Abstract. By considering the master equation of the totally asymmetric exclusion process on a one-dimensional lattice and using two types of boundary conditions (i.e. interactions), two new families of multi-species reaction–diffusion processes, with particle-dependent hopping rates, are investigated. In these models (i.e. reaction–diffusion and drop–push systems), we have the case of distinct particles where each particle $A_\alpha$ has its own intrinsic hopping rate $v_\alpha$. They also contain the parameters that control the annihilation–diffusion rates (including pair annihilation and coagulation to the right and left). We obtain two distinct new models. It is shown that these models are exactly solvable in the sense of the Bethe ansatz. The two-particle conditional probabilities and the large-time behavior of such systems are also calculated.

Keywords: driven diffusive systems (theory)
1. **Introduction**

Our understanding of non-equilibrium statistical physics is much behind that of equilibrium theory. Asymmetric exclusion processes (ASEP) are paradigmatic models for systems which are far from equilibrium. Despite their greatly reduced complexity they capture various fields of physics like the study of shocks [1,2], the noisy Burgers’ equation [3], polymers in random media, dynamical models of interface growth [4], traffic models [5] and the kinetics of biopolymerization [6].

On the one-dimensional lattice, the simplest system is the totally asymmetric exclusion process (TASEP). In this model, each lattice site is occupied by at most one particle and all particles can only hop with equal rates to their right-neighboring site, provided this site is empty; otherwise the attempted move is rejected. In [7], TASEP has been solved by introducing a master equation which describes the evolution equation of the particles when they are not in neighboring sites, and a so-called boundary condition, which specifies the situation in which the probabilities go outside the physical regions. This happens when some of the particles are in adjacent sites and the non-physical probability terms appear in the master equation. The coordinate Bethe ansatz (BA) has been used to obtain the \(N\)-particle conditional probabilities of TASEP.

The interesting point is that one can find new solvable models (including interactions besides diffusion to right-neighboring sites) with other boundary conditions and the same master equation. In [8], a similar technique has been used to solve the so-called drop-push model. In this model the particle hops to the next-right site, even if it is occupied. It can hop by pushing all the neighboring particles to their next-right sites, with a rate
depending on the number of these particles. Some other generalizations of TASEP can be found in [9]–[11].

The other interesting point is the study of multi-species systems in which several kinds of particles move and interact (i.e. reaction–diffusion) on a lattice. In [12], it has been shown that more-than-one species systems are solvable in the sense that the $S$-matrix corresponding to them is factorizable into two-particle $S$-matrices. It has been found there that the criterion for this is that the interactions must be such that the $S$-matrix satisfies a kind of spectral Yang–Baxter (SYB) equation. The multi-species generalization of the reactions considered in [12] has been studied in [13] and the authors remarked that the SYB reduces to a non-spectral matrix equation. The drop–push reaction of [8] has been generalized to $p$ species in [14]. The most general totally asymmetric reaction–diffusion process, including the extended drop–push interactions, has been studied in [15].

The other family of diffusion systems is the process in which particles can hop to the next-right and the next-left neighboring sites. In [7], the single-species model with only simple diffusion to the right and left, i.e. partially asymmetric exclusion process (PASEP), has been studied and in [16] the one-species PASEP drop–push model has been described. The multi-species generalizations of PASEP models, in which each particle hops to the right at a rate $D_R$ and hops to the left at a rate $D_L$, have been studied in [17]. The $D_L = 0$ case of the models studied in [17] leads to the previous mentioned TASEP models.

In all of the above studies, particles hop randomly in continuous time on the integer lattice $\mathbb{Z}$. A hopping event occurs independently for each particle with identical rates. The model of one-dimensional PASEP, in which the rates of jump are chosen randomly at time zero and fixed for the rest of the evolution, was first introduced by Benjamini et al [18] who proved the existence of a critical density $\rho^*$ depending only on the distribution of hopping rates. They also proved for this case the hydrodynamic limit of an associated zero-range process (ZRP) which is obtained from TASEP by identifying particles with the sites of a new one-dimensional lattice and the interparticle distance (number of empty sites between particles $i$, $i + 1$) as the occupation number at site $i$ of that lattice.

This model has received renewed attention because of the occurrence of a condensation transition analogous to Bose–Einstein condensation and because of its close relationship with exclusion processes. An analogous transition also occurs if the rates are identical for each particle, but dependent on the lattice distance to the next particle, see [19] for a recent review (in terms of the ZRP) and [20]–[25] for the current developments.

The generalization of TASEP (only for one species) with constant hopping rates to the particle-dependent hopping rates (PDHR) has been studied in [26]. The authors considered the case that a particle in site $x_i$ has a hopping rate $v_i$ and used the BA to obtain in a determinant form the exact solution of the master equation for the conditional probabilities of TASEP with the PDHR. They also derived a determinant expression for the time-integrated current for a step-function initial state. In the sense of the BA, the two-species TASEP in which different particles hop with different rates and fast particles stochastically overtake slow ones, has been considered by Karimipour in [27].

In this paper we are going to study the multi-species TASEP with the PDHR. We obtain two new families of multi-species diffusion processes with the PDHR. The first family (i.e. reaction–diffusion models) has the following reactions:

\begin{align}
A_\alpha & \to 0A_\alpha \quad \text{with rate } v_\alpha, \\
A_\alpha A_\beta & \to A_\gamma A_\delta \quad \text{with rate } c^\alpha_\gamma c^\beta_\delta,
\end{align}

\noalign{\hfill (1)\hfill}

doi:10.1088/1742-5468/2009/02/P02001
and the second family (i.e. drop–push models) describes the following processes:

\begin{align}
A_\alpha \emptyset & \rightarrow \emptyset A_\alpha \quad \text{with rate } v_\alpha, \\
A_\alpha A_\beta \emptyset & \rightarrow \emptyset A_\gamma A_\delta \quad \text{with rate } b_{\gamma\delta}^{\alpha\beta}, \\
& \vdots
\end{align}

(2)

where the dots indicate the other drop–push reactions with \( n \)-adjacent particles. In these models, we have the case of distinct particles where each particle \( A_\alpha \) diffuses to the right with its own intrinsic hopping rate \( v_\alpha \). We also consider the annihilation–diffusion (including pair annihilation and coagulation to the right and left) extension of these above models. The annihilation processes are

\begin{align}
A_\alpha A_\beta & \rightarrow \emptyset A_\beta \quad \text{with rate } \delta_{\alpha\beta}, \\
A_\alpha A_\beta & \rightarrow A_\alpha \emptyset \quad \text{with rate } \gamma_{\alpha\beta}, \\
A_\alpha A_\beta & \rightarrow \emptyset \emptyset \quad \text{with rate } \eta_{\alpha\beta}.
\end{align}

(3)

We show that the reaction rates of these models (now including the annihilation processes) must satisfy some specific constraints and these models are exactly solvable in the sense of the BA, provided a non-spectral matrix equation is satisfied.

