An orbit-averaged generalised-Landau-kinetic equation for probability distribution of $N$-stars in finite dense weakly-coupled star clusters

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ABSTRACT
In secular evolution of finite dense star clusters, the statistical acceleration of stars is the most essential non-collective relaxation process whose effect can be mathematically modeled by Kandrup’s generalized-Landau (g-Landau) kinetic equation for distribution function of stars. Understanding of the g-Landau equation is of significance in finite system since only the equation can correctly define the total energy and number of stars in phase spaces in case the effect of gravitational polarization is neglected. The present paper shows a kinetic formulation of an orbit-averaged generalized Landau equation in action-angle spaces beginning with BBGKY hierarchy and shows the conservation laws, $H$-theorem and anti-normalization condition. Furthermore, the orbit-averaged g-Landau equation is rewritten for anisotropic spherical system. It is shown that the statistical acceleration can be replaced by typical acceleration of star for the relaxation process in secular evolution of any anisotropic spherical systems. The derivation of the equations is made by generalizing the formulation for the inhomogeneous Landau equation done in (Polyachenko 1982) to the g-Landau equation.

Key words: gravitation – methods: analytical – globular clusters: general–galaxies: general

1 INTRODUCTION

1.1 Fundamental of statistical acceleration

Imagine an ideally isolated star cluster of $N$-stars (i.e. $N$-particles of the equal mass $m$ interacting each other via Newtonian pair-wise force). The acceleration of ‘test’ star is only subject to the vector sum of forces due to pairwise-Newtonian potential forces from $N - 1$ ‘field’ stars;

$$a_1 = \sum_{i=2}^{N} a_{1i},$$

where the acceleration of star 1 (test star) at $r_1$ due to the potential $\phi_{1i}$ from ‘field’ star $i$ at $r_i$ is defined as

$$a_{1i} = -\nabla_1 \phi_{1i} = -\nabla_1 \left( -\frac{Gm}{|r_1 - r_i|} \right),$$

where $G$ is Newton gravitational constant. Direct $N$-body numerical simulations for evolution of star clusters of $N(= 10^5 \sim 10^7)$-stars are, however, numerically expensive. Hence one’s concern is also to rely on statistical description of stellar dynamics which is, as first approximation ($N \rightarrow \infty$), based on the ‘smooth’ mean field (m.f.) acceleration of star 1, determined by $(N-1)$ body distribution function $F(2, \cdots, N)$

$$A_1(1,t) = \sum_{2}^{N} \int a_{1i} F(1, \cdots, N) d2 \cdots dN = \left( 1 - \frac{1}{N} \right) \int a_{12} f(1,i) d2$$

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where the symmetry in permutation between the states of two stars is assumed and the number DF \( f(1,t) = N\bar{F}(1,t) \) is introduced for self-consistency. While the m.f. description is correct on time scales of a few of dynamical times, in secular evolution of dense star clusters, stars may undergo the (non-collective) relaxation process due to the finite-\( N \) effect that is characterized by the deviation of the actual acceleration from the m.f. acceleration of star 1

\[
a_1 - A_1(1,t) = \sum_{i=2}^{N} a_{1i} - \left(1 - \frac{1}{N}\right) \int a_{12} F(1,t) d2
\]  

(1.4)

Since, for simplicity (Kandrup 1981a), typical statistical description relies on two-body DF description, meaning one needs to simplify the many-body effect to the acceleration of star 1 only due to \( N - 1 \) identical stars 2;

\[
a_1 - A_1(1,t) \approx (N - 1) a_{12} - \left(1 - \frac{1}{N}\right) \int a_{12} f(2,t) d2 = (N - 1) \left( a_{12} - \frac{1}{N} \int a_{12} f(2,t) d2 \right) \approx (N - 1) \bar{a}_{12}
\]  

(1.5)

The factor \( \bar{a}_{12} \) is termed statistical acceleration of star 1; it corresponds to the 'discreteness fluctuation (Kandrup 1988)' in acceleration of star 1 due to the discreteness of the system and it was originally termed “statistical term” in (Gilbert 1968).

