Time Dependent Effective Actions at Finite Temperature∗

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Abstract

I study derivative expansions of effective actions at finite temperature, illustrating how the standard methods are badly defined at finite temperature. I then show that by setting up the initial conditions at a finite time, these problems are solved.

1 Standard approach

1.1 Zero temperature

I will work with a simple model of two real fields φ and η

\[ \mathcal{L}[\phi, \eta] = \frac{1}{2} \eta \Delta^{-1} \eta - \frac{1}{2} g \phi \eta^2 + \mathcal{L}_0[\phi], \]  

and integrating out the η field gives

\[ Z = \int D\phi D\eta \ e^{i \int d^4x \ \mathcal{L}[\phi, \eta]} = \int D\phi \ e^{i \int d^4x \ \mathcal{L}_0[\phi]} \ e^{i S_{\text{eff}}[\phi]}. \]  

Here the integration is exact and the effective action for φ is the classical part \( \mathcal{L}_0[\phi] \) plus

\[ S_{\text{eff}}[\phi] = \frac{i}{2} \text{Tr} \{ \ln [1 - g \phi(x) \Delta(x, y)] \}. \]  

This \( S_{\text{eff}}[\phi] \) contains all η fluctuations, both quantum and statistical, even though it can only be used to describe the behaviour of \( \phi \). The problem is that \( S_{\text{eff}}[\phi] \) is still too

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1This is a toy model of QED, where the η field plays the role of the electrons, and I am looking for an effective theory for the photon, here played by the \( \phi \) field. In the case of QED this would then be the Euler-Heisenberg effective action.
The truncation of the ln expansion is valid for weak coupling and/or weak fields. Expanding the ln gives $S_{\text{eff}}[\phi] = S_{\text{eff}}^{(1)} + S_{\text{eff}}^{(2)} + \ldots$ where

$$S_{\text{eff}}^{(2)} := -\frac{ig^2}{4} \int d^4x \int d^4y \{ \phi(x)\Delta(x,y)\phi(y)\Delta(y,x) \} \quad (4)$$

I will focus on $S_{\text{eff}}^{(2)}$ as this is quadratic in $\phi$, so it contains important effective mass and kinetic terms for $\phi$ and it is also the first term which shows all the features of the problem. The truncation of the ln expansion is valid for weak coupling and/or weak fields.

This term $S_{\text{eff}}^{(2)}$ is however non-local in $\phi$ so we expand $\phi(y)$ in terms of the field at $x$ to get infinite number of local terms

$$\phi(y) = \phi(x) + (y - x)^\mu \partial_\mu \phi(x) + \ldots = e^{i(y-x)^\mu P_\mu} \phi(x), \quad P_\mu = -i \partial_\mu \quad (5)$$

Truncating this derivative expansion gives a tractable if approximate effective action with finite number of local terms. This truncation is valid for fields varying slowly in time and space.

### 1.2 Finite Temperature

It is easiest in this case to work in time not energy variables, so the integrals over real times $t$ are replaced by integrals along a directed path $C$ in the complex time plane. As this is a dynamical problem, I will use the CTP (Closed Time Path) approach[1] where $C$ has three sections. The first, $C_1$ runs along the real axis from $t_i$ to $t_f$. $C_2$ is $C_1$ but running in the opposite direction. Finally there is the vertical section, $C_3$ running from $t_i$ to $t_i - i\beta$.

The usual assumptions made are $t_i \rightarrow -\infty$, $t_f \rightarrow +\infty$, and an equilibrium background field $\eta$. For the latter the propagator is

$$\Delta_c(\tau, \tau'; k) = \frac{-i}{2\pi} \int_{-\infty}^{\infty} dE \ e^{-iE(\tau - \tau')} \left[ \theta_C(\tau, \tau') + N(E) \right] \rho(E, k). \quad (6)$$

$\theta_C$ is the contour theta function[1] - $\theta_C(\tau, \tau')$ is 1 (0) if $\tau$ ($\tau'$) is further along $C$ than $\tau'$ ($\tau$). $N(E)$ is the Bose-Einstein distribution $N(E) := [\exp\{\beta E\} - 1]^{-1}$ and $\rho$ is the spectral function, which for a free relativistic scalar field is of the form $\rho(E, k) = \pi(\omega(k))^{-1}[\delta(E - \omega(k)) - \delta(E + \omega(k))]$. I will choose $\omega(k) = (k^2 + m^2)^{1/2}$ for the $\eta$ field dispersion relation but the arguments below work with an arbitrary form.

