = 4(0) + 0(1) = 0 = 0(2) = s_{2,1} r_{2,2} = s_{1 \triangledown 2, 2 \triangledown 2} r_{2,2}
\]

and for axiom (iii.iii)

\[
r_{1 \triangledown 2, 2 \triangledown 2} s_{2,2} = r_{1 \triangledown 2, 1 \triangledown 2} s_{2,2} = 3(1) = 3 = 0(4) = s_{1,1} r_{1,2} = s_{2} r_{1,2} s_{1} r_{1,2}.
\]

The condition (i.i), i.e.

\[
t_{x,x} + s_{x,x} = r_{x,x},
\]

etc.

The biquandle module axioms are motivated by the idea of enhancing biquandle colorings of oriented link diagrams with secondary “bead colorings” where the beads satisfy equations similar to the Alexander biquandle relations but with coefficients which depend on the biquandle colors at a crossing:

\[
c = t_{x,y} a + s_{x,y} b \\
d = r_{x,y} b
\]

The condition (i.i), i.e.
is required by Reidemeister move I

\[
\begin{align*}
\text{which is satisfied provided that } b &= (t_{x,x} + s_{x,x})a = r_{x,x}a. \\
\text{The requirement that } t_{x,y} \text{ and } r_{x,y} \text{ are invertible implies that the pair } (c, d) \text{ determines the pair } (a, b), \text{ the pair } (a, d) \text{ determines the pair } (b, c) \text{ and the pair } (b, c) \text{ determines the pair } (a, d). \text{ This suffices to guarantee that a bead coloring on one side of a Reidemeister II move corresponds to a unique bead coloring on the other side of the move.}
\end{align*}
\]

The Reidemeister III move yields the conditions (iii.i) through (iii.vi):

\[
\begin{align*}
f &= r_{x,y,z} r_{y} j = r_{x,y,z} r_{y} c \\
&= r_{y} r_{x,z} r_{x} k = r_{y} r_{x,z} r_{x} c \\
h &= t_{y} s_{x,z} d + s_{y} s_{x,z} k = t_{y} s_{x,z} s_{x} r_{x,y} b + s_{y} s_{x,z} r_{x,y} c \\
&= r_{y} r_{z,y} r_{z} e = r_{y} r_{z,y} r_{z} b + r_{y} r_{z,y} r_{z} c \\
g &= t_{x,y,z} y i + s_{x} s_{y,y,z} y j = t_{x,y,z} y t_{x,y} a + t_{x,y,z} y s_{x,y} b + s_{x} s_{y,y,z} y r_{y,z} c \\
&= t_{x,y,z} y i + s_{x} s_{y,y,z} y c = t_{x,y,z} y t_{x,y} a + s_{x} s_{y,y,z} y t_{x,y} b + (t_{x,y,z} y s_{x,z} + s_{x} s_{y,y,z} s_{y,z}) c
\end{align*}
\]
The moves $\Gamma_4, \Gamma'_4, \Gamma_6$ and $\Gamma_7$ do not impose additional conditions on the bead colorings.
The condition (i.i) also ensures that the space of bead colorings is unchanged by Yoshikawa $\Gamma_5$-moves:

Finally, the Yoshikawa $\Gamma_8$-move does not require any additional conditions; the beads $c = t_{x,y}^{-1}a - t_{x,y}^{-1}s_{x,y}b$, $d = r_{x,y}^{-1}b$, $e = t_{y,x}^{-1}b - t_{y,x}^{-1}s_{y,x}a$ and $f = r_{y,x}^{-1}a$ are uniquely determined by the biquandle colors $x, y \in X$ and beads $a, b \in R$

Thus by construction we have the following:
Theorem 2. Let $X$ be a finite biquandle, $R$ a commutative ring, $m = [t, s, r]$ an $X$-module over $R$, and $L$ an oriented marked graph diagram. Then for each $X$-coloring $L_f$ of $L$, the number $\phi_m(L_f)$ of bead colorings by $m$ is unchanged by $X$-colored Yoshikawa moves.

Corollary 3. Let $X$ be a finite biquandle, $R$ a commutative ring, $[t, s, r]$ an $X$-module over $R$, and $L$ an oriented marked graph diagram. Then the multiset

$$\Phi^M,m_{X}(L) = \{|\phi_m(L_f)| \mid f \in \text{Hom}(B(L), X)\}$$

and polynomial

$$\Phi^M,m_{X}(L) = \sum_{f \in \text{Hom}(B(L), X)} u^{|\phi_m(L_f)|}$$

are invariants of surface-links called the biquandle module multiset and biquandle module polynomial respectively.

Example 5. Let us illustrate the computation of the invariant for the surface-link $L = 6^0_1,1$ below using the biquandle $X$ and module $m$ from example 4. This module data gives us bead coloring rules

where the beads are elements of $\mathbb{Z}_5$. There are four $x$-colorings of $L$ as shown.

