The structure of a minimal $n$-chart with two crossings I: Complementary domains of $\Gamma_1 \cup \Gamma_{n-1}$

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Abstract

This is the first step of the two steps to enumerate the minimal charts with two crossings. For a label $m$ of a chart $\Gamma$ we denote by $\Gamma_m$ the union of all the edges of label $m$ and their vertices. For a minimal chart $\Gamma$ with exactly two crossings, we can show that the two crossings are contained in $\Gamma_\alpha \cap \Gamma_\beta$ for some labels $\alpha < \beta$. In this paper, we study the structure of a disk $D$ not containing any crossing but satisfying $\Gamma \cap \partial D \subset \Gamma_{\alpha+1} \cup \Gamma_{\beta-1}$.

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1 Introduction

Charts are oriented labeled graphs in a disk with three kinds of vertices called black vertices, crossings, and white vertices (see Section 2 for the precise definition of charts, black vertices, crossings, and white vertices). From a chart, we can construct an oriented closed surface embedded in 4-space $\mathbb{R}^4$ (see [4, chapter 14, chapter 18 and chapter 23]). A C-move is a local modification between two charts in a disk (see Section 2). A C-move between two charts induces an ambient isotopy between oriented closed surfaces corresponding to the two charts. Two charts are said to be C-move equivalent if there exists a finite sequence of C-moves which modifies one of the two charts to the other.

We will work in the PL or smooth category. All submanifolds are assumed to be locally flat. A surface link is a closed surface embedded in 4-space $\mathbb{R}^4$. A 2-link is a surface link each of whose connected component is a 2-sphere. A 2-knot is a surface link which is a 2-sphere. An orientable surface link is called a ribbon surface link if there exists an immersion of a 3-manifold $M$ into $\mathbb{R}^4$ sending the boundary of $M$ onto the surface link such that each connected component of $M$ is a handlebody and its singularity consists of ribbon singularities, here a ribbon singularity is a disk in the image of $M$ whose pre-image consists of two disks; one of the two disks is a proper disk of $M$ and the other is a disk in the interior of $M$. In the words of charts, a ribbon surface link is a surface link corresponding to a ribbon chart, a chart C-move equivalent to a chart without white vertices [2]. A chart is called a 2-link chart if a surface link corresponding to the chart is a 2-link.
In this paper, we denote the closure, the interior, the boundary, and the complement of (...) by $Cl(...)$, $Int(...)$, $\partial(...)$, ($...)^c$ respectively. Also for a finite set $X$, the notation $|X|$ denotes the number of elements in $X$.

At the end of this paper there is the index of new words and notations introduced in this paper.

Kamada showed that any 3-chart is a ribbon chart $[2]$. Kamada’s result was extended by Nagase and Hirota: Any 4-chart with at most one crossing is a ribbon chart $[5]$. We showed that any $n$-chart with at most one crossing is a ribbon chart $[7]$. We also showed that any 2-link chart with at most two crossings is a ribbon chart $[8], [9]$.

Let $\Gamma$ be a chart. For each label $m$, we define $\Gamma_m = \text{the union of all the edges of label } m \text{ and their vertices in } \Gamma$.

In this paper we investigate the structure of minimal charts with two crossings (see Section 2 for the precise definition of a minimal chart), and give us an enumeration of the charts with two crossings. The enumeration is much complicated than the one of 2-bridge links in $\mathbb{R}^3$, of course. We enumerate charts with two crossings as follows (see Section 9 and $[11]$): For any minimal $n$-chart $\Gamma$ with two crossings in a disk $D^2$, there exist two labels $1 \leq \alpha < \beta \leq n - 1$ such that $\Gamma_\alpha$ and $\Gamma_\beta$ contain cycles $C_\alpha$ and $C_\beta$ with $C_\alpha \cap C_\beta$ the two crossings and that for any label $k$ with $k < \alpha$ or $\beta < k$, the set $\Gamma_k$ does not contain a white vertex. If $\Gamma_\alpha$ or $\Gamma_\beta$ contains at least three white vertices, then after shifting all the free edges and simple hoops into a regular neighbourhood of $\partial D^2$ by applying C-I-M1 moves and C-I-M2 moves, we can find an annulus $A$ containing all the white vertices of $\Gamma$ but not intersecting any hoops nor free edge such that (see Fig. 1(a))

(1) each connected component of $Cl(D^2 - A)$ contains a crossing,

(2) $\Gamma \cap \partial A = (C_\alpha \cup C_\beta) \cap \partial A$, and $\Gamma \cap \partial A$ consists of eight points.

We can show the annulus $A$ can be split into mutually disjoint four disks $D_1, D_2, D_3, D_4$ and mutually disjoint four disks $E_1, E_2, E_3, E_4$ such that

(3) for each $i = 1, 3$ (resp. $i = 2, 4$) the tangle $(\Gamma \cap D_i, D_i)$ is an IO-tangle of label $\alpha$ (resp. label $\beta$) (see $[10]$ theorem 1.3] and $[11]$),

(4) for each $i = 1, 2, 3, 4$, the tangle $(\Gamma \cap E_i, E_i)$ is a net-tangle with $\Gamma \cap E_i \subset \cup_{j=\alpha+1}^{\beta-1} \Gamma_j$ as shown in Fig. 1(b) (see Theorem 1.1 and Theorem 1.2).

We count the number of edges between edges with black vertices in Fig. 1(b) to enumerate charts with two crossings. As important results, from the enumeration we can calculate the fundamental group of the exterior of the surface link represented by $\Gamma$, and the braid monodromy of the surface braid represented by $\Gamma$.

We shall define a net-tangle and related words, and we shall introduce two theorems in this paper.
Figure 1: (b) A tangle \((\Gamma \cap E_i, E_i)\) with \(\Gamma \cap E_i \subset \Gamma_2 \cup \Gamma_3 \cup \Gamma_4\) for the case \(\alpha = 1\) and \(\beta = 5\), here all the free edges and simple hoops are in a regular neighbourhood of \(\partial D^2\).

Figure 2: The gray area is a disk \(E\). The edge \(e_1\) is an I-edge for \(E\), and the edge \(e_2\) is an O-edge for \(E\).

If an edge \(e\) of a chart \(\Gamma\) is oriented from a vertex \(v_1\) to the other vertex \(v_2\), then we say that the edge \(e\) is outward at \(v_1\), and the edge \(e\) is inward at \(v_2\).

Let \(\Gamma\) be a chart, and \(D\) a disk. An edge \(e\) of the chart \(\Gamma\) is called an I-edge (resp. O-edge) for \(E\) provided that (see Fig. 2)

(i) the edge \(e\) possesses two white vertices, one is in \(\text{Int } E\) and the other in \(E^c\),

(ii) the edge \(e\) intersects \(\partial E\) by exactly one point, and

(iii) the edge \(e\) is inward (resp. outward) at the vertex in \(\text{Int } E\).

We often say just an I-edge instead of an I-edge for \(E\) if there is no confusion. Similarly we often say just an O-edge instead of an O-edge for \(E\).

Let \(\Gamma\) be a chart, and \(D\) a disk. The pair \((\Gamma \cap D, D)\) is called a tangle provided that

(i) \(\partial D\) does not contain any white vertices, black vertices nor crossings of \(\Gamma\),
(ii) if an edge of $\Gamma$ intersects $\partial D$, then the edge intersects $\partial D$ transversely,

(iii) $\Gamma \cap D \neq \emptyset$.

For a simple arc $X$, we set

$\partial X = $ the set of its two endpoints, and

$\text{Int } X = X - \partial X$.

Let $\Gamma$ be a chart. A tangle $(\Gamma \cap D, D)$ is called a net-tangle provided that

(i) the disk $D$ contains no crossing, hoop, nor free edge but a white vertex (see Section [2] for the definition of free edges and hoops),

(ii) there exist two labels $\alpha, \beta$ with $\alpha < \beta$ and $\Gamma \cap D \subset \bigcup_{i=\alpha}^{\beta} \Gamma_i$, and

(iii) there exist two arcs $L_\alpha, L_\beta$ on $\partial D$ with $L_\alpha \cap L_\beta$ two points such that

(a) $\Gamma \cap \partial D = (\Gamma_\alpha \cap \text{Int } L_\alpha) \cup (\Gamma_\beta \cap \text{Int } L_\beta)$,

(b) all the edges intersecting $L_\alpha$ are I-edges of label $\alpha$ or all the edges intersecting $L_\alpha$ are O-edges of label $\alpha$, and

(c) all the edges intersecting $L_\beta$ are O-edges of label $\beta$ or all the edges intersecting $L_\beta$ are I-edges of label $\beta$.

The pairs $(\alpha, \beta)$ and $(L_\alpha, L_\beta)$ are called a label pair and a boundary arc pair of the net-tangle respectively. If all the edges of label $\alpha$ intersecting $L_\alpha$ are I-edges (resp. O-edges), and if all the edges of label $\beta$ intersecting $L_\beta$ are O-edges (resp. I-edges), then the net-tangle is said to be upward (resp. downward) (see Fig. [3]). An upward or downward net-tangle with a label pair $(\alpha, \alpha + 1)$ is called an N-tangle.

An edge of a chart $\Gamma$ is called a terminal edge if it contains a white vertex and a black vertex. If a terminal edge is inward at its black vertex, then the edge is called an I-terminal edge, otherwise the edge is called an O-terminal edge.

For minimal charts, we shall show the following two theorems:

**Theorem 1.1** Let $\Gamma$ be a minimal chart, and $(\Gamma \cap D, D)$ a net-tangle with a label pair $(\alpha, \alpha + 1)$. Then we have the following.

(a) The tangle is an N-tangle.

(b) The number of the edges of label $\alpha$ intersecting $\partial D$ is equal to the number of the edges of label $\alpha + 1$ intersecting $\partial D$.

(c) There exists a terminal edge in $D$.

(d) The number of terminal edges of label $\alpha$ in $D$ is equal to the number of terminal edges of label $\alpha + 1$ in $D$. 

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Figure 3: An example of an upward net-tangle with a label pair \((\alpha, \alpha + 1)\). The gray area is a disk \(D\), the thick edges are of label \(\alpha\), and the thin edges are of label \(\alpha + 1\).

(e) If the tangle is upward (resp. downward), then all the terminal edges of label \(\alpha\) in \(D\) are I-terminal (resp. O-terminal) edges and all the terminal edges of label \(\alpha + 1\) in \(D\) are O-terminal (resp. I-terminal) edges.

Let \(E\) be a disk. A simple arc \(\ell\) in \(E\) is called a proper arc provided that \(\ell \cap \partial E = \partial \ell\).

There exists a special C-move called a C-I-M2 move (see Fig. 5 in Section 2 for C-I-M2 moves). Let \(\Gamma\) be a minimal chart, and \((\Gamma \cap D, D)\) a net-tangle with a label pair \((\alpha, \beta)\). Let \(\tilde{\Gamma}\) be a minimal chart such that \((\tilde{\Gamma} \cap D, D)\) is a tangle without crossing nor hoops. Then the chart \(\tilde{\Gamma}\) is said to be M2-related to \(\Gamma\) with respect to \(D\) provided that

(i) \(\Gamma \cap D^c = \tilde{\Gamma} \cap D^c\), and

(ii) the chart \(\tilde{\Gamma}\) is obtained from the chart \(\Gamma\) by a finite sequence of C-I-M2 moves in \(D\) each of which modifies two edges of some label \(i\) with \(\alpha < i < \beta\).

**Theorem 1.2** Let \(\Gamma\) be a minimal chart, and \((\Gamma \cap D, D)\) a net-tangle with a label pair \((\alpha, \beta)\). Then \(\Gamma\) is M2-related to a minimal chart \(\tilde{\Gamma}\) with respect to \(D\) such that there exist N-tangles \((\tilde{\Gamma} \cap D_\alpha, D_\alpha), (\tilde{\Gamma} \cap D_{\alpha+1}, D_{\alpha+1}), \ldots, (\tilde{\Gamma} \cap D_{\beta-1}, D_{\beta-1})\) equipped with

(a) for each \(i = \alpha, \alpha + 1, \ldots, \beta - 1\), the tangle \((\tilde{\Gamma} \cap D_i, D_i)\) is an N-tangle with the label pair \((i, i + 1)\),

(b) \(D = \cup_{i=\alpha}^{\beta-1} D_i\),
(c) for each \( i = \alpha, \alpha + 1, \ldots, \beta - 2 \), the intersection \( D_i \cap D_{i+1} \) is a proper arc of \( D \).

(d) all the \( N \)-tangles are upward or downward simultaneously.

Our paper is organized as follows: In Section 2 we introduce the definition of charts and its related words. In Section 3 we investigate a path of label \( m \) called a one-way path. In Section 4 we investigate terminal edges and one-way paths of label \( m \) in an \( N \)-tangle. In Section 5 we investigate a disk \( E \) such that \( \partial E \) consists of two one-way paths of label \( m \). In Section 6 we investigate a disk \( E \) such that \( \partial E \) consists of an arc and two one-way paths of label \( m \). In Section 7 we prove Theorem 1.1. In Section 8 we prove Theorem 1.2. In [11], we shall propose a normal form for a minimal chart with exactly two crossings so that we enumerate minimal charts with exactly two crossings. In Section 9 we give an outline of [11].

2 Preliminaries

In this section, we introduce the definition of charts and its related words.

Let \( n \) be a positive integer. An \( n \)-chart (a braid chart of degree \( n \) or a surface braid chart of degree \( n \)) is an oriented labeled graph in a disk, which may be empty or have closed edges without vertices satisfying the following four conditions (see Fig. 4):

(i) every vertex has degree 1, 4, or 6.

