FREGIER ELLIPSES

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Abstract. We introduce Fregier ellipses as generalization of the Fregier point and exhibit the amazing angles and areas invariants.

Keywords ellipse, Fregier point, Poncelet closure theorem.

MSC 51M04 and 51N20 and 51N35 and 68T20

1. Introduction

Given an ellipse $E$, the Frégier theorem states that given a point $M$ on the ellipse, all the $M$-rectangular triangles $MNL$ inscribed in the ellipse, have edges $NL$ intersecting at a single point, the “M-Frégier point”. This paper is about a more general setup where internal angle at $M$ vertex is $\theta$ between $0$ and $\pi$ and we prove some properties observed by Dan Reznik and displayed in videos [3, 4] about envelope $E'$ of $MN$ chords.

It should be noticed that the reverse problem, given an angle $\theta$ and a chord $NL$ of the ellipse $E$ find $M$, has at most two solutions. Get them using cyclic quadrilateral opposite angles characterization : set a triangle $NPL$ on $NL$ with angle $\pi - \theta$ at $P$, and intersect $NPL$ circumcircle with the ellipse.

Main results. the envelope of $NL$ edges is an ellipse $E' = E'_{M, \theta}$. The ellipse is degenerated to a point (M-Frégier point) for $\theta = \frac{\pi}{2}$. The locus of $E'$ center is a $E$-concentric ellipse and, amazingly, $E'$ has constant area with varying axes lengths, when $M$ is moving around $E$. Complementary “optic” property : angle of the two tangents from $M$ to $E'$ as constant $\pi - 2\theta$ value. For special value, $\theta = \pi/3$ or $\theta = 2\pi/3$, $MNL$ is circumscribing $E'$.

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Subsidiary result. when $E$ is a circle then $E'$ is a concentric circle (a “Fregier circle”), and given a Poncelet circle-ellipse configuration, the sum of areas of $P_1$-Fregier circles is the same for every $P_1 \ldots P_n$ orbit. We prove it for $n = 3$ and conjecture it for $n > 3$.

2. Computing the envelope

Algebraic computations in proofs are done using trilinear coordinates with $ABC$ reference triangle being the isosceles billiard 3-orbit with $A$ on ellipse minor axis. If $a$ and $b$ are semi-axis lengths then $AB = AC = \frac{2s}{3+h}$ and $BC = \frac{2(1+h)}{3+h}$, where $s$ is $ABC$ semi-perimeter and $h$ is positive root of $(\frac{a}{b})^2 = \frac{(1+h)(3-h)}{(1-h)(3+h)}$.

**Proposition 1.** Given a point $M$ on an ellipse $E$, the envelope of chords $NL$ producing a constant angle $\theta$ for inscribed triangles $MNL$ is included in an ellipse $E'$.

**Proof.** Assuming the envelope is a conic, five tangents are computed as repetition of construction given by Figure 2: select one side (picture : $AC$) of the reference triangle $ABC$ and erect an isosceles triangle (picture : $AEC$), then draw chords $MN$ and $ML$ as parallel segments from $M$ with lines $AE$ and $AC$.

Solving system of five equations for tangents provides an equation, and by duality, equation of $E'$, envelope for the tangents, is obtained.

Given real parameter $u$ we set $M = u + 1 : u(u + 1) : -u$.

Assuming one generic tangent line $1 : m : n$ is intersecting ellipse $E$ at two points $M_{\pm} = -2mn : (m - n + \pm \sqrt{(m - n)^2 - 2(m + n) + 1})n : -(m - n + 1 \pm \sqrt{(m - n)^2 - 2(m + n) + 1})m$.
Proof of proposition is done using CAS software: we compute algebraic value of difference between squared cosine of angle \(M_1M_2\) (using cosines law and \(M, M_1, M_2\) coordinates) and \(\cos^2 \theta\). Equation in \(u\) is deduced when that difference is zero. Equation is identical to the one got by duality, proving the proposition. \(\Box\)

In the general case, tangent points \(T_1\) and \(T_2\) are defining the polar line of \(M\) with respect to \(E'\). Angle \(T_1MT_2\) doesn’t depend from \(M\) as given in the following proposition.

