ON ESTIMATES FOR THE STOKES FLOW IN A SPACE OF BOUNDED FUNCTIONS

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ABSTRACT. We establish a new a priori $L^\infty$-estimate for the composition operator $S(t)\mathcal{P}\partial$ for the Stokes semigroup $S(t)$ subject to the Dirichlet boundary condition and the Helmholtz projection $\mathcal{P}$ for a large class of domains including bounded and exterior domains with $C^3$-boundaries.

1. INTRODUCTION AND MAIN RESULTS

We consider the Stokes equations in a domain $\Omega \subset \mathbb{R}^n, n \geq 2$:

\begin{align*}
\partial_t v - \Delta v + \nabla q &= 0 \quad \text{in } \Omega \times (0, T), \\
\operatorname{div} v &= 0 \quad \text{in } \Omega \times (0, T), \\
v &= 0 \quad \text{on } \partial\Omega \times (0, T), \\
v &= v_0 \quad \text{on } \Omega \times \{t = 0\}.
\end{align*}

Let $S(t) : v_0 \mapsto v(\cdot, t)$ denote the Stokes semigroup and $\mathcal{P}$ denote the Helmholtz projection. In the sequel, $\partial = \partial_j, j \in \{1, \cdots, n\}$, indiscriminately denotes the spatial derivatives. The goal of this paper is to establish a new a priori $L^\infty$-estimate for the composition operator $S(t)\mathcal{P}\partial$.

To state a result, let $C^\infty_c(\Omega)$ denote the space of all smooth functions with compact support in $\Omega$. Let $C_0^1(\Omega)$ denote the closure of $C^\infty_c(\Omega)$ in the Sobolev space $W^{1,\infty}(\Omega)$. One of our main results is the following:

Theorem 1.1. Let $\Omega$ be a bounded or an exterior domain in $\mathbb{R}^n, n \geq 2$, with $C^3$-boundary. For $\alpha \in (0, 1)$ and $T_0 > 0$, there exists a constant $C$ such that the estimate

\begin{equation}
\left\|S(t)\mathcal{P}\partial f\right\|_{L^\infty(\Omega)} \leq \frac{C}{t^\alpha} \left\|f\right\|_{L^\infty(\Omega)}^{1-\alpha} \left\|\nabla f\right\|_{L^\infty(\Omega)}^\alpha,
\end{equation}

holds for $f \in C_0^1 \cap W^{1,2}(\Omega)$ and $t \leq T_0$. When $\Omega$ is bounded, (1.5) holds for $T_0 = \infty$.

The composition operator $S(t)\mathcal{P}\partial$ as well as the Stokes semigroup $S(t)$ plays a fundamental role for studying the nonlinear Navier-Stokes equations. It is well known that $S(t)\mathcal{P}\partial$ acts as a bounded operator on $L^p$ ($1 < p < \infty$) and satisfies the estimate of the form

\begin{equation}
\left\|S(t)\mathcal{P}\partial f\right\|_{L^p(\Omega)} \leq C_p \frac{t}{t^2} \left\|f\right\|_{L^p(\Omega)},
\end{equation}

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for \( f \in W^{1,p}(\Omega) \) and \( t \leq T_0 \). Since the Helmholtz projection acts as a bounded operator on \( L^p \), the estimate (1.6) follows from the analyticity of the Stokes semigroup on \( L^p \) \([27]\). \([16]\). Although the Stokes semigroup is analytic on \( L^\infty \) as recently proved in \([4], [5]\) (also \([6]\)), the \( L^\infty \)-estimate (1.5) does not follow from the usual analytic semigroup theory since the Helmholtz projection \( \mathcal{P} \) is not bounded on \( L^\infty \).

In the sequel, we establish the a priori estimate for

\[
N(v, q)(x, t) = |v(x, t)| + t^\frac{1}{2} |\nabla v(x, t)| + t |\nabla^2 v(x, t)| + t |\partial_t v(x, t)| + t |\nabla q(x, t)|
\]

of the form

\[
(1.7) \quad \sup_{0 \leq t \leq T_0} t^{\frac{1}{2} + s + \frac{|s|}{2}} \|N(v, q)\|_{L^\infty(\Omega)}(t) \leq C \int_{\Omega} |f|^{(\alpha)}
\]

for all solutions \((v, q)\) of (1.1)–(1.4) for \( v_0 = \mathcal{P} \partial f \) with some constants \( T_0 \) and \( C \), where \( |f|^{(\alpha)} \) denotes the Hölder semi-norm of \( f \) in \( \Omega \), i.e.,

\[
|f|^{(\alpha)}_{\Omega} = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} \middle| x, y \in \Omega, \ x \neq y \right\}.
\]

Since the Hölder semi-norm \( |f|^{(\alpha)}_{\Omega} \) is estimated by \( \|f\|_{L^\infty(\Omega)}^{1-\alpha} \|\nabla f\|_{L^\infty(\Omega)}^\alpha \) for \( f \in C^\infty(\Omega) \), the estimate (1.7) follows from the a priori estimate (1.6). The solutions \((v, q)\) of the Stokes equations (1.1)–(1.4) are given by the Stokes semigroup \( S(t) \) and the Helmholtz projection \( \mathcal{P} \) on \( L^p \). We call \((v, q)\) \( L^p \)-solution. We prove Theorem 1.1 from the following:

**Theorem 1.2.** Let \( \Omega \) be a bounded or an exterior domain with \( C^3 \)-boundary. Let \( \alpha \in (0, 1) \) and \( p > n/(1 - \alpha) \). For \( T_0 > 0 \) there exists a constant \( C \) such that (1.7) holds for all \( L^p \)-solutions \((v, q)\) for \( v_0 = \mathcal{P} \partial f, \ f \in C^\infty_0(\Omega) \). Moreover, the estimate

\[
(1.8) \quad \sup_{0 \leq t \leq T_0} t^{\frac{1}{2} + s + \frac{|s|}{2}} \|\partial_t^s \partial_y^k S(t) \mathcal{P} \partial f\|_{L^\infty(\Omega)}(t) \leq C \|f\|_{L^\infty(\Omega)}^{1-\alpha} \|\nabla f\|_{L^\infty(\Omega)}^\alpha
\]

holds for \( f \in C^1_0 \cap W^{1,2}(\Omega) \) and \( 0 \leq 2s + |k| \leq 2 \).

We prove (1.7) by a blow-up argument. It is shown in \([4], [5]\) that a blow-up argument is applicable to prove an a priori estimate for (1.1)–(1.4) for not only bounded or exterior domains but domains for which some a priori estimate holds for the Neumann problem of the Laplace equation. We call such a domain **admissible** and it is proved in \([4], [5]\) that bounded and exterior domains of class \( C^3 \) are admissible. (A perturbed half space of class \( C^3 \) for \( n \geq 3 \) is also admissible \([11]\)). In order to establish (1.7), we introduce a stronger term of admissible called **strongly admissible**. The term of strongly admissible is explained later in the introduction. We prove that bounded and exterior domains of class \( C^3 \) are also strongly admissible.

We prove (1.7) for general strongly admissible, uniformly \( C^3 \)-domains by the \( \tilde{L}^p \)-theory developed in \([11], [12], [13]\). It is proved in these works that the Helmholtz projection yields a unique decomposition on \( \tilde{L}^p = L^p \cap L^2 \) \((p \geq 2)\) and the Stokes semigroup is analytic on \( \tilde{L}^p \) for general uniformly \( C^2 \)-domains. Thus, solutions of (1.1)–(1.4) exist in a general uniformly \( C^2 \)-domain. We prove (1.7) for their \( \tilde{L}^p \)-solutions. The following Theorem 1.3 is a general form of Theorem 1.2.
\textbf{Theorem 1.3.} Let $\Omega$ be a strongly admissible, uniformly $C^3$-domain. Let $\alpha \in (0, 1)$ and $p > n/(1 - \alpha)$. Then, the estimate (1.7) holds for all $L^p$-solutions $(v, q)$ for $v_0 = \nabla \partial f$, $f \in C_c^\infty(\Omega)$. Moreover, (1.8) holds for $f \in C_0^1 \cap W^{1,2}(\Omega)$.

\textbf{Remarks} 1.4. (i) It is proved in [4] that the Stokes semigroup $S(t)$ is a $C_0$-analytic semigroup on $C_{0,\sigma}(\Omega)$ for admissible, uniformly $C^3$-domains. Here, $C_{0,\sigma}(\Omega)$ denotes the $L^\infty$-closure of $C_c^\infty(\Omega)$, the space of all smooth solenoidal vector fields with compact support in $\Omega$. The estimate (1.8) implies that the composition $S(t)\nabla \partial$ is extendable to a bounded operator $S(t)\nabla \partial$ from $C_0^1(\Omega)$ to $C_{0,\sigma}(\Omega)$ for strongly admissible, uniformly $C^3$-domains.

(ii) The estimate (1.5) is consistent with the $L^p$-estimate $(1 < p < \infty)$,

$$\|S(t)\nabla \partial f\|_{L^p(\Omega)} \leq \frac{C_p}{t^{\frac{1}{p}}} \|f\|_{L^p(\Omega)}^{\frac{1 - \alpha}{p}} \|\partial f\|_{L^p(\Omega)}^\alpha,$$

for all $0 \leq \alpha \leq 1$, $f \in W^{1,p}(\Omega)$ and $t \leq T_0$. When $\Omega$ is the whole space, (1.5) is valid for $\alpha = 0$ [17], [24] (see [28], [8] for a half space).

(iii) The estimate (1.8) is fundamental for studying the Navier-Stokes equations on $L^\infty$. So far, $L^\infty$-type results for the Navier-Stokes equations are only known for the whole space and a half space (e.g., [17], [28], [8]) since the $L^\infty$-estimate for $S(t)\nabla \partial$ as well as $S(t)$ was unknown. Recently, a local existence theorem for the Navier-Stokes equations on $L^\infty$ is established in [3] based on main results of this paper.

Let us sketch the proof of the a priori estimate (1.7).

When $\Omega$ is the whole space, the Stokes semigroup agrees with the heat semigroup (i.e., $v = e^{t\Delta}f$, $\nabla q = 0$). We estimate $v = \partial e^{t\Delta}f$ by the Hölder semi-norm of $f$, i.e.,

$$\|\partial e^{t\Delta}f\|_\infty \leq \frac{C_{\alpha}}{t^{\frac{1}{p}}} \|f\|^{\alpha} \infty_{\mathbb{R}^n}.$$

Since the Hölder semi-norm of $f$ is estimated by $\|f\|^{\alpha}_{\mathbb{R}^n}$ (see Proposition 3.1), the estimate (1.7) holds for $0 < \alpha < 1$. (We are able to prove the case $\alpha = 0$ by estimating the Oseen kernel $K_t$, i.e., $e^{t\Delta}f = K_t * f$).

We prove (1.7) by a blow-up argument. For simplicity, we set $\gamma = (1 - \alpha)/2$. We prove the existence of constants $T_0$ and $C$ such that (1.7) holds for all $f \in C_c^\infty(\Omega)$. Suppose on the contrary that (1.7) were false. Then, there would exist a sequence of solutions for (1.1)–(1.4), $(v_m, q_m)$ for $v_0 = \nabla \partial f_m$ such that

$$\sup_{0 \leq t \leq 1/m} t^\gamma \|N(v_m, q_m)\|_{L^\infty(\Omega)}(t) > m \|f_m\|^{\alpha} \Omega.$$

We take a point $t_m \in (0, 1/m)$ such that

$$t_m^\gamma \|N(v_m, q_m)\|_{L^\infty(\Omega)}(t_m) \geq \frac{1}{2} M_m, \quad M_m = \sup_{0 \leq t \leq 1/m} t^\gamma \|N(v_m, q_m)\|_{L^\infty(\Omega)}(t).$$
and normalize \((v_m, q_m)\) by dividing by \(M_m\) to get \(\tilde{v}_m = v_m / M_m\), \(\tilde{q}_m = q_m / M_m\) and \(\tilde{f}_m = f_m / M_m\) satisfying

\[
\sup_{0 \leq t \leq t_m} t^\gamma \|N(\tilde{v}_m, \tilde{q}_m)\|_{L^\infty(\Omega)}(t) \leq 1,
\]

\[
t_m^\gamma \|N(\tilde{v}_m, \tilde{q}_m)\|_{L^\infty(\Omega)}(t_m) \geq \frac{1}{2},
\]

\[
\left[\tilde{f}_m\right]_{\Omega} < \frac{1}{m}.
\]

Then, we rescale \((\tilde{v}_m, \tilde{q}_m)\) around a point \(x_m \in \Omega\) such that

\[
t_m^\gamma N(\tilde{v}_m, \tilde{q}_m)(x_m, t_m) \geq \frac{1}{4}
\]

to get a blow-up sequence \((u_m, p_m)\) of the form

\[
u_m(x, t) = t_m^\gamma \tilde{v}_m(x + \frac{1}{t_m} x, t_m), \quad p_m(x, t) = t_m^{\gamma + \frac{1}{t_m}} \tilde{q}_m(x + \frac{1}{t_m} x, t_m),
\]

and

\[g_m(x) = t_m^\gamma \tilde{f}_m(x + \frac{1}{t_m} x).\]

The blow-up sequence \((u_m, p_m)\) satisfies (1.1)–(1.4) for \(u_{0,m} = \mathbb{P}_{\Omega_m} \partial g_m \) in \(\Omega_m \times (0, 1)\) and the rescaled domain \(\Omega_m\) expands to either the whole space or a half space as \(m \to \infty\).

The basic strategy is to prove a compactness of the blow-up sequence \((u_m, p_m)\) and a uniqueness of a blow-up limit. If \((u_m, p_m)\) converges to a limit \((u, p)\) strongly enough, one gets a bound from below \(N(u, p)(0, 1) \geq 1/4\). On the other hand, \((u, p)\) solves a limit problem for \(u(\cdot, 0) = 0\) in a suitable sense. If the limit \((u, p)\) is unique, \(u \equiv 0\) and \(\nabla p \equiv 0\) follow. This yields a contradiction. For the compactness of \((u_m, p_m)\), we apply the local Hölder estimates for (1.1)–(1.4) proved in [41] (to get an equi-continuity of \((u_m, p_m)\)). For the uniqueness of \((u, p)\), we extend a uniqueness theorem in a half space due to [28] for velocities which may not be bounded as \(t \downarrow 0\). When \(\Omega_m\) expands to the whole space, the uniqueness is reduced to the heat equation.

