Detection and Tracking of Low Observable Objects in a Sequence of Image Frames Using Particle Filter

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Abstract- A track-before-detect (TBD) particle filter-based method for detection and tracking of low observable maneuvering objects based on a sequence of image frames in the presence of noise and clutter is briefly studied in this short letter. At each time instance after receiving a frame of image, first, some preprocessing approaches are applied to the image. Then, it is sent to the detection and tracking algorithm which is based on a particle filter. Performance of the approach is evaluated for detection and tracking of an object in different scenarios including noise and clutter.

I. INTRODUCTION

There are several methods for detection and tracking of low observable targets. In conventional methods, first, target is detected based on thresholding the received image and then its position can be estimated with a higher accuracy using a tracking filter (internal boxes in figure (1)). However, if the SNR of the target is not high enough to be always detected based on one frame thresholding, several consecutive frames of measurements can be used to detect the target using signal accumulation over time. In other word, such methods try to postpone thresholding to accumulate more power from the target to be able to detect it with a higher probability of detection. Therefore, in such methods detection and tracking must be done simultaneously (large box in figure (1)).

Fig (1): Different methods for detection and tracking of low observable targets

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In the following, first the required image preprocessing is discussed in section II-A. Then, particle filter is briefly reviewed in section II-B. Detection and tracking method is presented in section II-C. Finally, several simulations are provided.

II. PROBLEM DESCRIPTION AND FORMULATION

II-A: Preprocessing

To match the image frames to the assumptions of the detection and tracking algorithm, some preprocessing is required as follows.

**Inverse Filtering**

The considered detection and tracking approach assumes that the target is a point target and not an extended one. So, if the target in the received frame of image is an extended one, it can be converted to a point target using inverse filtering. Conceptually, an extended target can be understood as a point target affected by a filter. So, to restore the target as a point one, we apply the inverse filtering (block diagram below). An example is shown in figure (2).
Clutter Suppression

If there is any clutter in the image frames the detection and tracking algorithm cannot perform well. So, in the preprocessing part it is required to suppress the clutter. The clutter suppression is done based on background subtraction since it is assumed that the movement of the clutter is less than targets and it is negligible.
Estimate of Variance

The variance of noise must be known for the detection and tracking algorithm. Without clutter, a maximum likelihood (ML) estimate of the variance of the noise can be calculated based on a frame. However, in presence of clutter the ML estimate of the variance of noise is calculated based on the processed image after clutter suppression.

Target Intensity Estimation in Presence of Intensity Fluctuations

To deal with target intensity fluctuations, object intensity is augmented to the state vector as a state variable to be estimated with the help of particle filter along the algorithm.

II-B: Detection and Tracking Algorithm

Target Model

Target dynamic model is as follows

$$x_k = Fx_{k-1} + v_k$$

$$F = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ (1)
In which $x$ is the state vector including position and velocity in 2D space, and $v$ is dynamic noise, and $F$ is the transition matrix.

Target may appear or disappear any time in the space. A variable is defined with Markov model for presence of target as follows with 1 indicating presence and 0 indicating absence of a target

$$E_k \in \{0,1\} \quad P_b = \Pr \{ E_k = 1 | E_{k-1} = 0 \} \quad P_d = \Pr \{ E_k = 0 | E_{k-1} = 1 \}$$

Therefore,

$$\Pi_E = \begin{bmatrix} 1 - P_b & P_b \\ P_d & 1 - P_d \end{bmatrix}$$

And the initial probability at the beginning is

$$\mu_1 = \Pr \{ E_1 = 1 \}$$

It is assumed that these probabilities are known, otherwise they can be estimated.

