Disjointness of representations arising in harmonic analysis on the infinite-dimensional unitary group

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Introduction

In the context of the problem of harmonic analysis on the group $U(\infty)$ a family of representations $T_{z,w}$ was constructed and studied in the papers [Olsh2] and [BO]. These representations depend on two complex parameters and provide a natural generalization of the regular representation for the case of “big” group $U(\infty)$. The representation $T_{z,w}$ does not change if $z$ or $w$ is replaced by $\overline{z}$ or $\overline{w}$, respectively. The structure of the decomposition of $T_{z,w}$ substantially depends on whether parameters are integers or not. We handle the latter case. Our aim is to prove the disjointness of the representations $T_{z,w}$.

Recall that two representations $T$ and $T'$ are called disjoint if they have no equivalent nonzero subrepresentations. In this paper we prove the following theorem:

**Theorem 1.** Suppose that all parameters $z, w, z', w'$ are not integers, \( \{z, \overline{z}\} \neq \{z', \overline{z} \}, \{w, \overline{w}\} \neq \{w', \overline{w}\} \) (all pairs are non-ordered), then representations $T_{z,w}$ and $T_{z',w'}$ are disjoint.

The authors of [KOV] studied a family $T_z$ of representations of the infinite symmetric group and obtained a result similar to Theorem 1. Our reasoning uses some ideas from that paper, but the case of the unitary group turned out to be much more complicated and we have to use additional arguments.

Disjointness of representations can be reduced to disjointness of certain probability measures $\rho_{z,w}$ on the space of paths in the Gelfand-Tsetlin graph, and we are mainly concerned with analysis of the measures $\rho_{z,w}$.

A path in the Gelfand-Tsetlin graph is a sequence $\{\lambda(N)\}$ of dominant weights of unitary groups $U(N)$, and each $\lambda(N)$ can be viewed as pair $(\lambda^+(N), \lambda^-(N))$ of Young diagrams. Thus, each measure generates two random sequences $\{\lambda^+(N)\}, \{\lambda^-(N)\}$ of Young diagrams forming a Markov growth.

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process. As an application of our method we prove that the length of the diagonal in \( \{ \lambda^\pm(N) \} \) grows at most logarithmically in \( N \).

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1 Spherical representations of \( U(\infty) \), Gelfand-Tsetlin graph, reduction to spectral measures

In this section we collect some general facts about representations of the group \( U(\infty) \); most of them can be found in the paper [Olsh2].

Consider the chain of the compact classical groups \( U(N), N = 1, 2, \ldots, \) which are embedded one into another in a natural way. Let \( U(\infty) \) be their union. Following the philosophy of [Olsh1], we form a \((G,K)\)-pair, where \( G \) is the group \( U(\infty) \times U(\infty) \) and \( K \) is the diagonal subgroup, isomorphic to \( U(\infty) \).

We are dealing with unitary representations \( T \) of the group \( G \) possessing a distinguished cyclic \( K \)-invariant vector \( \xi \). Such representations are called spherical representations of the pair \((G,K)\). They are completely determined by the corresponding matrix coefficients \( \psi(\cdot) = (T(\cdot)\xi,\xi) \). The \( \psi \)'s are called spherical functions. They are \( K \)-biinvariant functions on \( G \), which can be converted (via restriction to the subgroup \( U(\infty) \times \{e\} \subset G \)) to central functions \( \chi \) on \( U(\infty) \) (that are functions which are constant on conjugacy classes). Irreducible representations \( T \) are in one-to-one correspondence to extreme characters (i.e. extreme points in the convex set of all characters, which are central positive definite continuous functions on \( U(\infty) \) taking on a value of 1 at the unity element of the group).

Irreducible spherical representations of \((G,K)\) and extreme characters of \( U(\infty) \) admit a complete description. They depend on countably many continuous parameters. There is a bijective correspondence \( \chi^{(\omega)} \leftrightarrow \omega \) between extreme characters and points \( \omega \) of the infinite-dimensional domain

\[
\Omega \subset \mathbb{R}^{4\infty+2} = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \times \mathbb{R},
\]

where \( \Omega \) is the set of sextuples

\[
\omega = (\alpha^+, \beta^+; \alpha^-, \beta^-; \delta^+, \delta^-)
\]

such that

\[
\alpha^\pm = (\alpha^+_1 \geq \alpha^+_2 \geq \cdots \geq 0) \in \mathbb{R}^\infty, \quad \beta^\pm = (\beta^+_1 \geq \beta^+_2 \geq \cdots \geq 0) \in \mathbb{R}^\infty,
\]

\[
\sum_{i=1}^{\infty}(\alpha^+_i + \beta^+_i) \leq \delta^+, \quad \beta^+_1 + \beta^-_1 \leq 1.
\]
D. Voiculescu discovered an explicit formula for the functions $\chi^{(\omega)}(U)$, where $U \in U(\infty)$. We do not need it in the present paper, this formula can be found in \cite{Vo} or \cite{Olsh2}.

It was proved in \cite{Olsh2} that any character $\chi$ corresponds to a unique probability measure $\sigma$ on $\Omega$, such that

$$\chi(U) = \int_{\Omega} \chi^{(\omega)}(U)\sigma(d\omega), \quad U \in U(\infty).$$

We call $\sigma$ the spectral measure of the character $\chi$. The inverse statement is also true, every probability measure on $\Omega$ corresponds to a certain character of the group $U(\infty)$.

According to the general theory, the disjointness of two spherical representations $T$ and $T'$ is equivalent to the disjointness of the spectral measures $\sigma$ and $\sigma'$ corresponding to their characters. Our aim is to prove the disjointness of these measures.

The Gelfand-Tsetlin graph (also known as the graph of signatures) is a convenient tool for describing characters of $U(\infty)$. The vertices of the graph symbolize the irreducible representations of groups $U(N)$ while the edges encode the inclusion relations between irreducible representations of $U(N)$ and $U(N + 1)$. The $N$–th level of the graph, denoted by $\mathcal{GT}_N$, corresponds to the irreducible representations of $U(N)$. Elements of $\mathcal{GT}_N$ can be identified with dominant weights for $U(N)$, i.e., these are $N$–tuples of integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N)$ which are also called signatures. We join two signatures $\lambda \in \mathcal{GT}_N$ and $\mu \in \mathcal{GT}_{N+1}$ by an edge and write $\lambda \prec \mu$ if

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \cdots \geq \lambda_N \geq \mu_{N+1}.$$

In what follows it is convenient for us to represent signature $\lambda$ by two Young diagrams $\lambda^+$ and $\lambda^-$: the row lengths of $\lambda^+$ are the positive coordinates $\lambda_i$, while the row lengths of $\lambda^-$ are the absolute values of the negative coordinates. We call $\lambda^+$ and $\lambda^-$ a “positive” Young diagram and a “negative” Young diagram, respectively.

Note that when one moves to the next level of the graph, a horizontal strip is added to the “positive” Young diagram. Recall that a horizontal strip is a skew Young diagram containing at most one box in each column (see \cite{Mac} Chapter 1).

Recall that number $c(\Box) = j - i$ is called the content of the box lying at the intersection of the $i$–th row and $j$–th column.

