Nuclear structure and double beta decay

P. Vogel

Physics Department, California Institute of Technology
Pasadena, CA 91125, USA

ABSTRACT. The nuclear structure aspects of double beta decay are reviewed. First, the standard approach, based on the QRPA, is briefly described, and the successes and open problems are discussed. Next, the recent attempts to simplify the calculation are described. It is shown that the so-called “Operator Expansion Method” is not reliable since it underestimates the size of the matrix elements in those cases where an exact calculation is possible. The methods based on the spin-isospin SU(4) symmetry are also questioned, since SU(4) symmetry in real nuclei is badly broken. Finally, the situation in the A = 100 nuclear system is discussed. There, two double beta decay rates and three ordinary beta decay rates are experimentally known. QRPA can describe the situation qualitatively. However, the theory cannot explain quantitatively some of the experimental features, such as the transition connecting the ground states of $^{100}$Tc and $^{100}$Mo.

1. INTRODUCTION

In double beta decay two neutrons are transformed into two protons simultaneously, and two electrons and, possibly, two antineutrinos are emitted. The $2\nu$ decay, with four leptons in the final state, is a process allowed by the standard theory of beta decay. On the other hand, the neutrinoless decay ($0\nu$) violates the law of lepton number conservation and is possible only if neutrinos are massive Majorana particles. Its observation would signal an important departure from the Standard Model of Electroweak Interaction. A massive experimental effort is devoted to searches for both modes of double beta decay. In order to plan and correctly interpret the results of such experiments, one has to know the various nuclear matrix elements associated with the decay process. Since these matrix elements represent the only difficulty in the case of the $2\nu$ decay, we will concentrate in the following on this mode. The general theory and the experimental status of the double beta decay has been summarized many times, and we refer to the monograph [1] for references and details.
Double beta decay is the slowest nuclear transmutation observed. The order of magnitude estimate of the decay rate is easy to obtain:

$$\lambda_{2\nu} \approx \frac{2\pi}{\hbar} G_F^4 |M_{\text{clois}}|^2 T_{11} \frac{(4\pi)^4}{(2\pi)^{12}}$$

(1)

Here, the first factor comes from the Fermi Golden rule, the next one from the second order weak interaction, then follows the nuclear perturbation theory expression involving the corresponding energy denominator, and the remaining factors arise from the phase space integral over the four outgoing leptons. The quantity $T_0$ is the kinetic energy available to the leptons; this is the only quantity of the dimension of energy in the problem. Substituting in the above formula the typical values $T_0 = 2$ MeV and $\Delta E = 10$ MeV, one arrives at the half-life

$$T_{1/2}^{2\nu} \approx 3 \times 10^{18} |M_{\text{clois}}|^{-2} \text{years.}$$

(2)

The experimental half-lives are $10^{19}$–$10^{21}$ years and hence the dimensionless closure matrix elements are smaller than unity, and much smaller than the sum-rule unit [2]

$$|M_{\text{clois}}|^2 \approx 6(N - Z)(N - Z + 1).$$

(3)

The small value of the nuclear matrix element, when expressed in its “natural” units, suggests that the theoretical evaluation will be difficult since it will depend sensitively on small and poorly determined components of the nuclear wave function.

The above approximate expressions used closure approximation in which the nuclear energy denominators are replaced by a single typical value $\Delta E$. This is, in fact, a poor approximation. In a more careful evaluation, the nuclear matrix element involves a summation over all virtual intermediate $1^+$ states,

$$M_{\text{GT}}^{2\nu} = \sum_{m} \frac{\langle \sigma \tau_+ |m\rangle \langle m\sigma_+ |i\rangle}{E_m - (M_i + M_f)/2},$$

(4)

where $|f\rangle$ ($|i\rangle$) are the $0^+$ ground states of the final (initial) even-even nuclei, and $|m\rangle$ are the $1^+$ states in the intermediate odd-odd nucleus. It is worthwhile to point out that the last factor in the numerator of Eq. (4) represents the amplitude of the $\beta^-$ decay (or of the forward angle $(p,n)$ reaction) of the initial nucleus, while the first factor represents the amplitude of the $\beta^+$ decay (or of the $(n,p)$ reaction) of the final nucleus. The description of the $2\nu$ double beta decay, therefore, is equivalent to the description of the full beta strength functions (even including the signs of the corresponding amplitudes) of both the initial and final nuclei.

