A Revisit of the Velocity Averaging Lemma: On the Regularity of Stationary Boltzmann Equation in a Bounded Convex Domain

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Abstract
In the present work, we adopt the idea of velocity averaging lemma to establish regularity for stationary linearized Boltzmann equations in a bounded convex domain. Considering the incoming data, with four iterations, we establish regularity in fractional Sobolev space in space variable up to order $1^{-}$.

Keywords Boltzmann equation · Regularity · Averaging lemmas · Fractional Sobolev spaces

1 Introduction

Introduced by Ludwig Boltzmann in 1872, the Boltzmann equation describes the statistical behavior of a gas dynamic system. In the study of the Boltzmann equation, one of the basic and important problems is to understand the steady state solutions. In particular, the case where the gas interacts with boundary is of more physical interests. In this article, we are interested in the regularity theory of the stationary linearized Boltzmann equations where the gas is confined in a bounded convex domain. Namely, we study the regularity theory for the following equation

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for $v \in \mathbb{R}^3$ and $x \in \Omega$, where $\Omega \subset \mathbb{R}^3$ is a $C^2$ bounded strictly convex domain such that $\partial \Omega$ is of positive Gaussian curvatures. Here, $L$ represents the linearization of the collision operator. The collision operator in Boltzmann equation reads:

$$Q(F, G) = \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^\pi (F(v')G(\xi') - F(v)G(v_*)B(|v_* - v|, \theta))d\theta d\epsilon dv_*,$$

where $v$, $v_*$ and $v'$, $v'_*$ are pairs of velocities before and after the impact, and $B$ is called the cross section, depending on interaction between particles. $L$ is obtained by linearizing $Q$ around the standard Maxwellian

$$M(v) = \pi^{-\frac{3}{2}} e^{-|v|^2}$$

in the fashion

$$F = M + M^{\frac{1}{2}} f.$$ 

Hence, $L$ reads

$$L(f) = M^{-\frac{1}{2}}(v) \left[ Q \left( M^{\frac{1}{2}} f, M \right) + Q \left( M, M^{\frac{1}{2}} f \right) \right].$$

The widely used angular cutoff potential is a mathematical model introduced by Grad [23] by assuming

$$0 \leq B \leq C |v - v_*|^\gamma \cos \theta \sin \theta.$$ 

In this paper, we follow Grad’s idea and assume

$$B = |v - v_*|^\beta(\theta),$$

$$0 \leq \beta(\theta) \leq C \cos \theta \sin \theta,$$

$$0 \leq \gamma \leq 1.$$ 

The range of $\gamma$ we consider is corresponding to the hard sphere model, cutoff hard potential, and cutoff Maxwellian molecular gases. The detailed setting of the problem will be addressed later.

When it comes to the regularity theory of the Boltzmann equation, one of the most interesting and powerful mathematical tools is the velocity averaging lemma. The celebrated velocity averaging lemma reveals that the combination of transport and averaging in velocity yields regularity in space variable [20, 21]. This is one of the key features that DiPerna and Lions used to attack the Cauchy problem for Boltzmann equations [16]. For kinetic and stationary cases for the whole space, this subject is extensively studied [12–14, 27]. It is natural to adopt this technique to the study of regularity problem of linearized Boltzmann equation in the whole space [11]. In [21], in addition to the applications to the whole space domains, the authors also investigate the applications to bounded convex domains for transport equations. By adopting zero extension, they reduce the bounded domain case to the whole space case. In contrast with the whole space case in [11], which the regularity can be improved indefinitely by iterations, when applying the trick in [21] in a bounded domain, one can only proceed for one iteration. Notice that the main tool of velocity averaging lemma, namely, the method of Fourier transform, does not translate well on a bounded domain. In this article, we adopt Slobodeckij semi-norm as an alternative concept of Sobolev function class. This shifts the difficulty to singular integrals. To calculate these integrals, we encounter some estimates related to the geometry of the boundary. The way we do it is to properly compare convex domains with spherical domains in $\mathbb{R}^3$ so that we can build up estimates for convex
domains based on those for spherical domains. Another key feature to prove the boundedness of the singular integrals is the change of variables which will be addressed in Lemma 7.1 and Lemma 7.2. Considering the incoming data, with four iterations, we establish regularity in fractional Sobolev space in space variable up to order $1^-$.

Recall the velocity averaging lemma in [12, 21]. Suppose $u$ is an $L^2$ solution to the transport equation

$$v \cdot \nabla_x u = G(x, v), \quad (x, v) \in \mathbb{R}^n \times \mathbb{R}^n,$$

where $G \in L^2$. Let

$$\bar{u}(x) := \int_{\mathbb{R}^n} u(x, v) \psi(v) \, dv,$$

where $\psi$ is a bounded function with compact support. Then, we have

$$\bar{u}(x) \in \tilde{H}^{\frac{1}{2}}(\mathbb{R}^n).$$

Here, the Sobolev space is generalized to non-integer order via the Fourier transform as follows.

**Definition 1.1** We say $u : \mathbb{R}^3 \to \mathbb{R}$ is in $\tilde{H}^s_x(\mathbb{R}^3)$ if

$$\|u\|_{\tilde{H}^s_x(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\mathcal{F}(u)(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} < \infty,$$

where $\mathcal{F}(u)(\xi)$ is the Fourier transform of $u$, i.e.,

$$\mathcal{F}(u)(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} u(x) e^{-i\xi \cdot x} \, dx.$$

The velocity averaging lemma demonstrates that the regularity in the transport direction can be converted to the regularity in space variable after averaging with weight $\psi$.

First, we recapitulate the stationary linearized Boltzmann equation in the whole space,

$$v \cdot \nabla_x f(x, v) = L(f),$$

where $f$ is the velocity distribution function and $L$ is the linearized collision operator. The linearized collision operator under consideration can be decomposed into a multiplicative operator and an integral operator.

$$L(f) = -v(v)f + K(f),$$

where

$$K(f) = \int_{\mathbb{R}^3} k(v, v_s) f(v_s) \, dv_s.$$

Therefore, we can rewrite (1.10) as

$$v(v)f + v \cdot \nabla_x f = K(f).$$

Observing that the integral operator $K$ can serve as an agent of averaging, it is natural to imagine applying velocity averaging lemma to linearized Boltzmann equation. In case the source term $\Psi(x, v)$ is imposed, i.e.,

$$v(v)f + v \cdot \nabla_x f = K(f) + \Psi(x, v),$$

one can derive an integral equation
\[ f(x, v) = \int_0^\infty e^{-\nu(v)t} [K(f)(x - vt, v) + \Psi(x - vt, v)] dt \]
\[ = S(K(f) + \Psi) \]
\[ = SK(f) + S(\Psi), \quad (1.13) \]

where
\[ S(h)(x, v) := \int_0^\infty e^{-\nu(v)t} h(x - vt, t) dt. \quad (1.14) \]

Performing the Picard iteration, formally we can derive that
\[ f = \sum_{k=0}^{\infty} S(KS)^k(\Psi). \quad (1.15) \]

By carefully adapting the idea of velocity averaging lemma, we find every two iterations improve regularity in space of order \( \frac{1}{2} \). More precisely, the following lemma was proved in [11].

**Lemma 1.2** The operator \( SK : L^2_v(\mathbb{R}^3; \tilde{H}^s_x(\mathbb{R}^3)) \to L^2_v(\mathbb{R}^3; \tilde{H}^{s+\frac{1}{2}}_x(\mathbb{R}^3)) \) is bounded for any \( s \geq 0 \).

Here, the mixed fractional Sobolev space is defined as follows.

**Definition 1.3** We say \( u : \Omega \times \mathbb{R}^3 \to \mathbb{R} \) is in \( L^2_v(\mathbb{R}^3; \tilde{H}^s_x(\Omega)) \) if
\[ \| u \|_{L^2_v(\mathbb{R}^3; \tilde{H}^s_x(\Omega))} = \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( 1 + |\xi|^2 \right)^s |F(u)(\xi, v)|^2 d\xi dv \right)^{\frac{1}{2}} < \infty, \quad (1.16) \]

where \( F(u)(\xi, v) \) is the Fourier transform of \( u \) with respect to space variable \( x \).

Motivated by the successful application of velocity averaging lemma to the study of regularity issue for stationary linearized Boltzmann equation in the whole space, we consider to give a similar account for the regularity problem in a bounded domain. However, we immediately notice that Definition 1.3 does not work for bounded space because that the Fourier transform is involved. For bounded domains, we adopt the fractional Sobolev space through the Slobodeckij semi-norm [15].

**Definition 1.4** Let \( s \in (0, 1), \Omega \subset \mathbb{R}^3 \) open. We say \( f(x, v) \in L^2_v(\mathbb{R}^3; H^s_x(\Omega)) \) if \( f \in L^2(\Omega \times \mathbb{R}^3) \) and
\[ \int_{\mathbb{R}^3} \int_{\Omega} \int_{\Omega} \frac{|f(x, v) - f(y, v)|^2}{|x - y|^{3+2s}} dxdydv < \infty, \quad (1.17) \]

with
\[ \| f \|_{L^2_v(\mathbb{R}^3; H^s_x(\Omega))} = \left( \| f \|_{L^2(\Omega \times \mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} \int_{\Omega} \int_{\Omega} \frac{|f(x, v) - f(y, v)|^2}{|x - y|^{3+2s}} dxdydv \right)^{\frac{1}{2}}. \quad (1.18) \]

Notice that Definitions 1.3 and 1.4 of fractional Sobolev spaces are equivalent on the whole space. More precisely, we have
\[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|f(x, v) - f(y, v)|}{|x - y|^{3+2s}} dxdydv = \sigma(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\xi|^2 |F(f)(\xi, v)|^2 d\xi dv, \quad (1.19) \]
where
\[
\sigma(s) = 2 \int_{\mathbb{R}^3} \frac{1 - \cos \zeta}{|\zeta|^{3+2s}} d\zeta.
\] (1.20)

Notice that \( \sigma(s) \sim (1 - s)^{-1} \) as \( s \sim 1^- \).

Here, we shall first introduce our main result and then explain the multiple obstacles we encounter and how we overcome them. We consider a bounded convex domain which satisfies the following assumption.

**Definition 1.5** We say a \( C^2 \) bounded convex domain \( \Omega \) in \( \mathbb{R}^3 \) satisfies the positive curvature condition if \( \partial \Omega \) is of positive Gaussian curvature.

**Remark 1.6** Positive curvature condition implies uniform convexity, which would also imply strict convexity. If the domain is compact, then its being strictly convex is equivalent to being uniformly convex. On the contrary, a uniformly convex domain does not necessarily satisfy the positive curvature condition.

We consider the incoming boundary value problem for linearized Boltzmann equation in \( \Omega \),
\[
\begin{align*}
\nu \cdot \nabla_x f(x, v) &= L(f)(x, v), \quad \text{for } x \in \Omega, \ v \in \mathbb{R}^3, \\
f|_{\Gamma_-}(q, v) &= g(q, v), \quad \text{for } (q, v) \in \Gamma_-, 
\end{align*}
\] (1.21)
where \( \Gamma_- := \{(q, v) \in \partial \Omega \times \mathbb{R}^3 : n(q) \cdot v < 0\} \), and \( n(q) \) is the unit outward normal of \( \partial \Omega \) at \( q \). In this context, \( L \) satisfies one of hard sphere, cutoff hard, and cutoff Maxwellian potentials. The detailed assumption on \( L \) will be addressed in Sect. 3.

Regarding the existence result of boundary value problem (1.21), it has been studied by Guiraud [24] for convex domains and by Esposito, Guo, Kim, and Marra [18] for general domains. In the paper of Esposito, Guo, Kim, and Marra [18], they proved the solution is continuous away from the grazing set. With the cutoff assumption (1.7) on cross section \( B \), the interior Hölder estimate was established in [7] and later improved to interior pointwise estimate for first derivatives [8]. Recently, a higher regularity estimate up to \( C^{1,\alpha} \) for all \( \alpha < 1 \) was studied in [9] for the nonlinear Boltzmann equation with hard sphere potential. The weighted \( C^1 \) estimates for both steady solutions and dynamical solutions with Cercignani-Lampis boundary condition was studied in [10]. The main idea in [7–10] is to move the regularity in velocity to space through transport and collision by double iteration scheme along the trajectory. The fact \( K \) improves regularity in velocity is a key property used. This idea was inspired by the mixture lemma by Liu and Yu [31]. For more development in the direction of mixture lemma, see [29, 30, 32]. In contrast, in the present result, we do not need the smoothing effect of \( K \) in velocity; the integral operator \( K \) itself provides “velocity averaging” and therefore regularity. Regarding the regularity issue for the time dependent Boltzmann equation, in [26], the authors establish \( W^{1,p} \) estimate for \( p < 2 \) without the kinetic distance weight and provide a counterexample of \( H^1 \) regularity. One of the key estimates is nonlocal-to-local estimate, which deals with the interaction of integral operator \( K \) and the kinetic weight. We also refer the interested readers to [25] for BV regularity estimate in non-convex domains.

In this article, we assume the following two conditions on the incoming data \( g \).

**Assumption 1.7** There are positive constants \( a, C \) such that
\[
|g(q_1, v)| \leq C e^{-a|v|^2}
\] (1.22)
and

\[ |g(q_1, v) - g(q_2, v)| \leq C |q_1 - q_2|, \quad (1.23) \]

for any \((q_1, v) \in \Gamma_-\) and \((q_2, v) \in \Gamma_-\).

The main result of this paper is as follows.

**Theorem 1.8** Let bounded convex open domain \( \Omega \subset \mathbb{R}^3 \) satisfy the positive curvature condition in Definition 1.5, linearized collision operator \( L \) satisfy angular cutoff assumption (2.5), and collision frequency \( v \) satisfy (2.10). Suppose the incoming data \( g \) satisfies Assumption 1.7. Then, for any solution \( f \in L^2(\Omega \times \mathbb{R}^3) \) to stationary linearized Boltzmann equation (1.21), we have

\[ f \in L^2_v(\mathbb{R}^3; H^{1-\epsilon}_x(\Omega)), \quad (1.24) \]

for any \( 0 < \epsilon < 1 \).

**Remark 1.9** In this manuscript, to demonstrate our mathematical analysis neatly, we consider the regularity theory for the incoming boundary value problem. We think that our method can also be applied to other boundary conditions, for example, the diffuse reflection boundary condition. We leave this issue to be the future work.

We shall sketch the proof and reveal the difficulties induced by geometry and the method we tackle the problem.

**Definition 1.10** For \( x \in \bar{\Omega} \) and \( v \in \mathbb{R}^3 \), we define \( \tau_-(x, v) \) to be the backward exit time for \( x \) leaving \( \Omega \) with velocity \(-v\), and we define \( q_-(x, v) \) to be the corresponding point that the backward trajectory touches \( \partial \Omega \). Similarly, \( \tau_+(x, v) \) and \( q_+(x, v) \) are defined to be forward ones accordingly. More precisely,

\[
\begin{align*}
\tau_-(x, v) &:= \inf_{t > 0} \{ t : x - vt \not\in \Omega \}, \\
q_-(x, v) &:= x - \tau_-(x, v)v, \\
\tau_+(x, v) &:= \inf_{t > 0} \{ t : x + vt \not\in \Omega \}, \\
q_+(x, v) &:= x + \tau_+(x, v)v.
\end{align*}
\]

With the notations of Definition 1.10, one can rewrite (1.21) as the integral equation

\[
f(x, v) = e^{-v(v)\tau_-(x, v)} g(q_-(x, v), v) + \int_0^{\tau_-(x, v)} e^{-v(v)s} K(f)(x - sv, v) \, ds. \quad (1.25)
\]

Hereafter, we define

\[
\begin{align*}
(Jg)(x, v) &:= e^{-v(v)\tau_-(x, v)} g(q_-(x, v), v), \\
(S_{\Omega} f)(x, v) &:= \int_0^{\tau_-(x, v)} e^{-v(v)s} f(x - sv, v) \, ds.
\end{align*}
\]

Performing Picard iteration, we have

\[
f(x, v) = J(g) + S_{\Omega} K(f) = J(g) + S_{\Omega} K J(g) + S_{\Omega} K S_{\Omega} K(f) = J(g) + S_{\Omega} K J(g) + S_{\Omega} K S_{\Omega} K J(g) + S_{\Omega} K S_{\Omega} K S_{\Omega} K J(g) + S_{\Omega} K S_{\Omega} K S_{\Omega} K S_{\Omega} K(f) = g_0 + g_1 + g_2 + g_3 + f_4, \quad (1.28)
\]
where
\[ g_i := (S_\Omega K)^i J(g), \]
\[ f_i := (S_\Omega K)^i (f). \]

We observe that each \( g_i \) is directly under influence of boundary data and the geometry of the domain. Our strategy is to prove \( g_i \in L^2_v(\mathbb{R}^3; H^{1-\epsilon}_x(\mathbb{R}^3)) \). And, concerning the remaining term \( f_4 \), we shall match up the regularity of boundary terms.

