BILINEAR PAIRINGS ON TWO-DIMENSIONAL COBORDISMS AND
GENERALIZATIONS OF THE DELIGNE CATEGORY

MIKHAIL KHOVANOV AND RADMILA SAZDANOVIC

ABSTRACT. The Deligne category of symmetric groups is the additive Karoubi closure of the partition category. It is semisimple for generic values of the parameter $t$ while producing categories of representations of the symmetric group when modded out by the ideal of negligible morphisms when $t$ is a non-negative integer. The partition category may be interpreted, following Comes, via a particular linearization of the category of two-dimensional oriented cobordisms. The Deligne category and its semisimple quotients admit similar interpretations. This viewpoint coupled to the universal construction of two-dimensional topological theories leads to multi-parameter monoidal generalizations of the partition and the Deligne categories, one for each rational function in one variable.

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1. INTRODUCTION

The Deligne category $\text{Rep}(S_t)$ interpolates between the categories of finite-dimensional representations of the symmetric groups $S_n$, viewed as tensor categories, turning integer $n$ into an element $t$ of the ground field $[D]$, $[CO]$, $[EGNO]$ Section 9.12.1].

The Deligne category has a diagrammatic description, via the partition category $\text{Pa}_t$, as the Karoubi envelope of the additive closure of $\text{Pa}_t$. When $t = n$ is a non-negative integer, the Deligne category $\text{Rep}(S_n)$ has a non-trivial ideal of negligible morphisms, and the quotient by this ideal is naturally equivalent to the tensor category of finite-dimensional representations of the symmetric group.

Diagrams commonly used to describe partitions $[CO]$, $[C]$, $[HR]$, $[LS]$ can be thickened to two-dimensional surfaces or cobordisms $S$ between unions of circles. Circles appear as ”thickenings” of points on which the partitions are formed. Vice versa, any 2D cobordism $S$ gives rise to a partition upon ignoring closed components of $S$ and the genus of each connected component with boundary. This informal correspondence is depicted in Figure 1. Cobordisms boast higher variability than partitions, admitting components without boundary and allowing arbitrary genus of each component.

A precise connection between 2D cobordisms and the partition category was pointed out by Comes $[C]$ Section 2.2]: modding out the cobordism category by the relations that adding a handle is the identity and that a 2-sphere evaluates to $t$, see Figure 2 produces the partition category,
Figure 1. Schematic correspondence between set partitions and 2D cobordisms.

with parameter $t$ corresponding to the 2-sphere. Comes used this observation to derive a set of defining relations for the partition category from that of the cobordism category.

Figure 2. Handle removal and sphere evaluation skein relations on 2D cobordisms.

A family of 2-dimensional topological theories was recently introduced by one of the authors [Kh2], based on Blanchet, Habegger, Masbaum and Vogel’s universal construction [BHMV]. It starts with an evaluation of closed oriented surfaces, which may be described by power series

$$Z_{\alpha}(T) = \alpha_0 + \alpha_1 T + \alpha_2 T^2 + \cdots = \sum_{n \geq 0} \alpha_n T^n \in R[T],$$

where $R$ is a ground commutative ring or a field $k$, evaluating a connected component of genus $g$ to $\alpha_g$. This evaluation gives rise to state spaces $A_{\alpha}(k)$ for collections of $k$ circles. It can then be extended to produce a category $\text{Cob}_{\alpha}$ with objects non-negative integers $n$ and hom spaces $\text{Hom}_{\text{Cob}_{\alpha}}(n, m)$ being $R$-linear combinations of cobordisms between $n$ and $m$ circles modulo universal relations defined by the sequence $\alpha = (\alpha_0, \alpha_1, \ldots)$, also see below.

In the partition category $\text{Pa}_{\alpha}$, when two partitions are composed, each connected component of the composition that has no boundary points is evaluated to $t \in R$ and removed. In the correspondence between 2D cobordisms and partitions, these components give rise to cobordisms of various genera, which, in general, evaluate to $\alpha_g$, where $g$ is the genus of the cobordism.

To match the general $\alpha$-evaluation of 2D cobordisms to the evaluation by powers of $t$ in the partition and Deligne categories specialize the sequence $\alpha$ to

$$\alpha(t) = (t, t, t, \ldots), \quad \alpha_g(t) = t \quad \forall g \in \mathbb{Z}_+.$$

Then the additive Karoubi closure of the category $\text{Cob}_{\alpha(t)}$ is equivalent, as a tensor category, to the Deligne category

$$\text{Kar}(\text{Cob}_{\alpha(t)}) \cong \text{Rep}(S_t),$$

for $t \in k \setminus \mathbb{Z}_+$ (specializing to characteristic zero field $k$ as the ground ring). When $t = n \in \mathbb{Z}_+$, the quotient of the Deligne category by the ideal $J_n$ of negligible morphisms produces the category.
of finite-dimensional representations of the symmetric group $S_n$, equivalent to the above Karoubi closure for $t = n$,

$$\text{Kar}(\text{Cob}_{\alpha(n)}) \cong \text{Rep}(S_n)/J_n \cong k[S_n] - \text{mod.}$$

This observation allows to generalize the Deligne category and its semisimple quotients by taking a more general sequence $\alpha$ of elements of $R$ and then forming tensor category $\text{Cob}_\alpha$ and its additive Karoubi closure $\text{Kar}(\text{Cob}_{\alpha})$, also denoted $\text{Kob}_\alpha$. The latter is given by first allowing finite linear combinations of objects of $\text{Cob}_\alpha$, with suitably defined hom spaces, and then adding all idempotents in endomorphism rings of these linear combinations as additional objects.

