Optimal weak type estimates for
dyadic-like maximal operators

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Abstract

We provide sharp weak estimates for the distribution function of $M\phi$ when on $\phi$ we impose $L^1$, $L^q$ and $L^{p,\infty}$ restrictions. Here $M$ is the dyadic maximal operator associated to a tree $T$ on a non-atomic probability measure space.

Keywords: Dyadic, Maximal

1. Introduction

The dyadic maximal operator on $\mathbb{R}^n$ is defined by

$$M_d \phi(x) = \sup \left\{ \frac{1}{|Q|} \int_Q |\phi(u)| \, du : \ x \in Q, \ Q \subseteq \mathbb{R}^n \text{ is a dyadic cube} \right\} \quad (1.1)$$

for every $\phi \in L^1_{\text{loc}}(\mathbb{R}^n)$ where the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^n$ for $N = 1, 2, \ldots$.

It is well known that it satisfies the following weak type (1.1) inequality

$$|\{x \in \mathbb{R}^n : M_d \phi(x) > \lambda\}| \leq \frac{1}{\lambda} \int_{\{M_d \phi > \lambda\}} |\phi(u)| \, du \quad (1.2)$$

for every $\phi \in L^1(\mathbb{R}^n)$ and every $\lambda > 0$. 

Using (1.1) we easily get the following $L^p$ inequality
\[
\| M_d \phi \|_p \leq \frac{p}{p-1} \| \phi \|_p
\] (1.3)
for every $p > 1$ and every $\phi \in L^p(\mathbb{R}^n)$, which is proved to be best possible (see [2], [3] for the general martingales and [10] for the dyadic ones).

A way of studying the dyadic maximal operator is the introduction of the so called Bellman functions (see [8]).

Actually, we define for every $p > 1$
\[
B_p(f, F) = \sup \left\{ \frac{1}{|Q|} \int_Q (M_d \phi)^p : \text{Av}_Q(\phi^p) = F, \text{Av}_Q(\phi) = f \right\}
\] (1.4)
where $Q$ is a fixed dyadic cube, $\phi$ is nonnegative in $L^p(Q)$ and $f, F$ are such that $0 < f^p \leq F$.

$B_p(f, F)$ has been computed in [5]. In fact it has been shown that $B_p(f, F) = F \omega_p(f^p/F)^p$ where $\omega_p : [0, 1] \rightarrow \left[1, \frac{p}{p-1}\right]$ is the inverse function of $H_p(z) = -(p-1)z^p + pz^{p-1}$.

Actually this has been proved in a much more general setting of tree like maximal operators on non-atomic probability spaces. The result turns out to be independent of the choice of the measure space.

The study of these operators has been continued in [7] where the Bellman functions of them in the case $p < 1$ have been computed.

Actually, as in [5] and [7] we will take the more general approach. So for a tree $T$ on a non atomic probability measure space $X$, we define the associated dyadic maximal operator, namely
\[
M_T \phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| d\mu : x \in I \in T \right\}
\]
for every $\phi \in L^1(X, \mu)$.2
It is now known that $\mathcal{M}_T : L^{p,\infty} \to L^{p,\infty}$ is a bounded operator satisfying
\[
\|\mathcal{M}_T \phi\|_{p,\infty} \leq |||\phi|||_{p,\infty}. \tag{1.5}
\]

It is now interesting to see what happens if we replace the $L^p$-norm of $\phi$ in (1.4) by its $L^{p,\infty}$-norm, $|||\cdot|||_{p,\infty}$, given by
\[
|||\phi|||_{p,\infty} = \sup \left\{ \mu(E)^{-1 + \frac{1}{p}} \int_E |\phi| d\mu : E \text{ measurable subset of } X \text{ such that } \mu(E) > 0 \right\}.
\]

It is well known that $|||\cdot|||_{p,\infty}$ is a norm on $L^{p,\infty}$ equivalent to the quasi norm $\|\cdot\|_{p,\infty}$ defined by
\[
\|\phi\|_{p,\infty} = \sup \left\{ \lambda \mu(\{\phi \geq \lambda\})^{1/p} : \lambda > 0 \right\}.
\]

In fact
\[
\|\phi\|_{p,\infty} \leq |||\phi|||_{p,\infty} \leq \frac{p}{p-1}\|\phi\|_{p,\infty}, \quad \forall \phi \in L^{p,\infty}
\]
as can been seen in [4].

In fact in [9] it is proved that (1.5) is sharp allowing every value for the $L^1$-norm of $\phi$.

In the present paper we compute the following function
\[
S(f,A,F,\lambda) = \sup \left\{ \mu(\{\mathcal{M}_T \phi \geq \lambda\}) : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^q d\mu = A, \quad |||\phi|||_{p,\infty} = F \right\} \tag{1.6}
\]
for every $\lambda > 0$, $(f,A,F)$ on the domain of the extremal problem and $q$ fixed such that $1 < q < p$. That is we provide improvements of (1.3) given additionally $L^q$ and $L^{p,\infty}$ restrictions on $\phi$.

From this we obtain as a corollary that
\[
\sup \left\{ \|\mathcal{M}_T \phi\|_{p,\infty} : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^q d\mu = A, \quad |||\phi|||_{p,\infty} = F \right\} = F \tag{1.7}
\]
that is (1.5) is sharp allowing every value of the integral and the $L^q$-norm of $\phi$, for a fixed $q$ such that $1 < q < p$. As a matter of fact we prove that the supremum in both cases (1.6) and (1.7) is attained. These estimates are provided in Section 4, while in Section 3 the domain of the extremal problem is found. On Section 2 we give some preliminaries needed during this paper.

Finally we mention that all the above estimates are independent of the measure space and the tree $T$.

2. Preliminaries

Let $(X, \mu)$ be a non-atomic probability measure space. We state the following lemma which can be found in [1].

**Lemma 2.1** Let $\phi : (X, \mu) \rightarrow \mathbb{R}^+$ and $\phi^*$ the decreasing rearrangement of $\phi$, defined on $[0, 1]$. Then

$$\int_0^t \phi^*(u)du = \sup \left\{ \int_E \phi d\mu : E \text{ measurable subset of } X \text{ with } \mu(E) = t \right\}$$

for every $t \in [0, 1]$, with the supremum in fact attained. \[\square\]

We prove now the following:

**Lemma 2.2** Let $\phi : X \rightarrow \mathbb{R}^+$ be measurable and $I \subseteq X$ be measurable with $\mu(I) > 0$. Suppose that $\frac{1}{\mu(I)} \int_I \phi d\mu = s$. Then for every $t$ such that $0 < t \leq \mu(I)$ then exists a measurable set $E_t \subseteq I$ with $\mu(E_t) = t$ and $\frac{1}{\mu(E_t)} \int_{E_t} \phi d\mu = s$.

