TWO-DIMENSIONAL QCD
ON THE SPHERE AND ON THE CYLINDER

M. Caselle, A. D’Adda, L. Magnea and S. Panzeri

Istituto Nazionale di Fisica Nucleare, Sezione di Torino
Dipartimento di Fisica Teorica dell’Università di Torino
via P.Giuria 1, I-10125 Torino, Italy

Abstract

The partition functions of QCD2 on simple surfaces admit representations in terms of exponentials of the inverse coupling, that are modular transforms of the usual character expansions. We review the construction of such a representation in the case of the cylinder, and show how it leads to a formulation of QCD2 as a $c=1$ matrix model of the Kazakov-Migdal type. The eigenvalues describe the positions of $N$ Sutherland fermions on a circle, while their discretized momenta label the representations in the corresponding character expansion. Using this language, we derive some new results: we give an alternative description of the Douglas-Kazakov phase transition on the sphere, and we argue that an analogous phase transition exists on the cylinder. We calculate the large $N$ limit of the partition function on the cylinder with boundary conditions given by semicircular distributions of eigenvalues, and we find an explicit expression for the large $N$ limit of the Itzykson-Zuber integral with the same boundary conditions. (Talk given at the “Workshop on high energy physics and cosmology” at Trieste, July 1993.)

email address: Decnet=(31890::CASELLE,DADDA,MAGNEA,PANZERI)
internet=CASELLE(DADDA)(MAGNEA)(PANZERI)@TO.INFN.IT
1 Introduction

In the last year, following the papers of Gross and Taylor \cite{1}, there has been a renewed interest in the attempts to understand two dimensional QCD in the large $N$ limit as a string theory. In ref. \cite{1} it was shown that QCD2 is indeed a string theory by proving that the coefficients of the expansion of the QCD2 partition function in power series of $1/N$ can be interpreted in terms of mappings from a two dimensional surface onto a two dimensional target space. A different approach was pursued in refs. \cite{2,3,4}, where the equivalence of QCD2 on a cylinder and on a torus with a $c = 1$ matrix model was shown in various ways. In this talk we will describe the approach of \cite{4}, which leads to expressions for the partition functions that are “dual” to the usual character expansions, in the sense that they can be expanded in powers of the inverse coupling. This approach is inspired by the interpretation of the Kazakov-Migdal model as a model for high temperature QCD \cite{5}, which we review in Section 2. In Section 3 we will show, following ref \cite{4}, how QCD2 on a cylinder and on a torus is equivalent for any $N$ to a one-dimensional Kazakov-Migdal model, with the eigenvalues of the matter fields constrained to live on a circle rather than on a line. The generalization of these results to Yang-Mills theories based on other classical groups \cite{6} is also discussed, together with the equivalence of these theories with the zero coupling limit of Sutherland models \cite{7}. In Section 4, we derive some previously unpublished results concerning the large $N$ limit of QCD2 on the sphere and on the cylinder. We show how the phase transition on the sphere recently discovered by Douglas and Kazakov \cite{8} can be understood in our language, and we exhibit its close analogy with the phase transition of Gross-Witten \cite{9} and Wadia \cite{10}. Finally, applying the techniques of ref. \cite{11}, we argue that the partition function on the cylinder exhibits the same transition\footnote{We have been informed \cite{12} that the existence of this transition has been independently established by D. Gross and A. Matytsin.}

2 The Kazakov-Migdal model as high temperature lattice gauge theory

Let us consider the Kazakov-Migdal model \cite{13} with quadratic potential defined on a $d$-dimensional ipercubic lattice labeled by $x$:

$$ S = \sum_x N \text{Tr}[m^2 \phi^2(x) - \sum_\mu \phi(x)U(x, x + \mu)\phi(x + \mu)U^\dagger(x, x + \mu)], $$

