Computation of asymptotic expansions of turning point problems via Cauchy’s integral formula

T. M. Dunster $^{SDSU}$, A. Gil $^{UC}$, J. Segura $^{UC}$

SDSU: San Diego State University, USA
UC: Universidad de Cantabria, Spain

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In this talk we are dealing with asymptotic expansions for large $u$ for the solutions of second order differential equations

$$\frac{d^2 w}{dz^2} = (u^2 f(z) + g(z))w,$$

in two cases:

1. In regions where $f(z)$ does not have a zero (type I),
2. Around a simple zero of $f(z)$ (type II),

to which we add a third intermediate case.

The first type of expansions are also called Liouville-Green expansions and the second one uniform Airy-type expansions. We will see how to link these two separates cases through the intermediate case of LG Airy-type expansions.

We will show applications to the computation of Bessel functions and Laguerre polynomials in the complex plane.
We say that an expansion of the form

\[ h(u) \sim \sum_{n=0}^{\infty} a_n u^{-n}, \]

is an **asymptotic expansion** as \( u \to \infty \), if

\[ u^N \left( h(u) - \sum_{n=0}^{N-1} a_n u^{-n} \right), \quad N = 0, 1, 2, \ldots, \]

is a bounded function for large values of \( u \), with limit \( a_N \) as \( u \to \infty \), for any \( N \). This can also be written as

\[ h(u) = \sum_{n=0}^{N-1} a_n u^{-n} + O\left(u^{-N}\right), \quad u \to \infty. \]
Liouville-Green expansions

We start with the expansions of type I (Liouville-Green) for solutions of

$$\frac{d^2 w}{dz^2} = (u^2 f(z) + g(z))w$$

for large $u$, and $z$ away from turning points (zeros of $f(z)$).

Change of variable: $\left(\frac{d\xi}{dz}\right)^2 = f(z)$. Change of function: $W = f^{1/4} w$.  

$$\frac{d^2 W}{d\xi^2} = \left\{ u^2 + \phi(\xi) \right\} W, \quad \phi(\xi) = \frac{4f(z)f''(z) - 5f'^2(z)}{16f^3(z)} + \frac{g(z)}{f(z)}.$$  

This gives formal solutions

$$W_\pm \sim \exp\{\pm u\xi\} \sum_{s=0}^{\infty} (\pm 1)^s A_s(\xi) u^{-s}.$$  

with coefficients that can be determined recursively by

$$A_{s+1}(\xi) = -\frac{1}{2} A'_s(\xi) + \frac{1}{2} \int \phi(\xi) A_s(\xi) d\xi.$$  

Symbolic differentiation is needed. Nested integration occurs.
Liouville-Green expansions

An alternative: expansion in exponential form.

Formal solutions:

\[ W_\pm \sim \exp \left\{ \pm u\xi + \sum_{s=1} \left( \pm \frac{E_s(\xi)}{u^s} \right) \right\} \]

In these, the coefficients are given by

\[ E_s(\xi) = \int F_s(\xi) d\xi \quad (s = 1, 2, 3, \cdots), \]

where

\[ F_1(\xi) = \frac{1}{2} \phi(\xi), \quad F_2(\xi) = -\frac{1}{4} \phi'(\xi), \]

and

\[ F_{s+1}(\xi) = -\frac{1}{2} F_s'(\xi) - \frac{1}{2} \sum_{j=1}^{s-1} F_j(\xi) F_{s-j}(\xi) \quad (s = 2, 3, \cdots). \]

See F. W. J. Olver, Asymptotics and Special Functions, Chap. 10, Ex 2.1.

