ON VECTOR BUNDLES OVER HYPERKÄHLER TWISTOR SPACES

INDRANIL BISWAS AND ARTOUR TOMBERG

Abstract. We study the holomorphic vector bundles $E$ over the twistor space $\text{Tw}(M)$ of a compact simply connected hyperkähler manifold $M$. We give a characterization of the semistability condition for $E$ in terms of its restrictions to the holomorphic sections of the holomorphic twistor projection $\pi : \text{Tw}(M) \rightarrow \mathbb{C}P^1$. It is shown that if $E$ admits a holomorphic connection, then $E$ is holomorphically trivial and the holomorphic connection on $E$ is trivial as well. For any irreducible vector bundle $E$ on $\text{Tw}(M)$ of prime rank, we prove that its restriction to the generic fibre of $\pi$ is stable. On the other hand, for a K3 surface $M$, we construct examples of irreducible vector bundles of any composite rank on $\text{Tw}(M)$ whose restriction to every fibre of $\pi$ is non-stable. We have obtained a new method of constructing irreducible vector bundles on hyperkähler twistor spaces; this method is employed in constructing these examples.

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1. Introduction

Twistor theory was introduced by Penrose [Pe] in the 1960s as a way of relating physical fields on Minkowski space-time to complex analytic objects on the projective space $\mathbb{C}P^3$. The idea has proved to be rich not only for theoretical physics, but for mathematics as well, in particular for the study of the geometry of 4-manifolds. Given an oriented Riemannian 4-manifold $X$, one can associate to it its twistor space $Z$, which is a 6-manifold with an almost complex structure. This almost complex structure on $Z$ is integrable precisely if

2010 Mathematics Subject Classification. 32L25, 53C28, 32L10.
Key words and phrases. Hyperkähler manifold, twistor space, stability, twistor line, holomorphic connection.
the Riemannian metric on $X$ is self-dual \cite{AHS}. In the situation where the almost complex structure on $Z$ is integrable, a version of the twistor correspondence relates the conformal geometry of $X$ with the complex analytic geometry of $Z$.

The twistor theory of oriented Riemannian 4-manifolds can be generalized to the hyperkähler setting, where it takes the following form. We first recall that a Riemannian manifold $M$ is called hyperkähler if it has a triple of integrable almost complex structures $I, J, K$, which are parallel with respect to the Levi-Civita connection of $M$ and satisfy the quaternionic relations $I^2 = J^2 = K^2 = -\text{Id}, \quad IJ = -JI = K$. Such a manifold $M$ is endowed with a 2-sphere $S^2 = \{ (a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1 \}$ of induced complex structures, given by linear combinations $aI + bJ + cK$, where $(a, b, c) \in S^2$. In this situation, the twistor space of $M$, which we denote by $\text{Tw}(M)$, is a Hermitian manifold which is canonically diffeomorphic (but not biholomorphic) to the Cartesian product $M \times S^2$, and thus has natural projections

$$
\begin{align*}
\text{Tw}(M) & \xrightarrow{\sigma} M \\
& \xleftarrow{\pi} \mathbb{CP}^1,
\end{align*}
$$

the second of which is a holomorphic map, while the fibres of $\sigma$ are complex submanifolds identified with $\mathbb{CP}^1$. It should be clarified that $\sigma$ is not holomorphic.

In this setting, the holomorphic structure of the complex manifold $\text{Tw}(M)$ completely encodes the quaternionic structure of the hyperkähler manifold $M$. For example, $M$ can be recovered from $\text{Tw}(M)$ (see Theorem 1 in \cite{Hit}). More generally, there is a version of the twistor correspondence (see Theorem 5.12 in \cite{KV}) which associates to every vector bundle on $M$, with a connection whose curvature has type $(1, 1)$ with respect to any of the complex structures parametrized by $S^2$, a holomorphic vector bundle on $\text{Tw}(M)$ whose restrictions to all the fibres of the projection $\sigma$ in the diagram (1.1) are trivial, and vice versa. This bijective correspondence leads to an identification of the corresponding moduli spaces, and in this way studying vector bundles on the twistor space $\text{Tw}(M)$ can help in our understanding of the geometry of the original manifold $M$.

In the present paper, we pursue this idea further and study holomorphic vector bundles $E$ on the twistor space $\text{Tw}(M)$ of a compact simply connected hyperkähler manifold $M$, in particular, we investigate the relationship between their stability and the stability of their restrictions to the fibres of the projections $\sigma$ and $\pi$ in the diagram (1.1).

The fibres of $\sigma : \text{Tw}(M) \longrightarrow M$ in $\text{Tw}(M)$ are called horizontal twistor lines; more generally, holomorphic sections of $\pi$ are called twistor lines in $\text{Tw}(M)$. We show that the semistability of $E$ on $\text{Tw}(M)$ can be related to the “semistability” of its restrictions to the twistor lines in $\text{Tw}(M)$. More precisely, we show that if $E$ restricts semi-stably to the image of some twistor line $s : \mathbb{CP}^1 \longrightarrow \text{Tw}(M)$, then $E$ itself is semistable. On the other hand, if $E$ is semistable, then for some twistor line $s : \mathbb{CP}^1 \longrightarrow \text{Tw}(M)$, either the restriction $s^*E$ is semistable, or the slopes of the associated graded components of the Harder–Narasimhan filtration

$$
0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{n-2} \subsetneq E_{n-1} \subsetneq E_n = s^*E
$$

of $s^*E$ satisfy the condition $\mu(E_i/E_{i-1}) = \mu(E_{i+1}/E_i) + 1$ for all $1 \leq i \leq n - 1$ (Theorem 3.1). We also show that if $E$ is a holomorphic bundle on $\text{Tw}(M)$ admitting a holomorphic
connection $D$, then $E$ is holomorphically trivial, and $D$ is the trivial connection (Proposition 3.4).

Concerning the restrictions of a holomorphic bundle $E$ on $\text{Tw}(M)$ to the fibres of the twistor projection $\pi : \text{Tw}(M) \rightarrow \mathbb{CP}^1$, a result of Kaledin and Verbitsky shows that if $E$ restricts stably to the generic fibre of $\pi$, then it is an irreducible bundle on $\text{Tw}(M)$, in the sense that it does not have any nonzero proper subsheaf of lower rank (see Lemma 7.3 in [KV]). In the paper [Tomb3], the second author proved a partial converse to this result (see Theorem 2.12 in the present article, which is Theorem 4.1 in [Tomb3]), while the following question was posed about the full converse:

**Question** ([Tomb3, p. 2]). Given an irreducible bundle $E$ on the twistor space $\text{Tw}(M)$, will it always be stable on the generic fibre of the twistor projection $\pi : \text{Tw}(M) \rightarrow \mathbb{CP}^1$?

Note that if we replace “irreducible” by “stable”, the answer is negative: in [Tomb2], an example of a stable (but not irreducible) bundle on $\text{Tw}(M)$ with non-stable restrictions to the fibres of $\pi$ was constructed.

In the present article, we prove that an irreducible $E$ of prime rank on $\text{Tw}(M)$ does restrict stably to the generic fibre of $\pi : \text{Tw}(M) \rightarrow \mathbb{CP}^1$ (Theorem 5.2). However, we also show that for $M$ a K3 surface, there are examples of bundles $E$ on $\text{Tw}(M)$ of any composite rank which are irreducible but whose restrictions to all the fibres of $\pi$ are non-stable (see Theorem 7.1). This settles the question above in the negative, and also strengthens the result of [Tomb2].

The proof of Theorem 5.2 gives a new method of constructing irreducible bundles on the twistor space $\text{Tw}(M)$. The significance of this comes from the fact that irreducible bundles (which only exist on nonalgebraic manifolds) are notoriously difficult to study, and the main difficulty is in fact a lack of general mechanisms of constructing such bundles. There are only specific methods for particular classes of manifolds; for the case of surfaces, see [Toma] [ABT] [TT] [BM]. Our proof gives a new method of constructing irreducible bundles on a 3-dimensional complex manifold, namely the twistor space $\text{Tw}(M)$ of a K3 surface $M$.

## 2. Preliminaries

We begin by giving the definitions of the basic objects that we shall be working with, and also recalling the results that will be used in the subsequent sections.

**Definition 2.1.** A hyperkähler structure on a smooth manifold $M$ consists of a triple of integrable almost complex structures $I, J, K : TM \rightarrow TM$ satisfying

$$I^2 = J^2 = K^2 = -\text{Id}, \quad IJ = -JI = K,$$

together with a Riemannian metric $g$ on $M$ which is simultaneously Kähler with respect to $I, J, K$.

Together with the identity mapping, $I, J, K$ induce an action of the quaternion algebra $\mathbb{H}$ on the smooth tangent bundle $TM$, which is moreover parallel with respect to the Levi-Civita connection on $M$ associated to the Riemannian metric $g$. Any linear combination $A = aI + bJ + cK$, where $(a, b, c) \in \mathbb{R}^3$ with $a^2 + b^2 + c^2 = 1$, is an endomorphism of the tangent bundle $TM$ satisfying $A^2 = -\text{Id}_{TM}$, and thus defines an almost complex structure on $M$. This almost complex structure $A$ is actually integrable and the metric $g$ is
again Kähler with respect to this complex structure. In this way, we get a family of induced complex structures

\[ S^2 = \left\{ aI + bJ + cK \mid a^2 + b^2 + c^2 = 1 \right\} \]

on \( M \) parametrized by \( S^2 \). Consider the (topological) product manifold \( \text{Tw}(M) := M \times S^2 \). For every point \( (m, A) \in \text{Tw}(M) = M \times S^2 \), we have the tangent space decomposition

\[ T_{(m, A)} \text{Tw}(M) = T_m M \bigoplus T_A S^2. \]

Identifying \( S^2 \) with \( \mathbb{CP}^1 \) using the stereographic projection from \((0, 0, 1)\), we have the almost complex structure \( I_{S^2} : T S^2 \rightarrow T S^2 \) on \( S^2 \), while any \( A = (a, b, c) \in S^2 \) itself defines the almost complex structure \( A = aI + bJ + cK : TM \rightarrow TM \) on \( M \) mentioned earlier. The operator

\[ I : T \text{Tw}(M) \rightarrow T \text{Tw}(M), \]

which at the point \( (m, A) \) is the direct sum \( A(m) \bigoplus I_{S^2}(A) \), satisfies the equation \( I^2 = -\text{Id}_{T \text{Tw}(M)} \), and thus defines an almost complex structure on \( \text{Tw}(M) \). It can be shown that \( I \) is actually integrable [Sa].

**Definition 2.2.** The above complex manifold \((\text{Tw}(M), I)\) is called the **twistor space** of the hyperkähler manifold \( M \).

Thinking of \( S^2 \cong \mathbb{CP}^1 \) as the set of induced complex structures of \( M \) as above, the twistor space \( \text{Tw}(M) \) parametrizes these structures at points of \( M \). We have the canonical projections

\[ \begin{array}{ccc}
\text{Tw}(M) & \xrightarrow{\pi} & \mathbb{CP}^1 \\
\sigma & \xleftarrow{\sigma} & M 
\end{array} \]

With the complex structure of \( \text{Tw}(M) \) described above, it is easy to verify that the second projection \( \pi : \text{Tw}(M) \rightarrow \mathbb{CP}^1 \) is a holomorphic map. Holomorphic sections of \( \pi \) will be called the **twistor lines**, while the constant sections of the form

\[ I \mapsto (m, I) \in M \times \mathbb{CP}^1 = \text{Tw}(M), \]

where \( m \in M \), will be called the horizontal twistor lines. The hyperkähler metric \( g \) on \( M \) and the Fubini-Study metric \( g_{\mathbb{CP}^1} \) on \( \mathbb{CP}^1 \) together produce a natural Hermitian metric

\[ \sigma^*(g) + \pi^*(g_{\mathbb{CP}^1}) \]

on \( \text{Tw}(M) \). The Hermitian metric on \( \text{Tw}(M) \) thus obtained is not Kähler but satisfies the weaker property of being balanced (see [KV]), i.e., its Hermitian form \( \omega \) satisfies the equation \( d(\omega^{n-1}) = 0 \), where \( n \) is the complex dimension of \( \text{Tw}(M) \) (which is clearly \( \dim_{\mathbb{C}} M + 1 \)).

From now on, the original complex structures \( I, J, K \) will not play any vital role. We will denote an arbitrary induced complex structure on \( M \) by \( I \), and the resulting complex manifold by \( M_I \). As noted above, \( g \) is a Kähler metric on \( M_I \). These \( M_I \) are precisely the fibres of the holomorphic twistor projection \( \pi : \text{Tw}(M) \rightarrow \mathbb{CP}^1 \), and it will be useful to think of \( \text{Tw}(M) \) as the collection of Kähler manifolds \( M_I \) lying above the points \( I \in \mathbb{CP}^1 \) via the map \( \pi \). From now on, we shall assume throughout the article that \( M \) is compact.
Recall that a hyperkähler manifold $M$ has an action of the quaternion algebra $\mathbb{H}$ on its tangent bundle, which is parallel with respect to the Levi-Civita connection for its hyperkähler metric $g$. Restricting to the group of unitary quaternions in $\mathbb{H}$, we get an action of SU(2) on $TM$, hence also on the bundle of differential forms $\Omega^*_M$. Since the action is parallel, it commutes with the Laplace operator, and thus preserves harmonic forms. Applying Hodge theory, we get a natural action of SU(2) on the cohomology $H^*(M, \mathbb{C})$.

