ABSTRACT

A dynamical system, called a binary closed chain of contours, is studied. The dynamical system belongs to the class of Buslaev networks. The system contains $N$ contours. There are two cells and a particle in each contour. There are two adjacent contours for each contour. There is a common point of adjacent contours. This common point is called a node. The node is located between the cells. In the deterministic version of the system, at any discrete moment, each particle moves to the other cell of the contour if there is no delay. The delays are due to the fact that two particles may not pass through the common node simultaneously. If two particles try to cross the same node, then a competition occurs, and only one of these particles moves in accordance with a prescribed competition resolution rule. In the stochastic version of the system, each particle moves with the probability $1 - \varepsilon$, if the system is in the state such that, in the same state of the deterministic system, this particle moves. where $\varepsilon$ is a small value. We have obtained a competition resolution rule such that the system results in a state such that all particles move without delays in present time and in the future (a state of free movement), and the system results in the state of free movement over a minimum time. The expectation of the number of the $i$th particle transitions per a time unit is called the average velocity of this particle, $v_i$, $i = 1, \ldots, N$. For the stochastic version of the system, under the assumption that $N = 3$, we have proved the following. For the optimal rule, the average velocity of particles is equal to $v_1 = v_2 = 1 - 2\varepsilon + o(\varepsilon)$ ($\varepsilon \to 0$). For the left-priority rule, which is studied earlier, the average velocity of particles equals $v = v_1 = v_2 = \frac{6}{7} + o(\sqrt{\varepsilon})$.

Keywords  Dynamical systems · Buslaev networks · Binary chain · Optimal control · Competition resolution rule

1 Introduction

Introduction.

A class of mathematical models is formed by dynamical systems such that in these systems particles move on closed or infinite one-dimensional lattices. A class of mathematical traffic models is formed by dynamical systems such that, in these systems, particles move on closed or infinite one-dimensional lattices. For simple versions of these systems, results have been obtained, for example in [1]–[8]. More complicated systems of these class, in particularly, models with network structure, are investigated mainly by simulation. Models of these class can be interpreted in terms of cellular automata [9], asynchronous exclusion processes [10], or synchronous exclusion processes, [9]. In [11]–[12], traffic models are investigated such that, in these models, particles move in two perpendicular directions on a toroidal lattice.
In [13], a dynamical system is studied such that particles move in channels of this system under a prescribed plan.

In [14], the concept of cluster movement for mathematical models of traffic were introduced. In the discrete versions, the clusters are groups of particles located in adjacent cells and moving simultaneously. In the continuous versions the clusters are moving segments. A.P. Buslaev has introduced a class of dynamical systems called contour networks (Buslaev networks) [15]. A contour networks contains contours. There are common points of adjacent contours called nodes. There are particles or clusters on the contours. The particles (clusters) move in accordance with a prescribed rule. Delays occur at the nodes. The delays are due to that the particles (clusters) may not pass through the node simultaneously. Contour networks may be traffic models and may have other applications. In particular, contour networks also used for modeling the work of communication systems. In [15]–[23] analytical results were obtained for contour networks.

In [15], a dynamical system, called a closed binary chain of contours, is considered. There are two cells and a particle in each contour. There are two nodes in each contour. These nodes are common points for this node and two adjacent contours. If two particles try to pass through the same node simultaneously, then a competition occurs. The paper [16] studies binary chains with three rules for competition resolution. These rules are a stochastic rule in accordance with each of two competing particles passes through the node first (wins the competition) equiprobably, the left-priority rule (the particle, located on the left wins), and the rule such that, in accordance with this rule the particle, located on the contour with an even index, wins. The paper [17] studies a stochastic version of the system consider in [16]. If the stochastic system is in the state such that a particle moves in deterministic version, then the probability that the particle moves is \( 1 - \varepsilon \). The behavior of the system is studied under the assumption that \( \varepsilon \) tends to 0. A formula for the average velocity of particles has been obtained for the competition resolution rule in accordance the particles located in the contours with even indices win the competitions. In [18], a closed binary chain of contours is considered such that, in this system, no competing particle moves in the current moment and in the future. In [18], a generalization of the binary chain of contours is also considered. Namely, there are \( m \) cells and a particle is in each contour, where \( m \) is an even number. In [19], a closed chain of contour is considered. There are \( m \) cells and a cluster, containing \( l \) particles, is considered, where \( m \) is an even number, \( l < m \). A continuous counterpart of this system is considered in [20].

This paper consider a binary chain of contours. The optimal competition resolution rule has been found. Namely, the system with this rule results in a state of free movement from any initial state over minimum time interval. This rule is called the long cluster rule. A stochastic version of this system is also considered. In this version, the probability that an attempt of a particle to move is realized is \( 1 - \varepsilon \). The behavior of the system is studied under the assumption that \( \varepsilon \) tends to 0.