When $R$ is a field $k$, it follows from [Kh2] and goes back to a theorem of Kronecker that hom spaces in $\text{Cob}_\alpha$ are finite-dimensional iff the power series $Z_\alpha(T)$ in (1) can be represented as a rational function,

$$Z_\alpha(T) = \frac{P(T)}{Q(T)},$$

where $P(T), Q(T)$ are coprime polynomials with coefficients in $k$. To each such rational function we can assign an additive Karoubi-complete tensor (symmetric monoidal) category

$$\text{Kob}_\alpha := \text{Kar}(\text{Cob}_{\alpha})$$

with finite dimensional hom spaces. This category is a natural generalization of the Deligne category $\text{Rep}(S_t)$ for generic $t$ and of its semisimple quotients for $t = n \in \mathbb{Z}_+$. The Deligne category corresponds to the rational function

$$Z_{\alpha(t)}(T) = \frac{t}{1-T} = t + tT + tT^2 + \ldots$$

It should be extremely interesting to extend various results and constructions related to the Deligne category and its semisimple quotients to this large family of tensor categories $\text{Kob}_\alpha$ (as well as categories $\text{PKob}_\alpha$ defined in Section 4) parametrized by rational functions.

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2. **Category of two-dimensional cobordisms and its linearization categories**

**Category $\text{Cob}_2$.** Consider the symmetric monoidal category of 2-dimensional oriented cobordisms. We use the skeletal version of this category (one object in each isomorphism class), denoted $\text{Cob}_2$. Its objects are non-negative integers $n \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ and morphisms from $n$ to $m$ are diffeomorphism classes rel boundary of compact oriented 2-manifolds $S$ with a fixed diffeomorphism

$$\partial S \cong (- \sqcup_n \mathbb{S}^1) \sqcup (\sqcup_m \mathbb{S}^1),$$

where $\mathbb{S}^1$ is the oriented circle. In other words, the boundary of $S$ is separated into the *bottom* and *top* boundary, and identified, correspondingly, with disjoint unions of $n$ and $m$ circles. Composition is given by concatenation. Cobordisms may have connected components with no boundary. A example of a morphism (cobordism) from 3 to 4 is given in Figure 3.

Morphisms from $n$ to $m$ in $\text{Cob}_2$ can be enumerated as follows. A morphism $x$ may have some number of closed components of various genera. Counting these components gives a sequence $cl(x) = (a_0, a_1, \ldots, 0, 0, \ldots)$, where $a_k$ is the number of closed components of genus $k$ in $x$. All but finitely many terms in the sequence $cl(x)$ are zero. Connected components with boundary provide a decomposition of the set of $n + m$ boundary circles into nonempty subsets, where circles from the same subset are the boundaries of the same connected component. Furthermore, each such component has genus zero or higher, which counts the number of handles of the component.
Figure 3. A morphism in Cob$_2$. The cobordism is not embedded anywhere, so overlaps of components do not carry any information and can be reversed. We label top circles by $1', 2', \ldots, m'$ and bottom circles by $1, 2, \ldots, n$. In this example $m = 4$ and $n = 3$.

Denote by $D^m_n$ the set of decompositions of $n + m$ circles. A morphism $\lambda \in \text{Hom}_{\text{Cob}_2}(n, m)$ can be described by a decomposition $\lambda \in D^m_n$, an assignment of a nonnegative integer (genus or number of handles) to each set in the decomposition $\lambda$ and a choice of a sequence $cl(\lambda)$ as above describing genera of closed components of $x$.

Let us label bottom circles $1, \ldots, n$ and top circles $1', \ldots, m'$, from left to right. Cobordism $x$ induces a decomposition of the set

$$N^m_n := \{1, 2, \ldots, n, 1', 2', \ldots, m'\}.$$  

For the cobordism $x$ in Figure 3 we have $n = 3, m = 4$, the set is $N^4_3 = \{1, 2, 3, 1', 2', 3', 4'\}$, and its subsets corresponding to components with boundary are $\{1, 3, 1'\}, \{2, 3'\}$, and $\{2', 4'\}$. These components have genera $2, 0, 1$ correspondingly. The sequence $cl(x) = (2, 0, 0, 1, 0, \ldots)$, since $x$ has two closed components of genus zero (2-spheres) and one component of genus three.

**Category R\text{Cob}_2.** Fix a commutative ring $R$ and consider pre-additive $R$-linear category $R\text{Cob}_2$ freely generated by Cob$_2$. It has the same objects $n$ as Cob$_2$, and morphisms from $n$ to $m$ in $R\text{Cob}_2$ are linear combinations of morphisms from $n$ to $m$ in Cob$_2$ with coefficients in $R$ and with composition induced from that in Cob$_2$. One can think of this construction, for an arbitrary category $C$, as analogous to passing from a group $G$ to its group algebra $R[G]$ or from a semigroup $G$ to its semigroup algebra. It results in an idempotented ring $RC$ with a collection of mutually orthogonal idempotents (corresponding to identity morphisms), one for each object of $C$, as a substitute for the unit element, see [KS2].

**Category Cob$'_\alpha$ for a sequence $\alpha$.** A more interesting category is obtained if we choose an infinite sequence of elements

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$$

of $R$ and evaluate each closed component of genus $k$ to $\alpha_k$. For a closed surface $S$ denote

$$\alpha(S) = \prod_{k \geq 0} \alpha_k^{c_k},$$

where $c_k$ is the number of components of $S$ of genus $k$. The resulting monoidal category, denoted Cob$'_\alpha$ or Cob$'_\alpha$, has the same objects $n$ as the earlier categories. A morphism from $n$ to $m$ in Cob$'_\alpha$ is an $R$-linear combination of cobordisms from $n$ circles to $m$ circles in Cob$_2$ without closed components. Composition is given by concatenation followed by evaluating each closed component of genus $k$ to $\alpha_k$. We can informally refer to Cob$'_\alpha$ as the $\alpha$-prelinearization of the category Cob$_2$. 


