Frequency Fluctuations in Tunable Superconducting Microwave Cavities

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We present a model for measurements of the scattering matrix elements of tunable microwave cavities in the presence of resonant frequency fluctuations induced by fluctuations in the tuning parameter. We apply this model to the specific case of a two-sided cavity and find an analytic expression for the average scattering matrix elements. A key signature of this ‘fluctuating model’ is a subtle deformation of the trajectories swept out by scattering matrix elements in the complex plane. We apply this model to experimental data and report a direct observation of this deformation in the data. Despite this signature, we show that the fluctuating and non-fluctuating models are qualitatively similar enough to be mistaken for one another, especially in the presence of measurement noise. However, if one applies the non-fluctuating model to data for which frequency fluctuations are significant then one will find damping rates that appear to depend on the tuning parameter, which is a common observation in tunable superconducting microwave cavities. We propose this model as both a potential explanation of and remedy to this apparent phenomenon.

I. INTRODUCTION

In recent years, superconducting microwave cavities with a tunable resonant frequency have found wide-ranging applications. In the field of quantum information, they have been used as parametric oscillators to achieve single-shot readout of a superconducting qubit [1, 2] and as quantum buses for selective qubit coupling [3]. As standalone cryogenic microwave devices, they have been used as filters [4] and as parametric amplifiers operating near the quantum limit of added noise [5]. In the study of fundamental physics, they have been used to investigate the dynamical Casimir effect [6, 7].

In order to benchmark these devices, however, one must first accurately characterize their basic properties. Determining their damping rates (or equivalently their quality factors) is particularly important, since these affect the equations of motion governing the cavities such that inaccuracies will have far-reaching consequences. Unfortunately, a common issue with these devices is that measured damping rates appear to vary with the tuning parameter [8–13]. Although many have attributed this to causes specific to their devices, here we argue that the tunability itself is at least partially responsible for these observations.

The line of reasoning is as follows: fluctuations in the tuning parameter will necessarily induce fluctuations in the resonant frequency. These frequency fluctuations will in turn affect the measurement of scattering matrix elements, which is a standard method for extracting the damping rates of microwave cavities [14, 15]. Thus, to accurately determine the damping rates one must separate out the effect of the frequency fluctuations.

To this end we present a model for measurements of the scattering matrix elements of a microwave network containing a cavity with a fluctuating resonant frequency, which we apply to the specific case of a two-sided cavity and obtain an analytic expression for the average scattering matrix elements. We show that the fluctuating and non-fluctuating models are qualitatively similar enough to be mistaken for one another, but that applying the latter model to data generated by the former yields damping rates that appear to vary with the tuning parameter. The fluctuating model therefore serves as both a potential explanation of tuning-dependent damping rates and a method for extracting the correct damping rates when frequency fluctuations are significant.

The primary qualitative difference between the fluctuating and non-fluctuating models is a deformation of the trajectories swept out by the scattering matrix elements in the complex plane. We apply both models to experimental data and report a direct observation of this deformation in the data, indicating the presence of significant frequency fluctuations. This signature of the fluctuating model is subtle and easily obscured by measurement noise, however, so we conclude by discussing alternative means of corroborating the presence of frequency fluctuations and distinguishing the effect of such fluctuations from genuine damping.

II. TUNING-INDUCED FREQUENCY FLUCTUATIONS

Consider a cavity whose resonant frequency $\omega_0$ is a function of tuning parameter $x$ such that $\omega_0 = \omega_0(x)$. In general, this tuning parameter can never be perfectly fixed: it will always fluctuate around its mean value. To model this effect, let $x \rightarrow \bar{x} + \delta x$, where $\bar{x}$ is the mean value of $x$ and $\delta x$ is a Gaussian random variable with mean zero and variance $\sigma_x^2$, describing fluctuations of $x$ about $\bar{x}$. Expanding to lowest order in these fluctuations, we find

$$\omega_0(\bar{x} + \delta x) \approx \omega_0(\bar{x}) + \delta \omega_0(\bar{x})$$ (1)
Port 1

\[ a_{1i}^{\text{in}} \quad \rightarrow \quad a_{1i}^{\text{out}} \]

