Solution of heat equation (Partial Differential Equation) by various methods.

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Abstract

In the present paper we solved heat equation (Partial Differential Equation) by various methods. The methods used here are Laplace Transform method, method of separation of variables, Fourier Transform and MATLAB software. We reached the same solution at the end in Laplace Transform method, method of separation of variables, but by Fourier Transform we reached solution in different form that is in sine and cosine series form.

Keywords: Transform method, Complimentary function, Particular Integral, Method of separation of variable

1. Introduction

An equations involving partial derivatives which is called as partial differential equations, which is written in terms of heat conduction and wave propagation useful in many physical processes and produce the two basic partial differential equations which are known as the heat equation or diffusion equation and wave equation.

A partial differential equation are used in various applications in the study of gravitation, heat transfer, perfect fluids and quantum mechanics. Most of the authors solve partial differential equations by using Integral transforms which includes Fourier transform, Laplace transform.

Here, we consider classical method for solution of these equations, namely, Laplace transform method which provide an alternative method of solution for these equations. With the use of the Laplace transform theory, a linear partial differential equation with constant coefficients is transformed into an ordinary differential equation. The transformed equation is then solved by the classical methods and the inverse transform of this solution gives the required solution. The physical problems like transient and steady state analysis of heat conduction in solids, vibrations of continuous mechanical systems can be solved by the method of Laplace Transform. [1,2]

In the same manner, Fourier transform is also useful in solving these equations. With respect to the space variable, using the Fourier transform and thus reducing the partial differential equation to an ordinary one. The transformed equation is then solved by the classical method for the transformed variable, using the boundary conditions involved and the Fourier inverse transform of this solution gives
the required solution. All these methods are very accurate and efficient in development of finding the approximate solutions for the partial differential equations. [3,6]

2. Methodology
A partial differential equation is solved under given conditions and restrictions. The problems of partial differential equations may be broadly classified into the following two major classes:
1. Initial value problems
2. Boundary value problems:
We solved three problems by various methods and compared it.

3. Examples

3.1. Solve \( \frac{\partial \omega}{\partial v} = 2 \frac{\partial^2 \omega}{\partial u^2} \) using the initial conditions \( \omega (0,v)= \omega (2,v)=0 \) and \( \omega (u,0) = 3 \sin 6 \pi u \) [4,5]

3.1.1 Solution Using Laplace Transform

The given equation is \[ \frac{\partial \omega}{\partial v} = 2 \frac{\partial^2 \omega}{\partial u^2} \] (1)

Where \( \omega = \omega (u,v) \)

Let \( L [\omega (u,v)] = \overline{\omega} (u,v) \) so that \( L^{-1}[\overline{\omega} (u,v)] = \omega (u,v) \)

Taking Laplace transform of both sides of equation , we obtain

\[ L[\frac{\partial \omega}{\partial v}] = 2 L[\frac{\partial^2 \omega}{\partial u^2}] \]

\[ s \overline{\omega} (u,s) - \omega (u,0) = 2 \frac{d^2 \overline{\omega}}{du^2} \]

Using \( \omega (u,0) = 3 \sin 6 \pi u \), we get

\[ s \overline{\omega} (u,s) - 3 \sin 6 \pi u = 2 \frac{d^2 \overline{\omega}}{du^2} \]

We can write above equation as

\[ (2D^2 - s) \overline{\omega} (u,s) = -3 \sin 6\pi u \] (2)

Equation (2) represents a linear differential equation with constant coefficient .

Therefore complementary function of equation (2) is \( k_1 e^{-\frac{1}{\sqrt{2}} u} + k_2 e^{\frac{1}{\sqrt{2}} u} \)
Now particular integral = \( \frac{1}{2D^2 - s} (-3\sin 6\pi u) \)

\[ = \frac{3\sin 6\pi u}{s + 72\pi^2} \]

Complete solution is = CF + PI

\[ \omega (u, s) = k_1 e^{-\frac{\sqrt{3}}{2} u} + k_2 e^{\frac{\sqrt{3}}{2} u} + \frac{3\sin 6\pi u}{s + 72\pi^2} \]