**Lemma 2.3.** A differential form $\eta$ on a hyperkähler manifold $M$ is SU(2)-invariant if and only if it is of Hodge type $(p, p)$ with respect to all induced complex structures $M_I$.

**Proof.** This is proved in Proposition 1.2 of [Ve2].

Using Lemma 2.3, we can define vector bundles on $M$ which are simultaneously holomorphic with respect to all induced complex structures $I \in \mathbb{CP}^1$.

**Definition 2.4.** Let $M$ be a hyperkähler manifold and $B$ a $C^\infty$ vector bundle on $M$. A connection $\nabla$ on $B$ is called *hyperholomorphic* if it preserves a Hermitian metric on $B$ and its curvature $K(\nabla) \in H^0(M, \text{End}(B) \otimes \Omega^2_M)$ is SU(2)-invariant.

By Lemma 2.3, the SU(2)-invariance condition is equivalent to $K(\nabla)$ being a $C^\infty$ section of

$$\text{End}(B) \otimes \left( \bigcap_{I \in \mathbb{CP}^1} \Omega_{M_I}^{1,1} \right) \subseteq \text{End}(B) \otimes \Omega^2_M.$$  

This means that for any $I \in \mathbb{CP}^1$, the $(0, 1)$-part $\nabla_I^{0,1}$ of $\nabla$ with respect to $I$ induces a holomorphic structure on $B$ over $M_I$ [Ko, p. 9, Proposition 3.7]; we shall denote the corresponding holomorphic bundle by $E_I$. In this way, a hyperholomorphic connection $\nabla$ gives a family of holomorphic vector bundles $E_I$ over the Kähler manifolds $M_I$, $I \in \mathbb{CP}^1$, all with the same underlying $C^\infty$ vector bundle $B$. To assemble these bundles into one object, we can use the twistor formalism.

Recall that the twistor space Tw($M$) comes equipped with a (nonholomorphic) projection $\sigma : \text{Tw}(M) \longrightarrow M$. Given a hyperholomorphic bundle $(B, \nabla)$ on $M$, consider the pullback bundle with connection $(\sigma^*B, \sigma^*\nabla)$ on Tw($M$). By the considerations in the previous paragraph and the structure of Tw($M$), the curvature of the connection $\sigma^*\nabla$ on $\text{Tw}(M)$ is of type $(1, 1)$, hence its $(0, 1)$-part $(\sigma^*\nabla)^{0,1}$ defines a holomorphic structure on the topological bundle $\sigma^*B$ over Tw($M$), which we shall denote by $E$. The restriction of $E$ to the fibre $\pi^{-1}(I) = M_I$ of the holomorphic projection $\pi : \text{Tw}(M) \longrightarrow \mathbb{CP}^1$ is none other than the holomorphic bundle $E_I$ described in the previous paragraph. Here we will use the term “hyperholomorphic bundle” interchangeably to refer either to a $C^\infty$ bundle with connection $(B, \nabla)$ on $M$ as in the statement of Definition 2.3 or to the holomorphic bundle $E_I$ on $M_I$ obtained from it as described in the previous paragraph, or to the holomorphic bundle $E$ on Tw($M$) constructed above. In either of these contexts, the hyperholomorphic line bundles form a complex abelian Lie group under tensor product.

**Definition 2.5.** Let $M$ be hyperkähler and $I$ an induced complex structure. We say that $I$ is *generic* with respect to the hyperkähler structure on $M$ if all elements in

$$\bigoplus_p H^{p,p}(M_I) \cap H^{2p}(M, \mathbb{Z}) \subset H^*(M, \mathbb{C})$$
are SU(2)-invariant.

This terminology is justified: most induced complex structures are generic, in a sense made precise in the following proposition.

**Proposition 2.6.** Let \( M \) be a hyperkähler manifold. The subset \( S_0 \subset S^2 \) of generic induced complex structures is dense in \( S^2 \) and its complement \( S^2 \setminus S_0 \) is countable.

**Proof.** This is proved in Proposition 2.2 of [Ve1]. □

We now give the definition of stable vector bundles and torsionfree sheaves. Recall that a coherent sheaf \( \mathcal{F} \) on an arbitrary complex manifold is called torsionfree if the natural morphism into the double dual \( \mathcal{F} \to \mathcal{F}^{**} \) is injective. If it is an isomorphism, \( \mathcal{F} \) is called reflexive. We call the sheaf \( \mathcal{F} \) normal if for every open set \( U \) and every analytic subset \( A \subset U \) of complex codimension at least two, the restriction map \( \mathcal{F}(U) \to \mathcal{F}(U \setminus A) \) is an isomorphism.

**Definition 2.7.** Let \( Z \) be a compact balanced manifold of complex dimension \( n \), and let \( \omega \) denote the Hermitian form of its balanced metric. The degree of a coherent sheaf \( \mathcal{F} \) on \( Z \) is defined to be

\[
\deg \mathcal{F} := \int_Z c_1(\mathcal{F}) \wedge \omega^{n-1},
\]

where by \( c_1(\mathcal{F}) \) we mean any representative of the first Chern class of \( \mathcal{F} \) in the de Rham cohomology \( H^2(Z, \mathbb{C}) \) (the condition that \( \omega \) is balanced ensures that \( \deg \mathcal{F} \) does not depend on the choice of the representative of the first Chern class). If \( \mathcal{F} \) is a nonzero torsionfree coherent sheaf, the slope of \( \mathcal{F} \) is

\[
\mu(\mathcal{F}) := \frac{\deg \mathcal{F}}{\text{rk} \mathcal{F}}.
\]

A torsionfree sheaf \( \mathcal{F} \) is called stable (respectively, semistable) if for every subsheaf \( \mathcal{G} \subset \mathcal{F} \) with \( 0 < \text{rk} \mathcal{G} < \text{rk} \mathcal{F} \) we have

\[
\mu(\mathcal{G}) < \mu(\mathcal{F}) \quad \text{(respectively, } \mu(\mathcal{G}) \leq \mu(\mathcal{F}))
\]

while \( \mathcal{F} \) is called polystable if it is a direct sum of stable sheaves of the same slope.

A torsionfree sheaf \( \mathcal{F} \) is called irreducible if it has no proper subsheaves of lower rank.

Note that any irreducible sheaf is stable.

For a hyperkähler \( M \), as mentioned previously, its twistor space \( \text{Tw}(M) \) is a balanced manifold, and the fibres \( \pi^{-1}(I) = M_I \) of the twistor projection \( \pi : \text{Tw}(M) \to \mathbb{CP}^1 \) are Kähler. Moreover, if we denote by \( \omega \) the Hermitian form of the balanced metric on \( \text{Tw}(M) \), its restriction \( \omega_I \) to \( M_I \) is precisely the Kähler form of the Kähler metric on \( M_I \). Thus, given a holomorphic vector bundle \( E \) on \( \text{Tw}(M) \), it makes sense to talk both about its stability as a bundle on \( \text{Tw}(M) \), and also the stability of its restrictions \( E_I \) to the fibres \( M_I \) of \( \pi \).

**Definition 2.8.** A holomorphic vector bundle \( E \) on the twistor space \( \text{Tw}(M) \) is called fibrewise stable if its restriction \( E_I \) to each fibre \( M_I \) of the projection \( \pi : \text{Tw}(M) \to \mathbb{CP}^1 \) is stable. \( E \) is called generically fibrewise stable if \( E_I \) is stable for all \( I \) in a nonempty Zariski open subset of \( \mathbb{CP}^1 \). Similarly, \( E \) is called fibrewise simple if all the restrictions \( E_I \) are simple bundles, in the sense that \( \text{Hom}_{M_I}(E_I, E_I) = \mathbb{C} \), and it is called generically fibrewise simple if \( E_I \) is simple for all \( I \) in a nonempty Zariski open subset of \( \mathbb{CP}^1 \).
The following important result gives a topological characterization of bundles on $M$ admitting hyperholomorphic structures.

**Theorem 2.9.** Let $M$ be a hyperkähler manifold, and let $E_I$ be a stable holomorphic bundle on $M_I$, for some $I \in \mathbb{CP}^1$. If $c_1(E_I)$ and $c_2(E_I)$ are SU(2)-invariant, then there exists a unique hyperholomorphic connection on the underlying $C\infty$ bundle of $E_I$ which induces the holomorphic structure of $E_I$.

*Proof.* This is proved in Theorem 2.5 of [Ve2]. □

Any hyperholomorphic bundle on any $M_I$ has degree zero, as shown by the following lemma.

**Lemma 2.10.** An SU(2)-invariant 2-form $\beta$ on a hyperkähler manifold $M$ satisfies
\[
\int_M \beta \wedge \omega_I^{n-1} = 0
\]
for any induced complex structure $I$, where $\omega_I$ denotes the Kähler form on $M_I$.

*Proof.* This is a consequence of Lemma 2.1 of [Ve2]. □

It follows from Lemma 2.10 that any sheaf $F$ on any $M_I$ with SU(2)-invariant first Chern class $c_1(F) \in H^2(M, \mathbb{R})$ has degree zero, because the harmonic representative of $c_1(F)$ must be SU(2)-invariant as a two-form. In particular, if $S_0 \subseteq S^2 \approx \mathbb{CP}^1$ denotes the subset of generic complex structures of $M$ as in the statement of Proposition 2.6, then for any $I \in S_0$, all sheaves on $M_I$ have degree zero, and are thus semistable. The following proposition is a consequence of this.

**Proposition 2.11.** The holomorphic twistor projection $\pi : \text{Tw}(M) \rightarrow \mathbb{CP}^1$ establishes a bijective correspondence between divisors on $\mathbb{CP}^1$ and those on $\text{Tw}(M)$.

*Proof.* This is proved in Proposition 2.17 of [Tomb3]. □

In view of this bijective correspondence, given a divisor $D$ on $\mathbb{CP}^1$, we will denote by the same letter $D$ the corresponding divisor on $\text{Tw}(M)$, and vice versa. The corresponding line bundle on $\mathbb{CP}^1$ will be denoted by $\mathcal{O}_{\mathbb{CP}^1}(D)$, and on $\text{Tw}(M)$ by $\mathcal{O}_{\text{Tw}(M)}(D)$. For a sheaf $\mathcal{F}$ on $\text{Tw}(M)$, we define
\[
\mathcal{F}(D) := \mathcal{F} \otimes_{\mathcal{O}_{\text{Tw}(M)}} \mathcal{O}_{\text{Tw}(M)}(D).
\]

We finish this section by stating a theorem from the paper [Tomb3] on fibrewise stability of bundles on the twistor space of a simply connected hyperkähler manifold $M$, which establishes a partial converse to the result of Kaledin and Verbitsky proved in [KV] that was mentioned in the introduction.

**Theorem 2.12.** Let $M$ be a compact simply connected hyperkähler manifold, and let $E$ be a holomorphic vector bundle on the twistor space $\text{Tw}(M)$. If $E$ is generically fibrewise stable, then it is irreducible. If $E$ is irreducible of rank two or three, then $E$ is generically fibrewise stable. Also, if $E$ is irreducible and it is generically fibrewise simple, then $E$ is generically fibrewise stable.
Proof. The forward implication follows from the proof of Lemma 7.3 in [KV]. The two partial converses are proved in Theorem 4.1 of [Tomb3] and also Theorem 4.2.1 of [Tomb1]. □

In Section 5, we shall strengthen this result by showing that the converse holds for arbitrary vector bundles of prime rank. On the other hand, there are examples of irreducible bundles of any composite rank on Tw(M), for M a K3 surface, which are not generically fibrewise stable (this is proved in Section 7).

3. Semistability of bundles and holomorphic connections on Tw(M)

3.1. Semistability and restriction to twistor lines. Let M be a compact hyperkähler manifold, and Tw(M) its twistor space. Let D denote the component of the Douady space for Tw(M) that contains the horizontal twistor lines. Let

\[ \text{Sec}(\pi) \subset D \] (3.1)

be the Zariski open subset consisting of the holomorphic sections of the projection \( \pi : \text{Tw}(M) \rightarrow \mathbb{CP}^1 \), that is, twistor lines in Tw(M).

The following theorem does not require M to be simply connected.

Theorem 3.1. Let \( E \) be a torsionfree coherent analytic sheaf on Tw(M).

If for some element \( s \in \text{Sec}(\pi) \), the pulled back coherent analytic sheaf \( s^*E \rightarrow \mathbb{CP}^1 \) is torsionfree and semistable, then \( E \) is semistable.

If \( E \) is semistable, then one of the following two holds:

- There is a nonempty Zariski open subset \( U_s \subset \text{Sec}(\pi) \) such that for all \( s \in U_s \), the pulled back coherent analytic sheaf \( s^*E \rightarrow \mathbb{CP}^1 \) is torsionfree and semistable.
- For all element \( s \in \text{Sec}(\pi) \) such that the pulled back coherent analytic sheaf \( s^*E \rightarrow \mathbb{CP}^1 \) is torsionfree, the vector bundle \( s^*E \) is not semistable. Furthermore, there is a nonempty Zariski open subset \( U_0 \subset \text{Sec}(\pi) \) such that for all \( s \in U_0 \), the pulled back coherent analytic sheaf \( s^*E \rightarrow \mathbb{CP}^1 \) is torsionfree, and if

\[ 0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \mathcal{E}_2 \subsetneq \cdots \subsetneq \mathcal{E}_{n-2} \subsetneq \mathcal{E}_{n-1} \subsetneq \mathcal{E}_n = s^*E \]

is the Harder–Narasimhan filtration of \( s^*E \) (which is not semistable by the first sentence), then

\[ \mu(\mathcal{E}_i/\mathcal{E}_{i-1}) = \mu(\mathcal{E}_{i+1}/\mathcal{E}_i) + 1 \]

for all \( 1 \leq i \leq n-1 \).