We point out the difference between the cases for the whole space and a bounded domain. Let \( h \) be any measurable function defined in \( \Omega \times \mathbb{R}^3 \). We use the notation \( \tilde{h} \) to denote its zero extension in \( \mathbb{R}^3 \). And, let \( Z : \Omega \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) be the zero extension operator from \( \Omega \times \mathbb{R}^3 \) to \( \mathbb{R}^3 \times \mathbb{R}^3 \), namely,
\[ (Zh)(x, v) = \tilde{h}(x, v) = \begin{cases} h(x, v), & \text{if } x \in \Omega, \\ 0, & \text{otherwise}. \end{cases} \tag{1.31} \]

Suppose \( f \in L^2(\Omega \times \mathbb{R}^3) \). Since \( K \) is a local operator, \( K(\tilde{f})(y) = 0 \) for \( y \notin \Omega \). Therefore, \( SK(\tilde{f}) \) does not pick up any contributions from outside \( \Omega \). Thus,
\[ SK(\tilde{f}) \big|_{\Omega} = S_\Omega K(f). \tag{1.32} \]

Therefore, applying Lemma 1.2, we have

**Corollary 1.11** The operator \( K S_\Omega K : L^2(\Omega \times \mathbb{R}^3) \rightarrow L^2_v(\mathbb{R}^3; H^{1-\epsilon}_x(\Omega)) \) is bounded.

**Remark 1.12** Corollary 1.11 can be viewed as a variant of Theorem 4 in [21]. For the latter, the transport equation under consideration is, for given \( h \in L^2(\Omega \times \mathbb{R}^3) \),
\[ u(x, v) + v \cdot \nabla_x u(x, v) = h(x, v), \quad \text{for } x \in \Omega, \quad v \in \mathbb{R}^3. \tag{1.33} \]

On the other hand, the transport equation we consider here is, for given \( h \in L^2(\Omega \times \mathbb{R}^3) \),
\[ v(v)u(x, v) + v \cdot \nabla_x u(x, v) = Kh(x, v), \quad \text{for } x \in \Omega, \quad v \in \mathbb{R}^3. \tag{1.34} \]

However, if we want to further iterate, we have to take the geometric structure into consideration. As mentioned earlier, the method of Fourier transform does not apply to bounded domains. In this article, we adopt Slobodeckij semi-norm as an alternative concept of Sobolev function class. As a result, we have

**Lemma 1.13** The operator \( S_\Omega K S_\Omega K : L^2(\Omega \times \mathbb{R}^3) \rightarrow L^2_v(\mathbb{R}^3; H^{1-\epsilon}_x(\Omega)) \) is bounded.

Therefore, if we end iteration at \( f_2 \), we can already claim \( f \in L^2_v(\mathbb{R}^3; H^{1-\epsilon}_x(\Omega)) \).

Considering piling up the regularity, we need to investigate \( SKSK(\tilde{f}) \). Although \( SK(\tilde{f})(x) = S_\Omega K(f)(x) \) for \( x \in \Omega \), \( SK(\tilde{f})(y) \) in general not zero for \( y \notin \Omega \) and, in contrast with (1.32), will contribute to the next iteration. Therefore,
\[ SKSK(\tilde{f}) \big|_{\Omega} \neq S_\Omega K S_\Omega K(f). \tag{1.35} \]

It seemingly comes to the limit of this strategy. Surprisingly, we have

**Lemma 1.14** \( ZS_\Omega K S_\Omega K : L^2(\Omega \times \mathbb{R}^3) \rightarrow L^2_v(\mathbb{R}^3; H^{1-\epsilon}_x(\mathbb{R}^3)) \) is bounded for any \( \epsilon \in (0, \frac{1}{2}) \). Furthermore, there is a constant \( C \) independent of \( \epsilon \) and \( f \) such that
\[ \| S_\Omega K S_\Omega K f \|_{L^2_v(\mathbb{R}^3; H^{1-\epsilon}_x(\mathbb{R}^3))} \leq \frac{C}{\sqrt{\epsilon}} \| f \|_{L^2(\Omega \times \mathbb{R}^3)}. \tag{1.36} \]
That is, zero extension only reduces infinitesimal regularity. Therefore, after zero extension, we can repeat our strategy and obtain the desired result. We mention that, in [19], the authors proved the $L^6$ integrability gain of the fluid part of the Boltzmann solution. In their analysis, they also prove a version of velocity averaging lemma in bounded domains by an extension argument.

The rest of the article is organized as follows. Section 2 gives a brief overview of the Boltzmann equation and the details of our setting. In Sect. 3, we recall several basic properties of the linearized collision operator. In Sect. 4, we recapitulate the velocity averaging for linearized Boltzmann equation in the whole space. Section 5 provides some geometric properties for bounded convex domains that will be used in our estimates. In Sect. 6, we study the regularity of transport equation in bounded convex domains. Section 7 is devoted to the regularity via velocity averaging.

## 2 Boltzmann Equation

The Boltzmann equation reads

$$\frac{\partial}{\partial t} F + v \cdot \nabla_x F = Q(F, F).$$  \hfill (2.1)

Here, $F = F(t, x, v) \geq 0$ is the distribution function for the particles located in the position $x$ with velocity $v$ at time $t$. The collision operator $Q$ is defined as

$$Q(F, G) = \int_{\mathbb{R}^3} \int_{0}^{\frac{\pi}{2}} \int_{0}^{\pi} \left( F(v') G(v'_*) - F(v) G(v_*) \right) B(|v - v_*|, \theta) \, d\theta \, d\epsilon \, dv_*,$$  \hfill (2.2)

where $v'$ and $v'_*$ are the velocities after the elastic collision of two particles whose velocities are $v$ and $v_*$, respectively, before the encounter. Here, the cross-section $B$ is chosen according to the type of interaction between particles. The precise form of cross-section $B$ depends on the model that is being studied. To specify the properties of the cross-section, we adopt the following coordinates. We set

$$e_1 = \frac{v_* - v}{|v_* - v|},$$

and choose $e_2 \in S^2$ and $e_3 \in S^2$ such that $\{e_1, e_2, e_3\}$ forms an orthonormal basis for $\mathbb{R}^3$, and define

$$\alpha = \cos \theta e_1 + \sin \theta \cos \epsilon e_2 + \sin \theta \sin \epsilon e_3.$$

Then,

$$v' = v + \left( (v_* - v) \cdot \alpha \right) \alpha,$$  \hfill (2.3)

$$v'_* = v_* - \left( (v_* - v) \cdot \alpha \right) \alpha.$$  \hfill (2.4)

Throughout this article, regarding the cross-section, we apply Grad’s angular cutoff potential [23] by assuming

$$0 \leq B(|v - v_*|, \theta) \leq C|v - v_*|^\gamma \cos \theta \sin \theta,$$  \hfill (2.5)

where, as mentioned above, $\gamma$ depends on the model being studied. Our discussion includes hard sphere model ($\gamma = 1$), cutoff hard potential ($0 < \gamma < 1$), and cutoff Maxwellian
molecular gases \((\gamma = 0)\). Consider the stationary solution \(F = F(x, v)\) as a perturbation of the standard Maxwellian,

\[
M(v) = \pi^{-\frac{3}{2}} e^{-|v|^2},
\]

in the form

\[
F = M + M^{\frac{1}{2}} f. \tag{2.6}
\]

Plugging the expression (2.6) into (2.1) and discarding the nonlinear term, we arrive at the stationary linearized Boltzmann equation

\[
v \cdot \nabla_x f(x, v) = L(f)(x, v), \tag{2.7}
\]

with linearized collision operator \(L\), which reads

\[
L(f) = M^{-\frac{1}{2}} \left( Q(M, M^{\frac{1}{2}} f) + Q(M^{\frac{1}{2}} f, M) \right). \tag{2.8}
\]

Under the assumption (2.5), \(L\) can be decomposed into a multiplicative operator and an integral operator, see [23],

\[
L(f) = -\nu(v) f + K(f). \tag{2.9}
\]

Here, \(\nu\) is a function of velocity variable \(v\) only, called collision frequency. For our analysis, we assume there exist two positive constants \(\nu_0\) and \(\nu_1\), depending only on \(\gamma\), such that

\[
0 < \nu_0 (1 + |v|)^{\gamma} < \nu(v) < \nu_1 (1 + |v|)^{\gamma}, \tag{2.10}
\]

for all \(v \in \mathbb{R}^3\). Throughout this article, we always assume (2.10) holds.

**Remark 2.1** If we only assume (2.5) holds, then there exists a constant \(\nu_1 > 0\) such that

\[0 \leq \nu(v) \leq \nu_1 (1 + |v|)^{\gamma}.\]

Meanwhile, the lower bound estimate for \(\nu(v)\) is not guaranteed. However, we remark that (2.10) holds for a large class of models and is adopted in a substantial literature, such as [2, 7, 8, 22, 23].

The integral operator \(K\) reads

\[
K(f)(x, v) = \int_{\mathbb{R}^3} f(x, v_*) k(v_*, v) dv_*,
\]

where the collision kernel \(k\) is symmetric, that is, \(k(v, v_*) = k(v_*, v)\). Notice that the assumption of the cross-section here is different from and more general than that in [6, 8]. The significant difference is that the operator \(K\) in the case we consider does not guarantee to have regularity in velocity variables.

Under the decomposition (2.9), we then consider the boundary value problem

\[
\begin{aligned}
&v(v) f(x, v) + v \cdot \nabla_x f(x, v) = K(f)(x, v), \quad \text{for } x \in \Omega, \ v \in \mathbb{R}^3, \\
&f|\Gamma_- (q, v) = g(q, v), \quad \text{for } (q, v) \in \Gamma_-.
\end{aligned} \tag{2.11}
\]

Let us pause here to make clear that how trace is being defined. We follow [3, 4, 28]. For \(f \in L^p(\Omega \times \mathbb{R}^3)\) satisfying the equation

\[
v(v) f(x, v) + v \cdot \nabla_x f(x, v) = K(f)(x, v) \tag{2.12}
\]

in \(\Omega \times \mathbb{R}^3\), since \(K(f) \in L^p(\Omega \times \mathbb{R}^3)\) according to Proposition 3.3, we have

\[
v \cdot \nabla_x f(x, v) = -v(v) f(x, v) + K(f)(x, v) \in L^p_x(\Omega) \tag{2.13}
\]
for a.e. \( v \in \mathbb{R}^3 \). For such \( v \in \mathbb{R}^3 \), we fix \( q \in \Gamma_-(v) \), where
\[
\Gamma_-(v) := \{ q \in \partial \Omega : (q, v) \in \Gamma_- \},
\]
and consider the function
\[
h_{q,v}(s) = f(q + sv, v), \tag{2.14}
\]
for \( 0 < s < \tau_+(q, v) \), where \( \tau_+(q, v) \) is as defined in Definition 1.10. Noticing that
\[
\frac{dh_{q,v}(s)}{ds} = v \cdot \nabla_x f(q + sv, v), \tag{2.15}
\]
and the formulas for change of variables,
\[
\int_{\Omega} |f(x, v)|^p \, dx = \int_{\Gamma_-(v)} \int_0^{\tau_+(q,v)} |f(q + sv, v)|^p |v \cdot n(q)| \, ds \, \Sigma(q) \tag{2.16}
\]
\[
\int_{\Omega} |v \cdot \nabla_x f(x, v)|^p \, dx = \int_{\Gamma_-(v)} \int_0^{\tau_+(q,v)} \left| \frac{\partial}{\partial s} f(q + sv, v) \right|^p |v \cdot n(q)| \, ds \, \Sigma(q), \tag{2.17}
\]
we obtain \( h_{q,v} \in W^{1,p}_1(0, \tau_+(q, v)) \) for a.e. \( (q, v) \in \Gamma_- \), which would imply \( h_{q,v}(s) \) is an absolutely continuous function on \((0, \tau_+(q, v))\) after possibly being redefined on a set of measure zero. As a result, we define
\[
f(q, v) := \lim_{s \to 0^+} h_{q,v}(s) = \lim_{s \to 0^+} f(q + sv, v), \tag{2.18}
\]
for \( (q, v) \in \Gamma_- \).

### 3 Properties of the Linearized Collision Operator

In this section, we review some basic properties of the linearized collision operator \( L \), see [2, 23]. As mentioned earlier, under the assumption (2.5), \( L \) can be decomposed into a multiplicative operator and an integral operator,
\[
L(f) = -v(v) f + K(f). \tag{3.1}
\]
We assume there are two constants \( 0 < v_0 < v_1 \) such that the following inequality holds for all \( v \in \mathbb{R}^3 \),
\[
0 < v_0 \leq v_0 (1 + |v|)^\gamma < v(v) < v_1 (1 + |v|)^\gamma. \tag{3.2}
\]
The integral operator \( K \) is defined as
\[
K(f)(x, v) = \int_{\mathbb{R}^3} f(x, v_*) k(v_*, v) \, dv_*,
\]
and the collision kernel \( k \) is symmetric. Furthermore, by Caflisch [2], we have an upper bound for \( k \),
\[
|k(v, v_*)| \leq C \frac{1}{|v - v_*|} (1 + |v| + |v_*|)^{-(1-\gamma)} e^{-\frac{1}{8} \left( |v-v_*|^2 + |v_0^2-|v_*|^2|^2 \right)}, \tag{3.3}
\]
where \( C > 0 \) is a constant depending only on \( \gamma \). The following lemma by Caflisch [2] is crucial in our estimates.
Lemma 3.1 For any positive constants $\epsilon$, $a_1$ and $a_2$, there exists $C = C(\epsilon, a_1, a_2)$ depending only on $\epsilon$, $a_1$ and $a_2$ such that

$$\int_{\mathbb{R}^3} \frac{1}{|v - v_*|^{3-\epsilon}} e^{-a_1|v - v_*|^2 - a_2 \frac{(v^2 - |v|^2)^2}{|v - v_*|^2}} \, dv_* \leq C \frac{1}{1 + |v|} \leq C.$$ 

The above inequality combining with (3.3) immediately implies the following facts.

Corollary 3.2 For any $v \in \mathbb{R}^3$,

$$\int_{\mathbb{R}^3} |k(v, v_*)| \, dv_* \leq C \left( \frac{1}{1 + |v|} \right)^{2-\gamma} \leq C, \quad (3.4)$$

$$\int_{\mathbb{R}^3} |k(v, v_*)|^2 \, dv_* \leq C \left( \frac{1}{1 + |v|} \right)^{3-2\gamma} \leq C. \quad (3.5)$$

Proposition 3.3 The integral operator $K : L^p_0(\mathbb{R}^3) \to L^p_0(\mathbb{R}^3)$ is bounded for $1 \leq p \leq \infty$.

Proof If $1 < p < \infty$, for given $h \in L^p_0(\mathbb{R}^3)$, we have

$$|K h(v)|^p = \left( \int_{\mathbb{R}^3} h(v_*) k(v_*, v) \, dv_* \right)^p \leq \left( \int_{\mathbb{R}^3} |h(v_*)| |k(v_*, v)| \frac{1}{p} |k(v_*, v)| \frac{1}{p} \, dv_* \right)^p \leq \left( \int_{\mathbb{R}^3} |h(v_*)|^p |k(v_*, v)| \, dv_* \right)^p \left( \int_{\mathbb{R}^3} |k(v_*, v)| \, dv_* \right)^p \leq C \int_{\mathbb{R}^3} |h(v_*)|^p |k(v_*, v)| \, dv_* , \quad (3.6)$$

where we have applied Corollary 3.2 in the above derivation. Therefore, applying Corollary 3.2 again, we obtain

$$\int_{\mathbb{R}^3} |K h(v)|^p \, dv \leq C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |h(v_*)|^p |k(v_*, v)| \, dv_* \, dv \leq C \int_{\mathbb{R}^3} (|k(v_*, v)| \, dv) |h(v_*)|^p \, dv_* \leq C \int_{\mathbb{R}^3} |h(v_*)|^p \, dv_* , \quad (3.7)$$

The proof for the cases where $p = 1, \infty$ is straightforward. We should omit it. \qed

We see from Proposition 3.3 $K$ is a bounded operator from $L^p_0(\mathbb{R}^3)$ to $L^p_0(\mathbb{R}^3)$. In particular, we shall only use the case $p = 2$.

