ACYLINDRICAL ACCESSIBILITY FOR GROUPS ACTING ON $\mathbb{R}$-TREES

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Abstract. We prove an acylindrical accessibility theorem for finitely generated groups acting on $\mathbb{R}$-trees. Namely, we show that if $G$ is a freely indecomposable non-cyclic $k$-generated group acting minimally and $D$-acylindrically on an $\mathbb{R}$-tree $X$ then there is a finite subtree $T_\varepsilon \subseteq X$ of measure at most $2D(k-1) + \varepsilon$ such that $GT_\varepsilon = X$. This generalizes theorems of Z. Sela and T. Delzant about actions on simplicial trees.

1. Introduction

An isometric action of a group $G$ on an $\mathbb{R}$-tree $X$ is said to be $D$-acylindrical (where $D \geq 0$) if for any $g \in G$, $g \neq 1$ we have $\text{diam} \text{ Fix}(g) \leq D$, that is any segment fixed point-wise by $g$ has length at most $D$. For example the action of an amalgamated free product $G = A *_C B$ on the corresponding Bass-Serre tree is 2-acylindrical if $C$ is malnormal in $A$ and 1-acylindrical if $C$ is malnormal in both $A$ and $B$. In fact the notion of acylindricity seems to have first appeared in this context in the work of Karras and Solitar $[32]$, who termed it being $r$-malnormal.

Sela $[37]$ proved an important acylindrical accessibility result for finitely generated groups which, when applied to one-ended groups, can be restated as follows: for any one-ended finitely generated group $G$ and any $D \geq 0$ there is a constant $c(G, D) > 0$ such that for any minimal $D$-acylindrical action of $G$ on a simplicial tree $X$ the quotient graph $X/G$ has at most $c(G, D)$ edges. This fact plays an important role in Sela’s theory of JSJ-decompositions for word-hyperbolic groups $[38]$ and thus in his solution of the isomorphism problem for torsion-free hyperbolic groups $[36]$. Moreover, acylindrical splittings feature prominently in relation to the Combination Theorem of Bestvina-Feighn $[7, 9]$ and its various applications and generalizations $[17, 20, 26, 33]$. Unlike other kinds of accessibility results, such as Dunwoody accessibility $[20, 21]$, Bestvina-Feighn generalized accessibility $[5, 6]$ and strong accessibility (introduced by Bowditch $[13]$ and proved by Delzant and Potyagailo $[14]$), acylindrical accessibility holds for finitely generated and not just finitely presented groups.

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Delzant [18] obtained a relative version of Sela’s theorem for finitely presented groups with respect to a family of subgroups. In particular, he showed that the constant \( c(G, D) \) above can be chosen to be \( 12DT \), where \( T \) is the number of relations in any finite presentation of \( G \) where all relators have length three. Weidmann [40] used the theory of Nielsen methods for groups acting on simplicial trees to show that for any \( k \)-generated one-ended group one can choose \( c(G, D) = 2D(k - 1) \). In the present paper we obtain an analogue of this last result for groups acting on \( \mathbb{R} \)-trees.

Before formulating our main result let us recall the notion of Nielsen equivalence:

**Definition 1.1 (Nielsen equivalence).** Let \( G \) be a group and let \( M = (g_1, \ldots, g_n) \in G^n \) be an \( n \)-tuple of elements of \( G \). The following moves are called elementary Nielsen moves on \( M \):

1. **(N1)** For some \( i, 1 \leq i \leq n \) replace \( g_i \) by \( g_i^{-1} \) in \( M \).
2. **(N2)** For some \( i \neq j, 1 \leq i, j \leq n \) replace \( g_i \) by \( g_i g_j \) in \( M \).
3. **(N3)** For some \( i \neq j, 1 \leq i, j \leq n \) interchange \( g_i \) and \( g_j \) in \( M \).