A key step of the proof is to get a sufficiently strong initial condition for the blow-up limit \((u, p)\) in order to apply a uniqueness theorem. If the initial data \(u_{0,m}\) does not involve the Helmholtz projection \(\mathbb{P}_{\Omega_m}\), it is easy to see \(u_{0,m} \to 0\) (in a suitable weak sense) as \([g_m]_{\Omega_m}^{(\alpha)} \to 0\) and \(m \to \infty\). However, it is non-trivial whether \(u_{0,m} = \mathbb{P}_{\Omega_m} \partial g_m \to 0\) as \([g_m]_{\Omega_m}^{(\alpha)} \to 0\) because of the term \(\nabla \Phi_{0,m} = Q_{\Omega_m} \partial g_m\) where \(Q_{\Omega_m} = I - \mathbb{P}_{\Omega_m}\). When \(\Omega\) is the whole space, the projection \(Q_{\mathbb{R}^n}\) has an explicit form by the fundamental solution of the Laplace equation \(\Delta\). In fact, \(\nabla \Phi_{1,m} = Q_{\mathbb{R}^n} \partial f\) agrees with \(-\nabla \text{div} h\) for \(h = E + \partial f\) so the Hölder semi-norm of \(\nabla h\) is estimated by \([f]_{\mathbb{R}^n}^{(\alpha)}\) and

\[
[\Phi_{1,m}]_{\mathbb{R}^n}^{(\alpha)} \leq C \alpha \left[ f \right]_{\mathbb{R}^n}^{(\alpha)}.
\]

Since the Hölder estimate (1.9) is scale invariant, it is inherited to \(\nabla \Phi_{1,m} = Q_{\mathbb{R}^n} \partial g_m\). We need a corresponding estimate for \(\nabla \Phi_0 = Q_{\Omega} \partial f\). For this purpose, we consider the Neumann problem of the Laplace equation

\[
\Delta \Phi = 0 \quad \text{in} \ \Omega, \quad \frac{\partial \Phi}{\partial n} = \text{div}_\Omega (A n) \quad \text{on} \ \partial \Omega
\]
for skew-symmetric matrix-valued functions $A \in C^q(\Omega)$ for $\alpha \in (0, 1)$, where $\text{div}_\alpha \Omega$ denotes the surface divergence on $\partial \Omega$ and $n = n_\Omega$ denotes the unit outward normal vector field on $\partial \Omega$. For a skew-symmetric $A$, $An$ is a tangential vector field on $\partial \Omega$. Moreover, $\text{div}_\alpha \Omega(An) = 0$ if $A$ is constant. We call $\Omega$ admissible for $\alpha \in (0, 1)$ if the a priori estimate

$$
\sup_{x \in \Omega} d^{1-\alpha}_\Omega(x) |\nabla \Phi(x)| \leq C\left[A\right]_\Omega^{(\alpha)},
$$

holds for all solutions of (1.10). Here, $d_\Omega(x)$ denotes the distance from $x \in \Omega$ to $\partial \Omega$. When $\alpha = 0$, we replace the right-hand side by the sup-norm of $A$ on $\partial \Omega$ and call the corresponding notion admissible for $\alpha = 0$. We call $\Omega$ strongly admissible if $\Omega$ is admissible for all $\alpha \in [0, 1)$. In this paper, we prove that bounded and exterior domains of class $C^3$ are strongly admissible.

The estimate (1.11) implies a scale invariant estimate corresponding to (1.9). We decompose $\nabla \Phi_0 = \Omega \partial f$ into two terms $\nabla \Phi_1 = \Omega f \partial f$ and $\nabla \Phi_2$ (by the zero extension of $f$ to $\mathbb{R}^n \setminus \Omega$). Then, $\Phi_2$ solves the Neumann problem (1.10) for $A = \nabla h - \nabla^T h$. We estimate $\nabla \Phi_0 = \nabla \Phi_1 + \nabla \Phi_2$ through the estimate (1.11) by

$$
\left[\Phi_1\right]_{\mathbb{R}^n}^{(\alpha)} + \sup_{x \in \Omega} d^{1-\alpha}_\Omega(x) |\nabla \Phi_2(x)| \leq C\left[A\right]_\Omega^{(\alpha)}.
$$

Since (1.12) is scale invariant, it is inherited to $\nabla \Phi_{0,m} = \nabla_{\Omega_m} g_m$ so $\nabla \Phi_{0,m}$ tends to zero as $[g_m]_{\Omega_m}^{(\alpha)} \rightarrow 0$. This yields a sufficiently strong initial condition $u(\cdot, 0) = 0$ for the blow-up limit $(u, p)$.

Actually, we used the estimate (1.11) for $\alpha = 0$ in order to prove analyticity of the Stokes semigroup on $C_{0,\omega}$ by a similar blow-up argument [4]. Since the pressure $p_m$ solves the Neumann problem (1.10) for $A = -\nabla u_m + \nabla^T u_m$, the a priori estimate (1.11) for $\alpha = 0$ implies a scale invariant estimate for $\nabla p_m$ in terms of velocity on $L^\infty$ (harmonic-pressure gradient estimate). The harmonic-pressure gradient estimate implies a necessary time Hölder continuity of $p_m$ for the compactness of $(u_m, p_m)$ and a decay condition $\nabla p \rightarrow 0$ as $x_m \rightarrow \infty$ for the uniqueness of the blow-up limit $(u, p)$.

This paper is organized as follows. In Section 2, we define the term of strongly admissible and prove that bounded and exterior domains of class $C^3$ are strongly admissible. In Section 3, we prove the Hölder-type estimate (1.12). In Section 4, we recall the $L^p$-theory and review the local Hölder estimates for the Stokes equations. In Section 5, we prove a uniqueness theorem for the Stokes equations in a half space. In Section 6, we prove Theorem 1.3. After the proof of Theorem 1.3, we complete the proof of Theorem 1.2 and 1.1. In Appendix A, we review $L^1$-type results for the Stokes equations in a half space used in Section 5.

2. Strongly admissible domains

In this section, we introduce the term of strongly admissible and prove that bounded and exterior domains of class $C^3$ are strongly admissible (Theorems 2.9 and 2.11). The proof is by a blow-up argument and parallel to the case $\alpha = 0$ as in the previous works [4], [5].

2.1. A priori estimates for the Neumann problem.
Let \( \Omega \) be a domain in \( \mathbb{R}^n, n \geq 2, \partial \Omega \neq \emptyset \). We say that \( \partial \Omega \) is \( C^k (k \geq 1) \) if for each \( x_0 \in \partial \Omega \), there exist constants \( \alpha, \beta, K \) and a \( C^k \)-function \( h = h(y') \) such that (up to rotation and translation if necessary) we have

\[
U(x_0) \cap \Omega = \left\{ (y', y_n) \in \mathbb{R}^n \mid h(y') < y_n < h(y') + |y'| < \alpha \right\},
\]

\[
U(x_0) \cap \partial \Omega = \left\{ (y', y_n) \in \mathbb{R}^n \mid y_n = h(y'), |y'| < \alpha \right\},
\]

\[
\sup_{|l| \leq k, |y'| < \alpha} |\partial^l_y h(y')| \leq K, \quad \nabla h(0) = 0, \quad h(0) = 0,
\]

with the neighborhood of \( x_0 \),

\[
U(x_0) = \left\{ (y', y_n) \in \mathbb{R}^n \mid h(y') - \beta < y_n < h(y') + \beta, |y'| < \alpha \right\}.
\]

Here, \( \partial^l_y = \partial^l_{y_1} \cdots \partial^l_{y_n} \) for a multi-index \( l = (l_1, \ldots, l_n) \) and \( \partial_x = \partial / \partial x \) as usual and \( \nabla' \) denotes the gradient in \( \mathbb{R}^{n-1} \). If \( h \) is just Lipschitz continuous, we call \( \partial \Omega \) Lipschitz boundary.

Moreover, if we are able to take uniform constants \( \alpha, \beta, K \) independent of each \( x_0 \in \partial \Omega \), we call \( \Omega \) a uniformly \( C^k \)-domain (Lipschitz domain) of type \( \alpha, \beta, K \) as defined in [26] I.3.2. In order to distinguish \( \alpha, \beta, K \) from Hölder exponents, we also write \( \alpha', \beta', K' \).

We begin with the term of admissible for \( \alpha = 0 \). Let \( \Omega \) be a domain in \( \mathbb{R}^n \) with \( C^1 \)-boundary. We consider the Neumann problem of the Laplace equation,

\[
\Delta \Phi = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial \Phi}{\partial n} = \text{div}_{\partial \Omega} (An) \quad \text{on} \quad \partial \Omega,
\]

for skew-symmetric matrix-valued functions \( A \), where \( \text{div}_{\partial \Omega} = \text{tr} \nabla_{\partial \Omega} \) denotes the surface divergence on \( \partial \Omega \) and \( \nabla_{\partial \Omega} = \nabla - n(n \cdot \nabla) \) is the gradient on \( \partial \Omega \) for \( n = n_\Omega \). Since \( A = (a_{ij}) \) is skew-symmetric, \( An = (\sum^n_{j=1} a_{ij} n_j) \) is a tangential vector field on \( \partial \Omega \). Let \( BC(\overline{\Omega}) \) denote the space of all bounded and continuous functions in \( \overline{\Omega} \). Let \( BC_{sk}(\overline{\Omega}) \) denote the space of all skew-symmetric matrix-valued functions \( A \in BC(\overline{\Omega}) \). We call \( \Phi \in L^1_{\text{loc}}(\overline{\Omega}) \) a solution of (2.1) for \( A \in BC_{sk}(\overline{\Omega}) \) if \( \Phi \) satisfies

\[
\sup_{x \in \Omega} d_{\Omega}(x)|\nabla \Phi(x)| < \infty,
\]

and

\[
\int_\Omega \Phi \Delta \varphi dx = \int_{\partial \Omega} An \cdot \nabla \varphi d\mathcal{H}
\]

for all \( \varphi \in C^2_c(\overline{\Omega}) \) satisfying \( \partial \varphi / \partial n = 0 \) on \( \partial \Omega \), where \( d\mathcal{H} \) denotes the surface element of \( \partial \Omega \).

The term of admissible for \( \alpha = 0 \) is defined by an a priori estimate for (2.1).

**Definition 2.1** (Admissible for \( \alpha = 0 \)). Let \( \Omega \) be a domain in \( \mathbb{R}^n \) with \( C^1 \)-boundary. We call \( \Omega \) admissible for \( \alpha = 0 \) if there exists a constant \( C = C_{\Omega} \) such that the a priori estimate

\[
\sup_{x \in \Omega} d_{\Omega}(x)|\nabla \Phi(x)| \leq C \|A\|_{L^\infty(\partial \Omega)}
\]

holds for all solutions of (2.1) for \( A \in BC_{sk}(\overline{\Omega}) \).
Remark 2.2. The term of admissible is first introduced in [4] by using the Helmholtz projection \( \mathbb{P} \) on \( L^p = L^p \cap L^2 \) for uniformly \( C^1 \)-domains. We call \( \Omega \) admissible in the sense of [4] Definition 2.3] if there exists a constant \( C \) such that the estimate
\[
\sup_{x \in \Omega} d_{\Omega}(x)(\mathbb{Q}_{\Omega} \nabla \cdot f)(x) \leq C \|f\|_{L^{p}(\partial \Omega)}
\]
holds for all matrix-valued functions \( f = (f_{ij}) \in C^{1}(\overline{\Omega}) \) such that \( \nabla \cdot f = (\sum_{i} \partial_{ij} f_{ij}) \in L^{p} \) \( (p \geq n) \). In fact, if there exists a constant \( C \) such that the estimate (2.5) follows from (2.4). Although the term of admissible for \( \alpha = 0 \) is stronger than admissible, we are able to prove that bounded and exterior domains of class \( C^3 \) are also admissible for \( \alpha = 0 \) by a blow-up argument as in [4, 5] (see also Remark 2.10).

We define the term of admissible for \( \alpha \in (0, 1) \). Let \( C^{\alpha}(\overline{\Omega}) \) denote the space of all \( \alpha \)-th Hölder continuous functions \( A \) in \( \overline{\Omega} \). Let \( C^{\alpha}_{sk}(\overline{\Omega}) \) denote the space of all skew-symmetric matrix-valued functions \( A \in C^{\alpha}(\overline{\Omega}) \). We call \( \nabla \Phi \in L^{1}_{\text{loc}}(\overline{\Omega}) \) a solution of (2.1) for \( A \in C^{\alpha}_{sk}(\overline{\Omega}) \) if \( \Phi \) satisfies
\[
\sup_{x \in \Omega} d^{1-\alpha}_{\Omega}(x) |\nabla \Phi(x)| < \infty,
\]
and
\[
\int_{\Omega} \nabla \Phi \cdot \nabla \varphi \, dx = -\int_{\partial \Omega} A_{n} \cdot \nabla \varphi \, dH
\]
for all \( \varphi \in C^{1}_{c}(\overline{\Omega}) \).

We define the term of strongly admissible by a priori estimates for \( \alpha \in [0, 1) \).

Definition 2.3 (Strongly admissible). Let \( \Omega \) be a domain in \( \mathbb{R}^{n} \) with \( C^{1} \)-boundary. We call \( \Omega \) admissible for \( \alpha \in (0, 1) \) if there exists a constant \( C = C_{\alpha, \Omega} \) such that the a priori estimate
\[
\sup_{x \in \Omega} d^{1-\alpha}_{\Omega}(x) |\nabla \Phi(x)| \leq C [A]^{(\alpha)}_{\Omega}
\]
holds for all solutions of (2.1) for \( A \in C^{\alpha}_{sk}(\overline{\Omega}) \). We call \( \Omega \) strongly admissible if \( \Omega \) is admissible for all \( \alpha \in [0, 1) \).

Remarks 2.4. (i) The constants in (2.4) and (2.8) are invariant of dilation and translation of \( \Omega \).

(ii) A half space is strongly admissible. Let \( E \) denote the fundamental solution of the Laplace equation, i.e., \( E(x) = C_{n}/|x|^{n-2} \) for \( n \geq 3 \) and \( E(x) = -1/(2\pi) \log |x| \) for \( n = 2 \) with the constant \( C_{n} = (an(n-2))^{-1} \) and the volume of \( n \)-dimensional unit ball \( a \). Since solutions of the Neumann problem (2.1) are expressed by
\[
\partial_{\nu} \Phi(x) = 2 \int_{\partial \Omega^{n}} \partial_{\nu} E(x - y') \text{div}_{\partial \Omega^{n}} w(y') \, dy' = -2 \int_{\partial \Omega^{n}} \nabla_{y'} \partial_{\nu} E(x - y') \cdot (a_{n}(y') - a_{n}(x')) \, dy',
\]
for \( w = An_{\mathbb{R}^n_+} = a_n \), it follows that

\[
|\partial_x \Phi(x)| \leq C |A|^{(\alpha)}_{\mathbb{R}^n_+} \int_{\partial \mathbb{R}^n_+} \frac{|x' - y'|^\alpha}{(|x' - y'|^2 + x_n^2)^{\frac{3}{2}}} \, dy'
= C_{\alpha} x_n^{1-\alpha} |A|^{(\alpha)}_{\mathbb{R}^n_+} \int_{\partial \mathbb{R}^n_+} \frac{|z'|^\alpha}{(|z'|^2 + 1)^{\frac{3}{2}}} \, dz'.
\]

The right-hand side is finite for \( \alpha \in [0, 1) \) so (2.8) is valid for \( \Omega = \mathbb{R}^n_+ \) and \( x_n = d_{\mathbb{R}^n_+}(x) \).

