EUCLIDEAN COMPLETE HYPERSURFACES OF A MONGE-AMPÈRE EQUATION

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ABSTRACT. We study the Monge-Ampère Equation

\[ \det D^2 u = u^p, \quad \forall x \in \Omega \]

for some \( p \in \mathbb{R} \). A solution \( u \) of (0.1) is called to be Euclidean complete if it is an entire solution defined over the whole \( \mathbb{R}^n \) or its graph is a large hypersurface satisfying the large condition

\[ \lim_{x \to \partial \Omega} u(x) = \infty \]

on boundary \( \partial \Omega \) in case of \( \Omega \neq \mathbb{R}^n \). In this paper, we will give various sharp conditions on \( p \) and \( \Omega \) classifying the Euclidean complete solution of (0.1). Our results clarify and extend largely the existence theorem of Cirstea-Trombetti (Calc. Var., 31, 2008, 167-186) for bounded convex domain and \( p > n \).

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1. Introduction

The Monge-Ampère Equation

\[(1.1) \det D^2 u = f(u), \ \forall x \in \Omega \]

with a semilinear term \(f\) has been studied extensively, see for examples [2, 5, 11, 15, 16, 21–25] about the existence and regularity of Dirichlet problems and the references therein. A problem concerning the solutions of (1.1) satisfying the boundary blow-up condition \(u = \infty\) on \(\partial \Omega\) has been proposed and studied by Cheng-Yau [6, 7] for exponential nonlinear term \(f(u) = e^u\) due to their applications in geometry. It was later been considered by Lazer-McKenna [17] for nonlinear term \(f(u) = u^p\) of power law. Some other results can be found in [3, 12, 18, 19, 26].

Throughout this paper, a solution \(u\) of (1.1) is called to be Euclidean complete if it is an entire graph defined over the whole Euclidean space \(\mathbb{R}^n\) or it is a large graph satisfying the large condition

\[(1.2) \lim_{x \to \partial \Omega} u(x) = \infty\]

on boundary \(\partial \Omega\) in case of \(\Omega \neq \mathbb{R}^n\). A first satisfactory existence result was shown by Cirstea-Trombetti in [3]. Using a hypothesis \((H_1)\)

\[
\int_1^\infty \frac{du}{\sqrt{F(u)}} < \infty, \ 	ext{where} \ F(u) \equiv \int_0^u f_0^{1/n}(s) ds
\]

together with another mild assumption

\[(H_2) \begin{cases} f_0^{1/n} \text{ is locally Lipschitz continuous on } [0, \infty), \\
\text{positive and non-decreasing on } (0, \infty) \text{ with } f_0(0) = 0,
\end{cases}\]

they have proven the following result.

**Theorem 1.1.** Let \(\Omega\) be a smooth, strictly convex, bounded domain in \(\mathbb{R}^n\) with \(n \geq 2\). Suppose that there exists a function \(f_0\) on \([0, \infty)\) such that:

\(A1\) \(f_0(u) \leq f(u)\) for every \(u > 0\);

\(A2\) \(f_0\) satisfies assumptions \((H1)\) and \((H2)\);

\(A3\) \(f_0^{1/n}\) is convex on \((0, \infty)\).

Then (1.1) admits a strictly convex large solution on \(\Omega\).

This result was later generalized to Hessian equations by Huang in [14], and was generalized to Monge-Ampère equations with weight by Zhang, Du-Zhang in [8, 26, 28]. See also [10, 20, 27] for some other extensions. The first part of this paper is devoted to prove the counterpart of Theorem 1.1 as following.
**Theorem 1.2.** Supposing that $f$ is an unbounded function satisfying

$$\limsup_{u \to \infty} \frac{f(u)}{u^n} < \infty,$$

then for any bounded smooth convex domain $\Omega \subset \mathbb{R}^n$, there is no convex large solution to (1.1).

Unlike the case of bounded domain, when one considers the entire solution defined on whole $\Omega = \mathbb{R}^n$, the situation changes dramatically. In fact, we have the following sharp non-existence result for $p > n$.

**Theorem 1.3.** Supposing that $f \in C((0, \infty))$ is a positive monotone non-decreasing function on $u$ and satisfies

$$f(u) \geq Au^p, \quad \forall u > 0$$

for some $p > n$ and $A > 0$, then there is no positive entire convex solution of (1.1) on $\mathbb{R}^n$.

With the help of the technics developed in [8], we can also prove the following sharp existence result.

**Theorem 1.4.** Considering (0.1) for $p < n$ and $\Omega = \mathbb{R}^n$, there are infinitely many positive convex solutions which are affine inequivalent. More precisely, for any $a_0 > 0$, there exists at least one positive convex solution $u$ of (0.1) satisfying

$$u(0) = a_0, \quad Du(0) = 0.$$

Finally, we turn to the case of unbounded domain $\Omega \neq \mathbb{R}^n$ and prove the following result for $p > n$.

**Theorem 1.5.** Supposing that $f \in C((0, \infty))$ is a positive monotone non-decreasing function on $u$ and satisfies (1.4) for some $p > n, n \geq 2$ and $A > 0$, then there is no positive entire convex solution of (1.1) for ideal domain $\Omega$.

As illustrated in Theorem 1.3-1.5, it’s natural to conjecture that for $p < n$ and unbounded domain $\Omega \neq \mathbb{R}^n$, there exist some positive convex solutions which are Euclidean complete. However, focusing on the case of $n = 2$ and $p \in (0, 1/2)$, we have shown the following surprising non-existence result.

**Theorem 1.6.** Considering (0.1) for $n = 2, p \in (0, 1/2)$ and unbounded domain $\Omega \neq \mathbb{R}^2$, there is no positive convex solution which is Euclidean complete.

The contents of this paper are organized as follows: We will prove Theorem 1.2 in Sect. 2, and prove Theorem 1.3 in Sect. 3. Using the technics
developed in [8], we show Theorem 1.4 in Sect. 4-5. Finally, the proofs of Theorem 1.5 and 1.6 were presented in Sect. 6 and Sect. 7 respectively.

2. Non-existence of large solution on bounded domain

Considering the Dirichlet problem

\[
\begin{cases}
\det D^2 u = f, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

let’s first recall a lemma in [1].

**Lemma 2.1.** Consider (2.1) where \( \Omega \) is a convex domain satisfying that

\[
B_\sigma \subset \Omega \subset B_{\sigma^{-1}}
\]

for some positive constant \( \sigma \) and

\[
0 \leq f(x) \leq C_1, \ \forall x \in \Omega.
\]

Then the convex solution \( u \) of (2.1) satisfies that

\[
u(x) \geq -C_2 \text{dist}^\gamma(x, \partial \Omega), \ \forall x \in \Omega,
\]

where \( \gamma = 2/n \) for \( n \geq 3 \) and \( \gamma \in (0, 1) \) for \( n = 2 \), \( C_2 = C_2(n, \sigma, C_1) \).

As a counterpart of Theorem 1.1 we have the following nonexistence result.

**Theorem 2.1.** Supposing that \( f \) is an unbounded function satisfying

\[
\limsup_{u \to \infty} \frac{f(u)}{u^n} < \infty,
\]

then for any bounded domain \( \Omega \subset \mathbb{R}^n \), there is no convex large solution to (0.1).

