Cyclic Homology of $DG$ Coalgebras and a Künneth Formula

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In this paper we extend the cyclic homology functor, and in particular the periodic cyclic homology, to the category of $DG$ (= differential graded) coalgebras. We are partly motivated by the question of products and coproducts in periodic cyclic homology of algebras. As an application, we will show how one can start from the classical shuffle map in homological algebra and algebraic topology, interpreted in [HMS] as a morphism of $DG$ coalgebras, and build a theory of products and coproducts in (periodic, negative, etc.) cyclic homology. Along these lines we also recover a Künneth isomorphism

$$\hat{S} : CC_\ast(A_1) \hat{\otimes} CC_\ast(A_2) \xrightarrow{\sim} CC_\ast(A_1 \otimes A_2),$$

relating the periodic cyclic complexes of unital ($DG$) algebras over a field of characteristic zero. Here $CC_\ast$ denotes the periodic cyclic complex and $\hat{\otimes}$ denotes the topologically completed tensor product of complexes. We note that a homotopy equivalent result, with a different choice for $CC_\ast$ and $\hat{S}$, can be found in the works of Cuntz and Quillen [CQ$_2$], Puschnigg [P] and more recently Bauval [B]. Our method is based on ideas of Cuntz and Quillen in cyclic homology (cf. [CQ$_1$] and reference therein) and in a sense is dual to them.

An interesting problem suggests itself. It would be interesting to see how one can use a similar approach to define the cyclic cohomology of ($DG$) Hopf algebras and compare it with the definition of periodic cyclic cohomology of Hopf algebras as recently defined by Connes and Moscovici [CM].

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1 Bar and cobar constructions and the shuffle map

The present form of the bar and cobar constructions first appeared in a paper by Husemoller, Moore and Stasheff [HMS] under the name of algebraic classifying space and loop space constructions. The reason for these names is that these constructions provide models for

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singular chains on the classifying space of groups and singular cochains on the (based) loop space of simply connected spaces. In the same paper the shuffle map is introduced as a morphism of DG coalgebras. This is crucial for applications to periodic cyclic homology. The bar construction, denoted here by $B$, is a functor from the category of DG algebras to the category of DG coalgebras. It has a left adjoint, namely the cobar construction, denoted here by $B^c$, from the category of DG coalgebras to DG algebras. Let us recall their definitions.

Let $A = \bigoplus_{i \geq 0} A_i$ be a positively graded DG algebra over a ground ring $k$. We assume its differential $d$ has degree $-1$. Let $A[1]$ denote the suspension of $A$ defined by $A[1]_n = A_{n-1}$. The bar construction of $A$, denoted by $BA$, is a DG coalgebra whose underlying graded coalgebra is $BA = T^c A[1]$, the cofree coaugmented counital coalgebra generated by the graded vector space $A[1]$. We have $(BA)_0 = k$, $(BA)_1 = A_0$ and in general

$$(BA)_n = \bigoplus_{s=0}^{n} \bigoplus_{i_1 + \ldots + i_r + r = s} A_{i_1} \otimes \ldots \otimes A_{i_r} .$$

The differential of $BA$ is $d + b'$, where $d$ and $b'$ are defined by

$$d(a_1, \ldots, a_n) = \sum_{i=1}^{n} (-1)^{\varepsilon_i}(a_1, \ldots, da_i, \ldots, a_n)$$

and

$$b'(a_1, \ldots, a_n) = \sum_{i=1}^{n-1} (-1)^{\varepsilon_i + |a_i|}(a_1, \ldots, a_ia_{i+1}, \ldots, a_n) ,$$

where

$$\varepsilon_i = \sum_{j=1}^{i-1} |a_j| + i - 1 .$$

Computation shows that $d^2 = b'^2 = db' + b'd = 0$, so that $d + b'$ is a differential of degree $-1$. It is moreover a graded coderivation of $BA$.

Morphisms of DG coalgebras into $BA$ are defined via twisting cochains and vice versa. More precisely, let $C$ be a DG coalgebra. A degree -1 map $\theta : C \to A$ is called a twisting cochain if

$$\delta \theta + \theta^2 = 0 ,$$

where, $\delta \theta = [d, \theta]$ in the Hom complex differential of $\theta$ and $\theta^2 = m \circ (\theta \otimes \theta) \Delta$, where $\Delta$ is the coproduct of $C$ and $m$ is the multiplication of $A$. The universal twisting cochain is the map $\theta : BA \to A$ defined by $\theta(a) = a$ for $a \in A$ and zero otherwise. There is a 1-1 correspondence between morphism of DG coalgebras $\widehat{\theta} : C \to BA$ of degree zero and
twisting cochains \( \theta : C \rightarrow A \). In this correspondence, \( \theta \) is simply the corestriction of \( \hat{\theta} \) to \( A \). Conversely, given \( \theta \), we have

\[
\hat{\theta} = \sum_{n=1}^{\infty} \hat{\theta}_n
\]

where

\[
\hat{\theta}_n = \theta \otimes^{\otimes n} \circ \Delta^{(n)},
\]

and \( \Delta^{(n)} \) is the \( n-th \) iteration of the coproduct.