We say that \( M = (g_1, \ldots, g_n) \in G^n \) and \( M' = (f_1, \ldots, f_n) \in G^n \) are Nielsen-equivalent, denoted \( M \sim_N M' \), if there is a chain of elementary Nielsen moves which transforms \( M \) to \( M' \).

It is easy to see that if \( M \sim_N M' \) then \( M \) and \( M' \) generate the same subgroup of \( G \). For this reason Nielsen equivalence is a very useful tool for studying the subgroup structure of various groups.

We prove the following statement which can be regarded as an “acylindrical accessibility” result for finitely generated groups acting on real trees. Indeed, our theorem says that there is a bound on the size of a “fundamental domain” for a minimal \( D \)-acylindrical isometric action of a \( k \)-generated group on an \( \mathbb{R} \)-tree:

**Theorem 1.2.** Let \( G \) be a freely indecomposable finitely generated group acting by isometries on an \( \mathbb{R} \)-tree \( X \). Let \( D \geq 0 \). Suppose that \( G \neq 1 \) is not infinite cyclic, that the action of \( G \) is \( D \)-acylindrical, nontrivial (does not have a fixed point) and minimal (that is \( X \) has no proper \( G \)-invariant subtrees).

Let \( \varepsilon > 0 \) be an arbitrary real number. Then any finite generating \( k \)-tuple \( Y \) of \( G \) is Nielsen-equivalent to a \( k \)-tuple \( S \) such that:

1. There is a finite subtree \( T_\varepsilon \subseteq X \) of measure at most \( 2D(k - 1) + \varepsilon \) such that \( GT_\varepsilon = X \);
2. for some \( x \in T_\varepsilon \) we have

\[
\max\{d(x, sx) | s \in S\} \leq 2D(k - 1) + \varepsilon.
\]

By the measure of \( T_\varepsilon \) we mean the sum of the lengths of intervals in any subdivision of \( T_\varepsilon \) as a disjoint union of finitely many intervals. This is equal to the 1-dimensional Hausdorff measure of \( Y \). If \( X \) is a simplicial tree and
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\(T_\varepsilon\) is a simplicial subtree, then the measure of \(T_\varepsilon\) is the number of edges in \(T_\varepsilon\).

Theorem 1.2 immediately implies the following since actions with trivial arc stabilizers are 0-acylindrical:

**Corollary 1.3.** Let \(G\) be a finitely generated freely indecomposable non-cyclic group which acts by isometries on an \(\mathbb{R}\)-tree \(X\) with trivial arc stabilizers. Then for any \(\varepsilon > 0\) and any finite generating tuple \(Y\) of \(G\) there is a tuple \(S\) Nielsen-equivalent to \(Y\) and a point \(x \in X\) such that \(d(x, sx) \leq \varepsilon\) for all \(s \in S\).

Thus if the action of \(G\) is 0-acylindrical, that is arc stabilizers are trivial, then a finite generating set of \(G\) can be made by Nielsen transformations to have arbitrarily small translation length. Not surprisingly our methods let us recover the same bound \(c(G, D) = 2D(k - 1)\) on the complexity of acylindrical accessibility splittings as the one given in [40]. The main ingredient is a theory of Nielsen methods for groups acting on hyperbolic spaces that we systematically developed in [29, 30]. This theory is analogous to Weidmann’s treatment of actions on simplicial trees [40], but the case of arbitrary hyperbolic spaces is technically much more complicated. Note that the proof of Theorem 1.2 completely avoids the Rips machinery for groups acting on \(\mathbb{R}\)-trees [8, 27] and Theorem 1.2 makes no traditional stability assumptions about the action. Rather, we use the fact that an \(\mathbb{R}\)-tree is \(\delta\)-hyperbolic for any \(\delta > 0\), which allows us to make a limiting argument for \(\delta\) tending to zero.

