Revisiting $1^{-+}$ Light Hybrid from Monte-Carlo Based QCD Sum Rules

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We reanalyze the $1^{-+}$ light hybrid from QCD sum rules with a Monte-Carlo based on uncertainty analysis. With 30% uncertainties in the accepted central values for QCD condensates and other input parameters, we obtain a prediction on the $1^{-+}$ hybrid mass of 1.71 ± 0.22 GeV, which covers the mass of $\pi_1(1600)$. We also study the correlations between the input and output parameters of QCD sum rules.

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There are several mesons with exotic quantum numbers (e.g., $\pi_1(1400)$ and $\pi_1(1600)$ with $J^{PC} = 1^{-+}$) listed in the Review of Particle Physics.\textsuperscript{[1]} It is important to pay special attention to these mesons due to the fact that they cannot be accommodated in the conventional quark model. In recent years, studies on their possible structures have become an important focus in hadronic physics. For $1^{-+}$ mesons, two possible structures have been studied in the literature: four-quark states\textsuperscript{[2–5]} and hybrid states\textsuperscript{[6–10]} (see Ref.\textsuperscript{[11]} for a recent review on studies of $1^{-+}$ mesons).

The possible hybrid structures of $1^{-+}$ mesons have been studied by using many different methods, including a flux-tube model, lattice QCD and QCD sum rules. The QCD sum rule method (QCDSR) is an important nonperturbative method in hadronic physics. Since introduced by Shifman \textit{et al.,}\textsuperscript{[12,13]} QCD sum rules have given numerous predictions on hadron properties.\textsuperscript{[14]} However, it has also been argued that there are some shortcomings in traditional QCD sum rule analysis methodologies (see Ref.\textsuperscript{[15]} for a detailed discussion). Mainly, the continuum threshold $s_0$ could not be completely constrained in the framework of traditional QCD sum rules, which causes a significant uncertainty. To overcome these shortcomings, Leinweber introduced a new Monte-Carlo based uncertainty analysis into QCD sum rules and restudied the $\rho$ meson and nucleon.\textsuperscript{[15]} Later, this procedure was used for predicting the decuplet baryon spectrum.\textsuperscript{[16]} Nucleon axial vector coupling constants\textsuperscript{[17]} and hadron magnetic moments.\textsuperscript{[18,19]} All these studies were fruitful. Recently, a mathematica package MathQCDSR was provided\textsuperscript{[20]} to facilitate the uncertainty analysis.

In this Letter, we restudy the $1^{-+}$ light hybrids by using this new procedure.

To predict the properties of $1^{-+}$ light hybrid in QCD sum rules, the relevant two-point correlator is necessary. In the present case, it can be written as

$$H_{\mu\nu}(q^2) = i \int d^4x e^{i q x} \langle 0 | T^{\text{ren}}_{\mu\nu}(x) | \bar{\psi} q (0) \rangle = (q_\mu q_\nu - g_{\mu\nu} q^2) H_0(q^2) + q_\mu q_\nu H_s(q^2),$$

where the current $J^{\text{ren}}_{\mu\nu} = (1 + \frac{2}{\tau} - 1) g_{\nu\mu} F^\alpha_{\mu\nu} T^\alpha q$ is the renormalized hybrid current. We focus on the operator product expand (OPE) of the invariant correlator $H_0$, which corresponds to the contribution from $1^{-+}$ state. After Borel transformation, it can be written as\textsuperscript{[8–10,21–26]}

$$H_0^{\text{OPE}}(\tau) = a_{11}^2 - \frac{2}{\tau^3} + a_{12}^2 \frac{2}{\tau^3} (2 \gamma E - 3 + 2 \ln(\tau \mu^2)) + b_{11}^2 - \frac{1}{\tau} + b_{12}^2 \frac{2}{\tau} (\gamma E - \ln(\tau \mu^2)) + c_{11}^2 + c_{12}^2 (-\gamma E - \ln(\tau \mu^2)) + d_{11}^2 \tau,$$

where $\gamma E$ is Euler’s constant, $\mu$ is the renormalization point for the current, and $\tau$ is the Borel-transform parameter. The coefficients $a - d$ in Eq. (2) for the isospin $I = 1$ state are as follows:\textsuperscript{[23,24,26]}

