DISSIPATIVITY AND POSITIVE OFF-DIAGONAL PROPERTY OF OPERATORS ON ORDERED BANACH SPACES

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Abstract. In this paper, we provide a sublinear function \( p \) on ordered Banach spaces, which depends on the order structure of the space. With respect to this \( p \), we study the relation between \( p \)-contractivity of positive semigroups and the \( p \)-dissipativity of its generators. The positive off-diagonal property of generators is also studied in ordered vector spaces.

1. Introduction

It is known that on a Banach space \( X \), the positivity and the contractivity of a semigroup \( (T(t))_{t \geq 0} \) can be characterized by means of dissipativity of its generator \( A \) with respect to some appropriate sublinear functions, see [2, Theorem 2.6]. This property is also proved in the case of \( X \) being an ordered Banach space with normal positive cone and with respect to a canonical half-norm on \( X \), see [5, Proposition 7.13]. In Banach lattices, moreover, the positivity of \( (T(t))_{t \geq 0} \) can be characterized by its generator \( A \) which satisfies the “positive off-diagonal” property (it is called “positive minimum principle” in [3, Theorem 1.11]). This is also studied in an ordered Banach space with a positive cone with nonempty interior, [5, Theorem 7.27]. For more positive off-diagonal properties of operators on ordered normed spaces, we refer the reader to [4, 6, 7, 10].

The goal of this paper is to investigate the positivity and contractivity of semigroups through the dissipativity of generators with respect to corresponding sublinear functions on ordered vector spaces. So the choice of sublinear functions on ordered vector spaces is crucial. In Section 3 we give two different ways of defining the sublinear functions on ordered vector spaces. One is given by a regular norm and the other one is obtained through a dual function. The latter one turns out to be more effective in studying the positivity and contractivity of semigroups on ordered Banach spaces in Section 4, these are also our main results. Moreover, in Section 5 we will show a representation for positive functions on Archimedean pre-Riesz spaces, which will be used to study the positive off-diagonal property of operators on the pre-Riesz space \( C^1[0,1] \).

2. Preliminaries

In this section, we will collect some basic terminology, which we mainly refer to [1, 2, 9]. Let \( X \) be a real vector space, and \( K \subseteq X \). \( K \) is said to be a cone in \( X \) if \( x, y \in K \), \( \lambda, \mu \in \mathbb{R}^+ \) implies \( \lambda x + \mu y \in K \) and \( K \cap (-K) = \{0\} \). A partial order in \( X \)
is induced by $x \leq y$ whenever $y - x \in K$, we then say $(X, K)$ is a (partially) ordered vector space. $(X, K, \| \cdot \|)$ is called an ordered Banach space whenever it is norm complete. The space of all bounded linear operators on $X$ is denoted by $L(X)$, the positive operators in $L(X)$ are $L(X)^+ := \{ T \in L(X) : Tx \geq 0, \forall 0 \leq x \in X \}$. Then $(L(X), L(X)^+)$ is an ordered vector space. The domain of an operator $T$ on $X$ is denoted by $\mathcal{D}(T)$. A one-parameter semigroup of operators is written usually as $(T(t))_{t \geq 0}$, and it is called strongly continuous if the map $t \mapsto T(t)$ is continuous for the strong topology on $L(X)$.

Let $(X, K, \| \cdot \|)$ be an ordered normed vector space. A function $p$ on $X$ is called sublinear if $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$ and $p(\lambda x) = \lambda p(x)$ for all $x \in X$, $\lambda \geq 0$. It is clear that $p(0) = 0$. If a sublinear function $p$ on $X$ satisfies $p(x) + p(-x) > 0$ whenever $x \neq 0$ in $X$, then $p$ is called a half-norm. If there exists a constant $M > 0$ such that $p(x) + p(-x) \geq M\|x\|$ for all $x \in X$, then $p$ is called a strict half-norm. Let $x \in X$, define

$$\psi(x) = \text{dist}(-x, K) = \inf\{\|x + y\| : y \in K\}$$

(2.1)