2. Standard Approach – QRPA

Ideally, the method of choice for evaluation of the matrix element $M_{\text{GT}}^{2\nu}$ is the nuclear shell model. However, among nuclei which are double beta decay candidates only $^{48}\text{Ca}$
TABLE I. Experimental and calculated matrix elements $M_{GT}^{2\nu}$ (me$^{-1}$) and halflives in years. The calculations are from [6] for $\alpha' = -390$ MeV-fm$^3$. (This value of the particle-particle coupling constant describes reasonably well the $\beta^+$ decay of semimagic nuclei.)

| Initial nucleus | $M_{GT}^{2\nu}$ | $T_{1/2}$ | $M_{GT}^{2\nu}$ | $T_{1/2}$ |
|-----------------|-----------------|----------|-----------------|----------|
| $^{76}\text{Ge}$ | 0.13 | $0.9 \times 10^{21}$ | 0.12 | $1.3 \times 10^{21}$ |
| $^{82}\text{Se}$ | 0.072 | $1.1 \times 10^{20}$ | 0.069 | $1.2 \times 10^{20}$ |
| $^{100}\text{Mo}$ | 0.15 | $1.2 \times 10^{19}$ | 0.21 | $6.0 \times 10^{18}$ |
| $^{128}\text{Te}$ | 0.022 | $6.0 \times 10^{24}$ | 0.073 | $5.5 \times 10^{23}$ |
| $^{130}\text{Te}$ | 0.013 | $2.6 \times 10^{21}$ | 0.049 | $2.2 \times 10^{20}$ |

is amenable to the shell model treatment without severe truncations. The calculated $2\nu$ decay rate in $^{48}\text{Ca}$ is very close to the experimental half-life limit of $3.6 \times 10^{19}$ years [3]. It would be very important to repeat that experiment, which is quite old now, and determine the actual decay rate.

In heavier nuclei the shell model treatment is impossible; the dimensionalities are simply too large to allow it. Thus, it is necessary to judiciously select a relatively small number of the most relevant configurations. The method of choice in recent years has been the Quasiparticle Random Phase Approximation (QRPA). In QRPA, one takes into account the features of nuclear structure that are most relevant for the evaluation of the double beta decay rates. These include the smearing of the Fermi surface described by the BCS method; the neutron-neutron and proton-proton particle-particle interactions are included in this way. The other important ingredient is the spin-dependent neutron-proton force. The particle-hole part of that force is responsible for the concentration of Gamow-Teller strength in the giant GT resonance. The particle-particle part of that force is responsible, among other things, for the reduction of the total $\beta^+$ strength, as seen e.g. in the $(sd)$ shell [4]. The same particle-particle part of the neutron-proton force also suppresses the $2\nu$ double beta decay matrix elements, and brings them to a closer agreement with the experimental values.

In the past several years QRPA has been used extensively in the evaluation of double beta rates for the $2\nu$ decay mode [5–8], as well as for the $0\nu$ mode. The calculations have been able to explain the smallness of the $2\nu$ matrix elements. However, it turns out that the actual value of the matrix element depends sensitively on the strength of the particle-particle force in the $S = 1, T = 0$ channel, $g_{pp}$. For realistic forces of finite range the matrix elements vanish for $g_{pp} \approx 1$, i.e., near the expected value of this coupling constant. This feature makes the actual value of these matrix elements rather uncertain.

Nevertheless, QRPA is at least semiquantitatively successful as shown in Table I. There we show results of the calculations performed at Caltech using the zero-range $\delta$ force.

It is encouraging that the halflives of $^{76}\text{Ge}$ and $^{100}\text{Mo}$ in Table I are, actually, predictions. On the other hand, one can see that the calculations overestimate the strongly suppressed matrix elements in the two tellurium isotopes. However, the ratio of the two matrix elements is reproduced correctly.
Despite this obvious success, QRPA has various problems. One, common to many areas of nuclear structure, is our inability to derive the effective nuclear force from first principles. The QRPA results depend, clearly, on the choice of the force. For example, in Ref. [9] it is shown that the double beta decay matrix elements change drastically if one replaces the bare $G$ matrix by the renormalized $G + G_{3p1h}$ matrix. In order to reduce this dependence one has to make sure that the chosen interaction correctly reproduces the essential features of nuclear spectra, such as the pairing gaps, energies of the giant GT resonance, and the rates of known ordinary beta decays. When this is done, the extreme dependence on the choice of interaction is drastically reduced.

