On Manin’s conjecture for singular del Pezzo surfaces of degree four, I

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Abstract

This paper contains a proof of the Manin conjecture for the singular del Pezzo surface

\[ X : x_0 x_1 - x_2 - x_0 x_4 - x_1 x_2 + x_3^2 = 0, \]

of degree four. In fact, if \( U \subset X \) is the open subset formed by deleting the unique line from \( X \), and \( H \) is the usual height function on \( \mathbb{P}^4(\mathbb{Q}) \), then the height zeta function \( \sum_{x \in U(\mathbb{Q})} H(x)^{-s} \) is analytically continued to the half-plane \( \Re(s) > 9/10 \).

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1 Introduction

Let \( Q_1, Q_2 \in \mathbb{Z}[x_0, \ldots, x_4] \) be a pair of quadratic forms whose common zero locus defines a geometrically integral surface \( X \subset \mathbb{P}^4 \). Then \( X \) is a del Pezzo surface of degree four. We assume henceforth that the set \( X(\mathbb{Q}) = X \cap \mathbb{P}^4(\mathbb{Q}) \) of rational points on \( X \) is non-empty, so that in particular \( X(\mathbb{Q}) \) is dense in \( X \) under the Zariski topology. Given a point \( x = [x_0, \ldots, x_4] \in \mathbb{P}^4(\mathbb{Q}) \), with \( x_0, \ldots, x_4 \in \mathbb{Z} \) such that \( \gcd(x_0, \ldots, x_4) = 1 \), we let \( H(x) = \max_{0 \leq i \leq 4} |x_i| \). Then \( H : \mathbb{P}^4(\mathbb{Q}) \to \mathbb{R}_{\geq 0} \) is the height attached to the anticanonical divisor \(-K_X\) on \( X \), metrized by the choice of norm \( \max_{0 \leq i \leq 4} |x_i| \). A finer notion of density is provided by analysing the asymptotic behaviour of the quantity

\[ N_{U,H}(B) = \# \{ x \in U(\mathbb{Q}) : H(x) \leq B \}, \]

as \( B \to \infty \), for appropriate open subsets \( U \subseteq X \). Since every quartic del Pezzo surface \( X \) contains a line, it is natural to estimate \( N_{U,H}(B) \) for the open subset \( U \) obtained by deleting the lines from \( X \). The motivation behind this paper is to consider the asymptotic behaviour of \( N_{U,H}(B) \) for singular del Pezzo surfaces of degree four.

A classification of quartic del Pezzo surfaces \( X \subset \mathbb{P}^4 \) can be found in the work of Hodge and Pedoe [8, Book IV, §XIII.11], showing in particular that there are only finitely many isomorphism classes to consider. Let \( \tilde{X} \) denote the minimal desingularisation of \( X \), and let \( \text{Pic} \tilde{X} \) be the Picard group of \( \tilde{X} \). Then Manin has stated a very general conjecture [6] that predicts the asymptotic
behaviour of counting functions associated to suitable Fano varieties. In our setting it leads us to expect the existence of a positive constant \( c_{X,H} \) such that

\[
N_{U,H}(B) = c_{X,H} B (\log B)^{\rho - 1} (1 + o(1)),
\]

as \( B \to \infty \), where \( \rho \) denotes the rank of \( \text{Pic} \bar{X} \). The constant \( c_{X,H} \) has received a conjectural interpretation at the hands of Peyre [9], which in turn has been generalised to cover certain other cases by Batyrev and Tschinkel [2] and Salberger [11]. A brief discussion of \( c_{X,H} \) will take place in §2.

There has been rather little progress towards the Manin conjecture for del Pezzo surfaces of degree four. The main successes in this direction are to be found in work of Batyrev and Tschinkel [1], covering the case of toric varieties, and in the work of Chambert-Loir and Tschinkel [5], covering the case of equivariant compactifications of vector groups. It is our intention to investigate the distribution of rational points in the special case that \( X \) is defined by the pair of equations

\[
x_0x_1 - x_2^2 = 0, \\
x_0x_4 - x_1x_2 + x_3^2 = 0.
\]

Then \( X \subset \mathbb{P}^4 \) is a del Pezzo surface of degree four, with a unique singular point \([0,0,0,0,1]\) which is of type \( D_5 \). Furthermore, \( X \) contains precisely one line \( x_0 = x_2 = x_3 = 0 \). It turns out that \( X \) is an equivariant compactification of \( G_2^a \), so that the work of Chambert-Loir and Tschinkel [5, Theorem 0.1] ensures that the asymptotic formula (1.1) holds when \( U \subset X \) is taken to be the open subset formed by deleting the unique line from \( X \). Nonetheless there are several reasons why this problem is still worthy of attention. Firstly, in making an exhaustive study of \( X \) it is hoped that a template will be set down for the treatment of other singular del Pezzo surfaces. In fact no explicit use is made of the fact that \( X \) is an equivariant compactification of \( G_2^a \), and the techniques that we develop in this paper have already been applied to other surfaces [3, 4]. Secondly, in addition to improving upon Chambert-Loir and Tschinkel’s asymptotic formula for \( N_{U,H}(B) \), the results that we obtain lend themselves more readily as a bench-test for future refinements of the Manin conjecture, such as that recently proposed by Swinnerton-Dyer [12] for example.

Let \( X \subset \mathbb{P}^4 \) be the \( D_5 \) del Pezzo surface defined above, and let \( U \subset X \) be the corresponding open subset. Our first result concerns the height zeta function

\[
Z_{U,H}(s) = \sum_{x \in U(\mathbb{Q})} \frac{1}{H(x)^s},
\]

that is defined when \( \Re(s) \) is sufficiently large. The analytic properties of \( Z_{U,H}(s) \) are intimately related to the asymptotic behaviour of the counting function \( N_{U,H}(B) \). For \( \Re(s) > 0 \) we define the functions

\[
E_1(s + 1) = \zeta(6s + 1)\zeta(5s + 1)\zeta(4s + 1)^2\zeta(3s + 1)\zeta(2s + 1),
\]

\[
E_2(s + 1) = \frac{\zeta(14s + 3)\zeta(13s + 3)^3}{\zeta(10s + 2)\zeta(9s + 2)\zeta(8s + 2)^3\zeta(7s + 2)^3\zeta(19s + 4)}.
\]

It is easily seen that \( E_1(s) \) has a meromorphic continuation to the entire complex plane with a single pole at \( s = 1 \). Similarly it is clear that \( E_2(s) \) is holomorphic
and bounded on the half-plane $\Re(s) \geq 9/10 + \varepsilon$, for any $\varepsilon > 0$. We are now ready to state our main result.

**Theorem 1.** Let $\varepsilon > 0$. Then there exists a constant $\beta \in \mathbb{R}$, and functions $G_1(s), G_2(s)$ that are holomorphic on the half-plane $\Re(s) \geq 5/6 + \varepsilon$, such that for $\Re(s) > 1$ we have

$$Z_{U,H}(s) = E_1(s)E_2(s)G_1(s) \frac{12/\pi^2 + 2\beta}{s - 1} + G_2(s).$$

In particular $(s - 1)^6Z_{U,H}(s)$ has a holomorphic continuation to the half-plane $\Re(s) > 9/10$. The function $G_1(s)$ is bounded for $\Re(s) \geq 5/6 + \varepsilon$ and satisfies $G_1(1) \neq 0$, and the function $G_2(s)$ satisfies

$$G_2(s) \ll (1 + |\Im(s)|)^{6(1-\Re(s)) + \varepsilon}$$
on this domain.

An explicit expression for $\beta$ can be found in (5.24), whereas the formulae (6.1)–(6.4) can be used to deduce an explicit expression for $G_1$. There are several features of Theorem 1 that are worthy of remark. The first step in the proof of Theorem 1 is the observation

$$Z_{U,H}(s) = s \int_1^\infty t^{-s-1} N_{U,H}(t)dt. \quad (1.5)$$

Thus we find ourselves in the situation of establishing a preliminary estimate for $N_{U,H}(B)$ in order to deduce the analytic properties of $Z_{U,H}(s)$ presented in Theorem 1, before then using this information to deduce an improved estimate for $N_{U,H}(B)$. This will be given in Theorem 2 below. With this order of events in mind we highlight that the term $\frac{12}{\pi^2}(s - 1)^{-3}$ appearing in Theorem 1 corresponds to an isolated conic contained in $X$. Moreover, whereas the first term $E_1(s)E_2(s)G_1(s)$ in the expression for $Z_{U,H}(s)$ corresponds to the main term in our preliminary estimate for $N_{U,H}(B)$, and arises through the approximation of certain arithmetic quantities by real-valued continuous functions, the term involving $\beta$ has a more arithmetic interpretation. Indeed, it will be seen to arise purely out of the error terms produced by approximating these arithmetic quantities by continuous functions. Finally we make the observation that under the assumption of the Riemann hypothesis $E_2(s)$ is holomorphic for $\Re(s) > 17/20$, so that $Z_{U,H}(s)$ has an analytic continuation to this domain.

We now come to how Theorem 1 can be used to deduce an asymptotic formula for $N_{U,H}(B)$. We shall verify in §2 that the following result is in accordance with the Manin conjecture.

**Theorem 2.** Let $\delta \in (0, 1/12)$. Then there exists a polynomial $P$ of degree 5 such that

$$N_{U,H}(B) = BP(\log B) + O(B^{1-\delta}),$$

for any $B \geq 1$. Moreover the leading coefficient of $P$ is equal to

$$\frac{\tau_\infty}{28800} \prod_p \left(1 - \frac{1}{p}\right)^6 \left(1 + \frac{6}{p} + \frac{1}{p^2}\right),$$

where

$$\tau_\infty = \int_0^1 \int_{-1}^{1/v} \left(\min\{\sqrt{u^3 + 1}, 1/v^3\} - \sqrt{\max\{u^3 - 1, 0\}}\right) du dv. \quad (1.6)$$
The deduction of Theorem 2 from Theorem 1 will take place in §7, and amounts to a routine application of Perron’s formula. Although we choose not to give the details here, it is in fact possible to take \( \delta \in (0, 1/11) \) in the statement of Theorem 2 by using more sophisticated estimates for moments of the Riemann zeta function in the critical strip. By expanding the height zeta function \( Z_{U,H}(s) \) as a power series in \((s-1)^{-1}\), one may obtain explicit expressions for the lower order coefficients of the polynomial \( P \) in Theorem 2. It would be interesting to obtain refinements of Manin’s conjecture that admit conjectural interpretations of the lower order coefficients.

The principal tool in the proof of Theorem 1 is a passage to the universal torsor above the minimal desingularisation \( \tilde{X} \) of \( X \). Although originally introduced to aid in the study of the Hasse principle and Weak approximation, universal torsors have recently become endemic in the context of counting rational points of bounded height. In §4 we shall establish a bijection between \( U(\mathbb{Q}) \) and integer points on the universal torsor, which in this setting has the natural affine embedding

\[
v_2y_0^2y_4 - v_0y_1y_2^2 + v_3y_3^2 = 0.
\]

Once we have translated the problem to the universal torsor, Theorem 1 will be established using a range of techniques drawn from classical analytic number theory. In particular Theorem 1 seems to be the first instance of a height zeta function that has been calculated via a passage to the universal torsor.

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## 2 Conformity with the Manin conjecture

In this section we shall show that Theorem 2 agrees with the Manin conjecture. For this we need to calculate the value of \( c_{U,H} \) and \( \rho \) in (1.1). We therefore review some of the geometry of the surface \( X \subset \mathbb{P}^4 \), as defined by the pair of quadratic forms

\[
Q_1(x) = x_0x_1 - x_2^2, \quad Q_2(x) = x_0x_4 - x_1x_2 + x_3^2, \quad (2.1)
\]

where \( x = (x_0, \ldots, x_4) \).

Let \( \tilde{X} \) denote the minimal desingularisation of \( X \), and let \( \pi : \tilde{X} \to X \) denote the corresponding blow-up map. We let \( E_6 \) denote the strict transform of the unique line contained in \( X \), and let \( E_1, \ldots, E_5 \) denote the exceptional curves
of $\pi$. Then the divisors $E_1, \ldots, E_6$ satisfy the Dynkin diagram

$$
\begin{array}{cccccccc}
E_6 & \rightarrow & E_5 & \rightarrow & E_4 & \rightarrow & E_3 & \rightarrow & E_2 & \rightarrow & E_1
\end{array}
$$

and generate the Picard group of $\hat{X}$. In particular we have $\rho = 6$ in (1.1), which agrees with Theorem 2.