We denote by $\chi^\lambda$ the irreducible character of $U(N)$, indexed by the signature $\lambda \in \mathcal{GT}_N$. Given an arbitrary character $\chi$ of the group $U(\infty)$ we may expand the restriction of $\chi$ on $U(N)$ into a convex combination of the functions $\chi^\lambda(\cdot)/\chi^\lambda(e)$. The coefficients of this expansion determine a probability distribution $M_N(\lambda)$ on the discrete set $\mathcal{GT}_N$. In this way, we get a bijection between characters and certain sequences of probability distributions $\{M_N\}_{N=1,2,\ldots}$, which are called coherent systems (because of certain coherency relations connecting $M_N$ and $M_{N+1}$). Additional information can be found in \cite{Olsh2}. 3
A path in the graph of signatures $\mathbb{G}\mathbb{T}$ is a finite or infinite sequence $t = (t(1), t(2), \ldots)$, such that $t(N) \in \mathbb{G}\mathbb{T}_N$, and for every $N = 1, 2, \ldots$, $t(N)$ and $t(N + 1)$ are joined by an edge. In what follows finite or infinite paths are denoted by letters $t$, $\tau$. Usually $\tau = (\tau(1), \tau(2), \ldots)$ and $\tau_i(N)$ stands for the length of the $i$-th row of the signature $\tau(N)$.

We can define a topology on the set $T$ of all infinite paths in $\mathbb{G}\mathbb{T}$. A base of this topology consists of cylindrical sets $C_\tau$, where $\tau = (\tau(1), \ldots, \tau(N))$ is a finite path and $C_\tau = \{ t \in T | t(1) = \tau(1), \ldots, t(N) = \tau(N) \}$.

Now we may take $\sigma$–algebra of Borel sets generated by this topology and consider measures on this $\sigma$–algebra.

A measure on $T$ is called central if the measure of a cylindrical set $C_\tau$, $\tau = (\tau(1), \ldots, \tau(N))$ depends solely on $\tau(N)$. Consider an arbitrary signature $\lambda \in \mathbb{G}\mathbb{T}_N$. The set of finite paths terminating at $\lambda$ consists of exactly $\text{Dim}_N(\lambda) = \chi^\lambda(e)$ elements. Given a coherent system on the graph of signatures we may construct a certain central measure $\rho$ on the set $T$ by setting

$$\rho(C_\tau) = \frac{M_N(\tau(N))}{\text{Dim}_N(\tau(N))}$$

where $C_\tau$ is an arbitrary cylindrical set as above and $\tau(N)$ is the endpoint of $\tau$. Correctness of this definition follows from the properties of coherent systems.

Thus, we obtain a sequence of bijections connecting characters of the group $U(\infty)$, probability measures $\sigma$ on $\Omega$ (spectral measures), coherent systems $\{M_N\}$, and central measures $\rho$ on $T$.

Let us show that the disjointness of spectral measures $\sigma$ and $\sigma'$ follows from the disjointness of the corresponding central measures $\rho$ and $\rho'$ (the similar proposition in the case of the infinite symmetric group was proved in [KOV]).

**Proposition 2.** Assume that central measures $\rho$ and $\rho'$ on the graph of signatures are disjoint, then the corresponding spectral measures $\sigma$ and $\sigma'$ on $\Omega$ are also disjoint.

**Proof.** Assume not true. Denote by $\tilde{\sigma} = \sigma \land \sigma'$ the greatest lower bound of the measures $\sigma$ and $\sigma'$ (we say that a measure $\mu$ is less than a measure $\nu$ if $\mu(A) \leq \nu(A)$ for an arbitrary measurable set $A$). The existence of such measure can be easily verified. It is evident that if $\sigma$ and $\sigma'$ are not disjoint, then $\tilde{\sigma}$ is a non-zero measure.

Next, observe that the correspondence $\sigma \leftrightarrow \rho$ between probability spectral measures on $\Omega$ and central measures on $T$ can be extended to arbitrary finite measures, which are not necessarily probability measures.

Denote by $\tilde{\rho}$ the central measure corresponding to $\tilde{\sigma}$.

There is an integral representation for the value of this measure on a cylindrical set. Let $\omega \in \Omega$. Let $\rho_\omega$ be the central measure on $T$ corresponding
to the extreme character $\chi^{(\omega)}(U)$. Denote by $f_{\tau(N)}(\omega)$ the value of $\rho_\omega$ on the cylindrical set $C_\tau$, where $\tau(N)$ is the endpoint of $\tau$. We have

$$\tilde{\rho}(C_\tau) = \int_{\Omega} f_{\tau(N)}(\omega) \tilde{\sigma}(d\omega).$$

From the last formula, nonnegativity of $f_{\lambda}(\omega)$, and inequalities $\tilde{\sigma} \leq \sigma, \tilde{\sigma} \leq \sigma'$ it follows that $\tilde{\rho} \leq \rho$ and $\tilde{\rho} \leq \rho'$. Consequently, $\rho \wedge \rho' \neq 0$ and the measures $\rho$ and $\rho'$ are not disjoint.

In what follows we are going to prove disjointness of the central measures on the graph of signatures corresponding to the characters of the representations $T_{z,w}$ and $T_{z',w'}$. Let us denote these measures by $\rho_{z,w}$ and $\rho_{z',w'}$.

It was shown in [Olsh2] that the coherent system corresponding to the representation $T_{z,w}$ is given by

$$M_{N}^{z,w}(\lambda) = (S_N(z,w))^{-1} \prod_{i=1}^{N} \left| \frac{1}{\Gamma(z - \lambda_i + i)\Gamma(w + N + 1 + \lambda_i - i)} \right|^2 \cdot \text{Dim}_N^2(\lambda),$$

where $S_N(z,w)$ is the normalization constant whose explicit value is not important for us. Denote by $P_{N}^{z,w}(\tau)$ the measure $\rho_{z,w}$ of an arbitrary cylindrical set $C_\tau$, provided that $\lambda$ is the endpoint of the path $\tau$. The following formula holds:

$$\rho_{z,w}(C_\tau) = \frac{P_{N}^{z,w}(\tau(N))}{M_{N}^{z,w}(\tau(N))}.\text{Dim}_N(\tau(N)).$$

The measure $\rho_{z,w}$ is called the $(z,w)$–measure below.

From this point on we forget about representations $T_{z,w}$, all further arguments are based on the last two formulas.

2 Scheme of proof

Our aim is to prove that the $(z,w)$–measures are pairwise disjoint.

Consider two pairs $(z,w)$ and $(z',w')$, assume that all four numbers are non-integral, $z \neq z', z \neq \overline{z}$, $w \neq w'$, $w \neq \overline{w}$. Let us study the ratio $\frac{P_{N}^{z,w}(\lambda)}{P_{N}^{z',w'}(\lambda)}$. Let $Q$ be the set of paths along which this ratio tends to a finite non-zero limit. Clearly, this is a Borel set. The disjointness of the measures follows from the equalities $\rho_{z,w}(Q) = \rho_{z',w'}(Q) = 0$ (see [S, Chapter 7, Section 6, Theorem 2]).

In order to show that the measure of the set $Q$ is equal to zero, we introduce a new set $G$, which consists of paths, that are not contained in any “thick hook”. Denote temporarily $Q_0 = G \cap Q$. We prove below that the measure of $Q_0$ is equal to zero (sections 3 and 4) and then we show that the $(z,w)$–measure of the set $G$ is equal to 1 (section 5). Clearly, these two facts imply vanishing of the $(z,w)$–measure of the set $Q$.

Proof of the first part (the estimate for the measure of $Q_0$) contains several technical details and we want to provide some main ideas first. It possible to
pass from $Q_0$ to a family of sets $Q^l_0$. For every $Q^l_0$ we construct a collection of maps $\{f_i\}$, which take $Q^l_0$ into a collection $\{Q^{l_i}\}$ of pairwise disjoint sets. The measure of each $Q^{l_i}$ is equal to the measure of $Q^l_0$. Since there exist infinitely many such sets $Q^{l_i}$ we conclude that the measure of each one is equal to zero. Hence, the measure of $Q_0$ is also equal to zero.