(iii) The a priori estimates (2.4) and (2.8) can be viewed as the \( L^\infty \)-estimates of the Poisson semigroup

\[
\|\partial_{\text{tan}} e^{sA}g\|_{L^\infty(\mathbb{R}^{n-1})} \leq \frac{C}{s} \|g\|_{L^\infty(\mathbb{R}^{n-1})},
\]

(2.9)

\[
\|\partial_{\text{tan}} e^{sA}g\|_{L^\infty(\mathbb{R}^{n-1})} \leq \frac{C}{s^{1-\alpha}} |g|^{(\alpha)}_{\mathbb{R}^n_+} s > 0.
\]

(2.10)

Here, the Poisson semigroup \( e^{sA} \) is defined by

\[
e^{sA}g = -2 \int_{\partial \mathbb{R}^n_+} \frac{\partial E}{\partial s}(x' - y', s)g(y')dy',
\]

and \( \partial_{\text{tan}} = \partial_j \) for \( j \in \{1, \cdots, n-1\} \). The Poisson semigroup is an analytic semigroup on \( L^p(\mathbb{R}^{n-1}) \) for \( 1 \leq p \leq \infty \) and its generator is given by \( A = -(-\Delta_{\text{tan}})^{1/2} \) (see [7, Example 3.7.9]). Since \( p = e^{sA}g \) satisfies \( \Delta p = 0 \) in \( \{x_n > 0\} \) and \( p = g \) on \( \{x_n = 0\} \), solutions of the Neumann problem (2.1) are expressed by

\[
\Phi(x', x_n) = -\int_{x_n}^{\infty} e^{sA} \text{div}_{\partial \mathbb{R}^n_+} w ds.
\]

The estimate (2.4) follows from (2.9). Similarly, (2.8) follows from (2.10).

(iv) For a skew-symmetric constant matrix \( A = (a_{ij}) \), the surface divergence of \( An \) vanishes, i.e., \( \text{div}_{\partial \Omega}(An) = 0 \), in the sense that

\[
\int_{\partial \Omega} An \cdot \nabla \varphi d\mathcal{H} = 0 \quad \text{for} \ \varphi \in C^1_c(\overline{\Omega}).
\]

In fact, it follows that

\[
\int_{\partial \Omega} An \cdot \nabla \varphi d\mathcal{H} = \sum_{i,j} \int_{\partial \Omega} a_{ij} n^j \partial_i \varphi d\mathcal{H}
= \sum_{i,j} \int_{\Omega} a_{ij} \partial_j \partial_i \varphi dx
= \sum_{i,j} \int_{\Omega} a_{ji} \partial_j \varphi dx = -\sum_{i,j} \int_{\Omega} a_{ij} \partial_j \partial_i \varphi dx.
\]

The right-hand side equals zero.
2.2. Uniqueness of the Neumann problem.

We prove the uniqueness of the Neumann problem (2.1) in order to prove the a priori estimate (2.8) by a blow-up argument.

**Lemma 2.5.** Let $\nabla \Phi \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfy

$$
\int_{\mathbb{R}^n} \nabla \Phi \cdot \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in C^1_0(\mathbb{R}^n).
$$

Assume that

$$
\sup_{x \in \mathbb{R}^n} x_n^{1-\alpha} |\nabla \Phi(x)| < \infty,
$$

for some $\alpha \in (0, 1)$. Then, $\nabla \Phi \equiv 0$.

**Proof.** We consider the even extension of $\Phi$ to $\mathbb{R}^n$, i.e.,

$$
\hat{\Phi}(x', x_n) = \begin{cases} 
\Phi(x', x_n) & \text{for } x_n \geq 0, \\
\Phi(x', -x_n) & \text{for } x_n < 0.
\end{cases}
$$

Then, $\hat{\Phi}$ is weakly harmonic in $\mathbb{R}^n$. In fact, for $\tilde{\varphi} \in C^2_c(\mathbb{R}^n)$, the function $\hat{\Phi}$ satisfies

$$
\int_{\mathbb{R}^n} \hat{\Phi}(x) \Delta \tilde{\varphi}(x) \, dx = \int_0^\infty \int_{\mathbb{R}^{n-1}} \Phi(x', x_n) \Delta \tilde{\varphi}(x', x_n) \, dx \, dx + \int_{-\infty}^0 \int_{\mathbb{R}^{n-1}} \Phi(x', -x_n) \Delta \tilde{\varphi}(x', x_n) \, dx \, dx
$$

$$
= \int_0^\infty \int_{\mathbb{R}^{n-1}} \Phi(x', x_n) \Delta \tilde{\varphi}(x', x_n) \, dx \, dx.
$$

Since $\varphi(x', x_n) = \tilde{\varphi}(x', x_n) + \tilde{\varphi}(x', -x_n)$ is $C^2$ in $\mathbb{R}^n$ and satisfies $\partial \varphi / \partial x_n = 0$ on $\{x_n = 0\}$, the right-hand side equals zero by (2.11). We multiply the mollifier $\eta_\varepsilon$ by $\Phi$ and observe that $\Phi_\varepsilon = \hat{\Phi} \ast \eta_\varepsilon$ is harmonic (and smooth) in $\mathbb{R}^n$. By (2.12), $\nabla \Phi_\varepsilon$ is bounded in $\mathbb{R}^n$ and decays as $x_n \to \infty$. We apply the Liouville theorem and conclude that $\nabla \Phi_\varepsilon \equiv 0$. Since $\nabla \Phi_\varepsilon \to \nabla \Phi$ locally uniformly in $\mathbb{R}^n$, $\nabla \Phi \equiv 0$ follows. \qed

We next prove the uniqueness theorem for bounded and exterior domains. Note that $\nabla \Phi \in L^p_{\text{loc}}(\Omega)$ for $1 \leq p < 1/(1 - \alpha)$ provided that $d_\Omega^{1-\alpha} \nabla \Phi \in L^\infty(\Omega)$. In particular, $\nabla \Phi \in L^p(\Omega)$ when $\Omega$ is bounded.

**Lemma 2.6.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $C^2$-boundary. Let $\nabla \Phi \in L^1_{\text{loc}}(\overline{\Omega})$ satisfy

$$
\int_{\Omega} \nabla \Phi \cdot \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in C^1(\overline{\Omega}).
$$

Assume that

$$
\sup_{x \in \Omega} d_\Omega^{1-\alpha}(x) |\nabla \Phi(x)| < \infty,
$$

for some $\alpha \in (0, 1)$. Then, $\nabla \Phi \equiv 0$. 

Proof. We consider the Neumann problem,
\[ \Delta \varphi = \text{div } g \quad \text{in } \Omega, \]
\[ \frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \partial \Omega. \]

For \( g \in C_c^\infty \), there exists a solution \( \varphi \in W^{2,q}_\text{loc} \) for \( q \in (1, \infty) \) [14] Chapter III.1. In particular, \( \varphi \) is in \( C^1(\Omega) \). Since \( \nabla \Phi \in L^p(\Omega) \) for \( 1 \leq p < 1/(1-\alpha) \) by (2.14), it follows that
\[
\int_\Omega \Phi \text{div } g \, dx = \int_\Omega \Phi \Delta \varphi \, dx = -\int_\Omega \nabla \Phi \cdot \nabla \varphi \, dx = 0.
\]

We proved \( \nabla \Phi \equiv 0 \).

\hfill \Box

Lemma 2.7. Let \( \Omega \) be an exterior domain in \( \mathbb{R}^n \), \( n \geq 2 \), with \( C^2 \)-boundary. Let \( \nabla \Phi \in L^1_\text{loc}(\overline{\Omega}) \) satisfy
\[
(2.15) \quad \int_\Omega \nabla \Phi \cdot \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in C^1_c(\overline{\Omega}).
\]

Assume that
\[
(2.16) \quad \sup_{x \in \Omega} d_{\Omega}^{1-\alpha}(x) \left| \nabla \Phi(x) \right| < \infty,
\]
for some \( \alpha \in (0, 1) \). Then, \( \nabla \Phi \equiv 0 \).

Proof. We first estimate \( \Phi(x) \) as \( |x| \to \infty \) by using (2.16). We may assume \( 0 \in \Omega^c \). We take a constant \( R_0 > \text{diam } \Omega^c \) and observe that \( |x| \leq 2d_{\Omega}(x) \) for \( |x| \geq 2R_0 \). It follows from (2.16) that
\[
\sup_{|x| \geq 2R_0} |x|^{1-\alpha} \left| \nabla \Phi(x) \right| < \infty.
\]

By a fundamental calculation, we estimate
\[
(2.17) \quad \left| \Phi(x) \right| \leq C_1 |x|^\alpha + C_2 \quad \text{for } |x| \geq 2R_0,
\]
with some constants \( C_1 \) and \( C_2 \) independent of \( x \).

We consider the Neumann problem,
\[ \Delta \varphi = \text{div } g \quad \text{in } \Omega, \]
\[ \frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \partial \Omega. \]

For \( g \in C_c^\infty \), there exists a solution \( \varphi \in W^{2,q}_\text{loc} \) satisfying \( \nabla \varphi \in L^q \) for \( q \in (1, \infty) \) [14]. In order to substitute \( \varphi \) into (2.15), we cutoff the function \( \varphi \) as \( |x| \to \infty \). Let \( \theta \in C_c^\infty(0, \infty) \) be a smooth cut-off function satisfying \( \theta \equiv 1 \) in \( [0, 1] \) and \( \theta \equiv 0 \) in \( [2, \infty) \). Set \( \theta_m(x) = \theta(|x|/m) \) for \( m \geq R_0 \) so that \( \theta_m \equiv 1 \) for \( |x| \leq m, \theta_m \equiv 0 \) for \( |x| \geq 2m \) and \( \text{spt } \nabla \theta_m \subset D_m \) for \( D_m = \{ m < |x| < 2m \} \). Since \( \varphi_m = \varphi \theta_m \in C^1_c(\overline{\Omega}) \cap W^{2,q}_\text{loc} \) satisfies \( \partial \varphi_m/\partial n = 0 \) on \( \partial \Omega \) and
\[
\Delta \varphi_m = \Delta \varphi \theta_m + 2 \nabla \varphi \cdot \nabla \theta_m + \varphi \Delta \theta_m
\]
\[ = \text{div } g_m - g \cdot \nabla \theta_m + 2 \nabla \varphi \cdot \nabla \theta_m + \varphi \Delta \theta_m
\]

\hfill \Box
for \( g_m = g \theta_m \), it follows from (2.15) that
\[
\int_\Omega \Phi \text{div } g_m \, dx = \int_\Omega \Phi (\Delta \varphi_m + g \cdot \nabla \theta_m - 2 \nabla \varphi \cdot \nabla \theta_m - \varphi \Delta \theta_m) \, dx
\]
\[
= \int_\Omega \Phi (g \cdot \nabla \theta_m - 2 \nabla \varphi \cdot \nabla \theta_m - \varphi \Delta \theta_m) \, dx =: I_m + II_m + III_m.
\]
Since \( g \) is compactly supported in \( \Omega \), \( g_m = g \) and \( I_m \equiv 0 \) for sufficiently large \( m \geq R_0 \). We shall show that \( II_m, III_m \to 0 \) as \( m \to \infty \). By the cut-off function estimate \( \|\nabla \theta_m\|_\infty \leq C/m^{|k|} \) for \( |k| \geq 0 \) and (2.17), it follows that
\[
|II_m| \leq \frac{C}{m^{1-\alpha/q}} \|\nabla \varphi\|_{L^q(D_m)}^2,
\]
with the constant \( C \) is dependent of \( m \geq 2R_0 \), where \( 1/q + 1/q' = 1 \).

By a similar way, we estimate \( III_m \). By the Poincaré inequality \([10, 5.8.1]\), we estimate
\[
\|\varphi - (\varphi)\|_{L^q(D_m)} \leq mC\|\nabla \varphi\|_{L^q(D_m)}
\]
with some constant \( C \) independent of \( m \), where \( (\varphi) \) denotes the average of \( \varphi \) in \( D_m \). Since \( \Delta \theta_m \) is supported in \( D_m \), it follows that
\[
|III_m| \leq \frac{C}{m^{1-\alpha/q}} \|\nabla \varphi\|_{L^q(D_m)}^2,
\]
The function \( \nabla \varphi \) is \( L^q \)-integrable in \( \Omega \) for all \( q \in (1, \infty) \). In particular, \( \nabla \varphi \in L^q \) for \( q \in (1, n/(n-1+\alpha)] \) and \( 1-\alpha - n/q' \geq 0 \). Thus, \( |II_m| + |III_m| \to 0 \) as \( m \to \infty \). We proved \( \nabla \Phi \equiv 0 \).

The proof is now complete. \( \Box \)

In the next subsection, we apply the following extension theorem in order to prove the a priori estimate (2.8) by a blow-up argument (see, e.g., \([5\] Lemma A.1\) for a proof).

**Proposition 2.8.** Let \( \Phi \) be a harmonic function in \( \mathbb{R}^n \setminus \{0\}, n \geq 2 \). Let \( \alpha \in (0, 1) \). Assume that
\[
\sup \left\{ |x|^{1-\alpha} |\nabla \Phi(x)| \bigg| x \in B_0(1), \ x \neq 0 \right\} < \infty.
\]
Then, \( \Phi \) is extendable to a harmonic function in \( \mathbb{R}^n \).

2.3. Blow-up arguments.

Since bounded and exterior domains of class \( C^3 \) are admissible for \( \alpha = 0 \) as in Remark 2.2, we prove the a priori estimate (2.8) for \( \alpha \in (0, 1) \).

**Theorem 2.9.** A bounded domain of class \( C^3 \) is strongly admissible.

**Proof.** We argue by contradiction. Suppose that (2.8) were false for any choice of constants \( C \). Then, there would exist a sequence of solutions of (2.1), \( \tilde{\Phi}_m \) for \( \tilde{\Phi}_m \in C^\alpha_{sk}(\overline{\Omega}) \) such that
\[
M_m = \sup_{x \in \Omega} d^{1-\alpha}(x)|\nabla \tilde{\Phi}_m(x)| > m\tilde{\Phi}_m^{(\alpha)}_{\Omega}.
\]
Divide both sides by $M_m$ and observe that $\Phi_m = \tilde{\Phi}_m / M_m$ and $A_m = \tilde{A}_m / M_m$ satisfy

\begin{equation}
\sup_{x \in \Omega} d_\Omega^{1-\alpha}(x) |\nabla \Phi_m(x)| = 1, \tag{2.18}
\end{equation}

\begin{equation}
[A_m]_\Omega^{(\alpha)} \leq \frac{1}{m}. \tag{2.19}
\end{equation}

We take a point $x_m \in \Omega$ such that

\begin{equation}
d_\Omega^{1-\alpha}(x_m) |\nabla \Phi_m(x_m)| \geq \frac{1}{2}. \tag{2.20}
\end{equation}

Since $\Omega$ is bounded, there exists a subsequence of $\{x_m\} \subset \Omega$ (still denoted by $\{x_m\}$) such that $x_m \to x_\infty \in \overline{\Omega}$ as $m \to \infty$. Then, the proof is divided into two cases whether $x_\infty \in \Omega$ or $x_\infty \in \partial \Omega$.

