BOX-SPLINES ORTHOGONAL PROJECTIONS

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Abstract. Let $P$ be orthogonal projection on B-splines of degree $r-1$ with equally spaced knots. Sweldens and Piessens proved that $P(x^r) - x^r$ is Bernoulli polynomial. We generalize Sweldens ans Piessens’s result for box-splines. It gives the opportunity to define the seminorm of Sobolev space in terms of the asymptotic formula for the error in orthogonal projection. In second part we deal with similar problems in $BV(\mathbb{R}^d)$. It is a modification of Boschariev’s asymptotic formula for the functions of bounded variation. 41A15, 41A35, 41A60. Keywords: box spline, Bernoulli spline, asymptotic formula, orthogonal projection, function of bounded variation BV, Marcinkiewicz’s average.

1. Introduction

Let $W^k_p(\mathbb{R}^d)$ denote the Sobolev spaces, $1 \leq p < \infty$ with the norm

$$
\|f\|_{k,p} = \sum_{|\beta| \leq k} \|D^\beta f\|_p,
$$

where

$$
D^\beta f = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}}, \quad \beta = (\beta_1, \cdots, \beta_d),
$$

$$
|\beta| = \beta_1 + \cdots + \beta_d.
$$

and

$$
\|f\|_p = \left(\int_{\mathbb{R}^d} |f|^p\right)^{1/p}.
$$

Let $V = \{v_1, v_2, \cdots, v_n\}$ denote a set of not necessarily distinct, non zero vectors in $\mathbb{Z}^d \setminus \{0\}$, such that

$$
\text{span}\{V\} = \mathbb{R}^d.
$$

We call such set admissible. The box spline denoted by $B_V(\cdot)$ corresponding to $V$ is defined by requiring that

$$
(1.1) \quad \int_{\mathbb{R}^d} f(x)B_V(x)\,dx = \int_{[0,1]^n} f(Vu)\,du
$$
holds for any continuous function $f$ defined on $\mathbb{R}^d$, see reference [6]. As usual

$$V u = u_1 v_1 + \cdots + u_n v_n.$$ 

The Fourier transform is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i \xi \cdot t} \, dt.$$ 

Here and subsequently "·" denotes the scalar product in $\mathbb{R}^d$. From (1.1) by simple calculation we get that

\begin{equation}
\hat{B}_V(x) = \prod_{v \in V} g(x \cdot v),
\end{equation}

where

$$g(t) = \frac{1 - e^{-2\pi i t}}{2\pi i t}.$$ 

We denote by $\# V$ the cardinality of the set $V$. For an admissible set $V$ let

\begin{equation}
g_V = \max \{ r : \text{span}\{V \setminus W\} = \mathbb{R}^d \text{ for all } W \subset V, \#W = r \}.
\end{equation}

This parameter determines the smoothness of a box splines

$$B_V(\cdot) \in C^{g_V-1}(\mathbb{R}^d) \setminus C^{g_V}(\mathbb{R}^d).$$

Let us define

$$S_{L^2}(hV) = \overline{\text{span}}\{ B_V(\cdot/h - \alpha) : \alpha \in \mathbb{Z}^d \},$$

where $h > 0$ and the closure is taken in $L^2(\mathbb{R}^d)$. The orthogonal projection from $L^2(\mathbb{R}^d)$ onto $S_{L^2}(hV)$ is denoted by $P_h$. Denoting by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\mathbb{R}^d)$, the orthogonal projection onto $S_{L^2}(hV)$ can be written by ($P = P_1$):

\begin{equation}
P_h = \sigma_h \circ P \circ \sigma_{1/h},
\end{equation}

where

$$\sigma_h f(x) = f(x/h).$$

A family $V \subset \mathbb{Z}^d$ is unimodular if for all $W \subset V$ with $\#W = d$ we have $|\det W| \leq 1$. Set

$$[x]^\beta = x^\beta$$

and

$$\gamma \leq \beta \iff \gamma_j \leq \beta_j, j = 1, \ldots, d.$$
2. Box-spline orthogonal projections

Let us define

\[ L_\beta(x) = P(\mathbb{I}_\beta)(x) - x^\beta, \quad x \in \mathbb{R}^d. \]

Note that in the univariate case \( L_\beta \) is a Bernoulli spline for \( |\beta| = \varrho_V + 1 \), see [18]. In [11] it was proved in a particular case that \( L_\beta \) is linear combination of Bernoulli splines. In this section we generalize this results, see Theorem 2.5 below. Applying this result we simplify the asymptotic formula for orthogonal projection calculated in [5]. Our method in the case of \( L^2(\mathbb{R}^d) \) implies Theorem 2.2 of [3].