Proof. Since Tw(M) is topologically the product of M and \( \mathbb{CP}^1 \), we have

\[ H^2(\text{Tw}(M), \mathbb{R}) = H^2(M, \mathbb{R}) \oplus H^2(\mathbb{CP}^1, \mathbb{R}) \] (3.2)

For any torsionfree coherent analytic sheaf \( \mathcal{F} \) on Tw(M), it is clear that all its restrictions \( \mathcal{F}_I = \mathcal{F}|_{\pi^{-1}(I)} \) for all \( I \in \mathbb{CP}^1 \) have the same first Chern class, and hence the harmonic representative of \( c_1(\mathcal{F}_I) \in H^2(M, \mathbb{Z}) \) is SU(2)-invariant as a consequence of Lemma 2.3 From Lemma 2.10 and the discussion following it, we know that the degree of \( \mathcal{F}_I \) is zero for all \( I \in \mathbb{CP}^1 \). Consequently, using (3.2) it follows that

\[ \text{deg}(\mathcal{F}) = \text{deg}(s^*\mathcal{F}) \cdot \text{Vol}(M), \] (3.3)

where \( s : \mathbb{CP}^1 \rightarrow \text{Tw}(M) \) is any element of Sec(\( \pi \)) defined in (3.1).
From (3.3) it follows immediately that \( E \) is semistable if for the general element \( s \in \text{Sec}(\pi) \), the pulled back coherent analytic sheaf \( s^*E \to \mathbb{CP}^1 \) is torsionfree and semistable. Now the openness of the semistability condition (see [Ma, p. 635, Theorem 2.8(B)] for it), implies that if \( s_0^*E \) is torsionfree and semistable for some \( s_0 \in \text{Sec}(\pi) \), then \( s^*E \) is torsionfree and semistable for the general element \( s \in \text{Sec}(\pi) \). Therefore, \( E \) is semistable if \( s^*E \) is torsionfree and semistable for some \( s \in \text{Sec}(\pi) \).

Now assume that \( E \) is semistable. Consider all \( s \in \text{Sec}(\pi) \) such that the pulled back coherent analytic sheaf \( s^*E \to \mathbb{CP}^1 \) is torsionfree. Their locus is a nonempty Zariski open subset of \( \text{Sec}(\pi) \). This Zariski open subset of \( \text{Sec}(\pi) \) will be denoted by \( D^1 \).

Assume that for some \( s_0 \in D^1 \), the torsionfree sheaf \( s_0^*E \) is semistable. Then from the openness of semistability condition it follows that there is a nonempty Zariski open subset \( U_{s_0} \subset D^1 \) such that for all \( s \in U_{s_0} \), the pulled back coherent analytic sheaf \( s^*E \to \mathbb{CP}^1 \) is torsionfree and semistable.

Therefore, assume that \( s_0^*E \) is not semistable for every \( s_0 \in D^1 \). Consequently, there is a nonempty Zariski open subset

\[ U'_{s_0} \subset D^1 \]

such that for every \( s_0 \in U'_{s_0} \), the Harder–Narasimhan filtration of \( s_0^*E \) is independent of \( s_0 \). In other words, the Harder–Narasimhan filtrations of all \( s_0^*E \), \( s_0 \in U'_{s_0} \), have equal length, and the rank and degree of the \( i \)-th term in the filtration are independent of \( s_0 \in U'_{s_0} \) for all \( i \).

Take an element \( s \in U'_{s_0} \). Let

\[ 0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \mathcal{E}_2 \subsetneq \cdots \subsetneq \mathcal{E}_{n-2} \subsetneq \mathcal{E}_{n-1} \subsetneq \mathcal{E}_n = s^*E \] (3.4)

be the Harder–Narasimhan filtration of \( s^*E \).

Let \( \mathcal{N} \) be the normal bundle of \( s(\mathbb{CP}^1) \subset \text{Tw}(M) \). We know that

\[ \mathcal{N} = \mathcal{O}_{\mathbb{CP}^1}(1)^{\oplus 2d}, \]

where \( 4d \) is the real dimension of \( M \) (see [Hit, p. 142, Theorem 1(2)])). From this it follows that the evaluation homomorphism

\[ \varepsilon : \mathbb{CP}^1 \times H^0(\mathbb{CP}^1, \mathcal{N}) \to \mathcal{N}, \]

that sends any \((x, v) \in \mathbb{CP}^1 \times H^0(\mathbb{CP}^1, \mathcal{N})\) to \( v(x) \in \mathcal{N}_x \), is surjective, and moreover

\[ \text{kernel}(\varepsilon) = \mathcal{O}_{\mathbb{CP}^1}(-1)^{\oplus 2d}. \] (3.5)

For every \( 1 \leq i \leq n - 1 \), there is a natural homomorphism

\[ \Psi_i^s : \text{kernel}(\varepsilon) \otimes \mathcal{E}_i \to (s^*E)/\mathcal{E}_i \] (3.6)

(see (3.4)); these homomorphisms \( \Psi_i^s \) correspond to the infinitesimal deformation of the subsheaves of the Harder–Narasimhan filtrations.

From the given condition that \( E \) is semistable it can be deduced that the homomorphism \( \Psi_i^s \) is not identically zero for the general element \( s \in U'_{s_0} \). Indeed, if \( \Psi_i^s = 0 \) for all \( s \in U'_{s_0} \), then there is a subsheaf

\[ \mathcal{E} \subset E \]
such that $\mathcal{E}_i = s^* \mathcal{E}$ for all $s \in \mathcal{U}'_s$. Now using (3.3), and the properties of the Harder–Narasimhan filtration, it follows that the subsheaf $\mathcal{E}$ of $E$ contradicts the semistability condition for $E$.

Since the homomorphism $\Psi^s_i$ in (3.6) is nonzero for the general element $s \in \mathcal{U}'_s$, using (3.5) it follows that $\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) = \mu(\mathcal{E}_{i+1}/\mathcal{E}_i) + 1$ (3.7) for all $s \in \mathcal{U}'_s$ such that $\Psi^s_i \neq 0$. In other words, there is a nonzero Zariski open subset $\mathcal{U}_s^0 \subset \mathcal{U}'_s$ such that (3.7) holds. This completes the proof of the theorem. □

A simply connected compact hyperkähler manifold is called irreducible if it is not a product of hyperkähler manifolds.

**Proposition 3.2.** Let $M$ be a compact simply connected irreducible hyperkähler manifold. Then the holomorphic tangent bundle $T^{1,0} \text{Tw}(M)$ of the twistor space $\text{Tw}(M)$ is stable.

**Proof.** Consider the projection $\pi$ in (2.1). We have the short exact sequence of holomorphic vector bundles on $\text{Tw}(M)$

$$0 \longrightarrow T^{1,0}_{\pi} \longrightarrow T^{1,0} \text{Tw}(M) \xrightarrow{d\pi} \pi^* T^{1,0} \mathbb{CP}^1 \longrightarrow 0,$$

where $T^{1,0}_{\pi}$ is the relative holomorphic tangent bundle for the projection $\pi$, and $d\pi$ is the differential of $\pi$.

Firstly, the relative tangent bundle $T^{1,0}_{\pi}$ is irreducible. Indeed, for any $I \in \mathbb{CP}^1$, the restriction $T^{1,0}_{\pi}|_{\pi^{-1}(I)}$ is stable, because $M$ is simply connected and irreducible; note that $M_I = \pi^{-1}(I)$ admits a Kähler–Einstein metric [Ya] (Calabi’s conjecture). Hence the vector bundle $T^{1,0}_{\pi}$ is fibrewise stable, and by the forward implication of Theorem 2.12, we conclude that it is irreducible.

Assume that $T^{1,0} \text{Tw}(M)$ is not stable. Let

$$F \subsetneq T^{1,0} \text{Tw}(M)$$

be a nonzero subsheaf such that $(T^{1,0} \text{Tw}(M))/F$ is torsionfree and

$$\mu(F) \geq \mu(T^{1,0} \text{Tw}(M)).$$

Since $T^{1,0}_{\pi}$ is irreducible, from (3.8) we now conclude the following:

1. either $F = T^{1,0}_{\pi}$,
2. $F$ is a subsheaf of $T^{1,0} \text{Tw}(M)$ of rank one such that the composition

$$F \hookrightarrow T^{1,0} \text{Tw}(M) \xrightarrow{d\pi} \pi^* T^{1,0} \mathbb{CP}^1$$

is not identically zero.

Firstly observe that $\mu(T^{1,0}_{\pi}) < \mu(T^{1,0} \text{Tw}(M))$, because the slope of the restriction of $T^{1,0}_{\pi}$ to a horizontal twistor line is strictly less than the slope of the restriction of $T^{1,0} \text{Tw}(M)$ to a horizontal twistor line. Therefore, $T^{1,0}_{\pi}$ does not destabilize $T^{1,0} \text{Tw}(M)$.

Secondly, it can be shown that there is no rank one subsheaf

$$F' \subset T^{1,0} \text{Tw}(M)$$

such that the composition

$$F \hookrightarrow T^{1,0} \text{Tw}(M) \xrightarrow{d\pi} \pi^* T^{1,0} \mathbb{CP}^1$$
is not identically zero. To prove this, restrict the exact sequence in (3.8) to \( M_I = \pi^{-1}(I) \). This produces the short exact sequence

\[
0 \to T^{1,0} M_I \to (T^{1,0} \Tw(M))|_{M_I} \overset{d\pi}{\to} M_I \times T^{1,0}_I \CP^1 \to 0,
\]

where \( M_I \times T^{1,0}_I \CP^1 \to M_I \) is the trivial holomorphic line bundle with fibre \( T^{1,0}_I \CP^1 \). This short exact sequence does not split holomorphically. Indeed, the obstruction class to its splitting, which lies in \( \text{Hom}(T_I \CP^1, H^1(M_I, TM_I)) \), is the Kodaira–Spencer homomorphism for the family \( \Tw(M) \). Consequently, there is no rank one subsheaf

\[
F' \subset T^{1,0} \Tw(M)
\]

such that the composition

\[
F \hookrightarrow T^{1,0} \Tw(M) \overset{d\pi}{\to} \pi^* T^{1,0} \CP^1
\]

is not identically zero. This completes the proof. \( \Box \)

**Remark 3.3.** Note that the restriction of \( T^{1,0} \Tw(M) \) to a twistor line decomposes as \( \O_{\CP^1}(2) \oplus \O_{\CP^1}(1)^{\oplus 2d} \). Also, [Tom02] gives an example of a stable rank 2 bundle on \( \Tw(M) \) for \( M \) a K3 surface whose restriction to a twistor line is not semistable. Therefore, the converse of the first part of Theorem 3.1 does not hold.

### 3.2. Holomorphic connections.

Let \( E \) be a holomorphic vector bundle on a complex manifold \( Z \). A **holomorphic connection** on \( E \) is a holomorphic differential operator

\[
D : E \to E \otimes \Omega^1_Z
\]

of order one satisfying the Leibniz identity which says that

\[
D(fs) = f \cdot D(s) + s \otimes df,
\]

where \( s \) is any locally defined holomorphic section of \( E \) and \( f \) is any locally defined holomorphic function on \( Z \) [At]. Let \( \overline{\partial}_E : E \to E \otimes \Omega^1_Z \) be the Dolbeault operator defining the holomorphic structure on \( E \). Then for any holomorphic connection \( D \) on \( E \), the differential operator \( D + \overline{\partial}_E \) is a usual connection on the holomorphic vector bundle \( E \). Since the differential operator \( D \) is holomorphic, the curvature \( (D + \overline{\partial}_E)^2 \) of the connection \( D + \overline{\partial}_E \) is a holomorphic section of \( \End(E) \otimes \Omega^2_Z \).

As before, let \( M \) be a compact hyperkähler manifold and \( \Tw(M) \) the corresponding twistor space. For the following proposition we do not assume that \( M \) is simply connected.

**Proposition 3.4.** Let \( E \) be a holomorphic vector bundle on \( \Tw(M) \) equipped with a holomorphic connection \( D \). Then the curvature of \( D \) vanishes identically.

**Proof.** Let

\[
\mathcal{K}(D) := (D + \overline{\partial}_E)^2 \in H^0(\Tw(M), \End(E) \otimes \Omega^2_{\Tw(M)})
\]

be the curvature of the connection \( D + \overline{\partial}_E \). To show that \( \mathcal{K}(D) \) vanishes identically, let

\[
s : \CP^1 \hookrightarrow \Tw(M)
\]

be a horizontal twistor line. The holomorphic vector bundle \( s^* \Omega^1_{\Tw(M)} \to \CP^1 \) will be denoted by \( \mathbb{V} \). We note that

\[
s^* \mathcal{K}(D) \in H^0(\CP^1, \End(s^* E) \otimes \bigwedge^2 \mathbb{V}) = H^0(\CP^1, (s^* \End(E)) \otimes \bigwedge^2 \mathbb{V}); \quad (3.9)
\]
to clarify, $s^*\mathcal{K}(D)$ is the pullback of the section $\mathcal{K}(D)$ and not the restriction of the differential form.