**Sensor Model**

The received intensity in each pixel is modeled as follows

$$z_k^{(i,j)} = \begin{cases} h_k^{(i,j)}(x_k) + w_k^{(i,j)} & \text{if target present} \\ w_k^{(i,j)} & \text{if target absent} \end{cases}$$

In which

$$h_k^{(i,j)}(x_k) = \begin{cases} I_k & \text{if target is in cell (i, j)} \\ 0 & \text{otherwise} \end{cases}$$

$$\Delta_x \times \Delta_y : \text{Cell dimension } \quad (i\Delta_x, j\Delta_y) \quad , \quad i = 1, \ldots, n \quad , \quad j = 1, \ldots, m$$

$$w_k^{(i,j)} : N(0, \sigma^2) \quad (\text{Observation noise in pixel (i, j) at time k}) \quad I_k : \text{Target Intensity}$$

And the observation of each image frame is as follows

$$Z_k = \{ z_k^{(i,j)} : i = 1, \ldots, n \quad , \quad j = 1, \ldots, m \}$$

And all the image frames since the beginning to time $k$ is denoted as
\[ Z^k = \{Z_i, \text{ } i = 1, \ldots, k \} \]

**Detection and Tracking**

This method estimates the joint density of dynamic state and presence probability [1]-[10]. Then, if based on the presence probability estimate it is decided that there is a target in the space, target state vector can be estimated based on output of the filter. The presence probability also is calculated based on the output of the filter at every time instance. Joint density of presence probability and target state vector based on all the observations since the beginning to the current time presented in [3] is as follows

\[
p(x_k, E_k | Z^k) = p(x_k | E_k, Z^k) p(E_k | Z^k) \tag{3}
\]

Since density of state vector is calculated only if target is present, and since sum of presence and absence probabilities is 1, one can just compute the following terms

\[
p(x_k | E_k = 1, Z^k) \quad p(E_k = 1 | Z^k)
\]

**State Vector Density Function**

Density function can be expanded based on the previous time instance as follows

\[
p(x_k | E_k = 1, Z^k) = p(x_k | E_k = 1, E_{k-1} = 1, Z^k) p(E_{k-1} = 1 | E_k = 1, Z^k) \]

\[
+ p(x_k | E_k = 1, E_{k-1} = 0, Z^k) p(E_{k-1} = 0 | E_k = 1, Z^k) \tag{4}
\]

The first density in RHS is called survival density and the second one is called new-born density. Then based on Bayes formula

\[
p(x_k | E_k = 1, E_{k-1} = 1, Z^k) = \frac{p(Z_k | x_k, E_k = 1) p(x_k | E_k = 1, E_{k-1} = 1, Z^{k-1})}{p(Z_k | E_k = 1, E_{k-1} = 1, Z^{k-1})} \tag{5}
\]

The numerator and denominator of the equation (5) are divided by \( p(Z_k | E_k = 0) \) and the likelihood function of the frame based on the likelihood function at each pixel is as follows

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\(^2\) Note that we present the approach of [3] in which there is no logic to handle maneuvering targets. However, we derived a multiple model extension of [3] to handle detection and tracking of maneuvering objects. We skip the details of that derivation and its formulation. But that extension is used in the simulations.
\[ L(Z_k|x_k, E_k = 1) = \prod_i \prod_j l(z_k^{(i,j)}|x_k, E_k = 1) \]  \hspace{1cm} (6)

Then
\[ p(x_k|E_k = 1, E_{k-1} = 1, Z^k) = \frac{L(Z_k|x_k, E_k = 1)p(x_k|E_k = 1, E_{k-1} = 1, Z^{k-1})}{L(Z_k|E_k = 1, E_{k-1} = 1, Z^{k-1})} \]  \hspace{1cm} (7)

Since a point target just has a contribution to one pixel (or an extend target just has contribution to its neighborhoods), the likelihood function can be simplified for that (those) pixels because other pixels are 1.