Proposition 3.4 For any $v \in \mathbb{R}^3$ and $\epsilon \in (0, 2)$, there is a constant $C$ independent of $v$ such that

$$\int_{\mathbb{R}^3} \frac{1}{|v_*|^{2-\epsilon}} |k(v, v_*)| \, dv_* \leq C. \quad (3.8)$$

Proof Let $v \in \mathbb{R}^3$ and $\epsilon \in (0, 2)$ be given. From (3.3), we have

$$|k(v, v_*)| \leq C \frac{1}{|v - v_*|} e^{-\frac{1}{8} |v - v_*|^2}. \quad \Box$$
It therefore suffices to prove that
\[
\int_{\mathbb{R}^3} \frac{1}{|v_*|^{2-\epsilon}} e^{-\frac{1}{8}|v-v_*|^2} \, dv_* \leq C
\] (3.9)
for some $C$. If $|v_*| \leq |v - v_*|$, then
\[
\frac{1}{|v_*|^{2-\epsilon}} \leq \frac{1}{|v_*|^{3-\epsilon}}.
\]
If $|v_*| > |v - v_*|$, then
\[
\frac{1}{|v_*|^{2-\epsilon}} \leq \frac{1}{|v - v_*|^{3-\epsilon}}.
\]
In any case,
\[
\int_{\mathbb{R}^3} \frac{1}{|v_*|^{2-\epsilon}} e^{-\frac{1}{8}|v-v_*|^2} \, dv_* \\
\leq \int_{\mathbb{R}^3} \frac{1}{|v_*|^{3-\epsilon}} e^{-\frac{1}{8}|v-v_*|^2} \, dv_* + \int_{\mathbb{R}^3} \frac{1}{|v - v_*|^{3-\epsilon}} e^{-\frac{1}{8}|v-v_*|^2} \, dv_*.
\] (3.10)
Clearly,
\[
\int_{\mathbb{R}^3} \frac{1}{|v - v_*|^{3-\epsilon}} e^{-\frac{1}{8}|v-v_*|^2} \, dv_* = \int_{\mathbb{R}^3} \frac{1}{|w|^{3-\epsilon}} e^{-\frac{1}{8}|w|^2} \, dw \leq C.
\]
Notice that
\[
\int_{\mathbb{R}^3} \frac{1}{|v_*|^{3-\epsilon}} e^{-\frac{1}{8}|v-v_*|^2} \, dv_* \\
\leq \int_{|v_*| \leq 1} \frac{1}{|v_*|^{3-\epsilon}} e^{-\frac{1}{8}|v-v_*|^2} \, dv_* + \int_{|v_*| > 1} \frac{1}{|v_*|^{3-\epsilon}} e^{-\frac{1}{8}|v-v_*|^2} \, dv_* \\
\leq \int_{|v_*| \leq 1} \frac{1}{|v_*|^{3-\epsilon}} \, dv_* + \int_{\mathbb{R}^3} e^{-\frac{1}{8}|v-v_*|^2} \, dv_* \\
\leq C.
\] (3.11)
We deduce (3.9).
\[\square\]
In Sect. 6, we will need the following estimate. The case for hard sphere is proved in [5]. More generally, the proof therein can be extended to the cases for cutoff hard and cutoff Maxwellian potentials.

**Lemma 3.5** Suppose that a function $h$ satisfies the following estimate for some constant $a \in [0, \frac{1}{4})$,
\[
|h(v)| \leq e^{-a|v|^2}.
\] (3.12)

Then there exists a positive constant $C_a$ such that
\[
|K(h)(v)| \leq C_a e^{-a|v|^2},
\] (3.13)
where $C_a$ is a constant depending on $a$.\[\square\] Springer
4 Velocity Averaging for Linearized Boltzmann Equation in the Whole Space

Lemma 1.2, which was first investigated in [11], plays an important role in our analysis. For readers’ convenience, we recapitulate the proof of Lemma 1.2 in this section. The idea of Lemma 1.2 originated from the velocity averaging lemma in [21], which can be proved by the method of Fourier transform. Roughly speaking, the $L^2$ averaging lemma can be stated as that if $u \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ is a solution to the transport equation in $\mathbb{R}^n \times \mathbb{R}^n$ with a source term $f \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$,

$$u + v \cdot \nabla_x u = f,$$  

(4.1)

then, for any $\psi \in L^\infty(\mathbb{R}^n)$, a bounded and compactly supported function, the velocity average of $u$ satisfies

$$\int_{\mathbb{R}^n} u(\cdot, v)\psi(v) \, dv \in \tilde{H}^{1/2}(\mathbb{R}^n).$$  

(4.2)

By analogy with the above result, we consider the transport equation in $\mathbb{R}^3 \times \mathbb{R}^3$ with a source term $K(f) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$,

$$\nu(v)u + v \cdot \nabla_x u = K(f)$$  

(4.3)

(Notice that $f \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ implies $K(f) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ according to Proposition 3.3). Recall from (1.14)

$$(Sf)(x, v) := \int_0^\infty e^{-\nu(v)s} f(x - sv, v) \, ds.$$  

(4.4)

A direct calculation shows that $S$ is a bounded operator from $L^p(\mathbb{R}^3 \times \mathbb{R}^3)$ to $L^p(\mathbb{R}^3 \times \mathbb{R}^3)$ for $1 \leq p \leq \infty$ and the $L^2$ solution of (4.3) is expressed as

$$u(x, v) = (SK)(f)(x, v).$$  

(4.5)

If we regard the integral operator $K$ as a kind of velocity averaging, then it is natural to anticipate $K(u) = KS(K)(f)$ has certain regularity in the space variable in view of (4.2). And it turns out we indeed have Lemma 1.2. In the following proof, we adopt the idea of Fourier transform in [21] with a careful application of Cauchy–Schwarz inequality. Furthermore, we also take advantage of (3.2) for the function $\nu$ to gain some integrability (see (4.13)) since the support of $k(v, v^*)$ is not bounded. We also remark that Lemma 1.2 holds for any function $\nu$ satisfying (3.2) and any kernel $k$ satisfying (3.4) and (3.5).

**Proof of Lemma 1.2** We denote the Fourier transform in $x$ of a function $h$ by $\mathcal{F}h$, the corresponding variable by $\xi$. Since the operator $K$ does not involve the space variable, a direct calculation shows that, for $f \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$, we have

$$\mathcal{F}(Sf)(\xi, v) = \frac{\mathcal{F}f(\xi, v)}{v(v) + i(v \cdot \xi)},$$  

(4.6)

$$\mathcal{F}(Kf)(\xi, v) = K(\mathcal{F}f)(\xi, v).$$  

(4.7)

For the case where $f \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, one can approximate $f$ by a sequence of functions in $L^1 \cap L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ to obtain (4.6)–(4.7). To prove Lemma 1.2, it suffices to show the boundedness of integral

$$I := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\xi|^{2r+1} |\mathcal{F}KSf(\xi, v)|^2 \, d\xi \, dv$$  

(4.8)
for given $f \in L^2_v(\mathbb{R}^3; \tilde{H}_x^2(\mathbb{R}^3))$. Using identities (4.6), (4.7) and Cauchy–Schwarz inequality yields

$$I = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\xi|^{2s+1} \left| \int_{\mathbb{R}^3} \mathcal{F} S K f(\xi, v_\ast) k(v_\ast, v) dv_\ast \right|^2 d\xi dv$$

$$= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\xi|^{2s+1} \left( \int_{\mathbb{R}^3} \frac{\mathcal{F} K f(\xi, v_\ast)}{v(v_\ast) + i(v_\ast \cdot \xi)} k(v_\ast, v) dv_\ast \right)^2 d\xi dv$$

$$\leq \iint |\xi|^{2s+1} \left( \int |\mathcal{F} K f(\xi, v_\ast)|^2 (1 + |v_\ast|^{3-2\gamma} |k(v_\ast, v)| dv_\ast \right)$$

$$\times \left( \int \frac{|k(v_\ast, v)|}{(v(v_\ast)^2 + (v_\ast \cdot \xi)^2)(1 + |v_\ast|^{3-2\gamma})} dv_\ast \right) d\xi dv. \quad (4.9)$$

By Corollary 3.2, we have

$$|\mathcal{F} K f(\xi, v_\ast)|^2 = \left| \int_{\mathbb{R}^3} \mathcal{F} f(\xi, w) k(w, v_\ast) dw \right|^2$$

$$\leq \left( \int_{\mathbb{R}^3} |\mathcal{F} f(\xi, w)|^2 dw \right) \left( \int_{\mathbb{R}^3} |k(w, v_\ast)|^2 dw \right)$$

$$\leq C \frac{\int_{\mathbb{R}^3} |\mathcal{F} f(\xi, w)|^2 dw}{(1 + |v_\ast|^{3-2\gamma})}, \quad (4.10)$$

which leads to

$$\int |\mathcal{F} K f(\xi, v_\ast)|^2 (1 + |v_\ast|^{3-2\gamma}) |k(v_\ast, v)| dv_\ast \leq C \iint |\mathcal{F} f(\xi, w)|^2 |k(v_\ast, v)| dw dv_\ast$$

$$\leq C \int |\mathcal{F} f(\xi, w)|^2 dw, \quad (4.11)$$

where the second inequality follows from Corollary 3.2. Therefore, it follows that

$$I \leq C \iiint |\xi|^{2s+1} |\mathcal{F} f(\xi, w)|^2 \frac{|k(v_\ast, v)|}{(v(v_\ast)^2 + (v_\ast \cdot \xi)^2)(1 + |v_\ast|^{3-2\gamma})} dv_\ast d\xi dw$$

$$\leq \iiint |\xi|^{2s+1} |\mathcal{F} f(\xi, w)|^2 \frac{1}{(v(v_\ast)^2 + (v_\ast \cdot \xi)^2)(1 + |v_\ast|^{3-2\gamma})} dv_\ast d\xi dw, \quad (4.12)$$

where we have used Corollary 3.2 again in the last line. We observe that

$$\frac{1}{v(v_\ast)^2 + (v_\ast \cdot \xi)^2} = \frac{1}{(v(v_\ast)^2 + (v_\ast \cdot \xi)^2)^{2/3}} \cdot \frac{1}{(v(v_\ast)^2 + (v_\ast \cdot \xi)^2)^{1/3}} \leq \frac{1}{(v_0^2 + (v_\ast \cdot \xi)^2)^{2/3}} \cdot \frac{1}{v(v_\ast)^{3/2}} \leq C \frac{1}{(v_0^2 + (v_\ast \cdot \xi)^2)^{2/3}} \cdot \frac{1}{(1 + |v_\ast|^{5-2\gamma})}. \quad (4.13)$$

As a result, we have

$$I \leq C \iiint |\xi|^{2s+1} \frac{|\mathcal{F} f(\xi, w)|^2}{(v_0^2 + (v_\ast \cdot \xi)^2)^{2/3}(1 + |v_\ast|^{5-2\gamma})} dv_\ast d\xi dw. \quad (4.14)$$
Denote the component of $v_*$ parallel to $\xi$ and the component perpendicular to $\xi$ respectively by
\[
\begin{align*}
v_* \| \xi &= v_* \cdot \frac{\xi}{|\xi|} \\
v_* \bot \xi &= v_* - (v_* \| \xi) \frac{\xi}{|\xi|}.
\end{align*}
\]
(4.15)

Consequently, we deduce
\[
I \leq C \iint \int |\xi|^{2s+1} \frac{|\mathcal{F} f(\xi, w)|^2}{(v_0^2 + |\xi|^2 v_*^2)^{2/3}} (1 + |v_* \| \xi|)^{\frac{5}{3} - \gamma} d v_* \| \xi \| |\xi| d w d \xi
\leq C \iint \int |\xi|^{2s+1} \frac{|\mathcal{F} f(\xi, w)|^2}{(v_0^2 + |\xi|^2 v_*^2)^{2/3}} d v_* \| \xi \| d w d \xi
\leq C \| f \|^2_{L_2^2(\mathbb{R}^3, H^s(\mathbb{R}^3))},
\]
(4.16)

where the second inequality follows since $5 - \frac{7\gamma}{3} > 2$ for $0 \leq \gamma \leq 1$ and $v_* \bot \xi$ is two-dimensional plane. This completes the proof. \qed

## 5 Geometric Properties of Bounded Convex Domains

In this section, we introduce some auxiliary geometric results involving bounded convex domains. Throughout this section, $\Omega \subset \mathbb{R}^3$ denotes a $C^2$ bounded strictly convex domain. Our strategy is to properly compare convex domain $\Omega$ with spherical domains in $\mathbb{R}^3$ so that we can build up estimates for convex domains based on those for spherical domains.

Let $n(q)$ denote the unit outward normal of $\partial \Omega$ at $q \in \partial \Omega$, and $\hat{v} = \frac{v}{|v|}$ denote the unit vector with the direction $v \in \mathbb{R}^3$. We start with several geometric notations we shall frequently use.

**Definition 5.1** Let $\text{diam } \Omega$ be the diameter of bounded domain $\Omega$. Namely, we define
\[
\text{diam } \Omega := \sup_{(x_1, x_2) \in \Omega \times \Omega} |x_1 - x_2|.
\]
(5.1)

For an interior point $x \in \Omega$, let
\[
d_x := d(x, \partial \Omega)
= \inf_{q \in \partial \Omega} |x - q|
\]
(5.2)
be the distance from $x$ to $\partial \Omega$. Furthermore, the absolute value of the component of velocity $v$ passing through the surface $\partial \Omega$ at $q_-(x, v)$ is denoted by
\[
N_-(x, v) := |n(q_-(x, v)) \cdot \hat{v}|.
\]
(5.3)

In a similar fashion, we can define the corresponding forward concept by
\[
N_+(x, v) := |n(q_+(x, v)) \cdot \hat{v}|.
\]

The following two propositions from [8] concern estimates for backward trajectory.
Proposition 5.2 Let \( x \) be an interior point of \( \Omega \) and \( v \in \mathbb{R}^3 \). Then
\[
|x - q(x, v)| \geq \frac{d_x}{N(x, v)}.
\]

Proposition 5.3 Let \( x, y \) be interior points of \( \Omega \) and \( v \in \mathbb{R}^3 \). If
\[
|x - q(x, v)| \leq |y - q(y, v)|,
\]
then
\[
|x - q(x, v)| - |y - q(y, v)| \leq \frac{2|x - y|}{N(x, v)}.
\]

Definition 5.4 We say a bounded convex domain \( \Omega \) satisfies uniform interior sphere condition (resp. uniform sphere-enclosing condition) if there is a constant \( r_1 = r_1(\Omega) > 0 \) (resp. \( R_1 = R_1(\Omega) > 0 \)) such that for any boundary point \( q \in \partial \Omega \), there is a sphere \( S_i(q) = \partial B_{r_1}(y) \) (resp. \( S_o(q) = \partial B_{R_1}(y') \)) such that \( B_{r_1}(y) \subset \Omega \) (resp. \( \Omega \subset B_{R_1}(y') \)) with \( q \in S_i(q) \) (resp. \( q \in S_o(q) \)). (See Fig. 1.)

Proposition 5.5 Let \( D \subset \mathbb{R}^2 \) be a disk of radius \( r \) and \( A, B \) two points on \( \partial D \). Suppose there is a regular parametrized curve \( \alpha(s) \) connecting \( \alpha(s_0) = A \) to \( \alpha(s_1) = B \) contained in the circular segment bounded by chord \( \overline{AB} \) and minor arc \( \overline{AB} \). We further assume that \( \alpha(s) \) is convex, i.e. the curvature \( k(s) \) at \( \alpha(s) \) is positive everywhere, \( \alpha(s) \) is tangent to \( D \) at \( A \), and \( \alpha(s) \) leaves \( D \) at \( B \) as Fig. 2 shows. Then, there exists \( s_0 \in (s_0, s_1) \) such that \( k(s_0) \leq \frac{1}{r} \).

