On The Number of Optimal Linear Index Codes
For Unicast Index Coding Problems

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Abstract—An index coding problem arises when there is a single source with a number of messages and multiple receivers each wanting a subset of messages and knowing a different set of messages a priori. The noiseless Index Coding Problem is to identify the minimum number of transmissions (optimal length) to be made by the source through noiseless channels so that all receivers can decode their wanted messages using the transmitted symbols and their respective prior information. Recently, it is shown that different optimal length codes perform differently in a noisy channel. Towards identifying the best optimal length index code one needs to know the number of optimal length index codes. In this paper we present results on the number of optimal length index codes making use of the representation of an index coding problem by an equivalent network code. Our formulation results in matrices of smaller sizes compared to the approach of Kotter and Medard [6]. Our formulation leads to a lower bound on the minimum number of optimal length codes possible for all unicast index coding problems [1] which is met with equality for several special cases of the unicast index coding problem. A method to identify the optimal length codes which lead to minimum-maximum probability of error is also presented.

I. INTRODUCTION

We consider the index coding problem first introduced by Birk and Kol in [3]. In an index coding (IC) problem, there is a single source with a set of messages and a set of receivers, each wanting a subset of messages and knowing a subset of messages (side-information) a priori. A general index coding problem can be formulated as follows: There are $n$ messages, $x_1, x_2, \ldots, x_n$ and $m$ receivers. Each receiver wants a subset of messages, $W_i$, and knows a subset of messages $K_i$. For a general unicast problem, $W_i \cap W_j = \emptyset$, for $i \neq j$.

A single uniprior IC problem is a scenario where each receiver knows a single unique message (not known to other receivers) a priori and a unicast problem is one where each receiver wants a unique set of messages, i.e., the intersection of the messages wanted by any two receivers is nullset. The general scenario is called group-icast IC problem. A single unicast is when the size of each of those wanted sets in a unicast problem is one. Without loss of generality a unicast problem can be reduced to a single unicast problem by increasing the number of receivers by splitting any receiver with more than one message into multiple receivers wanting only one message with identical side information. One needs to identify the minimum number of transmissions to be made so that all receivers can decode their wanted messages using the transmitted bits and their respective prior information. Ong and Ho in [11] gave an algorithm which finds the optimal length of a uniprior index coding problem. When a general unicast IC problem is modified to a single unicast IC problem one has $n = m$. In this paper we consider single unicast IC problem but the results apply to a general unicast problem as well.

El Rouayheb et. al. in [4] found that every index coding problem can be reduced to an equivalent network coding problem. An algebraic representation of network codes in terms of matrices representing the input mixing, topology and the output mixing operations was given by Koetter and Medard in [6]. In this paper we present a similar algebraic characterization of a single unicast IC problem after reducing it to an equivalent network code. Harvey et.al in [9] proposed an algorithm for network codes for multicast problems, which is based on a new algorithm for maximum-rank completion of mixed matrices. Our problem is not a multicast problem. Hence the results in [9] cannot be applied.

For a given index code a receiver may not use all the transmissions from the source. In fact, different receivers may use different number of transmissions of the source. It has been shown in [8] that there can be several linear optimal index codes in terms of lowest number of transmissions for an IC problem, but among them one needs to identify the linear optimal index code which minimizes the maximum number of transmissions that is required by any receiver in decoding its desired message. The motivation for this comes from the fact that each of the transmitted symbols is error prone in a wireless scenario and lesser the number of transmissions used in decoding the desired message, lesser will be its probability of error. Hence among all the codes with the same length, the one for which the maximum number of transmissions used by any receiver is the minimum, will have minimum-maximum error probability. This has already been discussed in [8] for single uniprior IC problems where a method to find a best linear solution in terms of minimum-maximum error probability among all codes with the optimal length is given. The contributions of this paper may be summarized as follows:

- A transfer matrix approach similar to that of Kotter and Medard [6], but with component matrices of much smaller sizes, is presented which enables identification of the optimal length for any unicast problem in terms of the component matrices of the transfer matrix.
- A lower bound on the number of optimal length codes for any single unicast index coding problem is obtained. This bound is shown to be exact for the following special cases:
(i) Single uniprior single unicast IC problems,
(ii) Single uniprior unicast IC problems and
(iii) Single unicast uniprior IC problems.

• For single uniprior unicast problems and single uniprior unicast problems, we obtain the length of optimal linear ICs by reducing the problem to that of single uniprior single unicast IC problem.

• A criterion for optimal linear index codes with minimum-maximum probability of error is presented in terms of the component matrices for any single unicast problem.

The remaining content is organized as follows: In Section II the equivalence of index coding problem to network coding problem is discussed and in Section III we obtain the input-mixing matrix, transfer matrix and the output-mixing matrices for the index coding problem and show that they can be partitioned into submatrices corresponding to the side-information and the index code used. Bounds on the number of optimal linear index codes is presented in Section IV and the number of codes with optimal length is discussed in Section V. In Section VI a method to identify optimal codes with minimum-maximum error probability is described and simulation results are presented.

II. EQUIVALENT NETWORK CODING PROBLEM

Throughout the paper we assume that the operations are over the finite field with two elements \( \mathbb{F}_2 \). But the results easily carry over to other finite field. Any single unicast problem can be represented by an equivalent network coding problem as in Fig. 1. This was proposed by El Rouayheb et. al. in [4].

Suppose the length of the index code, not necessarily optimal, be \( c \). Each of the messages \( x_1, x_2, \ldots, x_n \) is represented by a source node and \( g_1, g_2, \ldots, g_c \) represent the broadcast channel and \( l_1, l_2, \ldots, l_c, l'_1, l'_2, \ldots, l'_c \) represent the intermediate nodes. It is assumed that when two or more edges have the same tail node with only one incoming edge to it they carry the same message. Hence, \( l'_i \) transmits to its outgoing edges whatever it gets through \( g_i \). The source nodes \( X_i \) transmit their respective messages as such through their outgoing edges. The minimum possible value of \( c \) among all linear solutions of an IC problem is the optimal length. The dashed lines represent the connection between a receiver node and a source message node whose message is known to the receiver apriori, i.e., they represent the side information possessed by the receivers.

For every single unicast problem, we can find a graph like Fig. 1 which we will denote as \( G \). The graph \( G \) can be represented as \( G = (V, E) \), where \( V = \{x_1, x_2, \ldots, x_n, l_1, l_2, \ldots, l_c, l'_1, l'_2, \ldots, l'_c, R_1, R_2, \ldots, R_n\} \) is the vertex set and \( E \) is the edge set. It is easy to see that \(|E| = (2n + 1)c + \sum_{i=1}^{n} |K_i| \). An edge connecting vertex \( v_1 \) to \( v_2 \) is denoted by \((v_1, v_2)\) where \( v_1 \) is the tail of the edge and \( v_2 \) is the head of the edge. For an edge \( e \), \( Y(e) \) represents the bit passing through that edge. We can get a transfer matrix \( M_{n \times n} \) (which is shown in Section III) such that \( Z = [z_1 \ z_2 \ \ldots \ z_n]^T \), the vector of output messages from all the receivers, can be expressed as

\[
Z = M X,
\]

where \( X = [x_1 \ x_2 \ \ldots \ x_n]^T \), the vector of input messages. Hence, the considered index code of length \( c \) gives a solution in \( c \) number of transmissions if \( M \) is the identity matrix.