2. Description of the system

We consider system containing \( N \) contours, Fig. 1. The indices of the contours are \( 0, 1, \ldots, N - 1 \). There are two cells on each contour. These cells are the lower cell 0 and the upper cell 1. We say that the contour \( i \) is in the state \( j \) if the particle of this contour (particle \( i \)) is in the cell \( j, j = 1, 2 \). For each contour, there are two adjacent contours. One of these adjacent contours is located on the left, and the other adjacent contour is on the right. The contour \( i - 1 \) (subtraction by modulo \( N \)) is located to the left of the contour \( i \), and the contour \( i + 1 \) (addition by modulo \( N \)) is located to the right of the contour \( i \). There is a common point of the nodes \( i, i + 1 \) called the node \( (i, i + 1) \) (addition by modulo \( N \)). Passing from the cell 0 to the cell 1, the particle of the contour \( i \) (the particle \( i \)) crosses the node \( (i, i + 1) \), and, passing from the cell 0 to the cell 1, the particle \( i \) crosses the cell \( (i - 1, i) \) (counter-clockwise movement), \( i = 0, 1, \ldots, N - 1 \).

![Figure 1: Binary closed chain of contours](image-url)
In the deterministic version, at any discrete moment, each particle passes to the other cell of the contour if there is no delay. If two particles try to pass through the same node, then a *competition* occurs. In this case, only one of the competing particles moves chosen in accordance with a prescribed rule for competition resolution.

A state of the system is a cyclic vector 

\[(d_0, d_1, \ldots, d_{N-1}),\]

where \(d_i\) is the index of the cell in which the particle \(i\) is located, \(i = 0, 1, \ldots, N - 1\). A set of ‘‘1’’s, located in the adjacent positions of the cyclic state is called a *1-cluster*. The clusters of ‘‘1’’s are separated by ‘‘0’’s. The number of ‘‘1’’ in a 1-cluster is called the length of this 1-cluster. A 0-cluster and its length is defined analogously.

The initial state of the system is prescribed.

### 3. Description of the stochastic system

In the stochastic version, the probability that a particle moves is \(1 - \varepsilon\) if the system is in a state such that, in the deterministic system, the particle moves. If, in the corresponding state of the deterministic system, a particle does not move, the particle does not move in the stochastic system.

### 4. Description of the long cluster rule

Let \(l_0(x)\) be the length of the longest 0-cluster under the system is in the state \(x\) and \(l_1(x)\) be the length of the longest 1-cluster under this assumption. If the system is in the state \(x\) and \(l_0(x) > l_1(x)\), then, in accordance with the *long cluster rule*, for any pair of competing particle, the particle located on the left, i.e., the particle, passing from the cell 0 to the cell 1, moves. If \(l_0(x) < l_1(x)\), then, in accordance with the *long cluster rule*, for any pair of competing particle, the particle located on the right, i.e., the particle, passing from the cell 1 to the cell 0, moves. If \(l_0(x) = l_1(x)\), then we suppose that the competing particle located on the left moves.

In the system with the long cluster rule either, at any step, the length of the longest 0-cluster is decreased by 1 or the the length of the longest 1-cluster is decreased by 1.

### 5. Concept of the average velocity

Let \(H(t)\) be the expectation of the total number of transitions of particles in the time interval \((0, t)\). The limit

\[v_i = \lim_{t \to \infty} \frac{H(t)}{t}, \quad i = 0, 1, \ldots, N - 1,\]

is called the *average velocity of the particle* \(i\), \(i = 0, 1, \ldots, N - 1\).

The value

\[v = \frac{1}{N} \sum_{i=0}^{N-1} v_i\]

is called the *average velocity of particles*.

We say that the system is in a state of free movement at time \(t_0\) if all particles move at any time \(t \geq t_0\). If the system results in a state of free movement, then \(v_0 = v_1 = \ldots = v_{N-1}\).

### 6. Optimization property of the long cluster rule

Denote by \(S_0\) the long cluster rule.

Suppose

\[l(x(t)) = \min(l_0(x(t)), l_1(x(t))),\]

where \(x(t)\) is the state of the system at time \(t\).

**Lemma 1.** For any competition competition resolution rule,

\[l(x(t + 1)) \geq l(x(t)) - 1.\]  \(\quad (1)\)
Proof. Suppose \( l(x(t)) \geq 1 \), and, in the vector \( x(t) \), the longest 0-cluster occupies the positions \( i_0, 1_0 + 1, \ldots, i_0 + s - 1 \) (addition by modulo \( N \)).

Then, at time \( t \), the particle \( i_0 + s - 1 \) is a competing particle and the particles \( i_0, 1_0 + 1, \ldots, i_0 + l_0 - 1 \) pass to the cells with the index 1.

