Myers–Perry black holes with scalar hair and a mass gap: Unequal spins

Carlos Herdeiro a,*, Jutta Kunz b, Eugen Radu a, Bintoro Subagyo c

a Departamento de Física da Universidade de Aveiro and CIDMA, Campus de Santiago, 3810-183 Aveiro, Portugal
b Institut für Physik, Universität Oldenburg, Postfach 2503, D-26111 Oldenburg, Germany
c Department of Physics, Institut Teknologi Sepuluh Nopember, Indonesia

A R T I C L E   I N F O
Article history:
Received 15 May 2015
Received in revised form 23 June 2015
Accepted 24 June 2015
Available online 26 June 2015
Editor: M. Cvetič

A B S T R A C T
We construct rotating boson stars and Myers–Perry black holes with scalar hair (MPBHsSH) as fully non-linear solutions of five dimensional Einstein gravity minimally coupled to a complex, massive scalar field. The MPBHsSH are, in general, regular on and outside the horizon, asymptotically flat, and possess angular momentum in a single rotation plane. They are supported by rotation and have no static limit. Such hairy BHs may be thought of as bound states of boson stars and singly spinning, vacuum MPBHs and inherit properties of both these building blocks. When the horizon area shrinks to zero, the solutions reduce to (in a single plane) rotating boson stars; but the extremal limit also yields a zero area horizon, as for singly spinning MPBHs. Similarly to the case of equal angular momenta, and in contrast to Kerr black holes with scalar hair, singly spinning MPBHsSH are disconnected from the vacuum black holes, due to a mass gap. We observe that for the general case, with two unequal angular momenta, the equilibrium condition for the existence of MPBHsSH is \( w = m_1 \Omega_1 + m_2 \Omega_2 \), where \( \Omega_i \) are the horizon angular velocities in the two independent rotation planes and \( w, m_i, i = 1, 2 \), are the scalar field’s frequency and azimuthal harmonic indices.

© 2015 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP3.

1. Introduction and motivation

Apart from vacuum and electro-vacuum, scalar-vacuum is the simplest model that may be considered in Einstein gravity. In its simplest form, this theory corresponds to couple (minimally) to gravity one or more real massless scalar fields with standard kinetic terms and without self-interactions. Unlike electro-vacuum, however, such scalar-vacuum does not yield any new stationary, asymptotically flat and regular black hole (BH) solutions, as compared to pure vacuum. This conclusion is based on a four dimensional no-scalar-hair theorem [1] (see [2] for a review). The physical rationale is twofold. Firstly, scalar fields do not have an associated Gauss law, albeit they may have a local conservation law, for instance, if there is a global symmetry. Thus, if some amount of scalar field falls into a BH, then, at least classically, no memory of it is expected to be found in the exterior spacetime. Secondly, some amount of a free, minimally coupled scalar energy placed in the neighbourhood of a BH is expected to either disperse to infinity or be absorbed by the BH. And neither of these fates endows the BH spacetime with an eternally lingering scalar field in the vicinity of the event horizon.

A minimal addition to scalar-vacuum, however, produces a remarkable change of affairs. Adding a mass term in a theory with two equally massive real scalar fields, or equivalently, with a single massive complex scalar field, new regular, asymptotically flat BH solutions exist, both in four spacetime dimensions \( (D = 4) \) – Kerr BHs with scalar hair [3–5] – and in \( D = 5 \) – Myers–Perry BHs with scalar hair (MPBHsSH) [6] (see also the recent work [7] for a \( D = 4 \) generalization). The underlying physics justifying the existence of non-trivial scalar fields in these two examples has clear differences. In the Kerr case, the new solutions can be inferred at the linear level due to the existence of test field scalar clouds at the threshold of superradiant instabilities [8,9,3]. In the Myers–Perry case, by contrast, there are no superradiant instabilities for a massive scalar field [10]; the scalar hair found in [6] is intrinsically non-linear and originates a mass gap between the hairy and the vacuum Myers–Perry BHs. But similarities exist: in both cases i) the gravitational theory admits asymptotically flat, everywhere
regular, solitonic solutions without a horizon, boson stars [11,12], for which the scalar field has a harmonic time dependence with frequency \( \omega \); ii) the hairy BH solutions can be regarded as adding a rotating BH horizon within a spinning boson star [4], with the hairy BHs inheriting properties of both these building blocks; in particular whereas the \( D = 4 \) boson stars continuously connect to Minkowski spacetime, the \( D = 5 \) boson stars (with two equal rotations) already possess a mass gap with respect to the Minkowski vacuum [13]; iii) a central condition for the existence of all known scalar hairy BHs relies on the identification of the horizon null generator with the Killing vector field that preserves the rotating boson star solution [14].

The \( D = 5 \) case studied in Ref. [6] pertained solutions with two complex scalar fields and two equal angular momenta parameters, as this choice leads to a co-dimension one problem and thus considerable technical simplification. The corresponding \( D = 5 \) Myers–Perry BHs are akin to the \( D = 4 \) Kerr solution; in particular they are both a two parameter family of solutions – characterized, say, by the ADM mass, \( M \), and horizon angular velocity, \( \Omega_2 \) – and have a regular, finite area, extremal limit. In both cases the hairy BHs, just as the boson stars, have (a) monochromatic scalar field(s) whose frequency \( \omega \) is fixed by \( \Omega_2 \) and (in \( D = 4 \)) an azimuthal winding number.

