D-branes in Gepner models and supersymmetry

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Abstract

Boundary states corresponding to wrapped D-branes in Calabi-Yau compactifications of type II strings are discussed using Gepner models. In particular boundary conditions corresponding to D-0 branes and D-instantons in four dimensions are investigated. The boundary states constructed by Recknagel and Schomerus are analyzed in the light-cone gauge and the broken and conserved space-time supersymmetry charges are found. The geometrical interpretation of these algebraically constructed boundary states is clarified in some simple cases. Moreover, the action of mirror symmetry and other discrete symmetries of the Gepner model on the boundary states are discussed. As an application the boundary states are used to calculate instanton induced corrections to metric on the hypermultiplets in the $N = 2$ effective action.

August 1998

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1. Introduction

The compactification of type II string theories on a Calabi-Yau threefold produces four dimensional $N = 2$ supersymmetric theories. Many interesting phenomena like mirror symmetry, special geometry and string duality arise in these theories. Dirichlet branes provide a simple realization of non-perturbative solitonic objects carrying Ramond-Ramond ($R \otimes R$) charges in type II and type I string theories. The description of D-branes wrapping submanifolds of Calabi-Yau manifolds is an important ingredient for non-perturbative string theory. One example is Strominger’s resolution of the conifold singularity. Euclidean wrappings of branes on cycles in Calabi-Yau manifolds were discussed by Becker, Becker and Strominger. Using the world-volume description of branes geometric criteria for the wrapping cycles to preserve half the supersymmetry were found. The wrapping of D-branes on such ‘supersymmetric cycles’ was analyzed further by Ooguri, Oz and Yin. In these references the Calabi Yau compactifications are discussed in a sigma model framework where only the long wavelength (massless) excitations are taken into account.

On the other hand an exactly solvable model of a compactification on a complex manifold of dimension $2k, k = 1, 2, 3$ with holonomy group $SU(k)$ was constructed by Gepner. In this construction the internal $c = 3k, N = 2$ superconformal field theory (SCFT) is made of a tensor product of $N = 2$ minimal models with the correct total central charge. Since the Gepner model is exactly solvable many techniques for constructing boundary states in rational conformal field theories can be applied in this context. In particular Recknagel and Schomerus constructed boundary states for Gepner models. With these boundary states it may be possible to discuss many interesting problems in a precise manner including the massive string excitations. In addition, D-branes have played an important role in recent development in black hole physics. In this respect, the boundary states in Gepner models may give useful insight for understanding black holes in string theory in Calabi-Yau compactifications. In this paper we will discuss only trivial, i.e. $U(1)$, Chan-Paton factors which corresponds to the wrapping of a single D-brane.

However, the physical aspects of the boundary states in the string theory context have not been fully explored yet. In this paper, we will address this issue and analyze the boundary states of further from a space-time perspective. In particular we will use space-time supersymmetry in the light-cone gauge to construct the broken and unbroken supersymmetries associated with the wrapped brane. Furthermore the geometric interpretation of the boundary states in Gepner models will be discussed. We will find what D-branes those algebraically constructed boundary states represent in some simple cases. Gepner models have interesting discrete symmetries which are inherited from the symmetries of the minimal models. One important example is mirror symmetry. We will

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3 Gepner models in type I theory context were discussed in [3].
discuss how these symmetries are realized in the boundary states. We will also find some non-perturbative effects induced by D-instantons.

The organization of the paper is as follows. In section 2 we will give a brief review of $N = 2$ space-time supersymmetry and its relation to the internal $(2,2)$ superconformal symmetry. In section 3 the Gepner model construction is reviewed, see [12] for more details. In section 4 the boundary states are presented. In section 5 the case of the Gepner models corresponding to compactifications on $T^2$ is discussed and the geometric interpretation of the boundary states is given. Broken and unbroken supersymmetries are analyzed and a criterion for mutually supersymmetric D-brane configuration is found in section 6. In section 7 the action of mirror symmetry and T-duality (c-map) on the boundary state is discussed. The more complicated case of the $(k = 3)^5$ Gepner model which corresponds to a special quintic hypersurface is discussed in section 8. In section 9 some non-perturbative effects induced by D-instantons are calculated using the boundary states and the broken supersymmetry charges. The last section contains some comments about issues not discussed in the main part of the paper and conclusions. In appendix A, we give some calculation of the partition functions from the boundary states for simple Gepner models. In appendix B, we discuss geometrical interpretation of the boundary conditions for the space-time fields by using the free field realizations of $N = 2$ minimal models.

2. $N = 2$ compactification and space time supersymmetry

An $N = 2$ super conformal algebra with central charge $c$ contains apart from the stress tensor $T$ also two supercurrents $G^\pm$ and a $U(1)$ current $J$ [13]. The current $J$ can be expressed in term of a free bosonic current in the following way

$$J = i \sqrt{\frac{c}{3}} \partial H. \quad (1)$$

The $N = 2$ algebra admits an automorphism called spectral flow [14], which is given by twisting with respect to the $U(1)$ current. Spectral flow by half a unit connects the Neveu-Schwarz (NS) and the Ramond (R) sectors of the theory. The generators of the spectral flow are

$$U_{+1/2} = \exp\left(\frac{i}{2} \sqrt{\frac{c}{3}} H\right), \quad U_{-1/2} = \exp\left(-\frac{i}{2} \sqrt{\frac{c}{3}} H\right). \quad (2)$$

A field $\Phi$ of the $N = 2$ SCFT has a specific conformal dimensions $h$ and $U(1)$ charge $q$. The properties of the $N = 2$ algebra imply that there is a lower bound for the conformal dimension $h \geq 1/2 \ | q \ |$ for every field $\Phi$. Fields which saturate this bound $h = 1/2 \ | q \ |$ are primary. The field is annihilated by either $G^+_{-1/2}$ or $G^-_{-1/2}$ and called chiral (c) or antichiral (a) respectively. (Anti)chiral primaries have nonsingular OPE’s among themselves and form a finite ring, called (anti)chiral ring [15].
An internal conformal field theory with central charge \( c_{\text{int}} = 12 - 3(D-2)/2 \) is used to compactify type II superstring theory to \( D \) non-compact dimensions. The compactified theory will be space-time supersymmetric only if the internal conformal field theory has (at least) \( N = 2 \) superconformal invariance \([16]\).

Hence compactification to four dimensions \((D = 4)\) gives an internal SCFT with \( c = 9 \) and a transverse SCFT with \( c = 3 \). The transverse SCFT describes the propagation of the string in four dimensional Minkowski space in the light-cone gauge, where all string excitations can be described in terms of a free complex boson \( X = X^1 + iX^2 \) and fermion \( \psi = \psi^1 + i\psi^2 \). This system is the simplest example of an \( N = 2 \) SCFT with \( c = 3 \) where the superconformal tensors are given by

\[
T = \frac{1}{2} \partial X \partial X^* + \frac{1}{2} \psi \partial \psi^* + \frac{1}{2} \psi^* \partial \psi, \quad G^+ = \psi \partial X^*, \quad G^- = \psi^* \partial X, \quad J_{\text{ext}} = \psi^* \psi. \tag{3}
\]

Using (1) the \( U(1) \) currents \( J_{\text{ext}} \) and \( J_{\text{int}} \) for the transverse \((c = 3)\) and internal \((c = 9)\) SCFT can be expressed in terms of a free boson \( \phi \) and a free boson \( H \) respectively,

\[
J_{\text{ext}} = i \partial \phi, \quad J_{\text{int}} = i \sqrt{3} \partial H. \tag{4}
\]

The transverse \( U(1) \) charge determines the helicity of the transverse state in the light-cone gauge. The operators implementing the spectral flow by \( \eta = \pm 1/2 \) are given by \( SO(2) \) spin fields which connect the NS sector and the R sector,

\[
S = \exp(+i \frac{1}{2} \phi), \quad S^\dagger = \exp(-i \frac{1}{2} \phi). \tag{5}
\]

Space-time supersymmetry is achieved by imposing a generalized GSO projection which keeps only states with odd integer charge with respect to the total \( U(1) \) current given by the sum of the internal and space-time \((1)\) currents,

\[
J_{\text{tot}} = J_{\text{ext}} + J_{\text{int}} = i \partial \phi + i \sqrt{3} \partial H. \tag{6}
\]

In the light-cone gauge the four supersymmetry charges \( Q^a, a = 1, \cdots, 4 \) can be divided into linearly realized ones for which \( \Gamma^+ Q = 0 \) and nonlinearly realized ones for which \( \Gamma^- Q = 0 \). The linear supercharges are constructed from the transverse spin fields \((3)\) and the spectral flow operator \((2)\)

\[
Q = \sqrt{p^+} \oint \exp(+i \frac{1}{2} \phi) \exp(i \frac{\sqrt{3}}{2} H), \tag{7}
\]

\[
Q^\dagger = \sqrt{p^+} \oint \exp(-i \frac{1}{2} \phi) \exp(-i \frac{\sqrt{3}}{2} H).
\]
The nonlinear supercharges are given by
\[
S = \frac{1}{\sqrt{p^+}} \oint \exp(+i\frac{1}{2}\phi) \exp(i\frac{\sqrt{3}}{2}H) \partial X,
\]
\[
S^\dagger = \frac{1}{\sqrt{p^+}} \oint \exp(-i\frac{1}{2}\phi) \exp(-i\frac{\sqrt{3}}{2}H) \partial X^*.
\]
(8)
The supercharges satisfy the anti-commutation relations of $N = 1$ supersymmetry algebra in light-cone coordinates,
\[
\{Q, Q^\dagger\} = p^+, \quad \{S, S^\dagger\} = p^-, \quad \{Q^\dagger, S\} = p, \quad \{Q, S^\dagger\} = p^*.
\]
(9)
Where $p^-$ is determined by the physical state conditions in terms of the zero mode of the energy momentum tensor and $p = p_1 + ip_2, p^* = p_1 - ip_2$ denotes the transverse momenta in a complex basis.

| rep | $\lambda$ | $\Delta_{ext}$ | $q_{int}$ | $\Delta_{int}$ |
|-----|-----------|-------------|-----------|---------------|
| o   | 0         | 0           | $\pm 1$   | 1/2           |
| v   | $\pm 1$   | 1/2         | 0         | 0             |
| s   | +1/2      | 1/8         | 1/2,-3/2  | 3/8           |
| c   | -1/2      | 1/8         | -1/2,+3/2 | 3/8           |

Table 1: Massless left-moving sectors for a $c = 9$ compactification. $\lambda$ and $\Delta_{ext}$ denote external charge and dimension whereas $q$ and $\Delta_{int}$ denote internal charge and dimension.

The spectrum of type II string theories is a tensor product of left-moving and right-moving sectors. In the following we shall use unbarred fields for the left-movers and barred fields for the right-movers. The massless states in $N = 2 \mathrm{D} = 4$ compactifications are the ones which have left and the right conformal dimension $\Delta_{tot} = \Delta_{ext} + \Delta_{int} = 1/2, \bar{\Delta}_{tot} = \bar{\Delta}_{ext} + \bar{\Delta}_{int} = 1/2$. Such states are labeled by the left and right-moving helicities $\lambda, \bar{\lambda}$ and $U(1)$ charges $q, \bar{q}$.

Type IIA and type IIB strings differ by the relative sign of the internal and external $U(1)$ charge in the total $U(1)$ current for the right-movers. The left-moving current is given by (8) and the right moving current for IIB and IIA is given by
\[
\text{IIB} : \quad \bar{J}_{tot} = i\bar{\partial} \bar{\phi} + i\sqrt{3} \bar{\partial} \bar{H},
\]
\[
\text{IIA} : \quad \bar{J}_{tot} = i\bar{\partial} \bar{\phi} - i\sqrt{3} \bar{\partial} \bar{H}.
\]
(10)
The difference between IIB and IIA is the reversal of the sign of $\bar{H}$. This can be traced back to the fact that IIB in ten dimensions is chiral whereas IIA is not. The massless
spectrum is determined by tensoring the states given in table 1 for left and right-movers and projecting onto odd integer $U(1)$ charges using the currents (6) and (10).