Symbolic differentiation is needed. No nested integration occurs.
Bessel equation: \( t^2 y''(t) + ty'(t) + (\nu^2 - t^2)y(t) = 0 \). \( w(z) = \sqrt{z}y(\nu z) \) satisfies:

\[
w''(z) = \left\{ \nu^2 f(z) + g(z) \right\} w, \quad f(z) = -1 + z^{-2}, \quad g(z) = 1/(4z^2)
\]

Formal solutions:

\[
W \sim \exp \left\{ \pm \nu \xi + \sum_{s=1}^{\infty} (\pm 1)^s \frac{E_s(\xi)}{\nu^s} \right\}, \quad \xi(z) = \log \frac{1 + \sqrt{1-z^2}}{z} - \sqrt{1-z^2}.
\]

In particular, for the \( J \)-Bessel function:

\[
J_\nu(\nu z) \sim \frac{1}{\sqrt{2\nu \pi}} (1 - z^2)^{-1/4} \exp \left\{ -\nu \xi + \sum_{s=1}^{\infty} (-1)^s \frac{E_s(\xi)}{\nu^s} \right\}, \quad E_s(+i\infty) = 0
\]

The points where the relative error is greater than \( 10^{-12} \) are shown (for \( \nu = 100 \) and 14 terms)
Airy-type expansions around the turning points

\[
d^2 w/dz^2 = \left\{ u^2 f(z) + g(z) \right\} w,
\]

where \( u \to \infty \) and \( f(z) \) has a simple zero (turning point) at \( z = z_0 \).

For an approximation around the turning point, we set

\[
\zeta \left( \frac{d\zeta}{dz} \right)^2 = f(z) \to \frac{2}{3} \zeta^{3/2} = \int_{z_0}^{z} f^{1/2}(t) dt = \xi, \quad W = \zeta^{-1/4} f^{1/4}(z) w
\]

and we have

\[
d^2 W/d\zeta^2 = \left\{ u^2 \zeta + \psi(\zeta) \right\} W, \quad \psi(\zeta) = \frac{5}{16\zeta^2} + \zeta \phi(\xi)
\]

The natural approximants are **Airy functions**, solutions of \( y''(x) = xy(x) \), and more specifically

\[
Ai_j(x) = Ai(e^{-2\pi ij/3} x), \quad j = 0, \pm 1, \quad x = u^{2/3} \zeta
\]

which are solutions of

\[
d^2 W/d\zeta^2 = u^2 \zeta W.
\]
**Airy functions**: solutions of $y''(x) = xy(x)$. $Ai_j(x) = Ai(e^{-2\pi ij/3} x)$, $j = 0, \pm 1$, $x \in \mathbb{C}$.

$Ai_j(x)$ recessive as $x \to \infty$ in the sector $S_j$.

$$
Ai_j(x) = Ai(e^{-2\pi ij/3} x),
$$

where $c_s$ and $a_s$ are easily computable positive rational coefficients.

Airy expansions via Cauchy’s formula

$$
Ai(x) \sim \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-x} \sum_{s=0}^{\infty} (-1)^s c_s \chi^{-s}, \quad \chi = \frac{2}{3} x^{3/2}, \quad |\arg x| \leq \pi - \delta
$$

or alternatively

$$
Ai(x) \sim \frac{1}{2\sqrt{\pi}} x^{-1/4} \exp \left( -\chi + \sum_{i=1}^{\infty} (-1)^s a_s \frac{s}{s \chi^s} \right), \quad |\arg x| \leq \pi - \delta
$$

where $c_s$ and $a_s$ are easily computable positive rational coefficients.
Airy-type expansions around the turning points

\[
\frac{d^2 W}{d\zeta^2} = u^2 \zeta W \Rightarrow \text{Ai}_j(u^{2/3} \zeta), \ j = 0, \pm 1
\]

\[
\frac{d^2 W}{d\zeta^2} = \left\{ u^2 \zeta + \psi(\zeta) \right\} W \Rightarrow A(u, z) \text{Ai}_j(u^{2/3} \zeta) + B(u, z) \text{Ai}'_j(u^{2/3} \zeta)
\]

Formal solutions

\[
W_j(u, \zeta) \sim \text{Ai}_j(u^{2/3} \zeta) \sum_{s=0}^{\infty} \frac{a_s(\zeta)}{u^{2s}} + \text{Ai}'_j(u^{2/3} \zeta) u^{-4/3} \sum_{s=0}^{\infty} \frac{b_s(\zeta)}{u^{2s}}, \ j = 0, \pm 1
\]

