Unital with many involutory translations
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Abstract
If every point of a unital is fixed by a non-trivial translation and at least one translation has order two then the unital is classical (i.e., hermitian).

1 Introduction
A unital \( U = (U, B) \) of order \( q \) is a 2-(\( q^3 + 1, q + 1, 1 \))-design; i.e., an incidence structure with \( |U| = q^3 + 1 \) such that every block \( B \in \mathcal{B} \) has exactly \( q + 1 \) points, and any two points in \( U \) are joined by a unique block in \( \mathcal{B} \). It follows that every point is incident with exactly \( q^2 \) blocks. Without loss of generality, we will assume throughout that the blocks are subsets of \( U \). The classical examples of unitals are the hermitian ones: For any prime power \( q \), the points of the hermitian unital \( \mathcal{H}(F_{q^2}, F_q) \) are the absolute points with respect to a suitable polarity of the projective plane \( \text{PG}(2, F_{q^2}) \) over the field of order \( q^2 \), see \([22, II.8, pp. 57–63]\) or \([3, Ch. 2]\). The blocks are induced by secant lines in \( \text{PG}(2, F_{q^2}) \).
A translation (with center \( c \)) of a unital \( U \) is an automorphism of \( U \) that fixes the point \( c \) and every block through \( c \). We write \( \Gamma_{[c]} \) for the group of all translations with center \( c \). Much is known for the case where there are at least two points \( u, v \) such that \( \Gamma_{[u]} \) and \( \Gamma_{[v]} \) both have order \( q \), see \([15]\). If there even exist three non-collinear points with that property then the unital is the classical hermitian one, see \([13]\). We show (Theorem 4.3) that the same conclusion holds if \( \Gamma_{[c]} \) has even order for every point \( c \). In fact, it suffices that every point is the center of some non-trivial translation, and that at least one of these translations is an involution.
Polar unitals in Figueroa planes of even order provide examples where the centers of involutory translations form a proper subset not contained in a block (namely, a subunital that is in fact hermitian), see \( \text{Theorem 5.3} \). Examples where the set of centers is contained in a block exist in abundance, see Section 6.

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2 Transitivity

The following elementary observation can be traced back to Gleason; cp. [6] pp. 190 f], see [13, 4.1] for a (very short) proof.

**Lemma 2.1.** Let $p$ be a prime, and let $H$ be a group acting on a finite set $X$. Assume that for every $x \in X$ there exists in $H$ an element of order $p$ which fixes $x$ but no other element of $X$. Then $H$ is transitive on $X$.

Now let $U = (U, \mathcal{B})$ be a unital. A translation of order $n$ is called an $n$-translation, and $\Gamma^{[n]}$ denotes the set of all $n$-translations of $U$ with center $c$. For each $n$, let $\Omega_n$ be the (possibly empty) set of all centers of $n$-translations, and $\Gamma^{[n]} := \left\{ \bigcup_{c \in \Omega_n} \Gamma^{[1]}_c \right\}$.

Recall from [13, Theorem 1.3] that a non-trivial translation of a unital fixes no point apart from its center. If $\Omega$ is not empty, the size of any $\text{Aut}(U)$-orbit in $\Omega_n$ is therefore congruent 1 modulo $n$ (because a cyclic group of size $n$ acts semi-regularly on the complement of a point in $\Omega_n$), and every $\text{Aut}(U)$-orbit outside $\Omega_n$ has size divisible by $n$ (because the same cyclic group acts semi-regularly on that set).

**Lemma 2.2.** Let $p$ be a prime. If $H \leq \text{Aut}(U)$ contains a $p$-translation then $H$ is transitive on the set of all centers of $p$-translations in $H$. In particular, we have:

1. The group $\Gamma^{[p]}$ generated by all $p$-translations is transitive on $\Omega_p$.
2. For each block $B$ of $U$ the group generated by all $p$-translations with center on $B$ is transitive on the set $\Omega_p \cap B$ of centers of $p$-translations in $B$. Consequently, the stabilizer of $B$ in $\Gamma^{[p]} \leq \text{Aut}(U)$ is transitive on $\Omega_p \cap B$.
3. If $n > 1$ is an integer such that $\Omega_n$ is not empty, then the group $\Gamma^{[n]}$ is transitive on $\Omega_n$ and $\Omega_{k^n} = \Omega_n$ holds for each divisor $k > 1$ of $n$.