(ii) the labels of edges are in \( \{1, 2, \ldots, n-1\} \).

(iii) in a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward, and these six are labeled \( i \) and \( i+1 \) alternately for some \( i \), where the orientation and label of each arc are inherited from the edge containing the arc.

(iv) for each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels \( i \) and \( j \) of the diagonals satisfy \( |i-j| > 1 \).

We call a vertex of degree 1 a \textit{black vertex}, a vertex of degree 4 a \textit{crossing}, and a vertex of degree 6 a \textit{white vertex} respectively. Among six short arcs in a small neighborhood of a white vertex, a central arc of each three consecutive arcs oriented inward (resp. outward) is called a \textit{middle arc} at the white vertex (see Fig. 4(c)). For each white vertex \( v \), there are two middle arcs at \( v \) in a small neighborhood of \( v \). An edge \( e \) is said to be \textit{middle at} a white vertex \( v \) if it contains a middle arc at \( v \).
Now \emph{C-moves} are local modifications of charts as shown in Fig.\,5 (cf. \cite{11}, \cite{4} and \cite{12}). We often use C-I-M2 moves, C-I-M3 moves, C-II moves and C-III moves.

An edge of a chart is called a \emph{free edge} if it contains two black vertices.

Let $\Gamma$ be a chart. Let $e_1$ and $e_2$ be edges of $\Gamma$ which connect two white vertices $w_1$ and $w_2$ where possibly $w_1 = w_2$. Suppose that the union $e_1 \cup e_2$ bounds an open disk $U$. Then $\text{Cl}(U)$ is called a \emph{bigon} of $\Gamma$ provided that any edge containing $w_1$ or $w_2$ does not intersect the open disk $U$ (see Fig.\,6). Since $e_1$ and $e_2$ are edges of $\Gamma$, neither $e_1$ nor $e_2$ contains a crossing.

Let $\Gamma$ be a chart. Let $w(\Gamma), f(\Gamma), c(\Gamma)$, and $b(\Gamma)$ be the number of white vertices of $\Gamma$, the number of free edges of $\Gamma$, the number of crossings of $\Gamma$, and the number of bigons of $\Gamma$ respectively. The 4-tuple $(c(\Gamma), w(\Gamma), -f(\Gamma), -b(\Gamma))$ is called a \emph{c-complexity} of the chart $\Gamma$. The 4-tuple $(w(\Gamma), c(\Gamma), -f(\Gamma), -b(\Gamma))$ is called a \emph{w-complexity} of the chart $\Gamma$. The 3-tuple $(c(\Gamma)+w(\Gamma), -f(\Gamma), -b(\Gamma))$ is called a \emph{cw-complexity} of the chart $\Gamma$ (see \cite{2} for complexities of charts).

A chart $\Gamma$ is said to be \emph{c-minimal} (resp. \emph{w-minimal} or \emph{cw-minimal}) if its \emph{c-complexity} (resp. \emph{w-complexity} or \emph{cw-complexity}) is minimal among the charts which are C-move equivalent to the chart $\Gamma$ with respect to the lexicographical order of the 4-tuple (or 3-tuple) of the integers.

In this paper, if a chart is c-minimal, w-minimal or cw-minimal, then we say that the chart is minimal.

A \emph{hoop} is a closed edge of a chart $\Gamma$ that contains neither crossings nor white vertices. Therefore a hoop decomposes $\Gamma$ into disjoint pieces: an inside, an outside and itself. A hoop is said to be \emph{simple} if one of the complementary domains of the hoop does not contain any white vertices.

For any chart in a disk $D^2$ we can move free edges and simple hoops into a regular neighbourhood of $\partial D^2$ by C-I-M2 moves and ambient isotopies of $D^2$ as shown in Fig.\,7. Even during argument, if free edges or simple hoops appear, we immediately move them into a regular neighbourhood of $\partial D^2$. Thus we assume the following \cite{7}, \cite{10} assumption 1):

\textbf{Assumption 1} For any chart in a disk $D^2$, all the free edges and simple hoops are in a regular neighbourhood of $\partial D^2$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{chart_example.png}
\caption{(a) a black vertex. (b) a crossing. (c) a white vertex. Each arc with three transversal short arcs is a middle arc at the white vertex.}
\end{figure}
Figure 5: For the C-III move, the edge containing the black vertex does not contain a middle arc at a white vertex in the left figure.

Let $\Gamma$ be a chart in $D^2$, and $X$ the union of all the free edges and simple hoops. Now $X$ is in a regular neighbourhood $N$ of $\partial D^2$ in $D^2$ by Assumption 1. Define

$\text{Main}(\Gamma) = \Gamma - X$.

Let $\hat{D} = \text{Cl}(D^2 - N)$. Then $\Gamma \cap \hat{D} = \text{Main}(\Gamma)$. Hence $(\Gamma \cap \hat{D}, \hat{D})$ is a tangle without free edges and simple hoops.

Assumption 2 In this paper, our arguments are done in the disk $\hat{D}$, otherwise mentioned.

Remark 2.1 ([10, remark 2.3]) Let $\Gamma$ be a minimal chart. Then we have the following:

1. if an edge of $\Gamma$ contains a black vertex, then the edge is a terminal edge or a free edge.

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Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. A simple closed curve in $\Gamma_m$ is called a **loop** if it contains exactly one white vertex. Note that loops may contain crossings of $\Gamma$.

Let $E$ be a disk, and $\ell_1, \ell_2, \ell_3$ three arcs on $\partial E$ such that each of $\ell_1 \cap \ell_2$ and $\ell_2 \cap \ell_3$ is one point and $\ell_1 \cap \ell_3 = \emptyset$ (see Fig. 8(a)), say $p = \ell_1 \cap \ell_2$, $q = \ell_2 \cap \ell_3$. Let $\Gamma$ be a chart in a disk $D^2$. Let $e_1$ be a terminal edge of $\Gamma$. A triplet $(e_1, e_2, e_3)$ of mutually different edges of $\Gamma$ is called a **consecutive triplet** if there exists a continuous map $f$ from the disk $E$ to the disk $D^2$ such that (see Fig. 8(b) and (c))

(i) the map $f$ is injective on $E - \{p, q\}$,
(ii) $f(\ell_3)$ is an arc in $e_3$, and $f(\text{Int } E) \cap \Gamma = \emptyset$, $f(\ell_1) = e_1$, $f(\ell_2) = e_2$,
(iii) $f(p)$ and $f(q)$ are white vertices.

If the label of $e_3$ is different from the one of $e_1$ then the consecutive triplet is said to be **admissible**.

**Remark 2.2** Let $(e_1, e_2, e_3)$ be a consecutive triplet. Since $e_2$ is an edge of $\Gamma$, the edge $e_2$ **MUST NOT** contain a crossing.

**Lemma 2.3** [Consecutive Triplet Lemma] ([7, lemma 1.1], [10, lemma 3.1])

Any consecutive triplet in a minimal chart is admissible.

Let $\Gamma$ be a chart. A tangle $(\Gamma \cap D, D)$ is called an **NS-tangle of label $m$** (new significant tangle) provided that

(i) if $i \neq m$, then $\Gamma_i \cap \partial D$ is at most one point,
(ii) $\Gamma \cap D$ contains at least one white vertex, and
(iii) for each label $i$, the intersection $\Gamma_i \cap D$ contains at most one crossing.
Lemma 2.4 ([10] theorem 1.3) In a minimal chart, there does not exist an
NS-tangle of any label.

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. Let $E$ be a disk with $\partial E \subset \Gamma_m$. Then the disk $E$ is called a 2-color disk provided that $\Gamma \cap E \subset \Gamma_m \cup \Gamma_{m-1}$ or $\Gamma \cap E \subset \Gamma_m \cup \Gamma_{m+1}$.

Lemma 2.5 ([10] corollary 3.4(b)) Let $\Gamma$ be a minimal chart, and $m$ a label
of $\Gamma$. If $E$ is a 2-color disk with $\partial E \subset \Gamma_m$, then $E$ does not contain any
terminal edge.

Lemma 2.6 ([10] lemma 7.6(b)) Let $\Gamma$ be a minimal chart, and $m$ a label
of $\Gamma$. If $E$ is a 2-color disk with $\partial E \subset \Gamma_m$ but without free edges nor simple
hoops, then $\Gamma_m \cap E$ is connected.

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. A simple closed curve in $\Gamma_m$ is
called a cycle of label $m$.

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. Let $C$ be a cycle of label $m$ in $\Gamma$
bounding a disk $E$. Then an edge $e$ of label $m$ is called an outside edge for
$C$ provided that

(i) $e \cap C$ consists of one white vertex or two white vertices, and

(ii) $e \not\subset E$.

For a cycle $C$ of label $m$, we define

$\mathcal{W}(C) = \{w \mid w$ is a white vertex in $C\}$,

$\mathcal{W}^\text{Mid}_O(C,m) = \{w \in \mathcal{W}(C) \mid$ there exists an outside edge for $C$ middle at $w\}$.

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. Let $E$ be a disk with $\partial E \subset \Gamma_m$. Then the disk $E$ is called a 3-color disk provided that

(i) the disk $E$ does not contain any crossings, and

(ii) $\Gamma \cap E \subset \Gamma_{m-1} \cup \Gamma_m \cup \Gamma_{m+1}$.
Lemma 2.7 ([10] corollary 4.4]) Let $\Gamma$ be a minimal chart. Let $C$ be a cycle of label $m$ in $\Gamma$ bounding a 3-color disk $E$ without free edges nor simple hoops. If $\Gamma_m \cap E$ connected, then $|W_{O}^{Mid}(C, m)| \geq 2$.

Combining Lemma 2.6 and Lemma 2.7 we have the following lemma.

Lemma 2.8 Let $\Gamma$ be a minimal chart, and $m, k$ labels of $\Gamma$ with $|m - k| = 1$. Let $(\Gamma \cap D, D)$ be a tangle with $\Gamma \cap D \subset \Gamma_m \cup \Gamma_k$ but without free edges nor simple hoops. Then for any cycle $C$ of label $m$ in $D$, we have $|W_{O}^{Mid}(C, m)| \geq 2$.

Lemma 2.9 [Boundary Condition Lemma] ([8] lemma 4.1, [10] lemma 11.1]) Let $(\Gamma \cap D, D)$ be a tangle in a minimal chart $\Gamma$ such that $D$ does not contain any crossing, free edge, nor simple hoop. Let $a = \min\{i \mid \Gamma_i \cap \partial D \neq \emptyset\}$ and $b = \max\{i \mid \Gamma_i \cap \partial D \neq \emptyset\}$. Then $\Gamma_i \cap D = \emptyset$ except for $a \leq i \leq b$.

3 One-way Paths

In this section we investigate a path in $\Gamma_m$.

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. A simple arc $P$ in $\Gamma$ is called a path provided that the end points of $P$ are vertices of $\Gamma$. In particular, if the path $P$ is in $\Gamma_m$, then $P$ is called a path of label $m$. Suppose that $v_0, v_1, \ldots, v_p$ are all the vertices in a path $P$ situated in this order on $P$. For each $i = 1, \ldots, p$, let $e_i$ be the edge in $P$ with $\partial e_i = \{v_{i-1}, v_i\}$. Then the $(p + 1)$-tuple $(v_0, v_1, \ldots, v_p)$ is called a vertex sequence of the path $P$, and the $p$-tuple $(e_1, e_2, \ldots, e_p)$ is called an edge sequence of the path $P$.

Let $P$ be a path in a chart with an edge sequence $(e_1, e_2, \ldots, e_p)$. An edge $e$ of the chart is called a side-edge for $P$ if $e \not\subseteq P$ but $e \cap (e_2 \cup e_3 \cup \cdots \cup e_{p-1}) \neq \emptyset$.

Let $m$ be a label of a chart $\Gamma$, and $P$ a path of label $m$ with a vertex sequence $(v_0, v_1, \ldots, v_p)$ and an edge sequence $(e_1, e_2, \ldots, e_p)$. The path $P$ is called a one-way path if for each $i = 1, 2, \ldots, p$, the edge $e_i$ is oriented from $v_{i-1}$ to $v_i$. The path $P$ is called an M&M path if the edge $e_1$ is middle at $v_0$ and the edge $e_p$ is middle at $v_p$. The path $P$ is called a dichromatic path if there exists a label $k$ with $|m - k| = 1$ such that any vertex of the path is contained in $\Gamma_m \cap \Gamma_k$.

Lemma 3.1 In a minimal chart, for any label $m$ there does not exist any dichromatic M&M one-way path of label $m$.

Proof. Suppose that there exists a dichromatic M&M one-way path of label $m$ in a minimal chart $\Gamma$. Let $P$ be a dichromatic M&M one-way path of label $m$ containing the least number of edges amongst all the dichromatic M&M one-way paths in the chart. Set $(v_0, v_1, \ldots, v_p)$ and $(e_1, e_2, \ldots, e_p)$ the vertex sequence and the edge sequence of $P$.

We claim that $p = 1$. For, if $p > 1$, let $e$ be the side-edge of label $m$ for $P$ containing the vertex $v_1$. If the edge $e$ is outward at $v_1$, then $e_1$ is middle
at \( v_1 \). The path with the edge sequence \((e_1)\) is a dichromatic M&M one-way path of label \( m \) whose length is shorter than \( P \). This is a contradiction. If the edge \( e \) is inward at \( v_1 \), then \( e_2 \) is middle at \( v_1 \). The path with the edge sequence \((e_2, e_3, \cdots, e_p)\) is a dichromatic M&M one-way path whose length is shorter than \( P \). This is a contradiction. Thus \( p = 1 \).

