Moufang symmetry I.  
Generalized Lie and Maurer-Cartan equations

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Abstract

The differential equations for a local analytic Moufang loop are established. The commutation relations for the infinitesimal translations of the analytic Moufang are found. These commutation relations can be seen as a (minimal) generalization of the Maurer-Cartan equations.

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1 Moufang loops

A Moufang loop [6] (see also [3, 2, 7]) is a set $G$ with a binary operation (multiplication) $\cdot : G \times G \to G$, denoted also by juxtaposition, so that the following three axioms are satisfied:

1) in the equation $gh = k$, the knowledge of any two of $g, h, k \in G$ specifies the third one uniquely,

2) there is a distinguished element $e \in G$ with the property $eg = ge = g$ for all $g \in G$,

3) the Moufang identities

$$g(h \cdot gk) = (gh \cdot g)k$$

$$(kg \cdot h)g = k(g \cdot hg)$$

$$(gh)(kg) = g(hk \cdot g)$$

hold in $G$.

It must be noted that the Moufang identities (1.1a–c) are equivalent (see e.g [3, 2]), which means that each of the above identities can be considered as a defining identity of the Moufang loop.

Recall that a set with a binary operation is called a groupoid. A groupoid $G$ with axiom 1) is called a quasigroup. If axioms 1) and 2) are satisfied, the groupoid (quasigroup) $G$ is called a loop. The element $e$ in axiom 2) is called the unit (element) of the (Moufang) loop $G$.

In a (Moufang) loop, the multiplication need not be neither associative nor commutative. Associative (Moufang) loops are well known and called groups. The associativity and commutativity laws read, respectively,

$$g(hk) = (gh)k, \quad gh = hg, \quad \forall g, h, k \in G$$

The associative commutative (Moufang) loops are called the Abelian groups. The most familiar kind of loops are those with the associative law, and these are called groups. A (Moufang) loop $G$ is called commutative if the commutativity law holds in $G$, and (only) the commutative associative (Moufang) loops are said to be Abelian.
The most remarkable property of the Moufang loops is their *diassociativity*: in a Moufang loop $G$ every two elements generate an associative subloop (group) $[6]$. In particular, from this it follows that

$$g \cdot gh = g^2h, \quad hg \cdot g = hg^2, \quad gh \cdot g = g \cdot hg, \quad \forall g, h \in G$$

(1.2)

The first and second identities in (1.2) are called the left and right *alternativity*, respectively, and the third one is said to be *flexibility*. Note that these identities follow from the Moufang identities as well. Due to flexibility, the Moufang identity (1.1c) can be rewritten in the nice mnemonic form as follows:

$$(gh)(kg) = g(hk)g, \quad \forall g, h, k \in G$$

The unique solution of the equation $xg = e$ ($gx = e$) is called the left (right) *inverse* element of $g \in G$ and is denoted as $g_R^{-1}$ ($g_L^{-1}$). It follows from the diassociativity of the Moufang loop that

$$g_R^{-1} = g_L^{-1} = g^{-1}, \quad \forall g \in G$$

2 Analytic Moufang loops and Mal’tsev algebras

A Moufang loop $G$ is said to be *analytic* $[5]$ if $G$ is a finite dimensional real, analytic manifold so that both the Moufang loop operation $G \times G \to G$: $(g, h) \mapsto gh$ and the inversion map $G \to G$: $g \mapsto g^{-1}$ are analytic ones. Dimension of $G$ will be denoted as $\dim G = r$. The local coordinates of $g \in G$ are denoted (in a fixed chart of the unit element $e \in G$) by $g^1, \ldots, g^r$, and the local coordinates of the unit $e$ are supposed to be zero: $e^i = 0$, $i = 1, \ldots, r$. One has the evident initial conditions

$$(ge)^i = (eg)^i = g^i, \quad i = 1, \ldots, r$$

As in the case of the Lie groups $[8]$, we can use the Taylor expansions

$$(gh)^i = h^i + u^i_j(h)g^j + \frac{1}{2!}u^i_{jk}(h)g^jg^k + \cdots$$

$$= g^i + v^i_j(g)h^j + \frac{1}{2!}v^i_{jk}(g)h^jh^k + \cdots$$

$$= g^i + h^i + a^i_jg^jh^k + b^i_{jkl}g^jg^k h^l + d^i_{jkl}g^jh^k h^l + \cdots$$

