On maximal function of discrete rough truncated Hilbert transforms

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Abstract
We prove the weak type (1,1) estimate for maximal function of the truncated rough Hilbert transform considered in Paluszyński (ASNSCS 910:679-704, 2019), Paluszyński (CM 164(2):305-325, 2021).

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Singular integral operators · Hilbert transform

Mathematics Subject Classification
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1 Introduction and statement of the results

Let $N$ be a dyadic integer, $1 < \alpha \leq 1 + \frac{1}{1000}$ and let

$$\mu_N(x) = \sum_{N \leq m \leq 2N} \phi \left( \frac{m}{N} \right) \frac{\delta_0(x - [m^\alpha])}{m}$$

(1)

where $\delta_0$ denotes the unit point mass at zero, and $\phi \in C^\infty_c(1, 2)$. For fixed positive $\theta < 1$ and $\tilde{\mu}_N(x) = \mu_N(-x)$ we define

$$H_M f(x) = \sum_{M^\theta \leq N \leq M} (\mu_N - \tilde{\mu}_N) \ast f(x).$$

(2)

The operators $H_M$ can be viewed as a discrete analog of rough singular integral operators, see eg. [3]. They have appeared in the $\ell^{1,\infty}$ invertibility problem for discrete singular integral operators considered by the authors in [14, 15]. In particular, it has been proved in [15] that $H_M$ is of weak type $(1,1)$, uniformly with $M$.

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The second named author, Jacek Zienkiewicz has passed away, tragically, on January 9, 2023.
In the current paper we consider the following maximal variant of $H_M$

$$H_M^* f(x) = \max_{B \leq M} \left| \sum_{M^0 \leq N \leq B} (\mu_N - \tilde{\mu}_N) \ast f(x) \right|. \quad (3)$$

It is by now classical, that operators $H_M$ and $H_M^*$ are bounded on $\ell^p$, $p > 1$. However, their behavior on $\ell^1$ still seems to be an open question. In this direction we prove the following partial result.

**Theorem 1** The maximal Hilbert operators $H_M^* f$ defined by (3) are of weak type $(1,1)$, uniformly in $M$.

One of the motives to investigate this question is the natural difference in the analysis of singular integral operators in the discrete and the continuous settings. In the discrete case, the analytic properties of singular integral operators and ergodic averages along sequences of integers are a delicate consequence of arithmetic properties of these sequences. This subject has received considerable attention and we include some of the references. We refer in particular to [21] for a nice recap of the background. The proof of Theorem 1 is a refined version of the argument from [22] and also employs ideas from [3, 18]. In this direction, we refer to our previous work [14, 15, 22] for a more complete list of references and motivation.

## 2 Definitions

Let $A$, $N$ be positive dyadic integers, $\lambda > 0$. For a set $A$ we will denote its indicator function by $1_A(x)$, and its cardinality by $|A|$. For a function $f$ we denote $f^{\lambda N}(x) = f(x) \cdot 1_{\{|f| < \lambda N\}}(x)$. We put $f^{\lambda N}_\infty(x) = f(x) - f^{\lambda N}(x)$ and $f^{\lambda N}_{\overline{f}} = f1_{\{|f| \sim \lambda N \}}$. This last notation may be misleading, but we will try to avoid any ambiguity in what follows.

Let $\{Q\}$ be any collection of disjoint intervals in $\mathbb{Z}$. We define the conditional expectation operator

$$E_{\{Q\}} f(x) = \sum_Q \frac{1_Q(x)}{|Q|} \sum_{y \in Q} f(y) \quad (4)$$

In particular, we will consider the family of dyadic intervals $J$ of equal length $|J| \approx M^{\theta - \epsilon}$. Its expectation operator (4) will be denoted by $E_{\{J\}}$. We will write $E_A f$ if the collection contains only one interval $A$.

We will use the following variant of the Calderón-Zygmund decomposition

**Lemma 1** Let $f \geq 0$, $f \in \ell^1$, and $\lambda > 0$. Then there exists a family of disjoint dyadic intervals $Q \subset \mathbb{Z}$ such that

$$\lambda \leq \frac{1}{|Q|} \sum_{x \in Q} f(x) \leq 2\lambda,$$

and for any $x \notin \bigcup_Q Q$ we have $f(x) \leq \lambda$.

In what follows we will call the above intervals $Q$ Calderón-Zygmund cubes. We note that for the family of Calderón-Zygmund cubes we have $\|E_{\{Q\}} f(x)\|_{\ell^\infty} \leq 2\lambda$. If the family $\{Q\}$ is fixed we will abbreviate the notation $E_{\{Q\}} f(x)$ to $Ef(x)$. 

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We will estimate $|\{x : |H^*_M f_0(x)| > \lambda\}|$, where we assume $f_0 \geq 0$ and $\|f_0\|_{L^1} = 1$. $C$ will denote a constant, which can vary on each of its occurrences. For $\phi \in C_c^\infty$ and a number $s$ we denote $\phi_s(x) = \phi(\frac{x}{s})$. If $I$ is an interval with center $x_I$, we denote $\phi_I(x) = \phi_I(x - x_I)$.

**3 Lemmas**

We begin with the following

**Lemma 2** Fix dyadic integers $N_1, N_2$, an interval $J$ and $\phi \in C_c^\infty$. The interval $J$ is of the size specified above, but not necessarily dyadic. Let

$$K_{N_1, N_2}(x_1, x_2) = \sum_{y \in J} \phi_J(y) \mu_{N_1}(x_1 - y) \mu_{N_2}(x_2 - y)$$

(6)

Then

1. $K_{N_1, N_2}(x_1, x_2)$ is supported on the set

$$\{(x_1, x_2) : |x_1 - x_J| \leq CN_1^\alpha \land |x_2 - x_J| \leq CN_2^\alpha \}.$$

2. If $N_2 \neq N_1$ we have

$$|K_{N_1, N_2}(x_1, x_2)| \leq \frac{C|J|}{(N_1 N_2)^\alpha}$$

(7)

and for some $\delta > 0$

$$|K_{N_1, N_2}(x_1 + h, x_2) - K_{N_1, N_2}(x_1, x_2)| \leq \frac{C|J|}{(N_1 N_2)^\alpha} \left( \frac{|h|}{N_1^\alpha} \right)^{\delta}$$

