A CHERN-WEIL APPROACH TO DEFORMATIONS OF PAIRS
AND ITS APPLICATIONS

KWOKWAI CHAN AND YAT-HIN SUEN

ABSTRACT. We revisit the theory of deformations of pairs \((X, E)\), where \(X\) is a compact complex manifold and \(E\) is a holomorphic vector bundle over \(X\), from an analytic viewpoint à la Kodaira-Spencer. By introducing and exploiting an auxiliary differential operator, we derive the Maurer-Cartan equation and DGLA governing the deformation problem, and express them in terms of differential-geometric notions such as the connection and curvature of \(E\), obtaining a Chern-Weil–type refinement of the classical results that the tangent space and obstruction space of the moduli problem are respectively given by the first and second cohomology groups of the Atiyah extension of \(E\) over \(X\).

We also investigate circumstances where deformations of pairs are unobstructed using our analytic approach. Our results are then applied to study the jumping of the dimension of the cohomology group \(H^1(X, \text{End}(T_X))\), answering a question of Huybrechts \cite{Huy} in the affirmative in the case of algebraic K3 surfaces.

CONTENTS

1. Introduction 1
Acknowledgment 4
2. Connections, curvature and the Atiyah class 4
3. Maurer-Cartan equations 5
3.1. Deformations of complex structures and holomorphic vector bundles 5
3.2. Deformations of holomorphic pairs and the operator \(\bar{D}_t\) 6
3.3. DGLA and the Maurer-Cartan equation 11
4. First order deformations 14
5. Obstructions and Kuranishi family 16
6. A proof of completeness 19
7. Unobstructed deformations and applications 21
7.1. Unobstructed deformations of pairs 22
7.2. Applications to a question of Huybrechts \cite{Huy} 23
Appendix A. Comparison with the algebraic approach 29
References 32

1. INTRODUCTION

The theory of deformations of pairs \((X, E)\), where \(X\) is a compact complex manifold and \(E\) is a holomorphic vector bundle over \(X\), has been studied using both algebraic \cite{Kod} \cite{Spa} \cite{Mau} \cite{Ati} and analytic \cite{Kod} \cite{Spa} approaches and is well-understood

Date: August 27, 2014.
among experts. In this paper, we revisit this problem from a viewpoint à la Kodaira-Spencer [11, 12, 10], emphasizing the use of differential-geometric notions such as connections and curvatures of \( E \) and the induced differential operators. What we obtain is a Chern-Weil–type refinement of the classical results.

To illustrate our strategy, recall that a family of deformations \( \{X_t\}_{t \in \Delta} \) of a compact complex manifold \( X \) over a small ball \( \Delta \) can be represented by elements \( \{\varphi_t\}_{t \in \Delta} \subset \Omega^{0,1}(T_X) \), where \( T_X \) is the holomorphic tangent bundle of the complex manifold \( X \). While the Dolbeault operator \( \bar{\partial} : \Omega^0 \to \Omega^{0,1} \) on \( X_t \) is not easy to write down explicitly, one may consider the more convenient operator

\[
\bar{\partial} + \varphi_t \cdot \bar{\partial} : \Omega^0 \to \Omega^{0,1}
\]

which has the same kernel as \( \bar{\partial} \) and hence also defines the same space of holomorphic functions on \( X_t \).

Now given a family of deformations \( \{(X_t, E_t)\}_{t \in \Delta} \) of a holomorphic pair \((X, E)\), we obtain a family of elements \( \{\varphi_t\}_{t \in \Delta} \subset \Omega^{0,1}(T_X) \) since \( \{X_t\} \) is in particular a family of deformations of \( X \). By choosing a hermitian metric on \( E \) and considering the associated Chern connection, we define a differential operator

\[
\bar{D}_t : \Omega^{0,q}(E) \to \Omega^{0,q+1}(E)
\]

which satisfies the Leibniz rule and \( \bar{D}_t^2 = 0 \) (see Section 3 for details). While \( \bar{D}_t \) is not the Dolbeault operator \( \bar{\partial}_{E_t} \) on the holomorphic bundle \( E_t \), its kernel gives precisely the space of holomorphic sections of \( E_t \) over \( X_t \). Furthermore, it determines a family of elements

\[
A_t := \bar{D}_t - \bar{\partial}_{E_t} - \varphi_t \cdot \bar{\partial} \in \Omega^{0,1}(\text{End}(E)).
\]

Conversely, given any family of pairs of elements \( A_t \in \Omega^{0,1}(\text{End}(E)), \varphi_t \in \Omega^{0,1}(T_X) \), we can set

\[
\bar{D}_t := \bar{\partial}_{E_t} + \varphi_t \cdot \bar{\partial} + A_t.
\]

The upshot is the following Newlander-Nirenberg–type theorem for deformations of pairs:

**Theorem 1.1** (=Theorem 3.12). Given \( \varphi_t \in \Omega^{0,1}(T_X) \), \( A_t \in \Omega^{0,1}(\text{End}(E)) \), if the induced differential operator \( \bar{D}_t \) defined above satisfies \( \bar{D}_t^2 = 0 \), then it defines a holomorphic pair \((X_t, E_t)\).

Applying this, we derive the Maurer-Cartan equation:

**Theorem 1.2** (=Theorem 3.20). Given a holomorphic pair \((X, E)\) and a smooth family of elements \( \{(A_t, \varphi_t)\}_{t \in \Delta} \subset \Omega^{0,1}(\mathcal{E}) \). Then \( (A_t, \varphi_t) \) defines a holomorphic pair \((X_t, E_t)\) (i.e. an integrable complex structure \( J_t \) on \( X \) together with a holomorphic bundle structure on \( E \) over \((X, J_t)\)) if and only if the Maurer-Cartan equation

\[
\bar{\partial}_\mathcal{E}(A_t, \varphi_t) + \frac{1}{2}(A_t, \varphi_t), (A_t, \varphi_t)] = 0
\]

is satisfied. Here, \( \mathcal{E} \) is the Atiyah extension of \( E \) equipped with the Dolbeault operator \( \bar{\partial}_\mathcal{E} \) and the bracket \([\cdot, \cdot]\) is defined in terms of connections and curvatures on \( E \) in Proposition 3.13.

Moreover, the triple \((\Omega^{0,\ast}(\mathcal{E}), \bar{\partial}_\mathcal{E}, [\cdot, \cdot])\) forms a differential graded Lie algebra \((DGLA)\) which turns out to be naturally isomorphic to the one obtained via algebro means [19, 15] (see Appendix A).
From this we deduce that the space of first order deformations of $(X,E)$ is given by the first cohomology group $H^0_{\partial E} \cong H^1(X,E)$ (Section 4), and that the obstruction theory is captured by the Kuranishi map

$$Ob_{(X,E)} : H^1(X,E) \to H^2(X,E), \sum_{i=1}^m t^i(A_i, \varphi_i) \mapsto H[(A_t, \varphi_t), (A_t, \varphi_t)],$$

whence obstructions lie inside the second cohomology group $H^0_{\partial E} \cong H^2(X,E)$ (Section 5). We also prove the existence of a locally complete (or versal) family (see Theorem 6.2 cf. [20]) using an analytic method originally due to Kuranishi [19].

Remark 1.3. After we posted an earlier version of this paper on the arXiv, Carl Tipler informed us that the paper [6] of L. Huang had a large overlap with most of our results in Sections 3 to 6. Hence the reader may regard those sections as an exposition of known results. Nevertheless, we would like to point out that we have more detailed expositions of first order deformations (Section 4) and the proof of existence of Kuranishi families (Section 6) than Huang’s paper; also we have a comparison with the algebraic approach (Appendix A) showing in particular that the isomorphism class of the DGLA is independent of the choice of hermitian metric on $E$. For these reasons and also for making our paper more self-contained, we retain these sections in this new version.

Next we apply this analytic approach to look for situations where deformations of holomorphic pairs are unobstructed (Section 7). The main tool is the following proposition relating deformations of the pair $(X,E)$ to that of $X$ and $E$ which first appeared in [7, Appendix A] without proof:

**Proposition 1.4** (=Proposition 7.1). Denote the Kuranishi obstruction maps of the deformation theory of $X$, $E$ and $(X,E)$ by $Ob_X$, $Ob_E$ and $Ob_{(X,E)}$ respectively. Then we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^1(X, End(E)) & \xrightarrow{\iota^*} & H^1(X,E) & \xrightarrow{\pi^*} & H^1(X,T_X) & \delta & \cdots \\
& & \downarrow{Ob_E} & & \downarrow{Ob_{(X,E)}} & & \downarrow{Ob_X} & & \\
& & \cdots & \longrightarrow & H^2(X, End(E)) & \xrightarrow{\iota^*} & H^2(X,E) & \xrightarrow{\pi^*} & H^2(X,T_X) & \delta & \cdots
\end{array}
$$

Here, the connecting homomorphism $\delta$ is given by contracting with the Atiyah class:

$$\delta(\varphi) = \varphi \llbracket [F_\varphi] \rrbracket.$$

Applying this proposition, we obtain results which generalize some of those in the recent work of X. Pan [18] (where only the case when $E$ is a line bundle was considered). We also prove that when $X$ is a K3 surface and $E$ is a good bundle over $X$ with $c_1(E) \neq 0$ (Proposition 7.4), deformations of pairs $(X,E)$ are unobstructed.

Finally, we investigate a question raised by Huybrechts in [7], namely, whether the dimension of the cohomology group $H^1(X, End(T_X))$ is invariant under small deformations of projective Calabi-Yau manifolds. By applying the Chern-Weil approach to the deformation theory of pairs, we are able to solve this question in the affirmative in the case of algebraic K3 surfaces.

**Theorem 1.5** (=Theorem 7.13). Suppose that $X$ is an algebraic K3 surface. Then the dimension of the cohomology group $H^1(X_t, \text{End}(T_{X_t}))$ is invariant under any small algebraic deformation of $X$. 
Acknowledgment

The authors are grateful to Conan Leung, Si Li and Yi Zhang for various illuminating and useful discussions. Thanks are also due to Carl Tipler for pointing out the paper [6] and to Richard Thomas for interesting comments and suggestions on an earlier draft of this paper. The work of the first named author described in this paper was substantially supported by grants from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CUHK404412 & CUHK400213).

2. Connections, curvature and the Atiyah class

In this section, we review some basic notions in the theory of holomorphic vector bundles over complex manifolds and fix our notations. Excellent references for these materials include the textbooks [5, 8].

Let $E$ be a complex vector bundle over a smooth manifold $X$. For $k \geq 0$, we denote by $\Omega^k$ the sheaf of $k$-forms and by $\Omega^k(E)$ the sheaf of $E$-valued $k$-forms over $X$. Recall that a connection on $E$ is a $\mathbb{C}$-linear sheaf homomorphism $\nabla : \Omega^0(E) \to \Omega^1(E)$ satisfying the Leibniz rule:

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s$$

for $f \in \Omega^0$ and $s \in \Omega^0(E)$. We extend $\nabla$ naturally to $\nabla : \Omega^k(E) \to \Omega^{k+1}(E)$ by defining

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s$$

for $\alpha \in \Omega^k$ and any $s \in \Omega^0(E)$. The curvature $F_{\nabla} = \nabla \circ \nabla : \Omega^0(E) \to \Omega^2(E)$ of $\nabla$ can then be regarded as a global $\text{End}(E)$-valued 2-form. Also, $\nabla$ induces a natural connection on $\text{End}(E)$ by

$$(\nabla A)(s) = \nabla(As) - A(\nabla s),$$

where $A \in \Omega^0(\text{End}(E))$ and $s \in \Omega^0(E)$, and we have the Bianchi identity

$$\nabla F_{\nabla} = 0.$$

Now suppose that $X$ is a complex manifold. For $p, q \geq 0$, we denote by $\Omega^{p,q}$ the sheaf of $(p,q)$-forms and by $\Omega^{p,q}(E)$ the sheaf of $E$-valued $(p,q)$-forms over $X$. Recall that a holomorphic structure on a complex vector bundle $E$ over $X$ is uniquely determined by a $\mathbb{C}$-linear operator $\bar{\partial}_E : \Omega^p(E) \to \Omega^{p,1}(E)$ satisfying the Leibniz rule and the integrability condition $\bar{\partial}_E^2 = 0$. If we further equip $E$ with a hermitian metric $h$, then there exists a unique connection $\nabla$ on $E$ which is hermitian (i.e. $dh(s_1, s_2) = b(\nabla s_1, s_2) + b(s_1, \nabla s_2)$ for any $s_1, s_2 \in \Omega^0(E)$) and compatible with the holomorphic structure on $E$ (i.e. $\nabla^{0,1} = \bar{\partial}_E$, where $\nabla^{0,1} = \Pi^{0,1} \circ \nabla$ and $\Pi^{p,q} : \Omega^{p,q}(E) \to \Omega^{p,q}(E)$ is the natural projection map). $\nabla$ is usually called the Chern connection on $(E, h)$. The curvature $F_{\nabla}$ of the Chern connection on $(E, h)$ is real and of type $(1,1)$, so the Bianchi identity implies that $\bar{\partial}_{\text{End}(E)} F_{\nabla} = 0$, and thus this defines a class

$$[F_{\nabla}] \in H^{1,1}(X, \text{End}(E)),$$

called the Atiyah class of $E$ [1].

Lemma 2.1. The Atiyah class is well defined.
Proof. We need to check that the Atiyah class is independent of the choice of hermitian metrics on $E$. Let $h$ and $h'$ be two hermitian metrics on $E$, and let $\nabla$ and $\nabla'$ be the corresponding Chern connections. Then there exists $a \in \Omega^{1,0}(\text{End}(E))$ such that $\nabla' = \nabla + a$. Then

$$F_{\nabla'} = (\nabla + a) \circ (\nabla + a) = F_{\nabla} + \nabla \circ a + a \circ \nabla + a \wedge a = F_{\nabla} + \nabla a + a \wedge a.$$  

Because $F_{\nabla'} \in \Omega^{1,1}(\text{End}(E))$, we have $\nabla^{1,0}a + a \wedge a = 0$ so that

$$F_{\nabla'} = F_{\nabla} + \bar{\partial}_{\text{End}(E)}a,$$

and hence $[F_{\nabla'}] = [F_{\nabla}]$ as cohomology classes. □

3. Maurer-Cartan equations

In this section, we start our study of the deformation theory of pairs $(X, E)$. Our goal is to derive the Maurer-Cartan equation which governs this deformation problem.

3.1. Deformations of complex structures and holomorphic vector bundles. We begin by a brief review of the classical theory of deformations of complex structures and holomorphic vector bundles.

We first recall that a family of deformations $\pi : \mathcal{X} \to \Delta$ of a compact complex manifold $X$ can be represented by a family of sections $\varphi_t \in \Omega^{0,1}(T_X)$, where $T_X$ is the holomorphic tangent bundle of $X$ (or the $i$-eigenbundle of the almost complex structure defining $X$), satisfying the Maurer-Cartan equation

$$\bar{\partial} T_X \varphi_t + \frac{1}{2} [\varphi_t, \varphi_t] = 0.$$  

An essential ingredient in the proof is the Newlander-Nirenberg Theorem [17] which states that any integrable almost complex structure comes from a complex structure.

Proposition 3.1. Define an operator $\bar{\partial} + \varphi_t \circ \partial : \Omega^0 \to \Omega^{0,1}$ by $f \mapsto \bar{\partial} f + \varphi_t \circ (\partial f)$, where $\circ$ denotes the contraction or interior product. Then a local smooth function $f$ on $X_t$ is holomorphic if and only if $(\bar{\partial} + \varphi_t \circ \partial) f = 0$, i.e.

$$\bar{\partial} f = 0 \iff (\bar{\partial} + \varphi_t \circ \partial) f = 0,$$

where $\bar{\partial}$ is the $\bar{\partial}$-operator of the complex manifold $X_t$.

