Purity of branch and critical locus

Rolf Källström

July 6, 2010

Abstract

To a dominant morphism \( X/S \to Y/S \) of Noetherian integral \( S \)-schemes one has the inclusion \( C_{X/Y} \subset B_{X/Y} \) of the critical locus in the branch locus of \( X/Y \). Starting from the notion of locally complete intersection morphisms, we give conditions on the modules of relative differentials \( \Omega_{X/Y}, \Omega_{X/S}, \) and \( \Omega_{Y/S} \) that imply bounds on the codimensions of \( C_{X/Y} \) and \( B_{X/Y} \). These bounds generalise to a wider class of morphisms the classical purity results for finite morphisms by Zariski-Nagata-Auslander, and Faltings and Grothendieck, and van der Waerden’s purity for birational morphisms.

Introduction

In this paper \( \pi : X/S \to Y/S \) denotes a dominant morphism locally of finite type of Noetherian integral \( S \)-schemes of relative dimension \( d_{X/Y} \). Let \( \Omega_{X/Y} \) be the sheaf of relative differentials, i.e.

\[
\Omega_{X/Y} = \text{Coker}(\pi^*(\Omega_{Y/S}) \to \Omega_{X/S}),
\]

and dually let \( C_{X/Y} = \text{Coker}(d\pi : T_{X/S} \to T_{X/S} \to Y/S) \) be the critical module of \( \pi \), where \( d\pi \) is the tangent morphism of \( \pi \). The critical locus of \( \pi \) is the support \( C_{\pi} \) of \( C_{X/Y} \), and the branch locus \( B_{\pi} \) is the set of points \( x \) where the stalk \( \Omega_{X/Y,x} \) is not free; we abuse the terminology since \( B_{\pi} \) is the set of ramification points as defined in [20] only when \( \Omega_{X/Y} \) is torsion and hence \( B_{\pi} = \text{supp} \Omega_{X/Y} \).

These two subsets of \( X \), satisfying \( C_{\pi} \subset B_{\pi} \), exert much control on the morphism \( \pi \). If \( B_{\pi} = \emptyset \), then \( \pi \) is smooth (as defined in [20 Def. 17.3.1]) if it is flat and generically smooth, and if moreover \( \pi \) is finite and \( Y \) is normal, then \( \pi \) is étale [3 Sec. 4]; see [21] for a discussion of the relation between the branch locus and the non-smoothness locus. If \( C_{\pi} = \emptyset \), and \( \pi \) is either flat or \( Y/S \) is smooth, then tangent vector fields on \( Y \) lift (locally) to tangent vector fields on \( X \), so according to Zariski’s lemma (Char \( X = 0 \)) the morphism \( \pi \) is locally analytically trivial. It is therefore a natural problem to find upper bounds on the codimensions of \( B_{\pi} \) and \( C_{\pi} \), so that \( B_{\pi} = \emptyset \) or \( C_{\pi} = \emptyset \) can be controlled in low codimensions. The best situation is when \( \text{codim}_X C_{\pi} \leq 1 \) and \( \text{codim}_X B_{\pi} \leq 1 \) (when nonempty), and we say that \( C_{\pi} \) and \( B_{\pi} \) are pure (of codimension 1), respectively. Let \( F_i(M) \) denote the ith Fitting ideal of a module and the relative dimension \( d_{X/Y} \) be defined as the Krull dimension of the generic fibre of \( \pi \). If \( X/S \) and \( Y/S \) are smooth there is a duality relation

\[
F_i(C_{X/Y}) = F_{d_{X/Y}+i}(\Omega_{X/Y}) \tag{*}
\]

\[\text{2000 Mathematics Subject Classification: Primary: 14A10, 32C38; Secondary: 17B99 (Secondary)}\]
(Prop. 1.3), so in particular $C_π = B_π$ in this case. A simple notable fact is that $\text{codim}_X C_π \leq 1$ if the image of the tangent morphism $\text{Im}(dπ)$ satisfies Serre’s condition $(S_2)$, and that this holds in particular when $X$ is smooth and $X$ is generically separably algebraic; hence by $(*)$ $\text{codim}_X B_π \leq 1$ when $X/S$ and $Y/S$ are smooth. In general we shall see that $C_π$ is pure “more often” than $B_π$ (Th. 1.4). Our method of establishing purity results for $C_π$ and $B_π$ is by assuming a good behaviour of the modules $Ω_{X/S}$ and $Ω_{Y/S}$. Say that $π$ is a differentially complete intersection morphisms (d.c.i.) at a point $x$ if the projective dimension $p. d. Ω_{X/Y,x} \leq 1$. This is inspired by a result due to Ferrand and Vasconcelos [13, 31] that in the case of generically smooth domains over a field, which, when extended to a relative situation, states that locally complete intersection morphisms $X/Y$ that are smooth at all the associated points in $X$ are d.c.i., and that the converse holds if $p. d. C_π(x), O_{X,x} < ∞$ for each point $x \in X$; we include a complete proof (Th. 2.1). In Section 2 we work out some basic results for d.c.i. morphisms, demonstrating that not only is the class of d.c.i. morphisms larger, for singular $Y$, than the class of locally complete intersection morphisms, but it is also in some respects easier to work with. This is exemplified by a more general base-change theorem (Th. 2.3) for locally complete intersection morphisms than in [5, 5.11], using a simpler argument. Recalling that base-changes $X_1/Y_1$ are smooth when $X/Y$ is smooth, it is interesting to see that $X_1/Y_1$ is d.c.i. when $X/Y$ is d.c.i. and smooth at all points are are images of associated points in $X_1$; this also often implies that $X_1/Y_1$ is a locally complete intersection.

In Theorem 3.3 we give general upper bounds of $\text{codim}_X B_π^{(i)}$ and $\text{codim}_X C_π^{(i)}$, where $B_π^{(i)}$ and $C_π^{(i)}$ are higher order branch and critical loci ($B_π^{(0)} = B_π$ and $C_π^{(0)} = C_π$ are the most important sets), defined by certain Fitting ideals, in terms of defect numbers of $X/S$ and $Y/S$, the more precise the more assumptions on $π$ are made. Let $δ_{X/Y}$ be the relative smoothness defect, $χ_2(M)$ a partial Euler characteristic, and $β_1(M)$ the second Betti number of a module (see Section 3 for precise definitions). We state a similar but somewhat more precise result:

**Theorem 3.8** Let $π : X/S → Y/S$ be a generically smooth morphism of Noetherian integral $S$-schemes such that $Ω_{X/S}$ and $Ω_{Y/S}$ are coherent.

1. $\text{codim}_X^{+} B_π^{(i)} \leq (d_{X/Y} + i + 1)(i + 1 + δ_{Y/S} + χ_2(Ω_{X/S})).$

   In particular, if $X/S$ is d.c.i., then

   \[ \text{codim}_X^{+} B_π^{(i)} \leq (d_{X/Y} + i + 1)(i + 1 + δ_{Y/S}). \]

2. Assume that $X/S$ and $Y/S$ are d.c.i., then

   \[ \text{codim}_X^{+} B_π^{(i)} \leq (d_{X/Y} + i + 1)(i + 1 + χ_2(Ω_{X/Y})) \leq (d_{X/Y} + i + 1)(i + 1 + β_1(Ω_{X/Y})). \]

3. Assume that $X/S$ is smooth and that each restriction to fibres $X_s → Y_s, s ∈ S, is generically smooth, then

   \[ \text{codim}_X^{+} B_π \leq δ_{X/Y} + d_{X/Y}. \]

The proof of (1) results from a decomposition property for d.c.i. morphisms in Theorem 2.14 combined with a bound on the height of determinant ideals by Eagon and Northcott [14]. When $X/Y$ itself is a d.c.i., then $χ_2(Ω_{X/Y}) = 0$, and the first inequality in (2) results in an inequality due to Dolgachev (when $i = 0$). The proof of (3) is based on a refinement of the Eagon-Northcott
bound by Eisenbud, Ulrich and Huneke [12]. Clearly, (1) is more interesting than (2) and (3) if we have more knowledge of the ramification of $X/S$ and $Y/S$ than the ramification of $X/Y$.

One may reflect a moment over the usefulness of our relative setting, bearing in mind that the non-smoothness and branch loci are unions of the corresponding loci for morphisms of fibres $X_s \to Y_s$, taking one point $s$ in $S$ at a time [20 Props 17.8.1, 17.8.2]. It is nevertheless natural to consider the schemes over $S$, since, for example, if $S$ is defined over a field $k$ it is not necessary that $X/k$ and $Y/k$ satisfy very strong requirements, say necessary to apply Zariski-Nagata-Auslander or Grothendieck purity (described below), as long as the fibres $X_s$ and $Y_s$ are sufficiently nice.

In Section 4 we discuss the special case $d_{X/Y} = 0$. On the one hand, if $\pi : X \to Y$ is a finite morphism we have classical result by Zariski, Nagata and Auslander [1][30][33], stating that $B_\pi$ is pure when $\pi$ is finite, $X$ is normal, and $Y$ is regular. This was generalised by Grothendieck, Faltings, Griffith, Cutkosky, and Kantorovitz [10][15][17][18][23], allowing some singularities in $Y$. On the other hand, for birational morphisms van der Waerden’s purity theorem states that $B_\pi$ is pure when $Y$ is normal and a certain condition (W) is satisfied (e.g. $Y$ is Q-factorial). It is therefore natural to ask as in [20 Rem. 21.12.14, (v)], if the two types of purity, one for finite and another for birational morphisms, can be used together so that one gets purity for generically finite morphisms from the mere existence of a factorisation into a birational and a finite morphism $X \to Y' \to Y$, where $Y'$ is the normalisation of $Y$ in $X$. However, $Y'$ need not satisfy (W); more precisely, the complement of the branch locus $B_\pi$ of the finite morphism need not be affine (see Remark 4.2).

We work out necessary conditions in Theorem 4.3 to ensure that this phenomenon does not occur. Moreover, we get other quite general positive results when $\pi$ is only separably algebraic, which is thus less than what is required both in the Zariski-Nagata-Auslander purity theorem and van der Waerden’s purity theorem. The first is (1) in Theorem 3.5 implying $\text{codim}_XB_\pi \leq 1$ when $X/S$ d.c.i. and $Y/S$ is smooth. However, it is often important to allow that either $X/S$ or $Y/S$ is not d.c.i.. First in arbitrary characteristic, Theorem 4.1 gives $\text{codim}_XB_\pi \leq 1$, still assuming that $X/S$ is d.c.i., together with a regularity condition at points of low height in $X$ and $Y$, respectively, and that the canonical map $\pi^*(\Omega_{Y/S}) \to \Omega_{X/S}$ be injective. When the base scheme $S$ is defined over the rational numbers we have Theorem 4.4 again ensuring $\text{codim}_XB_\pi \leq 1$, which can be regarded as a generalised relative version of the Zariski-Nagata-Auslander purity theorem in the sense that the conditions on $X$ and $Y$ are of a similar type, namely that $Y/S$ is smooth and $X$ satisfies (S2), while $\pi$ is only generically algebraic and not necessarily finite. In general, even if $\pi$ is finite, $B_\pi$ need not be pure of codimension 1 when $Y/S$ is non-smooth and the bound on $\text{codim}B_\pi$ will depend on the type of singularities. For example, we include a simple argument for Cutkosky’s bound $\text{codim}_XB_\pi \leq 2$ when $\pi$ is finite, $X$ is normal, and $Y$ is a local complete intersection (Prop. 4.1): it is really a direct consequence of Grothendieck’s purity theorem. Our rather general bounds that arise from Theorems 3.5 and 4.4 in terms of defect numbers of $X/S$ and $Y/S$ are interesting to compare to a bound by Faltings and Cutkosky in terms of the regularity defect of $Y$, where the latter is applicable only when $\pi$ is finite. Not only is our bound easier to get and more general in that $\pi$ need not be finite, in the finite case it improves the Faltings-Cutkosky bound when $\pi$ is defined over some base $S$ and the singularities of $X$ and $Y$ are fibered over $S$ (see discussion after Proposition 4.1).

**Generalities:** All schemes are assumed to be Noetherian and we conform to the notation in EGA. The height $\text{ht}(x)$ of a point $x$ in $X$ is the same as the Krull dimension of the local ring $\mathcal{O}_{X,x}$, and the dimension of $X$ is $\dim X = \sup \{ \text{ht}(x) \mid x \in X \}$. A point $x$ is a **maximal point** in a subset $T$ of $X$ if for each point $y$ in the closure of $x$ in $T$ we have $\text{ht}(x) \leq \text{ht}(y)$, i.e., if $x \in T$ specialises to $x$. 

