GAUSSIAN FLUCTUATIONS OF REPRESENTATIONS
OF WREATH PRODUCTS

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ABSTRACT. We study the asymptotics of the reducible representations of the wreath products $G \wr S_q = G^q \rtimes S_q$ for large $q$, where $G$ is a fixed finite group and $S_q$ is the symmetric group in $q$ elements; in particular for $G = \mathbb{Z}/2\mathbb{Z}$ we recover the hyperoctahedral groups. We decompose such a reducible representation of $G \wr S_q$ as a sum of irreducible components (or, equivalently, as a collection of tuples of Young diagrams) and we ask what is the character of a randomly chosen component (or, what are the shapes of Young diagrams in a randomly chosen tuple). Our main result is that for a large class of representations the fluctuations of characters (and fluctuations of the shape of the Young diagrams) are asymptotically Gaussian. The considered class consists of the representations for which the characters asymptotically almost factorize and it includes, among others, the left regular representation therefore we prove the analogue of Kerov’s central limit theorem for wreath products.

1. Introduction

1.1. Formulation of the problem: asymptotics of representations. One of the classical problems in the theory of asymptotic combinatorics concerns the limit behavior of representations of large groups. The above formulation of the problem is very general and vague, so let us be more specific. An example of a question which fits into the above category is the following one: sequences $(G_q)$ and $(\rho_q)$ are given, where for each $q \in \mathbb{N}$ we know that $G_q$ is a finite group and $\rho_q$ is a finite-dimensional representation of $G_q$; we decompose $\rho_q$ as a sum of irreducible components and we ask about asymptotic properties of a random summand. Of course, if the irreducible representations of the groups $G_q$ do not have a nice common structure then it is not clear which asymptotic properties of a random summand could be studied and for this reason we should restrict our attention to ‘nice’ groups $G_q$ for which the irreducible representations can be nicely described in a uniform way.

In this article we are concerned with the case when $G_q = G \wr S_q = G^q \rtimes S_q$ is the wreath product of a fixed finite group $G$ by a symmetric group $S_q$. However, before we come to this topic in Section 1.3 we will
have a closer look (in Section 1.2 below) on a more developed case of the symmetric groups.

1.2. Asymptotics of representations of the symmetric groups. The simplest example of such ‘nice’ groups is the sequence of the symmetric groups $S_q$—in this case the irreducible representations of $S_q$ are enumerated by the Young diagrams with $q$ boxes. Our problem therefore asks about the statistical properties of some random Young diagrams as their size tends to infinity. This problem was studied in much detail: Logan and Shepp [LS77] and Vershik and Kerov [VK77] studied the situation when $\rho_q$ is the left-regular representation (which corresponds to, so called, Plancherel measure on the irreducible representations of $S_q$) and they proved that the corresponding Young diagrams concentrate (after some simple geometric rescaling) around some limit shape. Biane [Bia98, Bia01] generalized this result and he proved the concentration of the shapes for a very large class of representations of $S_q$; he also found a connection between the asymptotic theory of the representations of the symmetric groups and Voiculescu’s free probability theory [VDN92].

The above results can be viewed as an analogue of the law of large numbers, therefore it was very tempting to check if some kind of central limit theorem could be true. The first result of this kind was found by Kerov [Ker93b] (see also Ivanov and Olshanski [IO02]) who proved that for the left-regular representations the fluctuations of a random Young diagram around the limit shape are asymptotically Gaussian. Hora [Hor02, Hor03] found the non-commutative aspect of this result. In a previous paper [Sni06b] we proved that an analogue of the Kerov’s central limit theorem holds true for a large class of representations $\rho_q$ with, so called, approximate factorization of characters.

1.3. The main result: asymptotics of representations of the wreath products. It seems that our understanding of the asymptotic properties of the symmetric groups is reaching a level of maturity and for this reason the time has come to have a look on some other classical series of finite groups and in this article we will concentrate on wreath products $G \wr S_q$, where $G$ is a fixed finite group.

Since $G \wr S_q = G^q \rtimes S_q$ is equal to a semidirect product of a symmetric group $S_q$ therefore the structures of the groups $G \wr S_q$ and $S_q$ and of their irreducible representations are very closely related to each other; for this reason our strategy in this article is to provide a setup in which the known results and methods concerning $S_q$ [Sni06b] could be directly applied to $G \wr S_q$. We will do it in Section 2.
Section 3 contains the main result, namely that the methods which we used for the study of the symmetric groups \cite{Sni06b} can be extended to the wreath products. To be more explicit: we will introduce a class of representations with \textit{approximate factorization of characters}; this class will turn out to be big enough to contain a lot of natural examples and will be closed under some natural operations on representations such as induction, restriction, outer and tensor product. We will prove that for the representations with approximate factorization of characters an analogue of the Kerov’s theorem \cite{Ker93b} holds true, in other words the fluctuations of a randomly chosen irreducible component are asymptotically Gaussian.

2. Preliminaries

2.1. Partial permutations and disjoint products. Ivanov and Kerov \cite{IK99} defined a partial permutation as a pair $\alpha = (\pi, A)$, where $A$ (called support of $\alpha$) is any subset of $\{1, \ldots, q\}$ and $\pi : \{1, \ldots, q\} \to \{1, \ldots, q\}$ is a bijection which is equal to identity outside of $A$. The natural product of partial permutations is given by

$$(\pi_1, A_1)(\pi_2, A_2) = (\pi_1 \pi_2, A_1 \cup A_2).$$

Partial permutations form a semigroup $\mathcal{P}S_q$; in this article we are interested also in the corresponding semigroup algebra $\mathbb{C}(\mathcal{P}S_q)$ which should be regarded as an analogue of the permutation group algebra $\mathbb{C}(S_q)$ equipped with some additional structure.

