POISSON TYPE GENERATORS FOR $L^1(\mathbb{R})$

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Abstract. We characterize the discrete sets $\Lambda \subseteq \mathbb{R}$ such that
$\{\varphi(t - \lambda), \lambda \in \Lambda\}$ span $L^1(\mathbb{R})$, $\varphi$ being an $L^1(\mathbb{R})$-function whose Fourier transform behaves like $e^{-2\pi|\xi|}$.

1. Introduction

The study of the generators by translations for $L^p(\mathbb{R})$ has been a classical topic of study in harmonic analysis. Results in [Bru06] and [BOU06] characterize the discrete sets $\Lambda \subseteq \mathbb{R}$ for which there exists a function $\varphi \in L^1(\mathbb{R})$ with the property that $\{\varphi(t - \lambda), \lambda \in \Lambda\}$ span $L^1(\mathbb{R})$ as those having infinite Beurling-Malliavin density. In [Ole97] and [OU04] we can find results proving that in $L^2(\mathbb{R})$ there are more sets with this property, and that a characterization in terms of densities is not possible.

Given a function $\varphi$, a natural problem is to characterize the discrete sets $\Lambda$ such that $\{\varphi(t - \lambda), \lambda \in \Lambda\}$ span $L^p(\mathbb{R})$. There are very few complete results of this kind. In [BrM07] Bruna and Melnikov give a complete characterization for the Poisson function:

$$P(t) = \frac{1}{\pi} \frac{1}{1 + t^2}.$$  

Theorem 1.1 (Bruna, Melnikov). The translates $\{P(t - \lambda), \lambda \in \Lambda\}$ span $L^p(\mathbb{R})$, $1 \leq p < \infty$ if and only if

$$\sum_{\lambda \in \Lambda} e^{-\pi |\lambda|} = \infty.$$  

(1)

In the $L^2(\mathbb{R})$ case, by Fourier transform, $\{\varphi(t - \lambda), \lambda \in \Lambda\}$ span $L^2(\mathbb{R})$ if and only if the set of exponentials $\{e^{2\pi i \lambda \xi}, \lambda \in \Lambda\}$ span the weighted space $L^2(\mathbb{R}, |\varphi|^2)$, and hence the above characterization holds

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for any function $\varphi$ such that $|\hat{\varphi}| \simeq e^{-2\pi|\xi|}$. The aim of this note is to give a generalization of this type for the $L^1(\mathbb{R})$ case.

**Theorem 1.2.** Assume $\varphi \in L^1(\mathbb{R})$ has non-vanishing Fourier transform satisfying

$$Ae^{-2\pi|\xi|} \leq |\hat{\varphi}(\xi)| \leq Be^{-2\pi|\xi|}$$

for some constants $A, B$. Assume also that $|\hat{\varphi}'(\xi)| = O(e^{-2\pi|\xi|})$. Then $\{\varphi(t - \lambda), \lambda \in \Lambda\}$ span $L^1(\mathbb{R})$ if and only if condition 1.1 holds.

In fact the proof will show that condition 1.1 is necessary if $Ae^{-2\pi|\xi|} \leq |\hat{\varphi}(\xi)|$ and it is sufficient if both $\hat{\varphi}(\xi)$ and $(\hat{\varphi})'(\xi)$ are $O(e^{-2\pi|\xi|})$.

2. **Proof of the theorem**

Notice first that if certain translates of $\varphi \in L^1(\mathbb{R})$ span $L^1(\mathbb{R})$, then obviously $\hat{\varphi}(\xi) \neq 0$ for all $\xi$. In fact, non-vanishing of $\hat{\varphi}$ characterizes (as a consequence of Wiener’s Tauberian theorem) those $\varphi$ such that all its translates span $L^1(\mathbb{R})$. Analogously, for $p = 2$, a necessary condition in order than some translates of $\varphi$ span $L^2(\mathbb{R})$ is that $\hat{\varphi}(\xi) \neq 0$ for almost all $\xi$, this being equivalent to the fact that all translates of $\varphi$ span $L^2(\mathbb{R})$.