3
and \( \text{ht}(x_1) \leq \text{ht}(x) \), then \( x_1 = x \). Denote by \( \text{Max}(T) \) the set of maximal points of \( T \), so \( \text{Max}(X) \) consists of points of height 0. A property on \( X \) is \textit{generic} if it holds for all points in \( \text{Max}(X) \). \textbf{We assume that} \( X \) \textit{is generically reduced}, so \( k_{X,\xi} = \mathcal{O}_{X,\xi} \) when \( \xi \in \text{Max}(X) \) (so if the nilradical of \( \mathcal{O}_X \) is non-zero, then all its associated points are embedded points in \( X \)). Put

\[
\text{codim}^+_X T = \sup \{ \text{ht}(x) \mid x \in \text{Max}(T) \},
\]

\[
\text{codim}^-_X T = \inf \{ \text{ht}(x) \mid x \in \text{Max}(T) \},
\]

so \( \text{codim}^-_X T \leq \text{ht}(x) \leq \text{codim}^+_X T \) when \( x \in \text{Max}(T) \) (in the introduction we mean \( \text{codim} = \text{codim}^- \)); one may call \( \text{codim}^+_X T \) and \( \text{codim}^-_X T \) the upper and lower codimension of \( T \) in \( X \), respectively. If \( T \) is the empty set, put \( \text{codim}^+_X T = -1 \) and \( \text{codim}^-_X T = \infty \), since we are interested in lower and higher bounds of \( \text{codim}^+_X T \), respectively. For a coherent \( \mathcal{O}_X \)-module \( M \) we put \( \text{depth}_T M = \inf \{ \text{depth} M_x \mid x \in T \} \). \textbf{We define the} \textit{relative dimension} of a morphism locally of finite type \( \pi : X \to Y \) at a point \( x \in X \) as the infimum of the dimension of the vector space of Kähler differentials at all maximal points \( \xi \) that specialise to \( x \), i.e.

\[
d_{X/Y,x} = \inf \{ \dim_{k_{X,\xi}} \Omega_{X/Y,\xi} \mid x \in \xi^-, \xi \in \text{Max}(X) \}.
\]

Put \( d_{X/Y} = \inf \{ \dim_{k_{X,\xi}} \Omega_{X/Y,\xi} \mid \xi \in \text{Max}(X) \} \). If \( X/Y \) is generically smooth \( X \), and the numbers \( d_{X/S,\xi} \) and \( d_{Y/S,\pi(\xi)} \) do not depend on the choice of maximal point \( \xi \in \text{Max}(X) \) that specialises to \( x \), then \( d_{X/Y,x} = d_{X/S,x} - d_{Y/S,\pi(x)} \). Recall that \( \dim_{k_{\xi}} \Omega_{X/Y,\xi} = \dim_{k_{\xi}} \Omega_{X/\pi(\xi)} \) is the same as the transcendence degree and dimension of a \( p \)-basis of \( k_{\xi}/k_{\pi(\xi)} \) in characteristic 0 and \( p \), respectively, and these numbers are equal when the field extension is separable and finitely generated. Thus \( d_{X/Y,x} = 0 \) when \( x \in \xi^-, \xi \in \text{Max}(X) \), \( \mathcal{O}_{X,\xi} = k_{X,\xi} \) and \( k_{X,\xi}/k_{\pi(\xi)} \) is separably algebraic, and if moreover \( X \) is integral, then \( d_{X/Y,x} = \deg k_{X,\xi}/k_{\pi(\xi)} \).

1 \ Critical scheme and branch scheme

Assume that \( X \) is a connected scheme and \( \pi : X \to Y \) be a locally of finite type dominant morphism. The first fundamental exact sequence of quasi-coherent \( \mathcal{O}_X \)-modules

\[
0 \to \Gamma_{X/Y/S} \to \pi^*(\Omega_{Y/S}) \to \Omega_{X/Y} \to 0.
\]

contains the kernel \( \Gamma_{X/Y/S} \) of \( p \), which is called the imperfection module of \( X/Y/S \). Denoting the image by \( \mathcal{V}_{X/Y/S} \) we have two short exact sequences

\[
0 \to \Gamma_{X/Y/S} \to \pi^*(\Omega_{Y/S}) \to \mathcal{V}_{X/Y/S} \to 0,
\]

\[
0 \to \mathcal{V}_{X/Y/S} \to \Omega_{X/S} \to \Omega_{X/Y} \to 0.
\]

Consider a chain of morphisms \( X \xrightarrow{i} X_r \xrightarrow{p} Y \to S \) and put \( \pi = p \circ i \). There is an exact sequence \cite[Th. 20.6.17]{EGA}.

\[
0 \to \Gamma_{X_r/Y/S} \to \Gamma_{X/Y/S} \to \Gamma_{X/X_r/S} \to i^*(\Omega_{X_r/Y}) \to \Omega_{X/Y} \to \Omega_{X/X_r} \to 0,
\]

where

\[
\Gamma_{X_r/Y/S} = \text{Ker}(i^*(p^*(\Omega_{Y/S}))) \to i^*(\Omega_{X_r/Y/S}).
\]
We will study the Fitting ideals $F_{iX/Y}(\Omega_{X/Y})$, $i \geq 0$, defining the $i$th branch scheme $B^{(i)}_\pi$ (Kähler different, Jacobians), so there is a finite decreasing filtration of $B_\pi$

\[ \subset B^{(i)}_\pi \subset B^{(i-1)}_\pi \subset \cdots \subset B^{(0)}_\pi = B_\pi. \]

Here $B_\pi$ is the branch scheme and its underlying topological space is the branch locus.

**Remark 1.1.** (1) If $Y/S$ is smooth and $X/Y$ is generically smooth, then $\Gamma_{X/Y/S} = 0$, but $\Gamma_{X/Y/S}$ is normally non-zero, also in characteristic 0 if $Y/S$ is non-smooth. Example: $A = k[x,y]/(x^2 + y^3)$, $B = k[x',y']/(x'^2 + y')$ and let $A \to B$ be defined by $x' = xy$ and $y' = y$ (a chart of the strict transform with respect to the blow-up of the origin). The torsion submodule of $\Omega_{A/k}$ is $A(2ydx - 3xdy)$ and $\Gamma_{B/A/k} = B(2ydx - 3xdy) \subset B \otimes_A \Omega_{A/k}$.

(2) The two middle terms in (1.2) are quasi-coherent so $\Gamma_{X/Y/S}$ is quasi-coherent, and if $Y/S$ is of finite type, then $\Gamma_{X/Y/S}$ is coherent, since $X$ is a noetherian space. Also, if only $X/Y$ is locally of finite type, then $\Gamma_{X/Y/S}$ is coherent; this is proven using the sequence (1.4). Assume that $X, Y, S$ are locally of finite type, then $\Gamma_{X/S}$ is coherent, since $\Gamma_{Y/S}$ is coherent; this is proven using the sequence (1.4).

(3) We have $B^{(i)}_\pi = X$ when $i < 0$ and $B^{(i)}_\pi = \emptyset$ when $i \geq \text{sup}_x \{ \beta_0(\Omega_{X/Y,x}) \} - d_{X/Y}$. Hence $\text{codim}_X B_\pi \leq 1$, always, and if $X/Y$ is generically smooth, then $B_\pi = B^{(0)}_\pi$. If $d_{X/Y} = 0$, then $B_\pi = \text{supp} \Omega_{X/Y}$, and $B_\pi$ is the locus of points where $X/Y$ is not formally unramified, while if $d_{X/Y} > 0$, then $B_\pi$ is a subset of the non-smoothness locus of $X/Y$ (if $\pi$ is locally of finite type and flat, then $B_\pi$ is the non-smoothness locus).

The dual of $p$ induces a homomorphism of $\mathcal{O}_Y$-modules, the tangent morphism of $\pi$, $d\pi : T_{X/S} = \text{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{O}_X) \to T_{X/S \to Y/S}$ where the $\mathcal{O}_X$-module of ‘derivations from $\mathcal{O}_Y$ to $\mathcal{O}_X$’ is

\[ T_{X/S \to Y/S} = \text{Hom}_{\mathcal{O}_X}(\pi^*(\Omega_{Y/S}), \mathcal{O}_X) = \text{Hom}_{\mathcal{O}_Y}(\pi^{-1}(\Omega_{Y/S}), \mathcal{O}_X), \]

and is part of the exact sequence

\[ 0 \to T_{X/Y} \to T_{X/S} \xrightarrow{d\pi} T_{X/S \to Y/S} \to \mathcal{C}_{X/Y} \to 0, \]

(1.5)

where $\mathcal{C}_{X/Y}$ is the critical module of $\pi$. The critical set $C_\pi = \text{supp} \mathcal{C}_{X/Y}$ is endowed with a structure of a scheme (still denoted $C_\pi$), defined by the Fitting ideal $F_0(\mathcal{C}_{X/Y})$; we say that $\pi$ is submersive at a point $x$ in $X$ if $x \notin C_\pi$. Note that $T_{X/S \to Y/S} = \pi^*(T_{Y/S})$ when either $\pi$ is flat or $\Omega_{Y/S}$ is locally free of finite rank. Let $C^{(i)}_\pi$ be the scheme that is defined by the Fitting ideal $F_i(C_\pi)$, giving a finite decreasing filtration of the critical scheme $C_\pi$

\[ \subset C^{(i)}_\pi \subset \cdots \subset C^{(1)}_\pi \subset C^{(0)}_\pi = C_\pi. \]

**Remark 1.2.** If $\Omega_{X/S}$ is locally free it is straightforward to see that the space of $B^{(i)}_\pi$ is given by a rank condition on the induced map of fibres of the map $p$, while if $T_{Y/S}$ is locally of finite rank, the space of $C^{(i)}_\pi$ is given by a rank condition on the induced map of fibres of the map $d\pi$.

**Proposition 1.3.** If $X/S$ and $Y/S$ are generically smooth and $B_{X/S} = \emptyset$, $B_{Y/S} = \emptyset$, then

\[ F_i(\mathcal{C}_{X/Y}) = F_{d_{X/Y}+i}(\Omega_{X/Y}). \]

5
To be clear, note that the requirements in Proposition \[1.3\] are satisfied when \(X/S\) and \(Y/S\) are smooth, and thus in this situation \(C^{(i)} = B^{(i)}\). The reason is that \(C_{X/Y} \) and \(\Omega_{X/Y} \) are transposed modules of one another, so for the proof one needs a relation between the Fitting ideals of a module and its transpose.

**Lemma 1.4.** Let \(\phi : G_1 \to G_2\) be a homomorphism of locally free \(\mathcal{O}_X\)-modules of finite ranks \(g_1\) and \(g_2\), respectively. Let \(\phi^* : G_2^* \to G_1^*\) be the homomorphism of dual modules. Then

\[
F_1(\operatorname{Coker} \phi) = F_{g_1-g_2+i}(\operatorname{Coker} \phi^*).
\]

**Proof.** \(F_1(\operatorname{Coker} \phi)\) is the image of the map \(\wedge^{g_2-i}G_1 \otimes_{\mathcal{O}_Y} (\wedge^{g_2-i}G_2)^* \to \mathcal{O}_X\) induced by the map \(\wedge^{g_2-i}\phi : \wedge^{g_2-i}G_1 \to \wedge^{g_2-i}G_2\) and \(F_1(\operatorname{Coker} \phi^*)\) is the image of the map \(\wedge^{g_1-r}G_2^* \otimes_{\mathcal{O}_Y} \wedge^{g_1-r}G_1 \to \mathcal{O}_X\) induced by the map \(\wedge^{g_1-r}\phi^* : \wedge^{g_1-r}G_2^* \to \wedge^{g_1-r}G_1^*\). When \(g_2 - i = g_1 - r\), i.e. \(r = g_1 - g_2 + i\) we get a commutative diagram

\[
\begin{array}{ccc}
\wedge^{g_2-i}G_1 \otimes_{\mathcal{O}_Y} (\wedge^{g_2-i}G_2)^* & \longrightarrow & \mathcal{O}_X \\
\downarrow & & \downarrow \\
(\wedge^{g_2-i}G_1^*)^* \otimes_{\mathcal{O}_Y} \wedge^{g_2-i}G_2^* & \longrightarrow & \mathcal{O}_X
\end{array}
\]

where the left vertical homomorphism exists because there are canonical maps \(\wedge^{g_2-i}G_1 \to (\wedge^{g_2-i}G_1^*)^*\) and \(\wedge^{g_2-i}G_2^* \to (\wedge^{g_2-i}G_2)^*\), and the latter is an isomorphism because \(G_2\) is locally free (they both are isomorphisms since \(G_1\) also is locally free).

Proof of Proposition \[1.3\] The assumptions give that \(\Omega_{X/S}\) and \(\Omega_{Y/S}\) are locally free of ranks \(d_{X/S}\) and \(d_{Y/S}\), respectively, so \(d\pi : G_1 = T_{X/S} \to G_2 = \pi^*(T_{Y/S})\) is a homomorphism of locally free \(\mathcal{O}_X\)-modules, where \(\operatorname{Coker}(d\pi) = C_{X/Y}\) and \(\operatorname{Coker}(d\pi^*) = \Omega_{X/Y}\), so the result follows from Lemma \[1.4\] noting that \(d_{X/Y} = d_{X/S} - d_{Y/S}\).

**Remark 1.5.** Recall that the kernel and cokernel of a biduality morphism \(M \to M^{**}\) of a coherent \(\mathcal{O}_X\)-module \(M\) can be expressed using the transposed module \(D(M)\), locally defined up to local projective equivalence by \(D(M) = \operatorname{Coker}(\phi^*)\), where \(\phi\) is a local presentation \(F_1 \xrightarrow{\phi} F_0 \to M \to 0\). Then we have the exact sequence

\[
0 \to \text{Ext}^1_{\mathcal{O}_X}(D(M), \mathcal{O}_X) \to M \to M^{**} \to \text{Ext}^2_{\mathcal{O}_X}(D(M), \mathcal{O}_X) \to 0;
\]

see \[2\]. Note also that when the projective dimension \(p.d. M_x \leq 1\) for each point \(x\), then \(D(M)\) is locally projectively equivalent to \(\text{Ext}^1_{\mathcal{O}_X}(M, \mathcal{O}_X)\).