(8)

$$|K_{N_1, N_2}(x_1, x_2 + h) - K_{N_1, N_2}(x_1, x_2)| \leq \frac{C|J|}{(N_1 N_2)^\alpha} \left( \frac{|h|}{N_2^\alpha} \right)^{\delta}$$

(9)

3. If $N_2 = N_1 = N$ then there exists a kernel $Err_{N, J}(x_1, x_2)$ supported on the set

$$\{(x_1, x_2) : |x_1 - x_2| \leq CN^{1-\epsilon} \}$$

satisfying

$$\sum_J |Err_{N, J}(x_1, x_2)| \leq \frac{C}{N^{\alpha}}$$

(10)

such that

$$K_{N, N}(x_1, x_2) = \tilde{K}_{N, N}(x_1, x_2) + Err_{N, J}(x_1, x_2) + C_{N, J} \delta_0(x_1 - x_2)$$

(11)

where

$$|C_{N, J}| \leq \frac{C|J|}{N^{1+\alpha}}$$

and the kernel $\tilde{K}_{N, N}(x_1, x_2)$ satisfies (7), (8), (9).

4. Let $Q$ be an interval of length $|Q| \geq M^{\alpha-1+\epsilon}$, for some $\epsilon > 0$, and let

$$K_N(x) = \frac{1}{|Q|} \mu_N \ast 1_Q(x).$$
Then, for some \( \delta > 0 \)
\[
\sum_x |K_N(x + h) - K_N(x)|^2 \leq \frac{C}{N^\alpha} \left( \frac{|h|}{N^\alpha} \right)^\delta
\]  
(12)

**Proof** Statement 0 is obvious. Statements 1 and 2 follow by inspection of the proofs of the Lemma (3.1) in [22] and the Lemma in the appendix of [15]. We briefly sketch the necessary modifications of the argument. Since the statement is translation invariant we can assume that \( J \) is centered at 0, with length \( \sim M^{\alpha - \epsilon} \). We can assume \( N_1 \geq N_2 \).

Let
\[
Q = N_1^{1-\delta_1}, \quad R = Q^{\delta_2}, \quad H = Q^{\delta_3}
\]
where \( \delta_1, \delta_2, \delta_3 \) are small positive constants, which will be specified later. We define
\[
Z_{jQ} = \frac{1}{\alpha|x_1 - x_2 - (jQ)^\alpha|^{\alpha-1}}.
\]

One can easily check that under the conditions of the lemma the denominator never vanishes. In fact one can check, that if
\[
\alpha - \delta_1 < \alpha \theta \quad (13)
\]
then the denominator of \( Z_{jQ} \) is comparable to \( N_1^\alpha \). It what follows we will fix \( \delta_1 = \frac{1}{100} \). That, together with the choice of \( \theta \) close to 1 will ensure (13). \( \theta \) can be assumed to be arbitrarily close to 1 without loss of generality (see note in the Appendix of [15]). We define 1 - periodic functions \( \tilde{\Psi}_a \) and \( \Psi_a \) by the conditions:
\[
0 \leq \Psi_{jQ} \leq 1, \quad \Psi_{jQ}(t) = 1 \text{ for } 1 - Z_{jQ} \leq t \leq 1
\]
(14)
\[
\text{supp}(\Psi_{jQ}(t)) \subset \left\{ t : 1 - Z_{jQ} \left( 1 + \frac{1}{R} \right) \leq t \leq 1 + Z_{jQ} \left( 1 + \frac{1}{R} \right) \right\} + \mathbb{Z},
\]
(15)
where \( \mathbb{Z} \) is the set of integers,
\[
\Psi_{jQ}(t) \in C^\infty \text{ and } |\partial^k \Psi_{jQ}(t)| \leq C \left( RZ_{jQ}^{-1} \right)^k \text{ for } k \leq 4
\]
(16)
\[
0 \leq \tilde{\Psi}_H \leq 1, \quad \tilde{\Psi}_H \in C^\infty, \quad |\partial^k \tilde{\Psi}_H| \leq CH^k \text{ for } k \leq 4
\]
(17)
\[
\text{supp} \tilde{\Psi}_H \subset \left[ 0, \frac{2}{H} \right] + \mathbb{Z} \quad \text{and} \quad \sum_{0 \leq P < H} \tilde{\Psi}_H \left( \Theta - \frac{P}{H} \right) = 1
\]
(18)

Now, we briefly adopt the following convention: The symbol \( a \leq_N b \) means \( b - a = O\left( \frac{1}{N^{\alpha + \eta}} \right) + C \) where \( \eta > 0 \) is some constant depending on \( \alpha \) and independent of \( N \), and \( C > 0 \). Similarly as in [22] we have
\[
K_{N_1, N_2}(x_1, x_2) \leq N_1
\]
(19)
\[
\leq N_1 \sum_{0 \leq P < H} \sum_{j} \sum_{s=0}^{Q-1} \tilde{\Psi}_H \left( (jQ + s)^\alpha - \frac{P}{H} \right)
\]
\[
\times \Psi_{jQ} \left( (x_1 - x_2 - (jQ + s)^\alpha - \frac{P}{H})^\frac{1}{2} \right)
\]
\[
\times \phi_N(x_1 - (jQ + s)^\alpha - x_2) \phi_J(x_1 - (jQ + s)^\alpha)
\]
Now we replace $\Psi_{jQ}$ by its Fourier series. By the definition, we have
\[
c_\alpha \left(1 - \frac{1}{R}\right)(x_1 - x_2 - (jQ)^\alpha)^{-\frac{\alpha-1}{\alpha}} \leq \hat{\Psi}_{jQ}(0) \leq c_\alpha \left(1 + \frac{1}{R}\right)(x_1 - x_2 - (jQ)^\alpha)^{-\frac{\alpha-1}{\alpha}}.
\]
Since $\sum_{0 \leq P < H} \hat{\Psi}_H(\Theta - \frac{P}{H}) = 1$, the right hand side expression corresponding to $\hat{\Psi}_{jQ}(0)$ becomes
\[
D(x_1, x_2) = \sum_{j,s} \hat{\Psi}_{jQ}(0) \phi_{N_1^s}((jQ + s)^\alpha) \phi_{N_2^s}((x_1 - (jQ + s)^\alpha - x_2)
\]
\[
\times \phi_j(x_1 - (jQ + s)^\alpha)
\]
\[
\leq \left(1 + \frac{1}{R}\right) \sum_m \frac{c_\alpha}{(x_1 - x_2 - (jQ)^\alpha)^{\frac{\alpha-1}{\alpha}}} \phi_{N_1^s}(m^\alpha) \phi_{N_2^s}(x_1 - m^\alpha - x_2)
\]
\[
\times \phi_j(x_1 - m^\alpha)
\]
\[
\leq \left(1 + \frac{1}{R}\right) \int_{R_+^\alpha} \frac{c_\alpha}{(x_1 - x_2 - t^\alpha)^{\frac{\alpha-1}{\alpha}}} \phi_{N_1^s}(t^\alpha) \phi_{N_2^s}(x_1 - t^\alpha - x_2)
\]
\[
\times \phi_j(x_1 - t^\alpha) dt
\]
\[
= \left(1 + \frac{1}{R}\right) |J| \int_{R_+} \frac{c_\alpha}{(x_2 - |J||u|)^{\frac{\alpha-1}{\alpha}}} \phi_{N_1^s}(x_1 - |J||u|) \phi_{N_2^s}(|J||u| - x_2)
\]
\[
\times \phi(|u|) \frac{\alpha |u|}{(x_1 - |J||u|)^{\frac{\alpha-1}{\alpha}}}
\]
\[
= \left(1 + \frac{1}{R}\right) |J| F(x_1, x_2)
\]
By similar arguments, one can obtain the lower estimate
\[
D(x_1, x_2) \geq \left(1 - \frac{1}{R}\right) |J| F(x_1, x_2)
\]
Now the function of $F(x_1, x_2)$ can be easily shown to satisfy the estimates $|\partial_{x_1} F(x_1, x_2)| \leq \frac{C}{N_1^\alpha - N_2^\alpha - \tau}$. This immediately yields (7), (8), (11).

