RANKIN-SELBERG INTEGRALS FOR PRINCIPAL SERIES REPRESENTATIONS OF GL(n)

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Abstract. We prove that the local Rankin–Selberg integrals for principal series representations of the general linear groups agree with certain simple integrals over the Rankin–Selberg subgroups, up to certain constants given by the local gamma factors.

1. Introduction and the main results

Rankin–Selberg integrals provide a powerful tool in the study of automorphic representations and L-functions. Explicit calculations of the local Rankin–Selberg integrals are often desirable for arithmetic applications. The goal of this note is to show that the local Rankin–Selberg integrals for principal series representations of the general linear groups agree with certain simple integrals over the Rankin–Selberg subgroups, up to certain explicit constants given by the local gamma factors.

Fix an arbitrary local field \( k \). Let \( G := \text{GL}_n(k) \) \((n \geq 2)\). Let \( B = AN \) be the Borel subgroups of \( G \) of the upper-triangular matrices, where \( N \) is the unipotent radical of \( B \), and \( A \) is the subgroup of the diagonal matrices. Similarly, let \( \bar{B} = A\bar{N} \) be the Borel subgroup of \( G \) of the lower-triangular matrices, where \( \bar{N} \) is the unipotent radical of \( \bar{B} \).

The first Rankin-Selberg subgroup of \( G \) is the group \( R \) consisting of all matrices of the form
\[
\begin{bmatrix}
a & u \\
0 & h
\end{bmatrix} \in G
\]
such that \( a \in k^\times \), \( h \) is upper-triangular unipotent, and \( u \) is a row vector whose first entry equals 0. We put
\[N' := R \cap N\]
and \[A' := R \cap A \cong k^\times\]
so that \( R = A'N' \).

Fix a non-trivial unitary character \( \psi : k \to \mathbb{C}^\times \), and equip \( k \) with the self-dual Haar measure \( dx \) associated to \( \psi \). Write \( | \cdot |_k \) for the normalized absolute value on \( k \). We equip the following Haar measures on \( k^\times \), \( N \) and \( R \) respectively:
\[
\bullet \quad d^x a := |a|_k^{-1} da, \quad a \in k^\times;
\]

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\begin{itemize}
  \item \(du := \prod_{1 \leq i < j \leq n} du_{i,j}, \quad u = [u_{i,j}]_{1 \leq i,j \leq n} \in N;\)
  \item \(dg := \prod_{1 \leq i < j \leq n, j \neq 2} dg_{i,j} \cdot d^x g_{1,1}, \quad g = [g_{i,j}]_{1 \leq i,j \leq n} \in R.\)
\end{itemize}

Using \(\psi\) we define the following character of \(N\): 
\[
\psi_N : N \to \mathbb{C}^\times, \quad [u_{i,j}]_{1 \leq i,j \leq n} \mapsto \psi \left( \sum_{i=1}^{n-1} u_{i,i+1} \right).
\]

Write \(\text{Hom}(A, \mathbb{C}^\times)\) for the set of all characters of \(A\), which is a complex Lie group of dimension \(n\). Let \(\sigma = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_n \in \text{Hom}(A, \mathbb{C}^\times)\), where \(\sigma_1, \sigma_2, \ldots, \sigma_n\) are characters of \(k^\times\). View \(\sigma\) as a character of \(\bar{B}\) through the trivial extension to \(\bar{N}\), and define the principal series representation 
\[
I(\sigma) := \text{Ind}_{\bar{B}}^G \sigma.
\]

Recall that \(I(\sigma)\) consists of smooth functions \(f : G \to \mathbb{C}\) such that 
\[
f(\bar{b} \cdot g) = \sigma(\bar{b}) \cdot \bar{\rho}(\bar{b}) \cdot f(g), \quad \text{for all } \bar{b} \in \bar{B}, \ g \in G,
\]
where 
\[
\bar{\rho} = |\cdot|_k^{1-n} \otimes |\cdot|_k^{1-n} \otimes \cdots \otimes |\cdot|_k^{n-1} \in \text{Hom}(A, \mathbb{C}^\times),
\]
and that \(G\) acts on \(I(\sigma)\) through the right translations. In the non-archimedean case, \(I(\sigma)\) is equipped with the finest locally convex topology such that all seminorms on it are continuous. In the archimedean case, \(I(\sigma)\) is a Fréchet space under the smooth topology.