Proof. Let $z^1, \ldots, z^n$ be local holomorphic coordinates on $X$ (where $n$ is the complex dimension of $X$). Then $\varphi_t$ is of the form

$$\varphi_t = \sum_{i,j} \varphi^{i}_t(z,t) d\bar{z}^i \otimes \frac{\partial}{\partial z^j}.$$  

Hence $T^{0,1}_{X_t}$ is locally spanned by

$$\frac{\partial}{\partial \bar{z}^i} + \sum_j \varphi^{i}_t(z,t) \frac{\partial}{\partial z^j}.$$  

The result follows. □
Notice that \( \bar{\partial} + \varphi_\alpha \partial \) is not the same as \( \bar{\partial} \) but their kernels coincide. Hence \( \bar{\partial} + \varphi_\alpha \partial \) completely determines the local holomorphic functions with respective to the complex structure \( J_t \) on \( X_t \). The same idea will be applied to deformations of holomorphic pairs.

Next we recall the deformation theory of holomorphic vector bundles. Let \( E \to X \) be a complex vector bundle over a complex manifold \( X \). It is a standard fact in complex geometry that \( E \) admits a holomorphic structure if and only if there exists a linear operator \( \bar{\partial}_E : \Omega^{0,q}(E) \to \Omega^{0,q+1}(E) \) satisfying \( \bar{\partial}_E^2 = 0 \) and the Leibniz rule

\[
\bar{\partial}_E (\alpha \otimes s) = \bar{\partial} \alpha \otimes s + (-1)^{|\alpha|} \alpha \wedge \bar{\partial}_E (s),
\]

for any \( \alpha \in \Omega^{0,q}(E) \) and smooth section \( s \) of \( E \) (we call this the linearized version of the Newlander-Nirenberg Theorem; see e.g. \cite[Theorem 2.6.26]{S} or \cite[Theorem 3.2]{W}). Hence if we have a family of holomorphic vector bundles \( \mathcal{V} \to \Delta \) (or \( \{ E_t \}_{t \in \Delta} \)) on \( X \), then we have a family of Dolbeault operators \( \bar{\partial}_{E_t} \), whose squares are zero and all satisfy the Leibniz rule.

**Proposition 3.2.** Given a family of deformations \( \{ E_t \}_{t \in \Delta} \) of \( E \), the element \( A_t := \bar{\partial}_{E_t} - \bar{\partial}_E \in \Omega^{0,1}(\text{End}(E)) \) and satisfies the Maurer-Cartan equation

\[
\bar{\partial}_{\text{End}(E)} A_t + \frac{1}{2} [A_t, A_t] = 0
\]

for all \( t \in \Delta \). Conversely, if we are given a family \( \{ A_t \}_{t \in \Delta} \subset \Omega^{0,1}(\text{End}(E)) \) such that \( A_t \) satisfies the Maurer-Cartan equation for each \( t \), then \( \{ (E, \bar{\partial}_E + A_t) \}_{t \in \Delta} \) defines a family of deformations of \( E \).

**Proof.** Note that

\[
(\bar{\partial}_E + A_t)^2 = \bar{\partial}_E A_t + A_t \bar{\partial}_E + A_t \wedge A_t = \bar{\partial}_{\text{End}(E)} A_t + \frac{1}{2} [A_t, A_t].
\]

The result follows from the linearized version of the Newlander-Nirenberg Theorem. \( \square \)

### 3.2. Deformations of holomorphic pairs and the operator \( \bar{D}_t \).

**Definition 3.3.** A holomorphic pair \((X, E)\) consists of a compact complex manifold \( X \) together with a holomorphic vector bundle \( E \) over \( X \).

**Definition 3.4.** Let \((X, E)\) be a holomorphic pair. A family of deformations of \((X, E)\) over a small ball \( \Delta \) centered at the origin in \( \mathbb{C}^d \) consists of a proper and submersive holomorphic map \( \pi : \mathcal{X} \to \Delta \) (a family of deformations of \( X \) over \( \Delta \)) and a holomorphic vector bundle \( \mathcal{V} \to \mathcal{X} \) such that \( \pi^{-1}(0) = X \) and \( \mathcal{V}|_{\pi^{-1}(0)} = E \). For \( t \in \Delta \), we denote by \((X_t, E_t)\) the holomorphic pair parametrized by \( t \).

By the theorem of Ehresmann, if \( \Delta \) is chosen to be small enough, the family \( \mathcal{X} \) is smoothly trivial, i.e. one can find a diffeomorphism \( F : \mathcal{X} \to \Delta \times X \). Restricting to a fiber \( \mathcal{X}_t \subset \mathcal{X} \), one can push forward the complex structure on \( \mathcal{X}_t \) to define \( J_t \) on \( X_t := \{ t \} \times X \) via \( F \). One can also trivialize \( \mathcal{V} \) as \( \Delta \times E \) by a smooth bundle isomorphism \( P \) and the holomorphic structure on \( E_t := \{ t \} \times E \) is induced from that on \( V|_{\mathcal{X}_t} \) via the map \( P \). Hence we can assume that our family is a smoothly trivial family \( \Delta \times E \to \Delta \times X \) over a small ball \( \Delta \) in \( \mathbb{C}^d \) centered at the origin.

Now let \( \{ (X_t, E_t) \}_{t \in \Delta} \) be a family of deformations of \((X, E)\). By definition, \( \{ X_t \}_{t \in \Delta} \) is a family of deformations of \( X \), so it can be represented by an analytic
family of sections $\varphi_t \in \Omega^{0,1}(T_X)$ satisfying the Maurer-Cartan equation \[[1\]]$. Define the operator $\tilde{D}_t : \Omega^{0,q}(E) \to \Omega^{0,q+1}(E)$ by

$$\tilde{D}_t(s^k e_k) = (\bar{\partial} + \varphi_t \partial) s^k \otimes e_k,$$

where $\{e_k\}$ is a local holomorphic frame of $E_t$.

**Proposition 3.5.** The linear operator $\tilde{D}_t$ is well-defined and satisfies the Leibniz rule

$$\tilde{D}_t(\alpha \otimes s) = (\bar{\partial} + \varphi_t \partial)\alpha \otimes s + (-1)^{\ell_\alpha} \alpha \wedge \tilde{D}_t(s)$$

for any $s \in \Omega^0(E)$ and $\alpha \in \Omega^0(X)$. Also, $\tilde{D}_t(s) = 0$ if and only if $\bar{\partial}_{E_t}(s) = 0$.

**Proof.** To prove well-definedness, we need to show that $\tilde{D}_t$ is independent of the choice of a local holomorphic frame $\{e_k\}$ of $E_t$. So suppose $\{f_j\}$ is another local holomorphic frame of $E_t$. Let $\tau^k_j$ be local holomorphic functions on $X_t$ such that $f_j = \tau^k_j e_k$. Then for a local section $s = s^k e_k = \tilde{s}^j f_j$, we have $\tilde{s}^j = s^k \tau^k_j$ and thus

$$\tilde{D}_t(\tilde{s}^j f_j) = (\bar{\partial} + \varphi_t \partial) \tilde{s}^j \otimes f_j = (\bar{\partial} + \varphi_t \partial)(s^k \tau^k_j) \otimes f_j$$

$$= (\bar{\partial} + \varphi_t \partial)s^k \otimes \tau^k_j f_j = \tilde{D}_t(s^k e_k).$$

Hence $\tilde{D}_t$ is well-defined. Notice that if we choose another trivialization $(F', P')$ of $\mathcal{V} \to \mathcal{X}$, with holomorphic structure on $\Delta \times E \to \Delta \times X$ induced from $\mathcal{V} \to \mathcal{X}$ via $(F', P')$,

$$\Delta \times E \xrightarrow{P^{-1}} \mathcal{V} \xrightarrow{P'} \Delta \times E$$

$$\Delta \times E \xrightarrow{F^{-1}} \mathcal{X} \xrightarrow{F'} \Delta \times X$$

then $P' \circ P^{-1}$ is an isomorphism between holomorphic bundles and so $\tilde{D}_t$ is also independent of the choice of trivializations.

The Leibniz rule for $\tilde{D}_t$ is clear since $\bar{\partial}$ and $\partial$ both satisfy the usual Leibniz rule. Finally, for a smooth section $s$ of $E$, if we write $s = s^k e_k$ locally with $\{e_k\}$ a local holomorphic frame of $E_t$, then we have

$$\tilde{D}_t(s) = 0 \iff (\bar{\partial} + \varphi_t \partial) s^k = 0 \iff \bar{\partial}_t s^k = 0 \iff \bar{\partial}_{E_t}(s) = 0.$$

We claim that $\tilde{D}_t^2 = 0$. By our definition of $\tilde{D}_t$, for any smooth function $f : X \to \mathbb{C}$ and local nowhere vanishing holomorphic section $e$ of $E_t$, we have

$$\tilde{D}_t^2(f e) = (\bar{\partial} + \varphi_t \partial)^2 f \otimes e.$$

To compute the right hand side, we need the following

**Lemma 3.6.** For any $\varphi \in \Omega^{0,p}(T_X)$ and $\alpha \in \Omega^1(E)$, we have the Leibniz rule

$$\bar{\partial}_E(\varphi \cdot \alpha) = \bar{\partial}_{T_X} \varphi \cdot \alpha - (-1)^p \varphi \cdot \bar{\partial}_{E} \alpha.$$

**Proof.** Writing $\varphi = \varphi^j dz^j \otimes \frac{\partial}{\partial z^j}$, we have

$$\varphi \cdot \alpha = \varphi^j dz^j \otimes \alpha \left( \frac{\partial}{\partial z^j} \right).$$
Let \( \alpha_t := \alpha(\frac{\partial}{\partial t}) ) \in \Omega^0(E) \). Then
\[
\bar{\partial} E(\varphi, \alpha) = (\bar{\partial} \varphi_j^i \wedge d\bar{z}^j) \otimes \alpha_t + (-1)^p \varphi_j^i d\bar{z}^j \wedge \bar{\partial} E \alpha_t
= \bar{\partial} T \varphi, \alpha_t + (-1)^p \varphi_j^i d\bar{z}^j \wedge \bar{\partial} E \alpha_t.
\]
To compute the last term, first note that the contraction of \( \frac{\partial}{\partial z^i} \) with \( \alpha \) is taken in the \((1,0)\)-part, we can therefore assume \( \alpha = \alpha^k_t d\bar{z}^i \otimes e_k \), where \( \{e_k\} \) is a local holomorphic frame of \( E \). So we have
\[
\varphi_j^i d\bar{z}^j \wedge \bar{\partial} E \alpha_t = \varphi_j^i d\bar{z}^j \wedge \bar{\partial} E (\alpha^k_t e_k) = \varphi_j^i d\bar{z}^j \wedge \bar{\partial} \alpha_t^k \otimes e_k
= -\varphi_j^i d\bar{z}^j \wedge \left( \frac{\partial}{\partial z^i} \bar{\partial} \alpha_t^k \wedge d\bar{z}^j \right) \otimes e_k
= -\varphi_j^i d\bar{z}^j \wedge \frac{\partial}{\partial z^i} \bar{\partial} E (\alpha^k_t d\bar{z}^i \otimes e_k) = -\varphi_j^i \bar{\partial} E \alpha_t,
\]
and hence the desired formula. \( \square \)

We can now compute \( \bar{D}_t^2 \).

**Lemma 3.7.** For any smooth function, \( f : X \rightarrow \mathbb{C} \), we have the equality
\[
(\bar{\partial} + \varphi_t \bar{\partial})^2 f = \left( \bar{\partial} T \varphi_t + \frac{1}{2} [\varphi_t, \varphi_t] \right) \bar{\partial} f.
\]

**Proof.** First, we have
\[
(\bar{\partial} + \varphi_t \bar{\partial})^2 f = \bar{\partial}(\varphi_t \bar{\partial} f) + \varphi_t \bar{\partial} \bar{\partial} f + \varphi_t \bar{\partial}(\varphi_t \bar{\partial} f).
\]
By Lemma 3.6 the first term is given by
\[
\bar{\partial}(\varphi_t \bar{\partial} f) = \bar{\partial} T \varphi_t \bar{\partial} f + \varphi_t \bar{\partial} \bar{\partial} f.
\]
Since \( \bar{\partial} \bar{\partial} = -\bar{\partial}^2 \), we have
\[
\bar{\partial}(\varphi_t \bar{\partial} f) + \varphi_t \bar{\partial} \bar{\partial} f = \bar{\partial} T \varphi_t \bar{\partial} f.
\]
For the last term, by writing \( \varphi_t = \varphi^l_m d\bar{z}^m \otimes \frac{\partial}{\partial z^l} \) in local coordinates, we have
\[
\varphi_t \bar{\partial} f = \varphi^l_m \frac{\partial f}{\partial z^l} d\bar{z}^m,
\]
and so
\[
\varphi_t \bar{\partial}(\varphi_t \bar{\partial} f) = \varphi^l_j \varphi^m_l \frac{\partial f}{\partial z^l} \frac{\partial}{\partial z^m} d\bar{z}^j \wedge d\bar{z}^m + \varphi^l_j \varphi^l_m \frac{\partial^2 f}{\partial z^l \partial z^m} d\bar{z}^j \wedge d\bar{z}^m.
\]
But
\[
\varphi^l_j \varphi^m_l \frac{\partial^2 f}{\partial z^l \partial z^m} d\bar{z}^j \wedge d\bar{z}^m = -\varphi^m_l \varphi^l_j \frac{\partial^2 f}{\partial z^m \partial z^l} d\bar{z}^j \wedge d\bar{z}^m,
\]
so we obtain
\[
\varphi_t \bar{\partial}(\varphi_t \bar{\partial} f) = \varphi^l_j \varphi^m_l \frac{\partial f}{\partial z^l} \frac{\partial}{\partial z^m} d\bar{z}^j \wedge d\bar{z}^m = \frac{1}{2} [\varphi_t, \varphi_t] \bar{\partial} f.
\]
The result follows. \( \square \)

As \( \{X_t\}_{t \in \Delta} \) is an honest family of deformations of \( X \), the Maurer-Cartan equation \( \square \) for \( \varphi_t \) holds. Hence we have

**Proposition 3.8.** \( \bar{D}_t^2 = 0 \).

From the viewpoint of Proposition 3.1 it is natural to compare the operator \( \bar{D}_t \) with \( \bar{\partial} E + \varphi_t \bar{\partial} \nabla \).
Proposition 3.9. $A_t := \bar{D}_t - \bar{\partial}_E - \varphi_t \cdot \nabla \in \Omega^{0,1}(\text{End}(E))$.

Proof. Let $f$ be a smooth function and $s$ a smooth section of $E$. Using the Leibniz rules, and the fact that the contraction is only taken in the $(1,0)$-part, we have

$$A_t(f s) = (\bar{\partial} + \varphi_t \cdot \partial) f \otimes s + f \bar{D}_t(s) - \bar{\partial} f \otimes s - f \bar{\partial}_E(s) - \varphi_t \cdot \nabla (f s)$$

$$= (\varphi_t \cdot \partial) f \otimes s + f \bar{D}_t(s) - f \bar{\partial}_E(s) - \varphi_t \cdot \partial f \otimes s - f \varphi_t \cdot \nabla (s) = f A_t(s).$$

In the other direction, suppose we are now given elements $A_t \in \Omega^{0,1}(\text{End}(E))$ and $\varphi_t \in \Omega^{0,1}(T_X)$, parameterized by $t \in \Delta$, we can then define an operator $\bar{D}_t : \Omega^0(E) \to \Omega^{0,1}(E)$ by

$$\bar{D}_t := \bar{\partial}_E + \varphi_t \cdot \nabla + A_t.$$ 

We extend $\bar{D}_t$ to $\Omega^{0,\alpha}(E)$ in the obvious way, so that the Leibniz rule

$$\bar{D}_t(\alpha \otimes s) = (\bar{\partial} + \varphi_t \cdot \partial) \alpha \otimes s + (-1)^{|\alpha|} \alpha \wedge \bar{D}_t s$$

holds. We want to show that if $\bar{D}_t^2 = 0$, then $(A_t, \varphi_t)$ defines a holomorphic pair $(X_t, E_t)$. First of all, we have

Proposition 3.10. If $\bar{D}_t^2 = 0$, then $X_t$ is a complex manifold.