## 2 Differentially complete intersections

Ferrand and Vasconcelos \[13, 31\] have shown that if \(X/k\) is a reduced scheme locally of finite type and generically smooth (i.e. the residue fields at all maximal points are separable over \(k\)), then \(X/k\) is a locally complete intersection if and only if the projective dimension \(p.d. \Omega_{X/k,x} \leq 1\) at each point \(x\); see also \[24\] \(\S9\). Because of this result there are two natural notions of “locally complete intersection morphisms”. The first is well-known in the case when \(\pi : X \to Y\) is “smoothable”: there exists a locally defined factorisation \(X \to Z \to Y\), where \(X/Z\) is a regular immersion and
Z/Y is formally smooth, i.e. the ideal of X in Z is locally defined by a regular sequence; this was further developed in [5] to general morphisms employing “Cohen-factorisations”, proving that there is an alternative definition by the vanishing of certain André-Quillen homology groups. We continue to call such morphisms locally complete intersection morphisms (l.c.i.), but we shall however have more use for a second possibility. Say that a dominant morphism \( \pi : X \to Y \) is a differential complete intersection (d.c.i.) at a point \( x \) if \( \text{p.d. } \Omega_{X/Y,x} \leq 1 \), and that \( \pi \) is a d.c.i. if it is d.c.i. at each point \( x \); we then also write \( \text{p.d. } \Omega_{X/Y} \leq 1 \). Let \( x \) be a specialisation of a point \( \xi \) in X. Since \( \Omega_{X/Y,\xi} = \mathcal{O}_{X,\xi} \otimes_{\mathcal{O}_{X,x}} \Omega_{X/Y,x} \), it is evident that a morphism is a d.c.i. at \( \xi \) if it is d.c.i. at \( x \), hence it suffices to check the closed points in \( X \) to see if a morphism is d.c.i. If the first syzygy of the quasi-coherent module \( \Omega_{X/Y} \) is coherent it is clear that the set \( \{ x \in X \mid \pi \text{ is a d.c.i. at } x \} \) is open.

Recall also that \( \Omega_{X/Y,x} = \mathcal{O}_{X,x}/\mathcal{O}_{Y,\pi(x)} \) (see [22] for a proof not using the fact that \( \mathcal{O}_{X,x} \to \mathcal{O}_{X,\xi} \) is etale), so d.c.i. is a property of the morphism of local rings \( \mathcal{O}_{Y,\pi(x)} \to \mathcal{O}_{X,x} \). If \( \pi \) is smoothable and l.c.i. at \( x \) then it is l.c.i. at \( \xi \), but the proof of this assertion is not as immediate as for the d.c.i. property; for non-smoothable morphisms this localisation property for l.c.i. morphisms need not hold, see [5] 5.3, 5.12.

**Theorem 2.1. (Ferrand, Vasconcelos)** Let \( \pi : X/S \to Y/S \) be morphism locally of finite type. Consider the conditions for a point \( x \) in \( X \):

1. \( \pi \) is l.c.i. at \( x \).
2. \( \pi \) is d.c.i. at \( x \).

If \( X/Y \) is smooth at all associated points in \( X \) then (1) \( \Rightarrow \) (2). If \( X/Y \) is generically smooth and \( \text{p.d. } \mathcal{O}_{Y,\pi(x)} \mathcal{O}_{X,x} < \infty \) then (2) \( \Rightarrow \) (1).

**Remark 2.2.** Kunz gives a part of the proof in the above relative situation [24] Th. 9.2], but it seems that the possibility of embedded points is overlooked (flatness is included in his definition of locally complete intersection, but this does not rule out this possibility).

We record a situation where no embedded points are present in \( X \), so the above smoothness conditions can be expressed more concretely as \( X \) being geometrically reduced over the maximal points in \( Y \). The proof is immediate.

**Lemma 2.3.** If \( Y \) is Cohen-Macaulay and \( X/Y \) is l.c.i., then \( X \) is Cohen-Macaulay, and hence contains no embedded points.

The proof of the following lemma in [31] is perhaps a little succinct, so we include an argument.

**Lemma 2.4. (Vasconcelos)** Let \( J \) be a proper ideal of a Noetherian local ring \( A \), such that \( \text{p.d. } \mathcal{A} J < \infty \). If \( J/J^2 = \Gamma \oplus K \) where \( \Gamma \) is free of rank \( l \) over \( A/J \), then \( J \) contains a regular sequence of length \( l \), and if \( K \neq 0 \) it contains a regular sequence of length \( l+1 \).

**Proof.** Put \( B = A/J \). Since \( \text{p.d. } \mathcal{A} J < \infty \), hence \( \text{p.d. } \mathcal{A} B < \infty \), and since \( A \) is local, \( B \) has a finite free resolution as \( \mathcal{A} \)-module. Since \( \text{Ann}_B(B) = J \neq 0 \) by Auslander-Buchsbaum’s theorem [3] \( J \) contains an \( \mathcal{A} \)-regular element, so \( J \not\subseteq P \) for each associated prime \( P \) of \( A \). By prime avoidance there exists an element \( x \in J \) such that \( x \not\in P \) for all associated primes (so again \( x \) is a regular element) and \( x \not\in \mathfrak{m}_A J \). The image \( \bar{x} \) of \( x \) for the projection \( J/J^2 \to \Gamma \) satisfies \( \bar{x} \not\in \mathfrak{m}_B B \), so it can be complemented to a basis \( \{ \bar{x}_1 = \bar{x}, \bar{x}_2, \ldots, \bar{x}_l \} \) of the free \( B \)-module \( \Gamma \), hence \( B\bar{x} \) is a free summand of \( \Gamma \subseteq J/J^2 \). Select \( x_i \in J \) that project to \( \bar{x}_i \), \( i = 2, \ldots, l \).
Now put $A^* = A/(x)$ and $J^* = J/(x)$. Since $x$ is $A$-regular it is also $J$-regular, so by [27] Lem. 2, §18 p. d. $A, J/xJ < \infty$. Since $x \notin \mathfrak{m}_A J$ it follows that the natural map $J/xJ \to J^*$ splits (see proof of [Th 19.2, loc cit]). Therefore p. d. $A^*, J^*/xJ < \infty$. Since $B\bar{x}$ is a free summand of $\Gamma$ and hence a free direct summand of $J/J^2$, it follows that $J^*/(J^*)^2 = J/(Ax + J^2) \cong \Gamma^* \oplus K^*$, where $\Gamma^*$ is a free module of rank $l - 1$, generated by $x_i \text{mod}(x + J^2), i = 2, \ldots, l$. If $K = 0$ we see by induction that $I$ is generated by a regular sequence of length $l$. If $K \neq 0$ again by induction it follows that $I$ contains a regular sequence of length $l + 1$.

**Proof of Theorem 2.1** (1) $\Rightarrow$ (2): There exists locally a factorisation $X \overset{i}{\to} X_r \to Y$ where $X/X_r$ is a regular immersion and $X_r/Y$ is smooth. Letting $I$ be the ideal of $X$ in $X_r$ we get the exact sequence

$$0 \to K \to I/I^2 \to i^*(\Omega_{X_r/Y}) \to \Omega_{X_r/Y} \to 0.$$ 

Since $X/X_r$ is a regular immersion, so $I$ is locally generated by a regular sequence, the $\mathcal{O}_{X,r}$-module $I_x/I_x^2$ is free of some rank $l$. Note in passing, $\Omega_{X_r/Y,x}$ being free of rank $d_{X_r/Y,x}$ it follows that rank $\Omega_{k_{\xi}/k_{\pi(t)}} = l - d_{X_r/Y}$ for each $\xi \in \text{Max}(X)$. Moreover, $\Omega_{X/Y,x}$ is free when $x$ is an associated point in $X$, implying that $I_x/I_x^2 = \Gamma \oplus K_x$, where $\Gamma$ is free of rank $l$. Since $I_x$ does not contain a regular sequence of length $r + 1$, Lemma 2.4 implies $K = 0$, and since $K$ is a submodule of a locally free $\mathcal{O}_X$-module, $K = 0$. Since $I/I^2$ and $\Omega_{X_r/Y}$ are locally free it follows that p. d. $\Omega_{X/Y,x} \leq 1$ at each point $x$.

(2) $\Rightarrow$ (1): There exists locally a factorisation $X \overset{i}{\to} X_r \to Y$ where $X_r/Y$ is smooth and $X/X_r$ is a closed immersion. Consider the exact sequence

$$0 \to \Gamma_{X/X_r/Y} \to i^*(\Omega_{X_r/Y}) \to \Omega_{X_r/Y} \to 0.$$ 

Since p. d. $\Omega_{X_r/Y,x} \leq 1$ at each point $x$ and $\Omega_{X_r/Y,x}$ is free, it follows that $\Gamma_{X/X_r/Y,x}$ is free. Putting $l = d_{X_r/Y,x} - d_{X/Y,x}$, since $X/Y$ and $X_r/Y$ are generically smooth, rank $\Gamma_{X/X_r/Y} = l$. Combining with the previous exact sequence we get the split exact sequence

$$0 \to K \to I_x/I_x^2 \to \Gamma_{X/X_r/Y,x} \to 0,$$

where $K$ is torsion. Put $A = \mathcal{O}_{X,r}$, $J = I_x$ and $\Gamma = \Gamma_{X/X_r/Y,x}$, so $J/J^2 = \Gamma \oplus K_x$, where $\Gamma$ is $A$-free of rank $l$, and as the maximal length of an $A$-regular sequence in $J$ satisfies $d_{X,Y} \leq \dim A - \dim \mathcal{O}_{X,x} = l$ Lemma 2.4 implies $K = 0$; we only have to note that p. d. $\mathcal{O}_{X,x} < \infty$ implies p. d. $\mathcal{O}_{X,x} < \infty$ and hence p. d. $J < \infty$. □

**Remark 2.5.** (1) If $\pi$ is not generically smooth, then (1) does not imply (2) in Theorem 2.1. Example: $A = k[t]/(x^2)$ is l.c.i. over $k$, but p. d. $A, \Omega_{A/k} = p. d. A = k = \infty$.

(2) For a regular base $Y$, generically smooth d.c.i. morphisms are the same as generically smooth locally complete intersection morphisms, but if p. d. $\mathcal{O}_{Y,x}$ is $\infty$ for some point $x$, then (2) does not imply (1) in Theorem 2.1. Example: $A = k[t]/(t^2)$, $k$ is a field, $R = A[x,y]$ and $I = (t + x, t + y^2)$. Put $B = R/I$ and consider the natural map $A \to B$. Then

(a) In the ring $B$ we have $txy(t + x) = t(x^2y + t) + I$ and $I$ cannot be generated by a regular sequence.

(b) The $B$-module $I/I^2$ is free of rank 2, so by (a) and p. d. $I = \infty$.

(c) The sequence $0 \to I/I^2 \to B \otimes_R \Omega_{R/A} \to \Omega_B = \frac{k(t, x, y)}{(t, x, y)} \to 0$ is exact also to the left.
Therefore \( p.d. \Omega_{B/A} \leq 1 \) while \( A \to B \) is not l.c.i. See also Theorem 2.14 (3-4).

**Lemma 2.6.** Let \( \pi : X \to Y \) be a locally of finite type morphism that is smooth at all associated points in \( X \). Assume either:

1. \( X/S \) and \( Y/S \) are smooth.
2. \( X \) and \( Y \) are regular schemes.

Then \( \pi \) is d.c.i.

(1) was first observed by Dolgachev [11]; in Theorem 2.14 we will give another necessary condition for \( X/Y \) to be d.c.i. Clearly, (1) \( \Rightarrow \) (2) when \( X \) and \( Y \) are geometrically regular over a field.

**Proof.** (1): Since \( \pi^* (\Omega_{Y/S}) \) is locally free and \( \pi \) is smooth all associated points, implying that the imperfection module \( \Gamma_{X/Y/S} \) is 0 at all associated points, hence \( \Gamma_{X/Y/S} = 0 \) in the exact sequence (1.1). Since moreover \( \Omega_{X/S} \) is locally free, it follows that \( p.d. \Omega_{X/y,x} \leq 1 \) at each point \( x \). (2):

There exists locally a factorisation \( X \xrightarrow{i} X_r \xrightarrow{p} Y \), where \( i \) is a closed immersion and \( p \) is smooth. Since \( p \) is smooth and \( Y \) is regular, it follows that \( X_r \) is regular. Since \( X \) is regular and \( i \) is a closed immersion, it must be a regular immersion. Therefore \( \pi \) is l.c.i., hence by Theorem 2.1 \( \pi \) is d.c.i.

We give necessary conditions to conclude that a morphism is d.c.i. when all its fibres are d.c.i.

**Proposition 2.7.** Let \( \pi : X \to Y \) be a flat dominant morphism locally of finite type of Noetherian schemes, which is smooth at all associated points of \( X \). Assume either of the conditions:

1. \( \Omega_{X/Y} \) is \( Y \)-flat.
2. Each fibre \( X_y/k_{Y,y}, y \in Y \), is generically smooth (i.e. generically geometrically reduced).

If the fibre \( X_y/k_{Y,y} \) is d.c.i., then \( \pi \) is d.c.i. at each point \( x \) in \( X_y \subset X \). Hence if each (closed) fibre \( X_y \) is d.c.i., then \( \pi \) is d.c.i.

Next easy lemma is standard, and a similar assertion holds for higher bounds on the projective dimension.

**Lemma 2.8.** Let \( \pi : X \to Y \) be a flat morphism of schemes and \( M \) be a coherent \( \mathcal{O}_X \)-module, flat over \( Y \). Let \( x \) be a point in \( X \), \( X_y \) be the fibre over \( y = \pi(x) \), and \( M_{X_y} \) the restriction of \( M \) to \( X_y \). If \( p.d. M_{X_y,x} \leq 1 \), then \( p.d. M_x \leq 1 \).

**Proof.** Locally there exists a presentation \( 0 \to L \to F \to M \to 0 \) where \( F \) is locally free of finite rank. It suffices to see that \( L_x \) is free. Applying \( k_{Y,y} \otimes_{\mathcal{O}_{Y,y}} L_x \) to the exact sequence, by assumption \( k_{Y,y} \otimes_{\mathcal{O}_{Y,y}} L_x \) is free over \( \mathcal{O}_{X_y,x} \), since \( M_x \) is flat over \( \mathcal{O}_{Y,y} \) and \( p.d. \mathcal{O}_{X_y,x} k_{Y,y} \otimes_{\mathcal{O}_{Y,y}} M_x \leq 1 \). Selecting a basis \( k_{Y,y} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \cong k_{Y,y} \otimes_{\mathcal{O}_{Y,y}} L_x \), arising from a homomorphism \( u : \mathcal{O}_{X,x} \to L_x \) of \( \mathcal{O}_{X,x} \)-modules, so \( u \) is surjective by Nakayama’s lemma. Since \( M_x \) and \( \mathcal{O}_{X,x} \) are flat, hence \( L_x \) is flat over \( \mathcal{O}_{Y,y} \), we conclude that \( u \) is an isomorphism [27 Th. 22.5].