III. TRANSFER FUNCTION MATRICES FOR AN INDEX CODE

For a general single unicast problem, we can find a matrix \( M_{n \times n} \) in (1) such that \( M \) is a product of three matrices as

\[
M = B \ F \ A
\]

where the matrix \( A \) relates the input messages and the messages flowing through the outgoing edges of all the source nodes, the matrix \( F \) relates to the messages sent in the broadcast channel and the side information possessed by the the receivers, and the matrix \( B \) describes the decoding operations done at the receivers. All these three matrices can be partitioned in to two parts one corresponding to the index codeword transmission and the other corresponding to

\[2\]We are not following Koetter and Medard’s approach [2]. The approach in [6] would have given matrix \( A \) of order \(|E| \times n\), \( F \) of order \(|E| \times |E|\) and \( B \) of order \((n \times |E|) \) where \(|E| = (2n+1)c + \sum_{i=1}^{n} |K_i| \). Our formulation results in matrices \( A, F \) and \( B \) for a given index coding problem, where \(|E| = nc + \sum_{i=1}^{n} |K_i| \). Notice that our matrices \( A, F \) and \( B \) are of much smaller sizes compared to the sizes one deals with by following the approach in [6].
only the side information. Now we proceed to describe these partitioning.

**Partitioning of \( A \):** The matrix \( A \) satisfies the relation

\[
Y = AX,
\]

where \( Y^T \) is as in (2), with \( K_{i,j} \) denoting the index of \( j \)-th message in the side information set of receiver \( R_i \) and \( X = [x_1 \ x_2 \ x_3 \ldots \ x_n]^T \) is the vector of input messages. The vector \( Y \) is the vector of messages flowing through the outgoing edges of all the source nodes and is of order \((nc + \sum_{i=1}^{n} |K_i|) \times n \). The matrix \( A \) is of order \((nc + \sum_{i=1}^{n} |K_i|) \times n \) and it can be split in the form,

\[
A = \begin{bmatrix} A_{BC} \\ A_{SI} \end{bmatrix}
\]

where \( A_{BC} \) is of order \( nc \times n \) and \( A_{SI} \) is of order \( \sum_{i=1}^{n} |K_i| \times n \) (the subscript \( BC \) stands for "broadcast part" and the subscript \( SI \) for "side information part"). The matrix \( A_{BC} \) is a matrix formed by row-concatenation of matrices \( A_i, i = 1,\ldots,n \) where each \( A_i \) is a \( c \times n \) matrix in which all elements in the \( i \)-th column are ones and the rest all are zeros as given in (5).

\[
A_i = \begin{bmatrix}
1 & 0 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & 0 & \ldots & 0 \\
\vdots & & & & & \\
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & 0 & \ldots & 0 \\
\vdots & & & & & \\
0 & 0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & 0 & \ldots & 1 \\
\vdots & & & & & \\
0 & 0 & 0 & 0 & \ldots & 1
\end{bmatrix}
\]

Each \( A_i \) corresponds to the message passed by the source node \( x_i \) to the intermediate nodes, \( l_j, j = 1,\ldots,c \).

The matrix \( A_{SI} \) has only one non-zero element (which is one) in each row. This matrix corresponds to the side information possessed by the receivers and each successive set of \( |K_i| \) rows correspond to the side information possessed by \( R_i \) for \( i = 1 \) to \( n \). In each set of \( |K_i| \) rows, each row is distinct and has only one non-zero element which occupies the respective column position of one of the messages in the prior set of \( R_i \).

Notice that matrix \( A \) is fixed for a fixed \( c \) and does not depend on the index code except on its length.

**Partition of the matrix \( F \):** The matrix \( F \) which relates the message symbols to the transmission symbols sent in the broadcast channel and the side information possessed by the the receivers is of order \((nc + \sum_{i=1}^{n} |K_i|) \times (nc + \sum_{i=1}^{n} |K_i|) \).

It satisfies

\[
Y' = FX = FY = FAX,
\]

where \( Y'^T \) is as in (11), and is the vector of messages flowing to each of the receiver. We can observe that \( F \) can be split into four block matrices as

\[
F = \begin{bmatrix} F_{BC} & 0 \\ 0 & I \end{bmatrix}.
\]

The Matrix \( F_{BC} \) is a square matrix of order \( nc \) which is of the form given in (6) and \( I \) is the identity matrix of size \( \sum_{i=1}^{n} |K_i| \). The elements \( \beta_{X_i,l_j}, \forall i = 1,\ldots,n \) and \( j = 1,\ldots,c \) belong to \( F_2 \). All the \( ((i - 1)n + 1)\)-th to \( ((i - 1)n + n)\)-th row are identical for \( i = 1,2,\ldots,c \). If any one of these rows is denoted by \( t_i \), then we have the \( i \)-th transmission symbol of the code given by

\[
g_i = t_i A_{BC} X = \sum_{j=1}^{n} \beta_{X_i,l_j} x_j
\]

for \( i = 1,2,\ldots,c \). Notice that we are using \( g_i \) to denote both the \( i \)-th symbol of the index codeword as well as the edge carrying that symbol.

**Partitioning of matrix \( B \):** The matrix \( B \) that describes the decoding operations done at the receivers is of order \( n \times (nc + \sum_{i=1}^{n} |K_i|) \). It satisfies the relation,

\[
Z = BY' = BFAX.
\]
\[ Y^T = [Y((l'_1, R_1)) Y((l'_1, R_2)) \ldots Y((l'_1, R_n)) \\
Y((l'_2, R_1)) Y((l'_2, R_2)) \ldots Y((l'_2, R_n)) \\
\vdots \\
Y((l'_n, R_1)) Y((l'_n, R_2)) \ldots Y((l'_n, R_n)) \\
Y((x_{K_11}, R_1)) Y((x_{K_12}, R_1)) \ldots Y((x_{K_1|K_1|}, R_1)) \\
Y((x_{K_21}, R_2)) Y((x_{K_22}, R_2)) \ldots Y((x_{K_2|K_2|}, R_2)) \\
\vdots \\
Y((x_{K_{n1}}, R_n)) Y((x_{K_{n2}}, R_n)) \ldots Y((x_{K_{n|K_n|}}, R_n)) \]  

(5)

\[
F_{BC} = 
\begin{bmatrix}
\beta_{X_1,l_1} & 0 & \ldots & 0 & \beta_{X_2,l_1} & 0 & \ldots & 0 & \beta_{X_n,l_1} & 0 & \ldots & 0 \\
\beta_{X_1,l_1} & 0 & \ldots & 0 & \beta_{X_2,l_1} & 0 & \ldots & 0 & \beta_{X_n,l_1} & 0 & \ldots & 0 \\
\vdots & & & & & & & & & & & \\
\beta_{X_1,l_1} & 0 & \ldots & 0 & \beta_{X_2,l_1} & 0 & \ldots & 0 & \beta_{X_n,l_1} & 0 & \ldots & 0 \\
0 & \beta_{X_1,l_2} & \ldots & 0 & 0 & \beta_{X_2,l_2} & \ldots & 0 & 0 & \beta_{X_n,l_2} & \ldots & 0 \\
0 & \beta_{X_1,l_2} & \ldots & 0 & 0 & \beta_{X_2,l_2} & \ldots & 0 & 0 & \beta_{X_n,l_2} & \ldots & 0 \\
\vdots & & & & & & & & & & & \\
0 & \beta_{X_1,l_2} & \ldots & 0 & 0 & \beta_{X_2,l_2} & \ldots & 0 & 0 & \beta_{X_n,l_2} & \ldots & 0 \\
0 & \beta_{X_1,l_2} & \ldots & 0 & 0 & \beta_{X_2,l_2} & \ldots & 0 & 0 & \beta_{X_n,l_2} & \ldots & 0 \\
\vdots & & & & & & & & & & & \\
0 & \beta_{X_1,l_2} & \ldots & 0 & 0 & \beta_{X_2,l_2} & \ldots & 0 & 0 & \beta_{X_n,l_2} & \ldots & 0 \\
0 & \beta_{X_1,l_2} & \ldots & 0 & 0 & \beta_{X_2,l_2} & \ldots & 0 & 0 & \beta_{X_n,l_2} & \ldots & 0 \\
\vdots & & & & & & & & & & & \\
0 & \beta_{X_1,l_2} & \ldots & 0 & 0 & \beta_{X_2,l_2} & \ldots & 0 & 0 & \beta_{X_n,l_2} & \ldots & 0 \\
\end{bmatrix}
\]

