Quasi-degenerate neutrinos and tri-bi-maximal mixing

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Abstract. Assuming high-energy tri-bi-maximal mixing we study the radiative running of leptonic mixing angles and obtain limits on the high-energy scale from requiring consistency with the observed mixing. We construct a model in which a non-Abelian discrete family symmetry leads both to a quasi-degenerate neutrino mass spectrum and to near tri-bi-maximal mixing.

1. Introduction
This article is a proceedings submission corresponding to a parallel talk for the DISCRETE '08 conference. This article and the talk are based on [1]. Neutrino-oscillation data [2; 3] is presently consistent with just three light neutrinos with near tri-bi-maximal (TBM) mixing between flavours [4–8]. If TBM mixing is assumed the mixing matrix is of the form:

\[
V_{PMNS} = \begin{pmatrix}
-\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\
\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \\
\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}}
\end{pmatrix}
\] (1)

The nature of the mass spectrum is consistent with either a normal or an inverted hierarchy. The magnitude of the mass squared difference between neutrinos is reasonably well determined, but the absolute scale of mass is not, being consistent with both a strongly hierarchical spectrum or a quasi-degenerate (QD) spectrum.

Radiative running is especially important for QD neutrinos, as the effects on mixing angles are larger for QD neutrinos than in the hierarchical case [9; 10]. More recent studies of neutrino mixing angle’s running include [11–14]. We discuss radiative corrections to TBM mixing, assuming that they arise through new physics, such as a family symmetry, at a high-energy scale. Specifically, we set the angles to their TBM values at high-energy scales, run the angles to low-energy and iterate the process to find the highest-energy scale that still keeps the low-energy angles within current experimental bounds. The process is then repeated for different points of the parameter space, and the results are presented as a contour plot in the \( m_{\nu_i} - \tan \beta \) plane (\( i = 1 \) for normal and \( i = 3 \) for inverted hierarchy).
The underlying question raised by the observed near TBM mixing is the origin of the pattern and the reason it is so different from quark mixing. Models based on family symmetries, particularly discrete non-Abelian family symmetries, have been constructed to explain this pattern, e.g. [15; 16]. In these models the difference between the quark and lepton sector follows naturally from the see-saw mechanism together with a strongly hierarchical right-handed neutrino Majorana mass spectrum. However these models apply to hierarchical neutrino mass spectrums. We discuss how a discrete non-Abelian family symmetry can also give rise to near TBM mixing for the case of a QD masses.

2. Radiative corrections to TBM mixing
Family-symmetry models are typically constructed at some high scale, \( M_F \), at which the model specifies relationships among parameters. To compare the predictions to low-energy data, radiative effects should be considered through the use of the renormalization group equations. When there is a strong hierarchy, it is often the case that these running effects do not change the mixing angles by much [11–14]. In the case of QD neutrinos, however, the mixing angles can change a lot with the energy scale, to the point of erasing any special structure arranged by a family symmetry. For model-building purposes it is important to know the highest-energy scale at which we can start with TBM mixing and still be consistent with mixing-angle data after running the angles down to the low-energy scale.

The Standard Model (SM) suffers from the hierarchy problem associated with the need to keep electroweak breaking much below the Planck scale. This problem is evaded if the theory is supersymmetric, with supersymmetry broken close to the electroweak scale. For this reason we consider the radiative corrections to neutrino masses and mixing in the context of the minimal supersymmetric extension of the Standard Model (MSSM). We specify the low-energy boundary conditions of the renormalization group equations to be consistent with the three gauge coupling constants and the quark and lepton masses [17]. We assume an effective SUSY scale of \( M_S = 500 \) GeV. We use the SM renormalization-group equations below \( M_S \) and the MSSM renormalization-group equations above \( M_S \). The only boundary condition set at the family-symmetry breaking scale \( M_F \) is exact TBM mixing for the leptons \(^2\). The neutrino masses are set at the low-energy boundary relative to the lightest neutrino mass state (\( \nu_1 \) with a normal hierarchy and \( \nu_3 \) with an inverted hierarchy).

We keep \( |\Delta m^2_{12}| \) the solar mass difference and \( |\Delta m^2_{23}| \) the atmospheric mass difference.

Figure 1 has two contour plots. For the normal hierarchy the plot shows \( \nu_1 \) versus \( \tan \beta \) and for the inverted hierarchy shows \( \nu_3 \) versus \( \tan \beta \). The solar mixing angle \( \theta_{12} \) is the most sensitive to radiative corrections. Exact TBM mixing gives \( \tan^2 \theta_{12} = 0.5 \), and our 4\( \sigma \) requirement at low energy translates to \( \tan^2 \theta_{12} = 0.47 \pm 0.2 \) [3].