Proof. Using the Leibniz rule, we have for any smooth function and sections of $E$ that

$$0 = \bar{D}_t^2(f s) = (\bar{\partial} + \varphi_t \cdot \partial)^2 f \otimes s.$$

Hence $(\bar{\partial} + \varphi_t \cdot \partial)^2 = 0$, which is equivalent to the Maurer-Cartan equation by Lemma 3.7. Therefore, the almost complex structure defined by $\varphi_t$ is integrable.

We now need to show that $E$ also admits holomorphic structure over $X_t$. To prove this, we first make the following assertion:

Any smooth sections of $E$ can locally be written as $s^k e_k$, where $\{e_k\} \subset \ker(D_t)$. Then define $\bar{\partial}_E$ by $\bar{\partial}_E(s^k e_k) := \bar{\partial}_t s^k \otimes e_k$.

To check that it is well-defined, suppose we have another local basis $\{f_j\} \subset \ker(\bar{D}_t)$, then there exist $h^k_j$ such that $f_j = h^k_j e_k$. Applying $\bar{D}_t$, we have

$$(\bar{\partial} + \varphi_t \cdot \partial)h^k_j \otimes e_k = 0.$$

Since $\{e_k\}$ is assumed to be a local basis, we have $(\bar{\partial} + \varphi_t \cdot \partial)h^k_j = 0$, which is equivalent to $\bar{\partial}_t h^k_j = 0$. Hence

$$\bar{\partial}_E(\alpha \otimes s) = \bar{\partial}_t \alpha \otimes s + (-1)^{|\alpha|} \alpha \wedge \bar{\partial}_E s$$

and $\bar{\partial}_E^2 = 0$ since $\varphi_t$ defines an integrable complex structure on $X$. Hence by the linearized version of the Newlander-Nirenberg Theorem, $E_t = (E, \bar{\partial}_E)$ is a holomorphic vector bundle over $X_t$.

It remains to prove that our assertion is correct:

Lemma 3.11. $\ker(\bar{D}_t)$ generates $\Omega^0(E)$ locally.
Proof. Let us first fix a smooth local frame \( \{ \sigma_k \} \) of \( E \) over a coordinate neighborhood \( U \subset X \). What we need are coordinate changes \( (f_j^i(z,t)) \in \Gamma_{sm}(U,GL_r(\mathbb{C})) \) such that \( f_j^i \in \ker(\hat{D}_t) \). Writing \( \hat{D}_t \sigma_i = \tau^k_i \otimes \sigma_k \) with \( \tau^k_i \in \Omega^{0,1}(X) \), the existence of \( (f_j^i(z,t)) \) is equivalent to

\[
0 = \hat{D}_t (f_j^i \sigma_i) = (\partial f_j^i + \phi_t \partial f_j^i) \otimes \sigma_i + f_j^i \tau^k_i \otimes \sigma_k.
\]

This in turn is equivalent to the following system of PDEs

\[
(\partial + \phi_t \partial) f_j^i = 0
\]

subject to the condition:

\[
\hat{D}_t^2 = 0 \iff (\partial + \phi_t \partial) \tau_j^k = \tau^k_i \wedge \tau^k_j.
\]

We will show that this system is solvable, following the line of proof in [16, Theorem 3.2] (linearized version of the Newlander-Nirenberg Theorem).

First of all we set

\[
N := U \times \mathbb{C}^r, \ T := \text{span}\{dz^\alpha - \varphi_t \partial dz^\alpha, dw_i - \tau^k_i w_k\}.
\]

We want to show that \( d(T) \subset \Omega^0(\wedge^1 N) \wedge T \). First we have

\[
d(\partial - \varphi_t \partial) dz^\alpha = -\partial \varphi^\alpha_\beta dz^\beta \wedge dz^\gamma - \partial \varphi^\beta_\gamma dz^\gamma \wedge dz^\alpha = \partial \varphi^\alpha_\beta dz^\beta \wedge dz^\gamma - \partial \varphi^\beta_\gamma dz^\gamma \wedge dz^\alpha.
\]

Then applying the Maurer-Cartan equation \([11] \) gives

\[
d(\partial - \varphi_t \partial) dz^\alpha = \partial \varphi^\alpha_\beta dz^\beta \wedge dz^\gamma + \varphi^\mu_\alpha dz^\alpha \wedge \partial \varphi^\beta_\gamma dz^\beta = \partial \varphi^\alpha_\beta dz^\beta \wedge dz^\gamma.
\]

Secondly,

\[
d(\partial - \varphi_t \partial) dw_i - \tau^i_k w_k = -\partial \tau^i_k dw_i - \partial \tau^i_k w_i + \varphi^i_k \wedge dw_i = -\partial \varphi_t \partial \tau^i_k dw_i - \varphi^i_k \wedge dw_i = -\varphi^i_k \wedge dw_i.
\]

Hence by the Newlander-Nirenberg Theorem, we obtain holomorphic coordinates \((\zeta^\alpha, u^i)\) on \( N \) and smooth functions \( F^\beta_\alpha(\zeta^i) = F^\beta_\alpha(z,t), F^i_\alpha(\zeta^i, u^t) = F^i_\alpha(z,u,t) \), and \( F^i_\alpha(\zeta^i, u^t) = F^i_{\alpha}(z,w,t) \) such that

\[
\begin{align*}
\{ \ & d\zeta^\alpha = F^\beta_\alpha(z,t)(dz^\beta - \varphi_t \partial dz^\beta), \\
\ & du^i = F^i_\alpha(z,w,t)(dw_i - \tau^k_i w_k) + F^i_{\alpha}(z,w,t)(dz^\alpha - \varphi_t \partial dz^\alpha). 
\end{align*}
\]
Since \( \{dz^\alpha - \varphi_\beta dz^\alpha, dw_i - \tau^k_i \} \) and \( \{d\zeta^i, du^i\} \) are basis of \( T \), we see that the \((n + r) \times (n + r)\)-matrix
\[
\begin{pmatrix}
(F_\beta)_{ij} & (F_i^a)_{ij} \\
O_{r \times n} & (F_i^j)
\end{pmatrix}
\]
is invertible for all \((z, w, t)\). It follows that \((F_i^j)\) is also invertible for all \((z, w, t)\).

Applying the exterior differential on \( N \) and evaluating at \( w = 0 \), we have
\[
0 = dF_i^j \wedge dw_i + \varphi_t \tau^k_i \wedge dw_i + dF_{i\alpha} \wedge dz^\alpha - dF_{i\alpha} \wedge \varphi_t dz^\alpha - F_{i\alpha} d\varphi_t dz^\alpha.
\]
Comparing the \( dz \wedge dw \)-component on both sides gives \( \partial_z F_i^j \wedge dw_i + \partial_w F_{i\alpha} \wedge dz^\alpha = 0 \), which implies, by contracting with \( \varphi_t \), that
\[
\varphi_t dF_i^j \wedge dw_i + \partial_w F_{i\alpha} \wedge \varphi_t dz^\alpha = 0.
\]
Then by comparing the \( dz \wedge dw \)-component, we have
\[
\bar{\partial} F_i^j \wedge \partial_z w_i + \varphi_t \tau^k_i \wedge dw_i = 0.
\]
Together with the formula we just obtained, we arrive at
\[
(\bar{\partial} + \varphi_t \partial) F_i^j (z, 0) + F_i^k (z, 0) \tau^k_i = 0.
\]
The result now follows by setting \( f_i^j (z, t) := F_i^j (z, 0, t) \).

In summary, we have proved the following

**Theorem 3.12.** Given \( A_t \in \Omega^{0,1}(\text{End}(E)) \) and \( \varphi_t \in \Omega^{0,1}(T_X) \). If the induced differential operator \( \bar{\partial}_t : \Omega^{0,q}(E) \to \Omega^{0,q+1}(E) \) satisfies \( \bar{\partial}_t^2 = 0 \) and the Leibniz rule
\[
\bar{\partial}_t(\alpha \otimes s) = (\bar{\partial} + \varphi_t \partial_\alpha \otimes s + (-1)^{|\alpha|} \alpha \wedge \bar{\partial}_t(s),
\]
then \( E \) admits a holomorphic structure over the complex manifold \( X_t \), which we will denote by \( E_t \to X_t \) or just \( E_t \).

### 3.3. DGLA and the Maurer-Cartan equation

To simplify notations, from this point on, we will denote the vector bundles \( \text{End}(E) \) and \( \text{Hom}(T_X, \text{End}(E)) \) by \( Q \) and \( H \) respectively unless specified otherwise.

We are now ready to derive the Maurer-Cartan equation governing the deformations of pairs. Given \( A_t \in \Omega^{0,1}(\text{End}(E)) \), \( \varphi_t \in \Omega^{0,1}(T_X) \) such that the induced differential operator \( \bar{\partial}_t \) satisfies \( \bar{\partial}_t^2 = 0 \), we have
\[
(\bar{\partial}_t + \varphi_t \partial + A_t)^2 = \bar{\partial}_t^2 = 0.
\]

Let us expand the left hand side:
\[
(\bar{\partial}_t + \varphi_t \partial + A_t)^2 = \bar{\partial}_t(\varphi_t \partial + A_t) + \varphi_t \partial (\varphi_t \partial + A_t) + \varphi_t \partial A_t + A_t \bar{\partial}_t + \varphi_t \partial A_t + A_t \partial_t + A_t \partial_\alpha + \varphi_t \partial A_t + A_t \wedge A_t.
\]
Applying Lemma 3.6 to the term \( \bar{\partial}_t(\varphi_t \partial) \), we get
\[
(\bar{\partial}_t + \varphi_t \partial + A_t)^2 = \bar{\partial}_t(\varphi_t \partial + A_t) + \varphi_t \partial (\varphi_t \partial + \nabla \bar{\partial}_t) + \varphi_t \partial A_t + \varphi_t \partial A_t + A_t \partial_\alpha + A_t \wedge A_t.
\]
Since $\nabla$ is the Chern connection, we have $F_\nabla = \partial E + \nabla \partial E$, and so

$$(\partial E + \phi E + \nabla A)^2 = (\partial E + \phi E + \nabla (\phi E)) + \partial Q A_t + \phi E F_\nabla + \phi E A Q A_t + A_t \wedge A_t.$$ 

Note that the curvature $F_\nabla$ is given by

$$F_\nabla(\phi, \psi) = \phi E \nabla (\psi E) - (-1)^{|\phi||\psi|}\psi E \nabla (\phi E) + [\phi, \psi] E \nabla$$

for $\phi, \psi \in \Omega^{0,q}(T\c X)$. Hence

$$2\phi E \nabla (\phi E) = F_\nabla(\phi, \phi) + [\phi, \phi] E \nabla.$$ 

However, $\phi E \in \Omega^{0,1}(T\c X)$ and $F_\nabla$ is of type-(1,1), we must have $F_\nabla(\phi, \phi) = 0$. Therefore,

$$\phi E \nabla (\phi E) = \frac{1}{2} [\phi, \phi] E \nabla.$$ 

As a whole we obtain

$$(\partial E + \phi E + \nabla A)^2 = (\partial E + \phi E + \nabla A) \nabla + \partial Q A_t + \phi E F_\nabla + \phi E A Q A_t + \frac{1}{2} [A_t, A_t].$$

But $X_t$ is integrable, $\phi_t$ satisfies the Maurer-Cartan equation (1), and so

$$(\partial E + \phi E + A)^2 = \partial Q A_t + \phi E F_\nabla + A Q A_t + \frac{1}{2} [A_t, A_t].$$

Hence we conclude that $\tilde{D}_t^2 = 0$ is equivalent to the following two equations

$$(\partial E + \phi E + \nabla A)^2 = \partial Q A_t + \phi E F_\nabla + \phi E A Q A_t + \frac{1}{2} [A_t, A_t] = 0,$$

$$(\partial E + \phi E + \nabla A)^2 = \partial Q A_t + \phi E F_\nabla + \phi E A Q A_t + \frac{1}{2} [A_t, A_t] = 0.$$ 

Now we put $\mathcal{E} := Q \oplus T\c X$, and define a bracket $[-,-]: \Omega^{0,p}(\mathcal{E}) \times \Omega^{0,q}(\mathcal{E}) \rightarrow \Omega^{0,p+q}(\mathcal{E})$ by

$$[(A, \phi), (B, \psi)] := (\phi E \nabla B - (-1)^{|\phi|q}\psi E \nabla A) \wedge [A, B], [\phi, \psi]).$$

The following proposition can be proven by straightforward, but tedious, computations which we omit:

**Proposition 3.13.** The bracket $[-,-]: \Omega^{0,p}(\mathcal{E}) \times \Omega^{0,q}(\mathcal{E}) \rightarrow \Omega^{0,p+q}(\mathcal{E})$ defined by

$$[(A, \phi), (B, \psi)] := (\phi E \nabla B - (-1)^{|\phi|q}\psi E \nabla A) \wedge [A, B], [\phi, \psi])$$

satisfies

1. $[(A, \phi), (B, \psi)] = (-1)^{|\phi|}(B, \phi), (A, \psi)]$,
2. $[(A, \phi), [(B, \psi), (C, \tau))] = [[(A, \phi), (B, \psi)], (C, \tau)] + (-1)^{|\phi|}(B, \psi), [(A, \phi), (C, \tau)])$,

for $(A, \phi) \in \Omega^{0,p}(\mathcal{E}), (B, \psi) \in \Omega^{0,q}(\mathcal{E})$ and $(C, \tau) \in \Omega^{0,r}(\mathcal{E})$.

On the other hand, we let $B \in \Omega^{0,1}(H)$ be the form defined by

$$B \wedge \phi := (-1)^{|\phi|}\phi E F_\nabla.$$ 

**Proposition 3.14.** $B \in \Omega^{0,1}(H)$ is $\partial_H$-closed.

**Proof.** This follows from the Bianchi identity $\partial Q F_\nabla = 0$: For any $v \in T\c X$,

$$(\partial H B)(v) = \partial Q (Bv) + B(\partial E v) = -\partial Q (v E F_\nabla) + \partial E v E F_\nabla = v E \partial Q F_\nabla = 0.$$ 

$\square$
Then we define a differential operator \( \tilde{\partial}_{E_B} : \Omega^0(\mathcal{E}) \to \Omega^{0,1}(\mathcal{E}) \) on \( \mathcal{E} \) by
\[
\tilde{\partial}_{E_B} := \begin{pmatrix} \tilde{\partial}Q & B \\ O & \tilde{\partial}_{TX} \end{pmatrix}.
\]

**Proposition 3.15.** \((\mathcal{E}, \tilde{\partial}_{E_B})\) defines a holomorphic vector bundle over \( X \) whose holomorphic structure depends only on the class \([B]\).

**Proof.** Clearly \( \tilde{\partial}_{E_B} \) satisfies the Leibniz rule, so it suffices to prove that \( \tilde{\partial}_{E_B}^2 = 0 \). But \( \tilde{\partial}_{E_B}^2 = 0 \) if and only if \( \tilde{\partial}_H B = 0 \) which holds by Proposition 3.14. This proves the first part of the proposition.

To see the second part, suppose that \( B' - B = \tilde{\partial}_H f \) for some \( f \in \text{Hom}(T_X, Q) \). Define the smooth bundle isomorphism \( F : \mathcal{E} \to \mathcal{E} \) by
\[
F : (A, v) \mapsto (A - f v, v),
\]
and extend to \( \mathcal{E} \)-valued \( p \)-forms. We compute
\[
\tilde{\partial}_{E_B} F(A, v) = (\tilde{\partial}Q(A - f v) + B'v, \tilde{\partial}_{TX} v) = (\tilde{\partial}Q A - \tilde{\partial}Q f v + Bv + \tilde{\partial}_H f v, \tilde{\partial}_{TX} v) = (\tilde{\partial}Q A + Bv - f \tilde{\partial}_{TX} v, \tilde{\partial}_{TX} v) = F \tilde{\partial}_{E_B}(A, v).
\]
Hence \( F \) in fact defines a holomorphic bundle isomorphism between \((\mathcal{E}, \tilde{\partial}_{E_B})\) and \((\mathcal{E}, \tilde{\partial}_{E_B'})\). Since the curvature \( F_{\mathcal{E}} \) differs by an exact \( \text{End}(E) \)-valued 1-form if another metric was used, this shows that the holomorphic structure of \( \mathcal{E} \) only depends on the class \([B]\) but not the metric. \( \square \)

**Remark 3.16.** Under the Dolbeault isomorphism
\[
H^1(X, \text{Hom}(T_X \otimes Q)) \cong H^{1,1}(X, Q),
\]
the class \([B]\) corresponds to the Atiyah class \([F_{\mathcal{E}}]\). Hence the holomorphic structure of \( \mathcal{E} \) depends only on the Atiyah class of \( E \).

