Symmetry and collective motion

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This work presents the basic elements of the formalism involved in the treatment of Hamiltonian dynamical systems with symmetry and the geometrical description of collective motion.

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1 Introduction

The continuous symmetries of physical systems are reflected not only by conservation laws, but also by constraints on the probability, or particle distribution functions, and phase-space reduction.

Similarly to the cohomology groups [1], defined in terms of finite coverings by open sets, the statistical distributions are defined with respect to the partition of the phase-space in "elementary cells". In general, this space is the cotangent fibration over a configuration space which is homogeneous for the action of a kinematical symmetry group, such as \(SO(3, \mathbb{R})\) or Galilei. When a many-particle equilibrium distribution of minimum energy breaks to some extent this symmetry, then it defines an "intrinsic frame", and the low-energy dynamics of the system can be described in terms of collective modes.

This work presents\(^1\), following [2, 3] as main references, the elements of the formalism involved in the treatment of Hamiltonian dynamical systems with symmetry. The "momentum map" provided by the set of invariant observables is presented in Section 2. The related Marsden-Weinstein reduction of the phase-space is discussed in Section 3. The geometrical approach to collective motion is formulated and illustrated in examples in Section 4.

2 Symplectic actions and the moment map

Let \(M\) be a smooth manifold. An action to the left of the Lie group \(G\) on \(M\) is a smooth map \(\Phi : G \times M \to M\) such that

\[\begin{align*}
&i) \forall m \in M, \Phi(e, m) = m, \\
&ii) \forall g, h \in G, \forall m \in M, \Phi(g, \Phi(h, m)) = \Phi(gh, m).
\end{align*}\]

The orbit of \(m \in M\) is \(G \cdot m = \{\Phi(g, m) / g \in G\}\).

The action is transitive if \(G \cdot m = M, \forall m \in M\). In this case \(M\) is called a homogeneous space. The isotropy group of \(\Phi\) at \(m \in M\) is

\[G_m = \{g \in G / \Phi(g, m) = m\}.\]

\(^1\)the next sections are based on the notes of the seminars "Lie algebras" and "Dynamical symmetries and collective variables" given in 1987-1989 at the Institute of Atomic Physics from Bucharest.
This subgroup is closed because $\tilde{\Phi}_m : G \mapsto M$, $\tilde{\Phi}_m(g) \equiv \Phi(g, m)$ is continuous, and $G_m = \tilde{\Phi}_m^{-1}(m)$.

Let $\Phi$ be an action of $G$ on the symplectic manifold $(M, \omega)$. This action is symplectic if $\omega$ is invariant to $\Phi_g (\Phi_g(m) \equiv \Phi(g, m))$,

$$\Phi^*_g \omega = \omega, \quad \forall g \in G.$$ Let $\Phi : G \times M \mapsto M$ be a smooth action, and $\xi \in T_eG \equiv \mathfrak{g}$. Then $\Phi^\xi : R \times M \mapsto M$ defined by

$$\Phi^\xi(t, m) = \Phi(e^{t\xi}, m), \quad t \in R,$$

is an $R$-action on $M$, and $\Phi^\xi$ is a current on $M$. The corresponding vector field given by

$$\xi_M(m) = \frac{d}{dt} |_{t=0} \Phi(e^{t\xi}, m)$$

is the infinitesimal generator of the action induced by $\xi$, and

$$\xi_M(\Phi_g m) = T_m \Phi_g (Ad_{g^{-1}} \xi)_M(m)$$

with $Ad_g \xi = T_e R_{g^{-1}} L_g \xi$ (Appendix 1).

**Proposition 1.** Let $\Phi$ be an action to the left of $G$ on $M$, and $\tilde{\Phi}_m : G \mapsto M$ defined by $\tilde{\Phi}_m(g) = \Phi(g, m), \forall m \in M$. Then:

i) for any $\omega \in \Omega^q(M)$, $\Phi_g$-invariant ($\Phi^*_g \omega = \omega$), one can define a map $^2 \Sigma : M \mapsto \Lambda^q(\mathfrak{g})$,

$$\Sigma(m) = \tilde{\Phi}^*_m \omega.$$ ii) the map $\Sigma$ is a $G$-morphism (equivariant):

$$\Sigma \circ \Phi_g = Ad_{g^{-1}} \Sigma.$$  

**Theorem 1.** Any symplectic action of a Lie group $G$ on $(M, \omega)$ defines a $G$-morphism $\Sigma : M \mapsto Z^2(\mathfrak{g})$ such that:

i) $\Sigma(M)$ is a union of $G$-orbits in $Z^2(\mathfrak{g})$.

ii) if the action of $G$ on $M$ is transitive, then $\Sigma(M)$ is a single orbit.

Let $\Sigma \in Z^2(\mathfrak{g})$, $\mathfrak{g} \equiv T_eG$, considered as a 2-form on $G$, and

$$g_\Sigma = \{ \xi \in \mathfrak{g} / L_\xi \Sigma = 0 \} \quad \text{and} \quad h_\Sigma = \{ \xi \in \mathfrak{g} / i_\xi \Sigma = 0 \}.$$

$^2\Lambda^k(\mathfrak{g})$ is the set of $k$-forms on $\mathfrak{g}$ (the left-invariant subset of $\Omega^k(G)$), $Z^k(\mathfrak{g}) = \{ \omega \in \Lambda^k(\mathfrak{g}), d\omega = 0 \}$, $H^k(\mathfrak{g}) = Z^k(\mathfrak{g}) / B^k(\mathfrak{g}) = \text{Ker } d(\Lambda^k) / d\Lambda^{k-1}$.  

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where $L_{\xi} = i_\xi d + d i_\xi$ denotes the Lie derivative. Thus, $\mathfrak{g}_\Sigma$ is the isotropy algebra of $\Sigma$, while the connected group $H_\Sigma$ generated by $\mathfrak{h}_\Sigma$ is the leaf through the identity $e \in G$ of the characteristic (null) foliation of $\Sigma$.

**Proposition 2.** For $\Sigma \in Z^2(\mathfrak{g})$, if $H_\Sigma$ is closed in $G$, then $M_\Sigma = G/H_\Sigma$ is a symplectic manifold, and the orbit in $Z^2(\mathfrak{g})$ associated with $M_\Sigma$ is the orbit of $G$ through $\Sigma$.

**Proof:** Let $\pi : G \mapsto M\Sigma$ be the projection $\pi(g) = gH_\Sigma$. Considering $H_\Sigma \equiv m \in M_\Sigma$, then $\pi_m(g) = gm = \Phi_m(g)$, and $\pi = \Phi_m; \Sigma = \pi^*_m \omega \nabla$.

**Proposition 3.** The $G$-orbits through the elements $\Sigma \in Z^2(\mathfrak{g})$ having $H_\Sigma$ closed are covering spaces for symplectic manifolds $(M,\omega)$ on which $G$ acts transitively.

**Proof:** Let $M = G/H$. If $i_{\eta_M} \omega = 0$ then $\eta_M = 0$ and $\eta \in \mathfrak{h}$. Thus, if $\Sigma = \pi^* \omega$ then $\mathfrak{h}_\Sigma = \{\xi \in \mathfrak{h}\} = \mathfrak{h}$, $H_\Sigma$ is the connected component of $H$, and $M_\Sigma = G/H_\Sigma$ is covering space for $M \nabla$.

**Example.** (Kostant-Souriau): If $\Sigma = d\beta$, $\beta \in \mathfrak{g}^*$, then $H_\Sigma$ is closed, and $H_\beta$ is the connected component of the group $H_\beta = \{g \in G/Ad_g^{* -1} \beta = \beta\}$.

Thus, $G/H_\Sigma$ is covering space for the orbit $G \cdot \beta \simeq G/H_\beta$.

**Theorem 2,** (Kostant-Souriau): If $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = \{0\}$, then any symplectic manifold, homogeneous for $G$, covers a $G$-orbit in $\mathfrak{g}^*$ [4].

**Proof.** If $H^2(\mathfrak{g}) = \{0\}$ then $\forall \omega \in Z^2(\mathfrak{g})$ has the form $\omega = d\beta$, $\beta \in \mathfrak{g}^*$, and $H_\omega$ is closed. However, if $H^1(\mathfrak{g}) = \{0\}$ then $\forall \beta \in Z^1(\mathfrak{g})$, $\beta = 0$, so that $\forall \omega \in Z^2(\mathfrak{g})$, $\omega = d\beta$ with $\beta$ unique $\nabla$.

**Proposition 4.** Let us consider $\omega \in Z^2(\mathfrak{g})$, left-invariant on $G$. Then:

i) the subalgebra $\mathfrak{h}_\omega$ of minimal dimension is commutative.

ii) if $M$ is a homogeneous symplectic space with maximal dimension, then $\forall m \in M$, the connected component of the isotropy group $G_m$ is abelian.

**Proof.** i) Assume $\omega_t = \omega + t\theta$ with $\theta \in \Omega^1(G)$, left-invariant. If $dim(\mathfrak{h}_\omega)$ is minimal, then $dim(\mathfrak{h}_\omega) = dim(\mathfrak{h}_\omega)$, and $\forall \xi \in \mathfrak{h}_\omega$ can be extended to a curve $\xi_t = \xi + t\eta + 0(t^2) \in \mathfrak{h}_\omega$. As $i_{\xi_t} \omega_t = 0$, $\theta([\xi, \eta]) = 0$ $\forall \theta$, such that $[\xi, \eta] = 0$.

ii) If $M \simeq G/H$ has maximal dimension then $H$ has minimal dimension, and $\mathfrak{h} = \mathfrak{h}_\omega$ is abelian $\nabla$.

Let $\Phi : G \times M \mapsto M$ be a symplectic action of $G$ on $(M,\omega)$. The action
Φ is strongly symplectic\(^3\) (Hamiltonian) if there exists a (comomentum) map
\[ \lambda : g \mapsto F(M), \]
such that \( i_{\xi_M} \omega = d\lambda \). A strongly symplectic action Φ is strongly Hamiltonian if \( \lambda \) is a \( G \)-morphism (equivariant), namely
\[ \Phi^{-1}_g \lambda = \lambda \circ Ad_g, \]
and the diagram
\[ 0 \mapsto R \mapsto F(M) \mapsto \text{ham}(M) \mapsto 0 \]
commutes.

Let \( \Phi : G \times M \mapsto M \) be a symplectic action on the connected symplectic manifold \((M, \omega)\). Then \( J : M \mapsto g^* \) is a moment map for the action \( \Phi \) if \( \forall \xi \in g \)
\[ i_{\xi_M} \omega = d\hat{J}(\xi) \]
where \( \hat{J}(\xi) : M \mapsto R \) is defined by \( \hat{J}(\xi)_m = J_m \cdot \xi \equiv \langle J_m, \xi \rangle \). In particular, a Hamiltonian action \( \Phi : G \times M \mapsto M \) defines a moment map by \( \hat{J}(\xi) = \lambda_\xi \).