(6)

\[
B_{BC} = 
\begin{bmatrix}
\epsilon_{y_1,R_1} & 0 & 0 & \ldots & 0 & \epsilon_{y_2,R_1} & 0 & 0 & \ldots & 0 & \epsilon_{y_3,R_1} & 0 & 0 & \ldots & 0 \\
0 & \epsilon_{y_1,R_2} & 0 & \ldots & 0 & \epsilon_{y_2,R_2} & 0 & \ldots & 0 & \epsilon_{y_3,R_2} & 0 & 0 & \ldots & 0 \\
\vdots & & & & & & & & & & & & & & \\
0 & 0 & \ldots & \epsilon_{y_1,R_n} & 0 & 0 & \ldots & \epsilon_{y_2,R_n} & 0 & \ldots & \epsilon_{y_3,R_n} & 0 & 0 & \ldots & 0 \\
\end{bmatrix}
\]

(9)

\[
F_{BC} = 
\begin{bmatrix}
\beta_{X_1,l_1} & 0 & \beta_{X_2,l_1} & 0 & \beta_{X_3,l_1} & 0 \\
\beta_{X_1,l_1} & 0 & \beta_{X_2,l_1} & 0 & \beta_{X_3,l_1} & 0 \\
\beta_{X_1,l_1} & 0 & \beta_{X_2,l_1} & 0 & \beta_{X_3,l_1} & 0 \\
0 & \beta_{X_1,l_2} & 0 & \beta_{X_2,l_2} & 0 & \beta_{X_3,l_2} \\
0 & \beta_{X_1,l_2} & 0 & \beta_{X_2,l_2} & 0 & \beta_{X_3,l_2} \\
0 & \beta_{X_1,l_2} & 0 & \beta_{X_2,l_2} & 0 & \beta_{X_3,l_2} \\
\end{bmatrix}
\]

(13)

\[
B = 
\begin{bmatrix}
\epsilon_{(l'_1,R_1)} & 0 & 0 & \epsilon_{(l'_1,R_1)} & 0 & 0 & \epsilon_{(x_2,R_1)} & 0 & 0 \\
0 & \epsilon_{(l'_1,R_2)} & 0 & 0 & \epsilon_{(l'_1,R_2)} & 0 & 0 & \epsilon_{(x_3,R_2)} & 0 \\
0 & 0 & \epsilon_{(l'_1,R_3)} & 0 & \epsilon_{(l'_1,R_3)} & 0 & 0 & \epsilon_{(x_1,R_3)} & 0 \\
\end{bmatrix}
\]

(14)
In terms of the symbols,
\[ z_j = \sum_{i=1}^c \epsilon_{i',R_i} g_i = \sum_{i=1}^c \epsilon_{i',R_i} \left( \sum_{k=1}^n \beta_{x_k,i} x_k \right) \]
(16)
gives the symbols obtained by the receivers after all the operations.

The matrix \( B \) can be split into two block matrices as
\[ B = \begin{bmatrix} B_{BC} & B_{SI} \end{bmatrix} , \]
where \( B_{BC} \) is a matrix of order \( n \times nc \) and in every row only \( c \) elements are non-zero and the non-zero elements correspond to whether or not \( R_i \) uses that particular transmission to decode its wanted message. The matrix \( B_{SI} \) is of order \( n \times \sum_{i=1}^n | K_i | \). It relates to the side information possessed by the receivers. In this matrix all elements except the \( i \)-th element in every successive set of \( | K_i | \) columns are zeros, for all \( i = 1 \) to \( n \). The rest of the elements are either one or zero and it depends on the messages used by a receiver to decode its wanted message. The matrix \( B_B \) is as in (9). The elements \( \epsilon_{i',R_i} \) for \( j = 1, \ldots, c \) and \( i = 1, \ldots, n \) belong to \( \mathbb{F}_2 \). From (9), (10) and (15), we get
\[ Z = B F A X . \]  
(18)

So,
\[ M = B F A . \]  
(19)

An index coding problem is solvable with \( c \) number of transmissions if we can find variables \( \beta ' \)'s in \( F_{BC} \) which define the code and the variables \( \epsilon ' \)'s in \( B_{BC} \) which define the decoding operations such that \( M \) is an identity matrix.

The following example illustrates the partitioning of the matrices \( A, F \) and \( B \).

**Example 1.** Let \( m = n = 3 \). Each \( R_i \) wants \( x_i \) and knows \( x_{i+1} \), where \( + \) is mod-3 addition. The optimal length of a linear IC solution for this problem is 2, which is established in Example 1 (continued) Section IV by way of showing that length one codes do not exist. The graph \( G \) for \( c = 2 \) is as in Fig. 2.

The \( A \) matrix is as below.
\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \]  
(20)

The last three rows of the above matrix is \( A_{SI} \). The matrix \( F_{BC} \) is as in (13) and \( B \) matrix is as in (14). There are three linear codes which are optimal in terms of length. They are
\[ \mathcal{C}_1 = \{ x_1 \oplus x_2, x_2 \oplus x_3 \} , \]
\[ \mathcal{C}_2 = \{ x_1 \oplus x_3, x_3 \oplus x_2 \} , \]
\[ \mathcal{C}_3 = \{ x_1 \oplus x_3, x_1 \oplus x_2 \} . \]

For the codes \( \mathcal{C}_1, \mathcal{C}_2 \) and \( \mathcal{C}_3 \), the matrices \( F_{BC} \) and \( B \) are as in (22), (23) and (24) respectively. The matrix \( B_{SI} \) is given by
\[ B_{SI} = \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix} \]  
(21)

where \( x \) is either 0 or 1.