We know that \( L_\beta \) is a periodic piecewise polynomial and from Lemma 3.4 in [11] we have:

**Lemma 2.1.** Let \( |\beta| \leq \varrho_V + 1 \). Then

\[ L_\beta(x) = \left( \frac{1}{2\pi i} \right)^{|\beta|} \sum_{\alpha \in \mathbb{Z}^d, \alpha \neq 0} D^\beta \widehat{B}_V(\alpha) e^{2\pi i \alpha \cdot x}. \]

The series converges in every point of continuity of \( L_\beta \).

In fact the problem of the convergence appears only for box splines with \( \varrho_V = 0 \) and on the boundary of that box-splines. By Theorem [2.5] and Remark 2.9 we write \( L_\beta \) for \( |\beta| = 1 \) as a linear combination of a Bernoulli spline \( B^1 = x - 1/2 \), where the Fourier series of \( B^1 \) converges also in the point of discontinuity of \( B^1 \) (i.e \( x = 0 \)) to zero.

Define a set \( \Lambda \),

\[ \Lambda = \{ U \subset V : \#U = \varrho_V + 1, \text{span}\{V \setminus U\} \neq \mathbb{R}^d \}. \]

Let \( U \in \Lambda \). If for all \( v \in V \setminus U \)

\[ v \cdot \alpha = 0 \]

we will denote that \( \alpha \perp (V \setminus U) \). Note that the vectors from \( V \setminus U \) span a hyperplane in \( \mathbb{R}^d \) i.e. \( d - 1 \)-dimensional subspace. From definition of the set \( \Lambda \) we get that for all \( \alpha \neq 0 \) such that \( \alpha \perp (V \setminus U) \)

\[ v \cdot \alpha \neq 0 \quad \text{for all} \quad v \in U. \]

**Definition 2.2.** Let us define Bernoulli splines [15] for \( U \in \Lambda \) by

\[ B(V, U)(x) = \sum_{\alpha \neq 0, \alpha \perp (V \setminus U)} \prod_{v \in U} \frac{1}{2\pi i \alpha \cdot v} e^{2\pi i \alpha \cdot x}, \]

where \( \alpha \in \mathbb{Z}^d \) and \( x \in \mathbb{R}^d \).
Lemma 2.3. Let $V$ be unimodular and let $\alpha \in \mathbb{Z}^d \setminus \{0\}$. Let $D^\beta \widehat{B}_V(\alpha) \neq 0$ for given $|\beta| = \varrho_V + 1$. Then if $\# U_\alpha \geq \varrho_V + 1$ where $U_\alpha = \{v \in V : \alpha \cdot v \neq 0\}$ then $\# U_\alpha = \varrho_V + 1$.

Proof. Note that by (1.2)

$$D^\beta \widehat{B}_V(x) = \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} D^\gamma \prod_{v \in U_\alpha} g(x \cdot v) D^{\beta - \gamma} \prod_{v \in V \setminus U_\alpha} g(x \cdot v)$$

Since $\# U_\alpha \geq |\beta| = \varrho_V + 1$ then for $\gamma < \beta$

$$D^\gamma \prod_{v \in U_\alpha} g(x \cdot v) \bigg|_{x = \alpha} = 0.$$ 

since $v \cdot \alpha \neq 0$ for $v \in U_\alpha$ and $g(v \cdot \alpha) = 0$. Since $g(0) = 1$, (2.5) shows that

$$D^\beta \widehat{B}_V(\alpha) = D^\beta \prod_{v \in U_\alpha} g(x \cdot v) \bigg|_{x = \alpha} \neq 0.$$ 

(2.6) implies theorem. \qed

Lemma 2.4. Let $V$ be unimodular and let $\alpha \in \mathbb{Z}^d \setminus \{0\}$. Let $D^\beta \widehat{B}_V(\alpha) \neq 0$ for given $|\beta| = \varrho_V + 1$. Then $U_\alpha \in \Lambda$.

Proof. Note that $U_\alpha \neq \emptyset$ since $V$ spans $R^d$. If $\# U_\alpha \geq \varrho_V + 1$ then from Lemma 1.4 we get that $\# U_\alpha = \varrho_V + 1$. Moreover $\alpha \perp (V \setminus U_\alpha)$, hence $\text{span}\{V \setminus U_\alpha\} \neq R^d$, it follows that $U_\alpha \in \Lambda$.