The predicted density in the numerator of (7) can be written based on dynamic model as follows
\[ p(x_k|E_k = 1, E_{k-1} = 1, Z^{k-1}) = \int p(x_k|x_{k-1}, E_k = 1, E_{k-1} = 1) p(x_{k-1}|E_{k-1} = 1, E_k = 1, Z^{k-1}) dx_{k-1} \]  \hspace{1cm} (8)

The second density in (4) can be written as
\[ p(x_k|E_k = 1, E_{k-1} = 0, Z^k) \propto L(Z_k|x_k, E_k = 1) p(x_k|E_k = 1, E_{k-1} = 0) \]  \hspace{1cm} (9)

This density is related to a newborn target.

The other term in RHS of (4) can be written based on Bayes formula as follows
\[ p(E_{k-1} = 1|E_k = 1, Z^k) = \frac{p(Z_k|E_k = 1, E_{k-1} = 1, Z^{k-1}) p(E_k = 1|E_{k-1} = 1) p(E_{k-1} = 1|Z^{k-1})}{p(Z_k, E_k = 1|Z^{k-1})} \]
\[ \propto L(Z_k|E_k = 1, E_{k-1} = 1, Z^{k-1})(1 - p_d) p_{k-1} \]  \hspace{1cm} (10)

The first term of RHS is the same as the denominator of (7) which is the normalizing term. So, it can be written as
\[ L(Z_k|E_k = 1, E_{k-1} = 1, Z^{k-1}) = \int L(Z_k|x_k, E_k = 1) p(x_k|E_k = 1, E_{k-1} = 1, Z^{k-1}) dx_{k-1} \]  \hspace{1cm} (11)

The other term in (4) can be calculated in the same way as follows
\[ p(E_{k-1} = 0|E_k = 1, Z^k) \propto L(Z_k|E_k = 1, E_{k-1} = 0) p_b (1 - p_k) \]  \hspace{1cm} (12)
The likelihood in (12) can be written in the same way as follows

\[ L(z_k | E_k = 1, E_{k-1} = 0) = \int L(z_k | x_k, E_k = 1) p(x_k | E_k = 1, E_{k-1} = 0) dx_k \]  

Therefore, it is possible to recursively estimate the posterior density of target state vector.

**Probability of Target Presence**

In this subsection, probability of target presence is estimated and used as a test statistic. Then, by applying a threshold (e.g., 0.6) on the estimated probability, the algorithm can decide if there is a target in the space or not.

The probability of target presence based on all observations since the beginning to the current time can be estimated as follows

\[ p(E_k = 1 | Z_k) = p(E_k = 1, E_{k-1} = 1 | Z_k) + p(E_k = 1, E_{k-1} = 0 | Z_k) \]

\[ \propto L(z_k | E_k = 1, E_{k-1} = 1, Z_{k-1})(1 - p_d) p_{k-1} + L(z_k | E_k = 1, E_{k-1} = 0) p_b (1 - p_{k-1}) \]  

(14)

in which the likelihood terms have already been calculated in the calculation of target state density. For normalizing the probability of (14), the following probability is also calculated

\[ p(E_k = 0 | Z_k) \propto p_d p_{k-1} + (1 - p_b) (1 - p_{k-1}) \]  

(15)

The above terms can be calculated using a particle filter. Below, a particle filter and some of its properties are reviewed [5], [11]-[15].

The above approach of [3] does not have any logic to handle maneuvering targets. We derived a multiple model extension of the above approach to handle detection and tracking of maneuvering targets. We skip the details of the derivation and its formulation. Later, we implement our multiple model extension of the above detection and tracking approach in simulations.

**Particle Filter**

The idea of particle filter for density estimation is based on Monte Carlo integration. Particle filter estimates a density function based by considering some points at the most important spots of the density sequentially. So sometimes it is called sequential importance sampling (SIS) which is explained more in the following. In other word, particle filter estimates the density with the
help of some particles and their corresponding weights. For our problem we need to estimate the density of state vector (based on the dynamic and measurement models) which can be done based on particle filter as follows [5], [11]-[15],

\[ p(x_k \mid Z_k) \approx \sum_{i=1}^{N} w_k^i \delta(x_k - x_k^i) \]  

(16)

In which the Dirac delta function is used and also

\[ \tilde{w}_k^i \propto \frac{p(z_k \mid x_k^i) p(x_k^i \mid x_{k-1}^i)}{q(x_k^i \mid x_{k-1}^i, z_k)} \]  

(17)

In which the numerator can be calculated based on sensor and target models (in which \( x \) is state vector and \( z \) is observation vector). The denominator is a design function for generation of particles which is called proposal function or importance function. In this paper the proposal function is designed to be the second term in the numerator of (7) which is just related to target dynamic model. This particle filter is called SIR (sequential importance resampling).

