Possible evidences of Kaluza-Klein particles in a scalar model with spherical compactification

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Abstract

Possible experimental manifestations of the contribution of heavy Kaluza-Klein particles, within a simple scalar model in six dimensions with spherical compactification, are studied. The approach is based on the assumption that the inverse radius $L^{-1}$ of the space of extra dimensions is of the order of the scale of the super-symmetry breaking $M_{SUSY} \sim 1 \div 10 \text{ TeV}$. The total cross section of the scattering of two light particles is calculated to one loop order and the effect of the Kaluza-Klein tower is shown to be noticeable for energies $\sqrt{s} \geq 1.4L^{-1}$.

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1 Introduction

Many of the modern approaches extending the Standard Model include the hypothesis of multidimensionality of the space-time (e.g. Kaluza-Klein type theories, supergravity, superstring theory; see for instance [1] and [2], and references therein). The extra dimensions are supposed to be compactified, i.e. "curled up" to a compact manifold of a characteristic scale $L$.

Various aspects of multidimensional models of gravity and particle interactions at the classical level have been intensively studied (see, for example, [3], [4] for reviews). Some issues related to quantum features of Kaluza-Klein theories were considered in [5] - [9]. One of the interesting problems is the search and calculation of characteristic effects related to multidimensional nature of Kaluza-Klein theories. In this paper we consider one of the effects of this kind, which is essentially quantum.

In many models the scale of compactification $L$ is assumed to be (or appears to be) of the order of the inverse Planck mass $M_{Pl}^{-1}$ (see, for example, [10] and the reviews [4]). In this case additional dimensions could reveal themselves only as peculiar gravitational effects or at an early stage of the evolution of the Universe. On the other hand, there are some arguments in favour of a larger compactification scale. One of them comes from Kaluza-Klein cosmology and stems from the fact that the density of heavy Kaluza-Klein particles cannot be too large, in order not to exceed the critical density of the Universe. Estimates obtained in ref. [11] give the bound $L^{-1} < 10^6$ GeV.

Other arguments are related to results of papers [12] and [13]. In many of the above mentioned theories there is usually a much lower supersymmetry breaking scale $M_{SUSY}$. This scale can be naturally related to the compactification scale since, as it is known, supersymmetry in principle lowers under compactification of a part of the space-time dimensions [1]. Also, as it was shown in ref. [13] relating these two scales may lead to cancellation of unwanted threshold corrections in superstring theories for certain compactifications. In its turn multidimensional models, because of their non-renormalizability, apparently should be considered as some kind of low energy effective theory stemming from more fundamental theory, like superstring theory. This gives additional motivations for considering Kaluza-Klein models with $L^{-1} \sim M_{SUSY}$.

No natural mechanism providing compactification of the space of extra dimensions with such scale is known so far. In the present paper – having the above mentioned arguments in mind – we would like to study physical consequences in a multidimensional model assuming that a compactification of this kind is indeed possible. Our aim is to find an effect which (in principle) could be measured experimentally and that, because of the hypothesis made that $L^{-1}$ is of the order of a few TeV, could actually be observed at future experiments in supercolliders. This could yield decisive evidence about the validity of the Kaluza-Klein hypothesis within a given model. In performing such analysis, we must start by understanding what kind of effects can be observed and how they depend on the topology of the space of extra dimensions in general, rather than on the phenomenological details of the given model. That is why we will here restrict ourselves to a simple $\phi^4$ scalar model with two extra dimensions.

As is well known, by doing mode expansion, a multidimensional model on the space-time $M^4 \times K$ (where $K$ is a compact manifold) can be represented as an effective theory on $M^4$ with an infinite set of particles, which is often referred to as the Kaluza-Klein tower of particles or modes. The spectrum of the four-dimensional theory depends on the
topology and geometry of \( K \). The sector of the lowest state (of the zero mode) describes light particles (in the sense that their masses do not depend on \( L^{-1} \)) and coincides with the dimensionally reduced theory. Higher modes correspond to heavy particles with masses \( \sim L^{-1} \) called pyrgons (from the Greek \( \piυργος \), for ladder). It is the contribution of pyrgons to physical quantities that might give evidence about the existence of extra dimensions.

Because of its multidimensional character, the complete theory is non-renormalizable. As we have mentioned before, this can be understood as not being a basic difficulty if the theory is a low-energy limit of some more fundamental one. Whereas in a complete finite quantum theory renormalization counterterms of the low energy limit could be calculated, in the effective non-renormalizable theory, we are discussing here, the corresponding coupling constants must be considered as phenomenological ones. Fortunately, the contribution of these counterterms to the finite part of the amplitude is of order \( (s/\Lambda^2)^n \) [3], where \( \sqrt{s} \) is the energy of the colliding particles, \( \Lambda \) can be regarded as the characteristic scale of the complete theory (\( \Lambda \sim M_{Pl} \) in the case of superstrings), and \( n \geq 1 \). Thus, when \( \sqrt{s} \leq M_C \ll \Lambda \) these contributions can be neglected.

As was demonstrated in refs. [14], [15], in spite of the infinite number of fields in the theory on \( M^4 \) — and related to this non-renormalizability — the contributions of the heavy particles to the renormalized quantities decouple when the energy of the process is small enough so that \( sL \ll 1 \). This ensures that the low-energy limit of a Kaluza-Klein model is just the dimensionally reduced theory including zero modes only. When energies are comparable to \( L^{-1} \), the contributions of the heavy modes are not negligible. One may expect that, because of the infinite number of states in the Kaluza-Klein model, some accumulation of contributions of heavy modes takes place, which leads to a noticeable effect even for energies \( \sqrt{s} < 2M_1 \) (\( M_1 \) is the mass of the first heavy mode), when direct production of the excited states is still not possible. Of course, because of the decoupling, actually only a few of the lowest states (with masses up to about \( 10L^{-1} \)) will contribute essentially at these energies.

The purpose of our paper is to demonstrate that a noticeable effect does indeed appear. With this aim, we will choose a simple scalar \( \phi^4 \)-model on the six-dimensional space-time \( M^4 \times S^2 \). The space of extra dimensions is the two-dimensional sphere \( S^2 \) of radius \( L \) and with an \( SO(3) \)-invariant metric on it. Study of quantum effects on the spheres (mainly calculation of the Casimir effect and the effective potential) can be found, for example, in [3], [7]. In this paper we calculate the total cross section for the \( (2 \text{ light particles}) \rightarrow (2 \text{ light particles}) \) scattering process. Pyrgons do not contribute at tree level in this model. Actually, the discussion given in Sect. 2 shows that this statement is rather general. Thus, the 1-loop correction is important for having the effect due to presence of heavy Kaluza-Klein modes. We will calculate the 1-loop contribution and analyze it in a wide range of energies of the scattering particles. A similar problem but for the torus compactification (i.e. when the space of extra dimensions \( K \) is the two-dimensional torus \( T^2 \)) was considered in ref. [16]. We will use the results of that paper to compare with the results obtained in the present one, in order to understand to which extent the cross section may depend on the topology of the space of extra dimensions. We believe that our results are rather generic, and that comparable effects could be detected in more realistic theories constructed in the framework of the Kaluza-Klein approach. We would like to mention that some results on calculation of 1-loop Feynman diagrams on the space-time \( M^4 \times T^N \) can be found in [1].

An alternative attitude was taken in ref. [17]. There the authors used the re-
results of high energy experiments to get an upper bound on the size $L$ of the space of extra dimensions, assuming that even the first heavy Kaluza-Klein mode is not observed experimentally. Our philosophy in the present paper is different: we assume that $L^{-1} \sim M_{\text{SUSY}} \sim 1 \div 10$ TeV and look for possible experimental evidence of heavy Kaluza-Klein modes. It is clear that our results can be also used for obtaining bounds on $L$, provided that no experimental evidence of extra dimensions is seen in future experiments for this range of energies.

The paper also contains a section of more mathematical character, in which some zeta function regularization techniques [18] are described that provide a rather elegant way of treating sums over Kaluza-Klein modes which appear in the theory (for a review of these methods see [19]). Though there is some literature on the technique of performing calculations on the spheres (see, for example, [1], [4], [6] and references therein), we think that this part of our work is interesting in its own right, since some of the formulas that appear there are brand new and provide a non-trivial, alternative way of to deal with spherical compactification. Some general results on calculations on curved space-time can be found in the review [20].

