Non-semisimple Levin–Wen models and Hermitian TQFTs from quantum (super)groups

Nathan Geer$^1$ | Aaron D. Lauda$^2$ | Bertrand Patureau-Mirand$^3$ | Joshua Sussan$^{4,5}$

$^1$Mathematics & Statistics, Utah State University, Logan, Utah, USA
$^2$Mathematics & Physics, University of Southern California, Los Angeles, California, USA
$^3$UMR 6205, LMBA, Université de Bretagne-Sud, Vannes, France
$^4$Department of Mathematics, CUNY Medgar Evers, Brooklyn, New York, USA
$^5$Mathematics Program, The Graduate Center, CUNY, New York, New York, USA

Correspondence
Nathan Geer, Mathematics, and Statistics, Utah State University, Logan, Utah 84322, USA.
Email: nathan.geer@gmail.com

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Abstract
We develop the categorical context for defining Hermitian non-semisimple topological quantum field theories (TQFTs). We prove that relative Hermitian modular categories give rise to modified Hermitian Witten–Reshetikhin–Turaev TQFTs and provide numerous examples of these structures coming from the representation theory of quantum groups and quantum superalgebras. The Hermitian theory developed here for the modified Turaev–Viro TQFT is applied to define new pseudo-Hermitian topological phases that can be considered as non-semisimple analogs of Levin–Wen models.

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1 | INTRODUCTION

There is a rich interplay between two-dimensional topological phases in quantum mechanical systems and topological quantum field theories (TQFTs). This interaction is further enriched as topological structures inherent in TQFTs lead to novel features, such as nonabelian braiding statistics for low-energy excitations when expressed in the corresponding quantum mechanical models. This exciting interplay has led to a number of proposals for realizing fault-tolerant
quantum computation by exploiting the topological features arising in these models [17, 34, 37, 42, 44], what is collectively called topological quantum computation (TQC).

Both the Turaev–Viro (TV) TQFTs [56] and the Witten–Reshetikhin–Turaev (WRT) TQFTs [52, 58] have appeared in approaches to TQC and the study of topological phases. All of these connections rely on unitarity of the TQFT to produce physically meaningful quantum mechanical models. Both TV and WRT TQFTs have the common feature that the categorical data defining them, spherical categories and modular tensor categories, respectively, are based on semisimple categories. The purpose of this article is to build a theory of Hermitian structures on a new class of TQFTs based on non-semisimple categories; this key structure leads to new physically relevant applications of non-semisimple TQFTs.

1.1 Beyond semisimple unitary TQFTs

The primary examples of semisimple categories used to define (2+1)-dimensional TQFTs come from the representation theory of quantum groups where the quantum parameter is set to a root of unity. These categories are not semisimple; instead, a semisimplification procedure is performed that throws away an infinite number of representations, leaving only a finite collection remaining. This process is essential as it removes representations with vanishing quantum dimensions, which cannot be input into either the TV or WRT TQFTs. Kirillov [40] and Wenzl [57] showed that the semisimple categories coming from the representation theory of quantum groups give rise to Hermitian and unitary categories in the sense of Turaev [55].

It was an open question how to construct a TQFT that retains non-semisimple information. A crucial step in this direction was achieved in [31] (also see [20, 22]) where a modified trace was introduced, leading to nontrivial invariants of links colored by representations that classically led to trivial invariants. The modified trace was utilized in [32] where a non-semisimple TV invariant was constructed using the notion of a relative spherical category. A non-semisimple WRT invariant [8] and TQFT [4] were constructed using the notion of a relative modular category. An extended TQFT was formulated by De Renzi [11]. These non-semisimple constructions use a group \( G \) as input.

The key examples of such non-semisimple categories have an infinite number of nonisomorphic simple objects with vanishing quantum dimensions. Nevertheless, these non-semisimple TQFTs have remarkable properties, often proving more powerful than their semisimple analogs. For example, non-semisimple TQFTs lead to mapping class group representations with the notable property that the action of a Dehn twist has infinite order, and thus, the representations could be faithful (see [4]). This is in contrast with the usual quantum mapping class group representations where all Dehn twists have finite order and the representations are thus not faithful. Also, after projectivization, these TQFTs correspond to the Lyubashenko projective mapping class group representations given in [45] (also see [12, 13]).

Given that the non-semisimple TQFTs appear to be much more powerful than their semisimple counterparts, it suggests that these TQFTs can have similarly impactful applications to physical phenomena and the construction of topological phases. We mentioned above the importance of unitarity for applying TQFTs to physical phenomena in the usual approach, but mentioned our work here will be focused on Hermitian structures for non-semisimple categorical data. The reason is that the non-semisimple TQFTs do not lead to Hermitian Hamiltonians. Instead, what naturally arises are pseudo-Hermitian Hamiltonians. Such quantum mechanical systems have received a great deal of interest in the physics literature, as it has been observed that
many physically relevant systems exhibit non-Hermitian Hamiltonians [5, 59, 61]. The pseudo-Hermicity condition ensures that these systems have positive spectrum, normalizable wave functions, and time evolution given by the exponential of the Hamiltonian that is self-adjoint with respect to an indefinite inner product [48–50]. These inner products are also invariant under the time-evolution generated by the Schrödinger equation. Essentially, these theories are unitary, but for an indefinite inner product.

### 1.2 Hermitian non-semisimple TQFTs

The goal of this article is to develop a theory of Hermitian structures that can be applied to relative spherical categories and relative modular tensor categories, giving rise to Hermitian non-semisimple WRT invariants and allowing for the construction of pseudo-Hermitian topological phases from the non-semisimple TV theory (discussed more in the next section). In particular, in the WRT setting, we prove the following.

**Theorem 1** *(Theorem 6.6).* A relative Hermitian \(\mathcal{G}\)-modular category gives rise to a modified Hermitian WRT TQFT.

Using ideas of [40, 55, 57], the authors of the current paper gave a proof of concept for this approach by placing a Hermitian structure on a non-semisimple category of representations coming from the unrolled quantum group for \(\mathfrak{sl}_2\) [23]. This led to the first example of a Hermitian-modified WRT TQFT. Here, we vastly generalize this result. While many of the techniques developed for \(\mathfrak{sl}_2\) generalize, in [23], we relied on an explicit description of the category for \(\mathfrak{sl}_2\) that allowed us to use certain representations in the earlier works that are inaccessible here. Thus, we must develop additional theory needed to deduce the relevant Hermitian data in a general Hopf algebraic setup giving rise to many new examples of Hermitian \(\mathcal{G}\)-modular categories.

**Theorem 2** *(Theorems 7.7, 7.8, and 7.17).* Finite-dimensional simple complex Lie algebras as well as \(\mathfrak{sl}(m|n)\) provide examples of relative Hermitian \(\mathcal{G}\)-modular categories and relative Hermitian \(\mathcal{G}\)-spherical categories.

### 1.3 Turaev–Viro TQFT and topological phases

Turaev–Viro TQFTs appear prominently in Kitaev’s toric code [42] and Levin–Wen’s construction [44] of two-dimensional topological phases using “string net” models defined on dual graphs of triangulated surfaces. These models produce exactly solvable gapped Hamiltonians exhibiting a topologically degenerate ground state. We refer to these physical systems as LW-models. They are constructed from algebraic data we call LW-data, essentially defining a spherical category. The topologically dependent ground state of the LW Hamiltonian gives rise to a topological invariant of the surface it is defined on, and is known to be isomorphic to the vector space assigned to the surface coming from the TV TQFT [41, 43].

In [24], the authors extended the results in [23] by defining a Hermitian structure on the non-semisimple TV TQFT associated to the category of representations for the semirestricted quantum group for \(\mathfrak{sl}_2\), showing that this category was a Hermitian \(\mathcal{G}\)-relative spherical category. We also showed that using certain algebraic data, that we call \(\mathcal{G}\)-LW data, we could define a
pseudo-Hermitian Levin–Wen model. A number of simplifications made this example easier to define, including multiplicity one results in the relevant non-semisimple category. Here, we generalize this work by extending the definition of $G$-LW data and proving the following.

**Theorem 3** (Sections 2.2 and 2.3, Theorems 2.5 and 2.6). To $G$-LW data, a triangulated surface $\Sigma$ with dual triangulation $\Gamma$, and a cohomology class $[\Phi] \in H^1(\Sigma, G)$, there exists a Hilbert space $\mathcal{H} = H(\Gamma, \Phi)$ equipped with a pseudo-Hermitian Hamiltonian $H : \mathcal{H} \to \mathcal{H}$. This Hamiltonian is gapped (so there is a finite energy gap between each excited state), local, and exactly solvable.

**Theorem 4** (Theorems 5.5 and 5.6). A relative Hermitian $G$-spherical category gives rise to $G$-LW data. Furthermore, the modified Hermitian TV invariant for a surface $\Sigma$ associated to $G$ is isomorphic to the ground state of the Hamiltonian $H$ from the previous theorem.

The original construction from [24] based on semirestricted quantum $\mathfrak{sl}_2$ relied upon a forgetful functor between the unrolled quantum group and the semirestricted quantum group for $\mathfrak{sl}_2$. The auxiliary role of the unrolled quantum group is generalized here, allowing us to utilize data from a relative modular category to produce data needed to define Hermitian structures on relative spherical categories. Indeed, in all of the examples considered above, we need this auxiliary braided category to define the Hermitian structure.

**Theorem 5** (Theorem 4.14). Under certain conditions, a relative Hermitian $G$-modular category gives rise to a relative Hermitian $G$-spherical category.

In this paper, we show that this theorem applies to a large set of examples, including the categories of modules over the unrolled quantum groups associated to finite-dimensional simple complex Lie algebras and $\mathfrak{sl}(m|n)$. In these examples, the modules used in the construction only exist when the quantum parameter is a root of unity. In a different direction, one can consider deformed modules over quantum $\mathfrak{sl}(m|n)$ called perturbative modules. These modules exist before specializing the quantum parameter to a root of unity and they form a graded category that is not semisimple where a generic module has vanishing quantum dimension. In [25, 26], it is shown that the one can define an $m$-trace on these perturbative modules and specialize $q$ to a root of unity; this forces some modules to have zero $m$-trace. In the case of $\mathfrak{sl}(2|1)$, then taking the quotient by negligible morphisms of modules with zero $m$-trace, it is shown that the resulting category is a relative $G$-spherical category (see Theorem 5.6.1 of [25]). Thus, the category leads to a modified TV invariant of 3-manifolds with additional structures (in particular, cohomology classes on surfaces and cobordisms, see [29]). In this paper, we show that the corresponding Levin–Wen Hamiltonian is pseudo-Hermitian. The techniques of this paper were developed with this example in mind.

Let us say a few words about why we think this example is interesting. Most of the examples of this paper are based on modules that only appear after the quantum parameter is specialized to a root of unity, that is, when the quantum parameter is specialized to a root of unity, new central elements of the algebra appear, which allow for new families of simple modules. Instead, the perturbative modules are deformations of classical $\mathfrak{sl}(2|1)$-modules specialized to a root of unity. A theory based on deformations of classic modules should have connections with constructions in mathematical physics: Mikhaylov and Witten in [47] consider supergroup Chern–Simons (CS) theories (also see [1, 10, 18, 35, 38, 46, 51, 53]). The motivating example of [47] is a path integral related to classical modules over $\mathfrak{gl}(1|1)$ (see [46]). As
in the case of Witten’s path integral interpretation of the Jones polynomial [58], in [33], it is shown that a De Renzi–Blanchet–Costantino–Geer–Patureau–Mirand-style TQFT associated to the category of perturbative quantum $\mathfrak{gl}(1|1)$-modules recovers all the computable properties of the path integral given in [46]. However, it is not clear how to define a path integral for higher rank Lie superalgebras (in particular, see [47, Section 3.2] for many questions about supergroup CS theories). The results of [33] suggest that a solution to Conjecture 7.19 would lead to a family of Hermitian-modified TV invariants that should correspond to not yet constructed higher rank supergroup CS path integrals. Thus, since Conjecture 7.19 is true for $\mathfrak{sl}(2|1)$, Theorem 7.20 gives a pseudo-Hermitian relative LW-model associated to $\mathfrak{sl}(2|1)$ that should correspond to a higher rank supergroup CS path integral and therefore have interesting mathematical physics applications. In particular, this should be a combinatorial version of a (super) CS theory with a nonstandard continuous gauge group. Such generalizations of CS theory have recently drawn much interest, for example, see [2, 9, 14–16, 36] and the references within.

1.4 | Outline

We begin in Section 2 with the definition of a relative $\mathcal{G}$-LW-system. Given this data and a triangulated, compact, connected, oriented surface, we define a Hilbert space and a Hamiltonian acting on this state space. Section 3 contains a definition of sesquilinear pivotal categories. Hopf superalgebras equipped with a dagger map provide examples of such categories. In Section 4, we review the notion of a modified trace and define relative Hermitian modular and relative Hermitian spherical categories. We also give a construction on how to obtain a relative Hermitian spherical category from a relative Hermitian modular category. Section 5 contains a result showing that a relative Hermitian-spherical category leads to a relative $\mathcal{G}$-LW-system. In Section 6, we show how to obtain a Hermitian WRT type TQFT. We conclude with Section 7 where we show how to obtain relative Hermitian-spherical categories from quantum groups for finite-dimensional simple complex Lie algebras and $\mathfrak{sl}(m|n)$.

2 | RELATIVE $\mathcal{G}$-LW-SYSTEM

Let $k$ be a field equipped with an involutive automorphism $k \mapsto \overline{k}$. We follow and generalize the system Hilbert space described in [24].

2.1 | Basic input

The input for a relative $\mathcal{G}$-LW-system is a tuple $(\mathcal{G}, \mathcal{X}, (I_g)_{g \in \mathcal{G}}, \delta, b, d, N, \beta, \gamma)$, called $\mathcal{G}$-LW data, where:

1. $(\mathcal{G}, +)$ is an abelian group.
2. $\mathcal{X} \subset \mathcal{G}$ is symmetric (i.e., $-\mathcal{X} = \mathcal{X}$) and small (i.e., for any finite subset $Y \subset \mathcal{G}, \mathcal{G} \not\subset Y + \mathcal{X}$).
3. $I_g$ is a set for a generic element $g \in \mathcal{G} \setminus \mathcal{X}$. If $j \in I_g$, we call $g = |j| \in \mathcal{G}$ the degree of $j$.
4. Define $A = \cup_{g \in \mathcal{G} \setminus \mathcal{X}} I_g$. Then, we assume that $A$ is equipped with an involution $\ast$ such that $I_g^\ast = I_{-g}$. 
(5) The fusion multiplicity map $\delta : A^3 \to \mathbb{N} : (i, j, k) \mapsto \delta_{ijk}$ has dihedral symmetry:

$$\delta_{ijk} = \delta_{jki} = \delta_{k^*j^*i^*},$$

and $\delta_{ijk} = 0$ unless $|i| + |j| + |k| = 0$.

(6) The modified dimension $\rho : A \to k^*$ and map $b : A \to k^*$ satisfy

$$\tilde{b} = b, \quad \tilde{d} = d, \quad b(i^*) = b(i), \quad d(i^*) = d(i).$$

Furthermore, for $g, g_1, g_2 \in \mathcal{C} \setminus \mathcal{X}$, with $g + g_1 + g_2 = 0$, we have

$$b(j) = \sum_{j_1 \in I_{g_1}, j_2 \in I_{g_2}} b(j_1) b(j_2) \delta_{j^*j_1j_2},$$

for all $j \in I_g$.

(7) There are basic data maps $\beta : A \to k^*$ and $\gamma : A^3 \times \{1, ..., \delta(A^3)\} \to k^*$ satisfying

$$\gamma = \gamma, \quad \beta = \beta, \quad \beta(i^*) = \beta(i)$$

and for $1 \leq n \leq \delta_{ijk}$, we have

$$\gamma^n_{i,j,k} \gamma^n_{k^*,j^*,i^*} \beta(i) \beta(j) \beta(k) = 1. \quad (2.1)$$

(8) There are modified $6j$ symbols $N : A^6 \times (\mathbb{N}^*)^4 \to k$ with the property that $N^{j_1,j_2,j_3}_{j_4,j_5,j_6} (a_1,a_2,a_3,a_4) = 0$ unless $1 \leq a_1 \leq \delta_{j_1,j_2,j_3}$, $1 \leq a_2 \leq \delta_{j_3,j_4,j_5}$, $1 \leq a_3 \leq \delta_{j_5,j_6,j_1}$, and $1 \leq a_4 \leq \delta_{j_6,j_1,j_2}$.

The data satisfy the following conditions:

$$N^{j_1,j_2,j_3}_{j_4,j_5,j_6} (a_1,a_2,a_3,a_4) = N^{j_2,j_3,j_4}_{j_5,j_6,j_1} (a_1,a_3,a_2,a_4), \quad \delta_{k,p} \delta_{a_1,a_1'} \delta_{a_2,a_2'} \frac{N^{j_1,j_2,j_3}_{j_4,j_5,j_6} (a_1,a_2,a_3,a_4)}{d(k)}, \quad (2.2)$$

$$\sum_{j,c_1 \neq c_2,c_3} d(j) N^{j_1,j_2,j_3}_{j_4,j_5,j_6} (a_1,a_2,a_3,a_4) N^{j_4,j_5,j_6}_{j_1,j_2,j_3} (a_1,a_3,a_2,a_4) = \sum_{c_4} N^{j_1,j_2,j_3}_{j_4,j_5,j_6} (a_1,a_3,a_2,a_4) N^{j_4,j_5,j_6}_{j_1,j_2,j_3} (a_1,a_3,a_2,a_4), \quad (2.3)$$

$$\sum_{n,a_3,a_4} d(n) N^{j_{i_1}j_{i_2}}_{i_{l_1}i_{l_2}} (a_1,a_2,a_3,a_4) N^{k_{j_1}j_{j_2}}_{k_{j_3}j_{j_4}} (a_1,a_3,a_2,a_4) = \delta_{k,p} \delta_{a_1,a_1'} \delta_{a_2,a_2'} \frac{N^{j_1,j_2,j_3}_{j_4,j_5,j_6} (a_1,a_2,a_3,a_4)}{d(k)}, \quad (2.4)$$

$$N^{j_1,j_2,j_3}_{j_4,j_5,j_6} (a_1,a_2,a_3,a_4) = N^{j_2,j_3,j_4}_{j_5,j_6,j_1} (a_1,a_2,a_3,a_4) \gamma^a_1 \gamma^a_2 \gamma^a_{j_1,j_2,j_3} \gamma_{j_3,j_4,j_5} \gamma_{j_5,j_6,j_1} \gamma_{j_1,j_2,j_3} \prod_{l=1}^6 \beta(j_l). \quad (2.5)$$

### 2.2 Hermitian state space

Let $\Sigma$ be a compact, connected, oriented surface. Let $\mathcal{T}$ be a triangulation of $\Sigma$ and $\Gamma$ be a finite trivalent graph dual to $\mathcal{T}$. Each vertex of the graph $\Gamma$ acquires a cyclic ordering compatible with the orientation of $\Sigma$. A $G$-coloring of $\Gamma$ is a map $\Phi$ from the set of oriented edges of $\Gamma$ to $\mathcal{C}$ such that the following hold.
(1) $\Phi(-e) = -\Phi(e)$ for any oriented edge $e$ of $\Gamma$, where $-e$ is $e$ with opposite orientation.

(2) If $e_1, e_2, e_3$ are edges of a vertex $v$ of $\Gamma$ with a cyclic ordering compatible with the orientation of $\Sigma$, and each edge is oriented toward the vertex $v$, then $\Phi(e_1) + \Phi(e_2) + \Phi(e_3) = 0$.