The algebra $\mathbb{C}(\mathcal{P}S_q)$ can be also equipped with a different product $\bullet$, called disjoint product, given on generators by

$$(\pi_1, A_1) \bullet (\pi_2, A_2) = \begin{cases} (\pi_1 \pi_2, A_1 \cup A_2) & \text{if } A_1 \cap A_2 = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

2.2. Conjugacy classes for symmetric groups. Let integer numbers $k_1, \ldots, k_m \geq 1$ be given. We define the normalized conjugacy class indicator to be a central element in the group algebra $\mathbb{C}(S_q)$ given by \cite{KO94, Bia03, Sni06a, Sni06b}

$$\Sigma_{k_1, \ldots, k_m} = \sum_{a} (a_{1,1}, a_{1,2}, \ldots, a_{1,k_1}) \cdots (a_{m,1}, a_{m,2}, \ldots, a_{m,k_m}),$$

where the sum runs over all one–to–one functions $a : \{\{r, s\} : 1 \leq r \leq m, 1 \leq s \leq k_r\} \to \{1, \ldots, q\}$ and $(a_{1,1}, a_{1,2}, \ldots, a_{1,k_1}) \cdots (a_{m,1}, a_{m,2}, \ldots, a_{m,k_m})$ denotes the product of disjoint cycles. Of course, if $q < k_1 + \cdots + k_m$ then the above sum runs over the empty set and $\Sigma_{k_1, \ldots, k_m} = 0$. 
In other words, if \( k_1 \geq \cdots \geq k_m \) we consider a Young diagram with the rows of the lengths \( k_1, \ldots, k_m \) and all ways of filling it with the elements of the set \( \{1, \ldots, q\} \) in such a way that no element appears more than once. Each such a filling can be interpreted as a permutation when we treat the rows of the Young tableau as disjoint cycles. It follows that \( \Sigma_{k_1, \ldots, k_m} \) is a linear combination of permutations which in the cycle decomposition have cycles of length \( k_1, \ldots, k_m \) (and, additionally, \( q - (k_1 + \cdots + k_m) \) fix-points). Each such a permutation appears in \( \Sigma_{k_1, \ldots, k_m} \) with some positive integer multiplicity depending on the symmetry of the tuple \( k_1, \ldots, k_m \).

It is natural to regard \( \Sigma_{k_1, \ldots, k_m} \) as an element of \( \mathbb{C}(PS_q) \), where we treat each summand on the right-hand-side of (2.1) as a partial permutation with the support equal to the set \( \{a_{r,s}\} \).

2.3. Algebra of conjugacy classes. The sequence of the algebras \( \mathbb{C}(PS_q) \) forms an inverse system with the maps \( \mathbb{C}(PS_{q+1}) \to \mathbb{C}(PS_q) \) equal to restrictions. We consider a vector space spanned by the family of elements \( \lim \leftarrow \Sigma_{k_1, \ldots, k_m} \in \lim \leftarrow \mathbb{C}(PS_q) \). It is easy to check that both the natural product and the disjoint product of two such elements is again in this form therefore the above vector space is an algebra both when as the product we take the natural product (we will denote this algebra by \( \mathfrak{A} \)) and when as the product we take the disjoint product (we will denote this algebra by \( \mathfrak{A}^\bullet \)).

2.4. Wreath product and its representations. For a finite group \( G \) its wreath product with the symmetric group \( G \wr S_q \) is a semidirect product \( G^q \rtimes S_q \), where \( S_q \) acts on \( G^q \) by permuting the factors.

We denote by \( \hat{G} \) the set of the (equivalence classes of) irreducible representations of the finite group \( G \). For an irreducible representation \( \zeta \in \hat{G} \) we denote by \( p_\zeta \in \mathbb{C}(G) \) the corresponding minimal central projection.

Let a Young diagram \( \lambda \) with \(|\lambda|\) boxes be given and let \( \rho_\lambda : S_{|\lambda|} \to \text{End}(W_\lambda) \) be the corresponding irreducible representation of the symmetric group. For an irreducible representation \( \zeta : G \to \text{End}(V_\zeta) \) we consider the tensor power representation \( \zeta^{\otimes |\lambda|} \) of the group \( G^{|\lambda|} \) acting on the vector space

\[
(2.2) \quad (V_\zeta)^{\otimes |\lambda|}.
\]

Please notice that (2.2) is also a representation of \( S_{|\lambda|} \), where the symmetric groups acts by permuting the factors; in this way (2.2) is a representation of the semidirect product \( G^{|\lambda|} \rtimes S_{|\lambda|} = G \wr S_q \).

Also, \( W_\lambda \) is a representation of \( G^{|\lambda|} \rtimes S_{|\lambda|} \), where \( S_{|\lambda|} \) acts on this space by \( \rho_\lambda \) and \( G^{|\lambda|} \) acts trivially as identity.

Thus, the tensor product

\[
(2.3) \quad W_\lambda \otimes (V_\zeta)^{\otimes |\lambda|}
\]
is also a representation of $G \wr S_q$. It will be the basic building block in the construction of all irreducible representations of $G \wr S_q$. The following well known result gives a complete classification of the irreducible representations of the wreath products.

**Proposition 2.1.** Let $\Lambda : \hat{G} \to \mathbb{Y}$ be a function on the set of irreducible representations of $G$ valued in the set of the Young diagrams $\mathbb{Y}$ such that

$$\sum_{\zeta \in \hat{G}} |\Lambda(\zeta)| = q.$$

The representation

$$\rho_{\Lambda} = \left( \bigotimes_{\zeta \in \hat{G}} W_{\Lambda(\zeta)} \otimes V_{\zeta}^{\otimes |\Lambda(\zeta)|} \right) \uparrow_{G^q \rtimes S_q} \prod_{\zeta \in \hat{G}} (G_{|\Lambda(\zeta)|} \rtimes S_{|\Lambda(\zeta)|})$$

is irreducible; furthermore every irreducible representation of $G^q \rtimes S_q$ is of this form.

**Example 2.2.** We consider the case when $G = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. Then there are two irreducible representations of $\mathbb{Z}/2\mathbb{Z}$, namely $\mathbb{Z}/2\mathbb{Z} = \{\zeta_1, \zeta_2\}$, where $\zeta_1$ is the trivial representation and $\zeta_2$ is the “alternating” representation

$$\zeta_1(0) = 1, \quad \zeta_1(1) = 1,$$

$$\zeta_2(0) = 1, \quad \zeta_2(1) = -1.$$

The spaces $V_1, V_2$ on which these representations act are one-dimensional.

In the case when $\zeta = \zeta_i$ the representation (2.3) of $(\mathbb{Z}/2\mathbb{Z}) \wr S_{|\lambda|}$ takes a simpler form

$$W_{\lambda} \otimes (V_i)^{\otimes |\lambda|} \cong W_{\lambda_i},$$

on which the group $S_{|\lambda|}$ acts by $\rho_{\lambda}$ and the elements $(z_1, \ldots, z_{|\lambda|}) \in (\mathbb{Z}/2\mathbb{Z})^{|\lambda|}$ act either trivially as identity (if $\zeta = \zeta_1$) or by multiplying by $(-1)^{z_1 + \cdots + z_{|\lambda|}}$ (if $\zeta = \zeta_2$).

