Correlation functions for the 2D random bonds Potts Models.

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We study the spin-spin and energy-energy correlation functions for the 2D Ising and 3-states Potts model with random bonds at the critical point. The procedure employed is the renormalisation group approach of the perturbation series around the conformal field theories representing the pure models. For the Ising model, we obtain a crossover in the amplitude for the correlation functions which doesn’t change the critical exponent. For the 3-state Potts model, we found a shift in the critical exponent produced by randomness. A comparison with numerical data is discussed briefly.

1. INTRODUCTION

For models with random bonds, the main problem is to determine if the randomness leaves unchanged the critical properties of the pure system or if the singularities of the thermodynamical functions are modified. First results, which suggest an intermediate situation, have been obtained by Harris and Lubensky, Grinstein and Luther and Khmelnitskii using the standard \( \phi^4 \) theory. Other cases, like long-range correlated quenched defects have also been considered. First results for this model were obtained by Dotsenko and Dotsenko who showed that near the critical point, this model can be represented by an \( n = 0 \) Gross-Neveu model. With this technique, they found that the specific heat singularity get smoothed as \( \ln(\ln(|t|)) \) where \(|t|\) is the reduced temperature. Calculation of spin-spin correlation function by this technique which involves non-local fermionic representation of \( \langle \sigma \sigma \rangle \), was later questioned by Shalaev and Shankar who gave arguments that the asymptotic behavior of this correlation function is unchanged by the randomness. Let’s also mention some other approaches. In the situation was questioned by Ziegler who claims that non-perturbative effects introduce an intermediate phase around the critical point of the pure model. In an other approach, Mussardo and Simonetti studied the exact formulation of the random bond Ising model in term of \( S \)-matrix. On the other hand, numerical simulations of the
Ising model \([16,15]\) seem to confirm the theoretical predictions of the specific heat and spin-spin correlation function asymptotic behavior.

For the Potts model, the situation is more clear. Here, the Harris criterion makes some precise predictions: randomness is relevant and changes the critical behavior. Using conformal field theory techniques, Ludwig \([17]\) also perturbatively computed a shift in the critical exponent of the energy operator in the case of the random Potts model (this critical exponent for the Ising model is also unchanged).

The paper is organized as follows. In section 2 we introduce the model. In section 3, we briefly present the methods for computing correlation functions, referring to \([18]\) for more details. Results are presented in section 4 while in section 5 a discussion of the results is presented and a comparison with numerical data obtained recently.

2. THE MODEL

The partition function for a \(q\)-state Potts model with a fixed configuration of disorder \(J_{ij}\) is given by

\[
Z(J_{ij}) = \sum_{\{\sigma_i\}} e^{-\beta \sum_{<i,j>} \delta_{\sigma_i,\sigma_j} J_{ij}} \tag{1}
\]

Here, \(\sigma_i\) corresponds to the value of the spin at the location \(i\) on the lattice and takes the values \(\sigma_i = 0, \cdots, q - 1\)

and \(J_{ij}\) corresponds to the coupling constants between neighboring spins. These coupling constant can be separated in two parts

\[
J_{ij} = J_0 + \Delta_{ij} \tag{3}
\]

\(J_0\) is the coupling constant of the pure model (without disorder) while \(\Delta_{ij}\) is the random part of the couplings. This partition function can also be written under the form

\[
Z(J_{ij}) = \sum_{\{\sigma_i\}} e^{-S_0 - \beta \sum_{<i,j>} \delta_{\sigma_i,\sigma_j} \Delta_{ij}} \tag{4}
\]

where we have explicitly separate the non random part (to which corresponds a action associated to a conformal field theory at the critical temperature) and a random part. Thus, using the fact that the \(\delta_{\sigma_i,\sigma_j}\) corresponds, in the continuum limit of the pure Potts model, to the energy field \(\epsilon(z)\), we can also express the partition function like

\[
Z(m) = Tr(\sigma) e^{-S_0 - \int m(z) \epsilon(z) d^2 z} \tag{5}
\]

and \(m(z)\) is the continuum limit of \(\Delta_{ij}\). Under this form we recognize a conformal field theory with a perturbation term \(\int d^2 z m(z) \epsilon(z)\).