Since \( e_1 \) is middle at \( v_0 \) and \( v_1 \), we can eliminate the two vertices \( v_0 \) and \( v_1 \) by two C-I-M2 moves and a C-I-M3 move in a regular neighbourhood of the edge \( e_1 \) (see Fig. 9). This contradicts that the given chart is minimal. This proves Lemma 3.1.

Let \( \Gamma \) be a chart, and \( m \) a label of \( \Gamma \). Let \( C \) be a cycle of label \( m \). Let \( v_0, v_1, \cdots, v_{p-1} \) be all the vertices in \( C \), and \( e_1, e_2, \cdots, e_p \) all the edges in \( C \). Then the cycle \( C \) is called a one-way cycle provided that for each \( i = 1, 2, \cdots, p \), the edge \( e_i \) is oriented from \( v_{i-1} \) to \( v_i \), where \( v_p = v_0 \). We consider a loop as a one-way cycle.

**Lemma 3.2** Let \( \Gamma \) be a minimal chart, \( m, k \) labels of \( \Gamma \) with \( |m - k| = 1 \), and \((\Gamma \cap D, D)\) a tangle with \( \Gamma \cap D \subset \Gamma_m \cup \Gamma_k \) but without free edges nor simple hoops. Then \( D \) does not contain any one-way cycle of label \( m \).

**Proof.** Suppose that \( D \) contains a one-way cycle \( C \) of label \( m \). Now

1. for a white vertex \( v \) in \( \Gamma_m \) contained in three edges \( e_1, e_2, e_3 \) of label \( m \), if \( e_1 \) is inward at \( v \) and if \( e_2 \) is outward at \( v \), then \( e_3 \) is not middle at \( v \).

Hence we have

2. \[ \mathcal{W}^\text{Mid}_O(C, m) = \emptyset. \]

On the other hand, \( |\mathcal{W}^\text{Mid}_O(C, m)| \geq 2 \) by Lemma 2.8. This contradicts (2). Therefore there does not exist a one-way cycle of label \( m \) in \( D \). This proves Lemma 3.2.

**Lemma 3.3** Let \( \Gamma \) be a minimal chart, and \((\Gamma \cap D, D)\) an N-tangle or a net-tangle with a label pair \((\alpha, \alpha + 1)\). Let \( m = \alpha \) or \( m = \alpha + 1 \). Then \( D \) does not intersect any one-way cycle of label \( m \).

**Proof.** Suppose that \( D \) intersects a one-way cycle \( C \) of label \( m \). Then
Figure 10: The path $P$ is a one-way path with a vertex sequence $(v_0, v_1, v_2, v_3)$.

(1) the one-way cycle $C$ does not intersect $\partial D$.

For, if $C$ intersects $\partial D$, then $C$ contains an I-edge and an O-edge. On the other hand, by Condition (iii) of the definition of a net-tangle, all the edges of label $m$ intersecting $\partial D$ are I-edges or all the edges of label $m$ intersecting $\partial D$ are O-edges. This is a contradiction. Hence the one-way cycle $C$ does not intersect $\partial D$.

Since the one-way cycle $C$ intersects $D$, Statement (1) implies that

(2) the one-way cycle $C$ is contained in $D$.

On the other hand, by Condition (ii) of the definition of a net-tangle, we have $\Gamma \cap D \subset \Gamma_\alpha \cup \Gamma_{\alpha+1}$. Hence $D$ does not contain any one-way cycle of label $m$ by Lemma 3.2. This contradicts (2). Therefore Lemma 3.3 holds. □

Let $P$ be a one-way path in a chart, and $e$ a side-edge for $P$ not a loop. Let $v$ be a vertex in $e \cap \text{Int} P$, and $N$ a regular neighbourhood of $v$. The edge $e$ is said to be locally right-side (resp. locally left-side) at $v$ if the arc $e \cap N$ is situated right (resp. left) side of $P$ with respect to the direction of $P$ (see Fig. 10). If the edge $e$ is locally right-side at a vertex $v$ and inward (resp. outward) at $v$, then the edge is called a locally right-side edge inward (resp. outward) at $v$. Similarly if the edge $e$ is locally left-side at a vertex $v$ and inward (resp. outward) at $v$, then the edge is called a locally left-side edge inward (resp. outward) at $v$. In Fig. 10, the edge $e_2$ is a locally right-side edge inward at $v_1$, the edge $e_3$ is a locally right-side edge outward at $v_1$, the edge $e_4$ is a locally left-side edge inward at $v_2$, the edge $e_5$ is a locally left-side edge outward at $v_2$. But $e_1$ is not locally left-side at $v_0$ nor $e_6$ is not locally right-side at $v_3$.

Let $\Gamma$ be a chart, and $m$ a label of the chart. A one-way path $P$ of label $m$ is said to be upward right-selective (resp. upward left-selective) if any edge of label $m$ locally right-side (resp. left-side) at a vertex in $\text{Int} P$ is inward.
at the vertex (see Fig. 11(a) and (b)). A one-way path $P$ of label $m$ is said to be downward right-selective (resp. downward left-selective) if any edge of label $m$ locally right-side (resp. left-side) at a vertex in $\text{Int } P$ is outward at the vertex (see Fig. 11(c) and (d)).

Let $\Gamma$ be a chart, $P$ a path, and $E$ a disk. If each edge in $P$ intersects $\text{Int } E$ and if $P \cap E$ is connected, then we say that the path $P$ is dominated by the disk $E$ or the disk $E$ dominates the path $P$.

Let $e$ be an edge of label $m$ in a chart, and $E$ a disk. Let $P$ be an upward right-selective (resp. left-selective) one-way path of label $m$ starting from $e$ dominated by $E$. The path $P$ is said to be maximal with respect to $E$ if the path is not contained in another upward right-selective (resp. left-selective) one-way path of label $m$ starting from $e$ dominated by $E$. Similarly let $P$ be a downward right-selective (resp. left-selective) one-way path leading to $e$ dominated by $E$. The path $P$ is said to be maximal with respect to $E$ if the path is not contained in another downward right-selective (resp. left-selective) one-way path of label $m$ leading to $e$ dominated by $E$.

Lemma 3.4 Let $\Gamma$ be a minimal chart, and $(\Gamma \cap D, D)$ an $N$-tangle or a net-tangle with label pair $(\alpha, \alpha + 1)$. Set $m = \alpha$ or $m = \alpha + 1$. Let $e$ be an edge of label $m$ in $D$, and $P$ a one-way path of label $m$ dominated by $D$ with a vertex sequence $(v_0, v_1, \cdots, v_p)$. Then we have the following.

(a) Suppose that $P$ is an upward right-selective or left-selective one-way path starting from $e$. If $v_p$ is a white vertex in $\text{Int } D$, then $P$ is not maximal.

(b) Suppose that $P$ is a downward right-selective or left-selective one-way path leading to $e$. If $v_0$ is a white vertex in $\text{Int } D$, then $P$ is not maximal.
Proof. Statement (a). By Lemma 3.3 there does not exist a loop of label \( m \) containing \( v_p \). Now \( e_p \) is inward at \( v_p \). Since \( v_p \) is a white vertex, there are two edges of label \( m \) incident to \( v_p \) different from \( e_p \). There are two cases:

Case 1. The two edges are outward at \( v_p \).

Case 2. One of the two edges is inward at \( v_p \), and the other edge is outward at \( v_p \).

Case 1. If \( P \) is right-selective (resp. left-selective), let \( e' \) be the one of the two edges outward at \( v_p \) such that the other edge is locally left-side (resp. right-side) edge at \( v_p \) as a side-edge for the path \( P \cup e' \) (we select the ’right’ (resp. ’left’) edge \( e' \) with respect to the direction of the path \( P \), see Fig. 11(a)) and (b)).

Now \( P \cap e' = v_p \). For, if \( P \cap e' \) contains a vertex \( v_j \) for some \( 0 \leq j < p \), then \( e_j \cup e_j+2 \cup \cdots \cup e_p \cup e' \) is a one-way cycle of label \( m \) intersecting \( D \). This contradicts Lemma 3.3.

Thus the path \( P \cup e' \) is an upward right-selective (resp. left-selective) one-way path of label \( m \) starting from \( e' \) dominated by \( D \). Hence \( P \) is not maximal.

Case 2. There exists only one edge of label \( m \) outward at \( v_p \), say \( e' \). By the similar way as the one in Case 1, we can show that if \( P \) is right-selective (resp. left-selective), then \( P \cup e' \) is an upward right-selective (resp. left-selective) one-way path starting from \( e \) dominated by \( D \). Hence \( P \) is not maximal.

Thus Statement (a) holds. Similarly we can show Statement (b) (see Fig. 11(c) and (d)). □

4 N-tangles

In this section we investigate a terminal edge and a one-way path of label \( m \) in an N-tangle.

Let \( \Gamma \) be a minimal chart, \( v \) a white vertex, and \( e \) a terminal edge inward (resp. outward) at \( v \). Then the two edges inward (resp. outward) at \( v \) different from \( e \) are called the sibling edges of \( e \).

Lemma 4.1 Let \( \Gamma \) be a minimal chart, \( (\Gamma \cap D, D) \) an N-tangle or a net-tangle with a label pair \( (\alpha, \alpha + 1) \). Then we have the following.

(a) The sibling edges of an I-terminal edge in \( D \) are O-edges.

(b) The sibling edges of an O-terminal edge in \( D \) are I-edges.

Proof. By Condition (ii) of the definition of a net-tangle, we have

(1) \( \Gamma \cap D \subset \Gamma_\alpha \cup \Gamma_{\alpha+1} \).
Let $e$ be a terminal edge in $D$, $v$ the white vertex contained in the terminal edge $e$, and $e^*$ a sibling edge of $e$. Suppose $e^* \subset D$. Since $D$ does not contain a crossing,

(2) the edge $e^*$ does not contain a crossing.

Since any loop is a one-way cycle, the edge $e^*$ is not a loop by Lemma 3.3. Let $v^*$ be the vertex of $e^*$ different from $v$. Since $e$ is a terminal edge, the sibling edge $e^*$ is not a terminal edge. Thus the vertex $v^*$ is a white vertex by (2). Now one of the two edges $e, e^*$ is of label $\alpha$, and the other is of label $\alpha + 1$. Thus there exists a non-admissible consecutive triplet $(e, e^*, \tilde{e})$ by (1) and (2), here $\tilde{e}$ is an edge containing the vertex $v^*$, and of the same label as the one of $e$. This contradicts Consecutive Triplet Lemma (Lemma 2.3). Thus $e^* \not\subset D$.

Now the edge $e^*$ is an I-edge or an O-edge by Condition (iii) of the definition of a net-tangle. Therefore $e^*$ is an I-edge (resp. O-edge), if $e$ is inward (resp. outward) at $v$. Namely, $e^*$ is an I-edge (resp. O-edge), if $e$ is an O-terminal (resp. I-terminal) edge. This proves Lemma 4.1. \hfill \square

**Lemma 4.2** Let $\Gamma$ be a minimal chart, and $(\Gamma \cap D, D)$ a net-tangle with a label pair $(\alpha, \alpha + 1)$. Then the tangle is an $N$-tangle.

**Proof.** By Condition (ii) of the definition of a net-tangle, we have

(1) $\Gamma \cap D \subset \Gamma_\alpha \cup \Gamma_{\alpha+1}$.

If $\Gamma \cap \partial D = \emptyset$, then the tangle is an NS-tangle. This contradicts Lemma 2.4. Thus $\Gamma \cap \partial D \neq \emptyset$.

Let $e_1$ be an edge intersecting $\partial D$. If necessary, change orientation of all edges in the chart so that we can assume that

(2) the edge $e_1$ is an I-edge.

Let $m$ be the label of the edge $e_1$, here $m = \alpha$ or $m = \alpha + 1$. Then by Condition (iii) of the definition of a net-tangle,

(3) all the edges of label $m$ intersecting $\partial D$ are I-edges.

Let $P$ be the maximal upward right-selective one-way path of label $m$ starting from $e_1$ with respect to $D$. Let $(v_0, v_1, \ldots, v_p)$ be a vertex sequence and $(e_1, e_2, \ldots, e_p)$ an edge sequence of $P$. Now the edge $e_p$ is oriented from $v_{p-1}$ to $v_p$. Since there is no O-edge of label $m$ by (3), we have $v_p \in \text{Int} D$. Hence Lemma 3.4(a) implies that the vertex $v_p$ is not a white vertex, because $P$ is maximal with respect to $D$. Since $D$ does not contain any crossing by Condition (i) of the definition of a net-tangle, the vertex $v_p$ is a black vertex. Namely $e_p$ is a terminal edge contained in $D$. Since $e_p$ is inward at $v_p$, the edge $e_p$ is an I-terminal edge. Thus the sibling edges of the I-terminal edge is O-edges by Lemma 4.1(a). Let $\beta$ be the label of the sibling edges, here $\beta \neq m$. Hence by Condition (iii) of the definition of a net-tangle,
(4) all the edges of label $\beta$ intersecting $\partial D$ are O-edges.

