Collapse transition of the interacting prudent walk

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1. Interacting self-avoiding walk

2. Interacting partially directed self-avoiding walk

3. Interacting prudent walk

4. Methods and Proofs
1 Interacting self-avoiding walk
1.1) Set of configurations in dimension $d \geq 2$

$$\Omega_{L}^{SAW,d} = \left\{ w = (w_i)_{i=0}^{L} : \begin{array}{l} w_0 = 0, w_i - w_{i-1} \in \{ \pm e_1, \ldots, \pm e_d \}, \\
w \text{satisfies the self-avoiding condition} \end{array} \right\}$$
With every $w \in \Omega^{SAW,d}_L$, we associate an hamiltonian that sums the self-touchings performed by $w$, i.e.,

$$H_L(w) = \sum_{0 \leq i < j \leq L} 1\{|u_i - u_j| = 1\},$$

with $u_k = w_{k-1} + \frac{w_k - w_{k-1}}{2}$ center of the $k$-th step ($k \leq L$).
1.3) Free energy

The coupling parameter is $\beta \in [0, \infty[$ and ISAW model is then defined by

$$P_{\beta,L}(w) = \frac{e^{\beta H_L(w)}}{Z_{\beta,L}}, \quad w \in \Omega_{L}^{SAW,d},$$

and the free energy

$$F^{SAW}(\beta) := \lim \inf_{L \to \infty} \frac{1}{L} \log Z_{\beta,L}.$$
1.4) Main open questions

- Existence of \( F^{\text{SAW}}(\beta) \) for \( \beta > 0 \) : only proven for \( \beta \) small [Ueltschi (2002), Hammond & Helmuth (2019)]

- Phase transition conjectured at some \( \beta_c(d) > 0 \) between an extended phase \( \mathcal{E} = [0, \beta_c(d)) \) and a collapsed phase \( \mathcal{C} = [\beta_c(d), \infty) \).
  [Saleur (87), Duplantier & Saleur (87)]

- Typical extension: a typical path \( w \) sampled from \( P_{\beta,L} \) is expected to scale:
  - in \( \mathcal{E} \) as \( ||w_L|| \asymp L^{\nu_{\text{SAW}}} \),
  - in \( \mathcal{C} \) as \( ||w_L|| \asymp L^\chi \) with \( \chi < \nu_{\text{SAW}} \).
  [Brak, Owczarek, Prellberg (93)]
**Problem**: Self-avoiding walk is very complicated object! ⇒ relax the self-avoiding constraint or consider directed path.

- **Interacting Weakly Self-Avoiding Walk**  
  [v.d. Hofstadt, Klenke & Koenig (2001-2002), Bauerschmidt, Slade & Wallace (2016)]

- **Interacting Partially Directed Self-Avoiding Walk**  
  [Zwanzig, Lauritzen (1969)]  
  [Whittington, Brak, Owzcareck, Prellberg]  
  [Carmona, P., Nguyen (2012, 2016, 2019)]

- **Interacting Prudent Walk**  
  [P. & Torri (2018)]
Interacting partially directed self-avoiding walk
2.2) 2-dimensional Interacting partially directed self-avoiding walk \[\text{[Zwanzig & Lauritzen (1968)]}\]

\[
\Omega_{PDSAW}^{L} = \left\{ w = (w_i)^L_{i=0} : \begin{array}{l}
    w_0 = 0, \ w_i - w_{i-1} \in \{\uparrow, \rightarrow, \downarrow\}, \\
    w \text{ satisfies the self-avoiding condition}
\end{array} \right\}
\]

\[
\mathbf{F}_{PDSAW}^{IP} (\beta) = \lim_{L \to \infty} \frac{1}{L} \log Z_{\beta,L}^{IPDSAW} \in [\beta, \infty).
\]
2.2.1) Collapse transition

Theorem (Brack et al (1993), Nguyen & P. (2013))

Computation of \( \beta_c^D \in (0, \infty) \) such that
- Collapsed phase \( \beta > \beta_c^D \),
- Extended phase \( \beta < \beta_c^D \).

Phase transition second order with expo. 3/2.

2.2.2) Path properties The scaling limit of the path is identified in each regime.