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2. The main technical tool

Our main tool is a technical result (see Theorem 2.4 below) obtained by Kapovich and Weidmann in [30]. It is motivated by the Kurosh subgroup theorem (see [34, 3]) for free products, which states that a subgroup of a free product \(\ast_{i \in I} A_i\) is itself a free product of a free group and subgroups that are conjugate to subgroups of the factors \(A_i\).

**Definition 2.1.** Suppose that \((X, d)\) is an \(\mathbb{R}\)-tree and that \(U\) is a finitely generated group that acts on \(X\) by isometries. Put \(E(U) := \{x \in X \mid ux = x\text{ for some } u \in U - 1\}\). If \(U\) does not fix a point of \(X\), let \(X_U\) be the minimal \(U\)-invariant subtree of \(X\). If \(U\) fixes a point of \(X\), let \(X_U\) denote the set of all points of \(X\) fixed by \(U\).

Finally, put \(X(U)\) to be the smallest \(U\)-invariant subtree containing \(X_U \cup E(U)\). Thus \(X(U)\) is a nonempty \(U\)-invariant subtree of \(X\).

We generalize the notion of Nielsen equivalence as follows. The objects which correspond to the tuples of elements of \(G\) are the \(G\)-tuples:

**Definition 2.2** (\(G\)-tuple). Let \(G\) be a group.
Let $n \geq 0$, $m \geq 0$ be integers such that $m + n > 0$. We will say that a tuple $M = (U_1, \ldots, U_n; H)$ is a $G$-tuple if $U_i$ is a non-trivial subgroup of $G$ for $i \in \{1, \ldots, n\}$ and $H = (h_1, \ldots, h_m) \in G^m$ is an $m$-tuple of elements of $G$. We will denote $\overline{M} = U_1 \cup \cdots \cup U_n \cup \{h_1, \ldots, h_m\}$ and call $\overline{M}$ the underlying set of $M$. Note that $\overline{M}$ is nonempty since $m + n > 0$.

By analogy with the Kurosh subgroup theorem we will sometimes refer to the subgroups $U_i$ as elliptic components of $M$. This is justified since in most applications of our methods, in particular the proof of Theorem 2.4 the subgroups $U_i$ are generated by sets of elements with short translation length. We should stress, however, that $U_i$ need not be fixing a point of a tree on which $G$ acts. We will also refer to $H$ as the hyperbolic component of $M$. We have the following notion of equivalence for $G$-tuples which generalizes the classical Nielsen equivalence.

**Definition 2.3** (Equivalence of $G$-tuples). We will say that two $G$-tuples $M = (U_1, \ldots, U_n; H)$ and $M' = (U'_1, \ldots, U'_n; H')$ are equivalent if $H = (h_1, \ldots, h_m)$ and $H' = (h'_1, \ldots, h'_m)$ and $M'$ can be obtained from $M$ by a chain of moves of the following type:

1. For some $1 \leq j \leq n$ replace $U_j$ by $g U_j g^{-1}$ where $g \in \langle \{h_1, \ldots, h_m\} \cup U_1 \cup \cdots \cup U_{j-1} \cup U_{j+1} \cup \cdots \cup U_n \rangle$.
2. For some $1 \leq i \leq n$ replace $h_i$ by $h'_i = g_1 h_i g_2$ where $g_1, g_2 \in \langle \{h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_m\} \cup U_1 \cup \cdots \cup U_n \rangle$.

Our main technical tool is Theorem 2.4 stated below. It is a corollary of Theorem 2.4 in [30] that deals with arbitrary group actions on $\delta$-hyperbolic geodesic metric spaces.

**Theorem 2.4** (Kapovich-Weidmann). Let $G$ be a group acting on an $\mathbb{R}$-tree $(X, d)$ by isometries.

Let $M = (U_1, \ldots, U_n; H)$ be a $G$-tuple where $H = (h_1, \ldots, h_m)$ and let

$$U = \langle \overline{M} \rangle = \langle \{h_1, \ldots, h_m\} \cup \bigcup_{i=1}^n U_i \rangle \leq G.$$ 

Let $\varepsilon > 0$ be an arbitrary positive number.