It remains to discuss the conjectured value of the constant $c_{X,H}$ in (1.1). For this we shall follow the presentation of Batyrev and Tschinkel [2, §3.4]. Let $\Lambda_{\text{eff}}(\hat{X}) \subset \text{Pic} \hat{X} \otimes \mathbb{Z} \mathbb{R}$ be the cone of effective divisors on $\hat{X}$, and let

$$\Lambda_{\text{eff}}^\vee(\hat{X}) = \{ s \in \text{Pic}^\vee \hat{X} \otimes \mathbb{Z} \mathbb{R} : \langle s, t \rangle \geq 0 \ \forall \ t \in \Lambda_{\text{eff}}(\hat{X}) \}$$

be the corresponding dual cone, where $\text{Pic}^\vee \hat{X}$ denotes the dual lattice to $\text{Pic} \hat{X}$. Then if $d_t$ denotes the Lebesgue measure on $\text{Pic}^\vee \hat{X} \otimes \mathbb{Z} \mathbb{R}$, we define

$$\alpha(\hat{X}) = \int_{\Lambda_{\text{eff}}^\vee(\hat{X})} e^{-\langle -K_{\hat{X}}, t \rangle} dt,$$

where $-K_{\hat{X}}$ is the anticanonical divisor of $\hat{X}$. Thus $\alpha(\hat{X})$ measures the volume of the polytope obtained by intersecting $\Lambda_{\text{eff}}^\vee(\hat{X})$ with a certain affine hyperplane.

Next we discuss the Tamagawa measure on the closure $\overline{\hat{X}(\mathbb{Q})}$ of $\hat{X}(\mathbb{Q})$ in $\hat{X}(\mathbb{A}_\mathbb{Q})$, where $\mathbb{A}_\mathbb{Q}$ denotes the adele ring. Write $L_p(s, \text{Pic} \hat{X})$ for the local factors of $L(s, \text{Pic} \hat{X})$. Furthermore, let $\omega_\infty$ denote the archimedean density of points on $X$, and let $\omega_p$ denote the usual $p$-adic density of points on $X$, for any prime $p$. Then we may define the Tamagawa measure

$$\tau_H(\hat{X}) = \lim_{s \to 1} ((s - 1)^\rho L(s, \text{Pic} \hat{X})) \omega_\infty \prod_p \frac{\omega_p}{L_p(1, \text{Pic} \hat{X})},$$

where $\rho$ denotes the rank of $\text{Pic} \hat{X}$ as above. With these definitions in mind, the conjectured value of the constant in (1.1) is equal to

$$c_{X,H} = \alpha(\hat{X}) \beta(\hat{X}) \tau_H(\hat{X}),$$

where $\beta(\hat{X}) = \#H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Pic} \hat{X} \otimes \overline{\mathbb{Q}}) = 1$, since $\hat{X}$ is split over the ground field $\mathbb{Q}$.

We begin by calculating the value of $\alpha(\hat{X})$, for which we shall follow the approach of Peyre and Tschinkel [10, §5]. We need to determine the cone $\Lambda_{\text{eff}}(\hat{X})$ and the anticanonical divisor $-K_{\hat{X}}$. To determine these we may use the Dynkin diagram to write down the intersection matrix

|   | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$
|---|---|---|---|---|---|---|
| $E_1$ | -2 | 1 | 1 | 1 | 0 | 0
| $E_2$ | 1 | -2 | 0 | 0 | 0 | 1
| $E_3$ | 1 | 0 | -2 | 0 | 0 | 0
| $E_4$ | 1 | 0 | 0 | -2 | 1 | 0
| $E_5$ | 0 | 0 | 0 | 1 | -2 | 0
| $E_6$ | 0 | 1 | 0 | 0 | 0 | -1
Lemma 1. We have
\[ \tau = \frac{1}{5} \frac{1}{\sqrt{\theta(x)}} = \frac{1}{3} \frac{1}{\sqrt{\theta(x)}}. \]

But then it follows that \( \Lambda_{\text{eff}}(\tilde{X}) \) is generated by the divisors \( E_1, \ldots, E_6 \). Furthermore, on employing the adjunction formula \(-K_{\tilde{X}} \cdot D = 2 + D^2\) for \( D \in \text{Pic} \tilde{X} \), we easily deduce that
\[ -K_{\tilde{X}} = 6E_1 + 5E_2 + 3E_3 + 4E_4 + 2E_5 + 4E_6. \]

It therefore follows that
\[ \alpha(\tilde{X}) = \text{Vol}\{(t_1, t_2, t_3, t_4, t_5, t_6) \in \mathbb{R}_{\geq 0}^6 : 6t_1 + 5t_2 + 3t_3 + 4t_4 + 2t_5 + 4t_6 = 1\} \]
\[ = \frac{1}{4} \text{Vol}\{(t_3, t_2, t_3, t_4, t_5) \in \mathbb{R}_{\geq 0}^5 : 6t_1 + 5t_2 + 3t_3 + 4t_4 + 2t_5 \leq 1\}, \]
whence in fact
\[ \alpha(\tilde{X}) = 1/345600. \quad (2.4) \]

It remains to calculate the value of \( \tau_H(\tilde{X}) \) in (2.3).

**Lemma 1.** We have \( \tau_H(\tilde{X}) = 12\tau_{\infty} \), where \( \tau_{\infty} \) is given by (1.6) and
\[ \tau = \prod_p \left(1 - \frac{1}{p}\right)^6 \left(1 + \frac{6}{p} + \frac{1}{p^2}\right). \quad (2.5) \]

**Proof.** Recall the definition (2.2) of \( \tau_H(\tilde{X}) \). Our starting point is the observation that \( L(s, \text{Pic} \tilde{X}) = \zeta(s)^6 \). Hence it easily follows that
\[ \lim_{s \to 1} ((s - 1)^6L(s, \text{Pic} \tilde{X})) = \lim_{s \to 1} ((s - 1)^6\zeta(s)^6) = 1. \]

Furthermore we plainly have
\[ L_p(1, \text{Pic} \tilde{X})^{-1} = \left(1 - \frac{1}{p}\right)^6, \quad (2.6) \]
for any prime \( p \).

We proceed by employing the method of Peyre [9] to calculate the value of the archimedean density \( \omega_{\infty} \). It will be convenient to parameterise the points via the choice of variables \( x_0, x_1, x_4 \), for which we first observe that the Leray form \( \omega_L(\tilde{X}) \) is given by \( (\text{det}Q_{x_2})^{-1}dx_0dx_1dx_4 \), since
\[ \det \begin{pmatrix} \frac{\partial Q_2}{\partial x_2} & \frac{\partial Q_2}{\partial x_2} \\ \frac{\partial Q_2}{\partial x_3} & \frac{\partial Q_2}{\partial x_3} \end{pmatrix} = -4x_2x_3. \]

Now in any real solution to the pair of equations \( Q_1(x) = Q_2(x) = 0 \), the components \( x_0 \) and \( x_1 \) must necessarily share the same sign. Taking into account the fact that \( x \) and \(-x\) represent the same point in \( \mathbb{P}^3 \), we therefore see that
\[ \omega_{\infty} = 2 \int_{\{x \in \mathbb{R}^3 : Q_1(x) = Q_2(x) = 0, \ 0 \leq x_0, x_1, x_3, |x_4| \leq 1\}} \omega_L(\tilde{X}). \]

We write \( \omega_{\infty,-} \) to denote the contribution to \( \omega_{\infty} \) from the case \( x_2 = -\sqrt{x_0x_1} \), and \( \omega_{\infty,+} \) for the contribution from the case \( x_2 = \sqrt{x_0x_1} \). But then it follows that
\[ \omega_{\infty,+} = \frac{1}{2} \iint \frac{dx_0dx_1dx_4}{\sqrt{x_0^2x_1(x_0^{-1/2}x_1^{3/2} - x_4)}}. \]
where the triple integral is over all \(x_0, x_1, x_4 \in \mathbb{R}\) such that
\[
0 \leq x_0, x_1, |x_4| \leq 1, \quad x_1^{3/2} \geq x_0^{1/2}, \quad 1 + x_0 x_4 \geq x_1^{3/2} x_0^{1/2}.
\]
The change of variables \(u = x_1^{1/2} / x_0^{1/6}\), and then \(v = x_0^{1/6}\), therefore yields
\[
\omega_{\infty,+} = 6 \int \int \frac{du dx_4}{\sqrt{u^3 - x_4}}
\]
where the triple integral is now over all \(u, v, x_4 \in \mathbb{R}\) such that
\[
0 \leq v \leq 1, \quad 0 \leq u \leq 1/v, \quad \max\{-1, u^3 - 1/v^6\} \leq x_4 \leq \min\{1, u^3\}.
\]
On performing the integration over \(x_4\), a straightforward calculation leads to the equality
\[
\omega_{\infty,+} = 12 \int_0^1 \int_0^{1/v} \left( \min\{\sqrt{u^3 + 1}, 1/v^3\} - \sqrt{\max\{u^3 - 1, 0\}} \right) du dv.
\]
The calculation of \(\omega_{\infty,-}\) is similar. On following the steps outlined above, one is easily led to the equality
\[
\omega_{\infty,-} = 12 \int_0^1 \int_{-1}^0 \min\{\sqrt{u^3 + 1}, 1/v^3\} du dv.
\]
Once taken together, these equations combine to show that \(\omega_\infty = 12 \tau_\infty\), where \(\tau_\infty\) is given by (1.6).

It remains to calculate the value of \(\omega_p = \lim_{r \to \infty} p^{-3r} N(p^r)\), where we have written \(N(p^r) = \#\{x \ (\text{mod} p^r) : Q_1(x) \equiv Q_2(x) \equiv 0 \ (\text{mod} p^r)\}\). To begin with we write
\[
x_0 = p^{k_0} x_0', \quad x_1 = p^{k_1} x_1'
\]
with \(p \nmid x_0' x_1'\). Now we have \(p^r \mid x_2^2\) if and only if \(k_0 + k_1 \geq r\), and there are at most \(p^{r/2}\) square roots of zero modulo \(p^r\). When \(k_0 + k_1 < r\), it follows that \(k_0 + k_1\) must be even and we may write
\[
x_2 = p^{(k_0 + k_1)/2} x_2',
\]
with \(p \nmid x_2'\) and
\[
x_0' x_1' - x_2'^2 \equiv 0 \ (\text{mod} \ p^{r-k_0-k_1}).
\]
The number of possible choices for \(x_0', x_1', x_2'\) is therefore
\[
h_p(r, k_0, k_1) = \begin{cases} 
\phi(p^{r-k_0})\phi(p^{-(k_0+k_1)/2})p^{k_0} & \text{if } k_0 + k_1 < r, \\
O(p^{5r/2-k_0-k_1}) & \text{if } k_0 + k_1 \geq r.
\end{cases}
\]
It remains to determine the number of solutions \(x_3, x_4\) modulo \(p^r\) such that
\[
p^{k_0} x_0' x_4 - p^{k_0/2 + 3k_1/2} x_3' x_2' + x_3^2 \equiv 0 \ (\text{mod} \ p^r). \tag{2.7}
\]
In order to do so we distinguish between three basic cases: either \(k_0 + k_1 < r\) and \(k_0 \leq 3k_1\), or \(k_0 + k_1 < r\) and \(k_0 > 3k_1\), or else \(k_0 + k_1 \geq r\). For the first two of these cases we must take care only to sum over values of \(k_0, k_1\) such that
We shall denote by $N_i(p^r)$ the contribution to $N(p^r)$ from the
ith case, for $1 \leq i \leq 3$, so that

$$N(p^r) = N_1(p^r) + N_2(p^r) + N_3(p^r).$$  \hfill (2.8)

We begin by calculating the value of $N_1(p^r)$. For this we write $x_3 = p^{k_3}x_3'$, with $k_3 = \min\{r/2,\lfloor k_0/2\rfloor\} = \lfloor k_0/2\rfloor$. The number of possibilities for $x_3'$ is $p^{r-k_0} - 1$, each one leading to a congruence of the form

$$x_0'x_4 - p^{3k_1/2 - k_0/2}x_1'x_2' + p^{2\lfloor k_0/2\rfloor - k_0}x_3'^2 \equiv 0 \pmod{p^{r-k_0}}.$$  

Modulo $p^{r-k_0}$, there is one choice for $x_4$, and so there are $p^{r+k_0 - \lfloor k_0/2\rfloor} = p^{r+k_0}$ possibilities for $x_3$ and $x_4$. On summing these contributions over all the relevant values of $k_0, k_1$, we therefore obtain

$$N_1(p^r) = \sum_{k_0+k_1<r, \ k_0,k_1 \geq 0 \atop k_0 \leq 3k_1, \ 2|(k_0+k_1)} p^{r-k_0} - 1 \text{ possibilities for } x_3.$$

as $r \to \infty$.

