$D$-dimensional Eckart+deformed Hylleraas potential: 
Bound state solutions

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Abstract. By using an exactly-solvable multiparameter exponential-type potential, the 
bound-state solutions of the $D$-dimensional Eckart + Hylleraas potential are directly derived as
a particular case. Although our proposal accepts different approximations to the centrifugal
term, its usefulness is exemplified in the frame of the Taskin and Kocak approach. This
fact enables us to compare our results with specific potentials found in literature which are
obtained here as particular cases from our proposal. That is, instead of solving an specific
exponential-type potential, by resorting each time to a specialized method, the energy spectra
and wavefunctions are found straightforwardly by means of the simple choice of the involved
parameters. Furthermore, our proposal can be used as alternative way in the search of
bound-state solutions to new exponential-type potentials besides that one can study different
approximations to the term $1/r^2$.

PACS: 0.365Ge, 03.65.Fd, 03.65Ca.
Keywords: Point canonical transformation, Schrödinger equation, Exponential-type 
potentials. Bound states.

1. Introduction

The search of analytical bound state solutions of the Schrödinger equation for exponential-type
potentials, has been of great interest as shown by many cases of specific potentials studied by
means of different approaches to the centrifugal term and by using various methods such as
the Nikiforov-Uvarov [1], asymptotic iteration [2], supersymmetric quantum mechanics [3], the
path integral approach [4], numerical calculations [5] and many others. In particular, the bound
state solutions of the $D$--dimensional Eckart-Hylleraas potential has been recently obtained
by using the Nikiforov-Uvarov method [6]. Such a procedure, is one of the most popular
approaches extensively used with many other specific exponential-type potentials such as the
mixed potentials of Deng-Fan + Eckart [7], Hylleraas + Rosen Morse [8] or by other different
approaches to the five-parameter exponential-type potential model of Chun-Sheng et al [9]

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multiparameter exponential-type potential adapted to the $D-$dimensions case in the frame of the Taskin and Kocak approximation to the centrifugal term [11] such as shown in section III. In section IV, as worked examples, we consider some particular cases directly obtained from the proposal by selecting specific values of the involved parameters. As will be shown, instead of solving a specific exponential-type potential by means of a specialized solution method, a mixed exponential-type potential such as the one considered in this work has the advantage that it can be used to find the energy spectra and eigenfunctions of single potentials in $D-$dimensions.

II. Schrödinger equation in $D$-dimensions.

The Schrödinger equation in $D$-dimensions is given, in natural units ($\hbar^2 = 1$), by [12]

$$
\left[r^{1-D} \frac{\partial}{\partial r}(r^{D-1} \frac{\partial}{\partial r}) - \frac{\Lambda^2(\Omega)}{r^2} + E - V(r)\right] \psi_{n\ell m}(r, \Omega) = 0
$$

(1)

where $\psi_{n\ell m}(r, \Omega)$ is the solution in the $D$-dimensional coordinates $(r, \Omega) = (r, \theta_1, \theta_2, \theta_3, \ldots \theta_{D-2}, \phi)$ and $\Lambda^2(\Omega)$ is the $D$-dimensional angular momentum operator. Assuming the separation of variables $\psi_{n\ell m}(r, \Omega) = R_{n\ell}(r)Y^m_l(\Omega)$ Eq.(1) is rewritten as

$$
-\frac{d^2R_{n\ell}(r)}{dr^2} - \frac{D-1}{r} \frac{dR_{n\ell}(r)}{dr} + \frac{\ell_D(\ell_D + 1)}{r^2}R_{n\ell}(r) + V(r)R_{n\ell}(r) = E R_{n\ell}(r)
$$

(2)

such that $\Lambda^2(\Omega)Y^m_l(\Omega) = \ell_D(\ell_D + 1)Y^m_l(\Omega)$, with $\ell_D = \sqrt{\frac{1}{4} + \ell(\ell + D - 2)} - \frac{1}{2}$. Furthermore, if we set $R_{n\ell}(r) = r^{\ell_D-1} \psi_{n\ell}(r)$ Eq.(2) becomes

$$
-\frac{d^2\psi_{n\ell}(r)}{dr^2} + \left[V(r) + \frac{L_s}{r^2}\right] \psi_{n\ell}(r) = E\psi_{n\ell}(r),
$$

(3)

where $L_s = L(L + 1) = \ell_D(\ell_D + 1) - (D - 1)(3 - D)/4$ and, as expected, the case $D = 3$ implies $L = \ell_D = \ell$.