Proposition 2. For \(T_1T_2\) the \(E'\)-polar line of \(M\), angle \(T_1MT_2\) is constant with value \(\pi - 2\theta\).

Proof. Line 1: \(m: n\) is going through \(M\) when \(n = \frac{(mu+1)(u+1)}{u}\).

The two tangent lines to \(E'\) through \(M\) have trilinear coordinates: \((k_1 \pm k_2wu)u : k_3 \pm k_2w : (k_4 \pm k_2(u+1)w)(u+1)\) with

\[
\begin{align*}
  k_1 &= (h^2u-hu^2+h^2+2hu+u^2+2h+1)(u+2) \\
  k_2 &= hu+u^2+h+u+1 \\
  k_3 &= 2hu^3-h^2u^2+3hu^2-2u^3-h^2-3u^2-2h-3u-1 \\
  k_4 &= h^2u^2+hu^3+2hu^2+3hu^2-u^2+h^2+6hu+2h+1 \\
  w &= (\sqrt{3+h})(\sqrt{1-h})\cos \theta
\end{align*}
\]

We deduce trilinear coordinates of \(T_1\) and \(T_2\) by intersecting the two tangent lines with \(E\) and the result in the proposition follows from computation of \(\cos^2 (T_1MT_2)\) by cosines law. \(\Box\)

3. Locus of center and area

Refer to Figure 3 for yellow colored locus center \(K\) (depending on \(M\) and \(\theta\)) of ellipse \(E'\).

Proposition 3. For a varying \(M\) point the locus of \(K\) center of \(E'\) is an ellipse concentric with ellipse \(E\).

Proof. Trilinear coordinates \(\alpha : \beta : \gamma\) of center of ellipse \(K\) are retrieved directly by duality and from the equation of \(E'\):

\[
\begin{align*}
  \alpha &= -(hu+h+2)h-h(u+2)u+u^2+u+1)(3-h^2) + (hu+u^2+h+u+1)(1-h)w^2 \\
  \beta &= ((hu+h+2)h-h(u+4)u+u^2+u+1)(3-h^2) + (hu+u^2+h+u+1)(1-h)w^2 \\
  \gamma &= ((hu+h+2)h-hu^2+u^2+u+1)(3-h^2) + (hu+u^2+h+u+1)(1+h)w^2
\end{align*}
\]

Plugging five values (0, ±1, ±2) for \(u\), we get five points \(K_1, \ldots, K_5\). From them equation of conic locus of \(K\) follows:

\[
k_1\alpha^2 + k_2\beta^2 + k_3\gamma^2 + k_4\beta\gamma + k_5\gamma\alpha + k_6\alpha\beta = 0
\]

with

\[
k_2 = k_3 = ((1+h)(3-h)w^2 + (3-h^2)(3+h)(1-h))(w^2 + 3-h^2)
\]

\[
k_1 = k_2(1+h)^2
\]

\[
k_4 = -((1+h)(3-h)w^2 + 2(3-h^2)^2)(1+2h-h^2)w^2 - (h^4 - 4h^2 + 4h + 3)(3-h^2)^2
\]

\[
k_5 = k_6 = -((1+h)(3-h)w^2 + 2(3-h^2)^2)(1-h)w^2 - (h^4 + 2h^3 - 2h + 3)(3-h^2)^2
\]

Because \(k_1, k_2, k_3\) have same signs, the conic is an ellipse and its center is \(O = 1-h : 1+h : 1+h\), center of ellipse \(E\). \(\Box\)
Proposition 4. Squared area of $E'$ is $k^2$ times squared area of $E$ where:

$$k^2 = \frac{\rho^3 (\rho + 4)^3 \cos^2 \theta}{((\rho + 1)^2 + (2\rho - 1) \cos^2 \theta)^3}$$

with $\rho = (1 - h^2)/2$.

Proof. Extend $LN$ segment shown on Figure 1 and intersect with the two $E'$-tangent lines through $M$ to get a triangle $MNL$ inscribing the ellipse $E'$. The area is computed as an $MNL$ inellipse area by formula given in [5].