**Proof.** The group $H$ acts on the set of all centers of $p$-translations in $H$, and Lemma 2.1 applies. For assertions 1 and 2 we specialize $H = \Gamma^{[p]}$ and $H = \left\langle \bigcup_{c \in B} \Gamma^{[1]}_c \right\rangle$, respectively. Assertion 3 follows from the fact that that $\Omega_n$ is contained in the $\text{Aut}(U)$-orbit $\Omega_k$.

**Definitions 2.3.** Assume that $U$ admits a translation of order $k > 1$. Let $p_k$ be the smallest prime divisor of $k$. Then $\Omega_k = \Omega_{p_k}$ holds by Lemma 2.2.1. Put $K := \{ p_k \mid k \in \mathbb{N}, |\Gamma^{[k]}| > 1 \}$. Clearly, we obtain a disjoint union $U = \mathcal{O} \cup \bigcup_{p \in K} \Omega_p$, where $\mathcal{O} := \{ x \in U \mid \Gamma^{[e]}_x = \{ \text{id} \} \}$.

We write $\mathcal{B}_k$ for the set of all blocks of $U$ that contain at least two points of $\Omega_k$, and consider the incidence structure $\mathcal{U}_k := (\Omega_k, \mathcal{B}_k, \in)$. If $\Omega_k$ is not contained in a block then $\mathcal{U}_k$ is a non-trivial linear space.

From Lemma 2.2.1 and Lemma 2.2.2 we know that $\Gamma^{[k]}$ induces a transitive group of automorphisms of $\mathcal{U}_k$, and that for each block $B \in \mathcal{B}_k$ the stabilizer of $B$ in $\Gamma^{[k]}$ acts transitively on $\Omega_k \cap B$.

Recall that a substructure of an incidence structure (with points and blocks) is called **ideally embedded** if the pencils of blocks in the substructure are the same as the pencils in the larger structure.

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Lemma 2.4. Assume that \( U \) is a unital such that every point is the center of some non-trivial translation. For each \( p \in K \), the linear space \( U_p \) is then ideally embedded in \( U \). In particular, the set \( \Omega_p \) is not contained in a block. The action of \( \text{Aut}(U) \) on \( \Omega_p \) is faithful.

If there exists \( p \in K \) such that \( |\Omega_p \cap B| \) is constant for \( B \in \mathcal{B}_p \), then \( \Omega_p = U \), and \( \Gamma^{[p]} \) is transitive on \( U \). In particular, this happens if \( \text{Aut}(U) \) is two-transitive on \( \Omega_p \).

Proof. Consider \( p \in K \), and a block \( B \in \mathcal{B} \) through a point \( x \in \Omega_p \). It suffices to consider the case where there exists a point \( y \in B \setminus \Omega_p \). By our assumption, there exists a non-trivial translation \( \tau \) with center \( y \). Now \( x \tau \) is a point different from \( x \), and lies in \( \Omega_p \cap B \). So \( \Omega_p \) is ideally embedded in \( U \). Clearly, this implies that \( \Omega_p \) is not contained in any block.

Assume that \( \alpha \in \text{Aut}(U) \) fixes every point in \( \Omega_p \). Then \( \alpha \) fixes each block that meets \( \Omega_p \) because \( \Omega_p \) is ideally embedded. Each point \( x \) outside \( \Omega_p \) lies on more than one block meeting \( \Omega_p \), so \( \alpha \) fixes every point of \( U \), and is trivial.

Now assume that \( b := |\Omega_p \cap B| \) is constant for \( B \in \mathcal{B}_p \); then \( |\Omega_p| = 1 + q^b \). If there exists \( z \in U \setminus \Omega_p \), then, by our assumption, there is a translation of prime order \( r \) and center \( z \). Now \( r \) divides both \( q \) and \( |\Omega_p| \), and we obtain a contradiction.

The Figueroa unitals (see Theorem 5.2) of even order are examples that show that the condition \( \mathcal{U} = \emptyset \) is necessary in Lemma 2.4; in those unitals \( \mathcal{U}_2 \) is not ideally embedded, and there exists an automorphism of order 3 that acts trivially on \( \Omega_2 \). See also Proposition 4.2.

3 Results from group theory

Lemma 3.1. Let \( G \) be a transitive permutation group on some finite set \( \Omega \) with more than one element. Suppose that the stabilizer \( G_x \) of some \( x \in \Omega \) contains an involution \( \tau \) that is semi-regular on \( \Omega \setminus \{x\} \), and that \( G \) has a transitive normal subgroup \( M \) of odd order. Then the following are equivalent:

1. \( M \) is abelian.
2. \( M \) acts regularly (i.e., sharply transitively) on \( \Omega \).
3. The involution \( \tau \) acts semi-regularly on \( M \) by conjugation.