The single angular momentum \( D = 5 \) Myers–Perry solution, by contrast, is singular in the extremal limit, while the generic solution with two angular momenta is characterized by two different horizon angular velocities \( \Omega_1, \Omega_2 \). We would therefore like to understand if this more general case can still accommodate scalar hair and if so how the scalar field frequency relates the two angular velocities. In this paper we shall clarify both these issues. We show that the equilibrium condition for the general case with two non-vanishing angular momenta is:

\[
w = m_1 \Omega_1 + m_2 \Omega_2 ,
\]

where \( m_i \) are the two azimuthal quantum numbers in the scalar field ansatz, cf. eq. (2.8) below. Actually to reach this conclusion it is not necessary to solve the fully non-linear systems, as condition (1.1) can be derived from regularity at the horizon. We will then focus our analysis of the fully non-linear system on the case with a single angular momentum parameter, and we shall derive both the corresponding boson star solutions and the hairy BHs. The former solutions again show the property of their cousins with two equal angular momenta in [6]: they do not trivialize in the limit of maximal allowed frequency and exhibit a mass gap with respect to Minkowski spacetime. The latter solutions have a domain of existence delimited, in particular, by extremal solutions which are singular, in agreement with the behaviour of the hairless singly spinning Myers–Perry BHs. This reinforces the picture of these hairy BHs as “horizons inside classical lumps”, the classical lumps being boson stars in this case, wherein such a bound state inherits properties of both the solitonic limit and of the corresponding vacuum BH solutions.

This paper is organized as follows. In Section 2 we present a general model, including \( N \) complex scalar fields minimally coupled to gravity and the general ansatz for a solution with two different angular momenta. Various quantities of interest are described and the boundary conditions for the numerical implementation are presented. In particular, a near-horizon analysis immediately leads to condition (1.1), from regularity. In Section 3 we perform the analysis of single angular momentum solutions, starting with boson stars and addressing subsequently hairy BHs. Finally, in Section 4 we provide some final remarks.

2. The general model

2.1. Action and matter content

We shall consider a model with \( N \) complex scalar fields \( \psi^{(i)} \) coupled to Einstein gravity in \( D = 5 \),

\[
S = \int d^5 x \sqrt{-g} \left( \frac{1}{16 \pi G} R - \sum_{i=1}^{N} L^{(i)} \right) ,
\]

where \( G \), that will be set to unity, is Newton’s constant and the Lagrangian density for each of the scalar fields is

\[
L^{(i)} = \frac{1}{2} \partial_\alpha \psi^{(i)} \partial^\alpha \psi^{(i)} + U(|\psi^{(i)}|) .
\]

Thus, the scalar fields do not interact with one another. \( U(|\psi^{(i)}|) \) is the \( I \)-th scalar field potential. Variation of the action (2.2) with respect to the metric yields the Einstein equations:

\[
\mathcal{L}^{(i)} = \frac{1}{2} g^{ab} \left( \partial_\nu \psi^{(i)} \right) \partial^\nu \psi^{(i)} + 2 \partial_\alpha \psi^{(i)} \partial^\alpha \psi^{(i)} - g_{ab} \mathcal{L}^{(i)} .
\]

is the energy–momentum tensor of the \( I \)-th scalar field. There are also \( N \) Klein–Gordon equations, obtained by varying the action with respect to each of the scalar fields

\[
\nabla^2 \psi^{(i)} = \frac{dU^{(i)}}{d\psi^{(i)}} \psi^{(i)} .
\]

2.2. The general ansatz

To better understand the metric ansatz, split five dimensional Minkowski space as \( \mathbb{M}^{1,4} = \mathbb{R}_t \times \mathbb{R}_{\phi_1} \times \mathbb{R}_{\phi_2} \). Thus, the four-dimensional Euclidean space is split into two 2-planes each parameterized with polar coordinates. The corresponding coordinate transformation between Cartesian and bi-polar coordinates in \( \mathbb{R}^4 \) is \( x^1 = \rho \sin \phi_1, x^2 = \rho \cos \phi_1, x^3 = \sigma \sin \phi_2, x^4 = \sigma \cos \phi_2 \), where \( \rho, \sigma \) are polar radial coordinates in the 2-planes, \( 0 \leq \rho, \sigma < \infty \) and \( 0 \leq \phi_1 < 2\pi \) are azimuthal angles. Rotations around \( \phi_1 \) and \( \phi_2 \) generate two independent angular momenta. The generic rotating solutions depend on both \( \rho \) and \( \sigma \); however, the numerics and the description of solutions simplify by introducing a (hyper-)spherical radial coordinate in \( \mathbb{R}^4 \), \( r \), and an angle \( \theta \), such that the polar radii become projections of \( r \) into each of the two 2-planes: \( \rho = r \sin \theta, \sigma = r \cos \theta \), with \( 0 \leq r < \infty, 0 \leq \theta \leq \pi/2 \). Then \( \partial_{\phi_1} \) (\( \partial_{\phi_2} \)) generates rotations in the plane \( \theta = \pi/2 \) (\( \theta = 0 \)) and \( \mathbb{M}^{1,4} \) is written

\[
ds_{\mathbb{M}^{1,4}}^2 = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2
\]

\[
= -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi_1^2 + \cos^2 \theta d\phi_2^2) .
\]