This paper is organized as follows. We first introduce two types of boundary conditions that can be used to obtain two new families of \( p \)-species reaction–diffusion processes with PDHR in section 2. We use the type 1 boundary condition and the type 2 boundary condition, in terms of two \( p^2 \times p^2 \) matrices \( c \) and \( b \), respectively, to generalize \( p \)-species reaction–diffusion and drop–push systems with identical hopping rates into PDHR systems. In section 3 we consider the annihilation process extension of these new models. In section 4 we investigate the BA solution of the new models and discuss under what conditions, one can use the BA to find exact solutions. We show that the matrix \( \tilde{c}(\tilde{b}) \) (a version of \( c(b) \) that constructs the \( S \)-matrix and determines the coefficient of the BA) must satisfy a non-spectral matrix equation. Then in section 5 we show that, for \( p = 2 \) (for both models), the specific class of parameters which corresponds to the reaction rates together with the annihilation–diffusion rates satisfies a non-spectral matrix equation. Finally in section 6 we study the conditional probabilities of these models and, especially for two-particle systems of section 5, we obtain the exact expressions and the large-time behavior of such systems.

2. TASEP generalization

2.1. Boundary condition

Consider a \( p \)-species system with particles \( A_1, A_2, \ldots, A_p \). The basic objects we are interested in are the probabilities \( P_{\alpha_1, \ldots, \alpha_N}(x_1, \ldots, x_N; t) \) for finding at time \( t \) the particle of type \( \alpha_1 \) at site \( x_1 \), the particle of type \( \alpha_2 \) at site \( x_2 \), etc. We take the physical region of coordinates as \( x_1 < x_2 < \cdots < x_N \). The master equation for a totally asymmetric
exclusion process, with particle-dependent hopping rates, is

$$\frac{\partial}{\partial t}P_{\alpha_1...\alpha_N}(x_1, \ldots, x_N; t) = \sum_{i=1}^{N} v_{\alpha_i}P_{\alpha_1...\alpha_N}(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_N; t)$$

$$- \left( \sum_{i=1}^{N} v_{\alpha_i} \right) P_{\alpha_1...\alpha_N}(x_1, \ldots, x_N; t).$$

(4)

This equation describes a collection of $N$ particles where the $\alpha_i$th particle drifts to the next-right site by rate $v_{\alpha_i}$, where $v_{\alpha_i}$s are finite non-zero real numbers. This master equation is only valid for $x_i < x_{i+1} - 1$. For $x_i = x_{i+1} - 1$, there will be some terms with $x_i = x_{i+1}$ on the right-hand side of (4), which are outside the physical region. But one can assume that (4) is valid for all the physical regions $x_i < x_{i+1}$ by imposing certain boundary conditions for $x_i = x_{i+1}$. Different boundary conditions introduce different interactions for particles. Following the same argument given in [17], it can be easily seen that the master equation (4) leads to the following relation for two-particle probabilities:

$$\frac{\partial}{\partial t} \sum_{x_2} \sum_{x_1 < x_2} P_{\alpha_1\alpha_2}(x_1, x_2; t) = \sum_{x} v_{\alpha_2} P_{\alpha_1\alpha_2}(x, x; t) - \sum_{x} v_{\alpha_1} P_{\alpha_1\alpha_2}(x, x + 1; t).$$

(5)

This equation leads us to take $P_{\alpha_1\alpha_2}(x, x; t)$ as the following two types of boundary conditions:

$$v_{\alpha_2} P_{\alpha_1\alpha_2}(x, x) = \sum_{\beta} c_{\alpha_1\alpha_2}^{\beta_1\beta_2} P_{\beta_1\beta_2}(x, x + 1) \quad \text{type 1}$$

(6)

$$v_{\alpha_2} P_{\alpha_1\alpha_2}(x, x) = \sum_{\beta} b_{\alpha_1\alpha_2}^{\beta_1\beta_2} P_{\beta_1\beta_2}(x - 1, x) \quad \text{type 2}$$

(7)

where $\beta$ stands for $(\beta_1, \beta_2)$, and $b$ and $c$ are $p^2 \times p^2$ matrices which determine the interactions. In the probabilities which appear in (6) and (7), we have suppressed all the other coordinates and the time $t$ for simplicity. We consider the processes in which the number of particles is constant in time; in other words we exclude the creation and annihilation processes (in fact in this step). Therefore if we sum (5) over $\alpha_1$ and $\alpha_2$, the left-hand side becomes zero and results in

$$- \sum_{x} \sum_{\alpha_1} \alpha_2 P_{\alpha_1\alpha_2}(x, x + 1) + \sum_{x} \sum_{\beta} \left( \sum_{\alpha_1 \alpha_2} c_{\alpha_1\alpha_2}^{\beta_1\beta_2} \right) P_{\beta_1\beta_2}(x, x + 1) = 0$$

(8)

in which (6) has been used. Clearly (8) gives

$$\sum_{\alpha_1 \alpha_2} c_{\alpha_1\alpha_2}^{\beta_1\beta_2} = v_{\beta_1} \quad \text{constraint for type 1}.$$  

(9)

In the same way, if we use (7), the sum over the elements of each column of matrix $b$ results in

$$\sum_{\alpha_1 \alpha_2} b_{\alpha_1\alpha_2}^{\beta_1\beta_2} = v_{\beta_1} \quad \text{constraint for type 2}.$$  

(10)

Remember that the constraints (9) and (10) are the consequence of the conservation of probabilities so they can be interpreted as the probability conservation equations.
2.2. $P$-species reaction–diffusion systems with PDHR

Following the same steps as [17], we first consider $\dot{P}_{\alpha_1\alpha_2}(x, x + 1)$. Using (4) and the type 1 boundary condition (6), it is found

$$\dot{P}_{\alpha_1\alpha_2}(x, x + 1) = v_{\alpha_1} P_{\alpha_1\alpha_2}(x - 1, x + 1) + \sum_{\beta \neq \alpha} c_{\alpha_1\beta_2} P_{\beta_1\beta_2}(x, x + 1)$$

$$- v_{\alpha_2} P_{\alpha_1\alpha_2}(x, x + 1) - \sum_{\beta \neq \alpha} c_{\alpha_1\beta_2} P_{\alpha_1\alpha_2}(x, x)$$

in which we have used (9). It can be written as (in fact the diagonal elements are not reaction rates)

$$c_{\alpha_1\beta_2} = v_{\alpha_1} - \sum_{\beta \neq \alpha} c_{\alpha_1\beta_2},$$

(12)

It is seen that the evolution equation (11) describes the following two-particle interactions:

$$A_{\alpha} \emptyset \rightarrow \emptyset A_{\alpha} \quad \text{with rate } v_{\alpha},$$

$$A_{\alpha} A_{\beta} \emptyset \rightarrow \emptyset A_{\gamma} A_{\delta} \quad \text{with rate } c_{\alpha\beta}^{\gamma\delta}. \quad (13)$$

It is simple to show that our formalism is consistent for more-than-two-particle systems. In this model, in addition to interactions, we have the case of distinct particles where each particle $A_{\alpha}$ has its own intrinsic hopping rate $v_{\alpha}$. It is more general than the TASEP model and has been studied in [13], in which all particles have the equal rate.