It is known that there is a unique element \(\lambda := \lambda_\sigma \in \text{Hom}_N(I(\sigma), \psi_N)\) such that 
\[
\lambda(f) = \int_N f(u) \psi_N^{-1}(u) \, du
\]
for all \(f \in I(\sigma)\) such that \(f|_N \in \mathcal{S}(N)\) (see [W92, Theorem 15.4.1]). Here and henceforth, we write \(\mathcal{S}(X)\) for the space of Schwartz functions on \(X\) when \(X\) is a Nash manifold (see [AG08]), and the space of compactly supported locally constant functions on \(X\) when \(X\) is a totally disconnected locally compact topological space. All functions are complex valued unless otherwise specified.

For every \(a \in k^\times\), write 
\[
[a] := \text{diag}(a, 1, \cdots, 1) \in G.
\]

Then \(A' = \{[a] \mid a \in k^\times\}\). For every \(f \in I(\sigma)\) and \(s \in \mathbb{C}\), the local Rankin–Selberg integral is define to be
\[
Z_s(f) := \int_{k^\times} \lambda([a].f) |a|_k^{s-\frac{n-1}{2}} \, d^x a.
\]

For every \(s \in \mathbb{C}\), define a character 
\[
\psi_s : R \to \mathbb{C}^\times, \quad u' \cdot [a] \mapsto \psi_N(u') \cdot |a|_k^{\frac{n-1}{2} - s}, \quad u' \in N', \ a \in k^\times.
\]
For every character $\omega$ of $k^\times$, let $L(s, \omega)$ denote the local $L$-function of $\omega$. Write

$$L(s, \sigma) := \prod_{i=1}^{n} L(s, \sigma_i),$$

which is a meromorphic function on $\mathbb{C}$. Note that $\frac{1}{L(s, \sigma)}$ is an entire function.

Some basic properties of the Rankin–Selberg integrals are summarized in the following proposition. See [J09, Section 5.3] and [JPSS83, Section 8.3].

**Proposition 1.1.** There is a real number $C_\sigma$ with the following properties.

- For all $s \in \mathbb{C}$ with the real part $\text{Re}(s) > C_\sigma$, the integral $\Lambda_s(f)$ converges absolutely for all $f \in I(\sigma)$.
- The map

$$\{ s \in \mathbb{C} \mid \text{Re}(s) > C_\sigma \} \times I(\sigma) \to \mathbb{C}, \quad (s, f) \mapsto \Lambda_s(f),$$

extends to the multiplication of $L(s, \sigma)$ with a continuous map

$$Z^\circ : \mathbb{C} \times I(\sigma) \to \mathbb{C}$$

that is holomorphic on the first variable and linear on the second variable. Moreover, for every $s \in \mathbb{C}$,

$$Z^\circ(s, \cdot) \in \text{Hom}_R(I(\sigma), \psi_s).$$

On the other hand, we set

$$w_1 = \text{diag} \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, 1, \ldots, 1 \right) \in G,$$

and define the integral

$$\Lambda_s(f) := \int_{R} f(w_1 g) \psi_s^{-1}(g) d_r g, \quad s \in \mathbb{C}, f \in I(\sigma).$$

For each $i = 1, 2, \ldots, n$, write $\nu_i$ for the real number such that $|\sigma_i| = |\cdot|_w^\nu$ as positive characters of $k^\times$. Put

$$\Omega_\sigma := \{ s \in \mathbb{C} \mid -\nu_2 < \text{Re}(s) < 1 - \nu_1 \}.$$ 

**Theorem 1.2.** Assume that

$$\begin{cases} \max\{\nu_1, \nu_2\} < \nu_3 < \cdots < \nu_{n-1} < \nu_n, \\ \nu_1 < \nu_2 + 1. \end{cases}$$

Then for all $s \in \Omega_\sigma$, the integral $\Lambda_s(f)$ converges absolutely for all $f \in I(\sigma)$. Moreover, the map

$$\Omega_\sigma \times I(\sigma) \to \mathbb{C}, \quad (s, f) \mapsto \Lambda_s(f)$$

is continuous, holomorphic on the first variable, and linear on the second variable.

Under the assumptions of Theorem 1.2, we get an element $\Lambda_s \in \text{Hom}_R(I(\sigma), \psi_s)$ for every $s \in \Omega_\sigma$. It is clear that $\Lambda_s \neq 0$. 

