New Bound States for Modified Vibrational-Rotational Structure of Supersingular Plus Coulomb Potential of the Schrödinger Equation in One-Electron Atoms

Abdelmadjid Maireche

Laboratory of Physics and Material Chemistry, Physics Department, Sciences Faculty, University of M'sila-M'sila Algeria

abmaireche@gmail.com, abmaireche@yahoo.fr

Keywords: The Vibrational-Rotational Structure of Supersingular plus Coulomb potential, noncommutative space and phase, star product and Boopp's shift method.

Abstract. In this study, three-dimensional modified time-independent Schrödinger equation of modified vibrational-rotational structure of supersingular plus Coulomb (v.r.s.c) potential was solved using Boopp’s shift method instead to apply star product, in the framework of both noncommutativity three dimensional real space and phase (NC: 3D-RSP). We have obtained the explicit energy eigenvalues for ground state and first excited state for interactions in one-electron atoms. Furthermore, the obtained corrections of energies are depended on infinitesimal parameters \((\theta, \chi)\) and \(\{\vec{\theta}, \vec{\sigma}\}\) which are induced by position-position and momentum-momentum noncommutativity, respectively, in addition to the discreet atomic quantum numbers: \(j = l \pm 1/2, s = \pm 1/2, l\) and the angular momentum quantum number \(m\). We have also shown that, the usual states in ordinary three dimensional spaces for ordinary vibrational-rotational structure of supersingular plus Coulomb potential are canceled and has been replaced by new degenerated \(2(2l+1)\) sub-states in the extended new quantum symmetries of (NC: 3D-RSP).

1. Introduction

The exact solutions of the wave equations in non-relativistic and relativistic quantum mechanics (non-relativistic and relativistic spinless particles) in the case of ordinary commutative space with central and non-central potentials are very important for describing atoms, nuclei,…. various methods have been applied to solve the ordinary Schrödinger equation by means of asymptotic iteration method, improved (AIM), Laplace integral transform, factorization method, proper quantization rule, exact quantization rule, Nikiforov-Uvarov method, supersymmetry quantum mechanics (SUSYQM), In the recent years, authors have analytically derived exact solutions for diverse potentials [1-32]. It is well-known, that, the ordinary quantum mechanics based to the ordinary canonical commutations relations (CCRs) in both Schrödinger (time-independent operators) and Heisenberg pictures (time dependent operators) respectively, as:

\[
[x_i, p_j] = i \delta_{ij} \quad \text{and} \quad [x_i, x_j] = [p_i, p_j] = 0 \tag{1}
\]

and:

\[
[x_i(t), p_j(t)] = i \delta_{ij} \quad \text{and} \quad [x_i(t), x_j(t)] = [p_i(t), p_j(t)] = 0 \tag{2}
\]

where the two operators \((x_i(t), p_j(t))\) in Heisenberg picture are related to the corresponding two operators \((x_i, p_j)\) in Schrödinger picture from the two projections relations:

\[
x_i(t) = \exp(iH_{vrx}(t-t_0))x_i \exp(-iH_{vrx}(t-t_0)) \quad \text{and} \quad p_i(t) = \exp(iH_{vrx}(t-t_0))p_i \exp(-iH_{vrx}(t-t_0)) \tag{3}
\]

Here \(H_{vrx}\) denote to the ordinary quantum Hamiltonian operator for studied potential. Furthermore, much considerable effort has been expanded on the solutions of Schrödinger, Dirac and Klein-
Gordon equations to noncommutative quantum mechanics, to search a profound interpretation in microscopic scales, which based to new noncommutative canonical commutations relations (NNCCR) in both Schrödinger and Heisenberg pictures, respectively, as follows [33-67]:

\[
\begin{align*}
[\hat{x}_i, \hat{p}_j] = i\delta_{ij}, \quad [\hat{x}_i, \hat{x}_j] = i\partial_y \quad \text{and} \quad [\hat{p}_i, \hat{p}_j] = i\partial_y
\end{align*}
\]

and:

\[
\begin{align*}
[\hat{x}_i(t), \hat{p}_j(t)] = i\delta_{ij}, \quad [\hat{x}_i(t), \hat{x}_j(t)] = i\partial_y \quad \text{and} \quad [\hat{p}_i(t), \hat{p}_j(t)] = i\partial_y
\end{align*}
\]

where the two new operators \((\hat{x}_i(t), \hat{p}_j(t))\) in Heisenberg picture are related to the corresponding two new operators \((\hat{x}_i, \hat{p}_j)\) in Schrödinger picture from the two projections relations:

\[
\begin{align*}
\hat{x}_i(t) &= \exp(iH_{nc-vnc}(t-t_0)) \ast \hat{x}_i \ast \exp(-iH_{nc-vnc}(t-t_0)) \quad \text{and} \quad \hat{p}_i(t) = \exp(iH_{nc-vnc}(t-t_0)) \ast \hat{p}_i \ast \exp(-iH_{nc-vnc}(t-t_0))
\end{align*}
\]

Here \(H_{nc-vnc}\) denote to the new quantum Hamiltonian operator in the symmetries (NC: 3D-RSP). The very small two parameters \(\theta^{\mu\nu}\) and \(\bar{\theta}^{\mu\nu}\) (compared to the energy) are elements of two antisymmetric real matrices and (*) denote to the new star product, which is generalized between two arbitrary functions \(f(x, p)\) to \(\hat{f}(\hat{x}, \hat{p})\) and \(g(x, p)\) to \(\hat{g}(\hat{x}, \hat{p})\) to \(\hat{f}(\hat{x}, \hat{p})\hat{g}(\hat{x}, \hat{p}) = (f \ast g)(x, p)\) instead of the usual product \((fg)(x, p)\) in ordinary three dimensional spaces [39-57]:

\[
\hat{f}(\hat{x}, \hat{p})\hat{g}(\hat{x}, \hat{p}) = (f \ast g)(x, p) = \exp(\frac{i}{2} \theta^{\mu\nu} \partial^\mu \partial^{\nu} f(x, p)) \cdot \exp(-\frac{i}{2} \bar{\theta}^{\mu\nu} \partial^\mu \partial^{\nu} g(x, p)) + O(\theta^2, \bar{\theta}^2)
\]

where \(\hat{f}(\hat{x}, \hat{p})\) and \(\hat{g}(\hat{x}, \hat{p})\) are the new function in NC: 2D-RSP and (NC: 3D-RSP), the two covariant derivatives \(\partial^\mu f(x, p)\) are denoted to the \(\partial^\mu (\partial^l f(x, p))\), respectively, the two following terms \(-\frac{i}{2} \theta^{\mu\nu} \partial^\mu f(x, p)\partial^{\nu} g(x, p)\) and \(-\frac{i}{2} \bar{\theta}^{\mu\nu} \partial^\mu f(x, p)\partial^{\nu} g(x, p)\) are induced by (space-space) and (phase-phase) noncommutativity properties, respectively, and \(O(\theta^2, \bar{\theta}^2)\) stands for the second and higher order terms of \(\theta\) and \(\bar{\theta}\), a Boopp’s shift method can be used, instead of solving any quantum systems by using directly star product procedure [34-58]:

\[
[\hat{x}_i, \hat{x}_j] = i\partial_y \quad \text{and} \quad [\hat{p}_i, \hat{p}_j] = i\partial_y
\]

The 6 generalized positions and momentum coordinates in the noncommutative three dimensions quantum mechanics \((\hat{x}, \hat{y}, \hat{z})\) and \((\hat{p}_x, \hat{p}_y, \hat{p}_z)\) are depended with corresponding 6 usual generalized positions and momentum coordinates in the usual three dimensions quantum mechanics \((x, y, z)\) and \((p_x, p_y, p_z)\) by the following four relations, respectively, as follows [34-53]:

\[
\begin{align*}
\hat{x} &= x - \frac{\theta_{12}}{2} p_y - \frac{\theta_{13}}{2} p_z, \quad \hat{y} = y - \frac{\theta_{21}}{2} p_x - \frac{\theta_{23}}{2} p_z \\
\hat{z} &= z - \frac{\theta_{31}}{2} p_x - \frac{\theta_{32}}{2} p_y
\end{align*}
\]
\[
\begin{aligned}
\hat{p}_x &= p_x - \frac{\theta_{12}}{2} y - \frac{\theta_{13}}{2} z, \\
\hat{p}_y &= p_y - \frac{\theta_{21}}{2} x - \frac{\theta_{23}}{2} z, \\
\hat{p}_z &= p_z - \frac{\theta_{31}}{2} x - \frac{\theta_{32}}{2} y.
\end{aligned}
\] (10)