The next step consists in performing the average over the disorder i.e. over \(m(z)\). In fact, we need to compute the average over the free energy, i.e. over \(ln Z(m)\). This is done easily by using the replica method. Taking the partition function of \(n\) identical copies of the system and analytically continuing to the limit \(n \to 0\) gives the quenched free energy

\[
-\beta F = ln(Z) = \lim_{n \to 0} \frac{Z^n - 1}{n}
\]

where:

\[
Z^n = \prod_{a=1}^{n} Tr_{\{\sigma_i\}} e^{-\sum_{a=1}^{n} S_{0,a} - \int m(z) \sum_{a=1}^{n} \epsilon_a(z) d^2 z} \tag{6}
\]

the average of \(Z^n\) is made with a Gaussian distribution for \(m(z)\):

\[
\mathbb{Z}^n = \int dm(z) Z^n e^{-\frac{1}{2m_0} (m(z) - m_0)^2}
\]

which gives:

\[
\mathbb{Z}^n = \prod_{a=1}^{n} Tr_{\{\sigma_i\}} e^{-S} \tag{7}
\]

and

\[
S = \sum_{a=1}^{n} S_{0,a} + g_0 \int \sum_{a,b=1}^{n} \epsilon_a(z) \epsilon_b(z) d^2 z
\]

\[
- m_0 \int \sum_{a=1}^{n} \epsilon_a(z) d^2 z \tag{8}
\]

The terms in \(\int \sum_{a,b=1}^{n} \epsilon_a(z) \epsilon_b(z) d^2 z\) containing the same replica label produce irrelevant operators.
and can be omitted. Then \( S \) is reduced to
\[
S = \sum_{a=1}^{n} S_{a,a} + g_0 \int \sum_{a \neq b} \varepsilon_a(z) \varepsilon_b(z) d^2z - m_0 \int \sum_{a=1}^{n} \varepsilon_a(z) d^2z
\] (9)
In the limit \( m_0 \to 0 \), this model corresponds to a conformal field theory perturbed by a term quadratic in the \( \varepsilon \) operator. Then the “evolution” of the coupling constants \( g_0 \) and \( m_0 \) under a renormalisation group (R.G.) transformation can be analyzed as well as the behavior of the correlation functions. In the calculation of correlation functions \( \langle O(0) O(R) \rangle \), where \( O \) is some local operator, we will proceed perturbatively:
\[
\langle O(0) O(R) \rangle = \langle O(0) O(R) \rangle_0 + \frac{1}{2} \langle S_l^2 O(0) O(R) \rangle_0 + \ldots
\] (10)
where \( \langle \cdot \rangle_0 \) means the expectation value taken with respect to \( S_0 \) and
\[
S_l = \int H_l(z) d^2z = g_0 \int \sum_{a \neq b} \varepsilon_a(z) \varepsilon_b(z) d^2z \] (11)
The operator \( O \) is then renormalised as
\[
O \to O(1 + A_1 g_0 + A_2 g_0^2 + A_3 g_0^3 + \cdots) \equiv Z_0 O
\]
The integrals of correlation functions involved in the calculation can be performed by analytic continuation with the Coulomb gas representation of a conformal field theory [19] where the central charge is \( c = \frac{1}{2} + \epsilon' \). The \( \epsilon' \) term corresponds to a short distance regulator for the integrals. In addition, we also used an infrared (I.R.) cut-off \( r \).

The result is then expressed as an \( \epsilon' \) series with coefficients depending on \( r \). The limit \( \epsilon' \to 0 \) corresponds to the pure Ising model at the critical point while the Potts model is obtained for some finite value of \( \epsilon' \). We recall here some notations of the Coulomb-gas representation for the vertex operators [19]. The central charge \( c \) will be characterized in the following by the parameter \( \alpha_+^2 = \frac{2\alpha_+}{2\alpha_+ - 1} = \frac{4}{3} + \epsilon \) with
\[
c = 1 - 24\alpha_0^2 \quad ; \quad \alpha_{\pm} = \alpha_0 \pm \sqrt{\alpha_0^2 + 1}
\] (12)
Note that for the pure 2D Ising model \( \alpha_+^2 = \frac{4}{3} \) and \( c = \frac{1}{2} \) while for the 3-state Potts model \( \alpha_+^2 = \frac{9}{5}, \quad c = \frac{9}{8} \) and \( \epsilon = -\frac{2}{15} \). The vertex operators are defined by
\[
V_{nm}(x) = e^{i\alpha_{nm} \phi(x)}
\] (13)
where \( \phi(x) \) is a free scalar field and where the \( \alpha_{nm} \) are given by
\[
\alpha_{nm} = \frac{1}{2} (1 - n)\alpha_- + \frac{1}{2} (1 - m)\alpha_+
\] (14)
The conformal dimension of an operator \( V_{nm}(x) \) is \( \Delta_{nm} = -\alpha_{nm} \alpha_{nm} \) with
\[
\alpha_{nm} = 2\alpha_0 - \alpha_{nm} = \frac{1}{2} (1+n)\alpha_- + \frac{1}{2} (1+m)\alpha_+ \] (15)
The spin field \( \sigma \) can be represented by the vertex operator \( V_{p,p-1} \) whereas \( V_{1,2} \) corresponds to the energy operator \( \varepsilon \). In the same way, we associate \( e^{i\alpha_+ \phi(x)} \) to the screening charge operator \( V_+ \).

