Abstract

In this paper we give an overview of exactly solved edge-interaction models, where the spins are placed on sites of a planar lattice and interact through edges connecting the sites. We only consider the case of a single spin degree of freedom at each site of the lattice. The Yang–Baxter equation for such models takes a particular simple form called the star-triangle relation. Interestingly all known solutions of this relation can be obtained as particular cases of a single ‘master solution’, which is expressed through the elliptic gamma function and have continuous spins taking values on the circle. We show that in the low-temperature (or quasi-classical) limit these lattice models reproduce classical discrete integrable systems on planar graphs previously obtained and classified by Adler, Bobenko and Suris through the consistency-around-a-cube approach. We also discuss inversion relations, the physical meaning of Baxter’s rapidity-independent parameter in the star-triangle relations and the invariance of the action of the classical systems under the star-triangle (or cube-flip) transformation of the lattice, which is a direct consequence of Baxter’s $Z$-invariance in the associated lattice models.

Keywords: Yang–Baxter equation, star-triangle relation, classical discrete evolution equations, exactly solved lattice models, 3D consistency

* Dedicated to Professor Rodney Baxter on the occasion of his 75th birthday.
1. Introduction

The Yang–Baxter equation plays an exceptional role in statistical mechanics and quantum field theory. In particular, there exist many integrable models of statistical mechanics constructed from solutions of the Yang–Baxter equation. These models describe the interactions of discrete or continuous spins (or fields) arranged on a two-dimensional lattice. Here we only consider edge-interaction models, where the spins are placed on sites of the lattice and interact through edges connecting the sites. The Yang–Baxter equation for such models takes its simplest ‘star-triangular’ form [1]. The most important models in this class include the Kashiwara–Miwa [2] and chiral Potts [3–5] models where spins take a finite set of integer values (both models also contain the Ising model [6] and Fateev–Zamolodchikov $\mathbb{Z}_2$-model [7] as particular cases). There are also important continuous spin models, including the Zamolodchikov ‘fishing-net’ model [8], which describes certain planar Feynman diagrams in quantum field theory, and the Faddeev–Volkov model [9], connected with the quantization [10] of discrete conformal transformations [11].

Interestingly, all these models, describing rather different physical problems, can be obtained from particular cases of a single master solution of the star-triangle relation found in [12]. The master solution involves Boltzmann weights parameterized through the elliptic-gamma function [13, 14] and continuous spins taking values on a circle. It is worth noting that all the solutions mentioned above possess a positivity property and, therefore, can be directly interpreted as local Boltzmann weights for physical solvable edge interaction models on arbitrary planar graph. In this paper we review main properties of these models and establish their connection to classical discrete integrable evolution equations, previously obtained and classified by Adler, Bobenko and Suris (ABS) [15].

Before going into details of this correspondence it is useful to refer to other recent remarkable appearances of the star-triangle relation (and Yang–Baxter equation, in general) in different and seemingly unrelated areas of physics and mathematics. In particular, there are deep connections to the theory of elliptic hypergeometric functions [16–18], topological quantum field theory [19–21] and calculations of superconformal indices connected with electromagnetic dualities in 4D $\mathcal{N} = 1$ superconformal Yang–Mills theories [22]. Indeed, as found in [23–26], the 4D superconformal quiver gauge theories are closely related to previously known 2D lattice models [12, 27] and also lead to rather non-trivial new ones [28–31]. More generally, new advances were achieved in understanding the algebraic structure of solutions of the Yang–Baxter equation, see, e.g., [32–44]. Moreover, the 3D approach [45, 46] to quantum $R$-matrices resulted in an extremely concise expression [47, 48] of the higher-spin $R$-matrix of the XXZ model in terms of $q$-Hahn polynomials. Soon after it was shown [49] that this $R$-matrix happens to satisfy a stochasticity property and defines local transition probabilities of the most general integrable totally asymmetric simple exclusion process on a line.

The key to obtaining the classical integrable equations from the lattice spin models lies in the low-temperature (or quasi-classical) limit. Indeed, all our lattice spin models contain a parameter, which can be identified with the temperature (in the language of statistical mechanics) or with the Planck constant (in the language of Euclidean quantum field theory). We denote this parameter by the letter $\hbar$. Consider a general nearest neighbor edge-interaction model defined on a planar graph $\mathcal{G}$. Denote its set of sites (vertices) as $V(\mathcal{G})$ and the set of edge as $E(\mathcal{G})$. In the quasi-classical limit, $\hbar \to 0$, the appropriately scaled spin variables $\{s_i\}$, $i \in V(\mathcal{G})$ always become continuous. The leading asymptotics of the partition function (for fixed boundary conditions)
\[
Z = \exp\left(-\frac{A(x^{(cl)})}{\hbar}\right)(1 + O(\hbar)), \quad \hbar \to 0,
\]

is expressed in terms of an action \( A(x) \) evaluated on the solution of the classical equation of motion
\[
\frac{\partial A(x)}{\partial x_i} \bigg|_{x_i = x_i^{(cl)}} = 0, \quad i \in \mathcal{V}(\mathcal{G}).
\]

The action is defined as a sum over edges of the lattice
\[
A(x) = \sum_{(ij) \in \mathcal{E}(\mathcal{G})} \mathcal{L}_{ij}(x_i, x_j),
\]

where the function \( \mathcal{L}_{ij}(x_i, x_j) \), which can be identified with the Lagrangian density of the associated classical system, is determined by the leading order quasi-classical asymptotics of the edge Boltzmann weight. For each site \( i \in \mathcal{V}(\mathcal{G}) \) equation (1.2) gives a constraint that involves only the spin \( x_i \), and each of the nearest neighbor spins connected by an edge to site \( i \). These equations give a rather general form of the so-called discrete Laplace system of equations on \( \mathcal{G} \) [50]. Note, that the positivity property of the underlying lattice models in many cases automatically leads to a convexity property for the variational equations (1.2). From the classical theory side convex variational principles for the ABS equations were studied in [51].

For lattice models the star-triangle relation may be considered as an equation that connects the partition function of a 3-edge 'star' graph, consisting of one internal spin \( x_0 \) connected to three boundary spins \( x_1, x_2, x_3 \), and a 3-edge 'triangle' graph with vertices \( x_1, x_2, x_3 \). Since this relation defines an equality, in the leading order quasi-classical expansion (1.1) one obtains a classical star-triangle relation [10]
\[
A_\bullet(x^{(cl)}_0, x_1, x_2, x_3) = A_\circ(x_1, x_2, x_3),
\]

which equates the actions of the star and triangle graphs. Here \( x^{(cl)}_0 \) is the solution to the equation
\[
\frac{\partial A_\bullet(x)}{\partial x_0} \bigg|_{x_0 = x^{(cl)}_0} = 0.
\]

which is the only equation of motion (1.2) arising in the case of the 3-edge star graph. This equation provides a constraint on the four spin variables \( x_0, x_1, x_2, x_3 \). Its precise form is, of course, model-dependent. Quite remarkably, exactly the same constraints arise in the ABS classification of the classical discrete integrable equations on quad-graphs. More precisely, equation (1.5) can be interpreted [50] as the so-called three-leg form of the discrete evolution equation for an elementary quarilateral. We show that for all known solutions of the star-triangle equation the constraint equations (1.5) always reduces those appearing in the ABS classification list.

Another important property of the classical star-triangle relation (1.4), is that this relation implies the invariance of the action functional \( A(x) \), under 'star-triangle' transformations of the lattice [10, 12, 50], which is a natural counterpart of Baxter’s \( Z \)-invariance for lattice models. The classical star-triangle relation (1.4) and the associated constraint equation (1.5) are also related [50] to the 3D consistency relation [15] and more generally to the multiform Lagrangian structures and multidimensional consistency [50, 52], further studied in the recent papers [53–58].
This paper is structured as follows. Section 2 gives an overview of integrable models of statistical mechanics and their properties, including details of the inversion relations for edge interaction models. Section 3 describes how one obtains discrete integrable equations in general, by considering the quasi-classical limit of an edge interaction model whose Boltzmann weights possess the crossing symmetry. Special consideration are given to understanding physical regimes of the resulting equations. Section 4 contains a review of all exactly solved edge interaction models, including explicit definitions of the Boltzmann weights, the partition function per edge (in the thermodynamic limit), the corresponding classical Lagragian and its relation to the ABS list [15] of integrable quad equations. Details of calculations are presented in six appendices. The main results of the paper are briefly summarized in conclusion.

2. Star-triangle relation

This introductory section summarizes important facts about the star-triangle relation and solvable edge-interaction models on general planar graphs. It contains a brief review of relevant results of [10, 59–61], as well as some new additions to the inversion relation technique [62–64].

2.1. Edge-interaction models

A general solvable edge-interaction model on a planar graph can be defined in the following way [59, 60]. Consider a planar graph \( G \), of the type shown in figure 1, where its sites (or vertices) are drawn by open circles and the edges by bold lines. The same figure also contains another graph \( L \), shown by thin lines, which is the medial graph for \( G \). The faces of \( L \) are shaded alternatively; the sites of \( G \) are placed on the unshaded faces. We assume that for each line of \( L \) one can assign a direction, so that all the lines head generally from the bottom of the

![Figure 1](image-url). The planar graph \( G \) (shown by open circles and bold edges) and its medial graph \( L \) (shown by thin edges and alternatively shaded faces). Reproduced with permission from [10], © 2007 Elsevier.
graph to the top. They can go locally downwards, but there can be no closed directed paths in \( L \). This means that one can always distort \( L \), without changing its topology, so that the lines always head upwards\(^4\). For further reference, let \( F(\mathcal{G}) \), \( E(\mathcal{G}) \) and \( V(\mathcal{G}) \) denote respectively the set of faces, edges and sites (vertices) of \( \mathcal{G} \), and \( V_{\text{int}}(\mathcal{G}) \) the set of interior sites of \( \mathcal{G} \). The latter correspond to interior faces of \( L \) (with a closed boundary).

Now we define a statistical mechanical spin model on \( \mathcal{G} \). To each line \( \ell \) of \( L \) associate its own ‘rapidity’ variable \( p_\ell \), taking real values. At each site \( i \) of \( \mathcal{G} \) place a spin \( s_i \), which take values in some set. For the purpose of this introduction, it is convenient to assume that the spins are discrete and take a finite number \( N \) of different values \( s_i = 0, 1, \ldots, N - 1 \). In the following sections we will also consider discrete spins, taking arbitrary integer values \( s_i \in \mathbb{Z} \) and continuous spins, taking arbitrary values \( s_i \in \mathbb{R} \) on the real line.

Two spins interact only if they are connected by an edge. This means that each edge is assigned a Boltzmann weight that depends only on spins at the ends of the edge. Usually, this weight depends on some global parameters of the model (for instance, the temperature-like variables), which are the same for all edges. Here we consider the case when the edge weight also depends on a local parameter, given by the difference of two rapidity variables associated with the edge. The detailed construction is as follows. The edges of \( \mathcal{G} \) are either of the first type in figure 2, or the second, depending on the arrangement of the directed rapidity lines with respect to the edges. For each edge introduce a ‘rapidity difference variable’ \( \alpha_e \) defined as

\[
\alpha_e = \begin{cases} 
 p - q, & \text{for an edge of the first type,} \\
 \eta - p + q, & \text{for an edge of the second type,} 
\end{cases}
\]

where \( p \) and \( q \) are the rapidities, arranged as in figure 2. The constant \( \eta \) is a model-dependent parameter. To each edge assign a Boltzmann weight factor \( W(\alpha_e \mid a, b) \), which depends on the rapidity difference variable \( \alpha_e \) and the spins \( a, b \) at the ends of the edge. Here we consider ‘reflection-symmetric’ models\(^6\).

\(^4\) This assumption puts some restrictions on the topology of the planar graph \( \mathcal{G} \), but still allows enough generality for our considerations here.

\(^5\) The only known solvable edge-interaction model which does not have this ‘difference property’ is the chiral Potts model [4, 65]. Its quasi-classical limit has been studied in [66] and will not be considered here.

\(^6\) However there are also the Gamma function models with asymmetric Boltzmann weights, which are separately addressed in section 4.6.
where the weight $a_{ab}$ is unchanged by interchanging the spins $a$ and $b$. The property that the weight functions for the edges of two types in figure 2 are obtained from each other by the transformation $\alpha \to \eta - \alpha$ of rapidity difference variable $\alpha$ is called the crossing symmetry. By this reason the parameter $\eta$ in (2.1) is usually called the ‘crossing parameter’.

In general, there may also be a single-spin self-interaction with a rapidity-independent weight $S(a)$ for each spin $a$. The partition function is defined as

$$Z = \sum_{\{\sigma\}} \prod_{i} S(\sigma_{i}) \prod_{ij} W(\alpha_{ij} | \sigma_{i}, \sigma_{j}),$$

where the first product is over all sites $i \in V(\mathcal{G})$ and the second is over all edges $(ij) \in E(\mathcal{G})$. The sum is taken over all possible values of the interior spins (in the case of continuous spins, the sum is replaced by an integral). The boundary spins are kept fixed.

### 2.2. Star-triangle relation

Integrability of the model requires that the Boltzmann weights satisfy the star-triangle relation [1]. For the reflection-symmetric case (2.2) this relation reads

$$\sum_{\sigma} S(\sigma_{i}) W(\sigma_{i} - \alpha_{1} | a, \sigma) W(\sigma_{i} - \alpha_{2} | b, \sigma) W(\sigma_{i} - \alpha_{3} | c, \sigma) = R_{123} W(\alpha_{1} | b, c) W(\alpha_{2} | a, c) W(\alpha_{3} | a, b),$$

where

$$\alpha_{1} + \alpha_{2} + \alpha_{3} = \eta,$$

and $R_{123}$ is some scalar factor independent of the spins $a, b, c$. For continuous spins the sum in (2.4) is replaced by an integral. The star-triangle relation equates partition functions of the ‘star’ and ‘triangle’ graphs shown in figure 3, where the external spins $a, b$ and $c$ are fixed. The variables $\alpha_{1}, \alpha_{2}, \alpha_{3}$ in (2.4) are calculated, according to the rule (2.1), from the rapidity variables $p_{1}, p_{2}, p_{3}$ shown in figure 3

$$\alpha_{1} = p_{2} - p_{3}, \quad \alpha_{2} = \eta - p_{1} + p_{3}, \quad \alpha_{3} = p_{1} - p_{2}.$$
2.3. The factor $R_{123}$

As shown in [61, 67] the factor $R_{123}$ appearing in (2.4) can be conveniently expressed in terms of the weights $W(\alpha \mid a, b)$. The result applies in the case of discrete spins. It is based on a property that the quantity

$$ P(\alpha) = \prod_b W(\alpha \mid a, b), $$

is spin-independent, i.e., the same for all values of the spin $a = 0, 1, \ldots, N - 1$. It seems that this additional requirement is not a simple corollary of the symmetry (2.2) or the star-triangle (2.4) relations themselves, but it is certainly true for all their currently known discrete spin solutions, particularly for those considered in this paper. Define the function

$$ f(\alpha)^N = P(\alpha)^{-1} \det S(a) W(\eta - \alpha) \mid a, b\rangle_{0 \leq a, b \leq N-1}. $$

Now, regard the spin $c$ as fixed and consider each side of (2.4) as the element $(a, b)$ of some matrix. Taking the determinant of this matrix one obtains (to within an undetermined factor of an $N$th root of unity)

$$ R_{123} = f(\alpha)f(\alpha_2)/(\eta - \alpha_3). $$

Because of (2.7) the result is independent of the value of the fixed spin $c$. Repeat the same calculations two more times, replacing the spin $c$ with the spin $a$ or $b$. One obtains another two expressions for $R_{123}$, similar to (2.9), but with permuted indices 1, 2, 3 in the rhs. All three expressions are consistent only if the quantity

$$ \kappa^2 = f(\alpha)f(\eta - \alpha) $$

is independent of the variable $\alpha$. Taking this into account one can write (2.9) in a symmetric form

$$ R_{123} = f(\alpha_1)f(\alpha_2)f(\alpha_3)/\kappa^2. $$

The results (2.10) and (2.11) hold for any solutions of the star-triangle relation, having the property (2.7).