If one (and then any) of these conditions is satisfied then \( \langle \tau \rangle M \) is a generalized dihedral group; i.e., conjugation by \( \tau \) inverts each element of \( M \), and \( \langle \tau \rangle M = \tau^M \cup M \).

Proof. If \( M \) is abelian then the stabilizer \( M_x \) also fixes each element of the orbit \( \Omega \) of \( x \) under \( M \). Hence \( M_x \) is trivial, and the action of \( M \) is regular. If \( M \) acts regularly, we identify \( m \in M \) with the image \( x^m \). Semi-regularity of \( \tau \) on \( \Omega \) then translates into semi-regularity of the automorphism induced by conjugation with \( \tau \) on \( M \). The map \( m \mapsto m^{-1}m^\tau \) is then injective, and bijective since \( M \) is finite. Write \( m \in M \) as \( m = k^{-1}k^\tau \) with \( k \in M \), and then calculate \( m^{-1} = (k^\tau)^{-1}k = (k^{-1})^\tau k = (k^{-1}k)^\tau = m^\tau \). Thus the automorphism induced by \( \tau \) is the anti-automorphism \( m \mapsto m^{-1} \), and \( M \) is abelian.
If \( \tau \) induces inversion on \( M \) then \( m^{-1} \tau m = \tau (m^{-1} \tau) m = \tau m^2 \) holds for each \( m \in M \). As \( M \) has odd order, this means that the coset \( \tau M \) equals the conjugacy class \( \tau^M \), and consists of involutions.

The following result has been proved in [18, Theorem 2]; cp. [2, Theorem 2].

**Theorem 3.2.** Assume that the group \( G \) acts transitively (but not necessarily faithfully) on some finite set \( \Omega \) with more than one element. Suppose that the stabilizer \( G_x \) of some \( x \in \Omega \) has a normal subgroup \( Q \) of even order that is semi-regular on \( \Omega \setminus \{x\} \). Then the normal closure \( S := \langle Q^G \rangle \) either has a transitive normal subgroup of odd order, or acts two-transitively on \( \Omega \) as one of the groups \( \text{SL}_2(F_{2e}) \), \( \text{Sz}(2^{2e-1}) \), or \( \text{PSU}_3(F_{2e}|F_{2e}) \) for suitable \( e \geq 2 \); the action is the usual two-transitive one.

The group \( S \) itself is then \( S = Q N \) where \( N \) denotes the largest normal subgroup of odd order in \( S \), or \( S \) is isomorphic to \( \text{SL}_2(F_{2e}) \), \( \text{Sz}(2^{2e-1}) \), \( \text{SU}_3(F_{2e}|F_{2e}) \) or \( \text{PSU}_3(F_{2e}|F_{2e}) \), according to the group induced on \( \Omega \).

We add some information contained in [18] —in particular, see [18, Lemma 3]— but not in the statement of the theorem referred to above.

**Proposition 3.3.** The kernel of the action of \( G \) as considered in Theorem 3.2 is \( K = C_G(S) \).

If \( S \) has a transitive normal subgroup \( N \) of odd order then \( Q \) acts fixed-point-freely on \( N/K \), so the group \( N/K \) induced by \( N \) on \( \Omega \) is a sharply transitive abelian group in that case. The group \( Q \) is then (isomorphic to) a Frobenius complement.

If \( S \) induces a non-solvable group on \( \Omega \) then either \( K \cap S \) is trivial, or \( |K \cap S| = 3 \); the latter case can only occur if \( S \cong \text{SU}_3(F_{2e}|F_{2e}) \) with odd \( e > 1 \).

The Sylow subgroups of a Frobenius complement are either cyclic or generalized quaternion groups; see [10, 10.3.1]. So Proposition 3.3 yields:

**Corollary 3.4.** If \( S = \langle Q^G \rangle \) has a transitive normal subgroup \( N \) of odd order then the group \( Q \) contains exactly one involution. That involution induces inversion on \( N \).

### 4 Unitals with involutory translations

By Lemma 2.2 Hering’s result (see Theorem 4.2) applies to the permutation group induced by \( \Gamma[2] \) on \( \Omega_2 \) if this set has more than one element.

The following proposition is a small part of the results in [25]. For the reader’s convenience, we include a proof of the facts that we need here.