Cavity

\[ a \quad \rightarrow \quad H = \hbar \omega_0 \sigma \]

\[ \vec{a}_2 \quad \rightarrow \quad a_{2o} \]

Port 2

\[ \kappa_1 \quad \rightarrow \quad \kappa_2 \]

FIG. 1. Schematic of a two-sided linear cavity with decay rates \( \kappa_1 \) and \( \kappa_2 \).

where the second term

\[ \delta \omega_0(\pi) = \frac{\partial \omega_0(\pi)}{\partial x} \delta x \]  

is itself a Gaussian random variable with mean zero and variance \( \sigma_{\omega_0}^2 \), given by

\[ \sigma_{\omega_0}^2 = \left| \frac{\partial \omega_0(\pi)}{\partial x} \right|^2 \sigma_x^2 \]  

which varies with \( \pi \).

Now, let \( S_{ij}(\Delta) \) be the scattering matrix elements of the microwave network containing our cavity [16], describing an experiment in which the cavity is driven by a coherent signal on port \( j \) and measured on port \( i \), where \( \Delta = \omega - \omega_0 \) is the detuning of the drive from resonance. In general, measuring these scattering matrix elements involves a process of averaging, whether implicitly through the time-scale associated with the measurement or explicitly through the incorporation of multiple independent measurements. As a result, in the presence of frequency fluctuations what will actually be measured is

\[ S_{ij}(\Delta; \sigma_{\omega_0}) = \int_{-\infty}^{\infty} S_{ij}(\Delta - \Omega) P(\Omega; \sigma_{\omega_0}) d\Omega \]  

where \( P(\Omega; \sigma_{\omega_0}) \) is the Gaussian probability density function associated with drawing the value \( \Omega \) from the random variable \( \sigma_{\omega_0} \), given by

\[ P(\Omega; \sigma_{\omega_0}) = \frac{1}{\sqrt{2\pi \sigma_{\omega_0}}} \exp \left[ -\Omega^2 / 2\sigma_{\omega_0}^2 \right] . \]

It is worth noting that in the limit \( \sigma_{\omega_0} \to 0 \) we have \( S_{ij}(\Delta; 0) = S_{ij}(\Delta) \), as one would expect. The effect of this averaging is a deformation of the path swept out by each scattering matrix element in the complex plane (as the detuning \( \Delta \) is swept), and this deformation will vary with \( \pi \) through the dependence of \( \sigma_{\omega_0} \) on \( \partial \omega_0(\pi)/\partial x \). This will in turn lead to apparent changes in the damping rates, extracted from measurements of \( S_{ij} \), that will vary with \( \pi \).

III. TWO-SIDED CAVITY

To show this explicitly, we consider a generic two-sided linear cavity as depicted schematically in Fig. [3]. We expect this to be an appropriate model for a wide variety of systems, provided they are operated at sufficiently low powers to be treated linearly. Using input-output theory [17], we can relate the fields at the ports of the network to the internal field of the cavity according to

\[ a_{1o}^{\text{in}}(t) - a_{1o}^{\text{in}}(t) = -\sqrt{\kappa_1} a(t) \]

\[ a_{2o}^{\text{out}}(t) - a_{2o}^{\text{out}}(t) = -\sqrt{\kappa_2} a(t) \]  

where \( \kappa_1 \) and \( \kappa_2 \) are the damping rates associated with their respective ports, such that the internal cavity field obeys the equation of motion

\[ \dot{a}(t) = -i\omega_0 a(t) - \frac{\kappa_1 + \kappa_2}{2} a(t) + \sqrt{\kappa_1} a_{2o}^{\text{in}}(t) + \sqrt{\kappa_2} a_{2o}^{\text{in}}(t) . \]