Let us assume that $\# U_\alpha < \varrho_V + 1$. But $\alpha \perp (V \setminus U_\alpha)$, hence $\text{span}\{V \setminus U_\alpha\} \neq R^d$. This is a contradiction with definition of $\varrho_V$ see (1.3). \qed

Theorem 2.5. Let $V$ be unimodular. Let $|\beta| = \varrho_V + 1$. Then $L_\beta$ is a linear combination of Bernoulli splines

$$(2.7) \quad L_\beta(x) = P(|\beta|)(x) - x^\beta = \sum_{U \in \Lambda} C(\beta, U) B(V, U)(x)$$

where constants

$$C(\beta, U) = D^\beta \left( \prod_{v \in U} (x \cdot v) \right).$$

Proof. By Lemma 2.1 and Lemma 2.5 we get

$$L_\beta(x) = \left( \frac{1}{2\pi i} \right)^{\varrho_V + 1} \sum_{U \in \Lambda} \sum_{\alpha \neq 0, \alpha \perp (V \setminus U)} D^\beta \widehat{B}_V(\alpha) e^{2\pi i \alpha \cdot x}.$$
By (2.6) we get that

\[(2.8) \quad D^\beta \widehat{B}_V(\alpha) = \prod_{v \in U_\alpha} \frac{1}{v \cdot \alpha} D^\beta \left( \prod_{v \in U_\alpha} (x \cdot v) \right) \bigg|_{x = \alpha}.\]

Note that

\[C(\beta, U_\alpha) = D^\beta \left( \prod_{v \in U_\alpha} (x \cdot v) \right) \bigg|_{x = \alpha} = D^\beta \left( \prod_{v \in U_\alpha} (x \cdot v) \right)\]

Consequently

\[L_\beta(x) = \sum_{U \in \Lambda} C(\beta, U) B(V, U)(x)\]

Let us recall the results from [5] and [11].

**Theorem 2.6.** Let \(1 \leq p < \infty\). Let \(V\) be unimodular. Let \(f \in W_{\rho V+1}^p(\mathbb{R}^d)\). Then

\[(2.9) \quad \lim_{h \to 0^+} \left\| \frac{f - P_h f}{h^{\rho V+1}} \right\|_p^p = \int_{\mathbb{R}^d} \left( \int_{[0,1]^d} \left| \sum_{|\beta| = \rho V+1} \frac{1}{\beta!} D^\beta f(t) L_\beta(x) \right|^p dx \right) dt.\]

Now we want to examine the right part of (2.9).

**Theorem 2.7.** Let \(V\) be unimodular then

\[\sum_{|\beta| = \rho V+1} L_\beta(x) \frac{D^\beta f(t)}{\beta!} = \sum_{U \in \Lambda} D_U f(t) B(V, U)(x),\]

where

\[D_U = \prod_{v \in U} D_v\]

and \(D_v\) is the directional derivative.

**Proof.** From Lemma 2.1 and 2.5 we get

\[\sum_{|\beta| = \rho V+1} L_\beta(x) \frac{D^\beta f(t)}{\beta!} = \sum_{|\beta| = \rho V+1} \left( \frac{1}{2\pi i} \right)^{\rho V+1} \sum_{\alpha \in \mathbb{Z}^d, \alpha \neq 0} D^\beta \widehat{B}_V(\alpha) e^{2\pi i \alpha \cdot x} D^\beta f(t) \frac{1}{\beta!} = \left( \frac{1}{2\pi i} \right)^{\rho V+1} \sum_{|\beta| = \rho V+1} \sum_{U \in \Lambda} \sum_{\alpha \neq 0, \alpha \perp (V \setminus U)} D^\beta \widehat{B}_V(\alpha) e^{2\pi i \alpha \cdot x} D^\beta f(t) \frac{1}{\beta!}.\]
Note that the sets $V \setminus U$, where $U \in \Lambda$ are disjoint. Tedious calculation shows that

\[
\sum_{|\beta| = \rho V + 1} D^\beta \left( \prod_{v \in U_\alpha} (x \cdot v) \right) \left| \frac{D^\beta f(t)}{\beta!} \right|_{x=\alpha} = D_{U_\alpha} f(t).
\]