Thus there is a 1-cluster of length not less than \( l_0(x(t)) - 1 \) in the vector \( x(t + 1) \). Therefore,

\[
l_1(x(t + 1)) \geq l_0(x(t)) - 1.
\]

The proof of the inequality

\[
l_0(x(t + 1)) \geq l_1(x(t)) - 1
\]

is analogous. Using (2), (3) we get (1). Lemma 1 has been proved.

Lemma 2. Suppose \( l(x(t)) \geq 1 \). Then, for the long cluster rule \( S_0 \) and any state \( x \), the following equality is true

\[
l(x(t + 1)) = l(x(t)).
\]

Proof. Suppose that, in the cyclic vector \( x(t) \) the longest 0-cluster is in the positions \( i_0, 1_0 + 1, \ldots, i_0 + s - 1 \) (addition by modulo \( N \)). In accordance with the rule \( S_0 \), at time \( t + 1 \) the particles \( i_0 - 1, i_0 + s - 1 \) are in the cells 0 and the particles \( i_0, 1_0 + 1, \ldots, i_0 + l_0 - 2 \) are in the cells 1. From this and Lemma 1, Lemma 2 follows.

Assume that \( a(x, S) \) is the number of steps such that over these steps the system results in a state of free movement for the initial state \( x \) and the competition resolution rule \( S \). If the system with the competition resolution rule \( S \) does not results in a state of free movement, then we suppose that \( a(x, S) = \infty \).

Theorem 1. For any competition resolution rule \( S \) and initial state \( x \) the inequality

\[
a(x, S) = l(x) \leq a(x, S_0)
\]

is true, i.e., the long cluster rule minimizes the time interval over that the system results to a state of free movement from any initial state.

Theorem 1 follows from Lemmas 1 and 2.

7. Stochastic system with three contours and left-priority rule

Suppose that \( N = 3 \). There are 8 states

\[
\begin{align*}
E_0 &= (0, 0, 0), \quad E_1 = (0, 0, 1), \quad E_2 = (0, 1, 0), \quad E_3(0, 1, 1), \\
E_4 &= (1, 0, 0), \quad E_5 = (1, 0, 1), \quad E_6 = (1, 1, 0), \quad E_7(1, 1, 1),
\end{align*}
\]

Assume that the state space is divided into three subsets

\[
\begin{align*}
G_1 &= \{E_0, \}, \\
G_2 &= \{E_7, \} \\
G_3 &= \{E_3, E_5, E_6, \} \\
G_4 &= \{E_1, E_2, E_4, \}
\end{align*}
\]

Due to symmetry, the probability of the transition from the set \( G_i \) to the state \( G_j \) does not depend on the system state belonging to the state \( G_i, i, j = 1, 2, 3, 4 \). Thus \( G_0, G_1, G_2, G_3 \) are states of a Markov chain (macrostates). Thus the process of the system work is a Markov chain such that the states of this chain form a unique non-periodic communicating class, and therefore there exist positive stationary probability, and these probabilities do not depend on the initial state, [24].

Denote by \( p_{ij} \) the probability that the system passes from the macrostate \( G_i \) to the set \( G_j \) over one step, \( i, j = 1, 2, 3 \). We have

\[
\begin{align*}
p_{11} &= o(\varepsilon), \quad p_{12} = 1 - 3\varepsilon + o(\varepsilon), \quad p_{13} = 3\varepsilon + o(\varepsilon), \quad p_{14} = o(\varepsilon), \\
p_{21} &= 1 - 3\varepsilon + o(\varepsilon), \quad p_{22} = o(\varepsilon), \quad p_{23} = o(\varepsilon), \quad p_{24} = 3\varepsilon + o(\varepsilon), \\
p_{31} &= 0, \quad p_{32} = \varepsilon + o(\varepsilon), \quad p_{33} = 1 - 2\varepsilon + o(\varepsilon), \quad p_{34} = \varepsilon + o(\varepsilon), \\
p_{41} &= 0, \quad p_{42} = 1 - 2\varepsilon + o(\varepsilon), \quad p_{43} = 2\varepsilon o(\varepsilon), \quad p_{44} = o(\varepsilon)
\end{align*}
\]
The states of the Markov chain form a unique non-periodic communicating class and, hence, [30], there exist stationary probabilities of all states independent of the initial state. Denote by \( p_i \) the stationary probability of the macrostate \( G_i, i = 1, 2, 3, 4 \). Using (1)–(4), we get equations for the stationary probabilities of the macrostates

\[
\begin{align*}
p_1 &= (1 - 3\varepsilon)p_2 + o(\varepsilon), \\
p_2 &= (1 - 3\varepsilon)p_1 + \varepsilon p_4 + (1 - 2\varepsilon)p_4 + o(\varepsilon), \\
p_3 &= 3\varepsilon p_1 + (1 - 2\varepsilon)p_3 + 2\varepsilon p_4 + o(\varepsilon), \\
p_4 &= 3\varepsilon p_2 + \varepsilon p_3 + o(\varepsilon), \\
p_1 + p_2 + p_3 + p + 4 &= 1.
\end{align*}
\]