The paper is organized as follows. In Sect. 2 we describe the model, choose the renormalization condition and discuss the general structure of the 1-loop results. In Sect. 3, some useful asymptotic expansions of the corresponding (regularizing) zeta function are derived. Methods for the numerical evaluation of the 1-loop contribution are devised and the amplitude and cross section for a wide range of energies is calculated. Sect. 4 contains the analysis of the total cross section in the multidimensional model, and its comparison with the cross section for theories with a finite number of heavy particles, as well as with the cross section obtained for the torus compactification. Concluding remarks are presented in Sect. 5.

2 Description of the model, mode expansion and renormalization

Let us consider a one component scalar field on the $(4 + 2)$-dimensional manifold $E = M^4 \times S^2$, where $M^4$ is Minkowski space-time and $S^2$ is a two-dimensional sphere of radius $L$. In spite of its simplicity this model captures some interesting features of both the classical and quantum properties of multidimensional theories. The action is given by

$$S = \int_E d^4x d\Omega \left[ \frac{1}{2} \left( \frac{\partial \phi(x, \theta)}{\partial x^\mu} \right)^2 + \frac{1}{2} g^{ij} \frac{\partial \phi(x, \theta)}{\partial \theta^i} \frac{\partial \phi(x, \theta)}{\partial \theta^j} - \frac{1}{2} m_0^2 \phi^2(x, \theta) - \frac{\hat{\lambda}}{4!} \phi^4(x, \theta) \right], \quad (1)$$

where $x^\mu, \mu = 0, 1, 2, 3$, are the coordinates on $M^4$, $\theta^1$ and $\theta^2$ are the standard angle coordinates on $S^2$, $0 < \theta^1 < \pi, 0 < \theta^2 < 2\pi$, and $d\Omega$ is the integration measure on the sphere. The metric $g_{ij}$ is the standard $SO(3)$-invariant metric on the two-dimensional sphere:

$$ds^2 = g_{ij} d\theta^i d\theta^j = L^2 [(d\theta^1)^2 + \sin^2 \theta^1 (d\theta^2)^2].$$

To re-interpret this model in four-dimensional terms we make an expansion of the field $\phi(x, \theta)$,

$$\phi(x, \theta) = \sum_{lm} \phi_{lm}(x) Y_{lm}(\theta), \quad (2)$$
where \( l = 0, 1, 2, \ldots, m = -l, -l + 1, \ldots, l-1, l \) and \( Y_{lm}(\theta) \) are the eigenfunctions of the Laplace operator on the internal space, i.e. spherical harmonics, satisfying

\[
\Delta Y_{lm} = -\frac{l(l+1)}{L^2} Y_{lm},
\]

and

\[
\int d\Omega Y^*_{lm} Y_{lm'} = \delta_{l,l'} \delta_{m,m'}.
\]

Substituting this expansion into the action and integrating over \( \theta \), one obtains

\[
S = \int_{M^4} d^4x \left\{ \frac{1}{2} \left( \frac{\partial \phi_0(x)}{\partial x^\mu} \right)^2 - \frac{1}{2} m_0^2 \phi_0^2(x) - \frac{\lambda_1}{4!} \phi_0^4(x) + \sum_{l>0} \sum_m \left[ \frac{\partial \phi^*_{lm}(x)}{\partial x^\mu} \frac{\partial \phi_{lm}(x)}{\partial x_\mu} - M^2 \phi^*_{lm}(x) \phi_{lm}(x) \right] - \frac{\lambda_1}{2} \phi_0^2(x) \sum_{l>0} \sum_m \phi^*_{lm}(x) \phi_{lm}(x) \right\} - S'_{\text{int}},
\]

where the four-dimensional coupling constant \( \lambda_1 \) is related to the multidimensional one \( \hat{\lambda} \) by \( \lambda_1 = \hat{\lambda}/\text{volume}(S^2) \). In \( S'_{\text{int}} \) includes all terms containing third and fourth powers of \( \phi_{lm} \) with \( l > 0 \). We see that \( S'_{\text{int}} \) includes one real scalar field \( \phi_0 \equiv \phi_{00}(x) \) describing a light particle of mass \( m_0 \), and an infinite set (“tower”) of massive complex fields \( \phi_{lm}(x) \) corresponding to heavy particles, or pyrgons, of masses given by

\[
M^2_l = m_0^2 + l(l+1)M,
\]

where \( M = L^{-1} \) is the compactification scale.

Let us consider the 4-point Green function \( \Gamma^{(\infty)} \) with external legs corresponding to the light particles \( \phi_0 \). The index \( (\infty) \) indicates that the whole Kaluza-Klein tower of modes is taken into account. Diagrams which contribute to this function in the tree and one-loop approximation are presented in Figs. 1 and 2. Terms included in \( S'_{\text{int}} \) in (4) are not relevant for our computations.

Let us first analyze the tree level contribution. The first diagram in Fig. 1 is exactly the same as in the case of the dimensionally reduced theory whose action is given by the first line in eq. (4), the second diagram appears due to extra divergences in the theory and is discussed below. Heavy modes do not contribute at this level. This property is rather general and is valid for all processes of the type \( (n \text{ light particles}) \rightarrow (k \text{ light particles}) \) at least for all scalar multidimensional theories with polynomial interactions.

The reason for this is rather simple. Suppose that we have a theory on \( M^4 \times K \), where \( K \) is a compact space, with a polynomial interaction. The analysis of tree graphs shows that in order to obtain a contribution of heavy modes at the tree level, one needs to have at least one vertex at which \( q \) light modes \( \phi_0(x) \) (with \( q < n + k \) for \( n + k > 2 \)) interact with one heavy mode \( \phi_N(x) \), where the generalized index \( N \) corresponds to a non-zero eigenvalue of the Laplace operator on \( K \). After substituting the expansion of the multidimensional field over the eigenfunctions of this operator (analogous to (2)) into the original multidimensional action, the interaction term

\[
\int_{M^4} d^4x \phi_N(x) (\phi_0(x))^q,
\]
corresponding to this vertex, will always appear multiplied by a factor
\[
\int_K d\Omega N(\theta) Y_0 \cdots Y_0 \cdot \text{times}
\]

But since the eigenfunction \( Y_0 \) corresponding to the zero eigenvalue of the Laplace operator on the compact manifold is a constant, the above integral will always be zero, due to the orthonormality condition. This implies, for example, that the process of decay of a heavy mode with number \( N \) into \( q \) zero modes is forbidden for the class of models under consideration. In the case \( K = S^2 \) the feature discussed here is a manifestation of the conservation of angular momentum.

Let us analyze now the 1-loop correction in our model. It is easy to check that, owing to the infinite sum of diagrams (see Fig. 2), the Green function to one loop order is quadratically divergent. This is certainly a reflection of the fact that the original theory is actually six-dimensional and, therefore, non-renormalizable. Thus, the divergencies cannot be removed by renormalization of the coupling constant in (3) alone. We must also add a counterterm \( \lambda_{2B} \phi^2(x, y) \Box^{(4+d)} \phi^2(x, y) \), where \( \Box^{(4+d)} \) is the d’Alambertian on \( E \) and \( \lambda_{2B} \) has mass dimension two. The second diagram in Fig. 1 corresponds to the contribution of this counterterm. Of course, for calculation of other Green functions, or higher-order loop corrections, other types of counterterms are necessary, but we are not going to discuss them here. Hence, the Lagrangian we will use for our investigation is

\[
\mathcal{L} = \frac{1}{2} \left( \frac{\partial \phi_0(x)}{\partial x} \right)^2 + \frac{m_0^2}{2} \phi_0^2(x) + \sum_{l>0} \sum_m \left[ \frac{\partial \phi^{*} m(x)}{\partial x^\mu} \frac{\partial \phi_m(x)}{\partial x_\mu} - M_l^2 \phi^{*} m(x) \phi_m(x) \right] - \frac{\lambda_{1B}}{4!} \phi_0^{*} (x) - \frac{\lambda_{1B}}{2} \phi_0^2 (x) \sum_{l>0} \sum_m \phi^{*} m(x) \phi_m(x) - \frac{\lambda_{2B}}{4!} \phi_0^{*} (x) \Box \phi_0^2 (x),
\]

where \( \lambda_{1B} \) and \( \lambda_{2B} \) are bare coupling constants. To regularize the four-dimensional integrals corresponding to the 1-loop diagrams of Fig. 2, we will employ dimensional regularization, which is performed, as usual, by making analytical continuation of the integrals to \( (4 - 2\epsilon) \) dimensions. \( \kappa \) will be a mass scale set up by the regularization procedure. The sums over \( l \) will be regularized by means of the zeta function technique [18].