There is the obvious "evaluation" or "reduction" functor $R\text{Cob}_2 \rightarrow \text{Cob}_\alpha'$, which is the identity on objects, that evaluates (or reduces) each closed component of genus $k$ to $\alpha_k$. The hom space $\text{Hom}_{\text{Cob}_\alpha'}(n, m)$ is a free $R$-module with a basis of cobordisms without closed components. Basis elements are parametrized by partitions in $D^m_n$ with a non-negative integer (genus) assigned to each part of the partition.

**Remark:** Object 0 associated to the empty 1-manifold $\emptyset_1$ is the unit object of monoidal categories $\text{Cob}_2, R\text{Cob}_2$ and $\text{Cob}_\alpha'$. Commutative monoid of endomorphisms $\text{End}_{\text{Cob}_2}(0)$ is freely generated by isomorphism classes of closed oriented connected surfaces, one for each genus $g \geq 0$, and can be identified with the free abelian monoid on these generators. Commutative rings

$$\text{End}_{R\text{Cob}_2}(0) \cong \text{End}_{\text{Cob}_\alpha'}(0) \cong R[\text{End}_{\text{Cob}_2}(0)]$$

are the semigroup algebras of that monoid.

**Generating functions and the bilinear form.** Sequence $\alpha$ is conveniently encoded by the generating function

$$Z_\alpha(T) = \sum_{n \geq 0} \alpha_n T^n \in R[[T]].$$

For a closely related construction see [Kh2], where to such $Z_\alpha(T)$ there is associated a family of $R$-modules $A_\alpha(n)$, for each $n \geq 0$, constructed via a bilinear form on the space of linear combinations of oriented 2-manifolds with boundary the disjoint union of $n$ circles $\sqcup_n S^1$. Namely, one considers the free $R$-module $\text{Fr}(n)$ with a basis $\{[S]\}_S$ of oriented compact surfaces $S$ with $\partial S \cong \sqcup_n S^1$ (with the diffeomorphism fixed). On $\text{Fr}(n)$ there is an $R$-bilinear form $(\cdot, \cdot)_n$ given on pairs of generators $S_1, S_2$ by gluing the two surfaces along the common boundary and evaluating via $\alpha$:

$$([S_1], [S_2])_n = \alpha((-S_1) \sqcup \partial S_2)$$

The state space of $n$ circles is the quotient of $\text{Fr}(n)$ by the kernel of this bilinear form:

$$A_\alpha(n) := \text{Fr}(n)/\ker((\cdot, \cdot)_n).$$

This collection of $R$-modules is naturally a representation of the category $\text{Cob}_\alpha'$, when the latter is viewed as an idempotented $R$-algebra with a system of mutually-orthogonal idempotents $\{1_n\}_{n \in \mathbb{N}}$. Namely, to the object $n$ of $\text{Cob}_\alpha'$ associate the $R$-module $A_\alpha(n)$. To a morphism given by a cobordism $x \in \text{Cob}_2$ from $n$ to $m$ associate an $R$-module map

$$x_\alpha : A_\alpha(n) \rightarrow A_\alpha(m),$$

obtained directly from the construction in [Kh2], via the evaluations of $x$ capped off by various oriented surfaces with $n$ and $m$ circles as the boundary. These morphisms over all $x \in \text{Cob}_2$ provide a representation of $\text{Cob}_2$ and $\text{Cob}_\alpha'$ on the direct sum of $R$-modules

$$A_\alpha := \bigoplus_{n \geq 0} A_\alpha(n).$$

Monoidal structures on $\text{Cob}_2$ and $\text{Cob}_\alpha'$ are not used in this construction. $A_\alpha$ is a representation of the idempotented $R$-algebra underlying category $\text{Cob}_\alpha'$, in the sense of [KS1] [KS2].

**Category $\text{Cob}_\alpha$ as a quotient by negligible morphisms.** The action of $\text{Cob}_\alpha'$ can be quotiented down to a smaller category. Categories $R\text{Cob}_2$ and $\text{Cob}_\alpha'$ admit trace maps. Namely, given an element $x \in \text{Hom}(n, n)$, a finite linear combination of cobordisms with $n$ bottom and $n$ top circles, close up opposite circles $i$ and $i'$, $1 \leq i \leq n$ by annuli to get a linear combination of closed cobordisms $\tilde{x}$ and then evaluate the result via $\alpha$:

$$\text{Tr}(x) = \alpha(\tilde{x}) \in R,$$

see Figure 4.
For morphisms \( x \in \text{Hom}(n, m) \) and \( y \in \text{Hom}(m, n) \) we have \( \text{Tr}(xy) = \text{Tr}(yx) \).