where $\phi(x)$ is an Hermitian $N \times N$ matrix defined on the sites of the lattice, and $U(x, x + \mu)$ is a unitary $N \times N$ matrix, defined on the links, which plays the role of the gauge field as in the usual lattice discretization of Yang-Mills theories. The appearance of matter fields in the adjoint representation of the gauge group suggests the possibility that they may be thought of as the remnants of the gauge
field along an extra compactified dimension, in the limit where the compactification radius tends to zero. Finite temperature lattice gauge theory in an example where this situation is realized, with time as the extra compactified dimension, and the temperature $T$ as the inverse of the compactification radius. The corresponding Wilson action is given by

$$S_W = \frac{2N_c}{g^2} \sum_n \text{Re} \left\{ T \sum_i \text{tr}(U_{n;0i}) + \frac{1}{T} \sum_{i<j} \text{tr}(U_{n;ij}) \right\},$$

where $n \equiv (x,t)$ denotes the space-time position and $U_{n;0i}$, $U_{n;ij}$ are the time-like (space-like) plaquette variables.

In the large $T$ limit the spatial plaquettes are of order $\frac{1}{T}$ and the timelike plaquettes are frozen around $U_{n;0i} = 1$. The action (2) is invariant under a $\mathbb{Z}_N$ symmetry consisting in multiplying each timelike link by a space independent element of the center of the group. The order parameter of such symmetry is the Polyakov loop, defined by

$$P(x) = \text{tr} \prod_{t=1}^{N_t}(U_{x,t,0}).$$

At high temperature this symmetry is broken (deconfinement phase): the time-like links $U_{x,t,0}$ and the Polyakov loop are frozen around one of the $\mathbb{Z}_N$ invariant vacua, for instance 1, hence $\langle P(x) \rangle = 1$. In this situation it makes sense to expand $P(x)$ around its vacuum expectation value and consider small fluctuations. It was shown in ref. [5] that, in the large $T$ limit, if one

1. neglects the contributions coming from spacelike plaquettes,

2. expands $P(x) = e^{i\frac{\phi(x)}{\sqrt{T}}}$ in powers of $\frac{1}{\sqrt{T}}$,

one finds that the finite temperature lattice gauge theory in $d+1$ dimensions of eq. (2) coincides, up to terms of order $\frac{1}{T}$, with the Kazakov-Migdal model of eq.(1), in $d$ dimensions and with $m^2 = d$.

In the large $T$ limit the Wilson term corresponding to the spatial plaquettes appears as a small (order $\frac{1}{T}$) correction to the Kazakov-Migdal action. Something similar was proposed in ref. [14], where a small Wilson term is added to the Kazakov-Migdal model in order to break the local $\mathbb{Z}_N$ symmetry and avoid superconfinement. The results of ref. [14] are in complete agreement with the large temperature expansion proposed in [5]: two phases are present, according to whether the functional integral is dominated by configurations with small eigenvalues of $\Phi(x)$ (phase 1) or large eigenvalues of $\Phi(x)$ (phase 2). In phase 1 the coefficient of the Wilson term remains small in the continuum limit and the Wilson action can be consistently treated as a perturbation of the K-M action. However, as a theory of induced QCD this fails to explain the dominance of planar diagrams in the large $N$ limit. Instead, from the point of view of the large $T$ expansion,
small eigenvalues of $\Phi(x)$ imply that $< P(x) > = 1$. We are in the high temperature
deconfining phase and the spatial plaquettes are consistently depressed by the $\frac{1}{T}$
factor. In this phase foam-like diagrams are expected to give a large contribution
and we have no planar diagram dominance at large $N$. Quite the opposite happens
in phase 2 where, according to [14], the coefficient of the Wilson term is of order
$N$ in the continuum limit, so that the Wilson action is not any more a small per-
urbation of the K-M term. From our point of view, if the eigenvalues of $\Phi(x)$ are
large, the eigenvalues of $P(x)$ are distributed on the unit circle, and $< P(x) >= 0$.
The system is in the confined phase and $T$ is below the critical temperature, hence
the spatial plaquettes contribution cannot be neglected. This is the phase where
the usual large $N$ behaviour is expected to hold.