**Case 1** $x_\infty \in \Omega$. We take a point $x_0 \in \Omega$ and observe from (2.19) that $\hat{A}_m(x) = A_m(x) - A_m(x_0)$ converges to zero uniformly in $\overline{\Omega}$ as $m \to \infty$. Since $A_m(x_0)$ is skew-symmetric, we replace $A_m$ to $\hat{A}_m$, i.e.,

$$
\int_{\Omega} \nabla \Phi_m \cdot \nabla \varphi \, dx = -\int_{\partial \Omega} \hat{A}_m n \cdot \nabla \varphi \, dH
$$

for all $\varphi \in C^1(\overline{\Omega})$ by Remarks 2.4 (iv). By (2.18), there exists a subsequence of $\{\nabla \Phi_m\}$ (still denoted by $\{\nabla \Phi_m\}$) such that $\nabla \Phi_m$ converges to a limit $\nabla \Phi$ weakly on $L^p(\Omega)$ for $1 \leq p < 1/(1-\alpha)$. Since $\Phi_m$ is harmonic in $\Omega$, $\nabla \Phi_m$ subsequently converges to $\nabla \Phi$ locally uniformly in $\Omega$. Sending $m \to \infty$ implies

$$
\int_{\Omega} \nabla \Phi \cdot \nabla \varphi \, dx = 0.
$$

We apply Lemma 2.6 and conclude that $\nabla \Phi \equiv 0$. This contradicts $d_\Omega^{1-\alpha}(x_\infty) |\nabla \Phi(x_\infty)| \geq 1/2$. So Case 1 does not occur.

**Case 2** $x_\infty \in \partial \Omega$. By rotation and translation of $\Omega$, we may assume $x_m = (0, \cdots, 0, d_m)$ and $x_\infty = 0 \in \partial \Omega$. We rescale $\Phi_m$ around the point $x_m \in \Omega$ to get a blow-up sequence,

$$
\Psi_m(x) = d_m^{-\alpha} \Phi_m(x_m + d_m x),
$$

$$
B_m(x) = d_m^{-\alpha} A_m(x_m + d_m x).
$$

The blow-up sequence $\Psi_m$ satisfies the Neumann problem (2.1) for $B_m \in C^\alpha_{sk}(\overline{\Omega}_m)$ in the rescaled domain

$$
\Omega_m = \Omega - \{x_m\} \frac{d_m}{d_m}.
$$

Note that the distance from the origin to the boundary $\partial \Omega_m$ is normalized as one, i.e., $d_{\Omega_m}(0) = 1$ since we rescale $\Phi_m$ by $d_m = d_{\Omega_m}(x_m)$. The rescaled domain $\Omega_m$ expands to the half space $\mathbb{R}^{n+1}_+ = \{(x', x_n) \in \mathbb{R}^n \mid x_n > -1\}$. In fact, we consider the neighborhood of $x_\infty = 0 \in \partial \Omega$,

$$
\Omega_{loc} = \{(x', x_n) \in \mathbb{R}^n \mid h(x') < x_n < h(x') + \beta', \, |x'| < \alpha'\},
$$

with some constants $\alpha'$, $\beta'$, $K'$ and the $C^2$-function $h$ satisfying $h(0) = 0$, $\nabla h(0) = 0$ and $\|h\|_{C^2([|x'|<\alpha'])} \leq K'$. Then, $\Omega_{loc} \subset \Omega$ is rescaled to

$$
\Omega_{loc,m} = \{(x', x_n) \in \mathbb{R}^n \mid h_m(x') - 1 < x_n < h_m(x') - 1 + \frac{\beta'}{d_m}, \, |x'| < \frac{\alpha'}{d_m}\},
$$

where
where \( h_m(x') = d_m^{-1}h(d_m x') \). Since \( \nabla' h(0) = 0 \), \( \Omega_{\text{loc}, m} \) expands to the half space \( \mathbb{R}^n_{+,-1} \) as \( m \to \infty \).

We take an arbitrary \( \varphi \in C^1_c(\mathbb{R}^n_{+,-1}) \) and extend it as a compactly supported \( C^1 \)-function in \( \mathbb{R}^n \) (see, e.g., [10, 5.4]). Then, \( \Psi_m \) and \( B_m \) satisfy

\[
\int_{\Omega_m} \nabla \Psi_m \cdot \nabla \varphi \, dx = - \int_{\partial \Omega_m} B_m n_{\Omega_m} \cdot \nabla \varphi \, d\mathcal{H}.
\]

The estimates (2.18)–(2.20) are inherited to

\[
\sup_{x \in \Omega_m} d_m^{1-\alpha}(x) \left| \nabla \Psi_m(x) \right| \leq 1,
\]

\[
[B_m]_{\Omega_m}^{(\alpha)} \leq \frac{1}{m},
\]

\[
|\nabla \Psi_m(0)| \geq \frac{1}{2}.
\]

We set \( \hat{B}_m(x) = B_m(x) - B_m(x_0) \) by some \( x_0 \in \Omega_m \). Then, \( \hat{B}_m \) satisfies

\[
|\hat{B}_m(x)| \leq \frac{1}{m} |x - x_0|^\alpha \quad \text{for } x \in \Omega_m.
\]

Since \( B_m(x_0) \) is skew-symmetric, we replace \( B_m \) to \( \hat{B}_m \) in (2.21). There exists a subsequence of \( \{\Psi_m\} \) (stilled denoted by \( \{\Psi_m\} \)) such that \( \nabla \Psi_m \to \nabla \Psi \) weakly in \( L^p_{\text{loc}}(\mathbb{R}^n_{+,-1}) \) for \( 1 \leq p < 1/(1 - \alpha) \). Moreover, \( \nabla \Psi_m \to \nabla \Psi \) locally uniformly in \( \mathbb{R}^n_{+,-1} \). Sending \( m \to \infty \) implies

\[
\int_{\mathbb{R}^n_{+,-1}} \nabla \Psi \cdot \nabla \varphi \, dx = 0.
\]

We apply Lemma 2.5 and conclude that \( \nabla \Psi \equiv 0 \). This contradicts \( |\nabla \Psi(0)| \geq 1/2 \). Thus, Case 2 does not occur.

We reached a contradiction. The proof is now complete. \( \square \)

**Remark 2.10.** (Boundary regularity) We are able to prove the a priori estimate (2.4) for \( C^3 \)-bounded domains by a similar blow-up argument (see [4]). Since \( \nabla \Phi \) may not be integrable up to the boundary under the bound (2.2), we used the weak form (2.3). This is the reason why we need \( C^3 \) to prove (2.4) by a blow-up argument. However, \( C^3 \) is not optimal for (2.4). In fact, it is proved in [20, Lemma 6.2] (independently of [4, 5, 1]) that (2.4) is valid for \( C^{1,2} \)-bounded domains by using the Green function. Thus, bounded domains of class \( C^2 \) are also strongly admissible.

We next prove the a priori estimate (2.8) for \( C^2 \)-exterior domains.

**Theorem 2.11.** An exterior domain of class \( C^3 \) is also strongly admissible.

**Proof.** We argue by contradiction. Suppose that (2.8) were false. Then, there would exist a sequence of solutions for (2.1), \( \Phi_m \) for \( A_m \in C^\alpha_s(\overline{\Omega}) \) and a sequence of points \( \{x_m\} \subset \Omega \) such
that
\begin{equation}
\sup_{x \in \Omega} d_{\Omega}^{1-\alpha}(x) |\nabla \Phi_m(x)| \leq 1, \tag{2.22}
\end{equation}
\begin{equation}
[A_m]^{(\alpha)}_{\Omega} \leq \frac{1}{m}, \tag{2.23}
\end{equation}
\begin{equation}
d_{\Omega}^{1-\alpha}(x_m) |\nabla \Phi_m(x_m)| \geq \frac{1}{2}. \tag{2.24}
\end{equation}

The proof is divided into two cases depending whether $d_m = d_{\Omega}(x_m)$ converges or not.

**Case 1** $\lim_{m \to \infty} d_m < \infty$. We may assume $x_m \to x_\infty \in \overline{\Omega}$ as $m \to \infty$ by choosing a subsequence. Then, Case 1 is divided into two cases whether $x_\infty \in \Omega$ or $x_\infty \in \partial \Omega$.

(i) $x_\infty \in \Omega$. The proof reduces to the uniqueness in the exterior domain $\Omega$. As Case 1 in the proof of Theorem 2.9, there exists a subsequence of $\{\Phi_m\}$ (still denoted by $\{\Phi_m\}$) such that $\nabla \Phi_m \to \nabla \Phi$ weakly in $L^p_{\text{loc}}(\overline{\Omega})$ for $1 \leq p < 1/(1 - \alpha)$ and $\nabla \Phi_m \to \nabla \Phi$ locally uniformly in $\Omega$ as $m \to \infty$. By (2.23), sending $m \to \infty$ implies
\[
\int_{\Omega} \nabla \Phi \cdot \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in C^1_c(\overline{\Omega}).
\]
We apply Lemma 2.7 and conclude that $\nabla \Phi \equiv 0$. This contradicts $d_{\Omega}^{1-\alpha}(x_\infty)|\nabla \Phi(x_\infty)| \geq 1/2$. So the case (i) does not occur.

(ii) $x_\infty \in \partial \Omega$. The proof reduces to the uniqueness in a half space. By the same rescaling argument as Case 2 in the proof of Theorem 2.9, we are able to prove that the case (ii) does not occur.

**Case 2** $\lim_{m \to \infty} d_m = \infty$. We may assume $\lim_{m \to \infty} d_m = \infty$. The proof reduces to the whole space. We rescale $\Phi_m$ around the point $x_m \in \Omega$ to get a blow-up sequence,
\[
\Psi_m(x) = d_m^{-\alpha} \Phi_m(x_m + d_m x) \quad \text{for } x \in \Omega_m := \frac{\Omega - \{x_m\}}{d_m}.
\]
Since we rescale $\Phi_m$ by $d_m = d_{\Omega}(x_m)$, the distance from the origin to $\partial \Omega_m$ is normalized as one, i.e., $d_{\Omega_m}(0) = 1$. We take a point $a_m \in \partial \Omega_m$ such that $|a_m| = d_{\Omega_m}(0) = 1$. By choosing a subsequence of $\{a_m\}$, we may assume $a_m \to a \in \mathbb{R}^n$ as $m \to \infty$. Since $d_m \to \infty$, the rescaled domain $\Omega_m$ approaches to $\mathbb{R}^n \setminus \{a\}$. It follows from (2.22) and (2.24) that
\[
\sup_{x \in \Omega_m} d_{\Omega_m}^{1-\alpha}(x) |\nabla \Psi_m(x)| \leq 1,
\]
\[
|\nabla \Psi_m(0)| \geq \frac{1}{2}.
\]
Since $\Psi_m$ is harmonic in $\Omega_m$, there exists a subsequence of $\{\Psi_m\}$ (still denoted by $\{\Psi_m\}$) such that $\nabla \Psi_m \to \nabla \Psi$ locally uniformly in $\mathbb{R}^n \setminus \{a\}$. Then, the limit $\Psi$ is harmonic in $\mathbb{R}^n \setminus \{a\}$ and satisfies
\[
\sup \{|x - a|^{1-\alpha} |\nabla \Psi(x)| \mid x \in \mathbb{R}^n, \ x \neq a\} \leq 1.
\]
We apply Proposition 2.8 and observe that $\Psi$ is harmonic at $x = a$. By applying the Liouville theorem, we conclude that $\nabla \Psi \equiv 0$. This contradicts $|\nabla \Psi(0)| \geq 1/2$ so Case 2 does not occur.

We reached a contradiction. The proof is now complete. \qed
3. A scale invariant H"older-type estimate for the Helmholtz projection

The goal of this section is to prove the H"older-type estimate (1.12) (Lemma 3.3). We divide \( \nabla \Phi = Q_\Omega \partial f \) into two terms \( \nabla \Phi_1 = Q_{\mathbb{R}^n} \partial f \) and \( \nabla \Phi_2 \). We estimate \( \nabla \Phi_1 \) by an estimate of the Newton potential and \( \nabla \Phi_2 \) by the a priori estimate (2.8). In what follows, we identify \( g \in C_0^\infty(\Omega) \) and its zero extension to \( \mathbb{R}^n \setminus \overline{\Omega} \).

**Proposition 3.1.** For \( g \in C_0^\infty(\mathbb{R}^n) \), set \( h = E_g \). Then, \( h \in C_0^\infty(\mathbb{R}^n) \) satisfies \( \nabla^2 h \in L^2(\mathbb{R}^n) \) and \( -\Delta h = g \) in \( \mathbb{R}^n \). Moreover, \( -\nabla \text{div} h \) agrees with \( Q_{\mathbb{R}^n} g \) and the estimate

\[
\left[ \nabla^2 h \right]^{(\alpha)}_{\mathbb{R}^n} + \left[ Q_{\mathbb{R}^n} g \right]^{(\alpha)}_{\mathbb{R}^n} \leq C_{\alpha} \left[ g \right]^{(\alpha)}_{\mathbb{R}^n}
\]

holds for \( \alpha \in (0, 1) \) with some constant \( C_{\alpha} \) independent of \( g \).

**Proof.** The first assertion is well known (see [19, Lemma 4.4]). We prove the latter assertion. Set \( \nabla \Psi = -\nabla \text{div} h \) and \( \nabla \Phi = Q_{\mathbb{R}^n} g \). Since \( \Phi \) and \( \Psi \) satisfy the Poisson equation \( \Delta \Phi = \text{div} g \) in \( \mathbb{R}^n \), \( \tilde{\Phi} = \Phi - \Psi \) is weakly harmonic in \( \mathbb{R}^n \). Since \( \nabla \tilde{\Phi} \in L^2(\mathbb{R}^n) \), \( \tilde{\Phi} \) is harmonic and smooth in \( \mathbb{R}^n \). By the mean-value formula, it follows that

\[
\nabla \tilde{\Phi}(x) = \int_{B(r)} \nabla \tilde{\Phi}(y) dy \quad \text{for} \quad x \in \mathbb{R}^n, \ r > 0.
\]

Applying the H"older inequality yields

\[
|\nabla \tilde{\Phi}(x)| \leq \frac{1}{a^{1/2} r^{1/2}} \left\| \nabla \tilde{\Phi} \right\|_{L^2(\mathbb{R}^n)},
\]

where \( a \) denotes the volume of the unit ball. By sending \( r \to \infty \), \( \nabla \Phi \equiv 0 \) follows. We proved \( Q_{\mathbb{R}^n} g = -\nabla \text{div} h \). \( \square \)

We next show that \( \nabla \Phi = Q_\Omega g - Q_{\mathbb{R}^n} g \) solves the Neumann problem (2.1).

**Proposition 3.2.** Let \( \Omega \) be a uniformly \( C^1 \)-domain in \( \mathbb{R}^n \). Let \( \alpha \in (0, 1) \). Set \( \nabla \Phi = Q_\Omega g - Q_{\mathbb{R}^n} g \) and \( h = E_g \) for \( g \in C_0^\infty(\Omega) \). Then, \( \Phi \) is a solution of the Neumann problem (2.1) for

\[
A = \nabla h - \nabla^T h.
\]

Moreover, the estimate

\[
\sup_{x \in \Omega} d_\Omega^{1-\alpha}(x) |\nabla \Phi(x)| \leq C \left[ \nabla h - \nabla^T h \right]^{(\alpha)}_{\Omega}
\]

holds provided that \( \Omega \) is admissible for \( \alpha \). The constant \( C = C_{\alpha, \Omega} \) is invariant of dilation and translation of \( \Omega \).
Lemma 3.3. Let \( \Omega \) be a strongly admissible, uniformly \( C^1 \)-domain. Set \( \nabla \Phi_1 = \nabla Q_\Omega \partial f \) and \( \nabla \Phi_2 = \nabla Q_\Omega \partial f - \nabla \Phi \) for \( f \in C_c^\infty(\Omega) \). Then, the estimate
\[
\left[ \Phi_1 \right]^{(\alpha)} + \sup_{x \in \Omega} d_{\Omega}^{1-\alpha}(x) |\nabla \Phi_2(x)| \leq C \left[ f \right]^{(\alpha)}_{\Omega}
\]
holds for $\alpha \in (0, 1)$. The constant $C = C_{\alpha, \Omega}$ is invariant of translation and dilation of $\Omega$.