QRPA is based on a harmonic approximation; it is the theory of small amplitude collective motion. Yet, in its application to the charge-changing modes like beta decay, one finds that for realistic forces the collective amplitudes are large, near the breakdown point where energies become imaginary and a phase transition occurs. It is not clear how large are the effects caused by the approach to instability.

It is important to stress that the above mentioned difficulties affect the matrix elements of the $2\nu$ decay much more than the matrix elements of the $0\nu$ decay. Hence, these latter, and potentially more important matrix elements, are less uncertain.

3. ATTEMPTS OF SIMPLIFICATION

There have been several recent attempts to evaluate the $2\nu$ double beta decay matrix element in a different, and seemingly simpler way. Below we comment on two of them.

The “Operator Expansion Method” (OEM), proposed in Ref. [10], and used in a number of more recent papers, tries to avoid the necessity of evaluating the sum over the intermediate states in Eq. (4). By first expanding the energy denominator $1/(E_m - (M_i + M_f)/2)$ in an infinite power series, simplifying each term, and resumming the series, one arrives at the expression

$$M_{GT}^{OEM} = \left\langle 1 \left| \sum_{k,j} M_{k,j}(r) \tau_k^+ \tau_j^+ \right| \right.$$  \hspace{1cm} (5)

where $M(r)$ is a position and spin dependent operator, whose form depends on the nuclear hamiltonian, and the sum is over all nucleon pairs. Thus, OEM avoids summation over the intermediate states, but requires evaluation of a complicated matrix element, involving radial functions with poles. (The divergence is avoided by taking the principal values of the corresponding integrals.)

How can one decide whether the assumptions used in deriving the OEM expression, Eq. (5), are valid? There are various ways to approach this question. One of them will be described elsewhere in these proceedings (see the contribution of O. Civitarese). Our approach [11] is the comparison of the OEM results and the exact results in cases that are realistic, but allow an exact evaluation.

The power series expansion mentioned above involves multiple commutators of the hamiltonian and the Gamow-Teller operator. In the OEM the terms in these commutators
TABLE II. Overlaps (in percent) between the $J^* = 0^+$ ground states of the Ca isotopes (characterized by the neutron number) and the eigenstates of the SU(4) Casimir operator corresponding to its lowest eigenvalue and to the sum of the lowest and next eigenvalues.

| N:  | 22 | 24 | 26 | 28 | 30 | 32 | 34 | 36 | 38 |
|-----|----|----|----|----|----|----|----|----|----|
| First eigenvalue | 75 | 50 | 32 | 20 | 19 | 17 | 26 | 42 | 64 |
| Sum 1 + 2        | 100| 98 | 90 | 78 | 75 | 71 | 83 | 93 | 100|

involving the kinetic energy are neglected. Also, terms involving more than two-body operators are neglected. It is far from clear that one can neglect all these terms.

In Ref. [11] we compare the OEM and exact results for a number of cases involving eight nucleons in the $f_{7/2}$ shell (a simplified model of $^{48}$Ca). In all these cases the OEM matrix elements are significantly smaller than the exact ones. This conclusion remains true independently of the range of the nucleon force, but becomes particularly striking for the zero range $\delta$-force where the OEM matrix element vanishes, but the exact calculation gives a finite matrix element.

We have also performed calculation involving the full $pf$ shell. With the simple Serber-Yukawa force we obtain the matrix element $M_{\text{GT}} = 0.0428$ MeV, while the OEM gives $-0.0115$, again confirming the trend. The uncontrolled approximations in the OEM are, in fact, unnecessary; there exist a well-tested, and efficient algorithm for evaluating the fully interacting nuclear Green's function appearing in double beta decay [11]. Thus, whenever a consistent evaluation of the initial and final nuclear wave functions is feasible, the calculation of $M_{\text{GT}}$ is also possible.

Another approach to the evaluation of the double beta decay matrix elements is based on the approximate validity of the Wigner spin-isospin SU(4) symmetry in nuclei [12]. The leading irreducible representation of SU(4) for the ground state of the initial nucleus is $[y, y, 0]$, while for the final nucleus it is $[(y - 2), (y - 2), 0]$, where $y = (N_i - Z_i)/2$. In the closure approximation the $2\nu$ operator does not connect states belonging to the different irreducible representations of SU(4) (since it contains only group generators) and hence the corresponding matrix element vanishes in the limit of exact symmetry. If the symmetry is only slightly broken, one can use a perturbation theory expansion and calculate the (small) admixtures of $[y, y, 0], T = y - 2$ into $[(y - 2), (y - 2), 0], T = y - 2$. This is, basically, the approach suggested in Ref. [12].

