BRAIDED MATRIX STRUCTURE OF $q$-MINKOWSKI SPACE AND $q$-POINCARE GROUP

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Abstract We clarify the relation between the approach to $q$-Minkowski space of Carow-Watamura et al. with an approach based on the idea of $2 \times 2$ braided Hermitean matrices. The latter are objects like super-matrices but with Bose-Fermi statistics replaced by braid statistics. We also obtain new R-matrix formulae for the $q$-Poincaré group in this framework.

1 Introduction

This paper is a comment on the approach to $q$-Minkowski space developed in [1][2][3] based on the idea of the spinor decomposition of Minkowski space vectors. From this point of view there was found a suitable algebra of (non-commuting) co-ordinate functions as well as a $q$-Lorentz quantum group associated to them.

Meanwhile, since 1990 a systematic approach to $q$-deformations has been developed by the first author (S.M.) based on his idea of a braided matrix. This is like a super-matrix but with the usual $\pm 1$ statistics between co-ordinates replaced by braid statistics, typically controlled by an R-matrix. A fairly complete theory of braided geometry is available by now, modelled on the ideas of supersymmetry, and we refer to [4] for a review of 30–40 papers on this topic. The main idea in this approach is that $q$ should be viewed as a deformation of the notion of $\otimes$, i.e. it expresses how two independent copies of a system fail to commute. It is obvious that the $2 \times 2$ braided matrices introduced in [5] could be taken as a definition of $q$-Minkowski space from this point of view[6][7]. There are three $16 \times 16$ R-matrices in this approach and we recall them in Section 2.

Here we map the $q$-Minkowski space and $q$-Lorentz group of [1][2][3] into this braided setting. Part of this identification is clear because the algebra for $q$-Minkowski space there is easily
seen to be similar to that of $2 \times 2$ braided Hermitean matrices. But the underlying $16 \times 16$ R-matrices and $q$-Lorentz groups appear at first sight quite different in the two approaches (and were obtained quite differently), requiring a lot of care to sort out. We do this in Section 3.

In Section 4 we use the braided theory to obtain a new braided-matrix $q$-Poincaré group as a construction that goes beyond [1]–[3]. Indeed, the 30–40 papers on braided geometry mentioned above can all be applied at once to $q$-Minkowski space in this form. For this reason, we believe the detailed identification provided in the present paper to be an important one.

2 Braided-matrix approach

A braided matrix as introduced by the first author in [5] means an algebra with a matrix of generators $\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ say, which can be multiplied in the sense that

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

obeys the same relations as $a, b, c, d$ and another identical copy $a', b', c', d'$ provided (which is the new idea) the two copies are treated with suitable braid statistics between them.

The $2 \times 2$ braided matrices we need are the algebra $BM_q(2)$

$$ba = q^2 ab, \quad ca = q^{-2} ac, \quad da = ad, \quad bc = cb + (1 - q^{-2})a(d - a)$$
$$db = bd + (1 - q^{-2})ab, \quad cd = dc + (1 - q^{-2})ca$$

and the multiplicative braid statistics between them was found also in [5]

$$a'a = aa' + (1 - q^2)bc', \quad a'b = ba', \quad a'c = ca' + (1 - q^2)(d - a)c'$$
$$a'd = da' + (1 - q^2)bc', \quad b'a = ab' + (1 - q^2)b(d' - a'), \quad b'b = q^2 bb', \quad \text{etc.}$$

In all there are 16 such relations. In particular, we showed in [5] that

$$t = q^{-1}a + qd, \quad \det = ad - q^2 cb$$

are central and bosonic in the sense

$$x't = tx', \quad x'\det = \det x' \quad \forall x \in BM_q(2).$$

A braided (co)vector space as introduced by the first author in [8] means an algebra with a (co)vector of generators $\mathbf{x}$ say, which can be added in the sense that $\mathbf{x}'' = \mathbf{x} + \mathbf{x}'$ obeys the same relations as $\mathbf{x}$ and another identical copy $\mathbf{x}'$ provided again that the two copies are treated with suitable braid statistics between them. This applies for example to the quantum plane
The $2 \times 2$ braided matrices above have this additive structure too as found by the second author (U.M.). Thus,

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

obeys the same relations provided now that the two copies have the additive braid statistics

$$a'a = aa', \quad a'b = q^{-2}ba', \quad a'c = ca' + (1 - q^{-2})ac', \quad a'd = q^{-2}da' + (1 - q^{-2})bc' + (1 - q^{-2})^2aa', \quad b'a = ab' + (1 - q^{-2})ba', \quad b'b = bb', \quad \text{etc.}$$

Again there are $16$ of these and (2) holds for our central element $\det$, although not for $t$.

