Relativistic spin operator and Lorentz covariant reduced spin density matrix in a new representation space

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The spin reduced matrix defined by taking partial trace with the momentum variable has been shown not to transform covariantly under Lorentz transformations. In this paper, I suggest another representation space of the Poincaré group, the elements of which are in a sense “apparent wave functions”. In this new Hilbert space, the naive spin observable $\frac{1}{2} \tau$ from nonrelativistic quantum mechanics on the old Hilbert space is represented as the Newton-Wigner spin operator. Also on this space, one can naturally associate to each state a Lorentz covariant spin reduced density matrix, which is then given a new interpretation.

I. INTRODUCTION

In the seminal paper [1], the authors considered a massive spin-$\frac{1}{2}$ single particle state space represented as a tensor product system

$$\mathcal{H} := L^2 (\mathbb{R}^3, \mu) \otimes \mathbb{C}^2$$

(1)

where $d\mu(p) = \frac{d^3 p}{(2\pi)^3 \sqrt{p^2 + m^2}}$ is a Lorentz invariant measure on the mass shell (cf. [2]) and $L^2 (\mathbb{R}^3, \mu)$ is the space of momentum wave functions. So, an element $\psi(p) \in \mathcal{H}$ can be written as

$$\psi(p) = \begin{pmatrix} a_1(p) \\ a_2(p) \end{pmatrix}.$$  

(2)

A Lorentz transformation $(\Lambda, a) \in \mathbb{R}^4 \ltimes SL(2, \mathbb{C})$ acts on this Hilbert space as

$$[U(\Lambda, a) \psi](p) = e^{-ip \cdot a''} D(W(\Lambda, \Lambda^{-1}p)) \psi(\Lambda^{-1}p)$$

(3)

where $D(\cdot)$ is a little group representation and $W(\Lambda, p)$ is the Wigner rotation matrix, the precise definitions of which will be given in section II (cf. [2]).

On this space, they formed the density matrix corresponding to $\psi$ as

$$\rho(p', p'') = \begin{pmatrix} a_1(p') a_1(p'')^* & a_1(p') a_2(p'')^* \\ a_2(p') a_1(p'')^* & a_2(p') a_2(p'')^* \end{pmatrix}.$$  

(4)

and defined the reduced density matrix for spin as

$$\tau = \int_{\mathbb{R}^3} \rho(p, p) d\mu(p)$$

(5)

and showed that the von Neumann entropy of this reduced density matrix is not invariant under Lorentz transformations, showing that the reduced matrix has no invariant meaning in special relativity.

In the present paper, after examining Eq. (3) in detail for the purpose of developing some notations and relations, I suggest, in section III a different representation of the group $\mathbb{R}^4 \ltimes SL(2, \mathbb{C})$ on a different Hilbert space which are yet isomorphic to Eqs. (1) and (3), the elements of which can be thought of as “apparent wave functions”. Section IV presents some features of this new representation. In particular, I show that the naive spin operator $\frac{1}{2} \tau$ on the old representation space (Eqs. (1) and (3), is represented as the Newton-Wigner spin operator on this newly defined representation space, which is a pleasant coincidence since this operator has been proposed to be the right relativistic spin operator by several authors (see, e.g. [3]) and references therein). In addition, we will see that one can naturally associate to each state from the old space a Lorentz covariant reduced density matrix for spin, which turns out to be the covariant reduced density matrix introduced in [3].

II. THE OLD REPRESENTATION SPACE

Suppose an inertial observer (call her Alice) has prepared a single particle in the state $\psi \in \mathcal{H}$ as in Eq. (2). Consider another inertial observer (call him Bob) whose frame is Lorentz transformed by $\Lambda \in SL(2, \mathbb{C})$ (See Eq. (19)) from the original observer. Then, the particle is seen to be in the state $U(\Lambda) \psi \in \mathcal{H}$ in Bob’s reference frame. A formula for this transformed state is given by Eq. (3), which will be the focus of this section.