**Proposition 4.1.** Let \( (X, L) \) be a linear space with \( v \) points such that each line has \( k \) points. Assume that \( k > 2 \). Consider a group \( G \) isomorphic to one of the groups \( \text{PSL}_2(F_q) \), \( \text{Sz}(q) \) or \( \text{PSU}_3(F_q|F_q) \), for some prime power \( q \). If \( G \) acts on \( X \) in its usual two-transitive action and by automorphisms of \( (X, L) \), then either \( v \equiv k \) (and there is just one line in \( L \)), or \( G \cong \text{PSU}_3(F_q^2|F_q) \) and \( (X, L) \cong \mathcal{H}(F_q^2|F_q) \) is isomorphic to the hermitian unital of order \( q \).
Proof. We assume that there is more than one line, and discuss the three different cases separately.

(PSL) The usual two-transitive action of $G \cong \text{PSL}_2(\mathbb{F}_q)$ is the natural action on the projective line $\mathbb{P} \cup \{\infty\}$ via fractional linear transformations. Then $v = q + 1$, and the stabilizer of two points $x, y \in X$ has two orbits of length 1 and at most two other orbits, each of length $\frac{q^2 - 1}{\gcd(2, q - 1)}$. The line joining $x$ and $y$ contains at least one of those orbits, and $k - 1 \geq \frac{q^2 - 1}{\gcd(2, q - 1)} + 1 > \frac{q}{2}$. So there is no space left for a second line through $x$, and $v = k$ follows.

The Suzuki groups and the unitary groups need a closer look; we will use the following facts about linear spaces with $v$ points and $k > 2$ points per line:

The number of lines per point in $(X, \mathcal{L})$ is $r = \frac{v - 1}{k - 1}$. We assume that the linear space has more than one line, so $r > 1$. Fisher’s inequality (see [6] 1.3.8, p. 20) then says $r \geq k$, and equality holds precisely if the linear space is a projective plane. In the latter case, we have $v = k^2 - k + 1$. If $r > k$ then $v - 1 \geq (k + 1)(k - 1) = k^2 - 1$.

(Sz) The usual two-transitive action of $G \cong \text{Sz}(q)$ is its action on the Suzuki-Tits ovoid; see [29] Section 21. Then $v = q^2 + 1$ and $q$ is a power of 2. Hence we cannot have $v - 1 = k^2 - k$, and $v - 1 \geq k^2 - 1$ follows.

For any two points $x, y \in X$, the stabilizer $G_{x,y}$ acts on $X$ with two orbits of length 1, every other orbit has length $q - 1$; see [29] Lemma 21.5. The block joining $x$ and $y$ is thus the union of $\{x, y\}$ with a collection of such orbits of length $q - 1$, and $k \geq (q - 1) + 2 = q + 1$. Now the inequality $q^2 \geq v - 1 \geq k^2 - 1 \geq (q + 1)^2 - 1$ yields a contradiction.

(PSU) The usual two-transitive action of $G \cong \text{PSU}_3(\mathbb{F}_q)$ is its action on the points of the hermitian unital of order $q$; see [21], [33], [22] Theorem 5.2. Then $v = q^3 + 1$, the fact that $q$ is a prime power yields $v - 1 \neq k^2 - k$, and we have $q^3 = v - 1 \geq k^2 - 1$.

If $q = 2$ then $v = 9$ and $k > 2$ together with $k^2 - 1 \leq v - 1$ yields $k = 3$. Then $(X, \mathcal{L})$ is determined uniquely, we have $(X, \mathcal{L}) \cong \mathcal{H}(F_4/F_3) \cong \mathcal{A}G(2, F_3)$. We assume $q > 2$ from now on.

The stabilizer $G_{x,y}$ of two points $x, y \in X$ has two orbits of length 1, one orbit of length $q - 1$, and every other orbit has length $\frac{q^2 + 1}{\gcd(3, q + 1)}$; see [33] p.499. So there are integers $s \in \{0, 1\}$ and $t \geq 0$ such that $k - 1 = s(q - 1) + t\frac{q^2 + 1}{\gcd(3, q + 1)} + 1$.

Aiming at a contradiction, we assume $t > 0$. Then $k \geq \frac{q^2 - 1}{\gcd(3, q + 1)} + 2 \geq \frac{q^2}{3}$. From $q^3 = v - 1 \geq k^2 - 1 \geq (\frac{q^2}{3})^2 - 1$ we thus obtain $0 \geq q^3 - 9q^2 + 10q^2 + 16 > q^2(q(q - 9) + 10)$. So $q < 8$, and $q \in \{3, 4, 5, 7\}$.