Theorem 1.3. For all but countably many $s \in \mathbb{C}$,
\[
\dim \text{Hom}_R(I(\sigma), \psi_s) = 1.
\]

In the non-archimedean case, Theorem 1.3 is proved in [JPSS83, Proposition 2.11] in a more general setting. We will prove Theorem 1.3 in the archimedean case by using the theory of Schwartz homologies in [CS21].

For every $s \in \Omega_\sigma$, Theorem 1.3 implies that $\Lambda_s$ equals $Z_s$ up to a scalar multiplication. More precisely, we will prove the following result.

Theorem 1.4. Let the notation and assumptions be as in Theorem 1.2. Then
\[
(2) \quad \Lambda_s(f) = \gamma(s, \sigma_1, \psi) \cdot Z_s(f)
\]
for all $f \in I(\sigma)$ and $s \in \Omega_\sigma$.

Remark. Here $\gamma(s, \sigma_1, \psi)$ denotes the usual local gamma factor which will be recalled later. The right hand side of (2) is holomorphic in $s \in \Omega_\sigma$, which can be seen from the equalities
\[
\gamma(s, \sigma_1, \psi) \cdot Z_s(f) = \varepsilon(s, \sigma_1, \psi) \cdot \frac{L(1-s, \sigma_1^{-1})}{L(s, \sigma_1)} \cdot L(s, \sigma) \cdot Z^\circ(s, f)
\]
\[
= \varepsilon(s, \sigma_1, \psi) \cdot L(1-s, \sigma_1^{-1}) \cdot \prod_{i=2}^{n} L(s, \sigma_i) \cdot Z^\circ(s, f).
\]

Here $Z^\circ$ is as in Proposition 1.1.

We recall the definition of local gamma factor following [T79, J79, K03]. Given a character $\omega$ of $k^\times$, the Tate’s local zeta integral ([T50]) is defined by
\[
Z(s, \omega, f) = \int_{k^\times} f(x)\omega(x)|x|^s_k \, dx, \quad f \in \mathcal{S}(k),
\]
which converges absolutely when $\text{Re}(s) > -\text{ex}(\omega)$. Here $\text{ex}(\omega)$ is the real number such that $|\omega| = |\cdot|^{\text{ex}(\omega)}_k$ as positive characters of $k^\times$. The local epsilon factor attached to $\omega$ and $\psi$ will be denoted by $\varepsilon(s, \omega, \psi)$, which is defined by the local functional equation ([T50])
\[
(3) \quad \frac{Z(1-s, \omega^{-1}, \mathcal{F}_\psi(f))}{L(1-s, \omega^{-1})} = \varepsilon(s, \omega, \psi) \cdot \frac{Z(s, \omega, f)}{L(s, \omega)}, \quad f \in \mathcal{S}(k),
\]
where $\mathcal{F}_\psi(f) \in \mathcal{S}(k)$ is the Fourier transform of $f$ with respect to $\psi$ defined by
\[
\mathcal{F}_\psi(f)(x) := \int_{k} f(y)\psi(xy) \, dy, \quad x \in k.
\]

The meromorphic function
\[
\gamma(s, \omega, \psi) := \varepsilon(s, \omega, \psi) \cdot \frac{L(1-s, \omega^{-1})}{L(s, \omega)}
\]
is called the local gamma factor attached to $\omega$ and $\psi$. 
The functional $\Lambda_s$ takes a simpler form than $Z_s$, and we expect that Theorem 1.4 will be useful for the study of global L-functions.

2. Proof of Theorem 1.2

Let $f \in I(\sigma)$. We first rewrite $\Lambda_s(f)$ formally as an integral over $N$. Let $N_\alpha$ be the root subgroup of $N$ corresponding to the positive simple root $\alpha := e_1 - e_2$, that is, $N_\alpha$ consists of the matrices of the form

$$u(x) := \text{diag}\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, 1, \cdots, 1\right), \quad x \in k.$$ 

Then $N = N_\alpha N' = N'N_\alpha$. In view of the equalities

$$[a]^{-1}w_1[a] = [a]^{-1}u(1)[a] = u(a^{-1}), \quad a \in k^\times,$$

we have that

$$\Lambda_s(f) = \int_{k^\times} \int_{N'} f(wu'[a])\psi_s^{-1}(u'[a])du'd^\times a$$

$$= \int_{k^\times} \int_{N'} f([a] \cdot u(a^{-1}) \cdot [a]^{-1}u'[a])\psi_s^{-1}(u'[a])du'd^\times a$$

$$= \int_{k^\times} \int_{N'} \sigma_1(a)|a|_{k}^{-s} f(u(a^{-1}) \cdot [a]^{-1}u'[a])\psi_s^{-1}(u'[a])du'd^\times a.$$

