Strong Rates of Convergence for Space-Time Discretization of the Backward Stochastic Heat Equation, and of a Linear-Quadratic Control Problem for the Stochastic Heat Equation

Andreas Prohl† and Yanqing Wang‡

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Abstract

We introduce a time-implicit, finite-element based space-time discretization scheme for the backward stochastic heat equation, and for the forward-backward stochastic heat equation from stochastic optimal control, and prove strong rates of convergence. The fully discrete version of the forward-backward stochastic heat equation is then used within a gradient descent algorithm to approximately solve the linear-quadratic control problem for the stochastic heat equation driven by additive noise.

Keywords: Strong error estimate, backward stochastic heat equation, stochastic linear quadratic problem,

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1 Introduction

Let $D \subseteq \mathbb{R}^d$ be a bounded domain with $C^2$ boundary, $T > 0$, and a (deterministic) function $\tilde{X} \equiv \{\tilde{X}(t); t \in [0,T]\} \in C([0,T];H^1 \cap H^2)$ be given. Our goal is to numerically approximate the $L^2$-valued, $\mathbb{F}$-adapted control process $U^* \equiv \{U^*(t); t \in [0,T]\}$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ that minimizes the functional $(\alpha \geq 0)$

$$J(X, U) = \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \|X(t) - \tilde{X}(t)\|_{L^2}^2 + \|U(t)\|_{L^2}^2 \right) \, dt + \alpha \|X(T) - \tilde{X}(T)\|_{L^2}^2 \right]$$

subject to the (controlled forward) stochastic heat equation (SPDE for short)

$$(1.2) \quad \begin{cases} \ dX(t) = \left[ \Delta X(t) + U(t) \right] \, dt + \sigma(t)dW(t) & \forall \, t \in [0,T], \\ X(0) = X_0, \end{cases}$$

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†Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, D-72076 Tübingen, Germany. e-mail: prohl@na.uni-tuebingen.de.

‡Corresponding author. School of Mathematics and Statistics, Southwest University, Chongqing 400715, China. e-mail: yqwang@amss.ac.cn.
which is supplemented by homogeneous Dirichlet boundary condition. Here \( W \equiv \{ W(t); t \in [0, T] \} \) is an \( \mathbb{R}^m \)-valued Wiener process, and \( \sigma \equiv \{ \sigma(t); t \in [0, T] \} \) is \( L_2^2(\Omega; L^2(0, T; H_0^1 \cap H^2)) \), which both are given on the same filtered probability space. For every given \( U \in L_2^2(\Omega; L^2(0, T; L^2)) \), there exists a unique \( H_0^1 \)-valued strong (variational) solution \( X \equiv X(U) \) in (1.2) such that \( X(0) = X_0 \in L^2(\Omega; H_0^1) \), and a unique minimizer \( (X^*, U^*) \in L_2^2(\Omega; C([0, T]; H^1) \cap L^2(0, T; H_0^1) \times L^2(0, T; L^2)) \) of the stochastic optimal control problem: ’minimize (1.1) subject to (1.2’)’ – which we below refer to as SLQ; see e.g. [2].

We consider problem SLQ as a prototype example of a (linear-quadratic) stochastic optimal control problem involving a stochastic PDE, for which corresponding numerical analyses so far are rare in the existing literature; see e.g. [23, 8]. This is in contrast to the deterministic counterpart problem LQ which involves a linear PDE, where optimal rates of convergence are available for (finite element based) space-time discretization of related optimality conditions (see e.g. [17, 16, 21, 18, 11]), which may then be used as part of a gradient descent algorithm with step size control [12] to approximate the minimizing tuple \( (X^*, U^*) \), which here consists of deterministic state and control functions. If compared to problem LQ, problem SLQ owns some distinctive characters and additional difficulties caused by the driving Wiener process in the SPDE (1.2), which make the generalization of the numerical results for the deterministic control problem to SLQ a non-trivial task. For example, a crucial difficulty consists in solving the adjoint equation in the context of SLQ, which here is a backward stochastic PDE (BSPDE for short) of the form

\[
\begin{align*}
    dY(t) & = \left[ -\Delta Y(t) + \left( X(t) - \bar{X}(t) \right) \right] dt + Z(t) dW(t) \quad \forall t \in [0, T], \\
    Y(T) & = -\alpha \left( X(T) - \bar{X}(T) \right),
\end{align*}
\]

having a solution tuple \( (Y, Z) \in L_2^2(\Omega; C([0, T]; H_0^1) \cap L^2(0, T; H_0^1 \cap H^2)) \times L_2^2(\Omega; L^2(0, T; H_0^1) \times L^2(0, T; L^2)) \); cf. [7]. The adjoint variable \( Y \) is then related to the optimal control by Pontryagin’s maximum principle, which in the case of problem SLQ is

\[
0 = U^*(t) - Y(t) \quad \forall t \in (0, T).
\]

The combination of equations (1.2), (1.3), and (1.4) then uniquely determines the optimal process tuple \( (X^*, U^*) \) of problem SLQ.

The convergence analysis of space-time discretization BSPDEs is a recent research subject, and available results are rare as well. A first error analysis for an (abstract) time-space discretization based on the implicit Euler method for the above BSPDE (1.3) is [23], where the error depends on the ratio of time discretization and Galerkin parameters. That work heavily draws conclusions with the help of Malliavin calculus, and the (space-)time regularity of solutions to the underlying BS(P)DE. In [8], the authors derive rates of convergence for a conforming finite element semi-discretization, and discuss its actual implementation. The proofs in [8] use simple variational arguments, resting on improved regularity properties of the variational solution, Itô’s formula, and approximation results for the finite element method. However, the interplay of spatial and temporal discretization errors is left open in [8], in particular the relevant question regarding unconditional convergence rates, which allow discretization parameters w.r.t. time and space to independently tend to zero, and general regular space-time meshes.

We address this issue in Section 3 as our first goal in this work: its derivation requires to study the time regularity of the solution \( (Y, Z) \) to BSPDE (2.10) in particular, which seems not available...
in the literature so far, and uses Malliavin calculus for that matter for the solution component $Z$, in particular. For this purpose, we borrow related arguments from [26, 13] where BSDEs are studied, and variational arguments.

The second goal in this work is addressed in Section 4, where strong error estimates for a space-time discretization of the coupled forward-backward SPDE (1.2)-(1.3) (FBSPDE for short) are shown. These results extend available ones (cf. [8]) in the literature in several aspects: the obtained strong convergence rates for the used finite element based space-time discretization (4.15) holds for arbitrary times $T$ — and is not only a semi-discretization in space where optimal rates are obtained in [8] for small times $T$ via a contraction argument. The numerical analysis uses variational arguments to first bound the error in the optimal controls, and here exploits the unique solvability of (the discretization of) problem SLQ, as well as the related sufficient and necessary optimality conditions; in a second step, error bounds for the optimal state, and its adjoint are based on stability properties of the state equation, and the adjoint equation available from Section 3.

To solve a BSPDE computationally requires huge computational resources (see [8]), and it is even more computationally demanding (in terms of computational storage requirements and computational times) to solve the coupled FBSPDE. Consequently, an alternative numerical strategy to the space-time discretization of FBSPDE is useful to make accurate computations feasible, which decouples the computation of (approximating iterates of) the solution parts from a SPDE from that of a BSPDE per iteration. A simple fixed-point method on the level of optimality conditions to accomplish this goal is known to converge only for small times $T > 0$ (cf. [8]); instead, we may again return to the fully discretized problem SLQ$_{\delta T}$ (4.13)-(4.14) and exploit its character as a minimization problem to initiate a gradient descent method to successively determine approximations of the optimal control; this method is detailed in Section 5, and a convergence order is shown for this iteration which is the final goal in this work.

The rest of this paper is organized as follows. In Section 2, we introduce notations, and review relevant properties of the problems BSPDE (2.10) and FBSPDE considered in this work. In Section 3, we prove strong error estimates for a space-time discretization of BSPDE. By virtue of the obtained error estimates, in Section 4, we prove a convergence rate for a space-time discretization of FBSPDE, which is related to problem SLQ. Convergence of the related iterative gradient descent method towards the minimizer $U^*$ of SLQ is shown in Section 5.

2 Preliminaries

2.1 Notation — involved processes and the finite element method

Let $(\mathbb{K}, (\cdot, \cdot)_{\mathbb{K}})$ be a separable Hilbert space. By $\| \cdot \|_{L^2}$ resp. $(\cdot, \cdot)_{L^2}$, we denote the norm resp. the scalar product in $L^2 := L^2(D)$. The norm in $H^1_0 := H^1_0(D)$, $H^2 := H^2(D)$ is denoted by $\| \cdot \|_{H^1_0}$, $\| \cdot \|_{H^2}$ respectively. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space, where $\mathbb{F}$ is the filtration generated by the $\mathbb{R}^m$-valued Wiener process $W$, which is augmented by all $\mathbb{P}$-null sets. Below, we set $m = 1$ for simplicity. The space of all $\mathbb{F}$-adapted processes $X : \Omega \times [0, T] \to \mathbb{K}$ satisfying $\mathbb{E}[\int_0^T \| X(t) \|_{H^1_0}^2 \, dt] < \infty$ is denoted by $L^2_H(\Omega; L^2(0, T; \mathbb{K}))$; the space of all $\mathbb{F}$-adapted processes $X : \Omega \times [0, T] \to \mathbb{K}$ satisfying $\mathbb{E}[\sup_{t \in [0, T]} \| X(t) \|_{\mathbb{K}}^2] < \infty$ is denoted by $L^2(\Omega; C([0, T]; \mathbb{K}))$.

We partition the bounded domain $D \subset \mathbb{R}^d$ via a regular triangulation $\mathcal{T}_h$ into elements $K$ with
maximum mesh size \( h := \max \{ \text{diam}(K) : K \in \mathcal{T}_h \} \), and consider spaces

\[
\mathbb{V}_h^1 := \{ \phi \in H^1_{01} : \phi|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h \}, \quad \mathbb{V}_h^0 := \{ \phi \in L^2 : \phi|_K \in \mathbb{P}_0(K) \quad \forall K \in \mathcal{T}_h \},
\]

where \( \mathbb{P}_i(K) \) denotes the space of polynomials of degree \( i \) \( (i = 0, 1) \). The \( L^2 \)-projection \( \Pi_h^1 : L^2 \to \mathbb{V}_h^1 \) is defined by \( (\Pi_h^1 \xi - \xi, \phi_h)_{L^2} = 0 \) for all \( \phi_h \in \mathbb{V}_h^1 \). We define the discrete Laplacean \( \Delta_h : \mathbb{V}_h^1 \to \mathbb{V}_h^1 \)
by \( (-\Delta_h \xi_h, \phi_h)_{L^2} = (\nabla \xi_h, \nabla \phi_h)_{L^2} \) for all \( \xi_h, \phi_h \in \mathbb{V}_h^1 \).

We use approximation estimates for the projection \( \Pi_h^1 \), and an inverse estimate (cf. [5]) to conclude that

\[
(2.1) \quad \| \Delta_h \Pi_h^1 \xi \|_{L^2} \leq C \| \nabla^2 \xi \|_{L^2} \quad \forall \xi \in H_0^1 \cap H^2,
\]

since

\[
\| \Delta_h \Pi_h^1 \xi \|_{L^2}^2 = - (\nabla [\Pi_h^1 \xi - \xi], \nabla \Delta_h \Pi_h^1 \xi)_{L^2} - (\nabla \xi, \nabla \Delta_h \Pi_h^1 \xi)_{L^2} \\
\leq C h \| \nabla^2 \xi \|_{L^2} \| \nabla \Delta_h \Pi_h^1 \xi \|_{L^2} + (\Delta \xi, \Delta_h \Pi_h^1 \xi)_{L^2} \\
\leq C (\| \nabla^2 \xi \|_{L^2} + \| \Delta \xi \|_{L^2}) \| \Delta_h \Pi_h^1 \xi \|_{L^2}.
\]

We denote by \( I_r = \{ t_n \}_{n=0}^N \subset [0, T] \) a time mesh with maximum step size \( \tau := \max \{ t_{n+1} - t_n : n = 0, 1, \ldots, N - 1 \} \), and \( \Delta_n W = W(t_n) - W(t_{n-1}) \) for all \( n = 1, \ldots, N \). For simplicity, we choose a uniform partition, i.e. \( \tau = T/N \) and \( \tau \leq 1 \). The results in this work still hold for quasi-uniform partitions.