A morphism \( x \in \text{Hom}(n, m) \) in the category \( \text{Cob}'_\alpha \) (or in \( R\text{Cob}_2 \)) is called \textit{negligible} if for any \( y \in \text{Hom}(m, n) \) the trace \( \text{Tr}(yx) = 0 \). Denote by \( J(n, m) \subset \text{Hom}(n, m) \) the subset of negligible morphisms from \( n \) to \( m \). This subset is an \( R \)-submodule of \( \text{Hom}(n, m) \), and the union of \( J(n, m) \), over all \( n, m \geq 0 \), is the tensor ideal \( J_\alpha \) of \( \text{Cob}'_\alpha \). Define the category \( \text{Cob}_\alpha \) to be the quotient of \( \text{Cob}'_\alpha \) by this ideal,

\[
\text{Cob}_\alpha := \text{Cob}'_\alpha / J_\alpha.
\]

This category has objects \( n \in \mathbb{Z}_+ \), and

\[
\text{Hom}_{\text{Cob}_\alpha}(n, m) = \text{Hom}_{\text{Cob}'_\alpha}(n, m) / J(n, m).
\]

Category \( \text{Cob}_\alpha \) is an \( R \)-linear tensor category with duals and a non-degenerate trace: for any \( x \in \text{Hom}_{\text{Cob}_\alpha}(n, m), x \neq 0 \), there is \( y \in \text{Hom}_{\text{Cob}_\alpha}(n, m) \) such that \( \text{Tr}(yx) \neq 0 \). For information about ideals of negligible morphisms and corresponding quotient categories we refer the reader to [EO, BW].

Starting with the category \( R\text{Cob}_2 \) instead of \( \text{Cob}'_\alpha \) in this construction will result in the quotient category isomorphic to \( \text{Cob}_\alpha \).

The functor of modding out an \( R \)-linear tensor category with duals by the ideal of negligible morphisms is essentially the same operation as used in the universal construction [BHMV, Kh2], where one mods out by the kernel of the bilinear form. Thus, in the example above, there are isomorphisms of \( R \)-modules

\[
\text{Hom}_{\text{Cob}_\alpha}(n, m) \cong \text{Hom}_{\text{Cob}_\alpha}(0, n + m) \cong A_\alpha(n + m),
\]

with the first isomorphism given by bending the \( n \) bottom circles up, see Figure 5.

The space \( J(0, n + m) \) of negligible morphisms in \( \text{Hom}_{\text{Cob}_\alpha}(0, n + m) \) is exactly the kernel of the bilinear form on \( \text{Hom}_{\text{Cob}_\alpha}(0, n + m) \) constructed via the formula (13) for \( n + m \) boundary circles, implying the second isomorphism above. It is easy to rewrite composition of morphisms in \( \text{Cob}_\alpha \) via these isomorphisms and suitable cobordism maps.

We refer to category \( \text{Cob}_\alpha \) as \( \alpha \)-linearization of \( \text{Cob}_2 \) (and of related categories \( R\text{Cob}_2 \) and \( \text{Cob}'_\alpha \)). These categories are part of the package of the universal construction or pairing, see [BHMV, Kh2] and closely related [FKNSWW], and can be defined in any dimension and in a variety of

**Figure 4.** Closing up a linear combination \( x \) of \((n, n)\) cobordisms into a linear combination \( \hat{x} \) of closed cobordisms.
situations, given an evaluation of closed manifolds or similar objects (foams [Kh1, RW], manifolds with embedded submanifolds [FKNSWW, KR], or other decorations).

When commutative ring $R$ is a field $k$, it is observed in [Kh2] that the spaces $A_\alpha(n)$ are finite-dimensional for all $n$ (equivalently, for some $n \geq 1$) iff the generating function (12) is a rational function in $T$. Equivalently, representation (16) of $\text{Cob}_\alpha$ is locally finite-dimensional, in a similar sense, see [KS1, KS2]. The case when the function $Z_\alpha(T)$ is rational seems especially interesting, for many reasons.

$R$-modules $A_\alpha(n)$ and maps between them induced by cobordisms, see the discussion around (15), define a representation of $\text{Cob}_\alpha$ viewed as an idempotented ring

$$B_\alpha = \bigoplus_{n,m \geq 0} 1_m \text{Hom}_{\text{Cob}_\alpha}(n,m) 1_n,$$

see [KS1, KS2] for a general discussion. On the corresponding representation $A_\alpha$ in (15) idempotent $1_n$ acts as the projector onto $A_\alpha(n)$, and an element $x \in \text{Hom}_{\text{Cob}_\alpha}(n,m)$ acts by the corresponding map $A_\alpha(n) \to A_\alpha(m)$. When $R$ is a field and $Z_\alpha(T)$ is rational, this representation is locally finite-dimensional.

**Additive closure and the Karoubi envelope.** It is useful to consider the additive Karoubi envelope $\text{Kob}_\alpha$ of $\text{Cob}_\alpha$. First form the finite additive closure $\text{Cob}_\alpha^\oplus$ of $\text{Cob}_\alpha$ by taking formal finite direct sums of objects $n$ of $\text{Cob}_\alpha$, and extending to morphisms in the obvious way. The additive closure has the zero object $0$ different from the object $0$. The latter is associated to the empty 1-manifold and comes from the corresponding object of $\text{Cob}_\alpha$. Endomorphisms of the object $0$ is the zero $R$-algebra, while $\text{End}_{\text{Cob}_\alpha}(0) = \text{End}_{\text{Cob}_\alpha}(0) \cong R$.

Denote the resulting category by $\text{Cob}_\alpha^\oplus$. Next, let $\text{Kob}_\alpha$ be the Karoubi envelope of $\text{Cob}_\alpha^\oplus$. The six types of categories we’ve encountered so far are listed below:

$$\text{Cob}_2 \to \text{RCob}_2 \to \text{Cob}_\alpha' \to \text{Cob}_\alpha \to \text{Cob}_\alpha^\oplus \to \text{Kob}_\alpha.$$

The first arrow consists of allowing $R$-linear combinations of cobordisms. In the second arrow we evaluate closed surfaces of genus $k$ to fixed elements $\alpha_k$ of $R$, over all $k \geq 0$. We can refer to this procedure as $\alpha$-prelinearization. The third arrow consists of modding out $\text{Cob}_\alpha^\oplus$ by the ideal of negligible morphisms.