There is also a general R-matrix construction for $n \times n$ braided matrices $B(R)$ and their properties\[5\]\[7\], with the above as an example when one takes the $sl_2$ R-matrix

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

One can give the formulae directly in terms of such an $R$, or equivalently in terms of the following three multi-index R-matrices

\begin{align}
R^I_{\cdot J}K_L &= R^{-1d}_{\cdot a}R^{i_0}_{\cdot b}R^{k_1}_{\cdot c}R^{j_0}_{\cdot d}R^{e}_{\cdot i_1}R^{c}_{\cdot j_1}R^{l_0}_{\cdot l_1}d \\
R^\cdot I_{\cdot J}K_L &= R^{j_0}_{\cdot a}R^{1a}_{\cdot b}R^{-1i_0}_{\cdot d}R^{k_1}_{\cdot c}R^{b}_{\cdot i_1}R^{c}_{\cdot j_1}R^{e}_{\cdot j_1}R^{l_0}_{\cdot l_1}d \\
R^+ I_{\cdot J}K_L &= R^{j_0}_{\cdot a}R^{k_1}_{\cdot b}R^{1a}_{\cdot c}R^{i_0}_{\cdot d}R^{b}_{\cdot i_1}R^{c}_{\cdot j_1}R^{e}_{\cdot j_1}R^{l_0}_{\cdot l_1}d
\end{align}

where we used the multi-index notation $I = (i_0, i_1)$ etc, and where $\tilde{R}$ is given by transposition in the second two indices, inversion and transposition again. The first two multi-index R-matrices were introduced in [3] where they appear as $\Psi'$ and $\Psi$, while the third was introduced in [7].

Then the braided matrices $B(R)$ as an algebra have the relations [3]

$$R_{21}u_1Ru_2 = u_2R_{21}u_1R, \quad \text{i.e.} \quad u_Ju_L = u_Ku_JR^I_{\cdot J}K_L$$

where the second form is with $u_I = u^{i_0}_{i_1}$ etc. This second vector form is a general feature of these equations and it is for this reason that we said in [3] that $B(R)$ is braided-commutative with respect to $R$. We also studied the more familiar matrix form in [4][10] and explained its connection with the FRT description of standard quantum groups with $u = l^+Sl^-$. From this point of view, (8) are the ‘quantum-Lie algebra’ relations in [11][12]. We return to this in Section 5.

This describes the algebra. The novel braided-matrix multiplication property in this R-matrix setting is that if $u, u'$ obey (8) then [3]

$$u'' = uu', \quad R^{-1}u'Ru_2 = u_2R^{-1}u'R, \quad \text{i.e.} \quad u'_Ju_L = u_Ku_JR^I_{\cdot J}K_L$$
obeys (8) again. The novel braided-matrix addition property in this setting is that also
\[ u'' = u + u', \quad R^{-1}u'_1Ru_2 = u_2R_{21}u'_1R, \quad \text{i.e.} \quad u'_j u_L = u_K u'_j R_+ L J K \]
obes (8) as well.

This approach works in some generality: the multiplicative structure for any bi-invertible solution \( R \) of the QYBE and the additive one for any Hecke solution. In the example of \( q \)-Minkowski space, we see that its basic ‘ring’ structure of addition and multiplication is described by three \( 16 \times 16 \) \( R \)-matrices: \( R \) for the relations of \( B(R) \), \( R_+ \) for the multiplicative braid statistics and \( R_+ \) for the additive braid statistics.

Finally, as well as introducing these ideas, examples, and general \( R \)-matrix constructions for them, we have a sound mathematical foundation for them under the concept of a braided-Hopf algebra or ‘braided group’. The formalisation of the braided-multiplication property is a braided-coproduct
\[
\Delta_* u = u \otimes u, \quad \Psi_*(R^{-1}u_1 \otimes Ru_2) = u_2 R^{-1} \otimes u'_1 R
\]
and the formalisation of the braided-addition is a ‘braided-coaddition’
\[
\Delta_+ u = u \otimes 1 + 1 \otimes u, \quad \Psi_+(R^{-1}u_1 \otimes Ru_2) = u_2 R_{21} \otimes u'_1 R.
\]
The \( \otimes \) in the two cases is no ordinary tensor product algebra but a non-commuting one with relations between the two copies (as above) expressed mathematically by the braid-transposition operator \( \Psi \). So the braided matrices form a braided group in two ways, one with \( \Delta_* \) and braiding \( \Psi_* \) (and no antipode) and the other with \( \Delta_+ \) and the braiding \( \Psi_+ \) and braided-antipode \( Su = -u \). This mathematical formulation of braided groups is rather powerful as it allows all constructions to be done in a systematic way using braid and tangle diagrams. See [4] for an introduction to this novel technique.