**Proof.** Consider the measure space $(I, \mu/I)$ and let $\psi : I \rightarrow \mathbb{R}^+$ be the restriction of $\phi$ on $I$ that is $\psi = \phi/I$. Then if $\psi^* : [0, \mu(I)] \rightarrow \mathbb{R}^+$ is the decreasing rearrangement of $\psi$, we have that

$$\frac{1}{t} \int_0^t \psi^*(u)du \geq \frac{1}{\mu(I)} \int_0^{\mu(I)} \psi^*(u)du = s \geq \frac{1}{t} \int_{\mu(I)-t}^{\mu(I)} \psi^*(u)du. \quad (2.1)$$
Since $\psi^*$ is decreasing we get the inequalities in (2.1), while the equality is obvious since
\[ \int_0^{\mu(I)} \psi^*(u)du = \int_I \phi d\mu. \]
From (2.1) it is easily seen that there exists $r \geq 0$ such that $t + r \leq \mu(I)$ with
\[ \frac{1}{t} \int_r^{t+r} \psi^*(u)du = s. \] (2.2)
It is also easily seen that there exists $E_t$ measurable subset of $I$ such that
\[ \mu(E_t) = t \quad \text{and} \quad \int_{E_t} \phi d\mu = \int_r^{t+r} \psi^*(u)du \] (2.3)
since $(X, \mu)$ is non-atomic.

From (2.2) and (2.3) we get the conclusion of the lemma. □

We now call two measurable subsets of $X$ almost disjoint if $\mu(A \cap B) = 0$.

We give now the following

**Definition 2.1** A set $\mathcal{T}$ of measurable subsets of $X$ will be called a tree if the following conditions are satisfied.

(i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have that $\mu(I) > 0$.

(ii) For every $I \in \mathcal{T}$ there corresponds a finite or countable subset $C(I) \subseteq \mathcal{T}$ containing at least two elements such that:

(a) the elements of $C(I)$ are pairwise almost disjoint subsets of $I$.

(b) $I = \bigcup C(I)$.

(iii) $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}(m)$ where $\mathcal{T}_0 = \{X\}$ and

\[ \mathcal{T}_{(m+1)} = \bigcup_{I \in \mathcal{T}_m} C(I). \]
(iv) \( \lim_{m \to +\infty} \sup_{I \in \mathcal{T}(m)} \mu(I) = 0. \) \( \square \)

From [5] we get the following

**Lemma 2.3** For every \( I \in \mathcal{T} \) and every \( \alpha \) such that \( 0 < \alpha < 1 \) there exists subfamily \( \mathcal{F}(I) \subseteq Y \) consisting of pairwise almost disjoint subsets of \( I \) such that
\[
\mu \left( \bigcup_{J \in \mathcal{F}(I)} J \right) = \sum_{J \in \mathcal{F}(I)} \mu(J) = (1 - \alpha) \mu(I). \quad \square
\]

Let now \((X, \mu)\) be a non-atomic probability measure space and \( \mathcal{T} \) a tree as in Definition 1.1. We define the associated maximal operator to the tree \( \mathcal{T} \) as follows: For every \( \phi \in L^1(X, \mu) \) and \( x \in X \), then
\[
\mathcal{M}_T \phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| d\mu : x \in I \in \mathcal{T} \right\}.
\]

### 3. The domain of the extremal problem

Our aim is to find the exact allowable values of \((f, A, F)\) for which there exists \( \phi : (X, \mu) \to \mathbb{R}^+ \) measurable such that
\[
\int_X \phi d\mu = f, \quad \int_X \phi^q d\mu = A \quad \text{and} \quad |||\phi|||_{p, \infty} = F. \quad (3.1)
\]
We find it in the case where \( F = 1 \).

For the beginning assume that \((f, A)\) are such that there exist \( \phi \) as in (3.1). We set \( g = \phi^* : [0, 1] \to \mathbb{R}^+ \). Then
\[
\int_0^1 g = f, \quad \int_0^1 g^q = A \quad \text{and} \quad |||g|||_{p, \infty}^{[0,1]} = 1
\]
where
\[
|||g|||_{p, \infty}^{[0,1]} = \sup \left\{ |E|^{-1 + \frac{1}{p}} \int_E g : E \subset [0, 1], \text{ Lebesgue measurable such that } |E| > 0 \right\}.
\]
This is true because of the definition of the decreasing rearrangement of $\phi$ and Lemma 2.1. In fact since $g$ is decreasing $|||g|||_{p,\infty}$ is equal to

$$\sup \left\{ t^{-1+\frac{1}{p}} \int_0^t g : 0 < t \leq 1 \right\}.$$ 

Of course, we should have that $0 < f \leq 1$ and $f'^q \leq A$. We give now the following

**Definition 3.1** If $n \in \mathbb{N}$, and $h : [0, 1) \to \mathbb{R}^+$, $h$ will be called $\frac{1}{2^n}$-step if it is constant on each interval

$$\left[ \frac{i-1}{2^n}, \frac{i}{2^n} \right), \ i = 1, 2, \ldots, 2^n . \quad \square$$

Now for $n \in \mathbb{N}$ and $0 < f \leq 1$ fixed we set

$$\Delta_n(f) = \left\{ h : [0, 1) \to \mathbb{R}^+ : g \text{ is a } \frac{1}{2^n}\text{-step function,} \right\}$$

$$\int_0^1 g = f, \ |||g|||_{p,\infty} \leq 1 \}.$$ 

Then

$$\Delta_n = \Delta_n(f) \subset L^{p,\infty}([0, 1])$$

where we use the $||| \cdot |||_{p,\infty}^{[0,1]}$ norm for functions defined on $[0, 1]$. $\Delta_n$ is also convex, that is

$$h_1, h_2 \in \Delta_n \Rightarrow \frac{h_1 + h_2}{2} \in \Delta_n .$$

Additionally we have the following

**Lemma 3.1** $\Delta_n$ is compact subset of $L^{p,\infty}([0, 1]) = Y$ where the topology on $Y$ is that endowed by $||| \cdot |||_{p,\infty}^{[0,1]}$.