For type IIA, the right-moving supersymmetry charges are then similarly given by reversing the sign of $\bar{H} \rightarrow -\bar{H}$ in (7), (8). This implies that the supersymmetry $Q$ maps $\lambda \rightarrow \lambda + 1/2$ and $q \rightarrow q + 3/2$ and $\bar{Q}$ maps $\bar{\lambda} \rightarrow \bar{\lambda} + 1/2$ and $\bar{q} \rightarrow \bar{q} - 3/2$. In contrast, for IIB the left and right signs are the same and $Q$ maps maps $\lambda \rightarrow \lambda + 1/2$ and $q \rightarrow q + 3/2$ and $\bar{Q}$ maps $\bar{\lambda} \rightarrow \bar{\lambda} + 1/2$ and $\bar{q} \rightarrow \bar{q} + 3/2$. The action of the supersymmetries relate the scalars in the $NS \otimes NS$ sector to the $R \otimes R$ sector. $R \otimes R$ states with $\lambda + \bar{\lambda} = 0$ belong into hypermultiplets whereas states with $\lambda + \bar{\lambda} = \pm 1$ fall into vector multiplets. Table 2 shows which chiral rings give vector and hypermultiplets in the massless sector. Note that the role of the chiral rings is interchanged for IIA and IIB.

| type  | vector | hyper |
|-------|--------|-------|
| IIA   | (a,c)+(c,a) | (c,c)+(a,a) |
| IIB   | (c,c)+(a,a) | (a,c)+(c,a) |

Table 2: Hyper and vector multiplets for type IIA/B

Using the sigma-model description of Calabi-Yau compactification the elements of the (anti)chiral rings can be associated with elements of the cohomology classes of the Calabi-Yau manifold and it can be shown that the $(c,c)$ ring is in one to one correspondence with $H_{2,1}$ and the $(a,c)$ ring is in one to one correspondence with $H_{1,1}$ [17].

3. Review of Gepner models

Gepner models [3] are exactly soluble supersymmetric compactifications of type II and heterotic strings which use tensor products of $N = 2$ minimal models to construct the internal SCFT. The $N = 2$ minimal models are unitary representations of the $N = 2$ SCFT which are labeled by an integer $k = 1, 2, \cdots$ where the central charge is given by

$$c = \frac{3k}{k + 2}. \quad (11)$$

Primary fields $\Phi_{l,m,s}^i$ are labeled by three integers $l, m, s$ with the ranges:

$$l = 0, 1, \cdots, k, \quad m = -(k + 1), \cdots, k + 2, \quad s = 0, 2, \pm 1. \quad (12)$$

together with constraint $l + m + s \in 2Z$. The field identifications $(l, m, s) \sim (l, m, s + 4)$ and $(l, m, s) \sim (l, m + 2(k + 2), s)$ imply that $m$ is defined modulo $2(k + 2)$ and $s$ is defined

\footnote{Note that states with $s = 2$ are really descendants. Nevertheless splitting each module into subsets with $s = 0$ and $s = 2$ is a very useful bookkeeping device.}
modulo 4. The labels \((l, m, s)\) can be brought into the ‘standard range’ by another field identification \((l, m, s) \sim (k - l, m + k + 2, s + 2)\). The conformal dimension \(h\) and \(U(1)\) charge \(q\) of the primary fields (with \((l, m, s)\) in the standard rage) are given by

\[
\begin{align*}
    h &= \frac{l(l + 2) - m^2}{4(k + 2)} + \frac{s^2}{8}, \\
    q &= \frac{m}{k + 2} - \frac{s}{2}.
\end{align*}
\]  

(13)

A Gepner model is constructed by tensoring \(n\) minimal models with \(k_i, i = 1, \cdots, n\) such that the sum of the central charges of the \(n\) minimal models is equal to

\[
\sum_{i=1}^{n} \frac{3k_i}{k_i + 2} = c_{int}.
\]  

(14)

The total currents \(T, G^\pm, J\) of the tensor product are given by the sum of the currents of each minimal model. The external theory is given by \(D - 2\) free bosons and a level one \(SO(D - 2)\) current algebra. The primary fields can be labeled by two vectors

\[
\lambda = (l_1, \cdots, l_n), \quad \mu = (s_0; m_1, \cdots, m_n; s_1, \cdots, s_n).
\]  

(15)

Here \(s_0 = 0, 2, +1, -1\) labels the four characters corresponding to \(o, v, s, c\) conjugancy classes of the \(SO(D - 2)\) current algebra. Gepner constructed a supersymmetric partition function for the tensor product by using charge projections (generalizing the GSO projection) and adding twisted sectors to achieve modular invariance. This ‘\(\beta\)-method’ uses the \(2n + 1\) dimensional vectors: \(\beta_0\) which has 1 everywhere and \(\beta_i, i = 1, \cdots, n\) which has 2 in the first and \(n + 1 + j\) entry and is zero everywhere else. An inner product of two \(2n + 1\) dimensional vectors is defined by

\[
\mu \cdot \bar{\mu} = -\frac{(D - 2)}{8} s_0 \bar{s}_0 - \sum_{j=1}^{n} \frac{s_j \bar{s}_j}{4} + \sum_{j=1}^{n} \frac{m_j \bar{m}_j}{2(k_j + 2)}.
\]  

(16)

Note that with the help of this inner product the total \(U(1)\) charge of a primary field is given by \(q_\mu = 2 \beta_0 \cdot \mu\). The GSO projection is then implemented by projecting onto states with an odd integer charge \(q_\mu\). In order to preserve the \(N = 1\) superconformal invariance all fields in the tensor product have to be in the same sector (R or NS). This can be achieved by projecting onto states which satisfy \(\beta_j \cdot \mu \in \mathbb{Z}\) for \(j = 1, \cdots, n\). Gepner constructed a modular invariant partition function by including twisted sectors,

\[
Z = \frac{1}{2^n} \left| \eta(q) \right|^{-\frac{(D-2)}{2}} \sum_{b_0, b_j, \lambda, \mu} \sum_{\beta} \frac{(-1)^{b_0}}{b_0} \chi^\lambda(\bar{q}) \chi^\lambda_{\mu + b_0 \beta_0 + \sum_j b_j \beta_j}(\bar{q}).
\]  

(17)

5 Here and in the following we display the formulae for \(D = 4\) and \(D = 8\). The construction for \(D = 6\) is slightly different and will not be needed in this paper.
Here \( b_j = 0, 1; b_0 = 0, 1, \ldots, K - 1; K = \text{lcm}(4, 2(k_j + 2)) \) and \( q = e^{2\pi i \tau} \). \( \chi^\lambda_\mu \) are the characters corresponding to the primaries \( \Phi^\lambda_\mu \). In \((14)\) the diagonal affine \( SU(2) \) invariant is used which exists for all levels \( k_j \). Other choices according to the ADE classification of affine \( SU(2) \) invariants are possible and lead to different models \([15]\). The notation \( \sum_\beta \) indicates the summation over the \( \beta \) projected range \( \lambda, \mu \) and the \((-1)^{b_0} \) imposes the connection between spin and statistics. Note that the supersymmetries \([7]\) have a very simple action on the characters \( \chi^\lambda_\mu \); acting with \( Q \) corresponds to \( \mu \rightarrow \mu + \beta_0 \) and acting with \( Q^\dagger \) corresponds to \( \mu \rightarrow \mu - \beta_0 \).

Evidence for the equivalence of a Gepner model to compactification on a Calabi-Yau manifold was first presented in \([5]\). It was shown that massless spectrum and the discrete symmetries of Gepner models and certain hypersurfaces in weighted projective spaces and orbifolds thereof are the same. Using \((13)\) it is easy to see that chiral and antichiral primaries in the Gepner model correspond (up to field identification) to fields \( \Phi^{\lambda,\lambda}_{\mu,\mu} \) and \( \Phi^{\lambda,\lambda}_{\mu,-\mu} \) with \( l_i = m_i, s_i = 0 \) respectively. The massless fields satisfy \( 2\beta_0 \cdot \mu = \pm 1 \). Furthermore the Yukawa couplings are the same \([19]\). The equivalence was put onto firmer footing using a linear sigma model \([20]\) which interpolates between the CY sigma model and the LG orbifolds which in turn are well known to be equivalent to minimal models \[21]\).

In the case of \( D = 8 \) (\( c = 3 \)) only three Gepner models exist, \((k = 2)^2, (k = 1)^3\) and \((k_1 = 1, k_2 = 4)\). The first one corresponds to a compactifications on a \( SU(2)^2 \) torus and the last two correspond to a compactification on a \( SU(3) \) torus. For \( D = 4 \) (\( c = 9 \)) a large (but still finite) number of models exist corresponding to Calabi-Yau compactifications. As an specific example we will use the \((k = 3)^5\) model which corresponds to the quintic hypersurface in \( CP^4 \).

4. Boundary states for Gepner models

Dirichlet branes \([1]\) provide a surprisingly simple realization of non-perturbative objects in closed string theories. D-branes can be described by the boundary state formalism \([22]\) where open string boundary conditions are enforced on the closed string fields. Consistent boundary conditions for superstrings require that the \( N = 1 \) superconformal invariance is unbroken. This implies continuity conditions for the stress tensor and its superpartner on the boundary. In the simplest case of the upper half plane \( \mathcal{H} = \{ z \mid \text{Im}(z) > 0 \} \), we require \( T(z) = \bar{T}(\bar{z}) \) and \( G(z) = \pm \bar{G}(\bar{z}) \) at the boundary \( z = \bar{z} \). Via a conformal transformation which maps the upper half plane into a semi-infinite cylinder this can be related to conditions on a boundary state \( | B \rangle \).

If the conformal theory forms an extended algebras boundary conditions relating the left and right-moving currents in the extended algebra \( W, \bar{W} \) have to be specified \([1]\). To construct boundary states for rational conformal field theories one first defines Ishibashi
states $|i\rangle\rangle$ for every primary field defining an irreducible highest weight representation $\mathcal{H}_i$ of the algebra which satisfy

$$\left(W_n - (-1)^{hw} \bar{W}_{-n}\right) |i\rangle\rangle = 0. \quad (18)$$

In [6] it was shown that an Ishibashi state can be constructed using an anti-unitary operator $U$ which acts on the modes of the right-moving current $\bar{W}$ in the following way, $U \bar{W}_n U^{-1} = (-1)^{hw} \bar{W}_n$. When $\bar{W}$ is a fermionic operator, we need to take into account the anti-commutativity to prove (18), and thus the action of $U$ needs extra phase $(-1)^F$ where $F$ is the fermion-number operator. Such an operator $U$ is closely related to the chiral CPT operator. Explicit form of the Ishibashi state is given by

$$|i\rangle\rangle = \sum_N |i,N\rangle \otimes U |\bar{i},\bar{N}\rangle. \quad (19)$$

where $N$ denotes the sum over the basis of $\mathcal{H}_i$. In the second step a boundary state can be constructed from a complete set of Ishibashi states

$$|\alpha\rangle = \sum_i B_i^\alpha |i\rangle. \quad (20)$$

There are constraints on $B_i^\alpha$ which come from the fact that a boundary state $|\alpha\rangle$ has to define an open string boundary condition. This implies that the modular transform of the cylinder amplitude $Z_{\alpha\beta}(q) = \langle \beta | e^{-\pi t H_{cl}} | \alpha \rangle$ is related to an open string partition function $Z_{\alpha\beta}(\tilde{q}) = Tr_{\alpha\beta} e^{-\pi \tau H_{op}}$ via a modular transformation $t = 1/\tau$. The consistency of open string partition function demands that it contains the characters of the unbroken symmetry algebra with integer multiplicities and this imposes nonlinear constraints on the matrix $B_i^\alpha$. A solution to these constraints was found by Cardy [7],

$$B_i^\alpha = \frac{S_{\alpha i}}{\sqrt{S_{0 i}}}. \quad (21)$$

Here $S_{\alpha i}$ is the modular $S$-matrix and 0 denotes the vacuum representation.