The $G$-colorings of $\Gamma$ form a group isomorphic via Poincaré duality to the group of $G$-valued simplicial 1-cocycles on $\mathcal{T}$. We denote by $[\Phi] \in H^1(\Sigma, G)$ the associated cohomology class. A $G$-coloring of $\Gamma$ is admissible if $\Phi(e) \in G \setminus \mathcal{X}$ for any oriented edge $e$ of $\Gamma$. A state of an admissible $G$-coloring $\Phi$ is a map $\sigma$ assigning to every oriented edge $e$ of $\Gamma$ an element $\sigma(e) \in I_{\Phi(e)}$ such that $\sigma(-e) = \sigma(e)^*$ and to each vertex $v$ with incident edges $e_1, e_2, e_3$, the natural number $\sigma_0(v)$ where $0 \leq \sigma_0(v) \leq \delta_{\sigma(e_1), \sigma(e_2), \sigma(e_3)}$. Denote by St$(\Phi)$ the set of such states. For an admissible $G$-coloring $\Phi$, we define the Hilbert space $\mathcal{H} = \mathcal{H}(\Gamma, \Phi)$ as the span of all elements corresponding to the states of $\Phi$:

$$\mathcal{H}(\Gamma, \Phi) = \bigoplus_{\sigma \in \text{St}(\Phi)} \mathbb{C}\langle \Gamma, \sigma \rangle.$$ 

We write $\sigma^*$ for the state of the $G$-coloring $-\Phi$ assigning $\sigma(e)^*$ to each oriented edge $e$ of $\Gamma$.

### 2.3 Inner products

We equip $\mathcal{H} = \mathcal{H}(\Gamma, \Phi)$ with a complete Hilbert space structure by defining a positive-definite Hermitian inner product

$$\langle \cdot | \cdot \rangle_+ : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C},$$

in which the states $|\Gamma, \sigma\rangle$ form an orthonormal basis:

$$\langle \Gamma, \sigma | \Gamma, \sigma' \rangle_+ = \delta_{\sigma, \sigma'}.$$ (2.7)

We will see that from the TQFT perspective, this is not the most natural inner product. The most natural inner product can be obtained from this one utilizing a Hermitian operator $\eta : \mathcal{H} \to \mathcal{H}$. The invertible operator $\eta$ is defined by

$$\eta : \mathcal{H}(\Gamma, \Phi) \longrightarrow \mathcal{H}(\Gamma, \Phi), \quad |\Gamma, \sigma\rangle \mapsto \frac{\prod_{v \in \Gamma_0} \gamma(\sigma(v)) \prod_{e \in \Gamma_1} \beta(\sigma(e))}{\prod_{v \in \Gamma_0} \gamma(\sigma(v)) \prod_{e \in \Gamma_1} \beta(\sigma(e))} |\Gamma, \sigma\rangle. \tag{2.8}$$

where $\sigma(v) = (j_1, j_2, j_3, \sigma_0(v)) \in I^3 \times \mathbb{N}$, where $1 \leq \sigma_0(v) \leq 3$, $j_k = \sigma(e_k)$, and $e_1, e_2, e_3$ are the three ordered edges adjacent to $v$ oriented toward $v$. It is clear from the orthogonality of states $|\Gamma, \sigma\rangle$ and the fact that $\gamma(\sigma(v))$, $d(\sigma(e))$, and $\beta(\sigma(e))$ are all fixed under the field involution, that $\eta$ is Hermitian with respect to the form $\langle \cdot | \cdot \rangle_+$

$$\langle \psi | \eta \phi \rangle_+ = \langle \eta \psi | \phi \rangle_+ \tag{2.9}$$

for all $\psi, \phi \in \mathcal{H}$.

Define a new inner product on $\mathcal{H} = \mathcal{H}(\Gamma, \Phi)$ via

$$\langle \cdot | \cdot \rangle := \langle \cdot | \eta^{-1} \cdot \rangle_+,$$ (2.10)

$$\langle \Gamma, \sigma | \Gamma, \sigma' \rangle = \frac{\prod_{v \in \Gamma_0} \gamma(\sigma(v)) \prod_{e \in \Gamma_1} \beta(\sigma(e))}{\prod_{v \in \Gamma_0} \gamma(\sigma(v)) \prod_{e \in \Gamma_1} \beta(\sigma(e))} \cdot \delta_{\sigma, \sigma'}. \tag{2.10}$$
It is not difficult to show that this new pairing is Hermitian (see [24, Section II.C]). However, it is in general indefinite (see [24, Sections II.C and V]). In what follows, for an operator $F$, we let $F^\dagger$ be the adjoint with respect to this indefinite form.

If a linear operator $H : \mathcal{H} \rightarrow \mathcal{H}$ is Hermitian with respect to the form $\langle \cdot \mid \cdot \rangle$, so that $\langle \psi \mid H \phi \rangle = \langle H \psi \mid \phi \rangle$, then the Hermitian conjugate $H^\#$ of $H$ with respect to the form $\langle \cdot \mid \cdot \rangle_+$ satisfies

$$H^\# = \eta^{-1} H \eta. \quad (2.11)$$

Thus, a Hamiltonian, or Hermitian operator $H : \mathcal{H} \rightarrow \mathcal{H}$ with respect to the inner product $\langle \cdot, \cdot \rangle$, then becomes pseudo-Hermitian with respect to the inner product $\langle \cdot, - \rangle_+$. See [24, Section II.C] for more details.

### 2.4 Levin–Wen Hamiltonian and local operators

Associated to each vertex $v \in \Gamma_0$, we have a vertex operator $Q_v : \mathcal{H}(\Gamma, \Phi) \rightarrow \mathcal{H}(\Gamma, \Phi)$ that acts locally to impose the branching constraints from Section 2.1 by acting as a delta function on a single vertex $v$ of the graph $\Gamma$ via

$$Q_v \begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix} = \begin{pmatrix} j_2 \\ j_1 \\ j_3 \end{pmatrix} \text{ if } 1 \leq c \leq \delta_{j_1 j_2 j_3}, \text{ and } 0 \text{ otherwise}, \quad (2.12)$$

and acting as the identity everywhere else in $\mathcal{H}(\Gamma, \Phi)$. It is straightforward to see that these operators are mutually commuting projectors, so that

$$Q_v^2 = Q_v, \quad Q_v Q_{v'} = Q_{v'} Q_v \quad \text{for } v, v' \in \Gamma_0.$$

Furthermore, since $Q_v$ is a delta function, it is immediate that these operators are Hermitian with respect to the inner product $\langle \cdot \mid \cdot \rangle$ from (2.10).

If $g \in G$ and $p$ is a plaquette in $\Sigma \setminus \Gamma$, then we write $g.\delta p$ for the coloring that sends oriented edges of $\delta p$ to $g$, their opposites to $-g$, and other edges to 0.

For $g \in G \setminus \mathcal{X}$, let $\Phi$ be an admissible $G$-coloring such that $\Phi + g.\delta p$ is also admissible. Let $s \in I_g$ and define the operator $B^g_p : \mathcal{H}(\Gamma, \Phi) \rightarrow \mathcal{H}(\Gamma, \Phi + g.\delta p)$ that acts on the boundary edges of the plaquette $p$ and is given on a plaquette with $n$ sides by:

$$B^g_p \begin{pmatrix} j_1 \\ \cdots \\ j_n \end{pmatrix} := \sum_{j_1', \ldots, j_n'} \sum_{A_1, \ldots, A_n} \sum_{c_1, \ldots, c_n} \prod_{i=1}^n d(j_i') \sum_{j_1', \ldots, j_n'} N^g_{j_1 j_2, k_1 k_2, \ldots, k_{n-1}} (c_i A_i C_i).$$

Note that the conditions on $6j$ symbols in Section 2.1 ensure that $\deg(j_i') = \deg(j_i) - \deg(s)$; otherwise, the right-hand side is zero in the above formula.
Then, we define $G$-indexed plaquette operators

$$B_{p}^g = \sum_{s \in I_g} b(s)B_{p}^s : \mathcal{H}(\Gamma, \Phi) \to \mathcal{H}(\Gamma, \Phi + g \delta p).$$ \hfill (2.13)

**Proposition 2.1.** If $g_1$, $g_2$, and $g_1 + g_2$ are generic, then

$$B_{p}^{g_1}B_{p}^{g_2} = B_{p}^{g_1+g_2}.$$ \hfill (2.13)

Moreover, if $g_1$ and $g_2$ are generic, then $B_{p}^{g_1}B_{p}^{-g_1} = B_{p}^{g_2}B_{p}^{-g_2}.$

**Proof.** This is a straightforward generalization of [24, Proposition 3.1]. \hfill □

**Proposition 2.2.** Let $p$ and $p'$ be two plaquettes that share exactly one common edge $e_1$ in a state $|\Gamma, \sigma\rangle$ as in (2.14). Then, $B_{p'}^{t}B_{p}^{s} = B_{p}^{t}B_{p'}^{s}.$

\begin{equation}
|\Gamma, \sigma\rangle := \ldots \quad p \quad \ldots \quad p' \quad \ldots \hfill (2.14)
\end{equation}

**Proof.** This is a straightforward generalization of [24, Proposition 3.2]. \hfill □

The next result follows easily from the previous proposition and is a generalization of [24, Corollary III.3].

**Corollary 2.3.** Let $p$ and $p'$ be two plaquettes. Then, $B_{p'}^{t}B_{p}^{s} = B_{p}^{s}B_{p'}^{t}.$

**Proof.** If $p$ and $p'$ are identical, or if they share no edges, the result is obvious. The case that they share exactly one edge is Proposition 2.2. \hfill □

**Proposition 2.4.** The Hermitian adjoint of $B_{p}^{s}$ is $B_{p}^{s\ast}$. In particular,

$$(B_{p}^{g})^{\dagger} = B_{p}^{-g}.$$  

**Proof.** This is a straightforward generalization of [24, Proposition III.4]. \hfill □

Define the plaquette operator

$$B_{p} := B_{p}^{g}B_{p}^{-g} : \mathcal{H}(\Gamma, \Phi) \to \mathcal{H}(\Gamma, \Phi).$$ \hfill (2.15)

* A priori, this operator depends on $g$, but Proposition 2.1 and the smallness of the singular set $\lambda'$ guarantees that the operator is independent of $g$.  

The next result follows from the facts above. See the arguments in [24, Theorem III.6] for more details.

**Theorem 2.5.** The plaquette operators have the following properties.

1. The plaquette operators are projectors $B_p^2 = B_p$.
2. The plaquette operators are mutually commuting:
   $$B_p B_{p'} = B_{p'} B_p.$$  
3. The operators $B_p$ are Hermitian with respect to the indefinite norm (2.10).

It is also straightforward to see that the plaquette operators commute with the vertex operators from (2.12), so that we can define a local commuting projector Hamiltonian

$$H = - \sum_p B_p - \sum_v Q_v.$$  

We can shift this Hamiltonian so that the ground state has zero energy.

$$H = \sum_p (1 - B_p) + \sum_v (1 - Q_v).$$  

**Theorem 2.6.** The Hamiltonian from (2.17) has the following properties.

1. The Hamiltonian is Hermitian with respect to the indefinite inner product.
2. The Hamiltonian is gapped and local.
3. The spectrum of $H$ is $\mathbb{Z}_{\geq 0}$, with ground state corresponding to the simultaneous $+1$-eigenspace of the operators $B_p$ and $Q_v$ over all vertices $v$ and plaquettes $p$.

**Proof.** The first claim follows from Theorem 2.5. Since the Hamiltonian is constructed from mutually commuting projectors acting locally on the graph $\Gamma$, the second and third claims follow. □

\section{Sesquilinear Pivotal Categories}

\subsection{Definitions}

A tensor $k$-category is a tensor category $\mathcal{C}$ such that its hom-sets are left $k$-modules, the composition and tensor product of morphisms are $k$-bilinear, and the canonical $k$-algebra map $k \to \text{End}_{\mathcal{C}}(\mathbb{1}), k \mapsto k \text{Id}_\mathbb{1}$ is an isomorphism (where $\mathbb{1}$ is the unit object). Note that in this definition, $\mathcal{C}$ need not be abelian, for example, we do not expect the category $C^{\text{Herm}}$ in Section 4.3 to be abelian.

**Definition 3.1.** Let $C$ be a tensor $k$-category. A not necessarily involutive conjugation assigns for each $V, W \in C$ a bijection $f \mapsto f^\dagger$ between $\text{Hom}(V, W)$ and $\text{Hom}(W, V)$ such that

$$\mu f + \lambda g = \mu f^\dagger + \lambda g^\dagger, \quad (f \otimes g)^\dagger = f^\dagger \otimes g^\dagger, \quad (f \circ g)^\dagger = g^\dagger \circ f^\dagger.$$  

(3.1)
These relations imply $\text{Id}_V^\dagger = \text{Id}_V$.

**Definition 3.2.** A tensor category is **pivotal** if it has dual objects and duality morphisms

\[ \text{coev}_V : 1 \to V \otimes V^* , \quad \text{ev}_V : V^* \otimes V \to 1, \quad \text{coev}_V : 1 \to V^* \otimes V \quad \text{and} \quad \text{ev}_V : V \otimes V^* \to 1, \]

which satisfy compatibility conditions (see, e.g., [3, 21]). In particular, it has natural isomorphisms

\[ \{ \phi_V = (\text{ev}_V \otimes \text{id}_{V^*})(\text{id}_V \otimes \text{coev}_V^*) : V \to V^{**} \} \]

called the pivotal structure.

**Definition 3.3.** A **sesquilinear pivotal category** is a pivotal category $\mathcal{C}$ equipped with a not necessarily involutive conjugation satisfying:

\[ \overset{\to}{\text{coev}}^\dagger_{V, W} = \overset{\to}{\text{ev}}_{V, W}, \quad \overset{\to}{\text{ev}}^\dagger_{V, W} = \overset{\to}{\text{coev}}_{V, W}, \quad (3.2) \]

for any object $V$ of $\mathcal{C}$. These identities imply that $\phi_V^\dagger = \phi_V^{-1}$.

Recall that the pivotal category $\mathcal{C}$ is ribbon if it has a braiding $(c_{V,W})_{V,W \in \mathcal{C}}$ that satisfies for any object $V \in \mathcal{C}$,

\[ (\text{Id} \otimes \overset{\to}{\text{ev}}_V)(c_{V,V} \otimes \text{Id})(\text{Id} \otimes \text{coev}_V) = (\overset{\to}{\text{ev}}_V \otimes \text{Id})(\text{Id} \otimes c_{V,V})(\overset{\to}{\text{coev}}_V \otimes \text{Id}). \]

This endomorphism of $V$ is called the twist $\vartheta_V$.

**Definition 3.4.** A **sesquilinear ribbon category** is a sesquilinear pivotal category $\mathcal{C}$ that is ribbon with a braiding satisfying:

\[ c_{V,W}^\dagger = c_{V,W}^{-1}, \quad (3.3) \]

for any objects $V, W$ of $\mathcal{C}$. This identity implies that $\vartheta_V^\dagger = \vartheta_V^{-1}$.

**Definition 3.5.** A sesquilinear pivotal (or ribbon) category is called a **Hermitian pivotal (or ribbon) category** if for any morphism $f$ in $\mathcal{C}$, one has

\[ (f^\dagger)^\dagger = f. \quad (3.4) \]

### 3.2 Sesquilinear super vector spaces

Let us call a sesquilinear super vector space a finite-dimensional super $k$-vector space $V$ equipped with a nondegenerate graded sesquilinear form $(\cdot | \cdot)_V$, such that for any $v_1, v_2, v_3 \in V$ and any
\[ \lambda \in k, \]
\[ (\mu v_1 + \lambda v_2 \mid v_3) = \overline{\mu}(v_1 \mid v_3) + \overline{\lambda}(v_2 \mid v_3) \text{ and } (v_1 \mid \mu v_2 + \lambda v_3) = \mu(v_1 \mid v_2) + \lambda(v_1 \mid v_3), \]
and \((v_1 \mid v_2) = 0\) if \(v_1, v_2\) are homogeneous with opposite parity.

Let \(k^\dagger\text{-SVect}\) be the category of sesquilinear super vector spaces with even \(k\)-linear maps. Let \(V\) and \(W\) be objects in \(k^\dagger\text{-SVect}\) with sesquilinear forms \((\cdot \mid \cdot)_V\) and \((\cdot \mid \cdot)_W\), respectively. Define \((\cdot \mid \cdot)_p\) on \(V \otimes W\) by
\[ (v \otimes w \mid v' \otimes w')_p = (-1)^{|v'||w|}(v \mid v')_V(w \mid w')_W \tag{3.5} \]
for \(v, v' \in V\) and \(w, w' \in W\), and \(|x|\) means the degree of an element \(x\). Then, \(V \otimes W\) is an object of \(k^\dagger\text{-SVect}\) with the sesquilinear form \((\cdot \mid \cdot)_p\). Any basis \(\{e_i\}_i\) of \(V\) has a right sesquilinear dual basis \(\{e'_i\}_i\) such that \((e_i \mid e'_j)_V = \delta_{ij}\). Then, the dual vector space \(V^* = \text{Hom}_k(V, k)\) is equipped with the sesquilinear form given by
\[ (\varphi \mid \psi)_{V^*} = \sum_i \overline{\varphi(e_i)}\psi(e'_i). \tag{3.6} \]
This does not depend on the choice of \(\{e_i\}\). The category \(k^\dagger\text{-SVect}\) is pivotal with duality morphisms given by
\[ \text{coev}_V : k \to V \otimes V^*, \quad 1 \mapsto \sum_i v_i \otimes v_i^*, \quad \text{ev}_V : V^* \otimes V \to k, \quad f \otimes v \mapsto f(v), \]
\[ \text{coev}_V : k \to V^* \otimes V, \quad 1 \mapsto \sum_i (-1)^{|v_i||v_i^*|} v_i \otimes v_i, \quad \text{ev}_V : V \otimes V^* \to k, \quad v \otimes f \mapsto (-1)^{|f||v|}f(v). \]
The dagger is the right adjoint for the sesquilinear form. That is, given \(f : V \to W\), then \(f^\dagger : W \to V\) is the unique map defined by
\[ (f(v) \mid w)_W = (v \mid f^\dagger(w))_V \tag{3.7} \]
for all \((v, w) \in V \times W\). Equipped with these maps, we have the following result.

**Proposition 3.6.** The category \(k^\dagger\text{-SVect}\) is a sesquilinear pivotal category.

We note that if \(V, W\) are sesquilinear spaces, \(\text{Hom}_{k^\dagger\text{-SVect}}(V, W) = \text{Hom}_k(V, W)_0\) is the space of even linear maps. The dagger admits a natural extension for all homogeneous linear maps \(f\) by
\[ (f(v) \mid w)_W = (-1)^{|f||v|}(v \mid f^\dagger(w))_V. \tag{3.8} \]
With this definition, one easily check that the adjoint of a composition is given by
\[ (fg)^\dagger = (-1)^{|f||g|}g^\dagger f^\dagger. \tag{3.9} \]
We conclude with some considerations about symmetry.

**Definition 3.7.** A sesquilinear form on \(W\) is called **super Hermitian** if it has super Hermitian symmetry: for any \(v_1, v_2 \in W\), \((v_2 \mid v_1) = (-1)^{|v_1||v_2|}(v_1 \mid v_2)\). It is **Hermitian** if for any \(v_1, v_2 \in W\), \((v_2 \mid v_1) = \overline{(v_1 \mid v_2)}\).
Lemma 3.8. Let $V$ and $W$ be sesquilinear super vector spaces.

1. If the forms on $V$ and $W$ are super Hermitian, then the dagger is involutive, meaning that for any homogeneous linear map $f : V \rightarrow W$, $f^{††} = f$.

2. If the forms on $V$ and $W$ are Hermitian, then the dagger is superinvolutive, meaning that for any homogeneous linear map $f : V \rightarrow W$, $f^{††} = (-1)^{|f|} f$.