In scattering experiments we are generally interested in the steady state response of the cavity as a function of drive frequency \( \omega \), to which end we Fourier transform Eq. (7) and solve for the cavity response algebraically

\[ \tilde{a}(\omega) = \frac{\sqrt{\kappa_1} \tilde{a}_{1o}^{\text{in}}(\omega) + \sqrt{\kappa_2} \tilde{a}_{2o}^{\text{in}}(\omega)}{i(\omega_0 - \omega) + (\kappa_1 + \kappa_2)/2} \]  

where a tilde denotes the Fourier transform of the given mode operator. Plugging this result into the Fourier transform of Eq. (4), we find a linear relationship between the input and output mode operators

\[ \tilde{a}_{1o}^{\text{out}}(\omega) = \tilde{a}_{1o}^{\text{in}}(\omega) - \frac{\kappa_1 \tilde{a}_{1o}^{\text{in}}(\omega) + \sqrt{\kappa_1 \kappa_2} \tilde{a}_{2o}^{\text{in}}(\omega)}{i(\omega_0 - \omega) + (\kappa_1 + \kappa_2)/2} \]  

\[ \tilde{a}_{2o}^{\text{out}}(\omega) = \tilde{a}_{2o}^{\text{in}}(\omega) - \frac{\sqrt{\kappa_1 \kappa_2} \tilde{a}_{1o}^{\text{in}}(\omega) + \kappa_2 \tilde{a}_{2o}^{\text{in}}(\omega)}{i(\omega_0 - \omega) + (\kappa_1 + \kappa_2)/2} \]

that defines the scattering matrix according to

\[ \begin{bmatrix} \tilde{a}_{1o}^{\text{out}}(\omega) \\ \tilde{a}_{2o}^{\text{out}}(\omega) \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} \tilde{a}_{1o}^{\text{in}}(\omega) \\ \tilde{a}_{2o}^{\text{in}}(\omega) \end{bmatrix} \]

where the complex conjugate of each matrix element is taken, by convention, so that its phase corresponds to a counter-clockwise rotation in phase space of the output quadrature operators relative to the inputs. For ease of notation let \( \kappa_{\text{tot}} = \kappa_1 + \kappa_2 \) be the total damping rate of the cavity and let \( \xi = \kappa_1/\kappa_2 \) be the coupling ratio, in terms of which the scattering matrix elements can be expressed

\[ S_{21}(\Delta; \kappa_{\text{tot}}, \xi) = \frac{-2\sqrt{\xi}}{1 + \xi} \frac{1}{1 + 2\Delta/\kappa_{\text{tot}}} \]

\[ S_{12}(\Delta; \kappa_{\text{tot}}, \xi) = S_{21}(\Delta; \kappa_{\text{tot}}, \xi) \]

\[ S_{11}(\Delta; \kappa_{\text{tot}}, \xi) = 1 + \frac{\xi S_{21}(\Delta; \kappa_{\text{tot}}, \xi)}{1 + \xi} \]

\[ S_{22}(\Delta; \kappa_{\text{tot}}, \xi) = 1 + \frac{1}{\sqrt{\xi}} S_{21}(\Delta; \kappa_{\text{tot}}, \xi) \]

where \( \Delta = \omega - \omega_0 \) is the detuning, as before.
Since all the scattering matrix elements are interrelated, we will restrict our focus to the reflection coefficient $S_{11}(\Delta; \kappa_{\text{tot}}, \xi)$. When expressed in the form

$$S_{11}(\Delta; \kappa_{\text{tot}}, \xi) = 1 - \frac{\xi}{1 + \frac{2\xi}{\sqrt{2\pi} \sigma_{\omega}}} \left[ 1 + e^{-\frac{2\pi}{\sqrt{2\pi} \sigma_{\omega}} (2\Delta/\kappa_{\text{tot}})} \right]$$  \hspace{1cm} (12)$$

it is plain to see that $S_{11}(\Delta; \kappa_{\text{tot}}, \xi)$ sweeps out a circular trajectory as a function of detuning such that $\kappa_{\text{tot}}$ and $\xi$ have simple geometric interpretations: $\kappa_{\text{tot}}$ determines the rate at which the trajectory is traversed and is the natural frequency scale of the system, while $\xi$ determines the center and radius of the trajectory.