**Proof.** By assumption (2.3), there exists an increasing sequence \( u_k \to \infty \) such that

\[
f(u_k) = \sup_{u \leq u_k} f(u), \ \lim_{k \to \infty} u_k^{-n} f(u_k) = \mathcal{A} \in [0, \infty).
\]

We claim that

\[
u(x) = \infty, \ \forall x \in \Omega \setminus \Sigma_B \equiv \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) \leq B \}
\]

holds for any \( B < (C_2 \mathcal{A}^{1/n})^{-1/\gamma} \). In fact, setting

\[
\Omega_k \equiv \{ x \in \Omega \mid u(x) < u_k \}
\]
and rescaling \( v(x) = f^{-1/n}(u_k)u(x) \), we have \( \Omega_k \) is monotone in \( k \) and

\[
\det D^2v = f(u)/f(u_k) \leq 1, \quad \forall x \in \Omega_k.
\]

Since

\[
\emptyset \neq \Omega_{k_0} \subset \Omega_k \subset \Omega, \quad \forall k \geq k_0 + 1
\]

for some large \( k_0 \in \mathbb{N} \), it follows from (2.6) and Lemma 2.1 that

\[
f^{-1/n}(u_k)u(x) - u_k f^{-1/n}(u_k) \geq -C_2 \text{dist}^{\gamma}(x, \partial \Omega_k)
\]

\[
\geq -C_2 B^{\gamma}, \quad \forall x \in \Omega_k \cap \Sigma_B
\]

or equivalent

\[
u(x) \geq u_k - C_2 B^{\gamma} f^{-1/n}(u_k), \quad \forall x \in \Omega_k \cap \Sigma_B.
\]

Sending \( k \to \infty \) in (2.7) and using (2.4) for \( B < (C_2 A^{1/n})^{-1/\gamma} \), we conclude that \( u \) is infinite everywhere in \( \Sigma_B \). The claim holds true. By an bootstrapping argument we reach the conclusion that \( u \) is infinite everywhere in the whole domain \( \Omega \). The proof of the theorem was done. \( \square \)

3. Non-existence of entire solution on \( \mathbb{R}^n \)

In this section, we will show the non-existence of entire solution as following.

**Theorem 3.1.** Supposing that \( f \in C((0, \infty)) \) is a positive monotone non-decreasing function on \( u \) and satisfies

\[
f(u) \geq Au^p, \quad \forall u > 0
\]

for some \( p > n \) and \( A > 0 \), then there is no positive entire convex solution of (1.1) on \( \mathbb{R}^n \).

Before proving the theorem, we need the following existence result by Lazer-McKenna [17].

**Lemma 3.1.** Letting \( \Omega \) be a bounded smooth convex domain in \( \mathbb{R}^n \) and \( f(u) = u^p \) for some \( p > n \), there exists an unique large convex solution \( u \in C^\infty(\Omega) \) of (1.1) satisfying

\[
C_3^{-1} \text{dist}^{-\alpha}(x, \partial \Omega) \leq u(x) \leq C_3 \text{dist}^{-\alpha}(x, \partial \Omega)
\]

for some positive constant \( C_3 = C_3(n, p, \Omega) \), where \( \alpha = \frac{n+1}{p-n} \).
Proof of Theorem 3.1} By (3.1), each positive convex solution \( u \) of

\[
\det D^2 u = \mathcal{A} u^p, \quad \forall x \in \Omega
\]

is a sup-solution of (1.1). By Lemma 3.1 we denote \( u_R \) to be the unique convex large solution of (1.1) for \( f(u) = u^p \) and \( \Omega = B_R \) and rescale \( u_R \) by

\[
v_R(x) \equiv R^{2/n} u_R(Rx), \quad \forall x \in B_1.
\]

Since

\[
\begin{cases}
\det D^2 v_R = v_R^p, & \forall x \in B_1, \\
\lim_{x \to \partial B_1} v_R(x) = \infty,
\end{cases}
\]

it follows from Lemma 3.1 again that

\[
C_3^{-1} \text{dist}^{-\alpha}(x, \partial B_1) \leq v_R(x) \leq C_3 \text{dist}^{-\alpha}(x, \partial B_1), \quad \forall x \in B_1
\]

or equivalent

\[
C_3^{-1} R^{2/p + \alpha} \text{dist}^{-\alpha}(y, \partial B_R) \leq u_R(y) \leq C_3 R^{2/p + \alpha} \text{dist}^{-\alpha}(y, \partial B_R), \quad \forall y \in B_R.
\]

Noting that \( w_R(y) \equiv \mathcal{A}^{1/p} u_R(y) \) is a solution to

\[
\begin{cases}
\det D^2 w_R = \mathcal{A} w_R^p, & \forall y \in B_R, \\
\lim_{y \to \partial B_R} w_R(y) = \infty,
\end{cases}
\]

and thus a sup-solution to (1.1), using the relation

\[
\int_0^1 U^{ij}(t) dt D_{ij}(w_R - u) = \mathcal{B}(w_R - u), \quad \forall y \in B_R
\]

\[
w_R - u \geq 0, \quad \forall y \in \partial B_R
\]

for

\[
\mathcal{B} \equiv \frac{f(w_R) - f(u)}{w - u} \geq 0
\]

and \([U^{ij}]\) being the cofactor-matrix of \( tD^2 w_R + (1 - t)D^2 u \), it inferred from maximum principle of elliptic equation that \( u \leq w_R \) in \( B_R \). As a result,

\[
u(y) \leq C_3 \mathcal{A}^{1/p} R^{2/p + \alpha} \text{dist}^{-\alpha}(y, \partial B_R), \quad \forall y \in B_R.
\]

Fixing \( y \) and sending \( R \) tends to infinity, it’s inferred from (3.5) that \( u \) is identical to zero for \( p > n \). The proof was done. \( \square \)
4. Schauder’s fix point scheme and entire hypersurface for $p < n$

Letting $u = u(r)$ be a radial symmetric solution to (0.1), there holds that

\begin{equation}
    u'' \left( \frac{u'}{r} \right)^{n-1} = u^p.
\end{equation}

Noting that for $p < n$,

\[ u(x) = \beta_{n,p} |x|^\alpha, \quad \alpha \equiv \frac{2n}{n-p}, \quad \beta_{n,p} \equiv \left[ \alpha^n \alpha - 1 \right]^{\frac{1}{p-n}} \]

is a nonnegative convex entire solution of (1.1) for $f(u) = u^p$ with one degenerate singular point $x = 0$, it cannot be expected to prove an A-priori positive lower bound and regularity for entire solution of (1.1) as that in [3] for large solution. However, one may also ask the following question of existence of positive strictly convex entire solution of (1.1).

**Question.** Considering (0.1) for $p < n$ and $\Omega = \mathbb{R}^n$, whether it admits a smooth positive convex entire solution?

We will give a confirm answer in the following theorem.

**Theorem 4.1.** Considering (0.1) for $p < n$ and $\Omega = \mathbb{R}^n$, there are infinitely many positive entire convex solutions which are affine inequivalent. More precisely, for any $a_0 > 0$, there exists at least one positive entire convex solution $u$ of (1.1) satisfying

\[ u(0) = a_0, \quad Du(0) = 0. \]

Before proving the theorem, let’s check an easy necessary condition for positive entire solution.