In order to complete the argument we need to estimate the summands corresponding to coefficients $\hat{\Psi}_{jQ}(k), k \neq 0$.

For $j, P$ fixed, we are left with the estimates for
\[
\sum_{s=0}^{Q-1} \hat{\Psi}_H((jQ + s)^\alpha - \frac{P}{H})
\times \left(\Psi_{jQ}\left(\left(x_1 - x_2 - (jQ + s)^\alpha - \frac{P}{H}\right)^\frac{1}{\alpha}\right) - \hat{\Psi}_{jQ}(0)\right)
\times \phi_{N_1^s}((jQ + s)^\alpha) \phi_{N_2^s}(x_1 - (jQ + s)^\alpha - x_2) \phi_j(x_1 - (jQ + s)^\alpha)
\]
Let $N_1 \neq N_2$ or $|x_1 - x_2| \geq CN_1$, and $|J| \geq Q$. Then the above sum is exactly of the form considered in [22]. Let $\delta_1 = \frac{1}{100}$. Using van der Corput Lemma we get an estimate with additional $Q^{-\eta}$ factor, where $\eta > 0$ is independent of $\alpha$. We fix $\delta_2, \delta_3$ sufficiently small independent of $\alpha$. Then summing with respect to $j, P$ completes the job. We note, that
choosing \( \theta \) sufficiently close to 1 we can ensure required \( |J| \geq Q \). We omit further details, and refer the reader to [22].

The estimate (10) follows from
\[
|\mu_N \ast \tilde{\mu}_N(x)| \leq \frac{C}{N^\alpha} \text{ for } x \neq 0,
\]
proved in [22].

The Hölder’s estimate (12) has been proved in [14], Lemma 3.5. The proof there is carried out for a smooth function \( \varphi \) in the place of characteristic function, but it carries over (with small modification in part III, see [14]).

We fix small \( \epsilon > 0 \), \( 1 - \epsilon < \theta < 1 \), the function \( f_0 \geq 0 \), \( \lambda > 0 \) and the set of the Calderón-Zygmund cubes \( \{Q\} \) associated with \( f_0 \) by Lemma 1. By the \( \ell^2 \) boundedness of the maximal rough Hilbert transform \( H^*_M \) and Lemma 1 we can assume that \( f_0 = 0 \) away from the \( \bigcup Q \). Now we modify \( f_0 \) putting
\[
f_0 = 0 \text{ on each Calderón-Zygmund cube with } |Q| \leq M^{\theta-2\epsilon}.
\]
(20)

We will denote new function again by \( f_0 \). In the remark at the end of the paper we explain why this procedure do not bring any loss of the generality.

We perform further reductions. First we have
\[
\max_B \left| \sum_{N \leq B} (\mu_N - \tilde{\mu}_N) \ast f_0 \right| \leq \max_B \left| \sum_{N \leq B} (\mu_N - \tilde{\mu}_N) \ast (f_0^{\lambda N}) \right|
\]
\[
+ \max_B \left| \sum_{N \leq B} (\mu_N - \tilde{\mu}_N) \ast (Ef_0^{\lambda N}) \right|
\]
\[
+ \max_B \left| \sum_{N \leq B} (\mu_N - \tilde{\mu}_N) \ast (f_0^{\lambda N} - Ef_0^{\lambda N}) \right|
\]
\[
+ \max_B \left| \sum_{N \leq B} (\mu_N - \tilde{\mu}_N) \ast Ef_0 \right| = I + II + III + IV
\]

Now, \( I \) has been estimated in [15, 22] using support properties. The estimates for the \( IV \) follow since \( |Ef_0| \leq C\lambda \) and the maximal Hilbert transform \( H^*_M \) is bounded on \( \ell^2 \). The term \( II \) requires some care. First we note, that \( G_M = \sum_{M^\theta \leq N \leq M} (\mu_N - \tilde{\mu}_N) \ast (Ef_0^{\lambda N}) \in \ell^2 \) and \( \|G_M\|_{\ell^2} \leq C\lambda \|f_0\|_{\ell^1} \). This has been proved in [15] page 22 with \( f_0^{\lambda N} \) replaced by \( f_0^{\lambda N} \). The proof of our statement is exactly the same, and we do not present any details.

Next, the key observation is the following property of the Calderón-Zygmund cubes in our setting

**Lemma 3**

1. Let \( Q \) be the Calderón-Zygmund cube which contains the point of the support of \( f_0^{\lambda N} \). Then \( |Q| \geq CN \).

2. Let \( Q \) be the Calderón-Zygmund cube which contains the point of the support of \( f_0^{\lambda N} \). Then \( |Q| \geq CNA^{-1} \).