For $q \in \{3, 4, 7\}$ we have $\gcd(3, q + 1) = 1$, and our assumption $t > 0$ yields $q^3 \geq ((q^2 - 1 + 1)^2 = q^4$. This is impossible. For $q = 5$ and $t > 1$, we obtain the contradiction $5^3 \geq (2^{(7-1)} + 1)^2 = 17^2$. So $t = 1$, and $k - 1 = 4s + 8 + 1$ divides $v - 1 = 5^3$. Both cases for $s \in \{0, 1\}$ lead to a contradiction.

Therefore, we have $t = 0$ and $s = 1$; and the block through $x$ and $y$ is the union of $\{x, y\}$ with the unique orbit of length $q - 1$ under $G_{x,y}$. This means that $(X, \mathcal{L})$ is isomorphic to the hermitian unital $\mathcal{H}(F_4/F_3)$.

Therefore, $(X, \mathcal{L})$ is isomorphic to the hermitian unital $\mathcal{H}(F_4/F_3)$.

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Proposition 4.2. Let \( U \) be a unital of order \( q \), and assume that \( \Omega_2 \) is not contained in a block. If the group induced by \( \Gamma^{[2]} \) on \( \Omega_2 \) does not have a normal subgroup that acts regularly on \( \Omega_2 \) then that induced group is isomorphic to \( \text{PSU}_3(\mathbb{F}_{2^e}, [\mathbb{F}_{2^e}]^e) \), and \( U_2 = (\Omega_2, \mathcal{B}_2) \) is isomorphic to the hermitian unital of order \( 2^e \), with \( e > 1 \).

Proof. From Theorem 3.2 we know that the group induced by \( \Gamma^{[2]} \) on \( \Omega_2 \) is isomorphic to one of the groups \( \text{SL}_2(\mathbb{F}_{2^e}), \text{Sz}(2^{2e-1}) \), or \( \text{PSU}_3(\mathbb{F}_{2^e}, [\mathbb{F}_{2^e}]^e) \) (for suitable \( e \geq 2 \)), with the usual two-transitive action. From Proposition 1.1 we then know that \( \Gamma^{[2]} \cong \text{PSU}_3(\mathbb{F}_{2^e}, [\mathbb{F}_{2^e}]^e) \), and \( U_2 \) is the hermitian unital of order \( 2^e \).

The situation in Proposition 4.2 actually occurs, for example, in polar unitals of Figueroa planes of even order, see Theorem 5.2 and Theorem 5.3 below. Our Proposition 4.2 is a version for unitals (with translations) of a result [19, Theorem 5.2] on projective planes (and elations).

Theorem 4.3. Let \( U = (U, B) \) be a unital of order \( q \), assume that every point of \( U \) is the center of some non-trivial translation, and that there exists a translation of order 2. Then \( q \) is a power of 2, and the unital \( U \) is the hermitian unital of order \( q \).

Proof. If \( q = 2 \) then the unital is the hermitian unital of order 2, see [35, 10.16]. We assume \( q > 2 \) in the rest of the proof.

Case A: Assume first that \( \Omega_2 = U \), i.e., every point of \( U \) is the center of some involutory translation. If the group \( \Gamma^{[2]} \) generated by the involutory translations does not have a regular normal subgroup then Proposition 1.2 says that \( \Gamma^{[2]} \cong \text{PSU}_3(\mathbb{F}_{2^e}, [\mathbb{F}_{2^e}]^e) \), and \( U \) is the hermitian unital of order \( 2^e \).

(Alternatively, this can be derived more directly from Theorem 3.2: the groups \( \text{SL}_2(\mathbb{F}_{2^e}) \) are excluded since they are triply transitive; the orbits of the two-point-stabilizers in \( \text{Sz}(2^{2e-1}) \) are too large to yield blocks of the unital. For the groups \( \text{PSU}_3(\mathbb{F}_{2^e}, [\mathbb{F}_{2^e}]^e) \) the block through two points consists of those two together with the unique shortest orbit of their stabilizer. Hence \( U \) coincides with the hermitian unital \( \mathcal{H}(\mathbb{F}_{2^e}, [\mathbb{F}_{2^e}]^e) \) of order \( 2^e \).