Proof. Let $s = 0$ if the form is Hermitian and $s = 1$ if the form is super Hermitian, so that $(x|y) = (-1)^s |x||y|$. For $f : V \rightarrow W$, one has

$$\forall (v, w) \in V \times W, \ (f(v)|w)_W = (-1)^{|f||v|}(v|f^{†}(w))_V = (-1)^{|f||v|}(-1)^{|v|}(f^{††}(w)|v)_V$$

$$= (-1)^{|f||v|}(-1)^{|v|}(-1)^{|f||w|}(w|f^{††}(v))_W = (-1)^{|f||w|}(-1)^{|v|}(-1)^{|f||w|}(-1)^{|w|}(f^{††}(v)|w)_W.$$ But if this scalar is nonzero, then $|f| + |v| + |w| = 0$. Then, $(-1)^{|f||w|}(-1)^{|f||w|} = (-1)^{|f|}$ and $(-1)^{|v|}(-1)^{|w|} = (-1)^{|f|}$. So, overall we get $f^{††} = \begin{cases} (-1)^{|f|} f & \text{if } s = 0 \\ f & \text{if } s = 1 \end{cases}$. □

3.3 | Hopf superalgebras

Let $H$ be a ribbon Hopf superalgebra over $k$ with multiplication $m : H \otimes H \rightarrow H$, unit $\eta : k \rightarrow H$, coproduct $\Delta : H \rightarrow H \otimes H$, counit $\varepsilon : H \rightarrow k$, and antipode $S : H \rightarrow H$. Recall that in a Hopf superalgebra, the antipode satisfies

$$S(xy) = (-1)^{|x||y|}S(y)S(x). \quad (3.10)$$

We denote by $R, \theta, \text{ and } g$ the R-matrix, twist, and pivot of $H$, respectively. All these structure maps and elements are assumed to be of even parity. Strictly speaking, many of the examples we consider in Section 7 are not ribbon Hopf algebras, but rather Hopf algebras equipped with operators $R, \theta, g$ acting on the category $H$-$\text{mod}$ of finite-dimensional $H$-modules and satisfying the requisite relations as operators. Put another way, these are not elements of $H$, rather they lie in an appropriate completion of $H$, giving well-defined operators on the category of modules. In what follows, by abuse of notation, we sometimes refer to elements in this more general situation.

Assume that $H$ is equipped with anti-superautomorphism that is a bijective map $\dagger : H \rightarrow H$ that is antilinear, anti-superalgebra that also is a coalgebra antisupermorphism $^\dagger$. Thus, we assume for any $a \in k$ and $x, y \in H$,

$$(ax)^\dagger = \bar{a}x^\dagger, \quad (xy)^\dagger = (-1)^{|x||y|}y^\dagger x^\dagger \text{ and } \Delta(x^\dagger) = (\dagger \otimes \dagger)(\tau(\Delta x)), \quad (3.11)$$

where $\tau : H \otimes H \rightarrow H \otimes H$ is the super flip map given by $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$. We let $R_{21} = \tau(R)$. Finally, we also assume that

$$(\dagger \otimes \dagger)(R) = \tau(R^{-1}) \text{ and } g^\dagger = g^{-1}. \quad (3.12)$$

We can deduce that for any $x \in H$, we have $\varepsilon(x^\dagger) = \varepsilon(x), S(x^\dagger) = S(x)^\dagger, \theta^\dagger = \theta^{-1}$ and $1^\dagger = 1 \in H$. For the second fact, see [54].

$^\dagger$In the non-supercase, the convention of the coproduct corresponds to the notion of a twisted star Hopf algebra in [7].
Recall that the twist on an $H$-module $V$ is the endomorphism $\theta_V$ given by the action of $\theta^{-1}$. The dual of an $H$-module $V$ is the superspace $V^*$ equipped with the following action

$$(x \cdot \phi)(v) := (-1)^{|x||\phi|} \phi(S(x)v).$$

(3.13)

The category of finite-dimensional $H$-modules with even morphisms is known to be a pivotal $k$-tensor category (see, e.g., [6, p.163]) where the duality morphisms for an $H$-module $V$ are given by

$$\text{coev}_V : k \to V \otimes V^*, \quad 1 \mapsto \sum_i v_i \otimes v_i^*,$$

$$\text{ev}_V : V^* \otimes V \to k, \quad f \otimes v \mapsto f(v),$$

$$\text{coev}_V : k \to V^* \otimes V, \quad 1 \mapsto \sum (-1)^{|v_i|} v_i^* \otimes g^{-1} v_i,$$

$$\text{ev}_V : V \otimes V^* \to k, \quad v \otimes f \mapsto (-1)^{|f||v|} f(gv).$$

(3.14)

A sesquilinear $H$-module is a finite-dimensional $H$-module $V$ equipped with a nondegenerate sesquilinear form $(\cdot | \cdot)_V$ compatible with $\dagger$:

$$(uv | v')_V = (-1)^{|u||v'|}(u^\dagger v')_V,$$

(3.15)

where $u \in H$ and $v, v' \in V$. Define $H^\dagger$-mod to be the category of sesquilinear $H$-modules with even $H$-module morphisms. The dual of a sesquilinear $H$-module inherits the sesquilinear form given in (3.6) that can be shown to be compatible with $\dagger$.

Let $V, W$ be modules in $H^\dagger$-mod. Define $(\cdot | \cdot)_p$ on $V \otimes W$ by

$$(v \otimes w | v' \otimes w')_p = (-1)^{|w'||w|}(v | v')_V(w | w')_W$$

(3.16)

for $v, v' \in V$ and $w, w' \in W$. The tensor product $V \otimes W$ can be given two different sesquilinear structures. The first one is canonical and is given by defining

$$(v_1 \otimes w_1 | v_2 \otimes w_2)_W = (v_1 \otimes w_1 | R(v_2 \otimes w_2))_p,$$

(3.17)

where $(\cdot | \cdot)_p$ is defined in (3.16). This sesquilinear form is not compatible with the pivotal structure given in (3.14).

The second one depends on the choice of a half twist which we now recall from [23]. A half twist $\sqrt{\theta}$ for $H$-mod is a natural isomorphism of the identity functor whose square is the twist such that $\sqrt{\theta}_V^* = (\sqrt{\theta}_V)^*$. We now assume that $H$-mod has a fixed choice of half twist. By [23, Proposition 4.12], a half twist exists if $k$ is an algebraically closed field of characteristic 0. For objects $W_1, W_2$ of $H$-mod, define the isomorphism $X : W_1 \otimes W_2 \to W_2 \otimes W_1$ by

$$X_{W_1, W_2} = (\sqrt{\theta}_{W_2 \otimes W_1})^{-1} c_{W_1, W_2} (\sqrt{\theta}_{W_1} \otimes \sqrt{\theta}_{W_2}).$$

(3.18)

The forms given in (3.16) and (3.17) are not compatible with $\dagger$ but we have the following modification.

**Proposition 3.9.** Let $W_1, W_2$ be objects of $H^\dagger$-mod. Then, $W_1 \otimes W_2$ is a sesquilinear $H$-module with sesquilinear structure given by

$$(v | v') = (v | \tau(X_{W_1 \otimes W_2} v'))_p,$$

where $\tau$ is the twist.
where \( \tau : W_2 \otimes W_1 \to W_1 \otimes W_2 \) is the super flip map given by \( \tau(v \otimes w) = (-1)^{|w||v|}w \otimes v \).

This defines a monoidal structure on \( H^\dagger\)-mod.

Proof. This proof is reproduced from [23], but we now include all the relevant signs arising from super considerations. Let \( u \in H \) and write \( \Delta(u) = u_1 \otimes u_2 \). Consider \( v, v' \in W_1 \otimes W_2 \) with \( v = v_1 \otimes v_2 \), \( v' = v'_1 \otimes v'_2 \), and \( v'' = \mathbb{X}_{W_1 \otimes W_2}(v') = v''_1 \otimes v''_2 \in W_2 \otimes W_1 \). Note that we have omitted all summation symbols and will continue to do so throughout the course of the proof. Then, we have

\[
(\nu v | v') = (\nu v | \tau(X_{W_1 \otimes W_2} v'))_p = (\nu v | \tau(v''))_p \\
= (-1)^{|u_2||v_1|}(-1)^{|v_2'||v_1'||}(u_1 v_1 \otimes u_2 v_2 | v_1'' \otimes v_2'')_p \\
= (-1)^{|u_2||v_1|}(-1)^{|v''_2||v''_1|}(u_1 v_1 | v''_1 v_1') w_1 (u_2 v_2 | v''_2 v_2') w_2 \\
= (-1)^{|u_2||v_1|}(-1)^{|v''_2||v''_1|}(-1)^{|u_2|v_2||v''_1|}(-1)^{|u_1|v_1 + |v_2|v_2'} (v_1 | u_1 v''_1 v_1') w_1 (v_2 | u_2 v''_2 v_2') w_2 \\
= (-1)^{|u||v|}(-1)^{|u_1|v_2|}(-1)^{|v''_2||v''_1|}(-1)^{|u_2|v_2||v''_1|}(-1)^{|u_1|v''_1|v_2|} (v_1 \otimes v_2 | u_1 v''_1 \otimes u_2 v''_2)_p \\
= (-1)^{|u||v|}(-1)^{|v''_2||v''_1|+|u_2|v_2|} (v_1 \otimes v_2 | u_1 v''_1 \otimes u_2 v''_2)_p \\
= (-1)^{|u||v|}(-1)^{|v''_2||v''_1|+|u_2|v_2|} (v_1 \otimes v_2 | \tau(u_2 v''_2 \otimes u_1 v''_1))_p \\
= (-1)^{|u||v|}(|v_1 \otimes v_2 | \tau((-1)^{|u_1|v_2} u_2 \otimes u_1 v''_1))(v''_2 \otimes v''_1))_p \\
= (-1)^{|u||v|}(|v_1 \otimes v_2 | \tau(\Delta(u^\dagger) X_{W_1 \otimes W_2} (v'_1 \otimes v'_2)))_p \\
= (-1)^{|u||v|}(|v | \tau o X_{W_1 \otimes W_2} (u^\dagger v'))_p \\
= (-1)^{|u||v|}(|v | u^\dagger v'),
\]

where we used \( \Delta(u^\dagger) = (-1)^{|u_2|u_1} u_2^\dagger \otimes u_1^\dagger \). Hence, we have defined a compatible sesquilinear form on \( W_1 \otimes W_2 \).

To check that this defines a tensor product on \( H^\dagger\)-mod, we need to check the unital and associativity axioms, which, in turn, follow from the properties of \( \mathbb{X} \). The unital axiom follows from the fact that

\[
(W \sim W \otimes \mathbb{X} \rightarrow \mathbb{I} \otimes W \sim W) = \text{Id}_W = (W \sim \mathbb{I} \otimes W \rightarrow W \otimes \mathbb{I} \sim W),
\]

which is a direct consequence of the definition of \( \mathbb{X} \). The associativity axiom follows from the equality

\[
X_{V' \otimes V''}(X_{V', V''} \otimes \text{Id}_{V''}) = X_{V', V''} \otimes \text{Id}_{V'} : V \otimes V' \otimes V'' \rightarrow V'' \otimes V' \otimes V,
\]

which is proved in [23, Lemma 4.13]. \(\square\)
Theorem 3.10. $H^\dagger$-mod equipped with a choice of half twist $\sqrt{\theta}$ is a sesquilinear pivotal $\kappa$-category with a not necessarily involutive conjugation determined by:

$$\forall f \in \text{Hom}_{H^\dagger}\text{-mod}(V, W), \forall v \in V, \forall w \in W, \ (f(v)|w)_W = (v|f^\dagger(w))_V.$$  

Furthermore, $c_{V, W} = c_{V, W}^{-1}$ where $c_{V, W}$ is the braiding defined by $c_{V, W}(v \otimes w) = \tau(R(v \otimes w))^\dagger$.

Proof. To check that $H^\dagger$-mod is a sesquilinear pivotal category, we need to show the relations in (3.2) hold.

We first compute the adjoint of $\longrightarrow\text{ev}_V$. To do this, we fix a basis $\{e_i\}_i$ of $V$ and let $\{e'_i\}_i$ be the right sesquilinear dual basis such that $(e_i | e'_j)_V = \delta^j_i$. Then, for all basis elements $e_j$ and $e^*_k$, we have

$$\left(e^*_k \otimes e_j, \text{coev}_V (1)\right)_{V*} = \left(e^*_k \otimes e_j, \tau X_{V*, V} \text{coev}_V (1)\right)_V = \left(e^*_k \otimes e_j, \tau \text{coev}_V (1)\right)_V$$

where

$$\sum_i (-1)^{|e'_i| + |e_j|} e'_i \otimes e^*_i = \sum_i (-1)^{|e'_i| + |e_j|} e'_i \Delta_{i,j} = \sum_i (-1)^{|e'_i| + |e_j|} \delta_{k,l} \delta_{i,l} \delta_{i,j} = \delta_{k,j}$$

where the second equality follows from [23, Lemma 4.13]. Hence, $\text{coev}_V = \text{ev}_V^\dagger$.

Next, we compute the adjoint of $\text{coev}_V$. To do this, we use that for any vector $v \in V$, $(v, e'_i)_V = e^*_i(v)$. Then, for all basis elements $e_j$ and $e^*_k$, we have

$$\left(e_j \otimes e^*_k, \text{coev}_V (1)\right)_{V*} = \left(e_j \otimes e^*_k, \tau X_{V*, V} \text{coev}_V (1)\right)_V = \left(e_j \otimes e^*_k, \tau \text{coev}_V (1)\right)_V$$

where

$$\sum_i (-1)^{|e'_i| + |e_j|} (e^*_i \otimes e'_i) = \sum_i (-1)^{|e'_i| + |e_j|} (e^*_i \otimes e'_i) \delta_{k,l} \delta_{i,l} = \delta_{k,j}$$

where in the second equality we use [23, Lemma 4.13], in the fourth equality, we use the fact that the pivotal element $g$ is group-like so is necessarily of degree zero, and in the second to last equality, we use the fact that $|e^*_k| = |e_j|$ if $e^*_k(g e_j)$ is nonzero. So, $\text{ev}_V = \text{coev}_V$.

Next, using that the evaluations are determined by the zigzag relations, we have

$$\text{Id}_V = \left(\text{Id}_V \otimes \text{ev}_V (\text{coev}_V \otimes \text{Id}_V)\right) = \left(\text{coev}_V \otimes \text{Id}_V (\text{Id}_V \otimes \text{coev}_V)\right)^\dagger$$

\footnote{If, in addition, $(f^\dagger)^\dagger = f$, then these conditions imply that $H^\dagger$-mod is a Hermitian ribbon category in the sense of [23, Definition 3.1].}
so that coev\(_V = \text{ev} \_V\) and similarly, coev\(_W = \text{ev} \_W\).

Finally, we check that \(c_{V,W}^\dagger = c_{V,W}^{-1}\). To do this, we will need the fact that

\[
X_{V,W}c_{V,W}^{-1} = c_{W,V}X_{W,V}.
\]

This follows since \(X\) commutes with \(\Delta(\sqrt{\theta})\), \(\sqrt{\theta} \otimes \sqrt{\theta}\), and \(X\), and because by the definition of \(X\), we have

\[
c_{V,W}^{-1} = \sqrt{\theta} V \otimes \sqrt{\theta} W X_{V,W}^{-1} (\sqrt{\theta} W \otimes V)^{-1}.
\]

Then,

\[
\left( v \otimes w, c_{V,W}^{-1}(w' \otimes v') \right)_{V \otimes W} = (v \otimes w, \tau X_{V,W}c_{V,W}^{-1}(w' \otimes v'))_p
\]

\[
= (v \otimes w, \tau c_{W,V}^{-1}X_{W,V}(w' \otimes v'))_p
\]

\[
= (v \otimes w, \tau (R^{-1}\tau)X_{W,V}(w' \otimes v'))_p
\]

\[
= (v \otimes w, R^{-1}_{21}X_{W,V}(w' \otimes v'))_p
\]

\[
= (v \otimes w, R^{\dagger} \otimes \dagger X_{W,V}(w' \otimes v'))_p
\]

\[
= (R(v \otimes w), X_{W,V}(w' \otimes v'))_p
\]

\[
= (\tau R(v \otimes w), \tau X_{V,W}(w' \otimes v'))_p
\]

\[
= (c_{V,W}(v \otimes w), w' \otimes v')_{V \otimes W},
\]

where \(R^{-1}\tau\) and \(R^{-1}_{21} = \tau (R)^{-1}\) are the actions by the corresponding elements.

Finally, we note by assumption that \(g^\dagger = g^{-1}\). \(\square\)

We now consider existence and symmetry of compatible sesquilinear forms. If \(V\) is a \(k\)-vector space, let \(\overline{V} = \{ \overline{v} : v \in V \}\) be the same additive group with antilinear scalar multiplication, that is, for any \(k \in \mathbb{k}, v \in V\), one has \(k \overline{v} = \overline{k v}\). Let \(x \mapsto \dagger x\) be the inverse isomorphism of \(\dagger : H \to H\).

Then, if \(V\) is an \(H\)-module, we can define a module \(\widetilde{V}^\dagger\) with underlying vector space \(\overline{V}\) and action given by \(\forall x \in H, \forall v \in V\),

\[
x.\overline{v} = \dagger S(x).v.
\]

To see that this is a well-defined action, observe

\[
xy.\overline{v} = \dagger S(xy).v = \dagger S(x).y.\overline{v} = (\tau^{-1}xy.\overline{v}) = (\tau^{-1}x.\overline{v}).y = x. (y.\overline{v}),
\]

where we used that \((\tau^{-1}xy) = (-1)^{|x||y|} x. y.\overline{v} = x. (y.\overline{v}).

**Lemma 3.11.** Let \(V\) be an \(H\)-module. Then, there exists a bijection between the set of compatible sesquilinear forms \(f\) on \(V\) and the set of even isomorphisms of \(H\)-modules \(\varphi : \widetilde{V}^\dagger \rightarrow V^*\).
Proof. The sesquilinear form associated to an isomorphism \( \varphi : \tilde{V}^\dagger \sim \to V^* \) is given by

\[
\begin{align*}
f : V \times V & \longrightarrow \tilde{V}^\dagger \otimes V \xrightarrow{\varphi \otimes \text{id}} V^* \otimes V \xrightarrow{\text{ev}} \mathbb{k},
\end{align*}
\]

where we note that the first \( \times \) above is not a tensor product, and \( \text{id} \) is not linear. The compatibility follows from:

\[
\begin{align*}
f(x, v_1, v_2) :&= \text{ev} (\varphi(x \cdot v_1) \otimes v_2) = \text{ev} (\varphi(S^{-1}(x^\dagger) \cdot v_1) \otimes v_2) \\
&\overset{(3.19)}{=} \text{ev} (\varphi(S^{-1}(x^\dagger) \cdot \tilde{v}_1) \otimes v_2) \\
&\overset{\varphi \text{ intertwines}}{=} \text{ev} (S^{-1}(x^\dagger) \cdot \varphi(\tilde{v}_1) \otimes v_2) \\
&\overset{\text{def of ev}}{=} [S^{-1}(x^\dagger) \cdot \varphi(\tilde{v}_1)](v_2) \\
&\overset{(3.13)}{=} (-1)^{|x||v_1|} [\varphi(\tilde{v}_1)](x^\dagger \cdot v_2) \\
&= (-1)^{|x||v_1|} \text{ev} (\varphi(\tilde{v}_1) \otimes x^\dagger \cdot v_2) = (-1)^{|x||v_1|} f(v_1 \cdot x^\dagger \cdot v_2),
\end{align*}
\]

where we used that \( S \) and its inverse commute with \( ^\dagger \) and its inverse, and that \( |\varphi(v_1)| = |v_1| \) since \( \varphi \) is an even map. Conversely, given a sesquilinear form \( f \), \( \varphi \) is defined by \( \varphi(v_1) = f(v_1 \cdot \cdot \cdot) \). \( \square \)

**Lemma 3.12.** Assume that \( \mathbb{k} = \mathbb{C} \) and \( H \) has a superinvolutive dagger. If \( V \) is an irreducible sesquilinear module, then the form on \( V \) is proportional to a Hermitian form.

