ON CLASSICAL SOLUTIONS OF TIME-DEPENDENT FRACTIONAL MEAN FIELD GAME SYSTEMS

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Abstract. In this paper we study parabolic Mean Field Game systems with nonlocal/fractional diffusion. Such models come from games where the noise is non-Gaussian and the resulting controlled diffusion process anomalous. Here the noise is modeled by pure jump Levy processes that are \( \sigma \)-stable like. The corresponding diffusion operators include the fractional Laplacians \((-\Delta)_{\sigma}^{2}\), the generators of tempered stable and non-symmetric processes widely used in Finance, and sums of such operators. Our main result is existence and uniqueness of classical solutions of Mean Field Game systems for \( \sigma \in (1, 2) \). This corresponds to the subcritical or strong diffusion case where the equations are nondegenerate. We consider both local and nonlocal couplings. The proof we give is an extension of the fixed point argument introduced by P.-L. Lions. The new ingredients are fractional heat kernel estimates, regularity results for fractional Bellman and Fokker-Planck equations, and results on (very) weak solutions of fractional Fokker-Planck equations.

1. Introduction

In this paper we study fractional Mean Field Game (MFG) systems of the form

\[
\begin{align*}
-\partial_t u - \mathcal{L} u + H (x, u, Du) &= F (x, m (t)) & \text{in} \; (0, T) \times \mathbb{T}^d, \\
\partial_t m - \mathcal{L}^* m - \text{div} (m D_p H (x, u, Du)) &= 0 & \text{in} \; (0, T) \times \mathbb{T}^d, \\
m (0, x) &= m_0 (x), & u (x, T) &= G (x, m (T)),
\end{align*}
\]

where \( \mathbb{T}^d = \mathbb{R}^d \setminus \mathbb{Z}^d \) is the torus, \( H \) the (nonlinear) Hamiltonian, \( F \) and \( G \) source term and terminal condition, and \( m_0 \) an initial condition. Furthermore, \( \mathcal{L} \) and its adjoint \( \mathcal{L}^* \) are fractional diffusion operators of the form

\[
\mathcal{L} u (x) = \int_{\mathbb{R}^d} u (x + z) - u (x) - D u (x) \cdot z 1_{|z| < 1} \, d \mu (z),
\]

where \( \mu \) is a nonnegative Radon measure satisfying the Levy-condition \( \int_{\mathbb{R}^d} 1 \land |z|^2 \, d \mu (z) < \infty \) that is somehow comparable to the Levy measure of \( \sigma \)-stable processes \( K |z|^{-d-\sigma} \, dz \) for \( \sigma \in (1, 2) \) (see (A0) and (A0') below). This defines a large class of uniformly elliptic operators \( \mathcal{L} \) and includes fractional Laplacians, generators of processes used in Finance.
which typically are both tempered and nonsymmetric, and many others including operators with non-absolutely continuous Lévy measures.

Problem (1) is posed on the torus, or equivalently, on $[0,1]^d$ with periodic boundary conditions. It consists of a backward in time fractional Hamilton-Jacobi-Bellman (HJB) equation and a forward in time fractional Fokker-Planck (FP) equation. Since $\sigma \in (1,2)$ (from (A0') or (A0)), the system is uniformly parabolic and expected to have smooth solutions. The system is coupled through the $F$ and $G$ terms, and we consider two different types of couplings:

(i) **Nonlocal coupling** where $F, G : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ and the input is a probability density function $m(\cdot, t)$.

(ii) **Local coupling** where $F : \mathbb{T}^d \times [0, +\infty) \to \mathbb{R}$ and the input is a real number $m(x,t)$.

In this case we assume no coupling through $G$.

In the first case we write $F = F(x, m(t))$ and assume that $F$ is a smoothing operator on $m$, while in the second case we use the notation $F = f(x, m(t, x))$.

**Main results.** Under structure and regularity assumptions on $F, G, H, m_0$, we show:

(i) **Existence of smooth solutions of (1)** for nonlocal and local coupling (Theorems 2.5 and 7.1).

(ii) **Uniqueness of smooth solutions of (1)** for nonlocal and local coupling (Theorems 2.6 and 7.3).

The proofs follow the PDE-approach of Lions [22, 7], and existence and regularity is much more involved than uniqueness. Existence for MFG with nonlocal coupling is proved using a fixed point argument to decouple the two equations, an argument that crucially relies on new high order regularity estimates for fractional Hamilton-Jacobi-Bellman and Fokker Planck equations. These estimates are obtained from very general fractional heat kernel estimates in combination with semigroup/Duhamel representation formulas for the solutions in the spirit of [12, 13, 18]. The regularity results are of independent interest.

Existence for MFG with local coupling follows from an approximation procedure, the results for nonlocal coupling, and a bootstrapping argument. This result is conditional in the sense that it relies on a uniform $L^\infty$ bound on the approximation of $m$ which we take as an assumption in this paper. Such bounds holds for local equations [5], when $\mathcal{L} = -(\Delta)^{\frac{\alpha}{2}}$ [14], and are expected to hold under the assumption of this paper. We refer to Section 7 for a discussion on this point.

Mean Field Games is a relatively new field of mathematics and was introduced more or less at the same time by Lasry and Lions [21] and Caines, Huang and Malhamé [17], and has a wide range of applications for example in finance. One of the motivations behind MFG is to approximate Nash equilibria of $N$-player differential games when the number of agents $N$ is large [8]. Heuristically the MFG can be understood in the following way: A large number of identical players wants to minimize some cost functional that depends on their own state and the distribution of the states of the others. The mean field game system arise when the players are in a Nash equilibrium and the number of players tends to
infinity. In this paper each player controls a stochastic differential equation (SDE),
\[ dX_t = \alpha_t dt + dL_t, \]
with the aim of minimizing the cost functional
\[ \mathbb{E} \left[ \int_0^T \left( L(X_s, \alpha_s) + F(X_s, m(s)) \right) ds + G(X_T, m(T)) \right] \]
with respect to the control \( \alpha_s \). The noise \( L_t \) is a \( \sigma \)-stable like pure jump Lévy process, \( L \) is the Legendre transform of \( H \) with respect to the second variable, \( F \) and \( G \) are running and terminal costs, and \( m \) the density or distribution of the states of the other players.

If \( u \) is the value function of the player, then formally the optimal feedback control will be
\[ \alpha_t^* = -D_2H(x, Du) \]
and \( u \) satisfy the HJB equation in (1). The probability density function \( \tilde{m}(t) \) of the optimally controlled \( X_t^* \) will then satisfy the Fokker-Planck equation in (1). Since the players are identical (and under symmetry conditions on the couplings), the density function \( m \) for all players will be equal to \( \tilde{m} \) and hence also satisfies the Fokker-Planck equation in (1). This is a heuristic explanation for (1).

In our case, what differs from the standard MFG formulation is the type of noise used in the model. In many real world applications, jump processes or anomalous diffusions will better model the observed noise than Gaussian processes [23, 11, 26, 2]. Note that in one of the largest application areas, Finance, the observed jump processes are not symmetric and \( \sigma \)-stable and hence to not correspond to fractional Laplacians. Rather they are typically non-symmetric and tempered. Tempered means that the Levy measure decays exponentially at infinity. A typical example is the one-dimensional CGMY process [11] where
\[ \frac{d\mu}{dz}(z) = \frac{C}{|z|^\{1+Y\}} e^{-Gz^+ - Mz^-} \]
for \( C, G, M > 0 \) and \( Y \in (0, 2) \). Such models are covered in the results of this paper. Our results also cover anisotropic operators with different powers \( \sigma \) in different directions and the Riesz-Feller operators. We refer to Section 3 for a discussion, results, and examples.

There is already some work on MFG with fractional diffusion. In [9] the authors analyze a stationary MFG system with fractional Laplace diffusion and both non-local and local couplings. Well-posedness of time-fractional MFG systems, i.e. systems with with fractional time-derivatives, are studied in [6]. Fractional parabolic Bertrand and Carnout MFGs are studied in the recent paper [15]. These one space dimensional equations have a different and more complicated structure than ours, and the principal terms are the local second order terms. Moreover, during the rather long preparation of this paper we learned that M. Cirant and A. Goffi were working on a similar problem. Their results have now been published in [10]. A major technical difference compared to our paper is the additional convexity and coercivity assumptions on \( H \) which gives semiconcavity and gradient bounds for solutions of the fractional HJB equation that are independent of the diffusion. In particular, the \( H(x, Du) \)-term is bounded, weak energy solutions well-defined also for \( \sigma \in (0, 1] \), and vanishing viscosity and the uniform gradient bounds give existence for all \( \sigma \in (0, 2) \)! Nice and precise regularity results are given in terms of Bessel potentials and Hölder spaces. Our existence results cover also the case when \( H \) is not convex or coercive at the price of stronger smoothness assumptions, and more importantly, a stronger reliance on uniform ellipticity.
and then the restriction to \( \sigma \in (1, 2) \). We give results for the local coupling case, and as mentioned above, we cover many other diffusion operators than the fractional Laplacians, including tempered and non-symmetric operators that are popular in Finance. Because our setup is different, most of our proofs and arguments are quite different those in [10].

This paper is organized as follows: We state our assumptions and existence and uniqueness results for systems with nonlocal coupling in Section 2. To prove these results, we first establish fractional heat kernel estimates in Section 3. Using these estimates and Duhamel representation formulas, we prove regularity results for fractional Hamilton-Jacobi equations in Section 4. In Section 5 we establish results for fractional Fokker-Planck equations, both regularity of classical solutions and estimates on (very) weak solutions. In Section 6 we prove the existence result for nonlocal coupling. The local coupling case is treated in Section 7, and finally, in Appendix A, we prove a technical lemma.

2. Results for fractional MFG systems with nonlocal coupling

Here we state our assumptions and the existence and uniqueness results for classical solutions of the system (1) with nonlocal coupling. Note that throughout this paper we identify functions on the torus \( \mathbb{T}^d \) with their periodic extensions to \( \mathbb{R}^d \).

The fractional operator \( \mathcal{L} \) in (2). We use the following assumptions:

(A0’) (Levy condition) \( \mu \geq 0 \) is a Radon measure satisfying \( \int_{\mathbb{R}^d} 1 \wedge |z|^2 \, d\mu(z) < \infty \).

(Uniform ellipticity) There are constants \( \sigma \in (1, 2) \) and \( C > 0 \) such that

\[
\frac{1}{C} \frac{1}{|z|^{d+\sigma}} \leq \frac{d\mu}{dz} \leq C \frac{1}{|z|^{d+\sigma}} \quad \text{for } |z| \leq 1.
\]

This assumption is satisfied by generators \( \mathcal{L} \) for pure jump stochastic processes whose infinite activity part is close to \( \alpha \)-stable. Some examples are \( \alpha \)-stable processes, tempered \( \alpha \)-stable processes, and the nonsymmetric CGMY process in Finance [11, 2]. A more general condition is:

(A0) \( \mu \geq 0 \) is a Radon measure satisfying \( \int_{\mathbb{R}^d} 1 \wedge |z|^2 \, d\mu(z) < \infty \).

There are \( \sigma \in (1, 2) \) and \( K > 0 \) such that the heat kernels \( K_{\sigma} \) and \( K_{\sigma}^* \) of \( \mathcal{L} \) and \( \mathcal{L}^* \) satisfy for \( K = K_{\sigma}, K_{\sigma}^* : K \geq 0, ||K(t, \cdot)||_{L^1(\mathbb{R}^d)} = 1, \) and

\[
||D^\beta K(t, \cdot)||_{L^1(\mathbb{R}^d)} \leq Kt^{-\frac{|\beta|}{d}} \quad \text{for } t \in (0, T)
\]

and any multi-index \( \beta \in \mathbb{N}_0^d \) where \( D \) is the gradient in \( \mathbb{R}^d \).

As an example, we will prove that \( \mathcal{L} = -\left(-\frac{\partial^2}{\partial x^2}\right)^{\frac{\sigma_1}{2}} \cdots -\left(-\frac{\partial^2}{\partial x^2}\right)^{\frac{\sigma_d}{2}}, \sigma_1, \ldots, \sigma_d \in (1, 2), \) satisfies (A0) with \( \sigma = \min_i \sigma_i \). The heat kernel is a transition probability and fundamental solution. See Section 3 for the precise definition, a proof that (A0’) implies (A0), examples, and further extensions.
Definition 2.1. (Adjoint). The adjoint of $\mathcal{L}$ is the operator $\mathcal{L}^*$ such that
\[ \langle \mathcal{L}f, g \rangle_{L^2(\mathbb{T}^d)} = \langle f, \mathcal{L}^*g \rangle_{L^2(\mathbb{T}^d)} \quad \text{for all} \quad f, g \in C^2(\mathbb{T}^d). \]

Lemma 2.2. The adjoint operator $\mathcal{L}^*$ is given by
\[ \mathcal{L}^*u(x) = \int_{\mathbb{R}^d} u(x + z) - u(x) - Du(x) \cdot z 1_{|z| < 1} \, dz, \]
where $\mu^*(B) = \mu(-B)$ for all Borel sets $B \subset \mathbb{R}^d$.

Note that $\mathcal{L}^*$ is an operator of the same form as $\mathcal{L}$, but with Levy measure $\mu^*$.

Proof. Let $f, g \in C^2(\mathbb{T}^d)$ and $r > 0$. By Fubini,
\[
\int_{\mathbb{T}^d} \left[ \int_{|z| > r} f(x + z) - f(x) - Df(x) \cdot z 1_{|z| < 1} \, d\mu(z) \right] g(x) \, dx
= \int_{|z| > r} \int_{\mathbb{T}^d} f(x + z) g(x) \, dx \, d\mu(z) - \int_{\mathbb{T}^d} \int_{|z| > r} f(x) g(x) \, d\mu(z) \, dx
- \int_{|z| > r} \int_{\mathbb{T}^d} \sum_{i=1}^d f_i(x) z_i g(x) \, dx \, d\mu(z).
\]
We change variables $x' = x + z$ in the first integral, use integration by parts in the last, substitute $z' = -z$ in all integrals, and send $r \to 0$. Since $\mu^*(B) = \mu(-B)$, we find that
\[
\langle \mathcal{L}f, g \rangle_{L^2(\mathbb{T}^d)} = \lim_{r \to 0} \int_{\mathbb{T}^d} f(x) \left[ \int_{|z| > r} g(x + z) - g(x) - Dg(x) \cdot z \, d\mu(-z) \right] \, dx = \langle f, \mathcal{L}^*g \rangle.
\]
\]

It immediately follows that Assumption (A0') also holds for $\mu^*$. We have the following a priori bounds for the operators $\mathcal{L}$ and $\mathcal{L}^*$.