**IV. Bounds on the number of optimal linear ICs**

We have analyzed the structures of the three matrices in the previous section. We need \( M = B F A \) to be \( I_n \), the identity matrix. Here for a fixed length \( c \), \( A \) is fixed and as can be verified all the columns of \( A \) are independent and hence the rank of \( A \) is \( n \). So, the rows of \( I_n \) lie in the row space of \( A \). Therefore, the equation
\[ T_{n \times (nc + \sum_{i=1}^n | K_i |)} A = I_n \]
has at least one solution for the variables of the matrix \( T \) which is a left-inverse of \( A \). Observe that the number of free variables in \( T \) is \( n^2c - n^2 + n \sum_{i=1}^n | K_i | \) and the number of pivot variables is \( n^2c \) [7]. Hence the number of left inverses of \( A \) is \( n^2c - n^2 + n \sum_{i=1}^n | K_i | . \) We need to find a matrix \( T \) which is a left inverse of \( A \) as well as is a product \( BF \) of some \( B \) and \( F \)
in the required form. Let us denote the set of all left inverses of \( A \) which are of the form \( BF \) in the required form, by \( S(c) \) since it is a function of \( c \). Note that \( B \) and \( F \) are needed to be in the form that includes the constraints imposed by the side-information of the index coding problem. Since \( BF = \left[ \begin{array}{cc} B_{BC} B_{SI} & F_{BC} \\ 0 & I \end{array} \right] \), let \( T = \left[ \begin{array}{c} T_{BC} T_{SI} \end{array} \right] \). This gives \( T_{SI} = B_{SI} \). \( T_{SI} = B_{SI} \) (26)

So the positions which are to be occupied by zeros in \( B_{SI} \) are needed to be zeros in \( T_{SI} \) also. Therefore, \( T_{SI} \) which is of order \( n \times \sum_{i=1}^{n} | K_i | \) has \( (n-1)(\sum_{i=1}^{n} | K_i |) \) zeroes and when the rest of the elements of \( T_{SI} \) are fixed, \( B_{SI} \) also gets fixed. Let the left inverses of \( A \) that satisfy the constraint (26) be denoted by \( S'(c) \). Clearly \( S(c) \subseteq S'(c) \). As the rank of \( A \) is \( n \), the total number of left inverses of \( A \) with restrictions given by (26) is

\[
|S'(c)| = 2^{n^2-n^2+\sum_{i=1}^{n} |K_i|}.
\]

Since \( S(c) \subseteq S'(c) \), we have,

\[
|S(c)| \leq 2^{n^2-n^2+\sum_{i=1}^{n} |K_i|}.
\]

We need to identify the elements in the set \( S'(c) \) which also belong to \( S(c) \). First of all, when we fix \( T \), the submatrix \( B_{SI} \) gets fixed. So, for a pair \( (B, F) \) which belongs to set \( S(c) \), we have

\[
B_{BC} F_{BC} = T_{BC}.
\]

From (29) we get relations of the form,

\[
\left[ \begin{array}{c} \epsilon_{i'_1, R_1} \\ \vdots \\ \epsilon_{i'_n, R_n} \end{array} \right] = \beta_{X_{k'}, l'_{i}} = \left[ T_{col'i_{k-1}c+1} \right]
\]

\[\forall k \in \{1, 2, ..., n\} \text{ and } \forall i \in \{1, 2, ..., c\} \text{ where } T_{col'i} \text{ is the } i\text{-th column of } T_{BC} \].

For \( i = 1, 2, ..., c \), we define the \( n \times n \) matrix \( R_i \) as consisting of the \( n \) columns \( \{T_{col'i}, T_{col'i+1}, ..., T_{col'(n-1)c+1} \} \) of the matrix \( T_{BC} \). Notice that the matrix \( R_i \) consists of transmissions to all the receivers from the \( i \)-th transmission of the index code. Also note that

\[
R_i = \left[ \begin{array}{c c c c} \epsilon_{i'_1, R_1} \\ \vdots \\ \epsilon_{i'_n, R_n} \end{array} \right] \left[ \begin{array}{c} \beta_{X_1, l_i} \\ \beta_{X_2, l_i} \\ \vdots \\ \beta_{X_n, l_i} \end{array} \right] \]

and

\[
R_i = \left[ \begin{array}{c c c c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right] \left[ \begin{array}{c} \epsilon_{i'_1, R_1} \\ \vdots \\ \epsilon_{i'_n, R_n} \end{array} \right] \left[ \begin{array}{c} \beta_{X_1, l_i} \\ \beta_{X_2, l_i} \\ \vdots \\ \beta_{X_n, l_i} \end{array} \right] \left[ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right].
\]

Equation (32) above shows that the contribution of the \( i \)-th transmission of an index code is completely captured by the matrix \( R_i \).

**Lemma 1.** Any matrix \( T \) which belongs to \( S'(c) \) also belongs to \( S(c) \) if and only if \( R_i \) is a all-zero matrix or is a rank one matrix, for all \( i = 1, 2, ..., c \).

**Proof.** (Only-if part): If \( T \in S(c) \), then from (31), it follows that either \( R_i \) is a all-zero matrix or its rank is one.

Proof of ‘if part’ : For a \( T \in S'(c) \), for all \( i \), one can always find values for variables \( \epsilon \)'s and \( \beta \)'s satisfying (31). Hence one can get a pair \( (B, F) \) such that (29) is satisfied by substituting these values. Hence \( T \in S(c) \). This completes the proof.

However for a \( T \in S(c) \), if \( R_i \) is a zero matrix, then either all the \( \beta \)'s or \( \epsilon \)'s corresponding to \( R_i \) in (31) are zeros. When all the \( \beta \)'s are zeros, the \( \epsilon \)'s can take any of the \( 2^n \) values possible and vice versa. Hence the number of possibilities for a zero matrix is \( 2^{n+1} - 1 \). Hence the total number of \( (B, F) \) possible for a \( T \) matrix is \( (2^{n+1} - 1)^\lambda \), where \( \lambda, 0 \leq \lambda \leq c \).
is the number of zero matrices among $R_i$'s for $i = 1, 2, \ldots, c$.

**Theorem 1.** A length $c$ is optimal for a linear index coding problem if and only if for all the matrices $T \in S(c)$ none of the $n$ number of $R_i$ matrices is a null matrix. In other words, $\lambda = 0$ for every $T$ in $S(c)$.

**Proof.** Proof for 'only if' part: We need to prove that if there exists a $T \in S(c)$ whose $\lambda \neq 0$ for a particular length $c$, then $\lambda$ is not the optimal transmission length. Since $\lambda \neq 0$, let $1 \leq i \leq c$ be such that $R_i$ is the all-zero matrix. Then we have either all the $\beta_i$'s or all the $\epsilon_i$'s corresponding to $R_i$ are zeros. If all the $\epsilon_i$'s are zeros, that means that one particular transmission is not used by any of the receivers. Else, if all the $\beta_i$'s corresponding are zeroes, then no message is transmitted in one particular transmission. In either case, we can remove at least one transmission. Hence $c$ is not the optimal length. The proof for 'if part' is by contradiction. Assume that a length $c$ exists such that it is feasible but not optimal and all the matrices in $S(c)$ have $\lambda = 0$. Assume further that $c' = c - r$ for some $r > 0$, is the optimal length. Then take one feasible solution with length $c'$ and add extra $nr$ rows to the corresponding $F_B$ matrix and some extra $nc$ all zero columns to $B_B$. Let us call the new matrices $F_B'$ and $B_B'$. Let $g_i', i = 1, \ldots, c$ be the set of broadcast messages given by $F_B'$ and $g_i$ be those which are given by $F_B$. One can observe that $\{g_1', g_2', \ldots, g_c'\}$ is nothing but $\{g_1, g_2, \ldots, g_c\}$ plus some additional information. Hence when one sends $\{g_1', g_2', \ldots, g_c'\}$, the receivers get whatever they would have got if $\{g_1, g_2, \ldots, g_c\}$ was sent. Hence even if they do not use the extra transmissions given by $F_B'$, they will be able to decode their wanted messages. Hence the product of $F_B'$ and $B_B'$ matrices should belong to $S(c)$ (as it is a feasible index code) and has $\lambda \neq 0$, which is a contradiction. Hence $c$ is the optimal length.

Theorem 1 is illustrated in Example-1 (continued) and Example-2 below.

