Volume conjecture, geometric decomposition and deformation of hyperbolic structures

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Abstract

In this paper, we study the generalized volume conjectures of the colored Jones polynomials of links with complements containing more than one hyperbolic pieces. First of all, we construct an infinite family of prime links by considering the cabling on the figure eight knot by the Whitehead chains. The complement of these links consists of two hyperbolic pieces in their JSJ decompositions. We show that at the \((N + \frac{1}{2})\)-th root of unity, the exponential growth rates for the colored Jones polynomials for these links give the simplicial volume of the link complement. As an application, we prove the volume conjecture for the Turaev-Viro invariants for these links complements.

We also generalize the volume conjecture for link whose complement has more than one hyperbolic pieces in another direction. By considering the iterated Whitehead double on the figure eight knot and the Hopf link, we construct two infinite families of prime links. Furthermore, we assign certain "natural" incomplete hyperbolic structures on the hyperbolic pieces of the complements of these links and propose that the sum of their volume coincides with the exponential growth rate of certain sequence of values of colored Jones polynomials of the links.

1 Introduction

This paper aims to study the asymptotics of the \(\tilde{M}\)-th colored Jones polynomials at the \((N + \frac{1}{2})\)-th root of unity for links with more than one hyperbolic pieces in their JSJ decomposition, where \(\tilde{M}\) is a sequence of multi-color in \(N\). Our results combine the volume conjecture for non-hyperbolic links and the generalized volume conjecture for hyperbolic links. First, by considering the cabling of the figure eight knot by Whitehead chains, we find an infinite family of prime link with 2 hyperbolic pieces in its complement satisfying the volume conjecture at \((N + \frac{1}{2})\)-th root of unity. Moreover, by studying the iterated Whitehead double on the figure eight knot and Hopf link, we establish the relationship between the limiting ratio of the multi-color \(\tilde{s} = \lim_{N \to \infty} \tilde{M}/(N + \frac{1}{2})\) and the exponential growth rates of the corresponding colored Jones polynomials. Based on these results, we propose the generalized volume conjecture for link whose complement has only hyperbolic pieces. Finally, we propose another generalization for the generalized volume conjecture for link whose complement has both hyperbolic and Seifert fiber pieces.

1.1 Brief review of volume conjectures

The volume conjecture relates quantum invariants of knots and 3-manifolds with their topological and geometrical structures. For a hyperbolic knot, the volume conjecture suggests that the exponential growth rate of the \(N\)-th normalized colored Jones polynomials of a hyperbolic knot evaluated at the \(N\)-th root of unity captures the hyperbolic volume of the knot complement.

Conjecture 1. [8, 14] Let \(K\) be a hyperbolic knot and \(J_N(K; t)\) be the \(N\)-th normalized colored Jones polynomials of \(K\) evaluated at \(t\). We have

\[
\lim_{N \to \infty} \frac{2\pi}{N} \log |J_N(K; e^{2\pi i})| = \text{Vol}(S^3 \setminus K),
\]

where \(\text{Vol}(S^3 \setminus K)\) is the hyperbolic volume of the knot complement, and the normalization is chosen so that for the unknot \(U\), \(J_N(U; t) = 1\) for any \(N \in \mathbb{N}\).
Later, J. Murakami and H. Murakami extends the above volume conjecture to non-hyperbolic link and suggests that the exponential growth rate captures the simplicial volume of the link complement.

**Conjecture 2.** [14] Let \( L \) be a link and \( J'_N(L; t) \) be the \( N \)-th normalized colored Jones polynomials of \( L \) evaluated at \( t \), where \( N = (N, \ldots, N) \). We have
\[
\lim_{N \to \infty} \frac{2\pi}{N} \log |J'_N(L; e^{\frac{2\pi i}{N}})| = v_3 ||S^3\setminus L||,
\]
where \( ||S^3\setminus L|| \) is the simplicial volume of the link complement.

On the other hand, for 3-manifold, Q. Chen and T. Yang propose a version of volume conjecture for the Turaev-Viro invariants of finite volume hyperbolic three manifolds with or without boundary.

**Conjecture 3.** [3] For every hyperbolic 3-manifold \( M \) with finite volume, we have
\[
\lim_{r \to \infty} \frac{2\pi}{r \ odd} r \log (TV_r(M, e^{\frac{2\pi i}{r}})) = \text{Vol}(M)
\]

It turns out that when the 3-manifold is the complement of a link \( L \) in \( S^3 \), the Turaev-Viro invariants of the manifold \( S^3 \setminus L \) is related to the colored Jones polynomials \( J'_N(L; t) \) of the link \( L \) as follows.

**Theorem 1.** [5] Let \( L \) be a link in \( S^3 \) with \( n \) components. Then given an odd integer \( r = 2N + 1 \geq 3 \), we have
\[
TV_r \left( S^3 \setminus L, e^{\frac{2\pi i}{r}} \right) = 2^{n-1} \left( \frac{2 \sin \left( \frac{2\pi}{r} \right)}{\sqrt{r}} \right)^2 \sum_{1 \leq M \leq \frac{r-1}{2}} \left| J_M \left( L, e^{\frac{2\pi i}{r}} \right) \right|^2
\]

Here, \( J_M(L, t) \) is the unnormalized colored Jones polynomials of the link \( L \), i.e.
\[
J_N(U, t) = \frac{t^{N/2} - t^{-N/2}}{t^2 - t^{-2}}
\]
for the unknot \( U \) and any \( N \in \mathbb{N} \).

Theorem 1 provides a bridge between the volume conjecture of the colored Jones polynomials of links and the volume conjecture of the Turaev-Viro invariants of link complements. In particular, in [5], the volume conjecture of the Turaev-Viro invariants is extended to non-hyperbolic link complement and this generalization is proved for all knots with zero simplicial volume.

**Conjecture 4.** [5] For every link \( L \) in \( S^3 \), we have
\[
\lim_{t \to \infty} \frac{2\pi}{r \ odd} r \log (TV_r(S^3 \setminus L, e^{\frac{2\pi i}{r}})) = v_3 ||S^3\setminus L||
\]

Due to Theorem 1, it is natural to study the analogue of Conjecture 1 and 2 at the new root \( t = e^{\frac{2\pi i}{N+\frac{1}{2}}} \).

In [5], R. Detcherry, E. Kalfagianni and T. Yang ask the following question.

**Conjecture 5.** (Question 1.7 in [5]) Is it true that for any hyperbolic link \( L \) in \( S^3 \), we have
\[
\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_N(L, e^{\frac{2\pi i}{N+\frac{1}{2}}})| = \text{Vol}(S^3 \setminus L) \ ?
\]

Similar to Conjecture 2, we may extend the above conjecture to non-hyperbolic link as follow.

**Conjecture 6.** Let \( L \) be a link and \( J'_N(L; t) \) be the \( N \)-th colored Jones polynomials of \( L \) evaluated at \( t \). We have
\[
\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J'_N(L; e^{\frac{2\pi i}{N+\frac{1}{2}}})| = v_3 ||S^3\setminus L||,
\]
where \( ||S^3\setminus L|| \) is the simplicial volume of the link complement.
Another direction of generalizing the volume conjecture is to study the asymptotic behavior of the values of the $N$-th colored Jones polynomials near the $N$-th root of unity. It is expected that the exponential growth rate has to do with the volume of the link complement with certain incomplete hyperbolic metric \[6, 7, 13\]. Unfortunately, at $t = e^{\frac{2\pi i}{N}}$, this generalization is known to be false for the colored Jones polynomials of links in $S^3$ (see for example \[22\]). For knot, due to the choice of the root of unity, the values of the $N$-th colored Jones polynomials of the figure eight knot grow only polynomially at $t = e^{\frac{2\pi i r}{N}}$ for rational number $r \in (\frac{5}{6}, \frac{7}{6})$ \[11\]. Nevertheless, if we allow the ambient manifold to be the connected sum of $S^2 \times S^1$, this generalization is known to be true for the fundamental shadow links \[4\].

Recently, in \[22\], the author proposes the following version of generalized volume conjecture at the new root of unity $t = e^{\frac{2\pi i}{N} + 1}$. In particular, Conjecture 7 implies Conjecture 5.

**Conjecture 7.** (Generalized volume conjecture for link) Let $L$ be a hyperbolic link with $n$ component. Let $M_1, M_2, \ldots, M_n$ be sequence of positive integers in $N$. Let $s_i = \lim_{N \to \infty} \frac{M_i}{N + \frac{1}{2}}$. Then there exists $\delta_L > 0$ such that whenever the limiting ratio $s_i \in (1 - \delta_L, 1]$, we have

$$\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_{M_1, M_2, \ldots, M_n}(L, e^{\frac{2\pi i}{N + \frac{1}{2}}})| = \text{Vol}(S^3 \setminus L, u_i = 2\pi i (1 - s_i)),$$

where $\text{Vol}(S^3 \setminus L, u = 2\pi i (1 - s))$ is the volume of $S^3 \setminus L$ equipped with the hyperbolic structure such that the logarithm of the holonomy of the meridian around the $i$-th component is given by $u_i = 2\pi i (1 - s_i)$.

Conjecture 7 has been proved by Thomas Au and the author in the case of the figure eight knot \[21\] and proved by the author for the Whitehead link and some special case for the Whitehead chains \[22\]. So far there are limited results about Conjecture 2 and 6 for prime links whose complement have more than one hyperbolic pieces. To the best of the author knowledge, the only known result for Conjecture 2 is the cabling of the figure eight knot by the Borromean ring \[23\].

### 1.2 Construction of the links

Since the generalized volume conjecture is only well-established for the figure eight knot $4_1$ \[21\], the Whitehead link $WL$ and some special case for Whitehead chains $W_{a,b,c,d}$ \[22\], in order to provide new examples of links that may satisfy (some sort of) generalized volume conjecture, it is natural to do some operations on these links to construct new family of links and study the asymptotics of the corresponding colored Jones polynomials. In this paper, the operations that we are going to use are cabling by the Whitehead chains, the iterated Whitehead double and the Hopf union.

The first family of links $W_{a,2,c,d}(4_1)$ is obtained as follows. Consider the figure eight knot and the Whitehead link with two belts $W_{0,2,1,0}$ as shown in Figure 1 and Figure 2. Consider the standard genus 1 Heegaard splitting of $S^3$. Note that if we remove the tubular neighborhood of one of the belt of $W_{0,2,1,0}$, we obtain a Whitehead link sitting inside a torus (Figure 3).

![Figure 1: The figure eight knot 4_1](image1)

![Figure 2: The Whitehead link with two belts W_{0,2,1,0}](image2)

![Figure 3: The Whitehead link inside a red torus](image3)
We can then cable this torus along $4_1$ as shown in Figure 4. Moreover generally, let $W_{a,2,c,d}$ be the Whitehead chains with $a$ twists, 2 belt, $c$ clasps and $d$ mirror clasps [19]. By the same reason, we can cable the solid torus with Whitehead chains $W_{a,2,c,d}$ onto the tubular neighborhood of $4_1$. The resulting link is denoted by $W_{a,2,c,d}(4_1)$ (see for example Figure 5).

Figure 4: The link $W_{0,2,1,0}(4_1)$
Figure 5: The link $W_{1,2,1,1}(4_1)$

We can also consider the iterated cabling and construct the second family of links as follows. Consider the link $W_{0,2,1,0}(4_1)$. This link consists of two components, the first component is a belt and the second component is the Whitehead double of the figure eight knot. We consider the iterated Whitehead double along the second component, and denote this family of links by $W_{p,1,1,0}(W_{0,2,1,0}(4_1))$, where $p \geq 0$. Figure 6 shows the diagram for $W_{1,1,1,0}(W_{0,2,1,0}(4_1))$. 

Figure 6: The link $W_{1,1,1,0}(W_{0,2,1,0}(4_1))$.
The clasps inside the green and blue dotted circles are from the first and second cabling respectively.

Figure 7: Cut along the red incompressible torus

Note that these links have more than one hyperbolic pieces under the JSJ decomposition. For example, for the link $W_{1,1,1,0}(W_{0,2,1,0}(4_1))$, first of all we cut along the red torus shown in Figure 7. Then we obtain the manifolds $S^3 \setminus W_{0,2,1,0}(4_1)$ and $S^3 \setminus W_{0,1,1,0}$ (Figure 8 and 9).
For the component $S^3 \setminus W_{0,2,1,0}(4_1)$, we can embed a green incompressible torus (Figure 10). Cutting along the green torus yields the manifolds $S^3 \setminus 4_1$ and $S^3 \setminus W_{0,2,1,0}$ (Figure 11 and 12).

Finally, by embedding another incompressible torus containing the two belts, we can further decompose $S^3 \setminus W_{0,2,1,0}(4_1)$ into $S^3 \setminus W_{0,1,1,0}(W_{0,2,1,0}(4_1))$ and $S^3 \setminus W_{0,1,1,0}(W_{0,2,1,0}(4_1))$ are given by

$$S^3 \setminus W_{a,2,c,d}(4_1) \cong (S^3 \setminus 4_1) \cup (S^3 \setminus W_{a,1,c,d}) \cup P$$

$$S^3 \setminus W_{0,1,1,0}(W_{0,2,1,0}(4_1)) \cong (S^3 \setminus 4_1) \cup S^3 \setminus W_{0,1,1,0} \cup \cdots \cup S^3 \setminus W_{0,1,1,0} \cup P$$

with simplicial volumes

$$||S^3 \setminus W_{a,2,c,d}(4_1)|| = ||S^3 \setminus 4_1|| + ||S^3 \setminus W_{a,1,c,d}||$$

$$||S^3 \setminus W_{0,1,1,0}(W_{0,2,1,0}(4_1))|| = ||S^3 \setminus 4_1|| + p||S^3 \setminus W_{0,1,1,0}||$$

respectively. In particular, $S^3 \setminus W_{0,1,1,0}(W_{0,2,1,0}(4_1))$ contains $p + 1$ hyperbolic pieces and one Seifert fiber piece.

The third family of links $W_{\alpha,\beta}$ is constructed by doing the iterated Whitehead double on the Hopf link as follows. Consider the Hopf link denoted by $W_0^0$ as shown in Figure 13. The link $W_{\beta}^\alpha$ is obtained by doing the Whitehead double $\alpha$ times along the first component (colored by red) and $\beta$ times along the
second component (colored by blue). Under this notation, the Whitehead link is denoted by \( W^1_0 \) or \( W^0_1 \). An example of the link \( W^2_1 \) is shown in Figure 14.

\[ \text{Figure 13: Hopf link} \quad \text{Figure 14: The link } W^2_1. \text{ The red component is obtained by doing the Whitehead double twice and the blue component is obtained by doing the Whitehead double once} \]

Similar to previous discussion, the JSJ decomposition of the manifold \( S^3 \setminus W^\alpha_\beta \) is given by

\[
S^3 \setminus W^\alpha_\beta \cong \bigcup_{\alpha + \beta \text{ times}} S^3 \setminus W^0_1,1,0
\]

with simplicial volume

\[
||S^3 \setminus W^\alpha_\beta|| = (\alpha + \beta)||S^3 \setminus W^0_1,1,0||
\]

In particular, \( S^3 \setminus W^\alpha_\beta \) contains \((\alpha + \beta)\) hyperbolic pieces and no Seifert fiber piece.

Finally, motivated by the property of the colored Jones polynomials, we consider another type of construction of link called Hopf union. Let \( K_1 \) and \( K_2 \) be two knots. Consider the connected sum of \( K_1 \) with the first (red) component of the Hopf link and the connected sum of \( K_2 \) with the second (blue) component of the Hopf link. We call the resulting link by the Hopf union of \( K_1 \) and \( K_2 \) and denote it by \( K_1 \# W^0_0 \# K_2 \). Figure 15 below shows the example of \( 4_1 \# W^0_0 \# 4_1 \). To obtain the JSJ decomposition of \( S^3 \setminus K_1 \# W^0_0 \# K_2 \), we embed two tori as shown in Figure 16.

\[ \text{Figure 15: The link } 4_1 \# W^0_0 \# 4_1 \quad \text{Figure 16: Embed the red and blue tori to the link complement} \]

Then one can show that the JSJ decomposition of \( S^3 \setminus (K_1 \# W^0_0 \# K_2) \) is given by

\[
S^3 \setminus (K_1 \# W^0_0 \# K_2) \cong (S^3 \setminus W^0_0) \cup (S^3 \setminus K_1) \cup P \cup (S^3 \setminus K_2) \cup P
\]

with simplicial volume

\[
||S^3 \setminus (K_1 \# W^0_0 \# K_2)|| = ||S^3 \setminus K_1|| + ||S^3 \setminus K_2||
\]

6
1.3 Main results

1.3.1 Volume conjectures for the quantum invariants of $W_{a,b,c,d}(4_1)$

First of all, we study the asymptotics of the colored Jones polynomials and the Turaev-Viro invariants of $W_{a,2,c,d}(4_1)$, where $a \in \mathbb{Z}$, $c, d \in \mathbb{Z}_{\geq 0}$ with $c + d \geq 1$. The first result can be stated as follows.

**Theorem 2.** Conjecture 6 is true for all $W_{a,2,c,d}(4_1)$ with $a \in \mathbb{Z}$, $c + d \geq 1$.

As a consequence, we can prove that

**Corollary 1.** Conjecture 6 is true for all $S^3 \setminus W_{a,b,c,d}(4_1)$ with $a \in \mathbb{Z}$, $b \in \mathbb{N}$, $c + d \geq 1$.

In particular, this gives an infinite family of prime links with 2 hyperbolic pieces satisfying Conjecture 6 and Conjecture 4.

1.3.2 Generalized volume conjectures for the colored Jones polynomials of prime links whose complement consist of only hyperbolic pieces

Next, we study the asymptotics of the $(M_1, M_2)$-th colored Jones polynomials of the links $W^1_1$ and $W^2_1$. We have the following results.