**Proof.** $(Y, ||| \cdot |||_{p,\infty})$ is a Banach space. So, especially a metric space. So, we just need to prove that $\Delta_n$ is sequentially compact.
Let now \((h_i)_i \subset \Delta_n\). It is now easy to see by a finite diagonal argument that there exists \((h_{i_j})_j\) subsequence and \(h : [0, 1] \to \mathbb{R}^+\) such that \(h_{i_j} \to h\) uniformly on \([0, 1]\). Then obviously \(\int_0^1 h = f\), \(||h||_{p, \infty}^{[0,1]} \leq 1\), so \(h \in \Delta_n\). Additionally

\[
|||h_{i_j} - h|||_{p, \infty}^{[0,1]} = \sup \left\{ |E|^{1+\frac{1}{p}} \int_E |h_{i_j} - h| : |E| > 0 \right\} 
\leq \sup |(h_{i_j} - h)(t)| \ t \in [0, 1]
\]

as \(j \to \infty\). That is \(h_{i_j} \overset{Y}{\rightharpoonup} h \in \Delta_n\). Consequently, \(\Delta_n\) is a compact subset of \(L^{p, \infty}([0, 1])\). □

We give now the following known

**Definition 3.2** For a closed convex subset \(K\) of a topological vector space \(Y\), and for a \(y \in K\) we say that \(y\) is an extreme point of \(K\), if whenever \(y = \frac{x+z}{2}\), with \(x, z \in K\) it is implied that \(y = x = z\). We write \(y \in \text{ext}(K)\).

□

**Definition 3.3** For a subset \(A\) of a topological vector space \(Y\) we set

\[
\text{conv}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_i \geq 0, x_i \in A, \ n \in \mathbb{N}^*, \ \sum_{i=1}^{n} \lambda_i = 1 \right\}.
\]

We call \(\text{conv}(A)\) the convex hull of \(A\). □

We state now the following well known

**Theorem 3.1** (Krein–Milman) Let \(K\) be a convex, compact subset of a locally convex topological vector space \(Y\) then \(K = \text{conv}(\text{ext}(K))^Y\) that is \(K\) is the closed convex hull of it’s extreme points. □

According now to Lemma 3.1 we have that

\[
\Delta_n = \text{conv}[\text{ext}(\Delta_n)]^{L^{p,\infty}([0,1])}.
\]

We find now the set \(\text{ext}(\Delta_n)\).
Lemma 3.2. Let \( g \in \text{ext}(\Delta_n) \). Then for every \( i \in \{1, 2, \ldots, 2^n\} \) such that \( \left( \frac{i}{2^n} \right)^{1 - \frac{1}{p}} \leq f \) we have that

\[
\sup \left\{ |E|^{-\frac{1}{p}} \int_E g : |E| = \frac{i}{2^n} \right\} = 1.
\]

Proof. We prove it first when \( i = 1 \) and \( \left( \frac{1}{2^n} \right)^{1 - \frac{1}{p}} \leq f \). It is now easy to see that \( g \in \text{ext}(\Delta_n) \Leftrightarrow g^* \in \text{ext}(\Delta_n) \). So we just need to prove that

\[
\int_0^{2^n} g^* = \left( \frac{1}{2^n} \right)^{1 - \frac{1}{p}}.
\]

We write

\[
g^* = \sum_{i=1}^{2^n} \alpha_i \xi_{I_i} \quad \text{with} \quad I_i \left[ \frac{i - 1}{2^n}, \frac{i}{2^n} \right)
\]

and \( \alpha_1 \geq \alpha_{i+1} \) for every \( i \in \{1, 2, \ldots, 2^n - 1\} \).

Suppose now that \( \alpha_1 < 2^{n/p} \), and that \( \alpha_1 > \alpha_2 \) (the case \( \alpha_1 = \alpha_2 \) is handled in an analogous way).

For a suitable \( \varepsilon > 0 \) we set

\[
g_1 = \sum_{i=1}^{2^n} \alpha_i^{(1)} \xi_{I_i}, \quad g_2 = \sum_{i=1}^{2^n} \alpha_i^{(2)} \xi_{I_i} \quad \text{where} \quad \begin{cases} 
\alpha_1^{(1)} = \alpha_1 + \varepsilon, & \alpha_2^{(1)} = \alpha_2 - \varepsilon \\
\alpha_1^{(2)} = \alpha_1 - \varepsilon, & \alpha_2^{(2)} = \alpha_2 + \varepsilon 
\end{cases}
\]

and \( \alpha_k^{(1)} = \alpha_k^{(2)} = \alpha_k \) for every \( k > 2 \).

Since \( \alpha_1 < 2^{n/p} \) we can find small enough \( \varepsilon > 0 \) such that \( g_i \) satisfy \( \|g_i\|_{L^{0, \infty}} \leq 1 \), for \( i = 1, 2 \). Indeed, for \( i = 1 \), we need to prove that for small enough \( \varepsilon > 0 \)

\[
\int_0^t g_1 \leq t^{1 - \frac{1}{p}} \quad \text{(3.2)}
\]

for every \( t \in [0, 1) \), since \( g_1 \) is decreasing.

(3.2) is now obviously true for \( t \geq \frac{2^n}{2^n} \) since

\[
\int_0^t g_1 = \int_0^t g^* \quad \text{for every such} \quad t \quad \text{(3.3)}
\]
(3.2) is also true for \( t = 0, \frac{1}{2^n} \). But then it remains true for every \( t \in \left( 0, \frac{1}{2^n} \right) \) since the function \( t \mapsto \int_0^t g_1 \) represents a straight line on \([0, \frac{1}{2^n}]\) and \( t^{1 - \frac{1}{p}} \) is concave there. Analogously for the interval \([\frac{1}{2^n}, \frac{2}{2^n}]\). That is we proved \( \|g_1\|_{[0,1]} \leq 1 \).

Obviously, \( \int_0^1 g_i = f \), so that \( g_i \in \Delta_n \), for \( i = 1, 2 \). But \( g^* = \frac{g_1 + g_2}{2} \), with \( g_i \neq g \) and \( g_i \in \Delta_n \), \( i = 1, 2 \), a contradiction since \( g^* \in \text{ext}(\Delta_n) \). So,
\[
\alpha_1 = 2^{n/p} \quad \text{and} \quad \int_0^{1/2} g^* = \left( \frac{1}{2^n} \right)^{1 - \frac{1}{p}},
\]
what we wanted to prove. In the same way we prove that for \( i \in \{1, 2, \ldots, 2^n\} \) such that
\[
\left( \frac{i + 1}{2^n} \right)^{1 - \frac{1}{p}} \leq f, \quad \text{if} \quad \int_0^{i/2^n} g^* = \left( \frac{i}{2^n} \right)^{1 - \frac{1}{p}} \quad \text{then} \quad \int_{i/2^n}^{i+1/2^n} g^* = \left( \frac{i + 1}{2^n} \right)^{1 - \frac{1}{p}}.
\]
The lemma is now proved. \( \square \)

Let now \( g \in \text{ext}(\Delta_n) \) and \( k = \max \left\{ i \leq 2^n : \left( \frac{i}{2^n} \right)^{1 - \frac{1}{p}} \leq f \right\} \), so if we suppose that \( f < 1 \) we have that
\[
\left( \frac{k}{2^n} \right)^{1 - \frac{1}{p}} \leq f < \left( \frac{k + 1}{2^n} \right)^{1 - \frac{1}{p}}.
\]
By Lemma 3.2
\[
\int_0^{k/2^n} g^* = \left( \frac{k}{2^n} \right)^{1 - \frac{1}{p}}.
\]
But by using the reasoning of the previous lemma it is easy to see that
\[
\int_0^{k+1/2^n} g^* = f,
\]
which gives
\[
\int_{k/2^n}^{k+1/2^n} g^* = f - \left( \frac{k}{2^n} \right)^{1 - \frac{1}{p}} \Rightarrow \alpha_{k+1} = 2^n \cdot f - 2^{n/p} \cdot k^{1 - \frac{1}{p}}.
\]
Additionally \( \alpha_i = 0 \) for \( i > k + 1 \).