$$= \inf\{\|y\| : x \leq y, y \in X\},$$

then $\psi$ is a half-norm in $X$. This half-norm $\psi$ is called the canonical half-norm. A seminorm $p$ on $X$ is called regular if for all $x \in X$ one has

$$p(x) := \inf\{p(y) : y \in X \text{ such that } -y \leq x \leq y\}.$$ 

A regular seminorm on an ordered vector space will be denoted by $\| \cdot \|_r$. If $(X, K)$ is an ordered vector space equipped with a seminorm $p$, then for $x \in X$, $\|x\|_r := \inf\{p(y) : y \in X, -y \leq x \leq y\}$ is called the regularization of the original seminorm $p$. Moreover, $\| \cdot \|_r$ is a seminorm, and $\| \cdot \| \leq p(\cdot)$ on $K$. A seminorm $p$ on $X$ is called monotone if for every $x, y \in X$ such that $0 \leq x \leq y$ one has that $p(x) \leq p(y)$. If $(M, \| \cdot \|)$ is any normed vector space, we denote by $M'$ the continuous linear functionals on $M$.

Let $(X, K, \| \cdot \|)$ be an ordered normed vector space, $p$ be a sublinear function on $X$. A bounded operator $T$ on $X$ is called $p$-contractive if $p(Tx) \leq p(x)$ for all $x \in X$. Similarly, a semigroup $(T(t))_{t \geq 0}$ is called $p$-contractive if $T(t)$ is $p$-contractive for all $t \geq 0$. The subdifferential of $p$ at point $x \in X$, denoted by $dp(x)$, is defined as

$$dp(x) = \{ x' \in K' : \langle y, x' \rangle \leq p(y) \text{ for all } y \in X, \langle x, x' \rangle = p(x) \}.$$ 

Definition 2.1. An operator $A : D(A) \subseteq X \to X$ is called $p$-dissipative if for all $x \in D(A)$ there exists $y' \in dp(x)$ such that $\langle Ax, y' \rangle \leq 0$; $A$ is called strictly $p$-dissipative if for all $x \in D(A)$ the inequality $\langle Ax, y' \rangle \leq 0$ holds for all $y' \in dp(x)$.

Let $(X, K, \| \cdot \|)$ be a Banach lattice, $p$ be a canonical half-norm on $X$. We have that $p(x) = \|x^+\|$ for all $x \in X$. In fact, let $x \in X$, $p(x) \leq \|x^+\|$ is obvious. Since $|x + y| \geq \langle x + y, x' \rangle \geq x^+$ for any $0 \leq y \in X$, we have that $\|x + y\| \geq \|x^+\|$. So $p(x) \geq \|x^+\|$. To make a difference from the above, we will use distinctive notations. Let $N^+ : X \to \mathbb{R}$ be the function given by $N^+(x) = \|x^+\|$. Then the subdifferential of $N^+$ at point $x \in X$ is

$$dN^+(x) = \{ x' \in K' : \|x'\| \leq 1, \langle x, x' \rangle = \|x^+\| \}.$$ 

An operator $T$ on $X$ is called (strictly) dispersive if $T$ is (strictly) $N^+$-dissipative. We continue by some notations of pre-Riesz theory. An ordered vector space $(X, K)$ is called directed if for every $x, y \in X$ there exists $z \in X$ such that $z \geq x, z \geq y$, and is called Archimedean if for every $x, y \in X$ with $nx \leq y$ for all $n \in \mathbb{N}$ one
has $x \leq 0$. We say that $X$ is a pre-Riesz space if for every $x, y, z \in X$ the inclusion $(x + y, x + z) \subseteq (y, z)$ implies $x \in K$, where $(x, y) = \{z \in X : z \geq x, z \geq y\}$.

Every directed Archimedean ordered vector space is pre-Riesz [9]. A linear map $T: X \rightarrow Y$ between two ordered vector spaces is called bipositive if for every $x \in X$ one has $x \geq 0$ if and only if $i(x) \geq 0$.