2.3. $P$-species drop–push systems with PDHR

In the same way, we first consider $\dot{P}_{\alpha_1\alpha_2}(x, x + 1)$ by using (4) and the type 2 boundary condition (7). The result is

$$\dot{P}_{\alpha_1\alpha_2}(x, x + 1) = v_{\alpha_1} P_{\alpha_1\alpha_2}(x - 1, x + 1) + \sum_{\beta} b_{\alpha_1\beta_2} P_{\beta_1\beta_2}(x - 1, x)$$

$$- v_{\alpha_2} P_{\alpha_1\alpha_2}(x, x + 1) - \sum_{\beta} b_{\alpha_1\beta_2} P_{\alpha_1\alpha_2}(x, x + 1)$$

in the above equation we have directly used constraint (10) as

$$\sum_{\beta} b_{\alpha_1\beta_2} = v_{\alpha_1}.$$ \hspace{1cm} (15)

So in agreement with (14), the allowed processes are

$$A_{\alpha} \emptyset \rightarrow \emptyset A_{\alpha} \quad \text{with rate } v_{\alpha},$$

$$A_{\alpha} A_{\beta} \emptyset \rightarrow \emptyset A_{\gamma} A_{\delta} \quad \text{with rate } b_{\alpha\beta}^{\gamma\delta}. \quad (16)$$

In this model any particle can hop to the right site, with rate dependent on the type of particle, if that site is empty. If the right site is occupied, the left particle can still hop to that site by pushing the right one, but in the mean time there is a probability that the types of the particles change. It is important to note that all elements of $b$ (including the diagonal elements) should be nonnegative (since they are rates). We will consider more-than-two-particle systems when we introduce the type 2 model.
3. Annihilation–diffusion processes

The annihilation process is a diffusion-limited reaction–diffusion process. In this process, the particles annihilate pairwise or coagulate to the right and left whenever they meet each other. Now we add the annihilation–diffusion to the previous reactions of both models. Note that the annihilations appear only in the sink terms of the evolution equation as, if we consider the initial state with \( n \) particles, no annihilation processes can lead to a \( n \)-particle state at any later time. So we do not have the conservation of probabilities (particles) and one can enter a sink term, \( \lambda_{\beta_1 \beta_2} \), into the conservation equations (constraints) as

\[
\sum_{\alpha} c^{\beta_1 \beta_2}_{\alpha \alpha_2} = v_{\beta_1} - \lambda_{\beta_1 \beta_2} \tag{17}
\]

\[
\sum_{\alpha} b^{\beta_1 \beta_2}_{\alpha \alpha_2} = v_{\beta_1} - \lambda_{\beta_1 \beta_2} \tag{18}
\]

and if we use these modified constraints in the calculation of \( \dot{P}_{\alpha_1 \alpha_2}(x, x + 1) \), we find the same equation as (11) or (14) except for an extra term \( \lambda_{\alpha_1 \alpha_2} P_{\alpha_1 \alpha_2}(x, x + 1) \) which is added to the sink terms. So \( \lambda_{\alpha_1 \alpha_2} \) is the sum of the rates of all annihilation processes with initial state \((\alpha_1, \alpha_2)\) and therefore it is a positive quantity. These processes are

\[
\begin{align*}
A_{\alpha} A_{\beta} &\rightarrow A_{\alpha} A_{\beta} \quad \text{with rate } \delta_{\alpha \beta}, \\
A_{\alpha} A_{\beta} &\rightarrow A_{\alpha} \emptyset \quad \text{with rate } \gamma_{\alpha \beta}, \\
A_{\alpha} A_{\beta} &\rightarrow \emptyset \emptyset \quad \text{with rate } \eta_{\alpha \beta},
\end{align*}
\]

and the relation between the above rates is

\[
\lambda_{\alpha \beta} = \delta_{\alpha \beta} + \gamma_{\alpha \beta} + \eta_{\alpha \beta}. \tag{20}
\]

One should note that the \( p \)-species reaction–diffusion and drop–push systems, including the annihilations, have three kinds of processes. The first one is the pure diffusion with the PDHR which occurs when a particle is adjacent to a hole, while the second and third ones are the reactions and the annihilations, respectively, which occur when two particles are adjacent to each other. The master equation for the diffusion with the PDHR is (4) and for this case the master equation does not contain anything about both the reactions and the annihilations. The effect of reactions has been coded in the boundary condition which is in fact part of the master equation. Also the effect of annihilations has been coded in the modified constraint (17) or (18).

4. The Bethe ansatz solution

4.1. Solvability criteria

To solve the master equation (4) with two types of boundary conditions, we consider the modified Bethe ansatz of the form [27]

\[
P_{\alpha_1, \ldots, \alpha_N}(x; t) = \left( \prod_{i=1}^{N} v_{\alpha_i} e^{-v_{\alpha_i} t} \right) e^{-E_N t} \psi_{\alpha_1, \ldots, \alpha_N}(x; t), \tag{21}
\]
with $\Psi$ as the Bethe wavefunction:

$$\Psi(x) = \sum_\sigma A_\sigma e^{i\sigma(p) \cdot x}. \quad (22)$$

$\Psi$ is a tensor of rank $N$ with components $\psi_{\alpha_1 \cdots \alpha_N}(x)$, where the sum is taken over all permutations $\sigma$ of $(1, 2, 3, \ldots, N)$. Inserting (21) in (4) therefore, the above eigenfunctions correspond to the eigenvalues as follows:

$$E_N = -\sum_{j=1}^N e^{-ip_j}. \quad (23)$$

The next step is to determine the coefficients $A_\sigma$. Inserting (21) in the type 1 boundary condition (6) gives

$$v_{\alpha_2} e^{-(E_2 + v_{\alpha_1} + v_{\alpha_2})} \psi_{\alpha_1 \alpha_2}(x, x) = \sum_\beta c_{\alpha_1 \alpha_2}^\beta e^{-(E_2 + v_{\beta_1} + v_{\beta_2})} v_{\beta_1} v_{\beta_2} \psi_{\beta_1 \beta_2}(x, x + 1). \quad (24)$$