The change of variable $u' \mapsto [a]u'[a]^{-1}$ does not affect the value of $\psi_s(u'[a])$, and we have that

$$\left| \det (\text{Ad}([a]))_{\text{Lie } N'} \right|_k = |a|_k^{n-2}.$$

Hence

$$\Lambda_s(f) = \int_{k^\times} \int_{N'} \sigma_1(a)|a|_{k}^{-s} f(u(a^{-1}) \cdot u')\psi_s^{-1}(u'[a])du'd^\times a$$

$$(4)$$

$$= \int_{k^\times} \int_{N'} \sigma_1^{-1}(a)|a|_{k}^{1-s} f(u(a) \cdot u')\psi_N^{-1}(u')du'd^\times a,$$

where we made the change of variable $a \mapsto a^{-1}$ in the last equality.

Introduce an open subset of $N$,

$$N^* := \{ u \in [u_{i,j}] \in N \mid u_{1,2} \neq 0 \},$$

and for $s \in \mathbb{C}$ define a function $h_s$ on $N^*$ by

$$h_s(u) := \left| u_{1,2} \right|_k^{-s}, \quad u = [u_{i,j}] \in N^*.$$
Let $K$ be the standard maximal compact subgroup of $G$, namely
$$
K := \begin{cases} 
O(n), & \text{if } k \cong \mathbb{R}; \\
U(n), & \text{if } k \cong \mathbb{C}; \\
GL_n(\mathfrak{o}_k), & \text{if } k \text{ is nonarchimedean,}
\end{cases}
$$
where $\mathfrak{o}_k$ denotes the ring of integers in $k$. Put
$$
\|f\|_K := \max_{k \in K} |f(k)|.
$$
Then for $a \in k^\times$ and $u' \in N'$ we have that
$$
(6) \quad \left| \sigma_1^{-1}(a)|a|_k^{-s} f(u(a) \cdot u') \psi_N^{-1}(u') \right| \leq \|f\|_K \cdot \varphi_\nu(u(a) \cdot u') \cdot h_{\Re(s)}(u(a) \cdot u'),
$$
where $\nu := | \cdot |_k^{\omega_1} \otimes \cdots \otimes | \cdot |_k^{\omega_n} \in \text{Hom}(A, \mathbb{C}^\times)$, and $\varphi_\nu$ is the spherical vector of $I(\nu)$ satisfying that $\varphi_\nu|_K \equiv 1$. Note that $\varphi_\nu$ is positive valued. Define the integral
$$
\eta_\nu(s) := \int_N \varphi_\nu(u) h_s(u) du.
$$
To prove Theorem 1.2, we first prove the following result.

**Proposition 2.1.** Assume that (1) holds. Then the integral $\eta_\nu(s)$ converges absolutely and uniformly on compact subsets of $\Omega_\sigma$.

**Proof.** Let $G_\alpha = \text{GL}_2$ embedded into the top-left corner of $G$, with Iwasawa decomposition
$$
G_\alpha = \tilde{N}_\alpha A_\alpha K_\alpha,
$$
where $\tilde{N}_\alpha$ is the lower-triangular unipotent subgroup, $A_\alpha$ is the diagonal torus and $K_\alpha$ is the standard maximal compact subgroup of $G_\alpha$. Write the Iwasawa decomposition of $u(x)$, $x \in k$, accordingly as
$$
u_\nu(x) = \tilde{n}(x) \cdot a(x) \cdot k(x).
$$
Let $W$ be the subgroup of permutation matrices in $G$, and $s_\alpha \in W$ be the simple reflection corresponding to $\alpha$. Since $N_\alpha$ and $s_\alpha$ normalize $N'$, we see that $\tilde{N}_\alpha = \text{Ad}(s_\alpha)(N_\alpha)$ normalizes $N'$ as well. It is clear that for $x \in k$ the map
$$
\text{Ad}(\tilde{n}(x)) : N' \rightarrow N'
$$
is unimodular. Let $\Phi(N', A)$ be the set of roots of $A$ in $N'$. The roots in $\Phi(N', A)$ add up to
$$
\mu := 2\rho - \alpha,
$$
where $\rho$ acts on $A$ by the character $| \cdot |_k^{\omega_1} \otimes \cdots \otimes | \cdot |_k^{\omega_n}$. Then we have that
$$
| \det(\text{Ad}(a(x)))|_{\text{Lie } N'}|_k = |\mu(a(x))|_k.
It follows that we have formally