The problem can be seen in the energy representation as the derivative expansion of $S_{\text{eff}}^{(2)}$ can be rewritten in terms of the small four-momentum expansion of the bubble diagram $B(P)$

$$S_{\text{eff}}^{(2)} = \int d^4x \left[ B(P = 0) \phi(x)^2 + \left( \frac{\partial B(P)}{\partial E} \right)_{P=0} \phi(x) \frac{\partial}{\partial \tau} \phi(x) \right.
\left. + \frac{1}{2} \frac{\partial^2 B(0)}{\partial E^2} \phi(x) \frac{\partial^2}{\partial \tau^2} \phi(x) + \ldots \right], \quad (7)$$

$$B(P) := \int_{\beta} d^4K \Delta(K)\Delta(K + P). \quad (8)$$
However, the equilibrium $B$ does not have a unique momentum expansion at any $T > 0$, something known in the non-relativistic context at least since the work of Abrahams and Tsuneto in 1966[2] and for relativistic fields by Fujimoto in 1984[3, 4]. There is therefore a serious problem when trying to take the standard analysis to non-zero temperatures.

2 A new approach

The $T > 0$ analysis above followed the usual approach without too much attention to the physics of the problem. I will therefore try to work through it more carefully. I produce a slightly different result, one which will produce one solution to all these problems. I will work with time rather than energy variables, and to assume that the $\eta$ field starts and remains in equilibrium for all time, so that the propagator $\Delta$ is unchanged. The key lies in the values given to the c-number valued $\phi$ field in the effective action.

First for times lying on $C_3$, the field $\phi$ represents contributions coming from the initial density matrix $\rho$. Expressing the density matrix in the Heisenberg picture, $\rho_H$, in terms of interaction picture operators gives

$$\hat{\rho}_H := e^{-\beta \hat{H}_{\text{free,init}} \hat{U}_{\text{init}}}, \quad \hat{U}_{\text{init}} := T_C \exp\{-i \int_{C_3} d\tau \hat{H}_{\text{int,init}}\}.$$  (9)

where the Hamiltonian at the initial time and in the interaction picture is split as $\hat{H}_{\text{init}} = \hat{H}_{\text{free,init}} + \hat{H}_{\text{int,init}}$. The factor of $\hat{U}_{\text{init}}$ then generates Feynman diagrams with vertices running over times equal to $t_i$ plus a variable pure imaginary time component. It is therefore crucial not to use the derivative expansion to express the fields coming from $\hat{U}_{\text{init}}$ in terms of field values at later real times. Instead for $\tau \in C_3$ I set $\phi(\tau, \vec{k}) = \phi_i(\vec{k})$, a time independent constant initial field, to be specified as part of the initial conditions.

This is to be contrasted with the real-time fields $\phi(\tau)$ for $\tau \in C_1 + C_2$. These are not associated with the initial density matrix so I enforce no special initial conditions on these fields. However, I will use the derivative expansion on these real-time fields. Also I will not force the field to have the same value at the same real-time value if it is representing points on $C_1$ and $C_2$ unless we wish to refer to classical field configurations.