In [4],[23] it was shown that two different boundary conditions for $U(1)$ current $J$ and the superconformal generators $G^\pm$ are consistent with $N = 1$ superconformal invariance. The two cases are called A and B boundary conditions, referring to the two possible topological twists of the $N = 2$ theory [24]. The A boundary conditions are defined by

$$(J_n - \bar{J}_{-n}) |B\rangle = 0, \quad (G^-_r + i\eta \bar{G}^+_r) |B\rangle = 0, \quad (22)$$

whereas the B type boundary conditions are defined by

$$(J_n + \bar{J}_{-n}) |B\rangle = 0, \quad (G^+_r + i\eta \bar{G}^-_r) |B\rangle = 0. \quad (23)$$
The choice of $\eta = \pm 1$ corresponds to a choice of spin structure. The anti-unitary operator $U$ in (19) acts on the operators of the $N = 2$ algebra in the following way

$$U^{-1} J_n U = -J_n, \quad U^{-1} G^\pm_r U = -i \eta \bar{G}^\mp_r (-1)^F. \quad (24)$$

It is therefore easy to see that an Ishibashi state for an $N = 2$ minimal model (19) using $U$ imposes $B$ boundary conditions which satisfy $q = -\bar{q}$. On the other hand it follows from (22) that $A$ boundary conditions satisfy $q = \bar{q}$. Such boundary conditions are obtained from $B$ boundary conditions by a twist $\Omega$ which undoes the charge reversal caused by $U$. This is given by the mirror automorphism $\Omega$ of the $N = 2$ algebra,

$$\Omega^{-1} J_n \Omega = -J_n, \quad \Omega^{-1} G^\pm_r \Omega = \bar{G}^\mp_r. \quad (25)$$

An Ishibashi state including the additional twist can be written as

$$| i \rangle \rangle = \sum_N | i, N \rangle \otimes U \Omega | \tilde{i}, \tilde{N} \rangle. \quad (26)$$

The $N = 2$ space-time supersymmetric compactifications contain two $N = 2$ SCFT, the transverse $c = 3$ and the internal $c = 9$ SCFT. Hence $A$ and $B$ boundary conditions can be imposed separately on the two factors, which combination for type IIA/B is consistent is determined by the GSO projection.

The $c = 3$ part of the SCFT given by a free complex boson and fermion (3) and will be discussed first. As shown in (1) boundary states in the light-cone gauge impose Dirichlet boundary conditions on the light-cone coordinates $X^+, X^-$. Hence we are dealing with D-instantons which have fixed boundary conditions in the time direction.

When Dirichlet boundary conditions are imposed on the two transverse coordinates $X^1, X^2$ the resulting boundary state describes a $p = -1$ brane in four dimensions, i.e. an event which is localized in the four transverse directions $X^\mu = y^\mu, \mu = +, -, 1, 2$. Denoting $X = X^1 + i X^2$ and $\psi = \psi^1 + i \psi^2$ this condition is equivalent to

$$(\partial X - \bar{\partial} X) | B \rangle = 0, \quad (\psi - i \eta \bar{\psi}) | B \rangle = 0. \quad (27)$$

It is easy to see that with the definitions of the $N = 2$ algebra given in (3) the D-instanton boundary conditions correspond to the $B$ boundary conditions (23) for the $c = 3$ system. The boundary state for a free boson which imposes (27) is constructed using coherent states. Note that imposing Neuman boundary conditions for both $X^1, X^2$, also realizes $B$ boundary conditions. We will not discuss D1 branes in this paper since new subtleties (similar to D7-branes in ten dimensions) arise due to the fact that there are
only two transverse dimensions. One can also consider boundary conditions which impose Dirichlet boundary conditions on $X^1$ and Neuman boundary conditions on $X^2$.

$$\left( \partial X - \bar{\partial} X^* \right) | B \rangle = 0, \quad \left( \psi - i\eta \bar{\psi}^* \right) | B \rangle = 0. \quad (28)$$

Such boundary conditions correspond to an (Euclidean) D0-brane from the four-dimensional perspective. As discussed in [11] such a configuration can be related to the standard D0-brane by a double Wick rotation. It is easy to see that (28) imposes $A$ boundary conditions (22) on the $c=3 \ N = 2$ SCFT.

The boundary states for the internal $c = 9$ SCFT are constructed by tensoring Ishibashi states for the minimal models, subject to the charge projection and appearance of twisted sectors. For $A$ boundary conditions an Ishibashi state associated with a primary field of a minimal model labeled by $l, m, s$ is given by

$$| l, m, s \rangle \rangle = \sum_N | l, m, s, N \rangle \otimes U \bar{\Omega} | l, m, s, N \rangle. \quad (29)$$

For $B$ boundary conditions we get

$$| l, m, s \rangle \rangle = \sum_N | l, m, s, N \rangle \otimes U | l, m, s, N \rangle. \quad (30)$$

A boundary state satisfying $A$ and $B$ boundary conditions for the tensor product can be constructed by the product of the boundary states (29) and (30) respectively. It is important to note that an Ishibashi state exists only if the primary $\Phi_{\lambda,\lambda}^{\mu,\mu}$ (for $A$ boundary conditions) and $\Phi_{\mu,-\mu}^{\mu,-\mu}$ (for $B$ boundary conditions) appear in the partition function (17) of the Gepner model. Note that this implies that the boundary Ishibashi states satisfy the charge projection condition and that the states in (30) come from the twisted sectors in (17).

In [10] boundary states in Gepner models corresponding to $A$ and $B$ boundary conditions were constructed by applying Cardy’s construction [7] for each factor of the tensor product of $n$ minimal $N = 2$ theories. A boundary state is then labeled by a vector $\alpha = (\lambda', \mu')$ where $\lambda' = (l'_1, \cdots, l'_n)$ and $\mu' = (s'_0; m'_1, \cdots, s'_n)$ and given by

$$| \alpha \rangle = \frac{1}{\kappa_\alpha} \sum_{\lambda, \mu} B_{\lambda, \mu}^{\alpha} | \lambda, \mu \rangle \rangle. \quad (31)$$

The normalization constant $1/\kappa_\alpha$ can be determined by Cardy’s condition [10]. The factor $B_{\lambda, \mu}^{\alpha}$ is the product of $B_i^{\alpha}$ in (21) using the modular $S$-matrix for the $N = 2$ minimal models [10]:

$$B_{\lambda, \mu}^{\alpha} = e^{i\pi s'_0/2} e^{-i\pi s_0' \alpha_0} \prod_{j=1}^N \frac{\sin \left( \frac{\pi (l_j+1)(l_j'+1)}{k_j+2} \right)}{\sin^{1/2} \left( \frac{\pi (l_j+1)}{k_j+2} \right)} e^{i\pi m_j m_j'} e^{-i\pi s_{j}' \alpha'_j}. \quad (32)$$
The fact that either A or B boundary conditions can be imposed for the transverse $c = 3$ and the internal $c = 9$ system leads to four distinct boundary states. Since projection on odd integer $U(1)$ charge couples the two sectors in order to achieve space-time supersymmetry two choices are consistent with IIA and two with IIB.

|       | D-1 | D0 |
|-------|-----|----|
| IIA   | $B \otimes A$ | $A \otimes B$ |
| IIB   | $B \otimes B$ | $A \otimes A$ |

**Table 3:** Boundary conditions for type IIA/B in Gepner models

From table 3 and table 2 it is easy to see that the massless states appearing in the D0 brane boundary states lie in vector multiplets. This fact is in agreement with the interpretation of D0-branes as charged black holes in $N = 2$ supergravity. For the D-instanton the massless components of the boundary state lie in hypermultiplets which means that the instanton provides a source for the charge associated with the shift of RR-scalars, in analogy with the D-instanton in ten dimensions [25].

It is important to note that the solution (31) is constructed by tensoring the Ishibashi states where all minimal models satisfy either A or B boundary conditions, hence no mixed boundary conditions are allowed. There are possible generalization since the conditions (22) and (23) only have to be satisfied for the currents of the $c = 9$ SCFT which are sums of the currents of the minimal models. Hence if there is an automorphism $\mathcal{V}$ of the right-moving algebra which leaves the currents invariant

$$\mathcal{V} \bar{T} \mathcal{V}^{-1} = \bar{T}, \quad \mathcal{V} \bar{G}^{\pm} \mathcal{V}^{-1} = \bar{G}^{\pm}, \quad \mathcal{V} \bar{J} \mathcal{V}^{-1} = \bar{J}.$$  \hspace{1cm} (33)

but does not leave the each individual current $\bar{T}_i, \bar{G}_i^{\pm}, \bar{J}_i$ of the minimal models invariant, a more general Ishibashi state can be constructed by

$$| \lambda, \mu \rangle = \sum_N | \lambda, \mu, N \rangle \otimes U \mathcal{V} | \lambda, \mu, N \rangle.$$  \hspace{1cm} (34)

In the Gepner model a $B$ boundary state can contain such an Ishibashi state only if the field $\Phi^{\lambda, \nu \lambda}_{\mu, -\nu \mu}$ appears in the twisted sector of the partition function. One example of such an automorphism of the $N = 2$ algebra is a permutation of minimal models with the same $k$. It might also be possible to apply the more general ideas of [26] in this context. It would be interesting to analyze such boundary states further, but for the rest of this paper the boundary states defined by (31) will be used.
5. \( c = 3 \) Gepner models and \( T^2 \) compactifications

The simplest Gepner models arise for \( c_{\text{int}} = 3 \) and correspond to compactifications on special \( T^2 \). There are only three cases denoted by \( (k = 1)^3, (k = 1, k = 4) \) both of which correspond to an \( SU(3) \) torus and \( (k = 2)^2 \) which gives an \( SU(2)^2 \) torus. In this section we will treat the \( (k = 2)^2 \) Gepner model in detail. The \( c = 3 \) Gepner models are somewhat trivial, but their simplicity makes explicit computations easier and some of the general aspects of Gepner models are already manifest in these cases.

5.1. torus partition function

The boundary state \( | B \rangle = | B \rangle_{\text{osc}} \times | B \rangle_0 \) for a D-brane compactified on a \( T^2 \) contains an oscillator part \( | B \rangle_{\text{osc}} \) and an zero mode part \( | B \rangle_0 \) which contains a sum over momenta and winding modes.

In ten dimensions the cylinder partition function for a supersymmetric D-brane is given by

\[
Z(q) = \langle B | q^{L_0 + \bar{L}_0 - c/12} | B \rangle. \tag{35}
\]

The oscillator part is given by a summing over the \( NS \otimes NS \) and \( R \otimes R \) sector of the boundary state together with the insertion of a GSO projection operator \( 1/2(1 + (-1)^F) \);

\[
Z_{\text{osc}}(q) = \frac{1}{\eta^8} \left( \chi_v^{(8)} - \chi_c^{(8)} \right). \tag{36}
\]

Here the characters of the \( SO(2d) \) current algebra are given by

\[
\chi_o^{(2d)} = \frac{1}{2\eta^d} \left( \theta_3^d + \theta_4^d \right), \quad \chi_v^{(2d)} = \frac{1}{2\eta^d} \left( \theta_3^d - \theta_4^d \right), \quad \chi_s/c^{(2d)} = \frac{1}{2\eta^d} \theta_2^d, \tag{37}
\]

and \( \eta = q^{1/24} \prod (1 - q^n) \) is the Dedekind \( \eta \)-function. The cylinder partition function (36) vanishes because of supersymmetry.

The zero mode part of the boundary state \( | B \rangle_0 \) is defined by sum over a sublattice of the momentum and winding lattice

\[
p_L^i = \frac{G_{ij}}{\sqrt{2}} (m_j + (B_{jk} + G_{jk})n_k), \quad p_R^i = \frac{G_{ij}}{\sqrt{2}} (m_j + (B_{jk} - G_{jk})n_k), \tag{38}
\]

where \( i, j, k = 1, 2 \) and \( G_{ij}, B_{ij} \) are the metric and antisymmetric tensor background fields on the \( T^2 \). The boundary state is then defined by \( | B \rangle_0 = \sum_{p_L, p_R \in \Lambda} | p_L, p_R \rangle \) such that

\[
(p_L^i + R^i_j p_R^j) | B \rangle_0 = 0. \tag{39}
\]

Note that unlike in the heterotic compactification in the case of type II compactification these tori do not lead to an enhancement of gauge symmetry.
here the matrix $R^i_j$ defines the D-brane boundary conditions and $\Lambda$ is a maximally two
dimensional sublattice of the four dimensional lattice (38). For boundary conditions which
impose Dirichlet boundary conditions on one and Neumann on the other direction on the
torus the matrix $R$ takes the form

$$R^i_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}.$$  \hspace{1cm} (40)

Note that in general only for special values of the rotation angle $\theta$ the conditions (39) will
have nontrivial solutions.

The zero mode part of the partition function is then given by

$$Z_0(q) = \sum_{p_L, p_R \in \Lambda} q^{\frac{1}{2}(p_L^2 + p_R^2)}.$$ \hspace{1cm} (41)

A Poisson resummation of (41) transforms $Z_0$ to the open string channel.