Then either $U = U_1 \ast \ldots \ast U_n \ast F(H)$ or there exists a $G$-tuple $M' = (U'_1, \ldots, U'_n; H')$ with $H' = (h'_1, \ldots, h'_m)$ such that $M'$ is equivalent to $M$ and at least one of the following holds:

1. $d(X(U_i), X(U'_j)) < \varepsilon$ for some $1 \leq i < j \leq n$.
2. $d(X(U_i), h'_j X(U'_j)) < \varepsilon$ for some $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$.
3. There exists a point $x \in X$ such that $d(x, h_j x) < \varepsilon$ for some $j \in \{1, \ldots, m\}$.

**Remark 2.5.** The statement of Theorem 2.4 in [30] does not use the tree $X(U)$ defined above but, rather, the tree $X_{\delta}(U)$ (where $\delta \geq 0$) defined as follows.
Let
\[ E_\delta(U) := \{ x \in X : d(ux, u) \leq 100\delta \text{ for some } u \in U, u \neq 1 \}. \]

Then the tree \( X_\delta(U) \) is defined as \( E_\delta(U) \) if \( U \) fixes a point and as the smallest \( U \)-invariant subtree of \( X \) containing \( X_U \) and \( E_\delta(U) \) otherwise.

Note that \( X(U) \subseteq X_\delta(U) \) by construction. Let \( x \in E_\delta(U) \) and let \( y \in X(U) \) be such that \( d(x, y) = d(x, X(U)) \). Let \( u \in U \) be such that \( d(x, ux) \leq 100\delta \). Note that the choice of \( y \) guarantees that there exists no \( z \in [x, y] \) with \( y \neq z \) and \( uz = z \) as otherwise \( z \in X(U) \) and \( d(x, z) < d(x, y) \). It follows that \( d(x, ux) = 2d(x, y) + d(y, uy) \) and hence \( d(x, y) \leq 50\delta \). Thus we have shown that \( E_\delta(U) \) is contained in the \( 50\delta \)-neighborhood of \( X(U) \).

This implies that, whether or not \( U \) fixes a point, for any \( \delta \geq 0 \) the trees \( X(U) \) and \( X_\delta(U) \) are \( 50\delta \)-Hausdorff close.

Since an \( \mathbb{R} \)-tree is \( \delta \)-hyperbolic for any \( \delta > 0 \), Theorem 2.4 of [30] now directly implies Theorem 2.4 above by taking the limit \( \delta \to 0 \).

We deploy Theorem 2.4 in the proof of Theorem 1.2 for a "generator transfer" process to analyze a freely indecomposable subgroup generated by a finite set \( Y = \{ y_1, \ldots, y_k \} \) with \( k \) elements. First we start with a \( G \)-tuple \( M_1 = (; H_Y) \) where \( H_Y = (y_1, \ldots, y_k) \). We then construct a sequence of \( G \)-tuples \( M_1, M_2, \ldots \) by repeatedly applying Theorem 2.4 in order to either "drag" elements of the "hyperbolic" components of \( G \)-tuples into their "elliptic" components or to join two elliptic components to form one new elliptic component. A simple observation shows that the length of the sequence \( M_1, M_2, \ldots \) is bounded by \( 2k-1 \). The desired result is then obtained by analyzing the terminal member of this sequence.

3. Groups acting on real trees

We define "generating trees" exactly as in [10].

**Definition 3.1** (Generating tree). Let \( U = \langle S \rangle \) be a group acting by isometries on an \( \mathbb{R} \)-tree \( X \). We say that a tree \( T_U \subset X \) is an \( S \)-generating tree of \( U \) if \( Y \cap sY \neq \emptyset \) for all \( s \in S \). We further say that \( T_U \) is a generating tree of \( U \) if \( T_U \) is an \( S \)-generating tree for \( U \) for some generating set \( S \) of \( U \).