The curved spacetimes we shall be considering contain corrections to the metric tensor (2.6), which is only approached asymptotically. In general, we assume solely that the line element possesses three commuting Killing vectors, \( \xi = \partial_t, \eta_1 = \partial_{\phi_1}, \eta_2 = \partial_{\phi_2} \). A suitable metric parametrization for a BH spacetime reads

\[
\text{[1]} A version of this ansatz has been employed in the construction of \( D = 5 \) counterparts of the Kerr–Newman solution [15], generalizing the one used in [16] to construct the first four dimensional spinning hairy BHs in the literature.
\[ ds^2 = -F_0(r, \theta)N(r)dt^2 + F_1(r, \theta) \left( \frac{dr^2}{N(r)} + r^2 d\theta^2 \right) + F_2(r, \theta) r^2 \sin^2 \theta \left[ d\varphi_1 - W_1(r, \theta)dt \right]^2 + F_3(r, \theta) r^2 \cos^2 \theta \left[ d\varphi_2 - W_2(r, \theta)dt \right]^2 + F_4(r, \theta) r^2 \sin^2 \theta \cos^2 \theta \left[ W_2(r, \theta) d\varphi_1 - W_1(r, \theta) d\varphi_2 \right]^2 , \]

(2.7)

too many of seven metric functions, \( F_0, F_1, F_2, F_3, F_4, W_1, W_2 \) and also
\[ N(r) = 1 - \frac{r_H}{r} , \]

where the parameter \( r_H \geq 0 \) corresponds to the position of the BH horizon in this coordinate system.

The particular parametrization just described for the line element is compatible with an ansatz for the matter fields of the form:
\[ \psi^{(i)} = \phi^{(i)}(r, \theta)e^{i(m^{(i)}_1 \varphi_1 + m^{(i)}_2 \varphi_2 - w^{(i)} t)} , \]

(2.8)

where \( m^{(i)}_1, m^{(i)}_2 \in \mathbb{Z} \) are azimuthal harmonic indices in both planes of rotation and \( w^{(i)} > 0 \) is the \( l \)-th scalar field frequency. Observe that the three aforementioned Killing vector fields, \( \xi = \partial_t \), \( \eta_1 = \partial_{\varphi_1} \), and \( \eta_2 = \partial_{\varphi_2} \), do not preserve, independently the scalar fields; rather, \( \psi^{(i)} \) are only preserved by the \( 2 \)-parameter family of helicoidal Killing fields \( \xi + \alpha_1 \partial_{\varphi_1} + \alpha_2 \partial_{\varphi_2} \)
\[ w^{(i)} = m^{(i)}_1 \alpha_1 + m^{(i)}_2 \alpha_2 . \]

(2.9)

2.3. Global charges and other physical quantities

We shall now present a set of physical quantities and relations that apply to the boson stars and the MPBHsSH that shall be obtained in the next section.

The solutions approach Minkowski spacetime at infinity. Then, as usual, the ADM mass \( M \) and the ADM angular momenta \( J_i \) can be read off from the asymptotics of particular metric functions,
\[ \mathcal{E}_t = -1 + \frac{8M}{3\pi t^2} + \ldots , \quad \mathcal{E}_\varphi_{1,t} = - \frac{4\mathcal{J}_1}{\pi t^2} \sin^2 \theta + \ldots , \]
\[ \mathcal{E}_\varphi_{2,t} = - \frac{4\mathcal{J}_2}{\pi t^2} \cos^2 \theta + \ldots . \]

(2.10)

For the line element (2.7), the event horizon \( H \) is a surface of constant radial coordinate, \( r = r_H \); \( H \) is a Killing horizon of the Killing vector field
\[ \chi = \xi + \Omega_1 \eta_1 + \Omega_2 \eta_2 , \]

(2.11)

which is null on \( H \) and orthogonal to it. Here, \( \Omega_1 = W_1|_{r_H} \) and \( \Omega_2 = W_2|_{r_H} \) denote the horizon angular velocities with respect to rotation in the \( \theta = \pi/2 \) and \( \theta = 0 \) plane, respectively.

MPBHsSH have Hawking temperature
\[ T_H = \frac{1}{2\pi r_H} \sqrt{\left. \frac{F_0}{F_1} \right|_{r_H}}, \]

(2.12)

and horizon area (related to the entropy by \( S = A_H/4 \))
\[ A_H = (2\pi)^2 r_H^2 \int_0^{\pi/2} d\theta \sin \theta \cos \theta \]
\[ \left. \times \sqrt{F_1 (F_2 F_3 + F_4 (\sin^2 \theta F_2 W_1^2 + \cos^2 \theta F_3 W_2^2))} \right|_{r_H} . \]

(2.13)

The Lagrangian of each scalar field has a global \( U(1) \) symmetry which introduces \( N \) conserved currents \( \rho^{(i)} = -i \varphi_i^{(i)} \varphi \overline{\varphi}_i^{(i)} - \Psi^{(i)} \varphi_i^{(i)} \Psi^{(i)} \), with \( \rho_i^{(1)} = 0 \). Thus the solutions carry also \( N \) conserved Noether charges – in the sense of obeying local continuity equations, but not (global) Gauss laws – obtained by integrating the Noether charge density, \( j_i \), on a spacelike slice \( \Sigma \).
\[ Q^{(i)} = \int_{\Sigma} d\tau d\varphi_1 d\varphi_2 \sqrt{-g} j^{(i)} . \]

(2.14)