The coefficients satisfy (see Olver’s book, chap. 11):

\[
b_s(\zeta) = \frac{1}{2\zeta^{1/2}} \int_0^\zeta \{ \psi(t) a_s(t) - a_s''(t) \} \frac{dt}{t^{1/2}},
\]

and

\[
a_{s+1}(\zeta) = -\frac{1}{2} b_s(\zeta) + \frac{1}{2} \int \psi(\zeta) b_s(\zeta) d\zeta.
\]

Not easy! Instead of closed form expressions, normally one considers some approximations, like series around the turning point.
Alternative: compute directly $A(u, z)$ and $B(u, z)$ by some other means.

An iterative approach is considered in N. M. Temme, Numerical algorithms for uniform Airy-type asymptotic expansions, Numer. Algor. (1997).

The approach here is different. We start from

$$W_{-1}(u, z) = A(u, z)\text{Ai}_{-1}(u^{2/3} \zeta) + B(u, z)\text{Ai}'_{-1}(u^{2/3} \zeta),$$

$$W_{1}(u, z) = A(u, z)\text{Ai}_{+1}(u^{2/3} \zeta) + B(u, z)\text{Ai}'_{+1}(u^{2/3} \zeta).$$

Solve for the coefficients (using the Wronskian relation for Airy functions):

$$A(u, z) = -2\pi i \left\{ W_{-1}(u, z)\text{Ai}'_{+1}(u^{2/3} \zeta) - W_{1}(u, z)\text{Ai}'_{-1}(u^{2/3} \zeta) \right\},$$

$$B(u, z) = 2\pi i \left\{ W_{-1}(u, z)\text{Ai}_{+1}(u^{2/3} \zeta) - W_{1}(u, z)\text{Ai}_{-1}(u^{2/3} \zeta) \right\}.$$

We have $A(u, z)$ and $B(u, z)$. Problem: precisely we want to compute $W_j(u, z)$!!

But we know how to compute away from the turning point: use LG expansions.
Using LG asymptotics, we can write:

\[ A(u, z) \sim \exp \left\{ \sum_{j=1}^{\infty} \frac{E_{2j}(\xi) + \tilde{a}_{2j}\xi^{-2j}/(2j)}{u^{2j}} \right\} \times \cosh \left\{ \sum_{j=0}^{\infty} \frac{E_{2j+1}(\xi) - \tilde{a}_{2j+1}\xi^{-2j-1}/(2j+1)}{u^{2j+1}} \right\} \]

and

\[ B(u, z) \sim \frac{1}{u^{1/3} \zeta^{1/2}} \exp \left\{ \sum_{j=1}^{\infty} \frac{E_{2j}(\xi) + a_{2j}\xi^{-2j}/(2j)}{u^{2j}} \right\} \times \sinh \left\{ \sum_{j=0}^{\infty} \frac{E_{2j+1}(\xi) - a_{2j+1}\xi^{-2j-1}/(2j+1)}{u^{2j+1}} \right\} \]

The coefficients \( E_s \) and the constants \( a_s \) are those appearing before. The constants \( \tilde{a}_s \) correspond to the LG asymptotics for the derivative of the Airy function.
Theorem

For the differential equation

$$\frac{d^2w}{dz^2} = \left\{ u^2 f(z) + g(z) \right\} w,$$

assume $u$ is positive and large, $f(z)$ has a simple zero at $z = z_0$, and $f(z)$ and $g(z)$ are analytic in a domain $D$ containing $z_0$. Further assume that $f(z)$ does not vanish in the disk $D(z_0, \rho) := \{ z : 0 < |z - z_0| < \rho \} \subset D$. Define variables $\xi$ and $\zeta$ by

$$\xi = \frac{2}{3} \zeta^{3/2} = \int_{z_0}^{z} f^{1/2}(t) dt,$$

and let $\text{Ai}_j(u^{2/3}\zeta)$ $(j = 0, \pm 1)$ denote the Airy functions $\text{Ai}(u^{2/3}\zeta e^{-2\pi i j/3})$. Then there exist three numerically satisfactory solutions of the differential equation, given by

$$w_j(u, z) = \zeta^{1/4} f^{-1/4}(z) \left\{ \text{Ai}_j(u^{2/3}\zeta) A(u, z) + \text{Ai}_j'(u^{2/3}\zeta) B(u, z) \right\}.$$