**Theorem 3.** Let $s_i = \lim_{N \to \infty} \frac{M_i}{N + \frac{1}{2}}$ for $i = 1, 2$. Then we have

$$
\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_{M_1,M_2}(W^1_1, e^{\frac{2\pi i}{N + \frac{1}{2}}})| = \text{Vol}(S^3 \setminus W_1^1; u_1 = 0, u_2 = 2\pi i(1 - s_1)) + \text{Vol}(S^3 \setminus W_2^1; u_1 = 0, u_2 = 2\pi i(1 - s_2))
$$

(1)

and

$$
\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_{M_1,M_2}(W^2_1, e^{\frac{2\pi i}{N + \frac{1}{2}}})| = \text{Vol}(S^3 \setminus W_1^1; u_1 = 0, u_2 = 2\pi i(1 - s_1)) + \text{Vol}(S^3 \setminus W_2^1; u_1 = 0, u_2 = 2\pi i(1 - s_2)) + \text{Vol}(S^3 \setminus W_L)
$$

(2)

where $u_1, u_2$ are the logarithm of the holonomy of the meridian of the belt and the clasp respectively, and $\text{Vol}(S^3 \setminus W_L; u_1 = 0, u_2 = 2\pi i(1 - s))$ is the volume of the Whitehead link complement with incomplete hyperbolic structure parametrized by $u_1, u_2$.

As a consequence, we have

**Corollary 2.** Conjecture 6 and Conjecture 4 are true for $W^1_1$ and $W^2_1$.

Recall that $S^3 \setminus W^1_1$ and $S^3 \setminus W^2_1$ consist of only hyperbolic pieces. From Equation (1) and (2) in Theorem 3 we can see that the relationship between the color of a link component and the corresponding choice of hyperbolic structure is in the same spirit as that in Conjecture 6. Moreover, along the incompressible torus, the metric is always chosen to be complete. We call this assignment of hyperbolic structures to each hyperbolic pieces the **natural hyperbolic structure associated to the coloring** (or natural hyperbolic structure in short). Furthermore, recall that the colored Jones polynomials of $L_1 \# L_2$ along each component colored by $i$ is given by

$$
[i]J_{M_i}(L_1 \# L_2, t) = J_{\tilde{M}_i}(L_1, t)J_{\tilde{M}_2}(L_2, t),
$$

where $\tilde{M}_1$ and $\tilde{M}_2$ are the restrictions of the color $\tilde{M}$ to $L_1$ and $L_2$ respectively. Therefore, we restrict our attention to prime links. The above observation leads to the following conjecture.

**Conjecture 8.** Let $L$ be a n components prime link with complement having only hyperbolic structure. Then there exists $\delta > 0$ such that for any coloring $\tilde{M} = (M_1, \ldots, M_n)$ with the limiting ratio $1 - \delta < s_i \leq 1$, $i = 1, 2, \ldots, n$, we have

$$
\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_{\tilde{M}}(L, e^{\frac{2\pi i}{N + \frac{1}{2}}})| = \frac{1}{2\pi} \text{Vol}(S^3 \setminus L; u_i = 2\pi i(1 - s_i)),
$$

where $\text{Vol}(S^3 \setminus L; u_i = 2\pi i(1 - s_i))$ is the sum of the volume of the hyperbolic pieces with natural hyperbolic structure.
By Theorem 3, we have

**Corollary 3.** Conjecture [3] is true for the links \( W_1^2 \) and \( W_2^2 \).

**Remark 1.** Actually, most of the arguments in the proof of Theorem 3 work for all \( W_3^\alpha \) with \( \alpha, \beta \geq 1 \). In the proof, we apply the saddle point approximation. In particular, we need to verify that the critical point is non-degenerate. For small \( \alpha, \beta \), this can be done by direct calculation of the determinant of the Hessian of the potential function at the critical point. However, at this point, we do not have an argument to show that the determinant is non-zero for all \( \alpha, \beta \). Nevertheless, if this condition holds, then given any \( n \geq 2 \), the link \( W_3^n \) with \( \alpha + \beta = n \) will be a prime link whose complement consists of \( n \) hyperbolic pieces and satisfying Conjecture [3] and [4]. See Section 3 for more details.

### 1.3.3 Generalized volume conjecture for the colored Jones polynomials whose complement consists of hyperbolic pieces and Seifert fiber pieces

So far we discussed the generalized volume conjecture for links whose complement consists of only hyperbolic pieces. It is natural to consider a similar version of generalized volume conjecture for links whose complement consists of hyperbolic pieces together with Seifert fiber pieces.

**Theorem 4.** Let \( s = \lim_{N \to \infty} \frac{M}{N + \frac{1}{2}} \). Then we have

\[
\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_{N,M}(W_{0,2,1,0}(4_1), e^{\frac{2\pi i}{N + \frac{1}{2}}})| = \text{Vol}(S^3 \setminus WL; u_1 = 0, u_2 = 2\pi i(1 - s)) + \text{Vol}(S^3 \setminus 4_1) \tag{3}
\]

\[
\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_{N,M}(W_{0,1,1,0}(W_{0,2,1,0}(4_1)), e^{\frac{2\pi i}{N + \frac{1}{2}}})| = \text{Vol}(S^3 \setminus WL; u_1 = 0, u_2 = 2\pi i(1 - s)) + \text{Vol}(S^3 \setminus WL) + \text{Vol}(S^3 \setminus 4_1) \tag{4}
\]

From Equations (3) and (4), around the tubular neighborhood of the link components, we observe the same relationship between the color of the link component and the choice of hyperbolic structure. Besides, if the color of all the link components inside the Seifert fiber pieces are colored by \( N \), then around the incompressible torus, the hyperbolic structure is always chosen to be complete.

To conclude, we say that a sequence of colors \( \bar{M} = (M_1, \ldots, M_n) \) is a natural coloring if the color of the link complements inside the Seifert fiber pieces are \( N \). Once we have a natural coloring, we call the assignment of hyperbolic structures to each hyperbolic pieces discussed in the previous paragraph the natural hyperbolic structure associated to the natural coloring (or natural hyperbolic structure in short). We may ask the following question:

**Question 1.** Let \( L \) be a \( n \)-components prime link whose complement consists of hyperbolic pieces together with Seifert fiber pieces. Does there exist \( \delta > 0 \) such that for any natural coloring \( \bar{M} = (M_1, \ldots, M_n) \) with the limiting ratio \( 1 - \delta < s_i \leq 1, i = 1, 2, \ldots, n \), we have

\[
\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_{\bar{M}L}(L, e^{\frac{2\pi i}{N + \frac{1}{2}}})| = \frac{1}{2\pi} \text{Vol}(S^3 \setminus L; u_i = 2\pi i(1 - s_i)),
\]

where \( \text{Vol}(S^3 \setminus L; u_i = 2\pi i(1 - s_i)) \) is the sum of the volume of the hyperbolic pieces with natural hyperbolic structure?

In order to study Question 1, we consider the Hopf union of knots and introduce the following lemma, which can be proved by using skein theory (see Appendix A).

**Lemma 1.** Let \( M_1, M_2 \in \mathbb{N} \) and let \( K_1, K_2 \) be two knots. The modulus of the colored Jones polynomials of \( K_1 \# W_0^0 \# K_2 \) is given by

\[
|J_{M_1,M_2}(K_1 \# W_0^0 \# K_2, t)| = \left| \frac{M_1 \cdot M_2}{|M_1||M_2|} J_{M_1}(K_1, t) J_{M_2}(K_2, t) \right|, \tag{5}
\]

where \( [n] = \frac{t^n - t^{-n}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \).
Suppose $|J_{M_1 M_2}(K_1 \# W_0^0 \# K_2, e^{\frac{2\pi i}{N+\frac{1}{2}}} )| \neq 0$. Then we have
\[
\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_{M_1 M_2}(K_1 \# W_0^0 \# K_2, e^{\frac{2\pi i}{N+\frac{1}{2}}} )| \\
= \lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_{M_1}(K_1, e^{\frac{2\pi i}{N+\frac{1}{2}}} )| + \lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_{M_2}(K_2, e^{\frac{2\pi i}{N+\frac{1}{2}}} )|
\]
In particular, if the volume conjecture (or the generalized volume conjecture) for $K_1$ and $K_2$ are true for the sequences $M_1$ and $M_2$, i.e.
\[
\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_{M_1}(K_1, e^{\frac{2\pi i}{N+\frac{1}{2}}} )| = \text{Vol}(S^3 \setminus K_1, u = 2\pi i(1 - s))
\]
and
\[
\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_{M_2}(K_2, e^{\frac{2\pi i}{N+\frac{1}{2}}} )| = \text{Vol}(S^3 \setminus K_2, u = 2\pi i(1 - s))
\]
then we have
\[
\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_{M_1 M_2}(K_1 \# W_0^0 \# K_2, e^{\frac{2\pi i}{N+\frac{1}{2}}} )| = \text{Vol}(S^3 \setminus K_1, u = 2\pi i(1 - s)) + \text{Vol}(S^3 \setminus K_2, u = 2\pi i(1 - s))
\]
This result has two immediate consequences. First of all, note that when $t = e^{\frac{2\pi i}{N+\frac{1}{2}}}$, we have
\[
[N^2] = \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = -\frac{\sin(\frac{\pi N}{2N+1})}{\sin(\frac{\pi N}{2N+1})} \neq 0
\]
Therefore, we have

**Theorem 5.** Conjecture [4] for $K_1$ and $K_2$ imply Conjecture [6] for $K_1 \# W_0^0 \# K_2$.

In particular, we have

**Corollary 4.** Conjecture [4] is true for $4_1 \# W_0^0 \# 4_1$.

The second consequence is that the answer to Question [1] is ‘no’ in general. A counter example can be constructed as follows. Let $p > 1$ be an odd number. Consider the case where $2N + 1 = p^2$ and $M_1 = M_2 = p(p - 1)$. Note that the limiting ratios $s_i = \lim_{p \to \infty} \frac{p(p - 1)}{p^2} = 1$. However, when $t = e^{\frac{2\pi i}{N+\frac{1}{2}}}$, since
\[
[M_1 M_2] = \frac{\frac{M_1 M_2}{2} - \frac{M_1 M_2}{2}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} = \frac{\frac{2\pi i p^2 - 2\pi i p^2}{p^2}}{e^{\frac{2\pi i}{p^2}} - e^{-\frac{2\pi i}{p^2}}} = 0,
\]
we have $|J_{M_1 M_2}(K_1 \# W_0^0 \# K_2, e^{\frac{2\pi i}{N+\frac{1}{2}}} )| = 0$ for any knots $K_1, K_2$. As a result, $4_1 \# W_0^0 \# 4_1$ is a counter example to Question [1].

Nevertheless, we can always choose a special sequence of colors to avoid the problem mentioned above. Consider the case where $2N + 1$ is a prime number. Let $M_1 M_2 = d (\text{mod } 2N + 1)$. Then
\[
[M_1 M_2] = 0 \iff \sin \left( \frac{2\pi M_1 M_2}{2N + 1} \right) = 0 \iff \sin \left( \frac{2\pi d}{2N + 1} \right) = 0 \iff d = 0
\]
Since $2N + 1$ is a prime, $d = 0$ implies $2N + 1 \mid M_1$ or $M_2$, which is impossible when the limiting ratio is close to 1.

To conclude, we propose the following weaker version of generalized volume conjecture for links.

**Conjecture 9.** Let $L$ be a $n$ components prime link. Then there exists $\delta > 0$ such that for any natural coloring $\bar{M} = (M_1, \ldots, M_n)$ with the limiting ratio $1 - \delta < s_i \leq 1$, $i = 1, 2, \ldots, n$, we have
\[
\limsup_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log |J_{\bar{M}}(L, e^{\frac{2\pi i}{N+\frac{1}{2}}} )| = \text{Vol}(S^3 \setminus L; u_i = 2\pi i(1 - s_i)),
\]
where $\text{Vol}(S^3 \setminus L; u_i = 2\pi i(1 - s_i))$ is the sum of the volume of the hyperbolic pieces with natural hyperbolic structures.
From Theorem 4, we have

**Corollary 5.** Conjecture 9 is true for $W_{0,2,1,0}(4_1)$ and $W_{0,1,1,0}(W_{0,2,1,0}(4_1))$.

**Remark 2.** Similar to Remark 1, most of the argument in the proof of Theorem 4 works for the whole family $W_{0,1,1,0}(W_{0,2,1,0}(4_1)), p \geq 0$.

### 1.4 Further questions

In the context of Conjecture 9, for the natural hyperbolic structure, the colors of the link components inside the Seifert fiber piece are always equal to $N$. We can ask:

**Question 2.** What is the exponential growth rates for the values of the colored Jones polynomials of links with other coloring?

Besides, we know that the colored Jones polynomials of link is a special case of the Reshetikhin-Turaev invariants, which form a topological quantum field theory (TQFT) [2]. Since the link complement is obtained by gluing along the boundary tori of each of the pieces in the JSJ decomposition, it is natural to ask:

**Question 3.** Is there any TQFT interpretation for Conjecture 8?

### 1.5 Outline of this paper

In Section 2, we study the volume conjectures for the cabling of the figure eight knot by the Whitehead chains. In Section 2.1 and 2.2, in order to illustrate the technique of proving the volume conjecture, we first prove Theorem 2 and Corollary 1 for the link $W_{0,2,1,0}(4_1)$. In Section 2.3, we give a summary of the technique we used in the previous subsection. Theorem 2 and Corollary 1 will be proved in Section 2.4. Theorem 3 and 4 will be proved in Section 3 and 4 respectively. Finally, the proof of Lemma 1 will be included in Appendix A.

### 1.6 Acknowledgements

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## 2 Cabling of the figure eight knot by Whitehead chains

### 2.1 Volume conjecture for $J_{N,N}(W_{0,2,1,0}(4_1), e^{2\pi i N})$

First of all, we compute the $(N,N)$-th colored Jones polynomials of the link $W_{0,2,1,0}(4_1)$. Recall that the $M$-th normalized colored Jones polynomials of the figure eight knot and the $(N,N)$-th colored Jones polynomials of the Whitehead chains $W_{0,2,1,0}$ are given by [24, 19]

\[
J'_M(4_1, t) = \sum_{k=0}^{M-1} t^{-kM} \prod_{l=1}^{k} \left( 1 - t^{M-l} \right) \left( 1 - t^{l+1} \right)
\]

\[
J_{N}(W_{0,2,1,0}, t) = \left( \frac{t^{N(2n+1)/2} - t^{-(2n+1)/2}}{t^2 - t^{-2}} \right)^2 \cdot C(n, t; N) \cdot \left( \frac{t^{N(2n+1)/2} - t^{-(2n+1)/2}}{t^{(2n+1)/2} - t^{-(2n+1)/2}} \right)^2
\]

where

\[
C(n, t; N) = \sum_{l=0}^{N-1} t^{-N(l+n)} \prod_{j=1}^{n} \left( 1 - t^{-j} \right) \left( 1 - t^{l+j} \right)
\]
By using the method in [24], the $(N, N)$-th colored Jones polynomials of the link $W_{0,2,1,0}(4_1)$ is given by

$$J_N(W_{0,2,1,0}(4_1), t) = t^{N^2 - 1} \frac{N(N+1)}{2} \sum_{n=0}^{N-1} \frac{t^{2n+1}/2 - t^{-2n+1}/2}{t^2 - t^{-2}} \cdot C(n, t; N) \cdot \left( \frac{t^{N(2n+1)/2} - t^{-(2n+1)/2}}{t^{(2n+1)/2} - t^{-(2n+1)/2}} \right) \cdot J_{2n+1}^1(4_1, t)$$

$$= t^{N^2 - 1} + \frac{N(N+1)}{2} \sum_{n=0}^{N-1} \frac{t^{N(2n+1)/2} - t^{-(2n+1)/2}}{t^2 - t^{-2}} \times t^{-N(l+n)}$$

$$\times \left( \prod_{j=1}^{n} \frac{(1-t^{N-l-j})(1-t^{l+j})}{1-t^j} \right) \left( t^{-k(2n+1)} \prod_{l=1}^{k} \left( 1 - t^{2n+1-l} \right) \right)$$

When $t = e^{\frac{2\pi i}{N+\frac{3}{2}}}$, we have the following upper bound estimates.

**Lemma 2.** [21] For $k, l \in \{1, 2, \ldots, M-1\}$, let

$$g_M(k) = \prod_{l=1}^{k} \left| \left( \frac{M+1}{2} - t - \frac{M+1}{2} \right) \left( \frac{M+1}{2} - t^M \right) \right| = \prod_{l=1}^{k} \left| \left( 1 - t^{M-l} \right) \left( 1 - t^{M+l} \right) \right|$$

For each $M$, let $k_M \in \{1, 2, \ldots, M-1\}$ such that $g_M(k_M)$ achieves the maximum among all $g_M(k)$. Assume that $\frac{M}{N+\frac{3}{2}} \to s$ and $\frac{k_M}{N+\frac{3}{2}} \to \nu_s$ as $N \to \infty$. Then we have

$$\lim_{r \to \infty} \frac{1}{N+\frac{3}{2}} \log(g_M(k_M)) = -\frac{1}{2\pi} \left( \Lambda(\pi(\nu_s - s)) + \Lambda(\pi(\nu_s + s)) \right) \leq \frac{\text{Vol}(S^3\setminus4_1)}{2\pi}.$$

Furthermore, the equality holds if and only if $(s = 1$ and $\nu_s = \frac{5}{3})$ or $(s = \frac{1}{2}$ and $\nu_s = \frac{1}{3})$.

**Lemma 3.** [22] For any $n \in \{1, 2, \ldots, M-1\}$, $l \in \{1, 2, \ldots, M - 1 - n\}$, let

$$c_M(n, l; t) = \prod_{j=1}^{n} \left| \left( 1 - t^{M-l-j} \right) \left( 1 - t^{l+j} \right) \right|$$

For each $M$, let $n_M \in \{1, 2, \ldots, M-1\}$ and $l_M \in \{1, 2, \ldots, M - 1 - n\}$ such that $c_M(n_M, l_M)$ achieves the maximum among all $c_M(n, l)$. Assume that $\frac{M}{N+\frac{3}{2}} \to s \in [0, 1]$, $\frac{n_M}{N+\frac{3}{2}} \to \nu_s$ and $\frac{l_M}{N+\frac{3}{2}} \to \nu_l$. Then we have

$$\lim_{N \to \infty} \frac{1}{N+\frac{3}{2}} \log(c_M(n, l; e^{\frac{2\pi i}{N+\frac{3}{2}}})) \leq \frac{\text{Vol}(S^3\setminus WL)}{2\pi}$$

Furthermore, the equality holds if and only if $s = 1$, $\nu_s = \frac{1}{2}$ and $\nu_l = \frac{1}{4}$.