9
Proof of Proposition 2.7. (1): This follows immediately from Lemma 2.8 noting that \( k_{Y,y} \otimes_{O_Y} \Omega_{X/Y,x} \).

(2): We can assume that \( X/Y \) is a subscheme of a smooth scheme \( X'/Y \) so there is the short exact sequence
\[
0 \to \Lambda \to O_X \otimes_{O_{X'}} \Omega_{X'/Y} \to \Omega_{X/Y} \to 0,
\]
and the assertion is that \( \Lambda_x \) is free over \( O_{X,x} \) when \( x \in \pi^{-1}(y) \). Since \( X'/Y \) is smooth it follows that the two terms to the right are coherent, so \( \Lambda_x \) is of finite type. Let \( I \) be the defining ideal of \( X \) in \( X_r \) (defined locally). We have a surjective map \( I_x/I_x^2 \to \Lambda_x \to 0 \). Since p.d. \( \langle \Omega_{X,y}/k_{Y,y,x} \rangle \leq 1 \) for each point \( x \in X_y \) if follows by Theorem 2.7 that \( X_y/k_{Y,y} \) is l.c.i., since \( X_y/k_{Y,y} \) is generically smooth; hence since \( \pi \) is flat, \( f \) is generated by a regular sequence so \( I_x/I_x^2 \) is free over \( O_{X,x} \). Since \( X/Y \) is smooth at each associated point \( \xi \) in \( X \), and therefore \( d_\xi \) is injective, it follows that \( d_x \) is injective; hence \( \Lambda_x = I_x/I_x^2 \) is free. Therefore p.d. \( \Omega_{X/Y,x} \leq 1 \).

The class of d.c.i. morphisms behaves almost as well under base-change as the class of smooth morphisms. Consider a base change diagram over some scheme \( S \):

\[
\begin{array}{ccc}
X_1 & \xrightarrow{j} & X \\
\downarrow{\pi_1} & \Downarrow{\pi} & \Downarrow{\pi} \\
Y_1 & \rightarrow & Y,
\end{array}
\]

where \( X_1 = X \times_Y Y_1 \).

**Theorem 2.9.** Let \( \pi : X/S \to Y/S \) be a dominant morphism locally of finite type where \( X/S \) and \( Y/S \) are Nœtherian, and assume that \( \pi \) is smooth at all points in \( j(X_1) \subset X \) that are images of associated points in \( X_1 \). If \( \pi \) is d.c.i. then \( \pi_1 : X_1 \to Y_1 \) is d.c.i.

Note that when \( \pi \) is flat, then \( \pi \) is smooth at the associated points of \( j(X_1) \subset X \) if and only if \( \pi_1 \) is smooth at the associated points of \( X_1 \) [19, Th 19.7.1].

**Lemma 2.10.** Let \( j : X \to Y \) be a morphism of schemes and \( M \) a coherent \( O_Y \)-module satisfying p.d. \( M_y \leq 1 \) at each point \( y \) in the image \( j(X) \). Assume also that \( M_y \) is flat over \( O_{Y,y} \) when \( y \) is an associated point in \( j(X) \). Then p.d. \( j^*(M)_x \leq 1 \) at each point \( x \) in \( X \).

**Proof.** Let \( x \) be a point in \( X \) and set \( B = O_{X,x} \) and \( A = O_{Y,\pi(x)} \). If \( 0 \to F^1 \to F^0 \to M_{j(x)} \to 0 \) is exact, where \( F_1,F_0 \) are free, then \( 0 \to Tor^1_A(B,M_{j(x)}) \to B \otimes_A F^1 \to B \otimes_A F^0 \xrightarrow{b} j^*(M)_x \to 0 \) is exact. By assumption the support of the \( B \)-module \( Tor^1_A(B,M_{j(x)}) \) does not contain any associated point of \( B \) and \( B \otimes_A F^1 \) is free; therefore \( Tor^1_A(B,M_{j(x)}) = 0 \), implying p.d. \( j^*(M)_x \leq 1 \).

**Proof of Theorem 2.9.** This follows from Lemma 2.10 noting that \( \Omega_{X_1/Y_1} = j^*(\Omega_{X/Y}) \).

**Theorem 2.7 immediately implies the following corollary to Theorem 2.9**

**Corollary 2.11.** Make the same assumptions as in Theorem 2.7 and assume moreover that
\[
p.d. O_{Y,\pi(x)} \Omega_{X_1,x} < \infty
\]
for all \( x \) in \( X_1 \) (e.g. \( Y_1 \) is regular). Then \( \pi_1 \) is l.c.i.
Remark 2.12. Assume that \( \pi \) in the above diagram is l.c.i.. In [5, 5.11], one essentially requires that either \( Y_1/Y \) or \( X/Y \) to be flat to infer that \( \pi_1 \) is l.c.i.. Corollary 2.11 implies that it suffices that \( \pi \) be smooth at the associated points of \( j(X_1) \) when \( Y_1 \) is regular (Th. 2.1). It is easy to see that \( X_1/Y_1 \) is d.c.i. when \( X/Y \) is d.c.i. and flat.

Proposition 2.13. Let \( X \) and \( Y \) be Noetherian schemes and \( \pi : X/S \to Y/S \) be a morphism locally of finite type which is smooth at all associated points in \( X \). Assume either of the following conditions:

1. \( \pi \) is l.c.i. (or d.c.i. and \( p.d.\mathcal{O}_{Y,\pi(x)} \mathcal{O}_{X,x} < \infty \), \( x \in X \))

2. \( X/S \) is d.c.i.

Then \( \Gamma_{X/Y}/S = 0 \).

Proof. There exists locally a factorisation of \( \pi \) of the form \( X \to X_r \to Y \) where \( X_r/Y \) is smooth and \( X/X_r \) is closed immersion. Hence \( \Omega_{X/X_r} = 0 \) and \( \Gamma_{X_r/Y}/S = 0 \), and since \( \Omega_{X/Y} \) is locally free we also get \( \Gamma_{X_r/Y}/S = 0 \). Therefore the exact sequence (1.4) gives the short exact sequence

\[
0 \to \Gamma_{X/Y}/S \to \Gamma_{X/X_r}/S \to \delta \to \Gamma_{X/X_r}/Y \to 0.
\]

(1): Since \( X/Y \) is d.c.i., so \( p.d.\Omega_{X/Y,x} \leq 1 \) at each point \( x \in X \), and \( i^*(\Omega_{X_r/Y}) \) is locally free, it follows that \( \Gamma_{X/X_r}/Y \) is locally free. Since \( X/Y \) is generically smooth it follows that \( \Gamma_{X/Y}/S \) is torsion, but we want \( \Gamma_{X/Y}/S = 0 \). Let \( I \) be the ideal of \( X \) in \( X_r \). There exist surjections \( I/I^2 \to \Gamma_{X/X_r}/Y \to 0 \) and \( I/I^2 \to \Gamma_{X/X_r}/S \to 0 \), such that \( \delta \circ \beta = \alpha \). To conclude that \( \Gamma_{X/Y}/S = 0 \) it suffices to see that \( \alpha \) is injective, so \( \delta \) is an isomorphism. First, \( X/Y \) is smooth at all associated points, implying \( \text{Ker}(\alpha) = 0 \) at all such points. Second, since \( X/Y \) is a l.c.i., so \( I \) is locally generated by a regular sequence and \( I/I^2 \) is locally free, it follows that \( \text{Ker}(\alpha) = 0 \).

(2): If \( X/S \) is d.c.i. then since \( i^*(\Omega_{X_r/S}) \) is locally free, it follows that \( \Gamma_{X/X_r}/S \) is locally free. Since \( \pi \) is smooth at the associated points, it follows that \( \Gamma_{X/Y}/S \) is 0 at such points, and since \( \Gamma_{X/Y}/S \subset \Gamma_{X/X_r}/S \) the assertion follows. 

We have the following composition and decomposition properties.

Theorem 2.14. Let \( X \xrightarrow{f} Y \xrightarrow{g} S \) be a composition of dominant morphisms, locally of finite type.

1. If \( f \) is l.c.i., \( g \) is d.c.i., and \( f \) is smooth at all associated points of \( X \), then \( g \circ f \) is d.c.i.

2. Assume that \( f \) is generically smooth and \( g \) is smooth. Then \( g \circ f \) is d.c.i. if and only if \( f \) is d.c.i.

3. Assume that \( g \) and \( g \circ f \) are l.c.i., \( f \) is smooth at points that map to maximal points of \( Y \), and \( g \) is smooth at maximal points of \( f(X) \). Then \( p.d.\Omega_{X,Y,x} \leq 2 \), \( x \in X \).

4. Assume that \( X \) and \( Y \) are Cohen-Macaulay, \( f \) is flat and generically smooth along each fibre. Consider the conditions:

(a) \( g \circ f \) is d.c.i.

(b) \( f \) is d.c.i. and \( g \) is d.c.i. at all points in \( f(X) \).
Remark 2.15. In [3] (5.6), (5.7) there are results that can be compared to Theorem 2.1. Making the assumption that $\mathcal{O}_X$ be of finite flat dimension over $\mathcal{O}_{Y,\pi(x)}$ for each $x \in X$, Avramov gets a stronger decomposition property for l.c.i. morphisms, namely that $f$ and $g$ are l.c.i. if and only if $f \circ g$ is l.c.i. This result depends on two other results that are fundamental albeit hard to get: (i) The vanishing of all higher André-Quillen homology groups characterises l.c.i. morphisms; (ii) a certain connecting morphism of the Zariski-Jacobi long exact sequence is trivial [3] (4.7)]. Note that (3) does not rely on flatness; if $f$ is l.c.i. then $f$ has finite flat dimension.

Proof. (1): By Proposition 2.13 $\Gamma_{X/Y/S} = 0$. Since $X/Y$ is smooth at the associated points $\xi$, hence it is flat at $\xi$, so $\text{Tor}_{\mathcal{O}_{Y,\pi(\xi)}}(\mathcal{O}_{X,\xi}, \Omega_{Y/S,\pi(\xi)}) = 0$. This implies that p.d. $\pi^*(\Omega_{Y/S})_x \leq 1$ for each point $x \in X$, since $g$ is d.c.i.; see also the proof of Lemma 2.10. Since p.d. $\Omega_{X/Y,x} \leq 1$ it follows from the exact sequence (1.1) that p.d. $\Omega_{X/S,x} \leq 1$.

(2): Since $X/Y$ is smooth at all associated points in $X$ and since $f^*(\Omega_{Y/S})$ is locally free it follows that $\Gamma_{X/Y/S} = 0$. Again we get the exact the exact sequence

$$0 \to f^*(\Omega_{Y/S}) \to \Omega_{X/S} \to \Omega_{X/Y} \to 0,$$

implying that $X/S$ is d.c.i. if and only if $X/Y$ is d.c.i..

(3): By Proposition 2.13 $\Gamma_{X/Y/S} = 0$, so we again have the exact sequence (2.1). The assertion then follows from Lemma 2.10.

(4): First note that $X$ being Cohen-Macaulay, all its associated points are maximal, and that both (a) and (b) implies $\Gamma_{X/Y/S} = 0$, by Proposition 2.13. Let $y$ be a point in $Y$ and $j : X_y \to X$ and $p : X_y \to \text{Spec} k_{Y,y}$ be the canonical morphisms. Then $j^*(\pi^*(\Omega_{Y/S})) = p^*(k_{Y,y} \otimes_{\mathcal{O}_{Y,y}} \Omega_{Y/S})$ is free, and since $j^*(\Omega_{X/Y}) = \Omega_{X_y/k_{Y,y}}$ is generically free, while $X_y$ contains no embedded points since $X_y$ is Cohen-Macaulay, for $X$ and $Y$ are Cohen-Macaulay and $X/Y$ is flat. Therefore we get the exact sequence

$$0 \to j^*(\pi^*(\Omega_{Y/S})) \to j^*(\Omega_{X/S}) \to \Omega_{X_y/k_{Y,y}} \to 0,$$

hence p.d. $\Omega_{X_y/k_{Y,y}} = p.d. j^*(\Omega_{X/S})$ when $x \in X_y$. By Lemma 2.10 p.d. $j^*(\Omega_{X/S}) \leq 1$, hence by Proposition 2.7 p.d. $\Omega_{X/Y,x} \leq 1$ when $x \in X$. The exact sequence (2.1) then implies that $g$ is d.c.i. This proves (a)⇒(b). (b)⇒(a) follows from (1).

3 Purity for general $d_X/Y$

For a coherent $\mathcal{O}_X$-module $M$ with local presentation $\mathcal{O}_X^n \xrightarrow{\phi} \mathcal{O}_X^m \to M \to 0$ we let $I_i(\phi)$ be the ith determinant ideal of $\phi$, $i \leq \min\{m, n\}$ and the ith Fitting ideal $F_i(M) = I_{n-i}(\phi)$. It is a fundamental problem to compute the height of $F_i(M)$ for a given $M$. An important step is to find an upper bound in the form of refined versions of Krull’s principal ideal theorem. This problem is addressed in the literature where bounds are determined in terms of the integers $n, m$, the rank of $\phi$ and more precisely the analytic spread of $M$, together with some measure of the singularity defect of $X$ (embedding dimension). General bounds of the height in terms of $m$ and $n$ are in general quite crude for a given $M$ and depend on the presentation of $M$. The bounds we use arise from minimal presentations and from exact sequences, based on either the classical Eagon-Northcott formula involving the first two Betti numbers, but not the Euler number (generic rank), or a refinement due
Remark 3.2.

