Z-EIGENVALUE INCLUSION THEOREMS FOR TENSORS

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Abstract. In this paper, we establish Z-eigenvalue inclusion theorems for general tensors, which reveal some crucial differences between Z-eigenvalues and H-eigenvalues. As an application, we obtain upper bounds for the largest Z-eigenvalue of a weakly symmetric nonnegative tensor, which are sharper than existing upper bounds.

1. Introduction. Let C (R) denote the set of all complex (real) numbers. Consider an m-order n-dimensional tensor A consisting of n^m entries in R:

A = (a_{i_1 i_2 \ldots i_m}), \forall a_{i_1 i_2 \ldots i_m} \in R, 1 \leq i_1, i_2, \ldots, i_m \leq n.

A is called nonnegative (positive) if a_{i_1 i_2 \ldots i_m} \geq 0 (a_{i_1 i_2 \ldots i_m} > 0).

For an n-dimensional column vector x = [x_1, x_2, \ldots, x_n]^T, real or complex, we define an n-dimensional column vector:

A x^{m-1} = (\sum_{i_2, \ldots, i_m=1}^{n} a_{i_2 \ldots i_m} x_{i_2} \ldots x_{i_m})_{1 \leq i \leq n}.

The following definitions about eigenvalues of tensors were introduced in [8] [11].

Definition 1.1. Let A be an m-order n-dimensional tensor. We say that (\lambda, x) \in C \times (C^n \setminus \{0\}) is an eigenvalue-eigenvector of A if

A x^{m-1} = \lambda x^{[m-1]},

where x^{[m-1]} = [x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1}]^T. (\lambda, x) is called an H-eigenpair if both of them are real.

Definition 1.2. Let A be an m-order n-dimensional tensor. We say that (\lambda, x) \in C \times (C^n \setminus \{0\}) is an E-eigenpair of A if

A x^{m-1} = \lambda x \land x^T x = 1.

(\lambda, x) is called a Z-eigenpair if both of them are real.

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Both $H$-eigenvalue and $Z$-eigenvalue problems have attracted a lot of researchers due to their wide applications in medical resonance imaging [11, 13, 15], data analysis [6], higher-order Markov chains [12], positive definiteness of even-order multivariate forms in automatical control [13]. There is abundant literature on the development of $H$-eigenvalues for nonnegative tensors [2, 4, 8, 9, 12, 14, 20, 22, 23]. In particular, $H$-eigenvalue inclusion theorems were established for general tensors [11, 14] and the bounds of the largest $H$-eigenvalue were given for nonnegative tensors [11, 13, 19] 20.

The $Z$-eigenvalue problem plays a fundamental role in best rank-one approximation, which has numerous applications in engineering and higher order statistics [5, 21]. Some effective algorithms for finding $Z$-eigenvalue and the corresponding eigenvector of tensors have been implemented [3, 6, 10, 16]. Generally, we cannot judge that $Z$-eigenvalues generated by above algorithms are the largest $Z$-eigenvalue. Hence, it is important to estimate the largest $Z$-eigenvalue and investigate $Z$-eigenvalue inclusion set for general tensors. Chang et al. [3] proposed upper bounds for $Z$-spectral radius of nonnegative tensors. Furthermore, Song et al. [17] improved the upper bounds for $Z$-spectral radius based on the relationship between the Gelfand formula and the spectral radius. For weakly symmetric and positive tensors, He et al. [7] derived the Ledermann-like upper bound for the largest $Z$-eigenvalue. Compared with relatively mature characterizations of inclusion set for $H$-eigenvalue [11, 14], characterizations of inclusion set for $Z$-eigenvalue are still underdeveloped for general tensors. This stimulates us to establish some inclusion theorems to identify the distribution of $Z$-eigenvalues.

In this paper, we first reveal the differences between $H$-eigenvalue inclusion set and $Z$-eigenvalue inclusion set by Example 1. By choosing different components of an eigenvector, we obtain exact characterizations on $Z$-eigenpairs. Based on these characterizations, we establish sharp $Z$-eigenvalue inclusion theorems. Then, we discuss relations among these inclusion theorems and show that Theorem 3.2, Theorem 3.3 and Theorem 3.4 are different by Example 2. Finally, we apply these inclusion theorems to estimate upper bounds of the largest $Z$-eigenvalue for weakly symmetric nonnegative tensors. These bounds are tighter than existing bounds of [3, 7, 17].

This paper is organized as follows. In Section 2, we recall some preliminary results and introduce some existing results. In Section 3, we establish sharp $Z$-eigenvalue inclusion theorems and give comparisons among these eigenvalue inclusion sets. In Section 4, we apply these inclusion theorems to estimate upper bounds of the largest $Z$-eigenvalue for weakly symmetric nonnegative tensors.

2. Notation and preliminaries. In this section, we shall present some definitions and important properties related to $Z$-eigenvalues of a tensor, which are needed in the subsequent analysis.

**Definition 2.1.** Let $\mathcal{A}$ and $\mathcal{I}$ be $m$-order $n$-dimensional tensors.

(i) [14] We define $\sigma(\mathcal{A})$ the $Z$-spectrum of $\mathcal{A}$ by the set of all $Z$-eigenvalues of $\mathcal{A}$. Assume $\sigma(\mathcal{A}) \neq \emptyset$. Then the $Z$-spectral radius of $\mathcal{A}$ is denoted by

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$ 

(ii) [14] We call $\mathcal{I}$ a unit tensor if its entries are

$$\delta_{i_1i_2\ldots i_m} = \begin{cases} 1, & \text{if } i_1 = i_2 = \ldots = i_m \\ 0, & \text{otherwise}. \end{cases}$$
Theorem 3.1. Let eigenvalue inclusion sets.

