Abstract. In [TV19a] the equivariant quantum differential equation (qDE) for a projective space was considered and a compatible system of difference qKZ equations was introduced; the space of solutions to the joint system of the qDE and qKZ equations was identified with the space of the equivariant K-theory algebra of the projective space; Stokes bases in the space of solutions were identified with exceptional bases in the equivariant K-theory algebra. This paper is a continuation of [TV19a].

In this paper we describe the relation between solutions to the joint system of the qDE and qKZ equations and the topological-enummerative solution to the qDE only, defined as a generating function of equivariant descendant Gromov-Witten invariants. The relation is in terms of the equivariant graded Chern character on the equivariant K-theory algebra, the equivariant gamma class of the projective space, and the first Chern class of the tangent bundle of the projective space.

We also show that the Stokes matrix assigned to a Stokes basis in the space of solutions equals the Gram matrix of the equivariant Grothendieck-Euler-Poincaré pairing wrt to the corresponding exceptional basis in the equivariant K-theory algebra.

We identify the Stokes bases in the space of solutions with explicit full exceptional collections in the equivariant derived category of coherent sheaves on the projective space, where the elements of those exceptional collections are just line bundles on the projective space and exterior powers of the tangent bundle of the projective space.

These statement are equivariant analogs of results of G. Cotti, B. Dubrovin, D. Guzzetti, and S. Galkin, V. Golyshev, H. Iritani.

Key words: Equivariant quantum differential equation, qKZ difference equations, derived category of equivariant coherent sheaves, equivariant K-theory, exceptional collections.

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1. Introduction

1.1. In this paper we consider the equivariant quantum differential equation \((qDE)\) of a complex projective space \(\mathbb{P}^{n-1}\), endowed with the diagonal action of the torus \(T = (\mathbb{C}^*)^n\).
This equation is the ordinary differential equation

$$\left( q \frac{d}{dq} - x_{q,z} \right) I(q, z) = 0, \quad (1.1)$$

where the unknown function $I(q, z)$ takes values in the equivariant cohomology algebra $H^n_{\mathbb{T}}(\mathbb{P}^{n-1}, \mathbb{C})$, and $x_{q,z} : H^n_{\mathbb{T}}(\mathbb{P}^{n-1}, \mathbb{C}) \to H^n(\mathbb{P}^{n-1}, \mathbb{C})$ is the operator of quantum multiplication by the equivariant first Chern class of the tautological line bundle on $\mathbb{P}^{n-1}$.

The $qDE$ depends on the equivariant parameters $z = (z_1, \ldots, z_n)$ corresponding to the factors of the torus $\mathbb{T}$. The $qDE$ has two singularities: a regular singularity at $q = 0$ and an irregular singularity at $q = \infty$.

In [TV19a] a compatible system of difference equations, called the $qKZ$ equations, was introduced,

$$I(q, z_1, \ldots, z_i - 1, \ldots, z_n) = K_i(q, z)I(q, z), \quad i = 1, \ldots, n, \quad (1.2)$$

where $K_i$’s are suitable linear operators. In [TV19b] solutions to the joint system of the $qDE$ and $qKZ$ equations were constructed in the form of $q$-hypergeometric integrals. In [TV19a] the space of solutions was identified with the space of the equivariant $K$-theory algebra $K^*_0(\mathbb{P}^{n-1})$. The Stokes bases of the $qDE$ at its irregular singular point were described in terms of the exceptional bases of the equivariant $K$-theory and a suitable braid group action on the set of exceptional bases. In this paper we continue this study.

1.2. In this paper we establish relations between the monodromy data of the joint system of the $qDE$ and $qKZ$ equations for $\mathbb{P}^{n-1}$ and characteristic classes of objects of the derived category of equivariant coherent sheaves $\mathcal{D}^b_{\mathbb{T}}(\mathbb{P}^{n-1})$. Here are the three main results of the paper.

The first result is about the identification of solutions with objects in equivariant derived category. As mentioned above, in [TV19a] solutions of the joint system of $qDE$ and $qKZ$ equations were identified with $K$-theoretical classes in $K^*_0(\mathbb{P}^{n-1})\mathbb{C}$, and that Stokes bases of solutions correspond to exceptional bases of $K^*_0(\mathbb{P}^{n-1})\mathbb{C}$. We prove that this identification can be upgraded at a categorical level: Stokes bases of the joint system of $qDE$ and $qKZ$ equations are in correspondence with an explicit family of $\mathbb{T}$-full exceptional collections in the derived category of $\mathbb{T}$-equivariant coherent sheaves $\mathcal{D}^b_{\mathbb{T}}(\mathbb{P}^{n-1})$.

See Theorem 7.26, Corollary 7.27 and Theorem 10.13.

The second result is about the relation between central connection matrices and characteristic classes of objects on equivariant derived category. The $qDE$ has a distinguished basis of solutions, whose expansion at $q = 0$ is a generating function for equivariant descendants Gromov-Witten invariants of $\mathbb{P}^{n-1}$, see [Giv96]. We call this basis the equivariant topological-enumerative solution of the $qDE$.

Given an object $E$ in $K^*_0(\mathbb{P}^{n-1})\mathbb{C}$, denote by $\Psi_E$ the corresponding solution of the joint system of $qDE$ and $qKZ$ equations. Express $\Psi_E$ as a linear combination of elements of the equivariant topological-enumerative solution. In our B-Theorem (Theorem 8.3) and Corollary 8.5, we prove that the coefficients in this linear combination are the components of a characteristic class, called B-class, of the corresponding equivariant $K$-theoretical object $E$. The B-class of a coherent sheaf $F$ is a characteristic class which can be described interns of the equivariant graded Chern character of $F$, the equivariant first
Chern class of the tangent bundle of $\mathbb{P}^{n-1}$, and another important characteristic class the equivariant Gamma-class of $\mathbb{P}^{n-1}$.

Notice that our B-Theorem is a refinement of Gamma Theorem of [TV19a], [TV19b].

The third result is on the relation between Stokes matrices and Gram matrices of the equivariant Grothendieck-Euler-Poincaré pairing in $K$-theory. In Theorem 11.6, we prove that $T$-full exceptional collections attached to Stokes bases in complementary Stokes sectors are left/right dual to each other. This is proved by a detailed analysis of the braids connecting them. From this it follows that the Stokes matrix coincides with the Gram matrix of the Grothendieck-Euler-Poincaré pairing wrt exceptional collections.

1.3. In the non-equivariant limit, the $qDE$ plays a crucial role in the analytical study of the quantum cohomology of $\mathbb{P}^{n-1}$, seen as a Frobenius manifold [Dub92, Dub96]. According to the analytical theory of Frobenius manifolds developed by B. Dubrovin, a semisimple Frobenius manifold can be locally identified with the space of isomonodromic deformation parameters of an ordinary differential equation. Denoting by $\lambda$ the coordinate in the complex plane, this isomonodromic system has two singularities: a regular one at $\lambda = 0$, and an irregular one at $\lambda = \infty$. The monodromy data of this isomonodromic system define local invariants of the Frobenius manifold. From their knowledge the whole structure can be reconstructed through a Riemann-Hilbert problem [Dub99, Guz01, CDG17, CDG18]. In the case of quantum cohomology of $\mathbb{P}^{n-1}$, the isomonodromic equation is related to the $qDE$ at $z = 0$ by a shearing transformation (see Proposition 9.5). It follows that the monodromy data of the $qDE$ of $\mathbb{P}^{n-1}$ at $z = 0$ allow to reconstruct the whole Gromov-Witten theory of $\mathbb{P}^{n-1}$.

In occasion of the 1998 ICM in Berlin, Dubrovin conjectured that, for a Fano variety $X$, many properties of the isomonodromic system attached its quantum cohomology, in particular its monodromy, Stokes and central connection matrices, are encoded in the geometry of objects of the derived category of coherent sheaves $\mathcal{D}(X)$ [Dub98].

In particular, it was claimed that the Stokes matrices of this system at $\lambda = \infty$ should be identified with Gram matrices of the Grothendieck-Euler-Poincaré pairing of full exceptional collections in $\mathcal{D}(X)$. In the case of $\mathbb{P}^{n-1}$, this was proved by D. Guzzetti in [Guz99] (see also Section 6 of [CDG18] for some refinements of his results).

Also in this non-equivariant case, the isomonodromic system attached to the quantum cohomology of $X$ admits a topological-enumerative solution. The central connection matrix is the base change matrix relating the Stokes bases of solutions of at $\lambda = \infty$ with the topological-enumerative solution. In the original formulation of the conjecture, Dubrovin predicted that the entries of the central connection matrix should be related to graded Chern characters of objects of exceptional collections in derived category $\mathcal{D}(X)$. Later it was realized that in that relation enters also the Gamma-class of $X$. See [KKP08], [Dub13, CDG17, CDG18], [GGI16, GI15].

The current statement of the conjecture is formulated in [CDG18], Conjecture 5.2, where a $\Lambda$-class is introduced. The $\Lambda$-class is a non-equivariant analog of our $B$-class. Notice that part 3.b of that statement is equivalent to Gamma-Conjecture II of S. Galkin, V. Golyshev and H. Iritani [GGI16]. This conjecture has been proved for Grassmannians,
[CDG18], [GGI16]. See Section 5.6 of [CDG18] for a detailed discussion on the relation of these two works.

The results of the present paper are an equivariant analog to the statements of Conjecture 5.2 of [CDG18] in the case of $\mathbb{P}^{n-1}$, and Gamma Conjecture II of [GGI16].

1.4. This paper is related to the general theory of D. Maulik and A. Okounkov in [MO19] connecting quantum groups and equivariant quantum cohomology of Nakajima quiver varieties. In that context, it was realized that the $qDE$s of these varieties admit some compatible difference equations, called the $qKZ$ equations.

A special case of Nakajima varieties, namely, the case of the cotangent bundles $T^*\mathcal{F}_\lambda$ of partial flag varieties $\mathcal{F}_\lambda$ was considered in [GRTV13] and [RTV15] $^1$. In those papers the $qDE$s and $qKZ$ equations for cotangent bundles were identified with the dynamical differential equations and $qKZ$ difference equations, associated in representation theory with the evaluation module $\mathbb{C}^N(z_1) \otimes \cdots \otimes \mathbb{C}^N(z_n)$ of the Yangian $Y(gl_N)$. This identification lead to two constructions of solutions of the joint system of the $qDE$ and $qKZ$ equations for the cotangent bundles. One construction in [TV14] gave solutions in the form of multidimensional hypergeometric integrals and another construction in [TV19b] gave solutions in the form of multidimensional $q$-hypergeometric integrals.

Also in [TV19b] a suitable limit of the $qDE$s for cotangent bundles of partial flag varieties was considered. In that limit the $qDE$s for cotangent bundles turn into the $qDE$s for the partial flag varieties themselves. Moreover, in that limit the $qKZ$ equations for cotangent bundles survive and turn into new systems of difference equations compatible with the $qDE$s for partial flag varieties. These new systems of difference equations were also called the $qKZ$ equations. Furthermore, it was shown in [TV19b] that the $q$-hypergeometric solutions to the joint systems of the $qDE$s and $qKZ$ equations for cotangent bundles have a limit when the $qDE$s and $qKZ$ equations turn into the $qDE$s and $qKZ$ equations for partial flag varieties.

The special case of that limit was considered in [TV19a] for the partial flag variety $\mathbb{P}^{n-1}$. In [TV19a] the $q$-hypergeometric solutions to the joint system of the $qDE$ and $qKZ$ equations for $\mathbb{P}^{n-1}$ were applied to study the monodromy properties of the $qDE$ for $\mathbb{P}^{n-1}$.

1.5. The paper is organized as follows. In Section 2 basic notions are given on derived category of equivariant coherent sheaves and equivariant Helix theory. In Section 3 we describe the equivariant derived category and $K$-theory of $\mathbb{P}^{n-1}$. In Section 4 we introduce the equivariant cohomology of $\mathbb{P}^{n-1}$. In Section 5 we discuss the equivariant Gromov-Witten theory of $\mathbb{P}^{n-1}$ and we introduce the $qDE$ and $qKZ$ difference equations. In Section 6 we study properties of the $qDE$ and we introduce its topological-enumerative solution. In Section 7 we study solutions of the joint system of $qDE$ and $qKZ$ equations, their integral representations, their asymptotics and we describe the corresponding objects and exceptional collections in the derived category. In Section 8 we prove our $B$-Theorem. In Section 9 we describe the structure of formal solutions of the joint system of $qDE$ and $qKZ$ difference equations. In Section 10 we study Stokes

$^1$Note that the partial flag varieties themselves are not Nakajima varieties
bases of solutions, their normalizations and we show that they correspond to $\mathbb{T}$-full exceptional collections. In Section 11 we prove that the Stokes matrices coincide with the Gram matrices of the equivariant Grothendieck-Euler-Poincaré pairing. In Section 12 we study the specialization of the $q\text{DE}$ at points $z$ such that $(e^{2\pi\sqrt{-1}z_1}, \ldots, e^{2\pi\sqrt{-1}z_n})$ are roots of unity. We show that the monodromy group of the $q\text{DE}$ is $\mathbb{Z}_n$ only for this specialization of the equivariant parameters $z$. In Section 13 we discuss relations of our results with non-equivariant analogues of [CDG18]. In Appendix A we summarize the main aspects of the theory of formal and analytic Frobenius manifolds. In Appendix B we prove an identity relating $q\text{KZ}$ operators with the operator of quantum multiplication. In Appendix C we prove a Theorem of formal reduction for a compatible system of differential and difference equations.

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2. EQUIVARIANT EXCEPTIONAL COLLECTIONS AND BASES

General references for this Section are [GM03], [CG10], [GK04].

2.1. Basic notions. Let $G$ be a linear algebraic reductive group over $\mathbb{C}$. We denote by

- $\text{Rep}(G)$ the category of finite dimensional complex representations of $G$,
- $R(G) := K_0(\text{Rep}(G))$ (resp. $R(G)_C := R(G) \otimes_{\mathbb{Z}} \mathbb{C}$) the ring of finite dimensional complex representations of $G$ with integer (resp. complex) coefficients.

In particular, for a complex torus $\mathbb{T} := (\mathbb{C}^*)^n$ we have $R(\mathbb{T})_C = \mathbb{C}[Z_1^{\pm 1}, \ldots, Z_n^{\pm 1}]$. For short, we set $Z := (Z_1, \ldots, Z_n)$ and $\mathbb{C}[Z^{\pm 1}] := \mathbb{C}[Z_1^{\pm 1}, \ldots, Z_n^{\pm 1}]$.

Let $X$ be a smooth complex projective variety endowed with the action of $G$. We denote by

1. $D^b(X)$ its derived category of coherent sheaves,
2. $D^b_G(X)$ its derived category of $G$-equivariant coherent sheaves,
3. $K_0(X)$ (resp. $K_0(X)_C$) its Grothendieck group (resp. complexified),
4. $K_0^G(X)$ (resp. $K_0^G(X)_C$) its $G$-equivariant Grothendieck group (resp. complexified).

Any complex of $G$-equivariant quasi-coherent complexes admit flat and injective resolutions. From this one can deduce that on $D^b_G(X)$ all standard derived functors are well defined. In particular, we have a well defined left derived tensor product $\otimes : D^b_G(X) \times D^b_G(X) \to D^b_G(X)$, and any $f : X \to Y$, morphism of smooth projective $G$-varieties, induces left and right derived functors $Lf^* : D^b_G(Y) \to D^b_G(X)$ and $Rf_* : D^b_G(X) \to D^b_G(Y)$. It is possible to show that all the standard properties of the derived tensor product, the
derived pull-back and push-forward functors are valid in the equivariant setting. Moreover, all these equivariant derived functors are compatible with their non-equivariant versions via the forgetful functor ².

The structural morphism \( \pi: X \to \text{Spec}(\mathbb{C}) \) endows \( K_0(X) \) and \( K^G_0(X) \) with a \( \mathbb{C} \)-algebra and an \( R(G) \)-algebra structures, respectively. In addition, it induces several push-forward morphisms

\[
\pi_*: K_0(X) \to K_0(\text{Spec } \mathbb{C}) \cong \mathbb{Z}, \quad \pi^G_*: K^G_0(X) \to R(G). 
\]

and functors

\[
R\pi_*: \mathcal{D}^b(X) \to \mathcal{D}^b(\mathbb{C}), \quad R\pi^G_*: \mathcal{D}^b_G(X) \to \mathcal{D}^b(\text{Rep}(G)),
\]

which fit into the diagram

\[
\begin{array}{ccc}
\mathcal{D}^b_G(X) & \xrightarrow{R\pi^G_*} & \mathcal{D}^b(\text{Rep}(G)) \\
\delta_X \downarrow & & \downarrow \delta_{\text{pt}} \\
\mathcal{D}^b(X) & \xrightarrow{R\pi_*} & \mathcal{D}^b(\mathbb{C})
\end{array}
\]

where \( \delta_X, \delta_{\text{pt}} \) denote the forgetful functors. If \( V \in \text{Ob}(\mathcal{D}^b(X)) \) we call a \( G \)-equivariant structure on \( V \) any object \( V' \in \text{Ob}(\mathcal{D}^b_G(X)) \) such that \( \Phi(V') = V \).

2.2. Equivariant Grothendieck-Euler-Poincaré Characteristic. The push-forward morphisms

\[
\pi_*: K_0(X) \to K_0(\text{Spec } \mathbb{C}) \cong \mathbb{Z}, \quad \pi^G_*: K^G_0(X) \to R(G),
\]

are respectively given by

\[
\pi_* (V) := \sum (-1)^i \text{rk } H^i(X,V) \in \mathbb{Z}, \\
\pi^G_* (V) := \sum (-1)^i [H^i(X,V)] \in R(G),
\]

where \([H^i(X,V)]\) denotes the \( R(G) \)-class of the cohomology space \( H^i(X,V) \) seen as a representation of \( G \). These morphisms define the Grothendieck-Euler-Poincaré characteristic of (the isomorphism class of) an object \( V \), and its equivariant version respectively. They will be denoted by \( \chi, \chi^G \):

\[
\chi(V) := \pi_* (V), \quad \chi^G(V) := \pi^G_* (V).
\]

In both cases, using the duality involutions

\[
(-)^*: K_0(X) \to K_0(X), \quad E \mapsto E^*, \tag{2.1}
\]

\[
(-)^*: K^G_0(X) \to K^G_0(X), \quad E \mapsto E^*, \tag{2.2}
\]

we can define a non-symmetric pairing, called Grothendieck-Euler-Poincaré pairing (or also Mukai pairing):

\[
\chi(E,F) := \chi(E^* \otimes F), \quad \chi^G(E,F) := \chi^G(E^* \otimes F). \tag{2.3}
\]

²For the translation of the theory of derived functors from the non-equivariant setting to the equivariant one, the reader may consult [CG10], Chapter 5, [BL94] for the topological setting, [VV10], Section 1.5 and also [LH09].
These pairings naturally extend to the complexified algebras $K_0(X)_\mathbb{C}$ and $K^G_0(X)_\mathbb{C}$. In the non-equivariant case, the pairing $\chi$ is $\mathbb{C}$-bilinear, whereas in the equivariant case the pairing $\chi^G$ is $R(G)_\mathbb{C}$-sesquilinear wrt the duality involution naturally defined on $R(G)_\mathbb{C}$:

$$(-)^*: R(G)_\mathbb{C} \to R(G)_\mathbb{C}, \quad [V] \mapsto [V^*].$$

That is $\chi^G(\rho_1 E_1, \rho_2 E_2) = \rho_1^* \rho_2^* \chi^G(E_1, E_2)$ for $E_1, E_2 \in K^G_0(X)$ and $\rho_1, \rho_2 \in R(G)$.

In this paper we will also need to consider the involutive operation on $(n \times n)$-matrices

$$(-)^*: M_n(R(G)_\mathbb{C}) \to M_n(R(G)_\mathbb{C}),$$

which consists in applying (2.4) at each entry. If $A \in M_n(R(G)_\mathbb{C})$ we will also define the matrix $A^\dagger \in M_n(R(G)_\mathbb{C})$ as follows:

$$(A^\dagger)_{\alpha,\beta} := A^*_{\beta,\alpha}, \quad \alpha, \beta = 1, \ldots, n. \quad (2.5)$$

In the case of the torus $\mathbb{T}$, under the isomorphism $R_G(\mathbb{T})_\mathbb{C} \cong \mathbb{C}[\mathbb{Z}^{\pm 1}]$, the duality involution acts as follows:

$$f(Z)^* = f(Z^{-1}),$$

where for $f(Z) = f(Z_1, \ldots, Z_n) \in \mathbb{C}[\mathbb{Z}^{\pm 1}]$ we denote $f(Z^{-1}) = f(Z_1^{-1}, \ldots, Z_n^{-1})$.

2.3. Exceptional collections in $\mathcal{D}^b_G(X)$ and their mutations. Given two objects $E, F \in \text{Ob}(\mathcal{D}^b_G(X))$, we define

$$\text{Hom}^*_\mathcal{D}_G(E, F) := R\pi_*^G(E^* \otimes F) \in \text{Ob}(\mathcal{D}^b(\text{Rep}(G))),$$

where $E^* := R\mathcal{H}om(E, \mathcal{O}_X)$ is the ordinary dual sheaf of $E$.

**Definition 2.1.** An object $E \in \text{Ob}(\mathcal{D}^b_G(X))$ is called an exceptional object if and only if

$$\text{Hom}^*_\mathcal{D}_G(E, E) \cong \mathbb{C}_G,$$

where $\mathbb{C}_G$ denotes the object of $\mathcal{D}^b(\text{Rep}(G))$ given by the trivial complex one dimensional representation of $G$, concentrated in degree zero.

An ordered collection $(E_1, \ldots, E_n)$ is said to be an exceptional collection if and only if

- all objects $E_i$’s are exceptional objects,
- and $\text{Hom}^*_\mathcal{D}_G(E_j, E_i) = 0$ for $j > i$.

The definitions above are the natural equivariant versions of the standard notions of exceptional objects and collections in $\mathcal{D}^b(X)$. The following result, due to A. Elagin, gives an insight on the relationships between ordinary exceptional collections in $\mathcal{D}^b(X)$ and equivariant exceptional collections in $\mathcal{D}^b_G(X)$. Before stating Elagin’s result, let us recall that there is a naturally defined operations of tensor product between objects of $\mathcal{D}^b_G(X)$ and $\mathcal{D}^b(\text{Rep}(G))$: if $E \in \text{Ob}(\mathcal{D}^b_G(X))$ and $V^* \in \text{Ob}(\mathcal{D}^b(\text{Rep}(G)))$, the tensor product $V^* \otimes E$ is defined as the object of $\mathcal{D}^b_G(X)$ given by

$$\bigoplus_i E[-i] \otimes V^i.$$

(2.7)

This extends the obvious operation of tensor product between objects of $\text{Coh}_G(X)$ and $\text{Rep}(G)$. If $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are subcategories of $\mathcal{D}^b_G(X)$, we denote by $\langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ the smallest full triangulated subcategory of $\mathcal{D}^b_G(X)$ containing $\mathcal{A}_1, \ldots, \mathcal{A}_n$. 
Definition 2.2. Let $\mathcal{E} := (E_1, \ldots, E_n)$ be an exceptional collection in $\mathcal{D}^b_G(X)$. We say that $\mathcal{E}$ is $G$-full if
\[\mathcal{D}^b_G(X) = \left\langle E_1 \otimes \mathcal{D}^b(\text{Rep}(G)), \ldots, E_n \otimes \mathcal{D}^b(\text{Rep}(G)) \right\rangle.\] (2.8)

Remark 2.3. The exceptional collection $(E_1, \ldots, E_n)$ is thus $G$-full if and only if the collection $(E_1 \otimes \mathcal{D}^b(\text{Rep}(G)), \ldots, E_n \otimes \mathcal{D}^b(\text{Rep}(G)))$ realizes a so-called semi-orthogonal decomposition of $\mathcal{D}^b_G(X)$. See e.g. [Huy06], Chapter 1.

Remark 2.4. Our definition of $G$-fullness is different from the definition of fullness of exceptional collections in triangulated categories. In the paper [BO18], L. Borisov and D. Orlov studied bounded derived category for $T$-equivariant coherent sheaves for smooth toric varieties and Deligne-Mumford stacks. In particular, they described and explicitly constructed full exceptional collections in these categories. Notice that their exceptional collections have an infinite set of objects, while we collect an infinite set of objects in one symbol $E_i \otimes \mathcal{D}^b(\text{Rep}(G))$.

Theorem 2.5 ([Ela09], Theorem 2.6). Assume that $(E_1, \ldots, E_n)$ is a full exceptional collection of $\mathcal{D}^b(X)$, and also that each object $E_i$ admits a $G$-equivariant structure $\mathcal{E}_i$. Then, $(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ is a $G$-full exceptional collection in $\mathcal{D}^b_G(X)$.

Being thus important to know under which conditions an exceptional object of $\mathcal{D}^b(X)$ admits a $G$-equivariant structure, we recall the following result of A. Polishchuk.

Theorem 2.6 ([Pol11], Lemma 2.2). Let $X$ be a smooth projective complex variety endowed with the action of a linear algebraic connected reductive group $G$ with $\pi_1(G)$ torsion free. If $E \in \mathcal{D}^b(X)$ is an exceptional object, then $E$ admits a $G$-equivariant structure, which is unique up to tensoring by a character of $G$.

In the present paper we will focus on the case $G = T$, and the assumption of Proposition 2.6 applies.

Definition 2.7 (Mutations of objects). Let $E \in \text{Ob}(\mathcal{D}^b_G(X))$ be an exceptional object. For any $F \in \text{Ob}(\mathcal{D}^b_G(X))$ we can define two new objects
\[\mathbb{L}_E F, \quad \mathbb{R}_E F,\]
called left and right mutations of $X$ with respect to $E$. These two objects are defined through the distinguished triangles
\[\mathbb{L}_E F[-1] \rightarrow \text{Hom}_G^\bullet(F, E) \otimes E \rightarrow j_*^* F \rightarrow \mathbb{L}_E F,\] (2.9)
\[\mathbb{R}_E F \rightarrow F \rightarrow j^* \text{Hom}_G^\bullet(F, E)^* \otimes E \rightarrow \mathbb{R}_E F[1],\] (2.10)
where $j^*, j_*$ denote the natural evaluation and coevaluation morphisms.

Remark 2.8. As in the non-equivariant case, it can be shown that the objects $\mathbb{L}_E F, \mathbb{R}_E F$ are uniquely defined (up to unique isomorphism) by the distinguished triangles above. The key property is the exceptionality of $E$. We leave the details to the reader (see Section 3.3 of [CDG18]).
Lemma 2.9. Let $E \in \text{Ob}(\mathcal{D}_G^b(X))$ be an exceptional object. We have
\[ \text{Hom}^*_{\mathcal{G}}(E, L_E F) = 0, \quad \text{Hom}^*_{\mathcal{G}}(R_E F, E) = 0, \]
for all objects $F \in \text{Ob}(\mathcal{D}_G^b(X))$.

Proof. Apply the functor $\text{Hom}^*_{\mathcal{G}}(E, -)$ (resp. $\text{Hom}^*_{\mathcal{G}}(-, E)$) to the distinguished triangle (2.9) (resp. (2.10)), and use the exceptionality of $E$. □

Definition 2.10. Let $\mathcal{E} := (E_i)_{i=1}^n$ be an exceptional collection in $\mathcal{D}_G^b(X)$. For any integer $i$, with $0 < i < n$, we define two new collections
\[ \mathbb{L}_i(\mathcal{E}) := (E_1, \ldots, L_{E_i} E_{i+1}, E_i, \ldots, E_n), \]
\[ \mathbb{R}_i(\mathcal{E}) := (E_1, \ldots, E_{i+1}, R_{E_i} E_{i+1}, E_i, \ldots, E_n). \]

Proposition 2.11. For any $i$, with $0 < i < n$, the exceptional collections $\mathbb{L}_i(\mathcal{E}), \mathbb{R}_i(\mathcal{E})$ are exceptional. Moreover, the mutation operators $\mathbb{L}_i, \mathbb{R}_i$ satisfy the following identities:
\[ \mathbb{L}_i \mathbb{R}_i = \mathbb{R}_i \mathbb{L}_i = \text{Id}, \]
\[ \mathbb{R}_i \mathbb{R}_j = \mathbb{R}_j \mathbb{R}_i, \quad |i - j| > 1, \quad \mathbb{R}_{i+1} \mathbb{R}_i \mathbb{R}_{i+1} = \mathbb{R}_i \mathbb{R}_{i+1} \mathbb{R}_i. \]

Proof. The same as in the non-equivariant case. See e.g. [GK04], and Section 3.3 of [CDG18]. □

Denote by $\tau_1, \ldots, \tau_{n-1}$ the generators of the braid group $B_n$, satisfying the relations
\[ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \quad \tau_i \tau_j = \tau_j \tau_i, \quad |i - j| > 1. \]
By identities (2.11)-(2.12), we have a well-defined left action of $B_n$ on the set of exceptional collections of length $n$, by identifying the action of the generator $\tau_i$ with $\mathbb{R}_i$. For our purposes, it is convenient to slightly modify this natural action, by setting
\[ \tau_i(\mathcal{E}) := \mathbb{R}_{n-i}(\mathcal{E}), \quad i = 1, \ldots, n-1, \]
for any exceptional collection $\mathcal{E} = (E_1, \ldots, E_n)$.

Remark 2.12. Formula (2.13) is in agreement with the notations of [TV19a]. See Remark 3.3.

2.4. Dual Exceptional Collections and Helices.

Definition 2.13 (Dual exceptional collections). Let $\mathcal{E} = (E_1, \ldots, E_n)$ be an exceptional collection. We define the left and right dual exceptional collections $^\vee \mathcal{E}$ and $\mathcal{E}^\vee$ as the collections
\[ ^\vee \mathcal{E} := \beta(\mathcal{E}), \quad \beta := \tau_1 (\tau_2 \tau_1) \ldots (\tau_{n-2} \ldots \tau_1)(\tau_{n-1} \tau_{n-2} \ldots \tau_1), \]
\[ \mathcal{E}^\vee := \beta^{-1}(\mathcal{E}). \]
Proposition 2.14. Let $\mathcal{E} = (E_1, \ldots, E_n)$ be an exceptional collection, $^\vee \mathcal{E} = (^\vee E_1, \ldots, ^\vee E_n)$ and $\mathcal{E}^\vee = (E_1^\vee, \ldots, E_n^\vee)$ its left and right dual exceptional collections, respectively. The following orthogonality relations hold true:

$$\text{Hom}^\bullet_G(E_h, E_k^\vee) = \begin{cases} C_G, & h = n - k + 1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\text{Hom}^\bullet_G(^\vee E_k, E_h) = \begin{cases} C_G, & h = n - k + 1, \\ 0, & \text{otherwise}. \end{cases}$$

Moreover, for any $F \in \text{Ob}(\mathcal{D}_G^b(X))$ we have the functorial isomorphism

$$\text{Hom}^\bullet_G(^\vee E_k, F) \cong \text{Hom}^\bullet_G(F, E_k^\vee)^*.$$  

(2.16)

Proof. The argument is the same as in the non-equivariant case. See, e.g. Section 3.6 of [CDG18].

Given an exceptional collection $\mathcal{E}$, we can also introduce an infinite family of exceptional objects called the helix generated by $\mathcal{E}$.

Definition 2.15 (Helix). Let $\mathcal{E} = (E_1, \ldots, E_n)$ be an exceptional collection. We define the helix generated by $\mathcal{E}$ to be the infinite family of objects $(E_i)_{i \in \mathbb{Z}}$ defined by the iterated mutations

$$E_{i+n} := R_{E_{i+n-1}} \cdots R_{E_{i+1}} E_i, \quad E_{i-n} := L_{E_{i-n+1}} \cdots L_{E_{i-1}} E_i, \quad i \in \mathbb{Z}.$$ 

Such a helix is said to be of period $n$. Any family of $n$ consecutive objects $(E_i, \ldots, E_{i+n})$ is called a foundation of the helix.

2.5. Exceptional bases in equivariant $K$-theory. In this Section we focus on the $K$-theoretical counterpart of the notion of exceptional collections introduced in Definition 2.1 and of the action of the braid group on them.

Definition 2.16. An element $e \in K_0^G(X)$ is said to be exceptional if and only if

$$\chi^G(e, e) = C_G.$$ 

A basis $\varepsilon := (e_i)_{i=1}^n$ of $K_0^G(X)$ as a $R(G)$-module, is said to be exceptional if and only if

$$\chi^G(e_i, e_i) = C_G, \quad \chi^G(e_j, e_i) = 0, \quad \text{for } j > i. \quad (2.17)$$

The following result is a $K$-theoretical analogue of Theorem 2.5.