Relation (24) is the boundary condition for $\Psi$ but it does not have the form of boundary condition (6) (because of the time dependence and the elements of power $x$). To build a modified form of (24) (similar to (6)), we should omit the time dependence and the elements of power $x$ from both sides of this equation. After finding a modified form, we can put (22) in it to obtain the coefficients $A_\sigma$. Obviously to obtain a modified desired form one arrives at

$$v_{\alpha_1} + v_{\alpha_2} = v_{\beta_1} + v_{\beta_2}, \quad (25)$$

$$v_{\alpha_1} v_{\alpha_2} = v_{\beta_1} v_{\beta_2}. \quad (26)$$

One can write these equations as

$$v_{\alpha_1} + v_{\alpha_2} = v_{\beta_1} + v_{\beta_2}, \quad (27)$$

$$v_{\alpha_1}^2 + v_{\alpha_2}^2 = v_{\beta_1}^2 + v_{\beta_2}^2. \quad (28)$$

These equations have a physical interpretation in terms of an elastic collision. In the elastic collision of equal masses (supposing that $v_{\beta_i}$s are the velocities before collision and $v_{\alpha_i}$s are the velocities after collision) the conservation of momentum and kinetic energy are the same as (27) and (28), respectively. Solving these simultaneous equations we get

$$\{v_{\alpha_1} = v_{\beta_1}, v_{\alpha_2} = v_{\beta_2}\} \quad \text{and} \quad \{v_{\alpha_1} = v_{\beta_2}, v_{\alpha_2} = v_{\beta_1}\}. \quad (29)$$

The two above solutions imply that the only two states for $\vec{\beta}$ are $\vec{\beta} = (\alpha_1, \alpha_2)$ and $\vec{\beta} = (\alpha_2, \alpha_1)$. These states introduce a system with the PDHR and the exchange–reaction processes as we will show later on. So the matrix $c$ has non-zero elements provided $\vec{\beta} = (\alpha_1, \alpha_2)$ and $\vec{\beta} = (\alpha_2, \alpha_1)$ are our states and therefore (24) reduces to

$$\psi_{\alpha_1 \alpha_2}(x, x) = c_{\alpha_1 \alpha_2}^{\alpha_1 \alpha_2} \psi_{\alpha_1 \alpha_2}(x, x + 1) + \left(\frac{v_{\alpha_1}}{v_{\alpha_2}}\right) c_{\alpha_1 \alpha_2}^{\alpha_2 \alpha_1} \psi_{\alpha_2 \alpha_1}(x, x + 1)$$

$$= \sum_\beta \tilde{c}_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} \psi_{\beta_1 \beta_2}(x, x + 1) \quad (30)$$

doi:10.1088/1742-5468/2009/02/P02001
where

\[
\tilde{c}_{\alpha_1\alpha_2}^{\beta_1\beta_2} = \begin{cases} 
\frac{v_{\alpha_1} \alpha_1}{v_{\alpha_2} \alpha_2} c_{\alpha_1\alpha_2} & \text{for } \tilde{\beta} = \tilde{\alpha} \\
\frac{v_{\alpha_1} \alpha_2}{v_{\alpha_2} \alpha_1} c_{\alpha_2\alpha_1} & \text{for } \tilde{\beta} = (\alpha_2, \alpha_1).
\end{cases}
\]

(31)

In the same way, for the type 2 boundary condition (7), one arrives at

\[
\tilde{b}_{\alpha_1\alpha_2}^{\beta_1\beta_2} = \begin{cases} 
\left(\frac{1}{v_{\alpha_1} \alpha_1} v_{\alpha_2} \alpha_2\right) b_{\alpha_1\alpha_2} & \text{for } \tilde{\beta} = \tilde{\alpha} \\
\left(\frac{1}{v_{\alpha_2} \alpha_1} v_{\alpha_1} \alpha_2\right) b_{\alpha_2\alpha_1} & \text{for } \tilde{\beta} = (\alpha_2, \alpha_1).
\end{cases}
\]

(32)

In the $p$-species case, $c(\tilde{c})$ has $p!(p-1)$ non-diagonal non-zero elements where these elements are the exchange rates of the $p$-species particles but $b(\tilde{b})$ has $p!(p-1)+p^2$ non-zero elements ($p^2$ diagonal and $p!(p-1)$ non-diagonal) which describe the rates of the drop–push interactions. One should note that the annihilation rates are not the original elements of the matrix $c(b)$ and they enter the matrix when we impose the constraint (17) (constraint (18)) to this matrix. Now we can write (24) in the compact notation as

\[
\Psi(\ldots, x_k = x, x_{k+1} = x, \ldots) = \tilde{c}_{k,k+1} \Psi(\ldots, x_k = x, x_{k+1} = x+1, \ldots),
\]

(33)

where

\[
\tilde{c}_{k,k+1} = 1 \otimes \cdots \otimes 1 \otimes \tilde{c}^{k,k+1} \otimes 1 \otimes \cdots \otimes 1.
\]

(34)

Note that, however, (29) leads to the modified desired form of (24) but in itself implies some loss of generality on the elements of the matrices $b$ and $c$. Now the coefficients $A_\sigma$ can be determined by putting (22) in (33), which gives

\[
[1 - e^{i\sigma(p_{k+1})} \tilde{c}_{k,k+1}] A_\sigma + [1 - e^{i\sigma(p_k)} \tilde{c}_{k,k+1}] A_{\sigma \sigma_k} = 0.
\]

(35)

From this one obtains

\[
A_{\sigma \sigma_k} = S_{k,k+1}^{(1)}(\sigma(p_k), \sigma(p_{k+1})) A_\sigma,
\]

(36)

where the matrix $S^{(1)}$ is defined through

\[
S^{(1)}(z_1, z_2) = -(1 - z_1 \tilde{c})(1 - z_2 \tilde{c})^{-1}
\]

(37)

and the definition of $S_{k,k+1}^{(1)}$ is similar to that of $\tilde{c}_{k,k+1}$ in (34). We have also used $z_k = e^{ip_k}$. The same procedure for the type 2 boundary condition (7) results in

\[
S^{(2)}(z_1, z_2) = -(1 - z_2^{-1} \tilde{b})(1 - z_1^{-1} \tilde{b})^{-1}.
\]

(38)

Equation (36) allows one to compute all the $A_\sigma$s in terms of $A_1$ (which is set to unity). As the generators of the permutation group satisfy $\sigma_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \sigma_{k+1}$, so one also needs

\[
A_{\sigma_k \sigma_{k+1} \sigma_k} = A_{\sigma_{k+1} \sigma_k \sigma_{k+1}}.
\]