Lemma 2.3. ($L^p$-bounds) If (A0') holds, then for all $p \in [1, \infty]$, $0 < \sigma < 2$, and $r \in (0, 1)$,
\[
\|\mathcal{L}u\|_{L^p(\mathbb{T}^d)} \leq C (\|D^2 u\|_{L^p} r^{2-\sigma} + \|Du\|_{L^p} \Gamma(\sigma, r) + \|u\|_{L^p} r^{-\sigma})
\]
where
\[
\Gamma(\sigma, r) = \begin{cases} 1, & 0 < \sigma < 1, \\ |\ln r|, & \sigma = 1, \\ r^{1-\sigma}, & 1 < \sigma < 2. \end{cases}
\]

Proof. We only consider the case $p < \infty$ and split $\mathcal{L}u$ into three parts, $L_1 = \int_{B_r} u(x + z) - u(x) - Du(x) \cdot z \, d\mu(z)$, $L_2 = -\int_{B \setminus B_r} Du(x) \cdot z \, d\mu(z)$, and $L_3 = \int_{\mathbb{R}^d \setminus B_r} u(x + z) - u(x) \, d\mu(z)$.
Using Taylor expansions, Minkowski’s integral inequality, and (A0’,)
\[
\|L_1\|_{L^p(T^d)} \leq \left( \int_{T^d} |D^2u(x)|^p \, dx \right)^{1/p} \int_{B_r} |z|^2 \frac{C}{|z|^{d+\sigma}} \, dz \leq C \|D^2u\|_{L^p(T^d)} r^{2-\sigma},
\]
\[
\|L_2\|_{L^p(T^d)} \leq \left( \int_{T^d} |Du(x)|^p \, dx \right)^{1/p} \int_{B_1 \setminus B_r} |z|^2 \frac{C}{|z|^{d+\sigma}} \, dz \leq C \|Du\|_{L^p(T^d)} \Gamma(\sigma, r),
\]
\[
\|L_3\|_{L^p(T^d)} \leq 2 \left( \int_{T^d} |u(x)|^p \, dx \right)^{1/p} \left( \int_{B_1 \setminus B_r} + \int_{\mathbb{R}^d \setminus B_1} \right) d\mu(z) \leq C \|u\|_{L^p(T^d)} r^{-\sigma}.
\]
Summing these estimates we obtain (3).

Remark 2.4. (a) When \( \mu \) is symmetric, \( \int_{B_1 \setminus B_r} Du(x) \cdot z \, d\mu(z) = 0 \), and we can set \( \Gamma = 0 \) in (3). For the fractional Laplacian where \( d\mu(z) = \frac{1}{|z|^{d+\sigma}} \, dz \), estimate (3) even holds for all \( r > 0 \) (with \( \Gamma = 0 \)). Minimizing with respect to \( r \), we then obtain
\[
\|(-\Delta)^{\sigma/2}u\|_{L^p(T^d)} \leq C \|D^2u\|_{L^p(T^d)}^{\sigma/2} \|u\|_{L^p(T^d)}^{1-\sigma/2}.
\]
(b) When \( \sigma \in (0,1) \), a similar argument shows that \( \|Lu\|_{L^p(T^d)} \leq C \|Du\|_{L^p(T^d)}^{1-\sigma} + \|u\|_{L^p(T^d)}^{-\sigma} \) and we find that \( \|(-\Delta)^{\sigma/2}u\|_{L^p(T^d)} \leq C \|Du\|_{L^p(T^d)}^{\sigma/2} \|u\|_{L^p(T^d)}^{1-\sigma/2} \).

**Fractional MFG with nonlocal coupling.** The system of equations is given by
\[
\begin{cases}
-\partial_t u - Lu + H(x, u, Du) = F(x, m(t)) & \text{in } (0, T) \times T^d, \\
\partial_t m - L^* m - \text{div} (mD_p H(x, u, Du)) = 0 & \text{in } (0, T) \times T^d, \\
m(x, 0) = m_0(x), & u(x, T) = G(x, m(T)) & \text{in } T^d,
\end{cases}
\]
where \( \sigma \in (1,2) \) is fixed, the functions \( F, G : T^d \times P(T^d) \to \mathbb{R} \) are non-local coupling functions, and \( H : T^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) is the Hamiltonian. Here
\[
P(T^d) := \text{the set of Borel probability measures on } T^d,
\]
which is a (compact) metric space with Kantorovich-Rubinstein distance,
\[
d_1(m_1, m_2) = \sup_{\phi \text{ 1-Lipschitz}} \left\{ \int_{T^d} \phi(x)(m_1 - m_2) \, dx \right\},
\]
where \( \phi \text{ 1-Lipschitz} \) means that \( |\phi(x) - \phi(y)| \leq 1|x - y| \). Our assumptions are then:

(A1) \( F \) and \( G \) are continuous.

(A2) There exists a \( C_0 > 0 \) such that for all \( (x_1, m_1), (x_2, m_2) \in T^d \times P(T^d) \):
\[
|F(x_1, m_1) - F(x_2, m_2)| \leq C_0 \left[ |x_1 - x_2| + d_1(m_1, m_2) \right],
\]
\[
|G(x_1, m_1) - G(x_2, m_2)| \leq C_0 \left[ |x_1 - x_2| + d_1(m_1, m_2) \right].
\]
(A3) There exist constants $C_F, C_G > 0$, such that
\[ \sup_{m \in P(\mathbb{T}^d)} \| F (\cdot, m) \|_{C^2_b(\mathbb{T}^d)} \leq C_F \quad \text{and} \quad \sup_{m \in P(\mathbb{T}^d)} \| G (\cdot, m) \|_{W^{3,\infty}(\mathbb{T}^d)} \leq C_G. \]

(A4) For every $R > 0$ there is $C_R > 0$ such that for $x \in \mathbb{T}^d$, $u \in [-R, R], p \in B_R, \alpha \in \mathbb{N}_0^N, |\alpha| \leq 3,$
\[ |D^\alpha H (x, u, p)| \leq C_R. \]

(A5) For every $R > 0$ there is $C_R > 0$ such that for $x, y \in \mathbb{T}^d, u \in [-R, R], p \in \mathbb{R}^d$:
\[ |H (x, u, p) - H (y, u, p)| \leq C_R (|p| + 1) |x - y|. \]

(A6) There exists $\gamma \in \mathbb{R}$ such that for all $x \in \mathbb{T}^d, u, v \in \mathbb{R}, u \leq v, p \in \mathbb{R}^d$,
\[ H (x, v, p) - H (x, u, p) \geq \gamma (v - u). \]

(A7) The probability measure $m_0$ is absolutely continuous with respect to the Lebesgue measure with a density still denoted $m_0 \in W^{2,\infty}(\mathbb{T}^d)$.

**Theorem 2.5.** *(Existence of classical solution)* Assume (A0), (A1)–(A7). Then there exists a classical solution $(u, m)$ of (5).

The proof will given in Section 6. It is an extension of the fixed point argument of P.-L. Lions [22, 7] and requires a series of estimates for fractional Hamilton-Jacobi and Fokker-Planck equations given in Sections 4 and 5. For uniqueness, we add the following assumptions:

(A8) $F$ and $G$ satisfy monotonicity conditions:
\[ \int_{\mathbb{T}^d} (F (x, m_1) - F (x, m_2)) d (m_1 - m_2) (x) \geq 0 \quad \forall m_1, m_2 \in P(\mathbb{T}^d), \]
\[ \int_{\mathbb{T}^d} (G (x, m_1) - G (x, m_2)) d (m_1 - m_2) (x) \geq 0 \quad \forall m_1, m_2 \in P(\mathbb{T}^d). \]

(A9) The Hamiltonian $H = H (x, p)$ and is uniformly convex with respect to $p$:
\[ \exists C > 0, \quad \frac{1}{C} I_d \leq D_{pp}^2 H (x, p) \leq CI_d. \]

**Theorem 2.6.** Assume (A1)-(A9). Then there is at most one classical solution of the MFG system (5).

**Proof.** The proof is essentially the same as the proof in the College de France lectures of P.-L. Lions [22, 7]. Let $(u_1, m_1)$ and $(u_2, m_2)$ be two classical solutions, and set $\tilde{u} = u_1 - u_2$ and $\tilde{m} = m_1 - m_2$. By (5) and integration by parts,
\[ \frac{d}{dt} \int_{T^d} \tilde{u} \tilde{m} \, dx = \int_{T^d} \frac{\partial}{\partial t} (\tilde{u} \tilde{m}) \, dx = \int_{T^d} (\partial_t \tilde{u}) \tilde{m} + \tilde{u} (\partial_t \tilde{m}) \, dx \]

\[ = \int_{T^d} \left[ \left( - \mathcal{L} \tilde{u} + H(x, Du_1) - H(x, Du_2) - F(x, m_1) + F(x, m_2) \right) \tilde{m} \right. \]

\[ + \left. \tilde{u} \mathcal{L}^* \tilde{m} - \langle D \tilde{u}, m_1 D_p H(x, Du_1) - m_2 D_p H(x, Du_2) \rangle \right] \, dx. \]

By the definition of the adjoint, \( \int_{T^d} (\mathcal{L} \tilde{u}) \tilde{m} - \tilde{u} (\mathcal{L}^* \tilde{m}) \, dx = 0 \), and from (A8) we get

\[ \int_{T^d} (-F(x, m_1) + F(x, m_2)) \, d(m_1 - m_2)(x) \geq 0 \quad \forall m_1, m_2 \in P(T^d). \]

For the remaining terms on the right hand side, we use a Taylor expansion and (A9),

\[ \int_{T^d} \left[ -m_1 \left( H(x, Du_1) - H(x, Du_2) - \langle D_p H(x, Du_1), Du_2 - Du_1 \rangle \right) \right. \]

\[ - \left. m_2 \left( H(x, Du_2) - H(x, Du_1) - \langle D_p H(x, Du_2), Du_1 - Du_2 \rangle \right) \right] \, dx \]

\[ \leq - \int_{T^d} \frac{m_1 + m_2}{2C} |Du_2 - Du_1|^2 \, dx. \]

Integrating from 0 to T, using the fact that \( \tilde{m}(t = 0) = 0 \) and \( \tilde{u}(t = T) = G(x, m_1(T)) - G(x, m_2(T)) \),

\[ \int_0^T \frac{d}{dt} \int_{T^d} \tilde{u} \tilde{m} \, dx \, dt = \int_{T^d} (G(x, m_1(T)) - G(x, m_2(T))) (m_1(x, T) - m_2(x, T)) \, dx \geq 0, \]

where we used (A8) again. Combining all the estimates we find that

\[ 0 \leq - \int_0^T \int_{T^d} \frac{m_1 + m_2}{2C} |Du_1 - Du_2|^2 \, dx \, dt \]

Hence since the integrand is nonnegative it must be zero and \( Du_1 = Du_2 \) on the set \( \{ m_1 > 0 \} \cup \{ m_2 > 0 \} \). This means that \( m_1 \) and \( m_2 \) solve the same equation (the divergence terms are the same) and hence are equal by uniqueness (see Lemma 26 in Section 5). Then also \( u_1 \) and \( u_2 \) solve the same equation and \( u_1 = u_2 \) by standard uniqueness for nonlocal HJB equations (see e.g. [19]). \( \square \)

### 3. Fractional heat kernel estimates

Here we introduce fractional heat kernels and prove \( L^1 \)-estimates of their spatial derivatives. These estimates are used for the regularity results of Section 4 and 6. Taking Fourier transform of (2), a direct calculation (see [2]) shows that

\[ \mathcal{F}(\mathcal{L} u) = \hat{\mathcal{L}}(\xi) \hat{u}(\xi), \]
where the symbol or Fourier multiplier is given by
\begin{equation}
\hat{L}(\xi) = \int_{\mathbb{R}^d} \left( e^{ix \cdot \xi} - 1 - i \xi \cdot z 1_{|z| < 1} \right) d\mu(z).
\end{equation}

We can split \( \hat{L} \) into a singular and a non-singular part,
\begin{equation}
\hat{L}(\xi) = \left( \int_{|z| < 1} + \int_{|z| \geq 1} \right) \left( e^{ix \cdot \xi} - 1 - i \xi \cdot z 1_{|z| < 1} \right) d\mu(z) = \hat{L}_s(\xi) + \hat{L}_n(\xi).
\end{equation}

Note that since \( \mu \geq 0, \) \( \text{Re} \hat{L} = \int (\cos(z \cdot \xi) - 1) d\mu \leq 0. \)

The heat kernel of an elliptic operator \( A \) is the fundamental solution of \( u_t - Au = 0. \) We will need the heat kernels \( K_\sigma \) and \( \tilde{K}_\sigma \) of \( L \) and \( L_s: \)
\begin{equation}
K_\sigma(t, x) = \mathcal{F}^{-1} (e^{t \mathcal{L}(\cdot)}) \quad \text{and} \quad \tilde{K}_\sigma(t, x) = \mathcal{F}^{-1} (e^{t \mathcal{L}_s(\cdot)}).
\end{equation}

By the Levy-Kinchine theorem (Theorem 1.2.14 in [2]), \( K_\sigma \) and \( \tilde{K}_\sigma \) are probability measures for \( t > 0: \)
\begin{equation}
K_\sigma, \tilde{K}_\sigma \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^d} K_\sigma(x, t) \, dx = 1 = \int_{\mathbb{R}^d} \tilde{K}_\sigma(x, t) \, dx.
\end{equation}

When (A0') holds, \( \text{Re} \tilde{L} \) and \( \text{Re} \hat{L}_s \leq -c|\xi|^\sigma \) for \( |\xi| \geq 1, \) and \( K_\sigma \) and \( \tilde{K}_\sigma \) are absolutely continuous since \( |e^{t\mathcal{L}(\cdot)}| \) decays exponentially at infinity. An immediate consequence of assumption (A0) is existence for the initial value problem.