In spite of this qualitative agreement with the physical picture, the naive large
$T$ limit outlined above fails for $d > 1$. We know in fact that the K-M model with
a quadratic potential does not admit a continuum limit in that case [13]. On the
other hand, it is known in the theory of finite temperature QCD that the original
picture [10] of a complete dimensional reduction does not work due to the infrared
divergences in the spacial plaquettes term, and one finds instead a three dimensional
gauge theory coupled with a scalar field $\phi$ in the adjoint representation, with a non
trivial potential $V(\phi)$ (see for instance [17, 18, 19]).

The fundamental point for what follows is that none of these problems occur in
$d = 1$: the continuum limit in the K-M model is possible with quadratic potential
at $m^2 = 1$, and there are no spatial plaquettes, so that no approximation is done
by neglecting them. Besides, we know that two dimensional QCD depends only
on the area and the topology of the surface on which it is defined, so at finite
temperature it should be independent of the temperature $T$. Hence, we expect the
infinite temperature limit to give actually the exact result. This is shown in the
next section by working directly on the functional integral of the continuum theory.

3 QCD2 as a $c = 1$ matrix model

We shall study first the partition function $K_2(g_1, g_2; t)$ of QCD2 on a cylinder, with
fixed holonomies (Polyakov loops) $g_1$ and $g_2$ at the two boundaries (say $x = 0$ and
$x = 2\pi$). It is defined by the functional integral

$$K_2(g_1, g_2; t) = \int D A_{\mu} D F e^{-S(t)} \delta (W(0), g_1) \delta (W(2\pi), g_2) \psi(g_1) \psi(g_2),$$

where the action $S(t)$, as a result of the invariance of the theory under area pre-
serving diffeomorphisms, can be written as\footnote{The coordinates on the cylinder are denoted by $x, \tau$. Notice that we choose the time direction (with coordinate $\tau$) as the compactified one.}

$$S(t) = \frac{2N}{t} \int_0^{2\pi} dx d\tau \text{Tr}\{F^2 - iF(\partial_0 A_1 - \partial_1 A_0) - iF[A_0, A_1]\}.$$
Here $F$ and $A$ are independent fields and $t = \tilde{g}^2 A$, with $A$ the area. $W(x)$ denotes the Polyakov loop
\[ W(x) = \mathcal{P} e^{i \int_0^{2\pi} dx A_0(x,\tau)}, \tag{6} \]
and $\hat{\delta}(g,h)$ denotes the conjugation invariant delta function on the group manifold. Finally $\psi(g_1)$ and $\psi(g_2)$ are just normalization factors that depend only on the eigenvalues of $g_1$ and $g_2$ and are chosen in such a way that the sewing of two cylinders corresponds to a group integration.

