ABSTRACT. Let $\mu$ be a probability measure on $\text{GL}_d(\mathbb{R})$ which has a finite exponential moment and generates a proximal and strongly irreducible semi-group. Denote by $S_n := g_n \cdots g_1$ the associated random matrix product, where $g_j$ are i.i.d.’s with law $\mu$. We consider $\mathbb{R} \times \mathbb{P}^{d-1}$-valued random variables of the form $(\sigma(S_n, x), S_n x)$ and $(\log |f(S_n v)|, S_n x)$, where $\sigma$ is the norm cocycle and $f(S_n v)$ with $v \in \mathbb{R}^d$, $x = [v] \in \mathbb{P}^{d-1}$, $f \in (\mathbb{R}^d)^*$ encode the coefficients of $S_n$. We provide optimal Berry-Esseen bounds for such variables using a large class of observables on $\mathbb{R}$ and Hölder continuous target functions on $\mathbb{P}^{d-1}$. A Local Limit Theorem for these observables is also discussed.

Keywords: products of random matrices, Berry-Esseen bound, optimal rate, Local Limit Theorem.

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1. INTRODUCTION

Let $\mu$ be a probability measure on $G := \text{GL}_d(\mathbb{R})$ with $d \geq 2$. Consider the random walk on $G$ induced by $\mu$ and given by

$$S_n := g_n \cdots g_1,$$

where $n \geq 1$ and the $g_j$’s are independent and identically distributed (i.i.d.) random elements of $G$ with law $\mu$. The study of the statistical properties of various natural functions associated with $S_n$ is a well-developed topic with strong recent activity. We refer to [BL85, BQ16b] for an overview of the fundamental results.

In this work, we shall assume that $\mu$ has a finite exponential moment, that is, there exists an $\varepsilon > 0$ such that

$$\mathbb{E}(N(g)^\varepsilon) = \int_G N(g)^\varepsilon \, d\mu(g) < \infty,$$

where $N(g) := \max\left(\|g\|, \|g^{-1}\|\right)$. We shall also assume that the closed semi-group generated by the support of $\mu$, denoted by $\Gamma_\mu$, is (1) proximal, that is, it contains an element having a unique eigenvalue of maximal modulus, which is simple, and (2) strongly irreducible, that is, the action of $\Gamma_\mu$ on $\mathbb{R}^d$ does not preserve a finite union of proper linear subspaces. The last two assumptions are standard, easy to check and hold generically. They are necessary in order to obtain most of the meaningful limit theorems for the associated random walk, in particular the ones in this paper.

The group $G$ acts naturally on the real projective space $\mathbb{P}^{d-1}$. Recall that the norm cocycle is the function $\sigma : G \times \mathbb{P}^{d-1} \to \mathbb{R}$ given by

$$\sigma(g, x) = \sigma_g(x) := \log \frac{\|gv\|}{\|v\|}, \quad \text{for } v \in \mathbb{R}^d \setminus \{0\}, x = [v] \in \mathbb{P}^{d-1} \text{ and } g \in G.$$
and the first Lyapunov exponent associated with \( \mu \) is the following limit, which is known to exist

\[
\gamma := \lim_{n \to \infty} \frac{1}{n} E \left( \log \| S_n \| \right) = \lim_{n \to \infty} \frac{1}{n} \int \log \| g_n \cdots g_1 \| \, d\mu(g_1) \cdots d\mu(g_n).
\]

Furstenberg-Kesten’s theorem \([FK60]\) says that the sequence \( \frac{1}{n} \log \| S_n \| \) converges to \( \gamma \) almost surely. This is the analogue of the Law of Large Numbers for sums of real i.i.d.’s. The Central Limit Theorem (CLT) for the norm cocycle has been obtained by Le Page in \([LP82]\), see also Benoist-Quint \([BQ16a]\) for a new proof which holds for measures with a finite second moment. The theorem says that, for every \( x \in \mathbb{P}^{d-1} \), we have that

\[
\frac{1}{\sqrt{n}} \left( \sigma(S_n, x) - n \gamma \right) \rightarrow \mathcal{N}(0; \varrho^2)
\]

in law as \( n \to \infty \), where \( \varrho > 0 \) is a constant and \( \mathcal{N}(0; \varrho^2) \) denotes the centered normal distribution with variance \( \varrho^2 \). Le Page also obtained a “coupled” CLT for the random variable \( (\sigma(S_n, x), S_n x) \in \mathbb{R} \times \mathbb{P}^{d-1} \), in the spirit of Theorem 1.1 below. A CLT for the coefficients of \( S_n \) is much more recent, see \([BQ16b]\).

Once the CLT has been established, it is natural to ask whether we can estimate the speed of convergence in (1.2). The goal of the present work is to obtain such bounds for various test functions on the couples \( (\sigma(S_n, x), S_n x) \) and \( (\log | S_n |, S_n x) \), where \( S_n \) stands for the \( (i, j) \)-entry of \( S_n \), as announced in \([DKW21c]\). In analogy with the case of sums of real i.i.d.’s, the best bound one can hope for is \( O(1/\sqrt{n}) \). For the norm cocycle, some cases were obtained by Le Page \([LP82]\) and Xiao-Grama-Liu \([XGL19]\), see below for more details. For the coefficients, this question was treated by the authors in \([DKW21b]\). Later, a version for the couple \( (\log | S_n |, \varphi(S_n, x)) \) was announced in \([XGL21]\). We deal here with more general test functions which include these known cases. See also \([CDMP21a, CDMP21b, DKW21b]\) for partial results under lower moment conditions.

The above results for the coefficients of \( S_n \) are delicate to obtain. The main challenge comes from the appearance of singularities in the phase space \( \mathbb{P}^{d-1} \), see e.g. \([GQX20]\). In the recent work \([DKW21c]\), we have introduced new ideas allowing us to bypass this difficulty and obtain sharp bounds. As mentioned in that paper, our method can be pushed further to produce similar Berry-Esseen bounds for more general test functions on \( \mathbb{R} \times \mathbb{P}^{d-1} \). This is the content of the present work.

Recall that, under our assumptions, \( \mu \) admits a unique stationary measure, also known as Furstenberg measure. This is the unique probability measure \( \nu \) on \( \mathbb{P}^{d-1} \) satisfying

\[
\int_{\mathbb{P}^{d-1}} g \, d\nu(g) = \nu.
\]

Denote by \( \mathcal{H} \) the space consisting of bounded Lipschitz functions \( \psi \) on \( \mathbb{R} \) such that \( \psi \) is bounded and belongs to \( L^1(\mathbb{R}) \). Define a norm on \( \mathcal{H} \) by

\[
\| \psi \|_{\mathcal{H}} := \| \psi \|_{\infty} + \| \psi' \|_{\infty} + \| \psi' \|_{L^1}.
\]

For an interval \( J \subset \mathbb{R} \), set \( \psi_J(t) := 1_{t \in J} \cdot \psi(t) \). We now state our main results, using the notations introduced above.

**Theorem 1.1.** Let \( \mu \) be a probability measure on \( \text{GL}_d(\mathbb{R}) \). Assume that \( \mu \) has a finite exponential moment and that \( \Gamma_\mu \) is proximal and strongly irreducible. Let \( \gamma, \varrho \) and \( \nu \) be as
above. Then, there are constants \(0 < \alpha < 1\) and \(C > 0\) such that, for any \(\psi \in \mathcal{H}, \varphi \in \mathcal{C}^\alpha(\mathbb{P}^{d-1}), x \in \mathbb{P}^{d-1}\), any interval \(J \subset \mathbb{R}\), and all \(n \geq 1\), we have

\[
\left| \mathbb{E}\left( \psi_j \left( \frac{\sigma(\mathbb{S}_n,x) - n\varphi}{\sqrt{n}} \right) \cdot \varphi(\mathbb{S}_n x) \right) - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{P}^{d-1}} \varphi \, d\nu \int_J e^{-\frac{s^2}{2\sigma^2}} \psi(s) \, ds \right| \leq \frac{C}{\sqrt{n}} \|\psi\|_{\mathcal{H}} \|\varphi\|_{\mathcal{C}^\alpha}.
\]

We now state the analogous result for the coefficients of \(\mathbb{S}_n\). For \(v \in \mathbb{R}^d\) and \(f \in (\mathbb{R}^d)^*\), its dual space, we denote by \(\langle v, f \rangle := f(v)\) their natural coupling. Observe that the \((i, j)\)-entry of a matrix \(g\) is given by \(\langle ge_j, e_i^*\rangle\), where \((e_k)_{1 \leq k \leq d}\) and \((e_k^*)_{1 \leq k \leq d}\) denote the canonical basis of \(\mathbb{R}^d\) and \((\mathbb{R}^d)^*\), respectively.

**Theorem 1.2.** Let \(\mu\) be a probability measure on \(\text{GL}_d(\mathbb{R})\). Assume that \(\mu\) has a finite exponential moment and that \(\Gamma_\mu\) is proximal and strongly irreducible. Let \(\gamma, \varrho\) and \(v\) be as above. Then, there are constants \(0 < \alpha < 1\) and \(C > 0\) such that, for any \(\psi \in \mathcal{H}, \varphi \in \mathcal{C}^\alpha(\mathbb{P}^{d-1}), x := [v] \in \mathbb{P}^{d-1}, y := [f] \in (\mathbb{P}^{d-1})^*,\) any interval \(J \subset \mathbb{R}\), and all \(n \geq 1\), we have

\[
\left| \mathbb{E}\left( \psi_j \left( \frac{\log |S_n^{v,j}| - n\gamma}{\sqrt{n}} \right) \cdot \varphi(S_n x) \right) - \frac{1}{\sqrt{2\pi}} \int_{\mathbb{P}^{d-1}} \varphi \, d\nu \int_J e^{-\frac{s^2}{2\sigma^2}} \psi(s) \, ds \right| \leq \frac{C}{\sqrt{n}} \|\psi\|_{\mathcal{H}} \|\varphi\|_{\mathcal{C}^\alpha}.
\]

The above theorems can be seen as mixed versions of the corresponding limit theorem for the norm cocycle (resp. the coefficients of \(\mathbb{S}_n\)) with the equidistribution property of orbits \(S_n x\) towards the Furstenberg measure \(\nu\) on \(\mathbb{P}^{d-1}\), see the comments after Theorem 2.3.

As mentioned before, particular cases of both Theorems 1.1 and 1.2 can be found in the literature. When \(\psi = 1\) and \(\varphi = 1\), Theorem 1.1 is due to Le Page, see [LP82, BL85], and for \(\psi = 1\) and general \(\varphi\), it is due to Xiao-Grama-Liu [XGL19]. Due to the singularities of the variables \(\log |S_n^{v,j}|\), the case of coefficients is a considerably harder problem and requires refined techniques. The case \(\psi = 1\) and \(\varphi = 1\) was obtained very recently by the authors in [DKW21c] and, as mentioned in that paper, the method developed there turns out to be effective in a more general setting. Using some similar argument, Xiao-Grama-Liu have recently obtained the same result when \(\psi = 1\) and \(\varphi\) is general [XGL21].

The difficulty in proving Berry-Esseen bounds in the presence of non-constant target functions \(\varphi\) is explained in [XGL19, Subsection 1.2]. Roughly speaking, when applying the Berry-Esseen lemma [Fel71, XVI.3] directly, the integrand displays a singularity at the origin that cannot be easily handled. To overcome this, Xiao-Grama-Liu consider a complex contour around the origin and use Cauchy integral formula together with the so called saddle point method. In the present work, inspired by [DKW21c], we observe that the above mentioned singularity is actually “removable” by considering instead the Cauchy principal value of the integral along the imaginary line, see Lemma 2.8 below. This provides a simple solution to the above technical difficulty. In particular, we are not required to consider perturbations of the Markov operator along the real axis.

It is not difficult to deduce from our theorems some weaker Berry-Esseen bounds with rate \(O(\log n/\sqrt{n})\) for the random variable \(\log|\mathbb{S}_n|\) and for the spectral radius of \(\mathbb{S}_n\) and similar test functions. Since this rate is likely not optimal, we choose not to consider these questions in this article. Using the techniques developed here and the results of [DKW21b], we can also obtain analogues of Theorems 1.1 and 1.2 for random matrices in \(\text{GL}_2(\mathbb{R})\) or
GL₂(ℂ) under a finite third moment condition, which is the minimal moment condition one should require for this problem.

From [DKW21c, Th. B], [DKW21b, Th. A] and [LP82], we know that the spectral techniques used in the proof of optimal Berry-Esseen bounds are also useful in proving Local Limit Theorems. In particular, the methods of this article allow us to obtain the following result.

**Theorem 1.3.** Let µ be a probability measure on GL₂(ℝ). Assume that µ has a finite exponential moment and that Γµ is proximal and strongly irreducible. Let γ, g and ν be as above. Then, for any x := [v] ∈ ℙᵈ⁻¹, y := [f] ∈ (ℙᵈ⁻¹)∗ and any continuous function Φ with compact support on ℝ × ℙᵈ⁻¹, we have

\[
\lim_{n \to \infty} \sup_{t \in ℝ} \left| \sqrt{2πn} \, g \left( \Phi \left( t + \log \left| \frac{⟨f, S_nν⟩}{∥f∥∥ν∥} - nγ, S_nx \right| \right) \right) - e^{-\frac{1}{2}t^2} \int_{ℝ×ℙᵈ⁻¹} \Phi(s, w) \, dν(w) \right| = 0.
\]

We do not present the proof of Theorem 1.3 here. This is because, using a standard approximation argument, it can be reduced to the case where Φ(t, w) = ψ(t) · φ(w), which can be treated using the arguments in [DKW21c, Th. B], see also [DKW21b, Th. A] and [LP82]. Theorem 1.3 also holds for larger families of functions Φ, i.e. those which can be approximated in L¹(Leb ⊗ ν), both from above and below, by continuous functions with compact support.