For \( \nu_1 > 0.1 \) eV, the neutrino spectrum is referred to as quasi-degenerate (QD) [19]. If cosmological observations are considered, they constrain the sum of the neutrinos \( \sum_i \nu_i \leq 0.42 \) eV at the 95\% confidence level [20]. This implies \( \nu_1 \leq 0.14 \) eV which excludes the right half of each plot. The remaining allowed narrow strip is consistent with the non-observation of neutrinoless double beta decay \( \beta\beta_0 \nu \), which places a limit of \( m_{ee} < 0.34 \) eV (uncertainties in nuclear matrix element weaken this bound by about a factor of 3).

3. A discrete non-Abelian family symmetry model of QD neutrinos with TBM mixing
As stressed in [21] an underlying \( SO(3) \) family symmetry readily leads to a near degenerate neutrino mass spectrum. In their model the chiral superfields, \( L^i \) (where \( i \) is the \( SO(3) \) family

\(^2\) We ignore the small departures from TBM at the high scale which may arise from diagonalising the charged-lepton mass matrix [12; 18].
index), contain the lepton doublets and transform as triplets under the $SO(3)$ group. The chiral superfields containing the conjugates of the right-handed electron, muon and tau, respectively $e^c$, $\mu^c$ and $\tau^c$, are $SO(3)$ singlets. This type of configuration is also used in [22], using $SO(3)$ to enable a QD spectrum (but not TBM mixing) and [23], using $SO(3)$ to obtain TBM mixing (although in a hierarchical spectrum). The effective Majorana neutrino mass is constrained by the symmetry and comes from the superpotential

$$W_{\text{eff}} = y_0 (L^i L^i) H_u H_u / M$$

where $H_u$ is the supermultiplet containing the Higgs field whose vacuum expectation value (VEV), $\langle H_u \rangle = v$, is responsible for up quark masses in the MSSM and $M$ is the messenger scale associated with the mechanism generating this dimension 5 term (in the Type II see-saw it is the mass of the exchanged isotriplet Higgs field).

The important point to be taken from eq.(2) is that the family symmetry forces the three light neutrinos to be degenerate. Small departures from degeneracy result when the $SO(3)$ family symmetry is broken. In what follows we will show how this can naturally lead to a mass mixing matrix which gives near TBM mixing. This is done through the breaking of the family symmetry by the non-vanishing vacuum expectation values (VEVs) of familon fields, denoted as $\phi_A$, where the $A = 3, 23, 123$ labels three distinct fields and serves as a reminder of their VEV directions which are given by

$$\langle \phi_3 \rangle = \begin{pmatrix} 0 & 0 & a \end{pmatrix}, \quad \langle \phi_{23} \rangle = \begin{pmatrix} 0 & -b & b \end{pmatrix}, \quad \langle \phi_{123} \rangle = \begin{pmatrix} c & c & c \end{pmatrix}$$

where $a, b$ and $c$ are complex parameters. Table 1 lists the full set of supermultiplets and their symmetry properties under the $SO(3)$ symmetry extended by a further set of symmetries $G = Z_{3R} \times Z_2 \times U_\tau(1)$ which limit the terms that can appear in the superpotential. $Z_{3R}$ is a discrete $R$–symmetry which ensures the familon fields are moduli and cannot appear in the superpotential except coupled to “matter” fields carrying non-zero $R$–charge. The $U_\tau(1)$

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**Figure 1.** Contours of $\log_{10}(M_F)$. $M_F$ is the highest-energy scale at which we can set TBM and have the neutrino mixing within $4\sigma$ of the low-energy observed values. In white regions $M_F$ is greater than $10^{15}$ GeV.
symmetry is introduced to distinguish the third family of leptons from the first two. In practice it also explains why the mixing in the charged-lepton sector is different from that in the neutrino sector which leads to near tri-bi-maximal mixing.

### Table 1. Assignment of the fields under the $SO(3)$ family symmetry.

| Field | $SO(3)$ | $Z_{3R}$ | $U_{\tau}(1)$ | $Z_2$ |
|-------|---------|---------|--------------|-------|
| $L^i$ | 3       | 1       | 0            | +     |
| $e^c$ | 1       | 1       | 0            | +     |
| $\mu^c$ | 1 | 1       | 0            | -     |
| $\tau^c$ | 1 | 1       | -1           | +     |
| $H_{u,d}$ | 1 | 0       | 0            | +     |
| $\phi^i_3$ | 3 | 0       | 1            | +     |
| $\phi^i_{23}$ | 3 | 0       | 0            | -     |
| $\phi^i_{123}$ | 3 | 0       | 0            | +     |
| $X$ | 1 | 2       | 0            | -     |

The special structure of the VEVs in eq(3) is what will generate TBM mixing and is clearly the most important aspect of the model. This can happen naturally if the underlying family symmetry is not $SO(3)$ but a discrete non-Abelian subgroup. We will discuss below the nature of this symmetry and the vacuum alignment leading to eq(3) (the $X$ field of Table 1 is introduced to facilitate this vacuum alignment), but first we show that it does generate approximate TBM mixing.