$$a_{11} = -\frac{\alpha_s(\mu)}{240 \pi^3} \left( 1 + \frac{1301}{240} \frac{\alpha_s(\mu)}{\pi} \right),$$
$$a_{12} = \frac{\alpha_s(\mu)}{240 \pi^3} \left( 1 + \frac{1301}{240} \frac{\alpha_s(\mu)}{\pi} \right),$$
$$b_{11} = -\frac{1}{30 \pi} \frac{\alpha_s G^2}{8} \left( 1 - \frac{145}{72} \frac{\alpha_s(\mu)}{\pi} \right),$$
$$b_{12} = -\frac{1}{30 \pi} \frac{\alpha_s G^2}{8} \frac{\alpha_s(\mu)}{9} \frac{\alpha_s(\mu)}{\pi},$$
$$c_{11} = -\frac{4 \pi}{9} \frac{\alpha_s(\mu)}{\pi},$$
$$c_{12} = -\frac{4 \pi}{9} \frac{\alpha_s(\mu)}{\pi}. $$

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where $\alpha_s(\mu) = 4\pi/(9 \ln(\mu^2/A_{\text{QCD}}^2))$ is the running coupling constant for three flavors.

To obtain predictions for the ground state in QCDSR, the simple single narrow resonance spectral density ansatz $\text{Im} \Pi_{\text{phen}}(s) = \pi f^2 \delta(s-m^2) + \text{Im} \Pi_{\text{ESC}}(s)\theta(s-s_0)$ is typically used, where $s_0$ is the continuum threshold that separates the contribution from excited states (ESC), and $f$ and $m$ denote the coupling of the resonance to the current and the mass of the resonance, respectively. Based on this assumption, we can obtain the phenomenological representation of the correlator $\Pi_{\text{phen}}(\tau, s_0, f, m)$ via the dispersion relation.\cite{12,13,14} The spectral density for excited states $\text{Im} \Pi_{\text{ESC}}$ can be chosen as $\text{Im} \Pi_{\text{OPE}}(s)$, and following usual conventions, these contributions are placed on the OPE side of the sum rules. For the present case, we obtain

$$\Pi_{\text{OPE-ESC}}(s_0, \tau) = a_{11} \frac{-2}{\tau^3} (1 - \rho_2(s_0\tau)) + 2a_{12} F_1(s_0, \tau) + a_{12} \frac{2}{\tau} (2\gamma_E - 3 + 2 \ln(\tau \mu^2)) + b_{11} \frac{-1}{\tau} (1 - \rho_0(s_0\tau)) + b_{12} \left[2 \left(\gamma_E + \ln(\tau \mu^2)\right) + 2F_2(s_0, \tau)\right] + c_{11} + c_{12} (-\gamma_E - \ln(\tau \mu^2) - E_1(s_0\tau)) + d_{11} \tau,$$ \hspace{1cm} (3)

where $\rho_0(x) = e^{-x} \sum_{k=0}^{n} \frac{x^k}{k!}$, $E_1$ is the exponential integral function, and $F_1(s_0, \tau) = (2\rho_0(s_0\tau) + \rho_1(s_0\tau) + 2E_1(s_0\tau) + 2\rho_2(s_0\tau) \ln(s_0/m^2))/\tau^3$, $F_2(s_0, \tau) = (E_1(s_0\tau) + \rho_0(s_0\tau) \ln(s_0/m^2))/\tau$. Then the master equation for QCDSR can be written as

$$\Pi_{\text{OPE-ESC}}(s_0, \tau) = f^2 e^{-m^2 \tau}.$$ \hspace{1cm} (4)

Physical properties of $1^+ \rightarrow$ light hybrid, i.e., $m, f^2$ and $s_0$, should satisfy Eq. (4).