MPBHsSH satisfy a Smarr-type relation
\[ M = M^{(\psi)} + \frac{3}{2} \left( \Omega_1 S + \Omega_1 J_1 + \Omega_2 J_2 \right. \]
\[ \left. - \Omega_1 \sum_m m^{(i)}_1 Q^{(i)} - \Omega_2 \sum_m m^{(i)}_2 Q^{(i)} \right) , \]

(2.15)

where \( M^{(\psi)} \) measures the energy stored in the scalar field outside the horizon:
\[ M^{(\psi)} = - \frac{3}{2} \int_{\Sigma} d\chi \sqrt{-g} \left( r^{(H)}_t - \frac{1}{3} r^{(H)a} a \right) . \]

(2.16)

A Smarr-type relation involving only horizon quantities also exists
\[ \frac{2}{3} M_H = T_H S + \Omega_1 J_{1,H} + \Omega_2 J_{2,H} . \]

(2.17)

with
\[ J_{i,H} = J_i - \sum_l m^{(i)}_l Q^{(i)} . \]

(2.18)

Finally, MPBHsSH satisfy the first law of thermodynamics
\[ dM = T_H dS + \Omega_1 dJ_1 + \Omega_2 dJ_2 . \]

(2.19)

2.4. Boundary conditions

As for the case of Kerr BHs with scalar hair [3], and for the case of MPBHsSH with two equal angular momenta [6], there are no exact solutions in closed form of the above system with a non-trivial scalar field. The problem can, however, be tackled numerically, by solving a set of elliptic equations with given boundary conditions.

To obtain asymptotically flat solutions with finite mass, we impose the boundary conditions at infinity
\[ F_0 = F_1 = F_2 = F_3 = F_4 = 1 , \quad W_1 = W_2 = 0 , \quad \phi^{(i)} = 0 . \]

(2.20)

At \( \theta = 0, \pi/2 \) the metric functions satisfy Neumann boundary conditions. The boundary conditions for the scalar field amplitude \( \phi^{(i)} \) are more complicated. In the generic case with \( m^{(1)}_1 \neq 0, m^{(1)}_2 \neq 0 \) and \( \phi^{(1)} \) vanishes at \( \theta = 0, \pi/2 \). However, for \( m^{(1)}_1 = 0, m^{(1)}_2 = 0 \), the scalar field amplitude \( \phi^{(1)} \) vanishes at \( \theta = 0 \) only, and satisfies Neumann boundary condition at \( \theta = \pi/2 \); the case \( m^{(1)}_1 = 0, m^{(1)}_2 \neq 0 \) follows immediately, mutatis mutandis.

The boundary conditions on the horizon take a simpler form in terms of a new radial variable \( x = \sqrt{r^2 - r_H^2} \) (which is also employed in numerics): \( \left. \partial_r F_i \right|_{r_H} = \left. \partial_r \phi^{(i)} \right|_{r_H} = 0 \). Of central importance, regularity at the horizon implies that the following resonance condition should be satisfied for each scalar field
\[ w^{(i)} = m^{(i)}_1 \Omega_1 + m^{(i)}_2 \Omega_2 . \]

(2.21)
Comparing with (2.9) this condition singles out a particular helicoidal Killing vector field within the family that preserves the full ansatz (2.7) plus (2.8), corresponding to \( \alpha_1 = \Omega_r \), in other words, the one that coincides with the BH horizon generator.

Finally, the metric functions should satisfy the elementary flatness conditions, guaranteeing absence of conical singularities on the axes:

\[
(F_2 + F_4 W_2^2 - F_1 = 0)_{\theta = 0}, \quad (F_3 + F_4 W_1^2 - F_1 = 0)_{\theta = \pi/2}.
\]

(2.22)

3. Single angular momentum solutions

We shall now specify the general ansatz (2.7)–(2.8) and the general model (2.2), by focusing on the following special case:

i) We consider a single \( (N = 1) \) massive but non-self-interacting scalar field, such that \( U (|\psi^{(i)}|) = \mu^2 |\psi^{(i)}|^2 \), where \( \mu \) is the scalar field mass and \( i = 1 \). Thus, from now on we shall drop the superscript \( l = 1 \), as there will only be a single complex scalar field.

ii) We focus on solutions with rotation on a single plane. Then, one can set

\[
m_1 \neq 0, \quad m_2 = 0
\]

(3.23)
in the scalar field ansatz (2.8), which in particular implies that \( T^{i}_{i \theta} = 0 \), and thus one can consistently set \( F_4 = W_2 = 0 \) in the line-element (2.7). For simplicity of notation, in the following we shall drop the subscript \( 1 \) referring to the plane of rotation (e.g. \( m_1 \to m \)).