### 2.2 The stochastic heat equation — strong convergence rates for a space-time discretization

For a given \( U \in L_2^2(\Omega; L^2(0, T; \mathbb{L}^2)) \), and \( X_0 \in \mathbb{H}_0^1 \) in (1.2), there exist a strong solution \( X \in L_2^2(\Omega; C([0, T]; \mathbb{H}_0^1)) \cap L_2^2(0, T; \mathbb{H}_0^1 \cap \mathbb{H}^2) \) solving \( \mathbb{P} \)-a.s. for all \( t \in [0, T] \)

\[
( X(t), \phi )_{L^2} - ( X_0, \phi )_{L^2} + \int_0^t \left[ (\nabla X(s), \nabla \phi)_{L^2} - (U(s), \phi)_{L^2} \right] \, ds \\
= \int_0^t (\sigma(s), \phi )_{L^2} \, dW(s) \quad \forall \phi \in \mathbb{H}_0^1,
\]

and a constant \( C \equiv C(D, T) > 0 \) such that

\[
(2.3) \quad \mathbb{E} \left[ \sup_{t \in [0, T]} \| X(t) \|_{\mathbb{H}_0^1}^2 + \int_0^T \| X(t) \|_{\mathbb{H}^2}^2 \, dt \right] \leq C \mathbb{E} \left[ \| X_0 \|_{\mathbb{H}_0^1}^2 + \int_0^T \| U(t) \|_{\mathbb{H}^2}^2 \, dt \right].
\]

A finite element discretization of (2.2) then reads: For all \( t \in [0, T] \), find \( X_h \in L_2^2(\Omega; C([0, T]; \mathbb{V}_h^1)) \) such that \( \mathbb{P} \)-a.s. and for all times \( t \in [0, T] \)

\[
( X_h(t), \phi_h )_{L^2} - ( X_0, \phi_h )_{L^2} + \int_0^t (\nabla X_h(s), \nabla \phi_h)_{L^2} - (U(s), \phi_h)_{L^2} \, ds \\
= \int_0^t (\sigma(s), \phi_h )_{L^2} \, dW(s) \quad \forall \phi_h \in \mathbb{V}_h^1.
\]
Equation (2.4) may be recast into the following SDE system,

\begin{equation}
\begin{aligned}
(2.5)
\begin{cases}
    dX_h(t) &= \left[\Delta_h X_h(t) + \Pi_h^1 U(t)\right] dt + \Pi_h^1 \sigma(t) dW(t), \\
    X_h(0) &= \Pi_h^1 X_0.
\end{cases}
\end{aligned}
\end{equation}

The derivation of a strong error estimate is standard, and uses the improved (spatial) regularity properties of the strong variational solution,

\begin{equation}
\begin{aligned}
(2.6)
    &\sup_{t \in [0,T]} \mathbb{E}\left[\|X_h(t) - X(t)\|_{L^2}^2\right] + \mathbb{E}\left[\int_0^T \|\nabla [X_h(t) - X(t)]\|_{L^2}^2 dt\right] \leq C h^2.
\end{aligned}
\end{equation}

We now consider a time-implicit discretization of (2.4) on a partition \( I_r \) of \([0,T]\). The problem then reads: For every \( 0 \leq n \leq N - 1 \), find a solution \( X_{h}^{n+1} \in L^2_{F_{\tau_{n+1}}}((\Omega; V_0^1)) \) such that \( \mathbb{P}\)-a.s.

\begin{equation}
\begin{aligned}
(2.7)
    (X_{h}^{n+1} - X_{h}^{n}, \phi_h)_{L^2} + \tau \left( [\nabla X_{h}^{n+1}, \nabla \phi_h]_{L^2} - (U(t_n), \phi_h)_{L^2}\right) &= (\sigma(t_n), \phi_h)_{L^2} \Delta_{n+1} W,
\end{aligned}
\end{equation}

where \( \Delta_{n+1} W := W(t_{n+1}) - W(t_n) \). The verification of the error estimate (see [24])

\begin{equation}
\begin{aligned}
(2.8)
    \max_{0 \leq n \leq N} \mathbb{E}\left[\|X_h(t_n) - X_n^2\|_{L^2}^2\right] + \tau \sum_{n=1}^N \mathbb{E}\left[\|\nabla [X_h(t_n) - X_n^2]\|_{L^2}^2\right] \leq C \tau
\end{aligned}
\end{equation}

rests on stability properties of the implicit Euler, as well as the bound

\begin{equation}
\begin{aligned}
(2.9)
    &\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{E}\left[\|X_h(t) - X_h(t_n)\|_{L^2}^2\right] dt \leq C \tau,
\end{aligned}
\end{equation}

which requires additional regularity properties of involved data, i.e., \( X_0 \in H_0^1 \cap H^2 \), as well as \( \sigma \in L^2(\Omega; L^2(0, T; H_0^1 \cap H^2)), U \in L^2(\Omega; L^2(0, T; H_0^1)) \), such that

\begin{equation}
\begin{aligned}
&\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{E}\left[\|U(t) - U(t_n)\|_{L^2}^2 + \|\sigma(t) - \sigma(t_n)\|_{L^2}^2\right] dt \leq C \tau,
\end{aligned}
\end{equation}

and the \( H^1 \)-stability of the \( L^2 \)-projection \( \Pi_h^1 \); cf. [6, 4].

2.3 The backward stochastic heat equation — a finite element based spatial discretization

Let \( Y_T \in L^2_{F_T}(\Omega; H_0^1) \) and \( f \in L^2(\Omega; L^2(0, T; L^2)) \). A strong solution to the backward stochastic heat equation

\begin{equation}
\begin{aligned}
(2.10)
\begin{cases}
    dY(t) &= \left[-\Delta Y(t) + f(t)\right] dt + Z(t) dW(t), \\
    Y(T) &= Y_T
\end{cases}
\end{aligned}
\end{equation}

is a pair of square integrable \( \mathbb{P} \)-adapted processes

\begin{equation}
\begin{aligned}
(Y, Z) \in L^2_{d}(\Omega; C([0, T]; H_0^1) \cap L^2(0, T; H_0^1 \cap H^2) \times L^2(0, T; H_0^1))
\end{aligned}
\end{equation}
such that \( \mathbb{P} \)-a.s. for all times \( t \in [0, T] \)

\[
(Y_T, \phi)_{L^2} - (Y(t), \phi)_{L^2} - \int_t^T \left[ (\nabla Y(s), \nabla \phi)_{L^2} + (f(s), \phi)_{L^2} \right] ds
= \int_t^T (Z(s), \phi)_{L^2} dW(s) \quad \forall \phi \in \mathbb{H}^1_0,
\]

(2.11)

and there exists a constant \( C \equiv C(D, T) > 0 \) such that

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \|Y(t)\|_{H^1}^2 \right] + \mathbb{E} \left[ \int_0^T \|Y(t)\|_{H^2}^2 + \|Z(t)\|_{H^1}^2 \, dt \right] \leq CE \left[ \|Y_T\|_{H^1}^2 + \int_0^T \|f(t)\|_{L^2}^2 \, dt \right].
\]

(2.12)

The existence of a strong solution to (2.10) in the above sense, as well as its uniqueness are shown in [7].

We now consider a finite element discretization of the BSPDE (2.10). Let \( Y_{T,h} \in L^2_P(\Omega; \mathbb{V}^1_h) \) be an approximation of \( Y_T \). The problem BSPDE\(_h\) then reads: Find \( (Y_h, Z_h) \in L^2_P(\Omega; C([0, T]; \mathbb{V}^1_h)) \times L^2_P(\Omega; L^2(0, T; \mathbb{V}^1_h)) \) such that \( \mathbb{P} \)-a.s. for all \( t \in [0, T] \)

\[
(Y_{T,h}, \phi_h)_{L^2} - (Y_h(t), \phi_h)_{L^2} - \int_t^T (\nabla Y_h(s), \nabla \phi_h)_{L^2} + (f(s), \phi_h)_{L^2} \, ds
= \int_t^T (Z_h(s), \phi_h)_{L^2} dW(s) \quad \forall \phi_h \in \mathbb{V}^1_h.
\]

(2.13)

Equation (2.13) is equivalent to the following system of BSDEs:

\[
\begin{align*}
\frac{dY_h(t)}{dt} &= [-\Delta_h Y_h(t) + \Pi^1_h f(t)] \, dt + Z_h(t) dW(t) \quad \forall t \in [0, T] \\
Y_h(T) &= Y_{T,h}.
\end{align*}
\]

(2.14)

The existence and uniqueness of a solution tuple \( (Y_h, Z_h) \) e.g. follows from [9, Theorem 2.1]. Moreover, there exists \( C \equiv C(f, T) > 0 \) such that

\[
\sup_{t \in [0, T]} \mathbb{E} \left[ \|\nabla Y_h(t)\|_{L^2}^2 \right] + \mathbb{E} \left[ \int_0^T \|\Delta_h Y_h(t)\|_{L^2}^2 + \|\nabla Z_h(t)\|_{L^2}^2 \, dt \right]
\leq C \mathbb{E} \left[ \|\nabla Y_{T,h}\|_{L^2}^2 + \int_0^T \|f(t)\|_{L^2}^2 \, dt \right];
\]

(2.15)

cf. [8, Lemma 3.1]. — The following result is taken from [8, Theorem 3.2], whose proof exploits the bounds (2.12).

**Theorem 2.1.** Let \( Y_T \in L^2_P(\Omega; \mathbb{H}^1_0) \), \( Y_{T,h} \in L^2_P(\Omega; \mathbb{V}^1_h) \). Let \( (Y, Z) \) be the solution to (2.11), and \( (Y_h, Z_h) \) solve (2.13). There exists \( C \equiv C(Y_T, f, T) > 0 \) such that

\[
\sup_{t \in [0, T]} \mathbb{E} \left[ \|Y(t) - Y_h(t)\|_{L^2}^2 \right] + \mathbb{E} \left[ \int_0^T \|\nabla [Y(t) - Y_h(t)]\|_{L^2}^2 + \|Z(t) - Z_h(t)\|_{L^2}^2 \, dt \right]
\leq C \left( \mathbb{E} \left[ \|Y_T - Y_{T,h}\|_{L^2}^2 \right] + h^2 \right).
\]

Choosing \( Y_{T,h} = \Pi^1_h Y_T \) thus leads to an error estimate for the spatial semi-discretization (2.14).
2.4 Temporal discretization of the backward stochastic heat equation — the role of the Malliavin derivative

The numerical analysis of a temporal discretization of (2.14) requires Malliavin calculus to bound temporal increments \( \mathbb{E}\|Z_h(t) - Z_h(s)\|_{L^2} \) in terms of \(|t - s|\), where \( s, t \in [0, T] \). We therefore recall the definition of the Malliavin derivative of processes, and the crucial connection between the Malliavin derivative of \( Y_h \) and \( Z_h \) from (2.14). For further details, we refer to [20, 9].

Let us recall that \( \mathcal{F}_T = \sigma\{W(t); 0 \leq t \leq T\} \), and that \( \mathbb{K} \) denotes a separable Hilbert space. We define the Itô isometry \( W : L^2(0, T; \mathbb{R}) \to L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}) \) by

\[
W(h) = \int_0^T h(t) \, dW(t).
\]

For \( \ell \in \mathbb{N} \), we denote by \( C^\infty_p(\mathbb{R}^\ell) \) the space of all smooth functions \( g : \mathbb{R}^\ell \to \mathbb{R} \) such that \( g \) and all of its partial derivatives have polynomial growth. Let \( \mathcal{P} \) be the set of \( \mathbb{R} \)-valued random variables of the form

(2.16) \[
F = g(W(h_1), \cdots, W(h_\ell))
\]

for some \( g \in C^\infty_p(\mathbb{R}^\ell), \ell \in \mathbb{N}, \) and \( h_1, \ldots, h_\ell \in L^2(0, T; \mathbb{R}) \). To any \( F \in \mathcal{P} \) we define its \( \mathbb{R} \)-valued Malliavin derivative \( D F := \{ D_t F; 0 \leq t \leq T\} \) process via

\[
D_tF = \sum_{i=1}^\ell \frac{\partial g}{\partial x_i}(W(h_1), \cdots, W(h_\ell))h_i(t).
\]

In general, we can define the \( k \)-th iterated derivative of \( F \) by \( D^k F = D(D^{k-1} F) \), for any \( k \in \mathbb{N} \).

Now we extend the derivative operator to \( \mathbb{K} \)-valued variables. For any \( k \in \mathbb{N} \), and \( u \) in the set of \( \mathbb{K} \)-valued variables:

\[
\mathcal{P}_\mathbb{K} = \left\{ u = \sum_{j=1}^n F_j \phi_j : F_j \in \mathcal{P}, \ \phi_j \in \mathbb{K}, \ n \in \mathbb{N} \right\},
\]

we can define the \( k \)-th iterated derivative of \( u \) by

\[
D^k u = \sum_{j=1}^n D^k F_j \otimes \phi_j.
\]

For \( p \geq 1 \), we define the norm \( \| \cdot \|_{k,p} \) via

\[
\|u\|_{k,p} := \left( \mathbb{E}\|u\|_{\mathbb{K}}^p + \sum_{j=1}^k \|D^j u\|_{(L^2(0,T;\mathbb{R}))^{\otimes j} \otimes \mathbb{K}}}^p \right)^{\frac{1}{p}}.
\]

Then \( \mathbb{D}^{k,p}(\mathbb{K}) \) is the completion of \( \mathcal{P} \) under the norm \( \| \cdot \|_{k,p} \).

We may now express \( Z_h \) in BSDE (2.14) in terms of the Malliavin derivative of \( Y_h \).

Lemma 2.2 ([9], Prop. 5.3). Suppose that \( Y_{T,h} \in \mathbb{D}^{1,2}(\mathbb{L}^2), \ f \in L^2(\Omega; L^2(0,T;\mathbb{L}^2)), \) and

\[
\mathbb{E}\left[ \int_0^T \|D_\theta Y_{T,h}\|_{L^2}^2 \, d\theta \right] + \mathbb{E}\left[ \int_0^T \int_0^T \|D_\theta f(t)\|_{L^2}^2 \, dt \, d\theta \right] < \infty.
\]
Let \((Y_h, Z_h)\) be the solution to BSDE (2.14). Then

\[ (Y_h, Z_h) \in L^2_T \left( \Omega; C([0, T]; \mathbb{D}^{1,2}(L^2)) \times L^2(0, T; \mathbb{D}^{1,2}(L^2)) \right), \]

and its Malliavin derivative \((D_\theta Y_h, D_\theta Z_h)\) solves

\[ \begin{align*}
D_\theta Y_h(t) - D_\theta Y_h(T) + \int_t^T -\Delta_h D_\theta Y_h(s) + \Pi^1_h D_\theta f(s) \, ds \\
\quad = - \int_t^T D_\theta Z_h(s) \, dW(s) \quad 0 \leq \theta \leq t \leq T, \\
D_\theta Y_h(t) = D_\theta Z_h(t) = 0 \quad 0 \leq t < \theta \leq T.
\end{align*} \tag{2.17} \]

Moreover, \(\{D_\theta Y_h(t) : 0 \leq t \leq T\}\) is a version of \(\{Z_h(t) : 0 \leq t \leq T\}\).

### 3 Strong rates of convergence for a space-time discretization of the BSPDE (2.10)

In this section, we introduce the time discretization scheme (3.6) to approximate the solution \((Y_h, Z_h)\) to the BSPDE\(_h\) (2.14) by a finite sequence \(\{(Y^n_h, Z^n_h)\}_{n=0}^{N-1}\) on a mesh \(I_\tau\). The main results are Theorems 3.4 and 3.6 in Subsection 3.2. Their derivation crucially hinges on the time regularity of the solution \((Y_h, Z_h)\) to (2.14), which is provided in the subsequent Subsection 3.1.

#### 3.1 Uniform bounds for temporal increments of the solution \((Y_h, Z_h)\) to (2.14)

We start with the derivation of uniform estimates for \(Y_h\) which control its temporal increments. We note again that all involved generic constants \(C > 0\) do not depend on \(h\).