Let us now specify the renormalization scheme. It is known that the results of the calculation of physical quantities at finite orders of perturbation theory depend on the renormalization scheme. Since our goal is basically to study the difference between the contributions corresponding to the complete Kaluza-Klein tower of particles and to the light particle only, we should choose our scheme such that the results are minimally affected by the renormalization procedure. A reasonable way to do this is to impose the condition that the physical amplitude for the \( (2 \text{ light particles}) \rightarrow (2 \text{ light particles}) \) scattering process of the complete theory and the amplitude of the same process in the four-dimensional theory with the zero mode only (i.e. the theory given by the Lagrangian (2) with all non-zero modes and the last term being omitted) must coincide at some normalization (subtraction) point corresponding to low energies. As subtraction point

\[1\]

The authors thank C. Nash for useful discussions on this issue.
we will choose the following point in the space of invariant variables built up out of the external four-momenta \( p_i \) \((i = 1, 2, 3, 4)\) of the scattering particles:

\[
\begin{align*}
p_1^2 &= p_2^2 = p_3^2 = p_4^2 = m_0^2, \\
p_{12}^2 &= s = \mu_s^2, \\
p_{13}^2 &= t = \mu_t^2, \\
p_{14}^2 &= u = \mu_u^2,
\end{align*}
\]

(8)

where \( p_{ij}^2 = (p_i + p_j)^2 \), \((j = 2, 3, 4)\), and \( s, t \) and \( u \) are the Mandelstam variables. Since the subtraction point is located on the mass shell, it satisfies the standard relation \( \mu_s^2 + \mu_t^2 + \mu_u^2 = 4m_0^2 \). The renormalization prescription formulated above can be written as

\[
\Gamma^{(\infty)}(p_{ij}^2; m_0, M, \lambda_{1B}, \lambda_{2B}, \epsilon)\big|_{s.p.} = \Gamma^{(0)}(p_{ij}^2; m_0, \lambda'_{1B}, \epsilon)\big|_{s.p.} = g\kappa^{2\epsilon},
\]

(9)

\[
\left[ \frac{\partial}{\partial p_{12}^2} + \frac{\partial}{\partial p_{13}^2} + \frac{\partial}{\partial p_{14}^2} \right] \Gamma^{(\infty)}\big|_{s.p.} = \left[ \frac{\partial}{\partial p_{12}^2} + \frac{\partial}{\partial p_{13}^2} + \frac{\partial}{\partial p_{14}^2} \right] \Gamma^{(0)}\big|_{s.p.} + \frac{\lambda_2}{4\kappa^{2+2\epsilon}}.
\]

(10)

Here \( \Gamma^{(0)} \) is the four-point Green function of the four-dimensional theory with the zero mode field only (i.e., the dimensionally reduced theory), \( \lambda'_{1B} \) being its bare coupling constant. In the first line we have written down the dependence of the Green functions on the momentum arguments and parameters of the theory explicitly, and we have taken into account that to one-loop order they depend on \( p_{12}^2, p_{13}^2, \) and \( p_{14}^2 \) only. The label \( s.p. \) means that the corresponding quantities are taken at the subtraction point \( (5) \). \( g \) and \( \lambda_2 \) are renormalized coupling constants. The last one is included for the sake of generality only, and we will see that our result does not depend on it.

For the usual \( \lambda \phi_4^4 \)-theory in four dimensions, the renormalization condition given by \( (8) \) alone is sufficient, whereas both conditions \( (9) \) and \( (10) \) are necessary for subtracting the ultraviolet divergences in the theory \( (9) \), because of the presence of additional divergences due to its multidimensional character.

To one-loop order, the Green functions of the complete theory and of the theory with only zero mode, are given by

\[
\Gamma^{(\infty)}(p_{ij}^2; m_0, M, \lambda_{1B}, \lambda_{2B}, \epsilon) = \lambda_{1B} + \lambda_{2B} p_{12}^2 + p_{13}^2 + p_{14}^2 + \lambda_{1B}^2 \left[ K_0(p_{ij}^2; m_0, \epsilon) + \Delta K(p_{ij}^2; m_0, M, \epsilon) \right],
\]

(11)

\[
\Gamma^{(0)}(p_{ij}^2; m_0, \lambda'_{1B}, \epsilon) = \lambda'_{1B} + \lambda_{1B}^2 K_0(p_{ij}^2; m_0, \epsilon).
\]

Here

\[
K_0(p_{ij}^2; m_0, \epsilon) \equiv K_{00}(p_{ij}^2; m_0^2, \epsilon), \quad \Delta K(p_{ij}^2; m_0, M, \epsilon) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} K_{lm}(p_{ij}^2; M_l^2, \epsilon),
\]

(12)

and \( K_{lm} \) is the contribution of the mode \( \phi_{lm} \) with mass \( M_l = \sqrt{m_0^2 + M^2l(l+1)} \) (see eq. \( (2) \)) to the one-loop diagram of two light particles scattering

\[
K_{lm}(p_{ij}^2; M_l^2, \epsilon) = -\frac{i}{32\pi^4 M_l^{2\epsilon}} \left[ I \left( \frac{p_{12}^2}{M_l^2}, \epsilon \right) + I \left( \frac{p_{13}^2}{M_l^2}, \epsilon \right) + I \left( \frac{p_{14}^2}{M_l^2}, \epsilon \right) \right].
\]

(13)

\( K_0 \) corresponds to the first diagram in Fig. 1, and \( \Delta K \) to the contribution of the sum of the term in Fig. 2. Here we assume that \( \lambda_{2B} \sim \lambda_{1B}^2 \), so that the one loop diagrams proportional to
\(\lambda_1 B \lambda_2 B\) or \(\lambda_2^2 B\) can be neglected. It can be shown that this hypothesis is consistent (see ref. [4]). Notice that if we did not make this assumption, control on the proliferation of divergences would become very hard. The function \(I\) in the formula above is the standard one-loop integral

\[
I \left( \frac{p^2}{M^2}, \epsilon \right) = M^{2\epsilon} \int d^{4-2\epsilon} q \frac{1}{(q^2 + M^2)((q - p)^2 + M^2)} \\
= \frac{i\pi^{2-\epsilon} \Gamma(\epsilon) M^{2\epsilon}}{d_x} \int_0^1 dx \frac{1}{[M^2 - p^2 x(1-x)]^\epsilon}.
\]

(14)

Let us also introduce the sum of the one-loop integrals over all Kaluza-Klein modes

\[
\Delta I \left( \frac{p^2}{M^2}, \frac{m_0}{M}, \epsilon \right) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left( \frac{M^2}{M^2_l} \right)^\epsilon I \left( \frac{p^2}{M^2_l}, \epsilon \right),
\]

(15)

Performing the renormalization, we obtain the following expression for the renormalized four-point Green function

\[
\Gamma^{(\infty)}_R \left( \frac{p^2_1}{\mu^2_1}; \frac{\mu^2_0}{M^2}; \frac{m_0}{M}; \frac{\kappa}{M}; g, \lambda_2 \right) = \lim_{\epsilon \to 0} \kappa^{-2\epsilon} \Gamma^{(\infty)} \left( \frac{p^2_1}{\mu^2_1}; m_0, M, \lambda_1 B (g, \lambda_2), \lambda_2 B (g, \lambda_2), \epsilon \right)
\]

\[
= \lim_{\epsilon \to 0} \left\{ g + \lambda_2 \frac{p^2_{12} + p^2_{13} + p^2_{14} - \mu^2_2 - \mu^2 - \mu^2_0}{12\kappa^2} + g^2 \kappa^{2\epsilon} \left[ K_0 \left( \frac{p^2_1}{\mu^2_1}; m_0, \epsilon \right) - K_0 \left( \mu^2_0; m_0, \epsilon \right) \right]

- \left[ \frac{p^2_{12} + p^2_{13} + p^2_{14} - \mu^2_2 - \mu^2 - \mu^2_0}{3} \right] \left( \frac{\partial}{\partial p^2_{12}} + \frac{\partial}{\partial p^2_{13}} + \frac{\partial}{\partial p^2_{14}} \right) \Delta K \left( \frac{p^2_1}{\mu^2_1}; m_0, M, \epsilon \right) \bigg|_{s.p.} \right\}
\]