The leading terms in the superpotential responsible for neutrino masses that are invariant under the family symmetries are given by

$$W_\nu = y_0 (L^i L^i) H_u H_u + y_3 (\phi^i_{123} L^i)^2 H_u H_u + y_0 (\phi^i_{23} L^i)^2 H_u H_u.$$  \(4\)

where we have suppressed the messenger scale. Note that due to the $Z_2$ factor there are no cross terms involving $\phi_{23} \phi_{123}$ [24; 25] and due to the $U_{\tau}(1)$ factor there is no term involving $\phi_3$. As in eq(2), the QD mass scale is set by the first term of eq(4). For near degeneracy, the other terms must be relatively small ($y_3 c^2, y_0 b^2 \ll y_0$, still suppressing the messenger scale).

The charged-lepton masses come from the superpotential

$$W_e = \lambda_e (L^i L^i) e^c H_d + \lambda_\mu (L^i L^i) \mu^c H_d + \lambda_\tau (L^i L^i) \tau^c H_d.$$  \(5\)

The $m_\mu/m_\tau$ ratio is given by $\lambda_\mu \langle \phi^i_{23} \rangle / \lambda_\tau \langle \phi^i_3 \rangle$. Using this the mixing between the second and third families of charged leptons is small of $O(m_\mu/m_\tau)$. Similarly one may see that the mixing between the first and second families is of $O(m_\mu/m_\mu)$ and that between the first and third families is of $O(m_\mu/m_\tau)$, both very small. Ignoring the small corrections from the charged-lepton sector, the light neutrino mass eigenstates are proportional to the combinations $\phi^i_{123} L^i H_u$ and $\phi^i_{23} L^i H_u$.

3. From eq(3) we see that these are given by

$$\nu_0 = \frac{1}{\sqrt{2}} (\nu_\mu - \nu_\tau)$$  \(6\)

3 In finding the mass eigenstates with a complex Majorana mass matrix, one needs to be careful to diagonalize $M_\nu M_\nu^T$ and not just $M_\nu$. Because $M_\nu$ is symmetric, it can also be diagonalized by an orthogonal transformation $OM_\nu O^T$. In general $O \neq U_{\nu}$, and the square of the eigenvalues of $M_\nu$ are not the same as those of $M_\nu M_\nu^T$ [26].
\[ \nu_\odot = \frac{1}{\sqrt{3}} (\nu_e + \nu_\mu + \nu_\tau) \]

where \( \nu_e, \mu, \tau \) are the components of \( L^e, L^\mu, L^\tau \) respectively (selected by the VEV of \( H_u \)). Ignoring the small charged-lepton mixings discussed above, \( \nu_e, \mu, \tau \) can be identified with the current eigenstates. If \( b \) and \( c \) are real and positive, and \( m_\odot = y_0 c^2 v^2 < m_0 = y_0 b^2 v^2 \), one can see from eq(4) and eq(6) that we obtain the normal hierarchy, in which \( \nu_0 \) may be identified with the atmospheric neutrino with bi-maximal mixing while \( \nu_\odot \) may be identified with the solar neutrino with tri-maximal mixing. The normal hierarchy persists for a range of complex \( b \) and \( c \) values in the neighbourhood of the real solution. An inverted hierarchy is possible and viable if \( b, c \) are approximately imaginary and real, respectively.

Although here we are working at the effective Lagrangian level, we already noted that \((L^i L^j)HH\) naturally arises from the \( SO(3) \) invariant Type II see-saw mechanism. The other two neutrino mass terms can arise from Type I see-saw through exchange of appropriate heavy right-handed Majorana neutrinos, in a manner similarly to that discussed for a \( SU(3) \) based model in [27]. Being of different origin it can readily happen that the common mass, \( m_0 = y_0 v^2 \), is much larger than \( m_\odot \) and \( m_\odot \).

4. Discrete non-Abelian symmetry and vacuum alignment

We turn now to a discussion as to how the pattern of VEVs displayed in eq(3) is dynamically generated. This can be achieved relatively simply if the underlying family symmetry is a discrete non-Abelian subgroup of \( SO(3) \). A very simple example is given by \( A_4 \equiv \Delta(12) \), belonging to the \( \Delta(3n^2) \) family of groups [28]. The \( \Delta(12) \) invariant terms in the potential are those invariant under the group elements of the semi-direct product \( Z_3 \ltimes Z_2 \) (which generate the group \( \Delta(12) \)). Since \( \Delta(12) \) is a subgroup of \( SO(3) \), all \( SO(3) \) invariants are allowed by the discrete subgroup. Thus the terms of eq(4) and eq(5) are allowed. The discrete subgroup allows additional terms, but these are all higher dimensional and consequently small provided the VEVs of eq(3) are small relative to the relevant messenger mass. Thus the lepton mass and mixing structure discussed is a consequence of the non-Abelian discrete group even though the \( SO(3) \) structure used above to motivate it is only approximate.