The following lemma is an immediate consequence of the definitions; it also justifies the term "generating tree".

**Lemma 3.2.** Let \( X \) be an \( \mathbb{R} \)-tree and let \( U = \langle S \rangle \) be a group acting on \( X \) by isometries. Let \( T_U \subset X \) be an \( S \)-generating tree of \( U \). Then the following hold:

1. The set \( UT_U \) is connected and \( U \)-invariant, and hence is a subtree of \( X \).
2. If \( U \) does not fix a point of \( X \), then \( UT_U \) contains the minimal \( U \)-invariant subtree \( X_U \) of \( X \).
We can now observe that for acylindrical actions the fix point set \( E(U) \) cannot be too far from the minimal \( U \)-invariant tree \( X_U \), i.e. that \( X(U) \) and \( X_U \) are close. The following is a simple exercise.

**Lemma 3.3.** Let \( U \) be a group acting by isometries on an \( \mathbb{R} \)-tree \( X \) and suppose this action is \( D \)-acylindrical for some \( D \geq 0 \). Then the following hold:

1. Let \( y \in X \) be such that \( d(y, X_U) = R \). Then for any \( u \in U, u \neq 1 \) we have \( d(y, uy) \geq 2R - 2D \).
2. The set \( E(U) \) is contained in the \( D \)-neighborhood of \( X_U \). Moreover, \( X(U) \) and \( X_U \) are \( D \)-Hausdorff close.
3. If \( U \) fixes a point \( x \in X \) then for any \( y \in X(U) \) we have \( d(x, y) \leq D \).

Before we proceed with the proof of Theorem 1.2 let us recall some more notions from [30].

**Definition 3.4** (Partitioned tuple). Let \( G \) be a group. If \( Y \) is an \( n \)-tuple of elements of \( G \) we will say that \( n \) is the length of \( Y \) which we denote by \( L(Y) \). We will say that \( M = (Y_1, \ldots, Y_p; H) \) is a partitioned tuple in \( G \) if \( p \geq 0 \) and \( Y_1, \ldots, Y_p, H \) are finite tuples of elements of \( G \) such that at least one of these tuples has positive length and such that for any \( i \geq 1 \) the tuple \( Y_i \) has positive length.

We will call the sum of the lengths of \( L(Y_1) + \cdots + L(Y_p) + L(H) \) the length of \( M \) and denote it by \( L(M) \). We further call the \( L(M) \)-tuple, obtained by concatenating the tuples \( Y_1, \ldots, Y_p, H \), the tuple underlying \( M \).

**Definition 3.5** (Complexity). Let \( M = (Y_1, \ldots, Y_n; H) \) be a partitioned tuple. As in [10, 30], we define the complexity of \( M \) to be the pair \((L(H), n)\). Thus the complexity is an element of \( \mathbb{N}^2 \) where \( \mathbb{N} = 0, 1, 2, \ldots \). We order \( \mathbb{N}^2 \) by setting \((m, n) \leq (m', n')\) if \( m < m' \) or if \( m = m' \) and \( n \leq n' \). This gives a well-ordering on \( \mathbb{N}^2 \) and allows us to compare complexities.

**Remark 3.6.** To any partitioned tuple \( M = (Y_1, \ldots, Y_p; H) \) we associate the \( G \)-tuple \( \tau = (U_1, \ldots, U_p; H) \), where \( U_i = \langle Y_i \rangle \). Suppose now that \( \tau \) is equivalent to a \( G \)-tuple \( \tau' = (U'_1, \ldots, U'_p, H') \). The definition of equivalence of \( G \)-tuples implies that there is a partitioned tuple \( M' = (Y'_1, \ldots, Y'_p; H') \) with associated \( G \)-tuple \( \tau' \) such that the tuples underlying \( M \) and \( M' \) are Nielsen-equivalent. Moreover, we can choose \( Y'_i \) to be conjugate to \( Y_i \) for each \( 1 \leq i \leq p \).