Note that in the Ising case the \( \sigma \) operator can also be represented by the \( V_{21} \) operator (since both operators coincide in the limit \( \epsilon \to 0 \)). So, we can represent our spin operator by \( V_{k,k-1} \) where \( k = \frac{2+3\lambda}{1+3\lambda} \). We have \( \lambda = 2 \) for \( V_{21} \) and \( \lambda = \frac{1}{2} \) for \( V_{p,p-1} \).

3. Renormalisation Group Equations

In this section, we will deal with the computation of correlation functions of operators \( \varepsilon \) and \( \sigma \). To compute them, one needs to determine the effect of the random coupling on the operators \( \varepsilon \) and \( \sigma \) and compute the renormalised operators \( \varepsilon' \) and \( \sigma' \). This means that we want to compute the functions \( Z_\varepsilon \) and \( Z_\sigma \) such that
\[
\varepsilon' = Z_\varepsilon \varepsilon \quad \text{and} \quad \sigma' = Z_\sigma \sigma
\] (16)
A convenient way to define \( Z_\varepsilon \) and \( Z_\sigma \) is to consider the more general action
\[
\sum_{a=1}^{n} S_{0,a} - g_0 \int \sum_{a,b=1}^{n} \varepsilon_a(z) \varepsilon_b(z) d^2z + m_0 \int \sum_{a=1}^{n} \varepsilon_a(z) d^2z - \lambda \int \sum_{a=1}^{n} \sigma_a(z) d^2z
\] (17)
This merely corresponds to the action used in (8) with an additional coupling of the \( \sigma \)
field. Then, with the help of the operator algebra (O.A.) coming from contractions between $\varepsilon$ and $\sigma$ operators, we will compute the effect of the $g_0 \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z)d^2z$ term (that we will denote $g_0(\varepsilon\varepsilon)$ in the following for simplicity) on the coupling terms $m_0 \int \sum_{a=1}^{n} \varepsilon_a(z)d^2z$ and $h_0 \int \sum_{a=1}^{n} \sigma_a(z)d^2z$. More precisely, we will compute

\[
\sum_{i} (g_0(\varepsilon\varepsilon))^i m_0 \int \sum_{a=1}^{n} \varepsilon_a(z)d^2z = \sum_{i} (g_0(\varepsilon\varepsilon))^i h_0 \int \sum_{a=1}^{n} \sigma_a(z)d^2z 
\]

and

\[
\sum_{i} (g_0(\varepsilon\varepsilon))^i h_0 \int \sum_{a=1}^{n} \sigma_a(z)d^2z = \sum_{i} (g_0(\varepsilon\varepsilon))^i h_0 \int \sum_{a=1}^{n} \sigma_a(z)d^2z 
\]

$m$ and $h$ being the renormalised coupling constants. Obviously, this computation will be perturbatively made only up to some finite power in $g_0$. In fact, the first step of the computation will be to determine the renormalised $g$ constant on which $Z_\varepsilon$ and $Z_\sigma$ depend.