Theorem 2.17 ([Pol11], Lemma 2.1). Let $(E_1, \ldots, E_n)$ be a full exceptional collection in $\mathcal{D}^b(X)$. If each object $E_i$ admits a $G$-equivariant structure, then the classes $([E_i])_{i=1}^n$ form an exceptional basis of $K_0^G(X)$ as $R(G)$-module.
Proposition 2.18. Let $E \in \text{Ob}(\mathcal{D}_G^b(X))$ be an exceptional object. For any $F \in \text{Ob}(\mathcal{D}_G^b(X))$ we have
\[
[L_E F] = [F] - \chi^G(E, F) \cdot [E], \quad [R_E F] = [F] - \chi^G(F, E)^* \cdot [E].
\] (2.18)

Proof. From the distinguished triangle (2.9), and recalling equation (2.7), we deduce
\[
[L_E F] = [F] - \left[ \bigoplus_i E[-i] \otimes H^i(X, E^* \otimes F) \right] = [F] - \left( \sum_i (-1)^i H^i(X, E^* \otimes F) \right) [E].
\]
Analogously, from the distinguished triangle (2.10), we deduce
\[
[R_E F] = [F] - \left[ \bigoplus_i E[-i] \otimes H^i(X, F^* \otimes E) \right]^* = [F] - \left( \sum_i (-1)^i H^i(X, F^* \otimes E) \right)^* [E].
\]
This completes the proof. □

We can thus introduce the following definition.

Definition 2.19. Let $e \in K^G_0(X)$ be an exceptional element. Given $f \in K^G_0(X)$, we define its left and right mutations wrt $e$ as the elements
\[
L_e f := f - \chi^G(e, f) \cdot e, \quad R_e f := f - \chi^G(f, e)^* \cdot e.
\] (2.19)

Lemma 2.20. Let $e \in K^G_0(X)$ be an exceptional element. We have
\[
\chi^G(e, L_e f) = 0, \quad \chi^G(R_e f, e) = 0,
\]
for any $f \in K^G_0(X)$.

Proof. The proof is straightforward. □

Definition 2.21. Let assume that $K^G_0(X)$ is a free $R(G)$-module of finite rank. Assume that $K^G_0(X)$ admits an exceptional basis $\varepsilon := (e_i)_{i=1}^n$. For any $0 < i < n$ we define new exceptional bases
\[
L_i \varepsilon := (e_1, \ldots, e_{i-1}, L_{e_i} e_{i+1}, e_i, e_{i+2}, \ldots, e_n), \quad R_i \varepsilon := (e_1, \ldots, e_{i-1}, R_{e_{i+1}} e_i, e_{i+2}, \ldots, e_n).
\] (2.20) (2.21)
This defines an action of the braid group $B_n$ on the set of exceptional bases: the action of the generators $\tau_i$'s, with $i = 1, \ldots, n-1$, is identified with the action of the mutations $R_{n-i}$'s, respectively.
2.6. Dual exceptional bases. Let $\varepsilon := (\varepsilon_i)_{i=1}^n$ be an exceptional basis of $K^G_0(X)$. We define the left and right dual exceptional bases $\varepsilon^\vee$ and $\varepsilon^\vee$, through the mutations

$$\varepsilon^\vee := \beta(\varepsilon), \quad \beta := \tau_1(\tau_2\tau_1)\ldots(\tau_{n-2}\ldots\tau_1)(\tau_{n-1}\tau_{n-2}\ldots\tau_1),$$ (2.22)

$$\varepsilon^\vee := \beta^{-1}\varepsilon.$$ (2.23)

Proposition 2.22. Let $\varepsilon = (\varepsilon_i)_{i=1}^n$ be an exceptional basis of $K^G_0(X)$, $\varepsilon^\vee = (\varepsilon^\vee_i)_{i=1}^n$ and $\varepsilon^\vee = (\varepsilon^\vee_i)_{i=1}^n$ its left and right dual exceptional basis. The following orthogonality relations hold true

$$\chi^G(e_h, e^\vee_k) = \delta_{h+k,n+1}, \quad \chi^G(\varepsilon^\vee_k, e_h) = \delta_{h+k,n+1},$$ (2.24)

for $k = 1, \ldots, n$. In particular, for any $v \in K^G_0(X)$ the following identities hold true

$$v = \sum_{h=1}^n \chi^G(v, e^\vee_h) e_{n+1-h}, \quad v = \sum_{h=1}^n \chi^G(\varepsilon^\vee_h, v) e_{n+1-h}.$$ (2.25)

Proof. Let us prove the first identity of (2.24). From Lemma 2.20, we immediately have

$$\chi^G(e_h, e^\vee_k) = 0, \quad \text{for } h = 1, \ldots, n - k.$$

If $e, f \in K^G_0(X)$ are such that $e$ is exceptional and $\chi^G(f, e) = 0$, then for any $v \in K^G_0(X)$ we have $\chi^G(f, v) = \chi^G(f, L_e v)$. By iteration of this identity, we deduce

$$\chi^G(e_h, e^\vee_k) = 0, \quad \text{for } h = n - k + 2, \ldots, n,$$

and

$$\chi^G(e_{n-k+1}, e^\vee_k) = \chi^G(e_{n-k+1}, e_{n-k+1}) = 1.$$

The second identity of (2.24) is proved in an analogous way. The identities (2.25) follow from the sesquilinearity of $\chi^G$. \qed

Corollary 2.23. Let $\varepsilon = (\varepsilon_i)_{i=1}^n$ be an exceptional basis of $K^G_0(X)$, and let $G$ be the Gram matrix of $\chi^G$ wrt $\varepsilon$. The Gram matrix of $\chi^G$ wrt both $\varepsilon^\vee$ and $\varepsilon^\vee$ is equal to

$$J \cdot (G^\dagger)^{-1} \cdot J,$$

where $J_{\alpha,\beta} = \delta_{\alpha+\beta,n+1}$.

Proof. From formula (2.24) it follows that the matrix $X = (X^\dagger_k)_{j,k=1}^n$ defined by $e^\vee_k = \sum_{j=1}^n X^\dagger_k e_j$, satisfies $GX = J$. Hence the Gram matrix wrt $\varepsilon^\vee$ is given by

$$X^\dagger \cdot G \cdot X = J \cdot (G^\dagger)^{-1} \cdot J.$$

The case of $\varepsilon^\vee$ is analogous. \qed
2.7. **Serre functor and canonical operator.** A Serre functor $\mathcal{K}: \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ is a functor defined (uniquely up to canonical isomorphism) by the condition

$$\text{Hom}^\bullet_\mathcal{K}(E, F)^* \cong \text{Hom}^\bullet_\mathcal{K}(F, K(E)), \quad E, F \in \text{Ob}(\mathcal{D}^b(X)).$$

(2.26)

We can take

$$\mathcal{K} = (\omega_X \otimes -)[\dim \mathbb{C} X]: \mathcal{D}^b(X) \to \mathcal{D}^b(X),$$

(2.27)

where $\omega_X$ denotes the canonical sheaf of $X$. Analogously, in the equivariant case a Serre functor $\mathcal{K}: \mathcal{D}^b_G(X) \to \mathcal{D}^b_G(X)$ is defined by the condition

$$\text{Hom}^\bullet_\mathcal{K}(E, F)^* \cong \text{Hom}^\bullet_\mathcal{K}(F, K(E)), \quad E, F \in \text{Ob}(\mathcal{D}^b_G(X)).$$

(2.28)

We can take

$$\mathcal{K} = (\omega^G_X \otimes -)[\dim \mathbb{C} X]: \mathcal{D}^b_G(X) \to \mathcal{D}^b_G(X),$$

(2.29)

where $\omega^G_X$ is the $G$-equivariant canonical sheaf of $X$. By abuse of language, we will call (2.27) (and its equivariant version (2.29)) the **Serre functor** in $\mathcal{D}^b(X)$ (and $\mathcal{D}^b_G(X)$, respectively).

From Serre duality (2.28), it also follows the **Serre periodicity** property

$$\text{Hom}^\bullet_\mathcal{K}(E, F) \cong \text{Hom}^\bullet_\mathcal{K}(K(E), K(F)), \quad E, F \in \text{Ob}(\mathcal{D}^b_G(X)).$$

(2.30)

**Proposition 2.24.** Let $\mathcal{E} = (E_i)_{i=1}^n$ be an exceptional collection of length $n$ of $\mathcal{D}^b_G(X)$. The following operations are equivalent, i.e. produce the same exceptional collection when applied to $\mathcal{E}$:

1. to act on $\mathcal{E}$ with the braid $(\tau_1 \ldots \tau_{n-1})^{-n}$,
2. to take the double right-dual exceptional collection $(\mathcal{E}^\vee)\vee$,
3. to apply the Serre functor to each object of $\mathcal{E}$.

**Proof.** The equivalence of points (1) and (2) follows from the well-known identity of braids in $B_n$

$$(\tau_1 \ldots \tau_{n-1})^n = \beta^2,$$

(2.31)

where $\beta := \tau_1(\tau_2\tau_1) \ldots (\tau_{n-2} \ldots \tau_1)(\tau_{n-1}\tau_{n-2} \ldots \tau_1)$ is the braid that appears in (2.14) and (2.22). See e.g. [KT08], Theorem 1.24. The equivalence of (2) and (3) follows from the functorial isomorphism (2.16). \hfill $\square$

**Remark 2.25.** Notice that the element (2.31) of $B_n$ is the generator of its center $Z(B_n)$. See [KT08], Theorem 1.24.

The $K$-theoretical version of the Serre functor is the so-called **canonical operator** $k: \mathcal{K}^G_0(X) \to \mathcal{K}^G_0(X)$, defined through the identity

$$\chi^G(e, f)^* = \chi^G(f, k(e)), \quad e, f \in \mathcal{K}^G_0(X).$$

(2.32)

The $K$-theoretical analogous to (2.30), i.e.

$$\chi^G(e, f) = \chi^G(k(e), k(f)), \quad e, f \in \mathcal{K}^G_0(X),$$

(2.33)

does that the canonical operator $k$ is a $\chi^G$-isometry.
Proposition 2.26. Let \( \varepsilon = (e_i)_{i=1}^n \) be a basis of \( K^G_0(X) \), and let \( G \) be the Gram matrix of \( \chi^G \) wrt \( \varepsilon \). The matrix of the canonical operator \( k: K^G_0(X) \to K^G_0(X) \) wrt the basis \( \varepsilon \) is equal to
\[
G^{-1}G^\dagger. \tag{2.34}
\]

Proof. It follows from identity (2.32), written in matrix notation. \( \square \)

3. Equivariant Derived Category, Exceptional Collections and \( K \)-theory of \( \mathbb{P}^{n-1} \)

3.1. Symmetric functions. Consider the algebra \( \mathbb{C}[Z^{\pm 1}] = \mathbb{C}[Z_1^{\pm 1}, \ldots, Z_n^{\pm 1}] \) of Laurent polynomials in \( n \) indeterminates. The elementary and complete symmetric functions are defined as the elements
\[
s_k(Z) := \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^k Z_{i_j}, \quad k = 1, \ldots, n, \tag{3.1}
\]
\[
m_k(Z) := \sum_{i_1 \geq 0, \ldots, i_n \geq 0, i_1 + \cdots + i_n = k} Z_1^{i_1} \cdots Z_n^{i_n}, \quad k \in \mathbb{Z}_{>0}. \tag{3.2}
\]
Put also \( s_0 = 1, m_0 = 1 \). Then we have
\[
\sum_{i=0}^k (-1)^i m_i(Z) s_{k-i}(Z) = 0, \quad k \in \mathbb{Z}_{>0}. \tag{3.3}
\]

3.2. Torus action. Let \( n \geq 2 \). We consider the diagonal action of \( \mathbb{T} = (\mathbb{C}^*)^n \) on the space \( \mathbb{C}^n \). Such an action induces an action of \( \mathbb{C} \) on \( \mathbb{P}^{n-1} \), the projective space parametrizing the one dimensional subspaces \( F \subset \mathbb{C}^n \). If \( (u_1, \ldots, u_n) \) denote the standard basis of \( \mathbb{C}^n \), denote by \( pt_I \in \mathbb{P}^{n-1} \), with \( I = 1, \ldots, n \), the point corresponding to the coordinate line spanned by \( u_I \). The points \( pt_I, I = 1, \ldots, n \) are the fixed points of the \( \mathbb{T} \)-action.

3.3. Derived category. The action of \( \mathbb{T} \) on \( \mathbb{C}^n \) induces naturally a \( \mathbb{T} \)-structure on the structural sheaf \( O_{\mathbb{P}^{n-1}} \) and the tautological line bundle \( O(-1) \) on \( \mathbb{P}^{n-1} \). Any vector bundle obtained from \( O_{\mathbb{P}^{n-1}} \) and \( O(-1) \) through tensorial operations inherits a “natural” \( \mathbb{T} \)-structure.

The derived category \( D^b(\mathbb{P}^{n-1}) \) admits a well-known full exceptional collection, the Beilinson exceptional collection
\[
\mathcal{B} := (O, O(1), \ldots, O(n-1)).
\]

Such an exceptional collection, with its natural \( \mathbb{T} \)-structure, is an exceptional collection in \( D^b_\mathbb{T}(\mathbb{P}^{n-1}) \). Its \( K \)-theoretical counterpart \( \left([O(i-1)]\right)_{i=1}^n \) defines an exceptional basis of \( K^\mathbb{T}_0(\mathbb{P}^{n-1}) \) (in accordance with Theorem 2.5, Theorem 2.6 and Theorem 2.17).
3.4. **Equivariant K-theory.** The equivariant $K$-theory algebra $K^T_0(\mathbb{P}^{n-1})_C$ admits the following presentation

$$K^T_0(\mathbb{P}^{n-1})_C \cong \mathbb{C}[X^\pm 1, Z^\pm 1]/(\prod_{j=1}^{n}(X - Z_j)), \quad (3.4)$$

where the variable $X$ corresponds to the tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{P}^{n-1}$, and the variables $Z_1, \ldots, Z_n$ are the equivariant parameters corresponding to the factors of the torus $T = (\mathbb{C}^*)^n$.

Under the presentation (3.4), the duality involution (2.2) is given by

$$f(X, Z)^* := f(X^{-1}, Z^{-1}), \quad f \in K^T_0(\mathbb{P}^{n-1})_C.$$  

By Atiyah-Bott equivariant localization theorem [AB84], the equivariant Grothendieck-Euler-Poincaré pairing $\chi^T$ is given by the formula

$$\chi^T(f, g) = \sum_{a=1}^{n} f(Z_a^{-1}, Z^{-1})g(Z_a, Z) - \sum_{a=1}^{n} \text{Res}_{X=Z_a} f(X^{-1}, Z^{-1})g(X, Z)/X\prod_{j=1}^{n}(1 - X/Z_j). \quad (3.5)$$

**Remark 3.1.** By putting $Z_i = 1$, for $i = 1, \ldots, n$, in (3.4) and (3.5), we obtain the presentation of the non-equivariant $K$-theory of $\mathbb{P}^{n-1}$ and its non-equivariant Grothendieck-Euler-Poincaré pairing.

The class of the $T$-equivariant canonical sheaf $[\omega^T_{\mathbb{P}^{n-1}}]$ is obtained by twisting the class $X^n = [\mathcal{O}(-n)]$ with a character of $T$:

$$[\omega^T_{\mathbb{P}^{n-1}}] = \frac{X^n}{\prod_{j=1}^{n} Z_j} \in K^T_0(\mathbb{P}^{n-1}). \quad (3.6)$$

**Lemma 3.2.** For $i, j \in \mathbb{Z}$ we have

$$\chi^T([\mathcal{O}(i)], [\mathcal{O}(j)]) = \begin{cases} m_{j-i}(Z^{-1}), & i \leq j, \\ 0, & j < i < j + n, \\ (-1)^{n-1}m_{i-j-n}(Z)\prod_{j=1}^{n} Z_j, & i \geq j + n. \end{cases} \quad \square$$

**Remark 3.3.** In the paper [TV19a], instead of the non-degenerate pairing $\chi^T$ on $K^T_0(\mathbb{P}^{n-1})_C$, it is studied another non-symmetric pairing $A$ defined by the formula

$$A(f, g) := \pi_* \left( f^* \cdot g \cdot (-1)^{n-1} \frac{X^n}{\prod_{j=1}^{n} Z_j} \right), \quad (3.7)$$

where $f, g \in K^T_0(\mathbb{P}^{n-1})_C$ and $\pi: \mathbb{P}^{n-1} \to \text{Spec}(\mathbb{C})$. In the same article, a notion of exceptional bases of $K^T_0(\mathbb{P}^{n-1})_C$ wrt the pairing $A$, analogous to Definition 2.16, is given...
From (2.29), (2.32) and (3.6) we deduce the following relationships between $A$ and $\chi^T$:

$$A(f, g) = \chi^T(g, f)^*.$$  \tag{3.8}

This implies, in particular, that $A$-exceptional bases of $K_0^T(\mathbb{P}^{n-1})$ are exactly $\chi^T$-exceptional bases, although ordered in the opposite order. Moreover, if we denote by $L^A, R^A$ (resp. $L^{\chi^T}, R^{\chi^T}$) the morphisms of left/right mutations wrt $A$ (resp. $\chi^T$), then

$$L^A = R^{\chi^T}, \quad R^A = L^{\chi^T}. \tag{3.9}$$

### 3.5. Diophantine constraints on Gram matrices

In this Section, we show that the Gram matrices of the pairing $\chi^T$ wrt exceptional bases of $K_0^T(\mathbb{P}^{n-1})$ must satisfy several Diophantine constraints.

Given $G \in GL(n, \mathbb{Z}[\mathbb{Z}^{\pm 1}])$, let us denote

$$p_G(\lambda) := \det \left( \lambda \cdot 1 - G^{-1}G^\dagger \right) \in \mathbb{Z}[\mathbb{Z}^{\pm 1}, \lambda]. \tag{3.10}$$

**Lemma 3.4.** We have

$$p_{G^{-1}}(\lambda) = p_G(\lambda)^*, \tag{3.11}$$

where for any $f(\mathbb{Z}, \lambda) \in \mathbb{Z}[\mathbb{Z}^{\pm 1}, \lambda]$ we define $f(\mathbb{Z}, \lambda)^* := f(\mathbb{Z}^{-1}, \lambda)$.

**Proof.** Notice that

$$p_{G^{-1}}(\lambda) = \det \left( (\lambda \cdot 1 - G^{-1}G^\dagger)^\dagger \right) = p_G(\lambda)^*. \quad \square$$

**Theorem 3.5.** Let $\varepsilon := (e_i)_{i=1}^n$ be a basis of $K_0^T(\mathbb{P}^{n-1})$, and let $G$ be the Gram matrix of $\chi^T$ wrt $\varepsilon$. The following identity holds true:

$$p_G(\lambda) = \sum_{j=0}^n (-1)^j \lambda^{n-j} s_j \left( (-1)^{n-1} \frac{Z_i^n}{s_n(\mathbb{Z})}, \ldots, (-1)^{n-1} \frac{Z_n^n}{s_n(\mathbb{Z})} \right). \tag{3.12}$$

**Proof.** From the presentation (3.4) and equation (3.6), it is readily seen that the eigenvalues of the canonical operator $k$ are

$$(-1)^{n-1} \frac{Z_i^n}{s_n(\mathbb{Z})}, \ldots, (-1)^{n-1} \frac{Z_n^n}{s_n(\mathbb{Z})}.$$ 

Then, identity (3.12) follows from Proposition 2.26. \hfill \square

If we expand (3.10) in powers of $\lambda$, i.e.

$$p_G(\lambda) = \sum_{j=0}^n (-1)^j \lambda^{n-j} p_j(G), \tag{3.13}$$

for suitable polynomial functions $p_j(G)$ of the entries of $G$ and $G^\dagger$, from the identity (3.12) we deduce the validity of $n$ constraints:

$$p_j(G) = s_j \left( (-1)^{n+1} \frac{Z_i^n}{s_n(\mathbb{Z})}, \ldots, (-1)^{n+1} \frac{Z_n^n}{s_n(\mathbb{Z})} \right), \quad j = 1, \ldots, n. \tag{3.14}$$
If $G$ is a Gram matrix of $\chi^T$, then $\det G = 1$, and we have

$$p_g \left( \frac{1}{\lambda} \right) = \frac{(-1)^n}{\lambda^n} p_g(\lambda)^*, \quad (3.15)$$

so that

$$p_{n-j}(G) = p_j(G)^*, \quad j = 0, \ldots, n. \quad (3.16)$$

Thus, we are left with $\left[ \frac{n}{2} \right]$ constraints for the entries of $G$. Let us write the constraints for $n = 2, 3, 4$.

**Proposition 3.6.** Let $\varepsilon := (e_1, e_2)$ be an exceptional basis of $K^0_0(\mathbb{P}^1)$, and let

$$G = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}, \quad g \in \mathbb{Z}[Z_1^{\pm 1}, Z_2^{\pm 1}],$$

be the Gram matrix of $\chi^T$ wrt $\varepsilon$. Then, the Laurent polynomial $g$ is a solution of the equation

$$gg^* = \frac{(Z_1 + Z_2)^2}{Z_1 Z_2}. \quad (3.17)$$

All the solutions of (3.17) are of the form

$$g(Z_1, Z_2) = Z_1^\alpha Z_2^\beta (Z_1 + Z_2) \in \mathbb{Z}[Z_1^{\pm 1}, Z_2^{\pm 1}], \quad (3.18)$$

where $\alpha, \beta \in \mathbb{Z}$.

**Proof.** We have $p_g(\lambda) = \lambda^2 + (gg^* - 2) \lambda + 1$, and the only non-trivial constraint (3.14) is

$$gg^* - 2 = \frac{Z_1^2 + Z_2^2}{Z_1 Z_2}, \quad (3.19)$$

which coincides with (3.17). Notice that $g$ is a solution of (3.17) if and only if $\gamma := g \cdot s_1(Z_1, Z_2)^{-1}$ is a solution of $\gamma \gamma^* = 1$, whose solutions are $\gamma(Z_1, Z_2) = Z_1^\alpha Z_2^\beta$, with $\alpha, \beta \in \mathbb{Z}$. $\square$

**Remark 3.7.** From Lemma 3.2, it follows that the matrix $G$ corresponding to the solution (3.18) coincide with the Gram matrix wrt the exceptional basis $([p \otimes O], \quad [q \otimes O(1)])$, where $p, q \in R(T) \cong \mathbb{Z}[Z_1^{\pm 1}, Z_2^{\pm 1}]$ are two characters of $T$ such that $p^* q = Z_1^{a+1} Z_2^{b+1}$.

**Proposition 3.8.** Let $\varepsilon = (e_1, e_2, e_3)$ be an exceptional basis of $K^7_0(\mathbb{P}^2)$, and let

$$G = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b, c \in \mathbb{Z}[Z_1^{\pm 1}, Z_2^{\pm 1}, Z_3^{\pm 1}],$$

be the Gram matrix of $\chi^T$ wrt $\varepsilon$. Then, the triple $(a, b, c)$ is a solution of the Markov-type equations

\[
\begin{align*}
aa^* + bb^* + cc^* - ab^* c &= 3 - \frac{Z_1^2 + Z_2^2 + Z_3^2}{Z_1 Z_2 Z_3}, \\
Z_1^2 Z_3^2 + Z_1^3 Z_3^3 + Z_2^3 Z_3^3 &\quad Z_1^2 Z_2^2 Z_3^2.
\end{align*}
\tag{3.20}
\]

Notice that $(a, b, c) = (s_1(\mathbf{Z}), s_2(\mathbf{Z}), s_1(\mathbf{Z}))$ gives a solutions of (3.20) and (3.21). Properties of the Markov-type equations (3.20)-(3.21) and its solutions see in [CV].

**Remark 3.9.** In the non-equivariant case, Gram matrices wrt exceptional bases are upper triangular matrices with entries $(a, b, c)$ satisfying the equation $a^2 + b^2 + c^2 - abc = 0$. See [GK04], [Bon04]. This equation is a variant of the famous Markov equation. See [Aig13]. By putting $Z_1 = Z_2 = Z_3 = 1$ in both (3.20) and (3.21), we re-obtain this variant of Markov equation. Moreover, solutions of (3.20) and (3.21) reduce to integer Markov triples, up to a factor 3. For example, the solution $(a, b, c) = (s_1(\mathbf{Z}), s_2(\mathbf{Z}), s_1(\mathbf{Z}))$ reduces to (3, 3, 3).

**Proposition 3.10.** Let $\varepsilon = (e_1, e_2, e_3, e_3)$ be an exceptional basis of $K_0^\infty(\mathbb{P}^3)$, and let

\[
\mathcal{G} = \begin{pmatrix}
1 & a & b & c \\
0 & 1 & d & e \\
0 & 0 & 1 & f
\end{pmatrix}, \quad a, b, c, d, e, f \in \mathbb{Z}[Z_1^{\pm 1}, Z_2^{\pm 1}, Z_3^{\pm 1}, Z_4^{\pm 1}],
\]

be the Gram matrix of $\chi^T$ wrt $\varepsilon$. Then, $(a, b, c, d, e, f)$ is a solution of the equations

\[
\begin{align*}
aa^* + bb^* + cc^* + dd^* + ee^* + ff^* \\
- ab^* d - ac^* e + a^* ce^* - b^* cf^* - d^* e f^* + a^* cf^* + a^* cd^* f^* \\
&= 4 + \frac{Z_2 Z_4 Z_3}{Z_1^3} + \frac{Z_1 Z_4 Z_3}{Z_2^3} + \frac{Z_1 Z_2 Z_4}{Z_3^3} + \frac{Z_1 Z_2 Z_3}{Z_4^3},
\end{align*}
\tag{3.22}
\]

\[
\begin{align*}
-2aa^* - 2bb^* - 2cc^* - 2dd^* - 2ee^* - 2ff^* \\
+ ab^* d + ac^* e + a^* cd^* - b^* cf^* + bc^* f + de^* f + d^* e f^* \\
- ab^* e f^* - a^* be^* f - bc^* d^* e + b^* cde^* + a^* f f^* + bb^* ee^* + cc^* dd^* \\
&= -6 + \frac{Z_2^2 Z_4^2}{Z_1^3 Z_3^3} + \frac{Z_2^3 Z_4}{Z_1^3 Z_3^2} + \frac{Z_2^3 Z_3}{Z_1^3 Z_4^2} + \frac{Z_2^3 Z_2^2}{Z_1^3 Z_4^2} + \frac{Z_2^3 Z_4}{Z_1^3 Z_3^2} + \frac{Z_2^3 Z_3}{Z_1^3 Z_4^2} + \frac{Z_2^3 Z_4}{Z_1^3 Z_3^2} + \frac{Z_2^3 Z_3}{Z_1^3 Z_4^2},
\end{align*}
\tag{3.23}
\]
\[\begin{align*}
&aa^* + bb^* + cc^* + dd^* + ee^* + ff^* \\
&- ab^*d - ac^*e - bc^*f - de^*f + ac^*df \\
&= 4 + \frac{Z_1^3}{Z_2Z_3Z_4} + \frac{Z_2^3}{Z_1Z_3Z_4} + \frac{Z_3^3}{Z_1Z_2Z_4} + \frac{Z_4^3}{Z_1Z_2Z_3}.
\end{align*}\] (3.24)

Remark 3.11. In the non-equivariant case, the Diophantine constraints on Gram matrices wrt exceptional collection reduce to
\[a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - abd - ace - bcf + def + acdf = 8,\] (3.25)
\[(af - be + cd)^2 = 16.\] (3.26)

See e.g. [Bon04]. These constraints are re-obtained by putting \(Z_1 = Z_2 = Z_3 = Z_4 = 1\) in (3.22)-(3.23)-(3.24).

4. EQUIVARIANT COHOMOLOGY OF \(\mathbb{P}^{n-1}\)

4.1. Equivariant Cohomology. We consider the \(T\)-equivariant cohomology algebra \(H^*_T(\mathbb{P}^{n-1}, \mathbb{C})\). Let us denote
- by \(x\) the first equivariant Chern class of the tautological line bundle \(\mathcal{O}(-1)\) on \(\mathbb{P}^{n-1}\) with its standard \(T\)-structure,
- by \(y = (y_1, \ldots, y_{n-1})\) the equivariant Chern roots of the quotient bundle \(\mathcal{Q}\) (if \(F \subset \mathbb{C}^n\) is the line represented by \(p \in \mathbb{P}^{n-1}\), then the fiber \(\mathcal{Q}_p\) is the quotient \(\mathbb{C}^n / F\)),
- by \(z = (z_1, \ldots, z_n)\) the equivariant parameters corresponding to the factors of the torus \(T\),
- by \(\Omega\) the complement in \(\mathbb{C}^n\) of the union of the hyperplanes \(z_i - z_j = m, \ i, j = 1, \ldots, n, \ i \neq j, \ m \in \mathbb{Z}\).

It is well known that
\[H^*_T(\mathbb{P}^{n-1}, \mathbb{C}) \cong \mathbb{C}[x, z] / \left\langle \prod_{i=1}^{n} (x - z_i) \right\rangle,\] (4.1)
\[\cong \mathbb{C}[x, y, z]^{S_{n-1}} / \left\langle (u - x) \prod_{j=1}^{n-1} (u - y_j) - \prod_{a=1}^{n} (u - z_a) \right\rangle,\] (4.2)

where \(\mathbb{C}[x, y, z]^{S_{n-1}}\) is the algebra of polynomials in \(x, y, z\) symmetric in the variables \(y_1, \ldots, y_{n-1}\). The equivariant cohomology \(H^*_T(\mathbb{P}^{n-1}, \mathbb{C})\) is endowed with a module structure over the ring \(H^*_T(pt, \mathbb{C}) \cong \mathbb{C}[z]\). By setting all the equivariant parameter \(z_i\)’s equal to zero in (4.1), we obtain the presentation of the classical cohomology algebra
\[H^*(\mathbb{P}^{n-1}, \mathbb{C}) \cong \mathbb{C}[x] / \langle x^n \rangle.\]

We denote by
\[t^*: H^*_T(\mathbb{P}^{n-1}, \mathbb{C}) \to H^*(\mathbb{P}^{n-1}, \mathbb{C}), \ f(x, z) \mapsto f(x, 0),\] (4.3)
the non-equivariant limit morphism\(^3\).

Following the notations of [TV19a], we will use three different bases of \(H^*_T(\mathbb{P}^{n-1}, \mathbb{C})\):

1. the standard basis \((1, x, \ldots, x_{n-1})\), where \(x_\alpha := x^\alpha\);
2. the basis \((g_1, \ldots, g_n)\) defined by
   \[
g_i := \prod_{a=i+1}^{n} (x - z_a), \quad i = 1, \ldots, n - 1, \quad \text{and} \quad g_n := 1; \tag{4.4}\]
3. the idempotent basis \((\Delta_1, \ldots, \Delta_n)\) defined by the Lagrange interpolating polynomials
   \[
   \Delta_i := \prod_{j \neq i} \frac{x - z_j}{z_i - z_j}, \quad i = 1, \ldots, n, \tag{4.5}\]

From the presentation (4.1), it is easily seen that the polynomials \(\Delta_i\)'s are indeed the idempotents for the algebra, namely we have
\[
\Delta_i \cdot \Delta_j = \delta_{ij} \Delta_i. \tag{4.6}\]

4.2. Poincaré pairing and \(D\)-matrix. Let us denote by
\[
\eta: H^*_T(\mathbb{P}^{n-1}, \mathbb{C}) \times H^*_T(\mathbb{P}^{n-1}, \mathbb{C}) \to H^*_T(\text{pt}, \mathbb{C}) \cong \mathbb{C}[z] \tag{4.7}\]
the equivariant Poincaré metric given by equivariant integration
\[
\eta(v, w) := \int_{\mathbb{P}^{n-1}} v \cdot w = \sum_{a=1}^{n} \frac{v(z_a, z)w(z_a, z)}{\prod_{j \neq a}(z_a - z_j)}, \quad v, w \in H^*_T(\mathbb{P}^{n-1}, \mathbb{C}). \tag{4.8}\]

The equivariant cohomology \(H^*_T(\mathbb{P}^{n-1}, \mathbb{C})\), endowed with the equivariant Poincaré metric \(\eta\), is a Frobenius algebra over the ring \(H^*_T(\text{pt}, \mathbb{C}) \cong \mathbb{C}[z]\) (see Appendix A). From the compatibility of the product with the Poincaré metric, it is easily seen that the vectors are pairwise orthogonal:
\[
\eta(\Delta_i, \Delta_j) = \eta(\Delta_i \cdot \Delta_j, 1) = \int_{\mathbb{P}^{n-1}} \Delta_i \delta_{ij} = \delta_{ij} \chi_i, \tag{4.9}\]
where
\[
\chi_i := \frac{1}{\prod_{j \neq i}(z_i - z_j)}. \tag{4.10}\]

We define the matrix \(D = (D_{j\alpha})\) as the matrix describing the base change
\[
x_\alpha = \sum_{j=1}^{n} D_{j\alpha} \Delta_j, \quad \alpha = 0, \ldots, n - 1. \]

\(^3\)Recall that this morphism is induced in cohomology by the inclusion \(\iota: \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}_T := \mathbb{P}^{n-1} \times_T ET.\) See [AB84], Section 2.
Lemma 4.1. We have

\[ D_{j\alpha} = z_j^\alpha, \quad \alpha = 0, \ldots, n-1, \quad j = 1, \ldots, n. \]

Thus \( D \) is the Vandermonde matrix

\[
D = \begin{pmatrix}
1 & z_1 & z_1^2 & \cdots & z_1^{n-1} \\
1 & z_2 & z_2^2 & \cdots & z_2^{n-1} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & z_n & z_n^2 & \cdots & z_n^{n-1}
\end{pmatrix}.
\]

Its inverse \( D^{-1} \) is given by

\[(D^{-1})_{\alpha j} = \begin{cases} 
(-1)^\alpha \frac{s_j^\alpha(z)}{\prod_{m \neq j} (z_j - z_m)}, & 0 \leq \alpha < n-1, \\
1 \frac{1}{\prod_{m \neq j} (z_j - z_m)}, & \alpha = n-1,
\end{cases}\]

where

\[ s_j^\alpha(z) := \sum_{1 \leq m_1 < \cdots < m_{n-k} \leq n} z_{m_1} \cdots z_{m_{n-k}}. \]

Proof. From the identity \( x = z_1 \Delta_1 + \cdots + z_n \Delta_n \), using the equation (4.6), we obtain the identity \( x_\alpha = z_1^\alpha \Delta_1 + \cdots + z_n^\alpha \Delta_n \).

\[ \square \]

Lemma 4.2. Let \( \eta = (\eta_{\alpha \beta})_{\alpha, \beta} \), with

\[ \eta_{\alpha \beta} := \eta(x_\alpha, x_\beta), \]

be the Gram matrix of the equivariant Poincaré form. The following identities hold true:

\[ \eta_{\alpha \beta} = \begin{cases} 
0 & \text{if } \alpha + \beta < n-1, \\
1 & \text{if } \alpha + \beta = n-1, \\
m_{\alpha + \beta - n + 1}(z) & \text{if } \alpha + \beta > n-1.
\end{cases}\]

Proof. It readily follows from the identity \( D^T \cdot \text{diag}(\chi_1, \ldots, \chi_n) \cdot D = \eta. \)

\[ \square \]

4.3. Extension of scalars. Denote

- by \( \mathcal{M}_\mathbb{C}^n \) the ring of meromorphic functions in the parameters \( z \),
- by \( \mathcal{O}_\Omega \) the ring of holomorphic functions on the domain \( \Omega \).