By abuse of notations, we will now write \( \tilde{\partial}_{E_B} \) as \( \tilde{\partial}_{\mathcal{E}} \), keeping in mind that a hermitian metric on \( E \) has been chosen.

**Definition 3.17.** The holomorphic vector bundle \((\mathcal{E}, \tilde{\partial}_{\mathcal{E}})\), which is an extension of \( Q \) by \( T_X \), is called the Atiyah extension of \( E \).

Again by direct computations, one can prove that the bracket \([-,-]\) and the Dolbeault operator \( \tilde{\partial}_{\mathcal{E}} \) are compatible with each other:

**Proposition 3.18.** We have
\[
\tilde{\partial}_{\mathcal{E}} [(A, \varphi), (B, \psi)] = [\tilde{\partial}_{\mathcal{E}} (A, \varphi), (B, \psi)] + (-1)^p [(A, \varphi), \tilde{\partial}_{\mathcal{E}} (B, \psi)]
\]
for \((A, \varphi) \in \Omega^{0,p}(\mathcal{E})\) and \((B, \psi) \in \Omega^{0,*}(\mathcal{E})\).

Propositions 3.13 and 3.18 together say that \((\Omega^{0,*}(\mathcal{E}), \tilde{\partial}_{\mathcal{E}}, [-,-])\) defines a differential graded Lie algebra (DGLA).

**Remark 3.19.** In the appendix, we will prove that there exists a natural isomorphism between the complex \((\Omega^{0,*}(\mathcal{E}), \tilde{\partial}_{\mathcal{E}})\) and the one obtained using algebraic methods [13][15] intertwining our bracket \([-,-]\) with the algebraic one. This gives alternative proofs of Propositions 3.13 and 3.18 and shows that our DGLA is naturally isomorphic to the one derived using algebraic methods. In particular, the isomorphism class of our DGLA is independent of the choice of the hermitian metric we used to define the Chern connection \( \nabla \).
Using the bracket \([-,-]\) and the Dolbeault operator \(\bar{\partial}_E\), we can now rewrite the two equations (2) as the following Maurer-Cartan equation:

\[
\bar{\partial}_E(A_t, \varphi_t) + \frac{1}{2}[(A_t, \varphi_t), (A_t, \varphi_t)] = 0,
\]

which governs the deformation of pairs. We summarize our results by the following

**Theorem 3.20.** Given a holomorphic pair \((X, E)\) and a smooth family of elements \(\{(A_t, \varphi_t)\}_{t \in \Delta} \subset \Omega^{0,1}(E)\). Then \((A_t, \varphi_t)\) defines a holomorphic pair \((X_t, E_t)\) (namely, an integrable complex structure \(J_t\) on \(X_t\) together with a holomorphic bundle structure on \(E_t\) over \((X_t, J_t)\)) if and only if the Maurer-Cartan equation

\[
\bar{\partial}_E(A_t, \varphi_t) + \frac{1}{2}[(A_t, \varphi_t), (A_t, \varphi_t)] = 0
\]

is satisfied.

### 4. First order deformations

The Maurer-Cartan equation (3) implies that a first order deformation \((A_1, \varphi_1)\) (the linear term of the Taylor series expansion of a family \((A_t, \varphi_t)\)) is \(\bar{\partial}_E\)-closed:

\[
\bar{\partial}_E(A_1, \varphi_1) = 0,
\]

and hence defines a cohomology class in the Dolbeault cohomology group \(H^{0,1}_{\bar{\partial}_E} \cong H^1(X, \mathcal{E})\). To determine the space of first order deformations of a holomorphic pair \((X, E)\), it remains to identify isomorphic deformations.

**Definition 4.1.** Two deformations \(\mathcal{V} \rightarrow \mathcal{X}\), \(\mathcal{V}' \rightarrow \mathcal{X}'\) of \((X, E)\) are said to be isomorphic if there exists a biholomorphism \(F: \mathcal{X} \rightarrow \mathcal{X}'\) and a holomorphic bundle isomorphism \(\Phi: \mathcal{V} \rightarrow \mathcal{V}'\) covering \(F\) such that \(F|_{\mathcal{X}} = id_{\mathcal{X}}\) and \(\Phi|_{\mathcal{E}} = id_{\mathcal{E}}\).

**Proposition 4.2.** Suppose \(\mathcal{V} \rightarrow \mathcal{X}\) and \(\mathcal{V}' \rightarrow \mathcal{X}'\) are isomorphic 1-real parameter family of deformations of \((X, E)\). If we denote by \((A_t, \varphi_t)\) and \((A'_t, \varphi'_t)\) the elements that represent the families \(\mathcal{V} \rightarrow \mathcal{X}\) and \(\mathcal{V}' \rightarrow \mathcal{X}'\) respectively, then there exists \((\Theta_1, v) \in \Omega^0(\mathcal{E})\) such that

\[
(A'_t, \varphi'_t) = (A_t, \varphi_t) + t\bar{\partial}_E(-\Theta_1, v) + R((A_t, \varphi_t), t(\Theta_1, v)),
\]

where the error \(R\) depends smoothly on \(t, A(t), \varphi(t), \Theta_1, v\) and first partial derivatives of \(\Theta, v\). Moreover, \(R\) is of order \(s^2\) in the sense that

\[
R(s(A, \varphi), s(\Theta, v)) = s^2R_1((A, \varphi), (\Theta, v), s),
\]

for some map \(R_1\) which depends smoothly (with respect to the Sobolov norm; see Section \(\mathbb{D}\) for its precise definition) in \(s, (A, \varphi) \in \Omega^{0,1}(\mathcal{E})\) and \((\Theta, v) \in \Omega^0(\mathcal{E})\).

**Proof.** As before, let \(v \in \Omega^0(T_X)\) be the vector field which generates the 1-parameter family of diffeomorphisms \(F_t: X \rightarrow X\) of the underlying smooth manifold \(X\). Since

\[
dF_t(\text{Graph}(\varphi_t : T_X^{0,1} \rightarrow T_X^{1,0})) = \text{Graph}(\varphi'_t : T_X^{0,1} \rightarrow T_X^{1,0}),
\]

we already have

\[
\varphi'_t = \varphi_t + t\bar{\partial}_X v + R(\varphi_t, tv).
\]

Hence it remains to show that

\[
A'_t = A_t + t(\bar{\partial}_Q(-\Theta_1) - v \cdot F_v) + R((A_t, \varphi_t), t(\Theta_1, v)),
\]

for some \(\Theta_1 \in \Omega^0(\mathcal{Q})\).
We define an endomorphism of $E$ as follows: Fix $p \in X$ and the fiber $E_p$ of $E$. Let $P_{\gamma(t)} : E_p \rightarrow E_{F_t(p)}$ be the parallel transport along $t \mapsto \gamma_p(t) := F_t(p)$. Define $\Theta_t := P_{\gamma(t)}^{-1} \Phi_t : E_p \rightarrow E_p$. Then $\Theta_t$ defines a bundle endomorphism of $E$. Let us write
\[
\Theta_t = \Theta_0 + t\Theta_1 + O(t^2),
\]
\[
A'_t = A'_0 + tA'_1 + O(t^2),
\]
Since $\Phi_0 = \text{id}_E$, we have $\Theta_0 = \text{id}_E$ and $A'_0 = 0$. We need to compute $A'_1$.

Now let $e_t$ be a local holomorphic section of $E_t \subset V$. Since $\Phi_t$ is holomorphic, $\Phi_t(e_t)$ is a holomorphic section of $E'_t \subset V'$, so that $D_t\Phi_t(e_t) = 0$, i.e.
\[
\partial_t\Phi_t(e_t) = -\varphi'(t)_*\nabla\Phi_t(e_t) - A'_t\Phi_t(e_t).
\]
We want to compute the first derivatives of both sides of this equation with respective to $t$ at $t = 0$.

First note that $\Phi_t = P_{\gamma(t)}\Theta_t$, so we have
\[
\frac{\partial}{\partial t}\Phi_t(e_t)|_{t=0} = -v_*\nabla e + \Theta_1(e) + \frac{\partial}{\partial t}e_t|_{t=0},
\]
where $e = e_0$. Hence
\[
\frac{\partial}{\partial t}\partial_t\Phi_t(e_t)|_{t=0} = \bar{\partial}_E\left(\frac{\partial}{\partial t}\Phi_t(e_t)|_{t=0}\right)
\]
\[
= -\bar{\partial}_E(v_*\nabla e) + \bar{\partial}_E(\Theta_1(e)) + \bar{\partial}_E\left(\frac{\partial}{\partial t}e_t|_{t=0}\right).
\]
For the term, $-\varphi'_t_\ast\nabla\Phi_t(e_t)$, we have
\[
\frac{\partial}{\partial t}(-\varphi'_t_\ast\nabla\Phi_t(e_t))|_{t=0} = -\varphi'_t_\ast\nabla e - \bar{\partial}_E v_*\nabla e
\]
\[
= -\varphi'_t_\ast\nabla e - \bar{\partial}_E(v_*\nabla e) - v_*\bar{\partial}_E \nabla e.
\]
Since $e = e_0$ is holomorphic with respective to $V|_{\pi^{-1}(0)}$, we have
\[
\bar{\partial}_E \nabla e = F_{V}(e).
\]
Moreover, since $e_t$ is holomorphic with respect to $E_t$, we have $\bar{\partial}_e e_t = 0$, that is,
\[
\bar{\partial}_E e_t = -\varphi_t_\ast\nabla e_t - A_t e_t.
\]
Differentiate with respective to $t$ and set $t = 0$, we obtain
\[
-\varphi_1_\ast\nabla e = \bar{\partial}_E\left(\frac{\partial}{\partial t}e(t)|_{t=0}\right) + A_1 e.
\]
Hence
\[
\frac{\partial}{\partial t}(-\varphi'_t_\ast\nabla\Phi_t(e_t))|_{t=0} = \bar{\partial}_E\left(\frac{\partial}{\partial t}e_t|_{t=0}\right) - \bar{\partial}_E(v_*\nabla e) - v_*F_{V} e + A_1 e.
\]
For the term $-A'_t\Phi_t(e_t)$, we have
\[
\frac{\partial}{\partial t}(-A'_t\Phi_t(e_t))|_{t=0} = -A'_1 e.
\]
As a whole we obtain the formula
\[
\bar{\partial}_E(\Theta_1(e)) = -v_*F_{V}(e) + (A_1 - A'_1)(e).
\]
Since $e$ is holomorphic with respective to $V|_{\pi^{-1}(0)}$, $\bar{\partial}_E(\Theta_1(e)) = (\bar{\partial}_Q\Theta_1)(e)$, so that
\[
A'_1 = A_1 + \bar{\partial}_Q(-\Theta_1) - v_* F_{V}$. 

Therefore we have
\[
\frac{\partial}{\partial t}(A'_t - A_t)|_{t=0} = A'_1 - A_1 = \bar{\partial}Q(-\Theta_1) - v \cdot F_{\nabla'},
\]

or in other words,
\[
A'_t = A_t + t(\bar{\partial}Q(-\Theta_1) - v \cdot F_{\nabla'}) + O(t^2).
\]

Since \(A'_t\) is completely determined by \((A_t, \varphi_t)\) and \(t(\Theta_1, v)\), we have
\[
A'_t = A_t + t(\bar{\partial}Q(-\Theta_1) - v \cdot F_{\nabla'}) + R((A_t, \varphi_t), t(\Theta_1, v)),
\]

where \(R\) is of order \(t^2\) and depends smoothly on \(t, A_t, \varphi_t, \Theta_1, v\). Moreover, since the equation
\[
\bar{\partial}_E \Phi_t(e_t) = -\varphi'_t \cdot \nabla \Phi_t(e_t) - A'_t \Phi_t(e_t)
\]
depends smoothly on first order partial derivatives of \(\Theta_1\) and \(v\), we see that the error \(R\) also depends smoothly on first order partial derivatives of \(\Theta_1\) and \(v\).

Finally, \(R\) satisfies
\[
R(s(A, \varphi), st(\Theta, v)) = s^2 R_1((A, \varphi), (\Theta, v), s),
\]

for some map \(R_1\) which depends smoothly in \(s, (A, \varphi) \in \Omega^{0,1}(E)\) and \((\Theta, v) \in \Omega^0(E)\). This follows from the fact that
\[
R((A, \varphi), (\Theta, v)) \to 0 \ \text{as} \ (\Theta, v) \to 0, \ \text{and}
\]
\[
R((A, \varphi), (\Theta, v)) \to R((\Theta, v)) \ \text{as} \ (A, \varphi) \to 0,
\]

with \(R(s(\Theta, v)) = s^2 R((\Theta, v))\).

\[
\text{Corollary 4.3.} \text{ If } \mathcal{V} \to \mathcal{X} \text{ and } \mathcal{V}' \to \mathcal{X}' \text{ are isomorphic deformations of } (X, E), \text{ then the first order terms } (A_1, \varphi_1) \text{ and } (A'_1, \varphi'_1) \text{ of the corresponding families } (A_t, \varphi_t) \text{ and } (A'_t, \varphi'_t) \text{ respectively differ by an } \bar{\partial}_E \text{-exact form.}
\]

\[
\text{Proof.} \text{ We have } A'_1 - A_1 = \bar{\partial}Q(-\Theta_1) - v \cdot F_{\nabla'} \text{ and } \varphi'_1 - \varphi_1 = \bar{\partial}_E v, \text{ whence}
\]
\[
(A'_1, \varphi'_1) - (A_1, \varphi_1) = \bar{\partial}_E(-\Theta_1, v).
\]

We conclude that the space of first order deformations of a holomorphic pair \((X, E)\) is precisely given by the Dolbeault cohomology group \(H^{0,1}_{\bar{\partial}_E} \cong H^1(X, E)\).

5. Obstructions and Kuranishi family

Now given a first order deformation \([(A_1, \varphi_1)] \in H^{0,1}_{\bar{\partial}_E} \cong H^1(X, E)\), it is standard in deformation theory to ask whether one can find a family \((A_t, \varphi_t)\) integrating \((A_1, \varphi_1)\) to give an actual family of deformations. To study this problem, we use a method due to Kuranishi [13].

We need to review several operators commonly used in Hodge theory. We first choose a hermitian metric \(g\) on \(X\) and \(h\) on \(E\), so that we can define a hermitian product \((\cdot, \cdot)\) on \(\Omega^{0,\ast}(E)\). Define the formal adjoint of \(\bar{\partial}_E\) with respective to \((\cdot, \cdot)\) by
\[
(\bar{\partial}_E \alpha, \beta) = (\alpha, \bar{\partial}_E^* \beta).
\]

Then the Laplacian is defined by
\[
\Delta_E := \bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E.
\]
This is an elliptic self-adjoint operator and thus has a finite dimensional kernel \(H^p(X, E)\), consisting of harmonic forms. We have the standard isomorphism from Hodge theory:

\[ H^p(X, E) \cong H^0_{\bar{\partial}} \cong \Omega^{p, 0}(X, E). \]

Take a completion of \(\Omega^{0, *}(E)\) with respective to \((\cdot, \cdot)\) to get a Hilbert space \(L^*\), and let \(H^* : L^* \to \Omega^{p, 0}(X, E)\) be the harmonic projection. The Green’s operator \(G : L^* \to L^*\) is defined by

\[ I = H + \Delta_E G = H + G\Delta_E. \]

It commutes with \(\bar{\partial}_E\) and \(\bar{\partial}_E^*\).