Proof We may assume \( \alpha(s) \) is parametrized by arc length. Let \( e_1 \) denote the vector \( \frac{\overrightarrow{AB}}{|A - B|} \). We choose \( e_2 \in S^1 \) such that \( \{e_1, e_2\} \) forms a positively oriented orthonormal basis. Denote the signed angle from vector \( e_1 \) to tangent vector \( q \) by \( \theta(s) \). Therefore, \(|A - B| = 2r|\sin \theta(s_0)|\). On the other hand, since \( \alpha'(s) = \cos \theta(s)e_1 + \sin \theta(s)e_2 \), we obtain
\[
2r|\sin \theta(s_0)| = |A - B| = \int_{s_0}^{s_1} \alpha'(s) \cdot e_1 \, ds = \int_{s_0}^{s_1} \cos \theta(s) \, ds.
\]
Suppose \( k(s) > \frac{1}{r} \) for any \( s \in (s_0, s_1) \). Then, for any \( s \in (s_0, s_1) \),

\[
\frac{d\theta(s)}{ds} = k(s) > \frac{1}{r},
\]

or

\[
\frac{ds(\theta)}{d\theta} < r.
\]

Hence, we have

\[
2r |\sin \theta(s_0)| = \int_{s_0}^{s_1} \cos \theta(s) \, ds
\]

\[
= \int_{\theta_0}^{\theta_1} \cos \theta \frac{ds(\theta)}{d\theta} \, d\theta
\]

\[
< r \int_{\theta_0}^{\theta_1} \cos \theta \, d\theta
\]

\[
(\sin \theta_1 - \sin \theta_0),
\]

where \( \theta_0 = \theta(s_0) \) and \( \theta_1 = \theta(s_1) \). Since \( \frac{\pi}{2} \geq |\theta_0| \geq \theta_1 > 0 \), we notice that

\[
r (\sin \theta_1 - \sin \theta_0) \leq 2r |\sin \theta_0|,
\]

which leads to a contradiction. \( \square \)

**Proposition 5.6** If \( \Omega \) satisfies the positive curvature condition in Definition 1.5, then it also satisfies both uniform interior sphere condition and uniform sphere-enclosing condition.

**Proof** Since \( \partial \Omega \) is \( C^2 \) and compact, there is a tubular neighborhood near \( \partial \Omega \). That is, there exists a number \( \epsilon > 0 \) such that whenever \( q_1, q_2 \in \partial \Omega \) the segments of the normal lines of length \( 2\epsilon \), centered at \( q_1 \) and \( q_2 \), are disjoint. This fact can be found in [17], for example. Let \( r_1 = \frac{\epsilon}{2} \). For a given point \( q \in \partial \Omega \), therefore we can consider the sphere \( B_{r_1}(y) \subset \Omega \) such that \( q \in \partial B_{r_1}(y) \). Clearly, \( S_1(q) = \partial B_{r_1}(y) \) satisfies the desired property.

Concerning uniform sphere-enclosing condition, we shall proceed the proof by contradiction. We denote the principal curvatures of \( \partial \Omega \) at \( q \) by \( k_1(q) \) and \( k_2(q) \) (\( k_1(q) \leq k_2(q) \)).
By positive curvature property of $\partial \Omega$, we can choose $k_0$ such that $0 < k_0 < \min_{q \in \partial \Omega} k_1(q)$. We claim that $R_1 = \frac{1}{k_0}$ is a desired radius for the uniform sphere-enclosing condition. For a given point $q \in \partial \Omega$, there is $y'$ such that $\overrightarrow{y'q}/n(q)$ and $|y' - q| = R_1$. Under suitable rotations and translations, $\partial \Omega$ can be locally expressed as the graph of a function $z = h(x, y)$ at $q = (0, 0, 0)$, where

$$h(x, y) = \frac{1}{2}(k_1(q)x^2 + k_2(q)y^2) + o(x^2 + y^2). \quad (5.11)$$

Since $B_{R_1}(y')$ has constant normal curvature $k_0$, $\overrightarrow{BR_1}(y')$ contains a neighborhood of $q$ in $\partial \Omega$ by (5.11). Suppose $\Omega \not\subset B_{R_1}(y')$, then there exists $q_1 \in \partial \Omega \cap \partial B_{R_1}(y')$ such that the plane $E$ containing $q$, $y'$, $q_1$ has the intersection curve with $\partial \Omega$, $\alpha(s)$, satisfies the condition of Proposition 5.5 with $A = q$, $B = q_1$, and $r = R_1$. Then Proposition 5.5 implies that there is a point $q_s = \alpha(s) \in \partial \Omega$ such that the curvature of the curve $\alpha(s)$ $k(s_s) \leq \frac{1}{R_1}$. Hence,

$$k_1(q_s) \leq k(q_s) \leq \frac{1}{R_1} = k_0, \quad (5.12)$$

which is a contradiction. Therefore, we define $S_o(q) = \partial B_{R_1}(y')$. This completes the proof.

**Remark 5.7** The above proof shows that $\frac{1}{\min_{q \in \partial \Omega} k_1(q)}$ is a uniform radius for sphere-enclosing condition for every small $\epsilon > 0$. Letting $\epsilon \to 0^+$, we obtain the smallest and optimal radius $R_1 = \frac{1}{\min_{q \in \partial \Omega} k_1(q)}$.

The following lemma in [8] is useful in our estimates.

**Lemma 5.8** Suppose $\Omega$ satisfies the positive curvature condition in Definition 1.5. Then there exists a constant $C = C(\Omega)$ such that, for any interior point $x \in \Omega$, we have

$$\int_{\partial \Omega} \frac{1}{|x - q|^2} d\Sigma(q) \leq C(|\log(d_x)| + 1), \quad (5.13)$$

where $\Sigma(q)$ is the surface element of $\partial \Omega$ at point $q \in \partial \Omega$.

The following proposition concerns an estimate for chords in a bounded convex domain.

**Proposition 5.9** For a given bounded convex domain $\Omega$ satisfying positive curvature condition in Definition 1.5, there exists a constant $C = C(\Omega)$ such that for any $x \in \tilde{\Omega}$ and $v \in \mathbb{R}^3$, we have

$$|q_-(x, v) - q_+(x, v)| \leq CN_-(x, v). \quad (5.14)$$

**Proof** For given $x \in \tilde{\Omega}$ and $v \in \mathbb{R}^3$, write $q_- = q_-(x, v)$, $q_+ = q_+(x, v)$, and $\theta_- = \arccos N_-(x, v)$ for simplicity. In view of Proposition 5.6, there exists sphere $S_o(q_-)$ as defined in Definition 5.4. Denote the other intersection of half-line $q_-q_+$ and $S_o(q_-)$ by $q_1$. Therefore, we have

$$|q_- - q_+| \leq |q_- - q_1| \leq 2R_1 \cos \theta_- = 2R_1 N_-(x, v). \quad (5.15)$$

\[\boxtimes\]
Remark 5.10 From the above proof, we have

\[ |q_- - q_+| \leq 2R_1 \cos \theta_- . \]  
\[ (5.16) \]

Replacing \( v \) with \( -v \) in Proposition 5.9 yields

\[ |q_- - q_+| \leq 2R_1 \cos \theta_+ , \]
\[ (5.17) \]

where \( \theta_+ = \arccos N_+(x, v) \).

Proposition 5.11 For a given circle \( C \) in \( \mathbb{R}^2 \) centered at \( O \) with radius \( r \) and two given points \( A, B \) on \( C \). Let \( N \) be the arc midpoint of \( \widehat{AB} \) (the minor arc) and \( M \) be the midpoint of \( \widehat{AB} \). Then for any \( Y \in \overline{AM} \) (resp. \( Y \in \overline{BM} \)), we have

\[ d(Y, C) \geq \frac{1}{\sqrt{2}} |Z - Y| , \]
\[ (5.18) \]

where \( Z \) is the point on \( \overline{AN} \) (resp. \( \overline{BN} \)) such that \( (Z - Y) \perp (A - B) \) (See Fig. 3).

Proof Without loss of generality, we may assume \( Y \in \overline{AM} \). Let \( \theta \) denote the angle \( \angle OAM \).

Suppose half-line \( \overrightarrow{OY} \) meets \( C, \overline{AN} \) at \( P, Q \), respectively. Thus, \( |P - Y| = d(Y, C) \).

We also notice that

\[ \angle YZQ = \angle ONA = \frac{\pi}{4} + \frac{\theta}{2} . \]
\[ (5.19) \]

Therefore, \( \sin \angle YZQ \geq \frac{1}{\sqrt{2}} \). By the law of sines, we have

\[ |P - Y| \geq |Q - Y| = \frac{\sin \angle YZQ}{\sin \angle YQZ} |Z - Y| \]
\[ \geq \frac{1}{\sqrt{2}} |Z - Y| . \]
\[ (5.20) \]

We are now in a position to prove the following lemma.
**Lemma 5.12** Suppose $\Omega \subset \mathbb{R}^3$ satisfies the positive curvature condition in Definition 1.5. Then, there exists a constant $C = C(\Omega)$ such that for any $y \in \Omega$ and $\hat{v} \in S^2$, we have
\[
\int_{0}^{1} |q_+(y, \hat{v}) - y| d_{y+r\hat{v}} \leq C, \quad \forall \epsilon \in \left[0, \frac{1}{2}\right), \tag{5.21}
\]
and
\[
\int_{0}^{1} |q_+(y, \hat{v}) - y| d_{y+r\hat{v}} \leq \frac{C}{\epsilon} d_{y}^{-\frac{1}{2}+\epsilon}, \quad \forall \epsilon \in \left(0, \frac{1}{2}\right). \tag{5.22}
\]

**Proof** The idea of proof is that we first show the lemma holds for balls. For general cases, we employ Proposition 5.6 to compare the convex domain $\Omega$ with balls. For given $\epsilon \in [0, \frac{1}{2})$, $y \in \Omega$ and $\hat{v} \in S^2$. We denote $q_- = q_-(y, \hat{v})$ and $q_+ = q_+(y, \hat{v})$ for simplicity. We claim
\[
\int_{0}^{1} |q_+-q_-| d_{q_-+r\hat{v}} \leq C. \tag{5.23}
\]

First, we shall prove that (5.23) holds for the case of balls in $\mathbb{R}^3$. Let $B \subset \mathbb{R}^3$ be an open ball with radius $\rho$ centered at $O$ and $A, B$ be two points on $\partial B$. For $\hat{v} = \frac{AB}{|AB|}$, the following integral is bounded by some constant $C = C(\rho)$,
\[
\int_{0}^{1} d(A + r\hat{v}, \partial B)^{-\frac{1}{2}+\epsilon} dr \leq C. \tag{5.24}
\]

Let $C$ be the intersection circle of $\partial B$ and the plane passing through $A, B$ and $O$. Denote the midpoint of $AB$ by $M$ and write $\theta = \angle OAB$. For $0 \leq r \leq |A - M|$, it follows by Proposition 5.11 that
\[
d(A + r\hat{v}, C) \geq \frac{1}{\sqrt{2}} |Z - Y|
= \frac{1}{\sqrt{2}} \tan \left(\frac{\pi}{4} - \frac{\theta}{2}\right) r
= \frac{1}{\sqrt{2}} \frac{1 - \sin \theta}{\cos \theta} r, \tag{5.25}
\]
where $Y = A + r\hat{v}$ and $Z$ is as defined in Proposition 5.11. Therefore, we obtain
\[
\int_{0}^{1} d(A + r\hat{v}, \partial B)^{-\frac{1}{2}+\epsilon} dr = \int_{0}^{1} d(A + r\hat{v}, C)^{-\frac{1}{2}+\epsilon} dr
\leq C \int_{0}^{\rho \cos \theta} \left(\frac{\cos \theta}{1 - \sin \theta}\right)^{\frac{1}{2}-\epsilon} r^{-\frac{1}{2}+\epsilon} dr
\leq C \rho^{\frac{1}{2}+\epsilon} \left(\frac{\cos \theta}{1 - \sin \theta}\right)^{\frac{1}{2}-\epsilon}
\leq C \max\{\rho, 1\}, \tag{5.26}
\]
where the last inequality follows from the fact $\cos \theta \leq \sqrt{2(1 - \sin \theta)}$. The situation $|A - M| \leq r \leq |A - B|$ can be treated similarly. Therefore, (5.24) follows.

For general cases, we make a comparison with sphere cases. According to Proposition 5.6, there are spheres $S_i(q_-$) and $S_i(q_+)$ with radii $r_1 = r_1(\Omega)$ as defined in Definition 5.4. Denote
the other intersection of $S_i(q_-)$ and $\overline{q-q_+}$ by $q_1$, the other intersection of $S_i(q_+)$ and $\overline{q-q_+}$ by $q_2$. For $0 \leq r \leq |q_1 - q_-|$, we notice that

$$d(q_- + r \hat{v}, \partial \Omega) \geq d(q_- + r \hat{v}, S_i(q_-)).$$

(5.27)

For $|q_2 - q_-| \leq r \leq |q_+ - q_-|$, similarly we have

$$d(q_- + r \hat{v}, \partial \Omega) \geq d(q_- + r \hat{v}, S_i(q_+)).$$

(5.28)

Therefore, in the case of $\overline{q-q_1} \cup \overline{q_2q_+} = \overline{q-q_+}$, by (5.24) there exists a constant $C = C(r_1)$ such that

$$\int_0^{|q_+ - q_-|} d_{q_- + r \hat{v}}^{-\frac{1}{2}+\epsilon} dr \leq \int_0^{|q_1 - q_-|} d_{q_- + r \hat{v}}^{-\frac{1}{2}+\epsilon} dr + \int_{|q_2 - q_-|}^{|q_+ - q_-|} d_{q_- + r \hat{v}}^{-\frac{1}{2}+\epsilon} dr \leq \int_0^{|q_1 - q_-|} d(q_- + r \hat{v}, S_i(q_-))^{-\frac{1}{2}+\epsilon} dr \leq C.$$  

(5.29)

If $\overline{q-q_1} \cup \overline{q_2q_+}$ does not cover $\overline{q-q_+}$, we consider the convex hull of $S_i(q_-) \cup S_i(q_+)$, denoted by $S$. Clearly, one can see that

$$d(q_- + r \hat{v}, \partial \Omega) \geq d(q_- + r \hat{v}, \partial S).$$

(5.30)

From solid geometry, one can show that:

For $\left|\frac{q_1+q_+}{2} - q_-\right| \leq r \leq \left|\frac{q_1+q_+}{2} - q_-\right|$, if $N_-(y, \hat{v}) \leq N_+(y, \hat{v})$ (resp. $N_-(y, \hat{v}) > N_+(y, \hat{v})$) or equivalently $\theta_- \geq \theta_+$ (resp. $\theta_- < \theta_+$), where $\theta_\pm = \arccos N_\pm(y, \hat{v})$, then we have the inequality

$$d(q_- + r \hat{v}, \partial S) \geq d \left( \frac{q_1 + q_+}{2}, S_i(q_-) \right) = r_1 (1 - \sin \theta_-)$$

(5.31)

(resp. $d(q_- + r \hat{v}, \partial S) \geq d \left( \frac{q_1 + q_+}{2}, S_i(q_+) \right) = r_1 (1 - \sin \theta_+)$.)

(5.32)

See Fig. 4 for two-dimensional cases.

Recalling that $|q_- - q_+| \leq 2 R_1 \cos \theta_\pm$ from Remark 5.10, thus we deduce that, if $N_-(y, \hat{v}) \leq N_+(y, \hat{v})$,

$$\int_0^{|q_+ - q_-|} d_{q_- + r \hat{v}}^{-\frac{1}{2}+\epsilon} dr \leq \int_0^{|q_1 - q_-|} d_{q_- + r \hat{v}}^{-\frac{1}{2}+\epsilon} dr \leq C(r_1) + 2 R_1 r_1^{-\frac{1}{2}+\epsilon} \cos \theta_- (1 - \sin \theta_-)^{-\frac{1}{2}+\epsilon} + C(r_1) \leq C(r_1, R_1).$$

(5.33)

The inequality holds for $N_-(y, \hat{v}) > N_+(y, \hat{v})$ similarly.