We will now specialize on the $(k = 2)^2$ Gepner model which is equivalent to the
$SU(2)^2$ torus. This is simply the unit square torus with $G_{ij} = \delta_{ij}, B_{ij} = 0$. The momentum
lattice (38) is then given by

$$p^i_L = \frac{1}{\sqrt{2}} (m^i + n^i), \quad p^i_R = \frac{1}{\sqrt{2}} (m^i - n^i), \quad i = 1, 2.$$ \hspace{1cm} (42)

The $A$ boundary condition corresponding a D0 brane wrapping on the $X^1$ cycle is defined
by (40) with $\theta = 0$ and the conditions on the momenta turn out to be

$$p^i_L \parallel B \rangle = p^i_R \parallel B \rangle, \quad p^2_R \parallel B \rangle = -p^2_R \parallel B \rangle.$$ \hspace{1cm} (43)

The lattice $\Lambda$ is defined by (12) where $n^1 = m^2 = 0$ and $(n^2, m^1) \in Z^2$. The zero mode
part of the cylinder partition function is then given by

$$Z_{0,\theta=0}^\theta(q) = \sum_{m_1, n_2} q^{1/4m_1^2} q^{1/4n_2^2} = (\theta_3^2(q) + \theta_2^2(q)).$$ \hspace{1cm} (44)

After a modular transformation this turns into the zero mode part of the open string
partition function

$$\frac{1}{\eta^2} Z_{0,\theta=\pi/4}^\theta(q) = \frac{1}{\eta^2} \left( \theta_3^2(q) + \theta_2^2(q) \right),$$ \hspace{1cm} (45)

where $\tilde{q} = e^{-2\pi i/\tau}$. For later comparison with the partition function of the $(k = 2)^2$
Gepner models another boundary condition, which is given by (10) with $\theta = \pi/4$, will
be important. Repeating the analysis above in this case it follows that the open string
partition function is given by

$$\frac{1}{\eta^2} Z_{0,\theta=\pi/4}^\theta(q) = \frac{1}{\eta^2} \theta_3^2(q).$$ \hspace{1cm} (46)

The cylinder partition function is the product of the oscillator (39) and the zero
mode part (11). In order to compare it with the open string partition function for the
Gepner model, we make a modular transformation and decompose the oscillator part in
terms of $SO(6) \times SO(2)$ characters. Then we get

$$Z = \frac{1}{2} \left( \frac{\chi_0^{(6)}}{\eta^6} \theta_3 - \theta_4 + \frac{\chi_v^{(6)}}{\eta^6} \theta_3 + \theta_4 - \frac{\chi_s^{(6)}}{\eta^6} \theta_2 - \frac{\chi_c^{(6)}}{\eta^6} \theta_2 \right) Z_{0,\theta=0,\pi/4}^\theta(q).$$ \hspace{1cm} (47)

The partition functions for $\theta = n\pi/2 (+\pi/4)$ are given by $Z_{0,\theta=0,\pi/4}^\theta(q)$.  \hspace{1cm} (13)
5.2. Gepner model

The open string partition function \( Z(q) \) should be compared with one from the boundary states corresponding to A boundary conditions in the Gepner model,

\[
Z(q) = \langle \alpha | q^{\frac{1}{2} (L_0 + \bar{L}_0 - c/12)} | \alpha \rangle = \sum_{\lambda, \mu} B_{\lambda, \mu}^{\alpha} B_{\lambda, -\mu}^{\alpha} \chi_{\lambda, \mu}(q). \tag{48}
\]

A modular transformation into the open string channel gives

\[
Z(\tilde{q}) = \sum_{\lambda, \mu} \sum_{\tilde{\lambda}, \tilde{\mu}} B_{\lambda, \mu}^{\alpha} B_{\lambda, -\mu}^{\alpha} S_{\lambda, \tilde{\mu}}^{\lambda, \tilde{\mu}} \tilde{\chi}_{\lambda, \mu}, \tag{49}
\]

where \( \sum_{\text{ev}} \) stands for the constraints \( l_i + m_i + s_i = 2Z \). This expression can be evaluated using the explicit form of \( B_{\lambda, \mu}^{\alpha} \) and the modular matrix \( S_{\lambda, \tilde{\mu}}^{\lambda, \tilde{\mu}} \) for the Gepner models. The result can be found in [10]. Here we only need the result with the same boundary conditions on both ends of the cylinder,

\[
Z_{\alpha \alpha}(\tilde{q}) = \sum_{\lambda, \mu} \sum_{v_0=0}^{K-1} \sum_{v_1, \ldots, v_n=0,1} (-1)^{s_0} \delta_{s_0, 2v_0+2} \sum_{s_0} \prod_{j=1}^{n} N_{l_j}^{l_j} \delta_{s_j, 2v_0+2v_j} \chi_{\lambda, \mu}(\tilde{q}), \tag{50}
\]

up to a factor. \( \delta_{m,n} \) are non-zero for \( m = n \pmod{k} \). \( N_{l_1}^{l_2} \) is the matrix appearing in the fusion rules among the primaries with spin \( l_{1,2}/2 \) in the SU(2)_k WZW model; \( \phi_{l_1/2} \times \phi_{l_2/2} \sim \sum_{l_2} N_{l_1}^{l_2} \phi_{l_2/2} \). Namely, \( N_{l_1}^{l_2} = 1 \) for \( 0 \leq l_2 \leq \min(2l_1, 2k - 2l_1) \) and otherwise vanishing. Note that the open string partition function \( Z_{\alpha \alpha} \) for two identical D-branes only depends on \( \lambda = (l_1', \ldots, l_n') \) in \( \alpha = (\lambda', \mu') \).

We will now specialize in the \((k = 2)^2\) case. Then the matrix \( N_{l_1}^{l_2} \) is a 3 \times 3 matrix given by

\[
N_{l_1}^{l_2} = \delta_{l_1,0} + \delta_{l_1,1}, \delta_{l_2,2}, \tag{51}
\]

and hence the open string partition function depending on \( l_1, l_2 \) is given by

\[
Z_{l_1, l_2}(\tilde{q}) = \sum_{v_0, v_1} N_{l_1}^{l_1} N_{l_2}^{l_2} (-1)^{v_0} \chi_{l_1, v_0, v_0}^{2v_0+2v_1+2v_2} \chi_{l_1, v_0, v_0}^{2v_0+2v_1+2v_2}. \tag{52}
\]

Here \( \chi^{s_0} \) denotes the SO(6) character and \( \chi_{m, s}^{l} \) is the character of the \( k = 2 \) minimal model corresponding to the primary \( \Phi_{m, s}^{l} \). Note that the form of (51) implies that \( l'_i = 0 \) and \( l''_i = 2 \) give the same partition function. Using the field identification for the characters \( \chi_{m, s}^{l} = \chi_{m+4, s+2}^{l-2} \) it is also easy to see that there are essentially only two choices of \((l'_1, l'_2)\) for non-vanishing partition functions, namely, (a) \((l'_1, l'_2) = (0, 0)\) and (b) \((l'_1, l'_2) = (1, 0)\).\(^7\)

\(^7\) \((l'_1, l'_2) = (1, 1)\) gives the partition function twice that of \((l'_1, l'_2) = (1, 0)\).
To compare the Gepner model partition function (52) with the toroidal one (47) it is only necessary (because of supersymmetry) to compare \( \chi^6_0 \) part of (47) with the \( v_0 + 2v_1 + 2v_2 = 2 \) (mod 4) part of (52). We denote this part by \( \chi^6_0 Z^{v_0}_{l'_1, l'_2} \). After some calculation, we then find that

\[
Z^{0,0}_{0,0}(\tilde{q}) = \frac{1}{4\eta^3} (\theta_3 - \theta_4) Z^{0,0}_{0} (\tilde{q}) ,
\]

\[
Z^{0,0}_{1,0}(\tilde{q}) = \frac{1}{2\eta^3} (\theta_3 - \theta_4) Z^{\theta = \pi/4}_{0} (\tilde{q}) .
\]

(53)

We relegate some details of the calculation to appendix A. This gives the identification between our algebraically constructed boundary states and D-branes wrapping around geometrical cycles; case (a) represents the ‘short’ branes along \( \theta = n\pi/2 \) whereas case (b) represents the ‘long’ branes along \( \theta = \pi/4 + n\pi/2 \).

Furthermore, similar argument can be applied to other \( c = 3 \) Gepner models. (See also appendix A.) In the \( (k = 1)^3 \) case, we have only one independent choice given by \( (l'_1, l'_2, l'_3) = (0, 0, 0) \). It turns out that the corresponding boundary state represents the D-brane wrapping around \( \theta = n\pi/3 \) cycles of the \( SU(3) \) torus. In the \( (k = 1, k = 4) \) case, we have three independent choices, \( (l'_k = 1, l'_k = 4) = (a) (0, 0), (b) (0, 1) \) and (c) \( (0, 2) \). Case (a) gives the ‘short’ D-branes wrapping around \( \theta = n\pi/3 \) while case (b) gives the ‘long’ D-branes around \( \theta = \pi/6 + n\pi/3 \). The partition function for case (c) is the sum of the partition functions for case (a) and (b). The geometrical interpretation of the last case is not completely clear.

For the \( (k = 2)^2 \) and \( (k = 1)^3 \) case, we know explicit relations between the space-time bosons and fermions and the free fields realizing the minimal models. Using them we have discussed the geometrical interpretation of the boundary conditions for the space-time fields in appendix B. The above results are consistent with the possible types of D-branes from the open string channel argument (appendix B.1). From the closed string channel argument (appendix B.2), the boundary conditions for space-time fields are given by \( m'_i \) in \( \alpha \) and independent of \( l'_i \). This is complementary to the results in this section which are independent of \( m'_i \); we can choose \( l'_i \) and \( m'_i \) which are compatible with the two arguments. This implies that there may be some selection rules for allowed parameter \( \alpha \). In principle, the consistency conditions of the string theory such as sewing constraints (see, e.g., [10]) may give the solution.

6. Conserved and broken supersymmetry charges

For definiteness we will consider a boundary state representing a D-instanton in IIB compactification, which means that \( B \) type boundary conditions are imposed on the internal theory. The boundary state \( | \alpha \rangle \) is labeled by \( \alpha = (\lambda', \mu') \). The fact that the boundary state corresponds to a D-brane wrapping a supersymmetric cycle implies that
four of the eight four dimensional supersymmetries are unbroken by the boundary state. Since the boundary state $| \alpha \rangle$ contains only states with $q = -\bar{q}$ we have to consider combinations $Q_L - e^{-i\phi} Q_R$ and $Q_L + e^{i\phi} Q_R^\dagger$ as the unbroken supersymmetry. Here $e^{i\phi}$ is a phase which can be absorbed into the definition of $Q_R$ but will be kept in the following.

Acting with $Q_L$ shifts the left $\mu$ by $\beta_0$;

$$Q_L | \alpha \rangle = \sum B^\alpha_{\lambda,\mu} Q_L | \lambda, \mu; \lambda, -\mu \rangle$$

$$= \sum B^\alpha_{\lambda,\mu} | \lambda, \mu + \beta_0; \lambda, -\mu \rangle,$$

up to a factor.\(^8\) Here we have explicitly denoted the left- and right- primaries in the Ishibashi states by $| \lambda, \mu; \lambda, -\mu \rangle$. On the other hand acting with $Q_R$ on $| \alpha \rangle$ shifts the right-moving $\mu$ by $\beta_0$;

$$Q_R | \alpha \rangle = \sum B^\alpha_{\lambda,\mu} Q_R | \lambda, \mu; \lambda, -\mu \rangle$$

$$= \sum B^\alpha_{\lambda,\mu} (-1)^{s_0 - 1/2} | \lambda, \mu; \lambda, -\mu + \beta_0 \rangle.$$

The factor $(-1)^{s_0}$ accounts for the fact that the right-moving supercharge $Q_R$ acting on the fermionic part of the boundary state pick up an extra minus sign. The supercharge picks up the other factor $-i$ when it goes though $U$ in front of the right states (see also the appendix).