to introduce the *auxiliary functions* $u^i_j$ and $v^i_j$ and the *structure constants*

$$c^i_{jk} = a^i_{jk} - b^i_{kji} = -c^i_{kj}$$

Due to such a parametrization, one has the initial conditions

$$u^i_j(e) = v^i_j(e) = \delta^i_j, \quad \tilde{u}^i_{jk}(e) = \tilde{v}^i_{jk}(e) = 0$$

and the symmetry property

$$\tilde{u}^i_{jk} = \tilde{u}^i_{kj}, \quad \tilde{v}^i_{jk} = \tilde{v}^i_{kj}$$

(2.1)

Also, it follows from axiom 1) of the Moufang loop that

$$\det(u^i_j) \neq 0, \quad \det(v^i_j) \neq 0$$
The tangent algebra of $G$ can be defined similarly to the tangent (Lie) algebra of the Lie group $[8]$. Geometrically, this algebra is the tangent space $T_e(G)$ of $G$ at $e$. The product of $x, y \in T_e(G)$ will be denoted by $[x, y] \in T_e(G)$. In coordinate form,

$$[x, y]^i \equiv c^i_{jk} x^j y^k = -[y, x]^i, \quad i = 1, \ldots, r$$

The tangent algebra will be denoted by $\Gamma \equiv \{T_e(G), [\cdot, \cdot]\}$. The latter algebra need not be a Lie algebra. In other words, there may be a triple $x, y, z \in T_e(G)$, such that the Jacobi identity fails:

$$J(x, y, z) \equiv [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \neq 0$$

Instead, for all $x, y, z \in T_e(G)$, we have $[5]$ a more general identity

$$[[x, y], [z, x]] + [[[x, y], z], x] + [[[y, z], x], x] + [[[z, x], x], y] = 0$$

called the Mal’tsev identity. The tangent algebra is hence said to be the Mal’tsev algebra. The Mal’tsev identity concisely reads $[9]

$$[J(x, y, x), x] = J(x, y, [x, z])$$

from which it can be easily seen that every Lie algebra is a Mal’tsev algebra as well. It has been shown in [4] that every finite-dimensional real Mal’tsev algebra is the tangent algebra of some analytic Moufang loop.

### 3 Associators

In a (Moufang) loop, due to non-associativity, the elements $g \cdot hk$ and $gh \cdot k$ need not coincide. The nonassociativity of $G$ can be measured by the formal functions

$$a^i(g, h, k) \equiv (g \cdot hk)^i - (gh \cdot k)^i, \quad i = 1, \ldots, r$$

which are called the associators of $G$. One has the evident initial conditions

$$a^i(e, h, k) = a^i(g, e, k) = a^i(g, h, e) = 0, \quad i = 1, \ldots, r$$

The associators $a^i$ are considered as the generating expressions in the following sense. At first, define the first-order associators $l^i_j$, $r^i_j$, $m^i_j$ by

$$a^i(g, h, k) \equiv l^i_j(h, k) g^j + O(g^2)$$

$$\equiv m^i_j(g, k) h^j + O(h^2)$$

$$\equiv r^i_j(g, h) k^j + O(k^2)$$

These functions can be easily calculated and the result reads

$$l^i_j(g, h) = -u^j_s(g) \frac{\partial (gh)^i}{\partial g^s} + u^i_s(gh)$$

$$r^i_j(g, h) = -v^j_s(gh) + v^i_s(h) \frac{\partial (gh)^i}{\partial h^s}$$

$$m^i_j(g, h) = -v^j_s(g) \frac{\partial (gh)^i}{\partial g^s} + u^i_s(h) \frac{\partial (gh)^i}{\partial h^s}$$