3. Half-norms

In this section, we will introduce two different seminorms on ordered vector spaces. One is induced by the regular norms, and the other one involves the order structure which will be used in the following section.

**Proposition 3.1.** Let $(X, K, \| \cdot \|)$ be an ordered normed vector space, and $\| \cdot \|_r$ the regularization of $\| \cdot \|$, and let $p$ be defined as

$$p(x) = \inf \{ \|y\|_r : y \in X, y \geq 0, y \geq x\}, \forall x \in X.$$  \hspace{1cm} (3.1)

Then $p$ is a strict half-norm on $X$. Moreover, if $\psi$ on $X$ be defined by $\psi(x) = \inf \{\langle y, x \rangle : y \in K, y \geq 0, y \geq x\}$, in particular, $p(x) = \psi(x) = 0$ for $x \in (-K)$.

**Proof.** Let $x \neq 0$ be in $X$. Since $\| \cdot \|_r$ is the regularization of $\| \cdot \|$, we have that $\|x\|_r := \inf \{\|y\| : y \in X, y \leq x \leq y\}$ with $y \neq 0$. As $\| \cdot \|_r$ is a norm, $\|y\|_r \neq 0$. So we have that $\|x\|_r > 0$. If $y \geq 0, y \geq x$ and $z \geq 0, z \geq -x$, then $-(y + z) \leq x \leq (y + z)$. Hence, $\|x\|_r \leq \|y + z\|_r$. So $\|x\|_r \geq \|y + z\|_r \geq \|x\|_r > 0$.

Hence, by passing to the infimum in each term we get $p(x) + p(-x) \geq \|x\|_r > 0$. Thus $p$ is a strict half-norm.

Moreover, if $0 \leq x \in X$, we can take $y = x \geq 0$ in (3.1), then $p(x) = \|x\|_r = \psi(x)$. It is clear that $p(x) = \psi(x) = 0$ for $x \leq 0$ in $X$. \hfill $\square$

**Proposition 3.2.** Let $(X, K, \| \cdot \|)$ be an ordered normed vector space, and $\| \cdot \|_r$, the regularization of $\| \cdot \|$, and let $p$ be defined by (3.1). If $T \in L(X)^+$ is a contractive operator with respect to the regular norm $\| \cdot \|_r$, then $T$ is $p$-contractive.

**Proof.** Let $T \in L(X)^+$. By the assumption, we have that $\|Ty\|_r \leq \|y\|_r$ for all $0 \leq y \in X$. Then for $x \in X$, $p(x) = \inf \{\|y\|_r : y \geq 0, y \geq x\} \geq \inf \{\|Ty\|_r : y \geq 0, y \geq x\}$. Since $T$ is a positive operator, we have that $y \geq 0, y \geq x$ implies $Ty \geq 0, Ty \geq Tx$. Hence, $\inf \{\|Ty\|_r : y \geq 0, y \geq x\} \geq \inf \{\|Ty\|_r : Ty \geq 0, Ty \geq Tx\} = p(Tx)$. So $T$ is $p$-contractive. \hfill $\square$

We continue with a different approach of sublinear functions on ordered vector spaces, which turns out to be more useful in dealing with the contractivity and positivity of semigroups.

Let $\phi \in X'$, define $p_\phi$ on $X$ by

$$p_\phi(x) = \inf \{\langle y, \phi \rangle : y \in X, y \geq 0, y \geq x\}, \forall x \in X.$$  \hspace{1cm} (3.2)

**Proposition 3.3.** Let $X$ be an ordered vector space, and let $\phi \in X'$. If $p_\phi$ is defined by (3.2), then $p_\phi$ is sublinear. Moreover, $p_\phi(x) = \langle x, \phi \rangle$ if $x \geq 0$, and $p_\phi(x) = 0$ if $x \leq 0$. 