**Proof of Theorem 1.2**

Recall that in Theorem 1.2 \( Y \) is a given \( k \)-tuple generating \( G \). Clearly, it is enough to prove the statement of Theorem 1.2 under the assumption that \( Y \) is not Nielsen-equivalent to a tuple containing \( 1 \in G \). Thus we will assume that every \( k \)-tuple Nielsen-equivalent to \( Y \) consists of nontrivial elements.

Now in order to prove Theorem 1.2 it suffices to establish:
Claim. There exist a \( k \)-tuple \( S \) Nielsen-equivalent to \( Y \) and an \( S \)-generating tree \( T_e \) of measure at most \( 2D(k-1) + \varepsilon \).

Let \( N = (S_1, \ldots, S_n; H) \) be a partitioned tuple of elements of \( G \). Let \( k_i = L(S_i) \) for \( 1 \leq i \leq n \). We say that \( N \) is good if \( U_i = \langle S_i \rangle \neq 1 \) for all \( i \geq 1 \) and if for each \( U_i, i \geq 1 \) there exists an \( S_i \)-generating tree \( T_i \) of measure at most \( 2D(k_i - 1) + \frac{2k_i - 1}{2k} \varepsilon \).

We define \( N_1 \) to be the partitioned tuple \( N_1 = (; Y) \). Clearly \( N_1 \) is good.

Choice of \( M \). Let \( M = (S_1, \ldots, S_n; H) \) with \( H = (h_1, \ldots, h_m) \) be a partitioned tuple of minimal complexity among all partitioned good tuples with the underlying tuple being Nielsen equivalent to \( Y \). The partitioned tuple \( N_1 \) satisfies the above qualifying constraints and hence such an \( M \) exists.

We will show that \( M = (S_1; -) \). This would immediately imply the Claim.

Suppose that \( M \) is not of this type. Recall that \( G \) is freely indecomposable and not infinite cyclic. It follows from Theorem 2.4 and Remark 3.6 that there exists a good partitioned tuple \( M' = (S'_1, \ldots, S'_n; H') \) of the same complexity as \( M \) with \( H' = (h'_1, \ldots, h'_m) \) such that the partitioned tuple of \( M' \) is Nielsen equivalent to \( Y \) and such that the following holds. If we denote \( U'_i = \langle S'_i \rangle \) for \( 1 \leq i \leq n \) then at least one of the following occurs:

1. \( d(\langle X(U'_1) \rangle, \langle X(U'_2) \rangle) \leq \frac{\varepsilon}{2k} \);
2. \( d(\langle X(U'_1) \rangle, h'_m \langle X(U'_2) \rangle) \leq \frac{\varepsilon}{2k} \);
3. \( d(y, h'_m y) \leq \frac{\varepsilon}{2k} \) for some \( y \in X \).

Recall that \( k_i = L(S_i) = L(S'_i) \) for \( 1 \leq i \leq n \).

Case 1. Suppose that \( d(\langle X(U'_1) \rangle, \langle X(U'_2) \rangle) \leq \frac{\varepsilon}{2k} \). Choose \( x_1 \in X(U'_1) \) and \( x_2 \in X(U'_2) \) such that \( d(x_1, x_2) \leq \frac{\varepsilon}{2k} \). By part (2) of Lemma 3.3 there is \( y_i \in X_{U'_i} \) such that \( d(y_i, x_i) \leq D \) for \( i = 1, 2 \). It follows that \( d(y_i, y_2) \leq 2D + \frac{\varepsilon}{2k} \).

By assumption \( M' \) is good, we can choose an \( S'_i \)-generating tree \( T_{U'_i} \) for \( U'_i \) of measure at most \( 2D(k_i - 1) + \frac{2k_i - 1}{2k} \varepsilon \).