Theorem (Carmona, Nguyen & P. (2016), C. & P. (2016, 2019))

- \( \beta < \beta_c^D \): horizontal extension of the path \( \sim L \) and vertical extension \( \sim \sqrt{L} \)
- \( \beta = \beta_c^D \): horizontal extension of the path \( L^{2/3} \) and vertical extension \( \sim L^{1/3} \)
- \( \beta > \beta_c^D \): Limiting Wulff shape, horizontal extension and vertical extension \( \sqrt{L} \).
3 Interacting prudent walk
3.1) Prudent paths [Debierre & Turban (1987)]

\[ \Omega_{L}^{\text{Pr}} = \left\{ w = (w_i)_{i=1}^{L} : w_0 = 0, w_i \in \{←, ↑, →, ↓\}, \right. \]

\[ \left. w \text{ satisfies the prudent condition} \right\} \]

- **Combinatoric viewpoint** [Detheridge & Guttman (2008), Bousquet-Melou (2010), Beaton & Iliev (2015)]
- **Probabilistic approach** [Beffara, Friedli & Velenik (2009), P., Sun & Torri (2017) (Scaling Limit of the Kinetic and of the Uniform prudent walk)].
3.2) Families of Prudent paths

- 1-sided (partially directed)

![1-sided (partially directed) diagram]

- 2-sided (North-East)

![2-sided (North-East) diagram]

- 4-sided (Prudent paths)

![4-sided (Prudent paths) diagram]
- 1-sided (partially directed)
- 2-sided (North-East)
- 4-sided (Prudent walk)

**Open Question** (Bousquet-Melou): Exponential growth rate of number of configurations $\mu_{Pr}$:

$$\lim_{L \to \infty} \frac{1}{L} \log |\Omega_L^{Pr}| = \mu_{Pr}.$$ 

**Conjecture**: $\mu_{Pr} = \mu_{NE}$. 
3.3) Interacting Prudent Walk (IPRW)

\[ \Omega_{L}^{\text{Pr}} = \left\{ w = (w_i)_{i=1}^{L} : w_0 = 0, \ w_i \in \{\leftarrow, \uparrow, \rightarrow, \downarrow\}, \ w \text{ satisfies the prudent condition} \right\} \]

\[ F_{\text{Pr}}(\beta) = \lim_{L \to \infty} \frac{1}{L} \log Z_{L,\beta}^{\text{Pr}} \quad \text{and} \quad F_{\text{NE}}(\beta) = \lim_{L \to \infty} \frac{1}{L} \log Z_{L,\beta}^{\text{NE}} \]

with

\[ Z_{L,\beta}^{\text{Pr}} = \sum_{w \in \Omega_{L}^{\text{Pr}}} e^{\beta H_L(w)} \quad \text{and} \quad Z_{L,\beta}^{\text{NE}} = \sum_{w \in \Omega_{L}^{\text{NE}}} e^{\beta H_L(w)} \]
3.4) Results

Theorem (P. & Torri, (2018))

1. For any $\beta \geq 0$ the Free Energy exists and

$$F^{Pr}(\beta) = F^{NE}(\beta) \in [\beta, \infty).$$

Consequence (at $\beta = 0$) : $\mu_P = \mu_P^{NE}$ !

2. There exists a critical point $\beta^{Pr}_c \in (0, \infty)$ such that

$$-F^{Pr}(\beta) > \beta \text{ for every } \beta > \beta^{Pr}_c,$$

$$-F^{Pr}(\beta) = \beta \text{ for every } \beta \leq \beta^{Pr}_c.$$

3. $\beta^{Pr}_c \geq \beta^{PD}_c$.  

4. $F^{SAW}(\beta) > \beta$, for all $\beta \geq 0$. 
4 Methods and Proofs
4.1) Proofs

1. Existence critical point $\beta_{c}^{NE}$ such that $F^{NE}(\beta) = \beta$, for all $\beta \geq \beta_{c}^{NE}$.

2. $F^{Pr}(\beta) = F^{NE}(\beta)$ for all $\beta \geq 0$. 
4.2) Phase transition for the NE-model

Decompose each NE path into partially directed subpaths (called blocks).

4.3) Decomposition of a path into oriented blocks
Critical Point: $\beta_c : F^{\text{NE}}(\beta) = \beta$ for any $\beta > \beta_c$.