Before we embark on this job, let’s set some notations and mathematical facts which will be used throughout the paper. First, I use the Minkowski metric

$$\eta = \text{diag}(1, -1, -1, -1)$$

(6)

and the Pauli matrices

$$\tau^0 = I, \quad \tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(7)

Fix a positive number $m > 0$. Consider the following diffeomorphism from $\mathbb{R}^3$ onto the mass shell

$$X := \{ p \in \mathbb{R}^4 : p^0 > 0, (p)^2 = m^2 \}$$

(8)

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given by
\[ \mathbf{p} \mapsto \left( \sqrt{m^2 + \mathbf{p}^2}, \mathbf{p} \right). \] (9)

Denoting \( p^0 = \sqrt{m^2 + \mathbf{p}^2} \), we will identify \( \mathbf{p} \) with the four vector \( \mathbf{p}^\mu = (p^0, \mathbf{p}) \in X \) via this diffeomorphism throughout this paper even though it is not explicitly stated (in fact, this was already done in Eq. (3)).

\( X \) is a Lorentz invariant subset, i.e. \( \Lambda p \in X \) for all \( p \in X \) and \( \Lambda \in SO^1(1,3) \). The measure \( \mu \) on \( X \) defined by
\[ d\mu(p) = \frac{d^3\mathbf{p}}{(2\pi)^3 \sqrt{m^2 + \mathbf{p}^2}} \] (10)
is a Lorentz invariant measure (cf. [8]), that is, for all integrable functions \( f \) on \( X \) and \( \Lambda \in SO^1(1,3) \),
\[ \int_X f(\Lambda p) d\mu(p) = \int_X f(p) d\mu(p). \] (11)

Note that each \( p \in X \) satisfies the following relation.
\[ p_\mu p^\mu = m^2 \] (12)

Denote
\[ \mathbf{p} = \begin{pmatrix} p_0 + p_3 \\ p_1 + ip_2 \\ p_0 - p_3 \end{pmatrix} = p_\mu \tau^\mu \] (13)
\[ \tilde{\mathbf{p}} = \begin{pmatrix} p_0 - p_3 \\ -p_1 + ip_2 \\ p_0 + p_3 \end{pmatrix} = p_\mu \tau^\mu \] (14)

where of course the usual convention of index raising and lowering via \( \eta \) as well as Einstein’s summation convention is in action. (These notations are borrowed from the book [9] with a slight modification.)

A direct calculation shows
\[ pp = m^2 I = \tilde{p} \tilde{p} \] (15)

and also, these two matrices are positive matrices with square roots
\[ \sqrt{\mathbf{p}} = \frac{1}{\sqrt{2(m + p_0)}} (\mathbf{p} + ml) \] (16a)
\[ \sqrt{\tilde{\mathbf{p}}} = \frac{1}{\sqrt{2(m + p_0)}} (\tilde{\mathbf{p}} + ml) \] (16b)

meaning that these two matrices are positive and their squares are \( \mathbf{p} \) and \( \tilde{\mathbf{p}} \), respectively. It is easy to see that
\[ \sqrt{\mathbf{p}} \sqrt{\tilde{\mathbf{p}}} = ml = \sqrt{\tilde{\mathbf{p}}} \sqrt{\mathbf{p}} \] (17)

This fact may be expressed as
\[ \left( \frac{\mathbf{p}}{m} \right)^{-1} = \left( \frac{\tilde{\mathbf{p}}}{m} \right) \] (18)

Denote the proper Lorentz group (i.e. the connected component of \( O(1,3) \)) as \( SO^1(1,3) \). Let
\[ \kappa : SL(2, \mathbb{C}) \to SO^1(1,3) \] (19)
be the standard double covering homomorphism (cf. [8]) which is defined as, for an arbitrary four vector \( x^\mu \in \mathbb{R}^4 \),
\[ \kappa(\Lambda)x^\sim = \Lambda \tilde{x} \Lambda^* \] (20a)
\[ \kappa(\Lambda)x_\sim = \Lambda^{-1} \tilde{x} \Lambda^{-1} \] (20b)