**Proof** The lemma follows immediately from the upper inequality in (5). \( \square \)
From (20) and (12) we infer that the function $\mu_N * 1_Q$ is Hölder regular. Consequently we can estimate $II$ in the similar manner as the maximal Calderón-Zygmund operator. First observe that by (12) the following estimates hold
\[
\sum_{N \leq B} |\phi_{B^*} * (\mu_N - \tilde{\mu}_N)(x)| \leq C_{B^*} \|\phi_{B^*}\|_1
\]
and consequently
\[
\left| \sum_{N \geq B} (\mu_N - \tilde{\mu}_N) * (\mathcal{F}_{0,\infty}^N(x) - \phi_{B^*} \ast G_M(x)) \right| \leq C_M \mathcal{F}(x) + C(M(\mathcal{F}_0)^2)^{1/2}(x)
\]
where $M$ denotes the classical Hardy-Littlewood maximal function. By weak type $(1,1)$ of $M$ we obtain $II \leq C \lambda$. We leave the details (which are standard arguments in the Calderón-Zygmund theory) to the reader.

So the main term is $III$. We further decompose $f_0$ writing down
\[
f_0^{\lambda N} = \sum_{A \text{-dyadic}} f_{0,A}^{\lambda N}
\]
and obtain the following estimate
\[
|III| \leq \max_B \left| \sum_{N \leq B} \sum_{1 \leq A \leq N} (\mu_N - \tilde{\mu}_N) * (f_{0,A}^{\lambda N} - \mathcal{F}^N_{0,A}(x)) \right|
\]
For $i = 1, 2$ we write
\[
b_{s,i}^{A,N}(x) = \sum_{Q \in \mathcal{D}_{A,N,s}^i} (1_Q(x)f_{0,A}^{\lambda N}(x) - \mathcal{F}_{0,A}^{\lambda N}(x)),
\]
\[
f_{s,i}^{A,N}(x) = \sum_{Q \in \mathcal{D}_{A,N,s}^i} 1_Q(x)f_{0,A}^{\lambda N}(x),
\]
where the families of cubes $\mathcal{D}_{A,N,s}^i, i = 1, 2$ are defined as follows
\[
\mathcal{D}_{A,N,s}^1 = \{ Q : |Q| \sim (2^{-s}N)^a, 2^i > A \}
\]
\[
\mathcal{D}_{A,N,s}^2 = \{ Q : |Q| \sim (2^{-s}N)^a, 2i \leq A \}
\]
Furthermore, for $i = 1, 2$ we let
\[
f_{i}^{A,N}(x) = \sum_{s \geq 0} f_{s,i}^{A,N}(x),
\]
and for $i = 2$
\[
f_{s,2}^{N}(x) = \sum_{A : A \geq 2^s} f_{s,2}^{A,N}(x).
\]
From the above definitions, for fixed $A, N$ we have
\[
\bigcup_{s \geq 0} (\mathcal{D}_{A,N,s}^1 \cup \mathcal{D}_{A,N,s}^2) = \{ Q : |Q| \leq N^a \}
\]
We note that \( \text{supp} \mu_N \ast 1_Q \subset Q^{**} \) if \( |Q| \geq N^\alpha \), and by (5) we have \( \sum |Q^{**}| \leq \frac{1}{\lambda} \). Hence it will suffice to estimate

\[
H_i^s(x) = \max_{B} \left| \sum_{N \leq B} \sum_{A} \sum_{s \geq 0} (\mu_N - \tilde{\mu}_N) \ast b_{s,i}^A (x) \right|
\]

(28)

**Lemma 4** Let \( f_i^{A,N} \) be defined in (25) and let \( \beta_N \geq 0 \) be a sequence of numbers. We define a sequence of integers \( N_j, 1 \leq j \leq j_{\text{max}} \) by

\[
N_j = \max \left\{ 2^k : \sum_{N \leq 2^k} \beta_N \leq j \lambda_0 \right\}
\]

(29)

whenever the maximum exists. Assuming

\[
\beta_N \leq \lambda_0
\]

we see that \( N_j \)'s form a strictly increasing sequence of dyadic integers. Let \( x \in J \) be such that

\[
\max_B \left| \sum_{N \leq B} \mu_N \ast f_i^{A,N} - \sum_{N \leq B} \beta_N \right| \geq 4\lambda_0
\]

(30)

Then

\[
\max_j \left| \sum_{N \leq N_j} \mu_N \ast f_i^{A,N} - \sum_{N \leq N_j} \beta_N \right| \geq \lambda_0
\]

(31)

The same is true for the functions \( f_s^{N} \) in the place of \( f_i^{A,N} \). We call \( j_{\text{max}} \) the largest value of \( j \) in this case.

**Proof** Fix \( B \) maximising the estimate in (30) and the unique \( j \) such that \( N_j < B \leq N_{j+1} \). Assume, that

\[
\left| \sum_{N \leq N_j} \mu_N \ast f_i^{A,N} - \sum_{N \leq N_j} \beta_N \right| \leq \lambda_0
\]

(32)

Then

\[
\left| \sum_{N_j < N \leq B} \mu_N \ast f_i^{A,N} - \sum_{N_j < N \leq B} \beta_N \right| \geq 3\lambda_0
\]

(33)

and by (29) we must have

\[
\sum_{N_j < N \leq B} \mu_N \ast f_i^{A,N} - \sum_{N_j < N \leq B} \beta_N \geq 3\lambda_0
\]

(34)

Since \( f_i^{A,N} \geq 0 \), again by (29)

\[
\sum_{N_j < N \leq N_{j+1}} \mu_N \ast f_i^{A,N} - \sum_{N_j < N \leq N_{j+1}} \beta_N \geq \sum_{N_j < N \leq B} \mu_N \ast f_i^{A,N} - \sum_{N_j < N \leq B} \beta_N - \lambda_0 \geq 2\lambda_0
\]
Applying (32) we get
\[ \left| \sum_{N \leq N_{j+1}} \mu_N * f_i^{A,N} - \sum_{N \leq N_{j+1}} \beta_N \right| \geq \lambda_0 \]  \tag{35}

The proof for functions \( f_{s,2}^N \) is similar. \( \square \)

**Remark.** Under the assumptions of the above lemma we can split the set of dyadic naturals into two collections
\[ C_1 = \{ N : \beta_N \leq \lambda_0 \} \quad \text{and} \quad C_2 = \{ N : \beta_N > \lambda_0 \}, \]
such that
\[ \max_j \left| \sum_{N \leq N_j, N \in C_1} \mu_N * f_i^{A,N} - \sum_{N \leq N_j, N \in C_1} \beta_N \right| \geq \frac{1}{2} \lambda_0 \]  \tag{36}
or
\[ \max_j \left| \sum_{N \leq N_j, N \in C_2} \mu_N * f_i^{A,N} - \sum_{N \leq N_j, N \in C_2} \beta_N \right| \geq \frac{1}{2} \lambda_0 \]  \tag{37}
and the cardinality of \(|C_2| \leq j_{A}^A\).