Now, $s^*D$ is a holomorphic connection on the holomorphic vector bundle $s^*E \rightarrow \mathbb{C}P^1$. But any holomorphic connection on a Riemann surface $Y$ is flat (curvature vanishes identically) because $\Omega_Y^{2,0} = 0$. Therefore, $(s^*E, s^*D)$ is given by a representation of the fundamental group. Since $\mathbb{C}P^1$ is simply connected, we conclude that the holomorphic vector bundle $s^*E$ is holomorphically trivial. Fix a holomorphic trivialization $\mathcal{O} \oplus r_{\mathbb{C}P^1} \sim s^*E$, where $r = \text{rank}(E)$. Using this trivialization, $s^*\mathcal{K}(D)$ in (3.9) is a holomorphic section

$$s^*\mathcal{K}(D) \in H^0(\mathbb{C}P^1, \bigwedge^2 \mathcal{V})^r.$$  

As before, let $N$ denote the normal bundle of $s(\mathbb{C}P^1) \subset \text{Tw}(M)$. Recall that $N = \mathcal{O}_{\mathbb{C}P^1}(1)^{\oplus 2d}$, where $4d$ is the real dimension of $M$. From this it follows immediately that

$$\mathcal{V}^* = \mathcal{O}_{\mathbb{C}P^1}(1)^{\oplus 2d} \oplus \mathcal{O}_{\mathbb{C}P^1}(2).$$

Therefore, we have

$$H^0(\mathbb{C}P^1, \bigwedge^2 \mathcal{V}) = 0.$$

Hence from (3.10) it follows that $s^*\mathcal{K}(D) = 0$. This implies that the curvature $\mathcal{K}(D)$ vanishes identically. □

**Corollary 3.5.** Let $M$ be a simply connected compact hyperkähler manifold. Let $(E, D)$ be a holomorphic bundle, on the corresponding twistor space $\text{Tw}(M)$, equipped with a holomorphic connection. Then the vector bundle $E$ is holomorphically trivial and $D$ is the trivial connection on it.

**Proof.** Since $\text{Tw}(M)$ is simply connected, this follows from Proposition 3.4 □

### 4. Finite base extensions of the twistor projection

Let $M$ be a compact simply connected hyperkähler manifold, $\text{Tw}(M)$ its twistor space and $\pi : \text{Tw}(M) \rightarrow \mathbb{C}P^1$ the natural holomorphic twistor projection. In this section, we will examine the fibre product

$$\text{Tw}(M)_X \xrightarrow{\varphi} \text{Tw}(M)$$

$$\pi_X \downarrow \quad \pi$$

$$X \xrightarrow{f} \mathbb{C}P^1,$$

where $f : X \rightarrow \mathbb{C}P^1$ is an arbitrary holomorphic branched cover of $\mathbb{C}P^1$ by a smooth projective curve $X$. Observe that we have $\pi_*\mathcal{O}_{\text{Tw}(M)} = \mathcal{O}_{\mathbb{C}P^1}$, since the fibres $M_I$ of the map $\pi$ are connected, and for a similar reason there is an isomorphism $\pi_*\mathcal{O}_{\text{Tw}(M)_X} = \mathcal{O}_{X}$. Also, the maps $\pi$ and $\pi_X$ both induce embeddings of the corresponding Picard groups since they both admit holomorphic sections.

We would like to relate the Picard group of $\text{Tw}(M)_X$ to the Picard group of $\text{Tw}(M)$. In general, Picard groups of fibred products cannot be described in a nice way in terms of the
Picard groups of the factors, but in our particular case we do have such a description. We first describe Pic\(Tw(M)\).

**Proposition 4.1.** The following isomorphism
\[
\text{Pic } Tw(M) \cong \text{Pic } \mathbb{CP}^1 \oplus H^2(M, \mathbb{Z})_{\text{inv}}
\]
holds, where
\[
H^2(M, \mathbb{Z})_{\text{inv}} \subseteq H^2(M, \mathbb{Z})
\]
is the subgroup of SU(2)-invariant cohomology classes. More precisely, Pic\(Tw(M)\) is the direct sum of its subgroup \(\pi^*(\text{Pic } \mathbb{CP}^1)\) and the subgroup of hyperholomorphic line bundles on Tw\((M)\).

**Proof.** First, we will show that Pic\(Tw(M)\) is discrete.

To prove this, note that since \(M\) is simply connected, we have
\[
H^1(M, \mathbb{C}) = 0,
\]
and applying Hodge theory, for any induced complex structure \(I \in \mathbb{CP}^1\), we conclude that
\[
H^0,1(M_I) = H^1(M_I, \mathcal{O}_{M_I}) = 0.
\]
By Grauert’s theorem (Theorem 10.5.5 in [GR]), it follows that
\[
R^1\pi_*\mathcal{O}_{Tw(M)} = 0
\]
and, as mentioned above, \(\pi_*\mathcal{O}_{Tw(M)} = \mathcal{O}_{\mathbb{CP}^1}\). So we have
\[
H^1(\mathbb{CP}^1, R^1\pi_*\mathcal{O}_{Tw(M)}) = H^1(\mathbb{CP}^1, \pi_*\mathcal{O}_{Tw(M)}) = 0.
\]
Examining the Leray spectral sequence of the twistor projection \(\pi : Tw(M) \rightarrow \mathbb{CP}^1\) for the sheaf \(\mathcal{O}_{Tw(M)}\), we see that
\[
H^1(Tw(M), \mathcal{O}_{Tw(M)}) = 0.
\]
Next consider the exponential sequence of sheaves on Tw\((M)\)
\[
0 \rightarrow 2\pi\sqrt{-1}\mathbb{Z} \rightarrow \mathcal{O}_{Tw(M)} \xrightarrow{\exp} \mathcal{O}_{Tw(M)}^* \rightarrow 0.
\]
Let
\[
H^1(Tw(M), \mathcal{O}_{Tw(M)}) \rightarrow \text{Pic } Tw(M) = H^1(Tw(M), \mathcal{O}_{Tw(M)}^*)
\]
\[
\rightarrow H^2(Tw(M), 2\pi\sqrt{-1}\mathbb{Z})
\]
be the corresponding long exact sequence of cohomologies. Using (4.2) in this exact sequence we conclude that the above homomorphism
\[
\text{Pic } Tw(M) = H^1(Tw(M), \mathcal{O}_{Tw(M)}^*) \rightarrow H^2(Tw(M), 2\pi\sqrt{-1}\mathbb{Z})
\]
is injective. It follows from this that Pic\(Tw(M)\) is discrete. More precisely, the holomorphic structure of a holomorphic line bundle on Tw\((M)\) is completely determined by its underlying topological structure.

Since Tw\((M)\) is topologically the product of \(\mathbb{CP}^1\) and \(M\), we have
\[
H^2(Tw(M), \mathbb{Z}) = H^2(\mathbb{CP}^1, \mathbb{Z}) \oplus H^2(M, \mathbb{Z}).
\]
As noted above, the group homomorphism \(\pi^* : \text{Pic } \mathbb{CP}^1 \rightarrow \text{Pic } Tw(M)\) is injective, and so we can think of \(\text{Pic } \mathbb{CP}^1 \cong \mathbb{Z}\) as a subgroup of Pic\(Tw(M)\); it corresponds to the first summand \(H^2(\mathbb{CP}^1, \mathbb{Z})\) in (4.3). On the other hand, it follows from Lemma [2,3], Theorem [2,9] and the simple connectedness of \(M\) that the group of hyperholomorphic line bundles on \(M\) is isomorphic to \(H^2(M, \mathbb{Z})_{\text{inv}}\). The corresponding hyperholomorphic line bundles on Tw\((M)\)
can thus be identified with the subgroup $H^2(M, \mathbb{Z})_{\text{inv}} \subseteq H^2(M, \mathbb{Z})$ of the second summand in (4.3).

It only remains to observe that for an arbitrary holomorphic line bundle $L$ on $\text{Tw}(M)$, the second part of $c_1(L)$ according to the decomposition (4.3) lies in $H^2(M, \mathbb{Z})_{\text{inv}}$. Indeed, observe that the restrictions $L_I$ of $L$ to the fibres $M_I = \pi^{-1}(I)$ of the projection $\pi : \text{Tw}(M) \rightarrow \mathbb{CP}^1$ are all isomorphic topologically, and $c_1(L_I) \in H^{1,1}(M_I) \cap H^2(M, \mathbb{Z})$, so $c_1(L_I)$ must be $SU(2)$-invariant, as a consequence of Lemma 2.3.

The maps $f, \pi$ in the diagram (4.4) induce group homomorphisms from $\text{Pic} \mathbb{CP}^1$ to $\text{Pic} X$, $\text{Pic} \text{Tw}(M)$, respectively, and both of these are injective. Taking the product of these monomorphisms $\text{Pic} \mathbb{CP}^1 \rightarrow \text{Pic} X \oplus \text{Pic} \text{Tw}(M)$, $L \mapsto (f^*L, \pi^*L)$, we can thus think of $\text{Pic} \mathbb{CP}^1$ as a subgroup of $\text{Pic} X \oplus \text{Pic} \text{Tw}(M)$.

**Proposition 4.2.** In the diagram (4.4),
\[ \text{Pic} \text{Tw}(M)_X \cong \left( \text{Pic} X \oplus \text{Pic} \text{Tw}(M) \right) / \text{Pic} \mathbb{CP}^1. \]

**Proof.** There is a natural homomorphism
\[ \text{Pic} X \oplus \text{Pic} \text{Tw}(M) \rightarrow \text{Pic} \text{Tw}(M)_X \]
\[ (L_1, L_2) \mapsto \pi_X^*L_1 \otimes \varphi^*L_2^* \] (4.4)
We first show that the kernel is $\text{Pic} \mathbb{CP}^1$. Suppose $(L_1, L_2) \in \text{Pic} X \oplus \text{Pic} \text{Tw}(M)$ is such that $\pi_X^*L_1 \cong \varphi^*L_2$ on $\text{Tw}(M)_X$. This means that the restriction of $\varphi^*L_2$ on $\text{Tw}(M)_X$ to any fibre of the morphism $\pi_X$ is trivial, hence the same can be said about the restriction of $L_2$ on $\text{Tw}(M)$ to any fibre of $\pi$. By Proposition 4.1, the line bundle $L_2$ must be of the form $\pi^*L'$ for some line bundle $L'$ on $\mathbb{CP}^1$. Then on $\text{Tw}(M)_X$ we have
\[ \pi_X^*L_1 \cong \varphi^*L_2 \cong \varphi^*(\pi^*L') = \pi_X^*(f^*L'). \]

But, as noted previously, $\pi_X : \text{Pic} X \rightarrow \text{Pic} \text{Tw}(M)_X$ is injective, hence $L_1 \cong f^*L'$ on $X$. So $(L_1, L_2) = (f^*L', \pi^*L')$ lies in the subgroup $\text{Pic} \mathbb{CP}^1 \subseteq \text{Pic} X \oplus \text{Pic} \text{Tw}(M)$.

We now show that the map in (4.4) is surjective. Let $L$ be an arbitrary holomorphic line bundle on $\text{Tw}(M)_X$. Note that, for any point $P \in X$, the fibre $\pi_X^{-1}(P)$ is just the manifold $M_{f(P)}$, where $f(P) \in \mathbb{CP}^1$ is the corresponding induced complex structure on $M$. It follows that, when we take the restriction $L_P$ of $L$ to the fibre $\pi_X^{-1}(P)$, the first Chern class $c_1(L_P) = \eta \in H^2(M, \mathbb{Z})$ (which is the same for all $P$) must be an element of $H^2(M, \mathbb{Z})_{\text{inv}}$. By Proposition 4.1 there exists a hyperholomorphic line bundle $\tilde{L}$ on $\text{Tw}(M)$ corresponding to $\eta \in H^2(M, \mathbb{Z})_{\text{inv}}$, and taking its pullback to $\text{Tw}(M)_X$, we have that $L \otimes \varphi^*\tilde{L}^*$ restricts trivially to all fibres of $\pi_X$. It remains to show that the line bundle $L' := L \otimes \varphi^*\tilde{L}^*$ on $\text{Tw}(M)_X$ comes from $X$. For any $P \in X$, we have
\[ H^0\left(\pi_X^{-1}(P), L_P\right) = H^0\left(M_{f(P)}, \mathcal{O}_{M_{f(P)}}\right) = \mathbb{C}. \]

By Grauert’s theorem (Theorem 10.5.5 in [GR]), it follows that $\pi_X_*L'$ is a line bundle on $X$ and its fibre over $P \in X$ is isomorphic to the above. Taking the pullback of $\pi_X_*L'$ back to $\text{Tw}(M)_X$, we have a natural morphism of line bundles
\[ \pi_X^*(\pi_X_*L') \rightarrow L', \]
and it is easy to see that it is an isomorphism over every fibre of $\pi_X$. Hence it is an isomorphism everywhere on $\text{Tw}(M)_X$, and we are done. \qed
5. Irreducible bundles on \( \text{Tw}(M) \) of prime rank

In this section, we shall extend Theorem 2.12 by showing that, for a compact simply connected hyperkähler manifold \( M \), any irreducible bundle of prime rank on the twistor space \( \text{Tw}(M) \) is generically fibrewise stable with respect to the twistor projection \( \pi : \text{Tw}(M) \to \mathbb{CP}^1 \). The strategy of proof consists of showing that any such bundle is generically fibrewise simple, thus reducing to the case already covered by Theorem 2.12.

Let \( E \) be a holomorphic vector bundle on the twistor space \( \text{Tw}(M) \). In this section we will be concerned with morphisms of the form \( F : E \to E(D) \), where \( D \) is a divisor on \( \text{Tw}(M) \). For any divisor \( D' \geq D \), the composition of \( F \) with the natural monomorphism \( E(D) \to E(D') \) will be denoted by the same letter \( F : E \to E(D') \), and we will think of it as essentially the same morphism. Using this idea, given two morphisms \( F_1 : E \to E(D_1) \), \( F_2 : E \to E(D_2) \), we can think of their sum \( F_1 + F_2 \) as a morphism of the form \( E \to E(D') \), where \( D' \) is any divisor containing both \( D_1 \) and \( D_2 \).