The function $\Lambda : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Y}$ can be identified with a pair of Young diagrams $\lambda_1, \lambda_2$, where $\lambda_i = \Lambda(\zeta_i)$. The representation (2.4) can be written more explicitly as

$$\rho_{\lambda_1, \lambda_2} = \left( W_{\lambda_1} \otimes W_{\lambda_2} \right) \uparrow_{(\mathbb{Z}/2\mathbb{Z})^{|\lambda_1|} \rtimes S_{|\lambda_1|} \times (\mathbb{Z}/2\mathbb{Z})^{|\lambda_2|} \rtimes S_{|\lambda_2|}}.$$

2.5. **Normalized trace.** For a matrix $x \in \mathcal{M}_n(\mathbb{C})$ we define its normalized trace

$$\text{tr} x = \frac{1}{n} \text{Tr} x,$$

where $\text{Tr} x$ denotes the usual trace of $x$. In this way the normalized trace of the identity matrix fulfills $\text{tr} 1 = 1$. 
2.6. **The main homomorphism.** The following Lemma is of great importance for this article: it provides a homomorphism from the tensor product $\mathfrak{A}^{|\widehat{G}|}$ to $\mathbb{C}(G \wr S_q)$; in this way every representation of the wreath product $G \wr S_q$ defines a representation of the tensor product $\mathfrak{A}^{|\widehat{G}|}$ and the questions concerning the asymptotic behavior of the wreath products are reduced to the corresponding questions concerning the symmetric groups.

For an irreducible representation $\zeta \in \widehat{G}$, the corresponding minimal central projection $p_\zeta \in \mathbb{C}(G)$ and a partial permutation $(\pi, A) \in \mathcal{P}S_q$ we set

$$\phi_\zeta(\pi, A) = (r_1 \times \cdots \times r_q)\pi \in \mathbb{C}(G^q \rtimes S_q),$$

where

$$r_m = \begin{cases} p_\zeta & \text{if } m \in A, \\ 1 & \text{otherwise}. \end{cases}$$

**Lemma 2.3.**

1. The map $\phi_\zeta$ extends to a homomorphism of algebras $\phi_\zeta : \mathbb{C}(\mathcal{P}S_q) \to \mathbb{C}(G^q \rtimes S_q)$, where we consider $\mathcal{P}S_q$ equipped with the natural product;

2. Let us fix for a moment some numbering of the irreducible representations $\hat{G} = \{\zeta_1, \ldots, \zeta_{|\widehat{G}|}\}$. The map $\phi : (\mathbb{C}(\mathcal{P}S_q))^{|\widehat{G}|} \to \mathbb{C}(G^q \rtimes S_q)$ given on simple tensors by

$$\phi((\pi_1, A_1) \otimes \cdots \otimes (\pi_{|\widehat{G}|}, A_{|\widehat{G}|})) = \prod_{1 \leq j \leq |\widehat{G}|} \phi_{\zeta_j}(\pi_j, A_j)$$

is a homomorphism of algebras.

3. Let $\rho_\Lambda$ be an irreducible representation of $G \wr S_q$, as presented in Proposition 2.2. For any $a_1, \ldots, a_{|\overline{\Lambda}|} \in \mathfrak{A}$

$$\text{tr} \rho_\Lambda(\phi(a_1 \otimes \cdots \otimes a_{|\overline{\Lambda}|})) = \prod_{1 \leq j \leq |\overline{\Lambda}|} \text{tr} \rho_{\Lambda(\zeta_j)}(a_j),$$

where $\rho_{\Lambda(\zeta_j)}$ is the irreducible representation of $S_{|\Lambda(\zeta_j)|}$ corresponding to the Young diagram $\Lambda(\zeta_j)$.

**Proof.** If a permutation $\pi$ leaves a set $A$ invariant then the elements $r_1 \times \cdots \times r_q$ and $\pi \in \mathbb{C}(G^q \rtimes S_q)$ contributing to (2.5) commute. Let $(\pi^{(1)}, A^{(1)}), (\pi^{(2)}, A^{(2)}) \in \mathcal{P}S_q$ and $(\pi^{(3)}, A^{(3)}) := (\pi^{(1)}, A^{(1)})(\pi^{(2)}, A^{(2)})$;
it follows that

\[
\phi_\zeta(\pi^{(1)}, A^{(1)}) \phi_\zeta(\pi^{(2)}, A^{(2)}) = \\
\pi^{(1)}(r_1^{(1)} \times \cdots \times r_q^{(1)})(r_1^{(2)} \times \cdots \times r_q^{(2)})\pi^{(2)} = \\
\pi^{(1)}(r_1^{(3)} \times \cdots \times r_q^{(3)})\pi^{(2)} = \pi^{(1)}(r_1^{(3)} \times \cdots \times r_q^{(3)}) = \\
\phi_\zeta(\pi^{(3)}, A^{(3)})
\]

which finishes the proof of point (1).

In order to prove point (2) it is enough to prove that if \(\zeta^{(1)} \neq \zeta^{(2)}\) then the elements \(\phi_{\zeta^{(1)}}(\pi^{(1)}, A^{(1)})\) and \(\phi_{\zeta^{(2)}}(\pi^{(2)}, A^{(2)})\) commute. This holds true because when the sets \(A^{(1)}, A^{(2)}\) are disjoint then permutations \(\pi^{(1)}, \pi^{(2)}\) commute; when the sets \(A^{(1)}, A^{(2)}\) are not disjoint then

\[
\psi_{\zeta^{(1)}}(\pi^{(1)}, A^{(1)}) \psi_{\zeta^{(2)}}(\pi^{(2)}, A^{(2)}) = 0 = \phi_{\zeta^{(2)}}(\pi^{(2)}, A^{(2)}) \phi_{\zeta^{(1)}}(\pi^{(1)}, A^{(1)}).
\]