Shifting the summation variable $\mu$ in the second line of (55) and using the form of $B^\alpha_{\lambda,\mu}$ we can show that

$$B^\alpha_{\lambda,\mu + \beta_0} = B^\alpha_{\lambda,\mu} i (-1)^{s_0} e^{i\pi \beta_0 \cdot \mu'}.$$

Hence condition that $Q_L - e^{-i\phi} Q_R$ is killed by the boundary state $| \alpha \rangle$ relates the phase $\phi$ to the $U(1)$ charge of $\alpha$;

$$\phi = \pi Q_\alpha = 2\pi \beta_0 \cdot \mu'.$$

(57)

For one boundary state $| \alpha \rangle$ such a phase $\omega$ can be absorbed into the definition of the supercharges. Hence for any boundary state $| \alpha \rangle$ constructed by the procedure in section 3 four unbroken supersymmetries can be found. The importance of this phase becomes clear when we consider more than one boundary state. The boundary states $| \alpha_1 \rangle$ and $| \alpha_2 \rangle$ are mutually supersymmetric only if the two associated phases are related by

$$Q_{\alpha_1} = Q_{\alpha_2} + 2Z.$$

(58)

This is the same condition derived in \(^{10}\) by demanding that the open string partition function is space-time supersymmetric, i.e. satisfies the open string $U(1)$ projection. Since the supercharges are space-time fermions, the relative phases appearing in the boundary conditions may have geometrical meaning. In the simple three cases with $c_{int} = 3$, the interpretation of those phases is in agreement with the discussions in section 5 and appendix B.

\(^8\) Some states in the second line vanish.
7. Orbifolding and mirror map for boundary states

The mirror automorphism \([25]\) maps a \(N = 2\) SCFT into an equivalent one by reversing the right-moving \(U(1)\) charges. Although this operation is rather simple on the level of the conformal field theory it is the basis of mirror symmetry \([27],[28]\).

Since the string compactification is given by a product of a \(c = 3\) and \(c_{\text{int}} = 9\) \(N = 2\) SCFT for \(D = 4\), we firstly discuss the ‘mirror-map’ for the \(c = 3\) system where it is simply realized as a T-duality. For the free boson and fermion system the mirror map can be realized in the following way

\[
\bar{\partial}X \to \bar{\partial}X^*, \quad \bar{\psi} \to \bar{\psi}^*.
\]

Such a transformation is T-duality in the \(X^2\) direction since \(\bar{\partial}X^1 \to \bar{\partial}X^1\) and \(\bar{\partial}X^2 \to -\bar{\partial}X^2\). This operation was named ‘c-map’ in \([29]\). The T-duality relates IIA on \(R^3 \times S^1 \times \mathcal{M}\) to IIB on \(R^3 \times \tilde{S}^1 \times \mathcal{M}\) where \(S^1\) and \(\tilde{S}^1\) denotes circles of radius \(R\) and \(\alpha'/R\) respectively. Note that this map leaves the compactification manifold \(\mathcal{M}\) untouched. This map provides a relation between the special Kähler manifold of the vector and quaternionic manifold of the hypermultiplet moduli space since after a compactification on \(S^1\) both a vector and hyper multiplet contain four scalars and a T-duality on \(S^1\) relates the two (the universal hypermultiplet is mapped to the gravity multiplet). The map \((59)\) transforms the boundary conditions \((27)\) into \((28)\) and vice versa, hence it maps D-instantons of type II A/B into D0 branes of type II B/A respectively. Such a mapping between four dimensional D-instantons and four dimensional black holes has been discussed on the level of supergravity solutions in \([30]\).

It is well known that every \(N = 2\) minimal model at level \(k\) has a discrete \(Z_{k+2} \times Z_2\) symmetry. One can use the \(Z_{k+2} \times Z_2\) symmetry to define a twisted character in the following way

\[
Z[x, y, a, b] = \sum_{l,m,s,m=2y,s=s-2b} e^{-2\pi i x (\frac{m+m}{k+2})} e^{2\pi i a \frac{s+s}{4}} \chi_{m,s}^l \chi_{m,s}^{*l}.
\]

Here the twists are labeled by \(x, y = 0, \cdots, k - 1\) and \(a, b = 0, 1\). It is easy to see that the orbifolded partition function

\[
Z_{\text{orb}} = \frac{1}{2(k+2)} \sum_{x,y=0}^{k-1} \sum_{a,b=0,1} Z[x, y, a, b] = \sum_{l,m,s} \chi_{m,s}^l \chi_{m,s}^{*l} \chi_{m,s}^{*l}.
\]

is equal to an isomorphic partition function where the signs of the right-moving \(m\) and \(s\) are reversed.

The orbifolding procedure removes states from the partition function and adds new twisted sectors. This means that in general Ishibashi states implementing A boundary
conditions get removed, whereas new Ishibashi states for B boundary conditions will appear. The end result of the orbifolding is that boundary states $|\alpha\rangle$ for A and B boundary conditions get exchanged.

In the full Gepner model matter is complicated by the fact that the $\beta$-projections involve already an orbifoldization. In order to preserve the space-time supersymmetry one has to pick a subgroup of $\prod_i (Z_{k_i+2} \otimes Z_2)$ which preserves the projection on odd integer charge. This process has been discussed in detail in [27] and the result is given by the mirror map which reverses the right-moving $U(1)$ charges and maps (29) into (30).

Note that the mirror map has to map a type IIA into a type IIB configuration and it does not change the boundary condition for the transverse $c = 3$ part. Geometrically A boundary conditions correspond to branes wrapped on middle dimensional supersymmetric cycles, whereas B boundary conditions correspond to branes wrapped on even dimensional supersymmetric cycles [4]. The mirror map provides a mapping of $H_3(\mathcal{M})$ to $\sum_{i=0}^{3} H_{2i}(\mathcal{M})$. In the Gepner model both boundary states are characterized by the matrix $B_{\lambda,\mu}^\alpha$ with the same $\alpha$. Hence given a geometric interpretation of the boundary states labeled by $\alpha$ establishes an explicit realization of this map.

**Fig. 1:** Relations between different D-branes in four dimensions

From the geometric point of view mirror symmetry is much more nontrivial than the c-map since it relates compactification manifolds of different topology since the Hodges numbers get exchanged. On the level of the conformal field theory they are the same thing applied to the two factors of the theory.

**8. The quintic and $(k=3)^5$ Gepner Model**

The first example for the equivalence of a Gepner model and Calabi-Yau compactification given in [3] is the $(k=3)^5$ model which corresponds to the quintic hypersurface in $CP^4$ defined by

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0 .$$  (62)
The analysis of the massless spectrum of the \((k = 3)\) model shows that the \((c, c)\) ring contains 101 elements, whereas the \((a, c)\) ring is one-dimensional. This is in agreement with the Betti numbers \(h_{2,1} = 101\) and \(h_{1,1} = 1\) of the quintic hypersurface. All the information of the boundary state \(| \alpha \rangle\) in the Gepner model is encoded in the matrix \(B^\alpha_{\lambda, \mu}\) defined in (32).

As discussed in section 7 a minimal model of level \(k\) (with diagonal partition function) has a discrete symmetry group \(Z_{k+2}\). If more than one factor has the same value of \(k\) there is in addition a permutation symmetry of these factors. The discrete symmetry for the \((k = 3)\) is therefore \((S_5 \times Z_5^5)/Z_5\). In the Gepner model the quotient by \(Z_5\) is generated by \(2\beta_0\). The quintic has the same set of discrete symmetries which is generated by the permutation of \(z_i\) and the transformations \(z_i \rightarrow e^{2\pi i n_i/5}z_i, i = 1, \cdots, 5\). Note that an overall phase \(z_i \rightarrow \lambda z_i, i = 1, \cdots, 5\) is immaterial.

An element of discrete symmetry group \(g \in G\) acts on the boundary state in a certain way \(g | \alpha \rangle = | g\alpha \rangle\), i.e., the symmetry maps the boundary state into a new one which is characterized by the vector \(g\alpha\). An element of the \(j\)-th \(Z_5\) in \(G\) acts on a primary field \(\Phi_{\mu, \mu}^{\lambda, \lambda}\) in the following way

\[
g^\lambda_{\mu, \mu} = e^{\frac{2\pi i}{5} m_j} \Phi_{\mu, \mu}^{\lambda, \lambda}. \tag{63}\]

Hence the action on the boundary state is given by

\[
g | \alpha \rangle = \sum B^\alpha_{\lambda, \mu} g | \lambda, \mu \rangle \]
\[
= \sum B^\alpha_{\lambda, \mu} e^{\frac{2\pi i}{5} m_j} | \lambda, \mu \rangle \]
\[
= \sum B^{\alpha g}_{\lambda, \mu} | \lambda, \mu \rangle. \tag{64}\]

Using (32) one can show that \(g\alpha\) is obtained by shifting \(m'_j \rightarrow m'_j + 2\).

A permutation \(\pi \in S_5\) is acting in the following way on \((\lambda, \mu)\),

\[
g_\pi (l_1, \cdots, l_n) = (l_{\pi(1)}, \cdots, l_{\pi(n)}),
\]
\[
g_\pi (s_0; m_1, \cdots, m_n; s_1, \cdots, s_n) = (s_0; m_{\pi(1)}, \cdots, m_{\pi(n)}; s_{\pi(1)}, \cdots, s_{\pi(n)}). \tag{65}\]

Using (32) it is easy to show that \(g_\pi \alpha\) is

\[
g_\pi \alpha = ((l'_{\pi^{-1}(1)}, \cdots, l'_{\pi^{-1}(n)}), (s'_0; m'_{\pi^{-1}(1)}, \cdots, m'_{\pi^{-1}(n)}; s'_{\pi^{-1}(1)}, \cdots, s'_{\pi^{-1}(n)})). \tag{66}\]

The discrete symmetry group \(G\) acts on the set of boundary states labeled by \(\alpha\). Thus identifying these transformation with the discrete symmetries of the quintic hypersurface provides some information about the geometric interpretation of the boundary states \(| \alpha \rangle\). Note that applying the open string partition function in the \(c_{int} = 9\) case it follows that the choice of \(\mu'\) does not influence the spectrum of open strings stretched between two identical branes. This is in agreement with the interpretation of the \(Z_{k+2}\) symmetries acting on \(m'_i\) as geometric transformations which change just the orientation of the brane.

For the \(T^2\) compactification, we can read off the geometrical meaning of \(m'_i\) as discussed in appendix. Since \(m'_i \rightarrow m'_i + 2\) under the \(Z_{k+2}\) symmetries, they rotate the ‘short’(‘long’) branes to other ‘short’(‘long’) branes. This confirms the above geometrical interpretation for the quintic case.
9. D-instanton effects in four dimensions

D-instantons in the four noncompact dimensions are realized as wrapped Euclidean D-branes. From table 3 it is easy to see that for type IIA compactifications D-instantons are given by internal euclidean D2 branes wrapping middle dimensional cycles and for type IIB D-p branes wrapping $p + 1$ dimensional cycles with $p = -1, 1, 3, 5$. The boundary state formalism in the light-cone gauge is well suited for the description of such euclidean wrapped branes [11]. The D-instanton is localized at a space-time point $Y^\mu$ in the four non-compact dimensions and it is convenient to express the $Y$ dependence of the full boundary state via Fourier transformation

$$| B, Y \rangle = \int d^4 p \ e^{i p Y} \ | B, p \rangle.$$  (67)

The inclusion of D-instantons give non-perturbative corrections to certain amplitudes in string theory whose presence are often demanded by dualities.

Wrapped euclidean branes were first discussed by Becker, Becker and Strominger [3]. For Calabi-Yau compactifications of type II strings D-instantons correct the metric of the hypermultiplet moduli space. An important example of such an effect is IIA near conifold where large instanton correction due to euclidean wrapped D2 branes smooth out the classical singularity in the moduli space [31].

The fact that a D-instanton breaks half of the eight space-time supersymmetries which are left unbroken by the compactification implies that amplitudes in the presence of a D-instanton are nonzero only if the fermionic zero modes associated with the broken supersymmetries are soaked up. Very similarly to the case of D-instantons in ten dimensional IIB string theory discussed in [25] this leads to new instanton induced t’Hooft like terms in the effective action. In the case of euclidean wrapped D-branes on CY four of the eight supersymmetries are broken by a D-instanton and hence the simplest instanton term corresponds to a four fermion term [3].