**Example 1.** (continued) We illustrate Theorem 1 for the problem in Example 1. We will prove $c = 1$ is not possible in this case. With $n = 3$ and $c = 1$, from [23] there can be at most are $2^{12}$ matrices in $S(c)$. The matrix $B_{S1}$ is of the form

\[
\begin{bmatrix}
    x & 0 & 0 \\
    0 & x & 0 \\
    0 & 0 & x
\end{bmatrix}
\]

where $x$ can be either 0 or 1. From [27], we have $2^3 = 8$ matrices that belong to $S'(1)$. We found these by brute force among $2^{12}$ matrices which has zeros at places which are occupied by zeros strictly in the corresponding $B_{S1}$. These eight matrices are given below.

\[
\begin{bmatrix}
    1 & 1 & 0 & 1 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
    1 & 1 & 0 \\
    0 & 0 & 1 \\
    0 & 0 & 1
\end{bmatrix}
\]

Since $c = 1$ there is only one $R_1$ which is the $3 \times 3$ submatrix consisting of the first 3 columns in each of the 8 matrices above. Clearly none of these matrices have rank one. Hence, there does not exist a solution with $c = 1$.

**Example 2.** Let $n = m = 3$ and $R_i$ wants $x_i$, \( \forall i \in \{1, 2, 3\} \). $R_1$ knows $x_2$ and $x_3$. $R_2$ knows $x_3$. $R_3$ knows $x_1$. We will show that the optimal value of $c$ is 2. For $c = 1$, size of $S'(c) = 16$ (from [27]). The 16 matrices which belong to $S'(1)$ have been found by brute force among $2^{12}$ matrices which has zeros at places, which are to be occupied strictly by zeros in the corresponding $B_{S1} = \begin{bmatrix} x & x & 0 \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$ where $x$ stands for either 0 or 1. These 16 matrices are shown below.

\[
\begin{bmatrix}
    1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 1 \\
    0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 1 \\
    1 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
    1 & 1 & 1 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]
at least Theorem 2. which is equal to (36). F this case. For the case to the set $S(3)$.  

$$T = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Since $c = 1$, there is only one $R_1$ for each of the 16 matrices above and this $R_1$ is the $3 \times 3$ submatrix consisting of the first 3 columns. It is easily seen that all these matrices have rank more than 1. Hence $c = 1$ is not a feasible length for this case. For the case $c = 3$, the following matrix $T$ belongs to the set $S(3)$.

$$T = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(34)

For this matrix $T$, we have 

$$R_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, R_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

We see that $\lambda = 1$ due to the all-zero matrix $R_3$ and also the rank of the other two matrices is one. Hence $c = 3$ is not optimal. Therefore, $c = 2$ should be the optimal length.

V. MINIMUM NUMBER OF CODES POSSIBLE FOR AN OPTIMAL $c$

In this subsection, we find a lower bound on the number of linear codes which are optimal in terms of length or equivalently in terms of bandwidth, for a single unicast index coding problem. For the optimal $c$, the number of matrices which are left inverses of $A$ and is a product of some $B$ and $F$ gives the number of codes possible with that length, which is also the size of the set $S(c)$. But for any $T \in S(c)$,

$$T_{BC}A_{BC} = B_{BC}F_{BC}A_{BC} = I - T_{SI}A_{SI} = I - B_{SI}A_{SI}$$

which is equal to (36).

**Theorem 2.** The number of linear index coding solutions having optimal length $c$ for a single unicast IC problem is at least

$$\frac{1}{c!} \prod_{i=1}^{c-1} (2^c - 2^i)$$

(37)

**Proof.** Consider (35) and (36). Here if both RHS of (35) and the second matrix in (36) are fixed, a solution which is the first matrix in (36) will exist only if the row space of RHS of (35) is spanned by the rows of the second matrix in (36). But the rank of the second matrix in (36) is at most $c$. Hence this is possible only if the rank of the RHS matrix in (35) is less than or equal to $c$. The number of possible $B_{SI}$ matrices is $2^{\sum_{i=1}^{c} k_i}$. Since $c$ is the optimal length, there should be at least one $B_{SI}$ such that RHS of (35) is of rank $c$. For any such RHS of (35), we can take the second matrix in (36) in

$$(2^c - 1) \prod_{i=1}^{c-1} \left( 2^c - 2^i \right)$$

ways such that the row spaces of both the matrices are same. Each such matrix is an index code, which is feasible, and each row of the matrix represents a transmission. As order of transmissions does not matter, we need to neglect those matrices which are row-permuted versions of one another. Hence, total number of distinct transmission schemes possible is $\frac{1}{c!} \prod_{i=0}^{c-1} (2^c - 2^i)$. But there may be more than one $B_{SI}$ matrices which are of rank $c$ and whose row spaces are different. Hence the total number of index codes possible can be more than (37) as we take into account all possible basis sets of each of the different row spaces. (Example 3 is such a case.) Hence (37) is a lower bound on the number of index codes possible. □

Note that all possible matrices occupying RHS of (35) are exactly the collection of matrices which fits the index coding problem as per the definition of a fitting matrix in [2]. Hence algebraically we have proved the already established result in [2] that the optimal length of a linear solution is the minimum among the ranks of all the matrices which fits the IC problem. However, we will see subsequently that the matrices $A_{SI}$ and $I - B_{SI}A_{SI}$ will be useful in finding the optimal length and the number of optimal linear ICs in some special cases much more easily than using fitting matrices. Moreover, combining equations (12), (35) and (36), we see that the different bases of the $I - B_{SI}A_{SI}$ matrix can be used to obtain all possible linear optimal ICs. The elements of each such basis are precisely the codewords of the code determined by the chosen basis. This is illustrated in detail in Example 5.

Example 3. This example is to illustrate Theorem 2. Let $m = n = 4$. $R_i$ wants $x_i$ and knows $x_{i+1}$, where $+$ is modulo-4 addition. Here all possible matrices of the form $(I - B_{SI}A_{SI})$ as given by (33), denoted by $L_i$, $i = 1, \ldots, 16$ are shown below. Only $L_5$ satisfies the requirement given in the proof of Theorem 2. The set of all optimal index codes is given by the collection of all possible basis of the row space of this matrix. They are 28 in number. We list out those codes in Table 1.

| $L_1$ | $L_2$ | $L_3$ | $L_4$ | $L_5$ | $L_6$ | $L_7$ | $L_8$ |
|-------|-------|-------|-------|-------|-------|-------|-------|
| $[1 \ 0 \ 0 \ 0]$ | $[1 \ 0 \ 0 \ 0]$ | $[1 \ 1 \ 0 \ 0]$ | $[1 \ 1 \ 0 \ 0]$ | $[1 \ 1 \ 0 \ 0]$ | $[1 \ 0 \ 0 \ 0]$ | $[1 \ 0 \ 0 \ 0]$ | $[1 \ 0 \ 0 \ 0]$ |
| $[0 \ 1 \ 0 \ 0]$ | $[0 \ 1 \ 0 \ 0]$ | $[0 \ 1 \ 0 \ 0]$ | $[0 \ 1 \ 0 \ 0]$ | $[0 \ 1 \ 0 \ 0]$ | $[0 \ 1 \ 0 \ 0]$ | $[0 \ 1 \ 0 \ 0]$ | $[0 \ 1 \ 0 \ 0]$ |
| $[0 \ 0 \ 1 \ 0]$ | $[0 \ 0 \ 1 \ 0]$ | $[0 \ 0 \ 1 \ 0]$ | $[0 \ 0 \ 1 \ 0]$ | $[0 \ 0 \ 1 \ 0]$ | $[0 \ 0 \ 1 \ 0]$ | $[0 \ 0 \ 1 \ 0]$ | $[0 \ 0 \ 1 \ 0]$ |
| $[0 \ 0 \ 0 \ 1]$ | $[0 \ 0 \ 0 \ 1]$ | $[0 \ 0 \ 0 \ 1]$ | $[0 \ 0 \ 0 \ 1]$ | $[0 \ 0 \ 0 \ 1]$ | $[0 \ 0 \ 0 \ 1]$ | $[0 \ 0 \ 0 \ 1]$ | $[0 \ 0 \ 0 \ 1]$ |
optimal length $c$ for a single unicast IC problem is given by

$$
\frac{\mu}{c!} \sum_{i=0}^{c-1} (2^c - 2^i)
$$

(38)

where $\mu$ is the number of distinct row spaces of $c$-rank RHS matrix of $[25]$ obtainable from all possible choices of $B_{S1}$ matrices out of the $2^{N-1} |K_i|$ possible ones.