The essence of the maps $f_i$ is a small reorganization of paths. It turns out, that it is enough to change only one level of a path to generate a rather big fluctuation from the hypothetical limit value of the ratio $\frac{P_z,w}{P'_z,w}$. This observation is the main idea underlying the proof of the theorem.

Each map $f_i$ is very simple. We fix in advance some large integer $k$. Each $f_i$ modifies every path as follows: Instead of adding a box with content $k$ to the “positive” Young diagram at level $N$, we do that at level $N+1$. Although, not every path can be modified in this way, we will show later that for almost all paths from the set $Q^l_0$ maps $f_i$ can be correctly defined.

Important moment in the argument is the following: the probability of adding a horizontal strip containing $m$ boxes with the contents $k,k+1,\ldots,k+m-1$ at level $N$ does not exceed $c(k,m)/N^m$ (we use this statement only in the case $m = 1$, but it also holds in the more general case).

3 Modification of paths

We proceed to the detailed proof.

Let us say that a path $\tau = (\tau(1),\tau(2),\ldots)$ in the graph of signatures is contained in a left thick hook, if there exist numbers $i$ and $j$ such that the inequality $\tau_i(N) < j$ holds for every $N$. A path is contained in a right thick hook, if there exist numbers $i$ and $j$ such that the inequality $\tau_{N-i}(N) > j$ holds for every $N$.

Denote by $G^+$ the set of all paths that are not contained in any left thick hook and by $G^-$ the set of all paths, that are not contained in any right thick hook. In section 5 we show that almost every path (with respect to the $(z,w)$–measure) is not contained in any two-sided thick hook, i.e, either it is not contained in any left thick hook or it is not contained in any right thick hook. Thus, $\rho_{z,w}(G^+ \cup G^-) = 1$.

**Theorem 3.** If $\{z, \overline{z}\} \neq \{z', \overline{z}'\}$, then

$$\rho_{z,w}(Q \cap G^+) = \rho_{z',w'}(Q \cap G^+) = 0.$$ 

If $\{w, \overline{w}\} \neq \{w', \overline{w}'\}$, then

$$\rho_{z,w}(Q \cap G^-) = \rho_{z',w'}(Q \cap G^-) = 0.$$ 

In particular this theorem implies that if $z \neq z'$, $z \neq \overline{z}'$, $w \neq w'$ and $w \neq \overline{w}'$, then the corresponding measures are disjoint.

We give a proof of the first part of the theorem, the second part can be proved similarly.
Let us denote by $h_N$ the double ratio

$$h_N = \frac{P_N^{z,w}}{P_{N+1}^{z',w'}} \cdot \frac{P_N^{z',w'}}{P_{N+1}^{z,w}}.$$ 

Note that if $\frac{P_N^{z,w}}{P_N^{z',w'}}$ tends to a finite limit, then $h_N \to 1$.

Denote by $A$ the set of paths along which $h_N \to 1$. Our aim is to prove that the $(z,w)$–measure of the set $A \cap G^+$ is equal to zero ($\rho_{z,w}(A \cap G^+) = 0$).

Fix $\delta > 0$. We say that there is a $\delta$–fluctuation, or simply a fluctuation, at the level $k$ provided that $|h_k - 1| > \delta$. Denote by $A^\delta_t$ the set of such paths that starting from the level $t$ there are no $\delta$–fluctuations in them. We will show that if we choose sufficiently small $\delta$, then $\rho_{z,w}(A^\delta_t \cap G^+) = 0$ for every $t$. It is clear that this statement implies our theorem.

**Theorem 4.** There exists $\delta_0$ such that for every positive $\delta \leq \delta_0$ and any $t$

$$\rho_{z,w}(A^\delta_t \cap G^+) = 0.$$ 

**Proof.** Recall once again that the content of the box lying at the intersection of the $i$-th row and the $j$-th column of a Young diagram is the integer $c(\square) = j - i$.

It is evident that a path is contained in a left thick hook if and only if for every $k$ a box with the content $k$ is added to the “positive” Young diagram only finitely many times.

The idea of the subsequent argument is the following: adding a box with a relatively small positive content while moving to the next level along the path may cause a fluctuation. Suppose $\tau$ is a finite fragment of a path, there are no fluctuations in $\tau$, and a box with the content $k$ is added while moving along $\tau$. Then for almost any $\tau$ there exists another fragment $\tau'$, such that $\tau$ and $\tau'$ have the same starting point and endpoint, and there is a fluctuation in $\tau'$. Since infinitely many boxes with the content $k$ are being added along the typical path, the last claim also implies a stronger assertion. If a fragment $\tau$ is long enough, than we may construct arbitrary many such fragments $\tau'$. In turn, this implies that measure of the set of paths without fluctuations is equal to zero.

So fix a positive integer $k$, its value will be specified later. We introduce a family of sets of paths $B_{\varepsilon}$ which exhaust the set $G^+$. In what follows we construct certain maps on these sets which “create fluctuations”. The main technical part of the proof is contained in the following proposition.

**Proposition 5** (about admissible paths). Assume that $\Re(k + w) > 0$. Fix an arbitrary $\varepsilon > 0$. There exist a sequence of integers $N_1 < N_2 < N_3 < \ldots$ and a set of paths $B_{\varepsilon} \subset G^+$ such that $\rho_{z,w}(G^+ \setminus B_{\varepsilon}) < \varepsilon$, and for any $n > 0$ and any path from the set $B_{\varepsilon}$ the following holds

1. For every $i$, there is a number $n_i$, such that $N_i \leq n_i < N_{i+1}$ and the first addition of a box with content $k$ at a level from $N_i, N_i + 1, \ldots$ if at level $n_i$. 


2. There is no addition of a box with content $k + 1$ at the level $n_i$ (i.e. no other box is added to the same row)

3. There is no addition of a box with content $k - 1$ at the level $n_i + 1$ (consequently, no other box is added to the same column)

We give the proof of this proposition in the next section. Now we are going to use it to prove Theorem 4.

Denote by $H$ the intersection of the sets $B_\varepsilon$ and $A_\delta^t$. As the sets $B_\varepsilon$ exhaust $G^+$, it is sufficient to prove that for small enough $\delta$, arbitrary $\varepsilon > 0$, and large enough number $t$, the measure of $H$ is equal to zero.

A path from the set $H$ can be informally described in the following way. Starting from some level it has no fluctuations but boxes with content $k$, which were defined in the proposition, are still added.

Now fix $H$ and the sequence of integers $N_i$ corresponding to it. Choose some large enough number $N_p$ from the $N_i$'s. Choose an integer $q > p$ and consider the levels from $N_p$ to $N_q$ of the graph of signatures. We call a sequence $(τ(N_p), τ(N_p + 1), \ldots, τ(N_q))$ a $[N_p, N_q]$-fragment of path $τ = (τ(1), τ(2), \ldots)$. Fix an arbitrary signature $λ^p$ at level $N_p$ and an arbitrary signature $λ^q$ at $N_q$. We want to study $[N_p, N_q]$–fragments of such paths $τ$. It is sufficient to prove that the fraction of the $[N_p, N_q]$–fragments of paths from the set $H$ in all such $[N_p, N_q]$–fragments tends to zero when $q \to \infty$ (uniformly in $λ^p$ and $λ^q$). Obviously, this implies that the measure of the set $H$ is equal to zero.