**Notations.** Throughout this article, the symbols ≲ and ≥ stand for inequalities up to a multiplicative constant. The dependence of these constants on certain parameters (or lack thereof), if not explicitly stated, will be clear from the context. We denote by E the expectation and P the probability.

2. Preliminary results

This section contains some known results needed for the proof of the main theorems. For the details, the read may refer to [BQ16b, BL85, DKW21c, LP82]. We always assume that µ has a finite exponential moment and that Γµ is proximal and strongly irreducible, see the Introduction.

2.1. Large deviation estimates and regularity of the stationary measure. We equip ℙᵈ⁻¹ with the distance

\[d(x, w) := \sqrt{1 - \left( \frac{⟨v_x, v_w⟩}{∥v_x∥∥v_w∥} \right)^2}, \quad \text{where} \quad v_x, v_w ∈ ℝ^d \setminus \{0\}, \quad x = [v_x], \quad w = [v_w] ∈ ℙᵈ⁻¹.\]

Note that d(x, w) is the sine of the angle between the lines x and w in ℝᵈ and (ℙᵈ⁻¹, d) has diameter one. Denote by B(w, r) the associated open ball of center w and radius r in ℙᵈ⁻¹.

Let (ℙᵈ⁻¹)∗ be the projectivization of (ℝᵈ)∗. For y ∈ (ℙᵈ⁻¹)∗, we denote by H_y the kernel of y, which is a (projective) hyperplane in ℙᵈ⁻¹. We’ll need the following large deviation estimates.

**Proposition 2.1** ([BQ16b]–Theorem 12.1 and Lemma 14.11). For any ε > 0 there exist c > 0 and n₀ ∈ ℕ such that, for all ℓ ≥ n ≥ n₀, x ∈ ℙᵈ⁻¹ and y ∈ (ℙᵈ⁻¹)∗, one has

\[\mu^{n₀}\{ g ∈ G : |σ(g, x) - nγ| ≥ ne \} ≤ e^{-cn}.\]
such that
\[ \mu_{\operatorname{st}} \{ g \in G : d(gx,H_y) \leq e^{-cn} \} \leq e^{-cn}. \]

For a hyperplane \( H \) in \( \mathbb{P}^{d-1} \) and \( r > 0 \), we denote \( \mathbb{B}(H,r) := \{ x \in \mathbb{P}^{d-1} : d(x,H) < r \} \), which is a “tubular” neighborhood of \( H \). The stationary measure \( \nu \) satisfies the following.

**Proposition 2.2** ([Gu90], [BQ16b]–Theorem 14.1). There are constants \( C > 0 \) and \( \eta > 0 \) such that
\[ \nu(\mathbb{B}(H_y,r)) \leq C r^n \quad \text{for every} \quad y \in (\mathbb{P}^{d-1})^* \quad \text{and} \quad 0 \leq r \leq 1. \]

### 2.2. The Markov operator and its perturbations

The **Markov operator** associated to \( \mu \) is the operator
\[ \mathcal{P}\varphi(x) := \int_G \varphi(gx) \, d\mu(g), \]
acting on functions on \( \mathbb{P}^{d-1} \). For \( z \in \mathbb{C} \), consider the perturbation \( \mathcal{P}_z \) of \( \mathcal{P} \) (also called complex transfer operator) given by
\[ \mathcal{P}_z \varphi(x) := \int_G e^{z \sigma(g,x)} \varphi(gx) \, d\mu(g), \]
where \( \sigma(g,x) \) is the norm cocycle defined in (1.1). Notice that \( \mathcal{P}_0 = \mathcal{P} \) and a direct computation using the cocycle relation \( \sigma(g_2 g_1, x) = \sigma(g_2, g_1 x) + \sigma(g_1, x) \) gives that

\[ \mathcal{P}_z^n \varphi(x) = \int_G e^{z \sigma(g,x)} \varphi(gx) \, d\mu^\otimes n(g), \]

where \( \mu^\otimes n \) is the convolution power of \( \mu \), obtained by projecting the product measure \( \mu^\otimes n \) on \( G^n \) to \( G \) via the map \( (g_n, \ldots, g_1) \mapsto g_n \cdots g_1 \).

We now state some fundamental results of Le Page about the spectral properties of the above operators acting on some Hölder spaces. For \( 0 < \alpha < 1 \), we denote by \( \mathcal{C}^\alpha(\mathbb{P}^{d-1}) \) the space of Hölder continuous functions on \( \mathbb{P}^{d-1} \) equipped with the norm
\[ \| \varphi \|_{\mathcal{C}^\alpha} := \| \varphi \|_{\infty} + \sup_{x \neq y \in \mathbb{P}^{d-1}} \frac{|\varphi(x) - \varphi(y)|}{d(x,y)^\alpha}. \]

Recall that the essential spectrum of an operator is the subset of the spectrum obtained by removing its isolated points corresponding to eigenvalues of finite multiplicity. The essential spectral radius \( \rho_{\text{ess}} \) is the radius of the smallest disc centred at the origin which contains the essential spectrum.

**Theorem 2.3.** ([LP82], [BL85] V.2) There exists an \( 0 < \alpha_0 < 1 \) such that, for all \( 0 < \alpha \leq \alpha_0 \), the operator \( \mathcal{P} \) acts continuously on \( \mathcal{C}^\alpha(\mathbb{P}^{d-1}) \) with a spectral gap. More precisely, the essential spectral radius of \( \mathcal{P} \) satisfies \( \rho_{\text{ess}}(\mathcal{P}) < 1 \) and \( \mathcal{P} \) has a single eigenvalue of modulus \( \lambda \geq 1 \) located at 1, which is isolated and of multiplicity one.

It follows directly from the above theorem that
\[ \| \mathcal{P}^n - \mathcal{N} \|_{\mathcal{C}^\alpha} \leq C \lambda^n \]
for some constants \( C > 0 \) and \( 0 < \lambda < 1 \), where \( \mathcal{N} \) is the projection defined by \( \varphi \mapsto \left( \int_{\mathbb{P}^{d-1}} \varphi \, d\nu \right) \cdot 1 \). The following result gives the decomposition of \( \mathcal{P}_z \) for \( z \) near the origin, see e.g. [BL85] V.4.

**Proposition 2.4.** Let \( \mu \) and \( \alpha_0 \) be as in Theorem 2.3. There exists \( b > 0 \) such that for \( |\operatorname{Re} z| < b \), the operators \( \mathcal{P}_z \) act continuously on \( \mathcal{C}^\alpha(\mathbb{P}^{d-1}) \) for all \( 0 < \alpha \leq \alpha_0 \). Moreover, the family of operators \( z \mapsto \mathcal{P}_z \) is analytic near \( z = 0 \).
In particular, there exists an \( \epsilon_0 > 0 \) such that, for \( |z| \leq \epsilon_0 \), one has a decomposition

\[
P_z = \lambda_z N_z + Q_z,
\]

where \( \lambda_z \in \mathbb{C} \), \( N_z \) and \( Q_z \) are bounded operators on \( C^\alpha(\mathbb{P}^{d-1}) \) and

1. \( \lambda_0 = 1 \) and \( N_0 = \int_{\mathbb{P}^{d-1}} \varphi \, d\nu \), which is a constant function, where \( \nu \) is the unique \( \mu \)-stationary measure;
2. \( \rho := \lim_{n \to \infty} \|P_n^0 - N_0\|_{C^\alpha}^{1/n} < 1 \);
3. \( \lambda_z \) is the unique eigenvalue of maximum modulus of \( P_z \), \( N_z \) is a rank-one projection and \( N_z Q_z = Q_z N_z = 0 \);
4. the maps \( z \mapsto \lambda_z \), \( z \mapsto N_z \) and \( z \mapsto Q_z \) are analytic;
5. \( |\lambda_z| \geq 2 + \rho^3 \) and for every \( k \in \mathbb{N} \), there exists a constant \( c > 0 \) such that

\[
\left\| \frac{d^k Q^n_z}{dz^k} \right\|_{C^\alpha} \leq c \left( 1 + \frac{2\rho^3}{3} \right)^n \quad \text{for every} \quad n \geq 0;
\]
6. for \( z = i\xi \in i\mathbb{R} \), we have

\[
\lambda_{i\xi} = 1 + i\gamma \xi - \frac{\rho^2}{2} \xi^2 + O(|\xi|^3) \quad \text{as} \quad \xi \to 0,
\]

where \( \gamma \) is the first Lyapunov exponent of \( \mu \) and \( \rho > 0 \) is a constant.

The above constant \( \rho^2 > 0 \) coincides with the variance in the Central Limit Theorem for the norm cocycle (1.2), see [BL85, BQ16b, DKW21b]. As a consequence, we have the following estimates, see [DKW21b, Lemma 4.9] and [LP82, Lemma 9].

**Lemma 2.5.** Let \( \epsilon_0 \) be as in Proposition 2.4. There exists \( 0 < \xi_0 < \epsilon_0 \) such that, for all \( n \in \mathbb{N} \) large enough, one has

\[
|\lambda_{\xi n}^{\frac{n}{\xi_n}}| \leq e^{-\frac{2\rho^2}{3} \xi_n^2} \quad \text{for} \quad |\xi| \leq \xi_0 \sqrt{n},
\]

\[
\left| e^{-i\xi \sqrt{\pi} \gamma} \lambda_{\xi n}^{\frac{n}{\xi_n}} - e^{-\frac{2\rho^2}{3} \xi_n^2} \right| \leq \frac{c}{\sqrt{n}} |\xi|^3 e^{-\frac{2\rho^2}{3} \xi_n^2} \quad \text{for} \quad |\xi| \leq \sqrt{n},
\]

\[
\left| e^{-i\xi \sqrt{\pi} \gamma} \lambda_{\xi n}^{\frac{n}{\xi_n}} - e^{-\frac{2\rho^2}{3} \xi_n^2} \right| \leq \frac{c}{\sqrt{n}} e^{-\frac{2\rho^2}{3} \xi_n^2} \quad \text{for} \quad \sqrt{n} \leq |\xi| \leq \xi_0 \sqrt{n},
\]

where \( c > 0 \) is a constant independent of \( n \).

### 2.3. Fourier transform and characteristic function.

Recall that the Fourier transform of an integrable function \( f \), denoted by \( \widehat{f} \), is defined by

\[
\widehat{f}(\xi) := \int_{-\infty}^{\infty} f(u) e^{-iu\xi} \, du
\]

and the inverse Fourier transform is

\[
f(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{iu\xi} \, d\xi.
\]

Note that the Fourier transform of \( \widehat{f}(\xi) \) is \( 2\pi f(-u) \) and the Fourier transform exchanges the convolution and product: \( \widehat{f_1 \ast f_2} = \widehat{f_1} \cdot \widehat{f_2} \).
Lemma 2.6 ([DKW21b]–Lemma 2.2). There is a smooth strictly positive even function \( \vartheta \) on \( \mathbb{R} \) with \( \int_{\mathbb{R}} \vartheta(u) du = 1 \) such that its Fourier transform \( \hat{\vartheta} \) is a smooth even function supported by \([-1, 1]\). Moreover, for \( 0 < \delta < 1 \) and \( \vartheta_\delta(u) := \delta^{-2} \vartheta(u/\delta^2) \), we have that \( \hat{\vartheta}_\delta \) is supported by \([-\delta^{-2}, \delta^{-2}]\), \( |\hat{\vartheta}_\delta| \leq c \) and \( \int_{|u| \geq \delta} \vartheta_\delta(u) du \leq c\delta^2 \) for some constant \( c > 0 \) independent of \( \delta \).

We will need the following elementary lemma.

Lemma 2.7. Let \( \psi \) be a Lipschitz function on \( \mathbb{R} \) with Lipschitz norm bounded by 1. Then there is a constant \( C > 0 \) independent of \( \psi \) and \( \delta \) such that

\[
|\psi \ast \vartheta_\delta(t) - \psi(t)| \leq C\delta^2 \quad \text{for all} \quad t \in \mathbb{R}.
\]

Proof. By definition and Lemma 2.6, \( |\psi \ast \vartheta_\delta(t) - \psi(t)| \) is equal to

\[
\left| \int \psi(t - u)\vartheta_\delta(u) du - \int \psi(t)\vartheta_\delta(u) du \right| = \left| \int (\psi(t - u) - \psi(t))\vartheta_\delta(u) du \right|
\]

\[
\leq \left| \int_{|u| \leq \delta} (\psi(t - u) - \psi(t))\vartheta_\delta(u) du \right| + \left| \int_{|u| \geq \delta} (\psi(t - u) - \psi(t))\vartheta_\delta(u) du \right|
\]

\[
\leq \int_{|u| \leq \delta} |u| \vartheta_\delta(u) du + \int_{|u| \geq \delta} 2\vartheta_\delta(u) du = \delta^2 \int_{|s| \leq \delta^{-1}} |s| \vartheta(s) ds + \int_{|u| \geq \delta} 2\vartheta_\delta(u) du \lesssim \delta^2.
\]

The proof of the lemma is finished. \( \square \)

For a real random variable \( X \) with cumulative distribution function \( F \) (c.d.f. for short), we will denote by \( \phi_F(\xi) \) its conjugate characteristic function, that is

\[
\phi_F(\xi) := \mathbb{E}(e^{\text{i}\xi X}).
\]

The following version of Berry-Esseen lemma was obtained in [DKW21c, Corollary 3.2]. See also [Fel71, XVI.3]. As mentioned below, this allows us to avoid the integration of the characteristic functions on complex contours, which is a main technical difficulty in [XGL19].