**Lemma 3.1.** Suppose that \(Y_{T,h} \in L^2_T(\Omega; \mathbb{H}_1^1), f \in L^2_F(\Omega; L^2(0, T; \mathbb{H}_1^1)), I_\tau\) is a time partition of \([0, T]\). Let \((Y_h, Z_h)\) be the solution to (2.14). Then

(i) \[ \sum_{n=0}^{N-1} \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} \|Y_h(t) - Y_h(t_n)\|_{L^2}^2 \, dt \right] \leq C_T \mathbb{E} \left[ \|Y_{T,h}\|_{\mathbb{H}_1^1}^2 + \int_0^T \|f(t)\|_{L^2}^2 \, dt \right]. \]

(ii) Assume further \(\sup_{h > 0} \mathbb{E}[\|\Delta_h Y_{T,h}\|_{L^2}^2] < \infty\). Then

\[ \sum_{n=0}^{N-1} \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} \|\nabla (Y_h(t) - Y_h(t_n))\|_{L^2}^2 \, dt \right] \leq C_T \mathbb{E} \left[ \|\Delta_h Y_{T,h}\|_{\mathbb{H}_1^1}^2 + \|\nabla Y_{T,h}\|_{\mathbb{H}_1^1}^2 + \int_0^T \|f(t)\|_{L^2}^2 \, dt \right]. \]

(iii) Assume further \(\sup_{h > 0} \mathbb{E}[\|\Delta_h Y_{T,h}\|_{\mathbb{H}_1^1}^2] < \infty\) and \(f \in L^2_F(\Omega; L^2(0, T; \mathbb{H}_1^1 \cap \mathbb{H}_2^2))\). Then

\[ \sum_{n=0}^{N-1} \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} \|\Delta_h (Y_h(t) - Y_h(t_n))\|_{L^2}^2 \, dt \right] \leq C_T \mathbb{E} \left[ \|\Delta_h Y_{T,h}\|_{\mathbb{H}_1^1}^2 + \int_0^T \|f(t)\|_{L^2}^2 \, dt \right]. \]

Here, the constant \(C > 0\) only depends on \(Y_{T,h}, f\) and \(T\).
Proof. We only prove (i). The other statements can be proved in a similar vein.

By BSDE (2.14), we get
\[
N^{-1} \sum_{n=0}^{N-1} \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} \|Y_h(t) - Y_h(t_n)\|^2_{L^2} \, dt \right] \leq C_T \int_0^T \mathbb{E} \left[ \|\Delta_h Y_h(t)\|^2_{L^2} + \|\Pi_h f(t)\|^2_{L^2} + \|Z_h(t)\|^2_{L^2} \right] \, dt.
\]

Applying Itô’s formula for \(\|Y_h\|^2_{L^2}\) and \(\|\nabla Y_h\|^2_{L^2}\) in (2.14), we see
\[
\begin{align*}
\mathbb{E} \left[ \int_0^T \|Z_h(t)\|^2_{L^2} \, dt \right] &\leq C \mathbb{E} \left[ \|Y_{T,h}\|^2_{L^2} + \int_0^T \|\Pi_h f(t)\|^2_{L^2} \, dt \right], \\
\mathbb{E} \left[ \int_0^T \|\Delta_h Y_h(t)\|^2_{L^2} \, dt \right] &\leq C \mathbb{E} \left[ \|\nabla Y_{T,h}\|^2_{L^2} + \int_0^T \|\Pi_h f(t)\|^2_{L^2} \, dt \right].
\end{align*}
\]

Then (i) can be deduced by the above estimates. 

Lemma 3.2. Suppose that \(Y_{T,h} \in \mathbb{D}^{1,2}(\mathbb{H}_0^1)\), and \(f \in L^2_\mathbb{P}(\Omega; L^2(0, T; L^2))\) satisfy
\[
\sup_{0 \leq t \leq T} \mathbb{E} \left[ \|D_t Y_{T,h}\|^2_{H^1_0} \right] + \sup_{0 \leq t \leq T} \mathbb{E} \left[ \|D_{\theta} Y_{T,h}\|^2_{L^2} \right] < C,
\]
\[
\sup_{0 \leq t \leq T} \mathbb{E} \left[ \int_0^T \|D_t f(\tau)\|^2_{L^2} \, d\tau \right] + \sup_{0 \leq \theta \leq T \leq t \leq T} \mathbb{E} \left[ \int_0^T \|D_{\theta} D_t f(\tau)\|^2_{L^2} \, d\tau \right] \leq C,
\]
and for any \(s, t \in [0, T]\) with \(s \leq t\),
\[
\mathbb{E} \left[ \| (D_t - D_s) Y_{T,h}\|^2_{L^2} \right] + \mathbb{E} \left[ \int_s^T \| (D_t - D_s) f(\tau)\|^2_{L^2} \, d\tau \right] \leq C |t - s|.
\]

Then, it holds that
\[
\mathbb{E} \left[ \|Z_h(t) - Z_h(s)\|^2_{L^2} \right] \leq C |t - s|.
\]

Proof. By Lemma 2.2, we know that \(Z_h(t) = D_t Y_h(t)\) for all \(0 \leq t \leq T\), and therefore, for \(0 \leq s \leq t \leq T\),
\[
\frac{1}{2} \mathbb{E} \left[ \|Z_h(t) - Z_h(s)\|^2_{L^2} \right] \leq \mathbb{E} \left[ \|D_t Y_h(t) - D_s Y_h(t)\|^2_{L^2} \right] + \mathbb{E} \left[ \|D_s Y_h(t) - D_s Y_h(s)\|^2_{L^2} \right].
\]

In what follows, we estimate the two terms on the right-hand side of (3.2) independently.

Step 1. Fix two \(0 \leq \theta_2 \leq \theta_1 \leq t \leq T\) and define \(\delta_\theta = D_{\theta_1} - D_{\theta_2}\). By (2.17), we have the BSDE
\[
\delta_\theta Y_h(t) - \delta_\theta Y_h(T) + \int_t^T \left[ -\Delta_h \delta_\theta Y_h(s) + \Pi_h \delta_\theta f(s) \right] \, ds = -\int_t^T \delta_\theta Z_h(s) \, dW(s) \quad \forall t \in [\theta_1, T].
\]

Itô’s formula and Poincaré’s inequality lead to
\[
\begin{align*}
\mathbb{E} \left[ \|\delta_\theta Y_h(t)\|^2_{L^2} \right] &+ \int_t^T \mathbb{E} \left[ \|\nabla \delta_\theta Y_h(s)\|^2_{L^2} + \|\delta_\theta Z_h(s)\|^2_{L^2} \right] \, ds \\
&\leq \mathbb{E} \left[ \|\delta_\theta Y_h(T)\|^2_{L^2} \right] + \int_t^T \mathbb{E} \left[ \|\Pi_h \delta_\theta f(s)\|^2_{L^2} \right] \, ds.
\end{align*}
\]
Taking $\theta_2 = s$ and $\theta_1 = t$ and using (3.1) then lead to
\[
\mathbb{E}\left[\|D_t Y_h(t) - D_s Y_h(t)\|^2_{L^2}\right] \leq \mathbb{E}\left[\|D_t - D_s\| Y_h(t)\|^2_{L^2} + \int_t^T \|D_t - D_s\| f(\tau)\|^2_{L^2} \, d\tau\right]
\]
\[
(3.4)
\]
\[
\leq C|t - s|.
\]

Step 2. By (2.17), Itô’s isometry together with Poincaré’s inequality,
\[
\mathbb{E}\left[\|D_s Y_h(t) - D_s Y_h(s)\|^2_{L^2}\right] = \mathbb{E}\left[\int_s^t \left[-\Delta_h Y_h(\tau) + \Pi_h f(\tau)\right] \, d\tau + \int_s^t D_s Z_h(\tau) \, dW(\tau)\right]^{\frac{1}{2}}
\]
\[
\leq 2|t - s| \int_s^T \mathbb{E}\left[\|\Delta_h D_s Y_h(\tau)\|^2_{L^2} + \|\Pi_h D_s f(\tau)\|^2_{L^2}\right] d\tau + 2 \int_s^T \mathbb{E}\left[\|D_s Z_h(\tau)\|^2_{L^2}\right] d\tau
\]
\[
\leq C|t - s| \mathbb{E}\left[\|\nabla D_s Y_h(T)\|^2_{L^2} + \int_s^T \|\Pi_h D_s f(\tau)\|^2_{L^2} \, d\tau\right]
\]
\[
+ C|t - s| \sup_{0 < \theta < T} \sup_{0 < t < T} \mathbb{E}\left[\|D_0 D_t Y_{T,h}\|^2_{L^2} + \int_{\theta < t} \|D_0 D_t f(\tau)\|^2_{L^2} \, d\tau\right].
\]

Inserting (3.5) and (3.4) into (3.2) then settles the proof of the lemma.

3.2 A time-implicit space-time discretization of the BSPDE (2.10)

We use an implicit time discretization on the mesh $I_\tau$ to approximate BSPDE$_{t\tau}$ (2.14); we refer to it as BSPDE$_{h\tau}$, and the discretization reads as follows: For every $0 \leq n \leq N - 1$, find $(Y^n_h, Z^n_h) \in L^2_{F_{t_n}}(\Omega; \mathcal{V}^1_h \times \mathcal{V}^1_h)$ such that
\[
[1 - \tau \Delta_h] Y_h^n = \mathbb{E}\left[Y_{n+1}^n \big| \mathcal{F}_n\right] - \tau \Pi_h f(t_n),
\]
\[
Z_h^n = \frac{1}{\tau} \mathbb{E}\left[Y_{n+1}^n \Delta h W \big| \mathcal{F}_n\right],
\]
\[
Y_h^n = Y_{T,h}.
\]

We introduce an auxiliary BSDE system on each time interval $[t_n, t_{n+1}]$ for the convergence analysis of (3.6): Find $(\overline{Y}_{h,n}, \overline{Z}_{h,n}) \in L^2_{F_{t_n}}(\Omega; C([t_n, t_{n+1}]; \mathcal{V}_h^1) \times L^2(t_n, t_{n+1}; \mathcal{V}_h^1))$ such that
\[
\overline{Y}_{h,n}(t) - \overline{Y}_{h,n}(t_{n+1}) + \int_{t_n}^{t_{n+1}} \left[-\Delta_h Y_h^n + \Pi_h f(t_n)\right] \, ds = -\int_{t_n}^{t_{n+1}} \overline{Z}_{h,n}(s) \, dW(s)
\]
\[
(3.7)
\]
\[
\overline{Y}_{h,n}(T) = Y_{T,h}.
\]

We now construct $(Y_h^n, Z_h^n) \in L^2_{F_t}(\Omega; C([0, T]; \mathcal{V}_h^1) \times L^2(0, T; \mathcal{V}_h^1))$ via $(Y_h^n |_{[t_n, t_{n+1}]}, Z_h^n |_{[t_n, t_{n+1}]}) := (\overline{Y}_{h,n}, \overline{Z}_{h,n})$. Note that the integrand in the drift is evaluated with the help of the solution part $(Y_h^n |_{t_n})_{n=0}^{N-1}$ from (3.7).

Lemma 3.3. Let $\{(Y^n_h, Z^n_h)\}_{n=0}^{N-1}$ solve (3.6), and $(\overline{Y}_h, \overline{Z}_h)$ solve (3.7). For all $0 \leq n \leq N - 1$,
\[
Y^n_h = \overline{Y}_h(t_n), \quad Z^n_h = \frac{1}{\tau} \mathbb{E}\left[\int_{t_n}^{t_{n+1}} \overline{Z}_h(s) \, ds \big| \mathcal{F}_{t_n}\right].
\]
Proof. The first identity is immediate; the second follows from multiplication of (3.7) with the admissible $f_{t_n}^{t_{n+1}} 1 \, dW(s)$, and application of conditional expectation $\mathbb{E}[\cdot | \mathcal{F}_{t_n}]$. \hfill \Box

We may now prove a strong error estimate for the first component of $(Y_h, Z_h)$ that solves (2.14).

**Theorem 3.4.** Suppose that $Y_{T,h} \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{H}_0)$, $\Delta h Y_{T,h} \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{L}^2)$, $f \in L^2(\Omega; L^2(0, T; \mathbb{H}_0))$ as well as

$$
\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{E}[|f(t) - f(t_n)|^2_{\mathbb{L}^2}] \leq C \tau .
$$

Let $(Y_h, Z_h)$ solve (2.14), and $\{(Y_h^n, Z_h^n)\}_{n=0}^{N-1}$ solve (3.6). There exists a constant $C \equiv (Y_{T,h}, f, T) > 0$ such that

$$
\max_{0 \leq n \leq N} \mathbb{E}[\|Y_h(t_n) - Y_h^n\|_{\mathbb{L}^2}^2] + \tau \sum_{n=0}^{N-1} \mathbb{E}[\|\nabla(Y_h(t_n) - Y_h^n)\|_{\mathbb{L}^2}^2] \leq C \tau .
$$

Proof. Consider $(\mathbf{Y}_h, \mathbf{Z}_h)$ from (3.7), and define $\{e_n\}_{n=0}^{N-1}$, where each $e_n = Y_h(t_n) - \mathbf{Y}_h(t_n)$ is a $V^1_h$-valued random variable. Subtracting (3.7) from (2.14) yields \mathbb{P}\text{-a.s.}

$$
e_n - e_{n+1} - \int_{t_n}^{t_{n+1}} \Delta h e_n \, ds = \int_{t_n}^{t_{n+1}} \Delta h [Y_h(s) - Y_h(t_n)] - \Pi_h[f(s) - f(t_n)] \, ds
$$

$$+ \int_{t_n}^{t_{n+1}} [Z_h(s) - \mathbf{Z}_h(s)] \, dW(s).$$

Fixing one realization $\omega \in \Omega$, testing with the admissible $e_n(\omega) \in V^1_h$, using binomial formula, and then taking expectation, and Poincare’s and Young’s inequality lead to

$$
\frac{1}{2} \mathbb{E} \left[ \|e_n\|_{\mathbb{L}^2}^2 - \|e_{n+1}\|_{\mathbb{L}^2}^2 + \|e_n - e_{n+1}\|_{\mathbb{L}^2}^2 + 2 \int_{t_n}^{t_{n+1}} \|\nabla e_n\|_{\mathbb{L}^2}^2 \, ds \right]
$$

$$\leq \int_{t_n}^{t_{n+1}} \mathbb{E} \left[ \frac{1}{2} \|\nabla [Y_h(s) - Y_h(t_n)]\|_{\mathbb{L}^2}^2 + \|f(s) - f(t_n)\|_{\mathbb{L}^2}^2 \right] \, ds
$$

$$+ \frac{1}{2} \int_{t_n}^{t_{n+1}} \mathbb{E} \left[ \|\nabla e_n\|_{\mathbb{L}^2}^2 + \|e_n - e_{n+1}\|_{\mathbb{L}^2}^2 \right] \, ds + \frac{\tau}{2} \mathbb{E}[\|e_{n+1}\|_{\mathbb{L}^2}^2]^2.
$$

Subsequently, the discrete Gronwall inequality leads to

$$
\max_{0 \leq n \leq N} \mathbb{E}[\|e_n\|_{\mathbb{L}^2}^2] \leq 2e^{CT} \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{E} \left[ \|\nabla [Y_h(s) - Y_h(t_n)]\|_{\mathbb{L}^2}^2 + \|f(s) - f(t_n)\|_{\mathbb{L}^2}^2 \right] \, ds.
$$

Then, summing up over all steps of (3.10) yields

$$
\sum_{n=0}^{N-1} \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} \|\nabla e_n\|_{\mathbb{L}^2}^2 \, ds \right]
$$

$$\leq \tau \sum_{n=0}^{N-1} \mathbb{E}[\|e_{n+1}\|_{\mathbb{L}^2}^2] + 2 \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{E} \left[ \|\nabla [Y_h(s) - Y_h(t_n)]\|_{\mathbb{L}^2}^2 + \|f(s) - f(t_n)\|_{\mathbb{L}^2}^2 \right] \, ds.
$$

Then, (3.11), (3.12) together with (ii) of Lemma 3.1, and 3.3 lead to the desired estimate. \hfill \Box
By Theorems 2.1, 3.4 and Lemma 3.1 (i), we thus get the following convergence rate for the approximation \( \{Y_h^n\}_{n=0}^N \) of the first solution component \( Y \) to (2.11) via the space-time discretization scheme (3.6),

\[
\max_{0 \leq n \leq N} \mathbb{E} \left[ \|Y(t_n) - Y_h^n\|_{L^2}^2 \right] + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{E} \left[ \|\nabla(Y(t) - Y_h^n)\|_{L^2}^2 \right] \, dt \leq C (\tau + h^2) .
\]

**Remark 3.5.** If the drift term of (2.10) is \(-\Delta Y(t, x) + f(t, x, Y(t, x))\), with a Lipschitz non-linearity \( f \), we may apply a similar procedure to get the above convergence rate. However, the above strategy is not clear to be successful if \( Z \) appears in the drift term.