This comultiplication and coaddition provided by (9)–(10) are the new feature of the braided-matrix approach to \( q \)-Minkowski space. Thus the element \( \det \) was introduced in [3] as the braided determinant and is characterised by being multiplicative. Likewise, \( t \) is the braided trace and is characterised by being additive:
\[
\Delta_* \det = \det \otimes \det, \quad \Delta_+ t = t \otimes 1 + 1 \otimes t.
\]
In the interpretation of this algebra of \( 2 \times 2 \) braided matrices as \( q \)-Minkowski space \( \mathbb{R}^{1,3}_q \), we clearly should take \( \det \) as the metric or norm. In a basis
\[
t = q^{-1}a + qd, \quad x = \frac{b + c}{2}, \quad y = \frac{b - c}{2i}, \quad z = d - a
\]
this is
\[
\det = \frac{q^2}{(q^2 + 1)^2} t^2 - q^2 x^2 - q^2 y^2 - \frac{(q^4 + 1)q^2}{2(q^2 + 1)^2} z^2 + \left(\frac{q^2 - 1}{q^2 + 1}\right)^2 \frac{q^2}{2} t z.
\]

Also from the theory of braided-geometry comes a natural $\ast$-structure characterised by
\[
(\ast \otimes \ast) \circ \Delta_\ast = \tau \circ \Delta_\ast \circ \ast, \quad (\ast \otimes \ast) \circ \Delta_\ast = \Delta_\ast \circ \ast
\]
where $\tau$ is permutation. It is given by $u$ Hermitean or equivalently, $t, x, y, z$ are real in the sense $t^* = t$ etc. Again, this is a general feature that works for any $R$-matrix of real-type in the sense $\overline{R} = R_{21} \otimes t$, which holds in the present case for real $q$. The analysis is in [8].

Next, the $R$-matrix $R$ determines equally well a quantum vector algebra with generators $v^I$ and a braided-addition law [8]. It has relations and additive braid statistics
\[
v^I v^K = R_{IJKL} v^L v^J, \quad v'^I v^K = R_{IJKL}^+ v^L v'^J.
\]
This leads to a quantum metric $g^{IJ}$ with inverse $g_{IJ}$ and providing an isomorphism [8]
\[
v^I = g^{IJ} u_J, \quad u_I = g_{IJ} v^J
\]
between the vectors and covectors. It also obeys
\[
R_{IJKL}^+ g_{JL} = g_{IK}, \quad g_{IJ} R_{IJKL} = g_{LJ}, \quad \det = u_I u_J g^{IJ}.
\]

The above structure then leads to a natural $q$-Lorentz group. We take the quantum matrix FRT bialgebra $A(\mathbb{R}_+)$ and, denoting the quantum matrix generator by $\Lambda$, we define the quantum group $O_q(1, 3)$ as this bialgebra modulo the relation
\[
\Lambda^I J \Lambda^K L g^{IJL} = g^{IK}.
\]
This $q$-Lorentz group coacts covariantly on the algebra (8) and its vector version
\[
u \mapsto u \Lambda, \quad \Lambda^{-1} v, \ \text{i.e.} \quad u_J \mapsto u_J \otimes \Lambda^I J, \quad v^I \mapsto v^I \otimes S \Lambda^I J.
\]
This $q$-Lorentz group respects the algebra $B(R)$ but not yet is additive structure $\Delta_\ast$. According to the general scheme in [8], in order for the braided addition of 4-vectors to be covariant one must extend the $q$-Lorentz group by a dilatation element $\varsigma$ say, with
\[
[\varsigma, \Lambda^I J] = 0, \quad \Delta \varsigma = \varsigma \otimes \varsigma
\]
which is to be included in the coaction. This is the origin of the dilatation element in our approach [8] (where it was denoted by $g$). It is needed whenever $\lambda \neq 1$ where $\lambda R_+$ is the quantum group normalisation as explained in [8].
Because the additive structure is fully covariant under this extended $O_q(1, 3)$, we can make at once a semidirect product Hopf algebra $O_q(1, 3) \rtimes \mathbb{R}_q$ which is the $q$-Poincaré group in our approach [8, Theorem 6]. In the present case it comes out as generated by covectors $p$ say and $q$-Lorentz transformations, with quantum group structure