Since \(a_1, \ldots, a_{|G|} \in \mathcal{A}\) therefore \(\phi(a_1 \otimes \cdots \otimes a_{|G|}) \in \mathbb{C}(G^{q} \rtimes S_q)\) is central and Frobenius reciprocity can be applied to calculate the characters of the induced representation \((2, 4)\). This shows that the left-hand side of \((2.6)\) is equal to the character of the representation

\[
\bigotimes_{1 \leq j \leq |\mathcal{G}|} W_{\Lambda(\zeta_j)} \otimes \Lambda_{\zeta_j}^{\otimes |\Lambda(\zeta_j)|} (1)
\]

of the group

\[
G_\Lambda := \prod_{1 \leq j \leq |\mathcal{G}|} (G^{\Lambda(\zeta_j)} \rtimes S_{|\Lambda(\zeta_j)|}) (2)
\]

evaluated on the element \(\phi(a_1 \otimes \cdots \otimes a_{|\mathcal{G}|}) \downarrow_{G_\Lambda} \). Notice that the element of the form

\[
\phi((\pi_1, A_1) \otimes \cdots \otimes (\pi_{|\mathcal{G}|}, A_{|\mathcal{G}|})) \downarrow_{G_\Lambda}
\]

represents on this space as

\[
\begin{cases}
\bigotimes_{1 \leq j \leq |\mathcal{G}|} (\Lambda(\zeta_j))(\pi_j) \otimes 1^{\otimes |\Lambda(\zeta_j)|} & \text{if } A_j \text{ is a subset of the set of the elements permuted by subgroup } S_{|\Lambda(\zeta_j)|} \text{ of } G_\Lambda \text{ for each } j; \\
0 & \text{otherwise.}
\end{cases}
\]

It follows that the element \(\phi(a_1 \otimes \cdots \otimes a_{|\mathcal{G}|}) \downarrow_{G_\Lambda}\) represents as

\[
\bigotimes_{1 \leq j \leq |\mathcal{G}|} [(\Lambda(\zeta_j))(a_j) \otimes 1^{\otimes |\Lambda(\zeta_j)|}]
\]

which finishes the proof of point (3). \(\square\)
We define the natural product on $\mathfrak{A} \otimes \hat{G}$ by
\[
(a_1 \otimes \cdots \otimes a_{|\hat{G}|})(b_1 \otimes \cdots \otimes b_{|\hat{G}|}) = a_1 b_1 \otimes \cdots \otimes a_{|\hat{G}|} b_{|\hat{G}|};
\]
and the disjoint product $\bullet$ by
\[
(a_1 \otimes \cdots \otimes a_{|\hat{G}|}) \bullet (b_1 \otimes \cdots \otimes b_{|\hat{G}|}) = (a_1 \bullet b_1) \otimes \cdots \otimes (a_{|\hat{G}|} \bullet b_{|\hat{G}|}).
\]

Thus the map $\phi$ described in the above Lemma defines a homomorphism of algebras $\phi : \mathfrak{A} \otimes \hat{G} \to \mathbb{C}(G \wr S_q)$, where as the product in $\mathfrak{A} \otimes \hat{G}$ we take the natural product.

We consider the map $\tilde{\phi}_{\zeta_i} : \mathfrak{A} \to \mathfrak{A} \otimes \hat{G}$ given by
\[
\tilde{\phi}_{\zeta_i}(x) = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1;
\]
in this way $\phi_{\zeta_i} = \phi \circ \tilde{\phi}_{\zeta_i}$. From the following on we will not use any explicit numbering of the representations in $\hat{G}$.

2.7. Elements of the group algebra as random variables. Let us fix some finite-dimensional representation $\rho_q$ of the wreath product $G \wr S_q$. We can treat any commuting family of elements of the group algebra $\mathbb{C}(G \wr S_q)$ as a family of random variables equipped with the mean value given by the normalized character:
\[
\mathbb{E}X := \chi_{\rho_q}(X) = \text{tr} \rho_q(X).
\]
It should be stressed that in the general case we treat elements of $\mathbb{C}(G \wr S_q)$ as random variables only on a purely formal level; in particular we do not treat them as functions on some Kolmogorov probability space.

Lemma 2.3 allows us to consider a representation $\rho_q \circ \phi$ of $\mathfrak{A} \otimes \hat{G}$; for simplicity we will denote this representation by the same symbol $\rho_q$. In this way as random variables we may take the elements of the algebra $\mathfrak{A} \otimes \hat{G}$. This algebra can be equipped either with the natural product (2.7) and we denote the resulting cumulants (called natural cumulants) by $k(X_1, \ldots, X_n)$ or with the disjoint product $\bullet$ (2.8) and we denote the resulting cumulants (called disjoint cumulants) by $k^*(X_1, \ldots, X_n)$.

2.8. Canonical probability measure on Young diagrams associated to a representation. There is a special case when it is possible to give a truly probabilistic interpretation to the mean value (2.9); it is when for the family of random variables we take the center of $\mathbb{C}(G \wr S_q)$ (respectively, the algebra $\mathfrak{A} \otimes \hat{G}$ equipped with the natural product). The center of $\mathbb{C}(G \wr S_q)$ is isomorphic (via Fourier transform) to the algebra of functions on irreducible representations of $G \wr S_q$ and the expected value (2.9) corresponds under this isomorphism to the probability measure on irreducible representations (or, equivalently, on functions $\Lambda : \hat{G} \to \mathbb{Y}$ as described in Proposition 2.1) such
that the probability of a given irreducible representation is proportional to the total dimension of all irreducible components of this type in \( \rho_q \).

2.9. **Generalized Young diagrams.** Let \( \lambda \) be a Young diagram. We assign to it a piecewise affine function \( \omega^\lambda : \mathbb{R} \to \mathbb{R} \) with slopes \( \pm 1 \), such that \( \omega^\lambda(x) = |x| \) for large \( |x| \) as it can be seen on the example from Figure 2.2. By comparing Figure 2.1 and Figure 2.2 one can easily see that the graph of \( \omega^\lambda \) can be obtained from the graphical representation of the Young diagram by an appropriate mirror image, rotation and scaling by the factor \( \sqrt{2} \). We call \( \omega^\lambda \) the generalized Young diagram associated with the Young diagram \( \lambda \) [Ker93a, Ker98, Ker99].

The class of generalized Young diagrams consists of all functions \( \omega : \mathbb{R} \to \mathbb{R} \) which are Lipschitz with constant 1 and such that \( \omega(x) = |x| \) for large \( |x| \) and of course not every generalized Young diagram can be obtained by the above construction from some Young diagram \( \lambda \).

The setup of generalized Young diagrams is very useful in the study of the asymptotic properties since it allows us to define easily various notions of convergence of the Young diagram shapes.

2.10. **Functionals of the shape of Young diagrams.** The main result of this article is that the fluctuations of the shape of some random Young diagrams converge (after some rescaling) to a Gaussian distribution. Since the space of (generalized) Young diagrams is infinite-dimensional therefore we need to be very cautious when dealing with such statements. In fact, we will consider a family of functionals on Young diagrams and we show that the joint distribution of each finite set of these functionals converges to the Gaussian distribution.