Lemma 2.8. Let \( F \) be a c.d.f. of some real random variable and let \( H \) be a differentiable real-valued function on \( \mathbb{R} \) with derivative \( h \) such that \( H(-\infty) = 0, H(\infty) = 1, |h(u)| \leq m \) for some constant \( m > 0 \). Let \( D > 0 \) and \( 0 < \delta < 1 \) be real numbers such that \( |F(u) - H(u)| \leq D\delta^2 \) for \( |u| \geq \delta^{-2} \). Assume moreover that \( h \in L^1, \hat{h} \in \mathcal{C}' \) and that \( \phi_F \) is differentiable at zero. Then, there exist a constant \( C > 0 \) (resp. \( \kappa > 1 \)) independent of \( F, H, \delta \) (resp. \( D, F, H, \delta \)), such that

\[
\sup_{u \in \mathbb{R}} |F(u) - H(u)| \leq \frac{1}{\pi} \sup_{|u| \leq \delta^{-2}} \left| \int_{-\delta^{-2}}^{\delta^{-2}} \frac{\Theta_u(\xi)}{\xi} d\xi \right| + C\delta^2
\]

\[
= \frac{1}{\pi} \sup_{|u| \leq \delta^{-2}} \left| \int_{0}^{\delta^{-2}} \frac{\Theta_u(\xi) - \Theta_u(-\xi)}{\xi} d\xi \right| + C\delta^2,
\]

where \( \Theta_u(\xi) := e^{\text{i}u\xi} (\phi_F(\xi) - \hat{h}(\xi)) \hat{\vartheta}_\delta(\xi) \), and \( \vartheta_\delta \) is defined in Lemma 2.6.

3. Berry-Esseen bound for norm cocycle

This section is devoted to prove Theorem 1.1. It suffices to prove the theorem for \( J = (-\infty, b] \), as the case of an arbitrary interval can be obtained by taking complements...
and intersections. In order to simplify the notation, for fixed \( x \in \mathbb{P}^{d-1} \), we define two functionals
\[
\mathcal{E}_n(\Phi) := E\left( \Phi\left( \frac{\sigma(S_n, x) - n\gamma}{\sqrt{n}}, S_n x \right) \right)
\]
and
\[
\mathcal{R}(\Phi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{P}^{d-1}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} \Phi(s, w) \, ds \, d\nu(w).
\]
Here \( \Phi(t, w) \) is a function on \( \mathbb{R} \times \mathbb{P}^{d-1} \). Then, the conclusion of Theorem \( \ref{thm:1.2} \) for \( J = (-\infty, b] \) corresponds to
\[
(3.1) \quad \left| \mathcal{E}_n(1_{(-\infty, b]} \psi \cdot \varphi) - \mathcal{R}(1_{(-\infty, b]} \psi \cdot \varphi) \right| \leq \frac{C}{\sqrt{n}} \| \psi \|_{\mathcal{H}} \| \varphi \|_{\mathcal{E}^\alpha}.
\]

The following lemma will allow us later on to consider only the case when \( |b| \leq \epsilon \sqrt{n} \).

**Lemma 3.1.** For any \( \epsilon > 0 \), there exists a constant \( C_\epsilon > 0 \) such that for all \( n \geq 1 \),
\[
\mathcal{E}_n(1_{|t| \geq \sqrt{n}\epsilon}) \leq \frac{C_\epsilon}{\sqrt{n}} \quad \text{and} \quad \mathcal{R}(1_{|t| \geq \sqrt{n}\epsilon}) \leq \frac{C_\epsilon}{\sqrt{n}}.
\]

**Proof:** The second assertion is a simple property of normal distribution. For the first one, by Proposition \( \ref{prop:2.1} \), there exists a constant \( c > 0 \) depending on \( \epsilon \) such that
\[
\mathcal{E}_n(1_{|t| \geq \sqrt{n}\epsilon}) = P\left( \left| \frac{\sigma(S_n, x) - n\gamma}{\sqrt{n}} \right| \geq \sqrt{n}\epsilon \right) \leq e^{-c\epsilon}. \]
The lemma follows. \( \square \)

We first prove Theorem \( \ref{thm:1.1} \) in the case where \( \psi = 1 \). We note that this case was obtained in [XGL19]. However, we give here an independent proof that will be needed later. In contrast with [XGL19], we do not need to resort to the smoothing inequality on complex contours and on the saddle point method developed there. In particular, we only need to deal with purely imaginary perturbations of the Markov operator.

**Proposition 3.2.** Theorem \( \ref{thm:1.1} \) holds for \( \psi = 1 \).

We now begin the proof of the above proposition. From [LP82, BL85], we know that Theorem \( \ref{thm:1.1} \) holds for \( \psi = 1 \) and \( \varphi = 1 \). Since the problem is linear in \( \varphi \) we can assume, without loss of generality, that \( \frac{1}{2} \leq \varphi \leq 2 \), \( N_0 \varphi = \int_{\mathbb{P}^{d-1}} \varphi \, d\nu = 1 \) and \( \| \varphi \|_{\mathcal{E}^\alpha} \leq 2 \), because such functions span the space \( \mathcal{E}^\alpha(\mathbb{P}^{d-1}) \). Notice that, in particular, we have \( \| \varphi \|_{\mathcal{E}^\alpha} \geq 1 \). By Proposition \( \ref{prop:2.4} \)(5), there exist constants \( c > 0 \) and \( 0 < \beta < 1 \) such that \( \| Q^n_0 \|_{\mathcal{E}^\alpha} \leq c\beta^n \). Therefore,
\[
\left| \int_G \varphi(gx) \, d\mu^{*n}(g) - N_0 \varphi \right| = |P^n_0 \varphi(x) - N_0 \varphi| = |Q^n_0 \varphi(x)| \leq c\beta^n \| \varphi \|_{\mathcal{E}^\alpha}.
\]
Let \( d_{n,x} := \left( P^n_0 \varphi(x) \right)^{-1} \), which will be used as a normalization factor. We have
\[
\frac{1}{2} \leq d_{n,x} \leq 2 \quad \text{and} \quad |d_{n,x} - 1| \leq 2c\beta^n \| \varphi \|_{\mathcal{E}^\alpha} \quad \text{for } n \text{ large enough}.
\]

For \( n \in \mathbb{N} \), consider the function
\[
F_n(b) := d_{n,x} \mathcal{E}_n(1_{(-\infty, b]} \varphi) := d_{n,x} E\left( 1_{x(S_n x) \leq b} \varphi(S_n x) \right).
\]
Notice that $F_n$ is non-decreasing, right-continuous, $F_n(-\infty) = 0$ and
\[ F_n(\infty) = d_{n,x} E \left( \varphi(S_n x) \right) = d_{n,x} P^n_0 \varphi(x) = 1. \]
Therefore, $F_n$ is the c.d.f. of some probability distribution. It is not hard to see its conjugate characteristic function is given by
\[ \phi_{F_n}(\xi) = d_{n,x} \int_G \varphi(g x) e^{-i\xi \frac{\sigma(g, x) - \mu(x)}{\sqrt{n}}} \, d\mu^* (g) = d_{n,x} e^{i\xi \sqrt{\pi n} P^n_{-\frac{i\xi}{\sqrt{n}}} \varphi(x)}, \]
where we have used (2.1). From Proposition 2.4, $\xi \mapsto \mathcal{P}_\xi$ is an analytic family of operators acting on $C^\alpha(\mathbb{R}^{d-1})$. In particular, $\phi_{F_n}$ is differentiable near 0.

Let $\xi_0$ be the constant in Lemma 2.5. By Lemma 3.1 applied to $\epsilon = \xi_0$ and the definition of $F_n$, there exists a constant $D > 0$ such that
\[ |F_n(b) - H(b)| \leq \frac{D}{\sqrt{n}} \| \varphi \|_\infty \quad \text{for all} \quad |b| \geq \xi_0 \sqrt{n}. \]

Thus, $F_n$ and $H$ satisfy the conditions of Lemma 2.8 with $\delta_n := (\xi_0 \sqrt{n})^{-1/2}$. Let $\kappa > 1$ be the constant appearing in that lemma. For simplicity, by taking a smaller $\xi_0$ is necessary, one can assume that $2\kappa \xi_0 \leq 1$. Then, Lemma 2.8 gives that
\[ \sup_{b \in \mathbb{R}} \left| F_n(b) - H(b) \right| \leq \frac{1}{\pi} \sup_{|b| \leq \sqrt{n}} \left| \int_0^{\xi_0 \sqrt{n}} \frac{\Theta_b(\xi) - \Theta_b(-\xi)}{\xi} \, d\xi \right| + \frac{C}{\sqrt{n}} \| \varphi \|_\infty, \]
where $C > 0$ is a constant independent of $n$ and
\[ \Theta_b(\xi) := e^{ib\xi} \left( \phi_{F_n}(\xi) - \hat{h}(\xi) \right) \hat{\vartheta}_{\delta_n}(\xi) = e^{ib\xi} \left( d_{n,x} e^{i\xi \sqrt{\pi n} P^n_{-\frac{i\xi}{\sqrt{n}}} \varphi(x) - e^{-\frac{2\xi^2}{2}}} \right) \hat{\vartheta}_{\delta_n}(\xi), \]
where we have used (3.2).

**Lemma 3.3.** We have
\[ \sup_{|b| \leq \sqrt{n}} \left| \int_0^{\xi_0 \sqrt{n}} \frac{\Theta_b(\xi) - \Theta_b(-\xi)}{\xi} \, d\xi \right| \leq \frac{\| \varphi \|_{C^\alpha}}{\sqrt{n}}. \]

**Proof.** We can assume that $n$ is large enough. Put
\[ \Theta_b^{(1)}(\xi) := d_{n,x} e^{ib\xi} \left( e^{i\xi \sqrt{\pi n} P^n_{-\frac{i\xi}{\sqrt{n}}} \varphi(x) - e^{-\frac{2\xi^2}{2}}} \right) \hat{\vartheta}_{\delta_n}(\xi) \]
and
\[ \Theta_b^{(2)}(\xi) := (d_{n,x} - 1) e^{ib\xi} e^{-\frac{2\xi^2}{2}} \hat{\vartheta}_{\delta_n}(\xi). \]
Observe that $\Theta_b = \Theta_b^{(1)} + \Theta_b^{(2)}$. 

We deal with $\Theta_b^{(2)}$ first. Recall that $|d_{n,x} - 1| \leq 2c\beta^\alpha \|\varphi\|_{\infty^n}$, $\|\widehat{\delta_n}\|_{\infty^1} \lesssim 1$ and $|b| \leq \sqrt{n}$. By the mean value theorem, we get that

$$\left| \int_0^{\xi_0\sqrt{n}} \frac{\Theta_b^{(2)}(\xi) - \Theta_b^{(2)}(-\xi)}{\xi} \, d\xi \right| \leq 2\xi_0\sqrt{n} \sup_{|\xi| \leq \xi_0\sqrt{n}} |(\Theta_b^{(2)})'(\xi)| \lesssim \sqrt{n} \cdot |d_{n,x} - 1| \cdot |b| \lesssim n\beta^\alpha \|\varphi\|_{\infty^n} \lesssim \frac{\|\varphi\|_{\infty^n}}{\sqrt{n}}.$$  

In order to estimate $\Theta_b^{(1)}$, we use the decomposition

$$\mathcal{P}_{-\frac{\xi}{\sqrt{n}}} \varphi(x) = \lambda_{-\frac{\xi}{\sqrt{n}}} N_{-\frac{\xi}{\sqrt{n}}} \varphi(x) + Q_{-\frac{\xi}{\sqrt{n}}} \varphi(x)$$

from Proposition 2.4. Then, we can write

$$\Theta_b^{(1)} = d_{n,x}(\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4),$$

where

$$\Lambda_1(\xi) := e^{ib\xi} \left( e^{i\xi \sqrt{n}} \lambda_{-\frac{\xi}{\sqrt{n}}} N_{-\frac{\xi}{\sqrt{n}}} \varphi(x) - e^{i\xi \sqrt{n}} \lambda_{-\frac{\xi}{\sqrt{n}}} N_0 \varphi(x) \right) \widehat{\delta_n}(\xi),$$

$$\Lambda_2(\xi) := e^{ib\xi} \left( e^{i\xi \sqrt{n}} \lambda_{-\frac{\xi}{\sqrt{n}}} N_0 \varphi(x) - e^{-\frac{\xi^2}{2}} \right) \widehat{\delta_n}(\xi),$$

$$\Lambda_3(\xi) := e^{ib\xi} \left( e^{i\xi \sqrt{n}} Q_{-\frac{\xi}{\sqrt{n}}} \varphi(x) - e^{i\xi \sqrt{n}} Q_0 \varphi(x) \right) \widehat{\delta_n}(\xi)$$

and

$$\Lambda_4(\xi) := e^{ib\xi} e^{i\xi \sqrt{n}} Q_0 \varphi(x) \widehat{\delta_n}(\xi).$$

We omit here the dependence on $b$ in order to ease the notation. Recall that $N_0 \varphi = 1$.