Proof. Let \( \phi \in X' \), and \( x, y \in X \). By the definition, we have that \( p_\phi(x + y) = \inf \{ \langle z, \phi \rangle: z \in X, z \geq 0, z \geq x + y \} \). Let \( z_1 + z_2 = z \), we have that \( \inf \{ \langle z, \phi \rangle: z \in X, z \geq 0, z \geq x + y \} = \inf \{ \langle z_1 + z_2, \phi \rangle: z_1 + z_2 \in X, z_1 + z_2 \geq 0, z_1 + z_2 \geq x + y \} \). Since \( z_1 \in X, z_2 \in X, z_1 \geq 0, z_2 \geq 0, z_1 \geq x, z_2 \geq y \) implies \( z_1 + z_2 \in X, z_1 + z_2 \geq 0, z_1 + z_2 \geq x + y \). We have that \( \inf \{ \langle z_1 + z_2, \phi \rangle: z_1 + z_2 \in X, z_1 + z_2 \geq 0, z_1 + z_2 \geq x + y \} \leq \inf \{ \langle z_1 + z_2, \phi \rangle: z_1 \in X, z_2 \in X, z_1 \geq 0, z_2 \geq 0, z_1 \geq x, z_2 \geq y \} \). It is clear that \( \inf \{ \langle z_1 + z_2, \phi \rangle: z_1 \in X, z_2 \in X, z_1 \geq 0, z_2 \geq 0, z_1 \geq x, z_2 \geq y \} = \inf \{ \langle z_1, \phi \rangle: z_1 \in X, z_1 \geq 0, z_1 \geq x \} + \inf \{ \langle z_2, \phi \rangle: z_2 \in X, z_2 \geq 0, z_2 \geq y \} = p_\phi(x) + p_\phi(y) \). So we have the subadditivity. The positive homogeneity of \( p_\phi \) is obvious. So \( p_\phi \) is sublinear.

It is clear that \( p_\phi(x) = \langle x, \phi \rangle \) for \( x \geq 0 \), and \( p_\phi(x) = 0 \) for \( x \leq 0 \). □

**Proposition 3.4.** If \( X \) is a vector lattice, \( \phi \in X' \) and \( p_\phi \) is defined by (3.2). Then \( p_\phi(x^+) = p_\phi(x) \) for \( x \in X \).

**Proof.** Let \( X \) be a vector lattice, \( \phi \in X' \) and \( x \in X \). By (3.2), we have

\[
p_\phi(x^+) = \inf \{ \langle y, \phi \rangle: y \geq x^+, y \geq 0 \}
= \inf \{ \langle y, \phi \rangle: y \geq x^+, y \geq 0 \}
\geq \inf \{ \langle y, \phi \rangle: y \geq x, y \geq 0 \} = p_\phi(x).
\]

Because \( y \geq x \) and \( y \geq 0 \) implies \( y \geq x^+ \), and \( \phi \) is monotone, we have that \( \{ \langle y, \phi \rangle: y \geq x, y \geq 0 \} \subseteq \{ \langle y, \phi \rangle: y \geq x^+, y \geq 0 \} \), and hence \( \inf \{ \langle y, \phi \rangle: y \geq x, y \geq 0 \} \geq \inf \{ \langle y, \phi \rangle: y \geq x^+, y \geq 0 \} \). So we have \( p_\phi(x) \geq p_\phi(x^+) \). □

We note that the continuous differential function space \( X = C^1[0,1] \) and the Sobolev space \( X = W^{n,p} \) are (pointwise) ordered vector spaces but not lattices. They are norm complete with respect to \( \| \cdot \|_\infty \) and \( \| \cdot \|_{W^{n,p}} \) respectively. However, these norms are not monotone, but one could still define a sublinear function \( p_\phi \) by (3.2). Next, we will study the contractivity of \( (T(t))_{t \geq 0} \) with respect to \( p_\phi \) given by (3.2) on ordered Banach space in the following section.

4. Contractivity and positivity of semigroups on ordered Banach spaces

In this section, we will study the relation between the \( p_\phi \)-contractivity of \( (T(t))_{t \geq 0} \) and the strictly \( p_\phi \)-dissipativity of its generator \( A \) for \( X \) being an ordered Banach space, the sublinear function \( p_\phi \) on \( X \) is defined by (3.2). Firstly, we will give a sufficient condition under which \( (T(t))_{t \geq 0} \) is contractive with respect to \( p_\phi \) on an ordered Banach space.

