Aerodynamic forces exerted on a body in viscous incompressible fluid resulting from vorticity generation on the body surface

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Abstract. The vorticity evolution equation following from the Navier-Stokes equations for three-dimensional viscous incompressible fluid flows is represented in the divergent form. The partial derivative of the vorticity with respect to time is expressed as the divergence of a tensor called the vorticity transfer tensor. The concept of vorticity flux generated by a surface is defined. An expression for the pressure force in terms of the generated vorticity flux is obtained under the no-slip condition for three-dimensional bodies of constant and variable shape.

1. Introduction
The close connection between the aerodynamic forces exerted on a body and the attached vorticity and its fluxes shedding from the body surface is well known. This is the subject of the well-known Kutta-Joukowski theorem on the lifting force and the Sedov and Belotserkovskii formulas. In the studies of the authors mentioned above the formulas for the forces were obtained for potential flows. In [1-2] generalizations of these formulas to include the case of unsteady vortex ideal fluid separation flow were given. In [3-6] expressions for the aerodynamic forces in terms of the vorticity fluxes in a viscous incompressible fluid were obtained on the basis of the representation of evolution of the vorticity field as the motion of vortices in the case of unsteady plane-parallel and unswirled axisymmetric flows. Integral formulas relating the pressure distribution over the flow field to the motion of vortices were also obtained. These formulas were widely used to solve problems of unsteady flows past bodies [see 7-13 and others] and made it possible to design the effective method of solving the conjugate problems of interconnected motion of bodies and fluid [5,14,15]. In the present study the expressions for the forces in terms of the vorticity fluxes generated by the body surface is generalized to include the case of arbitrary three-dimensional flows of a viscous incompressible fluid.
In using the vortex methods, the expression for the aerodynamic forces in terms of the vorticity fluxes is most relevant. These methods are underlain by the vorticity evolution equation which can be obtained by applying the curl operator to the Navier-Stokes equation

\[ \frac{\partial}{\partial t} = -\nabla \times \left( \nabla \times \frac{V}{\nu} \nabla \times \right), \]

\[ = \nabla \times V, \quad \nabla V = 0 \] 

(1)
where $V$ and $\Omega$ are the fluid velocity and vorticity, respectively, and $\nu$ is the kinematic viscosity coefficient.

There is no pressure in these equations; therefore, the expression for the forces written directly in terms of the characteristics of vortex field is of importance.

For plane-parallel and unswirled axisymmetric flows the equation (1) can be written in the divergent form [16], [17]:

$$\frac{\partial \Omega}{\partial t} = -\nabla \cdot (U \Omega), \quad U = V + V_d,$$

where the vector $V_d$, called the diffusion velocity [18], is equal to

$$V_d = \begin{cases} \frac{-\nu \nabla \Omega}{\Omega} & \text{plane-parallel flows} \\ \frac{-\nu \nabla (r \Omega)}{r \Omega} & \text{axisymmetric flows} \end{cases}$$

The vectors $J = \nabla \Omega$ and $J_d = V_d \Omega$ represent the convective and diffusive vorticity transfer vectors.

The divergent form of writing the vorticity evolution equation was the basis for the development of the Lagrangian vortex methods of simulating two-dimensional viscous incompressible fluid flows in which the vorticity field was represented by a set of vortex particles of given circulation [18,14]. The particles move at a velocity $U$, their circulation being not varied. The fluid velocity $V$ can be determined from the vortex particle distribution using the Biot-Savart formula. New particles with a circulation $\Gamma_i^{\text{new}}$, where $i$ is the node number, are generated in the nodes of contours of the surfaces in each time step. The values of $\Gamma_i^{\text{new}}$ are determined from a system of linear equations representing the boundary conditions on the surface (no-flow or no-slip conditions). The value of $\Gamma_i^{\text{new}}$ is connected with the diffusive vorticity flux by the relation $\Gamma_i^{\text{new}} = J_d \mathbf{n} \Delta t \Delta l$, where $\mathbf{n}$ is the normal vector to the contour and $\Delta l$ is the length of a contour segment. Thus, the vorticity flux generated by the surface is known in each time step.