We will estimate each term separately. For $\Lambda_1$, recall that $|\widehat{\delta_n}| \lesssim 1$, so the integral involving $\Lambda_1$ is bounded by

$$\left| \int_{-\xi_0\sqrt{n}}^{\xi_0\sqrt{n}} \frac{\Lambda_1(\xi)}{\xi} \, d\xi \right| \lesssim \int_{-\xi_0\sqrt{n}}^{\xi_0\sqrt{n}} \frac{1}{|\xi|} \cdot \left| \lambda_{-\frac{\xi}{\sqrt{n}}} N_{-\frac{\xi}{\sqrt{n}}} \varphi(x) - \lambda_{-\frac{\xi}{\sqrt{n}}} N_0 \varphi(x) \right| \, d\xi.$$  

By Proposition 2.4, $\xi \mapsto N_{\xi} \varphi$ is analytic. It follows that

$$\left\| N_{-\frac{\xi}{\sqrt{n}}} \varphi - N_0 \varphi \right\|_{\infty^n} \lesssim \frac{|\xi|}{\sqrt{n}} \|\varphi\|_{\infty^n}.$$  

Using that $|\lambda_{-\frac{\xi}{\sqrt{n}}}| \leq e^{-\frac{\xi^2}{2}}$ for $|\xi| \leq \xi_0\sqrt{n}$ from Lemma 2.5, we obtain

$$\int_{-\xi_0\sqrt{n}}^{\xi_0\sqrt{n}} \frac{\Lambda_1(\xi)}{\xi} \, d\xi \lesssim \int_{-\xi_0\sqrt{n}}^{\xi_0\sqrt{n}} \frac{1}{|\xi|} \cdot e^{-\frac{\xi^2}{2}} \frac{|\xi|}{\sqrt{n}} \|\varphi\|_{\infty^n} \, d\xi \lesssim \frac{\|\varphi\|_{\infty^n}}{\sqrt{n}}.$$  

Now we estimate $\Lambda_2$. Recall that $N_0 \varphi = 1$ by assumption. Therefore, the integral involving $\Lambda_2$ is bounded by

$$\int_{-\xi_0\sqrt{n}}^{\xi_0\sqrt{n}} \frac{\Lambda_2(\xi)}{\xi} \, d\xi \lesssim \int_{-\xi_0\sqrt{n}}^{\xi_0\sqrt{n}} \frac{1}{|\xi|} \cdot e^{i\xi \sqrt{n}} \lambda_{-\frac{\xi}{\sqrt{n}}} - e^{-\frac{\xi^2}{2}} \, d\xi.$$  

Splitting the last integral along the intervals $|\xi| \leq \frac{\xi_0}{\sqrt{n}}$ and $\frac{\xi_0}{\sqrt{n}} < |\xi| \leq \xi_0\sqrt{n}$ and using Lemma 2.5, we see that it is bounded by a constant times

$$\int_{|\xi| \leq \frac{\xi_0}{\sqrt{n}}} \frac{1}{|\xi|} \cdot \frac{1}{\sqrt{n}} |\xi|^b e^{-\frac{\xi^2}{2}} \, d\xi + \int_{\frac{\xi_0}{\sqrt{n}} < |\xi| \leq \xi_0\sqrt{n}} \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}} e^{-\frac{\xi^2}{2}} \, d\xi \lesssim \frac{1}{\sqrt{n}}.$$
In order to estimate $\Lambda_3$, observe that, for $z$ small, the norm of the operator $Q_z^n - Q_0^n$ is bounded by a constant times $|z| n^{\tau n}$ for some $0 < \tau < 1$. This can be seen by writing the last difference as $\sum_{\ell=0}^{n-1} Q_z^{n-\ell-1} (Q_z - Q_0) Q_0^n$, applying Proposition 2.4(5) and using the fact that $||Q_z - Q_0||_{\psi^\alpha} \lesssim |z|$. Therefore,

$$\left| e^{i\xi \sqrt{n} \tau} Q^n_{\psi} \varphi(x) - e^{i\xi \sqrt{n} \tau} Q^n_0 \varphi(x) \right| \lesssim \frac{|\xi|}{\sqrt{n}} n^{\tau n} \| \varphi \|_{\psi^\alpha} = |\xi| \sqrt{n^{\tau n}} \| \varphi \|_{\psi^\alpha}.$$ 

This gives that

$$\left| \int_{-\xi_0 \sqrt{n}}^{\xi_0 \sqrt{n}} \frac{\Lambda_3(\xi)}{\xi} \, d\xi \right| \lesssim \frac{1}{|\xi_0 \sqrt{n}|} \left| e^{i\xi \sqrt{n} \tau} Q^n_{\psi} \varphi(x) - e^{i\xi \sqrt{n} \tau} Q^n_0 \varphi(x) \right| \, d\xi \lesssim \frac{1}{\sqrt{n}} \left| e^{i\xi \sqrt{n} \tau} \varphi \right|_{\psi^\alpha} \lesssim \frac{||\varphi||_{\psi^\alpha}}{\sqrt{n}}.$$ 

It remains to estimate $\Lambda_4$. Recall that $|Q^n_0 \varphi(x)| \leq c\beta^n \| \varphi \|_{\psi^\alpha}$, $|b| \leq \sqrt{n}$ and $||\varphi||_{\psi^\alpha} \leq 1$. Therefore, $|\Lambda_4(\xi)| \lesssim \sqrt{n^{\beta n}} \| \varphi \|_{\psi^\alpha}$. Using the mean value theorem, we get that

$$\left| \int_{-\xi_0 \sqrt{n}}^{\xi_0 \sqrt{n}} \frac{\Lambda_4(\xi) - \Lambda_4(-\xi)}{\xi} \, d\xi \right| \leq 2\xi_0 \sqrt{n} \sup_{|\xi| \leq \xi_0 \sqrt{n}} |\Lambda_4(\xi)| \lesssim n b \beta^n \| \varphi \|_{\psi^\alpha} \lesssim \frac{||\varphi||_{\psi^\alpha}}{\sqrt{n}}.$$ 

Finally, since $|d_{x,n}| \leq 2$ for $n$ large, we conclude that

$$\sup_{|b| \leq \sqrt{n}} \left| \int_{0}^{\xi_0 \sqrt{n}} \frac{\Theta_b(1)(\xi) - \Theta_b(1)(-\xi)}{\xi} \, d\xi \right| \lesssim \frac{||\varphi||_{\psi^\alpha}}{\sqrt{n}}.$$ 

Together with the estimate for $\Theta_b(2)$ obtained before, this gives the desired estimate for $\Theta_b$. The proof of the lemma is complete. 

We can now complete the proof of Proposition 3.2.

End of the proof of Proposition 3.2 Lemma 3.3 and (3.3) give that

$$\sup_{b \in \mathbb{R}} |F_n(b) - H(b)| \lesssim \frac{||\varphi||_{\psi^\alpha}}{\sqrt{n}}.$$ 

Therefore, for $b \in \mathbb{R}$, we have

$$|d_{n,x} \mathcal{E}_n (1_{(-\infty,b]} \cdot \varphi) - \mathcal{R}(1_{(-\infty,b]} \cdot \varphi)| = |F_n(b) - H(b)| \lesssim \frac{||\varphi||_{\psi^\alpha}}{\sqrt{n}}.$$ 

Finally, using again $|d_{n,x} - 1| \leq 2 c \beta^n \| \varphi \|_{\psi^\alpha}$, we obtain

$$|\mathcal{E}_n (1_{(-\infty,b]} \cdot \varphi) - \mathcal{R}(1_{(-\infty,b]} \cdot \varphi)| \lesssim \frac{||\varphi||_{\psi^\alpha}}{\sqrt{n}}$$

which corresponds to the conclusion of Theorem 1.1 for $\varphi$ and $\psi = 1$, see (3.1). The proof of the proposition is finished. 

We now proceed to the general case of Theorem 1.1. We will use Proposition 3.2 as an intermediate step. The following regularization procedure will be used. For every $n \geq 1$, consider a smooth cut-off function $\tau_n$ on $\mathbb{R}$ such that

$$\tau_n = 0 \text{ on } (-\infty, -\sqrt{n} - 2], \quad \tau_n = 1 \text{ on } [-\sqrt{n}, \infty) \quad \text{and} \quad |\tau_n'| \leq 1 \text{ on } \mathbb{R}.$$
Recall from the introduction that $\mathcal{H}$ is the space of bounded Lipschitz functions $\psi$ on $\mathbb{R}$ such that $\psi'$ is bounded and belongs to $L^1(\mathbb{R})$, equipped with the norm
\[
\|\psi\|_\mathcal{H} := \|\psi\|_\infty + \|\psi'\|_\infty + \|\psi'\|_{L^1}.
\]

**Lemma 3.4.** Let $\tau_n$ be as above and $b \in \mathbb{R}$ be such that $|b| \leq \sqrt{n}$. Let $\psi$ be a function in $\mathcal{H}$ such that $\|\psi\|_\mathcal{H} \leq 1$ and $\psi(b) = 0$. Then, we have
\[
\left| \hat{1}_{(-\infty,b]}(\psi \tau_n)(\xi) \right| \leq 2\sqrt{n} + 2 \quad \text{and} \quad \left| \hat{1}_{(-\infty,b]}(\psi \tau_n')(\xi) \right| \leq 5/|\xi| \quad \text{for} \quad \xi \neq 0.
\]

**Proof.** The assumption $\|\psi\|_\mathcal{H} \leq 1$ implies that $\|\psi\|_\infty \leq 1$, $\|\psi'\|_\infty \leq 1$ and $\|\psi'\|_{L^1} \leq 1$. By the definition of $\tau_n$, we have $|(\psi \tau_n)'| \leq |\psi'|$ on $\mathbb{R} \setminus (-\sqrt{n} - 2, -\sqrt{n})$. Also, $|(\psi \tau_n)'(t)| \leq 2$ there, since $|\tau_n'| \leq 1$. Therefore,
\[
(3.4) \quad \int_{-\infty}^{\infty} |(\psi \tau_n)'(u)| \, du \leq \int_{-\infty}^{\infty} |\psi'(u)| \, du + \int_{-\sqrt{n} - 2}^{-\sqrt{n}} 2 \, du \leq 1 + 4 = 5.
\]

By the definition of Fourier transform,
\[
\left| \hat{1}_{(-\infty,b]}(\psi \tau_n)(\xi) \right| \leq \|1_{(-\infty,b]} \psi \tau_n\|_{L^1} \leq 2\sqrt{n} + 2
\]
since $|b| \leq \sqrt{n}$. This gives the first inequality.

For the second inequality, using integration by parts, $(1_{(-\infty,b]}(\psi \tau_n))(\xi)$ is equal to
\[
\int_{-\infty}^{b} (\psi \tau_n)(u) e^{iu\xi} \, du = \left[ (\psi \tau_n)(u) \cdot e^{iu\xi} \right]_{-\infty}^{b} - \int_{-\infty}^{b} (\psi \tau_n)'(u) \cdot e^{iu\xi} \, du.
\]
Notice that $(\psi \tau_n)(-\infty) = (\psi \tau_n)(b) = 0$, where we have used the assumption that $\psi(b) = 0$. Therefore, the first term on the right hand side vanishes when $\xi \neq 0$. Using (3.4), we conclude that
\[
\left| \hat{1}_{(-\infty,b]}(\psi \tau_n)(\xi) \right| \leq 5/|\xi| \quad \text{for} \quad \xi \neq 0,
\]
yielding the second estimate. \hfill \square

**Proof of Theorem 1.1.** We can assume $\|\psi\|_\mathcal{H} \leq 1$ and $\|\varphi\|_{C^0} \leq 1$. In particular, $\psi$ is a Lipschitz function on $\mathbb{R}$ such that $\|\psi\|_\infty \leq 1$, $\|\psi'\|_\infty \leq 1$ and $\|\psi'\|_{L^1} \leq 1$. As before, we can assume $J = (-\infty, b]$. Since Theorem 1.1 holds when $\psi$ is a constant function, by Proposition 3.2, we can subtract $\psi(b)$ from $\psi$ and assume that $\psi(b) = 0$. In this case, $1_{(-\infty,b]}\psi$ is a Lipschitz function on $\mathbb{R}$ with Lipschitz norm bounded by 1.

For $n \geq 1$, let $\tau_n$ be the smooth cut-off function defined above. Then, the function $1_{(-\infty,b]}(\psi \tau_n)$ is Lipschitz with Lipschitz norm bounded by 2. Fix an $x \in \mathbb{R}^{d-1}$. Recall that our goal is to prove (3.1). By Lemma 3.1 applied to $\epsilon = 1$, it is enough to consider the case when $|b| \leq \sqrt{n}$ and prove (3.1) for the function $1_{(-\infty,b]}(\psi \tau_n)(t) \cdot \varphi(w)$ instead of $(1_{(-\infty,b]}(\psi))(t) \cdot \varphi(w)$.