In order to calculate the functional integral (4) it is convenient to choose the gauge $\partial_0 A_0 = 0$. In this gauge all the non zero modes of the Fourier expansion in $\tau$ of $A_0$ are set to zero. In [4] it is shown that the corresponding Faddeev-Popov determinant is exactly cancelled as a result of the integration over the non zero modes of $A_1$ and $F$, so that in the end one obtains the following expression for $K_2$:
\[ K_2(g_1, g_2; t) = \int DBDA e^{-\frac{N_A}{2} \text{Tr} \int_0^{2\pi} dx [\partial B - i[A,B]]^2} \times \hat{\delta}(W(0), g_1) \hat{\delta}(W(2\pi), g_2) \psi(g_1) \psi(g_2). \tag{7} \]
where $B(x)$ and $A(x)$ (matrix fields on the algebra) denote the static modes of the $A_0(x,\tau)$ and $A_1(x,\tau)$ gauge fields respectively. Eq (7) is the first result of our analysis: the dimensional reduction of the previous section is exact and the result is, as expected, a KM model in one continuous dimension (the spatial dimension of the cylinder), where $A(x)$ and $B(x)$ play respectively the role of gauge field and matter field. There is, however, one important difference between (7) and the usual one-dimensional KM model: the boundary conditions depend on $e^{2\pi i B}$ rather than $B$. This means that the eigenvalues of $B$ are defined on a circle rather than on a line or, as explained in [1, 3], that the matter field $B$ enters the action through the unitary matrix defined by $e^{2\pi i B}$. At this point, by means of standard matrix model techniques, the matrix $B(x)$ can be diagonalized, the functional integral over its eigenvalues performed and one finally obtains
\[ K_2(g_1, g_2; t) = \sum_P \frac{t^{-\frac{N_A}{2}}}{J(\theta)J(\phi)} \sum_{\{l_i\}} (-1)^P \exp \left[ -\frac{N}{2t} \sum_{i=1}^{N} \left( \phi_i - \theta_{P(i)} + 2\pi l_i \right)^2 \right]. \tag{8} \]
where $\theta_i$ and $\phi_i$ are the invariant angles of $g_1$ and $g_2$, respectively\(^3\), while $J(\sigma)$ is, up to a phase the Vandermonde determinant for a unitary matrix,
\[ J(\sigma) = \prod_{i < j} 2 \sin \frac{\sigma_i - \sigma_j}{2} \tag{9} \]
\(^3\)For the $SU(N)$ group the invariant angles $\theta_i$ and $\phi_i$ as well as the integers $l_i$ are constrained by $\sum_i \theta_i = \sum_i \phi_i = \sum_i l_i = 0$. These constraints are not there in the $U(N)$ case, where however an extra factor $(-1)^{(N-1)\sum_i l_i}$ is present at the r.h.s.
In (8) the normalization factor $\psi(g)$ has also been determined. Up to a constant, it is given by $\psi(g) = J(\theta)/\Delta(\theta)$, where $\theta_i$ are the invariant angles of $g$ and $\Delta(\theta)$ is the usual Vandermonde determinant of a hermitian matrix.

The formula (8) is one of the main results of our analysis. It provides for $K_2$ an expansion in exponentials of $1/t$, which is related to the well known character expansion by a non trivial modular transformation [20], as explicitly checked in [4].

By taking the $\theta_i \to 0$ limit of eq. (8), one obtains the modular inversion for the kernel on the disk, $K_1(\phi;t)$, which has been known for many years [21],

$$K_1(\phi; t) = N_1(N) \times t^{-\frac{N^2-1}{2}} \prod_{l_i=-\infty}^{+\infty} \sum_{i<j=1}^{N} \frac{\phi_i - \phi_j + 2\pi(l_i - l_j)}{2\sin \frac{1}{2} [\phi_i - \phi_j + 2\pi(l_i - l_j)]} \times \exp \left[ -\frac{N}{2t} \sum_{i=1}^{N} (\phi_i + 2\pi l_i)^2 \right].$$

(10)

The kernel on the cylinder, as given by eq. (8), can be interpreted as follows: let $\theta_i$ and $\phi_i$ be respectively the initial and final positions of an ensemble of $N$ particles on a circle, and let $t$ be the time elapsed in the transition from the initial to the final configuration. Then $K_2(g_1, g_2; t)$ can be interpreted as the transition probability for such process. Notice that, because of the Vandermonde determinants, the wave functions are completely antisymmetric under the exchange of pairs of these particles, which must therefore be fermions. That QCD2 would prove to be equivalent to a quantum theory of free fermions on a circle was already to be expected from eq. (7). It is known in fact that one dimensional KM model describes the singlet sector of an $SU(N)$ matrix model with $c = 1$ [22, 23], whose interpretation in terms of free fermions has been known for some time.