Proof. Let \( V_0 \) and \( H_0 \) be the degree zero subspaces of \( V \) and \( H \), respectively. First note that \( V_0 \) is an irreducible \( H_0 \)-module because any \( H_0 \)-submodule \( W_0 \) of \( V_0 \) would generate an \( H \)-submodule of \( V \), namely, \( H \cdot V_0 \). Then one can reproduce the proof of [23, Lemma 4.3] to show that up to rescaling, we can assume that the restriction of \( (\cdot | \cdot) \) on \( H_0 \times H_0 \) is Hermitian. Similarly, we can assume that the form on \( H_1 \times H_1 \) is \( \lambda \) times a Hermitian form. Next, since \( V_0 \) is not an \( H \)-submodule, there exists \( v_0 \in V_0, v_1 \in V_1, h_1 \in H_1 \) such that

\[
1 = (h_1 v_0 | v_1) = (v_0 | h_1 v_1) = (h_1 v_1 | v_0) = -(v_1 | h_1^\dagger v_0) = -\lambda^{-1}(h_1^\dagger v_0 | v_1) = \overline{\lambda} \lambda^{-1},
\]

so the form is also Hermitian on \( V_1 \). \( \square \)

A module \( W \) of \( H^\dagger \)-mod is called Hermitian if its sesquilinear form is Hermitian. The orthogonal direct sum of two sesquilinear modules is a sesquilinear module. Reciprocally, we say that a sesquilinear module is split if it is a direct sum of sesquilinear modules with the induced sesquilinear forms. We say that a sesquilinear module is semisplit if all its summands can be equipped with compatible sesquilinear forms (which are not necessarily the restrictions of the original sesquilinear form).
Proposition 3.13. Let $W_1, W_2$ be Hermitian modules of $H^\dagger$-mod that are indecomposable in $H$-mod. If $W_1 \otimes W_2$ is semisplit, then $W_1 \otimes W_2$ is Hermitian.

Proof. Consider a direct sum decomposition $W_1 \otimes W_2 = \bigoplus_i V_i$ with $V_i$ indecomposable and the dual decomposition $V_i^\dagger = \{ v' \in W_1 \otimes W_2 | (v', \bigoplus_{j \neq i} V_j) = \{0\} \}$. Fix a factor $V_i$. By Lemma 3.11, since $W_1 \otimes W_2$ is semisplit, we have $V_i^\dagger \simeq V_i^*$. Similarly, since $(\cdot, \cdot)$ restricts to a nondegenerate sesquilinear pairing on $V_i \times V_i$, we have $V_i^* \simeq V_i^\dagger$ where the isomorphism is given by

$V_i \rightarrow V_i^*$

$\tilde{v} \mapsto (\cdot, v)\|_{V_i}$.

So, in particular, $V_i \simeq V_i'$ and the half twist has the same value $\langle \sqrt{\theta}_{V_i} \rangle = \langle \sqrt{\theta}_{V_i'} \rangle$, which we denote by $\sqrt{\theta}_0 \in \mathbb{C}$. Let us call $\sqrt{\theta}_1 = \langle \sqrt{\theta}_{W_1} \rangle$ and $\sqrt{\theta}_2 = \langle \sqrt{\theta}_{W_2} \rangle$ and $\theta_j = \sqrt{\theta}_j^2$. Let $(v, v') \in V_i \times V_i'$. Then,

$$(v'|v) = (v'|\tau X(v))_p$$

$$= (v'|\tau \sqrt{\theta}_0^{-1} \sqrt{\theta}_1 \sqrt{\theta}_2 (v' | \sqrt{\theta}_0 \Delta^p(\theta))R(sqr(\theta_1^{-1} \theta^{-1}) \otimes \sqrt{\theta}_2^{-1} \theta^{-1})(v))_p$$

$$= \sqrt{\theta}_0^{-1} \sqrt{\theta}_1 \sqrt{\theta}_2 \left( (sqr(\theta_1^{-1} \theta^{-1})) \otimes \sqrt{\theta}_2^{-1} (\theta^{-1})^\dagger \right) R_{21}^{-1} \sqrt{\theta}_0 \Delta(\theta^\dagger))v'|v)_p$$

$$= \left( \sqrt{\theta}_0 \sqrt{\theta}_1 \sqrt{\theta}_2^{-1} (sqr(\theta_1 \theta) \otimes \sqrt{\theta}_2 \theta) \right) R_{21}^{-1} \sqrt{\theta}_0^{-1} \Delta(\theta^{-1}))v'|v)_p$$

$$= \left( \tau(\sqrt{\theta}_0^{-1} \sqrt{\theta}_1 \sqrt{\theta}_2^{-1} R_{21}^{-1} \tau \sqrt{\theta}_0 \Delta(\theta^\dagger))v'|v)_p$$

$$= (\tau X(v')|v)_p = (v|\tau X(v'))_p = (v|v').$$

In the above equalities, the elements $\theta, \theta^{-1}$, and so on, are all acting on modules. Note that the fifth equality above follows from $\sqrt{\theta}_j = +\sqrt{\theta}_j^{-1}$. □

4 | RELATIVE HERMITIAN-MODULAR AND HERMITIAN-SPHERICAL CATEGORIES

In this section, we recall notions related to generically semisimple categories and consider Hermitian structures on these categories.
4.1 Modified trace in a Hermitian pivotal category

4.1.1 Modified traces on projective modules

Let $C$ be a pivotal $k$-category and let $\text{Proj}$ be the full subcategory of $C$ consisting of projective objects.

For any objects $V, W$ of $C$, and any endomorphism $f$ of $V \otimes W$, set

$$\text{ptr}_L(f) = (\text{ev}_V \otimes \text{Id}_W) \circ (\text{Id}_V \otimes f) \circ (\text{coev}_V \otimes \text{Id}_W) \in \text{End}_C(W),$$

and

$$\text{ptr}_R(f) = (\text{Id}_V \otimes \text{ev}_W) \circ (f \otimes \text{Id}_W) \circ (\text{Id}_V \otimes \text{coev}_W) \in \text{End}_C(V).$$

**Definition 4.1.** A modified trace on $\text{Proj}$ (or $m$-trace) is a family of linear functions $\{t_V : \text{End}_C(V) \to k\}$ where $V$ runs over all objects of $\text{Proj}$, such that the following conditions hold.

1. If $U \in \text{Proj}$, and $W \in \text{Ob}(C)$, then for any $f \in \text{End}_C(U \otimes W)$, we have

$$t_{U \otimes W}(f) = t_U(\text{ptr}_R(f)).$$

2. If $U \in \text{Proj}$, and $W \in \text{Ob}(C)$, then for any $f \in \text{End}_C(W \otimes U)$, we have

$$t_{W \otimes U}(f) = t_U(\text{ptr}_L(f)).$$

3. If $U, V \in \text{Proj}$, then for any morphisms $f : V \to U$, and $g : U \to V$ in $C$, we have

$$t_V(g \circ f) = t_U(f \circ g).$$

If $t$ satisfies only (4.3) and (4.5), it is called a right trace (and the notion of a left trace is similar). When the category is ribbon, there is no difference between a right trace and a trace on $\text{Proj}$. If $C$ is unimodular, there exists up to a scalar a unique right trace on $\text{Proj}$; it is nondegenerate (cf. [22, Theorem 5.5]), in the following way. Let $V, W \in C$ with $V$ projective. Then, the pairing $\langle \cdot, \cdot \rangle_{V,W} : \text{Hom}_C(W, V) \otimes \text{Hom}_C(V, W) \to \mathbb{C}$ given by

$$\langle f, g \rangle_{V,W} = t_V(fg)$$

is nondegenerate. It is symmetric in the following sense. If $W$ is also projective, then

$$\langle g, f \rangle_{W,V} = \langle f, g \rangle_{V,W}. \quad (4.6)$$

If $W$ is not projective, then we take (4.6) as a definition.

4.1.2 Renormalized ribbon graph evaluations

We now recall the process of renormalizing colored ribbon graphs invariants first used in [27, 28]. Let $C$ be a $k$-linear pivotal category. Following Turaev ([55]), by a $C$-colored ribbon graph,
we mean an oriented surface obtained as the thickening of a graph whose oriented edges are colored by objects of $C$ and whose vertices are divided in two sets: the boundary vertices that are univalent and the internal vertices that are thickened to coupons colored by morphisms of $C$. Let $\text{Gr}_C$ be the category of $C$-colored ribbon graphs embedded in $\mathbb{R} \times [0, 1]$ and $G : \text{Gr}_C \to C$ be the “planar” Reshetikhin–Turaev $\mathbb{k}$-linear functor (see [32]). When $C$ is ribbon, we also denote by $G$ the original Reshetikhin–Turaev functor from the category of $C$-colored ribbon graphs embedded in $\mathbb{R}^2 \times [0, 1]$ to $C$.

Let $C$ be a $\mathbb{k}$-linear pivotal (resp. ribbon) category and let $T \subset S^2$ (resp. $T \subset S^3$) be a closed $C$-colored ribbon graph (i.e., with no boundary vertices). Let $e$ be an edge of $T$ colored with a projective object $V$ of $C$. Cutting $T$ at a point of $e$, we obtain a $C$-colored ribbon graph $T_V$ in $\mathbb{R} \times [0, 1]$ (resp., in $\mathbb{R}^2 \times [0, 1]$) where $G(T_V) \in \text{End}(V)$. We call $T_V$ a cutting presentation of $T$. A trace $t = \{t_V\}_{V \in \text{Proj}}$ determines an isotopy invariant $G'$ of $C$-colored ribbon graphs determined by

$$G'(T) = t_V(T_V),$$

where $T_V$ is any cutting presentation of $T$ (see [20]).

Let $A$ be the set of simple projective objects of $C$. Define a function $d : A \to \mathbb{k}\mathbb{^*}$ by $d(V) = t_V(\text{Id}_V)$ so that $d(V) = d(V^*)$ for all $V \in A$. If $T_V$ is as above with $V \in A$, then $G(T_V) \in \text{End}(V) = \mathbb{k}\text{Id}_V$. Let $\langle T_V \rangle \in \mathbb{k}$ denote the isotopy invariant of $T_V$ defined from the equality $G(T_V) = \langle T_V \rangle \text{Id}_V$. Then, $G'(T)$ can be computed by:

$$G'(T) = d(V)\langle T_V \rangle.$$  

(4.8)

4.1.3 Modified trace and pairing of morphisms

Let $C$ be a Hermitian pivotal category and assume that $C$ has a trace on projective objects. We say that a trace is real if for any endomorphism $f$ of a projective object, one has

$$t(f^\dagger) = \overline{t(f)}.$$  

Lemma 4.2. Assume that there exists a simple projective module in $C$. Then the trace is proportional to a real one. In particular, there is a nonzero real trace on $\text{Proj}$ for which $d(V) = d(V)$ for any $V \in \text{Proj}$.

Proof. Let $V_0$ be a simple projective module. Then the nondegeneracy of the trace implies that $d(V_0) \neq 0$. Up to rescaling $t$, we can assume that $d(V_0)$ is real. We now show that $t$ is real. This follows as in [23, Lemma 4.19] from the unicity of the trace and the fact that $f \mapsto t(f^\dagger) := \overline{t(f^\dagger)}$ is first also a trace, thus proportional to $t$, and second $t^\dagger(\text{Id}_{V_0}) = d(V_0) = d(V_0) = t(\text{Id}_{V_0}) \neq 0$. From this, we get $t = t^\dagger$. The last statement follows from $d(V) = t_V(\text{Id}_V)$ and $t_V^\dagger = \text{Id}_V$. □

From now on, we will always assume if a Hermitian pivotal category is equipped with a trace, then it is real. Then one has the following.

Proposition 4.3. For any projective object $V$ and any object $W$ of $C$, the following are true.

(1) There exists a nondegenerate bilinear pairing
\[ \text{Hom}(W, V) \otimes_\mathbb{k} \text{Hom}(V, W) \to \mathbb{k} \]
\[
\begin{align*}
f \otimes g & \mapsto \gamma_{V}(fg). \\
(2) & \text{ There exists a nondegenerate Hermitian form on } \text{Hom}(V, W) \text{ given by} \\
(f | g) & = \gamma_{V}(f^\dagger g).
\end{align*}
\]

### 4.2 Generically semisimple categories

Here, we recall the definition of generically semisimple categories that lead to TV type and Witten–Reshetikhin–Turaev-type non-semisimple topological invariants.

Let \( \mathbb{k} \) be a field and let \( \mathcal{G} \) be an abelian group. A pivotal \( \mathbb{k} \)-category is \( \mathcal{G} \)-graded if for each \( g \in \mathcal{G} \), we have a nonempty full subcategory \( \mathcal{C}_g \) of \( \mathcal{C} \) such that the following hold.

1. \( \mathbb{I} \in \mathcal{C}_e \), (where \( e \) is the identity element of \( \mathcal{G} \)).
2. \( \mathcal{C} = \bigoplus_{g \in \mathcal{G}} \mathcal{C}_g \).
3. if \( V \in \mathcal{C}_g, V' \in \mathcal{C}_{g'} \), then \( V \otimes V' \in \mathcal{C}_{g+g'} \).
4. if \( V \in \mathcal{C}_g, V' \in \mathcal{C}_{g'} \) and \( \text{Hom}_{\mathcal{C}}(V, V') \neq 0 \), then \( g = g' \).

For a subset \( \mathcal{X} \subset \mathcal{G} \), we say:

1. \( \mathcal{X} \) is symmetric if \( -\mathcal{X} = \mathcal{X} \).
2. \( \mathcal{X} \) is small in \( \mathcal{G} \) if the group \( \mathcal{G} \) cannot be covered by a finite number of translated copies of \( \mathcal{X} \), in other words, for any \( g_1, \ldots, g_n \in \mathcal{G} \), we have \( \bigcup_{i=1}^{n} (g_i + \mathcal{X}) \neq \mathcal{G} \).

1. A \( \mathbb{k} \)-category \( \mathcal{C} \) is semisimple if all its objects are semisimple.
2. A \( \mathbb{k} \)-category \( \mathcal{C} \) is finitely semisimple if it is semisimple and has finitely many isomorphism classes of simple objects.
3. A \( \mathcal{G} \)-graded category \( \mathcal{C} \) is a generically\( \mathcal{G} \)-semisimple category (resp. generically finitely\( \mathcal{G} \)-semisimple category) if there exists a small symmetric subset \( \mathcal{X} \subset \mathcal{G} \) such that for each \( g \in \mathcal{G} \setminus \mathcal{X} \), \( \mathcal{C}_g \) is semisimple (resp. finitely semisimple). By a generic simple object, we mean a simple object of \( \mathcal{C}_g \) for some \( g \in \mathcal{G} \setminus \mathcal{X} \).

#### 4.2.1 Relative Hermitian-spherical categories

We now introduce the notion of an \((\mathcal{X}, d)\)-relative\( \mathcal{G} \)-spherical category that is equivalent to the one given in [32].

**Definition 4.4.** Let \((\mathcal{C}, \mathcal{G}, \mathcal{X})\) be a generically finitely \( \mathcal{G} \)-semisimple category \( \mathbb{k} \)-linear pivotal category. Let \( \mathcal{A} \) be the class of all simple generic objects. We say that \( \mathcal{C} \) is \((\mathcal{X}, d)\)-relative \( \mathcal{G} \)-spherical if the following hold.

1. There exists a trace \( \{ t_V : \text{End}_{\mathcal{C}}(V) \to \mathbb{k} \} \) where \( V \) runs over all objects of \( \text{Proj} \) so that the function \( d : \mathcal{A} \to \mathbb{k}^* \) is given by \( d(V) = t_V(\text{Id}_V) \).
There exists a map \( \mathcal{S} : \mathcal{A} \to \mathcal{J}^* \) such that
\[
\mathcal{S}(V) = \mathcal{S}(V^*) \quad \text{and} \quad \mathcal{S}(V) = \mathcal{S}(V')
\]
for any isomorphic objects \( V, V' \in \mathcal{A} \) and for any \( g_1, g_2, g_1 + g_2 \in \mathcal{G} \setminus \mathcal{X} \) and \( V \in C_{g_1 + g_2} \), we have
\[
b(V) = \sum_{V_1 \in \text{irr}(C_{g_1}), V_2 \in \text{irr}(C_{g_2})} b(V_1) b(V_2) \dim_k(\text{Hom}_{\mathcal{C}}(V, V_1 \otimes V_2)), \tag{4.9}
\]
where \( \text{irr}(C_{g_1}) \) denotes a representing set of the isomorphism classes of simple objects of \( C_{g_1} \).

If \( C \) is a category with the above data, for brevity, we say that \( C \) is a relative \( G \)-spherical category.

The map \( b \) always exists when \( k \) is a field of characteristic 0 and \( C \) is a category whose objects are finite-dimensional \( k \)-vector spaces. In particular, in [32], it is shown that, for any \( g \in \mathcal{G} \setminus \mathcal{X} \), the map
\[
b(V) = \dim_k(V) / \left( \sum_{V' \in \text{irr}(C_g)} \dim_k(V')^2 \right) \quad \tag{4.10}
\]
is well defined and satisfies all the properties above. In particular, if for any generic degree \( g \), \( C_g \) has \( N \) simple objects of dimension \( d \), then \( b \) can be chosen to be the constant map \( b(V) = \frac{1}{N_0} \).

**Definition 4.5.** A relative \( G \)-spherical category is called a relative\( G \)-Hermitian-spherical category, if it is simultaneously a Hermitian pivotal category, a relative \( G \)-spherical category and the trace \( t \) is real.

### 4.2.2 Relative Hermitian-premodular categories

In [11], De Renzi gives the notion of a relative modular category and shows that such categories lead to TQFTs. A \( G \)-relative modular category is a ribbon category that is a generically \( G \)-semisimple category \( \mathcal{C} \) with symmetric small set \( \mathcal{X} \) has a free realization and nonzero m-trace satisfying compatibility conditions. Let us recall this definition.

Let \( Z \) be an abelian group. A free realization of \( Z \) in a ribbon category \( \mathcal{C} \) is a monoidal functor \( \sigma : Z \to \mathcal{C} \), where \( Z \) also denotes the discrete category over \( Z \) with tensor product given by the group operation \( + \), satisfying \( \sigma(k) = id_{\sigma(k)} \) for every \( k \in Z \), and inducing a free action on isomorphism classes of simple objects of \( \mathcal{C} \) by tensor product with \( \sigma(k) \). Recall the definition of a generically semisimple category at the beginning of Subsection 4.2.

**Definition 4.6** [11]. If \( G \) and \( Z \) are abelian groups, and if \( \mathcal{X} \subset \mathcal{G} \) is a small symmetric subset, then a \((Z, \mathcal{X})\)-relative \( G \)-premodular category is a generically semisimple category \((\mathcal{C}, G, \mathcal{X})\) that is ribbon, \( k \)-linear, has a free realization \( \sigma : Z \to C_0 \), and a nonzero m-trace \( t \) on the projective objects of \( C \). The data are subject to the following conditions.

1. **Finitely many orbits.** For every \( g \in \mathcal{G} \setminus \mathcal{X} \), the homogeneous subcategory \( C_g \) is semisimple and dominated by \( \Theta(C_g) \otimes \sigma(Z) \) for some finite set \( \Theta(C_g) = \{ V_i \in C_g \mid i \in I_g \} \) of simple projective objects with epic evaluation. Here, dominated means that the identity morphism of every object in the category can be written as a linear combination of maps that factor through objects in the dominating set.
(2) **Compatibility.** There exists a bilinear map \( \psi : G \times Z \to k^* \) such that

\[
c_{\sigma(k),V} \circ c_{V,\sigma(k)} = \psi(g,k) \cdot \text{id}_{V \otimes \sigma(k)}
\]

for every \( g \in G \), for every \( V \in C_g \), and for every \( k \in Z \).

If \( C \) is a category with the above data, for brevity, we say that \( C \) is a relative \( G \)-premodular category. If \( C \) is a \((Z, \mathcal{X})\)-premodular \( G \)-category, then the associated **Kirby color of index** \( g \in G \setminus \mathcal{X} \) is the formal linear combination of objects

\[
\Omega_g := \sum_{i \in I} d(V_i) \cdot V_i.
\]

In particular, there exist constants \( \Delta_{-\Omega}, \Delta_{+\Omega} \in k \), called **stabilization coefficients**, which realize the skein equivalences of Figure 1, and which are independent of both \( V \in C_g \) and \( g \in G \setminus \mathcal{X} \). We say that the relative premoidal category \( C \) is **non-degenerate** if \( \Delta_{-\Omega} \Delta_{+\Omega} \neq 0 \).