SU(4) is clearly not an exact symmetry. It is broken, first of all, by the spin-orbit force. Also, the residual nucleon-nucleon force is not exactly space exchange Majorana force (although it contains a large component of such a force). In order to test the validity of the SU(4) symmetry quantitatively, we have evaluated the direct overlaps between the exact nuclear wave functions, obtained by the shell model diagonalization of a realistic interaction, and the states belonging to various irreducible representations of SU(4) [13]. In Table II an example of the result obtained is shown. As one can see, the lowest representation has quite poor overlap, only 20% for $^{48}$Ca. Based on these (and similar results for the entire $sd$ shell) we are rather pessimistic about the prospects of using conclusions based on SU(4) symmetry in heavier, more complex nuclei.
4. DOUBLE BETA DECAY IN MASS $A = 100$ SYSTEM

From the point of view of nuclear structure the decays connecting $^{100}$Mo, $^{100}$Tc, and $^{100}$Ru are particularly interesting [14]. The ground state to ground state $2\nu$ double beta decay of $^{100}$Mo is relatively fast [15] (see Table I). In addition, there is an indication for the double beta decay to the excited $0^+$ state at 1130 keV in $^{106}$Ru [16]. After removing the effect of the lepton phase space, one finds that the two nuclear matrix elements are of a comparable magnitude. Moreover, since the ground state of the intermediate odd-odd nucleus $^{100}$Tc is $1^+$, there are three ordinary beta and electron capture transitions connecting this state with the $0^+$ states in the initial and final nuclei and with the excited $0^+$ of the final nucleus. Thus, it is worthwhile to ask whether all that experimental information, involving five pieces of data, can be understood theoretically.

As seen in Table I, QRPA is able to explain the ground state to ground state $2\nu$ decay reasonably well. The excited $0^+$ state appears to be a member of a triplet $0^+, 2^+, 4^+$ near twice the energy of the first $2^+$ state. This suggests that one should treat the excited $0^+$ state as a two-phonon quadrupole vibrational state. When this is done, and the corresponding wave function is calculated using the charge-conserving QRPA, one obtains the $2\nu$ matrix element for the excited state that is indeed of similar magnitude as the ground state to ground state matrix element. However, for $\alpha'_1 = -390$ MeV·fm$^3$ both calculated matrix elements are somewhat larger than the experimental values. It is important to note that the matrix element to the excited state is enhanced by the particle-particle force, unlike the ground state transition which is suppressed.

Next one wants to consider the three transitions from the $1^+$ ground state of $^{100}$Tc. One of them, leading to the ground state of the final nucleus, is of the $\beta^+$ type and hence it is reduced by the particle-particle force. The other two are of the "$\beta^-$" type and are, on the other hand, enhanced by the particle-particle force. In particular, the transition connecting the initial nucleus with the intermediate one has theoretically more Gamow-Teller strength than in experiment, by a factor of 2-3.

The enhancement of the $\beta^-$ like transitions by the particle-particle force is the consequence of the attractive nature of this interaction. For the case of $^{100}$Tc, the lowest $1^+$ state is dominantly $[9g_{9/2}^2\pi g_{7/2}]^{11+}$. The particle-hole interaction reduces its strength as it concentrates most of the Gamow-Teller strength in the giant resonance. The particle-particle force acts in two ways. The enhanced ground state correlations reduce the strength of the leading component, just as in the $\beta^+$ case. However, the attractive force admixes coherently other configurations to the leading one. Ultimately, this tendency is stronger, leading to the overestimate pointed out earlier. Thus, QRPA in its present form is not capable of explaining all data, and in particular, cannot give correctly the above $\beta^-$ strength. This effect, described in detail in Ref. [14] has not been considered properly before in the attempts to describe globally beta decays of most nuclei [17].

The whole above discussion illustrates the strengths and weaknesses of our present understanding of the Gamow-Teller transitions in heavier nuclei, and the intimate relation between the proper description of the ordinary and double beta decay.
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