For weakly-coupled star clusters, the the mean square of discrete fluctuation in the statistical acceleration is less significant than that of m.f. acceleration (e.g. Kandrup 1988)

\[
< \bar{a} \cdot \bar{a} > \approx \Theta( < A \cdot A > /N)
\]  

(1.6)

\[
< \bar{a} > = 0
\]  

(1.7)

The statistical acceleration of a star in a dense star cluster has been of essence to discuss the evolution of the cluster, the fundamental nature of the statistical acceleration, however, has not been fully understood. This is since the kinetic formulation is not complete yet, as explained in section 1.2.

1.2 Background of statistical acceleration in relaxational evolution of star clusters

Chandrasekhar’s classical works (Chandrasekhar & von Neumann 1942) casted on the stochastic nature of gravitational cumulative two-body relaxation in an infinite homogeneous system, correspondingly the following approximation is taken

\[
< \bar{a} \cdot \bar{a} >_{\text{A\neq0}} = < a \cdot a >_{\text{A=0}}
\]  

(1.8)

where the subscripts \( A \neq 0 \) and \( A = 0 \) represents inhomogeneous- and homogeneous- background stars in the two-body relaxation process respectively. Chandrasekhar employed Holtsmark distribution of Newtonian-force strength to model the stochastic nature of irregular force on test star (Chandrasekhar 1943); the approximation, equation (1.8) is of importance to understand the ‘dominant’ relaxation effect is due to the nearest neighbor stars. Yet, as later on it was experimented based on the \( N \)-body numerical simulation for finite inhomogeneous system, one must consider the effect of potentials due to the rest of distant stars, which results in Coulomb logarithm (Ambartsumian 1938)

\[
< a \cdot a > \sim \Theta \left( \ln \left[ \frac{N}{1} \right] \right)
\]  

(1.9)

One of the most concerns in modern star-cluster kinetic theory for the secular evolution is the effect of ‘non-dominant’ relaxation described by equation (1.6). The benchmark of kinetic theory is due to (Gilbert 1968) where the currently most basic kinetic equation for \( N \) stars in finite inhomogeneous star cluster was derived including the effects of statistical acceleration (action on test star from field stars) and gravitational polarization (the reaction).

\[
< \bar{a} \cdot \bar{a} >_{\text{A\neq0}} < a \cdot a >_{\text{A=0}}
\]  

(1.10)

A correct kinetic formulation of the statistical acceleration of stars \( < \bar{a} \cdot \bar{a} >_{\text{A\neq0}} \) without any approximation was first rendered in (Kandrup 1981a). Kandrup discussed the importance of the statistical acceleration in relaxation process for anisotropic stochastic system (Kandrup 1981b) and for the finite-\( N \) effect on fluctuation in smooth m.f. potential force (Kandrup 1988). Those works are held only at formal-expression level and did not explain the fundamental properties (\( H \)-theorem, conservation laws, \( \cdots \)) of the generalised Landau equation. As generally done, to understand the fundamental properties of an kinetic equation for stellar dynamics, one needs to employ an orbit-averaging of the kinetic equations and to find the explicit expressions in action-angle spaces as done for inhomogeneous Landau equation (Polyachenko & Shukhman 1982; Chavanis 2008) and inhomogeneous Balescu-Lanard equations (Heyvaerts 2010; Chavanis 2012). Hence, the present paper aims at establishing a kinetic formulation of orbit-averaged g-Landau equation. Especially, it is shown that the importance of the g-Landau kinetic equation originates from a point of view of fundamental statistical mechanics and the g-Landau kinetic equations is the only one kinetic equation that correctly approximates the (Gilbert 1968)’s kinetic equation among the existing kinetic equation for finite stellar systems while the inhomogeneous Landau and Balescu-Lenard equations are not.

The present paper is organized as follows. In section 2, the basic kinetic theory is reviewed. Section 3 explains the importance of the g-Landau equation for the evolution of finite star clusters. In section 4, the explicit form of orbit-averaged g-Landau equation in action-angle spaces is shown. In section 5, the basic properties of the g-Landau equation are discussed. In section 6 the g-Landau equation is rewritten for spherical star clusters. Section 7 is Conclusion.
2 BBGKY HIERARCHY FOR DISTRIBUTION FUNCTION OF STARS IN WEAKLY COUPLED DENSE STAR CLUSTERS

Assuming the star cluster of concern is a weakly-coupled system, the secular evolution of the system may be correctly described by the BBGKY hierarchy. After in section 2.1 the BBGKY hierarchy in phase spaces \((r, p)\) is explained, the hierarchy is rewritten in action-angle variable to focus on the motion of test star that follows integrable Hamiltonian.