Both rings are endowed with a natural module structure over the ring \( H^*_\mathbb{T}(\mathbb{P}^n, \mathbb{C}) \cong \mathbb{C}[z] \).

Set

\[ H^\Omega_\mathbb{T}(\mathbb{P}^n-1) := H^*_\mathbb{T}(\mathbb{P}^n-1, \mathbb{C}) \otimes H^*_\mathbb{T}(\text{pt}, \mathbb{C}) \mathcal{O}_\Omega, \quad (4.11) \]
\[ H_T^{\text{mer}}(\mathbb{P}^{n-1}) := H^*_T(\mathbb{P}^{n-1}, \mathbb{C}) \otimes H^*_T(\text{pt}, \mathbb{C}) \mathcal{M}_C^n. \] (4.12)

By localization, a class \( \alpha \) in \( H_T^\Omega(\mathbb{P}^{n-1}) \) (resp. \( H_T^{\text{mer}}(\mathbb{P}^{n-1}) \)) is uniquely determined by the restrictions \( \alpha|_{pt_I} \) at fixed points: these are elements of \( O_\Omega \) (resp. \( \mathcal{M}_C^n \)).

4.4. **Equivariant Characteristic classes.** Consider a \( \mathbb{T} \)-equivariant vector bundle \( V \) of rank \( r \) on \( \mathbb{P}^{n-1} \), with equivariant Chern roots \( \xi_1, \ldots, \xi_r \).

**Definition 4.3.** We define the **graded equivariant Chern character** of \( V \) as the characteristic class
\[ \text{Ch}_T(V) := \sum_{j=1}^r \exp(2\pi \sqrt{-1} \xi_j) \in H_T^{\text{mer}}(\mathbb{P}^{n-1}). \]

**Example 4.4.** If \( V = \mathcal{O}(k) \), \( k \in \mathbb{Z} \), the graded Chern character \( \text{Ch}_T(V) \) is the class
\[ \text{Ch}_T(\mathcal{O}(k)) = \exp(-2\pi \sqrt{-1} k x). \]
This is the class of \( H_T^{\text{mer}}(\mathbb{P}^{n-1}) \) whose restriction at the fixed point \( pt_I \) is \( \exp(-2\pi \sqrt{-1} k z_I) \), for \( I = 1, \ldots, n \).

**Definition 4.5.** Given any meromorphic function \( F \) on \( \mathbb{C} \), holomorphic at 0, with Taylor expansion of the form
\[ F(t) = 1 + \sum_{n=1}^{\infty} F_n t^n, \]
we define the \( \hat{F} \)-class of \( V \) as the characteristic class
\[ \hat{F}(V) := \prod_{j=1}^n F(\xi_j) \in H_T^{\text{mer}}(\mathbb{P}^{n-1}). \]

**Definition 4.6 (Gamma classes).** The \( \hat{\Gamma} \)-classes of \( V \) are defined as the characteristic classes of \( V \) obtained from the Taylor expansions
\[ \Gamma(1 \pm t) = \exp \left( \mp \gamma t + \sum_{k=2}^{\infty} (\mp 1)^{k-1} \frac{k \zeta(k)}{k} t^k \right). \]

Definitions 4.3, 4.5, 4.6 naturally extends to objects of the equivariant derived category \( D_T^b(\mathbb{P}^{n-1}) \).

**Remark 4.7.** If \( V = T \mathbb{P}^{n-1} \), by abuse of language the \( \hat{F} \)-class of \( V \) is called the \( \hat{F} \)-class of \( \mathbb{P}^{n-1} \). We will denote it by \( \hat{F}_{\mathbb{P}^{n-1}} \). Such a class is given by
\[ \hat{F}_{\mathbb{P}^{n-1}} = \prod_{a=1}^n F(y_a - x). \]
This is the class of \( H_T^{\text{mer}}(\mathbb{P}^{n-1}) \) whose restriction at the fixed point \( pt_I \) is
\[ \prod_{a=1}^n F(z_a - z_I). \]
If \( F(x) \) defines a meromorphic function with poles at points of \( \mathbb{Z} \), then \( \hat{F}_{\mathbb{P}^{n-1}} \) can be seen as an element of \( H_T^\Omega(\mathbb{P}^{n-1}) \). This is the case for \( \hat{\Gamma}_{\mathbb{P}^{n-1}} \).
5. Equivariant Quantum Cohomology of $\mathbb{P}^{n-1}$

5.1. Equivariant Gromov-Witten Invariants. For a given $d \in H_2(\mathbb{P}^{n-1}, \mathbb{Z}) \cong \mathbb{Z}$ and given integers $g, m \geq 0$, denote by $\overline{M}_{g,m}(\mathbb{P}^{n-1}, d)$ the moduli stack of genus $g$ stable maps to $\mathbb{P}^{n-1}$ with degree $d$ and $m$ marked points. We will assume that either $d > 0$ of $2g + m > 2$ so that $\overline{M}_{g,m}(\mathbb{P}^{n-1}, d)$ is non-empty. The $\mathbb{T}$-action on $\mathbb{P}^{n-1}$ induces a $\mathbb{T}$-action on $\overline{M}_{g,m}(\mathbb{P}^{n-1}, d)$. Given $m$ cohomological classes

$$\gamma_1, \ldots, \gamma_m \in H^*_\mathbb{T}(\mathbb{P}^n, \mathbb{C}),$$

and integers $d_1, \ldots, d_m \in \mathbb{Z}_{\geq 0}$, we define the genus $g$, degree $d$, $\mathbb{T}$-equivariant descendant Gromov-Witten invariants of $\mathbb{P}^{n-1}$ to be the polynomials

$$\langle \tau_{d_1}(\gamma_1), \ldots, \tau_{d_m}(\gamma_m) \rangle_{g,n,d}^{\mathbb{P}^{n-1}, \mathbb{T}} := \left( \int_{[\overline{M}_{g,m}(\mathbb{P}^{n-1}, d)]_{\text{vir}}} \prod_{j=1}^{m} \psi_j^{d_j} \text{ev}_j(\gamma_j) \right) \in H^*_\mathbb{T}(\text{pt}, \mathbb{C}) \cong \mathbb{C}[z],$$

(5.1)

where

- $[\overline{M}_{g,m}(\mathbb{P}^{n-1}, d)]_{\text{vir}} \in \mathbb{A}^*_D(\overline{M}_{g,m}(\mathbb{P}^{n-1}, d))$, with $D_{\text{vir}} := nd + (n-4)(1-g) + m$, is the equivariant fundamental class$^4$,
- the map $\text{ev}_j: \overline{M}_{g,m}(\mathbb{P}^{n-1}, d) \to \mathbb{P}^{n-1}$ is the evaluation at the $j$-th marked point, which is $\mathbb{T}$-equivariant,
- the classes $\psi_j \in \mathbb{A}^*_D(\overline{M}_{g,m}(\mathbb{P}^{n-1}, d))$ denote any equivariant lift of the first Chern classes of the tautological cotangent line bundles $L_j$ on $\overline{M}_{g,m}(\mathbb{P}^{n-1}, d)$.

We refer the interested reader to the expository article [LS17], and references therein, for details. If all $d_i$’s are zero, then the polynomials above are called primary equivariant Gromov-Witten invariants.

Remark 5.1. The reader can obtain the definition of non-equivariant descendant Gromov-Witten invariants of $\mathbb{P}^{n-1}$ (of genus $g$ and degree $d$) by taking the non-equivariant limit of all the classes appearing in equation (5.1). See for example [CK99].

Remark 5.2. If $\gamma_i \in H^2_T(\mathbb{P}^{n-1}, \mathbb{C})$, with $\alpha_i \geq 0$ for $i = 1, \ldots, m$, it can be proved that the polynomial $\langle \tau_{d_1}(\gamma_1), \ldots, \tau_{d_m}(\gamma_m) \rangle_{g,n,d}^{\mathbb{P}^{n-1}, \mathbb{T}}$ defined in (5.1) is homogeneous of degree $\sum_{j=1}^{m} (\alpha_j + d_j) = D_{\text{vir}}$. In particular,

$$\langle \tau_{d_1}(\gamma_1), \ldots, \tau_{d_m}(\gamma_m) \rangle_{g,n,d}^{\mathbb{P}^{n-1}, \mathbb{T}} = 0 \quad \text{if} \quad \sum_{j=1}^{m} (\alpha_j + d_j) < D_{\text{vir}}.$$

Thus, the non-equivariant limit of the polynomial $\langle \tau_{d_1}(\gamma_1), \ldots, \tau_{d_m}(\gamma_m) \rangle_{g,n,d}^{\mathbb{P}^{n-1}, \mathbb{T}}$ (obtained by setting $z_i = 0$) is non-zero only if $\sum_{j=1}^{m} (\alpha_j + d_j) = D_{\text{vir}}$. Such a limit coincide with the non-equivariant Gromov-Witten invariant $\langle \tau_{d_1}(\iota^* \gamma_1), \ldots, \tau_{d_m}(\iota^* \gamma_m) \rangle_{g,n,d}^{\mathbb{P}^{n-1}, \mathbb{T}} \in \mathbb{C}$.

$^4$Its existence is ensured by the properness of $\overline{M}_{g,m}(\mathbb{P}^{n-1}, d)$. See [LS17], Section 3 and references therein.
5.2. Equivariant Gromov-Witten potential. Consider the standard basis \((x_\alpha)_{\alpha=0}^{n-1}\) of \(H^*_T(\mathbb{P}^{n-1}, \mathbb{C})\), seen as a \(H^*_T(\text{pt}, \mathbb{C})\)-module. Denote by \(t := (t^0, \ldots, t^{n-1})\) the corresponding dual coordinates on \(H^*_T(\mathbb{P}^{n-1}, \mathbb{C})\), so that the generic element of \(H^*_T(\mathbb{P}^{n-1}, \mathbb{C})\) is

\[
\gamma = \sum_{\alpha=0}^{n-1} t^\alpha x_\alpha.
\]

Consider the generating function \(F_{0}^{\mathbb{P}^{n-1}, T}(t) \in H^*_T(\text{pt}, \mathbb{C})[t^0, \ldots, t^n] \cong \mathbb{C}[z][t]\), called equivariant Gromov-Witten potential of \(\mathbb{P}^{n-1}\), defined by

\[
F_{0}^{\mathbb{P}^{n-1}, T}(t) := \sum_{m=0}^{\infty} \sum_{d=0}^{\infty} \frac{1}{m!} \langle \gamma, \ldots, \gamma \rangle_{\mathbb{P}^{n-1}, T}^{\mathbb{P}^{n-1}, T}.
\]

\[
= \sum_{m=0}^{\infty} \sum_{d=0}^{\infty} \sum_{\alpha_1, \ldots, \alpha_m=0}^{n-1} \frac{t^{a_1} \cdots t^{a_m}}{m!} \langle x_{\alpha_1}, \ldots, x_{\alpha_m} \rangle_{\mathbb{P}^{n-1}, T}^{\mathbb{P}^{n-1}, T}. \tag{5.2}
\]

**Theorem 5.3** ([Giv96], Theorem 3.1). The function \(F_{0}^{\mathbb{P}^{n-1}, T}(t)\) satisfies the WDVV-equations

\[
\frac{\partial^3 F_{0}^{\mathbb{P}^{n-1}, T}}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\mu} \frac{\partial^3 F_{0}^{\mathbb{P}^{n-1}, T}}{\partial t^\nu \partial t^\beta \partial t^\lambda} = \frac{\partial^3 F_{0}^{\mathbb{P}^{n-1}, T}}{\partial t^\nu \partial t^\beta \partial t^\lambda} \eta^{\mu} \frac{\partial^3 F_{0}^{\mathbb{P}^{n-1}, T}}{\partial t^\alpha \partial t^\beta \partial t^\lambda}.
\]

5.3. Equivariant Quantum Cohomology. The big equivariant quantum product \(\ast\) defined by

\[
x_\alpha \ast x_\beta = \sum_{\lambda, \mu} \frac{\partial^3 F_{0}^{\mathbb{P}^{n-1}, T}}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\mu} x_\mu, \tag{5.3}
\]

endows \(H^*_T(\mathbb{P}^{n-1}, \mathbb{C})[t]\) with a Frobenius algebra structure, namely a commutative, associative algebra with unit (the element 1) whose product is compatible with the equivariant Poincaré metric (4.8), that is

\[
\eta(a \ast b, c) = \eta(a, b \ast c), \quad a, b, c \in H^*_T(\mathbb{P}^{n-1}, \mathbb{C}). \tag{5.4}
\]

This algebra structure on \(H^*_T(\mathbb{P}^{n-1}, \mathbb{C})[t]\) is called big equivariant quantum cohomology of \(\mathbb{P}^{n-1}\). It furnishes an example of formal Frobenius manifold (see Appendix A). The quantum product (5.3) has to be thought as a deformation of the product in classical cohomology. In literature, it is customary to denote the big quantum product also by \(\ast_t\) in order to emphasize its dependence on many parameters \(t^i\)’s.

5.4. Quantum Connection. The quantum connection of the equivariant quantum cohomology of \(\mathbb{P}^{n-1}\) is defined by the formula

\[
\nabla^\text{quant}_{\alpha, \kappa} : H^*_T(\mathbb{P}^{n-1}, \mathbb{C})[t] \to H^*_T(\mathbb{P}^{n-1}, \mathbb{C})[t], \quad \alpha = 0, \ldots, n - 1, \tag{5.5}
\]

\[
\nabla^\text{quant}_{\alpha, \kappa} := \kappa \frac{\partial}{\partial t^\alpha} - x_\alpha \ast_t, \quad \alpha = 0, \ldots, n - 1, \tag{5.6}
\]
and where \( \kappa \in \mathbb{C}^* \) is a “spectral” parameter. The associativity of the quantum multiplication \(*_q\) (i.e. Theorem 5.3) is equivalent to the flatness condition of the quantum connection \( \nabla^{\text{quant}}_{\alpha,\kappa} \), for all \( \kappa \in \mathbb{C}^* \):

\[
[\nabla^{\text{quant}}_{\alpha,\kappa}, \nabla^{\text{quant}}_{\beta,\kappa}] = 0, \quad \alpha, \beta = 0, \ldots, n - 1, \quad \kappa \in \mathbb{C}^*. \tag{5.7}
\]

The system of equations for flat sections of the quantum connection is called the system of \textit{equivariant quantum differential equations}.

5.5. \textbf{Small equivariant quantum product for} \( \mathbb{P}^{n-1} \).

\textbf{Definition 5.4.} The \textit{small} quantum product of \( \mathbb{P}^{n-1} \) is obtained by specializing the parameters \( t^i \)'s of the big quantum product (5.3) as follows: \( t^i = 0 \) for \( i \neq 1 \).

The only deformation parameter being \( t^1 \), it is customary to put \( q := \exp(t^1) \) and to denote by \(*_q\) the small quantum product. Following the notations of [TV19a], in this paper we will denote by \(*_{q,z}\) the small quantum product, underlining its implicit dependence on the equivariant parameters \( z \).

A detailed study of equivariant Gromov-Witten invariants of \( \mathbb{P}^{n-1} \) (and more general flag varieties) and its small quantum cohomology can be found for example in [GK95], [Kim96], [Mih06]. For a fixed \( t^1 \in \mathbb{C} \), the small quantum product operator

\[
x *_{q,z} : H^*_T(\mathbb{P}^{n-1}, \mathbb{C}) \to H^*_T(\mathbb{P}^{n-1}, \mathbb{C}),
\]

is the \( \mathbb{C}[z] \)-linear morphism defined by the identities

\[
x *_{q,z} x_j = x_{j+1}, \quad j = 0, \ldots, n - 2,
\]

\[
x *_{q,z} x_{n-1} = q + \sum_{i=1}^{n} (-1)^{i-1} s_i(z) x_{n-i},
\]

where \( q := \exp(t^1) \) and \( s_i(z) \) are the elementary symmetric polynomials in \( z \).

\textbf{Remark 5.5.} In the basis \((g_1, \ldots, g_n)\), the operator \( x *_{q,z} \) is given by

\[
x *_{q,z} g_i = z_i g_i + g_{i-1}, \quad i = 2, \ldots, n, \tag{5.11}
\]

\[
x *_{q,z} g_1 = z_1 g_1 + q g_n. \tag{5.12}
\]

5.6. \textbf{R-Matrices and qKZ operators}. For \( a, b \in \{1, \ldots, n\} \), with \( a \neq b \), we define a family of \( \mathbb{C}[z] \)-linear operators, called \textit{R-matrices},

\[
R_{ab}(u) : H^*_T(\mathbb{P}^{n-1}, \mathbb{C}) \to H^*_T(\mathbb{P}^{n-1}, \mathbb{C}),
\]

depending on a parameter \( u \in \mathbb{C} \), and defined by the formulae

\[
R_{ab}(u) g_i := g_i, \quad i \neq a, b,
\]

\[
R_{ab}(u) g_b := g_a, \quad R_{ab}(u) g_a := g_b + u g_a.
\]

The \( R \)-matrices satisfy the Yang-Baxter equation

\[
R_{ab}(u - v) R_{ac}(u) R_{bc}(v) = R_{bc}(v) R_{ac}(u) R_{ab}(u - v),
\]

for \( a, b, c \) all distinct, and the inversion relation

\[
R_{ab}(u) R_{ba}(-u) = 1.
\]
Define the operators $E_1, \ldots, E_n$ such that
\[ E_i g_j := \delta_{ij} g_j. \]

Define the $qKZ$ operators $K_1, \ldots, K_n$ by the formula
\[ K_i : H^*_{\mathbb{P}^{n-1}}(\mathbb{C}) \to H^*_{\mathbb{P}^{n-1}}(\mathbb{C}), \]
\[ K_i := R_{i,i-1}(z_i - z_{i-1} - 1) \cdots R_{i,1}(z_i - z_1 - 1) q^{-E_i} R_{i,n}(z_i - z_n) \cdots R_{i,i+1}(z_i - z_{i+1}). \quad (5.14) \]

5.7. **Equivariant qDE and qKZ difference equations.** Consider the vector bundle $H$ over the base space $\mathbb{C}^n$, with fiber over $z_0$ given by the equivariant cohomology algebra (4.1) specialized at $z = z_0$, i.e.
\[ H^*_{\mathbb{P}^{n-1}}(\mathbb{C})|_{z=z_0}. \]

Denote by $\text{pr}: \mathbb{C}^* \times \mathbb{C}^n \to \mathbb{C}^n$ the natural projection, and consider the pull-back vector bundle $\text{pr}^*H$. The quantum connection described in Section 5.4 defines a differential operator
\[ \nabla_q \delta_{\pi^*}\kappa := \kappa q \frac{d}{dq} - x^* q, \]
acting on sections $I(q, z)$ of the vector bundle $\text{pr}^*H$. Following [TV19a], in this paper we fix $\kappa = 1$.

**Definition 5.6.** Fix $q \in \mathbb{C}^*$, $z, z' \in \mathbb{C}^n$. We define the isomorphism of vector spaces
\[ \Pi_{z,z'}: \text{pr}^* H_{q,z} \to \text{pr}^* H_{q,z'}, \quad x_\alpha \mapsto x_\alpha, \quad (5.17) \]
for $\alpha = 0, \ldots, n - 1$.

**Definition 5.7.** Fix $q \in \mathbb{C}^*$, $z, z' \in \mathbb{C}^n$. For $i = 1, \ldots, n$, define the isomorphisms of vector spaces
\[ \Theta_{z,z'}: \text{pr}^* H_{q,z} \to \text{pr}^* H_{q,z'}, \quad g_j(z) \mapsto g_j(z'), \quad (5.18) \]
for $j = 1, \ldots, n$, where $g_j$’s are the elements of bases (4.4).

For $z = (z_1, \ldots, z_n), i = 1, \ldots, n$, we will use the following notations:
\[ z_i^\pm := (z_1, \ldots, z_i \pm 1, \ldots, z_n) \in \mathbb{C}^n, \]
\[ \Pi_{z,i} := \Pi_{z,z_i^\pm}: \text{pr}^* H_{q,z} \to \text{pr}^* H_{q,z_i^\pm}, \]
\[ z_i^\pm z_i := \Pi_{z,z_i^\pm}: \text{pr}^* H_{q,z_i^\pm} \to \text{pr}^* H_{q,z_i^\pm}, \]
\[ \Theta_{z,i} := \Theta_{z,z_i^\pm}: \text{pr}^* H_{q,z} \to \text{pr}^* H_{q,z_i^\pm}, \]
\[ \Theta_{z,i} := \Theta_{z,z_i^\pm}: \text{pr}^* H_{q,z_i^\pm} \to \text{pr}^* H_{q,z_i^\pm}. \]

For every fiber $\text{pr}^*H_{q,z}$ we have $qKZ$-operators $K_1, \ldots, K_n$ defined by equation (5.14).

**Definition 5.8.** Fix $q \in \mathbb{C}^*$, $z \in \mathbb{C}^n$. The $qKZ$-discrete connection on the bundle $H$ is given by the datum of the isomorphisms of vector spaces
\[ \Theta_{z,i} \circ K_i(q, z): \text{pr}^* H_{q,z} \to \text{pr}^* H_{q,z_1, \ldots, z_i - 1, \ldots, z_n}. \quad (5.24) \]
The (small) equivariant quantum differential equation (qDE for short) of $\mathbb{P}^{n-1}$ is the differential equation

$$
\nabla_q \frac{d}{dq_i} 
\sum_{\kappa=1}^n I (q, z) = \left( q \frac{d}{dq} - x^*_q, z \right) I (q, z) = 0,
$$

(5.25)

where $I$ is a section of the vector bundle $pr^* H$.

The system of difference equations

$$
I (q, z_1, \ldots, z_i - 1, \ldots, z_n) = \left[ \Theta_{z_i}^- \circ K_i (q, z) \right] I (q, z), \quad i = 1, \ldots, n,
$$

(5.26)

is called the system of qKZ difference equations.

**Theorem 5.9** ([TV19a], Theorem 3.1). The joint system of equation (5.25) and (5.26) is compatible.

**Remark 5.10.** qKZ difference equations (5.26), can be written in the equivalent form

$$
I (q, z_1, \ldots, z_i + 1, \ldots, z_n) = \left[ \Theta_{z_i}^+ \circ K_i (q, z) \right] I (q, z), \quad i = 1, \ldots, n,
$$

(5.27)

where the operators

$$
K_i (q, z)^+ := + \Theta_{z_i} \circ K_i (q, z_i)^+ \circ \Theta_{z_i}^-.
$$

(5.28)

act on the fiber $pr^* H_{q,z}$. In terms of $R$-matrices we have

$$
K_i (q, z_i^+)^{-1} = R_{i+1,i} (z_{i+1} - z_i - 1) \cdots R_{n,i} (z_n - z_i - 1) q^{E_i} R_{1,i} (z_1 - z_i) \cdots R_{i-1,i} (z_{i-1} - z_i).
$$

(5.29)

### 5.8. Frobenius manifold structure on the Quantum Cohomology of $\mathbb{P}^{n-1}$

We consider the non-equivariant limit of the objects described in Section 5.2, Section 5.3, and Section 5.4. The non-equivariant limit morphism $\iota^*: H^*_T (\mathbb{P}^{n-1}, \mathbb{C}) \to H^* (\mathbb{P}^{n-1}, \mathbb{C})$ maps the $\mathbb{C}[z]$-basis $(x_\alpha)^{n-1}_\alpha$ of $H^*_T (\mathbb{P}^{n-1}, \mathbb{C})$ to the C-basis $x_\alpha^{n-1}_\alpha$ of $H^* (\mathbb{P}^{n-1}, \mathbb{C})$. We denote the dual coordinates on $H^* (\mathbb{P}^{n-1}, \mathbb{C})$ by the notation $t := (t^0, \ldots, t^{n-1})$. Consider the Gromov-Witten potential $F^{\mathbb{P}^{n-1}}_0 \in \mathbb{C}[t]$, obtained as the non-equivariant limit of $F^{\mathbb{P}^{n-1},T}_0 \in \mathbb{C}[z][t]$, namely

$$
F^{\mathbb{P}^{n-1}}_0 (t) := \sum_{m=0}^{\infty} \sum_{d=0}^{\infty} \sum_{\alpha_1, \ldots, \alpha_m = 0}^{n-1} \frac{t^{\alpha_1} \cdots t^{\alpha_m}}{m!} \langle t^* x_{\alpha_1}, \ldots, t^* x_{\alpha_m} \rangle^{\mathbb{P}^{n-1}}_{0,m,d}.
$$

(5.30)

See Remark 5.1 and Remark 5.2. It is known that the Gromov-Witten potential $F^{\mathbb{P}^{n-1}}_0 (t)$ is convergent. Denote by $M \subseteq H^* (\mathbb{P}^{n-1}, \mathbb{C})$ the domain of convergence of $F^{\mathbb{P}^{n-1}}_0 (t)$. The domain $M$ is endowed with an analytic Frobenius manifold structure (see Appendix A). Such a structure is defined by the datum of the following tensors$^5$.

$^5$In what follows, $TM$ and $T^* M$ denote the holomorphic tangent and cotangent bundles of $M$. 
(1) A symmetric non-degenerate $\mathcal{O}_M$-bilinear form $\eta_{\mathcal{C}} \in \Gamma(\text{Sym}^3 T^* M)$ given by the non-equivariant Poincaré metric:\footnote{Tangent spaces to $M$ are canonically identified with $H^\bullet(\mathbb{P}^{n-1}, \mathbb{C})$.}

$$\eta_{\mathcal{C}}(v, w) := \int_{\mathbb{P}^{n-1}} v \cdot w, \quad v, w \in H^\bullet(\mathbb{P}^{n-1}, \mathbb{C}). \quad (5.31)$$

The metric $\eta_{\mathcal{C}}$ is clearly flat, the Gram matrix in the coordinates $t$ being the constant anti-diagonal matrix $(\eta_{\mathcal{C}})_{\alpha \beta} = \delta_{\alpha + \beta, n-1}$. Denote by $\nabla$ the Levi-Civita connection of $\eta_{\mathcal{C}}$.

(2) A $(0, 3)$-tensor $c \in \Gamma(\text{Sym}^3 T^* M)$ defined by

$$c := \nabla^3 F_{0}^{\mathbb{P}^{n-1}}. \quad (5.32)$$

We define the tensor $c^\sharp$ by raising an index using the metric $\eta_{\mathcal{C}}$: in flat coordinates $t$ we have

$$(c^\sharp)^{\alpha}_{\beta\gamma} := (\eta_{\mathcal{C}})^{\alpha \mu} c_{\beta\mu\gamma}, \quad \alpha, \beta, \gamma = 0, \ldots, n-1. \quad (5.33)$$

(3) A distinguished vector field $E \in \Gamma(TM)$, defined by

$$E := c_1(\mathbb{P}^{n-1}) + \sum_{\alpha=0}^{n-1} \left(1 - \frac{1}{2} \deg(t^x x_\alpha)\right) t^\alpha \frac{\partial}{\partial t^\alpha}, \quad (5.34)$$

where $\deg(v) := p$ if $v \in H^p(\mathbb{P}^{n-1}, \mathbb{C})$. We call $E$ the Euler vector field.

The tensor $c^\sharp$ defines a product $*$ of vector fields, the big quantum product:

$$v * w := c^\sharp(-, v, w), \quad v, w \in \Gamma(TM). \quad (5.35)$$

The product $*$ is commutative, associative, with unit $1 \in H^\bullet(\mathbb{P}^{n-1}, \mathbb{C})$ (the constant vector field) and compatible with $\eta_{\mathcal{C}}$: for any $p \in M$, the 4-tuple $(T_p M, \eta_{\mathcal{C}}|_p, *_p, 1)$ is a complex Frobenius algebra. The Euler vector field is a Killing-conformal vector field whose flow preserves the tensor $c^\sharp$:

$$\mathcal{L}_E \eta_{\mathcal{C}} = (3 - n) \eta_{\mathcal{C}}, \quad \mathcal{L}_E c^\sharp = c^\sharp. \quad (5.36)$$

The potential $F_{0}^{\mathbb{P}^{n-1}}(t)$, and consequently the whole Frobenius manifold structure, admits a maximal analytic continuation to a covering of $M$. It is customary to refer to this maximal analytic continuation of the Frobenius structure on $M$ as the quantum cohomology of $\mathbb{P}^{n-1}$, denoted by $QH^\bullet(\mathbb{P}^{n-1})$.

Consider the semisimple part $M_{ss}$ of $M$, namely the subset of points $p$ whose corresponding Frobenius algebra $T_p M$ is without nilpotents. Denote by $(\pi_1, \ldots, \pi_n)$ the idempotent tangent vectors at $p \in M_{ss}$, and introduce the normalized frame $(f_1, \ldots, f_n)$ by

$$f_i := \eta_{\mathcal{C}}(\pi_i, \pi_i)^{-\frac{1}{2}} \cdot \pi_i, \quad i = 1, \ldots, n, \quad (5.37)$$

for some choice of the square roots. Denote by $U(p)$ and $V(p)$ the matrices, wrt the frame $(f_1, \ldots, f_n)$, of the morphisms

$$U(p): T_p M \to T_p M, \quad v \mapsto E|_p *_p v, \quad (5.38)$$

$$V(p): T_p M \to T_p M, \quad v \mapsto C|_p *_p v, \quad (5.39)$$

The product $*$ is compatible with $U(p)$ and $V(p)$.

$$U(p)_* c^\sharp = U(p) \circ (U(p)|_p)_* c^\sharp \circ U(p)^{-1}, \quad (5.40)$$

$$V(p)_* c^\sharp = V(p) \circ (V(p)|_p)_* c^\sharp \circ V(p)^{-1}. \quad (5.41)$$
\( \mu(p): T_pM \to T_pM, \quad v \mapsto \frac{3-n}{2}v - \nabla_v E|_p. \)  

(5.39)

It is easily seen that they satisfy

\[
U(p)^T = U(p), \quad V(p)^T + V(p) = 0. 
\]

(5.40)

According to the analytic theory of Frobenius manifolds, there is a local identification of \( M_{ss} \) with the space of parameters of isomonodromic deformations of the ordinary differential equation

\[
\frac{d}{dz} Y(z, p) = \left( U(p) + \frac{1}{z} V(p) \right) Y(z, p), \quad z \in \mathbb{C}^*,
\]

(5.41)

for a \( n \times n \)-matrix valued function \( Y \). See Appendix A for a detailed discussion.

6. Equivariant \( qDE \) of \( \mathbb{P}^{n-1} \) and its Topological-Enumerative Solution

6.1. Equivariant quantum differential equation. We consider the equivariant quantum differential equation (5.25) written wrt the standard basis \((x_\alpha)_{\alpha=0}^{n-1}\), namely

\[
\frac{dY}{dq} = A(q, z)Y, \quad A(q, z) := \frac{1}{q} \begin{pmatrix}
0 & \ldots & 0 & q + (-1)^{n-1}s_n(z) \\
1 & 0 & \ldots & 0 & (-1)^{n-2}s_{n-1}(z) \\
0 & 1 & \ldots & 0 & (-1)^{n-3}s_{n-2}(z) \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & 1 & s_1(z)
\end{pmatrix}. 
\]

(6.1)

We have

\[
A(q, z) = A_0 + \frac{1}{q} A_1(z),
\]

(6.2)

where

\[
A_0 := \begin{pmatrix}
0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0
\end{pmatrix}, \quad A_1(z) := \begin{pmatrix}
0 & \ldots & 0 & (-1)^{n-1}s_n(z) \\
1 & 0 & \ldots & 0 & (-1)^{n-2}s_{n-1}(z) \\
0 & 1 & \ldots & 0 & (-1)^{n-3}s_{n-2}(z) \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & 1 & s_1(z)
\end{pmatrix}.
\]

The eigenvalues of the matrix \( A_1(z) \) are exactly \( z_1, \ldots, z_n \), as it easily follows from Viète formulae. Notice that \( A_1(z) \) denotes the matrix of equivariant multiplication

\[
H^*_\mathbb{P}(\mathbb{P}^{n-1}, \mathbb{C}) \to H^*_\mathbb{P}(\mathbb{P}^{n-1}, \mathbb{C}), \quad f \mapsto x \cdot f,
\]

whereas \( A_0 \) represents the quantum correction terms of the product. Moreover, we have

\[
D \cdot A_1(z) \cdot D^{-1} = Z := \text{diag}(z_1, \ldots, z_n),
\]

(6.3)

since the classes \( \Delta_i \)'s are the idempotents of the equivariant cohomology algebra.