**Lemma 4.1.** A necessary condition ensuring

\begin{equation}
    u(r) = \Sigma_{j=0}^{\infty} \frac{a_j}{j!} r^j, \quad \forall r \in [0, \delta]
\end{equation}

to be a radial symmetric positive entire solution is given by

\begin{equation}
    a_0, a_2 > 0, \quad a_2 = a_0^{p/n}, \quad a_{2j-1} = 0, \quad \forall j \in \mathbb{N}.
\end{equation}

**Proof.** The lemma can be easily verified by substituting $u$ into (4.1) at $r = 0$. \hfill \Box

At first step, we will introduce an iteration scheme to prove a local existence result for (4.1). Letting

\[ a \equiv (a_0, a_1, \cdots, a_{2\kappa}) \in \mathbb{R}^{2\kappa}, \quad \forall \kappa \in \mathbb{N} \]
to be a vector satisfying (4.3) and $\sigma, \delta \in (0, 1)$, we define
\[
\Gamma_{a,\delta,\sigma,\kappa} \equiv \left\{ u \in C^{2\kappa}([0, \delta]) \mid u^{(j)}(0) = a_j, \ u^{(j)}(r) \in [a_j - \sigma, a_j + \sigma], u^{(2\kappa)}(0) = a_{2\kappa}, \right. \\
\left. u^{(2\kappa)}(r) \in [a_{2\kappa} - 1, a_{2\kappa} + 1], \ \forall r \in [0, \delta], \ j = 0, 1, \ldots, 2\kappa - 1 \right\}
\]
to be a convex subset of $C^{2\kappa-1}([0, \delta])$ endowed with norm $\| \cdot \|_{C^{2\kappa-1}([0, \delta])}$. For any function $\varphi \in \Gamma_{a,\delta,\sigma,\kappa}$, we define $\xi \equiv T\varphi$ to be the solution determined by
\[
\begin{cases}
\xi''(r) = \varphi^l\left(\frac{r}{\varphi'}\right)^{l-1}, & \forall r \in [0, \delta] \\
\xi(0) = a_0, \xi'(0) = 0, \xi''(0) = a_2
\end{cases}
\]
(4.4)

Next, we will show that $T$ is a continuous and compact mapping from $\Gamma_{a,\delta,\sigma,\kappa}$ to $\Gamma_{a,\delta,\sigma,\kappa}$. Thus, it’s inferred from the following variant of Schauder’s fix point lemma that $T$ admits a fix point in the closure of $\Gamma_{a,\delta,\sigma,\kappa}$.

**Lemma 4.2.** (Schauder’s fix point lemma) Letting $C$ be a convex subset of $B$–space $X$ and $T : X \to X$ be a continuous and compact mapping from $C$ to $C$, then there exists a fix point of $T$ on the closure of $C$.

The proofs of Proposition 4.1 were divided into several lemmas. For any positive function $f \in C^k([0, \delta])$, we have $f^{-1}$ belongs also to $C^k([0, \delta])$. Therefore, supposing that $f^{(l)}(0) \equiv f_i$, $\forall l = 0, 1, \ldots, k$,

let’s introduce a new notation for conjugate indices
\[
\overline{f}_i \equiv \overline{f}_i(f_0, \ldots, f_i) \equiv \frac{d^lf^{-1}}{dr^l} \biggr|_{r=0} \in \mathbb{R}
\]
which is determined by $f_i, i = 0, 1, \ldots, l$ for each $l \leq k$. We need the following lemma.

**Lemma 4.3.** Letting $\varphi \in C^k([0, \delta])$ be a positive function satisfying
\[
\varphi(0) = \varphi_0 = 0, \ \varphi^{(l)}(0) = \varphi_l \in \mathbb{R}, \ \forall l = 1, 2, \ldots, k,
\]
(4.5)
the function defined by
\[
\phi(r) \equiv \begin{cases} 
\frac{r}{\varphi(r)}, & r \in (0, \delta] \\
\frac{1}{\varphi_1}, & r = 0
\end{cases} \in C^k([0, \delta])
\]
satisfies that
\[
\phi_l \equiv \phi^{(l)}(0) = \left(\frac{\varphi_{l+1}}{l+1}\right), \ \forall l = 0, 1, \ldots, k - 1.
\]
(4.6)
**Proof.** By Taylor’s expansion,

\[
\varphi(r) = \sum_{j=1}^{k-1} \varphi_j r^j + \int_0^r \frac{\varphi^{(k)}(t)}{(k-1)!} (r-t)^{k-1} dt.
\]

Therefore,

\[
f(r) \equiv \frac{\varphi(r)}{r} = \sum_{j=1}^{k-1} \frac{\varphi_j}{j!} r^{j-1} + \frac{\int_0^r \varphi^{(k)}(t)(r-t)^{k-1} dt}{r} \in C^{k-1}([0, \delta]).
\]

Taking derivatives \(\frac{d^l f}{dr^l}, l = 1, 2, \cdots, k-1\) on \(f\), we obtain that

\[
\frac{d^l f}{dr^l} = \frac{\varphi_{l+1}}{l+1} + \frac{\varphi_{l+2}}{l+2} r + \cdots + \frac{\varphi_{k-1}}{(k-1)(k-2-l)!} u^{k-l-2} + \sum_{q=0}^{l} \left[ (-1)^q C_l^q \frac{\int_0^r \varphi^{(k)}(t)(r-t)^{k-l-1} dt}{r^{l+1}} \right], \quad \forall l = 1, 2, \cdots, k-2
\]

and

\[
\frac{d^{k-1} f}{dr^{k-1}} = \sum_{q=0}^{k-1} (-1)^q C_l^{k-1} \int_0^r \frac{\varphi^{(k)}(t)(r-t)^{k-l-1} dt}{r^{l+1}}
\]

Noting that for \(l = 1, 2, \cdots, k-2\),

\[
\lim_{r \to 0^+} \frac{d^l f}{dr^l}(r) = \frac{\varphi_{l+1}}{l+1},
\]

we have \(f \in C^{k-2}([0, \delta])\) and

\[
f_l \equiv \frac{d^l f}{dr^l}(0) = \frac{\varphi_{l+1}}{l+1}, \quad \forall l = 0, 1, \cdots, k-2
\]

by Cauchy’s theorem. To show \(f \in C^{k-1}([0, \delta])\), one needs only use (4.8) to deduce that

\[
\lim_{r \to 0^+} \frac{d^k f}{dr^k}(r) = \lim_{r \to 0^+} \varphi^{(k)}(r) \sum_{q=0}^{k-1} (-1)^q C_l^{k-1} \frac{C_l^q}{q+1} = \frac{\varphi_k}{k}
\]

by integral intermediate value theorem, and thus conclude that \(f \in C^{k-1}([0, \delta])\) and

\[
f_{k-1} \equiv \frac{d^{k-1} f}{dr^{k-1}}(0) = \frac{\varphi_k}{k}.
\]

So, (4.6) follows from conjugations of (4.9) and (4.10). □

**Lemma 4.4.** For any positive function \(f \in C^{k}([0, \delta])\), one has

\[
\overline{f}_l \equiv \left. \frac{d^l f^{-1}}{dr^l} \right|_{r=0} = -\frac{f_l}{f_0^2} + \psi_l(f_0, f_1, \cdots, f_{l-1}), \quad \forall l = 1, \cdots, k
\]

for some rational functions \(\psi_l\) on \((f_0, f_1, \cdots, f_l)\).
The conclusion of the lemma follows from an easy calculation and induction on \( l \). As a corollary of Lemma 4.3 and 4.4, one gets that

**Corollary 4.1.** Under the assumptions of Lemma 4.3, there holds

\[
(4.12) \quad \phi_0 = \frac{1}{\varphi_1}, \quad \phi_l = -\frac{\varphi_{l+1}}{(l+1)\varphi_1^2} + \psi(\varphi_0, \varphi_1, \ldots, \varphi_l), \quad \forall l = 1, 2, \ldots, k - 1
\]

for some rational function \( \psi \) on \((f_0, f_1, \ldots, f_l)\).