From the above we obtain the following
Corollary 3.1 Let \( g \in \text{ext}(\Delta_n) \). Then \( g^* = \sum_{i=1}^{2^n} \alpha_i \xi_i \), where
\[
\alpha_i = 2^{n/p} \left( i^{1-\frac{1}{p}} - (i-1)^{1-\frac{1}{p}} \right) \quad \text{for} \quad i = 1, 2, \ldots, k
\]
and
\[
\alpha_{k+1} = 2^n f - 2^{n/p} \cdot k^{1-\frac{1}{p}}, \quad \alpha_i = 0, \quad i > k + 1,
\]
where
\[
k = \max \left\{ i \leq 2^n : \left( \frac{i}{2^n} \right)^{1-\frac{1}{p}} \leq f \right\}.
\]

Remark 3.1 Actually it is easy to see that the above functions described in Corollary 3.1 are exactly the extreme points of \( \Delta_n \).

We estimate now the \( L^q \)-norm of every \( g \in \text{ext}(\Delta_n) \).

We state it as

Lemma 3.3 Let \( g \in \text{ext}(\Delta_n) \) and \( A = \frac{1}{0} \int g^q \), then \( A \leq \Gamma f^{p-q/p-1} + E_n(f) \)
where
\[
\Gamma = \left( \frac{p-1}{p} \right)^q \frac{p}{p-q} \quad \text{and} \quad E_n(f) = \frac{\alpha_{k+1}^q}{2^n} = \frac{(2^n f - 2^{n/p} k^{1-\frac{1}{p}})^q}{2^n}.
\]

Proof. For \( g \) we write \( g^* = \sum_{i=1}^{2^n} \alpha_i \xi_i \), where \( \alpha_i \) are given in Corollary 3.1.

Then
\[
A = \int_0^1 (g^*)^q = \left[ \left( \sum_{i=1}^k \alpha_i^q \right) + \alpha_{k+1}^q \right] \cdot \frac{1}{2^n}.
\]

Now for \( i \in \{1, 2, \ldots, k\} \)
\[
\alpha_i^q = \left[ 2^{n/p} \left( i^{1/\frac{1}{p}} - (i-1)^{1/\frac{1}{p}} \right) \right]^q = \left\{ 2^n \left[ \left( \frac{i}{2^n} \right)^{1-\frac{1}{p}} - \left( \frac{i-1}{2^n} \right)^{1-\frac{1}{p}} \right] \right\}^q
\]
\[
= \left[ 2^n \int_{i-1/2^n}^{i/2^n} \psi \right]^q
\]

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where $\psi : (0, 1] \to \mathbb{R}^+$ is defined by $\psi(t) = \frac{n-1}{p} t^{-1/p}$. By (3.5) and in view of Holder’s inequality we have that for $i \in \{1, 2, \ldots, k\}$

$$
\alpha_i^q \leq 2^n \int_{i-1/2^n}^{i/2^n} \psi^q.
$$

(3.6)

Summing up relations (3.6) we have that

$$
\sum_{i=1}^{k} \alpha_i^q \leq 2^n \int_0^{k/2^n} \psi^q = 2^n \cdot \Gamma \cdot \left( \frac{k}{2^n} \right)^{1-\frac{q}{p}}.
$$

(3.7)

Additionally from the definition of $k$ we have that

$$
\left( \frac{k}{2^n} \right)^{1-\frac{1}{p}} \leq f \Rightarrow k^{1-\frac{q}{p}} \leq (2^n)^{1-\frac{q}{p}} \cdot f^{p-q/p-1}.
$$

(3.8)

From (3.4), (3.7) and (3.8) we obtain

$$
A \leq \left[ 2^n \cdot \Gamma \cdot f^{p-q/p-1} + \alpha_{k+1}^q \right] \frac{1}{2^n} = \Gamma f^{p-q/p-1} + \mathcal{E}_n(f)
$$

and Lemma 3.3 is proved.

**Corollary 3.2** For every $g \in \Delta_n$

$$
A \leq \Gamma f^{p-q/p-1} + \mathcal{E}_n(f), \text{ where } A = \int_0^1 g^q.
$$

**Proof.** This is true, of course, for $g \in ext(\Delta_n)$, and so also for $g \in \text{conv} (ext(\Delta_n))$, since $t \mapsto t^q$ is convex for $q > 1$ on $\mathbb{R}^+$. It remains true for $g \in \text{conv} (ext(\Delta_n))^{L_{p,\infty}([0,1])}$ using a simple continuity argument. In fact we just need the continuity of the identity operator if it is viewed as:

$I : L_{p,\infty}([0,1]) \to L^q([0,1])$. See [4].

Using now Krein - Milman Theorem the Corollary is proved.

We have now the following
**Corollary 3.3** Let \( \phi : (X, \mu) \to \mathbb{R}^+ \) such that
\[
\int_X \phi \, d\mu = f, \quad \int_X \phi^q \, d\mu = A, \quad ||\phi||_{p, \infty} \leq 1.
\]
Then
\[
f^q \leq A \leq \Gamma f^{p-q/p-1}.
\]

**Proof.** Let \( g = \phi^* : [0, 1] \to \mathbb{R}^+ \). There exist \( \phi_n^{1/2n} \)-simple functions, for every \( n \) such that \( g_n \leq g_{n+1} \leq g \) and \( g_n \) converges almost everywhere to \( g \). But then by defining
\[
f_n = \int_0^1 \phi_n, \quad A_n = \int_0^1 \phi_n^q
\]
we have that
\[
g_n \in \Delta_n(f_n) \text{ so that } A_n \leq \Gamma f_n^{p-q/p-1} + \mathcal{E}_n(f_n).
\]
(3.9)