**Lemma 5** Fix an interval \( J \), \( A \) and \( i \). Let \( b_{s,i}^{A,N} \) be defined in (22), \( x \in J \) and
\[ \max_B \left| \sum_{N \leq B} \sum_s \mu_N * b_{s,i}^{A,N}(x) \right| \geq 4 \lambda_0 \]  \tag{38}
Then there exists the sequence of nonnegative numbers \( \beta_N \) independent on \( x \in J \), such that (30) holds, or (the definition of \( E_R \) below) \( E_R(x) \geq \lambda_0 \). The sequence \( \beta_N \) depends on \( J \).

If \( \{ N_j \} \) is a sequence of numbers such that (30) holds for (40) then
\[ \max_j \left| \sum_{N \leq N_j} \sum_s \mu_N * b_{s,i}^{A,N}(x) \right| \geq \frac{\lambda_0}{8} \]  \tag{39}
or \( E_R(x) \geq \lambda_0 \).

**Proof** By the definition we have \( \sum_s \mu_N * b_{s,i}^{A,N}(x) = \mu_N * f_i^{A,N}(x) - F_{A,N}(x) \), where \( F_{A,N}(x) = \mu_N * E f_i^{A,N}(x) \). Then we put
\[ \beta_N = E_{\{ J \}} F_{A,N}(x) \]  \tag{40}
where \( E_{\{ J \}} \) is the conditional expectation operator defined by (4), and \( x \in J \). By (20) we have \(|Q| \geq N^{1-3\epsilon} \). We will show that the error function
\[ E_R(x) = \sum_N \left| F_{A,N}(x) - E_{\{ J \}} F_{A,N}(x) \right| \]  \tag{41}
satisfies
\[ \|E_R\|_{\ell^1} \leq C M^{-\delta \epsilon} \|f_0\|_{\ell^1}. \]  \tag{42}
where $\delta_e$ is some small constant depending on the $\epsilon$ in the definition of $J$. Observe

$$F_{A,N}(x) = \mu_N \ast \left( \sum_Q \frac{1}{|Q|} \sum_Q f_i^{A,N} \right)(x)$$

$$= \sum_Q \frac{C_{N,Q}}{|Q|} \mu_N \ast 1_Q(x)$$

$$= \sum_Q \rho_{N,Q}(x) \cdot C_{N,Q},$$

where

$$\rho_{N,Q} = \mu_N \ast \frac{1}{|Q|} \quad C_{N,Q} = \sum_{x \in Q} f_i^{A,N}(x).$$

Consequently, for $x \in J$

$$E R(x) \leq \sum_N \sum_Q C_{N,Q} \left| E_{[J]} \rho_{N,Q}(x) - \rho_{N,Q}(x) \right|$$

$$\leq \sum_N \sum_Q C_{N,Q} \frac{1_{J}(x)}{|J|} \sum_{h \in J} \left| \rho_{N,Q}(h) - \rho_{N,Q}(x) \right|$$

$$\leq \sum_N \sum_Q C_{N,Q} \frac{1_{J}(x)}{|J|} \sum_{h \in J_0^*} \left| \rho_{N,Q}(h + x) - \rho_{N,Q}(x) \right|,$$

where $J_0^*$ is the cube centered at 0, with size double that of $J$. Thus (recall, that $|J| \sim M^{\theta - \epsilon}$)

$$\| E R \|_{\ell^1} \leq \sum_N \sum_Q \frac{C_{N,Q}}{|J|} \sum_{x \in J} \sum_{h \in J_0^*} \left| \rho_{N,Q}(h + x) - \rho_{N,Q}(x) \right|$$

$$\leq \frac{1}{|J|} \sum_N \sum_Q C_{N,Q} \sum_{h \in J_0^*} \sum_x \left| \rho_{N,Q}(h + x) - \rho_{N,Q}(x) \right|$$

$$\leq C \frac{1}{|J|} \sum_N \sum_Q C_{N,Q} \sum_{h \in J_0^*} \left( \frac{|h|}{N^\alpha} \right)^{\delta}$$

$$\leq C \left( \frac{M^{\theta - \epsilon}}{M^{\theta \alpha}} \right)^{\delta} \sum_N \sum_Q C_{N,Q}$$

$$\leq CM^{-\epsilon \delta} \| f_0 \|_{\ell^1}.$$

We have used an $\ell^1$ Hölder estimate for $\rho_{N,Q}$. It follows from the $\ell^2$ estimate (12) by Cauchy-Schwarz inequality. We have also used

$$\sum_N \sum_Q C_{N,Q} \leq \| f_0 \|_{\ell^1}.$$

Finally, for the case $i = 2$ we define, similarly to $F_{A,N}$,

$$F_{s,2}^N(x) = \mu_N \ast E_{[J]} f_{s,2}^N(x),$$

where $f_{s,2}^N$ was defined in (26). The resulting $\beta_N = E_{[J]} F_{s,2}^N(x)$ and the error function $E R$ can be treated in the same way.
Lemma 6 Let
\[ A_A = \{ x \in \mathbb{Z} : \sum_N E_{(j)} F_{A,N}(x) \geq \lambda A^2 \} , \]
\[ A^s = \{ x \in \mathbb{Z} : \sum_N E_{(j)} F_{s,N}^N(x) \geq \lambda 2^{s^e} \} . \]

Then we have \(|A_A| \leq \frac{1}{\lambda A^2} , |A^s| \leq \frac{1}{\lambda 2^{s^e}} ,\) moreover for each \( J \), \( J \subset A_A \) or \( J \cap A_A = \emptyset \) and \( J \subset A^s \) or \( J \cap A^s = \emptyset \).