A nondegenerate relative \( G \)-premodular category gives rise to the construction of a 3-manifold invariant (see, e.g., [8, Section 4.3], where it is called relative \( G \)-modular rather than the modern term relative \( G \)-premodular). A relative \( G \)-modular category is a relative \( G \)-premodular category that satisfies the modularity condition given in [11]. Note that relative \( G \)-modular categories are automatically nondegenerate. Such a category gives rise to a TQFT. The exact form of the modularity condition is not critical for this paper. Next, we give a property that implies that the TQFT is Hermitian.

**Definition 4.7.** A **relative Hermitian-(pre)modular category** is a Hermitian ribbon category that is also a relative (pre)modular category, such that the trace on projectives is real.

### 4.3 Relative Hermitian-premodular category in a sequilinear one

Let \( C \) be a sesquilinear pivotal category. A collection of objects is called a **collection of Hermitian objects** in \( C \) if the following hold (in the following, we call an element of the collection a **Hermitian object**).

1. For any two Hermitian objects \( V, W \) and any \( f \in \text{Hom}_C(V, W) \), we have \((f^\dagger)^\dagger = f\).
2. For any simple \( V \in C \), \( V \) is isomorphic to a Hermitian object.
3. If the tensor product of two Hermitian simple objects is semisimple, then it is Hermitian.
If $S$ is a class of objects in a pivotal category $C$, then we define the category generated by $S$ as the full subcategory of $C$ that has as objects, all tensor products of the form:

$$X_1 \otimes X_2 \otimes \cdots \otimes X_p \quad \text{where} \quad p \in \mathbb{N} \text{ and } X_i \in S \cup S^*.$$  

**Theorem 4.8.** If $C$ is a generically semisimple, sesquilinear pivotal category that has a collection of Hermitian objects where $\overrightarrow{ev}_V$ is an epimorphism for each generic simple Hermitian object $V$, then the simple Hermitian objects generate a subcategory $C^{\text{Herm}}$ of $C$ that is a generically semisimple, Hermitian pivotal category.

**Proof.** The generically semisimple sesquilinear pivotal structure of $C$ provides the same structure on $C^{\text{Herm}}$. Thus, to prove the theorem, we need to show that $(f^\dagger)^\dagger = f$ for any morphism $f$ in $C^{\text{Herm}}$. Let $f : W_1 \otimes \cdots \otimes W_m \to W'_1 \otimes \cdots \otimes W'_n$ be a morphism in $C^{\text{Herm}}$ where $W_1, W_2, \ldots, W_n$ are Hermitian objects.

We first consider the following “regular” case: Assume $m, n \geq 1$ and for any $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$, $V_i \otimes W_i$ and $V_j \otimes W_j$ are in generic degrees. Then we prove $(f^\dagger)^\dagger = f$ by induction on $k = m + n$.

The base case is $k = 2$, which means that $m = n = 1$. Then by the first axiom of a collection of Hermitian objects, $(f^\dagger)^\dagger = f$ holds.

Assume true for $k \geq 2$ and consider a morphism $f$ where $m + n = k + 1$.

If $m \geq 2$, then since $W_1 \otimes W_2$ is generic, it is isomorphic to a direct sum of simple Hermitian modules $U_i$. Note that $(U_i, W_3, \ldots, W_m)$ still satisfies the “regular” case condition. Let $g_i : W_1 \otimes W_2 \to U_i$ be the projection morphisms corresponding to this direct sum which satisfy $(g_i^\dagger)^\dagger = g_i$. Then there exist morphisms $h_i : U_i \otimes W_3 \otimes \cdots \otimes W_m \to W'_1 \otimes \cdots \otimes W'_n$ such that $f = \sum_i (g_i \otimes Id_{W_3 \otimes \cdots \otimes W_m}) h_i$. Now by induction, $h_i$ satisfies $(h_i^\dagger)^\dagger = h_i$ and thus $(f^\dagger)^\dagger = f$.

If $n \geq 2$, then we can proceed similarly with the $W'_i$. Since $W'_1 \otimes W'_2$ is generic, it is isomorphic to a direct sum of simple Hermitian modules $U_i$. Let $g_i : U_i \to W'_1 \otimes W'_2$ be the injection morphisms corresponding to this direct sum that satisfy $(g_i^\dagger)^\dagger = g_i$. Then there exist morphisms $h_i : W_1 \otimes \cdots \otimes W_m \to U_i \otimes W'_1 \otimes \cdots \otimes W'_n$ such that $f = \sum_i (g_i \otimes Id_{W'_1 \otimes \cdots \otimes W'_n}) h_i$. Again by induction, $(h_i^\dagger)^\dagger = h_i$ thus $(f^\dagger)^\dagger = f$.

Hence, the theorem is proven in the “regular” case. Now for the general case, let $V$ be a Hermitian simple generic object such that for any $i \in \{1, \ldots, m\}$, and any $j \in \{1, \ldots, n\}$, $V \otimes W_1 \otimes \cdots \otimes W_i$, and $V \otimes W'_1 \otimes \cdots \otimes W'_j$ are in generic degrees (such a $V$ exists because $C$ is generically semisimple with respect to a small set $\mathcal{A}$, and by assumption, any simple is isomorphic to a Hermitian object). Then, $Id_V \otimes f$ is in the “regular” case so that $((Id_V \otimes f)^\dagger)^\dagger = Id_V \otimes f$.

But $((Id_V \otimes f)^\dagger)^\dagger = Id_V \otimes (f^\dagger)^\dagger$ and since $\overrightarrow{ev}_V$ is an epimorphism, we have

$$Id_V \otimes (f^\dagger)^\dagger = Id_V \otimes f \quad \text{if and only if} \quad (f^\dagger)^\dagger = f. \quad (4.12)$$

This completes the proof. \square

**Corollary 4.9.** If a relative premodular category $C$ is a sesquilinear ribbon category with a collection of Hermitian objects containing $\sigma(Z)$, then $C^{\text{Herm}}$ is a Hermitian relative premodular category.
4.4 Relative Hermitian-spherical category associated to a relative premodular category

We describe some machinery on how to obtain a relative Hermitian-spherical category from a relative Hermitian-premodular category. Assume that $C$ is a relative Hermitian-premodular category with grading group $(G, +)$, singular set $\mathcal{X}$, and translation group $(Z, +)$. Let $U = \{ z \in k : zz = 1 \}$. Note that applying the dagger to (4.11), we get that the bilinear map $\psi$ from Definition 4.6 (2) takes values in $U$.

We assume the following additional hypothesis:

$$\exists \sqrt{\psi} : G \times Z \to U, \text{ which is bilinear and } \sqrt{\psi}^2 = \psi, \quad (4.13)$$

$$\forall \lambda, \mu \in Z, \ c_{\sigma_\lambda, \sigma_\mu} = \text{Id}_{\sigma_{\lambda+\mu}}. \quad (4.14)$$

Remark 4.10.

1. If $Z/2Z$ is finite, then one can replace $Z$ with $2Z$. Then, the category is still relative premodular and one can define $\sqrt{\psi}(g, 2\lambda) := \psi(g, \lambda)$.
2. Identity (4.14) with $\partial \sigma_\lambda = \text{Id}$ implies that $\dim_C(\sigma_\lambda) = 1$.
3. A third consequence of (4.14) and $\sigma_\lambda \otimes \sigma_{-\lambda} = \mathbb{1}$ is the existence of canonical isomorphisms

\[ (\text{Id}_{\sigma_{-\lambda}} \otimes \text{ev}_{\sigma_\lambda})(\text{Id}_{\sigma_{-\lambda} \otimes \sigma_\lambda} \otimes \text{Id}_{\sigma^*_\lambda}) = \text{(ev}_{\sigma_\lambda} \otimes \text{Id}_{\sigma_{-\lambda}})(\text{Id}_{\sigma^*_\lambda} \otimes \text{Id}_{\sigma_\lambda \otimes \sigma_{-\lambda}}) \colon \sigma_\lambda^* \cong \sigma_{-\lambda}. \]

Through these isomorphisms, all (co)evaluation morphisms of the objects $\sigma_\lambda$ correspond to the identity of $\mathbb{1}$. We use these isomorphisms to freely identify $\sigma_\lambda^*$ and $\sigma_{-\lambda}$.

Definition 4.11. For any $\lambda \in Z$, and any $V \in C_g$, we let

\[ X_{\sigma_\lambda, V} = \sqrt{\psi}(g, \lambda)^{-1} c_{\sigma_\lambda, V} \quad \text{and} \quad X_{V, \sigma_\lambda} = \sqrt{\psi}(g, \lambda)^{-1} c_{V, \sigma_\lambda}. \]

Then, $X_{\cdot, \sigma_\lambda}$ is a half braiding and $X_{\cdot, \sigma_\lambda}^\dagger = X_{\sigma_\lambda, \cdot}^{-1} = X_{\sigma_\lambda, \cdot}^\times$.

Definition 4.12. Let $C^Z$ be the category with the same objects as $C$ and whose morphisms are the $Z$-graded vector spaces given by

\[ \text{Hom}_{C^Z}(V, W) = \bigoplus_{\lambda \in Z} \text{Hom}_C(V, \sigma_\lambda \otimes W). \]

For a morphism $f : V \to W$, we denote by $f_{\lambda} \in \text{Hom}_C(V, \sigma_\lambda \otimes W)$ its degree $\lambda$ component. The composition of morphisms $f : V_1 \to V_2$ with $g : V_2 \to V_3$ in degree $\lambda$ is the essentially finite sum

\[ (g \circ f)_{\lambda} = \sum_{\lambda' + \lambda'' = \lambda} (\text{Id}_{\sigma_{\lambda'}} \otimes g_{\lambda''})f_{\lambda'}. \quad (4.15) \]

The tensor product of $f : V_1 \to V_2$ and $h : V_3 \to V_4$ is given by

\[ (f \otimes h)_{\lambda} = \sum_{\lambda' + \lambda'' = \lambda} \left( \text{Id}_{\sigma_{\lambda'}} \otimes X_{V_2, V_3} \otimes \text{Id}_{V_4} \right)(f_{\lambda'} \otimes h_{\lambda''}). \quad (4.16) \]
Proposition 4.13. There is a unique structure of a pivotal category on $C^Z$ such that the embedding $F^Z : C \to C^Z$ given by considering all morphisms of $C$ as degree zero morphisms of $C^Z$ is a pivotal functor. The category $C^Z$ has a natural Hermitian structure compatible with $F^Z$.

Proof. The associativity of composition comes from the strict relation $\sigma_{\lambda_1} \otimes \sigma_{\lambda_2} = \sigma_{\lambda_1+\lambda_2}$. The associativity and the functoriality of the tensor product come from the fact that $X_{*,\sigma}$ are half braidings and Equation (4.14) ensures that $f \otimes g = (f \otimes \text{Id})(\text{Id} \otimes g)$. The functor $F^Z$ is clearly monoidal because $X_{*,\lambda}$ is the identity functor. The category $C^Z$ inherits the evaluation and coevaluation morphisms of Definition 3.2 from $C$.

To see that $C^Z$ is pivotal, we need to check that the left dual $f^*$ and right dual $^*f$ of a $C^Z$-morphism $f : V \to W$ are equal. Let us recall the definitions of $f^*$ and $^*f$:

$$f^* = (\ev_W \otimes \text{Id}_{V^*})(\text{Id}_{W^*} \otimes f \otimes \text{Id}_{V^*})(\text{Id}_{W^*} \otimes \text{coev}_V),$$

and

$$^*f = (\text{Id}_{V^*} \otimes \ev_W)(\text{Id}_{V^*} \otimes f \otimes \text{Id}_{W^*})(\text{coev}_V \otimes \text{Id}_{W^*}).$$

To compute these morphisms in $C^Z$, we need to use the definitions of the composition and tensor products for $C^Z$ given in (4.15) and (4.16). For example,

$$(f^*)_\lambda = \left(\left(\text{Id}_{V^*} \otimes \ev_W\right)(X_{*,\sigma}(\text{Id}_{W^*} \otimes f))(\text{Id}_{W^*} \otimes \text{coev}_V)\right) \otimes \text{Id}_{V^*} \otimes \text{coev}_V,$$

where $\ev_W$ denotes the evaluation in $C^Z$ and the last composition of maps is in $C$. It follows that as morphisms of $C$, we have

$$(f^*)_\lambda = \left(\left(\text{Id}_{V^*} \otimes \ev_W\right)(X_{*,\sigma}(\text{Id}_{W^*} \otimes f))(\text{Id}_{W^*} \otimes \text{coev}_V)\right) \otimes \text{Id}_{V^*} \otimes \text{coev}_V = (\text{Id}_{\sigma_\lambda} \otimes (f_{-\lambda})^*)(X_{*,\sigma}(\text{Id}_{W^*} \otimes f))(\text{Id}_{W^*} \otimes \text{coev}_{\sigma_\lambda})$$

$$= (\text{Id}_{\sigma_\lambda} \otimes (f_{-\lambda})^*)(X_{V^*,\sigma}(\text{Id}_{V^*} \otimes f))(\text{Id}_{V^*} \otimes \text{coev}_{\sigma_\lambda})$$

where the last equality follows from the naturality of $X_{*,\sigma}$ and $\theta_{\sigma_\lambda} = \text{Id}$.

In order to construct the Hermitian structure on $C^Z$, let $f \in \text{Hom}_{C^Z}(V_1, V_2)$. Now define a morphism $f^\dagger$ in $C^Z$ by

$$(f^\dagger)_\lambda := (\text{Id}_{\sigma_\lambda} \otimes (f_{-\lambda})^*) \circ (\text{coev}_{\sigma_{-\lambda}} \otimes \text{Id}_{V^*}) : V_2 \to \sigma_\lambda \otimes V_1. \quad (4.17)$$

In graphical terms,

$$(f^\dagger)_\lambda := \begin{tikzpicture}
\node (V2) at (0,0) {$V_2$};
\node (V1) at (1,1) {$V_1$};
\node (sigma) at (0.5,0.5) {$\sigma_\lambda$};
\draw (V2) edge[->] node[above]{$f_{-\lambda}^\dagger$} (V1);
\end{tikzpicture} : V_2 \to \sigma_\lambda \otimes V_1. \quad (4.18)$$
Using this graphical calculus, it is straightforward to check that the dagger is now a contravariant functor on $C^Z$.

In order to check that the functor is monoidal, let $f : V_1 \to V_2$ and $h : V_3 \to V_4$. Unraveling the definitions, we get

$$
(f \otimes h)^\dagger_\lambda = \sum_{\lambda' + \lambda'' = -\lambda} (\text{Id}_{\sigma_{-\lambda''}} \otimes ((f_{\lambda'})^\dagger \otimes (h_{\lambda''})^\dagger)) \circ (\text{Id}_{\sigma_{\lambda'}} \otimes X_{V_2,\sigma_{\lambda'}} \otimes \text{Id}_{V_4})
$$

$$\times \circ (\text{coev}_{\sigma_{\lambda'}} \otimes \text{coev}_{\sigma_{\lambda''}} \otimes \text{Id}_{V_2 \otimes V_4}).
$$

(4.19)

On the other hand,

$$
(f^\dagger \otimes h^\dagger)^\dagger_\lambda = \sum_{\lambda' + \lambda'' = \lambda} (\text{Id}_{\sigma_{\lambda'}} \otimes X_{V_1,\sigma_{\lambda'}} \otimes (h_{-\lambda''})^\dagger) \circ (\text{Id}_{\sigma_{\lambda'}} \otimes (f_{-\lambda'})^\dagger \otimes \text{coev}_{-\lambda''} \otimes \text{Id}_{V_4})
$$

$$\times \circ (\text{coev}_{-\lambda'} \otimes \text{Id}_{V_2 \otimes V_4}).
$$

(4.20)

One could most easily show $(f \otimes h)^\dagger_\lambda = (f^\dagger \otimes h^\dagger)^\dagger_\lambda$ in a graphical way by manipulating these expressions using the fact $X^\dagger = X$, using the naturality of $X_{\sigma_{\lambda''},\sigma_{\lambda'}}$, and reindexing the sum.

The conditions of Definition 3.3 are satisfied in $C^Z$ because they are satisfied in $C$ by assumption.

The embedding $F^Z$ is not a full subcategory since $C^Z$ has more morphisms. As a consequence, the braiding of $C$ is no longer dinatural in $C^Z$ but we will see that $C^Z$ is generically finitely semisimple.

**Theorem 4.14.** If $C$ is a relative Hermitian-premodular category with $\sqrt{\psi}$ as in (4.13), and the braiding assumption as in (4.14), then if $C^Z$ has a map $b$ as in Equation (4.9), it is a relative Hermitian-spherical category.

**Proof.** We note that a $Z$-orbit of a simple object in $C$ becomes an isomorphism class of a simple object in $C^Z$. Hence, $C^Z = \bigoplus_{g \in G} C^Z_g$ is finitely semisimple in each degree $g \in G \setminus \mathcal{X}$. 

\[ \square \]

## 5 | RELATIVE LW-DATA FROM RELATIVE HERMITIAN-SPHERICAL CATEGORIES

In this section, we show how the notion of a modified trace on a generically Hermitian category can be used to define the basic data needed for the non-semisimple Levin–Wen model from Section 2.

### 5.1 | Basic data and basis

Let $C$ be a relative $G$-Hermitian-spherical category with a family of representative isomorphism classes of generic simples objects $(V_i)_{i \in I}$. For $g \in G \setminus \mathcal{X}$, let $I_g$ be the finite set

$$
I_g = \{ i \in I : V_i \in C_g \}
$$

(5.1)
and call $g$ the degree of $i \in I_g$. For $(i, j, k) \in I^3$, define the multiplicity space

$$\mathcal{H}^{ijk} = \text{Hom}_C(l, V_i \otimes V_j \otimes V_k).$$

We call this dimension the branching level:

$$\delta_{ijk} = \dim_k(\mathcal{H}^{ijk}). \quad (5.2)$$

Note that $\delta_{ijk}$ is zero unless the sum of the degrees of $V_i, V_j$, and $V_k$ is zero in $G$. There is an involution on $I$, $i \mapsto i^*$ uniquely determined by $V_{i^*} \simeq V_i^*$. Basic data in $C$ is a family $(V_i)_{i \in I}$ as above equipped with isomorphisms $(w_i : V_i \rightarrow V_i^*)_{i \in I}$ satisfying $w_{i^*} = (w_i)^* \circ \phi_{V_i^*}$, or equivalently,

$$\overrightarrow{ev}_{V_i}(w_i \otimes \text{Id}_{V_i}) = \overrightarrow{ev}_{V_{i^*}}(\text{Id}_{V_{i^*}} \otimes w_i) : V_i^* \otimes V_i \rightarrow l. \quad (5.3)$$

The basic data induce nondegenerate bilinear pairings

$$\Theta = \left( \Theta_{ijk} : \mathcal{H}^{k^* j^* i^*} \otimes \mathcal{H}^{ijk} \rightarrow k, \quad y' \otimes y \mapsto \text{t}_{V_i \otimes V_j \otimes V_k} \left( y(y')^*(w_i \otimes w_j \otimes w_k) \right) \right)_{ijk \in I^3},$$

which satisfy $\Theta_{ijk}(y', y) = \Theta_{k^* j^* i^*}(y, y')$. This pairing is also invariant by pivotal cyclic permutation as we now see. Define $\text{rot}_{ijk} : \mathcal{H}^{ijk} \rightarrow \mathcal{H}^{kji}$ by

$$\text{rot}_{ijk}(y) = \left( \overrightarrow{ev}_{V_i} \otimes \text{Id}_{V_j \otimes V_k} \otimes \text{Id}_{V_i} \right) \left( \text{Id}_{V_{ij}^*} \otimes y \otimes \text{Id}_{V_i} \right) \left( \text{coev}_{V_i} \right).$$

Let $y \in \mathcal{H}^{ijk}$ and $y' \in \mathcal{H}^{k^* j^* i^*}$. Then, $\Theta_{jk}(y', \text{rot}_{ijk}(y)) = \Theta_{ijk}(\text{rot}_{k^* j^* i^*}(y'), y)$.