Consider 3 order 2 dimensional tensor values of a general tensor.

A polynomial Z cannot capture all Z-λ. However, \( |\{i, j, k\} \leq r_i(A)\), \( r_i(A) = \sum_{\delta \in \mathbb{C}} a_{iij} \in \mathbb{Z} \) and \( \Gamma(A) \) investigated the distribution of H-eigenvalue and established the Gersgorin H-eigenvalue inclusion theorem of real supersymmetric tensors. These results can be generalized to general tensors \([11, 20]\).

Lemma 2.2. ([11, 20]) Let \( A \) be a complex tensor of order \( m \) and dimension \( n \). Then,

\[ \sigma_H(A) \subseteq \Gamma(A) = \bigcup_{i \in \mathbb{N}} \Gamma_i(A), \]

where \( \sigma_H(A) \) is the set of all H-eigenvalues of tensor \( A \) and \( \Gamma_i(A) = \{ z \in C : |z - a_{i...i}| \leq r_i(A)\}, r_i(A) = \sum_{\delta \in \mathbb{C}} |a_{iij} \in \mathbb{Z} \} \) and \( N = \{1, 2, ..., n\} \).

The following example shows that Lemma 2.2 cannot be generalized to Z-eigenvalues of a general tensor.

Example 1. Consider 3 order 2 dimensional tensor \( A = (a_{ijk}) \) defined by

\[ a_{ijk} = \begin{cases} a_{111} = 2; a_{222} = 1; \\ a_{112} = a_{121} = a_{211} = -\frac{1}{3}; \\ a_{ijk} = 0, \text{ otherwise}. \end{cases} \]

By simple computation, we get some eigenpairs of \( A \) as follows: \( \{(\lambda, x) : (\lambda_1 = -2.8309, x^1 = (-0.9319, 0.3626), (\lambda_2 = -1, x^2 = (0, -1)), (\lambda_3 = 1, x^3 = (0, 1))\}\).

According to Lemma 2.1, we have

\[ \Gamma(A) = \bigcup_{i \in \mathbb{N}} \Gamma_i(A) = [-\frac{2}{3}, \frac{14}{3}] \cup \left[ -\frac{2}{3}, \frac{14}{3} \right] = [-\frac{2}{3}, \frac{14}{3}]. \]

However, \( \lambda_1 = -2.8309 \notin [-\frac{2}{3}, \frac{14}{3}] \) and \( \lambda_2 = -1 \notin [-\frac{2}{3}, \frac{14}{3}] \). This implies that \( \Gamma(A) \) cannot capture all Z-eigenvalues of \( A \).

3. Z-eigenvalue inclusion theorems. In this section, we establish Z-eigenvalue inclusion theorems of tensor \( A \), which is completely different from H-eigenvalue inclusion theorems. Furthermore, we establish comparisons among different Z-eigenvalue inclusion sets.

Theorem 3.1. Let \( A \) be a tensor with order \( m \) and dimension \( n \geq 2 \). Then, all Z-eigenvalues of \( A \) are located in the union of the following sets:

\[ \sigma(A) \subseteq \mathcal{K}(A) = \bigcup_{i \in \mathbb{N}} \mathcal{K}_i(A), \]

where \( \mathcal{K}_i(A) = \{ z \in C : |z| \leq R_i(A)\} \) and \( R_i(A) = \sum_{i \in \mathbb{N}} |a_{iij} \in \mathbb{Z} \} \).

Proof. Let \( \lambda \) be a Z-eigenvalue of \( A \) with corresponding eigenvector \( x \), i.e.,

\[ Ax^{m-1} = \lambda x, \|x\| = 1, \]

where \( x = [x_1, x_2, ..., x_n]^T \). Since \( x \) is an eigenvector and \( \|x\| = 1 \), it has at least one nonzero component. Define \( x_t \) as a component of \( x \) with the largest absolute value, i.e., \( |x_t| \geq |x_j| \) for all \( j \in \mathbb{N} \). It follows from the definition of Z-eigenvalues that

\[ \lambda x_t = \sum_{i \in \mathbb{N}} a_{iij} x_i, \]

where \( x_i \) is an eigenvector and \( \|x_i\| = 1 \), it has at least one nonzero component. Define \( x_t \) as a component of \( x \) with the largest absolute value, i.e., \( |x_t| \geq |x_j| \) for all \( j \in \mathbb{N} \). It follows from the definition of Z-eigenvalues that

\[ \lambda x_t = \sum_{i \in \mathbb{N}} a_{ij} x_i, \]

where \( x_i \) is an eigenvector and \( \|x_i\| = 1 \), it has at least one nonzero component. Define \( x_t \) as a component of \( x \) with the largest absolute value, i.e., \( |x_t| \geq |x_j| \) for all \( j \in \mathbb{N} \). It follows from the definition of Z-eigenvalues that

\[ \lambda x_t = \sum_{i \in \mathbb{N}} a_{ij} x_i, \]
Taking the modulus in \( (2) \) and dividing both sides by \(|x_t|\), we get

\[
|\lambda| \leq \sum_{i_2, \ldots, i_m \in N} |a_{i_2 \ldots i_m}| \frac{|x_{i_2}|}{m-\sqrt{|x_t|}} \cdots \frac{|x_{i_m}|}{m-\sqrt{|x_t|}} \leq \sum_{i_2, \ldots, i_m \in N} |a_{i_2 \ldots i_m}| = R_t(A),
\]

since \(|x_t| \leq m^{-1}|x_t|\). This implies \(\lambda \in \mathcal{K}(A)\).