Next, we discuss the cobar construction. Let \( C = \bigoplus_{i \geq 0} C_i \) be a positively graded DG coalgebra with a differential \( d \) of degree -1. We assume \( C_0 = k \). Let \( C[-1] \) be the graded space defined by \( C[-1]_i = 0 \) if \( i < 0 \), and \( C[-1]_i = C_{i+1} \) for \( i \geq 0 \). The cobar construction of \( C \), denoted by \( B^c C \), is the DG algebra whose underlying graded algebra is \( B^c C = TC[-1] \), the augmented unital free algebra generated by \( C[-1] \). We have \( (B^c C)_0 = \bigoplus_{n \geq 0} C^\otimes n \), and

\[
(B^c C)_n = \bigoplus_{r=1}^{n} \bigoplus_{i_1+i_2+\ldots+i_r = n+r} C_{i_1} \otimes \cdots \otimes C_{i_r}.
\]

The differential of \( B^c C \) is \( d + b' \), where \( d \) and \( b' \) are defined by

\[
d(a_1, \ldots, a_n) = \sum (-1)^{\varepsilon_i} (a_1, \ldots, da_i, \ldots, a_n)
\]

and

\[
b' = \sum 1 \otimes \cdots \otimes \tilde{\Delta} \otimes \cdots \otimes 1,
\]

where \( \varepsilon_i \) is the same as before and \( \tilde{\Delta} : C[-1] \rightarrow C[-1] \otimes C[-1] \) is the unique morphism of complexes of degree -1 such that \( \tilde{\Delta} \circ s^{-1} = (s^{-1} \otimes s^{-1}) \Delta \). Here \( s : C \rightarrow C[-1] \) is the desuspension.

We have \( d^2 = b'^2 = db' + b'd = 0 \), so that \( d + b' \) is a differential of degree -1. It is moreover a graded derivation of \( B^c C \).

Let \( A_1 \) and \( A_2 \) be unital algebras (no differential, no grading). The shuffle map is a morphism of DG coalgebras

\[
S : BA_1 \otimes BA_2 \rightarrow B(A_1 \otimes A_2)
\]

defined as follows. Let \( \theta_i : BA_i \rightarrow A_i, i = 1, 2 \), be universal twisting cochains of \( A_i \). Define a twisting cochain \( \theta : BA_1 \otimes BA_2 \rightarrow A_1 \otimes A_2 \) by

\[
\theta = \theta_1 \otimes \varepsilon \eta + \varepsilon \eta \otimes \theta_2,
\]
where $\eta$ is the counit map of $BA_i$ and $\varepsilon$ is the unit map of $A_i$. Note that $\theta$ is zero except on linear span of tensors of the type $a_1 \otimes 1$, or $1 \otimes a_2$. Checking the twisting cochain condition for $\theta$ is easy. By universal property of the bar construction we obtain $S$.

A simple computation using (1) gives the following explicit formula for $S$:

$$S((a_1, \ldots, a_p) \otimes (b_1, \ldots, b_q)) = \sum_{\sigma \in S_{p,q}} sgn(\sigma)\sigma(a'_1, \ldots, a'_p, b'_1, \ldots, b'_q),$$

where $a'_i = a_i \otimes 1$ and $b'_i = 1 \otimes b_i$, and $S_{p,q}$ is the set of all $(p, q)$ shuffle permutations in the symmetric group $S_{p+q}$.

We need to know that $S$ is a quasi-isomorphism. This is trivial for unital algebras over a field, since it is well known that $b'$ is acyclic for unital algebras and one can use the Künneth formula for tensor product of complexes over a field. We mention, however, that $S$ has an explicit homotopy inverse $A : B(A_1 \otimes A_2) \to BA_1 \otimes BA_2$, the Alexander-Whitney map (see, e.g. [M]), which proves that $S$ is a quasi-isomorphism over any ground ring. Same proof works when $A_1$ and $A_2$ are unital DG algebras. It is important, however, to note that $A$ is not a morphism of DG coalgebras. This makes finding an explicit formula for the inverse of the product map in periodic cyclic homology a difficult task. See however [B] where this problem is successfully solved. An alternative approach would be to extend $A$ to an $A_\infty$-morphism of DG coalgebras which in theory one knows to exist and let it act on cyclic complexes.