Given \( \varphi \in \Gamma_{a, b, c, \kappa} \) for some \( a \in \mathbb{R}^\kappa \) satisfying (4.3), the function \( \xi = T\varphi \) satisfies that

\[
(4.13) \quad \xi''(r) = \varphi^p\left(\frac{r}{\varphi}\right)^{n-1} = \varphi^p\varphi^{n-1}, \quad \phi(r) \equiv \begin{cases} \frac{r}{\varphi'}(r), & r \in (0, \delta] \\ \frac{1}{\varphi_2}, & r = 0 \end{cases}
\]

Using Corollary 4.1, for \( \varphi \) replaced by \( \varphi' \) and taking the derivative \( \frac{d^q}{dr} \) on (4.13), it yields that

\[
\left. \frac{d^{l+2}\xi}{dr^{l+2}} \right|_{r=0} = \sum_{q=0}^l C_l^q \frac{d^q\varphi^p}{dr^q} \left. \frac{d^{l-q}\varphi^{n-1}}{dr^{l-q}} \right|_{r=0}
\]

\[
= (n-1)\varphi^p\varphi_0^{n-2} \left( -\frac{\varphi_{l+2}}{(l+1)\varphi_2^2} \right) + \psi_*(a_0, a_1, \ldots, a_{l+1})
\]

\[
= -\frac{(n-1)a_0^m a_2^n}{l+1} \left. \frac{d^{l+2}\varphi}{dr^{l+2}} \right|_{r=0} + \psi_*(a_0, a_1, \ldots, a_{l+1})
\]

for some rational function \( \psi_* \) on \((\varphi_0, \varphi_1, \ldots, \varphi_{l+1})\). It’s remarkable that for \( a \in \mathbb{R}^\kappa \) satisfying (4.3), there holds

\[
(4.14) \quad \left. \frac{d^{2l-1}\xi}{dr^{2l-1}} \right|_{r=0} = 0, \quad \forall l = 1, 2, \ldots, \kappa.
\]

Moreover, by taking higher derivatives on (4.4) and applying the Cauchy’s intermediate value theorem repeatedly, it’s not hard to see that the following proposition holds true.

**Proposition 4.1.** Given \( \varphi \in \Gamma_{a, b, c, \kappa} \) for some \( a \in \mathbb{R}^\kappa \) satisfying (4.3), the function \( \xi = T\varphi \) satisfies that

\[
(4.15) \quad \left. \frac{d^{l+2}\xi}{dr^{l+2}} \right|_{r=0} = -\left( \frac{(n-1)a_0^m a_2^n}{l+1} \left. \frac{d^{l+2}\varphi}{dr^{l+2}} \right|_{r=0} + \psi_*(a_0, a_1, \ldots, a_{l+1}) \right)
\]

and

\[
(4.16) \quad \left. \frac{d^{l+2}\xi}{dr^{l+2}} \right|_{r=0} = -\left\{ \left( \frac{(n-1)a_0^m a_2^n}{l+1} + \sigma_1(l, \varphi) \right) \left. \frac{d^{l+2}\varphi}{dr^{l+2}} \right|_{r=0} + \psi_*(a_0, a_1, \ldots, a_{l+1}) + \sigma_2(l, \varphi) \right\}
\]
for \( r \in [0, \delta] \) and \( l = 1, 2, \ldots, 2\kappa - 2 \), where the functions

\[ |\sigma_1(l, \varphi)| \leq \sigma_1(l, a, \sigma), |\sigma_2(l, \varphi)| \leq \sigma_2(l, a, \sigma) \]

are small as long as \( \sigma \) is small for each fixed \( a \in \mathbb{R}^{2\kappa} \).

**Remark.** Without special indication, the functions on left hand side of (4.16) take values on \( r \), while the functions on right hand side take values on \( 0 < \cdots < r_3 < r_2 < r_1 < r \), which come from the Cauchy’s intermediate value theorem.

As a result, for any given \( a_0, a_2 \) satisfying (4.3), one can uniquely determine the whole vector \( a \) by iterating (4.17)

\[ a_{l+2} = -\frac{(n-1)a_0^pa_2^{-n}}{l+1}a_{l+2} + \psi_s(a_0, a_1, \cdots, a_{l+1}), \quad \forall l = 1, 2, \cdots, 2\kappa - 2 \]

successively using (4.15). As mentioned above by (4.14), the whole vector \( a \) satisfies also (4.3). So, we arrive at the following proposition.

**Proposition 4.2.** For some vector \( a \in \mathbb{R}^{2\kappa} \) determined by (4.17) and \( a_0, a_2 \), if one chooses \( \kappa \in \mathbb{N} \) so large that

\[ \frac{(n-1)a_0^pa_2^{-n}}{2\kappa - 1} < 1 \]

and then \( \sigma \) small, finally \( \delta \) small, the mapping \( T \) is a continuous and compact mapping from \( \Gamma_{a, \delta, \sigma, \kappa} \) to \( \Gamma_{a, \delta, \sigma, \kappa} \).

**Proof.** By (4.16), one has

\[ \xi^{(2\kappa)} = -\left\{ \frac{(n-1)a_0^pa_2^{-n}}{2\kappa - 1} + \sigma_1(2\kappa - 2, \varphi) \right\} \varphi^{(2\kappa)} \]

\[ +\psi_s(a_0, a_1, \cdots, a_{2\kappa-1}) + \sigma_2(2\kappa - 2, \varphi). \]

(4.19)

Another hand, by iterative formula (4.17) of \( a \), one also has

\[ a_{2\kappa} = -\frac{(n-1)a_0^pa_2^{-n}}{2\kappa - 1}a_{2\kappa} + \psi_s(a_0, a_1, \cdots, a_{2\kappa-1}). \]

(4.20)

Subtracting (4.20) from (4.19), we obtain

\[ (\xi^{(2\kappa)} - a_{2\kappa}) = -\left\{ \frac{(n-1)a_0^pa_2^{-n}}{2\kappa - 1} + \sigma_1(2\kappa - 2, \varphi) \right\}(\varphi^{(2\kappa)} - a_{2\kappa}) \]

\[ +\sigma_2(2\kappa - 2, \varphi) - a_{2\kappa}\varphi_1(2\kappa - 2, \varphi). \]

(4.21)

By (4.18), if one chooses \( \sigma \) so small that

\[ \vartheta \equiv \frac{(n-1)a_0^pa_2^{-n}}{2\kappa - 1} + \sigma_1(2\kappa - 2, a, \sigma) \in (0, 1) \]

(4.22)
and

$$|\sigma_2(2\kappa - 2, a, \sigma)| + |a_2\sigma_1(2\kappa - 2, a, \sigma)| < 1 - \vartheta,$$

it’s inferred from (4.21) that

$$\xi^{(2\kappa)}(r) \in [a_{2\kappa} - 1, a_{2\kappa} + 1]$$

Therefore, if $\delta$ is chosen small, one has

$$u^{(j)}(0) = a_j, \quad u^{(j)}(r) \in [a_j - \sigma, a_j + \sigma],$$

$$\forall j = 0, 1, \ldots, 2\kappa - 1, \quad r \in [0, \delta].$$

Combining (4.19), (4.24) and (4.25), we conclude that $T$ is a continuous and compact mapping from $\Gamma_{a, \delta, \sigma, \kappa}$ to $\Gamma_{a, \delta, \sigma, \kappa}$. The proof was done. □

As a corollary, we have the following local existence result of positive entire solution of (4.1).