Recall from Proposition 2.11 that the twistor projection \( \pi : \text{Tw}(M) \to \mathbb{CP}^1 \) identifies divisors on \( \text{Tw}(M) \) with divisors on \( \mathbb{CP}^1 \), and thus in what follows we will denote the corresponding divisors by the same letter. In particular, the field of meromorphic functions on \( \text{Tw}(M) \) can be identified with \( K(\mathbb{CP}^1) \), the function field of \( \mathbb{CP}^1 \). Thus, given an element \( \eta \in K(\mathbb{CP}^1) \) with divisor of poles \( D' \), we can think of it as a holomorphic section of the line bundle \( \mathcal{O}_{\text{Tw}(M)}(D') \) on \( \text{Tw}(M) \), and vice versa. In this manner, given a morphism \( F : E \to E(D) \) on \( \text{Tw}(M) \), we can think of the product \( \eta \cdot F \) as a morphism \( \eta \cdot F : E \to E(D+D') \).

Now let \( E \) be an irreducible bundle on \( \text{Tw}(M) \). Taking the pushforward of the endomorphism bundle \( \text{End} \, E = E^* \otimes E \) by the twistor projection, \( \pi_* (\text{End} \, E) \) is a vector bundle, being a torsionfree sheaf on \( \mathbb{CP}^1 \), and hence holomorphically decomposes as a sum of line bundles by the Birkhoff-Grothendieck theorem. Note that \( E \) is simple bundle on \( \text{Tw}(M) \), in the sense that

\[
\text{Hom}_{\text{Tw}(M)}(E, E) = \mathbb{C}.
\]

This is because the irreducibility of \( E \) clearly implies that it is stable, and stable bundles are simple (see Theorem 1.2.9 in Chapter 2 of [OSS]). Hence we have

\[
H^0(\mathbb{CP}^1, \pi_* (\text{End} \, E)) = H^0(\text{Tw}(M), \text{End} \, E) = \text{Hom}_{\text{Tw}(M)}(E, E) = \mathbb{C}.
\]

It follows that in the direct sum decomposition of \( \pi_* (\text{End} \, E) \), there is exactly one summand of the form \( \mathcal{O}_{\mathbb{CP}^1} \), while all other summands (if any) are negative line bundles. It’s not hard to see that \( \pi_* (\text{End} \, E) = \mathcal{O}_{\mathbb{CP}^1} \) occurs precisely when \( E \) is generically fibrewise simple, while if it is not, \( \pi_* (\text{End} \, E) \) also has negative summands.

As noted above, the only endomorphisms \( E \to E \) of an irreducible \( E \) on \( \text{Tw}(M) \) are homotheties, but if we look at morphisms \( E \to E(D) \) for various divisors \( D \), we get more possibilities. Using the projection formula,

\[
\text{Hom}_{\text{Tw}(M)}(E, E(D)) = H^0(\text{Tw}(M), (\text{End} \, E)(D)) = H^0(\mathbb{CP}^1, [\pi_* (\text{End} \, E)](D)).
\]

With the description of \( \pi_* (\text{End} \, E) \) given in the previous paragraph, we see that if \( E \) is generically fibrewise simple, the only morphisms \( E \to E(D) \) are multiplications of \( \text{Id}_E \) by meromorphic functions from \( K(\mathbb{CP}^1) \), while if it is not, we can always find a morphism \( F : E \to E(D) \) for some \( D > 0 \) coming from a negative summand of \( \pi_* (\text{End} \, E) \), which will not be a multiplication by an element of \( K(\mathbb{CP}^1) \).
Definition 5.1. For any morphism \( F : E \to E(D) \) on \( \text{Tw}(M) \), the trace of \( F \), denoted \( \text{Tr} F \), is a global section of the line bundle \( \mathcal{O}_{\text{Tw}(M)}(D) \) determined by the composition

\[
\mathcal{O}_{\text{Tw}(M)} \to \text{End } E \to \mathcal{O}_{\text{Tw}(M)}(D),
\]

where the first map is induced by the identity morphism \( \text{Id}_E \) over the field \( K \), while the second map is induced by \( F \). The characteristic polynomial of \( F \), denoted \( \text{char}_F \), is a polynomial over the field \( K(\mathbb{CP}^1) \), which takes \( t \in K(\mathbb{CP}^1) \) to

\[
\text{char}_F(t) = \det (t \cdot \text{Id}_E - F) = \sum_{i=0}^{r} (-1)^i \text{Tr}(A^i F)t^{r-i},
\]

where we view \( \text{Tr}(A^i F) \in H^0(\text{Tw}(M), \mathcal{O}_{\text{Tw}(M)}(iD)) \) as an element of \( K(\mathbb{CP}^1) \). A root \( \eta \in K(\mathbb{CP}^1) \) of \( \text{char}_F(t) \) (if it exists) is called an eigenvalue of \( F \).

Since \( \text{char}_F \) is a polynomial with meromorphic functions as coefficients, evaluating these coefficients at any point \( x \in \text{Tw}(M) \setminus \text{Supp} D \) gives a polynomial \( \text{char}_F|_x \) over the field \( \mathbb{C} \). It’s not hard to see that \( \text{char}_F|_x \) is simply the characteristic polynomial of the linear map \( F_x : E_x \to E(D)_x \cong E_x \). Similarly, if \( \eta \in K(\mathbb{CP}^1) \) is an eigenvalue of \( \text{char}_F(t) \) as in the above definition, then at any \( x \in \text{Tw}(M) \) outside \( \text{Supp} D \) where \( \eta \) is defined, the evaluation \( \eta(x) \) is an eigenvalue of \( F_x \).

The main result of this section follows.

Theorem 5.2. Let \( M \) be a compact simply connected hyperkähler manifold, and let \( E \) be a holomorphic vector bundle on its twistor space \( \text{Tw}(M) \). If \( E \) is irreducible and the rank of \( E \) is prime, then \( E \) is generically fibrewise stable.

Proof. Let \( E \) be an irreducible bundle on \( \text{Tw}(M) \) and suppose \( \text{rk } E \) is a prime number. If \( E \) is generically fibrewise simple, then an application of Theorem [2,12] gives that \( E \) is generically fibrewise stable.

To prove by contradiction, we assume that \( E \) is not generically fibrewise simple.

As discussed in the beginning of this section, this implies that there exists a divisor \( D \) and a morphism \( F : E \to E(D) \), which is not a multiplication by a meromorphic function. Consider the characteristic polynomial of the morphism \( F \),

\[
\text{char}_F(t) = t^r + c_1 t^{r-1} + \ldots + c_{r-1} t + c_r.
\]

We can write

\[
\text{char}_F(t) = p_1(t)^{n_1} \cdots p_s(t)^{n_s}, \tag{5.1}
\]

where \( p_1(t), \ldots, p_s(t) \) are distinct irreducible polynomials over the function field \( K(\mathbb{CP}^1) \), and \( n_1, \ldots, n_s \) are positive integers. Plugging \( F \) into \( \text{char}_F(t) \), we get that

\[
\text{char}_F(F) = F^r + c_1 F^{r-1} + \ldots + c_{r-1} F + c_r.
\]

Here, the powers \( F^i \) are morphisms \( E \to E(iD) \); for example, \( F^2 \) is the composition

\[
E \xrightarrow{F} E(D) \xrightarrow{F(D)} E(2D),
\]

and similarly for higher powers. Recalling from the definition of \( \text{char}_F \) that the coefficients \( c_{r-1} \) are global sections of \( \mathcal{O}_{\text{Tw}(M)}([r-i]D) \), we see that \( \text{char}_F(F) \) is a well-defined morphism \( E \to E(rD) \). Over any point \( x \in \text{Tw}(M) \setminus \text{Supp} D \),

\[
\text{char}_F(F)|_x = \text{char}_{F_x}(F_x),
\]
which is zero by the Cayley-Hamilton theorem, and since $E(rD)$ has no torsion, we conclude that $\text{char}_F(F) : E \longrightarrow E(rD)$ is zero globally as well.

Recalling the decomposition (5.1), we can write the morphism $\text{char}_F(F) : E \longrightarrow E(rD)$ as the composition

$$E \xrightarrow{p_1(F)^{n_1}} E(D_1) \xrightarrow{p_2(F)^{n_2}} E(D_2) \longrightarrow \cdots \longrightarrow E(D_{s-1}) \xrightarrow{p_s(F)^{n_s}} E(D_s),$$

where $D_1, \ldots, D_s$ are some divisors. As noted above, this composition is zero. But since $E$ is irreducible, the only possible morphisms from $E$ to any torsionfree sheaf on $\text{Tw}(M)$ are monomorphisms and the zero morphism, and the same can be said about the vector bundles $E(D_1), \ldots, E(D_s)$. So one of the morphisms in the above composition must be zero. Rearranging the polynomials $p_i$ if necessary, we can assume that $p_1(F)^{n_1} = 0$. Writing $p_1(F)^{n_1}$ as the composition of the morphism $p_1(F)$ with itself $n_1$ times, and repeating the exact same argument, we conclude that $p_1(F) = 0$.

We now claim that if $p_1(F) = 0$ then $p_1(t)$ is the only irreducible polynomial in the decomposition (5.1), in other words, $s = 1$.

To prove the above claim, let $N \supseteq K(\mathbb{C}P^1)$ be a splitting field of $\text{char}_F(t)$. Note that

(i) each $p_i(t)$ splits into distinct linear factors over $N$, since it’s separable as we’re working in characteristic 0;
(ii) since $p_1(t), \ldots, p_s(t)$ are all distinct and irreducible, no two of them have a common linear factor over $N$.

What this means geometrically is that for any point $x \in \text{Tw}(M) \setminus \text{Supp } D$ outside a divisor, each restriction $p_1|_x(t), \ldots, p_s|_x(t)$ has no repeated roots as a polynomial over $\mathbb{C}$, and no two of them have a common root. Since each root of each $p_i|_x(t)$ is an eigenvalue of

$$F_x : E_x \longrightarrow E(D)_x \cong E_x,$$

the fact that

$$p_1|_x(F_x) = p_1(F)|_x = 0$$

implies that $F_x$ has no eigenvalues other than the roots of $p_1|_x(t)$. This means that there can be no $p_i$ other than $p_1$. This completes the proof of the claim.

Relabeling, the decomposition in (5.1) can be written as

$$\text{char}_F(t) = p(t)^n,$$  \hspace{1cm} (5.2)

where $p(t)$ is an irreducible polynomial over the field $K(\mathbb{C}P^1)$, and $n$ is a positive integer.

We now use the fact that the rank of $E$ is prime. Note that $\deg \text{char}_F(t) = \text{rk } E$, so we have

$$\text{rk } E = n \cdot \deg p(t).$$

There are two cases, which we consider separately.

**Case n = \text{rk } E.** In other words, $p(t)$ is a linear polynomial in $5.2$ of the form

$$p(t) = t - \eta,$$

where $\eta \in K(\mathbb{C}P^1)$ is some meromorphic function. But we know that $p(F) = 0$, so in this case $F = \eta$. This contradicts the fact that $F$ was chosen not to be a multiplication by a meromorphic function.
Case $n = 1$. In other words, $\text{char}_F(t) = p(t)$ is an irreducible polynomial in (5.2). We pass to the splitting field $N \supseteq K(\mathbb{CP}^1)$ of $\text{char}_F(t)$. As noted previously, over $N$ the polynomial $\text{char}_F(t)$ splits into distinct linear factors:

$$\text{char}_F(t) = (t - \eta_1)(t - \eta_2) \cdots (t - \eta_r), \quad \eta_i \in N .$$

(5.3)

Let $f : X \longrightarrow \mathbb{CP}^1$ be the unique branched cover corresponding to the field extension $K(\mathbb{CP}^1) \subseteq N$, where $X$ is a smooth curve. Consider the fibred product

$$
\begin{array}{ccc}
\text{Tw}(M)_X & \xrightarrow{\varphi} & \text{Tw}(M) \\
\pi_X \downarrow & & \downarrow \pi \\
X & \xrightarrow{f} & \mathbb{CP}^1
\end{array}
$$

(5.4)

Pulling back $E$ and the morphism $F : E \longrightarrow E(D)$ by $\varphi$, we get a morphism $\varphi^*F : \varphi^*E \longrightarrow \varphi^*E(\varphi^*D)$ on $\text{Tw}(M)_X$. It’s not hard to see that the characteristic polynomial of $\varphi^*F$ is given by (5.3), where this time we think of $\eta_1, \cdots, \eta_r$ as meromorphic functions on $\text{Tw}(M)_X$ coming from $X$. Look at the morphism

$$
\varphi^*E \xrightarrow{\varphi^*F - \eta_1} \varphi^*E(\tilde{D})
$$

where $\tilde{D}$ denotes a divisor on $\text{Tw}(M)_X$ containing $\varphi^*D$ and the poles of $\eta_1$. Since $\eta_1, \cdots, \eta_r$ are all distinct, they generically have distinct values on $\text{Tw}(M)_X$, and so for all $y$ in $\text{Tw}(M)_X$ outside a divisor, the eigenspace of the linear map

$$(\varphi^*F)_y : (\varphi^*E)_y \longrightarrow \left(\varphi^*E(\tilde{D})\right)_y \cong (\varphi^*E)_y$$

corresponding to the eigenvalue $\eta_1(y)$ has dimension one. It follows from this that the kernel of the above morphism $\varphi^*F - \eta_1$ is a sheaf of rank one. It is clearly torsionfree and since its cokernel is also torsionfree, it is normal (by Lemma 1.1.16 of Chapter 2 in [OSS], and hence it is a line bundle (see Lemma 1.1.12 and Lemma 1.1.15 of Chapter 2 in [OSS]). In short, we have a line subsheaf

$$L \hookrightarrow \varphi^*E$$

on $\text{Tw}(M)_X$.