Proof Immediate from the Chebyshev inequality. The second part follows since \( E_{(j)} F_{A,N} \) is constant on each \( J \).

By the above lemma 6 we can consider only the intervals \( J \) with \( j_{\text{max}}^A \leq CA^3 \) and \( j_{\text{max}}^s \leq 2^{3s^e} \) for every \( A , s \). This will allow us to apply the Rademacher-Menschov classical estimate for the maximal function by the sum of a small number of square functions.

Lemma 7 Let \( \{ a_i \}_{1 \leq i \leq D} \) be a sequence of numbers. Then
\[ \max_{1 \leq i \leq D} \left| \sum_{1 \leq j \leq i} a_j \right|^2 \leq C \sum_{k \leq \log D} \sum_s \sum_{2^k s \leq j \leq 2^{k(s+1)}} |a_j|^2 \]
(43)

Proof This is a well known fact, see [12] Lemma 1.

For fixed \( N \) we denote by \( \mathcal{J}_N \) the family of dyadic intervals of size \( 8N^\alpha \leq |I| < 16N^\alpha \). Then, for \( N \) fixed, \( \cup_{j \in \mathcal{J}_N} I = \mathbb{Z} \). Moreover, any two \( I_1 \in \mathcal{J}_{N_1} , I_2 \in \mathcal{J}_{N_2} \) either have empty intersection, or one is a subset of the other. For given interval \( I \) we denote by \( I^* \) an interval concentric with \( I \), with a larger diameter. The exact ratio of diameters depends on constants appearing in Lemma 2, and will be obvious from the context.

Lemma 8 Let \( b_{s,i}^{A,N} \) be defined as in (22). Fix \( J \) and \( i \). Let \( I_1 \in \mathcal{J}_{N_1} , I_2 \in \mathcal{J}_{N_2} , J \subset I_{N_1} \cap I_{N_2} \). Then for any fixed increasing sequence of integers \( \{ S_j \} \) we have
\[ \sum_y \sum_j \sum_s \sum_{S_j < N \leq S_{j+1}} \mu_N * b_{s,i}^{A,N}(y)^2 \phi_J(y) \leq \sum_{s_1,s_2} \left( \sum_{N_2 \leq N_1} \frac{|J|}{(N_1 N_2)^\alpha} \| b_{s_1,i}^{A,N_1} \|_{\ell^1(I_1^*)} \| b_{s_2,i}^{A,N_2} \|_{\ell^1(I_2^*)} \right)
+ \sum_{N_1,s} \left( \sum_{x \in I_1^*} |b_{s,i}^{A,N_1}(x)|^2 \right)
+ \sum_{N_1,s_1,s_2} \left( |\text{Err}_{N_1,J} b_{s_1,i}^{A,N_1} | , |b_{s_2,i}^{A,N_2} | \right)
= D_I(J) + D_{II}(J) + D_{III}(J) \]
(44)

The terms \( D_{II}(J) , D_{III}(J) \) appears only if \( N_1 = N_2 \). Moreover the RHS of (44) do not depend on the particular choice of the sequence \( S_j \).

Proof We expand the square as a double sum
\[ \sum_y \sum_j \sum_s \sum_{S_j < N \leq S_{j+1}} \mu_N * b_{s,i}^{A,N}(y)^2 \phi_J(y) = \sum_{s_1,s_2} \sum_j \sum_{S_j < N_1,N_2 \leq S_{j+1}} < K_{N_1,N_2}^{J} b_{s_1,i}^{A,N_1} , b_{s_2,i}^{A,N_2} > \]
(45)
and to each summand we apply the regularity estimate (7), (8) of $K_{N_1,N_2}^J$, see Lemma 2.

Let $s_1 \geq s_2$. If $N_1 \neq N_2$ then we apply (8) and the standard cancellation argument (we omit the details). This leads to

$$\|K_{N_1,N_2}^J b_{s_1,i}^{A,N_1}\|_{\ell^{\infty}} \leq 2^{-s_1 \delta} |J| (N_1 N_2)^{-\alpha} \|b_{s_1,i}^{A,N_1}\|_{\ell^1(I_j^*)}$$  \quad (46)

If $N_1 = N_2$, we obtain the same estimate for $\tilde{K}$ instead of $K$.

If $s_1 < s_2$ we repeat the above argument to the kernels conjugate to $K$, $\tilde{K}$, acting on $b_{s_2,i}^{A,N_2}$. The estimate of the first summand in (44) follows.

If $N_1 = N_2$, $s_1 \neq s_2$ then the supports of the functions $b_{s_1,i}^{A,N_1}$, $b_{s_2,i}^{A,N_2}$ are disjoint. Consequently, the $\delta_0$ term in (11) produce the second $D_{11}(J)$ term in (44), and the Lemma follows.

\[ \Box \]

4 Proof of theorem 1

We now return to the proof of Theorem 1. Recall, that we have reduced the proof to estimating the 2 operators $H_i^*$ given by (28), $i = 1, 2$. We proceed with the 2 cases. Case $i = 1$. Recall the definition (22):

$$b_{s,1}^{A,N}(x) = \sum_{Q \in \mathcal{D}_{A,N,s}} (1_Q(x) f_0^\lambda N(x) - E_Q f_0^\lambda N(x))$$

For a dyadic interval $J$ (recall that we only consider $J$’s such that $|J| \sim M^{\theta - \epsilon}$) we need to estimate

$$\left| \left\{ x \in J : \max_{B} \sum_{N \leq B} \sum_{s \geq 0} (\mu_N - \hat{\mu}_N) \ast b_{s,1}^{A,N}(x) > \lambda A^{-\epsilon} \right\} \right|$$

\[
\begin{align*}
\leq & \left| \left\{ x \in J : \max_{B} \sum_{N \leq B} \sum_{s \geq 0} \mu_N \ast b_{s,1}^{A,N}(x) > \frac{1}{2} \lambda A^{-\epsilon} \right\} \right| \\
+ & \left| \left\{ x \in J : \max_{B} \sum_{N \leq B} \sum_{s \geq 0} \hat{\mu}_N \ast b_{s,1}^{A,N}(x) > \frac{1}{2} \lambda A^{-\epsilon} \right\} \right|
\end{align*}
\]