If the group \( \Gamma^{[2]} \) has a regular normal subgroup \( N \), then \( N \) is abelian (see Proposition 3.3). Consider two blocks \( B, C \) through some point. Then the stabilizers \( N_B \) and \( N_C \) are subgroups of order \( q + 1 \) in \( N \), and \( N_B N_C \) is a subgroup of order \((q + 1)^2\). So \((q + 1)^2 \mid |N| = q^3 + 1\), and \( q + 1 \) divides \((q^3 + 1) / (q + 1) = q^2 - q + 1 = (q + 1)(q - 1) - (q - 2)\). This implies \( q - 2 = 0 \), contradicting our assumption \( q > 2 \) (in fact, \( \Gamma^{[2]} \) on the unital of order 2 does contain a regular normal subgroup, see Remark 4.4).

Case B: Now we assume \( \Omega_2 \neq U \), and aim at a contradiction. As every point is the center of a non-trivial translation, there is a prime \( p \) such that \( \Omega_p \) is not empty, and disjoint to \( \Omega_2 \). We know from Lemma 2.3 that \( \Omega_2 \) is not contained in a block, and that the group \( \Gamma^{[2]} \) generated by all involutory translations acts faithfully on \( \Omega_2 \).

The action of \( \Gamma^{[2]} \) on \( \Omega_2 \) is as in Theorem 1.2 and Proposition 3.3. If \( \Gamma^{[2]} \) is two-transitive on \( \Omega_2 \) then Lemma 2.4 yields \( \Omega_2 = U \), contradicting our present assumptions. If \( \Gamma^{[2]} \) is not two-transitive on \( \Omega_2 \) then \( \Gamma^{[2]} \) has an abelian normal subgroup \( A \) acting sharply transitively on \( \Omega_2 \), see
Theorem 3.2 From Corollary 3.4 we know that each one of the involutory translations acts by inversion on $A$, and that $\Gamma_{3^2}^r$ contains exactly one involution if $x \in \Omega_2$. We denote that involution by $j_x$. The order of $A$ equals that of $\Omega_2$, so it is odd, and divisible by $p$.

Consider $a \in A$ and an arbitrary point $x \in \Omega_2$. Then both $j_x$ and $a^{-1}j_xa$ fix the block joining $x$ and $x^a$. So $a^2 = j_xa^{-1}j_xa$ fixes that block, and so does $a$ because squaring is an automorphism of the abelian group $\langle a \rangle$ of odd order.

As $|A| = |\Omega_2|$ is divisible by $p$, we find an element $a \in A$ of order $p$. That $a$ fixes a point $c \in \Omega_2$ because $|\Omega_p| \equiv 1 \pmod{p}$. For any $x \in \Omega_2$, we have $a^2 = j_xa^{-1}j_xa$, and $c = c^a$ yields that $x^a$ lies on the block joining $x$ and $c$ because the translations $j_x$ and $a^{-1}j_xa$ fix each block through their respective center. So $a$ fixes every block joining a point of $\Omega_2$ with $c$. As $a$ fixes no point in $\Omega_2$, the point $c$ is the only point fixed by $a$.

Let $j$ be any involutory translation. As $c$ is fixed by $a^2 = ja^{-1}ja$, we obtain $c^j = c^{a^{-1}ja} = c^a$, and the point $c^j$ is fixed by $a$. Since $c$ is the only fixed point of $a$, we reach the contradiction that the involutory translation $j$ fixes $c \not\in \Omega_2$.

Remark 4.4. Let $p$ be a prime, and let $e$ be a positive integer. Then each non-trivial translation of the hermitian unital $H(F_{p^e}, F_{p^e})$ of order $p^e$ has order $p$. If $p^e > 2$ then the group $\Gamma_{p^e}^{[2]}$ generated by all translations is simple, coincides with $PSU_3(F_{p^e})$, and acts two-transitively on the point set of $H(F_{p^e}, F_{p^e})$; see [35], 10.15, 10.12.

On the hermitian unital of order 2, the group $\Gamma_{p^2}^{[2]}$ behaves in an exceptional way: then the group $\Gamma_{p^2}^{[2]}$ is solvable, and not two-transitive, but it is still the commutator subgroup of $PSU_3(F_2)$; see [35], 10.15 and the discussion on pp. 123f.

The smallest unital (of order 2) is isomorphic to the affine plane of order 3. What we call a translation of the unital is an affine homology in that plane; this explains the structure of $\Gamma_{p^2}^{[2]} \cong C_2 \times C_3$, a generalized dihedral group. That group is transitive on the point set, but not transitive on the block set (it preserves each parallel class in the affine plane).