**Proposition 3.1.** Assume (A0), \( u_0 \in L^\infty(\mathbb{R}^d), \) and let \( u(t, x) = K_\sigma(t, \cdot) * u_0(x). \) Then \( u \in C^\infty((0, T) \times \mathbb{R}^d) \) and \( u \) is a classical solution of
\begin{equation}
\partial_t u - Lu = 0 \quad \text{in} \quad \mathbb{R}^d \times (0, T), \quad u(0, x) = u_0(x) \quad \text{in} \quad \mathbb{R}^d.
\end{equation}

We first prove that sums of operators \( \mathcal{L}_i \) satisfying (A0) also satisfy (A0). Let
\begin{equation}
\mathcal{L} = \mathcal{L}_1 + \cdots + \mathcal{L}_N \quad \text{where} \quad \mathcal{L}_i u(x) = \int_{Z_i} u(x + z) - u(x) - Du(x) \cdot z 1_{|z| < 1} \, d\mu_i(z),
\end{equation}

\( Z_i \) is a \( d_i \)-dimensional subspace, \( \oplus_{i=1}^N Z_i = \mathbb{R}^d, \) and \( \mathcal{L}_i \) satisfy (A0) in \( Z_i: \)
\begin{enumerate}
\item[(A0'')] \( Z_i \simeq \mathbb{R}^{d_i} \subset \mathbb{R}^d \) is a subspace for \( i = 1, \ldots, N, \) and \( \oplus_{i=1}^M Z_i = \mathbb{R}^d \) for \( M \leq N. \)
\item[(b)] \( \mu_i \geq 0 \) is a Radon measure on \( Z_i \) satisfying \( \int_{Z_i} 1 \land |z|^2 \, d\mu_i(z) < \infty. \)
\item[(c)] There are \( \sigma_i \in (1, 2) \) and \( c_i > 0 \) such that the heat kernel \( K_i \) of \( \mathcal{L}_i \) satisfy
\end{enumerate}
\begin{equation}
||D_{z_i}^\beta K_i(t, \cdot)||_{L^1(Z_i)} \leq c_i t^{-\frac{|\beta|}{\sigma_i}} \quad \text{for} \quad t \in (0, T), \quad i = 1, \ldots, M.
\end{equation}

for any multi-index \( \beta \in \mathbb{N}_0^{d_i} \) where \( D_{z_i} \) is the gradient in \( Z_i. \)

**Theorem 3.2.** Assume (A0'') and \( \mathcal{L} \) is defined in (10). Then the heat kernel \( K \) of \( \mathcal{L} \) belongs to \( C^\infty \) and satisfies the heat kernel bound in (A0) with \( \sigma = \min_i \sigma_i, \) i.e.
\begin{equation}
||D_{x}^\beta K(t, \cdot)||_{L^1(\mathbb{R}^d)} \leq c_{\beta, T} t^{-\frac{|\beta|}{\sigma}} \quad \text{for} \quad t \in (0, T), \quad \beta \in \mathbb{N}_0^d.
\end{equation}
Proof. First note that in this case \( K(t) = \mathcal{F}^{-1}(e^{t\hat{L}_1} \cdots e^{t\hat{L}_N}) = K_1(t) * \cdots * K_N(t) \) where

\[
K_i(t) := \mathcal{F}^{-1}_{\mathbb{R}_+}(e^{t\hat{\mathcal{L}}_i}) = K_i(t)\delta_{0,Z_i}, \quad K_i(t) = \mathcal{F}^{-1}_{\mathbb{Z}_+}(e^{t\hat{\mathcal{L}}_i}),
\]

and \( \delta_{0,Z_i} \) is the delta-measure in \( Z_i^\perp \) centered at 0. For \( t \in (0,T) \), \((A0')\) (c) implies that

\[
\|D^\beta_{z_i}K_i(t)\|_{L^1(\mathbb{R}^d)} = \|D^\beta_{z_i}K_i(t,\cdot)\|_{L^1(Z_i)} \leq \frac{c_T t^{-|\beta|/|\sigma|}}{\sigma} (\sigma \leq \sigma_i)
\]

for some constant \( c_T > 0 \). Since \( K_i \) is a probability measure by the Levy-Kinchine theorem \([2, \text{Thm 1.2.14}]\), \( \|K_j(t)\|_{L^1(\mathbb{R}^d)} = \|K_j(t)\|_{L^1(Z_j)} = 1 \). By properties of mollifiers and Young’s inequality for convolutions it then follows that

\[
\|D^\beta_{z_i}K(t,\cdot)\|_{L^1(\mathbb{R}^d)} = \|K_1 * \cdots * D^\beta_{z_i}K_i * \cdots * K_N\|_{L^1(\mathbb{R}^d)} \leq \|D^\beta_{z_i}K_i\|_{L^1(Z_i)} \leq c_T t^{-|\beta|/\sigma}
\]

Since \( i = 1, \ldots, M \) was arbitrary and \( \bigoplus_{i=1}^M Z_i = \mathbb{R}^d \), the proof is complete. \( \square \)

Now we check that \((A0')\) implies \((A0)\).

**Theorem 3.3.** Assume \((A0')\) and \( \mathcal{L} \) is defined in (2). Then the heat kernels \( K \) and \( K^* \) of \( \mathcal{L} \) and \( \mathcal{L}^* \) belong to \( C^\infty \) and satisfies \((A0)\), in particular

\[
\|D^\beta_{z}K(t,\cdot)\|_{L^1(\mathbb{R}^d)} + \|D^\beta_{z}K^*(t,\cdot)\|_{L^1(\mathbb{R}^d)} \leq c_{\beta,T} t^{-|\beta|/\sigma} \quad \text{for} \quad t \in (0,T), \quad \beta \in \mathbb{N}_0^d.
\]

**Example 3.4.** In view of Theorems 3.2 and 3.3, assumption \((A0)\) is satisfied by e.g.

\[
\mathcal{L}_1 = -(-\Delta_{\mathbb{R}^d})^{\sigma_1/2} - (-\Delta_{\mathbb{R}^d})^{\sigma_2/2},
\]

\[
\mathcal{L}_2 = -\left( -\frac{\partial^2}{\partial x_1^2} \right)^{\sigma_1/2} \cdots - \left( -\frac{\partial^2}{\partial x_d^2} \right)^{\sigma_d/2},
\]

\[
\mathcal{L}_3 u(x) = \int_{\mathbb{R}} u(x+z) - u(x) - u'(x)z1_{|z|<1} \frac{C e^{-M_+z+Gz-}}{|z|^{1+y}} dz,
\]

where \( C, G, M > 0, Y \in (0,2) \), [CGMY model in Finance]

\[
\mathcal{L}_4 = \mathcal{L} + L \quad \text{where} \quad \mathcal{L} \text{ satisfy (A0) and} \quad L \text{ is any other Levy operator}.
\]

We can even take \( L \) to be any local Levy operator (e.g. \( \Delta \)) if we relax the definition of \( \mathcal{L}_i \) to \( \mathcal{L}_i u(x) = \text{tr}[a_i D^2 u] + b_i \cdot Du + \int_{Z_i} u(x+z) - u(x) - Du(x) \cdot z 1_{|z|<1} d\mu_i(z) \) for \( a_i \geq 0 \).

**Remark 3.5.** (a) \((A0)\) holds also for very non-symmetric operators where \( \mu \) has support in a cone in \( \mathbb{R}^d \). Examples are Riesz-Feller operators like

\[
\mathcal{L}_3 u(x) = \int_{z > 0} u(x+z) - u(x) - u'(x)z1_{z<1} \frac{dz}{z^{1+\alpha}}, \quad \alpha \in (0,2).
\]

We refer to [1] for results and discussion, see e.g. Lemma 2.1 (G7) and Proposition 2.3.

(b) More general conditions implying \((A0)\) can be derived from the very general results on derivatives of heat semigroups in [24] and heat kernels in [16]. Such conditions could include more non absolutely continuous and non-symmetric Levy measures.

We will now prove Theorem 3.3 and start by proving the result for \( \hat{K}_\sigma \).
Lemma 3.6. Assume (A0'). Then $\tilde{K}_t \in C^\infty$ and for all multi-indices $\beta$ there is $c > 0$ such that $\|D_x^\beta \tilde{K}_t\|_{L^1(\mathbb{R}^d)} \leq ct^{-|\beta|/\sigma}$ for all $t > 0$.

Proof. We verify the conditions of Theorem 5.6 of [16]. By (A0') assumption (5.5) in [16] holds with

$$\nu_0(|x|) = \begin{cases} \frac{1}{|x|^{d+\sigma}}, & |x| < 1, \\ 0, & |x| \geq 0. \end{cases}$$

Then we compute the integral $h_0$,

$$h_0(r) := \int_{\mathbb{R}^d} 1 \wedge \frac{|x|^2}{r^2} \nu_0(|x|) \, dx = \begin{cases} c_d(\frac{1}{2-\sigma} + \frac{1}{\sigma})r^{-\sigma} - \frac{c_d}{\sigma}, & r < 1, \\ c_d \frac{1}{2-\sigma} r^{-2}, & r \geq 1, \end{cases}$$

where $c_d$ is the area of the unit sphere. Note that $h_0$ is positive, strictly decreasing, and that $h_0(r) \leq \lambda^\sigma h_0(\lambda r)$ for $0 < \lambda \leq 1$ and every $r > 0$. Hence the scaling condition (5.6) in [16] with $C_{h_0} = 1$ for any $\theta_{h_0} > 0$. The inverse is given by

$$h_0^{-1}(\rho) = \begin{cases} \left( \frac{(2-\sigma)\rho}{c_d} \right)^{-\frac{1}{2}}, & \rho \leq \frac{c_d}{2-\sigma}, \\ \left( \frac{\rho}{c_d} + \frac{1}{\sigma} \right)^{-\frac{1}{2}} \left( \frac{\sigma(2-\sigma)}{2} \right)^{-\frac{1}{2}}, & \rho \geq \frac{c_d}{2-\sigma}. \end{cases}$$

In both cases $t \leq (2-\sigma)/c_d$ and $t \geq (2-\sigma)/c_d$, we then find that $h_0^{-1}(1/t) \leq (\tilde{c}t)^{1/\sigma}$, where $\tilde{c}$ only depends on $\sigma$ and $d$.

At this point we can use Theorem 5.6 in [16] to get the following heat kernel bound:

$$|\partial_x^\beta p(t, x + tb_{h_0^{-1}(1/t)})| \leq C_0[h_0^{-1}(1/t)]^{-|\beta|} Y_t(x) = C_{0,\sigma} t^{-\frac{|\beta|}{\sigma}} Y_t(x),$$

for any $t > 0$, where $b_r$ does not depend on $x$,

$$Y_t(x) = [h_0^{-1}(1/t)]^{-d} \wedge \frac{t K_0(|x|)}{|x|^d},$$

and

$$K_0(r) := r^{-2} \int_{|x| < r} |x|^2 \nu_0(|x|) \, dx = \frac{c_d}{2-\sigma} \cdot \begin{cases} r^{-\sigma}, & r < 1, \\ r^{-2}, & r \geq 1 \end{cases} \leq \frac{c_d}{2-\sigma} r^{-\sigma}.$$ 

An integration in $x$ then yields

$$\|\partial_x^\beta p(t, x)\|_{L^1(\mathbb{R}^d)} \leq C_{0,\sigma} \tilde{c} t^{-\frac{|\beta|}{\sigma}} \int_{\mathbb{R}^d} Y_t(x) \, dx.$$ 

We check that $Y_t \in L^1(\mathbb{R}^d)$. Since $h_0^{-1}(1/t) \leq \tilde{c} t^{1/\sigma}$ and $K_0(r) \leq \frac{c_d}{2-\sigma} r^{-\sigma}$, we can compute the minimum to find a constant $c_{\sigma,d} > 0$ such that

$$0 \leq Y_t(x) \leq \begin{cases} \frac{(\tilde{c}t)^{-d/\sigma}}{c_{\sigma,d} t^{d+1/\sigma}}, & \text{for } |x| < c_{\sigma,d} t^{1/\sigma}, \\ \frac{c_{\sigma,d} t^{-d/\sigma}}{|x|^{d+1/\sigma}}, & \text{otherwise}. \end{cases}$$
A direct computation shows that \( \int_{\mathbb{R}^d} Y_t(x) \, dx \leq C \), where \( C > 0 \) is a constant not depending on \( t \), and hence there is a \( c > 0 \) such that \( \int_{\mathbb{R}^d} \left| \partial_x p(t,x) \right| \, dx \leq ct^{-|\beta|/\sigma} \) for all \( t > 0 \).

**Proof of Theorem 3.3.** result for \( K_\sigma \) follows by Lemma 3.6 and a simple computation:

\[
\|D^\beta_x K_\sigma\|_{L^1} = \|D^\beta_x \mathcal{F}^{-1}(e^{t\hat{\mathcal{L}}_\sigma}e^{t\mathcal{L}_n})\|_{L^1} = \left\| \left(D^\beta_x \mathcal{F}^{-1}(e^{t\hat{\mathcal{L}}_\sigma})\right) \ast \mathcal{F}^{-1}(e^{t\mathcal{L}_n}) \right\|_{L^1} \\
\leq \|D^\beta_x \mathcal{F}^{-1}(e^{t\hat{\mathcal{L}}_\sigma})\|_{L^1} \int_{\mathbb{R}^d} \mathcal{F}^{-1}(e^{t\mathcal{L}_n}) \leq ct^{-|\beta|/\sigma} \cdot 1.
\]

The last integral is 1 since \( \mathcal{F}^{-1}(e^{t\mathcal{L}_n}) \) is a probability by e.g. Theorem 1.2.14 in [2]. Since \( \mathcal{L}_\beta \) is an operator of the same type as \( \mathcal{L} \) with a Levy measure \( \mu^* \) also satisfying \((A0')\) (cf. Lemma 2.2), the computations above show that \( K_\sigma \) also satisfy the same bound as \( K_\sigma \).

By interpolation we obtain estimates for fractional derivatives of the heat kernel.

**Proposition 3.7.** Assume \((A0), t \in [0,T], s, \sigma \in (0,2), \text{ and } |D|^s := (-\Delta)^{s/2} \). Then

\[
\|D|^s K_\sigma(t)\|_{L^1(\mathbb{R}^d)} \leq ct^{-\sigma s},
\]

and if \( s \in (0,1) \), then

\[
\|D|^s \partial_x K_\sigma(t)\|_{L^1(\mathbb{R}^d)} \leq ct^{-\sigma s + 1}.\]

**Proof.** By Remark 2.4 (a) with \( p = 1 \) and \((A0)\), we find that

\[
\int \|D|^s K_\sigma(t)\| \, dx \leq c\|D^2 K_\sigma(t)\|^\frac{\sigma}{2} \|K_\sigma\|^\frac{1-\sigma}{2} \leq \left(c t^{-\sigma s}/2\right)^{s/2} 1^{1-s/2} \leq ct^{-\sigma s}.
\]

The proof of the second part follows in a similar way from Remark 2.4 (b). \( \square \)

4. **Regularity for Fractional Hamilton-Jacobi Equations**

Here we prove regularity for solutions of the fractional Hamilton-Jacobi equation. In our proof we use heat kernel estimates (Section 3), a Duhamel formula, and a fixed point argument as in [18, 12]. The fractional Hamilton-Jacobi equation is given by

\[
\begin{cases}
\partial_t u - \mathcal{L}u + H(t,x,u,Du) = f(t,x) & \text{in } (0,T) \times \mathbb{R}^N, \\
u(0,x) = u_0(x) & \text{in } \mathbb{R}^N,
\end{cases}
\]

where \( \mathcal{L} \) is as before, \( \sigma \in (1,2) \), \( H \) is the Hamiltonian, and \( f \) is a source term.

**Assumptions.**

1. **(B1)** The functions \( H : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) and \( f : [0, +\infty) \times \mathbb{R}^N \rightarrow \mathbb{R} \) are continuous.

2. **(B2)** There exists \( \gamma \in \mathbb{R} \) such that for all \( t \in [0,T), x \in \mathbb{R}^N \), \( u,v \in \mathbb{R}, u < v, p \in \mathbb{R}^N \),

\( H(t,x,v) - H(t,x,u) \geq \gamma (v - u) \).