Theorem 3.1. (1) If \( \beta_i(M) < \beta_i(M_x) \) be a minimal free resolution, so \( \beta \) \( \chi \) \( p \beta \) to Eisenbud, Ulrich, and Huneke, where the Euler number and the first Betti number appear, but \( X \) is assumed to be regular.

Assume for simplicity that \( X \) is an integral scheme. Let \( \beta_i(M) = \text{sup} \{ \beta_i(M_x) \mid x \in X \} \) where \( \beta_i(x) = \beta_i(M_x) = \dim_{k_x} \text{Tor}_{i}^{O_{X,x}}(k_x, M_x) \) is the \( i \)th Betti number of the \( O_{X,x} \)-module \( M_x \). When \( p.d. M_x < \infty \) the Euler number is \( \chi(x) = \chi(M_x) = \sum (-1)^i \beta_i(x) \). Let \( \cdots \rightarrow G^1_x \rightarrow G^0_x \rightarrow M_x \rightarrow 0 \) be a minimal free resolution, so \( \beta_i(x) = \text{rank} G^i \), and the partial Euler numbers are defined by \( \chi_i(x) = \chi_i(M_x) = \sum_{j \geq i} (-1)^j \beta_j(x) \). Note that in general \( \chi_i(x) \neq \chi_i(\xi) \) when \( x \) is a specialisation of the point \( \xi \) and \( i \geq 1 \), while \( \chi(x) = \chi_0(x) = \chi(\xi) \) is the generic rank of \( M_x \) and \( (-1)^i \chi_i(x) \) is the generic rank of the \( i \)th syzygy in a minimal free resolution of \( M_x \), so they are positive integers.

**Theorem 3.1.** (1) If \( i < \beta_0(x) - \min \{ \beta_0(x), \beta_1(x) \} \), then \( F_i(M_x) = 0 \), and if \( F_i(M_x) \neq 0 \) then

\[
\text{ht}(F_i(M_x)) \leq (i + 1)(i + 1 + \beta_1(x) - \beta_0(x)).
\]

Here \( \beta_0(x) - \beta_1(x) = \chi(x) - \chi_2(x) \leq \chi(x) \), with equality if \( p.d. M_x \leq 1 \).

(2) Assume \( X \) is regular. Then

\[
\text{ht}(F_i(M_x)) \leq (i + 1)(i + 1 - \chi(x)) + \beta_0(x) - i - 1
\]

(3) Let \( 0 \rightarrow M^1 \rightarrow M^2 \rightarrow M \rightarrow 0 \) be a short exact sequence of coherent \( O_X \)-modules.

(a) \[
\text{ht}(F_i(M_x)) \leq (i + 1)(i + 1 + \beta_0(M^1_x) + \beta_1(M^2_x) - \beta_0(M^2_x))
\]

(b) If \( X \) is regular, then

\[
\text{ht}(F_i(M_x)) \leq (i + 1)(i + 1 + \chi(M^1_x) - \chi(M^2_x)) + \beta_0(x) - i - 1.
\]

**Remark 3.2.** (1) In Theorem 3.1 (2) strengthens (1) when

\[
\chi_2(x)(i + 1) \geq \beta_0(x) - i - 1.
\]

Note that \( \chi_2(x) = 0 \) if \( p.d. M_x \leq 1 \), so we get 0 in the left side, and the condition is \( \beta_0(x) \leq i + 1 \). Assume that \( M \) has generic rank \( r = \beta_0(\xi) = \chi(x) \) and consider the Fitting ideal \( F_r(M) \). If \( p.d. M_x \leq 1 \), (2) is never stronger than (1) for \( i = r \), but if \( \chi_2(x) \neq 0 \), then (2) is stronger than (1) when the minimal number \( \beta_0(x) \) of generators of \( M_x \) is not too high compared to \( r \), in the sense

\[
\beta_0(x) - r = -\chi_1(x) \leq 1 + \chi_2(x)(\chi(x) + 1) = 1 + \chi_2(x)(r + 1).
\]

For example, \( \beta_0(x) \geq 2(r + 1) \) suffices when the second syzygy of \( M_x \) has nonzero generic rank.

(2) It is tempting to ask under what conditions \( \text{ht}(F_i(M_x)) \leq (i + 1)(i + 1 - \chi(M_x)) \) holds when \( p.d. M_x < \infty \).

(3) (3)a is stronger than (1) when

\[
\chi_2(M_x) \geq \chi_2(M^2_x) - \chi_1(M^1_x).
\]
Proof. (1): Let $I_i(x)$ be the $i$th determinant ideal of the homomorphism $\phi_x: G^1_x \to G^0_x$, so $F_i(M_x) = I_{\beta_0(x)-i}$. That $F_i(M_x) = 0$ when $i < \beta_0(x) - \min\{\beta_0(x), \beta_1(x)\}$ follows since rank $G^1 = \beta_1(x)$ and rank $G^0 = \beta_0(x)$. The Eagon-Northcott bound (see [27, Th 19.7]) (see also [5, Th. 3.5] and [27, Th. 13.10])

$$\text{ht}(I_i) \leq (\min\{m, n\} - i + 1)(\max\{m, n\} - i + 1),$$

now implies

$$\text{ht}(I_i) \leq (\beta_1(x) - i + 1)(\beta_0(x) - i + 1),$$

giving the first inequality in (1). Since $\beta_0(x) - \beta_1(x) = x(x) - \chi_2(x)$ and $\chi_2(x) = (-1)^2\chi_2(x) \geq 0$ (see [27, Th 19.7]) the second inequality is also established. The last assertion follows since $\chi_2(x) = 0$ when $p. d. M_x \leq 1$.

(2): If $X$ is regular, the Eisenbud-Ulrich-Huneke bound [12, Th. A] implies $\text{ht}(I_i) \leq ((-1)^i\chi_1(x) - i + 1)(\beta_0(x) - i + 1) + i - 1$. Therefore

$$\text{ht} F_i(M_x) \leq (((-1)^i\chi_1(x) - (\beta_0(x) - i) + 1)(\beta_0(x) - (\beta_0(x) - i) + 1) + \beta_0(x) - i - 1 = (1 + i - \chi(x))(i + 1) + \beta_0(x) - i - 1.$$

(3): (a) There is an exact sequence $\mathcal{O}_{X, x}^{\beta_1(M_x^2)} \to \mathcal{O}_{X, x}^{\beta_0(M_x^2)} \to M_x^2 \to 0$ and a surjection $\mathcal{O}_{X, x}^{\beta_0(M_x^2)} \to M_x^2 \to 0$. Therefore we can assume $m \geq \beta_0(M_x^2) + \beta_0(M_x^2)$ and $n \leq \beta_0(M_x^2)$ in the Eagon-Northcott bound, hence $\text{ht}(F_i(M_x)) = \text{ht}(I_{n-i}) \leq (\min\{m, n\} - (n - i) + 1)(\max\{m, n\} - (n - i) + 1) = (i + 1)(m - n + i + 1) \leq (i + 1)(i + 1 + \beta_0(M_x^2) + \beta_1(M_x^2) - \beta_0(M_x^2))$. (b) follows from [2].

The following interpretation of Theorem 3.1 is useful:

**Corollary 3.3.** Let $X$ be an integral Nœtherian scheme and $M$ be a coherent $\mathcal{O}_X$-module. Let $j : U \to X$ be an open subset and put $V = X \setminus U$. Assume either of the conditions:

1. $\text{codim}_X V > (\sup_{x \in V} (\beta_1(M_x) - \beta_0(M_x))) + \text{rank} M + 1 \text{rank} M + 1$.
2. The projective dimension $\text{p. d.} M_x \leq 1$ at each point $x \in V$ and $\text{codim}_X V > \text{rank} M + 1$.

If $j^*(M)$ is locally free, it follows that $M$ is locally free. If $X$ moreover satisfies $(S_2)$, then $M = j_*(j^*(M))$.

**Remark 3.4.** In [20] §21.13 a couple $(X, V)$ is parafactorial if for any open set $\Omega$ the restriction functor $\mathcal{L}_\Omega \to \mathcal{L}|_{\Omega \cap U}$ is an equivalence of categories of invertible sheaves. In particular, this holds when $X = \text{Spec} A$, $A$ is factorial of dimension $\geq 2$ and $V = \{m_A\}$ [20, 21.6.13]. By Grothendieck’s finiteness theorem, if $U$ is Cohen-Macaulay and $j^*(M)$ is locally free, then $j_*j^*(M)$ is coherent when $\text{codim}_X V \geq 2$. Assuming $X$ is Cohen-Macaulay, $\dim X \geq 2$, and $\text{codim}_X V \geq 2$, we see that $j^*(M)$ is the restriction of a locally free sheaf if the maximal extension $j_*(j^*(M))$ satisfies either of the conditions (1) or (2) in Corollary 3.3.

**Proof.** The last assertion is evident so we have to prove that $M$ is locally free when $j^*(M)$ is locally free. Let $\mathcal{O}_X^{m_{x, x}} \twoheadrightarrow \mathcal{O}_{X, x}^{\beta_0(M_x)} \to M_x \to 0$ be a presentation at a point $x \in V$, where $m = \beta_1(M_x)$ and $n = \beta_0(M_x)$, and let $F_r(M_x) \subset \mathcal{O}_{X, x}$ be the $r$th Fitting ideal of $M_x$, where $r = \text{rank} M$. According to the Eagon-Northcott bound one gets as in the proof of Theorem 3.1 that $\text{ht}(F_r(M)) = \text{ht}(I_{n-r}) \leq (m - n + r + 1)(r + 1)$. If $\text{codim}_X V > (m - n + r + 1)(r + 1) = (\sup_{x \in V} (\beta_1(M_x) - \beta_0(M_x))) + \text{rank} M + 1 \text{rank} M + 1$, it follows that $F_r(M) = \mathcal{O}_X$. This proves (1). Assuming $\text{p. d.} M_x \leq 1$ it follows that the map $\phi_x$ is injective, hence $\text{rank} M = \beta_0(M_x) - \beta_1(M_x)$ for each point $x$, implying (2).
The relative embedding dimension of $X/S$ is
\[ \text{ed}_{X/S} = \beta_0(\Omega_{X/S}) = \sup \{ \dim_{k_x} k_x \otimes \Omega_{X/S} \mid x \in X \} = \sup \{ \text{embdim}_S k_s X_s \mid s \in S \}, \]
and the “smoothness defect” $\delta_{X/S} = \text{ed}_S X - d_{X/S}$ (this is the regularity defect when $S$ is the spectrum of a perfect field $k$ and $X = \text{Spec} A$ for a local $k$-algebra $A$). We also have the dual notion of defect of tangent space dimension
\[ \eta_{Y/S} = \beta_0(T_{Y/S}) = \sup \{ \dim_{k_y} k_y \otimes \mathcal{O}_{Y,y} T_{Y/S} \mid y \in Y \} - d_{Y/S}. \]
Clearly, $\eta_{Y/S} \geq 0$, and $\eta_{Y/S} = 0$ if and only if $T_{Y/S}$ is locally free and $Y/S$ is generically smooth.

The image of the tangent morphism is denoted $\mathcal{T}_{X/S} = \text{Im}(d \pi : T_{X/S} \rightarrow T_{X/S} \rightarrow Y/S)$. (3.1)

**Theorem 3.5.** Let $\pi : X/S \rightarrow Y/S$ be a generically smooth morphism of integral Noetherian $S$-schemes.

1. $C_\pi \subset B_\pi$, and if $\Omega_{X/S}$ is locally projective, then
   \[ \mathcal{C}_{X/Y} = \text{Ext}^1_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X). \]

   (see also Remark 1.3)

2. (a) If $X/S$ and $Y/S$ are locally of finite type, and $\pi$ is locally of finite type, then:
   \[ \text{codim}_X^+ B^{(i)}_\pi \leq (d_{X/Y} + i + 1)(i + 1 + \chi_2(\mathcal{O}_{X/Y})) \]
   and
   \[ \text{codim}_X^+ B^{(i)}_\pi \leq (d_{X/Y} + i + 1)(d_{X/Y} - \text{rank} \Omega_{X/S} + i + 1 + \beta_0(\mathcal{V}_{X/Y/S} + \beta_1(\Omega_{X/S})) \leq (d_{X/Y} + i + 1)(d_{X/Y} + i + 1 + \beta_0(\mathcal{V}_{X/Y/S} + \beta_1(\Omega_{X/S}) - d_{Y/S}) \leq (d_{X/Y} + i + 1)(\delta_{Y/S} - \beta_0(\mathcal{T}_{X/Y/S}) + \beta_1(\mathcal{V}_{X/Y/S} + \beta_1(\Omega_{X/S}) + i + 1). \] (A)

   The second inequality is an equality when $X/S$ is generically smooth.

(b) $\text{codim}_X^+ C^{(i)}_\pi \leq (d_{X/Y} + i + 1)(i + 1 + \chi_2(\mathcal{C}_{X/Y}))$

   and
   \[ \text{codim}_X^+ C^{(i)}_\pi \leq (i + 1)(\beta_0(\mathcal{T}_{X/S}) + \beta_1(T_{X/S \rightarrow Y/S}) - \text{rank}(T_{X/S \rightarrow Y/S}) + i + 1) \leq (i + 1)(\beta_0(\mathcal{T}_{X/S}) + \beta_1(T_{X/S \rightarrow Y/S}) - d_{Y/S} + i + 1) \leq (i + 1)(\eta_{X/S} - \eta_{Y/S} + \beta_1(\mathcal{T}_{X/S}) + \beta_1(T_{X/S \rightarrow Y/S}) + i + 1). \] (B)

   The second inequality is an equality when $Y/S$ is generically smooth.
(3) Assume that $\Omega_{X/S}$ and $C_{X/Y}$ are coherent. If $\mathcal{T}_{X/S}$ satisfies $(S_2)$, then $\text{codim}^+_X C_\pi \leq 1$. Assume moreover that $X$ is regular in codimension $\leq 1$ and $\Omega_{X/S,x}$ is free when $\text{ht}(x) \leq 1$. Then the maximal points of height $1$ in $C_\pi$ and $B_\pi$ coincide.