The quantity (2.10) is the ‘rapidity independent parameter’, defined by Baxter in [61]. To be more precise, Baxter defined the ‘invariant’ $I = \kappa^2/N$ (see equation (13) of [61]) and showed that $I = 1$ for self-dual and/or critical models. Here we give a different physical interpretation to the quantity $\kappa$ by identifying it with a single-site factor in the asymptotic
expression for the partition function in the large-lattice limit (see equation (2.25) below).

From (2.8) and (2.10) it is easy to see that $\kappa_i$ linearly depends on the normalization of the site weights $S(a)$, but does not depend on the normalization of the edge weights $W(\alpha | a, b)$. Indeed, it scales linearly with the site weights

$$S(a) \to \lambda S(a), \quad \kappa_i \to \lambda \kappa_i, \quad (2.12)$$

but remains unchanged if the edge weight $W(\alpha | a, b)$ is multiplied by any factor independent of the spins $a, b$.

2.4. Inversion relations

For all models considered here that satisfy the crossing symmetry, the weights can be normalized so that they satisfy

(a) the boundary conditions

$$W(\alpha | a, b)|_{\alpha=\eta} = \delta_{ab}/S(a), \quad W(\alpha | a, b)|_{\alpha=0} = 1, \quad \forall a, b, \quad (2.13)$$

Together with the star-triangle relations these conditions imply

(b) the inversion relations

$$W(\alpha | a, b)W(-\alpha | a, b) = 1, \quad \forall a, b, \quad (2.14)$$

$$S(a)\sum_c W(\eta + \alpha | a, c)S(c)W(\eta - \alpha | c, b) = f(\alpha)f(-\alpha)\delta_{ab}. \quad (2.15)$$

Note that (2.13) and (2.10) imply

$$f(0) = 1, \quad f(\eta) = \kappa^2. \quad (2.16)$$

The two inversion relations (2.14) and (2.15) are corollaries of the star-triangle relation (2.4) and boundary conditions (2.13). Indeed, comparing both sides of (2.4) with $\alpha_3 = \eta$ and taking into account (2.13), one concludes that the product $W(\alpha_1 | a, c)W(-\alpha_1 | a, c)$ is
independent of the spins $a$, $c$. Without loss of generality this product could be normalized by the condition (2.14). Substituting this condition back into (2.4) (still keeping $\alpha_1 = \eta$) and using (2.11) and (2.16), one immediately obtains (2.15). The inversion relations (2.14) and (2.15) are represented pictorially in figures 4 and 5. They can be used to calculate the partition function (2.3) in the large-lattice limit. For example, for a regular square lattice of $M$ sites there are only two different rapidities $p$ and $q$. Correspondingly, half of the edges have the rapidity difference variable (2.1) equal to $\alpha = p - q$ and the other half to $\eta - p + q = \eta - \alpha$. Let

$$\kappa(\alpha) = Z^{1/M}, \quad M \to \infty, \quad (2.17)$$

be the partition function per site, then one can show that [62–64]

$$\kappa(\alpha)\kappa(-\alpha) = f(\alpha)f(-\alpha), \quad \kappa(\alpha) = \kappa(\eta - \alpha). \quad (2.18)$$

Together with an appropriate analyticity assumption (typically log $\kappa(\alpha)$ is analytic and bounded in a suitable domain including the segment $0 \leq \alpha \leq \eta$) these equations uniquely determine $\kappa(\alpha)$, see [62–64] for further details.

We have found that for all models with crossing symmetry considered here (we believe that it is a general property of solvable edge-interaction models), the quantity $\kappa(\alpha)$ can be factorized as

$$\kappa(\alpha) = \kappa_s \kappa_e(\alpha)\kappa_e(\eta - \alpha), \quad (2.19)$$

where $\kappa_s$ is defined in (2.10) and the function $\kappa_e(\alpha)$ satisfies the functional equations

$$\kappa_e(\alpha)\kappa_e(-\alpha) = 1, \quad \kappa_e(\eta - \alpha)/\kappa_e(\alpha) = f(\alpha)/\kappa_s, \quad (2.20)$$

$$\kappa_e(\eta - \alpha)\kappa_e(\eta + \alpha) = f(\alpha)f(-\alpha)/\kappa_s^2. \quad (2.21)$$

where the third equation is a corollary of the first two. It is easy to see that if $\kappa_s$ and $\kappa_e(\alpha)$ satisfy (2.10) and (2.20) then $\kappa(\alpha)$ solves (2.18).

There are exactly two edges (one of each type) for each site of a regular square lattice. Correspondingly, the partition function per site (2.19) is a product of the rapidity independent single-site factor $\kappa_s$ and two single-edge factors $\kappa_e(\alpha)$ and $\kappa_e(\eta - \alpha)$ (see equation (2.25) below for the generalization of (2.19) for an arbitrary graph).

### 2.5. Canonical normalization of the Boltzmann weights

It is possible to further refine the normalization of the weights to simplify the scalar factors in the star-triangle relation (2.4), and in the second inversion relation (2.15). Indeed, it is easy to see that if one rescales the weight functions

$$W(\alpha \mid a, b) \to \mathcal{W}(\alpha \mid a, b) = \frac{1}{\kappa_e(\alpha)} W(\alpha \mid a, b), \quad S(a) \to S(a) = \frac{1}{\kappa_s} S(a), \quad (2.22)$$

in the definitions (2.8), (2.10), (2.11) and (2.20) then all the associated scalar factors become equal to one,

$$\hat{\kappa}_s = 1, \quad \hat{R}_{123} \equiv \hat{f}(\alpha) \equiv \hat{\kappa}_e(\alpha) \equiv 1. \quad (2.23)$$

In what follows we will refer to this distinguished normalization as the canonical normalization of weights. For further reference, define a rescaled partition function $\tilde{Z}$ obtained from (2.3) by the substitution (2.22)

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The partition function (2.24) possesses remarkable invariance properties [59, 60, 68]. It remains unchanged (up to simple $f(\alpha_{ij})$ and $\kappa_{ij}$ factors) by continuously deforming the lines of $\mathcal{L}$ with their boundary positions kept fixed, as long as the graph $\mathcal{G}$ remains directed. In particular, no closed directed paths are allowed to appear. It is easy to see that all such transformations reduce to a combination of the moves shown in figures 3–5, corresponding to the star-triangle (2.4) and inversion relations (2.14) and (2.15). In general the partition function acquires simple $f(\alpha_{ij})$ and $\kappa_{ij}$ factors under these moves, however for the canonical normalization (2.24) the invariance is strict (all extra factors become equal to one in this case, see (2.23)). Given that the graphs $\mathcal{L}$ and $\mathcal{G}$ can undergo rather drastic changes, the above $Z$-invariance statement is rather non-trivial.

The partition function (2.24) depends on the exterior spins and the rapidity variables $p_1, p_2, \ldots, p_L$. Of course, it also depends on the graph $\mathcal{L}$, but only on a relative ordering (permutation) of the rapidity lines at the boundaries and not on their arrangement inside the graph. Naturally, this graph can be identified with an element of the permutation group. Then the partition function $Z$ can be regarded as a group representation matrix, acting non-trivially on the spins at the lower and upper boundaries (it acts as an identity on the leftmost and rightmost spins in figure 1, corresponding to unbounded faces).

Although the above $Z$-invariance holds for arbitrary values of rapidity variables $p_1, p_2, \ldots, p_L$, it is convenient to distinguish the physical regime, when all the rapidity differences $\alpha_{ij}$ lie in the interval $0 < \alpha_{ij} < \eta$. In this case all Boltzmann weights entering (2.3) and (2.24) are real and positive and the corresponding partition functions have a straightforward interpretation in statistical mechanics. Note that for the physical regime the graph $\mathcal{L}$ cannot contain more than one intersection for the same pair of the rapidity lines.

Consider a generic graph $\mathcal{G}$ with a large number of sites, $M$, and a large number of edges, $N \sim 2M$. Then the number of boundary sites is on the order of $2M^{1/2}$. Assume that the corresponding boundary spins are kept finite. Then, following [59], one can show that the leading asymptotics of the partition function (2.24) at large $M$ has the form

$$
\log Z = M \log \kappa_{i} + \sum_{(ij) \in E(\mathcal{G})} \log \kappa_{ij}(\alpha_{ij}) + O(\sqrt{M}),
$$

(2.25)

where the factors $\kappa_{i}$ and $\kappa_{ij}(\alpha)$ are defined in (2.10) and (2.20). Note that the factors are universal; they are independent of the graph $\mathcal{G}$. This result holds for any $Z$-invariant system with positive Boltzmann weights for a large graph $\mathcal{G}$ with sites in general position. Evidently, for the canonically normalized partition function (2.24), the leading term in (2.25) (the bulk free energy) vanishes.

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7 Actually, these restrictions can be removed if one properly defines ‘reflected’ rapidities for downward going lines (see section 3 of [60]), but we will not elaborate this point here here.
2.7. Rapidity graphs and rhombic tilings

Consider some additional combinatorial and geometric structures associated with the graph $\mathcal{L}$. First, if the unshaded faces in the above definition of $\mathcal{G}$ are replaced by the shaded ones, one obtains another graph $\mathcal{G}^\ast$, which is dual to $\mathcal{G}$. Each site of $\mathcal{G}^\ast$ corresponds to a face of $\mathcal{G}$ and vice versa. Obviously, both graphs $\mathcal{G}$ and $\mathcal{G}^\ast$ have the same medial graph $\mathcal{L}$. Assign the difference variables $\alpha_{\cdot\cdot} \in \mathcal{G}^\ast$ to the edges of $\mathcal{G}^\ast$ by the same rule (2.1). Note that there is a one-to-one correspondence between the edges of $\mathcal{G}^\ast$ and $\mathcal{G}$. Moreover, if $e \in E(\mathcal{G})$ is of the first type then the corresponding edge $e^\ast \in E(\mathcal{G}^\ast)$ is of the second (and vice versa). In other words for the corresponding edges $\alpha_e + \alpha_{\cdot\cdot} = \eta$. Let $\text{star}(i)$ denote the set of edges meeting at the site $i$. It is easy to show that for any interior site of $\mathcal{G}$

$$\sum_{(ij) \in \text{star}(i)} \alpha_{ij} = 2\eta, \quad i \in V_{\text{int}}(\mathcal{G}).$$  \hfill (2.27)$$

Indeed, consider a face of $\mathcal{L}$, containing the site $i$. It is bounded by the directed thin lines forming the graph $\mathcal{L}$, see figure 1. Vertices of this face correspond to the edges of $\mathcal{G}$ that meet at the site $i$. By construction, the lines of $\mathcal{L}$ are always heading upwards, so there must be exactly one lowest and one highest vertex for each face of $\mathcal{L}$. These two vertices correspond to the second type edges in figure 2. The remaining vertices correspond to edges of the first type. Taking this into account, one immediately arrives to (2.27). A similar sum rule holds for the dual graph $\mathcal{G}^\ast$,

$$\sum_{(kl) \in \text{star}(k)} \alpha_{kl}^\ast = 2\eta, \quad k \in V_{\text{int}}(\mathcal{G}^\ast).$$  \hfill (2.28)$$

Consider yet another graph $\mathcal{L}^\ast$, dual to $\mathcal{L}$. The set of sites of $\mathcal{L}^\ast$ consists of those of the graph $\mathcal{G}$ and of its dual $\mathcal{G}^\ast$. These sites are shown in figure 6 by white and black dots, respectively.
The edges of \( L^* \) always connect one white and one black site. The faces of \( L^* \) correspond to the vertices of \( L \). The latter are of degree four, therefore \( L^* \) is a ‘quad-graph’ (a planar graph with quadrilateral faces). The edges of \( G \) and \( G^* \) are diagonals of these quadrilaterals (see figure 7). Evidently, there are exactly two white and two black vertices in each face. Remarkably, the graph \( L^* \) admits a rhombic embedding into the plane. In other words this graph can be drawn so that all its edges are line segments of the same length and, hence, all of its faces are rhombi, as shown in figure 6. The corresponding theorem [69] states that such an embedding exists if and only if (a) no two lines of \( L \) cross more than once \(^8\) and (b) no line of \( L \) crosses itself or is periodic. Note, that in the physical regime these conditions are obviously satisfied.

Assume that the edges of the quadrilaterals are of unit length, and consider them as vectors in the complex plane. To precisely specify a rhombic embedding one needs to provide the angles between these vectors. A rapidity line always crosses opposite (equal) edges of a rhombus. Therefore, all edges crossed by the same rapidity line \( p \) are given by one vector \( \zeta_p \), \( |\zeta_p| = 1 \). Thus, if the original rapidity graph \( L \) has \( L \) lines, there will be only \( L \) different edge vectors. Choose them as \( \zeta_p = e^{-ip}/\eta \), where \( p_1, p_2, \ldots, p_L \) are the corresponding rapidity variables. Each face of \( L^* \) is crossed by exactly two rapidity lines \( p_k \) and \( p_{\ell} \). To this face associate a rhombus with the edges \( \zeta_{p_k} \) and \( \zeta_{p_{\ell}} \), as shown in figure 7. Its diagonals are edges of \( G \) and \( G^* \). The rhombus angles are precisely the (rescaled) difference variables \( \pi \alpha_p/\eta \) and \( \pi \alpha_{p_{\ell}}/\eta \) assigned to these edges (this is true for both types of rhombi shown in figure 7). In the physical regime all these angles are in the range from 0 to \( \pi \). So the rhombi will have positive area and will not overlap. The sum rules (2.27) and (2.28) guarantee that the resulting ‘rhombic tiling’ is flat with no cusps at the sites of \( L^* \).

3. Quasi-classical expansion

3.1. Boltzmann weights, partition function, Lagrangian and action

The models considered below contain a free parameter, which can be identified with the Planck constant (in the language of Euclidean quantum field theory) or with the temperature

\(^8\) The rapidity lines forming the graph \( L \) are called ‘train tracks’ in [69].
We denote this parameter by the letter $\beta$. Its precise definition is model-dependent (see below), but it is always possible to choose $\beta$ in such a way that the canonically normalized Boltzmann weights (2.22) have the following quasi-classical asymptotics when $\beta \to 0$,

$$W(\alpha \mid \sigma_1, \sigma_2) = \exp \left\{ -\frac{\mathcal{L}(\alpha \mid x_1, x_2)}{\beta} - \mathcal{L}_1(\alpha \mid x_1, x_2) + O(\beta) \right\},$$

$$\sum_{\sigma} S(\sigma) \ldots = \int \exp \left\{ -\frac{\mathcal{C}(x)}{\beta} + \mathcal{C}_1(x) + O(\beta) \right\} \ldots \frac{dx}{\sqrt{2\pi\beta}},$$

where $x_i$ denotes appropriately scaled spin variables, $x_i = \text{const} \sigma_i$, which in this limit, always become continuous. The functions $\mathcal{L}(\alpha \mid x_1, x_2)$, $\mathcal{L}_1(\alpha \mid x_1, x_2)$, $\mathcal{C}(x)$ and $\mathcal{C}_1(x)$ are independent of $\beta$. As functions of complex variables, they are, in general, multi-valued functions of the spins $x_1, x_2$ and the spectral parameter $\alpha$.

One important thing which could be affected in the quasi-classical limit is the assignment of the edge variables $a_{ij}$ defined by (2.21). We assume the rapidity variables $\beta_{i,j}$ remain unchanged in the limit. However the parameter $h$ entering (2.21) for the edges of the second type could depend on $\beta$. The models discussed here fall into two different classes, where the parameter $h$ either vanishes linearly in $\beta$, or remains finite as $\beta \to 0$.