3. **(B3)** For every \( R > 0 \) there is \( C_R > 0 \) such that for \( t \in [0,T) \), \( x,y,p \in \mathbb{R}^N \), \( u \in [-R,R] \),

\( |f(t,x) - f(t,y)| + |H(t,x,u,p) - H(t,y,u,p)| \leq C_R(|p| + 1)|x - y| \).
For every $R > 0$ there is $C_R > 0$ such that for $t \in [0, T), x, y \in \mathbb{R}^N, u, v \in [-R, R], p, q \in B_R,$

$$|H(t, x, u, p) - H(t, x, v, q)| \leq C_R (|u - v| + |p - q|).$$

There exists $C_0 > 0$ such that for $t \in [0, T), x, y \in \mathbb{R}^N, u, v \in [-R, R], p, q \in B_R,$

$$|H(t, x, u, p) - H(t, x, v, q)| \leq C_R (|u - v| + |p - q|).$$
Note that $f$ needs less regularity than $H$.

**Theorem 4.4.** Assume $k \geq 2$, (A0), (B1)–(B8) hold, and $H,f,u_0$ are 1-periodic in $x$. Then (11) has a unique classical solution $u$ that satisfies
\[
\partial_t u, u, Du, \ldots, D^k u \in C_b \left( (0,T) \times \mathbb{R}^N \right),
\]
with
\[
\|u_t\|_{L^\infty} + \|u\|_{L^\infty} + \|Du\|_{L^\infty} + \ldots + \|D^k u\|_{L^\infty} \leq c,
\]
where $c$ is a constant depending only on $\sigma, T, N, k$, and the constants in (B6)–(B8).

To have more regularity in time, we assume

(B9) There is $\alpha \in (0,1]$ such that for every $R > 0$ there is $L_R > 0$ such that for $s,t \in [0,T], x,y \in \mathbb{R}^N, u,v \in [-R,R], p,q \in B_R,$
\[
|H(s,x,u,p) - H(t,y,v,q)| \leq L_R(|s-t|^{\alpha} + |x-y| + |u-v| + |p-q|),
\]
\[
|f(s,x) - f(t,y)| \leq L_R(|s-t|^{\alpha} + |x-y|).
\]

**Theorem 4.5.** If $k \geq 3$, (A0), (B1)–(B9) hold, and $H,f,u_0$ are 1-periodic in $x$. Then the unique 1-periodic classical solution $u$ of (11) satisfies
\[
|u(t,x) - u(s,y)| + |Du(t,x) - Du(s,y)| + |D^2 u(t,x) - D^2 u(s,y)|
\]
\[
+ |\partial_t u(t,x) - \partial_t u(s,y)| + |\mathcal{L} u(t,x) - \mathcal{L} u(s,y)| \leq \omega(|t-s| + |x-y|),
\]
where $\omega$ only depends on $\sigma, T, N$, and the constants in (B6)–(B9).

**Remark 4.6.** In [18], Imbert shows that when $\mathcal{L} = -(-\Delta)^{\sigma/2}$, $f \equiv 0$, and $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, there exists a unique classical solution $u$ satisfying
\[
u(t,x) = \psi(v)(t,x)
\]
\[
u(0,x) = u_0(x)
\]
\[
u(t,x) = K(t,\cdot) \ast u_0(\cdot)(x) - \int_0^t K(t-s,\cdot) \ast (H(s,\cdot) v(s,\cdot) + \mathcal{L} v(s,\cdot)) ds.
\]

Note that solutions of this equation are fixed points of $\psi$. We now show that (13) has a smooth solution by finding a smooth fixed point of $\psi$ on $[0,T_0]$ for $T_0 > 0$ sufficiently small. Note that any $C^{1,2}_b((0,T_0) \times \mathbb{R}^d)$ fixed point of (13) is a classical solution of (11). Hence if we can prove enough regularity for $v$, Theorem 4.4 and 4.5 will follow on $(0,T_0)$. 

4.1. **Short time regularity by a Duhamel formula.** If $K$ is the fractional heat kernel defined in (9), then a solution $v$ of (11) is formally a solution of the Duhamel formula
\[
v(t,x) = \psi(v)(t,x)
\]
\[
v(t,x) = K(t,\cdot) \ast u_0(\cdot)(x) - \int_0^t K(t-s,\cdot) \ast (H(s,\cdot) v(s,\cdot) + \mathcal{L} v(s,\cdot)) ds,
\]

Note that solutions of this equation are fixed points of $\psi$. We now show that (13) has a smooth solution by finding a smooth fixed point of $\psi$ on $[0,T_0]$ for $T_0 > 0$ sufficiently small. Note that any $C^{1,2}_b((0,T_0) \times \mathbb{R}^d)$ fixed point of (13) is a classical solution of (11). Hence if we can prove enough regularity for $v$, Theorem 4.4 and 4.5 will follow on $(0,T_0)$. 


Proposition 4.7. Assume $k \geq 2$, (A0), (B1)-(B8), $v_0 \in W^{k-1,\infty}(\mathbb{R}^N)$ with $\|v_0\|_{W^{k-1,\infty}} \leq R_0$, and $R_1 = (1 + K)R_0 + 1$ with $K$ defined in (A0). Then there exists $T_0 \in (0, T)$ depending on $R_1$, $\sigma$, and the constants in (A0), (B1)-(B7) such that $\psi$ in (13) has a unique fixed point $v \in C_b((0, T_0) \times \mathbb{R}^N)$. Moreover, $v, Dv, \ldots, D^{k-1}v, t^{1/\sigma}D^kv \in C_b((0, T_0) \times \mathbb{R}^N)$ with

$$\|v\|_{L^\infty} + \ldots + \|D^{k-1}v\|_{L^\infty} + \|t^{1/\sigma}D^kv\|_{L^\infty} \leq R_1.$$  

Proof. By Banach’s fixed point theorem, the proof is complete if we can show that there is a $T_0 > 0$ such that $\psi(u)$ is a contraction mapping on the Banach (sub) space

$$X = \{v : v, Dv, \ldots, D^{k-1}v, t^{1/\sigma}D^kv \in C_b((0, T_0) \times \mathbb{R}^N) \text{ and } \|v\|_k \leq R_1\},$$

where $\|v\|_k = \|v\|_{k-1} + \sum_{|\beta| = k} \|t^{1/\sigma}D^\beta v\|_\infty$ and $\|v\|_k = \sum_{0 \leq |\beta| \leq k} \|D^\beta v\|_\infty$.

We start by showing that $\psi$ maps $X$ to itself. Take a $v \in X$ and recall that $\|v_0\|_{W^{k-1,\infty}} \leq R_0$. For $i = 1, \ldots, N$ and $\beta \in \mathbb{N}^N$, $|\beta| \leq k - 2$ we have

$$\partial^\beta_x \partial_{x_i} \psi(v) = K(t) \ast \partial^\beta_x \partial_{x_i} v_0(x) - \int_0^t \partial_{x_i} K(t - s) \ast \partial^\beta_x \left(H(\cdot, \cdot, v, Dv) - f\right)(s, x) \, ds,$$

while for $|\beta| = k - 1$,

$$t^{1/\sigma} \partial^\beta_x \partial_{x_i} \psi(v) = t^{1/\sigma} \partial_{x_i} K(t) \ast \partial^\beta_x v_0(x)$$

$$- t^{1/\sigma} \int_0^t \partial_{x_i} K(t - s) \ast \partial^\beta_x \left(H(\cdot, \cdot, v, Dv) - f\right)(s, x) \, ds.$$

In view of (A0) and an argument like in the proof of Proposition 3.1 in [12], we can conclude that $K(t, \cdot) \ast w$ and $\int_0^t \partial_x K(t - s, \cdot) \ast F(s, \cdot) \, ds$ are well-defined, bounded and continuous for any bounded functions $w$ and $F$. In view of the regularity of $v$, $H$, and $f$, it then follows that all above derivatives of $\psi(v)$ appearing in $X$ are well-defined, bounded, and continuous. In particular by (A0), for $t \in (0, T)$

$$\|t^{1/\sigma} \partial_{x_i} K(t) \ast \partial^\beta_x v_0\|_{C_b} \leq K\|\partial^\beta_x v_0\|_{C_b}.$$  

For sufficiently small $T_0 \in (0, T)$, we now show that (i) $\psi$ maps $X$ into itself and (ii) it is a contraction. Note that if $\|u\|_k, \|v\|_k \leq R_1$, then there exists a constant $C_{R_1} > 0$ depending only on $R_1$ and the constants in (B5), (B6) and (B7), such that

$$\left| \partial^\beta_x \left[H(s, x, u(s, x), Du(s, x))\right] + |D^\beta f(s, x)| \right|$$

$$\leq \begin{cases} C_{R_1}, & 0 \leq |\beta| \leq k - 2 \\ C_{R_1} \left(1 + s^{-1/\sigma}\right), & |\beta| = k - 1, \end{cases}$$

$$\left| \partial^\beta_x \left[H(s, x, u, Du)\right] - \partial^\beta_x \left[H(s, x, v, Dv)\right] \right|$$

$$\leq \begin{cases} C_{R_1}\|u - v\|_{|\beta|+1}, & 0 \leq |\beta| \leq k - 2 \\ C_{R_1} \left(1 + s^{-1/\sigma}\right)\|u - v\|_k, & |\beta| = k - 1. \end{cases}$$
Note that in (16) we differentiate \( H \) up to \( k \) times (one of these due to the difference), and that this derivative is locally bounded by assumption (B6). For \( t \leq T \), (A0) yields that, 
\[
\int_0^t \int_{\mathbb{R}^N} |K(t - s, x)| \, dx \, ds = t \leq T_0, \quad \int_0^t \int_{\mathbb{R}^N} |\partial_x K(t - s, x)| \, dx \, ds \leq k(\sigma)T_0^{1-1/\sigma},
\]
and 
\[
\int_0^t s^{-1/\sigma} \int_{\mathbb{R}^N} |\partial_x K(t - s, x)| \, dx \, ds \leq \gamma(\sigma)T_0^{1-1/\sigma},
\]
where \( k(\sigma) = K_{\sigma-1} \) and \( \gamma(\sigma) = K\int_0^1 (1 - \tau)^{-1/\sigma} \tau^{-1/\sigma} \, d\tau \). Then using Young’s inequality for convolutions, we find that
\[
\|\psi(v)\|_k = \|\psi(v)\|_\infty + \sum_{i=1}^N \left( \|\partial_x^i \psi(v)\|_\infty + \sum_{1 \leq |\beta| \leq k-2} \|\partial_x^\beta \partial_t \psi(v)\|_\infty + \sum_{|\beta| = k-1} \|t^{1/\sigma} \partial_x^\beta \partial_t \psi(v)\|_\infty \right)
\leq (1 + K)R_0 + C_{R_1} \left( T_0 + \sum_{i=1}^N \left( k(\sigma)T_0^{1-1/\sigma} + \sum_{1 \leq |\beta| \leq k-2} k(\sigma)T_0^{1-1/\sigma} + \sum_{|\beta| = k-1} k(\sigma)T_0 + \gamma(\sigma)T_0^{1-1/\sigma} \right) \right).
\]
Now we take \( T_0 \in (0,T) \) so small that \( c(T_0) \leq 1/2 \). It then follows that \( \psi \) maps \( X \) into itself since by the definition of \( R_1 \),
\[
\|\psi(v)\|_k \leq (1 + K)R_0 + \frac{1}{2} \leq R_1.
\]
We also have a contraction. For \( u, v \in X \), we use (16) and \( \|u\|_1 \leq \|u\|_{k-1} \leq \|u\|_k \) to find
\[
\|\psi(u) - \psi(v)\|_k \leq C_{R_1} \left( T_0 \|u - v\|_1 + \sum_{i=1}^N \left( k(\sigma)T_0^{1-1/\sigma}\|u - v\|_1 + \sum_{1 \leq |\beta| \leq k-2} k(\sigma)T_0^{1-1/\sigma}\|u - v\|_{|\beta|+1} \right) \right) + \sum_{|\beta| = k-1} \left( k(\sigma)T_0 + \gamma(\sigma)T_0^{1-1/\sigma} \right) \|u - v\|_{|\beta|+1}) \right).
\]
\[
\leq c(T_0)\|u - v\|_k \leq \frac{1}{2}\|u - v\|_k.
\]
An application of Banach’s fixed point theorem in \( X \) now concludes the proof. \( \square \)

Now we prove time and mixed time-space regularity results that will be needed later. As a consequence, we conclude that the solution of (13) is a classical solution of (11).

**Proposition 4.8.** Assume \( T_0 > 0 \), (A0), (B1)–(B8) hold with \( k \geq 2 \), \( v \) satisfies (13), and \( v, \text{Dv}, \text{D}^2v \in C_b((0,T_0) \times \mathbb{R}^N) \).
(a) Then $v_t \in C_b((0, T_0) \times \mathbb{R}^N)$ and $\|v_t\|_{\infty} \leq c$, where $c$ depends only on $\sigma$, $T_0$, $N$ and the constants in (B6)-(B8).

Assume in addition $k \geq 3$ and $D^3v \in C_b((0, T_0) \times \mathbb{R}^N)$.

(b) Then $v, Dv, D^2v \in UC((0, T_0) \times \mathbb{R}^N)$ with modulus $\omega(t - s, x - y) = C(|t - s|^\frac{1}{\sigma} + |x - y|)$, where $C > 0$ only depends on $\sigma$, $T_0$, $N$, the constants in (B6)-(B9), and $\|D^k v\|_{\infty}$, $\|D^l f\|_{\infty}$ for $k = 0, 1, 2, 3$, $l = 0, 1, 2$.

(c) Moreover, $\partial_t v, L v \in UC((0, T_0) \times \mathbb{R}^N)$, where the modulus is dependent only on $\sigma$, $T_0$, $N$, the constants in (B6)-(B9), and $\|D^k v\|_{\infty}$, $\|D^l f\|_{\infty}$ for $k = 0, 1, 2$, $l = 0, 1$.

From part (a) and differentiation of the Duhamel formula (13), we find that $v$ is a classical solution of the fractional Hamilton-Jacobi equation (11).

**Corollary 4.9.** Under the assumptions of Proposition 4.8 (a), $v$ is a classical solution of (11) on $(0, T_0) \times \mathbb{R}^N$.

To show the remaining regularity results, we use the Duhamel formula

$$v(t, x) = K(t, \cdot) * v_0(\cdot)(x) - \int_0^t K(t - s, \cdot) * g(s, \cdot)(x) ds,$$

corresponding to the equation

$$\partial_t v(t, x) - Lv(t, x) + g(t, x) = 0.$$

We will use the following technical result that is proved in Appendix A.

**Lemma 4.10.** Assume (A0), $g, \nabla g \in C_b((0, T) \times \mathbb{R}^N)$, and let

$$\Phi(g)(t, x) = \int_0^t K(t - s, \cdot) * g(s, \cdot)(x) ds.$$

(a) $\Phi(g)(t, x)$ is $C^1$ w.r.t. $t \in (0, T)$ and $\partial_t \Phi(g)(t, x) = g(t, x) + L[\Phi(g)](t, x)$.