5 Figueroa unitals

Examples 5.1. Let $r$ be a prime power. [9] has constructed a projective plane $\text{Fig}(r^3)$ of order $r^3$ with a pappian subplane $\mathbb{D}$ of order $r$; see [20] for the case of general $r$, and [11] for a synthetic construction. The plane $\text{Fig}(r^3)$ is not desarguesian unless $r = 2$. The full group of automorphisms of the subplane $\mathbb{D}$ extends to a group $G \cong \text{PGL}_3(F_r)$ of automorphisms of $\text{Fig}(r^3)$. If $r > 2$ then the full automorphism group $\text{Aut}(\text{Fig}(r^3))$ is the direct product of a group of order 3 with said group $G$; see [20], cp. [17]. The cyclic factor is generated by a planar automorphism $\alpha$ (of order 3) that is used in the construction of $\text{Fig}(r^3)$. Dempwolff [8] Theorem B] has noted that every elation of the subplane $\mathbb{D}$ (as an element of the group $G$) is induced by an elation of $\text{Fig}(r^3)$.
If } r \text{ is a perfect square, say } r = q^2, \text{ then there is a polarity } \pi \text{ of the Figueroa plane of order } q^6, \text{ and the absolute points of } \pi \text{ carry a unital } U_{\text{Fig}}(q^6) \text{ of order } q^3, \text{ see } [3]. \text{ The unital } U_{\text{Fig}}(q^6) \text{ is not hermitian; see } [24]. \text{ In fact, there are ONan configurations, see } [24]. \text{ The intersection } H \text{ of } U_{\text{Fig}}(q^6) \text{ with the subplane } D \text{ is isomorphic to the hermitian unital of order } q.

The centralizer of the polarity } \pi \text{ in } \text{Aut}(\text{Fig}(q^6)) \text{ is the direct product of } \langle \alpha \rangle \text{ with the centralizer of } \pi \text{ in } G \cong \text{PΓL}_3(F_q^2). \text{ In particular, every translation of } H \text{ is induced by an elation of the subplane } D, \text{ and thus by an elation } \psi \text{ of } \text{Fig}(q^6) \text{ in the centralizer of } \pi. \text{ The restriction of } \psi \text{ to } U_{\text{Fig}}(q^6) \text{ is then a translation of } U_{\text{Fig}}(q^6).

We obtain:

**Theorem 5.2.** In the unital } U_{\text{Fig}}(q^6) \text{ of order } q^3, \text{ there is a hermitian subunital } H \text{ of order } q \text{ such that every point of } H \text{ is the center of a group of order } q \text{ consisting of translations. In particular, the point set of } H \text{ is contained in } \Omega_9.

Let } G \text{ be the group generated by all elations of the Figueroa plane } \text{Fig}(q^6) \text{ that leave } U_{\text{Fig}}(q^6) \text{ invariant. Then } G \text{ is isomorphic to the commutator group of } \text{PSU}_3(F_q^2|F_q), \text{ and acts two-transitively on the point set of } H.

**Theorem 5.3.** Let } q \text{ be a power of } 2. \text{ Then every involutory translation of } U_{\text{Fig}}(q^6) \text{ has its center in } H, \text{ and is induced by an elation of the Figueroa plane } \text{Fig}(q^6) \text{ that leaves } U_{\text{Fig}}(q^6) \text{ invariant. In particular, we have } H = U_2 = (\Omega_2, E_2), \text{ the subunital } H \text{ is invariant under } \text{Aut}(U_{\text{Fig}}(q^6)), \text{ and every non-trivial translation is an involution.}

**Proof.** We write } q = 2^a \text{ with } a \in \mathbb{N}; \text{ then } U_{\text{Fig}}(q^6) \text{ has order } 2^{3a} \text{ and } H \text{ is a unital of order } 2^a. \text{ The order of any translation of } U_{\text{Fig}}(q^6) \text{ divides } q, \text{ and is thus a power of } 2. \text{ If the translation is not trivial then Lemma } 2.2 \text{ yields that its center lies in } \Omega_2.

From Theorem 5.2, we know that the point set of } H \text{ is contained in } \Omega_2. \text{ Assume that } \Omega_3 \text{ is not contained in the subunital } H. \text{ Then Proposition 4.4 yields that } \Gamma^{[2]} \text{ induces a group isomorphic to } \text{PSU}_3(F_{q^{2a}}|F_{q^a}) \text{ on } \Omega_2, \text{ and } U_2 \text{ is isomorphic to the hermitian unital of order } 2^e. \text{ Thus } 2^a < 2^e \leq 2^{3a}, \text{ and } U_2 \cong H(F_{q^{2a}}|F_{q^a}) \text{ contains the subunital } H \cong H(F_{q^{2a}}|F_{q^a}). \text{ According to } [16], \text{ the embedding of the unital } H \text{ is given by an embedding of quadratic field extensions. This leaves only the possibility } e = 3a, \text{ but then } U_{\text{Fig}}(q^6) = U_2 \text{ is hermitian, a contradiction.} \text{ So } U_2 = H, \text{ every translation of } U_{\text{Fig}}(q^6) \text{ has its center in } H, \text{ and } H \text{ is invariant under all translations of } U_{\text{Fig}}(q^6).