The Schrödinger equation given in Eq.(3) has been solved by different authors by means of the Nikiforov-Uvarov (NU) method [1] applied to specific potentials [13]. With the same purpose, in the next section we will consider a straightforward approach that can be applied to any exponential-type potential. In particular, the usefulness of our proposition is exemplified with the application to the Eckart+deformed Hylleraas potential [6].

III. Bound state solutions of the Eckart+deformed Hylleraas potential.

Recently, by using the canonical transformation method applied to the hypergeometric differential equation (DE) it was obtained a class of multiparameter exponential-type potential given by [14]

$$
V(r) = \frac{qA \exp(-r/k)}{1 - q \exp(-r/k)} + \frac{qB \exp(-r/k)}{(1 - q \exp(-r/k))^2} + \frac{q^2C \exp(-2r/k)}{(1 - q \exp(-r/k))^2},
$$

(4)

such that for the Schrödinger equation

$$
-\frac{d^2\psi_n(r)}{dr^2} + V(r)\psi_n(r) = E_n\psi_n(r),
$$

(5)
it has the eigenvalues

$$E_n = -\frac{1}{4k^2} \left( \frac{n^2 + (h + 1)(2n + 1)/2 + k^2(A + B)}{n + (h + 1)/2} \right)^2$$

(6)

and eigenfunctions

$$\psi_n(r) = (q \exp(-r/k))^{(c-1)/2}(1 - q \exp(-r/k))^{(b+1)/2} \, _2F_1(-n, b; c; q \exp(-r/k)).$$

(7)

where $a, b$ and $c$ are given from the hypergeometric DE, with the $A, B$ and $C$ parameters related accordingly to

$$A + B = \frac{2ab - c(a + b + 1 - c)}{2k^2} \quad \text{and} \quad A - C = \frac{(c - 1)^2 - (a - b)^2}{4k^2},$$

(8)

such that

$$b = \frac{(h + 1)(h - a) - 2k^2(A + B)}{h + 1 - 2a}, \quad c = \frac{2a(h - a) - 2k^2(A + B)}{h + 1 - 2a}$$

(9)

and

$$c = a + b - h, \quad \text{with} \quad h = \left(1 + 4k^2(B + C)\right)^{1/2}.$$  

(10)

Consequently, in order to take advantage of above result for obtaining bound state solutions of exponential-type potentials in $D$-dimensions, we propose $q = 1$ and the definitions

$$A = A + \frac{L_s}{k^2}, \quad B = B + \frac{L_s}{k^2}, \quad C = C + \frac{L_s}{k^2}.$$ 

(11)

in such a way that the potential of Eq.(4) is rewritten as

$$V(r) = \frac{A \exp(-r/k)}{1 - \exp(-r/k)} + \frac{B \exp(-r/k)}{(1 - \exp(-r/k))^2} + \frac{C \exp(-2r/k)}{(1 - \exp(-r/k))^2} + L_s T_c$$

(12)

where the term

$$T_c = \frac{(\alpha + \beta) \exp(-r/k) + (\gamma - \alpha) \exp(-2r/k)}{k^2(1 - \exp(-r/k))^2}$$

(13)

may be identified with an approximation to $\frac{1}{r^2}$. However, before to do that, we want to notice that, as shown recently [15,16], the choice of the $A, B$ and $C$ parameters leads to many of most popular exponential-type potentials as particular cases of Eq.(4). So, one can make extensive this feature to the $A, B$ and $C$ parameters in the effective potential for which, to fulfill the purpose of this work, let us consider the selection $A = \frac{V_0(a - 1)}{b} - V_1$, $B = V_2$, $C = 0$, $k = \frac{1}{2\gamma}$, $\alpha = \omega k^2$, $\beta = \lambda k^2$ and $\gamma = 0$ that allows to write Eq.(14) as

$$V_{eff}(r) = \left(\frac{V_0(a - 1)}{b} - V_1\right) \frac{\exp(-2\alpha r)}{1 - \exp(-2\alpha r)} + V_2 \frac{\exp(-2\alpha r)}{(1 - \exp(-2\alpha r))^2} + \frac{L_s}{r^2}$$

(14)

where the parameters $\alpha, \beta$ and $\gamma$ have permitted to use the approximation $T_c \approx r^{-2}$ of Taskin and Kocak [11].