From Lemma 2 and 3 we have

$$\limsup_{N \to \infty} \frac{1}{N+\frac{3}{2}} \log \left| J_N(W_{0,2,1,0}(4_1), e^{\frac{2\pi i}{N+\frac{3}{2}}}) \right| \leq \frac{\text{Vol}(S^3\setminus WL) + \text{Vol}(S^3\setminus4_1)}{2\pi} = \frac{\nu_4 ||W_{0,2,1,0}(4_1)||}{2\pi}.$$

Let

$$D_\delta = \left\{ (n, l, k) \mid \left| \frac{n}{N+\frac{3}{2}} - \frac{1}{2} \right| \leq \frac{1}{4}, \left| \frac{l}{N+\frac{3}{2}} - \frac{1}{4} \right| \leq \frac{k}{N+\frac{3}{2}} - \frac{5}{6} < \delta \right\}$$

At the end we will show that the exponential growth rate is exactly $\frac{\nu_4 ||W_{0,2,1,0}(4_1)||}{2\pi}$. In particular, we can write

$$J_N(W_{0,2,1,0}(4_1), e^{\frac{2\pi i}{N+\frac{3}{2}}}) \sim \frac{N(N-1)}{2} \sum_{D_\delta} \frac{t^{N(2n+1)/2} - t^{-(2n+1)/2}}{t^2 - t^{-2}} \times t^{-N(l+n)}$$

$$\times \left( \prod_{j=1}^{n} \frac{(1-t^{N-l-j})(1-t^{l+j})}{1-t^{j}} \right) \left( t^{-k(2n+1)} \prod_{l=1}^{k} \left( 1 - t^{2n+1-l} \right) \right)$$ (6)
Next, recall that the quantum dilogarithm function $\varphi^h(z)$ is defined by

$$
\varphi^h(z) = \int_{\Omega} e^{(2z-\pi)x} dx,
$$

where

$$
\Omega = (-\infty, -\epsilon] \cup \{ z \in \mathbb{C} | |z| = \epsilon, \Im z > 0 \} \cup [\epsilon, \infty)
$$

and

$$
z \in \left\{ z \in \mathbb{C} | -\frac{\pi h}{2} < \Re z < \pi + \frac{\pi h}{2} \right\}
$$

For any $z \in \mathbb{C}$ with $0 < \Re z < \pi$, the quantum dilogarithm function satisfies the functional equation that

$$
1 - e^{2iz} = \exp \left( \varphi^h \left( z - \frac{\pi h}{2} \right) - \varphi^h \left( z + \frac{\pi h}{2} \right) \right)
$$

Furthermore, it satisfies

$$
\lim_{h \to 0} 2\pi i h \varphi^h(z) = \text{Li}_2(e^{2iz})
$$

Put $h = \frac{1}{N + \frac{1}{2}}$. Using the result of [21] and [22], we have

$$
J_N(W_{0,2,1,0}(4i); e^{\frac{\pi i}{2N + 1}})
\approx -\frac{N + \frac{1}{2}}{\pi} \exp \left( -\varphi^h \left( \frac{\pi}{2N + 1} \right) \right) \sum_{D_4} \sin \left( \pi \left( \frac{n + \frac{1}{2}}{N + \frac{1}{2}} \right) \right)
\times \frac{\exp \left( \varphi^h \left( \frac{2N-4n-2}{2N+1} \pi \right) \right)}{\exp \left( \varphi^h \left( \frac{n - 2N+4n}{2N+1} \pi \right) \right)} \exp \left( (N + \frac{1}{2}) \Phi_{N,N} \left( W_{0,2,1,0}(4i); \frac{n}{N + \frac{1}{2}}, \frac{l}{N + \frac{1}{2}}, \frac{k}{N + \frac{1}{2}} \right) \right)
$$

where $\Phi_{N,N}(W_{0,2,1,0}(4i); z_1, z_2, z_3)$ is given by

$$
\Phi_{N,N}(W_{0,2,1,0}(4i); z_1, z_2, z_3) = \Phi_{N,N}(WL; z_1, z_2) + \Psi_N(4i; z_1, z_3)
$$

$$
\Phi_{N,N}(WL; z_1, z_2) = \frac{1}{2\pi i} \left\{ \frac{2\pi i}{N + \frac{1}{2}} \left[ \varphi^h \left( \frac{N\pi}{N + \frac{1}{2}} - \pi z_1 - \pi z_2 - \frac{\pi}{2N + 1} \right) 
\right. 
- \varphi^h \left( \frac{N\pi}{N + \frac{1}{2}} - \pi z_2 - \frac{\pi}{2N + 1} \right) 
\left. 
+ \varphi^h \left( \pi z_2 + \frac{\pi}{2N + 1} \right) - \varphi^h \left( \pi z_1 + \pi z_2 + \frac{\pi}{2N + 1} \right) 
\right.
\right. 
+ \varphi^h \left( \pi z_1 + \frac{\pi}{2N + 1} \right) \right\}
$$

$$
\Psi_N(4i; z_1, z_3) = \frac{1}{2\pi i} \left\{ \frac{2\pi i}{N + \frac{1}{2}} \left[ \varphi^h \left( -\pi z_3 + 2\pi z_1 + \frac{3\pi}{2N + 1} \right) 
\right. 
- \varphi^h \left( \pi z_3 - \pi + 2\pi z_1 - \frac{\pi}{2N + 1} \right) \right. 
\right. 
- 2\pi i (1 - 2z_1) z_3 \right\}
$$

Note that the quantum dilogarithm function $S_z(z)$ in [21] is related to the quantum dilogarithm function $\varphi^h(z)$ by

$$
\exp(\varphi^h(z)) = S_{zh}(2z - \pi)
$$
Let $N \to \infty$. We have

$$
\Phi(W_{0,2,1,0}(4_1); z_1, z_2, z_3) = \Phi(WL; z_1, z_2) + \Phi(4_1; z_1, z_3)
$$

(10)

$$
\Phi(WL; z_1, z_2) = \frac{1}{2\pi i} \left[ \text{Li}_2(e^{-2\pi i z_1 - 2\pi i z_2}) - \text{Li}_2(e^{-2\pi i z_2}) + \text{Li}_2(e^{2\pi i z_2}) - \text{Li}_2(e^{2\pi i z_1} + 2\pi i z_2) - \text{Li}_2(e^{2\pi i z_1}) \right]
$$

(11)

$$
\Psi(4_1; z_1, z_3) = \frac{1}{2\pi i} \left[ \text{Li}_2(e^{-2\pi i z_1 + 4\pi i z_2}) - \text{Li}_2(e^{2\pi i z_1} + 4\pi i z_2) - 2\pi i(1 - 2z_1)z_3 \right]
$$

(12)

where the functions is defined on the domain

$$
D = \left\{ (z_1, z_2, z_3) \mid |z_1 - \frac{1}{2}|, |z_2 - \frac{1}{4}|, |z_3 - \frac{5}{6}| < \epsilon \right\}
$$

for some sufficiently small $\epsilon > 0$.

It is important to note that the function $\Psi(4_1; z_1, z_3)$ is the same as the function $\Phi(s)(z)$ studied in [21] with $s = 2z_1$ and $z = z_2$. This suggests that the parameter $2z_1$ can be understood as choosing the hyperbolic structure of the figure eight knot complement. In particular, to get the complete hyperbolic structure of the figure eight knot complement, we should take $z_1 = \frac{1}{2}$.

By considering the Taylor series expansion, we have

$$
\exp \left( \left( N + \frac{1}{2} \right) \Phi_{N,N}(W_{0,2,1,0}(4_1); z_1, z_2, z_3) \right)

= E(W_{0,2,1,0}(4_1); z_1, z_2, z_3) \exp \left( \left( N + \frac{1}{2} \right) \Phi(W_{0,2,1,0}(4_1); z_1, z_2, z_3) \right) \left( 1 + O \left( \frac{1}{N + \frac{1}{2}} \right) \right),
$$

where

$$
E(W_{0,2,1,0}(4_1); z_1, z_2, z_3) = \exp \left( \log(1 - e^{-2\pi i z_1 - 2\pi i z_2}) - \log(1 - e^{-2\pi i z_2}) - \frac{1}{2} \log(1 - e^{-2\pi i z_2}) - \frac{1}{2} \log(1 - e^{2\pi i z_1}) - \frac{3}{2} \log(1 - e^{-2\pi i z_1 + 4\pi i z_2}) - \frac{1}{2} \log(1 - e^{2\pi i z_1 + 4\pi i z_2}) \right)
$$

Next, note that the critical point equations for the potential function $\Phi(WL(4_1); z_1, z_2, z_3)$ are given by

$$
0 = \frac{\partial \Phi(W_{0,2,1,0}(4_1); z_1, z_2, z_3)}{\partial z_1} = \frac{\partial \Phi(WL; z_1, z_2)}{\partial z_1} + \frac{\partial \Phi(4_1; z_1, z_3)}{\partial z_1}
$$

(13)

$$
0 = \frac{\partial \Phi(W_{0,2,1,0}(4_1); z_1, z_2, z_3)}{\partial z_2} = \frac{\partial \Phi(WL; z_1, z_2)}{\partial z_2}
$$

(14)

$$
0 = \frac{\partial \Phi(W_{0,2,1,0}(4_1); z_1, z_2, z_3)}{\partial z_3} = \frac{\partial \Phi(4_1; z_1, z_3)}{\partial z_3}
$$

(15)

Recall from [22] that $(z_1, z_2) = (\frac{1}{2}, \frac{1}{4})$ is the critical point of the potential function $\Phi(WL; z_1, z_2)$. As a result, if we put $(z_1, z_2) = (\frac{1}{2}, \frac{1}{4})$, the system of critical point equations become

$$
0 = \frac{\partial \Phi(4_1; \frac{1}{2}, z_3)}{\partial z_1} = 2[- \log(1 - e^{-2\pi i z_1}) + \log(1 - e^{-2\pi i z_1}) + 2\pi i z_3]
$$

(16)

$$
0 = \frac{\partial \Phi(4_1; \frac{1}{2}, z_3)}{\partial z_3} = \log(1 - e^{-2\pi i z_1}) + \log(1 - e^{-2\pi i z_1})
$$

(17)

Note that the second equation is exactly the critical point equation of the potential function $\Psi(4_1; \frac{1}{2}, z_3)$ (which is the function $\Phi^{(1)}(z)$ in [21]). In particular, since $z_3 = \frac{5}{6}$ is a critical point of $\Psi(4_1; \frac{1}{2}, z_3)$, we have

$$
\frac{\partial \Phi(4_1; \frac{1}{2}, \frac{5}{6})}{\partial z_3} = 0
$$

(18)
Finally, by direct computation, we have

\[
\frac{\partial \Phi(41; \frac{1}{2}, \frac{5}{6})}{\partial z_1} = 2[- \log(1 - e^{-5\pi i/3}) + \log(1 - e^{5\pi i/3}) + 10\pi i/3] = 8\pi i
\]  

(19)

As a result, the point \((z_1, z_2, z_3) = (\frac{1}{2}, \frac{1}{4}, \frac{5}{6})\) is indeed a critical point of the Fourier coefficient

\[
\Phi(W_{0,2,1,0}(41); z_1, z_2, z_3) = 8\pi iz_1
\]

with critical value

\[
\Phi \left( W_{0,2,1,0}(41); \frac{1}{2}, \frac{1}{4}, \frac{5}{6} \right) = \frac{\text{Vol}(S^3 \setminus WL) + i\frac{\pi^2}{4}}{2\pi} - 4\pi i
\]

(20)

Moreover, the Hessian of the potential function at the critical point is given by

\[
\text{Hess} \left( \Phi \left( \frac{1}{2}, \frac{1}{4}, \frac{5}{6} \right) \right) = 2\pi i \begin{pmatrix}
  i + \frac{5}{2} & i & i \\
  i & i & 2i \\
  2 + \sqrt{3}i & 2i & \sqrt{3}i
\end{pmatrix}
\]

(21)

with non-zero determinant.

Therefore, by the Poisson summation formula and saddle point approximation (Proposition 4.6 and Proposition 3.5 in [17]), we have

\[
J_{N,N}(W_{0,2,1,0}(41), t) \\
\sim \frac{e^{\frac{\pi^2}{12} (N^2 - \frac{5}{2} - \frac{4}{3})}}{\sqrt{-\det \text{Hess}(\Phi(\frac{1}{2}, \frac{1}{4}, \frac{5}{6}))}} \exp \left( -\varphi^h \left( \frac{\pi}{2N+1} \right) \right) \frac{(N + \frac{1}{2})^3}{\int_D \sin(\pi z_1) \exp \left( \varphi^h \left( \frac{\pi}{2N+1} + \pi z_1 \right) \right)}
\]

(22)

By Lemma A.3 in [17], we have

\[
\exp \left( \varphi^h \left( \frac{\pi}{2N+1} \right) \right) \sim e^{\frac{\pi}{12} (N + \frac{1}{2})^3/2} \exp \left( \frac{-N + \frac{1}{2} \pi^2 i}{2\pi} \right)
\]  

(23)

\[
\exp \left( \varphi^h \left( \frac{\pi}{2N+1} \right) \right) \sim e^{\frac{\pi}{12} (N + \frac{1}{2})^3/2} \exp \left( \frac{-N + \frac{1}{2} \pi^2 i}{2\pi} \right)
\]

(24)
In particular, we have
\[
\exp \left( -\varphi^h \left( \frac{\pi}{2N + 1} \right) \right) \sim e^{-\frac{4\pi^2}{2^2}} \left( N + \frac{1}{2} \right)^{-\frac{1}{2}} \exp \left( \frac{N + \frac{1}{2}}{2\pi} \frac{2^2 \pi^2}{6} \right) \quad (25)
\]
\[
\frac{\exp \left( \varphi^h \left( \frac{\pi}{2N + 1} \right) \right)}{\exp \left( \varphi^h \left( \frac{\pi}{2N + 1} \right) \right)} \sim (N + \frac{1}{2}) \quad (26)
\]
Besides, we have
\[
E \left( W_{0,2,1,0}(41); \frac{1}{2}, \frac{1}{4}, \frac{5}{6} \right) = -\frac{1}{2} e^{-\frac{\pi^2}{2}}
\]
Thus, we have
\[
J_{N,N}(W_{0,2,1,0}(41), t) \sim \frac{-(N + \frac{1}{2})^3 e^{-5\pi i/6}}{\sqrt{-\det \text{Hess}(\Phi(\frac{1}{2}, \frac{1}{4}, \frac{5}{6}))}} \exp \left( \frac{N + \frac{1}{2}}{2\pi^2} \left( v_3 ||W_{0,2,1,0}(41)|| + \frac{\pi^2}{4} \right) \right) \quad (27)
\]

### 2.2 Volume conjecture for $TV_r(S^3 \setminus W_{0,2,1,0}(41), e^{\frac{2\pi i}{N + \frac{1}{2}}})$

Next, by Lemma 2 and 3 for any $0 \leq \bar{M} \leq N$, we have
\[
\lim_{N \to \infty} \sup_{\bar{M}} \frac{2\pi}{N + \frac{1}{2}} \left| J_{\bar{M}} \left( W_{0,2,1,0}(41), e^{\frac{2\pi i}{N + \frac{1}{2}}} \right) \right| \leq v_3 ||W_{0,2,1,0}(41)||
\]
As a result, by Sandwich theorem, it is easy to show that
\[
\lim_{r \to \infty, r \text{ odd}} \frac{2\pi}{r} \log |TV_r(S^3 \setminus W_{0,2,1,0}(41))| = \lim_{r \to \infty, r \text{ odd}} \frac{2\pi}{r} \log \left( \sum_{1 \leq \bar{M} \leq \frac{r - 1}{2}} \left| J_{\bar{M}} \left( L, e^{\frac{2\pi i}{N + \frac{1}{2}}} \right) \right| \right)^2 = \lim_{N \to \infty, N + \frac{1}{2}} \frac{2\pi}{r} \log \left| J_{N} \left( W_{0,2,1,0}(41), e^{\frac{2\pi i}{N + \frac{1}{2}}} \right) \right| = v_3 ||W_{0,2,1,0}(41)|| \quad (28)
\]
In particular, since $S^3 \setminus W_{a,2,1,0}(41)$ are all homeomorphic for any $a \in \mathbb{Z}$, we have
\[
\lim_{r \to \infty, r \text{ odd}} \frac{2\pi}{r} \log |TV_r(S^3 \setminus W_{a,2,1,0}(41))| = \lim_{r \to \infty, r \text{ odd}} \frac{2\pi}{r} \log |TV_r(S^3 \setminus W_{0,2,1,0}(41))| = v_3 ||W_{a,2,1,0}(41)|| \quad (29)
\]

### 2.3 A summary of the strategy of proving the volume conjectures

In this subsection, we summarize the strategy we used to prove the volume conjectures. This strategy will be used repeatedly in the rest of this paper.

1. To compute the asymptotic expansion formula for $J_{N,N}(W_{0,2,1,0}(41), e^{\frac{2\pi i}{N + \frac{1}{2}}})$,
   
   (a) **(maximum estimation)** by using Lemma 2 and 3, we show that in order to study the AEF of $J_{N,N}(W_{0,2,1,0}(41), e^{\frac{2\pi i}{N + \frac{1}{2}}})$, we only need to consider a small neighborhood around the critical point;
   
   (b) **(Poisson Summation formula and the saddle point approximation)** by using the Poisson Summation formula and the saddle point approximation (Proposition 4.6 and Proposition 3.5 in [17]), we obtain the AEF for $J_{N,N}(W_{0,2,1,0}(41), e^{\frac{2\pi i}{N + \frac{1}{2}}})$. 