Like $\text{Cob}_\alpha'$, category $\text{RCob}_2$ also has the ideal $I_\alpha$ of negligible morphisms, via the trace given by $\alpha$. The composition of the second and third arrows above can also be described as the quotient of $\text{RCob}_2$ by this ideal

$$\text{RCob}_2 \to \text{RCob}_2/I_\alpha \cong \text{Cob}_\alpha.$$
The fourth and the fifth arrows in (22) are fully faithful functors. The second, third and fourth categories are pre-additive, the fifth category is additive and the last category is additive and Karoubi-complete. All six categories are tensor (symmetric monoidal) and these five functors are monoidal.

3. Partition category and the Deligne category

Recall that $R$ is a commutative ring. In the context of the partition category and the Deligne category ring $R$ is often taken to be a field $k$. Fix $t \in R$.

**Partition category.** Partition category $\text{Pa}_t$ extends the notion of the partition algebra that originally appeared in Martin [M] and Jones [J], see [HR, LS] for more information and references.

Objects $n$ of the partition category are non-negative integers and morphisms from $n$ to $m$ are $R$-linear combinations of decompositions $D^m_n$ (also called partitions) of the set $\mathbb{N}^m_n$, see discussion around formula (9).

Diagrammatically, partitions are often denoted by marking $n$ points on a horizontal line in the plane and $m$ points on a parallel line above it. One connects these $n + m$ points by arcs, and connected components of the resulting graph are the parts of the partition. Intersections of arcs are ignored. A partition usually has more than one such diagram. For instance, if $\{1, 3, 1'\}$ is a part of the partition, it can be described by two arcs $(1, 3), (1, 1')$ or two arcs $(1, 3), (3, 1')$, or all three arcs, also see Figure 9 left below demonstrating this indeterminacy. This diagrammatic description of partitions is standard in papers on the partition algebra and category, see for instance [LS]. Two examples of diagrammatic presentation are given in Figure 6.

![Figure 6](image)

**Figure 6.** Partitions $a = \{\{1, 3, 1'\}, \{2, 4, 6'\}, \{2', 4'\}, \{3', \{5'\}\} \in P^6_4$ and $b = \{\{1, 2'\}, \{2, 3\}, \{4\}, \{5\}, \{6, 4', 5'\}, \{1', 3'\}\} \in P^5_6$. Notice multiple ways to display the same partition. For the subset $\{1, 3, 1'\}$ we depicted edges $(1, 3)$ and $(1, 1')$. Another possibility is to depict edges $(1, 3)$ and $(3, 1')$ or edges $(1, 1')$ and $(3, 1')$. In choosing a diagram for a partition It is natural to at least minimize the number of bottom-top edges, showing only one such edge for each subset that contains both bottom and top points.

Composition is given by concatenating diagrams, see Figure 7 and treating points in the middle that connect to bottom or top as ‘pass through’ points that vanish from the concatenation but are used before that to create the new partition. If there exist a connected component that consists entirely of points in the middle part of the diagram, it is removed and what’s left is multiplied by $t$. This procedure is iterated until no such components are left.

Composition is then extended bilinearly to $R$-linear combinations of partitions. The resulting $R$-linear category $\text{Pa}_t$ is symmetric monoidal, with the tensor product given on partitions by placing their diagrams in parallel.
Noah Snyder’s diagrammatics for the partition category. Long time ago Noah Snyder [S] pointed out to one of us an alternative diagrammatics for the partition category, which will be treated in more detail in [H]. Figure 8 shows conventional diagrammatics versus the Snyder diagrammatics for the standard generating morphisms of the partition category. One difference is the use of trivalent vertex to depict the morphism from 2 to 1 corresponding to the partition \{1, 2, 1'\} and the dual morphism from 1 to 2. This trivalent vertex as well as other configurations can be freely rotated in the plane. As in the usual diagrammatics, one allows intersections of distinct parts of the partition, thinking of them as virtual intersections.

Figure 9 displays one benefit of the Snyder calculus: generating morphism (a) has essentially unique minimal presentation.

It is convenient to introduce cup and cap diagrams (as additional generators), defined in the top row of Figure 10 via the original generators. Isotopy relations on the generators are shown in the next two rows of Figure 10. Some other defining relations are shown in Figure 11. We leave it to the reader to convert a full set of relations as found in [C, Theorem 1] or [LS] into defining relations for the Snyder calculus.

The relation between 2D cobordisms and partitions is especially easy to see in the Snyder calculus. Thickening Snyder’s trivalent graphs when viewed as graphs in \( \mathbb{R}^3 \) rather than in \( \mathbb{R}^2 \) results in a surface with boundary that corresponds to the partition. Intersection points of different components of the graph should be disregarded, as before, for instance by pulling the components slightly apart in \( \mathbb{R}^3 \) before thickening (the embedding into \( \mathbb{R}^3 \) is then forgotten). An example of a matching between Snyder’s relations and diffeomorphisms of surfaces is shown in Figure 12.