$$p_I \varsigma = \lambda^{-1} p_I, \quad p_I \Lambda^J_K = \lambda \Lambda^J_N p_M R^{M}_I N^K$$

$$\Delta p_J = p_J \otimes \Lambda^I_J \varsigma + 1 \otimes p_I, \quad \lambda = q^{-1}.$$  

(17)

We explained in [8] how this general form arises as the bosonisation of any linear braided momentum group. Unlike previous attempts [13], the bosonisation construction is very general and it includes the case we need now. Finally, this $q$-Poincaré group coacts on $q$-Minkowski space by

$$u \mapsto u \Lambda \varsigma + p, \quad \text{i.e.} \quad u_J \mapsto u_J \otimes \Lambda^I_J \varsigma + 1 \otimes p_J.$$  

(18)

Another general application of the braided theory is that infinitesimal translation using again the addition law (10) defines at once the operators of differentiation according to the general theory of [14]. Thus

$$(\partial^I f)(u') = \left(u_I^{-1}(f(u + u') - f(u'))\right)_{u=0}$$

(19)

by which we mean to order the terms in $f(u + u')$ using the additive braid statistics (10) to put all the $u$ to the left, and then pick out the coefficient of $u_I$. It was shown in [14] that these $\partial^I$ always realise the vector algebra $v^I$ above. There are also right-differentials defined in a similar way by translation from the right.

3 Detailed comparison with the approach of Carow-Watamura et al

The main idea behind the approach of [1]–[3] is that of spinors. They considered two non-commuting copies of the 2-dimensional quantum plane, say $x_i$ and $y_j$ and considered the properties of the null-four vectors $X_I = x_{i_0} y_{i_1}$ say, where $I$ is a multi-index (they used upper indices but to match with the above conventions we write lower ones). These authors obtained in this way an approach based on two $16 \times 16$ R-matrices $R$ and $R + R_{11}$. The second was the Lorentz quantum group R-matrix introduced in [2] as a product of 4 R-matrices

$$R^{I}_{J} \quad \text{and} \quad \text{the first was found by decomposing it into projectors and taking a different combination of them.}$$

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We give here two ways to embed this pioneering work into the braided-matrix picture above. The first is to give the required change of basis explicitly, while the second is more abstract but works in general. The explicit change of basis is provided by the spinor metric $\epsilon_{ij}$ which was one of the key ingredients in [2] and which is given by

$$
\epsilon_{ij} = \begin{pmatrix} 0 & \frac{1}{\sqrt{q}} \\ \frac{1}{q} & 0 \end{pmatrix}
$$

in the relevant conventions. Its inverse is $\epsilon^{ij}$. We then define

$$
X_I = \epsilon_{ai} u^a_{i1}, \quad T^I_J = \epsilon_{a_0 b_0} \Lambda_{ai}^{b_1} e^{b_0}
$$

i.e.

$$
X_J = u_I E^{I}_J, \quad T^{I}_J = E^{-1} \Lambda E, \quad E^{I}_J = \epsilon_{i_0 j_0} \delta_{i_1}^{j_1}
$$

and find by an explicit calculation that

$$
R_I = (E^{-1} \otimes E^{-1}) R (E \otimes E), \quad R_I R_J = (E^{-1} \otimes E^{-1}) R_{I} (E \otimes E).
$$

In fact, we recover here $R_I$ in a 4 R-matrix form

$$
R_{I}^{J} K_{L} = R_{a_0 b_0}^{-1} R_{c_0 d_0}^{-1} R_{i_0 j_0}^{a d} R_{j_0 i_0}^{c d},
$$

which corresponds to the $16 \times 16$ matrix defined in [3] in terms of projectors.