To obtain \( k_1 \) and \( k_2 \), we employ the unused boundary conditions

\[ \omega (0,v) = \omega (2,v) = 0 \]

\[ \text{L}[\omega (0,v)] = \overline{\omega} (0,s) = 0 \]

\[ \text{L}[\omega (2,v)] = \overline{\omega} (2,s) = 0 \]

Using equation (4) in (3) we get \( k_1 = -k_2 \)

Using equation (5) in (3) we get \( k_1 = 0 \Rightarrow k_2 = 0 \)

Therefore \( \omega (u, s) = \frac{3\sin 6\pi u}{s + 72\pi^2} \)

Taking inverse of above equation we get the solution

\[ \omega (u, v) = 3\sin(6\pi u) e^{-72\pi^2 v} \]  (3)

3.1.2. Solution Using method of separation of variable

Let \( \omega = U V \)

\[ \frac{\partial \omega}{\partial v} = U V' \]

And \( \frac{\partial^2 \omega}{\partial u^2} = V U'' \)

Equation (1) gives \( U V' = c^2 V U' \)

\[ \Rightarrow \frac{1}{c^2} V' = \frac{U'}{U} = -P^2 \]

So \( \frac{V'}{V} = -c^2 P^2 \quad \text{and} \quad \frac{U'}{U} = -P^2 \)

Which on solving \( V = e^{-c^2 P^2} \) and \( U = k_2 \cos pu + k_3 \sin pu \)
Applying boundary conditions we get

$$\omega(u, v) = 3 \sin(6 \pi u) e^{-72 \pi^2 v}$$

(4)

3.1.3. Solution Using Fourier Transform Method

$$\frac{\partial \omega}{\partial v} = 2 \frac{\partial^2 \omega}{\partial u^2}, \quad \omega(u, v) = 0, \text{ for } 0 < u < 2$$

$$\omega(u, 0) = 3 \sin 6 \pi u$$

$$\frac{d \overline{\omega}}{dv} = -\frac{p^2 \pi^2}{4} \omega + \frac{p \pi}{2} \left[ \omega(0, v) - (-1)^p \right] \omega(2, v) = -\frac{p^2 \pi^2}{4} \overline{\omega}$$

$$\frac{d \overline{\omega}}{\omega} = -\frac{p^2 \pi^2}{4} dv$$

$$\log \overline{\omega} = -\frac{p^2 \pi^2}{4} v + \log A$$

$$\overline{\omega} = A e^{-\frac{p^2 \pi^2}{4}}$$

When $$t = 0$$, $$\overline{\omega} = A$$

Given $$\omega(u, 0) = 3 \sin 6 \pi u$$

By definition

$$\overline{\omega}(p, 0) = \int_0^3 3 \sin 6 \pi u \sin \frac{p \pi u}{2} du$$

$$= \frac{3}{2} \int_0^3 \left[ \cos(6 \pi - \frac{p \pi}{2})u - \cos(6 \pi + \frac{p \pi}{2})u \right] du$$

$$= \frac{3}{2} \left[ \sin(6 \pi - \frac{p \pi}{2})u - \sin(6 \pi + \frac{p \pi}{2})u \right]_0$$

$$= \frac{3}{2} \left[ \frac{-\sin p \pi}{(6 \pi - \frac{p \pi}{2})} - \frac{\sin p \pi}{(6 \pi + \frac{p \pi}{2})} \right]$$

$$= \frac{3}{2} \sin p \pi \left[ \frac{p \pi}{36 \pi - p^2 \pi^2} \right]$$

$$= \frac{3}{2} \sin p \pi \left[ \frac{p \pi}{36 \pi - p^2 \pi^2} \right]$$
We tried to develop a computer program (code) in Matlab programming of scientific computing and implement analytic solution for a heat equation. The main parts of the implementation of our analytic scheme are given as in the following fig:

\[
\begin{align*}
A &= \frac{3}{2} \sin p\pi \\
\omega(u,v) &= \sum_{p=2}^{\infty} \left( -\frac{3}{2} \sin \frac{p\pi}{2} \right) \sin \left( \frac{p\mu}{2} \right) e^{-\frac{p^2\pi^2}{4}} \\
(5)
\end{align*}
\]