3.1. The vacuum limit: Myers–Perry BHs

Setting \( \phi = 0 \) in (2.8), the model described in Section 2 admits as solutions MPBHs [17], which are exact solutions known in closed form. MPBHs with a singular angular momentum parameter (in \( D = 5 \)) can be written in the form of our ansatz (2.7), with:

\[
F_1 (r, \theta) = 1 + \frac{a^2}{r^2} \cos^2 \theta,
\]

\[
F_2 (r, \theta) = \left( 1 + \frac{a^2}{r^2} \right) \left( 1 + \frac{a^2 \sin^2 \theta}{(a^2 + r^2 + a^2 \cos^2 \theta)} \right), \quad F_3 (r, \theta) = 1,
\]

\[
F_0 (r, \theta) = \frac{1}{F_2}, \quad W_1 (r, \theta) = \frac{1}{F_2} \frac{a(r^2_H + a^2)}{r^2 + a^2 \cos^2 \theta},
\]

(3.24)

which apart from \( r_H \), contains the extra parameter \( a \) associated with rotation. Some physical quantities are given, in terms of the parameters \( r_H, a \), as

\[
M = \frac{3\pi}{4} (a^2 + r^2_H), \quad J = \frac{\pi}{2} a (a^2 + r^2_H), \quad T_H = \frac{r_H}{2\pi (a^2 + r^2_H)}.
\]

\[
A_H = 4\pi^2 r_H (a^2 + r^2_H), \quad \Omega_H = \frac{a}{a^2 + r^2_H}.
\]

The properties of these solutions have been extensively discussed in the literature. Here we mention only that the spinning BHs are continuously connected to the Schwarzschild–Tangherlini solution in the static limit; also, in contrast to the \( D = 4 \) Kerr metric, the zero temperature limit (which corresponds to \( r_H \to 0 \) for nonzero \( a \)) is singular in this case, with \( A_H \to 0 \) [18].

3.2. The solitonic limit: boson stars

Turning on the scalar field, we have found both solitonic (boson stars) and BH solutions. These cannot, however, be found in closed form and we have resorted to numerical methods. The numerical approach employed here is similar to that used in constructing \( D = 4 \) Kerr BHs with scalar hair described in [3]. As usual, dimensionless variables and global quantities are introduced by using natural units set by \( \mu \) (we recall \( G = 1 \)), e.g. \( r \to r_H, \phi \to \phi/\sqrt{8\pi} \) and \( w \to w/\mu \). Then, the numerical treatment of the model relies on only four input parameters: the horizon radius \( r_H \) (for BHs), the field frequency \( w \), the winding number \( m \) and the scalar field node number \( n \). In the following we shall only consider nodeless solutions \( (n = 0) \) corresponding to the fundamental state of boson stars and hairy BHs.

The equations for the \( F_0, F_1, F_2, F_3, W, \phi \) are solved by using a professional finite difference solver [19], which provides an error estimate for each unknown function. Other numerical tests were provided by the Smarr relation (2.15) and the first law (2.19), based on that, the typical numerical error for the solutions here is estimated to be around \( 10^{-3} \).

Setting \( r_H = 0 \) in (2.7) the horizon is replaced with a regular origin and one finds boson star solutions. Up to now, only co-dimension one problems have been studied: boson star solutions have been reported both within spherical symmetry [20], and with two equal angular momenta [13]; the latter are, however found for a model with two complex scalar fields. The boundary conditions at the origin are similar to those described above, except for the metric function \( W \), which satisfies now a Neumann boundary condition \( \partial_r W = 0 \).

The Noether charge and the angular momenta of these boson stars are not independent quantities; they are simply related by

\[
J = m Q,
\]

(3.25)

while the Smarr relation and the first law read

\[
M = M^\phi, \quad dM = m w d\Omega.
\]

(3.26)

Taking the scalar field frequency \( w \) as a control parameter, the numerical results show that, for any \( m \), boson stars exist for a limited range of frequencies, \( w_{\text{min}} < w < \mu \), with \( w_{\text{min}}(m) \) decreasing with \( m \). Fig. 1. A striking property of the \( D = 5 \) boson stars is that these do not possess a true vacuum limit. That is, in contrast to the \( D = 5 \) Anti-de Sitter case [14], or to the case of \( D = 4 \)

![Fig. 1. ADM mass vs. scalar field frequency diagram for boson stars, with \( m = 0, 1, 2, 3 \).](image-url)
spinning boson stars \[21,22\]. \(D = 5\) asymptotically flat solutions do not trivialize as \(w \to \mu\). Indeed, as noticed in \[13\] for the special case of \(D = 5\) boson stars with two equal angular momenta, as the frequency tends to the upper bound set by \(\mu\), the scalar field spreads and tends to zero while the geometry becomes arbitrarily close to the Minkowski one. The global charges of the solutions, however, remain finite and nonzero as \(w \to \mu\). Thus a mass (and charge) gap is found between the \(\phi = 0\) vacuum flat space ground state and the limiting configurations with a frequency \(w\) arbitrarily close to \(\mu\). This behaviour has been explained for spherically symmetric solutions and for the two equal angular momenta boson stars \[13\], observing the existence of a special scaling symmetry of the limiting solutions. It seems plausible that the results in \[13\] can be extended to the case of boson stars with a single angular momentum.

The results of the numerical integration for several values of \(m\) are displayed in Fig. 1. For completeness, we have included there also the case of spherically symmetric boson stars, which can also be studied within the general ansatz \(2.7\)–\(2.8\), by taking \(m_1 = 0\), \(F_1 = F_2 = F_3\), \(W_1 = 0\), and the surviving three independent functions, \(F_0\), \(F_1\) and \(\phi\) depending only on \(r\) (note also that in this case the scalar field does not vanish at \(r = 0\)).