$$\eta_\nu(s) = \int_k \int_{N'} \varphi_\nu(u' \cdot u(x)) h_s(u(x)) du' dx$$

(7)

$$= \int_k (\nu \cdot \bar{\rho} \cdot \mu)(a(x)) |x|_k^{-\nu_1 - s} dx \cdot \int_{N'} \varphi_\nu(u') du'.$$

Let $\Phi$ be the standard root system of $GL_n$. For $w \in W$, let $N'_w$ be the subgroup of $N$ corresponding to the set of positive roots

$$\Phi_w := \Phi + \cap w^{-1} \Phi -.$$  

Then it is clear that $N'_w = N_{w_0 s_o}$, where $w_0 \in W$ is the anti-diagonal matrix, and that $\Phi_{w_0 s_o} = \Phi(N', A)$. In view of this, the second integral in the last product in (7) is among the Harish-Chandra’s $c$-functions (see e.g. [L67, GGPS90] and [H84, Chapter IV]), and it converges if and only if

$$\langle \nu, \beta^\vee \rangle < 0$$

for all $\beta \in \Phi(N', A)$, where $\beta^\vee$ denotes the coroot of $\beta$, and $\langle , \rangle$ is the natural pairing between $X^*(A)_C$ and $X_*(A)_C$. The above condition reads

$$\max\{\nu_1, \nu_2\} < \nu_3 < \cdots < \nu_{n-1} < \nu_n.$$

In this case, up to a positive scalar depending on the Haar measure, the Gindikin-Karpelevich formula (see [L67, Section 4]) gives that

$$\int_{N'} \varphi_\nu(u') du' = \prod_{\beta \in \Phi(N', A)} \frac{\zeta_k(-\langle \nu, \beta^\vee \rangle)}{\zeta_k(1 - \langle \nu, \beta^\vee \rangle)}.$$

Here $\zeta_k(s)$ is the zeta function of $k$ (see [179, Section 3]) defined by

$$\zeta_k(s) = \begin{cases} \pi^{-s/2} \Gamma(s/2), & \text{if } k = \mathbb{R}; \\ 2(2\pi)^{-s} \Gamma(s), & \text{if } k = \mathbb{C}; \\ 1 - q_k^{-s}, & \text{if } k \text{ is nonarchimedean}, \end{cases}$$

with $\Gamma(s)$ the usual Gamma function, and $q_k$ the cardinality of the residue field of $\mathfrak{o}_k$ in the last case.

By the standard computation of Iwasawa decomposition for $GL_2(k)$, the first integral in the last product in (7) is

$$\int_k (\nu \cdot \bar{\rho} \cdot \mu)(a(x)) |x|_k^{-\nu_1 - s} dx$$

$$= \begin{cases} \int_k (1 + |x|_k^{\nu_1 - \nu_2 - 1}) |x|_k^{-\nu_1 - s} dx, & \text{if } k = \mathbb{R}; \\ \int_k (1 + |x|_k^{\nu_1 - \nu_2 - 1}) |x|_k^{-\nu_1 - s} dx, & \text{if } k = \mathbb{C}; \\ \int_{|x|_k \leq 1} |x|_k^{-\nu_1 - s} dx + \int_{|x|_k > 1} |x|_k^{-\nu_2 - s - 1} dx, & \text{if } k \text{ is nonarchimedean}. \end{cases}$$
It is easy to check that the above integral converges absolutely and uniformly on compact subsets of $\Omega_\sigma$. This proves the Proposition.

By (1), (3) and Proposition 2.1 we see that $\Lambda_s(f)$ converges absolutely and uniformly on compact subsets of $\Omega_\sigma$. By the Weierstrass Theorem in complex analysis (see [A78, Chap. 5, Theorem 1]), $\Lambda_s(f)$ is holomorphic on $\Omega_\sigma$.