(3.1a)
Next one can check the initial conditions

\[ l^i_j(e, g) = r^i_j(e, g) = m^i_j(e, g) = 0 \]
\[ \tilde{l}^i_j(g, e) = r^i_j(g, e) = m^i_j(g, e) = 0 \]

and define the second-order associators \( l^i_{jk}, \tilde{l}^i_{jk}, m^i_{jk}, \tilde{m}^i_{jk}, r^i_{jk}, \tilde{r}^i_{jk} \) by

\[ l^i_{jk}(g, h) = \tilde{l}^i_{jk}(h)g^k + O(g^2) \]
\[ = \tilde{r}^i_{jk}(g)h^k + O(h^2) \]
\[ r^i_{jk}(g, h) = \tilde{r}^i_{jk}(h)g^k + O(g^2) \]
\[ = \tilde{m}^i_{jk}(g)h^k + O(h^2) \]
\[ m^i_{jk}(g, h) = \tilde{m}^i_{jk}(h)g^k + O(g^2) \]
\[ = \tilde{m}^i_{jk}(g)h^k + O(h^2) \]

By calculating the latter, the result reads

\[ \tilde{l}^i_{jk}(g) = \tilde{m}^i_{jk}(g) \] (3.2a)
\[ = -u^i_{jk}(g) - a^i_{jk}u^i_k(g) + u^i_k(g) \frac{\partial u^i_j(g)}{\partial g^k} \] (3.2b)
\[ m^i_{jk}(g) = \tilde{r}^i_{jk}(g) \] (3.2c)
\[ = v^i_{jk}(g) + a^i_{jk}v^i_k(g) - v^i_k(g) \frac{\partial v^i_j(g)}{\partial g^k} \] (3.2d)
\[ r^i_{jk}(g) = \tilde{r}^i_{jk}(g) \] (3.2e)
\[ = v^i_j(g) \frac{\partial u^i_k(g)}{\partial g^s} - u^i_k(g) \frac{\partial v^i_j(g)}{\partial g^s} \] (3.2f)

Finally one has the initial conditions

\[ l^i_{jk}(e) = r^i_{jk}(e) = m^i_{jk}(e) = 0 \]
\[ \tilde{l}^i_{jk}(e) = \tilde{r}^i_{jk}(e) = \tilde{m}^i_{jk}(e) = 0 \]

and define the third-order associators \( l^i_{jkl}, \tilde{l}^i_{jkl}, m^i_{jkl}, \tilde{m}^i_{jkl}, r^i_{jkl}, \tilde{r}^i_{jkl} \) by

\[ l^i_{jkl}(g, h) = \tilde{l}^i_{jkl}g^l + O(g^2) \]
\[ = \tilde{m}^i_{jkl}g^l + O(g^2) \]
\[ m^i_{jkl}(g, h) = \tilde{m}^i_{jkl}g^l + O(g^2) \]
\[ \tilde{m}^i_{jkl}(g) = \tilde{m}^i_{jkl}g^l + O(g^2) \]
\[ r^i_{jkl}(g, h) = \tilde{r}^i_{jkl}g^l + O(g^2) \]
\[ \tilde{r}^i_{jkl}(g) = \tilde{r}^i_{jkl}g^l + O(g^2) \]

By calculating the latter one gets

\[ l^i_{jkl} = m^i_{jkl} = r^i_{jkl} = \tilde{l}^i_{jkl} = \tilde{r}^i_{jkl} = \tilde{m}^i_{jkl} \] (3.3a)
\[ = a^i_{jls}a^s_{kl} - a^i_{jk}a^i_{sl} + 2(d^i_{jkl} - b^i_{jkl}) \] (3.3b)
All third-order associators can be obtained from $l_{kl}^i$ via permutations of the lower indices. By using (3.3a,b) one can prove the Akivis identity \[1\]

$$l_{kl}^i + r_{jk}^i + m_{kjl}^i - \hat{l}_{kjl}^i - \hat{r}_{jkl}^i - \hat{m}_{jkl}^i = l_{kl}^i + \dot{l}_{kl}^i + \ddot{l}_{kl}^i - l_{kjl}^i - l_{jkl}^i$$ \[(3.4a)\]

$$c_{js}^i c_{kl}^j + c_{ks}^i c_{lj}^j + c_{ls}^i c_{jk}^j$$ \[(3.4b)\]