**IV. FLUCTUATING TWO-SIDED CAVITY**

In the case of a fluctuating resonant frequency, however, what we actually measure is given by Eq. (4), which now takes the form

$$\overline{S}_{11}(\Delta; \kappa_{\text{tot}}, \xi, \sigma_{\omega}) = 1 - \frac{2\xi}{1 + \frac{2\xi}{\sqrt{2\pi} \sigma_{\omega}}} \int_{-\infty}^{\infty} e^{-\Omega^2/2\sigma_{\omega}^2} d\Omega \left[ 1 + 2i(\Delta - \Omega)/\kappa_{\text{tot}} \right]$$

$$= 1 - \frac{\xi}{1 + \frac{2\xi}{\sqrt{2\pi} \sigma_{\omega}}} \int_{-\infty}^{\infty} e^{-\Omega^2/2\sigma_{\omega}^2} d\Omega \left( \frac{-2\Delta + i\kappa_{\text{tot}}}{2\sqrt{2\sigma_{\omega}}} \right) \hspace{1cm} (13)$$

where $w(z)$ is the Faddeeva function, which can be expressed in terms of the complementary error function as $w(z) = e^{-z^2} \text{erfc}(-iz)$ [15]. We note that this function shows up in a related context when analyzing the effect of Gaussian broadening on an ideally Lorentzian lineshape, in which case one obtains a Voigt profile [19], but here we are interested in the full complex trajectory swept out by the scattering matrix elements rather than the lineshapes of their squared magnitudes.

As illustrated in Fig. 2, the effect of these fluctuations is a deformation of the trajectory of $S_{11}$ in the complex plane. Compared to $S_{11}(\Delta; \kappa_{\text{tot}}, \xi)$, $\overline{S}_{11}(\Delta; \kappa_{\text{tot}}, \xi, \sigma_{\omega})$ is slightly oblong, its ‘radius’ is smaller (corresponding to an apparent decrease in $\xi$), and it traverses its path more slowly as a function of detuning (corresponding to an apparent increase in $\kappa_{\text{tot}}$). Furthermore, although there is a systematic deviation between the closest fit of $S_{11}(\Delta; \kappa_{\text{tot}}^\prime, \xi^\prime)$ to a trajectory generated by $\overline{S}_{11}(\Delta; \kappa_{\text{tot}}, \xi, \sigma_{\omega})$, this deviation is subtle. It is even more subtle, in fact, if one considers only the squared magnitude of $S_{11}$ rather than its full complex trajectory, a practice that is fairly common [5, 9, 14].

We have observed this oblong trajectory in measurements of the reflection coefficient $S_{11}$ of a device in our lab [20], which we present in Fig. 3. The device is similar to that discussed in [21], but without the mechanical resonator. It is a tunable superconducting quarter-wavelength coplanar waveguide cavity with one physical port and one virtual port modeling internal losses, such that the reflection coefficient $S_{11}(\Delta; \kappa_{\text{tot}}, \xi)$ agrees with that derived from microwave theory [18]. Clearly, the subtle systematic differences between the two models can easily be obscured by or misattributed to measurement noise.
Thus, presented with experimental data for which frequency fluctuations are significant, it is quite reasonable to believe that it is well-modeled by $S_{ij}(\Delta; \kappa_{tot}, \xi')$. If one tries to fit this model to the data, however, one will extract damping rates that seem to vary with $\sigma_{\omega_0}$, as illustrated in Fig. 4. In the case of a tunable cavity for which $\sigma_{\omega_0} = |\partial \omega_0(\mathfrak{s})/\partial x|\sigma_x$, one will therefore find damping rates that seem to vary with $\mathfrak{s}$, which is precisely the syndrome we set out to explain.