From graphs to surfaces. The lifting, discussed in this paper and in [C], from partitions, which are graph-like objects, to two-dimensional cobordisms is analogous, in some rather naive way, to passing from Feynman diagrams (graph-like objects) to strings (two-dimensional objects):

\[
\text{Feynman diagrams} \rightarrow \text{Strings} \\
\text{Partition diagrams} \rightarrow \text{2D cobordisms}
\]
Figure 8. Conventional and Snyder’s generators for the partition category. Object 0 is shown by a dashed line without dots on it. In (c) and (d) we indicated the loose end of a strand by the * symbol; other ways to depict the end are fine too. Likeng-Savage [LS] use a similar notation for the generators (c),(d). Element shown in (f) is a suitable composition of generators (c) and (d) and evaluates to $t$ in the partition category. Top and bottom dashed lines in the depiction of a diagram are optional and are not shown in the top row diagrams.

Figure 9. Multiple ways to depict generating morphism (a) in Figure 8 versus unique up to isotopy diagram in the Snyder graphical calculus.

Of course, the complexity of mathematics hidden in the top arrow structures is orders of magnitude higher than those in the bottom arrow, discussed in the present paper.

**Deligne category.** Let us specialize to ground ring $R = k$ a field of characteristic 0. The Deligne category $\text{Rep}(S_t)$ is the additive Karoubi envelope of the partition category $\text{Pa}_t$, $t \in k$,

$$\text{Rep}(S_t) = \text{Kar}(\text{Pa}_t^\otimes).$$

It is known to be semisimple when $t \notin \mathbb{Z}_+$. When $t = n \in \mathbb{Z}_+ \subset k$, the Deligne category admits a nontrivial ideal $J_n$ that consists of *negligible morphisms*. A morphism $x \in \text{Hom}(a,b)$ is negligible if for any $y \in \text{Hom}(b,a)$ the trace of the composition $\text{Tr}(yx) = 0$. The category $\text{Rep}(S_t)$ is a tensor category with duals, and the trace is straightforward to define. The trace in $\text{Pa}_t$ on a diagram $\lambda \in D_m^l$ is given by identifying points $i$ and $i'$, $1 \leq i \leq m$. If $r$ is the number of components in the resulting diagram, $\text{Tr}(\lambda) = t^r$. 

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Figure 10. Cup and cap diagrams and some isotopy relations in Snyder’s diagrammatics.

Figure 11. Some other defining relations in the Snyder calculus.

Figure 12. One of the defining relations versus surface diffeomorphism.

The quotient $\text{Rep}(S_n)/J_n$ is equivalent, as a tensor category, to the category $\mathbf{k}[S_n]^{-\text{mod}}$ of finite-dimensional representations of the symmetric group $S_n$,

\begin{equation}
\text{Rep}(S_n)/J_n \cong \mathbf{k}[S_n]^{-\text{mod}},
\end{equation}
The ideal $J_t$ of negligible morphisms in $\text{Rep}(S_t)$ is zero if $t \notin \mathbb{Z}_+$.  

4. Generalized Deligne categories

**Cobordisms and partitions.** Consider the category $\text{Cob}_2$ of two-dimensional cobordisms. Given a morphism $x$ from $n$ to $m$, disregard its closed components and ignore genera of connected components with boundary. A connected component with boundary defines a subset among the set of boundary circles of $x$. The latter set can be identified with $\{1, 2, \ldots, n, 1', 2', \ldots, m'\}$, see Figure 3. Consequently, the union of connected components of $x$ that have a non-empty boundary determines a partition in $D_n^m$. To a cobordism $x$ from $n$ to $m$ we associate this partition in $D_n^m$, denoted $p(x)$.

To extend this assignment to a functor

$$F : R\text{Cob}_2 \rightarrow \text{Pa}_t$$

let $|\text{cl}(x)|$ be the number of connected components of $x$ without boundary (closed components). Functor $F$ is identity on objects $n \in \mathbb{Z}_+$ of $R\text{Cob}_2$ and $\text{Pa}_t$ and

$$F(x) = t^{|\text{cl}(x)|}p(x)$$

on cobordisms. It is then extended $R$-linearly to linear combinations of cobordisms. Notice that $F$ ignores genera of all components of $x$. Clearly, $F$ is a tensor (symmetric monoidal) functor. This construction can be found in Comes [C, Section 2.2]. One can think of $\text{Pa}_t$ as the quotient of $R\text{Cob}_2$ by skein relations in Figure 2.

Recall categories $\text{Cob}'_\alpha$ and $\text{Cob}_\alpha$ introduced earlier and associated to a sequence $\alpha$, where a closed surface of genus $g$ evaluates to $\alpha_g \in R$. Let $\alpha(t) = (t, t, t, \ldots)$ be the constant sequence associated to $t \in \mathbb{k}$. The evaluation $\alpha(t)$ associates $t$ to any oriented connected closed surface irrespectively of its genus. Relations in Figure 2 hold in the category $\text{Cob}_{\alpha(t)}$ and they hold in $\text{Cob}'_{\alpha(t)}$ when restricted to closed components. Consequently, there are natural tensor functors

$$\text{Cob}'_{\alpha(t)} \xrightarrow{F'_t} \text{Pa}_t \xrightarrow{F''_t} \text{Cob}_{\alpha(t)}$$

between these three categories. These functors are identities on objects, $F'_t(n) = n$, $F''_t(n) = n$. The first functor forgets about handles of each component of a cobordism $S$, evaluates each closed component to $t$, and associates a partition of the set $\mathbb{N}^m$ in (9) to $S$ according to subsets of boundary circles bounded by connected components of $S$.

The second functor $F''_t$ exists by an earlier discussion, due to the definition of $\text{Cob}_{\alpha(t)}$ via the quotient by the kernel of a bilinear form. It identifies $\text{Cob}_{\alpha(t)}$ with the quotient of $\text{Pa}_t$ by the ideal of negligible morphisms.