Let us also choose signatures at levels $N_p + 1 \ldots N_q - 1$. Suppose that $τ(N_p) = λ^p, τ(N_p + 1) = λ^{p+1}, \ldots, τ(N_q) = λ^q$ and examine $[N_p, N_q]$–fragments of such paths $τ$. It is sufficient to prove that the fraction of such $[N_p, N_q]$–fragments, obtained from the paths of $H$, in all such $[N_p, N_q]$–fragments tends to zero, provided that $q$ tends to infinity.

Now fix a $[N_p, N_q]$–fragment, satisfying the above conditions. Consider a $[N_s, N_{s+1}]$–fragment, that is a subfragment of the above $[N_p, N_q]$–fragment, and define a path modification $f_s$ in the following way.

We know that a box with content $k$ is added at a level $N$ from $\{N_s, N_s + 1, \ldots, N_{s+1} - 1\}$, no boxes with bigger content are added to the same row at the same level, and no box is added to the same column at the next level. Modification $f_s$ consists in adding this box not at level $N$, but at level $N + 1$. Thus, we obtained some map, its domain of definition is the set $H$. It is clear that this map is injective. (To determine the preimage of a path one should choose the first addition of a box with content $k$ and simply move this addition to the previous level)

**Lemma 6.** It is possible to choose $k$, $δ$ and $p$ such that for any path $τ \in H$, $f_s(τ)$ does not belong to $H$. 


Proof. Consider the ratio of the densities at level $N$. The essence of our modification is that in the new path for a single $i$ the number $\lambda_i$ (which is the length of the $i$–th row) decreased by 1. Recall that

$$P_{N}^{z,w}(\lambda) = (S_{N}(z,w))^{-1} \cdot \prod_{i=1}^{N} \frac{1}{\Gamma(z - \lambda_i + i)\Gamma(w + N + 1 + \lambda_i - i)} \cdot \text{Dim}_{N}(\lambda)$$

Consequently, under the transform $\lambda_i \mapsto \lambda_i - 1$ this density is multiplied by

$$\frac{|w + N + 1 + k|}{z - k}^2 \cdot (\text{factors not depending on } z,w).$$

We conclude that the ratio of densities $\frac{P_{N}^{z,w}}{P_{N}^{z',w}}$ is multiplied by

$$\frac{|z' - k|}{|z - k|}^2 \cdot \frac{|w + N + 1 + k|}{|w' + N + 1 + k|} \quad (\ast)$$

Examine the first factor. Note that if $z \neq z'$ and $z \neq z'$, then one can find $k > 0$ and $\nu > 0$ such that

$$\left| \frac{|z' - k|}{|z - k|}^2 - 1 \right| > \nu$$

and $\Re(k + w) > 0$. Let us fix such $k$ and $\nu$.

Now consider the second factor in $(\ast)$. It is clear that this factor approaches 1 as $N$ tends to infinity. Assume that $N_p$ is so large that for any $N > N_p$ the factor differs from 1 by less than $\nu/100$.

Now we can conclude that the product $(\ast)$ differs from 1 by more than $\nu/2$.

Choose $\delta \leq \nu/100$. It is evident that for such $\delta$ the ratios of densities before and after applying $f_s$ can not both belong to the $\delta$–neighborhood of 1 (as $f_s$ multiplies the ratio of densities by the number $(\ast)$ that is far enough from 1).

Choose $p$ such that the inequality $N_p \geq t$ also holds. From all that has been said above it follows that in any path from $H$ there are no fluctuations at levels starting from $N_p$, but after applying $f_s$ such fluctuations inevitably appear. Thus the image with respect to $f_s$ of a path from the set $H$ does not belong to $H$.

Lemma 7. Suppose that the parameters are chosen as in Lemma 6. If $s \neq s'$, then the sets $f_s(H)$ and $f_{s'}(H)$ are mutually disjoint.

Proof. It follows from the proof of the last lemma that in the paths from $f_s(H)$ and $f_{s'}(H)$ fluctuations appear at distinct levels, thus, these paths are distinct.
of the paths from the set $H$ in all $[N_p, N_q]$-fragments (hence also the measure of the set $H$) does not exceed $\frac{E}{E + b(q-p)}$. Choosing the number $q$ large enough we can obtain arbitrary small values of the last expression. Thus, Theorem 4 is proved.

\[\square\]

4 Proof of the proposition about admissible paths

Proof of the proposition about admissible parts. Let us interpret the $(z, w)$-measure as a Markovian process on the Gelfand-Tsetlin graph with transition probabilities

$$p_{z,w}(\lambda|\mu) = \frac{P_{N+1}^{z,w}(\lambda)}{P_{N}^{z,w}(\mu)}, \quad \lambda \in \mathbb{GT}_{N+1}, \quad \mu \in \mathbb{GT}_{N}.$$ 

Thus, from now on we may speak about the random path $\tau$ or about the Markov chain $\tau(N).

Lemma 8. Assume that $\Re(k + w) > 0$. Fix an arbitrary signature $\mu$ at level $N$ (thus also fix the corresponding "positive" Young diagram). Consider the event that a box with content $k$ is added to the "positive" Young diagram during the transition to the $(N + 1)$st level along the path.

The conditional probability of this event, given that $\tau(N) = \mu$, is less than $c(k)N$.

The same estimate holds if we additionally condition a box with content $k-1$ to be added to the "positive" diagram at the same level.

Remark. Instead of $\mu$ we may fix the whole path $(\mu^1, \ldots, \mu^N)$ from the first level to the signature $\mu^N$, and condition on $\tau(1) = \mu^1, \ldots, \tau(N) = \mu^N$. The centrality property of the measure implies that this change does not affect either our statement or its proof.

Proof. Note that for the fixed $\mu$ there is at most one box with content $k$ that can be added. Denote the coordinates of this box by $(i, j)$. Let us compare the conditional probability of the event $\tau_i(N+1) = j - 1$, given that $\tau(N) = \mu$, with the conditional probability of the event $\tau_i(N+1) \geq j$. We will show that the latter differs from the former by the factor $\frac{c(k)}{N}$, which clearly implies the lemma.

Besides fixing signature $\mu$, fix additionally all row lengths at level $N+1$, except the $i$-th row, in an arbitrary way. Thus, now we condition on $\tau(N) = \mu$, $\tau_1(N+1) = \lambda_1, \ldots, \tau_{i-1}(N+1) = \lambda_{i-1}, \tau_{i+1}(N+1) = \lambda_{i+1}, \ldots, \tau_{N+1}(N+1) = \lambda_{N+1}$.