The fermionic system that describes QCD2 turns out to be a particular case of a family of integrable one dimensional quantum mechanical models first investigated by Calogero [24] and Sutherland [7]. In fact, one can easily check from eq. (8) that $K_2(g_1, g_2; t)$ satisfies the differential equation

$$\left( N \frac{\partial}{\partial t} - \frac{1}{2} J^{-1}(\phi) \sum_i \frac{\partial^2}{\partial \phi_i^2} J(\phi) \right) K_2(\phi, \theta; t) = 0.$$ 

(11)

By taking the limit $\theta_i \to 0$ in eq. (11) one finds that also the kernel on a disk $K_1(\phi; t)$ obeys the same equation. Upon redefinition of the kernel by $K_1 \to J(\phi) K_1$, eq. (11) becomes the (euclidean) free Schrödinger equation for $N$ fermions on a circle. The boundary conditions are determined by this redefinition, and are just the boundary conditions that apply to the fermions in the zero coupling limit of the Sutherland integrable model related to the Lie algebra $A_{N-1}$ [7].

The same equation, but with $J(\phi)$ replaced by $\Delta(\phi)$, is satisfied by the corresponding quantities in one dimensional hermitian matrix models, where the place of $K_2$ is taken by the Itzykson-Zuber integral (see for instance the recent paper by Matytsin [11]). Such models are then related to the zero coupling limit of the
Calogero integrable model \[24\], in the same way as QCD2 is related to the Sutherland model.

All quantities of relevance to QCD2 can readily be reinterpreted in the Sutherland language, as discussed in more detail in \[6\]. For example, we already mentioned that $K_2(\theta, \phi; t)$ can be interpreted as the propagator from an initial configuration $\theta_i$ to a final one $\phi_i$ in a time $t$. Similarly, the kernel on the disk $K_1(\phi; t)$ corresponds to the transition amplitude from the configuration $\theta_i = 0$ to $\phi_i$, and the partition function on the sphere is just the amplitude for the process in which all the fermions start at the origin and come back there after a time $t$. The meaning of the modular transformation relating the expression (8) to the usual character expansion is also clear: the integers labelling the unitary representations in the character expansion correspond to discretized momenta of the fermions on the circle, while eq. (8) gives the corresponding coordinate representation.

The partition function of QCD2 on a torus can be obtained from eq. (8) by identifying $\theta_i$ with $\phi_i$ and then integrating with the group invariant measure $\prod_i d\theta_i J_2(\theta)$. The integral is gaussian in the invariant angles $\theta_i$, although with a complicated combinatorial structure due to the sum over all permutation in eq. (8). This can be disentangled by decomposing each permutation in cycles and by calculating the integrals corresponding to cyclic permutations. The result has a particularly simple form if one considers the grand canonical partition function at $\tilde{t} = \frac{1}{N}$ fixed. For $SU(N)$ for instance one gets

$$Z_{SU(N)}(q) = \sum_N Z_{SU(N)}(N, \tilde{t}) q^N = \left( \frac{\tilde{t}}{4\pi} \right)^{1/2} \int_0^{2\pi} d\beta \prod_{n=-\infty}^{+\infty} \left( 1 + q e^{-\frac{\tilde{t}}{N}(n-\beta)^2} \right).$$

The interpretation of this formula as the grand canonical partition function of a gas of free fermions with energy levels that go like $n^2$ for $n \to \infty$ is evident. The parameter $\beta$ is the momentum corresponding to the center of mass of the fermions, which in $SU(N)$ is localized (see footnote following eq. (8)). Hence the corresponding momentum $\beta$ is completely undetermined.

All the results obtained in this section can be generalized to the case of an arbitrary simple gauge group \[6\]. One obtains that the matrix model describing a two dimensional Yang-Mills theory on a cylinder is a KM model where the matter fields are in the fundamental representation of the gauge group. Since the string interpretation of this matrix model requires expanding the group matrices as exponentials of the matrices on the algebra, one has an orientable string theory for the $SU(N)$ group, while if the gauge group is $SO(N)$ or $Sp(2N)$ the worldsheet of the string may be both orientable or nonorientable \[25\].
4 Large N phase transitions on the sphere and on the cylinder