The non-vanish 9 commutators in (NC-3D: RSP) can be determined, as follows:

\[
\begin{aligned}
[\hat{x}, \hat{p}_x] &= [\hat{y}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = i, \\
[\hat{x}, \hat{y}] &= i \theta_{12}, [\hat{x}, \hat{z}] = i \theta_{13}, [\hat{y}, \hat{z}] = i \theta_{23}, \\
[\hat{p}_x, \hat{p}_y] &= i \theta_{12}, [\hat{p}_y, \hat{p}_z] = i \theta_{23}, [\hat{p}_z, \hat{p}_x] = i \theta_{13}
\end{aligned}
\] (11)

which allow us to getting the two operators \( \hat{r}^2 \) and \( \hat{p}^2 \) in (NC-3D: RSP), respectively, as follows [34-53]:

\[
\hat{r}^2 = r^2 - \hat{L}\hat{\Theta} \quad \text{and} \quad \hat{p}^2 = p^2 + \hat{\vec{\rho}}^2
\] (12)

where the two couplings \( \hat{L}\hat{\Theta} \) and \( \hat{\vec{\rho}}^2 \) are given by, respectively \( (\Theta_v \equiv \Theta / 2) \):

\[
\hat{L}\hat{\Theta} \equiv L_x \Theta_{12} + L_y \Theta_{23} + L_z \Theta_{13} \quad \text{and} \quad \hat{\vec{\rho}}^2 \equiv L_x \vec{\rho}_{12} + L_y \vec{\rho}_{23} + L_z \vec{\rho}_{13}.
\] (13)

It is-well known, that, in quantum mechanics, the three components \( (L_x, L_y \text{ and } L_z) \) are determined, in spherical coordinates:

\[
L_x = i \left( \sin\theta \frac{\partial}{\partial \theta} + \cos\theta \frac{\partial}{\partial \phi} \right), \quad L_y = -i \left( -\cos\theta \frac{\partial}{\partial \theta} + \sin\theta \frac{\partial}{\partial \phi} \right) \quad \text{and} \quad L_z = \frac{1}{i} \frac{\partial}{\partial \phi}.
\] (14)

The study of vibrational-rotational structure of supersingular plus Coulomb potential has now become a very interest field due to their applications in different fields [32], the bound state solutions of the non-relativistic Schrödinger equation, with the modified vibrational-rotational structure of supersingular plus Coulomb potential has not been obtained yet. This is the priority for this work. The modified vibrational-rotational structure of supersingular plus Coulomb potential used in this frame work takes the form:

\[
V_{nc-\text{vrs}}(r, A, Z, \Theta, \tilde{\Theta}) = \frac{A}{r^4} - \frac{Z}{r} + \frac{\hat{\vec{\rho}}^2}{2m_0} - \left( \frac{2A}{r^6} - \frac{Z}{2r^3} \right) \hat{L}\hat{\Theta}.
\] (15)

The crucial purpose of this paper is to determine the energy levels of above potential in (NC: 3D-RSP) symmetries using the generalization Boopp’s shift method based on mentioned formalisms on above equations to discover the new symmetries and a possibility to obtain another applications to this potential in different fields. It is worth to mention that, the noncommutative idea was introduced firstly by H. Snyder [33]. In the recent years, the problem of finding exact solutions of the non-relativistic modified Schrödinger equation in noncommutative spaces and phases for a number of special potential has been a line of great interest [34-67]. The organization scheme of the study is given as follows: In next section, we briefly review the Schrödinger equation with vibrational-rotational structure of supersingular plus Coulomb potential on based to Ref. [32]. The Section 3, devoted to studying the three deformed Schrödinger equation by applying both Boopp’s shift method to the vibrational-rotational structure of supersingular plus Coulomb potential. In the fourth section and by applying standard perturbation theory we find the quantum spectrum of the excited states in (NC-3D: RSP) for spin-orbital interaction corresponding the ground states and first
excited states. In the next section, we derive the magnetic spectrum for studied potential. In the sixth section, we resume the global spectrum and corresponding noncommutative Hamiltonian for vibrational-rotational structure of supersingular plus Coulomb potential. Conclusions are drawn in Sect 6.

2. Review the eigenfunctions and the energy eigenvalues for vibrational-rotational structure of supersingular plus Coulomb potential in ordinary three dimensional spaces

Our goal in this section is to review the essential steps, which give the solutions of time independent Schrödinger equation for a fermionic particle like electron of rest mass \( m_0 \) and its energy \( E \) moving in vibrational-rotational structure of supersingular plus Coulomb potential [32]:

\[
V(r) = \frac{A}{r^4} - \frac{Z}{r}
\]

where \( A \) play the role of positive constant coefficient and \( Z \) is the nuclear charge. The (v.r.s.c) potential plays a basic role in chemical and molecular physics since it can be used to calculate the molecular vibration-rotation energy spectrum of linear and non-linear systems. The above potential is the sum of Colombian \( \left( -\frac{Z}{r} \right) \) and vibrational-rotational structure of supersingular potential \( \left( \frac{A}{r^4} \right) \), if we insert this potential into the non-relativistic Schrödinger equation; we obtain the following equation in three dimensional spaces as follows [32]:

\[
\left\{ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{l(l+1)}{r^2} - \frac{A}{r^4} + \frac{Z}{r} \right\} R(r) = ER(r)
\]

here \( R(r) \) is the radial function. As it is mentioned in ref. [32], the solution of above second order differential equation in the coordinate basis turns out that there is a family of solutions:

\[
R_{n,l}(r, \beta) = N_{n,l} e^{-\alpha/r - \beta/r^n} r^l M\left(-n_r, 2(l+1) + \frac{2\beta r}{n} \right)
\]

where \( n = n_r + l + 1 \) is the principal quantum number, \( \alpha = \sqrt{\lambda} \) while \( \beta \) and \( M\left(-n_r, 2l + 2; \frac{2\beta r}{n} \right) \) are the charge parameter and confluent hypergeometric functions reducing to a polynomial of degree \( n_r \), \( 0, 1, 2, \ldots \) respectively. The normalized wave function \( \psi_{n,l}(r, \theta, \phi) \) expressed in terms of the radial function and spherical harmonic functions read as [32]:

\[
\psi_{n,l}(r, \theta, \phi) = N_{n,l} e^{-\alpha/r - \beta/r^n} r^l M\left(-n_r, 2l + 2; \frac{2\beta r}{n} \right) Y_l^m(\theta, \phi)
\]

here \( N_{n,l} \) is a normalized constant and the corresponding eigenvalues \( E_{\alpha = 0, l}(\beta) \) is determined from relation [32]:

\[
E_{\alpha = 0, l}(\beta) = \left( \frac{\beta}{l+1} \right)^2 - \frac{2\alpha}{l+1} \left( \frac{\beta}{l+1} \right)^2 K_{2l+3} \left\{ \frac{(\alpha \beta)^l}{l+1} \right\} + \frac{2(\beta - Z)}{l+1} K_{2l+2} \left\{ \frac{(\alpha \beta)^l}{l+1} \right\} + (2\beta - Z) \frac{(\alpha \beta)^l}{l+1} K_{2l+1} \left\{ \frac{(\alpha \beta)^l}{l+1} \right\}
\]

Here \( K_{2l}(b) \) is the modified Bessel function and the first term corresponds to the atomic hydrogen atom and the successive terms is the fraction involves the expectation values of inverse power of \( r \), \( \frac{1}{r^p} \) for \( p = 3, 2, 1 \).
3. Theoretical framework

This section is devoted to constructing of non relativistic modified Schrödinger equation (m.s.e) in (NC-3D: RSP) for (v.r.s.c) potential; to achieve this subject, we apply the essentials following steps [34-55]:

- Ordinary three dimensional Hamiltonian operator \( \hat{H}_{nc-vrsc} (p_i, x_i) \) will be replace by new three dimensional Hamiltonian operators \( \hat{H}_{nc-vrsc} (\hat{p}_i, \hat{x}_i) \), in (NC-3D: RSP),

- Ordinary complex wave function \( \psi(x) \) will be replacing by new complex wave function \( \hat{\psi}(\hat{r}) \),

- Ordinary energies \( E_{v=0}(\beta) \), in three dimensional spaces will be replaced by new values \( E_{nc-vrsc} \), in (NC-3D: RSP) symmetries.