**Proposition 5.** Let \( M \) be a symplectic connected manifold and \( \Phi : G \times M \mapsto M \) a symplectic action with the moment map \( J \). For any \( a \in G \) and \( \xi \in g \) one defines the map \( \hat{Z}_a(\xi) : M \mapsto R \)
\[ \hat{Z}_a(\xi) = \Phi^{-1}_a \hat{J}(\xi) - \hat{J}(Ad_a \xi). \]
Then
i) \( \hat{Z}_a(\xi) \) is a constant on \( M \) and defines a map \( Z : G \mapsto g^*, \langle Z_a, \xi \rangle = \hat{Z}_a(\xi) \), called coadjoint cocycle.
ii) \( Z \) satisfies the identity \( Z_{ab} = Z_a + Ad_{a^{-1}}Z_b \).

**Proof.** \( M \) is connected and \( d\hat{Z}_a(\xi) = 0 \) on \( M \), such that \( \hat{Z}_a(\xi) \) is a constant \( \nabla \).

\(^3\)If \( H^1(M, R) = \{0\} \) then every closed 1-form is exact, and every symplectic action is strongly symplectic.
One should note that the cocycle \( Z : G \mapsto g^* \) defines an infinitesimal cocycle \( \omega \in Z^2(g) \), by
\[
\omega(\xi, \eta) = \frac{d}{dt} \big|_{t=0} \hat{Z}_{\exp(\xi t)}(\eta) = \{ \hat{J}(\xi), \hat{J}(\eta) \} - \hat{J}([\xi, \eta]) .
\] (1)

A coadjoint cocycle \( \Delta : G \mapsto g^* \) is coboundary if there exists \( \mu \in g^* \) such that
\[
\Delta_a = \mu - Ad_{a^\ast} \mu .
\]
Similarly to (1), \( \Delta \) defines an infinitesimal coboundary \( \delta \in B^2(g) \) by
\[
\delta(\xi, \eta) = \frac{d}{dt} \big|_{t=0} \hat{\Delta}_{\exp(\xi t)}(\eta) = -\mu \cdot [\xi, \eta] ,\; \delta = d\mu .
\]

The set of the equivalence classes \([Z]\) of the cocycles-mod-coboundaries is the (deRham) cohomology space \( H^2(G, \mathbb{R}) \) of \( G \).

**Proposition 6.** Let \( J_1 \) and \( J_2 \) be two moment maps for the symplectic action \( \Phi : G \times M \mapsto M \), with cocycles \( Z_1 \) and \( Z_2 \). Then \([Z_1] = [Z_2]\), so that to any symplectic action which admits a moment map it corresponds a well defined cohomology class.

**Proposition 7.** Let \( \Phi : G \times M \mapsto M \) be a symplectic action with a moment map. Then
\begin{itemize}
  \item [i)] \( \Phi \) defines a cohomology class \([c_\Phi] \in H^2(g, \mathbb{R})\) which measures the obstruction to find an equivariant moment map \((J \circ \Phi_g = Ad_{g^{-1}} \cdot J)\).
  \item [ii)] an equivariant moment map \( J \) exists only iff \([c_\Phi] = 0\).
  \item [iii)] when \([c_\Phi] = 0\) the set of all possible equivariant moment maps is parameterized by \( H^1(g) \).
\end{itemize}

**Examples**

I. If \( g \) is a semisimple Lie algebra (Appendix 2) then \( H^1(g) = H^2(g) = 0 \), and any symplectic action of \( G \) admits an unique equivariant moment map.

II. Let \( G = H \times V \) be the Galilei group, expressed as a semidirect product between \( H = SO(3, \mathbb{R}) \times \mathbb{R} \) and \( V = \mathbb{R}^6 \). Thus, an element \( \Gamma \in G \),
\[
\Gamma = \begin{bmatrix} \hat{A} & V & X \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} ,
\]

\(4 \) \( b_k(M) = \dim H^k(M, \mathbb{R}), \ k = 0, 1, 2, \ldots \) are the Betti numbers, and \( \chi(M, \mathbb{R}) = \sum_k (-1)^k b_k(M) \) is the Euler-Poincaré characteristic of \( M \).
is specified by $\hat{A} \in SO(3, \mathbb{R})$, $t \in \mathbb{R}$, $V \in \mathbb{R}^3$ and $X \in \mathbb{R}^3$. Such an element acts on a point $(x_0, t_0)$ of the space-time manifold $M = \mathbb{R}^3 \times \mathbb{R}$ by

$$\Phi_T(x_0, t_0) = (\hat{A}x_0 + X + Vt_0, t + t_0).$$

The algebra $g$ of $G$ has the form $g = k + p$, where $k = \{\hat{\xi} \in so(3)\}$, $p = \{(v, x, \tau) \in \mathbb{R}^7\}$, $k \cap p = \{0\}$, $[k, k] \subset k$, $[k, p] \subset p$ according to

$$[\hat{\xi}, (v, x, \tau)] = (\hat{\xi}v, \hat{\xi}x, 0),$$

and $[p, p] \subset p$,

$$[(v, x, \tau), (v', x', \tau')] = (0, \tau'x - \tau v', 0).$$

Proposition 8. If $G$ is the Galilei group, then $H^2(g)$ is 1-dimensional, and up to a coboundary any cocycle $\Sigma \in Z^2(g)$ is given by

$$\Sigma((\hat{\xi}, v, x, \tau), (\hat{\xi}', v', x', \tau')) = m(v \cdot x' - v' \cdot x).$$

Theorem 3. Let $\Phi$ be a transitive Hamiltonian action of $G$ on $M$ with the equivariant moment map $J$. Then:

i) $J$ maps $M$ on an orbit of $G$ in $g^*$.

ii) $J$ is an immersion, and $M$ is a covering space for $J(M)$.

Proof. (i) $\Phi^*_g \cdot J = J \circ \text{Ad}_g$, with $\Phi_g$ transitive. (ii) Let $T_mJ \cdot \eta_M = 0$, $\eta \in g$, $\eta_M \in T_mM$. Then

$$\langle T_mJ \cdot \eta_M, \xi \rangle = 0, \ \forall \xi \in g.$$

If $dJ \xi \cdot \eta_M = 0$ then $\omega(\xi_M, \eta_M) = 0 \ \forall \xi \in g$. However $\{\xi_M(m)/\xi \in g\} = T_mM$ and $\omega$ is nondegenerate, such that $\eta_m = 0$. Thus, $\text{Ker}(TJ) = \{0\}$, and $J$ is an immersion $\triangledown$.

3 Dynamical systems with symmetry

3.1 Introduction

The description of space and time in classical mechanics in terms of a class of reference frames which are equivalent with respect to formulation of the

\footnote{The result for the Lorentz group is presented in [5].}
principles of mechanics extends the notion of geometrical symmetry to the one of symmetry of equations of classical mechanics. This symmetry has a kinematical character, because is independent of the physical system under consideration, and the effect of symmetry transformations is a change in the reference frame. As the set of geometrical symmetry transformations is a group, the kinematical symmetry transformations form a Lie group, called Galilei group in the nonrelativistic case, or Poincaré in relativistic mechanics. The existence of these symmetry groups is reflected by the existence of a set of observables \( \{ J_\alpha \}_\alpha \) which remain invariant during the time-evolution of the physical system. The fields \( \{ X_\alpha \}_\alpha \) on the classical phase-space \((M, \omega)\), \(i_{X_\alpha} \omega = dJ_\alpha\), generated by \( \{ J_\alpha \}_\alpha \), commute with the field \( X_H\), \(i_{X_H} \omega = dH\) of the Hamilton function, and \( H \) is a constant on the leaves \( J_\alpha^{-1}(\mu_\alpha) \subset M \) determined by regular values \( \{ \mu_\alpha \}_\alpha \). As a result, the "intrinsic" dynamics provided by \( H \) is defined on a phase-space \( \tilde{M} \), called "reduced phase-space", with a dimension smaller than of the initial phase-space \( M \).

The existence of additional invariant observables, beside the ones provided by the kinematical symmetry, implies the existence of a larger symmetry group, specific for the physical system under consideration, called dynamical symmetry group (in the case of nonrelativistic Kepler problem this is \( SO(3, 1) \) for states with positive energy, \( H = E > 0 \), and \( SO(4, R) \) for bound states, \( H = E < 0 \)). Its existence implies a lower dimensionality of the reduced phase-space, and allows a partial description of the intrinsic dynamics in terms of the group action. When the reduced phase-space \( \tilde{M} \) with respect to the action of the dynamical symmetry group \( G \) is a point, the system is called completely integrable, and the Hamiltonian current on \( M \) is provided by the action of \( G \), or one of its subgroups. In general, a similar result is obtained for an equilibrium point on \( \tilde{M} \) of the reduced Hamiltonian, or for systems with constrains which "freeze" the dynamics on \( \tilde{M} \). In such a situation the system has a collective behaviour, because the Hamiltonian current on \( M \) arises by the action of \( G \), and can be described completely using a number of variables smaller than the dimension of \( M \).

### 3.2 The reduced phase-space

Let \((M, \omega)\) be a symplectic manifold, \( \Phi : G \times M \mapsto M \) a symplectic action of \( G \) on \( M \), \( J : M \mapsto g^* \) a moment map for \( \Phi \) such that \( J \circ \Phi_g = Ad_{g^{-1}}^*J \),
and

\[ G_\mu = \{ g \in G/\text{Ad}_{g^{-1}}^* \mu = \mu \} , \]

with \( \mu \) a regular value\(^6\) of \( J \). Then:

\( i) P_\mu = J^{-1}(\mu) \) is a submanifold of \( M \).

\( ii) \forall p \in J^{-1}(\mu), T_p(G_\mu \cdot p) = T_p(G \cdot p) \cap T_pP_\mu, \) such that \( G_\mu \) acts on \( J^{-1}(\mu) \).

The proof of (i) follows from the Sard’s theorem, while for (ii) \( J \circ \Phi_g = \text{Ad}_{g^{-1}}^*J \) shows that \( \xi_M(p) \in T_pJ^{-1}(\mu) \) iff \( \xi_M^g(\mu) = 0 \), namely \( \xi \in \mathfrak{g}_\mu \nabla \).

An action \( \Phi : G \times M \mapsto M \) is called proper if the map \( \Psi : G \times M \mapsto M \times M \) defined by \( \Psi(g, m) = (m, \Phi_g(m)) \) is proper\(^7\). For such an action \( \forall m \in M, G_m = \{ a \in G/\Phi_a(m) = m \} \) is compact. The action is called free if \( G_m = \{ e \} \).

If \( \mu \) is a regular value of \( J \), then the action of \( G_\mu \) on \( J^{-1}(\mu) \) is locally free, and provides a foliation having as leaves the orbits of \( G_\mu \). If the action of \( G_\mu \) is proper, then the orbits \( G_\mu \cdot p, p \in P_\mu, \) are closed submanifolds in \( J^{-1}(\mu) \). If the action of \( G_\mu \) is free, \( \forall p \in P_\mu, G_p = \{ e \} \subset G_\mu, \) and there exists a submanifold \( S \subset P_\mu, \) containing \( p, \) having the properties:

\( i) S \) is closed in \( G_\mu(S) \).