Now (3) and (4) imply that if $m = \alpha$, then the tangle is upward, else the tangle is downward. Hence the tangle is an N-tangle. Therefore this proves Lemma 4.2. □

From now on, for an N-tangle with a label pair $(\alpha, \alpha + 1)$ and a boundary arc pair $(L_\alpha, L_{\alpha + 1})$, we denote by $s_I$ the label of the I-edges, and by $s_O$ the label of the O-edges. Further, we denote by $L_I$ (resp. $L_O$) the one of the arcs $L_\alpha, L_{\alpha + 1}$ which intersects I-edges (resp. O-edges). The pair $(s_I, s_O)$ is called the IO-label pair of the N-tangle. The pair $(L_I, L_O)$ is called the boundary IO-arc pair of the N-tangle. Then we do not need to distinguish upward N-tangles and downward N-tangles anymore. Namely, according to our new rule,

(I) any upward net-tangle with a label pair $(\alpha, \alpha + 1)$ is an N-tangle with an IO-label pair $(s_I, s_O)$ (here $s_I = \alpha, s_O = \alpha + 1$), and

(II) any downward net-tangle with a label pair $(\alpha, \alpha + 1)$ is also an N-tangle with an IO-label pair $(s_I, s_O)$ (here $s_I = \alpha + 1, s_O = \alpha$).

**Remark 4.3** Let $\Gamma$ be a chart, and $(\Gamma \cap D, D)$ an N-tangle with an IO-label pair $(s_I, s_O)$.

(1) There is neither O-edge of label $s_I$ nor I-edge of label $s_O$.

(2) Any terminal edge intersecting $D$ is contained in Int $D$ by Condition (iii) of a net-tangle and Condition (i) of an I-edge and an O-edge.

(3) An edge is contained in Int $D$ if and only if its two vertices are in $D$.

**Lemma 4.4** Let $\Gamma$ be a minimal chart, and $(\Gamma \cap D, D)$ an N-tangle with an IO-label pair $(s_I, s_O)$. Then we have the following.

(a) Any terminal edge of label $s_I$ in $D$ is an I-terminal edge.

(b) Any terminal edge of label $s_O$ in $D$ is an O-terminal edge.

**Proof.** **Statement (a).** Suppose that there exists an O-terminal edge $e$ of label $s_I$ in $D$. Then the two sibling edges of $e$ are of label $s_O$. Since any O-terminal edge is outward at its black vertex, the edge $e$ is inward at a white vertex $v$, and so are the two sibling edges. Hence by Lemma 4.1(b), the two sibling edges are I-edges of label $s_O$. This contradicts Remark 4.3(1). Hence Statement (a) holds. Similarly we can show Statement (b). □

**Lemma 4.5** Let $\Gamma$ be a minimal chart, and $(\Gamma \cap D, D)$ an N-tangle with an IO-label pair $(s_I, s_O)$. Let $e$ be an edge intersecting $D$, and $P$ a path dominated by $D$ with an edge sequence $(e_1, e_2, \cdots, e_p)$.
(a) Suppose that $P$ is an upward right-selective or left-selective one-way path starting from $e$.

(i) Suppose that $P$ is of label $s_I$. Then $P$ is maximal with respect to $D$ if and only if $e_p$ is a terminal edge.

(ii) Suppose that $P$ is of label $s_O$. Then $P$ is maximal with respect to $D$ if and only if $e_p$ is an $O$-edge for $D$.

(b) Suppose that $P$ is a downward right-selective or left-selective one-way path leading to $e$.

(i) Suppose that $P$ is of label $s_I$. Then $P$ is maximal with respect to $D$ if and only if $e_1$ is an I-edge for $D$.

(ii) Suppose that $P$ is of label $s_O$. Then $P$ is maximal with respect to $D$ if and only if $e_1$ is a terminal edge.

Proof. We give a proof only for the case that $P$ is upward right-selective. Let $(v_0, v_1, \ldots, v_p)$ be the vertex sequence of the one-way path $P$.

Statement (a)(i). Suppose that the path $P$ is of label $s_I$ and maximal with respect to $D$. The edge $e_p$ is of label $s_I$, and oriented from $v_{p-1}$ to $v_p$.

If $v_p$ is outside $D$, then the edge $e_p$ is an $O$-edge of label $s_I$. This contradicts Remark 4.3(1). Hence the vertex $v_p$ lies in Int $D$. Since $P$ is maximal, the vertex $v_p$ is not a white vertex by Lemma 3.4(a). Since $D$ does not contain a crossing by Condition (i) of a net-tangle, the edge $e_p$ is a terminal edge.

Conversely, if $e_p$ is a terminal edge, then there does not exist an edge of label $s_I$ outward at $v_p$. Hence $P$ is maximal.

Statement (a)(ii). Suppose that the path $P$ is of label $s_O$ and maximal with respect to $D$. The edge $e_p$ is of label $s_O$ and oriented from $v_{p-1}$ to $v_p$.

Suppose that $v_p$ is contained in $D$. If $v_p$ is a black vertex, then $e_p$ is an $I$-terminal edge of label $s_O$. This contradicts Lemma 3.4(b). Thus the vertex $v_p$ is a crossing or a white vertex. Since $D$ does not contain a crossing by Condition (i) of the definition of a net-tangle, the vertex $v_p$ is a white vertex in $D$. Since $P$ is maximal, this contradicts Lemma 3.4(a). Hence $v_p$ is outside $D$. Since $e_p$ is oriented from $v_{p-1}$ to $v_p$, the edge $e_p$ is an $O$-edge.

Conversely, if $e_p$ is an $O$-edge, then it is clear that $P$ is maximal. □

5 Spindles

In this section we investigate a disk $E$ such that $\partial E$ consists of two one-way paths of label $m$.

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. Let $P^*$ be an upward right-selective one-way path of label $m$ with a vertex sequence $(v_0^*, v_1^*, \ldots, v_s^*)$, and $\tilde{P}$ an upward left-selective one-way path of label $m$ with a vertex sequence $(\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_t)$. Suppose that $v_0^* = \tilde{v}_0$, $v_s^* = \tilde{v}_t$, and $P^* \cap \tilde{P} = \{v_0^*, v_s^*\}$. A
disk $E$ is called a *spindle* for $\Gamma$ with a path pair $(\tilde{P}, P^*)$ provided that (see Fig. 12(a))

(i) $\Gamma \cap E \subset \Gamma_m \cup \Gamma_h$ for some label $h$ with $|m - h| = 1$,

(ii) $\partial E = P^* \cup \tilde{P}$,

(iii) the edge of label $m$ outward at $v_0^*$ and the edge of label $m$ inward at $v_0^*$ are contained in $\text{Cl}(E^c)$,

(iv) for a point $x \in \text{Int } P^*$, three points $v_0^*, x, v_s^*$ counterclockwise situate on $\partial E$ in this order.

The pair $(m, h)$ is called a *label pair* of the spindle $E$.

**Remark 5.1** Let $E$ be a spindle for a minimal chart $\Gamma$ with a path pair $(\tilde{P}, P^*)$ and a label pair $(m, h)$.

(1) The paths $\tilde{P}, P^*$ are dichromatic paths.

(2) For any regular neighbourhood $\tilde{D}$ of $E$, the pair $(\Gamma \cap \tilde{D}, \tilde{D})$ is a tangle with $\Gamma \cap \tilde{D} \subset \Gamma_m \cup \Gamma_h$.

(3) The spindle $E$ does not contain a one-cycle of label $m$ nor $h$ by (2) and Lemma 3.2.

(4) Since the spindle $E$ is a 2-color disk, the spindle $E$ does not contain a crossing nor a terminal edge by the definition of a 2-color disk and by Lemma 2.5.

Let $P$ be a path in a chart with a vertex sequence $(v_0, v_1, \cdots, v_p)$ and an edge sequence $(e_1, e_2, \cdots, e_p)$. For two integers $i, j$ with $0 \leq i < j \leq p$, we denote the path $e_{i+1} \cup e_{i+2} \cup \cdots \cup e_j$ by $P[v_i, v_j]$. 

![Figure 12:](image-url)
Lemma 5.2 Let $E$ be a spindle for a minimal chart $\Gamma$ with a path pair $(\tilde{P}, P^*)$ and a label pair $(m, h)$. Let $e$ be an edge of label $m$ in $E$ with $e \cap \partial E$ a white vertex $v$.

(a) If $e$ is outward at $v$, then in $E$ there exists an upward right-selective (resp. left-selective) one-way path $P$ of label $m$ starting from $e$ with $P \cap \partial E$ two white vertices.

(b) If $e$ is inward at $v$, then in $E$ there exists a downward right-selective (resp. left-selective) one-way path $P$ of label $m$ leading to $e$ with $P \cap \partial E$ two white vertices.

Proof. Statement (a). Let $P$ be a maximal upward right-selective (resp. left-selective) one-way path of label $m$ starting from $e$ with respect to $E$ with a vertex sequence $(v_0, v_1, \ldots, v_p)$. Suppose that $P \cap \partial E = v_0$. Since $E$ is a 2-color disk, the disk $E$ does not contain a crossing nor a terminal edge by Remark 5.1(4). Hence $v_p$ is a white vertex in $\text{Int} E$. Since $E$ does not contains any one-way cycle of label $m$ by Remark 5.1(3), we can show that $P$ is not maximal by the similar way as the one of Lemma 3.4(a). This is a contradiction. Hence $P \cap \partial E$ contains at least two points. Let $i = \min\{j \mid j > 0, v_j \in \partial E\}$. Then $P[v_0, v_j]$ is a desired one. Thus Statement (a) holds. Similarly we can show Statement (b). This proves Lemma 5.2 □

Let $\Gamma$ be a chart, and $(\Gamma \cap D, D)$ a tangle. For each white vertex in $D$, there are six short oriented arcs with orientations inherited from the ones of the edges containing the short arcs. Then for each white vertex, three consecutive arcs are inward at the vertex and the other three consecutive arcs are outward at the vertex. For each tangle $(\Gamma \cap D, D)$, we always assume that

(1) each short arc is contained in $\text{Int} D$, and

(2) the interiors of all the short arcs are mutually disjoint.

Each short arc is called an IS-arc (resp. an OS-arc) of the tangle if the short arc is inward (resp. outward) at the white vertex.

A spindle is said to be minimal if it does not contain another spindle.

Lemma 5.3 For any minimal chart there does not exist a spindle.

Proof. Suppose that there exists a spindle for a minimal chart $\Gamma$. Then there exists a minimal spindle $E$ for $\Gamma$ with a path pair $(\tilde{P}, P^*)$ and a label pair $(m, h)$. Let $(v_0^*, v_1^*, \ldots, v_s^*)$ and $(\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_t)$ be the vertex sequences of $P^*$ and $\tilde{P}$ respectively.

Claim 1. The disk $E$ does not dominate an edge of label $m$ outward at a vertex in $P^* \cup \text{Int} \tilde{P}$.

Proof of Claim 1. Suppose that $E$ dominates an edge $\hat{e}$ of label $m$ outward at $v_i^*$ for some $i \in \{1, 2, \ldots, s-1\}$. By Lemma 5.2(a) there exists an upward
left-selective one-way path $P$ of label $m$ starting from $\hat{e}$ with $P \cap \partial E$ two white vertices. Let $v$ be the one of the two white vertices different from $v^*_i$. Then $v \in \{v^*_i, v^*_2, \cdots, v^*_{s-1}\} \cup \{\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_{t-1}\}$.

Suppose that $v = \tilde{v}_j$ for some $j \in \{1, 2, \cdots, t-1\}$. Then the disk bounded by $P^*[v^*_i, v^*_j] \cup \partial \tilde{P}[\tilde{v}_j, \tilde{v}_i]$ is a spindle in $E$ (see Fig. 12(b)). This contradicts that $E$ is a minimal spindle.

Suppose that $v = v^*_j$ for some $j \in \{1, 2, \cdots, s-1\}$. Then $i < j$ or $j < i$. If $i < j$, then the disk bounded by $P^*[v^*_i, v^*_j] \cup P$ is a spindle in $E$ (see Fig. 12(c)). This contradicts that $E$ is a minimal spindle. If $j < i$, then $P^*[v^*_j, v^*_i]$ is a dichromatic M&M one-way path (see Fig. 12(d)). This contradicts Lemma 3.1. Hence the edge $\hat{e}$ is not outward at a vertex in Int $\tilde{P}^*$. Similarly we can show that $\tilde{E}$ does not dominate an edge of label $m$ outward at a vertex in Int $\tilde{P}$. Therefore Claim 1 holds.

Claim 2. The disk $E$ does not dominate an edge of label $m$ inward at a vertex in $P^* \cup$ Int $\tilde{P}$.

Proof of Claim 2. Suppose that $E$ dominates an edge $\hat{e}$ of label $m$ inward at $\hat{v}$ in Int $P^* \cup$ Int $\tilde{P}$. By Lemma 5.2(b) there exists a one-way path $P$ of label $m$ leading to $\hat{e}$ with $P \cap \partial E$ two white vertices. Let $v$ be the one of the two white vertices different from $\hat{v}$. Thus $E$ dominates an edge of label $m$ outward at $v \in$ Int $P^* \cup$ Int $\tilde{P}$. This contradicts Claim 1. Hence the edge $\hat{e}$ is not inward at a vertex in Int $P^* \cup$ Int $\tilde{P}$. Hence Claim 2 holds.

Claim 3. There does not exist any white vertex in Int $E$.

For, if there exists a white vertex in Int $E$, let $A$ be a regular neighbourhood of $\partial E$ in $E$. Then $(A - \partial E) \cap \Gamma_m = \emptyset$ by Claim 1 and Claim 2. Set $D' = Cl(E - A)$. Then the pair $(\Gamma\cap D', D')$ is an NS-tangle of label $h$. This contradicts Lemma 2.4. Hence Claim 3 holds.

Now, Claim 1 and Claim 2 imply that

(1) any side-edge of label $m$ for $P^*$ or $\tilde{P}$ lies outside $E$.

Since there is no terminal edge nor crossing in $E$ by Remark 5.1(4), each edge of label $h$ in $E$ contains exactly one IS-arc and one OS-arc. Hence

(2) the number of IS-arcs of label $h$ in $E$ equals the number of OS-arcs of label $h$ in $E$.