(16)

where we denote \(\mu^2_s = \mu^2_t = \mu^2\) and \(\mu^2_u = \mu^2_u\). The r.h.s. of this expression is regular in \(\epsilon\), and after calculating the integrals and the sums over \(l\) and \(m\), we take the limit \(\epsilon \to 0\). The above expression is rather general and valid for an arbitrary subtraction point. It simplifies if the relation \(\mu^2_s + \mu^2_t + \mu^2_u = 4m_0^2\), fulfilled by the subtraction point (8), is taken into account. Before doing this, let us discuss the structure of the renormalized Green function for a subtraction point when all subtraction scales are equal: \(\mu^2_s = \mu^2_t = \mu^2_u = \mu^2\). Then eq. (16) can be written as

\[
\Gamma^{(\infty)}_R \left( \frac{p^2_1}{\mu^2_1}; \mu^2_0; \frac{m_0}{M}; \frac{\kappa}{M}; g, \lambda_2 \right)
\]

\[
= g + \lambda_2 \frac{p^2_{12} + p^2_{13} + p^2_{14} - 3\mu^2}{12\kappa^2}

+ g^2 \lim_{\epsilon \to 0} \kappa^{2\epsilon} \left\{ K_0 \left( \frac{p^2_1}{\mu^2_1}; m_0, \epsilon \right) - K_0 \left( \mu^2_0; \mu^2, \mu^2; m_0, \epsilon \right) - \frac{i}{32\pi^4 M^{2\epsilon}} \left[ \left( \Delta I \left( \frac{p^2_1}{\mu^2_1}; \frac{m_0}{M}, \epsilon \right) \right) \right]

- \left[ \frac{p^2_{12} - \mu^2}{3} \right] \left( \frac{\partial}{\partial p^2_{12}} \Delta I \left( \frac{p^2_1}{\mu^2_1}; \frac{m_0}{M}, \epsilon \right) \bigg|_{p^2_{12} = \mu^2} \right)

+ \left( p^2_{12} \to p^2_{13} \right) + \left( p^2_{12} \to p^2_{14} \right) \right\}.
\]
As it was mentioned above, the contribution $\Delta I$ of the heavy modes contains quadratic divergences in momenta (in the framework of the dimensional regularization used here this means that it contains singular terms $\sim 1/\epsilon$ and $\sim p^2/\epsilon$). The renormalization prescription amounts to a subtraction of the first two terms of the Taylor expansion of this contribution at the point $\mu^2$, which is sufficient to remove the divergences. The contribution $K_0$ of the zero mode sector (the dimensionally reduced theory) diverges only logarithmically and subtraction of the first term of the Taylor expansion, imposed by the renormalization prescription (8), is sufficient to make it finite in the limit $\epsilon \to 0$.

Finally, let us write down the expression for the four-point Green function of the complete theory (i.e. with all the Kaluza-Klein modes) renormalized according to the conditions (9) and (10) at the subtraction point (8), and taken at a momentum point which lies on the mass shell of the light particle

$$\Gamma_\beta^{(\infty)}(s, t, u; g) = (s, t, u; M, g)$$

The variables $s$, $t$ and $u$ are not independent, since they satisfy the well known Mandelstam relation $s + t + u = 4m_0^2$.

The formula above is rather remarkable. It turns out that on mass shell, due to cancellations between the $s$-, $t$- and $u$-channels, the contribution proportional to $\lambda_2$ and the terms containing derivatives of the one-loop integrals vanish. Thus, heavy Kaluza-Klein modes contribute to the renormalized Green function on the mass shell in exactly the same way as the light particle in the dimensionally reduced theory does. Indeed, it can be easily checked that the additional non-renormalized divergences arising from the infinite summation in $\Delta K$ cancel among themselves when the three scattering channels are summed up together.

## 3 Calculation of the one-loop contribution

In this section we will analyze the one-loop contribution of the heavy Kaluza-Klein modes, develop methods for its numerical evaluation, and present results for the amplitude of $(2 \text{ light particles}) \rightarrow (2 \text{ light particles})$ scattering.

Starting point of the analysis is the expression

$$\Delta I\left(\frac{p^2}{M^2}, \frac{m_0}{M}, \epsilon\right) = i\pi^{2-\epsilon}\Gamma(\epsilon) \int_0^1 dx \sum_{l, m}^\prime \left(\frac{M^2}{M_l^2}\right)^\epsilon \left[1 - \frac{p^2 x(1 - x)}{M_l^2}\right]^{-\epsilon},$$

(see (14) and (15)), where the prime means that the term for $l = 0$ is absent from the summatory. One could think of considering two different limits corresponding to two possible expansions of the binomial (low and high momentum expansions, in principle):

$$\begin{align*}
(i) &\quad (1 - y)^{-\epsilon} = \sum_{k=0}^{\infty} \frac{\Gamma(k + \epsilon)}{k! \Gamma(\epsilon)} y^k, & y &\equiv \frac{p^2 x(1 - x)}{M_l^2}, \\
(ii) &\quad (1 - y)^{-\epsilon} = (-y)^{-\epsilon} (1 - y^{-1})^{-\epsilon} = (-1)^{-\epsilon} \sum_{k=0}^{\infty} \frac{\Gamma(k + \epsilon)}{k! \Gamma(\epsilon)} y^{-k-\epsilon}.
\end{align*}$$

(21)
There is no problem in doing the small-momentum expansion (i), which is valid for $|y| < 1$. In fact, since the maximum of $x(1-x)$ when $0 \leq x \leq 1$ is attained at $1/4$ (for $x = 1/2$), this formula is valid whenever $p^2 < 4M_{l,m}^2$. Nevertheless, the “high-momentum expansion” (ii) is much more difficult to perform. Actually, it is not possible to express its range of validity, $|y| < 1$, in terms of a simple inequality involving $p^2$ and $M_{l,m}^2$. As it stands, eq. (21) is useless: we must first integrate over $x$, in order to get rid of this unwanted dependence and then the formula yielding the desired expansion for $p^2 \geq 4M_{l,m}^2$ is different, according to different ranges of variation of $p^2$ in terms of $M_{l,m}^2$.

Taking into account eq. (6), we will see that the infinite sum over $l$ gives rise to a certain derivative of a generalized, inhomogeneous Epstein zeta-function, which we will manage to continue analytically after some work. On the other hand, in case (i) the $x$-integral yields just beta function factors.

3.1. Expansion for small momentum.

The sums and integrals involved in the low-momentum expansion of eq. (19) can be performed in the following order:

$$\Delta I \left( \frac{p^2}{M^2}, m_0, \epsilon \right) = i\pi^{2-\epsilon} \sum_{k=0}^{\infty} \frac{\Gamma(k + \epsilon)}{k!} B(k + 1, k + 1) \left( \frac{p^2}{M^2} \right)^k S_{k+\epsilon}, \quad (22)$$

where we have defined

$$S_{k+\epsilon} = \sum'_{l,m} \left( \frac{M_{l,m}^2}{M^2} \right)^{-k-\epsilon}, \quad (23)$$

$M^2 = (L^{-1})^2$ being the constant, leading mass which invariably appears in $M_l^2$ (see (3) for the sphere compactification), and where we have used

$$\int_0^1 dx [x(1-x)]^s = B(s + 1, s + 1), \quad (24)$$

$B(s,t) = \Gamma(s)\Gamma(t)/\Gamma(s+t)$ being Euler’s beta function.