We now give the relationship in a more conceptual way which, if desired, extends to the same level of generality as the braided approach above. We make the comparison by embedding the various $q$-Lorentz groups into the $*$-quantum group $SU_q(2) \bowtie SU_q(2)$ and comparing their images there. In general the spinorial $q$-Lorentz group is $A \bowtie A$ as generated by two quantum matrices $s, t$ say with their usual FRT relations of $A(R)$ and cross-relations and $*$-structure

$$
R t_1 s_2 = s_2 t_1 R, \quad t^i_1 s^j_2 = S s^j_2 t^i_1, \quad s^i_1 t^j_2 = S t^j_2 s^i_1
$$

The factorisation construction $\bowtie$ needed here is from [15] where this quantum group was studied, while the $*$-structure is as used in [2]. One can also work at the bialgebra level by working with

$$
t^\dagger \equiv S s
$$

as the abstract generator in place of $s$. So $A \bowtie A$ is generated by $t^\dagger, t$ with appropriate relations and $*$-structure. We gave the $*$-structure more abstractly in [8, Sec. 4] and also that there is a dual-quasitriangular structure or ‘universal R-matrix functional’

$$
R_M((a \otimes b) \otimes (c \otimes d)) = R^{-1}(d_{(1)} \otimes a_{(1)}) R^{-1}(c_{(1)} \otimes a_{(2)}) R(b_{(1)} \otimes d_{(2)}) R(b_{(2)} \otimes c_{(2)})
$$

for all $a, b, c, d \in A$ (this is actually the canonical inverse-transpose of the one in [8]). We used the usual $a_{(1)} \otimes a_{(2)}$ notation for the coproduct. We also showed that this quantum group was
the dual of the `twisted square’ in [14] and hence in some nice cases isomorphic to the dual of
the quantum double [16] as in the approach of [18]. To this we add the observation that this
bialgebra has a second dual-quasitriangular structure [7]

\[ R_L((a \otimes b) \otimes (c \otimes d)) = R^{-1}(d_{(1)} \otimes a_{(1)}) R(a_{(2)} \otimes c_{(1)}) R(b_{(2)} \otimes c_{(2)}) R(b_{(1)} \otimes d_{(2)}). \]  

(23)

In these formulas, \( R \) is the dual quasitriangular structure of \( A \) and is real in the sense of [8, Prop. 13] as is the case for \( SU_q(2) \) at real \( q \). Then (23) is also real while (22) is anti-real.

The vectorial \( q \)-Lorentz group of Section 2 then maps into this spinorial one by

\[ \Lambda^I_{\ J} = t^i_{\ ja} t^{i_1}_{\ j_1}, \]  

(24)

giving the transformation law

\[ u^i_{\ j} \mapsto u^a_{\ b} t^{i_1}_{\ a} t^b_{\ j}, \quad \text{i.e.,} \quad u \mapsto t^i u t^j \]  

(25)

while the original embedding of [2] is

\[ T^I_{\ J} = s^i_{\ j_0} t^{i_1}_{\ j_1}, \quad X_{ij} \mapsto X_{ab} s^a_{\ i} t^b_{\ j}. \]

Moreover, one has consistency in that the above dual-quasitriangular structures recover the specific R-matrices above. Thus

\[ R = R_M(\Lambda_1 \otimes \Lambda_2), \quad R_+ = R_L(\Lambda_1 \otimes \Lambda_2) \]

\[ tR = R_M(T_1 \otimes T_2), \quad II R = R_L(T_1 \otimes T_2) \]

This is a general construction, but comparing the specific realisations of \( T \) and \( \Lambda \) in this quantum group \( A \rtimes A \) gives the matrix \( E \) or the spinor metric \( \epsilon \) for the standard case.

In summary, one could say that the braided-matrix picture of \( q \)-Minkowski space and the original spinor picture differ by a change of basis. The latter spinorial approach adds the spinor metric (we do not need it in the braided matrix approach) while the braided matrix approach adds the concepts of braided comultiplication (which in turn determines the metric) and braided coaddition. They come out as \( \Delta_+, X = X \otimes 1 + 1 \otimes X \) (or \( X + X' \) where the two copies have the additive braid statistics \( II R \)), and

\[ \Delta_+ X_{ij} = X_{ia} \otimes \epsilon^{ab} X_{bj} \]  

(26)

with respect to multiplicative braid statistics

\[ III R = (E^{-1} \otimes E^{-1}) R_*(E \otimes E), \quad III R^I_{\ J} R^K_{\ L} = R^{-1k_1}_{\ ia} R^{i_1}_{\ ac} R^{a}_{\ il} R^{-1k_0}_{\ db} R^{c}_{\ j_0} R^{d}_{\ l_0}. \]

Moreover, the braided approach avoids the use of projectors and thereby stays in a general R-matrix form.