Hermitian basic data is basic data together with a basis

$$(y_n^{ijk})_{n=1, \ldots, \delta_{ijk}}$$

of $\mathcal{H}^{ijk}$ such that

$$\Theta_{ijk}(y_m^{k^* j^* i^*}, y_n^{ijk}) = \delta^n_m \quad (5.4)$$

$$\Theta_{n^{ijk}}(y_n^{ijk}) = \begin{cases} 0 \text{ if } m \neq n, \\ y_n^{ijk} \in k^* \text{ if } m = n, \end{cases} \quad (5.5)$$

$$\text{rot}_{ijk}(y_n^{ijk}) = y_n^{kij}. \quad (5.6)$$

We call $i, j, k$ the string labels and $n$ the branching label of $y_n^{ijk}$. By convention, for $n > \delta_{ijk}$, we set $y_n^{ijk} = 0 \in \mathcal{H}^{ijk}$.

Equation (5.5) determines $\langle \cdot \rangle$-invariant scalars $\gamma_n^{ijk} \in k^*$. Another family of such scalars $\beta(i) \in k^*$ is defined by

$$w_i^\dagger = \beta(i)w_i^{-1} \text{ so that } (w_i \mid w_j) = \beta(i) \text{ d}(V_i). \quad (5.7)$$

**Proposition 5.1.** If $\mathcal{X}$ contains all the order 2 and order 3 elements of $G$, then the relative $G$-Hermitian-spherical category $C$ has Hermitian basic data.
Proof. Let $A$ be the set of generic labels and choose a total order on $A$. The assumption on order 2 elements implies for $i \in A$, $i^* \neq i$. For each $i \in A$ with $i < i^*$, choose an isomorphism $w_i : V_i \to V_i^{*}$ and define

$$w_{i^*} = (\text{Id}_{V_i^*} \otimes \text{ev}_{V_i})(\text{Id}_{V_i^*} \otimes w_i \otimes \text{Id}_{V_i^{*}})(\text{coev}_{V_i} \otimes \text{Id}_{V_i^{*}}) : V_i^* \to V_i^{*}.$$ 

This ensures that (5.3) is satisfied for $i$ and $i^*$.

Choose now a total order on $A_3 = \{(i, j, k) \in A^3 : \delta_{ijk} > 0\}$. The assumption on order 3 elements implies that for any $i \in A$, $(i, i, i) \notin A_3$. The dihedral group $D_{2 \times 3}$ acts on $A_3$ by $(i, j, k) \mapsto (j, k, i)$ and $(i, j, k) \mapsto (k^*, j^*, i^*)$. This action is free, so there are six triples in each orbit. For each orbit, pick $(i, j, k)$ to be the smallest element of this orbit and choose an orthogonal basis $(\gamma_{ijk}^n)$ of $\mathcal{H}_{ijk}$ for the Hermitian form. Define $\gamma_{ijk}^n = (\gamma_{ijk}^n \mid \gamma_{ijk}^n)$.

Now, since $(y_{ijk}^n)$ is an orthogonal basis, $(\gamma_{ijk}^n)^{\dagger} = \Theta_{ijk}(y_{ijk}^m \mid y_{ijk}^n)$ is the corresponding dual basis of $\mathcal{H}_{ijk}$. So, $(y_{ijk}^n)^{\dagger} = y_{ijk}^n y_{ijk}^{k^* j^* i^*}$ and it is enough to show that $(y_{ijk}^n)^{\dagger}$ is orthogonal. This is true because

$$\left(\left(y_{ijk}^n\right)^{\dagger} \mid \left(y_{ijk}^m\right)^{\dagger}\right) = \Theta_{k^* j^* i^*}(\left(y_{ijk}^n\right)^{\dagger}, \left(y_{ijk}^m\right)^{\dagger}) = \frac{\Theta_{ijk}(y_{ijk}^n, y_{ijk}^m)}{\beta(i) \beta(j) \beta(k)} = \frac{\left(y_{ijk}^n \mid y_{ijk}^m\right)}{\beta(i) \beta(j) \beta(k)}. \quad \square$$

### 5.2 Invariant of labeled unitrivalent graphs

Let $C$ be a relative $\mathcal{C}$-Hermitian-spherical category equipped with basic data as in Section 5.1. A $C$-labeled unitrivalent graph is an unoriented graph $\Gamma$ embedded in the unit disc along with

- a map $\sigma_1$: oriented edges $\to I$ with $\sigma_1(\varepsilon) = \sigma_1(\varepsilon)^*$, and
- a map $\sigma_0$: trivalent vertices $\to \mathbb{N}$ such that if all edges labeled $(i, j, k)$ of a trivalent vertex $v$ are oriented inward toward the vertex, then $\sigma_0(v) \in [1, \delta_{ijk}]$.

To each $C$-labeled unitrivalent graph, we can associate a $C$-colored ribbon graph $\tilde{\Gamma}$ by

- add a bivalent vertex $v_\varepsilon$ that bisects each unoriented edge $\varepsilon$ and orienting the two resulting edges $\bar{\varepsilon}$ and $\bar{\varepsilon}$ away from the vertex $v_\varepsilon$.
- label each bivalent vertex $v_\varepsilon$ with a coupon labeled $w_{\sigma_1(\varepsilon)}$, which is well defined by (5.3) and the equalities $\sigma_1(\bar{\varepsilon}) = \sigma_1(-\bar{\varepsilon}) = \sigma_1(\bar{\varepsilon})^*$.
- replace each resulting trivalent vertex $v$ with inward oriented edges $(i, j, k)$ by a coupon labeled $y_{\sigma_0(v)}^{ijk}$.
Following the unit circle clockwise, we see the univalent vertices of \( \tilde{\Gamma} \) form a word \(((V_{i_1}, +), \ldots, (V_{i_n}, +))\). To each such \( C \)-colored ribbon graph \( \tilde{\Gamma} \), the RT-functor produces an element \( G(\tilde{\Gamma}) \in \text{Hom}_C(\mathbb{I}, V_{i_1} \otimes \cdots \otimes V_{i_n}) \). We can then define an operation on labeled univalent graphs via

\[
\langle \Gamma \rangle := G(\tilde{\Gamma}),
\]

(5.8)

which only depends on \( \Gamma \). For a closed (without univalent vertices) graph \( \Gamma \), we modify the definition of \( \langle \Gamma \rangle \) to

\[
\langle \Gamma \rangle' := G'(\tilde{\Gamma})
\]

(5.9)

using a cutting presentation and the modification from (4.8). Since \( C \) is relative \( G \)-spherical, then the graph evaluation (5.9) extends to a well-defined evaluation for spherical ribbon graphs, that is, \( C \)-ribbon graphs in \( S^2 \), see [29, Lemma 9].

### 5.3 Conjugations for labeled unitrivalent graphs

Now suppose that \( C \) is a relative \( G \)-Hermitian-spherical category equipped with basic data as in Section 5.1. There is a noninvolutive semilinear conjugation operator \( \Gamma \mapsto \Gamma^{\dagger} \) on the space generated by labeled planar uni-trivalent networks as follows. The definition of this operator is inspired by the \( \dagger \) functor and Equation (5.8). The image of a graph \( \Gamma \) with state \( \sigma \), with set of trivalent vertices \( \Gamma_0 \), with set of univalent vertices \( \Gamma'_0 \) and with set of edges \( \Gamma_1 \) is given by

\[
\Gamma^{\dagger} = c(\Gamma) \Gamma^{\dagger},
\]

(5.10)

where \( \Gamma^{\dagger} \) is the mirror image of \( \Gamma \) with the dual state and the complex coefficient \( c(\Gamma) \) is given by

\[
c(\Gamma) = \prod_{v \in \Gamma_0} \gamma(\sigma(v)) \prod_{u \in \Gamma'_0} \beta(\sigma_1(u))^{-1} \prod_{e \in \Gamma_1} \beta(\sigma_1(e)),
\]

where \( \sigma_1(u) \) is \( \sigma_1 \) of the unique edge adjacent to \( u \) and \( \gamma(\sigma(v)) = \gamma_{\sigma_1(u)}^{\sigma_1(e_1)\sigma_1(e_2)\sigma_1(e_3)} \) where \( e_1, e_2, e_3 \) are the three ordered edges adjacent to \( v \) oriented toward \( v \). A network with no univalent vertices is closed.

**Remark 5.2.** Note that for an oriented edge \( e \), \( \beta(\sigma(-e)) = \beta(\sigma(e)) \). Thus, \( \beta(\sigma(e)) \) make sense even if we forget the orientation of \( e \).

**Proposition 5.3.** Let \( C \) be a relative Hermitian-pivotal (resp. relative Hermitian-spherical) category, then for \( \Gamma \) a closed planar (or spherical) network, the evaluation of the network \( \Gamma^{\dagger} \) is the conjugate of the value of \( \Gamma \)

\[
\langle \Gamma^{\dagger} \rangle' = \overline{\langle \Gamma \rangle}'.
\]

(5.11)
5.4 Scalar 6j-symbols in relative Hermitian-spherical categories

Now assume that \( C \) is relative Hermitian-spherical equipped with basic data. We define scalar 6j symbols from the tetrahedral graph ribbon graph \( \tilde{\Gamma} \)

\[
N^j_{j_1j_2j_3}(a_1a_2a_3) = G^j_{j_1j_2j_3}(a_1a_2a_3)
\]

where the coupons at vertices are filled with

\[
y_{a_1} = y_{j_1j_2j_3}^j, \quad y_{a_2} = y_{j_3j_4j_5}^j, \quad y_{a_3} = y_{j_5j_6j_1}^j, \quad y_{a_4} = y_{j_6j_4j_2}^j.
\]

**Proposition 5.4.** The 6j symbols associated with a relative Hermitian-spherical category \( C \) satisfy the equalities (2.1) and (2.2)–(2.5).

**Proof.** The definition of modified 6j symbols in the relative spherical category \( C \) implies that they have the symmetries of an oriented tetrahedron implying (2.2). Equation (2.3) follows from [29, Theorem 11] and (2.4) from [29, Theorem 12]. Finally, (2.1) and (2.5) follow as in the proof of [24, Appendix A.15].

5.5 Relative \( \mathcal{G} \)-LW-data from relative Hermitian-spherical

Finally, we explain how the data of an \((\mathcal{X}, d)\)-relative \( \mathcal{G} \)-Hermitian-spherical category provide \((\mathcal{X}, d)\)-relative \( \mathcal{G} \)-spherical LW-data in the sense of Section 2.1.

**Theorem 5.5.** Let \( C \) be a \((\mathcal{X}, d)\)-relative Hermitian-spherical category where \( \mathcal{X} \) contains all elements of \( \mathcal{G} \) of order 2 and 3. Then, \( C \) determines relative LW-data

\[
(\mathcal{G}, \mathcal{X}, (I_g)_{g \in \mathcal{G}}, \delta, b, d, N, \beta, \gamma),
\]

where \( I_g \) is as in (5.1), \( \delta \) is defined in (5.2), \( b, d \) come from the relative spherical data, \( N \) are the modified 6j symbols from (5.12), \( \beta \) are defined in (5.7), and \( \gamma \) are defined in the proof of Proposition 5.1.

**Proof.** This is immediate from the definitions and Proposition 5.4.

There is a modified TV TQFT associated to the data of an \((\mathcal{X}, d)\)-relative \( \mathcal{G} \)-spherical category [29]. This is a \((2+1)\)-dimensional TQFT defined on a 3-manifold \( M \) with the additional data of an isotopy class of an embedded graph \( Y \) in \( M \) and a cohomology class \([\Phi] \in H^1(M, \mathcal{G})\). Given a triangulation \( T_M \) of \( M \), the graph \( Y \) is assumed to be Hamiltonian, meaning that it intersects every vertex of the triangulation exactly once, and its edges are part of the triangulation. When the manifold \( M \) has boundary, the triangulation of \( M \) induces a triangulation \( T \) on the boundary \( \Sigma \).
and the cohomology class \([\Phi] \in H^1(M, G)\) restricts to \([\Phi|_{\Sigma}] \in H^1(\Sigma, G)\). In [29], it is assumed that the graph \(Y\) is embedded transversely to the boundary surface. Thus, the modified TV invariant for the surface \(\Sigma\) includes the additional data of a finite set of marked points \(m \subset \Sigma\) on the surface corresponding to where the graph intersects the surface.

Assume that the set of vertices in the triangulation \(\Sigma\) is exactly the set of marked points. As explained in [24, Section F], we have the following.

**Theorem 5.6.** The degenerate ground state of the Hamiltonian in (2.17) defined on a surface \(\Sigma\) is isomorphic to the vector space assigned to the surface by the modified TQFT from [29] defined from the corresponding relative \(G\)-Hermitian-spherical category:

\[
\text{Ker } H \simeq TV(\Sigma, m, [\Phi]),
\]

where \([\Phi] \in H^1(\Sigma, G)\) and the cardinality of \(m\) is the number of vertices in the triangulation \(T\) of \(\Sigma\).

In particular, the ground state of the system is a topological invariant of the surface \(\Sigma\) equipped with the set of points \(m\) and the cohomology class \([\Phi] \in H^1(\Sigma, G)\).

## 6 | HERMITIAN SURGERY TQFT

We review a WRT-style TQFT having a Hermitian structure.

### 6.1 | Hermitian structure on decorated cobordisms

Let \(C\) be a relative \(G\)-Hermitian-modular category. We follow Turaev [55, I.II.5.1] and define the dagger of a \(C\)-colored ribbon graph \(T\) as the following transformation: invert the orientation of the surface of \(T\), reverse directions of bands and annuli of \(T\), exchange bottom and top bases of coupons, and replace the colors \(f\) of coupons with \(f^\dagger\). The colors of edges do not change.

The boundary of a \(C\)-colored ribbon graph is a set of \(C\)-colored framed points where a framed point \(p = (V, \epsilon)\) is a point equipped with a sign \(\epsilon\), a nonzero vector (its framing) tangent to the surface \(T\) and to the direction of the band, and a color that is an object \(V \in C\). Let \(G(p) = V\) if \(\epsilon = +\) and \(G(p) = V^*\) if \(\epsilon = -\), where \(G\) is the Reshetikhin–Turaev functor from Section 4.1.2. The conjugate \(\overline{p}\) of a \(C\)-colored framed point \(p\) is obtained by changing the sign and framing to their opposites.

Recall the category of decorated cobordisms first introduced in [11], for a short description also see [13, Section 1.3].

**Definition 6.1 (Objects of \(\text{Cob}\)).** A decorated surface is a 4-tuple \(\overline{\Sigma} = (\Sigma, \{p_i\}, \omega, \mathcal{L})\) where

- \(\Sigma\) is a closed, oriented surface that is an ordered disjoint union of connected surfaces each having a distinguished base point \(*\);
- \(\{p_i\}\) is a finite (possibly empty) set of framed points colored with objects of \(C\) whose framings are tangent to the surface \(\Sigma\);
- \(\omega \in H^1(\Sigma \setminus \{p_1, \ldots, p_k\}, *; G)\) is a cohomology class;
- compatibility condition: letting \(\omega(m_i) = a_i \in G \setminus \mathcal{X}\) where \(m_i\) is a positively oriented circle around \(p_i\), then we require \(F(p_i) \in C_{a_i}\).
• \( \mathcal{L} \) is a Lagrangian subspace of \( H_1(\Sigma; \mathbb{R}) \).

A decorated surface is **admissible** if each component of \( \tilde{\Sigma} \) either has a framed point colored by a projective object or it contains a closed curve \( \gamma \) such that \( \omega(\gamma) \notin \mathcal{X} \). Objects of \( \text{Cob} \) are admissible decorated surfaces.

**Definition 6.2** (Morphisms of \( \text{Cob} \)). Let \( \tilde{\Sigma} = (\Sigma, \{p_i\}, \omega, \mathcal{L}) \) be admissible, decorated surfaces. A decorated cobordism from \( \tilde{\Sigma}_- \) to \( \tilde{\Sigma}_+ \) is a 5-tuple \( \tilde{M} = (M, T, f, \omega, n) \) where

- \( M \) is an oriented 3-manifold with boundary \( \partial M \);
- \( f : \Sigma_- \sqcup \Sigma_+ \to \partial M \) is a diffeomorphism preserving the orientation, and the image under \( f \) of the base points of \( \Sigma_- \sqcup \Sigma_+ \) is denoted by \( * \);
- \( T \) is a \( \mathcal{C} \)-colored ribbon graph in \( M \) such that \( \partial T = \{f(p^-_i)\} \sqcup \{f(p^+_i)\} \);
- \( \omega \in H^1(M \setminus T, *; \mathbb{G}) \) is a cohomology class relative to the base points on \( \partial M \), such that the restriction of \( \omega \) to \( (\partial M \setminus \partial T) \cap \Sigma_\pm \) is \( (f^{-1})^*(\omega_\pm) \);
- the coloring of \( T \) is compatible with \( \omega \), that is, each oriented edge \( e \) of \( T \) is colored by an object in \( C_{\omega(m_e)} \) where \( m_e \) is the oriented meridian of \( e \);
- all the edges of \( T \) are colored by Hermitian objects of \( \mathcal{C} \);
- \( n \) is an arbitrary integer called the signature-defect of \( \tilde{M} \).

We can summarize the first four items by saying that \( \partial \tilde{M} = \tilde{\Sigma}^-_* \sqcup \tilde{\Sigma}^+_* \) where the dual of a decorated surface \( \tilde{\Sigma} = (\Sigma, \{p_i\}, \omega, \mathcal{L}) \) is defined to be \( \tilde{\Sigma}^* = (\Sigma, \{p_i\}, \omega, \mathcal{L}) \). A decorated cobordism is **admissible** if each component of \( M \) contains either a component of \( T \) with an edge colored by a projective object or it contains a closed curve \( \gamma \) such that \( \omega(\gamma) \notin \mathcal{X} \). This condition is automatically satisfied by components of \( \tilde{M} \) with nonempty boundary. Hence, morphisms of \( \text{Cob} \) are orientation preserving diffeomorphism classes of admissible decorated cobordisms.

**Proposition 6.3.** Let \( \tilde{M} = (M, T, f, \omega, n) : \tilde{\Sigma}_- \to \tilde{\Sigma}_+ \) be a cobordism in \( \text{Cob} \). Then, the following defines a cobordism \( \tilde{M}^\dagger : \tilde{\Sigma}_+ \to \tilde{\Sigma}_- \) in \( \text{Cob} \):

\[
\tilde{M}^\dagger = (\overline{M}, T^\dagger, \overline{f}, \omega, -n),
\]

where \( \overline{M} \) is \( M \) with opposite orientation, and \( \overline{f} = \tilde{\Sigma}_+ \sqcup \tilde{\Sigma}_- \to \partial \tilde{M} \) is the same set-theoretic map as \( f \).

Furthermore, with the above assignment, \( \text{Cob} \) is a Hermitian ribbon category.

**Proof.** The proof is the same as the proof of [23, Proposition 5.4]. \( \square \)

### 6.2 The 3-manifold invariant

Let \( \mathcal{C} \) be a relative \( \mathcal{G} \)-Hermitian-modular category with stabilization coefficients \( \Delta_{\pm \Omega} \). If \( T_{-\Omega_g}^V \) and \( T_{+\Omega_g}^V \) are the graphs on the left side of the equalities in Figure 1 (here the open strand is colored with \( V \)), then

\[
\Delta_{\pm \Omega} \text{Id}_V = G(T_{\pm \Omega_g}^V) = G((T_{\pm \Omega_g}^V)^\dagger) = G(T_{\mp \Omega_g}^{V*})^\dagger = (\Delta_{\mp \Omega} \text{Id}_{V^*})^\dagger = \overline{\Delta_{\mp \Omega}} \text{Id}_V.
\]

Thus, for a relative \( \mathcal{G} \)-Hermitian-modular category, we have \( \Delta_{\pm \Omega} = \overline{\Delta_{\mp \Omega}} \).
Fix a choice of a square root $D_\Omega \in \mathbb{k}$ of $\Delta_\Omega \Delta_\Omega$ (this might require one to replace $\mathbb{k}$ with a quadratic extension) and set $\delta_\Omega = \frac{D_\Omega}{\Delta_\Omega} = \frac{\Delta_\Omega}{D_\Omega}$. Notice that since $\Delta_\pm \Omega = \bar{\Delta}_\mp \Omega$, we are able to deduce that $\bar{D}_{\Omega} = D_{\Omega}$ and $\bar{\delta}_{\Omega} = \delta_{\Omega}^{-1}$.