**Remark 1.** When \(m = 2\), exact eigenvalue inclusion theorems can be found in [18].

**Theorem 3.2.** Let \(A\) be a tensor with order \(m\) and dimension \(n \geq 2\). Then, all \(Z\)-eigenvalues of \(A\) are located in the union of the following sets:

\[
\sigma(A) \subseteq \mathcal{L}(A) = \bigcup_{i \in N} \bigcap_{N,j \neq i} \mathcal{L}_{i,j}(A),
\]

where \(\mathcal{L}_{i,j}(A) = \{ z \in \mathbb{C} : |z| = (R_i(A) - |a_{ij \ldots j}|) |z| \leq |a_{ij \ldots j}| R_j(A) \}\).

**Proof.** Let \(\lambda\) be a \(Z\)-eigenvalue of \(A\) with corresponding eigenvector \(x\). Similar to the proof of Theorem 3.1, we define \(x_t\) as a component of \(x\) with the largest modulus such that \(|x_t| \geq |x_j|\) for all \(j \in N\). Note that

\[
|\lambda||x_t| \leq \sum_{\delta_{i_2 \ldots i_m} = 0} |a_{i_2 \ldots i_m}| \frac{|x_{i_2}|}{m-\sqrt{|x_t|}} \cdots \frac{|x_{i_m}|}{m-\sqrt{|x_t|}} + |a_{ts \ldots s}| |x_t|^{m-1}, \quad s \in N, s \neq t \quad (3)
\]

Dividing both sides by \(|x_t|\) in \( (3) \) and from \(|x_s|^{m-1} \leq |x_s|\), we have

\[
|\lambda| \leq \sum_{\delta_{i_2 \ldots i_m} = 0} |a_{i_2 \ldots i_m}| \frac{|x_{i_2}|}{m-\sqrt{|x_t|}} \cdots \frac{|x_{i_m}|}{m-\sqrt{|x_t|}} + \frac{|a_{ts \ldots s}|}{|x_t|} |x_t|^{m-1} \quad (4)
\]

\[
\leq \sum_{\delta_{i_2 \ldots i_m} = 0} |a_{i_2 \ldots i_m}| + \frac{|a_{ts \ldots s}|}{|x_t|} |x_t| = (R_t(A) - |a_{ts \ldots s}|) + \frac{|a_{ts \ldots s}|}{|x_t|} |x_t|.
\]

If \(|x_s| = 0\), then \(|\lambda| - (R_t(A) - |a_{ts \ldots s}|) \leq 0\), and it is obvious that \(\lambda \in \mathcal{L}_{i,s}(A) \subseteq \mathcal{L}(A)\). Otherwise, \(|x_s| > 0\). Moreover, similar to \( (3) \), we obtain

\[
|\lambda||x_s| \leq \sum_{i_2, \ldots, i_m \in N} |a_{i_2 \ldots i_m}| |x_{i_2}| \cdots |x_{i_m}| \leq \sum_{i_2, \ldots, i_m \in N} |a_{i_2 \ldots i_m}| |x_t|^{m-1}, \quad (5)
\]

Dividing both sides by \(|x_s|\) in \( (5) \) and from \(|x_t| \leq 1\), one has

\[
|\lambda| \leq \sum_{i_2, \ldots, i_m \in N} \frac{|a_{i_2 \ldots i_m}|}{|x_s|} |x_t|^{m-1} \quad (6)
\]

Multiplying \( (4) \) with \( (6) \), we see

\[
|\lambda| - (R_t(A) - |a_{ts \ldots s}|) |x_s| \leq |a_{ts \ldots s}| R_s(A) |x_t|, \quad |x_s| \leq |x_t|,
\]

which implies \(\lambda \in \mathcal{L}_{i,s}(A)\). From the arbitrariness of \(s\), we have \(\lambda \in \bigcap_{j \in N,j \neq t} \mathcal{L}_{t,j}(A)\). Furthermore, \(\lambda \in \bigcup_{i \in N} \bigcap_{j \in N,j \neq t} \mathcal{L}_{i,j}(A)\).

**Corollary 3.1.** Let \(A\) be a tensor with order \(m\) and dimension \(n \geq 2\). Then,

\[
\sigma(A) \subseteq \mathcal{L}(A) \subseteq \mathcal{K}(A).
\]
Proof. For any $\lambda \in \mathcal{L}(A)$, without loss of generality, there exists $t \in N$ such that $\lambda \in \mathcal{L}_{t,s}(A)$, that is,

\[
[|\lambda| - (R_t(A) - |a_{ts...s}|)]|\lambda| \leq |a_{ts...s}|R_s(A), \forall s \neq t. \tag{7}
\]

If $|a_{ts...s}|R_s(A) = 0$, then $\lambda = 0$ or $|\lambda| - (R_t(A) - |a_{ts...s}|) \leq 0$. Hence, $\lambda \in \mathcal{K}_t(A) \cup \mathcal{K}_s(A)$. Otherwise, it follows from (7) that

\[
\frac{|\lambda| - (R_t(A) - |a_{ts...s}|)}{|a_{ts...s}|} \leq 1.
\]

Furthermore,

\[
\frac{|\lambda| - (R_t(A) - |a_{ts...s}|)}{|a_{ts...s}|} \leq 1
\]

or

\[
\frac{|\lambda|}{R_s(A)} \leq 1,
\]

which implies $\lambda \in \mathcal{K}_t(A) \cup \mathcal{K}_s(A) \subseteq \mathcal{K}(A)$. \hfill $\square$

In the following theorem, choosing $x_s$ as a component of $x$ with the second largest modulus, we obtain some sharp results for $\sigma(A)$.