8
4 Spinorial q-Poincaré group

We now continue with further results in the braided-matrix approach. By using the change of basis provided in Section 3, our results can be viewed equally well as new results in the approach of [1]–[3]. In fact, the basis \{u^{ij}\} has a number of advantages over the \{X_{ij}\} basis whenever braided-matrix multiplication (9) is needed implicitly or explicitly. A general rule is that for vector multi-index descriptions either basis is fine but the \textbf{u}-basis has an advantage if we really want to work with the original spinorial 4 \times 4 R-matrix.

Here we apply the techniques of braided geometry [4] to obtain the q-Poincaré group (17) in the spinorial representation (24) for the q-Lorentz group part. This gives a picture of the q-Poincaré group as generated by two copies \textbf{s}, \textbf{t} of SU\textsubscript{q}(2) and the momentum braided group of translations as a copy of q-Minkowski space with its additive structure (10). At issue is the commutation relations between \(p_I\) and the \textbf{s}, \textbf{t}. In the braided-matrix approach this follows as a new application of the general bosonisation theorem in [8]. The idea comes from the Jordan-Wigner transform whereby a super-matrix is realised as an ordinary matrix by ‘bosonising’.

Firstly, we extend the spinorial q-Lorentz group by a dilatation element \(\varsigma\) as in (16), commuting with \textbf{s}, \textbf{t}. We also extend the dual-quasitriangular structure \(R_L\) in (23) by \(R_L(\varsigma^a \otimes \varsigma^b) = \lambda^{-ab}\). This describes the extended quantum group \(\tilde{SU}_q(2)\tilde{\triangleright} SU_q(2)\) which is a double-cover of the vector Lorentz group \(O_{q}(1,3)\) in Section 2. It coacts as in (25) with an extra \(\varsigma\) factor.

Then one can show that q-Minkowski space is a braided-Hopf algebra fully covariant under this extended spinorial q-Lorentz group. Its braiding is that induced by \(R_L\). Hence by the bosonisation theorem as applied in [8, Theorem 6] we obtain an ordinary Hopf algebra q-Poincaré group as a semidirect product \((\tilde{SU}_q(2)\tilde{\triangleright} SU_q(2))\ltimes \mathbb{R}^1{,}3\). Following the same steps to compute this as in [8] we first define the right action of the extended q-Lorentz group on \(\mathbb{R}^1{,}3\) by evaluating \(R_L\) against its coaction. This gives

\[
\begin{align*}
\rho^j_{\alpha} t^k_{\beta} &= \rho^a_{\alpha} R_L ((S_{\alpha} s_{\beta}^i ) t^j_{\gamma} \otimes t^k_{\delta}) = \rho^a_{\alpha} R^{-1} (t^c_{\gamma}\otimes S_{\alpha} t^i_{\beta}) R(t^j_{\gamma} \otimes t^c_{\delta}) = \lambda \rho^a_{\alpha} R^{-1}_{c,i} R_{j,l} t^c_{\gamma} t^i_{\beta}, \\
\rho^j_{\alpha} s^k_{\beta} &= \rho^a_{\alpha} R_L ((S_{\alpha} s_{\beta}^i ) t^j_{\gamma} \otimes s^k_{\delta}) = \rho^a_{\alpha} R(St_{\alpha}^i \otimes t^k_{\beta}) R(t^j_{\gamma} \otimes t^c_{l}) = \rho^a_{\alpha} R^{-1}_{c,i} R_{j,l} t^c_{\gamma} t^i_{\beta}
\end{align*}
\]

i.e. \(p_1 \lhd t_2 = \lambda R_{21} p_1 R, \quad p_1 \lhd s_2 = R^{-1} p_1 R, \quad p_\varsigma = \lambda^{-1} \varsigma p\)

This right action gives the semidirect product algebra structure of the q-Poincaré group, while the coaction itself gives the semidirect coproduct. The result is that our spinorial q-Poincaré group has the cross-relations and coproduct

\[
\begin{align*}
\Delta p &= p \otimes t^\dagger t_\varsigma + 1 \otimes p, \quad \text{i.e.} \quad \Delta p^i_{j} = p^a_{\beta} \otimes (S_{\beta} s^i_{\alpha}) t^j_{\gamma} + 1 \otimes p^i_{j}
\end{align*}
\]

(27)

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where $t^\dagger(\ )t$ has a space for the matrix indices of $p$ to be inserted. There is also a counit $\epsilon p = 0$ and antipode. As far as we know, this R-matrix construction is new. It is the double-cover of the vectorial $q$-Poincaré group in Section 2. It coacts on $q$-Minkowski space, which becomes a comodule algebra as in (18) but in the spinor version with (24).