Proof. By Proposition 3.1, $\nabla \Phi_1 = \mathbb{Q}_{\mathbb{R}^n} \partial f$ agrees with $-\nabla \text{div} h$ for $h = E \ast \partial f$ and $f \in C_c^\infty(\Omega)$. Moreover, by an integration by parts we estimate

$$\left[ \nabla h \right]_{\mathbb{R}^n}^{(\alpha)} \leq C_{\alpha}^{*} \left[ f \right]_{\mathbb{R}^n}^{(\alpha)} = C_{\alpha}^{*} [f]_{\Omega}^{(\alpha)}.$$  

(3.5)

Since $\Phi_1$ agrees with $-\text{div} h$ up to an additive constant, we have

$$\left[ \Phi_1 \right]_{\mathbb{R}^n}^{(\alpha)} \leq C_{\alpha}^{*} [f]_{\Omega}^{(\alpha)}.$$  

Since

$$\left[ \nabla h - \nabla^T h \right]_{\Omega}^{(\alpha)} \leq \left[ \nabla h - \nabla^T h \right]_{\mathbb{R}^n}^{(\alpha)} \leq 2 \left[ \nabla h \right]_{\mathbb{R}^n}^{(\alpha)} \leq 2 C_{\alpha}^{*} [f]_{\Omega}^{(\alpha)},$$

by (3.5), we apply (3.2) and estimate $\nabla \Phi_2$. Since the constants $C_{\alpha}^{*}$ and the constant in (3.2) are invariant of dilation of $\Omega$, so is $C = C_{\alpha, \Omega}$. The proof is complete. \(\blacksquare\)

4. Local Hölder estimates for the Stokes equations

In this section, we review the local Hölder estimates for the Stokes equations (Lemma 4.3). We recall the $\bar{L}^p$-theory for the Stokes equations in a general uniformly $C^2$-domain.

Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \geq 2$. We define the space $\bar{L}^p(\Omega)$ by

$$\bar{L}^p(\Omega) = L^p(\Omega) \cap L^2(\Omega) \quad \text{for } 2 \leq p < \infty.$$  

The space $\bar{L}^p(\Omega)$ is a Banach space equipped with the norm

$$\| f \|_{\bar{L}^p(\Omega)} = \max \{ \| f \|_{L^p(\Omega)}, \| f \|_{L^2(\Omega)} \}.$$  

Let $L^p_c(\Omega)$ denote the $L^p$-closure of $C_c^\infty(\Omega)$. We define $G^p(\Omega) = \{ \nabla \varphi \in L^p(\Omega) \mid \varphi \in L^p_{\text{loc}}(\Omega) \}$. We define $\bar{L}^p_c(\Omega)$ and $\bar{G}^p(\Omega)$ by a similar way. It is proved in [11] that for each $f \in \bar{L}^p$, there exists a unique decomposition $f = f_0 + \nabla \varphi$ by $f_0 \in \bar{L}^p_c$ and $\nabla \varphi \in \bar{G}^p$ satisfying

$$\| f_0 \|_{\bar{L}^p(\Omega)} + \| \nabla \varphi \|_{\bar{L}^p(\Omega)} \leq C \| f \|_{\bar{L}^p(\Omega)}$$

for uniformly $C^1$-domains $\Omega$ in $\mathbb{R}^3$. Thus, the Helmholtz projection $\mathbb{P}_{\Omega} : f \mapsto f_0$ and $\mathbb{Q}_{\Omega} = I - \mathbb{P}_{\Omega}$ exist and are bounded on $\bar{L}^p$. Moreover, it is proved that the Stokes semigroup $S(t)$ is an analytic semigroup on $\bar{L}^p$ for uniformly $C^2$-domains. The result is extended to the $n$-dimensional case for $n \geq 2$ in [12], [13]. Thus, solutions of the Stokes equations are given by the Stokes semigroup and the Helmholtz projection defined on $\bar{L}^p$. This means $\bar{L}^p$-solutions.

We estimate Hölder norms of $\bar{L}^p$-solutions $(v, q)$ by applying the a priori estimate (2.4) for $\nabla q$. 


Proposition 4.1. Let \( \Omega \) be a uniformly \( C^2 \)-domain in \( \mathbb{R}^n, n \geq 2 \). Let \((v, q)\) be an \( L^p \)-solution of (1.1)–(1.4) for \( p \geq n \). Then, the pressure \( q \) is a solution of the Neumann problem (2.1) for
\[
A = -\nabla_v + \nabla^T v.
\]

Moreover, the estimate
\[
\sup_{x \in \Omega} d_\Omega(x) \left| \nabla q(x, t) \right| \leq C \left\| \nabla_v - \nabla^T v \right\|_{L^\infty(\partial \Omega)}(t)
\]
holds for \( t \in (0, T) \) provided that \( \Omega \) is admissible for \( \alpha = 0 \). The constant \( C = C_\Omega \) is invariant of dilation and translation of \( \Omega \).

Proof. Although the assertion is essentially proved in [4], we give a proof in order to make the paper self-contained. We take an arbitrary \((1.1)–(1.4)\) for \( p \geq n \). Then, the pressure \( q \) is a solution of the Neumann problem (2.1) for
\[
\sup_{x \in \Omega} d_\Omega(x) \left| \nabla q(x, t) \right| \leq C \left\| \nabla_v - \nabla^T v \right\|_{L^\infty(\partial \Omega)}(t)
\]
holds for \( t \in (0, T) \) provided that \( \Omega \) is admissible for \( \alpha = 0 \). The constant \( C = C_\Omega \) is invariant of dilation and translation of \( \Omega \).

Proof. Although the assertion is essentially proved in [4], we give a proof in order to make the paper self-contained. We take an arbitrary \( \phi \in C^1_c(\overline{\Omega}) \) satisfying \( \partial \phi / \partial n = 0 \) on \( \partial \Omega \). We may assume \( \phi \in C^\infty_c(\overline{\Omega}) \). By \( \text{div} \, v = 0 \) in \( \Omega \) and \( v \cdot n = 0 \) on \( \partial \Omega \), it follows that
\[
\int_{\partial \Omega} A \cdot \nabla \phi d\mathcal{H} = \int_{\partial \Omega} (-\nabla_v + \nabla^T v)n \cdot \nabla \phi d\mathcal{H}
\]
\[
= \sum_{i,j=1}^n \int_{\partial \Omega} (-\partial_j v^j + \partial_i \nabla^i) n^i \partial_i \phi d\mathcal{H}
\]
\[
= \sum_{i,j=1}^n \int_{\Omega} ((-\partial_j v^j + \partial_i \nabla^j) \partial_i \phi + (-\partial_j v^j + \partial_i \nabla^j) \partial_j \partial_i \phi) dx
\]
\[
= -\int_{\Omega} \Delta v \cdot \nabla \phi dx = -\int_{\Omega} (v_t + \nabla q) \cdot \nabla \phi dx = \int_{\Omega} q \Delta \phi dx.
\]
So the pressure \( q \) satisfies (2.3). Since \( q \) is harmonic in \( \Omega \), by the same way as in the proof of Proposition 3.2, we estimate
\[
d_\Omega^2(x) \left| \nabla q(x) \right| \leq C_s \left\| \nabla q \right\|_{L^p(\Omega)} \quad \text{for} \quad x \in \Omega,
\]
and all \( s \in [1, \infty) \), where the time-variable of \( q = q(x, \cdot) \) is suppressed. Since
\[
d_\Omega(x) \left| \nabla q(x) \right| = d_\Omega^2(x) \left| \nabla q(x) \right| d_\Omega^{1-\frac{2}{p}}(x)
\]
\[
\leq C_s \left\| \nabla q \right\|_{L^p(\Omega)} d_\Omega^{1-\frac{2}{p}}(x),
\]
we take \( s = p \geq n \) for \( d_\Omega(x) \leq 1 \) and \( s = 2 \) for \( d_\Omega(x) \geq 1 \) to estimate
\[
\sup_{x \in \Omega} d_\Omega(x) \left| \nabla q(x) \right| \leq C_p \left\| \nabla q \right\|_{L^p(\Omega)}
\]
with \( C_p = \max\{C_p, C_\Omega\} \). Since \( \mathcal{Q}_\Omega \) acts as a bounded operator on \( L^p \), \( \nabla q \) is in \( L^p \). Thus, \( q \) is a solution of (2.1) for \( A = -\nabla_v + \nabla^T v \). The estimate (4.1) follows from (2.4) with a dilation invariant constant \( C = C_\Omega \). The proof is complete. \( \square \)

Remark 4.2. For \( \alpha \in (0, 1) \) and \( p \geq n/(1 - \alpha) \), the pressure \( q \) (defined on \( \mathcal{Q}^p \)) is a solution of the Neumann problem (2.1) and the estimate
\[
\sup_{x \in \Omega} d_\Omega^{1-\alpha}(x) \left| \nabla q(x, t) \right| \leq C_\alpha \left[ \nabla_v - \nabla^T v \right]^{(\alpha)}_{\Omega}(t)
\]
holds provided that \( \Omega \) is strongly admissible.
We recall the following notation for Hölder semi-norms of space-time functions [21]. Let \( f = f(x, t) \) be a real-valued or an \( \mathbb{R}^n \)-valued function defined in \( Q = \Omega \times (0, T) \). For \( \mu \in (0, 1) \), we set the Hölder semi-norms

\[
[f]^{(\mu)}_{\mu, Q} = \sup_{x \in \Omega} [f]^{(\mu)}_{(0, T)}(x), \quad [f]^{(\mu)}_{\mu, \Omega} = \sup_{t \in (0, T)} [f]^{(\mu)}_{\Omega}(t).
\]

In the parabolic scale for \( \mu \in (0, 1) \), we set

\[
[f]^{(\mu, \frac{\mu}{2})}_{\mu, \Omega} = [f]^{(\mu)}_{\mu, \Omega} + [f]^{(\mu)}_{\mu, \Omega}.
\]

We estimate local Hölder norms for solutions of the Stokes equations both interior and up to boundary of \( \Omega \). In the interior of \( \Omega \), \( \nabla q \) is smooth for spatial variables since \( q \) is harmonic in \( \Omega \). Moreover, \( \nabla q \) is Hölder continuous for a time variable by (4.1). We thus estimate Hölder norms of \( \partial_t v \) and \( \nabla^2 v \) by the parabolic regularity theory [21]. A corresponding estimate up to the boundary is more involved. By combining (4.1) and the Schauder estimate for the Stokes equations [27, 29], we estimate Hölder norms of \( \partial_t v, \nabla^2 v, \nabla q \) up to the boundary. We estimate Hölder norms of \( \partial_t v, \nabla^2 v, \nabla q \) by

\[
N_{\delta, T} = \sup_{\delta \leq t \leq T} \|N(v, q)\|_{L^p(\Omega)}(t) \quad \text{for } \delta > 0.
\]

The following Hölder estimate is proved in [4, Theorem 3.4].

**Lemma 4.3.** Let \( \Omega \) be an admissible, uniformly \( C^3 \)-domain of type \( \alpha, \beta, K \) in \( \mathbb{R}^n \).

(i) (Interior Hölder estimates) Take \( \mu \in (0, 1), \delta > 0, T > 0, R > 0 \). Then, there exists a constant \( C = C(\delta, R, d, \mu, T, C_\Omega) \) such that the a priori estimate

\[
[\nabla^2 v]^{(\mu, \frac{\mu}{2})}_{\mu, Q} + [v]^{(\mu, \frac{\mu}{2})}_{\mu, Q} + [\nabla q]^{(\mu, \frac{\mu}{2})}_{\mu, Q} \leq CN_{\delta, T}
\]

holds for all \( \tilde{L}^p \)-solutions \((v, q)\) for \( p > n \) in \( Q' = B_{x_0}(R) \times (2\delta, T) \) provided that \( B_{x_0}(R) \subset \Omega \) and \( x_0 \in \Omega \), where \( d \) denotes the distance from \( B_{x_0}(R) \) to \( \partial \Omega \) and \( C_\Omega \) is the constant in (4.1).

(ii) (Estimates near the boundary) There exists \( R_0 = R_0(\alpha, \beta, K) > 0 \) such that for any \( \mu \in (0, 1), \delta \in (0, T) \) and \( R \leq R_0 \), there exists a constant

\[
C = C(\delta, \mu, T, R, \alpha, \beta, K, C_\Omega)
\]

such that (4.2) holds for all \( \tilde{L}^p \)-solutions \((v, q)\) for \( p > n \) in \( Q' = \Omega_{x_0, R} \times (2\delta, T) \) for \( \Omega_{x_0, R} = B_{x_0}(R) \cap \Omega \) and \( x_0 \in \partial \Omega \).

5. **Uniqueness in a half space**

The goal of this section is to prove the uniqueness for the Stokes equations (1.1)–(1.4) in a half space (Theorem 5.1). The uniqueness theorem on \( L^\infty \) is known for continuous velocity at time zero [28]. However, a blow-up limit may not be continuous nor even bounded as \( t \downarrow 0 \). Thus, we need a stronger uniqueness theorem in order to apply it for a blow-up limit. We prove a uniqueness theorem under suitable sup-bounds for velocity and pressure gradient.
Theorem 5.1. Let $v \in C^{2,1}(\mathbb{R}^n_+ \times (0, T))$ and $\nabla q \in C(\mathbb{R}^n_+ \times (0, T))$ satisfy (1.1)–(1.3),

\[
\sup_{0 < t \leq T} t^\gamma \|N(v, q)\|_{L^n(\mathbb{R}^n_+)}(t) < \infty,
\]

\[
\sup_{0 < t \leq T} \left\{ t^{\gamma + \frac{1}{2}} x_n |\nabla q(x, t)| \right\} x \in \mathbb{R}^n_+, \ 0 < t \leq T \right\} < \infty,
\]

for some $\gamma \in [0, 1/2)$. Assume that $(v, q)$ satisfies

\[
\int_0^T \int_{\mathbb{R}^n_+} (v \cdot (\partial_t \varphi + \Delta \varphi) - \nabla q \cdot \varphi) dx dt = 0,
\]

for all $\varphi \in C^\infty_c(\mathbb{R}^n_+ \times [0, T])$. Then, $v \equiv 0$ and $\nabla q \equiv 0$.

We prove Theorem 5.1 from the following stronger assertion.