For simplicity an initial condition $\phi(t_i, \vec{p}) = 0$ is assumed here. Under this simplification, in this simple model I have then

$$S_{\text{eff}}^{(2)}[\phi] = -\frac{1}{2} \int_{C_a} d\tau \int d^3\vec{p} \sum_{a,b=1}^2 \phi_a(\tau, \vec{p}) B_{ab} \phi_b(-\vec{p}),$$  (10)

where $B$ is a bubble diagram given by

$$B_{ab}(\tau, \tau_i, \vec{p}, E'') = \frac{ig^2}{2} \int_{C_b} d\tau' \frac{d^3\vec{k}}{(2\pi)^3} \Delta_{ab}(\tau, \tau', \vec{k}) \Delta_{ba}(\tau', \tau, \vec{p} + \vec{k}) \times \exp\{-iE''(\tau' - \tau)\},$$  (11)

$$\phi_a(\tau) := \phi(\tau), \quad \Delta_{ab}(\tau, \tau', \vec{k}) := \Delta_c(\tau, \tau', \vec{k}) \quad \tau \in C_a, \tau' \in C_b.$$  (12)

Note that $-iE'' \equiv \partial/\partial t$ ($t \in \mathbb{R}$) and $B_{ab}(a, b = 1, 2)$ are operators acting on $\phi_b(b = 1, 2)$ only (not $b = 3$ due to initial conditions). $B$ satisfies the algebraic identities $B_{11} + B_{12} + B_{21} + B_{22} = 0$ and $B_{13} + B_{23} = 0$.  

3
As the only physical solution for field expectation values at real times is \( \phi_1(t) = \phi_2(t) \), it is useful to rewrite this as

\[
S_{\text{eff}}^{(2)}[\phi] = -\frac{1}{2} \int_{C_1} dt \int d^3 \vec{p} \ (\bar{\phi}(t, \vec{p}), \phi(t, \vec{p})) \begin{pmatrix} 0 & B_{\text{adv}} \\ B_{\text{ret}} & B_{\text{fluc}} \end{pmatrix} \begin{pmatrix} \bar{\phi}(t, -\vec{p}) \\ \phi(t, -\vec{p}) \end{pmatrix},
\]

where \( \bar{\phi} = (\phi_1 + \phi_2)/\sqrt{2}, \phi_\delta = (\phi_1 - \phi_2)/\sqrt{2} \) at any one time and three-momentum. \( B_{\text{ret}} = B_{11} + B_{12} = -B_{22} - B_{21} \) is one key object of interest, as it is this term which appears in the equations of motion so I will focus on this term. The result is

\[
B_{\text{ret}} = \frac{g^2}{8} \int d^3 \vec{k} \sum_{s_0, s_1 = \pm 1} \frac{s_0 s_1}{\omega \Omega} (1 + N(s_0 \omega) + N(s_1 \Omega)) \frac{(1 - e^{i(t_i - t)A})}{A}.
\]

The Landau damping terms come from the \( s_0 = -s_1, \omega - \Omega \) factors. In the limit of interest for derivative expansion \( E, p \to 0 \) so that \( \Omega \to \omega \) and thus these are dangerous as the denominator \( A \to 0 \). In my case though I have a crucial \( t_i \) dependent term in the numerator which ensures my numerator also goes to zero in this limit and my expression is well behaved. Thus my \( B_{\text{ret}} \) has unique derivative series as its analytic about \( E, p \to 0, A = 0 \).

I do not get the traditional equilibrium result for \( B_{\text{ret}} \) and the difference is the unusual factor \( \exp\{i(t_i - t)A\} \). In equilibrium calculations using pure imaginary time methods, the real external energy \( E \) is replaced by \( E - i\epsilon \) (\( \epsilon \) is a real positive infinitesimal) during analytic continuation[1]. Then one takes the \( t_i \to -\infty \) limit. This \( t_i \) dependent term is then removed but the integrand is then singular in the \( \epsilon \gg E, p \to 0 \) limit.