Next we calculate $N_2(p^r)$, for which we shall not use the above calculation for $h_p(r, k_0, k_1)$. On writing $x_3 = p^{k_3}x_3'$, with

$$k_3 = \min\{r/2, \lfloor k_0/4 + 3k_1/4\rfloor\} = \lfloor k_0/4 + 3k_1/4\rfloor,$$

we observe that $k_0/2 + 3k_1/2$ must be even since $p \nmid x_1'x_2'$. Thus $k_3 = k_0/4 + 3k_1/4$ and $p \nmid x_3'$. In this way (2.7) becomes

$$x_0'x_4 - x_1'x_2' + x_3'^2 \equiv 0 \pmod{p^{r-k_0/2 - 3k_1/2}},$$

which thereby implies that $x_3'^2 \equiv x_1'x_2' \pmod{p^{r-k_0/2 - 3k_1/2}}$. At this point we recall the auxiliary congruence $x_2'^2 \equiv x_0'x_1'$ (mod $p^{r-k_0-k_1}$) that is satisfied by $x_0', x_1', x_2'$. We proceed by fixing values of $x_2'$ and $x_3'$, for which there are precisely $(1 - 1/p)^2p^{r-k_0/2 - k_1/2}p^{r-k_1}$ choices. But then $x_1'$ is fixed modulo $p^{r-k_0/2 - 3k_1/2}$, and so there are $p^{r-k_1-(k_0/2 - 3k_1/2)}$ possibilities for $x_1'$. Finally we deduce from the remaining two congruences that there are $p^{k_3}$ ways of fixing $x_0'$, and $p^{k_0}$ ways of fixing $x_4$. Summing over the relevant values of $k_0$ and $k_1$ we therefore obtain

$$N_2(p^r) = \sum_{k_0+k_1<r, \ k_0,k_1 \geq 0 \atop k_0 \geq 3k_1, \ 4|(k_0-k_1)} (1 - 1/p)^2p^{3r-(k_0-k_1)/4} = 2p^{3r-1}(1 + o(1)),$$

as $r \to \infty$.

Finally we calculate the value of $N_3(p^r)$. In this case we write $x_2 = p^{r/2}x_2'$. But then a similar calculation ultimately shows that $N_3(p^r) = o(p^{3r})$ as $r \to \infty$. On combining our estimates for $N_1(p^r), N_2(p^r), N_3(p^r)$ into (2.8), we therefore deduce that

$$N(p^r) = p^{3r} \left(1 + \frac{6}{p} + \frac{1}{p^2} \right) (1 + o(1))$$

as $r \to \infty$, whence

$$\omega_p = \lim_{r \to \infty} p^{-3r}N(p^r) = 1 + \frac{6}{p} + \frac{1}{p^2}$$

for any prime $p$. We combine this with (2.6), in the manner indicated by (2.2), in order to deduce (2.5). \qed

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We end this section by combining (2.4) and Lemma 1 in (2.3), in order to deduce that the conjectured value of the constant in (1.1) is equal to

\[ c_{X,H} = \frac{1}{28800} \tau_{\infty}, \]

where \( \tau_{\infty} \) is given by (1.6) and \( \tau \) is given by (2.5). This agrees with the value of the leading coefficient obtained in Theorem 2.

3 Congruences

In this section we shall collect together some of the basic facts concerning congruences that will be needed in the proof of Theorem 1. We begin by discussing the case of quadratic congruences. For any integers \( a, q \) such that \( q > 0 \), we define the arithmetic function \( \eta(a; q) \) to be the number of positive integers \( n \leq q \) such that \( n^2 \equiv a \pmod{q} \). When \( q \) is odd it follows that \( \eta(a; q) = \sum_{d \mid q} \mu(d) |d^{a/d}| \), where \( (a/d) \) is the usual Jacobi symbol. On noting that \( \eta(a; 2^\nu) \leq 4 \) for any \( \nu \in \mathbb{N} \), it easily follows that \( \eta(a; q) \leq 2^{\omega(q)} + 1 \) (3.1) for any \( q \in \mathbb{N} \). Here \( \omega(q) \) denotes the number of distinct prime factors of \( q \).

Turning to the case of linear congruences, let \( \kappa \in [0, 1] \) and let \( \vartheta \) be any arithmetic function such that

\[ \sum_{d=1}^{\infty} \left( \vartheta * \mu \right)(d) \frac{|d^{a/d}|}{d^\kappa} < \infty, \]

where \( (f * g)(d) = \sum_{e \mid d} f(e) g(d/e) \) is the usual Dirichlet convolution of any two arithmetic functions \( f, g \). Then for any coprime integers \( a, q \) such that \( q > 0 \), and any \( t \geq 1 \), we deduce that

\[ \sum_{n \leq t \atop n \equiv a \pmod{q}} \vartheta(n) = \sum_{d=1}^{\infty} \left( \vartheta * \mu \right)(d) \sum_{m \leq t/d \atop \gcd(m,q) = 1} 1 \]

\[ = \frac{t}{q} \sum_{d=1}^{\infty} \frac{(\vartheta * \mu)(d)}{d} + O\left(t^\kappa \sum_{d=1}^{\infty} \frac{1}{d^\kappa} \right), \]

on using the equality \( \vartheta = (\vartheta * \mu) * 1 \) and the trivial estimate \( \lfloor x \rfloor = x + O(x^\kappa) \) for any \( x > 0 \). We summarise this estimate in the following result.

**Lemma 2.** Let \( \kappa \in [0, 1] \), let \( \vartheta \) be any arithmetic function such that (3.2) holds, and let \( a, q \in \mathbb{Z} \) be such that \( q > 0 \) and \( \gcd(a, q) = 1 \). Then we have

\[ \sum_{n \leq t \atop n \equiv a \pmod{q}} \vartheta(n) = \frac{t}{q} \sum_{d=1}^{\infty} \frac{(\vartheta * \mu)(d)}{d} + O\left(t^\kappa \sum_{d=1}^{\infty} \frac{1}{d^\kappa} \right). \]
Define the real-valued function $\psi(t) = \{t\} - 1/2$, where $\{t\}$ denotes the fractional part of $t \in \mathbb{R}$. Then $\psi$ is periodic with period 1. When $\vartheta(n) = 1$ for all $n \in \mathbb{N}$ we are able to refine Lemma 2 considerably.

**Lemma 3.** Let $a, q \in \mathbb{Z}$ be such that $q > 0$, and let $t_1, t_2 \in \mathbb{R}$ such that $t_2 \geq \max\{0, t_1\}$. Then

$$\#\{t_1 < n \leq t_2 : n \equiv a \pmod{q}\} = \frac{t_2 - t_1}{q} + r(t_1, t_2; a, q),$$

where

$$r(t_1, t_2; a, q) = \psi\left(\frac{t_1 - a}{q}\right) - \psi\left(\frac{t_2 - a}{q}\right).$$

**Proof.** Write $a = b + qc$ for some integer $0 \leq b < q$. Then it is clear that

$$\#\{t_1 < n \leq t_2 : n \equiv b \pmod{q}\} = \left\lfloor \frac{t_2 - b}{q} \right\rfloor - \left\lfloor \frac{t_1 - b}{q} \right\rfloor,$$

whence

$$\#\{t_1 < n \leq t_2 : n \equiv a \pmod{q}\} - \frac{t_2 - t_1}{q} = r(t_1, t_2; b, q).$$

We complete the proof of Lemma 3 by noting that $r(t_1, t_2; b, q) = r(t_1, t_2; a, q)$, since $\psi$ has period 1.

We shall also need to know something about the average order of the function $\psi$. We proceed by demonstrating the following result.

**Lemma 4.** Let $\varepsilon > 0$, $t \geq 0$ and let $X \geq 1$. Then for any $b, q \in \mathbb{Z}$ such that $q > 0$ and $\gcd(b, q) = 1$, we have

$$\sum_{0 \leq x < X} \psi\left(\frac{t - bx^2}{q}\right) \ll \varepsilon (qX)^\varepsilon \left(X^\frac{1}{q^{1/2}} + q^{1/2}\right).$$

**Proof.** Throughout this proof we shall write $e(t) = e^{2\pi i t}$ and $e_q(t) = e^{2\pi i t / q}$. In order to establish Lemma 4 we shall expand the function $f(k) = \psi((t - k)/q)$ as a Fourier series. Thus we have

$$f(k) = \sum_{0 \leq \ell < q} a(\ell) e_q(\ell k),$$

for any $k \in \mathbb{Z}$, where the coefficients $a(\ell)$ are given by

$$a(\ell) = \frac{1}{q} \sum_{0 \leq j < q} f(j) e_q(-j\ell).$$

Let $\|\alpha\|$ denote the distance from $\alpha \in \mathbb{R}$ to the nearest integer. We proceed by proving the estimates

$$a(\ell) \ll \begin{cases} q^{-1}, & \ell = 0, \\ q^{-1} \ell/q \|^{-1}, & \ell \neq 0. \end{cases}$$

(3.3)
This is straightforward. To verify the estimate for \( a(0) \) we simply note that
\[
a(0) = \frac{1}{q} \sum_{0 \leq j < q} \left( \frac{t-j}{q} \right) - \frac{1}{2}
\]
\[
= \frac{1}{q} \sum_{0 \leq j \leq t} \left( \frac{t-j}{q} \right) + \frac{1}{q} \sum_{t < j < q} \left( \frac{t-j+q/2}{q} \right) \ll \frac{1}{q}.
\]

Similarly, when \( \ell \neq 0 \) we have
\[
a(\ell) = \frac{1}{q} \sum_{0 \leq j \leq t} \left( \frac{t-j-q/2}{q} \right) e(-j\ell) + \frac{1}{q} \sum_{t < j < q} \left( \frac{t-j+q/2}{q} \right) e(-j\ell)
\]
\[
= \frac{1}{q} \sum_{0 \leq j < q} \frac{-j}{q} e(-j\ell) - \frac{1}{2q} \sum_{0 \leq j \leq t} e(-j\ell) + \frac{1}{2q} \sum_{t < j < q} e(-j\ell) \ll \frac{1}{q \| \ell/q \|},
\]
as required.

In view of the above we therefore obtain
\[
\sum_{0 \leq x < X} \psi \left( \frac{t-bx^2}{q} \right) = \sum_{0 \leq \ell < q} a(\ell) \sum_{0 \leq x < X} e_q(\ell x^2)
\]
\[
= a(0) [X] + \sum_{m|q} \mu(n) \sum_{1 \leq \ell' < q/m} \gcd(\ell', q/m) = 1 a(\ell'm) \sum_{0 \leq x < X} e_q(\ell' x^2).
\]

But here the inner sum can plainly be estimated using Weyl’s inequality, and so has size
\[
\ll \epsilon X \left( \frac{m^{1/2}X}{q^{1/2}} + \frac{q^{1/2}}{m^{1/2}} \right).
\]

On employing (3.3), we therefore deduce that
\[
\sum_{0 \leq x < X} \psi \left( \frac{t-bx^2}{q} \right) \ll \epsilon \frac{X}{q} + \sum_{m|q} \frac{m^{1/2}X}{q} \sum_{1 \leq \ell' < q/m} \frac{q^{-1/2}X + q^{1/2}}{q \| \ell' m/q \|}
\]
\[
\ll \epsilon (qX)^{2\epsilon} \left( \frac{X}{q^{1/2}} + q^{1/2} \right),
\]
which completes the proof of Lemma 4.

Let \( \epsilon > 0 \) and let \( t \geq 0 \). Then for any \( b, q \in \mathbb{Z} \) such that \( q > 0 \) and \( \gcd(b, q) = 1 \), we may deduce from Lemma 4 that
\[
\sum_{0 \leq x < q} \psi \left( \frac{t-bx^2}{q} \right) \ll \epsilon q^{1/2+\epsilon}.
\]

But then it follows from an application of Möbius inversion that
\[
\sum_{0 \leq x < q} \psi \left( \frac{t-bx^2}{q} \right) = \sum_{n|q} \mu(n) \sum_{0 \leq x' < q/n} \psi \left( \frac{t/n-bnx^2}{q/n} \right)
\]
\[
= \sum_{n|q} \mu(n) \sum_{m=\gcd(n,q/n)} \psi \left( \frac{t/(mn)-bnx^2/m}{q/(mn)} \right),
\]
whence (3.4) yields
\[
\sum_{0 \leq x < q \atop \gcd(x,q) = 1} \psi\left(\frac{t - bx^2}{q}\right) \ll \varepsilon q^{1/2 + \varepsilon} \sum_{n|q} \left(\frac{\gcd(n,q/n)}{n}\right)^{1/2} \ll \varepsilon q^{1/2 + 2\varepsilon}.
\]

This therefore establishes the following result, on re-defining the choice of \(\varepsilon\).