The functionals mentioned above are given as follows: for a Young diagram \( \lambda \) and the corresponding generalized Young diagram \( \omega \) we denote \( \sigma(x) = \frac{\omega(x)^2}{2} \) [Bia98 IO02] and consider the family of maps

\[
\bar{p}_n(\lambda) = \int_{\mathbb{R}} x^n \sigma''(x) \, dx.
\]
Since $\sigma''$ makes sense as a distribution and $\sigma$ is compactly supported hence the collection $(\bar{p}_n(\lambda))_n$ determines the Young diagram $\lambda$ uniquely.

2.11. **Transition measure of a Young diagram.** To any generalized Young diagram $\omega$ we can assign the unique probability measure $\mu^\omega$ on $\mathbb{R}$, called transition measure of $\omega$, the Cauchy transform of which

$$G_{\mu^\omega}(z) = \int_{\mathbb{R}} \frac{1}{z-x} d\mu^\omega(x)$$

is given by

$$\log G_{\mu^\omega}(z) = -\frac{1}{2} \int_{\mathbb{R}} \log(z-x) \omega''(x) dx = -\frac{1}{2} \int_{\mathbb{R}} \frac{1}{z-x} \omega'(x) dx$$

for every $z \in \mathbb{R}$. For a Young diagram $\lambda$ we will write $\mu^\lambda$ as a short hand of $\mu^{\omega^\lambda}$. This definition may look artificial but it turns out [Ker93a, OV96, Bia98, Oko00] that it is equivalent to natural representation-theoretic definitions which arise by studying the representation $\rho_q$ together with the inclusion $S_q \subset S_{q+1}$.

For $p > 0$ and a Young diagram $\lambda$ we consider the rescaled (generalized) Young diagram $\omega^{p\lambda}$ given by $\omega^{p\lambda} : x \mapsto p \omega^\lambda \left( \frac{x}{p} \right)$. Informally speaking, the symbol $p\lambda$ corresponds to the shape of the Young diagram $\lambda$ geometrically scaled by factor $p$ (in particular, if $\lambda$ has $q$ boxes then $p\lambda$ has $p^2q$ boxes). It is easy to see that (2.12) implies that the corresponding transition measure $\mu^{p\lambda}$ is a dilation of $\mu^\lambda$:

$$\mu^{p\lambda} = D_p \mu^\lambda.$$
This nice behavior of the transition measure with respect to rescaling of Young diagrams makes it a perfect tool for the study of the asymptotics of symmetric groups $S_q$ as $q \to \infty$.

2.12. **Free cumulants of the transition measure.** Cauchy transform of a compactly supported probability measure is given at the neighborhood of infinity by a power series

$$G_\mu(z) = \frac{1}{z} + \sum_{n \geq 2} M_n z^{-n-1},$$

where $M_n = \int x^n d\mu$ are the moments of the measure $\mu$. It follows that on some neighborhood of infinity $G_\mu$ has a right inverse $K_\mu$ with respect to the composition of power series given by

$$K_\mu(z) = \frac{1}{z} + \sum_{n \geq 1} R_n z^{n-1},$$

convergent on some neighborhood of $0$. The coefficients $R_i = R_i(\mu)$ are called free cumulants of measure $\mu$. Free cumulants appeared implicitly in Voiculescu’s R–transform [Voi86] and their combinatorial meaning was given by Speicher [Spe97].

Free cumulants are homogenous in the sense that if $X$ is a random variable and $c$ is some number then

$$R_i(cX) = c^i R_i(X)$$

and for this reason they are very useful in the study of asymptotic questions.

Each free cumulant $R_n$ is a polynomial in the moments $M_1, M_2, \ldots, M_n$ of the measure and each moment $M_n$ can be expressed as a polynomial in the free cumulants $R_1, \ldots, R_n$; in other words the sequence of moments $M_1, M_2, \ldots$ and the sequence of free cumulants $R_1, R_2, \ldots$ contain the same information about the probability measure. The functionals of Young diagrams considered in (2.10) have a nice geometric interpretation but they are not very convenient in actual calculations. For this reason we will prefer to describe the shape of a Young diagram by considering a family of functionals

$$\lambda \mapsto R_n(\mu^\lambda)$$

given by the free cumulants of the transition measure. Equation (2.12) shows that functionals $\tilde{p}_k$ from the family (2.10) can be expressed as polynomials in the functionals from the family (2.14) and vice versa.

Please note that the first two cumulants of a transition measure do not carry any interesting information since

$$R_1(\mu^\lambda) = M_1(\mu^\lambda) = 0,$$
\[ R_2(\mu^\lambda) = M_2(\mu^\lambda) = q, \]

where \( q \) denotes the number of the boxes of the Young diagram \( \lambda \).

Above we treated the free cumulant \( R_i \) as a function on Young diagrams, but it also can be viewed (via Fourier transform) as a central element in \( \mathbb{C}(S_q) \).

### 3. The main result: representations with approximate factorization of characters

#### 3.1. The main theorems.

For a permutation \( \sigma \in G^q \rtimes S_q \) we identify the coset \( \sigma G^q \) as an element of \( (G^q \rtimes S_q)/G^q = S_q \). For a permutation \( \pi \in S_q \) we denote by \( |\pi| \) the minimal number of factors needed to write \( \pi \) as a product of transpositions. For \( l \leq q \) we treat \( G^l \rtimes S_l \) as a subgroup of \( G^q \rtimes S_q \).

The following theorem is the main result of this article.