We now use the results obtained in Section 4. By Proposition 4.2, we can write

$$L \cong \varphi^*L' \otimes \pi_X^*L'' ,$$

where $L'$ is some line bundle on $\text{Tw}(M)$ and $L''$ is some line bundle on $X$. Taking the pushforward of the sheaf monomorphism $L \cong \varphi^*L' \otimes \pi_X^*L'' \hookrightarrow \varphi^*E$ by $\varphi$, we have a sheaf monomorphism

$$\varphi_* (\varphi^*L' \otimes \pi_X^*L'') \hookrightarrow \varphi_* (\varphi^*E)$$

on $\text{Tw}(M)$. Applying the projection formula to the two sides, this morphism can be expressed in the following form:

$$L' \otimes \varphi_* (\pi_X^*L'') \hookrightarrow E \otimes \varphi_* (\mathcal{O}_{\text{Tw}(M)_X}) .$$

(5.5)

In the diagram (5.4) we have an isomorphism

$$\pi^*(f_*(\mathcal{F})) \cong \varphi_* (\pi_X^*(\mathcal{F}))$$
for any torsionfree sheaf on $X$ (see Theorem III.3.4 and its corollaries in [BS]). Taking $\mathcal{F}$ to be $L''$ and $\mathcal{O}_X$, and using the Birkhoff-Grothendieck theorem, we can write

$$\varphi_*(\pi_X^* L'') \cong \pi^*(f_*(L'')) \cong \pi^* \left( \bigoplus_{l=1}^d \mathcal{O}_{\mathbb{CP}^1}(A_l) \right) = \bigoplus_{l=1}^d \mathcal{O}_{\text{Tw}(M)}(A_l),$$

$$\varphi_*(\mathcal{O}_{\text{Tw}(M)X}) \cong \varphi_*(\pi_X^*(\mathcal{O}_X)) \cong \pi^*(\mathcal{O}_X) \cong \pi^* \left( \bigoplus_{l=1}^d \mathcal{O}_{\mathbb{CP}^1}(B_l) \right) = \bigoplus_{l=1}^d \mathcal{O}_{\text{Tw}(M)}(B_l),$$

where $A_1, \ldots, A_d, B_1, \ldots, B_d$ are some divisors on $\mathbb{CP}^1$. In view of this, we can write the morphism (5.5) as

$$L'(A_1) \oplus \cdots \oplus L'(A_d) \hookrightarrow E(B_1) \oplus \cdots \oplus E(B_d),$$

which we can think of as a $d \times d$ matrix of morphisms. Since this is a monomorphism, there must be some $1 \leq j \leq d, 1 \leq k \leq d$, such that the $(j, k)$-th constituent morphism

$$L'(A_k) \rightarrow E(B_j)$$

is nonzero, which contradicts the irreducibility of $E$.

In this way, both possible cases lead to contradictions. This means that we could not have chosen the morphism $F : E \rightarrow E(D)$ in the first place to be anything other than a multiplication by a meromorphic function. Hence $E$ must be generically fibrewise simple, and an application of Theorem 2.12 completes the argument. \hfill \Box

6. The moduli space of fibrewise stable bundles

The moduli space of fibrewise stable bundles on the twistor space $\text{Tw}(M)$ of a hyperkähler manifold $M$ can be interpreted in terms of rational curves in the twistor space of a certain dual variety $\hat{M}$. This identification of moduli spaces is due to Kaledin and Verbitsky [KV]. We present it here, and slightly extend it with a technical result which will be used in the next section. Given any complex analytic space $X$ with a morphism $X \rightarrow \mathbb{CP}^1$, the fibred product of $X \rightarrow \mathbb{CP}^1$ with the twistor projection $\pi : \text{Tw}(M) \rightarrow \mathbb{CP}^1$ will again be denoted by $\text{Tw}(M)_{X}$, as in the diagram

$$\begin{array}{ccc}
\text{Tw}(M)_X & \xrightarrow{\varphi} & \text{Tw}(M) \\
\pi_X \downarrow & & \downarrow \pi \\
X & \xrightarrow{f} & \mathbb{CP}^1
\end{array}$$

By analogy with $\text{Tw}(M)$, we will call a holomorphic bundle on $\text{Tw}(M)_X$ fibrewise stable if all its restrictions to the fibres of $\pi_X : \text{Tw}(M)_X \rightarrow X$ are stable. For the rest of this section, we fix a topological complex vector bundle $B$ on $M$ whose first two Chern classes $c_1(B), c_2(B)$ are $\text{SU}(2)$-invariant. Recalling that the twistor space $\text{Tw}(M)$ comes equipped with a nonholomorphic projection $\sigma : \text{Tw}(M) \rightarrow M$, we can take the pullback bundle $\sigma^*(B)$ on $\text{Tw}(M)$. Any holomorphic vector bundle on $\text{Tw}(M)$ that we consider in this section will be assumed to have underlying topological structure $\sigma^*(B)$. Similarly, holomorphic bundles on $\text{Tw}(M)_X$ will be assumed to have underlying topological structure $\varphi^*(\sigma^*(B))$.

Let $I \in \mathbb{CP}^1$ be any induced complex structure on $M$, and let $\tilde{M}$ denote the moduli space of stable holomorphic bundles on $M_I$ with underlying topological structure $B$. Although
\( \hat{M} \) need not be smooth or reduced, the complex analytic space structure on \( \hat{M} \) induces an almost complex structure on the real Zariski tangent spaces of points of \( \hat{M} \), which we denote by \( \hat{I} \).

Now let \( J \in \mathbb{CP}^1 \) be any other induced complex structure on \( M \). By Theorem 2.9, any stable bundle on \( M_J \) with underlying topological structure \( B \) is induced by a unique hyperholomorphic connection on \( B \). In turn, such a connection uniquely induces a holomorphic bundle on \( M_J \), which is stable since hyperholomorphic connections are Yang-Mills (Theorem 2.3 in [Ve2]). It follows from this that the underlying set of the moduli space of stable bundles on \( M_J \) can be identified with \( \hat{M} \), and we denote the corresponding almost complex structure on \( \hat{M} \) by \( \hat{J} \).

In [Ve2], it is shown that if \((I, J, K = IJ)\) is a quaternionic triple on \( M \), \((\hat{I}, \hat{J}, \hat{K} = \hat{I}\hat{J})\) will be a quaternionic triple on \( \hat{M} \), and moreover there exists a metric on the real Zariski tangent space of \( \hat{M} \), compatible with this quaternionic structure, which gives \( \hat{M} \) the structure of a singular hyperkähler variety; see [Ve2] for the precise definition and proof. Following [KV], we will call \( \hat{M} \) the Mukai dual of \( M \). In case \( M \) is a hyperkähler surface, it follows from the work of Mukai [Mu] that \( \hat{M} \) is actually smooth.

Although in general the Mukai dual \( \hat{M} \) is singular (and noncompact), one can still construct its twistor space, in the same way that we did for \( M \) in Section 2. The twistor space \( \text{Tw}(\hat{M}) \) is a complex analytic space parametrizing the induced complex structures at points of \( \hat{M} \); it is singular if \( \hat{M} \) is. To ease notation, in the rest of this section we will denote the Mukai dual twistor space \( \text{Tw}(\hat{M}) \) by \( \hat{Z} \). Just like the usual twistor space, \( \hat{Z} \) comes equipped with a holomorphic twistor projection

\[
\hat{\pi} : \hat{Z} \longrightarrow \mathbb{CP}^1,
\]

whose holomorphic sections will be called twistor lines. The set \( \text{Sec}(\hat{\pi}) \) of such twistor lines has the structure of a complex analytic space as a subset of the Douady space of rational curves in \( \hat{Z} \).

Now let \( \mathcal{M}_{\text{fib}}^s \) denote the set of fibrewise stable bundles on the original twistor space \( \text{Tw}(M) \). Since any such bundle is irreducible by Theorem 2.12, it is in particular stable, so we have a set-theoretic inclusion

\[
\mathcal{M}_{\text{fib}}^s \subset \mathcal{M}^s,
\]

where \( \mathcal{M}^s \) denotes the moduli space of stable holomorphic bundles on \( \text{Tw}(M) \). In fact, since stability is an open property, we know that \( \mathcal{M}_{\text{fib}}^s \) is an open subset of \( \mathcal{M}^s \), and thus it inherits from \( \mathcal{M}^s \) the structure of a complex analytic space. Let \( E \) be any element of \( \mathcal{M}_{\text{fib}}^s \). From the discussion above, for any \( I \in \mathbb{CP}^1 \), the moduli space of stable bundles on \( M_I \) has underlying set \( \hat{M} \). In this way, \( E \) defines a (set-theoretic) map

\[
\mathbb{CP}^1 \longrightarrow \hat{M} \quad I \mapsto E_I,
\]

where \( E_I \) is the restriction of \( E \) to the fibre \( \pi^{-1}(I) = M_I \), and this map in turn defines a (set-theoretic) section of the Mukai dual twistor projection \( \hat{\pi} : \hat{Z} \longrightarrow \mathbb{CP}^1 \). For example, it’s not hard to see that if \( E \) is hyperholomorphic, the resulting section is just a horizontal
A twistor line. In general, the section of \( \hat{\pi} \) induced by \( E \) will be holomorphic, and will thus be a twistor line. Furthermore, any twistor line can be obtained in this way from a unique \( E \), as the next result shows.

**Theorem 6.1.** Any fibrewise stable bundle on \( \text{Tw}(M) \) induces in a natural way a twistor line in the Mukai dual twistor space \( \hat{Z} = \text{Tw}(\hat{M}) \), and the resulting map

\[
\mathcal{M}_{\text{fib}}^s \xrightarrow{\cong} \text{Sec}(\hat{\pi})
\]

is an isomorphism of complex analytic spaces. Moreover, there exists an open cover \( \{U_\alpha\} \) of \( \hat{Z} \), together with holomorphic vector bundles \( E_\alpha \) on the corresponding open neighborhoods \( \hat{\pi}^{-1}(U_\alpha) \subseteq \text{Tw}(M) \hat{Z} \), with the following property: for any complex analytic space \( X \), morphism \( X \rightarrow \mathbb{CP}^1 \) and fibrewise stable bundle \( F \) on \( \text{Tw}(M)_X \), there exists a unique map \( g : X \rightarrow \hat{Z} \) over \( \mathbb{CP}^1 \), such that, for every index \( \alpha \),

\[
F \cong \psi^*E_\alpha \text{ over } \pi_1^{-1}(g^{-1}(U_\alpha)),
\]

where the map \( \psi : \text{Tw}(M)_X \rightarrow \text{Tw}(M) \hat{Z} \) is induced by \( g \), as in the diagram

\[
\begin{array}{cccc}
\text{Tw}(M)_X & \xrightarrow{\psi} & \text{Tw}(M) \hat{Z} & \xrightarrow{\hat{\pi}} \mathbb{CP}^1 \\
\pi_X \downarrow & & \pi \downarrow & \\
X & \xrightarrow{g} & \hat{Z} & \xrightarrow{\hat{\pi}} \mathbb{CP}^1
\end{array}
\]

**Proof.** See Section 7 of [KV]. \( \square \)

In other words, \( \hat{Z} \) is a coarse moduli space of fibrewise stable bundles on \( \text{Tw}(M) \) which locally admits a universal family \( E_\alpha \), but these local universal families \( \{E_\alpha\} \) need not come from a single global universal family.

Now look at the special case that \( f : X \rightarrow \mathbb{CP}^1 \) is a branched cover of \( \mathbb{CP}^1 \) by a smooth curve \( X \). By Theorem 6.1 above, a fibrewise stable bundle on the fibred product \( \text{Tw}(M)_X \) gives rise to a *multisection* of the Mukai dual twistor projection \( \hat{\pi} : \hat{Z} \rightarrow \mathbb{CP}^1 \) over \( X \), i.e., a morphism

\[
\begin{array}{c}
\hat{Z} \\
\hat{\pi} \downarrow
\end{array} \xrightarrow{f} \mathbb{CP}^1
\]

We thus get a map

\[
\mathcal{M}_{\text{fib},X}^s \rightarrow \text{Sec}_X(\hat{\pi}),
\]

(6.2)

where \( \mathcal{M}_{\text{fib},X}^s \) denotes the moduli space of fibrewise stable bundles on \( \text{Tw}(M)_X \), and \( \text{Sec}_X(\hat{\pi}) \) the space of multisections of \( \hat{\pi} \) over \( X \). In contrast to the result of Theorem 6.1, this map need not be injective. However, it is surjective, as the following lemma shows.

**Lemma 6.2.** Let \( g : X \rightarrow \hat{Z} \) be a multisection of the Mukai dual twistor projection \( \hat{\pi} : \hat{Z} \rightarrow \mathbb{CP}^1 \). There exists a fibrewise stable bundle on \( \text{Tw}(M)_X \) which gets mapped to \( g \) via the map (6.2).
Proof. The argument is the same as in the proof of Lemma 7.5 in [KV]. The map \( g : X \to \hat{Z} \) induces a map \( \psi : \text{Tw}(M)_X \to \text{Tw}(M)_{\hat{Z}} \) of fibred products, as in the diagram \((6.1)\).