By the monotone convergence theorem \( f_n \to f, \ A_n \to A \). Moreover
\[
\mathcal{E}_n(f_n) = \frac{(2^n f_n - k_n^{1-p} 2^n/p)^q}{2^n}
\]
where \( k_n \) satisfy
\[
\left( \frac{k_n}{2^n} \right)^{1-\frac{q}{p}} \leq f_n < \left( \frac{k_n + 1}{2^n} \right)^{1-\frac{1}{p}}.
\]
As a consequence
\[
\mathcal{E}_n(f_n) = (2^n)^{q-1} \left[ f_n - \left( \frac{k_n}{2^n} \right)^{1-\frac{1}{p}} \right]^q < (2^n)^{q-1} \left[ \left( \frac{k_n + 1}{2^n} \right)^{1-\frac{1}{p}} - \left( \frac{k_n}{2^n} \right)^{1-\frac{1}{p}} \right]^q
\]
\[
\leq (2^n)^{q-1} \left[ \left( \frac{1}{2^n} \right)^{1-\frac{q}{p}} \right]^q = \left( \frac{1}{2^{1-\frac{q}{p}}} \right)^n \to 0, \quad \text{as } n \to \infty
\]
where in the second inequality we used the known
\[
(t + s)^\alpha \leq t^\alpha + s^\alpha \text{ for } t, s \geq 0, \ 0 < \alpha < 1.
\]
Now (3.9) gives the corollary. \( \square \)

In fact the converse of Corollary 3.3 is also true.
**Theorem 3.2** For $0 < f \leq 1$, $A > 0$ the following are equivalent

i) $f^q \leq A \leq \Gamma f^{p-q/p-1}$

ii) $\exists \phi : (X, \mu) \to \mathbb{R}^+$ such that

$$\int_X \phi d\mu = f, \quad \int_X \phi^q d\mu = A, \quad |||\phi|||_{p,\infty} \leq 1.$$ □

We prove first the following

**Lemma 3.4** Let $\alpha \in (0, 1)$ and $(f, A)$ such that

$$f \leq \alpha^{\frac{1}{p}}$$ \hspace{1cm} (3.10)

$$f^q \leq \alpha^{q-1} A$$ \hspace{1cm} (3.11)

$$A \leq \Gamma f^{p-q/p-1} A.$$ \hspace{1cm} (3.12)

Then there exists $g : [0, \alpha] \to \mathbb{R}^+$ such that

$$\int_0^\alpha g = f, \quad \int_0^\alpha g^q = A, \quad \text{and} \quad |||g|||_{[0,\alpha]} = 1$$

where

$$|||g|||_{[0,\alpha]} = \sup \left\{ E^-1 + \frac{1}{p} \int_E g : \text{such that } |E| > 0 \right\}$$

**Proof.** We search for a $g$ of the form

$$g := \begin{cases} \frac{t^{-1/p}}{\mu_2}, & 0 < t \leq c_1 \\ \mu_2, & c_1 < t \leq \alpha \end{cases}$$

for suitable constant $c_1 \mu_2$.

We must have that

$$\int_0^\alpha g = f \iff c_1^{\frac{1}{p}} + \mu_2(\alpha - c_1) = f.$$ \hspace{1cm} (3.13)
Additionally $g$ must satisfy
\begin{equation}
\int_0^\alpha g^q = A \iff \Gamma c_1^{1 - \frac{q}{p}} + \mu_2^q(\alpha - c_1) = A. \tag{3.14}
\end{equation}

(3.13) gives
\begin{equation}
\mu_2 = \frac{f - c_1^{1 - \frac{1}{p}}}{a - c_1} \tag{3.15}
\end{equation}
so (3.14) becomes
\begin{equation}
\Gamma c_1^{1 - \frac{q}{p}} + \frac{(f - c_1^{1 - \frac{1}{p}})^q}{(a - c_1)^{q-1}} = A. \tag{3.16}
\end{equation}

We search for a $c_1 \in (0, \alpha)$ such that
\[ T(c_1) = A \text{ where } T : [0, \alpha) \to \mathbb{R}^+ \]
defined by
\[ T(t) = \Gamma t^{1 - \frac{q}{p}} + \frac{(f - t^{1 - \frac{1}{p}})^q}{(a - t)^{q-1}}. \]

Observe that $T(0) = \frac{f^q}{a^{q-1}} \leq A$ because of (3.11) and that $T(f^{p/p-1}) = \Gamma f^{p-q/p-1} \geq A$. Now because of the continuity of $T$, we have that there exists $c_1 \in (0, f^{p/p-1}]$ such that $T(c_1) = A$. Then $c_1 \in (0, \alpha)$ because of (3.10), and if we define $\mu_2$ by (??), we guarantee (3.13) and (3.14).

We need to prove now that $|||g|||_{[0,\alpha]} = 1$.

Obviously, because of the form of $g$, $|||g|||_{[0,\alpha]} \geq 1$. So we have to prove that
\begin{equation}
\int_0^t g \leq t^{1-\frac{1}{p}}, \quad \forall t \in (0, \alpha]. \tag{3.17}
\end{equation}

This is of course true for $t \in [0, c_1]$. For $t \in (c_1, \alpha]$
\[ \int_0^t g = c_1^{1 - \frac{1}{p}} + \mu_2(t - c_1) =: G(t). \]
Since \( G(c_1) = c_1^{1 - \frac{1}{p}} \), \( G(\alpha) = f < \alpha^{1 - \frac{1}{p}} \) and \( t \mapsto t^{1 - \frac{1}{p}} \) is concave on \((c_1, \alpha] \)
(3.17) is true. Thus Lemma 3.4 is proved. □

We have now the

**Proof of Theorem 3.2:** We have to prove the direction i) \( \Rightarrow \) ii).

Indeed if \( f^q \leq A \leq \Gamma f^{p/q - 1} \) and \( f < 1 \) we apply Lemma 3.4.

If \( f^q = A \), with \( 0 < f \leq 1 \) we set \( g \) by \( g(t) = f \), for every \( t \in [0, 1] \) while
if \( f = 1 \leq A \leq \Gamma \) a simple modification of Lemma 3.4 gives the result. □

We conclude Section 3 with the following theorem which can be proved easily using all the above.

**Theorem 3.3** For \( f, A \) such that \( f < 1, A > 0 \) the following are equivalent:

i) \( f^q \leq A \leq \Gamma f^{p/q - 1} \)

ii) \( \exists \phi : (X, \mu) \rightarrow \mathbb{R}^+ \) such that

\[
\int_X \phi d\mu = f, \quad \int_X \phi^q d\mu = A, \quad \|\phi\|_{p, \infty} = 1.
\]

□

**Remark 3.2** Theorem 3.3 is completed if we mention that for \( f = 1 \) the following are equivalent:

i) \( f = 1 \leq A \leq \Gamma \)

ii) \( \exists \phi : (X, \mu) \rightarrow \mathbb{R}^+ \) such that \( \int_X \phi d\mu = 1, \int_X \phi^q d\mu = A, \quad \|\phi\|_{p, \infty} = 1 \). □

4. The Extremal Problem

Let \( \mathcal{M}_T = \mathcal{M} \) the dyadic maximal operator associated to the tree \( T \), on
the probability non-atomic measure space \((X, \mu)\).
Our aim is to find

\[ T_{f,A,F}(\lambda) = \sup \left\{ \mu(\{M\phi \geq \lambda\}) : \phi \geq 0, \int_X \phi d\mu = f, \int_Z \phi^q d\mu = A, \right\} \]

for all the allowable values of \( f, A, F \).