In the particular case of the spherical compactification, this sum reads

$$S_{k+\epsilon} = \sum_{l=1}^{\infty} (2l + 1) \left[ l(l+1) + \frac{m_0^2}{M^2} \right]^{-k-\epsilon} = 2 \sum_{l=1}^{\infty} (l+1/2) \left[ (l+1/2)^2 + \left( \frac{m_0^2}{M^2} - \frac{1}{4} \right) \right]^{-k-\epsilon}, \quad (25)$$

and can be written exactly as

$$S_{k+\epsilon} = \frac{1}{1-k-\epsilon} \frac{\partial}{\partial a} F(s; a, b) \bigg|_{s = k + \epsilon, a = \frac{1}{2}, b = \frac{m_0^2}{M^2} - \frac{1}{4}}, \quad (26)$$

where the function $F$ is defined to be

$$F(s; a, b) \equiv \sum_{l=1}^{\infty} [(l + a)^2 + b]^{1-s}, \quad (27)$$

and is related to the most simple example of —what is called— an Epstein-Hurwitz zeta function. Some useful and mathematically elegant expressions for these functions have
been obtained in refs. [21] (see also [19]). In particular, use of Jacobi’s theta function
identity yields [21]

\[ F(s; a, b) = \frac{b^{1-s}}{\Gamma(s-1)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n + s - 1)}{n!} b^{-n} \zeta(-2n, a) + \frac{\sqrt{\pi}}{2} \frac{b^{3/2-s}}{\Gamma(s-1)} \frac{\Gamma(s-3/2)}{\Gamma(s-1)} \]

+ \frac{2\pi^{s-1}}{\Gamma(s-1)} b^{3/4-s/2} \sum_{n=1}^{\infty} n^{s-3/2} \cos(2\pi na) K_{s-3/2}(2\pi n \sqrt{b}), \]

(28)

which, in spite of the equality sign, should be understood as an asymptotic expression, not as a convergent series expansion. Later we will obtain explicitly the optimal cut of this series, which has been numerically studied in detail in Ref. [22]. Taking now into account eq. (26), we obtain

\[ S_{k+\epsilon} = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!} \frac{\Gamma(n + k + \epsilon - 1)}{\Gamma(k + \epsilon)} b^{1-n-k-\epsilon} \tilde{\zeta}(-2n, 1/2), \]

(29)

where

\[ \tilde{\zeta}(s, a) \equiv \frac{\partial}{\partial a} \zeta(s, a), \]

(30)

\[ \zeta(s, a) \] being the Hurwitz (also called Riemann generalized) zeta function defined for \( s > 1 \) by the series

\[ \zeta(s, a) \equiv \sum_{l=0}^{\infty} \frac{1}{(l + a)^s}. \]

(31)

Actually, the following simple relation holds

\[ \tilde{\zeta}(s, a) = -s \zeta(s + 1, a), \]

(32)

and for the few first terms of \( S_{k+\epsilon} \) (providing the best cut of the asymptotic series), we obtain

\[ S_{k+\epsilon} = b^{1-k-\epsilon} \left[ \frac{1}{k + \epsilon - 1} + 2b^{-1} \zeta(-1, 1/2) - 2(k + \epsilon)b^{-2} \zeta(-3, 1/2) \right. \]

\[ \left. + (k + \epsilon)(k + 1 + \epsilon)b^{-3} \zeta(-5, 1/2) - \cdots \right]. \]

(33)

We have used the fact that the coefficient \( -\tilde{\zeta}(0, 1/2) = 1 \), actually

\[ \tilde{\zeta}(0, a) = \frac{\partial}{\partial a} \zeta(0, a) = \frac{\partial}{\partial a} \left( \frac{1}{2} - a \right) = -1. \]

(34)

The same result is obtained from (32) in the limit \( s \to 0 \).

Putting everything together, we can write

\[ \Delta I(\frac{p^2}{M^2}, \frac{m_0}{M}, \epsilon) = i\pi^{2-\epsilon} \Gamma(\epsilon) \left\{ \left[ \frac{b^{1-\epsilon}}{\epsilon - 1} + 2b^{-\epsilon} \zeta(-1, 1/2) - 2\epsilon b^{-1-\epsilon} \zeta(-3, 1/2) \right. \right. \]

\[ + \epsilon(1 + \epsilon)b^{-2-\epsilon} \zeta(-5, 1/2) - \frac{\epsilon(1 + \epsilon)(2 + \epsilon)}{3} b^{-3-\epsilon} \zeta(-7, 1/2) + \cdots \right\} \]

\[ + \frac{p^2}{M^2} B(2, 2) \left[ b^{-\epsilon} + 2\epsilon b^{-1-\epsilon} \zeta(-1, 1/2) - 2\epsilon(1 + \epsilon)b^{-2-\epsilon} \zeta(-3, 1/2) \right. \]

\[ + \frac{\epsilon(1 + \epsilon)(2 + \epsilon)}{3} b^{-3-\epsilon} \zeta(-7, 1/2) + \cdots \right\}. \]
3.2. Expansion for arbitrary momentum

In a physical setting, obtain convergent series for very small values of \(m\). On the contrary, for large values of \(m_0/M\) the above series is useful. In a physical setting, \(m_0^2 \ll M^2\), so that \(b \simeq -1/4\). In the next subsection we will obtain convergent series for \(m_0/M\) small (even \(m_0 = 0\) is allowed) and valid for all finite \(p^2\).

3.2. Expansion for arbitrary momentum.

In the general case we must follow a completely different strategy. We shall first perform the \(\epsilon\)-expansion and then integrate over the \(x\)-variable in eq. (14):

\[
I(\frac{p^2}{M^2}, \epsilon) = i\pi^{2-\epsilon} \Gamma(\epsilon) - i\pi^2 J(\frac{p^2}{4M^2}) + O(\epsilon),
\]

where \(J(z)\) is the finite part of the 1-loop integral

\[
J(z) = \int_0^1 dx \ln[1 - 4zx(1 - x)],
\]

which is equal to

\[
J(z) = \begin{cases} 
J_1(z) = 2\sqrt{\frac{z}{1-z}} \ln(\sqrt{1-z} + \sqrt{-z}) - 2, & \text{for } z \leq 0, \\
J_2(z) = 2\sqrt{\frac{z}{1-z}} \arctan\sqrt{\frac{z}{1-z}} - 2, & \text{for } 0 < z \leq 1, \\
J_3(z) = -i\pi \sqrt{\frac{z}{1-z}} + 2\sqrt{\frac{z}{1-z}} \ln(\sqrt{z} + \sqrt{z-1}) - 2, & \text{for } z > 1.
\end{cases}
\]

Now we can proceed with the summation over \(l, m\). Taking the degeneracy in (19) into account and using \(\zeta\)-regularization for the sums, we get

\[
\Delta I(\frac{p^2}{M^2}, \frac{m_0}{M}, \epsilon) = i\pi^{2-\epsilon} \Gamma(\epsilon)[2\zeta(-1, \frac{1}{2}) - 1] - 2i\pi^2 \sum_{l=1}^{\infty} (l + \frac{1}{2}) \ln[(l + \frac{1}{2})^2 + b] - 2i\pi^2 \Delta J(\frac{p^2}{4M^2}, \frac{m_0}{M}) + O(\epsilon).
\]
In particular, for \( p^2 < 0 \)

\[
\Delta J\left( \frac{p^2}{4M^2}; \frac{m_0}{M} \right) = \sum_{l=1}^{\infty} (l + \frac{1}{2})J_1\left( \frac{p^2}{4M^2} \right),
\]

while for \( p^2 > 0 \)

\[
\Delta J\left( \frac{p^2}{4M^2}; \frac{m_0}{M} \right) = \sum_{l=1}^{l^*(p)} (l + \frac{1}{2})J_3\left( \frac{p^2}{4M^2} \right) + \sum_{l=l^*(p)+1}^{\infty} (l + \frac{1}{2})J_2\left( \frac{p^2}{4M^2} \right)
\]

which contains in general an imaginary part. Here \( l^*(p) \) is the maximum value of \( l \) which satisfies the inequality \( 4M^2(l + 1/2)^2 < p^2 - 4m_0^2 \). If such \( l \) does not exist or is smaller than 1, we put \( l^*(p) = 0 \) and the first sum in eq. (43) is absent. As we have already mentioned, the divergent sums over \( l \) are understood as being regularized by the zeta-function regularization procedure. These formally divergent series, which are independent of \( p^2 \) or are linear in \( p^2 \), do not contribute to physical (renormalized) quantities, see the discussion in Sect. 2.

The calculation is now carried out in connection with the zeta function regularization procedure. In fact, after expanding the functions under the summation signs in powers of \( m \) to the low-momentum series that was obtained before (see eq. (35)), now for small values of \( |p^2/4M^2| < 1 \), we put \( l^*(p) = 0 \), and the first sum in eq. (43) is absent. As we have already mentioned, the divergent sums over \( l \) are understood as being regularized by the zeta-function regularization procedure. These formally divergent series, which are independent of \( p^2 \) or are linear in \( p^2 \), do not contribute to physical (renormalized) quantities, see the discussion in Sect. 2.

The calculation is now carried out in connection with the zeta function regularization procedure. In fact, after expanding the functions under the summation signs in powers of \( u = p^2/(4M^2) \) we are faced up with summations over the \( l \)-index, which give rise to Hurwitz zeta functions. As we clearly see from expression (41) above, the number of terms contributing to each sum changes with \( p \). Thus, different explicit series are obtained for the different ranges of \( M^2/p^2 \). Specifically, for the first ranges we get the following series.