From Theorem \ref{thm:6.1}, we know that there are open sets \( U_\alpha \subseteq \hat{Z} \) covering \( \hat{Z} \), and universal bundles \( \mathcal{E}_\alpha \) on the corresponding open neighborhoods \( \pi_{\hat{Z}}^{-1}(U_\alpha) \subseteq \text{Tw}(M)_{\hat{Z}} \). Passing from \( \hat{Z} \) to \( X \), we have the open sets \( g^{-1}(U_\alpha) \subseteq X \), and the pullbacks of the bundles \( \mathcal{E}_\alpha \) by \( \psi \) on the corresponding open neighborhoods \( \pi_X^{-1}(g^{-1}(U_\alpha)) \subseteq \text{Tw}(M)_X \). For simplicity, we will denote the preimage neighborhood \( g^{-1}(U_\alpha) \) by \( U_\alpha \) as well, and the pullback of the bundle \( \mathcal{E}_\alpha \) again by \( \mathcal{E}_\alpha \). If we can somehow glue these \( \mathcal{E}_\alpha \) into a bundle on \( \text{Tw}(M)_X \), it’s clear that it will be fibrewise stable, and that it will be mapped to \( g \) via the map \((6.2)\).

By compactness of \( X \), we can choose a finite sub-cover \( U_1, \ldots, U_n \) of \( \{U_\alpha\} \). The corresponding bundles \( \mathcal{E}_1, \ldots, \mathcal{E}_n \) are isomorphic on overlaps by their universal property. In other words, for any \( 1 \leq i, j \leq n \), we have isomorphisms
\[
\begin{align*}
\phi_{ij} : \mathcal{E}_j|_{\pi_X^{-1}(U_i \cap U_j)} & \cong \mathcal{E}_i|_{\pi_X^{-1}(U_i \cap U_j)}.
\end{align*}
\]

For any \( i, j, k \), the composition
\[
\phi_{ij} \circ \phi_{jk} \circ \phi_{ki} : \mathcal{E}_j|_{\pi_X^{-1}(U_i \cap U_j \cap U_k)} \cong \mathcal{E}_i|_{\pi_X^{-1}(U_i \cap U_j \cap U_k)}
\]
need not equal the identity map. However, for any point \( P \in U_i \cap U_j \cap U_k \), the restriction \( \mathcal{E}_i|_{\pi_X^{-1}(P)} \) is a stable bundle, hence in particular simple, i.e.,
\[
\text{Hom}\left( \mathcal{E}_i|_{\pi_X^{-1}(P)} : \mathcal{E}_i|_{\pi_X^{-1}(P)} \right) = \mathbb{C}.
\]

It follows from this that we have
\[
\phi_{ij} \circ \phi_{jk} \circ \phi_{ki} = \theta_{ijk} \text{Id}_{\mathcal{E}_i}, \quad \theta_{ijk} \in \mathcal{O}_X^*(U_i \cap U_j \cap U_k).
\]

The collection \( \{\theta_{ijk}\} \) is a Čech 2-cocycle defining an element of the cohomology group \( H^2(X, \mathcal{O}_X^*) \). Thus the bundles \( \mathcal{E}_i \) together with the isomorphisms \( \phi_{ij} \) define a twisted sheaf on \( \text{Tw}(M)_X \) in the sense of Căldăraru [Ca], and it’s not hard to verify that the \( \mathcal{E}_i \) glue into an actual sheaf if and only if the element of \( H^2(X, \mathcal{O}_X^*) \) defined by the collection \( \{\theta_{ijk}\} \) is zero. But since \( X \) is a curve, \( H^2(X, \mathcal{O}_X^*) = 0 \), as can be seen from the following portion of the long exact cohomology sequence of the exponential sheaf sequence of \( X \):
\[
\ldots \to H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X^*) \to H^3(X, \mathbb{Z}) \to \ldots
\]

This completes the proof. \( \square \)

7. Irreducible bundles on \( \text{Tw}(M) \) of composite rank

We note that Theorem \ref{thm:5.2} gives some hope that the full converse of Theorem \ref{thm:2.12} might be true, in other words, that an arbitrary irreducible bundle \( E \) on the twistor space \( \text{Tw}(M) \) of a compact simply connected hyperkähler manifold \( M \) is generically fibrewise stable. This, however, turns out not to be true, and in this section we will construct examples of irreducible but nowhere fibrewise stable bundles on \( \text{Tw}(M) \) of any composite rank. We will carry out the construction on the twistor space of a K3 surface.

Recall that a K3 surface is a compact simply connected complex surface \( M \) with trivial canonical bundle. A nonzero holomorphic section of the canonical bundle of \( M \) is a holomorphic symplectic form, making it into a Kähler holomorphic symplectic manifold. As a
consequence of Yau’s theorem proving Calabi’s conjecture, \cite{Ya}, the manifold \( M \) admits a hyperkähler structure.

The main result of this section follows.

**Theorem 7.1.** Let \( M \) be a K3 surface. There exist examples of irreducible but nowhere fibrewise stable bundles of any composite rank on its twistor space \( \text{Tw}(M) \).

Before going ahead with the proof, we give a concise overview of the argument. The construction will be carried out in the following three steps:

1. Given any composite number, we write it as a product \( dr \), where \( d \) is prime. We choose a topological complex vector bundle \( B \) on \( M \) of rank \( r \) that admits stable structures in every induced complex structure of \( M \), so that the corresponding Mukai dual variety \( \hat{M} \) is nonempty.

2. We choose a branched cover of \( \mathbb{CP}^1 \) by a smooth curve \( f : X \rightarrow \mathbb{CP}^1 \), in such a way that the Mukai dual twistor projection \( \hat{\pi} : \hat{Z} = \text{Tw}(\hat{M}) \rightarrow \mathbb{CP}^1 \) admits a multisection over \( X \) which does not come from a twistor line in \( \hat{Z} \). Applying Lemma \ref{lemma:multisection}, this gives rise to a fibrewise stable bundle \( F \) on the fibred product \( \text{Tw}(M)_X \), as in the diagram

\[
\begin{array}{ccc}
\text{Tw}(M)_X & \xrightarrow{\varphi} & \text{Tw}(M) \\
\downarrow{\pi_X} & & \downarrow{\pi} \\
X & \xrightarrow{f} & \mathbb{CP}^1
\end{array}
\]

The bundle \( F \) will have the property that for generic \( I \in \mathbb{CP}^1 \), the restrictions of \( F \) to the fibres of \( \pi_X \) corresponding to distinct elements in \( f^{-1}(I) \) are nonisomorphic as bundles on \( M_I \).

3. Taking the pushforward \( E := \varphi_*F \) by the map \( \varphi \) in the diagram above, we show that \( E \) is an irreducible bundle on \( \text{Tw}(\hat{M}) \) of rank \( dr \) which is nowhere fibrewise stable.

**Proof of Theorem 7.1.** Take any composite number, and write it as a product \( dr \), where \( d \) is some prime number. We fix once and for the rest of the proof a branched cover \( f : X \rightarrow \mathbb{CP}^1 \) as follows. Let \( X = \mathbb{CP}^1 \), and choose any local coordinate \( z \) about any point in \( \mathbb{CP}^1 \). The map \( f : X = \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \) is given by \( f(z) = z^d \).

**Step 1.** We first choose a topological complex vector bundle on \( M \) of rank \( r \) that admits stable holomorphic structures in every induced complex structure of \( M \). Fix \( I \in \mathbb{CP}^1 \). Using Serre’s construction, one can show that there exists a stable bundle of rank \( r \) on \( M_I \) with first Chern class zero (see Theorem 5.1.6 in \cite{HL}). By Theorem 2.9 we know that such bundle comes from a unique hyperholomorphic connection \( \nabla \) on its underlying topological bundle \( B \). Since hyperholomorphic connections are Yang-Mills (Theorem 2.3 in \cite{Ve2}), \( \nabla \) gives rise to stable holomorphic structures on \( B \) in all other induced complex structures of \( M \). It follows from this that the Mukai dual variety \( \hat{M} \) associated to \( B \) is nonempty. As was mentioned in the previous section, \( \hat{M} \) is smooth since \( M \) is a surface, and is thus a (noncompact) hyperkähler manifold. As in the previous section, let \( \hat{Z} = \text{Tw}(\hat{M}) \) denote the twistor space of \( \hat{M} \), and \( \hat{\pi} : \hat{Z} \rightarrow \mathbb{CP}^1 \) its holomorphic twistor projection.
Step 2. Recall that sections of the twistor projection $\hat{\pi} : \hat{Z} \to \mathbb{CP}^1$ are called twistor lines, and the set of twistor lines in $\hat{Z}$ is denoted by $\text{Sec}(\hat{\pi})$. We will also be interested in the multisections of $\hat{\pi} : \hat{Z} \to \mathbb{CP}^1$ over $f : X \to \mathbb{CP}^1$, and the set of such multisections will be denoted by $\text{Sec}_X(\hat{\pi})$. Viewed as Douady spaces of morphisms, both $\text{Sec}(\hat{\pi})$ and $\text{Sec}_X(\hat{\pi})$ have a complex analytic structure, and composition with $f : X \to \mathbb{CP}^1$ induces an analytic map

$$\text{Sec}(\hat{\pi}) \to \text{Sec}_X(\hat{\pi})$$

(7.1)

We would like to show that this map is not surjective. We do this by examining the induced natural short exact sequence of holomorphic bundles on $\hat{Z}$:

$$0 \to T_{\hat{\pi}}^{1,0} \to T^{1,0}\hat{Z} \xrightarrow{d\hat{\pi}} \hat{\pi}^*T^{1,0}\mathbb{CP}^1 \to 0.$$  

Here $T_{\hat{\pi}}^{1,0}$ is the relative holomorphic tangent bundle for the projection $\hat{\pi}$, and $d\hat{\pi}$ is the differential of $\hat{\pi}$. Pulling this sequence back to $\mathbb{CP}^1$ via $s : \mathbb{CP}^1 \to \hat{Z}$, we get:

$$0 \to s^*T_{\hat{\pi}}^{1,0} \to s^*T^{1,0}\hat{Z} \xrightarrow{s^*d\hat{\pi}} T^{1,0}\mathbb{CP}^1 \to 0.$$  

(7.2)

The Zariski tangent space of $\text{Sec}(\hat{\pi})$ at $[s]$ can be identified with the space of global sections of the bundle $s^*T_{\hat{\pi}}^{1,0}$,

$$T_{[s]}\text{Sec}(\hat{\pi}) \cong H^0 \left( \mathbb{CP}^1, s^*T_{\hat{\pi}}^{1,0} \right)$$

(see Section 2.3 in [De]), and similarly,

$$T_{[s\circ f]}\text{Sec}_X(\hat{\pi}) \cong H^0 \left( X, f^* \left( s^*T_{\hat{\pi}}^{1,0} \right) \right).$$

We now describe the structure of the vector bundle $s^*T_{\hat{\pi}}^{1,0}$ on $\mathbb{CP}^1$. Since $T\mathbb{CP}^1 \cong \mathcal{O}_{\mathbb{CP}^1}(2)$ and the normal bundle of the twistor line $s : \mathbb{CP}^1 \to \hat{Z}$ is isomorphic to $\mathcal{O}_{\mathbb{CP}^1}(1)^{\oplus n}$, where $n$ is the complex dimension of $\hat{M}$ (see Theorem 1(2) in [Hit]), the sequence (7.2) has the form

$$0 \to s^*T_{\hat{\pi}}^{1,0} \to \mathcal{O}_{\mathbb{CP}^1}(2) \oplus \mathcal{O}_{\mathbb{CP}^1}(1)^{\oplus n} \to \mathcal{O}_{\mathbb{CP}^1}(2) \to 0.$$  

Here the $\mathcal{O}_{\mathbb{CP}^1}(2)$ term in the middle gets mapped identically to $\mathcal{O}_{\mathbb{CP}^1}(2)$ on the right. It follows from this that $s^*T_{\hat{\pi}}^{1,0} \cong \mathcal{O}_{\mathbb{CP}^1}(1)^{\oplus n}$, and recalling that $X = \mathbb{CP}^1$ and that $f : X \to \mathbb{CP}^1$ is a degree $d$ map, we have $f^* \left( s^*T_{\hat{\pi}}^{1,0} \right) \cong \mathcal{O}_{\mathbb{CP}^1}(d)^{\oplus n}$. In view of this, the map of Zariski tangent spaces

$$T_{[s]}\text{Sec}(\hat{\pi}) \to T_{[s\circ f]}\text{Sec}_X(\hat{\pi})$$

induced by the morphism (7.1) takes the form

$$H^0 \left( \mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(1)^{\oplus n} \right) \to H^0 \left( X, \mathcal{O}_{\mathbb{CP}^1}(d)^{\oplus n} \right).$$

Since $d > 1$, we see that this map cannot be surjective.
It follows from this that the multisection \( s \circ f : X \to \hat{Z} \) can be deformed to a multisection \( g : X \to \hat{Z} \), as in the diagram

\[
\begin{array}{c}
\hat{Z} \\
\uparrow g \\
X \\
\downarrow f \\
\hat{Z} \\
\end{array}
\]

that does not factor through \( f \), that is, does not come from from a twistor line in \( \hat{Z} \). Since the degree of \( f \) is a prime number \( d \), it follows that the map \( g \) must be injective. Applying Lemma 6.2 to \( g \), we get a fibrewise stable bundle \( F \) of rank \( r \) on the fibred product \( \text{Tw}(M)_X \), as in the diagram

\[
\begin{array}{c}
\text{Tw}(M)_X \\
\downarrow \pi_X \\
X \\
\downarrow f \\
\text{CP}^1. \\
\end{array}
\]

By choice of \( g \), the bundle \( F \) will have the property that for generic \( I \in \text{CP}^1 \), the restrictions of \( F \) to the fibres of \( \pi_X \) corresponding to distinct elements in \( f^{-1}(I) \) are nonisomorphic stable bundles on \( M_I \) of degree zero.