3.2.1 Solution Using Laplace Transform

Solve \( \frac{\partial \omega}{\partial v} = \frac{2}{\partial u^2} \frac{\partial^2 \omega}{\partial u^2} \), for \( 0 < u < 4, v > 0 \) given \( \omega(0,v) = 0 ; \omega(4,v) = 0 ; \)

\( \omega(u,0) = 3\sin \pi u - 2\sin 5\pi u. \)

The given equation is \( \frac{\partial \omega}{\partial v} = \frac{2}{\partial u^2} \frac{\partial^2 \omega}{\partial u^2} \) \( (6) \)

Taking Laplace transform of both sides of equation, we obtain

\[
\begin{align*}
L[\frac{\partial \omega}{\partial v}] &= 2 L[\frac{\partial^2 \omega}{\partial u^2}] \\
\mathcal{L}[\omega(u,s)] - \omega(u,0) &= 2 \frac{d^2 \omega}{du^2} \\
\text{Using } \omega(u,0) &= 3\sin \pi u - 2\sin 5\pi u,
\end{align*}
\]
we get \( s \overline{\omega}(u,s) - 3 \sin \pi u + 2 \sin 5\pi u = 2 \frac{d^2 \overline{\omega}}{du^2} \)

We can write above equation as

\[(2D^2 - s) \overline{\omega}(u,s) = -3 \sin \pi u + 2 \sin 5\pi u\]

Equation represents a linear differential equation with constant coefficient.

Therefore complementary function of equation is \( k_1 e^{\sqrt[2]{2}u} + k_2 e^{-\sqrt[2]{2}u} \)

Now particular integral \( = \frac{1}{2D^2 - s} (-3 \sin \pi u + 2 \sin 5\pi u) \)

\[= \frac{3 \sin \pi u - 2 \sin 5\pi u}{2\pi^2 + s} - \frac{2 \sin 5\pi u}{50\pi^2 + s}\]

Complete solution is \( = CF + PI \)

\[\overline{\omega}(u,s) = k_1 e^{\sqrt[2]{2}u} + k_2 e^{-\sqrt[2]{2}u} + \frac{3 \sin \pi u - 2 \sin 5\pi u}{2\pi^2 + s} - \frac{2 \sin 5\pi u}{50\pi^2 + s}\]

(7)

To obtain \( k_1 \) and \( k_2 \), we employ the unused boundary conditions

\( \omega (0,v) = \omega (4,v) = 0 \)

\[L [\omega (0,v)] = \overline{\omega} (0,s) = 0\]

(8)

\[L [\omega (2,v)] = \overline{\omega} (2,s) = 0\]

(9)

Using equation (8) in (7) we get \( k_1 = -k_2 \)

Using equation (9) in (7) we get \( k_1 = 0 \Rightarrow k_2 = 0 \)

Therefore \( \overline{\omega} (u,s) = \frac{3 \sin \pi u - 2 \sin 5\pi u}{2\pi^2 + s} - \frac{2 \sin 5\pi u}{50\pi^2 + s} \)

Taking inverse of above equation we get the solution

\( \omega (u,v) = 3 e^{-2\pi v} \sin \pi u - 2 e^{-50\pi v} \sin 5\pi v \)

(10)

3.2.2. Solution Using method of separation of variable

Let \( \omega = UV \)

So \( \frac{\partial \omega}{\partial v} = UV' \)
And \( \frac{\partial^2 \omega}{\partial u^2} = V U^* \)

Equation (6) gives \( UV' = 2VU^* \)

\[ \Rightarrow \frac{1}{2} \frac{V'}{V} = \frac{U'}{U} = -P^2 \]

So \( \frac{V'}{V} = -2P^2 \) and \( \frac{U'}{U} = -P^2 \)

Which on solving \( V = k_1 e^{-P^2v} \) and \( U = k_2 \cos pu + k_3 \sin pu \)

Applying boundary conditions we get the same solution

\[ \therefore \omega(u, v) = 3e^{-2\pi v} \sin \pi u - 2e^{-50\pi v} \sin 5\pi u \] (11)