Lemma 5.2. Let $v \in C^{2,1}(\mathbb{R}^n_+ \times (0, T))$ and $\nabla q \in C(\mathbb{R}^n_+ \times (0, T))$ satisfy (1.1)–(1.3),

\[
\sup_{0 < t \leq T} t^\gamma \|v\|_{L^n(\mathbb{R}^n_+)}(t) < \infty,
\]

\[
\sup_{0 < t \leq T} \left\{ t^{\gamma + \frac{1}{2}} (x_n^2 + t)^{\frac{1}{2}} |\nabla q(x, t)| \right\} x \in \mathbb{R}^n_+, \ 0 < t \leq T \right\} < \infty,
\]

for $\gamma \in [0, 1/2)$, and $\nabla v$ is bounded in $\mathbb{R}^n_+$ for $t \in (0, T)$. Assume that $(v, q)$ satisfies (5.3) for all $\varphi \in C^\infty_c(\mathbb{R}^n_+ \times [0, T])$. Then, $v \equiv 0$ and $\nabla q \equiv 0$.

The uniqueness of the Stokes equations on $L^n(\mathbb{R}^n_+)$ was first proved by V. A. Solonnikov based on a duality argument [28, Theorem 1.1]. However, the result was restricted to continuous velocity at time zero. Recently, the author proved some uniqueness theorem without assuming continuity of velocity at time zero. In the sequel, we give a proof for Lemma 5.2 based on the proof in [2].

We sketch the proof of Lemma 5.2. An essential step is to prove $\partial_{\text{tan}} v \equiv 0$. Once we know $\partial_{\text{tan}} v \equiv 0$, then $\nabla q \equiv 0$ and $v \equiv 0$ easily follow. In fact, the divergence-free condition for the velocity implies

\[
\frac{\partial v^n}{\partial x_n} = - \sum_{j=1}^{n-1} \frac{\partial v^j}{\partial x_j} = 0.
\]

By the Dirichlet boundary condition, $v^n \equiv 0$ and $\partial q/\partial x_n \equiv 0$ follows. Thus, $\nabla q = \nabla q(x', t)$ is independent of the $x_n$-variable. The condition (5.5) implies $\nabla q \equiv 0$. So $v \equiv 0$ follows from the uniqueness of the heat equation in $\mathbb{R}^n_+ \times (0, T)$.

Let $C^\infty_{c, \sigma}(\mathbb{R}^n_+ \times (0, T))$ denote the space of all smooth solenoidal vector fields with compact support in $\mathbb{R}^n_+ \times (0, T)$. We prove

\[
\int_0^T \int_{\mathbb{R}^n_+} \partial_{\text{tan}} v(x, t) \cdot f(x, t) dx dt = 0
\]

for all $f \in C^\infty_{c, \sigma}(\mathbb{R}^n_+ \times (0, T))$. Then, $\partial_{\text{tan}} v \equiv 0$ follows from the de Rham’s theorem.

Proposition 5.3. Let $u \in L^n(\mathbb{R}^n_+) \cap C^1(\mathbb{R}^n_+)$ satisfy $\text{div} \ u = 0$ in $\mathbb{R}^n_+$, $u = 0$ on $\{x_n = 0\}$ and

\[
\int_{\mathbb{R}^n_+} u \cdot f dx = 0 \quad \text{for all } f \in C^\infty_{c, \sigma}(\mathbb{R}^n_+).
\]
Then, \( u \equiv 0 \).

**Proof.** By the de Rham’s theorem (e.g., [25] Theorem 1.1), there exists a function \( \Phi \in C^2(\mathbb{R}^n_+ \times (0, T)) \) such that \( u = \nabla \Phi \). Since \( \nabla u = 0 \) in \( \mathbb{R}^n_+ \) and \( u^0 = 0 \) on \( \{x_n = 0\} \), the function \( \Phi \) is harmonic in \( \mathbb{R}^n_+ \) and \( \partial \Phi / \partial x_n = 0 \) on \( \{x_n = 0\} \). We extend \( \Phi \) to \( \mathbb{R}^n \) by the even extension, i.e.,

\[
\tilde{\Phi}(x', x_n) = \begin{cases} 
\Phi(x', x_n) & \text{for } x_n \geq 0, \\
\Phi(x', -x_n) & \text{for } x_n < 0.
\end{cases}
\]

Then, \( \tilde{\Phi} \in C^2(\mathbb{R}^n) \) is harmonic in \( \mathbb{R}^n \). We apply the Liouville theorem for \( \nabla \tilde{\Phi} \in L^\infty(\mathbb{R}^n) \) and conclude that \( \nabla \tilde{\Phi} \) is constant. Since \( \nabla \tilde{\Phi} \) is vanishing on \( \{x_n = 0\} \), \( \nabla \tilde{\Phi} \equiv 0 \) follows.

In the sequel, we prove (5.6). We consider the dual problem,

\[
\begin{align}
-\partial \psi - \Delta \psi + \nabla \pi &= \partial_{\mathrm{tan}} f \quad \text{in } \mathbb{R}^n_+ \times (0, T), \\
\text{div } \psi &= 0 \quad \text{in } \mathbb{R}^n_+ \times (0, T), \\
\psi &= 0 \quad \text{on } \partial \mathbb{R}^n_+ \times (0, T), \\
\psi &= 0 \quad \text{on } \mathbb{R}^n_+ \times \{t = T\}.
\end{align}
\]

It is proved in [2] (see Proposition A.1) that solutions \((\psi, \pi)\) exist and satisfy \( \psi \in S \) and \( \nabla \pi \in L^\infty(0, T; L^1(\mathbb{R}^n_+)) \), where

\[
S = \left\{ \psi \in C^\infty(\mathbb{R}^n_+ \times [0, T]) \mid \psi, \nabla \psi, \nabla^2 \psi, \partial_{\mathrm{tan}} \psi, x_n^{-1} \psi \in L^\infty(0, T; L^1(\mathbb{R}^n_+)), \right. \\
\left. \psi = 0 \text{ on } \{x_n = 0\} \cup \{t = T\} \right\}.
\]

It is noted that the solution \( \psi \) is in \( L^\infty(0, T; L^1) \) by \( \nabla S(t)f \in L^1(\mathbb{R}_+^n \times (0, T)) \) (2 Proposition 3.1) although \( S(t)f \notin L^1 \) for general \( f \) (i.e., there exists some \( f \in L^2(\mathbb{R}_+^n) \) such that \( S(t)f \notin L^1 \) \([2], [23]\)).

We complete the proof of Lemma 5.2 and then give a proof for the following Proposition 5.4 later in Appendix A.

**Proposition 5.4.** Under the assumption of Lemma 5.2, the initial condition (5.3) is extendable for all \( \psi \in S \).

**Proof of Lemma 5.2.** For \( f \in C^\infty(\mathbb{R}_+^n \times (0, T)) \), there exists a smooth solution \((\psi, \pi)\) for (5.7)-(5.10) satisfying \( \psi \in S \) and \( \nabla \pi \in L^\infty(0, T; L^1) \) by Proposition A.1. Since the condition (5.3) is extendable for all test functions in \( S \) by Proposition 5.4, it follows that

\[
\int_0^T \int_{\mathbb{R}^n_+} v \cdot \partial_{\mathrm{tan}} f \, dx \, dt = \int_0^T \int_{\mathbb{R}^n_+} v \cdot (-\Delta \psi + \nabla \pi) \, dx \, dt \\
= \int_0^T \int_{\mathbb{R}^n_+} (v \cdot \nabla \pi - \nabla q \cdot \psi) \, dx \, dt.
\]

Since \( v(\cdot, t) \in L^\infty(\mathbb{R}_+^n) \cap C^2(\mathbb{R}_+^n) \) satisfies \( \text{div } v = 0 \) in \( \mathbb{R}_+^n \), \( v^0 = 0 \) on \( \{x_n = 0\} \) and \( \nabla \pi(\cdot, t) \in L^1(\mathbb{R}_+^n) \), the first term vanishes (see, e.g., [2] Proposition 2.3). Similarly, the second term vanishes. Thus, (5.6) holds for all \( f \in C^\infty(\mathbb{R}_+^n \times (0, T)) \).
We apply Proposition 5.3 for $u = \partial_t \tan v$ and conclude that $\partial_t \tan v \equiv 0$. So $\nabla q \equiv 0$ and $v \equiv 0$. The proof is complete.

**Proof of Theorem 5.1.** Since $(x_n^2 + t)^{1/2} \leq x_n + t^{1/2}$, the condition (5.5) is satisfied by (5.1) and (5.2). The assertion follows from Lemma 5.2.

---

**6. A priori estimates for the Stokes flow**

We prove Theorem 1.3. We first show that $\tilde{L}^p$-solutions are sufficiently regular near $t = 0$ and then prove (1.7) by a blow-up argument. After the proof of (1.7), we prove the estimate (1.8) by approximation.

6.1. Regularity of $\tilde{L}^p$-solutions.

**Proposition 6.1.** Let $\Omega$ be a uniformly $C^3$-domain in $\mathbb{R}^n$, $n \geq 2$. Let $p > n$. Then, $\tilde{L}^p$-solutions $(v, q)$ for $v_0 \in \tilde{L}^p(\Omega)$ are bounded and Hölder continuous in $\Omega \times [\delta, T]$ for each $\delta > 0$ up to second orders. Moreover, for $\gamma \in (0, 1)$,

$$t^\gamma \|N(v, q)\|_{L^\infty(t)} \in C[0, T] \quad \text{and} \quad \lim_{t \downarrow 0} t^\gamma \|N(v, q)\|_{L^\infty(t)} = 0,$$

provided that $p > n/(2\gamma)$.

**Proof.** We set

$$\tilde{N}(v, q)(x, t) = N(v, q)(x, t) + t^{3/2} |\nabla \partial_t v(x, t)| + t^{3} |\nabla^2 v(x, t)| + t^{2} |\nabla^2 q(x, t)|.$$

We shall show that

$$\sup_{0 \leq t \leq T} \|\tilde{N}(v, q)\|_{L^p(t)} \leq C \|v_0\|_{L^p(\Omega)}$$

for $\tilde{L}^p$-solutions $(v, q)$ for $p > n$, where $L^p_{ul}(\Omega)$ denotes the uniformly local $L^p$ space and is equipped with the norm

$$\|f\|_{L^p_{ul}(\Omega)} = \sup_{x_0 \in \Omega} \|f\|_{L^p(\Omega_{x_0,r})}, \quad \Omega_{x_0,r} = B_{x_0}(r) \cap \Omega.$$

We define the space $W^{1,p}_{ul}(\Omega)$ by a similar way. For simplicity, we suppress the subscript for $r = 1$, i.e., $\|f\|_{L^p_{ul,1}} = \|f\|_{L^p_{ul}}$ and $\Omega_{x_0,1} = \Omega_{x_0}$.

We observe from (6.2) that $(v, q)$ is bounded and Hölder continuous in $\Omega \times [\delta, T]$ for $\delta > 0$. In fact, by the Sobolev embedding we estimate

$$\sup_{0 \leq t \leq T} \|\tilde{N}(v, q)\|_{L^\infty(\Omega)} \leq C \|v_0\|_{L^p(\Omega)}.$$

Thus, $(v, q)$ is bounded in $\Omega \times [\delta, T]$. Moreover, $\partial_t v, \nabla^2 v, \nabla q$ are Hölder continuous in $\Omega \times [\delta, T]$. We observe that $\nabla^2 v(\cdot, t)$ is Hölder continuous in $[\delta, T]$. Let $A$ denote the generator of $S(t)$ on $L^p_{ul}$ and $D(A)$ denote the domain of $A$ in $L^p_{ul}$. We may assume $v_0 \in D(A)$. For $t > s \geq \delta$, it follows from (6.3) that

$$\|\nabla^2 v(t) - \nabla^2 v(s)\|_{L^\infty(\Omega)} \leq \int_s^t \|\nabla^2 S(r)A v_0\|_{L^\infty(\Omega)} \, dr \leq C |t - s| \|A v_0\|_{L^p_{ul}(\Omega)}.$$
Thus, $\nabla^2 v(x, t)$ is Hölder continuous in $[\delta, T]$. By a similar way, we are able to prove that $\partial_t v$ and $\nabla q$ are Hölder continuous in $[\delta, T]$. We proved that $(v, q)$ is bounded and Hölder continuous in $\overline{\Omega} \times [\delta, T]$. In particular, $\|N(v, q)\|_{\infty}(t) \in C(0, T]$.

We prove (6.1) by applying the interpolation inequality,
\[
\|\varphi\|_{L^\infty(\Omega)} \leq \frac{C}{r^p} \left( \|\varphi\|_{L^p_{ul, \Omega}(\Omega)} + r \|\nabla \varphi\|_{L^p_{ul, \Omega}(\Omega)} \right)
\]
for $\varphi \in W^{1,p}_0(\Omega)$ and $r \leq r_0$ (see [22, Lemma 3.1]). We may assume $r_0 \leq 1$. We substitute $\varphi = v$ and $r = t^{1/2}$ into (6.4) to estimate
\[
\|v\|_{L^\infty(\Omega)} \leq \frac{C}{t^{p/2}} \left( \|v\|_{L^p_{ul, \Omega}(\Omega)} + t^{1/2} \|\nabla v\|_{L^p_{ul, \Omega}(\Omega)} \right)
\]
by (6.2). By a similar way, we apply (6.4) for $\nabla v$, $\nabla^2 v$, $\partial_t v$, $\nabla q$ and observe that
\[
\sup_{0 < t \leq 1} t^p \|N(v, q)\|_{\infty}(t) < \infty.
\]
Thus, $t^p \|N(v, q)\|_{\infty}(t)$ is continuous in $[0, T]$ and takes zero at $t = 0$ provided that $p > n/(2\gamma)$.

It remains to show (6.2). By estimates of $S(t)$ and $P$ on $L^p$ [13, Theorem 1.3], it follows that
\[
\sup_{0 \leq t \leq T} \|N(v, q)\|_{L^p(\Omega)}(t) \leq C\|v_0\|_{L^p(\Omega)}.
\]
Moreover, we have
\[
\sup_{0 \leq t \leq T} t^{3/2} \|\nabla \partial_t v\|_{L^p(\Omega)}(t) \leq C\|v_0\|_{L^p(\Omega)},
\]
since $\partial_t v = Ae^{\delta t}v_0 = e^{t/\gamma}Ae^{\delta t}v_0$. We estimate the uniformly local $L^p$-norms of $\nabla^3 v$ and $\nabla^2 q$. For $x_0 \in \Omega$, we take a $C^3$-bounded domain $\Omega'$ such that $\Omega_{x_0} \subset \Omega' \subset \Omega$ and set the average of $q$ in $\Omega'$ by
\[
(q) = \frac{1}{\Omega'} \int_{\Omega'} q \, dx.
\]
By the Poincaré inequality [10, 5.8.1], we estimate
\[
\|q - (q)\|_{L^p(\Omega')} \leq C\|\nabla q\|_{L^p(\Omega')}.
\]
We shift $q$ to $\hat{q} = q - (q)$. By the higher-order regularity theory [14, Chapter IV.4 and 5] for the stationary Stokes equations (for each $t > 0$),
\[
-\Delta v + \nabla \hat{q} = -\partial_t v \quad \text{in } \Omega,
\]
\[
\text{div } v = 0 \quad \text{in } \Omega,
\]
\[
v = 0 \quad \text{on } \partial \Omega,
\]
we estimate
\[
\|\nabla^3 v\|_{L^p(\Omega_{x_0})} + \|\nabla^2 q\|_{L^p(\Omega_{x_0})} \leq C \left( \|\partial_t v\|_{W^{3,p}(\Omega')} + \|v\|_{W^{1,p}(\Omega')} + \|\hat{q}\|_{L^p(\Omega')} \right),
\]
with some constant $C$ independent of $x_0 \in \Omega$. Since $x_0 \in \Omega$ is an arbitrary point, by (6.5)–(6.8) we obtain
\[
\sup_{0 \leq t \leq T} \left( t^2 \|\nabla^3 v\|_{L^p_{ul, \Omega}(\Omega)}(t) + t^3 \|\nabla^2 q\|_{L^p_{ul, \Omega}(\Omega)}(t) \right) \leq C\|v_0\|_{L^p(\Omega)}.
\]
We proved (6.2). The proof is complete. □

6.2. A blow-up argument.

Now, we prove the a priori estimate (1.7) by a blow-up argument. For $\alpha \in (0, 1)$ we set

$$\gamma = \frac{1 - \alpha}{2}.$$ 

Then, $t^\gamma \|N(v, q)\|_\infty(t)$ is continuous in $[0, T]$ and takes zero at $t = 0$ for $\tilde{L}^p$-solutions $(v, q)$ for $v_0 \in \tilde{L}^p$ provided that $p > n/(2\gamma)$ by (6.1).