The equation (6.1) has a regular singularity at \( q = 0 \) and an irregular singularity (of Poincaré rank 1) at \( q = \infty \).
6.2. Levelt Solution.

**Theorem 6.1.** There exist unique functions $G_k(z)$, meromorphic on $\mathbb{C}^n$ and regular on $\Omega$, such that the gauge transformation

$$Y(q, z) = G(q, z)\tilde{Y}(q, z), \quad G(q, z) := D^{-1}\left(1 + \sum_{k=1}^{\infty} G_k(z)q^k\right),$$

transforms equation (6.1) into

$$\frac{d}{dq}\tilde{Y} = \frac{1}{q}Z\tilde{Y}.$$  \hspace{1cm} (6.5)

The function $G(q, z)$ is meromorphic on $\mathbb{C} \times \mathbb{C}^n$, and regular on $\mathbb{C} \times \Omega$.

**Proof.** Let us look for a (formal) gauge transformation

$$Y = D^{-1}G\tilde{Y}, \quad G(q, z) := \sum_{k=0}^{\infty} G_k(z)q^k,$$

which puts the system (6.1) into the simplest normal form

$$\frac{d\tilde{Y}}{dq} = \frac{1}{q}Z\tilde{Y}.$$  \hspace{1cm} (6.6)

This implies the following equation for $G$

$$DA_0D^{-1}G - \frac{dG}{dq} + \frac{1}{q}[Z, G] = 0,$$

which reduces to the following recurrence equations in terms of the coefficients $G_k$’s:

$$ZG_0 = G_0Z,$$  \hspace{1cm} (6.7)

$$DA_0D^{-1}G_k + [Z, G_{k+1}] - (k + 1)G_{k+1} = 0.$$  \hspace{1cm} (6.7)

Equation (6.6) is satisfied if and only if $G_0$ is diagonal. Since $(z_1, \ldots, z_n) \in \Omega$, from equation (6.7) we deduce that $G_{k+1}$ is uniquely determined by the value of $G_k$. Indeed, the linear operator

$$\varphi_k: M_n(\mathbb{C}) \to M_n(\mathbb{C}), \quad X \mapsto [Z, X] - (k + 1)X,$$

has eigenvalues $z_i - z_j - (k + 1), \quad i, j = 1, \ldots, n$, which are nonzero by assumption. Hence, $\varphi_k$ is invertible, and we deduce

$$G_{k+1} = \varphi_k^{-1}(-DA_0D^{-1}G_k).$$  \hspace{1cm} \square
Corollary 6.2. For $G_k$’s as in Theorem 6.1, the function
\[ Y_0(q, z) = D^{-1} \left( 1 + \sum_{k=1}^{\infty} G_k(z) q^k \right) q^Z, \]
is a solution of the equation (6.1). For each fixed $z \in \Omega$, the function $Y_0$ is a fundamental system of solutions.

We call the fundamental solution $Y_0$ the Levelt fundamental solution.

Fix $(q, z)$ and increase the argument of $q$ by $2\pi$. The analytic continuation of the solutions of (6.1) along this curve produces the monodromy operator $M_0(z)$ on the space of solutions.

Corollary 6.3. The Levelt fundamental solution $Y_0(q, z)$ is an eigenbasis for the monodromy operator $M_0(z)$. The matrix of the monodromy operator $M_0(z)$ wrt the solution $Y_0(q, z)$ is
\[ M_0(z) = \exp \left( 2\pi \sqrt{-1} Z \right). \]

Proof. We have
\[ Y_0 \left( e^{2\pi \sqrt{-1}} q, z \right) = Y_0(q, z) M_0(z), \quad M_0(z) = \exp \left( 2\pi \sqrt{-1} Z \right). \]

6.3. Topological-enumerative solution. Denote $t := (t^0, \ldots, t^{n-1})$.

Definition 6.4 ([Giv96, Giv98]). The $T$-equivariant $S$-operator is the element
\[ S \in \text{End}(H^\bullet_T(\mathbb{P}^{n-1}, \mathbb{C}))[t][\kappa^{-1}] \]
defined by the formula
\[ \eta(S(t, \kappa)u, v) = \eta(u, v) + \sum_{d=0}^{\infty} \sum_{m=0}^{\infty} \sum_{a_1, \ldots, a_m = 0}^{n-1} \frac{t^{a_1} \cdots t^{a_m}}{m!} \langle u, x_{a_1}, \ldots, x_{a_m}, \frac{v}{\kappa - \psi} \rangle_{\mathbb{P}^{n-1}, T, 0, m+2, d}, \]
where $u, v \in H^\bullet_T(\mathbb{P}^{n-1}, \mathbb{C})$, $\psi := c_1(L_{m+2})$ and the term $\frac{1}{\kappa - \psi}$ has to be expanded in power series $\sum_{j=0}^{\infty} \psi^j \kappa^{-j-1}$.

The (big) $T$-equivariant $J$-function of $\mathbb{P}^{n-1}$ is the cohomology-valued function defined by the identity
\[ \eta(J(t, \kappa), a) = \eta(1, S(t, \kappa)a), \quad a \in H^\bullet_T(\mathbb{P}^{n-1}, \mathbb{C}). \]
Theorem 6.5 ([Giv96, Giv98]). For any \( \kappa \in \mathbb{C}^* \), and any \( \alpha \in H_T^\bullet(\mathbb{P}^{n-1}, \mathbb{C}) \), the cohomology class \( S(t, \kappa)\alpha \) is a flat section of the quantum connection \( \nabla^\text{quant}_{\frac{\partial}{\partial t}, \kappa} \), namely it satisfies the following system of PDEs
\[
\kappa \frac{\partial}{\partial t} S(t, \kappa)\alpha = x_\beta *_{\gamma} S(t, \kappa)\alpha, \quad \gamma = \sum_{\mu=0}^{n-1} t^\mu x_\mu, \quad \beta = 0, \ldots, n-1. \tag{6.10}
\]

Proof. The validity of equations (6.10) is equivalent to the topological recursion relations in genus 0 for Gromov-Witten invariants with descendants [Wit90]. For the proof in the non-equivariant case, see for example [Dub92], Lecture 6 of [Dub96], Lecture 2 of [Dub99], Chapter 10 of [CK99], Section 7 of [CDG17]. For the adaptation to the equivariant case see for example Section 6 of [Giv96], Sections 1 and 2 of [Giv98]. \( \square \)

Corollary 6.6. The equivariant quantum differential equation (6.1) admits a unique solution of the form
\[
Y_{\text{top}}(q, z) = \Phi(q, z)q^{A_1}(z), \tag{6.11}
\]
where
\[
\Phi(q, z) = 1 + \sum_{j=1}^{\infty} \Phi_j(z)q^j.
\]
The coefficients \( \Phi_j \) are holomorphic on \( \Omega \), and they are related to Gromov-Witten invariants through the equation
\[
\Phi_j(z)^\lambda_\alpha = \left\langle x_\mu, \frac{x_\alpha}{1 - \psi}\right\rangle_{0,2,j}^{\mathbb{P}^{n-1}, \mathbb{T}} \eta^{\mu\lambda}, \quad j \in \mathbb{N}_{>0}, \quad \alpha, \lambda = 0, \ldots, n-1. \tag{6.12}
\]
Furthermore, we have
\[
Y_{\text{top}}(q, z) = Y_o(q, z) \cdot D,
\]
where \( Y_o \) is the solution of (6.1) described in Theorem 6.2.

We will refer to solution \( Y_{\text{top}}(q, z) \) as the topological-enumerative solution.

Proof. Using the Divisor axiom for descendant Gromov-Witten Invariants, it can be shown that the restriction \( \mathcal{S}(t^1, \kappa) \) of the \( \mathcal{S} \)-operator at point of the small quantum locus, i.e.
\[
t^\alpha = 0, \quad \text{for } \alpha \neq 1,
\]
reduces to
\[ S(t^1, \kappa)a = \exp\left(\frac{t^1 x_1}{\kappa}\right) \cdot a + \sum_{d=1}^{\infty} e^{t^1 d} \left\langle x_\mu, \exp\left(\frac{t^1 x_1}{\kappa - \psi}\right) \cdot a\right\rangle_{\mathbb{P}^{n-1}, T} \eta^{\mu \lambda} x_\lambda. \]

See [CK99], Chapter 10, for a detailed proof. By setting \( q = \exp(t^1) \) and \( \kappa = 1 \), we obtain the solution (6.11) of the equivariant quantum differential equation of \( \mathbb{P}^{n-1} \). Notice that
\[
Y_{\text{top}}(q, z_1, \ldots, z_n) = \left(1 + \sum_{j=1}^{\infty} \Phi_j(z)q^j\right) q^{A_1(z)}
\]
where in the last line we used (6.3). The uniqueness of a solution of the form (6.11) thus follows from the uniqueness of the solution \( Y_o \) (Theorem 6.2).

6.4. **Scalar equivariant quantum differential equation.** Let \( Y \) be a fundamental solution of the differential equation (6.1). Then, the matrix \( \hat{Y} := \eta \cdot Y \cdot \eta^{-1} \) is a solution of the equation
\[
\frac{d\hat{Y}}{dq} = A(q, z) \hat{Y}. \tag{6.13}
\]
This follows from the Frobenius algebra property (5.4). Clearly, equation (6.13) is reduced to the scalar differential equation
\[
\vartheta^n q \phi = \left(q + (-1)^{n-1} s_n(z)\right) \phi + \sum_{j=1}^{n-1} (-1)^{n-j-1} s_{n-j}(z) \vartheta^j q \phi, \quad \vartheta_q := \frac{d}{dq}, \tag{6.14}
\]
that will be called the *scalar equivariant quantum differential equation* of \( \mathbb{P}^{n-1} \). Given \( n \) linearly independent solutions \( (\phi_1, \ldots, \phi_n) \) of (6.14) one can reconstruct a fundamental system \( \hat{Y} \) of solutions of the equation (6.13) by setting
\[
(\hat{Y})^k_h := \vartheta^h q \phi_k, \quad h = 0, \ldots, n-1, \quad k = 1, \ldots, n.
\]

**Remark 6.7.** In the non-equivariant limit \( z_1 = \cdots = z_n = 0 \), the equation (6.14) reduces to
\[
\vartheta^h q \phi = q \phi,
\]
which coincides with the quantum differential equation of \( \mathbb{P}^{n-1} \)
\[
\vartheta^h s \phi = (ns)^h \phi, \quad \vartheta_s := s \frac{d}{ds},
\]
under the identification \( q = s^n \). The monodromy of this equation has been studied in details in [Guz99, CDG18].
Theorem 6.8. The matrix
\[ \hat{Y}(q, z) := \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ \vartheta q a_1 & \vartheta q a_2 & \ldots & \vartheta q a_n \\ \vdots & \vdots & & \vdots \\ \vartheta^{n-1} a_1 & \vartheta^{n-1} a_2 & \ldots & \vartheta^{n-1} a_n \end{pmatrix}(D^{-1})^T, \] (6.15)
where
\[ a_j(q, z) := q^j \left( 1 + \sum_{d=1}^{\infty} q^d \prod_{i=1}^{n} \prod_{m=1}^{d} (z_j - z_i + m) \right), \quad j = 1, \ldots, n, \] (6.16)
is a fundamental system of solutions of the differential equation (6.13). The corresponding solution of the equivariant differential equation (6.1) is the topological-ensemble solution, i.e.
\[ Y_{\text{top}}(q, z) = \eta^{-1} \cdot \hat{Y} \cdot \eta. \]

Proof. Equation (6.9) implies that the components, wrt to the standard basis \((x_\alpha)_{\alpha=1}^{n}\), of the \(J\)-function, restricted to the small equivariant quantum locus, are solutions of the scalar equivariant quantum differential equation (6.14). The small equivariant \(J\)-function of \(\mathbb{P}^{n-1}\), computed by A. Givental [Giv96], B.J. Lian, K. Liu, S.-T. Yau [LLY97] is given by the formula
\[ J(q, z) = q^x \left( 1 + \sum_{d=1}^{\infty} q^d \prod_{i=1}^{n} \prod_{m=1}^{d} (x - z_i + m) \right). \]
We have
\[ x = \sum_{j=1}^{n} z_j \Delta_j, \quad 1 = \sum_{j=1}^{n} \Delta_j, \quad \frac{1}{\sum_i \alpha_i \Delta_i} = \sum_i \frac{1}{\alpha_i \Delta_i}, \]
for any \(\alpha_i \in \mathbb{C}^*\). We deduce
\[ J(q, z) = q^x \left( 1 + \sum_{d=1}^{\infty} q^d \prod_{i=1}^{n} \prod_{m=1}^{d} \frac{1}{\sum_{j=1}^{n} (z_j - z_i + m) \Delta_j} \right) \]
\[ = q^x \left( 1 + \sum_{d=1}^{\infty} q^d \frac{1}{\sum_{j=1}^{n} \prod_{i=1}^{n} \prod_{m=1}^{d} (z_j - z_i + m) \Delta_j} \right) \]
\[ = q^x \left( 1 + \sum_{d=1}^{\infty} \sum_{j=1}^{n} q^d \frac{\Delta_j}{\prod_{i=1}^{n} \prod_{m=1}^{d} (z_j - z_i + m)} \right) \]
\[ q^d \sum_{j=1}^{n} \left( 1 + \sum_{d=1}^{\infty} q^d \prod_{i=1}^{n} \prod_{m=1}^{d} (z_j - z_i + m) \right) \Delta_j \]

\[ = \sum_{j=1}^{n} q^{z_j} \left( 1 + \sum_{d=1}^{\infty} q^d \prod_{i=1}^{n} \prod_{m=1}^{d} (z_j - z_i + m) \right) \Delta_j \]

\[ = \sum_{\alpha=0}^{n} \left\{ \sum_{j=1}^{n} q^{z_j} \left( 1 + \sum_{d=1}^{\infty} q^d \prod_{i=1}^{n} \prod_{m=1}^{d} (z_j - z_i + m) \right) (D^{-1})_{\alpha j} \right\} x_{\alpha}. \]

If we define

\[ a_j(q, z) := q^{z_j} \left( 1 + \sum_{d=1}^{\infty} q^d \prod_{i=1}^{n} \prod_{m=1}^{d} (z_j - z_i + m) \right), \quad j = 1, \ldots, n, \]

then the matrix

\[ \hat{Y}(q, z) := \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ \vartheta_q a_1 & \vartheta_q a_2 & \ldots & \vartheta_q a_n \\ \vdots & \vdots & & \vdots \\ \vartheta_q^{n-1} a_1 & \vartheta_q^{n-1} a_2 & \ldots & \vartheta_q^{n-1} a_n \end{pmatrix} (D^{-1})^T \]

is a solution of the differential system (6.13), and the corresponding solution

\[ \eta^{-1} \cdot \hat{Y} \cdot \eta \]

is the topological solution of equation (6.1).

\[ \square \]

7. Solutions of the equivariant qDE and qKZ difference equations

7.1. q-Hypergeometric Solutions. In this section we describe a fundamental system of solutions of the joint system of equations (5.25) and (5.26) described in [TV19a].

**Definition 7.1** (Master and Weight function). We define the *master function* \( \Phi \) and the \( H^*_T(\mathbb{P}^{n-1}, \mathbb{C}) \)-valued *weight function* \( W \) by the formulae

\[ \Phi(t, q, z) := e^{\pi \sqrt{-1} \sum_{a=1}^{n} z_a} \left( e^{-\pi \sqrt{-1} n} q \right)^t \prod_{a=1}^{n} \Gamma(z_a - t), \quad W(t, y) := \prod_{j=1}^{n-1} (y_j - t). \quad (7.1) \]

Recall that \( y_1, \ldots, y_{n-1} \) denote the equivariant Chern roots of the natural quotient bundle \( Q \) on \( \mathbb{P}^{n-1} \).

**Remark 7.2.** Notice the difference in the definition (7.1) of the master function \( \Phi \) with respect to [TV19b] and [TV19a]. In Section 4.1 of [TV19a] the master function is defined as

\[ \Phi(t, q, z) := \left( e^{\pi \sqrt{-1} (2-n)} q \right)^t \prod_{a=1}^{n} \Gamma(z_a - t), \]
i.e. differing from (7.1) by an extra factor \( \exp(\pi \sqrt{-1}(2t + \sum_{i=1}^{n} z_i)) \). In [TV19b] the general case of partial flag varieties is considered. The master function in Section 11.4 of [TV19b] (see formula 11.16), specialized to the case of projective spaces, is

\[
\Phi(t, q, z) := e^{-\pi \sqrt{-1} \sum_{i=1}^{n} z_i \left( e^{-\pi \sqrt{-1} n q} \right)^t \prod_{a=1}^{n} \Gamma(z_a - t)}.
\]

Thus it differs from the function \( \Phi \) in (7.1) by a factor \( \exp(2\pi \sqrt{-1} \sum_{i=1}^{n} z_i) \).

**Definition 7.3 (Jackson Integrals).** We define the Jackson integrals \( \Psi_J \), for \( J = 1, \ldots, n \), to be the \( H^*_\mathbb{C}(\mathbb{P}^{n-1}, \mathbb{C}) \)-valued functions defined on \( \hat{\mathbb{C}}^* \times \Omega \) by the formula

\[
\Psi_J(q, y, z) := -\sum_{r=0}^{\infty} \text{Res}_{t=z_J+r} \Phi(t, q, z) W(t, y).
\]  

(7.2)

**Theorem 7.4 ([TV19b]).** The functions \( \Psi_J(q, y, z) \) with \( J = 1, \ldots, n \) are holomorphic on \( \hat{\mathbb{C}}^* \times \Omega \). Each of them is a solution of the equivariant quantum differential equation (5.25) and of qKZ-difference equations (5.26). These functions form a basis of solutions of this joint system of equations.

We will call the solutions \( \Psi_J \) the \( q \)-hypergeometric solutions.

**Remark 7.5.** Notice that in [TV19a], because of a different normalization of the master function \( \Phi \), the system of qKZ equations differs from the one considered in [TV19b] and in this article by a sign in the rhs of (5.26).

**Corollary 7.6 ([TV19b]).** The \( q \)-hypergeometric solutions \( \Psi_J \) admit the following expansion

\[
\Psi_J(q, y, z) = e^{\pi \sqrt{-1} \sum_{i=1}^{n} z_i \left( e^{-\pi \sqrt{-1} n q} \right)^t \prod_{a\neq J} \Gamma(1 + z_a - z_J)} \left( \Delta_J + \sum_{k=1}^{\infty} \Psi_{J,k}(z) q^k \right),
\]

where the classes \( \Psi_{J,k}(z) \) are rational functions in \( (z_1, \ldots, z_n) \), regular on \( \Omega \).

If we introduce the matrix \( Y_{q \text{-hyp}} = ([Y_q]^{\lambda})_{\lambda,J} \) defined by the identities

\[
\Psi_J = \sum_{\lambda} [Y_q]^{\lambda}_{J} x_\lambda, \quad J = 1, \ldots, n,
\]

we obtain a fundamental matrix solution of equation (6.1).
Theorem 7.7. The connection matrix $C$ relating the topological-enumerative solution with the $q$-hypergeometric one, i.e. such that

$$Y_{q,\text{hyp}}(q, z) = Y_{\text{top}}(q, z) \cdot C,$$

is given by

$$C = D^{-1} \cdot \text{diag} \left( e^{\pi \sqrt{-1} (-nz_j + \sum_{i=1}^{n} z_i)} \prod_{a \neq j} \Gamma(1 + z_a - z_j) \right)_{j=1}^{n}. \quad (7.3)$$

This is the matrix attached to the morphism

$$\rho: H^*_T(\mathbb{P}^{n-1}, \mathbb{C}) \rightarrow H^*_T(\mathbb{P}^{n-1}, \mathbb{C}), \quad v \mapsto v \cdot e^{\pi \sqrt{-1} c_1(\mathbb{P}^{n-1})} \cdot \hat{\Gamma}^+_{\mathbb{P}^{n-1}}, \quad (7.4)$$

where we fix the bases

- $(\Delta_j)_{j=1}^{n}$ in the domain of $\rho$,
- the standard basis $(x_{a})_{a=0}^{n-1}$ in the target space of $\rho$.

Proof. The proof follows from Corollary 6.6 and Corollary 7.6. Notice that

$$c_1(\mathbb{P}^{n-1}) = \sum_{i=1}^{n} z_i - nx,$$

$$\hat{\Gamma}^+_{\mathbb{P}^{n-1}} = \prod_{a=1}^{n-1} \Gamma(1 + y_a - x).$$

Each term of the entries of the diagonal matrix in (7.3) can be indentified with the multiplication by these classes wrt the basis $(\Delta_i)$. \hfill \Box

Remark 7.8. It follows from Theorem 7.7 that neither $Y_{o}$ nor $Y_{\text{top}}$ are solutions of $qKZ$ difference equations (5.26). Indeed, the connection matrix $C$ given by (7.3) is not 1-periodic in the equivariant parameters $z_i$'s.

7.2. Identification of solutions with $K$-theoretical classes. Following [TV19a], we introduce the symbols

$$\hat{T} := \exp(2\pi \sqrt{-1} t), \quad \hat{Z}_J := \exp(2\pi \sqrt{-1} z_J), \quad J = 1, \ldots, n. \quad (7.5)$$

Definition 7.9. Let $Q(X, Z) \in \mathbb{C}[X^{\pm 1}, Z^{\pm 1}]$ be a Laurent polynomial. We define

$$\Psi_{Q}(q, y, z) := \sum_{J=1}^{n} Q(\hat{Z}_J, \hat{Z}) \Psi_{J}(q, y, z).$$

Clearly, $\Psi_{Q}$ is a solution of the joint system of equations (5.25) and (5.26). If $Q(X, Z) = X^m$, we denote the corresponding solution $\Psi_{Q}$ by $\Psi_{m}$. 

Remark 7.10. Notice the difference of notations wrt [TV19a]: in loc. cit. the solution $\Psi_{m}$ is the solution attached to the polynomial $Q(X, Z) = X^{m-1}$. 

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Theorem 7.11 ([TV19a]). There is a well-defined morphism from $K^*_0(\mathbb{P}^{n-1})_\mathbb{C}$ to the space of solutions of the joint system of equations (5.25) and (5.26), defined by the association
\[ Q \mapsto \Psi_Q, \]
under the isomorphism (3.4).

Corollary 7.12 ([TV19a], Corollary 4.4). For any $k \in \mathbb{Z}$, we have
\[ \sum_{i=0}^{n} (-1)^{n-i} s_{n-i}(\hat{Z}) \Psi^{k+i}(q, y, z) = 0, \] (7.6)
where $s_i(\hat{Z})$ are the elementary symmetric polynomials in $\hat{Z}$.

Theorem 7.13 ([TV19b], Theorem 11.3). For any $k$, $(\Psi^{k+i}(q, y, z))_{i=0}^{n-1}$ is a basis of the space of solutions of the joint system (5.25)-(5.26).

Remark 7.14. The idea that space of solutions of $qDE$ and $qKZ$ is naturally identified with the $K$-algebra space can be observed in [TV97a], [TV97b] and was implicitly discussed there.

7.3. Module $S_n$ of solutions. Let us now select a subspace of the space of solutions of the joint system of equations (5.25)-(5.26).

Definition 7.15. We define the space $S_n$ of solutions of the joint system (5.25) and (5.26) of the form
\[ \sum_{m=1}^{n} Q_m(\hat{Z}) \Psi^m(q, y, z), \quad Q_m \in \mathbb{C}[\mathbb{Z}^{\pm 1}]. \] (7.7)
The space $S_n$ admits a structure of $\mathbb{C}[\mathbb{Z}^{\pm 1}]$-module, the multiplication by $Q(Z)$ being defined as the multiplication by $Q(\hat{Z})$.

By Corollary 7.12, the module $S_n$ contains all the solutions $\Psi^m(q, y, z)$, $m \in \mathbb{Z}$. From Theorem 7.13, we also deduce the following result.

Corollary 7.16 ([TV19a], Corollary 4.6). The module $S_n$ contains a basis of solutions of the joint system of equations (5.25)-(5.26). Moreover, the map $\theta : K^*_0(\mathbb{P}^{n-1})_\mathbb{C} \to S_n$ defined by
\[ \theta(X^m) := \Psi^m(q, y, z), \quad m \in \mathbb{Z}, \] (7.8)
defines an isomorphism of $\mathbb{C}[\mathbb{Z}^{\pm 1}]$-modules.
Using the isomorphism $\theta$ we define a sesquilinear form on the module $S_n$ as the image of the $\chi^T$-form on $K^\ast_0(\mathbb{P}^{n-1})_\mathbb{C}$. The notions of exceptional bases and the action of the braid group on them can be lifted to $S_n$.

7.4. Integral representations for solutions. For $p \in \mathbb{C}$, let us denote by $C(p)$ the parabola in $\mathbb{C}$ defined by the equation

$$C(p) := \{p + t^2 + t\sqrt{-1}: t \in \mathbb{R}\}.$$ 

Given a point $z \in \Omega$, take $p$ such that all the points $z_1, \ldots, z_n$ line inside $C(p)$. The value of the integral (7.9) below does not depend on a particular choice of $p$, so we will simply denote $C(p)$ by $C(z)$.

**Lemma 7.17** ([TV19b], Lemma 11.5). For any Laurent polynomial $Q(X, Z)$ we have

$$\Psi^m(q, y, z) = \frac{1}{2\pi \sqrt{-1}} \int_{C(z)} Q(\hat{T}, \hat{Z}) \Phi(t, q, z) W(t, y) dt,$$  \hspace{1cm} (7.9)

where the integral converges for any $(q, z) \in \hat{\mathbb{C}}^\ast \times \Omega$. In particular, we have

$$\Psi^m(q, y, z) = \int_{C(z)} e^{2\pi \sqrt{-1} \text{Im}(t)} e^{-\pi \sqrt{-1} \text{Im}(t)} q^n \prod_{a=1}^{n} \Gamma(z_a - t) \prod_{j=1}^{n-1} (y_j - t) dt.$$  \hspace{1cm} (7.10)

**Remark 7.18.** These formulae differ from the corresponding ones in [TV19a]. See also Remark 7.2 and Remark 7.10.

7.5. Coxeter element, and elements $\gamma_n, \delta_{n, odd}, \delta_{n, even} \in B_n$. The Coxeter element of $B_n$ is the braid

$$C := \tau_1 \tau_2 \ldots \tau_{n-1} \in B_n.$$  \hspace{1cm} (7.11)

For any $n \geq 3$, let

$$\ell_n := \begin{cases} n - 1, & \text{for } n \text{ odd,} \\ n - 2, & \text{for } n \text{ even.} \end{cases}$$

Set $\gamma_2 := 1$, and for $n \geq 3$,

$$\beta_k := \tau_k \tau_{k+1} \ldots \tau_{n-1}, \hspace{1cm} \gamma_n := \beta_{\ell_n} \beta_{\ell_{n-2}} \ldots \beta_2.$$ 

Define also

$$\delta_{n, odd} := \tau_1 \tau_3 \ldots \tau_{n-2}, \hspace{1cm} \delta_{n, even} := \tau_2 \tau_4 \ldots \tau_{n-1}, \hspace{1cm} \text{for odd,}$$

$$\delta_{n, odd} := \tau_1 \tau_3 \ldots \tau_{n-1}, \hspace{1cm} \delta_{n, even} := \tau_2 \tau_4 \ldots \tau_{n-2}, \hspace{1cm} \text{for even.}$$

The elements $C, \gamma_n, \delta_{n, odd}, \delta_{n, even}$ satisfy the following relation.

**Lemma 7.19** ([TV19a], Lemma 6.3). We have the following identity in $B_n$:

$$\delta_{n, even} \delta_{n, odd} \gamma_n = \gamma_n C.$$  \hspace{1cm} (7.12)
7.6. **Exceptional bases** $Q_k, Q'_k, Q''_k, \tilde{Q}_k, \tilde{Q}'_k, \tilde{Q}''_k$. For any $k \in \mathbb{Z}$, we define the basis $Q_k$ of solutions of the joint system (5.25)-(5.26) to be the basis
\[ Q_k := (\Psi^{k+n-1}, \ldots, \Psi^{k+1}, \Psi^k). \] (7.13)

**Lemma 7.20.** The basis $Q_k$ is an exceptional basis of $\mathcal{S}_n$. Via the isomorphism $\theta: K_0^T(\mathbb{P}^{n-1})_\mathbb{C} \to \mathcal{S}_n$, it is identified with the exceptional basis of $K_0^T(\mathbb{P}^{n-1})_\mathbb{C}$
\[ ([\mathcal{O}(-k - n + 1)], \ldots, [\mathcal{O}(-k - 1)], [\mathcal{O}(-k)]), \]
obtained from the Beilinson basis $([\mathcal{O}(i)])_{i=0}^{n-1}$ and twisting it by $- \otimes [\mathcal{O}(-k - n + 1)]$.

**Proof.** It follows from Corollary 7.16. \qed

For any $k \in \mathbb{Z}$, we define the exceptional bases $Q'_k$ and $Q''_k$ through the mutations
\[ Q'_k := \gamma_n Q_k, \quad Q''_k := \delta_{n, \text{odd}} Q'_k. \] (7.14)

**Proposition 7.21** ([TV19a], Lemma 6.6, Corollary 7.2). The basis $Q_k$ and $Q_{k-1}$ are related by the so-called modified Coxeter map: this means that $Q_{k-1}$ is obtained from $CQ_k$ by multiplying its last element by $(-1)^{n+1}\mathcal{S}_n(\mathbb{Z}^{-1})$. Moreover, the basis $Q'_{k-1}$ is obtained from the basis $\delta_{n, \text{even}} Q''_k$ by multiplying its last vector by $(-1)^{n+1}\mathcal{S}_n(\mathbb{Z}^{-1})$.

**Remark 7.22.** Our bases $Q_k, Q'_k, Q''_k$ have the same elements of the bases $Q_{k-1}, Q'_{k-1}, Q''_{k-1}$ of [TV19a], but ordered in the opposite way. See Remark 3.3.

Introduce three more families of exceptional bases of $\mathcal{S}_n$, denoted by $\tilde{Q}_k, \tilde{Q}'_k, \tilde{Q}''_k$. We start by defining
\[ \tilde{Q}_0 := Q_0, \quad \tilde{Q}_k := C^{-k} \tilde{Q}_0, \quad k \in \mathbb{Z}. \]

For any $k \in \mathbb{Z}$, define
\[ \tilde{Q}'_k := \gamma_n \tilde{Q}_k, \quad \tilde{Q}''_k := \delta_{n, \text{odd}} \tilde{Q}'_k. \] (7.15)

Using identity (7.12), we deduce
\[ \tilde{Q}'_{k-1} = \delta_{n, \text{even}} \tilde{Q}''_k, \quad k \in \mathbb{Z}. \]

We represent the relationship between the bases $\tilde{Q}_k, \tilde{Q}'_k, \tilde{Q}''_k$ through the diagram
\[ \begin{array}{ccc}
\tilde{Q}'_k & \xrightarrow{\delta_{n, \text{odd}}} & \tilde{Q}''_k \\
\gamma_n \downarrow & & \gamma_n \downarrow \\
\tilde{Q}_k & \xrightarrow{C} & \tilde{Q}_{k-1}
\end{array} \] (7.16)

which is commutative by Lemma 7.19.

**Remark 7.23.** Notice that the pre-images of the objects of the bases $\tilde{Q}_0, \tilde{Q}_k$ via the isomorphism $\theta: K_0^T(\mathbb{P}^{n-1})_\mathbb{C} \to \mathcal{S}_n$ are $K$-classes of two parallel foundations of the helix generated by the Beilinson exceptional collection $(\mathcal{O}(-n+1), \ldots, \mathcal{O}(-1), \mathcal{O})$. 

For any \( \ell, m \in \mathbb{Z} \) such that \( 0 \leq m - \ell \leq n \), denote
\[
\Psi^m(\ell) := \Psi^m - s_1(Z)\Psi^{m-1} + \cdots + (-1)^{m-\ell}s_{m-\ell}(Z)\Psi^\ell. \quad (7.17)
\]
From Section 6.3 of [TV19a], Remark 7.10 and Remark 7.22, we deduce the following explicit formulæ for \( Q'_k \) and \( Q''_k \).