**Proof.** The proof of this follows from that of Theorem 2. □

Note that $\mu = 1$ for Example 1 and Example 2. The following example is a case with $\mu = 2$.

**Example 4.** This example illustrates Corollary 7. Let $m = n = 4$. $R_i$ wants $x_i$ and knows $x_{i+1}$ where $i$ is modulo-4 operation. $x_3$ knows $x_1$ also. The optimal length is $c = 3$ and it can be checked that $\mu = 2$ and the corresponding $B_{S1}$ matrices are $B_{S1,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and $B_{S1,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. These have been obtained

---

**Table I: All possible optimal linear solutions for Example 3**

| Code | Encoding |
|------|----------|
| $E_1$ | $x_1 + x_2, x_2 + x_3, x_3 + x_4$ |
| $E_2$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ |
| $E_3$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_1 + x_4$ |
| $E_4$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_1 + x_3$ |
| $E_5$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |
| $E_6$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |
| $E_7$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_1 + x_3 + x_4$ |
| $E_8$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_1 + x_3 + x_4$ |
| $E_9$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_1 + x_4$ |
| $E_{10}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_1 + x_4$ |
| $E_{11}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_1 + x_3 + x_4$ |
| $E_{12}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_1 + x_3 + x_4$ |
| $E_{13}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_1 + x_3 + x_4$ |
| $E_{14}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |
| $E_{15}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |
| $E_{16}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |
| $E_{17}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |
| $E_{18}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |
| $E_{19}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |
| $E_{20}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |
| $E_{21}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |
| $E_{22}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |
| $E_{23}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |
| $E_{24}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |
| $E_{25}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |
| $E_{26}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |
| $E_{27}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |
| $E_{28}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3 + x_4$ |

**Table II: Optimal linear solutions corresponding to $B_{S1,1}$ for Example 4**

| Code | Encoding | $t_{max}$ ($T$) |
|------|----------|----------------|
| $E_1$ | $x_1 + x_2, x_2 + x_1, x_4$ | 3 |
| $E_2$ | $x_1 + x_2, x_2 + x_1, x_4$ | 2 |
| $E_3$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 2 |
| $E_4$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_5$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 2 |
| $E_6$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 2 |
| $E_7$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_8$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_9$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 2 |
| $E_{10}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{11}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{12}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{13}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{14}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{15}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{16}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{17}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{18}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{19}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{20}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{21}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{22}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{23}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{24}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{25}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{26}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{27}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
| $E_{28}$ | $x_1 + x_2, x_2 + x_3, x_1 + x_4$ | 3 |
Consider a single unicast-single uniprior problem with $n$ receivers and $n$ messages. By construction, $B_{SI} = xI_{n \times n}$, which is a diagonal matrix with entries $x$ each one of which can take values from $\{0, 1\}$. Since the $j^{th}$ receiver does not know $x_i$ a priori, $(i, j)^{th}$ element in $A_{SI}$ is 0 for all $i \in \{1, 2, \ldots, n\}$. The matrix $A_{SI}$ is a permutation matrix which has no 1s on the diagonal. It permutes the columns of $B_{SI}$ such that $j^{th}$ column in $B_{SI}A_{SI}$ is not identical to the $j^{th}$ column of $B_{SI}$ for any $j \in \{1, 2, \ldots, n\}$. So, the matrix $B_{SI}A_{SI}$ has 0s on its diagonal. This implies that the $n \times n$ matrix $I - B_{SI}A_{SI}$ has 1s along its diagonal. Its $i^{th}$ row corresponds to the $i^{th}$ receiver $R_i$ and $i^{th}$ column corresponds to the message $x_i$. In a given row $r_i$, there are at most two 1s. The 1 on the diagonal corresponds to the message $x_i$ wanted by $R_i$ and the other 1 (say $(i,j)^{th}$ position) corresponds to the message $x_j$ known a priori by $R_i$.

Information flow graph $G$ on $n$ nodes can be constructed as follows. By convention, node $i$ corresponds to the receiver that knows message $x_i$. It has one incoming edge $(j, i)$ originating from the node $j$ where $x_j$ is the message wanted by the receiver $R_i$ and known by the receiver $R_j$. It has one outgoing edge $(i,k)$ terminating at node $k$ where $x_i$ is the message wanted by the receiver $R_k$ and known by $R_i$.

Identify the receiver that knows $x_1$. Suppose $i^{th}$ row has $x$ in the 1st column. This means $R_i$ knows $x_1$. Draw arc $(i, 1)$. Search along the 1st row to identify the message known a priori by $R_i$. Suppose it is $x_j$. Draw arc $(1, j)$. Now, search along the $j^{th}$ row to identify the message known a priori by $R_j$. Since the problem is single unicast, it can be either $x_k$ or $x_1$. If $x_k$, draw arc $(j, k)$ else draw arc $(j, 1)$. Each node has only one incoming edge since every receiver wants a unique message. Each node has only one outgoing edge as the message known a priori by the corresponding receiver is demanded by some other node. Since there are a finite number of nodes, we can conclude that the information-flow graph $G$ for a single unicast-single uniprior problem will be either one cycle of $n$ nodes or a set of disjoint cycles.

It was shown in (1) that for any single-uniprior problem represented by an information-flow graph $G(V, A)$, after executing the Pruning Algorithm, we have

$$ I^+(G) = \sum_{i=1}^{N_{sub}} (V(G'_{sub,i}) - 1) + A(G' \setminus G'_{sub}) $$

where $I^+(G)$ is the optimal length of the index code, $A(G' \setminus G'_{sub})$ is the number of arcs in $A' \setminus A_{sub} : G' = G'_{sub} \cup (G \setminus G'_{sub})$ where $G'_{sub} = \bigcup_{i=1}^{N_{sub}} G'_{sub,i}$ is a graph consisting of non-trivial strongly connected components $\{G'_{sub,i}\}$ and $G' \setminus G'_{sub} = (V' \setminus V'_{sub}) \setminus (A' \setminus A'_{sub})$.