(b) If $\beta \in (\sigma - 1, 1)$ and $g \in UC((0, T) \times \mathbb{R}^N)$, then

$$|\partial_t \Phi(g)(t, x) - \partial_t \Phi(g)(s, y)| + |L \Phi(g)(t, x) - L \Phi(g)(s, y)|$$

$$\leq 2(1 + c) \|g\|_{C_b \cap C_{b, x}} |x - y|^{1 - \beta}$$

$$+ 2(1 + c) \|g\|_{C_b \cap C_{b, x}} \omega_g(|t - s|)^{1 - \beta} + \tilde{c} \|g\|_{C_b} |t - s|^{\frac{\sigma - 1}{\sigma}},$$

where $c = \frac{\sigma}{\sigma - 1} T^{\frac{\sigma - 1}{\sigma}} K \int_{|z| < 1} |z|^{1 + \beta} d\mu(z) + 4T \int_{|z| \geq 1} d\mu(z)$,

$$\tilde{c} = 2 \frac{\sigma}{\sigma - 1} K \int_{|z| < 1} |z|^{1 + \beta} d\mu(z) K + 2T^{\frac{\beta}{\sigma}} \int_{|z| \geq 1} d\mu(z),$$

and $K = \max_{s, t \in [0, T]} |t^{\frac{\sigma - 1}{\sigma}} - s^{\frac{\sigma - 1}{\sigma}} /[t - s]^{\frac{\sigma - 1}{\sigma}}$.

Note that $c, \tilde{c}$, and $K$ are finite. We have the following results for (17) and (18).
Lemma 4.11. Assume (A0), \( v \) is given by (17), and \( v, \nabla v, D^2 v, g, \nabla g \in C_b \left( (0, T) \times \mathbb{R}^N \right) \).

(a) \( \partial_t v \in C_b \left( (0, T) \times \mathbb{R}^N \right) \), and \( v \) solves equation (18) pointwise.

(b) If in addition \( g \in UC((0, T) \times \mathbb{R}^N) \), then \( \partial_t v \) and \( \mathcal{L} v \) are uniformly continuous and for any \( t, s \in [0, T] \), \( k = 0, 1, 2 \),
\[
|\partial_t v(t, x) - \partial_t v(s, y)| + |\mathcal{L} v(t, x) - \mathcal{L} v(s, y)| \leq \omega(|t - s| + |x - y|),
\]
where \( \omega \) only depends on \( \omega_b, \|g\|_{\infty}, \|g\|_{C_{0,t} C_{b,x}}^1, \|Dv_0\|_{\infty}, \|D^2 v_0\|_{\infty}, \sigma, T, \) and \( \mu \).

Proof. (a) By the assumptions and Proposition 3.1 and Lemma 4.10 (a), we can differentiate the right hand side of (17). Differentiating and using the two results then leads to
\[
\partial_t v = \partial_t (K(t) * v_0) - \partial_t \int_0^t K(t - s) * g(s) \, ds
= \mathcal{L} (K(t) * v_0) - g(t) - \mathcal{L} \int_0^t K(t - s) * g(s) \, ds
= -g(t) + \mathcal{L} \left( K(t) * v_0 - \int_0^t K(t - s) * g(s) \, ds \right)
= -g(t) + \mathcal{L} v(t).
\]

Thus we end up with (18) and the proof of (a) is complete.

(b) By (17), \( v \) is the sum of two convolution integrals. The regularity of the second integral follows from Lemma 4.10 (b). The regularity of the first integral follows by similar but much simpler arguments, this time with no derivatives on the kernel \( K \) (and hence two derivatives on \( v_0 \)). We omit the details. \( \square \)

Proof of Proposition 4.8. (a) In view of the assumptions, the result follows from Lemma 4.11 with \( g(t, x) = H(t, x, v(t, x), Dv(t, x)) - f(t, x) \).

(b) By (a) and Corollary 4.9, \( v \) solve (11). Let \( w = \partial^2_{x_i x_j} v \) and \( w^\varepsilon = w * \rho_\varepsilon \) for a standard mollifier \( \rho_\varepsilon \). Convolving equation (11) with \( \rho_\varepsilon \) and then differentiating twice \( (\partial_t, \partial_{x_i}) \), we find that
\[
\partial_t w^\varepsilon - \mathcal{L} w^\varepsilon + \partial^2_{x_i x_j} \left( H(t, x, v, Dv) * \rho_\varepsilon \right) = \partial_{x_i} \partial_{x_j} f * \rho_\varepsilon.
\]
By Taylor expansion \( \|\mathcal{L} w^\varepsilon\|_{\infty} \leq c \|w^\varepsilon\|_{C^2_b} \) (see the proof of Lemma 2.3), and then by properties of convolutions,
\[
\|\mathcal{L} w^\varepsilon\|_{\infty} \leq c \sum_{k=2}^4 \|D^k v^\varepsilon\|_{\infty} \leq \frac{c}{\varepsilon} \|D\rho\|_{L_1} \|D^3 v\|_{\infty} + c(\|D^3 v\|_{\infty} + \|D^2 v\|_{\infty}).
\]
It follows that \( |\partial_t w^\varepsilon| \leq \frac{\tilde{c}}{\varepsilon} + K \), where \( \tilde{c} = c \|D\rho\|_{L_1} \|D^3 v\|_{\infty} \) and \( K > 0 \) is a constant only depending on \( \|v\|_{\infty}, \|Dv\|_{\infty}, \|D^2 v\|_{\infty}, \|D^3 v\|_{\infty}, \|D^2 f\|_{\infty} \) and \( C_R > 0 \) from assumption.
(B6)$_2$ with $R = \max(\|v\|_{\infty}, \|Dv\|_{\infty})$. We then find that
\[
\|w(t) - w(s)\|_{\infty} \leq \|w^\epsilon(t) - w(t)\|_{\infty} + \|w^\epsilon(t) - w^\epsilon(s)\|_{\infty} + \|w^\epsilon(s) - w(s)\|_{\infty}
\leq 2\|Dw\|_{\infty} \cdot \epsilon + \|\partial_t w^\epsilon\|_{\infty} |t - s| \leq 2\|D^3v\|_{\infty} \cdot \epsilon + (\frac{\epsilon}{\epsilon} + K)|t - s|
= C|t - s|^\frac{1}{2} + K|t - s|,
\]
when we take $\epsilon = |t - s|^\frac{1}{2}$. Since $w$ is bounded, this implies Hölder 1/2 regularity in time. The spatial continuity follows from $|w(t,x) - w(t,y)| \leq \|D^3v\|_{\infty}|x - y|$. In total, we get (recalling that $w = \partial_x, \partial_{x_j} v$),
\[
|D^2v(s,x) - D^2v(t,y)| \leq C(|t - s|^\frac{1}{2} + |x - y|),
\]
where $C > 0$ is only dependent on $\sigma$, $T$, $N$, the constants of (B6)-(B9), and $\|D^k v\|_{\infty}$ and $\|D^l f\|_{\infty}$ for $k = 0, 1, 2, 3$ and $l = 0, 1, 2$.

(c) Since $v$, $Dv$ and $D^2v$ are uniformly continuous, by Taylor expansion (as in the proof Lemma 2.3) it follows that $Cv$ is uniformly continuous with a modulus only depending on the moduluses of $v$, $Dv$ and $D^2v$. Then by the equation $\partial_t v$ will be uniformly continuous as well. \hfill \Box

4.2. Global regularity and proofs of Theorems 4.4 and 4.5. From the local in time estimates, we now construct a classical solution $u$ of (11) on the whole interval $(0, T) \times \mathbb{R}^N$. By Theorem 4.3, there is a unique 1-periodic viscosity solution $u$ of (11) on $(0, T)$. To show that this solution is smooth, we proceed in steps.

1) By Lemma 4.7 we find a $T_0 > 0$ and a unique solution $v$ of (13) satisfying
\[
v, Dv, \cdots, D^k v \in C_b((0, T_0) \times \mathbb{R}^N).
\]
When $k \geq 2$, Corollary 4.9 then gives that $v$ is a classical solution of (11) on $(0, T_0)$. Since classical solutions are viscosity solutions, $v$ coincides with the unique viscosity solution $u$ on $(0, T_0)$. (Note that solutions of (13) are 1-periodic by uniqueness for (13) and periodicity of the data).

2) Take an arbitrary $t_0 \in [0, T)$ and take the value of the viscosity solution $u$ of (11) as initial condition for (13) at $t = t_0$. Then $v(t_0, x) = u(t_0, x)$ and by Lemma 4.3,
\[
\|v(t_0, \cdot)\|_{W^{1,\infty}(\mathbb{R}^N)} \leq M_T.
\]
We start by applying Lemma 4.7 with $k = 1$ (translate time $t \to t - t_0$, apply the theorem, and translate back) to obtain a $T_1 > 0$, independent of $t_0$, such that on $(t_0, t_0 + T_1)$,
\[
\text{we have a unique solution } v \text{ of (13) satisfying } v, \nabla v, (t - t_0)^{1/\sigma} D^2 v \in C_b. \text{ Then}
\]
\[
v, \nabla v, D^2 v \in C_b \left( (t_0 + \delta_1, t_0 + T_1) \times \mathbb{R}^N \right)
\]
for any $\delta_1 \in (0, T_1)$. Now take $v(t_0 + \delta_1, \cdot)$ as initial condition. By Lemma 4.7 again we find a $T_2 > 0$ such that on the interval 

$$(t_0 + \delta_1, t_0 + \delta_1 + T_2)$$

there exists a unique solution $v$ of (13) such that for any $\delta_2 \in (0, T_2)$,

$$v, \nabla v, D^2 v, t^{1/\sigma} D^3 v \in C_b((t_0 + \delta_1 + \delta_2, t_0 + \delta_1 + T_2)).$$

3) Take $\delta_i = \frac{1}{2^{i+1}} \min (T_0, \ldots, T_{i-1})$ for $i = 1, \cdots, k - 1$, and let $\delta := \delta_1 + \cdots + \delta_{k-1} > 0$. Note that $\delta \leq \frac{1}{2} \min (T_0, \ldots, T_{k-2})$. Let us iterate the argument in step 2) until $i = k - 1$ to find that there is $T_{k-1} > 0$ such that on the interval 

$$(t_0 + \delta, t_0 + \delta + T_{k-1})$$

there exists a unique solution $v$ of (13) satisfying

$$v, Dv, \cdots, D^k v \in C_b((t_0 + \delta, t_0 + \delta + T_{k-1}) \times \mathbb{R}^N).$$

For $k \geq 2$ this is a classical solution with $\partial_t u \in C_b$ by Proposition 4.8. Since $t_0$ was arbitrary, we therefore conclude that for any $\tilde{t} \in (0, T)$, there exists $\tilde{T} > 0$ such that

$$u, Du, \cdots, D^k u, \partial_t u \in C_b((\tilde{t}, \tilde{t} + \tilde{T}) \times \mathbb{R}^N).$$

Note that $\tilde{T} > 0$ can be chosen independently of $\tilde{t} \in (0, T)$ because of (20) and (B6)–(B8).

4) We use step 1) and 3) to show that the viscosity solution $u$ in fact is a classical solution on all of $(0, T)$. Simply combine the results for the overlapping intervals

$$[0, T_0), \frac{1}{2}T_0 + (0, \tilde{T}), \frac{1}{2}T_0 + \tilde{T} + (0, \tilde{T}), \cdots, \frac{1}{2}T_0 + k\tilde{T} + (0, \tilde{T})$$

where $\frac{1}{2}T_0 + k\tilde{T} > T$. This concludes the proof of Theorem 4.4.

5) Theorem 4.5 follows from Theorem 4.4 and Proposition 4.8.

5. Fractional Fokker-Planck equations

Here we give regularity results for classical solutions of the fractional Fokker-Planck equation. Then we define weak solutions and obtain time estimates using a stochastic interpretation. The equation is given by

$$\begin{cases}
\partial_t m - \mathcal{L}^* m + \text{div} (b(t, x)m) = 0 & \text{in } (0, T) \times \mathbb{R}^N, \\
m(0, x) = m_0(x) & \text{in } \mathbb{R}^N,
\end{cases}$$

where $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\mathcal{L}^*$ satisfies (A0).

Lemma 5.1. Assume (A0) and $b, Db \in C_b((0, T) \times \mathbb{R}^N)$ and $m$ is a classical solution of (21). Then for $t \in [0, T]$,

$$\|m(t, \cdot)\|_\infty \leq e^{\|\text{div} b\|_\infty T} \|m_0\|_\infty.$$
Proof. Standard estimate since in non-divergence form we have (the linear!) equation

$$\partial_t m - L^* m + \sum_{i=1}^d (b_i \partial_t m + \partial_i b_i m) = 0,$$

with all coefficients being continuous and bounded by the assumptions. \qed

**Proposition 5.2.** Assume (A0) holds, $b, Db, D^2 b \in C_b((0, T) \times \mathbb{R}^N)$, $m_0 \in C^2_b(\mathbb{R}^d)$, and $b, m_0$ are 1-periodic.

(a) There exists a unique 1-periodic classical solution $m$ of (21) satisfying

$$\|m(t)\|_{L^\infty} + \|m\|_{L^\infty} + \|Dm\|_{L^\infty} + \|D^2 m\|_{L^\infty} \leq c,$$

where $c$ is a constant depending only on $\sigma$, $T$, $N$, and $\|D^k b\|_{\infty}$ for $k = 0, 1, 2$.

(b) There exists a modulus $\tilde{\omega}$ only depending on $\|D^k m\|_{\infty}$, $\|D^k b\|_{\infty}$ for $k = 0, 1, 2$, and (A0), such that for $s, t \in [0, T]$ and $x, y \in \mathbb{R}^N$,

$$|m(t, x) - m(s, y)| + |Dm(t, x) - Dm(s, y)| \leq \tilde{\omega}(|t - s| + |x - y|).$$

(c) If in addition $b, Db \in UC((0, T) \times \mathbb{R}^N)$, then there exists a modulus $\omega$ only depending on $\tilde{\omega}$, $\omega_b$, $\omega_{Db}$, $\|Db\|_{\infty}$, $m_0$, $T$, $\sigma$, and $N$, such that for $s, t \in [0, T]$ and $x, y \in \mathbb{R}^N$,

$$|\mathcal{L}^* m(t, x) - \mathcal{L}^* m(s, y)| + |\partial_t m(t, x) - \partial_t m(s, y)| \leq \omega(|s - t| + |x - y|).$$

**Proof.** (a) The proof uses a Banach fixed point argument based on the Duhamel formula

$$m(t, x) = \tilde{\psi}(m)(t, x)$$

$$:= K^*(t, \cdot) * m_0(\cdot)(x) - \sum_{i=1}^N \int_0^t \partial_{x_i} K^*(t - s, \cdot) * (b_i m)(s, \cdot) \, ds,$$

and is similar to the proof of Theorem 4.4. Here $K^*$ is the heat kernel of $\mathcal{L}^*$. It is essentially a corollary to Proposition 5.1 in [12] (but in our case the we have more regular initial contition and hence no blowup of norms when $t \to 0^+$).