In the following we shall consider type IIA strings for definiteness. Table 3 shows that this corresponds to A boundary conditions for the internal $c = 9$ SCFT. The eight supersymmetry charges can be expressed in the light-cone gauge notation as

$$Q^1_\pm = \frac{1}{\sqrt{2}}(Q \pm i \bar{Q}^\dagger), \quad Q^2_\pm = \frac{1}{\sqrt{2}}(Q^\dagger \pm i \bar{Q}), \quad S^1_\pm = \frac{1}{\sqrt{2}}(S \pm i \bar{S}^\dagger), \quad S^2_\pm = \frac{1}{\sqrt{2}}(S^\dagger \pm i \bar{S}),$$  (68)

where the linearly and nonlinearly realized supercharges are defined in (7) and (8).

The four unbroken supersymmetries which annihilate the boundary state are given by $Q^1_+, Q^2_+, S^1_+, S^2_+$ whereas the broken supersymmetries are given by $Q^1_-, Q^2_-, S^1_-, S^2_-$.\footnote{We have absorbed phases such as $i e^{-i \phi}$ in section 6 into the definition of the right supercharges.}
Note that hermitian conjugation relates the broken and unbroken supersymmetries, i.e. \((Q_1^\dagger) = Q_2\) etc, which is consistent with the fact that this operation maps instantons into anti-instantons. The broken supersymmetries are associated with the fermionic collective coordinates in the instanton background and they are related to the bosonic collective coordinates (given by the translation of the instanton-coordinate) by the unbroken supersymmetries

\[ Q_1 S^2_+ | B \rangle = \oint \frac{1}{2} (\partial X + \bar{\partial} X) | B \rangle = i \frac{\partial}{\partial Y^*} | B \rangle, \]
\[ Q_2 S^1_+ | B \rangle = \oint \frac{1}{2} (\partial X^* + \bar{\partial} X^*) | B \rangle = i \frac{\partial}{\partial Y} | B \rangle, \]
\[ Q_2 S^1_+ Q_1^- | B \rangle = p^+ | B \rangle = i \frac{\partial}{\partial Y^-} | B \rangle, \]
\[ S^2_+ S^1_+ | B \rangle = p^- | B \rangle = i \frac{\partial}{\partial Y^+} | B \rangle, \]

and similarly for other combination of broken and unbroken supercharges. Note that the derivative \(i \partial/\partial Y^\mu\) with respect to the instanton coordinate \(Y^\mu\) induces an infinitesimal translation and hence corresponds to the bosonic zero modes. The broken supersymmetries are combined with wave functions; \(\eta^1 S^1_-, \eta^2 S^2_-, \epsilon^1 Q_1^-, \epsilon^2 Q_2^-\). The integration over the fermionic collective coordinates is then given by \(\int d^2 \varepsilon d^2 \eta\). Instanton induced interactions are given to lowest order in the string coupling constant by disk diagrams with one closed string vertex and several broken supersymmetry generators inserted. The simplest such term would be a four fermion term with four disk (This is the analog of the sixteen fermion term in ten dimensional IIB superstrings \[25\]). The inclusion of disconnected diagrams is necessary for consistency of string perturbation theory in an D-instanton background \[32\],[33]. Such diagrams are lowest order processes and the experience from ten dimensional D-instantons and special examples where exact instanton contributions are available in four dimensions \[34\] indicate that there are contributions from higher charged D-instantons and an infinite number of higher order perturbative fluctuations around the D-instanton. Such perturbative fluctuations around the D-instanton appear as higher genus worldsheets with D-instanton boundary conditions in this framework.

An \(N = 2\) hypermultiplet contains on shell four fermionic and four bosonic states. In the light-cone gauge one can decompose a Dirac spinor \(\Psi\) into two two-component spinors \(\psi^1\) and \(\dot{\psi}\) which satisfy \(\Gamma^+ \psi = 0\) and \(\Gamma^- \dot{\psi} = 0\).

The fermionic states in the hypermultiplets can be created by acting with the supersymmetry charges on the scalar \(| \lambda, \mu; \lambda, \mu \rangle\). Here \(\lambda, \mu\) denote a scalar state in the NS-NS sector with \(s_0 = \bar{s}_0 = 0\) and hence \(q = \bar{q} = \pm 1\). It is understood that this expression includes the momentum dependence. There are \(2h_{2,1} + 2\) such states in a Gepner model corresponding to a Calabi-Yau compactification with Betti number \(h_{2,1}\) and we shall use the label \(I\) to distinguish them;

\[ | \psi^{1^I}_1 \rangle = \frac{1}{k^+} \psi^{1^I}_1 Q^1_- | \lambda, \mu; \lambda, \mu \rangle, \quad | \dot{\psi}^{2^I}_1 \rangle = \frac{1}{k^-} \psi^{2^I}_1 Q^2_- | \lambda, \mu; \lambda, \mu \rangle. \]
The states (70) are two physical polarizations. In order to obtain the four fermionic states in a hypermultiplet (70), $(\lambda, \mu)$ has to be combined with a charge conjugate field $(\lambda, -\mu)$. Since the label $I = 1, \cdots, 2h_{2,1} + 2$ this gives $h_{2,1} + 1$ hypermultiplets.

The disk diagram with the insertion of a fermionic state $\psi^{1,2\, I}$ and one fermionic zero mode will be denoted $A^{1,2\, I}_{\psi}$ and can be calculated by acting with a broken supersymmetry on the boundary state and evaluating the overlap with the states (70). Using (69) to evaluate the resulting one point function for the linearly realized supercharges gives

$$\langle \dot{\psi}^{1 I} | \epsilon^1 \eta^1_+ | B \rangle = \epsilon^1 \dot{\psi}^{1 I} \langle \lambda, \mu | B \rangle, \quad (71)$$

$$\langle \dot{\psi}^{2 I} | \epsilon^2 \eta^2_+ | B \rangle = \epsilon^2 \dot{\psi}^{2 I} \langle \lambda, \mu | B \rangle. \quad (72)$$

The nonlinearly realized supercharges give

$$\langle \dot{\psi}^{1 I} | \eta^1 S^1_- | B \rangle = \eta^1 \frac{k}{k^+} \dot{\psi}^{1 I} \langle \lambda, \mu | B \rangle = \eta^1 \dot{\psi}^{1 I} \langle \lambda, \mu | B \rangle, \quad (73)$$

$$\langle \dot{\psi}^{2 I} | \eta^2 S^2_- | B \rangle = \eta^2 \frac{k^*}{k^+} \dot{\psi}^{2 I} \langle \lambda, \mu | B \rangle = \eta^2 \dot{\psi}^{2 I} \langle \lambda, \mu | B \rangle. \quad (74)$$

Where the Dirac equation $\Gamma^\mu p_\mu \Psi = 0$ is used to relate the $\psi$ and $\dot{\psi}$ components of the spinor:

$$\frac{k}{k^+} \dot{\psi}^1 = \psi^1, \quad \frac{k^*}{k^+} \dot{\psi}^2 = \psi^2. \quad (75)$$

The four fermionic zero modes of the D-instanton can then be saturated by four disk amplitudes given in (71) and (72). The result can be expressed in a covariant form

$$\int d^2 \epsilon d^2 \eta A_{\psi^I} A_{\psi^J} A_{\psi^K} A_{\psi^L} = (\bar{\psi}^I \dot{\psi}^J) (\bar{\psi}^K \dot{\psi}^L) R_{IJKKL}. \quad (76)$$

Using the fact that the overlap $\langle \lambda, \mu | B \rangle = B^\alpha_{\lambda, \mu}$ the tensor $R_{IJKKL}$ is given in terms of the matrix $B^\alpha_{\lambda, \mu}$ by

$$R_{IJKKL} = B^\alpha_{\lambda^I, \mu^I} B^{\alpha}_{\lambda^J, \mu^J} B^\alpha_{\lambda^K, \mu^K} B^\alpha_{\lambda^L, \mu^L}. \quad (77)$$

Here $\lambda^I, \mu^I$ labels the states in the hypermultiplet coming from massless states in the NS-NS sector of the Gepner model and we have omitted an inessential normalization factor. The four fermion term (76) agrees with the result in (3) which was obtained with different methods.

Supersymmetry relates this four fermion term to other instanton induced terms in the effective action which can also be calculated directly using the boundary states and the broken supersymmetry charges. A correction of the hypermultiplet metric is produced by two disks with a vertex operator for a scalar in the hypermultiplet and two broken susy generators on each disk.
For a massless scalar in the hypermultiplet we insert a vertex operator \( V_{\Phi^i} = \Phi_{\mu, \bar{\mu}}^{\lambda, \bar{\lambda}} e^{ikX} \) at the center of the disk. The scalars in the hypermultiplet \( \Phi^i \) are labeled by \( i = 1, \cdots, 4(h_2,1 + 1) \) denoting the scalars of the NS-NS sector with \( q = \bar{q} = \pm 1 \) and the scalars in the RR sector related by spectral flow. A disk diagram with such a state inserted together with two broken supercharges can be written in the cylinder frame as

\[
A_{\Phi^i} = \langle \lambda, \mu; \lambda, \mu | Q_- S_2^2 | B \rangle = \frac{1}{2} \langle \lambda, \mu; \lambda, \mu | (Q - i\bar{Q}^\dagger)(S^\dagger - i\bar{S}) | B \rangle \\
= \langle \lambda, \mu; \lambda, \mu | (QS^\dagger - S^\dagger Q) | B \rangle \\
= k^* \langle \lambda, \mu; \lambda, \mu | B \rangle \\
= k^* B_{\lambda,\mu}^\alpha. 
\]

(76)

Where we used the boundary conditions on the supersymmetry charges imposed by \(| B \rangle\) to turn all the supersymmetry charges into left-moving ones. Here and in the following we dropped the wavefunction such as \( \Phi^i \) associated with the scalars in the hypermultiplet for notational ease. On the second disk another hypermultiplet vertex operator \( V_{\Phi^j} = \Phi_{\bar{\mu}, \bar{\mu}}^{\bar{\lambda}, \lambda} e^{ikX} \) together with the two remaining broken supercharges have to be inserted;

\[
A_{\Phi^j} = \langle \bar{\lambda}, \bar{\mu}; \bar{\lambda}, \bar{\mu} | Q_2^2 S_1^- | B \rangle = \langle \bar{\lambda}, \bar{\mu}; \bar{\lambda}, \bar{\mu} | (Q^\dagger S - SQ^\dagger) | B \rangle \\
= k \langle \bar{\lambda}, \bar{\mu}; \bar{\lambda}, \bar{\mu} | B \rangle \\
= k B_{\bar{\lambda},\bar{\mu}}^\alpha. 
\]

(77)

Putting together the two disk amplitudes (76) and (77) and integrating over the fermionic collective coordinates gives

\[
A = \int d^2 \epsilon d^2 \eta A_{\Phi^i} A_{\Phi^j} = \frac{\partial}{\partial X} \Phi^i \frac{\partial}{\partial X^*} \Phi^j B_{\lambda,\mu}^\alpha B_{\bar{\lambda},\bar{\mu}}^\alpha. 
\]

(78)

Different combinations of the fermionic zero modes on the two disks produce similar terms, i.e., when \( Q_1^- \) and \( Q_2^- \) are inserted on one disk and \( S_1^+ \) and \( S_2^+ \) on the other we get

\[
A_{\Phi^i} = \langle \lambda, \mu; \lambda, \mu | Q_-^1 Q_-^2 | B \rangle \\
= \langle \lambda, \mu; \lambda, \mu | (QQ^\dagger - Q^\dagger Q) | B \rangle \\
= k^+ B_{\lambda,\mu}^\alpha. 
\]

(79)
and
\[
A_{\Phi} = \langle \bar{\lambda}, \bar{\mu}; \bar{\lambda}, \bar{\mu} \mid | S_1 S_2 | B \rangle \\
= \langle \bar{\lambda}, \bar{\mu}; \bar{\lambda}, \bar{\mu} \mid (SS^\dagger - S^\dagger S) | B \rangle \\
= \frac{k k^*}{k^*} \langle \bar{\lambda}, \bar{\mu}; \bar{\lambda}, \bar{\mu} \mid B \rangle \\
= k B^\alpha_{\bar{\lambda}, \bar{\mu}}.  
\] (80)

Integration over the fermionic zero modes like in (78) then gives
\[
A = \int d^2 \epsilon d^2 \eta A_{\Phi} A_{\Phi} = \frac{\partial}{\partial X^+} \Phi^i \frac{\partial}{\partial X^-} \Phi^j B^\alpha_{\lambda, \mu} B^\alpha_{\bar{\lambda}, \bar{\mu}}.  
\] (81)