**Lemma 5.** Let \(\varepsilon > 0\) and let \(t \geq 0\). Then for any \(b, q \in \mathbb{Z}\) such that \(q > 0\) and \(\gcd(b, q) = 1\), we have
\[
\sum_{0 \leq x < q \atop \gcd(x,q) = 1} \psi\left(\frac{t - bx^2}{q}\right) \ll \varepsilon q^{1/2 + \varepsilon}.
\]

## 4 Preliminary manoeuvres

We begin this section by introducing some notation. For any \(n \geq 2\) we let \(\mathbb{Z}^{n+1}\) denote the set of primitive vectors in \(\mathbb{Z}^{n+1}\), where \(v = (v_0, \ldots, v_n) \in \mathbb{Z}^{n+1}\) is said to be primitive if \(\gcd(v_0, \ldots, v_n) = 1\). Moreover we shall let \(\mathbb{Z}^{n+1}_*\) (resp. \(\mathbb{Z}^{n+1}_*\)) denote the set of vectors \(v \in \mathbb{Z}^{n+1}\) (resp. \(v \in \mathbb{Z}^{n+1}\)) such that \(v_0 \cdots v_n \neq 0\). Finally we underline the fact that throughout our work \(N\) is always taken to denote the set of positive integers.

The proof of Theorem 1 rests upon establishing a preliminary asymptotic formula for the counting function \(N_{U, H}(B)\). Recall the definition (2.1) of the quadratic forms \(Q_1, Q_2\). Our first task in this section is to relate \(N_{U, H}(B)\) to the quantity
\[
N(Q_1, Q_2; B) = \# \{ x \in \mathbb{Z}_5^* : 0 < x_0, x_1, x_3 \leq B, |x_4| \leq B, Q_1(x) = Q_2(x) = 0 \}.
\]

In fact we shall establish the following result rather easily.

**Lemma 6.** Let \(B \geq 1\). Then we have
\[
N_{U, H}(B) = 2N(Q_1, Q_2; B) + \frac{12}{\pi^2} B + O(B^{2/3}).
\]

**Proof.** It is clear that any solution to the pair of equations \(Q_1(x) = Q_2(x) = 0\) which satisfies \(x_0 = 0\), must in fact correspond to a point lying on the line \(x_0 = x_2 = x_3 = 0\) contained in \(X\). On noting that \(x\) and \(-x\) represent the same point in projective space, we therefore deduce that
\[
N_{U, H}(B) = \frac{1}{2} \# \{ x \in \mathbb{Z}_5^* : ||x|| \leq B, Q_1(x) = Q_2(x) = 0, x_0 \neq 0 \},
\]
where \(||x|| = \max_{0 \leq i \leq 4} |x_i|\). We proceed to consider the contribution from the vectors \(x \in \mathbb{Z}^5\) for which \(||x|| \leq B\) and
\[
Q_1(x) = Q_2(x) = 0, \quad x_1x_2x_3x_4 = 0.
\]

Note first that \(x_1 = 0\) if and only if \(x_2 = 0\) in (4.1), since \(x_0 \neq 0\). Thus if we consider the contribution from those vectors for which \(x_1x_2 = 0\), it follows that we must count integers \(|x_0|, |x_3|, |x_4| \leq B\) for which \(\gcd(x_0, x_3, x_4) = 1\) and
\[ x_0 x_4 + x_2^2 = 0. \] Now either \( x_3 = 0 \), in which case \( x = (1, 0, 0, 0, 0) \) since \( x \) is primitive and \( x_6 \neq 0 \), or else the primitivity of \( x \) implies that \( x = (a^2, 0, 0, \pm ab, -b^2) \) for coprime non-zero integers \( a, b \). Hence the overall contribution from this case is clearly \( 12B/\pi^2 + O(B^{1/2}) \).

Suppose now that \( x_3 = 0 \) and \( x_1 x_2 \neq 0 \) in (4.1). Then we must count the number of mutually coprime non-zero integers \( x_0, x_1, x_2, x_4 \), with modulus at most \( B \), such that \( x_0 x_1 = x_2^2 \) and \( x_0 x_4 = x_1 x_2 \). Since we are only interested in an upper bound it clearly suffices to count non-zero integers \( x_0, x_1, x_4 \), with modulus at most \( B \), such that \( \text{gcd}(x_0, x_1, x_4) = 1 \) and \( x_0 x_1 = x_4^2 \). Then it follows that \( (x_0, x_1, x_4) = \pm(a^3, ab^2, -b^3) \) for coprime non-zero integers \( a, b \), whence the overall contribution is \( O(B^{2/3}) \). Finally the case \( x_4 = 0 \) and \( x_1 x_2 \neq 0 \) in (4.1) is handled in much the same way, now via the parameterisation \( x = \pm(a, b, a^2 b^2, ab^3, 0) \). This therefore establishes that

\[
N_{u,t,H}(B) = \frac{1}{2} \# \{ x \in \mathbb{Z}_*^5 : ||x|| \leq B, Q_1(x) = Q_2(x) = 0 \} + \frac{12}{\pi^2} B + O(B^{2/3}).
\]

We complete the proof of Lemma 6 by choosing \( x_0 > 0 \) and \( x_3 > 0 \). This then forces the inequality \( x_1 > 0 \), whence \( ||x|| = \max\{x_0, x_1, x_3, |x_4|\} \).

We now turn to the task of establishing a bijection between the points counted by \( N(Q_1, Q_2; B) \) and integral points on the universal torsor above the minimal desingularisation of \( X \). Let \( x \in \mathbb{Z}_*^5 \) be any vector counted by \( N(Q_1, Q_2; B) \). In particular it follows that \( x_0, x_1, x_3 \) are positive. We begin by considering solutions to the equation \( Q_1(x) = 0 \). But it is easy to see that there is a bijection between the set of integers \( x_0, x_1, x_2 \) such that \( x_0 x_1 > 0 \) and \( x_0 x_1 = x_2^2 \), and the set of \( x_0, x_1, x_2 \) such that

\[
x_0 = z_0^2 z_2, \quad x_1 = z_1^2 z_2, \quad x_2 = z_0 z_1 z_2,
\]

for non-zero integers \( z_0, z_1, z_2 \) such that \( z_0, z_2 > 0 \) and

\[
\text{gcd}(z_0, z_1) = 1. \tag{4.2}
\]

We now substitute these values into the equation \( Q_2(x) = 0 \), in order to obtain

\[
x_4 z_0^2 z_2 - z_0 z_1^2 z_2 + x_3^2 = 0. \tag{4.3}
\]

It is clear that \( z_0 z_2 \) divides \( x_3^2 \). Hence we write

\[
z_0 = v_0 v_3 y_0''^2, \quad z_2 = v_2 v_3 y_2''^2,
\]

for \( v_0, v_2, v_3, y_0'', y_2'' \in \mathbb{N} \) such that the products \( v_0 v_3 \) and \( v_2 v_3 \) are square-free, with \( \text{gcd}(v_0, v_2) = 1 \). In particular the product \( v_0 v_2 v_3 \) is clearly square-free. We easily deduce that \( v_0 v_2 v_3 y_0''^2 y_2''^2 \) must divide \( x_3 \), whence there exists \( y_3'' \in \mathbb{N} \) such that \( x_3 = v_0 v_2 v_3 y_0''^2 y_2''^2 y_3'' \). Combining the various coprimality conditions arising from (4.2) and the definitions of \( v_0, v_2 \) and \( v_3 \), we therefore obtain

\[
|\mu(v_0 v_2 v_3)| = 1, \quad \text{gcd}(v_0 v_3 y_0'', z_1) = 1, \tag{4.4}
\]

where \( \mu(n) \) denotes the M"obius function for any non-zero integer \( n \). On making the appropriate substitutions into (4.3), we deduce that

\[
v_0 v_3 x_4 y_0''^2 - v_2 v_3 y_2''^2 z_1^3 + v_0 v_2 y_3''^2 = 0. \tag{4.5}
\]
At this point it is convenient to deduce a further coprimality condition which follows from the assumption made at the outset that \( \gcd(x_0, \ldots, x_4) = 1 \). Recalling the various changes of variables that we have made so far, it is easily checked that \( \gcd(x_0, x_1, x_2, x_3) = v_2 v_3 y_2'' \gcd(y_2'', v_0 y_0'' y_3'') \). Hence we find that
\[
\gcd(v_2 v_3 y_2'', x_4) = 1. \tag{4.6}
\]
Now it follows from (4.5) that \( v_0 \) divides \( v_2 v_3 y_2'' z_1^3 \). But then since \( v_0 \) is square-free, we may conclude from (4.4) that \( v_0 \mid y_2'' \). Similarly we deduce from (4.4) and (4.6) that \( v_2 \mid y_0'' \) and \( v_3 \mid y_3'' \). Thus there exist \( y_0', y_2', y_3' \in \mathbb{N} \) and \( y_1, y_4 \in \mathbb{Z}_* \) such that
\[
y_0'' = v_2 y_0', \quad z_1 = y_1, \quad y_2'' = v_0 y_2', \quad y_3'' = v_3 y_3', \quad x_4 = y_4.
\]
Substituting the above into (4.5) we therefore obtain the equation
\[
v_2 y_0'^2 y_4 - v_0 y_1^2 y_2'^2 + v_3 y_3'^2 = 0. \tag{4.7}
\]
Moreover we may combine (4.4) and (4.6) to get
\[
|\mu(v_0 v_2 v_3)| = 1, \quad \gcd(v_0 v_2 v_3 y_0', y_1) = \gcd(v_0 v_2 v_3 y_2', y_4) = 1. \tag{4.8}
\]
Finally we write \( v_1 = \gcd(y_0', y_2', y_3') \). Thus there exist \( y_0, y_2, y_3 \in \mathbb{N} \) such that
\[
y_0' = v_1 y_0, \quad y_2' = v_1 y_2, \quad y_3' = v_1 y_3,
\]
and we obtain the final equation
\[
v_2 y_0'^2 y_4 - v_0 y_1^2 y_2'^2 + v_3 y_3'^2 = 0. \tag{4.9}
\]
It remains to collect together the coprimality conditions that have arisen from this last change of variables. First however we take a moment to deduce three further coprimality conditions
\[
\gcd(y_0, y_2) = 1, \quad \gcd(y_0, y_3) = 1, \quad \gcd(y_2, y_3) = 1. \tag{4.10}
\]
To do so we simply use the obvious fact that \( \gcd(y_0, y_2, y_3) = 1 \). Suppose that \( p \) is any prime divisor of \( y_2 \) and \( y_3 \). Then we clearly have \( p^2 \mid v_2 y_0'^2 y_4 \) in (4.9). This is impossible by (4.8) and the fact that \( \gcd(y_0, y_2, y_3) = 1 \). From this we may establish the second relation in (4.10). Indeed, if \( p \mid y_0, y_3 \) then clearly \( p^2 \mid v_0 y_1^2 y_2'^2 \), which is impossible by (4.8) and the fact that \( \gcd(y_2, y_3) = 1 \). One checks the first relation in (4.10) in a similar fashion. Combining (4.8) with (4.10) we therefore deduce that
\[
\gcd(y_3, v_0 y_0 y_2) = \gcd(y_4, v_0 v_1 v_2 v_3 y_2) = 1 \tag{4.11}
\]
and
\[
|\mu(v_0 v_2 v_3)| = 1, \quad \gcd(y_1, v_0 v_1 v_2 v_3 y_0) = \gcd(y_0, y_2) = 1. \tag{4.12}
\]
In fact it will be necessary to reformulate these coprimality conditions somewhat. We claim that once taken together with (4.9), the relations (4.11) and (4.12) are equivalent to
\[
\gcd(y_3, v_0 y_0 y_2) = \gcd(y_4, v_1 v_2) = 1. \tag{4.13}
\]
and
\[
\begin{align*}
gcd(y_1, v_0 v_1 v_2 v_3 y_0) &= 1, \quad (4.14) \\
|\mu(v_0 v_2 v_3)| &= 1, \quad \gcd(v_2 v_3 y_0, y_2) = \gcd(v_0 v_3, y_0) = 1. \quad (4.15)
\end{align*}
\]

We first show how (4.9), (4.11) and (4.12) imply (4.9), (4.13), (4.14) and (4.15). Suppose that \( p \) is any prime divisor of \( v_0 \) and \( y_3 \). Then (4.9) implies that \( p \mid v_2 y_0^2 y_4 \) which is easily seen to be impossible via (4.11) and (4.12). Thus \( \gcd(y_3, v_0) = 1 \). Now suppose that \( p \) is a prime divisor of \( v_3 \) and \( y_2 \). Then \( p \mid v_2 y_0^2 y_4 \) which is also impossible, and so \( \gcd(v_3, y_2) = 1 \). The supplementary conditions \( \gcd(v_2, y_2) = \gcd(v_0 v_3, y_0) = 1 \) easily follow from the relations \( \gcd(v_0 y_2, v_3 y_3) = \gcd(v_3, y_1) = 1 \), in addition to (4.9). The converse is established along similar lines.