As in the proof corresponding proof for the HJB equation, we first show short-time $C^1$-regularity using the Duhamel formula. Let $R_0 = \|m_0\|_{\infty}$, $R_1 = (1 + NK)R_0 + 1$, and the Banach (sub) space

$$X = \{ m : m, t^{1/\sigma} Dm \in C_b((0, T_0) \times \mathbb{R}^N) \text{ and } \|m\| \leq R_1 \},$$

where $\|m\| = \|m\|_{\infty} + \sum_{i=1}^N \|t^{1/\sigma} \partial_i m\|_{\infty}$. Then if $k(\sigma)$ and $\gamma(\sigma)$ are defined in the proof of Proposition 4.7, we find that

$$\tilde{\psi}(m)(t, x) \leq \|K^*\|_{L^1} \|m_0\|_{\infty} + \sum_{i=1}^N \int_0^t \|\partial_{x_i} K^*(t - s, \cdot)\|_{L^1} \|b_i\|_{\infty} \|m\|_{\infty} \, ds$$

$$\leq R_0 + NK(\sigma)T_0^{1-\frac{1}{\sigma}} \|b\|_{\infty} R_1,$$
and
\[ |t^{1/\sigma} \partial_{x_j} \tilde{\psi}(m)(t, x)| \]
\[ \leq t^{1/\sigma} |\partial_{x_j} K^*|_{L^1} |m_0|_\infty + \sum_{i=1}^N t^{1/\sigma} \int_0^t \|\partial_{x_i} K^*(t-s, \cdot)\|_{L^1} \| (m \partial_j b_i + b_i \partial_j m) \|_\infty ds \]
\[ \leq KR_0 + \sum_{i=1}^N t^{1/\sigma} \int_0^t K(t-s)^{-1/\sigma} \left[ \|m\|_\infty \|\partial_j b_i\|_\infty + s^{-1/\sigma} \|b_i\|_\infty \|s^{1/\sigma} \partial_j m\|_\infty \right] ds \]
\[ \leq KR_0 + \left[ k(\sigma) T_0 \| Db\|_\infty + \gamma(\sigma) T_0^{-1/\sigma} \|b\|_\infty \right] N R_1, \]
Computing the full norm, we get
\[ \| \tilde{\psi}(m) \| \]
\[ \leq \left( 1 + NK \right) R_0 + \left[ N k(\sigma) T_0^{1 - \frac{2}{\sigma}} \|b\|_\infty + N^2 \left[ k(\sigma) T_0 \| Db\|_\infty + \gamma(\sigma) T_0^{-1/\sigma} \|b\|_\infty \right] \right] R_1. \]

We take \( T_0 > 0 \) so small that \( c(T_0) \leq 1/2 \). Then it follows that \( \tilde{\psi} \) maps \( X \) into itself by the definition of \( R_1 \). It is also a contraction since for \( m_1, m_2 \in X \), it easily follows that
\[ \| \tilde{\psi}(m_1) - \tilde{\psi}(m_2) \| \leq c(T_0) \| m_1 - m_2 \|. \]
An application of Banach’s fixed point theorem in \( X \) then concludes the proof. Note that we only needed \( m_0 \in C_b \) and \( b, Db \in C_b \) to obtain the result. In a similar way we find that
\[ b, Db, \ldots, D^k b \in C_b((0, T) \times \mathbb{R}^N) \]
\[ \Rightarrow m, \ldots, D^{k-1} m, t^{\frac{1}{2}} D^k m \in C_b((0, T_0) \times \mathbb{R}^N), \]for \( T_0 > 0 \) sufficiently small.
Using the same technique as for the Hamilton-Jacobi equation, and the uniform \( \| m \|_\infty \) bound from Lemma 5.1, we obtain the result for the whole time interval \((0, T)\).

(b) Part (b) follows in a similar way as part (b) in Theorem 4.8. We omit the details.

(c) From parts (a), (b), and the assumptions, the function \( g(t, x) = \text{div}(mb) \) satisfies \( g, \nabla g \in C_b((0, T) \times \mathbb{R}^N) \) and \( g \in UC((0, T) \times \mathbb{R}^N) \). Lemma 4.10 (b) (with \( K^* \) instead of \( K \)) then gives that \( \partial_t \Phi(g), \mathcal{L}^* \Phi(g) \in UC((0, T) \times \mathbb{R}^N) \) with modulus \( \omega \) only dependent on \( \sigma, T, N, \| g \|_\infty, \| \nabla g \|_\infty \) and \( \omega_g \). A similar, but simpler argument shows that \( \partial_t K^*_t * m_0 = \mathcal{L}^* K^*_t * m_0 \in UC((0, T) \times \mathbb{R}^N) \). Since \( m = K^*_t * m_0 - \Phi(g) \), this concludes the proof. \( \square \)

Next we look at distributional or very weak solutions of (21) on the torus \( \mathbb{T}^d \).

Definition 5.3. A function \( m \in L^1([0, T] \times \mathbb{T}^d) \) is a distributional solution of (21) if for any \( \phi \in C_c^\infty([0, T] \times \mathbb{T}^d) \),
\[ 0 = \int_{\mathbb{T}^d} \phi(0, x) \ dm_0(x) + \int_0^T \int_{\mathbb{T}^d} \left( \partial_t \phi + \mathcal{L} \phi + \langle b, D \phi \rangle \right) \ dm(t)(x). \]
If \( m \) is a strong or classical solution of (21), then by multiplying with a test function, integrating by parts and using Lemma 2.2 we see that \( m \) is also a distributional solution.

**Lemma 5.4.** Assume (A0) and \( b, Db, D^2b \in C_b((0, T) \times \mathbb{R}^N) \). Then there is at most one distributional solution of (21).

**Proof.** Let \( m_1, m_2 \) be two distributional solutions, and define \( w := m_1 - m_2 \). Then by substracting the two equations and using \( m_0 - m_0 = 0 \),

\[
0 = \int_0^T \int_{\mathbb{T}^d} (\partial_t \phi + \mathcal{L} \phi + \langle b, D\phi \rangle) \, dw (t) \, (x).
\]

Now consider the equation

\[
\partial_t \phi + \mathcal{L} \phi + \langle b, D\phi \rangle = \psi
\]

for \( \psi \in C_{1,2}^1 \left([0, T] \times \mathbb{T}^d \right) \). By Theorem 4.4 there is a unique solution of this equation with terminal conditions \( \phi(x, T) = 0 \). A standard mollification argument shows that there is \( \phi_\varepsilon \in C_{1,2}^\infty \) such that \( \phi_\varepsilon \to \phi \) in \( C_{1,2}^1 \), and by inserting \( \phi_\varepsilon \) in (26), we find that

\[
0 = \int_0^T \int_{\mathbb{T}^d} (\partial_t \phi_\varepsilon + \mathcal{L} \phi_\varepsilon + \langle b, D\phi_\varepsilon \rangle) \, w \, dx \, dt
\]

\[
\to \int_0^T \int_{\mathbb{T}^d} (\partial_t \phi + \mathcal{L} \phi + \langle b, D\phi \rangle) \, w \, dx \, dt = \int_0^T \int_{\mathbb{T}^d} \psi w \, dx \, dt.
\]

Since \( \psi \in C_{1,2}^1 \left([0, T] \times \mathbb{T}^d \right) \) was arbitrary, we conclude that \( w \) a.e. = 0. \( \square \)

We continue by introducing the stochastic differential equation (SDE) related to the fractional Fokker-Planck equation (21):

\[
\begin{aligned}
&\left\{
\begin{array}{l}
dX_t = b(X_t, t) \, dt + dL_t, \quad t \in [0, T], \\
X_0 = Z_0
\end{array}
\right.
\end{aligned}
\]

where \((L_t)\) is a \( d \)-dimensional \( \sigma \)-stable pure jump Lévy process. \( L_t \) can be written as

\[
dL_t = \int_{|z|<1} z\tilde{N} \, (dt, dz) + \int_{|z|\geq1} zN \, (dt, dz),
\]

where \( N, \nu, \) and \( \tilde{N} \, (dt, dz) = N \, (dt, dz) - \nu \, (dz) \, dt \), are the Poisson random measure, the Lévy measure (satisfying (A0)), and the compensated Poisson random measure. Let \( m(t) = \mathcal{L}(X_t) \) be the law of the random variable \( X_t \), i.e. \( \mathbb{E} [\phi(t, X_t)] = \int_{\mathbb{T}^d} \phi(t, x) \, dm(t) \, (x) \). By Theorem 6.2.9 in [2] there exists a unique strong solution \( X_t \) of (27).

**Lemma 5.5.** If \( X_t \) is a strong solution of (27) such that \( \mathcal{L}(Z_0) = m_0 \), then \( m(t) := \mathcal{L}(X_t) \) is a distributional solution of (21).
Proof. The proof follows from Itô’s formula (see Applebaum [2], Theorem 4.4.7). If \( \phi \in C^\infty_c ([0, T) \times \mathbb{T}^d) \), then
\[
\begin{align*}
\phi (t, X_t) &= \phi (0, Z_0) \\
&+ \int_0^t [\partial_t \phi (s, X_s) + \langle b (X_s, s), D\phi (s, X_s) \rangle] ds \\
&+ \int_0^t \int_{|z| \geq 1} [\phi (s-, X_{s-} + z) - \phi (s-, X_{s-})] \tilde{N} (ds, dz) \\
&+ \int_0^t \int_{|z| \geq 1} [\phi (s-, X_{s-} + z) - \phi (s-, X_{s-})] \nu (dz) dt \\
&+ \int_0^t \int_{|z| < 1} [\phi (s-, X_{s-} + z) - \phi (s-, X_{s-})] \tilde{N} (ds, dz) \\
&+ \int_0^t \int_{|z| < 1} [\phi (s-, X_{s-} + z) - \phi (s-, X_{s-}) - \langle \nabla \phi (s-, X_{s-}), z \rangle] \nu (dz) ds
\end{align*}
\]

We take the expectation and note that the \( \tilde{N} \)-terms vanish since they are Martingales with expectation 0. Then since \( m(t) \) is the Law of \( X_t \) and recalling (2), then (25) holds and \( m \) is a distributional solution of (21). \( \square \)

By definition of the metric \( d_1 \) in (6) and the stochastic interpretation we get a time regularity estimate for \( m(t) \).

**Lemma 5.6.** Let \( \sigma \in (1, 2) \) and \( m \) be the distributional solution of the fractional Fokker-Planck equation (21). Then there exists a constant \( c_0 > 0 \) such that
\[
d_1 (m(t), m(s)) \leq c_0 (1 + \| b \|_\infty)|t - s|^{\frac{1}{\sigma}} \quad \forall s, t \in [0, T].
\]

**Proof.** We use the SDE (27) to obtain the estimate:
\[
d_1 (m(s), m(t)) = \sup_{\phi \in \mathcal{F}^{1-Lip}} \left\{ \int_{T^d} \phi (x)(m(s) - m(t))(dx) \right\} = \sup_{\phi \in \mathcal{F}^{1-Lip}} \mathbb{E} [\phi (X_t) - \phi (X_s)]
\]
\[
\leq \mathbb{E} [\| X_t - X_s \|] \leq \mathbb{E} \left[ \int_s^t |b(\tau, X_\tau)| d\tau + |L_t - L_s| \right].
\]

Note that \( \mathbb{E} \left[ \int_s^t |b(\tau, X_\tau)| d\tau \right] \leq \| b \|_\infty|s - t| \), and that by the definition of \( \tilde{N} \),
\[
\mathbb{E} \left| \int_s^t dL_t \right| \leq \mathbb{E} \left| \int_s^t \int_{|z| < r} z\tilde{N}(d\tau, dz) \right| + \mathbb{E} \left| \int_s^t \int_{r < |z| < 1} z\tilde{N}(d\tau, dz) \right|
\]
\[
+ \mathbb{E} \left| \int_s^t \int_{r < |z| < 1} z\nu(dz) d\tau \right| + \mathbb{E} \left| \int_s^t \int_{|z| \geq 1} z\tilde{N}(d\tau, dz) \right|.
\]
Let us now estimate the four integrals on the right hand side. By the Itô-Lévy isometry (see e.g. [2], p. 223), a change to polar coordinates, and (A0),

$$
\mathbb{E} \left[ \left| \int_s^t \int_{|z| \leq r} z \tilde{N}(d\tau, dz) \right|^2 \right] \overset{\text{Itô-Lévy iso.}}{=} \int_s^t \int_{|z| \leq r} |z|^2 \nu(dz) d\tau \leq \frac{r^{2-\sigma}}{2-\sigma} |s-t|.
$$

Thus by Cauchy-Schwartz the first integral is bounded by $\sqrt{(2-\sigma)^{-1}r^{2-\sigma}|s-t|}$. For the third integral, since $N \geq 0$, $\mathbb{E}(N(dt, dz)) = \nu(dz)dt$, and (A0) holds,

$$
\mathbb{E} \left| \int_s^t \int_{r<|z|<1} z N(d\tau, dz) \right| \leq \int_s^t \int_{r<|z|<1} |z| \nu(dz) d\tau \leq c \int_s^t \int_r^1 \frac{d\rho}{\rho^\sigma} \leq cr^{1-\sigma}|s-t|.
$$

Such an estimate also holds for the second integral, and since $\int_{|z| \geq 1} |z| \nu(dz) < \infty$, a similar argument shows that the last integral is of order $c|s-t|$. By inspection, the optimal choice of $r$ is $r = |s-t|^{1/\sigma}$ and it follows that there is a constant $c > 0$ such that

$$
\mathbb{E} \left| \int_s^t dL_t \right| \leq c|s-t|^{1/\sigma}.
$$

We conclude that $d_1(m(t), m(s)) \leq c_0(\|b\|_{\infty} + 1)|s-t|^{1/\sigma}$ and the proof is complete. \(\square\)

6. Classical solutions of fractional MFG – the proof of Theorem 2.5

In this section we prove Theorem 2.5 – existence for fractional MFG systems with nonlocal coupling – using fixed point and compactness arguments that rely on the regularity results of the previous sections. Recall $\mathcal{P}$ and $d_1$ as defined in and above (6), and note that $(\mathcal{P}(\mathbb{T}^d), d_1)$ is a compact metric space by Prokhorov’s theorem. We follow [22, 7] and use Schauder’s fixed point theorem to prove existence of solutions. We work in the space $C^0([0,T], \mathcal{P}(\mathbb{T}^d))$ with metric $d(\mu, \nu) = \sup_{t \in [0,T]} d_1(\mu(t), \nu(t))$ and consider the subset

$$
\mathcal{C} := \left\{ \mu \in C^0([0,T], \mathcal{P}(\mathbb{T}^d)) : \sup_{s \neq t} \frac{d_1(\mu(s), \mu(t))}{|s-t|^\frac{1}{\sigma}} \leq C_1 \right\},
$$

for some constant $C_1 > 0$ to be determined. It is easy to see that $\mathcal{C}$ is convex and closed in $C^0([0,T], \mathcal{P}(\mathbb{T}^d))$. It is also compact by the Arzéla-Ascoli theorem.