3.1. Renormalisation of the coupling constant $g$.

The renormalisation of the coupling constant $g$ will be determined directly by a perturbative computation. $g$ is also given by the O.A. producing

\[
g_0(\varepsilon\varepsilon) = g_0(\varepsilon\varepsilon)^2 + g_0(\varepsilon\varepsilon)^3 + \cdots = g \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z)d^2z 
\]

with $g = g_0 + A_2 g_0^2 + A_3 g_0^3 + \cdots$ where $A_2$ comes from

\[
\frac{1}{2} \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z)d^2z \int \sum_{c,d=1}^{n} \varepsilon_c(z)\varepsilon_d(z)d^2z = A_2 \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z)d^2z + \cdots 
\]

and $A_3$ from

\[
\frac{1}{6} \left( \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z)d^2z \right)^3 = A_3 \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z)d^2z + \cdots 
\]

The computation of $A_2$ is made by contracting two $\varepsilon$ operators. We obtain (with $b = c \neq a, d$)

\[
\frac{1}{2} \int \sum_{a,b=1}^{n} \varepsilon_a(x)\varepsilon_b(x)d^2x \int \sum_{c,d=1}^{n} \varepsilon_c(y)\varepsilon_d(y)d^2y = 2(n-2) \int_{|x-y|<r} <\varepsilon(x)\varepsilon(y)> d^2y \times \int \sum_{a,d=1}^{n} \varepsilon_a(x)\varepsilon_d(x)d^2x = 4\pi(n-2) \int_{|y<r} \frac{dy}{y^{1+3\varepsilon}} \int \sum_{a,d=1}^{n} \varepsilon_a(x)\varepsilon_d(x)d^2x = -4\pi(n-2) \frac{\pi^{-3\varepsilon}}{3\varepsilon} \int \sum_{a,d=1}^{n} \varepsilon_a(x)\varepsilon_d(x)d^2x
\]

Thus $A_2 = -4\pi(n-2) \frac{\pi^{-3\varepsilon}}{3\varepsilon}$. Computations at higher order will become very complicated. In the following, we will just present the results, referring to [18] for all the technical details. For the computation of $g(r)$, calculations were made up to the third order with the following result:

\[
g(r) = r^{-3\varepsilon}(g_0 - g_0^2 4\pi(n-2) \frac{\pi^{-3\varepsilon}}{3\varepsilon}) + g_0^3 8\pi^2(n-2) \frac{r^{-6\varepsilon}}{3\varepsilon} \left( 1 + \frac{2(n-2)}{3\varepsilon} \right)
\]

Note that we multiply the result by $r^{-3\varepsilon}$ in order to obtain a dimensionless coupling constant $g(r)$. From there, we can compute directly the $\beta$-function:

\[
\beta(g) = \frac{dg}{d\ln(r)} = -3\varepsilon g(r) + 4\pi(n-2)g^2(r) - 16\pi^2(n-2)g^3(r) + O(g^4(r))
\]
Finally, taking the limit \( n \to 0 \), we obtain for the \( \beta \)-function up to the third order:

\[
\beta(g) = -3\epsilon g - 8\pi g^2 + 32\pi^2 g^3 \tag{25}
\]

From this \( \beta \)-function, we determine immediately the effect of the disorder on the model. For the Ising model (\( \epsilon = 0 \)), the infrared fixed point is \( g = 0 \), while for the 3-state Potts model (\( \epsilon < 0 \)), a new infrared fixed point is reached with \( g_c = -\frac{3\epsilon}{8\pi} + \frac{9\epsilon^2}{16\pi} + O(\epsilon^3) \).

### 3.2. Renormalisation of \( \sigma \) and \( \epsilon \)

In order to be able to compute the correlation functions of \( \sigma \) and \( \epsilon \), the second step is to determine the effect of the renormalisation on these operators. One needs to compute the multiplicative functions \( Z_\sigma \) and \( Z_\epsilon \). This will be made by computing the renormalised coupling terms

\[
m \int \sum_{a=1}^{n} \varepsilon_a(z)d^2z = (m_0Z_\epsilon) \int \sum_{a=1}^{n} \varepsilon_a(z)d^2z
\]

and

\[
h \int \sum_{a=1}^{n} \sigma_a(z)d^2z = (h_0Z_\sigma) \int \sum_{a=1}^{n} \sigma_a(z)d^2z. \tag{26}
\]