Let us now look at (5.22). We mimic the above proof for (5.23), since the steps proceed in the same way. For given $\epsilon \in (0, \frac{1}{2})$, $y \in \Omega$ and $\hat{v} \in S^2$, this time we claim

$$\int_0^{|q_+ - q_-|} d_{q_- + r \hat{v}}^{-1+\epsilon} dr \leq \frac{C}{\epsilon} d_{y}^{-\frac{1}{2}+\epsilon}$$

(5.34)

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for some constant $C = C(r_1, R_1)$. For the cases of open balls in $\mathbb{R}^3$, we adopt the above notations and deduce that

$$
\int_0^{|A-M|} d(A + r\hat{v}, \partial B)^{-1+\epsilon} dr = \int_0^{|A-M|} d(A + r\hat{v}, C)^{-1+\epsilon} dr \\
\leq C \int_0^\rho \cos \theta \left( \frac{\cos \theta}{1 - \sin \theta} \right)^{1-\epsilon} r^{-1+\epsilon} dr \\
\leq C \frac{\rho^\epsilon \cos \theta}{\epsilon (1 - \sin \theta)^{1-\epsilon}} \\
\leq C \frac{\max\{\rho, 1\}}{\epsilon (1 - \sin \theta)^{\frac{1}{2} - \epsilon}} \\
\leq \frac{C}{\epsilon} d(M, C)^{-\frac{1}{2}+\epsilon} \\
\left( \leq \frac{C}{\epsilon} d(Y, C)^{-\frac{1}{2}+\epsilon} \text{ for any } Y \in \overline{AB} \right), \quad (5.35)
$$

where we have used $\cos \theta \leq \sqrt{2(1 - \sin \theta)}$. For general cases, we denote $q_3 = \frac{q_+ + q_1}{2}$ and $q_4 = \frac{q_+ + q_1}{2}$ for convenience. By considering the convex hull $S$ of $S_i(q_-) \cup S_i(q_+)$ again, if $N_-(y, \hat{v}) \leq N_+(y, \hat{v})$, we deduce

$$
\int_0^{|q_1-q_-|} d^{-1+\epsilon}_{q_-+r\hat{v}} dr \leq \int_0^{|q_1-q_-|} d_{q_-+r\hat{v}} dr \\
+ \int_0^{|q_2-q_-|} d_{q_-+r\hat{v}} dr \\
\leq \frac{C}{\epsilon} d((q_3, S_i(q_-)))^{-\frac{1}{2}+\epsilon} \\
+ 2R_1 r_1^{-1+\epsilon} \cos \theta_- (1 - \sin \theta_-)^{-1+\epsilon} + \frac{C}{\epsilon} d(q_4, S_i(q_+))^{-\frac{1}{2}+\epsilon} \\
\leq \frac{C}{\epsilon} \left( d(q_3, S_i(q_-))^{-\frac{1}{2}+\epsilon} + d(q_4, S_i(q_+))^{-\frac{1}{2}+\epsilon} \right), \quad (5.36)
$$
where we have used \( \cos \theta_\perp \leq \sqrt{2(1 - \sin \theta_\perp)} \) and \( d(q_3, S_t(q_-)) = r_1(1 - \sin \theta_\perp) \) in the last inequality. To complete the proof for (5.34), we consider sphere \( S_\perp(q_-) \) defined in Definition 5.4. Denote the other intersection of half-line \( q_--q_5 \) and \( S_\perp(q_-) \) by \( q_6 \), the midpoint of the line segment \( q_--q_5 \) by \( q_6 \). Let \( q_7 \) be the intersection of \( \partial \Omega \) and a line segment that realizes \( d(y, S_\perp(q_-)) \). Then, we have

\[
d_y = d(y, \partial \Omega) \leq |q_7 - y| \leq d(y, S_\perp(q_-)) \leq d(q_6, S_\perp(q_-)) = R_1(1 - \sin \theta_\perp). \tag{5.37}
\]

Notice that the last inequality above is from the fact that \( q_6 \) has the maximum distance to \( S_\perp(q_-) \) over the points on \( q_--q_5 \). Observing that \( q_- \) and centers of \( S_\perp(q_-) \) and \( S_t(q_-) \) lie on the same line, we obtain the last equality of (5.37) from the geometry of \( S_0(q_-) \). Hence, it follows that

\[
d(q_3, S_t(q_-))^{-\frac{1}{2}+\epsilon} = r_1^{-\frac{1}{2}+\epsilon}(1 - \sin \theta_\perp)^{-\frac{1}{2}+\epsilon}
= \left( \frac{r_1}{R_1} \right)^{-\frac{1}{2}+\epsilon} d(q_6, S_\perp(q_-))^{-\frac{1}{2}+\epsilon}
\leq C d_y^{-\frac{1}{2}+\epsilon}. \tag{5.38}
\]

Similarly, we have

\[
d(q_4, S_t(q_+))^{-\frac{1}{2}+\epsilon} \leq C d_y^{-\frac{1}{2}+\epsilon}. \tag{5.39}
\]

We conclude that

\[
\int_0^{|q_+ - q_-|} d_{q_--r_\theta}^{-1+\epsilon} \, dr \leq C \epsilon d_y^{-\frac{1}{2}+\epsilon}. \tag{5.40}
\]

The inequality holds for \( N_- (y, \hat{v}) > N_+ (y, \hat{v}) \) similarly. This completes the proof. \( \square \)

**Remark 5.13** One can see that the behavior of integral \( \int_0^{|q_+ - q_-|} d_{q_--r_\theta}^{-s} \, dr \) depends heavily on boundedness of \( \Theta(\theta; s) = \frac{\cos \theta}{(1 - \sin \theta)^s} \) for \( \theta \in (0, \frac{\pi}{2}) \). When \( s \leq \frac{1}{2} \), there is a constant \( C \) independent of \( \theta \) such that \( \Theta(\theta; s) \leq C \). On the other hand, there is no such constant whenever \( s > \frac{1}{2} \). We also notice that the integral blows up as \( s \to 1- \).

**Corollary 5.14** Suppose \( \Omega \subset \mathbb{R}^3 \) satisfies the positive curvature condition in Definition 1.5. Then, there exists a constant \( C = C(\Omega) \) such that, for any small \( \epsilon > 0 \), we have

\[
\int_\Omega \frac{1}{d_x^{1-\epsilon}} \, dx \leq \frac{C}{\epsilon}. \tag{5.41}
\]

**Proof** Let \( y \) be an interior point of \( \Omega \) and \( \{u, v, w\} \) be an orthonormal basis for \( \mathbb{R}^3 \). We note that

\[
\int_\Omega \frac{1}{d_x^{1-\epsilon}} \, dx
= \int_{|q_+(y, u) - y|}^{q_+(y, u + r, v) - (y + r u)} \int_{|q_+(y, u + r, v) - (y + r u) + s v|}^{q_+(y, u + r, v + s v) - (y + r u + s v)} d_{q_+ r u + s v + t w}^{-1+\epsilon} \, dt \, ds \, dr.
\]

According to Lemma 5.12, it follows that

\[
\int_{|q_+(y, u) - y|}^{q_+(y, u + r, v) - (y + r u)} \int_{|q_+(y, u + r, v) - (y + r u) + s v|}^{q_+(y, u + r, v + s v) - (y + r u + s v)} d_{q_+ r u + s v + t w}^{-1+\epsilon} \, dt \, ds \, dr
\]

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To estimate

Consequently, we have

We partition the domain of integration

Definition 1.5 and

Hereafter, \( \Omega \) denotes a bounded convex domain satisfying positive curvature condition in Definition 1.5 and \( g \) denotes an incoming data satisfying Assumption 1.7.

**Lemma 6.1** Let \( Jg \) as defined by (1.26). Then \( Jg \in L^2_v(\mathbb{R}^3; H^1_t(\Omega)) \) for any small \( \epsilon > 0 \).

**Proof** To prove the lemma, for given \( \epsilon > 0 \) and an incoming data \( g \) satisfying Assumption 1.7, it suffices to show the boundedness of integral

\[
\int \int I_{\Omega} \frac{|Jg(x, v) - Jg(y, v)|^2}{|x-y|^{5-2\epsilon}} \, dx \, dy \, dv. \tag{6.1}
\]

We partition the domain of integration \( D := \mathbb{R}^3 \times \Omega \times \Omega \) into \( D_1, D_2 \) as below.

\[
D_1 := \{(x, y, v) \in D : |x - q_-(x, v)| \leq |y - q_-(y, v)|\}, \tag{6.2}
\]

\[
D_2 := \{(x, y, v) \in D : |x - q_-(x, v)| > |y - q_-(y, v)|\}. \tag{6.3}
\]

In view of symmetry of \( D_1, D_2 \), we only need to calculate the integral (6.1) over \( D_1 \). Notice that

\[
|Jg(x, v) - Jg(y, v)|^2 \leq 2e^{-2v(y)\tau_-(x, v)}|g(q_-(x, v), v) - g(q_-(y, v), v)|^2 + 2|g(q_-(y, v), v)|^2[e^{-v(y)\tau_-(x, v)} - e^{-v(y)\tau_-(y, v)}]^2. \tag{6.4}
\]

Consequently, we have

\[
\int_{D_1} \frac{|Jg(x, v) - Jg(y, v)|^2}{|x-y|^{5-2\epsilon}} \, dx \, dy \, dv \leq I_1 + I_2, \tag{6.5}
\]

where

\[
I_1 := \int_{D_1} \frac{2e^{-2v(y)\tau_-(x, v)}|g(q_-(x, v), v) - g(q_-(y, v), v)|^2}{|x-y|^{5-2\epsilon}} \, dx \, dy \, dv, \tag{6.6}
\]

\[
I_2 := \int_{D_1} \frac{2|g(q_-(y, v), v)|^2[e^{-v(y)\tau_-(x, v)} - e^{-v(y)\tau_-(y, v)}]^2}{|x-y|^{5-2\epsilon}} \, dx \, dy \, dv. \tag{6.7}
\]

To estimate \( I_1 \), by Hölder continuity (1.23) and the fact from condition (1.22) that

\[
|g(q_-(x, v), v) - g(q_-(y, v), v)| \leq Ce^{-a|v|^2}, \tag{6.8}
\]
we obtain
\[
e^{-2v(x,y)}|g(q_-(x,v),v) - g(q_-(y,v),v)|^2
= e^{-2v(x,y)}|g(q_-(x,v),v) - g(q_-(y,v),v)|^{2-\epsilon}
\times |g(q_-(x,v),v) - g(q_-(y,v),v)|^{\epsilon}
\leq C e^{-2v(x,y)}|q_-(x,v) - q_-(y,v)|^{2-\epsilon} e^{-\epsilon|v|^2}.
\] (6.9)

According to Proposition 5.3, we have
\[
|q_-(x,v) - q_-(y,v)| \leq \frac{1}{N_\nu(x,v)}|x - y|,
\] (6.10)

whenever \(|x - q_-(x,v)| \leq |y - q_-(y,v)|\). Proposition 5.2 implies that
\[
e^{-2v(x,y)} \leq \frac{1}{v_\nu(x,v)^{1-\epsilon}}
\leq \frac{1}{v_\nu(x,v)^{1-\epsilon}} \cdot \left(\frac{N_\nu(x,v)|x - q_-(x,v)|}{d_x}\right)^{1-\epsilon}
\leq \frac{N_\nu(x,v)^{-\epsilon}|v|^{1-\epsilon}}{v_\nu(x,v)^{1-\epsilon} d_x^{1-\epsilon}}.
\]

Combining (6.8),(6.9),(6.10), and (6.11), we have
\[
I_1 \leq C \int_{D_1} \frac{|v|^{1-\epsilon} e^{-\epsilon|v|^2}}{v_\nu(x,v)^{1-\epsilon} N_\nu(x,v)^{1-\epsilon} d_x^{1-\epsilon}}\, dx dy dv
\leq C \int_{\Omega} \int_{\mathbb{R}_3} \frac{|v|^{1-\epsilon} e^{-\epsilon|v|^2}}{v_\nu(x,v)^{1-\epsilon} N_\nu(x,v)^{1-\epsilon} d_x^{1-\epsilon}}\, dy dv dx
\leq C \int_{\Omega} \int_{\mathbb{R}_3} \frac{|v|^{1-\epsilon} e^{-\epsilon|v|^2}}{d_x^{1-\epsilon} N_\nu(x,v)}\, dv dx.
\]

(6.12)

For fixed \(x \in \Omega\), we introduce a change of variable \(v = (x-z)l\) with \(z \in \partial \Omega\) and \(0 \leq l < \infty\). For any local chart of \(\partial \Omega\), say \(z = \phi(\alpha, \beta)\), we note that
\[
v(\alpha, \beta, l) = (x - \phi(\alpha, \beta))l
\]

and the Jacobian for \(v\) is given by
\[
|\det J_v(\alpha, \beta, l)| = |\partial v \cdot (\partial_\alpha v \times \partial_\beta v)|
= l^2 \left| (x - \phi(\alpha, \beta)) \cdot (\phi_\alpha \times \phi_\beta) \right|
= l^2 \left| (x - \phi(\alpha, \beta)) \cdot n(\phi(\alpha, \beta)) \right| |\phi_\alpha \times \phi_\beta|.
\]

(6.13)

Covering \(\partial \Omega\) by finitely many such coordinate charts, we obtain the formula for change of variables
\[
\int_{\mathbb{R}_3} h(v)\, dv = \int_{\partial \Omega} \int_0^\infty h((x-z)l) l^2 |(x-z) \cdot n(z)|\, dl d\Sigma(z).
\]

(6.14)
for any \( h \in L^1(\mathbb{R}^3) \). Therefore, by the change of variables, we obtain

\[
\int \int_{\Omega \times \mathbb{R}^3} \frac{|v|^{1-\epsilon} e^{-\epsilon a|v|^2}}{d_x^{1-\epsilon} N_-(x, v)} d\nu d\Sigma \leq \int \int_{\Omega \times \mathbb{R}^3} \frac{e^{-\epsilon a|x-z|^2 |x-z|^{1-\epsilon}}}{d_x^{1-\epsilon} |x-z|^2} |(x-z) \cdot n(z)| d\nu d\Sigma d \tau \quad \text{(6.15)}
\]

Letting \( s = |x-z|/l \) yields

\[
I_1 \leq C \int \int_{\Omega \times \mathbb{R}^3} \frac{e^{-\epsilon a s^2 |s|^{3-\epsilon}}}{d_x^{1-\epsilon} |s|^2} d\nu d\Sigma d \tau 
\]

\[
= C \int \int_{\Omega \times \mathbb{R}^3} \frac{1}{d_x^{1-\epsilon} |x-z|^2} d\nu d\Sigma d \tau 
\]

\[
\leq C \int \int_{\Omega \times \mathbb{R}^3} |\log(d_x)| + 1 \quad \text{d} \nu d\Sigma d \tau 
\]

\[
\leq C \epsilon. \quad \text{(6.16)}
\]

The third inequality above follows from Lemma 5.8 and the last inequality follows from Corollary 5.14 and the fact

\[
\frac{|\log(d_x)|}{d_x^{1-\epsilon}} \leq \frac{1}{d_x^{1-\epsilon/2}} \quad \text{(6.17)}
\]

for small \( d_x > 0 \).