We will use \( X \) to denote the ordered Banach space in the following of this section, and \( p_\phi \) on \( X \) is defined by (3.2) with respect to \( \phi \in X' \).

**Theorem 4.1.** Let \( A: X \supseteq D(A) \rightarrow X \) be a \( p_\phi \)-dissipative operator. If \( (I - \lambda A) \) is invertible for some \( \lambda > 0 \), then \( (I - \lambda A)^{-1} \) is \( p_\phi \)-contractive.

**Proof.** For a fixed \( x \in D(A) \), let \( \psi \in d_{p_\phi}(x) \) be such that \( \langle Ax, \psi \rangle \leq 0 \). Then \( \langle y, \psi \rangle \leq p_\phi(y) \) for all \( y \in X \), and \( \langle x, \psi \rangle = p_\phi(x) \). Since \( \psi \) is positive on \( X \), we have that \( \langle y, \psi \rangle \leq p_\phi(y) \) for some \( \lambda > 0 \) one has that

\[
p_\phi((\lambda I - A)x) \geq \| (\lambda I - A)x, \psi \| \geq \text{Re}(\langle (\lambda I - A)x, \psi \rangle)
= \text{Re}(\langle Ax, \psi \rangle - (Ax, \psi)) \geq \text{Re}(Ax, \psi) = \text{Re}(\lambda x, \psi) = \lambda p_\phi(x) = \lambda p_\phi(x).
\]
Consider dissipative for all $\lambda > 0$. So if $(I - \lambda A)^{-1}$ is invertible, then $(I - \lambda A)^{-1}$ is $p_\phi$-contractive for some $\lambda > 0$.

Observe that $(I - \lambda A)^{-1} = \frac{1}{\lambda} (\frac{1}{\lambda} I - A)^{-1} = \frac{1}{\lambda} R(\frac{1}{\lambda}, A)$.

If a strongly continuous semigroup $(T(t))_{t \geq 0}$ is generated by the operator $A$, then for $x$ in $X$, one has
\[
T(t)x = \lim_{n \to \infty} \left[ \frac{n}{t} R(\frac{n}{t}, A) \right]^n x = \lim_{n \to \infty} \left( I - \frac{t}{n} A \right)^{-n} x.
\]

Hence, by Theorem 4.1, if $A$ is $p_\phi$-dissipative, then $T(t)$ is $p_\phi$-contractive for every $t \geq 0$. As a consequence we have the following corollary.

**Corollary 4.2.** If a strongly continuous semigroup $(T(t))_{t \geq 0}$ is generated by a $p_\phi$-dissipative operator $A$, and $(I - \lambda A)$ is invertible for some $\lambda > 0$. Then $T(t)$ is $p_\phi$-contractive for every $t \geq 0$.

Next, we will study the positivity of strongly continuous semigroup $(T(t))_{t \geq 0}$ on an ordered Banach space. The generator $A$ of $(T(t))_{t \geq 0}$ is required to be $p_\phi$-dissipative for all $\phi$ in a total subset of an ordered Banach space.

**Definition 4.3.** A nonempty subset $\Phi \subseteq K'$ is called total in $X$ if $\phi(x) \geq 0$ for each $\phi \in \Phi$ implies $x \geq 0$.

The following example shows that the intersection of a subdifferential set and a total subset on an ordered Banach space $X = C^1[0, 1]$ is nonempty.