**Lemma 1.** Assume $h \in L^1(\mathbb{R})$ and that $\hat{h}(\xi) \neq 0$ for all $\xi$. Then, if $f \in L^p(\mathbb{R}), 1 \leq p \leq \infty$, and the convolution $f \ast h$ is zero, then $f = 0$. The same holds if $h \in L^2(\mathbb{R})$ and $\hat{h}(\xi) \neq 0$ almost everywhere.

**Proof.** For $1 \leq p \leq 2$, the Fourier transform of $f$ is a function in $L^q, \frac{1}{p} + \frac{1}{q} = 1$, and the Fourier transform of $f \ast h$ is $\hat{f} \hat{h}$, so the lemma follows. In the general case, we consider the closed subspace $E$ of $L^1(\mathbb{R})$ consisting of functions $g$ such that $f \ast g = 0$; since $E$ is translation invariant and contains $h$, Wiener’s tauberian theorem implies that $E$ is the whole $L^1(\mathbb{R})$, and this implies $f = 0$. When $h \in L^2(\mathbb{R})$ we use Beurling’s theorem describing all closed translation-invariant subspaces of $L^2(\mathbb{R})$ to reach the same result.

We note in passing that the version of Wiener or Beurling theorem for $L^p(\mathbb{R}), 1 < p < 2$, is still unknown.

In the following, we assume that $\varphi \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ satisfies $\hat{\varphi}(\xi) \neq 0$ for all $\xi$ or else $\varphi \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ with $\hat{\varphi}(\xi) \neq 0$ for almost all $\xi$. By duality, $\{\varphi(t - \lambda), \lambda \in \Lambda\}$ span $L^p(\mathbb{R})$ if and only if $f \in L^q(\mathbb{R})$ and

$$\bar{\varphi} \ast f(\lambda) = \int_{\mathbb{R}} f(t)\varphi(t - \lambda) \, dt = 0 \quad \forall \lambda \in \Lambda$$
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implies $f = 0$. Here $\tilde{\varphi}(t) = \varphi(-t)$. By the lemma, $f = 0$ is equivalent to $\varphi * f = 0$, whence $\{\varphi(t - \lambda), \lambda \in \Lambda\}$ span $L^p(\mathbb{R})$ if and only if $\Lambda$ is a uniqueness set for the space

$$E^q_\varphi = \{F = f * \tilde{\varphi}; f \in L^q(\mathbb{R})\}$$

meaning that $F \in E^q_\varphi, F(\lambda) = 0, \lambda \in \Lambda$, implies $F = 0$.

**Lemma 2.** Assume $h \in L^1(\mathbb{R})$ and $\hat{h}(\xi) \neq 0$ for every $\xi$ (respectively, $h \in L^2(\mathbb{R})$ with $\hat{h}(\xi) \neq 0$ almost everywhere). Then if $\{\varphi(t - \lambda), \lambda \in \Lambda\}$ span $L^1(\mathbb{R})$ then $\{(\varphi * h)(t - \lambda), \lambda \in \Lambda\}$ span $L^1(\mathbb{R})$ (respectively $L^2(\mathbb{R})$).

**Proof.** For $f \in L^q(\mathbb{R})$,

$$\int_\mathbb{R} f(t) (\varphi * h)(t - \lambda) dt = \int_\mathbb{R} (\tilde{h} * f)(x) \varphi(x - \lambda) dx,$$

whence the result follows from the lemma above. \hfill \Box

**Lemma 3.** If $\phi(t) = P * \hat{P}(t)$ then $\{\phi(t - \lambda), \lambda \in \Lambda\}$ span $L^p(\mathbb{R}), 1 \leq p < \infty$ if and only if [3.1] holds