At this point we may summarise our argument as follows. Let \( T \subset \mathbb{Z}^9 \) denote the set of \((v, y) = (v_0, v_1, v_2, v_3, y_0, \ldots, y_4) \in \mathbb{N} \times \mathbb{Z}_p^5\) such that \( y_0, y_2, y_3 \geq 0 \), (4.9), and (4.13)–(4.15) hold. Then for any \( x \in \mathbb{Z}^5 \) counted by \( N(Q_1, Q_2; B) \), we have shown that there exists \( (v, y) \in T \) such that
\[
\begin{align*}
x_0 &= v_0^4 v_1^2 v_2^5 v_3^3 y_0^2 y_2^2, \\
x_1 &= v_0^2 v_1^2 v_2 v_3 y_0^2 y_2^2, \\
x_2 &= v_0^3 v_1^3 v_2^3 y_0 y_2^2, \\
x_3 &= v_0^2 v_1^3 v_2^2 y_0 y_2 y_3, \\
x_4 &= y_4.
\end{align*}
\]

Conversely, given any \((v, y) \in T\) the point \( x \) given above will be a solution of the equations \( Q_1(x) = Q_2(x) = 0 \), with \( x \in \mathbb{Z}^5 \). To see the primitivity of \( x \) we first recall that once taken together with (4.9), the coprimality relations (4.13)–(4.15) are equivalent to (4.11) and (4.12). But then it follows that \( \gcd(x_0, x_1, x_2, x_3) \) divides \( v_0^2 v_1^2 v_2 v_3 y_2 y_4 \). Finally an application of (4.11) and (4.12) yields
\[
\gcd(x_0, \ldots, x_4) = \gcd(\gcd(x_0, x_1, x_2, x_3), x_4) \leq \gcd(v_0^2 v_1^2 v_2 v_3 y_2^2, y_4) = 1,
\]
as claimed. Let us define the function \( \Psi : \mathbb{R}^9 \to \mathbb{R}_{\geq 0} \), given by
\[
\Psi(v, y) = \max \{ v_0^4 v_1^2 v_2^5 v_3^3 y_0^2 y_2^2, v_0^2 v_1^2 v_2 v_3 y_0^2 y_2^2, v_0^3 v_1^3 v_2^3 y_0 y_2^2, v_0^2 v_1^3 v_2^2 y_0 y_2 y_3, |y_4| \}.
\]
We have therefore established the following result.

**Lemma 7.** Let \( B \geq 1 \). Then we have
\[
N(Q_1, Q_2; B) = \# \{(v, y) \in T : \Psi(v, y) \leq B\}.
\]

It will become clear in subsequent sections that the equation (4.9) is a crucial ingredient in our proof of Theorem 1. In fact (4.9) is an affine embedding of the universal torsor above the minimal desingularisation of \( X \). Thus Derenthal, in work to appear, has established the isomorphism
\[
\text{Cox}(\tilde{X}) = \text{Spec}(\mathbb{Q}[v, y]/(v_2 y_0^2 y_4 - v_0 y_1^3 y_2^2 + v_3 y_3^2)),
\]
where \( \text{Cox}(\tilde{X}) \) is the Cox ring of \( \tilde{X} \).
5 The final count

In this section we estimate $N(Q_1, Q_2; B)$, which we shall then combine with Lemma 6 to provide an initial estimate for $N_{U,H}(B)$. Before proceeding with this task, it will be helpful to first outline our strategy. In view of (4.9) it is clear that for any $(v, y) \in T$, the inequality $|y_4| \leq B$ is equivalent to

$$-Bv_2y_0^2 \leq v_3y_3^2 - v_0y_1^2y_2^2 \leq Bv_2y_0^2. \quad (5.1)$$

We henceforth write $\Phi(v, y)$ to denote the condition obtained by replacing the term $|y_4|$ by $|(v_3y_3^2 - v_0y_1^2y_2^2)/(v_2y_0^2)|$ in the definition of $\Psi(v, y)$.

The basic idea behind our method is simply to view the equation (4.9) as a congruence

$$v_3y_3^2 \equiv v_0y_1^2y_2^2 \pmod{v_2y_0^2}.$$  

Since we will have $\gcd(v_0y_1^2y_2^2, v_2y_0^2) = 1$ when $(v, y) \in T$, by (4.9), (4.13) and (4.14), there exists a unique positive integer $\varrho \leq v_2y_0^2$ such that

$$\gcd(\varrho, v_2y_0^2) = 1, \quad v_3\varrho^2 \equiv v_0y_1 \pmod{v_2y_0^2},$$

and

$$y_3 \equiv \varrho y_1y_2 \pmod{v_2y_0^2}.$$  

The fact that $y_3$ and $y_4$ satisfy the coprimality conditions (4.13) complicates matters slightly, and makes it necessary to first carry out a Möbius inversion.

Next we analyse the inequality $\Phi(v, y) \leq B$. In doing so it will be convenient to define the quantities

$$V_1 = \left( \frac{B}{v_0v_1^2v_3^2y_0^2y_2} \right)^{1/6} \quad (5.2)$$

and

$$Y_1 = \left( \frac{Bv_2y_0^2}{v_0y_2^2} \right)^{1/3}, \quad Y_2 = \left( \frac{B}{v_0^2v_1^2v_2^2y_0^3} \right)^{1/2}, \quad Y_3 = \left( \frac{Bv_2y_0^2}{v_3} \right)^{1/2}. \quad (5.3)$$

Moreover, we shall need to define the real-valued functions

$$f_-(u, v) = \sqrt{\max\{u^3 - 1, 0\}}, \quad f_+(u, v) = \min\{\sqrt{u^3 + 1}/v^3\},$$

and

$$f(u, v) = f_+(u, v) - f_-(u, v). \quad (5.4)$$

In view of the inequality $v_0^2v_1^2v_2^2v_3^3y_0y_2y_3 \leq B$ that is implied by $\Phi(v, y) \leq B$, we plainly have $y_1 \leq V_1^3Y_3^3/v_1^3$. A little thought therefore reveals that once combined with the inequalities in (5.1), we have

$$Y_3f_-(y_1/Y_1, v_1/V_1) \leq y_3 \leq Y_3f_+(y_1/Y_1, v_1/V_1). \quad (5.5)$$

Using the inequality $v_0^2v_1^2v_2^3v_3^3y_0^2y_2^2 \leq B$, and deducing from (5.1) that $y_1 \geq -Y_1$, we also see that

$$-Y_1 \leq y_1 \leq \frac{V_1Y_1}{v_1}. \quad (5.6)$$
Next it follows from the inequality $\Phi(v, y) \leq B$ that
\[
v_0^4v_1^6v_2^5v_3^4v_6^2 \leq B, \tag{5.7}
\]
whence
\[
1 \leq y_2 \leq Y_2 \tag{5.8}
\]
and $1 \leq v_1 \leq V_1$. In particular we must have $V_1 \geq 1$, and so we may deduce the further inequality
\[
V_1y_1 \leq V_1^3y_1 = \frac{B^{5/6}}{v_0^{7/3}v_2^{13/6}v_3^{5/2}y_0^{4/3}y_2^{5/3}}. \tag{5.9}
\]
This will turn out to be useful at the end of §5.1.

After having taken care of the contribution $S$, say, from the variables $y_3$ and $y_4$ in §5.1, we will proceed in §5.2 by summing $S$ over non-zero integers $y_1$ such that (5.6) holds and positive integers $y_2$ such that (5.8) holds, subject to certain conditions. We shall denote this contribution by $S'$. Finally, in §5.3, we shall obtain an estimate for $N_{U,H}(B)$ by summing $S'$ over the remaining values of $v_0, v_1, v_2, v_3, y_0, y_1$ subject to certain constraints, and then applying Lemma 6.

During the course of the ensuing argument, in which we establish estimates for $S, S'$ and finally $N_{U,H}(B)$, it will be convenient to handle the overall contribution from the error term in each estimate as we go.

5.1 Summation over the variables $y_3$ and $y_4$

We begin by summing over the variables $y_3, y_4$. Let $(v, y_0, y_1, y_2) \in \mathbb{N}^4 \times \mathbb{Z}^3$ satisfy (4.14), (4.15) and be constrained to lie in the region defined by the inequalities (5.6), (5.7) and $y_0, y_2 > 0$. As indicated above, we shall denote the double summation over $y_3$ and $y_4$ by $S$. In order to take care of the coprimality condition $\gcd(y_4, v_1v_2) = 1$ in (4.13), we apply a Möbius inversion to get
\[
S = \sum_{k_4 \mid v_1v_2} \mu(k_4)S_{k_4},
\]
where the definition of $S_{k_4}$ is as for $S$ but with the extra condition $k_4 \mid y_4$ and without the coprimality condition $\gcd(y_4, v_1v_2) = 1$. Thus it follows that $S_{k_4}$ is equal to the number of non-zero integers $y_3$ contained in the region (5.5), such that $\gcd(y_3, v_0y_0y_2) = 1$ and
\[
v_3y_3^2 = v_0y_0^3y_2^2 \pmod{k_4v_2y_0^2}.
\]
Now it straightforward to deduce from (4.9), (4.14), (4.15) and the coprimality relation $\gcd(y_3, v_0y_0y_2) = 1$, that
\[
\gcd(v_0y_0^3y_2^2, k_4v_2y_0^2) = \gcd(v_0y_0^3y_2^2, k_4)
\]
\[
= \gcd(v_0y_0^3y_2^2, v_1v_2, v_3y_3^2)
\]
\[
= \gcd(\gcd(v_0y_0^3y_2^2, v_1), v_3y_3^2)) = 1,
\]
for any $k_4$ dividing $v_1v_2$ and $y_4$. Similarly one sees that $\gcd(v_3, k_4v_2y_0^2) = 1$, for any such $k_4$. We shall therefore only be interested in summing over divisors $k_4 \mid v_1v_2$ for which $\gcd(k_4, v_0y_3y_1y_2) = 1$. In fact it suffices to sum over all
divisors $k_4 \mid v_1v_2$ for which $\gcd(k_4, v_0v_3y_2) = 1$, since any divisor of $v_1v_2$ is coprime to $y_1$ by (4.14). Under this understanding it is now clear that there exists a unique positive integer $\varrho$, with $\varrho \leq k_4v_2y_0^2$ and $\gcd(\varrho, k_4v_2y_0^2) = 1$, such that

$$v_3\varrho^2 \equiv v_0y_1 \pmod{k_4v_2y_0^2}, \quad y_3 \equiv \varrho y_1y_2 \pmod{k_4v_2y_0^2}.$$ 

Our investigation has therefore led to the equality

$$S = \sum_{\gcd(k_4, v_0v_3y_2) = 1} \mu(k_4) \sum_{\varrho \leq k_4v_2y_0^2} S_{k_4}(\varrho),$$

where

$$S_{k_4}(\varrho) = \# \left\{ y_3 \in \mathbb{Z}_*: \gcd(y_3, v_0y_2) = 1, (5.5) \text{ holds}, \quad y_3 \equiv \varrho y_1y_2 \pmod{k_4v_2y_0^2} \right\}.$$ 

Here the coprimality relation $\gcd(y_0, y_3) = 1$ follows from the relations (4.14), (4.15) and $\gcd(\varrho, k_4v_2y_0^2) = 1$.