Finally, before proceeding with numerical calculations, a renormalization-group (RG) improvement of the sum rules, i.e., substitutions $\mu^2 \rightarrow 1/\tau$ in Eq. (4), is needed.\cite{27} In addition, the anomalous dimensions for condensate $\langle O \rangle$ in $\Pi_{\text{OPE-ESC}}$ should also be implemented by multiplying $(\alpha_s/\pi)$ or approximately RG-invariant (e.g., $\langle \alpha_s G^2 \rangle$) or approximately RG-invariant (e.g., $\langle m_q \bar{q}q \rangle$, $\langle \bar{q}q \rangle$), where anomalous dimensions are $-23/27$ and $10/27$, respectively.\cite{12,14,24} The coupling constant $f$ should also be multiplied by factor $L(m(\tau))$, where $\gamma_j = -32/81$ is used to incorporate the anomalous dimension of the current;\cite{24,26} $f$ then receives its value at the hybrid mass shell.

Obviously, due to the truncation of the OPE and the simplified assumption for the phenomenological spectral density, Eq. (4) is valid for some $\tau$ but not for all the $\tau$, thus requiring a sum rule window in which the validity of Eq. (4) can be established. Specifically, to ensure convergence of the OPE,\cite{15} the contributions of the highest dimensional operators (HDO) in the OPE should not be too large (less than 10% of the total OPE contributions), which will give an upper bound $\tau_{\text{max}}$ for the sum rule window. Meanwhile, the continuum contributions should not be larger than the pole contributions, otherwise we cannot rely on the single narrow resonance ansatz. It is required that the ratio of ESC/total contributions < 50% gives the lower bound $\tau_{\text{min}}$. However, an ESC is dependent on $s_0$, which is to be determined in a QCD sum rule analysis, thus we cannot determine $\tau_{\text{min}}$ before carrying out the QCD sum rule analysis. In practice, we will initially ‘guess’ a $\tau_{\text{min}}$, and after finishing the analysis we can check our initial choice and adjust it iteratively until it is consistent with the results of the analysis.

The least-square method is appropriate for matching the two sides of Eq. (4) in the sum rule window. However, the condensates appearing in the OPE are not known accurately, thus these parameter uncertainties will lead to uncertainties in the OPE. Due to the fact that the uncertainties in the OPE are not equal at different points in the sum rule window, a weighted-least-square method is more appropriate.

Following Leinweber’s procedure, we first need to estimate the standard deviation $\sigma_{\text{OPE}}(\tau)$ of $\Pi_{\text{OPE}}(\tau)$ at any point $\tau$ in the sum rule window. This can be carried out by randomly generating 200 sets of Gaussian distributed input parameters (condensates and $A_{\text{QCD}}$) with given uncertainties (200 samples are enough to establish the stable standard deviation). After obtaining $\sigma_{\text{OPE}}(\tau)$, the phenomenological output parameters $s_0, f^2$ and $m$ can be obtained by minimizing a weighted $\chi^2$, which is defined as

$$\chi^2 = \sum_{j=1}^{mn} \left(\frac{\Pi_{\text{OPE}}(\tau_j) - \Pi_{\text{phen}}(s_0, s_0, f, m)}{\sigma_{\text{OPE}}(\tau_j)}\right)^2.$$ \hspace{1cm} (5)

The points $\tau_j$ are selected to be $\tau_j = \tau_{\text{min}} + (\tau_{\text{max}} - \tau_{\text{min}}) \times (j-1)/(mn-1)$, i.e., we divide the sum rule window into $(mn-1)$ even parts. In the present case, we find that the fitting results do not change provided $nB \geq 8$, thus we set $nB = 21$ for simplicity.

We then generate a set of 2000 Gaussian distributed input parameters with given uncertainties, and for each set we minimize $\chi^2$ to obtain a set of fitted phenomenological output parameters. Finally, we select the physical results (it is natural that constraints between output parameters exist, such as $s_0 > m^2$, $m_B > 0$, $m_B^2 > m^2$, $m_B > m^2$).
and results that violate these constraints should be excluded) from the set of fitted values.