**Proposition 6.2.** Let $\Omega$ be a strongly admissible, uniformly $C^3$-domain. For $\alpha \in (0, 1)$ and $p > n/(1 - \alpha)$, there exist some constants $T_0$ and $C$ such that (1.7) holds all $\tilde{L}^p$-solutions for $v_0 = \tilde{\partial}f, f \in C^\infty_c(\Omega)$.

**Proof.** We argue by contradiction. Suppose on the contrary that (1.7) were false for any choice of constants $C$ and $T_0$. Then, there would exist a sequence of $\tilde{L}^p$-solutions $(v_m, q_m)$ for $v_{0,m} = \tilde{\partial}f_m, f_m \in C^\infty_c(\Omega)$ such that

$$\sup_{0 \leq t \leq 1/m} t^\gamma \|N(v_m, q_m)\|_\infty(t) > m [f_m]_\Omega^{(a)}.$$ 

We take a point $t_m \in (0, 1/m)$ such that

$$t_m^\gamma \|N(v_m, q_m)\|_\infty(t_m) \geq \frac{1}{2} M_m, \quad M_m = \sup_{0 \leq t \leq 1/m} t^\gamma \|N(v_m, q_m)\|_\infty(t),$$

and divide $(v_m, q_m)$ by $M_m$ to get $\tilde{v}_m = v_m/M_m, \tilde{q}_m = q_m/M_m$ and $\tilde{f}_m = f_m/M_m$ satisfying

$$\sup_{0 \leq t \leq t_m} t^\gamma \|N(\tilde{v}_m, \tilde{q}_m)\|_\infty(t) \leq 1,$$

$$\left[\tilde{f}_m\right]_\Omega^{(a)} \leq \frac{1}{m},$$

$$t_m^\gamma \|N(\tilde{v}_m, \tilde{q}_m)\|_\infty(t_m) \geq \frac{1}{2}.$$ 

We take a point $x_m \in \Omega$ such that

$$t_m^\gamma N(\tilde{v}_m, \tilde{q}_m)(x_m, t_m) \geq \frac{1}{4},$$ 

and rescale $(\tilde{v}_m, \tilde{q}_m)$ around $(x_m, t_m)$ to get a blow-up sequence

$$u_m(x, t) = t_m^{\frac{\alpha}{2}} \tilde{v}_m(x_m + t_m x, t_m t), \quad p_m(x, t) = t_m^{\frac{\alpha}{2}} \tilde{q}_m(x_m + t_m x, t_m t),$$

and

$$g_m(x) = t_m^{\frac{\gamma}{2}} \tilde{f}_m(x_m + t_m x).$$

The blow-up sequence $(u_m, p_m)$ satisfies (1.1)–(1.4) in $\Omega_m \times (0, 1)$ for $u_{0,m} = \tilde{\partial}g_m$ and

$$\Omega_m = \frac{\Omega - \{x_m\}}{t_m^{\frac{1}{\alpha}}}.$$
The estimates for \( (v_m, q_m) \) are inherited to
\[
(6.9) \quad \sup_{0 \leq t \leq 1} t^\gamma \| N(u_m, p_m) \|_{L^\infty(\Omega_m)}(t) \leq 1,
\]
\[
(6.10) \quad \left[ g_m \right]_{\Omega_m}^{(a)} \leq \frac{1}{m},
\]
\[
(6.11) \quad N(v_m, q_m)(0, 1) \geq \frac{1}{4}.
\]
We set \( c_m = d_m / t_m^\gamma \) for \( d_m = d_\Omega(x_m) \). Then, the proof is divided into two cases depending on whether \( \{c_m\} \) converges or not.

**Case 1** \( \lim_{m \to \infty} c_m = \infty \). We may assume \( \lim_{m \to \infty} c_m = \infty \). In this case, the rescaled domain \( \Omega_m \) expands to the whole space. In fact, for each \( R > 0 \) we observe that
\[
\inf \{ d_{\Omega_m}(x) \mid |x| \leq R \} \to \infty \quad \text{as} \quad c_m \to \infty.
\]
We take an arbitrary \( \varphi \in C^\infty_c(\mathbb{R}^n \times [0, 1]) \). We may assume that \( \varphi \) is supported in \( \Omega_m \times [0, 1] \). Since \( (u_m, p_m) \) satisfies (1.1) in \( \Omega_m \times (0, 1) \) for \( u_{0,m} = \partial g_m - \nabla \Phi_{0,m} \) and \( \nabla \Phi_{0,m} = Q_{\Omega, \partial g_m} \), it follows that
\[
(6.12) \quad \int_0^1 \int_{\mathbb{R}^n} (u_m \cdot (\partial \varphi + \nabla p_m \cdot \varphi) - \nabla p_m : \varphi) \, dx \, dt = \int_{\Omega_m} (g_m \cdot \partial \varphi_0 - \Phi_{0,m} \div \varphi_0) \, dx,
\]
where \( \varphi_0(x) = \varphi(x, 0) \).

We apply Lemma 4.3 and observe that \( \partial_t u_m, \nabla^2 u_m, \nabla p_m \) are equi-continuous in the interior of \( \Omega_m \times (0, 1) \). There exists a subsequence of \( \{(u_m, p_m)\} \) (still denoted by \( \{(u_m, p_m)\} \) such that \( (u_m, p_m) \) converges to a limit \((u, p)\) locally uniformly in \( \mathbb{R}^n \times (0, 1) \) together with \( \nabla u_m, \nabla^2 u_m, \partial_t u_m, \nabla p_m \). Moreover, it follows from (4.1) and (6.9) that
\[
(6.13) \quad \sup \left\{ t^{\gamma + \frac{1}{2}} d_{\Omega_m}(x) | \nabla p_m(x, t) | \mid x \in \Omega_m, 0 < t \leq 1 \right\} \leq C,
\]
with some constant \( C \) independent of \( m \). Since \( \Omega_m \) expands to the whole space, \( \nabla p_m \) converges to zero locally uniformly in \( \mathbb{R}^n \times (0, 1) \), i.e., \( \nabla p \equiv 0 \).

We apply Lemma 3.3 for \( \nabla \Phi_{1,m} = Q_{\mathbb{R}^n, \partial g_m} \) and \( \nabla \Phi_{2,m} = Q_{\Omega, \partial g_m} - Q_{\mathbb{R}^n, \partial g_m} \) to estimate
\[
(6.14) \quad \left[ \Phi_{1,m} \right]_{\mathbb{R}^n}^{(a)} + \sup_{x \in \Omega_m} d_{\Omega_m}^{1-a}(x) | \nabla \Phi_{2,m}(x) | \leq C \left[ g_m \right]_{\Omega_m}^{(a)},
\]
with some constant \( C \) independent of \( m \). By (6.10) and (6.14), the right-hand side of (6.12) vanishes as \( m \to \infty \). Thus, the limit \( u \) satisfies
\[
\int_0^1 \int_{\mathbb{R}^n} u \cdot (\partial_t \varphi + \nabla \varphi) \, dx \, dt = 0.
\]
By the uniqueness of the heat equation, we conclude that \( u \equiv 0 \) (and \( \nabla p \equiv 0 \)). This contradicts \( N(u, p)(0, 1) \geq 1/4 \) so Case 1 does not occur.

**Case 2** \( \lim_{m \to \infty} c_m < \infty \). By choosing a subsequence, we may assume \( \lim_{m \to \infty} c_m = c_0 \) for some \( c_0 \geq 0 \). In this case, the rescaled domain \( \Omega_m \) expands to a half space. Since \( d_\Omega(x_m) = c_m t_m^{1/2} \to 0 \), the points \( \{x_m\} \) accumulate to the boundary. By rotation and translation of \( \Omega \), we may assume \( x_m = (0, d_m) \) and \( 0 \in \partial \Omega \). We consider the neighborhood of the origin denoted by
\[
\Omega_{loc} = \left\{ (x', x_n) \in \mathbb{R}^n \mid h(x') < x_n < h(x') + \beta, |x'| < \alpha' \right\},
\]
with some constants $\alpha', \beta', K'$ and a $C^3$-function $h$ satisfying $h(0) = 0$, $\nabla' h(0) = 0$ and $\| h \|_{C^3([|x'| < \alpha])} \leq K'$. Since $\Omega_{\text{loc}} \subset \Omega$ is rescaled to

$$
\Omega_{\text{loc},m} = \left\{ (x',x_n) \in \mathbb{R}^n \left| h_m(x') - c_m < x_n < h_m(x') - c_m + \frac{\beta'}{t_m^2}, \ |x'| < \frac{\alpha'}{t_m} \right. \right\},
$$

where $h_m(x') = t_m^{-1/2} h(t_m^{1/2} x')$, $\Omega_{\text{loc},m}$ expands to the half space $\mathbb{R}^n_{+, -c_0} = \{(x',x_n) \mid x_n > -c_0\}$.

We take an arbitrary $\varphi \in C_c^{\infty}(\mathbb{R}^n_{+, -c_0} \times [0,1])$ and observe that $\varphi$ is supported in $\Omega_{\text{loc},m} \times [0,1)$ for sufficiently large $m$. Since $(u_m, p_m)$ satisfies (1.1), it follows that

$$
(6.15) \quad \int_0^1 \int_{\Omega_{\text{loc},m}} (u_m \cdot (\partial \varphi + \Delta \varphi) - \nabla p_m \cdot \varphi) \, dx \, dt = \int_{\Omega_m} (g_m \cdot \partial \varphi_0 - \Phi_{0,m} \text{div} \varphi_0) \, dx.
$$

We apply Lemma 4.3 and observe that $\partial_t u_m$, $\nabla^2 u_m$, $\nabla p_m$ are equi-continuous up to the boundary of $\Omega_m \times (0,1]$. There exists a subsequence denoted by $\{(u_m, p_m)\}$ such that $(u_m, p_m)$ converges to a limit $(u, p)$ locally uniformly in $\mathbb{R}^n_{+, -c_0} \times [0,1]$ together with $\nabla u_m$, $\nabla^2 u_m$, $\partial_t u_m$, $\nabla p_m$. By (6.13), the limit $p$ satisfies

$$
\sup \left\{ \int_t^{t+1} \int_{\mathbb{R}^n_{+, -c_0}} (u \cdot (\partial \varphi + \Delta \varphi) - \nabla p \cdot \varphi) \, dx \, dt \mid x \in \mathbb{R}^n_{+, -c_0}, \ 0 < t \leq 1 \right\} \leq C.
$$

By (6.10), (6.14) and sending $m \to \infty$, the right-hand side of (6.15) vanishes as in Case 1. Thus, the limit $(u, p)$ satisfies

$$
\int_0^1 \int_{\mathbb{R}^n_{+, -c_0}} (u \cdot (\partial \varphi + \Delta \varphi) - \nabla p \cdot \varphi) \, dx \, dt = 0.
$$

We apply Theorem 5.1 and conclude that $u \equiv 0$ and $\nabla p \equiv 0$. This contradicts $N(u, p)(0,1) \geq 1/4$ so Case 2 does not occur.

We reached a contradiction. The proof is now complete. \qed

6.3. Approximation.

We prove the estimate (1.8) by interpolation and approximation. After the proof of Theorem 1.3, we give a proof for Theorems 1.2 and 1.1.

**Proposition 6.3.** Let $\Omega$ be a domain in $\mathbb{R}^n$. Then, the estimate

$$
(6.16) \quad \left[ f \right]_{\Omega}^{(\alpha)} \leq 2 \| f \|_{l^\infty(\Omega)}^{1-\alpha} \| \nabla f \|_{l^\infty(\Omega)}^\alpha
$$

holds for $f \in C_c^{\infty}(\Omega)$ and $\alpha \in (0,1)$.

**Proof.** We identify $f \in C_c^{\infty}(\Omega)$ and its zero extension to $\mathbb{R}^n \setminus \Omega$. For arbitrary $x, y \in \mathbb{R}^n$, $x \neq y$, we estimate

$$
\frac{|f(x) - f(y)|}{|x - y|^\alpha} = \left| \frac{f(x) - f(y)}{|x - y|} \right|^{1-\alpha} \left| \frac{f(x) - f(y)}{|x - y|} \right|^\alpha
\leq 2 \| f \|_{l^\infty(\mathbb{R}^n)}^{1-\alpha} \| \nabla f \|_{l^\infty(\mathbb{R}^n)}^\alpha.
$$

Since $f$ is supported in $\Omega$, (6.16) follows. \qed
Proposition 6.4. Let $\Omega$ be a domain with Lipschitz boundary.

(i) When $\Omega$ is bounded,

\[ C^1_0(\Omega) = \left\{ f \in C^1(\overline{\Omega}) \mid f = 0, \nabla f = 0 \text{ on } \partial \Omega \right\}. \]

(ii) When $\Omega$ is unbounded,

\[ C^1_0(\Omega) = \left\{ f \in C^1(\overline{\Omega}) \mid f \text{ and } \nabla f \text{ are vanishing on } \partial \Omega \text{ and as } |x| \to \infty \right\}. \]

Moreover, $C^\infty_c(\Omega)$ is dense in $C^1_0 \cap W^{1,2}(\Omega)$.

Proof. When $\Omega$ is a bounded Lipschitz domain, we are able to prove (i) by decomposing $\Omega$ into star-shaped domains. Moreover, $C^\infty_c$ is dense in $C^1_0 \cap W^{1,2} = C^1_0$.