Then the weights are normalized

\[ w_k^i = \frac{\tilde{w}_k^i}{\sum_{j=1}^{N} \tilde{w}_k^j} \]

**Degeneracy**

In applying the above formula, a problem happens which is called degeneracy. Conceptually, this problem, which is called degeneracy, can be explained as follows. After some time, the normalized weights of the particles all are negligible except one of them. Therefore, on the one hand much computation is done for updating particles with almost zero weight, and on the other it makes the filter not so effective. There is a measure based on which one can say how the degeneracy is serious. The measure is as follows

\[ N_{eff} = \frac{1}{\sum_{i=1}^{N} (w_k^i)^2} \]  

(17)

In which the summation is over the number of particles. It is always between 1 and \( N \) (the number of particles). The smaller the value of this measure, the more serious degeneracy one will have in the algorithm. To come up with a solution for this problem, an algorithm, which is called resampling, is used. This method is briefly described in the following.
Resampling

When degeneracy is serious, resampling is needed. Conceptually, resampling tries to delete the particles with negligible weights and increase the particles with higher weights. Finally, all the particles will have equal weights. In other word, one can say the particles at the output of resampling is generated based on the following density function

\[
p(x_k | Z_k) \approx \sum_{i=1}^{N} w'_i \delta(x_k - x'_k)
\]  

(18)

Figure (2) shows how resampling works. Table (1) is the resampling algorithm.

Considering resampling, particle filter can be applied based on the following algorithm (Table (2)). Some particle filters apply resampling at every time instance.

![Resampling Process](image)

**Fig (2): Resampling process. The larger the weight, the more chance for a particle to become double [5]**

**Table (1): Resampling [5]**

\[
\left\{ \begin{array}{c} x_k^i, w_k^i, i \end{array} \right\}_{j=1}^{N} = RESAMPLE \left\{ \begin{array}{c} x_k^{i-1}, w_k^{i-1} \end{array} \right\}_{j=1}^{N}
\]

- Initiate the CSW: \( c_1 = w_k^1 \)
- FOR \( i = 2 : N \)
  - Construct CSW: \( c_i = c_{i-1} + w_k^i \)
- END FOR
- Start at the bottom of the CSW: \( i = 1 \)
- Draw a starting point: \( u_i \sim u[0, N^{-1}] \)
- FOR \( j = 1 : N \)
- Move along the CSW: \( u_j = u_1 + N^{-1}(j - 1) \)
- WHILE \( u_j > c_i \)
  - \( i = i + 1 \)
- END WHILE
- Assign sample: \( x_k^i = x_k^i \)
- Assign weight: \( w_k^i = N^{-1} \)
- Assign parent: \( i^j = i \)
- END FOR

Table (3): Generic Particle Filter [5]
\[
\{x_k^i, w_k^i\}_{i=1}^N = PF\{x_{k-1}^i, w_{k-1}^i\}_{i=1}^N, z_k
\]
- Filtering via SIS
\[
\{x_k^i, w_k^i\}_{i=1}^N = SIS\{x_{k-1}^i, w_{k-1}^i\}_{i=1}^N, z_k
\]
- Calculate \( N_{\text{eff}} \)
- If \( N_{\text{eff}} < N_{\text{thr}} \)
  - Resampling using the systematic algorithm
\[
\{x_k^i, w_k^i, i^j\}_{i=1}^N = RESAMPLE\{x_k^i, w_k^i\}_{i=1}^N
\]
- END IF

The process of filtering by particle filter can be illustrated by figure (3).
II-C: Applying TBD Detection and Tracking Algorithm Using Particle Filter

Now, we can implement the TBD detection and tracking method [3] described in section II-A with the help of particle filter [1]-[10]. Below, we present the algorithm for detection and tracking using a particle filter.