In these, the coefficient functions $A(u, z)$ and $B(u, z)$ are analytic at $z = z_0$, and possess the previous asymptotic expansions in a domain that includes $D(z_0, \rho)$. The integration constants for the odd coefficients must be selected so that $\zeta^{1/2} E_{2j+1}(\xi)$ $(j = 0, 1, 2, \cdots)$ is meromorphic as a function of $\zeta$ at $\zeta = 0$. 

Javier Segura (Universidad de Cantabria)
We have LG expansions for the coefficients $A(u, z), B(u, z)$, with which we can approximate

$$w_j(u, z) = \zeta^{1/4} f^{-1/4}(z) \left\{ A(u, z) \text{Ai}_j(u^{2/3} \zeta) + B(u, z) \text{Ai}'_j(u^{2/3} \zeta) \right\}.$$ 

But we can not compute close to the turning point with the LG expansions.

By now, we just have a **LG Airy-type** expansion (valid all around the TP, but not too close).

But...
But because the coefficients $A(u, z)$ and $B(u, z)$ are analytic in a domain containing the turning point we can consider the Cauchy integral formula

$$A(u, z) = \frac{1}{2\pi i} \oint \frac{A(u, t)}{t-z} dt, \quad B(u, z) = \frac{1}{2\pi i} \oint \frac{B(u, t)}{t-z} dt,$$

with $A(u, t)$ and $B(u, t)$ approximated by LG-asymptotics along the Cauchy contour.

For the numerical integration the trapezoidal rule gives an exponential convergence rate.

Parametrization: $t(\theta) = z_c + Re^{i\theta}, \theta \in [0, 2\pi]$ (a circuit enclosing the turning point).

$$A(u, z) = \frac{1}{2\pi} \int_0^{2\pi} F(\theta) d\theta, \quad F(\theta) = A(u, t(\theta)) \frac{t(\theta) - z_c}{t(\theta) - z}$$

We write $\theta_j = 2\pi j/N, j = 0, \ldots, N$ and then

$$A(u, z) \approx \frac{1}{N} \sum_{j=1}^{N} F(2\pi j/N)$$
Bessel functions

With the transformations and coefficients shown previously for the Bessel equation, the computation of the coefficients is straightforward. We show an example of application of the Cauchy integral.

![Graph showing error in computation of H_ν^{(1)}(νz) for z = 1 + 0.1i and various selections of the number of coefficients. The minimal possible error (close to 10^{-26}) is due to the discretization of the Cauchy integral with N = 500 points.](image)

Error in the computation of $H_ν^{(1)}(νz)$ for $z = 1 + 0.1i$ and various selections of the number of coefficients. The minimal possible error (close to $10^{-26}$) is due to the discretization of the Cauchy integral with $N = 500$ points.

The trapezoidal rule has exponential convergence and the smallest reachable error roughly depends on the number of points over the Cauchy contour as $10^{-N/20}$. 
Laguerre polynomials

\[ w = z^{(\alpha+1)/2} e^{-uz/2} L_n^{(\alpha)}(uz), \ u = n + 1/2 \] satisfies

\[
\frac{d^2 w}{dz^2} = \left\{ u^2 f(z) + g(z) \right\}, \ f(z) = \frac{(z-z_1)(z-z_2)}{4z^2}, \ g(z) = -\frac{1}{4z^2}
\]

where \( \alpha = u(a^2 - 1) (a \geq 0) \) and

\[ z_1 = (a - 1)^2, \ z_2 = (a + 1)^2. \]

The TPs coalesce when \( a = 0 (\alpha = -(n + 1/2)) \). \( z_1 \) coalesces with \( z = 0 \) when \( a = 1 (\alpha = 0) \).