## 2 Periodic cyclic homology of DG coalgebras

Periodic cyclic homology of DG algebras was defined by Goodwillie and others (see [G] and references therein). One of the main results of [G] is the fact that if $f : A \to B$ is a morphism of DG algebras which is a quasi-isomorphism of complexes, then the induced map on periodic cyclic homology $f_* : HP_*(A) \to HP_*(B)$ is an isomorphism. We need the analogue of this result for DG coalgebras. We also need the fact that for DG coalgebras of finite cohomological dimension (1 and 2 in our applications), the periodic cyclic complex is quasi-isomorphic to some higher order versions of the $X$-complex.

Let $C$ be a DG coalgebra. In [Kh], we have carefully defined the DG coalgebra of (noncommutative) differential forms over $C$, and denoted it by $(\Omega C, d)$. Its construction is dual to the corresponding construction for algebras and DG algebras.

We need to adopt some basic definitions and constructions from [CQ1] to our DG coalgebraic set up. Let $C$ be a DG coalgebra and let $(\Omega C, d)$ denote the DG coalgebra of universal codifferential forms over $C$. Let $\eta : C \to k$ be the counit of $C$. We have $\Omega^n C = C \otimes \overline{C}^\otimes n$, where $\overline{C} = \text{Ker } \eta$. Let $b : \Omega^* C \to \Omega^{*+1} C$ be the analogue of the Hochschild boundary operator and let $N$ be the number operator which multiplies a differential form by its degree. Let

$$\Omega^{\text{norm}} C = \ker \{(b + dN)^2 : \Omega C \to \Omega C\}.$$
Equipped with the differential $b + dN$ and with its natural $\mathbb{Z}/2$ grading, $(\Omega^{\text{norm}} C, b + dN)$ can be regarded as a supercomplex. There is a decreasing filtration $\{ F^n \Omega^{\text{norm}} C \}_{n \geq 2}$ on $\Omega^{\text{norm}} C$, where $F^n$ consists of forms of degree at least $n$. The successive quotient complexes $\Omega^{\text{norm}} C / F^n$ approximate the normalized cyclic bicomplex for DG coalgebras. We need only the first two quotients, denoted by $X(C)$ and $X^2(C)$. These are the supercomplexes

\[
X(C) : \quad C \xleftarrow{b} \Omega^1 C_z \xrightarrow{d} \\
X^2(C) : \quad C \bigoplus \Omega^2 C_z \xleftarrow{b + 2d} \Omega^1 C,
\]

where $\Omega^1 C_z$ denotes the cocommutator subspace and $\Omega^1 C = \Omega^{\text{norm}, 1} C$. Note that $\Omega^1 C_z \subset \hat{\Omega}^1 C$.

We are mostly interested in the total complexes of these bicomplexes which we denote by $\hat{X}(C)$ and $\hat{X}^2(C)$. Here total means taking direct products. When working with $X$-complex of DG algebras, total means taking direct sums.

We define the periodic cyclic complex of a DG coalgebra $C$ by

\[
CC_*(C) = \hat{X}(B^* C).
\]

We denote the homology of this complex by $HP_*(C)$. The motivation for this definition is as follows. In [Q1], D. Quillen computed $\hat{X}(B A)$, when $A$ is an algebra, and showed that it is isomorphic with the total complex of Connes-Tsygan bicomplex of $A$. The same proof easily extends to the case of DG algebras. Our definition of periodic cyclic complex of DG coalgebras is simply the dual of this definition-theorem for DG algebras.

The following result was first proved by T. Goodwillie for DG algebras [G].

**2.1 Proposition.** Let $f : C \to D$ be a morphism of DG coalgebras such that $f$ is a quasi-isomorphism of complexes. Then $f_* : HP_*(C) \to HP_*(D)$ is an isomorphism.

Let $C$ be a DG coalgebra. Note that we have a natural morphism of complexes

\[
I : \hat{X}(C) \to \hat{X}^2(C),
\]

obtained from the inclusion $C \to C \bigoplus \Omega^2 C_z$ and $\Omega^1 C_z \to \hat{\Omega}^1 C$. In general, there is no natural map $\hat{X}^2(C) \to \hat{X}(C)$. However, it is shown in [Kh] that if $C = BA$ is the bar construction then $I$ is a homotopy equivalence and a homotopy inverse

\[
R : \hat{X}^2(BA) \to \hat{X}(BA)
\]

is constructed. There is a dual statement for DG coalgebras, but we don’t need it in this paper.