Denote by $p_m$ the ratio of the conditional probability of the event $\tau_i(N+1) = j - 1 + m$ to the conditional probability of the event $\tau_i(N+1) = j - 1$. Our
problem reduces to verifying the inequality \( \sum_{m=1}^{\infty} p_m < \frac{c(k)}{n} \). Using the definition of the transition probability and Weyl’s dimension formula we obtain:

\[
\text{Dim}_{N+1}(\lambda) = \prod_{1 \leq i < j \leq N+1} \frac{\lambda_i - \lambda_j + j - i}{j - i},
\]

\[
p_m = \frac{\Gamma(z-k+1)\Gamma(w+N+1+k)}{\Gamma(z-k+1-m)\Gamma(w+N+1+k+m)} \prod_{a=1}^{i-1} \frac{\lambda_a - a - (k-1 + m)}{\lambda_a - a - (k-1)} \prod_{a=i+1}^{N+1} \frac{(k-1 + m) - (\lambda_a - a)}{(k-1) - (\lambda_a - a)}.
\]

Consequently,

\[
p_m \leq \frac{\Gamma(z-k+1)\Gamma(w+N+1+k)}{\Gamma(z-k+1-m)\Gamma(w+N+1+k+m)} \prod_{a=1}^{N+1} \frac{m + (a-i)}{a - i} \prod_{a=i+1}^{N+1} \frac{m + a}{a - a} = \frac{|(k-z)_m|^2}{|(w+N+1+k)_m|^2} \cdot \frac{(m+N)!}{N!m!} \cdot \frac{|(w+N+1+k)_m|^2}{|(w+N+1+k)_m|^2} \cdot \frac{(N+1)_m}{\prod_{a=1}^{N} |\Re(k-z)| + |3z|}_m \cdot \frac{1}{|\Re(w+N+1+k)|_m!}
\]

Note that, if \( \Re(k+w) > 0 \), then the second factor is less than 1, thus,

\[
p_m \leq \frac{\left((|\Re(k-z)| + |3z|)_m\right)^2}{|\Re(w+N+1+k)|_m!} \cdot \frac{(m+N)!}{N!m!} \cdot \frac{|(w+N+1+k)_m|^2}{|(w+N+1+k)_m|^2} \cdot \frac{(N+1)_m}{\prod_{a=1}^{N} |\Re(k-z)| + |3z|}_m \cdot \frac{1}{|\Re(k-w+2+k)|_m!}
\]

It means that

\[
\sum_{m=1}^{\infty} p_m \leq \frac{c(k)}{N} F(|\Re(k-z)| + |3z| + 1, |\Re(k-z)| + |3z| + 1, |\Re(w+N+2+k)|; 1)
\]

Here \( F(a, b, c; 1) \) is the hypergeometric function \( _2F_1 \) with parameters \( a, b, c \) and the argument 1.

In our case \( a = b = |\Re(k-z)| + |3z| + 1, c = |\Re(w+N+2+k)| \).
Using Gauss’ formula for the value of the hypergeometric function at 1 (see, for instance, [BE]) we obtain

$$\sum_{m=1}^\infty p_m \leq \frac{c(k)}{N} \frac{\Gamma(c)\Gamma(c-2a)}{\Gamma(c-a)\Gamma(c-a)}$$

When $N \to \infty$, the parameter $c$ tends to infinity, while $a$ does not change. Thus, $\frac{c(k)}{N} \frac{\Gamma(c)\Gamma(c-2a)}{\Gamma(c-a)\Gamma(c-a)} \to 1$. Consequently,

$$\sum_{m=1}^\infty p_m \leq \frac{c_1(k)}{N}.$$

Lemma 9. Choose two large enough integers $m$ and $n$. Consider the following event:

1. A box with content $k$ is added to a “positive” Young diagram at a level from \(\{n, n+1, \ldots, m\}\).

2. Once such a box is added, either a box with content $k + 1$ is added at the same level or a box with content $k - 1$ is added at the next level.

The probability of this event does not exceed \(\frac{c_1(k)}{n}\).

Proof. Let us represent the event “a box with content $k$ is added at a level from \(\{n, \ldots, m\}\)” as the disjoint union of the events “the first addition of a box with content $k$ at a level from \(\{n, \ldots, m\}\) is at level $i$” ($n < i < m$). Denote the last event by $L^i$. Let $\tau^i$ be a path going from the first level of the graph of signatures to level $i$, such that the first addition of a box with content $k$ at a level from \(\{n, \ldots, m\}\) is at level $i$. Then let us write $L^i$ as the disjoint union of the elementary events $C^i_{\tau^i}$ corresponding to the various $\tau^i$.

Applying the previous lemma for every $C^i_{\tau^i}$ and then summing up all the estimates completes the proof.

Now we continue the proof of the proposition.

Note that Lemma 9 implies the following: If $N_1$ is large enough and the $N_i$’s increase at least as fast as $2^i$, then the second condition and the third condition of the proposition hold on a set with the measure arbitrarily close to 1 automatically.

Given $\varepsilon$ we choose $N_1$ so large that the measure of the set, where the second or the third conditions of the proposition might not hold, does not exceed $\varepsilon/2$. Denote the complement of this set by $S$.

Let $A_1 = S \cap G^+$. Obviously, $\rho_{z,w}(G^+ \setminus A_1) \leq \varepsilon/2$.

Denote by $A_2^R \subset A_1 \ (R > N_1)$ the subset containing paths such that a box with content $k$ is added at a level from \(\{N_1, N_1 + 1, \ldots, R\}\). It is clear that $A_2^R \subset A_2^{R+1}$. We know that along every path from the set $A_1 \subset G^+$ boxes with
content $k$ are added infinitely many times. Consequently, the sets $A_2^R$ exhaust $A_1$. Thus, we can choose a number $N_2 > 2 \cdot N_1$ such that 
\[ \rho_{z,w}(A_1 \setminus A_2^{N_2}) < \frac{\varepsilon}{4}. \]

Let $A_2 = A_2^{N_2}$

Further we choose a number $N_3 > 2 \cdot N_2$ and the set $A_3 \subset A_2$ in a similar way. The inequality $\rho_{z,w}(A_2 \setminus A_3) < \frac{\varepsilon}{8}$ holds. And so on.

The intersection of all sets $A_i$ is the desired set $B_{\varepsilon}$.

\[
5 \quad \text{The measure of the set of paths contained in a thick hook}
\]

We say that a path $\tau = (\tau(1), \tau(2), \ldots)$ is contained in a thick hook if one can find four numbers $i, j, l, m$ such that $\tau_i(N) < j$ and $\tau_{N-l}(N) > m$ for every $N$.

**Theorem 10.** The $(z,w)$–measure of the set of all paths contained in a thick hook is equal to zero.

**Proof.** Denote by $R(i, j, l, m, t)$ the set of paths $\tau$ satisfying for every $N \geq t$

1. $\tau_i(N) = j - 1$,
2. $\tau_{i-1}(N) \geq j$,
3. $\tau_{N-l}(N) = m + 1$,
4. $\tau_{N-l+1}(N) \leq m$.

To verify the claim it is sufficient to prove that for any quintuple $(i, j, l, m, t)$,
\[ \rho_{z,w}(R(i, j, l, m, t)) = 0 \] holds.

It is convenient to view $\tau(N)$ as a Markov chain again.

**Lemma 11.** Suppose $N$ is large enough. Denote by $p_N$ the conditional probability of the event “$\tau_i(N + 1) = j - 1$ and $\tau_{N-l}(N + 1) = m + 1$” given that $\tau_i(N) = j - 1$, $\tau_{i-1}(N) \geq j$, $\tau_{N-l}(N) = m + 1$, $\tau_{N-l+1}(N) \leq m$.

Then $p_N$ does not exceed $1 - \frac{\varepsilon}{8}$.

**Proof.** Consider all possible signatures at level $N + 1$ which are joined by an edge with a given signature $\mu$ at level $N$. For every signature $\lambda$ at level $N + 1$, such that $\lambda_i = j - 1$, we introduce a new signature $\lambda'$ as follows. All row lengths of $\lambda'$ except for the $i$th row, are the same as in $\lambda$, while $\lambda_i = j$.

Let us compare the conditional probabilities of the events $\tau(N + 1) = \lambda'$ and $\tau(N + 1) = \lambda$, given that $\tau(N) = \mu$.

This conditional probabilities differ by the factor
\[
\frac{P_{N}^{z,w}(\lambda')}{P_{N}^{z,w}(\lambda)} = \left| \frac{z - j + i}{w + N + 1 + j - i} \right|^2 \cdot \frac{\text{Dim}_{N+1}(\lambda')}{\text{Dim}_{N+1}(\lambda)}. \]
If $N$ is large enough, then the first factor in the right-hand side is greater than $\text{const}/N^2$.