**Proof.** It is clear that this condition is sufficient, since $\phi$ is a convolution of $P$ with a function of $L^1(\mathbb{R})$ and we can apply lemma [2]. For the necessity we will revise the proof of theorem [1.1]. By duality, we must see that if $\sum_{\lambda \in \Lambda} e^{-\pi |\lambda|} < \infty$ then we can find $g \in L^q(\mathbb{R}), \neq 0$ such that:

$$\int_\mathbb{R} g(t) \phi(t - \lambda) dt = 0 \quad \forall \lambda \in \Lambda,$$

where we can think that $g$ is real. The above integral equals

$$\int_\mathbb{R} g(t)(P * \hat{P})(t - \lambda) dt = \frac{1}{\pi} \int_\mathbb{R} (g * \hat{P})(t) \frac{1}{1 + (t - \lambda)^2} dt$$

Now we complexify this expression:

$$F(z) = \frac{1}{\pi} \int_\mathbb{R} \frac{f(t)}{(t - z)^2 + 1} dt$$

with $f = g * \hat{P}$. When $f$ ranges in $L^q(\mathbb{R})$, $F$ ranges in the space $E^q(B)$ which in [BrM07] is shown to be the space of holomorphic functions in $B = \mathbb{R} z < 1$ such that:

$$\|F\| = \sup_{|g| < 1} \int_\mathbb{R} |\Re F(x + iy)|^q dx = \|F\|_q < \infty$$

$$F(z) = F(\overline{z}), \quad z \in B.$$ 

For $E^\infty(B)$ the first condition is replaced by $\Re F$ bounded. Hence we have to find $F \in E^q(B)$ such that $F(\lambda) = 0$ for every $\lambda \in \Lambda$ and that
it can be written as (2) with \( f = g \ast \hat{P} \) for some \( g \in L^q(\mathbb{R}) \). We use the Fourier transform to see that:

\[
F(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{-2\pi|\xi|} e^{2\pi i x \xi} d\xi.
\]

By analytical continuation we obtain:

\[
F(z) = \int_{\mathbb{R}} \hat{f}(\xi) e^{-2\pi|\xi|} e^{2\pi i z \xi} d\xi,
\]

where we want that \( \hat{f}(\xi) = \frac{1}{\pi} \frac{\hat{g}(\xi)}{1 + \xi^2} \) with \( g \in L^q(\mathbb{R}) \). That is, we search \( F \in E^q(B) \) that can be written as:

\[
F(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\hat{g}(\xi)}{1 + \xi^2} e^{-2\pi|\xi|} e^{2\pi i z \xi} d\xi
\]

with \( g \in L^q(\mathbb{R}) \). Since

\[
F''(z) = \int_{\mathbb{R}} \hat{f}(\xi)(2\pi i \xi)^2 e^{-2\pi|\xi|} e^{2\pi i z \xi} d\xi,
\]

this amounts to \( F'' \in E^q(B) \). So we have reduced the problem to find \( F \in E^q(B) \) such that \( F(\lambda) = 0 \) for \( \lambda \in \Lambda \) and such that \( F'' \in E^q(B) \).

Now, as in [BrM07], we translate the problem to the disk. The conformal map from \( B \) to the disk is given by

\[
w = \Phi(z) = \frac{e^{\pi z} - 1}{e^{\pi z} + 1}.
\]

Let \( \Gamma = \Phi(\Lambda) \subset \mathbb{R} \cap \mathbb{D} \), as shown in [BrM07] finiteness of \( \sum_{\lambda \in \Lambda} e^{-\frac{\pi}{2}|\lambda|} \) is equivalent to

\[
\sum_{\gamma \in \Gamma} \log \frac{1}{|\gamma|} < \infty,
\]

that is the Blaschke condition. This guarantees that the product

\[
\beta(w) = \prod_{\gamma \in \Gamma} \frac{w - \gamma}{1 - \gamma w}
\]

is convergent (it is necessary to multiply by \( w \) if \( 0 \in \Gamma \)).