(39)
This, in terms of $S$-matrices, becomes
\[
S_{12}(z_2, z_3)S_{23}(z_1, z_3)S_{12}(z_1, z_2) = S_{23}(z_1, z_2)S_{12}(z_1, z_3)S_{23}(z_2, z_3).
\] (40)

Writing the $S$-matrix as the product of the permutation matrix $\Pi$ and an $R$-matrix,
\[
S_{k,k+1} =: \Pi_{k,k+1}R_{k,k+1},
\] (41)
equation (40) is transformed to
\[
R_{23}(z_2, z_3)R_{13}(z_1, z_3)R_{12}(z_1, z_2) = R_{12}(z_1, z_2)R_{13}(z_1, z_3)R_{23}(z_2, z_3).
\] (42)
This is the spectral Yang–Baxter equation. The Bethe ansatz solution exists if the scattering matrix satisfies (40): in other words the matrix $\tilde{c}$ in (37) and $\tilde{b}$ in (38) are acceptable only if the resulting $S$-matrices satisfy (40). This is a very restricted condition and needed for having the solvability. The $S$-matrices (37) and (38) are exactly the ones considered in [13] and [14], respectively. Using the fact that $S^{(1)}$ is a binomial of degree one with respect to $z_2$ and $S^{(2)}$ is a binomial of degree one with respect to $z_1^{-1} = e^{-i\pi_1}$, it can be shown that the SYB equation (40) for $S^{(1)}$ and $S^{(2)}$ reduces to the non-spectral matrix equations
\[
\tilde{c}_{12}[\tilde{c}_{12}, \tilde{c}_{23}] = [\tilde{c}_{12}, \tilde{c}_{23}]\tilde{c}_{23},
\] (43)
and
\[
\tilde{b}_{23}[\tilde{b}_{23}, \tilde{b}_{12}] = [\tilde{b}_{23}, \tilde{b}_{12}]\tilde{b}_{12},
\] (44)
which are the same as [13] and [14], respectively. The above equations are much simpler than (40). So it is far simpler to seek the solutions of these equations than to seek those of (40).

### 4.2. Type 1 model

The master equation (4) with the boundary condition
\[
v_{\alpha_2}P_{\alpha_1\alpha_2}(x, x) = \sum_{\beta} c_{\alpha_1\alpha_2}^{\beta_1\beta_2} P_{\beta_1\beta_2}(x, x + 1)
\] (45)
and relation (17) describe consistently the following reactions:
\[
A_\alpha \emptyset \rightarrow \emptyset A_\alpha \quad \text{with rate } v_\alpha, \\
A_\alpha A_\beta \rightarrow A_\beta A_\alpha \quad \text{with rate } c_{\alpha\beta}^{\alpha\beta}, \\
A_\alpha A_\beta \rightarrow \emptyset A_\beta \quad \text{with rate } \delta_{\alpha\beta}, \\
A_\alpha A_\beta \rightarrow A_\alpha \emptyset \quad \text{with rate } \gamma_{\alpha\beta}, \\
A_\alpha A_\beta \rightarrow \emptyset \emptyset \quad \text{with rate } \eta_{\alpha\beta}.
\] (46)

A matrix $c$, or the above reactions, corresponds to an exactly solvable exchange reaction–diffusion system on a one-dimensional lattice, provided $c$ satisfies (17) and $\tilde{c}$ satisfies (43) and the non-diagonal elements are nonnegative.
4.3. Type 2 model

The boundary condition
\[ v_{\alpha_2} P_{\alpha_1 \alpha_2}(x, x) = \sum_{\beta} b_{\alpha_1 \alpha_2}^{\beta} P_{\alpha_1 \alpha_2}(x-1, x) \] (47)

and (18) with the master equation (4) introduce the reactions as follows:
\[ A_{\alpha} \emptyset \rightarrow \emptyset A_{\alpha} \quad \text{with rate } v_{\alpha}, \]
\[ A_{\alpha} A_{\beta} \emptyset \rightarrow \emptyset A_{\beta} A_{\alpha} \quad \text{with rate } b_{\alpha \beta}, \]
\[ A_{\alpha} A_{\beta} \emptyset \rightarrow \emptyset A_{\beta} \emptyset \quad \text{with rate } \delta_{\alpha \beta}, \]
\[ A_{\alpha} A_{\beta} \emptyset \rightarrow A_{\alpha} \emptyset \emptyset \quad \text{with rate } \gamma_{\alpha \beta}, \]
\[ A_{\alpha} A_{\beta} \emptyset \rightarrow \emptyset \emptyset \emptyset \quad \text{with rate } \eta_{\alpha \beta}. \] (48)

In brief, one can find the solvable systems corresponding to \( b \) if the matrix \( b \) satisfies a constraint (18) and \( \tilde{b} \) satisfies (44), and all the permissible elements of \( b \) (including the diagonal) are nonnegative.

For more-than-two-particle systems, we now consider a system consisting of \( N \) particles of various species, with the evolution equation
\[ \dot{P}(x_1, \ldots, x_N) = v_{\alpha_1} P(x_1 - 1, \ldots, x_N) + \cdots + v_{\alpha_N} P(x_1, \ldots, x_N - 1) \]
\[ - (v_{\alpha_1} + \cdots + v_{\alpha_N}) P(x_1 + \cdots + x_N) \] (49)
in the whole physical region and the boundary condition
\[ \dot{P}(\ldots, x_k = x, x_{k+1} = x, \ldots) = \tilde{b}_{k,k+1} P(\ldots, x_k = x - 1, x_{k+1} = x, \ldots), \] (50)
where \( \tilde{b}_{k,k+1} \) is defined like (34) and \( P \) is an \( N \) tensor, the components of which are probabilities. It is seen that in this system, apart from the simple diffusion, there is a reaction between a block of \( n+1 \) adjacent particles
\[ A_{\alpha_0} \cdots A_{\alpha_n} \emptyset \rightarrow \emptyset A_{\gamma_0} \cdots A_{\gamma_n} \quad \text{with rate } (\tilde{b}_{n-1,n} \cdots \tilde{b}_{0,1})_{\gamma_0 \cdots \gamma_n}, \] (51)
which comes from the fact that
\[ P(x_0 = x, \ldots, x_{n-1} = x + n - 1, x_n = x + n - 1) \]
\[ = (\tilde{b}_{n-1,n} \cdots \tilde{b}_{0,1}) P(x_0 = x - 1, \ldots, x_{n-1} = x + n - 2, x_n = x + n - 1). \] (52)

Note the order of the matrices \( \tilde{b} \). This order suggests that, if a collection of \( n+1 \) particles are adjacent, there is a probability that the first particle pushes the second and changes the type of the second (and itself) and then it is the second (modified) particle that interacts with the third.