The area of $E'$ doesn’t depend from $u$ (parameter defining $M$) hence the area is constant when $M$ is moving around the ellipse.

When the outer ellipse $E$ is a circle, $h = 0$ and $\rho = 1/2$ giving $k^2 = \cos^2 \theta$.

4. Special theta values

One special angle value is $\theta = \pi/2$ where ellipse $E'$ is reduced to $K$ the M-Fregier point.

Another special angle value is $\theta = \pi/4$, because the ellipse $E'$ is “viewed” from $M$ point on $E$ ellipse with a $\pi/2$ angle, $M$ is on the $E'$ orthoptic circle of the caustic. In that case, $E'$ semi-axis lengths $(a_1, b_1)$ can be easily deduced from the formula for $E'$ squared area $(\pi^2 a_1^2 b_1^2)$ and $KM^2 = a_1^2 + b_1^2$. For generic angle $\theta$, use “roulette” construction of orthoptic circle point $Q$ given in [2]: set any tangency point, and draw the tangent and its reflection with respect to $K$ defining a plane band with width $D(u, \theta)$, then draw point $Q$ on the normal at half distance $D/2$ to tangency point outside caustic.

Finally we have $\theta$ special values, $\pi/3$ and $2\pi/3$, with a family of Poncelet configurations.

Proposition 5. For special values, $\theta = \pi/3$ and $\theta = 2\pi/3$, the ellipse $E'$ is inscribed in the $MNL$ triangle.

Proof. Intersect tangents from $M$ to $E'$ with ellipse $E$ to get two points $T_1, T_2$. Line $T_1T_2$ is tangent to $E'$ if and only if $\cos^2 \theta = 1/4$, happening for $\theta = \pi/3$ and
\( \theta = 2\pi/3 \). If \( \theta = \pi/3 \) circumcircle of \( MT_1T_2 \) intersects ellipse \( \mathcal{E} \) a fourth time at point \( M' \) viewing \( T_1T_2 \) and \( \mathcal{E}' \) at \( 2\pi/3 \) angle.

\[ k^2 = \frac{\rho^3(\rho + 4)^{3/4}}{((\rho + 1)^2 + (2\rho - 1)/4)^{3/2}} = \frac{16\rho^3(\rho + 4)^3}{(4\rho^2 + 10\rho + 3)^3} \]

5. FREGIER CIRCLES

The Poncelet circle-ellipse (orange color) concentric configuration is an affine transformation of the Poncelet ellipse-ellipse (green and purple colors on Figure 4) concentric configuration by scaling distances on the major axis.

Given a cyclic n-orbit \( P_1 \ldots P_i \ldots P_n \) there are \( n \) Frégier concentric circles defined by couples \((P_i, \theta_i)\) where \( \theta_i \) is internal angle \( P_{i-1}P_iP_{i+1} \).

**Proposition 6.** When \( n = 3 \), the sum of squared areas of Fregier circles for an n-orbit is a Poncelet invariant : \( \pi b^2(1 - \rho/2) \) in the billiard Poncelet configuration with specular reflection on ellipse border.

**Proof.** Because squared area of the \((P_i, \theta_i)\) Frégier circle is \( \cos^2 \theta_i \) times squared area of outer circle, it suffices to prove that, the sum of squared cosines is the same for any n-orbit. Following the idea given in [1], we write the sum as:

\[ \sum \cos^2 \theta_i = 1/2 \sum (1 + \cos (P_{i-1}OP_{i+1})) = n - 1/(4b^2) \sum (P_{i-1}P_{i+1})^2 \]

from the circle angle theorem applied to chord \( P_{i-1}P_{i+1} \). Equivalently we need to prove that the sum of squared lengths for short diagonals is constant.
When $n = 3$ the short diagonals are simply the $P_1P_2P_3$ triangle edges.