**Theorem 3.3.** Let $A$ be a tensor with order $m$ and dimension $n \geq 2$. Then, all $Z$-eigenvalues of $A$ are located in the union of the following sets:

\[
\sigma(A) \subseteq \mathcal{M}(A) = \bigcup_{i,j \in N, i \neq j} \mathcal{M}_{i,j}(A) \bigcup \mathcal{H}_{i,j}(A),
\]

where $\mathcal{M}_{i,j}(A) = \{ z \in \mathcal{C} : |z| - (R_i(A) - |a_{ij...j}|) \leq |a_{ij...j}|(R_j(A) - P_j(A)) \}$, $P_j(A) = \sum_{i_2, \ldots, i_m \in N} |a_{i_2...i_m}|$ and $\mathcal{H}_{i,j}(A) = \{ z \in \mathcal{C} : |z| - (R_i(A) - |a_{ts...s}|) < 0, |z| < P_j(A) \}$.

Proof. Let $\lambda$ be a $Z$-eigenvalue of $A$ with corresponding eigenvector $x$, i.e., $Ax^{m-1} = \lambda x$. Since $x$ is an eigenvector, it has at least one nonzero component. Let $|x_i| \geq \max\{ |x_k| : k \in N, k \neq s, k \neq t \}$. Obviously, $|x_t| > 0$. Similar to the characterization of (4), we get

\[
|\lambda| \leq (R_i(A) - |a_{ts...s}|) + |a_{ts...s}| \frac{|x_s|}{|x_t|}. \tag{8}
\]

If $|x_s| = 0$, then $(|\lambda| - (R_t(A) - |a_{ts...s}|)) < 0$. For $|\lambda| \geq P_s(A)$, one has $\lambda \in \mathcal{M}_{t,s}(A)$; For $|\lambda| < P_s(A)$, we have $\lambda \in \mathcal{H}_{t,s}(A)$. Otherwise, $|x_s| > 0$. Moreover, noting that $x_s$ is a component of $x$ with the second largest modulus, we have

\[
|\lambda||x_s| \leq \sum_{i_2, \ldots, i_m \in N} |a_{i_2...i_m}| |x_{i_2}| \cdots |x_{i_m}|
\]

\[
= \sum_{i_2, \ldots, i_m \in N, \ t \in \{i_2, \ldots, i_m\}} |a_{i_2...i_m}| |x_{i_2}| \cdots |x_{i_m}| + \sum_{i_2, \ldots, i_m \in N, \ t \notin \{i_2, \ldots, i_m\}} |a_{i_2...i_m}| |x_{i_2}| \cdots |x_{i_m}|
\]

\[
\leq \sum_{i_2, \ldots, i_m \in N, \ t \in \{i_2, \ldots, i_m\}} |a_{i_2...i_m}| |x_t|^{m-1} + \sum_{i_2, \ldots, i_m \in N, \ t \notin \{i_2, \ldots, i_m\}} |a_{i_2...i_m}| |x_s|^{m-1}. \tag{9}
\]
Dividing both sides by $|x_s|$ in (9), one has
\[
|\lambda| \leq \sum_{i_2, \ldots, i_m \in N} |a_{si_2\ldots i_m}| |x_t|^{|x_t| - 2} |x_t| |x_s| + \sum_{i_2, \ldots, i_m \in N} |a_{si_2\ldots i_m}| |x_s|^{m-2}
\]
\[
\leq \sum_{i_2, \ldots, i_m \in N} |a_{si_2\ldots i_m}| |x_t| |x_s| + \sum_{i_2, \ldots, i_m \in N} |a_{si_2\ldots i_m}| |x_s|,
\]
(10)
since $|x_t| \leq 1, |x_s| \leq 1$. When $|\lambda| - (R_t(A) - |a_{ts\ldots s}|) \geq 0$ or $|\lambda| - P_s^t(A) \geq 0$ holds, multiplying inequalities (8) with (10), we have
\[
[|\lambda| - (R_t(A) - |a_{ts\ldots s}|)] (|\lambda| - \sum_{i_2, \ldots, i_m \in N} |a_{si_2\ldots i_m}|) \leq |a_{ts\ldots s}| \sum_{i_2, \ldots, i_m \in N} |a_{si_2\ldots i_m}|,
\]
with the consequence that
\[
||\lambda| - (R_t(A) - |a_{ts\ldots s}|)| - P_s^t(A) \leq |a_{ts\ldots s}| (R_s(A) - P_s^t(A)).
\]
This shows $\lambda \in M_{t,s}(A) \subseteq M(A)$.

When $|\lambda| - (R_t(A) - |a_{ts\ldots s}|) < 0$ and $|\lambda| - P_s^t(A) < 0$ hold, one has $\lambda \in H_{t,s}(A) \subseteq M(A)$. So, the result holds. $\square$

Now, we give a proof to show $M(A) \subseteq K(A)$.