And the last step corresponds to replace the ordinary old product by new star product (\( \ast \)), which allow us to constructing the modified three dimensional Schrödinger equation in (NC-3D: RSP) symmetries for modified (v.r.s.c) potential:

\[
\hat{H}_{nc-vrsc} (\hat{p}_i, \hat{x}_i) \ast \hat{\psi}(\hat{r}) = E_{nc-vrsc} \hat{\psi}(\hat{r})
\]

It is worth to emphasize that the Boopp’s shift method permits to use the ordinary product without star product for modified Schrödinger equation, in (NC-3D: RSP) symmetries for modified (v.r.s.c) potential as follows:

\[
\hat{H}_{nc-vrsc} (\hat{p}_i, \hat{x}_i) \hat{\psi}(\hat{r}) = E_{nc-vrsc} \hat{\psi}(\hat{r})
\]

where the new operator of Hamiltonian \( \hat{H}_{nc-vrsc} (\hat{p}_i, \hat{x}_i) \) can be expressed in three general varieties: both noncommutative space and noncommutative phase (NC-3D: RSP), only noncommutative space (NC-3D: RS) and only noncommutative phase (NC: 3D-RP) as, respectively:

\[
\begin{align*}
H_{nc-vrsc} (\hat{p}_i, \hat{x}_i) &\equiv H \left( \hat{p}_i = p_i - \frac{1}{2} \partial_x x_j; \hat{x}_i = x_i - \frac{1}{2} \partial_p p_j \right) & \text{for NC-3D: RSP} \\
H_{nc-vrsc} (\hat{p}_i, \hat{x}_i) &\equiv H \left( \hat{p}_i = p_i; \hat{x}_i = x_i \right) & \text{for NC-3D: RS} \\
H_{nc-vrsc} (\hat{p}_i, \hat{x}_i) &\equiv H \left( \hat{p}_i = p_i - \frac{1}{2} \partial_y x_j; \hat{x}_i = x_i \right) & \text{for NC-3D: RP}
\end{align*}
\]

In recently work, we are interest with the first variety which presented by eq. (23), after straightforward calculations, we can obtain the five important terms, which will be used to determine the modified (v.r.s.c) potential in (NC: 3D-RSP), as:

\[
\frac{A}{r^4} = \frac{A}{r^4} - \frac{2A\bar{L}\bar{O}}{r^6}, \quad \frac{Z}{r} = \frac{Z}{r} - \frac{Z\bar{L}\bar{O}}{2r^3}
\]

and:

\[
\frac{\hat{p}^2}{2m_0} = \frac{p^2}{2m_0} + \frac{\bar{L}\bar{O}}{2m_0}
\]

which allow us to obtaining the global potential operator \( H_{nc-vrsc} (\hat{p}_i, \hat{x}_i) (\hat{p}_i, \hat{x}_i) \) for (v.r.s.c) potential in (NC: 3D-RSP), as:
\[ H_{n\text{-vrc}}(\hat{p}_i, \hat{x}_i)(\hat{p}_i, \hat{x}_i) = \frac{A}{r^4} - \frac{Z}{r} + \frac{p^2}{2m_0} + \frac{L\vec{\Theta}}{2m_0} - \left( \frac{2A}{r^6} - \frac{Z}{2r^3} \right) \vec{L}\vec{\Theta} \] (28)

It’s clearly, that the four first terms are given the ordinary inverse-square potential and kinetic energy in (3D) spaces, while the rest terms are proportional’s with infinitesimal parameter \( \Theta \), thus, we can considered as a perturbations terms, we noted by \( \hat{H}_{n\text{-vrc-so-per}}(r, A, Z, \Theta, \vec{\Theta}) \) for (NC: 3D-RSP) symmetries as:

\[ \hat{H}_{n\text{-vrc-so-per}}(r, A, Z, \Theta, \vec{\Theta}) = \frac{\vec{L}\vec{\Theta}}{2m_0} - \left( \frac{2A}{r^6} - \frac{Z}{2r^3} \right) \vec{L}\vec{\Theta} \] (29)

4. The exact spin-orbital Hamiltonian and the corresponding spectrum for modified (v.r.s.c) potential for excited \( n^\text{th} \) states for one-electron atoms in (NC: 3D- RSP) symmetries

4.1. The exact spin-orbital Hamiltonian for modified (v.r.s.c) potential for one-electron atoms in (NC: 3D- RSP) symmetries:

Again, the perturbative terms \( \hat{H}_{n\text{-vrc-so-per}}(r, A, Z, \Theta, \vec{\Theta}) \) can be rewritten to the equivalent new physical form:

\[ \hat{H}_{n\text{-vrc-so-per}}(r, A, Z, \Theta, \vec{\Theta}) = \frac{1}{2} \left\{ \frac{\vec{\Theta}}{2m_0} - \Theta \left( \frac{2A}{r^6} - \frac{Z}{2r^3} \right) \right\} \vec{S}\vec{L} \] (30)

We have choses the infinitesimal vector \( \vec{\Theta} \) parallel to spin operator \( \vec{S} \), which allow us to replace \( \vec{L}\vec{\Theta} \) by spin–orbital coupling \( \vec{S}\vec{L} \), however, the local equivalent potential \( \hat{H}_{n\text{-vrc-so-per}}(r, A, Z, \Theta, \vec{\Theta}) \) can be rewritten to the following new equivalent form for modified (v.r.s.c) potential:

\[ \hat{H}_{n\text{-vrc-so-per}}(r, A, Z, \Theta, \vec{\Theta}) = \frac{1}{2} \left\{ \frac{\vec{\Theta}}{2m_0} - \Theta \left( \frac{2A}{r^6} - \frac{Z}{2r^3} \right) \right\} \vec{S}^2 - \vec{S}^2 \] (31)

To the best of our knowledge, we just replace the coupling spin-orbital \( \vec{S}\vec{L} \) by the expression \( \frac{1}{2} \left( \vec{J}^2 - \vec{L}^2 - \vec{S}^2 \right) \), in quantum mechanics. It should be underlined that \( (H_{n\text{-vrc}}(\hat{p}_i, \hat{x}_i)(\hat{p}_i, \hat{x}_i), J^2, L^2, S^2 \text{ and } J_z) \) forms a complete of conserved physics quantities and the eigenvalues of the spin orbital coupling operator are:

\[
p_z(j = l \pm 1/2, l, s = 1/2) = \begin{cases} \left( \frac{l + 1}{2} \right) \left( \frac{l + 1}{2} + 1 \right) + l(l + 1) - \frac{3}{4} = p, & \text{for } j = l + \frac{1}{2} \Rightarrow \text{polarization – up} \\
\left( l - \frac{1}{2} \right) \left( \frac{l - 1}{2} + 1 \right) + l(l + 1) - \frac{3}{4} = p, & \text{for } j = l - \frac{1}{2} \Rightarrow \text{polarization – down} 
\end{cases}
\] (32)

Which allows us to form a diagonal \( (3 \times 3) \) matrixes, with non null elements are \( \left[ (\hat{H}_{n\text{-vrc-so}})_{11}, (\hat{H}_{n\text{-vrc-so}})_{22} \right] \) and \( (\hat{H}_{n\text{-vrc-so}})_{33} \) for (v.r.s.c) potential in (NC: 3D-RSP) symmetries, as:

\[
(\hat{H}_{n\text{-vrc-so}})_{11} = p_x \left( \frac{\vec{\Theta}}{2m_0} - \Theta \left( \frac{2A}{r^6} - \frac{Z}{2r^3} \right) \right) \text{ if } j = l + \frac{1}{2} \Rightarrow \text{spin up}
\]

\[
(\hat{H}_{n\text{-vrc-so}})_{22} = p_x \left( \frac{\vec{\Theta}}{2m_0} - \Theta \left( \frac{2A}{r^6} - \frac{Z}{2r^3} \right) \right) \text{ if } j = l - \frac{1}{2} \Rightarrow \text{spin down}
\]

\[
(\hat{H}_{n\text{-vrc-so}})_{33} = 0
\] (33)
Substituting equation (30) into equation (22) and then, the radial part of the modified Schrödinger equation, satisfying the following important equation:

$$\left\{ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{l(l+1)}{r^2} - \frac{A}{r^4} + \frac{Z}{r^4} - \frac{\theta}{2m_0} - \frac{\left( 2A/r^6 - \frac{Z}{2r^3} \right)}{2} \right\} R(r) = E_{nc\rightarrow vrc} R(r)$$

(34)

It is clearly that the above radial equation including the perturbative terms of Hamiltonian operator $\hat{H}_{nc\rightarrow vrc}(r,A,Z,\Theta,\bar{\theta})$ for modified (v.r.s.c) potential, which we are subject of discussion in next sub-section.