\( ii) G_\mu(S) \) is an open neighborhood of the orbit \( G_\mu \cdot p. \)

\( iii) \) if \( a \in G_\mu \) and \( \Phi_a(S) \cap S \neq \emptyset \) then \( a = e. \)

This last property shows that the leaves of \( G_\mu \) through the points of \( S \) may intersect at most once, so that the space of the orbits \( \bar{M} = J^{-1}(\mu)/G_\mu \) has the structure of a manifold (the coordinates on \( S \) can be chosen as local coordinates on \( M \))\(^8\). Moreover, \( \bar{M} \) is Hausdorff because the orbits are closed submanifolds in \( J^{-1}(\mu) \) (\( G_\mu \) is closed and \( \Phi \) is continuous). When (iii) does not hold \( \forall S, \) then \( G_p \neq \{ e \}. \)

Let \( P \) be a manifold and \( \omega \in \Omega^2(P). \) Then

\[ E_\omega = \{ v \in TP/i_\omega v = 0 \} \]

is called the characteristic distribution of \( \omega. \) If \( E_\omega \) is a subbundle of \( TP, \) then \( \omega \) is called regular on \( P. \) In such a case, the distribution \( E_\omega \) is integrable,

\(^6\)For \( f : M \mapsto N \) of class \( C^1, n \in N \) is a regular value of \( f \) if \( \forall m \in f^{-1}\{n\}, \) \( T_m f \) is surjective.

\(^7\)Which means that any compact subset \( K \subset M \times M \) is the image of a compact subset \( \Psi^{-1}(K) \subset G \times M, \) or that if the sequences \( x_i \) and \( \Phi_{g_i} x_i, i = 1, 2, 3, \ldots, \) are convergent, then \( g_i \) contains a convergent subsequence.

\(^8\)If the action of \( G \) on \( M \) is smooth, free and proper then \( M \mapsto M/G \) is a principal bundle with \( G \) as fiber and structure group, [2] p. 276.
and it defines a foliation $\mathcal{F}$ on $P$. Let $\bar{M} = P/\mathcal{F}$ be the space of leaves, obtained by the identification of the points from each leaf. The space $\bar{M}$ has the structure of a manifold if any point of a leaf is contained in a submanifold $S$, transversal to the leaf, such that $S$ intersects the leaf at most once. In this case, the local coordinates on $S$ can be used as local coordinates on $\bar{M}$, and $\bar{M}$ acquires the structure of a manifold.

**Proposition 9.** Let $G$ be a compact Lie group acting on the manifold $P$ such that its orbits provide a foliation of $P$. For $p \in P$, $G_p = \{ a \in G / \Phi_a(p) = p \}$, and $G_p^0$ denotes the connected component of $G_p$. Then the foliation of $P$ by $G$-orbits is fibration iff the representation of $\Gamma_p = G_p/G_p^0$ on the space $N_p$, normal to the orbit $G \cdot p$, is trivial $\forall p \in P$.

**Examples of one-dimensional foliations which are not fibrations**

1. **The linear current on the 2-dimensional torus**

Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus, with coordinates

$$[x, y] = (x, y) \bmod \mathbb{Z}^2 , \ (x, y) \in \mathbb{R}^2 .$$

An action $\Phi_t : T \mapsto T$, $t \in \mathbb{R}$, of $\mathbb{R}$ on $T$, can be defined by

$$\Phi_t[x, y] = [x + at, y + bt] ,$$

where $a, b$ are real constants. Thus, $\{ \Phi_t[x, y] / t \in \mathbb{R} \}$ is the leaf at $[x, y]$. When the ratio $a/b$ is irrational, any such orbit is dense on the torus.

Let $\bar{M}$ be the space of the leaves of the foliation, and $\pi : T \mapsto \bar{M},$

$$\pi[x, y] = \{ \Phi_t[x, y] / t \in \mathbb{R} \} ,$$

the projection on $\bar{M}$. Presuming that there exists a topology on $\bar{M}$ with respect to which $\pi$ is continuous, let $U$ be an open set on $\bar{M}$. If $U \neq \emptyset$, then $\pi^{-1}(U) = T$, so that $U = \bar{M}$. Thus, the only admissible topology on $\bar{M}$ is the trivial one.

2. **The $S^1$ orbits on the M"obius band**

Let $\varphi : \mathbb{Z} \times \mathbb{R}^2 \mapsto \mathbb{R}^2$ be the action of $\mathbb{Z}$ on $\mathbb{R}^2$ given by

$$\varphi_n(x, y) = A^n(x, y) , n \in \mathbb{Z}, \ A(x, y) = (x + 1, -y) .$$

The M"obius band is then $M = \mathbb{R}^2/\mathbb{Z}$, with points

$$[x, y] = \{ \varphi_n(x, y) / n \in \mathbb{Z} \} \in M .$$
Also, $M$ is a bundle over the circle $S^1$, with projection $\pi : M \mapsto S^1$,

$$\pi[x, y] = x \mod \mathbb{Z}.$$ 

Let $\Phi : G \times M \mapsto M$ be the action on $M$ of the circle $G = \mathbb{R}/2\mathbb{Z}$, given by

$$\Phi_t[x, y] = [x + t, y], \quad t \in \mathbb{R}.$$ 

The orbits of $G$ on $M$ are double coverings for the central circle $S^1 = \Phi_{r}[x, 0]$, and $M$ is a half-line. It can be parameterized by $y \geq 0$, but is not a manifold near $y = 0$.

**Theorem 4.** Let $(M, \omega)$ be a symplectic manifold, $G$ a Lie group acting symplectically on $M$, and $J : M \mapsto g^*$ the equivariant moment map. Let $\mu \in g^*$ be a regular value of $J$, and presume that $G_{\mu}$ acts freely and properly on $J^{-1}(\mu) \equiv P_{\mu}$. If $i_{\mu} : J^{-1}(\mu) \mapsto M$ is the inclusion, and $i_{\mu}^* \omega = \omega|_{P_{\mu}}$, then:

The leaves of the characteristic distribution of the form $i_{\mu}^* \omega$ on $P_{\mu}$ are orbits for $G_{\mu}$, and the orbit space $M = J^{-1}(\mu)/G_{\mu}$ has the structure of a symplectic manifold $(M, \tilde{\omega})$, $\pi^* \tilde{\omega} = i_{\mu}^* \omega$, with $\pi : P_{\mu} \mapsto M$ the projection $\pi(p) = \{G_{\mu}p\}$.

**Proof.** $\forall \xi \in g_{\mu}$, $i_{\xi} \omega|_{P_{\mu}} = 0$, and $\xi_{\mu} \in E_{i_{\mu}^* \omega}$. Conversely, let $v$ be an element of $T_{p}P_{\mu}$, but not of $T_{p}(G_{\mu} \cdot p)$, such that $i_{\mu} \omega|_{P_{\mu}} = 0$. If $F = T_{p}(G \cdot p) \cup T_{p}P_{\mu}$, then $F \cap F^\perp = T_{p}(G_{\mu} \cdot p)$. As $\omega$ is nondegenerate on $T_{p}M$, $F/F \cap F^\perp$ should be symplectic, with $\tilde{\omega}([f], [f']) = \omega(f, f')$, $f, f' \in F$. Thus, if $i_{\mu} \omega = 0$, $v \in T_{p}P_{\mu}$, then also $i_{[v]} \tilde{\omega} = 0$, so that $[v] = 0$ and $v \in T_{p}(G_{\mu} \cdot p)$.

**Examples**

1. Let $f_1, f_2, \ldots, f_n$ be $n$ functions in involution, $\{f_i, f_j\} = 0$, with respect to the Poisson bracket on a $2N$-dimensional manifold $M$. Considering $G = \mathbb{R}^n$, $\mu \in \mathbb{R}^n$ and $J = f_1 \times f_2 \times \ldots \times f_n$, then $G_{\mu} = G$ and $\dim(J^{-1}(\mu)/G) = 2N - 2n$. When $n = N = \dim M/2$ the system is called completely integrable.

2. Let $\mathcal{H}$ be a complex Hilbert space with the symplectic form $\omega(X, Y) = Im\langle X|Y \rangle$, and $\Phi_{\omega} : \mathcal{H} \mapsto \mathcal{H}, \Phi_{\omega}\psi = z\psi$, the action of $G = S^1 = \{z \in \mathbb{C}/|z| = 1\}$ on $\mathcal{H}$. Then

- the action $\Phi$ is symplectic.
- the moment map is $J_{\psi} = -i\langle \psi|\psi \rangle/2$.
- the reduced phase-space is the projective Hilbert space $P_{\omega}$.

**Proof.** The algebra of $G$ is $g = \{\xi \in \mathbb{C}/\xi = -\xi^*\}$, and as $\Phi_{e^{it} \psi} = e^{it} \psi$, $\psi \in \mathcal{H}$, $\xi_{\mathcal{H}}(\psi) = \xi \psi$. Also

$$T_{\psi}\Phi_{\omega}\psi = \frac{d}{dt}|_{t=0}\Phi_{\omega}(\psi + X_{\psi}t) = \frac{d}{dt}|_{t=0}z(\psi + X_{\psi}t) = zX_{\psi}$$
\( i) \Phi^*\omega(X, Y) = \omega(T_\psi\Phi_2 X, T_\psi\Phi_2 Y) = \omega(zX, zY) = \omega(X, Y), \) or \( \Phi^*\omega = \omega. \)

\( ii) \)

\[ i_\xi\omega(X) = \text{Im} \langle \xi | X \rangle = \text{Im} \langle \xi \psi | X \psi \rangle = i\xi \text{Re} \langle \psi | X \psi \rangle \equiv \frac{d}{dt}|_{t=0} \hat{J}(\xi) \psi + tX \psi. \]

However, for \( f(\psi) = \langle \psi | \psi \rangle, \)

\[ \frac{d}{dt}|_{t=0} f(\psi + tX) = 2 \text{Re} \langle \psi | X \rangle \]

such that \( \hat{J}(\xi) \psi = i\xi \langle \psi | \psi \rangle / 2. \)

\( iii) \) As \( g^* \cong g, \) if \( \mu \in g^* \) and \( \xi \in g \) then \( \langle \mu, \xi \rangle = \mu^* \xi, \) and \( J_\psi = -i\langle \psi | \psi \rangle / 2. \)

Thus,

\[ J^{-1}(\mu) = \{ \psi \in \mathcal{H} / \langle \psi | \psi \rangle = 2i\mu \}. \]

Because \( G_\mu = G = S^1 \) acts on \( J^{-1}(\mu), \)

\[ J^{-1}(\mu) / S^1 \cong \{ [\psi], \psi \in \mathcal{H} / \langle \psi | \psi \rangle = 1 \} \equiv P_H, \ [\psi] = \{ z\psi / z \in \mathbb{C}, |z| = 1 \}. \]

**Theorem 5.** With the assumptions of Theorem 4, let \((M, \omega)\) be a symplectic manifold, \( G \) a Lie group acting symplectically on \( M, \) \( J : M \rightarrow g^* \) the equivariant moment map, and \( H : M \mapsto \mathbb{R} \) the Hamilton function, invariant to the action of \( G \) (Appendix 3). If \( \mu \in g^* \) is a regular value of \( J, \) \( G_\mu \) acts freely and properly on \( J^{-1}(\mu) \equiv P_\mu, i_\mu : J^{-1}(\mu) \mapsto M \) is the inclusion, and \( i_\mu^* \omega = \omega|_{P_\mu}, \) then:

\( i) \) the current \( F_t \) of \( X_H \) leaves \( J^{-1}(\mu) \) invariant and commutes with the action of \( G_\mu \) on \( J^{-1}(\mu), \) such that it induces canonically a current \( \phi_t \) on \( M, \)

\[ \pi_\mu \circ F_t = \phi_t \circ \pi_\mu \quad . \] (5)

\( ii) \) the current \( \phi_t \) on \( M \) is Hamiltonian, generated by the function \( \tilde{H} : M \mapsto \mathbb{R}, \)

\[ \tilde{H} \circ \pi_\mu = H \circ i_\mu \]

called reduced Hamiltonian.