Since $P^*$ is upward right-selective, and since $\tilde{P}$ is upward left-selective, Statement (1) implies that any side-edge of label $m$ for $P^*$ or $\tilde{P}$ is inward at a vertex in $\partial E = P^* \cup \tilde{P}$. Namely for each vertex in $\partial E - v^*_s$, the disk $E$ contains exactly one edge of label $h$ outward at the vertex. Hence considering $v^*_0 = \tilde{v}_0$, the three claims Claim 1, Claim 2, Claim 3 imply that the number of OS-arcs of label $h$ in $E$ is $s + t - 1$. On the other hand, there exists only one IS-arc of label $h$ in $E$, which is inward at $v^*_s = \tilde{v}_s$. Thus Statement (2) implies that $s + t - 1 = 1$. Since $P^*$ and $\tilde{P}$ are paths, we have $s \geq 1$ and $t \geq 1$. Hence $s = t = 1$. Thus $P^*, \tilde{P}$ consist of the edges $e^*, \hat{e}$ respectively. Further, $E$ contains an edge $e$ of label $h$ connecting the two white vertices $v^*_0$.
and $v^*_1$ so that $e^*, e, \tilde{e}$ are three consecutive edges connecting the two vertices $v_0^*, v_s^*$. Hence we can eliminate the two white vertices by a C-I-M3 move. This contradicts that the chart is minimal. Therefore Lemma 5.3 holds. □

6 Half Spindles

In this section, we investigate a disk $E$ such that $\partial E$ consists of an arc and two one-way paths of label $m$.

Let $\Gamma$ be a chart, and $m$ a label of $\Gamma$. Let $P^*$ be an upward right-selective one-way path of label $m$ with a vertex sequence $(v_0^*, v_1^*, \cdots, v_s^*)$, and $\tilde{P}$ an upward left-selective one-way path of label $m$ with a vertex sequence $(\tilde{v}_0, \tilde{v}_1, \cdots, \tilde{v}_t)$ equipped with $P^* \cap \tilde{P} = v_s^* = \tilde{v}_t$. Let $E$ be a disk and $P^\dagger$ an arc on $\partial E$. The disk $E$ is called a half spindle for $\Gamma$ with a path triplet $(\tilde{P}, P^\dagger, P^*)$ provided that (see Fig. 13(a))

(i) $\Gamma \cap E \subset \Gamma_m \cup \Gamma_h$ for some label $h$ with $|m - h| = 1$,

(ii) $(P^* \cup \tilde{P}) \cap \text{Int} E = \emptyset$, and $E \not\ni v_0^*, \tilde{v}_0$,

(iii) the edge of label $m$ outward at $v_s^*$ is contained in $Cl(E^c)$,

(iv) $P^* \cap E$ is an arc containing $P^*[v_1^*, v_s^*]$, $\tilde{P} \cap E$ is an arc containing $\tilde{P}[\tilde{v}_1, \tilde{v}_t]$, and $P^\dagger = Cl(\partial E - (P^* \cup \tilde{P}))$,

(v) the three arcs $\tilde{P} \cap E, P^\dagger, P^* \cap E$ are counterclockwise situated on $\partial E$ in this order,

(vi) if an edge $e$ intersects $\text{Int} P^\dagger$, then it is an I-edge of label $m$ for $E$,

(vii) $s \geq 2$ and $t \geq 2$.

The pair $(m, h)$ is called a label pair of the half spindle $E$.  

Remark 6.1 Let $E$ be a half spindle for a minimal chart $\Gamma$ with a path triplet $(\tilde{P}, P^\dagger, P^*)$ and a label pair $(m, h)$.

1. The paths $\tilde{P}[\tilde{v}_1, \tilde{v}_t], P^*[v_1^*, v_s^*]$ are dichromatic paths.

2. For each regular neighbourhood $\hat{D}$ of $E$, the pair $(\Gamma \cap \hat{D}, \hat{D})$ is a tangle with $\Gamma \cap \hat{D} \subset \Gamma_m \cup \Gamma_h$.

3. The half spindle $E$ does not contain a one-way cycle of label $m$ nor $h$ by (2) and Lemma 3.2.

4. The half spindle $E$ does not contain a crossing nor a terminal edge of label $m$. For, there does not exist an I-edge nor an O-edge of label $h$ by Condition (vi) of the definition of a half spindle. If $E$ contains a terminal edge of label $m$, we can find a non-admissible consecutive triplet that contradicts Lemma 2.3.

We can show the following lemma by a similar argument as the one of Lemma 5.2.

Lemma 6.2 Let $E$ be a half spindle for a minimal chart $\Gamma$ with a path triplet $(\tilde{P}, P^\dagger, P^*)$ and a label pair $(m, h)$. Let $e$ be an edge of label $m$ with $e \cap \partial E$ a point.

(a) If $e$ is outward at a vertex on $\partial E$ or if $e$ intersects $P^\dagger$, then there exists an upward right-selective (resp. left-selective) one-way path $P$ of label $m$ starting from $e$ dominated by $E$ with $P \cap \partial E$ two points.

(b) If $e$ is inward at a vertex on $\partial E$, then there exists a downward right-selective (resp. left-selective) one-way path $P$ of label $m$ leading to $e$ dominated by $E$ with $P \cap \partial E$ two points.

A half spindle is said to be minimal if it does not contain another half spindle.

Lemma 6.3 For any minimal chart there does not exist a half spindle.

Proof. Suppose that there exists a half spindle for a minimal chart $\Gamma$. Then there exists a minimal half spindle $E$ with a path triplet $(\tilde{P}, P^\dagger, P^*)$ and a label pair $(m, h)$. Now

1. the disk $E$ does not contain a crossing nor a terminal edge of label $m$ by Remark 6.1(4), and

2. the disk $E$ does not contain a one-way cycle by Remark 6.1(3).
Let \((v_0^*, v_1^*, \ldots, v_s^*)\) and \((\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_t)\) be the vertex sequences of \(P^*\) and \(\tilde{P}\) respectively.

**Claim 1.** \(\Gamma_m \cap \text{Int } P^\dagger = \emptyset\).

**Proof of Claim 1.** Suppose \(\Gamma_m \cap \text{Int } P^\dagger \neq \emptyset\). Let \(e\) be an edge of label \(m\) intersecting \(\text{Int } P^\dagger\). By Lemma 6.2(a), there exists an upward right-selective one-way path \(P\) of label \(m\) starting from \(e\) with \(P \cap \partial E\) two points. Since there does not exist an O-edge intersecting \(\text{Int } P^\dagger\), the intersection \(P \cap (P^* \cup \tilde{P})\) is a vertex \(v\).

If \(v = v_i^* (1 \leq i < s)\), then \(P\) splits the disk \(E\) into two disks. Let \(E'\) be the one of the two disks intersecting \(\tilde{P}\). Then \(E'\) is a half spindle with a path triplet \((\tilde{P}, L, P \cup P^*[v_i^*, v_s^*])\) for some arc \(L \subset P^\dagger\) (see Fig. 13(b)). This contradicts that \(E\) is a minimal half spindle.

Suppose \(v = \tilde{v}_i (1 \leq i < t)\). Then \(P\) splits the disk \(E\) into two disks. Let \(E'\) be the one of the two disks not intersecting \(P^*\). Let \(P'\) be an upward left-selective one-way path of label \(m\) starting from \(e\) with \(P' \cap (P^* \cup \tilde{P})\) a vertex \(v'\). Then \(P' \subset E'\) and hence \(v' = \tilde{v}_j (1 \leq j \leq i < t)\). Thus \(P'\) splits \(E\) into two disks. Let \(E''\) be the one of the two disks intersecting \(P^*\). Then \(E''\) is a half spindle with a path triplet \((P' \cup \tilde{P}[\tilde{v}_j, \tilde{v}_i], L, P^*)\) for some arc \(L \subset P^\dagger\) (see Fig. 13(c)). This contradicts that \(E\) is a minimal half spindle.

Therefore Claim 1 holds.

By the help of Claim 1, we can show the following Claim 2 and Claim 3 by the same argument as the ones of Claim 1 and Claim 2 of Lemma 5.3 respectively.

**Claim 2.** The disk \(E\) does not dominate an edge of label \(m\) outward at a vertex in \(\text{Int } P^* \cup \text{Int } \tilde{P}\).

**Claim 3.** The disk \(E\) does not dominate an edge of label \(m\) inward at a vertex in \(\text{Int } P^* \cup \text{Int } \tilde{P}\).

By a similar argument as the one of Claim 3 in Lemma 5.3, the three claims Claim 1, Claim 2 and Claim 3 assure that

(3) there does not exist any white vertex in \(\text{Int } E\).

Now Claim 1 implies that there does not exist a terminal edge of label \(h\) in \(E\). Thus each edge of label \(h\) in \(E\) contains exactly one IS-arc and one OS-arc. Hence

(4) the number of IS-arcs of label \(h\) in \(E\) equals the one of OS-arcs of label \(h\) in \(E\).

Now, Claim 2 and Claim 3 imply that

(5) any side-edge of label \(m\) for \(P^*\) or \(\tilde{P}\) lies outside \(E\).
two claims Claim 2, Claim 3, and Statement (3) imply that the number of OS-arcs of label $h$ in $E$ is $s + t - 2$. On the other hand, there exists only one IS-arc of label $h$ in $E$, which is inward at $v^* = \tilde{v}_t$. Thus Statement (4) implies that $s + t - 2 = 1$, i.e. $s + t = 3$. On the other hand, by Condition (vii) for the definition of a half spindle, we have $s + t \geq 2 + 2 = 4$. This is a contradiction. Hence Lemma 6.3 holds. □

7 Proof of Theorem 1.1

In this section we prove Theorem 1.1.

Let $\Gamma$ be a chart. For each tangle $(\Gamma \cap D, D)$, we define

$E_I(D) =$ the number of I-edges for $D$,

$E_O(D) =$ the number of O-edges for $D$,

$T_I(D) =$ the number of I-terminal edges in $D$,

$T_O(D) =$ the number of O-terminal edges in $D$.

Lemma 7.1 Let $\Gamma$ be a minimal chart, and $(\Gamma \cap D, D)$ a tangle without crossing. If $\partial D$ does not intersect any terminal edge, then we have

$$E_I(D) + T_O(D) = E_O(D) + T_I(D).$$

Proof. Since $\partial D$ does not intersect any terminal edge, any terminal edge intersecting $D$ is contained in $D$. There are three IS-arcs and three OS-arcs around each white vertex in $D$. Hence

(1) the number of IS-arcs in $D$ is equal to the number of OS-arcs in $D$.

Let $S = \{e \mid e$ is an edge of $\Gamma$ containing two white vertices in $D\}$, and $L = \{\ell \mid \ell$ is a loop in $\Gamma$ containing a vertex in $D\}$. Then we have the following.

(2) Each I-edge contains exactly one IS-arc in $D$.

(3) Each O-edge contains exactly one OS-arc in $D$.

(4) Each I-terminal edge contains exactly one OS-arc.

(5) Each O-terminal edge contains exactly one IS-arc.

(6) Each edge in $S$ contains exactly one IS-arc and exactly one OS-arc.

(7) Each loop in $L$ contains exactly one IS-arc and exactly one OS-arc.

Thus Statement (1) implies

$$E_I(D) + T_O(D) + |S| + |L| = E_O(D) + T_I(D) + |S| + |L|. $$

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Therefore $E_I(D) + T_O(D) = E_O(D) + T_I(D)$. This proves Lemma 7.1. □

Let $\Gamma$ be a chart, and $(\Gamma \cap D, D)$ an N-tangle with an IO-label pair $(s_I, s_O)$. Let $P$ be a one-way path of label $s_I$ with a vertex sequence $(v_0, v_1, \cdots, v_p)$ and an edge sequence $(e_1, e_2, \cdots, e_p)$ such that

(i) $P \subset D$,

(ii) the vertex $v_0$ is a white vertex of a terminal edge $\tau_O$ of label $s_O$, and

(iii) the edge $e_p$ is a terminal edge $\tau_I$.

Then $P$ is called the principal path associated to a terminal edge $\tau_O$. The terminal edge $\tau_I$ is called a corresponding terminal edge for $\tau_O$. We also say that the path $P$ is the principal path connecting the terminal edge $\tau_O$ and the terminal edge $\tau_I$.

**Lemma 7.2** Let $\Gamma$ be a minimal chart, and $(\Gamma \cap D, D)$ an N-tangle with an IO-label pair $(s_I, s_O)$. Let $\tau_O$ be a terminal edge of label $s_O$, and $P$ a principal path of label $s_I$ associated to $\tau_O$ with a vertex sequence $(v_0, v_1, \cdots, v_p)$ and an edge sequence $(e_1, e_2, \cdots, e_p)$. Then we have the following.

(a) The edge $e_1$ is middle at $v_0$.

(b) Any side-edge of label $s_I$ for $P$ is inward at a vertex in Int $P$.

(c) The path $P$ is a maximal upward right-selective one-way path starting from $e_1$ with respect to $D$.

(d) The path $P$ is a maximal upward left-selective one-way path starting from $e_1$ with respect to $D$.

**Proof.** Statement (a). The terminal edge $\tau_O$ is inward at $v_0$ by Lemma 4.4(b). Thus the sibling edges of $\tau_O$ are of label $s_I$ and inward at $v_0$. Since $e_1$ is oriented from $v_0$ to $v_1$, the edge $e_1$ is not a sibling edge of $\tau_O$. Hence $e_1$ is middle at $v_0$.