4.2. The exact spin-orbital spectrum for modified (v.r.s.c) potential for ground states and first excited states for one-electron atoms in (NC: 3D- RSP) symmetries:

In this sub-section, we are going to study the modifications to the energy levels ($E_{nc\rightarrow vrc}(n_r,\Theta,\bar{\theta}), E_{nc\rightarrow vrc,D}(n_r,\Theta,\bar{\theta})$) for spin up and spin down, respectively, at first order of parameters $(\Theta, \bar{\theta})$, for excited states $n^{th}$, obtained by applying the standard perturbation theory, using eqs. (19) and (33) in (NC-3D: RSP) symmetries:

$$E_{nc\rightarrow vrc}(n_r,\Theta,\bar{\theta}) \equiv \left| N_{n_l} \right|^2 p_r \int e^{-2\alpha r/2\beta r/n} r^{2l} \left[ M \left( -n_r, 2l + 2; \frac{2\beta r}{n} \right) \right]^2 \left( \frac{\theta}{2m_0} - \frac{\left( 2A/r^6 - \frac{Z}{2r^3} \right)}{2} \right) r^2 dr$$

$$E_{nc\rightarrow vrc,D}(n_r,\Theta,\bar{\theta}) \equiv \left| N_{n_l} \right|^2 p_r \int e^{-2\alpha r/2\beta r/n} r^{2l} \left[ M \left( -n_r, 2l + 2; \frac{2\beta r}{n} \right) \right]^2 \left( \frac{\theta}{2m_0} - \frac{\left( 2A/r^6 - \frac{Z}{2r^3} \right)}{2} \right) r^2 dr$$

(35)

Now, we want to obtain the modified energy levels ($E_{nc\rightarrow vrc}(n_r = 0,\Theta,\bar{\theta}), E_{nc\rightarrow vrc,D}(n_r = 0,\Theta,\bar{\theta})$) and ($E_{nc\rightarrow vrc}(n_r = 1,\Theta,\bar{\theta}), E_{nc\rightarrow vrc,D}(n_r = 1,\Theta,\bar{\theta})$) corresponding the ground states and first excited states, a direct simplification gives:

$$E_{nc\rightarrow vrc}(n_r = 0,\Theta,\bar{\theta}) \equiv \left| N_{0_l} \right|^2 p_r \left( \sum_{i=1}^{3} T_{i-3}(n_r = 0) + \frac{\theta}{2m_0} T_{nc-p}(n_r = 0) \right)$$

$$E_{nc\rightarrow vrc,D}(n_r = 0,\Theta,\bar{\theta}) \equiv \left| N_{0_l} \right|^2 p_r \left( \sum_{i=1}^{3} T_{i-3}(n_r = 0) + \frac{\theta}{2m_0} T_{nc-p}(n_r = 0) \right)$$

(36)

and

$$E_{nc\rightarrow vrc}(n_r = 1,\Theta,\bar{\theta}) \equiv \left| N_{1_l} \right|^2 p_r \left( \sum_{i=1}^{3} T_{i-3}(n_r = 1) + \frac{\theta}{2m_0} T_{nc-p}(n_r = 1) \right)$$

$$E_{nc\rightarrow vrc,D}(n_r = 1,\Theta,\bar{\theta}) \equiv \left| N_{1_l} \right|^2 p_r \left( \sum_{i=1}^{3} T_{i-3}(n_r = 1) + \frac{\theta}{2m_0} T_{nc-p}(n_r = 1) \right)$$

(37)

where, the 6- terms: ($T_{i-3}(n_r = 0), T_{i-3}(n_r = 1), i = 1,2, T_{nc-p}(n_r = 0)$ and $T_{nc-p}(n_r = 1)$ are given by:

$$T_{1-3}(n_r = 0) = -2A \int e^{-2\alpha r/2\beta r/1} r^{2l-4} dr$$

$$T_{2-3}(n_r = 0) = Z \int e^{-2\alpha r/2\beta r/1} r^{2l-1} dr$$

$$T_{nc-p}(n_r = 0) = 0$$

(38)

and

$$T_{1-3}(n_r = 1) = -2A \int e^{-2\alpha r/2\beta r/1+2} r^{2l-4} dr - 2A \int e^{-2\alpha r/2\beta r/1+2} r^{2l-3} dr - 4A \int e^{-2\alpha r/2\beta r/1+2} r^{2l-2} dr$$

$$T_{2-3}(n_r = 1) = Z \int e^{-2\alpha r/2\beta r/1+2} r^{2l-1} dr + Z \int e^{-2\alpha r/2\beta r/1+2} r^{2l-1} dr + 2Z \int e^{-2\alpha r/2\beta r/1+2} r^{2l-1} dr$$

(39)

$$T_{nc-p}(n_r = 1) = \int e^{-2\alpha r/2\beta r/1+2} r^{2l+2} dr + \int e^{-2\alpha r/2\beta r/1+2} r^{2l+3} dr + 2 \int e^{-2\alpha r/2\beta r/1+2} r^{2l+4} dr$$
It is convenient to apply the following integral [68]:

\[
\int_0^{\infty} r^{v-1} \exp \left( - \frac{\lambda_2}{r} + \lambda_1 r \right) dr = 2 \left( \frac{\lambda_2}{\lambda_1} \right)^{\frac{1}{2}} K_v \left( 2 \sqrt{\frac{\lambda_2}{\lambda_1}} r \right)
\] (40)

where \((\lambda_1, \lambda_2)\) are positive numbers and \(2\sqrt{\lambda_1 \lambda_2} \frac{\alpha \beta}{2}\) and \(K_v\) the modified Bessel function of second kind and order \(v\) (as its mentioned previously). After straightforward calculations, we can obtain the explicitly results:

\[
T_{1-3}(n_r = 0) = -4 \lambda \left( \frac{\alpha}{\beta} (l+1) \right)^{\frac{2l+3}{2}} K_{2l+3} \left( \frac{4 \alpha \beta}{l+1} \right)
\]

\[
T_{2-3}(n_r = 0) = 2 \lambda \left( \frac{\alpha}{\beta} (l+1) \right)^{\frac{2l+1}{2}} K_{2l} \left( \frac{4 \alpha \beta}{l+1} \right)
\]

\[
T_{np}(n_r = 0) = 2 \left( \frac{\alpha}{\beta} (l+1) \right)^{\frac{2l+3}{2}} K_{2l+3} \left( \frac{4 \alpha \beta}{l+1} \right)
\]

and

\[
T_{1-3}(n_r = 1) = -4 \lambda \left( \frac{\alpha}{\beta} (l+2) \right)^{\frac{2l+3}{2}} K_{2l+3} \left( \frac{4 \alpha \beta}{l+2} \right) + \left( \frac{\alpha}{\beta} (l+2) \right)^{\frac{2l+1}{2}} K_{2l+1} \left( \frac{4 \alpha \beta}{l+2} \right) - 8 \lambda \left( \frac{\alpha}{\beta} (l+2) \right)^{\frac{2l+1}{2}} K_{2l+1} \left( \frac{4 \alpha \beta}{l+2} \right)
\]

\[
T_{2-3}(n_r = 1) = 2 \lambda \left( \frac{\alpha}{\beta} (l+2) \right)^{\frac{2l+1}{2}} K_{2l+1} \left( \frac{4 \alpha \beta}{l+2} \right) + 4 \lambda \left( \frac{\alpha}{\beta} (l+2) \right)^{\frac{2l+3}{2}} K_{2l+3} \left( \frac{4 \alpha \beta}{l+2} \right)
\]