The multiple model extension of the detection and tracking approach mentioned in the previous section is implemented using a particle filter and used in simulations. We skip the details of its formulation.

Since the algorithm is recursive, assume that \( N_c \) particles \( \{x^{i}_{k-1}\}_{i=1,...,N_c} \) are available from previous time describing the density function of state vector corresponding to the last time (Note that the weights are equal after resampling). Also, it is assumed that the estimate of the probability of target presence corresponding to the last time is available. Then, the recursive algorithm for detection and tracking of target is as follows

- Newborn particles are generated based on a proposal function (which can be a uniform density over those spots of the space with higher probability of including target)

  \[
  x^{(b)j}_{k} \sim q(x^{i}_{k}, E^{k} = 1, E^{k-1} = 0, z^{k}) \tag{19}
  \]

The corresponding weights are calculated as follows

\[
\tilde{W}^{(b)j}_{k} = \frac{L(z^{j}_{k} | x^{(b)j}_{k}, E^{(b)j}_{k} = 1) p(x^{(b)j}_{k} | E^{(b)j}_{k} = 1, E^{(b)j}_{k-1} = 0)}{N_b q(x^{(b)j}_{k} | E^{(b)j}_{k} = 1, E^{(b)j}_{k-1} = 0, z^{k})} \tag{20}
\]

And after normalizing
Continuing (survival) particles are generated. The considered proposal function is the dynamic model of target

\[
\tilde{w}_k^{(c)i} = \frac{1}{N_c} L(z_k^{(c)i}, E_k^{(c)i} = 1)
\]  

Following coefficients are calculated for the calculation of target presence probability

\[
\tilde{M}_b = p_b \left[ 1 - p_{k-1} \right] \sum_{j=1}^{N_b} \tilde{w}_k^{(b)j}
\]

\[
\tilde{M}_c = [1 - p_d] p_{k-1} \sum_{j=1}^{N_c} \tilde{w}_k^{(c)j}
\]

After normalization

\[
M_b = \frac{\tilde{M}_b}{(\tilde{M}_b + \tilde{M}_c)}
\]

\[
M_c = \frac{\tilde{M}_c}{(\tilde{M}_b + \tilde{M}_c)}
\]

Probability of presence at current time

\[
p_k = \frac{\tilde{M}_b + \tilde{M}_c}{\tilde{M}_b + \tilde{M}_c + p_d p_{k-1} + [1 - p_b][1 - p_{k-1}]}
\]

Also, these weights are computed for calculation of posterior density of target state vector.

\[
\hat{w}_k^{(b)i} = M_b w_k^{(b)i}
\]

\[
\hat{w}_k^{(c)i} = M_c w_k^{(c)i}
\]
Then two sets of particles (newborn and continuation) are considered together

$$\{(x^{(t)i}, \hat{w}^{(t)i}) | i = 1, ..., N_t, t = c, b\}$$  

(31)

- Finally, resampling is applied to the whole particles so that $N_c + N_b$ particles are reduced to $N_c$ particles.

After doing these steps, $\{(\alpha_k^i, \frac{1}{N_c}) | i = 1, ..., N_c\}$ particles estimate density function of the state vector and the probability of target presence. Then, comparing this probability with a threshold (e.g., 0.6), the algorithm decides if there is a target in the space or not.

In the next section, this algorithm is applied to different scenarios to illustrate its performance in different situations.

III. SIMULATIONS

To illustrate the performance of the algorithm in detection and tracking, different scenarios have been considered including noise, clutter, synthetic image, real image, point and extended target with/without rotation.