The first $X$-coloring has system of bead coloring equations yielding coloring matrix

$$
\begin{align*}
t_{11}d + s_{11}c &= a \\
(-1) 0 & s_{11} t_{11} 0 0 0 \\
r_{11}c &= b \\
0 -1 & r_{11} 0 0 0 \\
t_{11}c + s_{11}d &= e \\
0 0 & t_{11} s_{11} r_{11} 0 0 \\
r_{11}d &= a \\
-1 0 0 & r_{11} 0 0 0 \\
t_{21}e + s_{21}g &= f \\
0 0 0 & 0 t_{21} r_{21} 0 \\
r_{21}g &= a \\
-1 0 0 0 0 & 0 r_{21} \\
s_{12}b + t_{12}g &= a \\
-1 s_{12} 0 0 0 0 & t_{12} \\
r_{12}b &= f \\
0 r_{12} 0 0 0 & -1 0
\end{align*}
$$
Then substituting the values from our chosen $X$-module we obtain matrix over $\mathbb{Z}_5$

\[
\begin{bmatrix}
4 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \leftrightarrow \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 1 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

and this $X$-coloring has a one dimensional space of bead colorings contributing $u^5$ to the invariant. Computing the other three, we obtain invariant value $2u^5 + 2u^{25}$. We observe that this contains more information than the biquandle counting invariant $\Phi^Z_X(6_{11}) = 4$ alone, since it separates the colorings into two sets: two with 25 bead colorings and two with 5. In particular this invariant distinguishes this surface-link from the unlink of a torus and sphere which has invariant value $4u^5$.

**Example 6.** Let $X$ be the biquandle

\[
\begin{array}{c|c|c}
\sigma & 1 & 2 \\
1 & 2 & 2 \\
2 & 1 & 1 \\
\end{array}
\begin{array}{c|c|c}
\tau & 1 & 2 \\
1 & 2 & 2 \\
2 & 1 & 1 \\
\end{array}
\]

from example 4 and $m$ the biquandle module over $\mathbb{Z}_5$ given by the matrices

\[
\begin{bmatrix}
3 & 4 \\
4 & 3 \\
\end{bmatrix}, \begin{bmatrix}
0 & 0 \\
0 & 0 \\
\end{bmatrix}, \begin{bmatrix}
3 & 1 \\
1 & 3 \\
\end{bmatrix}.
\]

Our python computations give the following $\Phi^Z_X$ values for orientable surface-links of small $ch$-index with
orientations as shown.

The results are in the table:

| $\Phi^m_X(L)$ | $L$            |
|----------------|----------------|
| $2u^3$         | $8_1, 9_1, 10_1, 10_2, 10_3$ |
| $2u^5 + 2u^{25}$ | $6^{0,1}, 10^{0,1}$ |
| $2u + 2u^{25}$ | $8_1^{0,1}, 10_1^{1,1}$ |
| $4u^{25}$      | $9_1^{0,1}, 10_1^{0,1}$ |
| $4u^{25} + 4u^{125}$ | $10_1^{0,0,1}$ |

We can observe that this particular pair of biquandle and module do not distinguish the surface-knots in this small sample, but are effective at distinguishing the surface-links from each other.

**Example 7.** For our final example we computed $\Phi^m_X$ for the orientable surface-links of small $ch$-index with respect to the $X$-modules over $\mathbb{Z}_3$ given by the matrices

$$m_1 = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

and

$$m_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

for the biquandle

| $\Gamma$ | 1 | 2 | 3 |
|----------|---|---|---|
| 1        | 2 | 2 | 2 |
| 2        | 2 | 1 | 1 |
| 3        | 3 | 3 | 3 |

| $\sigma$ | 1 | 2 | 3 |
|----------|---|---|---|
| 1        | 2 | 3 | 1 |
| 2        | 3 | 1 | 2 |
| 3        | 1 | 2 | 3 |
The results are in the table.

| \( L \) | \( \Phi_{m_1}^n(L) \) | \( \Phi_{m_2}^n(L) \) |
|-------|----------------|----------------|
| \( 2_1 \) | \( 3u^3 \) | \( 3u^3 \) |
| \( 0_1^{0,1} \) | \( 3u^9 \) | \( 3u^3 \) |
| \( 8_1 \) | \( 9u^3 \) | \( 9u^9 \) |
| \( 8_1^{1,1} \) | \( 3u^9 \) | \( 3u^3 \) |
| \( 9_1 \) | \( 9u^3 \) | \( 9u^9 \) |
| \( 9_1^{0,1} \) | \( 3u^9 \) | \( 3u^3 \) |
| \( 10_1 \) | \( 3u^3 \) | \( 3u^3 \) |
| \( 10_2 \) | \( 9u^3 \) | \( 9u^9 \) |
| \( 10_3 \) | \( 3u^3 \) | \( 3u^3 \) |
| \( 10_1^1 \) | \( 9u^3 \) | \( 9u^9 \) |
| \( 10_2^{0,1} \) | \( 3u^9 \) | \( 3u^3 \) |
| \( 10_2^{1,1} \) | \( 3u^9 \) | \( 3u^9 \) |
| \( 10_1^{0,0,1} \) | \( 9u^{27} \) | \( 9u^9 \) |

5 Questions

We end with some questions and directions for future research.

Faster methods for finding biquandle modules would be desirable; our current approach fills in entries in the \([t,s,r]\) matrix using the module conditions and works well enough for small biquandles and small rings, but other methods will be necessary for finding biquandle modules over larger finite and infinite rings.

As in [5], biquandle modules over polynomial rings should be of interest. Such a module effectively defines Alexander invariants for biquandle-colored links. This suggests natural questions such as:

- Which, if any, skein relations are satisfied by various biquandle modules?
- What kinds of categorifications can be defined for these invariants?
- What additional enhancements can be defined in the case of biquandle module invariants of surface-links?

et cetera.

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