**Proof.** (i) As \( J^{-1}(\mu) \) is invariant to \( F_t, \) the projection of \( F_t \) on \( M \) defines the current \( \phi_t \) by (5), and

\[ (\phi_t \circ \pi_\mu)^* \omega = F_t^* \pi_\mu^* \omega = F_t^* i_\mu^* \omega = i_\mu^* \omega = \pi_\mu^* \omega \quad . \]
Thus, $\pi^*_\mu(\phi_t\omega - \bar{\omega}) = 0$, and as $\pi_\mu$ is surjective, $\phi_t\omega = \bar{\omega}$.

(ii) Let $v \in TM$ and $[v] = T\pi_\mu v \in \tilde{T}M$. Then

$$d\bar{H}([v]) = \pi^*_\mu d\bar{H}(v) = d(H \circ \pi_\mu)(v) = d(H \circ i_\mu)(v) = \bar{\omega}(X_H, v) = \pi^*_\mu \omega(X_H, v) = \pi^*_\mu \omega(T\pi_\mu X_H, T\pi_\mu v) = \omega([X_H], [v]) = i_{[X_H]}\omega([v]),$$

such that $i_{[X_H]}\omega = d\bar{H}$. Thus, $\phi_t$ has $[X_H] = T\pi_\mu X_H$ as generator and $\bar{H}$ as Hamiltonian $\nabla$.

**Proposition 10.** Let $c_t$, with $c_0 = m_0 \in J^{-1}(\mu)$, be an integral curve for $X_H$, and $[c_t]$ the integral curve for $X_H$, presumed to be known. Let $d_t \in J^{-1}(\mu)$, with $d_0 = m_0$, a smooth curve such that $[d_t] = [c_t]$. Then $c_t$ is of the form

$$c_t = \Phi_{a_t} d_t,$$

where $a_t \in G_\mu$ is provided by the equation $\dot{a}_t = T_{c_t L a_t} \xi(t)$ in which $\xi(t) \in g_\mu$ is the solution of

$$\xi_M(d_t) = X_H(d_t) - \dot{d}_t.$$

**Proof.** If $c_t = \Phi_{a_t} d_t$ then

$$X_H(c_t) = \dot{c}_t = T_{d_t} \Phi_{a_t} \dot{d}_t + T_{a_t} \Psi_{d_t}(a_t) \dot{a}_t,$$

with $\Psi_m : G \mapsto M$, $\Psi_m(a) = \Phi_a m$. As

$$T_{a_t} \Psi_m \dot{a} = T\Psi_m T R_a T R_{a^{-1}} \dot{a} = T_{c_t} (\Psi_m \circ R_{a^{-1}}) T R_{a^{-1}} \dot{a} = T_{e}\Psi_{\Phi_{a^{-1}}} T R_{a^{-1}} \dot{a} = (T R_{a^{-1}}) M(\Phi_{a^{-1}}),$$

(6) becomes

$$X_H(\Phi_{a_t} d_t) = T_{d_t} \Phi_{a_t} \dot{d}_t + (T R_{a^{-1}}) M(\Phi_{a_t} d_t).$$

(7) However, $X_H(\Phi_{a_t} d_t) = T_{d_t} \Phi_{a} X_H(d_t)$ from the $\Phi_a$-invariance of $X_H$, and

$$\xi_M(\Phi_{a_t} d_t) = T_{d_t} \Phi_{a} (\text{Ad}_{a^{-1}} \xi)_M(d_t), \quad \text{Ad}_a \xi \equiv T_{a} R_{a^{-1}} L_a \xi,$$

such that (7) takes the form

$$X_H(d_t) = \dot{d}_t + (T_{a} L_{a^{-1}}) M(d_t),$$

or $\xi_M(d_t) = X_H(d_t) - \dot{d}_t$ with $\xi = T_{a} L_{a^{-1}} \dot{a} \nabla$. 

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3.3 Stability

A point \( m \in M \) is a relative equilibrium if \( \pi_\mu(m) \in \bar{M} \) is a fixed point for the Hamiltonian system \( X_\bar{H} \) on \( \bar{M} \). If \( \pi_\mu(m) \) is on a periodic orbit for \( X_\bar{H} \) then \( m \in M \) is called relative periodic.

**Proposition 11.** Let \( m_0 \in J^{-1}(\mu) \) be a point of relative equilibrium for the Hamiltonian system \( X_H \) on \( M \), \( X_H = \dot{F}_t \in \chi(M) \). Then there exists a one-parameter subgroup \( a_t \) of \( G_\mu \) such that 
\[
    m_t = F_t(m_0) = \Phi_{a_t}(m_0), \quad a_t \in G_\mu,
\]
and \( X_H(m_t) = \xi_M(m_t), \xi \in g_\mu \).

It is important to remark that if \( m \in M \) is a relative equilibrium, or relative periodic of small amplitude, then the Hamiltonian system \( X_H \) on \( M \) has a collective behaviour\(^9\).

Let \( H \) be a Hamilton function on \( (M, \omega) \), and \( G \) a Lie group acting on \( M \) with the moment map \( J : M \mapsto g^* \), so that \( \Phi_\mu^* H = H \), \( a \in G \). One defines the energy-moment map \( H \times J : M \mapsto \mathbb{R} \times g^* \) by
\[
    (H \times J)(m) = (H(m), J(m)).
\]
The previous results indicate that \( \forall C \in \mathbb{R} \times g^*, C \equiv (E, \mu) \), the set \( I_C = (H \times J)^{-1}(C) \) is invariant to the current of \( X_H \). The topological structure of the current is determined by:
- the topological type of \( I_C \), \( \forall C \in \mathbb{R} \times g^* \).
- the bifurcation set for the map \( H \times J \).
- the current of \( X_H \) on each \( I_C \).
- the decomposition of \( H^{-1}(E) \) in submanifolds \( I_C \).

The knowledge of the bifurcation set and of the current of \( X_H \) on each \( I_C \) may provide the values of \( C \) for which the system has a collective dynamics.

4 Collective dynamics

A classical collective model for the Lie group \( G \) is a phase-space \( (M, \omega) \) on which \( G \) acts symplectically and transitively ((\( M, \omega \)) is \( G \)-elementary). Let \(^9\)

\[\text{If } \omega = -d\theta \text{ and } \Phi_\mu^* \theta = \theta \text{ then } L_{\xi_M} \theta = i_{\xi_M} d\theta + d i_{\xi_M} \theta = 0, \text{ and } i_{\xi_M} \omega = d \tilde{J}(\xi) \text{ with } \tilde{J}(\xi) = \theta(\xi_M).\]
Φ : G × M \mapsto M be a symplectic action of G on (M, ω) with the moment map J : M \mapsto g^*. A Hamiltonian H on M is called collective if it has the form

\[ H = h_c \circ J = J^*h_c \]  

(8)

where h_c : g^* \mapsto \mathbb{R} is a smooth function. If J(M) is a cotangent fibration, \( J(M) = T^*Q \), then (8) defines a physical collective model.

**Proposition 12.** Let Φ : G × M \mapsto M be a Hamiltonian action of the Lie group G on (M, ω) with the moment map J : M \mapsto g^*, and H = h_c \circ J, h_c : g^* \mapsto \mathbb{R}, a collective Hamiltonian. Then the trajectory \( m_t = X_H(m_t), m_0 \in M \), is completely determined by the trajectory \( \gamma_t \) of the Hamiltonian system defined by h_c on the orbit \( \mathcal{O} = G : J(m_0), \gamma_t = J(m_t) \).

**Proof.** \( \forall \mu \in g^* \), there exists \( L_{h_c}(\mu) \in g \), \( L_{h_c} : g^* \mapsto g \), defined by

\[ \langle \dot{\mu}, L_{h_c}(\mu) \rangle = \frac{d}{dt}\big|_{t=0} h_c(\mu + t\dot{\mu}) \] .

Thus,

\[ i_{X_H} \omega(v) = dH(v) = J^*dh_c(v) = d(h_c)_{J(m)}(T_mJv) = \]

\[ \langle T_mJv, L_{h_c}(J_m) \rangle = d\hat{J}(L_{h_c}(J_m)) \cdot v = i_{(L_{h_c}J_m)^*} \omega(v) \]

such that \( X_H(m) = (L_{h_c} \circ J(m))_M(m) \). If \( \gamma_t \in \mathcal{O} \) then \( L_{h_c} \gamma_t \subset g \) and \( a_t \) defined by \( \dot{a}_t = T_eL_{a_t}(L_{h_c} \gamma_t) \) is a curve in G. Considering \( m_t = \Phi_{a_t}m_0 \), then

\[
\dot{m}_t = \dot{\Phi}_{a_t}m_0 = (T_{a_t}L_{a_t^{-1}}\dot{a}_t)_M(m_t) = \\
(L_{h_c} \gamma_t)_M(m_t) = (L_{h_c} \circ J(m_t))_M(m_t) = X_H(m_t) \]

\( \triangledown \).

In applications, to obtain the trajectory \( m_t \) at \( m_0 \) are necessary several elements:

1. the orbit \( \mathcal{O} = Ad_{a_t^{-1}}J(m_0) \subset g^* \).
2. the trajectory \( \gamma_t \) on \( \mathcal{O} \) for the Hamiltonian system induced by \( h_c \).
3. the curve \( \xi_t \subset g, \xi_t = L_{h_c}(\gamma_t) \).
4. the trajectory \( a_t \subset G \) provided by \( \dot{a}_t = T_eL_{a_t}\xi_t \).

The result is then \( m_t = \Phi_{a_t}(m_0) \).

It is important to remark that a relative equilibrium for a G-invariant Hamiltonian system moves on \( G_\mu \)-collective trajectories, corresponding to orbits \( \gamma_t \subset g^* \), degenerate in critical points of some \( h_c \). Thus, in such situations the G-invariance implies \( G_\mu \)-collectivity.
Proof. (i) Thus, the rigid body orbit \( G \) acts freely and properly on \( (\tilde{\mathcal{M}}, \tilde{\omega}) \) with respect to the action of \( G \) is a homogeneous space for a certain group \( G_0 \). This is the case for instance when:
- \( M = T^*Q \) and \( G \) acts transitively on \( Q \).
- \( M = T^*Q \) and \( G \) acts freely on \( Q \).
- \( M = T^*G \).

\textbf{Theorem 6.} (Kirillov-Kostant-Souriau) Let \( G \) be a Lie group, \( L : G \times G \mapsto G \) the action of \( G \) on \( G \) by left translations, and \( \Phi^L : G \times T^*G \mapsto T^*G \) the action induced by \( L \) on \( T^*G \) with the moment map \( \mathcal{J}^L : T^*G \mapsto g^* \). Then (notation is explained in Appendix 1):

(i) the reduced phase-space \((J^L)^{-1}(\mu)/G_\mu\) can be identified naturally with the orbit \( G \cdot \mu = \{Ad^*_a \cdot \mu, a \in G\} \) of \( \mu \) in \( g^* \).