### A. An explicit formula for Wigner rotations

The Wigner rotation \( W(\Lambda, p) \) is defined as (cf. [2])
\[ W(\Lambda, p) = L(\Lambda p)^{-1} \Lambda L(p) \] (21)

where \( L(p) \in SO^1(1,3) \) is a boosting that maps \( k^\mu = (m, 0, 0, 0) \in X \) to \( p^\mu = (\sqrt{m^2 + \mathbf{p}^2}, \mathbf{p}) \in X \). Since we are working on the double covering group \( SL(2, \mathbb{C}) \), we need to find what matrix would do this boosting. If we observe
\[ \sqrt{\frac{p_\mu k}{m}} \sqrt{\frac{p_\mu}{m}} = \sqrt{\frac{p_\mu}{m}} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \sqrt{\frac{p_\mu}{m}} = \tilde{p} \] (22)

and invoke Eq. (20), we see that
\[ L(p) = \sqrt{\frac{p_\mu}{m}} \in SL(2, \mathbb{C}) \] (23)

So, using Eq. (18), we obtain
\[ W(\Lambda, p) = \sqrt{\frac{(\Lambda p)^\sim}{m}} \Lambda \sqrt{\frac{p_\mu}{m}} \in SU(2). \] (24)

This is an explicit formula for the Wigner rotations in the universal covering group \( SU(2) \) of \( SO(3) \).

### B. The transformation law for state vectors in the old representation space

The little group representation corresponding to a massive particle with spin \( \frac{1}{2} \) is the identity representation \( SU(2) \to GL(2, \mathbb{C}) \) (cf. [2] and Chapter 16–17 of [10]). So, we see
\[ D(R) = R, \quad \forall R \in SU(2). \] (25)

Substituting Eqs. (24) and (25) into Eq. (23), we obtain the transformation law for state vectors \( \psi \in \mathcal{H} \):
\[ [U(\Lambda, a)] \psi (p) = e^{-ip_\mu a^\mu} \sqrt{\frac{p_\mu}{m}} \Lambda \sqrt{\frac{(\Lambda p)^\sim}{m}} \psi (\Lambda^{-1} p) \] (26)

where of course \( \Lambda^{-1} p = \kappa(\Lambda^{-1})p \) for \( \Lambda \in SL(2, \mathbb{C}) \) and \( p \in X \). This practice of omitting \( \kappa \) when acted upon an element \( p \in X \) will be done without mention.
III. A NEW REPRESENTATION SPACE

A. The definitions

Let \( \mathcal{H}' \) be the set of all measurable functions

\[
f(p) = \left( \frac{f_1(p)}{f_2(p)} \right)
\]

defined on \( X \) which satisfy

\[
\int_X d\mu(p) \left( \frac{f_1(p)}{f_2(p)} \right)^\dagger \frac{p}{m} \left( \frac{f_1(p)}{f_2(p)} \right) < \infty
\]

(28)

It can be shown that \( \mathcal{H}' \) is a Hilbert space with this inner product (in mathematical language, it is the \( L^2 \) section space of the \( \text{Hermitian bundle} \times \mathbb{C}^2 \rightarrow X \) with metric \( h_{\mu\nu}(u, v) = u^\dagger \frac{p}{m} v \).

Define a linear map \( \alpha : \mathcal{H} \rightarrow \mathcal{H}' \) as

\[
[\alpha(g)](p) = \sqrt{\frac{p}{m}} g(p)
\]

(29)

which is a Hilbert space isomorphism since, for \( g, h \in \mathcal{H} \),

\[
\langle \alpha(g), \alpha(h) \rangle = \int_X d\mu(p) \left( \sqrt{\frac{p}{m}} g(p) \right)^\dagger \frac{p}{m} \left( \sqrt{\frac{p}{m}} h(p) \right) = \int_X d\mu(p) g(p)^\dagger h(p) = \langle g, h \rangle
\]

(30)

where we used Eq. (18) in the second equality.

Using this Hilbert space isomorphism, we can transfer the representation \( U : G \rightarrow U(\mathcal{H}) \) into \( U' : G \rightarrow U(\mathcal{H}') \) by

\[
U'(\Lambda, \alpha) := \alpha \circ U(\Lambda, \alpha) \circ \alpha^{-1}
\]

(31)

so that when acted upon a vector \( \phi \in \mathcal{H}' \),

\[
|U'(\Lambda, \alpha)\phi\rangle(p) = e^{-ip_{\mu}a^\mu} \Lambda \phi(\Lambda^{-1}p)
\]

(32)

where we have used Eqs. (23) and (24).