Define a fixed point map $\psi$ as follows: For any $\mu \in \mathcal{C}$, let $\psi(\mu) = m$ where $m$ is the unique classical solution of

$$
\begin{aligned}
\partial_t m - \mathcal{L}^* m - \text{div}(D_p H(x, u, Du)m) &= 0, \\
m(0, \cdot) &= m_0(\cdot),
\end{aligned}
$$

and $u$ is the unique classical solution of

$$
\begin{aligned}
-\partial_t u + H(x, u, Du) &= F(x, \mu), \\
u(x, T) &= G(x, \mu(T)).
\end{aligned}
$$
Note that $\mu$ goes into the data of the $u$ equation. For convenience, we define the sets

$$U := \{u : u \text{ solves (31) given } \mu \in \mathcal{C}\}, \quad \mathcal{M} := \{m : m \text{ solves (30) given } u \in U\}.$$ 

We first show that with a suitable choice of $C_1$, $\psi$ maps $\mathcal{C}$ into itself. By the definition of $\mathcal{C}$ and assumption (A2) we see that $F(\cdot, \mu(t))$ is $1/\sigma$ Hölder continuous in $t$ and Lipschitz in $x$. Note that the $L^\infty$ and Lipschitz bounds of $F$ are independent of $C_1$ but not the time-Hölder constant. All bounds are independent of $\mu$. By assumptions (A0)–(A7) and Theorems 4.4 and 4.5, there exists a unique solution $u$ of (31) with the following regularity

$$\begin{align*}
\|u\|_\infty, \|Du\|_\infty, \cdots, &\|D^3u\|_\infty, \|\partial_t u\|_\infty \leq U_1, \\
\partial_t u, u, Du, D^2u, \mathcal{L}u &\text{ equicontinuous with modulus } \omega,
\end{align*}$$

where $U_1$ depends on $d, \sigma$ and the spatial regularity of $F, G$ and $H$ (the constants in (A1)-(A7)), and $\omega$ depends on $C_1$ in (29) in addition to the quantities that $U_1$ depends on. By the uniform bound (A3), $U_1$ is independent of $\mu$ (see Theorems 4.4 and 4.5).

For any $u \in U$ there exists a unique $m$ solving (30) by Proposition 5.2. Moreover, from Proposition 5.2 part (a)-(c),

$$\begin{align*}
\|m\|_\infty, \|Dm\|_\infty, \|D^2m\|_\infty, \|\partial_t m\|_\infty &\leq M_1, \\
\partial_t m, m, Dm, \mathcal{L}^*m &\text{ equicontinuous with modulus } \bar{\omega},
\end{align*}$$

where $M_1$ depends on $U_1$ and the local regularity of $H$ and hence is uniform in $\mu$. Also $\bar{\omega}$ depends on $U_1$, the local regularity of $H$, the modulus $\omega$, but not on $\mu$. By Lemma 5.6 we have the estimate

$$d_1(m(s), m(t)) \leq c_0(1 + \|D_p H(\cdot, Du)\|_\infty) |s - t|^{\frac{1}{2}}.$$ 

Since $\|Du\|_\infty \leq U_1$ and assumption (A4) holds, $D_p H(x, Du)$ is bounded by some constant $C_2 > 0$ which is independent of $\mu$. Hence we take the constant in $\mathcal{C}$ to be $C_1 = c_0(1 + C_2)$ and get that $\psi$ maps $\mathcal{C}$ into itself.

Next we show that $\psi$ is a continuous map. For any convergent sequence $\mu_n \in \mathcal{C}$, the sequence $\psi(\mu_n)$ converges to $\psi(\mu)$. The convergence is taken in the sense of $C^0([0, T], P(\mathbb{T}^d))$. We use the following well-known Lemma.

**Lemma 6.1.** Let $(X, d)$ a metric space, $K \subset X$ a compact subset and $(x_n) \subset K$ a sequence such that all convergent subsequences have the same limit $x^* \in K$. Then $x_n \to x^*$.

The sets $\mathcal{U}$ and $\mathcal{M}$ are compact in $X_1 := \{f : f, Df, D^2f, f_t, \mathcal{L}f \in C_0\}$ and $X_2 := \{f : f, Df, f_t, \mathcal{L}^*f \in C_0\}$, respectively, thanks to the uniform bounds (33) and (35) and the Arzela-Ascoli theorem. Let $(\mu_n)_n$ be a convergent sequence with limit $\mu$, and denote by $(u_n)_n, (m_n)_n$ the corresponding sequences of solutions of equations (31) and (30). By compactness and Lemma 6.1, convergence of e.g. $(u_n)_n$ follows if we can show that every convergent subsequence has the same limit.
Let \((\mu_{n_k}, u_{n_k})\) be any convergent subsequence and \((\mu, \tilde{u})\) the corresponding limit. Since \(u_{n_k} \to \tilde{u}\) in \(X_1\) and \(u_{n_k}\) solves (31) with \(\mu_{n_k}\) as input,
\[
\begin{align*}
&[- \partial_t \tilde{u} - \mathcal{L} \tilde{u} + H(x, D\tilde{u}) - F(x, \mu)] \\
&\leq \|\partial_t u_{n_k} - \partial_t \tilde{u}\|_\infty + \|\mathcal{L} u_{n_k}(t, \cdot) - \mathcal{L} \tilde{u}(t, \cdot)\|_\infty \\
&+ \|H(x, Du_{n_k}) - H(x, D\tilde{u})\| + \|F(x, \mu_{n_k}(t)) - F(x, \mu(t))\| \\
&\to 0,
\end{align*}
\]
and \(|\tilde{u}(T, x) - G(x, \mu(T))| \leq \|\tilde{u} - u_{n_k}\|_\infty + \|G(x, \mu_{n_k}(T)) - G(x, \mu(T))\| \to 0.\) Here we also used the continuity assumptions on \(H, F\) and \(G\), see (A2) and (A4). Hence \(\tilde{u}\) solves equation (31) with \(\mu\) as input, and \(\tilde{u} = u\) by uniqueness. By Lemma 6.1 we then conclude that the full sequence \(u_n \to u\) in \(X_1\).

A similar argument shows that \(m_n\) converges in \(X_2\) to the unique classical solution \(m\) of (30), and it easily follows that \(m_n \to m\) in \(C^0([0, T], \mathcal{P}(\mathbb{T}^d))\).

We have shown that \(\psi\) is a continuous map from \(\mathcal{C}\) into itself. By Schauders fixed point theorem, there then exists an \(m\) such that \(\psi(m) = m\). This fixed point is a classical solution of the fractional Mean Field Game system (5), and the proof of Theorem 2.5 is complete.

7. Fractional MFG systems with local coupling

In this section we introduce fractional MFG with local coupling and prove uniqueness and conditional existence results. For the existence part the idea is to approximate by a MFG system with nonlocal coupling and use the previous regularity results, bootstrap and compactness arguments to pass to the limit.

**Fractional MFG with Local Coupling.** The system of equations is given by
\[
\begin{align*}
\begin{cases}
-\partial_t u - \mathcal{L} u + H(x, Du) = f(x, m(t, x)) & \text{in } (0, T) \times \mathbb{T}^d \\
\partial_t m - \mathcal{L}^*m - \text{div} (m D_p H(x, Du)) = 0 & \text{in } (0, T) \times \mathbb{T}^d \\
m(0) = m_0, u(x, T) = g(x),
\end{cases}
\end{align*}
\]
where the coupling term \(f\) only depends on the value of \(m\) at \((x, t)\), and we say there is a local coupling between the Hamilton-Jacobi and the Fokker-Planck equation.

**Assumptions.**

(A3′) (Regularity of \(f\) and \(g\)) \(f \in C^2(\mathbb{T}^d \times [0, \infty))\) and \(g \in C^3(\mathbb{T}^d)\).

(A4′) (Uniform bounds on \(H\)) \(H \in C^3(\mathbb{T}^d \times \mathbb{R}^d)\) is globally Lipschitz.

(m-bnd) The solution \(m_\epsilon\) of the approximate MFG system (38) below satisfy \(\|m_\epsilon\|_\infty \leq K\) for some \(K > 0\) independent of \(\epsilon\).

**Theorem 7.1.** Assume (A0), (A3′), (A4′), (A7), and (m-bnd). Then there exists a classical solution \((u, m)\) of the MFG system (36).
Remark 7.2. (a) Assumption (m-bnd) holds for local diffusion operators, a result can be found e.g. in Theorem 2.1 in [5]. It also holds when $\mathcal{L} = -(\Delta)^\frac{\alpha}{2}$ and $\alpha \in (1, 2)$, see Theorem 7.12 in [14]. We have not been able to find more general nonlocal results, but we expect (m-bnd) to hold under the other assumptions of Theorem 7.1. We plan to come back to this in future works.

(b) The $L^\infty$ bound of 5.1 does not imply (m-bnd). It is independent of $\epsilon$ only when $u_\epsilon$ is semiconcave uniformly in $\epsilon$ which is not good enough for the proof below.

The proof is given after the next result. For uniqueness we follow [22, 7] and look at the more general MFG system

\begin{equation}
\begin{cases}
-\partial_t u - \nu \mathcal{L} u + H(x, Du, m) = 0 & \text{in } \mathbb{T}^d \times (0, T) \\
\partial_t m - \nu \mathcal{L}^* m - \text{div}(m D_p H(x, Du(t, x), m)) = 0 & \text{in } \mathbb{T}^d \times (0, T) \\
m(0) = m_0, u(x, T) = G(x),
\end{cases}
\end{equation}

where $\nu > 0$, $H = H(x, p, m)$ is convex in $p$.

Assumptions.

(A10) \[
\begin{bmatrix}
m \partial^2_{pp} H \\
\frac{1}{2} m (\partial^2_{pm} H)^T \\
-\partial_m H
\end{bmatrix} > 0 \text{ for all } (x, p, m) \text{ with } m > 0.
\]

Note that whenever $H(x, p, m) = \tilde{H}(x, p) - F(x, m)$, we recover assumption (A9).

Theorem 7.3. If $H = H(x, p, m) \in C^2$ and (A10) holds, then (36) has at most one classical solution.

Proof. In view of adjointness of $\mathcal{L}$ and $\mathcal{L}^*$, the proof is the same as in [22, 7].

To prove existence we follow Lions [22, 7]: Approximate (36) by a system with nonlocal coupling, prove uniform a priori estimates, and conclude by a compactness argument. Let $\epsilon > 0$, $0 \leq \phi \in C_c^\infty$ with $\int_{\mathbb{T}^d} \phi = 1$, $\phi_\epsilon := \frac{1}{\epsilon^d} \phi(x/\epsilon)$, and define for all $\mu \in P(\mathbb{T}^d)$,

$F_\epsilon(x, \mu) := f(x, \mu * \phi_\epsilon(x))$.

For fixed $\epsilon > 0$, $F_\epsilon$ is a nonlocal coupling function satisfying (A1)–(A3) (since $\|D^\beta (\mu * \phi_\epsilon)\|_\infty \leq \|\mu\|_1 \|D^\beta \phi_\epsilon\|_\infty = \|D^\beta \phi_\epsilon\|_\infty$). Moreover, for $\mu \in L^1$,

$F_\epsilon(x, \mu) \rightarrow f(x, \mu(x))$ \quad in \quad $L^1_{\text{loc}}$ (and a.e. along subsequences).

We look at the approximate MFG system with nonlocal coupling given by $F_\epsilon$,

\begin{equation}
\begin{cases}
-\partial_t u_\epsilon - \mathcal{L} u_\epsilon + H(x, Du_\epsilon) = F_\epsilon(x, m_\epsilon(t)) & \text{in } (0, T) \times \mathbb{T}^d, \\
\partial_t m_\epsilon - \mathcal{L}^* m_\epsilon - \text{div}(m_\epsilon D_p H(x, Du_\epsilon)) = 0 & \text{in } (0, T) \times \mathbb{T}^d, \\
m(0) = m_0, u(x, T) = g(x),
\end{cases}
\end{equation}

Since (A1)-(A7) obviously hold for (38), by Theorem 2.5 it has at least one pair of classical solutions $(u_\epsilon, m_\epsilon)$. 
**Proof of Theorem 7.1.** We prove that \((u_\epsilon, m_\epsilon)\) has a convergent subsequence whose limit is a classical solution of (36). We start by deriving uniform \(\epsilon\) a priori estimates. Note that by assumption (m-bnd) and Theorem 4.3 (b),

\[
\|m_\epsilon\|_\infty \leq K,
\|u_\epsilon\|_\infty \leq \|g\|_\infty + (T-t)\|F_\epsilon(\cdot, m_\epsilon(t))\|_\infty,
\]

where \(K\) is does not depend on \(\epsilon\). Since \(F\) is also continuous in \((x,t)\), we have

\[
\|Du_\epsilon\|_\infty \leq C
\]

by Corollary 7 from [3] for \(C \geq 0\) independent of \(\epsilon\) (\(C\) depends on \(F_\epsilon\) only through its \(C_b\)-norm). The original proof is for the right-hand side \(f\) not dependent on \(t\). If \(f = f(x,t)\) is continuous in \(x\) and \(t\), then the proof is exactly the same. This last result uses elliptic regularity and not the comparison principle.

To prove more uniform in \(\epsilon\) regularity, we will use the following Duhamel formulas for \(Du_\epsilon\) and \(m_\epsilon\) (see Sections 4 and 5),

\[
m_\epsilon(t) = K_\sigma^\epsilon(t) * m_0 - \sum_{i=1}^{d} \int_0^t \partial_i K_\sigma^\epsilon(t-s) * m_\epsilon[D_pH(\cdot, Du_\epsilon(s))]ds,
\]

\[
Du_\epsilon(t) = K_\sigma^\epsilon(t) * Du_0 - \int_0^t D_x K_\sigma^\epsilon(t-s) * (H(\cdot, Du_\epsilon(s)) - F_\epsilon(\cdot, m_\epsilon(s, \cdot)))ds
\]

where \(K_\sigma^\epsilon(t) = K_\sigma(\cdot, t, x)\) and \(K_\sigma^\epsilon(t) = K_\sigma^\epsilon(\cdot, x)\) are the fractional heat kernels in \(\mathbb{R}^d\) corresponding to \(L\) and \(L^*\). Recall the heat kernel estimates of Proposition 3.7, and note that since \(H\) is Lipschitz in the last variable, \(\|D_pH(\cdot, Du_\epsilon)\|_\infty \leq C\) independent of \(\epsilon\). Let \(s \in (0, \sigma - 1)\) and apply \(|D|^s\) to (39),

\[
|D|^s m_\epsilon(t) \leq K_\sigma(t) * |D|^s m_0 - \sum_{i=1}^{d} \int_0^t |D|^s \partial_i K_\sigma(t-s) * \left[ m_\epsilon D_pH(\cdot, Du_\epsilon(s)) \right]_i ds.
\]

We compute the \(L^\infty\)-norm to find that \(|D|^s m_\epsilon\) is bounded, and then by Proposition 2.9 in [25], \(m_\epsilon \in C^\delta, \sigma - 1 + s - \delta\) for any \(\delta > 0\). Since \(\|Du_\epsilon\|_\infty \leq C\) independent of \(\epsilon\), a similar argument shows that also \(Du_\epsilon \in C^\delta, \sigma - 1 - \delta\).