For $x, y, z \in T_e(G)$ define \[1\] their trilinear product $(x, y, z) \in T_e(G)$ by \[(x, y, z)^i = l_{jkl}^i x^j y^k z^l, \quad i = 1 \ldots, r\]

Then the Akivis \[3.4\] identity reads \[1\]

$$J(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) - (x, z, y) - (z, y, x) - (y, x, z)$$ \[(3.5)\]

4 Minimality conditions and generalized Lie equations

Differentiate the Moufang identities (1.1a–c) in local coordinates with respect to $g^j$ at $g = e$. Then, by redenoting the variables we obtain for the first-order associators the following relations:

$$l_j^i(g, h) = -m_j^i(g, h)$$ \[(4.1a)\]

$$r_j^i(g, h) = -m_j^i(g, h)$$ \[(4.1b)\]

$$l_j^i(g, h) = r_j^i(g, h)$$ \[(4.1c)\]

Thus the Moufang identities give rise to the constraints

$$l_j^i(g, h) = r_j^i(g, h) = -m_j^i(g, h)$$ \[(4.2)\]

These constraints generalize in the natural way the differential equations of the Lie group, the Lie equations \[8\]. If $G$ is associative, i.e a (local) Lie group, then one has

$$l_j^i(g, h) = r_j^i(g, h) = -m_j^i(g, h) = 0$$ \[(4.3)\]

By comparing formulae (4.2) and (4.3) one can say that the Moufang loops have the property that their associativity is spoiled in the minimal way. The constraints (4.2) are hence called the first-order minimality conditions.

Define the new auxiliary functions $w_j^i$ by

$$u_j^i(g) + v_j^i(g) + w_j^i(g) = 0$$ \[(4.4)\]

Then constraints (1.1a–c) can be rewritten as the differential equations, called the generalized Lie equations

$$w_j^i(g) \frac{\partial (gh)^i}{\partial g^s} + u_j^i(h) \frac{\partial (gh)^i}{\partial h^s} + u_j^i(gh) = 0$$

$$v_j^i(g) \frac{\partial (gh)^i}{\partial g^s} + w_j^i(h) \frac{\partial (gh)^i}{\partial h^s} + v_j^i(gh) = 0$$

$$u_j^i(g) \frac{\partial (gh)^i}{\partial g^s} + v_j^i(h) \frac{\partial (gh)^i}{\partial h^s} + w_j^i(gh) = 0$$

In view of (4.4) these equations are linearly dependent.
5 Generalized Maurer-Cartan equations

Differentiate constraints (4.1a–c) with respect to \( g^k \) and \( h^k \) at \( g = e \) and \( h = e \), respectively. Then, redenoting the variables one obtains

\[
l'^i_{jk}(g) = r'^i_{jk}(g) = m'^i_{jk}(g) = -m'^j_{kj}(g) \tag{5.1}
\]

If \( G \) is associative (i.e., a local Lie group), we have

\[
l'^i_{jk}(g) = r'^i_{jk}(g) = m'^i_{jk}(g) = -m'^j_{kj}(g) = 0
\]

Constraints (5.1) are called the second-order minimality conditions for the Moufang loop \( G \). Let us consider the latter more closely.

It follows from the skew-symmetry \( l_{ijk} = -l_{kji} \) and \( m_{ijk} = -m_{kij} \), respectively, that

\[
2\tilde{u}^i_{jk} = u^i_k \frac{\partial u^i_j}{\partial g^s} + u^i_j \frac{\partial u^i_k}{\partial g^s} - (a^i_{jk} + a^i_{kj}) u^i_s \tag{5.2a}
\]

\[
2\tilde{v}^i_{jk} = v^i_k \frac{\partial v^i_j}{\partial g^s} + v^i_j \frac{\partial v^i_k}{\partial g^s} - (a^i_{jk} + a^i_{kj}) v^i_s \tag{5.2b}
\]