2. To prove the volume conjecture for $TV_r(S^3 \setminus W_{a,2,1,0}(4_1))$, 

(a) by using Lemma 2 and 3 as well as Theorem 1, we show that

$$\limsup_{r \to \infty} \frac{2\pi}{r} \log |TV_r(S^3 \setminus W_{a,2,1,0}(4_1))| \leq v_3 ||W_{a,2,1,0}(4_1)||$$

(b) together with the result that

$$\lim_{N \to \infty} \frac{2\pi}{N + \frac{1}{2}} \log \left| J_{N,N} \left( W_{0,2,1,0}(4_1), e^{\frac{2\pi i}{N + \frac{1}{2}}} \right) \right| = v_3 ||W_{a,2,1,0}(4_1)||$$

by Theorem 1 and the Sandwich theorem we have

$$\lim_{r \to \infty} \frac{2\pi}{r} \log |TV_r(S^3 \setminus W_{a,2,1,0}(4_1))| = v_3 ||W_{a,2,1,0}(4_1)||$$

2.4 Volume conjectures for $W_{a,2,c,d}(4_1)$

By the same method discussed in Section 2.1 for $t = e^{\frac{2\pi i}{N + \frac{1}{2}}}$, the $N$-th colored Jones polynomials for $W_{a,2,c,d}(4_1)$ is given by

$$J_N(W_{a,2,c,d}(4_1), t) = t^{\frac{N^2 - 1}{2} + N(N-1)} \sum_{n=0}^{N-1} \frac{t^{(2n+1)/2} - t^{-(2n+1)/2}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \cdot t^{\frac{n\pi}{2}} \cdot C(n, t; N) \cdot C(n, t^{-1}; N)$$

$$\times \prod_{\gamma=1}^{c} \prod_{j=1}^{n} \frac{(1 - t^{N-l_{\gamma,j}})(1 - t^{l_{\gamma,j} + j})}{1 - t^{j}} \times \prod_{\gamma=1}^{d} \prod_{j=1}^{n} \frac{(1 - t^{N-l_{\gamma,j}'})(1 - t^{-l_{\gamma,j}'} - j)}{1 - t^{-j}}$$

$$\times \left( t^{-k(2n+1)} \prod_{j=1}^{k} (1 - t^{2n+1} - j)(1 - t^{2n+1} + j) \right)$$

Let $D_3$ to be the set

$$D_3 = \left\{ (n,l_1, \ldots, l_c, l_1', \ldots, l_d', k) \mid \left| \frac{n}{N + \frac{1}{2}} - \frac{1}{2} \right|, \left| \frac{l_1}{N + \frac{1}{2}} - \frac{1}{4} \right|, \ldots, \left| \frac{l_d'}{N + \frac{1}{2}} - \frac{1}{4} \right|, \left| \frac{k}{N + \frac{5}{6}} \right| < \delta \right\}$$
Similar to the arguments in Section 2.1, we have

\[
J_N(W_{a, 2, c, d}(4_1), e^\frac{2\pi i}{N} z) \sim \frac{(-1)^a e^{-\frac{2\pi i}{N} \left( \frac{N^2 - 1}{2} c - d (N+1) \right)}}{\sin(\frac{\pi}{N+\frac{1}{2}})} \times \exp\left(-\varphi^h\left(2\pi i \left( \frac{\pi}{2(N+\frac{1}{2})} \right)\right)\right) \times \sum_{D_k} \exp\left(\varphi^h\left(\pi \frac{n + \frac{1}{2}}{N+\frac{1}{2}}\right)\right) \times \exp\left((N + \frac{1}{2}) \Phi_N \left( W_{a, 2, c, d}(4_1); \frac{n_1}{N + \frac{1}{2}}, \frac{l_1}{N + \frac{1}{2}}, \ldots, \frac{l_c}{N + \frac{1}{2}}\right)\right)
\]

where

\[
\Phi_N \left( W_{a, 2, c, d}(4_1); z_1, z_2, \ldots, z_{c+1}, z_{c+2}, \ldots, z_{c+d+1}, z_{c+d+2}\right) = \Phi_N \left( W_{a, 2, c, d}; z_1, z_2, \ldots, z_{c+1}, z_{c+2}, \ldots, z_{c+d+1}\right) + \Psi_N(4_1; z_1, z_{c+d+1})
\]

with

\[
\Psi_N(4_1; z_1, z_{c+d+1}) = \frac{1}{2\pi i} \left\{ a \left[ 2\pi i \left( z_1 - \frac{1}{2} \right) \right] \left[ 2\pi i \left( z_1 - \frac{1}{2} + \frac{1}{N + \frac{1}{2}} \right) \right] \right\} + \sum_{\gamma=1}^c \psi_{N, N}(z_1, z_{\gamma+1}) + \sum_{\gamma=1}^d \kappa_{N, N}(z_1, z_{\gamma+1})
\]

\[
\kappa_{N, N}(z_1, z_{\gamma+1}) = \frac{1}{2\pi i} \left\{ -\frac{2\pi i}{N + \frac{1}{2}} \left[ \varphi^h \left( \pi \frac{z_1 + \pi z_{\gamma+1} + \frac{1}{2}}{2 N + 1}\right) - \varphi^h \left( \pi \frac{z_1 + \pi z_{\gamma+1} \frac{1}{2}}{2 N + 1}\right) - \varphi^h \left( \pi \frac{z_1 + \pi z_{\gamma+1} + \frac{1}{2}}{2 N + 1}\right) \right]\right\}
\]

\[
\Psi_N(4_1; z_1, z_{c+d+2}) = \frac{1}{2\pi i} \left\{ \frac{1}{N + \frac{1}{2}} \left[ \varphi^h \left( \left( -z_{c+d+2} + 2 z_1 + \frac{3}{2 N + 1} \right) \pi \right) \right] - 2\pi i \left( 1 - 2 z_1 \right) z_{c+d+2}\right\}
\]

Take \(N \to \infty\), we have

\[
\Phi \left( W_{a, 2, c, d}(4_1); z_1, z_2, \ldots, z_{c+1}, z_{c+2}, \ldots, z_{c+d+1}, z_{c+d+2}\right) = \Phi \left( W_{a, 2, c, d}; z_1, z_2, \ldots, z_{c+1}, z_{c+2}, \ldots, z_{c+d+1}\right) + \Psi(4_1; z_1, z_{c+d+2})
\]

with

\[
\Phi \left( W_{a, 2, c, d}; z_1, z_2, \ldots, z_{c+1}, z_{c+2}, \ldots, z_{c+d+1}\right) = \frac{1}{2\pi i} \left\{ a \left[ 2\pi i \left( z_1 - \frac{1}{2} \right) \right]^2 \right\} + \sum_{\gamma=1}^c \psi(z_1, z_{\gamma+1}) + \sum_{\gamma=1}^d \kappa(z_1, z_{\gamma+1})
\]

\[17\]
\[\psi_i(z_1, z_{\gamma+1}) = \frac{1}{2\pi i} \left[ \text{Li}_2\left( e^{-2\pi iz_1-2\pi iz_{\gamma+1}} \right) - \text{Li}_2\left( e^{-2\pi iz_1+2\pi iz_{\gamma+1}} \right) \right. \\
+ \text{Li}_2\left( e^{2\pi iz_{\gamma+1}} \right) - \text{Li}_2\left( e^{2\pi iz_1+2\pi iz_{\gamma+1}} \right) + \text{Li}_2\left( e^{2\pi iz_1} \right) \] 
for \( \gamma = 1, \ldots, c, \)

\[\kappa_i(z_1, z_{\gamma+1}) = \frac{1}{2\pi i} \left[ -\text{Li}_2\left( e^{2\pi iz_{\gamma+1}} \right) + \text{Li}_2\left( e^{2\pi iz_{\gamma+1}+2\pi iz_1} \right) - \text{Li}_2\left( e^{-2\pi iz_{\gamma+1}} \right) - \text{Li}_2\left( e^{-2\pi iz_1} \right) \right. \\
+ \text{Li}_2\left( e^{2\pi iz_1+2\pi iz_{\gamma+1}} \right) + \text{Li}_2\left( e^{2\pi iz_1} \right) \] 
for \( \gamma = 1, \ldots, d, \)

\[\Psi(4_i; z_1, z_{c+d+2}) = \frac{1}{2\pi i} \left[ \text{Li}_2\left( e^{-2\pi iz_1+2\pi iz_{c+d+2}+4\pi iz_1} \right) - \text{Li}_2\left( e^{2\pi iz_1+2\pi iz_{c+d+2}+4\pi iz_1} \right) \right. \\
- 2\pi i(1 - z_1)z_{c+d+2} \] 
where the functions are defined on

\[D = \left\{ (z_1, z_2, \ldots, z_{c+1}, z_{c+2}, \ldots, z_{c+d+1}, z_{c+d+2}) \mid \begin{array}{c}
|z_1 - \frac{1}{2}|,
|z_2 - \frac{1}{2}|, \ldots, |z_{c+1} - \frac{1}{2}|, |z_{c+2} - \frac{1}{2}|, |z_{c+d+1} - \frac{5}{6}| < \epsilon
\end{array} \right\} \]

for some sufficiently small \( \epsilon > 0. \)

By considering the Taylor series expansion, we have

\[\exp\left(\left(N + \frac{1}{2}\right) \Phi_N(W_{a,2,c,d}; z_1, z_2, \ldots, z_{c+1}, z_{c+2}, \ldots, z_{c+d+1})\right) = E(W_{a,2,c,d}; z_1, z_2, \ldots, z_{c+1}, z_{c+2}, \ldots, z_{c+d+1})\]

\[\exp\left(\left(N + \frac{1}{2}\right) \Phi(W_{a,2,c,d}; z_1, z_2, \ldots, z_{c+1}, z_{c+2}, \ldots, z_{c+d+1})\right) \left(1 + O\left(\frac{1}{N + \frac{1}{2}}\right)\right),\]

where

\[E(W_{a,2,c,d}; z_1, z_2, \ldots, z_{c+1}, z_{c+2}, \ldots, z_{c+d+1}) = \exp\left(2\pi ai\left(z_1 - \frac{1}{2}\right)\right)\]

\[+ \sum_{\gamma=1}^{c} \left( \log(1 - e^{-2\pi iz_1-2\pi iz_{\gamma+1}}) - \log(1 - e^{-2\pi iz_{\gamma+1}}) \right) - \frac{1}{2} \log(1 - e^{2\pi iz_{\gamma+1}})\]

\[+ \frac{1}{2} \log(1 - e^{2\pi iz_1+2\pi iz_{\gamma+1}}) - \frac{1}{2} \log(1 - e^{2\pi iz_1})\]

\[+ \sum_{\gamma=1}^{d} \left( \log(1 - e^{2\pi iz_{\gamma+1}}) - \log(1 - e^{2\pi iz_1}) \right) - \frac{1}{2} \log(1 - e^{-2\pi iz_{\gamma+1}})\]

\[+ \frac{1}{2} \log(1 - e^{-2\pi iz_1})\]

\[- \frac{3}{2} \log(1 - e^{-2\pi iz_1}) \left(1 + O\left(\frac{1}{N + \frac{1}{2}}\right)\right)\]

Note that the critical point equations for \(\Phi(W_{a,2,c,d}; z_1, z_2, \ldots, z_{c+1}, z_{c+2}, \ldots, z_{c+d+1})\) are given by

\[\begin{align*}
0 &= 2\pi ai\left(z_1 - \frac{1}{2}\right) + \sum_{\gamma=1}^{c} \frac{\partial \psi_i(z_1, z_{\gamma+1})}{\partial z_1} + \sum_{\gamma=1}^{d} \frac{\partial \kappa_i(z_1, z_{\gamma+1})}{\partial z_1} + \frac{\partial \Psi(4_i; z_1, z_3)}{\partial z_1} \\
0 &= \frac{\partial \psi_i(z_1, z_{\gamma+1})}{\partial z_{\gamma+1}} \quad \text{for } \gamma = 1, \ldots, c \\
0 &= \frac{\partial \kappa_i(z_1, z_{\gamma+1})}{\partial z_{\gamma+1}} \quad \text{for } \gamma = 1, \ldots, d \\
0 &= \frac{\partial \Psi(4_i; z_1, z_{c+d+2})}{\partial z_{c+d+2}}
\end{align*}\]
Using the same arguments in Section 2.1, it is straightforward to verify that

\[(z_1, z_2, \ldots, z_c, z_{c+1}, \ldots, z_{c+d+2}) = \left(\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{4}, \frac{5}{6}\right)\]

is the solution of critical point of the Fourier coefficient

\[\Phi(W_{a,2,c,d}(4_1); z_1, z_2, \ldots, z_{c+1}, z_{c+2}, \ldots, z_{c+d+1}, z_{c+d+2}) = 8\pi i z_1\]

with critical value 22

\[
\Phi\left(W_{a,2,c,d}(4_1); \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{4}, \frac{5}{6}\right) = \frac{1}{2\pi i} \sum_{\gamma=1}^{c} \psi_{\gamma} \left(\frac{1}{2}, \frac{1}{4}\right) + \frac{1}{2\pi i} \sum_{\gamma=1}^{d} \kappa_{\gamma} \left(\frac{1}{2}, \frac{1}{4}\right) + \Psi\left(4_1; \frac{5}{6}\right) - 4\pi i
\]

Moreover, by direct computation, the Hessian of the potential function evaluated at the critical point is a \((c + d + 2) \times (c + d + 2)\) matrix given by

\[
\text{Hess}\left(\Phi\left(W_{a,2,c,d}(4_1); \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{4}, \frac{5}{6}\right)\right) = 2\pi i \begin{pmatrix}
(c + d)i - \frac{c - d}{2} + 2a & i & i & \ldots & i & 2 + 2\sqrt{3}i \\
i & 2i & 0 & \ldots & 0 & 0 \\
i & 0 & 2i & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
i & 0 & 0 & \ldots & 2i & 0 \\
2 + 2\sqrt{3}i & 0 & 0 & \ldots & 0 & \sqrt{3}i
\end{pmatrix}
\]

In particular, the determinant of the above matrix is given by

\[
\text{det}\left(\Phi\left(W_{a,2,c,d}(4_1); \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{4}, \frac{5}{6}\right)\right) = (2\pi i)^{c+d+2} \sqrt{3}i \begin{pmatrix}
(c + d)i - \frac{c - d}{2} + 2a & i & i & \ldots & i & 2 + 2\sqrt{3}i \\
i & 2i & 0 & \ldots & 0 & 0 \\
i & 0 & 2i & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
i & 0 & 0 & \ldots & 2i & 0 \\
2 + 2\sqrt{3}i & 0 & 0 & \ldots & 0 & \sqrt{3}i
\end{pmatrix} - (1)^{c+d+1}(2 + 2\sqrt{3}i) \neq 0
\]
Besides, by direct computation,

\begin{equation*}
E \left( W_{a,b,c,d} ; \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{4}, \frac{5}{6} \right) = \left( -\frac{1}{2} \right)^{c+d} e^{-\frac{7\pi(c-d)}{4}}
\end{equation*}

By the argument discussed in Section 2.3, we have

\begin{equation*}
J_N (W_{a,b,c,d}(1), e^{\frac{2\pi i}{2})}
\sim \frac{\sin \left( \frac{\pi}{N+\frac{1}{2}} \right)}{N \to \infty} \exp \left( -\frac{\pi}{2(N+\frac{1}{2})} \right) \exp \left( \frac{\pi}{2(N+\frac{1}{2})} \right)
\times \sum_{\ell} \sin \left( \frac{\pi}{N+\frac{1}{2}} \right) \exp \left( \frac{\pi}{2(N+\frac{1}{2})} \right)
\times \exp \left( \left( N + \frac{1}{2} \right) \Phi \left( W_{a,b,c,d}(1); \frac{n}{N+\frac{1}{2}}, \frac{l_1}{N+\frac{1}{2}}, \ldots, \frac{l_c}{N+\frac{1}{2}}, \frac{l'_1}{N+\frac{1}{2}}, \ldots, \frac{l'_d}{N+\frac{1}{2}} \right) \right)
\sim \frac{-\exp \left( \frac{\pi}{2(N+\frac{1}{2})} \right) \exp \left( \frac{\pi}{2(N+\frac{1}{2})} \right)}{N \to \infty}
\times \left( N + \frac{1}{2} \right)^{c+d+2} \int \sin \left( \pi z_1 \right) \exp \left( \frac{\pi}{2N+1} + 2\pi z_1 \right)
\times \exp \left( \left( N + \frac{1}{2} \right) \Phi \left( W_{a,b,c,d}(1); z_1, \ldots, z_{c+d+2} \right) - 8\pi z_1 \right) dz_1 \ldots dz_{c+d+2}
\end{equation*}

This completes the proof of Theorem 2. To prove Corollary 1 for $W_{a,b,c,d}(1)$, note that by Theorem 1, we have

\begin{equation*}
TV_e \left( S^3 \setminus W_{a,b,c,d}(1), e^{\frac{2\pi i}{2}} \right) = 2^{n-1} \left( \frac{2 \sin \left( \frac{\pi}{2} \right)}{\sqrt{\pi}} \right)^2 \sum_{1 \leq M \leq \omega - 1} |J_M (W_{a,b,c,d}(1), e^{\frac{2\pi i}{2}}) |^2
\geq 2^{n-1} \left( \frac{2 \sin \left( \frac{\pi}{2} \right)}{\sqrt{\pi}} \right)^2 |J_{\bar{N}} (W_{a,b,c,d}(1), e^{\frac{2\pi i}{2}}) |^2
\end{equation*}

Corollary 1 then follows from the argument in Section 2.3.
3 Iterated Whitehead double on Hopf link

In this Section, we study the asymptotics of the $(M_1, M_2)$-th colored Jones polynomials of the link $W_\alpha$ at $(N + \frac{1}{2})$-th root of unity.

3.1 Explicit formula for the $J_{M_1, M_2}(W_\alpha, t)$

First of all, the formula of the colored Jones polynomials of the link $W_\alpha$ is given as follows.