We find it in the case where \( F = 1 \).

We write \( T_{f,A}(\lambda) \) for \( T_{f,A,1}(\lambda) \).

In order to find \( T_{f,A}(\lambda) \) we find first the following

\[ T^{(1)}_{f,A}(\lambda) = \sup \left\{ \mu(\{M\phi \geq \lambda\}) : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^q d\mu = A, \right\} \]

(4.1)

The domain of this extremal problem is the following:

\[ D = \left\{ (f, A) : 0 < f \leq 1, \ f^q \leq A \leq \Gamma f^{p-q/p-1} \right\}. \]

Obviously, \( T^{(1)}_{f,A}(\lambda) = 1 \), for \( \lambda \leq f \).

Now for \( \lambda > f \) and \((f, A) \in D\).

Let \( \phi \) as in (4.1). Consider the decreasing rearrangement of \( \phi \), \( g = \phi^* : [0, 1] \to \mathbb{R}^+ \). Then

\[ \int_0^1 g = f, \ \int_0^1 g^q = A, \ \|\phi\|_{p,\infty} \leq 1. \]

Consider also \( E = \{M\phi \geq \lambda\} \subseteq X \).

Then \( E \) is the almost disjoint union of elements of \( T \), let \((I_j)_j\). In fact we just need to consider the elements \( I \) of \( T \), maximal under the condition

\[ \frac{1}{\mu(I)} \int_I \phi d\mu \geq \lambda. \quad (4.2) \]
We, then, have $E = \bigcup_j I_j$ and $\int_E \phi d\mu \geq \lambda \mu(E)$ because of (4.2). Then according to Lemma 2.1 we have that $\int_0^\alpha g \geq \alpha \lambda$ where $\alpha = \mu(E)$. That is we proved that
\[
T^{(1)}_{f,A}(\lambda) \leq \Delta_{f,A}(\lambda)
\] (4.3)

where
\[
\Delta_{f,A}(\lambda) = \sup \left\{ \alpha \in (0,1] : \exists g : [0,1] \to \mathbb{R}^+ \text{ with } \int_0^1 g = f, \int_0^1 g^q = A, \right. \\
\left. \quad |||g|||_{p,\infty} \leq 1 \quad \text{and} \quad \int_0^\alpha g \geq \alpha \lambda \right\}.
\] (4.4)

We prove now the converse inequality in (4.3) by proving the following

**Lemma 4.1** Let $g$ be as in (4.4) for a fixed $\alpha \in (0,1]$. Then there exists $\phi : (X,\mu) \to \mathbb{R}^+$ such that
\[
\Phi d\mu = f, \quad \int_X \phi d\mu = A, \quad |||\phi|||_{p,\infty} \leq 1 \quad \text{and} \quad \mu(\{M\phi \geq \lambda\}) \geq \alpha.
\]

**Proof.** Lemma 2.3 guarantees the existence of a sequence $(I_j)_j$ of pairwise almost disjoint elements of $\mathcal{T}$ such that
\[
\mu(\cup I_j) = \sum \mu(I_j) = \alpha.
\] (4.5)

Consider now the finite measure space $([0,\alpha],|\cdot|)$ where $|\cdot|$ is the Lebesgue measure. Then since $\int_0^\alpha g \geq \alpha \lambda$ and (4.5) holds, applying Lemma 2.2 repeatedly, we obtain the existence of a sequence $(A_j)$ of Lebesgue measurable subsets of $[0,\alpha]$ such that the following hold:

$(A_j)_j$ is a pairwise disjoint family, $\cup A_j = [0,\alpha]$, $|A_j| = \mu(I_j)$, $\frac{1}{|A_j|} \int_{A_j} g \geq \lambda$. Then we define $g_j : [0,|A_j|] \to \mathbb{R}^+$ by $g_j = (g/A_j)^\ast$. Define also for every $j$ a measurable function $\phi_j : I_j \to \mathbb{R}^+$ so that $\phi^\ast_j = g_j$. The existence of such
a function is guaranteed by the fact that \((I_j, \mu/I_j)\) is non-atomic. Here we mean

\[
\mu/I_j(A) = \mu(A \cap I_j) \quad \text{for every} \quad A \subseteq I_j.
\]

Since \((I_j)\) is almost pairwise disjoint family we produce a \(\phi^{(1)} : \cup I_j \to \mathbb{R}^+\) measurable such that \(\phi^{(1)}/I_j = \phi_j\). We set now \(Y = X \setminus \cup I_j\) and \(h : [0, 1 - \alpha] \to \mathbb{R}^+\) by \(h = (g/[\alpha, 1])^*\). Then since \(\mu(Y) = 1 - \alpha\) there exists \(\phi^{(2)} : Y \to \mathbb{R}^+\) such that \((\phi^{(2)})^* = h\).

Set now \(\phi = \begin{cases} 
\phi^{(1)}, & \text{on } \cup I_j \\
\phi^{(2)}, & \text{on } Y.
\end{cases}\)

It is easy to see from the above construction that \(\int_X \phi d\mu = f, \int_X \phi^q d\mu = A\) and \(|||\phi|||_{p, \infty} \leq 1\).

Additionally

\[
\frac{1}{\mu(|I_j|)} \int_{I_j} \phi d\mu = \frac{1}{|A_j|} \int_{A_j} g \geq \lambda \quad \text{for every} \quad j
\]

that is

\(\{M\phi \geq \lambda\} \supseteq \cup I_j, \quad \text{so} \quad \mu(\{M\phi \geq \lambda\}) \geq \alpha\)

and the lemma is proved. \(\Box\)

It is now not difficult to see that we can replace the inequality \(\int_0^\alpha g \geq \alpha \lambda\) in the definition of \(\Delta_{f,A}(\lambda)\) by equality, thus giving \(S_{f,A}(\lambda)\), in such a way that (4.3) remains true, that is

\[
T^{(1)}_{f,A}(\lambda) = \Delta_{f,A}(\lambda) = S_{f,A}(\lambda).
\] (4.6)

This is true since if \(g\) is as in (4.4) there exists \(\beta \geq \alpha\) such that \(\int_0^\beta g = \beta \lambda\).