**Corollary 4.2.** Under assumptions of Proposition 4.2, there exists a local smooth positive entire solution $u \in \Gamma_{a, \delta, \sigma, \kappa}$ of (4.1) on $[0, \delta]$ satisfying

$$u^{(j)}(0) = a_j, \quad \forall j = 0, 1, \ldots, 2\kappa$$

for each $a$ determined by (4.17) and $a_2 = a_0^{p/n}$.

5. **Long time existence and validity of Theorem 4.1**

Letting $u$ be the solution derived in Corollary 4.2, we will show that the solution exist for all $r > 0$, provided $p < n$.

**Lemma 5.1.** Letting $u$ be a solution of (4.1) derived in Corollary 4.2 on $[0, R]$ for some $R > 0$, if there holds

$$\sup_{r \in [0, R]} u^p(r) \leq C_1 < \infty$$

for some positive constant $C_1$, we have

$$\sup_{r \in [0, R]} (u' + u'')(r) \leq C_2 < \infty$$

for positive constant $C_2 = C_2(C_1, n, p, R, a_0, \delta, \sigma, \kappa)$.

**Proof.** Noting that by (4.1), the solution $u$ satisfies that

$$u''(r) > 0, \quad u'(r) > 0, \quad \forall r \in (0, R]$$
in the whole life time. Since \( u' \) is monotone increasing and \( u \in \Gamma_{a, \delta, \sigma, \kappa} \), one has

\[
(5.3) \quad u'(r) \geq u'(\delta) = \int_0^\delta u''(s)ds \geq (a_2 - \sigma)\delta > 0, \quad \forall r \in [\delta, R].
\]

Therefore,

\[
u''(r) = u^p \left( \frac{r}{u'} \right)^{n-1} \leq C_1 R^{n-1}[(a_2 - \sigma)\delta]^{1-n} \equiv C_3 < \infty, \quad \forall r \in [\delta, R]
\]

and

\[
u''(r) \leq a_2 + \sigma < \infty, \quad \forall r \in [0, \delta].
\]

That's to say, \( u'' \) is bounded from above by an A-priori bound \( C_4 \) depending only on \( C_1, n, p, R, a_0, \delta, \sigma, \kappa \). Integrating over \([\delta, R]\) and using

\[
u'(r) = \int_0^r u''(s)ds \leq C_4 R, \quad \forall r \in [0, R].
\]

The proof of (5.2) was done. \( \square \)

**Lemma 5.2.** Let \( u \) be a solution of (4.1) derived in Corollary 4.2. If \( p < n \), there exists a positive constant \( C_5 = C_5(n, p, R, a_0, \delta, \sigma, \kappa) \) such that

\[
(5.4) \quad u^p(r) \leq C_5, \quad \forall r \in [0, R]
\]

holds for any \( R > 0 \).

**Proof.** The case \( p \leq 0 \) is easy since the lower bound of \( u \) follows from the monotonicity of \( u \) and definition of \( \Gamma_{a, \delta, \sigma, \kappa} \). To solve the case \( p \in (0, n) \), let’s rewrite (4.1) by

\[
u'' = u^p \left( \frac{r}{u'} \right)^{n-1} \leq R^{n-1} u^p (u')^{1-n}
\]

\[
\Rightarrow (u')^n u'' \leq R^{n-1} u^p u'
\]

\[
\Rightarrow \frac{d}{dr} \left\{ \frac{1}{n+1} (u')^{n+1} - \frac{R^{n-1}}{p+1} u^{p+1} \right\} \leq 0.
\]

After integrating over \([\delta, R]\), we conclude that

\[
u' \leq C_6 (u + 1)^{n+1}, \quad \forall r \in [\delta, R]
\]

for some positive constant \( C_6 = C_6(n, p, R, a_0, \delta, \sigma, \kappa) \). Thus, one get

\[
(u + 1)^{\frac{np}{n+1}}(r) \leq C_6 \frac{n-1}{n+1} (R - \delta) + C_7, \quad \forall r \in [\delta, R]
\]

for some positive constant \( C_7 = C_7(n, p, R, a_0, \delta, \sigma, \kappa) \). (5.4) follows since \( p \in (0, n) \) and \( u \) is A-priori bounded on \([0, \delta]\). \( \square \)
Complete the proof of Theorem 4.1. Lemma 5.1 and 5.2 ensure local solutions \( u \) of (4.2) derived from Corollary 4.2 would not be blowup in finite \( r \), and hence give the derived entire positive entire solutions of (1.1). Moreover, these solutions are affine independent. □

6. Euclidean complete solution on unbounded domain \( \Omega \neq \mathbb{R}^n \)

In this section, we assume that \( \Omega \neq \mathbb{R}^n \) is an unbounded convex domain, which satisfies the following hypothesis of “ideal domain”:

(H3) There exist a nonempty bounded Lipschitz convex domain \( \Omega_0 \subset \mathbb{R}^n \) and sequences \( a_k \in \Omega, \lambda_k \to \infty \) such that

\[
\Omega_k = \lambda_k (\Omega_0 - a_k) \subset \Omega, \quad \forall k \in \mathbb{N}
\]

and

\[
\inf_{k \in \mathbb{N}} \text{dist}(a_\infty, \partial \Omega_k) > 0
\]

holds for some \( a_\infty \in \bigcap_{k=1}^{\infty} \Omega_k \). We will call \( \Omega \) to be an “ideal domain” and call \( a_\infty \) to be an “ideal center” of \( \Omega \). It’s clearly that any unbounded convex set \( \Omega \subset \mathbb{R}^n \) containing an infinite cone

\[
C = \{ \lambda x \in \mathbb{R}^n \mid x \in C_0, \lambda \in \mathbb{R}^+ \},
\]

which is defined for a nonempty domain \( C_0 \subset \mathbb{R}^n \), are all ideal domains. We will prove the following nonexistence result for ideal domain.

Theorem 6.1. Supposing that \( f \in C((0, \infty)) \) is a positive monotone non-decreasing function on \( u \) and satisfies (3.1) for some \( p > n, n \geq 2 \) and \( A > 0 \), then there is no positive entire convex solution of (1.1) for ideal domain \( \Omega \).

In the definition of ideal domain, \( \Omega_0 \) is only bounded Lipschitz convex domain. Therefore, Lemma 3.1 can not be applied directly to \( \Omega_0 \). Fortunately, we have the following variant version of Lemma 3.1.

Lemma 6.1. Letting \( \Omega \) be a bounded Lipschitz convex domain in \( \mathbb{R}^n \) and \( f(u) = u^p \) for some \( p > n \), there exists a large convex solution \( u \in C^\infty(\Omega) \) of (1.1) satisfying (3.2) for some positive constant \( C_3 = C_3(n, p, \Omega) \), where \( \alpha = \frac{p}{p-n} \).