5. Two-particle system and exact solution

Let us focus on the type 1 model. We consider the \( p = 2 \) case, but the argument can be easily applied to arbitrary \( p \). Taking \( A_1 = A \) and \( A_2 = B \), one may label the two-particle states \((\alpha_1, \alpha_2)\) as follows:
\[ |1\rangle = (1, 1), \quad |2\rangle = (1, 2), \quad |3\rangle = (2, 1), \quad |4\rangle = (2, 2). \] (53)
The most general matrix $c$ that satisfies (17) can be written as

$$
c = \begin{pmatrix}
  v_1 - \lambda_1 & 0 & 0 & 0 \\
  0 & v_1 - \lambda_2 - c_{32} & c_{23} & 0 \\
  0 & c_{32} & v_2 - \lambda_3 - c_{23} & 0 \\
  0 & 0 & 0 & v_2 - \lambda_4
\end{pmatrix}
$$

(54)

and therefore $\tilde{c}$ is (see (31))

$$
\tilde{c} = \begin{pmatrix}
  v_1 - \lambda_1 & 0 & 0 & 0 \\
  0 & v_1 - \lambda_2 - c_{32} & \left(\frac{v_1}{v_2}\right)c_{23} & 0 \\
  0 & \left(\frac{v_2}{v_1}\right)c_{32} & v_2 - \lambda_3 - c_{23} & 0 \\
  0 & 0 & 0 & v_2 - \lambda_4
\end{pmatrix},
$$

(55)

where, according to (20), $\lambda_i = \delta_i + \gamma_i + \eta_i$. Now, we must check under what conditions $\tilde{c}$ satisfies (43). If one writes (43) as $RHS - LHS = 0$, then one has 64 equations that must be solved for 16 variables. By using a symbolic manipulator (e.g. MAPLE), one obtains the most general solution. We express the class of parameters of this solution in the following two sets as the given parameters (Gp) and the arbitrary parameters (Ap):

$$
\begin{align*}
\text{Gp} : & \{ c_{32} = 0, \delta_1 = (\delta_2 + \gamma_2 + \eta_2) - (\gamma_1 + \eta_1) \}, \\
\text{Ap} : & \{ v_1, v_2, c_{23}, (\gamma_1, \eta_1), (\delta_2, \gamma_2, \eta_2), (\delta_3, \gamma_3, \eta_3), (\delta_4, \gamma_4, \eta_4) \}.
\end{align*}
$$

(56)

Since all reaction rates must be nonnegative, for $\delta_1 \geq 0$, one should choose $(\lambda_2, \gamma_1, \eta_1)$ as $\lambda_2 \geq (\gamma_1 + \eta_1)$. The interactions introduced by (54) are

$$
\begin{align*}
A\emptyset & \overset{\delta_1}{\rightarrow} \emptyset A \\
B\emptyset & \overset{\gamma_1}{\rightarrow} \emptyset B \\
BA & \overset{c_{23}}{\rightarrow} AB \\
AA & \overset{\delta_2}{\rightarrow} A \emptyset \\
AA & \overset{\gamma_2}{\rightarrow} A \emptyset \\
AA & \overset{\eta_2}{\rightarrow} \emptyset \emptyset \\
AB & \overset{\delta_3}{\rightarrow} \emptyset B \\
AB & \overset{\gamma_3}{\rightarrow} A \emptyset \\
AB & \overset{\eta_3}{\rightarrow} \emptyset \emptyset \\
BA & \overset{\delta_4}{\rightarrow} \emptyset A \\
BA & \overset{\gamma_4}{\rightarrow} B \emptyset \\
BA & \overset{\eta_4}{\rightarrow} \emptyset \emptyset.
\end{align*}
$$

(57)

The model built on the reactions (57) is integrable. This is a remarkable model because the rate of all the interactions is arbitrary except $AA \rightarrow \emptyset \emptyset$, where the rate of this process, $\delta_1$, depends on some other processes.
Some subcases of this model

In the above model, if we abandon the annihilation processes ($\lambda_i = 0$), we have the first subcase as

\[
\begin{align*}
A\emptyset &\overset{v_1}{\rightarrow} \emptyset A \\
B\emptyset &\overset{v_2}{\rightarrow} \emptyset B \\
BA &\overset{v_{23}}{\rightarrow} AB,
\end{align*}
\]

which is the generalization of the reactions studied in [27] in which $c_{23} = v_2 - v_1$. It has been showed in [26] that, in the case of the single-species particles, each particle has a distinct hopping rate according to its site instead of its type (a particle in the position $x_i$ has a hopping rate $v_i$). Now if we consider a single-species system including the annihilation processes, the boundary condition is

\[
v_2 P(x, x) = (v_1 - \lambda) P(x, x + 1)
\]

with

\[
\lambda = \delta + \gamma + \eta
\]

and the reactions are

\[
\begin{align*}
A\emptyset &\overset{v_1}{\rightarrow} \emptyset A \\
AA &\overset{\delta}{\rightarrow} \emptyset A \\
AA &\overset{\gamma}{\rightarrow} A\emptyset \\
AA &\overset{\eta}{\rightarrow} \emptyset \emptyset
\end{align*}
\]

the reactions (61) make the second subcase. This subcase with $v_1 = 1$ and $\eta = 0$ has been considered in [28].

In the two-species case for the type 2 model, the most general matrix $b$ that satisfies (18) and $\tilde{b}$, which is constructed from the matrix $b$, can be written as

\[
b = \begin{pmatrix}
v_1 - \lambda_1 & 0 & 0 & 0 \\
0 & v_1 - \lambda_2 - b_{23} & b_{23} & 0 \\
0 & b_{23} & v_2 - \lambda_3 - b_{23} & 0 \\
0 & 0 & 0 & v_2 - \lambda_4
\end{pmatrix}
\]

and

\[
\tilde{b} = \begin{pmatrix}
v_1 - \lambda_1 \\
0 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
v_1 - \lambda_2 - b_{23} & b_{23} \\
b_{23} & v_2 - \lambda_3 - b_{23} \\
0 & 0 \\
0 & 0 & v_2 - \lambda_4
\end{pmatrix}
\]

The matrix $\tilde{b}$ satisfies (44) if we have two sets of Gp and Ap as

\[
\text{Gp : } \left\{ b_{23} = 0, \delta_1 = \frac{v_1}{v_2} (\delta_3 + \gamma_3 + \eta_3) - (\gamma_1 + \eta_1) \right\},
\]

\[
\text{Ap : } \{ (v_1, v_2, b_{23}, \gamma_1, \eta_1), (\delta_2, \gamma_2, \eta_2), (\delta_3, \gamma_3, \eta_3), (\delta_4, \gamma_4, \eta_4) \}.
\]
This case is different from that case we considered in the type 1 model. In the matrix \( c \) of the type 1 model the diagonal elements do not describe any interactions but in this model all elements of the matrix \( b \) are reaction rates. For example, \( c_{11} \) is not a reaction rate but \( b_{11} = v_1 - \lambda_1 \) is the reaction rate of \( AA\emptyset \rightarrow \emptyset AA \). So one must choose the arbitrary parameters in such a manner that all elements of \( b \) (reaction rates) are nonnegative.