Statement (b). Suppose that there exists a side-edge $e^*$ of label $s_I$ for $P$ outward at a vertex $v_i$ in Int $P$. Since the edge $e_{i+1}$ is of label $s_I$ and outward at $v_i$, the edge $e_i$ is middle at $v_i$. Hence $P[v_0, v_i]$ is a dichromatic M&M one-way path. This contradicts Lemma 3.1. Thus Statement (b) holds.

Now Statement (b) implies that the path $P$ is an upward right-selective and left-selective one-way path. Further, Lemma 4.5(a)(i) implies that $P$ is maximal with respect to $D$. Hence Statement (c) and (d) hold. This proves Lemma 7.2. □

**Lemma 7.3** Let $\Gamma$ be a minimal chart, and $(\Gamma \cap D, D)$ an N-tangle with an IO-label pair $(s_I, s_O)$. Let $\tau_O$ be a terminal edge of label $s_O$ in $D$. Then there exist the principal path of label $s_I$ associated to the terminal edge $\tau_O$ and the corresponding terminal edge for $\tau_O$. 26
Proof. Let \( v_0 \) be the white vertex of \( \tau_O \), and \( e_1 \) the edge of label \( s_I \) middle at \( v_0 \). Since \( \tau_O \) is inward at \( v_0 \) by Lemma 4.4(b), the edge \( e_1 \) is outward at \( v_0 \).

Let \( P \) be a maximal right-selective one-way path of label \( s_I \) starting from \( e_1 \) with respect to \( D \). Let \((e_1, e_2, \ldots, e_p)\) be an edge sequence of \( P \). Then \( e_p \) is a terminal edge by Lemma 4.5(a)(i). Therefore \( P \) is a principal path of label \( s_I \) associated to the terminal edge \( \tau_O \), and \( e_p \) is a corresponding terminal edge for \( \tau_O \). This proves Lemma 7.3.

Remark 7.4 In the proof of the lemma above, the edge \( e_1 \) is uniquely determined by the terminal edge \( \tau_O \). Hence in Lemma 7.3 above, the maximal right-selective one-way path starting from \( e_1 \) is uniquely determined by \( \tau_O \), and so is the corresponding terminal edge for \( \tau_O \).

Let \( \Gamma \) be a minimal chart, and \((\Gamma \cap D, D)\) an \( \text{N-tangle} \) with an \( \text{IO-label pair} \) \((s_I, s_O)\). Let
\[
\begin{align*}
T_O(D) &= \text{the set of all the terminal edges of label } s_O \text{ in } D, \\
T_I(D) &= \text{the set of all the terminal edges of label } s_I \text{ in } D.
\end{align*}
\]
Then by Lemma 4.4 we have \( T_O(D) = |T_O(D)| \) and \( T_I(D) = |T_I(D)| \).

By Lemma 7.3 and Remark 7.4, we define a map \( f_D : T_O(D) \to T_I(D) \) by for each \( \tau_O \in T_O(D) \),
\[
f_D(\tau_O) = \text{the corresponding terminal edge for } \tau_O.
\]

Let \( \Gamma \) be a minimal chart, and \((\Gamma \cap D, D)\) an \( \text{N-tangle} \) with an \( \text{IO-label pair} \) \((s_I, s_O)\). For a terminal edge \( \tau \) in \( D \) with sibling edges \( e^*, e^{**} \), the union \( e^* \cup e^{**} \) splits the disk \( D \) into two disks by Lemma 4.1. We denote by \( \Delta(\tau) \) the one of the two disks containing the terminal edge \( \tau \). Let \( T_O(D) = \{\tau_1, \tau_2, \cdots, \tau_s\} \), and \( T_I(D) = \{\tilde{\tau}_1, \tilde{\tau}_2, \cdots, \tilde{\tau}_t\} \), here \( s = T_O(D) \) and \( t = T_I(D) \).

Let \( w_1, w_2, \cdots, w_s \) be the white vertices of \( \tau_1, \tau_2, \cdots, \tau_s \) respectively, and \( \tilde{w}_1, \tilde{w}_2, \cdots, \tilde{\tau}_t \) the white vertices of \( \tilde{\tau}_1, \tilde{\tau}_2, \cdots, \tilde{\tau}_t \) respectively. Set
\[
D^\dagger = \text{Cl}(D - \bigcup_{i=1}^s \Delta(\tau_i)) - \bigcup_{j=1}^t \Delta(\tilde{\tau}_j)).
\]
The set \( D^\dagger \) is called a fundamental region of the \( \text{N-tangle} \). Renumbering the terminal edges we always assume that

1. the vertices \( \tilde{w}_1, \tilde{w}_2, \cdots, \tilde{\tau}_t \) are situated clockwise on \( \partial D^\dagger \) in this order, and

2. the vertices \( w_1, w_2, \cdots, w_s \) are situated counterclockwise on \( \partial D^\dagger \) in this order.

The tuple \((D^\dagger; w_1, w_2, \cdots, w_s; \tilde{w}_1, \tilde{w}_2, \cdots, \tilde{\tau}_t)\) is called the fundamental information for the \( \text{N-tangle} \).

Lemma 7.5 Let \( \Gamma \) be a minimal chart, and \((\Gamma \cap D, D)\) an \( \text{N-tangle} \) with an \( \text{IO-label pair} \) \((s_I, s_O)\). Then the principal paths associated to the terminal edges of label \( s_O \) in \( D \) are mutually disjoint.

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Proof. Let \((D^1; w_1, w_2, \ldots, w_s; \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_t)\) be the fundamental information for the N-tangle, and \(\mathbb{T}_O(D) = \{\tau_1, \tau_2, \ldots, \tau_s\}\), here \(s = T_O(D)\) and \(t = T_I(D)\).

Now each principal path associated to a terminal edge of label \(s_O\) in \(D\) splits the set \(D^1\). Thus to prove Lemma 7.5, it is sufficient to show that for each \(i = 1, 2, \ldots, s - 1\), two principal paths associated to the terminal edges \(\tau_i\) and \(\tau_{i+1}\) are disjoint.

Suppose that for some integer \(i (1 \leq i < s)\), there exist the two principal paths \(\tilde{P}, P^*\) associated to the terminal edges \(\tau_i, \tau_{i+1}\) with \(\tilde{P} \cap P^* \neq \emptyset\).

Let \((\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_p), (v^*_1, v^*_2, \ldots, v^*_q)\) be vertex sequences of the principal paths \(\tilde{P}, P^*\) respectively, here \(\tilde{v}_1 = w_i, v^*_1 = w_{i+1}\). Let \(h = \min\{i \mid \tilde{v}_i \in P^*\}, k = \min\{i \mid v^*_i \in \tilde{P}\}\).

Then \(h \geq 2\) and \(k \geq 2\). Further \(\tilde{v}_h = v^*_k\) and \(\tilde{P}[\tilde{v}_1, \tilde{v}_h] \cap P^*[v^*_1, v^*_k] = v^*_k\).

Let \((L^1, L^2)\) be a boundary IO-arc pair of the N-tangle \((\Gamma \cap D, D)\), and \(E\) the closure of the connected component of \(\Delta(D)^1 - (\tilde{P}[\tilde{v}_1, \tilde{v}_h] \cup P^*[v^*_1, v^*_k])\) with \(E \cap \partial D \subseteq L^1\) (see Fig. 14). Let \(\tilde{e}, e^*\) be the siblings of \(\tau_i\) and \(\tau_{i+1}\) intersecting \(E - \{v^*_1, \tilde{v}_1\}\) respectively. Set \(\tilde{v}_0\) and \(v^*_0\) the vertices of \(\tilde{e}\) and \(e^*\) outside \(D\) respectively. Then the path with the vertex sequence \((\tilde{v}_0, \tilde{v}_1, \ldots, \tilde{v}_h)\) is an upward left-selective one-way path by Lemma 7.2(d), and the path with the vertex sequence \((v^*_0, v^*_1, \ldots, v^*_k)\) is an upward right-selective one-way path by Lemma 7.2(c). Thus \(E\) is a half spindle. This contradicts Lemma 6.3. Therefore Lemma 7.5 holds.

\[\square\]

**Lemma 7.6** Let \(\Gamma\) be a minimal chart, and \((\Gamma \cap D, D)\) an N-tangle with an IO-label pair \((s_I, s_O)\). Then the map \(f_D : \mathbb{T}_O(D) \rightarrow \mathbb{T}_I(D)\) is bijective.

Proof. Since the map \(f_D : \mathbb{T}_O(D) \rightarrow \mathbb{T}_I(D)\) is injetive by Lemma 7.5, we have

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Figure 14: The thick edges are of label \(s_I\), and the thin edges are of label \(s_O\).
By changing all the orientations of the edges of the chart and setting \( \tilde{s}_I = s_O, \tilde{s}_O = s_I \), we obtain a new chart \( \tilde{\Gamma} \) and a new N-tangle \( (\tilde{\Gamma} \cap \tilde{D}, \tilde{D}) \) with the IO-label pair \((\tilde{s}_I, \tilde{s}_O)\), here \( \tilde{D} = D \). Now

(2) all the terminal edges of label \( s_I \) of the old N-tangle change to all the terminal edges of label \( \tilde{s}_O \) of the new N-tangle, and

(3) all the terminal edges of label \( s_O \) of the old N-tangle change to all the terminal edges of label \( \tilde{s}_I \) of the new N-tangle.

Then for the new N-tangle, instead of (1), we have \( |T_O(\tilde{D})| \leq |T_I(\tilde{D})| \).

Namely

(4) \( |T_I(D)| \leq |T_O(D)| \).

Thus we have \( |T_I(D)| = |T_O(D)| \) by (1) and (4). Therefore the map \( f_D \) is bijective. Hence this proves Lemma 7.6.

8 Proof of Theorem 1.2

In this section we prove Theorem 1.2.

Let \( \Gamma \) be a chart. For a subset \( X \) of \( \Gamma \), let

\[ B(X) = \text{the union of all the disk bounded by a cycle in } X, \text{ and} \]

\[ T(X) = \text{the union of all the terminal edge intersecting } X \cup B(X). \]

The set \( X \cup B(X) \cup T(X) \) is called the SC-closure of \( X \) and denoted by \( SC(X) \).

Remark 8.1 Each connected component of the SC-closure \( SC(X) \) is simply connected.
Let $\Gamma$ be a chart, and $(\Gamma \cap D, D)$ a net-tangle with a label pair $(\alpha, \beta)$ and a boundary arc pair $(L_\alpha, L_\beta)$. Let (see Fig. 15)

\begin{align*}
\mathcal{V} &= \bigcup \{v \mid v \text{ is a vertex in } D \cap \Gamma_\alpha \cap \Gamma_{\alpha + 1}\}, \\
\mathcal{E}_1 &= \bigcup \{e \mid e \text{ is an edge of label } \alpha \text{ intersecting } L_\alpha\}, \\
\mathcal{E}_2 &= \bigcup \{e \mid e \text{ is an edge connecting two white vertices in } \mathcal{V}\}, \\
Y &= \text{the connected component of } (L_\alpha \cup \mathcal{V} \cup \mathcal{E}_1 \cup \mathcal{E}_2) \cap D \text{ containing } L_\alpha, \\
N(\Gamma, D, \alpha) &= \text{a regular neighbourhood of the SC-closure } SC(Y) \text{ in } D, \\
L^*_\alpha &= \text{Cl}(\partial D \cap N(\Gamma, D, \alpha)), \text{ and} \\
L^*_{\alpha + 1} &= \text{Cl}(\partial N(\Gamma, D, \alpha) - L^*_\alpha).
\end{align*}

Then $N(\Gamma, D, \alpha)$ is simply connected by Remark 8.1. Thus the set $N(\Gamma, D, \alpha)$ is a disk. The number $|\Gamma_{\alpha + 1} \cap \partial N(\Gamma, D, \alpha)|$ is called the $(D, \alpha + 1)$-number of the chart and denoted by $n(\Gamma, D, \alpha + 1)$. Let $N(L^*_{\alpha + 1})$ be a regular neighbourhood of $L^*_{\alpha + 1}$ in $D$, and $S(\Gamma, D, \alpha + 1)$ the set of all the chart M2-related to $\Gamma$ with respect to $D$ by a finite sequence of C-I-M2 moves in $N(L^*_{\alpha + 1})$. A chart in $S(\Gamma, D, \alpha + 1)$ is $(\Gamma, D, \alpha + 1)$-minimal if its $(D, \alpha + 1)$-number is minimal amongst $S(\Gamma, D, \alpha + 1)$.

**Remark 8.2** Let $\Gamma$ be a chart, and $(\Gamma \cap D, D)$ a net-tangle with a label pair $(\alpha, \beta)$ and a boundary arc pair $(L_\alpha, L_\beta)$.  

![Figure 15](image_url)
If an edge intersects $\partial N(\Gamma, D, \alpha)$, then the edge transversely intersects $\partial N(\Gamma, D, \alpha)$.

(2) $L_{\alpha+1}^*$ is a proper arc of $D$.

(3) $\partial N(\Gamma, D, \alpha) = L_{\alpha}^* \cup L_{\alpha+1}^*$, $\Gamma \cap L_{\alpha}^* = \Gamma_\alpha \cap L_{\alpha}^*$, and $\Gamma \cap L_{\alpha+1}^* = \Gamma_{\alpha+1} \cap L_{\alpha+1}^*$.

**Proof of Theorem 1.2** We prove Theorem 1.2 by induction on the number $|\alpha - \beta|$. Let $\Gamma$ be a minimal chart, and $(\Gamma \cap D, \alpha)$ a net-tangle with a label pair $(\alpha, \beta)$ and a boundary arc pair $(L_\alpha, L_\beta)$.