(4) Assume that $X/S$ and $Y/S$ are smooth. If $\mathcal{T}_{X/S}$ satisfies $(S_2)$, then $\text{codim}^+_X B_\pi \leq 1$.

(5) Assume $\pi$ is d.c.i., locally of finite type, and generically smooth. Then

$$\text{codim}^+_X B^{(i)}_\pi \leq (d_{X/Y} + i + 1)(i + 1).$$

Remark 3.6. (1) [11] is due to Dolgachev (11) (when $i = 0$). If $\pi$ is generically separably algebraic, then (5) $\Rightarrow$ (4) by Lemma 2.6.

(2) By (3), $\text{codim}^+_X C_\pi \leq 1$ essentially follows when Serre’s $(S_2)$-property holds for the $\mathcal{O}_X$-modules $\mathcal{T}_{X/S}$ and $\mathcal{O}_X$. If $X/Y$ is generically separably algebraic, so $\mathcal{T}_{X/S} = T_{X/S}$, then if $\mathcal{O}_X$ satisfies $(S_2)$ the module $\mathcal{T}_{X/S}$ also satisfies $(S_2)$. We get $\text{codim}^+_X B_\pi \leq 1$ in this case only when $X/S$ and $Y/S$ are smooth, while (5) gives this for any d.c.i.

(3) If $X/S$ is smooth, then $\beta_1(\Omega_{X/S}) = 0$. If $Y/S$ is smooth, then $\beta_1(\mathcal{T}_{X/S-Y/S}) = 0$. It follows from the proof that the last inequalities (A) and (B) can be improved by using higher Betti numbers, but the formulas are perhaps sufficiently long already.

Proof of Theorem 3.5. (1) Combining the dual of the exact sequences (1.2) (1.3), noting that $\Gamma^*_{X/Y/S} = 0$ since $\pi$ is generically smooth, one gets the exact sequence

$$0 \to C_{X/Y} \to \text{Ext}^1_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X) \to \text{Ext}^1_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{O}_X),$$

implying $C_\pi \subset B_\pi$; it also implies the other assertion when $\Omega_{X/S}$ is locally projective (by quasi-coherence).

(2) (a): Apply Theorem 3.1 first to the module $M = \Omega_{X/Y}$, considering the Fitting ideal $F_{i+d_{X/Y}}(\Omega_{X/Y})$, to get the first inequality, then apply it to the exact sequence (1.2) to get the first inequality in (A). The second inequality follows since $-\text{rank}_{\mathcal{O}_{X/S}} d_{X/Y} \leq d_{X/S} + d_{X/Y} = -d_{Y/S}$, and this is an equality when $X/S$ is generically smooth since then $\text{rank}_{\mathcal{O}_{X/Y}} \Omega_{X/Y} = d_{X/Y}$. The last inequality follows after considering the Betti numbers of the members in the exact sequence (1.2), giving $\beta_0(V_{X/Y/S}) \leq \beta_0(\Omega_{X/S}) - \beta_0(\Gamma_{X/Y/S}) + \beta_1(\Omega_{X/Y/S}) = \text{embdim}_S Y - \beta_0(\Gamma_{X/Y/S}) + \beta_1(\Omega_{X/Y/S}) = \delta_{Y/S} + d_{Y/S} - \delta_{Y/S} = \beta_1(V_{X/Y/S})$. (b): Again apply Theorem 3.1 but this time to the short exact sequence one gets from (1.5) (using $\mathcal{T}_{X/S}$). This immediately gives the first inequality. The first inequality in (B) follows since $\text{rank}_{\mathcal{O}_{X/S}} \mathcal{T}_{X/S-Y/S} = s_{Y/S} \geq d_{Y/S}$, and this is an equality when $Y/S$ is generically smooth. To get the last inequality consider the Betti numbers of the members in the exact sequence, giving $\beta_0(\mathcal{T}_{X/S}) \leq \beta_0(\mathcal{T}_{X/S}) - \beta_0(\mathcal{T}_{X/S}) = \eta_{X/S} - d_{X/S} - \eta_{X/Y} - d_{X/Y} + \beta_1(\mathcal{T}_{X/S}) = \eta_{X/S} - \eta_{X/Y} + d_{X/Y} + \beta_1(\mathcal{T}_{X/S})$.

(3) Clearly $\text{codim}^+_X C_\pi \leq 1$ follows if $C_{X/Y}$ has no associated points of height $\geq 2$. Suppose the contrary, that there exists an associated point $x$ of height $\geq 2$, so $k_x \subset C_{X/Y,x}$. Letting $\mathcal{T}_{X/S-Y/S,x}$ be the pre-image of $k_x$ in $\mathcal{T}_{X/S-Y/S,x}$, the short exact sequence

$$0 \to \mathcal{T}_{X/S,x} \to \mathcal{T}_{X/S-Y/S,x} \to C_{X/Y,x} \to 0$$

pops back to

$$0 \to \mathcal{T}_{X/S,x} \to \mathcal{T}_{X/S-Y/S,x} \to k_x \to 0.$$
Since \( T_{X/Y,x} \) satisfies (S2), so \( \text{Ext}^1_{\mathcal{O}_X}(k_x, T_{X/Y,x}) = 0 \), the above sequence is split exact; hence \( T^a_{\mathcal{O}_X, Y} \subset T_{X/Y} \) has non-zero torsion. But \( \mathcal{O}_{X,x} \) is integral, hence \( T_{X/Y} = 0 \). Since \( \mathcal{O}_{X,x} \) is integral, hence \( T_{X/Y} = 0 \) is torsion free, which gives a contradiction.

To see that points of height 1 in \( C_\pi \) and \( B_\pi \) are equal, by (1) it suffices to see that if \( x \) is a maximal point of \( B_\pi \) of height 1, then it belongs to \( C_\pi \). Since \( X/S \) is smooth in codimension \( \leq 1 \) the dual of the exact sequence (1) (or apply (1) again) implies the first equality in

\[
\begin{align*}
C_\pi &= \text{Ext}^1_{\mathcal{O}_X}(X/Y, \mathcal{O}_X) = \text{Ext}^1_{\mathcal{O}_{X,x}}(X/Y, \mathcal{O}_{X,x}) \neq 0; \\
\end{align*}
\]

the second equality follows since \( \Omega_{X/Y} \) is quasi-coherent. The inequality signs follows since \( x \in B_\pi \) implies that \( \Omega_{X/Y,x} \) is not free over the regular ring \( \mathcal{O}_{X,x} \) of global homological dimension 1.

Remark 3.7. Assume in (5) that \( X/Y \) is locally free, then the assertion follows from Proposition (1) and (3). When \( \text{p.d.} \Omega_{X/Y,x} \leq 1 \) then \( \chi_2(\Omega_{X/Y,x}) = 0 \), so the assertion follows from (2).

Theorem 3.8. Let \( \pi : X/S \to Y/S \) be a generically smooth morphism of Noetherian integral \( S \)-schemes such that \( \Omega_{X/S} \) and \( \Omega_{Y/S} \) are coherent.

(1) \[ \text{codim}^+_X B^\pi = (d_{X/Y} + i + 1)(i + \delta_{Y/S} + \chi_2(\Omega_{X/S})). \]

In particular, if \( X/S \) is d.c.i., then

\[ \text{codim}^+_X B^\pi = (d_{X/Y} + i + 1)(i + \delta_{Y/S}). \]

(2) Assume that \( X/S \) and \( Y/S \) are d.c.i.. Then

\[ \text{codim}^+_X B^\pi = (d_{X/Y} + i + 1)(i + \delta_{X/Y}). \]

(3) Assume that \( X/S \) is smooth and each restriction to fibres \( X_s \to Y_s \), \( s \in S \), is generically smooth. Then

\[ \text{codim}^+_X B^\pi \leq \delta_{X/Y} + d_{X/Y}. \]

Proof. (1): We have the exact sequence

\[ \pi^*(\Omega_{Y/S}) \to \Omega_{X/S} \to \Omega_{X/Y} \to 0. \]

Noting that \( \beta_1(\Omega_{X/S,x}) - \beta_2(\Omega_{X/S,x}) = -\chi(\Omega_{X/S,x}) + \chi_2(\Omega_{X/S,x}) = -d_{X/Y} + \chi_2(\Omega_{X/S,x}) \) and \( \beta_0(\pi^*(\Omega_{Y/S})) \leq \delta_{Y/S} + d_{Y/S} \), Theorem 3.1 (3), implies the first assertion. If \( X/S \) is d.c.i. (if we like, by Proposition 2.13 we can then insert 0 to to the left in the exact sequence), then \( \chi_2(\Omega_{X/S}) = 0 \), implying \( \text{codim}^+_X B^\pi \) as the second assertion.

(2): By Theorem 2.14 (5) \( \text{p.d.} \Omega_{X/Y} \leq 2 \), hence \( \chi_2(\Omega_{X/Y,x}) - \beta_2(\Omega_{X/Y,x}) = \chi(\Omega_{X/Y,x}) - \beta_0(\Omega_{X/Y,x}) + \beta_1(\Omega_{X/Y,x}) = -\delta_{X/Y,x} + \beta_1(\Omega_{X/Y,x}) \leq \delta_{X/Y} \), so the assertion follows from Theorem 3.3 (2).

(3): Since \( X/S \) is smooth, each fibre \( X_s \) is regular, and since \( X_s \to Y_s \) is generally smooth, so \( \Omega_{X/Y,s} \) has generic rank \( d_{X/Y,s} \). By Theorem 3.3 (2), with \( M = \Omega_{X/Y,s} \), we get \( \text{codim}_{X_s} B_{X_s/Y_s} \leq \delta_0(\Omega_{X/Y,s}) - 1 = d_{X/Y,s} + \delta_{X/Y,s}. \) Since \( \text{codim}_{X} B^\pi \leq \sup\{\text{codim}_{X_s} B_{X_s/Y_s} \mid s \in S \} \), \( d_{X/Y} = \sup\{d_{X/Y,s} \mid s \in S \} \) (see generalities) and \( \delta_{X/Y} = \sup\{\delta_{X/Y,s} \} \) the assertion follows.
4 Purity when $d_{X/Y} = 0$

Theorem 3.8 and [4] (5) in Theorem 3.9 contain sufficient conditions to imply $\text{codim}_X^+ B_\pi \leq 1$ when the relative dimension $d_{X/Y} = 0$. These results rely mainly on establishing that $\pi$ be d.c.i., while Lemma 2.6 and Proposition 2.7 give sufficient, but rather restrictive conditions ensuring this. We aim for more precise results when $d_{X/Y} = 0$, which is possible since maximal associated points of $\Omega_{X/Y}$ of high height cannot occur if $\Omega_{X/S}$ is void of associated points of high height, and if moreover $\mathcal{V}_{X/Y/S}$ has depth $\geq 2$ at such points. On the other hand, it is quite difficult to find upper bounds on the height of the associated points of $\Omega_{X/S}$. For instance, the rather natural assumption that $X$ satisfies $(S_2)$ and $(R_1)$ does not imply that $\Omega_{X/S}$ satisfies $(S_1)$, and hence that $\Omega_{X/S}$ is torsion free since $X$ is integral. For example, the (non-zero) torsion of $\Omega_{X/k}$ was computed in [1] when $X/k$ is a singular toric variety. On the other hand, by the Auslander-Buchsbaum formula, a point $x$ cannot be associated if the projective dimension satisfies $p. d. \mathcal{O}_{X,x} \Omega_{X/S,x} < \text{depth} \mathcal{O}_{X,x}$.

Purity for finite morphisms $\pi$ is well-studied. It started with Zariski [33] who proved that $\text{codim}_X^+ B_\pi \leq 1$ when $X/k$ and $Y/k$ are of finite type over a perfect field $k$, $X$ is normal, and $Y$ regular. Nagata [30] proved $\text{codim}_X^+ B_\pi \leq 1$ when $Y$ is regular, $X$ is normal. Auslander [4] gave a module theoretic proof of Nagata’s result. Grothendieck states purity in the following way [13]: A local Noetherian ring $(A, m)$ is pure if the restriction map of étale covers $Et(\text{Spec } A) \to Et(\pi_\ast \text{Spec } A \setminus m)$ is an equivalence of categories, and a scheme $Y$ is pure at a point $y$ if $(\mathcal{O}_{Y,y}, m_y)$ is pure. Grothendieck proves, using Lefschetz conditions on $\mathcal{O}_Y$-modules for $Y = \text{Spec } A$, that if $A$ is a Noetherian complete intersection of dimension $\geq 3$, then $A$ is pure (regular Noetherian rings of dimension $\geq 2$ are pure). It would be interesting to find a proof of Grothendieck’s theorem such that étale covers of $\text{Spec } A \setminus m$ can be constructively extended to an étale cover of $\text{Spec } A$ (as is the case when $A$ is regular of dimension 2). According to Theorem 3.8 this is equivalent to finding an extension $X \to \text{Spec } A$ of an étale morphism $X_0 \to \text{Spec } A \setminus \{m\}$ such that $X$ is d.c.i.

Grothendieck’s theorem implies the following result.