By Weyl’s dimension formula
\[
\dim_{N+1}(\lambda) = \prod_{1 \leq i < j \leq N+1} \frac{\lambda_i - \lambda_j + j - i}{j - i}.
\]

Consequently,
\[
\frac{\dim_{N+1}(\lambda')}{\dim_{N+1}(\lambda)} = \prod_{1 \leq p < i} \frac{\lambda_p - p + j + i}{\lambda_p - p - (j - 1) + i} \cdot \prod_{N+1 \geq p > i} \frac{j - i - (\lambda_p - p)}{j - i - 1 - (\lambda_p - p)}. \tag{**}
\]

Note that here the first product is bounded from below by the constant $2^{-(i-1)}$.

Further, all factors in the second product are greater than 1 and less than 2. Moreover, if $i < p < N - l$, then $m < \lambda_p < j$. Thus, $\lambda_p - p$ attains all values from $\{j - i - n, \ldots, j - i - 2\}$, except for $s$ values; and $s$ can be bounded from above by a number not depending on $N$. We conclude that (**) can be bounded from below by the following expression:
\[
\text{const} \cdot \prod_{g=j-i-2}^{j-i-N} \frac{j - i - g}{j - i - g - 1} = \text{const} \cdot \frac{2\cdot3\cdot \ldots \cdot N}{N - 1} = \text{const} \cdot N.
\]

Therefore, the conditional probabilities differ by a factor which is less than $\text{const}/N$. Consequently, $p_N \leq 1 - \frac{\epsilon}{N}$. \qed

We conclude that
\[
\rho_{z,w}(R(i,j,l,m,t)) \leq \prod_{N=t+1}^{\infty} p_N \leq \prod_{N=t+1}^{\infty} \left(1 - \frac{\epsilon}{N}\right) \to 0.
\]
\qed

The author thinks that the theorem about the disjointness of the measures also holds in a more general case.

It seems quite plausible that the following stronger statement holds: the $(z, w)$–measure of the set of paths contained in a left thick hook is equal to zero. (Recall, that a path $\tau$ is contained in a left thick hook, if there exist two such numbers $i$ and $j$, that $\tau_i(N) < j$ for every $N$)

However, at the moment there is no simple and direct proof of this statement. It seems likely that the statement can be verified using arguments of the paper [RO2] about limit behavior of the intermediate Frobenius coordinates for the ”positive” and ”negative” Young diagrams, corresponding to a signature (tail kernel).

If our conjecture is true, Theorem 1 can be extended to some additional cases. For example, we can state that the representations $T_{z,w}$ and $T_{z,w'}$, where $\{w, w'\} \neq \{w', w\}$ are also disjoint.
6 Appendix. Growth of a diagonal

Let $\tau$ be an infinite path in the graph of signatures. Recall that, there exists a unique Young diagram $\lambda^+(N, \tau)$, corresponding to the signature $\tau(N)$. We call this diagram a “positive” Young diagram.

As a corollary from Lemma 8, we prove that for almost every (with respect to the $(z, w)$–measure) path the length of the diagonal of the “positive” Young diagram grows at most logarithmically in the number of the level.

Denote by $\tilde{s}_N(\tau)$ the length of the main diagonal of $\lambda^+(N, \tau)$.

If we view the $(z, w)$–measure as a probability measure on the set of all paths $T$, then $\tilde{s}_N$ is a sequence of random variables.

**Theorem 12.** There exists a positive constant $c(z, w)$ such that almost surely with respect to the $(z, w)$–measure

$$\lim_{N \to \infty} \frac{\tilde{s}_N}{\log(N)} \leq c(z, w).$$

**Proof.** First we collect some necessary general definitions and facts.

Denote by $X$ the set $\{0, 1\}^\infty$ of infinite sequences of 0’s and 1’s. We equip $X$ with the topology of the direct product. Consider the sigma-algebra of Borel sets on $X$.

Given $A \subset X$ denote by $A_i$ the $i$th term of the sequence $A$. We also employ the notation $A_i = q_i(A)$.

There is a natural partial order on $X$: $A \leq B$, if $A_i < B_i$ for all $i$.

We say that a real-valued function $f$ on the space of sequences is increasing, if from $A \leq B$ follows $f(A) \leq f(B)$. Denote by $\mathcal{M}$ the family of all continuous increasing functions.

Consider two probability measures $\mu_1$ and $\mu_2$ on $X$. We write $\mu_1 \leq \mu_2$ if

$$E_{\mu_1} f \leq E_{\mu_2} f$$

for all functions $f \in \mathcal{M}$. Here $E_{\mu} f$ is the expectation of the function $f$ with respect to the measure $\mu$.

Suppose $A \subset X$. In what follows we identify $A$ with the event “$x \in A$” and denote by $\text{Prob}_\mu A$ the probability of this event with respect to the probability measure $\mu$. We also denote by $\text{Prob}_\mu \{A|B\}$ a conditional probability of the event $A$ given the event $B$.

**Proposition 13.** Suppose $\mu_1$ and $\mu_2$ are two probability measures on the partially ordered compact metric space $X$. A necessary and sufficient condition for $\mu_1 \leq \mu_2$ is that there exists a probability measure $\eta$ on $X \times X$ which satisfies

$$\text{Prob}_\eta \{(x, y) : x \in A\} = \text{Prob}_{\mu_1} A,$$

$$\text{Prob}_\eta \{(x, y) : y \in A\} = \text{Prob}_{\mu_2} A,$$

for all Borel sets $A \subset X$, and

$$\text{Prob}_\eta \{(x, y) : x \leq y\} = 1$$

The measure $\eta$ is called a coupling measure for $\mu_1$ and $\mu_2$. 
Proof. We won’t need necessity of the condition, its proof can be found in the book [Ligg] (theorem 2.4 of the second chapter). Let us proof the sufficiency of the condition.

Let $f$ be an increasing function on $X$. The definition of the measure $\eta$ implies that $f(x) \leq f(y)$ almost surely with respect to the measure $\eta$ on the set of pairs $(x, y) \in X \times X$. It follows that

$$E_{\mu_1} f = \int f(x) d\mu_1 = \int f(x) d\eta \leq \int f(y) d\eta = \int f(y) d\mu_2 = E_{\mu_2} f$$

Lemma 14. Suppose $\mu$ and $\nu$ are two probability measures on $X$, such that $\nu$ is a Bernoulli measure (i.e. product–measure) and the following estimates hold

$$\text{Prob}_\nu\{q_n = 1\} \geq \text{Prob}_\mu\{q_n = 1 \mid A\},$$

for all events $A$, which belong to the $\sigma$–algebra generated by the first $n - 1$ coordinate functions $q_1, \ldots, q_{n-1}$.

Then $\mu \leq \nu$.

Proof. Let us construct a coupling measure $\eta$ for $\mu$ and $\nu$. The support of the coupling measure $\text{supp}(\eta)$ consists of pairs $(A, B) \in X \times X$, such that $A \leq B$. It is clear that there are 3 possibilities for a pair $(A_i, B_i)$: $(0, 0), (0, 1), (1, 1)$. Thus,

$$\text{supp}(\eta) = \{Z\}^\infty,$$

where $Z = \{(0, 0), (0, 1), (1, 1)\}$.

It suffices to define the measure $\eta$ on the cylindrical sets. The base of a cylindrical set is a subset of $Z^n$. Thus, we should define a family of measures $\{\eta_n\}$ ($\eta_i$ is a probability measure on $Z^i$) compatible with natural projections $Z^n \rightarrow Z^{n-1}$.