If \( n = 2h + 1 \), we have
- the basis \( Q'_k \) is the basis in which the solutions \( \Psi^k, \ldots, \Psi^{k+h} \) stay at the positions \( 2h+1, 2h-1, \ldots, 1 \), and the solutions \( \Psi^{k+2h}(k+1), \Psi^{k+2h-1}(k+2), \ldots, \Psi^{k+h+1}(k+h) \) stay at the positions \( 2h, 2h-2, \ldots, 6, 4, 2 \), respectively;
- the basis \( Q''_k \) is the basis in which the solutions \( \Psi^k, \ldots, \Psi^{k+h} \) stay at the positions \( 2h, 2h-2, \ldots, 6, 4, 2, 1 \), and the solutions \( \Psi^{k+2h}(k), \Psi^{k+2h-1}(k+1), \ldots, \Psi^{k+h+1}(k+h-1) \) stay at the positions \( 2h+1, \ldots, 5, 3 \).

If \( n = 2h \), we have
- the basis \( Q'_k \) is the basis in which the solutions \( \Psi^k, \ldots, \Psi^{k+h} \) stay at the positions \( 2h, 2h-2, \ldots, 4, 2, 1 \), and the solutions \( \Psi^{k+2h-1}(k+1), \Psi^{k+2h-2}(k+2), \ldots, \Psi^{k+h+1}(k+h-1) \) stay at the positions \( 2h-1, \ldots, 7, 5, 3 \);
- the basis \( Q''_k \) is the basis in which the solutions \( \Psi^k, \ldots, \Psi^{k+h-1} \) stay at the positions \( 2h-1, \ldots, 5, 3, 1 \), and the solutions \( \Psi^{k+2h-1}(k), \Psi^{k+2h-2}(k+1), \ldots, \Psi^{k+h}(k+h-1) \) stay at the positions \( 2h, \ldots, 6, 4, 2 \).

The basis \( \tilde{Q}'_k, \tilde{Q}''_k \) can be obtained from the bases \( Q'_k, Q''_k \) by application of the following rule.

**Rule 7.24.** The basis \( \tilde{Q}'_k \) (respectively, \( \tilde{Q}''_k \)) is obtained by the basis \( Q'_k \) (resp., \( Q''_k \)) by substituting any solution \( \Psi^m(\ell) \) with
\[
((-1)^{n+1}s_n(Z))^a\Psi^m(\ell),
\]
where \( a \in \mathbb{Z} \) is such that
\[
0 \leq m + an \leq n - 1.
\]

**Example 7.25.** Let \( n = 5 \) and \( k = -1 \). We have
\[
Q'_{-1} = (\Psi^1, \Psi^2(1), \Psi^0, \Psi^3(0), \Psi^{-1}),
\]
\[
\tilde{Q}'_{-1} = (\Psi^1, \Psi^2(1), \Psi^0, \Psi^3(0), s_5(Z)\Psi^{-1}),
\]
\[
Q''_{-1} = (\Psi^1, \Psi^0, \Psi^2(0), \Psi^{-1}, \Psi^3(-1)),
\]
\[
\tilde{Q}''_{-1} = (\Psi^1, \Psi^0, \Psi^2(0), s_5(Z)\Psi^{-1}, \Psi^3(-1)).
\]

**Theorem 7.26.** Via the isomorphism \( \theta \colon K^T_0(\mathbb{P}^{n-1})_\mathbb{C} \rightarrow S_n \), the solution \( \Psi^m(\ell) \) corresponds (up to signs) to the \( K \)-class of the exceptional object
\[
\bigwedge^{m-\ell} T(-m) := \left(\bigwedge^{m-\ell} T\right) \otimes \mathcal{O}(-m),
\]
where \( T \) denotes the tangent sheaf of \( \mathbb{P}^{n-1} \) endowed with its natural \( T \)-equivariant structure.
Proof. Let $V = \mathbb{C}^n$ be the diagonal representation on $\mathbb{T}$ described in Section 3.2. Consider the well known Euler exact sequence, together with its exterior powers
\begin{equation}
0 \longrightarrow \mathcal{O} \longrightarrow V \otimes \mathcal{O}(1) \longrightarrow \mathcal{T} \longrightarrow 0,
\end{equation}
\begin{equation}
0 \longrightarrow \mathcal{T} \longrightarrow \Lambda^2 V \otimes \mathcal{O}(2) \longrightarrow \Lambda^2 \mathcal{T} \longrightarrow 0,
\end{equation}
\vdots
\begin{equation}
0 \longrightarrow \Lambda^{h-1} \mathcal{T} \longrightarrow \Lambda^h V \otimes \mathcal{O}(h) \longrightarrow \Lambda^h \mathcal{T} \longrightarrow 0,
\end{equation}
\vdots
\begin{equation}
0 \longrightarrow \Lambda^{n-2} \mathcal{T} \longrightarrow \Lambda^{n-1} V \otimes \mathcal{O}(n-1) \longrightarrow \mathcal{O}(n) \longrightarrow 0.
\end{equation}
Each morphism in (7.18) is $\mathbb{T}$-equivariant. In equivariant $K$-theory we have
\begin{equation}
\left[ \Lambda^h \mathcal{T} \right] = \left[ \Lambda^h V \otimes \mathcal{O}(h) \right] - \left[ \Lambda^{h-1} \mathcal{T} \right] = s_h(\mathbb{Z})[\mathcal{O}(h)] - \left[ \Lambda^{h-1} \mathcal{T} \right],
\end{equation}
for $h = 1, \ldots, n$. By induction, we obtain
\begin{equation}
\left[ \Lambda^h \mathcal{T} \right] = \pm \sum_{j=0}^{h} (-1)^j s_j(\mathbb{Z})[\mathcal{O}(j)],
\end{equation}
where the sign is $(+)$ for even $h$, and $(-)$ for odd $h$. The result follows from identity (7.17).

Corollary 7.27. For any $k \in \mathbb{Z}$, via the isomorphism $\theta: K^\mathbb{T}_0(\mathbb{P}^{n-1})_\mathbb{C} \rightarrow \mathcal{S}_n$, the bases $Q'_k, Q''_k$ correspond (up to signs) to the $K$-class of the following $\mathbb{T}$-full exceptional collections:

1. If $n$ is odd, the basis $Q'_k$ corresponds to
\begin{align*}
&\mathcal{O}\left(-k - \frac{n-1}{2}\right), \mathcal{T}\left(-k - \frac{n-1}{2} - 1\right), \mathcal{O}\left(-k - \frac{n-1}{2} + 1\right), \Lambda^2 \mathcal{T}\left(-k - \frac{n-1}{2} - 2\right), \ldots, \\
&\ldots, \Lambda^{n-4} \mathcal{T}\left(-k - n + 2\right), \mathcal{O}(k-1), \Lambda^{n-2} \mathcal{T}\left(-k - n + 1\right), \mathcal{O}(k),
\end{align*}
and the basis $Q''_k$ corresponds to
\begin{align*}
&\mathcal{O}\left(-k - \frac{n-1}{2}\right), \mathcal{O}\left(-k - \frac{n-1}{2} + 1\right), \Lambda^2 \mathcal{T}\left(-k - \frac{n-1}{2} - 1\right), \mathcal{O}\left(-k - \frac{n-1}{2} + 2\right), \ldots, \\
&\ldots, \mathcal{O}(k-1), \Lambda^{n-3} \mathcal{T}\left(-k - n + 2\right), \mathcal{O}(k), \Lambda^{n-1} \mathcal{T}\left(-k - n + 1\right).
\end{align*}

2. If $n$ is even, the basis $Q'_k$ corresponds to
\begin{align*}
&\mathcal{O}\left(-k - \frac{n}{2}\right), \mathcal{O}\left(-k - \frac{n}{2} + 1\right), \Lambda^2 \mathcal{T}\left(-k - \frac{n}{2} - 1\right), \mathcal{O}\left(-k - \frac{n}{2} + 2\right), \ldots, \\
&\ldots, \Lambda^{n-4} \mathcal{T}\left(-k - n + 2\right), \mathcal{O}(k-1), \Lambda^{n-2} \mathcal{T}\left(-k - n + 1\right), \mathcal{O}(k),
\end{align*}
and the basis $Q'_k$ corresponds to
\[
O\left(-k - \frac{n}{2} + 1\right), T\left(-k - \frac{n}{2}ight), O\left(-k - \frac{n}{2} + 2\right), \wedge^3 T\left(-k - \frac{n}{2} - 1\right), \ldots,
\]
\[
\ldots, O(-k-1), \wedge^{n-1} T\left(-k - n + 2\right), O(-k), \wedge^{n-2} T\left(-k - n + 1\right).
\]
The bases $\tilde{Q}'_k, \tilde{Q}''_k$ are obtained by $Q'_k, Q''_k$ by twisting the $\mathbb{T}$-equivariant structure of all objects $O(-m)$ and $\wedge^{m-\ell} T(-m)$, by tensoring them with the representation
\[
\bigwedge^{n-a} V \otimes \cdots \otimes \bigwedge^{n} V, \quad a \geq 0,
\]
\[
\bigwedge^{n-a} V^* \otimes \cdots \otimes \bigwedge^{n} V^*, \quad a < 0,
\]
where $a \in \mathbb{Z}$ is such that
\[
0 \leq m + an \leq n - 1,
\]
and $V \cong \mathbb{C}^n$ is the diagonal representation of $\mathbb{T}$.

7.7. Asymptotic expansions of bases $Q'_k$ and $Q''_k$ on sectors $\mathcal{V}'_k$ and $\mathcal{V}''_k$. From the integral representations (7.10), it is easy to obtain the following asymptotic expansion, by applying the steepest descent method. Following the notations of [TV19a], let us introduce coordinates $(r, \phi)$ on the universal cover $\hat{\mathbb{C}}^*$ of the punctured $s$-plane $\mathbb{C}^*$ as follows:
\[
q = s^n, \quad s = r e^{-2\pi\sqrt{-1}\phi}, \quad r > 0, \quad \phi \in \mathbb{R}.
\]

Lemma 7.28 ([TV19a], Lemma 5.1). For $m \in \mathbb{Z}$ and $\phi \in \mathbb{R}$ such that
\[
\frac{m}{n} - 1 < \phi < \frac{m}{n},
\]
we have the asymptotic expansion as $s \to \infty$
\[
\Psi^m(s^n, y, z) = \frac{(2\pi)^{\frac{m-1}{4}}}{\sqrt{n}} e^{\pi\sqrt{-1}\sum_{i=1}^{n} z_i} \left(e^{-\pi\sqrt{-1}\sum_{i=1}^{n} \zeta_m s} \right) \sum_{i=1}^{n} z_i + \frac{n+1}{2} e^{\pi\sqrt{-1}\sum_{i=1}^{n} z_i + \frac{n+1}{2}} \left(1 + O\left(\frac{1}{s}\right)\right),
\]
where $\arg\left(e^{-\pi\sqrt{-1}\sum_{i=1}^{n} \zeta_m s}\right) = 2\pi\frac{m}{n} - \pi - 2\pi\phi$, so that $|\arg\left(e^{-\pi\sqrt{-1}\sum_{i=1}^{n} \zeta_m s}\right)| < \pi$.

Consider the sectors
\[
\mathcal{V}'_k := \left\{ s \in \hat{\mathbb{C}}^* : \frac{2k - n - 1}{2n} < \phi < \frac{k}{n} \right\},
\]
\[
\mathcal{V}''_k := \left\{ s \in \hat{\mathbb{C}}^* : \frac{2k - n - 2}{2n} < \phi < \frac{2k - 1}{2n} \right\},
\]
for $k \in \mathbb{Z}$.

Let us recall the main result of [TV19a] concerning the asymptotic expansion of the bases $Q'_k$, $Q''_k$.

**Theorem 7.29** ([TV19a], Theorem 7.1). The elements of the basis of solutions $Q'_k$ (resp. $Q''_k$) can be reordered to a basis $(I_m(s^n, y, z))_{n=0}^{n-1}$ with asymptotic expansion

$$I_m(s^n, y, z) = \frac{(2\pi)^{\frac{n+1}{2}}}{\sqrt{n}} e^{\pi \sqrt{-1} \sum_{i=1}^{n} z_i} \left( e^{-\pi \sqrt{-1} \sum_{i=1}^{n} \frac{s_i}{m} s} \sum_{i=1}^{n} z_i + \frac{n+1}{2} \right) e^{ns\zeta_m} \left( 1 + O \left( \frac{1}{s} \right) \right),$$

(7.27)

for $s \to \infty$ and $s \in \mathcal{V}'_k$ (resp $s \in \mathcal{V}''_k$). Here, for defining $e^{-\pi \sqrt{-1} \sum_{i=1}^{n} \frac{s_i}{m} s}$, the following choice of the branch of $\log \left( e^{-\pi \sqrt{-1} \sum_{i=1}^{n} \frac{s_i}{m} s} \right)$ is done: for every $m$ the argument of

$$e^{-\pi \sqrt{-1} \sum_{i=1}^{n} \frac{s_i}{m} s}$$

is chosen so that $|\arg(e^{-\pi \sqrt{-1} \sum_{i=1}^{n} \frac{s_i}{m} s})| < \pi$ when $\phi$ tends

1. to $\frac{k}{n}$ inside $\left( \frac{2k-n-1}{2n}, \frac{k}{n} \right)$, for the case of $Q'_k$,
2. to $\frac{2k-1}{2n}$ inside $\left( \frac{2k-n-2}{2n}, \frac{2k-1}{2n} \right)$, for the case of $Q''_k$.

In both cases, the argument of $e^{-\pi \sqrt{-1} \sum_{i=1}^{n} \frac{s_i}{m} s}$ is continuous for $\phi$ in the intervals above.

In terms of the bases $\tilde{Q}'_k$ and $\tilde{Q}''_k$ we can recast this result as follows.

**Corollary 7.30.** The elements of the basis of solutions $\tilde{Q}'_k$ (resp. $\tilde{Q}''_k$) can be reordered to a basis $(I_m(s^n, y, z))_{n=0}^{n-1}$ with asymptotic expansion

$$I_m(s^n, y, z) = \frac{(2\pi)^{\frac{n+1}{2}}}{\sqrt{n}} e^{-\pi \sqrt{-1} \sum_{i=1}^{n} \frac{s_i}{m} s} \left( \sum_{i=1}^{n} z_i + \frac{n+1}{2} \right) e^{ns\zeta_m} \left( 1 + O \left( \frac{1}{s} \right) \right),$$

(7.28)

for $s \to \infty$ and $s \in \mathcal{V}'_k$ (resp $s \in \mathcal{V}''_k$). Here for defining $\sum_{i=1}^{n} \frac{s_i}{m} s$ with asymptotic expansion

$$e^{-\pi \sqrt{-1} \sum_{i=1}^{n} \frac{s_i}{m} s}$$

the principal determination of the argument of $\sum_{i=1}^{n} \frac{s_i}{m} s$ is chosen, i.e. $\arg(\sum_{i=1}^{n} \frac{s_i}{m} s) = 2\pi \frac{m}{n} - 2\pi \phi$.

**Proof.** From Lemma 7.28 and Theorem 7.29, we have that, for any $m \in \mathbb{Z}$, the element $\Psi_m(\ell)$ of $Q'_k$ admits the following expansion on $\mathcal{V}'_k$:

$$\Psi_m(\ell)(s^n, y, z) = \frac{(2\pi)^{\frac{n+1}{2}}}{\sqrt{n}} e^{\pi \sqrt{-1} s_1(z)} e^{-\pi \sqrt{-1} \frac{s_1}{m} s} \left( \sum_{i=1}^{n} \frac{s_i}{m} s \right) e^{ns\zeta_m} \left( 1 + O \left( \frac{1}{s} \right) \right),$$

where $a \in \mathbb{Z}$ is such that

$$0 \leq m + an \leq n - 1.$$
Thus, we have the following asymptotic expansion for $s \to \infty$ and $s \in \mathcal{V}_k'$:

$$(c_n^{an})^{s_1(z) + \frac{n-1}{2}} \Psi^m(\ell)(s^n, y, z) = \frac{(2\pi)^{n-1}}{\sqrt{n}} e^{-\pi \sqrt{-1} \frac{n-1}{2} \left(\zeta^{m+an}_n s_1(z) + \frac{n-1}{2} e^{ns\zeta^m_n}\right)} \left(1 + O\left(\frac{1}{s}\right)\right).$$

Notice that

$$(c_n^{an})^{s_1(z) + \frac{n-1}{2}} \Psi^m(\ell) = \left((-1)^{n-1} s_n(\hat{Z})\right)^a \Psi^m(\ell)$$

is the element of $\bar{Q}'_k$ obtained the element $\Psi^m(\ell)$ of $Q'_k$ by applying the Rule 7.24. This proves the statement for $\bar{Q}'_k$. The same argument applies for $\bar{Q}''_k$. \hfill \Box

8. CENTRAL CONNECTION MATRICES AND $B$-CLASSES

In this Section we prove one of the main result of the paper, the $B$-Theorem. This is an equivariant analogue, for $\mathbb{P}^{n-1}$, of point (3.b) of the statement of Conjecture 5.2 of [CDG18], and Gamma-Conjecture II of [GGI16]. It is also a refinement of the Gamma Theorem of [TV19b].

8.1. Morphism $B$.

**Lemma 8.1.** Let $V \in \mathcal{K}_{T_0}(\mathbb{P}^{n-1}) \subset \mathbb{C}$ and $Q(Z) \in \mathbb{C}[Z^{\pm 1}]$. We have

$$\text{Ch}_T(Q(Z)V) = Q(\hat{Z})\text{Ch}_T(V).$$

**Proof.** By additivity it is sufficient to prove the theorem for a monomial $Q(Z) = Z^{a_1} \cdots Z^{a_n}$. If $(\xi_i)_i$ are the equivariant Chern roots of $V$, then $(\xi_i + \sum_{j=1}^n \alpha_j z_j)_i$ are the equivariant Chern roots of $Q(Z)V$. \hfill \Box

**Definition 8.2.** We define the morphism of complex vector spaces

$$B: \mathcal{K}_{T_0}(\mathbb{P}^{n-1}) \subset \mathbb{C} \to H_{\Omega T_0}(\mathbb{P}^{n-1})$$

by

$$B(F) := \text{Ch}_T(F) \exp \left(\pi \sqrt{-1} \Gamma_{c_1(\mathbb{P}^{n-1})}\right) \hat{\Gamma}^+_{\mathbb{P}^{n-1}}. \quad (8.1)$$

8.2. $B$-Theorem. Consider the cohomology-valued functions $(\Psi_{\text{top},m})_{m=1,\ldots,n}$ defined by

$$\Psi_{\text{top},m}(q, z) := \sum_{\mu=0}^{n-1} [Y_{\text{top}}(q, z)]_m^\mu x_\mu.$$

The functions $\Psi_{\text{top},m}(q, z)$ define a basis of solutions of the equivariant quantum differential equation (5.25).

**Theorem 8.3** ($B$-Theorem). Let $X \in \mathcal{K}_{T_0}(\mathbb{P}^{n-1}) \subset \mathbb{C}$ and let $\theta(X) \in S_n$ be the corresponding solution of the joint system of equations (5.25) and (5.26). The meromorphic functions $h_{X,j}(z)$, with $j = 1, \ldots, n$, defined by the identity

$$\theta(X) = \sum_{j=1}^n h_{X,j}(z) \Psi_{\text{top},j}, \quad (8.2)$$
are the components of the equivariant cohomology class $B(X)$ wrt the basis $(x_\alpha)^n_{\alpha=0}$, i.e.

$$B(X) = \sum_{j=1}^{n} h_{X,j}(z)x_{j-1}. \quad (8.3)$$

**Proof.** We prove the Theorem for a basis in $K^T_0(\mathbb{P}^{n-1})_C$. The result follows by linearity and Lemma 8.1. Consider the basis $(\mathcal{O}(-k-n+1), \ldots, \mathcal{O}(-k-1), \mathcal{O}(-k))$ in $K^T_0(\mathbb{P}^{n-1})_C$. Its image in $S_n$ under the morphism $\theta$ is the basis $(\Psi^{k+n-1-m})^m_{m=0}$. For any $k \in \mathbb{Z}$, let $Y_{TV,k} = (Y_{TV,k})^\lambda_{\lambda,m} x_\lambda$ be the matrices defined by

$$\Psi^{k+n-1-m} = \sum_\lambda (Y_{TV,k})^\lambda_{\lambda,m} x_\lambda, \quad m = 0, \ldots, n-1.$$

For $z \in \Omega$, the matrix $Y_{TV,k}$ is a fundamental system of solutions of the equation (6.1). The central connection matrix $C_{TV,k}$ connecting the basis $Y_{TV,k}$ with the topological enumerative solution $Y_{top}$, i.e.

$$Y_{TV,k} = Y_{top} \cdot C_{TV,k},$$

is

$$C_{TV,k} = C \cdot \text{diag} \left( \exp(2\pi \sqrt{-1}(k+n-1-m)z_{m+1}) \right)^{n-1}_{m=0},$$

where $C$ is given by (7.3). This shows that $C_{TV,k}$ is the matrix of the morphism $B$ with respect the bases $(\mathcal{O}(-k-n+1), \ldots, \mathcal{O}(-k-1), \mathcal{O}(-k))$ and $(x_\alpha)^n_{\alpha=0}$. This concludes the proof. □

**Remark 8.4.** This is an important statement. The relation between $\theta(X)$ and the equivariant-topological solution is the equivariant version of the point 3.b of the statement of Conjecture 5.2 of [CDG18]. See also [Dub98], [KKP08], [GGI16]. Notice that our $B$-theorem is more general, in the sense that it is valid for arbitrary solutions in $S_n$, and not just elements of Stokes bases. Notice that this is also a refinement of Gamma Theorem of [TV19b].

Consider the space $\text{Sol}_n$ of solutions $I(q, z)$ of the equivariant quantum differential equation (5.25) that are holomorphic w.r.t. $z$ in $\Omega$. The space $\text{Sol}_n$ is a module over $\mathcal{O}_\Omega$. Since elements of $S_n$ can be seen as element of $\text{Sol}_n$, the isomorphism $\theta : K^T_0(\mathbb{P}^{n-1}) \to S_n$ induces a map

$$\bar{\theta} : K^T_0(\mathbb{P}^{n-1}) \to \text{Sol}_n.$$

Denote by $S(q)$ the specialization of the $S$-operator $S(t, \kappa)$ to the points

$$t^\alpha = 0, \quad \alpha \neq 1, \quad t^1 = \log q, \quad \kappa = 1.$$

The operator $S(q)$ allows us to associate an element of $\text{Sol}_n$ to any element of $H^*_T(\mathbb{P}^{n-1}, \mathbb{C})$. By extension of scalars, there is an induced morphism

$$S(q) : H^\Omega_\ast(\mathbb{P}^{n-1}) \to \text{Sol}_n.$$
Corollary 8.5. The following diagram is commutative:

\[
\begin{array}{ccc}
K_0^\mathbb{C}(\mathbb{P}^{n-1}) & \xrightarrow{E} & H_\mathbb{C}^0(\mathbb{P}^{n-1}) \\
\downarrow{\theta} & & \downarrow{S(q)} \\
\text{Sol}_n & & \\
\end{array}
\]

9. Formal Solutions of the System of \(qDE\) and \(qKZ\) equations

In the next two Sections we prove the following Theorem.

Theorem 9.1. Consider the joint system of \(qDE\) and \(qKZ\) equations for \(\mathbb{P}^{n-1}\). As \(q \to \infty\), this system is formally equivalent, up to gauge equivalence and change of variable \(q = s^n\), to the system

\[
\frac{dZ}{ds} = UZ, \quad (9.1)
\]

\[
Z(s, z_1, \ldots, z_j - 1, \ldots, z_n) = K_j Z(s, z), \quad j = 1, \ldots, n, \quad (9.2)
\]

where

\[
U := \operatorname{diag}(n\zeta_0^n, \ldots, n\zeta^{n-1}_n),
\]

\[
K_j := \operatorname{diag}(\zeta_0^n, \zeta_1^n, \ldots, \zeta^{j-1}_n, \ldots, \zeta^{(n-1)}_n), \quad j = 1, \ldots, n,
\]

where \(\zeta_n := \exp\left(\frac{2\pi\sqrt{-1}}{n}\right)\).

That is, after a formal transformation, the system of \(qDE\) and \(qKZ\) equations becomes a system with constant coefficients and separated variables. Moreover, it splits into the union of systems of rank one.

System (9.1)-(9.2) admits the basis of solutions

\[
Z_i(s, z) = \exp\left(n\zeta_i^{i-1}s + \frac{2(i-1)\pi\sqrt{-1}}{n}\sum_{a=1}^{n} z_a\right)
\begin{pmatrix}
0 \\
\vdots \\
1_i \\
0
\end{pmatrix},
\]

for \(i = 1, \ldots, n\). All other solutions of system (9.1)-(9.2) are linear combinations of these basis solutions with 1-periodic coefficients in \(z_1, \ldots, z_n\).

The formal transformation which realizes the reduction to system (9.1)-(9.2), will be described in details in the following paragraphs.
9.1. Shearing transformation. The singularity \( q = \infty \) of the equivariant quantum differential equation (6.1) is irregular of Poincaré rank 1. It is known ([Was65], [MS16], [LR16]) that (6.1) admits a formal solution

\[
Y_{\text{form}}(q, z) = \Phi(q^\frac{1}{\nu}, z) q^{\Lambda(z)} \exp \left( P(q^\frac{1}{\nu}, z) \right),
\]

where

- \( \nu \in \mathbb{N}^* \), the degree of ramification of the singularity,
- \( \Phi \) is an \( n \times n \) matrix-valued formal power series in \( q^{-\frac{1}{\nu}} \) of the form
  \[
  \Phi(q^\frac{1}{\nu}, z) = \sum_{j=0}^{\infty} \Phi_j(z) q^{-j \cdot \frac{1}{\nu}}, \quad \text{det} \Phi_0(z) \neq 0,
  \]
- \( \Lambda \) is an \( n \times n \) matrix depending only on \( z \) (the so-called exponent of formal monodromy),
- \( P = \text{diag}(p_1, \ldots, p_n) \) where each \( p_j(q^{\frac{1}{\nu}}, z) \) is a polynomial in \( q^{\frac{1}{\nu}} \) of the form
  \[
  p_j(q^{\frac{1}{\nu}}, z) = \sum_{\ell=1}^{N_j} p_{j\ell}(z) q^{\ell \cdot \frac{1}{\nu}}, \quad N_j \geq 1.
  \]

To find the formal solution \( Y_{\text{form}} \), let us perform the gauge transformation of (6.1) defined by

\[
Y(q, z) = H(q) \cdot \tilde{T}(q, z), \quad H(q) := \text{diag} \left( 1, q^{-\frac{1}{\nu}}, \ldots, q^{-\frac{n-1}{\nu}} \right), \quad \tilde{T}(q, z) := \tilde{T}(s^n, z),
\]

called shearing transformation ([Was65], Section 19). The function \( \tilde{T} \) satisfies the following differential equation

\[
\frac{d\tilde{T}}{dq} = A_{sh}(q, z) \tilde{T}, \quad A_{sh} = \mathcal{H}^{-1} \cdot A \cdot \mathcal{H} - \mathcal{H}^{-1} \frac{d\mathcal{H}}{dq}. \quad (9.5)
\]

Explicitly, the entries of \( A_{sh} \) are given by

\[
(A_{sh})_{\alpha\beta} = q^{\frac{1-n}{n}} \delta_{\alpha-\beta,1} + q^{\frac{1-n}{n}} \delta_{\alpha,1} \delta_{\beta,n} \]
\[
+ \sum_{j=1}^{n} (-1)^{j+1} s_j(z) \delta_{\alpha+j,n+1} \delta_{\beta,n} q^{\frac{1-n-j}{n}} + \delta_{\alpha,\beta} \frac{1}{n} q^{-1},
\]

for \( \alpha, \beta = 1, \ldots, n \). With the change of variable \( q = s^n \), the function \( T(s, z) := \tilde{T}(s^n, z) \) is a solution of the equation

\[
\frac{d}{ds} T(s, z) = B(s, z) T(s, z), \quad B(s, z) := ns^{n-1} A_{sh}(s^n, z). \quad (9.6)
\]

**Lemma 9.2.** We have the following expansion for the coefficient \( B(s, z) \):

\[
B(s, z) = B_0 + \frac{1}{s} B_1(z) + \sum_{j=2}^{n} \frac{1}{s^j} B_j(z), \quad (9.7)
\]
where

\[
\mathcal{B}_0 := \begin{pmatrix}
0 & \cdots & 0 & n \\
n & 0 & \cdots & 0 \\
0 & n & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
n & 0 & \cdots & 0
\end{pmatrix},
\]

(9.8)

\[
\mathcal{B}_1(z) := \begin{pmatrix}
0 & 1 & 2 & \cdots & n-2 & n-1 + ns_1(z) \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix},
\]

(9.9)

\[
\mathcal{B}_j(z) := \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

(9.10)

By the shearing transformation (9.4) and the change of variable \( q = s^n \), we have reduced the equivariant quantum differential equation of \( \mathbb{P}^{n-1} \) to the equation

\[
\frac{dT}{ds} = \mathcal{B}(s, z)T.
\]

(9.11)

Equation (9.11) still admits an irregular singularity at \( s = \infty \) of Poincaré rank 1. There is a main difference between equations (6.1) and (9.11): the matrix of coefficients \( \mathcal{B}(s, z) \) has leading term \( \mathcal{B}_0 \) with distinct eigenvalues

\[
u_k := n\zeta_n^{k-1}, \quad \zeta_n := \exp\left(\frac{2\pi\sqrt{-1}}{n}\right), \quad k = 1, \ldots, n.
\]

(9.12)

On the contrary, the coefficient matrix \( \mathcal{A}(q, z) \) has a nilpotent leading term \( \mathcal{A}_0(z) \).

9.2. The \( \mathcal{D}_q \)-matrix. Let \((e_1, \ldots, e_n)\) be the standard basis of \( \mathbb{C}^n \). Consider on \( \mathbb{C}^n \) the symmetric and non-degenerate \( \mathbb{C} \)-bilinear form defined, wrt the standard basis, by the anti-diagonal matrix

\[
(\eta_{kl})_{\alpha\beta} = \delta_{n+1,\alpha+\beta}.
\]

(9.13)

For fixed \( z \in \mathbb{C}^n \), consider the \( \mathbb{C} \)-linear endomorphisms \( \mathcal{B}_0, \mathcal{B}_1(z) \in \text{End}(\mathbb{C}^n) \) defined, in the standard basis, by the matrices \( \mathcal{B}_0 \) and \( \mathcal{B}_1(z) \) of equations (9.8)-(9.9). Introduce the
matrix $D_q \in GL(n, \mathbb{C})$

$$
(D_q)_{i\alpha} := \frac{1}{\sqrt{n}} \exp \left( \frac{(i - 1)(2\alpha - 1)^{-1/2}}{n} \right), \quad \alpha, i = 1, \ldots, n, \\
(9.14)
$$

with inverse

$$
(D_q^{-1})_{i\alpha} = \frac{1}{\sqrt{n}} \exp \left( \frac{(i - 1)(1 - 2\alpha)^{-1/2}}{n} \right), \quad \alpha, i = 1, \ldots, n. \\
(9.15)
$$

**Lemma 9.3.** Define the basis $(f_1, \ldots, f_n)$ of $\mathbb{C}^n$ by

$$
f_j := \sum_{\alpha=1}^{n} (D_q^{-1})_{\alpha j} e_\alpha, \quad j = 1, \ldots, n. \\
(1)
$$

(1) The basis $(f_1, \ldots, f_n)$ is orthonormal wrt the bilinear form $\eta_{kl}$.

(2) The basis $(f_1, \ldots, f_n)$ is an eigenbasis of the operator $B_0$.

(3) For any fixed $z \in \mathbb{C}^n$, we have that

$$
\eta_{kl}(B_1(z)f_i, f_i) = s_1(z) + \frac{n - 1}{2}, \quad i = 1, \ldots, n. \\
(9.18)
$$

**Proof.** The statements are equivalent to the identities

$$
(D_q^{-1})^T \eta_{kl} D_q^{-1} = 1, \quad (9.16)
$$

$$
D_q B_0 D_q^{-1} = \text{diag}(u_1, \ldots, u_n), \quad (9.17)
$$

$$
D_q B_1(z) D_q^{-1} = \left( s_1(z) + \frac{n - 1}{2} \right) \cdot 1 + B^{\text{od}}(z), \quad (9.18)
$$

where $B^{\text{od}}$ is an off-diagonal matrix, i.e. $(B^{\text{od}})_{ii} = 0$. A straightforward computation shows the validity of these identities. \hfill \Box

**Remark 9.4.** Consider the analytic Frobenius manifold $M := QH^\bullet(\mathbb{P}^{n-1})$. The tangent space $T_0 M \cong H^\bullet(\mathbb{P}^{n-1}, \mathbb{C})$ can be identified with $\mathbb{C}^n$ by fixing the frame $(1, x, \ldots, x_{n-1})$, where $x$ denote the non-equivariant hyperplane class. Under this identification,

- the Gram matrix of the non-equivariant Poincaré metric coincides with the matrix $\eta_{kl}$ of equation (9.13),
- the matrix of the operator $U(0): T_0 M \to T_0 M$ of multiplication by the Euler vector field coincides with the matrix $B_0$ of equation (9.8).