We give a proof for (ii) for unbounded domains $\Omega$. For $f \in C^1(\overline{\Omega})$ satisfying

\[ f = 0, \quad \nabla f = 0 \quad \partial \Omega, \]

\[ \lim_{|x| \to \infty} f(x) = 0, \quad \lim_{|x| \to \infty} \nabla f(x) = 0, \]

we prove that there exists a sequence $\{f_m\} \subset C^\infty_c$ such that

\[ \lim_{m \to \infty} \|f - f_m\|_{W^{1,\infty}(\Omega)} = 0. \]

Here, we write $\lim_{|x| \to \infty} f(x) = 0$ in the sense that $f(x_m) \to 0$ as $m \to \infty$ for any sequence $\{x_m\} \subset \Omega$ such that $|x_m| \to \infty$. This condition is equivalent to

\[ \lim_{R \to \infty} \sup_{x \in \Omega} \{f(x) \mid x \in \Omega, |x| \geq R\} = 0. \]

Let $\theta \in C^\infty_c[0,\infty)$ be a smooth cutoff function such that $\theta \equiv 1$ in $[0, 1/2]$ and $\theta \equiv 0$ in $[1, \infty)$. We set $\theta_m(x) = \theta(|x|m)$ so that $\theta_m \in C^\infty_c(\mathbb{R}^n)$ satisfies $\theta_m \equiv 1$ for $|x| \leq m/2$ and $\theta_m \equiv 0$ for $|x| \geq m$. We observe that $\tilde{f}_m = f \theta_m$ satisfies $\tilde{f}_m \in C^1(\overline{\Omega})$ and $\text{spt } \tilde{f}_m \subset \overline{\Omega} \cap \{|x| \leq m\}$. Since $f$ and $\nabla f$ are vanishing on $\partial \Omega$, $\tilde{f}_m$ satisfies $\tilde{f}_m = 0$ and $\nabla \tilde{f}_m = 0$ on $\partial \Omega$. Moreover, $\tilde{f}_m$ converges to $f$ uniformly in $\overline{\Omega}$ since $f$ is decaying as $|x| \to \infty$, i.e.,

\[ \|f - \tilde{f}_m\|_{L^\infty(\Omega)} = \|f(1-\theta_m)\|_{L^\infty(\Omega)} \]

\[ \leq \sup \{f(x) \mid x \in \Omega, |x| \geq m\} \]

\[ \to 0 \quad \text{as } m \to \infty. \]

By a similar way, $\nabla \tilde{f}_m$ converges to $\nabla f$ uniformly in $\overline{\Omega}$. Thus, we have

\[ \lim_{m \to \infty} \|f - \tilde{f}_m\|_{W^{1,\infty}(\Omega)} = 0. \]

We set $\Omega_m = \Omega \cap B_0(m)$. We may assume that $\Omega_m$ has Lipschitz boundary by taking a bounded Lipschitz domain $\Omega'_m \supset \Omega_m$ if necessary. Since $\tilde{f}_m \in C^1(\overline{\Omega}_m)$ by the assertion (i), for each $m \geq 1$ there exists $\{f_{m,k}\} \subset C^\infty_c(\Omega_m)$ such that

\[ \lim_{k \to \infty} \|\tilde{f}_m - f_{m,k}\|_{W^{1,\infty}(\Omega_m)} = 0, \]

i.e., for an arbitrary $\varepsilon > 0$ there exists $K = K_{m,\varepsilon}$ such that

\[ \|\tilde{f}_m - f_{m,k}\|_{W^{1,\infty}(\Omega_m)} \leq \varepsilon \quad \text{for } k \geq K_{m,\varepsilon}. \]
We set \( f_m = \tilde{f}_{m,k} \) for \( k = K_{m,e} \). Then, \( f_m \in C_c^\infty(\Omega) \) satisfies
\[
\| f - f_m \|_{W^{1,\infty}(\Omega)} \leq \| f - \tilde{f}_m \|_{W^{1,\infty}(\Omega)} + \| \tilde{f}_m - f_m \|_{W^{1,\infty}(\Omega)}
\]
\[
\leq \| f - \tilde{f}_m \|_{W^{1,\infty}(\Omega)} + \varepsilon.
\]
It follows that
\[
\lim_{m \to \infty} \| f - f_m \|_{W^{1,\infty}(\Omega)} \leq \varepsilon.
\]
Since \( \varepsilon \) is an arbitrary constant, letting \( \varepsilon \downarrow 0 \) yields (6.17). If in addition \( f \in W^{1,2} \), \( f_m \to f \) in \( W^{1,2} \) so \( C_c^\infty \) is dense in \( C_0^1 \cap W^{1,2} \). The proof is complete. \( \square \)

**Proof of Theorem 1.3.** It follows from (1.7) and (6.16) that
\[
(6.18) \quad \sup_{\partial \Omega} |N_v(t)| \leq C \| f \|_{W^{1,\infty}}^{1-q} \| \nabla f \|_\infty^q
\]
for all \( \tilde{L}^p \)-solutions for \( v_0 = \nabla f \), \( f \in C_c^\infty \) for some \( T_0 > 0 \). Since \( v = S(t)\partial f \) and \( S(t) \) is an analytic semigroup on \( C_{0,\sigma} \) [4], we are able to extend \( T_0 \) up to an arbitrary time. Since \( C_c^\infty \) is dense in \( C_0^1 \cap W^{1,2} \) by Proposition 6.4, we are able to extend (6.18) for \( f \in C_0^1 \cap W^{1,2} \). We proved (1.8). The proof is complete. \( \square \)

**Remark 6.5.** We used \( \tilde{L}^p \)-theory in order to establish (1.7) since \( L^p \)-theory may not be available for general unbounded domains (see [13] for \( L^p \)-theory for uniformly \( C^3 \)-domains). If \( L^p \)-theory is available for uniformly \( C^3 \)-domains \( \Omega \) (e.g., bounded or exterior domains), the statement of Theorem 1.3 is valid by replacing \( \tilde{L}^p \) to \( L^p \).

**Proof of Theorems 1.2 and 1.1.** Since bounded and exterior domains of class \( C^3 \) are strongly admissible, Theorem 1.2 holds. It remains to show (1.5) for all \( t > 0 \) for bounded domains. It is shown in [4] Remark 5.4 (i) that the maximum of \( S(t)v_0 \) for \( v_0 \in C_{0,\sigma} \) exponentially decays as \( t \to \infty \), i.e.,
\[
\| S(t)v_0 \|_\infty \leq C e^{-\nu t} \| v_0 \|_\infty \quad \text{for } t \geq 0
\]
with some constants \( \nu > 0 \) and \( C > 0 \). It follows that
\[
\| S(t)\partial f \|_\infty = \| S(t-1)S(1)\partial f \|_\infty
\]
\[
\leq C e^{-\nu(t-1)} \| f \|_\infty^{1-q} \| \nabla f \|_\infty^q \quad \text{for } t \geq 1.
\]
Thus, the estimate (1.5) is valid for all \( t > 0 \) for bounded domains. We proved Theorem 1.1. \( \square \)

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Appendix A. $L^1$-type results for the Stokes equations in a half space

In Appendix A, we recall an existence result for the dual problem (5.7)–(5.10) on $L^1$ and give a proof for Proposition 5.4.

**Proposition A.1.** For $f \in C^\infty_c(\mathbb{R}^n_+ \times [0, T))$, there exists a smooth solution $(\varphi, \pi)$ of (5.7)–(5.10) in $\mathbb{R}^n_+ \times [0, T]$ satisfying $\varphi \in S$ and $\nabla \pi \in L^\infty(0, T; L^1(\mathbb{R}^n_+))$.

**Proof.** The assertion is essentially proved in [2] Proposition 2.4. It is proved that smooth solutions $(\varphi, \pi)$ exist and satisfy $\partial_t \varphi + \nabla \pi \in L^\infty(0, T; L^1_1(\mathbb{R}^n_+))$ for $0 \leq 2s + |k| \leq 2$, $\varphi = 0$ on $\{x_n = 0\} \cup \{t = T\}$ and

$$\partial_n \varphi \in L^\infty(0, T; L^\infty(\mathbb{R}^n_+; L^1(\mathbb{R}^n_+))).$$

Here, $L^\infty(\mathbb{R}^n_+; L^1(\mathbb{R}^n_+))$ denotes the space of all essentially bounded functions $g(\cdot, x_n) : \mathbb{R}^n_+ \to L^1(\mathbb{R}^n_+)$ and is equipped with the norm $\|g\|_{L^\infty(\mathbb{R}^n_+; L^1(\mathbb{R}^n_+))} = \text{ess-sup}_{x_n > 0} \|g\|_{L^1(\mathbb{R}^n_+)}(x_n).

The solution $\varphi$ satisfies $\partial_n \varphi \in L^\infty(0, T; L^1_1)$ (i.e., $\varphi \in S$). In fact, by $\varphi = 0$ on $\{x_n = 0\}$ and

$$\|\varphi\|_{L^1_1(\mathbb{R}^n_+)} \leq \|\partial_n \varphi\|_{L^\infty(\mathbb{R}^n_+; L^1(\mathbb{R}^n_+))},$$

it follows that

$$\|\partial_n \varphi\|_{L^\infty(\mathbb{R}^n_+; L^1(\mathbb{R}^n_+))} \leq \|\partial_n \varphi\|_{L^\infty(\mathbb{R}^n_+; L^1_1)}(x_n) dx_n \leq \|\partial_n \varphi\|_{L^\infty(\mathbb{R}^n_+; L^1(\mathbb{R}^n_+))} + \|\varphi\|_{L^1_1(\mathbb{R}^n_+)}.$$

so $\partial_n \varphi \in L^\infty(0, T; L^1_1)$ and $\varphi \in S$.

We give a proof for Proposition 5.4.

**Proposition A.2.** Under the assumption of Lemma 5.2, the condition (5.3) is extendable for all $\varphi \in C^\infty_c(\mathbb{R}^n_+ \times [0, T])$ satisfying $\varphi = 0$ on $\{x_n = 0\} \cup \{t = T\}$.

**Proof of Proposition 5.4.** We cutoff $\varphi \in S$ as $|x| \to \infty$. Let $\theta \in C^\infty_c[0, \infty)$ be a smooth cutoff function satisfying $\theta \equiv 1$ in $[0, 1]$ and $\theta \equiv 0$ in $[2, \infty)$. We set $\theta_m(x) = \theta(|x|/m)$ for $m \geq 1$ and $\varphi_m = \varphi \theta_m$. Then, (5.3) holds for $\varphi_m$ by Proposition A.2, i.e.,

$$0 = \int_0^T \int_{\mathbb{R}^n_+} (v \cdot (\partial_t \varphi_m + \Delta \varphi_m) - \nabla q \cdot \varphi_m) dx dt$$

$$= (v, \partial_t \varphi_m) + (v, \Delta \varphi_m) + (\nabla q, \varphi_m).$$

Here, $(\cdot, \cdot)$ denotes the product on $\mathbb{R}^n_+ \times (0, T)$. Since $\varphi \in S$ satisfies $\partial_t \varphi + \nabla \pi \in L^\infty(0, T; L^1_1)$ (0 $\leq 2s + |k| \leq 2$), the first two terms converges to $(v, \partial_t \varphi + \Delta \varphi)$ as $m \to \infty$. Since $\varphi$ satisfies $\partial_n \varphi \in L^\infty(0, T; L^1_1)$, the last term converges to $(-\nabla q, \varphi)$. Thus, the condition (5.3) is extendable for all $\varphi \in S$.

**Proof of Proposition A.2.** We show that the condition (5.3) is extendable for all $\varphi \in C^\infty_c(\mathbb{R}^n_+ \times [0, T])$ satisfying $\varphi = 0$ on $\{x_n = 0\}$. Let $\theta \in C^\infty_c[0, \infty)$ be the smooth cut-off function as above and set $\rho_m(x_n) = 1 - \theta_m(x_n)$ by $\theta_m(x_n) = \theta(m x_n)$ for $m \geq 1$ so that $\rho_m \in C^\infty_c[0, \infty)$ satisfies $\rho_m \equiv 0$ for $x_n \leq 1/m$ and $\rho_m \equiv 1$ for $x_n \geq 2/m$. We substitute $\varphi_m = \varphi \rho_m$ into (5.3) and observe.
We proved that the condition (5.3) is extendable for all \( x_n = 0 \), the last term converges to \((-\nabla q, \varphi)\).

We show that the second term converges to \((\varphi, \Delta \varphi)\). Since \( \varphi \) is vanishing on \( \{x_n = 0\} \), the last term converges to \((-\nabla q, \varphi)\).

We observe that \( \varphi \) is vanishing on \( \{x_n = 0\} \). Since

\[
\Delta \tilde{\varphi}_m = \Delta \varphi \rho_m + 2\partial_n \varphi \partial_n \rho_m + \varphi \partial_n^2 \rho_m
\]

\[
= \Delta \varphi (1 - \tilde{\theta}_m) - 2\partial_n \varphi \partial_n \tilde{\theta}_m - \varphi \partial_n^2 \tilde{\theta}_m,
\]

it follows that

\[
\int_0^T \int_{\mathbb{R}^n} \nu \cdot \Delta (\varphi - \tilde{\varphi}_m) \, dx \, dt = \int_0^T \int_{\mathbb{R}^n} \nu \cdot (\Delta \varphi \tilde{\theta}_m + \partial_n \varphi \partial_n \tilde{\theta}_m + \varphi \partial_n^2 \tilde{\theta}_m) \, dx \, dt
\]

\[
=: I_m + II_m + III_m.
\]

The first term \( I_m \) converges to zero since \( \tilde{\theta}_m \) is supported in \( \{0 \leq x_n \leq 2/m\} \). We show that \( II_m \) converges to zero. By a similar way, we are able to show \( III_m \to 0 \) as \( m \to \infty \). We take \( R > 0 \) such that \( spt \varphi \subset B_0(R) \times [0, T] \) and set

\[
\eta^R_m(t) = \sup \left\{ |\nu(x, t)| \left| x \leq R, \frac{1}{m} \leq x_n < \frac{2}{m} \right. \right\}.
\]

We observe that \( \eta^R_m(t) \to 0 \) as \( m \to \infty \) for each \( t \in (0, T) \) since \( \nu \) is vanishing on \( \{x_n = 0\} \). Moreover, \( \eta^R_m(t) \) is estimated by \( C/t^\gamma \) with some constant \( C \) by (5.4). Since \( \partial_n \tilde{\theta}_m \) is supported in \([1/m \leq |x| \leq 2/m]\) and \( \|\partial_n \tilde{\theta}_m\|_{\infty} \leq m\|\partial_n \theta\|_{\infty} \), it follows that

\[
\|II_m\| \leq C m \int_0^T \eta^R_m(t) \, dt \int_{1/m}^{2/m} \|\partial_n \varphi\|_{L^1(\mathbb{R}^{n-1})} \, dx_n
\]

\[
\leq C \left( \sup \left\{ \|\partial_n \varphi\|_{L^1(\mathbb{R}^{n-1})} \right\} \right) \int_0^T \eta^R_m(t) \, dt \to 0 \quad \text{as} \quad m \to \infty.
\]

We proved that the condition (5.3) is extendable for all \( \varphi \in C^\infty_c(\mathbb{R}^n_+ \times [0, T]) \) satisfying \( \varphi = 0 \) on \( \{x_n = 0\} \). By a similar cut-off argument near \( t = T \), we are able to extend (5.3) for all \( \varphi \in C^\infty_c(\mathbb{R}^n_+ \times [0, T]) \) satisfying \( \varphi = 0 \) on \( \{x_n = 0\} \cup \{t = T\} \). The proof is now complete. \( \square \)

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