The first range, \( |p^2/4M^2| < 1 \), is somewhat special and deserves a careful treatment. According to the preceding analysis, only contributions in terms of a power series of \( p^2/(4M^2) \) arise in this case, and we arrive to a series expansion which is the alternative to the low-momentum series that was obtained before (see eq. (43)), now for small values of \( m_0^2/M^2 \), including the case \( m_0^2 = 0 \), i.e.

\[
\Delta I\left( \frac{p^2}{M^2}, \frac{m_0}{M}, \epsilon \right) = i\pi^{2-\epsilon} \frac{\Gamma(\epsilon)}{\Gamma(2-\epsilon)} \left[ 2\zeta(-1, \frac{1}{2}) - 1 \right] - 2i\pi^2 \sum_{l=1}^{\infty} (l + \frac{1}{2}) \ln \left[ (l + \frac{1}{2})^2 + b \right] + 4i\pi^2 \sum_{l=1}^{\infty} \left( \frac{u_l}{3} + \frac{2u_l^2}{15} + \frac{8u_l^3}{105} + \frac{16u_l^4}{315} + \frac{128u_l^5}{3465} + \frac{256u_l^6}{9009} + \cdots \right) + O(\epsilon),
\]

\[
u_l \equiv \frac{p^2}{4M_l^2}; \quad \nu_l \equiv \frac{p^2}{4M^2(l+1/2)^2 + b}, \quad b \equiv \frac{m_0^2}{M^2} - \frac{1}{4}, \quad |p^2| < 4m_0^2 + 8M^2.
\]

The \( l \)-sums yield again Epstein-Hurwitz zeta functions. In particular,

\[
2 \sum_{l=1}^{\infty} (l + 1/2)u_l^k = \frac{1}{2(1-k)} \frac{\partial}{\partial a} E_1^{(1)}(k-1; a, b) \Bigg|_{a=1/2} \left( \frac{p^2}{4M^2} \right)^k, \quad (45)
\]

where the superindices (1) mean ‘truncated’, in the sense that the first term in the definitions of these zeta functions (the one for \( n = 0 \)) is absent, namely

\[
E_1^{(1)}(k; a, b) = \sum_{n=1}^{\infty} [(n + a)^2 + b]^{-k} = F(k + 1; a, b),
\]
in terms of the function $F$ introduced before. Let us call
\[ h^{(1)}(k; 1/2, b) \equiv \left. \frac{1}{1-k} \frac{\partial}{\partial a} E_1^{(1)}(k; a, b) \right|_{a=1/2}. \]

The following bounds for these coefficients of the power series expansion will be useful. First, introducing the constants
\[ \alpha_k \equiv \sum_{l=1}^{\infty} (2l + 1)[l(l+1)]^{-k} < 2\zeta(2k-1), \]

we have
\[ \alpha_1 = 1.1544, \quad \alpha_2 = 0.9996, \quad \alpha_3 = 0.4041, \quad \alpha_4 = 0.1918, \quad \alpha_5 = 0.0944, \]
\[ \alpha_6 = 0.0470, \quad \alpha_7 = 0.0235, \quad \alpha_8 = 0.0117, \quad \alpha_9 = 0.0059, \ldots, \]

and
\[ m_0 = 0, \quad M \neq 0 \text{ arbitrary} : \quad h^{(1)}(k; 1/2, b) = \alpha_k, \]
\[ m_0 \neq 0 \quad \left\{ \begin{array}{l} m_0^2 \leq M^2/4 : \quad 2\zeta(1)(2k-1, 1/2) \leq h^{(1)}(k; 1/2, b) \leq \alpha_k, \\ m_0^2 \geq M^2/4 : \quad h^{(1)}(k; 1/2, b) \leq 2\zeta(1)(2k, 1/2). \end{array} \right. \]

We obtain, in particular
\[ h^{(1)}(1; 1/2, b)|_{b=0} \leq 0.93 \leq h^{(1)}(1; 1/2, b)|_{-1/4 \leq b \leq 0} \leq 1.15, \]
\[ h^{(1)}(2; 1/2, b)|_{b=0} \leq 0.85 \leq h^{(1)}(2; 1/2, b)|_{-1/4 \leq b \leq 0} \leq 0.90, \]
\[ h^{(1)}(3; 1/2, b)|_{b=0} \leq 0.29 \leq h^{(1)}(3; 1/2, b)|_{-1/4 \leq b \leq 0} \leq 0.40, \]
\[ h^{(1)}(4; 1/2, b)|_{b=0} \leq 0.12 \leq h^{(1)}(4; 1/2, b)|_{-1/4 \leq b \leq 0} \leq 0.19, \]

In few words, we see that with increasing $m_0$ the value of the coefficients decreases, starting from very reasonable values (the $\alpha_k$) for $m_0 = 0$. However, the fact that $b$ in eq. (44) can be negative (for small values of $m_0$) makes it difficult in practice to use expression (28) for the Epstein-Hurwitz zeta function. A more convenient (albeit lengthy) alternative is to perform a binomial expansion, which is absolutely convergent in $p^2/4M_1^2$ whenever $|b(2/3)^2| < 1$, i.e. $m_0^2/M_0^2 < 5/2$ [1], [21], [22], [24]. In this way, we obtain
\[
\Delta I\left(\frac{p^2}{M^2}, \frac{m_0}{M}, \epsilon\right) = 2i\pi^2 \left[ -\frac{11}{24} \pi^{-\epsilon} \Gamma(\epsilon) + 2\zeta(-1, 1/2) + \ln 2 \\
+ \sum_{k=1}^{\infty} \left( \frac{-1}{k} \right)^k \zeta^{(1)}(2k-1, 1/2) \left( \frac{m_0^2}{M^2} - \frac{1}{4} \right)^k \\
+ \sum_{k=1}^{\infty} c_k \left( \frac{m_0^2}{2M^2} + 1 \right)^k h^{(1)}(k; \frac{1}{2}, \frac{m_0^2}{2M^2} - \frac{1}{4}) \left( \frac{p^2}{4M_1^2} \right)^k + O(\epsilon^2) \right],
\]

where
\[ c_1 = \frac{2}{3}, \quad c_2 = \frac{3}{2}, \quad c_3 = \frac{2}{3}, \quad c_4 = \frac{5}{7}, \quad c_5 = \frac{2}{20}, \quad c_6 = \frac{9}{13}, \cdots \]

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For higher positive values of $p^2$ contributions of new type appear (corresponding to the first sum of eq. (43)). Thus, for $4(m_0^2 + 2M^2) \leq p^2 < 4(m_0^2 + 6M^2)$ we get

$$\Delta I \left( \frac{p^2}{M^2}, \frac{m_0}{M}, \epsilon \right) = i \pi^{2-\epsilon} \Gamma(\epsilon) \left[ 2\zeta(-1, \frac{1}{2}) - 1 \right] - \frac{2m_0^2}{M^2} \sum_{l=1}^{\infty} (l + \frac{1}{2}) \ln[(l + \frac{1}{2})^2 + b]$$

$$- \frac{2M^2}{p^2} \left[ \frac{1}{2} \ln \frac{p^2}{M^2} - \frac{M^2}{p^2} \right] - 2 + 2 \sum_{l=2}^{\infty} (l + \frac{1}{2}) J_2 \left( \frac{p^2}{4M^2} \right) + O(\epsilon)$$

$$= 2i \pi^2 \left[ -\frac{11}{24} \pi^{-\epsilon} \Gamma(\epsilon) + 2\zeta'(-1, 1/2) + \ln 2 \right.$$ 

$$+ \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \zeta^{(1)}(2k - 1, 1/2) \left( \frac{m_0^2}{M^2} - \frac{1}{4} \right)^k - \frac{3}{2} \sqrt{1 - \frac{4M^2}{p^2}}$$

$$+ \frac{1}{2} \sqrt{1 - \frac{4M^2}{p^2}} \ln \frac{p^2}{M^2} - \frac{M^2}{p^2} \left( 1 - \frac{M^2}{2p^2} + \ldots \right)$$

$$+ \sum_{k=1}^{\infty} 3^k c_k \left( \frac{m_0^2}{6M^2} + 1 \right)^k h^{(2)} \left( k; \frac{1}{2} \frac{m_0^2}{M^2} - \frac{1}{4} \right) \left( \frac{p^2}{4M^2} \right)^k + O(\epsilon^2) \right],$$

and so on, for the rest of the intervals for higher momentum. Here

$$h^{(2)}(k; 1/2, b) \equiv \frac{1}{1-k} \frac{\partial}{\partial a} E_1^{(2)}(k; a, b) \bigg|_{a=1/2},$$

being $E_1^{(2)}$ the truncated function

$$E_1^{(2)}(k; a, b) \equiv \sum_{n=2}^{\infty} [(n + a)^2 + b]^{-k}.$$  

(55)