Hence, the basis $(f_1, \ldots, f_n)$ above coincides with the orthonormalized idempotent frame introduced in the analytic theory of Frobenius manifolds (see Appendix A). Denoting by $(\pi_i)_{i=1}^n$ the idempotents of the Frobenius algebra $(T_0 M, \ast_0, \eta_{kl})$, the vector fields $f_i$ are defined as the normalized vectors

$$
f_i := \frac{1}{\eta(\pi_i, \pi_i)^{1/2}} \pi_i, \quad i = 1, \ldots, n, \\
(9.19)
$$
for some choice of signs of the square roots. In the standard notations of Dubrovin’s theory of Frobenius manifolds, the matrix $D_q$ is usually denoted by $\Psi$. Here we avoid this notation, the symbol $\Psi$ being already used for solutions of the joint systems (5.25)-(5.26).

We close this Section by commenting the close relationship of equation (9.11) and the isomonodromic system attached to the Frobenius manifold $QH^\bullet(\mathbb{P}^{n-1})$, described in Section 5.8.

**Proposition 9.5.** For $z = 0$, equation (9.11) is

$$\frac{d}{ds} T(s) = \left( B_0 + \frac{1}{s} B_1(0) \right) T(s). \quad (9.20)$$

If $T(s)$ is a solution of (9.20), then

$$Y(s) := s^{-\frac{n-1}{2}} D_q T(s)$$

is a solution of the equation

$$\frac{dY}{ds} = \left( U + \frac{1}{s} V \right) Y,$$

where $U$ and $V$ are the matrices introduced in Section 5.8, evaluated at the point $0 \in QH^\bullet(\mathbb{P}^{n-1})$.

**Proof.** if $z = 0$, all the coefficients $B_2, \ldots, B_n$ vanish, and the coefficient $B_1$ takes the form

$$B_1(0) = \text{diag}(0, 1, 2, \ldots, n) \quad (9.21)$$

Such a matrix is related to the matrix of the grading operator $\mu$ of the Frobenius manifold $QH^\bullet(\mathbb{P}^{n-1})$ (see equation (5.39)), written in coordinates wrt the basis $(\iota^* x_\alpha)_{\alpha=0}^{n-1}$, namely

$$\mu = \text{diag} \left( -\frac{n-1}{2}, -\frac{n-3}{2}, \ldots, 0, \ldots, \frac{n-3}{2}, \frac{n-1}{2} \right).$$

We have indeed that

$$B_1(0) - \frac{n-1}{2} = \mu.$$

As a consequence, the matrix $B^{od}$ of formula (9.18) in the classical limit is given by

$$B^{od}(0) = D_q \left( B_1(0) - \frac{n-1}{2} \right) D_q^{-1} = D_q \mu \ D_q^{-1} = V,$$

where the matrix $V$ is the antisymmetric matrix that appears in the coefficient of the isomonodromic system attached to $QH^\bullet(\mathbb{P}^{n-1})$, spacialized at the point $0 \in QH^\bullet(\mathbb{P}^{n-1})$:

$$\frac{dY}{ds} = \left( U + \frac{1}{s} V \right) Y.$$

The antisymmetry of $B^{od}$ however is lost for $z \neq 0$. \qed
9.3. **Formal Reduction of the System of qDE and qKZ equations.** By subsequent application of

- the shearing transformation (9.4),
- the change of variable \( q = s^n \),
- Theorem C.1 of Appendix C,

we obtain the following result.

**Theorem 9.6.** There exists a unique formal gauge \( G(s, z) \), of the form

\[
G(s, z) = \mathcal{H}(s^n)D_q^{-1}F(s, z)s^\Lambda(z),
\]

with

\[
\mathcal{H}(s^n) = \text{diag}(1, s^{-1}, \ldots, s^{-(n-1)}),
\]

\[
F(s, z) := 1 + \sum_{k=1}^{\infty} \frac{F_k(z)}{s^k},
\]

\[
\Lambda(z) := \left(s_1(z) + \frac{n-1}{2}\right) \cdot 1,
\]

such that the gauge transformation

\[
Y(s^n, z) = G(s, z)Z(s, z)
\]

maps the joint system of qDE and qKZ equations to the system of equations

\[
\frac{dZ}{ds} = UZ, \quad U = \text{diag}(\zeta_0, \ldots, \zeta_n^{n-1}),
\]

\[
Z(s, z_1, \ldots, z_j - 1, \ldots, z_n) = K_j(z)Z(s, z), \quad j = 1, \ldots, n, \quad K_j(z) \text{ diagonal.}
\]

Moreover, the coefficients \( F_k \)'s in (9.24) are polynomial in \( z \).

**Remark 9.7.** Denote by \( K_j(s^n, z) \) the matrix of the qKZ operators (5.14) wrt the basis \( (x_\alpha)_{\alpha=0}^{n-1} \). The formal gauge \( G(s, z) \) of Theorem 9.6 relates the qKZ-operators \( K_j \)'s with the modified ones \( \tilde{K}_j \)'s of (9.28) by the identity

\[
\tilde{K}_j(z) := sF(s, z_1, \ldots, z_j - 1, \ldots, z_n)^{-1}D_q\mathcal{H}(s^n)^{-1}K_j(s^n, z)\mathcal{H}(s^n)D_q^{-1}F(s, z).
\]

**Corollary 9.8.** The following identity holds true

\[
\tilde{K}_j(z) = \text{Res}_{s=0} \left( D_q \mathcal{H}(s) K_j(s^n, z) \mathcal{H}(s) D_q^{-1} \right),
\]

for any \( j = 1, \ldots, n. \)
Proof. Since the series $F(s, z)$ has the form (9.24), from (9.29) we deduce that
\[
K_j(z) = s \left( 1 + O \left( \frac{1}{s} \right) \right) \left( D_q H(s)^{-1} K_j(s^n, z) H(s) D_q^{-1} \right) \left( 1 + O \left( \frac{1}{s} \right) \right)
\]
\[= s \left( D_q H(s)^{-1} K_j(s^n, z) H(s) D_q^{-1} \right) \left( 1 + O \left( \frac{1}{s} \right) \right).
\]
Hence, we deduce that
\[
D_q H(s)^{-1} K_j(s^n, z) H(s) D_q^{-1} = \frac{K_j(z)}{s} + O \left( \frac{1}{s^2} \right).
\]
\[
\square
\]

9.4. Structure of formal solutions of the System of $qDE$ and $qKZ$ equations at $q = \infty$. In the previous paragraph, we have reduced, by a formal gauge equivalence, the original system of $qDE$ and $qKZ$ equations to a simplified system with diagonal coefficients (thus a union of systems of rank one) and with separated variables:
\[
dZ = UZ, \tag{9.31}
\]
\[
Z(s, z_1, \ldots, z_j - 1, \ldots, z_n) = K_j Z(s, z), \quad j = 1, \ldots, n, \tag{9.32}
\]
where the coefficients $K_j$ are (a priori) polynomial\(^7\) in $z$. Fundamental systems of solutions of the joint system (9.31)-(9.32) can be found of the form
\[
\tilde{Z}(s, z) = Z(s) C(z),
\]
where $\tilde{Z}(s)$ is a basis of solutions for (9.31) and $C(z)$ is a basis of solutions for (9.32). Moreover, being the coefficients of the system in diagonal form, we can assume $\tilde{Z}(s)$ and $C(z)$ to be diagonal.

**Theorem 9.9.** Consider the joint system of $qDE$ and $qKZ$ equations for $\mathbb{P}^{n-1}$, written in the basis $(x_\alpha)_{\alpha=0}^{n-1}$, namely
\[
d_Y^n Y(s^n, z) = n s^{n-1} A(s^n, z) Y(s^n, z), \tag{9.33}
\]
\[
Y(s^n, z_1, \ldots, z_j - 1, \ldots, z_n) = K_j(s^n, z) Y(s^n, z). \tag{9.34}
\]
This system admits a formal fundamental solution of the form
\[
Y_{\text{form}}(s^n, z) = H(s^n) D_q^{-1} F(s, z) s^\Lambda(z) e^{Us}, \tag{9.35}
\]
with
\[
F(s, z) = C(z) + \sum_{j=1}^{\infty} \frac{F_j(z)}{s^j}, \quad \text{C(z) diagonal}, \tag{9.36}
\]
if and only if $C(z)$ is a fundamental system of solutions of difference equations (9.32). Moreover, if such a formal solution exists, then it is unique.

\(^7\)Later we will show that actually all the operators $K_j$ are equal and constant. See Corollary 10.12.
Proof. The system (9.33)-(9.34) admits a unique formal solution of the form

$$Y_{\text{form}}(s^n, z) = H(s^n)D_q^{-1} \left( 1 + \sum_{k=1}^{\infty} \frac{F_k(z)}{s^k} \right) s^{\Lambda(z)} e^{Us} C(z),$$  \hspace{1cm} (9.37)

if and only if $C(z)$ is a basis of solutions of (9.32). If $C(z)$ is diagonal, then

$$[s^{\Lambda(z)} e^{Us}, C(z)] = 0,$$

and (9.37) can be re-written in the form (9.35). □

Remark 9.10. If we consider the system of $qDE$ and $qKZ$ equations written in matrix notation wrt a generic basis $(T_\alpha)_{\alpha=0}^{n-1}$ of $H_{\mathbb{C}}^\bullet(P^{n-1}, \mathbb{C})$, then the structure of the formal solution (9.35) is

$$Y_{\text{form}}(s^n, z) = J^{-1}H(s^n)D_q^{-1}F(s, z)s^{\Lambda(z)} e^{Us},$$

where $J = (J^\alpha_\beta)_{\alpha, \beta}$ is the matrix defined by

$$T_\beta = \sum_{\alpha} J^\alpha_\beta x_\alpha, \hspace{0.5cm} \beta = 0, \ldots, n - 1.$$

Example 9.11. Let us consider the case of $P^1$. The original system of $qDE$ and $qKZ$ equations is the following

$$\frac{d}{dq} Y(q, z) = \frac{1}{q} \begin{pmatrix} 0 & q - z_1 z_2 \\ 1 & z_1 + z_2 \end{pmatrix} Y(q, z),$$

$$Y(q, z_1 - 1, z_2) = \frac{1}{q} \begin{pmatrix} -z_2 & q - z_1 z_2 \\ 1 & z_1 \end{pmatrix} Y(q, z),$$

$$Y(q, z_1, z_2 - 1) = \frac{1}{q} \begin{pmatrix} -z_1 & q - z_1 z_2 \\ 1 & z_2 \end{pmatrix} Y(q, z).$$

Through a formal gauge transformation $Y(s^2, z) = G(s, z)Z(s, z)$, the system above can be reduced to the system

$$\frac{d}{ds} Z(s, z) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} Z(s, z),$$

$$Z(s, z_1 - 1, z_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Z(s, z),$$

$$Z(s, z_1, z_2 - 1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Z(s, z).$$

The formal gauge $G(s, z)$ is given by

$$G(s, z) = H(s^2)D_q^{-1} \left( 1 + \frac{F_1(z)}{s} + \frac{F_2(z)}{s^2} + \ldots \right) s^{\Lambda(z)},$$

where

$$H(s^2) = \text{diag} \left( 1, \frac{1}{s} \right), \hspace{0.5cm} D_q^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}, \hspace{0.5cm} \Lambda(z) = \left( \frac{1}{2} + z_1 + z_2 \right) \mathbb{1},$$
and the coefficients can be computed recursively as in the proof of Theorem 9.6. Here we give just the first four coefficients $F_1, F_2, F_3, F_4$:

$$F_1 = \begin{pmatrix} F_{1,1} & F_{1,2} \\ F_{1,2} & -F_{1,1} \end{pmatrix},$$

where

$$F_{1,1} = s_2(z) - \frac{1}{16} (2s_1(z) + 1)^2, \quad F_{1,2} = -\frac{\sqrt{-1}}{8} (2s_1(z) + 1),$$

$$F_2 = \begin{pmatrix} F_{2,1} & F_{2,2} \\ -F_{2,2} & F_{2,1} \end{pmatrix},$$

where

$$F_{2,1} = \frac{1}{512} \left( 4s_1(z)^2 - 16s_2(z) - 1 \right) \left( 4s_1(z)(s_1(z) + 2) - 16s_2(z) + 3 \right),$$

$$F_{2,2} = -\frac{\sqrt{-1}}{128} \left( 2s_1(z) + 3 \right) \left( 4s_1(z)^2 - 16s_2(z) - 1 \right),$$

$$F_3 = \begin{pmatrix} F_{3,1} & F_{3,2} \\ F_{3,2} & -F_{3,1} \end{pmatrix},$$

where

$$F_{3,1} = -\frac{(4s_1(z)^2 - 16s_2(z) - 9) (4s_1(z)^2 - 16s_2(z) - 1) (4s_1(z)(s_1(z) + 3) - 16s_2(z) + 5)}{24576},$$

$$F_{3,2} = -\frac{\sqrt{-1}}{4096} \left( 2s_1(z) + 5 \right) \left( 4s_1(z)^2 - 16s_2(z) - 9 \right) (4s_1(z)^2 - 16s_2(z) - 1),$$

$$F_4 = \begin{pmatrix} F_{4,1} & F_{4,2} \\ -F_{4,2} & F_{4,1} \end{pmatrix},$$

where

$$F_{4,1} = \frac{1}{1572864} \left( 4s_1(z)^2 - 16s_2(z) - 25 \right) \left( 4s_1(z)(s_1(z) + 4) - 16s_2(z) + 7 \right) \cdot \left( 4s_1(z)^2 - 16s_2(z) - 1 \right),$$

$$F_{4,2} = -\frac{\sqrt{-1}}{196608} \left( 2s_1(z) + 7 \right) \left( 4s_1(z)^2 - 16s_2(z) - 25 \right) \left( 4s_1(z)^2 - 16s_2(z) - 9 \right) \cdot \left( 4s_1(z)^2 - 16s_2(z) - 1 \right).$$

Notice that Corollary 9.8 allow us to compute directly the coefficients $K_1, K_2$:

$$\mathcal{D}_q \mathcal{H}(s)^{-1} K_1(s^2, z) \mathcal{H}(s) \mathcal{D}_q^{-1} = \begin{pmatrix} \frac{z_1 - z_2}{2s} + \frac{1}{s} & \frac{\sqrt{-1}(z_1 + z_2)}{2s^2} & -\frac{\sqrt{-1\sqrt{z_1 z_2}}}{2s^3} \\ -\frac{\sqrt{-1\sqrt{z_1 z_2}}}{2s^3} & -\frac{z_1 z_2}{2s^2} + \frac{1}{s} & \frac{\sqrt{-1}(z_1 + z_2)}{2s^2} \\ -\frac{\sqrt{-1\sqrt{z_1 z_2}}}{2s^3} & -\frac{\sqrt{-1}(z_1 + z_2)}{2s^2} & \frac{z_1 - z_2}{2s} + \frac{1}{s} \end{pmatrix},$$

$$\mathcal{D}_q \mathcal{H}(s)^{-1} K_2(s^2, z) \mathcal{H}(s) \mathcal{D}_q^{-1} = \begin{pmatrix} \frac{z_1 - z_2}{2s} + \frac{1}{s} & \frac{\sqrt{-1}(z_1 + z_2)}{2s^2} & -\frac{\sqrt{-1\sqrt{z_1 z_2}}}{2s^3} \\ -\frac{\sqrt{-1\sqrt{z_1 z_2}}}{2s^3} & -\frac{z_1 z_2}{2s^2} + \frac{1}{s} & \frac{\sqrt{-1}(z_1 + z_2)}{2s^2} \\ -\frac{\sqrt{-1\sqrt{z_1 z_2}}}{2s^3} & -\frac{\sqrt{-1}(z_1 + z_2)}{2s^2} & \frac{z_1 - z_2}{2s} + \frac{1}{s} \end{pmatrix}. $$
By Corollary 9.8, we obtain that

\[ K_1(z) = K_2(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Notice, in particular, that both \( K_1 \) and \( K_2 \) are equal and constant matrices: in Corollary 10.12 we will prove that this is a general property valid for all projective spaces \( \mathbb{P}^{n-1} \).

10. Stokes bases of the System of \( qDE \) and \( qKZ \) equations

10.1. Stokes rays, Stokes sectors. Although the solution \( Y_{\text{form}} \) described in Theorem 9.9 is purely formal (the series \( F(s, z) \) being not convergent, in general), it still contains information about genuine solutions of the equivariant quantum differential equation (6.1). The formal solutions prescribes indeed the asymptotics of genuine fundamental solutions of (6.1).

Following the notations of [TV19a], let us introduce coordinates \( (r, \phi) \) on the universal cover \( \hat{\mathbb{C}}^* \) of the punctured \( s \)-plane \( \mathbb{C}^* \) as follows:

\[ q = s^n, \quad s = re^{-2\pi\sqrt{-1}\phi}, \quad r > 0, \quad \phi \in \mathbb{R}. \]  

(10.1)

**Definition 10.1.** We call *Stokes ray* any ray in the universal cover \( \hat{\mathbb{C}}^* \) of the \( s \)-plane, defined by the equation

\[ \phi = \frac{k}{2n}, \quad k \in \mathbb{Z}. \]  

(10.2)

We will denote the ray (10.2) by \( R_k \).

**Remark 10.2.** The Stokes rays (10.2) coincide with the rays on which at least two of the exponentials \( e^{su_1}, e^{su_j} \) are asymptotically equivalent, i.e. along which

\[ \text{Re}(\zeta_n^{m_1}s) = \text{Re}(\zeta_n^{m_2}s), \quad \zeta_n = \exp\left(\frac{2\pi\sqrt{-1}}{n}\right), \]  

(10.3)

and \( m_1 \neq m_2 \) (mod \( n \)). See [TV19a] Section 5.3 and [CDG18] Section 6.4.

**Definition 10.3.** We call *Stokes sector* any open sector in \( \hat{\mathbb{C}}^* \) which contains exactly \( n \) consecutive Stokes rays \( R_k, \ldots, R_{k+n} \) for some \( k \in \mathbb{Z} \).

**Lemma 10.4.** Any open sector \( \mathcal{V} \subset \hat{\mathbb{C}}^* \) of width \( \pi + \delta \), i.e. of the form

\[ \mathcal{V} = \left\{ s \in \hat{\mathbb{C}}^*: a - \frac{1}{2} - \delta < \phi < a \right\}, \quad a \in \mathbb{R}, \]  

(10.4)

is a Stokes sector for \( \delta > 0 \) sufficiently small.

From the general theory of differential equations (see [Was65],[Sib90],[Fed87],[FIKN06]), we deduce the following result.
Theorem 10.5. Let $\mathcal{V} \subseteq \hat{\mathbb{C}}^*$ be a Stokes sector. There exists a unique solution $Y(s^n, z)$ of the equivariant quantum differential equation (6.1) satisfying the asymptotic condition

$$Y(s^n, z) \sim Y_{\text{form}}(s^n, z), \quad s \to \infty, \quad s \in \mathcal{V},$$

uniformly on compact subsets of $\Omega$. Here $Y_{\text{form}}(s^n, z)$ denotes the unique solution described in Theorem 9.9. The asymptotic expansion (10.5) can actually be extended to a sector wider than $\mathcal{V}$, up to the nearest Stokes rays.

Remark 10.6. In the same notations of Theorem 9.9, the precise meaning of the asymptotic relation (10.5) is the following:

$$\forall K \in \Omega, \forall h \in \mathbb{N}, \forall \mathcal{V}' \subseteq \mathcal{V}, \exists C_{K,h,\mathcal{V}}>0: \text{if } s \in \mathcal{V}' \setminus \{0\} \text{ then }$$

$$\sup_{z \in K} \left\| D_{\mathcal{A}} \cdot \mathcal{H}(s)^{-1} \cdot Y(s, z) \cdot \exp(-sU) s^{-\Lambda(z)} - \sum_{m=0}^{h-1} \frac{F_m(z)}{s^m} \right\| < C_{K,h,\mathcal{V}} |s|^h.$$ 

Here $\mathcal{V}$ denotes any unbounded closed sector of $\hat{\mathbb{C}}^*$ with vertex at 0, and $F_0(z) = C(z)$. Here, for defining $s^{\Lambda(z)}$, the principal branch of $\log s$ is chosen.

Lemma 10.7. The sectors $\mathcal{V}'_k, \mathcal{V}''_k$, defined by (7.25)-(7.26), are maximal Stokes sectors wrt to the inclusion, i.e.

1. they are Stokes sectors,
2. any Stokes sector $\mathcal{V}$ is contained in one (and only one) $\mathcal{V}'_k$ or $\mathcal{V}''_k$.

10.2. Stokes bases and Stokes matrices.

Definition 10.8. We say that a basis $(I_i(s^n, y, z))_{i=1}^n$ of solutions of the joint system (5.25) and (5.26) is a Stokes basis on a sector $\mathcal{V}$ if it can be reordered in such a way that the corresponding matrix-valued solution $Y(s^n, z)$, defined by

$$I_\beta(s^n, y, z) = \sum_{\alpha=0}^{n-1} Y(s^n, z)^\alpha_{\beta} x_\alpha, \quad \beta = 1, \ldots, n$$

satisfies the asymptotic expansion

$$Y(s^n, z) \sim Y_{\text{form}}(s^n, z), \quad s \to \infty, \quad s \in \mathcal{V},$$

uniformly on compact subsets of $\Omega$. Here the formal solution $Y_{\text{form}}(s^n, z)$ is the unique formal solution associated to a diagonal matrix $C(z)$ described in Theorem 9.9. The matrix $C(z)$ is called the normalization of the Stokes basis $(I_i(s^n, y, z))_{i=1}^n$.

By Theorem 10.5, if $\mathcal{V}$ is a Stokes sector, and $C(z)$ is a fixed normalization, two Stokes bases on $\mathcal{V}$ differ only for the order of their objects. Thus, by abuse of language, we will refer to the $\mathfrak{S}_n$-orbit of Stokes bases on $\mathcal{V}$ as the Stokes basis on $\mathcal{V}$. Furthermore, if $\mathcal{V} \subseteq \mathcal{V}'_k$ (or $\mathcal{V}''_k$, resp.) then the Stokes basis on $\mathcal{V}$ is actually the Stokes basis on $\mathcal{V}'_k$ (or $\mathcal{V}''_k$, resp.), by Theorem 10.5.
Notice that if \( V \) is a Stokes sector, then also \( e^{\pi \sqrt{-1} V} \) and \( e^{2\pi \sqrt{-1} V} \) are Stokes sectors. We can thus introduce the following definition.

**Definition 10.9.** Let \( Y(s^n, z) \) be the Stokes solution (with normalization \( C(z) \)) of the equivariant quantum differential equation (6.1) on the Stokes sector \( V \). Let \( Y_1(s^n, z) \), and \( Y_2(s^n, z) \) resp., be the Stokes solutions on \( e^{\pi \sqrt{-1} V} \), and on \( e^{2\pi \sqrt{-1} V} \). We define the **Stokes matrices** attached to \( V \) and \( C(z) \) as the matrices \( S_1, S_2 \) (depending on \( z \in \Omega \)) for which we have

\[
Y_1(s^n, z) = Y(s^n, z)S_1, \quad Y_2(s^n, z) = Y_1(s^n, z)S_2, \quad s \in \hat{\mathbb{C}}^*, \quad z \in \Omega. \tag{10.8}
\]

10.3. **Properties of Stokes matrices and Lexicographical order.** It is standard to show that the Stokes matrices \( S_1, S_2 \), computed wrt a Stokes sector \( V \), have the following properties:

1. \((S_k)_{\alpha\alpha} = 1, \quad \alpha = 1, \ldots, n, \quad k = 1, 2;\)
2. \(S_1, S_2\) have at least \( \frac{n^2-n}{2} \) entries equal to 0;
3. any reordering of the diagonal entries of \( U = \text{diag}(u_1, \ldots, u_n) \) change the Stokes matrix through a conjugation \( \Pi \cdot S_k \cdot \Pi^{-1} \), where \( \Pi \) is a permutation matrix;
4. it is possible to reorder the diagonal entries of \( U \), so that \( S_1 \) is in upper triangular form and \( S_2 \) in lower triangular form.

The order which realizes the upper triangular form of \( S_1 \) (and consequently the lower triangular form of \( S_2 \)) is unique, since \( u_i \neq u_j \) for \( i \neq j \), and it will be called the **lexicographical order** wrt the Stokes sector \( V \). For proofs and detailed discussions, the reader may consult [Was65], [MS16], [LR16], [FIKN06]. See also [CDG18], for a detailed analytical discussion of the isomonodromic deformations associated to Frobenius manifolds in the non-equivariant case.

**Proposition 10.10.** With respect the same notations of Definition 10.9, the following identities hold true for any \( s \in \hat{\mathbb{C}}^* \) and \( z \in \Omega \):

1. \( Y_2 \left( (e^{2\pi \sqrt{-1} s})^n, z \right) = Y(s^n, z) \cdot \exp \left( 2\pi \sqrt{-1} \Lambda(z) \right), \)
2. \( Y_2(s^n, z) = Y(s^n, z) \cdot S_1S_2, \)
3. \( Y \left( (e^{2\pi \sqrt{-1} s})^n, z \right) = Y(s^n, z) \cdot \exp \left( 2\pi \sqrt{-1} \Lambda(z) \right) \cdot (S_1S_2)^{-1}. \)

Here \( \Lambda(z) \) is the exponent of formal monodromy, i.e.

\[
\Lambda(z) = \Lambda(z) := \left( s_1(z) + \frac{n-1}{2} \right) \cdot 1.
\]

**Proof.** For (1), notice that

\[
Y_2 \left( (e^{2\pi \sqrt{-1} s})^n, z \right) \cdot \exp \left( -2\pi \sqrt{-1} \Lambda(z) \right)
\]

is a solution of (6.1) with asymptotic expansion \( Y_{\text{form}}(s^n, z) \) on the Stokes sector \( V \). Hence it must coincide with \( Y(s^n, z) \). Point (2) is a direct consequence of the definition of Stokes matrices. Point (3) follows from points (1) and (2). \( \square \)
10.4. Stokes bases \( \tilde{Q}'_k \) and \( \tilde{Q}''_k \).

**Proposition 10.11.** The basis \( \tilde{Q}'_k \) (resp. \( \tilde{Q}''_k \)) is a Stokes basis on \( V'_k \) (resp. \( V''_k \)) with normalization
\[
C(z) = (2\pi)^{n-1/2} e^{-\pi \sqrt{-1}} \left( \prod_{m=0}^{n-1} \right) \frac{(\zeta_m^m)^{s_1(z)+n/2}}{e^{m\pi \sqrt{-1}}(\zeta_m^m)^n}. \tag{10.9}
\]

**Proof.** It follows from Corollary 7.30, and formula (9.15) for \( D_{q}^{-1} \). \( \square \)

**Corollary 10.12.** The operators \( K_j(z) \), with \( j = 1, \ldots, n \) are all equal and independent of \( z \). Indeed, we have
\[
K_j = \text{diag} \left( \zeta_m^m \right)_{m=0}^{n-1}, \quad j = 1, \ldots, n.
\]

**Proof.** It follows from Theorem 9.9 and the explicit computation
\[
C(z_1, \ldots, z_j-1, \ldots, z_n) C(z) \tag{10.9}
\]

10.5. Stokes bases as \( T \)-full exceptional collections.

**Theorem 10.13.** Via the isomorphism \( \theta: K_0^T(\mathbb{P}^{n-1})_{\mathbb{C}} \rightarrow \mathcal{S}_n \), Stokes bases on Stokes sectors of the \( qDE \) of \( \mathbb{P}^{n-1} \) correspond to \( K \)-classes of \( T \)-full exceptional collections in \( D^k_T(\mathbb{P}^{n-1}) \).

**Proof.** Stokes bases correspond to \( T \)-full exceptional collections of Corollary 7.27. \( \square \)

In particular, the Stokes basis \( \tilde{Q}'_{1-n} \) corresponds (up to shifts) to the exceptional collection
\[
\left( \mathcal{O} \left( \frac{n}{2} \right) \right) \prod_{1} \mathcal{T} \left( \frac{n}{2} - 1 \right), \mathcal{O} \left( \frac{n}{2} + 1 \right) \prod_{3} \mathcal{T} \left( \frac{n}{2} - 2 \right), \mathcal{O}(n-1) \prod_{n-1} \mathcal{T} \tag{10.10}
\]
for \( n \) even, and
\[
\left( \mathcal{O} \left( \frac{n-1}{2} \right) \right) \mathcal{O} \left( \frac{n+1}{2} \right) \prod_{2} \mathcal{T} \left( \frac{n-3}{2} \right), \mathcal{O} \left( \frac{n+3}{2} \right) \prod_{4} \mathcal{T} \left( \frac{n-5}{2} \right) \ldots, \mathcal{O}(n-1) \prod_{n-1} \mathcal{T} \tag{10.11}
\]
for \( n \) odd. All other Stokes bases, and corresponding exceptional collections, are obtained by application of a braid of the form
\[
\ldots \delta_{n, \text{odd}} \delta_{n, \text{even}} \delta_{n, \text{odd}} \delta_{n, \text{even}},
\]
or

\[ \cdots \delta_{n, \text{even}}^{-1} \delta_{n, \text{odd}}^{-1} \delta_{n, \text{even}}^{-1} \delta_{n, \text{odd}}^{-1}. \]

**Remark 10.14.** Exceptional collections (10.10) and (10.11) are the natural equivariant lift in \( \mathcal{D}^b(\mathbb{P}^{n-1}) \) of the exceptional collections of Corollary 6.11 of [CDG18]. Also in this non-equivariant case, these collections are identified with a Stokes basis of the \( qDE \) of \( \mathbb{P}^{n-1} \) in a suitable Stokes sector. See Section 6 of [CDG18] for details.

**Remark 10.15.** All the objects on the \( \mathbb{T} \)-full exceptional collections attached with Stokes bases are endowed with their natural \( \mathbb{T} \)-equivariant structure, restriction of the natural \( GL(n, \mathbb{C}) \)-equivariant structure. Under the presentation (3.4), their \( K \)-theoretical classes in \( K^*_0(\mathbb{P}^{n-1}) \) are symmetric polynomials w.r.t. the equivariant parameters \( Z \).

## 11. Stokes matrices as Gram matrices of Exceptional collections

### 11.1. Musical notation for braids

In this Section we introduce a notation for braids, which will be very convenient for proving some identities in \( \mathcal{B}_n \). In such a diagrammatic notation, elements of \( \mathcal{B}_n \) will be represented as notes on a musical \((n-1)\)-line staff. The lines are enumerated from the bottom (1-st line) to the top ((n-1)-th line). The generator \( \tau_i \) will be represented as a whole note (a hollow oval note head, i.e. a small empty oval) on the \( i \)-th line. The inverse \( \tau_i^{-1} \) will be denoted by the note head of a quarter, with no stem (i.e. a small black oval).

The relations defining the braid group \( \mathcal{B}_n \) translate into the diagrammatic rules described in Figure 11.1.

![Figure 11.1. Braid relations in musical notation.](image)

### 11.2. A useful identity in \( \mathcal{B}_n \).

**Lemma 11.1.** For any \( n \geq 2 \), the following identities of braids hold true in \( \mathcal{B}_n \):

\[
\delta_{n, \text{odd}} \delta_{n, \text{even}} \delta_{n, \text{odd}} \delta_{n, \text{even}} \cdots = \delta_{n, \text{even}} \delta_{n, \text{odd}} \delta_{n, \text{even}} \delta_{n, \text{odd}} \cdots = \beta,
\]

(11.1)

where \( \beta \) is given by equation (2.22), i.e.

\[
\beta := \tau_1(\tau_2 \tau_1) \cdots (\tau_{n-2} \cdots \tau_1)(\tau_{n-1} \tau_{n-2} \cdots \tau_1).
\]
Proof. We prove the statement by induction on \( n \). For \( n = 2 \), the statement is obvious, being

\[
\delta_{2,\text{odd}} = \tau_1, \quad \delta_{2,\text{even}} = 1, \quad \beta = \tau_1.
\]

Then, it is sufficient to prove that for any \( n \) we can always realize the following factorization

\[
\underbrace{\delta_{n,\text{odd}}\delta_{n,\text{even}}\delta_{n,\text{odd}}\delta_{n,\text{even}} \ldots}_{n \text{ factors}} = \beta' \cdot (\tau_{n-1}\tau_{n-2} \ldots \tau_1), \quad (11.2)
\]

for a suitable braid \( \beta' \), on which the inductive step applies. Similarly for the braid \( \delta_{n,\text{even}}\delta_{n,\text{odd}}\delta_{n,\text{even}} \ldots \).

The musical diagram corresponding to the braid \( \delta_{n,\text{odd}}\delta_{n,\text{even}}\delta_{n,\text{odd}}\delta_{n,\text{even}} \ldots \) is the following, according to the parity of \( n \).

![Diagram](https://via.placeholder.com/150)

**Figure 11.2.** Diagrammatic notation for the braid \( \delta_{n,\text{odd}}\delta_{n,\text{even}}\delta_{n,\text{odd}}\delta_{n,\text{even}} \ldots \) according to the parity of \( n \).

We collected with a stem the notes corresponding to a same factor \( \delta_{n,\text{odd}} \) (or \( \delta_{n,\text{even}} \)). By commutativity, the order of the notes in any factor \( \delta_{n,\text{odd}} \) (or \( \delta_{n,\text{even}} \)) can be modified at will, and for this reason we simply collect them with a vertical stem. In the top line we have

- \( \frac{n}{2} \) notes, if \( n \) is even,
- \( \frac{n-1}{2} \) notes, if \( n \) is odd.