**Proposition 4.1.** Let $\pi : X \to Y$ be a finite morphism of normal Noetherian integral schemes, where $Y$ is a local complete intersection. Then

$$\text{codim}_X^+ B_\pi \leq 2.$$  

A somewhat more involved proof of this assertion, assuming $Y$ is excellent, is due to Cutkosky [10] Th. 5).

**Proof.** By normality, $\pi$ is formally étale if and only if it is formally unramified, i.e. $\Omega_{X/Y} = 0$. If $x$ is a maximal associated point of $\Omega_{X/Y}$ we argue that $\text{ht}(x) \leq 2$. Suppose $\text{ht}(x) \geq 3$. Points $x_1 \in \text{Spec } \mathcal{O}_{X,x} \setminus m_x$ specialize to the closed point $x$, and since $x$ is a maximal associated point it follows that $\text{Spec } \mathcal{O}_{X,x} \setminus \{m_x\}$ contains no associated point for $\Omega_{X/Y}$. Therefore $\Omega_{X/Y,x_1} = 0$ when $x_1 \in \text{Spec } \mathcal{O}_{X,x} \setminus \{m_x\}$, i.e. the morphism $\text{Spec } \mathcal{O}_{X,x} \setminus \{m_x\} \to \text{Spec } \mathcal{O}_{Y,\pi(x)} \setminus \{m_{\pi(x)}\}$ is étale. By Grothendieck purity described above it follows that $\Omega_{X/Y,x} = 0$, contradicting the assumption that $x$ is an associated point.

Let $D_\pi = \pi(B_\pi)$ be the discriminant set of a finite and generically separable morphism $\pi : X \to Y$, so $\text{codim}_Y^+ D_\pi = \text{codim}_X^+ B_\pi$. Assume $Y$ is a closed subscheme of a regular scheme $Y_r/k$ over a perfect field $k$. Perhaps a starting point for Faltings in [13] Th. 2] was that an étale covering of $Y \setminus D_\pi$ extends to an étale covering of a formal neighbourhood $U$ of $Y \setminus D_\pi$ in $U = Y_r \setminus D_\pi$ [13] Exp. X, Prop. 1.1]. He shows that such an étale covering, given by a coherent $\mathcal{O}_U$-algebra, actually comes
by the completion of a coherent $\mathcal{O}_Y$-algebra $\mathcal{A}$ (possibly ramified) when $\text{codim}_Y D_\pi$ is sufficiently high, and notices that $\mathcal{A}$ is normal, so by Zariski-Nagata-Auslander purity it is actually unramified. The allowed size of $D_\pi$ is $\text{codim}_Y D_\pi \geq \delta_Y/k + 2$ expressed in \[15, \text{Th. 2}\], where the regularity defect is $\delta_{Y/k} = \sup\{\dim_k m_y/m^2_y - \text{ht}(y) \mid y \in D_\pi\}$. This readily implies that if each maximal point $x$ in $B_\pi$ satisfies $\text{ht}(x) \geq \delta_{Y/k} + 2$, then actually $B_\pi = \emptyset$. Therefore

$$\text{codim}_X B_\pi \leq \delta_{Y/k} + 1.$$  \hfill (F-C)

This was observed by Cutkosky \[10, \text{Th. 6}\]. Theorem 3.8, (1), generalises this inequality to positive relative dimensions $d_{X/Y} \geq 0$, making the added assumption that $X/S$ is d.c.i., or (perhaps more generally) $\chi_2(\Omega_{X/S}) = 0$. Compare to Theorem 3.8, (3) which gives $\text{codim}_X B_\pi \leq \delta_{X/Y}$ when $X/k$ is smooth. We can make another comparison to Theorems 3.8 and 3.5, putting $d_{X/Y} = 0$, which improve (F-C) when either

$$\delta_{Y/k} > \delta_{Y/S} + \chi_2(\Omega_{X/S}), \quad \text{or} \quad \delta_{Y/k} > \delta_{Y/S} - \beta_0(\Gamma_{X/Y/S}) + \beta_1(\Omega_{X/Y/S}) + \beta_1(\Omega_{X/S}).$$

The latter inequality holds for example when $Y/k$ is non-smooth (i.e. $Y$ is non-regular) so $\delta_{Y/k} > 0$, while assuming $X/S$ is smooth, $\Gamma_{X/Y/S} = 0$, and $\delta_{Y/S} + \beta_1(\Omega_{X/S}) < \delta_{Y/k}$ ("$Y/S$ is more smooth than $Y/k$"). If $X/S$ and $Y/S$ are smooth, and $X/Y$ is generically smooth, one gets 0 in the right-hand side of the second inequality, but a bound in this situation, however, also follows from (5) in Theorem 3.5 and Lemma 2.6.

The purity results in \[4, 10, 15, 18, 23, 30, 33\] apply to finite morphisms, while Theorems 3.5 and 3.8 apply to a much larger class of morphisms, although the conclusion is weaker than in the purity results of Grothendieck-Cutkosky (and Zariski-Nagata-Auslander-Faltings) in the finite case when the morphism $X \to Y$ cannot be fibred over $S$ as stated ($X/S$ is required to be d.c.i. to rule out the existence of associated points of high height for $\Omega_{X/S}$). On the other hand, van der Waerden’s theorem \[20, \text{Th. 21.12.12}\] states that $\text{codim}_X B_\pi \leq 1$ when $\pi : X \to Y$ is locally of finite type and \textit{birational}, and $Y$ moreover satisfies the following condition (see also \[7\]):

\textbf{(W)} If $T$ is an irreducible closed subset with $\text{codim}_T T \leq 1$, then the inclusion morphism $Y \setminus T \to Y$ is affine.

The condition (W) is satisfied in particular when $Y$ is normal and its local divisor class groups are torsion, i.e. $Y$ is $\mathbb{Q}$-factorial.

It is convenient to single out the following class (F) of morphisms:

\textbf{(F)} $\pi : X/S \to Y/S$ is dominant, generically separably algebraic, and locally finite type, and the schemes $X, Y$ and $S$ are integral and Noetherian.

\textbf{Remark 4.2.} The remark \[20, \text{Rem. 21.12.14,(v)}\] contains a discussion of the possibility to combine van der Waerden purity with Zariski-Nagata-Auslander purity. Let $\pi$ be a morphism of the type (F), where $Y$ is pure in the sense of Grothendieck, e.g. regular. Letting $g : Y' \to Y$ be the integral closure of $Y$ in the fraction field of $X$ we get a factorisation $\pi = g \circ h$

$$X \xrightarrow{\pi} Y' \xrightarrow{g} Y$$

where $h$ is birational, and since $Y$ is normal, $g$ is finite. We have branch loci

$$B_\pi \subset X, \quad B_g \subset Y', \quad B_h \subset X,$$
and would like to know when $\text{codim}_X^{+} B_\pi \leq 1$. Clearly, $B_\pi \subset B_h \cup h^{-1}(B_g)$. Since $Y$ is pure we have $\text{codim}_X^{+} B_\pi \leq 1$, so $\text{codim}_Y^{+} B_\pi \leq 1$ would imply $\text{codim}_X^{+} B_\pi \leq 1$. To get $\text{codim}_X^{+} B_\pi \leq 1$ from the argument in the proof of \cite{20} Th. 21.12.12 one needs to see that the inclusion $Y' \setminus B_g \to Y'$ is affine (e.g. that $Y'$ satisfies (W)). This however need not be the case (it seems to be asked in \cite{20} Rem. 21.12.14. (v)) whether $Y' \setminus B_g$ is always affine in this situation.

**Example 4.3.** Let $\pi : X = \mathbb{A}^3_k \to Y = \mathbb{A}^3_k$ be defined by $(x_1, x_2, x_3) \mapsto (x_2 x_3 - x_1, x_2, x_1 x_3)$. We have $B_\pi = C_\pi = V(x_1 + x_2 x_3)$, so $\text{codim}_X^{+} B_\pi = 1$. The integral closure of $Y$ in the fraction field of $X$ is $Y' = V(x_1 x_2 - x_3 x_4) \subset \mathbb{A}^4_k$ (one can check that $Y'$ satisfies (S2) and (R1), hence it is normal). We have thus a factorisation $\pi = g \circ h$ where $h : \mathbb{A}^3_k \to Y'$ is defined by $(y_1, y_2, y_3) \mapsto (y_1, y_2 y_3, y_2, y_3)$ (birational) \cite{28} III.9, Ex O], and $g : Y' \to \mathbb{A}^3_k$ is defined by $(x_2, x_3, x_4) \mapsto (x_2 - x_1, x_3, x_4)$. Then $B_h = V((y_1, y_2))$, so $\text{codim}^{+}_{\mathbb{A}^3_k} B_h = 2$, $h(B_h) = (0, 0, 0, 0)$ and

$$B_g = V((x_1 - x_2, x_1 x_3 - x_3 x_4)) = V((x_1, x_2)).$$

Then $\text{codim}_X^{+} B_g = 1$, but the set $Y' \setminus B_g$ is not affine \cite{6}. The divisor class group $\text{Cl}(Y') = \mathbb{Z}$, with generator the class of $(x_1, x_2)$. \cite{9} see also \cite{10} Ch III, Prop 14.8. The fact that $\text{codim}^{+}_{\mathbb{A}^3_k} B_h = 2$ does not contradict $\text{codim}_X^{+} B_\pi \leq 1$, since $B_h \subset h^{-1}(B_g) = B_\pi$. Note also that $C_h = \emptyset$, in agreement with \cite{3} and \cite{4} in Theorem 5.5.

We will generalize in three theorems van der Waerden’s theorem to certain morphisms in the class (F). Theorem 4.4 sheds more light on Remark 4.2 by showing what is needed to make the finite and birational purity theorems work together. Morally, one needs to know that $X$ is (close to) UFD or that it be d.c.i., so in the other results these conditions are replaced by depth conditions, and we then make no reference to van der Waerden’s theorem. Theorem 4.5 can be thought of as a characteristic 0 relative version of Zariski-Nagata-Auslander purity (smooth $Y/S$), generalised to morphisms $\pi$ that need not be finite, but induce algebraic extensions of function fields. Theorem 4.6 applies to certain non-smooth bases $Y/S$ also in positive characteristic, at the price of a higher depth assumption on the source $X$.

**Theorem 4.4.** Let $\pi : X/S \to Y/S$ be a morphism of the type (F). Assume that $Y$ is pure at all points of height $\geq 2$, and that $X$ and $Y$ satisfies the condition (W). Then $\text{codim}_X^{+} B_\pi \leq 1$.

**Proof.** Suppose $x$ is a maximal point in $B_\pi$ of height $\geq 2$ in $X$, and put $y = \pi(x)$. We then have $\text{ht}(y) \geq 2$ by the dimension inequality. Let $X(y)^0 = \pi^{-1}(\text{Spec } \mathcal{O}_Y \setminus \{m_y\})$ and $X(y) = \pi^{-1}(\text{Spec } \mathcal{O}_Y)$. Since $Y$ is pure at $y$ the restriction of $\pi$ to the morphism $\pi(y) : X(y)^0 \to \text{Spec } \mathcal{O}_Y$ can be extended to an etale morphism $E(y) \to \text{Spec } \mathcal{O}_Y$. Then clearly $E(y)$ is birational to $X(y)$. The graph of the birational correspondence gives birational maps $p_1 : Z \to E(y)$ and $p_2 : Z \to X(y)$ where $\text{codim}_{E(y)} B_{p_1} \geq 2$ and $\text{codim}_{X(y)} B_{p_2} \geq 2$. Since $E(y)$ is etale over $\text{Spec } \mathcal{O}_y$ and $Y$ satisfies (W), it follows that $E(y)$ satisfies (W), hence by van der Waerden’s theorem $p_1$ is an isomorphism. By assumption $X$ satisfies (W), hence again by van der Waerden’s theorem $p_2$ is an isomorphism. This implies that $\pi$ is smooth at $x$, in contradiction to the assumption. Therefore there exist no maximal points in $B_\pi$ of height $\geq 2$. \hfill \Box

**Theorem 4.5.** Let $\pi : X/S \to Y/S$ be of the type (F), where moreover $Y/S$ is smooth, $X$ satisfies (S2), and $S$ is defined over the rational numbers. Then $\text{codim}_X^{+} B_\pi \leq 1$.

**Proof.** If on the contrary there exists a maximal point of height $\geq 2$, after localisation one may assume that $\text{ht}(x) \geq 2$ for each maximal point $x$ in $B_\pi = \text{supp } \Omega_{X/Y}$. Since $\Omega_{Y/S}$ is locally free
the exact sequence (1.1) can be complemented with $0 \to \Gamma$ if one moreover somehow can prove that the conditions in (3) hold, e.g. either that $\Gamma$ is torsion, so $\Omega_{X/S,x}$ is locally free, or that the canonical map $\delta$ if one knows $\delta$ is injective, for example by proving $I/I^2$ is injective in another way. Hence we have the exact sequence (1.1) is locally split exact; hence the quotient $\Omega_{X/S}/\Omega_{X/S}^t$ by the torsion sub-module is locally free. It follows that $T_{X/S}$ is locally free (of finite rank) and that the canonical map $\Omega_{X/S} \to T_{X/S}^*$ is surjective (see e.g. [25, Lem. 2]). Therefore, by [29] (see [26, §3, p. 880]), for any point $x$ in $X$ there exist elements $\{f_1, f_2, \ldots, f_r\}$ in $\mathcal{O}_{x,S}$ and derivations $\partial_1, \partial_2, \ldots, \partial_r$ ($r = \text{rank } \Omega_{X/S,x}/\Omega_{X,S,x}^t$) such that $\det(\partial_i(f_j))$ is invertible. Imitating the proof of the Zariski-Lipman-Nagata regularity criterion; see [26, Th. 30.1 and its Corollary], noting that the completion of the local ring $\mathcal{O}_{X,S}$ along the ideal $(f_1, f_2, \ldots, f_r)$ is reduced since $X/S$ is locally of finite type and integral, it follows that $\Omega_{X,S,x}$ is free, so $\Omega_{X/S,x}^t = 0$; hence $\Omega_{X/S,x} = 0$, contradicting the assumption that $x \in B_{\pi}$. Therefore $B_{\pi}$ contains no maximal points of height $\geq 2$.