(ii) if \( \tilde{R} : G \times G \mapsto G \) is the action to the right of \( G \) on \( G \), and \( \tilde{\Phi}^R : G \times T^*G \mapsto T^*G \) is the action induced by \( \tilde{R} \) on \( T^*G \), having the moment map \( \mathcal{J}^R : T^*G \mapsto g^* \), then the \( \Phi^L \)-invariant Hamiltonians are \( \tilde{\Phi}^R \)-collective.

\textbf{Example: the rigid body}

If \( G = SO(3) \), then \( g \cong g^* \cong \mathbb{R}^3 \), and \( \forall \mu \in g^* \), \( \mu = \sum_{i=1}^3 \mu_i f_i \), the orbit \( G \cdot \mu \) is the sphere of radius \( |\mu| \), \( \bar{\mu} = (\mu_1, \mu_2, \mu_3) \). The set \( \{f_i, i = 1, 2, 3\} \) of basis elements in \( g^* \) is defined with respect to a set \( \{\xi_i, i = 1, 2, 3\} \), of basis elements in \( g \), \( [\xi_i, \xi_j] = \epsilon_{ijk} \xi_k \), such that \( f_i(\xi_k) = \delta_{ik} \).

Let us consider \( H : T^*SO(3) \mapsto \mathbb{R} \) of the form \( H = h_c \circ \mathcal{J}^r \), with \( \mathcal{J}^r(a, \mu) = -\mu \) and \( h_c : g^* \mapsto \mathbb{R} \),

\[
h_c(\mu) = \frac{1}{2} \sum_{i=1}^3 \frac{\mu_i^2}{I_i} \quad .
\]

As \( \mathcal{J}^r(\xi) : T^*G \mapsto \mathbb{R} \) has the expression \( \mathcal{J}^r(\xi) = -\langle \mu, \xi \rangle = -\mu_i \), and the Poisson bracket satisfies \( \{\mathcal{J}^r(\xi), \mathcal{J}^r(\eta)\} = \mathcal{J}^r([\xi, \eta]) \) from equivariance, we get
\{\mu_i, \mu_j\} = -\epsilon_{ijk} \mu_k.

The equations of motion are

\[ \dot{\mu}_i = \{\mu_i, h_c\} = \sum_{k=1}^3 \frac{\mu_k}{I_k} \{\mu_i, \mu_k\} = -\sum_{k=1}^3 \epsilon_{ikl} \frac{\mu_k \mu_l}{I_k}, \]

or

\[ \dot{\mu}_1 = \mu_2 \mu_3 \left( \frac{1}{I_3} - \frac{1}{I_2} \right), \quad \dot{\mu}_2 = \mu_1 \mu_3 \left( \frac{1}{I_1} - \frac{1}{I_3} \right), \quad \dot{\mu}_3 = \mu_1 \mu_2 \left( \frac{1}{I_2} - \frac{1}{I_1} \right). \]

Introducing \( \omega_i = \langle f_i, L_{h_c}(\mu) \rangle = \mu_i/I_i \) we also get

\[ I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3), \quad I_2 \dot{\omega}_2 = \omega_1 \omega_3 (I_3 - I_1), \quad I_3 \dot{\omega}_3 = \omega_1 \omega_2 (I_1 - I_2), \]

Similar equations can be obtained considering the rigid body rotation as geodesic motion (Appendix 4).

When \( I_1 > I_2 > I_3 \) the critical points \( \vec{\kappa} = (\kappa_1, \kappa_2, \kappa_3) \) of \( h_c \) on \( G \cdot \mu \), \( (dh_c(T_n G \cdot \mu) = 0) \), are

\( (0, \pm |\vec{\mu}|, 0), \quad (\pm |\vec{\mu}|, 0, 0), \quad (0, 0, \pm |\vec{\mu}|) \).

For the trajectory in the chart \( \Psi(T^*G) \) one obtains

\[ m_t = \rho_{a_t}(a_0, \mu_0) = (a_0 a_t^{-1}, Ad_{a_t}^* \mu_0) \]

where \( a_t \) is provided by \( \dot{a}_t = T_c L_{a_t} \omega_t, \quad \omega_t = L_{h_c}(\mu_t) \).

As presented above, the Kirilov-Kostant-Souriau theorem concerns the case when on the configuration space \( Q \) are defined two actions of the group \( G \) which commute, \( \Phi^r \) and \( \Phi^\ell \), free and transitive, such that the action \( \varphi : G \times Q \mapsto Q, \quad \varphi_a = \Phi_a^r \Phi_a^\ell \) has a fixed point. In the example of the rigid body \( \Phi^\ell \) corresponds to the rotation of the laboratory frame, while \( \Phi^r \) to the rotation of the intrinsic frame.

**Theorem 7.** Let \( \Phi : H \times Q \mapsto Q \) be a transitive action of the Lie group \( H \subset Gl(V) \) on the manifold \( Q \), and \( \tau : H \times V \mapsto V \) a representation of \( H \) in the linear space \( V \). Assume that \( f : V \mapsto \mathcal{F}(Q) \) is a \( H \)-equivariant linear map, \( (\Phi_h^* f = f \circ \tau_h, \forall h \in H) \), which defines a Hamiltonian action of the semidirect product \( G = H \times V \) on \( T^*Q \),

\[ U^f_{(h,v)} \alpha_q = T_{\Phi_{h}^* q} \Phi_{h^{-1}}^* \alpha_q - df_v(\Phi_{h} q), \quad (h, v) \in G, \quad \alpha_q \in T^*Q, \]

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with the equivariant moment map \( J : T^*Q \mapsto h^* + V^* \). If \( \exists q_0 \in Q \) such that \( H_{q_0} = H_K \) (the action of \( G \) on \( T^*Q \) is transitive), then \( J(T^*Q) \) is a single \( G \)-orbit in \( g^* \), specifically the \( G \)-orbit through the point \((0, K)\).

**Proof.** The action (11) has the form

\[
U^f_{(h,v)} \equiv tf_v \check{\Phi}_h
\]

where

\[
\check{\Phi}_h \alpha_q = T_{\Phi_{h,q}}^* \check{\Phi}_{h^{-1} \alpha_q} , \quad tf_v \alpha_q = \alpha_q - df_v(q).
\]

If \( j : T^*Q \mapsto h^* \) denotes the moment map for the action of \( H \) on \( T^*Q \), \( \pi : T^*Q \mapsto Q \) is the projection, and \( v_{T^*Q}(\alpha_q) = -\pi^* df_v(\alpha_q) \), then the moment map for the action \( U^f \) is provided by

\[
\check{J}_{\alpha_q}(\xi, v) = j_{\alpha_q}(\xi) + \pi^* f_v(\alpha_q).
\]

Considering \( f_v(q_0) \equiv (K, v), K \in V^* \), then \( \forall q = \Phi_h(q_0) \) we get

\[
f_v(q) = (K, \tau_{h^{-1}} v)
\]

such that

\[
J_{\alpha_q} = (j_{\alpha_q}, \tau_{h^{-1}} K) ,
\]

where \( \alpha_q = \check{\Phi}_h \alpha_{q_0} \), and \( j_{\alpha_q} = j_{\check{\Phi}_h \alpha_{q_0}} = (Ad_{h_q}^*)^{-1} j_{p} \), with \( j_{p} = j_{q_0}, j^0 : T_{q_0}^* Q \mapsto h^* \). In the chart on \( T^*Q \) defined by

\[
\Psi : T^*Q \mapsto Q \times T_{q_0}^* Q , \quad \Psi(\alpha_q) = (q, \check{\Phi}_{h^{-1} \alpha_q} q) \equiv (q, p),
\]

(12) takes the form

\[
J'_{(q,p)} = (Ad_{h^{-1} j_{p}}^* \tau_{h^{-1}} K) .
\]

Let \( H_0 = \{ h \in H / \Phi h q_0 = q_0 \} \) be the isotropy group of \( q_0 \), and \( h_0 \) its algebra.

Thus, \( \forall \xi \in h_0, j_{p}^0(\xi) = 0 \), and in fact \( j_{p}^0 : T_{q_0}^* Q \mapsto h^*/h_0^* \equiv \tilde{h}_0^* \),

\[
\tilde{h}_0^* = \{ \mu \in h^*/\mu(\xi) = 0 \forall \xi \in h_0 \} .
\]

Let also

\[
H_K = \{ h \in H / \tau_{h^{-1}} K = K \}
\]

be the isotropy group of \( K \), and \( h_K \) its algebra. Presuming now that \( h_0 = h_K \), then \( h/h_0 = h/h_K, Q \simeq H \cdot K \), and as \( H \subset Gl(V) \) the action of \( G \) on \( T^*Q \) is
transitive. Thus, \( J(T^*Q) \) is a covering space for the \( G \)-orbit \( G \cdot (0, K) \).

For \( (\mu, K) \in \mathfrak{h}^* + \mathcal{V}^* \) and \( (h, v) \in H \times V \) we get

\[
Ad^*_{(h,v)}(\mu, K) = (Ad^*_{h^{-1}}\mu + (\tau^*_{h^{-1}}K) \odot v, \tau^*_{h^{-1}}K) ,
\]

where \( (h, v)^{-1} = (h^{-1}, -\tau_h^{-1}v) \), \( Ad^*_{h^{-1}}(K \odot v) = \tau^*_{h^{-1}}K \odot \tau_h v \), and \( (K \odot v)(\xi) \equiv \langle K, \xi_V(v) \rangle \) with

\[
\xi_V(v) = \frac{d}{dt}|_{t=0} \tau e^{t\xi}(v) \in T_vV \simeq V .
\]

If \( \pi_K : h \mapsto h/h_K \), and \( \mu_0 \in \mathfrak{h}^*/h_K^* \equiv \tilde{h}_K^* \), then

\[
\pi_K^*(\mu - \mu_0) = 0 \Leftrightarrow \exists v \in V , \mu - \mu_0 = K \odot v .
\]

Thus, \( \forall j_p^0 \in \tilde{h}_K^* \), \( \exists v \in V \) such that \( K \odot v = j_p^0 \), \( dim K \odot V = dim (H/H_0) = dim Q \), and

\[
J_{(q,p)}^*(Ad^*_{h^{-1},j_p^0,\tau^*_{h^{-1}}K} = (Ad^*_{h^{-1}}(K \odot v), \tau^*_{h^{-1}}K)) = Ad^*_{(h^{-1},-v)}(0, K) ,
\]

which shows that \( J_{T^*Q} \) is the single orbit \( J_{T^*Q} = Ad^*_{\tilde{h}_K^*}(0, K) \) \( \triangledown \).

One should note that when \( Q \) is the orbit \( Q = H \cdot K \subset \mathcal{V}^* \), then \( T^*Q \) is mapped by \( J \) on the \( G \)-orbit \( G \cdot (0, K) \), and any Hamiltonian on \( M = T^*Q \) is \( G \)-collective. Moreover, if \( (M, \omega) \) is an arbitrary phase-space, and \( J : M \mapsto V^* \), then the action \( \Phi \) defines a physical collective model. The most general coadjoint orbit of the semidirect product \( G = H \times V \) appears as reduced phase-space with respect to the action to the right of \( H_K \) on \( T^*H \).