For \((f, A) \in D\) we set

\[
G_{f,A}(\lambda) = \sup \left\{ \mu(\{M\phi \geq \lambda\}) : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^q d\mu = A \right\}.
\]

It is obvious that \(T^{(1)}_{f,A}(\lambda) \leq G_{f,A}(\lambda)\).
As a matter of fact $G_{f,A}(\lambda)$ has been computed in [3] and was found to be

$$G_{f,A}(\lambda) = \begin{cases} 1, & \lambda \leq f \\ \frac{f}{k}, & f < \lambda < \left(\frac{A}{f}\right)^{1/q} \\ k, & \left(\frac{A}{f}\right)^{1/q-1} \leq \lambda \end{cases}$$

(4.7)

where $k$ is the unique root of the equation

$$\frac{(f - \alpha \lambda)^q}{(1 - \alpha)^q - 1} + \alpha \lambda^q = A \text{ on } [0, \frac{f}{\lambda}], \text{ when } \lambda > \left(\frac{A}{f}\right)^{1/q-1}.$$

We have now the following

**Proposition 4.1** For $(f, A) \in D$, then

$$T_{f,A}^{(1)}(\lambda) \leq \min \left\{1, G_{f,A}(\lambda), \frac{1}{\lambda^p}\right\}.$$

**Proof.** We just need to see that $\mu(\{M\phi \geq \lambda\}) \leq \frac{1}{\lambda^p}$ for every $\phi$ such that $$|||\phi|||_{p,\infty} \leq 1.$$ But if $E = \{M\phi \geq \lambda\}$ we have by the definition of the norm $||| \cdot |||_{p,\infty}$ that $\int_E M\phi \leq \mu(E)^{1-\frac{1}{p}}$. But by (4.3) $\int_E M\phi \geq \lambda \mu(E)$, so that

$$\lambda \mu(E) \leq \mu(E)^{1-\frac{1}{p}} \Rightarrow \mu(E) \leq \frac{1}{\lambda^p}.$$

So Proposition 4.1 is true. □

We prove now the converse of Proposition 4.1 in three steps.

**Proposition 4.2** Let $(f, A) \in D$ and $\lambda$ such that

$$\frac{f}{\lambda} = \min \left\{1, G_{f,A}(\lambda), \frac{1}{\lambda^p}\right\}.$$

(4.8)

Then $T_{f,A}^{(1)}(\lambda) = \frac{f}{k}$.
Proof. We use Lemma 3.4 and equations (4.6). Because of (4.6) we need to find \( g : [0, 1] \to \mathbb{R}^+ \) such that

\[
\int_0^1 g = f, \quad \int_0^1 g^q = A, \quad \|g\|_{p,\infty} \leq 1 \quad \text{and} \quad \int_0^{f/\lambda} g = \frac{f}{\lambda} \cdot \lambda = f
\]

that is \( g \) should be defined on \([0, f/\lambda]\).

We apply Lemma 3.4 with \( \alpha = \frac{f}{\lambda} \).

In fact, since (4.8), is true we have that \( G_{f,A}(\lambda) = \frac{f}{\lambda} \), so, \( \lambda < \left(\frac{f}{A}\right)^{1/q-1} \) which gives (3.11), while \( \frac{f}{\lambda} \leq \frac{1}{\lambda^p} \) gives (3.10). In fact Lemma 3.4 works even with equality on (3.10) as it is easily can be seen. So, in view of (4.6) we have \( T_{f,A}(\lambda) \geq f/\lambda \) and the proposition is proved. \( \square \)

At the next step we have

**Proposition 4.3** Let \((f, A) \in D \) and \( \lambda \) such that

\[
k = \min \left\{ 1, G_{f,A}(\lambda) \frac{1}{\lambda^p} \right\}.
\]

Let (4.9)

Then \( T_{f,A}(\lambda) = k. \)

Proof. Obviously (4.9) gives \( \lambda \geq \left(\frac{f}{A}\right)^{1/q-1} \).

We prove that there exists \( g : [0, 1] \to \mathbb{R}^+ \) such that

\[
\int_0^k g = k\lambda, \quad \int_0^1 g = f, \quad \int_0^1 g^q = A \quad \text{and} \quad \|g\|_{p,\infty} \leq 1.
\]

(4.10)

For this purpose we define:

\[
g := \begin{cases} 
\lambda, & \text{on } [0, k] \\
\frac{f-k\lambda}{1-k}, & \text{on } (k, 1].
\end{cases}
\]

Then, obviously, the first two conditions in (4.10) are satisfied, while

\[
\int_0^1 g^q = \frac{(f-k\lambda)^q}{(1-k)^{q-1}} + k\lambda^q = A,
\]

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by the definition of $k$.

Moreover $|||g|||_{p,\infty} \leq 1$. This is true since $k\lambda \leq k^{1-\frac{q}{p}}$, $f \leq 1$ and the fact that $g$ is constant on each of the intervals $[0,k]$ and $(k,1]$. So we proved that $T^{(1)}_{f,A}(\lambda) \geq k$, that is what we wanted to prove. $\square$

At last we prove

**Proposition 4.4** Let $(f,A) \in D$ and $\lambda$ such that

$$\frac{1}{\lambda^p} = \min \left\{ 1, G_{f,A}(\lambda), \frac{1}{\lambda^p} \right\}. \quad (4.11)$$

Then $T^{(1)}_{f,A}(\lambda) = \frac{1}{\lambda^p}$.

**Proof.** As before we search for a function $g$ such that

$$\int_0^1 g = f, \quad \int_0^1 g^q = A, \quad |||g|||_{p,\infty} \leq 1 \text{ and } \int_0^{1/\lambda^p} g = \frac{1}{\lambda^p} \cdot \lambda = \frac{1}{\lambda^{p-1}}. \quad (4.12)$$

We define

$$\vartheta_\lambda = \frac{\Gamma}{\lambda^{p-q}} + \frac{(f - \frac{1}{\lambda^{p-1}})^q}{\left(1 - \frac{1}{\lambda^p}\right)^{q-1}},$$

and we consider two cases:

i) $\vartheta_\lambda > A$

We search for a function of the form

$$g := \begin{cases} 
(1 - \frac{1}{p})t^{-1/p}, & 0 < t \leq c_1 \\
\mu_2, & c_1 < t \leq \frac{1}{\lambda^p} \\
\mu_3, & \frac{1}{\lambda^p} < t < 1
\end{cases} \quad (4.13)$$

for suitable constants $c_1 \leq \frac{1}{\lambda^p}$, $\mu_2, \mu_3$. Then in view of (4.12) the following must hold:

$$c_1^{1-\frac{1}{p}} + \mu_2 \left(\frac{1}{\lambda^p} - c_1\right) = \frac{1}{\lambda^{p-1}}. \quad (4.14)$$
\[ c_1^{1-\frac{1}{p}} + \mu_2 \left( \frac{1}{\lambda^p} - c_1 \right) + \mu_3 \left( 1 - \frac{1}{\lambda^p} \right) = f \]  
(4.15)

\[ \Gamma c_1^{1-\frac{2}{q}} + \mu_2 \left( \frac{1}{\lambda^p} - c_1 \right) + \mu_3 \left( 1 - \frac{1}{\lambda^p} \right) = A. \]  
(4.16)

Notice that the condition \( \|\|g\|\|_{p,\infty} \leq 1 \) is automatically satisfied because of the form of \( g \) and the previous stated relations.