For $(s, f), (s', f') \in \Omega_\sigma \times I(\sigma)$, we have that

$$|\Lambda_s(f) - \Lambda_{s'}(f')| \leq |\Lambda_s(f - f')| + |\Lambda_s(f') - \Lambda_{s'}(f')|$$

$$\leq \|f - f'\|_K \cdot \eta(\text{Re}(s)) + \|f'\|_K \cdot \int_N \varphi(u) |h_s(u) - h_{s'}(u)| \, du.$$

It follows from the above inequalities and Proposition 2.1 that the map

$$\Omega_\sigma \times I(\sigma) \to \mathbb{C}, \quad (s, f) \mapsto \Lambda_s(f)$$

is continuous. This finishes the proof of Theorem 1.2.

3. $R$-orbits and proof of Theorem 1.3

In this section we prove Theorem 1.3 regarding the uniqueness of Rankin–Selberg periods.

3.1. $R$-orbits on the flag variety. Let $M := \bar{B}\backslash G$ be the flag variety of Borel subgroups of $G$ with base point $x_0 := B \in M$. For every $g \in G$, denote the $R$-orbit of $x_0 \cdot g \in M$ by

$$O_g := \{x_0 \cdot (gr) \mid r \in R\} \subset M.$$

We shall describe the $R$-orbits in $M$, or equivalently the $\bar{B}$-$R$ double cosets. To this end we make some preparations. For every $w \in W$ we have the opposite Bruhat cell

$$C_w := x_0 \cdot wB = x_0 \cdot wU_w,$$

where $U_w := N_{w_0w}$ is the subgroup of $N$ corresponding to the following set of positive roots (cf. [N])

$$\Psi_w := \Phi_{w_0w} = \Phi_+ \cap w^{-1} \Phi_+.$$

Then $C_1 = x_0 \cdot N$ is open dense in $M$, and we recall the Bruhat decomposition

$$M = \bigsqcup_{w \in W} C_w.$$

Introduce the following subsets of $W$:

$$W_+ := \{w \in W \mid w\alpha > 0\},$$

$$W_- := \{w \in W \mid w\alpha < 0\}.$$
Proposition 3.1. We have the following disjoint union of $R$-orbits

$$M = \left( \bigcup_{w \in W} \mathcal{O}_w \right) \bigcup \left( \bigcup_{w \in W_+} \mathcal{O}_{ww_1} \right).$$

Moreover it holds that

(i) if $w \in W_-$ then $C_w = \mathcal{O}_w$;
(ii) if $w \in W_+$ then $C_w = \mathcal{O}_w \sqcup \mathcal{O}_{ww_1}$ and $\mathcal{O}_{ww_1}$ is open dense in $C_w$.

Proof. By the Bruhat decomposition, it suffices to prove (i) and (ii). Put $N^*_\alpha := N_\alpha \cap N^* = N_\alpha \setminus \{1\} = \{u(a) \mid a \in k^\times\}$.

If $w \in W_-$, then

$$C_w = x_0 \cdot wN_\alpha N' = x_0 \cdot wN' = x_0 \cdot wR = \mathcal{O}_w.$$  

If $w \in W_+$, then

$$C_w = x_0 \cdot wN = x_0 \cdot wN' \sqcup x_0 \cdot wN^*_\alpha N'.$$

We have $x_0 \cdot wN' = \mathcal{O}_w$, and it is easy to verify that

$$x_0 \cdot wN^*_\alpha N' = x_0 \cdot wA'N^*_\alpha N' = x_0 \cdot wA'w_1A'N' = x_0 \cdot ww_1R = \mathcal{O}_{ww_1},$$

which finishes the proof. □

Example 3.2. If $n = 2$, then $R = k^\times$ acts on $M = \mathbb{P}^1(k)$ with three orbits $\{0\}$, $\{\infty\}$ and $k^\times$.

3.2. Proof of Theorem 1.3. Assume that $k$ is archimedean. Then we have the following result, which in particular implies Theorem 1.3.

Theorem 3.3. For all but countably many $s \in \mathbb{C}$, there is a topological linear isomorphism

$$H^S_i(R; I(\sigma) \otimes \psi_s^{-1}) \cong \begin{cases} \mathbb{C}, & \text{if } i = 0; \\ \{0\}, & \text{if } i \neq 0. \end{cases}$$

Here $H^S_i$ indicates the Schwartz homology studied in [CS21].