Note that here we have used the symmetry property (2.1) as well. Express \( u^i_{jk} \) and \( v^i_{jk} \) from (5.2a) and substitute into (3.2a) and (3.2b), respectively. The result reads

\[
\frac{\partial u^i_j}{\partial g^s} - u^i_s \frac{\partial u^i_j}{\partial g^s} = c^s_{jk} u^i_s + 2l'^i_{jk} \tag{5.3a}
\]

\[
\frac{\partial v^i_j}{\partial g^s} - v^i_s \frac{\partial v^i_j}{\partial g^s} = c^s_{kj} v^i_s + 2m'^i_{jk} \tag{5.3b}
\]

Now using the equalities \( l'^i_{jk} = r'^i_{jk}, m'^i_{jk} = -r'^i_{kj} \) and (3.2c) and (3.2d) for \( r'^i_{jk} \), one obtains the differential equations for the auxiliary functions \( u^i_j \) and \( v^i_j \):

\[
\frac{\partial u^i_j}{\partial g^s} - u^i_s \frac{\partial u^i_j}{\partial g^s} = c^s_{jk} u^i_s + 2 \left( v^i_j \frac{\partial v^i_k}{\partial g^s} - u^i_k \frac{\partial v^i_j}{\partial g^s} \right) \tag{5.5a}
\]

\[
\frac{\partial v^i_j}{\partial g^s} - v^i_s \frac{\partial v^i_j}{\partial g^s} = c^s_{kj} v^i_s + 2 \left( u^i_j \frac{\partial u^i_k}{\partial g^s} - v^i_k \frac{\partial u^i_j}{\partial g^s} \right) \tag{5.5b}
\]

called the generalized Maurer-Cartan equations. In a sense, the generalized Maurer-Cartan equations generalize the Maurer-Cartan equations \[8\] in the minimal way.

Finally, constraints \( r'^i_{jk} = -r'^i_{kj} \) read

\[
v^i_j \frac{\partial u^i_k}{\partial g^s} - u^i_s \frac{\partial u^i_k}{\partial g^s} = u^i_j \frac{\partial v^i_k}{\partial g^s} - v^i_s \frac{\partial v^i_k}{\partial g^s} \tag{5.4}
\]

The generalized Maurer-Cartan differential equations can be rewritten more concisely. For \( x \in T_e(G) \) introduce the infinitesimal translations:

\[
L_x = x^j u^i_j(g) \frac{\partial}{\partial g^i}, \quad R_y = x^j v^i_j(g) \frac{\partial}{\partial g^i} \in T_g(G)
\]
Then the generalized Maurer-Cartan equations (5.3) and differential equations (5.4) can be rewritten, respectively, as the commutation relations

\[ [L_x, L_y] = L_{[x,y]} - 2 [L_x, R_y] \] (5.5a)
\[ [R_x, R_y] = R_{[y,x]} - 2 [R_x, L_y] \] (5.5b)
\[ [L_x, R_y] = [R_x, L_y] \] (5.5c)

The Lie bracket \([A, B]\) for the vector fields \(A\) and \(B\) is defined in the usual way: \([A, B] = AB - BA\). Note that commutation relation (5.5c) can also easily be obtained from the identities

\[ [L_x, L_y] = - [L_y, L_x], \quad [R_x, R_y] = - [R_y, R_x] \]

In the case of associativity of \(G\) the (generalized) Maurer-Cartan equations read

\[ 2 [L_x, R_y] = L_{[x,y]} - 2 [L_x, L_y] = R_{[y,x]} - 2 [R_x, R_y] = 0 \]

6 Minimality conditions and Akivis identity

Differentiate (5.1) with respect to \(g^l (l = 1, \ldots, n)\) at \(g = e\). Then one gets the third-order minimality conditions

\[ \ell^i_{jkl} = \ell^i_{kjl} = - \ell^i_{ljk} = r^i_{jkl} = m^i_{jkl} = - \ell^i_{jkl} = - r^i_{jkl} = - m^i_{jkl} \]

Thus the associators of the third order of a local analytic Moufang loop have the property of total anti-symmetry with respect of the lower indexes. From this it follows that the Akivis identity (5.3) can be written as follows:

\[ J(x, y, z) = 6(x, y, z) \]

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