Concerning \( I_2 \), the condition (1.22) implies that

\[
|g(q_-(y, v), (x, v))|^2 |e^{-\nu(v)\tau_-(y, v)} - e^{-\nu(v)\tau_-(x, v)}|^2 
\]

\[
\leq C e^{-2\nu|v|^2 |e^{-\nu(v)\tau_-(y, v)} - e^{-\nu(v)\tau_-(x, v)}|^2} \quad \text{(6.18)}
\]

By the mean value theorem and Proposition 5.3, we obtain

\[
|e^{-\nu(v)\tau_-(x, v)} - e^{-\nu(v)\tau_-(y, v)}| \leq \nu(v) e^{-\nu(v)\tau_-(x, v)} |\tau_-(x, v) - \tau_-(y, v)| 
\]

\[
\leq \frac{2\nu(v) e^{-\nu(v)\tau_-(x, v)} |x-y|}{N_-(x, v)|v|} \quad \text{(6.19)}
\]

Proposition 5.2 implies that

\[
e^{-(2-\epsilon)v(v)\tau_-(x, v)} \leq \frac{1}{\nu(v)^{1-\epsilon} \tau_-(x, v)^{1-\epsilon}} 
\]

\[
\leq \frac{N_-(x, v)^{1-\epsilon} |v|^{1-\epsilon}}{\nu(v)^{1-\epsilon} d_x^{1-\epsilon}} \quad \text{(6.20)}
\]

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Taking (6.18), (6.19), and (6.20) into consideration, we obtain
\[ I_2 \leq C \int_{D_1} e^{-2a|v|^2} \nu(v) \, dx \, dy \, dv \tag{6.21} \]
Comparing (6.21) with (6.12), one can repeat the steps in (6.12), (6.15), and (6.16) to obtain the following boundedness,
\[ I_2 \leq \frac{C}{\epsilon}. \]
This completes the proof. \qed

Since \( K : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \) is a bounded operator regarding velocity variable, we can see that \( K \) would preserve regularity in space variable.

**Proposition 6.2** The operator \( K : L^2_v(\mathbb{R}^3; H^{1-\epsilon}_x(\Omega)) \to L^2_v(\mathbb{R}^3; H^{1-\epsilon}_x(\Omega)) \) is bounded for any small \( \epsilon > 0 \).

Next, we deal with regularity preservation of \( S_{\Omega} \). Recall
\[ S_{\Omega} h(x, v) = \int_{0}^{\tau_{-(x,v)}} e^{-\nu(v)s} h(x - sv, v) \, ds. \tag{6.22} \]

**Lemma 6.3** Suppose \( h \in L^2_v(\mathbb{R}^3; H^{1-\epsilon}_x(\Omega)) \) and there exist positive constants \( a \) and \( C \) such that \( |h(x, v)| \leq Ce^{-a|v|^2} \). Then \( S_{\Omega} h \in L^2_v(\mathbb{R}^3; H^{1-\epsilon}_x(\Omega)) \) for any small \( \epsilon > 0 \).

**Proof** To prove the lemma, for given \( \epsilon > 0 \) and \( h \in L^2_v(\mathbb{R}^3; H^{1-\epsilon}_x(\Omega)) \) with \( |h(x, v)| \leq Ce^{-a|v|^2} \), it suffices to show the boundedness of integral
\[ \int_{D_1} \frac{|S_{\Omega} h(x, v) - S_{\Omega} h(y, v)|^2}{|x - y|^{5-2\epsilon}} \, dx \, dy \, dv, \tag{6.23} \]
where \( D_1 \) as defined by (6.2). In domain \( D_1 \), we have
\[ |S_{\Omega} h(x, v) - S_{\Omega} h(y, v)|^2 \leq 2 \left| \int_{0}^{\tau_{-(x,v)}} e^{-\nu(v)s} \, (h(x - sv, v) - h(y - sv, v)) \, ds \right|^2 + 2 \left| \int_{\tau_{-(y,v)}}^{\tau_{-(x,v)}} e^{-\nu(v)s} \, h(y - sv, v) \, ds \right|^2. \tag{6.24} \]
Therefore, we have
\[ \int_{D_1} \frac{|S_{\Omega} h(x, v) - S_{\Omega} h(y, v)|^2}{|x - y|^{5-2\epsilon}} \, dx \, dy \, dv \leq I_1 + I_2, \tag{6.25} \]
where
\[ I_1 := \int_{D_1} \frac{2 \left| \int_{0}^{\tau_{-(x,v)}} e^{-\nu(v)s} \, (h(x - sv, v) - h(y - sv, v)) \, ds \right|^2}{|x - y|^{5-2\epsilon}} \, dx \, dy \, dv, \]
\[ I_2 := \int_{D_1} \frac{2 \left| \int_{\tau_{-(y,v)}}^{\tau_{-(x,v)}} e^{-\nu(v)s} \, h(y - sv, v) \, ds \right|^2}{|x - y|^{5-2\epsilon}} \, dx \, dy \, dv. \tag{6.26} \]
Concerning \( I_1 \), by Cauchy–Schwarz inequality, we obtain
\[
\left| \int_0^{\tau_-(x,v)} e^{-v(s)} (h(x - sv, v) - h(y - sv, v)) \, ds \right|^2 \leq \left( \int_0^{\tau_-(x,v)} e^{-v(s)} \, ds \right) \left( \int_0^{\tau_-(x,v)} e^{-v(s)} (h(x - sv, v) - h(y - sv, v))^2 \, ds \right)
\]
\[
\leq \frac{1}{v_0} \int_0^{\tau_-(x,v)} e^{-v(s)} \, ds \leq \frac{1}{v_0} \int_0^{\tau_-(x,v)} e^{-v(s)} (h(x - sv, v) - h(y - sv, v))^2 \, ds
\]
\[
\leq \frac{1}{v_0} \int_0^{\infty} e^{-v(s)} (\tilde{h}(x - sv, v) - \tilde{h}(y - sv, v))^2 \, ds, \tag{6.27}
\]
where \( \tilde{h} \in L_v^2(\mathbb{R}^3; H_s^{1-\epsilon}(\mathbb{R}^3)) \) is an extension of \( h \in L_v^2(\mathbb{R}^3; H_s^{1-\epsilon}(\Omega)) \) as defined in [15, Theorem 5.4].

**Remark 6.4** By Theorem 5.4 from [15], we know \( H^s(\Omega) \) can be continuously embedded in \( H^3(\mathbb{R}^3) \). That is, there exists a constant \( C = C(s, \Omega) \) such that, for given \( u \in H^s(\Omega) \), there is an extension \( \overline{u} \in H^3(\mathbb{R}^3) \) of \( u \), i.e., \( \overline{u}|_\Omega = u \), satisfying
\[
\| \overline{u} \|_{H^3(\mathbb{R}^3)} \leq C \| u \|_{H^s(\Omega)}.
\]
For a.e. fixed \( v \in \mathbb{R}^3 \), we then define \( \tilde{h}(x, v) = \overline{h}(\cdot, v)(x) \in L_v^2(\mathbb{R}^3; H_{s}^{1-\epsilon}(\mathbb{R}^3)) \).

Consequently, we deduce that
\[
I_1 \leq C \int_0^{\infty} \frac{e^{-v(s)} (\tilde{h}(x - sv, v) - \tilde{h}(y - sv, v))^2}{|x - y|^{5-2\epsilon}} \, ds \, dx \, dy \, dv
\]
\[
\leq C \int_0^{\infty} \frac{e^{-v(s)'} (\tilde{h}(x', v') - \tilde{h}(y', v'))^2}{|x' - y'|^{5-2\epsilon}} \, ds' \, dx' \, dy' \, dv'
\]
\[
\leq C \| \tilde{h} \|_{L_v^2(\mathbb{R}^3; H_{s}^{1-\epsilon}(\mathbb{R}^3))}^2 \leq C \| \tilde{h} \|_{L_v^2(\mathbb{R}^3; H_{s}^{1-\epsilon}(\Omega))}^2, \tag{6.28}
\]
where we have used the change of variables
\[
\begin{cases}
  v' = v, \\
  y' = y - sv, \\
  x' = x - sv, \\
  s' = s.
\end{cases}
\tag{6.29}
\]

Regarding \( I_2 \), we notice that
\[
\left| \int_0^{\tau_-(y,v)} e^{-v(s)} h(y - sv, v) \, ds \right|^2 \leq C \left| \int_0^{\tau_-(y,v)} e^{-v(s)} e^{-a|v|^2} \, ds \right|^2 = Ce^{-2a|v|^2} \frac{1}{v(y)^2} \left| e^{-v(s)} \tau_-(y,v) - e^{-v(s)} \tau_-(y,v) \right|^2
\]
\begin{align*}
\leq C e^{-2a|v|^2} \frac{1}{\nu(v)^2} |e^{-\nu(v)\tau_- (x, v)} - e^{-\nu(v)\tau_- (y, v)}|^{2-\epsilon} \\
\leq C e^{-2a|v|^2} \nu(v) e^{-\nu(v)\tau_- (x, v)} |\tau_- (x, v) - \tau_- (y, v)|^{2-\epsilon}.
\end{align*}

(6.30)

Proposition 5.2 implies that
\[ e^{-\nu(v)(2-\epsilon)\tau_- (x, v)} \leq C \frac{1}{\nu(v)^{1-\epsilon} |x-y|^{2-\epsilon}}. \]

(6.31)

By Proposition 5.3, we have
\[ |\tau_- (x, v) - \tau_- (y, v)|^{2-\epsilon} \leq C \frac{|x-y|^{2-\epsilon}}{\nu(v)^{1-\epsilon} |x-y|^{1-\epsilon}}. \]

(6.32)

Combining (6.30)–(6.32), we deduce
\[ I_2 \leq C \int_{D_1} e^{-2a|v|^2} \frac{N_- (x, v)^{1-\epsilon} |v|^{1-\epsilon}}{v(\nu(v))^{1-\epsilon}} dxdydv. \]

(6.33)

To show the boundedness of above integral on the right, comparing (6.33) with (6.12), one can repeat the steps in (6.12), (6.15), and (6.16) to conclude
\[ S_{\Omega, h} \in L^2_v(B; H^1_{\nu} (\Omega)). \]

Corollary 6.5 Let \( g_i \) be as defined by (1.29). Then, \( g_i \in L^2_v(B; H^1_{\nu} (\Omega)) \) for each \( i \geq 0 \).

**Proof** We prove \( g_i \in L^2_v(B; H^1_{\nu} (\Omega)) \) by induction on \( i \geq 0 \).

**Step 1** For \( i = 0 \), since \( g_0 = Jg \), \( g_0 \in L^2_v(B; H^1_{\nu} (\Omega)) \) follows by Lemma 6.1. For \( i = 1 \), we notice that
\[ |Jg(x, v)| \leq |g(q_-(x, v), v)| \leq Ce^{-a|v|^2}. \]

(6.34)

In view of Lemma 3.5, we have
\[ |KJg(x, v)| \leq Ce^{-a|v|^2}, \]

(6.35)

where we may assume \( a < \frac{1}{4} \). With this in mind, Lemma 6.3 guarantees that \( g_1 \in L^2_v(B; H^1_{\nu} (\Omega)) \). Moreover, by noticing that
\[ S_{\Omega, h}(e^{-a|v|^2})(x, v) = \int_0^{\tau_- (x, v)} e^{-\nu(v)s} e^{-a|v|^2} ds \leq \frac{1}{v_0} e^{-a|v|^2}, \]

(6.36)

we see that
\[ |g_1(x, v)| \leq Ce^{-a|v|^2}. \]

(6.37)

**Step 2** Suppose for some \( i \geq 2 \), we have
\[ g_{i-1} = (S_{\Omega, h})^{i-1} Jg \in L^2_v(B; H^1_{\nu} (\Omega)) \]

(6.38)

and
\[ |g_{i-1}(x, v)| \leq Ce^{-a|v|^2}. \]

(6.39)
Then Lemma 3.5 implies
\[ |Kg_{i-1}(x, v)| \leq Ce^{-a|v|^2}. \] (6.40)
Applying Lemma 6.3 again yields
\[ g_i \in L^2_v(\mathbb{R}^3; H_x^{1-\varepsilon}(\Omega)). \] (6.41)
Finally, we also notice that (6.36) implies
\[ |g_i(x, v)| \leq Ce^{-a|v|^2}. \] (6.42)
We conclude that \( g_i \in L^2_v(\mathbb{R}^3; H_x^{1-\varepsilon}(\Omega)) \) for each \( i \geq 0 \) by induction.

\section*{7 Regularity via Velocity Averaging}

This section is devoted to the regularity due to velocity averaging. In contrast to Sect. 4, we address the velocity averaging effect in bounded convex domains instead of the whole space. As mentioned in the introduction, the method of Fourier transform does not apply to bounded domains. To remedy this crux, we adopt Slobodeckij semi-norm as an alternative concept of Sobolev function class, see Definition 1.4. Hence, the difficulty shifts to estimates of singular integrals. To carry out this strategy, we introduce changes of coordinates, see Lemma 7.1 and Lemma 7.2, to obtain the boundedness of the aforementioned singular integrals.

We begin with the proof of Corollary 1.11. Recall from (1.31) that the notation \( \tilde{h} \) denotes the zero extension of \( h \) from \( \Omega \times \mathbb{R}^3 \) to \( \mathbb{R}^3 \times \mathbb{R}^3 \) and \( Z : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3 \) is the zero extension operator from \( \Omega \times \mathbb{R}^3 \) to \( \mathbb{R}^3 \times \mathbb{R}^3 \).

**Proof of Corollary 1.11** For given \( f \in L^2(\Omega \times \mathbb{R}^3) \), according to Lemma 1.2, we have \( KS\tilde{f} \in L^2_v(\mathbb{R}^3; H_x^{1/2}(\Omega)). \) Noticing
\[
(KSK\tilde{f}) \bigg|_{\Omega \times \mathbb{R}^3} = KS\tilde{f} = KS\tilde{f},
\] (7.1)
we see that
\[
\|KS\tilde{f}\|_{L^2_v(\mathbb{R}^3; H_x^{1/2}(\Omega))} \leq \|KS\tilde{f}\|_{L^2_v(\mathbb{R}^3; H_x^{1/2}(\mathbb{R}^3))} \leq C \|KS\tilde{f}\|_{L^2_v(\mathbb{R}^3; H_x^{1/2}(\mathbb{R}^3))} \leq C \|f\|_{L^2(\Omega \times \mathbb{R}^3)} = C \|f\|_{L^2(\Omega \times \mathbb{R}^3)}. \] (7.2)

**Lemma 7.1** We have the following formula for change of variables for any non-negative measurable function \( h = h(v, y, r) \).
\[
\int_{\mathbb{R}^3} \int_{\Omega} \int_{0}^{[y-q-(y,v)]} h(v, y, r) dr dy dv = \int_{\mathbb{R}^3} \int_{\Omega} \int_{0}^{[y'-q+(y',v')]} h(v', y' + r' \tilde{v}', r') dr' dy' dv'. \] (7.3)
Proof Let $A$ and $B$ denote the domain of integration on the left hand side and right hand side of (7.3), respectively. We consider the change of variables

$$\begin{cases}
v' = v, \\
y' = y - r\hat{v}, \\
r' = r.
\end{cases} \quad (7.4)$$

Let $X : A \rightarrow B$ be defined by $X(v, y, r) = (v, y - r\hat{v}, r)$ and $Y : B \rightarrow A$ be defined by $Y(v', y', r') = (v', y' + r'\hat{v'}, r')$.

**Step 1** $X : A \rightarrow B$ is well-defined.

To verify well-definedness of $X$, for given $(v, y, r) \in A$, $y - r\hat{v} \in \Omega$ clearly. We also notice that

$$0 < r < r + |y - q(y, v)| = |(y - r\hat{v}) - q(y, v)|. \quad (7.5)$$

Therefore, $X(v, y, r) \in B$ and $X$ is well-defined.

**Step 2** $Y : B \rightarrow A$ is well-defined.

To verify well-definedness of $Y$, for given $(v', y', r') \in B$, $y' + r'\hat{v'} \in \Omega$ clearly. We note that

$$0 < r' < r' + |y' - q(y', v')| = |(y' + r'\hat{v'}) - q(y' + r'\hat{v'}, v')|. \quad (7.6)$$

Therefore, $Y(v', y', r') \in A$ and $Y$ is well-defined.

Since $X \circ Y = Y \circ X = \text{id}$ and the Jacobian for the change of variables is clearly constant 1, we have (7.3).