We suppose that \( H \) is a holomorphic function of the disk. If \( F(z) = H(\Phi(z)) \) then \( F \in E^q(B) \) exactly when \( g(s) = F(s \pm i) = H \left( \frac{e^{\pi s(i+1)}}{ie^{\pi s(i+1)}} \right) \) is in \( L^q(\mathbb{R}) \), that is if

\[
\int_{\mathbb{R}} |g(s)|^q \ ds = \frac{1}{\pi} \int_{|z|=1} |H(z)|^q \frac{|dz|}{|1 - z^2|} < \infty.
\]
We need $F'' \in E^q(B)$ as well. Computing

$$F''(z) = H''(\Phi(z)) \left( \frac{\pi e^{\frac{\pi z}{2}}}{(e^{\frac{\pi z}{2}} + 1)^2} \right)^2 + H'(\Phi(z)) \frac{\pi^2 e^{\frac{\pi z}{2}}(1 - e^{\frac{\pi z}{2}})}{2 (e^{\frac{\pi z}{2}} + 1)^3}$$

and changing variables again we need

$$\int_{|z|=1} \left| H''(z) \frac{\pi i(z+1)(z-1)}{(i(z+1) + (z-1))^2} \right|^q \frac{|dz|}{|1-z^2|} < \infty \quad (4)$$

$$\int_{|z|=1} \left| H'(z) \frac{\pi i(z+1)(z-1)((z-1) - i(z+1))}{(i(z+1) + (z+1))^3} \right|^q \frac{|dz|}{|1-z^2|} < \infty \quad (5)$$

Therefore we have to find a holomorphic function $H$ in the disk with $H(\gamma) = 0$ for $\gamma \in \Gamma$, $H(\overline{z}) = H(z)$ and so that (3), (4) and (5) are fulfilled. We choose $H(z) = (1 - z^2)^n \beta(z)$ with $n$ big enough. We first bound the derivatives of $\beta$:

$$\beta'(z) = \sum_{\gamma \in \Gamma} \frac{1 - \gamma^2}{(1 - \gamma z)^2} \prod_{\lambda \in \Gamma, \lambda \neq \gamma} \frac{z - \lambda}{1 - \lambda z}.$$

The product is bounded by 1 independently of $\gamma$ for almost all $z$. Moreover, for $|z| = 1$ we have $|1 - \gamma z| = |z - \lambda| \geq \frac{1}{2} |1 - z^2|$ and $|1 - \gamma^2| \leq 2(1 - |\gamma|)$. Therefore

$$|\beta'(z)| \leq \frac{2}{|1 - z^2|^2} \sum_{\gamma \in \Gamma} |1 - \gamma^2| \leq \frac{4}{|1 - z^2|^2} \sum_{\gamma \in \Gamma} 1 - |\gamma| \leq \frac{2K}{|1 - z^2|^2},$$

where we have used the Blaschke condition. For the second derivative we have

$$\beta''(z) = 2 \sum_{\gamma_1 \neq \gamma_2 \in \Gamma} \frac{1 - \gamma_1^2}{(1 - \gamma_1 z)^2} \frac{1 - \gamma_2^2}{(1 - \gamma_2 z)^2} \prod_{\lambda \in \Gamma, \lambda \neq \gamma_1, \gamma_2} \frac{z - \lambda}{1 - \lambda z} +$$

$$+ 2 \sum_{\gamma \in \Gamma} \frac{1 - \gamma^2}{(1 - \gamma z)^3} \prod_{\lambda \in \Gamma, \lambda \neq \gamma} \frac{z - \lambda}{1 - \lambda z},$$

which similarly can be bound by

$$|\beta''(z)| \leq \frac{12K^2}{|1 - z^2|^4}. $$

Then, choosing $H(z) = (1 - z^2)^4 \beta(z)$ all required conditions are fulfilled and the proof is finished. □

**Lemma 4.** Let $\psi(t) = P(t) - P''(t)$. Then $\{\psi(t - \lambda), \lambda \in \Lambda\}$ span $L^p(\mathbb{R})$ if and only if condition (4) holds.
Proof. Notice that \( \hat{\psi}(\xi) = (1 + 4\pi^2 \xi^2)e^{-2\pi|\xi|} \). The space \( E^q_\psi \) consists of the functions

\[
F(z) = \langle f(t), \psi(t-z) \rangle = \int_{\mathbb{R}} \hat{f}(\xi)(1+4\pi^2 \xi^2)e^{-2\pi|\xi|}e^{-2\pi i \xi z} = G(z) - G''(z)
\]

with \( G \in E^q(B) \), that is with \( f \in L^q(\mathbb{R}) \). Clearly this space contains \( E^q(B) \), and so the condition (1) is necessary, for if the series converges we already know that there is \( H \in E^q(B) \) vanishing on \( \Lambda \).