Let \( i \in \{1, 2\} \). If \( U'_i \) does not fix a point then \( X_{U'_i} \subset U'_i \cap T_{U'_i} \) and there exists a \( u_i \in U'_i \) such that \( y_i \in u_i T_{U'_i} \). If \( U'_i \) fixes a point, then \( X_{U'_i} \) is the fixed set of \( U'_i \) and hence \( y_i \in X_{U'_i} \) is fixed by \( U'_i \). In this case we can assume that in \( M' \) we have \( U'_{U'_i} = \{y_i\} \). This is clearly an \( S'_i \)-generating tree for \( U'_i \) of measure zero. With \( u_i = 1 \) we also still have \( y_i \in u_i T_{U'_i} \).

Denote \( S''_i := u_i S'_i u_i^{-1} \) for \( i = 1, 2 \). Then \( \langle S''_i \rangle = \langle S'_i \rangle = U'_i \) and \( u_i T_{U'_i} \) is the \( S''_i \)-generating tree for \( U'_i \).

Moreover \( S''_i \) is Nielsen-equivalent to \( S'_i \) since \( u_i \in U'_i \) for \( i = 1, 2 \). Put \( V := \langle U'_1, U'_2 \rangle \). Then \( T_V = u_1 T_{U'_1} \cup \{y_1, y_2\} \cup u_2 T_{U'_2} \) is a generating tree for \( V \) with respect to \( S''_1 \cup S''_2 \). The measure of \( T_V \) is at most

\[
2D(k_1 - 1) + \frac{2k_1 - 1}{2k} \varepsilon + 2D(k_2 - 1) + \frac{2k_2 - 1}{2k} \varepsilon + 2D + \frac{\varepsilon}{2k} = 2D(k_1 + k_2 - 1) + \frac{2(k_1 + k_2) - 1}{2k} \varepsilon.
\]
Hence the partitioned tuple \( M'' := (S'_1 \cup S'_2, S'_3, \ldots, S'_n; H') \) is good. Since the underlying tuple of \( M'' \) is Nielsen equivalent to \( Y \) and since \( M'' \) has smaller complexity than \( M' \) and \( M \), we obtain a contradiction to the choice of \( M \).

**Case 2.** Suppose that \( d(X(U'_1), h'_m X(U'_1)) \leq \frac{\varepsilon}{2k} \). As in (1) we see that there exist \( y_1 \in X_{U'_1} \) and \( y_2 \in h'_m X_{U'_1} \) such that \( d(y_1, y_2) \leq 2D + \frac{\varepsilon}{2k} \).

**Subcase 2A.** Suppose first that \( U'_1 \) does not fix a point. Again, since \( M' \) is good, after replacing \( S'_1 \) by a conjugate (in \( U'_1 \)) tuple we can assume that there exists an \( S'_1 \)-generating tree \( T_{U'_1} \) for \( U'_1 \) of measure at most \( 2D(k_1 - 1) + \frac{2k_1 - 1}{2k} \varepsilon \) such that \( y_1 \in T_{U'_1} \). Clearly \( (h'_m)^{-1} y_2 \) lies in \( X_{U'_1} \). Since \( X_{U'_1} \subseteq U'_1 T_{U'_1} \), it follows that \( u'_2 (h'_m)^{-1} y_2 \in T_{U'_1} \) for some \( u'_2 \in U'_1 \).

Hence \( T_V = T_{U'_1} \cup [y_1, y_2] \) is a generating tree for the subgroup \( V = \langle S'_1, u'_2 (h'_m)^{-1} \rangle = \langle S'_1, h'_m \rangle \) with respect to the generating set \( S'_1 = S'_1 \cup \{ u'_2 (h'_m)^{-1} \} \).