**Lemma 7.2** We have the following formula for change of variables for any non-negative measurable function $h = h(v, y, x, r)$.

$$\int_{D_1} \int_{|x-y|} |y-q(y, v)| h(v, y, x, r) dr dx dy dv = \int_{\mathbb{R}^3} \int_{\Omega_{v', y'}} \int_{|x-y|} \min\{|x'-q(x', v')|, |y'-q(y', v')|\} h(v', y' + r'\hat{v'}, x' + r'\hat{v'}, r') dr dx dy dv', \quad (7.7)$$

where $D_1$ is as defined by (6.2) and $\Omega_{v', y'}$ is defined by (see Fig. 5)

$$\Omega_{v', y'} = \{x' \notin \Omega : \text{there exist } t^i(x', v'), i = 1, 2, \text{ such that } 0 < t^1(x', v') < t^2(x', v'), \text{ q}^i(x', v') := x' + t^i(x', v')\hat{v'} \in \partial \Omega, \quad t^i(x', v') < |y' - q(y', v')|. \quad (7.8)$$

**Remark 7.3** An alternative way to characterize $\Omega_{v', y'}$ is as follows. We first define

$$\Omega'_{v', y'} = \{z - t\hat{v'} : z \in \Omega, 0 < t < |y' - q(y', v')|\}. \quad (7.9)$$

Then, $\Omega_{v', y'}$ can be expressed as

$$\Omega_{v', y'} = \Omega'_{v', y'} \setminus \tilde{\Omega}. \quad (7.10)$$
Proof We denote the domain of integration on the left hand side and right hand side by \( \tilde{A} \) and \( \tilde{B} \), respectively. This time we consider

\[
\begin{align*}
v' &= v, \\
y' &= y - r \hat{v}, \\
x' &= x - r \hat{v}, \\
r' &= r.
\end{align*}
\]

(7.11)

Let \( \tilde{X} : \tilde{A} \rightarrow \tilde{B} \) be defined by \( \tilde{X}(v, y, x, r) = (v, y - r \hat{v}, x - r \hat{v}, r) \) and \( \tilde{Y} : \tilde{B} \rightarrow \tilde{A} \) be defined by \( \tilde{Y}(v', y', x', r') = (v', y' + r' \hat{v}', x' + r' \hat{v}', r') \).

Step 1 \( \tilde{X} : \tilde{A} \rightarrow \tilde{B} \) is well-defined.

For given \( (v, y, x, r) \in \tilde{A} \), clearly we have \( y - r \hat{v} \in \Omega \) and \( x - r \hat{v} \notin \Omega \) and there are two positive numbers \( t^1(x - r \hat{v}, v) < t^2(x - r \hat{v}, v) \) such that

\[
q^i(x - r \hat{v}, v) = x - r \hat{v} + t^i(x - r \hat{v}, v) \hat{v} \in \partial\Omega.
\]

Moreover, comparing the distances results in

\[
t^1(x - r \hat{v}, v) = |(x - r \hat{v}) - q^1(x - r \hat{v}, v)|
< |(x - r \hat{v}) - x|
= r
= |(y - r \hat{v}) - y|
< |(y - r \hat{v}) - q_+(y - r \hat{v}, v)|.
\]

(7.12)

On the other hand, we notice

\[
r = |(x - r \hat{v}) - x|
< |(x - r \hat{v}) - q^2(x - r \hat{v}, v)|.
\]

(7.13)

We conclude that \( \tilde{X}(v, y, x, r) \in \tilde{B} \) and therefore \( \tilde{X} \) is well-defined.

Step 2 \( \tilde{Y} : \tilde{B} \rightarrow \tilde{A} \) is well-defined.

For given \( (v', y', x', r') \in \tilde{B} \), since \( r' \leq |y' - q_+(y', v')| \), we have \( y' + r' \hat{v}' \in \Omega \). Likewise, since

\[
|x' - q^1(x', v')| < r < |x' - q^2(x', v')|,
\]

\[ Springer \]
we have \( x' + r' \hat{v} \in \Omega \). Comparing the distances yields
\[
|(x' + r' \hat{v}) - q_- (x' + r' \hat{v}, v')| < |(x' + r' \hat{v}) - x'|
\]
\[
= r'
\]
\[
= |(y' + r' \hat{v}) - y'|
\]
\[
< |(y' + r' \hat{v}) - q_- (y' + r' \hat{v}, v')|.
\]
\[(7.14)\]

Therefore, it follows that \( \tilde{Y} (v', y', x', r') \in A \) and therefore \( \tilde{Y} \) is well-defined.
Since \( \tilde{X} \circ \tilde{Y} = \tilde{Y} \circ \tilde{X} = \text{id} \) and the Jacobian for the change of variables is clearly constant 1, we conclude (7.7). This completes the proof. \( \square \)

**Proposition 7.4** Let \( h \) be a function belonging to \( L^2 (\Omega \times \mathbb{R}^3) \). Then, there exists a constant \( C \) such that
\[
|S_\Omega K h(y, v)|^2 \leq C \frac{1}{|v|} \int_{\mathbb{R}^3} |y - q_- (y, v)| |k(v, v_*)| |h(y - r \hat{v}, v_*)|^2 \, dr \, dv_*.
\]
\[(7.15)\]
\[
|K S_\Omega h(y, v)|^2 \leq C \int_{\mathbb{R}^3} \int_0^{\tau_- (y, v)} \frac{1}{|v_*|} |k(v, v_*)| |h(y - r \hat{v}_*, v_*)|^2 \, dr \, dv_*.
\]
\[(7.16)\]

**Proof** By Cauchy–Schwarz inequality, we see that
\[
|S_\Omega K h(y, v)|^2 = \left( \int_0^{\tau_- (y, v)} e^{-v_*(y, v)} K h(y - sv, v) \, ds \right)^2
\]
\[
\leq \left( \int_0^{\tau_- (y, v)} e^{-2v_*(y, v)} \, ds \right) \left( \int_0^{\tau_- (y, v)} |K h(y - sv, v)|^2 \, ds \right)
\]
\[
\leq C \int_0^{\tau_- (y, v)} |K h(y - sv, v)|^2 \, ds
\]
\[
= C \int_0^{\tau_- (y, v)} \frac{1}{|v|} |K h(y - r \hat{v}, v)|^2 \, dr,
\]
\[(7.17)\]

where \( r = s|v| \). Furthermore, Corollary 3.2 implies that
\[
|K h(y - r \hat{v}, v)|^2 \leq \left( \int_{\mathbb{R}^3} |k(v, v_*)| \, dv_* \right) \left( \int_{\mathbb{R}^3} |k(v, v_*)| |h(y - r \hat{v}, v_*)|^2 \, dv_* \right)
\]
\[
\leq C \int_{\mathbb{R}^3} |k(v, v_*)| |h(y - r \hat{v}, v_*)|^2 \, dv_*.
\]
\[(7.18)\]

Combining (7.17) and (7.18) yields (7.15).
To prove (7.16), we apply Cauchy–Schwarz inequality again to deduce that
\[
|K S_\Omega h(y, v)|^2 = \left( \int_{\mathbb{R}^3} \int_0^{\tau_- (y, v_*)} e^{-v_*(y, v_*)} k(v, v_*) h(y - sv_*, v_*) \, ds \, dv_* \right)^2
\]
\[
\leq \left( \int_{\mathbb{R}^3} |k(v, v_*)| \, dv_* \right) \left( \int_{\mathbb{R}^3} \int_0^{\tau_- (y, v_*)} |k(v, v_*)| |h(y - sv_*, v_*)|^2 \, ds \, dv_* \right)
\]
\[
\leq C \int_{\mathbb{R}^3} \int_{0}^{\tau- (y, v_*)} |k(v, v_*)||h(y - sv_*, v_*)|^2 \, ds \, dv_*
\]
\[
= C \int_{\mathbb{R}^3} \int_{0}^{\mid y - q - (y, v_*) \mid} \frac{1}{|v_*|} |k(v, v_*)||h(y - r v_*, v_*)|^2 \, dr \, dv_*.
\tag{7.19}
\]
where \( r = s|v_*| \).

We are now in a position to prove Lemma 1.13.

**Proof of Lemma 1.13** For given \( f \in L^2(\Omega \times \mathbb{R}^3) \), we recall \( f_i \) denotes the function \((S_\Omega K)^i f\).

To prove the lemma, it suffices to show
\[
\int_{D_1} \frac{|f_2(x, v) - f_2(y, v)|^2}{|x - y|^4} \, dx \, dy \, dv \leq C \| f \|^2_{L^2(\Omega \times \mathbb{R}^3)},
\tag{7.20}
\]
where \( D_1 \) is as defined by (6.2). According to Corollary 1.11, we have
\[
\| K f_i \|_{L^2_2(\mathbb{R}^3; H^2_0(\Omega))} \leq C \| f \|_{L^2(\Omega \times \mathbb{R}^3)}.
\tag{7.21}
\]
In domain \( D_1 \), we notice that \( \tau_-(x, v) \leq \tau_-(y, v) \). Therefore,
\[
|S_{\Omega} K f_1 (x, v) - S_{\Omega} K f_1 (y, v)|^2 \\
\leq 2 \int_{0}^{\tau_-(x, v)} e^{-v(s)} (K f_1 (x - sv, v) - K f_1 (y - sv, v)) \, ds \\
+ 2 \int_{\tau_-(x, v)}^{\tau_-(y, v)} e^{-v(s)} K f_1 (y - sv, v) \, ds.
\tag{7.22}
\]
Hence, we have
\[
\int_{D_1} \frac{|f_2(x, v) - f_2(y, v)|^2}{|x - y|^4} \, dx \, dy \, dv \leq I_1 + I_2,
\tag{7.23}
\]
where
\[
I_1 := \int_{D_1} \frac{2 \int_{0}^{\tau_-(x, v)} e^{-v(s)} (K f_1 (x - sv, v) - K f_1 (y - sv, v)) \, ds}{|x - y|^4} \, dx \, dy \, dv,
\tag{7.24}
\]
\[
I_2 := \int_{D_1} \frac{2 \int_{\tau_-(x, v)}^{\tau_-(y, v)} e^{-v(s)} K f_1 (y - sv, v) \, ds}{|x - y|^4} \, dx \, dy \, dv.
\tag{7.25}
\]
To estimate \( I_1 \), taking \( \epsilon = \frac{1}{2} \) and \( h = K f_1 \) in the steps of (6.27) and (6.28), in the same fashion, we conclude
\[
\int_{D_1} \frac{2 \int_{0}^{\tau_-(x, v)} e^{-v(s)} (K f_1 (x - sv, v) - K f_1 (y - sv, v)) \, ds}{|x - y|^4} \, dx \, dy \, dv \\
\leq C \| K f_1 \|^2_{L^2_2(\mathbb{R}^3; H^2_0(\Omega))} \\
\leq C \| f \|^2_{L^2(\Omega \times \mathbb{R}^3)},
\tag{7.26}
\]
\(\Box\)
where we have used (7.21). To deal with $I_2$, by Cauchy–Schwarz inequality again, we have
\[
\left| \int_{\tau_-(y,v)}^{\tau_-(x,v)} e^{-v(s)} Kf_1(y - s v, v) \, ds \right|^2 \leq \left( \int_{\tau_-(y,v)}^{\tau_-(x,v)} e^{-v(s)} \, ds \right) \left( \int_{\tau_-(y,v)}^{\tau_-(x,v)} |Kf_1(y - s v, v)|^2 \, ds \right). \tag{7.27}
\]

For small $\epsilon > 0$, \[\int_{\tau_-(y,v)}^{\tau_-(x,v)} e^{-v(s)} \, ds \leq \left( \int_0^{\infty} e^{-v(s)} \, ds \right)^{1-\epsilon} \left( \int_{\tau_-(y,v)}^{\tau_-(x,v)} e^{-v(s)} \, ds \right)^{\epsilon} \leq \frac{1}{v_0} |\tau_-(x, v) - \tau_-(y, v)|^{1-\epsilon} \leq C \frac{|x - y|^{1-\epsilon}}{N_-(x, v)^{1-\epsilon} |v|^{1-\epsilon}}, \tag{7.28}\]

where the last inequality follows from Proposition 5.3. On the other hand, the change of variable $r = s |v|$ results in
\[
\int_{\tau_-(y,v)}^{\tau_-(x,v)} e^{-v(s)} |Kf_1(y - s v, v)|^2 \, ds = \frac{1}{|v|} \int_{|x-q-(x,v)|}^{\infty} e^{-\frac{v(r)}{|v|}} |Kf_1(y - r \hat{v}, v)|^2 \, dr \leq \frac{1}{|v|} \int_{|x-q-(x,v)|}^{\infty} |Kf_1(y - r \hat{v}, v)|^2 \, dr. \tag{7.29}\]

Combining (7.27)–(7.29), we have
\[
\left| \int_{\tau_-(y,v)}^{\tau_-(x,v)} e^{-v(s)} Kf_1(y - s v, v) \, ds \right|^2 \leq C \frac{|x - y|^{1-\epsilon}}{N_-(x, v)^{1-\epsilon} |v|^{2-\epsilon}} \int_{|x-q-(x,v)|}^{\infty} |Kf_1(y - r \hat{v}, v)|^2 \, dr. \tag{7.30}\]

Therefore, it follows that
\[
I_2 \leq C \int_{D_1} \int_{|x-q-(x,v)|}^{\infty} \frac{|Kf_1(y - r \hat{v}, v)|^2}{N_-(x, v)^{1-\epsilon} |v|^{2-\epsilon} |x - y|^{3+\epsilon}} \, drdxdydv. \tag{7.31}\]

Letting $x' = x - r \hat{v}$ and $y' = y - r \hat{v}$, by Lemma 7.2, we obtain
\[
\int_{D_1} \int_{|x-q-(x,v)|}^{\infty} \frac{|Kf_1(y - r \hat{v}, v)|^2}{N_-(x, v)^{1-\epsilon} |v|^{2-\epsilon} |x - y|^{3+\epsilon}} \, drdxdydv
\]
\[
= \int_{\mathbb{R}^3} \int_{\Omega} \int_{\Omega_y} \int_{|q^1(x', v) - x'|} \min\{|q^2(x', v) - x'|, |q^3(y', v) - y'|\} \frac{|Kf_1(y', v)|^2}{N_-(q^1(x', v), v)^{1-\epsilon} |v|^2 \epsilon |x' - y'|^{3+\epsilon}} \, drdxdydv
\]
where we have utilized Proposition 5.9 by identifying $q^+(x', v) = q^1(x', v)$. Noticing that from (7.8), $\Omega_{v, y'}$ lies outside of $\Omega$, we have

$$\Omega_{v, y'} \subset (\mathbb{R}^3 \setminus \Omega) \subset \left(\mathbb{R}^3 \setminus B_{d_y}(y')\right).$$

With this in mind, we deduce

$$I_2 \leq C \int_{\mathbb{R}^3} \int_{\Omega_{v, y'}} \left| Kf_1(y', v) \right|^2 \frac{1}{|v|^{2-\epsilon}|x' - y'|^{3+\epsilon}} \, dx' \, dy' \, dv,$$

$$\leq C \int_{\Omega_{v, y'}} \left| Kf_1(y', v) \right|^2 \frac{1}{|v|^{2-\epsilon}|x' - y'|^{3+\epsilon}} \, dy' \, dv,$$

$$\leq C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \setminus B_{d_y}(y')} \left| Kf_1(y', v) \right|^2 \frac{1}{|v|^{2-\epsilon}|x' - y'|^{3+\epsilon}} \, dy' \, dv,$$

$$\leq C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left| Kf_1(y', v) \right|^2 \frac{1}{|v|^{2-\epsilon}|x' - y'|^{3+\epsilon}} \, dy' \, dv. \quad (7.33)$$

Proposition 7.4 and Lemma 7.1 imply

$$\int_{\mathbb{R}^3} \int_{\Omega} \frac{|Kf_1(y', v)|^2}{|v|^{2-\epsilon}d_y} \, dy' \, dv$$

$$\leq C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{0} \frac{|k(v, v_\ast)||Kf(y', v_\ast)||q_-(y', v_\ast)|}{|v_\ast||v|^{2-\epsilon}d_y} \, dr \, dv_\ast \, dy' \, dv,$$

$$= C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{0} \frac{|k(v, v_\ast)||Kf(y', v_\ast)|^2}{|v_\ast||v|^{2-\epsilon}d_y} \, dr \, dv_\ast \, dy'' \, dv,$$

$$\leq C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|k(v, v_\ast)||Kf(y'', v_\ast)|^2}{|v_\ast||v|^{2-\epsilon}} \, dv_\ast \, dy'' \, dv, \quad (7.34)$$

where $y'' = y' - r v_\ast$ and we have used Lemma 5.12 in the last line. Utilizing Proposition 3.4, we conclude

$$I_2 \leq C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|k(v, v_\ast)||Kf(y'', v_\ast)|^2}{|v_\ast||v|^{2-\epsilon}} \, dv \, dv_\ast \, dy''$$

$$\leq C \int_{\mathbb{R}^3} \int \frac{|Kf(y'', v_\ast)|^2}{|v_\ast|} \, dv_\ast \, dy''$$

$$\leq C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|v_\ast|} \left( \int_{\mathbb{R}^3} |k(v_\ast, w)| \, dw \right) \left( \int_{\mathbb{R}^3} |k(v_\ast, w)||f(y'', w)|^2 \, dw \right) \, dv_\ast \, dy''$$
where

\[ \leq C \int \int \int_{\Omega \times \mathbb{R}^3 \times \mathbb{R}^3} \frac{|k(v_\ast, w)||f(y'', w)|^2}{|v_\ast|} dv_\ast dw dy'' \]

\[ \leq C \int \int |f(y'', w)|^2 dw dy''. \quad (7.35) \]

This completes the proof. \( \square \)

Let us elaborate on Lemma 1.14 for a while. For \( f \in L^2(\Omega \times \mathbb{R}^3) \), in view of Lemma 1.13, the function \( S_\Omega K S_\Omega K f \) belongs to \( L^2(\mathbb{R}^3; H^{1/2}_{\varepsilon}(\Omega)) \). Now consider its zero extension \( \tilde{S}_\Omega K \tilde{S}_\Omega K f \) in the whole space \( \mathbb{R}^3 \times \mathbb{R}^3 \). We claim that \( S_\Omega K \tilde{S}_\Omega K f \) belongs to \( L^2(\mathbb{R}^3; H^{1/2-\varepsilon}_{\varepsilon}(\mathbb{R}^3)) \) for any \( \varepsilon \in (0, \frac{1}{2}) \) and

\[ \|S_\Omega K \tilde{S}_\Omega K f\|_{L^2(\mathbb{R}^3; H^{1/2-\varepsilon}_{\varepsilon}(\mathbb{R}^3))} \leq \frac{C}{\varepsilon} \|f\|_{L^2(\Omega \times \mathbb{R}^3)} \quad (7.36) \]

for some constant \( C \).