6. The conditional probability

Assume that the solvability condition (43) or (44) is satisfied. If at \( t = 0 \) we have the initial configuration \((y_1, \ldots, y_N)\), we require

\[
P(\vec{x}; t = 0) = \delta_{\vec{x}, \vec{y}}
\]

and the probability \( P(\vec{x}; t) \) becomes the conditional probability (the propagator) \( U(\vec{x}; t|\vec{y}; 0) \) and thus a complete solution of the problem. It seems that the propagator is

\[
U(\vec{x}; t|\vec{y}; 0) = \prod_{j=1}^{N} v_{x_j} x_j e^{-v_j t} \int \frac{d^N p}{(2\pi)^N} f(\vec{p}) e^{-E_N t} \sum_{\sigma} A_{\sigma} e^{i\sigma(\vec{p})\cdot \vec{x}}
\]

where the integration region for each \( p_i \) is \([0, 2\pi]\) and \( A_1 = 1 \). In the above expansion \( f(\vec{p}) \) is the coefficient of expansion which in the second equality we choose \( \prod_{j=1}^{N} v_{x_j} x_j e^{-i\vec{p}\cdot \vec{y}} \), and by setting \( p_j \to p_j + i\varepsilon \), where one should consider the limit \( \varepsilon \to 0^+ \), (66) reproduces the required initial condition (65). Using this propagator, one can of course write the probability at time \( t \) in terms of the initial value of the probability

\[
|P(\vec{x}; t)\rangle = \sum_{\vec{y}} U(\vec{x}; t|\vec{y}; 0)|P(\vec{y}; 0)\rangle.
\]

Note that, although \( S^{(1)} \) and \( S^{(2)} \) are similar to ones considered in [13, 14], the propagators \( U^{(1)} \) and \( U^{(2)} \) are different since the energy spectrum and the definition of the propagator of our models differ from those considered there. For the two-particle sector, like reactions (57), there is only one matrix in the expression of \( U^{(1)} \) (\( \hat{c} \) in \( U^{(1)} \) and \( \hat{b} \) in \( U^{(2)} \)). So it can be treated as a \( c \)-number. Following [13, 14] and [17], and by using (66), (36) and (37), the propagator for the two-particle system of the type 1 model is

\[
U^{(1)}(\vec{x}; t|\vec{y}; 0) = \prod_{j=1}^{2} v_{x_j} x_j e^{-v_j t} \int \frac{d^2 p}{(2\pi)^2} e^{-E_2 t - i\vec{p}\cdot \vec{y}}
\]

\[
\times \left[ e^{i(p_1 x_1 + p_2 x_2)} - \frac{1 - e^{i p_2 x_2}}{1 - e^{i p_1 x_1}} e^{i(p_2 x_1 + p_1 x_2)} \right],
\]

doi:10.1088/1742-5468/2009/02/P02001
where $E_2$ is obtained from (23). Using the variable $x_k = e^{i p_k}$, a simple contour integration yields

$$
U^{(1)}(x; t|y; 0) = \prod_{j=1}^{d} \frac{v_j^{x_j-y_j} e^{-v_j t}}{(x_1 - y_1)! (x_2 - y_2)!} \\
+ \sum_{l=1}^{\infty} \frac{t^{x_2-y_1+l}}{(x_2 - y_1 + l)! (x_1 - y_2)!} \tilde{c} \left( -1 + \frac{t \tilde{c}}{x_1 - y_2 + 1} \right)
$$

and in the same way, for a two-particle system of the type 2 model, one arrives at

$$
U^{(2)}(x; t|y; 0) = \prod_{j=1}^{d} \frac{v_j^{x_j-y_j} e^{-v_j t}}{(x_1 - y_1)! (x_2 - y_2)!} \\
+ \sum_{l=1}^{\infty} \frac{t^{x_2-y_1-l}}{(x_2 - y_1)! (x_1 - y_2 - l)!} \tilde{b} \left( -1 + \frac{t \tilde{b}}{x_2 - y_1 + 1} \right)
$$

Now we concentrate on the type 1 model or $U^{(1)}$. As we said, the most general $\tilde{c}$ for the two-species system is the matrix (55) with elements that are determined by (56). So we have

$$
\tilde{c} = \begin{pmatrix}
v_1 - \lambda_2 & 0 & 0 & 0 \\
0 & v_1 - \lambda_2 & \left( \frac{v_2}{v_1 c_23} \right) & 0 \\
0 & 0 & v_2 - \lambda_3 - c_23 & 0 \\
0 & 0 & 0 & v_2 - \lambda_4
\end{pmatrix}
$$

Note that only the eigenvalues of $\tilde{c}$ with modulus 1 cause singularities in $A_{\sigma}$ at $p_j = 0$ and therefore in the integrand (66). Corresponding to matrix (71), it is obvious that the eigenvalues of this matrix are

$$
v_1 - \lambda_2, \\
v_1 - \lambda_2, \\
v_2 - \lambda_3 - c_23, \\
v_2 - \lambda_4.
$$

Since all parameters that construct eigenvalues are arbitrary, the modulus of eigenvalues can be 1 or different from 1. To investigate the large-time behavior of the propagator $U^{(1)}$, it is useful to decompose the vector space on which $\tilde{c}$ acts into two subspaces invariant under the action of $\tilde{c}$, the first subspace corresponding to eigenvalues with modulus 1 and the second invariant subspace corresponding to eigenvalues with modulus different from 1. This decomposition can be done by introducing two projections $Q$ and $R$, satisfying

$$
Q + R = 1, \\
QR = RQ = 0, \\
[\tilde{c}, Q] = [\tilde{c}, R] = 0.
$$

$Q$ projects on the first subspace and $R$ projects on the second. Following [13], we multiply $U^{(1)}$ by $Q + R = 1$:

$$
U^{(1)}(x; t|y; 0) = U^{(1)} Q + U^{(1)} R.
$$
In the term multiplied by $R$, one can treat $\tilde{c}$ as a number with modulus different from 1. Thus in this term, there is no pole in $S^{(1)}$ and hence in $A_\sigma$. So the integrand in (66) is nonsingular at $p_j = 0$, which has the main contributions at large times. Setting $p_j = 0$ in $S^{(1)}$ as an approximation to arrive at

$$S^{(1)} \approx -1, \quad (75)$$

and

$$A_\sigma \approx (-1)^{|\sigma|}. \quad (76)$$

One can also approximate $E$ as

$$E(p) \approx \sum_{j=1}^{2} \left( -1 + ip_j + \frac{p_j^2}{2} \right). \quad (77)$$