Straightforwardly, this potential is identified with the Eckart +deformed Hylleraas potential given by $V_{EH}(r) = V_{eff}(r) + \frac{4V_0}{b}$, namely

$$V_{EH}(r) = \frac{V_0}{b} \left(\frac{a - \exp(-2\alpha r)}{1 - \exp(-2\alpha r)}\right) - V_1 \frac{\exp(-2\alpha r)}{1 - \exp(-2\alpha r)} + V_2 \frac{\exp(-2\alpha r)}{(1 - \exp(-2\alpha r))^2} + \frac{L_s}{r^2}.$$ 

(15)
where \( \frac{aV_0}{b} \) is a displacement. Consequently, according to our generalized proposal for exponential-type potentials, \( V_{EH}(r) \) has, from Eq.(6), the energy spectra

\[
E_{EH(n,L)} = -\alpha^2 \left( \frac{n^2 + (2n + 1)(h_L + 1)/2 + \frac{1}{4\alpha^2} \left( \frac{V_0(a-1)}{b} - V_1 + V_2 + (\omega + \lambda)L_s \right)}{n + (h_L + 1)/2} \right)^2 + \frac{aV_0}{b}
\]

(16)

that is in good agreement with the solution obtained from the Nikiforov-Uvarov method by Ikot et.al. \cite{6} matching their parameters

\[
\begin{align*}
\gamma &= \frac{V_0}{4\alpha^2}, \\
\theta &= \frac{1}{4\alpha^2} \left[ V_1 - \omega \left( \ell(\ell + D - 2) + (D - 1)(D - 3)/4 \right) \right], \\
\phi &= \frac{1}{4\alpha^2} \left[ V_2 + \lambda \left( \ell(\ell + D - 2) + (D - 1)(D - 3)/4 \right) \right], \\
\ell &= \ell(\ell + D - 2) + (D - 1)(D - 3)/4 = L_s, \\
\sigma &= \frac{1}{2} \left[ 1 + \sqrt{1 + 4\phi} \right] = \frac{b_L + 1}{2}.
\end{align*}
\]

Also, directly form Eq.(7), the eigenfunctions are given by

\[
\psi_{n,L}(r) = \left( \exp(-2\alpha r) \right)^{(c_L-1)/2} \left( 1 - \exp(-2\alpha r) \right)^{(h_L+1)/2} {}_2F_1(-n, b_L; c_L; \exp(-2\alpha r)),
\]

(18)

where \( h_L = \sqrt{1 + \frac{1}{\alpha^2} [V_2 + \lambda L_s]} \) and

\[
\begin{align*}
b_L &= \frac{(h_L + 1)(n + h_L) - \frac{1}{2\alpha^2} \left( \frac{V_0(a-1)}{b} - V_1 + V_2 + (\omega + \lambda) L_s \right)}{2n + h_L + 1}, \\
c_L &= \frac{-2n(h_L + n) - \frac{1}{2\alpha^2} \left( \frac{V_0(a-1)}{b} - V_1 + V_2 + (\omega + \lambda) L_s \right)}{2n + h_L + 1}.
\end{align*}
\]

(19)

(20)

**IV. Bound state solutions of some specific exponential-type potentials.**

As mentioned before, the selection of parameters involved in the proposed multiparameter exponential-type potential of Eq.(12), leads to various specific potentials as particular cases \cite{15}. The same is valid for the Eckart+deformed Hylleraas potential here considered. In fact, as shown in Table I, for specific parameters, it become as particular cases of \( V_{EH}(r) \) the standard \( V_E(r) \) Eckart, \( V_H(r) \) Hylleraas, \( V_{hu}(r) \) Hulthén and \( V_{RM}(r) \) Rosen-Morse potentials just for give some examples.

In table I, it should be used \( N_X = n^2 + (h_{X(L)} + 1)(2n + 1)/2 \) with \( X = E, H, Hu \) and \( RM \) for Eckart, Hylleraas, Hulthén and Rosen-Morse respectively such that

\[
h_E(L) = \sqrt{1 + \frac{1}{\alpha^2} (V_2 + \lambda L_s)} \quad \text{and} \quad h_H(L) = h_{Hu(L)} = h_{RM(L)} = \sqrt{1 + \frac{\lambda L_s}{\alpha^2}}.
\]

(21)

Also, the corresponding wavefunctions are derived from Eq.(18) as

\[
\psi_{X(n,L)}(r) = \left( \exp(-2\alpha r) \right)^{(c_{X(L)}-1)/2} \left( 1 - \exp(-2\alpha r) \right)^{(h_{X(L)}+1)/2} {}_2F_1(-n, b_{X(L)}; c_{X(L)}; \exp(-2\alpha r)),
\]

(22)
where the corresponding parameters $b_{X(L)}$ and $c_{X(L)}$ are obtained directly from Eqs.(19) and (20) for the respective involved parameters given in the Table I.