We call top factors those factors \( \delta_{n,\text{odd}} \)'s (or \( \delta_{n,\text{even}} \)'s) which contains the notes on the top line. In other words, the top factors are

(1) the factors \( \delta_{n,\text{odd}} \)'s for \( n \) even,
(2) the factors \( \delta_{n,\text{even}} \)'s for \( n \) odd.

The configuration (11.2) can be reached by filling the empty spaces between two notes in the last factor \( \delta_{n,\text{odd}} \) (or \( \delta_{n,\text{even}} \)), from the bottom to the top line. We perform this in several steps:

(1) Label by \( A_0 \) the first (from the left) elementary braid on the \((n-1)\)-th line. By a chain of elementary moves, the braid \( A_0 \) can be moved on the \((n-2)\)-th line, towards the right, and can be collected with the next top factor, as described in the following Figure 11.3.

We call \( A_1 \) the new note obtained from \( A_0 \). In this way, this factor is “overcharged” of notes (i.e. it contains notes \( \tau_i \) and \( \tau_{i+1} \) for some \( i \)), and we have an inclination of the stem, the order of the elementary braids being not anymore arbitrary.
(2) By the braid relations, the braid $A_1$ can be moved on the $(n - 3)$-th line, towards the right, and can be collected with the next factor (not a top factor), as described in the following picture.

Figure 11.3.

We call $A_2$ the new note obtained from $A_1$. Also in this case, we have an inclination of the stem.

(3) Starting from $A_j$ on the $(n - 1 - j)$-th line, iterate the procedure of point (2) in order to produce a new braid $A_{j+1}$ in the line $(n - 2 - j)$-th line, by overcharging the next factor.

(4) Stop when the final braid $A_{j+1}$ fills the empty space on the
- 1-st line if $n$ is even,
- 2-nd line if $n$ is odd.

(5) Iterate points (1),(2),(3),(4) and stop when the final braid $A_{j+1}$ fills the first empty space from the bottom line to the top.

By applying the procedure above, the factorization (11.2) is reached. The argument for $\delta_{n,\text{even}}\delta_{n,\text{odd}}\delta_{n,\text{even}} \ldots$ is similar. □
Example 11.2. Consider $n = 7$. The factorization (11.2) is obtained with the following moves. For simplicity, we remove all the stems of the notes.

![Diagram showing moves](image)

Figure 11.5. These are the moves described in the proof of Lemma 11.1 in order to obtain the factorization 11.2.

11.3. Stokes matrices as Gram matrices.

Lemma 11.3. The following identities among Stokes sectors hold true:

$$e^{\pi \sqrt{-1} \mathcal{V}_k'} = \begin{cases} 
\mathcal{V}_{k-\frac{n}{2}}', & \text{if } n \equiv 0 \pmod{2}, \\
\mathcal{V}_{k-\frac{n+1}{2}}', & \text{if } n \equiv 1 \pmod{2},
\end{cases}$$

$$e^{\pi \sqrt{-1} \mathcal{V}_k''} = \begin{cases} 
\mathcal{V}_{k-\frac{n}{2}}'', & \text{if } n \equiv 0 \pmod{2}, \\
\mathcal{V}_{k-\frac{n-1}{2}}'', & \text{if } n \equiv 1 \pmod{2}.
\end{cases}$$
Proof. It is readily obtained from the definition of \( V'_k \) and \( V''_k \).

Corollary 11.4. For any \( k \in \mathbb{Z} \), the Stokes basis on \( e^{\pi \sqrt{-1}} V'_k \) is obtained by acting (on the left) on the Stokes basis \( \tilde{Q}'_k \) with the braid

\[
\ldots \delta_{n, \text{odd}} \delta_{n, \text{even}} \delta_{n, \text{odd}} \delta_{n, \text{even}} \ldots \quad \text{n factors}
\]

For any \( k \in \mathbb{Z} \), the Stokes basis on \( e^{\pi \sqrt{-1}} V''_k \) is obtained by acting (on the left) on the Stokes basis \( \tilde{Q}''_k \) with the braid

\[
\ldots \delta_{n, \text{even}} \delta_{n, \text{odd}} \delta_{n, \text{even}} \delta_{n, \text{odd}} \ldots \quad \text{n factors}
\]

Proof. It follows from the definition of \( \tilde{Q}'_k \) and \( \tilde{Q}''_k \) (see also diagram (7.16)), from Proposition 10.11 and Lemma 11.3.

Corollary 11.5. Let \( \mathcal{V} \) be a Stokes sector of the qDE of \( \mathbb{P}^{n-1} \), and let \( \mathcal{E} \) be the full exceptional collection corresponding to the Stokes basis on \( \mathcal{V} \). The exceptional collection \( \mathcal{E}' \) corresponding to the Stokes sector \( e^{2\pi \sqrt{-1}} \mathcal{V} \) is a parallel foundation of the helix generated by \( \mathcal{E} \). More precisely, \( \mathcal{E}' \) is obtained by application of the inverse Serre functor to objects of \( \mathcal{E} \):

\[
(E_1, \ldots, E_n) \mapsto (E_1 \otimes (\omega^\mathbb{T}_{\mathbb{P}^{n-1}})^{-1}[-n + 1], \ldots, E_n \otimes (\omega^\mathbb{T}_{\mathbb{P}^{n-1}})^{-1}[-n + 1]),
\]

where \( \omega^\mathbb{T}_{\mathbb{P}^{n-1}} \) denotes the \( \mathbb{T} \)-equivariant canonical sheaf of \( \mathbb{P}^{n-1} \).

Proof. By Corollary 11.4, \( \mathcal{E}' \) is obtained from \( \mathcal{E} \) by mutation either with the braid

\[
\ldots \delta_{n, \text{odd}} \delta_{n, \text{even}} \delta_{n, \text{odd}} \delta_{n, \text{even}} \ldots \quad \text{2n factors}
\]

or with the braid

\[
\ldots \delta_{n, \text{even}} \delta_{n, \text{odd}} \delta_{n, \text{even}} \delta_{n, \text{odd}} \ldots \quad \text{2n factors}
\]

In both cases, by Lemma 11.1, the resulting braid is \( \beta^2 \), where

\[
\beta := \tau_1 (\tau_2 \tau_1) \ldots (\tau_{n-2} \ldots \tau_1)(\tau_{n-1} \tau_{n-2} \ldots \tau_1).
\]

It is well-known that \( \beta^2 = (\tau_1 \ldots \tau_{n-1})^n \) (see e.g. Theorem 1.24 of [KT08]). The result follows from Proposition 2.24.
Theorem 11.6. Let $S_1, S_2$ the Stokes matrices computed wrt a Stokes sector $V$ in lexicographical order. Let $\varepsilon$ be the exceptional basis of $K^p_0(\mathbb{P}^{n-1})_C$ associated with the Stokes basis on $V$ via the isomorphism $\theta: K^p_0(\mathbb{P}^{n-1})_C \to S_n$ defined in (7.8), and let $G$ be the Gram matrix of $\chi^T$ wrt $\varepsilon$. Let $J$ be the anti-diagonal matrix

$$J_{\alpha\beta} := \delta_{\alpha+\beta,n+1}, \quad \alpha, \beta = 1, \ldots, n.$$

(1) The Stokes matrix $S_1$ is equal to the Gram matrix of $\chi^T$ wrt the left dual exceptional basis $\tilde{\varepsilon}^\vee$ (\varepsilon), i.e.

$$S_1 = J (G^T)^{-1} J. \quad (11.8)$$

(2) The matrix $JS_2J$ is equal to the Gram matrix of $\chi^T$ wrt the exceptional basis $\varepsilon$, i.e.

$$S_2 = J G J. \quad (11.9)$$

Proof. By Lemma 10.7, $V$ is contained in one (and only one) $V'_k$ or $V''_k$. By Corollary 11.4, the Stokes basis on $e^{\sqrt{-1}\pi} V$ is obtained from the Stokes basis on $V$ by applying either the braid

$$\underbrace{\delta_{n,\text{odd}}\delta_{n,\text{even}}\delta_{n,\text{odd}}\delta_{n,\text{even}} \ldots}_{\text{n factors}},$$

or the braid

$$\underbrace{\delta_{n,\text{even}}\delta_{n,\text{odd}}\delta_{n,\text{even}}\delta_{n,\text{odd}} \ldots}_{\text{n factors}}.$$ 

Consequently, by Lemma 11.1, the exceptional basis associated to the Stokes basis on $e^{\sqrt{-1}\pi} V$ is $\tilde{\varepsilon}^\vee$. Both points (1) and (2) then follow from Proposition 2.22, more precisely from the second identity (2.25). \hfill \Box

Corollary 11.7. Let $S_1$ and $S_2$ be the Stokes matrices computed wrt a Stokes sector $V$. We have

$$S_2 = \left( S_1^T \right)^{-1}. \quad (11.10)$$

Remark 11.8. In the analytical theory of Frobenius manifolds [Dub96, Dub99, Dub98], [CDG17], the Stokes matrices of the associated isomonodromic system of differential equations (see Appendix A) satisfy an analogous identity, in which the $\dagger$-operator is replaced by transposition. See for example [CDG17], Theorem 2.10.

Corollary 11.9. Let $S_1$ and $S_2$ be the Stokes matrices computed wrt a Stokes sector $V$. Both $S_1$ and $S_2$ have entries in the ring of symmetric Laurent polynomials with integer coefficients, i.e. $S_1, S_2 \in M_n \left( \mathbb{Z}[\mathbb{Z}^{\pm1}]^S_n \right)$. 

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Proof. By Lemma 3.2, the Gram matrix associated with the Beilinson exceptional collection is with integer symmetric Laurent polynomial entries. The braid group action preserves this property.

Hence, the entries of the Stokes matrices of the equivariant quantum differential equation (6.1) must satisfy several Diophantine constraints in the ring \( \mathbb{C}[Z^\pm] \).

Corollary 11.10. Let \( S \) be a Stokes matrix of the equivariant quantum differential equation (6.1) computed wrt some Stokes sector \( \mathcal{V} \). The following identity holds true:

\[
\det \left( \lambda \cdot 1 - S^\dagger S^{-1} \right) = \sum_{j=0}^{n} (-1)^j \lambda^{n-j} s_j \left( (-1)^{n+1} \frac{Z_1^n}{s_n(Z)}, \ldots, (-1)^{n+1} \frac{Z_n^n}{s_n(Z)} \right). \quad (11.11)
\]

Proof. The Corollary immediately follows from Theorem 3.5 and Theorem 11.6. For an alternative proof (purely analytical), notice that without loss of generality we can assume that \( S \) is the matrix \( S_1 \) computed wrt \( \mathcal{V} \). From point (3) of Proposition 10.10, and equation (11.10) we deduce that

\[
\exp(2\pi \sqrt{-1} \Lambda(z))S^\dagger S^{-1} = M_0(z)^n,
\]

where \( M_0(z) \) is the monodromy operator of the equation (6.1). From Corollary 6.3 we deduce the constraint

\[
\det \left( \lambda \cdot 1 - (-1)^{n+1} s_n(Z)S^\dagger S^{-1} \right) = \sum_{j=0}^{n} (-1)^j \lambda^{n-j} s_j(Z^n), \quad (11.12)
\]

which is easily seen to be equivalent to equation (11.11).

12. Specialization of the \( qDE \) at the roots of unity

12.1. Specialization of equivariant \( K \)-theory at roots of unity. In the very interesting paper [Pol11], A. Polishchuck deduced several congruences for discrete invariants of exceptional objects in \( D_T^b(X) \) for a wide class of smooth projective complex varieties \( X \) (including Grassmannians and smooth quadrics) endowed with the action of \( T \). These constraints are deduced by a detailed study of cyclotomic specializations of the equivariant \( K \)-theory: in the case corresponding to projective spaces ([Pol11], Section 3), the specialization of \( K_0^T(\mathbb{P}^{n-1}) \) is obtained by setting

\[
Z_m = \zeta_m^{-1}, \quad m = 1, \ldots, n. \quad (12.1)
\]

We will denote by \( K_\zeta \) this specialization of the \( K \)-theory of \( \mathbb{P}^{n-1} \).

Theorem 12.1 ([Pol11], Theorem 1.1).

1. The Grothendieck-Euler-Poincaré pairing \( \chi_T^\zeta \) specializes to an Hermitian positive definite form \( \chi_\zeta \) on \( K_\zeta \).
If $E$ is an exceptional object in $\mathcal{D}^b(\mathbb{P}^{n-1})$ equipped with a $\mathbb{T}$-equivariant structure, then the class $[E]$ in $K_\zeta$ has length 1 wrt the Hermitian form $\chi_\zeta$.

(3) If $(E_1, E_2)$ is an exceptional pair in $\mathcal{D}^b(\mathbb{P}^{n-1})$, with both $E_1$ and $E_2$ equipped with a $\mathbb{T}$-structure, then the classes $[E_1], [E_2]$ in $K_\zeta$ are orthogonal wrt $\chi_\zeta$.

(4) If $(E_1, \ldots, E_n)$ is a full exceptional collection in $\mathcal{D}^b(\mathbb{P}^{n-1})$, with each $E_i$ equipped with a $\mathbb{T}$-structure, then each unit vector in $K_\zeta$ is of the form $\pm \zeta^n_i [E_i]$ for some $i$ and some $k$.

(5) The action of the braid group on the set of orthonormal exceptional bases of $K_\zeta$ reduces to the action by permutations of basis vectors.

12.2. **Stirling numbers and two useful combinatorial identities.** Stirling numbers of the first kind $\left[ \begin{array}{c} n+1 \\ k \end{array} \right]$ are defined recursively by

\[
\left[ \begin{array}{c} n+1 \\ k \end{array} \right] = n\left[ \begin{array}{c} n \\ k \end{array} \right] + \left[ \begin{array}{c} n \\ k-1 \end{array} \right],
\]

for $k > 0$, with the initial conditions

\[
\left[ \begin{array}{c} 0 \\ 0 \end{array} \right] = 1, \quad \left[ \begin{array}{c} 0 \\ n \end{array} \right] = \left[ \begin{array}{c} n \\ 0 \end{array} \right] = 0, \quad n > 0.
\]

Stirling numbers of the second kind $\left\{ \begin{array}{c} n \\ k \end{array} \right\}$ are defined recursively by

\[
\left\{ \begin{array}{c} n+1 \\ k \end{array} \right\} = k\left\{ \begin{array}{c} n \\ k \end{array} \right\} + \left\{ \begin{array}{c} n \\ k-1 \end{array} \right\},
\]

for $k > 0$, with the initial conditions

\[
\left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} = 1, \quad \left\{ \begin{array}{c} 0 \\ n \end{array} \right\} = \left\{ \begin{array}{c} n \\ 0 \end{array} \right\} = 0, \quad n > 0.
\]

Stirling numbers of the first and of the second kind are related by the well-known identity

\[
\sum_{j \geq 0} (-1)^{n-j}\left\{ \begin{array}{c} n \\ j \end{array} \right\} \left[ \begin{array}{c} j \\ k \end{array} \right] = \delta_{nk}.
\]

**Lemma 12.2.** Let $n \geq 2$ and $1 \leq k \leq n$. We have

\[
s_k \left( \begin{array}{c} 0, \frac{2}{n}, \ldots, \frac{n-1}{n} \end{array} \right) = \frac{1}{n^k} \begin{array}{c} n \\ n-k \end{array},
\]

\[
m_k \left( \begin{array}{c} 0, \frac{2}{n}, \ldots, \frac{n-1}{n} \end{array} \right) = \frac{1}{n^k} \left\{ \begin{array}{c} n+k-1 \\ n-1 \end{array} \right\}.
\]
Proof. It is sufficient to prove the identities
\begin{align*}
s_k(0,1,\ldots,n-1) &= \binom{n}{n-k}, \\
m_k(0,\ldots,n-1) &= \left\{ \frac{n+k-1}{n-1} \right\}. \tag{12.10}
\end{align*}
We prove these identities by induction on \( n \). For \( n = 2 \) both equations (12.9) and (12.10) hold true. Recall the following recurrence equations for elementary and complete symmetric polynomials: for \( k \geq 2 \)
\begin{align*}
s_k(z_1,\ldots,z_n) &= s_k(z_1,\ldots,z_{n-1}) + z_n s_{k-1}(z_1,\ldots,z_{n-1}) \tag{12.11} \\
m_k(z_1,\ldots,z_n) &= m_k(z_1,\ldots,z_{n-1}) + z_n m_{k-1}(z_1,\ldots,z_{n}). \tag{12.12}
\end{align*}
The equation (12.9) follows from (12.11) and the recurrence relation (12.2) for Stirling numbers of the first kind. Analogously, the equation (12.10) follows from (12.12) and the recurrence relation (12.4) for the Stirling numbers of the second kind. \( \square \)

**Lemma 12.3.** If \( \vartheta_s := s_d d/s \) denotes the Euler operator, we have the following identities of differential operators:
\begin{align*}
\vartheta^n_s &= \sum_{j=1}^{n} \binom{n}{j} s^j \frac{d^j}{ds^j}, \tag{12.13} \\
s^n d^n d/s^n &= \sum_{j=1}^{n} (-1)^{n-j} \binom{n}{j} \vartheta^j_s. \tag{12.14}
\end{align*}
Proof. The identity (12.13) is easily proved by induction on \( n \). The identity (12.14) follows from (12.13) and the identity (12.6). \( \square \)

12.3. **Scalar equivariant quantum differential equation at roots of unity.** Let us consider the specialization of the equivariant quantum cohomology obtained by setting the analogue conditions to (12.1), namely
\[ \zeta_m = \zeta_n^{-1}, \quad m = 1,\ldots,n. \tag{12.15} \]
These equations define the locus
\[ \mathcal{P} := \left\{ z \in \mathbb{C}^n : z = \left( k_1, k_2 + \frac{1}{n}, \ldots, k_n + \frac{n-1}{n} \right), \quad k \in \mathbb{Z}^n \right\}. \tag{12.16} \]
We have a distinguished point \( z_o \in \mathcal{P} \), namely
\[ z_o := \left( 0, \frac{1}{n}, \ldots, \frac{n-1}{n} \right). \tag{12.17} \]
Theorem 12.4. At \( z_o \), the scalar equivariant quantum differential equation (6.14) of \( \mathbb{P}^{n-1} \) for the function \( \phi(q) \) reduces to the linear equation with constant coefficients
\[
\frac{d^n}{ds^n} \varphi(s) = n^n \varphi(s), \quad \varphi(s) := \phi(s^n).
\] (12.18)

Proof. By the change of variable \( q = s^n \), the equation (6.14) reduces to
\[
\left[ \frac{1}{n^n} \frac{\partial^n}{\partial s^n} + \sum_{j=1}^{n-1} (-1)^{n-j} s_{n-j}(z) \frac{1}{n^j} \frac{\partial^j}{\partial s^j} - (s^n + (-1)^{n-1} s_n(z)) \right] \phi(s^n) = 0.
\] (12.19)

If \( z = z_o \), then by identity (12.7) the equation reduces to
\[
\left[ \frac{1}{n^n} \frac{\partial^n}{\partial s^n} + \sum_{j=1}^{n-1} (-1)^{n-j} \frac{1}{n^n} \frac{n!}{j!} \frac{\partial^j}{\partial s^j} - s^n \right] \phi(s^n) = 0.
\] (12.20)

Using identity (12.14), one obtains the equation
\[
\left( s^n \frac{d^n}{ds^n} - n^n s^n \right) \phi(s^n) = 0.
\] (12.21)

The equation (12.18) admits two natural bases of solutions, namely
(1) the basis \( (f_m(s))_{m=0}^{n-1} \) defined by
\[
f_m(s) := \exp(n \zeta^m s), \quad m = 0, \ldots, n - 1;
\] (12.22)
(2) the basis \( (g_m(s))_{m=0}^{n-1} \) defined by
\[
g_m(s) := \sum_{k=0}^{\infty} \frac{(ns)^{m+kn}}{(m+kn)!}, \quad m = 0, \ldots, n - 1.
\] (12.23)

Notice that the functions \( g_m(s) \) are real-valued for \( s \in \mathbb{R} \), and that they define a partition of the exponential function \( e^{ns} \), namely
\[
\sum_{m=0}^{n-1} g_m(s) = e^{ns}.
\] (12.24)

Lemma 12.5. The cyclic group \( \mathbb{Z}/n\mathbb{Z} \) acts on the space of solutions of equation (12.18) via the transformations \( T_k : s \mapsto \zeta^k s, \quad k = 1, \ldots, n \). The basis \( (f_m(s))_{m=0}^{n-1} \) is cyclically permuted by this action, while \( (g_m(s))_{m=0}^{n-1} \) is an eigenbasis.
Introduce the matrices
\[
\hat{Y}_f(s^n) := \left( \hat{Y}_f(s^n)_m^{h} \right)_{h,m=0,\ldots,n-1}, \quad \hat{Y}_f(s^n)_m^{h} := \frac{1}{\eta^h} \vartheta^h s f_m(s), \quad (12.25)
\]
\[
\hat{Y}_g(s^n) := \left( \hat{Y}_g(s^n)_m^{h} \right)_{h,m=0,\ldots,n-1}, \quad \hat{Y}_g(s^n)_m^{h} := \frac{1}{\eta^h} \vartheta^h g_m(s). \quad (12.26)
\]
Both \( \hat{Y}_f(s^n) \) and \( \hat{Y}_g(s^n) \) are solutions of the differential system (6.13), specialized at \( z = z_o \), see Section 6.4.

**Proposition 12.6.** The matrix-valued function \( \eta(z_o)^{-1}\hat{Y}_f(s^n) \) is a fundamental system of solutions of (6.1) of the form
\[
\eta(z_o)^{-1}\hat{Y}_f(s^n) = G(s) \exp(sU), \quad (12.27)
\]
where \( G(s) \) is a polynomial in \( s \) of degree \( n-1 \). Hence, \( \eta(z_o)^{-1}\hat{Y}_f(s^n) \) is a Stokes basis of the equivariant qDE (6.1) at \( z_o \). In particular, the corresponding formal series \( F(s,z_o) \) of the form (9.36) is actually convergent.

**Proof.** The matrix \( \eta(z_o)^{-1}\hat{Y}_f(s^n) \eta(z_o) \) is a fundamental system of solutions of (6.1) by the discussion in Section 6.4. It follows that also \( \eta(z_o)^{-1}\hat{Y}_f(s^n) \) is a fundamental system of solutions. The matrix \( G(s) \) is given by
\[
G(s) = \eta(z_o)^{-1} \cdot L(s), \quad L(s) = \left( L(s)_m^{h} \right)_{h,m=0}^{n-1}, \quad L(s)_m^{h} := \zeta^h_m s^h. \quad (12.28)
\]
The series \( F(s,z_o) \) is thus given by
\[
F(s,z_o) = D_q H(s^n)^{-1} \eta(z_o)^{-1} L(s) s^{1-n}, \quad (12.29)
\]
and it is thus convergent. The normalization of the Stokes basis can be readily computed from this formula. \( \Box \)

**Proposition 12.7.** The matrix-valued function \( \eta(z_o)^{-1}\hat{Y}_g(s^n) \) is a fundamental system of solutions of (6.1) of the form
\[
\eta(z_o)^{-1}\hat{Y}_g(s^n) = Y_o(s^n, z_o) \cdot C, \quad (12.30)
\]
where \( C \) is a diagonal matrix.

**Proof.** It follows from Lemma 12.5 and Corollary 6.3. We leave to the reader the explicit computation of the matrix \( C \). \( \Box \)

**Theorem 12.8.** The following conditions are equivalent:

1. \( z' \in \mathcal{P} \);
(2) the formal gauge transformation $G(s, z')$ of Theorem 9.6 is convergent;
(3) the Stokes phenomenon of the equivariant differential equation (6.1) specialized at $z = z'$ is trivial, i.e. all the Stokes matrices $S(z')$ are the identity matrix;
(4) the monodromy group $\mathcal{M}(z')$ of the equivariant quantum differential equation (6.1) specialized at $z = z' \in \Omega$ is $\mathbb{Z}_n$.

Proof. We prove that (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (1).

Assume $z' = (k_1, k_2 + \frac{1}{n}, \ldots, k_n + \frac{n-1}{n})$ for some $k \in \mathbb{Z}^n$. Then, we have to show that the series $F(s, z)$ in (9.22) is convergent for $z = z'$. From the identity (9.29) we deduce that

$$F(s, z_1, \ldots, z_j - 1, \ldots, z_n) = W_j(s, z)F(s, z)K_j^{-1},$$

where

$$W_j(s, z) := sD_q\mathcal{H}(s^n)^{-1}K_j(s^n, z)\mathcal{H}(s^n)D_q^{-1}, \quad j = 1, \ldots, n.$$ 

Hence, we have

$$F(s, z') = \left(\prod_{j=1}^n W_j(s, z_o)^{-k_j}\right) F(s, z_o) \left(\prod_{j=1}^n K_j^{-k_j}\right),$$

and the convergence of $F(s, z')$ follows from the convergence of $F(s, z_o)$.

If (2) holds then $Y(s^n, z') = G(s, z')e^{stU}$ is a solution of the equation (6.1) at $z = z'$, and the transformation $s \mapsto \zeta_n s$ cyclically permutes the exponents $u_1, \ldots, u_n$. Thus (4) holds true.

If (4) holds true, then from Corollary 6.3 we deduce that $\exp(2\pi \sqrt{-1}z_j')$ is a $n$-th root of unity, i.e. $z' \in P$.

The equivalence of (2) and (3) is obvious. □

13. Relations with the non-equivariant results of [CDG18] on projective spaces

Many of the results obtained in this paper are an equivariant analogue of results of Section 6 of [CDG18]. In that paper, the authors focused on the study of the monodromy data of the non-equivariant quantum differential equation (5.41) for all complex Grassmannians, and their relations with the geometry of the non-equivariant derived category of coherent sheaves and $K$-theory.

In this last Section, we limit ourselves to list some points of touch and analogies between these two works, with the hope and belief that this will help for a future study unifying all these results.

The main message of the present paper and of [CDG18] can be summarized as follows: the monodromy data of the (equivariant) qDE are encoded in characteristic classes of objects of the (equivariant) derived categories of coherent sheaves.

In both papers, the qDE admits some companion equations, such that the resulting joint systems are compatible. In [CDG18], it is studied a joint system of differential equations, whose geometrical meaning is a flatness condition for the extended deformed connection $\nabla$ of $QH^\bullet(\mathbb{P}^{n-1})$ (see Appendix A for details). In the present paper, on the
contrary, the object of study is a joint system of differential and difference equations ($qDE$ and $qKZ$ equations, respectively).

The main results of the present paper and of [TV19a] can be summarized as follows:

1. It is shown that (a suitable subspace of) solutions $\Psi_E$ of the joint system of $qDE$ and $qKZ$ equations can be labelled by objects $E$ of $D^b_T(\mathbb{P}^{n-1})$ and their $K$-theoretical classes, see Theorem 7.11 and Corollary 7.16. Stokes bases of solutions are labelled by $\mathbb{T}$-full exceptional collections, see Theorem 10.13.

2. Given a solution $\Psi_E$ of the joint system of $qDE$ and $qKZ$ equations, we can express it as a linear combination of topological solutions (solutions of the only $qDE$). The coefficients of this linear combination are given by the components of the $E$-class of the object $E$, see Theorem 8.3 and Corollary 8.5.

3. Stokes bases of solutions in two Stokes sectors $\mathcal{V}$ and $\mathcal{V}_{\pm\pi}$ are attached to exceptional collections that are left/right dual to each other. Consequently, the Stokes matrices of the $qDE$ are equal to Gram matrices of the Grothendieck-Euler-Poincaré pairing $\chi^T$. See Corollary 11.4 and Theorem 11.6.

Point (2) above is an equivariant analogue of Theorem 6.7 and Corollary 6.11 of [CDG18]. It is claimed that the central connection matrices of Stokes bases of (5.41) with the topological solution are given by the components of the $D$-class of objects of exceptional collections, where

$$
\mathcal{D}: K_0(\mathbb{P}^{n-1})_{\mathbb{C}} \to H^\bullet(\mathbb{P}^{n-1}, \mathbb{C}), \quad V \mapsto \frac{(\sqrt{-1})^{n-1}}{(2\pi)^{\frac{n-1}{2}}} \hat{\Gamma}_{\mathbb{P}^{n-1}} \exp \left( \pi \sqrt{-1} c_1(\mathbb{P}^{n-1}) \right) \operatorname{Ch}(V),
$$

(13.1)

where $n-1$ is the residue class modulo 2 of $n-1$, $\hat{\Gamma}_{\mathbb{P}^{n-1}}$ is the non-equivariant $\Gamma$-class of $\mathbb{P}^{n-1}$, and $\operatorname{Ch}(V)$ is the non-equivariant graded Chern character of $V$. Such a statement is equivalent to the $\Gamma$-conjecture II of S. Galkin, V. Golyshev, H. Iritani [GGI16] for projective spaces. Notice that in [GGI16] a different choice of normalization of the topological solution of the $qDE$ were done wrt the “natural one” in Frobenius manifolds theory. See Section 5.6 of [CDG18] for a detailed discussion.

**Remark 13.1.** Notice that, in the present paper, the constant factors in the $D$-class (13.1) have been incorporated in the normalization $C(z)$ of the Stokes bases. See Proposition 10.11.

Although not explicitly stated in [CDG18], solutions of the non-equivariant $qDE$ can be identified with (complexified) $K$-theoretical classes, the morphism $\mathcal{D}$ being an isomorphism. Under this identification, Stokes bases correspond to $K$-theoretical classes of full exceptional collections in $D^b(\mathbb{P}^{n-1})$. Remarkably, the exceptional collections found from the analysis of [CDG18] are the underlying ones to the $\mathbb{T}$-full exceptional collections found in this paper.

The identification of the Stokes matrices of the non-equivariant $qDE$ with Gram matrices of the Grothendieck-Euler-Poincaré pairing $\chi$ wrt exceptional collections follows from the results above, by application of the Hirzebruch-Riemann-Roch Theorem, see Corollary 5.8 of [CDG18]. Although not explicitly noticed by the authors of [CDG18], also in the non-equivariant case, the exceptional collections corresponding to Stokes bases
in complementary Stokes sectors, $\mathcal{V}$ and $e^{\pm \pi \sqrt{-1}} \mathcal{V}$, are right/left dual to each other. This furnishes a further proof of the equality of Stokes and Gram matrices.

APPENDIX A. FORMAL AND ANALYTIC FROBENIUS MANIFOLDS

In this Appendix we collect basic notions and results on both formal and analytic theory of Frobenius manifolds. Basic references are [Dub92, Dub96, Dub98, Dub99, CDG17, CDG18], [Man99], [Her02], [Sab08].

Let
- $R$ be a commutative ring with unit,
- $(A, *, e)$ be a finite rank commutative, associative $R$-algebra with unit $e$,
- $\eta: A \times A \to R$ be a non-degenerate, symmetric, $R$-bilinear form.

Definition A.1. The 4-tuple $(A, *, \eta, e)$ is called a Frobenius algebra over $R$ if the product $*$ is compatible with the form $\eta$, that is
\[ \eta(x \ast y, z) = \eta(x, y \ast z), \quad x, y, z \in A. \] (A.1)

A.1. Formal Frobenius Manifolds. Let $k$ be a commutative $\mathbb{Q}$-algebra, and let $H$ be a finite rank $k$-module endowed with a non-degenerate symmetric pairing $\eta$. Fix a $k$-basis $(x_\alpha)_{\alpha=1}^n$ of $H$, and denote by $t := (t^\alpha)_{\alpha=1}^n$ the corresponding coordinates. Consider the datum of formal power series $A_{\alpha \beta \gamma}^\lambda \in k[[t]]$ representing 3-tensor fields on $H$. We define a $k[[t]]$-algebra structure on $H \otimes_k k[[t]]$ whose product is defined by
\[ x_\alpha \ast x_\beta := \sum_\gamma A_{\alpha \beta \gamma}^\lambda x_\lambda, \quad \alpha, \beta, \gamma = 1, \ldots, n. \] (A.2)

We also extend the pairing $\eta$ to a $k[[t]]$-bilinear pairing on $H \otimes_k k[[t]]$. We define $\eta_{\alpha \beta} := \eta(x_\alpha, x_\beta)$, for $\alpha, \beta = 1, \ldots, n$.

These data define a formal Frobenius structure on $H \otimes_k k[[t]]$ provided the following conditions hold:

1. The product (A.2) is commutative, associative and compatible w.r.t. $\eta$, i.e.
\[ A_{\beta \gamma}^\alpha = A_{\gamma \beta}^\alpha, \quad \alpha, \beta, \gamma = 1, \ldots, n, \] (A.3)
\[ \sum_\gamma A_{\alpha \beta \gamma}^\delta A_{\gamma \delta}^\epsilon = \sum_\gamma A_{\alpha \beta \epsilon}^\delta A_{\gamma \delta}^\gamma, \quad \alpha, \beta, \delta, \epsilon = 1, \ldots, n, \] (A.4)
\[ A_{\alpha \beta \gamma} = A_{\beta \gamma \alpha}, \quad \alpha, \beta, \gamma = 1, \ldots, n, \] (A.5)
where $A_{\alpha \beta \gamma} := \sum_\lambda A_{\alpha \beta \lambda}^\gamma \eta_{\lambda \gamma}$.