So, the second term of (74) for large times results in

$$U^{(1)} R = \prod_{j=1}^{2} v_j^{x_j - y_j} e^{-v_j t} \left\{ e^{-[(x_1-y_1(t)^2+(x_2-y_2(t))^2/2t]} - \frac{1}{2\pi t} \right\} R \quad t \to \infty, \quad (78)$$

which is independent of $\tilde{c}$. So at large times the second term tends to zero faster than $t^{-1}$ and only the first term of (74) survives.

If one chooses the class of parameters in such a manner that the only eigenvalue of $\tilde{c}$ with modulus 1 is 1, then $U^{(1)}$ has a simple behavior for $t \to \infty$. In this case, $\tilde{c}Q = Q$, and one can simplify $U^{(1)}$ to find

$$U^{(1)}(x; t; y; 0) = \prod_{j=1}^{2} v_j^{x_j - y_j} e^{-v_j t} \left\{ \frac{t^{x_1-y_1}}{(x_1-y_1)! (x_2-y_2)!} - \frac{t^{x_2-y_2}}{(x_2-y_1+t)! (x_1-y_2)!} \right\} Q \quad t \to \infty. \quad (79)$$

This is the propagator corresponding to a single-species asymmetric simple exclusion process with the PDHR multiplied by $Q$. In fact (79) is the case of the model in [26], times $Q$.

### 7. Conclusion

We have defined two generalized totally asymmetric exclusion processes, in which we have $p$-species particles that interact and diffuse where each particle $A_\alpha$ has its own intrinsic hopping rate $v_\alpha$. We have introduced two types of boundary conditions in terms of two $p^2 \times p^2$ matrices $c$ and $b$, respectively, that led to two new models, i.e. reaction–diffusion and drop–push systems with the PDHR. At first we have assumed that these interactions preserve the total number of particles, so we have obtained the constraint $\sum_\alpha F^{\beta_1 \beta_2}_{\alpha_1 \alpha_2} = v_{\beta_1}$ (here and elsewhere $F$ denotes either $b$ or $c$). Next we have shown that, when one violates the conservation of particles, the mentioned constraint changes into $\sum_\alpha F^{\beta_1 \beta_2}_{\alpha_1 \alpha_2} = v_{\beta_1} - \lambda_{\beta_1 \beta_2}$.
and the annihilation processes add to the previous reactions of new models, where $\lambda_{\beta_1,\beta_2}$ is the sum of all annihilation processes. We have also shown that, only for the states $\vec{\beta} = \vec{\alpha}$ and $\vec{\beta} = (\alpha_2, \alpha_1)$, $F$ introduces the new model with the PDHR and has non-zero elements. These states resulted in that the matrix $\tilde{F}$, a version of $F$ that constructs the $S$-matrix and determines the coefficient $A_\sigma$, must satisfy the non-spectral equation matrix for solvability of our models in the sense of the Bethe ansatz. Then we have considered the two-particle systems and obtained the permissible reaction rates by checking the solution of the non-spectral matrix equation. Finally the conditional probability of our new models has been obtained and, for the two-particle sector, the exact expression and especially its large-time behavior have been calculated. We have shown that the propagator of a two-species reaction–diffusion system with the PDHR in large times is equivalent to the propagator of the single-species asymmetric simple exclusion process with the PDHR.

References

[1] Derrida B, Janowsky S A, Lebowitz J L and Speer E R, 1993 Europhys. Lett. 22 651
[2] Ferrari P A and Fontes L R G, 1994 Probab. Theory Relat. Fields 99 305
[3] Burgers J M, 1974 The Nonlinear Diffusion Equation (Boston, MA: Reidel)
[4] Krug J and Spohn H, 1991 Solids Far From Equilibrium ed C Godreche (Cambridge: Cambridge University Press)
[5] Nagel K, 1996 Phys. Rev. E 53 4655
[6] MacDonald C T, Gibbs J H and Pipkin A C, 1968 Biopolymers 6 1
[7] Schütz G M, 1997 J. Stat. Phys. 88 427
[8] Alimohammadi M, Karimipour V and Khorrami M, 1998 Phys. Rev. E 57 6370
[9] Sasamoto T and Wadati M, 1998 Phys. Rev. E 58 4181
[10] Sasamoto T and Wadati M, 1998 J. Phys. A: Math. Gen. 31 6057
[11] Povolotsky A M, Priezzhev V B and Hu C-K, 2003 J. Stat. Phys. 111 1149
[12] Alimohammadi M and Ahmadi N, 2000 Phys. Rev. E 62 1674
[13] Roshani F and Khorrami M, 2001 Phys. Rev. E 64 011101
[14] Roshani F and Khorrami M, 2003 Eur. Phys. J. B 36 99
[15] Alimohammadi M, 2004 Eur. Phys. J. B 42 415
[16] Alimohammadi M, Karimipour V and Khorrami M, 1999 J. Stat. Phys. 97 373
[17] Alimohammadi M and Naimi Y, 2005 J. Math. Phys. 46 053306
[18] Benjamini I, Ferrari P A and Landim C, 1996 Stoch. Process. Appl. 61 181
[19] Evans M R and Hanney T, 2005 J. Phys. A: Math. Gen. 38 R195
[20] Grosskinsky S, Schütz G M and Spohn H, 2003 J. Stat. Phys. 113 389
[21] Godrèche C, 2003 J. Phys. A: Math. Gen. 36 6313
[22] Levine E, Mukamel D and Schütz G M, 2005 J. Stat. Phys. 120 759
[23] Kaupuzs J, Mahnke R and Harris R J, 2005 Phys. Rev. E 72 056125
[24] Harris R J, Rákos A and Schütz G M, 2005 J. Stat. Mech. P08003
[25] Godrèche C and Luck J M, 2005 J. Phys. A: Math. Gen. 38 7215
[26] Rákos A and Schütz G M, 2006 Markov Process. Relat. Fields 12 323
[27] Karimipour V, 1999 Europhys. Lett. 47 501
[28] Roshani F and Khorrami M, 2002 J. Math. Phys. 43 2627

doi:10.1088/1742-5468/2009/02/P02001