We have considered two types of expansions:

1. **Expansions around** \( z_1 \) with \( 1 + \delta \leq a \leq a_1 < \infty \). The domain contains \( 0 \leq z \leq z_2 - \delta \). The case of negative \( \alpha \) (but \( z_1 \) no coalescing with \( z = z_2 \) or \( z = 0 \)) has also been considered.

2. **Expansions around** \( z_2 \) with \( 0 < a_0 \leq a \leq a_1 < \infty \): the TPs do not coalesce, but \( z_1 \) may coalesce with \( z = 0 \) (we therefore exclude it). Expansions hold in a complex domain including \( z_1 + \delta \leq z < \infty \). This case can include negative \( \alpha \) if \( a_0 < 1 \).

We give some results for the second case.
We compute expansions around $z_2$ for
\[
\frac{d^2 w}{dz^2} = \left\{ u^2 f(z) + g(z) \right\}, \quad f(z) = \frac{(z - z_1)(z - z_2)}{4z^2}, \quad g(z) = -\frac{1}{4z^2}.
\]

As before, we should consider:

1. The computation of the LG expansions (away from $z_2$). The transformation is the same, but with a different $f(z)$ (and therefore a different new variable). This provides the coefficients for the next step as well as LG expansions for $L_n^{(\alpha)}(z)$.

2. The computation of the analytical (and slowly varying coefficients) $A(u, z)$ and $B(u, z)$ from the previous LG expansions. The expression for the coefficients is the same as before, but with different LG coefficients. This gives the LG Airy-type expansion away from the turning point.

3. The computation of the Airy-type expansion close to the turning point with Cauchy integrals.
We don’t give all the details of the expansions.

The LG expansion (with matching at $+\infty$) is:

$$L_n^{(\alpha)}(uz) \sim \frac{(-1)^n u^{(u-1)/2}(u + \alpha)^{(u+\alpha)/2}}{e^u n!(euz)^{\alpha/2} \{(z - z_1)(z - z_2)\}^{1/4}} \exp \left\{ \frac{1}{2} uz - u \xi + \sum_{s=1}^{\infty} (-1)^s \frac{\hat{E}_s(z)}{u^s} \right\}$$

with $\hat{E}_s(+\infty) = 0$, and in terms of the corresponding new variables.

The LG Airy-type expansion away from the turning point

$$L_n^{(\alpha)}(uz) = H(u, z) \left\{ \text{Ai}(u^{2/3} \zeta)A(u, z) + \text{Ai}'(u^{2/3} \zeta)B(u, z) \right\}$$

with $H(u, z)$ a known (lengthy but simple) function and $A(u, z)$ and $B(u, z)$ very similar to the Bessel case (but with different LG coefficients $E_s$).

And around the turning point Cauchy integrals are used.

Let us illustrate the use of these expansions with some numerical results.
Relative errors $\epsilon$ for the LG expansion and the LG Airy-type expansion as a function of $\theta$, where $z = z_2 + R e^{i\theta}$, $\theta \in [0, \pi]$ and $R = 0.25(z_2 - z_1)$. 
Relative error of the LG Airy-type expansion for real values of $z$ as a function of $\rho = (z - z_2)/(z_2 - z_1)$. 
Relative error of the Airy-type expansion for real values of $z$ as a function of

$$\rho = \frac{z - z_2}{z_2 - z_1},$$

with the coefficients $A$ and $B$ computed by Cauchy integrals. The Cauchy contour is a circumference of radius $R = 0.7(z_2 - z_1)$ centered at $z_2$. 
Conclusions:

Methods for the computation of asymptotic approximations in the complex plane have been discussed.

The methods relate two different types of asymptotic approximations (away or around turning points)

Cauchy’s integral formula for computing the coefficient functions provides this link.

The new approach enlarges the range of validity of previous expansion (notably for Laguerre polynomials) and provides accurate approximations for complex variables near turning points.

For the future: sharp bounds, coalescing turning points, turning points and poles, ...

References

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THANK YOU!