**Theorem and Definition 3.1.** For each \( q \geq 1 \) let \( \rho_q \) be a representation of \( G \rtimes S_q \). We say that the sequence \( (\rho_q) \) has the character factorization property if it fulfills one (hence all) of the following equivalent conditions:

- For any permutations \( \sigma_1, \ldots, \sigma_n \in G^l \rtimes S_l \) with disjoint supports and such that \( \sigma_1 G^l, \ldots, \sigma_n G^l \in S_l \) are cycles

\[
 k(\sigma_1, \ldots, \sigma_n) q \frac{|\sigma_1 G^l| + \cdots + |\sigma_n G^l| + 2(n-1)}{2} = O(1); \tag{3.1}
\]

- For any integers \( l_1, \ldots, l_n \geq 1 \) and any irreducible representations \( \zeta_1, \ldots, \zeta_n \in \hat{G} \)

\[
 k^*(\tilde{\phi}_{\zeta_1}(\Sigma_{l_1}), \ldots, \tilde{\phi}_{\zeta_n}(\Sigma_{l_n})) q \frac{1 + \cdots + 1 + n + 2}{2} = O(1); \tag{3.2}
\]

- For any integers \( l_1, \ldots, l_n \geq 1 \) and any irreducible representations \( \zeta_1, \ldots, \zeta_n \in \hat{G} \)

\[
 k(\phi_{\zeta_1}(\Sigma_{l_1}), \ldots, \phi_{\zeta_n}(\Sigma_{l_n})) q \frac{1 + \cdots + 1 + n + 2}{2} = O(1); \tag{3.3}
\]

- For any integers \( l_1, \ldots, l_n \geq 2 \) and irreducible representations \( \zeta_1, \ldots, \zeta_n \in \hat{G} \)

\[
 k(\phi_{\zeta_1}(R_{l_1}), \ldots, \phi_{\zeta_n}(R_{l_n})) q \frac{1 + \cdots + 1 + 2(n-1)}{2} = O(1). \tag{3.4}
\]

Notice that in (3.2) we use disjoint cumulants which are well defined in \( \mathcal{A}^{\otimes |\hat{G}|} \) therefore the homomorphisms \( \tilde{\phi}_{\zeta_j} \) cannot be replaced by \( \phi_{\zeta_j} \). On the other hand, in (3.3), (3.4) we use the natural cumulants which are well defined both in \( \mathcal{A}^{\otimes |\hat{G}|} \) and in \( \mathbb{C}(G^q \rtimes S_q) \) therefore it does not matter if we use the homomorphism \( \tilde{\phi}_{\zeta_j} \) or the homomorphism \( \phi_{\zeta_j} \).

The above theorem is a straightforward generalization of the analogous result (Theorem and Definition 1 in [Sni06b]) concerning representations.
of $\mathfrak{A}$ to the case of the representations of $\mathfrak{S}^{\otimes |\mathfrak{G}|}$ and for this reason we skip its proof. Similarly, Theorem 3 in [Sni06b] has the following analogue for the wreath products:

**Theorem 3.2.** Let $(\rho_q)$ has the character factorization property. If the limit of one of the expressions (3.1)–(3.4) exists for $n \in \{1, 2\}$ then the limits of all of the expressions (3.1)–(3.4) exist for $n \in \{1, 2\}$.

These limits fulfill

$$(3.5) \quad c_{\zeta_1,1+1} := \lim_{q \to \infty} \mathbb{E}(\phi_{\zeta}(\sigma, A_{\sigma}))q^{\frac{l_1}{2}} = \lim_{q \to \infty} \mathbb{E}(\phi_{\zeta}(\Sigma_{1}))q^{\frac{l_1}{2}} = \lim_{q \to \infty} \mathbb{E}(\phi_{\zeta}(R_{l+1}))q^{\frac{l_1+1}{2}},$$

where $(\sigma, A_{\sigma}) \in \mathcal{PS}_q$ is a partial permutation equal to a cycle of length $l$ and

$$(3.6) \quad \lim_{q \to \infty} \text{Cov} \left( \phi_{\zeta}(R_{l_1+1}), \phi_{\zeta}(R_{l_2+1}) \right)q^{\frac{l_1+l_2}{2}} = \lim_{q \to \infty} \text{Cov} \left( \phi_{\zeta}(\Sigma_{l_1}), \phi_{\zeta}(\Sigma_{l_2}) \right)q^{\frac{l_1+l_2}{2}} = \lim_{q \to \infty} \text{Cov}^* \left( \phi_{\zeta}(\Sigma_{l_1}), \phi_{\zeta}(\Sigma_{l_2}) \right)q^{\frac{l_1+l_2}{2}} + \sum_{r \geq 1} \sum_{a_1, \ldots, a_r \geq 1} \sum_{b_1, \ldots, b_r \geq 1} \sum_{a_t+a_r=l_1, b_t+b_r=l_2} \frac{l_1l_2}{r} c_{\zeta_1,a_1+b_1} \cdots c_{\zeta_r,a_r+b_r} = \lim_{q \to \infty} \text{Cov} \left( \phi_{\zeta}(\sigma_1, A_{\sigma_1}), \phi_{\zeta}(\sigma_2, A_{\sigma_2}) \right)q^{\frac{l_1+l_2}{2}} - l_1l_2 c_{\zeta_1,l_1+1} c_{\zeta_2,l_2+1} + \sum_{r \geq 1} \sum_{a_1, \ldots, a_r \geq 1} \sum_{b_1, \ldots, b_r \geq 1} \sum_{a_t+a_r=l_1, b_t+b_r=l_2} \frac{l_1l_2}{r} c_{\zeta_1,a_1+b_1} \cdots c_{\zeta_r,a_r+b_r},$$

where $(\sigma_1, A_{\sigma_1}), (\sigma_2, A_{\sigma_2}) \in \mathcal{PS}_q$ are disjoint cycles of length $l_1, l_2$, respectively, and where the numbers $c_{\zeta_1}$ were defined in (3.5); also for any representations $\zeta_1 \neq \zeta_2 \in \hat{\mathfrak{G}}$

$$(3.7) \quad \lim_{q \to \infty} \text{Cov} \left( \phi_{\zeta_1}(R_{l_1+1}), \phi_{\zeta_2}(R_{l_2+1}) \right)q^{\frac{l_1+l_2}{2}} = \lim_{q \to \infty} \text{Cov} \left( \phi_{\zeta_1}(\Sigma_{l_1}), \phi_{\zeta_2}(\Sigma_{l_2}) \right)q^{\frac{l_1+l_2}{2}} = \lim_{q \to \infty} \text{Cov}^* \left( \phi_{\zeta_1}(\Sigma_{l_1}), \phi_{\zeta_2}(\Sigma_{l_2}) \right)q^{\frac{l_1+l_2}{2}} = \lim_{q \to \infty} \text{Cov} \left( \phi_{\zeta_1}(\sigma_1, A_{\sigma_1}), \phi_{\zeta_2}(\sigma_2, A_{\sigma_2}) \right)q^{\frac{l_1+l_2}{2}} - l_1l_2 c_{\zeta_1,l_1+1} c_{\zeta_2,l_2+1}.$$
By mimicking the line of the proof of Corollary 4 in [Sni06b] we get the following corollary.