**Corollary 3.2.** Let $A$ be a tensor with order $m$ and dimension $n \geq 2$. Then,
\[
\sigma(A) \subseteq M(A) \subseteq K(A).
\]

**Proof.** For any $\lambda \in M(A)$, we divide the proof into two parts.

(I) There exist $t, s \in N, s \neq t$ such that $\lambda \in M_{t,s}(A)$, that is
\[
||\lambda| - (R_t(A) - |a_{ts\ldots s}|)| - P_s^t(A) \leq |a_{ts\ldots s}| (R_s(A) - P_s^t(A)).
\]
(11) If $|a_{ts\ldots s}| (R_t(A) - P_s^t(A)) = 0$, then $|\lambda| = P_s^t(A)$ or $|\lambda| - (R_t(A) - |a_{ts\ldots s}|) \leq 0$, which shows $\lambda \in K_t(A) \cup K_s(A)$. Otherwise, (11) entails
\[
\frac{|\lambda| - (R_t(A) - |a_{ts\ldots s}|)}{|a_{ts\ldots s}|} \frac{|\lambda| - P_s^t(A)}{R_s(A) - P_s^t(A)} \leq 1.
\]
Furthermore,
\[
\frac{|\lambda| - (R_t(A) - |a_{ts\ldots s}|)}{|a_{ts\ldots s}|} \leq 1
\]
or
\[
\frac{|\lambda| - P_s^t(A)}{R_s(A) - P_s^t(A)} \leq 1,
\]
which implies $\lambda \in K_t(A) \cup K_s(A) \subseteq K(A)$.

(II) There exist $t, s \in N, s \neq t$ such that $\lambda \in H_{t,s}(A)$, that is
\[
|\lambda| - (R_t(A) - |a_{ts\ldots s}|) < 0
\]
and $|\lambda| - P_s^t(A) < 0$.

Obviously, $\lambda \in K_t(A) \cap K_s(A) \subseteq K(A).$ $\square$

Based on exact characterization of (3), we obtain another $Z$-eigenvalue inclusion theorems.
Theorem 3.4. Let \( A \) be a tensor with order \( m \) and dimension \( n \geq 2 \). Then, all \( Z \)-eigenvalues of \( A \) are located in the union of the following sets:

\[
\sigma(A) \subseteq N(A) = \bigcup_{i,j \in N, i \neq j} N_{i,j}(A),
\]

where \( N_{i,j}(A) = \{ z \in C : |z| - |R_i(A) - P_i(A)| \leq P_i(A)R_j(A) \} \) and \( P_i(A) = \sum_{\substack{i_2, \ldots, i_m \in N \\ t \notin \{i_2, \ldots, i_m\}}} |a_{i_2 \ldots i_m}|. \)

Proof. Let \( \lambda \) be a \( Z \)-eigenvalue of \( A \) with corresponding eigenvector \( x \). Let \(|x_s| \geq \max |x_k| : k \in N, k \neq s, k \neq t\). Obviously, \(|x_t| > 0\). From equality (1), we have

\[
|\lambda||x_t| \leq \sum_{\substack{i_2, \ldots, i_m \in N \\ t \in \{i_2, \ldots , i_m\}}} |a_{i_2 \ldots i_m}| |x_{i_2} \ldots x_{i_m}| + \sum_{\substack{i_2, \ldots, i_m \in N \\ t \notin \{i_2, \ldots, i_m\}}} |a_{i_2 \ldots i_m}| |x_{i_2} \ldots x_{i_m}|.
\]

Furthermore,

\[
|\lambda||x_t| \leq \sum_{\substack{i_2, \ldots, i_m \in N \\ t \in \{i_2, \ldots , i_m\}}} |a_{i_2 \ldots i_m}| |x_{i_2} \ldots x_{i_m}| + \sum_{\substack{i_2, \ldots, i_m \in N \\ t \notin \{i_2, \ldots, i_m\}}} |a_{i_2 \ldots i_m}| |x_{i_2} \ldots x_{i_m}|.
\]

since \(|x_t|^{m-1} \leq |x_{i_2} \ldots x_{i_m}| \leq |x_{i_2} | \leq 1, |x_s|^{m-1} \leq |x_{i_2} \ldots x_{i_m}| \leq 1\). Dividing both sides by \(|x_s|\) in the above inequality, we get

\[
|\lambda| \leq \sum_{\substack{i_2, \ldots, i_m \in N \\ t \in \{i_2, \ldots , i_m\}}} |a_{i_2 \ldots i_m}| + \sum_{\substack{i_2, \ldots, i_m \in N \\ t \notin \{i_2, \ldots, i_m\}}} |a_{i_2 \ldots i_m}| \frac{|x_{i_2} \ldots x_{i_m}|}{|x_t|}, \tag{12}
\]

If \(|x_s| = 0\), then \((|\lambda| - \sum_{\substack{i_2, \ldots, i_m \in N \\ t \in \{i_2, \ldots , i_m\}}} |a_{i_2 \ldots i_m}|) = (|\lambda| - (R_t(A) - P_t(A))) \leq 0\), that is, \(|\lambda| \leq (R_t(A))\). This shows \( \lambda \in N_{i,s}(A) \subseteq N(A) \). Otherwise, \(|x_s| > 0\). Multiplying (9) with (12) yields,

\[
(|\lambda| - \sum_{\substack{i_2, \ldots, i_m \in N \\ t \notin \{i_2, \ldots, i_m\}}} |a_{i_2 \ldots i_m}|) |\lambda| \leq \sum_{\substack{i_2, \ldots, i_m \in N \\ t \notin \{i_2, \ldots, i_m\}}} |a_{i_2 \ldots i_m}| R_s(A),
\]

equivalently,

\[
(|\lambda| - |R_t(A) - P_t(A)|) |\lambda| \leq P_t(A)R_s(A),
\]

which implies \( \lambda \in N_{i,s}(A) \subseteq N(A) \). \( \square \)