Alternatively, the whole expression can be expanded in terms of the truncated Hurwitz zeta functions

$$\zeta^{(k)}(s, 1/2) \equiv \zeta(s, 1/2) - \sum_{n=0}^{k-1} (n + 1/2)^{-s}, \quad k = 1, 2, 3, \ldots$$  

(56)

whose numerical values are:

$$\zeta^{(1)}(2, 1/2) \simeq 0.93, \quad \zeta^{(1)}(4, 1/2) \simeq 0.23, \quad \zeta^{(1)}(6, 1/2) \simeq 0.09,$$

$$\zeta^{(1)}(8, 1/2) \simeq 0.04, \quad \zeta^{(2)}(2, 1/2) \simeq 0.4904, \quad \zeta^{(2)}(4, 1/2) \simeq 0.0373,$$

$$\zeta^{(2)}(6, 1/2) \simeq 0.0048, \quad \zeta^{(2)}(8, 1/2) \simeq 0.0007, \quad \zeta^{(3)}(2, 1/2) \simeq 0.33036,$$

$$\zeta^{(3)}(4, 1/2) \simeq 0.01172, \quad \zeta^{(3)}(6, 1/2) \simeq 0.00073, \quad \zeta^{(3)}(8, 1/2) \simeq 0.00005.$$  

Furthermore, a useful asymptotic expression for the derivative $\zeta'(-1, 1/2)$ can be found in [23]. By analogous methods, expansions for $p^2 < 0$ can be obtained.
4 Calculation of the total cross section

In this section we calculate the total cross section $\sigma^{(\infty)}(s)$ of the scattering process (2 light particles) $\rightarrow$ (2 light particles), in the case when the whole Kaluza-Klein tower of heavy particles contribute, and we compare it with $\sigma^{(N)}(s)$, the cross section obtained for the case when only $N$ first modes are taken into account (i.e. modes with $l = 0, 1, 2, \ldots, N$).

With this notation, $\sigma^{(0)}(s)$ is the cross section in the dimensionally reduced model, i.e. when only the light particle contributes. Such comparison will be quite illuminative for understanding the relative contributions of the various heavy modes.

We have found that the quantity which describes the net effect due to the tower of heavy particles is the following ratio, which is built up of the total cross sections:

$$\Delta^{(\infty,0)} \left( \frac{s}{4M^2}, \frac{\mu_s^2}{M^2}, \frac{\mu_u^2}{M^2}, \frac{\mu_t^2}{M^2}, m_0 \right) \equiv 16\pi^2 \frac{\sigma^{(\infty)}(s) - \sigma^{(0)}(s)}{g \sigma^{(0)}(s)}.$$  \hspace{1cm} (57)

Using the expression for the 4-point Green function [18], renormalized according to [3] and [11], we calculate the corresponding total cross sections and obtain that, to leading order (i.e. 1-loop order) in the coupling constant $g$, the function (57) is equal to

$$\Delta^{(\infty,0)} \left( \frac{s}{4M^2}, \frac{\mu_s^2}{M^2}, \frac{\mu_u^2}{M^2}, \frac{\mu_t^2}{M^2}, m_0 \right) = -2 \left\{ \text{Re} J \left( \frac{s}{4M^2}, \frac{m_0}{M} \right) \right.$$  
$$+ \frac{2}{s - 4m_0^2} \int_{-(s-4m_0^2)}^0 du \text{Re} J \left( \frac{u}{4M^2}, \frac{m_0}{M} \right) - \Delta J \left( \frac{\mu_s^2}{4M^2}, \frac{m_0}{M} \right)$$  
$$- \Delta J \left( \frac{\mu_u^2}{4M^2}, \frac{m_0}{M} \right) - \Delta J \left( \frac{\mu_t^2}{4M^2}, \frac{m_0}{M} \right) \right\}. \hspace{1cm} (58)$$

Here we assume that $\mu_s^2, \mu_u^2, \mu_t^2 < 4m_0^2$.

For the numerical evaluation we take the zero mode particle to be much lighter than the first heavy mode and choose the subtraction point to be at the low energy interval, even in comparison with $m_0$. Recall that $\mu_s^2 + \mu_t^2 + \mu_u^2 = 4m_0^2$. We take

$$\frac{m_0^2}{M^2} = 10^{-4}, \quad \frac{\mu_s^2}{m_0^2} = 10^{-2}, \quad \frac{\mu_u^2}{m_0^2} = \frac{1}{2}(4m_0^2 - \mu_s^2). \hspace{1cm} (59)$$

First of all, we observe that if the parameters of our theory satisfy $\mu_s^2 \ll m_0^2 \ll M^2$ (notice that for our choice [58] these inequalities are indeed fulfilled) the dependence of the function $\Delta^{(\infty,0)}$ on $\mu_s^2/M^2$, $\mu_t^2/M^2$, $\mu_u^2/M^2$ and $m_0^2/M^2$ is very weak, and in practice it depends only on one dimensionless parameter. We choose this parameter to be $z = s/(4M^2)$.

Using the results of Sect. 3 we calculate the function $\Delta^{(\infty,0)}(z)$. Its plot in the range $0 < z < 1$ is presented in Fig. 1. We see that the contribution of the Kaluza-Klein tower of particles is considerable. Thus, $\Delta^{(\infty,0)} \approx 0.51$ for $s = 0.5(4M_t^2)$ and $\Delta^{(\infty,0)} \approx 0.12$ for $s = 0.25(4M_t^2)$. So, already for the energies much smaller than the threshold of the first heavy particle the effect is quite noticeable.

To be remarked is also the quick convergence of the sums over $l$ in (42) and (43), contributing to the function $\Delta^{(\infty,0)}(z)$. For instance, a few terms give already curves which do not change any more when adding more summands (we have checked indeed that the sum of 50 terms of the above series is, for all purposes, identical to the sum of the 20 first terms).
Of course, due to the convergence of the sums only the first few terms with low \( l \) give essential contributions whereas those corresponding to higher modes are quite negligible. To understand how many modes we really see from the plot of \( \Delta^{(\infty,0)} \) further analysis is needed. For this purpose, in Fig. 3 we also present the curve \( \Delta^{(1,0)}(z) \) characterizing the contribution of the first heavy mode only. We see that the difference is not small (0.3 for \( z = 0.5 \) and 0.05 for \( z = 0.25 \)), and the function \( \Delta^{(\infty,0)}(z) \) within a few per cent accuracy represents more than just the first mode.

To have more illustrative characteristics, we introduce the quantities

\[
\epsilon_N(z) \equiv \frac{\Delta^{(N,0)}(z)}{\Delta^{(\infty,0)}(z)},
\]

which show the relative contributions of the first \( N \) heavy Kaluza-Klein modes. The plots for some \( \epsilon_N(z) \) for \( 0 < z < 1 \) are presented in Fig. 4. We conclude that with an accuracy of about 5 \( \div \) 10\% the function \( \Delta^{(\infty,0)}(z) \) in the range \( s \sim 0.8M_T^2 \div 2.4M_T^2 \) actually shows the presence of at least 3 \( \div \) 4 first heavy modes in the theory.