**Proof of Lemma 1.14** Recall \( f_1 := S_\Omega K f \) and \( f_2 := S_\Omega K S_\Omega K f \). Since \( \tilde{f}_2(\cdot, v) \) vanishes outside of \( \Omega \), we have

\[ \int \int \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \frac{\tilde{f}_2(x, v) - \tilde{f}_2(y, v)^2}{|x - y|^{4-2\varepsilon}} dxdydv = I_1 + I_2 + I_3, \quad (7.37) \]

where

\[ I_1 := \int \int \int_{\mathbb{R}^3 \times \Omega \times \Omega} \frac{|f_2(x, v) - f_2(y, v)|^2}{|x - y|^{4-2\varepsilon}} dxdydv, \quad (7.38) \]

\[ I_2 := \int \int \int_{\mathbb{R}^3 \times \Omega \times \mathbb{R}^3 \setminus \Omega} \frac{|f_2(y, v)|^2}{|x - y|^{4-2\varepsilon}} dxdydv, \quad (7.39) \]

\[ I_3 := \int \int \int_{\mathbb{R}^3 \times \mathbb{R}^3 \setminus \Omega} \frac{|f_2(x, v)|^2}{|x - y|^{4-2\varepsilon}} dxdydv. \quad (7.40) \]

Regarding the boundedness of \( I_1 \), by Lemma 1.13, we have

\[ I_1 = \int \int \int_{\mathbb{R}^3 \times \Omega \times \Omega} \frac{|f_2(x, v) - f_2(y, v)|^2}{|x - y|^{4}} |x - y|^{2\varepsilon} dxdydv \]

\[ \leq \max\{\text{diam } \Omega, 1\} \int \int \int_{\mathbb{R}^3 \times \Omega \times \Omega} \frac{|f_2(x, v) - f_2(y, v)|^2}{|x - y|^{4}} dxdydv \]

\[ \leq C \|f\|^2_{L^2(\Omega \times \mathbb{R}^3)} \quad (7.41) \]

By symmetry, one can see that \( I_2 = I_3 \). Consequently, the remaining task is to deal with \( I_2 \). Here, we present the case when \( 0 < \varepsilon < \frac{1}{2} \). The case when \( \frac{1}{4} \leq \varepsilon < \frac{1}{2} \) is similar and easier. We observe that, for \( y \in \Omega \),

\[ \int_{\mathbb{R}^3 \setminus \Omega} \frac{1}{|x - y|^{4-2\varepsilon}} dx \leq \int_{\mathbb{R}^3 \setminus B_{\delta_y}(y)} \frac{1}{|x - y|^{4-2\varepsilon}} dx \leq C d_y^{-1+2\varepsilon}. \quad (7.42) \]
By Proposition 7.4, we have
\[ |f_2(y, v)|^2 = |S_\Omega K f_1(y, v)|^2 \leq C \int_{\Omega} \int_{\mathbb{R}^3} \frac{|y-q-(y, v)|}{|v|} |k(v, v_s)||f_1(y-r\hat{v}, v_s)|^2 dr dv_* \quad (7.43) \]

Therefore, first combining (7.42) and (7.43) and performing the change of variable \(y' = y-r\hat{v}\) as in Lemma 7.1, we deduce that
\[ I_2 \leq C \int_{\Omega} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|y-q-(y, v)|}{|v|} |k(v, v_s)||f_1(y', v_s)|^2 d_{y'}^{-1+2\epsilon} dr dy' dv_* \]
\[ = C \int_{\Omega} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|y-q'(y')v|}{|v|} |k(v, v_s)||f_1(y', v_s)|^2 d_{y'}^{-1+2\epsilon} dr dy' dv_* \]
\[ \leq C \int_{\Omega} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\frac{1}{|v|} |k(v, v_s)| dv) |f_1(y', v_s)|^2 d_{y'}^{-\frac{1}{2}+2\epsilon} dy' dv_* \]
\[ \leq C \int_{\Omega} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |f_1(y', v_s)|^2 d_{y'}^{-\frac{1}{2}+2\epsilon} dy' dv_* \quad (7.44) \]

where the third inequality follows from Lemma 5.12 and the last inequality follows from Proposition 3.4. Applying Proposition 7.4 again, we have
\[ |f_1(y', v_s)|^2 \leq C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |y-q-(y', v_s)| \frac{1}{|v_s|} |k(v_s, w)||f(y'-r\hat{v}_s, w)|^2 dr dw \quad (7.45) \]

Then, it follows that
\[ \int_{\Omega} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |f_1(y', v_s)|^2 d_{y'}^{-\frac{1}{2}+2\epsilon} dy' dv_* \]
\[ \leq C \int_{\Omega} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\frac{1}{|v_s|} |k(v_s, w)||f(y'-r\hat{v}_s, w)|^2 d_{y'}^{-\frac{1}{2}+2\epsilon} dr dw dy' dv_* \]
\[ = C \int_{\Omega} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\frac{1}{|v_s|} |k(v_s, w)||f(y'', w)|^2 d_{y''}^{-\frac{1}{2}+2\epsilon} dr dy'' dv_* dw \quad (7.46) \]

where \(y'' = y' - r\hat{v}_s\). Similarly, by Lemma 5.12 and Proposition 3.4, we conclude that
\[ I_2 \leq C \int_{\Omega} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |f_1(y', v_s)|^2 d_{y'}^{-\frac{1}{2}+2\epsilon} dy' dv_* \]
\[ \leq C \int_{\Omega} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\frac{1}{|v_s|} |k(v_s, w)||f(y'', w)|^2 d_{y''}^{-\frac{1}{2}+2\epsilon} dv_* dy'' dw \]
\[ \leq C \int_{\Omega} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |f(y'', w)|^2 dy'' dw \quad (7.47) \]

Combining (7.41) and (7.47) completes the proof. \(\square\)
Lemma 1.14 allows us to improve regularity results via Lemma 1.2.

**Corollary 7.5** The operator $KS\Omega K\Omega K\Omega K : L^2(\Omega \times \mathbb{R}^3) \rightarrow L^2_1(\mathbb{R}^3; H^{1-\epsilon}_x(\Omega))$ is bounded for any $\epsilon \in (0, \frac{1}{2})$.

**Proof** For given $f \in L^2(\Omega \times \mathbb{R}^3)$ and $\epsilon \in (0, \frac{1}{2})$, Lemma 1.14 implies

$$S\Omega K\Omega K\Omega Kf \in L^2_1(\mathbb{R}^3; H^{1-\epsilon}_x(\mathbb{R}^3)) = L^2_1(\mathbb{R}^3; \bar{H}^{1-\epsilon}_x(\mathbb{R}^3)).$$

(7.48)

By Lemma 1.2, we obtain

$$KSK (S\Omega K\Omega K\Omega Kf) \in L^2_1(\mathbb{R}^3; \bar{H}^{1-\epsilon}_x(\mathbb{R}^3)).$$

(7.49)

Since

$$\left(KSK (S\Omega K\Omega K\Omega Kf)\right)_{|\Omega \times \mathbb{R}^3} = KS\Omega K\Omega K\Omega Kf,$$

(7.50)

consequently $KS\Omega K\Omega K\Omega Kf$ belongs to $L^2_1(\mathbb{R}^3; H^{1-\epsilon}_x(\Omega))$. The boundedness of the operator is due to boundedness of $KSK$ and $ZS\Omega K\Omega K$.

We are now ready to prove the last lemma in this section.

**Lemma 7.6** The operator $S\Omega K\Omega K\Omega K\Omega K : L^2(\Omega \times \mathbb{R}^3) \rightarrow L^2_1(\mathbb{R}^3; H^{1-\epsilon}_x(\Omega))$ is bounded for any $\epsilon \in (0, \frac{1}{2})$. Therefore, $f_4 \in L^2_1(\mathbb{R}^3; H^{1-\epsilon}_x(\Omega))$.

**Proof** We shall prove this lemma in a similar fashion as we prove Lemma 1.13. To do so, for given $f \in L^2(\Omega \times \mathbb{R}^3)$ and $\epsilon \in (0, \frac{1}{2})$, it suffices to show

$$\int_{D_1} \frac{|f_4(x, v) - f_4(y, v)|^2}{|x - y|^{5-2\epsilon}} dxdydv \leq C \|f\|^2_{L^2(\Omega \times \mathbb{R}^3)},$$

(7.51)

where $D_1$ is as defined by (6.2). According to Corollary 7.5, it follows that

$$\|Kf_3\|_{L^2_1(\mathbb{R}^3; H^{1-\epsilon}_x(\Omega))} \leq C \|f\|_{L^2(\Omega \times \mathbb{R}^3)}.$$  

(7.52)

In domain $D_1$, we have

$$|S\Omega Kf_3(x, v) - S\Omega Kf_3(y, v)|^2 \leq 2 \int_{0}^{\tau_{-(x,v)}} e^{-\nu(v)s} (Kf_3(x - sv, v) - Kf_3(y - sv, v)) ds \left| \int_{\tau_{-(y,v)}}^{\tau_{-(x,v)}} e^{-\nu(v)s} Kf_3(y - sv, v) ds \right|^2.$$

(7.53)

Therefore, we have

$$\int_{D_1} \frac{|f_4(x, v) - f_4(y, v)|^2}{|x - y|^{5-2\epsilon}} dxdydv \leq I_1 + I_2,$$

(7.54)

where

$$I_1 := \int_{D_1} \frac{2 \left| \int_{0}^{\tau_{-(x,v)}} e^{-\nu(v)s} (Kf_3(x - sv, v) - Kf_3(y - sv, v)) ds \right|^2}{|x - y|^{5-2\epsilon}} dxdydv,$$

(7.55)
To estimate $I_1$, taking $h = Kf_3$ in the steps of (6.27) and (6.28), in the same fashion, we conclude

$$I_2 := \int_{D_1} 2 \left( \int_{\tau_-(x,v)}^{\tau^-(y,v)} e^{-v(s)} Kf_3(y - sv, v) ds \right)^2 |x - y|^{5-2\epsilon} dxdydv. \quad (7.56)$$

where we have utilized (7.52). Concerning $I_2$, we proceed as in (7.30)–(7.34) to deduce

$$I_2 \leq C \int \int \int \int_{\Omega \times \Omega} \frac{|Kf_3(y - r\hat{v}, v)|^2}{N_-(x, v)^{1-\epsilon}|v|^{2-\epsilon}|x - y|^{4-\epsilon}} drdxdydv$$

$$= C \int \int \int \int_{\Omega \times \Omega \times \Omega} |Kf_3(y', v)|^2 |v|^{2-\epsilon}|x' - y'|^{4-\epsilon} dx'dy'dv \quad (7.58)$$

$$\leq C \int \int \int \int_{\Omega \times \Omega \times \Omega} |Kf_3(y', v)|^2 |v|^{2-\epsilon} dy'dv$$

$$\leq C \int \int \int \int_{\Omega \times \Omega \times \Omega} |k(v, v_*)||Kf_2(y' - r\hat{v}, v_*)|^2 |v_*| |v|^{2-\epsilon} dy'dv \quad (7.59)$$

where we utilized (5.22) of Lemma 5.12 in the last line instead. Continuing as in (7.35) yields

$$I_2 \leq C \int \int \int \int_{\Omega \times \Omega \times \Omega} |k(v, v_*)||Kf_2(y'', v_*)|^2 |v_*| |v|^{2-\epsilon} dy''dv_*$$

$$\leq C \int \int \int_{\Omega \times \Omega \times \Omega} |Kf_2(y'', v_*)|^2 |v_*|^{2-\epsilon} dv_*dy'' \quad (7.59)$$
\[ \leq C \int_{\Omega} \int_{\mathbb{R}^3} \frac{1}{|v_\ast| d_2^{\frac{1}{2}-\epsilon}} \left( \int |k(v_\ast, w)| \, dw \right) \left( \int |k(v_\ast, w)||f_2(y'', w)|^2 \, dw \right) \, dv_\ast d'y'' \]
\[ \leq C \int_{\Omega} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|k(v_\ast, w)||f_2(y'', w)|^2}{|v_\ast| d_2^{\frac{1}{2}-\epsilon}} \, dv_\ast d'y'' \]
\[ \leq C \int_{\Omega} \int_{\mathbb{R}^3} |f_2(y'', w)|^2 \, d'y'' \cdot \tag{7.60} \]

The last line above is (7.46) with \( f_2 \) in place of \( f_1 \). In the same fashion, we can conclude that
\[ I_2 \leq C \int_{\Omega} \int_{\mathbb{R}^3} |f_1(z, w_\ast)|^2 \, dz \, dw_\ast \]
\[ \leq C \|f\|_{L^2(\Omega \times \mathbb{R}^3)}^2, \tag{7.61} \]

This completes the proof. \( \square \)

**Remark 7.7** From our calculation, we are not able to show that \( \tilde{f}_4 \in L^2_{x} (\mathbb{R}^3; H^1_{x} (\Omega)) \). As a result, we can not further improve regularity by the same method as in Corollary 7.5.

**Remark 7.8** From [1], for the domain \( \Omega \) under consideration and \( h \in L^2(\Omega) \), there exists a constant \( K > 0 \) such that
\[ \lim_{\epsilon \to 0} \epsilon \int_{\Omega} \int_{\Omega} \frac{|h(x) - h(y)|^2}{|x - y|^{5-2\epsilon}} \, dx \, dy = K \int_{\Omega} |\nabla h(x)|^2 \, dx. \]

If we keep tracking the effect of \( \epsilon \) accumulate in the process of applying Lemma 1.14, Corollary 5.14, and the property of \( \sigma (1 - \epsilon) \) in (1.19)–(1.20), we have
\[ \int_{\mathbb{R}^3} \int_{\Omega} \frac{|f_4(x, v) - f_4(y, v)|^2}{|x - y|^{5-2\epsilon}} \, dx \, dy \, dv = O \left( \frac{1}{\epsilon^3} \right), \quad \text{as } \epsilon \to 0. \]

This does not suffice to conclude \( f_4 \in L^2_{x} (\mathbb{R}^3; H^1_{x} (\Omega)). \)

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