Corollary 3.3. Let \( A \) be a tensor with order \( m \) and dimension \( n \geq 2 \). Then,

\[
\sigma(A) \subseteq L(A) = M(A) = N(A) \subseteq K(A).
\]

Proof. Similar to the proof of Corollary 3.1 and Corollary 3.2 \( \square \)

From Corollary 3.1 and Corollary 3.2 we observe \( \sigma(A) \subseteq L(A) = M(A) = N(A) \subseteq K(A) \) when \( n = 2 \). The following example shows that Theorems 3.2-3.4 are different, since \( L_{i,j}(A), M_{i,j}(A) \) and \( N_{i,j}(A) \) are not included each other.
Example 2. Consider 3 order 3 dimensional tensor $A = (a_{ijk})$ defined by
\[
a_{ijk} = \begin{cases} 
a_{111} = 1; & a_{112} = -1; & a_{131} = 1; & a_{133} = 1 \\
a_{211} = -1; & a_{222} = 2; & a_{232} = 1 \\
a_{311} = 1; & a_{323} = 1; & a_{333} = 3 \\
a_{ijk} = 0, & \text{otherwise.} \end{cases}
\]

According to Theorem 3.1, we get
\[
K(A) = \bigcup_{i \in N} K_i(A) = \{ \lambda \in C : |\lambda| \leq 5 \}.
\]

According to Theorem 3.2, one has
\[
L(A) = \bigcup_{i \in N, i \neq j} \bigcap_{j \in N, i \neq j} L_{i,j}(A) = \{ \lambda \in C : |\lambda| \leq 2 + 2\sqrt{2} \},
\]
where
\[
\begin{array}{|c|c|}
\hline
L_{1,2}(A) &= \{ \lambda \in C : |\lambda| \leq 4 \} \\
L_{2,1}(A) &= \{ \lambda \in C : |\lambda| \leq 4 \} \\
L_{3,1}(A) &= \{ \lambda \in C : |\lambda| \leq 2 + 2\sqrt{2} \} \\
\hline
L_{1,3}(A) &= \{ \lambda \in C : |\lambda| \leq \frac{5 + \sqrt{21}}{2} \} \\
L_{2,3}(A) &= \{ \lambda \in C : |\lambda| \leq \frac{5 + \sqrt{21}}{2} \} \\
L_{3,2}(A) &= \{ \lambda \in C : |\lambda| \leq 5 \} \\
\hline
\end{array}
\]

It follows from Theorem 3.3 that
\[
M(A) = \bigcup_{i,j \in N, i \neq j} M_{i,j}(A) = \{ \lambda \in C : |\lambda| \leq 5 \},
\]
where
\[
\begin{array}{|c|c|}
\hline
M_{1,2}(A) &= \{ \lambda \in C : 3 \leq |\lambda| \leq 4 \} \\
M_{2,1}(A) &= \{ \lambda \in C : 2 \leq |\lambda| \leq 4 \} \\
M_{3,1}(A) &= \{ \lambda \in C : 3 + \sqrt{3} \leq |\lambda| \leq 5 \} \\
\hline
M_{1,3}(A) &= \{ \lambda \in C : |\lambda| \leq \frac{7 + \sqrt{21}}{2} \} \\
M_{2,3}(A) &= \{ \lambda \in C : |\lambda| \leq \frac{7 + \sqrt{21}}{2} \} \\
M_{3,2}(A) &= \{ \lambda \in C : 3 \leq |\lambda| \leq 5 \} \\
\hline
\end{array}
\]

According to Theorem 3.4, we have
\[
N(A) = \bigcup_{i,j \in N, i \neq j} N_{i,j}(A) = \{ \lambda \in C : |\lambda| \leq 2 + 2\sqrt{2} \},
\]
where
\[
\begin{array}{|c|c|}
\hline
N_{1,2}(A) &= \{ \lambda \in C : |\lambda| \leq \frac{3 + \sqrt{21}}{2} \} \\
N_{2,1}(A) &= \{ \lambda \in C : |\lambda| \leq 4 \} \\
N_{3,1}(A) &= \{ \lambda \in C : |\lambda| \leq 2 + 2\sqrt{2} \} \\
\hline
N_{1,3}(A) &= \{ \lambda \in C : |\lambda| \leq \frac{3 + \sqrt{21}}{2} \} \\
N_{2,3}(A) &= \{ \lambda \in C : |\lambda| \leq \frac{3 + \sqrt{21}}{2} \} \\
N_{3,2}(A) &= \{ \lambda \in C : |\lambda| \leq 4 \} \\
\hline
\end{array}
\]

4. Bounds on the largest $Z$-eigenvalue of weakly symmetric nonnegative tensors. In this section, we give some sharp upper bounds for a weakly symmetric nonnegative tensor, which generalize the results of [3] [7] [17]. We start this section with some fundamental results of nonnegative tensors [3] [7] [17].