Another interesting question is how the contribution of the heavy Kaluza-Klein modes depends on the topology of the space of extra dimensions. Here we restrict ourselves only to comparison of the results for \( \Delta_S^{(\infty,0)}(z) \), which is calculated in this article ("S" stands for "spherical compactification"), with the behaviour of \( \Delta_T^{(\infty,0)}(z) \) for the toroidal compactification calculated in ref. [16] for the similar model but with the space-time being \( M^4 \times T^2 \), where \( T^2 \) is the two-dimensional torus. It is clear that the results of the comparison depend on the relation between the inverse radius of the sphere \( M_S = L^{-1} \) (for further discussion we have attached the index "S" to it) and the scale \( M_T \) equal to the inverse radius of the circles forming the torus \( T^2 = S^1 \times S^1 \). For the present analysis we assume that in both compactifications the dimensionally reduced models, i.e. zero mode sectors of the initial theories, coincide and that the masses of the first heavy modes are the same. These imply that for both cases the mass \( m_0 \) of the zero mode is the same and

\[
2M_S^2 = M_T^2.
\]

The plots of the functions \( \Delta_S^{(\infty,0)} \) and \( \Delta_T^{(\infty,0)} \) for \( 0 < z < 1 \) are represented in Fig. 5. Table 1 provides explicit values of these functions, corresponding to a sample of values of \( z \). The difference between the curves is quite noticeable (for example, \( \Delta_T^{(\infty,0)} - \Delta_S^{(\infty,0)} = 0.25 \) for \( z = 0.5 \) and is equal to 0.06 for \( z = 0.25 \)) and is due to the difference of the spectra of the Laplace operator on the sphere and on the torus. Namely, for the two-dimensional sphere \( S^2 \) the eigenvalues \( \lambda_l(S^2) \) of the Laplace operator, the squares of the masses of the Kaluza-Klein modes determined by them, and the multiplicities \( d_l(S^2) \) of the eigenvalues are given by (cf. eq. (5))

\[
(M_l^{(S)})^2 = m_0^2 + \lambda_l(S^2)M_S^2, \quad \lambda_l(S^2) = l(l + 1), \quad d_l^{(S)} = 2l + 1.
\]

For the two-dimensional torus \( T^2 \) the analogous are:

\[
(M_n^{(T)})^2 = m_0^2 + \lambda_n(T^2)M_T^2, \quad \lambda_n(T^2) = n_1^2 + n_2^2,
\]

where \( n = (n_1, n_2) \) is a two-vector labelling the eigenvalues, \( -\infty < n_i < \infty \) (\( i = 1, 2 \)), and \( d_{|n|}^{(T)} \) is the number of such vectors with the same length \(|n| = \sqrt{n_1^2 + n_2^2} \). Using the
expansions similar to (44) and eq. (58) we get that for $|z| < 1$
\[
\Delta_{K}^{(\infty,0)}(z) \approx \frac{4}{9} \left( \frac{M_{T}^{2}}{M_{K}^{2}} \right)^{2} \zeta(2|K|).
\]

Here the index $K$ labels the type of compactification, e.g. $K = S$ for the case of the sphere and $K = T$ for the case of the torus, and $\zeta(s|K)$ is the zeta-function of the Laplace operator on the manifold $K$ [18] (see also [26]).

\[
\zeta(s|K) = \sum_{n} \frac{1}{(\lambda_{n}(K))^{s}},
\]

where the prime means that the term corresponding to the zero eigenvalue is absent from the summation. For example, for $K = S$ this function can be expressed in terms of the derivative of the generalized Epstein-Hurwitz zeta-function (27):

\[
\zeta(s|S^{2}) = \sum_{l=1}^{\infty} \frac{d_{l}(S^{2})}{[\lambda_{l}(S^{2})]} = \sum_{l=1}^{\infty} \frac{2l + 1}{[l(l + 1)]^{s}} = -\frac{1}{s - 1} \frac{\partial}{\partial a} F(s; a, -\frac{1}{4})|_{a=1/2}.
\]

Taking into account the relation (61) between $M_{S}^{2}$ and $M_{T}^{2}$, we obtain an approximate expression relating the ratio of the contributions of the Kaluza-Klein towers of particles corresponding to the spherical and toroidal compactifications with the characteristics of the Laplace operator on these manifolds:

\[
\frac{\Delta_{S}^{(\infty,0)}(z)}{\Delta_{T}^{(\infty,0)}(z)} \approx \frac{4\zeta(2|S^{2}|)}{\zeta(2|T^{2}|)} \approx 0.66.
\]

Results of our numerical computations (Fig. 5) are with good accuracy in accordance with formula (66).

In spite of the fact that, from the point of view of high-energy physics, it seems rather unlikely that energies satisfying $\sqrt{s} > 2M_{1}$ will be available in the nearest future, for theoretical reasons and for the sake of completeness we have also calculated the function $\Delta_{S}^{(\infty,0)}(z)$ for $1 < z < 21$ (see Fig. 6). Peaks of the curve correspond very approximately (since $m_{0} \neq 0$) to the thresholds of creation of heavy mode particles, i.e. to the values $s = 4M_{l}^{2}$ or $z = l(l + 1)/2$ for $l = 1, 2, 3, 4, 5, 6$.
5 Conclusions

In this paper we have studied the behaviour of the total cross section for scattering of two light particles in an effective theory in four dimensions, obtained from the six-dimensional scalar theory through the spherical compactification of two extra dimensions.

Though our model cannot be directly termed as being physical, we do believe that the effect we have calculated is of a very general nature, and that it will also take place in more realistic theories. Thus, our results can be in principle used for comparison with actual experiments. The idea is the following.

We assume that the low-energy sector of the theory is already well determined. Therefore, the value of the renormalized coupling constant $g$ is known and the total cross section $\sigma^{(0)}(s)$ can be calculated with sufficient accuracy. Experimentally one should measure the total cross section $\sigma^{\exp}(s)$ and compute the quantity

$$\Delta^{\exp}(s) = -16\pi^2 \frac{\sigma^{\exp}(s) - \sigma^{(0)}(s)}{g\sigma^{(0)}(s)}.$$  

(cf. (57)). If above the threshold for the light particle, one has that $\Delta^{\exp}(s) = 0$, then there is no evidence of heavy Kaluza-Klein modes at given energies.

If, on the contrary, $\Delta^{\exp}(s) \neq 0$, then there will be evidence for the existence of heavier particles. The obvious next step would be to see which curve $\Delta^{(N,0)}(s)$ fits the experimental data best. If it is the curve with $N = \infty$ (or sufficiently large $N$), for a certain manifold $K$, this fact should be considered as an indirect evidence of the multidimensional nature of the interactions — at least within the framework of the given class of models and for that type of compactification.

Our calculations suggest that, indeed, the effect can be quite noticeable, even for energies below the threshold of the first heavy particle (see Fig. 1). The values of the parameters (59) that we have used for our computations can mimic a physical situation with, for example, $m_0 = 100$ GeV and $M = 10$ TeV, and with the charge $g$ renormalized at the low energy point $\sqrt{\mu^2} = 10$ GeV.

Our results also show that we can distinguish between different types of compactification of the extra dimensions. The main contribution, for energies below the threshold of the first heavy mass state, is basically determined by the zeta-function $\zeta(2|K)$, uniquely associated to the two-dimensional manifold $K$ through the spectrum of the Laplace operator on it. This provides, by the way, a further example of the relevance of the concept of zeta function in high-energy physics.

Of course, one should not forget that the effect studied here is of one-loop order and, apparently, rather hard to detect experimentally. Because of this, an interesting possibility in a more realistic model would be to consider specific processes for which the tree approximation is absent and the leading contribution is given by the one-loop diagrams even in the zero-mode sector of the theory. Such processes are obviously more sensitive to the heavy Kaluza-Klein modes.

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Figure captions

**Fig. 1** Tree diagrams contributing to the 4-point Green function $\Gamma^{(\infty)}$. The lines correspond to the light particle, the bar corresponds to derivatives with respect to external momenta.

**Fig. 2** 1-loop diagrams contributing to the 4-point Green function $\Gamma^{(\infty)}$. Thin lines correspond to the light particle with mass $m_0$. The thick line with the label $M_l$ corresponds to propagation of the particle with mass $M_l$.

**Fig. 3** Plots of the functions $\Delta^{(\infty,0)}(z)$ and $\Delta^{(1,0)}(z)$ in the interval $0 < z < 1$ for the sphere compactification.

**Fig. 4** Plots of the functions $\epsilon_N(z)$ defined by eq. (60) for $N = 1, 2, 3, 4, 5$; $\epsilon_\infty(z) \equiv 1$.

**Fig. 5** Plots of the functions $\Delta_{S}^{(\infty,0)}(z)$ and $\Delta_{T}^{(\infty,0)}(z)$ in the interval $0 < z < 1$ for compactifications of extra dimensions to the two-dimensional sphere $S^2$ and the two-dimensional torus $T^2$ respectively. In both cases the dimensionally reduced models are the same and scales characterizing these manifolds are chosen in such a way that $2M^2_S = M^2_T$, so that the masses of the first heavy particles of the Kaluza-Klein towers are equal.

**Fig. 6** Plot of the function $\Delta_{S}^{(\infty,0)}(z)$ with the sphere compactification for $0 < z < 21$, i.e. up to the threshold of the sixth heavy mode.
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