Moreover, the measure of \( T_V \) is at most

\[
2D(k_1 - 1) + \frac{2k_1 - 1}{2k} \varepsilon + 2D + \frac{\varepsilon}{2k} \leq 2D((k_1 + 1) - 1) + \frac{2(k_1 + 1) - 1}{2k} \varepsilon.
\]

This implies that the tuple \( M'' = (S''_1, S''_2, \ldots, S''_n; H'') \) is good, where \( H'' = (h'_1, \ldots, h'_{m-1}) \). Since \( M'' \) has smaller complexity than does \( M \), this contradicts the choice of \( M \).

**Subcase 2B.** Suppose now that \( U'_1 \) fixes a point. Then the assumption \( d(X(U'_1), h'_m X(U'_1)) \leq \frac{\varepsilon}{2k} \) implies that there exist \( y_1 \in X(U'_1) \) and \( y_2 \in h'_m X(U'_1) \) such that \( d(y_1, y_2) \leq \frac{\varepsilon}{2k} \). Note that since \( U'_1 \) fixes a point, \( X_{U'_1} \) is the fixed set of \( U'_1 \). Denote \( y'_2 = (h'_m)^{-1} y_2 \in X(U'_1) \). Let \( x \) be a point fixed by \( U'_1 \). By Lemma 3.3, we have \( d(y_1, x) \leq D \) and \( d(y'_2, x) \leq D \). Put

\[
K = [y'_2, x] \cup [x, y_1] \cup [y_1, y_2]
\]

Then \( x \) is fixed by \( U'_1 \) and hence by \( S'_1 \), while \( h'_m y'_2 = y_2 \). Thus \( K \) is an \( S'_1 \)-generating tree for the subgroup \( V = \langle S'_1, h'_m \rangle \) where \( S''_1 = (S'_1, h'_m) \). Note that \( L(S''_1) = L(S'_1) + 1 \geq 2 \) by construction. The tree \( K \) has measure at most

\[
2D + \frac{\varepsilon}{2k} \leq 2D(2 - 1) + \frac{2 \cdot 2 - 1}{2k} \varepsilon.
\]

Again put \( H'' = (h'_1, \ldots, h'_{m-1}) \) and \( M'' = (S''_1, S''_2, \ldots, S''_n; H'') \). Then \( M'' \) is good and has smaller complexity than \( M \). This contradicts the choice of \( M \).

**Case 3.** Suppose that \( d(y, h'_m y) \leq \frac{\varepsilon}{2k} \) for some \( y \in X \). In this case we replace \( M' \) by

\[
M'' = (S'_1, \ldots, S'_n, S'_{n+1}; H'')
\]

where \( S'_{n+1} = (h'_n) \) and \( H'' = (h'_1, \ldots, h'_{m-1}) \).

By assumption on \( Y \) we have \( h'_m \neq 1 \). Hence \( M'' \) is good and of smaller complexity than \( M \), which again yields a contradiction. \( \square \)
Remark 3.7. Essentially the same argument as in the proof of Theorem 1.2 implies a relative version of our main result. If $U_1, \ldots, U_n$ are subgroups of a group $G$, we will say that $G$ is freely indecomposable relative to $U_1, \ldots, U_n$ if there does not exist an action of $G$ on a simplicial tree $X$ with trivial edge stabilizers and without inversions such that each $U_i$ fixes a vertex. Essentially the same argument as in the proof of Theorem 1.2 implies the following relative version of our main result.

Suppose $G$ is freely indecomposable relative to $U_1, \ldots, U_n$ and that $G$ is generated by $U_1 \cup \cdots \cup U_n \cup \{s_1, \ldots, s_k\}$. Suppose $G$ acts $D$-acylindrically on an $\mathbb{R}$-tree in such a way that each $U_i$ fixes a point. Then for any $\epsilon > 0$ there exists a $G$-generating tree of measure at most $2D(n + k - 1) + \epsilon$.

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