B. A physical interpretation

Suppose Alice has prepared a state \( \phi \in \mathcal{H}' \) in the new space. Then, using Eq. (23), we see

\[
[\alpha^{-1}(\phi)](p) = L^{-1}(p)\phi(p) = [U'(L(p)^{-1}, 0)\phi] (k)
\]

(33)

where \( k^\mu = (m, 0, 0, 0) \). So, Eq. (33) gives a spin information which “an inertial observer moving with four velocity \( p^\mu/m \) with respect to Alice would get when he/she finds the particle is at rest”.

So, if we agree to receive \( \phi(p) \) as giving “spin information in the laboratory frame” (that is, the particle is seen to be moving with velocity \( p^\mu/m \), then the corresponding wave function \( [\alpha^{-1}(\phi)](p) \) gives “spin information in the particle-rest frame” (that is, the particle is seen to be at rest). In this sense, we can interpret the elements \( \phi \in \mathcal{H}' \) as “apparent wave functions” for the particle and \( \psi \in \mathcal{H} \) as “intrinsic wave functions” for the particle.

From now on, I will refer to \( \mathcal{H} \) as the intrinsic space and \( \mathcal{H}' \) as the apparent space.

C. A remark

After I had finished the draft of this paper, I found that a similar representation space was already used in [5] with the name “covariant basis”. What made me depressed was that a result similar to section IV A which regards the relativistic spin operator, was also discussed in [5].

But, I began to realize that the interpretations given in sections III B and IV B are new and give more insight to the result of [5]. Moreover, the present paper differs from [5] in that is used the Dirac bispinor representation to describe a particle with spin \( \frac{1}{2} \) rather than Eq. (1). Also, communicating the result of [5] to mathematically inclined researchers seemed good enough for me. So, I decided to publish it with this remark section added.

IV. SOME FEATURES OF THE NEW REPRESENTATION SPACE

A. Spin observable

With the help of the interpretation given in III B, we would like to call states in \( \mathcal{H} \) of the form \( (\begin{pmatrix} a_1(p) & 0 \\ 0 & a_2(p) \end{pmatrix}) \) as having spin up and down along the \( z \)-axis, respectively. So, we define the spin observable for the particle as

\[
S = \frac{1}{2} \tau
\]

(34)

on the intrinsic Hilbert space \( \mathcal{H} \), which is just the spin operator used in nonrelativistic quantum mechanics (e.g. Stern-Gerlach experiment). On the new Hilbert space \( \mathcal{H}' \), this spin operator becomes

\[
S' = \alpha \circ S \circ \alpha^{-1}
\]

(35)

In the appendix, it is shown that this is exactly the Newton-Wigner spin operator on the apparent space \( \mathcal{H}' \), i.e.

\[
S' = \frac{1}{m} \left( W - \frac{W^0P}{m + P^0} \right)
\]

(36)

where \( W \) is the Pauli-Lubanski vector defined as the spatial component of the four vector (cf. [6])

\[
W^\mu = \frac{1}{2} \epsilon^{\alpha\beta\mu} P_\nu J_{\alpha\beta}
\]

(37)
So, we may say that the intrinsic spin of the particle in the particle-rest frame is precisely the Newton-Wigner spin in the laboratory frame.