The discrete spin models (the Kashiwara–Miwa model and its reductions) fall into the first class, while all the continuous spin models (including the master model, Faddeev–Volkov and Zamolodchikov’s ‘fishing-net’ models) belong to the second class. As explained in section 2.6 in the physical regime (i.e., when all weights are positive) all the edge variables $\alpha_{ij}$ in (2.24) must be in the physical domain $0 < \alpha_{ij} < \eta$. However, if $\eta = O(\beta)$, this domain shrinks to a point when $\beta \to 0$. Therefore, unless all $\alpha_{ij} = 0$ and the model is trivial, there must be negative (or even complex) weights in the limit, which is an unphysical regime from the point of statistical mechanics.

The symmetry of Boltzmann weights (2.2) implies

$$\mathcal{L}(\alpha \mid x, y) = \mathcal{L}(\alpha \mid y, x),$$

while the inversion relations (2.14) and (2.15) imply

$$\mathcal{L}(\alpha \mid x, y) + \mathcal{L}(-\alpha \mid x, y) = 0,$$

$$\mathcal{L}(\eta_0 + \alpha \mid x, y) + \mathcal{L}(\eta_0 - \alpha \mid x, y) = -\mathcal{C}(x) - \mathcal{C}(y).$$

The above relations hold provided one chooses appropriate branches of the function $\mathcal{L}(\alpha \mid x, y)$, if the argument $\alpha$ lies outside the physical domain $0 < \alpha < \eta_0$. Note also, that for the unphysical regime, when $\eta_0 = 0$, the function $C(x)$ must vanish, so that the two relations coincide.

Substituting (3.1) into the partition function (2.24) one obtains,

$$Z = \int \exp \left\{ -A(x)/\beta - B(x) + O(\beta) \right\} \prod_i \frac{dx_i}{\sqrt{2\pi\beta}}, \quad \beta \to 0,$$
where the product is over all internal sites \(i \in V_{\text{int}}(\mathcal{G})\) of our graph \(\mathcal{G}\), and

\[
A(x) = \sum_{(ij) \in E(\mathcal{G})} \mathcal{L}(\alpha_{ij} \mid x_i, x_j) + \sum_{i \in V(\mathcal{G})} \mathcal{C}(x_i),
\]

\[
B(x) = \sum_{(ij) \in E(\mathcal{G})} \mathcal{L}(\alpha_{ij} \mid x_i, x_j) + \sum_{i \in V(\mathcal{G})} \mathcal{C}(x_i) + \text{terms proportional to } \partial \eta / \partial h,
\]

and the sums are taken over all edges and over all sites of \(\mathcal{G}\). As before, the external spins are kept fixed. The variables \(\alpha_{ij}\) in (3.6) and (3.7) are defined by (2.1) with \(\eta\) substituted by \(\eta_0\), given in (3.2). Calculating the integral (3.5) by the saddle point method one obtains\(^{10}\)

\[
\log Z = -\frac{1}{\hbar} A(x^{(cl)}) + B(x^{(cl)}) - \frac{1}{2} \log \det \left[ \frac{\partial^2 A(x)}{\partial x_i \partial x_j} \right]_{x=x^{(cl)}} + O(\hbar).
\]

The symbol \(x^{(cl)}\) denotes the stationary point of the action \(A(x)\), determined by the classical equations of motion

\[
\left. \frac{\partial A(x)}{\partial x_i} \right|_{x=x^{(cl)}} = 0, \quad i \in V_{\text{int}}(\mathcal{G}).
\]

Introducing a new function

\[
\psi(\alpha \mid x, y) = \frac{\partial}{\partial x} \mathcal{L}(\alpha \mid x, y).
\]

and using (3.6) one can write (3.9) explicitly as discrete Laplace-type equations

\[
\sum_{(ij) \in \text{star}(i)} \psi(\alpha_{ij} \mid x_i, x_j) + \frac{\partial}{\partial x_i} \mathcal{C}(x_i) = 0, \quad i \in V_{\text{int}}(\mathcal{G}),
\]

where \(\text{star}(i)\) denotes the set of edges meeting at the site \(i\). Note, that the parameters \(\alpha_{ij}\) entering this equation obeys the sum rule (2.27).

### 3.2. Classical star-triangle relation and invariance of the action

Let us now recall the \(Z\)-invariance properties of the partition function (2.24). Obviously, these properties must hold for each term of the quasi-classical expansion (3.8). In particular, the classical action \(A(x^{(cl)})\), evaluated on solutions of the classical equation of motion (3.9) (the leading term in the expansion (3.8)), remains invariant with respect to the star-triangle moves\(^{11}\) of the rapidity graph \(\mathcal{L}\), shown in figure 3

\[
A(x^{(cl)}) = \text{invariant under the star-triangle moves of } \mathcal{L}.
\]

This result was obtained in [10] and illustrated on the example of the Faddeev–Volkov model. The arguments of [10] are rather general and apply with no modifications to all integrable edge-interaction models, which admit the quasi-classical limit (3.1). Indeed, all mathematical relations required for the invariance (3.12) arise automatically from the quasi-classical expansion of the star-triangle relation. To be more precise this expansion generates an infinite number of non-trivial relations, one relation in each order of \(\hbar\). The statement (3.12) only requires the first of these relations arising in the leading order in \(\hbar\).

\(^{10}\) Note, that the saddle point method requires some modifications if the equations (3.9) are defined to a continuous set of stationary points. This situation occurs in the Zamolodchikov’s ‘fishing net’ model, which will be considered separately in section 4.7.

\(^{11}\) As explained in section 2.7 the star-triangle moves are equivalent to the ‘flipping cube’ moves of the quad-graph \(\mathcal{L}^*\) which is the dual to \(\mathcal{L}\).
Substituting (3.1) into (2.4) and taking into account (2.23), one obtains

$$\int \frac{dx_0}{\sqrt{2\pi \hbar}} \exp \left\{ -\frac{1}{\hbar} A_\bullet(x) - B_\bullet(x) \right\} = \exp \left\{ -\frac{1}{\hbar} A_\circ(x) - B_\circ(x) + O(\hbar) \right\},$$

(3.13)

where

$$A_\bullet(x) = \mathcal{L}(\eta_0 - \alpha_1 | x_0, x_1) + \mathcal{L}(\eta_0 - \alpha_2 | x_0, x_2) + \mathcal{L}(\eta_0 - \alpha_3 | x_0, x_3) + \mathcal{C}(x_0),$$

(3.14)

and

$$A_\circ(x) = \mathcal{L}(|x_2, x_3) + \mathcal{L}(|x_3, x_1) + \mathcal{L}(|x_1, x_2).$$

(3.15)

The symbol $x$ stands for the set $x = (x_0, x_1, x_2, x_3)$. Expressions for $B_\bullet$ and $B_\circ$ are defined in a similar way; they are just specializations of (3.7) for the star and triangular graphs in figure 3. Evaluating the integral (3.13) by the saddle point method one immediately obtains two non-trivial identities valid for arbitrary values of $x_1, x_2, x_3$. In the leading order in $\hbar$ one gets

$$A_\bullet(x_0^{(cl)}, x_1, x_2, x_3) = A_\circ(x_1, x_2, x_3),$$

(3.17)

where $x_0^{(cl)}$ is the stationary point of the integral in (3.13), i.e., the value of $x_0$, which solves the equation

$$\frac{\partial A_\bullet(x)}{\partial x_0} \bigg|_{x_0 = x_0^{(cl)}} = 0.$$  

(3.18)

In the order $O(\hbar^0)$ one gets [10]

$$\left\{ \frac{1}{2} \log \frac{\partial^2 A_\bullet(x)}{\partial x_0^2} + B_\bullet(x) \right\} \bigg|_{x_0 = x_0^{(cl)}} = B_\circ(x).$$

(3.19)

The last relation will not be used in what follows. It is presented here just to illustrate that the star-triangle relation has a consistent expansion in powers of $\hbar$.

From now on we will omit the superfix "(cl)" for the solution of (3.18) and assume $x_0 \equiv x_0^{(cl)}$. Writing (3.17) in full one obtains the classical star-triangle relation

$$\mathcal{L}(\eta_0 - \alpha_1 | x_0, x_1) + \mathcal{L}(\eta_0 - \alpha_2 | x_0, x_2) + \mathcal{L}(\eta_0 - \alpha_3 | x_0, x_3) + \mathcal{C}(x_0)$$

$$= \mathcal{L}(\alpha_1 | x_2, x_3) + \mathcal{L}(\alpha_2 | x_3, x_1) + \mathcal{L}(\alpha_3 | x_1, x_2),$$

(3.20)

which, as before, the arguments $\alpha_1$, $\alpha_2$, $\alpha_3$ obey the relation (3.16). The stationary point $x_0$ is determined by the equation

$$\psi(\eta_0 - \alpha_1 | x_0, x_1) + \psi(\eta_0 - \alpha_2 | x_0, x_2) + \psi(\eta_0 - \alpha_3 | x_0, x_3) + \frac{\partial}{\partial x_0} \mathcal{C}(x_0) = 0,$$

(3.21a)

with $\psi(\alpha | x, y)$ defined in (3.10). It is convenient to regard the last equation as a constraint on the four variables $x_0, x_1, x_2, x_3$, rather than an equation for $x_0$. The classical star-triangle relation (3.20) holds as long as this constraint is satisfied. Note that it can be re-written in three other equivalent forms. To do this one needs to differentiate (3.20) with respect to $x_1, x_2$ or $x_3$. There is no need to take into account the dependence of $x_0$ on $x_1, x_2, x_3$, since the expression (3.20) is stationary with respect to $x_0$. As a result one obtains
Note that the function $\psi(\alpha \mid x, y)$ satisfies a pair of functional equations\(^{12}\)

\[
\begin{align*}
\psi(\alpha \mid x, y) + \psi(\alpha + \beta \mid x, y) &= 0, \\
\psi(\alpha \mid x, y) + \psi(\alpha + \alpha \mid x, y) &= 0,
\end{align*}
\]

which simply follow from (2.14) and (2.15).

### 3.3. Consistency around a cube

In [15] ABS introduced a remarkable class of integrable discrete evolution equations. A distinguished feature of these equations is that their integrability properties are automatically satisfied due to the equations themselves (another way of describing this situation would be to say that the corresponding “Lax pair” is contained within the equations). The above equations are classical (not quantum) evolution equations for a complex scalar field, defined on vertices of a quad-graph. The later could be either a regular graph (e.g., a square lattice) or an irregular graph of type shown in figure 6. The four values of the field $x_0, x_1, x_2, x_3$ at the vertices of an elementary quadrilateral, as in figure 8, are constrained by one relation $Q(x_0, x_1, x_2, x_3) = 0$. In general, this relation varies for different quadrilaterals (see below). The integrability conditions for such system, are called the consistency-around-a-cube conditions. The list of all solutions of these conditions for the case of affine-linear constraints $Q(x_0, x_1, x_2, x_3)$ admitting the symmetries of the square were found in [15] (which we refer to below as the ABS list).

Recently, Bobenko and Suris [50] have shown that every equation from that list corresponds to a certain solution of the classical star-triangle relation. Previously this fact was established [10] for the Hirota equations, which is $Q_{3,0} = 0$ in the ABS list\(^{13}\). As shown in [50], a generic ABS equation is related to a more general, than (3.20), classical star-triangle relation, containing different functions $\mathcal{L}(\alpha \mid x, y)$ for different edges. In our setting this corresponds to systems without the crossing symmetry.

\(^{12}\) As noted before $C(x) \equiv 0$ when $\eta_0 = 0$.

\(^{13}\) This fact was also known to us for $Q_{3,1} = 0$ and $Q_4$ equations before the paper [50] has appeared.
In [50] solutions of the classical star-triangle relation were obtained from solutions of the consistency-around-a-cube conditions. Here we want to reverse the argument and consider a converse procedure. To do this we use an observation of [50] that the constraints (3.21), associated with the classical star-triangle relation, can be identified with the so-called three-leg form [15] of the equation $Q(x_0, x_1, x_2, x_{12}) = 0$ on an elementary quadrilateral. Note that the variable $x_0$ appears in every term of (3.21a). For this reason we will call this equation the ‘three-leg form centered at $x_0$’. The other three equivalent forms of this relation (3.21b)–(3.21d) are centered at $x_1$, $x_2$ and $x_3$, respectively.

The construction of a quad-graph $L^*$ and its (oriented) rapidity graph $L$ is explained in section 2.7. Here we assume the same notations. Recall that sites of $L^*$ are colored black and white (every quad-graph is bipartite). There are two types of quad-faces, differing by the position of white sites relative to the directed rapidity lines as shown in figures 7 and 8. Let $x_0, x_1, x_2, x_{12}$ be the fields at the corners of a face, and $p_1, p_2$ denote the rapidity variables arranged as in figure 8. Define two different constraints $Q_{12}$ and $\overline{Q}_{12}$,

\[
Q(p_1, p_2 \mid x_0, x_1, x_2, x_{12}) = \psi(p_1 - p_2 \mid x_2, x_1) + \psi(p_2 \mid x_2, x_0) - \psi(p_1 \mid x_2, x_{12}),
\]

\[
\overline{Q}(p_1, p_2 \mid x_0, x_1, x_2, x_{12}) = \psi(p_1 - p_2 \mid x_2, x_1) + \psi(\eta_0 - p_1 \mid x_2, x_{12}) - \psi(\eta_0 - p_2 \mid x_2, x_0),
\]

where $\psi(\alpha \mid x, y)$ satisfies (3.21). Below we will also use the abbreviated notations

\[
Q_{ij}(x_0, x_1, x_2, x_{12}) \equiv Q(p_i, p_j \mid x_0, x_1, x_2, x_{12}), \quad i, j = 1, 2, 3
\]

and similarly for $\overline{Q}_{ij}$.

**Consistency around a cube.** Let the rapidity variables $p_1, p_2, p_3$ take arbitrary values and the fields $x, x', x_1, x_2, x_3, x_{12}, x_{13}, x_{23}$ be arranged as shown in figure 9. Assume that the equations (3.21) are satisfied. Then the system of three equations

\[
Q_{12}(x_{12}, x_1, x_2, x_{13}, x_3) = 0, \quad Q_{13}(x_{12}, x_2, x, x_{23}) = 0, \quad Q_{23}(x_1, x_{12}, x_{13}, x) = 0,
\]

corresponding to the three faces of a cube, shown on the left side of figure 9, is consistent with the system of three equations

\[
Q_{12}(x_{12}, x_2, x_3, x') = 0, \quad Q_{13}(x_1, x', x_{13}, x_3) = 0, \quad Q_{23}(x', x_2, x_3, x_{23}) = 0.
\]

corresponding to the other three faces of a cube, shown on the right side of figure 9.

\[\text{Figure 9. Arrangement of the fields } x, x', x_1, x_2, x_3, x_{12}, x_{13}, x_{23} \text{ on the vertices of a cube.}\]
The proof is essentially identical to that of [15]. Equations (3.25) contain three relations for seven variables, leaving four degrees of freedom. For example, if $x_1, x_{12}, x_2, x_{23}$ are given, then $x, x_3, x_{13}$ are uniquely determined. This fixes all variables entering (3.26), except $x'$. To prove the consistency one needs to show that each of the three relations in (3.26) define the same value of $x'$. For instance, suppose that $x'$ satisfies $Q_{13} = 0$. Combining this equation with two equations from (3.25), one obtains

$$Q_{13}(x_1, x', x_{13}, x_3) - Q_{12}(x, x_{23}, x_3, x_3) - Q_{23}(x_1, x_{12}, x_{13}, x) = 0.$$  \hspace{1cm} (3.27)

Rewriting the last equation in the form centered at $x_{23}$, combining it with suitable forms of the equations $Q_{13} = 0$ and $Q_{12} = 0$ from (3.25) (also centered at $x_{23}$) and using the second relation in (3.22) one can readily deduce that the equation $Q_{23} = 0$ in (3.26) is satisfied. Similarly one can check that $Q_{12} = 0$ as well, this time one needs to use the first relation from (3.22), with the relevant equations centered at $x_{12}$.