### 4.1 Examples

1. The rigid body in external field

Let \( Q = SO(3, \mathbb{R}) = H \) and \( V = \mathbb{R}^3 \). For the action to the right of \( G = H \times V \) on \( T^*Q \) the moment map is

\[
J^*(h, \mu) = (-\mu, \tau^*_{h^{-1}}K) .
\]

If \( K \in \mathcal{V}^* \simeq \mathbb{R}^3 \) is a constant force field, and \( x \in V \simeq \mathbb{R}^3 \) is the center of mass position vector, then the Hamiltonian on \( T^*SO(3) \) is \( H = h_c \circ J^* \), with

\[
h_c(\mu, K) = \frac{1}{2} \sum_{i=1}^{3} \frac{\mu_i^2}{2I_i} - \langle K, x \rangle .
\]
2. The liquid drop model

Let $V = \{w \in GL(3, \mathbb{R})/w^T = w, \det w > 0\} \simeq \mathbb{R}^6$ be the space of the quadrupole moments $w_{ij} = \sum_{n=1}^{N} x_{ni}x_{nj}$, $i, j = 1, 2, 3$ for a system of $N$ particles. An action $\tau$ of $H = Sl(3)$ on $V$, $\tau : Sl(3) \times V \mapsto V$

$$\tau_h w = hwh^T, \ h \in Sl(3)$$

is induced by its action $x_n \mapsto h x_n$ on $\mathbb{R}^3_N$, and $Q = H \cdot K, K \in V^*$, is the 5-dimensional configuration space for the liquid drop.

Let $G$ be the 14-dimensional group $G = H \times V \equiv CM(3)$,

$$CM(3) = \{(h, w) = \begin{bmatrix} h & 0 \\ wh & (h^T)^{-1} \end{bmatrix}, \ h \in SL(3), w \in V \simeq \mathbb{R}^6\} .$$

Thus,

$$cm(3) = \{(\xi, \eta) = \begin{bmatrix} \xi & 0 \\ \eta & -\xi^T \end{bmatrix}, \ \xi \in sl(3), \eta \in V\} \subset sp(3, \mathbb{R}) \subset gl(3, \mathbb{R}) ,$$

and

$$cm(3)^* = \{(\mu, \nu)^b, \ tr\mu = 0, \ \nu = \nu^T/\mu \in sl(3), \nu \in V^*\}$$

with

$$\langle (\mu, \nu)^b, \begin{bmatrix} \xi & 0 \\ \eta & -\xi^T \end{bmatrix} \rangle \equiv \frac{1}{2}[tr(\mu^T \xi) + tr(\nu \eta)] .$$

For the element $(\alpha L_3, \beta I)^b$, $L_3 = e_{12} - e_{21}$, (Appendix 2), one obtains

$$Ad_{(h, w)}(\alpha L_3, \beta I)^b = (\alpha (h^T)^{-1}L_3h^T - 2\beta whh^T + \frac{2}{3}\beta Tr(whh^T), \beta hh^T)^b$$

such that when $\alpha \neq 0$ the stability group $G_{(\alpha, \beta)}$ of $(\alpha L_3, \beta I)^b$ is

$$G_{(\alpha, \beta)} = \{(SO(2), w_0I), w_0 \in \mathbb{R}\} ,$$

while for $\alpha = 0$

$$G_{(0, \beta)} = \{(SO(3), w_0I), w_0 \in \mathbb{R}\} .$$

Thus, as $CM(3)$ is 14-dimensional, when $\alpha \neq 0$ the coadjoint orbits are 12-dimensional, and 10-dimensional when $\alpha = 0$. The moment map $J : T^*Q \mapsto g^*$ is a diffeomorphism from the phase space $M = T^*Q$ to the coadjoint orbit.
of the element \((0, \beta I)^h\).

3. The relativistic particle

If \(H = SO(3, 1)\) is the Lorentz group, and \(V = \mathbb{R}^{3,1}\) is the Minkowski space, then \(G = H \times V\) is the Poincaré group. The Lie algebra \(so(3, 1)\) is generated by six \(4 \times 4\) real matrices, \(\{J_i, K_i, i = 1, 2, 3\}\), with nonvanishing elements \((J_i)_{kl} = \epsilon_{ikl}, i, k, l = 1, 2, 3, (K_i)_{\mu\nu} = \delta_{\mu0}\delta_{\nu0} + \delta_{\nu0}\delta_{\mu0}, \mu, \nu = 0, 1, 2, 3\), and commutation relations 
\[ [J_i, J_k] = -\epsilon_{ikm}J_m, \quad [K_i, K_m] = \epsilon_{lmi}J_l, \quad [J_m, K_i] = -\epsilon_{mil}K_l. \]

As \(^{10} so(3, 1) \cong sl(2, \mathbb{C})\), an element \(x = \alpha \cdot K + \beta \cdot J \in so(3, 1), \alpha, \beta \in \mathbb{R}^3\), corresponds to \(s = (\alpha + i\beta) \cdot \sigma/2 \in sl(2, \mathbb{C})\), by \(s \equiv (\sigma_1, \sigma_2, \sigma_3)\) denoting the Pauli matrices.

An element \(F \in g^*\) can be expressed in the form
\[ F = p_0 X^*_0 - p \cdot X^* + \kappa \cdot K^* + s \cdot J^* , \]
where the basis \((X^*_0, X^*_K, X^*_J)\) is defined by \(X^*_\mu(X_\nu) = \delta_{\mu\nu}\), \(K_i^* (K_i) = J_i^* (J_i) = \delta_{i0}\). Denoting \(f_0 \equiv (p_0, p, \kappa, s)\), the orbit \(G \cdot F\) contains \(f_0 \equiv (m_0c, 0, 0, s_0)\), where \(m_0\) and \(s_0\) are the “intrinsic” mass and angular momentum. As \(G_{f_0} \cong \mathbb{R} \times SO(2, \mathbb{R})\), \(G \cdot F\) contains \(f_0 \equiv (m_0c, 0, 0, s_0)\), where \(m_0\) and \(s_0\) are the “intrinsic” mass and angular momentum. As \(G_{f_0} \cong \mathbb{R} \times SO(2, \mathbb{R})\), \(G \cdot F\) contains \(f_0 \equiv (m_0c, 0, 0, s_0)\), where \(m_0\) and \(s_0\) are the “intrinsic” mass and angular momentum. As \(G_{f_0} \cong \mathbb{R} \times SO(2, \mathbb{R})\), \(G \cdot F\) contains \(f_0 \equiv (m_0c, 0, 0, s_0)\), where \(m_0\) and \(s_0\) are the “intrinsic” mass and angular momentum.

5 Appendix 1: Adjoint group actions

Let \(G\) be a Lie group, and \(M = T^* G\). The left and right translations on \(G\) are defined by \(L_a(h) = ah, R_a(h) = ha\), respectively, while \(\tilde{R}_a = R_{a^{-1}}\) is the action to the right.

For \(\xi_h \in T_h G\) one gets
\[ T_h L_a \xi_h \in T_{ah} G, \quad T_h R_a \xi_h \in T_{ha} G, \]
while for \(\alpha \in T^*_a G\),
\[ T_{ah} L_a^{-1} \alpha_h \in T^*_a G, \quad T_{ha^{-1}} R^*_a \alpha_h \in T^*_{ha^{-1}} G. \]
One denotes by $\Phi_\ell^a = TL_{a}^{-1}$ and $\Phi_r^a = TR_{a}^*$ the actions induced by $L$ and $R$ on $M$. Let $\Psi : M \mapsto G \times g^*$ be the map

$$\Psi(\alpha_a) = (a, T_e L_a^* \alpha_a) = (a, \Phi_\ell^a \alpha_a) \in G \times g^* .$$

In the chart $\Psi(M)$, (of intrinsic coordinates), the actions $\Phi_\ell$ and $\Phi_r$ are represented by the actions denoted $\lambda$ and $\rho$, respectively,

$$\Psi(\Phi_\ell^a \alpha_h) \equiv \lambda_a \Psi(\alpha_h) , \quad \Psi(\Phi_r^a \alpha_h) \equiv \rho_a \Psi(\alpha_h) ,$$

provided by

$$\lambda_a(h, \mu) = (ah, \mu) , \quad \rho_a(h, \mu) = (ha^{-1}, Ad_{a^{-1}}^* \mu) ,$$

where

$$Ad_{a^{-1}}^* = \Phi_r^a \Phi_\ell^a = T_e (L_a R_a)^* .$$

The projection on $TG$ of the infinitesimal generators for these actions are

$$\xi_\ell^G(a) = T_e R_a \xi , \quad \xi_r^G(a) = -T_e L_a \xi .$$

such that the moment maps at $\alpha_a = \Phi_\ell^a \mu$ are

$$J_\alpha^\ell \cdot \xi = \alpha_a(\xi_\ell^G(a)) = T_e R_a^* \alpha_a(\xi) = (Ad_{a^{-1}}^* \alpha) \cdot \xi ,$$

$$J_\alpha^r \cdot \xi = \Phi^a_r \mu(\xi^G(a)) = -(\Phi^a_\ell \Phi^a_r \mu)(\xi) = -\mu \cdot \xi ,$$

or

$$J_\alpha^\ell(a, \mu) = Ad_{a^{-1}}^* \mu , \quad J_\alpha^r(a, \mu) = -\mu .$$

These maps are equivariant, as

$$J^\ell(\lambda_a(h, \mu)) = Ad_{a^{-1}}^* J^\ell(h, \mu) , \quad J^r(\rho_a(h, \mu)) = Ad_{a^{-1}}^* J^r(h, \mu) .$$

6 Appendix 2: The classical semisimple Lie algebras

The root space decomposition $g = h + \sum_{a \neq 0} g_a$ of a complex semisimple Lie algebra $g$ with respect to the adjoint representation ($ad_x y = [x, y]$) of a
Cartan subalgebra $h$ provides the Weyl basis $\{h_i, e_\alpha, i = 1, \text{rank}(\mathfrak{g}), \alpha \in h^*\}$, with

$$[h, e_\alpha] = \alpha(h)e_\alpha \quad [e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})h_\alpha.$$ 

Here $(x, y) \equiv Tr(ad_x ad_y)$ is the (nondegenerate) Cartan-Killing form, $\alpha(h) \equiv (h_\alpha, h)$, $(\alpha, \beta) \equiv (h'_\alpha, h'_\beta)$. If $(\alpha_1, \alpha_2, ..., \alpha_i)$ is an ordered basis in $h^*$, then $eta = \sum_i c_i \alpha_i$ is positive ($\in \Sigma$) if the first $c_i \neq 0$ is positive, and simple if it is positive, but not a sum of positive roots. For the simple roots $\alpha_i$, $\alpha_j$, the elements of the Cartan matrix $A_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ are integers, so that the real span of the Weyl basis is the normal (or split) real form of $g$. If $V$ is a finite dimensional vector space, $\mathfrak{gl}(V)$ is reductive, and $[\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$ is semisimple. When $V = \mathbb{C}^n$, a basis of $\mathfrak{gl}(\mathbb{C}^n) \equiv \mathfrak{gl}(n, \mathbb{C})$ is provided by the set of $n^2$ real matrices $\{e_{pq}, p, q = 1, n\}$, with the single non-vanishing element, equal to 1, in the row $p$ and column $q$, $(e_{pq})_{ij} = \delta_{pi}\delta_{qj}$, and $[e_{pq}, e_{rs}] = \delta_{qr}e_{ps} - \delta_{ps}e_{rq}$, $\langle e_{pq}, e_{qq} \rangle = \sum_{ij}(\delta_{ip} - \delta_{jp})(\delta_{iq} - \delta_{jq}) = 2n\delta_{pq} - 2$. This basis can be represented using a set of $n$ boson operators $\{b_1^p, b_p, p = 1, n\}, \{b_p, b_1^p \} = \delta_{pq}$, as $e_{pq} = b_1^p b_q$, or a set of $n$ fermion operators, $\{c_1^p, c_p, p = 1, n\}, \{c_p, c_1^p \} = \delta_{pq}$, as $e_{pq} = c_1^p c_q$.