**Scenario 1: Detection and tracking of a maneuvering point target in noise and clutter background (synthetic image)**

In this scenario the mean of the intensity of target is about 5, and the variance of noise is 1 (6.5 dB SNR before preprocessing). The intensity of clutter is 10 which is much more than that of target. Markov model transition probabilities are 0.05 and 0.95.

As it can be seen in the figure (4), (5), and (6) when there is no target in the space particles are uniformly spread in the space. We consider a very low threshold at the very beginning because pixels with very low intensity are not desired. This is the reason that particles are not completely uniformly spread in the space). When there is a target in the space, according to noise and target intensity realizations, the target may/may not be visible, however, particles can recognize the target and gather around it (as it can be seen in figure (5) and (6)). Because of clutter we need to apply some preprocessing for clutter suppression (background subtraction), which remove clutter but decreases the SNR.
Fig (3): True trajectory of target

Fig (4): (left) a frame with clutter without target, (right) particles are uniformly spread

Fig (5): (left) a frame with clutter with target (it is not visible), (right) particles are centered on target
Scenario 2: Detection and tracking of a maneuvering point target in clutter and noise background (real image)

In this scenario the background image is a real one (figure (8)) and the same trajectory as the previous scenario. Mean of target intensity is 7, noise standard deviation is 0.5, and the maximum of clutter intensity is about 4.5 (Figure (9)).
The same scenario is considered with a very low SNR (mean of target intensity is 5, standard deviation of noise is 0.5, the maximum intensity of clutter is about 4.5). The results of detection and tracking are as follows (figure (10))
Scenario 3: Detection and tracking of a maneuvering extended target (without rotation) in clutter and noise background (real image)

In this scenario the background image is a real one (figure (11)). Mean of target intensity is 20, noise standard deviation is 0.1, and the maximum of clutter intensity is about 1. The results are shown in figure (13).

The same scenario is considered with a very low SNR (mean of target intensity is 10, standard deviation of noise is 0.1, the maximum intensity of clutter is about 1). The results of detection and tracking are as follows (figure (14))
Fig (12): (left) Frame (including extended target) after preprocessing, (right) Frame (including extended target) after preprocessing

Fig (13): (left) Detection probability, (right) RMSE for tracking

Fig (14): (left) Detection probability, (right) RMSE for tracking
Scenario 4: Detection and tracking of a maneuvering extended target (with rotation) in clutter and noise background (real image)

In this scenario the background image is a real one (figure (15)) and the trajectory is the same as scenario 1. Mean of target intensity is 6, noise standard deviation is 0.1, and the maximum of clutter intensity is about 1.7. The results are shown in figure (18).

Fig (15): Background with noise and clutter

Fig (16): (left) Extended target, (right) Extended target rotated
Scenario 4: Detection and tracking of a maneuvering extended target (with rotation) in clutter and noise background with target intensity fluctuations (real image)

In this scenario the background image is a real one (the same as figure (15)) and the trajectory is the same as scenario 1. Mean of target intensity is 9 with uniform fluctuations in the interval of 4 centered at the mean. Noise standard deviation is 0.1, and the maximum of clutter intensity is about 1.7. The results are shown in figure (19).
Conclusions and Future Work Directions:

A particle filter-based approach for detection and tracking of a low observable maneuvering point/extended target has been studied. Some preprocessing is required to make the image frames ready for detection and tracking of the target.

Detection and tracking of objects is a decision-estimation problem and, therefore, another approach for handling this problem is based on using a joint decision-estimation approach [16]-[18].

Information about the destination or waypoints for the trajectory of the target has not been incorporated in the above detection and tracking approach. A theoretical foundation of conditionally Markov (CM) sequences was presented in [19]-[26] and their dynamic models, their properties, and some tools were derived for their application to trajectory modeling with destination and waypoint information in [27]-[31]. As a future work, these CM models can be used in the above detection and tracking approach to model and incorporate destination/waypoint information and enhance the detection and tracking performance.
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