Putting together all these terms the instanton induced correction of the metric can then be expressed in a Lorentz covariant form
\[
A = \partial \mu \Phi^i \partial \mu \Phi^j g_{ij} e^{-S_{\text{inst}}}.  
\] (82)

It is easy to see that the correction to the metric due to the lowest order instanton process gives
\[
g_{ij} = B^\alpha_{\lambda, \mu} B^\alpha_{\bar{\lambda}, \bar{\mu}} \\
= e^{i \pi \frac{s_0^2 + \bar{s}_0^2}{2}} e^{-i \pi \frac{s_j^0 + \bar{s}_j^0}{2}} \prod_{j=1}^n \sin \left( \frac{\pi (l_j^0 + 1)(l_j^0 + 1)}{k_j + 2} \right) \sin^{1/2} \left( \frac{\pi (l_j^0 + 1)}{k_j + 2} \right) \\
\times \prod_{j=1}^n \frac{\sin \left( \frac{\pi (l_j^0 + 1)(l_j^0 + 1)}{k_j + 2} \right)}{\sin^{1/2} \left( \frac{\pi (l_j^0 + 1)}{k_j + 2} \right)} e^{i \pi m_j^0 (l_j^0 + 1) k_j + 2} e^{-i \pi \gamma_j^0 (l_j^0 + 1)^2} e^{i \pi \gamma_j^0 (l_j^0 + 1)} e^{i \pi \gamma_j^0 (l_j^0 + 1)}.  
\] (83)

Such an instanton induced correction has to be weighted by a factor $e^{-S_{\text{inst}}}$ where $S_{\text{inst}}$ is the action of the instanton. In the case of the wrapped branes the action is simply given by $S_{\text{inst}} = e^{-\phi} \text{Vol}(C)$ where $C$ is the cycle on which the (euclidean) D-brane is wrapped. The action can be read off from the one point function of the dilaton on the disk. The dilaton corresponds to the scalar state in the RR sector which has $q = \bar{q} = 0$ and is given by a linear combination of the states with $s_0 = \pm 1, \bar{s}_0 = \mp 1$ and $l_i = m_i = s_i = 0, i = 1, \cdots, n$. The overlap with the boundary state is then
\[
\langle s_0, 0, 0 \mid \alpha \rangle = B^\alpha_{0, 0} = \frac{i}{k^0_{\alpha}} \prod_{j}^n \sin \left( \frac{l_j^0 + 1}{k_j + 2} \right).  
\] (84)

The instanton induced terms in the effective action are of importance since sometimes the exact corrections can be obtained by other means like duality or symmetry constraints. The instanton calculations given above can then be used to determine D-instanton partition functions which in turn can be related to complicated matrix integrals.
10. Conclusions

In this paper we have analyzed the boundary states constructed by Recknagel and Schomerus [10]. Using the light-cone gauge description of boundary states, there are two $N = 2$ SCFT, one for the transverse and one for the internal degrees freedom. The consistent boundary conditions give D-instantons and D-0 branes (black holes) corresponding to euclidean wrapped D-branes for type IIA and IIB. The boundary states are characterized by a set of discrete labels $\alpha = (\mu', \lambda')$. The open string partition function of strings ending on identical branes depend only on $\lambda'$, whereas the relative phase of the left- and right-moving part of the conserved supersymmetry charges depends only on $\mu'$. This suggests existence of the selection rule for the allowed parameter $\alpha$ since both arguments give geometrical interpretations and they should be compatible. The understanding of this rule may give useful insight into the classification of the supersymmetric cycles. The discrete symmetries of the Gepner model map boundary states labeled by different $\mu'$ into each other. The construction of the D-brane boundary states used Ishibashi states satisfying A or B boundary conditions for each minimal model in the tensor product. It would be very interesting to generalize this construction and to find new (‘non-rational’) supersymmetric boundary states in Gepner models.

The Gepner model (and the associated boundary states) provide one point in the moduli space of compactifications (and branes wrapping cycles). It is important to note that mirror symmetry [27], [28] was established in this context and then extrapolated through the moduli space of compactifications. The dependence of D-brane boundary states on the Kähler and complex structure deformations was analyzed in [4] using the topologically twisted sigma model. It might be useful to consider the topologically twisted minimal models [36] to analyze the behavior of Gepner model boundary states under marginal deformations.

The boundary states and the space-time supersymmetry charges were used to calculate instanton induced corrections to the hypermultiplets in the four dimensional $N = 2$ effective action. This calculation uses the overlap of a boundary state and a closed string state, corresponding to the insertion of a closed string vertex operator on the disk.

We observed that the mirror automorphism (‘c-map’) of the transverse conformal field theory is given by a time-like T-duality which maps D-instantons into D0 branes corresponding to RR-charged black holes in $\mathcal{N} = 2$ supergravity. In this context the one point functions for a boundary state can be related to the properties of the corresponding black hole. It would be very interesting to use the exact solution provided by the Gepner model boundary states to calculate higher point functions on the disk which correspond to scattering off the black hole, absorption and Hawking radiation. We hope to report progress in this direction elsewhere.
Acknowledgments

We would like to thank A. Recknagel and V. Schomerus for very useful correspondence. We are also grateful to N. Ishibashi, A. Kato and H. Ooguri for useful correspondence and/or conversation. M.G. gratefully acknowledges the hospitality of the Theory Division at CERN while this work was completed. The work of M.G. was supported in part by DOE grant DE-FG02-91ER40671 and NSF grant PHY-9157482, whereas the work of Y.S. was supported in part by Japan Society for the Promotion of Science.

Appendix A. Calculation of the partition functions

In appendix A, we calculate the partition functions from the boundary states for the \( c = 3 \) Gepner models. For this purpose, we need the characters of the level \( k \) \( N = 2 \) minimal model in the NS sector \((s = 0, 2)\),

\[
\chi_{m,s}^l(q) = \sum_{j=0}^{k-1} c_{m+4j-s}^j(q) \theta_{2m+(4j-s)(k+2),2k(k+2)}(q),
\]

\[
\theta_{M,K}(q) = \sum_{n \in \mathbb{Z}} q^{K(n+\frac{M}{K})^2},
\]

where \( c_m^l \) are Hecke modular forms depending on \( k \). We will use the identities \( \theta_{M+2K,K} = \theta_{M,K}, c_m^l = c_{-m}^l = c_{k+m}^{k-l} = c_{m+2k}^l \) and \( c_m^l = 0 \) unless \( l + m \in 2\mathbb{Z} \).

A.1. \((k = 2)^2\) case

For \( k = 2 \), \( c_m^l \) reduce to the fermion characters

\[
\chi_0 = \frac{1}{2} \left( \sqrt{\frac{\theta_3}{\eta}} + \sqrt{\frac{\theta_4}{\eta}} \right), \quad \chi_{1/2} = \frac{1}{2} \left( \sqrt{\frac{\theta_3}{\eta}} - \sqrt{\frac{\theta_4}{\eta}} \right), \quad \chi_{1/16} = \sqrt{\frac{\theta_2}{2\eta}}.
\]

Precise relation is given by

\[
\eta(q)c_0^0(q) = \chi_0(q), \quad \eta(q)c_0^2(q) = \chi_{1/2}(q), \quad 2\eta(q)c_1^1(q) = \chi_{1/16}(q).
\]

Also, the following combinations of \( \theta_{M,K} \) are used;

\[
S_1(q) \equiv \theta_{0,16}(q) + \theta_{16,16}(q) = \frac{1}{2} \left( \theta_3(q^2) + \theta_4(q^2) \right),
\]

\[
S_2(q) \equiv \theta_{-8,16}(q) + \theta_{8,16}(q) = \frac{1}{2} \left( \theta_3(q^2) - \theta_4(q^2) \right),
\]

\[
S_3(q) \equiv \theta_{\pm 4,16}(q) + \theta_{\mp 12,16}(q) = \frac{1}{2} \theta_2(q^2).
\]
In this case, we have three independent non-vanishing partition functions labeled by 
\((l_1', l_2') = (0, 0), (1, 0), (1, 1)\). Using (A.2) and the duplication formulas for \(\theta_i\), we find that

\[
Z_{0,0}^o(\tilde{q}) = 4c_0^2c_0^2S_1S_2 + 2 \left( (c_0^2)^2 + (c_0^2)^2 \right) S_3^2
\]

\[
= \frac{1}{4\eta^3(\tilde{q})} \left( \theta_3(\tilde{q}) - \theta_4(\tilde{q}) \right) \left( \theta_3^2(\tilde{q}) + \theta_4^2(\tilde{q}) \right).
\]

(A.5)

In the \((1, 0)\) case, we have additional contribution

\[
\Delta Z = 4c_0^2c_0^2S_2^2 + 2 \left( (c_0^2)^2 + (c_0^2)^2 \right) S_1S_2.
\]

(A.6)

Therefore the total partition function is

\[
Z_{1,0}^o(\tilde{q}) = Z_{0,0}^o(\tilde{q}) + \Delta Z
\]

\[
= \frac{1}{2\eta^3(\tilde{q})} \left( \theta_3(\tilde{q}) - \theta_4(\tilde{q}) \right) \theta_3^2(\tilde{q}).
\]

(A.7)

We easily confirm that the partition function of the last case \((1, 1)\) is \(2Z_{1,0}^o\).

A.2. other cases

The other \(c = 3\) Gepner models correspond to the \(SU(3)\) torus. The metric and the anti-symmetric tensor in (38) are given by \(G_{11} = G_{22} = 1\) and \(G_{12} = B_{12} = 1/2\) in the coordinate system whose basis is along adjacent two roots. By a similar argument in section 5.1, the zero-mode part of the torus partition functions is given by

\[
Z^0_{SU(3),0}(\tilde{q}) = \sum_{m,n \in \mathbb{Z}} \tilde{q}^{2m^2 + n^2 + mn},
\]

(A.8)

where \(a = 1\) for the D-branes wrapping on the cycles \(\theta = n\pi/3\) of the \(SU(3)\) torus and \(a = 1/3\) for \(\theta = (2n + 1)\pi/6\).

In the \((k = 1)^3\) case, we have only one independent partition function labeled by 
\((l_1', l_2', l_3') = (0, 0, 0)\) on the Gepner model side. In addition, \(c_m^k\) reduce to \(\eta^{-1}\). Then using the identity

\[
\sum_{l,m,n \in \mathbb{Z}} q^{\frac{1}{2}(2l-1)^2} \left[ 3q^6 \{(6m+2l+2)^2+(6m+2l+2)^2\} + q^6 \{(6m+2l-1)^2+(6m+2l-1)^2\} \right]
\]

\[
= \sum_{l,m,n \in \mathbb{Z}} q^{2(l-1/2)^2 + m^2 + n^2 + mn},
\]

(A.9)

we obtain

\[
Z_{0,0,0}^o(\tilde{q}) = \frac{1}{\eta^3(\tilde{q})} \left( \theta_3(\tilde{q}) - \theta_4(\tilde{q}) \right) Z_{SU(3),0}^0(\tilde{q}).
\]

(A.10)

In the \((k = 1, k = 4)\) case, on the Gepner model side we have three independent partition functions labeled by 
\((l_1', l_2', l_3') = (0, 0), (0, 1), (0, 2)\). Since \(c_m^k\) for \(k = 4\) sector are complicated, it seems difficult to analytically deal with the partition functions. However, using computers we find that \(Z_{0,0}^o = Z_{0,0,0}^o/2\),

\[
Z_{0,1}^o(\tilde{q}) = \frac{1}{2\eta^3(\tilde{q})} \left( \theta_3(\tilde{q}) - \theta_4(\tilde{q}) \right) Z_{SU(3),0}^0(\tilde{q}),
\]

(A.11)

and \(Z_{0,2}^o = Z_{0,0}^o + Z_{0,1}^o\).
Appendix B. Geometrical interpretation of the boundary conditions

The \((k = 2)^2\) and \((k = 1)^3\) Gepner model correspond to \(SU(2)^2\) and \(SU(3)\) torus respectively. In these cases, one can explicitly construct the sigma-model variables for \(T^2\) using free field realization. This allows us to find the geometrical meaning of the boundary conditions.