**Corollary 3.3.** Let $(\rho_q)$ be as in Theorem 3.2 and let $\Lambda : \hat{G} \to \mathbb{Y}$ be a random function distributed according to the canonical probability measure associated to $\rho_q$, as described in Section 2.8.

1. **(Gaussian fluctuations of free cumulants)** Then the joint distribution of the centered random variables

$$r_{\zeta,i} = q^{-\frac{i-1}{2}}(R_{\zeta,i} - \mathbb{E}R_{\zeta,i}) = q^{-\frac{i-1}{2}}(\phi_{\zeta}(R_i) - \mathbb{E}\phi_{\zeta}(R_i))$$

converges to a Gaussian distribution in the weak topology of probability measures, where $R_{\zeta,i}$ denotes the $i$-th free cumulant of the transition measure $\mu^{\Lambda(\zeta)}$.

2. **(Gaussian fluctuations of characters)** Let $\sigma_i \in G \wr S_q$ be a permutation such that $\sigma_i G^q \in S_q$ is a cycle. Then the joint distribution of the centered random variables

$$q^{\frac{|\sigma_i G^q| + 1}{2}}(\chi_{\Lambda}(\sigma_i) - \mathbb{E}\chi_{\Lambda}(\sigma_i))$$

converges to a Gaussian distribution in the weak topology of probability measures, where $\chi_{\Lambda}(x) = \text{tr}\rho_{\Lambda}(x)$ denotes the normalized character associated to the irreducible representation $\rho_{\Lambda}$.

3. **(Gaussian fluctuations of the shape of the Young diagrams)** Then the joint distribution of the centered random variables

$$q^{-\frac{i-2}{2}}(\tilde{p}_{\zeta,i} - \mathbb{E}\tilde{p}_{\zeta,i})$$

converges to a Gaussian distribution in the weak topology of probability measures, where $\tilde{p}_{\zeta,i} = \tilde{p}_i(\Lambda(\zeta))$ is the functional of the shape of the Young diagram defined in (2.10).

**3.2. Examples.** All examples presented in this section not only have the character factorization property but additionally are as in Theorem 3.2, i.e. the limits of the means and of the covariances exist.

**Example 3.4.** Let $\rho : G \to \text{End}(V)$ be a fixed representation of $G$ and let $\rho_q = (V^\otimes q) \uparrow_{G^q \wr S_q}^{G^q \rtimes S_q}$ be the induced representation of $G^q \rtimes S_q$. In particular, if $\rho$ is the left-regular representation then $\rho_q$ is equal to the left regular representation of the wreath product $G \wr S_q$.

It is easy to check that that for any permutations $\sigma_1, \ldots, \sigma_n \in G \wr S_q$ with disjoint supports

$$k(\sigma_1, \ldots, \sigma_n)q^{\frac{|\sigma_1 G| + \cdots + |\sigma_n G| + (n-1)}{2}} = 0$$

if $n \neq 1$ or $\sigma_1 G \neq e$. 
It follows from condition (3.1) that the sequence \((\rho_{q})\) has the character factorization property and that the mean and the covariance of the free cumulants are given by

\[
\lim_{q \to \infty} \mathbb{E}(\phi_{\zeta}(R_{l+1})) q^{-\frac{l+1}{2}} = \begin{cases} 
c_{\zeta} & \text{if } l = 1, \\
0 & \text{if } l \geq 2,
\end{cases}
\]

where

\[c_{\zeta} = \frac{\text{(dimension of the representation } \zeta)\text{(multiplicity of } \zeta \text{ in } W)}{\text{(dimension of } W)}\]

and

\[
\lim_{q \to \infty} \text{Cov} \left( \phi_{\zeta_1}(R_{l_1+1}), \phi_{\zeta_2}(R_{l_2+1}) \right) q^{-\frac{l_1 + l_2}{2}} =
\begin{cases} 
lc_{\zeta} & \text{if } l_1 = l_2 = l \geq 2, \\
c_{\zeta}(1 - c_{\zeta}) & \text{if } l_1 = l_2 = 1, \\
0 & \text{if } l_1 \neq l_2,
\end{cases}
\]

and for \(\zeta_1 \neq \zeta_2\)

\[
\lim_{q \to \infty} \text{Cov} \left( \phi_{\zeta_1}(R_{l_1+1}), \phi_{\zeta_2}(R_{l_2+1}) \right) q^{-\frac{l_1 + l_2}{2}} =
\begin{cases} 
-c_{\zeta_1} c_{\zeta_2} & \text{if } l_1 = l_2 = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

**Example 3.5 (Irreducible representations).** Let \(c > 0\) be a constant and let \((\Lambda_{q})\) be a sequence of functions, \(\Lambda_{q} : \hat{G} \to \mathbb{Y}\) as in Proposition 2.1. We assume that \(\sum_{\zeta \in \hat{G}} |\Lambda_{q}(\zeta)| = q\) and that each diagram \(\Lambda_{q}(\zeta)\) has at most \(c\sqrt{q}\) rows and columns. Suppose that for each \(\zeta \in \hat{G}\) the shapes of rescaled Young diagrams \(q^{-\frac{1}{2}}\Lambda_{q}(\zeta)\) converge to some limit. The convergence of the shapes of Young diagrams implies convergence of the free cumulants and it follows that the sequence \((\rho_{\Lambda_{q}})\) of the corresponding irreducible representations has the characters factorization property.

In this example the cumulants (3.3) and (3.4) vanish for \(n \geq 2\) since the Young diagrams are non-random and the corresponding limits for \(n = 1\) are determined by the limit shapes of the Young diagrams.

The above two examples are the building blocks from which one can construct some more complex representations with the help of the operations on representations presented below.
Theorem 3.6 (Restriction of representations). Suppose that the sequence of representations \( (\rho_q) \) has the character factorization property. Let a sequence of integers \( (\tau_q) \) be given, such that \( \tau_q \geq q \) and the limit \( p = \lim_{q \to \infty} \frac{\tau_q}{q} \) exists.