When $G$ consists of only cycles, $A(G' \setminus G'_{sub}) = 0$. Thus, for a Single Unicast-Single Uniprior problem, the optimal length is given by

$$ c = \sum_{i=1}^{N_{sub}} (V(G'_{sub,i}) - 1) $$

Suppose that in the $B_{SI}$ matrix any 1 is replaced with 0, i.e., the corresponding side information is not used in decoding. This means a node is removed from the graph $G$. The resulting graph $G$ will have arcs apart from cycles. Consequently, $A(G' \setminus G'_{sub})$ component will be non-zero. This will increase the value of $I^+(G)$ which is the minimum number of transmissions required. Hence, we conclude that every side information bit must be used. Thus there is only

| Code | Encoding | $I_{max}(F)$ |
|------|----------|--------------|
| C20  | $x_1 + x_2, x_2 + x_1, x_4 + x_3$ | 3 |
| C21  | $x_1 + x_2, x_2 + x_1, x_3 + x_2$ | 2 |
| C22  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C23  | $x_1 + x_2, x_4, x_1 + x_4$ | 3 |
| C24  | $x_1 + x_2, x_1 + x_3, x_2 + x_3$ | 3 |
| C25  | $x_1 + x_2, x_2 + x_4, x_1 + x_4$ | 3 |
| C26  | $x_1 + x_2, x_1 + x_3, x_4 + x_3$ | 2 |
| C27  | $x_1 + x_2, x_2 + x_3, x_3 + x_2$ | 2 |
| C28  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C29  | $x_1 + x_2, x_1 + x_3, x_3 + x_2$ | 2 |
| C30  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C31  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C32  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C33  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C34  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C35  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C36  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C37  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C38  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C39  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C40  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C41  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C42  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C43  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C44  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C45  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C46  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C47  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C48  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C49  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C50  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C51  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C52  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C53  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C54  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C55  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |
| C56  | $x_1 + x_2, x_2 + x_3, x_4 + x_3$ | 2 |

TABLE III: Optimal linear solutions corresponding to $B_{SI,2}$ for Example 2.
one possible choice of $B_{SI}$ matrix in which all $x$ take value 1. So $\mu = 1$.

Note that we can easily compute the optimal length of the single unicast-single uniprior problem without going into the pruning algorithm of \(1\). This is done by inspecting the ‘cycles’ of $G$ from $I - B_{SI}A_{SI}$ matrix as described in the proof. Thus we have a simpler way of finding the optimal length $c$ for a Single Unicast-Single Uniprior IC problem.

There is yet another way of finding the optimal length $c$ for a single unicast-single uniprior problem using appropriate permutations corresponding to $A_{SI}$ of the given problem. We know that $A_{SI}$ is a permutation matrix that permutes the $n$ columns of $B_{SI}$. Every permutation on a finite set can be written as a cycle or as a product of disjoint cycles. Once we have the cycle decomposition of the permutation corresponding to $A_{SI}$, let $l_1, l_2, \ldots, l_k$ be the lengths of its disjoint cycles. The optimal length is given by

$$c = \sum_{i=1}^{k} (l_i - 1). \hspace{1cm} (41)$$

This means that for Single-Unicast-Single-Uniprior problems all the information are available in $A_{SI}$. These advantages are illustrated in the following two examples.

**Example 5.** Consider the Single Unicast-Single Uniprior problem given in Table IV with the number of receivers (equivalently, the number of messages), $n = 10$. For this problem,

$$B_{SI} = \begin{bmatrix}
    x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x
  \end{bmatrix}$$

$A_{SI}$ is a permutation matrix that permutes $SI_{10}$, let $SI_{10} \in \{x\}$. Every permutation on a finite set can be written as a cycle or as a product of disjoint cycles. The number of optimal index codes given by (41) is

$$c = \sum_{i=1}^{N_{sub}} (V(G'_{sub,i}) - 1) = 3 + 3 + 1 = 7$$

The number of optimal index codes given by (41) is

$$N_{OIC} = \frac{1}{7!} \prod_{i=0}^{7-1} (2^7 - 2^i) = 3.2510 \times 10^{10}$$

Note that $x \in \{0, 1\}$.

$$A_{SI} = \begin{bmatrix}
    0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
    0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
  \end{bmatrix}$$

The optimal length of the linear index code calculated from the information flow graph is

$$c = \sum_{i=1}^{N_{sub}} (V(G'_{sub,i}) - 1) = 3 + 3 + 1 = 7$$

**Example 6.** Consider the Single Unicast-Single Uniprior problem given in Table V with $n = 5$. For this problem,

$$B_{SI} = \begin{bmatrix}
    x & 0 & 0 & 0 & 0 \\
    0 & x & 0 & 0 & 0 \\
    0 & 0 & x & 0 & 0 \\
    0 & 0 & 0 & x & 0 \\
    0 & 0 & 0 & 0 & x
  \end{bmatrix}$$

The permutation corresponding to the $A_{SI}$ matrix is

$$A_{SI} = \begin{bmatrix}
    1 & 2 & 3 & 4 & 5 \\
    2 & 3 & 4 & 5 & 6 \\
    3 & 4 & 5 & 6 & 7 \\
    4 & 5 & 6 & 7 & 8 \\
    5 & 6 & 7 & 8 & 9 \\
    6 & 7 & 8 & 9 & 10
  \end{bmatrix}$$

which in terms of cycles is $(1 2 3 4)(6 7)(5 8 9 10)$ from which we get the optimal length, using (41) to be 7.

The information-flow graph obtained from the $I - B_{SI}A_{SI}$ matrix for this problem is shown in figure 3.

![Figure 3: Information flow graph for Example 5](image)

Optimal length of the linear index code calculated from the information flow graph is

$$c = \sum_{i=1}^{N_{sub}} (V(G'_{sub,i}) - 1) = 3 + 3 + 1 = 7$$

The number of optimal index codes given by (41) is

$$N_{OIC} = \frac{1}{5!} \prod_{i=0}^{7-1} (2^7 - 2^i) = 3.2510 \times 10^{10}$$
Table IV: Single Unicast Single Uniprior problem in Example 5
Note that $x \in \{0, 1\}.

\[
A_{SI} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
1 & x & 0 & 0 & 0
\end{bmatrix}
\]

The optimal length of the linear code, from the information flow graph is
\[
c = \sum_{i=1}^{N_{sub}} (V(G'_{sub,i}) - 1) = 2 + 1 = 3
\]

The number of optimal index codes
\[
N_{OIC} = \frac{1}{3!} \prod_{i=0}^{3-1} (2^3 - 2^i) = 28
\]
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where $c$ is the optimal length.

**Proof.** Let the number of messages be $n$ and the number of receivers be $m$. Let receiver $R_i$ know a single unique message $x_i$ a priori. Clearly $m \leq n$. If $m < n$, then there are $n - m$ messages which are not known to any of the receivers apriori and hence these messages need to be transmitted individually. We assume that this is done and after this, the remaining problem reduces to that of the case where $m = n = n_1$. Now, let the receiver $R_i$ want $W_i$ messages. Since the problem is unicast, unless each $W_i$ has only one element, there will be receivers having apriori information but not wanting any message. Such receivers can be removed from further consideration. After removing such receivers let the number of receivers in the problem be $m_1$. Now if $m_1 = n_1$ then we have a problem of single uniprior single unicast for which $\mu = 1$. On the contrary, if $m_1 < n_1$, then we repeat the process and eventually we will end up with a single uniprior single unicast problem and will have $\mu = 1$ by Corollary 2. This completes the proof. \(\Box\)

**Example 9.** Consider the Single Uniprior-Unicast problem given in Table IX with number of messages, $n = 10$ and number of receivers, $m = 8$.