We give here directly the result:

\[
r^{-\frac{4}{3} - a(\epsilon)} h(r) = h_0(1 + (n - 1)g_0^3\frac{r^{-6\epsilon}}{6} \times \frac{1}{1 - 2\lambda} \frac{\Gamma^2(\frac{4}{3})\Gamma^2(\frac{1}{3})}{\Gamma^2(\frac{4}{3})\Gamma^2(-\frac{1}{6})} - 12(n - 1)(n - 2)g_0^3\frac{r^{-6\epsilon}}{9\epsilon} \times \frac{1}{1 - \frac{8}{9}(2 - \lambda)\frac{\Gamma^2(\frac{3}{2})\Gamma^2(\frac{1}{3})}{\Gamma^2(\frac{3}{2})\Gamma^2(-\frac{1}{6})})}
\]

where \( a(\epsilon) \) is a function of \( \epsilon \) depending on which representation of the spin field we are taking in the Coulomb gas picture (see section 2). Its explicit form will be irrelevant in the following. The corresponding R.G. equation for \( Z_\sigma \) will be given by

\[
\frac{dln(Z_\sigma(r))}{dln(r)} = B_2 \int \sum_{a=1}^{n} \varepsilon_a(z)d^2z \tag{33}
\]
-3(n - 1)g^2(r)\pi^2 \epsilon \left[ 1 + \frac{4}{3} (2 - \lambda) \frac{\Gamma^2(-\frac{2}{3})\Gamma^2(\frac{1}{3})}{\Gamma^2(\frac{5}{3})\Gamma^2(\frac{1}{3})} \right] \\
+4(n - 1)(n - 2)\pi^3 g^3(r) 

4. Correlation Functions

We now have all the ingredients needed in order to compute the correlation functions. They will be calculated with the help of the R.G. equations, for the theory with \( m_0, h_0 \to 0 \). From the R.G. equations, we have:

\[
< \sigma(0) \sigma(R) >_{r,g(r)} = \frac{Z^2_Z(sR, g(s))}{Z^2_Z(r, g(r))} s^{-2\Delta s} < \sigma(0) \sigma(R) >_{r,g(sR)}
\]

This can be written as:

\[
< \sigma(0) \sigma(sR) >_{r,g(r)} = 2 \int_{g_0}^{g(s)} \frac{g(g') dg'}{g} s^{-2\Delta s} < \sigma(0) \sigma(R) >_{r,g(sR)}
\]

where we used the notation:

\[
\frac{dlnZ_r}{dlnr} = \gamma_r(g)
\]

and \( g(s) = g(sr); g_0 = g(r) \). We assume now \( r \) to be a lattice cut-off scale. In a similar way for \( \gamma_r(g) \sigma(R) \to \) the R.G. equation is:

\[
< \sigma(0) \sigma(sR) >_{r,g(r)} = 2 \int_{g_0}^{g(s)} \frac{g(g') dg'}{g} s^{-2\Delta s} < \sigma(0) \sigma(R) >_{r,g(sR)}
\]

with

\[
\frac{dlnZ_r}{dlnr} = \gamma_r(g)
\]

In equations (35)-(37), \( R \) is an arbitrary scale which can be fixed to one lattice spacing \( r \) of a true statistical model. The dependence of \( < \sigma(0) \sigma(r) >_{r,g(sR)} \) on \( s \) will then be negligible, assuming that there are not interactions on distances smaller than \( r \). Therefore, it reduces to a constant. Then, \( s \) will measure the number of lattice spacings between two spins in \( < \sigma(0) \sigma(sR) > \). In the following, we adopt the choice \( r = 1 \).

4.1. The Ising model

The Ising model corresponds to the case \( \epsilon \to 0 \) and so the \( \beta \) function is:

\[
\beta(g) = -8\pi g^2 + 32\pi^2 g^3 
\]

Therefore, we can see that the I.R. fixed point is located at \( g = 0 \). Also we have, by eqs. (36), (38) for \( n = 0 \), \( \epsilon = 0 \) and definitions (34), (35),

\[
\gamma_\epsilon(g) = -4\pi g + 8\pi^2 g^2 \\
\gamma_\sigma(g) = 8\pi g^3
\]

The integral for the \( \epsilon \) correlation function, eq. (35), gives:

\[
2 \int_{g_0}^{g(s)} \frac{g(g') dg}{g} \approx \int_{g_0}^{g(s)} (1 + 2\pi g) \frac{dg}{g} = 2\pi(g(s) - g_0) + ln\left(\frac{g(s)}{g_0}\right)
\]