In [3] the following expression was obtained for the pressure force $F_p$ exerted on a profile under the no-slip condition:

$$F_p = e_z \times \int \rho (J_d \mathbf{n}) dl + 2\mathbf{\Omega} \times \mathbf{r}_0 S + \mathbf{r}_0 S.$$

Here, $\mathbf{r}_0$ is the coordinates of the center of gravity of the domain inside the profile, $S$ is its area, and $\mathbf{\Omega}$ is its angular velocity. This formula can be simplified since the integral of the vorticity over the entire space vanishes and the vorticity inside the profile rotating with the angular velocity $\mathbf{\Omega}$ is equal to $2\mathbf{\Omega}$.

Consequently, the increment of circulation is equal to zero: $e_z \int (J_d \mathbf{n}) dl + 2\mathbf{\Omega} S = 0$. With regard to this equality, the expression (3) takes the form:

$$F_p = e_z \times \int (r - \mathbf{r}_0) (J_d \mathbf{n}) dl + \mathbf{r}_0 S.$$

In axisymmetric flow a similar formula can be represented as follows:

$$F_p = \pi e_z \int r^2 (J_d \mathbf{n}) dl + \mathbf{r}_0 \Lambda$$

where $e_z$ is the unit vector directed along the axis of symmetry and $\Lambda$ is the body volume.
2. Vorticity transfer in three-dimensional flows
In the case of an arbitrary three-dimensional flow there exists no representation of the vorticity evolution in the form of continuous motion of vortex tubes with conserved circulation. We will show that despite this fact the equation (1) can be written in the divergent form and the concept of vorticity transfer can be defined.

The curl of the vector \( \mathbf{Q} = \nabla \times \mathbf{V} + \nu \nabla \times \) on the right-hand side of Eq. (2) can be written in the form of divergence of the tensor

\[
\nabla \times \mathbf{Q} = \begin{pmatrix}
\frac{\partial}{\partial y} Q_z - \frac{\partial}{\partial z} Q_y \\
\frac{\partial}{\partial z} Q_x - \frac{\partial}{\partial x} Q_z \\
\frac{\partial}{\partial x} Q_y - \frac{\partial}{\partial y} Q_x
\end{pmatrix} = \nabla \cdot \mathbf{T} = \begin{pmatrix}
\frac{\partial}{\partial x} T_{yx} + \frac{\partial}{\partial y} T_{xy} + \frac{\partial}{\partial z} T_{xz} \\
\frac{\partial}{\partial y} T_{zx} + \frac{\partial}{\partial z} T_{yz} + \frac{\partial}{\partial x} T_{zy} \\
\frac{\partial}{\partial z} T_{xy} + \frac{\partial}{\partial x} T_{yz} + \frac{\partial}{\partial y} T_{zx}
\end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix}
0 & -Q_z & Q_y \\
Q_z & 0 & -Q_x \\
-Q_y & Q_x & 0
\end{pmatrix}
\]

According to the Stokes theorem, the integral of the function \( \nabla \cdot \mathbf{T} \) over a volume \( \tau \) bounded by a closed surface \( S \) is equal to the integral of the product \( \mathbf{n} \cdot \mathbf{T} \) over the surface \( S \), where \( \mathbf{n} \) is the outward normal to the surface \( S \). Consequently,

\[
\frac{\partial}{\partial t} \int_{\tau} \mathbf{Q} d\tau = -\oint_{\partial \tau} \mathbf{n} \cdot \mathbf{T} ds
\]

Hence we can see that the tensor \( \mathbf{T} \) and the vector \( \mathbf{n} \cdot \mathbf{T} ds \) on the right-hand side have the meanings of the vorticity transfer tensor and its flux across the elementary area \( ds \), respectively. We can readily verify that

\[
\mathbf{n} \cdot \mathbf{T} = \begin{pmatrix}
0 + n_x Q_z - n_y Q_y \\
-n_z Q_x + 0 + n_x Q_x \\
n_y Q_y - n_y Q_x + 0
\end{pmatrix} = \mathbf{n} \times \mathbf{Q} = \mathbf{n} \times (\nabla \times \mathbf{V}) + \mathbf{v} \mathbf{n} \times (\nabla \times \mathbf{V})
\]

The first and second terms on the right-hand side correspond to the convective and diffusive vorticity fluxes \( \mathbf{n} \cdot \mathbf{T}_c = \mathbf{n} \times (\nabla \times \mathbf{V}) \) and \( \mathbf{n} \cdot \mathbf{T}_d = \mathbf{v} \mathbf{n} \times (\nabla \times \mathbf{V}) \), respectively.