We now derive estimates for the approximation \( \{Z^n\}_{n=0}^{N-1} \) of the second solution component \( Z \) to (2.11), which uses the characterization \( Z_h(t) = D_t Y_h(t) \), and (2.17).

**Theorem 3.6.** Let \((Y_h, Z_h)\) solve (2.13), where data satisfy the assumptions in Lemma 3.1 (ii), Lemma 3.2, as well as

\[
\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{E} \left[ \|f(t) - f(t_n)\|_{L^2}^2 \right] \leq C \tau .
\]

Let \(\{ (Y_h^n, Z^n_h) \}_{n=0}^{N-1}\) solve (3.6). There exists a constant \( C \equiv (Y_T, f, T) > 0 \) such that

\[
\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{E} \left[ \|Z(t) - Z^n_h\|_{L^2}^2 \right] \, dt \leq C \tau .
\]

The proof begins with an estimate for \( Z_h - Z_h \), which exploits time regularity properties of the solution \((Y_h, Z_h)\) in stronger norms; cf. Lemma 3.1, (ii). Moreover, the following technical result is needed; see also [23].

**Lemma 3.7.** For any \( \varphi \in L^2_0(0, T; \mathbb{R}) \) and \( 0 \leq s < t \leq T \), define

\[
\varphi_0 = \frac{1}{t - s} \mathbb{E} \left[ \int_s^t \varphi(\tau) \, d\tau \bigg| \mathcal{F}_s \right].
\]

For any \( \xi \in L^2_0(\Omega; \mathbb{R}) \) there holds

\[
\mathbb{E} \left[ \int_s^t \|\varphi(\tau) - \varphi_0\|_{K}^2 \, d\tau \right] \leq \mathbb{E} \left[ \int_s^t \|\varphi(\tau) - \xi\|_{K}^2 \, d\tau \right].
\]

**Proof.** Let \( \{\phi_i\}_{i=1}^\infty \) be an orthonormal basis of \( K \), and \( \Pi_n \) be the projection from \( K \) to span\{\( \phi_i : i = 1, 2, \cdots, n \). For any \( \eta \in K \), one has

\[
\mathbb{E} \left[ \int_s^t \|\varphi(\tau) - \xi\|_{K}^2 \, d\tau \right] \geq \mathbb{E} \left[ \int_s^t \|\Pi_n(\varphi(\tau) - \xi)\|_{K}^2 \, d\tau \right]
\]

\[
=\mathbb{E} \left[ \int_s^t \|\Pi_n(\varphi(\tau) - \varphi_0)\|_{K}^2 + \|\Pi_n(\varphi_0 - \xi)\|_{K}^2 + 2\langle \Pi_n(\varphi(\tau) - \varphi_0), \Pi_n(\varphi_0 - \xi) \rangle_K \, d\tau \right]
\]

\[
=\mathbb{E} \left[ \int_s^t \|\Pi_n(\varphi(\tau) - \varphi_0)\|_{K}^2 + \|\Pi_n(\varphi_0 - \xi)\|_{K}^2 \, d\tau \right]
\]

\[
+ 2\mathbb{E} \left[ \left( \int_s^t \Pi_n(\varphi(\tau) \, d\tau - \mathbb{E} \left[ \int_s^t \Pi_n(\varphi(\tau) \, d\tau \bigg| \mathcal{F}_s \right] , \Pi_n(\varphi_0 - \xi) \right) \right)_K \right] \]
Since, \( \varphi_0 \) and \( \xi \) are \( \mathcal{F}_s \)-measurable, the last term vanishes, i.e.,
\[
\mathbb{E}\left[ \mathbb{E}\left[ (\int_s^t \Pi_n \varphi(\tau) \, d\tau - \mathbb{E}[\int_s^t \Pi_n \varphi(\tau) \, d\tau | \mathcal{F}_s], \Pi_n(\varphi_0 - \xi))_X \right] \right] \\
= \mathbb{E}\left[ (\int_s^t \Pi_n \varphi(\tau) \, d\tau - \mathbb{E}[\int_s^t \Pi_n \varphi(\tau) \, d\tau | \mathcal{F}_s], \Pi_n(\varphi_0 - \xi))_X \right] = 0.
\]
Therefore,
\[
\mathbb{E}\left[ \int_s^t \|\varphi(\tau) - \xi\|_X^2 \, d\tau \right] \geq \mathbb{E}\left[ \int_s^t \|\Pi_n(\varphi(\tau) - \varphi_0)\|_X^2 \, d\tau \right] + \mathbb{E}\left[ \int_s^t \|\Pi_n(\varphi_0 - \xi)\|_X^2 \, d\tau \right] \\
\geq \mathbb{E}\left[ \int_s^t \|\Pi_n(\varphi(\tau) - \varphi_0)\|_X^2 \, d\tau \right].
\]
By letting \( n \uparrow \infty \), we may therefore conclude
\[
\mathbb{E}\left[ \int_s^t \|\varphi(\tau) - \varphi_0\|_X^2 \, d\tau \right] = \lim_{n \to \infty} \mathbb{E}\left[ \int_s^t \|\Pi_n(\varphi(\tau) - \varphi_0)\|_X^2 \, d\tau \right] \leq \mathbb{E}\left[ \int_s^t \|\varphi(\tau) - \xi\|_X^2 \, d\tau \right],
\]
which completes the proof.

**Proof of Theorem 3.6.** **Step 1.** Claim: there exists a constant \( C \), which is independent of \( h, \tau \), such that
\[
(3.14) \quad \mathbb{E}\left[ \int_0^T \|Z_h(s) - Z_h(s)\|_{L^2}^2 \, ds \right] \leq C \tau.
\]
We recall the definition of \( \{e_n\}_{n=0}^{N-1} \) in the proof of Theorem 3.4, as well as equation (3.9), which we recast into the form
\[
(\mathbb{I} - \tau \Delta_h)e_n + \int_{t_n}^{t_{n+1}} Z_h(s) - Z_h(s) \, dW(s) \\
= e_{n+1} + \int_{t_n}^{t_{n+1}} \left[ \Delta_h (Y_h(s) - Y_h(t_n)) - \Pi_h^1 (f(s) - f(t_n)) \right] \, ds.
\]
Taking squares and afterwards expectations on both sides, by binomial formula, Itô isometry, and Young’s inequality, we arrive at
\[
\mathbb{E}\left[ \left\| (\mathbb{I} - \tau \Delta_h)e_n \right\|_{L^2}^2 + \int_{t_n}^{t_{n+1}} \|Z_h(s) - Z_h(s) \, dW(s) \|_{L^2}^2 \right] \\
= \mathbb{E}\left[ \left\| (\mathbb{I} - \tau \Delta_h)e_n \right\|_{L^2}^2 + \int_{t_n}^{t_{n+1}} \|Z_h(s) - Z_h(s) \|_{L^2}^2 \, ds \right] \\
\leq (1 + \tau) \mathbb{E}\left[ \left\| e_{n+1} \right\|_{L^2}^2 + (1 + \frac{1}{4\tau}) \tau \int_{t_n}^{t_{n+1}} \left\| \Delta_h [Y_h(s) - Y_h(t_n)] - \Pi_h^1 [f(s) - f(t_n)] \right\|_{L^2}^2 \, ds \right].
\]
Note that \( \left\| (\mathbb{I} - \tau \Delta_h)e_n \right\|_{L^2}^2 = \left\| e_n \right\|_{L^2}^2 + 2\tau \|\nabla e_n\|_{L^2}^2 + \tau^2 \|\Delta_h e_n\|_{L^2}^2 \). Summation over \( 0 \leq n \leq N - 1 \) then leads to
\[
\mathbb{E}\left[ \left\| e_0 \right\|_{L^2}^2 + 2\tau \sum_{n=0}^{N-1} \|\nabla e_n\|_{L^2}^2 + \int_0^T \|Z_h(s) - Z_h(s) \|_{L^2}^2 \, ds \right] \\
\leq \tau \sum_{n=0}^{N-1} \mathbb{E}\left[ \left\| e_{n+1} \right\|_{L^2}^2 + 2\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{E}\left[ \left\| \Delta_h [Y_h(s) - Y_h(t_n)] - \Pi_h^1 [f(s) - f(t_n)] \right\|_{L^2}^2 \right] \, ds \right].
\]

(3.15)
By the discrete version of Gronwall’s inequality, and Lemma 3.1, (iii), the right-hand side is bounded by $C \tau$. Hence, (3.14) is proved.

**Step 2.** We use the triangle inequality twice to deduce

$$\sum_{n=0}^{N-1} \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} \|Z_h(t) - Z_h^n\|_{L^2}^2 \, dt \right]$$

$$\leq 2 \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{E} \left[ \|Z_h(t) - \mathbb{Z}_h(t)\|_{L^2}^2 + \|\mathbb{Z}_h(t) - Z_h(t)\|_{L^2}^2 + |Z_h(t) - Z_h^n|_{L^2}^2 \right] \, dt$$

$$\leq 2 \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{E} \left[ 3\|Z_h(t) - \mathbb{Z}_h(t)\|_{L^2}^2 + 2\|Z_h(t) - Z_h(t_n)\|_{L^2}^2 \right] \, dt .$$

By the definition of $Z_h^n$, on taking $\xi = Z_h(t_n)$ in Lemma 3.7 we may further estimate by

$$\leq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{E} \left[ 6\|Z_h(t) - \mathbb{Z}_h(t)\|_{L^2}^2 + 4\|Z_h(t) - Z_h(t_n)\|_{L^2}^2 \right] \, dt .$$

We use (3.14) to bound the first term, and Lemma 3.2 is utilized to bound the last term.

**4 Strong rates of convergence for a space-time discretization of SLQ**

In this part, we discretize the original problem SLQ within two steps, starting with its semi-discretization in space (which is referred to as SLQ$_h$), which is then followed by a discretization in space and time (which is referred to as SLQ$_{h\tau}$). Our goal is to prove strong convergence rates in both cases. By [15], the SLQ problem is uniquely solvable, and its solution $(X^*, U^*)$ may be characterized by the following FBSDE with the unique solution $(X^*, Y, Z, U^*)$,

$$\begin{cases}
    dX^*(t) = (\Delta X^*(t) + U^*(t)) \, dt + \sigma(t) \, dW(t) & \forall t \in (0, T), \\
    dY(t) = (-\Delta Y(t) + [X^*(t) - \bar{X}(t)]) \, dt + \sigma(t) \, dW(t) & \forall t \in (0, T), \\
    X^*(0) = X_0, & Y(T) = -\alpha(X^*(T) - \bar{X}(T)),
\end{cases}
$$

(4.1)

with the condition

$$U^* - Y = 0 .$$

(4.2)

We remark that by (4.1)$_1$, $X^*$ may be written as $X^* = S(U^*)$, where

$$S : L^2_b(\Omega; L^2(0, T; \mathbb{L}^2)) \to L^2_b(\Omega; C([0, T]; \mathbb{H}^1) \cap L^2(0, T; \mathbb{H}^2))$$

is the bounded ‘control-to-state’ map. Moreover, we introduce the reduced functional

$$\widehat{\mathcal{J}} : L^2_b(\Omega; L^2(0, T; \mathbb{L}^2)) \to \mathbb{R} \quad \text{via} \quad \widehat{\mathcal{J}}(U) = \mathcal{J}(S(U), U) .$$

The first component of the solution to equation (4.1)$_2$ may be written as $Y = \mathcal{T}(X^*)$, where $\mathcal{T}$

$$\mathcal{T} : L^2_b(\Omega; C([0, T]; \mathbb{L}^2)) \to L^2_b(\Omega; C([0, T]; \mathbb{H}^1) \cap L^2(0, T; \mathbb{H}^1 \cap \mathbb{H}^2)) ,$$

which is also bounded. For every $U \in L^2_b(\Omega; L^2(0, T; \mathbb{L}^2))$, the Gateaux derivative $D\widehat{\mathcal{J}}(U)$ is also a bounded operator on $L^2_b(\Omega; L^2(0, T; \mathbb{L}^2))$ and takes the form

$$D\widehat{\mathcal{J}}(U) = U - \mathcal{T}(S(U)) .$$
4.1 Problem SLQ$_h$: Semi-discretization in space