**Lemma 4.** The $(M_1, M_2)$-th colored Jones polynomials for the link $W_\alpha$ is given by

$$J_{M_1, M_2}(W_\alpha, t) = \sum_{n_1=0}^{2M_1-1} \sum_{n_2=0}^{2n_1} \cdots \sum_{n_{\alpha-1}=0}^{2n_\alpha-2} \sum_{n_1'=0}^{2M_1'-1} \sum_{n_2'=0}^{2n_1'-1} \cdots \sum_{n_{\beta-1}=0}^{2n_\beta-1} C(n_1, t; M_1) \prod_{\gamma=2}^{\alpha-1} C(n_\gamma, t; 2n_{\gamma-1} + 1)$$

$$\cdot C(n_1', t; M_2) \prod_{\gamma=2}^{\beta} C(n_\gamma', t; 2n_{\gamma-1} + 1) \cdot J_{2n_{\alpha-1}+1, 2n_{\beta}+1}(WL, t)$$

$$= \sum_{n_1=0}^{2M_1-1} \sum_{n_2=0}^{2n_1} \cdots \sum_{n_{\alpha-1}=0}^{2n_\alpha-2} \sum_{n_1'=0}^{2M_1'-1} \sum_{n_2'=0}^{2n_1'-1} \cdots \sum_{n_{\beta-1}=0}^{2n_\beta-1} C(n_1, t; M_1) \prod_{\gamma=2}^{\alpha-1} C(n_\gamma, t; 2n_{\gamma-1} + 1)$$

$$\cdot C(n_1', t; M_2) \prod_{\gamma=2}^{\beta} C(n_\gamma', t; 2n_{\gamma-1} + 1)$$

$$= \sum_{n_1=0}^{2M_1-1} \sum_{n_2=0}^{2n_1} \cdots \sum_{n_{\alpha-1}=0}^{2n_\alpha-2} \sum_{n_1'=0}^{2M_1'-1} \sum_{n_2'=0}^{2n_1'-1} \cdots \sum_{n_{\beta-1}=0}^{2n_\beta-1} C(n_1, t; M_1) \prod_{\gamma=2}^{\alpha-1} C(n_\gamma, t; 2n_{\gamma-1} + 1)$$

$$\cdot C(n_1', t; M_2) \prod_{\gamma=2}^{\beta} C(n_\gamma', t; 2n_{\gamma-1} + 1)$$

(34)

where

$$C(n, t; N) = \sum_{l=0}^{N-1-n} t^{-N(t+l)} \prod_{j=1}^{n} \frac{1 - t^{N-l-j} (1 - t^{l+j})}{1 - t^j}$$

**Proof.** The formula follows from the method described in [21]. □
Thus,

\[
J_{M_1, M_2}(W^\alpha_\beta, t)
= \sum_{s_1=0}^{2M_1-1} \sum_{n_1=2}^{2n_{0-2}} \sum_{s_2=0}^{2M_2-1} \sum_{n_2=2}^{2n_{0-2}'} \sum_{s_3=0}^{2n_{0-3}'} \sum_{n_3=0}^{2n_{0-3}} \cdot \sum_{t_1=0}^{M_1-1} \sum_{n_1=0}^{2n_{0-2}-n_1} \sum_{t_2=0}^{M_2-1} \sum_{n_2=0}^{2n_{0-2}'-n_1} \sum_{t_3=0}^{M_1-1} \sum_{n_3=0}^{2n_{0-3}'} \sum_{t_4=0}^{M_2-1-n_3} \sum_{n_4=0}^{2n_{0-3}'} \sum_{t_5=0}^{M_1-1} \sum_{n_1=0}^{n_1} \sum_{t_6=0}^{M_2-1} \sum_{n_2=0}^{n_2} \sum_{t_7=0}^{M_1-1} \sum_{n_3=0}^{n_3} \sum_{t_8=0}^{M_2-1} \sum_{n_4=0}^{n_4} \sum_{t_9=0}^{M_1-1} \sum_{n_1=0}^{n_1} \sum_{t_10=0}^{M_2-1} \sum_{n_2=0}^{n_2} \sum_{t_11=0}^{M_1-1} \sum_{n_3=0}^{n_3} \sum_{t_12=0}^{M_2-1} \sum_{n_4=0}^{n_4}
\]

\[\cdot \left( M_2^2 - \frac{M_2}{4} - \frac{1}{4} \right) + \left( (2n_1')^2 + 3n_1 \right) + \cdots + \left( (2n_{0-2}')^2 + 3n_{0-2} \right) \]

\[\cdot \left( M_2^2 - \frac{M_2}{4} - \frac{1}{4} \right) + \left( (2n_1')^2 + 3n_1 \right) + \cdots + \left( (2n_{0-2}')^2 + 3n_{0-2} \right) \]

\[\cdot t^{-M_1(t_1+n_1) - (2n_1+1)(t_2+n_2) - \cdots - (2n_{0-2}+1)(t_{a-1}+n_{a-1})} \]

\[\cdot t^{-M_2(t_1'+n_1') - (2n_1'+1)(t_2'+n_2') - \cdots - (2n_{0-2}'+1)(t_{a-1}'+n_{a-1}')} \]

\[\cdot \frac{\prod_{j=1}^{n_1} \left( 1 - t^{M_1-1-j_1} \right) \left( 1 - t^{1+j_1} \right)}{1 - t^{n_1}} \cdot \frac{\prod_{j=1}^{n_1} \left( 1 - t^{M_2-1-j_1'} \right) \left( 1 - t^{1+j_1'} \right)}{1 - t^{n_1'}} \cdot \frac{\prod_{j=1}^{n_1} \left( 1 - t^{2n_1-1-j_1} \right) \left( 1 - t^{1+j_1} \right)}{1 - t^{2n_1}} \cdot \frac{\prod_{j=1}^{n_1} \left( 1 - t^{2n_1'-1-j_1'} \right) \left( 1 - t^{1+j_1'} \right)}{1 - t^{2n_1'}} \cdot \frac{t^{\frac{M_1}{2}} - t^{-\frac{M_1}{2}}}{t^{\frac{M_2}{2}} - t^{-\frac{M_2}{2}}} \cdot t^{-2n_{0-2}'(t_{a-1}'+n_{a-1}')/2} \cdot t^{-(2n_{0-2}+1)(t_{a-1}+n_{a-1})/2} \cdot t^{-2n_1(t_{a-1}+n_{a-1})/2} \cdot t^{-(2n_1'+1)(t_{a-1}'+n_{a-1}')/2}
\]

(35)

### 3.2 Maximum estimation

By Lemma 3, when \( t = e^{\frac{2\pi i}{N+\frac{1}{2}}} \), we know that

\[
\limsup_{N \to \infty} \frac{2\pi}{N+\frac{1}{2}} \log |J_{N,N}(W^\alpha_\beta, t)| \leq v_3 ||S^3 \mid W^\alpha_\beta|| = (\alpha + \beta) \Vol(S^3 \mid WL)
\]

Furthermore, consider the term

\[
\prod_{j=1}^{n_1} \frac{\left( 1 - t^{M_1-1-j_1} \right) \left( 1 - t^{1+j_1} \right)}{1 - t^{n_1}}
\]

Then for \( s_1 = \lim_{N \to \infty} \frac{M_1}{N+\frac{1}{2}} \) sufficiently close to 1, in order to compute the asymptotic expansion formula, it suffices to consider those \( n_1 \) and \( l_1 \) with

\[
\frac{n_1}{N+\frac{1}{2}} \sim \frac{1}{2} \quad \text{and} \quad \frac{l_1}{N+\frac{1}{2}} \sim \frac{1}{4}
\]

In this case, we have \( 2n_1 + 1 \sim 1 \). By considering the term

\[
\prod_{j=2}^{n_2} \frac{(1 - t^{2n_1+1-l_2-j_2})(1 - t^{l_2+j_2})}{1 - t^{l_2}}
\]

to compute the asymptotic expansion formula, it suffices to consider those \( n_2 \) and \( l_2 \) with

\[
\frac{n_2}{N+\frac{1}{2}} \sim \frac{1}{2} \quad \text{and} \quad \frac{l_2}{N+\frac{1}{2}} \sim \frac{1}{4}
\]
Inductively, for \( \gamma = 3, \ldots, \alpha - 1 \), it suffices to consider

\[
\frac{n_\gamma}{N + \frac{1}{2}} \sim \frac{1}{2} \quad \text{and} \quad \frac{l_\gamma}{N + \frac{1}{2}} \sim \frac{1}{4}
\]

Similarly, for \( \gamma = 1, 2, \ldots, \beta \) and \( s_2 = \lim_{N \to \infty} \frac{M_2}{N + \frac{1}{2}} \) sufficiently close to 1, we only need to consider the terms with

\[
\frac{n'_\gamma}{N + \frac{1}{2}} \sim \frac{1}{2} \quad \text{and} \quad \frac{l'_\gamma}{N + \frac{1}{2}} \sim \frac{1}{4}
\]

From now on, we restrict our attention to the sum of the terms satisfying the above conditions.

### 3.3 Potential function for \( J_{M_1, M_2}(W^\alpha_{\beta}; e^{\frac{2\pi i}{N + \frac{1}{2}}}) \)

Define

\[
\Psi_N(WL; z_2\gamma - 1, z_2\gamma + 1, z_2\gamma + 2)
\]

\[
= \frac{1}{2\pi i} \left( -2\pi i(z_2\gamma - 1)(2\pi i z_2\gamma + 1 + 2\pi i z_2\gamma + 2) + \frac{2\pi i}{N + \frac{1}{2}} \left[ \varphi^h \left( z_2\gamma - 1\pi - \pi z_2\gamma + 1 - \pi z_2\gamma + 2 - \frac{\pi}{2N + 1} \right) \right. \\
- \varphi^h \left( z_2\gamma - 1\pi - \pi z_2\gamma + 2 + \frac{\pi}{2N + 1} \right) + \left. \varphi^h \left( \pi z_2\gamma + 1 + \frac{\pi}{2N + 1} \right) \right]
\]

From the computation in [22], we have

\[
C(n_1, t; M_1) = e^{\frac{2\pi i}{N + \frac{1}{2}} \frac{M_1(M_1 - 1)}{2}} \exp \left( \frac{N + \frac{1}{2}}{2\pi i} \Psi_N \left( WL; \frac{M_1}{N + \frac{1}{2}}, \frac{n_1}{N + \frac{1}{2}} \right) \right)
\]

\[
C(n'_1, t; M_2) = e^{\frac{2\pi i}{N + \frac{1}{2}} \frac{M_2(M_2 - 1)}{2}} \exp \left( \frac{N + \frac{1}{2}}{2\pi i} \Psi_N \left( WL; \frac{M_2}{N + \frac{1}{2}}, \frac{n'_1}{N + \frac{1}{2}} \right) \right)
\]

\[
C(n_\gamma, t; 2n_\gamma - 1 + 1) = e^{6\pi i \left( \frac{n_\gamma - 1}{N + \frac{1}{2}} \right)} \exp \left( \frac{N + \frac{1}{2}}{2\pi i} \left( 4\pi i \left( \frac{n_\gamma - 1}{N + \frac{1}{2}} \right)^2 \right) \right)
\]

\[
\exp \left( \frac{N + \frac{1}{2}}{2\pi i} \Psi_N \left( WL; \frac{2n_\gamma - 1}{N + \frac{1}{2}}, \frac{n_\gamma}{N + \frac{1}{2}} \right) \right)
\]

\text{for } \gamma = 2, \ldots, \alpha - 1

\[
C(n'_\gamma, t; 2n'_\gamma - 1 + 1) = e^{6\pi i \left( \frac{n'_\gamma - 1}{N + \frac{1}{2}} \right)} \exp \left( \frac{N + \frac{1}{2}}{2\pi i} \left( 4\pi i \left( \frac{n'_\gamma - 1}{N + \frac{1}{2}} \right)^2 \right) \right)
\]

\[
\exp \left( \frac{N + \frac{1}{2}}{2\pi i} \Psi_N \left( WL; \frac{2n'_\gamma - 1}{N + \frac{1}{2}}, \frac{n'_\gamma}{N + \frac{1}{2}} \right) \right)
\]

\text{for } \gamma = 2, \ldots, \beta
Define

\[
\Xi^\pm_N(WL; z_{2\alpha-3}, z_{2\beta-1}, z_\zeta, z_{\zeta+1}) = \frac{1}{2\pi i} \left\{ \pm (2\pi i (2z_{2\alpha-3})) \left( 2\pi i \left( z_\zeta - \frac{1}{2} \right) \right) \right.
\]
\[
- (2\pi i (2z_{2\beta-1}))(2\pi i z_\zeta + 2\pi i z_{\zeta+1})
\]
\[
+ \frac{2\pi i}{N + \frac{1}{2}} \left[ \phi^h \left( 2z_{2\beta-1} - \pi z_\zeta - \pi z_{\zeta+1} - \frac{\pi}{2N + 1} \right) \right.
\]
\[
- \phi^h \left( 2z_{2\beta-1} - \pi z_\zeta + \pi z_{\zeta+1} - \frac{\pi}{2N + 1} \right)
\]
\[
+ \phi^h \left( \pi z_{\zeta+1} + \frac{\pi}{2N + 1} \right) - \phi^h \left( \pi z_\zeta + \pi z_{\zeta+1} + \frac{\pi}{2N + 1} \right)
\]
\[
+ \phi^h \left( \pi z_\zeta + \frac{\pi}{2N + 1} \right) \}
\]

and

\[
\Phi^\pm_{M_1, M_2} (W_\beta^\alpha; z_1, z_2, \ldots, z_{2\alpha-3}, z_{2\alpha-2}, z_1', z_2', \ldots, z_{2\beta-1}', z_{2\beta}', z_\zeta, z_{\zeta+1})
\]
\[
= \Psi_N(WL; \frac{M_1}{N + \frac{1}{2}}; z_1, z_2) + \sum_{\gamma=1}^{\alpha-2} \Psi_N(WL; 2z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2})
\]
\[
+ \Psi_N(WL; \frac{M_2}{N + \frac{1}{2}}; z_1', z_2') + \sum_{\gamma=1}^{\beta-2} \Psi_N(WL; 2z_{2\gamma-1}', z_{2\gamma+1}', z_{2\gamma+2}')
\]
\[
+ \Xi^\pm_N(WL; z_{\alpha-1}, z_{\beta-1}, z_\zeta, z_{\zeta+1})
\]

Then we can write

\[
J_{M_1, M_2}(W_\beta^\alpha, e^{\frac{2\pi i}{N + \frac{1}{2}}}) \sim \frac{(-1)^N e^{\frac{2\pi i}{N + \frac{1}{2}} \left( \frac{M_1(M_1-1)}{2} + \frac{M_2(M_2-1)}{2} \right)}}{2 \sin \left( \frac{\pi}{N + \frac{1}{2}} \right)} (I_+ + I_-),
\]

where

\[
I_\pm = \sum_{n_1=0}^{2M_1-1} \sum_{n_2=0}^{2n_1} \cdots \sum_{n_{\alpha-2}=0}^{2n_{\alpha-3}} \sum_{n_{\alpha-1}=0}^{2n_{\alpha-2}} \sum_{n_1'=0}^{2n_1} \cdots \sum_{n_{\beta-2}=0}^{2n_{\beta-3}} \sum_{n_1'=0}^{2n_1} \cdots \sum_{n_{\zeta}=0}^{2n_{\zeta}}
\]
\[
\times \left[ \sum_{l_1=0}^{M_1-1-n_1} \sum_{l_2=0}^{2n_{\alpha-2}-n_2} \cdots \sum_{l_{\alpha-1}=0}^{2n_{\alpha-3}} \sum_{l_1'=0}^{M_2-1-n_1} \sum_{l_2'=0}^{2n_{\beta-2}-n_2} \cdots \sum_{l_{\beta-1}=0}^{2n_{\beta-3}} \sum_{l_1'=0}^{M_2-1-n_1} \sum_{l_2'=0}^{2n_{\beta-2}-n_2} \cdots \sum_{l_{\zeta}=0}^{2n_{\zeta}}
\]
\[
\times e^{2\pi i \left( \frac{n_\zeta}{N + \frac{1}{2}} \right)} \right., \exp \left( \frac{N + \frac{1}{2}}{2\pi i} \Phi^\pm_{M_1, M_2} \left( \frac{W_\beta^\alpha}{N + \frac{1}{2}}; \frac{n_1}{N + \frac{1}{2}} \frac{l_1}{N + \frac{1}{2}} \cdots \frac{n_{\alpha-1}}{N + \frac{1}{2}} \frac{l_{\alpha-1}}{N + \frac{1}{2}} \frac{n_1'}{N + \frac{1}{2}} \frac{l_1'}{N + \frac{1}{2}} \cdots \frac{n_{\beta-1}}{N + \frac{1}{2}} \frac{l_{\beta-1}}{N + \frac{1}{2}} \frac{n_{\zeta}}{N + \frac{1}{2}} \frac{l_{\zeta}}{N + \frac{1}{2}} \right) \right)
\]

Take \( N \to \infty \), we have

\[
\Phi^\pm (W_\beta^\alpha; z_1, z_2, \ldots, z_{2\alpha-3}, z_{2\alpha-2}, z_1', z_2', \ldots, z_{2\beta-1}', z_{2\beta}', z_\zeta, z_{\zeta+1})
\]
\[
= \Psi(WL; \frac{M_1}{N + \frac{1}{2}}; z_1, z_2) + \sum_{\gamma=1}^{\alpha-1} \Psi(WL; 2z_{2\gamma-3}, z_{2\gamma-1}, z_{2\gamma})
\]
\[
+ \Psi(WL; \frac{M_2}{N + \frac{1}{2}}; z_1', z_2') + \sum_{\gamma=1}^{\beta-2} \Psi(WL; 2z_{2\gamma-3}', z_{2\gamma-1}', z_{2\gamma}')
\]
\[
+ \Xi^\pm(WL; 2z_{2\alpha-3}, 2z_{2\beta-1}, z_\zeta, z_{\zeta+1})
\]
with

\[ \Psi(WL; z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2}) \]

\[ = \frac{1}{2\pi i} \left( (2\pi i (z_{2\gamma-1} - 1))(2\pi i z_{2\gamma+1} + 2\pi i z_{2\gamma+2}) \right) \]

\[ + \log \left( e^{2\pi i (z_{2\beta-1} - 1)} - 2\pi i z_{2\beta+1} \right) - \log \left( e^{2\pi i (z_{2\gamma-1} - 1)} - 2\pi i z_{2\gamma+2} \right) \]

\[ + \log \left( e^{2\pi i (z_{2\gamma - 1})} - 2\pi i z_{2\gamma+2} \right) + \log \left( e^{2\pi i (z_{2\gamma + 1})} \right) \]  