From Fig. 1 we observe that the mass \(M\) decreases as \(w\) is decreased from the maximal value \(\mu\). After approaching the minimal value \(w_{\text{min}}\), a backbending in \(w\) is observed. Then, one expects an inspiralling behaviour of the curves, towards a limiting configuration at the center of the spiral, for a frequency \(w_{c1}/\mu\). This part of the diagram is difficult to explore numerically for spinning solutions, and so a second backbending is only clearly shown for \(m = 0, 1\). This inspiralling pattern appears to be generic for boson star solutions,\(^2\) being found also for boson stars in \(D = 4\) Einstein gravity and a scalar-tensor extension \(7\), for \(D = 5\) solutions with Anti-de Sitter asymptotics \(14\) and for \(D = 5\) asymptotically flat solutions \(6\). A similar diagram is found for \(J(w)\), showing that boson stars do not possess a slowly rotating limit.

The \(T^I_I\) component of the energy–momentum tensor of a typical boson star is shown in Fig. 2. There one can notice the existence of a maximum in the plane of rotation, for some nonzero value of \(r\).

### 3.3. Hairy black holes

In order to obtain MPBHsSH we consider \(r_H \neq 0\). Turning on this parameter, starting from any given boson star solution with frequency \(w\), can be regarded as adding a small BH at the center of the boson star. For a given \(\Omega_H\), the boson star with \(r_H = 0\) provides a good initial profile for hairy BHs with a small \(r_H\). By increasing \(r_H\) from zero, we obtain MPBHsSH with \(\Omega_H\) fixed by the scalar field frequency. It follows that the minimal frequency of the boson stars sets a lower bound on the horizon velocity of the hairy BHs, while the upper bound on the frequency is still set by \(\mu\), the scalar field mass.

Given this systematic construction technique, it is convenient to describe the domain of existence of the hairy BHs in terms of \(\Omega_H\). The emerging picture shows that, when varying the horizon size (via the parameter \(r_H\)), there are two possible types of sequences of BH solutions with a fixed \(\Omega_H\):

(S1) There are sequences of BH solutions that connect two different boson star solutions with the same frequency. Along these sequences, the BH solutions attain a maximal area at some point in between the two boson star solutions with the same scalar field frequency. Approaching these solutions, \(r_H \to 0\), the horizon area vanishes, the temperature diverges and \(J \to mQ\). For \(m = 1\), this occurs, for instance, for frequencies between the minimal boson star frequency \(w_{\text{min}}\) and \(w/\mu \approx 0.936\).

(S2) There are sequences of BH solutions that end in a zero temperature extremal BH with scalar hair. In contrast to both Kerr BHs with scalar hair \(3\) and the MPBHsSH studied in \(6\), these limiting configurations have vanishing horizon size and do not seem to possess a regular horizon. The global charges, however, are finite and nonzero in this case.

In Fig. 3 we show the domain of existence of MPBHsSH (the shaded blue region), for solutions with \(n = 0, m = 1\), as a function of frequency \(w\). This domain was obtained by extrapolating to the continuum the results from a set of around two thousand numerical solutions. This can safely be done for most of the parameter space. We cannot exclude, however, a more complicated picture for a small region around the center of the boson star spiral, which is rather difficult to explore numerically within our approach. We further remark that the set of extremal MPBHsSH which form a part of the boundary of the domain of existence have been obtained by extrapolation of the numerical results.\(^3\)

\(^2\) This part of the diagram appears to change, however, for solutions with two equal angular momenta in the \(D = 5\) Einstein–Gauss–Bonnet model \(21,24\).

\(^3\) Differently from the hairy BHs in \(3,6\), the direct construction of this set of extremal BHs presented unsurmountable difficulties, presumably due to their singular nature. Indeed, we observed that for the near extremal solutions, both the Ricci and the Kretschmann scalars take very large values on the horizon, in particular at \(\theta = \pi/2\).
responds

maximal

MPBHs.

interpolates

of

temperature

for

also

Fig. 4,
singular

and

minimal

is

(black

solutions

line),
three

velocity

Fig. 4.

This

AH vs.

/Ω1

4

line).

We

remark

that

a

similar

diagram

is

found

for

J(w).

Thus,

we

conclude

that

MPBHsSH

with

a

singular

angular

momentum

have

a

minimal

mass

and

angular

momentum.

In

particular

they

have

no

static

limit,

analogously

to

Kerr

BHs

with

scalar

hair

[3].

Fig. 3

focuses

on

m = 1;

based

on

preliminary

numerical

data,

we

are

confident

that

a

similar

pattern

for

the

domain

of

existence

of

MPBHsSH

occurs

for

other

values

of

m.

The

domain

describing

the

extremal

solutions

starts

at

a

non-zero

ADM

mass

at

the

maximal

frequency

w/μ → 1,

decreases

until

a

minimal

value

of

w/μ

(with

w/μ ≈ 0.936

for

m = 1),

backbends

and

keeps

decreasing,

reaches

a

minimal

value

of

the

ADM

mass

and

then,

we

conjecture,

proceed

to

inspiral

towards

a

central

value

where

it

meets

the

endpoint

of

the

boson

star

spiral

in

a

singular

solution.

Further

features

of

singly

spinning

MPBHsSH

are

shown

in

Fig. 4,

where

we

plot

their

domain

of

existence

in

a

horizon

area

A_H

vs.

temperature

diagram

(left

panel),

and

in

A_H

vs.

frequency

diagram

(right

panel).

The

results

for

vacuum

MPBHs

are

also

shown

for

comparison.

As

one

can

observe

from

the

left

panel,

for

a

given

frequency,

the

horizon

area

reaches

a

maximal

value

for

some

solution

with

nonzero

T_H. Let

us

take

two

qualitatively

distinct

examples.