### 6.1 Semisimple Lie algebras of type $A_n$ : $\mathfrak{sl}(n+1, \mathbb{C})$

$\mathfrak{sl}(n+1, \mathbb{C}) = \{\xi \in \mathfrak{gl}(n+1, \mathbb{C})/Tr\xi = 0\}$. If $e_{pq} \in \mathfrak{gl}(n+1, \mathbb{C}), (e_{pq})_{ik} = \delta_{ip}\delta_{kq}$, the the diagonal matrices $h_m, m = 1, n$, with

$$h_m = \sum_{i=1}^{n+1} c^i_m e_{ii} \quad \sum_{i=1}^{n+1} c^i_m = 0 \quad \sum_{i=1}^{n+1} c^i_m c^{i'}_{m'} = \delta_{mm'}$$

satisfy $(h_m, h_{m'}) = 2(n+1)\delta_{mm'}$, and provide an orthogonal basis in the Cartan subalgebra $h$ of $\mathfrak{sl}(n+1, \mathbb{C})$. For these elements $[h_m, e_{pq}] = \alpha_{pq}(m)e_{pq}$ with

$$\alpha_{pq}(m) = -\alpha_{qp}(m) = \sum_{i=1}^{n+1} c^i_m (\delta_{ip} - \delta_{iq}) = c^p_m - c^q_m.$$

Denoting $\beta_p \equiv \alpha_{pp+1}$, we get $\alpha_{pq} = \sum_{l=p}^{q-1} \beta_i$. These results provide:

- the set of roots:

$$\Delta = \{\alpha_{pq}, p \neq q\}$$
with \((n+1)^2-(n+1)\) elements.

- the set of positive roots:

\[
\Sigma = \{\alpha_{pq}, p < q\}
\]

with \(n(n+1)/2\) elements.

- the set of simple roots:

\[
\Pi = \{\beta_p = \alpha_{pp+1}, p = 1, n\}
\]

with \(n\) elements. A particular basis in \(\mathfrak{h}\) consists of the elements \(g_m = e_{mm} - e_{m+1m+1}, m = 1, n\). For this basis we get

\[
[g_m, e_{pq}] = (g_m, h'_{\alpha_{pq}})e_{pq}, \quad [e_{pq}, e_{qp}] = e_{pp} - e_{qq} = \sum_{i=p}^{q-1} g_i = 2(n+1)h'_{\alpha_{pq}}
\]

where

\[
h'_{\alpha_{pq}} = \frac{1}{2(n+1)} \sum_{i=p}^{q-1} g_i \quad , \quad (g_m, h'_{\alpha_{pq}}) = \delta_{mp} - \delta_{mq} + \delta_{m+1q} - \delta_{m+1p}
\]

As \((\beta_p, \beta_q) = (h'_{\beta_p}, h'_{\beta_q}) = (g_p, g_q)/4(n+1)^2 = (2\delta_{pq} - \delta_{pq+1} - \delta_{p+1q})/2(n+1)\), we get the Cartan matrix

\[
A_{mm'} = 2\frac{(\beta_m, \beta_{m'})}{(\beta_m, \beta_m)} = 2\delta_{mm'} - \delta_{mm'+1} - \delta_{m+1m'}
\]

and the Dynkin diagram\(^{11}\) [7] presented in Figure 1.

The compact form \(\mathfrak{su}(3)\) of \(\mathfrak{sl}(3, \mathbb{C})\) is presented in detail in [8].

### 6.2 Semisimple Lie algebras of type \(B_n : \mathfrak{so}(2n+1, \mathbb{C})\)

\(\mathfrak{so}(2n+1, \mathbb{C}) = \{\xi \in \mathfrak{gl}(2n+1, \mathbb{C})/\xi^T = -\xi\}\), with basis provided by \(n(2n+1)\) independent matrices \(\xi_{pq} = e_{pq} - e_{qp}, p, q = -n, ..., -1, 0, 1, ..., n\). The real

\(^{11}\)The circles \(\{\circ\}\) correspond to the simple roots \(\{\beta_i\}\) and the number of lines between \(\circ^i\) and \(\circ^j\) is equal to \(A_{ij}A_{ji} = 4\cos^2\theta_{ij}\), where \(\theta_{ij}\) is the angle between \(\beta_i\) and \(\beta_j\). Whenever \((\beta_i, \beta_i) < (\beta_j, \beta_j)\), the lines get an arrow pointing towards \(\circ^i\) [6].
span of this basis generates the compact real form $\mathfrak{so}(2n + 1, \mathbb{R})$, but to obtain the Weyl basis the representation should be changed to $w_\xi w^{-1}$, with $w_{ij} = [(1 + i)\delta_{ij} + (1 - i)\delta_{i,j}]/2$, $(w^{-1} = w^*)$. In the new representation $\mathfrak{so}(2n + 1, \mathbb{C})$ is generated by the elements

$$\{f_{pq} = -f_{-q-p} = e_{pq} - e_{-q-p}/p, q = -n, ..., -1, 0, 1, ..., n\},$$

while the real span of $f_{pq}$ generates $\mathfrak{so}(n+1)\mathfrak{so}(n)$. As $[f_{pq}, f_{kl}] = \delta_{qk}f_{pl} - \delta_{lp}f_{kq} + \delta_{p-k}f_{-l} + \delta_{-q}f_{k-p}$ we get

$$[f_{pp}, f_{kl}] = \alpha_{kl}(p)f_{kl},$$

with $\alpha_{kl}(p) = \delta_{pk} - \delta_{pl} + \delta_{p-l} - \delta_{p-k}$, and

$$[f_{kl}, f_{lk}] = f_{kk} - (1 - 2\delta_{k-l})f_{ll}.$$

These commutation relations provide:

- the set of roots:
  $$\Delta = \{\alpha_{kl}(p), k > -l, k \neq l\}$$
  with $2n^2$ elements.

- the set of positive roots:
  $$\Sigma = \{\alpha_{kl}(p) = \delta_{pk} - \delta_{pl}, l > k > 0\} \cup \{\alpha_{kl}(p) = -\delta_{p-k} - \delta_{pl}, -l < k < 0\} \cup \{\alpha_{0l}(p) = -\delta_{pl}, 1 \leq l \leq n\}$$
  with $n^2$ elements.

- the set of simple roots:
  $$\Pi = \{\beta_0 = \alpha_{0l}\} \cup \{\beta_k = \alpha_{kk+1}, k = 1, n - 1\}$$
  with $n$ elements. In terms of the simple roots

$$\alpha_{kl} = \sum_{i=|k|}^{l-1} \beta_i + 2 \sum_{i=0}^{|k|-1} \beta_i \text{ if } -l < k < 0,$$

$$\alpha_{0l} = \sum_{i=0}^{l-1} \beta_i,$$

and

$$\alpha_{kl} = \sum_{i=|k|}^{l-1} \beta_i \text{ if } l > k > 0.$$
Because
\[ h'_{\alpha_{kl}} = -\frac{f_{ll}}{2(n-1)} , \quad h'_{\alpha_{kk+1}} = \frac{f_{kk} - f_{kk+1}}{2(n-1)} , \quad \]
and
\[ (f_{pp}, f_{qq}) = \sum_{-l < k = -n}^{n} \alpha_{kl}(p)\alpha_{kl}(q) = 2(n-1)\delta_{pq} \]
we get
\[ \langle \beta_0, \beta_0 \rangle = \frac{1}{2(2n-1)} , \quad \langle \beta_k, \beta_k \rangle = -2(\beta_{k-1}, \beta_k) = \frac{1}{2n-1} , \quad k = 1, n - 1 , \]
such that the angles between the simple roots are specified by \( \cos \varphi_{01} = -1/\sqrt{2} \) and \( \cos \varphi_{kk+1} = -1/2, \quad k > 0. \) The Dynkin diagram associated to the Cartan matrix \( [2(\beta_i, \beta_j)/(\beta_i, \beta_j)] \) is represented in Figure 2.

In the case of \( \mathfrak{so}(5,\mathbb{C}) \), if \( \Sigma = \{ \alpha_{-12}, \alpha_{01}, \alpha_{02}, \alpha_{12} \} \), then \( \Pi = \{ \alpha_{01}, \alpha_{12} \} \). Some applications with \( \mathfrak{so}(5) \) as a subalgebra of \( \mathfrak{so}(8) \) are presented in [9, 10].

### 6.3 Semisimple Lie algebras of type \( C_n : \mathfrak{sp}(2n, \mathbb{C}) \)
\( \mathfrak{sp}(2n, \mathbb{C}) = \{ \xi \in \mathfrak{gl}(2n, \mathbb{C})/\xi^T J = -J \xi \} \), where \( J \) has the \( n \times n \) block form
\[ J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} . \]
Because any element \( \xi \in \mathfrak{sp}(2n, \mathbb{C}) \) can be expressed as
\[ \xi = \begin{bmatrix} a & b \\ c & -a^T \end{bmatrix} , \quad b = b^T , \quad c = c^T , \]
a basis in \( \mathfrak{sp}(2n, \mathbb{C}) \) is provided by \( n^2 \) independent matrices \( A \),
\[ A_{ij} = \begin{bmatrix} e_{ij} & 0 \\ 0 & -e_{ji} \end{bmatrix} , \]
\( n(n + 1)/2 \) independent matrices \( B \),
\[ B_{ij} = \begin{bmatrix} 0 & e_{ij} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & e_{ji} \\ 0 & 0 \end{bmatrix} , \]
and \(n(n + 1)/2\) independent matrices \(C_i\),

\[
C_{ij} = \begin{bmatrix}
0 & 0 \\
e_{ij} & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
e_{ji} & 0
\end{bmatrix}.
\]

The commutation relations are specified by

\[
\begin{align*}
[A_{ii}, A_{pq}] &= (\delta_{ip} - \delta_{iq})A_{pq}, \\
[A_{ii}, B_{pq}] &= (\delta_{ip} + \delta_{iq})B_{pq}, \\
[A_{ii}, C_{pq}] &= -(\delta_{ip} + \delta_{iq})C_{pq}, \\
[B_{pq}, C_{pq}] &= (2\delta_{pq} + 1)A_{pp} + A_{qq}, \\
[A_{pq}, A_{qp}] &= A_{pp} - A_{qq},
\end{align*}
\]

with \(i, p, q = 1, n\). Denoting \(\alpha_{pq}(i) = \delta_{ip} - \delta_{iq}, p > q\) and \(\alpha'_{pq}(i) = (\delta_{ip} + \delta_{iq}), p \geq q\), we get the set of roots:

\[
\Delta = \{\alpha_{pq}, -\alpha_{pq}, p, q = 1, n/p > q\} \cup \{\alpha'_{pq}, -\alpha'_{pq}, p, q = 1, n/p \geq q\},
\]

with \(2n^2\) elements.