Let $\vartheta_{\delta}$ be the function from Lemma 2.6 and take $\delta_n := (\xi_0 \sqrt{n})^{-1/2}$, where $\xi_0$ is a small constant satisfying Lemma 2.5. Since $1_{(-\infty,b]}(\psi \tau_n)$ is Lipschitz, Lemma 2.7 gives that
\[
\left| (1_{(-\infty,b]}(\psi \tau_n) \ast \vartheta_{\delta_n})(t) - 1_{(-\infty,b]}(\psi \tau_n)(t) \right| \lesssim 1/\sqrt{n} \quad \text{for all} \quad t \in \mathbb{R}.
\]

Put $\Psi_n := (1_{(-\infty,b]}(\psi \tau_n) \ast \vartheta_{\delta_n})$. Then,
\[
(3.5) \quad |E_n(\Psi_n(t) \cdot \varphi(x)) - E_n(1_{(-\infty,b]}(\psi \tau_n)(t) \cdot \varphi(x))| \lesssim 1/\sqrt{n}
\]
and
\begin{equation}
|\mathcal{R}(\Psi_n(t) \cdot \varphi(x)) - \mathcal{R}(1_{(-\infty,b]} \psi \tau_n(t) \cdot \varphi(x))| \lesssim 1/\sqrt{n}.
\end{equation}

It is clear that $\Psi_n$ is integrable and $\hat{\Psi}_n = (1_{(-\infty,b]} \psi \tau_n \cdot \vartheta_{d_n})$ is supported by $[-\delta_n^{-2}, \delta_n^{-2}]$. Using the inverse Fourier transform for $\Psi_n$ and applying Fubini's theorem, yields
\begin{align*}
\mathcal{E}_n(\Psi_n(t) \cdot \varphi(x)) &= \mathcal{E}_n(\hat{\Psi}_n(\xi) e^{i\xi \pi \tau_n(t) \cdot \varphi(x)} d\mu^m(g) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\Psi}_n(\xi) e^{i\xi \pi \tau_n(t) \cdot \varphi(x)} d\mu^m(g) d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\Psi}_n(\xi) \cdot e^{-i\xi \pi \tau_n(t) \cdot \varphi(x)} d\xi,
\end{align*}
where the last step we have used (2.1). Using that the inverse Fourier transform of $\sqrt{2\pi} e^{-\xi^2/2}$ is $e^{-\pi^2/2}$, we have that
\begin{align*}
\mathcal{R}(\hat{\Psi}_n(\xi) \cdot \varphi(w)) &= \frac{1}{\sqrt{2\pi} \rho} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2\rho^2}} \Psi_n(s) \cdot \varphi(w) \, ds \, d\nu(w) \\
&= \frac{1}{2\pi} (N_0 \varphi) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\xi s} e^{-\frac{\xi^2}{2}} \Psi_n(s) \, d\xi \, ds \\
&= \frac{1}{2\pi} (N_0 \varphi) \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2}} \Psi_n(\xi) \, d\xi.
\end{align*}

It follows that
\begin{align*}
\mathcal{E}_n(\Psi_n(t) \cdot \varphi(x)) - \mathcal{R}(\Psi_n(t) \cdot \varphi(w)) &= \frac{1}{2\pi} \int_{-\xi_0 \sqrt{m}}^{\xi_0 \sqrt{m}} \hat{\Psi}_n(\xi) \cdot [e^{-i\xi \pi \tau_n(t) \cdot \varphi(x)} - e^{-\frac{\xi^2}{2}} N_0 \varphi(x)] \, d\xi.
\end{align*}

Recall that the support of $\hat{\Psi}_n$ is contained in $[-\delta_n^{-2}, \delta_n^{-2}]$.

Using the decomposition $\mathcal{P}_z^n = \lambda_z^n N_z + Q_z^n$ for $z$ small, we see that
\begin{equation}
|\mathcal{E}_n(\Psi_n(t) \cdot \varphi(w)) - \mathcal{R}(\Psi_n(t) \cdot \varphi(w))| \leq \frac{1}{2\pi} (A_n + B_n + D_n),
\end{equation}
where
\begin{align*}
A_n := & \frac{1}{2\pi} \int_{-\xi_0 \sqrt{m}}^{\xi_0 \sqrt{m}} \hat{\Psi}_n(\xi) \cdot e^{-i\xi \pi \tau_n(t) \cdot \varphi(x)} - e^{-\frac{\xi^2}{2}} N_0 \varphi(x) \, d\xi, \\
B_n := & \frac{1}{2\pi} \int_{-\xi_0 \sqrt{m}}^{\xi_0 \sqrt{m}} \hat{\Psi}_n(\xi) \cdot e^{-i\xi \pi \tau_n(t) \cdot \varphi(x)} - e^{-i\xi \pi \tau_n(t) \cdot \varphi(x) + Q_0 \varphi(x)} \, d\xi \\
D_n := & \int_{-\xi_0 \sqrt{m}}^{\xi_0 \sqrt{m}} \hat{\Psi}_n(\xi) e^{-i\xi \pi \tau_n(t) \cdot \varphi(x)} \, d\xi.
\end{align*}

In order to estimate the above quantities, we first bound $\hat{\Psi}_n$. Since $\hat{\Psi}_n = (1_{(-\infty,b]} \psi \tau_n \cdot \vartheta_{d_n})$ and $|\vartheta_{d_n}| \leq 1$, we only need to estimate $(1_{(-\infty,b]} \psi \tau_n)(\xi)$. From Lemma 3.4, we see that
\begin{align*}
|1_{(-\infty,b]}(\psi \tau_n)(\xi)| \leq 2\sqrt{n} + 2 \quad \text{and} \quad |1_{(-\infty,b]}(\psi \tau_n)(\xi)| \lesssim 1/|\xi| \quad \text{for} \quad \xi \neq 0,
\end{align*}
so $|\hat{\Psi}_n(\xi)| \lesssim 1/|\xi|$. This allows us to repeat the arguments of Lemma 3.3 for $A_n$ and $B_n$. We conclude that $A_n, B_n$ are $\lesssim 1/\sqrt{n}$. 
We now estimate $D_n$. Recall that $|Q^n_0 \varphi(x)| \lesssim \beta^n$ for some $0 < \beta < 1$. Using that $\hat{\Psi}_n(\xi) \leq (1_{(-\infty, b]}\psi \tau_n)(\xi) \leq \sqrt{n} + 2$, we get

$$
D_n \leq \int_{-\xi_0 \sqrt{n}}^{\xi_0 \sqrt{n}} (2\sqrt{n} + 2)|Q^n_0 \varphi(x)| \, dx \lesssim \sqrt{n}(2\sqrt{n} + 2)\beta^n \lesssim \frac{1}{\sqrt{n}}.
$$

From the above estimates, we conclude that

$$
|\mathcal{E}_n(\Psi_n(t) \cdot \varphi(w)) - \mathcal{R}(\Psi_n(t) \cdot \varphi(w))| \lesssim A_n + B_n + D_n \lesssim 1/\sqrt{n}.
$$

Combining with (3.5) and (3.6), we deduce that (3.1) holds for $\Psi_n(t) \cdot \varphi$. As discussed in the beginning of the proof, this implies the same estimate for $(1_{(-\infty, b]}\psi) \cdot \varphi$. This completes the proof of the theorem.

\section{Berry-Esseen bound for matrix coefficients}

This section is devoted to the proof of Theorem 1.2. We follow the strategy of \[DKW21c\]. As in Section 3, we can assume that $J = (-\infty, b]$, since the other cases follow by taking complements and intersections. Fix $x := [v] \in \mathbb{P}^{d-1}$, $y := [f] \in (\mathbb{P}^{d-1})^*$. In order to simplify the notation, we let

$$
(4.1) \quad \tilde{\mathcal{E}}_n(\Phi) := \mathbb{E}\left( \Phi \left( \frac{\log |\langle S_n v, f \rangle|}{\| v \| \| f \|} - n\gamma, S_n x \right) \right),
$$

where $\Phi$ is a function on $\mathbb{R} \times \mathbb{P}^{d-1}$. Recall that $H_y$ is the hyperplane in $\mathbb{P}^{d-1}$ defined by the kernel of $y$. It is not hard to see that

$$
(4.2) \quad \log \frac{|\langle S_n v, f \rangle|}{\| v \| \| f \|} = \sigma(S_n, x) + \log d(S_n x, H_y),
$$

where the distance $d$ is introduced in Section 2. We will use the above identity to replace the study of $\log |\langle S_n v, f \rangle|$ to that of the couple of random variables $\sigma(S_n, x)$ and $\log d(S_n x, H_y)$. The approximation of unity that we now introduce, will allow us to split the above variables and work with a sum of functions of two separate variables $\sigma(S_n, x)$ and $S_n x$.

For integers $k \geq 0$ and $r > 0$, recall that

$$
\mathbb{B}(H_y, r) := \{ w \in \mathbb{P}^{d-1} : d(w, H_y) < r \}.
$$

Define

$$
\mathcal{F}_k := \{ w \in \mathbb{P}^{d-1} : e^{-k-1} < d(w, H_y) < e^{-k+1} \} = \mathbb{B}(H_y, e^{-k+1}) \setminus \mathbb{B}(H_y, e^{-k-1}).
$$

As mentioned above, the use of the following partition of unity will be essential. See [DKW21c] for its construction.

\begin{lemma}
There exist non-negative smooth functions $\chi_k$ on $\mathbb{P}^{d-1}$, $k \geq 0$, such that

\begin{enumerate}
    \item $\chi_k$ is supported by $\mathcal{F}_k$;
    \item if $w \in \mathbb{P}^{d-1} \setminus H_y$, then $\chi_k(w) \neq 0$ for at most two values of $k$;
    \item $\sum_{k \geq 0} \chi_k = 1$ on $\mathbb{P}^{d-1} \setminus H_y$;
    \item $\| \chi_k \|_{C^1} \leq 12e^k$.
\end{enumerate}
\end{lemma}

Similarly as for Theorem 1.1, we'll first prove Theorem 1.2 in the case where $\psi = 1$. We note that this case was recently obtained in [XGL21], but we give here an independent proof, since our arguments and notations will be needed later.
Proposition 4.2. Theorem 1.2 holds for $\psi = 1$.

We now begin the proof of Proposition 4.2. As in Section 3 we can assume that $1/2 \leq \varphi \leq 2$, $N_0 \varphi = 1$ and $\|\varphi\|_{\mathcal{E}^\alpha} \leq 2$, since the problem is linear on $\varphi$ and these functions span the space $\mathcal{E}^\alpha (\mathbb{P}^{d-1})$.

Fix a large constant $A > 0$. By Proposition 2.1, there exists a constant $c > 0$ such that for $\ell \geq m$ large enough, one has

$$\mu^{\ell} \{ g \in G : d(gx, H_y) \leq e^m \} \leq e^{-cm}.$$

Taking $\ell := n$ and $m := \lfloor A \log n \rfloor$ gives that

$$\mu^{n} \{ g \in G : d(gx, H_y) \leq n^{-A} \} \leq e^{-c[A \log n]} = n^{-c} e^{c} \leq e^c / \sqrt{n}$$

since $A$ is large. Then, $\log d(S_n, H_y) \leq -A \log n$ with probability smaller than $e^c / \sqrt{n}$. Therefore, it is enough to prove that

$$\left| \mathcal{L}_n^b(\varphi) - \frac{1}{\sqrt{2\pi} \varphi} \int_{-\infty}^{b} e^{-u^2/2\varphi} du \right| \lesssim \frac{\|\varphi\|_{\mathcal{E}^\alpha}}{\sqrt{n}},$$

where

$$\mathcal{L}_n^b(\varphi) := \mathbb{E} \left( 1_{\sigma(S_n, x) - k \varphi - n^{-1/2} \log d(S_n, H_y) > A \log n} \varphi(S_n x) \right).$$

Recall that $N_0 \varphi = \int_{\mathbb{P}^{d-1}} \varphi \, d\nu = 1$.

Let $\chi_k$ be as in Lemma 4.1. Then, for $w \in \mathbb{P}^{d-1}$,

$$\sum_{0 \leq k \leq A \log n - 1} (\chi_k \varphi)(w) \leq \mathbb{E} \left( 1_{\sigma(S_n, x) - k \varphi - n^{-1/2} \log d(S_n, H_y) > A \log n} \varphi(w) \right) \leq \sum_{0 \leq k \leq A \log n + 1} (\chi_k \varphi)(w).$$

Moreover, $\chi_k(w)$ is non-zero only when $-k - 1 \leq \log d(w, H_y) \leq -k + 1$. Thus,

$$\sum_{0 \leq k \leq A \log n - 1} \mathbb{E} \left( 1_{\sigma(S_n, x) - n^{-1/2} \log d(S_n, H_y) > A \log n} \varphi(S_n x) \right) \leq \mathcal{L}_n^b(\varphi) \leq \sum_{0 \leq k \leq A \log n + 1} \mathbb{E} \left( 1_{\sigma(S_n, x) - n^{-1/2} \log d(S_n, H_y) > A \log n} \varphi(S_n x) \right).$$

Define the functions $\Phi_n^*$ and $\Phi_{n, \xi}$ on $\mathbb{P}^{d-1}$ by

$$\Phi_n^*(w) := \varphi(w) - \sum_{0 \leq k \leq A \log n} (\chi_k \varphi)(w),$$

and

$$\Phi_{n, \xi}(w) := \sum_{0 \leq k \leq A \log n} e^{i\xi \frac{k}{\sqrt{n}}} (\chi_k \varphi)(w),$$

where $n \geq 1$ and $\xi \in \mathbb{R}$.

Let $d_{n, x} := \left( \mathcal{P}_n^\varphi(x) \right)^{-1}$ as in Section 3. As before, we have

$$\frac{1}{2} \leq d_{n, x} \leq 2 \quad \text{and} \quad |d_{n, x} - 1| \lesssim \beta^n \|\varphi\|_{\mathcal{E}^\alpha} \quad \text{for } n \text{ large enough},$$

where $0 < \beta < 1$. For $n \geq 1$, let

$$\tilde{F}_n(b) := d_{n, x} \sum_{0 \leq k \leq A \log n} \mathbb{E} \left( 1_{\sigma(S_n, x) - n^{-1/2} \log d(S_n, H_y) > A \log n} \chi_k \varphi(S_n x) \right) + d_{n, x} \mathbb{E} \left( 1_{\sigma(S_n, x) - n^{-1/2} \log d(S_n, H_y) > A \log n} \Phi_n^*(S_n x) \right).$$
Then, \( \tilde{F}_n \) is a non-decreasing, right-continuous functions with \( \tilde{F}_n(-\infty) = 0 \) and

\[
\tilde{F}_n(\infty) = d_{n,x} \sum_{k=1}^{\infty} P_k^n(x_k \varphi)(x) = d_{n,x} P_0^n \varphi(x) = 1.
\]

Therefore, it is the c.d.f. of some probability distribution.