We call the first summand $L_J$. It is enough to estimate $L_J$, since the second summand can be estimated analogously. We apply Lemma 5 (with $\lambda_0 = c \lambda A^{-\epsilon}$) and thus there exists a sequence $\{N_j\}_{j \leq \lambda A^3}$, depending on $J$ but independent of $x \in J$, the collections $C_1, C_2$ such that for some $v \in \{1, 2\}$ ($v = v(A,J)$

$$L_J \leq \left| \left\{ x \in J : \max_{j \leq \lambda A^3} \sum_{N \leq N_j} \sum_{s \geq 0} \mu_N \ast b_{s,1}^{A,N}(x) > \lambda A^{-\epsilon} \right\} \right|$$

\[
\begin{align*}
+ & \left| \left\{ x \in J : E R(x) > c \lambda A^{-\epsilon} \right\} \right| \left| A_{A} \cap J \right|
\end{align*}
\]

(regarding the range of $j$’s see Lemma 6 and remark that follows). Now apply Lemma 7, and obtain a $k \leq c \log A$ such that if we put $S_j = N_{2^k - j}$ we have

$$\sum_j L_J \leq \sum_j \left| \left\{ x \in J : \sum_j \sum_{s \leq S_j} \sum_{s \geq 0} \mu_N \ast b_{s,1}^{A,N}(x) \right\}^2 > \lambda^2 A^{-3\epsilon} \right|$$

\[
\begin{align*}
+ & \left| \left\{ x : E R(x) > c \lambda A^{-\epsilon} \right\} \right| + |A_A|.
\end{align*}
\]

\[ \Box \]
By (42) and Chebychev’s inequality, Lemmas 5 and 6, the two last summands have estimates

$$\|x : ER(x) > c\lambda A^{-\varepsilon}\| \leq c \frac{\|f_0\|_{\ell^1}}{\lambda N^\delta} \leq c \frac{1}{\lambda A^{\varepsilon}},$$

$$|A| \leq c \frac{1}{\lambda A^2},$$

since $\varepsilon$ can be chosen small enough and, as was pointed out before, only $A \leq N$ are relevant.

We turn to the first summand. Applying Lemma 8 and the Chebychev’s inequality we get

$$\sum_{J} \left\{ \left| \left\{ x \in J : \sum_j \sum_{s_j \leq |J|} \sum_{N \in C_j} \sum_{s \geq 0} \mu_N \ast b_{s,1}^A (x) \right| \right| \right|^2 > \lambda^2 A^{-3\varepsilon} \right\} \right|$$

$$\leq \sum_{s \geq t_A} \sum_{N \geq N^2} 2^{-5(s_1 + s_2)} \frac{A^{3\varepsilon} |J|}{\lambda^2 (N_1 N_2)^\alpha} \times \|b_{s_1,1}^A \|_{\ell^1(I^*_1)} \|b_{s_2,1}^A \|_{\ell^1(I^*_2)} +$$

$$+ \sum_{J} \sum_{s \geq t_A} \sum_{N \geq N^2} \frac{A^{3\varepsilon} |J|}{\lambda^2 N^{\alpha + 1}} \|b_{s_1,1}^A \|_{\ell^1(I^*_1)} \|b_{s_2,1}^A \|_{\ell^1(I^*_2)}$$

$$+ \sum_{J} \sum_{s_j \geq t_A} \sum_{i=1,2} \|b_{s_j,1}^A \|_{\ell^1(I^*_j)} \|b_{s_2,1}^A \|_{\ell^1(I^*_j)} \geq \lambda^2 A^{3\varepsilon} \right|$$

$$= I + II + III,$$

where $I_N(J)$ is the unique dyadic interval from the family $J_N$ which contains $J$. Again, we start with the first component $I$. Note, that the sum of $|J|$ of those $J$’s, which share the same $I_N^j(J)$ is equal to $|I_N^j(J)| \leq N_2^\alpha$. So,

$$I \leq \sum_{s \geq t_A} \sum_{N \geq N^2} 2^{-5(s_1 + s_2)} \sum_{N_1 \geq N_2} \sum_{I_N^1 \subset I^N_1} \sum_{I_N^2 \subset I^N_1} \frac{A^{3\varepsilon} |J|}{\lambda^2 N_1^\alpha} \times \|b_{s_1,1}^A \|_{\ell^1(I^*_1)} \|b_{s_2,1}^A \|_{\ell^1(I^*_2)}.$$

Further, for fixed $N_1$ and $I_N^1 \in J_{N_1}$ observe

$$\sum_{N_2 \leq N_1} \sum_{I \in \mathcal{J}_{N_2} \subset I_{N_1}} \|b_{s_2,1}^A \|_{\ell^1(I^*)}$$

$$= \sum_{N_2 \leq N_1} \sum_{I \in \mathcal{J}_{N_2} \subset I_{N_1}} \sum_{x \in I^*} \sum_{Q \in \mathcal{D}_{A,N_2,r_2}^1} \left| \left( Q \mathcal{F}_{0}^{\lambda N_2}(x) - E Q \mathcal{F}_{0}^{\lambda N_2}(x) \right) \right|$$

$$\leq 2 \sum_{N_2 \leq N_1} \sum_{I \in \mathcal{J}_{N_2} \subset I_{N_1}} \sum_{x \in I^*} \sum_{Q \in \mathcal{D}_{A,N_2,r_2}^1} \mathcal{F}_{0}^{\lambda N_2}(x)$$

$$\leq c \sum_{Q \in \mathcal{D}_{A,N_2,r_2}} \sum_{x \in Q} \mathcal{F}_{0}^{\lambda N_2}(x).$$
We have used the fact, that scales of cubes in $D^1_{A, s_2}$ are all smaller than $N^\alpha_2$, and also that for fixed $A, s_2$ the cubes in $D^1_{A, N_2, s_2}$ are disjoint. Consequently, again using disjointness and (5)

\[
\sum_{N_2 \leq N_1} \sum_{Q \in D^1_{A, N_2, s_2}} \sum_{x \in Q} 1_Q f_0^{\lambda N_2} (x) \leq C \sum_{Q \subset I^*_N} \sum_{x \in Q} \left( \sum_{N_2 \leq N_1} 1_Q f_0^{\lambda N_2} (x) \right) \leq C \sum_{Q \subset I^*_N} \sum_{x \in Q} f_0(x)
\]

\[
\leq C \sum_{Q \subset I^*_N} \lambda |Q| \leq C \lambda |I_N| \leq C \lambda N^\alpha_1
\]

Thus,

\[
I \leq c \sum_{s_i: 2s_i \geq A} 2^{-\delta (s_1 + s_2)} \sum_{N_1 \in J_{N_1}} \sum_{I \in J_{N_1}} A^{3e} \lambda \|b^A_{s_1, 1}\|_{\ell^2(I^*_N)} \leq c \sum_{N_1 \in J_{N_1}} \sum_{I \in J_{N_1}} A^{3e} \lambda \|f_0^A\|_{\ell^2(I^*_N)} \leq \frac{c}{\lambda A^2},
\]

using disjointness of the supports of $f_0^A$.

