Comment on “σ-meson: Four-quark versus two-quark components and decay width in a Bethe-Salpeter approach”

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Abstract

In a recent paper by N. Santowsky et al. [Phys. Rev. D 102, 056014 (2020)], covariant coupled equations were derived to describe a tetraquark in terms of a mix of four-quark states $2q2\bar{q}$ and two-quark states $q\bar{q}$. These equations were expressed in terms of vertices describing the disintegration of a tetraquark into identical two-meson states, into a diquark-antidiquark pair, and into a quark-antiquark pair. We show that these equations are inconsistent as they imply a $q\bar{q}$ Bethe-Salpeter kernel that is $q\bar{q}$-reducible.
In 2012, Heupel, Eichmann and Fischer (HEF) \cite{1} developed covariant equations describing a tetraquark using a model where the two-quark plus two-antiquark \((2q2\bar{q})\) system is described by four-body \((4q)\) Faddeev-like equations of Khvedelidze and Kvinikhidze \cite{2}, and where the dynamics is dominated by the formation of either two identical mesons or a diquark-antidiquark pair. These equations are represented graphically in Fig. 1 and relate the form factors \(\phi_M\) and \(\phi_D\) of the tetraquark, describing its disintegration into two identical mesons, and a diquark-antidiquark pair, respectively. As is evident from Fig. 1 the input interactions to these equations consist of vertices for the transitions between a meson \((M)\) and a quark-antiquark pair \((q\bar{q}\leftrightarrow M)\), a diquark \((D)\) and a quark-quark pair \((qq\leftrightarrow D)\), and between an antidiquark \(\bar{D}\) and an antiquark-antiquark pair \((\bar{q}\bar{q}\leftrightarrow \bar{D})\). Missing from these equations is the phenomenon of quark-antiquark annihilation which would result in coupling to two-body \((2q)\) \(q\bar{q}\) states.

There have since been two attempts to extend the equations of HEF to include coupling to \(q\bar{q}\) channels. The first of these was our derivation of 2014 \cite{3} where disconnected contributions were added to the usual connected part of the \(q\bar{q}\) interaction. The second was a recent derivation of Santowsky et al. (SEFWW) \cite{4} where coupling to \(q\bar{q}\) channels was included phenomenologically. The tetraquark equations of SEFWW are represented graphically in Fig. 2 and include an additional tetraquark form factor \(\Gamma^*\), describing the disintegration of a tetraquark into a \(q\bar{q}\) pair.

For the tetraquark equations of Fig. 2 to be meaningful, it is essential that the form factor \(\Gamma^*\) be identified with the residue of the \(q\bar{q}\) Green function \(G^{(2)}\) describing the formation of the tetraquark in the scattering of a quark from an antiquark; that is, as \(P^2 \to M^2\), where \(P\) is the total momentum of the \(q\bar{q}\) system and \(M\) is the mass of the tetraquark,

\[
G^{(2)} \to \frac{G_0^{(2)} \Gamma^* \Gamma^* G_0^{(2)}}{P^2 - M^2},
\]

where \(G_0^{(2)}\) is the fully disconnected \(q\bar{q}\) propagator corresponding to the independent propagation of \(q\) and \(\bar{q}\) in the \(s\) channel. This implies that \(\Gamma^*\) satisfies the bound state equation

\[
\Gamma^* = K_{ir} G_0^{(2)} \Gamma^*
\]

where \(K_{ir}\) is the \(q\bar{q}\)-irreducible Bethe-Salpeter kernel for the \(q\bar{q}\) system.

Here we would like to point out that the tetraquark equations of SEFWW cannot be correct as they imply a kernel \(K_{ir}\) that is \(q\bar{q}\)-reducible. To show this, we write the three coupled

\[
\phi_M = \phi_M + \phi_D
\]

\[
\phi_D = \phi_M
\]