Proof. The flag variety $M$ is naturally a $G$-Nash manifold, and we have

$$I(\sigma) = \Gamma^\zeta(M, E)$$

for a certain tempered $G$-vector bundle $E$ of rank one over $M$. Here $\Gamma^\zeta(M, E)$ is the Fréchet space of Schwartz sections of $E$ defined as in [CS21, Section 6.1].

Denote $U := \mathcal{O}_{w_1}$ the unique open $R$-orbit in $M$, and $Z := M \setminus U$ its complement. Then

$$\Gamma^\zeta(U, E|_U) \cong S(R).$$

For $z \in M$ we have its stabilizer in $R$ given by

$$R_z = R \cap \bar{z}^{-1}B\bar{z},$$

where $B$ is the Borel group.
where $\tilde{z}$ is an arbitrary representative of $z$ in $G$. Write

$$N^*_z := \frac{T^*_z(M)}{T^*_z(R_z)} \otimes_{\mathbb{R}} \mathbb{C}$$

(T$^*_z$ stands for the cotangent space)

for the complexified conormal space, and $\delta_{R/R_z} := (\delta_R)|_{R_z} \cdot \delta_{R_z}^{-1} : R_z \to \mathbb{C}^\times$ where $\delta$ stands for the modular character.

We claim that for all but countably many $s \in \mathbb{C}$, the condition of [CS21, Theorem 1.15] holds, namely for all $z \in Z$ and all integers $k \geq 0$, the trivial representation of $R_z$ does not occur as a subquotient of

$$(9) \quad E_z \otimes \text{Sym}^k(N^*_z) \otimes \delta_{R/R_z} \otimes \psi^{-1},$$

where $E_z$ is the fibre of $E$ at $z$. By Proposition 3.1, we may assume that $z = w$, $w \in W$ or $z = ww_1$, $w \in W_+ \setminus \{1\}$.

If $z = w$, $w \in W$, then $A' \subset R_z$.

If $z = ww_1$, $w \in W_+ \setminus \{1\}$, then $w$ maps at least one of the positive simple roots $\alpha_i = e_i - e_{i+1}$, $i = 2, \ldots, n - 1$ to a negative root. At this point we consider two cases separately:

- $w\alpha_i < 0$ for some $i \geq 3$. Then $w_1 \in N_\alpha$ commutes with $N_{\alpha_i}$, hence $N_{\alpha_i} \subset R_z$;
- $w\alpha_2 < 0$. Let $S$ be the subgroup of $R$ consisting of the matrices

$$\text{diag} \left( \begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} , 1, \ldots, 1 \right), \quad x \in k.$$ 

Then $w_1 Sw_1^{-1} = N_{\alpha_2}$ and hence $S \subset R_z$.

In conclusion, we observe that every $R$-orbit in $Z$ contains an element $z$ with the following property: the stabilizer $R_z$ contains a subgroup $S_z$ such that the trivial representation of $S_z$ does not occur in (9) as an irreducible subquotient, for all $k = 0, 1, 2, \ldots$, and all but countably many $s \in \mathbb{C}$. Thus the trivial representation of $R_z$ does not occur in (9) for all $k = 0, 1, 2, \ldots$, and all but countably many $s \in \mathbb{C}$. This finishes the proof of the claim.

By Theorem 1.15 and Example 1.16 of [CS21], for all but countably many $s \in \mathbb{C}$ we have topological linear isomorphisms

$$H^S_i(R; I(\sigma) \otimes \psi^{-1}) = H^S_i(R; \Gamma^S(M, E) \otimes \psi^{-1})$$

$$\cong H^S_i(R; \Gamma^S(U, E|_U) \otimes \psi^{-1})$$

$$\cong H^S_i(R; S(R) \otimes \psi^{-1})$$

$$\cong \begin{cases} \mathbb{C}, & \text{if } i = 0; \\ \{0\}, & \text{if } i \neq 0. \end{cases}$$

□
4. Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4. We adopt the notations in previous sections.

Let \( f \in I(\sigma) \). We first assume that \( f|_{w_1R} \in \mathcal{S}(w_1R) \). Then \( \Lambda_s(f) \) converges absolutely and uniformly on compact subsets of \( \mathbb{C} \), and hence it is entire. It is clear that for every \( s \in \mathbb{C} \) there exists such a function \( f \) with \( \Lambda_s(f) \neq 0 \).