In this section we report on some new results concerning mainly the large $N$ limit of the kernel on the cylinder and the partition function on the sphere, where a large $N$ phase transition was recently found by Douglas and Kazakov [8]. Unlike the authors of [8], we work in configuration space and consider the expression for the partition function on a sphere of area $A$, obtained by sewing disks of areas $t$ and $A - t$,

$$Z_{G=0}(N, A) = \int_0^{2\pi} d\theta_i J^2(\theta) K_1(\theta, t) K_1(\theta, A - t)$$

$$= \left( \frac{t(A - t)}{4N^2} \right)^{-\frac{N^2+1}{2}} \int_{-\infty}^{+\infty} d\lambda_i \sum_{\{n_i\}} \Delta(\lambda) \Delta(\lambda + 2\pi n_i) \times$$

$$\times \exp \left\{ -N \sum_i \left[ \frac{\lambda_i^2}{2t} + \frac{(\lambda_i + 2\pi n_i)^2}{2(A - t)} \right] \right\}, \quad (13)$$

where we used eq. (10) for $K_\infty$ and, in the first kernel, we replaced $\theta_i + 2\pi m_i$ with a variable $\lambda_i$ defined on the whole real axis. We want to find the eigenvalue distribution that, for any given $t$, dominates the integral in eq. (13) in the large $N$ limit. The crucial point is that, if the eigenvalue distribution is, for any $t$, confined in the interval $(-\pi, \pi)$, then we are entitled to consider in eq. (13) only the term corresponding to $n_i = 0$, for all $i$. The other terms in fact describe the winding of the eigenvalues, which is possible, at $N = \infty$, only if the eigenvalue distribution spreads at some $t$ over the whole circle. The term $n_i = 0$ describes a gaussian matrix model whose eigenvalue distribution is given by Wigner’s semicircle law

$$\rho(\lambda, r) = \frac{2}{\pi r^2} \sqrt{r^2 - \lambda^2}, \quad (14)$$

where the radius $r$ of the distribution is given in terms of $t$ and $A$ by

$$r = 2\sqrt{\frac{t(A - t)}{A}}. \quad (15)$$

The maximum value of the radius of the eigenvalue distribution occurs at $t = \frac{A}{2}$, and it is given by $r_{\text{max}} = \sqrt{A}$. According to the previous argument, $r_{\text{max}}$ must not exceed $\pi$, which implies $A < \pi^2$. In complete agreement with the results of ref. [8] we find then two phases: one for $A < \pi^2$, where the partition function on the sphere is completely described in the large $N$ limit by a gaussian matrix model, and the winding modes can be neglected. In this region the Sutherland and Calogero models are indistinguishable at large $N$. The other phase, for $A > \pi^2$, is characterized in configuration space by winding numbers different from zero. It is remarkable that,
in configuration space, the mechanism of the phase transition discovered by Douglas and Kazakov is the same as the one of the Gross-Witten-Wadia phase transition [3, 11].

This reasoning can be extended to the cylinder if one realizes that, from the point of view of the dynamics of the eigenvalues, the sphere is just a cylinder with special boundary conditions, namely that all eigenvalues must be concentrated at the origin at the initial and final times. To study the cylinder in the large $N$ limit, in the gaussian phase, one needs to generalize eq. (15), to find the time evolution of the radius of the semicircular eigenvalue distribution for arbitrary boundary conditions (that is allowing for non-zero initial and final radii). This can be done by making use of the results of a recent paper by Matytsin [11], which studies the time evolution of the density of eigenvalues in a system satisfying eq. (11), with $J(\phi)$ replaced by $\Delta(\phi)$. The result is that the evolution is determined by the Das-Jevicki hamiltonian [26], and the corresponding equations of motion can be written as the Hopf equation

$$\frac{\partial f}{\partial t} + f \frac{\partial}{\partial \lambda} f = 0,$$

(16)

where $f(\lambda, t) \equiv v(\lambda, t) + i\pi \rho(\lambda, t)$, and $\rho(\lambda, t)$ is the density of eigenvalues $\lambda_i$ at time $t$. According to our discussion, the replacement of $J(\phi)$ with $\Delta(\phi)$ in eq. (11) does not affect the large $N$ limit in the gaussian phase. Therefore, the semicircular distribution (14), with radius given by (15), must be a solution of (16). Indeed, if one inserts in (16) the ansatz (14), with $r = r(t)$, one finds that (16) is solved if

$$\ddot{r} + \frac{4}{r^3} = 0.$$

(17)