\[
T_{np}(n_r = 1) = 2 \left( \frac{\alpha}{\beta} (l+2) \right)^{\frac{2l+3}{2}} K_{2l+3} \left( \frac{4 \alpha \beta}{l+2} \right) + 2 \left( \frac{\alpha}{\beta} (l+2) \right)^{\frac{2l+1}{2}} K_{2l+1} \left( \frac{4 \alpha \beta}{l+2} \right) + 4 \lambda \left( \frac{\alpha}{\beta} (l+2) \right)^{\frac{2l+5}{2}} K_{2l+5} \left( \frac{4 \alpha \beta}{l+2} \right)
\]

which allow us to obtaining the exact modifications of fundamental energy levels \(E_{nc-peru}(n_r = 0, \Theta, \bar{\Theta})\), \(E_{nc-perD}(n_r = 0, \Theta, \bar{\Theta})\) and \(E_{nc-peru}(n_r = 1, \Theta, \bar{\Theta})\), \(E_{nc-perD}(n_r = 1, \Theta, \bar{\Theta})\) corresponding the ground states and first excited states:

\[
E_{nc-peru}(n_r = 0, \Theta, \bar{\Theta}) = |N_0| p_{r, \Theta} \left[ \varTheta T_{nr}(n_r = 0) + \frac{\bar{\Theta}}{2 m_0} T_{nr}(n_r = 0) \right]
\]

\[
E_{nc-perD}(n_r = 0, \Theta, \bar{\Theta}) = |N_0| p_{r, \Theta} \left[ \varTheta T_{nr}(n_r = 0) + \frac{\bar{\Theta}}{2 m_0} T_{nr}(n_r = 0) \right]
\]

and

\[
E_{nc-peru}(n_r = 1, \Theta, \bar{\Theta}) = |N_1| p_{r, \Theta} \left[ \varTheta T_{nr}(n_r = 1) + \frac{\bar{\Theta}}{2 m_0} T_{nr}(n_r = 1) \right]
\]

\[
E_{nc-perD}(n_r = 1, \Theta, \bar{\Theta}) = |N_1| p_{r, \Theta} \left[ \varTheta T_{nr}(n_r = 1) + \frac{\bar{\Theta}}{2 m_0} T_{nr}(n_r = 1) \right]
\]

(43)

(44)

Where, the two factors \(T_{nr}(n_r = 0)\) and \(T_{nr}(n_r = 1)\) are given by, respectively:

\[
T_{nr}(n_r = 0) = T_{1-3}(n_r = 0) + T_{2-3}(n_r = 0)
\]

\[
T_{nr}(n_r = 1) = T_{1-3}(n_r = 1) + T_{2-3}(n_r = 1)
\]

(45)
4.3. The exact magnetic spectrum for modified (v.r.s.c) potential for ground states and first excited states for one-electron atoms in (NC: 3D- RSP) symmetries:

Having obtained the exact modifications to the energy levels \( E_{nc-peru}(n_r = 0, \Theta, \bar{\Theta}) \), \( E_{nc-peru}(n_r = 1, \Theta, \bar{\Theta}) \) and \( E_{nc-peru}(n_r = 1, \Theta, \bar{\Theta}) \) corresponding the ground states and first excited states, produced with spin-orbital induced Hamiltonians operator \( \hat{H}_{vrcm-peru}(r, A, Z, \Theta, \bar{\Theta}) \), we now consider another interested physically meaningful phenomena, which produced from the perturbative terms of inverse-square potential related to the influence of an external uniform magnetic field, it’s sufficient to apply the following three replacements to describing these phenomena:

\[
\frac{1}{2m_0} - \left( \frac{2A}{r^6} - \frac{Z}{2r^3} \right) L \Theta \rightarrow \left[ \frac{\sigma}{2m_0} - \chi \left( \frac{2A}{r^6} - \frac{Z}{2r^3} \right) \right] \tilde{B}L
\]

\[
\theta \rightarrow \chi B, \Theta \rightarrow \chi B \quad \text{and} \quad \bar{\theta} \rightarrow \bar{\sigma}B
\]

Here \( \chi \) and \( \bar{\sigma} \) are infinitesimal real proportional’s constants, and we choose the uniform magnetic field parallel to the (Oz) axes, which allow us to introduce the modified new magnetic Hamiltonians \( \hat{H}_{vrcm-peru}(r, A, Z, \chi, \bar{\sigma}) \) in (NC: 3D-RSP) symmetries, as:

\[
\hat{H}_{vrcm-peru}(r, A, Z, \chi, \bar{\sigma}) = \left[ \frac{\sigma}{2m_0} - \chi \left( \frac{2A}{r^6} - \frac{Z}{2r^3} \right) \right] \tilde{B}L - \bar{\sigma}B
\]

Here \( \left( \tilde{B}L - \bar{\sigma}B \right) \) is the new modified Hamiltonian of Zeeman Effect and \( \left( -\bar{\sigma}B \right) \) denote to the ordinary Hamiltonian of Zeeman Effect in commutative space. To obtain the exact noncommutative magnetic modifications of energy \( E_{nc-perm}(n_r = 0, \Theta, \bar{\Theta}) \) and \( E_{nc-perm}(n_r = 1, \Theta, \bar{\Theta}) \) corresponding the ground states and first excited states, produced with spin-orbital induced Hamiltonians operator \( \hat{H}_{vrcm-peru}(r, A, Z, \Theta, \bar{\Theta}) \) for modified (v.r.s.c) potential, we make the following three simultaneously replacements, this to avoid repetition in the previous calculations:

\[
p_x (p_-) \rightarrow m,(\theta, \Theta) \rightarrow (\chi, \chi) \quad \text{and} \quad \bar{\theta} \rightarrow \bar{\sigma}B
\]

Into two Eqs. (43) and (44) to obtain the new values \( E_{nc-perm}(n_r = 0, \chi, \bar{\sigma}) \) and \( E_{nc-perm}(n_r = 1, \chi, \bar{\sigma}) \), respectively, as:

\[
E_{nc-perm}(n_r = 0, \chi, \bar{\sigma}) = \left| N_0 \right|^2 m \left( \chi T_{nc-\perp}(n_r = 0) + \frac{\sigma}{2m_0} T_{nc-p}(n_r = 0) \right)
\]

\[
E_{nc-perm}(n_r = 1, \chi, \bar{\sigma}) = \left| N_1 \right|^2 m \left( \chi T_{nc-\perp}(n_r = 1) + \frac{\sigma}{2m_0} T_{nc-p}(n_r = 1) \right)
\]

where \( m \) denote to the angular momentum quantum number, \(-l \leq m \leq +l\), which allow us to fixing \((2l+1)\) values for the orbital angular momentum quantum numbers.

5. Main Results

Our main results are summarized the eigenenergies of the modified Schrödinger equations obtained in this paper, the total modified energies \( (E_{nc-vrscm}(n_r = 0, \Theta, \bar{\Theta}, \chi, \bar{\sigma}), E_{nc-vrscm}(n_r = 0, \Theta, \bar{\Theta}, \chi, \bar{\sigma})) \) and \( E_{nc-vrscm}(n_r = 1, \Theta, \bar{\Theta}, \chi, \bar{\sigma}), E_{nc-vrscm}(n_r = 1, \Theta, \bar{\Theta}, \chi, \bar{\sigma}) \) of a particle fermionic with spin up and spin down are determined corresponding the ground states and first excited states, respectively, for modified (v.r.s.c) potential on based to the obtained new results (43), (44) and (50), in addition to
the original results (20) of energies corresponding ordinary commutative space, we obtain the
detailed energy behaviours of the system as:
\[ E_{n_{v-rsc}} \left( n_r = 0, \Theta, \vec{r}, \chi, \sigma \right) = E_{v = 0, l} \left( \beta \right) + \left| N_0 \right|^2 \left( p_r, \Theta + m \chi \right) T_{nc-l} (n_r = 0) + \frac{1}{2 m_0} \left( \partial p_r, \Theta + \sigma \chi \right) T_{nc-p} (n_r = 0) \]  \tag{51}