Turning now to the question of vacuum alignment, consider the leading terms in the potential for the triplet familon fields. Because of the \( R \)-symmetry, in the absence of the \( X \)-field, there are no \( F \)-terms involving just the familon fields coming from the superpotential. The leading \( D \)-terms consistent with symmetries of Table 1 are

\[ V(\phi) = \alpha m^2 \sum_i |\phi|^2 + \beta m^2 \sum_i |\phi|^2 + \gamma m^2 \sum_i |\phi|^4 + \delta m^2 \sum_i (\phi^*)^2 \]  
(7)

Here the quadratic term is driven by supersymmetry breaking and \( m \) is the gravitino mass. The coefficient includes radiative corrections which can drive it negative at some scale \( \Lambda \), triggering a VEV for \( \phi \). The remaining terms can arise through radiative corrections and only if supersymmetry is broken (hence the factor of \( m^2 \)). For details on how they can be generated refer to [1]. The first two terms are invariant under the larger group \( SO(3) \) and, if \( \alpha \) is negative, generate a VEV of the form \( \langle \phi \rangle = (r, s, t) \) where \( r^2 + s^2 + t^2 = x^2 \), with \( x^2 \) a constant of \( O(\Lambda^2) \). The third term, consistent with the non-Abelian family group, breaks \( SO(3) \) and splits the vacuum degeneracy. For negative \( \alpha \), the minimum for \( \gamma \) positive has \( |\langle \phi \rangle| = x(1, 1, 1)/\sqrt{3} \) while for \( \gamma \) negative \( |\langle \phi \rangle| = x(0, 0, 1) \). Finally the fourth term is \( SO(3) \) invariant and constrains the phases of the familon fields. For \( \delta \) negative and \( \gamma \) positive the minimum has \( \langle \phi \rangle = x(1, 1, 1)/\sqrt{3} \) where \( x \) can be complex. This provides a mechanism to generate the vacuum alignment of \( \phi_3 \) and \( \phi_{123} \) as each will have a potential of the form in eq(7) - as we are considering more than
one familon, we label the coefficients with the familon’s subscript to identify which term they correspond to. The structure of eq(3) results if $\gamma_3$ is positive and $\gamma_{123}, \delta_{123}$ are negative.

$\phi_{23}$ can get a VEV of the form in eq(3) partly through similar soft terms, but in a slightly more involved way that requires additional (alignment) fields (e.g. $X$) - refer to [1] for details.

5. Neutrinoless double-beta decay

![Figure 2. Neutrinoless double-beta decay plots, from [17].](image)

The implication for neutrinoless double-beta decay in this model is unambiguous because the relative phases of the familon fields are determined. The amplitude for neutrinoless double-beta decay is proportional to the magnitude of $\sum m_\nu U^2_{ei} \equiv m_{\beta\beta}$. For TBM mixing $U_{e\tau}$ vanishes. The relative phase between the remaining two terms is given by $\text{Arg}[m_0 + e^{2ip_{123}m_{\odot}}] - \text{Arg}[m_0]$ where $p_{123} = \text{Arg}[y_{\odot}\phi_{123}\phi_{123}/y_0]$. As $m_\odot < m_0$ the relative phase remains small. This corresponds to the upper branches of Figure 2 in the QD region. Complex phases in the VEVs induce other CP violations through the charged-lepton sector that do not significantly affect $m_{\beta\beta}$.

6. Conclusion

Attempts to explain the structure of fermion masses and mixings often rely on structure at a high scale, $M_F$, to generate the observed pattern. One possibility, consistent with neutrino oscillation, is that neutrinos are nearly degenerate. However, due to enhanced radiative corrections in this case, the observation of near TBM mixing is difficult to reconcile with such a high-scale mechanism. To keep the deviations from TBM mixing within experimental limits, it is necessary to limit the scale at which TBM mixing is generated. We have determined this scale for the MSSM and found significant bounds on $M_F$. To get close to the Grand Unified scale with QD neutrinos it is necessary to have very small $\tan \beta$.

Turning to the origin of the structure, we have constructed a model based on a discrete non-Abelian family symmetry which gives a QD neutrino spectrum and near TBM mixing. This relies on a natural mechanism for vacuum alignment of the familons which break the family symmetry. The mechanism predicts that neutrinoless double-beta decay should be maximal.

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