**Example 4.4.** Consider $X = C^1[0, 1]$, and $\Phi = \{\delta_t: t \in [0, 1]\}$, where $\delta_t(x) = x(t)$ for an arbitrary $x \in X$. Since $\delta_t(x) = x(t) \geq 0$ for all $t \in [0, 1]$ implies $x \geq 0$ in $X$. So $\Phi$ is total. We claim that for a given $t \in [0, 1]$ and $x \in X$, $dp_t(x) \cap \Phi \neq \emptyset$ for $p_t(x) = \inf\{\delta_t(y): y \geq 0, y \geq x\}$. In fact, let $t \in [0, 1]$ and $x \in X$, then $p_t(x) = \inf\{\delta_t(y): y \geq 0, y \geq x\} = \inf\{y(t): y \geq 0, y \geq x\} = x^+(t)$. (Here, as $X = C^1[0, 1]$ is a partially ordered vector space, the positive part is defined in such a way, $x^+ = \inf\{y(x): x \geq y, x \geq y\}, \forall x \in X$). So if $x(t) \leq 0$ then $p_t(x) = x^+(t) = 0$, and $dp_t(x) \cap \Phi = 0$. If $x \geq 0$, then $\delta_t(y) = y(t) \leq y^+(t) = p_t(y)$, and $\psi(x) = \delta_t(x) = x(t) = x^+(t) = p_t(x)$. So $\delta_t \in dp_t(x)$. Thus we have shown that $\Phi \subseteq dp_t(x)$.

**Theorem 4.5.** Let $A: X \supseteq D(A) \to X$ be the generator of strongly continuous semigroup $(T(t))_{t \geq 0}$. If $A$ is $p_\phi$-dissipative for all $\phi$ in a total set $\Phi$ and $(I - \lambda A)$ is invertible for some $\lambda \geq 0$, then $T(t)$ is positive for all $t \geq 0$.

**Proof.** Let $\phi \in \Phi$, select $x \leq 0$ in $X$. By Theorem 4.1, for the semigroup $(T(t))_{t \geq 0}$ is generated by the operator $A$ one has $p_\phi(T(t)x) \leq p_\phi(x)$ for all $t \geq 0$, which means
\[
\inf\{\langle y, \phi \rangle: y \geq T(t)x, y \geq 0\} \leq \inf\{\langle z, \phi \rangle: z \geq x, z \geq 0\}.
\]

Take $z = 0$, then the right side of the above inequality is $0$. It follows that
\[
\inf\{\langle y, \phi \rangle: y \geq T(t)x, y \geq 0\} \leq 0.
\]

Because of $y \geq T(t)x$, then $\langle y, \phi \rangle \geq \langle T(t)x, \phi \rangle$ for $\phi$ is positive. So
\[
\langle T(t)x, \phi \rangle \leq \inf\{\langle y, \phi \rangle: y \geq T(t)x, y \geq 0\} \leq 0.
\]

Since $\Phi$ is total, one has that $\langle T(t)x, \phi \rangle \leq 0$ for all $\phi \in \Phi$ implies $T(t)x \leq 0$. Thus $T(t)$ is positive. \qed
Remark 4.6. Due to [3] Theorem 1.2, if $A$ is densely defined on a Banach lattice, then $(T(t))_{t\geq 0}$ is positive and contractive if and only if $A$ is dispersive and $(\lambda I - A)$ is surjective for some $\lambda > 0$. By Theorem 4.1, we could generalize one direction of this conclusion to ordered Banach spaces. We will illustrate this through an example of a second derivative operator with Dirichlet boundary condition. This example originally comes from [3, Example 1.5].