\[ E_{n_{v-rsc}} \left( n_r = 0, \Theta, \vec{r}, \chi, \sigma \right) = E_{v = 0, l} \left( \beta \right) + \left| N_0 \right|^2 \left( p_r, \Theta + m \chi \right) T_{nc-s} (n_r = 0) + \frac{1}{2 m_0} \left( \partial p_r, \Theta + \sigma \chi \right) T_{nc-p} (n_r = 0) \]  \tag{52}

\[ E_{n_{v-rsc}} \left( n_r = 1, \Theta, \vec{r}, \chi, \sigma \right) = E_{v = 0, l} \left( \beta \right) + \left| N_1 \right|^2 \left( p_r, \Theta + m \chi \right) T_{nc-l} (n_r = 1) + \frac{1}{2 m_0} \left( \partial p_r, \Theta + \sigma \chi \right) T_{nc-p} (n_r = 1) \]  \tag{53}

\[ E_{n_{v-rsc}} \left( n_r = 1, \Theta, \vec{r}, \chi, \sigma \right) = E_{v = 0, l} \left( \beta \right) + \left| N_1 \right|^2 \left( p_r, \Theta + m \chi \right) T_{nc-s} (n_r = 1) + \frac{1}{2 m_0} \left( \partial p_r, \Theta + \sigma \chi \right) T_{nc-p} (n_r = 1) \]  \tag{54}

In this way, one can obtain the complete energy spectra for (v.r.s.c) potential in (NC: 3D-RSP)
symmetries. Know the following accompanying constraint relations:

- The original spectrum contains two possible values of energies in ordinary two–three
dimensional space which presented by equation (20).
- The quantum number \( m \) satisfied the interval: \(-l \leq m \leq +l\), thus we have \((2l+1)\) values for this
  quantum number.
- We have also two values for \( j = l + \frac{1}{2} \) and \( j = l - \frac{1}{2} \).

Allow us to deduce the important original results: every state in usually three dimensional space for
vibrational-rotational structure of supersingular plus Coulomb potential will be replace by
2\((2l+1)\)sub-states for modified (v.r.s.c) potential and then the degenerated state can be take
\[
\sum_{\ell = 0}^{2} (2l+1) = 2n^2 \quad \text{values in (NC: 3D-RSP) symmetries. It’s clearly, that the obtained eigenvalues of}
\]
energies are real and then the noncommutative diagonal Hamiltonian operator
\[ \hat{H}_{nc-rsc} \left( n_r, A, \Theta, \vec{r}, \chi, \sigma \right) \] is Hermitian, furthermore it’s possible to writing the elements \( \left( \hat{H}_{nc-rsc} \right)_{11}, \)
\( \left( \hat{H}_{nc-rsc} \right)_{22} \) and \( \left( \hat{H}_{nc-rsc} \right)_{33} \) as follows:
\[
\left( \hat{H}_{nc-rsc} \right)_{11} = -\frac{1}{2m_0} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{\sigma}{r^6} - \frac{Z}{r^3} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \left( \sin \theta \right)^2} \frac{\partial^2}{\partial \phi^2} \right] + \frac{A}{r^4} \frac{Z}{r^3} \quad \text{for} \quad j = \ell + 1/2 \Rightarrow \text{spin up}
\]
\[
\left( \hat{H}_{nc-rsc} \right)_{22} = -\frac{1}{2m_0} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{\sigma}{r^6} - \frac{Z}{r^3} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \left( \sin \theta \right)^2} \frac{\partial^2}{\partial \phi^2} \right] + \frac{A}{r^4} \frac{Z}{r^3} \quad \text{for} \quad j = \ell - 1/2 \Rightarrow \text{spin down}
\]
\[
\left( \hat{H}_{nc-rsc} \right)_{33} \quad \text{(Perturbative spin-orbital terms)} + \quad \text{(modified Zeeman Effect)}
\]
\[
\left( \hat{H}_{nc-rsc} \right)_{11} + \frac{1}{2m_0} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{\sigma}{r^6} - \frac{Z}{r^3} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \left( \sin \theta \right)^2} \frac{\partial^2}{\partial \phi^2} \right] + \frac{A}{r^4} \frac{Z}{r^3} \quad \text{for} \quad j = \ell + 1/2 \Rightarrow \text{spin up}
\]
\[
\left( \hat{H}_{nc-rsc} \right)_{22} + \frac{1}{2m_0} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{\sigma}{r^6} - \frac{Z}{r^3} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \left( \sin \theta \right)^2} \frac{\partial^2}{\partial \phi^2} \right] + \frac{A}{r^4} \frac{Z}{r^3} \quad \text{for} \quad j = \ell - 1/2 \Rightarrow \text{spin down}
\]
\[
(\hat{H}_{\text{nc-vrsc}})_{33} = -\frac{1}{2m_0} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 (\sin \theta)^2} \frac{\partial^2}{\partial \phi^2} \right] + \frac{A - Z}{r^{4/3}} \frac{Z}{2(r,v,s,c)\text{potential}}
\]  

The first term in the modified Hamiltonian operator \( \hat{H}_{\text{nc-vrsc}} \) of the fermionic particle, the second term represents the potential energy \( \hat{H}_{\text{vrsc-int}} \) in ordinary commutative space, the third term \( \hat{H}_{\text{vrsc-so-per}} \) represents the induced spin-orbital parts and the last term \( \hat{H}_{\text{vrsc-m-per}} \) is induced modified Zeeman effect, the last two terms have produced automatically from the position-position and momentum-momentum noncommutativity. On the other hand, the above obtain results allow us to constructing the diagonal anisotropic matrices \( \{\hat{H}_{\text{nc-vrsc}}\}_{33} \neq \{\hat{H}_{\text{nc-vrsc}}\}_{22} \neq \{\hat{H}_{\text{nc-vrsc}}\}_{33} \) of the Hamiltonian operators \( \hat{H}_{\text{nc-vrsc}} \) for modified (v.r.s.c) potential in (NC: 3D-RSP) symmetries is given below:

\[
\hat{H}_{\text{nc-vrsc}}(n_r, A, Z, \Theta, \bar{\Theta}, \chi, \bar{\chi}) - \hat{H}_{\text{nr-vrsc}}(n_r, A, Z, \Theta, \bar{\Theta}, \chi, \bar{\chi}) + \hat{H}_{\text{vrsc-int}}(n_r, A, Z, \Theta, \bar{\Theta}) + \hat{H}_{\text{vrsc-so-per}}(n_r, A, Z, \Theta, \bar{\Theta}) + \hat{H}_{\text{vrsc-m-per}}(n_r, A, Z, \Theta, \bar{\Theta}).
\]  

Thus, for a given potential energy function \( V_{\text{nc-vrsc}}(r, A, Z, \Theta) \) the modified Schrödinger equation is governed by the above Hamiltonian operator \( \hat{H}_{\text{nc-vrsc}}(n_r, A, Z, \Theta, \bar{\Theta}, \chi, \bar{\chi}) \), which allows us to obtain the original results for this investigation: the obtained Hamiltonian operator (58) can be describing atom which has two permanent dipoles: the first dipole moment produce the spin-orbital interaction terms \( \hat{H}_{\text{nr-vrsc-so-per}}(r, A, Z, \Theta, \bar{\Theta}) \) and the second is magnetic moment in external stationary magnetic field produce the modified Zeeman interaction terms \( \hat{H}_{\text{vrsc-m-per}}(r, A, Z, \chi, \bar{\chi}) \). Finally, we can write the important following original results:

\[
\begin{align*}
\{\hat{H}_{\text{nc-vrsc}}\}_{11} \Psi_{01}(r, \theta, \phi) &= E_{\text{nc-vrsc-ca}}(n_r = 0, \Theta, \bar{\Theta}, \chi, \bar{\chi}) \Psi_{01}(r, \theta, \phi) \\
\{\hat{H}_{\text{nc-vrsc}}\}_{11} \Psi_{11}(r, \theta, \phi) &= E_{\text{nc-vrsc-ca}}(n_r = 1, \Theta, \bar{\Theta}, \chi, \bar{\chi}) \Psi_{11}(r, \theta, \phi) \\
\{\hat{H}_{\text{nc-vrsc}}\}_{22} \Psi_{02}(r, \theta, \phi) &= E_{\text{nc-vrsc-ca},D}(n_r = 0, \Theta, \bar{\Theta}, \chi, \bar{\chi}) \Psi_{02}(r, \theta, \phi) \\
\{\hat{H}_{\text{nc-vrsc}}\}_{11} \Psi_{11}(r, \theta, \phi) &= E_{\text{nc-vrsc-ca},B}(n_r = 1, \Theta, \bar{\Theta}, \chi, \bar{\chi}) \Psi_{11}(r, \theta, \phi)
\end{align*}
\]

and

\[
\begin{align*}
\{\hat{H}_{\text{nc-vrsc}}\}_{33} \Psi_{03}(r, \theta, \phi) &= E_{\nu=0,j}(\beta, n_r = 0) \Psi_{03}(r, \theta, \phi) \\
\{\hat{H}_{\text{nc-vrsc}}\}_{33} \Psi_{13}(r, \theta, \phi) &= E_{\nu=0,j}(\beta, n_r = 1) \Psi_{13}(r, \theta, \phi)
\end{align*}
\]