Now we bootstrap to improve the regularity. Let \(k \in \{0, 1, 2\}\) and \(s \in (0, 1)\). Assume there is \(C \geq 0\) independent of \(\epsilon\) such that

\[
\|m_\epsilon\|_{C^k, \sigma} + \|Du_\epsilon\|_{C^k, \sigma} \leq C.
\]

We show that for any \(\delta \in (0, s)\) there exists \(\tilde{C} \geq 0\) independent of \(\epsilon\) such that

\[
\begin{cases}
\|m_\epsilon\|_{C^k, \sigma - 1 + s - \delta} + \|Du_\epsilon\|_{C^k, \sigma - 1 + s - \delta} \leq \tilde{C}, & \text{for } \sigma - 1 + s - \delta \leq 1, \\
\|m_\epsilon\|_{C^{k+1}, \sigma - 2 + s - \delta} + \|Du_\epsilon\|_{C^{k+1}, \sigma - 2 + s - \delta} \leq \tilde{C}, & \text{for } \sigma - 1 + s - \delta > 1.
\end{cases}
\]

We start with \(m_\epsilon\). Note that \(m_\epsilon D_pH(x, Du_\epsilon) \in C^{\delta, s}(\mathbb{R}^d)\) by (41), the chain rule and (A4). From Proposition 2.7 in [25] it follows that \(|D|^{s-\delta} D^k(m_\epsilon D_pH(x, Du_\epsilon)) \in C^{\delta, s}(\mathbb{R}^d)\).
for any $0 < \delta \ll s$. To show improved regularity we use $|D|^\alpha |D|^{s-\delta} D^k$ on both sides of the Duhamel formula (39),

$$|D|^\alpha |D|^{s-\delta} D^k m_\epsilon = K^*_\epsilon(t) * |D|^{s+\delta} D^k m_0$$

$$- \int_0^t |D|^\alpha D K^*_\epsilon(t-s) * |D|^{s-\delta} D^k (m_\epsilon D_p H(\cdot, D u_\epsilon)) ds. \tag{40}$$

Let $a = \sigma - 1 - \delta > 0$. Taking the $L^\infty$-norm, using Young’s inequality and the heat kernel estimates (A0),

$$\|D|^{\sigma-1+s-2\delta} D^k m_\epsilon\|_\infty \leq \|D|^{\sigma-1+s-2\delta} m_0\|_\infty + \int_0^t \left\| t^{\delta-\sigma} \right\| D|^{s-\delta} D^k m_\epsilon (D_p H(\cdot, D u_\epsilon)) \|_\infty \leq C \left( \|D|^{\sigma-1+s-2\delta} m_0\|_\infty + T^\frac{2}{\sigma} \|D|^{s-\delta} D^k (m_\epsilon D_p H(\cdot, D u_\epsilon)) \|_\infty \right).$$

The right hand side is bounded independent of $\epsilon$. By Proposition 2.9 in [25] this leads to

$$m_\epsilon \in \left\{ \begin{array}{ll}
C^{k,s+\sigma-1-3\delta}(\mathbb{R}^d), & \text{for } \sigma - 1 + s - 3\delta \leq 1, \\
C^{k+1, s+\sigma-2-3\delta}(\mathbb{R}^d), & \text{for } \sigma - 1 + s - 3\delta > 1.
\end{array} \right.$$

The Hölder norms are bounded by a constant $C$ independent of $\epsilon$ and $t \in [0, T]$. In a similar way we apply $|D|^{\sigma-1-\delta} |D|^{s-\delta} D^k$ to (40) and argue as before to get

$$Du_\epsilon \in \left\{ \begin{array}{ll}
C^{k,s+\sigma-1-3\delta}(\mathbb{R}^d), & \text{for } \sigma - 1 + s - 3\delta \leq 1, \\
C^{k+1, s+\sigma-2-3\delta}(\mathbb{R}^d), & \text{for } \sigma - 1 + s - 3\delta > 1,
\end{array} \right.$$ 

with the corresponding norms bounded independently of $\epsilon$ and $t \in [0, T]$. This concludes the proof of (42).

Now we iterate (42) to get $\|u_\epsilon(t)\|_{C^3_b(\mathbb{T}^d)} + \|m_\epsilon(t)\|_{C^3_b(\mathbb{T}^d)} \leq C$ independent of $\epsilon$. Then by a similar type of reasoning as in the non-local coupling case and Arzela-Ascoli, the sequence $(m_\epsilon, u_\epsilon)$ is compact in $X_1 \times X_2$ (see below Lemma 6.1 for the definitions). Thus we can extract a convergent subsequence, $(u_\epsilon_k, m_\epsilon_k) \rightarrow (u, m)$ in $X_1 \times X_2$. By a direct calculation the limit $(u, m)$ solves equation (36). This concludes the proof.

**Appendix A. Proof of Lemma 4.10**

**Proof.** a) The proof is exactly the same as in [18]. The difference is that $f$ only needs to be $C^1$ in space, since $D_x K$ is integrable in $t$.

b) **Part 1:** Uniform continuity in $x$ for $\mathcal{L}\Phi(f)$ and $\partial_x \Phi(f)$. By the definition of $\mathcal{L}$,

$$\mathcal{L}[\Phi(f)](t, x) = \int_0^t \mathcal{L}K(t-s, \cdot) * f(s, \cdot)(x) ds$$

$$= \int_0^t \int_{\mathbb{R}^N} \left[ \int_{\mathbb{R}^N} K(t-s, y + z) - K(t-s, y) - \nabla_z K(t-s, y) \cdot z 1_{|z|<1} d\mu(z) \right] f(s, x - y) dy ds$$

$$= \int_0^t \int_{\mathbb{R}^N} \int_{|z|<1} (\cdots) + \int_0^t \int_{\mathbb{R}^N} \int_{|z|>1} (\cdots) =: I_1(t, x) + I_2(t, x).$$
After a change of variables and $\|K(t, \cdot)\|_{L^1} = 1$, 
\[
|I_2(t, x_1) - I_2(t, x_2)| \leq \int_0^t \int_{|z| \geq 1} \int_{\mathbb{R}^N} K(t - s, y) \left[ f(s, x_1 - y + z) - f(s, x_1 - y) 
- f(s, x_2 - y + z) + f(s, x_2 - y) \right] dyd\mu(z)ds 
\leq 2t\|f\|_{C_{b,t}C_{b,x}^1 \mathbb{R}^N} |x_1 - x_2| \int_{|z| \geq 1} d\mu(z).
\]
Thus since and $\|I_2(t, \cdot)\|_{C_b} \leq 2t\|f\|_{C_{b,t}C_{b,x}^1 \mathbb{R}^N} \int_{|z| \geq 1} d\mu(z)$, 
\[
|I_2(t, x_1) - I_2(t, x_2)| \leq (2\|I_2(t, \cdot)\|_{C_b})^\beta |I_2(t, x_2) - I_2(t, x_2)|^{1-\beta} 
\leq 4t\|f\|_{C_{b,t}C_{b,x}^1 \mathbb{R}^N} \int_{|z| \geq 1} d\mu(z)|x_1 - x_2|^{1-\beta}.
\]

By the fundamental theorem, Fubini, and a change of variables, 
\[
I_1(t, x) = \int_0^t \int_{|z| < 1} \left[ \int_{\mathbb{R}^N} \int_0^1 \nabla_x K(t - s, y + \sigma z) - \nabla_x K(t - s, y) \right] \cdot z f(s, x - y) d\sigma dyd\mu(z)ds, 
\]
\[
= \int_0^t \int_0^1 \int_{\mathbb{R}^N} \nabla_x K(t - s, y) \cdot z \left[ f(s, x - y + \sigma z) - f(s, x - y) \right] d\mu(z)dyd\sigma ds.
\]

It follows that 
\[
I_1(t, x_1) - I_1(t, x_2) = \int_0^t \int_{|z| < 1} \nabla_x K(t - s, y) : \int_{|z| < 1} z \left[ f(s, x_1 - y + \sigma z) 
- f(s, x_2 - y + \sigma z) - (f(s, x_1 - y) - f(s, x_2 - y)) \right] d\mu(z)dyd\sigma ds.
\]

Since 
\[
|f(x_1 + \sigma z) - f(x_1) - f(x_2 + \sigma z) - f(x_2)|^{1-\beta+\gamma} \leq 2\|f\|_{C_{b,t}C_{b,x}^1 \mathbb{R}^N} |x_1 - x_2|^{1-\beta} \|f\|_{C_{b,t}C_{b,x}^1 \mathbb{R}^N} |\sigma z|^{\beta}.
\]
we see by Theorem 3.3 and (A0) that 
\[
|I_1(t, x_1) - I_1(t, x_2)| 
\leq 2 t \|f\|_{C_{b,t}C_{b,x}^1 \mathbb{R}^N} |x_1 - x_2|^{1-\beta} \|f\|_{C_{b,t}C_{b,x}^1 \mathbb{R}^N} |z|^{\beta+1} d\mu(z) 
\leq \mathcal{C} \frac{\sigma}{\gamma-1} T^{\frac{\gamma-1}{\gamma}} \int_{|z| < 1} |z|^{\beta+1} d\mu(z) \|f\|_{C_{b,t}C_{b,x}^1 \mathbb{R}^N} |x_1 - x_2|^{1-\beta}.
\]

Combining the above two estimates, we conclude that 
\[
|L[\Phi(f)](t, x_1) - L[\Phi(f)](t, x_2)| \leq \mathcal{C} \|f\|_{C_{b,t}C_{b,x}^1 \mathbb{R}^N} |x_1 - x_2|^{1-\beta},
\]

where $\mathcal{C}$ is a constant dependent on $\gamma$. 

\[\]
with \( c = \frac{\sigma}{\sigma - 1} T^{\sigma - 1} K \int_{|z| < 1} |z|^{1+\beta} d\mu(z) + 4T \int_{|z| \geq 1} d\mu(z) \). By part a), \( \partial_t \Phi(f)(t, x) = f(t, x) + \mathcal{L}[\Phi(f)](t, x) \). Since
\[
|f(t, x) - f(t, y)| \leq (2\|f\|_{C_b}^\beta |f(t, x) - f(t, y)|^{1-\beta} \leq 2\|f\|_{C_{b,t}C_{b,x}^1} |x - y|^{1-\beta},
\]
we then also get that
\[
|\partial_t \Phi[f](t, x_1) - \partial_t \Phi[f](t, x_2)| \leq (2 + c)\|f\|_{C_{b,t}C_{b,x}^1} |x_1 - x_2|^{1-\beta}.
\]

b) **Part 2**: Uniform continuity in time. First note that
\[
\mathcal{L}[\Phi(f)](s, x) = \int_0^t \mathcal{L} \Phi[(t - s, \cdot) * f(t - s, \cdot)] d\tau - \int_0^s \mathcal{L} \Phi[(s - t, \cdot) * f(s - t, \cdot)] d\tau
\]
\[
= \int_0^s \mathcal{L} \Phi[(s - t, \cdot) * f(t - s, \cdot)] d\tau + \int_s^t \mathcal{L} \Phi[(s - t, \cdot) * f(t - s, \cdot)] d\tau.
\]
Now we do as before: Split the \( z \)-domain in two parts, use the fundamental theorem and a change of variables to get
\[
\mathcal{L} \Phi[(t - s, \cdot) * f(t - s, \cdot)] = \int_0^1 \int_{\mathbb{R}^N} \int_{|z| < 1} \nabla_x K(t - s, y - z) \cdot z [f(t - s, y + \sigma z) - f(t - s, y)] d\mu(z) dy d\sigma.
\]
\[
+ \int_{\mathbb{R}^N} \int_{|z| \geq 1} K(t - s, y - z) [f(t - s, y + z) - f(t - s, y)] d\mu(z) dy.
\]
Then we apply the trick
\[
|f(t - s, y + \sigma z) - f(t - s, y) - f(s - t, y + \sigma z) + f(s - t, y)|
\]
\[
\leq 2\omega_f(|t - s|)^{1-\beta} \|f\|_{C_{b,t}C_{b,x}^1} \|z\|^\beta \quad \text{or} \quad 4\omega_f(|t - s|)^{1-\beta} \|f\|_{C_b}^\beta,
\]
and find using Theorem 3.3 and (A0) that
\[
\left| \int_0^s \mathcal{L} \Phi[(t - s, \cdot) * f(t - s, \cdot)] d\tau \right|
\]
\[
\leq \left[ \frac{\sigma}{\sigma - 1} \int_{|z| < 1} |z|^{1+\beta} d\mu(z) + 4s \int_{|z| \geq 1} d\mu(z) \right] \|f\|_{C_{b,t}C_{b,x}^1} \omega_f(|t - s|)^{1-\beta}.
\]
In a similar way we find that
\[
\left| \int_s^t \mathcal{L}K(\tau,\cdot) \ast f(t-\tau,\cdot) d\tau \right|
\leq \left[ 2 \frac{\sigma}{\sigma - 1} \left( (t^{\frac{\sigma - 1}{\sigma}} - s^{\frac{\sigma - 1}{\sigma}}) K \int_{|z| < 1} |z|^{1+\beta} d\mu(z) + 2(t - s) \int_{|z| \geq 1} d\mu(z) \right) \right] \|f\|_{C_b^\beta}.
\]
Combining all above estimates leads to
\[
\left| \mathcal{L} \Phi[f](t, x) - \mathcal{L} \Phi[f](s, x) \right| \leq c \|f\|_{C_b \cap C_{b,x}^{1,1}}^\beta \omega_f \|t - s\|^{1-\beta} + \bar{c} \|f\|_{C_b} \|t - s\|^{\frac{\sigma - 1}{\sigma}},
\]
where $c$ is defined above and in the Lemma and
\[
\bar{c} = 2 \frac{\sigma}{\sigma - 1} K \int_{|z| < 1} |z|^{1+\beta} d\mu(z) \max_{s,t\in[0,T]} \left[ \left( \frac{s^{\frac{\sigma - 1}{\sigma}} - t^{\frac{\sigma - 1}{\sigma}}}{t - s} \right)^{\frac{\sigma - 1}{\sigma}} \right] + 2T \frac{\beta}{2} \int_{|z| \geq 1} d\mu(z).
\]
Note that $\bar{c}$ is finite. Then since
\[
\partial_t \Phi[f](t, x) - \partial_t \Phi[f](s, x) = f(t, x) - f(s, x) + \mathcal{L} \Phi[f](t, x) - \mathcal{L} \Phi[f](s, x),
\]
and $|f(t, x) - f(s, x)| \leq (2\|f\|_{C_b}^\beta) \omega_f \|t - s\|^{1-\beta}$, the continuity estimate for $\partial_t \Phi[f]$ follows.

c) The proof follows by writing
\[
\partial_{x_i} \Phi(g)(t, x) = \int_0^t \partial_{x_i} K(\tau, z) g(t - \tau, x - z) dz d\tau,
\]
and then directly compute the difference $|\partial_{x_i} \Phi(g)(t, x) - \partial_{x_i} \Phi(g)(s, y)|$. 

\[\square\]

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