Let \( H(b) \) be the c.d.f. of the normal distribution \( \mathcal{N}(0; \varphi^2) \) as in Section \[3\]. As seen there, \( H(-\infty) = 0, H(\infty) = 1 \) and the derivative \( h \) of \( H \) is bounded by \( 1/(\sqrt{2\pi} \rho) \). Moreover,

\[
h(b) = \frac{1}{\sqrt{2\pi} \rho} e^{-\frac{b^2}{2\rho^2}} \quad \text{and} \quad \hat{h}(\xi) = e^{-\frac{3}{2} \xi^2}.
\]

The following results correspond to Lemmas 3.4 to 3.7 in \[DKW21c\] when \( \varphi = 1 \). The proofs given there can be easily adapted to our case. In particular, the presence of a general function \( \varphi \) and the constants \( d_{n,x} \) do not raise any issues.

**Lemma 4.3.** Let \( \mathcal{L}_n^b \) and \( \tilde{F}_n \) be as above. Then, there exists a constant \( C > 0 \) independent of \( n \), such that for all \( n \geq 1 \) and \( b \in \mathbb{R} \),

\[
\tilde{F}_n\left(b - \frac{1}{\sqrt{n}}\right) - \frac{C}{\sqrt{n}} \|\varphi\|_\infty \leq \mathcal{L}_n^b(\varphi) \leq \tilde{F}_n\left(b + \frac{1}{\sqrt{n}}\right) + \frac{C}{\sqrt{n}} \|\varphi\|_\infty.
\]

**Lemma 4.4.** The conjugate characteristic function of \( \tilde{F}_n \) is given by

\[
\phi_{\tilde{F}_n}(\xi) = d_{n,x} e^{i\xi \sqrt{\pi} \rho} P_n^{\xi}(\Phi_{n,\xi} + \Phi^*_n)(x).
\]

In particular, \( \phi_{\tilde{F}_n} \) is differentiable near zero.

**Lemma 4.5.** Let \( \Phi_{n,\xi}, \Phi^*_n \) be the functions on \( \mathbb{P}^{d-1} \) defined above. Then, the following identity holds

\[
(4.8) \quad \Phi_{n,\xi} + \Phi^*_n = \varphi + \sum_{0 \leq k \leq A \log n} \left(e^{i\xi \sqrt{n}} - 1\right) \chi_k \varphi.
\]

Moreover, \( \|\chi_k \varphi\|_{\mathcal{F}_{\alpha}} \lesssim e^{\alpha k} \|\varphi\|_{\mathcal{F}_{\alpha}} \) and there is a constant \( C > 0 \) independent of \( \xi \) and \( n \) such that

\[
(4.9) \quad \|\Phi_{n,\xi}\|_{\mathcal{F}_{\alpha}} \leq C n^{\alpha A} \|\varphi\|_{\mathcal{F}_{\alpha}} \quad \text{and} \quad \|\Phi^*_n\|_{\mathcal{F}_{\alpha}} \leq C n^{\alpha A} \|\varphi\|_{\mathcal{F}_{\alpha}}.
\]

In addition, \( \Phi^*_n \) is supported by \( \{w : \log d(w, H_y) \leq -A \log n + 1\} \).

Let \( \xi_0 \) be a small constant satisfying Lemma \[2.5\].

**Lemma 4.6.** Let \( \tilde{F}_n \) and \( H \) be as above. Then, \( \tilde{F}_n(b) - H(b) \lesssim \|\varphi\|_\infty/\sqrt{n} \) for \( |b| \geq \xi_0 \sqrt{n} \).

From now on, we fix \( 0 < \alpha < 1 \) small enough so that

\[
\alpha A \leq 1/6 \quad \text{and} \quad \alpha \leq \alpha_0,
\]

where \( 0 < \alpha_0 < 1 \) is the exponent appearing in Theorem \[2.3\]. Then, from the results of Subsection \[2.2\], the family \( \xi \mapsto P_{\xi} \) acts continuously on \( \mathcal{F}_{\alpha}(\mathbb{P}^{d-1}) \) for \( \xi \in \mathbb{R} \), it is analytic near \( 0 \) and \( P_{\xi} \) has a spectral gap.

Lemmas \[4.4 \] and \[4.6 \] imply that \( \tilde{F}_n \) and \( H \) satisfy the conditions of Lemma \[2.8 \] with \( \delta_n := (\xi_0 \sqrt{n})^{-1/2} \). Let \( \kappa > 1 \) be the constant appearing in that corollary. For simplicity, by taking a smaller \( \xi_0 \) is necessary, one can assume that \( 2\kappa \xi_0 \leq 1 \). Then, Lemma \[2.8 \] gives that

\[
(4.10) \quad \sup_{b \in \mathbb{R}} |\tilde{F}_n(b) - H(b)| \leq \frac{1}{\pi} \sup_{|b| \leq \sqrt{n}} \left| \int_0^{\xi_0 \sqrt{n}} \frac{\Theta \theta(-\xi)}{\xi} \, d\xi \right| + \frac{C}{\sqrt{n}} \|\varphi\|_\infty.
\]
where \( C > 0 \) is a constant independent of \( n \) and
\[
\Theta_b(\xi) := e^{ib\xi}(\phi_{F_n}(\xi) - \tilde{h}(\xi))\vartheta_{\delta_n}(\xi).
\]

We now estimate the integral in (4.10). In order to deal with \( \phi_{F_n} \), we'll use Lemmas 4.4 coupled with Proposition 2.4 and Lemma 2.5. It is then natural to introduce the function
\[
\tilde{h}_n(\xi) := d_{n,x}e^{-\frac{\xi^2}{2}}N_0(\Phi_{n,\xi} + \Phi_n^*).
\]
Notice that for every fixed \( n \) and \( \xi, N_0(\Phi_{n,\xi} + \Phi_n^*) \) is a constant independent of \( x \). Define also
\[
\Theta_b^{(1)}(\xi) := e^{ib\xi}(\phi_{F_n}(\xi) - \tilde{h}_n(\xi))\vartheta_{\delta_n}(\xi) \quad \text{and} \quad \Theta_b^{(2)}(\xi) := e^{ib\xi}(\tilde{h}_n(\xi) - \tilde{h}(\xi))\vartheta_{\delta_n}(\xi),
\]
so that \( \Theta_b = \Theta_b^{(1)} + \Theta_b^{(2)} \).

**Lemma 4.7.** We have
\[
\sup_{|\xi| \leq \sqrt{n}} \left| \int_0^{\xi_0 \sqrt{n}} \frac{\Theta_b^{(1)}(\xi) - \Theta_b^{(2)}(\xi)}{\xi} \, d\xi \right| \lesssim \frac{\|\varphi\|_{\mathcal{E}_n}}{\sqrt{n}}.
\]

**Proof.** Using Lemma 4.4, the decomposition of \( P_z \) from Proposition 2.4 and recalling that \( d_{n,x} \leq 2 \), we have
\[
\Theta_b^{(1)} = d_{n,x}(\Lambda_1 + \Lambda_2 + \Lambda_3) \leq 2(\Lambda_1 + \Lambda_2 + \Lambda_3),
\]
where
\[
\Lambda_1(\xi) := e^{ib\xi} \left( e^{i\xi\sqrt{n}}\gamma_n\sqrt{n}N_{-\xi \sqrt{n}}(\Phi_{n,\xi} + \Phi_n^*)(x) - e^{-\frac{\xi^2}{2}}N_0(\Phi_{n,\xi} + \Phi_n^*)(x) \right)\vartheta_{\delta_n}(\xi),
\]
\[
\Lambda_2(\xi) := e^{ib\xi} \left( e^{i\xi\sqrt{n}}\gamma_n\sqrt{n}Q_n^0(\Phi_{n,\xi} + \Phi_n^*)(x) - e^{i\xi\sqrt{n}}Q_n^0(\Phi_{n,\xi} + \Phi_n^*)(x) \right)\vartheta_{\delta_n}(\xi)
\]
and
\[
\Lambda_3(\xi) := e^{ib\xi} e^{i\xi\sqrt{n}}Q_n^0(\Phi_{n,\xi} + \Phi_n^*)(x)\vartheta_{\delta_n}(\xi).
\]
Notice that \( \Lambda_1(0) = \Lambda_2(0) = 0 \). So for \( j = 1, 2 \), we have
\[
\int_{\xi_0 \sqrt{n}}^{\xi_0 \sqrt{n}} \frac{\Lambda_j(\xi) - \Lambda_j(-\xi)}{\xi} \, d\xi = \int_{-\xi_0 \sqrt{n}}^{\xi_0 \sqrt{n}} \frac{\Lambda_j(\xi)}{\xi} \, d\xi.
\]

In order to estimate \( \Lambda_2 \) we recall that, for \( z \) small, the norm of the operator \( Q_n^0 - Q_n^0 \) is bounded by a constant times \( |z|n^\beta \) for some \( 0 < \beta < 1 \), see the proof of Lemma 3.3. Therefore, we have
\[
\left| Q_n^0(\Phi_{n,\xi} + \Phi_n^*)(x) - Q_n^0(\Phi_{n,\xi} + \Phi_n^*)(x) \right| \lesssim \frac{|\xi|}{\sqrt{n}}n^\beta\|\Phi_{n,\xi} + \Phi_n^*\|_{\mathcal{E}_n}.
\]
Using (4.9), we get that
\[
\left| \int_{-\xi_0 \sqrt{n}}^{\xi_0 \sqrt{n}} \frac{\Lambda_2(\xi)}{\xi} \, d\xi \right| \leq \int_{-\xi_0 \sqrt{n}}^{\xi_0 \sqrt{n}} \frac{1}{|\xi|} \left| Q_n^0(\Phi_{n,\xi} + \Phi_n^*)(x) - Q_n^0(\Phi_{n,\xi} + \Phi_n^*)(x) \right| \, d\xi
\lesssim \int_{-\xi_0 \sqrt{n}}^{\xi_0 \sqrt{n}} \frac{1}{|\xi|} \cdot |\xi|\sqrt{n^\beta}\|\Phi_{n,\xi} + \Phi_n^*\|_{\mathcal{E}_n} \, d\xi \lesssim \beta n^\alpha A1\|\varphi\|_{\mathcal{E}_n} \lesssim \frac{\|\varphi\|_{\mathcal{E}_n}}{\sqrt{n}}.
\]
We now estimate $\Lambda_3$ using its derivative $\Lambda_3'$. Recall that $\| \hat{\vartheta}_n \|_{\varphi^*} \leq 1, |b| \leq \sqrt{n}$ and $|Q_0^n(\Phi_{n,\xi} + \Phi_n^*)(x)| \leq \beta^n \|\Phi_{n,\xi} + \Phi_n^*\|_{\varphi^*}$, where $0 < \beta < 1$ is as before. A direct computation using the definition of $\Phi_{n,\xi}$ gives for $|\xi| \leq \xi_0 \sqrt{n}$,

$$|\Lambda_3'(\xi)| \leq \left| bQ_0^n(\Phi_{n,\xi} + \Phi_n^*)(x) \right| + \left| \sqrt{n} \gamma \cdot Q_0^n(\Phi_{n,\xi} + \Phi_n^*)(x) \right| + \sum_{0 \leq k \leq A \log n} \frac{k}{\sqrt{n}} e^{\xi_k \lambda_n} Q_0^n(\chi_k \varphi)(x) \right| + \left| Q_0^n(\Phi_{n,\xi} + \Phi_n^*)(x) \right| \cdot \| \hat{\vartheta}_n \|_{\varphi^*}$$

$$\lesssim \sqrt{n} \beta^n n^{\alpha A} \|\varphi\|_{\varphi^*} + \sum_{0 \leq k \leq A \log n} \frac{k}{\sqrt{n}} \beta^n n^{\alpha A} \|\varphi\|_{\varphi^*} + \beta^n n^{\alpha A} \|\varphi\|_{\varphi^*} \lesssim \sqrt{n} \beta^n n^{\alpha A} \|\varphi\|_{\varphi^*},$$

where we have used that $\|\chi_k \varphi\|_{\varphi^*} \leq e^{\alpha_k} \|\varphi\|_{\varphi^*}$ and $\|\Phi_{n,\xi} + \Phi_n^*\|_{\varphi^*} \leq n^{\alpha A} \|\varphi\|_{\varphi^*}$, see Lemma 4.5.