For the sufficiency we find a growth condition fulfilled by the second derivative of a function of \( E^q(B) \). There is a constant \( c_q \) such that whenever \( G \) is holomorphic in a disk \( D(a,R) \) of center \( a \) and radius \( R \) one has

\[
|G''(a)|^q \leq c_q \frac{1}{R^{2q+2}} \int_{D(a,R)} |G(z)|^q dA(z).
\]

Let \( G \in E^q(B) \). For \( z \in B \) we apply the above to the ball of center \( z \) and radius \( \frac{1-|y|}{2} \) (\( z = x + iy \)) to get

\[
\int_B (1-|y|)^2 q |G''(z)|^q \, dm(z) \leq c_q \int_B \int_{B(z, \frac{1-|y|}{2})} |G(w)|^q \, dm(w) \, dm(z).
\]

Apply Fubini and noticing that \( \{ z : w \in B(z, \frac{1-|y|}{2}) \} \subset B(w, 1-|3w|) \) and that for \( z \) in this set \( (1-|3w|)^2 \leq c(1-|y|)^2 \) we obtain

\[
\int_B (1-|y|)^2 q |G''(z)|^q \, dm(z) \leq c_q \int_B |G(w)|^q \, dm(w).
\]

This last integral is bounded since

\[
\int_B |G(z)|^q \, dm(z) = \int_{-1}^{1} \int_{-1}^{1} |G(x+iy)|^q \, dx \, dy \leq \int_{-1}^{1} \|G\|^q \, dy = 2\|G\|^q.
\]

This says that \( G'' \) is in a Bergman type space. Obviously \( G \) satisfies this condition too, and so the above holds with \( G \) replaced by \( F \in E^q_\psi \).

We next translate this integral to the disk.

We can check that if \( w = \Phi(z) = \frac{e^{2\pi i z} - 1}{e^{2\pi i} + 1} \) then \( 1-|y| \geq 1-|w| \). If \( H(w) = F(\Phi^{-1}(w)) \) then

\[
\int_D |H(w)|^q(1-|w|)^2q-1 \, dm(w) \leq \int_B (1-|w|)^2q |H(w)|^q \, dm(w)
\]

\[
\leq \int_B (1-|y|)^2q |F''(z)|^q \, dm(z).
\]
Therefore $H$ is in the Bergman space in the disk with weight $(1 - |w|)^{2r-1}$. The set of zeros contained in a diameter of a function of this space satisfies the Blaschke condition \[\text{[Kor75]}\]. Therefore the zeros of a function $F \in E^q$ satisfy $\sum_{\lambda \in \Lambda} e^{-\pi |\lambda|} < \infty$ and so (1) is sufficient. \[\square\]

Now, theorem 1.2 can be deduced using the previous lemmas. Assume that $\{\varphi(t - \lambda), \lambda \in \Lambda\}$ span $L^1(\mathbb{R})$ and that $A e^{-2\pi|\xi|} \leq |\hat{\varphi}(\xi)|$. Writing

$$\phi = P \ast \hat{P} = h \ast \varphi$$

with

$$\hat{h}(\xi) = \frac{e^{-2\pi|\xi|}}{\hat{\varphi}(\xi)(1 + \xi^2)}.$$ 

By lemma 2 (in the $L^2$ case), the functions $\{\phi(t - \lambda)\}_{\lambda \in \Lambda}$ span $L^2(\mathbb{R})$ and therefore by 3 (1) must hold. Assume next that $\hat{\varphi}(\xi)$ and $(\hat{\varphi})'(\xi)$ are $O(e^{-2\pi|\xi|})$ and that (1) holds; we write