In view of the fact that $\gcd(k_4, v_0v_3y_2) = 1$, it follows from (4.14) and (4.15) that $\gcd(\varrho y_1y_2, k_4v_2y_0^2) = 1$ in the definition of $S_{k_4}(\varrho)$. In order to estimate $S_{k_4}(\varrho)$ we may therefore employ Lemma 2 with $\kappa = 0$ and the characteristic function

$$\chi(n) = \begin{cases} 
1, & \text{if } \gcd(n, v_0y_2) = 1, \\
0, & \text{otherwise}. 
\end{cases}$$

Now it is easy to see that

$$\sum_{\gcd(d, k_4v_2y_0^2) = 1} \frac{(\chi * \mu)(d)}{d} = \prod_{p \mid v_0y_2} \left( 1 - \frac{1}{p} \right) = \prod_{p \mid v_0y_2} \left( 1 - \frac{1}{p} \right),$$

whence

$$S_{k_4}(\varrho) = \phi^*(v_0y_2) \frac{Y_3f(y_1/Y_1, v_1/V_1)}{k_4v_2y_0^2} + O(2^{\omega(v_0y_2)}).$$

Here, as throughout this paper, we use the notation

$$\phi^*(n) = \frac{\phi(n)}{n} = \prod_{p \mid n} \left( 1 - \frac{1}{p} \right) \quad (5.10)$$

for any $n \in \mathbb{N}$. Note that the number of positive integers $\varrho \leq k_4v_2y_0^2$ such that $\gcd(\varrho, k_4v_2y_0^2) = 1$ and

$$v_3\varrho^2 \equiv v_0y_1 \pmod{k_4v_2y_0^2},$$

is at most $\eta(v_0v_3y_1; k_4v_2y_0^2) \leq 2^{\omega(k_4v_2y_0^2)} + 1 \leq 2^{\omega(v_1v_2y_0)} + 1$ by (3.1). We have therefore established the following result.

**Lemma 8.** Let $(v, y_0, y_1, y_2) \in \mathbb{N}^5 \times \mathbb{Z}_+ \times \mathbb{N}$ satisfy (4.14), (4.15), (5.6) and (5.7). Then for any $B \geq 1$ we have

$$S = \frac{Y_3f(y_1/Y_1, v_1/V_1)}{v_2y_0^2} \Sigma(v, y_0, y_1, y_2) + O(2^{\omega(v_0y_2)} 4^{\omega(v_1v_2y_0)}),$$

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where

\[
\Sigma(v, y_0, y_1, y_2) = \phi^*(v_0y_2) \sum_{k_4 | v_1v_2} \frac{\mu(k_4)}{k_4} \sum_{g \in k_4v_2y_0^2 \mod \gcd(k_4, v_0v_3y_2) = 1} \frac{\omega}{v_3g^2 = v_0y_1 (\mod k_4v_2y_0^2)} 1. \tag{5.11}
\]

We close this section by showing that once summed over all \((v, y_0, y_1, y_2) \in \mathbb{N}^5 \times \mathbb{Z} \times \mathbb{N}\) satisfying (5.6) and (5.7), the error term in Lemma 8 is satisfactory. For this we shall make use of the familiar estimate

\[
\sum_{n \leq x} a^{\omega(n)} \ll x (\log x)^{\omega - 1},
\]

for any \(a \in \mathbb{N}\), in addition to estimates that follow from applying partial summation to it. In this way we therefore obtain the overall contribution

\[
\ll \sum_{v_0, v_1, v_2, v_3, y_0, y_2} \sum_{y_1 < y_1 \leq V_1/y_1} 2^{\omega(v_0y_2)} 4^{\omega(v_1v_2y_0)}
\]

\[
\ll \sum_{v_0, v_1, v_2, v_3, y_0, y_2} 2^{\omega(v_0y_2)} 4^{\omega(v_1v_2y_0)} V_1 Y_1 / v_1
\]

\[
\ll (\log B)^4 \sum_{v_0, v_2, v_3, y_0, y_2} 2^{\omega(v_0y_2)} 4^{\omega(v_2y_0)} V_1 Y_1.
\]

But now we may employ (5.9) to bound this quantity by

\[
\ll B^{5/6} (\log B)^4 \sum_{v_0, v_2, v_3, y_0, y_2} 2^{\omega(v_0y_2)} 4^{\omega(v_2y_0)} \ll B^{5/6} (\log B)^4.
\]

We shall see below that this is satisfactory.

### 5.2 Summation over the variables \(y_1\) and \(y_2\)

Our next task is to sum \(S\) over all non-zero integers \(y_1\) which satisfy (4.14) and (5.6), and all positive integers \(y_2\) which satisfy \(\gcd(y_2, v_2v_3y_0) = 1\) and (5.8). We therefore write

\[
S' = \frac{Y_3}{v_2y_0^2} \sum_{y_2 \leq Y_3} \sum_{\gcd(y_2, v_2v_3y_0) = 1} f(y_1/Y_1, v_1/V_1) \Sigma(v, y_0, y_1, y_2),
\]

where \(\Sigma(v, y_0, y_1, y_2)\) is given by (5.11).

Let \(t > 0\). We begin by establishing asymptotic formulae for the two quantities

\[
S(\pm t) = \phi^*(v_0y_2) \sum_{k_4 | v_1v_2} \frac{\mu(k_4)}{k_4} \sum_{g \in k_4v_2y_0^2 \mod \gcd(k_4, v_0v_3y_2) = 1} \frac{\omega}{v_3g^2 = v_0y_1 (\mod k_4v_2y_0^2)} S'_{k_4}(g; \pm t),
\]

where

\[
S'_{k_4}(g; +t) = \# \left\{ 0 < y_1 \leq t : \gcd(y_1, v_0v_1v_2v_3y_0) = 1, \ v_3g^2 = v_0y_1 (\mod k_4v_2y_0^2) \right\},
\]

\[
S'_{k_4}(g; -t) = \# \left\{ -t < y_1 \leq 0 : \gcd(y_1, v_0v_1v_2v_3y_0) = 1, \ v_3g^2 = v_0y_1 (\mod k_4v_2y_0^2) \right\}.
\]
Now it is clear that we have
\[ \gcd(v_3 q^2, k_4 v_2 y_0^2) = 1 \] (5.12)
in the definition of \( S_{k_4}^t(v; \pm t) \), since \( \gcd(k_4, v_3) = 1 \). In particular it follows that we may replace the coprimality relation appearing in \( S_{k_4}^t(v; \pm t) \) by \( \gcd(y_1, v_0 v_1 v_3) = 1 \). We shall treat this coprimality condition with a Möbius inversion. Thus we find that \( S(\pm t) \) is equal to
\[
\sum_{\gcd(k_4, v_0 v_1 v_3) = 1} \frac{\mu(k_4)}{k_4} \sum_{\gcd(k_1, k_4 v_1 v_3) = 1} \mu(k_1) \sum_{\gcd(t/k_1, k_4 v_2 y_0^2) = 1} S_{k_1, k_4}^t(v; \pm t),
\]
where
\[
S_{k_1, k_4}^t(v; \pm t) = \# \{ 0 \leq y_1 \leq t/k_1 : v_3 q^2 \equiv k_1 v_0 y_1 \pmod{k_4 v_2 y_0^2} \},
\]
\[
S_{k_1, k_4}^t(v; \mp t) = \# \{ -t/k_1 \leq y_1 \leq 0 : v_3 q^2 \equiv k_1 v_0 y_1 \pmod{k_4 v_2 y_0^2} \}.
\]
Here we have used (5.12) to deduce that we must only sum over values of \( k_1 \mid v_0 v_1 v_3 \) for which \( \gcd(k_1, k_4 v_2 y_0) = 1 \).

Let \((v, y_0) \in \mathbb{N}^2\) satisfy the constraints
\[ |\mu(v_0 v_1 v_3)| = \gcd(v_0 v_3, y_0) = 1, \quad v_0^4 v_1^6 v_2^5 y_0^3 \leq B, \] (5.13)
that follow from (4.15) and (5.7). We let \( b_\pm \leq k_4 v_2 y_0^2 \) be the unique positive integer such that
\[ b_\pm k_4 v_0 \equiv \pm v_3 \pmod{k_4 v_2 y_0^2}. \]
In particular it follows from (5.12) that \( \gcd(b_\pm, k_4 v_2 y_0^2) = 1 \), and we may therefore employ Lemma 3 to deduce that
\[
S_{k_1, k_4}^t(v; \pm t) = \frac{t}{k_1 k_4 v_2 y_0^2} + r(\pm t; b_\pm q^2),
\]
where
\[ r(\pm t; b_\pm q^2) = \psi\left(\frac{-b_\pm q^2}{k_4 v_2 y_0^2}\right) - \psi\left(\frac{t/k_1 - b_\pm q^2}{k_4 v_2 y_0^2}\right). \] (5.14)
Recall the definition (5.10) of \( \phi^* \) and observe that \( \phi^*(ab)\phi^*(\gcd(a, b)) = \phi^*(a)\phi^*(b) \), for any \( a, b \in \mathbb{N} \). We define
\[
\vartheta(v, y_0, v_2) = \begin{cases} 
\phi^*(v_0 v_1 v_2 y_2) \phi^*(v_0 v_1 v_2 y_3 v_0) & \text{if (4.15) holds},
\phi^*(\gcd(v_1, v_3)) & \text{otherwise}.
\end{cases}
\] (5.15)
Then a straightforward calculation reveals that
\[ S(\pm t) = \vartheta(v, y_0, v_2) t + \mathcal{R}(\pm t) \] (5.16)
for any non-zero \( t > 0 \), where
\[
\mathcal{R}(\pm t) = \phi^*(v_0 y_2) \sum_{\gcd(k_4, v_0 v_1 v_3) = 1} \frac{\mu(k_4)}{k_4} \sum_{\gcd(k_1, k_4 v_2 y_0) = 1} \mu(k_1) \sum_{\gcd(t/k_1, k_4 v_2 y_0) = 1} r(\pm t; b_\pm q^2).
\]
Here $r(\pm t; b_+ g^2)$ is given by (5.14) and the positive integers $b_-, b_+$ are uniquely determined by fixed choices of $k_1, k_4, v_0, v_2, v_3, y_0$, as outlined above.

We may now apply partial summation to estimate $S'$. Now it is clear that $S' = S'_- + S'_+$, where $S'_-$ denotes the contribution from $y_1$ contained in the interval $(-Y_1, 0)$ and $S'_+$ denotes the contribution from $y_1$ contained in the interval $(0, Y_1/v_1)$. We begin by estimating $S'_-$, for which we first deduce from (5.2) and (5.3) that

$$
\frac{v_1}{V_1} = \left( \frac{y_2}{Y_2} \right)^{1/3}, \quad Y_1 = \left( \frac{Bv_2y_0^2}{v_0y_2^2} \right)^{1/3} = \frac{Y'_1}{y_2^{2/3}},
$$
say. We may now apply (5.16), in conjunction with partial summation, in order to deduce that

$$
S'_- = \sum_{\gcd(y_2, v_2 v_3 y_0) = 1} \left( \frac{\vartheta(v, y_0, y_2)Y_3}{v_2y_0^2} \int_{-1}^{0} f(u, v_1/V_1) \, du \right) + R'_-,
$$

where

$$
R'_- = \frac{Y_3}{v_2y_0^2} \sum_{\gcd(y_2, v_2 v_3 y_0) = 1} \int_{0}^{1} f'(-u, (y_2/Y_2)^{1/3}) R(-uY'_1/y_2^{2/3}) \, du
$$

and

$$
= \frac{Y_3}{v_2y_0^2} \sum_{\gcd(k_4, v_0 v_2) = 1} \frac{\mu(k_4)}{k_4} \sum_{\gcd(k_1, k_4 v_2 y_0) = 1} \mu(k_1) \sum_{\gcd(g, k_4 v_2 y_0) = 1} \sum_{\gcd(y_2, k_4 v_2 v_3 y_0) = 1} \phi^*(v_0 y_2) \int_{0}^{1} f'(-u, (y_2/Y_2)^{1/3}) r(-uY'_1/y_2^{2/3}; b_- g^2) \, du.
$$

Define the arithmetic function

$$
\phi^* (n) = \prod_{p|n} \left( 1 + \frac{1}{p} \right)^{-1}.
$$

We estimate $R'_-$ via an application of Lemma 2 with $a = 0, q = 1$ and $\kappa = \varepsilon$. This gives

$$
\sum_{\gcd(y_2, k_4 v_2 v_3 y_0) = 1} \phi^*(v_0 y_2) = \frac{6}{\pi^2} \phi^*(k_4 v_0 v_2 v_3 y_0) t + O(2^\omega(v_1 v_2 v_3 y_0) t^\varepsilon).
$$