There are alternative solutions which work by keeping \( A \neq 0 \) in the zero momentum limit e.g. including thermalisation rates/complex dispersion relations for the \( \eta \) field[5, 6] or keeping the masses in the two propagators different[7]. However it is achieved, what is happening in all cases is that a long time scale is being introduced and this sets a regulator for this long time and long distance (small \( E \) and \( \vec{p} \)) problem. In my case, it is more obvious as it is an actual physical time \( t_i \) rather than an energy parameter which is performing the regulation.\(^2\)

There are several conclusions to be drawn from this work. First we have shown how to obtain a unique expansion for weak, slowly varying fields in a heat bath with the \( \omega - \Omega \) Landau damping terms giving the dominant contribution. In particular, when contributions from the vertical part are included this analysis does show how the usual free energy results are the lowest term in a consistent derivative expansion of an effective action, as found at zero temperature.\(^3\) This approach also solves a lack of analyticity problem inherent in linear response calculations. Though the analysis has been presented

\(^2\)Of course another side effect is that my results depend explicitly on \( t_i \), the time at which the initial conditions were setup. This should be expected as the problem is not an equilibrium one even if some of my fields are to an approximation in equilibrium. After all, looking at small frequency perturbations means that you are implicitly probing long time scales which will inevitably probe the time at which the system was initialised.

\(^3\)With the vertical contributions (none-zero if \( \phi \neq 0 \) initially) and a constant \( \phi \) field, I get usual results for free energy.
for a simple relativistic model, the principles are universal, e.g. they work for a BCS superconductor[8].

The biggest remaining problem is that there are time dependent U.V. divergences (∼ ln(t − tᵢ) in the equations of motion), presumably reflecting the fact that at tᵢ = t we have no new time scale in the game to set the scale for the low energy/long time behaviour of my fields.⁴

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⁴This latter problem will be avoided if other long time scales are included, such as damping in propagators[5, 6]. Perhaps one also needs to ensure that the initial conditions are set more consistently.
3 Appendix

3.1 CTP curve

3.2 Effective Potential

The effective potential should be the lowest term in the derivative expansion of the effective action, i.e. equal to the effective action for constant fields

$$S_{\text{eff}}^{(2)} = -i\phi_i^2 \sum_{a,b=1}^{3} \int_{C_a} d\tau \int d\vec{x} B_{ab}$$

(16)

The identities $B_{11} + B_{12} + B_{21} + B_{22} = 0$ and $B_{13} + B_{23} = 0$ then tell us that in this form only integrations along $C_3$ contribute to the total effective potential. Another way of seeing this is to take $t_f = t_i$ leaving $C = C_3$, $C_H = \emptyset$. Thus

$$V_{\text{eff}}^{(2)} \propto \phi_i^2 \int_{C_3} d\tau \int d\vec{x} B_{33}$$

(17)

However, I can get an alternative expression which is more relevant to my analysis of the effective action. Note[4] that in equilibrium $B_{ab}$ is independent of $\tau$ by time translation invariance. The integral over $\tau$ merely gives a factor of $\beta$ to the overall volume factor out front. Thus $\partial_\mu \phi = 0 \Rightarrow$

$$S_{\text{eff}}^{(2)} = \frac{1}{2}(-i\beta)V\phi_i^2 \sum_{b=1}^{3} B_{ab}(\tau, \vec{p} = 0, E'' = 0)$$

$$\forall \tau \in C_a, \ a = 1, 2, 3$$

(18)

Without loss of generality choose $a = 1, \tau \in C_1$ which gives

$$V_{\text{eff}}^{(2)}(\phi) \propto \phi_i^2 [(B_{11} + B_{12}) + B_{13}]$$

$$= \phi_i^2 [B_{\text{ret}} + B_{13}]$$

(19)

(20)
Note there are no ambiguities in the effective potential coming from the $E'', \vec{p} \to 0$ limit of $B$ with whatever $t_i \neq -\infty$ is used, so the lack of analyticity problem is in this context again solved. However it is essential to have both horizontal $B_{11}, B_{12}$ and vertical $B_{13}$ contributions to get correct answer. In $t_i \to -\infty$ both horizontal $B_{11}, B_{12}$ and vertical $B_{13}$ may be needed but great care is needed with regulators to analyse this limit, and this accounts for much of the confusion with real-time calculations of the effective potential where just $B_{\text{ret}}$ is often encountered.