Consequently using (2.8), (2.7) and (2.10) we get the theorem. \qed

**Remark 2.8.** Let $V$ be unimodular. Then the functions $B(V, U)$, $U \in \Lambda$ are orthogonal in $L^2([0,1]^d)$. Moreover since all norms in a finite dimension space are equivalent “$\sim$” we get

\[
\int_{[0,1]^d} \left( \int_{[0,1]^d} \left| \sum_{|\beta| = \rho V + 1} \frac{1}{\beta!} D^\beta f(t) L_\beta(x) \right|^p \, dx \right) \, dt = \int_{[0,1]^d} \left( \sum_{U \in \Lambda} D_U f(t) B(V, U)(x) \right|^p \, dx \, dt \sim \int_{[0,1]^d} \left| D_U f(t) \right|^p \, dt \int_{[0,1]^d} \left| B(V, U)(x) \right|^p \, dx.
\]

For $p = 2$ we have equality and we obtain the Theorem 2.2 [3], com. [10].

**Remark 2.9.** Note also that for all $U \in \Lambda$ there is a vector $\alpha_U \in \mathbb{Z}^d \setminus \{0\}$ such that

\[
\{ \alpha \in \mathbb{Z}^d \setminus \{0\} : \alpha \perp (V \setminus U) \} = \{ k\alpha_U : k \in \mathbb{Z} \setminus \{0\} \}.
\]

Thus

\[
B(V, U)(x) = B^{\rho V + 1}(\alpha_U \cdot x) \prod_{v \in U} \frac{1}{\alpha_U \cdot v},
\]

where $B^k$ is Bernoulli polynomial

\[
B^k(t) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{e^{2\pi int}}{(2\pi in)^k}.
\]

Consequently changing the variable we get

\[
\int_{[0,1]^d} \left| B(V, U)(x) \right|^p \, dx = \int_{[0,1]^d} \left| B^{\rho V + 1}(\alpha_U \cdot x) \right|^p \, dx \left( \prod_{v \in U} \frac{1}{\alpha_U \cdot v} \right)^p
\]

\[
= \int_0^1 \left| B^{\rho V + 1}(t) \right|^p \, dt \left( \prod_{v \in U} \frac{1}{\alpha_U \cdot v} \right)^p.
\]
3. Boschariev Theorem

In 1969 S.V. Boschkariev proved the asymptotic formula of the coefficients of Haar expansion for the functions of bounded variation in $< 0, 1 >$. In fact he proved that if $f$ is absolutely continuous and $a_k(f)$ are the coefficients of Haar expansion then

$$\lim_{n \to \infty} \sqrt{2^n} \sum_{k=1}^{2^n} |a_{2^n+k}(f)| = \frac{1}{4} \vee_0^1 f.$$

The approximation of the functions with bounded variation are now attracted many mathematicians [14], [19], [8]. Form our point of view we are interested of asymptotic formula between picture $f$ and it’s digital image $Pf$ in $L^1$ norm on $\mathbb{R}^d$, $d > 1$ where $P$ is orthogonal projection corresponding to a box spline

$$B_V(x) = \chi_{[0,1]^d},$$

where by $\chi_A$ we denote the characteristic function of the set $A$. From Theorem 2.2. and 2.7 we know that for $f \in W^1_1(\mathbb{R}^d)$

$$(3.1) \lim_{h \to 0^+} \int_{\mathbb{R}^d} \left| \frac{f - P_h f}{h} \right| dx = \int_{\mathbb{R}^d} dt \int_{[0,1]^d} dx \left| \frac{\partial f(t)}{\partial x_1} B^1(x_1) + \cdots + \frac{\partial f(t)}{\partial x_d} B^1(x_d) \right| \asymp |f|_{1,1},$$

and

$$|f|_{1,1} = \sum_{j=1}^{k} \left\| \frac{\partial f}{\partial x_j} \right\|_1 \asymp \int_{\mathbb{R}^d} |Df|,$$

$Df = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d})$. Consequently, by asymptotic formula we get the semi-norm of $W^1_1(\mathbb{R}^d)$. The challenge is to obtain similar result for $BV(\mathbb{R}^d)$.