Let \( \rho'_q \) denote the restriction of the representation \( \rho_{\tau_q} \) to the subgroup \( G \wr S_q \leq G \wr S_{\tau_q} \). Then the sequence \( (\rho'_q) \) has the factorization property of characters. The fluctuations of the free cumulants are determined by

\[
(3.12) \quad c'_{\zeta,l+1} := \lim_{q \to \infty} \mathbb{E}(R'_{\zeta,l+1})q^\frac{l+1}{2} = p^\frac{l-1}{2} \lim_{q \to \infty} \mathbb{E}(R_{\zeta,l+1})q^\frac{l+1}{2} = p^\frac{l-1}{2} c_{\zeta,l+1},
\]

(3.13) \[
\lim_{q \to \infty} \text{Cov}(R'_{\zeta_1,l_1+1}, R'_{\zeta_2,l_2+1})q^\frac{l_1+l_2}{2} = \]

\[
p^\frac{l_1+l_2}{2} \left[ \lim_{q \to \infty} \text{Cov}(R_{\zeta_1,l_1+1}, R_{\zeta_2,l_2+1})q^\frac{l_1+l_2}{2} - l_1l_2c_{\zeta_1,l_1+1}c_{\zeta_2,l_2+1} (p^{-1} - 1) + \sum_{r \geq 1} \sum_{a_1, \ldots, a_r \geq 1} \sum_{b_1, \ldots, b_r \geq 1} \frac{l_1l_2}{r!} c_{\zeta,a_1+b_1} \cdots c_{\zeta,a_r+b_r} (p^{-r} - 1) \right]
\]

for all \( \zeta \in \hat{G} \) and

(3.14) \[
\lim_{q \to \infty} \text{Cov}(R'_{\zeta_1,l_1+1}, R'_{\zeta_2,l_2+1})q^\frac{l_1+l_2}{2} = \]

\[
p^\frac{l_1+l_2}{2} \left[ \lim_{q \to \infty} \text{Cov}(R_{\zeta_1,l_1+1}, R_{\zeta_2,l_2+1})q^\frac{l_1+l_2}{2} - l_1l_2c_{\zeta_1,l_1+1}c_{\zeta_2,l_2+1} (p^{-1} - 1) \right]
\]

for all \( \zeta_1 \neq \zeta_2 \) and for all \( l_1, l_1, l_2 \geq 1 \), where the quantities \( R'_{\zeta_1,l}, c'_{\zeta,i} \) concern the representations \( (\rho'_q) \) while \( R_{\zeta_1,l}, c_{\zeta,i} \) concern the representations \( (\rho_q) \).

In particular, for \( p = 0 \) we recover the fluctuations of the induced representations from Example 3.4.

The proof is a straightforward application of Theorem and Definition 3.1; for details we refer to Theorem 8 in [Šni06b].

Theorem 3.7 (Outer product of representations). Suppose that for \( i \in \{1, 2\} \) the sequence of representations \( (\rho_q^{(i)}) \) has the character factorization property. Let sequences of positive integers \( \tau_q^{(i)} \) be given, such that \( \tau_q^{(1)} + \tau_q^{(2)} = q \) and the limits \( p^{(i)} := \lim_{q \to \infty} \frac{\tau_q^{(i)}}{q} \) exist.
Let \( \rho_q' = \rho^{(1)}_{r_q} \circ \rho^{(2)}_{r_q} \) denote the outer product of representations. Then the sequence \( \{\rho_q'\} \) has the factorization property of characters with (3.15)
\[
c_{\zeta,1+1} := \lim_{q \to \infty} E(R'_{\zeta,1+1}) q^{-\frac{1+1}{2}} = (p^{(1)})^{-\frac{1+1}{2}} c_{\zeta,1+1}^{(1)} + (p^{(2)})^{-\frac{1+1}{2}} c_{\zeta,1+1}^{(2)},
\]
and with an explicit (but involved) covariance of free cumulants. The appropriate disjoint covariance is given by

\[
\lim_{q \to \infty} \text{Cov}_{\rho_q'}(\phi_{\zeta_1}(\Sigma_{l_1}), \phi_{\zeta_2}(\Sigma_{l_2})) q^{-\frac{1+1}{2}} = (p^{(1)})^{-\frac{1+1}{2}} \lim_{q \to \infty} \text{Cov}_{\rho_q^{(1)}}(\phi_{\zeta_1}(\Sigma_{l_1}), \phi_{\zeta_2}(\Sigma_{l_2})) q^{-\frac{1+1}{2}} + (p^{(2)})^{-\frac{1+1}{2}} \lim_{q \to \infty} \text{Cov}_{\rho_q^{(2)}}(\phi_{\zeta_1}(\Sigma_{l_1}), \phi_{\zeta_2}(\Sigma_{l_2})) q^{-\frac{1+1}{2}}
\]
for all \( \zeta_1, \zeta_2 \in \hat{G} \).

The proof of this result also follows closely the proof of the analogous result for symmetric groups, namely Theorem 10 in [Sni06b].

**Theorem 3.8 (Induction of representations).** Suppose that the sequence of representations \( \{\rho_q\} \) has character factorization property. Let a sequence of integers \( r_q \) be given, such that \( r_q \leq q \) and the limit \( p = \lim_{q \to \infty} \frac{r_q}{q} \) exists.

Let \( \rho_q' = \rho_q^{(1)} \uparrow^{\text{G} \wr \text{S}_q \text{G}} \) denote the induced representation. Then the sequence \( \{\rho_q'\} \) has the characters factorization property with
\[
c_{\zeta,1+1} = \begin{cases} p \frac{1+1}{2} c_{\zeta,1+1} & \text{for } l \geq 2, \\ pc_{\zeta,2} + (1 - p) \left( \frac{\text{dimension of } \zeta}{|G|} \right)^2 & \text{for } l = 1, \end{cases}
\]
and with an explicit (but involved) covariance of free cumulants.

**Proof.** It is enough to adapt the proof of Theorem 3.7. □

**Theorem 3.9 (Tensor product of representations).** Suppose that for \( i \in \{1, 2\} \) the sequence of representations \( \{\rho_q^{(i)}\} \) has character factorization property. Then the tensor product \( \rho_q' = \rho_q^{(1)} \otimes \rho_q^{(2)} \) has the property of factorization of characters. Furthermore, the limit distribution and the fluctuations are the same as for the tensor representations (3.9) and (3.10) with
\[
c_{\zeta} = \lim_{q \to \infty} \left( \frac{\text{multiplicity of } \zeta \text{ in } (\rho^{(1)} \otimes \rho^{(2)}) \downarrow_{\text{G}}^{\text{G} \wr \text{S}_q} \right) \left( \frac{\text{dimension of } \zeta}{\text{dimension of } \rho^{(1)} \otimes \rho^{(2)}} \right).
\]
Proof again follows the proof of the analogous result concerning symmetric groups, namely Theorem 12 from [Sni06b].
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