Goal: Upper bound for $Z_{L,\beta}^{\text{NE}} : Z_{L,\beta}^{\text{NE}} \leq C(\beta)e^{\beta L}$ for $\beta$ large. Have to control

- Self-touchings in any block,
- Self-touchings between different blocks.
4.4) Partition function restricted to one block
4.4.1) Stretches representation of an oriented block

- \( N - 1 \) inter-stretches: increments along the orientation.
- \((\ell_1, \ldots, \ell_N) \in \mathbb{Z}^N\) sequence of stretches.
- Total length \(|\ell_1| + \cdots + |\ell_N| + N - 1|.

![Diagram showing inter-stretches and stretch orientation with numbers of increments in each stretch]
$$\mathcal{L}_{T,N}(d, f) = \{ (\ell_i)_{i=1}^N : \ell_1 = d, \ell_N = f, \sum_{i=1}^N |\ell_i| = T - N + 1 \}$$

Number self-touchings between $\ell_i$ and $\ell_{i+1}$:

$$\ell_i \lessdot \ell_{i+1} := \frac{1}{2} (|\ell_i| + |\ell_{i+1}| - |\ell_i + \ell_{i+1}|).$$

$$Z_{T,\beta}^{PD}(N; d, f) = \sum_{\ell \in \mathcal{L}_{T,N}(d, f)} e^{\beta \sum_{i=1}^{N-1} \ell_i \lessdot \ell_{i+1}}$$

$$= \frac{e^{\beta T - \frac{\beta}{2} |f| - \frac{\beta}{2} |d|}}{e^{\beta (N-1)}} \sum_{\ell \in \mathcal{L}_{T,N}(d, f)} \prod_{i=1}^{N-1} e^{-\frac{\beta}{2} |\ell_{i+1} + \ell_i|}$$
4.4.2) Auxiliary Random Walk
With each \((\ell_i)_{i=1}^N \in \mathcal{L}_{T,N}(d, f)\) we associate a random walk trajectory

\[
(\ell_1, \ldots, \ell_N) \Leftrightarrow (V_i)_{i=1}^N : V_i = (-1)^{i-1}\ell_i
\]

The increments of \(V\):

\[
V_i - V_{i-1} = (-)^{i-1}(\ell_{i-1} + \ell_i).
\]

One to one correspondence between \(\mathcal{L}_{T,N}(d, f)\) and

\[
\left\{ (V_i)_{i=1}^N : G_N(V) = L - N + 1, V_1 = d, V_N = (-1)^{N-1}f \right\}
\]

with \(G_N(V) = \sum_{i=1}^N |V_i|\).
\[ Z_{T,\beta}^{PD}(N; d, f) = e^{\beta T - \frac{\beta}{2}|f| - \frac{\beta}{2}|d|} \left( \frac{c\beta}{e^\beta} \right)^{N-1} \sum_{\ell \in \mathcal{L}_{T,N}(d,f)} e^{-\frac{\beta}{2}|\ell_{i+1} + \ell_i|} \prod_{i=1}^{N-1} c\beta \]

with

\[ \sum_{\ell \in \mathcal{L}_{T,N}(d,f)} \ldots = \sum_{\ell \in \mathcal{L}_{T,N}(d,f)} P_{\beta} \left( (V_i)_{i=2}^{N} = ((-1)^{i-1}\ell_i)_{i=2}^{N} \mid V_1 = d \right) \]

\[ Z_{T,\beta}^{PD}(N; d, f) = e^{\beta T - \frac{\beta}{2}|f| - \frac{\beta}{2}|d|} \left( \frac{c\beta}{e^\beta} \right)^{N-1} P_{\beta} \left( \begin{array}{c} V_N = (-1)^{N-1}f \\ G_N(V) = T - N + 1 \end{array} \mid V_1 = d \right) \]
4.5) Interactions between oriented blocks

Block $i + 1$ interacts with blocks $i - 1$ and $i$ and such interactions are bounded above by

$$(N_i + |f_{i-1}|) \wedge |d_{i+1}| \leq \frac{|f_{i-1}| + |d_{i+1}|}{2} + \frac{3}{4} (N_i - 1) - \frac{1}{4} |f_{i-1} + d_{i+1}|$$
4.6) Full partition function

Partition $\Omega^\text{NE}_L$ depending on

- $r$ number of oriented blocks.
- $T_1, \ldots, T_r$ length of each block \( (T_1 + \cdots + T_r = L) \)
- $N_1, \ldots, N_r$ numbers of stretches in each block
- \((d_i, f_i)_{i=1}^r\) length of first and last stretch in each block.