(46)

\[ \Xi^\pm(WL; z_{2\alpha-3}, z_{2\beta-1}, z_{\zeta}, z_{\zeta+1}) \]

\[ = \frac{1}{2\pi i} \left[ (2\pi i (z_{2\alpha-3} - 1))(2\pi i (z_{\zeta} - 1)) \right. \]

\[ - (2\pi i (z_{2\beta-1} - 1))(2\pi i z_{\zeta} + 2\pi i z_{\zeta+1}) \]

\[ + \log \left( e^{2\pi i (z_{2\beta - 1})} - 2\pi i z_{2\beta+1} \right) - \log \left( e^{2\pi i (z_{2\zeta - 1})} - 2\pi i z_{2\zeta+1} \right) \]

\[ + \log \left( e^{2\pi i z_{\zeta+1}} - 2\pi i z_{2\zeta+1} \right) + \log \left( e^{2\pi i z_{\zeta}} \right) \]  

(47)

Note that the function \( \Psi(WL; z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2}) \) is the potential function \( \Phi^{(s_1, s_2)}(z_1, z_2) \) in [22] with \( s_2 = 2z_{2\gamma-1}, z_1 = z_{2\gamma+1} \) and \( z_2 = z_{2\gamma+2} \). Besides, the function \( \Xi^\pm(WL; z_{2\alpha-1}, z_{2\beta-1}, z_{\zeta}, z_{\zeta+1}) \) is the potential function \( \Phi^{\pm(s_1, s_2)}(z_1, z_2) \) in [22] with \( s_1 = 2z_{2\alpha-1}, s_2 = 2z_{2\beta-1}, z_1 = z_{\zeta} \) and \( z_2 = z_{\zeta+1} \). By the maximum point estimation, it remains to study the potential function

\[ \Phi^\pm_{M_1, M_2}(W^\beta_\gamma; z_1, z_2, \ldots, z_{2\alpha-3}, z_{2\beta-2}, z_1', z_2', \ldots, z_{2\beta-1}', z_{\zeta}', z_{\zeta+1}') \]

defined on the domain

\[ D = \left\{ (z_1, z_2, \ldots, z_{2\alpha-3}, z_{2\beta-2}, z_1', z_2', \ldots, z_{2\beta-1}', z_{\zeta}', z_{\zeta+1}') \mid \right. \]

\[ \left| z_{2\gamma} - \frac{1}{2} \right|, \left| z_{2\gamma} - \frac{1}{4} \right| < \epsilon, \text{ where } \gamma = 1, \ldots, \alpha - 1 \]

\[ \left| z_{2\beta} - \frac{1}{2} \right|, \left| z_{2\beta} - \frac{1}{4} \right| < \epsilon, \text{ where } \gamma = 1, \ldots, \beta \]

\[ \left| z_{\zeta} - \frac{1}{2} \right|, \left| z_{\zeta+1} - \frac{1}{4} \right| < \epsilon \}

where \( \epsilon > 0 \) is any sufficiently small positive constant.

By considering the Taylor series expansion, we have

\[ \exp \left( \left(N + \frac{1}{2}\right) \Phi^\pm(W^\alpha_\gamma; z_1, z_2, \ldots, z_{2\alpha-3}, z_{2\alpha-2}, z_1', z_2', \ldots, z_{2\beta-1}', z_{\zeta}', z_{\zeta+1}') \right) \]

\[ = E(\Phi^\pm_{M_1, M_2}(W^\alpha_\gamma; z_1, z_2, \ldots, z_{2\alpha-3}, z_{2\alpha-2}, z_1', z_2', \ldots, z_{2\beta-1}', z_{\zeta}', z_{\zeta+1}')) \]

\[ \exp \left( \left(N + \frac{1}{2}\right) \Phi(W_{a, b, c, d}; z_1, z_2, \ldots, z_{c+1}, z_{c+2}, \ldots, z_{c+d+1}) \right) \left(1 + O\left(\frac{1}{N + \frac{1}{2}}\right)\right), \]
Altogether, we have

$$\hat{I}^{\pm}_N \sim \frac{(-1)^N e^{\pi i x} \left( \frac{M_1(M_1-1)}{4} + \frac{M_2(M_2-1)}{4} \right)}{2 \sin \left( \frac{x}{N+\frac{1}{2}} \right)} N^{2(\alpha+\beta)} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_{2\alpha+2\beta}=-\infty}^{\infty} \hat{F}^{\pm}(k_1, k_2, \ldots, k_{2\alpha+2\beta}) \quad (48)$$

where $\hat{F}^{\pm}(k_1, k_2, \ldots, k_{2\alpha+2\beta})$ is the $(k_1, k_2, \ldots, k_{2\alpha+2\beta})$-th Fourier coefficient given by

$$\hat{F}^{\pm}(k_1, k_2, \ldots, k_{2\alpha+2\beta}) = \int_D e^{\pm 2\pi i z \cdot \frac{1}{2}} E(z_1, z_2, \ldots, z_{2\alpha-3}, z_{2\alpha-2}, z_1', z_2', \ldots, z_{2\beta-1}', z_{2\beta}', z_1, z_2, \ldots, z_{2\beta-1}, z_1, z_2) \times$$

$$\exp \left( N + \frac{1}{2} \left( \Phi^{\pm}(W^{\beta}_{\alpha}; z_1, z_2, \ldots, z_{2\alpha-3}, z_{2\alpha-2}, z_1', z_2', \ldots, z_{2\beta-1}', z_{2\beta}', z_1, z_2, \ldots, z_{2\beta-1}, z_1, z_2) \right) + 2k_1 \pi i z_1 + \cdots + 2k_{2\alpha+2\beta} \pi i z \right) dz_1 \cdots dz \quad (49)$$

Note that the critical point equations of the potential function

$$\Phi^{\pm}(W^{\beta}_{\alpha}; z_1, z_2, \ldots, z_{2\alpha-3}, z_{2\alpha-2}, z_1', z_2', \ldots, z_{2\beta-1}', z_{2\beta}', z_1, z_2, \ldots, z_{2\beta-1}, z_1, z_2)$$
are given by

\[
0 = \frac{\partial \Psi(WL; \frac{N+1}{M}, z_1, z_2)}{\partial z_1} + \frac{\partial \Psi(WL; 2z_1, z_3, z_4)}{\partial z_1} = 0 \tag{50}
\]

\[
0 = \frac{\partial \Psi(WL; \frac{N+1}{M}, z_1, z_2)}{\partial z_2} \tag{51}
\]

\[
0 = \frac{\partial \Psi(WL; 2z_2\gamma-3, z_2\gamma-1, z_2\gamma)}{\partial z_{2\gamma-1}} + \frac{\partial \Psi(WL; 2z_2\gamma-1, z_2\gamma+1, z_2\gamma+2)}{\partial z_{2\gamma-1}} \quad \text{for } \gamma = 2, \ldots, \alpha - 2 \tag{52}
\]

\[
0 = \frac{\partial \Psi(WL; 2z_{2\alpha-3}, z_{2\alpha-3}, z_{2\alpha-2})}{\partial z_{2\alpha-3}} + \frac{\partial \Xi^+(WL; 2z_{2\alpha-3}, 2z_{2\beta-1}, z_\zeta, z_\zeta+1)}{\partial z_{2\alpha-3}} \tag{53}
\]

\[
0 = \frac{\partial \Psi(WL; 2z_2\gamma-3, z_2\gamma-1, z_2\gamma)}{\partial z_{2\gamma}} \quad \text{for } \gamma = 2, \ldots, \alpha - 1 \tag{54}
\]

\[
0 = \frac{\partial \Psi(WL; \frac{M}{N+M}, z'_1, z'_2)}{\partial z'_1} + \frac{\partial \Psi(WL; 2z'_1, z'_3, z'_4)}{\partial z'_1} \tag{55}
\]

\[
0 = \frac{\partial \Psi(WL; \frac{M}{N+M}, z'_1, z'_2)}{\partial z'_2} \tag{56}
\]

\[
0 = \frac{\partial \Psi(WL; 2z'_{2\gamma-3}, z'_{2\gamma-1}, z'_{2\gamma})}{\partial z'_{2\gamma-1}} + \frac{\partial \Psi(WL; 2z'_{2\gamma-1}, z'_{2\gamma+1}, z'_{2\gamma+2})}{\partial z'_{2\gamma-1}} \quad \text{for } \gamma = 2, \ldots, \beta - 1 \tag{57}
\]

\[
0 = \frac{\partial \Psi(WL; 2z'_{2\beta-3}, z'_{2\beta-1}, z'_{2\beta})}{\partial z'_{2\beta-1}} + \frac{\partial \Xi^+(WL; 2z_{2\alpha-3}, 2z_{2\beta-1}, z_\zeta, z_\zeta+1)}{\partial z'_{2\beta-1}} \tag{58}
\]

\[
0 = \frac{\partial \Psi(WL; 2z'_{2\gamma-3}, z'_{2\gamma-1}, z'_{2\gamma})}{\partial z'_{2\gamma}} \quad \text{for } \gamma = 2, \ldots, \beta - 1 \tag{59}
\]

\[
0 = \frac{\partial \Xi^+(WL; 2z'_{2\alpha-3}, 2z'_{2\beta-1}, z_\zeta, z_\zeta+1)}{\partial z_\zeta} \tag{60}
\]

\[
0 = \frac{\partial \Xi^+(WL; 2z'_{2\alpha-3}, 2z'_{2\beta-1}, z_\zeta, z_\zeta+1)}{\partial z_\zeta+1} \tag{61}
\]

### 3.4.4 Critical point of the potential function and the saddle point approximation

To solve this system of equations, recall from [22] that for the potential function $\Phi^{(1, i)}(z_1, z_2)$, when we put $z_1 = \frac{1}{2}$, the critical point equations become

\[
\begin{aligned}
\left(1 + \frac{B_2}{Z_2}\right)(1 + Z_2) &= B_2 \\
\left(1 + \frac{B_2}{Z_2}\right)(1 + Z_2) &= B_2 \\
(1 - \frac{B_2}{Z_2})(1 - Z_2) &= B_2
\end{aligned}
\]

Next, consider the equation

\[
\left(1 - \frac{B_2}{Z_2}\right)(1 - Z_2) = 2
\]

which is equivalent to

\[
Z_2^2 + (1 - B_2)Z_2 + B_2 = 0
\]

Let $Z_2(s) = \frac{-1 - B_2 + \sqrt{(1-B_2)^2 - 4B_2}}{2}$. Note that $Z_2(1) = i$. Furthermore, from [64],

\[
\left(\frac{B_2}{Z_2} + Z_2\right) = -1 + B_2, \tag{66}
\]

27
which implies
\[
(1 + \frac{B_2}{Z_2})(1 + Z_2) = (1 - \frac{B_2}{Z_2})(1 - Z_2) + 2 \left( \frac{B_2}{Z_2} + Z_2 \right) = 2B_2
\]

As a result, from (64) and (67), \((Z_1(s), Z_2(s)) = (-1, Z_2(s))\) is a holomorphic family of solution for (62) and (63) with \((Z_1(1), Z_2(1)) = (-1, i) = (e^{2\pi i(\frac{1}{2})}, e^{2\pi i(\frac{1}{2}))}\)). Let \(z_1(s), z_2(s)\) be holomorphic functions such that \(Z_1(s) = e^{2\pi i s}, z_1, z_2, \ldots, z_{2\alpha-3}, z_{2\alpha-2}, z_1, z_2, \ldots, z_{2\beta-1}, z_{2\beta}, z, z_{\zeta+1}\), note that if we put \(z_1 = z_3 = \frac{1}{2}, z_2 = z_2 \left( \frac{M_1}{N+1} \right), z_4 = \frac{1}{4}, \) we get
\[
\frac{\partial \Psi(WL; M_1, \frac{N+1}{2}, z_1, z_2)}{\partial z_1} + \frac{\partial \Psi(WL; 2z_1, z_3, z_4)}{\partial z_1} = -2(2\pi i)\left(\frac{1}{2} + \frac{1}{4}\right) - 2 \log(1 - e^{-2\pi i(3/4)}) + 2 \log(1 - e^{-2\pi i(1/4)}) = -2\pi i
\]

Besides, when \(z_{2\gamma-3} = z_{2\gamma-1} = z_{2\gamma+1} = \frac{1}{2}, z_{2\gamma} = z_{2\gamma+2} = \frac{1}{4}\), we have
\[
\frac{\partial \Psi(WL; 2z_{2\gamma-3}, z_{2\gamma-1}, z_{2\gamma})}{\partial z_{2\gamma-1}} + \frac{\partial \Psi(WL; 2z_{2\gamma-3}, z_{2\gamma+1}, z_{2\gamma+2})}{\partial z_{2\gamma-1}} = -2\pi i
\]
for \(\gamma = 2, \ldots, \alpha - 2\)
\[
\frac{\partial \Psi(WL; 2z_{2\gamma-3}, z_{2\gamma-1}, z_{2\gamma})}{\partial z_{2\gamma}} = 0
\]
for \(\gamma = 2, \ldots, \alpha - 1\)

and
\[
\frac{\partial \Psi(WL; 2z_{2\alpha-3}, z_{2\alpha-3}, z_{2\alpha-2})}{\partial z_{2\alpha-3}} + \frac{\partial \Xi(WL; 2z_{2\alpha-3}, 2z_{2\beta-1}, z_{\zeta}, z_{\zeta+1})}{\partial z_{2\alpha-3}} = 0
\]

By the same computation, when \(z_1' = z_3' = \frac{1}{4}, z_2' = z_2 \left( \frac{M_2}{N+2} \right), z_4' = \frac{1}{4}, \) we have
\[
\frac{\partial \Psi(WL; z_1', z_2')}{\partial z_1'} + \frac{\partial \Psi(WL; 2z_1', z_3', z_4')}{\partial z_1'} = -2(2\pi i)\left(\frac{1}{2} + \frac{1}{4}\right) - 2 \log(1 - e^{-2\pi i(3/4)}) + 2 \log(1 - e^{-2\pi i(1/4)}) = -2\pi i
\]
\[
\frac{\partial \Phi(WL; z_1', z_2')}{\partial z_2'} = 0
\]

Besides, when \(z_{2\gamma-3} = z_{2\gamma-1} = z_{2\gamma+1} = \frac{1}{2}, z_{2\gamma} = z_{2\gamma+2} = \frac{1}{4}\), we have
\[
\frac{\partial \Psi(WL; 2z_{2\gamma-3}, z_{2\gamma-1}, z_{2\gamma})}{\partial z_{2\gamma-1}} + \frac{\partial \Psi(WL; 2z_{2\gamma-3}, z_{2\gamma+1}, z_{2\gamma+2})}{\partial z_{2\gamma-1}} = -2\pi i
\]
for \(\gamma = 2, \ldots, \beta - 1\)
\[
\frac{\partial \Psi(WL; 2z_{2\gamma-3}, z_{2\gamma-1}, z_{2\gamma})}{\partial z_{2\gamma}} = 0
\]
for \(\gamma = 2, \ldots, \beta\)

and
\[
\frac{\partial \Psi(WL; 2z_{2\beta-3}, z_{2\beta-1}, z_{2\beta})}{\partial z_{2\beta-1}} + \frac{\partial \Xi(WL; 2z_{2\alpha-3}, 2z_{2\beta-1}, z_{\zeta}, z_{\zeta+1})}{\partial z_{2\beta-1}} = 0
\]

Finally, when \(z_{2\alpha-3} = z_{2\beta-1} = \frac{1}{2}, z_{\zeta} = \frac{1}{2}, z_{\zeta} = \frac{1}{4}, \) we have
\[
\frac{\partial \Xi(WL; 2z_{2\alpha-3}, 2z_{2\beta-1}, z_{\zeta}, z_{\zeta+1})}{\partial z_{\zeta}} = \frac{\partial \Xi(WL; 2z_{2\alpha-3}, 2z_{2\beta-1}, z_{\zeta}, z_{\zeta+1})}{\partial z_{\zeta+1}} = 0
\]
As a result, the point $\vec{z}_{M_1, M_2} = (z_1, \ldots, z_{2\alpha-2}, \hat{z}_1', \ldots, \hat{z}_{2\beta}', z_\zeta, z_{\zeta+1})$ with

\begin{align*}
  z_1 &= \frac{1}{2}, \quad z_2 = z_2 \left( \frac{M_1}{N + \frac{1}{2}} \right), \quad z_3 = z_5 = \cdots = z_{2\alpha-3} = \frac{1}{2}, \quad z_4 = z_6 = \cdots = z_{2\alpha-2} = \frac{1}{4} \\
  z_1' &= \frac{1}{2}, \quad z_2' = z_2' \left( \frac{M_2}{N + \frac{1}{2}} \right), \quad z_3' = z_5' = \cdots = z_{2\alpha-3}' = \frac{1}{2}, \quad z_4' = z_6' = \cdots = z_{2\beta}' = \frac{1}{4}
\end{align*}

is a critical point of the Fourier coefficient $\hat{F}(k_1, \ldots, k_{2\gamma-1+2\alpha-2})$ with $k_{2\gamma-1} = 1$ for $\gamma = 1, \ldots, \alpha - 2$, $k_{2\gamma-1+2\alpha-2} = 1$ for $\gamma = 1, \ldots, \beta - 1$ and $k_i = 0$ otherwise. Furthermore, we actually have

$$
\Phi^+(W^\alpha_\beta; \vec{z}_{M_1, M_2}) = \Phi^-(W^\alpha_\beta; \vec{z}_{M_1, M_2})
$$

with

\begin{align*}
\text{Re}\, \Phi^\pm(W^\alpha_\beta; \vec{z}_{M_1, M_2}) &= \text{Re} \left[ \Psi_N \left( WL; \frac{M_1}{N + \frac{1}{2}}, \frac{1}{2}, z \left( \frac{M_1}{N + \frac{1}{2}} \right) \right) + \sum_{\gamma=1}^{\alpha-2} \Psi_N \left( WL; 1, \frac{1}{2}, \frac{1}{4} \right) \\
&\quad + \Psi_N \left( WL; \frac{M_2}{N + \frac{1}{2}}, \frac{1}{2}, z \left( \frac{M_1}{N + \frac{1}{2}} \right) \right) + \sum_{\gamma=1}^{\beta-2} \Psi_N \left( WL; 1, \frac{1}{2}, \frac{1}{4} \right) \\
&\quad + \Xi^\pm \left( WL; 1, \frac{1}{2}, \frac{1}{4} \right) \right] \\
&= \frac{1}{2\pi} \left[ \text{Vol} \left( S^3 \setminus WL; s_1 = 1, s_2 = \frac{M_1}{N + \frac{1}{2}} \right) + \text{Vol} \left( S^3 \setminus WL; s_1 = 1, s_2 = \frac{M_2}{N + \frac{1}{2}} \right) \\
&\quad + (\alpha + \beta - 2) \text{Vol}(S^3 \setminus WL) \right]
\end{align*}