We will consider the asymptotic for $f = \chi_E$, the characteristic functions of the bounded open set $E$ with Lipschitz boundary. Let us recall the notation. A function $u \in L^1(\mathbb{R}^d)$ whose partial derivatives in the sense of distributions are measures (Radon signed measures) with finite variation is called a function with bounded variation i.e.

$$Du = (\mu_1, \mu_2, \ldots, \mu_d)$$

and

$$|\mu(\mathbb{R}^d)| < \infty, \quad i = 1, \ldots, d.$$
total variation \( \|Du\| \) may be regarded as a measure, if \( g \geq 0 \) and \( g \) is continuous then
\[
\|Du\|(g) = \sup \left\{ \int_{\mathbb{R}^d} (\text{Div} \Phi) u : \Phi = (\phi_1, \ldots, \phi_d) \in C_0^1(\mathbb{R}^d, \mathbb{R}^d), \right.
\]
\[
|\Phi(x)| = \sqrt{\phi_1(x)^2 + \cdots + \phi_d(x)^2} \leq g(x). \]

\textit{Caccioppoli set} in \( \mathbb{R}^d \) which are known also as \textit{set} \( E \) \textit{of finite perimeter} are defined by \( \chi_E \in BV(\mathbb{R}^d) \). If \( D\chi_E = (\mu_1, \mu_2, \ldots, \mu_d) \) note that \( \mu_i << \|D\chi_E\| \) then there is Radon-Nikodym derivative of \( D\chi_E \) with respect to \( \|D\chi_E\| \) i.e.
\[
\nu(x) = \nu(x, E) = -\frac{d\chi_E}{d\|D\chi_E\|},
\]
\( \nu(x) \) is called \textit{generalized exterior normal} to \( E \) at \( x \). We need also the notation of \textit{reduced boundary} of \( E \):
\begin{enumerate}
\item \( \|D\chi_E\|(B(x, r)) > 0 \) for all ball with arbitrary radius \( r > 0 \)
\item if
\[
\nu_r(x) = -\frac{D\chi_E(B(x, r))}{\|D\chi_E\|(B(x, r))},
\]
then the limit \( \nu(x) = \lim_{r \to 0^+} \nu_r(x) \) exists with \( |\nu(x)| = 1 \). It is also known that for every Borel set \( B \subset \partial^* E \), we have \( \|D\chi_E(B)||B) = H^{d-1}(B), \) where \( H^{d-1} \) is \((d - 1)\)-dimensional Hausdorff measure.

Note that it makes sense to consider modified right side of formula of (3.1) and from Remark 2.8 and 2.9 it is equivalent to \( |f|_{BV} \) i.e.
\[
(3.2) \quad \int_{\partial^* E} H^{d-1}(dt) \int_{[0,1]^d} dx |\nu(t) \cdot B(x)| \asymp H^{d-1}(\partial^* E) = |f|_{BV},
\]
where \( B(x) = (B^1(x_1), \ldots, B^1(x_d)) \). Unfortunately the limit of (3.1) for \( f = \chi_E \) may not exist, see example below. Consequently we introduce Marcinkiewicz average of the operators \( P_h \) \[\text{P}h\], which was used to prove equivalent norm in Hardy spaces. We restricted our considerations on \( h = 2^{-n} \) i.e. we will consider the averaging-projections \( P_{2^{-n}} \) with respect to dyadic cubes of side length \( 2^{-n} \). To pose correctly problems let us introduce for \( \tau \in \mathbb{R}^d \)
\[
(P^\tau)_{2^{-n}} f(x) = P_{2^{-n}} f(\cdot - \tau)(x + \tau).
\]
We start with a simply motivation. Let \( H^1 \) be 1-dimensional Hausdorff measure.
Example 3.1. Let $A = [a_1, b_1] \times [a_2, b_2]$. Then for $f = \chi_A$ and a.e. \( \tau \in \mathbb{R}^2 \) and $0 \leq \theta \leq 1/4$ there is a sequence \( \{n_k\} \) such that

$$
\lim_{n_k \to \infty} \int_{\mathbb{R}^2} 2^{n_k} |f - (P^\tau)_{2^{-n_k}} f| = 2\theta H^1(\partial A) = 2\theta |f|_{BV},
$$

The easy proof is left to reader.