To randomly generate Gaussian distribution input parameters, the central values and uncertainties of these parameters should be set first. In this study, we treat all condensates as independent parameters to distinguish the importance of different condensates. After reviewing the literature, we choose the central value of input parameters at \( \mu_0 = 1 \text{ GeV} \) as follows:

\[
\begin{align*}
A_{\text{QCD}} &= 0.20 \text{ GeV}, \\
\langle \alpha_s G^2 \rangle &= 0.095 \text{ GeV}^4, \\
m_q \langle \bar{q}q \rangle &= 0.007 \times (-0.236)^3 \text{ GeV}^4, \\
\langle g^3 G^3 \rangle &= 1.1 \times 0.095 \text{ GeV}^6, \\
\alpha_s \langle \bar{q}q \rangle^2 &= 1.8 \times 10^{-4} \text{ GeV}^4, \\
\langle \bar{q}q \rangle \langle \bar{g}gGq \rangle &= (-0.236)^6 \times 0.72 \text{ GeV}^8.
\end{align*}
\]

Then all sets of input parameters are generated with 10\% uncertainties, which is a typical uncertainty in QCDSR. For physical considerations, we add an additional constraint 0.10 GeV \( \leq A_{\text{QCD}} \leq 0.30 \text{ GeV} \) on \( A_{\text{QCD}} \). Any set of randomly generated input parameters that violates this constraint will be excluded from our set of input parameters.

After several numerical samples, we find that the appropriate sum rule window for the \( 1^{-+} \) light hybrid is \( \tau = 0.4 - 1.0 \text{ GeV}^{-2} \). By minimizing the \( \chi^2 \) function for each sample of input parameters, we finally obtain a 2000-member set of phenomenological output parameters which satisfy our physical constraints. In Fig. 1, we plot the histogram for these 2000 different \( 1^{-+} \) light hybrid masses obtained in the weighted-least-square fitting procedure. It is obvious that the distribution of \( m \) is very close to Gaussian.

The main results of our fitting procedure are as follows:

\[
\begin{align*}
s_0 &= 5.37^{+0.81}_{-0.62} \text{ GeV}^2, \\
m &= 1.73^{+0.10}_{-0.09} \text{ GeV}, \\
f^2 &= 0.0053^{+0.0013}_{-0.0009} \text{ GeV}^6,
\end{align*}
\]

where we have reported the median and the asymmetric standard deviations from the median\([28]\) for all physical output parameters.

The uncertainty of \( m \) is less than 6\%, implying that the fitted results are very stable with different input parameters. Figure 2 illustrates that the fits are qualitatively acceptable.

The continuum threshold \( s_0 \) now is an output rather than an input parameter. In principle, this result of \( s_0 \) can be extended to study excited states, although some new assumptions are necessary. For instance, if we assume \( s_0 = (m^2 + m_1^2)/2 \), where \( m_1 \) is the first excited state\([32-34]\) then the predicted mass of the first excited \( 1^{-+} \) hybrid is about 3 GeV.
fits, the correlations of the input and output parameters can also be obtained. From the scatter plot in Fig. 3, we find that there exists a very strong positive correlation between \( m \) and \( s_0 \) that may explain why a broad range of \( m \) can occur in traditional sum rule analyses by setting an appropriate \( s_0 \).[29] From Fig. 4 we find that there exists a weak negative correlation between \( m \) and \( \langle \alpha_s G^2 \rangle \); the Wilson coefficient and the value for \( \langle \alpha_s G^2 \rangle \) are thus crucial to our results. Further study should improve the present understanding on this point. Furthermore, we find that there exists weak positive correlations between \( m \) and the four-quark condensate, and for \( \Lambda_{QCD} \). Thus the Wilson coefficient and possible factorization violation effects for the four-quark condensate also play important roles in the result. It is somewhat unexpected that the present result is sensitive to \( \Lambda_{QCD} \) in contrast to the \( \rho \) meson case.[15] The possible explanation may come from the fact that the leading term of OPE is \( \alpha_s \)-dependent and the anomalous dimension is not zero in the current.

### Table 1. Fitting results with larger uncertainties for input parameters.

| Parameters with 30% uncertainties* | \( \langle \alpha_s G^2 \rangle \) | \( \alpha_s \langle \bar{q}q \rangle \) | \( \Lambda_{QCD} \) | \( \langle \alpha_s G^2 \rangle \), \( \alpha_s \langle \bar{q}q \rangle \), \( \Lambda_{QCD} \) all parameters |
|------------------------------------|----------------|----------------|-----------------|--------------------------------------------------|
| Output \( s_0 \) (GeV²)            | 5.24\(^{+1.13}_{-1.03} \) | 5.26\(^{+1.10}_{-1.19} \) | 5.56\(^{+1.07}_{-0.89} \) | 5.26\(^{+1.96}_{-1.49} \) \(^{+0.01}_{-0.03} \) |
| Output \( f^2 \) (GeV⁶)            | 1.71\(^{+0.21}_{-0.16} \) | 1.73\(^{+0.17}_{-0.16} \) | 1.73\(^{+0.13}_{-0.13} \) | 1.71\(^{+0.22}_{-0.22} \) \(^{+0.22}_{-0.22} \) |

*Other input parameters are still generated with 10% uncertainties.