Applying the mean value theorem to $\Lambda_3$ between $\xi$ and $-\xi$, yields

$$\left| \int_{\xi_0 \sqrt{n}}^{\xi_0 \sqrt{n}} \Lambda_3(\xi) - \Lambda_3(-\xi) \frac{d\xi}{\xi} \right| \leq 2 \xi_0 \sqrt{n} \sup_{|\xi| \leq \xi_0 \sqrt{n}} |\Lambda_3'(\xi)| \leq n \beta^n n^{\alpha A} \|\varphi\|_{\varphi^*} \lesssim \frac{\|\varphi\|_{\varphi^*}}{\sqrt{n}}.$$

It remains to estimate the term involving $\Lambda_1$. We have

$$\left| \int_{-\xi_0 \sqrt{n}}^{\xi_0 \sqrt{n}} \Lambda_1(\xi) \frac{d\xi}{\xi} \right| \leq \int_{-\xi_0 \sqrt{n}}^{\xi_0 \sqrt{n}} \frac{1}{\xi} \left| e^{\xi \sqrt{n} \gamma} \frac{\lambda_n}{\sqrt{n}} N_0(\Phi_{n,\xi} + \Phi_n^*)(x) - e^{-\frac{\xi^2}{2} n} N_0(\Phi_{n,\xi} + \Phi_n^*)(x) \right| d\xi.$$

We split the last integral into two integrals using

$$\Gamma_1(\xi) := e^{\xi \sqrt{n} \gamma} \frac{\lambda_n}{\sqrt{n}} N_0(\Phi_{n,\xi} + \Phi_n^*)(x) - e^{-\frac{\xi^2}{2} n} N_0(\Phi_{n,\xi} + \Phi_n^*)(x)$$

and

$$\Gamma_2(\xi) := e^{\xi \sqrt{n} \gamma} \frac{\lambda_n}{\sqrt{n}} N_0(\Phi_{n,\xi} + \Phi_n^*)(x) - e^{-\frac{\xi^2}{2} n} N_0(\Phi_{n,\xi} + \Phi_n^*)(x).$$

**Case 1.** $\sqrt{n} < |\xi| \leq \xi_0 \sqrt{n}$. In this case, by Lemma 2.5, we have

(4.11) $|\lambda_n| \leq e^{-\frac{\xi^2}{2}}$ and $|e^{\xi \sqrt{n} \gamma} \frac{\lambda_n}{\sqrt{n}} - e^{-\frac{\xi^2}{2}}| \lesssim \frac{1}{\sqrt{n}} e^{-\frac{\xi^2}{4}}.$

From the analyticity of $\xi \mapsto N_0$ (cf. Proposition 2.4, Lemma 4.5) and the fact that $\alpha A \leq 1/6$, one has

$$\left\| N_0(\Phi_{n,\xi} + \Phi_n^*) \right\|_{\varphi^*} \lesssim \frac{|\xi|}{\sqrt{n}} \|\Phi_{n,\xi} + \Phi_n^*\|_{\varphi^*} \lesssim \frac{|\xi|}{\sqrt{n}} n^{\alpha A} \|\varphi\|_{\varphi^*} \lesssim \frac{|\xi|}{\sqrt{n}} \sqrt{n} \|\varphi\|_{\varphi^*}.$$

Hence, using (4.11), we get

$$\int_{\sqrt{n} < |\xi| \leq \xi_0 \sqrt{n}} \frac{1}{|\xi|} |\Gamma_1(\xi)| d\xi \lesssim \int_{-\infty}^{\infty} \frac{1}{\sqrt{n}} \cdot e^{-\frac{\xi^2}{4}} \frac{|\xi|}{\sqrt{n}} |\varphi|_{\varphi^*} d\xi \lesssim \frac{\|\varphi\|_{\varphi^*}}{\sqrt{n}}.$$

Observe that $|\Phi_{n,\xi} + \Phi_n^*| \leq \Phi_{n,0} + \Phi_n^* = \varphi$, so $|N_0(\Phi_{n,\xi} + \Phi_n^*)| \leq N_0 \varphi = 1$. Therefore, using (4.11), we obtain

$$\int_{\sqrt{n} < |\xi| \leq \xi_0 \sqrt{n}} \frac{1}{|\xi|} |\Gamma_2(\xi)| d\xi \lesssim \int_{-\sqrt{n}}^{\sqrt{n}} \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n}} e^{-\frac{\xi^2}{4}} d\xi \lesssim \frac{1}{\sqrt{n}}.$$
Case 2. $|ξ| ≤ \sqrt{η}$. In this case, by Lemma 2.5, we have
\[(4.12) \quad |λ^{n/\sqrt{n}}| ≤ e^{-2ξ^2/3} \quad \text{and} \quad |eiξ\sqrt{n}λ^{n/\sqrt{n}} - e^{-2ξ^2/3}| ≤ \frac{1}{n}|ξ|^3 e^{-2ξ^2/3}.
\]
From Lemma 4.5 it follows that $\|Φ_{n,ξ} + Φ^*_n\|_{φ^α}$ is bounded by
\[\|φ\|_{φ^α} + \sum_{0 ≤ k ≤ A\log n} e^{iξ\sqrt{n} - 1} \cdot |χ_kφ|_{φ^α} ≤ \|φ\|_{φ^α} + \sum_{0 ≤ k ≤ A\log n} |ξ|^k e^{αk}\|φ\|_{φ^α} ≤ \|φ\|_{φ^α} + \frac{\sqrt{n}(log n)^2nαA}{√n}∥φ∥_{φ^α} ≤ \|φ\|_{φ^α},\]
where we have used that $\|χ_kφ\|_{φ^α} ≤ e^{αk}\|φ\|_{φ^α}$ and $αA ≤ 1/6$. It follows from the analyticity of $ξ → N_{ζ,ξ}$ that
\[\|N_{ζ,ξ} - N_0(Φ_{n,ξ} + Φ^*_n)\|_∞ ≤ \frac{|ξ|}{n}\|φ\|_{φ^α}.
\]
We conclude, using (4.12), that
\[\int_{|ξ| ≤ \frac{\sqrt{η}}{n}} \frac{1}{|ξ|} \cdot |Γ_1(ξ)| \, dξ ≤ \int_{|ξ| ≤ \frac{\sqrt{η}}{n}} \frac{1}{|ξ|} \cdot e^{-2ξ^2/3} \frac{|ξ|}{n}\|φ\|_{φ^α} \, dξ ≤ \frac{\|φ\|_{φ^α}}{n}.
\]
For $Γ_2$, using that $|N_0(Φ_{n,ξ} + Φ^*_n)| ≤ 1$ as before together with (4.12), gives
\[\int_{|ξ| ≤ \frac{\sqrt{η}}{n}} \frac{1}{|ξ|} \cdot |Γ_2(ξ)| \, dξ ≤ \int_{|ξ| ≤ \frac{\sqrt{η}}{n}} \frac{1}{|ξ|} \cdot \frac{1}{n}|ξ|^3 e^{-2ξ^2} \, dξ ≤ \frac{1}{n}.
\]
Together with Case 1, we deduce that
\[\left|\int_{-\sqrt{η}n}^{\sqrt{η}n} \frac{A_1(ξ)}{ξ} \, dξ\right| ≤ \|φ\|_{φ^α} \frac{1}{n}.
\]
The lemma follows.

Lemma 4.8. We have
\[\sup_{|η| ≤ \sqrt{η}} \left|\int_{0}^{\sqrt{n}} \Theta^{(2)}_k(ξ) - Θ^{(2)}_k(-ξ) \, dξ\right| ≤ \frac{\|φ\|_{φ^α}}{n}.
\]

Proof. Recall that $\tilde{h}_n(ξ) = d_{n,ξ}e^{-2ξ^2/3}N_0(Φ_{n,ξ} + Φ^*_n)$ and $\tilde{h}(ξ) = e^{-ξ^2/3}$. Put
\[Ω_1(ξ) := e^{iξ}\sqrt{n}(-N_0(Φ_{n,ξ} + Φ^*_n) - 1) \hat{d}_n(ξ)
\]
and
\[Ω_2(ξ) := e^{iξ}\sqrt{n}(d_{n,ξ} - 1)N_0(Φ_{n,ξ} + Φ^*_n) \hat{d}_n(ξ),
\]
so that $Θ^{(2)}_k = Ω_1 + Ω_2$.

We estimate $Ω_1$ first. Since $χ_κ$ is bounded by 1 and it is supported by $\mathcal{S}_κ \subset \mathbb{B}(H_y, e^{-k+1})$, we have
\[N_0(χ_κφ) = \int_{\mathbb{R}^{k-1}} χ_κφ \, dν \leq ν(\mathbb{B}(H_y, e^{-k+1}))\|φ\|_∞ \leq e^{-kη}\|φ\|_∞,
\]
where in the last step we have used Proposition 2.2.

Using the identity (4.8), the fact that $N_0φ = 1$, $\|\hat{d}_n\|_{φ^α} ≤ 1$ and Lemma 4.5, we get
\[\left|Ω_1(ξ)\right| = \left|e^{iξ}\sqrt{n}(-N_0(Φ_{n,ξ} + Φ^*_n) - 1) \hat{d}_n(ξ)\right| ≤ e^{ξ^2/3}\|N_0(Φ_{n,ξ} + Φ^*_n) - N_0φ\| \cdot \|\hat{d}_n(ξ)\|
\]
\[ e^{-\frac{x^2}{2}} \sum_{0 \leq k \leq A \log n} e^{i \frac{k}{\sqrt{n}} x} \leq e^{-\frac{x^2}{2}} \sum_{k \geq 0} |\xi| k \frac{e^{-k\eta}}{\sqrt{n}} \| \varphi \|_\infty \leq e^{-\frac{x^2}{2}} \frac{|\xi|}{\sqrt{n}} \| \varphi \|_\infty. \]

Therefore, the integral involving $\Omega_1$ is bounded by

\[ \int^{-\xi_0 \sqrt{n}}_{-\xi_0 \sqrt{n}} \Omega_1(\xi) \frac{1}{\xi} d\xi \leq \int^{-\xi_0 \sqrt{n}}_{-\xi_0 \sqrt{n}} \frac{1}{\xi} e^{-\frac{x^2}{2}} \frac{|\xi|}{\sqrt{n}} \| \varphi \|_\infty \| \psi \|_\infty d\xi \leq \| \varphi \|_\infty. \]

It remains to estimate $\Omega_2$. By a direct computation using the fact that $|\mathcal{N}_0(\Phi_n, \xi + \Phi_n^*)| \leq 1$, $|d_{n,x} - 1| \lesssim \beta^\alpha \| \varphi \|_\infty$, and $|b| \lesssim \sqrt{n}$, we have for $|\xi| \leq \xi_0 \sqrt{n}$,

\[ |\Omega_2(\xi)| \lesssim |d_{n,x} - 1| \left( |b\mathcal{N}_0(\Phi_n, \xi + \Phi_n^*)(x)| + |\sqrt{n}\mathcal{N}_0(\Phi_n, \xi + \Phi_n^*)(x)| \right) + \sum_{0 \leq k \leq A \log n} \mathcal{N}_0(\chi_k \varphi)(x) \right) \lesssim \sqrt{n} \beta^\alpha \| \varphi \|_\infty. \]

Thus, by the mean value theorem,

\[ \left| \int_{0}^{\xi_0 \sqrt{n}} \frac{\Omega_2(\xi) - \Omega_2(-\xi)}{\xi} d\xi \right| \lesssim 2\xi_0 \sqrt{n} \sup_{|\xi| \leq \xi_0 \sqrt{n}} |\Omega_2(\xi)| \lesssim n \beta^\alpha \| \varphi \|_\infty \lesssim \frac{\| \varphi \|_\infty}{\sqrt{n}}. \]

The lemma follows. \hfill \Box

**End of the proof of Proposition 4.2** Recall that our goal is to prove (4.4). Estimate (4.10) together with Lemmas 4.7 and 4.8 give that

\[ |\hat F_n(b) - H(b)| \leq \frac{\| \varphi \|_\infty}{\sqrt{n}} \quad \text{for all} \quad b \in \mathbb{R}. \]

Recall that $H(b) := \frac{1}{\sqrt{2\pi} e} \int_{-\infty}^{b} e^{-\frac{x^2}{2}} \mathrm{d}u$. Coupling the last estimate with Lemma 4.3 and the fact that $\sup_{b \in \mathbb{R}} |H(b) - H(b \pm 1/\sqrt{n})| \lesssim 1/\sqrt{n}$, we get $|\mathcal{L}_n(b) - H(b)| \lesssim \| \varphi \|_\infty / \sqrt{n}$, thus showing that (4.4) holds. Observe that all of our estimates are uniform in $x \in \mathbb{P}^{d-1}$ and $y \in (\mathbb{P}^{d-1})^*$. The proof of the proposition is complete. \hfill \Box

We will now use Proposition 4.2 to prove the general case of Theorem 1.2. We first prove the following lemma, which is an analog of Lemma 3.1.