Corollary 3.2. If $f = \sum_{j=1}^n d_j \chi_{A_j}$ where $d_j \in \mathbb{R}$ and $A_j$ are rectangles then for a.e. \( \tau \in \mathbb{R}^2 \)

$$
\lim_{n \to \infty} \sup \int_{\mathbb{R}^2} 2^n |f - (P^\tau)_{2^{-n}} f| = \frac{1}{2} |f|_{BV}.
$$

Let $K(x, 2^k)$ denote cube in center at $x$ of the side length $2^k$. By simple calculation we get:

Lemma 3.3. Let $E \subset \mathbb{R}^d$ be measurable bounded set. Then

$$(3.3) \quad \int_{\mathbb{R}^d} 2^n |\chi_E - (P^\tau)_{2^{-n}} \chi_E| = 2 \sum_{K \in Q_n} (2^{-n})^{d-1} |K \cap (E + \tau)| \left(1 - \frac{|K \cap (E + \tau)|}{|K|}\right),$$

where $Q_n$ is a collection of all dyadic cubes of side length $2^{-n}$.

$$(3.4) \quad \int_{[0,1]^d} \int_{\mathbb{R}^d} 2^n |\chi_E - (P^\tau)_{2^{-n}} \chi_E| = 2 \int_{\mathbb{R}^d} 2^n M_n(x) dx,$$

where

$$
M_n(\tau) = \frac{|K(\tau, 2^{-n}) \cap E|}{|K|} \left(1 - \frac{|K(\tau, 2^{-n}) \cap E|}{|K|}\right).
$$

For given vector $v \in \mathbb{R}^d \setminus \{0\}$ let us define

$$
F(v) = \int_{[0,1]^d} G_v(u) du.
$$

To define function $G_v$ we need for $u \in \mathbb{R}^d$

$$
\Pi_v(u) = \{ y \in \mathbb{R}^d : (y - u) \cdot v = 0 \},
$$

$$
\Pi^+_v(u) = \{ y \in \mathbb{R}^d : (y - u) \cdot v > 0 \},
$$

$$
\Pi^-_v(u) = \{ y \in \mathbb{R}^d : (y - u) \cdot v < 0 \}.
$$

Thus

$$
G_v(u) = \frac{|\Pi^+_v(u) \cap [0,1]^d|}{\Pi_v(u) \cap [0,1]^d}.
$$

Theorem 3.4. Let $E$ be open bounded with Lipschitz boundary. Then

$$
\lim_{n \to \infty} \int_{[0,1]^d} d\tau \int_{\mathbb{R}^d} 2^n |\chi_E - (P^\tau)_{2^{-n}} \chi_E| = \int_{\partial^* E} F(\nu(x)) H^{d-1}(dx),
$$

where $H^d$ is the d-dimensional Lebesgue measure.
We postpone the proof of the theorem. Now we want to present the main ideas. We consider \( d = 2 \). In the ergodic theory [9] pp. 69 it is known that for any continuous function \( f \) on 2-dimensional torus \( T \) or \( T^2 \) with Lebesgue’s measure \( dt \)

\[
\lim_{t \to \infty} \frac{1}{2t} \int_{-t}^{t} f(T^{s}_x) ds = \int_{T^2} f(t) dt
\]

uniformly for all \( x \in T^2 \), where \( T^{s}_x \) is a one-parameter group of translations \( (l = l(\alpha, x)) \)

\[
T^{s}_x = (x_1 + \cos(\alpha)s \mod 1, x_2 + \sin(\alpha)s \mod 1)
\]

and \( \cos(\alpha), \sin(\alpha) \) are rationally independent i.e. \( \tan(\alpha) \) is a irrational number. Note that \( \alpha \) is an angle between that line \( l \) and \( OX \). Let

\[
\nu(\alpha) = [-\sin \alpha, \cos \alpha].
\]

Then \( \nu(\alpha) \perp l \). Note also that \( \tan(\alpha) \) is a irrational number iff for all \( x \) the line \( t \to T^{t}_x \) is dense in \( T^2 \) iff there is \( x \) that the line \( t \to T^{t}_x \) is dense in \( T^2 \).

We will calculate the asymptotic formula for polygons \( E \) whose sides \( l_1(\alpha_1), l_2(\alpha_2), \ldots, l_k(\alpha_k) \) are not parallel to axis and each line determined by \( l_j \) is dense in \( T^2 \). It is crucial that the above theorem \( 3.5 \) is true also for the bounded functions \( G_{\nu(\alpha)} \) on \( T^2 = [0, 1)^2 \) and continuous on \( (0, 1)^2 \).

The function \( F(\nu(\alpha)) \) has following properties: \( F(\nu(\alpha)) = F(\nu(\alpha + \pi/2)) \) and if \( \pi/4 \leq \alpha \leq \pi/2 \) then \( F(\nu(\alpha)) = F(\nu(\pi/2 - \alpha)) \).