**Remark.** The above reasonings apply to two cases $\eta_0 = 0$ and $\eta_0 = \pi$. They correspond, respectively, to the unphysical and physical regimes from the point of the quasi-classical limit of a quantum model. Note that for $\eta_0 = 0$ the two constraints in (3.23) coincide, thanks to (3.22). Thus the consistency equations (3.25) and (3.26) in this case involve the only one constraint with different rapidity variables for different quadrilaterals.

4. Particular lattice models and their quasiclassical limits

The models considered here (except for the gamma function model of section 4.6), possess all of the properties discussed in the previous section, including the rapidity difference property, crossing symmetry, positivity, inversion relations (2.14), (2.15) and reflection symmetry (2.2).

4.1. Master solution to the star-triangle relation

The Boltzmann weights of the master solution are $\pi$-periodic, so spins take arbitrary values modulo $\pi$. It is convenient to regard them as

$$0 \leq x_i < \pi.$$  \hspace{1cm} (4.1)

Let $q$ and $p$ be elliptic nome

$$q = e^{i\pi\tau}, \quad p = e^{i\pi\tau'}.$$  \hspace{1cm} (4.2)

The crossing parameter is defined by

$$\eta = \frac{\pi}{2i} (\tau + \tau').$$  \hspace{1cm} (4.3)

In what follows, we use the standard notations [70] for Jacobi $\vartheta$-functions, e.g.

$$\vartheta_1(z \mid \tau) = 2q^{1/4} \sin(z) \prod_{n=1}^{\infty} \left(1 - e^{2\pi q^{2n}}(1 - e^{2\pi q^{2n}})(1 - q^{2n}) \right).$$  \hspace{1cm} (4.4)

Explicit expressions for the Boltzmann weights contain two special functions. The elliptic $\Gamma$-function is defined by
\[ \Gamma(z) = \prod_{n,m=0}^{\infty} \frac{1 - z^{-1}q^{2n+2}p^{2m+2}}{1 - zq^{2n}p^{2m}}, \] (4.5)

however, a more convenient notation is
\[ \Phi(z) = \Gamma(e^{-2i(z-\nu)}) = \exp\left\{ \sum_{n=0}^{\infty} \frac{e^{-2izn}}{n(q^n - q^{-n})(p^n - p^{-n})} \right\}. \] (4.6)

since \( \Phi(z)\Phi(-z) = 1 \). Another special function is
\[ K_\epsilon(\alpha) = \exp\left\{ \sum_{n=0}^{\infty} \frac{e^{4izn}}{n(q^n - q^{-n})(p^n - p^{-n})(q^n p^n + q^{-n} p^{-n})} \right\}. \] (4.7)

The function \( K_\epsilon(\alpha) \) satisfies the functional relations
\[ \frac{K_\epsilon(\eta - \alpha)}{K_\epsilon(\alpha)} = \Gamma(e^{-4i\eta}), \quad K_\epsilon(\alpha)K_\epsilon(-\alpha) = 1. \] (4.8)

These equations correspond to (2.20) with \( f(\alpha) = \Gamma(e^{-4i\alpha}) \) and \( \kappa_\epsilon = 1 \). The canonically normalized Boltzmann weight is defined by
\[ \mathcal{W}(\alpha \mid x, y) = K_\epsilon(\alpha)^{-1} \frac{\Phi(x - y + i\alpha)\Phi(x + y + i\alpha)}{\Phi(x - y - i\alpha)\Phi(x + y - i\alpha)}. \] (4.9)

This Boltzmann weight admits the symmetries:
\[ \mathcal{W}(\alpha \mid x, y) = \mathcal{W}(\alpha \mid y, x) = \mathcal{W}(\alpha \mid \pm x, \pm y). \] (4.10)

The one-point weight is defined by
\[ S(x) = \frac{1}{2\pi} e^{\eta/2} d_1(2x \mid \tau) d_1(2x \mid \tau'). \] (4.11)

The weights are \( \pi \)-periodic with respect to \( x, y \). The weights are positive when \( \eta \) is real and \( 0 \leq \alpha \leq \eta \). The weights satisfy the difference relations
\[ \mathcal{W}\left(\alpha \mid x - \frac{\pi\tau}{2}, y\right) = \mathcal{W}\left(\alpha \mid x + \frac{\pi\tau}{2}, y\right) \left( \frac{\partial_4(x - y + i\alpha \mid \tau)}{\partial_4(x - y - i\alpha \mid \tau)} \times \frac{\partial_4(x + y + i\alpha \mid \tau)}{\partial_4(x + y - i\alpha \mid \tau)} \right) \quad \text{and similar with } \tau \leftrightarrow \tau'. \] (4.12)

The weights have the canonical normalization
\[ \mathcal{W}(0 \mid x, y) = 1, \quad \mathcal{W}(\eta \mid x, y) = \frac{1}{2S(x)} (\delta(x - y) + \delta(x + y)), \] (4.13)

and satisfy the corresponding inversion relations
\[ \mathcal{W}(\alpha \mid x, y)\mathcal{W}(-\alpha \mid x, y) = 1, \] (4.14)

and
\[ \int_0^\pi dz S(z) \mathcal{W}(\eta - \alpha \mid x, z) \mathcal{W}(\eta + \alpha \mid z, y) = \frac{1}{2S(x)} (\delta(x - y) + \delta(x + y)). \] (4.15)
Finally the Boltzmann weights satisfy the star-triangle equation
\[ \int_{-\pi/2}^{\pi/2} dx \, S(x) \mathcal{W}(\eta - \alpha_1 \mid x_1, x) \mathcal{W}(\alpha_1 + \alpha_3 \mid x_2, x) \mathcal{W}(\eta - \alpha_3 \mid x_3, x) = \mathcal{W}(\alpha_1 \mid x_2, x_3) \mathcal{W}(\eta - \alpha_1 - \alpha_3 \mid x_1, x_3) \mathcal{W}(\alpha_3 \mid x_1, x_2). \] (4.16)

4.1.1. Classical limit of the master solution. The classical limit is the limit \( p \to 1 \). Let \( \tau' = i\hbar / \pi, \hbar \to 0 \). In this limit
\[ \Phi(z) = \exp \left\{ -\frac{1}{\hbar} \lambda_4(z \mid \tau) + O(1) \right\}, \] (4.17)
and
\[ K(\alpha) = \exp \left\{ -\frac{1}{\hbar} \lambda_4(2i\alpha \mid 2\tau) + O(1) \right\}, \] (4.18)
where
\[ \lambda_4(z \mid \tau) = \frac{1}{16} \int \frac{dz}{2} \log \tilde{\mathcal{L}}_4(x \mid \tau), \quad \tilde{\mathcal{L}}_4(x \mid \tau) = \prod_{n=1}^{\infty} (1 - e^{2\pi i q^{2n+1}})(1 - e^{-2\pi i q^{2n+1}}). \] (4.19)

\( \lambda_4(z \mid \tau) \) is even and \( \pi \)-periodic. The two-point Lagrangian
\[ \mathcal{W}(\alpha \mid x, y) = \exp \left\{ -\frac{1}{\hbar} \mathcal{L}(\alpha \mid x, y) + O(1) \right\}, \] (4.20)
and one-point Lagrangian
\[ S(x) = \exp \left\{ -\frac{1}{\hbar} \mathcal{C}(x) + O(\log \hbar) \right\}, \] (4.21)
are given by
\[ \mathcal{L}(\alpha \mid x, y) = \lambda_4(x - y + i\alpha) - \lambda_4(x - y - i\alpha) + \lambda_4(x + y + i\alpha) - \lambda_4(x + y - i\alpha) - \lambda_4(2i\alpha \mid 2\tau), \] (4.22)
and
\[ \mathcal{C}(x) = \left(2|x| - \frac{\pi}{2}\right)^2, \quad |x| < \frac{\pi}{2}, \] (4.23)
respectively. An equivalent expression for \( \mathcal{L}_\alpha(x, y) \) is
\[ \mathcal{L}(\alpha \mid x, y) = \frac{1}{i} \int_{0}^{x-y} dz \log \frac{\partial_4(z + i\alpha \mid \tau)}{\partial_4(z - i\alpha \mid \tau)} + \frac{1}{i} \int_{x/2}^{x+y} dz \log \frac{\partial_4(z + i\alpha \mid \tau)}{\partial_4(z - i\alpha \mid \tau)}. \] (4.24)
The crossing parameter in the classical limit is
\[ \eta_0 = \frac{\pi \tau}{2i}. \] (4.25)
Thus the classical limit corresponds to the equations labeled as \( Q_4 \) in [15] with \( \eta_0 \neq 0 \).
4.2. Kashiwara–Miwa model

In 1986 Kashiwara and Miwa [2] found an elliptic solution of the star-triangle relation (i.e., parameterized by elliptic functions of the rapidity variable) where spins take $N \geq 2$ distinct values

$$a, b, c, \ldots \in \{0, 1, \ldots N - 1\}, \quad N \geq 2. \quad (4.26)$$

Their model contains the Ising model as the special case $N = 2$. The Kashiwara–Miwa model can also be derived from the master solution in the limit

$$p \to e^{\pi/N}, \quad (4.27)$$

for details see the [12]. The model contains three parameters:

$$N \geq 2, \quad \zeta \in \mathbb{Z}, \quad \tau \in \mathbb{C}, \quad \text{Im} \tau > 0. \quad (4.28)$$

Define the two functions

$$r(\alpha | n) = \prod_{j=1}^{n} \frac{\vartheta_1\left(\eta \left(j - \frac{1}{2}\right) - \frac{1}{2} \alpha | \tau \right)}{\vartheta_1\left(\eta \left(j - \frac{1}{2}\right) + \frac{1}{2} \alpha | \tau \right)},$$

$$t(\alpha | n) = \prod_{j=1}^{n} \frac{\vartheta_4\left(\eta \left(j - \frac{1}{2}\right) - \frac{1}{2} \alpha | \tau \right)}{\vartheta_4\left(\eta \left(j - \frac{1}{2}\right) + \frac{1}{2} \alpha | \tau \right)}, \quad (4.29)$$

where

$$\eta = \pi/N, \quad (4.30)$$

is the crossing parameter which enters (2.1). The weights of the Kashiwara–Miwa model read

$$W(\alpha | a, b) = r(\alpha | a - b) t(\alpha | a + b + \zeta), \quad S(a) = \frac{\vartheta_4(2a + \zeta | \tau)}{\vartheta_4(0 | \tau)}, \quad (4.31)$$

where $a, b$ are integer spins. Note that the functions (4.29) are periodic in their second argument

$$r(\alpha | n) = r(\alpha | n + N), \quad t(\alpha | n) = t(\alpha | n + N), \quad (4.32)$$

therefore the weights (4.31) are periodic with respect to shifts of spins

$$W(\alpha | a, b) = W(\alpha | a + N, b) = W(\alpha | a, b + N), \quad S(a + N) = S(a). \quad (4.33)$$

Next, the function $r(\alpha | n) = r(\alpha | - n)$ is an even function of $n$, therefore the weights are unchanged by interchanging the spins $a$ and $b$,

$$W(\alpha | a, b) = W(\alpha | b, a). \quad (4.34)$$

The weights are real and positive when $\text{Re} \tau = 0$ and $\alpha$ is real and lies in the interval $0 < \alpha < \eta$.

An explicit expression for the factor $f(\alpha)$ in this case was conjectured in [61]

$$f(\alpha) = \kappa_x \prod_{j=1}^{[N/2]} \frac{\vartheta_1\left(\eta \left(j - \frac{1}{2}\right) + \frac{1}{2} \alpha | \frac{1}{2} \tau \right)}{\vartheta_1\left(\eta \left(j - \frac{1}{2}\right) - \frac{1}{2} \alpha | \frac{1}{2} \tau \right)}, \quad \kappa_x = g \sqrt{N}, \quad (4.35)$$

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where \([N/2]\) denotes the integer part of \(N/2\) and the constant

\[
g = \prod_{m=1}^{\infty} \left( \frac{1 + q^m}{1 - q^m} \right) \left( \frac{1 - q^{N_{\text{odd}}}}{1 + q^{N_{\text{odd}}}} \right), \quad q = e^{i\tau}, \tag{4.36}
\]

is determined from the requirement \(f(0) = 1\).

Solving (2.20) under the assumptions that \(\Re \alpha = 0\) and that \(\log \kappa_\nu(\alpha)\) analytic and bounded in the rectangle \(0 \leq \Re \alpha \leq \eta, 0 \leq \Im \alpha \leq \Im \tau\), one obtains

\[
\log \kappa_\nu(\alpha) = -i \frac{N - 1}{4\tau} \alpha - \sum_{k=1}^{\infty} \frac{q^k + q^{N_{\text{even}}}}{k (1 + q^k)(1 + q^{N_{\text{even}}})} w^{2k} - \bar{w}^{2k}, \tag{4.37}
\]

where \(w\) and \(\bar{q}\) are defined by

\[
w = \exp(-i\alpha/\tau), \quad \bar{q} = \exp(-2i\eta/\tau), \quad |\bar{q}| < 1. \tag{4.38}
\]

The expression (4.37) can be written in the exponential form

\[
\kappa_\nu(\alpha) = \exp(-i(N-1)/4\tau) z_1(w)/z_2(w), \tag{4.39}
\]

where

\[
z_Y(w) = \prod_{j=0}^{N-1} \prod_{k=1}^{\infty} \left( \frac{1 - \bar{q}^{2jN+2j+1}w^{2j+2}}{(1 - q^{2jN+2j+1}w^{2j+2}) (1 - q^{2jN+2j+1}w^{2j+2})} \right)^k, \tag{4.40}
\]

and \(z_Y(w)\) is defined by the last formula with \(N = 1\). Note, in particular, that for the Ising model, \(N = 2\), one obtains

\[
k_\nu(\alpha) = \exp(-i/4\tau) z_1(w)/z_2(w), \quad N = 2, \tag{4.41}
\]

where

\[
z_1(w) = \prod_{n=1}^{\infty} \left( \frac{1 - q^{2n-w^{-2}}}{1 - q^{2n-w^{-2}}} \right), \quad z_2(w) = \prod_{n=1}^{\infty} \left( \frac{1 - q^{4n-1}w^{-2}}{1 - q^{4n-1}w^{-2}} \right). \tag{4.42}
\]

### 4.2.1. Classical limit of the Kashiwara–Miwa model

The classical limit for the Kashiwara–Miwa model corresponds to \(N \to \infty\). Let

\[
h = \frac{2\pi}{N}, \quad x = h\sigma, \tag{4.43}
\]

where \(\sigma\) represents the original discrete spin, and \(x\) is the spin in the continuous limit. In the limit \(h = 2\eta \to 0\) the condition \(0 < \alpha < \eta\) makes the explicit classical limit trivial. Moreover, the regime of real \(\alpha\) becomes ill-defined since the poles of the Boltzmann weight condense to a branch along the real axis. However, the case of imaginary spectral parameter is well defined. Changing then \(\alpha \to i\alpha\), one obtains in the limit \(\hbar \to 0\)

\[
\mathcal{W}(i\alpha \left| \frac{x}{\hbar}, \frac{y}{\hbar} \right) \sim \exp \left\{ -\frac{i}{\hbar} \mathcal{L}(i\alpha \left| x, y \right) \right\},
\]

\[
\mathcal{W}(i\alpha \left| \frac{x}{\hbar}, \frac{y}{\hbar} \right) \sim \exp \left\{ -\frac{i}{\hbar} \mathcal{P}(i\alpha \left| x, y \right) \right\}. \tag{4.44}
\]
where

\[ \mathcal{Z}(i\alpha \mid x, y) = \mathcal{L}(-i\alpha \mid x, y) = -\mathcal{L}(i\alpha \mid x, y), \]  
(4.45)

and \( \mathcal{L}(i\alpha \mid x, y) = \mathcal{L}(i\alpha \mid y, x) \) is given by

\[ \mathcal{L}(i\alpha \mid x, y) = i \int_0^{x-y} \log \left( \frac{\phi_1 \left( \frac{1}{2} (z - i\alpha) \right)}{\phi_1 \left( \frac{1}{2} (z + i\alpha) \right)} \right) \, dz + i \int_0^{x+y} \log \left( \frac{\phi_1 \left( \frac{1}{2} (z - i\alpha) \right)}{\phi_1 \left( \frac{1}{2} (z + i\alpha) \right)} \right) \, dz 
+ \int_0^\infty \log \left| \frac{\phi_1 \left( \frac{1}{2} \left| \frac{z^2}{2} \right| \right)}{\phi_2 \left( \frac{1}{2} \left| \frac{z^2}{2} \right| \right)} \right| \, dz. \]  
(4.46)

The first integral here must be understood as

\[ i \int_0^\infty \log \left( \frac{\phi_1 \left( \frac{1}{2} (z - i\alpha) \right)}{\phi_1 \left( \frac{1}{2} (z + i\alpha) \right)} \right) \, dz = \pi|x| - \frac{x^2}{2} 
+ i \int_0^\infty \log \left( \frac{(e^{-\alpha - \epsilon}; q^2_\infty \left( e^{\alpha-\epsilon} q^2_\infty \right)}{(e^{-\alpha + \epsilon}; q^2_\infty \left( e^{\alpha + \epsilon} q^2_\infty \right))} \right) \, dz, \]  
(4.47)

where

\[ |x| \leq 2\pi, \quad 0 < \alpha. \]  
(4.48)

The one-point Lagrangian is zero, \( \mathcal{L}(x) = 0 \). The Lagrangians are canonically normalized and correspond to \( Q_4 \) with \( \eta_0 = 0 \).