Now (4.14) and (4.15) give

\[ \mu_3 = \frac{f - \frac{1}{\lambda^{p-1}}}{1 - \frac{1}{\lambda^p}}, \]  
(4.17)

and

\[ \mu_2 = \frac{1}{\lambda^{p-1}} - c_1^{1-\frac{1}{p}}, \]  
(4.18)

while (4.16) gives \( T(c_1) = A \) where \( T \) is defined on \( \left[ 0, \frac{1}{\lambda^p} \right] \) by

\[ T(c) = \Gamma c_1^{1-\frac{2}{q}} + \left( \frac{1}{\lambda^{p-1}} - c_1^{1-\frac{1}{p}} \right)^q \frac{(1 - \frac{1}{\lambda^p})}{(c - \frac{1}{\lambda^p})^{q-1}}. \]

Then

\[ T(0) = \frac{1}{\lambda^{p-q}} + \frac{(f - \frac{1}{\lambda^{p-1}})^q}{(1 - \frac{1}{\lambda^p})^{q-1}}. \]

It is now easy to see that \( T(0) \leq A \) by using that \( F : [0, f/\lambda] \to \mathbb{R}^+ \) defined by

\[ F(t) = \frac{(f - t\lambda)^q}{(1 - t)^{q-1}} + t\lambda^q \]

is increasing, and the definition of \( G_{f,A}(\lambda) \).

Moreover \( \lim_{c \to \frac{1}{\lambda^p}} T(c) = \vartheta_{\lambda} > A \), so by continuity of the function \( t \), we end case i). Now for
ii) \( \vartheta \lambda \leq A \) we search for a function of the form

\[
g := \begin{cases} 
(1 - \frac{1}{p})^{t^{-1/p}}, & 0 < t \leq c_1 \\
\mu_2, & c_1 < t \leq 1
\end{cases}
\]

where \( \frac{1}{\lambda^p} < c_1 \). Similar arguments as in case i) give the result. \( \square \)

From Propositions 4.1 - 4.4 we have now of course

**Theorem 4.1** For

\[
(f,A) \in D, \quad T_{f,A}^{(1)}(\lambda) = \min \left\{ 1, G_{f,A}(\lambda), \frac{1}{\lambda^p} \right\}.
\]

**Remark 4.1** Notice that \( T_{f,A}(\lambda) = T_{f,A}^{(1)}(\lambda) \) for every \( f,A \) such that \( f^q < A \leq \Gamma f^{p-a/p-1} \) and \( 0 < f \leq 1 \). Indeed suppose that \( \alpha = T_{f,A}^{(1)}(\lambda) \). Then there exists \( g : [0,1] \to \mathbb{R}^+ \) such that

\[
\int_0^1 g = f, \quad \int_0^1 g^q = A, \quad \int_0^\alpha g = \alpha \lambda \quad \text{and} \quad |||g|||_{p,\infty} \leq 1.
\] (4.19)

It is easy to see that for every \( \varepsilon > 0 \), small enough we can produce from \( g \) a function \( g_\varepsilon \) satisfying

\[
\int_0^{\alpha-\varepsilon} g_\varepsilon \geq (\alpha - \varepsilon) \lambda, \quad \int_0^1 g_\varepsilon = f, \quad \int_0^1 g_\varepsilon = A + \delta \varepsilon \quad \text{and} \quad |||g_\varepsilon|||_{p,\infty} = 1
\]

and \( \lim_{\varepsilon \to 0^+} \delta \varepsilon = 0 \). This and continuity reasons shows \( T_{f,A}(\lambda) = \alpha \).

iii) The case \( A = f^q \) can be worked out separately because there is essentially unique function \( g \) satisfying \( \int_0^1 g = f, \quad \int_0^1 g^q = f^q \), namely the constant function with value \( f \). \( \square \)

Scaling all the above we have that
Theorem 4.2 For $f, A$ such that $f^q < A \leq F^{p-q/p-1} F^{p(q-1)/p}$ and $0 < f \leq F$ the following hold

$$\sup \left\{ \mu(\{M\phi \geq \lambda \}) : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^q d\mu = A, \|\|\phi\|\|_{p,\infty} = F \right\}$$

$$= \min \left\{ 1, G_{f,A}(\lambda), \frac{F^p}{\lambda^p} \right\} (4.20)$$

and

$$\sup \left\{ \|M\phi\|_{p,\infty} : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^q d\mu = A, \|\|\phi\|\|_{p,\infty} = F \right\} = F. \quad \square$$

References

[1] C. Bennett, R. Sharpley, Interpolation of Operators, Academic Press.
[2] D. L. Burkholder, Martingales and Fourier Analysis in Banach spaces, C.I.M.E. Lectures (Varenna (Como), Italy, 1985), Lecture Notes in Mathematics 1206(1986), 61-108.
[3] D. L. Burkholder, Boundary value problems and sharp inequalities for martingale transforms, Ann. of Prob. 12(1984), 647-702.
[4] L. Grafakos, Classical and Modern Fourier Analysis, Pearson Education, Upper Saddle River, N.J., 2004.
[5] A. D. Melas, The Bellman functions of dyadic-like maximal operators and related inequalities, Adv. Math. 192(2005) 310-340.
[6] A. D. Melas, E. Nikolidakis, On weak type inequalities for dyadic maximal functions, J. Math. Anal. Appl. (2008) 404-410.
[7] A. D. Melas, E. Nikolidakis, Dyadic-like maximal operators on integrable functions and Bellman functions related to Kolcmgorov’s inequality, Transactions of the American Mathematical Society, vol. 362, No 3, March 2010, pages: 1571-1596.
[8] F. Nazarov, S. Treil, The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic Analysis, Algebra i Analyz 8 no. 5 (1996), 32-162.
[9] E. N. Nikolidakis, Extremal problems related to maximal dyadic-like operators, J. Math. Anal. Appl. 369(2010) 377-385.
[10] G. Wang, Sharp maximal inequalities for conditionally symmetric martingales and Brownian motion, Proc. Amer. Math. Soc. 112(1991) 579-586.