Table I. Specific potential models obtained directly as particular cases of $V_{EH}(r)$.

| Parameters          | Potential                                                                 | Energy Spectra                                                                 |
|---------------------|---------------------------------------------------------------------------|--------------------------------------------------------------------------------|
| $b \to \infty$ or $V_0 = 0$ | $V_E(r) = \frac{-V_1 \exp(-2\alpha r)}{1-\exp(-2\alpha r)} + \frac{V_2 \exp(-2\alpha r)}{(1-\exp(-2\alpha r))^2} L + \frac{L}{r^2}$ | $E_E = -\alpha^2 \left( \frac{N_E + \frac{1}{n^2}(-V_1+V_2+(\omega+\lambda)L_s)}{n+(h_{E(L)}+1)/2} \right)^2$ |
| $V_1 = V_2 = 0$    | $V_H(r) = \frac{V_0}{b} \left( \frac{a-\exp(-2\alpha r)}{1-\exp(-2\alpha r)} \right) + \frac{L}{r^2}$ | $E_H = -\alpha^2 \left( \frac{N_H + \frac{1}{n^2}(\frac{V_0(a-1)}{b}+(\omega+\lambda)L_s)}{n+(h_{H(L)}+1)/2} + \frac{aV_0}{b} \right)^2$ |
| $V_0 = V_2 = a = 0$ | $V_{Hu}(r) = \frac{-V_1 \exp(-2\alpha r)}{1-\exp(-2\alpha r)} + \frac{L}{r^2}$ | $E_{Hu} = -\alpha^2 \left( \frac{N_{Hu} + \frac{1}{n^2}(-V_1+(\omega+\lambda)L_s)}{n+(h_{Hu(L)}+1)/2} \right)^2$ |
| $V_1 = V_2 = 0, b = 1, a = -1$ | $V_{RM}(r) = \frac{-V_0(1+\exp(-2\alpha r))}{1-\exp(-2\alpha r)} + \frac{L}{r^2}$ | $E_{RM} = -\alpha^2 \left( \frac{N_{RM} + \frac{1}{n^2}(-2V_0+(\omega+\lambda)L_s)}{n+(h_{RM(L)}+1)/2} - V_0 \right)^2$ |

Finally, we want to notice that due to the fact that we assume $C = 0$ and $\gamma = 0$ other specific $D$–dimensional potentials such as the Manning-Rosen or Deng-Fan models can not be derived in the present work from $V_{EH}(r)$. However, they are particular cases contained into the multiparameter exponential-type potential (Eq.12) as already published elsewhere [15,16].

V Concluding remarks.

This work has been devoted to find the analytical bound-state solutions of the Schrödinger equation for the Eckart-deformed Hylleraas potential in $D$–dimensions. At difference with the celebrated Nikiforov-Uvarov method, our proposal has been adapted to the $D$–dimensional case of the exactly-solvable solutions of a class of exponential-type potentials. The effective potential that we are proposing accepts different approximations to the centrifugal term as well as different exponential-type potential models depending on the choice of the involved parameters $A$, $B$ and $C$. However, we have considered the Taskin - Kocak approximation to $1/r^2$ as well as specific choices of parameters to get the Eckart-deformed Hylleraas potential. From there, the particular cases of bound-state solutions in $D$–dimensions to the Hulthén, Eckart, Rosen-Morse and the single deformed Hylleraas potential are directly derived from the proposal. However, we stand why some other potentials such as the Manning-Rosen or Deng-Fan potential can not
be obtained from the Eckart-deformed Hylleraas model. By the contrary, these potentials are particular cases from the proposed multiparameter exponential-type potential in $D$—dimensions. In short, instead of obtaining the analytical bound-state solutions of the Schrödinger equation for an specific exponential-type potential in $D$—dimensions, by means of a specialized method, our proposal gives the general solution from where specific potentials can be obtained as particular cases.

Acknowledgments

This work was partially supported by the projects UAM-A-CBI-2232004 and 009. We are grateful with the SNI - Conacyt - México for the stipend received.

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