Note that the condition (4.45) is assumed for all classical models with \( \eta_0 = 0 \).

### 4.3. Hyperbolic limit of the master solution

The hyperbolic limit of the master solution, is the limit when \( q, p \to 1 \),

\[ \tau = i \frac{b}{T}, \quad \tau' = i \frac{b^{-1}}{T}, \quad T \to \infty. \]  
(4.49)

Such a limit was first considered by Spiridonov [17] and details of this limit can be found in appendix. There are two regimes of spin variables providing two hyperbolic solutions of the following star-triangle equation

\[ \int_{-\infty}^{\infty} dx \, S(x) \mathcal{W}(\eta - \alpha_1 \mid x_1, x) \mathcal{W}(\alpha_1 + \alpha_3 \mid x_2, x) \mathcal{W}(\eta - \alpha_3 \mid x_3, x) 
= \mathcal{W}(\alpha_1 \mid x_2, x_3) \mathcal{W}(\eta - \alpha_1 - \alpha_3 \mid x_1, x_3) \mathcal{W}(\alpha_3 \mid x_1, x_2). \]  
(4.50)

In this section set the crossing parameter to be

\[ \eta = \frac{1}{2} (b + b^{-1}), \quad \eta > 0. \]  
(4.51)

The spins now take values \( x_i \in \mathbb{R} \), and spectral parameters are restricted to \( 0 < \alpha_i < \eta \). It is convenient to use symmetric dilogarithm function

\[ \phi(z) = \exp \left\{ \frac{1}{4} \int_{-\infty}^{\infty} \frac{e^{-2zw}}{\sinh(bw) \sinh(b^{-1}w)} \, dw \right\}. \]  
(4.52)
which satisfies $\phi(z)\phi(-z) = 1$, and symmetric normalization function
\[
\kappa_c(\alpha) = \exp\left\{ \frac{1}{8} \int_{pv} \frac{e^{bw}}{\sinh(bw)\sinh(b^{-1}w)\cosh((b + b^{-1})w)} \, dw \right\},
\]
where $\int_{pv}$ denotes the principal value integral. The function $\kappa_c(\alpha)$ is a solution of (2.20) with $f(\alpha) = \phi(in - 2\alpha)$:
\[
\frac{\kappa_c(n - \alpha)}{\kappa_c(\alpha)} = \phi(in - 2\alpha), \quad \kappa(\alpha)\kappa(-\alpha) = 1.
\]

The first hyperbolic solution to the star-triangle equation is given by
\[
\mathcal{W}(\alpha | x, y) = \frac{1}{\kappa_c(\alpha)} \frac{\phi(x - y + i\alpha)}{\phi(x - y - i\alpha)} \frac{\phi(x + y + i\alpha)}{\phi(x + y - i\alpha)},
\]
with
\[
S(x) = 2 \sinh(2\pi bx)\sinh(2\pi b^{-1}x).
\]
The Boltzmann weights (4.55) retain the symmetries (4.10). This solution corresponds to the regime of small external spins in (4.16)
\[
x_i = \frac{\pi}{T} x_i.
\]
In this case only when the integrand is near zero
\[
x \to \frac{\pi}{T} x,
\]
will there be a contribution to the integral in the left-hand side of (4.16).

**4.3.1. Faddeev–Volkov model.** Another solution of the hyperbolic star-triangle equation corresponds to the Boltzmann weights for the Faddeev–Volkov model. In 1995 Faddeev and Volkov obtained [9] a solution of the star triangle relation which, in some sense, could be regarded as an analytic continuation the Fateev–Zamolodchikov solution to negative number of spin states $N$. Remarkably, the corresponding model of statistical mechanics has positive Boltzmann weights [10], its partition function in the large-lattice limit was calculated in [10, 71]. The Boltzmann weights for the Faddeev–Volkov model are given by
\[
\mathcal{W}(\alpha | x, y) = \frac{1}{\kappa_c(\alpha)} \frac{\phi(x - y + i\alpha)}{\phi(x - y - i\alpha)} \frac{\phi(x + y + i\alpha)}{\phi(x + y - i\alpha)}, \quad S(x) = 1,
\]
where $\kappa_c(\alpha)$ is defined by (4.53). It corresponds to the regime of external spins in (4.16)
\[
x_i \to \frac{\pi}{4} + \frac{\pi}{T} x_i,
\]
so that only vicinities of $x \sim \pm \pi/4$,
\[
x \to \pm \frac{\pi}{4} + \frac{\pi}{T} x,
\]
contribute to the integral.

The Boltzmann weights of the Faddeev–Volkov model are symmetric
\[
\mathcal{W}(\alpha | x, y) = \mathcal{W}(\alpha | y, x),
\]
and possess a self-duality property
\[ \mathcal{W}(\alpha | x, y) = \int_{\mathbb{R}} e^{2i(\alpha - y)z} \mathcal{W}(\alpha | z, 0). \] (4.63)

### 4.3.2. Classical limit of the hyperbolic models

The classical limit implies the re-scale of the variables
\[ \alpha \rightarrow \frac{\alpha}{\pi b}, \quad x \rightarrow \frac{x}{\pi b}, \quad h = \pi b^2. \] (4.64)

with \( b \rightarrow 0 \). The two-point Lagrangian for (4.55) is
\[
\mathcal{L}(\alpha | x, y) = \frac{1}{i} \left( \int_0^{\alpha - y} \log \frac{\cosh(z + i\alpha)}{\cosh(z - i\alpha)} \, dz + \int_0^{x+y} \log \frac{\cosh(z + i\alpha)}{\cosh(z - i\alpha)} \, dz \right) + 2 \int_0^\alpha \log 2 \cos(z) \, dz, \tag{4.65}
\]
and the one-point Lagrangian is
\[ \mathcal{L}(x) = -2\pi |x|. \] (4.66)

The crossing parameter in the classical limit is \( \eta_0 = \pi/2 \).

The two-point Lagrangian for the Faddeev–Volkov model (4.59) in the classical limit is
\[
\mathcal{L}(\alpha | x, y) = \frac{1}{i} \int_0^{\alpha - y} \log \frac{\cosh(z + i\alpha)}{\cosh(z - i\alpha)} \, dz + 2 \int_0^\alpha \log 2 \cos(z) \, dz. \tag{4.67}
\]

The one-point Lagrangian is zero.

Both hyperbolic models (4.55) and (4.59), in the classical limit correspond to \( Q_{1,\theta=1} \) and \( Q_{1,\theta=0} \) with \( \eta_0 = \pi/2 \) respectively.

### 4.4. Fateev–Zamolodchikov Z\(_N\)-model

Taking a straightforward trigonometric limit, \( \Im \tau \rightarrow +\infty \), in the Kashiwara–Miwa model one obtains from (4.31)
\[
W(\alpha | a, b) = \prod_{j=1}^{a-b} \frac{\sin \left( \eta \left( j - \frac{1}{2} \right) - \frac{1}{2} \alpha \right)}{\sin \left( \eta \left( j - \frac{1}{2} \right) + \frac{1}{2} \alpha \right)}, \quad S(a) \equiv 1, \quad \eta = \frac{\pi}{N}. \tag{4.68}
\]

The model contains only one integer parameter \( N \geq 2 \). As before the spins take the values \( a, b = 0, 1, \ldots, N-1 \) and the crossing parameter, \( \eta = \pi/N \), takes the same value as in (4.30). The resulting model is exactly the Fateev–Zamolodchikov model [7] obtained in 1982. Obviously, the weights (4.68) retain the symmetries (4.33) and (4.34). They also acquire an additional \( Z_N \)-symmetry as they only depend on the difference of spins \( a - b \pmod{N} \). The factor (2.8) for this case was calculated in [61],
\[
f(\alpha) = \kappa_s \prod_{j=1}^{[N/2]} \frac{\sin \left( \eta \left( j - \frac{1}{2} \right) + \frac{1}{2} \alpha \right)}{\sin \left( \eta j - \frac{1}{2} \alpha \right)}, \quad \kappa_s = \sqrt{N}. \tag{4.69}
\]

The weights are real and positive when \( 0 < \Re \alpha < \eta \). Taking the limit of (4.37) when \( \Im \tau \rightarrow +\infty \), one obtains
\[ \log \kappa_r(\alpha) = -\frac{1}{2} \int_0^\infty \frac{\sinh(N-1)\pi x}{\cosh \pi x \cosh N\pi x} \sinh 2N\alpha x \frac{dx}{x}. \]  

(4.70)

Finally, note that the weights \((4.68)\) are self-dual \([1]\)

\[ W(\eta - \alpha \mid a, b) = N^{-1} f(\alpha) \sum_{k=0}^{N-1} \omega^k(a-b) W(\alpha \mid k, 0), \quad \omega = e^{2\pi i/N}. \]  

(4.71)

The scalar factor in front of the sum can be calculated in the same way as the factor \(\mathcal{R}_{123}\) in section 2.3. Consider both sides of \((4.71)\) as an element \((a, b)\) of some matrix. Taking the determinant of this matrix one immediately obtains \((2.8)\).

4.4.1. Classical limit of the Fateev–Zamolodchikov model. The classical limit can be taken similarly to the classical limit for Kashiwara–Miwa model (in fact, it is just the trigonometric limit). The regime of imaginary \(\alpha\) and convention \((4.44)\) give the canonically normalized two-point Lagrangian

\[ L(\alpha \mid x, y) = i \int_0^{x-y} \log \frac{\sin \left(\frac{1}{2}(z - i\alpha)\right)}{\sin \left(\frac{1}{2}(z + i\alpha)\right)} dz + \int_0^\alpha \log \tanh \frac{z}{2} dz, \]  

(4.72)

where

\[ i \int_0^y \log \frac{\sin \left(\frac{1}{2}(z - i\alpha)\right)}{\sin \left(\frac{1}{2}(z + i\alpha)\right)} dz = \pi|x| - \frac{x^2}{2} - 2 \int_0^x \arctan \frac{\sin z}{e^{\alpha} - \cos z} dz, \]  

(4.73)

This classical limit corresponds to \(Q_{3,0=0}\) with \(\eta_0 = 0\).

4.5. A new trigonometric model with an infinite number of spin states

There exists another trigonometric limit of the Kashiwara–Miwa model. Consider the limit of the weights \((4.31)\) when

\[ \tau \to 0, \quad N \to \infty, \quad \alpha \to 0, \]  

(4.74)

but the variables

\[ \bar{\alpha} = -\alpha/\tau, \quad \bar{\eta} = -\frac{\pi}{N\tau}, \quad \text{Re} \bar{\eta} > 0, \]  

(4.75)

are kept fixed. It is convenient to define

\[ w = \exp(i\bar{\alpha}), \quad \bar{q} = \exp(2i\bar{\eta}), \quad |\bar{q}| < 1, \]  

(4.76)

which are exactly the same variables as in \((4.38)\). Introduce standard notations for \(q\)-products

\[ (a; q)_\infty = \prod_{k=0}^\infty (1 - a q^k), \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}, \]  

(4.77)

Making the Jacobi imaginary transformation in \((4.29)\) and \((4.31)\), dropping off associated exponentials of quadratic forms in the spins \(a, b\) (since they cancel out from the star-triangle relation) and taking the limit \((4.74)\) and \((4.75)\), one obtains
where

\[ S(a) = \frac{1}{2} (q^{a + \frac{1}{2}} + q^{-a - \frac{1}{2}}), \]

The spins now take (infinitely many) arbitrary integer values

\[ a, b, c, \ldots \in \mathbb{Z}. \]

The model contains two parameters: an integer \( \zeta \) and a complex parameter \( \tilde{q} \).

Note that by a uniform shift of all spins the parameter \( \zeta \) can be reduced to \( \zeta = 0 \) or \( \zeta = 1 \). The reflection symmetry of the weights (4.34) remains intact. The weights (4.78) are real and positive when \( \tilde{q} \) and \( w \) are real and \( 1 > w > \tilde{q}^\frac{1}{2} \).

The star-triangle relation for this trigonometric model reduces to the summation formula (F.1) for a particular balanced very-well-poised \( \psi/\psi \) series, defined in (F.4). The summation formula (F.1) appears to be new, see appendix F for further details.

The factor (2.8) simplifies to

\[ f(\bar{\alpha}) = \kappa_\alpha (q^2; q^2)_\infty, \quad \kappa_\alpha = (q_\alpha^2; q^2)_\infty, \quad w = e^{i\bar{\alpha}}, \]

while (4.39) reduces to

\[ \kappa_\alpha(\bar{\alpha}) = z_\alpha(w), \]

where \( z_\alpha(w) \) is defined (4.42) (the exponential factor from (4.39) is absent in (4.83) since it was removed from the weights (4.78)). Interestingly, the single-edge partition function (4.83) essentially coincides with that of the Ising model in (4.41). Indeed, apart from the trivial exponent which was absorbed into the normalization of weights, the two expressions differ by a rather simple factor \( z_\alpha(w) \), given by (4.42).

4.5.1. Classical limit of the trigonometric model. The classical limit corresponds to \( \tilde{q} \to 1 \),

\[ \tilde{q} = e^{-\hbar}, \quad x = /\hbar. \]

Assuming \( \bar{\alpha} \) to be real and using convention (4.44), one obtains the following two-point Lagrangian:

\[ \mathcal{L}(\bar{\alpha} | x, y) = i \int_0^{x-y} \log \frac{\sinh \left( \frac{\bar{\alpha}}{2}(z - i\bar{\alpha}) \right)}{\sinh \left( \frac{\bar{\alpha}}{2}(z + i\bar{\alpha}) \right)} \frac{dz}{z} + i \int_0^{x+y} \log \frac{\cosh \left( \frac{\bar{\alpha}}{2}(z - i\bar{\alpha}) \right)}{\cosh \left( \frac{\bar{\alpha}}{2}(z + i\bar{\alpha}) \right)} \frac{dz}{z} \]

This Lagrangian corresponds to \( Q_{3,\delta=1} \) with \( \eta_0 = 0 \).