The general solution of this differential equation is easily found, and it generalizes naturally eq. (15), as

$$r(t) = 2 \sqrt{\frac{(t + \alpha)(\beta - t)}{\alpha + \beta}}.$$

(18)

Given arbitrary boundary conditions $r(0) = \gamma_0$ and $r(A) = \gamma_1$, one can find from (18) the corresponding values of $\alpha$ and $\beta$. In particular, the sphere corresponds to $\gamma_0 = \gamma_1 = 0$, which implies $\alpha = 0$ and $\beta = A$, in agreement with (15). Eq. (14), with $r$ given by (18), gives then the classical trajectory for the density of eigenvalues that dominates, in the large $N$ limit, the functional integral corresponding to the kernel $K_2$, with area $A$ and boundary conditions given by Wigner’s distributions of radii $\gamma_0$ and $\gamma_1$. The actual value of $K_2$ in the large $N$ limit is given by

$$K_2(\gamma_0, \gamma_1; A) = e^{-N^2 S_{\text{class}}},$$

(19)

where $S_{\text{class}}$ is the action calculated on the classical trajectory,

$$S_{\text{class}} = \frac{1}{8} \int_0^A dt \left( r^2 + \frac{4}{r^2} \right).$$

(20)
with \( r \) given by (18).

Now it is easy to see that the same mechanism that leads to a phase transition on the sphere operates also on the cylinder. The solution (19) is the large \( N \) limit of the QCD2 kernel on a cylinder only if \( r(t) < \pi \) for any \( t \) on the trajectory. For any given boundary conditions (that is for any given \( \gamma_0 \) and \( \gamma_1 \)), the maximum value of \( r(t) \) increases as the area \( \mathcal{A} \) increases (this is because \( \ddot{r} < 0 \) from (17)), so it is bound to hit the value \( \pi \) at some critical value of \( \mathcal{A} \), which can easily be determined from the initial conditions. There a phase transition occurs, in analogy with case of the sphere.

Our solution for the large \( N \) limit of \( K_2 \) of course applies also to matrix models where the eigenvalues live on the real axis rather than on a circle, but in that case with no restrictions on the values of \( r(t) \), and no phase transitions. In other words, our solution applies without restrictions if one is dealing with a Calogero rather than a Sutherland model. In this case the solution (19) is equivalent to the calculation of the Itzykson-Zuber integral in the large \( N \) limit \([11]\), with Wigner’s distribution of the eigenvalues. We find

\[
\frac{1}{N^2} \ln I_{IZ}(\gamma_0, \gamma_1) = -\frac{1}{2} \ln \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{\gamma_0^2 \gamma_1^2}{4}} \right) - \frac{1}{2} \left( 1 - \sqrt{1 + \frac{\gamma_0^2 \gamma_1^2}{4}} \right) + O(1/N), \tag{21}
\]

where \( I_{IZ}(\gamma_0, \gamma_1) \) is the Itzykson-Zuber integral, calculated with respect to two sets of eigenvalues that in the large \( N \) limit have Wigner distributions of radii \( \gamma_0 \) and \( \gamma_1 \). Notice that, for \( \gamma_0 = \gamma_1 \), the expression (21) reduces to the one calculated by Gross in \([15]\).

We see that the “dual” expansions for the partition functions of QCD2, which have been determined so far for the disk, the sphere, the cylinder and the torus, are valuable tools for the study of the theory, both for finite \( N \) and in the large \( N \) limit. They provide a physical insight which is complementary to the one given by the character expansions, just as configuration space is complementary to momentum space, and they have a very direct interpretation in terms of Sutherland fermions, which leads to a simple understanding of the phase transitions on the sphere and on the cylinder.

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