We proceed by induction on $n$.

First we define the measure $\eta_1$ on $Z$.

In other words, we want to define 3 numbers $\eta(0, 0), \eta(0, 1), \eta(1, 1)$ in such a way that

$$\mu(0) = \eta(0, 0) + \eta(0, 1),$$

$$\mu(1) = \eta(1, 1),$$

$$\nu(0) = \eta(0, 0),$$

$$\nu(1) = \eta(0, 1) + \eta(1, 1).$$

(Here we denote by $\mu(a)$ the marginal distribution $\text{Prob}_\mu\{A_1 = a\}$, and $\eta(a, b) = \eta_1(\{(a, b)\}) = \text{Prob}_\eta\{A_1 = a, B_1 = b\}$)

We may view these relations as a system of linear equations defining 3 desired numbers. It is evident, that as $\nu(1) \geq \mu(1)$ under the hypothesis, the system has a positive solution that is not greater than 1.
Now we want to define the measure $\eta_2$ on $\mathbb{Z}^2$.
We know the numbers $\eta(0, 0), \eta(0, 1), \eta(1, 1)$, and we want to define 9 new numbers

\[
\begin{align*}
\eta(00, 00), & \quad \eta(00, 01), & \quad \eta(01, 01), \\
\eta(00, 10), & \quad \eta(00, 11), & \quad \eta(01, 11), \\
\eta(10, 10), & \quad \eta(10, 11), & \quad \eta(11, 11).
\end{align*}
\]

Let us define the first three of them using the following relations

\[
\begin{align*}
\eta(0, 0) &= \eta(00, 00) + \eta(00, 01) + \eta(01, 01), \\
Prob_\mu \{X_2 = 0 \mid X_1 = 0\} \cdot \eta(0, 0) &= \eta(00, 00) + \eta(00, 01), \\
Prob_\mu \{X_2 = 1 \mid X_1 = 0\} \cdot \eta(0, 0) &= \eta(00, 01), \\
Prob_\nu \{X_2 = 0 \mid X_1 = 0\} \cdot \eta(0, 0) &= \eta(00, 00), \\
Prob_\nu \{X_2 = 1 \mid X_1 = 0\} \cdot \eta(0, 0) &= \eta(00, 01) + \eta(01, 01).
\end{align*}
\]

By the hypothesis,

\[
Prob_\nu \{X_2 = 1 \mid X_1 = 0\} \geq Prob_\mu \{X_2 = 1\} \quad X_1 = 0,
\]

consequently, the system has a positive, not greater than 1, solution.

In a similar manner we define 6 remaining numbers using the relations

\[
\begin{align*}
\eta(0, 1) &= \eta(00, 10) + \eta(00, 11) + \eta(01, 11), \\
Prob_\mu \{X_2 = 0 \mid X_1 = 1\} \cdot \eta(0, 1) &= \eta(00, 10) + \eta(00, 11), \\
Prob_\mu \{X_2 = 1 \mid X_1 = 1\} \cdot \eta(0, 1) &= \eta(01, 11), \\
Prob_\nu \{X_2 = 0 \mid X_1 = 1\} \cdot \eta(0, 1) &= \eta(00, 10), \\
Prob_\nu \{X_2 = 1 \mid X_1 = 1\} \cdot \eta(0, 1) &= \eta(00, 11) + \eta(01, 11).
\end{align*}
\]

and

\[
\begin{align*}
\eta(1, 1) &= \eta(10, 10) + \eta(10, 11) + \eta(11, 11), \\
Prob_\mu \{X_2 = 0 \mid X_1 = 1\} \cdot \eta(1, 1) &= \eta(10, 10) + \eta(10, 11), \\
Prob_\mu \{X_2 = 1 \mid X_1 = 1\} \cdot \eta(1, 1) &= \eta(11, 11), \\
Prob_\nu \{X_2 = 0 \mid X_1 = 1\} \cdot \eta(1, 1) &= \eta(10, 10), \\
Prob_\nu \{X_2 = 1 \mid X_1 = 1\} \cdot \eta(1, 1) &= \eta(10, 11) + \eta(11, 11).
\end{align*}
\]

It is quite clear, the measure $\eta_2$ satisfies all required conditions. We derive some of them as an example

\[
\begin{align*}
Prob_\mu \{X_1 = 1, X_2 = 0\} &= Prob_\mu \{X_1 = 1\} \cdot Prob_\mu \{X_2 = 0\} \quad X_1 = 1 = \\
&= (\eta(1, 1)) \cdot \left( \frac{\eta(10, 10) + \eta(10, 11)}{\eta(1, 1)} \right) = \eta(10, 10) + \eta(10, 11).
\end{align*}
\]
\[ \Pr_{\nu}\{X_1 = 1, X_2 = 1\} = \Pr_{\nu}\{X_1 = 1\} \cdot \Pr_{\nu}\{X_2 = 1 | X_1 = 1\} = \]
\[ = (\eta(0,1) + \eta(1,1)) \cdot \frac{(\eta(10,11) + \eta(11,11)) + (\eta(00,11) + \eta(01,11))}{\eta(1,1) + \eta(0,1)} = \]
\[ = \eta(10,11) + \eta(11,11) + \eta(00,11) + \eta(01,11). \]

The remaining conditions follow similarly from the equations defining our 9 parameters.

The general inductive step \( n \to n + 1 \) is quite similar.

Now we return to our theorem.

The main diagonal of a Young diagram is formed by boxes of the diagram such that \( c(\square) = 0 \). Consider a secondary diagonal, corresponding to content of a box \( k \), i.e. the set of boxes with \( c(\square) = k \).

Choose \( k \) such that \( \Re(k + w) > 0 \). Denote the length of the secondary diagonal, corresponding to the box content \( k \), by \( s_N \). It is clear that \( \tilde{s}_N \leq s_N + k \).

Therefore, it is sufficient to prove the theorem for the random variables \( s_N \) instead of \( \tilde{s}_N \).

Note that since \( \Re(k + w) > 0 \) we can apply Lemma 8 to estimate the growth of the diagonal.

Introduce a family of random variables \( \xi_N = \xi_N(\tau) \) on \( T \). If a box with content \( k \) is added to the “positive” Young diagram when one moves from level \( N - 1 \) to level \( N \) of the Gelfand-Tsetlin graph along the path \( \tau \) (i.e., if the length of the secondary diagonal increases by one), then \( \xi_N(\tau) = 1 \). Otherwise, \( \xi_N(\tau) = 0 \). It is evident that \( s_N = \sum_{i=1}^{N} \xi_i \).

The joint distribution of the random variables \( \xi_i \) defines in a natural way a probability measure on the set of sequences \( X \). Denote this measure by \( \mu \).

The claim of Theorem 12 can be reformulated as a property of \( \mu \), as follows. We replace the probability space \( T \) by \( X \) and the \( (z,w) \)-measure by \( \mu \). Then the random variables \( \xi_i \) turn into the coordinate functions \( q_i \). The length of the secondary diagonal \( s_N \) turns into the sum \( q_1 + q_2 + \cdots + q_N \).

**Proposition 15.** There exists a constant \( c_1(z,w) \) such that for any collection \( \{a_1, \ldots, a_{N-1}\} \) of 0’s and 1’s the following estimate on the conditional probability holds

\[ \Pr\{\xi_N = 1 | \xi_1 = a_1, \ldots, \xi_{N-1} = a_{N-1}\} \leq \frac{c_1(z,w)}{N}. \]

**Proof.** Immediately follows from Lemma 8.

Consider the following family \( \{\nu_N\} \) of probability measures on \( \{0,1\} \):

\[ \nu_N(\{1\}) = \min(1, c_1(z,w)/N), \]
Here $c_1(z,w)$ is the constant from Proposition 15.