We may also estimate how the result will change if the uncertainties for input parameters are larger than 10%. Table 1 shows that the uncertainties for \( \langle \alpha_s G^2 \rangle \), \( \alpha_s \langle \bar{q}q \rangle \) and \( \Lambda_{QCD} \) play the most important roles in the weighted-least-square fit results, i.e., the uncertainties of output parameters are mainly determined by these three input parameters. Table 1 also demonstrates that the medians of \( f^2 \) and \( m \) change slightly while \( s_0 \) is somewhat sensitive to larger uncertainties for input parameters. It probably means that the continuum absorbs these uncertainties in some way.

Based on the above results, we conclude that a reliable mass prediction has been obtained for the \( 1^{+} \) light hybrid mass. Input parameters with 10% uncertainties cause about only 6% uncertainty in the mass, while 30% uncertainties cause about 13% uncertainty in the mass. We choose the latter as a cautious estimate, thus we predict the mass of the \( 1^{+} \) light hybrid is \( 1.71 \pm 0.22 \text{ GeV} \), this result favors \( \pi_1(1600) \) to be a hybrid. The uncertainty of \( f^2 \) seems much larger, while it could be easily understood by re-parameterizing \( f = m^2 f' \). Then the uncertainty of new decay constant \( f' \) is as small as the uncertainty of \( m \) (input parameters with 10% uncertainties cause about 6% uncertainty in \( f^2 \)).

Finally, to improve the over-simplified single narrow resonance spectral density, we may use a Breit–Wigner form \( \langle \frac{2m\Gamma}{(s - m^2)^2 + m^2\Gamma^2} \rangle \) instead, this modification replaces \( e^{-m^2\tau} \) in Eq. (4) with

\[
\frac{1}{\pi} \text{Im}[e^{-m^2\tau + im\Gamma\tau} \text{Ei}(m^2\tau - im\Gamma\tau)]
\]

where \( \Gamma \) is the width of the resonance and \( \text{Ei} \) is an exponential integral function. However, a four parameter fit \( (m, f^2, s_0 \text{ and } \Gamma) \) search always gives the same result as the three parameter fit \( (m, f^2 \text{ and } s_0) \), i.e., it gives a result with \( \Gamma = 0 \) automatically. If we set a ‘reasonable’ non-zero \( \Gamma \) as the input, then a three parameter fit \( (m, f^2 \text{ and } s_0) \) will give a result with a slightly increased value for the predicted mass. However, in the present case, the weighted \( \chi^2 \) will increase significantly, implying that the goodness of fit with non-zero \( \Gamma \) is worse than that with \( \Gamma = 0 \). We have checked that this situation not only occurs in the hybrid, but also occurs in the rho meson. These specific results are in agreement with the general argument presented in Ref. [30]. Obviously, the procedure is insensitive to the decay width at present. This fact may demonstrate that the present spectral density is still over-simplified, further improvement, e.g., considering the concrete decay modes of a hybrid, may give a solution to this problem.

In conclusion, we have reanalyzed the \( 1^{+} \) light hybrid from QCD sum rules with a Monte-Carlo based uncertainty analysis. Now \( s_0 \) is treated as an output parameter, thus we avoid some subjective factors. A stable result accommodates \( \pi_1(1600) \) as a light hybrid state. The result also shows that the QCD sum rule uncertainties arise not only from OPE truncation and the single resonance assumption, but also from sensitivity to the uncertainties of input parameters. Thus any correction, higher order \( \alpha_s \) to the coefficients of the condensates or higher dimension contributions would be valuable. By contrast, the decay width of the light hybrid is below the sensitivity of the present method, and hence the QCD sum-rules analysis is not dependent on the form of the spectral density.

### References

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