Suppose $|\alpha - \beta| = 1$. Then the tangle is a net-tangle with a label pair $(\alpha, \alpha + 1)$. Thus it is an N-tangle by Lemma 4.2. Hence Theorem 1.2 holds for the case $|\alpha - \beta| = 1$.

Suppose that $|\alpha - \beta| > 1$. Without loss of generality we can assume that

(1) the chart $\Gamma$ is $(\Gamma, D, \alpha + 1)$-minimal, and

(2) all the edges of label $\alpha$ intersecting $\partial D$ are I-edges.

We use all the notations in the definition of $(\Gamma, D, \alpha + 1)$-minimal. Now $\Gamma \cap \partial N(\Gamma, D, \alpha) = (\Gamma \cap L_\alpha^*) \cup (\Gamma \cap L_{\alpha+1}^*) \subset \Gamma_\alpha \cup \Gamma_{\alpha+1}$ by Remark 8.2.3

Thus by Boundary Condition Lemma (Lemma 2.9) we have

(3) $\Gamma \cap N(\Gamma, D, \alpha) \subset \Gamma_\alpha \cup \Gamma_{\alpha+1}$.

**Claim 1.** Any edge of label $\alpha + 1$ intersecting $\partial N(\Gamma, D, \alpha)$ is not an edge of a bigon.

**Proof of Claim 1.** Suppose that an edge $e$ of label $\alpha + 1$ intersecting $\partial N(\Gamma, D, \alpha)$ is an edge of a bigon. Then

(4) the edge $e$ transversely intersects $\partial N(\Gamma, D, \alpha)$ by Remark 8.2.1.

Since $|\alpha - \beta| > 1$, we have $\alpha < \alpha + 1 < \beta$. Thus

(5) $e \subset D$.

Let $e^*$ be the other edge of the bigon. Then $e^*$ intersects $N(\Gamma, D, \alpha)$. Further $e^*$ is of label $\alpha$ or $\alpha + 2$, because $e$ is of label $\alpha + 1$. Since no edge of label $\alpha + 2$ intersects $N(\Gamma, D, \alpha)$ by (3), the edge $e^*$ is of label $\alpha$. Hence the two white vertices of $e$ are in $\Gamma_\alpha \cap \Gamma_{\alpha+1}$. Hence $e \subset E_2 \subset N(\Gamma, D, \alpha)$ by (5). This contradicts Statement (4). Hence Claim 1 holds.

**Claim 2.** All the edges intersecting $L_{\alpha+1}^*$ are I-edges of label $\alpha + 1$ or all the edges intersecting $L_{\alpha+1}^*$ are O-edges of label $\alpha + 1$.

**Proof of Claim 2.** Suppose that there exist two edges $e_1, e_2$ of label $\alpha + 1$ intersecting $L_{\alpha+1}^*$ such that one of $e_1, e_2$ is an I-edge and the other an O-edge. Let $L$ be the subarc of $L_{\alpha+1}^*$ connecting the two edges $e_1$ and $e_2$. Without loss of generality we can assume that $\Gamma_{\alpha+1} \cap \text{Int} \ L = \emptyset$. Let $\hat{\Gamma}$ be a chart obtained from $\Gamma$ by applying a C-I-M2 move between $e_1$ and $e_2$ along the arc $L$ in $N(L_{\alpha+1}^*)$. Since each of $e_1$ and $e_2$ contains two white vertices, $\hat{\Gamma} \cap D$
does not contain a free edge nor a hoop. Further Claim 1 assures that $\tilde{\Gamma}$ is a minimal chart with $n(\Gamma, D, \alpha + 1) < n(\Gamma, D, \alpha + 1)$. This contradicts Statement (1). Thus Claim 2 holds.

By Claim 2, the tangle $(\Gamma \cap N(\Gamma, D, \alpha), N(\Gamma, D, \alpha))$ is a net-tangle with $\Gamma \cap N(\Gamma, D, \alpha) \subset \Gamma_\alpha \cup \Gamma_{\alpha+1}$. Thus it is an N-tangle by Lemma \ref{lemma:net-tangle}. Hence it is upward by (2). Let $D^* = Cl(D - N(\Gamma, D, \alpha))$ and $L^*_\beta = Cl(\partial D^* - L_{\alpha+1}^*)$. Then $\Gamma \cap \partial D^* \subset \Gamma_{\alpha+1} \cup \Gamma_\beta$. Thus $\Gamma \cap D^* \subset \bigcup_{i=\alpha+1}^{\beta} \Gamma_i$, by Boundary Condition Lemma (Lemma \ref{lemma:boundary-condition}). Thus $(\Gamma \cap D^*, D^*)$ is a net-tangle with a label pair $(\alpha + 1, \beta)$ and a boundary pair $(L_{\alpha+1}^*, L_\beta^*)$. Since $|\alpha + 1 - \beta| < |\alpha - \beta|$, the results follow by the induction hypothesis. This proves Theorem 1.2. □

9 2-Crossing Minimal Charts

This section is an outline of \cite{11}.

9.1 Simple IO-tangle

Let $\Gamma$ be a chart containing a white vertex. Define

$$\alpha(\Gamma) = \min \{ i \mid \Gamma_i \text{ contains a white vertex} \},$$

$$\beta(\Gamma) = \max \{ i \mid \Gamma_i \text{ contains a white vertex} \}.$$

A chart with exactly two crossings is called a 2-crossing chart.

**Lemma 9.1** \cite[lemma 1.1, cf.][lemma 6.3]{9} Let $\Gamma$ be a 2-crossing minimal chart in a disk $D^2$. Set $\alpha = \alpha(\Gamma), \beta = \beta(\Gamma)$. Then there exists a minimal chart $\Gamma'$ obtained from $\Gamma$ by applying C-I-M1 moves and C-I-M2 moves satisfying the following conditions.

(a) There exist two cycles $C_\alpha, C_\beta$ with $C_\alpha \subset \Gamma'_\alpha, C_\beta \subset \Gamma'_\beta$ such that $C_\alpha \cap C_\beta$ consists of the two crossings.

(b) There exists an annulus $A$ containing all the white vertices of $\Gamma'$ but not intersecting hoops nor free edges.

(c) Each connected component of $Cl(D^2 - A)$ contains a crossing.

(d) $\Gamma' \cap \partial A = (C_\alpha \cup C_\beta) \cap \partial A$, $\Gamma' \cap \partial A$ consists of eight points.

(e) $\Gamma'_\alpha \cap A$ consists of two connected components $X_1, X_3$ separated by $C_\beta$.

(f) $\Gamma'_\beta \cap A$ consists of two connected components $X_2, X_4$ separated by $C_\alpha$.

For each $i = 1, 2, 3, 4$, let $D^*_i$ be a regular neighbourhood of the SC-closure $SC(Cl(X_i))$ in the annulus $A$. Then the tangle $(\Gamma' \cap D^*_i, D^*_i)$ is called fundamental tangle of the 2-crossing chart $\Gamma'$. 

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Let $\Gamma$ be a chart, and $E$ a disk. Suppose that an edge $e$ of $\Gamma$ transversely intersects $\partial E$. Let $p$ be a point in $e \cap \partial E$, and $N$ a regular neighbourhood of $p$. Then the orientation of $e$ induces the one of the arc $e \cap N$. The edge $e$ is said to be \textit{locally inward} (resp. \textit{locally outward}) at $p$ with respect to $E$ if the oriented arc $e \cap N$ is oriented from a point outside (resp. inside) $E$ to a point inside (resp. outside) $E$. We often say that $e$ is locally inward (resp. outward) at $p$ instead of saying that $e$ is locally inward (resp. outward) at $p$ with respect to $E$ if there is no confusion.

Let $\Gamma$ be a chart, and $m$ a label of the chart. A tangle $(\Gamma \cap D, D)$ is called an \textit{IO-tangle of label $m$} provided that

(i) no terminal edge nor free edge intersects $\partial D$,

(ii) there exists a label $k$ with $|m - k| = 1$ and $\Gamma \cap D \subset \Gamma_m \cup \Gamma_k$,

(iii) there exist two arcs $L_I, L_O$ on $\partial D$ with $L_I \cap L_O = \partial L_I = \partial L_O = \Gamma_m \cap \partial D$,

(iv) for any point $p \in \Gamma \cap \text{Int} L_I$, there exists an edge of label $k$ locally inward at $p$, and

for any point $p \in \Gamma \cap \text{Int} L_O$, there exists an edge of label $k$ locally outward at $p$.

The pair $(L_I, L_O)$ is called a \textit{boundary IO-arc pair} of the tangle. An IO-tangle of label $m$ is said to be \textit{simple} if all the terminal edge in $D$ is of label $m$.

\textbf{Lemma 9.2} ([11, theorem 1.2]) Let $\Gamma$ be a 2-crossing minimal chart. If a fundamental tangle of $\Gamma$ contains at least two white vertices, then the tangle is a simple IO-tangle.

\section{Indices}

Let $\Gamma$ be a minimal chart, and $m$ a label of the chart. Let $(\Gamma \cap D, D)$ be a simple IO-tangle of label $m$ with a boundary IO-arc pair $(L_I, L_O)$. Let $\sigma_1, \sigma_2, \cdots, \sigma_s$ be all the O-terminal edges of label $m$ in $D$ and $\tilde{\sigma}_1, \tilde{\sigma}_2, \cdots, \tilde{\sigma}_t$ all the I-terminal edges of label $m$ in $D$. For each terminal edge $\tau$ in $D$, let $e^*, e^{**}$ be the sibling edges of the terminal edge. By the similar argument of the proof of Lemma 4.1, the union $e^* \cup e^{**}$ splits the disk $D$ into two disks. Let $\Delta(\tau)$ be the one of the two disks containing the terminal edge. Set $D^\dagger = \text{Cl}(D - \bigcup_{i=1}^s \Delta(\sigma_i) - \bigcup_{j=1}^t \Delta(\tilde{\sigma}_j))$.

Let $A_0, A_1, \cdots, A_s$ be the connected components of $L_I \cap \partial D^\dagger$ situated counterclockwise on $\partial D^\dagger$ in this order, and $B_0, B_1, \cdots, B_t$ the connected components of $L_O \cap \partial D^\dagger$ situated clockwise on $\partial D^\dagger$ in this order. Let $X_I$ be the union of all the I-edges for $D$ each of whose label is different from $m$, and $X_O$ the union of all the O-edges for $D$ each of whose label is different from $m$. For each $i = 0, 1, \cdots, s$ and $j = 0, 1, \cdots, t$, let
Then the \((s + 1)\)-tuple \(a(\Gamma, D) = (a_0, a_1, \ldots, a_s)\) and the \((t + 1)\)-tuple \(b(\Gamma, D) = (b_0, b_1, \ldots, b_t)\) are called the I-index and O-index of the simple IO-tangle respectively. The pair \(\text{Index}(\Gamma, D) = (a(\Gamma, D), b(\Gamma, D))\) is called the index of the simple IO-tangle. For the simple IO-tangle as the one shown in Fig. 16, the I-index is \((4, 3, 1)\) and the O-index is \((2, 5)\).

Similarly we can define an index of the N-tangle as follows.

Let \(\Gamma\) be a minimal chart. Let \((\Gamma \cap D, D)\) be an N-tangle with an IO-label pair \((s_I, s_O)\) and a boundary IO-arc pair \((L_I, L_O)\). Considering Theorem 1.1(d) and (e), let \(k = T_O(D) = T_I(D)\) and \(D^\dagger\) the fundamental region of the N-tangle. Let \(A_0, A_1, \ldots, A_k\) be the connected components of \(L_I \cap \partial D^\dagger\) situated counterclockwise on \(\partial D^\dagger\) in this order, and \(B_0, B_1, \ldots, B_k\) the connected components of \(L_O \cap \partial D^\dagger\) situated clockwise on \(\partial D^\dagger\) in this order. Let \(X_I\) be the union of all the I-edges for \(D\), and \(X_O\) the union of all the O-edges for \(D\). For each \(i = 0, 1, \ldots, k\), let \(a_i = |A_i \cap X_I|\), and \(b_i = |B_i \cap X_O|\). Then \((k + 1)\)-tuples \(a(\Gamma, D) = (a_0, a_1, \cdots, a_k)\) and \(b(\Gamma, D) = (b_0, b_1, \cdots, b_k)\) are called the I-index and O-index of the N-tangle respectively. The pair \(\text{Index}(\Gamma, D) = (a(\Gamma, D), b(\Gamma, D))\) is called the index of the N-tangle. According to Theorem 1.1(b), we have \(\sum_{i=0}^{k} a_i = \sum_{i=0}^{k} b_i\).

For the N-tangle as the one shown in Fig. 17, the I-index is \((3, 3, 3, 1)\) and the O-index is \((2, 3, 4, 1)\).

Let \(x = (x_1, x_2, \cdots, x_k)\) be an I-index or O-index of a tangle, then the
Figure 18: The dark gray area \( N \) is a regular neighbourhood of \( \partial D^2 \).

sum of components, \( \sum_{i=1}^{k} x_i \), is called the index sum, and denoted by \( ||x|| \).

### 9.3 A double hoops trick

Let \( \Gamma \) be a chart in a disk \( D^2 \). By Assumption 1 and Assumption 2, all the simple hoops and free edges of \( \Gamma \) are in a regular neighbourhood \( N \) of \( \partial D^2 \) in \( D^2 \). Set \( C^* = \partial N - \partial D^2 \). Let \( e \) be an edge in Main(\( \Gamma \)) of label \( m \) such that there exists an arc \( \ell \) in \( Cl(D^2 - N) \) connecting a point \( p \) in \( \text{Int} \ e \) and a point \( q \) in \( C^* \) with \( \ell \cap \text{Main}(\Gamma) = \{p\} \) (see Fig. 18(a)).