Example 4.7. Let $X = (C^1[0,1], \| \cdot \|_\infty)$ be an ordered Banach space, the densely defined operator $A$ be the second derivative operator with Dirichlet boundary condition. Then the domain satisfies $D(A) = \{ x \in C^4[0,1] : x(0) = x(1) = x''(0) = x''(1) \}$. For $x \in X$, we choose $t \in [0,1]$ such that $p_{\delta t}(x) = \langle x, \delta_t \rangle = x(t) = \sup_{s \in [0,1]} x(s) = \| x \|_\infty$. Then $\langle y, \delta_t \rangle = y(t) \leq \| y \|_\infty = p_{\delta t}(y)$ for all $y \in X$. Hence $\delta_t \in dp_{\delta t}(x)$. Moreover, since $x$ has a maximum in $X$, we have that $\langle Ax, \delta_t \rangle = x''(t) \leq 0$. So $A$ is $p_{\delta_t}$-dissipative. For $y \in X$ define the function $x_0(t) = \frac{1}{2} [e^t \int_t^1 e^{-s} g(s) ds - e^{-t} \int_t^1 e^s g(s) ds]$. Then there exist $m, n \in \mathbb{R}$ such that $x(t) = x_0(t) + me^t + ne^{-t}$ and $x(0) = x(1) = 0$, and then $x \in D(A)$. Since $x-x'' = x_0 - x_0'' = y$, we have $(I-A)$ is surjective. For $x \in D(A)$, suppose that $(I-A)x = 0$, then $x(t) = \alpha e^t + \beta e^{-t}$. Notice that $x(0) = x(1) = 0$ such that $\alpha = \beta = 0$, so $x(t) = 0$ for $t \in [0,1]$. So $(I-A)$ is injective. It follows from Corollary 4.2 that $A$ is the generator of a contractive semigroup. By Theorem 4.5 $A$ generates a strongly continuous positive semigroup $T(t)_{t\geq 0}$.

Remark 4.8. Note that in an ordered Banach space, specifically $C^1[0,1]$, it is hard to give the definition of the dispersivity of an operator $A$ with respect to the original norm, because $C^1[0,1]$ is not a lattice. However, by the above discussion, we still have flexibility to choose a function as in [3,2], which depends on a function $\phi$ in $X'$. This is also different from the arguments in [3] Example 1.5.

5. Positive off-diagonal property of operators on ordered vector spaces

In this section, we will introduce the positive off-diagonal property especially on pre-Riesz spaces, in particular $C^1(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ open. Explicitly, we investigate a representation theorem for positive linear functionals on Archimedean pre-Riesz spaces, which is also interesting independently.

Definition 5.1. Let $X$ be an ordered vector space. A linear operator $A : D(A) \subseteq X \to X$ is said to have the positive off-diagonal property if $\langle Ax, \phi \rangle \geq 0$ whenever $0 \leq x \in D(A)$ and $0 \leq \phi \in X'$ with $\langle x, \phi \rangle = 0$.

The motivation of the positive off-diagonal property comes from matrix theory, where the off-diagonal elements of the matrix $A = (a_{ij})$ are positive, i.e., $a_{ij} \geq 0$ for all $i \neq j$. It is shown in [3, Lemma 7.23] that on an ordered Banach space $X$ with an order unit $u$ such that $u \in D(A)$, the operator $A$ has the positive off-diagonal property and $Au \leq 0$ if and only if $A$ is $\Psi_u$-dissipative, where $\Psi_u$ is the order unit function, i.e. $\Psi_u(x) = \inf \{ \lambda \geq 0 : x \leq \lambda u \}, x \in X$. However, in general, the properties that $A$ has the positive off-diagonal property and $A$ is $p$-dissipative for a given $p$ on $X$ do not imply each other, as the following example shows.

Example 5.2. Let $X = \mathbb{R}^2$ and $p(x) = \sqrt{x_1^2 + x_2^2}$ for $x = (x_1, x_2)$. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then $A$ has the positive off-diagonal property. Take $x = (1, 0)$, then...
\(x' = (1, 0) \in dp(x)\) but \(\langle Ax, x' \rangle = 1\), so \(A\) is not \(p\)-dissipative. Let \(A = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\), and \(\mathcal{D}(A) = \{x = (x_1, x_2) \in X : x_1 \geq 0, x_2 = 0\}\). Take \(x' = (1, 0)\), then \(x' \in dp(x)\) for every \(x \in \mathcal{D}(A)\). It is obvious that \(\langle Ax, x' \rangle \leq 0\). So \(A\) is \(p\)-dissipative, but does not have positive off-diagonal property.

Next, we consider a representation theorem in pre-Riesz spaces.