Now, we need to compute \( g(s) \). The integration of equation \( \beta(g) = a_{ln(r)} \) gives:

\[
\int_{g_0}^{g(s)} \frac{dg}{g} = \int_{r}^{sr} dlnr
\]

with the following solution up to the second order:

\[
g(s) = g_0 \frac{\frac{g_0}{1 + 8\pi g_0 ln(s)} + \frac{4\pi g_0 ln(1 + 8\pi g_0 ln(s))}{1 + 8\pi g_0 ln(s)}}{1 + 8\pi g_0 ln(s)} + O(g_0^3)
\]

So, \( < \epsilon \epsilon > \) correlation function is given by:

\[
< \epsilon(0) \epsilon(s) >_{g_0} \sim \frac{g(s)}{g_0} (1 - 2\pi(g_0 - g(s))) s^{-2\Delta s} \sim \frac{1}{1 + 8\pi g_0 ln(s)} s^{-2\Delta s} (1 + \frac{4\pi g_0}{1 + 8\pi g_0 ln(s)}) \times (ln(1 + 8\pi g_0 ln(s)) - 4\pi g_0 ln(s)) + O(g_0^2)
\]

For the \( \sigma \) correlation function, the computation is similar. In fact, in that case, we have \( \gamma_\sigma(g) = \)
8π^3g^3. Thus, keeping only the first order of the β-function (i.e. β(g) = −8πg^2), we obtain:

\[ 2 \int_{g_0}^{g(s)} \frac{\gamma_\sigma(g)}{\beta(g)} dg = -2\pi^2 \int_{g_0}^{g(s)} g dg \]
\[ = -\pi^2 (g(s)^2 - g_0^2) + O(g_0^3) \quad (45) \]

The <σσ> correlation function is then found to be given by:

<σ(0)σ(s) > ~ (1 + π^2 (g_0^2 - g(s)^2)) s^{-2\Delta_\sigma} \quad (46)

The calculation of the g^3 term in the β function and the g^2 term in the renormalisation of ε was already done in [17], extending the one loop result of [8]. We recovered these higher order corrections using a different technique, which allowed us to calculate also the modified correlation function of the spin operators.

4.2. The Potts model

We consider here the 3-state Potts model. With our conventions this case corresponds to ε = −27/8. β(g) is given in eq.(25), and

\[ \gamma_\epsilon(g) = -4\pi g + 8\pi^2 g^2 \quad (47) \]
\[ \gamma_\sigma(g) = 3\pi^2 g \left( 1 + \frac{4}{3} (2 - \lambda) \frac{\Gamma^2(-\frac{2}{3}) \Gamma^2(\frac{1}{3})} {\Gamma^2(-\frac{1}{3}) \Gamma^2(-\frac{4}{3})} \right) g^2 \]
\[ + 8\pi^3 g^3 \quad (48) \]

At long distances, the integrals in eqs.(27), (28) will be dominated by the region g ≈ g_c, with g_c = \frac{\beta}{\epsilon} + \frac{2\pi^2}{3\epsilon} + O(\epsilon^3). This is different from the Ising model, because here γ_\epsilon(g_c) and γ_\sigma(g_c) have finite values for g = g_c. Thus,

\[ \int_{g_0}^{g(s)} \frac{\gamma_\epsilon(g)}{\beta(g)} dg \approx \gamma_\epsilon(g_c) \ln(s) \quad (49) \]

and

\[ \int_{g_0}^{g(s)} \frac{\gamma_\sigma(g)}{\beta(g)} dg \approx \gamma_\epsilon(g_c) \ln(s) \quad (50) \]

The correlation functions can then be deduced directly:

<ε(0)ε(s) > ~ s^{-2(2\Delta_\epsilon - 2\gamma_\epsilon(g_c))} \quad (51)

and

<σ(0)σ(s) > ~ s^{-2(2\Delta_\sigma - 2\gamma_\sigma(g_c))} \quad (52)

So, we can see that a direct consequence of the new IR fixed point is a modification of the critical exponents \Delta_\epsilon and \Delta_\sigma. A straightforward computation will give these new exponents:

\[ 2\Delta_\epsilon' = 2\Delta_\epsilon - 2\gamma_\epsilon(g_c) \]
\[ = 2\Delta_\epsilon + 8\pi g_c - 16\pi^2 g_c^2 \]
\[ = 2\Delta_\epsilon - 3\epsilon + \frac{9}{4} \epsilon^2 + O(\epsilon^3) \quad (53) \]