We begin with a spatial semi-discretization SLQ$_h$ of the problem SLQ stated in the introduction, which reads: Find an optimal pair $(X^*_h, U^*_h) \in L^2(\Omega; C([0, T]; \mathbb{V}^1_h))$ that minimizes the functional $(\alpha \geq 0)$

$$ J(X_h, U_h) = \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \|X_h(t) - \tilde{X}(t)\|^2_{L^2} + \|U_h(t)\|^2_{L^2} \right) dt + \alpha \|X_h(T) - \tilde{X}(T)\|^2_{L^2} \right] $$

subject to the equation

$$ dX_h(t) = \left[ \Delta_h X_h(t) + \Pi_h^0 U_h(t) \right] dt + \Pi_h^1 \sigma(t) dW(t) \quad \forall t \in [0, T], $$

$$ X_h(0) = \Pi_h^0 X_0. $$

The existence of a unique optimal pair $(X^*_h, U^*_h)$ follows from [25], as well as its characterization via Pontryagin’s maximum principle, i.e.,

$$ 0 = U^*_h(t) - \Pi_h^0 Y_h(t) \quad \forall t \in (0, T), $$

where the adjoint $(Y_h, Z_h) \in L^2(\Omega; C([0, T]; \mathbb{V}^1_h)) \times L^2(\Omega; L^2(0, T; \mathbb{V}^1_h))$ solves the BSPDE$_h$

$$ dY_h(t) = \left[ -\Delta_h Y_h(t) + \left[ X^*_h(t) - \Pi_h^1 \tilde{X}(t) \right] \right] dt + Z_h(t) dW(t) \quad \forall t \in [0, T], $$

$$ Y_h(0) = -\alpha (X^*_h(T) - \Pi_h^1 \tilde{X}(T)). $$

In [8], optimal error estimates have been obtained for $(X^*_h, Y_h, Z_h)$ with the help of a fixed point argument — which crucially exploits $T > 0$ to be sufficiently small. The goal in this section is to derive corresponding estimates for $(X^*_h, U^*_h, Y_h, Z_h)$ for arbitrary $T > 0$ via a variational argument which exploits properties of the reduced functional $\tilde{J} \equiv \tilde{J}(u)$ that is now defined: once an estimate for $\int_0^T \mathbb{E}\left[ \|U^* - U^*_h\|^2_{L^2} \right] dt$ has been obtained, we use the convergence analysis from Section 3 to derive estimates for $X^* - X^*_h$, as well as $Y - Y_h$ and $Z - Z_h$.

By the unique solvability property of (4.5), we associate to this equation the bounded solution operator

$$ S_h : L^2(\Omega; L^2(0, T; \mathbb{V}^0_h)) \rightarrow L^2(\Omega; C([0, T]; \mathbb{V}^1_h)), $$

which allows to introduce the reduced functional

$$ \tilde{J}_h : L^2(\Omega; L^2(0, T; \mathbb{V}^0_h)) \rightarrow \mathbb{R}, \quad \text{via} \quad \tilde{J}_h(U_h) = J(S_h(U_h), U_h). $$

The first solution component to equation (4.7) may be written as $Y_h = T_h(X^*_h)$, where

$$ T_h : L^2(\Omega; C([0, T]; \mathbb{V}^1_h)) \rightarrow L^2(\Omega; C([0, T]; \mathbb{V}^1_h)). $$

For every $U_h \in L^2(\Omega; L^2(0, T; \mathbb{V}^0_h))$, the Gateaux derivative $D \tilde{J}_h(U_h)$ is a bounded operator (uniformly in $h$) on $L^2(\Omega; L^2(0, T; \mathbb{V}^0_h))$ at $U_h$, and has the form

$$ D \tilde{J}_h(U_h) = U_h - \Pi_h^0 T_h(S_h(U_h)). $$

Let $U_h \in L^2(\Omega; L^2(0, T; \mathbb{V}^0_h))$ be arbitrary; it is due to the quadratic structure of the reduced functional (4.8) that

$$ \mathbb{E}\left[ \langle D^2 \tilde{J}_h(U_h) R_h, R_h \rangle_{L^2(0, T; L^2)} \right] \geq \mathbb{E}\| R_h \|^2_{L^2(0, T; L^2)} \quad \forall R_h \in L^2(\Omega; L^2(0, T; \mathbb{V}^0_h)). $$
As a consequence, on putting $R_h = U_h^* - \Pi_h^0 U^*$,

$$
(4.10) \quad \mathbb{E}[\|U_h^* - \Pi_h^0 U^*\|_{L^2(0,T;\mathbb{L}^2)}^2] \leq \mathbb{E}\left[ (D \tilde{J}(U_h^*) - D \tilde{J}(U_h^*))_{L^2(0,T;\mathbb{L}^2)} \right] = \mathbb{E}\left[ (D \tilde{J}(U_h^*) - D \tilde{J}(\Pi_h^0 U^*))_{L^2(0,T;\mathbb{L}^2)} \right].
$$

Note that $D \tilde{J}(U_h^*) = 0$ by (4.6), as well as $D \tilde{J}(U^*) = 0$ by (4.2), such that the last line equals

$$
\mathbb{E}\left[ (D \tilde{J}(U^*), U_h^* - \Pi_h^0 U^*)_{L^2(0,T;\mathbb{L}^2)} \right] = \mathbb{E}\left[ (D \tilde{J}(U^*), U_h^* - \Pi_h^0 U^*)_{L^2(0,T;\mathbb{L}^2)} \right] + \mathbb{E}\left[ (D \tilde{J}(\Pi_h^0 U^*), U_h^* - \Pi_h^0 U^*)_{L^2(0,T;\mathbb{L}^2)} \right] =: I + II.
$$

We use (4.3) to bound $I$ as follows,

$$
I = \mathbb{E}\left[ (U^* - \Pi_h^0 U^* + T(S(\Pi_h^0 U^*)) - T(S(U^*)), U_h^* - \Pi_h^0 U^*)_{L^2(0,T;\mathbb{L}^2)} \right] \leq \left( \mathbb{E}[\|U^* - \Pi_h^0 U^*\|_{L^2(0,T;\mathbb{L}^2)}^2] \right)^{1/2} I_a + \mathbb{E}[\|U_h^* - \Pi_h^0 U^*\|_{L^2(0,T;\mathbb{L}^2)}^2]^{1/2},
$$

where $I_a^2 = \mathbb{E}[\|T(S(U^* - S(\Pi_h^0 U^*))\|_{L^2(0,T;\mathbb{L}^2)}^2]$. By Poincaré’s inequality, and a stability bound (see also (2.12)) for the backward stochastic heat equation (2.10), as well as for the stochastic heat equation (2.2) (see also (2.3)),

$$
I_a^2 \leq C\mathbb{E}[\|S(U^* - S(\Pi_h^0 U^*))\|_{L^2(0,T;\mathbb{L}^2)}^2] \leq C\mathbb{E}[\|U^* - \Pi_h^0 U^*\|_{L^2(0,T;\mathbb{L}^2)}^2].
$$

By optimality condition (4.2), and the regularity properties of the solution to BSPDE (2.10), we know that already $U^* \in L^2(\Omega; C([0,T]; \mathbb{H}_0^1) \cap L^2(0,T; \mathbb{H}_0^1 \cap \mathbb{H}^2))$; as a consequence, the right-hand side of (4.12) may be bounded by $Ch^2$.

We use the representation (4.9) and properties of the projection $\Pi_h^0$ to bound $II$ via

$$
II = \mathbb{E}\left[ (T(S(\Pi_h^0 U^*)) - T(Sh(\Pi_h^0 U^*)), U_h^* - \Pi_h^0 U^*)_{L^2(0,T;\mathbb{L}^2)} \right] \leq II_a \times \left( \mathbb{E}[\|U_h^* - \Pi_h^0 U^*\|_{L^2(0,T;\mathbb{L}^2)}^2] \right)^{1/2},
$$

where $II_a^2 := \mathbb{E}[\|T(S(\Pi_h^0 U^*)) - Sh(\Pi_h^0 U^*))\|_{L^2(0,T;\mathbb{L}^2)}^2]$. We split $II_a^2$ into two terms

$$
II_{a,1}^2 = \mathbb{E}[\|T(S(\Pi_h^0 U^*)) - T(Sh(\Pi_h^0 U^*))\|_{L^2(0,T;\mathbb{L}^2)}^2], \quad \text{and} \quad II_{a,2}^2 = \mathbb{E}[\|T(Sh(\Pi_h^0 U^*)) - T(Sh(\Pi_h^0 U^*))\|_{L^2(0,T;\mathbb{L}^2)}^2].
$$

In order to bound $II_{a,1}^2$, we use stability properties for BSPDE (2.10), in combination with the error estimate (2.6) for (2.5) to conclude

$$
II_{a,1}^2 \leq C\mathbb{E}[\|S(\Pi_h^0 U^*) - Sh(\Pi_h^0 U^*))\|_{L^2(0,T;\mathbb{L}^2)}^2] \leq Ch^2.
$$

In order to bound $II_{a,2}^2$, we use the error estimate in Theorem 2.1 for BSPDE (2.10), in combination with stability properties of (2.5), and again the error estimate (2.6) for (2.5) to find

$$
II_{a,2}^2 \leq C\left( \mathbb{E}[\|S(\Pi_h^0 U^*)(T) - Sh(\Pi_h^0 U^*)(T)\|_{L^2}^2] + h^2 \right) \leq Ch^2.
$$
We now insert these estimates into (4.11) resp. (4.10) to obtain the bound
\[ \mathbb{E}[\|U_h^* - \Pi_h^0U^*\|^2_{L^2(0,T;L^2)}] \leq Ch^2. \]
By arguing as below (4.12), this settles part (i) of the following

**Theorem 4.1.** Let \((X^*, Y, Z, U^*)\) be the solution to problem SLQ, and \((X_h^*, Y_h, Z_h, U_h^*)\) be the solution to problem SLQ\(_h\). There exists \(C \equiv C(X_0, T) > 0\) such that

(i) \[ \mathbb{E}\left[\int_0^T \|U^*(t) - U_h^*(t)\|^2_{L^2} \, dt\right] \leq Ch^2; \]

(ii) \[ \sup_{0 \leq t \leq T} \mathbb{E}\left[\|X^*(t) - X_h^*(t)\|^2_{L^2} + \int_0^T \mathbb{E}\left[\|X^*(t) - X_h^*(t)\|^2_{\mathbb{H}_h}\right] \, dt\leq Ch^2; \]

(iii) \[ \sup_{0 \leq t \leq T} \mathbb{E}\left[\|Y(t) - Y_h(t)\|^2_{L^2} + \int_0^T \mathbb{E}\left[\|Y(t) - Y_h(t)\|^2_{\mathbb{H}_h} + \|Z(t) - Z_h(t)\|^2_{L^2}\right] \, dt\leq Ch^2. \]

**Proof.** Since \(U^* \in L^2_\mathcal{F}(\Omega; L^2(0,T;\mathbb{H}_h^1)), \) and (i), the first estimate of (ii) can be deduced as (2.6). Assertion (iii) now follows accordingly as Theorem 2.1, thanks to (ii). \( \blacksquare \)

### 4.2 Problem SLQ\(_h\): Discretization in space and time

In this part, we provide the temporal discretization of problem SLQ\(_h\), which was analyzed in Section 4.1. For this purpose, we use a mesh \(I_r\) covering \([0,T]\), and consider processes \((X_{h\tau}, U_{h\tau}) \in \mathbb{X}_{h\tau} \times \mathbb{U}_{h\tau} \subset L_\mathcal{F}^2(\Omega; L^2(0,T;\mathbb{V}_h^1)) \times L_\mathcal{F}^2(\Omega; L^2(0,T;\mathbb{V}_h^0)),\) where

\[
\mathbb{X}_{h\tau} := \{ X \in L^2_\mathcal{F}(\Omega; L^2(0,T;\mathbb{V}_h^1)) : X(t) = X(t_n), \forall t \in [t_n, t_{n+1}), n = 0, 1, \cdots, N - 1 \},
\]
\[
\mathbb{U}_{h\tau} := \{ U \in L^2_\mathcal{F}(\Omega; L^2(0,T;\mathbb{V}_h^0)) : U(t) = U(t_n), \forall t \in [t_n, t_{n+1}), n = 0, 1, \cdots, N - 1 \},
\]
and for any \(X \in \mathbb{X}_{h\tau}, U \in \mathbb{U}_{h\tau},\)
\[
\|X\|_{\mathbb{X}_{h\tau}} := \left( \tau \sum_{n=1}^N \mathbb{E}[\|X(t_n)\|^2_{L^2}] \right)^{1/2}, \quad \|U\|_{\mathbb{U}_{h\tau}} := \left( \tau \sum_{n=0}^{N-1} \mathbb{E}[\|U(t_n)\|^2_{L^2}] \right)^{1/2}.
\]

Problem SLQ\(_{h\tau}\) then reads as follows: Find an optimal pair \((X_{h\tau}^*, U_{h\tau}^*) \in \mathbb{X}_{h\tau} \times \mathbb{U}_{h\tau}\) which minimizes the cost functional
\[
J(\tau)(X_{h\tau}, U_{h\tau})
\]
(4.13)
\[
= \frac{\tau}{2} \sum_{n=1}^N \mathbb{E}[\|X_{h\tau}(t_n) - \tilde{X}(t_n)\|^2_{L^2}] + \frac{\tau}{2} \sum_{n=0}^{N-1} \mathbb{E}[\|U_{h\tau}(t_n)\|^2_{L^2}] + \frac{\alpha}{2} \mathbb{E}[\|X_{h\tau}(T) - \tilde{X}(T)\|^2_{L^2}],
\]
subject to the difference equation
\[
\begin{aligned}
X_{h\tau}(t_{n+1}) - X_{h\tau}(t_n) &= \tau \left[ \Delta_h X_{h\tau}(t_{n+1}) + \Pi_h^1 U_{h\tau}(t_n) \right] + \Pi_h^1 \sigma(t_n) \Delta_{n+1} W \\
X_{h\tau}(0) &= \Pi_h^1 X_0,
\end{aligned}
\]  
subject to the difference equation
(4.14)
\[
\begin{aligned}
X_{h\tau}(t_{n+1}) - X_{h\tau}(t_n) &= \tau \left[ \Delta_h X_{h\tau}(t_{n+1}) + \Pi_h^1 U_{h\tau}(t_n) \right] + \Pi_h^1 \sigma(t_n) \Delta_{n+1} W \\
X_{h\tau}(0) &= \Pi_h^1 X_0,
\end{aligned}
\]
where \(\Delta_{n+1} W = W(t_{n+1}) - W(t_n).\) The following result states the Pontryagin maximum principle for problem SLQ\(_{h\tau}\), which is later used to verify convergence rates for the solution to problem SLQ\(_{h\tau}\) towards the solution to SLQ.
Theorem 4.2. Problem \( \text{SLQ}_{h\tau} \) admits a unique minimizer \((X_{h\tau}^*, U_{h\tau}^*) \in \mathcal{X}_{h\tau} \times \mathbb{U}_{h\tau}\), which is (part of) the unique solution

\[
(X_{h\tau}^*, Y_{h\tau}, U_{h\tau}^*) \in [\mathcal{X}_{h\tau}]^2 \times \mathbb{U}_{h\tau}
\]
to the following forward-backward difference equation for \(0 \leq n \leq N - 1\),

\[
\begin{align*}
[1 - \tau \Delta_h]X_{h\tau}(t_{n+1}) &= X_{h\tau}^*(t_n) + \tau \Pi_{h\tau}^1 U_{h\tau}^*(t_n) + \Pi_{h\tau}^1 \sigma(t_n) \Delta_{n+1} W, \\
[1 - \tau \Delta_h]Y_{h\tau}(t_n) &= \mathbb{E} \left[ Y_{h\tau}(t_{n+1}) - \tau (X_{h\tau}^*(t_{n+1}) - \Pi_{h\tau}^1 X(t_{n+1})) | \mathcal{F}_n \right], \\
X_{h\tau}^*(0) &= \Pi_{h\tau}^1 X_0, \\
Y_{h\tau}(T) &= -\alpha (X_{h\tau}^*(T) - \Pi_{h\tau}^1 X(T)),
\end{align*}
\]