Indeed, the corresponding Dirichlet series is equal to

$$
\phi^*(v_0) \zeta(s) \prod_{p|k_4} \left( 1 - \frac{1}{p^{s+1}} \right) \prod_{p|k_4 v_2 v_3 y_0} \left( 1 - \frac{1}{p^s} \right).
$$

An application of partial summation therefore yields the estimate

$$
R'_- = \frac{\varphi_- (v, y_0) Y_2 Y_3}{v_2y_0^2} + O \left( 2^\omega(v_1 v_2) + \omega(v_0 v_2 v_3) + \omega(v_1 v_2 v_3 y_0) Y_2^\varepsilon Y_3^\varepsilon \right) + O_\varepsilon \left( B^\varepsilon Y_3^\varepsilon \right),
$$

(5.17)
where
\[
\varphi_-(\mathbf{v}, y_0) = \frac{18}{\pi^2} \sum_{\substack{k_4|v_1v_2 \\ \gcd(k_4, v_1v_2) = 1}} \frac{\mu(k_4) \phi_4(k_4v_0v_2v_3y_0)}{k_4} \sum_{\substack{k_1|v_0v_1v_3 \\ \gcd(k_1, k_4v_2v_3y_0) = 1}} \mu(k_1) 
\]
\[
\int_0^1 \int_0^1 (t^2 f'(-u, t) \sum_{\theta \in k_4v_2y_0^2} r(-v_0v_1^2v_2^2v_3y_0^2u/t^2; b-\theta^2))u \, dt. 
\]

Here we have used the trivial inequality $2^\omega(n) = O_e(n^\varepsilon)$ for any $n \in \mathbb{N}$. An application of Lemma 5 clearly reveals that
\[
\varphi_- (\mathbf{v}, y_0) \ll \varepsilon \left( v_2y_0^2 \right)^{1/2 + \varepsilon} 2^\omega(v_1v_2) + \omega(v_0v_1v_3),
\]
for any $\varepsilon > 0$. Our estimate (5.17) for $R_-'$ isn't terribly good when $Y_2$ is small. Fortunately, by inverting the order of summation over $\theta$ and $y_2$ we may use Lemma 5 to deduce the alternative estimate
\[
R_-' = \frac{\varphi_-(\mathbf{v}, y_0)Y_2Y_3}{v_2y_0^2} + O_e \left( 2^\omega(v_1v_2) + \omega(v_0v_1v_3) \right) 
\]
\[= \frac{\varphi_-(\mathbf{v}, y_0)Y_2Y_3}{v_2y_0^2} + O_e \left( B^\varepsilon \frac{Y_2Y_3}{(v_2y_0^2)^{1/2}} \right). \tag{5.18}
\]

Note here that the main term is dominated by the error term. On combining (5.17) and (5.18), however, we obtain the estimate
\[
R_-' = \frac{\varphi_-(\mathbf{v}, y_0)Y_2Y_3}{v_2y_0^2} + O_e \left( B^\varepsilon \frac{Y_2}{(v_2y_0^2)^{1/2}} \right). 
\]

Arguing in a similar fashion it is straightforward to deduce that
\[
S_+ = \sum_{y_2 \leq Y_2} \left( \frac{\partial(\mathbf{v}, y_0, y_2)Y_1Y_3}{v_2y_0^2} \int_0^{Y_1/v_1} f(u, v_1/V_1) \, du \right) + R_+', 
\]
where
\[
R_+' = \frac{\varphi_+(\mathbf{v}, y_0)Y_2Y_3}{v_2y_0^2} + O_e \left( B^\varepsilon \frac{Y_2}{(v_2y_0^2)^{1/2}} \right). 
\]
Here one finds that
\[
\varphi_+(\mathbf{v}, y_0) = \frac{18}{\pi^2} \sum_{\substack{k_4|v_1v_2 \\ \gcd(k_4, v_1v_2) = 1}} \frac{\mu(k_4) \phi_4(k_4v_0v_2v_3y_0)}{k_4} \sum_{\substack{k_1|v_0v_1v_3 \\ \gcd(k_1, k_4v_2v_3y_0) = 1}} \mu(k_1) 
\]
\[
\int_0^1 \int_0^1 (t^2 f'(u, t) \sum_{\theta \in k_4v_2y_0^2} r(-v_0v_1^2v_2^2v_3y_0^2u/t^2; b-\theta^2))u \, dt, 
\]
with $\varphi_+(\mathbf{v}, y_0) \ll \varepsilon \left( v_2y_0^2 \right)^{1/2 + \varepsilon} 2^\omega(v_1v_2) + \omega(v_0v_1v_3)$. 

We may now complete our estimate for $S'$. Recall the definition (5.4) of the function $f(u, v)$, and define

$$g(v) = \int_{-1}^{1/v} f(u, v)du.$$  

(5.19)

Then $g$ is a bounded differentiable function, whose derivative is also bounded on the interval $[0, \infty)$. Moreover let

$$\varphi(v, y_0) = \varphi_-(v, y_0) + \varphi_+(v, y_0).$$  

(5.20)

Then on combining our various estimates we have therefore established the following result.

**Lemma 9.** Let $(v, y_0) \in \mathbb{N}^5$ satisfy (5.13). Then for any $B \geq 1$ we have

$$S' = \sum_{y_2 \leq Y_2 \atop \gcd(y_2, v_2y_3y_0) = 1} \left( \frac{\vartheta(v, y_0, y_2)Y_1Y_3g(v_1/V_1)}{v_2y_0^2} + \frac{\varphi(v, y_0)Y_2Y_3}{v_2y_0^2} \right) + O_\varepsilon \left( B^6Y_3 \min \left\{ 1, \frac{Y_2}{(v_2y_0^2)^{1/2}} \right\} \right),$$  

where $\vartheta(v, y_0, y_2)$ is given by (5.15), $g$ is given by (5.19) and $\varphi(v, y_0)$ is given by (5.20) and satisfies

$$\varphi(v, y_0) \ll \varepsilon (v_2y_0^2)^{1/2} + 2\omega(v_1v_2) + \omega(v_0v_1v_3),$$  

(5.21)

for any $\varepsilon > 0$.

We end this section by showing that once summed over all $(v, y_0) \in \mathbb{N}^5$ satisfying (5.13), the error term in Lemma 9 is satisfactory. On recalling the definition (5.3) of $Y_2$ and $Y_3$, and then first summing over $y_0$, we easily obtain the satisfactory overall contribution

$$\ll \varepsilon B^{1/2+\varepsilon} \sum_{v_0, v_1, v_2, v_3, y_0} (v_2y_0^2)^{1/2} \frac{1}{v_3} \left( \frac{B^{1/2}}{v_0^2v_1^2v_2^2v_3^2y_0^3} \right)$$  

$$\ll \varepsilon B^{5/6+\varepsilon} \sum_{v_0, v_1, v_2, v_3} \frac{1}{v_0^{4/3}v_1^{2/3}v_2^{2/3}v_3} \ll \varepsilon B^{5/6+\varepsilon}.$$  

### 5.3 Summation over the remaining variables

In this section we complete our preliminary estimate for $N_{U,H}(B)$. It is clear from Lemma 9 that we have two distinct terms to deal with. We begin by deducing from (5.3) that

$$\frac{Y_1Y_3}{v_2y_0^2} = \frac{B^{5/6}n^{1/6}}{v_0v_1v_2v_3y_0y_2},$$  

in the statement of Lemma 9, with $n = v_0^4v_1^6v_2^5v_3^3y_0y_2^2$. Define the arithmetic function

$$\Delta(n) = B^{-5/6} \sum_{v_0, v_1, v_2, v_3, y_0, y_2 \atop v_0^4v_1^6v_2^5v_3^3y_0y_2^2 = n} \frac{\vartheta(v, y_0, y_2)Y_1Y_3}{v_2y_0},$$  

(5.22)

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where \( \vartheta(v, y_0, y_2) \) is given by (5.15). Recall the definition (5.19) of the function \( g \) and that of the counting function \( N(Q_1, Q_2; B) \) that appears in the statement of Lemma 6. Let \( \varepsilon > 0 \). We proceed by establishing the existence of a constant \( \beta \in \mathbb{R} \) for which

\[
N(Q_1, Q_2; B) = B^{5/6} \sum_{n \leq B} \Delta(n) g \left( \left( \frac{n}{B} \right)^{1/6} \right) + \beta B + O_\varepsilon \left( B^{5/6 + \varepsilon} \right). \tag{5.23}
\]

This follows rather easily from Lemma 9. Define the sum

\[
T(B) = \sum_{v, y_0 \text{ (5.13) holds}} \frac{\varphi(v, y_0) Y_2 Y_3}{v_2 y_0},
\]

for any \( B \geq 1 \). Then in view of the error terms that we have estimated along the way in §5.1 and §5.2, it is clearly enough to establish the existence of a constant \( \beta \in \mathbb{R} \) for which

\[
T(B) = \beta B + O_\varepsilon \left( B^{5/6 + \varepsilon} \right).
\]

On recalling (5.3), we see that

\[
Y_2 Y_3 = B \frac{v_0 v_1 v_2 v_3 y_0}{v_0^3 v_1^3 v_2^3 y_0^3}.
\]

Hence on taking \( \varepsilon < 1/3 \), it follows from (5.21) that

\[
T(B) - \beta B \ll \varepsilon B \sum_{v, y_0 \atop v_0 v_1 v_2 v_3 y_0 \geq B} \frac{(v_0 v_1 v_2 v_3 y_0)^\varepsilon}{v_0^3 v_1^3 v_2^3 v_3^3 y_0^3} \ll \varepsilon \left( B^{5/6} \right),
\]

with

\[
\beta = \sum_{v, y_0 \atop \gcd(v_0 v_1 v_2 y_0) = 1} \left| \frac{\mu(v_0 v_1 v_2 y_0)}{v_0^3 v_1^3 v_2^3 y_0^3} \right| \varphi(v, y_0) \tag{5.24}
\]

This therefore completes the proof of (5.23). On inserting this estimate into Lemma 6 we obtain the following result.

**Lemma 10.** Let \( \varepsilon > 0 \). Then for any \( B \geq 1 \) we have

\[
N_{U,H}(B) = 2B^{5/6} \sum_{n \leq B} \Delta(n) g \left( \left( \frac{n}{B} \right)^{1/6} \right) + \left( \frac{12}{\pi^2} + 2\beta \right) B + O_\varepsilon \left( B^{5/6 + \varepsilon} \right),
\]

where \( g \) is given by (5.19), \( \Delta \) is given by (5.22) and \( \beta \) is given by (5.24).

## 6 The height zeta function

For \( \Re(s) > 1 \) we recall the definition of the height zeta function (1.2), and the identity (1.5). Thus it follows from Lemma 10 that \( Z_{U,H}(s) = Z_1(s) + Z_2(s) \),
where

\[ Z_1(s) = 2s \int_{1}^{\infty} t^{-s-1/6} \sum_{n \leq t} \Delta(n) g \left( \left( \frac{n}{t} \right)^{1/6} \right) dt, \]

\[ Z_2(s) = \frac{12/\pi^2 + 2\beta}{s-1} + G_2(s), \]

and

\[ G_2(s) = s \int_{1}^{\infty} t^{-s-1} R(t) dt \]

for some function \( R(t) \) such that \( R(t) \ll t^{5/6+\varepsilon} \) for any \( \varepsilon > 0 \). But then it easily follows that \( G_2(s) \) is holomorphic on the half-plane \( \Re(s) \geq 5/6 + \varepsilon \), and satisfies \( G_2(s) \ll 1 + |3m(s)| \) on this domain. Finally an application of the Phragmèn-Lindelöf Theorem yields the finer upper bound

\[ G_2(s) \ll (1 + |3m(s)|) \left( 6(1 - \Re(s)) + \varepsilon \right). \]

on this domain.