4.6. Gamma-function limit

The Gamma-function limit of the hyperbolic star-triangle relations (4.50) for the Boltzmann weights (4.55) corresponds to...
and $b^2 = i\epsilon$, $\epsilon \to 0$. Let

$$W(\alpha \mid x, y) = \frac{\Gamma(i(x - y) - \alpha) \Gamma(i(x + y) - \alpha)}{\Gamma(i(x - y) + \alpha) \Gamma(i(x + y) + \alpha)}.$$

(4.87)

$$\mathcal{W}(\alpha \mid x, y) = \Gamma(\alpha + i(x - y))\Gamma(\alpha - i(x - y))\Gamma(\alpha + i(x + y))\Gamma(\alpha - i(x + y)),$$

(4.87)

and

$$S(x) = \frac{1}{\pi} x \sinh(2\pi x) = \frac{1}{2} \frac{1}{\Gamma(2\pi x)\Gamma(-2\pi x)}.$$

(4.88)

where $x, y \in \mathbb{R}$, and $0 < \alpha < \pi$. The star-triangle equation is

$$\int_{-\infty}^{\infty} dx \ S(x) \mathcal{W}(\alpha_1 \mid x_1, x)W(\alpha_1 + \alpha_3 \mid x_2, x) \mathcal{W}(\alpha_3 \mid x, x_3) = RW(\alpha_1 \mid x_2, x_3)\mathcal{W}(\alpha_1 + \alpha_3 \mid x_1, x_3)W(\alpha_3 \mid x_2, x_3),$$

(4.89)

where

$$R = 2\pi \frac{\Gamma(2\alpha_1)\Gamma(2\alpha_2)}{\Gamma(2(\alpha_1 + \alpha_2))}.$$

(4.90)

For fixed choices of $x_1, x_2, x_3, \alpha_1, \alpha_3$, the integrand in (4.89) is an even function with respect to the integration variable $x$. Note that in this case the star-triangle relation obtained by reversing the orientation of rapidity lines (see section 2) is unfortunately not satisfied.

There are some differences between this model and the models of the previous subsections which are worth mentioning. Firstly one of the weights in (4.87) is symmetric in the spins $\mathcal{W}(\alpha \mid x, y) = \mathcal{W}(\alpha \mid y, x)$, while the other one is not $W(\alpha \mid x, y) = W(\alpha \mid y, x)$. In general this non-symmetric weight has a non-vanishing imaginary component $\text{Im}(W(\alpha \mid x, y)) \neq 0$, and thus the underlying edge-interaction model is non-physical. Also in this case the weights do not possess the simple crossing symmetry property defined in section 2, and consequently the special properties of the canonical normalization defined in section 2.5 will not apply here.

The same limiting procedure (4.86), applied to the Boltzmann weights (4.59), gives the star-triangle equation (4.89) with

$$W(\alpha \mid x, y) = \frac{\Gamma(i(x - y) - \alpha)}{\Gamma(i(x - y) + \alpha)}, \quad S(x) = 1,$$

$$\mathcal{W}(\alpha \mid x, y) = \Gamma(\alpha + i(x - y))\Gamma(\alpha - i(x - y)),$$

(4.91)

and $R$ given by (4.90). Similar to the preceding model, the spins take values $x, y \in \mathbb{R}$ and the spectral parameter is restricted to $0 < \alpha < \pi$. In this case, one Boltzmann weight is symmetric $\mathcal{W}(\alpha \mid x, y) = \mathcal{W}(\alpha \mid y, x)$, but the other Boltzmann weight satisfies $W(\alpha \mid y, x) = W^*(\alpha \mid x, y)$, where $^*$ denotes the complex conjugate. In general $W(\alpha \mid x, y)$ has non-vanishing imaginary component $\text{Im}(W(\alpha \mid x, y)) \neq 0$, thus the model is non-physical.

In this case, because of the spin reflection symmetries satisfied by the Boltzmann weights, the star-triangle relation with rapidity lines reversed
\[
\int_{-\infty}^{\infty} \, dx \, S(x) \, W(\alpha_1 \mid x, x_1) \, W(\alpha_1 + \alpha_3 \mid x, x_2) \, \overline{W}(\alpha_3 \mid x_3, x) = RW(\alpha_1 \mid x_1, x_2) \, \overline{W}(\alpha_1 + \alpha_3 \mid x_3, x_1) \, W(\alpha_3 \mid x_1, x_2),
\]
(4.92)
is satisfied along with (4.89). However one may restrict consideration to (4.89) since both of these star-triangle relations give the same three-leg forms in the quasi-classical expansion.

4.6.1. **Classical limit of the Gamma function models.** Here there is an extra complication not present in the other models of this paper. In the case of all other models here, one may keep the unscaled spins \(x_1, x_2, x_3\) and spectral parameters \(\alpha_1, \alpha_3\) in the physical regime, and there is a unique maximum of the action functional \(A\), around which one makes the quasi-classical expansion.

For the Gamma function models one does not have a physical regime (one of the weights has non-vanishing imaginary component). As discussed in section 3, the classical equation of motion is evaluated on solutions of the additive three-leg form of one of the \(Q\)-type equations of the ABS list. However for \(Q_{1,\delta=1}\) and \(Q_2\), one does not have a physical regime where solutions of the additive three leg form encompass the solutions of the associated quad equations.

The classical limit of the model is just given by the Stirling approximation of Gamma-functions. The two-point Lagrangians coming from (4.89) are

\[
\mathcal{L}(\alpha \mid x, y) = 2 \int_{0}^{x-y} \arctan \frac{z}{\alpha} \, dz + 2 \int_{0}^{x+y} \arctan \frac{z}{\alpha} \, dz - 3 \alpha \log|\alpha|,
\]
(4.93)
and

\[
\mathcal{L}(\alpha \mid x, y) = -\mathcal{L}(\alpha \mid x, y) + 2\pi|x|.
\]
(4.94)

Note that \(\mathcal{L}\) is not symmetric, the correct ordering of spin arguments follow from (4.89). The Lagrangians are canonically normalized, coefficient (4.90) is taken into account. The one-point Lagrangian is

\[
\mathcal{C}(x) = 2\pi|x|.
\]
(4.95)

The classical limit of this model thus corresponds to \(Q_2\) with \(\eta_0 = 0\). Note that contrary to all other classical models with \(\eta_0 = 0\), \(\mathcal{L}\) is not symmetric, relation (4.45) does not hold and the one-point Lagrangian is not zero.

The two-point Lagrangians coming from (4.92) is

\[
\mathcal{L}(\alpha \mid x, y) = \pi|x|-2 \int_{0}^{x-y} \arctan \frac{z}{\alpha} \, dz + \alpha \log|\alpha|,
\]
(4.96)
relation (4.45) holds and one-point Lagrangian is zero. The classical limit of this model thus corresponds to \(Q_{1,\delta=1}\) with \(\eta_0 = 0\).

4.7. **Zamolodchikov’s ‘fishing net’ model**

In 1980 Zamolodchikov [8] obtained a solution of the star-triangle relation motivated by considerations of ‘fishing-net’ diagrams in quantum field theory. The model has continuous spins, which are vectors in \(D\)-dimensional Euclidean space \(x_1, x_2, \ldots \in \mathbb{R}^D\), where \(D \geq 1\). The canonically normalized weights read [8]

\[
\mathcal{W}(\alpha \mid x_1, x_2) = A(\alpha)|x_1 - x_2|^{-D/2}, \quad S(x) = 1, \quad x, x_1, x_2 \in \mathbb{R}^D,
\]
(4.97)
where \(|x_1 - x_2|\) is the Euclidean distance between the points \(x_1, x_2 \in \mathbb{R}^D\) and the crossing parameter is
The normalization coefficient in (4.97) is given by

\[
A(\alpha) = \frac{\pi^{-D_\alpha/2\pi}}{\Gamma(D/2 - D\alpha/2\pi)} \times \prod_{\ell = 1}^{\infty} \frac{\Gamma(D\ell - D/2 + D\alpha/2\pi) \Gamma(D\ell - D\alpha/2\pi) \Gamma(D\ell + D/2 - D\alpha/2\pi) \Gamma(D\ell + D\alpha/2\pi) \Gamma(D\ell - D/2)}{\Gamma(1/2 + \alpha/D + \alpha/2\pi) \Gamma(1/2 + \alpha/D - \alpha/2\pi)},
\]

(4.99)

for any values of \(D > 1\). The last formula simplifies for even \(D\)’s

\[
A(\alpha) = \left(\frac{D}{\pi}\right)^{D_\alpha/2\pi} \prod_{n=0}^{D_\alpha/2-1} \frac{\Gamma(1/2 + n/D + \alpha/2\pi) \Gamma(1/2 + n/D - \alpha/2\pi)}{\Gamma(1/2 + \alpha/D + \alpha/2\pi) \Gamma(1/2 + \alpha/D - \alpha/2\pi)}, \quad D = \text{even}.
\]

(4.100)

The function \(A(\alpha)\) is a ‘minimal solution’ (in the sense defined in section 4.2) of the functional equations (2.20), which in this case read

\[
A(\alpha)A(-\alpha) = 1, \quad A(\alpha)/A(\pi - \alpha) = \pi^{D(\pi - 2\alpha)/2\pi} \frac{\Gamma(D\alpha/2\pi)}{\Gamma(D(\pi - \alpha)/2\pi)}.
\]

(4.101)

The boundary conditions read

\[
\mathcal{W}(\alpha | x_1, x_2) = 1 + O(\alpha),
\]

\[
\mathcal{W}(\pi - \alpha | x_1, x_2) = \delta^D(x_1 - x_2) + O(\alpha), \quad \alpha \to 0,
\]

(4.102)

where \(\delta^D(x)\) is the \(D\)-dimensional \(\delta\)-function. The inversion relations read

\[
\lim_{\varepsilon \to 0^+} \int d^Dx \mathcal{W}(\pi - \alpha - \varepsilon | x_1, x) \mathcal{W}(\pi + \varepsilon | x_1, x_2) = \mathcal{W}(\delta^D(x_1 - x_2)),
\]

(4.103)

and \(\varepsilon\) is real and the integral is taken over the whole \(\mathbb{R}^D\). The self-duality relation reads

\[
\mathcal{W}(\pi - \alpha | x_1, x_2) = \int d^Dk e^{2\pi i(x_1 - x_2)} \mathcal{W}(\alpha | k, 0).
\]

(4.104)

The weights (4.97) satisfy the star-triangle relation (2.4) where the sum is replaced by the \(D\)-dimensional integral over \(\mathbb{R}^D\) and the parameter \(\eta = \pi\). The coefficient \(\mathcal{R}_{123} = 1\), because of the canonical normalization of weights. The star-triangle relation for this model

\[
\int d^Dx \mathcal{W}(\alpha | x, x_1) \mathcal{W}(\pi - \alpha - \beta | x, x_2) \mathcal{W}(\beta | x, x_3)
\]

\[
= \mathcal{W}(\alpha - \beta | x_2, x_3) \mathcal{W}(\alpha + \beta | x_1, x_3) \mathcal{W}(\pi - \beta | x_1, x_2),
\]

(4.105)

follows from Symanzik’s result [72].

4.7.1. Classical limit of the fishing net model. Consider the star-triangle equation for the fishing-net model in the limit \(D \to \infty\). External ‘spins’ \(x_i\) form a three-dimensional subspace \(\mathbb{E}^d \subset \mathbb{E}^D\), where \(d = 3\) but in fact we use only the condition \(d \ll D\). The central spin \(x_0\) can be decomposed as

\[
x_0 = \pi_0 + y, \quad \pi_0 \in \mathbb{E}^d, \quad y \in \mathbb{E}^{D-d}.
\]

(4.106)

The saddle point of the limit \(D \to \infty\) gives two equations for \(y\) and \(\pi_0\): the scalar equation for \(y^2\),
\[ \frac{-\eta}{y^2} + \sum_{i=1}^{3} \frac{\alpha_i}{(x_0 - x_i)^2 + y^2} = 0, \tag{4.107} \]

and equation for \( x_0 \) in \( \mathbb{R}^d \),
\[ \sum_{i=1}^{3} \alpha_i \frac{x_0 - x_i}{(x_0 - x_i)^2 + y^2} = 0. \tag{4.108} \]

Here
\[ \alpha_1 + \alpha_2 + \alpha_3 = 2\eta, \tag{4.109} \]

and the natural scale is \( \eta = \pi \). Equations (4.107) and (4.108) have the rational solution on which the other three-legs equations are satisfied:
\[ \alpha_i \frac{x_i - x_0}{(x_i - x_0)^2 + y^2} = (\eta - \alpha_j) \frac{x_j - x_k}{(x_j - x_k)^2} + (\eta - \alpha_k) \frac{x_k - x_j}{(x_k - x_j)^2}, \quad i, j, k = \text{perm}(1, 2, 3) \tag{4.110} \]

The crossing parameter \( \eta \) is free in equations (4.107) and (4.108), and in particular \( y^2 \sim \eta \). Thus, in the limit \( \eta \to 0 \) the variable \( y \) disappears and the result is the single saddle point equation in \( \mathbb{R}^d \),
\[ \sum_{i=1}^{3} \alpha_i \frac{x_0 - x_i}{(x_0 - x_i)^2} = 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 0, \tag{4.111} \]

providing all the others:
\[ \alpha_i \frac{x_i - x_0}{(x_i - x_0)^2} + \alpha_j \frac{x_i - x_k}{(x_i - x_k)^2} + \alpha_k \frac{x_k - x_j}{(x_k - x_j)^2} = 0, \quad i, j, k = \text{perm}(1, 2, 3). \tag{4.112} \]

In the application to the ABS-type systems, equations (4.111) and (4.112) are the three-legs equations with
\[ \psi(\alpha | x, y) = \alpha \frac{x - y}{(x - y)^2}, \quad x, y \in \mathbb{R}^d, \quad \eta = 0. \tag{4.113} \]

Equation (4.111) in ABS notations
\[ (\alpha_1 - \alpha_2) \frac{x_{12} - x}{(x_{12} - x)^2} = \alpha_1 \frac{x_1 - x}{(x_1 - x)^2} - \alpha_2 \frac{x_2 - x}{(x_2 - x)^2}, \tag{4.114} \]

uniquely defines the point \( x_{12} \) in terms of three points \( x_1, x_2 \), and \( x_3 \),
\[ x_{12} = x + (\alpha_1 - \alpha_2) \frac{\left((\alpha_1 - \alpha_2) \frac{x_1 - x}{(x_1 - x)^2} - \alpha_2 \frac{x_2 - x}{(x_2 - x)^2}\right)}{\left((\alpha_1 - \alpha_2) \frac{x_1 - x}{(x_1 - x)^2} - \alpha_2 \frac{x_2 - x}{(x_2 - x)^2}\right)^2}, \tag{4.115} \]

and gives thus the example of the vector extension of \( Q_{1,\delta=0} \) [73].