Let us present the function \( F \) for \( \alpha \in [\pi/4, \pi/2] \).

![Figure 1.](image)

**Theorem 3.5.** Let \( f = \chi_E \) where \( E \) is a polygon of sides \( l_1(\alpha_1), l_2(\alpha_2), \ldots, l_k(\alpha_k) \) not parallel to axis and \( \sin \alpha_j \) and \( \cos \alpha_j \) are rationally independent.
Uniformly for all \( \tau \in \mathbb{R}^2 \) we have

\[
\lim_{n \to \infty} \int_{\mathbb{R}^2} 2^n |f - (P_\tau)_{2-n} f| = 2 \sum_{j=1}^k |l_j| F(\nu(\alpha_j)).
\]

Hence

\[
\lim_{n \to \infty} \int_{[0,1]^2} d\tau \int_{\mathbb{R}^2} 2^n |f - (P_\tau)_{2-n} f| = 2 \sum_{j=1}^k |l_j| F(\nu(\alpha_j)).
\]

**Proof.** We need to calculate the limit of (3.3), (3.4). It is immediate consequence of (3.5).

It is interesting to compare the limit obtained in Theorem 3.5 with the left formula in (3.2). Taking to account that

\[
Df = \sum_{j=1}^k (-\sin \alpha_j, \cos \alpha_j) d\mu_j,
\]

where \( \mu_j \) is a Lebesgue measure on \( l_j \) we get

\[
\int_{\mathbb{R}^2} d\mu(t) \int_{[0,1]^2} dx |\phi_1(t) B^1(x_1) + \phi_2(t) B^1(x_2)|
\]

\[
= \sum_{j=1}^k |l_j| \int_{[0,1]^2} dx \left| -\sin \alpha_j B^1(x_1) + \cos \alpha_j B^1(x_2) \right|
\]

Let us compare

\[
\frac{\int_{[0,1]^2} dx \left| -\sin \alpha_j B^1(x_1) + \cos \alpha_j B^1(x_2) \right|}{2F(\nu(\alpha_j))}
\]

for \( \pi/4 < \alpha_j < \pi/2 \).

*Figure 2.*
there is a neighborhood of \( x \) \( U_r \) and a Lipschitz function \( \phi \) with a constant \( L \) such that
\[
U_r = B_r \times (-a, a),
\]
where \( B_r \) is a ball \( B_r = \{ x^* \in R^{d-1}, |x^*| < r \} \) and \( d > 0 \).
\[
\partial E \cap U_r = \{ (x^*, \phi(x^*)) : x^* \in B_r \},
\]
\[
E \cap U_r = \{ x_d < \phi(x^*) : x^* \in B_r, x_d \in (-a, a) \}.
\]
To prove the theorem it is sufficient to show that for any \( s < r \)
\[
\lim_{n \to \infty} \int_{D \times (-a, a)} 2^n M_n(x) dx = \int_{\partial E \cap U} F(\nu(x)) H^{d-1}(dx),
\]
where \( D = B_s, U = D \times (-a, a) \). Since \( \phi \) is a Lipschitz function for sufficient small \( n \) (dependent on \( s \))
\[
\int_{D \times (-a, a)} 2^n M_n(x) dx = \int_D dx^* \int_{\phi(x^*)-(L+1)2^{-n}}^{\phi(x^*)+(L+1)2^{-n}} 2^n M_n((x^*, t)) dt
\]
Changing variables \( t = \phi(x^*) + \tau 2^{-n} \) we get
\[
= \int_D dx^* \int_{-(L+1)2^{-n}}^{(L+1)2^{-n}} M_n((x^*, \phi(x^*) + \tau 2^{-n})) d\tau
\]
From Rademacher theorem \( \phi \) is differentiable at almost all points. Let us denote a set of these points by \( D^* \subset D \). Let us fix \( x^* \in D^* \). Note that \( x = (x^*, \phi(x^*)) \in \partial^* E \). Let \( K(x, 2^k) \) denote cube in center at \( x \) of the side length \( 2^k \). Let
\[
T_{n,x^*} = \{ t : K((x^*, t), 2^{-n}) \cap \Pi_{\nu(x)}(x) \neq \emptyset \}
\]
It is geometrically obvious that for bijection \( H(t) = (t - \phi(x^*))2^n \) there is \( \delta = \delta(x^*) > 0 \) such that \( H(T_{n,x^*}) = [-\delta, \delta] \). Moreover for all \( \tau \in [-\delta, \delta] \) and \( i = + \) or \( i = - \)
\[
\frac{|K((x^*, \phi(x^*) + \tau 2^{-n}), 2^{-n}) \cap \Pi_{\nu(x)}^i(x)|}{|K((x^*, \phi(x^*) + \tau 2^{-n}), 2^{-n})|}
\]
\[
= |K((x^*, \phi(x^*) + \tau), 1) \cap \Pi_{\nu(x)}^i(x)| =: h_{x^*}^i(\tau).
\]
Note that
\[
\frac{|K((x^*, \phi(x^*) + \tau 2^{-n}), 2^{-n}) \cap \Pi_{\nu(x)}^i(x)|}{2^{-dn}}
\]
\[
= \frac{|K((x^*, \phi(x^*) + \tau 2^{-n}), 2^{-n}) \cap \Pi_{\nu(x)}^i(x) \cap E|}{2^{-dn}}
\]
\[
+ \frac{|K((x^*, \phi(x^*) + \tau 2^{-n}), 2^{-n}) \cap \Pi_{\nu(x)}^i(x) \cap (R^d \setminus E)|}{2^{-dn}}.
\]
From Theorem 5.6.5 [20] we infer that
\[
\lim_{n \to \infty} \frac{|K((x^*, \phi(x^*) + \tau 2^{-n}), 2^{-n}) \cap \Pi_{\nu(x^*)}(x) \cap E|}{|K((x^*, \phi(x^*) + \tau 2^{-n}), 2^{-n})|} = h_{x^*}^-(\tau).
\]