FIG. 1. Tetraquark equations without coupling to \(q\bar{q}\) channels, as first developed in Ref. \cite{1}. Tetraquark form factors \(\phi_M\) (displayed in red) couple to two mesons (dashed lines), and tetraquark form factors \(\phi_D\) (displayed in blue) couple to diquark and antidiquark states (double-lines).
FIG. 2. The tetraquark equations of SEFWW [4] which include coupling to $q\bar{q}$ channels. In addition to the tetraquark form factors as in Fig. 1, these equations involve the tetraquark form factor $\Gamma^*$ (displayed in yellow) that couples to $q\bar{q}$ states (solid lines). The amplitude $K^{(2)}$ (displayed in light blue) represents the $q\bar{q}$ kernel in a theory without $q\bar{q}$ annihilation.

equations corresponding to Fig. 2 as

\[ \Phi_M = V_{MM} G_0^0 \Phi_M + V_{MD} G_0^0 \Phi_D + N_M G_0^{(2)} \Gamma^*, \] (3a)

\[ \Phi_D = V_{DM} G_0^0 \Phi_M + N_D G_0^{(2)} \Gamma^*, \] (3b)

\[ \Gamma^* = K^{(2)} G_0^{(2)} \Gamma^* + N_M G_0 G_0^{(2)} \Phi_M + K^{(2)} G_0^{(2)} N_D G_0^{(2)} \Phi_D, \] (3c)

where $V_{MM}$, $V_{MD}$, and $V_{DM}$, are quark-exchange potentials for the processes $MM \leftarrow MM$, $MM \leftarrow DD$, and $DD \leftarrow MM$, respectively, and where $N_M$, $N_D$, $\bar{N}_M$, and $\bar{N}_D$ describe the transitions between $4q$ and $2q$ states via the processes $MM \leftarrow q\bar{q}$, $DD \leftarrow q\bar{q}$, $q\bar{q} \leftarrow MM$, and $q\bar{q} \leftarrow DD$. Note that these equations also involve a $q\bar{q}$ kernel $K^{(2)}$ which should not be confused with $K^{(2)}$ as it does not contain terms that involve $2q \leftrightarrow 4q$ transitions.

Writing Eqs. (3) in matrix form as

\[ \Phi = VG^0 \Phi + NG_0^{(2)} \Gamma^*, \] (4a)

\[ \Gamma^* = K^{(2)} G_0^{(2)} (\Gamma^* + \bar{N} G^0 \Phi), \] (4b)

where

\[ \Phi = \begin{pmatrix} \Phi_M \\ \Phi_D \end{pmatrix}, \quad G^0 = \begin{pmatrix} G_0^0 & 0 \\ 0 & G_0^0 \end{pmatrix}, \] (5)

\[ N = \begin{pmatrix} N_M \\ N_D \end{pmatrix}, \quad \bar{N} = \begin{pmatrix} \bar{N}_M \\ \bar{N}_D \end{pmatrix}, \] (6)

and

\[ V = \begin{pmatrix} V_{MM} & V_{MD} \\ V_{DM} & 0 \end{pmatrix}, \] (7)

\[ ^{1} \text{For simplicity, we ignore all symmetry factors in Eqs. (3) as they do not affect our argument.} \]
one can solve Eq. (4a) for $\Phi$ and substitute the result into Eq. (4b) to obtain
\[
\Gamma^* = \left[ K^{(2)} + K^{(2)} G_0^{(2)} \bar{N} G^0 (1 - V G^0)^{-1} N \right] G_0^{(2)} \Gamma^*.
\] (8)

Comparison with Eq. (2) shows that
\[
K_{ir} = K^{(2)} + K^{(2)} G_0^{(2)} \bar{N} G^0 (1 - V G^0)^{-1} N,
\] (9)

which is in conflict with the very definition of a $q\bar{q}$ kernel since this expression for $K_{ir}$ is $q\bar{q}$ reducible (notice the presence of the $q\bar{q}$ propagator $G_0^{(2)}$). For this reason the tetraquark equations of Ref. [4] are inconsistent.

[1] W. Heupel, G. Eichmann, and C. S. Fischer, Tetraquark Bound States in a Bethe-Salpeter Approach, Phys. Lett. B718, 545 (2012), arXiv:1206.5129 [hep-ph].
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[4] N. Santowsky, G. Eichmann, C. S. Fischer, P. C. Wallbott, and R. Williams, $\sigma$-meson: Four-quark versus two-quark components and decay width in a Bethe-Salpeter approach, Phys. Rev. D 102, 056014 (2020) arXiv:2007.06495 [hep-ph].