If $\bar{M} = (M, T, f, \omega, n)$ is a closed decorated manifold (i.e., $\bar{M} \in \text{End}_{\text{Cob}}(\emptyset)$) that is connected, a surgery presentation of $\bar{M}$ is a $C$-colored ribbon graph $T \cup L \subset S^3$ such that $M$ is obtained from $S^3$ by surgery on $L$ and each component of $L$ is colored by a Kirby color $\Omega$ of degree $g$ equal to the value of the cohomology class on its meridian. Then, the invariant of $\bar{M}$ is given by

$$N_c(\bar{M}) = D_{\Omega}^{-1} \delta_{\Omega}^{-\sigma(L)+n} G'(L \cup T),$$

where $l \in \mathbb{N}$ is the number of components of $L$ and $\sigma \in \mathbb{Z}$ is the signature of the linking matrix $\text{lk}(L)$. The invariant is extended multiplicatively for disjoint unions.

**Lemma 6.4.** Let $\bar{M}$ be a closed decorated manifold, then

$$N_c(\bar{M}^\dagger) = N_c(\bar{M}).$$

**Proof.** If a framed link $L$ gives rise via surgery to the manifold $M$, then the manifold $\bar{M}$ may be constructed from $\bar{L}$ where $\bar{L}$ is the mirror image of $L$. It follows from the conjugation on the underlying Hermitian category that $G'(\bar{L} \cup T^\dagger) = G'(L \cup T)$.

If the linking matrix of $L$ has signature $\sigma$, then the signature of the linking matrix of $\bar{L}$ is $-\sigma$. Thus,

$$N_c(\bar{M}^\dagger) = N_c(M, T^\dagger, f, \omega, -n) = D_{\Omega}^{-1} \delta_{\Omega}^{-\sigma(L)-n} G'(\bar{L} \cup T^\dagger) = \frac{D_{\Omega}^{-1} \delta_{\Omega}^{-\sigma(L)+n} G'(L \cup T)}{\delta_{\Omega}^{-\sigma(L)-n} G'(\bar{L} \cup T)}$$

where the third equality follows from facts that $D_{\Omega}$ is real and $\bar{\delta}_{\Omega} = \delta_{\Omega}^{-1}$. \hfill \Box

**Corollary 6.5.** The pairing $\text{Cob}(\emptyset, \Sigma) \times \text{Cob}(\emptyset, \Sigma) \to \mathbb{C}$, given by

$$(\bar{M}_1, \bar{M}_2) \mapsto N_c(\bar{M}_1^\dagger \circ \bar{M}_2) \in \mathbb{C}$$

has Hermitian symmetry.

**Proof.** This follows directly from Lemma 6.4. \hfill \Box

### 6.3 The $(2 + 1)$-TQFT

Fix a choice $g_0 \in G \setminus X$ and a generic Hermitian simple object $V_{g_0}$ in $C_{g_0}$. For $k \in \mathbb{Z}$, let $\hat{S}_k$ be the decorated sphere

$$\hat{S}_k = (S^2, \{(V_{g_0}, 1), (\sigma(k), 1), (V_{g_0}, -1)\}, \omega, \{0\})$$
of Cob that is determined by the framed colored points \{(V_{g_0}, 1), (\sigma(k), 1), (V_{g_0}, -1)\} in \(S^2\). Now we define the state space associated to a decorated surface in the following way:

\[ \mathcal{V}(\Sigma) = \text{Span}_C \{ \text{Hom}_{\text{Cob}}(\emptyset, \Sigma) \}/K_{\Sigma}, \]

where \(K_{\Sigma}\) is the right kernel of the bilinear pairing given on generators

\[ \text{Span}_C \{ \text{Cob}(\Sigma, \emptyset) \} \otimes \text{Span}_C \{ \text{Cob}(\emptyset, \Sigma) \} \rightarrow \text{C} \]

\[ [\tilde{M}_1] \otimes [\tilde{M}_2] \mapsto N_C(\tilde{M}_1 \circ \tilde{M}_2) \]

\[ \forall(\Sigma) = \bigoplus_{k \in \mathbb{Z}} \forall_k(\Sigma) \text{ where } \forall_k(\Sigma) = \forall(\Sigma \cup \tilde{S}_k). \]

These state spaces are part of a \((2+1)\)-TQFT constructed in [11]. Since \(\dagger : \text{Cob}(\emptyset, \Sigma) \rightarrow \text{Cob}(\Sigma, \emptyset)\) is bijective, the pairing described in Corollary 6.5 descends to a nondegenerate Hermitian pairing on the state spaces \(\forall(\Sigma)\) of the TQFT.

**Theorem 6.6.** The TQFT \((\forall, N_C)\) is Hermitian. More specifically, for any decorated surface \(\Sigma\), there is a nondegenerate Hermitian pairing

\[ \langle \cdot, \cdot \rangle_{\forall(\Sigma)} \]

and for any \(y \in \forall(\Sigma_-)\) and for any \(x \in \forall(\Sigma_+)\), and for any decorated cobordism \(\tilde{M} : \Sigma_- \rightarrow \Sigma_+\) between decorated surfaces, there is an equality

\[ \langle x, \forall(\tilde{M})(y) \rangle_{\forall(\Sigma_+)} = \langle \forall(\tilde{M}^\dagger)(x), y \rangle_{\forall(\Sigma_-)} . \quad (6.2) \]

**Proof.** The fact that the pairing is nondegenerate and Hermitian follows from the discussion above.

The equality (6.2) follows from the functoriality of the TQFT and Lemma 6.4. The details follow as in [55, Theorem III.5.3]. \(\square\)

## 7 EXAMPLES FROM QUANTUM (SUPER)GROUPS

In this section, we provide examples of \(G\)-LW-systems coming from quantum groups and super quantum groups.

### 7.1 Complex finite-dimensional Lie algebras

Let \(\mathfrak{g}\) be a simple finite-dimensional complex Lie algebra. Let \(r\) be an odd integer such that \(r \geq 3\) (and \(r \not\in 3\mathbb{Z}\) if \(\mathfrak{g} = G_2\)).

Fix a set of simple roots \(\{\alpha_1, \ldots, \alpha_n\}\) of \(\mathfrak{g}\). Let \(\Delta\) and the \(\Delta^+\) be the corresponding sets of roots and positive roots, respectively. Let \(A = (a_{ij})\) be the Cartan matrix corresponding to the set of simple roots. Let \(D = (d_1, \ldots, d_n)\) be a diagonal matrix such that \(DA\) is symmetric and positive definite. Let \(\mathfrak{h}\) be the Cartan subalgebra of \(\mathfrak{g}\) spanned by elements \(H_1, \ldots, H_n\) where \(\alpha_i(H_j) = a_{ji}\). Denote by \(L_R\) (the root lattice), the \(\mathbb{Z}\)-module generated by the simple roots \(\{\alpha_i\}\). Let \((\cdot, \cdot)\) be the form on \(L_R\) determined by \(\langle \alpha_i, \alpha_j \rangle = d_ia_{ij}\). The weight lattice \(L_W\) is the \(\mathbb{Z}\)-module generated by the elements
of \( \mathfrak{h}^* \) that are dual to the elements \( \{ H_1, \ldots, H_n \} \). Let \( L'_R = \{ \beta \in \mathfrak{h}^* : \langle \beta, L_R \rangle \subset \mathbb{Z} \} \) be the lattice dual to \( L_R \). We have \( L_R \subset L_W \subset L'_R \subset \mathfrak{h}^* \). Let \( \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \in L_W \).

Let \( q = e^{2\pi i \frac{r}{\rho}} \) and for \( i = 1, \ldots, n \) set \( q_i = q^{d_i} \).

The Drinfeld–Jimbo quantum group \( U \) associated to \( \mathfrak{g} \) is the \( \mathbb{C} \)-algebra generated by \( K_\beta, X_i, X_{-i} \) for \( \beta \in L_W \) and \( i = 1, \ldots, n \) subject to relations:

\[
K_0 = 1, \quad K_\beta K_\gamma = K_{\beta + \gamma}, \quad K_\beta X_{\pm i} K_{-\beta} = q^{\pm \langle \beta, \alpha_i \rangle} X_{\pm i},
\]

\[
[ X_i, X_{-j} ] = \delta_{ij}, \quad \sum_{k=0}^{1-a_{ij}} (-1)^k \left[ 1 - \frac{a_{ij}}{k} \right] \frac{K_{\alpha_i} - K_{-\alpha_i}^{-1}}{q_i - q_i^{-1}} = 0, \text{ if } i \neq j.
\]

The quantum group \( U \) can be given the structure of a Hopf algebra with coproduct \( \Delta \), counit \( \varepsilon \), and antipode \( S \) defined by

\[
\Delta(X_i) = X_i \otimes K_{\alpha_i} + 1 \otimes X_i, \quad \varepsilon(X_i) = 0, \quad S(X_i) = -X_i K_{\alpha_i}^{-1},
\]

\[
\Delta(X_{-i}) = X_{-i} \otimes 1 + K_{\alpha_i}^{-1} \otimes X_{-i}, \quad \varepsilon(X_{-i}) = 0, \quad S(X_{-i}) = -K_{\alpha_i} X_{-i},
\]

\[
\Delta(K_\beta) = K_\beta \otimes K_\beta, \quad \varepsilon(K_\beta) = 1, \quad S(K_\beta) = K_{-\beta}.
\]

The unrolled quantum group \( U^H \) associated to \( \mathfrak{g} \) is the \( \mathbb{C} \)-algebra generated by \( K_\beta, X_i, X_{-i}, H_i \) for \( \beta \in L_W \) and \( i = 1, \ldots, n \) subject to relations (7.1), (7.2), and:

\[
[H_i, X_{\pm j}] = \pm a_{ij} X_{\pm j}, \quad [H_i, H_j] = [H_i, K_\beta] = 0.
\]

The algebra \( U^H \) is also a Hopf algebra with coproduct, counit, and antipode defined by (7.3), (7.4), (7.5), and

\[
\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \varepsilon(H_i) = 0, \quad S(H_i) = -H_i.
\]

The Hopf algebra \( U^H \) is pivotal with pivot \( g = K_{\frac{1}{2\rho}}^1 \).

Let \( \alpha_1, \ldots, \alpha_n \) be any ordering of the simple roots and let \( s_{i_1} \ldots s_{i_N} \) be a reduced decomposition of the longest element of the Weyl group. Then, \( \beta_1 = \alpha_1, \beta_2 = s_{i_1} \alpha_2, \ldots, \beta_N = s_{i_1} \ldots s_{i_{N-1}} \alpha_N \) is a total ordering of the set of positive roots. We call \( \beta_* = (\beta_1, \ldots, \beta_N) \) the convex order of \( \Delta^+ \). Let \( X_{\pm \beta_\gamma} \) be the corresponding positive and negative root vectors, respectively (see [6, Definition 8.1.4]). Denote by \( \mathcal{Z}^0 \) the commutative subalgebra

\[
\mathcal{Z}^0 = \mathbb{C}\langle X_{\pm \beta}, K_\gamma^\tau \mid \beta \in \beta_*, \gamma \in L_W \rangle.
\]

Let \( I = \text{Span}_\mathbb{C} \{ X_{\pm \beta} \mid \beta \in \beta_* \} \). \( I \) generates ideals in the algebras \( U \) and \( U^H \) that are Hopf ideals.

Correspondingly, we call \( \overline{U} \) and \( \overline{U}^H \) the quotient Hopf algebras.

A weight is a \( \mathbb{C} \)-algebra homomorphism \( \lambda \) from the \( \mathbb{C} \)-algebra generated by the \( H_i \) to \( \mathbb{C} \). For a \( \overline{U}^H \)-module \( M \), the weight space \( M_\lambda \) is the subspace of \( M \) where the \( H_i \) act as a scalar via \( \lambda \) and if \( \beta = \sum_i b_i \alpha_i \), then \( K_\beta \) acts as the scalar \( \prod_i q_i^{b_i \lambda(H_i)} = q^{\langle \beta, \lambda \rangle} \). A \( \overline{U}^H \)-module \( M \) is called a weight module if it is equal to a direct sum of its weight spaces.
Let $D$ be the category of finite-dimensional weight modules over $\overline{U}^H$. Let $G$ be the group $\mathfrak{h}^*_\mathbb{R}/L_\mathbb{R} \cong (\mathbb{U})^n$ where $\mathfrak{h}^*_\mathbb{R} = L_W \otimes \mathbb{R}$ is the set of real weights and $U \cong S^1$ is the group of $1 \times 1$ unitary matrices. For $g \in G$, let $D_g$ be the full subcategory of $D$ whose objects are the $\overline{U}^H$-weight modules whose weights are all in the class $g$ (mod $L_\mathbb{R}$). Define

$$\exp_q^<(x) = \sum_{0 \leq n < r} q^{n(n-1)/2} [n]! x^n \in \mathbb{C}[x].$$

(7.8)

**Lemma 7.1.** We have the identity $\exp_q^<(x) \exp_q^{-1}(-x) = 1$ modulo $x'$.

For a root $\beta$, let $q_\beta = q^{\langle \beta, \beta \rangle}$.

$$\hat{R} = \prod_{\beta \in \beta_*} \exp_q^<((q_\beta - q_\beta^{-1})X_\beta \otimes X_{-\beta}).$$

(7.9)

The following elements define operators on the category $D$ (alternatively, they are elements of the nuclear completion of $\overline{U}^H$, see [19])

$$H = q^{\sum_{i,j} d_i(A^{-1})_{ij}H_i \otimes H_j}, \quad R = H\hat{R}.$$ 

(7.10)

Consider the operator $u_H = q^{\sum_{i,j} -d_i(A^{-1})_{ij}H_i \otimes H_j}$. Note that $S(u_H) = u_H$ and if $M$ is a $\overline{U}^H$-module and $m \in M$ is an element of weight $\lambda$, then $u_H m = q^{-\langle \lambda, \lambda \rangle} m$. Also, note that if $x \in \overline{U}^H$ has weight $\alpha$, then $u_H x = q^{-\langle \alpha, \alpha \rangle} x K_{-\alpha}^2 u_H$. Then, conjugation $Ad_{u_H}$ by $u_H^{-1}$ is an automorphism of $\mathbb{C}[H_1, \ldots, H_n]$. Let $u = \mu o(Ad_{u_H} \circ S) \otimes \text{Id})(\hat{R})$, where $\mu$ is multiplication. Let $\nu$ and $\vartheta$ be the operators defined by $\nu = u_H u$ and $\vartheta = \nu K_{-\rho}^- = \nu g^{-1}$. We now recall the following key result.

**Proposition 7.2** [30, Theorem 19]. The category $D$ is a $\mathbb{C}$-linear ribbon category with twist $\vartheta_M : M \to M, m \mapsto \vartheta^{-1}(m)$.

**Lemma 7.3.** There is an antiautomorphism $\dagger$ of $\overline{U}^H$ (as in Equation (3.11)) given by

$$\dagger(F_i) = F_i, \quad \dagger(F_i) = E_i, \quad \dagger(K_\mu) = K_{-\mu}, \quad \dagger(H_i) = H_i.$$ 

(7.11)

We will write $x^\dagger$ in place of $\dagger(x)$ when no confusion is likely to arise.

**Proof.** This is a straightforward verification.

**Lemma 7.4.** For all finite-dimensional simple complex Lie algebras, one has the identities:

$$(\dagger \otimes \dagger)(R) = \tau(R^{-1}) \quad \text{and} \quad \dagger(g) = g^{-1}.$$ 

(7.12)

As a consequence, $\dagger(\vartheta) = \vartheta^{-1}$.

**Proof.** First, note that $(\exp_{q_\beta}^<(X))^{-1} = \exp_{q_\beta^{-1}}^<(-X)$ and $g = K_{2\rho}^{1-r}$. It is straightforward to check that $\dagger(X_\beta) = X_{-\beta}$. One way to see this is to use the fact that $\dagger$ commutes with Lusztig’s braid operators. The first result now follows.
It is clear that \( \dagger (g) = g^{-1} \) and we note further that the third equality is a consequence of the two first.

Let \( D^{\text{Sesq}} \) be the category of sesquilinear weight modules over \( \overline{U}^H \) with real weights (see Equation (3.15)). Recall that a sesquilinear module \( W \) is Hermitian if its sesquilinear form has Hermitian symmetry: if for any \( v_1, v_2 \in W \), \( (v_2|v_1) = (v_1|v_2) \). The next lemma is similar to [23, Proposition 4.5].

**Lemma 7.5.** Any simple module \( V \) in \( D \) with real highest weight has a sesquilinear structure. Moreover, if \( v_0 \) is a highest weight vector of \( V \), there is a unique form \( (\cdot, \cdot) \) on \( V \) such that \( (v_0, v_0) = 1 \) that is Hermitian.

**Proof.** Let \( V \) be a simple module in \( D \) with real weights. Since \( H_i \dagger = H_i \), we have that \( V^\dagger \) is a simple module with a character that is the conjugate of that of \( V^* \) (where the \( \overline{U}^H \)-action on \( V^\dagger \) is given in Equation (3.19)). Since the weights of the module \( V \) are real, \( V^\dagger \) and \( V^* \) have the same character, so they are isomorphic. Then, Lemma 3.11 applies. Finally, the form is Hermitian by Lemma 3.12.

Let \( Z = rL'_R \cap L_R \). For \( k \in Z \), define \( \sigma(k) \) to be the vector space \( \mathbb{C} \) with the action determined by \( E_i, F_i, H_i - k(H_i) \) act by zero. This module is Hermitian with form \( (z_1, z_2) = \overline{z_1}z_2 \).

**Proposition 7.6.** The category \( D^{\text{Sesq}} \) is a sesquilinear ribbon category and a relative modular category with grading group \( G = \mathfrak{h}_R^*/L_R \), symmetric small set

\[
\mathcal{X} = \{ \{ \xi \} \in G \mid \text{there exists } \alpha \in \Delta^+ \text{ satisfying } 2\langle \alpha, \xi \rangle \in \mathbb{Z} \},
\]

free realization \( \{ \sigma(k) \}_{k \in rL'_R \cap L_R} \), and bilinear map \( \psi : \mathcal{X} \times Z \to \mathbb{C}^* \) given by \( \langle \{ \xi \}, k \rangle \mapsto q^{2\langle \xi, k \rangle} \).

**Proof.** By [13, Theorem 1.3], the category \( D \) is a relative modular category with grading group \( \mathfrak{h}_R^*/L_R \) and the free realization given in the statement of the theorem we are proving. Since \( \mathfrak{h}_R^*/L_R \) is a subgroup of \( \mathfrak{h}_R^*/L_R \), this structure induces a relative modular structure on \( D^{\text{Sesq}} \). By Lemma 7.4, the dagger on \( \overline{U}^H \) satisfies Equation (3.12), and so by Theorem 3.10, the category \( D^{\text{Sesq}} \) is a sesquilinear ribbon \( \mathbb{C} \)-category.

Let \( S \) be the set of all objects of \( D^{\text{Sesq}} \) such that the modules are simple and their forms are Hermitian. As above, such modules are called Hermitian. Let \( D^{\text{Herm}} \) be the full subcategory \( D^{\text{Sesq}} \) generated by the set \( S \) as defined in Section 4.3. By Lemma 7.5, every simple module in \( D^{\text{Sesq}} \) is isomorphic to a simple module endowed with a Hermitian form. Lemma 3.8 implies that if \( V, W \) are objects of \( D^{\text{Herm}} \) and any \( f \in \text{Hom}_C(V, W) \), then \( (f^\dagger)^\dagger = f \). Finally, Proposition 3.13 implies that a semisplit tensor product of two Hermitian simple objects in \( D^{\text{Sesq}} \) is also Hermitian. Therefore, the set \( S \) is a collection of Hermitian objects as defined in Section 4.3. Thus, Corollary 4.9 applies and we have the following theorem.