3.2.3. Solution Using Fourier Transform Method

Since \( \omega(0, v) \) given, take finite Fourier cosine transform. [7, 9]

\[ \int_0^4 \frac{\partial \omega}{\partial v} \cos \frac{p\pi u}{4} du = 2 \int_0^4 \frac{\partial^2 \omega}{\partial u^2} \cos \frac{p\pi u}{4} du \]

\[ \frac{d\omega}{dv} = F_i(\frac{\partial^2 \omega}{\partial u^2}) = 2[-\frac{p^2 \pi^2}{16} \omega + \frac{p\pi}{4}[\omega(0, v) - (-1)^v \omega(4, v)]] \]

\[ = -\frac{p^2 \pi^2}{8} \omega_s \]

On solving we get

\[ \omega_s = Ae^{-p^2\pi^2/8} \]

Given \( \omega(u, 0) = 3\sin \pi u - 2\sin 5\pi u \)

Taking sine transform

\[ \bar{\omega}_s (p, 0) = \int_0^4 (3\sin \pi u - 2\sin 5\pi u) \sin \frac{p\pi u}{4} du \]

\[ = 0, \text{ if } p \neq 4 \text{ or } p \neq 20 \]

\[ \text{if } p = 4, \bar{\omega}_s (4, 0) = 6 \]

\[ \text{if } p = 20, \bar{\omega}_s (20, 0) = -4 \]
\[ \therefore \omega (u,v) = \frac{2}{4} \sum_{p=1}^{\infty} u_p(p,v) \sin \frac{p\pi u}{4} \]

\[ \therefore \omega (u,v) = \frac{1}{2} \left[ 6e^{-\frac{p^2\pi^2}{8}} \sin \pi u - 4e^{-\frac{p^2\pi^2}{8}} \sin 5\pi u \right] \]

Where \( p \) in the first term is 4 and \( p \) in the second term is 20

\[ \therefore \omega (u,v) = 3e^{-2\pi^2 v} \sin \pi u - 2e^{-50\pi^2 v} \sin 5\pi u \] (12)

For this problem following is the graphical representation using MATLAB.

**Figure 3.2.** Graphical representation of the problem using Fourier Transform Method.

3.3 Solve \( \frac{\partial \omega}{\partial u} = 2 \frac{\partial \omega}{\partial v} + \omega \) where \( \omega(u,0) = 6e^{-3u} \) (13)

3.3.1 Using the method of separation of variables Assume the solution \( \omega(u,v) = U(u)V(v) \)

Substituting in the given equation, we have

\[ U'V = 2UV' + UV \]

\[ (U' - U)V = 2VT' \]

Or \[ \frac{U' - U}{2U} = \frac{V'}{V} = k \]

\[ U' - U - 2kU = 0 \] Or \[ \frac{U'}{U} = 1 + 2k \] and \( \frac{V'}{V} = k \)

Solving above equations, we get

\[ U = ce^{(1+2k)u} \] and \[ V = ce^{kv} \]

Thus \( \omega(u,v) = UV = cce^{(1+2k)u}e^{kv} \)
Now \( 6 e^{-3u} = \omega(u,0) = cc' e^{(1+2k)u} \)

Therefore \( cc' = 6 \) and \( 1+2k = -3 \) or \( k = -2 \)

Substituting these values in we get \( \omega = 6e^{-3u} e^{-2v} \)

\[
\omega=6e^{-(3u+2v)} \quad \text{which is the required solution.} \quad (14)
\]

### 3.3.2. Alternative approach by Laplace Transform

\[
\frac{\partial \omega}{\partial u} = 2 \frac{\partial \omega}{\partial v} + \omega \quad \text{where} \quad \omega = \omega(u,v)
\]

Let \( L[\omega(u,v)] = \overline{\omega}(u,s) \) so that \( L^{-1}[\overline{\omega}(u,s)] = \omega(u,v) \)

Taking Laplace transform of both sides of above equation, we obtain

We get

\[
\frac{d\overline{\omega}}{du} = 2s \overline{\omega} - 12 e^{-3u} + \overline{\omega}
\]

\[
\frac{d\overline{\omega}}{du} - (2s+1)\overline{\omega} = -12 e^{-3u}
\]