**Theorem 5.3.** Let \(X\) be an Archimedean pre-Riesz space with order unit. Then there exists a compact Hausdorff space \(\Omega\) and a bipositive linear map \(i : X \to C(\Omega)\) such that \(i(X)\) is order dense in \(C(\Omega)\). Moreover, for every positive linear functional \(\phi\) on \(X\), there exists a regular Borel measure \(\mu\) on \(\Omega\) such that

\[
\phi(x) = \int_{\Omega} i(x)(\omega)d\mu(\omega), \quad x \in X, \quad \omega \in \Omega.
\]

**Proof.** The first part of this theorem follows from [8, Lemma 6].

For the second part, let \(C(\Omega)\) be the space of all continuous functions on \(\Omega\), where \(\Omega\) is a compact Hausdorff space. Let \(i : X \to C(\Omega)\) be a bipositive linear map such that \(i(X)\) is an order dense subspace of \(C(\Omega)\). So for a positive linear function \(\phi : X \to \mathbb{R}\), one has that \(\phi \circ i^{-1} : i(X) \to \mathbb{R}^+\) is a positive linear function on \(i(X)\). Since \(\mathbb{R}\) is Dedekind complete, and \(i(X)\) is a majorizing subspace of \(C(\Omega)\), by Kantorovich’s extension theorem (see [11, Theorem 1.32]), there exists an extension of \(\phi \circ i^{-1}\) to a positive function \(\psi : C(\Omega) \to \mathbb{R}\). By the Riesz representation theorem, for \(\psi\) on \(C(\Omega)\), there exists a unique regular Borel measure \(\mu\) on \(\Omega\) such that

\[
\psi(f) = \int_{\Omega} f(\omega)d\mu(\omega), \quad \forall f \in C(\Omega), \quad \omega \in \Omega.
\]

So for every \(x \in X\), one has \(\phi \circ i^{-1}(i(x)) = \psi(i(x))\). If we take \(f = i(x)\), then

\[
\phi(x) = \phi \circ i^{-1}(i(x)) = \psi(i(x)) = \int_{\Omega} i(x)(\omega)d\mu(\omega).
\]

Thus we get the conclusion. \(\square\)

We give an example to illustrate that the positive off-diagonal property of \(A\) can be generalized to a special kind of ordered vector space, in particular to the pre-Riesz space \(C^1[0, 1]\).

**Example 5.4.** Let \(C[0, 1]\) be the real continuous functions. Let \(X = C^1[0, 1]\) which is an Archimedean pre-Riesz space, then \(X\) is an order dense subspace of \(C[0, 1]\). Let \(A \in L(X)\) be a densely defined operator, we claim that \(A\) has positive off-diagonal property if and only if \(\langle Ax, t \rangle \geq 0\) whenever \(0 \leq x \in \mathcal{D}(A)\) and \(t \in [0, 1]\) with \(x(t) = 0\). In fact, first suppose that \(A\) has the positive off-diagonal property and \(0 \leq x \in \mathcal{D}(A), t \in [0, 1]\) with \(x(t) = 0\). Take \(0 \leq \delta_t \in X'\) to be the point evaluation at \(t\), then it follows from \(x(t) = \langle x, \delta_t \rangle = 0\) that \(\langle Ax, \delta_t \rangle \geq 0\), i.e. \(\langle Ax(t) \rangle \geq 0\). Conversely, assume \(0 \leq x \in \mathcal{D}(A)\), and \(0 \leq \phi \in X'\) with \(\langle x, \phi \rangle = 0\). Then Theorem 5.3 can be applied since \(C[0, 1]\) has an order unit, and then there exists a regular Borel measure \(\mu\) on \([0, 1]\) that represents \(\phi\), i.e. \(\langle x, \phi \rangle = \int_0^1 i(x)(t)d\mu(t)\). By assumption, we have \(i(x)(t) = 0\) and \(x(t) = 0\) for all \(t\) in the support of \(\mu\), then \(\langle Ax(t) \rangle \geq 0\) and \(i(Ax)(t) \geq 0\). Hence \(\langle Ax, \phi \rangle = \int_0^1 i(Ax)(t)d\mu(t) \geq 0\). This shows that \(A\) has the positive off-diagonal property.
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