2. The element $x_1$ is the identity of the product (A.2), i.e.
\[ A_{\lambda \alpha}^\beta = \delta_{\alpha}^\beta, \quad \alpha, \beta = 1, \ldots, n. \] (A.6)

3. The tensor $A$ satisfies the potentiality conditions
\[ \frac{\partial}{\partial t^\beta} A_{\alpha \beta}^\gamma = \frac{\partial}{\partial t^\alpha} A_{\beta \gamma}^\delta, \quad \alpha, \beta, \gamma, \delta = 1, \ldots, n. \] (A.7)

This means that $\eta$ induces an isomorphism $\eta': H \to H^*$, where $H^*$ is the dual module.
Condition (A.7), together with (A.3) and (A.5), implies that the formal power series $A_{\alpha \beta \gamma}$ are third derivatives of a potential function $\Phi \in H \otimes_k k[t]$:

$$A_{\alpha \beta \gamma} = \frac{\partial^3}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \Phi.$$  

(A.8)

The associativity condition (A.4) is equivalent to the Witten-Dijkgraaf-Verlinde-Verlinde equations (WDVV-equations)

$$\frac{\partial^3 \Phi}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \eta^\gamma_\lambda \frac{\partial^3 \Phi}{\partial t^\nu \partial t^\varphi \partial t^\mu} = \frac{\partial^3 \Phi}{\partial t^\nu \partial t^\varphi \partial t^\mu} \eta^\nu_\varphi \frac{\partial^3 \Phi}{\partial t^\lambda \partial t^\varphi \partial t^\alpha},$$  

(A.9)

for $\alpha, \beta, \mu, \nu = 1, \ldots, n$.

A.2. Analytic Frobenius Manifolds. We denote with $\bigodot^k$ the $k$-th symmetric tensor power of vector bundles, and with $(-)\flat$ the standard operation of lowering the index of a $(1,k)$-tensor using a fixed inner product.

**Definition A.2.** A Frobenius manifold structure on a complex manifold $M$ of dimension $n$ is defined by giving

(FM1) a symmetric nondegenerate $\mathcal{O}(M)$-bilinear tensor $\eta \in \Gamma \left( \bigodot^2 T^*M \right)$, called metric, whose corresponding Levi-Civita connection $\nabla$ is flat;

(FM2) a $(1,2)$-tensor $c \in \Gamma \left( TM \otimes \bigodot^2 T^*M \right)$ such that

(a) the induced multiplication of vector fields $X \circ Y := c(\ - , X, Y)$, for $X, Y \in \Gamma(TM)$, is associative,

(b) $c^\flat \in \Gamma \left( \bigodot^3 T^*M \right),$

(c) $\nabla c^\flat \in \Gamma \left( \bigodot^4 T^*M \right)$;

(FM3) a vector field $e \in \Gamma(TM)$, called the unity vector field, such that

(a) the bundle morphism $c(-, e, -) : TM \to TM$ is the identity morphism,

(b) $\nabla e = 0$;

(FM4) a vector field $E \in \Gamma(TM)$, called the Euler vector field, such that

(a) $\mathcal{L}_E c = c$,

(b) $\mathcal{L}_E \eta = (2 - d) \cdot \eta$, where $d \in \mathbb{C}$ is called the charge of the Frobenius manifold.

**Remark A.3.** The flatness condition of $\nabla$ is equivalent to the existence of local flat coordinates $\mathbf{t} = (t^1, \ldots, t^n)$, w.r.t which the metric $\eta$ is constant, and the covariant derivatives $\nabla_\alpha$ coincide with the partial derivatives $\frac{\partial}{\partial t^\alpha}$, $\alpha = 1, \ldots, n$.

**Remark A.4.** Because of the flatness of the Levi-Civita connection $\nabla$ and the conformal Killing condition (FM4-b), the Euler vector field is affine, i.e. $\nabla \nabla E = 0$. In local flat coordinates $\mathbf{t}$, this means that $E$ has the form

$$E = \sum_{\alpha=1}^n \left( (1 - q_\alpha) t^\alpha + r_\alpha \right) \frac{\partial}{\partial t^\alpha}, \quad q_\alpha, r_\alpha \in \mathbb{C}.$$  

(A.10)

up to an affine change of coordinates, we can choose flat coordinates $\mathbf{t}$ so that $\frac{\partial}{\partial t^1} \equiv e$ and $r_\alpha \neq 0$ only if $q_\alpha = 1$. 
Remark A.5. In flat coordinates $t$, let $\eta_{\alpha\beta} = \eta(\partial_\alpha, \partial_\beta)$, and $c^\gamma_{\alpha\beta} = c(dt^\gamma, \partial_\alpha, \partial_\beta)$, so that $\partial_\alpha \circ \partial_\beta = c^\gamma_{\alpha\beta} \partial_\gamma$. Condition (FM2) means that $c_{\alpha\beta\gamma} := \eta_{\alpha\rho} c^\rho_{\beta\gamma}$ and $\partial_\alpha c_{\beta\gamma\delta}$ are symmetric in all indices. This implies the local existence of a function $\Phi$ such that

$$c_{\alpha\beta\gamma} = \partial_\alpha \partial_\beta \partial_\gamma \Phi.$$  

The associativity of the algebra is equivalent to the WDVV-equations (A.9) for $\Phi$.

Axiom (FM4) is equivalent to

$$\eta_{\alpha\beta} = \partial_1 \partial_\alpha \partial_\beta \Phi, \quad \mathcal{L}_E \Phi = (3 - d)\Phi + Q(t),$$

with $Q(t)$ a quadratic expression in $t^\alpha$'s. Conversely, given an analytic solution $\Phi$ of the WDVV-equations, satisfying the quasi-homogeneity conditions above, a structure of Frobenius manifold is naturally defined on an open subset of the space of parameters $t^\alpha$'s.

Definition A.6. Introduce two $(1,1)$-tensors $U$ and $\mu$ on $M$ defined by

$$U(Y) := E \circ Y, \quad \mu(Y) := \frac{2-d}{2} Y - \nabla_X E$$

for all $X \in \Gamma(TM)$.

Remark A.7. For simplicity it will be assumed that the tensor $\nabla E \in TM \otimes T^*M$ is diagonalizable. In flat coordinates $t$ chosen as in Remark A.4, the operator $\mu$ is constant and in diagonal form

$$\mu = \text{diag}(\mu_1, ..., \mu_n), \quad \mu_\alpha = q_\alpha - \frac{d}{2} \in \mathbb{C}.$$

Let us consider the canonical projection $\pi : \mathbb{P}^1(\mathbb{C}) \times M \to M$, and the pull-back of the tangent bundle $TM$:

$$\xymatrix{
\pi^*TM \ar[d] \ar[r] & TM \ar[d] \\
\mathbb{P}^1(\mathbb{C}) \times M \ar[r]^\pi & M
}$$

We will denote by

1. $\mathcal{F}_M$ the sheaf of sections of $TM$,
2. $\pi^*\mathcal{F}_M$ the pull-back sheaf, i.e. the sheaf of sections of $\pi^*TM$
3. $\pi^{-1}\mathcal{F}_M$ the sheaf of sections of $\pi^*TM$ constant on the fibers of $\pi$.

All the tensors $\eta, e, c, E, U, \mu$ can be lifted to $\pi^*TM$, and their lift will be denoted with the same symbol. So, also the Levi-Civita connection $\nabla$ is lifted on $\pi^*TM$, and it acts so that

$$\nabla_\partial_\alpha Y = 0 \quad \text{for} \quad Y \in (\pi^{-1}\mathcal{F}_M)(M).$$

Let us now twist this connection by using the multiplication of vectors and the operators $U, \mu$.

Definition A.8. Let $\widehat{M} := \mathbb{C}^* \times M$. The deformed connection $\widehat{\nabla}$ on the vector bundle $\pi^*TM|_{\widehat{M}} \to \widehat{M}$ is defined by

$$\widehat{\nabla}_X Y = \nabla_X Y + z \cdot X \circ Y,$$
\[ \hat{\nabla}_{\partial_t} Y = \nabla_{\partial_t} Y + U(Y) - \frac{1}{z} \mu(Y) \]

for \( X, Y \in (\pi^*\mathcal{G}_M)(\hat{M}) \).

The crucial fact is that the deformed extended connection \( \hat{\nabla} \) is flat.

**Theorem A.9** ([Dub96],[Dub99]). The flatness of \( \hat{\nabla} \) is equivalent to the following conditions on \( M \)

- \( \nabla c^\beta \) is completely symmetric,
- the product on each tangent space of \( M \) is associative,
- \( \nabla \nabla E = 0 \),
- \( \mathcal{L}_{E c} = c \).

Because of this integrability condition, we can look for deformed flat coordinates \( \tilde{t} := (\tilde{t}_1, \ldots, \tilde{t}_n) \), with \( \tilde{t}^\alpha = \tilde{t}^\alpha(t, z) \). These coordinates are defined by \( n \) independent solutions of the equation

\[ \hat{\nabla} d\tilde{t} = 0. \]

Let \( \xi \) denote a column vector of components of the differential \( d\tilde{t} \). The above equation becomes the linear system

\[ \frac{\partial}{\partial t^\alpha} \xi = z C^T_{\alpha}(t) \xi, \] (A.11)

\[ \frac{\partial}{\partial z} \xi = (U^T(t) - \frac{1}{z} \mu^T) \xi, \] (A.12)

where \( C_{\alpha} \) is the matrix \( (C_{\alpha})^{\beta}_{\gamma} = c^\beta_{\alpha \gamma} \). We can rewrite the system in the form

\[ \frac{\partial}{\partial t^\alpha} \zeta = z C_{\alpha} \zeta, \] (A.13)

\[ \frac{\partial}{\partial z} \zeta = (U + \frac{1}{z} \mu) \zeta, \] (A.14)

where \( \zeta := \eta^{-1} \xi \). In order to obtain (A.13)-(A.14), we used the invariance of the product, encoded in the relations

\[ \eta^{-1} C^T_{\alpha} \eta = C_{\alpha}, \]

\[ U^T \eta = \eta U, \] (A.15)

and the \( \eta \)-skew-symmetry of \( \mu \)

\[ \mu \eta + \eta \mu = 0. \] (A.16)

Geometrically, \( \zeta \) is the \( \eta \)-gradient of a deformed flat coordinate. Monodromy data of system (A.14) define local invariants of the Frobenius manifold, as explained below.
Definition A.10. A point $p \in M$ is called semisimple if and only if the corresponding Frobenius algebra $(T_p M, c_p, \eta_p, e_p)$ is without nilpotents. Denote by $M_{ss}$ the open dense subset of $M$ of semisimple points.

On $M_{ss}$ there are $n$ well-defined idempotent vector fields $\pi_1, \ldots, \pi_n \in \Gamma(TM_{ss})$, satisfying

$$\pi_i \circ \pi_j = \delta_{ij} \pi_i, \quad \eta(\pi_i, \pi_j) = \delta_{ij} \eta(\pi_i, \pi_i),$$

(A.17)

for $i, j = 1, \ldots, n$.

Definition A.11. We call orthonormalized idempotent frame a frame $(f_1, \ldots, f_n)$ of $TM_{ss}$ defined by

$$f_i := \eta(\pi_i, \pi_i)^{-\frac{1}{2}} \pi_i, \quad i = 1, \ldots, n,$$

(A.18)

for arbitrary choices of signs of the square roots. The $\Psi$-matrix is the matrix $(\Psi_{\alpha i})_{i,\alpha=1}^n$ of change of tangent frames, defined by

$$\frac{\partial}{\partial t^\alpha} = \sum_{i=1}^n \Psi_{\alpha i} f_i, \quad \alpha = 1, \ldots, n.$$

(A.19)

The $\Psi$-matrix diagonalizes the $U$-operator,

$$\Psi U \Psi^{-1} = U := \text{diag}(u_1, \ldots, u_n),$$

(A.20)

and puts the operator $\mu$ in anti-symmetric form,

$$V := \Psi \mu \Psi^{-1}, \quad V^T + V = 0.$$

(A.21)

Definition A.12. We call Stokes rays at a point $p \in M_{ss}$ the oriented rays $R_{ij}(p)$ in $\mathbb{C}$ defined by

$$R_{ij}(p) := \left\{-\sqrt{-1}(\bar{u}_i(p) - \bar{u}_j(p))\rho: \rho > 0\right\},$$

(A.22)

where $(u_1(p), \ldots, u_n(p))$ is the spectrum of the operator $U(p)$ (with a fixed arbitrary labeling).

Definition A.13. Fix an oriented ray $\ell$ in the universal cover $\widehat{\mathbb{C}}^*$. Consider the open subset $X$ of points $p \in M$ defined by the following condition:

- the eigenvalues $u_i(p)$ are pairwise distinct,
- the projection of the the ray $\ell$ on $\mathbb{C}^*$ does not coincide with any Stokes ray $R_{ij}(p)$.

We call $\ell$-chamber of $M$ any connected component of $X$.

Equation (A.14) admits two singularities on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$:

1. a Fuchsian singularity at $z = 0$,
2. an irregular singularity (of Poincaré rank 1) at $z = \infty$.

Fix an oriented ray $\ell \equiv \{\arg z = \phi\}$ in $\widehat{\mathbb{C}}^*$, and a point $p$ (with coordinates $t$) inside an $\ell$-chamber of $M$. In order to describe the monodromy phenomenon of (A.14) at $p$, let us fix
• a fundamental solution $Z_0(t, z)$ in Levelt form at $z = 0$, namely of the form

$$Z_0(t, z) = F(t, z)z^R, \quad R = \sum_{k \geq 1} R_k, \quad F(t, z) = 1 + \sum_{j=1}^{\infty} F_j(t)z^j$$  \hspace{1cm} (A.23)

where $(R_k)_{\alpha\beta} \neq 0$ only if $\mu_\alpha - \mu_\beta = k$, and $F(t, z)$ convergent satisfying the orthogonality condition

$$F(t, -z)^T \eta F(t, z) = \eta, \quad z \in \hat{\mathbb{C}}^*;$$  \hspace{1cm} (A.24)

• two fundamental solutions $Z_{L/R}(t, z)$ uniquely determined by the asymptotic expansion

$$Z_{L/R}(t, z) \sim \Psi(t)^{-1}G(t, z)\exp(zU(t)), \quad G(t, z) = 1 + \sum_{k=1}^{\infty} \frac{1}{z^k}G_k(t),$$ \hspace{1cm} for $|z| \to \infty$ respectively in sectors

$\Pi_L^\varepsilon(\phi) := \left\{ z \in \hat{\mathbb{C}}^*: \phi - \varepsilon < \arg z < \phi + \pi + \varepsilon \right\},$ \hspace{1cm} (A.25)

$\Pi_R^\varepsilon(\phi) := \left\{ z \in \hat{\mathbb{C}}^*: \phi - \pi - \varepsilon < \arg z < \phi + \varepsilon \right\},$ \hspace{1cm} (A.26)

where $\varepsilon$ is a sufficiently small positive number.

**Definition A.14.** Given $Z_0(t, z), Z_{L/R}(t, z)$ as above, we define the Stokes and Central Connection matrices at $p$ by the identities

$$Z_L(t, z) = Z_R(t, z)S, \quad Z_R(t, z) = Z_0(t, z)C.$$  \hspace{1cm} (A.27)

The 4-tuple $(\mu, R, S, C)$ defines the monodromy data of equation (A.14) at $p$.

**Theorem A.15 ([Dub96, Dub99]).** The monodromy data $(\mu, R, S, C)$ are constant in each $\ell$-chamber.

**Appendix B. qKZ operators and quantum multiplication operators**

This appendix is devoted to a proof of an identity relating qKZ operators of $\mathbb{P}^{n-1}$ and operators of quantum multiplication $x_{*q,z}$. Notations of Section 5.7 will be used.

**B.1. Polynomials $\xi_\alpha$’s and expansion of weight function.**

**Definition B.1.** For $\alpha = 0, \ldots, n$ define the polynomials

$$\xi_\alpha(t, z) := \sum_{\ell=0}^{\alpha} (-1)^{\ell}t^{\alpha-\ell}s_\ell(z).$$  \hspace{1cm} (B.1)
Lemma B.2. Fix $i \in \{1, \ldots, n\}$. For $\alpha = 0, \ldots, n-1$, we have

$$(-1)^{\alpha} s_{1+\alpha}(z_i^+) + \xi_{1+\alpha}(t, z) - (z_i + 1)\xi_\alpha(t, z) = (t - z_i)\xi_\alpha(t, z_i^+).$$

Recall the notations of Section 5.7, where we set $z_i^+ := (z, \ldots, z_i + 1, \ldots, z_n)$.

Proof. For $\ell \geq 1$ we have $s_\ell(z) = s_\ell(z_i^+) - s_\ell(z_{i-1})$, where $s_\ell(z_{i-1})$ is the $(\ell - 1)$-th elementary symmetric polynomial in the $n-1$ variables $(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$. Thus, we have

$$(-1)^{\alpha} s_{1+\alpha}(z_i^+) + \xi_{1+\alpha}(t, z) - (z_i + 1)\xi_\alpha(t, z)$$

$$= t^{\alpha+1} + \sum_{m=0}^{\alpha} (-1)^{m-1} t^m \left[ s_{m+1}(z_i^+) - s_m(z) + (z_i + 1)(s_m(z_i^+) - s_m(z)) \right]$$

$$+ (-1)^{\alpha} s_{1+\alpha}(z_i^+)$$

$$= t^{\alpha+1} + \sum_{m=0}^{\alpha} (-1)^{m-1} t^m \left[ s_{m+1}(z_i^+) + z_i s_m(z_i^+) \right] + (-1)^{\alpha} s_{1+\alpha}(z_i^+)$$

$$= (t - z_i)\xi_\alpha(t, z_i^+).$$

□

Lemma B.3. We have the following expansion of the weight function

$$W(t, y) = \prod_{j=1}^{n-1} (y_j - t) = (-1)^{n-1} \sum_{\alpha=0}^{n-1} \xi_\alpha(t, z) x^{n-\alpha-1}. \tag{B.2}$$

Proof. We have

$$\left( \sum_{\alpha=0}^{n-1} \xi_\alpha(t, z) x^{n-\alpha-1} \right) (t - x) = \prod_{j=1}^{n} (t - z_j), \tag{B.3}$$

as it can be easily checked using the relation

$$\sum_{k=0}^{n} (-1)^k x^{n-k} s_{n-k}(z) = 0. \tag{B.4}$$

Hence, we deduce

$$\prod_{j=1}^{n-1} (t - y_j) = \sum_{\alpha=0}^{n-1} \xi_\alpha(t, z) x^{n-\alpha-1}. \tag{B.5}$$

□
B.2. Relation between quantum multiplication operator and \( qKZ \) operators.

Consider \( qKZ \) difference equations (5.26) written in the equivalent form

\[
I(q, z_1, \ldots, z_i + 1, \ldots, z_n) = \left[ \Theta^+_{z_i} \circ K_i(q, z) \right] I(q, z), \quad i = 1, \ldots, n, \tag{B.6}
\]

where we set

\[
K_i(q, z) := \Theta_{z_i} \circ K_i(q, z^+_{z_i})^{-1} \circ \Theta_{z_i}. \tag{B.7}
\]

See Remark 5.10. In the notations of Section 5.7, for \( i = 1, \ldots, n \) define the operators

\[
\tilde{K}_i(q, z) := \Pr^* H_{q, z} \rightarrow \Pr^* H_{q, z}, \tag{B.8}
\]

as the composition

\[
\Pr^* H_{q, z} \xrightarrow{K_i(q, z)} \Pr^* H_{q, z} \xrightarrow{\Theta_{z_i}^+} \Pr^* H_{q, z_i} \xrightarrow{\Pi_{z_i}} \Pr^* H_{q, z},
\]

**Theorem B.4.** For \( i = 1, \ldots, n \), the following identity holds true:

\[
\tilde{K}_i(q, z) = \Theta_{z_i}^+ \circ \left[ x_{q, z_i^+} - (z_i + 1) \right] \circ \Pi_{z_i}^+. \tag{B.9}
\]

**Proof.** Fix \( i \in \{1, \ldots, n\} \). We show that

\[
\Psi_J(q, y, z_i^+) = \left[ \left( x_{q, z_i^+} - (z_i + 1) \right) \circ \Pi_{z_i}^+ \right] \Psi_J(q, y, z) \tag{B.10}
\]

for \( J = 1, \ldots, n \), where \( \Psi_J(q, y, z) \) denotes the Jackson integral (7.2). Define the components \( B_J(q, z) \) and \( X_J(q, z) \) by

\[
\Psi_J(q, y, z_i^+) = \sum_{\alpha=0}^{n-1} B_J^\alpha(q, z) x^{n-\alpha-1}, \tag{B.11}
\]

\[
\left[ \left( x_{q, z_i^+} - (z_i + 1) \right) \circ \Pi_{z_i}^+ \right] \Psi_J(q, y, z) = \sum_{\alpha=0}^{n-1} X_J^\alpha(q, z) x^{n-\alpha-1}. \tag{B.12}
\]

We have to show the identities

\[
B_J^\alpha = X_J^\alpha, \quad \text{for } \alpha = 0, \ldots, n - 1. \tag{B.13}
\]

By Lemma B.3, we have

\[
B_J^\alpha(q, z) = (-1)^{n-1} \frac{e^{\pi \sqrt{-1}(1+\sum_{i=1}^{\alpha-1} z_i)}}{2\pi \sqrt{-1}} \int_{C_J(z_i^+)} e^{-\pi \sqrt{-1} t} q^\alpha \Gamma(z_i + 1 - t) \prod_{\alpha \neq i} \Gamma(z_i - t) \xi_\alpha(t, z_i^+) dt. \tag{B.14}
\]

Here \( C_J(z_i^+) \) is a loop encircling the points \( z_J + r \), with \( r \in \mathbb{N} \) if \( J \neq i \), and \( r \in \mathbb{N}^* \) if \( J = i \). We have

\[
X_J^\alpha(q, z) = (-1)^{n-1} \frac{e^{\pi \sqrt{-1} \sum_{k=1}^{\alpha} z_k}}{2\pi \sqrt{-1}} \int_{C_J(z)} e^{-\pi \sqrt{-1} t} q^\alpha \Gamma(z_i - t) \prod_{\alpha = 1}^n \Gamma(z_i - t) f_\alpha(t, z) dt, \tag{B.15}
\]

where for \( \alpha = 0, \ldots, n - 2 \)

\[
f_\alpha(t, z) := (-1)^\alpha s_{1+\alpha}(z_i^+) + \xi_{1+\alpha}(t, z) - (z_i + 1) \xi_\alpha(t, z), \tag{B.16}
\]
and
\[ f_{n-1}(t, z) := q + (-1)^{n-1}s_n(z_i^+) - (z_i + 1)\xi_{n-1}(t, z). \] (B.17)

By Lemma B.2, and the identity \( s\Gamma(s) = \Gamma(s + 1) \), identity (B.13) follows for \( \alpha = 0, \ldots, n-2 \).

Finally, we have that
\[ X^J_{n-1} = (-1)^{n-1}e^{\sqrt{-1}\sum z_k} \left( \int_{C_j(z)} e^{-\sqrt{-1}ntq^{t+1}} \prod_{a=1}^{n} \Gamma(z_a - t) dt \right) + \int_{C_j(z)} e^{-\sqrt{-1}ntq^t} \prod_{a=1}^{n} \Gamma(z_a - t) \tilde{f}_{n-1}(t, z) dt, \]
where we set
\[ \tilde{f}_{n-1}(t, z) := (-1)^{n-1}s_n(z_i^+) - (z_i + 1)\xi_{n-1}(z). \] (B.18)

In the first summand, we change variable of integration: \( t + 1 \mapsto t \). The first summand is then equal to
\[ \int_{C_j(z)} (-1)^{n}e^{-\sqrt{-1}ntq^t} \prod_{a=1}^{n} \Gamma(z_a - t + 1) dt \]
\[ = (-1)^{n} \int_{C_j(z)} e^{-\sqrt{-1}ntq^t} \prod_{a=1}^{n} \Gamma(z_a - t) \prod_{j=1}^{n} (z_j - t) dt \]
\[ = \int_{C_j(z)} e^{-\sqrt{-1}ntq^t} \prod_{a=1}^{n} \Gamma(z_a - t) \xi_n(t, z) dt. \]

Using once more the Lemma B.2, one concludes that (B.13) holds true also for \( \alpha = n - 1 \). \( \square \)

**Corollary B.5.** The following identities hold true:
\[ \tilde{K}_i(q, z) - \tilde{K}_j(q, z) = z_j - z_i, \quad i, j = 1, \ldots, n, \] (B.19)
\[ \tilde{K}_i(q, z)\tilde{K}_j(q, z) = \tilde{K}_j(q, z)\tilde{K}_i(q, z), \quad I, J = 1, \ldots, n. \] (B.20)

**Appendix C. Formal Reduction of a Joint System of Differential and Difference Equations**

Consider a joint system of differential and difference equations
\[ \frac{d}{ds}X(s, z) = A(s, z)X(s, z), \] (C.1)
\[ X(s, z_1, \ldots, z_i - 1, \ldots, z_n) = P_i(s, z)X(s, z), \] (C.2)
where \( A, P, X \) are \((m \times m)\)-matrix valued functions, rationals wrt \( s \in \mathbb{C} \) and holomorphic wrt \( z \in \mathbb{C}^n \). We assume the validity of the compatibility equations of (C.1)-(C.2)

\[
\frac{d}{ds} P_i(s, z) = A(s, z_1, \ldots, z_i - 1, \ldots, z_n) P_i(s, z) - P_i(s, z) A(s, z), \quad (C.3)
\]

\[
P_i(s, z_1, \ldots, z_i, \ldots, z_j - 1, \ldots, z_n) P_j(s, z) = P_j(s, z_1, \ldots, z_i - 1, \ldots, z_j, \ldots, z_n) P_i(s, z). \quad (C.4)
\]

This is a necessary condition for the existence of non-zero solutions of the joint system of equations. Assume that

1. the coefficients \( A(s, z), P_i(s, z) \) have the following form

\[
A(s, z) = \sum_{k=0}^{\infty} A_k(z) \frac{1}{s^k}, \quad P_i(s, z) = \sum_{k=0}^{\infty} P_k(z) \frac{1}{s^k}, \quad (C.5)
\]

where the matrices \( A_k(z), P_k(z) \) are holomorphic functions of \( z \);

2. the matrix \( A_0(z) \) is diagonalizable

\[
D(z) \cdot A_0(z) \cdot D(z)^{-1} = U(z), \quad U(z) := \text{diag}(u_1(z), \ldots, u_n(z)), \quad (C.6)
\]

with \( D(z) \) a holomorphic function;

3. the Jordan diagonal form \( U(z) \) is \( 1\)-periodic, that is

\[
U(z_1, \ldots, z_i - 1, \ldots, z_n) = U(z), \quad i = 1, \ldots, n, \quad (C.7)
\]

4. the eigenvalues \( u_i(z) \)'s are pairwise distinct for \( z \) varying in the complement of a set \( \Delta \subseteq \mathbb{C}^n \).

**Theorem C.1.** Consider a joint system of differential and difference equations (C.1)-(C.2) satisfying the Assumptions (1)-(2)-(3)-(4) above. Then, there exists a unique formal gauge transformation \( X(s, z) = G(s, z) Z(s, z) \), of the form

\[
G(s, z) = D(z)^{-1} F(s, z) s^{\Lambda_1(z)}, \quad (C.8)
\]

\[
F(s, z) := 1 + \sum_{m=1}^{\infty} F_k(z) \frac{1}{s^m}, \quad [\Lambda_1(z)]_{ij} := [D(z) \cdot A_1(z) \cdot D(z)^{-1}]_{ij} \delta_{ij}, \quad (C.9)
\]

with \( F_k \)'s regular functions on \( \mathbb{C}^n \setminus \Delta \), that transforms the joint system (C.1)-(C.2) into the joint system

\[
\frac{d}{ds} Z(s, z) = U(z) Z(s, z), \quad (C.10)
\]

\[
Z(s, z_1, \ldots, z_i, \ldots, z_n) = P_i(z) Z(s, z), \quad P_i \text{ diagonal matrices.} \quad (C.11)
\]

**Proof.** Firstly, we prove that there exists a unique formal gauge transformation \( X(s, z) = G(s, z) Z(s, z) \), of the form, which transforms the differential equation (C.1) into the differential equation (C.10). Secondly, we prove that the same gauge transformation automatically maps the difference equations (C.2) into the difference equations (C.11).
Thus, let us look for a gauge transformation $X(s, z) = G(s, z)Z(s, z)$, of the form
\[ G(s, z) = D(z)^{-1}F(s, z)s^{\Lambda_1(z)}, \quad (C.12) \]
\[ F(s, z) := 1 + \sum_{m=1}^{\infty} F_k(z) \frac{1}{s^m}, \quad [\Lambda_1(z)]_{ij} := \left[ D(z) \cdot A_1(z) \cdot D(z)^{-1} \right]_{ij} \delta_{ij}, \quad (C.13) \]
that transforms the equation (C.1) to the equation
\[ \frac{d}{ds}Z(s, z) = U(z)Z(s, z). \quad (C.14) \]
The function $Z(s, z)$ is a solution of the equation
\[ \frac{d}{ds}Z(s, z) = \left( G(s, z)^{-1}A(s, z)G(s, z) - G(s, z)^{-1}\frac{d}{ds}G(s, z) \right)Z(s, z). \quad (C.15) \]
Thus, the formal series $F(s, z)$ satisfies the equation
\[ U(z)F(s, z) + \left( \sum_{k=1}^{\infty} \frac{D(z)A_k(z)D(z)^{-1}}{s^k} \right)F(s, z) = \frac{d}{ds}F(s, z) + \frac{1}{s}F(s, z)A_1(z) + F(s, z)U(z). \]
For brevity of notations, set $\hat{A}_j := DAD^{-1}$ and $\hat{A}_1^{\text{od}} := \hat{A}_1 - \Lambda_1$ (so that $\hat{A}_1^{\text{od}}$ is an off-diagonal matrix). We obtain the first equation for the coefficients $F_k$’s:
\[ UF_1 + \hat{A}_1^{\text{od}} = F_1U. \quad (C.16) \]
For $\alpha \neq \beta$ we find that
\[ F_1(z)_{\alpha\beta} = \frac{1}{u_{\beta}(z) - u_{\alpha}(z)} \left( \hat{A}_1^{\text{od}}(z) \right)_{\alpha\beta}. \quad (C.17) \]
The subsequent equation is
\[ UF_2 + \hat{A}_2 + \hat{A}_1F_1 = -F_1 + F_1\Lambda + F_2U. \quad (C.18) \]
From its diagonal part we can find the diagonal entries of $F_1(z)$. From its off-diagonal part, we can compute the entries $F_2(z)_{\alpha\beta}$ with $\alpha \neq \beta$:
\[ F_2(z)_{\alpha\beta} = \frac{1}{u_{\beta}(z) - u_{\alpha}(z)} \left( F_1(z) - F_1(z)\Lambda(z) + \hat{A}_2(z) + \hat{A}_1(z)F_1(z) \right). \quad (C.19) \]
The subsequent equation is
\[ -2F_2 + [F_3, U] = -F_2\Lambda + \hat{A}_1F_2 + \hat{A}_2F_1 + \hat{A}_3, \quad (C.20) \]
from which we can uniquely determine the diagonal entries of $F_2$ and the off-diagonal part of $F_3$. At the $k$-th step, if we have already computed the off-diagonal part of $F_k$, we can compute its diagonal entries from the equation
\[ -kF_k + [F_{k+1}, U] = -F_k\Lambda + \hat{A}_{k+1} + \sum_{h+\ell=k+1} \hat{A}_hF_{\ell}. \]
From the off-diagonal part, we can compute all the entries \((F_{k+1})^\alpha\beta\) with \(\alpha \neq \beta\). The series \(F(s, z)\) is thus uniquely determined. The gauge transformation \(X(s, z) = G(s, z)Z(s, z)\) transforms the joint system (C.1)-(C.2) into the system
\[
\frac{dZ}{ds} = U(z)Z, \\
Z(s, z_1, \ldots, z_i - 1, \ldots, z_n) = P_i(s, z)Z(s, z),
\]
where we set
\[
P_i(s, z) := G(s, z_1, \ldots, z_i - 1, \ldots, z_n)^{-1}P_i(s, z)G(s, z)
\]
for \(i = 1, \ldots, n\). We claim that the matrices \(P_i\) are diagonal and they actually do not depend on \(s\). Indeed, the compatibility condition (C.3) for the transformed system imply
\[
\frac{d}{ds}P_i = [U, P_i], \quad i = 1, \ldots, n.
\]
Thus the entries of \(P_i\) are of the form
\[
P_i(s, z)_{\alpha\beta} = f_{\alpha\beta}(z) \exp((u_\alpha - u_\beta)s),
\]
where \(\alpha, \beta = 1, \ldots, n\), and \(f_{\alpha\beta}(z)\) is a function of \(z\). Since all the entries of the r.h.s of (C.21) are of the form
\[
s^m(z) \sum_{n=0}^{\infty} \frac{a_n(z)}{s^n},
\]
for suitable functions \(m(z), a_n(z)\), the operator \(P_i\) can be of the form (C.23) if and only if
\[
f_{\alpha\beta}(z) = 0, \quad \alpha \neq \beta.
\]
This concludes the proof. \(\square\)

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