Now let \(\eta_1, \ldots, \eta_m \in \Omega^1(X, E)\) be a basis and \(t_j \eta_j \in \Omega^1(X, E)\). Consider the equation

\[ \epsilon(t) = \epsilon_1(t) - \frac{1}{2} \bar{\partial}_E^* G[\epsilon(t), \epsilon(t)]. \]

We denote the Hölder norm by \(\| \cdot \|_{k, \alpha}\). The following estimates are obvious:

\[ \| \bar{\partial}_E^* \epsilon \|_{k, \alpha} \leq C_1 \| \epsilon \|_{k+1, \alpha}, \]

\[ \| [\epsilon, \delta] \|_{k, \alpha} \leq C_2 \| \epsilon \|_{k+1, \alpha} \| \delta \|_{k+1, \alpha}. \]

In [3], Douglis and Nirenberg proved the following nontrivial a priori estimate:

\[ \| \epsilon \|_{k, \alpha} \leq C_3 (\| \Delta_E \epsilon \|_{k-2, \alpha} + \| \epsilon \|_{0, \alpha}). \]

Applying these and following the proof of [10, Chapter 4, Proposition 2.3], one can deduce an estimate for the Green’s operator \(G\):

\[ \| G\epsilon \|_{k, \alpha} \leq C_4 \| \epsilon \|_{k-2, \alpha}, \]

where all \(C_i\)’s are positive constants which depend only on \(k\) and \(\alpha\).

Then by the same argument as in [10, Chapter 4, Proposition 2.4], or alternatively using an implicit function theorem for Banach spaces [13], we obtain a unique solution \(\epsilon(t)\) which satisfies the equation

\[ \epsilon(t) = \epsilon_1(t) - \frac{1}{2} \bar{\partial}_E^* G[\epsilon(t), \epsilon(t)], \]

and is analytic in the variable \(t\). Note that the solution \(\epsilon(t)\) is always smooth. Indeed, by applying the Laplacian to the above equation, we get

\[ \Delta_E \epsilon(t) + \frac{1}{2} \bar{\partial}_E^* G[\epsilon(t), \epsilon(t)] = 0. \]

Also, the solution \(\epsilon(t)\) is holomorphic in \(t\), so we have

\[ \sum_j \frac{\partial^2 \epsilon(t)}{\partial t_j \partial \bar{t}_j} = 0. \]

Now since the operator

\[ \Delta_E + \sum_j \frac{\partial^2}{\partial t_j \partial \bar{t}_j} \]

is elliptic, we see that \(\epsilon(t)\) is smooth by elliptic regularity.

Following Kuranishi [13] (see also [10, Chapter 4]), we have the following
Proposition 5.1. The solution $\epsilon(t)$ that satisfies
\begin{equation*}
\epsilon(t) = \epsilon_1(t) - \frac{1}{2} \overline{\partial}_t \partial_t G[\epsilon(t),\epsilon(t)]
\end{equation*}
solves the Maurer-Cartan equation if and only if $H[\epsilon(t),\epsilon(t)] = 0$, where $H$ is the harmonic projection.

Proof. Suppose the Maurer-Cartan equation holds. Then
\begin{equation*}
H[\epsilon(t),\epsilon(t)] = 2H \overline{\partial}_t \partial_t \epsilon(t) = 0.
\end{equation*}

Conversely, suppose that $H[\epsilon(t),\epsilon(t)] = 0$. We must show that
\begin{equation*}
\delta(t) := \overline{\partial}_t \partial_t \epsilon(t) + \frac{1}{2} \overline{\partial}_t \partial_t G[\epsilon(t),\epsilon(t)] = 0.
\end{equation*}
Recall that $\epsilon(t)$ is a solution to
\begin{equation*}
\epsilon(t) = \epsilon_1(t) - \frac{1}{2} \overline{\partial}_t \partial_t G[\epsilon(t),\epsilon(t)]
\end{equation*}
and $\epsilon_1(t)$ is $\overline{\partial}_t$-closed. By applying $\overline{\partial}_t$ to this equation, we get
\begin{equation*}
\overline{\partial}_t \partial_t \epsilon(t) = -\frac{1}{2} \overline{\partial}_t \partial_t G[\epsilon(t),\epsilon(t)].
\end{equation*}
Hence
\begin{equation*}
2\delta(t) = \overline{\partial}_t \partial_t G[\epsilon(t),\epsilon(t)] - [\epsilon(t),\epsilon(t)].
\end{equation*}
Using the Hodge decomposition on forms, we can write
\begin{equation*}
[\epsilon(t),\epsilon(t)] = H[\epsilon(t),\epsilon(t)] + \Delta_{\epsilon} G[\epsilon(t),\epsilon(t)] = \Delta_{\epsilon} G[\epsilon(t),\epsilon(t)].
\end{equation*}
Therefore, we have
\begin{equation*}
2\delta(t) = (\Delta_{\epsilon} G - \overline{\partial}_t \partial_t G)[\epsilon(t),\epsilon(t)] = \partial_t^2 \partial_t \partial_t G[\epsilon(t),\epsilon(t)] = 2\overline{\partial}_t \partial_t G[\partial_t \partial_t \epsilon(t),\epsilon(t)],
\end{equation*}
and hence
\begin{equation*}
\delta(t) = \overline{\partial}_t \partial_t G[\overline{\partial}_t \partial_t \epsilon(t),\epsilon(t)] = \overline{\partial}_t \partial_t G[\delta(t) - \frac{1}{2} [\epsilon(t),\epsilon(t)],\epsilon(t)] = \overline{\partial}_t \partial_t G[\delta(t),\epsilon(t)],
\end{equation*}
where we have used the Jacobi identity in the last equality. Using the estimate
\begin{equation*}
||[\xi,\eta]||_{k,\alpha} \leq C_{k,\alpha}||\xi||_{k+1,\alpha}||\eta||_{k+1,\alpha},
\end{equation*}
we get
\begin{equation*}
||\delta(t)||_{k,\alpha} \leq C_{k,\alpha}||\delta(t)||_{k,\alpha}||\epsilon(t)||_{k,\alpha}.
\end{equation*}
By choosing $|t|$ to be small enough so that $C_{k,\alpha}||\epsilon(t)||_{k,\alpha} < 1$, we must have $\delta(t) = 0$ for all $|t|$ small enough. This finishes the proof.

In the case when $H[\epsilon(t),\epsilon(t)]$ vanishes identically (which always holds if $H^2(X,\mathcal{E}) = 0$), we have the following

Corollary 5.2. If $H[\epsilon(t),\epsilon(t)] = 0$ for all $t$, then we have a complex analytic family $\mathcal{V} \to \mathcal{X}$.

Proof. If $H[\epsilon(t),\epsilon(t)] = 0$ for all $t$, then $\epsilon(t) = (A_t, \varphi_t)$ satisfies the Maurer-Cartan equation and so $(X_t, E_t)$ is holomorphic for each $t$. In particular, we obtain a deformation $\mathcal{X}'$ of $\mathcal{X}$. Let $\mathcal{V} := \Delta \times E$. A smooth section $\sigma : \mathcal{X}' \to \mathcal{V}$ of $\mathcal{V}$ on $\mathcal{X}'$ can be written as
\begin{equation*}
\sigma : (t, x) \mapsto (t, s(x, t)),
\end{equation*}
for some smooth map $s : \mathcal{X} \to E$. We define a Dolbeault operator $\bar{\partial}_V : \Omega^0_\mathcal{X}(\mathcal{V}) \to \Omega^0_{\mathcal{X}}(\mathcal{V})$ on $\mathcal{V}$ by

$$\bar{\partial}_V \sigma(t,x) = (t, \bar{\partial}_E s(t,x)).$$

Note that $\bar{\partial}_V$ is well-defined, if $\{e_k(t,x)\}$ are local holomorphic frame of $E_t \to X_t$, then we can write

$$\bar{\partial}_V \sigma(t,x) = (t, \bar{\partial}_V (s^k(t,x) e_k(t,x))) = \bar{\partial}_t s^k(t,x) \otimes e_k(t,x),$$

which is a smooth section of $\Omega^0_{\mathcal{X}}(\mathcal{V})$. Clearly, $\bar{\partial}^2_V = 0$ and hence $\mathcal{V}$ is a holomorphic vector bundle over $\mathcal{X}$. □

In general, the condition $H^2(X, E) = 0$ may not be satisfied. But we can define the (singular) analytic space $S := \{t \in \Delta : H[\epsilon(t), \epsilon(t)] = 0\}$ and form a family $\mathcal{V} \to \mathcal{X}$ over $S$, which is called the Kuranishi family of $(X, E)$. In particular, we see that the obstruction space is precisely given by the Dolbeault cohomology group $H^1_{\partial_V} \cong H^2(X, E)$, and the obstructions to deformations of a holomorphic pair $(X, E)$ is captured by the Kuranishi map

$$\text{Ob}_{(X, E)} : H^1(X, E) \to H^2(X, E), \sum_{i=1}^m t_i \eta_i \mapsto H[\epsilon(t), \epsilon(t)].$$

6. A PROOF OF COMPLETENESS

The goal of this section is to give a proof of the local completeness of a Kuranishi family for the deformation of the pair $(X, E)$. Existence of a locally complete (or versal) family for deformations of pairs was first proved by Siu-Trautmann [20]. Here we give another proof using Kuranishi’s method.

**Definition 6.1.** A family $\mathcal{V} \to \mathcal{X}$ over a analytic space $S$ is said to be locally complete (or versal) if for any family $\mathcal{V}' \to \mathcal{X}'$ over a sufficiently small ball $\Delta$, there exists a analytic map $f : \Delta \to S$ such that the family $\mathcal{V}' \to \mathcal{X}'$ is the pull-back of $\mathcal{V} \to \mathcal{X}$ via $f$.

Recall that for given $\epsilon_1(t) \in H^1(X, E)$, we have existence of solutions $\epsilon(t)$ to

$$\epsilon(t) = \epsilon_1(t) - \frac{1}{2} \partial^*_E G[\epsilon(t), \epsilon(t)]$$

and $\epsilon(t)$ satisfies the Maurer-Cartan equation if and only if $H[\epsilon(t), \epsilon(t)] = 0$. We then obtain an analytic family $\mathcal{V} \to \mathcal{X}$ over

$$S := \{t \in \Delta : H[\epsilon(t), \epsilon(t)] = 0\}.$$ 

The main theorem is as follows:

**Theorem 6.2.** The Kuranishi family $\mathcal{V} \to \mathcal{X}$ over $S$ is locally complete.

Before going into the details of the proof, we first introduce the Sobolev norm: One can endow $\mathcal{E}$ a hermitian metric $H$, induced from that of $E$ and $X$, and define the inner product

$$(\alpha, \beta)_k := \sum_{|I| \leq k} \int_X H(D^I \alpha, D^I \beta),$$
α, β ∈ Ω^0.*(E). The Sobolev norm is defined by

\[ |α| := (α, α)^{1/2}. \]

One has the estimate

\[ ||[α, β]|_k \leq C_k|α|_{k+1}|β|_{k+1}, \]

for some constant \( C_k > 0 \).

We take a completion of \( Ω^0.*(E) \) with respective to \( (·, ·)_k \) to get a Hilbert space \( L^*_k \). The harmonic projection \( H : L^*_k \to \mathbb{H}^*(X, E) \) and the Green’s operator \( G : L^*_k \to L^*_{k+2} \) satisfy the estimates

\[ |Hα|_k \leq C_k|α|_k, \]

\[ |\bar{∂}^*E Gα|_k \leq C_k|α|_{k-1}. \]

The following lemma will be useful in the proof of the completeness theorem.

**Lemma 6.3.** For fixed \( ε_1(t) \in \mathbb{H}^1(X, E) \), \( t \in S \), the equation

\[ ε(t) = ε_1(t) - \frac{1}{2} \bar{∂}^*E[I[ε_1(t), ε(t)]] \]

has only one small solution.

**Proof.** Suppose \( ε \) is another solution. Let \( δ := ε - ε(t) \). Then

\[ δ = -\frac{1}{2} \bar{∂}^*E[I[ε, ε] - [ε(t), ε(t)]] \]

\[ = -\frac{1}{2} \bar{∂}^*E[I[δ, ε(t)] + [ε(t), δ] + [δ, δ]] \]

\[ = -\frac{1}{2} \bar{∂}^*E[I[2δ, ε(t)] + [δ, δ]]. \]

Hence

\[ |δ|_k \leq C_k(|δ|_k|ε(t)|_k + |δ|^2_k) \leq C_k|δ|_k(|ε(t)|_k + |δ|_k). \]

For \( |ε(t)|_k \) and \( |ε|_k \) small, we can only have \( |δ|_k = 0 \).

We are now ready to prove the local completeness theorem.

**Proof of Theorem 6.2.** Let \( V' \to X' \) be a deformation of \((X, E)\). Let \( ε' \), be the element representing this deformation. We first prove that if \( \bar{∂}^*E ε' = 0 \), then there exists \( t \in S \) such that \( ε' = ε(t) \).

Note that \( ε' \) satisfies the Maurer-Cartan equation:

\[ \bar{∂}ε' + \frac{1}{2}[ε', ε'] = 0. \]

Applying \( \bar{∂}^*_E \), we get

\[ \bar{∂}^*_E \bar{∂}ε' + \frac{1}{2} \bar{∂}^*_E [ε', ε'] = 0. \]

Since \( \bar{∂}^*_E ε' = 0 \), we have

\[ Δ_E ε' + \frac{1}{2} \bar{∂}^*_E [ε', ε'] = 0. \]

Then using \( I = H + GΔ_E \), we get

\[ ε' = Hε' - \frac{1}{2} \bar{∂}^*_E G[ε', ε']. \]
Note that $H\epsilon' \in \mathbb{H}^1(X, \mathcal{E})$ and by the estimate $|H\epsilon'|_k \leq C_k|\epsilon'|$, we see that $|H\epsilon'|_k$ is small if $|\epsilon'|_k$ is small. Hence $H\epsilon' = \epsilon_1(t)$ for some $t \in \mathcal{S}$. Therefore, if the ball $\Delta$ is small enough, $\epsilon'$ is a solution to

$$
\epsilon' = \epsilon_1(t) - \frac{1}{2}\bar{\partial}_E G[\epsilon', \epsilon'].
$$

Therefore, $\epsilon' = \epsilon(t)$ for some $t \in \mathcal{S}$ by Lemma 6.3.

Now we prove that for any given small deformation $\mathcal{V}' \to \mathcal{X}'$, one can find an isomorphic deformation $\mathcal{V}'' \to \mathcal{X}''$ such that the element $\epsilon''$ which represents the family $\mathcal{V}'' \to \mathcal{X}''$ is $\partial_E^*\epsilon$-closed. This will prove the local completeness. Indeed, we will prove the following: Given a deformation $\epsilon'$, there exists $\eta \in Im(\bar{\partial}_E) \subset \Omega^1(\mathcal{E})$ such that the element $\epsilon''$, which represents the deformation $\mathcal{V}'' \to \mathcal{X}''$, is $\partial_E^*\epsilon$-closed.