For

Ω_H/μ = 0.945

the

sequence

of

solutions

interpolates

between

infinite

temperature

(a

boson

star)

and

zero

temperature

(an

extremal

MPBHSH),

corresponding

to

a

sequence

of

type

S2

above.

By

contrast,

for

Ω_H/μ = 0.925

the

sequence

interpolates

between

two

boson

stars

(hence

two

infinite

temperatures),

corresponding

to

a

sequence

of

type

S1

above.

Note

that

for

Ω_H/μ = 0.945

we

have

also

plotted

a

sequence

of

vacuum

MPBHs.

In

the

A_H

vs.

w

diagram,

the

set

of

critical

configurations

with

maximal

area

for

fixed

Ω_H

form

a

part

of

the

boundary

of

the

domain

of

existence.

The

remaining

boundary

is

given

by

the

set

of

boson

stars,

which

have

A_\text{H} = 0,

together

with

the

extremal

MPBHsSH, which

have

also

zero

horizon

area

and

the

set

of

maximally

bound

BHs.

From

Fig. 4

it

can

also

be

observed

that

there

is

continuity

between

vacuum

MPBHs

and

MPBHsSH

in

terms

of

horizon

quantities,

as

was

observed

in

[6]

for

the

two

equal

angular

momenta

case.

This

occurs

despite

the

mass

gap

between

the

two

families

of

solutions

in

terms

of

global

charges.

Finally,

in

Fig. 5

we

plot

the

phase

space

of

MPBHsSH,

i.e.

the

domain

of

existence

of

these

BHs

in

the

(J, M)-plane.

As

it

can

be

observed

they

exist

in

the

region

where

vacuum

MPBHs

exist

as

well.

As

such

there

is

non-uniqueness,

when

only

the

ADM

mass

and

angular

momentum

are

specified,

in

analogy

to

the

case

of

Kerr

BHs

with

scalar

hair

[3].

4 Further

remarks

In

this

paper

we

have

reported

the

first

construction

of

higher

dimensional

(D > 4)

boson

stars

and

scalar

hairy

BHs

with

a

single

angular

momentum

parameter

in

the

literature.

One

of

the

conclusions

of

our

study

is

the

confirmation

that

the

properties

of

scalar

hairy

BHs

within

this

large

family

of

solutions

anchored

on

conditions

of

type

(1.1)

are

inherited

from

their

"building

blocks", which

---

4 This

is

the

behaviour

found

also

for

MPBHs.

For

a

given

angular

velocity

Ω_H,

the

horizon

area

of

a

MPBH

approaches

a

maximal

value

for

a = 3/(4Ω_H)

where

T_H = \frac{\Omega_H}{2\pi c^3}.

The

horizon

area

decreases

for

larger

values

of

a,

and

approaches

zero

for

the

maximal

value

a \rightarrow Ω_H which

 corresponds

to

an

extremal

(singular)

configuration.
in the case considered here are $D = 5$ singly spinning boson stars and Myers–Perry BHs. Thus, MPBHsSH have a mass gap with respect to the vacuum MPBHs, as do boson stars with respect to Minkowski spacetime. Moreover, in the extremal limit, MPBHsSH yield a singular configuration as vacuum MPBHs do. This reinforces the picture that such hairy BHs can be viewed as bound states of “bald” BHs and solitonic configurations (boson stars) [4].

We have not considered here the general case with two non-vanishing angular momenta. In Ref. [6], however, MPBHsSH with two equal angular momenta were studied in a model with $N = 2$ complex scalar fields. Therein a special ansatz is used, originally proposed in [13], such that the spacetime isometry group is enhanced from $S^1 \times U(1)^2$ to $S^1 \times U(2)$. This enhancement is obtained by taking the same mass and frequency for both complex scalars, and requiring the fields to rotate with the lowest azimuthal harmonic index in different planes:

\[ w^{(1)} = w^{(2)} = w, \quad \mu^{(1)} = \mu^{(2)} = \mu, \quad m_1^{(1)} = m_2^{(2)} = 1, \quad m_2^{(1)} = m_1^{(2)} = 0, \]

such that the resonance condition (2.21) is fulfilled by each scalar. Then the $\theta$-dependence factorizes

\[ \phi^{(1)} = \phi(t) \sin \theta, \quad \phi^{(2)} = \phi(t) \cos \theta, \]

while the metric functions $F_i, W$ in (2.7) depend only on $r$, with $W_1 = W_2 = W(r)$, $F_1(t) = F_3(t) - F_2(t)$, and the problem is effectively co-dimension one.\(^5\) The general properties of these MPBHsSH with $J_1 = J_2$ are similar to those found in this work for MPBHsSH with a single $J$. The main difference concerns the extremal solutions, which, therein – and similarly to the behaviour of vacuum MPBHs with two equal angular momenta – have finite (and nonzero) horizon size and global charges and possess a regular horizon.

Based on the results in this paper and those in [6] one can make an educated guess for the general case with two non-vanishing and non-equal angular momenta. The domain of existence of such MPBHsSH will be bounded by the corresponding boson stars, by a set of marginally bound solutions – which have a mass gap with respect to the vacuum MPBHs – and the extremal limit will have the same properties as those of the corresponding vacuum MPBHs. A different state of affairs, however, will certainly be found in the asymptotically Anti-de Sitter case. Singly spinning MPBHs are afflicted by the superradiant instability of a massive scalar field and thus singly spinning MPBHsSH continuously connected to MPBHs in Anti-de Sitter should also exist, similarly to the equal angular momentum case [14].