- the set of positive roots:

\[
\Sigma = \{\alpha_{pq}, p, q = 1, n/p < q\} \cup \{\alpha'_{pq}, p, q = 1, n/p \leq q\},
\]

with \(n^2\) elements.

- the set of simple roots:

\[
\Pi = \{\beta_p = \alpha_{pp+1}, p = 1, n - 1\} \cup \{\beta_n = \alpha'_{nn}\}
\]

with \(n\) elements. In terms of the simple roots

\[
\alpha_{pq} = \sum_{i=p}^{q-1} \beta_i, \quad \alpha'_{qq} = \beta_n + 2 \sum_{i=q}^{n-1} \beta_i, \quad \alpha'_{pq} = (1 - \delta_{pq})\alpha_{pq} + \alpha'_{qq}.
\]

The Cartan subalgebra is generated by \(n\) independent diagonal matrices,

\[
h'_{\beta_p} = \frac{A_{pp} - A_{p+1p+1}}{4(n + 1)}, \quad p < n, \quad h'_{\beta_n} = \frac{A_{nn}}{2(n + 1)}.
\]
Figure 3. Dynkin diagram for the $C_n$ algebras

Because

$$(A_{ii}, A_{jj}) = 2\sum_{p>q} \alpha_{pq}(i)\alpha_{pq}(j) + 2\sum_{p\geq q} \alpha'_{pq}(i)\alpha'_{pq}(j) = 4(n+1)\delta_{ij}$$

we get

$$(\beta_p, \beta_q) = \frac{2\delta_{pq} - \delta_{p+1} - \delta_{p+1}q}{4(n+1)} , (\beta_p, \beta_n) = -\frac{\delta_{pn-1}}{2(n+1)} , p < n$$

and $(\beta_n, \beta_n) = 1/(n+1)$. The Dynkin diagram associated with the matrix $[2(\beta_i, \beta_j)/(\beta_i, \beta_i)]$ is represented in Figure 3.

### 6.4 Semisimple Lie algebras of type $D_n$ : $\mathfrak{so}(2n, \mathbb{C})$

$\mathfrak{so}(2n, \mathbb{C}) = \{ \xi \in \mathfrak{gl}(2n, \mathbb{C})/\xi^T = -\xi \}$ Considering the notation introduced above for $B_n$, in the Weyl basis $\mathfrak{so}(2n, \mathbb{C})$ is generated by the elements

$$\{f_{pq} = -f_{-q-p} = e_{pq} - e_{-q-p}/p, q = -n,...,-1,1,...,n\}$$,

having as real span $\mathfrak{so}(n, n)$. Because $h = \{f_{pp}, p = 1, n\}$, and

$$[f_{pq}, f_{kl}] = \delta_{qk}f_{pl} - \delta_{lp}f_{kq} + \delta_{p-k}f_{-lq} + \delta_{-q}f_{k-p}$$,

we get $[f_{pp}, f_{kl}] = \alpha_{kl}(p)f_{kl}$ with $\alpha_{kl}(p) = \delta_{pk} - \delta_{pl} + \delta_{p-l} - \delta_{p-k}$. Also

$$[f_{kl}, f_{lk}] = f_{kk} - (1 - 2\delta_{k-l})f_{ll}$$.

These relations provide:

- the set of roots:
  $$\Delta = \{\alpha_{kl}(p), k > -l, k \neq l/k, l = -n,...,-1,1,...n\}$$
  with $2n(n-1)$ elements.

- the set of positive roots $\alpha_{kl}$ with $k > |l|$ or $l > |k|$,
  $$\Sigma = \{\alpha_{kl}(p) = \delta_{pk} - \delta_{pl}, l > k > 0\} \cup \{\alpha_{kl}(p) = -\delta_{p|k|} - \delta_{pl}, -l < k < 0\}$$
Figure 4. Dynkin diagram for the $D_n$ algebras

with $n(n-1)$ elements.
- the set of simple roots:

$$\Pi = \{\beta_0 = \alpha_{-12}\} \cup \{\beta_k = \alpha_{kk+1}, k = 1, n-1\}$$

with $n$ elements. In terms of the simple roots

$$\alpha_{kl} = \sum_{i=k}^{l-1} \beta_i , \ l > k > 0 ,$$

$$\alpha_{-|k||k|+1} = \beta_0 + \beta_1 + \beta_k + 2 \sum_{i=2}^{k-1} \beta_i , \ k = 1, n-1 ,$$

and $\alpha_{-|k||k|} = \alpha_{-|k||k|+1} + \alpha_{|k||k|+11}, \ l > |k|$. Because

$$h'_{\alpha_{-12}} = -\frac{f_{11} - f_{22}}{4(n-1)} , \ \ h'_{\alpha_{kl}} = \frac{f_{kk} - f_{ll}}{4(n-1)}, l > k > 0 ,$$

and

$$(f_{pp}, f_{qq}) = \sum_{-l<k=-n}^{n} \alpha_{kl}(p)\alpha_{kl}(q) = 4(n-1)\delta_{pq}$$

we get $(\beta_0, \beta_1) = 0$,

$$(\beta_0, \beta_0) = (\beta_k, \beta_k) = -2(\beta_0, \beta_2) = -2(\beta_k, \beta_{k+1}) = \frac{1}{2(n-1)} , \ k = 1, n-1$$

such that the angles between the simple roots are specified by $\cos \varphi_{01} = 0$, $\cos \varphi_{02} = -1/2$ and $\cos \varphi_{kk+1} = -1/2$ for $k \geq 1$. The Dynkin diagram associated with the matrix $[2(\beta_i, \beta_j)/(\beta_i, \beta_i)]$ is represented in Figure 4.

Let $c_i^\dagger, c_i$ be the fermion creation and annihilation operators for a quantum many-body system described in terms of $n > 1$ single-particle states, specified by $i = 1, n$. Then

$$[c_i^\dagger c_j, c_k^\dagger c_k] = (\delta_{ij} + \delta_{ik})c_j^\dagger c_k^\dagger , \ [c_i^\dagger c_i, c_j c_k] = -(\delta_{ij} + \delta_{ik})c_j c_k ,$$

and $[c_i^\dagger c_i, c_j^\dagger c_k] = (\delta_{ij} - \delta_{ik})c_j^\dagger c_k$, so that the set $S = \{c_i^\dagger c_j, c_i c_j, c_i^\dagger c_j / i, j = 1, n\}$ generates an $\mathfrak{so}(2n)$ algebra. For a two-state system $\mathfrak{so}(4, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{C}) + \mathfrak{sl}(2, \mathbb{C})$ is generated by

$$J_+ = c_1^\dagger c_2 , \ J_0 = \frac{1}{2}(c_1^\dagger c_1 - c_2^\dagger c_2), \ J_- = c_2^\dagger c_1$$

29
and

\[ P_+ = c_1^\dagger c_2^\dagger , \quad P_0 = \frac{1}{2}(c_1^\dagger c_1 + c_2^\dagger c_2 - 1), \quad P_- = c_2 c_1 . \]

An application of the coherent states generated by \( S \) can be found in [11].

7 Appendix 3: Symplectic actions and invariance

Let \((M, \omega)\) be a symplectic manifold, \( \Phi_a \) a symplectic action of the Lie group \( G \) on \( M \), and \( X_H \) a Hamiltonian field on \( M \), \( i_{X_H} \omega = dH \). The equations of motion are called invariant to the action of \( G \) if

\[ \Phi_a^* X_H = X_H , \quad \forall a \in G . \]

When the equations of motion are invariant \( \Phi_a^* H - H \equiv \rho(a) \) is a constant on \( M \) and \( \rho(ab) = \rho(a) + \rho(b) \), \( \forall a, b \in G \), so that \( a \mapsto \rho(a) \) is a homomorphism of \( G \) into \( \mathbb{R} \). If \( G \) is compact then \( \rho(a) = 0 \), and \( H \) is invariant to the action of \( G \).

The Poisson bracket of the functions \( f, g \) on \( M \) is \( \{f, g\} = \omega(X_f, X_g) \) and

\[ \frac{d}{dt}(f \circ F_t) = \{f \circ F_t, H\} \]

or

\[ \dot{f} = df(X_H) = L_{X_H} f = \omega(X_f, X_H) = \{f, H\} . \]

8 Appendix 4: The rigid body and geodesic motion

Let \( \{x_i, i = 1, 2, 3\} \) be the space coordinates in the laboratory frame, and \( \{x'_k = \sum_i R_{ki} x_i\} \), the coordinates in the rotated (intrinsic) frame. The rotation matrix \( R \) can be taken of the form \( R = e^{\psi J_3} e^{\theta J_1} e^{\varphi J_3} \), where \((\psi, \theta, \varphi)\) are the Euler angles, and the \( \mathfrak{so}(3, \mathbb{R}) \) generators \((J_1, J_2, J_3)\) are \( 3 \times 3 \) matrices with elements \((J_i)_{kl} = \epsilon_{ilk} \), and commutation relations \([J_i, J_k] = -\epsilon_{ikl} J_l \).

If \((\psi, \theta, \varphi)\) depend on time, then

\[ \dot{R} = R \sum_i \omega_i J_i = \sum_i \omega_i' J_i R \]
defines the angular velocity components \( \omega_i, \omega'_i \) (\( \omega'_k = \sum R_{ki} \omega_i \)) in the laboratory, respectively in the intrinsic frame. Explicitly,

\[
\begin{align*}
\omega'_1 &= \dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi \\
\omega'_2 &= -\dot{\theta} \sin \psi + \dot{\phi} \sin \theta \cos \psi \\
\omega'_3 &= \dot{\psi} + \dot{\phi} \cos \theta .
\end{align*}
\]

For the rigid body the kinetic energy is

\[
T = \sum_{pq} I_{pq} \omega'_p \omega'_q / 2 = \sum_{ij} g_{ij} \omega_i \omega_j / 2,
\]

where \( I_{pq} = I_p \delta_{pq} \) is the intrinsic moment of inertia (constant), and \( g_{ij} = \sum_{pq} R_{ip}^{-1} R_{jq}^{-1} I_{pq} \). Thus, \( L_k = \partial T / \partial \omega_k \), \( L'_k = \partial T / \partial \omega'_k \), are the angular momentum components, and with

\[
\dot{g}_{ij} = \sum_{k,l=1}^3 \epsilon_{ikl} \omega_k g_{lj} + \epsilon_{jkl} \omega_k g_{li} ,
\]

the conservation law \( \dot{L}_k = 0, k = 1, 2, 3 \), yields the Euler equations

\[
L' = L' \times \omega' ,
\]

or \( g \dot{\omega} = L \times \omega \).

It is interesting to remark that if \( g_{ij} \) is considered as a metric tensor, and

\[
\Gamma^m_{ij} = \sum_{l=1}^3 g^{ml} (g_{il,j} + g_{jl,i} - g_{ij,l})
\]

as Christoffel symbols of a Riemannian connection, then for a geodesic \( \gamma(t) \), \( \dot{\gamma} = \eta \), the equation \( \eta_m = -\sum_{ij} \Gamma^m_{ij} \eta_i \eta_j \) takes the form \( g \dot{\eta} = 2(g \eta) \times \eta \).

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