**The Deligne category.** Starting with the functor $F''_t$ and passing to additive Karoubi closures results in a functor

$$F_t : \text{Rep}(S_t) \rightarrow \text{Kob}_{\alpha(t)}$$

from the Deligne category to the additive Karoubi closure $\text{Kob}_{\alpha(t)} = \text{Kar}(\text{Cob}_{\alpha(t)}^\oplus)$ of the category $\text{Cob}_{\alpha(t)}$. From the structure theory of the Deligne categories we can conclude that $F_t$ consists of taking the quotient of $\text{Rep}(S_t)$ by the ideal $J_t$ of negligible morphisms and induces an equivalence

$$\text{Rep}(S_t)/J_t \cong \text{Kob}_{\alpha(t)}.$$

Notice that there’s a difference in the order in which we take the additive Karoubi closure and mod out by negligible morphisms. On the Deligne category side, one first forms the additive Karoubi closure and then mods out by negligible morphisms. On the $\text{Kob}_{\alpha(t)}$ side, one first mods out by negligible morphisms to get the category $\text{Cob}_{\alpha(t)}$ and then forms the additive Karoubi closure. It is
not clear whether this may produce a discrepancy in more general cases, but for Deligne categories (and with \( R \) a field \( k \) of characteristic 0) this change of order results in equivalent categories and makes no difference.

**Generalizations.** We obtain an immediate generalization of the categories \( \text{Rep}(S_t)/J_t \) by changing from the constant sequence \( \alpha(t) \) in (2) to a more general sequence \( \alpha \). The most interesting case is when the generating function \( Z_{\alpha}(T) \) of \( \alpha \), see (11), is a rational function, a ratio of two coprime polynomials

\[
Z_{\alpha}(T) = \frac{P(T)}{Q(T)}
\]

with coefficients in \( k \). In this case categories \( \text{Cob}_{\alpha} \) and \( \text{Kob}_{\alpha} \) have finite-dimensional hom spaces. We can view \( \text{Kob}_{\alpha} \) as a natural generalization of the quotient category \( \text{Rep}(S_t)/J_t \). For generic \( t \), the ideal \( J_t \) is zero, and then the quotient category is the Deligne category.

**Theorem 1.** Categories \( \text{Kob}_{\alpha} \) are tensor \( k \)-linear Karoubi-closed additive categories. When \( Z_{\alpha}(T) \) is rational, morphism spaces in \( \text{Kob}_{\alpha} \) are finite dimensional.

It is an interesting project to investigate categories \( \text{Kob}_{\alpha} \) when the generating function \( Z_{\alpha}(T) \) is rational. Deligne category quotients are recovered for the rational function in (7).

Notice that categories \( \text{Kob}_{\alpha} \) deliver generalizations of the quotients \( \text{Rep}(S_t)/J_t \) rather than of Deligne categories \( \text{Rep}(S_t) \) themselves. To remedy this discrepancy, we instead pass from \( \text{Cob}'_{\alpha} \) to \( \text{Kob}_{\alpha} \) in one more step, when \( R = k \) is a field and the partition function \( Z_{\alpha}(t) \) is rational (31). Let

\[
N = \deg(P(T)), \quad M = \deg(Q(T)), \quad K = \max(N + 1, M),
\]

\[
Q(T) = 1 - e_1 T + e_2 T^2 + \ldots + (-1)^M e_M T^M,
\]

as in [Kh2] Section 2.4. Then in the state space \( A_{\alpha}(1) \) of a circle equality

\[
x^K - e_1 x^{K-1} + e_2 x^{K-2} - \ldots + (-1)^M e_M x^{K-M} = 0
\]

holds, where \( x \) denotes a 2-torus with one boundary component. Power \( x^k \) of \( x \) represents a surface of genus \( k \) with one boundary component, with multiplication in \( A_{\alpha}(1) \) given by the pants cobordism, see [Kh2]. Equation (34) gives a skein relation in category \( \text{Cob}_{\alpha} \) which reduces a collection of \( K \) handles on a single component to a linear combinations of collections of \( K - 1, K - 2, \ldots, K - M \) handles.

For rational \( \alpha \), start with the pre-additive category \( \text{Cob}'_{\alpha} \), see Section 2 and diagram (22) that shows the position of \( \text{Cob}'_{\alpha} \) in the chain of categories and functors associated with \( \alpha \). In \( \text{Cob}'_{\alpha} \) only closed components are reduced to elements \( \alpha_k \) of \( k \). Hom spaces \( \text{Hom}(n, m) \) in \( \text{Cob}'_{\alpha} \) are infinite-dimensional \( k \)-vector spaces, unless \( n = m = 0 \), with a basis of diffeomorphism classes rel boundary of all cobordisms without closed components. Thus, a basis element is described by a decomposition in \( D^n_m \) and a choice of genus for each connected component.

Define category \( \text{PCob}_{\alpha} \) to have the same objects \( n \geq 0 \) as \( \text{Cob}'_{\alpha} \) and morphism spaces to be quotients of those in \( \text{Cob}'_{\alpha} \) by the skein relations corresponding to the equation (33). That is, we set this linear combination of morphisms (cobordisms) to zero in the quotient category. Applying this relation we reduce a component which contains at least \( K \) handles to components with fewer handles. In particular, any morphism in \( \text{Cob}'_{\alpha} \) reduces to a \( k \)-linear combination of cobordisms with no closed components and at most \( K - 1 \) handles on each connected component. Diffeomorphism classes rel boundary of these cobordisms are in a bijection with elements of the set \( D^n_m(< K) \) of partitions in \( D^n_m \) with an integral weight between 0 and \( K - 1 \) associated to each part (number of handles of the component, on the cobordism side). Recall that to a partition \( x \) we associated cobordism \( p(x) \), see discussion preceeding formula (27). We can now extend this association, also
denoted \( p \), and assign to a partition \( x \) with non-negative integral weights of its parts the cobordism \( p(x) \) by starting with the cobordism for the partition without weights and adding the number of handles equal to the weight to each two-sphere with boundary holes.