V. DISTINGUISHING FREQUENCY FLUCTUATIONS FROM DAMPING

We emphasize that these fluctuations will always couple into the system as given by Eqs. 1 - 3, but they will not always be significant enough to require the use of the fluctuating model $\overline{S_{ij}}(\Delta; \sigma_{\omega_0})$. The relevant frequency scale for a two-sided cavity is $\kappa_{tot}$, as seen in Fig. 4 when $\sigma_{\omega_0} \ll \kappa_{tot}$, the apparent and actual damping rates coincide. As a benchmark in the intermediate case, one must have $\sigma_{\omega_0} \lesssim 0.17\kappa_{tot}$ in order for the apparent $\kappa'_{tot}$ to deviate from its actual value $\kappa_{tot}$ by less than 10%. We also note that the actual damping rates themselves may vary with $\mathfrak{s}$, whether explicitly or implicitly through a dependence on $\omega_0(\mathfrak{s})$. In the practical application of this model, therefore, it may be necessary both to corroborate the presence of appreciable frequency fluctuations and to distinguish these fluctuations from damping.

One method of measuring the magnitude of frequency fluctuations is by driving the cavity at its resonant frequency and measuring the scattered signal in the time domain [23][25]. In general, the phase of the scattering-matrix elements varies most strongly with the detuning at $\Delta = 0$, so fluctuations in $\omega_0$ will encode themselves as fluctuations in the phase of the scattered signal. The power spectral density of these phase fluctuations will thus correspond to the power spectral density of frequency fluctuations, provided that other sources of phase noise are small compared to those induced by frequency fluctuations.

To differentiate between frequency fluctuations and damping one may perform a 'ringdown' experiment by directly measuring the decay rate of oscillations in the cavity, usually through a homodyne or heterodyne detection scheme in the case of microwave frequency devices [26]. Although this measurement will only yield the total damping rate, it shouldn’t depend on frequency fluctuations so long as these fluctuations occur on timescales much longer than the lifetime of oscillations in the cavity. Thus, if the damping rates extracted from ringdown experiments are inconsistent with those extracted from fitting $S_{ij}(\Delta)$ to data from scattering experiments, then frequency fluctuations are likely significant enough to require use of the model $\overline{S_{ij}}(\Delta; \sigma_{\omega_0})$.

Finally, to validate the use of the model $\overline{S_{ij}}(\Delta; \sigma_{\omega_0})$ one can simply fit it to measurements of the scattering matrix elements and extract $\sigma_{\omega_0}$ as a function of the tuning parameter $\mathfrak{s}$. If a good model for $\omega_0(\mathfrak{s})$ is available and in good agreement with experiment, as is usually the case, one can then try fitting the extracted values of $\sigma_{\omega_0}(\mathfrak{s})$ to the trend $\sigma_{\omega_0}(\mathfrak{s}) = |\partial \omega_0(\mathfrak{s})/\partial x|\sigma_x$. If this can be done for some constant and physically reasonable value of $\sigma_x$ then one has strong evidence that the fluctuating model accurately accounts for the experimental results.

VI. CONCLUSION

In conclusion, we have shown how fluctuations in the tuning parameter induce fluctuations in the resonant frequency of tunable microwave cavities, and have presented a model for measurements of scattering matrix elements in the presence of such fluctuations that can be applied to any microwave network. For the specific case of a two-sided cavity, we have found an analytic expression for the average scattering matrix elements and shown that they are qualitatively similar to those that would be measured in the absence of these fluctuations. In neglecting to account for these fluctuations, however, we have shown that one may find damping rates that appear to vary with the tuning parameter. Finally, we have discussed several practical considerations in applying the model to experimental data and distinguishing frequency fluctuations from damping.

Although this model could apply to any tunable microwave cavity, we have focused our attention on superconducting microwave cavities for three reasons. First, these are the systems for which tuning-dependent damp-
ing rates have been commonly observed. Second, they tend to have very low damping rates [27, 28], so frequency fluctuations don’t have to be as large to significantly affect their apparent linewidths. Third, they tend to use tuning mechanisms (the flux through SQUID loops, commonly) that are likely to experience $1/f$ noise [29, 30]. Since the time it takes to measure scattering matrix elements using a vector network analyzer is typically on the order of hundreds of milliseconds or more, a significant amount of $1/f$ noise will couple into the resonant frequency during such a measurement. Altogether, these make tunable superconducting microwave cavities likely candidates for observing the effects predicted by this model.

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