**Theorem 4.6.** Let $\pi : X/S \to Y/S$ be a morphism of the type (F). Make also the assumptions:

1. $X$ satisfies (S), $X/S$ is d.c.i., and $\text{codim}_X B_{X/S} \geq 2$.
2. $\Omega_{Y/S,y}$ is free for $y \in D_{\pi}$ such that $\text{ht}(x) \leq 2$ when $y = \pi(x)$.
3. p.d. $\mathcal{V}_{Y/S,x} \leq 1$ for each point $x$.

Then $\text{codim}_X \pi B_{\pi} \leq 1$.

**Remark 4.7.** Note that $\Gamma_{X/Y/S} = 0$ when $X/Y$ is locally of finite type and d.c.i. (Prop. 2.13), but this case is superseded by Theorem 3.5, 6. Hence Theorem 4.6 is useful when one instead of knowing $X/Y$ be d.c.i., which may be hard to get when $Y/S$ is not d.c.i. (see Theorem 3.8). If one moreover somehow can prove that the conditions in (3) hold, e.g. either that $\Gamma_{X/Y/S}$ be locally free, or that the canonical map $\delta : \Gamma_{X/Y/S} \to \Gamma_{X/Y}$ be injective, for example by proving $I/I^2 \to \Gamma_{X/Y}$ is injective; see the proof of Proposition 2.13.

Note that since $\pi$ is generically smooth, $\Gamma_{X/Y/S}$ is torsion, so $\mathcal{V}_{X/S,y} = T_{X/S} S$. 

Proof of Theorem 4.6. Assume that $\text{codim}_X \pi B_{\pi} \geq 2$, so after localisation we may assume that each maximal point $x$ if $B_{\pi}$ has height $\text{ht}(x) \geq 2$. Since $\pi$ is generically separable $\text{Hom}_{\mathcal{O}_X}(\mathcal{C}_{X/Y}, \mathcal{O}_X) = 0$, we have from 1.3 the commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{Hom}_{\mathcal{O}_X}(T_{X/S} S, \mathcal{O}_X) & \longrightarrow & \text{Hom}_{\mathcal{O}_X}(T_{X/S}, \mathcal{O}_X) & \longrightarrow & \Lambda_{X/Y} & \longrightarrow & 0 \\
& & \alpha & \downarrow & \beta & \downarrow & \gamma & & \\
0 & \longrightarrow & \text{Hom}_{\mathcal{O}_X}(T_{X/S}, \mathcal{O}_X) & \longrightarrow & \text{Hom}_{\mathcal{O}_X}(T_{X/S}, \mathcal{O}_X) & \longrightarrow & \text{Hom}_{\mathcal{O}_X}(T_{X/Y}, \mathcal{O}_X) \\
\end{array}
$$

where $T_{X/S} = \text{Im}(dx)$ and $\Lambda_{X/Y}$ is defined by the diagram. Here $\beta$ is the identity and by the snake lemma $\text{Ker}(\gamma) = \text{Coker}(\alpha) = \text{Ext}^1_{\mathcal{O}_X}(\mathcal{C}_{X/Y}, \mathcal{O}_X)$. Hence we have the exact sequence

$$
0 \to \text{Ext}^1_{\mathcal{O}_X}(\mathcal{C}_{X/Y}, \mathcal{O}_X) \to \Lambda_{X/Y} \to \text{Hom}_{\mathcal{O}_X}(T_{X/S}, \mathcal{O}_X).
$$

As a maximal point $x$ of $C_{\pi}$ is maximal also in $B_{\pi}$ (Th. 2.13), we have $\text{ht}(x) \geq 2$, hence $\text{Ext}^1_{\mathcal{O}_X}(\mathcal{C}_{X/Y}, \mathcal{O}_X) = 0$, since $X$ satisfies (S). It is now straightforward to see that we get the
following commutative diagram (see Remark 15):

\[
\begin{array}{ccc}
\text{Ext}^2_{\mathcal{O}_X}(D(V_{X/Y}/S), \mathcal{O}_X) & \rightarrow & \text{Ext}^2_{\mathcal{O}_X}(D(\Omega_{X/S}), \mathcal{O}_X) \\
\rightarrow & & \rightarrow \\
\text{Hom}_{\mathcal{O}_X}(T_{X/S\rightarrow Y/S}, \mathcal{O}_X) & \rightarrow & \text{Hom}_{\mathcal{O}_X}(T_{X/S}, \mathcal{O}_X) \\
\rightarrow & & \rightarrow \\
\text{Hom}_{\mathcal{O}_X}(T_{X/Y}, \mathcal{O}_X) & \rightarrow & 0 \\
\text{Ext}^1_{\mathcal{O}_X}(D(V_{X/Y}/S), \mathcal{O}_X) & \rightarrow & \text{Ext}^1_{\mathcal{O}_X}(D(\Omega_{X/S}), \mathcal{O}_X) \\
\rightarrow & & \rightarrow \\
\text{Ext}^1_{\mathcal{O}_X}(D(\Omega_{X/Y}) \rightarrow \mathcal{O}_X). & & \\
\end{array}
\]

Since \( \pi \) is generically separably algebraic, so \( h = 0 \), by the serpent lemma one has the exact sequence

\[
0 \rightarrow \text{Ext}^1_{\mathcal{O}_X}(D(V_{X/Y}/S), \mathcal{O}_X) \xrightarrow{a} \text{Ext}^1_{\mathcal{O}_X}(D(\Omega_{X/S}), \mathcal{O}_X) \rightarrow \Omega_{X/Y} \rightarrow
\]

\[
\rightarrow \text{Ext}^2_{\mathcal{O}_X}(D(V_{X/Y}/S), \mathcal{O}_X) \xrightarrow{b} \text{Ext}^2_{\mathcal{O}_X}(D(\Omega_{X/S}), \mathcal{O}_X),
\]

whence \( B_x = \supp \text{Coker } a \cup \supp \text{Ker } b. \) By \textbf{[3]}, \( D(V_{X/Y}/S) \) is projectively equivalent to \( \text{Ext}^1_{\mathcal{O}_X}(V_{X/Y}, \mathcal{O}_X) \), and since \( \Omega_{Y/Sx} \) is free at points \( y \) when \( y = \pi(x) \) and \( ht(x) \leq 2 \), and therefore \( V_{X/Y,Sx} \) is free when \( ht(x) \leq 2 \). Therefore \( \text{codim}_X \supp D(V_{X/Y}/S) \geq 3 \); hence

\[
\text{Ext}^2_{\mathcal{O}_X}(D(V_{X/Y}/S), \mathcal{O}_X) = 0
\]

since \( X \) satisfies \( (S_3) \); hence \( \text{Ker } b = 0 \). Similarly, since \( X/S \) is d.c.i. and \( \Omega_{X/S,x} \) is free at points \( x \) with \( ht(x) \leq 1 \), we get

\[
\text{Ext}^1_{\mathcal{O}_X}(D(\Omega_{X/S}), \mathcal{O}_X) = \text{Ext}^1_{\mathcal{O}_X}(\text{Ext}^1_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{O}_X), \mathcal{O}_X) = 0;
\]

hence \( \text{Coker } a = 0 \). Therefore \( \Omega_{X/Y} = 0 \).

If \( \Gamma_{X/Y}/S = 0 \), the last assertion in \textbf{[3]} in Theorem \textbf{4.3} follows from Lemma \textbf{2.10} (If \( B_Y/S = 0 \), it is evident that \( \Gamma_{X/Y}/S = 0 \).) \( \square \)

**Remark 4.8.** If \( X/Y \) is a d.c.i., \( X \) satisfies \( (S_3) \), and each maximal point of \( B_x \) is of height \( \geq 2 \), then \( \Omega_{X/Y} \) is reflexive (it is actually locally free (Th. \textbf{3.5}[3])). This follows since \( D(\Omega_{X/Y}) \) is locally projectively equivalent to \( \text{Ext}^1_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X) \), implying \( \text{codim}_X^2 D(\Omega_{X/Y}) \geq 3 \) since \( \Omega_{X/Y} \) is free at points of height \( \leq 2 \); hence \( \text{Ext}^2_{\mathcal{O}_X}(D(\Omega_{X/Y}), \mathcal{O}_X) = \text{Ext}^2_{\mathcal{O}_X}(D(\Omega_{X/Y}), \mathcal{O}_X) = 0 \).

We state an easily workable excerpt of the above results that may prove useful to get that a morphism is étale.

**Corollary 4.9.** Let \( \pi : X/S \rightarrow Y/S \) be a morphism of integral Nœtherian \( S \)-schemes which is locally of finite type generically separably algebraic and dominant. Assume moreover that \( Y \) is normal, and that at points of height \( \leq 1 \), \( X \) is regular and \( X/S \) is smooth. The following are equivalent:

1. \( \pi \) is submersive.
(2) \( \pi \) is submersive at points of height 1.

(3) there exists an exceptional set \( E \subset X \) of codimension \( \geq 2 \) such that the restriction \( X \setminus E \to Y \setminus \pi(E) \) is formally étale.

Make one of the following additional assumptions:

(a) \( \pi \) is birational and locally of finite type, \( X \) and \( Y \) are normal, and \( Y \) satisfies the condition (W).

(b) \( \pi \) is quasi-finite, \( X \) is normal and \( Y \) is regular.

(c) The assumptions in Theorem 3.5 or Theorem 4.6 hold.

(d) \( \pi \) is a d.c.i. (see e.g. Lemma 2.6 and 2.7).

(e) \( X/S \) and \( Y/S \) are d.c.i.

Then it follows that (1 – 3) are equivalent to:

(4) \( \pi \) is étale.

**Remark 4.10.** It is not sufficient to know \( \text{codim}_X^+ B_\pi \leq 2 \) to get (4) from (1) as seen from the following well-known example: If \( \pi : X = \text{Spec} k[s, t] \to Y = \text{Spec} k[s^2, st, t^2] \), then \( B_\pi = \{(s, t)\} \) and \( C_\pi = \emptyset \). Here \( Y/k \) is d.c.i. and \( X/k \) is smooth, but \( \pi \) is not d.c.i. Assume that all morphisms \( X/S, Y/S, \) and \( X/Y \) are locally of finite type and assume that \( \pi : X/S \to Y/S \) is generically smooth. Let \( B_{X/S} \) be the branch locus of \( X/S \), so \( B_{X/S} \subset X^s \), where \( X^s \) is the locus of points where \( X/S \) fails to be smooth, and \( X^s = B_{X/S} \) when \( X/S \) is flat [20, §17]. It is easy to see that if \( Y/S \) is smooth, then \( B_{X/S} \subset B_\pi \); but \( B_{X/S} \) need not be contained in \( C_\pi \), by the above example.

**Proof.** (1) \( \Leftrightarrow \) (2) follows from (3) in Theorem 3.5, (3) \( \Rightarrow \) (4) is clear. (2) \( \Rightarrow \) (3) \( \Rightarrow \) (1). If \( x \in X \) is a point of height 1, then \( \Omega_{X/Y, x} = 0 \) by (3) in Theorem 3.5 since \( Y \) is normal it follows that \( \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \) is étale when \( \text{ht}(x) \leq 1 \) [20, Cor. 18.10.3]. (4) \( \Rightarrow \) (3) is evident, (1) \( \Rightarrow \) (4): Since \( Y \) is normal, \( \pi \) is étale if and only if \( \Omega_{X/Y} = 0 \). By (1) \( C_\pi = \emptyset \), hence by Theorem 3.5 (3), \( B_\pi \) contains no points of height 1. Therefore \( \pi \) is étale if \( \text{codim}_X^+ B_\pi \leq 1 \). (a) this follows from van der Waerden’s theorem, (b) follows from the Zariski-Nagata-Auslander theorem, (d) follows from (5) in Theorem 3.5 (e) follows from Theorem 3.5 .

Stay with the case \( d_{X/Y} = 0 \). It is natural to fix the general type of morphism as in (F) but one can make different assumptions in the source \( X \). Let \( C_S \) be a class of integral and Noetherian schemes over \( S \).

**Definition 4.11.** A Noetherian scheme \( Y/S \) is strongly (weakly) \( C_S \)-pure if for any morphism of the type (F) we have \( \text{codim}_X^+ B_\pi \leq 1 \) (\( \text{codim}_X^+ C_\pi \leq 1 \)) when \( X/S \) belongs to \( C_S \).

Let \( C_S^2 \) be the category of integral schemes \( X/S \) locally of finite type satisfying \( S_2 \), and \( C_S^1 \) the category of smooth schemes. For example, Theorem 3.5 implies that all Noetherian integral schemes \( Y/S \) are weakly \( C_S^2 \)-pure, and that smooth schemes are strongly \( C_S^1 \)-pure.

It should not be a big surprise that \( \text{codim}_X^+ C_\pi \leq 1 \) holds under weaker conditions than those needed to get \( \text{codim}_X^+ B_\pi \leq 1 \). For example, Griffith [17] has examples where \( B_\pi \) does not satisfy \( \text{codim}_X^+ B_\pi \leq 1 \) for finite morphisms \( \pi : X/S \to Y/S \) of the type (F), when \( X \) is normal and \( Y \) Gorenstein, but we know that \( Y/S \) is weakly \( C_S^2 \)-pure, so \( \text{codim}_X^+ C_\pi \leq 1 \).

**Department of Mathematics, University of Gävle, S-801 76, Sweden**

**E-mail address:** rkm@hig.se

23
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