Now denote by $\nu$ the direct product of the measures $\nu_N$.

Applying Proposition 15 and Lemma 14 we conclude that $\mu \leq \nu$.

Next, we prove that an analogue of the strong law of large numbers holds for the measure $\nu$. Thus $\nu$ satisfies the estimate similar to the one of Theorem 12.

**Proposition 16.** Almost surely with respect to the measure $\nu$

$$
\lim_{N \to \infty} \frac{q_1 + q_2 + \cdots + q_N}{\log(N)} = c_1(z,w)
$$

**Proof.** It is clear that $E_{\nu} q_i = c_1(z,w)/i$. Consequently,

$$
\lim_{N \to \infty} E_{\nu} \left( \frac{q_1 + q_2 + \cdots + q_N}{\log(N)} \right) = c_1(z,w)
$$

Denote $p_i = q_i - E_{\nu}(q_i)$. We have to prove that almost surely with respect to the measure $\nu$

$$
\lim_{N \to \infty} \frac{p_1 + p_2 + \cdots + p_N}{\log(N)} = 0
$$

**Lemma 17.** Suppose $b_i$ is an arbitrary numerical sequence such that the series $\sum_{i=1}^{\infty} b_i$ converges. Then

$$
\lim_{N \to \infty} \frac{b_1 \cdot \log(1) + b_2 \cdot \log(2) + \cdots + b_N \cdot \log(N)}{\log(N)} = 0
$$

**Proof.** We use the discrete Abel transform:

$$
\sum_{k=1}^{M} u_k v_k = \left( \sum_{i=1}^{M} u_i \right) v_M + \sum_{k=1}^{M-1} \left( \sum_{i=1}^{k} u_i \right) (v_k - v_{k+1}).
$$

Choose $\varepsilon > 0$ and apply the last formula for $M = N$, $u_i = b_i$ and $v_i = \frac{\log(i)}{\log(N)}$. 


We obtain

\[
\frac{b_1 \cdot \log(1) + b_2 \cdot \log(2) + \cdots + b_N \cdot \log(N)}{\log(N)} = \\
= \left( \sum_{i=1}^{N} b_i \right) + \sum_{k=1}^{N-1} \left( \sum_{i=1}^{k} b_i \right) \frac{\log(i) - \log(i+1)}{\log(N)} = \\
= \left( \sum_{i=1}^{N} b_i \right) + \sum_{k=1}^{N-1} \left( \sum_{i=1}^{k} b_i \right) \frac{\log(i) - \log(i+1)}{\log(N)} = \\
- \sum_{k=L}^{N-1} \left( \sum_{i=k+1}^{N} b_i \right) \frac{\log(i) - \log(i+1)}{\log(N)} + \sum_{k=1}^{L-1} \left( \sum_{i=1}^{k} b_i \right) \frac{\log(i) - \log(i+1)}{\log(N)} = \\
= \left( \sum_{i=1}^{N} b_i \right) \left( 1 + \sum_{k=1}^{N-1} \frac{\log(i) - \log(i+1)}{\log(N)} \right) - \\
- \sum_{k=L}^{N-1} \left( \sum_{i=k+1}^{N} b_i \right) \frac{\log(i) - \log(i+1)}{\log(N)} + \sum_{k=1}^{L-1} \left( \sum_{i=1}^{k} b_i \right) \frac{\log(i) - \log(i+1)}{\log(N)} = \\
= \left( \sum_{i=1}^{N} b_i \right) \frac{\log(L)}{\log(N)} - \sum_{k=L}^{N-1} \left( \sum_{i=k+1}^{N} b_i \right) \frac{\log(i) - \log(i+1)}{\log(N)} + \\
+ \sum_{k=1}^{L-1} \left( \sum_{i=1}^{k} b_i \right) \frac{\log(i) - \log(i+1)}{\log(N)} \tag{1}
\]

By the hypothesis the series \( \sum_{i=1}^{\infty} b_i \) converges, hence, we can choose such \( L \) that for every \( m, n > L \) one has \( \left| \sum_{i=m}^{n} b_i \right| \leq \varepsilon/6 \). This fact implies that the second term in the sum (1) does not exceed \( \varepsilon/3 \) in absolute value. If we now choose \( N \) large enough, then the first and the second terms do not exceed \( \varepsilon/3 \) too. Consequently, the sum can be bounded by \( \varepsilon \).

Introduce \( t_i = \frac{c}{\log(i)} \). The last lemma implies that it is sufficient to prove almost sure convergence of the series \( \sum_{i=1}^{\infty} t_i \). Let us estimate the variance of \( t_i \). We denote the variance by \( D_\nu \). Evidently, \( D_\nu q_i \leq c/i \). Therefore, \( D_\nu t_i \leq \frac{c}{\log(i)} \). Hence, the series \( \sum_{i=1}^{\infty} D_\nu t_i \) converges. Now we use the following lemma:

**Lemma 18.** Suppose \( \phi_i \) is a sequence of independent centered random variables such that \( \sum_{i=1}^{\infty} D_\nu \phi_i \) converges. Then the series \( \sum_{i=1}^{\infty} \phi_i \) converges almost surely.

See [S] Chapter 4, Section 2, Theorem 3]

This concludes the proof of Proposition 16.
It follows from Proposition 16 that for almost every with respect to the measure \( \nu \) sequence in \( X \), there exists such \( N \) that for every \( n > N \):

\[
\sum_{i=1}^{n} q_i \frac{1}{\log(n)} < c, \quad (***)
\]

where \( c \) is a constant depending solely on \( z, w \), and \( q_i \) is the \( i \)th coordinate function. Now we show that this statement also holds for the measure \( \mu \).

Let \( A_N \subset X \) be the set where the inequality (***) holds for all \( n > N \). It is clear that for every \( N \), \( A_N \subset A_{N+1} \). By Proposition 16

\[
\lim_{N \to \infty} \text{Prob}_\nu A_N = 1.
\]

**Lemma 19.**

\[
\text{Prob}_\mu A_N \geq \text{Prob}_\nu A_N
\]

**Proof.** Denote by \( B^i_N \subset X \) the set where inequality (***) holds for \( n = N, N+1, \ldots, N+i-1 \). The set \( A_N \) coincides with the intersection of the decreasing sequence of the sets \( B^i_N \):

\[
A_N = \bigcap_{i=1}^{\infty} B^i_N.
\]

Note that if the inequality (***) holds for some sequence \( A \) and \( B \leq A \) (with respect to the partial order on \( X \) defined above), then the inequality also holds for \( B \). Consequently, the indicator of the set \( B^i_N \) is a decreasing function. Thus, \( \nu \geq \mu \) implies that \( \text{Prob}_\mu B^i_N \geq \text{Prob}_\nu B^i_N \). Therefore

\[
\text{Prob}_\mu A_N = \lim_{i \to \infty} \text{Prob}_\mu B^i_N \geq \lim_{i \to \infty} \text{Prob}_\nu B^i_N = \text{Prob}_\nu A_N.
\]

It follows from the lemma that

\[
\lim_{N \to \infty} \text{Prob}_\mu A_N = 1.
\]

Hence, for almost every with respect to the measure \( \mu \), sequence in \( X \) there exists a number \( N \) such that for every \( n > N \) (***) holds. The last statement is equivalent to Theorem 12.

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