To establish Theorem 1 it therefore remains to analyse the function \( Z_1(s) \). Recall the definition (5.22) of \( \Delta \) and define the corresponding Dirichlet series

\[ F(s) = \sum_{n=1}^{\infty} \frac{\Delta(n)}{n^s}. \]

Then it is easily seen that

\[ Z_1(s) = 2sF(s-5/6) \int_{1}^{\infty} t^{-s-1/6} g(1/t^{1/6}) dt = F(s-5/6)G_{1,1}(s), \]

where

\[ G_{1,1}(s) = 12s \int_{0}^{1} v^{6s-6} g(v) dv. \] (6.1)

Recall the definition (5.19) of \( g \). Then a simple calculation reveals that \( G_{1,1}(1) = 12\tau_{\infty} \), in the notation of (1.6). Moreover, an application of partial integration yields

\[ G_{1,1}(s) = \frac{12s}{6s-5} \left( g(1) - \int_{0}^{1} v^{6s-5} g'(v) dv \right), \]

whence it is clear that \( G_{1,1}(s) \) is holomorphic and bounded on the half-plane \( \Re(s) \geq 5/6 + \varepsilon \) for any \( \varepsilon > 0 \).

We proceed by analysing the Dirichlet series \( F(s-5/6) \) in more detail. Define the function

\[ G_{1,2}(s) = \frac{F(s-5/6)}{E_1(s)E_2(s)}, \] (6.2)

for \( \Re(s) > 5/6 \) and let \( \varepsilon > 0 \). Here \( E_1(s) \) and \( E_2(s) \) are given by (1.3) and (1.4), respectively. In order to complete the proof of Theorem 1, with

\[ G_1(s) = G_{1,1}(s)G_{1,2}(s), \] (6.3)

it remains to establish that \( G_{1,2}(1) \neq 0 \) and that \( G_{1,2}(s) \) is holomorphic and bounded for \( \Re(s) \geq 5/6 + \varepsilon \). This is achieved for us in the following result.
Lemma 11. Let $\varepsilon > 0$. Then $G_{1,2}(s + 1)$ is holomorphic and bounded on the half-plane $H = \{ s \in \mathbb{C} : \Re(s) \geq -1/6 + \varepsilon \}$.

Proof. On writing
\[ G_{1,2}(s + 1) = \prod_p G_p(s + 1), \]
it will clearly suffice to show that $G_p(s + 1) = 1 + O_\varepsilon(1/p^{1+\varepsilon})$ uniformly on $H$. We begin the proof of Lemma 11 by observing that
\[
F(s + 1) = \sum_{(v, y_0, y_2) \in \mathbb{N}^6 \atop \gcd(v_3y, y_2) = 1} \phi^*(v_0v_1v_2y_2) \phi^*(v_0v_1v_2v_3y_6),
\]
Thus on writing $F(s + 1) = \prod_p F_p(s + 1/6)$ as a product of local factors, a straightforward calculation reveals that $F_p(s + 1/6)$ is equal to
\[
1 + \frac{1}{p^{2s+1} - 1} + \frac{1}{p^{4s+1} - 1} + (1 - 1/p)^2 \left( \frac{p^{2s+1}}{p^{2s+1} - 1} + \frac{1}{p^{4s+1} - 1} \right) + \left( \frac{p^{6s+1}}{p^{6s+1} - 1} \right),
\]
for any prime $p$. On collecting together factors of $(p^{2s+1} - 1)$ and $(p^{4s+1} - 1)$ we see that
\[
F_p(s + 1/6) \left( 1 - \frac{1}{p^{6s+1}} \right) = 1 - \frac{1}{p^{6s+1}} + \frac{1}{p^{4s+1}} + \frac{1}{p^{2s+1}} - \frac{1}{p^{6s+1}} - \frac{1}{p^{4s+1}} - \frac{1}{p^{2s+1}} - \frac{1}{p^{3s+1}},
\]
on $H$. We now record the obvious estimates
\[
\frac{1}{p^{2s+1} - 1} = \frac{1}{p^{2s+1}} + \frac{1}{p^{4s+2}} + O_\varepsilon \left( \frac{1}{p^{2s+6_\varepsilon}} \right),
\]
and
\[
1 + \frac{1}{p^{2s+1}} + \frac{1}{p^{4s+2}} + \frac{1}{p^{6s+3}} + O_\varepsilon \left( \frac{1}{p^{4s+16_\varepsilon}} \right),
\]
that all hold on $H$. Combining these estimates we therefore deduce that
\[
F_p(s + 1/6) \left( 1 - \frac{1}{p^{6s+1}} \right) = 1 + \frac{1}{p^{2s+1}} + \frac{1}{p^{4s+1}} + \frac{2}{p^{2s+1}} + \frac{1}{p^{4s+1}} + \frac{1}{p^{6s+1}} + \frac{1}{p^{8s+1}} + \frac{1}{p^{9s+2}} + \frac{1}{p^{13s+3}} + O_\varepsilon \left( \frac{1}{p^{1+\varepsilon}} \right).
\]
Write $E_{1,p}(s+1)$ for the Euler factor of (1.3) and write $E_{2,p}(s+1)$ for the Euler factor of (1.4). Then it is now a routine matter to deduce that
\[
\frac{F_p(s+1/6)}{E_{1,p}(s+1)} = 1 - \frac{3}{p^{s+2}} - \frac{3}{p^{2s+2}} - \frac{1}{p^{3s+2}} - \frac{1}{p^{10s+2}} + \frac{3}{p^{13s+3}} \\
+ \frac{1}{p^{14s+3}} + O\left(\frac{1}{p^{1+\varepsilon}}\right) \\
= E_{2,p}(s+1)\left(1 + O\left(\frac{1}{p^{1+\varepsilon}}\right)\right),
\]
on $\mathcal{H}$, which therefore completes the proof of Lemma 11.

It remains to combine the expression (6.4) for $F_{p}(s+1/6)$, with (1.3) and (6.2) in order to deduce that
\[
E_{2}(1)G_{1,2}(1) = \prod_p \left(1 - \frac{1}{p}\right)^6 \left(1 + \frac{6}{p} + \frac{1}{p^2}\right) \neq 0.
\]
This therefore completes the proof of Theorem 1.

7 Deduction of Theorem 2

In this section we shall deduce Theorem 2 from Theorem 1 and Lemma 10. Let $\varepsilon > 0$ and let $T \in [1, B]$. Then an application of Perron’s formula yields
\[
N_{U,H}(B) - \left(\frac{12}{\pi^2} + 2\beta\right)B = \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} E_1(s)E_2(s)G_1(s)\frac{B^s}{s}ds \\
+ O\left(\frac{B^{11/6+\varepsilon}}{T}\right).
\]
We apply Cauchy’s residue theorem to the rectangular contour $C$ joining the points $\kappa-iT$, $\kappa+iT$, $1+\varepsilon+iT$ and $1+\varepsilon-iT$, for any $\kappa \in [11/12, 1]$. We must calculate the residue of $E_1(s)E_2(s)G_1(s)B^s/s$ at $s = 1$. For $\Re(s) > 9/10$ Theorem 1 implies that the product $E_2(s)G_1(s)$ is holomorphic and bounded. In view of (1.3), we see that
\[
E_1(s) = \frac{1}{2880(s-1)^6} + O\left(\frac{1}{(s-1)^5}\right),
\]
as $s \to 1$. Hence it follows that
\[
\text{Res}_{s=1} \left\{E_1(s)E_2(s)G_1(s)\frac{B^s}{s}\right\} = \frac{E_2(1)G_1(1)}{5\cdot 2880} BQ_1(\log B),
\]
for some monic polynomial $Q_1$ of degree 5. Recall from (6.3) that $G_1 = G_{1,1}G_{1,2}$. Then we have already seen in the previous section that $G_1(1) = 12\tau_\infty$, in the notation of (1.6) and (2.5). Putting all of this together we have therefore shown that
\[
\frac{1}{2\pi i} \int_C E_1(s)E_2(s)G_1(s)\frac{B^s}{s}ds = \frac{\tau_\infty}{2880} BQ_2(\log B),
\]
for some monic polynomial $Q_2$ of degree 5. Define the difference
\[
E(B) = N_{U,H}(B) - \frac{\tau_\infty}{2880} BQ_2(\log B) - \left(\frac{12}{\pi^2} + 2\beta\right)B,
\]
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Then, in view of (7.1) and the fact that the product $E_2(s)G_1(s)$ is holomorphic and bounded for $\Re(s) > 9/10$, we deduce that

$$E(B) \ll \varepsilon \frac{B^{11/6 + \varepsilon}}{T} + \left( \int_{\kappa-iT}^{\kappa+iT} + \int_{\kappa-iT}^{1+\varepsilon-iT} + \int_{1+\varepsilon+iT}^{\kappa+iT} \right) \left| E_1(s) \frac{B^s}{s} \right| ds, \quad (7.2)$$

for any $\kappa \in [11/12, 1)$ and any $T \in [1, B]$. We begin by estimating the contribution from the horizontal contours. Recall the well-known convexity bound

$$\zeta(\sigma + it) \ll |t|^{(1-\sigma)/3+\varepsilon},$$

that is valid for any $\sigma \in [1/2, 1]$ and $|t| \geq 1$. Then it follows that

$$E_1(\sigma + it) \ll |t|^{(1-\sigma)+\varepsilon} \quad (7.3)$$

for any $\sigma \in [11/12, 1]$ and $|t| \geq 1$. This estimate allows us to deduce that

$$\int_{\kappa-iT}^{1+\varepsilon-iT} \left| E_1(s) \frac{B^s}{s} \right| ds \ll \varepsilon \int_{\kappa}^{1+\varepsilon} B^\sigma T^{7-8\sigma+\varepsilon} d\sigma \ll \varepsilon \frac{B^{1+\varepsilon}T^\varepsilon}{T} + B^\kappa T^{7-8\kappa+\varepsilon}. \quad (7.4)$$

One obtains the same estimate for the contribution from the remaining horizontal contour joining $\kappa + iT$ to $1 + \varepsilon + iT$.

We now turn to the size of the integral

$$\int_{\kappa-iT}^{\kappa+iT} \left| E_1(s) \frac{B^s}{s} \right| ds \ll B^\kappa \int_{-T}^{T} \frac{|E_1(\kappa + it)|}{1 + |t|} dt = B^\kappa I(T), \quad (7.5)$$

say. For given $0 < U \ll T$, we begin by estimating the contribution to $I(T)$ from each integral

$$\int_{U}^{2U} \frac{|E_1(\kappa + it)|}{1 + |t|} dt \ll \frac{1}{U} \int_{U}^{2U} |E_1(\kappa + it)| dt = \frac{J(U)}{U},$$

say. Let $\varepsilon > 0$ and let $k \in \mathbb{N}$. Then we define $\sigma_k$ to be the infimum of $\sigma$ such that

$$\frac{1}{T} \int_{1}^{T} |\zeta(\sigma + it)|^{2k} dt = O_\varepsilon(T^\varepsilon).$$

It therefore follows from the mean-value theorem in [13, §7.8] that

$$\int_{U}^{2U} |\zeta(\sigma + it)|^{2k} dt \ll \varepsilon U^{1+\varepsilon} \quad (7.6)$$

for any $\sigma \in (\sigma_k, 1]$, and any $U \geq 1$. We shall apply this estimate in the cases $k = 2$ and $k = 4$, for which we combine a result due to Heath-Brown [7] with well-known estimates for the fourth moment of $|\zeta(1/2 + it)|$ in order to deduce that

$$\sigma_k \leq \begin{cases} 
1/2, & k = 2, \\
5/8, & k = 4.
\end{cases} \quad (7.7)$$
Returning to our estimate for $J(U)$, for fixed $0 < U \ll T$ and any $\kappa \in [11/12, 1)$, we define $J(U; c) = \int_0^{2U} |\zeta(c\kappa - c + 1 + cit)|^4 dt$. Then we may apply Hölder’s inequality to deduce that

$$J(U) \leq J(U; 6)^{1/4} J(U; 5)^{1/4} J(U; 4)^{1/4} J(U; 3)^{1/8} J(U; 2)^{1/8}.$$ 

On combining (7.6), (7.7) and the fact that $\kappa \in [11/12, 1)$, we therefore deduce that $J(U) \ll U^{1+\varepsilon}$, on re-defining the choice of $\varepsilon$. Summing over dyadic intervals for $0 < U \ll T$ we obtain

$$\int_0^T \frac{|E_1(\kappa + it)|}{1 + |t|} dt \ll \varepsilon T^\varepsilon.$$

We obtain the same estimate for the integral over the interval $[-T, 0]$, and so it follows that $I(T) \ll \varepsilon T^\varepsilon$. We may insert this estimate into (7.5), and then combine it with (7.4) in (7.2), in order to conclude that

$$E(B) \ll \frac{B^{11/6+\varepsilon} T^\varepsilon}{T} + B^\kappa T^\varepsilon,$$

for any $T \in [1, B]$. We therefore complete the proof of Theorem 2 by taking $T = B$.

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