6. Conclusion

In this article, our study involves finding exact eigenvalues of the modified radial Schrödinger equation for new expansion of the modified vibrational-rotational structure of supersingular plus Coulomb potential \( V_{\text{nc-vrsc}}(r, A, Z, \Theta) \). The novelties of this study are the new results \( E_{\text{nc-vrsc-ca}}(n_r = 0, \Theta, \bar{\Theta}, \chi, \bar{\chi}), E_{\text{nc-vrsc-ca},D}(n_r = 0, \Theta, \bar{\Theta}, \chi, \bar{\chi}) \) and \( E_{\text{nc-vrsc-ca}}(n_r = 1, \Theta, \bar{\Theta}, \chi, \bar{\chi}), E_{\text{nc-vrsc-ca},D}(n_r = 1, \Theta, \bar{\Theta}, \chi, \bar{\chi}) \) for ground state and first excited state, respectively, of a particle fermionic with spin up and spin down and the corresponding noncommutative Hamiltonians \( \hat{H}_{\text{nc-vrsc}}(n_r, A, Z, \Theta, \Theta, \chi, \bar{\chi}) \) operator in the symmetries of (NC: 3D-RSP). We found that the energy eigenvalues depend on the dimensionality of the problem and new atomic quantum numbers \( j = l \pm 1/1, s = \pm 1/2, l \) and the angular
momentum quantum number in addition to four infinitesimals parameters \((\theta, \bar{\theta}, \varphi, \bar{\varphi})\) in the symmetries of (NC: 3D-RSP). We have also shown that, the studied one-electron atom has two permanent dipoles: the first dipole moment produce the spin-orbital interaction terms 
\[ H_{\text{spin-orbital}}(r, A, Z, \Theta, \varphi) \]
and the second is magnetic moment in external stationary magnetic field produce the modified Zeeman interaction terms 
\[ H_{\text{Zeeman}}(r, A, Z, \varphi) \]. The results of this research can be used in various applications, especially material science, including physical and chemical options.

Acknowledgements
The research leading to this article was supported with search Laboratory of Physics and Material Chemistry, Physics department, Sciences Faculty, University of M'sila-M’sila Algeria.

References
[1] S.-H. Dong, G.-H. Sun, The Schrödinger equation with a Coulomb plus inverse-square potential in D dimensions, Physica Scripta. 70(2-3) (2004) 94–97.
[2] D.L. Aronstein, C.R. Stroud, Jr., General series solution for finite square-well energy levels for use in wave-packet studies, Am. J. Phys. 68 (2000) 943–949.
[3] L. Buragohain, S.A.S. Ahmed, Exactly solvable quantum mechanical systems generated from the anharmonic potentials, Lat. Am. J. Phys. Educ. 4(1) (2010) 79–83.
[4] R.J. Lombard, J. Mars, C. Volpe, Wave equations of energy dependent potentials for confined systems, Journal of Physics G: Nuclear and Particle Physics. 34 (2007) 1879–1888.
[5] K.J. Oyewumi, E.A. Bangudu, Isotropic harmonic oscillator plus inverse quadratic potential in N-dimensional space, The Arabian Journal for Science and Engineering. 28(2A) (2003) 173.
[6] M.M. Nieto, Hydrogen atom and relativistic pi-mesic atom in N-space dimension, Am. J. Phys. 47 (1979) 1067–1072.
[7] S.M. Ikhdair, R. Sever, Exact polynomial eigensolutions of the Schrödinger equation for the pseudoharmonic potential, Journal of Molecular Structure: THEOCHEM. 806(1) (2007) 155–158.
[8] A.S. Ahmed, L. Buragohain, Generation of new classes of exactly solvable potentials, Phys. Scr. 80 (2009) 1–6.
[9] S.K. Bose, Exact solution of non-relativistic Schrödinger equation for certain central physical potentials, Nouvo Cimento B. 113 (1996) 299–328.
[10] G.P. Flesses, A. Watt, An exact solution of the Schrödinger equation for a multiterm potential, J. Phys. A: Math. Gen. 14 (1981) L315–L318.
[11] M. Yıldız, Energy levels and atomic lifetimes of Rydberg states in neutral Indium, Acta Physica Polonica A. 123(1) (2013) 25–30.
[12] S.H. Dong, Schrödinger equation with the potential \( V(r) = Ar^{-4} + Br^{-3} + Cr^{-2} + Dr^{-1} \), Physica Scripta. 64(4) (2001) 273–276.
[13] D. Mikulski et al., Exact solution of the Schrödinger equation with a new expansion of anharmonic potential with the use of the supersymmetric quantum mechanics and factorization method, J. Math. Chem. 53 (2015) 2018–2027.
[14] S.H. Dong, A new approach to the relativistic Schrödinger equation with central potential: Ansatz method, International Journal of Theoretical Physics. 40(2) (2001) 559–567.
A new Coulomb ring-shaped potential via generalized parametric Nikiforov-Uvarov method, Journal of Theoretical and Applied Physics. 7(1) (2013) 1–15.

B. Biswas, S. Deb Nath, Bound states for pseudoharmonic potential of the Dirac equation with spin and pseudo-spin symmetry via Laplace transform approach, Acta Physica Polonica A. 130(3) (2016) 692–696.

S.-H. Dong, G.-H. San, Quantum spectrum of some anharmonic central potentials: wave functions ansatz, Foundations of Physics Letters. 16(4) (2003) 357–367.

L. Buragohain, S.A.S. Ahmed, Exactly solvable quantum mechanical systems generated from the anharmonic potentials, Lat. Am. J. Phys. Educ. 4(1) (2010) 79-83.

S.M. Ikhdair, Exact solution of Dirac equation with charged harmonic oscillator in electric field: bound states, Journal of Modern Physics. 3(2) (2012) 170–179.

Zhang Tian-Yi, Zheng Neng-Wu, Theoretical study of energy levels and transition probabilities of boron atom, Acta Physica Polonica A. 116(2) (2009) 141–153.

H. Hassanabadi et al., Exact solutions of N-dimensional Schrödinger equation for a potential containing coulomb and quadratic terms, International Journal of the Physical Sciences. 6(3) (2011) 583–586.

S.-H. Dong, Z.-Q. Ma, G. Esposito, Exact solutions of the Schrödinger equation with inverse-power potential, Foundations of Physics Letters. 12(5) (1999) 465–473.

D. Agboola, Complete analytical solutions of the Mie-Type potentials in N-dimensions, Acta Physica Polonica A. 120 (2011) 371–377.

D. Shi-Hai, Exact solutions of the two-dimensional Schrödinger equation with certain central potentials, Int. J. Theor. Phys. 39 (2000) 1119–1128.

B.I. Ita, Solutions of the Schrödinger equation with inversely quadratic Hellmann plus Mie-type potential using Nikiforov-Uvarov method, International Journal of Recent Advances in Physics. 2(4) (2013).

P. Gupta, I. Mehrotra, Study of heavy quarkonium with energy dependent potential, Journal of Modern Physics. 3 (2012) 1530–1536.

B.I. Ita, A.I. Ikeuba, A.N. Ikot, Solutions of the Schrödinger equation with quantum mechanical gravitational potential plus harmonic oscillator potential, Commun. Theor. Phys. 61(2) (2014) 149.