The messages $x_{10}$ and $x_9$ have to be transmitted explicitly since they are not part of any receiver’s side-information. Once done, the receiver $R_7$ can be eliminated from further consideration as its demand has been met. The problem reduces to STAGE:2 with $m = 7$ receivers and $n = 8$ messages. Again, since $x_7$ is not part of any side-information it has to be transmitted explicitly. The problem now reduces to Single Unicast-Single Uniprior case with $n = m = 7$. The optimal length of this IC is

$$c = 2 + 1 + 4 = 7$$

The number of optimal index codes,

$$N_{OIC} = \frac{1}{7!} \prod_{i=0}^{6} (2^7 - 2^i) = 3.251 \times 10^{10}$$

**VI. Optimal Codes with Minimum-Maximum Error Probability**

There can be several linear optimal solutions in terms of least bandwidth for an IC problem but among them we try to identify the index code which minimizes the maximum number of transmissions that is required by any receiver in decoding its desired message. The motivation for this is that each of the transmitted symbols is error prone and the lesser the number of transmissions used for decoding the desired message, lesser will be its probability of error. Hence among all the codes with the same length of transmission, the one for which the maximum number of transmissions used by any receiver is the minimum, will have minimum-maximum error probability amongst all the receivers. We give a method to find the best linear solution in terms of minimum-maximum error probability among all the receivers and among all codes with the optimal length $c_{opt}$. For simplicity, throughout the rest of this section, the length $c$ will mean the optimal length. Each $T \in S(c)$ corresponds to a unique pair of encoding-decoding operations. For the same index code there can be more than one way of decoding at each receiver. For each set of decoding operations at the receivers, $T$ matrix differs. For a $T \in S(c)$, the corresponding matrix $T_{BC}$ has $n$ rows and $nc$ columns. Let $t_{i,j}, i = 1, 2, \ldots n$, denote the $i$-th row of $T_{BC}$. Denoting this $i$-th row as $[r_{i,1}r_{i,2} \ldots r_{i,n}]$, we define $t_{i,use}$ for this row as

$$t_{i,use} = \sum_{j=1}^{c} 1 - (I_{r_{i,j}=0}I_{r_{i,j+c}=0} \ldots I_{r_{i,j+(n-1)c}=0})$$

(43)

where $I_z$ is the indicator function which is one when event $z$ occurs. Note that $t_{i,use}$ is the number of transmissions that are used by the $i$-th receiver $R_i$ out of $c$ transmissions. Also define

$$t_{max}(T) = \max_i t_{i,use}$$

(44) and

$$T_{minmax} = \arg\{\min_{T \in S(c)} t_{max}(T)\}$$

(45)

Note that $T_{minmax}$ is not unique; there may be several such matrices in $S(c)$.

**Theorem 3.** For the optimal length $c$, any matrix $T_{minmax}$ in $S(c)$ gives an IC with the minimum-maximum error probability among all the receivers. Also, the matrix formed by taking every $n$-th row of the corresponding $F_{BC}$ matrix is an optimal linear solution in terms of minimum-maximum error probability.

**TABLE IX:** Single Uniprior-Unicast problem in Example 9

| Receiver, $R_i$ | Demand set, $W_i$ | Side-information, $K_i$ |
|----------------|-------------------|-------------------------|
| $R_1$          | $x_3$             | $x_1$                   |
| $R_2$          | $x_1$             | $x_2$                   |
| $R_3$          | $x_2$             | $x_3$                   |
| $R_4$          | $x_5x_{10}$       | $x_4$                   |
| $R_5$          | $x_4$             | $x_5$                   |
| $R_6$          | $x_6$             | $x_7$                   |
| $R_7$          | $x_9$             |                         |
| $R_8$          | $x_7x_6$          | $x_8$                   |

| Receiver, $R_i$ | Demand set, $W_i$ | Side-information, $K_i$ |
|----------------|-------------------|-------------------------|
| $R_1$          | $x_3$             | $x_1$                   |
| $R_2$          | $x_1$             | $x_2$                   |
| $R_3$          | $x_2$             | $x_3$                   |
| $R_4$          | $x_3$             | $x_4$                   |
| $R_5$          | $x_4$             | $x_5$                   |
| $R_6$          | $x_5$             | $x_6$                   |
| $R_7$          | $x_6$             |                         |
| $R_8$          | $x_7$             | $x_8$                   |

**TABLE IX:** Single Uniprior-Unicast problem in Example 9
probability using \( c \) number of transmissions.

Proof. For the fixed optimal length \( c \), \( B_{BC} \) matrix will be as given in (9). For any \( T \in S(c) \), the number of transmissions used by the \( i \)-th receiver is given by the number of non-zero entries in \( i \)-th row of \( B_{BC} \). When for example \( t \)-th \( \epsilon \) element in the \( i \)-th row of \( B_{BC} \) is zero, the \( i \)-th element of every \((t + (k - 1)c)\)-th column for \( k = 1 \) to \( n \), in \( T_{BC} \) turns 0. Hence the number of transmissions used by it is proportional to the \( t_{\text{use}} \). Therefore, our claim is proved. Moreover the corresponding \( F_{BC} \) is the matrix which decides the message flowing in the broadcast channels. So the matrix formed by taking every \( n \)-th row of \( F_{BC} \) is the corresponding Index code.

From the theorem above it is clear that to minimize the maximum probability of the receivers one needs to pick that \( T \in S(c) \) for which the the maximum number of nonzeros in a row in the corresponding \( B_{BC} \) matrix is minimized.

Example 10. Let \( m = n = 3 \). Each \( R_i \) wants \( x_i \) and knows \( x_{i+1} \), where + is mod-3 addition. The optimal length of a linear IC solution for this problem is 2. For Example 1, we found out the optimal IC’s: They are 1. \( C_1 : x_1 \oplus x_2, x_2 \oplus x_3 \)
2. \( C_2 : x_1 \oplus x_3, x_3 \oplus x_2 \)
3. \( C_3 : x_1 \oplus x_3, x_1 \oplus x_2 \). For all the three, the maximum number of transmissions used by any receiver is 2. This is verified below. We find the \( T_B \) matrices for each case as follows:

\[
T_{B,1} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}
\]

\[
T_{B,2} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\]

\[
T_{B,3} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}
\]

It can be verified that \( t_{\max}(T) \) for all the three is 2. Hence 2 transmissions at most are used by any receiver to decode in all the three cases. The BEP (Bit Error Probability) versus SNR curves for each of the three codes at various receivers are given in Fig.5 Fig.6 Fig.7. We considered BPSK modulation in Rayleigh faded channel. In Fig8 the worst case BEP curves for each of the three codes are plotted. We can see that the curves lie on top of each other which proves our claim that all the three codes are equally good in terms of minimum-maximum error probability.

Example 11. Let \( m = n = 4 \). \( R_i \) wants \( x_i \) and knows \( x_{i+1} \) where + is modulo-4 operation. \( R_3 \) knows \( x_1 \) also.

The optimal length \( c = 3 \). Tables 14 and 15 gives the \( t_{\max}(T) \) for each of the optimal linear codes. The minimum \( r_{\min}(T) \) is 2. The BEP versus SNR curve for \( C_{30} \) whose \( r_{\max}(T) = 2 \) is as in Fig 9. The BEP versus SNR curve for \( C_{29} \) whose \( t_{\max}(T) = 3 \) is as in Fig 10. The worst case Performance of both codes are plotted in Fig.11.

We can observe that the worst performance of \( C_{30} \) is better than worst performance of \( C_{29} \).
$K_j=0$, for $i \neq j, i, j = 1 \text{ to } n$. We would like to extend this work to find out least complexity algorithms which finds IC solutions by matrix completion. Harvey et al. in [9] gives such algorithms for multicast network codes. However what we have is a general problem and hence their results are not applicable. We have followed an approach which is different and simpler than Koetter and Medard’s [5] for this specific class of network coding problem.

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Fig. 11: Worst case BER versus SNR (db) curves for codes $C_{30}$ and $C_{29}$ for Example 4

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