$$\varphi = h \ast (P - P'')$$

with

$$\hat{h}(\xi) = \frac{\hat{\varphi}(\xi)}{e^{-2\pi|\xi|}(1 + 4\pi^2\xi^2)}.$$ 

The hypothesis on $\varphi$ implies that both $\hat{h}$ and $(\hat{h})'$ are in $L^2(\mathbb{R})$, whence both $h(x)$ and $xh(x)$ are in $L^2(\mathbb{R})$, so $h \in L^1(\mathbb{R})$. Since (1) holds, by lemma 4 the functions $\{\psi(t - \lambda), \lambda \in \Lambda\}$ span $L^1(\mathbb{R})$ and then lemma 2 implies that the functions $\{\varphi(t - \lambda), \lambda \in \Lambda\}$ span $L^1(\mathbb{R})$.

### 3. Generalizations and comments.

Using the same ideas we can also prove

**Theorem 3.1.** Fix $n \in \mathbb{N}$. Let $\varphi$ be a function for which there exists constants $A, B > 0$ such that:

$$A \frac{e^{-2\pi|\xi|}}{1 + \xi^{2n}} \leq |\hat{\varphi}(\xi)| \leq B(1 + \xi^{2n})e^{-2\pi|\xi|}. \tag{6}$$

We also suppose that:

$$|\hat{\varphi}'(\xi)| \leq C(1 + \xi^{2n}e^{-2\pi|\xi|}).$$

Then the set $\{\varphi(t - \lambda), \lambda \in \Lambda\}$ spans $L^1(\mathbb{R})$ if and only if condition 1.1 holds.

Only slight modifications are needed. For instance, one must use that the $2n$-th derivative of a function of $E^q(\mathcal{B})$ is in a Bergman type space (with a different weight); the Korenblum’s result quoted before applies to all the Bergman spaces as it is in fact true for functions in the
class $A^{-\infty}$. Another fact which is needed is that the $2n$-th derivative of the Blaschke product appearing above can be bounded by $\frac{K}{|1-z|^n}$.

As used in the proof, for the function $P$ the space $E^q_P$ is exactly $E^q(B)$. For the functions $\varphi$ considered here we have not exactly described this space, yet we can describe its uniqueness sets.

In [Zal78] the Gaussian function $G$ is considered. A complete characterization is not achieved. In fact, one can show that in this case the space $E^2_\varphi$ can be identified with the Fock space, for which the description of the uniqueness sets is an open question. It is known that a sufficient condition in order that the translates $\{G(t-\lambda), \lambda \in \Lambda\}$ span $L^2(\mathbb{R})$ is that the series $\sum_n \frac{1}{|\lambda_n|^{2+\varepsilon}}$ diverges for some $\varepsilon$, while it is necessary that it diverges for $\varepsilon = 0$.

REFERENCES

[Bru06] Bruna,J., On Translation and Affine Systems Spanning $L^1(\mathbb{R})$. J. Fourier Anal. Appl. 12 (2006), no. 1, 71–82.
[BrM07] Bruna,J., Melnikov,M., On Translates of the Poisson Kernel and Zeros of Harmonic Functions. Bull. London Math. Soc. 22 (2007), 317–326.
[BOU06] Bruna,J., Olevskii,A., Ulanovskii,A., Completeness in $L^1(\mathbb{R})$ of discrete translates. Revista Mat. Iberoamericana 22 (2006), no. 1, 1–16.
[Kor75] Korenblum,B., An extension of the Nevanlinna theory. Acta Math. 135 (1975), no. 3-4, 187–219.
[Ole97] Olevskii,A., Completeness in $L^2(\mathbb{R})$ of almost integer translates. C. R. Acad. Sci. Paris Sér.I Math. 324 (1997), no. 9, 987–991.
[OIU04] Olevskii,A., Ulanovskii,A., Almost Integer Translates. Do Nice Generators Exist? J. Fourier Anal. Appl. 10 (2004), no. 1, 93–104.
[Zal78] Zalik,R.A., On Approximation by Shifts and a Theorem of Wiener. Trans. Amer. Math Soc. 243 (1978), 299–308.