(4.15)

Together with

\[
U_{h\tau}^*(t_n) - \Pi_{h\tau}^0 Y_{h\tau}(t_n) = 0 \quad n = 0, 1, \ldots, N - 1.
\]

(4.16)

By (4.16), we can see that \(U_{h\tau}^*\) is càdlàg, and then \(U_{h\tau}^* \in \mathbb{U}_{h\tau}\). Inserting (4.16) into (4.15) leads to a coupled problem for \(\left\{ X_{h\tau}^*(t_{n+1}) \right\}_{n=0}^{N-1}, \left\{ Y_{h\tau}(t_n) \right\}_{n=0}^{N-1} \), where (4.15) is similar to (3.6). Note that no \(Z\)-component appears explicitly in (4.15), where the conditional expectation is used to compute the \(Y\)-component. It is in particular due to the need to compute conditional expectations in (4.15) that the optimality system (4.15)–(4.16) is still not amenable to an actual implementation, but serves as a key step towards a practical method which approximately solves \( \text{SLQ}_{h\tau} \) — which is proposed and studied in Section 5.

Proof. We divide the proof into three steps.

Step 1. Let \( A_0 := (1 - \tau \Delta_h)^{-1} \). For any \( U_{h\tau} \in \mathbb{U}_{h\tau} \), by equation (4.15), we have

\[
X_{h\tau}(t_n) = A_0 \left( X_{h\tau}(t_{n-1}) + \tau \Pi_{h\tau}^1 U_{h\tau}(t_{n-1}) + \Pi_{h\tau}^1 \sigma(t_{n-1}) \Delta_n W \right).
\]

(4.17)

Hence, by iteration we arrive at

\[
X_{h\tau}(t_n) = A_0^n X_{h\tau}(0) + \tau \sum_{j=0}^{n-1} A_0^{n-1-j} \Pi_{h\tau}^1 U_{h\tau}(t_j) + \sum_{j=1}^{n} A_0^{n+1-j} \Pi_{h\tau}^1 \sigma(t_{j-1}) \Delta_j W
\]

(4.18)

Here, \( \Gamma : \mathcal{Y}_{h} \to \mathcal{X}_{h\tau} \) and \( L : \mathbb{U}_{h\tau} \to \mathcal{X}_{h\tau} \) are bounded operators. Below, we use the abbreviations

\[
\tilde{\Gamma} \Pi_{h}^1 X_0 := \Gamma \Pi_{h}^1 X_0(T), \quad \tilde{L} U_{h\tau} := (LU_{h\tau})(T), \quad \tilde{f} = f(T).
\]

Claim: For any \( \xi \in \mathcal{X}_{h\tau} \), and any \( \eta \in L^2_{\mathbb{F}}(\Omega; \mathcal{Y}_{h}^1) \),

\[
L^* \xi = -\Pi_0^0 Y_0, \quad \tilde{L}^* \eta = -\Pi_0^0 Y_1,
\]

(4.20)

where \( (Y_0, Z_0) \) solves the following backward stochastic difference equation:

\[
\begin{cases}
Y_0(t_{n+1}) - Y_0(t_n) = \tau (-\Delta_h Y_0(t_n) + \xi(t_{n+1})) + \int_{t_n}^{t_{n+1}} Z_0(t) \, dW(t) \quad n = 0, 1, \ldots, N - 1, \\
Y_0(t_N) = Y_0(T) = 0,
\end{cases}
\]

(4.21)
and \((Y_1, Z_1)\) solves
\[
\begin{cases}
Y_1(t_{n+1}) - Y_1(t_n) = -\tau \Delta h Y_1(t_n) + \int_{t_n}^{t_{n+1}} Z_1(t) \, dW(t) & n = 0, 1, \cdots, N - 1, \\
Y_1(T) = -\eta.
\end{cases}
\]

**Proof of Claim:** The existence and the uniqueness of solutions to (4.21) are obvious. Note that
\[
(4.22) \quad Y_0(t_j) = \mathbb{E}[A_0 Y_0(t_{j+1}) - \tau A_0 \xi(t_{j+1}) | \mathcal{F}_{t_j}].
\]

With the similar procedure as that in (4.17), we conclude from (4.22) and (4.21),
\[
(4.23) \quad Y_0(t_j) = \mathbb{E}[A_0^{N-j} Y_0(t_N) | \mathcal{F}_{t_j}] - \mathbb{E}[\tau \sum_{k=j+1}^{N} A_0^{k-j} \xi(t_k) | \mathcal{F}_{t_j}]
\]

Let \(U_{h\tau} \in U_{h\tau}\) be arbitrary. By the definition of \(L\), (4.23) and the fact \(A_0 = A_0^\top\), we can calculate that
\[
\tau \sum_{n=1}^{N} \mathbb{E}\left[(L U_{h\tau})(t_n), \xi(t_n)\right]_{L^2} = \tau \sum_{n=1}^{N} \mathbb{E}\left[\tau \sum_{j=0}^{n-1} A_0^{n-j} \Pi_1^j U_{h\tau}(t_j), \xi(t_n)\right]_{L^2}
\]
\[
= \tau \sum_{j=0}^{N-1} \mathbb{E}\left[(\Pi_1^j U_{h\tau}(t_j), \xi(t_n))_{L^2} | \mathcal{F}_{t_j}\right].
\]

Since the second argument is \(\mathbb{V}_{h}^1\)-valued, we may skip the projection operator in the first argument, and may continue instead
\[
= \tau \sum_{j=0}^{N-1} \mathbb{E}\left[(U_{h\tau}(t_j), \Pi_1^0 Y_0(t_j))_{L^2}\right]
\]
Because of (4.23), the latter equals
\[
= \tau \sum_{j=0}^{N-1} \mathbb{E}\left[(U_{h\tau}(t_j), -\Pi_1^0 Y_0(t_j))_{L^2}\right],
\]
which is the first part of the claim.

The remaining part can be deduced from the fact that \(Y_1(t_j) = -\mathbb{E}[A_0^{N-j} \eta | \mathcal{F}_{t_j}]\) for \(j = 0, 1, \cdots, N - 1\), and the following calculation:
\[
\mathbb{E}\left[(\hat{L} U_{h\tau}, \eta)_{L^2}\right] = \mathbb{E}\left[\tau \sum_{j=0}^{N-1} A_0^{N-j} \Pi_1^j U_{h\tau}(t_j), \eta\right]_{L^2}
\]
\[
(4.24) \quad = \tau \sum_{j=0}^{N-1} \mathbb{E}\left[(\Pi_1^j U_{h\tau}(t_j), \mathbb{E}[A_0^{N-j} \eta | \mathcal{F}_{t_j}])_{L^2}\right]
\]
\[
= \tau \sum_{j=0}^{N-1} \mathbb{E}\left[(U_{h\tau}(t_j), -\Pi_1^0 Y_1(t_j))_{L^2}\right] \quad \forall U_{h\tau} \in U_{h\tau}.
\]

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Step 2. By (4.18) and (4.19), we can rewrite $J_\tau(X_{h\tau}, U_{h\tau})$ as follows:

$$
J_\tau(X_{h\tau}, U_{h\tau})
= \frac{1}{2} \left[ \|X_{h\tau} - \Pi_{h} \tilde{X}\|^2_{L^2(\Omega;L^2(0,T;\mathbb{R}^2))} + \|U_{h\tau}\|^2_{L^2(\Omega;L^2(0,T;\mathbb{R}^2))} + \alpha \|X_{h\tau}(T) - \tilde{X}(T)\|^2_{L^2(\Omega;\mathbb{R}^2)} \right]
$$

$$
= \frac{1}{2} \left[ \left\langle \Pi_{h} \tilde{X} \right, \Pi_{h} \tilde{X} \right\rangle_{L^2(\Omega;L^2(0,T;\mathbb{R}^2))} + \left\langle U_{h\tau}, U_{h\tau} \right\rangle_{L^2(\Omega;L^2(0,T;\mathbb{R}^2))} + \alpha \left\langle \Pi_{h} \tilde{X}(T), \Pi_{h} \tilde{X}(T) \right\rangle_{L^2(\Omega;\mathbb{R}^2)} \right]
$$

where $\Pi_{h} \tilde{X}(t) = \tilde{X}(t_n)$, for $t \in [t_n, t_{n+1})$, $n = 0, 1, \cdots, N - 1$. Rearranging terms then leads to

$$
= \frac{1}{2} \left[ \left\langle \Pi_{h} \tilde{X} \right, \Pi_{h} \tilde{X} \right\rangle_{L^2(\Omega;L^2(0,T;\mathbb{R}^2))} + \left\langle U_{h\tau}, U_{h\tau} \right\rangle_{L^2(\Omega;L^2(0,T;\mathbb{R}^2))} + \alpha \left\langle \Pi_{h} \tilde{X}(T), \Pi_{h} \tilde{X}(T) \right\rangle_{L^2(\Omega;\mathbb{R}^2)} \right]
$$

$$
= \frac{1}{2} \left[ \left\langle \Pi_{h} \tilde{X} \right, \Pi_{h} \tilde{X} \right\rangle_{L^2(\Omega;L^2(0,T;\mathbb{R}^2))} + \left\langle U_{h\tau}, U_{h\tau} \right\rangle_{L^2(\Omega;L^2(0,T;\mathbb{R}^2))} + \alpha \left\langle \Pi_{h} \tilde{X}(T), \Pi_{h} \tilde{X}(T) \right\rangle_{L^2(\Omega;\mathbb{R}^2)} \right]
$$

$$
= \frac{1}{2} \left[ \left\langle \Pi_{h} \tilde{X} \right, \Pi_{h} \tilde{X} \right\rangle_{L^2(\Omega;L^2(0,T;\mathbb{R}^2))} + \left\langle U_{h\tau}, U_{h\tau} \right\rangle_{L^2(\Omega;L^2(0,T;\mathbb{R}^2))} + \alpha \left\langle \Pi_{h} \tilde{X}(T), \Pi_{h} \tilde{X}(T) \right\rangle_{L^2(\Omega;\mathbb{R}^2)} \right]
$$

Since $N = 1 + L^*L + \alpha \tilde{L}^*\tilde{L}$ is positive definite, there exists a unique $U_{h\tau}^* \in U_{h\tau}$ such that

$$
NU_{h\tau}^* + H(\Pi_{h} X_{h\tau}, f, \tilde{X}) = 0.
$$

Therefore, for any $U_{h\tau} \in U_{h\tau}$ such that $U_{h\tau} \neq U_{h\tau}^*$,

$$
J_\tau(X_{h\tau}, U_{h\tau}) - J_\tau(X_{h\tau}^*, U_{h\tau}^*)
= \left( NU_{h\tau}^* + H(\Pi_{h} X_{h\tau}, f, \tilde{X}), U_{h\tau} - U_{h\tau}^* \right)_{L^2(\Omega;L^2(0,T;\mathbb{R}^2))}
$$

$$
= \frac{1}{2} \left( NU_{h\tau}^* - NU_{h\tau}^* + H(\Pi_{h} X_{h\tau}, f, \tilde{X}), U_{h\tau} - U_{h\tau}^* \right)_{L^2(\Omega;L^2(0,T;\mathbb{R}^2))}
$$

$$
= \frac{1}{2} \left( NU_{h\tau}^* - NU_{h\tau}^* + H(\Pi_{h} X_{h\tau}, f, \tilde{X}), U_{h\tau} - U_{h\tau}^* \right)_{L^2(\Omega;L^2(0,T;\mathbb{R}^2))}
$$

$$
> 0,
$$

which means that $U_{h\tau}^*$ is the unique optimal control, and $(X_{h\tau}^*, U_{h\tau}^*)$ is the unique optimal pair.

Step 3. By the definition of $N, H, L^*, \tilde{L}^*$, and properties (4.20) and (4.18), we can get

$$
0 = NU_{h\tau}^* + H(\Pi_{h} X_{h\tau}, f, \tilde{X})
$$

$$
= U_{h\tau}^* + L^* \left( \Pi_{h} X_{h\tau} + LU_{h\tau}^* + f - \tilde{X} \right) + \alpha \tilde{L}^* \left( \Pi_{h} X_{h\tau} + \tilde{L} U_{h\tau}^* + \tilde{f} - \tilde{X}(T) \right)
$$

$$
= U_{h\tau}^* - \Pi_{h} \left[ Y_0 \left( \cdot : X_{h\tau}^* - \tilde{X} \right) + Y_1 \left( \cdot : \alpha \left( X_{h\tau}^* - \tilde{X}(T) \right) \right) \right]
$$

which is (4.16). This completes the proof.
We are now ready to verify strong rates of convergence for the solution to $\text{SLQ}_{h_T}$; it is as in Section 4.1 that the reduced cost functional $\hat{J}_{h_T} : U_{h_T} \to \mathbb{R}$ is used, which is defined via

$$\hat{J}_{h_T}(U_{h_T}) = \mathcal{J}_{l_T}(S_{h_T}(U_{h_T}), U_{h_T}),$$

where $S_{h_T} : U_{h_T} \to \mathbb{X}_{h_T}$ is the solution operator to the forward equation (4.15)$_1$. Moreover, we use the solution operator $\mathcal{T}_{h_T} : \mathbb{X}_{h_T} \to \mathbb{X}_{h_T}$ for the first solution component of the backward equation (4.15)$_2$.