5. Conclusion

In this paper we review the exactly solved edge interaction models of statistical mechanics and establish their connection to classical discrete integrable evolution equations.
classified by Adler et al [15]. We only consider the case of a single spin degree of freedom at each site of the lattice. The Boltzmann weights for all such models can be obtained from different particular cases of the master solution of the star-triangle relation [12]. From the algebraic point of view this solution is related [35] to the modular double of the Sklyanin algebra. The corresponding classical evolution equations are denoted as $Q_4$, they are located at the top of the ABS list [15]. Similarly to the case of lattice model the simpler equations $Q_1, Q_2, Q_3$ can in principle be obtained as different particular cases of $Q_4$. However, since the limiting procedure is not always very transparent, we present separate considerations of the correspondence to lattice models for all particular cases. The main idea of this correspondence is related to the low-temperature (or quasi-classical) limit of the lattice model. We found that in this limit all known edge interaction models always reduce to equations from the ABS list. The correspondence is complete in the sense that for any equation in the ABS list there is at least one lattice counterpart. Here we only consider the $Q$-type equations and the lattice models with crossing symmetry. Apparently, the correspondence can easily be extended to whole ABS list, though the related lattice models are expected to be unphysical (i.e., to have negative Boltzmann weights, similar to the case of $Q_2$ and $Q_{1,\phi=-1}$ in this paper) and, therefore, are of a limited interest in statistical mechanics.

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Appendix A. List of classical star-triangle relations

For convenience here is a list of Lagrangian functions that appear in section 4 and satisfy the star-triangle relation (3.17).

Master solution: $Q_4, \eta = \frac{\pi \tau}{2}$

$\mathcal{C}(x) = \left(2|x| - \frac{\pi}{2}\right)^2$,

$\mathcal{L}(\alpha \mid x, y) = \frac{1}{i} \int_0^{x+y} \log \frac{\vartheta_4(z + i\alpha \mid \tau)}{\vartheta_4(z - i\alpha \mid \tau)} \, dz + \frac{1}{i} \int_{x/2}^{x+y/2} \log \frac{\vartheta_4(z + i\alpha \mid \tau)}{\vartheta_4(z - i\alpha \mid \tau)} \, dz$. (A.1)

Kashiwara–Miwa model: $Q_4, \eta = 0$

$\mathcal{C}(x) = 0$,

$\mathcal{L}(i\alpha \mid x, y) = i \int_0^{x-y} \log \frac{\vartheta_4\left(\frac{1}{2} (z - i\alpha) \mid \tau\right)}{\vartheta_4\left(\frac{1}{2} (z + i\alpha) \mid \tau\right)} \, dz + i \int_0^{x+y} \log \frac{\vartheta_4\left(\frac{1}{2} (z - i\alpha) \mid \tau\right)}{\vartheta_4\left(\frac{1}{2} (z + i\alpha) \mid \tau\right)} \, dz$

$+ \int_0^\alpha \log \left| \frac{\vartheta_2\left(\frac{1}{2} z \mid \frac{1}{\tau}\right)}{\vartheta_2\left(\frac{1}{2} z \mid \frac{1}{2\tau}\right)} \right| \, dz$. (A.2)
Hyperbolic limit of master solution: $Q_{3, \delta = 1}$, $\eta = \frac{\pi}{2}$

$$C(x) = -2\pi|x|,$$

$$\mathcal{L}(\alpha | x, y) = \frac{1}{i} \int_{0}^{x-y} \log \frac{\cosh(z + i\alpha)}{\cosh(z - i\alpha)} \, dz + \frac{1}{i} \int_{0}^{x+y} \log \frac{\cosh(z + i\alpha)}{\cosh(z - i\alpha)} \, dz + 2 \int_{0}^{\alpha} \log 2 \cos(z) \, dz.$$  

(A.3)

Trigonometric solution: $Q_{3, \delta = 1}$, $\eta = 0$

$$C(x) = 0,$$

$$\mathcal{L}(\alpha | x, y) = i \int_{0}^{x-y} \log \frac{\sinh\left(\frac{1}{2}\left(z - i\alpha\right)\right)}{\sinh\left(\frac{1}{2}\left(z + i\alpha\right)\right)} \, dz + i \int_{0}^{x+y} \log \frac{\cosh\left(\frac{1}{2}\left(z - i\alpha\right)\right)}{\cosh\left(\frac{1}{2}\left(z + i\alpha\right)\right)} \, dz + \int_{0}^{\alpha} \log|2\sin z| \, dz.$$  

(A.4)

Faddeev–Volkov model: $Q_{3, \delta = 0}$, $\eta = \frac{\pi}{2}$

$$C(x) = 0,$$

$$\mathcal{L}(\alpha | x, y) = \frac{1}{i} \int_{0}^{x-y} \log \frac{\cosh(z + i\alpha)}{\cosh(z - i\alpha)} \, dz + 2 \int_{0}^{\alpha} \log 2 \cos(z) \, dz.$$  

(A.5)

Fateev–Zamolodchikov model: $Q_{3, \delta = 0}$, $\eta = 0$

$$C(x) = 0,$$

$$\mathcal{L}(\alpha | x, y) = \pi|x| + i \int_{0}^{x-y} \log \frac{\sinh\left(\frac{1}{2}\left(z - i\alpha\right)\right)}{\sinh\left(\frac{1}{2}\left(z + i\alpha\right)\right)} \, dz + \int_{0}^{\alpha} \log \tanh\left(\frac{z}{2}\right) \, dz.$$  

(A.6)

Gamma function model: $Q_{2}$, $\eta = 0$

$$C(x) = 2\pi|x|,$$

$$\mathcal{L}(\alpha | x, y) = -2 \int_{0}^{x-y} \arctan\left(\frac{z}{\alpha}\right) \, dz - 2 \int_{0}^{x+y} \arctan\left(\frac{z}{\alpha}\right) \, dz + 3\alpha \log|\alpha| + 2\pi|x|,$$

$$\mathcal{Z}(\alpha | x, y) = 2 \int_{0}^{x-y} \arctan\left(\frac{z}{\alpha}\right) \, dz + 2 \int_{0}^{x+y} \arctan\left(\frac{z}{\alpha}\right) \, dz - 3\alpha \log|\alpha|.$$  

(A.7)

Gamma function model: $Q_{1, \delta = 1}$, $\eta = 0$

$$C(x) = 0,$$

$$\mathcal{L}(\alpha | x, y) = \pi|x| - 2 \int_{0}^{x-y} \arctan\left(\frac{z}{\alpha}\right) \, dz + \alpha \log|\alpha|.$$  

(A.8)

$D = 1$ fishing net model: $Q_{1, \delta = 0}$, $\eta = 0$

$$C(x) = 0,$$

$$\mathcal{L}(\alpha | x, y) = \alpha \log|x - y| - \frac{\alpha}{2} \log|\alpha|.$$  

(A.9)
Appendix B. Identity for the derivatives of the three-leg form

The next-to-leading order quasi-classical expansion of the star-triangle relation, gives the following new identity involving the derivative of a three leg equation

\[
\frac{\partial^2}{\partial x^2} (\mathcal{L}(\eta - \alpha | x, x_i) + \mathcal{L}(\alpha + \beta | x, x_2) + \mathcal{L}(\eta - \beta | x_3, x))|_{x=x_0} = 0.
\]  

(B.1)

The Lagrangians \( \mathcal{L}(\alpha | x, y) \) are listed in appendix A, and \( x_0 \) is the unique solution to the three leg equation

\[
\frac{\partial}{\partial x} (\mathcal{L}(\eta - \alpha | x, x_i) + \mathcal{L}(\alpha + \beta | x, x_2) + \mathcal{L}(\eta - \beta | x_3, x)) = 0,
\]  

(B.2)

for fixed choices of \( x_1, x_2, x_3, \alpha, \beta \). An example of the function \( \mathcal{L}^{(0)}(\alpha | x, y) \) for each of the equations in the \( Q \) list is given below.

Master solution \( (Q_0, \eta = 0) \):

\[
\mathcal{L}^{(0)}(\alpha | x, y) = \frac{-2i\eta_1(2x | \tau)\eta_1(2y | \tau)\eta((p^2, p^2)_{\infty})p^{1/4}}{\eta_1(x + y - i\alpha | \tau)\eta_1(x - y + i\alpha | \tau)\eta_1(x - y - i\alpha | \tau)}.
\]  

(B.3)

Hyperbolic limit of the master solution \( (Q_{3, \beta=1}, \eta = 0) \):

\[
\mathcal{L}^{(0)}(\alpha | x, y) = \frac{\sinh x \sinh y \sin \alpha}{2 \sinh \frac{x+y+i\alpha}{2} \sinh \frac{x-y-i\alpha}{2} \sinh \frac{x+y-i\alpha}{2} \sinh \frac{x-y+i\alpha}{2}}.
\]  

(B.4)

Faddeev–Volkov model \( (Q_{3, \beta=0}, \eta = 0) \):

\[
\mathcal{L}^{(0)}(\alpha | x, y) = \frac{\sin \alpha}{2 \sinh \frac{x+y+i\alpha}{2} \sinh \frac{x-y-i\alpha}{2}}.
\]  

(B.5)

Gamma function model \( (Q_2, \eta = 0) \):

\[
\mathcal{L}^{(0)}(\alpha | x, y) = \frac{8\eta \alpha}{(x+y+i\alpha)(x+y-i\alpha)(x+y+i\alpha)(x-y-i\alpha)}.
\]  

(B.6)

Gamma function model \( (Q_{1, \beta=1}, \eta = 0) \):

\[
\mathcal{L}^{(0)}(\alpha | x, y) = \frac{2\alpha}{(x+y+i\alpha)(x-y-i\alpha)}.
\]  

(B.7)

\( d = 1 \) Zamolodchikov fishing net model \( (Q_{1, \beta=0}, \eta = 0) \):

\[
\mathcal{L}^{(0)}(\alpha | x, y) = \frac{|\alpha|}{(x-y)^2}.
\]  

(B.8)

Appendix C. Limits of elliptic gamma-functions

The elliptic gamma-function \( \Phi(z) \) is defined by (4.6). Two particular limits should be considered. The classical limit corresponds to
In this limit the leading asymptotics are

$$\Phi(z) = \exp \left\{ -\frac{1}{\pi \tau'} \int_0^\infty \log \overline{\eta}_4(t \mid \tau) \, dt + O(1) \right\}.$$  \hspace{1cm} (C.2)

For the normalization function $K(\alpha)$ defined in (4.7)

$$K(\alpha) = \exp \left\{ -\frac{1}{\pi \tau'} \int_0^{2\pi} \log \overline{\eta}_4(t \mid 2\tau) \, dt + O(1) \right\}.$$  \hspace{1cm} (C.3)

Here the Jacobi theta function $\overline{\eta}_4$ is given by

$$\overline{\eta}_4(z \mid \tau) = \prod_{n=0}^{\infty} (1 - e^{2\pi i q^{2n+1}})(1 - e^{-2\pi i q^{2n+1}}).$$  \hspace{1cm} (C.4)

Another limit is the hyperbolic case. Let

$$\tau = \frac{b}{T}, \quad \tau' = i \frac{b^{-1}}{T}, \quad T \to \infty.$$  \hspace{1cm} (C.5)

Below we use the hyperbolic notations

$$\eta = \frac{1}{2} (b + b^{-1}), \quad q = e^{i\pi b}, \quad \bar{q} = e^{-i\pi b^{-1}}, \quad \bar{q} = i \exp \left\{ \frac{i\pi (b - b^{-1})}{2(b + b^{-1})} \right\}.$$  \hspace{1cm} (C.6)

$$(x, q)_\infty \overset{\text{def}}{=} \prod_{k=0}^{\infty} (1 - q^k x).$$  \hspace{1cm} (C.7)

In this limit

$$\log \Phi(z) = -\frac{T^2}{6} \left( 1 - \frac{z}{\pi} \right) \left( 1 - \frac{2z}{\pi} \right) + \frac{i}{12} (b^2 + b^{-2}) \left( \frac{\pi}{2} - z \right) + O(T^{-2}).$$  \hspace{1cm} (C.8)

This is a $\pi$-periodic function written in the interval $0 < z < \pi$. The regular term here has a jump discontinuity at $z = 0$, so there is another limit for small $z$:

$$\Phi \left( \frac{\pi}{T} z \right) = e^{-\pi T^2} \phi(z),$$  \hspace{1cm} (C.9)

where

$$\phi(z) = \exp \left\{ \frac{1}{4} \int_{\mathbb{R}^+} \frac{e^{-2\pi w}}{w \sinh(bw) \sinh(b^{-1}w)} \, dw \right\}.$$  \hspace{1cm} (C.10)

is the symmetric quantum dilogarithm. It is related to the usual quantum dilogarithm

$$\varphi(z) = \exp \left\{ \frac{1}{4} \int_{\mathbb{R} + i\mathbb{R}} \frac{e^{-2\pi w}}{w \sinh(bw) \sinh(b^{-1}w)} \, dw \right\},$$  \hspace{1cm} (C.11)

by

$$\phi(z) = e^{-\frac{\pi z + \frac{1}{\pi} \ln(1-2e^{i\alpha})} \varphi(z).}$$  \hspace{1cm} (C.12)

The normalization function $K(\alpha)$ has the limit

$$K \left( \frac{\pi}{T} \alpha \right) = e^{\pi T^2} K(\alpha),$$  \hspace{1cm} (C.13)
where
\[
\kappa(\alpha) = \exp \left\{ \frac{1}{8} \int_{\mathbb{R}} \frac{e^{2i\omega w}}{\sinh(bw)\sinh(b^{-1}w)\cosh((b + b^{-1})w)} \frac{dw}{w} \right\}. \tag{C.14}
\]
Removal of the principal value gives
\[
\kappa(\alpha) = e^{\pi\alpha^2 + \frac{i}{8}(1-8\alpha^2)}\Psi(2\alpha), \tag{C.15}
\]
where
\[
\Psi(z) = \exp \left\{ \frac{1}{8} \int_{\mathbb{R}+i0} \frac{e^{-2i\omega w}}{\sinh(bw)\sinh(b^{-1}w)\cosh((b + b^{-1})w)} \frac{dw}{w} \right\}. \tag{C.16}
\]
In the limit \(b^2 = i\epsilon\) and \(\epsilon \to 0\), the leading asymptotics of the above functions are
\[
\varphi(bz + i\eta) \sim \frac{1}{2\pi i\sqrt{\epsilon}} e^{-\pi/12z} \left(\frac{2\pi\epsilon}{\epsilon^{1/2}}\right)^i \Gamma(iz), \tag{C.17}
\]
\[
\phi(bz + i\eta) \sim \frac{1}{2\pi b} \left(\frac{2\pi\epsilon}{\epsilon^{1/2}}\right)^i \Gamma(iz), \tag{C.18}
\]
and
\[
\kappa(\eta - b\alpha) \sim \frac{1}{2\pi b} e^{\pi i\epsilon(\pi^2\epsilon^2(1 - b^2))} \Gamma(2\alpha). \tag{C.19}
\]
The classical limit for the hyperbolic functions \(\phi(z)\) and \(\kappa(\alpha)\) in the limit \(b \to 0\) are
\[
\phi\left(\frac{z}{\pi b}\right) = \exp \left\{ \frac{i}{\pi b^2} \int_0^z \log 2 \cosh x \, dx + O(b^2) \right\}, \tag{C.20}
\]
and
\[
\kappa\left(\frac{\alpha}{\pi b}\right) = \exp \left\{ \frac{2i}{\pi b^2} \int_0^{i\alpha} \log 2 \cosh x \, dx + O(1) \right\}. \tag{C.21}
\]

**Appendix D. \(D = 1\) fishnet model limit of master solution**

The algebraic limit of the master solution is related to the Zamolodchikov fishnet model of section 4.7, with \(D = 1\). This limit was previously considered by Rains [74, 75]. Recall the master solution of the star-triangle relation of section 4 in the equivalent following form
\[
\int_{0}^{2\pi} dx_0 \, S(x_0) W(\eta - \alpha_1 | x_1, x_0) W(\alpha_1 + \alpha_3 | x_2, x_0) W(\eta - \alpha_3 | x_3, x_0) = RW(\alpha_1 | x_2, x_3) W(\eta - \alpha_1 - \alpha_3 | x_1, x_3) W(\alpha_3 | x_2, x_1), \tag{D.1}
\]
where
\[
W(\alpha | x, y) = \frac{\Phi(x - y + i\alpha)\Phi(x + y + i\alpha)}{\Phi(x - y - i\alpha)\Phi(x + y - i\alpha)}, \tag{D.2}
\]
and
\[ S(x) = \frac{(p^2; p^2)_\infty (q^2; q^2)_\infty \Phi(2x - i\eta)\Phi(-2x - i\eta)}{4\pi}, \]
\[ R = \frac{\Phi(i\eta - 2i\alpha_1)\Phi(i\eta - 2i\alpha_3)}{\Phi(i\eta - 2i(\alpha_1 + \alpha_3))}. \tag{D.3} \]