Let us call a DG coalgebra of finite cohomological dimension if its underlying coalgebra has finite cohomological dimension. We need to know that if a DG coalgebra $C$ has finite cohomological dimension, then its periodic cyclic complex $CC_*(C)$ is quasi-isomorphic to a “small” complex. We need this only for cohomological dimensions 1 and 2.
2.2 Proposition. Let $C$ be a DG coalgebra. Then

1. The natural map $\hat{X}(C) \longrightarrow CC_\ast(C)$ is a quasi-isomorphism if $C$ has cohomological dimension 1.

2. The natural map $\hat{X}^2(C) \longrightarrow CC_\ast(C)$ is a quasi-isomorphism if $C$ has cohomological dimension 2.

3. A Proof of the K"unneth formula

Let $A_1$ and $A_2$ be unital algebras over a field of characteristic zero. In general, $X(A_1) \otimes X(A_2)$ can not be quasi-isomorphic to $X(A_1 \otimes A_2)$. However, when $A_1$ and $A_2$ are quasifree (in particular free) algebras then the cohomological dimension of $A_1 \otimes A_2$ is at most 2 and $X(A_1) \otimes X(A_2)$ is quasi-isomorphic to $X^2(A_1 \otimes A_2)$. This fact is due to Cuntz and Quillen [CQ]. In order to obtain explicit formulas, M. Puschnigg constructed a morphism of complexes [P],

$$P : X^2(A_1 \otimes A_2) \longrightarrow X(A_1) \otimes X(A_2),$$

and showed that $P$ is a quasi-isomorphism when $A_1$ and $A_2$ are quasi-free. In fact, an explicit right inverse to $P$ is the map defined in [CQ], for arbitrary algebras, $X(A_1) \otimes X(A_2) \longrightarrow X(A_1 \otimes A_2)$ combined with the canonical inclusion $X(A_1 \otimes A_2) \longrightarrow X^2(A_1 \otimes A_2)$.

Next we observe that the above constructions are completely functorial and extend to DG algebras and DG coalgebras. To be precise, let us define the completed tensor product $C \hat{\otimes} D$ of infinite product vector spaces $C = \prod_{i \geq 0} C_i$ and $D = \prod_{i \geq 0} D_i$ by

$$C \hat{\otimes} D = \prod_{n \geq 0} \bigoplus_{i+j=n} C_i \otimes D_j.$$

There is an obvious injection $C \otimes D \longrightarrow C \hat{\otimes} D$. Since the bar construction is free as a coalgebra, dualizing the above map $P$ we obtain a quasi-isomorphism of complexes

$$\hat{X}(BA_1) \hat{\otimes} \hat{X}(BA_2) \longrightarrow \hat{X}^2(BA_1 \otimes BA_2).$$

Consider the sequence of maps

$$\hat{X}(BA_1) \hat{\otimes} \hat{X}(BA_2) \xrightarrow{P} \hat{X}^2(BA_1 \otimes BA_2) \xrightarrow{S} \hat{X}^2(B(A_1 \otimes A_2)) \xrightarrow{R} \hat{X}(B(A_1 \otimes A_2)),$$

where $S$ is induced by the shuffle map $S : BA_1 \otimes BA_2 \longrightarrow B(A_1 \otimes A_2)$ and $R$ is the retraction introduced earlier.

3.1 Theorem. $P$, $S$ and $R$ are quasi-isomorphisms.
Proof. We only have to show that $\tilde{S}$ is a quasi-isomorphism. Consider the diagram

$$
\begin{array}{ccc}
\hat{X}^2(BA_1 \otimes BA_2) & \xleftarrow{\sigma} & \hat{X}(B^c(BA_1 \otimes BA_2)) = CC_*(BA_1 \otimes BA_2) \\
\downarrow S & & \downarrow \tilde{S} \\
\hat{X}^2(B(A_1 \otimes A_2)) & \xleftarrow{\sigma} & \hat{X}(B^cB(A_1 \otimes A_2)) = CC_*(B(A_1 \otimes A_2))
\end{array}
$$

where $\tilde{S}$ is the map induced from the DG coalgebra map $S : BA_1 \otimes BA_2 \rightarrow B(A_1 \otimes A_2)$ on periodic cyclic complexes. By proposition 2.1, $\tilde{S}$ is a quasi-isomorphism as $S$ is a quasi-isomorphism of complexes. By proposition 2.2, the horizontal arrows are also quasi-isomorphisms and hence $\overline{S}$ is a quasi-isomorphism.

It follows that the composition

$$
\hat{S} = R \circ \tilde{S} \circ P : CC_*(A_1) \hat{\otimes} CC_*(A_2) \rightarrow CC_*(A_1 \otimes A_2)
$$

is a quasi-isomorphism of complexes.

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