**Theorem 4.3.** Suppose that $X(0) \in \mathbb{H}^1_0 \cap \mathbb{H}^1$, and

$$\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{E} \left[ \|\sigma(t) - \sigma(t_n)\|^2_{\mathbb{H}^1_0} \right] + \|\hat{X}(t) - \hat{X}(t_n)\|^2_{\mathbb{L}^2} + \|\hat{X}(t) - \hat{X}(t_{n+1})\|^2_{\mathbb{L}^2} \, dt \leq C_T. \tag{4.25}$$

Let $(X^*_h, Y_h, Z_h, U^*_h)$ be the solution to problem $\text{SLQ}_h$, and $(X^*_{h_T}, Y_{h_T}, U^*_{h_T})$ be the solution to problem $\text{SLQ}_{h_T}$. There exists $C \equiv C(X_0, T) > 0$ such that

(i) \quad $\sum_{k=0}^{N-1} \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \|U_h^*(t) - U^*_{h_T}(t_k)\|^2_{\mathbb{L}^2} \, dt \right] \leq C_T$;

(ii) \quad $\max_{0 \leq k \leq N} \mathbb{E} \left[ \|X^*_h(t_k) - X^*_{h_T}(t_k)\|^2_{\mathbb{L}^2} \right] + \mathbb{E} \left[ \tau \sum_{k=1}^{N} \|X^*_h(t_k) - X^*_{h_T}(t_k)\|^2_{\mathbb{H}^1_0} \right] \leq C_T$;

(iii) \quad $\max_{0 \leq k \leq N} \mathbb{E} \left[ \|Y_h(t_k) - Y_{h_T}(t_k)\|^2_{\mathbb{H}^1_0} \right] + \mathbb{E} \left[ \tau \sum_{k=0}^{N-1} \|Y_h(t_k) - Y_{h_T}(t_k)\|^2_{\mathbb{H}^1_0} \right] \leq C_T$.

**Proof.** We divide the proof into three steps.

**Step 1.** We follow the argumentation in the proof of Theorem 4.1. For every $U_{h_T}, R_{h_T} \in U_{h_T}$, the first Gateaux derivative $D\hat{J}_{h_T}(U_{h_T})$, and the second Gateaux derivative $D^2\hat{J}_{h_T}(U_{h_T})$ satisfy

$$D\hat{J}_{h_T}(U_{h_T}) = U_{h_T} - \Pi_h T_{h_T}(S_{h_T}(U_{h_T})), \quad E\left( D^2\hat{J}_{h_T}(U_{h_T})R_{h_T}, R_{h_T} \right)_{L^2(0,T;\mathbb{H}^1_0)} \geq E\left( ||R_{h_T}||_{L^2(0,T;\mathbb{H}^1_0)}^2 \right). \tag{4.26}$$

Define the (piecewise constant) operator $\Pi_T : L^2(\Omega; C(0,T;\mathbb{H}^1_0)) \to U_{h_T}$ by

$$\Pi_T U_h(t) := U_h(t_n) \quad \forall t \in [t_n, t_{n+1}) \quad n = 0, 1, \cdots, N - 1.$$

By putting $R_{h_T} = U^*_{h_T} - \Pi_T U_h^*$ in (4.26), and applying the fact $D\hat{J}_{h_T}(U^*_h) = D\hat{J}_{h_T}(U_h^*) = 0$, we see that

$$E\left( ||U_{h_T} - \Pi_T U_h^*||_{L^2(0,T;\mathbb{H}^1_0)}^2 \right) \leq E\left( \left( D\hat{J}_{h_T}(U_{h_T}^*), U^*_h - \Pi_T U_h^* \right)_{L^2(0,T;\mathbb{H}^1_0)} \right) - E\left( \left( D\hat{J}_{h_T}(\Pi_T U_{h_T}^*), U^*_h - \Pi_T U_h^* \right)_{L^2(0,T;\mathbb{H}^1_0)} \right) \tag{4.27}$$

$$= E\left( \left( D\hat{J}_{h_T}(U^*_h) - D\hat{J}_{h_T}(\Pi_T U_{h_T}^*), U^*_h - \Pi_T U_h^* \right)_{L^2(0,T;\mathbb{H}^1_0)} \right) + E\left( \left( D\hat{J}_{h_T}(\Pi_T U_{h_T}^*), U^*_h - \Pi_T U_h^* \right)_{L^2(0,T;\mathbb{H}^1_0)} \right).$$

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Therefore,
\[
\mathbb{E} \left[ \| U_h^* - \Pi_r U_h^* \|^2_{L^2(0,T;L^2)} \right] 
\leq 2 \mathbb{E} \left[ \| D\tilde{\mathcal{F}}_h(U_h^*) - D\tilde{\mathcal{F}}_h(\Pi_r U_h^*) \|^2_{L^2(0,T;L^2)} \right] 
\leq 2 \mathbb{E} \left[ \| D\tilde{\mathcal{F}}_h(\Pi_r U_h^*) - D\tilde{\mathcal{F}}_h(\Pi_r U_h^*) \|^2_{L^2(0,T;L^2)} \right] 
=: 2I' + 2II'.
\]

We use (4.9) and (4.6), and stability properties of the projection \(\Pi^0_h\) to bound \(I'\) as follows,
\[
I' = \mathbb{E} \left[ \| U_h^* - \Pi_r U_h^* + \Pi^0_h T_h(S_h(\Pi_r U_h^*)) - \Pi^0_h T_h(S_h(U_h^*)) \|^2_{L^2(0,T;L^2)} \right] 
\leq 2 \mathbb{E} \left[ \| U_h^* - \Pi_r U_h^* \|^2_{L^2(0,T;L^2)} + \| T_h(S_h(\Pi_r U_h^*)) - T_h(S_h(U_h^*)) \|^2_{L^2(0,T;L^2)} \right].
\]

By stability properties of solutions to \(\text{BSPDE}_h\) (2.14), and the discretization (2.5) of \(\text{BSPDE}\), we obtain
\[
\mathbb{E} \left[ \| T_h(S_h(\Pi_r U_h^*)) - T_h(S_h(U_h^*)) \|^2_{L^2(0,T;L^2)} \right] 
\leq C \mathbb{E} \left[ \| (S_h(U_h^*) - S_h(\Pi_r U_h^*))(T) \|^2_{L^2} + \| S_h(U_h^*) - S_h(\Pi_r U_h^*) \|^2_{L^2(0,T;L^2)} \right] 
\leq C \| U_h^* - \Pi_r U_h^* \|^2_{L^2(0,L^2)}.
\]

By the optimality condition (4.6), estimate (2.15), and Theorem 4.1 (i) we have
\[
\| U_h^* - \Pi_r U_h^* \|^2_{L^2(0,L^2)} 
\leq C \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} \mathbb{E} \left[ \| -\Delta h Y_h(s) + (X_{h,h}^*(s) - \Pi_{h,h}^1 \tilde{X}(s)) \|^2_{L^2} + \| Z_h(s) \|^2_{L^2} \right] \, ds \, dt 
\leq C t \int_{0}^{T} \mathbb{E} \left[ \| -\Delta h Y_h(s) + (X_{h,h}^*(s) - \Pi_{h,h}^1 \tilde{X}(s)) \|^2_{L^2} + \| Z_h(s) \|^2_{L^2} \right] \, ds 
\leq C t.
\]

Next, we turn to \(II'\), for which we use the representations (4.9), (4.26) and the stability property of \(\Pi^0_h\) to conclude
\[
\mathbb{E} \left[ \| \Pi^0_h T_h(S_h(\Pi_r U_h^*)) - \Pi^0_h T_h(S_h(\Pi_r U_h^*)) \|^2_{L^2(0,T;L^2)} \right] 
\leq 2 \mathbb{E} \left[ \| T_h(S_h(\Pi_r U_h^*)) - T_h(S_h(\Pi_r U_h^*)) \|^2_{L^2(0,T;L^2)} \right] 
\leq 2 \mathbb{E} \left[ \| T_h(S_h(\Pi_r U_h^*)) - T_h(S_h(\Pi_r U_h^*)) \|^2_{L^2(0,T;L^2)} \right] 
=: 2 \left( I_{a,1} + I_{a,2} \right).
\]

In order to bound \(I_{a,1}\), we use stability properties for \(\text{SPDE}_h\) (2.5), \(\text{BSPDE}_h\) (4.7), in combination with the error estimate (2.8) for (2.5) to conclude
\[
I_{a,1} \leq C \mathbb{E} \left[ \| (S_h(\Pi_r U_h^*)) - S_{\tau_h}(\Pi_r U_h^*) \|^2_{L^2} + \| S_h(\Pi_r U_h^*) - S_{\tau_h}(\Pi_r U_h^*) \|^2_{L^2(0,T;L^2)} \right] \leq C t.
\]

To bound \(I_{a,2}\), it is easy to see
\[
I_{a,2} \leq 2 \sum_{n=0}^{N-1} \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} \| Y_h(t; S_h(\Pi_r U_h^*)) - Y_h(t_n; S_h(\Pi_r U_h^*)) \|^2_{L^2} \, dt \right] 
\leq 2T \max_{0 \leq n \leq N} \mathbb{E} \left[ \| Y_h(t_n; S_h(\Pi_r U_h^*)) - Y_h^n(\Pi_r U_h^*) \|^2_{L^2} \right].
\]
By (4.9), Lemma 3.1 (i), stable property of (4.15)$_1$, we can get

\[
\sum_{n=0}^{N-1} \mathbb{E} \left[ \int_{t_n}^{t_{n+1}} \left\| Y_h(t; S_{h\tau}(\Pi_{\tau} U_{h}^*)) - Y_h(t_n; S_{h\tau}(\Pi_{\tau} U_{h}^*)) \right\|^2_{L^2} \, dt \right]
\leq C\tau \left( \mathbb{E} \left\| S_{h\tau}(\Pi_{\tau} U_{h}^*)(T) \right\|^2_{H^0} + \left\| \tilde{X}(T) \right\|^2_{H^0} \right) + \mathbb{E} \left\| S_{h\tau}(\Pi_{\tau} U_{h}^*) \right\|^2_{L^2(\Omega; L^2(0,T; L^2))} + \left\| \tilde{X} \right\|^2_{L^2(0,T; L^2)}
\]

\[
\leq C\tau \left( \| X_0 \|^2_{H^0} + \left\| \tilde{X}(T) \right\|^2_{H^0} + \| \sigma \|^2_{L^2(\Omega; L^2(0,T; \mathbb{R}^d)))} + \| Y_h \|^2_{L^2(\Omega; L^2(0,T; H^0)))} + \left\| \tilde{X} \right\|^2_{L^2(0,T; L^2)} \right)
\]

\[
\leq C\tau .
\]

Utilizing Theorem 3.4 for BSPDE$_h$ (2.14) with \( \Pi_h f = S_{h\tau}(\Pi_{\tau} U_{h}^*) - \Pi_h \tilde{X} \) and (4.25), we can find that

\[
\max_{0 \leq n \leq N} \mathbb{E} \left[ \left\| Y_h(t_n; S_{h\tau}(\Pi_{\tau} U_{h}^*)) - Y_h^n(S_{h\tau}(\Pi_{\tau} U_{h}^*)) \right\|^2_{L^2} \right]
\leq C\tau \mathbb{E} \left[ \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left\| \nabla [Y_h(s; S_{h\tau}(\Pi_{\tau} U_{h}^*)) - Y_h(t_n; S_{h\tau}(\Pi_{\tau} U_{h}^*))] \right\|^2_{L^2} + \left\| \tilde{X}(s) - \tilde{X}(t_n) \right\|^2_{L^2} \, ds \right]
\leq C\tau \left[ \max_{0 \leq n \leq N} \mathbb{E} \left\| \nabla X_{h\tau}(t_n; S_{h\tau}(\Pi_{\tau} U_{h}^*)) \right\|^2_{L^2} + \mathbb{E} \int_0^T \left\| \nabla \Pi_h \tilde{X}(t) \right\|^2_{L^2} \, dt \right]
+ C\tau \mathbb{E} \left[ \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left\| \tilde{X}(s) - \tilde{X}(t_n) \right\|^2_{L^2} \, ds \right]
\leq C\tau .
\]

Here, we apply the representation of \( X_{h\tau} \) (4.18), the fact \( \tilde{X} \in L^2(0,T; H^0) \), and condition (4.25).

Now we insert above estimates into (4.28) to obtain assertion (i).