Proof. Assuming \( \Omega \) is a bounded Lipschitz convex domain, there exists a sequence of smooth convex domains \( \Omega_j \subset \Omega, j \in \mathbb{N} \) which is monotone...
increasing in $j$ and satisfies

$$\bigcup_{j=1}^{\infty} \Omega_j = \Omega.$$  

Applying Lemma 3.1 to $\Omega_j$, there exists a sequence of positive constants $C_{3,j}$ such that (3.2) holds for solution $u_j$ of (1.1) on $\Omega_j$. Noting that by maximum principle (1.1), we have $u_j$ is monotone decreasing in $j$ and $C_{3,j}$ can also be chosen to be monotone decreasing. As a result, passing to the limit $j \to \infty$ in (3.2), we obtain that the limiting solution

$$u(x) \equiv \lim_{j \to \infty} u_j(x), \quad \forall x \in \Omega$$

satisfying

$$u(x) = \lim_{j \to \infty} u_j(x) \leq \lim_{j \to \infty} C_{3,j} \text{dist}^{-\alpha}(x, \partial \Omega_j) \leq C_{3,\infty} \text{dist}^{-\alpha}(x, \partial \Omega)$$

for any $x \in \Omega$ and some positive constant $C_{3,\infty}$. To show the left inequality of (3.2), one needs only to choose a monotone shrinking bounded smooth convex domains $\tilde{\Omega}_j \supset \Omega$ satisfying

$$\bigcap_{j=1}^{\infty} \tilde{\Omega}_j = \Omega.$$  

Then, applying the monotone increasing of $\tilde{u}_j$ and $C_{3,j}^{-1}$, we arrive at

$$u(x) \geq \lim_{j \to \infty} \tilde{u}_j(x) \geq \lim_{j \to \infty} C_{3,j}^{-1} \text{dist}^{-\alpha}(x, \partial \Omega_j) \geq C_{3,\infty}^{-1} \text{dist}^{-\alpha}(x, \partial \Omega)$$

for any $x \in \Omega$ and some positive constant $C_{3,\infty}$. Choosing

$$C_3 \equiv \max \{C_{3,\infty}, C_{3,\infty}^{-1}\},$$

we conclude that $u$ satisfies (3.2) for $C_3$. □

Now, we can complete the proof of Theorem 6.1.

**Proof.** As in proof of Theorem 3.1, each positive convex solution $u$ of

$$(6.3) \quad \det D_x^2 u = Au^p, \quad \forall x \in \Omega$$

is a sup-solution of (1.1) due to (3.1). Letting $\Omega_0 \subset \Omega$, $a_k \in \Omega, \lambda_k \to \infty, k \in \mathbb{N}$ as given in definition of ideal domain, we set

$$\Omega_k \equiv \lambda_k(\Omega - a_k) \subset \Omega, \quad \forall k \in \mathbb{N}.$$  

By Lemma 6.1, we denote $u_0$ to be a convex large solution of (1.1) for $f(u) = u^p$ on $\Omega_0$, which satisfies that

$$C_{3}^{-1} \text{dist}^{-\alpha}(x, \partial \Omega_0) \leq u_0(x) \leq C_{3} \text{dist}^{-\alpha}(x, \partial \Omega_0), \quad \forall x \in \Omega_0$$

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Rescaling $u_0$ by
\[ u_0(x) \equiv \lambda_\frac{n}{p} u_k(\lambda_k(x - a_k)), \quad \forall x \in \Omega_0, \]
there hold
\[ C_3^{-1} \lambda_\frac{n}{p} \text{dist}^{-a}(y, \partial \Omega_k) \leq u_k(y) \leq C_3 \lambda_\frac{n}{p} \text{dist}^{-a}(y, \partial \Omega_k), \quad \forall y \in \Omega_k \]
and
\[ \begin{cases} 
\det D^2 u_k = u_k, \\
\lim_{x \to \partial \Omega_k} u_k(x) = \infty,
\end{cases} \quad \forall x \in \Omega_k, \]
Noting that $w_k(y) \equiv \mathcal{A}^\frac{1}{p} u_k(y)$ is a solution to
\[ \begin{cases} 
\det D^2 w_k = \mathcal{A} w_k, \\
\lim_{y \to \partial \Omega_k} w_k(y) = \infty,
\end{cases} \quad \forall y \in \Omega_k, \]
we have it is also a sup-solution to (1.1). Using the relation
\[ \int_0^1 U^{ij}(t) dt D_{ij}(w_k - u) = \mathcal{B}(w_k - u), \quad \forall y \in \Omega_k \]
for
\[ \mathcal{B} \equiv \frac{f(w_k) - f(u)}{w_k - u} \geq 0 \]
and $[U^{ij}]$ being the cofactor-matrix of $tD^2 w_k + (1 - t)D^2 u$, it is inferred from maximum principle of uniformly elliptic equation that $u \leq w_k$ in $\Omega_k$. As a result,
\[ u(y) \leq C_3 \mathcal{A}^\frac{1}{p} \lambda_\frac{n-1}{p} \text{dist}^{-a}(y, \partial \Omega_k), \quad \forall y \in \Omega_k. \]
Taking $y$ to be an ideal center $a_\infty$ of $\Omega$ and sending $k$ to be large, it’s inferred from (6.5) that $u(a_\infty)$ must be identical to zero since $p > n$, which contradicts with the positivity of $u$. The proof of the theorem was done. □

7. Wiping barrier and large solution on unbounded domain

Inspired of Theorems 3.1, 4.1 and 6.1, it’s naturally to ask the following question.

**Question:** For $n \geq 2$ and $p < n$, given any unbounded convex domain $\Omega \neq \mathbb{R}^n$, whether there is some Euclidean complete solution of (0.1) on $\Omega$? It would be surprising that we have the following non-existence result of large solution of (0.1).
Theorem 7.1. Considering (0.1) for \( n = 2, p \in (0, 1/2) \) and unbounded domain \( \Omega \neq \mathbb{R}^2 \), there is no positive convex solution which is Euclidean complete.

Our proofs of Theorem 7.1 are relied on a construction of zero barrier solution on some wiping domain defined by

\[
\Sigma_{\beta, r_0} = \{(x, y) \in \mathbb{R}^2 \mid e^{x^\beta y} \in (r_0, \infty)\}
\]

where \( r_0 \in (0, \infty) \) and \( \beta < 0 \). We need to use the following existence result in proof of Theorem 7.1.