Lemma 4.1. (Proposition 3.3 of [3]) Let $A$ be an $m$-order and $n$-dimensional nonnegative tensor. Then,
\[
\rho(A) \leq \max_{i \in N} \sqrt{n} R_i(A).
\]

Lemma 4.2. (Corollary 4.5 of [17]) Let $A$ be an $m$-order and $n$-dimensional nonnegative tensor. Then,
\[
\rho(A) \leq \max_{i \in N} R_i(A).
\]
Lemma 4.3. (Theorem 2.7 of [7]) Suppose that an $m$-order $n$-dimensional nonnegative tensor $A$ is weakly symmetric and positive, and $x$ is an eigenvector associated with the largest $Z$-eigenvalue $\rho(A)$ of $A$. Then,

$$\rho(A) \leq R - l(1 - \theta),$$

where $R = \max_{i \in N} R_i(A), l = \min_{i_1, \ldots, i_m \in N} a_{i_1 \ldots i_m}, \theta = \left(\frac{r}{R}\right)^\frac{1}{m}$.

Based on the assumption that $A$ is weakly symmetric, Chang et al. [3] established the equivalent relation between the largest $Z$-eigenvalue and $Z$-spectral radius of nonnegative tensors.

Lemma 4.4. (Theorem 3.11 of [3]) Assume $A$ is a weakly symmetric nonnegative tensor. Then,

$$\rho(A) = \lambda^*,$$

where $\lambda^*$ denotes the largest $Z$-eigenvalue.

We shall devote to finding sharper upper bounds of the largest $Z$-eigenvalue for a weakly symmetric nonnegative tensor.

Theorem 4.5. Suppose that an $m$-order $n$-dimensional nonnegative tensor $A$ is weakly symmetric. Then,

$$\rho(A) \leq \bar{u} = \max_{i \in N} \min_{j \in N, i \neq j} \frac{1}{2} (R_i(A) - a_{ij \ldots j} + \sqrt{(R_i(A) - a_{ij \ldots j})^2 + 4a_{ij \ldots j}R_j(A)}).$$

Proof. Using Lemma 4.4, without loss of generality, we assume that $\rho(A) = \lambda^*$ is the largest $Z$-eigenvalue of $A$. It follows from Theorem 3.2 that there exists $t \in N$ such that

$$[\rho(A) - (R_t(A) - a_{ts \ldots s})]\rho(A) \leq a_{ts \ldots s}R_s(A), \forall s \in N, s \neq t.$$

Then,

$$\rho(A) \leq \frac{1}{2}(R_t(A) - a_{ts \ldots s} + \sqrt{(R_t(A) - a_{ts \ldots s})^2 + 4a_{ts \ldots s}R_s(A)}).$$

Since $s \in N$ is chosen arbitrarily, it holds

$$\rho(A) \leq \min_{j \in N, i \neq j} \frac{1}{2} (R_i(A) - a_{ij \ldots j} + \sqrt{(R_i(A) - a_{ij \ldots j})^2 + 4a_{ij \ldots j}R_j(A)}).$$

Furthermore,

$$\rho(A) \leq \max_{i \in N} \min_{j \in N, i \neq j} \frac{1}{2} (R_i(A) - a_{ij \ldots j} + \sqrt{(R_i(A) - a_{ij \ldots j})^2 + 4a_{ij \ldots j}R_j(A)}).$$

Remark 2. It is necessary that $A$ is weakly symmetric. Example 3.4 of [3] is shown $\rho(A) \neq \lambda^*$ for a nonnegative tensor. It is noteworthy that Theorem 4.5 can be regarded as upper bound of $Z$-spectral radius when $A$ is not weakly symmetric.

Similar to Corollary 3.1, we have the following result.

Corollary 4.1. Suppose that an $m$-order $n$-dimensional nonnegative tensor $A$ is weakly symmetric. Then,

$$\rho(A) \leq \bar{u} \leq \max_{i \in N} R_i(A),$$

where $\bar{u}$ is defined in Theorem 4.5.
Suppose that an $m$-order $n$-dimensional nonnegative tensor $A$ is weakly symmetric. Then, $4a_{ij...j}R_j(A) \leq 4a_{ij...j}R_i(A)$, that is, 
\[
(R_i(A) - a_{ij...j})^2 + 4a_{ij...j}R_j(A) \leq (R_i(A) + a_{ij...j})^2,
\]
\[
\sqrt{(R_i(A) - a_{ij...j})^2 + 4a_{ij...j}R_j(A)} \leq (R_i(A) + a_{ij...j}).
\]
Furthermore, 
\[
\frac{1}{2}[R_i(A) - a_{ij...j} + \sqrt{(R_i(A) - a_{ij...j})^2 + 4a_{ij...j}R_j(A)] \leq R_i(A)}.
\]
Thus, 
\[
\min_{j \in N, j \neq i} \frac{1}{2}[R_i(A) - a_{ij...j} + \sqrt{(R_i(A) - a_{ij...j})^2 + 4a_{ij...j}R_j(A)] \leq \max_{i \in N} R_i(A),
\]
furthermore, 
\[
\max_{i \in N} \min_{j \in N, j \neq i} \frac{1}{2}[R_i(A) - a_{ij...j} + \sqrt{(R_i(A) - a_{ij...j})^2 + 4a_{ij...j}R_j(A)] \leq \max_{i \in N} R_i(A).
\]

(ii) For any $i, j \in N, i \neq j$, if $R_i(A) \leq R_j(A)$, note that 
\[
\frac{1}{2}[R_i(A) - a_{ij...j} + \sqrt{(R_i(A) - a_{ij...j})^2 + 4a_{ij...j}R_j(A)] \leq \frac{1}{2}(R_j[A] - a_{ij...j} + \sqrt{(R_j(A) - a_{ij...j})^2 + 4a_{ij...j}R_j(A]} = R_j(A),
\]
since $0 \leq R_i(A) - a_{ij...j} \leq R_j(A) - a_{ij...j}$. Moreover, 
\[
\max_{i \in N} \min_{j \in N, j \neq i} \frac{1}{2}[R_i(A) - a_{ij...j} + \sqrt{(R_i(A) - a_{ij...j})^2 + 4a_{ij...j}R_j(A)] \leq \max_{j \in N} R_j(A).
\]

It follows from (13) and (14) that this corollary holds. 