Be ABC the reference triangle for trilinear coordinates. Cartesian plane axes are defined by major and minor axis, with origin at O. Projection of $E = E(u)$ on major axis and scaling with ratio $b/a$ along it, is giving cartesian coordinates of $P$ as:

$$x(u) = \frac{-u(u + 2)s\sqrt{1 - h^2}}{(u^2 + (1 + h)(u + 1))\sqrt{9 - h^2}}$$

$$y(u) = \frac{((h - 1)u^2 + 2(1 + h)u + 2(1 + h))s\sqrt{1 - h}}{(((u^2 + (1 + h)(u + 1))(3 - h))\sqrt{3 + h}}$$

Five tangent lines from vertices $A, B, C$ to caustic ($ABC$ Mandart inellipse) are used to get its conic functions:

$$(h + 1)^2 : (1 - h)^2 : (1 - h)^2 : -2(1 - h)^2 : -2(1 + h)^2(1 - h) : -2(1 + h)^2(1 - h)$$

and by duality, tangent lines coordinates $\alpha : \beta : \gamma$ are verifying equation:

$$(1 + h)^2\beta\gamma + (1 - h)\gamma\alpha + (1 - h)\alpha\beta = 0$$

Be $M(X)$ a generic point different from $E_1(u)$ with $u = u_1$, then line $E_1M = uX : (u + 1)(X + 1)$ belongs to caustic tangents if its trilinear coordinates are satisfying the caustic dual equation. So $X$ is satisfying binomial equation:

$$X^2 - \mu X + \psi = 0$$

with

$$\mu = \frac{(1 - h)u^2 + (3 + h^2)u + (1 + h)^2}{(h - 1)(u + 1)u}$$

$$\psi = \frac{-(h + 1)^2}{(h - 1)u}$$

which has roots $u_2, u_3$ ($u_2 + u_3 = \mu; u_2u_3 = \psi$) defining the two tangents $E_1E_2$ and $E_1E_3$ from $E$ to the caustic.

It results from CAS computation that the sum of the 3-orbit squared edges lengths is

$$\sum_{\text{cyc}} (x(u_i + 1) - x(u_i))^2 + (y(u_i + 1) - y(u_i))^2 = 2b^2(\rho + 4)$$

and the sum of areas of Fregier circles has constant value $\pi b^2(1 - \rho/2)$.

**Conjecture 1.** Previous proposition is valid for every integer $n$ greater than 3.

**Dynamical Systems remark:** the Fregier circles are an example of generalized Poncelet configuration, where the $n$-concentric inner circles are in the same pencil defined by two of them. For $n = 3$, if $R$ is radius of outer circle and $r_1, r_2, r_3$ are radii of inner concentric circles, the proposition has a corollary for a 3-orbit $P_1P_2P_3$ existence condition (equivalent to $\rho$ defined between 0 and 1/2):

$$3/4 \leq (r_1^2 + r_2^2 + r_3^2)/R^2 \leq 1$$

6. **Videos and Symbols**

Animations illustrating some phenomena herein are listed on Table 1. Table 2 provides a quick-reference to the symbols used in this article.
Table 1. Videos of some focus-inversive phenomena. The last column is clickable and provides the YouTube code.

| id | Title                                                                 | youtu.be/              |
|----|----------------------------------------------------------------------|------------------------|
| 01 | Frégier phenomena i: Area-invariant envelope of chords.             | UCC5AT8dh8             |
| 02 | Frégier phenomena iii: Circular envelopes and a new invariant        | AzNxeBU2NTI            |

| symbol | meaning                                    |
|--------|--------------------------------------------|
| $\mathcal{E}, \mathcal{E}'$ | outer and inner ellipses                     |
| $O$    | center of $\mathcal{E}, \mathcal{E}'$       |
| $a, b$ | outer ellipse semi-axes’ lengths            |
| $\rho$ | ratio of inradius to circumradius $r/R$      |
| $x : y$| cartesian coordinates                        |
| $\alpha : \beta : \gamma$ | trilinear coordinates                        |
| $u, u_i$ | parameter of point on $\mathcal{E}$        |
| $h : s$ | elliptic billiard parameters                |
| $E_i$  | point on $\mathcal{E}$                    |
| $P_i$  | point on outer circle                        |
| $r_i$  | Frégier circle radius                       |

Table 2. Symbols used in the article.

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References

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