A. Ghoshal, Y.K. Ho, Ground states of helium in exponential-cosine-screened Coulomb potentials, Journal of Physics B: Atomic, Molecular and Optical Physics. 42(7) (2009).

S.M. Kuchin, N.V. Maksimenko, Theoretical estimations of the spin-averaged mass spectra of heavy quarkonia and Bc mesons, Universal Journal of Physics and Applications. 1(3) (2013) 295–298.

H. Egrifes, D. Demirhan, F. Buyukkilic, Exact solutions of the Schrödinger equation for the deformed hyperbolic potential well and the deformed four-parameter exponential type potential, Phys. Lett. A. 275 (2000) 229.

V.C. Aguilera-Navarro, E. Ley-Koo, S. Mateos-Cortés, Vibrational–rotational analysis of supersingular plus quadratic potential $A/r^4 + r^2$, Int. J. Theor. Phys. 40(10) (2001) 157–166.

V.C. Aguilera-Navarro, E. Ley-Koo, S. Mateos-Cortés, Vibrational–rotational structure of supersingular plus Coulomb potential $A/r^4 – Z/r$, Int. J. Theor. Phys. 36(01) (1997) 1809–1816.
[33] H. Snyder, The Quantization of space time, Phys. Rev. 71 (1946) 38–41.

[34] A. Maireche, Spectrum of hydrogen atom ground state counting quadratic term in Schrödinger equation, Afr. Rev Phys. 10 (2015) 177–183.

[35] A. Maireche, Deformed bound states for central fraction power potential: non relativistic Schrödinger equation, Afr. Rev Phys. 10 (2015) 97–103.

[36] A. Maireche, Spectrum of Schrödinger equation with H.L.C. potential in non-commutative two-dimensional real space, Afr. Rev Phys. 9 (2014) 479–485.

[37] A. Maireche, A study of Schrödinger equation with inverse sextic potential in 2-dimensional non-commutative space, Afr. Rev Phys. 9 (2014) 185–193.

[38] A. Maireche, A complete analytical solution of the Mie-type potentials in non-commutative 3-dimensional spaces and phases symmetries, Afr. Rev Phys. 11 (2016) 111–117.

[39] A. Maireche, Atomic spectrum for Schrödinger equation with rational spherical type potential in non-commutative space and phase, Afr. Rev Phys. 10 (2016) 373–381.

[40] A. Maireche, Nonrelativistic atomic spectrum for companied harmonic oscillator potential and its inverse in both NC-2D: RSP, International Letters of Chemistry, Physics and Astronomy. 56 (2015) 1–9.

[41] A. Maireche, New exact solution of the bound states for the potential family $V(r)=A/r^2-B/r+C/r^k$ (k=0,-1,-2) in both noncommutative three dimensional spaces and phases: non relativistic quantum mechanics, International Letters of Chemistry, Physics and Astronomy. 58 (2015) 164–176.

[42] A. Maireche, A new approach to the non relativistic Schrödinger equation for an energy-depended potential $V(r,E_k)=V_0(1+\eta E_k)r^2$ in both noncommutative three dimensional spaces and phases, International Letters of Chemistry, Physics and Astronomy. 60 (2015) 11–19.

[43] A. Maireche, A new study to the Schrödinger equation for modified potential $V(r)=ar^2+br^4+cr^6$ in nonrelativistic three dimensional real spaces and phases, International Letters of Chemistry, Physics and Astronomy. 61 (2015) 38–48.

[44] A. Maireche, A new nonrelativistic investigation for the lowest excitations states of interactions in one-electron atoms, Muonic, Hadronic and Rydberg atoms with modified inverse power potential, International Frontier Science Letters. 9 (2016) 33–46.

[45] A. Maireche, Deformed quantum energy spectra with mixed harmonic potential for nonrelativistic Schrödinger equation, J. Nano- Electron. Phys. 7(2) (2015) 02003-1 – 02003-6.

[46] A. Maireche, A Recent study of quantum atomic spectrum of the lowest excitations for Schrödinger equation with typical rational spherical potential at Planck's and nanoscales, J. Nano- Electron. Phys. 7(3) (2015) 03047-1 – 03047-7.

[47] A. Maireche, Quantum Hamiltonian and spectrum of Schrödinger equation with companied harmonic oscillator potential and its inverse in three dimensional noncommutative real space and phase, J. Nano- Electron. Phys. 7(4) (2015) 04021-104021-7.

[48] A. Maireche, New relativistic atomic mass spectra of quark (u, d and s) for extended modified Cornell potential in nano and Plank’s scales, J. Nano- Electron. Phys. 8(1) (2016) 01020-1 – 01020-7.

[49] A. Maireche, The nonrelativistic ground state energy spectra of potential counting Coulomb and quadratic terms in non-commutative two dimensional real spaces and phases, J. Nano- Electron. Phys. 8(1) (2016) 01021-1 – 01021-6.
[50] A. Maireche, New theoretical study of quantum atomic energy spectra for lowest excited states of central (PiHOIQ) potential in noncommutative spaces and phases symmetries at Plan’s and nanoscales, J. Nano- Electron. Phys. 8(2) (2016) 02027-1 – 02027-10.

[51] A. Maireche, A new nonrelativistic atomic energy spectrum of energy dependent potential for heavy quarkonium in noncommutative spaces and phases symmetries, J. Nano- Electron. Phys. 8(2) (2016) 02046-1 – 02046-6.

[52] A. Maireche, New exact bound states solutions for (C.F.P.S.) potential in the case of noncommutative three dimensional non relativistic quantum mechanics, Med. J. Model. Simul. 04 (2015) 060–072.

[53] A. Maireche, New bound state energies for spherical quantum dots in presence of a confining potential model at nano and Plank’s scales. NanoWorld J. 1(4) (2016) 122–129.

[54] A. Maireche, Quantum Schrödinger equation with Octic potential in non-commutative two-dimensional complex space, Life Sci. J. 11(6) (2014) 353–359.

[55] A. Maireche, New quantum atomic spectrum of Schrödinger equation with pseudo harmonic potential in both noncommutative three dimensional spaces and phases, Lat. Am. J. Phys. Educ. 9(1) (2015) 1301.

[56] A.E.F. Djemei, H. Smail, On quantum mechanics on noncommutative quantum phase space, Commun. Theor. Phys. 41 (2004) 837–844.

[57] S. Cai et al., Dirac oscillator in noncommutative phase space, Int. J. Theor. Phys. 49(8) (2010) 1699–1705.

[58] J. Lee, Star products and the Landau problem, Journal of the Korean Physical Society. 47(4) (2005) 571–576.

[59] A. Jahan, Noncommutative harmonic oscillator at finite temperature: a path integral approach, Brazilian Journal of Physics. 37(4) (2007) 144–146.

[60] A.F. Dossa, G.Y.H. Avossevou, Noncommutative phase space and the two dimensional quantum dipole in background electric and magnetic fields, Journal of Modern Physics. 4 (2013) 1400–1411.

[61] Z.-H. Yang et al., DKP oscillator with spin-0 in three dimensional noncommutative phase space, Int. J. Theor. Phys. 49 (2010) 644–657.

[62] Y. Yuan et al., Spin ½ relativistic particle in a magnetic field in NC phase space, Chinese Physics C. 34(5) (2010) 543.

[63] J. Mamat, S. Dulat, H. Mamatabdulla, Landau-like atomic problem on a non-commutative phase space, Int. J. Theor. Phys. 55 (2016) 2913–2918.

[64] B. Mirza et al., Relativistic oscillators in a noncommutative space in a magnetic field, Commun. Theor. Phys. 55 (2011) 405–409.

[65] Y. Xiao; Z. Long, S. Cai, Klein-Gordon oscillator in noncommutative phase space under a uniform magnetic field, Int. J. Theor. Phys. 50 (2011) 3105–3111.

[66] A.E.F. Djemaï, H. Smail, On quantum mechanics on noncommutative quantum phase space, Commun. Theor. Phys. 41 (2004) 837.

[67] A. Al-Jamel, Heavy quarkonia with Cornell potential on noncommutative space, Journal of Theoretical and Applied Physics. 5(1) (2011) 21–24.

[68] I.S. Gradshteyn, I.M. Ryzhik, Table of integrals, Series and Products, 7th. ed., Elsevier, 2007.