**Step 2.** For all \( k = 0, 1, \ldots, N \), we define \( e^k_{X} = X_{h}^k(t_k) - X_{h\tau}^k(t_k) \). Subtracting (4.14) from (4.5) leads to

\[
e^{k+1}_{X} - e^k_{X} = \tau \Delta_h e^{k+1}_{X} + \tau \Pi_h [U_h^k(t_k) - U_{h\tau}^k(t_k)] + \int_{t_k}^{t_{k+1}} \Pi_h [\sigma(s) - \sigma(t_k)] \, dW(s)
\]

\[
+ \int_{t_k}^{t_{k+1}} (\Delta_h [X_{h}^k(s) - X_{h\tau}^k(t_{k+1})] + \Pi_h [U_h^k(s) - U_{h\tau}^k(t_k)]) \, ds .
\]

Testing with \( e^{k+1}_{X} \), and using binomial formula, Poincaré’s inequality, independence, and absorption lead to

\[
\frac{1}{2} \mathbb{E} \left[ \left\| e^{k+1}_{X} \right\|^2_{L^2} - \left\| e^k_{X} \right\|^2_{L^2} + \frac{1}{2} \left\| e^{k+1}_{X} - e^k_{X} \right\|^2_{L^2} \right] + \frac{\tau}{2} \mathbb{E} \left[ \left\| \nabla e^{k+1}_{X} \right\|^2_{L^2} \right]
\leq C\tau \mathbb{E} \left[ \left\| U_h^k(t_k) - U_{h\tau}^k(t_k) \right\|^2_{L^2} \right] + C\tau \mathbb{E} \left[ \left\| \int_{t_k}^{t_{k+1}} \Pi_h [\sigma(s) - \sigma(t_k)] \, dW(s) \right\|^2_{L^2} \right]
+ C\tau \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \left\| \nabla [X_{h}^k(s) - X_{h\tau}^k(t_{k+1})] \right\|^2_{L^2} \, ds \right] + C\tau \mathbb{E} \left[ \int_{t_k}^{t_{k+1}} \left\| U_h^k(s) - U_{h\tau}^k(t_k) \right\|^2_{L^2} \, ds \right] .
\]
By taking the sum over all $0 \leq k \leq n$ and $0 \leq k \leq N - 1$, and noting that $e_0^n = 0$, we find that

$$
\max_{0 \leq n \leq N} \mathbb{E}[\|e^n_X\|^2_{L^2}] + \sum_{n=1}^N \tau \mathbb{E}[\|\nabla e^n_X\|^2_{L^2}] \\
\leq C \sum_{k=0}^{N-1} \mathbb{E}\left[\tau \|U^*_h(t_k) - U^*_{h^r}(t_k)\|^2_{L^2} + \int_{t_k}^{t_{k+1}} \left(\|\sigma(s) - \sigma(t_k)\|^2_{L^2} + \|\nabla [X^*_h(s) - X^*_h(t_{k+1})]\|_{L^2}^2 + \|U^*_h(s) - U^*_h(t_k)\|_{L^2}^2\right) ds\right].
$$

By (4.28), the first term on the right-hand side is bounded by $C\tau$. We use Itô isometry for the second term, and Hölder regularity in time of $\sigma$ to bound it equally. Adopting the method in (4.31), we can bound the third term by $C\tau \left(\|\Delta_h X^*_h(0)\|^2_{L^2} + \|\nabla \Pi^1_h X^*_h(t)\|^2_{L^2(\Omega;L^2(0,T;L^2))} + \|\sigma(t)\|^2_{L^2(\Omega;L^2(0,T;H^2))}\right)$. We use (4.31) to bound the last term by $C\tau$. That is assertion (ii).

**Step 3.** Firstly, we introduce an auxiliary BSDE

$$(4.32) \begin{cases}
Y_\tau(t_{n+1}) - Y_\tau(t_n) = \tau \left[-\Delta_h Y_\tau(t_n) + \left(X^*_h(t_{n+1}) - \Pi^1_h \tilde{X}(t_{n+1})\right)\right] \\
+ \int_{t_n}^{t_{n+1}} Z_\tau(t) \, dW(t) \quad n = 0, 1, \cdots, N - 1, \\
Y_\tau(T) = -\alpha \left(X^*_h(T) - \Pi^1_h \tilde{X}(T)\right).
\end{cases}
$$

It is easy to see that $Y_\tau = Y_{h^r}$. Define $e^n_Y = Y_h(t_n) - Y_\tau(t_n)$, $n = 0, 1, \cdots, N$. With the same argument as that in the proof of Theorem 3.4, we can deduce

$$
\max_{0 \leq n \leq N} \mathbb{E}[\|e^n_Y\|^2_{L^2}] + \sum_{n=1}^N \tau \mathbb{E}[\|\nabla e^n_Y\|^2_{L^2}] \\
\leq C \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \mathbb{E}\left[\|\nabla [Y_h(s) - Y_h(t_k)]\|_{L^2}^2 + \|X^*_h(s) - X^*_h(t_{k+1})\|_{L^2}^2 + \|\tilde{X}(s) - \tilde{X}(t_{k+1})\|_{L^2}^2\right] ds.
$$

Applying Lemma 3.1 (ii), the first integral term is bounded by

$$
C\tau \left\{\|\Delta_h X_h(0)\|^2_{L^2} + \|\Delta_h \Pi^1_h \tilde{X}(T)\|^2_{L^2} + \int_0^T \mathbb{E}[\|\nabla X_h(t)\|^2_{L^2} + \|\nabla \Pi^1_h \tilde{X}(t)\|^2_{L^2} + \|\Pi^1_h U^*_h(t)\|^2_{L^2}] dt\right\}.
$$

It remains to estimate the second integral term, which is bounded by

$$
C \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} \mathbb{E}\left[\|X^*_h(s) - X^*_h(t_{k+1})\|_{L^2}^2 + \|X^*_h(t_{k+1}) - X^*_h(t_k)\|_{L^2}^2\right] ds \\
\leq C\tau \left\{\|\nabla X_h(0)\|^2_{L^2} + \int_0^T \mathbb{E}[\|\Pi^1_h U^*_h(t)\|^2_{L^2} + \|\nabla \Pi^1_h \sigma(t)\|^2_{L^2}] dt\right\} \\
+ C \max_{0 \leq k \leq N} \mathbb{E}[\|X^*_h(t_k) - X^*_h(t_k)\|^2_{L^2}] \\
\leq C\tau.
$$

Assertion (iii) now follows from the above three statements and conditions on $X_0, \sigma, \tilde{X}$.  

}\end{document}
5 The gradient descent method to solve $\text{SLQ}_{h_T}$

By Theorem 4.2, solving minimization problem $\text{SLQ}_{h_T}$ is equivalent to solving the system of coupled forward-backward difference equations (4.15) and (4.16). We may exploit the variational character of problem $\text{SLQ}_{h_T}$ to construct a gradient descent method $\text{SLQ}_{h_T}^{\text{grad}}$ where approximate iterates of the optimal control $U^*_T$ in the Hilbert space $U_{h_T}$ are obtained; see also [19, 14].

**Algorithm 5.1. ($\text{SLQ}_{h_T}^{\text{grad}}$)** Let $U_{h_T}^{(0)} \in U_{h_T}$, and fix $\kappa > 0$. For any $\ell \in \mathbb{N}_0$, update $U_{h_T}^{(\ell)} \in U_{h_T}$ as follows:

1. Compute $X_{h_T}^{(\ell)} \in X_{h_T}$ by
   \[
   \begin{aligned}
   \left[ 1 - \tau \Delta_h \right] X_{h_T}^{(\ell)} (t_{n+1}) & = X_{h_T}^{(\ell)} (t_n) + \tau \Pi_h^1 U_{h_T}^{(\ell)} (t_n) + \Pi_h^1 \sigma (t_n) \Delta_n W \quad n = 0, 1, \ldots, N - 1, \\
   X_{h_T}^{(\ell)} (0) & = \Pi_h^1 X_0 .
   \end{aligned}
   \]

2. Use $X_{h_T}^{(\ell)} \in X_{h_T}$ to compute $Y_{h_T}^{(\ell)} \in X_{h_T}$ via
   \[
   \begin{aligned}
   \left[ 1 - \tau \Delta_h \right] Y_{h_T}^{(\ell)} (t_n) & = \mathbb{E} \left[ Y_{h_T}^{(\ell)} (t_{n+1}) - \tau \left( X_{h_T}^{(\ell)} (t_{n+1}) - \Pi_h^1 \tilde{X} (t_{n+1}) \right) \right] | F_{t_n} \\
   n & = 0, 1, \ldots, N - 1,
   \end{aligned}
   \]
   \[
   Y_{h_T}^{(\ell)} (T) = -\alpha \left( X_{h_T}^{(\ell)} (T) - \Pi_h^1 \tilde{X} (T) \right) .
   \]

3. Compute the update $U_{h_T}^{(\ell+1)} \in U_{h_T}$ via
   \[
   U_{h_T}^{(\ell+1)} = U_{h_T}^{(\ell)} - \frac{1}{\kappa} \left( U_{h_T}^{(\ell)} - \Pi_h^1 Y_{h_T}^{(\ell)} \right) .
   \]

Note that Steps 1 and 2 are now decoupled: the first step requires to solve a space-time discretization (2.7) of SPDE (2.2), while the second requires to solve the space-time discretization (3.6) of the BSPDE (4.1). We refer to related works on how to approximate conditional expectations [3, 10, 1, 22]; a similar method to $\text{SLQ}_{h_T}^{\text{grad}}$ to solve problem $\text{SLQ}_{h_T}$ has been proposed in [8].

We want to show convergence of $\text{SLQ}_{h_T}^{\text{grad}}$ for $\kappa > 0$ sufficiently large and $\ell \uparrow \infty$. For this purpose, we recall the notations $S_{h_T}, T_{h_T}, \tilde{J}_{h_T}$ introduced in Section 4.2. For this purpose, we first recall Lipschitz continuity of $D\tilde{J}_{h_T}$; since
\[
D^2 \tilde{J}_{h_T} (U_{h_T}) = \left( I + L^* L + \alpha \hat{L}^* \hat{L} \right) U_{h_T} ,
\]
where operators $L, \hat{L}$ are defined in (4.18), we find $K := \| I + L^* L + \alpha \hat{L}^* \hat{L} \|_{L (U_{h_T}; U_{h_T})}$, such that
\[
\| D\tilde{J}_{h_T} (U_{h_T}^1) - D\tilde{J}_{h_T} (U_{h_T}^2) \|_{U_{h_T}} \leq K \| U_{h_T}^1 - U_{h_T}^2 \|_{U_{h_T}} .
\]
Indeed, noting that $\| (I - \tau \Delta_h)^{-1} \|_{L (W^1, W^1)} \leq 1$, we conclude
\[
\| LU_{h_T} \|_{X_{h_T}}^2 = \sum_{n=1}^N \tau \mathbb{E} \left[ \| LU_{h_T} (t_n) \|_{L^2}^2 \right] = \sum_{n=1}^N \tau \mathbb{E} \left[ \| \sum_{j=0}^{n-1} ((I - \tau \Delta_h)^{-1})^{n-j} \Pi_h^1 U_{h_T} (t_j) \|_{L^2}^2 \right]
\]
\[
\leq T^2 \| U_{h_T} \|_{U_{h_T}}^2 ,
\]
\[
\| L \|_{U_{h_T}} \|_{X_{h_T}}^2 = \sum_{n=1}^N \tau \mathbb{E} \left[ \| L U_{h_T} (t_n) \|_{L^2}^2 \right] = \sum_{n=1}^N \tau \mathbb{E} \left[ \| \sum_{j=0}^{n-1} ((I - \tau \Delta_h)^{-1})^{n-j} \Pi_h^1 U_{h_T} (t_j) \|_{L^2}^2 \right]
\]
\[
\leq T^2 \| U_{h_T} \|_{U_{h_T}}^2 ,
\]
\[
\| L \|_{U_{h_T}} \|_{X_{h_T}}^2 = \sum_{n=1}^N \tau \mathbb{E} \left[ \| L U_{h_T} (t_n) \|_{L^2}^2 \right] = \sum_{n=1}^N \tau \mathbb{E} \left[ \| \sum_{j=0}^{n-1} ((I - \tau \Delta_h)^{-1})^{n-j} \Pi_h^1 U_{h_T} (t_j) \|_{L^2}^2 \right]
\]
\[
\leq T^2 \| U_{h_T} \|_{U_{h_T}}^2 ,
\]
and
\[ \| \hat{\mathcal{L}} U_{hT} \|_{L^2(U_{hT};\mathbb{L}^2)}^2 = \mathbb{E} \left[ \left\| \tau \sum_{j=0}^{N-1} \left( (1 - \tau \Delta_h)^{-1} \right)^{N-j} \Pi_h U_{hT}(t_j) \right\|_{\mathbb{L}^2}^2 \right] \leq T \| U_{hT} \|_{U_{hT}}^2. \]

Hence
\[ K = \| 1 + L^* L + a \hat{L}^* \hat{L} \|_{\mathcal{L}(U_{hT}, \hat{U}_{hT})} \leq 1 + \alpha T + T^2. \]

Since \( \text{SLQ}^{\text{grad}}_{hT} \) is the gradient descent method for \( \text{SLQ}_{hT} \), we have the following result.

**Theorem 5.2.** Suppose that \( \kappa \geq K \). Let \( \{ U_{hT}^{(t)} \}_{t \in \mathbb{N}_0} \subset U_{hT} \) be generated by \( \text{SLQ}^{\text{grad}}_{hT} \), and \( U_{hT}^* \) solve \( \text{SLQ}_{hT} \). Then

\[
\begin{aligned}
\hat{\mathcal{J}}_{hT}(U_{hT}^{(t)}) - \hat{\mathcal{J}}_{hT}(U_{hT}^*) &\leq \frac{2\kappa \| U_{hT}^{(0)} - U_{hT}^* \|_{U_{hT}}^2}{\ell}, \\
\| U_{hT}^{(t)} - U_{hT}^* \|_{U_{hT}}^2 &\leq \left( 1 - \frac{1}{\kappa} \right)^\ell \| U_{hT}^{(0)} - U_{hT}^* \|_{U_{hT}}^2, \quad \ell = 1, 2, \ldots.
\end{aligned}
\]

**Proof.** We know that \( D \hat{\mathcal{J}}_{hT} \) is Lipschitz continuous with constant \( K > 0 \). Also, \( \hat{\mathcal{J}}_{hT} \) is strongly convex. Hence, the gradient descent method in abstract form is the following iteration (see Algorithm 5.1, Step 3.)

\[ U_{hT}^{(t+1)} = U_{hT}^{(t)} - \frac{1}{\kappa} D \hat{\mathcal{J}}_{hT}(U_{hT}^{(t)}), \quad \ell = 0, 1, 2, \ldots. \]

By the proof of Theorem 4.2, we have obtained the following facts:

\[
\begin{aligned}
D \hat{\mathcal{J}}_{hT}(U_{hT}^{(t)}) = U_{hT}^{(t)} - \Pi_h^0 \mathcal{T}_{hT}(S_{hT}(U_{hT}^{(t)})), \\
\Pi_h^0 \mathcal{T}_{hT}(S_{hT}(U_{hT}^{(t)})) = -L^* \left( \Gamma \Pi_h^0 X_0 + LU_{hT}^{(t)} + f - \tilde{X} \right) \\
&\quad - a \hat{L}^* \left( \hat{\Gamma} \Pi_h^0 X_0 + \hat{L} U_{hT}^{(t)} + \hat{f} - \tilde{X}(T) \right),
\end{aligned}
\]

where \( L, \hat{L}, \Gamma, \tilde{X}, \hat{\Gamma} \) are defined in (4.18) and (4.19). Via (4.20), we have that \( \Pi_h^0 \mathcal{T}_{hT}(S_{hT}(U_{hT}^{(t)})) \) is just \( Y_{hT}^{(t)} \), the solution of Step 2 in Algorithm 5.1. Therefore, (5.1) is consistent with the gradient descent method \( \text{SLQ}^{\text{grad}}_{hT} \). The desired error estimates now follow by standard estimates for the gradient descent method (see, e.g. [19, Theorem 1.2.4]).

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