Computations in [Kh2, Section 2.4] imply that relation (34) is compatible with evaluation \( \alpha \) applied to closed cobordisms. In particular, no additional relations on cobordisms appear and elements of the set \( D^m_n(\leq K) \), converted to cobordisms, provide a basis of \( \Hom_{\PCob_\alpha}(n, m) \).

**Proposition 2.** The hom space \( \Hom(n, m) \) in \( \PCob_\alpha \) has a basis \( \{ p(x) \} \), over all \( x \in D^m_n(\leq K) \).

In particular, hom spaces in \( \PCob_\alpha \) are finite-dimensional. We can now insert category \( \PCob_\alpha \) into the chain of six categories in (22):

\[
\begin{align*}
\text{Cob}^2 & \longrightarrow \text{R} \text{Cob}^2 \longrightarrow \text{Cob}'_\alpha \longrightarrow \text{PCob}_\alpha \longrightarrow \text{Cob}_\alpha \longrightarrow \text{Cob}^\oplus_\alpha \longrightarrow \text{Kob}_\alpha.
\end{align*}
\]

It fits in between \( \text{Cob}'_\alpha \) and \( \text{Cob}_\alpha \). Category \( \PCob_\alpha \) is the quotient of \( \text{Cob}'_\alpha \) by the skein relation (34). Like every other category in this chain, it is tensor (symmetric monoidal). The trace form on \( \text{Cob}'_\alpha \) descends to that on \( \PCob_\alpha \). The quotient of \( \PCob_\alpha \) by the ideal of negligible morphisms relative to this trace form is isomorphic to \( \text{Cob}_\alpha \) (isomorphic and not just equivalent, since these categories are essentially skeletal and have very few objects). As we’ve mentioned, this insertion is possible when \( Z_\alpha(T) \) is a rational function and \( R \) is a field.

Category \( \PCob_\alpha \) generalizes the partition category \( \text{Pa}_t \). Partition category \( \text{Pa}_t \) is isomorphic to \( \PCob_\alpha(t) \) for the constant sequence \( \alpha(t) = (t, t, \ldots) \). More generally, choosing \( K \) in (32) fixes the size of homs in \( \PCob_\alpha \), analogously to independence of dimensions of homs in \( \text{Pa}_t \) on \( t \), given by the number of partitions (the Bell number). When \( K = 1 \), dimensions of hom spaces in \( \PCob_\alpha \) are also given by the number of partitions in \( D^m_n \), and for \( K > 1 \) by the number of weighted partitions as discussed above.

To get from \( \PCob_\alpha \) to the analogue of the Deligne category, pass to the additive Karoubi closure to get an additive and idempotent-complete category

\[
\text{PKob}_\alpha := \text{Kar}(\PCob_\alpha^\oplus).
\]

Chain of functors (35) can be upgraded to a commutative diagram of functors

\[
\begin{align*}
\text{Cob}_2 & \longrightarrow \text{R} \text{Cob}_2 \longrightarrow \text{Cob}'_\alpha \longrightarrow \text{PCob}_\alpha \longrightarrow \text{PCob}^\oplus_\alpha \longrightarrow \text{PKob}_\alpha \\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \\
\text{Cob}_\alpha & \longrightarrow \text{Cob}^\oplus_\alpha \longrightarrow \text{Kob}_\alpha
\end{align*}
\]

where the chain (35) is given by the left, left, down, left, left sequence of arrows. Two new categories are added in the upper right. Vertical down arrows are quotients by the ideals of negligible morphisms. Both squares in the diagram is commutative.

Notice that first modding out \( \PCob_\alpha \) by negligible morphisms to get \( \text{Cob}_\alpha \) and then taking the Karoubi envelope \( \text{Kob}_\alpha \) compared to first taking the Karoubi envelope \( \text{PKob}_\alpha \) and then modding out by negligible morphisms does not produce any extra idempotents. This is due to the easy to check idempotent lifting property that holds for any finite-dimensional algebra \( B \) over \( k \) and any 2-sided ideal \( J \subset B \) (not necessarily nilpotent). Any idempotent in \( B/J \) lifts to an idempotent in \( B \). Endomorphism algebras of objects in \( \PCob_\alpha^\oplus \) are finite-dimensional over \( k \). For the ideal \( J \) one would take the ideal of negligible endomorphisms of an object in \( \text{PCob}^\oplus_\alpha \).

To summarise, the chain of three categories and two functors (the partition category, the Deligne category, and its quotient by negligible morphisms)

\[
\text{Pa}_t \longrightarrow \text{Rep}(S_t) \longrightarrow \text{Rep}(S_t)/J_t
\]
generalizes to a similar chain

\[(39) \quad \text{PCob}_\alpha \longrightarrow \text{PKob}_\alpha \longrightarrow \text{Kob}_\alpha \]

for any sequence \(\alpha\) with rational power series \(Z_\alpha(t)\). Specializing to the constant series \(\alpha(t)\) and rational function \(t/(1 - T)\) recovers the original setup [58].

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