We construct a chart from \( \Gamma \) by a C-I-M1 move and C-I-M2 moves as follows. First we create a simple hoop \( H \) of label \( m \) surrounding the point \( q \) by a C-I-M1 move (see Fig. 18(b)), where \( H \) is oriented such that we can apply C-I-M2 move between \( e \) and \( H \). Next apply a C-I-M2 move to the hoop \( H \) along \( C^* \) (see Fig. 18(c) and (d)). Then we obtain two simple hoops parallel to \( C^* \); one hoop \( H_1 \) is in \( N \) and the other hoop \( H_2 \) is in \( D^2 - N \). Finally apply a C-I-M2 move between the edge \( e \) and \( H_2 \) along the arc \( \ell \) to get a new edge \( e^* \) of label \( m \) (see Fig. 18(e)). Let

\[ \Gamma^* = (\Gamma - e) \cup e^* \cup H_1. \]

Then \( \Gamma^* \) is a chart C-move equivalent to \( \Gamma \). We say that the chart \( \Gamma^* \) is obtained from \( \Gamma \) by a double hoops trick for the edge \( e \).

### 9.4 Normal form for Main(\( \Gamma \))

Let \( \Gamma \) be a 2-crossing minimal chart in a disk \( D^2 \) different from the chart in Fig. 19(a). Set \( \alpha = \alpha(\Gamma) \), and \( \beta = \beta(\Gamma) \). Then by Lemma 9.1, we can assume that
Figure 19: (a) the thin edges are of label $m + 1$. (b) $A = D_1 \cup E_1 \cup D_2 \cup E_2 \cup D_3 \cup E_3 \cup D_4 \cup E_4$.

1. There exist two cycles $C_\alpha, C_\beta$ with $C_\alpha \subset \Gamma_\alpha, C_\beta \subset \Gamma_\beta$ such that $C_\alpha \cap C_\beta$ consists of the two crossings,

2. There exists an annulus $A$ containing all the white vertices of $\Gamma$ but not intersecting hoops nor free edges,

3. Each connected component of $\text{Cl}(D^2 - A)$ contains a crossing,

4. $\Gamma \cap \partial A = (C_\alpha \cup C_\beta) \cap \partial A$, $\Gamma \cap \partial A$ consists of eight points,

5. $\Gamma_\alpha \cap A$ consists of two connected components $X_1, X_3$ separated by $C_\beta$,

6. $\Gamma_\beta \cap A$ consists of two connected components $X_2, X_4$ separated by $C_\alpha$.

Then we can show that each of $X_1, X_2, X_3, X_4$ contains at least two white vertices. For each $i = 1, 2, 3, 4$, let $D_i$ be a regular neighbourhood of the SC-closure $SC(X_i)$ in the annulus $A$. Then $(\Gamma \cap D_i, D_i)$ is a simple IO-tangle by Lemma 9.2. Without loss of generality we can assume that (see Fig. 19(b))

7. $\text{Cl}(A - (D_1 \cup D_2 \cup D_3 \cup D_4))$ consists of four disks, say $E_1, E_2, E_3, E_4$,

8. $D_1, E_1, D_2, E_2, D_3, E_3, D_4, E_4$ are situated on the annulus $A$ in this order,

9. An O-edge for $D_1$ is an I-edge for $E_1$.

Let $e_1^*$ be an O-edge for $D_1$. Let $C_0$ be a simple closed curve in Int $A$ containing the edge $e_1^*$ and intersecting each of the eight disks $D_1, E_1, D_2, E_2, D_3, E_3, D_4, E_4$ by a proper arc. The oriented edge $e_1^*$ induces the orientation of the simple closed curve $C_0$. Let $\ell$ be an arc with $\ell \cap A = E_1 \cap D_2$ connecting a point in $\partial A$ and a point in $\partial D^2$. 

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If the simple closed curve $C_0$ is oriented clockwise (see Fig. 20(a)), then apply the chart $\Gamma$ by a double hoops trick for each the edge of Main($\Gamma$) intersecting $\ell$ one by one from the outside (see Fig. 20(b) and (c)), we can assume

(10) the simple closed curve $C_0$ is oriented counterclockwise (see Fig. 21).

Figure 20:

Let $U$ be the connected component of $D^2 - A$ containing $\partial D^2$. Now consider the cycles $C_\alpha, C_\beta$ as non-oriented simple closed curves. The oriented edge $e_\alpha$ of $\Gamma_\alpha$ containing $\Gamma_\alpha \cap U$ induces the orientation of the simple closed curve $C_\alpha$. Similarly the oriented edge $e_\beta$ of $\Gamma_\beta$ containing $\Gamma_\beta \cap U$ induces the orientation of the simple closed curve $C_\beta$.

If necessary we apply the chart $\Gamma$ by a double hoops trick for the edge $e_\alpha$, we can assume that

(11) the simple closed curve $C_\alpha$ is oriented counterclockwise.
If necessary we apply the chart $\Gamma$ by a double hoops trick for the edge $e_\beta$ (see Fig. 20(c) and (d)) and if necessary we renumber $E_1, D_1, E_2, D_2, E_3, D_3, E_4, D_4$, we can assume that (see Fig. 21)

(12) $E_1$ does not intersect any of disks bounded by $C_\alpha$ nor $C_\beta$.

Define

$$\delta = \begin{cases} 
1 & \text{if the simple closed curve } C_\beta \text{ is oriented counterclockwise}, \\
2 & \text{otherwise}.
\end{cases}$$

Case 1: $\beta - \alpha < 2$. Then the chart is a ribbon chart (see [2]).

Case 2: $\beta - \alpha = 2$. Then we can assume that

(i) $D_1 \cup D_2 \cup D_3 \cup D_4 = A$,

(ii) for each $i = 1, 2, 3, 4$,

$$||a(\Gamma, D_{i+1})|| = ||b(\Gamma, D_i)||,$$

here $D_5 = D_1$.

We define the normal form for the chart $\Gamma$ by

$$F(\Gamma) = ((n, \beta - \alpha, \alpha, \delta), (||b(\Gamma, D_1)||, ||b(\Gamma, D_2)||, ||b(\Gamma, D_3)||, ||b(\Gamma, D_4)||);$$

$$b(\Gamma, D_1), a(\Gamma, D_2), b(\Gamma, D_2), a(\Gamma, D_3), b(\Gamma, D_3), a(\Gamma, D_4), b(\Gamma, D_4), a(\Gamma, D_1)).$$

Case 3: $\beta - \alpha \geq 3$. By Theorem [1.2] for each $i = 1, 2, 3, 4$ and $j = \alpha + 1, \alpha + 2, \ldots, \beta - 2$, there exists an N-tangle $(\Gamma \cap D_i(j), D_i(j))$ with a label pair $(j, j+1)$ such that

(i) $A = (\cup_{i=1}^4 D_i) \cup (\cup_{i=1}^4 \cup_{j=\alpha+1}^{\beta-2} D_i(j))$,

(ii) $E_i = \cup_{j=\alpha+1}^{\beta-2} D_i(j)$ for each $i = 1, 2, 3, 4$,

(iii) for each $i = 1, 2, 3, 4$,

$$||b(\Gamma, D_i)|| = ||a(\Gamma, D_i(\alpha + 1))|| = ||b(\Gamma, D_i(\alpha + 1))||$$

$$= ||a(\Gamma, D_i(\alpha + 2))|| = ||b(\Gamma, D_i(\alpha + 2))||$$

$$= \cdots = ||a(\Gamma, D_i(\beta - 2))|| = ||b(\Gamma, D_i(\beta - 2))||$$

$$= ||a(\Gamma, D_{i+1})||,$$

here $D_5 = D_1$.  

Figure 21: (a) $\delta = 1$ (b) $\delta = 2$. 

Case 1: $\beta - \alpha < 2$. Then the chart is a ribbon chart (see [2]).

Case 2: $\beta - \alpha = 2$. Then we can assume that

(i) $D_1 \cup D_2 \cup D_3 \cup D_4 = A$,

(ii) for each $i = 1, 2, 3, 4$,

$$||a(\Gamma, D_{i+1})|| = ||b(\Gamma, D_i)||,$$

here $D_5 = D_1$.  

We define the normal form for the chart $\Gamma$ by

$$F(\Gamma) = ((n, \beta - \alpha, \alpha, \delta), (||b(\Gamma, D_1)||, ||b(\Gamma, D_2)||, ||b(\Gamma, D_3)||, ||b(\Gamma, D_4)||);$$

$$b(\Gamma, D_1), a(\Gamma, D_2), b(\Gamma, D_2), a(\Gamma, D_3), b(\Gamma, D_3), a(\Gamma, D_4), b(\Gamma, D_4), a(\Gamma, D_1)).$$

Case 3: $\beta - \alpha \geq 3$. By Theorem [1.2] for each $i = 1, 2, 3, 4$ and $j = \alpha + 1, \alpha + 2, \ldots, \beta - 2$, there exists an N-tangle $(\Gamma \cap D_i(j), D_i(j))$ with a label pair $(j, j+1)$ such that

(i) $A = (\cup_{i=1}^4 D_i) \cup (\cup_{i=1}^4 \cup_{j=\alpha+1}^{\beta-2} D_i(j))$,

(ii) $E_i = \cup_{j=\alpha+1}^{\beta-2} D_i(j)$ for each $i = 1, 2, 3, 4$,

(iii) for each $i = 1, 2, 3, 4$,

$$||b(\Gamma, D_i)|| = ||a(\Gamma, D_i(\alpha + 1))|| = ||b(\Gamma, D_i(\alpha + 1))||$$

$$= ||a(\Gamma, D_i(\alpha + 2))|| = ||b(\Gamma, D_i(\alpha + 2))||$$

$$= \cdots = ||a(\Gamma, D_i(\beta - 2))|| = ||b(\Gamma, D_i(\beta - 2))||$$

$$= ||a(\Gamma, D_{i+1})||,$$
We define the normal form for the chart $\Gamma$ by $F(\Gamma) = ((n, \beta - \alpha, \alpha, \delta), (|b(\Gamma, D_1)|, |b(\Gamma, D_2)|, |b(\Gamma, D_3)|, |b(\Gamma, D_4)|); b(\Gamma, D_1), \text{Index}(\Gamma, D_1(\alpha + 1)), \text{Index}(\Gamma, D_1(\alpha + 2)), \ldots, \text{Index}(\Gamma, D_1(\beta - 2)), a(\Gamma, D_2); b(\Gamma, D_2), \text{Index}(\Gamma, D_2(\beta - 2)), \text{Index}(\Gamma, D_2(\beta - 3)), \ldots, \text{Index}(\Gamma, D_2(\alpha + 1)), a(\Gamma, D_3); b(\Gamma, D_3), \text{Index}(\Gamma, D_3(\alpha + 1)), \text{Index}(\Gamma, D_3(\alpha + 2)), \ldots, \text{Index}(\Gamma, D_3(\beta - 2)), a(\Gamma, D_4); b(\Gamma, D_4), \text{Index}(\Gamma, D_4(\beta - 2)), \text{Index}(\Gamma, D_4(\beta - 3)), \ldots, \text{Index}(\Gamma, D_4(\alpha + 1)), a(\Gamma, D_1))$.

For example, the normal form for the 5-chart in Fig. 22 is
$((5, 3, 1, 2), (8, 9, 8, 7); (1, 3, 4), (3, 3, 2), (2, 3, 3), (6, 2); (1, 3, 5), (2, 3, 4), (4, 3, 2), (2, 3, 3, 1); (2, 4, 2), (1, 3, 2, 2), (1, 2, 4, 1), (4, 3, 1); (2, 5), (2, 3, 2), (2, 3, 2), (5, 2)).$

We have the following.

1. Let $\Gamma$ and $\Gamma^*$ be 2-crossing minimal charts. If $F(\Gamma) = F(\Gamma^*)$ and $\Gamma - \text{Main}(\Gamma) = \Gamma^* - \text{Main}(\Gamma^*)$, then the two charts are C-move equivalent.

2. There does not exist a 2-crossing $w$-minimal chart $\Gamma$ with $\alpha(\Gamma) - \beta(\Gamma) \geq 3$. Hence $c$-minimality seems to be good for classifying 2-crossing
There does not exist a 2-crossing minimal chart representing a surface braid whose closure is a 2-knot \[9\text{, Theorem 1.2}\].

There exists a 2-crossing minimal 4-chart (see Fig. 19(a) and Fig. 1 in \[6\]).

But we do not know if there exists a 2-crossing \(c\)-minimal chart with \(\alpha(\Gamma) - \beta(\Gamma) \geq 3\).

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| 2-crossing chart            | p32     |
| 3-color disk                | p10     |
### Index of notations

| Notation                  | Page | Definition                                                                 |
|---------------------------|------|-----------------------------------------------------------------------------|
| $\Gamma_m$                | 2    | $T_I(D)$                                                                    |
| $\text{Main}(\Gamma)$     | 8    | $T_O(D)$                                                                    |
| $\mathcal{W}_O^{\text{Mid}}(C, m)$ | 10   | $\mathcal{I}_I(D)$                                                        |
| $P[v_i, v_j]$              | 19   | $\mathcal{O}_O(D)$                                                        |
| $E_I(D)$                   | 25   | $(D, \alpha + 1)$-number $n(\Gamma, D, \alpha + 1)$                       |
| $E_O(D)$                   | 25   | $(\Gamma, D, \alpha + 1)$-minimal                                          |
| $\alpha(\Gamma)$          |      | $\alpha(\Gamma), \beta(\Gamma)$                                          |

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