Let $\eta = (\Theta, v) \in \Omega^1(\mathcal{E})$, then the elements $\epsilon', \epsilon''$, which represent the deformations $\mathcal{V}' \to \mathcal{X}'$ and $\mathcal{V}'' \to \mathcal{X}''$ respectively, satisfy

$$
\epsilon'' = \epsilon' + \bar{\partial}_E \eta + R(\epsilon', \eta),
$$

where the error term $R$ satisfies $R(s\epsilon', s\eta) = s^2 R_1(\epsilon', \eta, s)$ as in Proposition 4.2. Hence $\bar{\partial}_E^*\epsilon'' = 0$ if and only if

$$
\bar{\partial}_E^*\epsilon' + \bar{\partial}_E \bar{\partial}_E \eta + \bar{\partial}_E R(\epsilon', \eta) = 0.
$$

If $\eta \in Im(\bar{\partial}_E)$, then

$$
\Delta_E(\eta) + \bar{\partial}_E^* R(\epsilon'(s), \eta) + \bar{\partial}_E^* \epsilon' = 0.
$$

Applying $G$, we get

$$
\eta + \bar{\partial}_E^* G R(\epsilon', \eta) + \bar{\partial}_E^* G \epsilon' = 0.
$$

Let $U_1 \subset L^1_k$ and $V_1 \subset L^0_k$ be neighborhoods around 0 such that $R(\epsilon', \eta)$ is defined. Define $F : U_1 \times V_1 \to L^0_k$ by

$$
F(\epsilon', \eta) := \eta + \bar{\partial}_E^* G R(\epsilon', \eta) + \bar{\partial}_E^* G \epsilon'.
$$

By the order condition on the error term $R$, the derivative of $F$ with respective to $\eta$ at $(0, 0)$ is the identity map. Hence by the implicit function theorem, there is a $C^\infty$ function $g$ such that $F(\epsilon', \eta) = 0$ if and only if $\eta = g(\epsilon')$. By the error condition again, the (second order) operator $|\bar{\partial}_E R(\epsilon', -)|_k$ is small if $|\epsilon'|_k$ is small. Hence

$$
\Delta_E + \bar{\partial}_E^* R(\epsilon', -) + \bar{\partial}_E^* \epsilon'
$$

is still a quasi-linear elliptic operator. By elliptic regularity, $\eta$ is smooth. This completes our proof. \hfill \Box

### 7. Unobstructed deformations and applications

In this section, we investigate various circumstances under which deformations of holomorphic pairs are unobstructed. We will also apply our results to show that the dimension of the cohomology group $H^1(X, \text{End}(T_X))$ is invariant under small deformations of an algebraic K3 surface, answering a question of Huybrechts [7] in the 2-dimensional case.
7.1. **Unobstructed deformations of pairs.** To begin with, note that we have an exact sequence of holomorphic vector bundles

\[ 0 \to \text{End}(E) \to E \to T_X \to 0 \]

by the construction of \( E \) (which shows that \( E \) is an extension of \( Q = \text{End}(E) \) by \( T_X \)). This induces a long exact sequence in cohomology groups:

\[ \cdots \to H^1(X, Q) \to H^1(X, E) \to H^1(X, T_X) \to H^2(X, Q) \to H^2(X, E) \to H^2(X, T_X) \to \cdots \]

and the first order term \((A_1, \varphi_1)\) defines a class \([ (A_1, \varphi_1) ] \in H^1(X, E) \).

The following proposition, which first appeared in [7, Appendix] without proof, describes the relations between the deformations of a pair \((X, E)\) and that of \(X\) and \(E\).

**Proposition 7.1.** Denote the Kuranishi obstruction maps of the deformation theory of \( X \), \( E \) and \((X, E)\) by \( \text{Ob}_X \), \( \text{Ob}_E \) and \( \text{Ob}_{(X, E)} \) respectively. Then we have the following commutative diagram:

\[
\begin{array}{ccc}
\cdots & \longrightarrow & H^1(X, Q) \\
\downarrow_{\text{Ob}_E} & & \downarrow_{\text{Ob}_{(X, E)}} \\
\cdots & \longrightarrow & H^1(X, E) \\
\downarrow_{\pi^*} & & \downarrow_{\pi^*} \\
\cdots & \longrightarrow & H^1(X, T_X) \\
\downarrow_{\delta} & & \downarrow_{\delta} \\
\cdots & \longrightarrow & \cdots
\end{array}
\]

Here, the connecting homomorphism \( \delta \) is given by contracting with the Atiyah class:

\[ \delta(\varphi) = \varphi \ast [F_\varphi]. \]

**Proof.** By definition,

\[ \iota^*[A] = [(A, 0)], \quad \pi^*([(A, \varphi)]) = [\varphi]. \]

The commutative diagram follows directly from the definitions of the obstructions \( \text{Ob}_X, \text{Ob}_E \) and \( \text{Ob}_{(X, E)} \). \( \square \)

**Remark 7.2.** In particular, for any \([(A, \varphi)] \in H^1(X, E)\) such that \( \text{Ob}_{(X, E)}(A, \varphi) = 0 \), meaning that \([(A, \varphi)]\) gives an unobstructed deformation of \((X, E)\), we have

\[ 0 = \text{Ob}_X \circ \pi^*[A, \varphi] = \text{Ob}_X([\varphi]), \]

i.e. the deformation of \( X \) must then be unobstructed along \([\varphi]\).

An immediate consequence of this proposition is the following slight generalization of a result in [13]:

**Proposition 7.3.** Suppose that \( \text{Ob}_X \circ \pi^* = 0 \) and the connecting homomorphism \( \delta : H^1(X, T_X) \to H^2(X, Q) \) is surjective, then deformations of the pair \((X, E)\) are unobstructed.

**Proof.** Surjectivity of \( \delta \) implies that the map \( \iota^* : H^2(X, Q) \to H^2(X, E) \) is a zero map, and hence the map \( \pi^* : H^2(X, E) \to H^2(X, T_X) \) is injective. But \( \pi^* \circ \text{Ob}_{(X, E)} = \text{Ob}_X \circ \pi^* = 0 \), so we have \( \text{Ob}_{(X, E)} = 0 \). \( \square \)

In the case when \( E = L \) is a line bundle, we recover the following
Corollary 7.4 ([13], Lemma 2.4). Let $X$ be a compact complex manifold with unobstructed deformations and $L$ be a holomorphic line bundle over $X$ such that the map
\[
\cup c_1(L) : H^1(X,T_X) \to H^2(X,\mathcal{O}_X)
\]
is surjective. Then deformations of the pair $(X,L)$ are unobstructed.

For example, if $X$ is an $n$-dimensional compact Kähler manifold with trivial canonical line bundle, then $X$ admits unobstructed deformations. If we further assume that $H^{0,2}(X) = 0$ (e.g. when the holonomy of $X$ is precisely $SU(n)$), then deformations of $(X,L)$ for any line bundle $L$ are unobstructed.

Definition 7.5. A holomorphic vector bundle $E$ over a compact complex manifold $X$ is said to be good if $H^2(X,Q_0) = 0$, where $Q_0$ is the trace-free part of $Q = \text{End}(E)$.

Proposition 7.6. Let $X$ be a compact complex surface with trivial canonical line bundle (e.g. a K3-surface), and let $E$ be a good bundle over $X$ with $c_1(E) \neq 0$. Then deformations of the pair $(X,E)$ are unobstructed.

Proof. By the theorem of Tian and Todorov [21,22], we have $\text{Ob}_X = 0$. Hence the condition $\text{Ob}_X \circ \pi^* = 0$ is automatic.

On the other hand, note that $Q^* = (E^* \otimes E)^* = E^* \otimes E = Q$ and similarly $Q_0^* = Q_0$. By Serre duality and the fact that $K_X \cong \mathcal{O}_X$, we have $H^0(X,Q_0) \cong (H^2(X,Q_0))^* = 0$ since $E$ is good. This implies that $H^0(X,Q) \cong H^0(X,\mathcal{O}_X) \cong \mathbb{C}$. Then applying Serre duality again gives
\[
H^2(X,Q) \cong (H^0(X,Q^* \otimes K_X))^* \cong (H^0(X,Q))^* \cong \mathbb{C}.
\]

In this case, the connecting homomorphism $\delta : H^1(X,T_X) \cong H^{1,1}(X) \to H^2(X,Q) \cong \mathbb{C}$ is simply given by
\[
\delta(\varphi) = \int_X \varphi \cup [\text{tr} F_{\nabla}] = -2\pi i \int_X \varphi \cup c_1(E).
\]

When $c_1(E) \neq 0$, $\delta$ is a nonzero map and hence surjective. Proposition 7.3 then says that any deformation of $(X,E)$ is unobstructed. \qed

7.2. Applications to a question of Huybrechts [7]. In this section, we apply the theory of deformation of pairs to study the jumping of the dimension of the cohomology group $H^1(X,\text{End}(T_X))$ under small deformations. In the case of projective Calabi-Yau manifolds, this was asked by Huybrechts in [7].

To begin with, let $\text{Def}(X)$ and $\text{Def}(X,E)$ be the deformation spaces of $X$ and of the pair $(X,E)$ respectively. Suppose we have a deformation $\{(X_t,E_t)\}_{t \in \text{Def}(X,E)}$ of $(X,E)$. Recall that we have a differential operator $\partial_t : \Omega^{0,*}(E) \to \Omega^{0,*,+1}(E)$ satisfying the Leibniz rule and $\partial_t^2 = 0$.

Lemma 7.7. Let $\partial_{E_t}$ be the Dolbeault operator of $E_t \to X_t$ and $P : \Omega^{0,p}(E) \to \Omega^{0,p}(E)$ be the natural projection, given by restricting the projection $P : \Omega^p(X) \to \Omega^{0,p}(X)$ to $\Omega^{0,p}(E)$. Then $\partial_{E_t}$ is identified with $\partial_t$ via $P$ i.e.,
\[
\partial_{E_t} P = P \partial_t.
\]

Proof. We first prove the case when $p = 0$ and $E = \mathcal{O}_X$. In this case, $A = 0$ and $P : \Omega^p(X) \to \Omega^{0,p}(X) = \Omega^0(X)$ is the identity map. We fix $x \in X$ and let $z^j$ be local
complex coordinates around \( x \). Let \( \bar{v}_j = \frac{\partial}{\partial z^j} + \varphi^k_j \frac{\partial}{\partial z^k} \) and \( \bar{\epsilon}^j \) be its dual vector. Then by the Maurer-Cartan equation satisfied by \( \varphi \),

\[
[\bar{v}_j, \bar{v}_k] = 0.
\]

Hence we have complex coordinates \( \zeta^j \) on \( X_t \) such that

\[
\frac{\partial}{\partial \zeta^j} = \bar{v}_j, \quad d\zeta^j = \bar{\epsilon}^j
\]

at the point \( x \). Then at \( x \),

\[
P(\bar{\partial} + \varphi_t \partial)f = \left( \frac{\partial f}{\partial \bar{z}^j} + \varphi^k_j \frac{\partial f}{\partial z^k} \right) P(d\bar{z}^j).
\]

We need to show that \( P(d\bar{z}^j) = \bar{\epsilon}^j \). We write

\[
d\bar{z}^j = c^j_k \bar{\epsilon}^j + d^j_k \epsilon^k.
\]

Then

\[
c^j_k = d\bar{z}^j(\bar{v}_k) = d\bar{z}^j \left( \frac{\partial}{\partial \bar{z}^j} + \varphi^k_j \frac{\partial}{\partial z^k} \right) = \delta^j_k.
\]

Hence \( P(d\bar{z}^j) = \bar{\epsilon}^j \). Therefore,

\[
P(\bar{\partial} + \varphi_t \partial)f = \bar{v}^j f \bar{\epsilon}^j = \frac{\partial f}{\partial \zeta^j} d\zeta^j = \bar{\partial}_t P f
\]

at \( x \). Since \( x \) is arbitrary, \( PD_t = \bar{\partial}_t P \) on \( X \). This proves the case when \( p = 0 \).

For \( p > 0 \), let \( \alpha = \alpha_j d\bar{z}^j \). Then

\[
P(\bar{\partial} + \varphi_t \partial)\alpha = P(\bar{\partial} + \varphi_t \partial)(\alpha_j) \wedge P(d\bar{z}^j) = \bar{\partial}_t P \alpha_j \wedge \bar{\epsilon}^j.
\]

We need to show that \( \bar{\partial}_t (\bar{\epsilon}^j) = 0 \) for all \( j \). Since \( \bar{\epsilon}^j \) is a local frame of \((T_{X_t}^{0,1})^*, d\bar{\epsilon}^j \in \Omega_{X_t}^{1,1}(X) \oplus \Omega_{X_t}^{0,2}(X) \). Hence, to prove that \( \bar{\partial}_t \bar{\epsilon}^j = 0 \), it is equivalent to show that \( d\bar{\epsilon}^j(\bar{v}_k, \bar{v}_l) = 0 \) for all \( k, l \). We compute:

\[
d\bar{\epsilon}^j(\bar{v}_k, \bar{v}_l) = \bar{v}_k \bar{\epsilon}^j(\bar{v}_l) - \bar{v}_l \bar{\epsilon}^j(\bar{v}_k) - \bar{\epsilon}^j([\bar{v}_k, \bar{v}_l]) = 0.
\]

Therefore,

\[
P(\bar{\partial} + \varphi_t \partial)\alpha = \bar{\partial}_t (P \alpha_j \wedge \bar{\epsilon}^j) = \bar{\partial}_t P \alpha.
\]

This proves the case when \( p > 0 \).

Now for a general holomorphic vector bundle \( E_t \), \( \bar{D}_t \) is locally given by

\[
\bar{D}_t(\alpha^j \otimes e_j) = (\bar{\partial} + \varphi_t \partial)\alpha^j \otimes e_j,
\]

where \( e_j \) is a local holomorphic frame of \( E_t \). In this case, \( P \) is given by

\[
P : \alpha^j \otimes e_j \mapsto P(\alpha^j) \otimes e_j.
\]

It is then obvious that the required relation follows from the \( O_X \) case. \( \square \)

For \( |t| \) small, \( P \) is an isomorphism, so we obtain the following

**Corollary 7.8.** For any holomorphic vector bundle \( E_t \to X_t \), we have the isomorphism

\[
H^\bullet(X_t, E_t) \cong H^\bullet(\Omega^{0,\bullet}(E), \bar{D}_t).
\]
This shows that jumping of the dimension \( \dim \mathbb{C} H^\bullet(X_t, E_t) \) is equivalent to the jumping of \( \dim \mathbb{C} H^\bullet(\Omega^0, \bullet)(E_t, \hat{D}_t) \).

In order to formulate the jumping of \( \dim \mathbb{C} H^\bullet(\Omega^0, \bullet)(E_t, \hat{D}_t) \) in a more concrete and rigorous way, we apply Grauert’s Direct Image Theorem [4] to obtain a complex of vector bundles \((V^\bullet, d^\bullet)\) over the deformation space \(\text{Def}(X, E)\) such that

\[
H^p(\Omega^0, \bullet)(E_t, \hat{D}_t) \cong H^p(X_t, E_t) \cong H^p(V^\bullet_t) = \frac{\ker(d^p)_t}{\text{Im}(d^{p-1})_t},
\]

for \( t \in \text{Def}(X, E) \).

Note that for fixed \( p \), \( \dim \mathbb{C} V^{p-1}_t \) is independent of \( t \). Therefore, the exact sequence

\[
0 \to \ker(d^{p-1})_t \to V^{p-1}_t \to \text{Im}(d^{p-1})_t \to 0
\]
suggests that the jumping of \( H^p(\Omega^0, \bullet)(E_t, \hat{D}_t) \cong H^p(V^\bullet_t) \) at \( t = 0 \) is due to the jumping of \( \ker(d^p)_t \) or \( \ker(d^{p-1})_t \) at \( t = 0 \). Since the jumping of \( \ker(d^p)_t \) at \( t = 0 \) is due to the failure of extending elements of \( \ker(d^0)_t \) to \( \ker(d^1)_t \), or equivalently, the failure of extending elements of \( \ker(\partial_E) \cap \Omega^0, \bullet(E) \) to \( \ker(\hat{D}_t) \cap \Omega^0, \bullet(E) \), we obtain the following

**Proposition 7.9.** The dimension \( \dim \mathbb{C} H^p(\Omega^0, \bullet)(E_t, \hat{D}_t) \) does not jump at \( t = 0 \) if and only if for any \( \alpha^q \in \ker(\partial_E : \Omega^0, q(E) \to \Omega^0, q+1(E)) \) where \( q = p - 1 \) or \( p \), there exist \( \alpha^q_t \in \Omega^0, q(E) \) such that \( [\alpha^q_0] = [\alpha^q_t] \in H^q(X, E) \) and

\[
\hat{D}(\alpha^q_t) = 0.
\]

We now focus on the case \( p = 1 \). Instead of proving directly that \( \dim \mathbb{C} H^1(X_t, \text{End}(T_{X_t})) \) does not jump at \( t = 0 \) for any deformation of \( X \), we first prove that in some nice cases, \( \dim \mathbb{C} H^1(X_t, E_t) \) does not jump at \( t = 0 \) for any deformation of \((X, E)\).