3. Expression of the pressure force acting on a body
The pressure force \( \mathbf{F}_p \) exerted on an individual body is equal to the integral over its surface \( S \):

\[
\mathbf{F}_p = -\oint_S \mathbf{n} p ds, \quad (5)
\]

Here, the normal vector \( \mathbf{n} \) is directed outward the body.

We will now demonstrate that the equality holds:

\[
-\oint_S \mathbf{n} p ds = \frac{1}{2} \oint_S \mathbf{r} \times (\mathbf{n} \times \nabla p) ds. \quad (6)
\]

For this purpose, we will use the integral relation \( \oint_S (\mathbf{n} \times \nabla) \times \mathbf{f} ds = \oint_C \mathbf{r} \times \mathbf{f} \) known from the vector analysis. This relation holds for a continuous vector-function \( \mathbf{f} \) which has continuous partial
derivatives on an open surface $S$ bounded by a contour $\mathcal{C}$. From this equality there apparently follows the vanishing of the integral over the closed surface $\oint_S (n \times \nabla) \times \mathbf{f} ds = 0$. Setting $\mathbf{f} = r \mathbf{p}$, we obtain

$$\oint_S (n \times \nabla) \times (r \mathbf{p}) ds = 0 \quad (7)$$

Applying the differential operator $(n \times \nabla) \times$ to the multipliers of the product $r \mathbf{p}$, we can write

$$(n \times \nabla) \times (r \mathbf{p}) = p (n \times \nabla) \times r - r \times (n \times \nabla p) \quad (8)$$

Using the identity $(\times \nabla) \times r = -2a$, which can readily be verified, developing the double vector product, and differentiating the components of the vector $r$, we obtain

$$(n \times \nabla) \times (r \mathbf{p}) = -2pn - r \times (n \times \nabla p)$$

Integrating this equality over the surface $S$ and taking (7) into account, we obtain

$$0 = -2\oint_S pnds - \oint_S r \times (n \times \nabla p) ds$$

Thus, the equality (6) is proved.

We will now express $\nabla p$ from the Navier-Stokes equation $\nabla p = -\rho \frac{D\mathbf{V}}{Dt} - \nu \rho \nabla \times$ . Under the no-slip condition the fluid velocity on the surface is equal to the velocity of the latter $\mathbf{V}_s$; consequently, in this case

$$\mathbf{F}_p = -\frac{\rho}{2} \oint_S r \times (n \times \mathbf{V}_s) ds - \frac{\nu}{2} \oint_S r \times (n \times (\nabla \times )) ds \quad (9)$$

As shown above, the expression $\mathbf{V}n \times (\nabla \times )$ represents the diffusive vorticity flux from the surface.

We will denote this flux as follows: $\mathbf{G} = \mathbf{n} \cdot \mathbf{T}_s = \mathbf{V}n \times (\nabla \times )$.

Consequently, under the no-slip condition on the body surface, the pressure force exerted on a deformable body arbitrarily traveling in a viscous incompressible fluid is equal to

$$\mathbf{F}_p = \frac{\rho}{2} \oint_S \mathbf{G} \times r ds - \frac{\rho}{2} \oint_S r \times (n \times (\nabla \times )) ds$$

In the case of the rigid body, the velocity of points on its surface can be expressed in terms of the translational and angular velocities

$$\mathbf{V}_s = \mathbf{r}_0 + \omega \times (r - \mathbf{r}_0),$$

where $\mathbf{r}_0$ is the center of gravity of the homogeneous body of equivalent shape

$$\mathbf{r}_0 = \frac{1}{\Lambda} \int r d\tau,$$

where $\Lambda$ is the body volume and $\tau$ is the spatial domain inside the body.