The elliptic gamma function \( \Phi \) is defined in (4.6), and \( (x; p)\infty = \prod_{n=0}^\infty (1 - xp^n) \). To obtain the algebraic limit the first step is to take the limit of the spectral parameters and elliptic nomes
\[ \alpha_i = \epsilon \alpha_i, \quad p = e^{i\pi \epsilon} > 0, \quad q = e^{-2\epsilon}, \quad \eta = \epsilon - \frac{i\pi \epsilon}{2}, \quad \epsilon \to 0. \tag{D.4} \]

As \( \epsilon \to 0 \), the asymptotics of the elliptic gamma function are given by\(^{14}\)
\[ \log \frac{\Phi(x + i\alpha \epsilon)}{\Phi(x - i\alpha \epsilon)} = -\alpha \log \left(\frac{\partial_4(x \mid \tau)}{(p^2, p^2)\infty}\right) + O(\epsilon), \]
\[ \log \frac{\Phi(x + i(\eta - \alpha \epsilon))}{\Phi(x - i(\eta - \alpha \epsilon))} = \frac{1}{4\epsilon} (\text{Li}_2(e^{2i\epsilon}) + \text{Li}_2(e^{-2i\epsilon})) \]
\[ - (1 - \alpha) \log \frac{|\partial_1(x \mid \tau)|}{p^{1/2}(p^2, p^2)\infty} + O(\epsilon). \tag{D.5} \]

This gives the following asymptotic of the Boltzmann weights of the master solution (4.9)
\[ \log W(\alpha \epsilon \mid x, y) = -\alpha \log \left(\frac{\partial_4(x \pm y \mid \tau)}{(p^2, p^2)\infty}\right) + O(\epsilon), \]
\[ \log W(\eta - \alpha \epsilon \mid x, y) = \frac{1}{\epsilon} \left(\frac{\pi^2}{6} - \frac{\pi}{2}[x+y]-\frac{\pi}{2}[x-y]+x^2+y^2\right) \]
\[ - (1 - \alpha) \log \frac{|\partial_1(x \pm y \mid \tau)|}{p^{1/2}(p^2, p^2)\infty} + O(\epsilon), \tag{D.6} \]

and
\[ \log S(x) = -\frac{1}{\epsilon} \left(\frac{\pi^2}{8} + 2\pi^2 - \pi|x|\right) - \frac{1}{2} \log(8\pi \epsilon) + \log p^{1/4} \log |\partial_1(2x \mid \tau)| + O(\epsilon), \]
\[ \log R = \frac{\pi^2}{24\epsilon} - \frac{1}{2} \log(8\pi \epsilon) - \log (p^2, p^2)\infty + \log \frac{\Gamma(\alpha_3)\Gamma(\alpha_3)}{\Gamma(\alpha_1 + \alpha_3)} + O(\epsilon). \tag{D.7} \]

Both sides of the star-triangle relation are exponentially suppressed by \( e^{-\pi/\epsilon} \) outside the region
\[ x_0 \in B \equiv \left[ \max(|x_1|, |x_3|), \min(|x_1|, |x_3|) \right] \cup \left[ \min(|x_1|, |x_3|), \max(|x_1|, |x_3|) \right]. \tag{D.8} \]

\(^{14}\) We understand \( |\partial_1(x \mid \tau)| = 2p^{1/4}\text{sin} \frac{\pi}{2}(p^2e^{2i\tau}; p^2)\infty(p^2e^{-2i\tau}; p^2)\infty(p^2; p^2)\infty \).
One then finds the following Boltzmann weights

\[ W(\alpha \mid x, y) = \vartheta_4(x \pm y \mid \tau)^{-\alpha}, \quad \overline{W}(\alpha \mid x, y) = |\vartheta_4(x \pm y \mid \tau)|^{-(1-\alpha)}, \]

\[ S(x) = |\eta^{2}(\tau) \vartheta_4(2x \mid \tau)|, \quad R = \frac{\Gamma(\alpha_1)\Gamma(\alpha_3)}{\Gamma(\alpha_1 + \alpha_3)}, \quad (D.9) \]

where \( \eta(\tau) = p^{1/2}(p^2; p^2)_\infty \), that satisfy the star-triangle relation

\[ \int_B dx_0 S(x_0) W(\alpha_1 \mid x_1, x_0) W(\alpha_1 + \alpha_3 \mid x_2, x_0) \overline{W}(\alpha_3 \mid x_3, x_0) = RW(\alpha_1 \mid x_2, x_3) \overline{W}(\alpha_1 + \alpha_3 \mid x_1, x_3) W(\alpha_3 \mid x_2, x_1), \quad (D.10) \]

The second step is to substitute the following function

\[ x_i = \frac{\vartheta_4(x_i \mid \tau)^2}{\vartheta_4(\tau)^2}, \quad i = 0, 1, 3, \quad y_2 = \frac{\vartheta_4(x_2 \mid \tau)^2}{\vartheta_4(\tau)^2}, \quad \frac{\partial^2 S(x_0)}{\partial^2 \vartheta_4(\tau)^2} dx_0 = \frac{1}{2} dy_0, \quad (D.12) \]

into the star-triangle relation (D.9), and after a change of variables one finds the Boltzmann weights

\[ W(\alpha \mid x, y) = |x - y|^{-\alpha}, \quad R = \frac{\Gamma(\alpha_1)\Gamma(\alpha_3)}{\Gamma(\alpha_1 + \alpha_3)}, \quad (D.13) \]

These Boltzmann weights satisfy the star-triangle relation

\[ \int_{x_0}^{x_3} dx_0 W(\eta - \alpha_1 \mid x_1, x_0) W(\alpha_1 + \alpha_3 \mid x_2, x_0) W(\eta - \alpha_3 \mid x_3, x_0), \quad (D.14) \]

\[ = RW(\alpha_1 \mid x_2, x_3) W(\eta - \alpha_1 - \alpha_3 \mid x_1, x_3) W(\alpha_3 \mid x_2, x_1) \quad (D.15) \]

with \( \eta = 1 \) and assumption \( x_1 < x_0 < x_3 < x_2 \), and is equivalent to the Selberg integral \([74, 75]\).

The star-triangle relation (D.13) is contained in the \( D = 1 \) Zamolodchikov fishing net model, the latter being given in terms of the same weights \( W \) in (D.13)\(^{15}\), only \( R \) for Zamolodchikov fishing net is given by

\[ R = \sqrt{\pi} \frac{\Gamma\left(\frac{\alpha_1}{2}\right)\Gamma\left(\frac{1-\alpha_1+\alpha_3}{2}\right)\Gamma\left(\frac{\alpha_3}{2}\right)}{\Gamma\left(\frac{1-\alpha_3}{2}\right)\Gamma\left(\frac{\alpha_1+\alpha_3}{2}\right)\Gamma\left(\frac{1-\alpha_1}{2}\right)}, \quad (D.16) \]

and the integration in (D.14) is taken over the real axis. This form of the star-triangle relation is more satisfactory from physical considerations, as the value of different spins should be independent of one another.

**Appendix E. Analytic properties of \( \varphi \) and \( \Psi \)**

The function \( \varphi(z) \). This function is defined by the integral (C.11). It has the following properties (with the notations defined in (C.6))

\(^{15}\) These weights differ from (4.97) only by a trivial rescaling \( \alpha \leftrightarrow \pi \alpha \).
(a) Simple poles and zeros

poles of \( \varphi(z) = \{i\lambda + imb + inb^{-1}, \ m, n \in \mathbb{Z}_{\geq 0}\} \),
zeros of \( \varphi(z) = \{-i\lambda + imb + inb^{-1}, \ m, n \in \mathbb{Z}_{\geq 0}\} \). \hfill (E.1)

(b) Functional relations

\[
\varphi(z) \varphi(-z) = e^{iz^2 - ir |(1 - 2\lambda^2)|/6}, \quad \frac{\varphi(z - i b^{1/2})}{\varphi(z + i b^{1/2})} = (1 + e^{2\pi b^{1/2}}).
\] \hfill (E.2)

(c) Asymptotics

\( \varphi(z) \approx 1 \), \( \Re(z) \rightarrow -\infty \); \( \varphi(z) \approx e^{iz^2 - ir |(1 - 2\lambda^2)|/6} \) \hfill (E.3)

where \( \Re(z) \) is kept finite.

(d) Product representation

\[
\varphi(z) = \frac{(-q e^{2\pi z b}; q^2)_{\infty}}{(-\bar{q} e^{2\pi \bar{z} b}; \bar{q}^2)_{\infty}}, \quad \Re b > 0.
\] \hfill (E.4)

(e) Pentagon relation. The function \( \varphi(z) \) satisfy the following operator identity [76]

\[
\varphi(\mathcal{P}) \varphi(\mathcal{X}) = \varphi(\mathcal{X}) \varphi(\mathcal{P} + \mathcal{X}) \varphi(\mathcal{P}), \quad \{\mathcal{P}, \mathcal{X}\} = \frac{1}{2\pi i},
\] \hfill (E.5)

where \([,]\) denotes the commutator. It can be re-written in the matrix form [77, 78]

\[
\int_{\mathbb{R}} \frac{\varphi(x + u)}{\varphi(x + v)} e^{2\pi i w} \, dw = e^{i\pi(1 + 4\lambda)/12 - 2\pi i (w + i\lambda)} \frac{\varphi(u - v - i\lambda)\varphi(w + i\lambda)}{\varphi(u - v + w - i\lambda)},
\] \hfill (E.6)

where the wedge of poles of \( \varphi(x + u) \) must lie in the upper half-plane, the wedge of zeros of \( \varphi(x + v) \) must lie in the down half-plane, and the integrand must decay when \( x \rightarrow \pm \infty \).

The function \( \Psi(z) \). This function is defined by the integral (4.53). It has the following properties.

(i) Simple poles and zeros

poles of \( \Psi(z) = \{2i\lambda + imb + inb^{-1}, \ m, n \in \mathbb{Z}_{\geq 0}\}, \)
\( m + n - |m - n| = 0 \mod 4 \}, \)
zeros of \( \Psi(z) = \{-2i\lambda + imb + inb^{-1}, \ m, n \in \mathbb{Z}_{\geq 0}\}, \)
\( m + n - |m - n| = 0 \mod 4 \}. \) \hfill (E.7)

(ii) Functional relations

\[
\Psi(z) \Psi(-z) = e^{iz^2/2 - ir |(1 - 8\lambda^2)|/12}, \quad \Psi(z + i\lambda)\Psi(z - i\lambda) = \varphi(z).
\] \hfill (E.8)

(iii) Asymptotics

\( \Psi(z) \approx 1 \), \( z \rightarrow -\infty \); \( \Psi(z) \approx e^{iz^2/2 - ir |(1 - 8\lambda^2)|/12}, \quad z \rightarrow +\infty \), \hfill (E.9)

where \( \Im(z) \) is kept finite.
Appendix F. Summation formula for a balanced very-well-poised $8\psi_8$ series

The weights (4.78) satisfy the star-triangle relation (2.4). Introducing the variables $v_i = e^{ib_i}$, $i = 1, 2, 3$, and denoting the integer spins $(a, b, c)$ as $(\sigma_1, \sigma_2, \sigma_3)$, and for shortness using $q$ instead of $\varphi$, one can rewrite (2.4) as an identity for basic hypergeometric series

$$
\sum_{n=-\infty}^{\infty} \frac{q^n + q^{3n+\zeta}}{1 + q^\zeta} \prod_{j=1}^{3} \frac{(q^{-\eta} v_j; q)_n}{(q^{-\eta_0} v_j; q)_n} = \left( \frac{q^2}{1 + q^\zeta} \right) \left( \frac{q^2 v_1^2 + q^2 v_2^2 + q^2 v_3^2}{q^2 v_1^2 + q^2 v_2^2 + q^2 v_3^2} \right) \prod_{i=1}^{3} \frac{(q^{1/2}/v_i; q)_{\eta_0} + \zeta}{(v_i; q)_{\eta_0} - \zeta}(q^{1/2}/v_i; q)_{\eta_0 - \zeta}(-q^{1/2}/v_i; q)_{\eta_0 + \zeta},
$$

where $(i, j, k) = \text{cycle}(1, 2, 3)$ in the second product is a cyclic permutation of $(1, 2, 3)$. The quantities $\sigma_1, \sigma_2, \sigma_3, \zeta \in \mathbb{Z}$ are arbitrary integers, the quantities $q, v_1, v_2$ are indeterminates and $v_3 \equiv q^{1/2}/v_1 v_2$.

Several remarks are in order. First, the identity (F.1) is unvariant under simultaneous shifts

$$
\zeta \rightarrow \zeta - 2d, \quad \sigma_i \rightarrow \sigma_i + d, \quad d \in \mathbb{Z}, \quad i = 1, 2, 3.
$$

This is especially obvious in the original star-triangle form of this relation. Thus, without loss of generality, one can assume that $\zeta = 0$ or $\zeta = 1$. Second, with the standard notation [79] for the bilateral basic hypergeometric series

$$
\sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_k; b_1, b_2, \ldots, b_k; q)_n}{(q; q)_n} z^n,
$$

the sum in the lhs of (F.1) can be written as

$$
\sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_k; b_1, b_2, \ldots, b_k; q)_n}{(q; q)_n} z^n,
$$

where we have denoted

$$
z^\frac{1}{2} = i q^{\frac{1}{4}}, \quad s_1 = q^{\sigma_1}, \quad s_2 = q^{\sigma_2}, \quad s_3 = q^{\sigma_3}, \quad \sigma_1, \sigma_2, \sigma_3, \zeta \in \mathbb{Z}.
$$

Examining relationships between arguments in (F.4) one concludes that (F.1) is a summation formula for a particular balanced very-well-poised series $8\psi_8$. The only similar result we

16 A bilateral basic hypergeometric series $\psi_k$ is well-poised if $a_1 b_1 = a_2 b_2 = \cdots = a_k b_k$ and very-well-poised if, in addition, $a_1 = -a_2 = q b_1 = -q b_2$. Further, the series $\psi_k$ is called balanced if $b_1 \cdots b_k = a_1 \cdots a_k q^k$ and $z = q$. For the most general balanced very-well-poised series $8\psi_8$ all six parameters $z, s_1, s_2, s_3, v_1, v_2$ in (F.4) should be arbitrary.
were able to find in the literature is given by Schlosser (see equation (2.5) in [80]). However, his formula is different, since it requires completely opposite conditions on the arguments in \((F.4)\). In our case the parameters \(v_1, v_2\) are arbitrary indeterminates, but \(s_1, s_2, s_3\) and \((-z)\) are restricted to integral powers of \(q\). In Schlosser’s case the parameters \(v_1^2, v_2^2\) are restricted to integral powers of \(q\), but \(s_1, s_2, s_3, z\) are arbitrary.

Finally note, that for positive values of \(v_1, v_2\) and \(q < 1\), such that

\[
q^2 < v_1 < 1, \quad q^2 < v_2 < 1, \quad q^2 < v_1 v_2 < 1, \quad (F.6)
\]

all terms of the series \((F.1)\) are positive for arbitrary \(\sigma_1, \sigma_2, \sigma_3, \zeta \in \mathbb{Z}\) (and the series is, of course, convergent). That is what makes this series important for applications in statistical mechanics.

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