**Step 3.** Let \( E = \varphi_* F \) be the pushforward of \( F \) by \( \varphi \) in the diagram (7.3). First, observe that \( E \) is locally free. Being a local statement on \( \text{Tw}(M) \), this follows from the fact that \( \varphi_* \mathcal{O}_{\text{Tw}(M)_X} \) is locally free, which itself follows from the fact that \( \varphi \) is proper and finite. Thus, \( E \) is a vector bundle, and it has rank \( dr \), since the degree of the map \( f : X \to \text{CP}^1 \) is \( d \), and the rank of \( F \) is \( r \). Second, the vector bundle \( E \) on \( \text{Tw}(M) \) is nowhere fibrewise stable. Indeed, for any \( I \in \text{CP}^1 \) outside the branch locus of \( f \), we know from the construction of \( F \) that the restriction \( E_I = E|_{\pi^{-1}(I)} \) decomposes as

\[
E_I \cong E_1 \oplus \cdots \oplus E_d ,
\]

where the \( E_i \) correspond to restrictions of \( F \) over points in \( f^{-1}(I) \), and are (nonisomorphic) stable bundles on \( M_I \) of rank \( r \) and degree zero. This shows that \( E_I \) is non-stable for all \( I \) outside a finite subset, and since non-stability is a closed property, it follows that the same must be true for all \( I \in \text{CP}^1 \). We emphasize that, unlike \( E_I \), the bundles \( E_i \) in the decomposition (7.4) live only on \( M_I \) and cannot be extended to the whole \( \text{Tw}(M) \).

It remains to show that \( E \) is irreducible as a bundle on \( \text{Tw}(M) \). Let \( G \subset E \) be any subsheaf of lower rank; we can assume that \( G \) is reflexive. Notice that the restriction \( G_I \) of \( G \) to any fibre \( \pi^{-1}(I) \subset \text{Tw}(M) \) has degree zero. This is certainly true for generic \( I \) in the sense of Definition 2.5, hence by continuity it is true for all \( I \in \text{CP}^1 \). Recalling that \( \dim_C \text{Tw}(M) = 3 \) since \( M \) is a surface, and that for reflexive sheaves the codimension of the singular locus is at least three (Lemma 1.1.10 in Chapter 2 of [OSS]), we see that \( G \) is a vector bundle outside a finite subset of \( \text{Tw}(M) \). Let

\[
\Delta = \text{Branch locus of } f \cup \text{Singularity set of } G.
\]

Fix \( I \in \text{CP}^1 \setminus \Delta \). The restriction of the sheaf inclusion \( G \subset E \) to the fibre \( \pi^{-1}(I) = M_I \) is a sheaf monomorphism

\[
G_I \hookrightarrow E_1 \oplus \cdots \oplus E_d ,
\]

(7.5)
where we have used the decomposition (7.4). Notice that both $G_I$ and the $E_i$ are vector bundles here.

We claim that there exists a choice of a subset $\{i_1, \cdots, i_t\} \subseteq \{1, \cdots, d\}$ such that the composition of (7.5) with the corresponding projection,

$$G_I \longrightarrow E_1 \oplus \cdots \oplus E_d \longrightarrow E_{i_1} \oplus \cdots \oplus E_{i_t},$$

is generically an isomorphism.

To prove the above claim, let $1 \leq j_1 \leq d$ be arbitrary, and look at the composition

$$G_I \longrightarrow E_1 \oplus \cdots \oplus E_d \longrightarrow \bigoplus_{l \neq j_1} E_l = E_1 \oplus \cdots \oplus \bigoplus_{l \neq j_1} E_l = E_j.$$ 

Let $K_{j_1}$ denote the kernel of this composition. We have the following diagram with exact rows:

$$
\begin{array}{cccccccc}
0 & \longrightarrow & K_{j_1} & \longrightarrow & G_I & \longrightarrow & \bigoplus_{l \neq j_1} E_l & \longrightarrow & 0 \\
& & \downarrow \gamma & & \downarrow \gamma & & \downarrow \gamma & & \\
0 & \longrightarrow & E_{j_1} & \longrightarrow & E_1 \oplus \cdots \oplus E_d & \longrightarrow & \bigoplus_{l \neq j_1} E_l & \longrightarrow & 0 \\
\end{array}
$$

Just like $G_I$, the sheaf $K_{j_1}$ has degree zero. Indeed, if we had $\deg K_{j_1} > 0$, then the composition $K_{j_1} \subset G_I \subset E_1 \oplus \cdots \oplus E_d$ would destabilize the polystable vector bundle $E_1 \oplus \cdots \oplus E_d$, while if $\deg K_{j_1} < 0$, the fact that $\deg G_I = 0$ would imply that the image of $G_I$ under the rightmost map in the first row of the above diagram would have positive degree, thus destabilizing $\bigoplus_{l \neq j_1} E_l$. Since $E_{j_1}$ is stable and also has degree zero, the condition $\deg K_{j_1} = 0$ implies that the left-most vertical arrow in the above diagram is either zero or generically an isomorphism. If the latter is true for every $j_1$ from 1 to $d$, then $rk G_I = rk [E_1 \oplus \cdots \oplus E_d]$, but this cannot be as $G$ was chosen to be a subsheaf of $E$ on $\text{Tw}(M)$ of lower rank. Fixing an index $j_1$ such that $K_{j_1} = 0$, the composition

$$G_I \longrightarrow E_1 \oplus \cdots \oplus E_d \longrightarrow \bigoplus_{l \neq j_1} E_l$$

must be a monomorphism. If $rk G_I = rk \bigoplus_{l \neq j_1} E_l$, we stop here. If not, we repeat the argument above with $\{1, \cdots, d\}$ replaced by $\{1, \cdots, \widehat{j_1}, \cdots, d\}$ to conclude the existence of an index $j_2 \in \{1, \cdots, \widehat{j_1}, \cdots, d\}$ such that the composition

$$G_I \longrightarrow \bigoplus_{l \neq j_1} E_l \longrightarrow \bigoplus_{l \neq j_1, j_2} E_l$$

is still a monomorphism. At a certain point, after having chosen some indices $j_1, j_2, \cdots, j_s$ in this manner, and letting $i_1, i_2, \cdots, i_t$ denote the other indices, we will arrive at a monomorphism $G_I \hookrightarrow E_{i_1} \oplus \cdots \oplus E_{i_t}$ with $rk G_I = rk [E_{i_1} \oplus \cdots \oplus E_{i_t}]$.

If $t = 0$, then $G_I = 0$, and so $G = 0$. If this happens for an arbitrary choice of a subsheaf $G \subset E$, then clearly $E$ must be irreducible. Assume for contradiction that $t > 0$.

We thus have that for some subset $\{i_1, \cdots, i_t\} \subseteq \{1, \cdots, d\}$, the map

$$G_I \longrightarrow E_{i_1} \oplus \cdots \oplus E_{i_t}$$ 

(7.6)
obtained as the composition of (7.5) with the corresponding projection is a monomorphism of sheaves with the property that \( \text{rk} G_I = \text{rk} [E_{i_1} \oplus \cdots \oplus E_{i_t}] \). But since also \( \text{deg} G_I = \text{deg} [E_{i_1} \oplus \cdots \oplus E_{i_t}] = 0 \), it must be that the corresponding map of line bundles
\[
\det G_I \longrightarrow \det (E_{i_1} \oplus \cdots \oplus E_{i_t})
\]
is an isomorphism. Recalling that both \( G_I \) and \( E_{i_1} \oplus \cdots \oplus E_{i_t} \) are vector bundles, it follows that (7.6) must also be an epimorphism, and hence an isomorphism. This proves the claim.

Identifying \( G_I \) with \( E_{i_1} \oplus \cdots \oplus E_{i_t} \) on \( M_I \) in the morphism (7.5), we also have that for any \( j \) in \( \{1, \cdots, d\} \backslash \{i_1, \cdots, i_t\} \), the composition
\[
G_I \cong E_{i_1} \oplus \cdots \oplus E_{i_t} \longrightarrow E_1 \oplus \cdots \oplus E_d \longrightarrow E_j
\]
is zero. This follows from the fact that by construction, the bundles \( E_1, \cdots, E_d \) are all stable on \( M_I \) of the same rank and degree, and are pairwise nonisomorphic.

It follows from all this that the choice of a subset \( \{i_1, \cdots, i_t\} \subseteq \{1, \cdots, d\} \) as described above is uniquely determined by the morphism (7.5). Moreover, it’s not hard to see that in a neighborhood \( U \) of \( I \) in \( \mathbb{C}P^1 \backslash \Delta \) which is evenly covered by the map \( f : X \longrightarrow \mathbb{C}P^1 \), carrying out the same procedure for every other point in \( U \) yields the same choice of subset of \( \{1, \cdots, d\} \). Thus, a nonzero subsheaf \( G \subset E \) gives a consistent choice of \( t \) sheets of the covering \( f : X \longrightarrow \mathbb{C}P^1 \) for every point in the nonempty Zariski open set \( \mathbb{C}P^1 \backslash \Delta \), which contradicts the fact that \( X = \mathbb{C}P^1 \) is connected. We must have that the only subsheaf of \( E \) of lower rank on \( \text{Tw}(M) \) is the zero subsheaf, i.e., \( E \) is irreducible. \( \square \)

**Acknowledgements**

The authors would like to thank Ajneet Dhillon, Jacques Hurtubise and Misha Verbitsky for the many valuable discussions and suggestions. The study of the second author has been funded within the framework of the HSE University Basic Research Program and the Russian Academic Excellence Project ’5-100’. The first author is supported by a J. C. Bose Fellowship.

**References**

[ABT] M. Aprodu, V. Brînzănescu and M. Toma, Holomorphic vector bundles on primary Kodaira surfaces, *Math. Zeit.* **242** (2002), 63–73.

[At] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181–207.

[AHS] M. F. Atiyah, N. Hitchin and I. Singer, Self-duality in four-dimensional Riemannian geometry, *Proc. Roy. Soc. London Ser. A* **362** (1978), 425–461.

[BS] C. Bănică, O. Stănășilă, *Algebraic methods in the global theory of complex spaces*, John Wiley & Sons (1976).

[BM] V. Brînzănescu and R. Moraru, Holomorphic rank-2 vector bundles on non-Kähler elliptic surfaces, *Ann. Inst. Fourier* **55** (2005), 1659–1683.

[Ca] A. Căldăraşu, Derived categories of twisted sheaves on Calabi-Yau manifolds, PhD thesis, Cornell University (2000).

[De] O. Debarre, *Higher-Dimensional Algebraic Geometry*, Springer Verlag, Universitext (2001).

[GR] H. Grauert and R. Remmert, *Coherent Analytic Sheaves*, Springer Verlag, Berlin (1984).

[Hit] N. J. Hitchin, Hyperkähler manifolds, *Astérisque* **206** (1992), 137–166.

[HL] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Cambridge University Press, (2010).
D. Kaledin and M. Verbitsky, Non-Hermitian Yang-Mills connections, Selecta Math. 4 (1998), 279–320.

S. Kobayashi, Differential geometry of complex vector bundles, Publications of the Mathematical Society of Japan. 15. Kanô Memorial Lectures, 5. Princeton University Press, Princeton, NJ, 1987.

M. Maruyama, Openness of a family of torsion free sheaves, J. Math. Kyoto Univ. 16 (1976), 627–637.

S. Mukai, Symplectic structure of the moduli space of sheaves on abelian or K3 surfaces, Invent. Math. 77 (1984), 101–116.

C. Okonek, M. Schneider and H. Spindler, Vector bundles on complex projective spaces, Progress in Mathematics vol. 3, Birkhäuser, (1980).

R. Penrose, Twistor algebra, Jour. Math. Phys. 8 (1967), 345–366.

S. Salamon, Quaternionic Kähler manifolds, Inv. Math. 67 (1982), 143–171.

A. Teleman and M. Toma, Holomorphic vector bundles on non-algebraic surfaces, Com. Ren. Acad. Sci. Paris 334 (2002), 1–6.

M. Toma, Stable bundle with small $c_2$ over 2-dimensional complex tori, Math. Zeit. 232 (1999), 511–525.

A. Tomberg, On the Geometry of Twistor Spaces of Hypercomplex and Hyperkähler Manifolds, PhD thesis, McGill University (2018).

A. Tomberg, Example of a Stable but Fiberwise Nonstable Bundle on the Twistor Space of a Hyper-Kähler Manifold, Math. Notes 105 no. 6 (2019), 941–945.

A. Tomberg, Families of stable bundles on the fibres of the hyperkähler twistor projection, arXiv:1908.05333 [math.AG] (2019).

M. Verbitsky, Hyperkähler embeddings and holomorphic symplectic geometry II, Geom. Funct. Anal. 5 (1995), 92–104.

M. Verbitsky, Hyperholomorphic bundles over a hyperkähler manifold, Journ. Alg. Geom. 5 (1996), 633–669.

S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I, Comm. on Pure and Appl. Math. 31 (1978), 339–411.