The acceleration $\mathbf{V}_s$ is equal to

$$\mathbf{V}_s = \mathbf{V}_0 + \dot{\omega} \times (r - \mathbf{r}_0) + \omega \times (\omega \times (r - \mathbf{r}_0))$$
Using the Stokes theorem, the first integral on the right-hand side of the equality (9) can be transformed into the volume integral

\[ \oint_S \mathbf{r} \times (\mathbf{n} \times \dot{\mathbf{V}}_s) \, ds = \int \mathbf{r} \times (\nabla \times \dot{\mathbf{V}}_s) \, d\tau + \int (\dot{\mathbf{V}}_s \times \nabla) \times r \, d\tau \]  

(10)

Using the relations \( \nabla \mathbf{r} = 3 \), \( (\mathbf{a} \nabla) \mathbf{r} = \mathbf{a} \), \( \nabla (\mathbf{ar}) = \mathbf{a} \), \( \nabla \times \mathbf{r} = 0 \), \( (\mathbf{a} \times \nabla) \times \mathbf{r} = -2\mathbf{a} \), which hold for any vector \( \mathbf{a} \), we will transform the integrands on the right-hand side of (10)

\[ \nabla \times \dot{\mathbf{V}}_s = \nabla \times (\dot{\mathbf{\omega}} \times \mathbf{r} + \omega (\omega \mathbf{r}) - \omega^2 \mathbf{r}) = \dot{\mathbf{\omega}} (\nabla \mathbf{r}) - (\dot{\omega} \nabla) \mathbf{r} - \omega \times \nabla (\omega \mathbf{r}) - \omega^2 \nabla \times \mathbf{r} = 2\dot{\mathbf{\omega}} \]

\( (\dot{\mathbf{V}}_s \times \nabla) \times \mathbf{r} = -2\dot{\mathbf{V}}_s \)

Substituting these expressions in (10), we obtain

\[ \oint_S \mathbf{r} \times (\mathbf{n} \times \dot{\mathbf{V}}_s) \, ds = 2 \int \mathbf{r} \times \dot{\mathbf{\omega}} d\tau - 2 \int \dot{\mathbf{V}}_s d\tau = -2\dot{\mathbf{\omega}} \times \mathbf{r}_0 - 2\ddot{\mathbf{r}}_0 \Lambda \]

Consequently, the pressure force exerted on a rigid body under the no-slip condition on its surface is equal to

\[ \frac{\mathbf{F}_p}{\rho} = \frac{1}{2} \oint_S \mathbf{G} \times \mathbf{r} \, ds + \dot{\mathbf{\omega}} \times \mathbf{r}_0 \Lambda + \ddot{\mathbf{r}}_0 \Lambda \]

(11)

In addition, since from the circulation conservation condition it follows that

\[ 2\dot{\mathbf{\omega}} \Lambda + \oint_S \mathbf{G} \, ds = 0, \]

the equality (11) takes the final form:

\[ \frac{\mathbf{F}_p}{\rho} = \frac{1}{2} \oint_S \mathbf{G} \times (\mathbf{r} - \mathbf{r}_0) \, ds + \ddot{\mathbf{r}}_0 \Lambda \]

(12)

In simulating the generated vorticity by vortex loops, the surface integral can be approximated by the sum of the contributions of each loop \( \mathbf{F}_i \)

\[ \frac{\rho}{2} \oint_S \mathbf{G} \times (\mathbf{r} - \mathbf{r}_0) \, ds \approx \sum_i \mathbf{F}_i \quad \mathbf{F}_i = -\frac{\rho s_i \Gamma_i}{\Delta t} \mathbf{n}_i \]

where \( \Gamma_i \) is the circulation of a discrete vortex forming the loop, \( s_i \) is its area, and \( \Delta t \) is the step in time.

4. Conclusion

Comparing the formulas (12) and (4), we can see that these formulas have the same structure and physical meaning since both the expression \( \mathbf{e}_z (\mathbf{J}_n \mathbf{n}) \) and the vector \( \mathbf{G} \) represent the diffusive vorticity fluxes from the body surface. The only difference consists in the presence of the factor 1/2 on the right-hand side of (12) related to the space dimension. Thus, for both two-dimensional and arbitrary three-dimensional flows of a viscous incompressible fluid the aerodynamic force can be explicitly expressed in terms of the vorticity flux generated by the surface and the acceleration. We should note that the vorticity flux depends implicitly on the body velocity and acceleration, as well as on the spatial vorticity distribution. In the case of using the vortex methods for calculating the flows, the vorticity flux can be determined from the no-flow condition on the surface. The expressions
obtained can also be useful in calculations of flow in the natural coordinates for checking the accuracy of the results obtained.

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