Thus

$$Z^\text{NE}_{L,\beta} \leq \sum_{r \leq L} \sum_{T,N,d,f} \prod_{i=1}^r Z^\text{PD}_{T_i,\beta}(N_i, d_i, f_i) e^{\beta(|f_i-2|+N_{i-1})\wedge|d_i|)}$$

and

$$\prod_{i=1}^r \leq \prod_{i=1}^r \frac{e^{\beta T_i}}{2} \left( \frac{|d_i| + |f_i|}{e^{\beta}} \right) \left( \frac{c_\beta}{e^{\beta}} \right)^{N_{i-1}} P_{\beta} \left( \frac{V_{N_i}}{G_{N_i}(V)} = \frac{T_i - N_i + 1}{f_i} \left| V_1 = d_i \right. \right)$$
\[
\prod_{i=1}^{r} \prod_{i=1}^{r} \left( \frac{c_{\beta}}{\beta} \right)^{N_{i} - 1} P_{\beta} \left( V_{N_{i}} = (-1)^{N_{i} - 1} f_{i} \right) \quad (c_{\beta}/2)
\]

so that \( f_{0} = d_{r+1} = 0 \)

\[
\prod_{i=1}^{r} \left( \frac{c_{\beta}}{\beta} \right)^{N_{1} + \cdots + N_{r} - r} c_{\beta}^{r/2}
\]

\[
\times \prod_{i=1}^{r} P_{\beta} \left( V_{N_{i}} = (-1)^{N_{i} - 1} f_{i} \right) \quad (c_{\beta}/2)
\]

\[
\times e^{\frac{\beta}{4} \mid f_{i-1} + d_{i+1} \mid}
\]
assume $r \in 2\mathbb{N}$:

\[
\prod_{i=1}^{r} P_{\beta} \left( \begin{array}{c}
V_{N_{i}} = (-1)^{N_{i}-1} f_{i} \\
G_{N_{i}}(V) = T_{i} - N_{i} + 1
\end{array} \right) \frac{e^{-\beta/4 |f_{i-1} + d_{i+1}|}}{c_{\beta/2}}
\]

\[
= \prod_{i=1}^{r/2} P_{\beta} \left( \begin{array}{c}
V_{N_{2i}} = (-1)^{N_{2i}-1} f_{2i} \\
G_{N_{2i}}(V) = T_{2i} - N_{2i} + 1
\end{array} \right) \frac{e^{-\beta/4 |f_{2i-2} + d_{2i}|}}{c_{\beta/2}}
\]

\[
= \prod_{i=1}^{r/2} P_{\beta} \left( \begin{array}{c}
V_{N_{2i-1}} = (-1)^{N_{2i-1}-1} f_{2i-1} \\
G_{N_{2i-1}}(V) = T_{2i-1} - N_{2i-1} + 1
\end{array} \right) \frac{e^{-\beta/4 |f_{2i-1} + d_{2i+1}|}}{c_{\beta/2}}
\]

Consequence: The summations over $T_1, \ldots, T_r$ and over $(d_i, f_i)_{i=1}^r$ disappear in the probabilities.
Conclusion: Since $N_i \geq 2$ for every $i \leq r$ (at least two stretches in an oriented block)

$$Z^{\text{NE}}_{L,\beta} \leq e^{\beta L} \sum_{r=1}^{L/4} \sum_{N_1+\ldots+N_r \leq L} c_r^{r/2} \left( \frac{c\beta}{e^{4\beta}} \right)^{N_1+\ldots+N_r-r}$$

$$\leq e^{\beta L} \sum_{r=1}^{\infty} \left[ c_{\beta/2} \sum_{N=1}^{\infty} \left( \frac{c\beta}{e^{4\beta}} \right)^N \right]^r$$

That is, $Z^{\text{NE}}_{L,\beta} \leq C(\beta)e^{\beta L}$ if $\beta$ is large enough. □