Finally we remark on two further possible generalizations. Firstly, as it is well known, $D = 5$ vacuum gravity admits other solutions with different horizon topologies, most notably black rings [26]. It seems plausible that black rings with scalar hair anchored to the condition (1.1) also exist, even if finding them numerically may be challenging. Secondly, going to $D > 5$, MPBHs exhibit yet a qualitatively new feature: the existence of ultra-spinning BHs. It would certainly be interesting to construct both singly spinning boson stars and singly spinning MPBHsSH in $D > 5$ to see if/how this new possibility impacts on such solutions.

Acknowledgements

J.K. would like to acknowledge support by the DFG Research Training Group 1620 “Models of Gravity” and by FP7, Marie Curie Actions, People, International Research Staff Exchange Scheme (IRSES-605096). The work of C.H. and E.R. has been supported by the grants PTDC/FIS/116625/2010, NRHEP-295189–FP7-PEOPLE-2011-IRSES and by the CIDMA strategic funding UID/MAT/04106/2013. B.S. acknowledges partial support from DIKTI research grant No. 0324.81/IT2/11/PN/08/2015.

References

[1] J.E. Chaze, Commun. Math. Phys. 19 (1970) 276–288.
[2] C.A.R. Herdeiro, E. Radu, arXiv:1504.08209 [gr-qc].
[3] C.A.R. Herdeiro, E. Radu, Phys. Rev. Lett. 112 (2014) 221101, arXiv:1403.2757 [gr-qc].
[4] C.A.R. Herdeiro, E. Radu, Int. J. Mod. Phys. D 23 (12) (2014) 1442014, arXiv:1405.3606 [gr-qc].
[5] C. Herdeiro, E. Radu, arXiv:1501.04319 [gr-qc].
[6] Y. Brihaye, C. Herdeiro, E. Radu, Phys. Lett. B 739 (2014) 1, arXiv:1408.5581 [gr-qc].
[7] B. Kleihaus, J. Kunz, S. Yazadjiev, Phys. Lett. B 744 (2015) 406, arXiv:1503.01672 [gr-qc].
[8] S. Hod, Phys. Rev. D 86 (2012) 104026, arXiv:1211.3202 [gr-qc]; S. Hod, Phys. Rev. D 86 (2012) 124002.
[9] S. Hod, Eur. Phys. J. C 73 (4) (2013) 2378, arXiv:1311.5298 [gr-qc].
[10] V. Cardoso, S. Yoshida, J. High Energy Phys. 0507 (2005) 009, arXiv:hep-th/0502206.
[11] F.E. Schunck, E.W. Mielke, Class. Quantum Gravity 20 (2003) R301, arXiv:0801.0307 [astro-ph].
[12] S.L. Liebling, C. Palenzuela, Living Rev. Relativ. 15 (2012) 6, arXiv:1202.5809 [gr-qc].
[13] B. Hartmann, B. Kleihaus, J. Kunz, M. List, Phys. Rev. D 82 (2010) 084022, arXiv:1008.3137 [gr-qc].
[14] O.J.C. Dias, G.T. Horowitz, J.E. Santos, J. High Energy Phys. 1107 (2011) 115, arXiv:1105.4167 [hep-th].
[15] J. Kunz, F. Navarro-Lerida, A.K. Petersen, Phys. Lett. B 614 (2005) 104, arXiv:gr-qc/0503010.
[16] B. Kleihaus, J. Kunz, Phys. Rev. Lett. 86 (2001) 3704, arXiv:gr-qc/0012081.
[17] R.C. Myers, M.J. Perry, Ann. Phys. 172 (1986) 304.
[18] J.M. Bardeen, G.T. Horowitz, Phys. Rev. D 60 (1999) 104030, arXiv:hep-th/9905095.
[19] W. Schömann, R. Weiß, J. Comput. Appl. Math. 27 (279) (1989) 279; M. Schauder, R. Weißband, W. Schömann, The CADSOL Program Package, In- terner Bericht Nr. 46/92, Universität Karlsruhe, 1992.
[20] D. Astefanesei, E. Radu, Nucl. Phys. B 665 (2003) 594, arXiv:gr-qc/0309131.
[21] S. Yoshida, V. Erioglu, Phys. Rev. D 56 (1997) 762.
[22] B. Kleihaus, J. Kunz, M. List, Phys. Rev. D 72 (2005) 064002, arXiv:gr-qc/0505143.
[23] Y. Brihaye, J. Riedel, Phys. Rev. D 89 (2014) 104060, arXiv:1310.7223 [gr-qc].
[24] L.J. Henderson, R.B. Mann, S. Stotyn, Phys. Rev. D 91 (2) (2015) 024009, arXiv:1403.1865 [gr-qc].
[25] S. Stotyn, M. Park, P. McGarrah, R.B. Mann, Phys. Rev. D 85 (2012) 044036, arXiv:1110.2223 [hep-th].
[26] R. Emparan, H.S. Reall, Phys. Rev. Lett. 88 (2002) 101101, arXiv:hep-th/0110026.

\(^5\) An explanation of this fact is given in Ref. [25], together with a generalization of the ansatz to higher odd dimensions.