(69)

Furthermore, assume that the determinant of the Hessian of the potential function at the critical point is non-zero. Then the asymptotic expansion formula for $I^\pm$ is given by

\begin{align*}
I^\pm_{N \to \infty} &= N^{2(\alpha+\beta)} \int_D e^{2\pi i (\zeta_2 - \frac{1}{4})} \\
&\quad \times E(\Phi^\pm_{M_1, M_2}(W^\alpha_\beta; z_1, z_2, \ldots, z_{2\alpha-3}, z_2, \hat{z}_1', \hat{z}_2', \ldots, \hat{z}_{2\beta}', z_\zeta, z_{\zeta+1})) \\
&\quad \times \exp \left( (N + \frac{1}{2}) \Phi(W^\alpha_\beta; z_1, z_2, \ldots, z_{2\alpha-3}, z_2, z_1', z_2', \ldots, z_{2\beta}', z_\zeta, z_{\zeta+1}) \right) \\
&\quad + \sum_{\gamma=1}^{\alpha-2} 2\pi i z_{2\gamma-1} + \sum_{\gamma=1}^{\beta-1} 2\pi i z_2 \right) \, dz_1 \cdots dz_{\zeta+1}
\end{align*}

(70)

\begin{align*}
&\sim N^{-\alpha+\beta} E(\vec{z}_{M_1, M_2}) \\
&= (N + \frac{1}{2})^{\alpha+\beta} \text{exp}((N + \frac{1}{2}) \Phi^\pm_{M_1, M_2}(W^\alpha_\beta; \vec{z}_{M_1, M_2}))
\end{align*}

(71)

and

\begin{align*}
J_{M_1, M_2}(W^\alpha_\beta, e^{\frac{2\pi i}{N+\frac{1}{2}}})
&\sim \frac{(-1)^{N^2 \frac{2\pi i}{N+\frac{1}{2}}} (M_1(M_1-1) \cdot M_2(M_2-1))}{2 \sin(\frac{\pi}{N+\frac{1}{2}}) (I_+ + I_-)} \\
&\sim \frac{(-1)^{N^2 \alpha+\beta-1} e^{\frac{2\pi i}{N+\frac{1}{2}}} (M_1(M_1-1) + M_2(M_2-1))}{2} \\
&\times \left[ \frac{E^+(\vec{z}_{M_1, M_2})}{\sqrt{\det(-\text{Hess} \Phi^+_M_{M_1, M_2}(W^\alpha_\beta; \vec{z}_{M_1, M_2}))}} + \frac{E^-(\vec{z}_{M_1, M_2})}{\sqrt{\det(-\text{Hess} \Phi^-_{M_1, M_2}(W^\alpha_\beta; \vec{z}_{M_1, M_2}))}} \right] \times (N + \frac{1}{2})^{\alpha+\beta+1} \text{exp}((N + \frac{1}{2}) \Phi^+_M_{M_1, M_2}(W^\alpha_\beta; \vec{z}_{M_1, M_2}))
\end{align*}

(72)

29
In this section, we are going to compute the $(N,N)$-th colored Jones polynomials of the family of links $W_{0,2,1,0}(4_1)$, where the first and the second $N$ are the colors of the belt and the iterated Whitehead double of the figure eight knot respectively. Again, we use the method described in [24] to obtain the following lemma.

**Lemma 5.** For any $p \geq 0$, the unnormalized colored Jones polynomials of $W_{0,1,1,0}(W_{0,2,1,0}(4_1))$ and that of $W_{0,1,1,0}(W_{0,2,1,0}(4_1))$ are related by

$$J_{N,N}(W_{0,1,1,0}(W_{0,2,1,0}(4_1)), t) = \sum_{n_1=0}^{N-1} C(n_1, t; N) \cdot J_{N,2n_1+1}(W_{0,2,1,0}(4_1), t)$$

(73)

**4 Iterated Whitehead double of $W_{0,2,1,0}(4_1)$**

**4.1 Explicit formula for the $J_{N,N}(W_{0,1,1,0}(W_{0,2,1,0}(4_1)), t)$**

In this section, we are going to compute the $(N,N)$-th colored Jones polynomials of the family of links $W_{0,1,1,0}(W_{0,2,1,0}(4_1))$, where the first and the second $N$ are the colors of the belt and the iterated Whitehead double of the figure eight knot respectively. Again, we use the method described in [24] to obtain the following lemma.

**Lemma 5.** For any $p \geq 0$, the unnormalized colored Jones polynomials of $W_{0,1,1,0}(W_{0,2,1,0}(4_1))$ and that of $W_{0,1,1,0}(W_{0,2,1,0}(4_1))$ are related by

$$J_{N,N}(W_{0,1,1,0}(W_{0,2,1,0}(4_1)), t) = \sum_{n_1=0}^{N-1} C(n_1, t; N) \cdot J_{N,2n_1+1}(W_{0,2,1,0}(4_1), t)$$

(73)

**Inductively, we have**

$$J_{N,N}(W_{0,1,1,0}(W_{0,2,1,0}(4_1)), t)$$

(74)
In particular, by Lemma 4, we have

\[ J_{N,N}(W_{0,1.0}(W_{0,2.1.0}(4_1)), t) = \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{2n_1+1} \cdots \sum_{n_p+1=0}^{2n_{p-1}+1} \frac{2n_p+1}{2n_p+1} \cdots \sum_{n_1=0}^{2n_1+1} \sum_{n_2=0}^{2n_1+1} \cdots \sum_{n_p+1=0}^{2n_{p-1}+1} \sum_{k=0}^{2n_{p+1}+1} \frac{t^{N(2n_{p+1}+1)/2} - t^{-N(2n_{p+1}+1)/2}}{t^{\frac{k}{2}} - t^{-\frac{k}{2}}} \]

\[ = \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{2n_1+1} \cdots \sum_{n_p+1=0}^{2n_{p-1}+1} \sum_{l_1=0}^{t-2n_{p-1}-2n_p-2n_{p-1}} \sum_{l_2=0}^{t-2n_{p-1}-2n_p-2n_{p-1}} \cdots \sum_{l_p+1=0}^{t-2n_{p-1}-2n_p-2n_{p-1}} \sum_{k=0}^{2n_{p+1}+1} \frac{t^{N(2n_{p+1}+1)/2} - t^{-N(2n_{p+1}+1)/2}}{t^{\frac{k}{2}} - t^{-\frac{k}{2}}} \]

4.2 Maximum estimation

By Lemma 2 and 3, when \( t = e^{\frac{2\pi j}{N+N}} \), we know that

\[ \lim_{N \to \infty} \frac{2\pi}{N+N} \log |J_{N,N}(W_{0,1.0}(W_{0,2.1.0}(4_1)), t)| \leq C_3 ||S^p \setminus W_{0,2.1.0}(4_1)|| \]

Furthermore, by considering the term

\[ \prod_{j_1=1}^{n_1} \frac{1 - t^{N-1} - j_1}{1 - t^{\frac{1}{2}}} \]

to compute the asymptotic expansion formula, it suffices to consider those \( n_1 \) and \( l_1 \) with

\[ \frac{n_1}{N+N} \sim \frac{1}{2} \quad \text{and} \quad \frac{l_1}{N+N} \sim \frac{1}{4} \]

In this case, we have \( 2n_1 + 1 \sim 1 \). By considering the term

\[ \prod_{j_2=1}^{n_2} \frac{1 - t^{2n_1+1} - l_2 - j_2}{1 - t^{\frac{1}{2}}} \]

to compute the asymptotic expansion formula, it suffices to consider those \( n_2 \) and \( l_2 \) with

\[ \frac{n_2}{N+N} \sim \frac{1}{2} \quad \text{and} \quad \frac{l_2}{N+N} \sim \frac{1}{4} \]

Inductively, for \( \gamma = 3, \ldots, p+1 \), it suffices to consider

\[ \frac{n_\gamma}{N+N} \sim \frac{1}{2} \quad \text{and} \quad \frac{l_\gamma}{N+N} \sim \frac{1}{4} \]

In particular, for \( 2n_{p+1} \sim N \), by considering the term

\[ \left( t^{-k(2n_{p+1}+1)} \prod_{l=1}^{k} \frac{1 - t^{2n_{p+1}+1} - j_1}{1 - t^{\frac{1}{2}}} \right) \]

to compute the asymptotic expansion formula, it suffices to consider

\[ \frac{k}{N+N} \sim \frac{5}{6} \]

31
Next, define
\[
\Phi_{N,N}(W_{0,2,1,0}^{p}(W_{0,2,1,0}(4_{1})); z_{1}, z_{2}, \ldots, z_{2p+2}, z_{2p+3}) = \Phi_{N,N}(W_{L}; z_{1}, z_{2}) + \sum_{\gamma=1}^{p} \Psi_{N}(W_{L}; 2z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2}) + \Psi(4_{1}; 2z_{2p+1}, z_{2p+3})
\] (76)

where the functions \(\Phi_{N,N}(W_{L}, z_{1}, z_{2})\) and \(\Phi(4_{1}; 2z_{2p+1}, z_{2p+3})\) are the functions defined before with formulas

\[
\Phi_{N,N}(W_{L}; z_{1}, z_{2}) = \frac{1}{2\pi i} \left\{ \frac{2\pi i}{N + \frac{1}{2}} \left[ \varphi^{h} \left( \frac{N\pi}{N + \frac{1}{2}} - \pi z_{1} - \pi z_{2} - \frac{\pi}{2N + 1} \right) - \varphi^{h} \left( \frac{N\pi}{N + \frac{1}{2}} - \pi z_{2} - \frac{\pi}{2N + 1} \right) + \varphi^{h} \left( \frac{\pi z_{1} + \pi z_{2} + \frac{\pi}{2N + 1}}{2N + 1} \right) \right] \right\}
\] (77)

\[
\Phi(4_{1}; 2z_{2p+1}, z_{2p+3}) = \frac{1}{2\pi i} \left\{ \frac{1}{N + \frac{1}{2}} \left[ \varphi^{h} \left( \left( -z_{2p+3} + 2z_{2p+1} + \frac{3}{2N + 1} \right) \pi \right) - \varphi^{h} \left( \left( z_{2p+3} - 1 + 2z_{2p+1} - \frac{1}{2N + 1} \right) \pi \right) \right] - 2\pi i \left( 1 - 2z_{2p+1} \right) z_{2p+3} \right\}
\] (78)

and the function \(\Psi_{N}(W_{L}; 2z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2})\) is defined by

\[
\Psi_{N}(W_{L}; 2z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2}) = \frac{1}{2\pi i} \left\{ \frac{2\pi i}{N + \frac{1}{2}} \left( \frac{2\pi i}{N + \frac{1}{2}} - 1 \right) \left( \frac{2\pi i}{N + \frac{1}{2}} \right) \left( \frac{\pi}{N + \frac{1}{2}} + \frac{n}{N + \frac{1}{2}} \right) \right\}
\] (79)

Take \(N \to \infty\), we have

\[
\Phi(W_{0,1,1,0}^{p}(W_{0,2,1,0}(4_{1})); z_{1}, z_{2}, \ldots, z_{2p+2}, z_{2p+3}) = \Phi(W_{L}; z_{1}, z_{2}) + \sum_{\gamma=1}^{p} \Psi(W_{L}; 2z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2}) + \Psi(4_{1}; 2z_{2p+1}, z_{2p+3})
\] (80)

where the functions \(\Phi(W_{L}, z_{1}, z_{2})\) and \(\Phi(4_{1}; 2z_{2p+1}, z_{2p+3})\) are the functions defined before with formulas

\[
\Phi(W_{L}; z_{1}, z_{2}) = \frac{1}{2\pi i} \left\{ \text{Li}_{2} \left( e^{-2\pi i z_{1}} - 2\pi i z_{2} \right) - \text{Li}_{2} \left( e^{-2\pi i z_{2}} \right) + \text{Li}_{2} \left( e^{2\pi i z_{1}} \right) - \text{Li}_{2} \left( e^{2\pi i z_{2}} + 2\pi i z_{2} \right) + \text{Li}_{2} \left( e^{2\pi i z_{1}} \right) \right\}
\] (81)

\[
\Psi(4_{1}; 2z_{2p+1}, z_{2p+3}) = \frac{1}{2\pi i} \left\{ \text{Li}_{2} \left( e^{-2\pi i z_{2p+3} + 4\pi i z_{2p+1}} \right) - \text{Li}_{2} \left( e^{2\pi i z_{2p+3} + 4\pi i z_{2p+1}} \right) - 2\pi i \left( 1 - 2z_{2p+1} \right) z_{2p+3} \right\}
\] (82)
and the function $\Psi(WL; 2z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2})$ is given by

$$
\Psi(WL; 2z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2}) = \frac{1}{2\pi i} \left[ (-2\pi i (2z_{2\gamma-1} - 1)) (2\pi iz_{2\gamma+1} + 2\pi iz_{2\gamma+2}) 
+ \text{Li}_2 \left( e^{2\pi i (2z_{2\gamma-1} - 1) - 2\pi iz_{2\gamma+1} - 2\pi iz_{2\gamma+2}} \right) 
- \text{Li}_2 \left( e^{2\pi i (2z_{2\gamma-1} - 1) - 2\pi iz_{2\gamma+2}} \right) 
+ \text{Li}_2 \left( e^{2\pi iz_{2\gamma+1} + 2\pi iz_{2\gamma+2}} \right) + \text{Li}_2 \left( e^{2\pi iz_{2\gamma+1}} \right) \right] 
$$

(83)

Here the function $\Phi(W_{0,1,1,0}^p(W_{0,2,1,0}(4_1)); z_1, z_2, \ldots, z_{2p+2}, z_{2p+3})$ is defined on the domain

$$
D = \left\{ (z_1, z_2, \ldots, z_{2p+1}, z_{2p+2}, z_{2p+3}) \mid \left| z_{2\gamma-1} - \frac{1}{2}, \left| z_{2\gamma} - \frac{1}{4}, \left| z_{2p+3} - \frac{5}{6} \right| < \epsilon, \gamma = 1, \ldots, p + 1 \right. \right\}
$$

Also, note that the function $\Psi(WL; 2z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2})$ is the potential function $\Phi^{(1,s_2)}(z_1, z_2)$ in [22], with $s_2 = 2z_{2\gamma-1}, z_1 = z_{2\gamma+1}$ and $z_2 = z_{2\gamma+2}$.