If } \tau \text{ is any non-trivial translation of } U_{\text{Fig}}(q^6) \text{ then the order of } \tau \text{ divides } q^3 = 2^{3a}. \text{ So the center } c \text{ of } \tau \text{ lies in } \Omega_2, \text{ and is thus a point of } H. \text{ The elations of } \text{Fig}(q^6) \text{ with center } c \text{ that leave } U_{\text{Fig}}(q^6) \text{ invariant form a group } \Theta \text{ that acts faithfully on } U_{\text{Fig}}(q^6) \text{ and induces the full group of translations of } H \text{ with center } c.

Let } B \text{ be a block through } c, \text{ and consider any point } x \in (\Omega_2 \cap B) \setminus \{c\}. \text{ Since the group of all translations of the hermitian unital } H \text{ is transitive on } (\Omega_2 \cap B) \setminus \{c\}, \text{ there exists an involutory elation } \vartheta \in \Theta \text{ with } x^\vartheta = x^{3^a}. \text{ Then } \tau \vartheta \text{ is a translation of } U_{\text{Fig}}(q^6) \text{ with center } c \text{ fixing } x \neq c, \text{ so } \tau \vartheta \text{ is trivial on } U_{\text{Fig}}(q^6), \text{ and } \tau = \vartheta. \blacksquare
6 Examples: unitals with few translations

There are unitals with no translations at all, e.g., the Ree unitals, or the presently known unitals of order 6. See [13, 1.8] and [12] for the Ree unitals; the unitals of order 6 are treated in [28, 5.1, p. 2888]; the information about the automorphisms given there suffices to see that there are no translations. Most of the unitals of order 3 and many unitals of order 4 found by computer do not admit automorphisms of order 3 or 2, respectively, let alone translations; see [11, 24, Table III, p. 301].

There are unitals of prime power order \( p^e \) where \( \Omega_p \) consists of a single point (and, obviously, \( \Omega_r \) is empty for every prime \( r \neq p \)). For instance, consider a Coulter-Matthews plane of order \( 3^e \) with even \( e \), defined by a suitable planar monomial; see [4]. Such a plane admits a unitary polarity; the absolute points carry a unital of order \( 3^{e/2} \), see [26, 5.2]. That unital has a point \( \infty \) with a translation group \( \Gamma[\infty] \) of order \( 3^e \), see [26, 6.2]. These groups are elementary abelian 3-groups.

If the unital is not classical (this surely happens if a certain Baer subplane is not desarguesian, see [26, 6.8]) then the point \( \infty \) is fixed by every automorphism of the unital, and \( \infty \) is the only center of any translation (for abstract automorphisms, this follows from a deep result [13], it does not suffice to note that the point \( \infty \) is fixed by every automorphism of the plane).

In planes over finite Dickson semifields, and in planes over twisted fields, one also finds non-classical unitals (polar and otherwise) with exactly one center of translations, see [23] and [14, 5.2], respectively.

For each order \( q = p^e \) with \( p \) prime, Möller gives a construction of unitals [32, 4.1] depending on the choice of a suitable family \( \mathcal{D} \) of subsets of \( SL_2(F_q) \) such that either \( \Omega_p \) is a block and \( \Gamma[p] \cong SL_2(F_q) \), or the unital is hermitian; see [31, 3.11]. Suitable families \( \mathcal{D} \) leading to non-hermitian unitals are known for \( q \in \{4, 8\} \), see [14, 2.1, 3.5] and [32, Sect. 3]. In [30, Corollary 3.8] it is proved: For \( q \) even, the set \( \Omega_2 \) in Grüning’s unital ([17], see [14, 5.5] for the description needed here) has size \( q + 1 \), and that unital admits exactly \( q + 1 \) non-trivial translations, each of order 2. In [30, 4.6, 4.7, 4.8] one finds unitals of order 4 with no translations, unitals of order 4 with \( |\Omega_2| = 1 \) and \( \Gamma[2] \cong C_2 \), and unitals of order 4 with \( |\Omega_2| = 1 \) and \( \Gamma[2] \cong C_2 \times C_2 \).

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