**Theorem 7.7.** The category \( D^{\text{Herm}} \) is a relative Hermitian-modular category with \( G = \mathfrak{h}_R^*/L_R \), \( \mathcal{X} \), \( \{ \sigma(k) \}_{k \in rL'_R \cap L_R} \) and \( \psi \) as in Proposition 7.6.
Let us increase $\mathcal{X}$ to $\mathcal{X}' = \{ [\xi] \in \mathcal{G} \mid \text{there exists } \alpha \in \Delta^+ \text{ satisfying } 6\langle \alpha, \xi \rangle \in \mathbb{Z} \}$. Then, $\mathcal{G} \setminus \mathcal{X}'$ has no elements of order 2 or 3. Recall the construction of the category $C^2$ from Definition 4.12.

**Theorem 7.8.** The category $(D^{\text{Herm}})^2$ is a relative Hermitian-spherical category with grading group $\mathcal{G} = h^*_R / L_R$, symmetric small set

$$\mathcal{X}' = \{ [\xi] \in \mathcal{G} \mid \text{there exists } \alpha \in \Delta^+ \text{ satisfying } 6\langle \alpha, \xi \rangle \in \mathbb{Z} \},$$

and where the map $b$ is the constant $r^{-\text{card}(\Delta_+)}(\text{card}(L_R / \mathbb{Z}))^{-1}$.

**Proof.** It suffices to show that the hypothesis of Theorem 4.14 are satisfied. Theorem 7.7 implies that $D^{\text{Herm}}$ is a relative Hermitian-premodular category. Let $\sqrt{\psi} : \mathcal{G} \times \mathbb{Z} \to \mathbb{U}$ be defined by

$$\sqrt{\psi}([\gamma], \lambda) = q^{\langle \gamma, \lambda \rangle}.$$

If $V$ is a module of degree $[\gamma]$, the $R$-matrix acts on $V \otimes \sigma(\lambda)$ and $\sigma(\lambda) \otimes V$ by the scalar $\sqrt{\psi}([\gamma], \lambda)$. In particular, $c_{\sigma(\lambda), \sigma(\mu)} = \text{Id}_{\sigma(\lambda + \mu)}$. Thus, $\sqrt{\psi}$ satisfies (4.13) and the braiding assumption in Equation (4.14).

Finally, generic simple modules of $D^{\text{Herm}}$ are free (generated by their highest weight vectors) over the negative Borel of $U_q \mathfrak{g}$ whose dimension is $r^{\text{card}(\Delta_+)}$ (by the Poincaré Birkhoff Witt theorem). Their isomorphism classes in generic degree $g$ are in bijection with the set of weights of degree $g$ on which $L_R$ acts simply transitively. But two such modules are isomorphic in $(D^{\text{Herm}})^2$ if and only if they belong to the same $\mathbb{Z}$ orbit. Thus, there are $\text{card}(L_R / \mathbb{Z})$ isomorphism classes in $(D^{\text{Herm}})^2$. Since the objects of the category $(D^{\text{Herm}})^2$ are vector spaces (with usual direct sum and tensor product of vector spaces), Formula (4.10) can be used to define the map $b$ that is the given constant map.

**Remark 7.9.** It is not clear if the category of weight modules over $\overline{U}$ (see [29]) is a relative Hermitian-spherical category. If so, it would be interesting to check if it is equivalent to the category $(D^{\text{Herm}})^2$.

### 7.2 The special linear quantum Lie superalgebra

Here, we consider two categories of modules associated to the unrolled quantum group of $\mathfrak{sI}(m|n)$. We use the notation of [26], see this paper for more details.

We assume that $m \neq n$ are nonzero positive integers, $r \geq r_k = m + n - 1$ is an odd integer, and $q$ is the root of unity $q = \exp(2i\pi / r)$. Let $A$ be the Cartan matrix of $\mathfrak{sI}(m|n)$:

$$A_{ii} = 2 \quad \text{for } i \neq m, \quad A_{mm} = 0, \quad A_{i,i\pm 1} = -1 \quad \text{for } i \neq m, \quad A_{m,m\pm 1} = \pm 1, \quad A_{ij} = 0 \quad \text{otherwise.}$$

(7.13)

Define $d_i = 1$ if $i < m$ and $d_i = -1$ if $i > m$. Then, $(d_i A_{ij})$ is a symmetric matrix.

Let $q$ an $r$th root of unity. Let $\Delta^\pm = \Delta^\pm_0 \cup \Delta^\pm_1$ to be the set of positive and negative roots, respectively. Here, the subscripts $\bar{0}$ and $\bar{1}$ denote the even and odd roots, respectively. Let $\rho_0$ and $\rho_1$ be half the sum of the positive even and odd roots, respectively. Let $\rho = \rho_0 - \rho_1$. A positive root is called simple if it cannot be decomposed into a sum of two positive roots. Denote by $L_R$ (the root lattice), the $\mathbb{Z}$-module generated by the simple roots $\{\alpha_i\}$, see [26].
Definition 7.10. Define $U_q\mathfrak{sl}(m|n)$ as the $\mathbb{C}$-algebra with generators $E_i, F_i, K_i, K_i^{-1}$ with $i = 1, \ldots, m + n - 1$ satisfying relations

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$K_i E_j K_i^{-1} = q^{d_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-d_{ij}} F_j,$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}, \quad E_i^2 = F_i^2 = 0,$$

and for $X_i = E_i, F_i$,

$$[X_i, X_j] = 0 \quad \text{if } |i - j| > 2,$$

$$X_i^2 X_j - (q + q^{-1}) X_i X_j X_i + X_j X_i^2 = 0 \quad \text{if } |i - j| = 1, \text{ and } i \neq m,$$

$$X_m X_{m-1} X_m X_{m+1} + X_m X_{m+1} X_m X_{m-1} + X_{m-1} X_m X_{m+1} X_m + X_{m+1} X_m X_{m-1} X_m - (q + q^{-1}) X_m X_{m-1} X_{m+1} X_m = 0.$$

All generators are even except for $E_m, F_m$ that are odd and $[-,-]$ is the supercommutator given by $[x, y] = xy - (-1)^{|x||y|}yx$. $U_q(\mathfrak{sl}(m|n))$ can be given the structure of a Hopf superalgebra with coproduct $\Delta$, counit $\varepsilon$, and antipode $S$ by

$$\Delta(E_i) = E_i \otimes 1 + K_i^{-1} \otimes E_i, \quad \varepsilon(E_i) = 0, \quad S(E_i) = -K_i E_i,$$

$$\Delta(F_i) = F_i \otimes K_i + 1 \otimes F_i, \quad \varepsilon(F_i) = 0, \quad S(F_i) = -F_i K_i^{-1},$$

$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\mp 1}, \quad \varepsilon(K_i^{\pm 1}) = 1, \quad S(K_i^{\pm 1}) = K_i^{-\mp 1}.$$

The unrolled quantum group $U^H_q \mathfrak{sl}(m|n)$ is the $\mathbb{C}$-algebra with generators $H_i, E_i, F_i, K_i^{\pm 1}$ with $i = 1, \ldots, m + n - 1$ and relations (7.14)-(7.19) plus the relations

$$[H_i, H_j] = [H_i, K_j^{\pm}] = 0, \quad [H_i, E_j] = a_{ij} E_j, \quad [H_i, F_j] = -a_{ij} F_j,$$

$$E_\alpha^r = F_\alpha^r = 0, \quad \text{for all } \alpha \in \Delta^+_0,$$

where $E_\alpha$ and $F_\alpha$, for $\alpha \in \Delta^+$, are the $q$-analogos of the Cartan–Weyl generators such that $E_{\alpha_i} = E_i$ and $F_{\alpha_i} = F_i$ for any simple root $\alpha_i$ (see [60, Section 5.1.1]). The $\mathbb{C}$-algebra $U^H_q \mathfrak{sl}(m|n)$ becomes a Hopf algebra with coproduct, counit, and antipode satisfying relations (7.20)–(7.22) and

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \varepsilon(H_i) = 0, \quad S(H_i) = -H_i.$$

We call a $U^H_q \mathfrak{sl}(m|n)$-module $V$ a weight module if it is finite-dimensional and the following hold.

(1) The elements $H_i, i = 1, \ldots, m + n - 1$ act semisimply on $V$. 


\( K_i = q^{d_i H_i} \) as operators on \( V \).

We denote by \( D \) the category whose objects are \( U^H_q \mathfrak{sl}(m|n) \) weight modules with real weights where the morphisms of \( D \) are equivariant linear maps that respect the parity (i.e., all morphisms are even). Let \( \mathfrak{h} := \text{Span}\{H_1, \ldots, H_{m+n-1}\} \). Any weight module \( V \) admits a basis \( \{v_j\}_{j \in I} \) and a family of linear functionals \( \{\lambda_j\}_{j \in I} \in \mathfrak{h}^* \) such that

\[
H_i v_j = \lambda_j(H_i)v_j
\]

for all \( i \) and \( j \). We call \( \lambda_j \) the weight of \( v_j \) and a weight of \( V \) if it is a weight for some \( v \in V \). We call a weight \( \lambda \in \mathfrak{h}^* \) of a module \( V \) perturbative if \( \lambda(H_i) \in \mathbb{Z} \) for all \( i \neq m \) and a \( U_q^{H} \mathfrak{sl}(m|n) \)-module is called perturbative if all of its weights are. Let \( D^{\text{pert}} \) be the full subcategory \( D \) of perturbative \( U_q^{H} \mathfrak{sl}(m|n) \)-modules.

Proposition 3.20 of [26] implies that both \( D \) and \( D^{\text{pert}} \) are pivotal categories where the pivotal element is defined to be \( K_\pi = \prod_i K_i^{n_i} \in U^H_q \mathfrak{sl}(m|n) \) where \( \pi = 2\rho - 2r_0 = \sum_i n_i \alpha_i \) for some \( n_i \).

Khoroshkin, Tolstoy [39], and Yamane [60] showed that the \( h \)-adic completion version of quantum \( \mathfrak{sl}(m|n) \) has an \( R \)-matrix. This leads to the following truncated \( R \)-matrix and braiding in \( D \) and \( D^{\text{pert}} \), see [26, Section 3.2] for details.

We follow the exposition of [39]. First, let \( (d_{ij}) \) be the inverse of the matrix \( (a_{ij}/d_j) \). Set

\[
\mathcal{H} = q^{\sum_i d_i H_i \otimes H_i}, \tag{7.24}
\]

Then, the \( R \)-matrix is of the form

\[
R = \tilde{R}\mathcal{H}, \tag{7.25}
\]

where \( \tilde{R} \) is described as follows. Consider the truncated quantum exponential:

\[
\exp_q < (x) := \sum_{n=0}^{r-1} \frac{x^n}{(n)_q!}.
\]

Let

\[
\tilde{R} = \prod_{\alpha \in \Delta^+} \exp_{q_{\alpha}} < (-1)^{\bar{\alpha}} a_{\alpha}^{-1} (q - q^{-1}) (E_{\alpha} \otimes F_{\alpha})), \quad (n)_q = \frac{1 - q^n}{1 - q},
\]

where again \( E_{\alpha} \) and \( F_{\alpha} \) are the \( q \)-analogues of the Cartan–Weyl generators (see [39, Definition 3.2]), \( q_{\alpha} = (-1)^{|\alpha|} q^{-\langle\alpha, \alpha\rangle} \) (here \( |\alpha| \) is the parity of \( E_{\alpha} \)) and \( a_{\alpha} \) is defined by

\[
[E_{\alpha}, F_{\alpha}] = a_{\alpha} (q^{h_{\alpha}} - q^{-h_{\alpha}})/(q - q^{-1})
\]

for \( \alpha \in \Delta^+ \) and \( h_{\alpha} := [e_{\alpha}, f_{\alpha}] \) is a relation in the Lie algebra. The ordering in the product of \( \tilde{R} \) is given by a chosen fixed normal ordering of \( \Delta^+ \).

Recall our conventions for antiautomorphisms from (3.11).

**Definition 7.11.** Consider the bijection \( \dagger \) of \( U_q^H(\mathfrak{g}) \) defined by

\[
\dagger(E_i) = F_i \quad i \neq m, \quad \dagger(E_m) = -F_m, \quad \dagger(F_i) = E_i, \quad \dagger(K_i) = K_i^{-1}, \quad \dagger(H_i) = H_i. \tag{7.26}
\]

Observe that this map is a superinvolution.
Lemma 7.12. The map $\dagger : U_q^H (g) \to U_q^H (g)$ is a antisuperautomorphism as in Equation (3.11).

Proof. First observe that the effect of the flip map and superflip map $\tau$ on the output of $\Delta$ on any generator is identical. It is therefore straightforward as in the non-supercase to see that $\Delta(X^\dagger) = (\tau \Delta(X))^\dagger$ for all generators $X$. \hfill $\square$

Lemma 7.13. One has the identity:

$$(\dagger \otimes \dagger)(R) = \tau (R^{-1}).$$ (7.27)

Proof. The result follows easily from $(\exp_{q^2}(X))^{-1} = \exp_{q^{-2}}(-X)$ and the fact that $\dagger(E_\alpha) = \pm F_\alpha$ and $\dagger(F_\alpha) = \pm E_\alpha$. Note that in the super case, a sign arising from $\tau$ gets canceled by a sign coming from the definition of $\dagger$ on odd elements. \hfill $\square$

Lemma 7.14. The dagger of the pivotal element is given by $K^\dagger = K^{-1}$.

Proof. This follows trivially from the definition of $\dagger$ on this super Hopf algebra. \hfill $\square$

Proposition 7.15. The category $\mathcal{D}$ of weight modules is a $\mathbb{C}$-sesquilinear ribbon category.

Proof. By [26], $\mathcal{D}$ is a ribbon category. Lemmas 7.13 and 7.14 show that the dagger involution is compatible with the braiding and pivotal structure, and consequently, the ribbon structure. \hfill $\square$

Lemma 7.16. Any simple module $V$ in $\mathcal{D}$ with real highest weight has a sesquilinear structure. Moreover, if $v_0$ is a highest weight vector of $V$, there is a unique form $(\cdot, \cdot)$ on $V$ such that $(v_0, v_0) = 1$ that is Hermitian.

Proof. The isomorphism class of a simple module $V$ of $\mathcal{D}$ is uniquely determined by its highest weight and the parity of its highest weight vector. Since the weights of the module $V$ are real, $V^\dagger$ and $V^*$ have the same character. Also, the highest weight space of $V^\dagger$ and $V^*$ both has the parity of the lowest weight space of $V$, so $V^\dagger \simeq V^*$ with an even isomorphism. Then, Lemma 3.11 applies. Finally, the form is Hermitian by Lemma 3.12. \hfill $\square$

Following [26], let $L_W$ be the weight lattice, which is the $\mathbb{Z}$-lattice generated by the fundamental dominant weights. Let $\mathcal{G}$ be the group $\mathfrak{h}_R^+ / L_R \simeq (\mathbb{U})^{m+n-1}$ where $\mathfrak{h}_R^+ = L_W \otimes \mathbb{R}$ is the set of real weights. Let $\mathcal{X}' = \{ g \in \mathcal{G} : C_g \text{ is not semisimple or } 6g = 0 \}$. Then, $\mathcal{G} \setminus \mathcal{X}'$ is a Zariski open dense subset of $\mathcal{G}$ and, in particular, $\mathcal{X}'$ is small. Let $Z = Z/2Z \times Z_0$ where $Z_0 = rL_W \cap L_R$. For $z = (p, \lambda) \in Z$, $\sigma(z)$ is the unique module structure on the vector space $\mathbb{C}$ with parity $p$, weight $\lambda$, and Hermitian form $(\langle v \mid v' \rangle = \bar{v}v'$. The bilinear pairing $\psi : \mathcal{G} \times Z \to \mathbb{C}$ is given by $\psi(\langle (\zeta, (p, k)) = q^{2\langle \zeta, k \rangle}$. The m-trace on projective modules is discussed in [26, Section 3.7]. The relative modularity of $D^{sesq}$ follows from [26, Theorem 3.37]. Equation (4.14) applies if we restrict the translation group to $Z_0 \simeq \{0\} \times Z_0 \subset Z$. The rest is similar to the Lie algebra case and we have the following.

Theorem 7.17. With the above data associated to $U_q \mathfrak{sl}(m|n)$ at an odd root of unity, we have the following.
(1) The category $D^\text{sesq}$ of sesquilinear real weight module is ribbon sesquilinear.
(2) Its subcategory $D^\text{Herm}$ generated by simple Hermitian modules $a$ is relative Hermitian-modular giving rise to a Hermitian $(2+1)$-TQFT.
(3) The associated category $(D^\text{Herm})^Z_0$ with the map $b$ is given by (4.10), is relative Hermitian-spherical giving rise to a relative $G$-LW-system.

Next, we define a relative premodular category from the subcategory of perturbative modules by taking a quotient with respect to the negligible morphism of a different $m$-trace than the $m$-trace used above, and on a different ideal than Proj, see [26, Section 4.3] for more details. An interesting feature of the associated ribbon graph invariants is that they are specializations at root of unity of invariants coming from quantum $U_q\mathfrak{sl}(m|n)$ for generic $q$ and, in particular, they are related to the perturbative expansion after setting $q = \exp(h)$. Let us quickly recall the categories involved.

There exists a replete collection of objects in $D^\text{op}$ called negligible, see [26, Definition 5.5]. A morphism $f$ of $D^\text{op}$ is called negligible if it factors through a negligible object. Note that we automatically have that $f^\dagger$ is negligible if and only if $f$ is negligible. Then, $D^\text{N}$ is a quotient of a full subcategory of $D^\text{op}$ by the negligible morphisms (Hom$_{D^\text{op}}(V, W)$ is the quotient of Hom$_{D^\text{op}}(V, W)$ by the vector space of negligible morphisms). Let $Z^\text{op} = \mathbb{Z}/2\mathbb{Z} \times Z_0^\text{op} \subset Z$ where $Z_0^\text{op} \simeq Z$ is the subgroup of $Z_0$ generated by $\frac{|m-n|}{\gcd(m,n)}r$ times the $m$th fundamental weight. Then, the realization $\sigma$ induces by restriction a functor $\sigma^\text{N} : Z^\text{op} \to D^\text{N}$. The categories $D^\text{op}$ and $D^\text{N}$ are $\mathcal{U}$-graded where a module is of degree $z$ if $K^r_m$ acts by the scalar $z$ on it. Let $\mathcal{U} = \{\pm 1\}$.

**Theorem 7.18** [26, Theorem 5.17]. $D^\text{N}$ is a premodular $\mathcal{U}$-category relative to $(Z^\text{op}, \mathcal{U}^\text{op})$.

Then, Equation (4.14) holds if we restrict the translation group to $Z_0 \simeq \{0\} \times Z_0 \subset Z$. Moreover, $\sqrt{\psi} : \mathcal{G} \times Z \to \mathcal{U}$ defined by $\sqrt{\psi}([\zeta], (p, k)) = q^{\langle \zeta, k \rangle}$ satisfies Equation (4.13). Thus, the category $(D^\text{N})^\text{Herm}Z_0$ satisfies all the hypothesis of Theorem 4.14 except the existence of a map $b$. We are lead to the following conjecture.

**Conjecture 7.19.** The category $(D^\text{N})^\text{Herm}Z_0$ admits a map $b$ satisfying Equation (4.9). A candidate is given on generic simple $V$ of degree $g$ by

$$b(V) = \text{oqdim}(V) / \sum_{V'} \text{oqdim}(V')^2,$$

where the sum is over isomorphism classes of simples of degree $g$ and oqdim is the ordinary (non-super) quantum dimension of the underlying $U_q\mathfrak{g}_0$-module given by $\text{oqdim}(V) = \text{tr}_V(K_{2p_0})$.

In [25], Conjecture 7.19 is shown to hold for $\mathfrak{sl}(2|1)$.

**Theorem 7.20.** If Conjecture 7.19 is true, the category $(D^\text{N})^\text{Herm}Z_0$ is relative Hermitian-spherical giving rise to a relative $\mathcal{U}$-LW-system.

In particular, $(D^\text{N})^\text{Herm}Z_0$ is relative Hermitian-spherical giving rise to a relative $\mathcal{U}$-LW-system for $\mathfrak{sl}(2|1)$.

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