Proposition 3.1 implies that \( f|_N \in \mathcal{S}(N) \). We unfold \( Z_s(f) \) as

\[
Z_s(f) = \int_{k} \int_{N} f(u[a])\psi^{-1}_N(u)|a|^{s-n+1} dud^\times a
\]

\[
= \int_{k} \int_{N} f([a] \cdot [a]^{-1}u[a])\psi^{-1}_N(u)|a|^{s-n+1} dud^\times a
\]

\[
= \int_{k} \int_{N} \sigma_1(a)f([a]^{-1}u[a])\psi^{-1}_N(u)|a|^{s-n+1} dud^\times a
\]

\[
= \int_{k} \int_{N'} \int_{k} \sigma_1(a)f([a]^{-1}u' \cdot u(x)[a])\psi^{-1}_N(u' \cdot u(x))|a|^{s-n+1} dxdud^\times a.
\]

Note that \( \psi_N \) restricted to \( N' \) is invariant under the change of variable \( u' \mapsto [a]u'[a]^{-1} \), and we have

\[
|\det(\text{Ad}([a])|_{\text{Lie } N})|_k = |a|^{n-1}.
\]

Hence

\[
Z_s(f) = \int_{k} \int_{N'} \int_{k} \sigma_1(a)f(u' \cdot u(x))\psi^{-1}_N(u')\psi^{-1}(ax)|a|^{s-n+1} dxdud^\times a.
\]

For \( u' \in N' \), define

\[
f_{u'}(x) := f(u' \cdot u(x)), \quad x \in k.
\]

Then \( f_{u'} \in \mathcal{S}(k) \) and we have that

\[
(10) \quad Z_s(f) = \int_{k} \int_{N'} \mathcal{F}_{\psi^{-1}}(f_{u'})(a)\sigma_1(a)|a|^s\psi^{-1}_N(u')dud^\times a.
\]

Note that the function

\[
(N' \times k \to \mathbb{C}, \quad (u', a) \mapsto \mathcal{F}_{\psi^{-1}}(f_{u'})(a)) \in \mathcal{S}(N' \times k).
\]

Thus by Tate’s thesis, the integral (10) is absolutely convergent when \( \text{Re}(s) > -\nu_1 \).

Assume that this is the case. Then we can exchange the order of integration to obtain that

\[
Z_s(f) = \int_{N'} \left( \int_{k} \mathcal{F}_{\psi^{-1}}(f_{u'})(a)\sigma_1(a)|a|^s d^\times a \right) \psi^{-1}_N(u')dud' = \int_{N'} \mathcal{Z}(s, \sigma_1, \mathcal{F}_{\psi^{-1}}(f_{u'}))\psi^{-1}_N(u')dud'.
\]
By (3), (4) and the Fourier inversion,
\[
Z_s(f) = \gamma(s, \sigma_1, \psi)^{-1} \int N' \int \int_{k^*} f(u' \cdot u(a)) \sigma_1^{-1}(a) |a|^{1-s} \psi_N^{-1}(u') d^k a d u'
\]

where in the second last equality we have made the change of variable \( u' \mapsto \text{Ad}(u(a))(u') \) and used invariance property of \( \psi_N \).

Recall from Proposition 1.1 that \( Z_s(f) / L(s, \sigma) \) is an entire function on \( s \in \mathbb{C} \). Note that
\[
\frac{1}{\gamma(s, \sigma_1, \psi) \cdot L(s, \sigma)} = \frac{1}{\varepsilon(s, \sigma_1, \psi) \cdot L(1 - s, \sigma_1^{-1}) \cdot \prod_{i=2}^n L(s, \sigma_i)},
\]

which is also an entire function on \( s \in \mathbb{C} \). It follows from the uniqueness of holomorphic continuation that
\[
\frac{Z_s(f)}{L(s, \sigma)} = \frac{\Lambda_s(f)}{\gamma(s, \sigma_1, \psi) \cdot L(s, \sigma)}
\]

for any \( s \in \mathbb{C} \).

We now consider an arbitrary \( f \in I(\sigma) \). By Theorem 1.3, we have that (11) holds for all but countably many \( s \in \Omega_\sigma \), hence it holds for all \( s \in \Omega_\sigma \) by continuity. This proves that
\[
\Lambda_s(f) = \gamma(s, \sigma_1, \psi) \cdot Z_s(f)
\]

for all \( s \in \Omega_\sigma \) and \( f \in I(\sigma) \).

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