and

\[ 2\Delta_\sigma' = 2\Delta_\sigma - 2\gamma_\sigma(g_c) \]
\[ = 2\Delta_\sigma - 6\pi^2 g_c^2 \left( 1 + \frac{4}{3} (2 - \lambda) \frac{\Gamma^2(-\frac{2}{3}) \Gamma^2(\frac{1}{3})} {\Gamma^2(-\frac{1}{3}) \Gamma^2(-\frac{4}{3})} \right) - 16\pi^3 g_c^3 \]
\[ = 2\Delta_\sigma - \frac{9}{8} (2 - \lambda) \frac{\Gamma^2(-\frac{2}{3}) \Gamma^2(\frac{1}{3})} {\Gamma^2(-\frac{1}{3}) \Gamma^2(-\frac{4}{3})} \epsilon^3 + O(\epsilon^4) \quad (54) \]

Here λ = \frac{1}{2} and the final result for the new critical exponent is:

\[ 2\Delta_\sigma' = 2\Delta_\sigma - \frac{27}{16} \frac{\Gamma^2(-\frac{2}{3}) \Gamma^2(\frac{1}{3})} {\Gamma^2(-\frac{1}{3}) \Gamma^2(-\frac{4}{3})} \epsilon^3 + O(\epsilon^4) \quad (55) \]

The value of the critical exponent for the energy operator was already computed, up to the second order, by Ludwig [17]. While for the pure case \Delta_\epsilon = 0.8, here we obtained \Delta_\epsilon' = 1.02 + O(\epsilon^3). The difference between these exponents is quite important and should be measured in a numerical simulation of this model. See next section for a discussion on this point. For the spin operator, deviation of the critical exponent from the pure case appears only at the third order, which we compute. This deviation is in fact very small. While in case of 3-state Potts model without disorder 2\Delta_\sigma = \frac{1}{16}, we obtain the new critical exponent 2\Delta_\sigma' = \frac{1}{16} + 0.00264 = 0.26931. Thus, the deviation corresponds to a modification of 1%.
5. Discussion

In case of Ising model spin-spin function the calculation of up to third order of the renormalisation group was needed to find the deviation from the perfect model case, in the form of the cross-over in the amplitude. This completes the observations of [10, 12], based on absence of renormalisation of this function in the first order, that asymptotically the spin-spin function has the same exponent as in case of the perfect lattice model.

Recently the numerical simulations of the random Ising model has been performed which measure directly the deviation of \(<\sigma \sigma>\) from the pure Ising model at the critical point [10]. These measurements were made for disorder such that \(8\pi g_0 \approx 0.3\). Deviations predicted by our computations are very small. They correspond to 0.1%. The deviations obtained in numerical simulations are around ten times larger, and they are of opposite sign, \(i.e.\), correspond to an extra decrease of the spin-spin function with distance \(r\). In [16], it has been checked that this decrease corresponds, within the accuracy of the measurements, to a factor function of the ratio \(r/L\), \(F(r/L)\), \(r\) being the distance between the spins and \(L\) is the lattice size. So they correspond to finite size effects, being different for perfect and random models. We would suggest, on the bases of our calculation of the \(r\) dependence of the spin-spin function on an infinite lattice, that numerical deviations will continue to be plotted by the same curve \(F(r/L)\), if one measures \(<\sigma \sigma>\) for different lattice sizes as it has been done in [16], until the accuracy reaches the value of the \(r\)-deviation which we calculated here. Only then the curves for different \(L\) will split.

Recently numerical simulations of the 3-state Potts model with disorder were performed by one of the authors [21]. In these simulations, \(\Delta_{\sigma'}\) was measured as well as \(\Delta_{\sigma\prime}\). For \(\Delta_{\sigma\prime}\), it is very difficult to obtain any conclusion. The measured value \(\Delta_{\sigma'}\) is very close to \(\Delta_{\sigma}\) and error bars do include the case \(\Delta_{\sigma'} = \Delta_{\sigma}\) up to a deviation of 1%, which is the deviation predicted by analytical computations presented in this work. For the energy-energy correlation function, the measured exponent is \(\Delta_{\sigma'} = 1.065 \pm 0.02\), to be compared with the prediction of our computations: \(\Delta_{\sigma'} = 1.02\). This is quite a satisfactory result if we remember that the result \(\Delta_{\sigma'} = 1.02\) is a perturbative computation while the pure model (around which we perturbed) has the value \(\Delta_{\sigma} = 0.8\).

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