To do this, we recall that, by choosing a harmonic basis \( \{ (A_i, \varphi_i) \}_{i=1}^m \) for \( H^1(X, \mathcal{E}) \), the obstruction map \( \text{Ob}((X, E) : H^1(X, \mathcal{E}) \to H^2(X, \mathcal{E}) \) of the deformation theory of \((X, E)\) is given by

\[
\text{Ob}((X, E)) : \sum_{i=1}^m t_i(A_i, \varphi_i) \mapsto H[(A_t, \varphi_t), (A_i, \varphi_i)],
\]

where \((A_t, \varphi_t)\) satisfies

\[
(A_t, \varphi_t) = \sum_{i=1}^m t_i(A_i, \varphi_i) - \frac{1}{2} \partial_E G((A_t, \varphi_t), (A_i, \varphi_i)).
\]

Moreover, \((A_t, \varphi_t)\) satisfies the Maurer-Cartan equation if and only if \( \text{Ob}((X, E)) = 0 \).

Suppose now that \( \text{Ob}((X, E)) = 0 \). Then we have

\[
\partial_E(A_t, \varphi_t) + \frac{1}{2} [(A_t, \varphi_t), (A_t, \varphi_t)] = 0.
\]

Differentiating \((A_t, \varphi_t)\) with respect to \( t_i \) and setting \( t = 0 \), we get

\[
\frac{\partial}{\partial t_i}|_{t=0}(A_t, \varphi_t) = (A_i, \varphi_i).
\]

Hence, for each \( i = 1, \ldots, m \), if we define \( (B_i, \psi_i)_i \) to be

\[
(B_i, \psi_i)_i = \frac{\partial}{\partial t_i}(A_i, \varphi_i),
\]

...
In fact, the deformation induced by this operator is isomorphic to the family \( \langle B_t, \psi_t \rangle \), satisfies
\[
\bar{\partial}_E (B_t, \psi_t) + [(A_t, \varphi_t), (B_t, \psi_t)] = 0
\]
and \( \{(B_0, \psi_0), \}^{m}_{i=1} \) forms a basis for \( H^1(X, \mathcal{E}) \).

Moreover, if \( T \in \mathcal{E} \), then \( \dim \mathcal{E} = 0 \) and the Leibniz rule
\[
\bar{D}_{\mathcal{E}} (f \otimes s) = (\bar{\partial} + \varphi_t \phi) f \otimes s + f \bar{D}_{\mathcal{E}} s.
\]
It follows that \( \bar{D}_{\mathcal{E}} \) defines a deformation \( \{(X_t, \mathcal{E}_t) \}_{t \in \text{Def}(X, \mathcal{E})} \) of the pair \( (X, \mathcal{E}) \).

In fact, \( \mathcal{E} \) is the Atiyah extension of the deformed bundle \( E_t \) on \( X_t \).

**Lemma 7.10.** Suppose \( Ob_t(X, \mathcal{E}) = 0 \). Then for any \( \langle [B, \psi] \rangle \in H^1(X, \mathcal{E}) \), there exists \( \langle B_t, \psi_t \rangle \) such that \( \bar{D}_{\mathcal{E}_t}(B_t, \psi_t) = 0 \) and \( \langle [B_0, \psi_0], [B, \psi] \rangle \). Hence all element in \( H^1(X, \mathcal{E}) \) admits an extension to \( H^1(X_t, \mathcal{E}_t) \) for any deformation of \( (X, \mathcal{E}) \).

**Proof.** Since this is true for the harmonic basis \( \{(B_0, \psi_0), \}^{m}_{i=1} \), it is true for any element in \( H^1(X, \mathcal{E}) \). \( \square \)

**Lemma 7.11.** Let \( X \) be a compact complex manifold and \( E \to X \) be a holomorphic vector bundle. Suppose the deformation of the pair \( (X, \mathcal{E}) \) is always unobstructed and \( \dim C H^0(X_t, \mathcal{E}_t) \) does not jump at \( t = 0 \) along any deformations of \( (X, \mathcal{E}) \). Then \( \dim C H^1(X_t, \mathcal{E}_t) \) does not jump at \( t = 0 \) along any deformation of \( (X, \mathcal{E}) \).

**Proof.** Since \( Ob_t(X, \mathcal{E}) = 0 \), Lemma 7.10 allows us to extend any element in \( H^1(X, \mathcal{E}) \) to \( H^1(X_t, \mathcal{E}_t) \). Since
\[
H^0(X_t, \mathcal{E}_t) = \ker (\bar{\partial}_{\mathcal{E}_t} : \Omega^0(\mathcal{E}) \to \Omega^0(\mathcal{E})) = \ker (\bar{D}_{\mathcal{E}_t} : \Omega^0(\mathcal{E}) \to \Omega^1(\mathcal{E})),
\]
the assumption that \( \dim C H^0(X_t, \mathcal{E}_t) \) does not jump at \( t = 0 \) simply means any \( \bar{\partial}_{\mathcal{E}_t} \)-closed section can be extended to a \( \bar{D}_{\mathcal{E}_t} \)-closed section. Now apply Proposition 7.9. \( \square \)

We are now going to prove that under certain assumptions, the dimension \( \dim C H^1(X_t, \text{End}(T_{X_t})) \) does not jump at \( t = 0 \) along any deformation of \( X_t \).

First of all, when \( E = T_X \), we have a canonical lift \( L : H^1(X, T_X) \to H^1(X, \mathcal{E}) \) defined by
\[
L : \varphi \mapsto (-\nabla \varphi - T(\varphi, \bullet), \varphi),
\]
where \( T : \Omega^{0,p}(T_X) \otimes \Omega^{0,q}(T_X) \to \Omega^{0,p+q}(T_X) \) is the graded torsion defined as
\[
T(\varphi, \psi) = \varphi \wedge \psi - (-1)^{pq} \psi \wedge \varphi - [\varphi, \psi].
\]
Moreover, if \( Ob_X = 0 \), then we have a Maurer-Cartan element \( \varphi_t \in \Omega^{0,1}(T_X) \) and we obtain a deformation of \( (X, T_X) \) by
\[
\bar{D}_t = \bar{\partial}_{T_X} + \varphi_t \otimes \nabla - \nabla \varphi_t - T(\varphi_t, \bullet) = \bar{\partial}_{T_X} + [\varphi_t, \bullet].
\]
In fact, the deformation induced by this operator is isomorphic to the family \( \{(X_t, T_{X_t}) \}_{t \in \text{Def}(X)} \), where \( T_{X_t} \) is the holomorphic tangent bundle of \( X_t \). Therefore, \( L \) induces a natural embedding
\[
\text{Def}(X) \subset \text{Def}(X, T_X).
\]
In the following theorem, by a Calabi-Yau n-fold we mean an n-dimensional compact Kähler manifold X with trivial canonical line bundle, i.e. $K_X \cong \mathcal{O}_X$ and such that $H^{0,p}(X) = 0$ for all $p \neq 0, n$.

**Theorem 7.12.** Suppose that X is a Calabi-Yau manifold such that deformations of the pair $(X,T_X)$ are unobstructed, then $\dim \mathbb{C} H^1(X,t,\text{End}(T_X))$ does not jump at $t = 0$ for any deformation of X.

**Proof.** Since the pair $(X,T_X)$ admits unobstructed deformations, Lemma 7.10 allows us to extend any element in $H^1(X,E)$ to an element in $H^1(X,E_t)$, where $t \in \text{Def}(X,T_X)$. Consider the Atiyah exact sequence of $(T_X)_t$ (caution: this is not the tangent bundle of X in general!) over $X_t$:

$$0 \to \text{End}((T_X)_t) \to E_t \to T_{X_t} \to 0,$$

which gives rise to the injective map $\iota_t^*: H^0(X_t,\text{End}((T_X)_t)) \to H^0(X_t,E_t)$.

Since the tangent bundle of a Calabi-Yau manifold is stable, we have $H^0(X,\text{End}_0(T_X)) = 0$ and so

$$H^0(X,E) \cong H^0(X,\text{End}(T_X)) \cong H^0(X,\mathcal{O}_X) = \mathbb{C}.$$

Also, since the identity map $\text{id}_{(T_X)_t}$ is always a non-zero holomorphic section of $H^0(X_t,\text{End}((T_X)_t))$ and $\iota_t^*: H^0(X_t,\text{End}((T_X)_t)) \to H^0(X_t,E_t)$ is injective, we get

$$1 \leq \dim \mathbb{C} H^0(X_t,E_t) \leq \dim \mathbb{C} H^0(X,E) = 1$$

for $|t|$ small. By Lemma 7.11 we conclude that $\dim \mathbb{C} H^1(X_t,E_t)$ does not jump at $t = 0$ along any deformation of $(X,T_X)$.

For $t \in \text{Def}(X) \subset \text{Def}(X,T_X)$, we have a family of canonical lifts $L_t: H^1(X_t,T_{X_t}) \to H^1(X_t,E_t)$, since $E_t$ is the Atiyah extension of $T_{X_t}$, for $t \in \text{Def}(X)$. So the map $\pi_t^*: H^1(X_t,E_t) \to H^1(X_t,T_{X_t})$ is surjective and we obtain the following exact sequence

$$0 \to H^1(X_t,\text{End}(T_{X_t})) \to H^1(X_t,E_t) \to H^1(X_t,T_{X_t}) \to 0.$$

Since $\dim \mathbb{C} H^1(X_t,T_{X_t}) = \dim \mathbb{C} H^{n-1,1}(X_t)$ does not jump at $t = 0$ for $t \in \text{Def}(X)$ with $|t|$ small, we see that $\dim \mathbb{C} H^1(X_t,\text{End}(T_{X_t}))$ does not jump at $t = 0$ for any deformation of X.

Finally we consider algebraic K3 surfaces, i.e. complex algebraic surfaces with trivial canonical line bundle and such that $H^1(X,\mathcal{O}_X) = 0$. Recall that for every K3 surface X, we have $\dim \mathbb{C} H^3(X,T_X) = 20$ (see e.g. [2]).

**Theorem 7.13.** Suppose that X is an algebraic K3 surface. Then the dimension $\dim \mathbb{C} H^1(X_t,\text{End}(T_{X_t}))$ does not jump at $t = 0$ for any small algebraic deformation of X.

**Proof.** Let $L$ be a line bundle on X with $c_1(L) \neq 0$. Define $E := T_X \otimes L$ and let $E$ be the Atiyah extension of $E$.

Note that as $c_1(E) = c_1(L) \neq 0$ and $H^2(X,\text{End}_0(E)) \cong H^0(X,\text{End}_0(T_X)) = 0$ ($T_X$ is stable), we can apply Proposition 7.4 to conclude that $\text{Ob}_{(X,E)} = 0$. Moreover, since any deformation of $(X,E)$ gives rise to a deformation of X via the map $\pi^*: H^1(X,E) \to H^1(X,T_X)$, we obtain the Atiyah exact sequence of $E_t$ on $X_t$:

$$0 \to \text{End}(E_t) \to E_t \to T_{X_t} \to 0,$$
for $t \in \text{Def}(X, E)$. Note that
\[ H^0(X, \mathcal{E}) \cong H^0(X, \text{End}(E)) \cong \mathbb{C}, \]
we have
\[ 1 \leq \dim_{\mathbb{C}} H^0(X_t, \text{End}(E_t)) \leq \dim_{\mathbb{C}} H^0(X_t, \mathcal{E}_t) \leq \dim_{\mathbb{C}} H^0(X, \mathcal{E}) = 1. \]
This shows that $\dim_{\mathbb{C}} H^0(X_t, \mathcal{E}_t)$ does not jump at $t = 0$ along any deformation of $(X, E)$. Hence it follows from Lemma 7.11 that $\dim_{\mathbb{C}} H^1(X_t, \mathcal{E}_t)$ does not jump at $t = 0$ along any deformation of $(X, E)$.

Recall that the Atiyah exact sequence of $L$ gives
\[ 0 \to H^1(X, \mathcal{E}_L) \to H^1(X, T_X) \to H^2(X, \text{End}(E)) \cong \mathbb{C} \to 0. \]
Since $c_1(L) \neq 0$, the connecting homomorphism $\delta : H^1(X, T_X) \to H^2(X, \text{End}(E)) \cong \mathbb{C}$ is surjective and so $\dim \ker(\delta) = 19$. Since $H^0(X, T_X) = 0$, for this 19-dimensional space, the map $\pi^* : H^1(X, \mathcal{E}_L) \to \ker(\delta)$ is an isomorphism. Since line bundles are always good, Proposition 7.6 can be applied again to conclude that the pair $(X, L)$ admits unobstructed deformations over a 19-dimensional space $\text{Def}^{19}(X)$ whose tangent space is given by $\ker(\delta)$. Hence we obtain a family $\{(X_t, L_t)\}_{t \in \text{Def}^{19}(X)}$.

Note that $c_1(L_t) \neq 0$ for all small $|t|$. Indeed, since $c_1(L) \neq 0$ and $\text{Pic}(X) \cong H^{1,1}(X) \cap H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{Z})$, $L$ is non-trivial as a smooth line bundle over $X$. Hence $L_t$ is also a non-trivial smooth line bundle over $X_t$ and so cannot be holomorphically trivial. Since for $|t|$ small, $\text{Pic}(X_t) \cong H^{1,1}(X_t) \cap H^2(X_t, \mathbb{Z})$, we must have $c_1(L_t) \neq 0$.

Now in the 19-dimensional space $\text{Def}^{19}(X)$, we have a deformation of $E$ given by $E_t = T_{X_t} \otimes L_t$, $t \in \text{Def}^{19}(X)$, where $L_t$ is constructed above. This gives an inclusion $\text{Def}^{19}(X) \subset \text{Def}(X, E)$. Consider the Atiyah sequence of $E_t$:
\[ 0 \to \text{End}(E_t) \to \mathcal{E}_t \to T_{X_t} \to 0. \]
The vanishing of $H^0(X_t, T_{X_t}) \cong H^{n-1,0}(X_t)$ and the surjectivity of $\delta_t : H^1(X_t, T_{X_t}) \to H^2(X_t, \text{End}(E_t)) \cong \mathbb{C}$ imply that for all $t \in \text{Def}^{19}(X)$, we have the following exact sequence
\[ 0 \to H^1(X_t, \text{End}(E_t)) \to H^1(X_t, \mathcal{E}_t) \to \ker(\delta_t) \to 0, \]
where $\ker(\delta_t) \subset H^1(X_t, T_{X_t})$ with $\dim_{\mathbb{C}} \ker(\delta_t) = 19$. Since $\dim_{\mathbb{C}} H^1(X_t, \mathcal{E}_t)$ does not jump at $t = 0$, so is $\dim_{\mathbb{C}} H^1(X_t, \text{End}(E_t)) = \dim_{\mathbb{C}} H^1(X, \text{End}(T_{X_t}))$ for $t \in \text{Def}^{19}(X)$.

Finally, it remains to see that $\text{Def}^{19}(X)$ is precisely the algebraic deformation space $\text{Def}^{\text{alg}}(X)$ of $X$. Recall that we have a family of holomorphic line bundles $L_t$ over $X_t$ with $c_1(L_t) \neq 0 \in H^{1,1}(X_t) \cap H^2(X_t, \mathbb{Z})$. Hence the rank of the Picard group Pic$(X)$ is bigger than or equal to 1. Therefore, the K3 surfaces $X_t$ must all be algebraic. Hence we obtain an embedding
\[ \text{Def}^{19}(X) \subset \text{Def}^{\text{alg}}(X). \]
Since both $\text{Def}^{19}(X)$ and $\text{Def}^{\text{alg}}(X)$ are 19-dimensional, we see that we must have $\text{Def}^{19}(X) = \text{Def}^{\text{alg}}(X).$ \qed
APPENDIX A. COMPARISON WITH THE ALGEBRAIC APPROACH

The aim of this appendix is to give an explicit comparison between the analytic approach we adopt here and the classical algebraic approach (see the book [19] for the deformation theory of \((X, L)\) where \(L\) is a holomorphic line bundle on \(X\), and the thesis [15] for the general case).