Proposition 7.1. Considering (0.1) for \( n = 2, p \in (0, 1/2) \) and \( \beta < 0 \), there exists a positive constant \( r_0 = r_0(p, \beta) \), such that (0.1) admits a positive convex smooth solution \( u \) on \( \Sigma_{\beta, r_0} \) satisfying

\[
u(x, y) = 0, \quad \forall (x, y) \in \Sigma_{\beta, r_0} = \{(x, y) \in \mathbb{R}^2 \mid e^{x^\beta y} = r_0\}.
\]

To verify the validity of Proposition 7.1, we look for solutions of the form

\[
u(x, y) = y^{\alpha} \varphi(e^{x^\beta y}), \quad x \in \mathbb{R}, \ y \in \mathbb{R}_+.
\]

Direct computation shows that

\[
u_{xx} = y^{\alpha}[r \varphi_r + r^2 \varphi_{rr}],
\]

\[
u_{xy} = y^{\alpha-1}[(\alpha + \beta)r \varphi_r + \beta r^2 \varphi_{rr}],
\]

\[
u_{yy} = y^{\alpha-2}[\alpha(\alpha - 1)r \varphi_r + \beta(2\alpha + \beta - 1)r \varphi_r + \beta^2 r^2 \varphi_{rr}].
\]

Therefore,

\[
det D^2 u = y^{2\alpha-2}\left\{r^2 \varphi_r \left[\alpha(\alpha - 1)r \varphi_r - \beta \varphi_r \right] + \alpha(\alpha - 1)r \varphi_r - (\alpha^2 + \beta) \varphi_r^2 \right\}
\]

\[
u^p = y^{\alpha p} \varphi^p.
\]

As a result, if one chooses \( \alpha = \frac{2}{2 - p} \), then (1.1) changes to

\[(7.1) \quad r^2 \varphi_{rr} \left[\alpha(\alpha - 1)r \varphi_r - \beta \varphi_r \right] = -\alpha(\alpha - 1)r \varphi_r + (\alpha^2 + \beta) \varphi_r^2 + \varphi^p.
\]

Setting \( \xi(\varphi) = r \varphi_r \), one has

\[\xi' \xi = r \varphi_r + r^2 \varphi_{rr} = \xi + r^2 \varphi_{rr},\]

and thus changes (7.1) to

\[(7.2) \quad \xi \xi' = \frac{\alpha^2 \xi^2 + \varphi^p}{\alpha(\alpha - 1) \varphi - \beta \xi^\alpha}.
\]

To recover \( \varphi \) from \( \xi \), we choose \( \varphi_1 = \varphi(r_1) \) for some \( r_1 > 0 \), and obtain that

\[(7.3) \quad \int_{\varphi_1}^{\varphi(r)} \frac{d\varphi}{\xi(\varphi)} = \int_{r_1}^{r} \frac{dr}{r} = \ln \frac{r}{r_1}.
\]
Now, let’s introduce a closed convex subset
\[ \Gamma_{\beta,q,\delta} \equiv \{ \xi \in C([0,\delta]) \mid \xi(\varphi) - \gamma_{\beta\varphi^{\frac{p+1}{3}}} \in [-\varphi^{q}, \varphi^{q}], \ \forall \varphi \in [0,\delta] \} \]
of \(C([0,\delta])\), where \(q \in \left(\frac{p+1}{3}, 1\right)\) is chosen later and
\[ (7.4) \quad \gamma_{\beta} \equiv \left[ \frac{3}{\beta(p + 1)} \right]^{\frac{1}{3}}. \]

For each \(\xi \in \Gamma_{\beta,q,\delta}\), we also define a mapping \(T \xi \equiv \zeta\) by
\[ (7.5) \quad \begin{cases} \zeta' = L\xi = \frac{\alpha^2(\gamma_{\beta\varphi^{\frac{p+1}{3}}} + \varphi^{q}) + \varphi^{p}(\gamma_{\beta\varphi^{\frac{p+1}{3}}} - \varphi^{q})^{-1}}{\alpha(\alpha - 1)\varphi - \beta\xi}, \\ \zeta(0) = 0, \ \forall \varphi \in (0,\delta). \end{cases} \]

**Lemma 7.1.** Supposing that \(p \in (0, 1/2)\) and \(\beta < 0\), if one chooses \(q\) close to 1 then chooses \(\delta\) small, then \(T\) is a continuous and compact mapping from \(\Gamma_{\beta,q,\delta}\) to \(\Gamma_{\beta,q,\delta}\).

**Proof.** To verify the compactness of \(T\), we use (7.5) to calculate
\[ \zeta' = L\xi \leq \frac{\alpha^2(\gamma_{\beta\varphi^{\frac{p+1}{3}}} + \varphi^{q}) + \varphi^{p}(\gamma_{\beta\varphi^{\frac{p+1}{3}}} - \varphi^{q})^{-1}}{\alpha(\alpha - 1)\varphi - \beta\xi}, \]
\[ \leq |\beta|^{-1} \gamma^{-2}_{\beta} \varphi^{\frac{p+1}{3}} + 2|\beta|^{-1} \gamma^{-3}_{\beta} \varphi^{-1} + o(\varphi^{-1}) \]
\[ \leq \frac{\gamma_{p}(p + 1)}{3} \varphi^{\frac{p+2}{3}} + q\varphi^{q-1}, \ \forall \varphi \in (0,\delta) \]
and
\[ \zeta' = L\xi \geq \frac{\alpha^2(\gamma_{\beta\varphi^{\frac{p+1}{3}}} + \varphi^{q}) + \varphi^{p}(\gamma_{\beta\varphi^{\frac{p+1}{3}}} - \varphi^{q})^{-1}}{\alpha(\alpha - 1)\varphi - \beta\xi}, \]
\[ \geq |\beta|^{-1} \gamma^{-2}_{\beta} \varphi^{\frac{p+1}{3}} - 2|\beta|^{-1} \gamma^{-3}_{\beta} \varphi^{-1} + o(\varphi^{-1}) \]
\[ \geq \frac{\gamma_{p}(p + 1)}{3} \varphi^{\frac{p+2}{3}} - q\varphi^{q-1}, \ \forall \varphi \in (0,\delta) \]
provided \(q\) is chosen to close 1 and then \(\delta\) is chosen small, where
\[ 2|\beta|^{-1} \gamma^{-3}_{\beta} = \frac{2(p + 1)}{3} < 1 \]
has been used. Integrating over \(\varphi\) yields that
\[ \zeta(\varphi) - \gamma_{\beta\varphi^{\frac{p+1}{3}}} \in [-\varphi^{q}, \varphi^{q}], \ \forall \varphi \in (0,\delta). \]
Consequently, \(T\) is a continuous and compact mapping from \(\Gamma_{\beta,q,\delta}\) to \(\Gamma_{\beta,q,\delta}\).
The proof was done. □
As a corollary, we obtain the following local solvability of (7.2).

**Corollary 7.1.** Under the assumptions of Lemma 7.1, the mapping \( T \) admits a fix point \( \zeta \) in \( \Gamma_{\beta,q,\delta} \) which is a local smooth solution of (7.2) satisfying (7.8).

Finally, let’s complete the proof of Proposition 7.1 with the help of the following long time existence and asymptotic results.

**Proposition 7.2.** (Long time existence) Under the assumptions of Corollary 7.1, the local solution \( \zeta \) exists for all time \( \varphi > 0 \) and preserves positivity and monotonicity.

**Proof.** It’s clear that the solution of (7.2) preserves positivity and monotonicity. To show that the solution exists for all \( \varphi \), one needs only using (7.2) to deduce that

\[
(\zeta^2)' \leq \frac{2\alpha}{\alpha - 1} \varphi^{-1} \zeta^2 + \frac{2}{\alpha(\alpha - 1)} \varphi^{\rho - 1} \]

\[
\Rightarrow \quad \left( \varphi^{-\frac{2n}{\alpha - 1}} \zeta^2 \right)' \leq \frac{2}{\alpha(\alpha - 1)} \varphi^{\rho - 1 - \frac{2n}{\alpha - 1}} \]

\[
\Rightarrow \quad \varphi^{-\frac{2n}{\alpha - 1}} \zeta^2 \leq \delta^{-\frac{2n}{\alpha - 1}} \zeta^2(\delta) - \frac{2p}{\alpha(\alpha - 1)(4 - p^2)} \left( \varphi^{\rho - \frac{2n}{\alpha - 1}} - \delta^{\rho - \frac{2n}{\alpha - 1}} \right), \quad \forall \varphi \geq \delta.
\]

So, \( \zeta \) is A-priori bounded from above and thus exists for all \( \varphi \). \( \square \)

The second proposition gives the asymptotic behavior of \( \zeta \) at infinity.