**B. A Lorentz covariant reduced density matrix for spin**

Suppose Alice has prepared $\phi \in \mathcal{H}'$. Form a function

$$\rho(p, p') = \phi(p)\phi(p')\dagger$$

which is NOT a density matrix on $\mathcal{H}'$ due to the presence of the nontrivial inner product (Eq. (39)). Nevertheless, in almost all cases of interest (e.g., when $\phi = (a_1, a_2)$ with $a_j$ Schwartz class functions), we can integrate this diagonally to obtain a $2 \times 2$-matrix

$$\tau := \int_X d\mu(p) \rho(p, p).$$

Again, this is NOT a partial trace operation since $\mathcal{H}'$ is not a tensor product system (again, due to Eq. (29)). Nonetheless, it is positive and has nonzero trace:

$$u^\dagger \tau u = \int_X d\mu(p) \left| u^\dagger \phi(p)\phi(p)\dagger u \right|^2 \geq 0 \quad \forall u \in \mathbb{C}^2$$

and

$$\text{Tr} \tau = \int_X d\mu(p) \text{Tr} \rho = \int_X d\mu(p) \left( |\phi_1(p)|^2 + |\phi_2(p)|^2 \right) > 0.$$

So, the matrix $\tau$ can be normalized to yield a density matrix which, however, we will not write since it is not relevant to our discussions.

Consider another observer Bob, in whose frame the state is $U'(\Lambda, a)\phi$. Then, due to the transformation law Eq. (22) and the Lorentz invariance of $\mu$ (Eq. (11)),

$$\tau_B = \int_X d\mu(p) \Lambda\phi(\Lambda^{-1}p)\phi(\Lambda^{-1}p)\dagger \Lambda^\dagger = \Lambda\tau\Lambda^\dagger.$$

So, we see that the newly defined reduced matrix is Lorentz covariant.

We can generalize this construction as follows. Suppose $\rho(p, p')$ is a density matrix on the intrinsic space $\mathcal{H}$. We pull this back to get a density matrix $\rho' := \alpha \circ \rho \circ \alpha^{-1}$ on $\mathcal{H}'$, which is an integral operator with symbol

$$\rho'(p, p') = \sqrt{\frac{p'}{m}} \rho(p, p') \sqrt{\frac{p}{m}}.$$  

Assuming that this is diagonally integrable, we form

$$\tau := \int_X d\mu(p) \rho'(p, p)$$

which is exactly the definition of the Lorentz covariant reduced matrix of $\mathcal{H}$ (apart from the fact that the Dirac spinor representation, which is equivalent to Eqs. (11) and (3), was used in that paper). It is easy to check that Eq. (39) is a special case of Eq. (41) when the density matrix $\rho$ is $\mathcal{H}$ is associated to the state $\alpha^{-1}\phi \in \mathcal{H}$.

As emphasized above, Eq. (41) is NOT the result of a partial trace operation. Yet, there is a vestige of partial trace operation. Given a matrix $C \in M_2(\mathbb{C})$, the map $\hat{C} : \mathcal{H}' \to \mathcal{H}'$ defined by

$$[\hat{C}\phi](p) = C \cdot \phi(p)$$

is a continuous linear operator. $\hat{C}$ is a reminiscent of an operator of the form $1 \otimes C$. On $\mathcal{H}$, this becomes

$$[\alpha^{-1}\hat{C}\alpha\psi](p) = \sqrt{\frac{p}{m}} C \sqrt{\frac{p'}{m}} \psi(p)$$

so that, with the notation of Eq. (13),

$$\text{Tr}_{\mathcal{H}'} \left( \rho^\dagger \hat{C} \right) = \text{Tr}_{\mathcal{H}} \left( \rho\alpha^{-1}\hat{C}\alpha \right) = \int_X d\mu(p) \text{Tr} \left( \rho(p, p) \sqrt{\frac{p}{m}} C \sqrt{\frac{p'}{m}} \right) = \text{Tr} \left( \tau C \right)$$

where in the second equality we used the fact that $\mathcal{H}$ is a tensor product system and hence the trace can be evaluated as an iteration of the traces over each factor.

Eq. (47) shows that $\tau$ can be regarded as a true reduced matrix for “apparent spin”. That is, it can be used to describe a sort of “local operation” (cf. Chapter 19 of Ref.) with apparent spin degree of freedom.

The point is that the construction of the covariant reduced density matrix of $\mathcal{H}$ becomes very natural in this apparent space. Also, the physical interpretation given in section 11 suggests that the matrix Eq. (44) contains “spin information in the laboratory frame” which is confirmed by the fact that this matrix indeed contains information about average polarization of the particle as well as average kinematical state (Ref.).