Using (5)–(9), we get

\[
\begin{align*}
p_1 &= \frac{2}{7} + o(\sqrt{\varepsilon}), \\
p_2 &= \frac{2}{7} + o(\sqrt{\varepsilon}), \\
p_3 &= \frac{3}{7} + o(\sqrt{\varepsilon}), \\
p_4 &= o(\sqrt{\varepsilon}).
\end{align*}
\]

If the system is in the macrostate \( G_1 \) or \( G_2 \), then all particles move. If the system is in the macrostate \( G_3 \), then a delay of one particle occurs. From this, using (10)–(13), we get the following proposition.

**Proposition 1.** If \( N = 3 \) and the left-priority competition resolution rule is prescribed, then the average velocity of any particle is equal to

\[ v = v_1 = v_2 = v_3 = \frac{6}{7} + o(\sqrt{\varepsilon}). \]

**Remark 1.** Suppose that \( N \) is an even number, and the competition resolution rule the particles located in the contours with even numbers win the competitions (odd-even rule). Then, in accordance with results, obtained in [21], the average velocities of particles are equal to

\[
\begin{align*}
v_i &= 1 - \varepsilon, \quad i = 0, 2, \ldots, N - 2, \\
v_i &= \frac{3}{4} - o(\sqrt{\varepsilon}), \quad i = 1, 3, \ldots, N - 1, \\
v &= \frac{7}{8} - o(\sqrt{\varepsilon}).
\end{align*}
\]

8. Stochastic system with rule of long cluster

We consider a stochastic closed chain of contours with the long cluster rule.

Suppose that the number of contours equals \( N \), and the probability that the particles moves with the particle moves, if the system is in the state such that the particle may move, is equal to \( 1 - \varepsilon \).

**Theorem 2.** For a stochastic closed chain, the average velocity of clusters is equal to

\[ v = 1 - \varepsilon + o(\varepsilon), \quad \varepsilon \to 0. \]

**Proof.** From the state \((0, \ldots, 0)\), the system results in any prescribed state with a positive probability, in particular, the system may stay in the state \((0, \ldots, 0)\). From any state, with a positive probability, the system results in the state \((1, \ldots, 1)\), and, from the state \((1, \ldots, 1)\), the system results in the state \((0, \ldots, 0)\).

Assume that the system state space is divided into sets \( S_0, S_1, \ldots, S_{\lfloor N/2 \rfloor} \), where \( \lfloor N/2 \rfloor \) is the integral part of the number \( N/2 \). The set \( S_i \) is the set of all \( x \) such that \( l(x) = i, \quad i = 0, 1, \ldots, \lfloor N/2 \rfloor \). Denote by \( P_i \) the stationary probability that the system is in a state belonging to \( S_i \).

If the system is in the set \( S_0 \), then the probability that, at next step, the system will be in this set is \( 1 - N\varepsilon + o(\varepsilon) \), the probability that the system will be in the set \( S_1 \) is \( N\varepsilon + o(\varepsilon) \), and the probability that the system will be in a state...
not belonging to \( S_0 \cup S_1 \) is \( o(\varepsilon) \), \( \varepsilon \to 0 \). The probability that, from the set \( S_i \), the system passes to the set \( S_{i-1} \), \( i = 2, \ldots, [N/2] \), is equal to \( 1 - o(\sqrt{\varepsilon}) \). From this, it follows to

\[
P_0 = 1 - N\varepsilon + o(\varepsilon),
\]

\[
P_1 = N\varepsilon + o(\varepsilon),
\]

\[
P_i = o(\varepsilon).
\]  

Using (14)–(16), we get

\[
v = P_0 + \frac{(N-1)P_1}{N} + o(\varepsilon) = 1 - N\varepsilon + o(\varepsilon).
\]

Theorem 2 has been proved.

2 Conclusion

For a deterministic binary chain of contours, we have obtained a competition resolution rule such that this rule is optimal in the following sense. From any initial state, the system results in a state of free movement over the minimum time interval. A stochastic version of the system is considered. In this version, the probability that an attempt of a particle to move is \( 1 - \varepsilon \). We have proved that the average velocity of particles tends to 1 as \( \varepsilon \to 0 \). Under the assumption that the number of contours equals 3, we study the asymptotic behavior of the system with the left-priority resolution rule as \( \varepsilon \to 0 \). For this rule, the average velocity of particles does not tend to 1 as \( \varepsilon \to 0 \).

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