We now turn to $II$ in (47), and proceed similarly.

\[
II = \sum_{J} \sum_{N} \sum_{s: 2^s \geq A} A^{3e} \frac{|J|}{\lambda^2 N^{\alpha+1}} \|b^A_{s, 1}\|^2_{\ell^2(I^*_N(J))} \leq c \sum_{J} \sum_{N} \sum_{s: 2^s \geq A} A^{3e} \frac{|J|}{\lambda^2 N^{\alpha+1}} \|f^A_{s, 1}\|^2_{\ell^2(I^*_N(J))} \leq c \sum_{J} \sum_{N} \sum_{s: 2^s \geq A} A^{3e} \frac{|J|}{\lambda^2 N^{\alpha+1}} \sum_{Q \subset I^*_N(J)} \|f^A_{s, 1}\|^2_{\ell^1(Q)} \leq c \sum_{J} \sum_{N} \sum_{s: 2^s \geq A} A^{3e} \frac{|J|}{\lambda^2 N^{\alpha+1}} \sum_{Q \subset I^*_N(J)} \|f^A_{s, 1}\|^2_{\ell^1(Q)}
\]
\begin{align*}
\leq c \sum_{N} \sum_{I_N} \frac{A^{3e}}{\lambda^2 N^\alpha + 1} \sum_{Q \subset I_N^*} (f_{s,1}^{A,N})_Q^2 \|f_{s,1}^{A,N}\|_{\ell^1(Q)} \sum_{J \subset I_N^*} |J| \\
\leq c \sum_{N} \frac{A^{3e}}{\lambda^2 N} \sum_{Q} (f_{s,1}^{A,N})_Q^2 \|f_{s,1}^{A,N}\|_{\ell^1(Q)} \\
\leq c \sum_{N} \frac{A^{3e}}{\lambda A} \sum_{Q} f_{s,1}^{A,N} \|f_{s,1}^{A,N}\|_{\ell^1} \\
= c \frac{A^{3e}}{\lambda A} \sum_{Q} f_{s,1}^{A,N} \|f_{s,1}^{A,N}\|_{\ell^1} \\
\leq c \frac{\|f_0\|_{\ell^1}}{\lambda} \\
= c \frac{1}{\lambda A^{1-3e}}.
\end{align*}

We finally consider the last summand in (47). We use the following two properties of the kernel $Err_{N,J}$ (Lemma 2): its support (which is within $cN^\alpha$ from the center of $J$ in both variables and, for any $x$, $y$, and also in the strip $|x − y| \leq CM$)

$$
\sum_{J} |Err_{N,J}(x, y)| \leq \frac{c}{N^\alpha}.
$$

which, by (5), leads to

$$
\sum_{J} \sum_{y} |Err_{N,J}(x, y)||b_{s_2,1}^{A,N}(y)| \leq \frac{c\lambda(N + (2^{-s}N)^\alpha)}{N^\alpha}.
$$

We have

$$
III = \sum_{J} \sum_{s_1,2^{s_1} \geq A} \sum_{i=1,2} \frac{A^{3e}}{\lambda^2} (|Err_{N,J}b_{s_1,1}^{A,N}, b_{s_2,1}^{A,N}|) \\
\leq C \frac{A^{3e}}{\lambda^2} \sum_{s_1,2^{s_1} \geq A} \sum_{i=1,2} \sum_{J} \sum_{x,y \in I_N^*(J)} |Err_{N,J}(x, y)||b_{s_1,1}^{A,N}(x)||b_{s_2,1}^{A,N}(y)| \\
\leq C \frac{A^{3e}}{\lambda^2} \sum_{s_1,2^{s_1} \geq A} \sum_{i=1,2} \sum_{J} \sum_{x,y \in I_N^*(J)} \frac{\lambda (N + (2^{-s_1}N)^\alpha)}{N^\alpha} \|b_{s_2,1}^{A,N}\|_{\ell^1(I_N^*)} \\
\leq C \frac{\|f_0\|_{\ell^1}}{\lambda}.
$$

\textbf{Case } i = 2. \text{ Recall (24)}

$$
\mathcal{D}_{A,N,s}^2 = \{Q : |Q| \sim (2^{-s}N)^\alpha, \ 2^s \leq A\},
$$

and (22)

$$
b_{s_2,2}^{A,N}(x) = \sum_{Q \in \mathcal{D}_{A,N,s}^2} \left(1_Q(x) f_{s_2}^{\lambda N} (x) − E_Q f_{s_2}^{\lambda N} (x)\right).
$$
We have

\[
\max_B \left| \sum_s \sum_{\substack{A \geq 2^s \leq B}} \mu_N * b_{s,2}^{A,N}(x) \right|^2 \leq c_\varepsilon \sum_s 2^{s \varepsilon} \max_B \left| \sum_{\substack{A \geq 2^s \leq B}} \mu_N * b_{s,2}^{A,N}(x) \right|^2.
\]

Hence, for some \(s\), we must have

\[
\max_B \left| \sum_{N \leq B} \mu_N * b_{s,2}^N(x) \right|^2 > 2^{-2s} \lambda_0^2,
\]

where

\[
b_{s,2}^N = \sum_{A : A \geq 2^s} b_{s,2}^{A,N}.
\]

Now, apply lemmas 4, 5, 6 exactly as in the case \(i = 1\). The Theorem follows

**Remark.** If \(|Q| \leq M^{\theta - 2\varepsilon}\) then \(A \geq M^\varepsilon\) by Lemma 3. By Lemma 8 or by [22] we have

\[
\sum_N \sum_{I_N} |\mu_N * b_Q^{A,N}(x)|^2 \leq \frac{C_\lambda}{M^{2\varepsilon}} \tag{50}
\]

This yields the desired maximal estimate since we have at most \(C \log M\) summands. We omit the details.

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