**Lemma 4.9.** There exists a constant $C > 0$ such that for all $n \geq 1$,

\[ \bar{\mathcal{E}}_n(1_{|t| \geq \sqrt{n}}) \leq C/\sqrt{n}. \]

**Proof.** We can assume that $n$ is large enough. Let $A > 0$ be as before. By (4.3),

\[ \log d(S_n x, H_y) \leq -A \log n \quad \text{with probability smaller than} \quad e^{c/\sqrt{n}}. \]

Therefore, by the definition of $\bar{\mathcal{E}}_n$, see (4.1), and identity (4.2), we have

\[ \bar{\mathcal{E}}_n(1_{|t| \geq \sqrt{n}}) \leq E \left( \frac{1}{\sqrt{n}} |\log d(S_n x, H_y)| - A \log n \right) \geq \frac{1}{\sqrt{n}} \log d(S_n x, H_y) - A \log n) + e^{c/\sqrt{n}}. \]
Since $\log d(S_n, x, H_y) > -A \log n \geq -n/2$ for $n$ large enough, the last quantity is bounded by

$$E\left(1_{|\sigma(S_n, x) - n\gamma| \geq n/2} \log d(S_n, x, H_y) + A \log n\right) + e^c/\sqrt{n} \leq P\left(|\sigma(S_n, x) - n\gamma| \geq n/2\right) + e^c/\sqrt{n}.$$

The result follows by applying Proposition 2.1 with $\epsilon = 1/2$. □

We continue the proof of Theorem 1.2. We can assume $\|\psi\|_{\mathcal{K}} \leq 1$. So, $\psi$ is a Lipschitz function on $\mathbb{R}$ such that $\|\psi\|_\infty \leq 1$, $\|\psi^{'}\|_1 \leq 1$ and $\|\psi^{'}\|_{L^1} \leq 1$. We can also assume that $\|\phi\|_{\mathcal{K}} \leq 1$. Recall that $J = (-\infty, b]$. By Proposition 4.2, Theorem 1.2 holds when $\psi$ is a constant function. Hence, we can subtract $\psi(b)$ from $\psi$ and assume that $\psi(b) = 0$. In this case, $1_{(-\infty,b]}\psi$ is a Lipschitz function on $\mathbb{R}$ with Lipschitz norm bounded by 1.

Let $\tau_n$ be the cut-off functions defined as in Section 3. By Lemma 4.9, we can assume that $|b| \leq \sqrt{n}$ and it is enough to prove the estimate in Theorem 1.2 for the function $(\psi\tau_n)(t) \cdot \varphi(w)$ instead of $\psi(t) \cdot \varphi(w)$.

Let $\vartheta_b$ be the function from Lemma 2.6 and take $\delta_n := (\xi_0\sqrt{n})^{-1/2}$, where $\xi_0$ is a small constant satisfying Lemma 2.5. Define

$$\Psi_n := (1_{(-\infty,b]}\psi\tau_n) \ast \vartheta_{\delta_n}$$

as in the proof of Theorem 1.1 in Section 3. By the same arguments given there, we have that (3.5) holds for $\xi_n$ instead of $\xi_{\mathcal{K}}$. Recall that

$$\mathcal{R}(\Phi) := \frac{1}{\sqrt{2\pi}} \int_{d-1} e^{-\frac{x^2}{2}} \Phi(s, w) \, ds \, d\nu(w)$$

for functions $\Phi(t, w)$ on $\mathbb{R} \times \mathbb{P}^{d-1}$. The estimate (3.6) also holds in the current setting.

Combining the identity (4.2) with the estimate (4.3), it is enough to prove that

$$(4.13) \quad |S_n - \mathcal{R}(\Psi_n(t) \cdot \varphi(w))| \leq 1/\sqrt{n},$$

where

$$S_n := E\left(\Psi_n \left(\frac{\sigma(S_n, x) + \log d(S_n, x, H_y) - n\gamma}{\sqrt{n}}\right) \mathbf{1}_{\log d(S_n, x, H_y) > -A \log n} \varphi(S_n, x)\right).$$

Let $\chi_k$ be the functions defined in Lemma 4.1 and $\Phi^*_n$ be the function on $\mathbb{P}^{d-1}$ defined in (4.6). Set

$$A_n := \sum_{0 \leq k \leq A \log n} E\left(\Psi_n \left(\frac{\sigma(S_n, x) - n\gamma - k}{\sqrt{n}}\right)(\chi_k \varphi)(S_n, x)\right) + \mathcal{E}_n(\Psi_n(t) \cdot \Phi^*_n(w)).$$

Lemma 4.10. We have $|A_n - S_n| \leq 1/\sqrt{n}$.

Proof: Since $\chi_k$ is supported by $\{w : -k - 1 < \log d(w, H_y) < -k + 1\}$ and $\sum \chi_k = 1$ on $\mathbb{P}^{d-1} \setminus H_y$, we have

$$\mathbf{1}_{\log d(S_n, x, H_y) > -A \log n} \varphi(S_n, x) = \mathbf{1}_{\log d(S_n, x, H_y) > -A \log n} \sum_{0 \leq k \leq A \log n + 1} \chi_k \varphi(S_n, x).$$

It follows that

$$S_n = \sum_{0 \leq k \leq A \log n + 1} E\left(\Psi_n \left(\frac{\sigma(S_n, x) + \log d(S_n, x, H_y) - n\gamma}{\sqrt{n}}\right) \mathbf{1}_{\log d(S_n, x, H_y) > -A \log n} (\chi_k \varphi)(S_n, x)\right).$$
On the other hand, since \(1_{(-\infty, 0]}\psi_{\tau_n}\) is Lipschitz with a bounded Lipschitz norm, \(\Psi_t^n\) is bounded by a constant. Therefore,

\[
\left| \Psi_n \left( \frac{(S_n, x) - n\gamma - k}{\sqrt{n}} \right) - \Psi_n \left( \frac{(S_n, x) + \log d(S_n, H_y) - n\gamma}{\sqrt{n}} \right) \right| \lesssim \left| \log d(S_n, H_y) + k \right|,
\]

which is bounded by \(1/\sqrt{n}\) when \(S_n x \in \text{supp}(\chi_k)\). Hence, for \(k_0 := [A \log n] + 1\), we have

\[
|A_n - S_n| \lesssim \sum_{0 \leq k \leq A \log n} E \left( \frac{1}{\sqrt{n}} (\chi_k \varphi)(S_n x) \right) + E_n(\Psi_n(t - k_0/\sqrt{n}) \cdot (\chi_k \varphi)(w)) + E_n(\Psi_n(t) \Phi_n^*(w)).
\]

Since \(\sum \chi_k \varphi \leq \|\varphi\|_\infty \leq 1\), the first term on the right hand side is bounded by \(1/\sqrt{n}\). For the remaining terms, observe that \(\chi_k \varphi\) and \(\Phi_n^*\) are supported by \(\{w : \log d(w, H_y) \leq -A \log n + 2\}\), see Lemma 4.5. Using that \(\chi_k \varphi\), \(\Psi_n\) and \(\Phi_n^*\) are bounded, we see that these terms are bounded by a constant times \(P(\log d(S_n, H_y) \leq -A \log n + 2)\). By the same argument as the ones showing (4.3), the last quantity is \(\lesssim 1/\sqrt{n}\). This ends the proof of the lemma.

We now resume the proof of Theorem 1.2. Clearly, \(\Psi_n\) is integrable and \(\widehat{\Psi_n}\) is supported by \([-\delta_n^{-2}, \delta_n^{-2}]\). Using the inverse Fourier transform for \(\Psi_n\), Fubini’s theorem and (2.1), we have

\[
E \left( \Psi_n \left( \frac{(S_n, x) - n\gamma - k}{\sqrt{n}} \right) \right) \cdot (\chi_k \varphi)(S_n x) = \int_G \Psi_n \left( \frac{(g, x) - n\gamma - k}{\sqrt{n}} \right) (\chi_k \varphi)(gx) \, d\mu^S_n(g)
\]

\[
= \frac{1}{2\pi} \int_G \int_{-\infty}^{\infty} \widehat{\Psi_n}(\xi) e^{i\xi \left( (g, x) - n\gamma - k \right) / \sqrt{n}} (\chi_k \varphi)(gx) \, d\mu^S_n(g) \, d\xi
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\Psi_n}(\xi) \cdot e^{-i\xi \sqrt{n} \gamma} e^{-i\xi \frac{\delta_n^2}{\sqrt{n}}} P_{\frac{\delta_n^2}{\sqrt{n}}} (\chi_k \varphi)(x) \, d\xi.
\]

Similarly,

\[
E_n(\Psi_n(t) \cdot \Phi_n^*(w)) = E \left( \Psi_n \left( \frac{(S_n, x) - n\gamma}{\sqrt{n}} \right) \right) \Phi_n^*(S_n x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\Psi_n}(\xi) \cdot e^{-i\xi \sqrt{n} \gamma} P_{\frac{\delta_n^2}{\sqrt{n}}} (\Phi_n \xi + \Phi_n^*)(x) \, d\xi.
\]

Since the support of \(\widehat{\Psi_n}\) is contained in \([-\delta_n^{-2}, \delta_n^{-2}]\), it follows that

\[
A_n = \frac{1}{2\pi} \int_{-\delta_n \sqrt{n}}^{\delta_n \sqrt{n}} \widehat{\Psi_n}(\xi) \cdot e^{-i\xi \sqrt{n} \gamma} P_{\frac{\delta_n^2}{\sqrt{n}}} (\Phi_n \xi + \Phi_n^*)(x) \, d\xi,
\]

where \(\Phi_n \xi\) is the function on \(\mathbb{R}^{d-1}\) defined by

\[
\Phi_n \xi(w) := \sum_{0 \leq k \leq A \log n} e^{-i\xi \frac{\delta_n^2}{\sqrt{n}}} (\chi_k \varphi)(w).
\]

Notice that the definition of \(\Phi_n \xi\) is slightly different from the one in the proof of Proposition 4.2, but it still satisfies the properties in Lemma 4.5.

From the proof of Theorem 1.1 in Section 3 that

\[
R(\Psi_n(t) \cdot \varphi(w)) = \frac{1}{2\pi} \mathcal{N}_0 \varphi \int_{-\infty}^{\infty} \widehat{\Psi_n}(\xi) e^{-\frac{\xi^2 \varphi}{2}} \, d\xi = \frac{1}{2\pi} \mathcal{N}_0 \varphi \int_{-\delta_n \sqrt{n}}^{\delta_n \sqrt{n}} \widehat{\Psi_n}(\xi) e^{-\frac{\xi^2 \varphi}{2}} \, d\xi.
\]

Therefore,

\[
A_n - R(\Psi_n(t) \cdot \varphi(w)) = \frac{1}{2\pi} \int_{-\delta_n \sqrt{n}}^{\delta_n \sqrt{n}} \widehat{\Psi_n}(\xi) \left[ e^{-i\xi \sqrt{n} \gamma} P_{\frac{\delta_n^2}{\sqrt{n}}} (\Phi_n \xi + \Phi_n^*)(x) - e^{-\frac{\xi^2 \varphi}{2}} \mathcal{N}_0 \varphi \right] \, d\xi.
\]
Lemma 4.11. We have \(|A_n - R(\Psi_n(t) \cdot \varphi(w))| \lesssim 1/\sqrt{n}.

Proof. Define

\[ \Lambda_1(\xi) := e^{-i\xi \sqrt{n} \gamma} N^\alpha_0(\Phi_{n,\xi} + \Phi^*_n)(x) - e^{-\frac{2\xi^2}{2}} N_0(\Phi_{n,\xi} + \Phi^*_n)(x), \]

\[ \Lambda_2(\xi) := e^{-i\xi \sqrt{n} \gamma} Q^n_0(\Phi_{n,\xi} + \Phi^*_n)(x) - e^{-i\xi \sqrt{n} \gamma} Q^n_0(\Phi_{n,\xi} + \Phi^*_n)(x), \]

\[ \Lambda_3(\xi) := e^{-i\xi \sqrt{n} \gamma} Q^n_0(\Phi_{n,\xi} + \Phi^*_n)(x) \]

and

\[ \Omega_1(\xi) := e^{-\frac{2\xi^2}{2}} N_0(\Phi_{n,\xi} + \Phi^*_n) - e^{-\frac{2\xi^2}{2}} N_0 \varphi. \]

Then,

\[ e^{-i\xi \sqrt{n} \gamma} P^n_0(\Phi_{n,\xi} + \Phi^*_n)(x) - e^{-\frac{2\xi^2}{2}} N_0 \varphi = \Lambda_1(\xi) + \Lambda_2(\xi) + \Lambda_3(\xi) + \Omega_1(\xi). \]

From Lemma 3.4, we see that \(|\hat{\Psi}_n(\xi)| \leq 1/|\xi|\) for \(\xi \neq 0\). Using this estimate, we can repeat the arguments in the proofs of Lemmas 4.7 and 4.8 and deduce that

\[ \int_{-\xi_0 \sqrt{n}}^{\xi_0 \sqrt{n}} \hat{\Psi}_n(\xi) \Lambda_1(\xi) \, d\xi, \quad \int_{-\xi_0 \sqrt{n}}^{\xi_0 \sqrt{n}} \hat{\Psi}_n(\xi) \Lambda_2(\xi) \, d\xi, \quad \int_{-\xi_0 \sqrt{n}}^{\xi_0 \sqrt{n}} \hat{\Psi}_n(\xi) \Omega_1(\xi) \, d\xi \]

are all \(\lesssim 1/\sqrt{n}\).

It remains to estimate the integral involving \(\Lambda_3\). Recall from Lemma 3.4 that \(|\hat{\Psi}_n(\xi)| \leq 2\sqrt{n} + 2\). Using \(|Q^n_0(\Phi_{n,\xi} + \Phi^*_n)(x)| \lesssim \beta^n \|\Phi_{n,\xi} + \Phi^*_n\|_{\mathcal{F}_0} \lesssim \beta^n n^{1/6}\) for some \(0 < \beta < 1\), see Lemma 4.5, we get

\[ \int_{-\xi_0 \sqrt{n}}^{\xi_0 \sqrt{n}} \hat{\Psi}_n(\xi) \Lambda_3(\xi) \, d\xi \lesssim \sqrt{n} (2\sqrt{n} + 2) \beta^n n^{1/6} \lesssim \frac{1}{\sqrt{n}}. \]

This ends the proof of the lemma. 

End of the proof of Theorem 1.2 As seen above, the theorem will follow if we show that (4.13) holds. This estimate is an immediate consequence of Lemmas 4.10 and 4.11. The proof of the theorem is complete.

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