**V. CONCLUSIONS**

We have defined a new representation space of the universal cover of the Poincaré group $\mathbb{R}^4 \rtimes SL(2, \mathbb{C})$, the state vectors of which may be interpreted as “apparent wave functions” for the particle. In this interpretation, the state vectors in the old representation space becomes “intrinsic wave functions” for the particle.

This dual point of view for a relativistic massive particle with spin suggests that the spin operator is merely $\frac{1}{2}\tau$ on the intrinsic space and it is represented as the Newton-Wigner spin operator in the apparent space.

Also, the definition of the Lorentz covariant matrix in $\mathcal{H}$ becomes very natural in this representation space and a sort of “local operation” with apparent spin degree of freedom becomes possible.
Appendix

To describe these operators Eqs. (36) and (37) on the representation space \((U', \mathcal{H}')\), we need the following formulae for the 10 standard mechanical operators on \((U', \mathcal{H}')\). Denoting purely symbolically

\[ x^\mu := \frac{1}{i} \frac{\partial}{\partial p_\mu}, \quad (A.1) \]

the 10 mechanical operators are given by

\[
[P^0 \psi] (p) = \left[ i \frac{\partial}{\partial t} U'(I, t\epsilon_0) \psi \right] (p) = p^0 \psi(p) \quad (A.2a)
\]

\[
[P^j \psi] (p) = \left[ -i \frac{\partial}{\partial x^j} U'(I, x\epsilon_j) \psi \right] (p) = p^j \psi(p) \quad (A.2b)
\]

\[
[J^j \psi] (p) = \left[ i \frac{\partial}{\partial \theta} U' \left( R'(\theta), 0 \right) \psi \right] (p) = \left[ \frac{1}{2} \tau^j - (p \times x)^j \right] \psi(p) \quad (A.3)
\]

\[
[K^j \psi] (p) = \left[ i \frac{\partial}{\partial u} U' \left( B^j(u), 0 \right) \psi \right] (p) = \left[ \frac{1}{2} \tau^j - (p^0 x^j - p^j x^0) \right] \psi(p) \quad (A.4)
\]

where the standard rotation and boosting matrices \(R^j(\theta)\) and \(B^j(u)\) are given by

\[
R^j(\theta) = \exp \left( -i \frac{\theta \tau^j}{2} \right) \quad (A.5)
\]

\[
B^j(u) = \exp \left( \frac{1}{2} u \tau^j \right) \quad (A.6)
\]

respectively.

Now, inserting these formulae into Eq. (37) while recalling \(J^i = \varepsilon_{ijk} J^k\) and \(K^j = J^0 j\) (cf. [11]), we obtain

\[
W^0 = P \cdot J = \frac{1}{2} p \cdot \tau = -\frac{1}{2} (p^0 I) \quad (A.7)
\]

\[
W = P^0 J - P \times K = \frac{1}{2} (p^0 \tau - ip \times \tau) \quad (A.8)
\]

which, after a bit of algebra, becomes

\[
W = \frac{1}{2} (\tau p + p I) \quad (A.9)
\]

Inserting Eqs. (A.2), (A.7) and (A.9) into Eq. (36) yields

\[
S_{NW} = \frac{1}{2m} \left( \tau p + p I + p - p_0 I \right) = \frac{1}{2m} \left( \tau p + p + m I \right) \quad (A.10)
\]

By the way, using Eq. (10) and the commutation relation \(\hat{p} \tau = \tau p + 2p I\) of the Pauli matrices, Eq. (35) becomes

\[
S' = \sqrt{\frac{p}{m}} S = \frac{1}{2 \sqrt{2m(m + p_0)}} (p + m I) \tau \sqrt{\frac{p}{m}} = \frac{1}{2 \sqrt{2m(m + p_0)}} (\tau (p + m I) + 2p I) \sqrt{\frac{p}{m}} = \frac{1}{2m} \left( \tau p + p + m I \right) \quad (A.11)
\]

which is exactly Eq. (A.10). Therefore, we conclude

\[
S' = S_{NW} \quad (A.12)
\]

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