We start with a definition

**Definition A.1.** A differential operator of order 1 on a vector bundle \(E\) is a linear map \(P : \Omega^0(E) \to \Omega^0(E)\) such that locally,

\[
P = (g_{ij}) + \sum_k h^k_{ij} \frac{\partial}{\partial x^k},
\]

with \((g_{ij})\) be a matrix with entries in \(\mathcal{O}_X(U_\alpha)\) and \(h^k_{ij} \in \mathcal{O}_X(U_\alpha)\).

A differential operator of order 1 is said to be with scalar principle symbol if \(h^k_{ij} = h^k \cdot I\).

In the algebraic approach, the role of the Atiyah extension \(E\) is replaced by the sheaf \(D^1(E)\) of scalar differential operators of order less than 1 on \(E\), namely, we have an exact sequence

\[
0 \to \End(E) \to D^1(E) \to T_X \to 0,
\]

where the surjective map \(\sigma : D^1(E) \to T_X\) is locally defined by the symbol

\[
\sigma(P) = \sum_k h^k \frac{\partial}{\partial x^k}.
\]

There is an obvious identification of \(D^1(E)\) with \(E\) as smooth vector bundles, but we will see that in fact \(D^1(E)\) can be given a holomorphic structure so that \(D^1(E)\) and \(E\) are isomorphic as holomorphic vector bundles.

First of all, locally on an open set \(U_\alpha\), we can write

\[
P|_{U_\alpha} = g_\alpha + d_\alpha.
\]

Let \(e_\alpha\) be local sections of \(E\), \(\{f_{\alpha\beta}\}\) be holomorphic transition functions of \(E\) and \(P_\alpha := P|_{U_\alpha}(e_\alpha)\). To get a global differential operator, we must have

\[
f_{\alpha\beta}P_\beta = P_\alpha f_{\alpha\beta}.
\]

Hence

\[
g_\beta = f_{\beta\alpha}g_\alpha + f_{\beta\alpha}(d_\alpha f_{\beta\alpha}), \quad d_\alpha = d_\beta.
\]

Define

\[
\tau_\alpha := g_\alpha - d_\alpha \cdot (\bar{h}_\alpha^{-1} \partial \bar{h}_\alpha),
\]

where \(h_\alpha\) is the Hermitian metric on \(E|_{U_\alpha}\).

**Lemma A.2.** The collection \(\{\tau_\alpha\}\) defines an endomorphism on \(E\). Moreover, \(\{\tau_\alpha\}\) is \(\partial E\)-closed.

**Proof.** We need to prove that

\[
f_{\alpha\beta} \tau_\beta = \tau_\alpha f_{\alpha\beta}.
\]
Using $h_\beta = f_{\beta\alpha} h_\alpha f_{\alpha\beta}$, we compute that
\[ f_{\alpha\beta} \tau_\beta = f_{\alpha\beta} (f_{\beta\alpha} g_\alpha f_{\alpha\beta} + f_{\beta\alpha} (d_\alpha f_{\beta\alpha}) - d_\beta (\bar{h}_\beta^{-1} \partial \bar{h}_\beta)) \]
\[ = g_\alpha f_{\alpha\beta} + d_\alpha f_{\beta\alpha} - d_\beta (\bar{h}_\beta^{-1} \partial \bar{h}_\beta)) \]
\[ = \tau_\alpha f_{\alpha\beta}, \]
as desired. Finally,
\[ \partial_\xi \tau_\alpha = \partial_Q g_\alpha - \partial_T \mathcal{D} \partial (\bar{h}_\alpha^{-1} \partial \bar{h}_\alpha) + d_\alpha \partial (\bar{h}_\alpha^{-1} \partial \bar{h}_\alpha) - d_\alpha \mathcal{D} F_\mathcal{D} = 0, \]
since $g_\alpha$ is a matrix with entries in $\mathcal{O}_X(U_\alpha)$, $d_\alpha$ is a holomorphic vector and $F_\mathcal{D} = \partial (\bar{h}_\alpha^{-1} \partial \bar{h}_\alpha)$.

Therefore, the map defined by
\[ \Phi : g_\alpha + d_\alpha \longmapsto (g_\alpha - d_\alpha \bar{h}_\alpha^{-1} \partial \bar{h}_\alpha, d_\alpha) \]
is an isomorphism between $D^1(E)$ and $\mathcal{E}$. So we can give $D^1(E)$ a holomorphic structure by pulling back that on $\mathcal{E}$ via $\Phi$. Hence we obtain

**Proposition A.3.** $D^1(E)$ carries a natural holomorphic structure so that it is isomorphic to the Atiyah extension $\mathcal{E}$. In particular, $H^p(X, D^1(E)) \cong H^p(X, \mathcal{E})$ for any $p$.

Together with the Lie bracket [15]
\[ [\omega \otimes P, \eta \otimes Q] := \omega \wedge \eta \otimes [P, Q] + \omega \wedge \mathcal{L}_{\sigma(P)} \eta \otimes Q - (-1)^{|\omega||\eta|} \eta \wedge \mathcal{L}_{\sigma(Q)} \omega \otimes P, \]
the triple $(\Omega^{0,\ast}(D^1(E)), \partial, [-, -])$ forms a DGLA. Note that the Lie derivative acts by
\[ \mathcal{L}_X \omega = d(i_X \omega) + i_X d\omega = i_X \partial \omega, \]
for any $\omega \in \Omega^{0,\ast}(X)$ and $X \in \Omega^{\mathbf{g}}(T_X)$.

**Theorem A.4.** The isomorphism $\Phi : D^1(E) \to \mathcal{E}$
\[ \Phi : g_\alpha + d_\alpha \longmapsto g_\alpha - \bar{h}_\alpha^{-1} d_\alpha \bar{h}_\alpha \]
intertwines with the brackets $[-, -]$ and $[-, -]_h$, i.e.
\[ \Phi[\varphi \otimes P, \psi \otimes Q] = [\varphi \otimes \Phi(P), \psi \otimes \Phi(Q)]_h. \]

**Proof.** We first prove that
\[ \Phi[P, Q] = [\Phi(P), \Phi(Q)]_h. \]
Write $P = g + d$ and $Q = g' + d'$ locally in a coordinate neighborhood $U \subset X$. Then
\[ [P, Q] = [g, g'] + dg' - d'g + [d, d'] \]
and so
\[ \Phi[P, Q] = ([g, g'] + dg' - d'g - \bar{h}_\alpha^{-1}[d, d'] \bar{h}_\alpha, [d, d']). \]
On the other hand,
\[ [\Phi(P), \Phi(Q)]_h = (\nabla^Q_d (g' - \bar{h}_\alpha^{-1} d' \bar{h}) - \nabla^Q_d (g - \bar{h}_\alpha^{-1} d \bar{h}) + [g - \bar{h}_\alpha^{-1} d', g - \bar{h}_\alpha^{-1} d \bar{h}], [d, d']) \]
Now, we compute
\[\nabla^Q_d (g' - \tilde{h}^{-1}d'\tilde{h}) - \nabla^Q_d (g - \tilde{h}^{-1}d\tilde{h}) = d(g' - \tilde{h}^{-1}d'\tilde{h}) + [\tilde{h}^{-1}d\tilde{h}, g' - \tilde{h}^{-1}d'\tilde{h}] - d' (g + \tilde{h}^{-1}d\tilde{h}) - [\tilde{h}^{-1}d\tilde{h}, g - \tilde{h}^{-1}d\tilde{h}] = dg' - d'g + [\tilde{h}^{-1}d\tilde{h}, g'] - [\tilde{h}^{-1}d'\tilde{h}, g] - \tilde{d}^{-1}h^{-1}d'\tilde{h} + d\tilde{h}^{-1}d\tilde{h} - 2[\tilde{h}^{-1}d\tilde{h}, \tilde{h}^{-1}d'\tilde{h}] \]
and
\[[g - \tilde{h}^{-1}d\tilde{h}, g' - \tilde{h}^{-1}d'\tilde{h}] = [g, g'] - [g, \tilde{h}^{-1}d'\tilde{h}] - [\tilde{h}^{-1}d\tilde{h}, g'] + [\tilde{h}^{-1}d\tilde{h}, \tilde{h}^{-1}d'\tilde{h}]\]
Therefore, their sum equals to
\[[g, g'] + dg' - d'g - \tilde{h}^{-1}d'\tilde{h} + d\tilde{h}^{-1}d\tilde{h} - [\tilde{h}^{-1}d\tilde{h}, \tilde{h}^{-1}d'\tilde{h}]\]
Finally,
\[[\tilde{h}^{-1}d\tilde{h}, \tilde{h}^{-1}d'\tilde{h}] = \tilde{h}^{-1}(d\tilde{h})\tilde{h}^{-1}(d'\tilde{h}) - \tilde{h}^{-1}(d'\tilde{h})\tilde{h}^{-1}(d\tilde{h}) = -(d\tilde{h}^{-1})(d'\tilde{h}) + (d'\tilde{h}^{-1})(d\tilde{h}) = -d(\tilde{h}^{-1}d'\tilde{h}) + \tilde{h}^{-1}d'd\tilde{h} - \tilde{h}^{-1}d'\tilde{h} = -d(\tilde{h}^{-1}d'\tilde{h}) + d'(\tilde{h}^{-1}d\tilde{h}) + \tilde{h}^{-1}[d, d']\tilde{h}.
Hence
\[\nabla^Q_d (g' - \tilde{h}^{-1}d'\tilde{h}) - \nabla^Q_d (g - \tilde{h}^{-1}d\tilde{h}) = [g, g'] + dg' - d'g - \tilde{h}^{-1}[d, d']\tilde{h},\]
which is the required equality.

To prove the general case, we have, by linearity and the case that we have proved, the End(E)-part of \(\Phi[\omega \otimes P, \eta \otimes Q]\) is equal to
\[\omega \wedge \eta \otimes [\tau(P), \tau(Q)]_h - \omega \wedge \eta \otimes [\tau(P), \sigma(Q)]_h + \omega \wedge \eta \otimes [\sigma(P), \tau(Q)]_h + \omega \wedge \eta \otimes [\sigma(P), \tau(Q)],\]
\(+ \omega \wedge L_{\tau(P)}\eta \otimes \tau(Q) - (-1)^{|\omega||\eta|}\eta \wedge L_{\sigma(Q)}\omega \otimes \tau(P),\)
where \(\tau(P) := pr_{\text{End}(E)} \circ \Phi(P).\) On the other hand, the End(E)-part of \([\Phi(\omega \otimes P), \Phi(\eta \otimes Q)]_h\) is equal to
\([(\omega \otimes \sigma(P)) \otimes \nabla^Q(\eta \otimes \tau(Q)) - (-1)^{|\omega||\eta|}\eta \otimes \sigma(Q)) \otimes \nabla^Q(\omega \otimes \tau(P)) + \omega \wedge \eta \otimes [\tau(P), \tau(Q)]_h,\)
The Leibniz rule for connections implies that
\[(\omega \otimes \sigma(P)) \otimes \nabla^Q(\eta \otimes \tau(Q)) = \omega \wedge L_{\sigma(P)}\eta \otimes \tau(Q) + \omega \wedge \eta \otimes \nabla^Q(P)\tau(Q) = \omega \wedge L_{\sigma(P)}\eta \otimes \tau(Q) + \omega \wedge \eta \otimes [\sigma(P), \tau(Q)].\]
Similarly, we have
\[(\eta \otimes \sigma(Q)) \otimes \nabla^Q(\omega \otimes \tau(P)) = \eta \wedge L_{\tau(Q)}\omega \otimes \tau(P) + (-1)^{|\omega||\tau|}\omega \wedge \eta \otimes [\sigma(Q), \tau(P)].\]
Putting these back into \([\Phi(\omega \otimes P), \Phi(\eta \otimes Q)]_h,\) we get
\[\Phi[\omega \otimes P, \eta \otimes Q] = [\Phi(\omega \otimes P), \Phi(\eta \otimes Q)]_h.\]
This proves our theorem. \(\square\)

**Remark A.5.** This theorem gives a proof of the required identities in Propositions 3.15 and 3.18 and the fact that the DGLA \((\Omega^0(\mathcal{E}), d_E, [-,-]_h)\) is independent of the choice of the hermitian metric \(h.\)
References

1. M. F. Atiyah, Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc. 85 (1957), 181–207. MR 0086359 (19,172c)
2. A. Beauville, Complex algebraic surfaces, second ed., London Mathematical Society Student Texts, vol. 34, Cambridge University Press, Cambridge, 1996. MR 1406314 (97e:14045)
3. A. Douglis and L. Nirenberg, Interior estimates for elliptic systems of partial differential equations, Comm. Pure Appl. Math. 8 (1955), 503–538. MR 0075417 (17,743b)
4. H. Grauert, Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen, Inst. Hautes Études Sci. Publ. Math. (1960), no. 5, 64. MR 0121814 (22 #12544)
5. P. Griffiths and J. Harris, Principles of algebraic geometry, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994, Reprint of the 1978 original. MR 1288523 (95d:14001)
6. L. Huang, On joint moduli spaces, Math. Ann. 302 (1995), no. 1, 61–79. MR 1329447 (96d:32021)
7. D. Huybrechts, The tangent bundle of a Calabi-Yau manifold—deformations and restriction to rational curves, Comm. Math. Phys. 171 (1995), no. 1, 139–158. MR 1341697 (96g:14032)
8. D. Huybrechts, Complex geometry, Universitext, Springer-Verlag, Berlin, 2005, An introduction. MR 2093043 (2005h:32052)
9. D. Huybrechts and R. P. Thomas, Deformation-obstruction theory for complexes via Atiyah and Kodaira-Spencer classes, Math. Ann. 346 (2010), no. 3, 545–569. MR 2578562 (2011b:14030)
10. K. Kodaira and J. Morrow, Complex manifolds, AMS Chelsea Publishing, Providence, RI, 2006, Reprint of the 1971 edition with errata. MR 2214741 (2006d:32001)
11. K. Kodaira and D. C. Spencer, On deformations of complex analytic structures. I, II, Ann. of Math. (2) 67 (1958), 328–466. MR 0112154 (22 #3009)
12. K. Kodaira, On deformations of complex analytic structures. III. Stability theorems for complex structures, Ann. of Math. (2) 71 (1960), 43–76. MR 0115189 (22 #5991)
13. M. Kuranishi, New proof for the existence of locally complete families of complex structures, Proc. Conf. Complex Analysis (Minneapolis, 1964), Springer, Berlin, 1965, pp. 142–154. MR 0176496 (31 #768)

14. S. Li, On the deformation theory of pair (X, E), preprint, [arXiv:0809.0344]
15. E. Martinecgo, Higher brackets and moduli space of vector bundles, Ph.D. thesis, Sapienza Università di Roma, 2009.
16. A. Moroianu, Lectures on Kähler geometry, London Mathematical Society Student Texts, vol. 69, Cambridge University Press, Cambridge, 2007. MR 2325063 (2008f:32028)
17. A. Newlander and L. Nirenberg, Complex analytic coordinates in almost complex manifolds, Ann. of Math. (2) 65 (1957), 391–404. MR 0088770 (19,577a)
18. X. Pan, Deformations of polarized manifolds with torsion canonical bundles, preprint, [arXiv:1310.7162]
19. E. Sernesi, Deformations of algebraic schemes, Grundlehren der Mathematischen Wissenschaften, vol. 334, Springer-Verlag, Berlin, 2006. MR 2247663 (2008e:14011)
20. Y.-T. Siu and G. Trautmann, Deformations of coherent analytic sheaves with compact supports, Mem. Amer. Math. Soc. 29 (1981), no. 238, iii+155. MR 609018 (82c:32023)
21. G. Tian, Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric, Mathematical aspects of string theory (San Diego, Calif., 1986), Adv. Ser. Math. Phys., vol. 1, World Sci. Publishing, Singapore, 1987, pp. 629–646. MR 915841
22. A. Todorov, The Weil-Petersson geometry of the moduli space of SU(n ≥ 3) (Calabi-Yau) manifolds. I, Comm. Math. Phys. 126 (1989), no. 2, 325–346. MR 1027500 (91f:32022)

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: kwchan@math.cuhk.edu.hk

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
E-mail address: yhsuen@math.cuhk.edu.hk