**Proposition 7.3.** (Asymptotic behavior) Under the assumptions of Corollary 7.1 for each \( \varepsilon > 0 \), there exists a positive constant \( \varphi_\varepsilon \) such that

(7.6)

\[
\left( \frac{\alpha}{|\beta|} - \varepsilon \right) \varphi \leq \zeta \leq \left( \frac{\alpha}{|\beta|} + \varepsilon \right) \varphi, \quad \forall \varphi \geq \varphi_\varepsilon
\]

holds.

We divided the proof into two lemmas.

**Lemma 7.2.** Under the assumptions of Corollary 7.1 there holds

(7.7)

\[
\zeta^2 \leq A_1 \varphi^{\frac{\rho}{\alpha}} - A_2 \varphi^\rho, \quad \forall \varphi \geq \delta,
\]

where

\[
A_1 \equiv \delta^{\frac{\rho}{\alpha}} \zeta^2(\delta) + \frac{2p}{\alpha(\alpha - 1)(4 - p^2)} \delta^{\rho - \frac{\rho}{\alpha}}
\]

\[
A_2 \equiv \frac{2p}{\alpha(\alpha - 1)(4 - p^2)}.
\]

Lemma 7.2 is a direct consequence of proof of Proposition 7.2.
Lemma 7.3. Under the assumptions of Corollary 7.1 for any \( \varepsilon > 0 \), there exists positive constant \( \varphi_2 = \varphi_2(\varepsilon) \) such that

\[
\zeta \geq \left( \frac{\alpha^2(2 - p)}{2|\beta|} - \varepsilon \right) \varphi, \quad \forall \varphi \geq \varphi_2.
\]  

Proof. By (7.2) and \( p < 2 \), there holds

\[
\zeta' \geq \frac{\alpha^2 \zeta}{\alpha(\alpha - 1)\varphi - \beta \zeta} \implies \frac{d\varphi}{d\zeta} \leq \frac{\alpha - 1}{\alpha} \zeta^{-1} \varphi + \frac{|\beta|}{\alpha^2}
\]

\[
\implies \left( \zeta^{-\frac{\alpha - 1}{\alpha}} \varphi \right)' \leq \frac{|\beta|}{\alpha^2} \zeta^{-\frac{\alpha - 1}{\alpha}} \implies \zeta^{-\frac{\alpha - 1}{\alpha}} \varphi \leq C_1 + \frac{2|\beta|}{\alpha^2(2 - p)} \zeta^{1 - \frac{\alpha - 1}{\alpha}}
\]

\[
\implies \varphi \leq C_1 \zeta^{\frac{\alpha - 1}{\alpha}} + \frac{2|\beta|}{\alpha^2(2 - p)} \zeta \leq \left( \frac{2|\beta|}{\alpha^2(2 - p)} + \varepsilon \right) \zeta + C_\varepsilon
\]

for each \( \varepsilon > 0 \), where \( C_\varepsilon \) is a positive constant depending on \( \varepsilon \). Therefore, for this given \( \varepsilon \), there exists \( \varphi_2 = \varphi_2(\varepsilon) \), such that (7.8) holds. \( \square \)

Now, one can complete the proof of Proposition 7.3.

Proof of Proposition 7.3. By (7.2) and Lemma 7.3 for each \( \varepsilon \) small, there exists \( \varphi_3 = \varphi_3(\varepsilon) \) such that

\[
\zeta' \leq \frac{(\alpha^2 + \varepsilon) \zeta}{\alpha(\alpha - 1)\varphi - \beta \zeta}, \quad \forall \varphi \geq \varphi_3.
\]

Solving this first order O.D.E. as in proof of Lemma 7.3, it yields the desired inequality (7.6) with the help of (7.8). \( \square \)

At the end of this section, let’s complete the proof of Proposition 7.1 and Theorem 7.1.

Proof of Proposition 7.1. Noting that the solution \( \zeta \) derived by Corollary 7.1 and Proposition 7.2 satisfies

\[
\zeta(\varphi) \sim \gamma_\beta \varphi^{\frac{p+1}{\alpha}}, \quad \forall \varphi \sim 0,
\]

there exists a positive constant \( r_0 \in (0, r_1) \), such that

\[
\int_{\varphi_1}^{0} \frac{d\varphi}{\zeta(\varphi)} = \ln \frac{r_0}{r_1}.
\]

Similarly, by Proposition 7.3

\[
\zeta(\varphi) \sim \left( \frac{\alpha}{|\beta|} \pm \varepsilon \right) \varphi, \quad \forall \varphi \sim \infty,
\]

there holds

\[
\int_{\varphi_1}^{\infty} \frac{d\varphi}{\zeta(\varphi)} = \infty.
\]
Therefore, the recovered convex solution $\varphi$ from (7.3) satisfies that

$$\varphi(r_0) = 0, \quad \varphi(\infty) = \infty.$$  

Using the relation $u(x,y) = y^\alpha \varphi(e^{x^\beta})$ and $\alpha > 0, \beta < 0$, one obtains a desired positive convex smooth solution $u$ of $\Sigma_{\beta, r_0}$. The proof of Proposition 7.1 was completed. □

**Proof of Theorem 7.1.** Suppose on the contrary, there exists a positive convex large solution $u$ of (0.1) on an unbounded convex domain $\Omega \neq \mathbb{R}^2$. Choosing a point $z_0 = (x_0, y_0) \in \partial \Omega$ and drawing a line $l_{z_0}$ tangential to $\partial \Omega$ at the point $z_0$, whose unit normal is given by $\nu$. The boundary of $\Sigma_{\beta, r_0}$ contains a curve $\varsigma_{\beta, r_0}$ and a straight line $l_{\beta, r_0}$. The curve $\varsigma_{\beta, r_0}$ divides the plane into two sides. The side of $\varsigma_{\beta, r_0}$ which contains $\Sigma_{\beta, r_0}$ will be called positive side of $\varsigma_{\beta, r_0}$ for short. Now, translating and rotating the wiping domain $\Sigma_{\beta, r_0}$ obtained in Proposition 7.1, $\Omega$ is divided into two portions by $\varsigma_{\beta, r_0}$. One may assume that one of the portions $\Sigma_*$, which lies on positive side of $\varsigma_{\beta, r_0}$, is bounded and lies strictly inside of $\Sigma_{\beta, r_0}$. Comparing with the zero barrier solution $u_*$ found in Proposition 7.1 on $\Sigma_*$, one gets

$$u(x,y) \geq u_*(x,y) / \varepsilon, \quad \forall (x,y) \in \Sigma_*$$  

for each $\varepsilon \in (0, 1)$, where we have used $u_* / \varepsilon$ is a subsolution of (0.1) and

$$u(x,y) \geq u_*(x,y) / \varepsilon, \quad \forall (x,y) \in \partial \Sigma_*.$$  

As a result, $u$ must be infinite everywhere in $\Sigma_*$. Contradiction holds. The proof of Theorem 7.1 was done. □

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