Based on Theorem 3.3, we obtain some sharp bounds of the largest $Z$-eigenvalue for a weakly symmetric nonnegative tensor.

**Theorem 4.6.** Suppose that an $m$-order $n$-dimensional nonnegative tensor $A$ is weakly symmetric.

Then, 
\[
\rho(A) \leq \bar{v} = \max_{i,j \in N, i \neq j} \left\{ \frac{1}{2}(R_i(A) - a_{ij...j} + P_j^t(A) + \Lambda_i^2(A)), R_i(A) - a_{ij...j}, P_j^t(A) \right\},
\]
where $\Lambda_i^t(A) = (R_i(A) - a_{ij...j} - P_j^t(A))^2 + 4a_{ij...j}(R_j(A) - P_j^t(A)).$

**Proof.** Suppose $\rho(A)$ is the largest $Z$-eigenvalue of $A$. We divide the proof into two parts.

(i) There exist $t, s \in N, s \neq t$ such that $\rho(A) \in \mathcal{M}_{t,s}(A)$, i.e., 
\[
[r(A) - (R_p(A) - a_{ts...s})][r(A) - P_s^t(A)] \leq a_{ts...s}(R_s(A) - P_s^t(A)).
\]

Then, solving for $\rho(A)$, we get 
\[
\rho(A) \leq \frac{1}{2}[R_i(A) - a_{ij...j} + P_j^t(A) + \Lambda_i^2(A)] \leq \max_{i,j \in N, i \neq j} \frac{1}{2}(R_i(A) - a_{ij...j} + P_j^t(A) + \Lambda_i^2(A)).
\]
(II) There exist $t, s \in \mathbb{N}, s \neq t$ such that $\rho(A) \in \mathcal{H}_{t,s}(A)$, i.e.,
$$\rho(A) < R_t(A) - a_{ij...j} \text{ and } \rho(A) < P_j(A).$$
So, the results hold. \(\square\)

**Corollary 4.2.** Suppose that an $m$-order $n$-dimensional nonnegative tensor $A$ is weakly symmetric. Then,
$$\rho(A) \leq \bar{v} \leq \max_{i \in \mathbb{N}} R_i(A),$$
where $\bar{v}$ is defined in Theorem 4.6.

**Proof.** Similar to the proof of Corollary 4.1, the conclusion holds. \(\square\)

From Theorem 3.4 and Corollary 3.3, we obtain some sharp bounds of the largest $Z$-eigenvalue for a weakly symmetric nonnegative tensor.

**Theorem 4.7.** Suppose that an $m$-order $n$-dimensional nonnegative tensor $A$ is weakly symmetric. Then,
$$\rho(A) \leq \bar{w} = \max_{i,j \in \mathbb{N}, i \neq j} \frac{1}{2} \{ R_i(A) - P_j(A) + \sqrt{(R_i(A) - P_j(A))^2 + 4P_j(A)R_i(A)} \}.$$ 

**Corollary 4.3.** Suppose that an $m$-order $n$-dimensional nonnegative tensor $A$ is weakly symmetric. Then,
$$\rho(A) \leq \bar{w} \leq \max_{i \in \mathbb{N}} R_i(A),$$
where $\bar{w}$ is defined in Theorem 4.7.

Now, we use Example 4.8 of [3] to show that bounds of Theorem 4.5-4.7 are tighter than the upper bounds in Lemma 4.1-4.3.

**Example 3.** Consider 4 order 2 dimensional tensor $A = (a_{ijkl})$ defined by
$$a_{ijkl} = \begin{cases} a_{1111} = \frac{1}{2}; a_{2222} = 3; \\ a_{ijkl} = \frac{1}{3}, \text{ otherwise.} \end{cases}$$

By simple computation, we obtain $(\rho(A), x) = (3.1122, (0.1700, 0.9854))$. According to Lemma 4.1, we have upper bound
$$\rho(A) \leq UP_{\text{Lemma 4.1}} = 7.5432;$$
According to Lemma 4.2, we have upper bound
$$\rho(A) \leq UP_{\text{Lemma 4.2}} = 5.3333;$$
According to Lemma 4.3, we have upper bound
$$\rho(A) \leq UP_{\text{Lemma 4.3}} = 5.2846.$$ 

Since Theorem 4.5, Theorem 4.6 and Theorem 4.7 are equivalent when $n = 2$, we have upper bound
$$\rho(A) \leq UP_{\text{Theorem 4.5-4.7}} = 5.1822,$$
which shows that our upper bound is tighter.

5. **Conclusion.** In this paper, we establish $Z$-eigenvalue inclusion theorems for general tensors and show that Gersgorin-type $H$-eigenvalue inclusion theorem do not apply to $Z$-eigenvalue of a general tensor. Moreover, using these $Z$-eigenvalue inclusion theorems, we obtain some new upper bounds for the largest $Z$-eigenvalue of a weakly symmetric nonnegative tensor and show that these upper bounds are sharper than the bounds in [3 7 17].
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