Equation is a linear differential equation in \( \overline{\omega}(u,s) \)

\[
\text{IF} = e^{-\int (2s+1)du} = e^{-(2s+1)u}
\]

Therefore, solution of equation is

\[
\overline{\omega}(u,s) e^{-(2s+1)u} = \int -12 e^{-3u} e^{-(2s+4)u} du + k
\]

\[
= -12 \int e^{-(2s+4)u} du + k
\]

\[
= -12 \frac{e^{-(2s+4)u}}{-(2s+4)} + k
\]

\[
= \frac{6}{s+2} e^{-(2s+4)u} + k
\]

\[
\overline{\omega}(u,s) = \frac{6e^{-3u}}{s+2} + k e^{(2s+1)u}
\]

To obtain the value of \( k \), we employ the unused boundary condition, namely \( u(u,v) \) is bounded for \( u>0 \),

\[
\text{i.e.} \quad \lim_{u \to +\infty} \omega(u,v) = \text{finite}
\]

Transforming this condition, we have
Using this condition in equation, we get

\[ \text{finite } = 0 + k \lim_{x \to \infty} e^{(2x+1)u} \]

The right hand side will be finite only if \( k = 0 \),

Therefore we obtain

\[ \omega(u, s) = \frac{6e^{-3u}}{s + 2} \]

Taking inverse Laplace transform, we get

\[ L^{-1} \left[ \omega(u, s) \right] = 6e^{-3u} L^{-1} \left[ \frac{1}{s + 2} \right] \]

\[ \omega(u, v) = 6e^{-3u} e^{-2v} = 6e^{-(3u+2v)} \]  \hspace{1cm} (15)

For the above problem also we tried to develop a computer program (code) in Matlab programming of scientific computing and implement analytic solution for a heat equation. The main parts of the implementation of our analytic scheme are given as in the following fig:

**Figure 3.3.** Graphical representation of the problem using Laplace Transform Method.
4. Result and discussion

4.1. Criteria for choosing of Laplace Transform

1) Both the independent variables or at least one of the independent variables should have the range from 0 to $\infty$. If for both the range $-\infty$ to $\infty$ one can not use Laplace Transform because LT has been defined $0$ to $\infty$ only.

2) If both variables have range from 0 to $\infty$ apply LT with respect to either variable.

3) If only one variable has range from 0 to $\infty$ apply LT with respect to that variable only.

4) Appropriate initial conditions must be specified at the lower limit of the variable which has the range from 0 to $\infty$ also one cannot use the Laplace Transform on partial differential equation if we do not know initial conditions.

4.2. Criteria for choosing of Fourier Transform

1) One of the independent variable should have the range from $-\infty$ to $\infty$ and apply Fourier Transform with respect to that variable only.

2) Both $v$ and $\frac{\partial v}{\partial x}$ must vanish as $x$ tends to $-\infty$ and $\infty$

If these two conditions not satisfied then we cannot apply Fourier Transform.

In examples it is found that the solution by using Laplace transform and separation of variable method are exactly same whereas by using Fourier Transform we got solution in the form of Fourier series. Also it is found that in third example 3.3 could not be solved by Fourier transform.

We shown graphical representation Using MATLAB we get the solution in terms of graph of examples 3.1,3.2,3.3.

In [6] Mehran Makhtoumi has discussed analytical solution using Fourier Series and numerical solutions of heat diffusion in one dimensional thin rod. [8] Yang, X.-J has used the theoretical approach to solve the differential equation in the steady heat-transfer problem. [4] E Firmansah* and M F Rosyid discussed Heat Equation with Stochastic Boundary Problems.

5. Conclusion

In this paper we have solved three partial differential equations by various methods like separation of variable method, Fourier transform method and Laplace transform method. Finally, we have shown the analytical solution graphically by using MATLAB. It is found that solution is same by using separation of variable method and Laplace transform method but by using Fourier transform we get solution in the form of Fourier series. If appropriate initial conditions specified in the problem then Laplace transform method is easy to find analytical solution.

In future work, we try to implement the numerical methods and also compare in heat equation in engineering applications.[8].

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