Let $D_e$ to be the collection of all $(n_1, \ldots, n_{p+1}, l_1, \ldots, l_{p+1}, k)$ such that

$$
\left( \frac{n_1}{N + \frac{1}{2}}, \cdots, \frac{n_{p+1}}{N + \frac{1}{2}}, \frac{l_1}{N + \frac{1}{2}}, \cdots, \frac{l_{p+1}}{N + \frac{1}{2}}, \frac{k}{N + \frac{1}{2}} \right) \in D_e
$$

By direct computation and the maximum point estimation, we have

$$
J_{N,N}(W_{0,1,1,0}^p(W_{0,2,1,0}(4_1)), e^{N/h}) \sim \frac{N + \frac{1}{2}}{\pi} \exp \left( -\varphi^h \left( \frac{\pi}{2N + 1} \right) \right) \sum_{D_e} \sin \left( \frac{\pi}{N + \frac{1}{2}} \left( n_{p+1} + \frac{1}{2} \right) \right) \exp \left( \frac{\varphi^h}{\pi} \left( -\frac{2N - 4n - 2\pi}{2N + 1} \right) \right) 
\times e^{N/h} \left( \frac{1 + n + \sum_{l=1}^{n_{p+1}} \left( l + 1 + n_{p+1} \right) \varphi^h}{N + \frac{1}{2}} \right) \exp \left( (N + \frac{1}{2}) \Phi_{N,N} \left( W_{0,1,1,0}^p(W_{0,2,1,0}(4_1)); \frac{n_1}{N + \frac{1}{2}}, \cdots, \frac{n_{p+1}}{N + \frac{1}{2}}, \frac{l_1}{N + \frac{1}{2}}, \cdots, \frac{l_{p+1}}{N + \frac{1}{2}}, \frac{k}{N + \frac{1}{2}} \right) \right)
$$

(84)

4.3 Poisson summation formula and saddle point approximation

Note that the critical point equations of the potential function

$$
\Phi(W_{0,1,1,0}^p(W_{0,2,1,0}(4_1)); z_1, z_2, \ldots, z_{2p+2}, z_{2p+3})
$$

are given by

$$
\begin{align*}
0 &= \frac{\partial \Phi(WL; z_1, z_2)}{\partial z_1} + \frac{\partial \Phi(WL; 2z_1, z_3, z_4)}{\partial z_3} \\
0 &= \frac{\partial \Phi(WL; z_1, z_2)}{\partial z_2} \\
0 &= \frac{\partial \Phi(WL; 2z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2})}{\partial z_{2\gamma}} + \frac{\partial \Phi(WL; 2z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2})}{\partial z_{2\gamma+1}} \\
0 &= \frac{\partial \Phi(WL; 2z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2})}{\partial z_{2\gamma+2}} \\
0 &= \frac{\partial \Phi(WL; 2z_{2p+1}, z_{2p+2}, z_{2p+3})}{\partial z_{2p+1}} \\
0 &= \frac{\partial \Phi(WL; 2z_{2p+1}, z_{2p+2}, z_{2p+3})}{\partial z_{2p+3}} \\
\end{align*}
$$

(85) – (90)
Note that when \( z_1 = z_3 = \frac{1}{2}, z_2 = z_4 = \frac{1}{4} \), we have
\[
\frac{\partial \Phi(WL; z_1, z_2)}{\partial z_1} + \frac{\partial \Psi(WL; 2z_1, z_3, z_4)}{\partial z_1} = -2(2\pi i)\left(\frac{1}{4} + \frac{1}{4} \right) - 2\log(1 - e^{-2\pi i(3/4)}) + 2\log(1 - e^{-2\pi i(1/4)}) = -2\pi i
\]
\[
\frac{\partial \Phi(WL; z_1, z_2)}{\partial z_2} = 0
\]
Similarly, when \( z_{2\gamma-1} = z_{2\gamma-1} = z_{2\gamma+1} = \frac{1}{2}, z_{2\gamma} = z_{2\gamma+2} = \frac{1}{4} \), we have
\[
\frac{\partial \Psi(WL; 2z_{2\gamma-3}, z_{2\gamma-1}, z_{2\gamma})}{\partial z_{2\gamma-1}} + \frac{\partial \Psi(WL; 2z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2})}{\partial z_{2\gamma-1}} = -2\pi i \quad \text{for} \quad \gamma = 2, \ldots, p
\]
\[
\frac{\partial \Psi(WL; 2z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2})}{\partial z_{2\gamma+2}} = 0 \quad \text{for} \quad \gamma = 1, \ldots, p
\]
Finally, when \( z_{2p-1} = z_{2p+1} = \frac{1}{2}, z_{2p+2} = \frac{1}{4}, z_{2p+3} = \frac{5}{6} \), we have
\[
\frac{\partial \Psi(WL; 2z_{2p-1}, z_{2p+1}, z_{2p+2})}{\partial z_{2p+1}} + \frac{\partial \Psi(WL; 2z_{2p+1}, z_{2p+2}, z_{2p+3})}{\partial z_{2p+1}} = \frac{\partial \Psi(WL; 2z_{2p+1}, z_{2p+2}, z_{2p+3})}{\partial z_{2p+3}} = 0
\]
As a result, the point
\[
\left( \frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2}, \frac{1}{4}, \frac{5}{6} \right)
\]
is a critical point of the Fourier coefficient
\[
\frac{\partial}{\partial z_1} \Phi \left( W_0^{p} \omega_1, 0, (W_0, 2, 1, 0)(4_1); z_1, z_2, \ldots, z_{2p+2}, z_{2p+3} \right) + \left( \sum_{\gamma=1}^{p} 2\pi i z_{2\gamma-1} \right) - 8\pi i z_{2p+1}
\]
Moreover, the critical value is given by
\[
\Phi \left( W_0^{p} \omega_1, 0, (W_0, 2, 1, 0)(4_1); z_1, z_2, \ldots, z_{2p+2}, z_{2p+3} \right) + \left( \sum_{\gamma=1}^{p} \pi i \right) - 4\pi i
\]
\[
= (p + 1) \Phi \left( W_0^{p} \omega_1, 0, (W_0, 2, 1, 0)(4_1); z_1, z_2, \ldots, z_{2p+2}, z_{2p+3} \right) + \left( \sum_{\gamma=1}^{p} \pi i \right) - 4\pi i
\]
\[
= \frac{1}{2\pi} \left[ (p + 1) \text{Vol}(S^3 \setminus WL) + \text{Vol}(S^3 \setminus 4_1) + i \left( \frac{p + 1}{4} \right)^2 \right] + (p - 4)\pi i
\]
As a result, by the Poisson Summation formula (Proposition 4.6 in \[17\]), we have
\[
J_{N,N}(W_0^{p} \omega_1, 0, (W_0, 2, 1, 0)(4_1), e^{\frac{2\pi i}{N}}) \sim \frac{N + \frac{1}{2}}{\pi} \exp \left( -\frac{4\pi^2}{N + 1} \right) \int\int_{D} \sin(\pi z_{2p+1}) \exp \left( -\pi + \frac{\pi}{2N + 1} + 2\pi z_{2p+1} \right) e^{2\pi i \left( \frac{z_{1} + z_{2}}{2} + \sum_{\gamma=1}^{p} (z_{2\gamma+1} + z_{2\gamma+2}) \right)}
\]
\[
\times \exp \left( (N + 1)^2 \Phi_4, N, (W_0^{p} \omega_1, 0, (W_0, 2, 1, 0)(4_1); z_1, z_2, \ldots, z_{2p+3}) \right)
\]
\[
+ \left( \sum_{\gamma=1}^{p} 2\pi i z_{2\gamma-1} \right) - 8\pi i z_{2p+1} \right) d z_1 d z_2 \ldots d z_{2p+3}
\]
(91)

34
By considering the Taylor series expansion, we have

\[
\exp \left( \left( N + \frac{1}{2} \right) \Phi_{N,N} \left( W_{0,1,1,0}^p (W_{0,2,1,0}^p (4_1)); z_1, z_2, \ldots, z_{2p+3} \right) \right) = E(W_{0,1,1,0}^p (W_{0,2,1,0}^p (4_1)); z_1, z_2, \ldots, z_{2p+3})
\]

\[
\exp \left( \left( N + \frac{1}{2} \right) \Phi \left( W_{0,1,1,0}^p (W_{0,2,1,0}^p (4_1)); z_1, z_2, \ldots, z_{2p+3} \right) \right) \left( 1 + O \left( \frac{1}{N + \frac{1}{2}} \right) \right),
\]

where

\[
E(W_{0,1,1,0}^p (W_{0,2,1,0}^p (4_1)); z_1, z_2, \ldots, z_{2p+3})
\]

\[
= \exp \left( \log(1 - e^{-2\pi i z_1 - 2\pi i z_2}) - \log(1 - e^{-2\pi i z_2}) - \frac{1}{2} \log(1 - e^{2\pi i z_2}) \right.
\]

\[
+ \frac{1}{2} \log(1 - e^{2\pi i z_1 + 2\pi i z_2}) - \frac{1}{2} \log(1 - e^{2\pi i z_1})
\]

\[
+ \sum_{\gamma=1}^{p} \left( \frac{1}{2} \log(1 - e^{-2\pi i z_{2\gamma-1} - 2\pi i z_{2\gamma+2}}) - \frac{1}{2} \log(1 - e^{-2\pi i z_{2\gamma+2}}) - \frac{1}{2} \log(1 - e^{2\pi i z_{2\gamma+2}}) \right)
\]

\[
+ \frac{1}{2} \log(1 - e^{2\pi i z_{2\gamma+1} + 2\pi i z_{2\gamma+2}}) - \frac{1}{2} \log(1 - e^{2\pi i z_{2\gamma+1}})
\]

\[
- \frac{3}{2} \log(1 - e^{-2\pi i z_{2p+3} + 4\pi i z_{2p+1}}) - \frac{1}{2} \log(1 - e^{2\pi i z_{2p+3} + 4\pi i z_{2p+1}}) \right)
\]

with

\[
E \left( W_{0,1,1,0}^p (W_{0,2,1,0}^p (4_1)); \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2}, \frac{1}{4}, \frac{5}{6} \right) = -2^{-\frac{22}{15} \pi i} e^{-\frac{22}{27} \pi i}
\]

Assume that the determinant \( \text{Hess}(\Phi_{N,N} (W_{0,1,1,0}^p (W_{0,2,1,0}^p (4_1)); \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2}, \frac{1}{4}, \frac{5}{6}) \neq 0 \). Then by using the same argument as before, the asymptotic expansion formula for \( J_{N,N} (W_{0,1,1,0}^p (W_{0,2,1,0}^p (4_1)) \) is given by

\[
J_{N,N} (W_{0,1,1,0}^p (W_{0,2,1,0}^p (4_1)), e^{\frac{2\pi i}{2p+3}})
\]

\[
\sim \frac{N + \frac{1}{2}}{2\pi i} e^{-(p+1) \frac{2\pi i}{2p+3}} \left( N + \frac{1}{2} \right)^{-\frac{1}{2p+3}} \exp \left( -\frac{N + \frac{1}{2}}{2\pi} \frac{(p+1) \pi^2 i}{6} \right) \times \exp \left( \frac{(p+1)^{3/2} \pi}{2\sqrt{\text{det Hess}(\Phi (W_{0,1,1,0}^p (W_{0,2,1,0}^p (4_1)); \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2}, \frac{1}{4}, \frac{5}{6}))}}{2\pi i} \right)
\]

\[
\times \exp \left( \left( N + \frac{1}{2} \right)^{3/2} \exp \left( \frac{1}{(p+1)^{3/2} \pi} \right) \right)
\]

\[
\sim \frac{N + \frac{1}{2}}{2\pi i} \right) e^{-(p+1) \frac{2\pi i}{2p+3}} \right) -2 e^{-\frac{23}{27} \pi i} \pi^{(2p+3)/2} \exp \left( (N + \frac{1}{2}) \pi i \right)
\]

\[
\times \exp \left( \left( N + \frac{1}{2} \right)^{3/2} \exp \left( \frac{1}{(p+1)^{3/2} \pi} \right) \right)
\]

\[
\times \exp \left( \frac{N + \frac{1}{2}}{2\pi i} (v_3 ||W_{0,1,1,0}^p (W_{0,2,1,0}^p (4_1))|| + i \text{CS}(W_{0,1,1,0}^p (W_{0,2,1,0}^p (4_1)))) \right)
\]

where

\[
v_3 ||W_{0,1,1,0}^p (W_{0,2,1,0}^p (4_1))|| = (p+1) \text{Vol}(S^3 \setminus W_1 L) + \text{Vol}(S^3 \setminus 4_1)
\]

\[
\text{CS}(W_{0,1,1,0}^p (W_{0,2,1,0}^p (4_1))) = (p+1) \text{CS}(W_{0,1,1,0}) + \text{CS}(4_1) = (p+1) \text{CS}(W_{0,2,1,0}) = \frac{(p+1)^2}{4}
\]
Finally, when \( p = 1 \), the Hessians of the potential function is given by

\[
\text{Hess} \left( \Phi_{N,N} \left( W_{0,1,1,0}^1(W_{0,2,1,0}(41)) : \frac{1}{2} : \frac{1}{2} : \frac{1}{2} : \frac{1}{6} \right) \right) = 2\pi i \begin{pmatrix}
\frac{1}{2} + 3i & i & -1 - i & -2 - 2i & 0 \\
i & 2i & 0 & 0 & 0 \\
-1 - i & 0 & \frac{7}{2} + i & i & 2 + 2\sqrt{3}i \\
-2 - 2i & 0 & \frac{i}{2} & 2i & 0 \\
0 & 0 & 2 + 2\sqrt{3}i & 0 & \sqrt{3}i
\end{pmatrix}
\]

with determinant not equal to zero.

### 4.4 Generalized volume conjecture for \( W_{0,1,1,0}^p(W_{0,2,1,0}(41)) \)

Next, we apply the ‘continuity argument’ described in [22] to study the generalized volume conjecture for \( W_{0,1,1,0}^p(W_{0,2,1,0}(41)) \). Let \( s = \lim_{N \to \infty} \frac{M}{N + \frac{1}{2}} \). From direct computation, the potential function for colored Jones polynomials \( J_{N,M}(W_{0,1,1,0}^p(W_{0,2,1,0}(41)), e^{\frac{2\pi i}{N}}) \) is given by

\[
\Phi^{(1,s)} \left( W_{0,1,1,0}^p(W_{0,2,1,0}(41)); z_1, z_2, \ldots, z_{2p+2}, z_{2p+3} \right) = \Phi^{(1,s)}(WL; z_1, z_2) + \sum_{\gamma=1}^p \Psi(WL; 2z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2}) + \Psi(41; z_{2p+1}, z_{2p+3}) \tag{93}
\]

where

\[
\Phi^{(1,s)}(WL; z_1, z_2) = \frac{1}{2\pi i} \left[ -(2\pi i s(s_2 - 1))(2\pi i z_1 + 2\pi i z_2) \right. \\
\left. + \text{Li}_2 \left( e^{2\pi i(s_2 - 1) - 2\pi i z_1 - 2\pi i z_2} \right) - \text{Li}_2 \left( e^{2\pi i(s_2 - 1) - 2\pi i z_1} \right) \right. \\
\left. + \text{Li}_2 \left( e^{2\pi i z_2} \right) - \text{Li}_2 \left( e^{2\pi i z_1 + 2\pi i z_2} \right) + \text{Li}_2 \left( e^{2\pi i z_1} \right) \right]
\]

\[
\Psi(WL; 2z_{2\gamma-1}, z_{2\gamma+1}, z_{2\gamma+2}) = \frac{1}{2\pi i} \left[ -(2\pi i(2z_{2\gamma-1} - 1))(2\pi i z_{2\gamma+1} + 2\pi i z_{2\gamma+2}) \right. \\
\left. + \text{Li}_2 \left( e^{2\pi i(2z_{2\gamma-1} - 1) - 2\pi i z_{2\gamma+1} - 2\pi i z_{2\gamma+2}} \right) - \text{Li}_2 \left( e^{2\pi i(2z_{2\gamma-1} - 1) - 2\pi i z_{2\gamma+1}} \right) \right. \\
\left. + \text{Li}_2 \left( e^{2\pi i z_{2\gamma+1}} \right) - \text{Li}_2 \left( e^{2\pi i z_{2\gamma+1} + 2\pi i z_{2\gamma+2}} \right) + \text{Li}_2 \left( e^{2\pi i z_{2\gamma+1}} \right) \right]
\]

\[
\Psi(41; z_{2p+1}, z_{2p+3}) = \frac{1}{2\pi i} \left[ \text{Li}_2 \left( e^{-2\pi i z_{2p+3} + 4\pi i z_{2p+1}} \right) - \text{Li}_2 \left( e^{2\pi i z_{2p+3} + 4\pi i z_{2p+1}} \right) \right. \\
\left. - 2\pi i \left( 1 - 2z_{2p+1} \right) \right] \tag{96}
\]

Next, similar to Section 3.4.2 we put \( Z_1 = e^{2\pi i z_1}, Z_2 = e^{2\pi i z_2} \) and \( B_2 = e^{2\pi i s_2} \). From straightforward computation, the point

\[(z_1, z_2, \ldots, z_{2p+2}, z_{2p+3}) = \left( \frac{1}{2}, z_2(s), \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2}, \frac{1}{4}, \frac{1}{6} \right) \]

is a critical point for the Fourier coefficient

\[
\Phi^{(1,s)} \left( W_{0,1,1,0}^p(W_{0,2,1,0}(41)); z_1, z_2, \ldots, z_{2p+2}, z_{2p+3} \right) + \left( \sum_{\gamma=1}^p 2\pi i z_{2\gamma-1} \right) - 8\pi i z_{2p+1}
\]

with

\[
\Re \Phi^{(1,s)} \left( W_{0,1,1,0}^{p+1}(W_{0,2,1,0}(41)), \frac{1}{2}, z_2(s), \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2}, \frac{1}{4}, \frac{1}{6} \right) = \frac{1}{2\pi i} \left[ \text{Vol}(\mathbb{S}^3 \setminus WL; u_1 = 0, u_2 = 2\pi i(1 - s)) + p \text{Vol}(\mathbb{S}^3 \setminus WL) + \text{Vol}(\mathbb{S}^3 \setminus 41) \right]
\]

By the ‘continuity argument’, Equations (3) and (1) in Theorem 3 follow from the same arguments as before. Corollary 2 for \( \mathbb{S}^3 \setminus W_{0,1,1,0}(W_{0,2,1,0}(41)) \) then follows directly from the method described in Section 2.3.
A Colored Jones polynomials under the Hopf union

Recall that in the $N$-th Temperley-Lieb algebra $TL_N$ over $\mathbb{Z}[t^{\pm \frac{1}{2}}]$, the Jones Wenzl idempotent $f_N$ satisfies the properties that for any $b \in TL_N$, $f_N \cdot b = C_b f_N$ for some constant $C_b \in \mathbb{Z}[t^{\pm \frac{1}{2}}]$ (see for example Lemma 13.2 in [9]). Suppose $D_K$ is a 1-tangle such that the closure of $D_K$ is a knot diagram of the knot $K$. Then we have

\[
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{hopf.png}} \\
= C
\end{array}
\]

where the black box is the $N$-th Jones Wenzl idempotent and $C \in \mathbb{Z}[t^{\pm \frac{1}{2}}]$ is a constant. By closing up the tangles on both sides, we get

\[
C = \frac{J_N(K,t)}{J_N(U,t)} = \frac{J_N(K,t)}{[N]}
\]

where $U$ is the unknot. As a consequence,

\[
|J_{M_1,M_2}(K_1 \# W_0^0 \# K_2, t)| = \left| J_{M_1,M_2}(W_0^0, t) J_{M_1}(K_1, t) J_{M_2}(K_2, t) \right|^{[M_1][M_2]}
\]

By Lemma 14.2 in [9], we have

\[
|J_{M_1,M_2}(W_0^0, t)| = \left| [M_1][M_2] J_{M_2}(U, t) \right| = \left| [M_1][M_2] \right| = \left| t^{\frac{M_1 M_2}{2}} - t^{\frac{-M_1 M_2}{2}} \right|
\]

This completes the proof of Lemma [1]

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