We have worked in an R-matrix formulation and quantum groups of function algebra type, but it is not hard to give the enveloping algebra version too by the same techniques. Thus the vector algebra (11) is represented by the $\partial^I$ and is dually-paired as a braided-Hopf algebra with the $\{p_I\}$ [8], the pairing being in the categorical sense explained in [4, Prop. 4.11]. The spinorial Lorentz group has enveloping algebra $U_q(su_2) \otimes_R U_q(su_2)$ along the lines of [10] with $*$-structure and $R_L$ dual to those above. Its extension $U_q(su_2) \otimes_R U_q(su_2) \otimes U(1)$ acts on the vector algebra and cross product by this (in other words the bosonisation of the vector algebra) constructs the enveloping algebra of the $q$-Poincaré group. By general theorems cf [3, Lemma 4.4] it will be a Hopf algebra and dual to the $q$-Poincaré group above. Details will be presented elsewhere.

Another variant is to work with right-handed derivatives $\overset{\leftarrow}{\partial^I}$ rather than usual left ones. Related to this, it is obvious that all formulae could be presented with upper and lower indices swapped (there are some reasons for our conventions above which do not show up in the classical case where algebras of functions are commutative.)

Finally, if one wants to stay entirely in the braided setting, there is a braided $q$-Lorentz group $BO_q(1,3)$ based on braided matrices $B(R_\frac{1}{2})$ rather than quantum matrices $A(R_\frac{1}{2})$ as above. The braided-semidirect product [4, Eq. 72] of $q$-Minkowski space by this gives the braided-Poincaré group. Its advantage over the one above is that the translation part appears as a sub-braided group. This too will be computed elsewhere.

5 Concluding remarks

We conclude with some comments about more novel aspects of our braided-matrix approach. Firstly, it is very general and applies just as well to other real-type Hecke R-matrices. For example, a non-standard variant of (4) gives the braided version $BM_q(1|1)$ of super-$2 \times 2$ matrices introduced in [5]. It provides a super-version of $q$-Minkowski space with some interesting features such as vanishing quantum-dimension.

Secondly, there is by now a systematic theory of braided-Lie algebras introduced in [10, Prop. 2.4] [19] and associated to any bi-invertible R-matrix. The one corresponding to (4) is computed in [19, Ex. 5.5] as braided $gl_{2,q}$. It turns out that the canonical braided-enveloping
algebra for this class of examples is isomorphic when \( q \neq 1 \) to the braided-matrix bialgebras \( B(R) \) with their multiplicative coproduct. A part of this (if one sets \( \det = 1 \) etc) is the known picture of \((8)\) as a description of quantum enveloping algebras\([11]\)\([12]\) but in the braided theory we do not have to make such restrictions. Thus for example, we have

\[
U(gl_{2,q}) \simeq \mathbb{R}^{1,3}_q
\]

as *-algebras. The non-commuting space-time co-ordinates \( t, x, y, z \) can be viewed equally well on the left hand side as generators of \( gl_{2,q} \). The time direction \( t \) corresponds to the \( u(1) \) direction. This remarkable possibility of the unification of a \( SU(2) \times U(1) \) symmetry in the form of \( gl_{2,q} \) with the co-ordinates of space-time itself is perhaps the main result of \([19]\) for physics. Its meaning was discussed further in \([20]\).

Thirdly, just as the braided matrices \( B(R) \) have these two interpretations, so the \( q \)-Lorentz group above has at least three interpretations. One is as above, another is as the quantum algebra of observables of a \( q \)-deformed particle moving on a hyperboloid in \( q \)-Minkowski space, which is the main result of \([1]\). A third is as a ‘frame bundle’ in quantum group gauge theory\([21, Ex. 5.6]\). These are all valid pictures of Drinfeld’s quantum double within quantum and braided geometry. In summary, one can say that \( q \)-deformation unifies concepts because structures which are quite different at \( q = 1 \) can become isomorphic when \( q \neq 1 \). This provides therefore a novel motivation for \( q \)-deforming physics.

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