and
\[
\lim_{n \to \infty} \frac{|K((x^*, \phi(x^*) + \tau 2^{-n}), 2^{-n}) \cap \Pi_{\nu(x^*)}(x) \cap E|}{|K((x^*, \phi(x^*) + \tau 2^{-n}), 2^{-n})|} = 0.
\]

Consequently for all \( \tau \in [-\delta, \delta] \)
\[
\lim_{n \to \infty} M_n(x^*, \phi(x^* + \tau 2^{-n})) = h_{x^*}^-(\tau)h_{x^*}^+(\tau).
\]

and \( \tau \not\in [-\delta, \delta] \)
\[
\lim_{n \to \infty} M_n(x^*, \phi(x^* + \tau 2^{-n})) = 0.
\]

From Lebesgue bounded convergence theorem
\[
\lim_{n \to \infty} \int_{(L+1)}^{-(L+1)} 2^n M_n((x^*, \phi(x^* + \tau 2^{-n}))d\tau = \int_{-\delta}^{\delta} h_{x^*}^-(\tau)h_{x^*}^+(\tau)
\]

Making use of the last relation, we obtain
\[
\int_D dx^* \int_{-(L+1)}^{(L+1)} 2^n M_n((x^*, \phi(x^* + \tau 2^{-n}))d\tau =
\]
\[
= \int_D dx^* \int_{-\delta(x^*)}^{\delta(x^*)} h_{x^*}^-(\tau)h_{x^*}^+(\tau)d\tau
\]
\[
= \int_D dx^* \sqrt{1 + |D\phi(x)|^2} \frac{1}{\sqrt{1 + |D\phi(x)|^2}} \int_{-\delta(x^*)}^{\delta(x^*)} h_{x^*}^-(\tau)h_{x^*}^+(\tau)d\tau
\]
\[
= \int_D \sqrt{1 + |D\phi(x)|^2} F(\nu(x^*, \phi(x^*)))dx^*
\]
\[
= \int_{\partial^* E} F(\nu(x))H^{d-1}(dx).
\]

By Theorem 5.7.3 [20] we get following corollary

**Corollary 3.6.** Let \( E \) be a set of finite perimeter. Then

\[
\liminf_{n \to \infty} \int_{[0,1]^d} d\tau \int_{\mathbb{R}^d} 2^n |\chi_E - (P^\tau)_{2^{-n}}\chi_E| \geq 2 \int_{\partial^* E} F(\nu(x))H^{d-1}(dx).
\]

It seems to be true that

\[
\liminf_{n \to \infty} \int_{[0,1]^d} d\tau \int_{\mathbb{R}^d} 2^n |\chi_E - (P^\tau)_{2^{-n}}\chi_E| \simeq H^{d-1}(\partial^* E).
\]
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