The tensor product of two \(k = 2\) \(N = 2\) minimal models are realized by two real bosons and real fermions as

\[
T(z) = -\frac{1}{2} \sum_{j=1}^{2} (\partial \varphi_j \partial \varphi_j + \psi_j \partial \psi_j), \\
J(z) = \frac{i}{\sqrt{2}} \partial (\varphi_1 + \varphi_2), \\
G^\pm(z) = \frac{1}{\sqrt{2}} \sum_j \psi_j e^{\pm i \sqrt{2} \varphi_j}.
\]

Using \(\varphi_i\) and \(\psi_i\), the chiral complex boson and fermion on the torus are given by \(\Psi(z) = e^{i H/\sqrt{2}}\), \(\partial X(z) = \sqrt{2} \left(\psi_1 e^{-i(\varphi_1 - \varphi_2)/\sqrt{2}} + \psi_2 e^{i(\varphi_1 - \varphi_2)/\sqrt{2}}\right)\),

where \(H = \sum_j \varphi_j\).

In the \((k = 1)^3\) case, the tensor product of the minimal models is realized by three real bosons as

\[
T(z) = -\frac{1}{2} \sum_{j=1}^{3} \partial \varphi_j \partial \varphi_j, \\
J(z) = \frac{i}{\sqrt{3}} \sum_j \partial \varphi_j, \\
G^\pm(z) = \sqrt{\frac{2}{3}} \sum_j e^{\pm i \sqrt{3} \varphi_j}.
\]

The complex boson and fermion on the torus are (see, e.g. \(\Psi(z) = e^{i H/\sqrt{3}}, \partial X(z) = \frac{1}{\sqrt{3}} \sum_j e^{i (H - 3 \varphi_j)/\sqrt{3}}\).

B.1. boundary conditions in the open string channel

We first discuss the geometrical meaning of the boundary conditions from the open string channel. A and B boundary conditions are given by

(A) \(J_n + \bar{J}_{-n} = 0\), \(G^\pm_r - \eta \bar{G}^\mp_r = 0\), \(L_n = \bar{L}_{-n}\),
(B) \(J_n - \bar{J}_{-n} = 0\), \(G^\pm_r - \eta \bar{G}^\mp_r = 0\), \(L_n = \bar{L}_{-n}\),

\(\text{(B.5)}\)

\footnote{Precisely, to get Gepner models we need to twist the tensor product of the minimal models (see section 3). The sigma-model variables discussed here and in the following are in the untwisted sectors.}
with \( \eta = \pm 1 \).

In the \((k = 2)^2\) case, these boundary conditions are translated into

\[
\varphi_j = \epsilon \bar{\varphi}_j + \pi n_j / \sqrt{2}, \quad \psi_j = (-1)^{m_j} \bar{\psi}_j,
\]

(B.6)

where \( \epsilon = -1 \) for A and +1 for B; \( m_j + n_j = 2Z \) for \( \eta = +1 \) and \( m_j + n_j = 2Z + 1 \) for \( \eta = -1 \). In terms of the sigma-model variables, these imply

\[
\Psi = e^{\pi i(n_1 + n_2)/2} \bar{\Psi}^*, \quad \partial X = (-1)^{n_1 + m_1} e^{\pi i(n_1 + n_2)/2} \partial \bar{X}^*,
\]

(B.7)

for A and similar expression with \( \bar{\Psi}, \bar{X} \) instead of \( \bar{\Psi}^*, \bar{X}^* \) for B. Thus A boundary conditions represent the D-branes wrapping around the cycles \( \theta = \arg X = \pi(n_1 + n_2)/4 \). These cycles are the shortest and the second shortest cycles of the SU(2)^2 torus. On the other hand, B boundary conditions represent the N-N or the D-D boundary conditions in both directions of the torus.

The \((k = 1)^3\) case has been discussed in [4]. Similarly to the above, we find that A and B boundary conditions give \( \varphi_j = \epsilon \bar{\varphi}_j + c_j \), where \( c_j = 2\pi n_j / \sqrt{3} \), \( n_j \in \mathbb{Z} \) for \( \eta = +1 \) and \( 2\pi (n_j + 1/2) / \sqrt{3} \) for \( \eta = -1 \). Geometrically these imply

\[
\Psi = \eta e^{2\pi i n/3} \bar{\Psi}^*, \quad \partial X = e^{2\pi i n/3} \partial \bar{X}^*,
\]

(B.8)

for A with \( n = \sum n_j \) and similar expressions with \( \bar{\Psi}, \bar{X} \) instead of \( \bar{\Psi}^*, \bar{X}^* \) for B. Thus A boundary conditions represent the D-branes wrapping around the cycles \( \theta = \pi n/3 \). These are the shortest cycles of the SU(3) torus. B boundary conditions correspond to the N-N or D-D boundary conditions.

### B.2. Boundary conditions in the closed string channel

Next, we discuss the boundary conditions in the closed string channel, i.e., on the boundary states. They are given by (22) and (23).

To analyze them, we first recall that generically the \( N = 2 \) minimal models are realized by parafermions and a free boson [39]:

\[
T(z) = T_{PF}(z) - \frac{1}{2} \partial \varphi \partial \bar{\varphi}, \quad J(z) = i \gamma^{-1} \partial \varphi, \quad G^+(z) = \sqrt{2} \gamma^{-1} \psi_1^{PF} e^{i \gamma \varphi}, \quad G^-(z) = \sqrt{2} \gamma^{-1} (\psi_1^{PF})^\dagger e^{-i \gamma \varphi},
\]

(B.9)

where \( T_{PF} \) is the energy-momentum tensor for the parafermions; \( \gamma = \sqrt{(k + 2)/k} = \sqrt{3/c} \); \( k \) is the level of the minimal model. In addition, the highest weight states corresponding to \( |l, m, s\rangle \) are written as

\[
\Phi^l_{m,s} = \phi^l_{m-s} e^{i \gamma_{m,s} \varphi}, \quad \gamma^k_{m,s} = \frac{m - s(k + 2)/2}{\sqrt{k(k + 2)}}.
\]

(B.10)
\( \phi^l_q \) are the primary fields in the parafermion theory and the above expression of \( \gamma_{m,s}^k \) is valid in the standard range. From these realizations, we find that the states in the module of \( |l, m, s \rangle \), i.e. \( |l, m, s; N \rangle \), take the form

\[
(PF \text{ modes}) \times (\text{non-zero modes of } \varphi) | \gamma_{m,s}^k + n \gamma \rangle, \quad n \in \mathbb{Z},
\] (B.11)

where \( |p \rangle \) is the momentum eigenstate of \( \varphi \).

For \( k = 1 \), the ‘parafermions’ are trivial, namely, \( \phi^l_{m-s} = 1 \), whereas for \( k = 2 \) they are usual fermions. Notice that the sigma-model fermions both in (B.2) and (B.4) take the form

\[
\Psi(z) = e^{i \gamma^{-1} H(z)}, \quad i \gamma^{-1} \partial H(z) = J(z).
\] (B.12)

For definiteness, we focus on the \( (k = 2)^2 \) case and \( A \) boundary conditions for the time being. By definition, \( A \) boundary states satisfy

\[
J_n | \alpha \rangle_A = \bar{J}_{-n} | \alpha \rangle_A.
\] (B.13)

Using this and (B.12), we find that

\[
e^{i H/\sqrt{2}(\sigma)} | \alpha \rangle_A = e^{-i \bar{H}/\sqrt{2}(\sigma)} e^{i(x+\bar{x})/\sqrt{2}} | \alpha \rangle_A,
\] (B.14)

where \( x = x_1 + x_2; \ x_j \) are the zero modes of \( \varphi_j = x_j - i \alpha_0^j \ln z + \cdots \) and we have set \( z = e^{i \sigma}, \ \bar{z} = e^{-i \sigma} \). From (B.11), we see that the zero-mode operator \( e^{i(x+\bar{x})/\sqrt{2}} \) acts only on the momentum eigenstates of \( H \) and \( \bar{H} \) at the base. Since

\[
\gamma_{m,s}^{k=2} + \frac{1}{\sqrt{2}} = \gamma_{m-2,s-2}^{k=2},
\] (B.15)

it follows that

\[
e^{i H/\sqrt{2}(\sigma)} | \alpha \rangle_A = e^{-i \bar{H}/\sqrt{2}(\sigma)} \frac{1}{k_0^\alpha} \sum_{\lambda, \mu} B_{\lambda, \mu}^\alpha | \lambda, \mu - 2 \eta_0 \rangle \rangle_A,
\] (B.16)

\[
= -i \eta e^{-i \bar{H}/\sqrt{2}(\sigma)} e^{+2\pi i \hat{Q}_\alpha} | \alpha \rangle_A.
\]

Here \( \eta_0 = (0; 2, 2; 2, 2) \),

\[
\hat{Q}_\alpha \equiv 2 \eta_0 \cdot \mu' = (m_1' + m_2')/4 - (s_1' + s_2')/2,
\] (B.17)
and $\alpha = (\lambda', \mu')$. We have also used $U e^{ipx_j} U^{-1} = (-1)^{p^2/2} e^{-ipx_j}$, $U c U^{-1} = c^*$ for a c-number, $-i\eta = (-1)^{-1/2}(-1)^{s_0}$, and the fact that the zero-mode operator picks up $(-1)^{s_0}$ when it goes through the left states. In terms of modes, this means that

$$\left( \bar{\Psi}_r + i\eta e^{+2\pi iQ_\alpha} \bar{\Psi}^*_{-r} \right) \mid \alpha \rangle_A = 0.$$  \hspace{1cm} (B.18)

For B boundary states, we obtain a similar expression with $\bar{\Psi}_{-r}$ instead of $\bar{\Psi}^*$.

Since $\Psi$ is a space-time vector, the phase appearing in $\bar{\Psi}_r$ has geometrical meaning. The phase coming from $s'_i$ may be understood as sign ambiguity associated to fermions or the $Z_2$ symmetry. The remaining phase is in accord with the phase in the previous section (B.7) and represents the D-branes wrapping around the cycles $\theta = \pi (m'_1 + m'_2)/2$ for A boundary states.

To apply this method to the $(k = 1)^3$ case is straightforward. As a result, we get expressions similar to the above with $\hat{Q}_\alpha = (m'_1 + m'_2 + m'_3)/3 - (s'_1 + s'_2 + s'_3)/2$. \hspace{1cm} (B.19)

This is in agreement with the open string channel argument.

Furthermore, we can discuss the boundary conditions for the supercharges in a generic Gepner model. This is because the supercharges are essentially the zero-modes of the spectral flow operators by half a unit and written as in (7),(8). Again they are expressed by the free bosons associated to the $U(1)$ currents and the analysis similar to the fermion case is possible. Note that the internal part of the spectral flow operator is nothing but $\Psi^{1/2}$ for the $(k = 2)^2$ and $(k = 1)^3$ case. For the supercharges, the zero-mode part shifts $\mu$ to $\mu - \beta_0$. Consequently, we obtain

$$\left( Q - e^{\pi iQ_\alpha} \bar{Q}^\dagger \right) \mid B \rangle = 0,$$  \hspace{1cm} (B.20)

for $A \otimes A$ boundary states in table 3, and a similar expression with $\bar{Q}$ instead of $\bar{Q}^\dagger$ for $B \otimes B$. This is the same as the result in section 6 without using free field realizations. The other cases for $A(B) \otimes B(A)$ and the non-linearly realized supercharges, $S, S^\dagger$, can be discussed similarly. The phase for the supercharges is half of the one in (B.18) up to the contribution from $s'_0$. This is consistent with the fact that $\Psi$ is a space-time vector while $Q$ is a space-time spinor.

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11 These follow from the fact that $\bar{\Psi} = \exp(i\hat{H}/\sqrt{2})$ and left states with $s_0 = \pm 1$ are fermionic and that $U$ is essentially the CPT operator. The first equation is in accord with $U \Phi(\sigma) U^{-1} = (-1)^h \Phi^*(-\sigma)$ with $h$ the dimension of $\Phi$. Also, we can translate the boundary conditions for $J$ and $G^\pm$ into those for $\varphi$ and $\psi^{EF}_i$. We can confirm that the action on $e^{ipx_j}$ is consistent with them. The action of $U$ is originally defined for $\bar{G}^\pm$ and $\bar{J}$, and hence there may be some ambiguity about the action on other operators such as $e^{ip\varphi_j}$ and $e^{ipx_j}$. However, we are interested only in the relative phases appearing in the boundary conditions for various boundary states. Therefore, such an ambiguity, even if it exists, can be absorbed into the definition of the complex field such as $\bar{\Psi}$ as long as it is just an overall phase.
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