2.1 BBGKY hierarchy in the phase space \((r, p)\) and discreteness parameter

The Hamiltonian in terms of phase-space variables \((r_i, p_i)\) for \(N\) stars of equal masses \(m\) may be written as

\[
H = \frac{\mathbf{p}^2}{2m} + m \sum_{j=1}^{N} \phi(r_{ij}) \quad (1 \leq i \leq N),
\]

where \(\phi(r_{ij})\) is a Newtonian gravitational potential

\[
\phi(r_{ij}) = \frac{Gm}{r_{ij}} \quad (1 \leq i, j \leq N \text{ with } i \neq j),
\]

where \(G\) is the gravitational constant. Assuming \(N\) stars are statistically identical and indistinguishable, \(N\)-body Liouville equation for the Hamiltonian \(H(r_i, p_i)\) reduces to the BBGKY hierarchy (e.g. Saslaw 1985; Landau & Lifshitz 1987; Liboff 2003)

\[
\partial_t f_s(1, \cdots, s) + \sum_{i=1}^{s} \left[ v_i \cdot \frac{\partial}{\partial r_i} + \sum_{a 
eq 1} a_{ij} \cdot \frac{\partial}{\partial \mathbf{q}_{ij}} \right] f_s(1, \cdots, s) + \sum_{i=1}^{s} \left[ \frac{\partial}{\partial \mathbf{q}_{ij}} \cdot \int f_{s+1}(1, \cdots, s+1) a_{i,s+1} d(s+1) \right] = 0.
\]

where \(1 \leq s \leq N\). The function \(f_s(1, \cdots, s)\) is the \(s\)-tuple DF, i.e. the probable number density of stars \(1, \cdots, s\) that can be found at phase spaces points \((r_1, p_1), \cdots, (r_s, p_s)\) respectively at time \(t\). To find self-consistent kinetic equation for one-body DF, one necessitates only the first two equations of the hierarchy

\[
(\partial_t + v_1 \cdot \nabla_1) f_1(t, 1) = -\partial_1 \cdot \int f_2(1, 2, t) a_{12} d^2
\]

\[
(\partial_t + v_1 \cdot \nabla_1 + v_2 \cdot \nabla_2 + a_{12} \cdot \partial_{12}) f_2(1, 2, t) = -\int \left[ a_{1,3} \cdot \partial_1 + a_{2,3} \cdot \partial_2 \right] f_3(1, 2, 3, t) d3.
\]

One may rewrite the single, double and triple DFs following Mayer cluster expansion (e.g. Mayer & MG 1940; Green 1956)

\[
f_1(t, 1) = f(1, t),
\]

\[
f_2(t, 1, 2, 3, \cdots, s) = f(1, 2, t) f(s, t) + \left( g(1, 2, t) - \frac{f(1, t) f(2, t)}{N} \right) \left( g(s, t) - \frac{f(s, t)}{N} \right),
\]

\[
f_3(t, 1, 2, 3) = f(1, 2, t) f(3, t) + \left( g(1, 2, t) - \frac{f(1, 2, t)}{N} \right) \left( g(3, t) - \frac{f(3, t)}{N} \right) f(1, t) + \left( g(3, 1, t) - \frac{f(3, 1, t)}{N} \right) f(2, t).
\]

The DFs and correlation functions for stars may depend on the number \(N\) as

\[
f(1, 2, t), f(2, 3, t) \propto N,
\]

\[
g(1, 2, t), g(2, 3, t), g(3, 1, t) \propto N(N - 1),
\]

\[
G \propto 1/N
\]

\[
m, v, r, A \propto 1
\]

where the normalization condition for DFs and correlation functions follows (Liboff 1966) and the scaling for physical quantities follows (Chavanis 2013b; Ito 2018). It is to be noted that the correlation function formulation (especially for weakly-couple systems) needs the anti-normalization condition (Liboff 1965)

\[
\int g(1, 2, t) d1 = 0 = \int g(1, 2, t) d2
\]

If the effect of the gravitational polarization is neglected, one obtains the two equations for DF \(f(1, t)\) and correlation function \(g(1, 2, t)\) that should compose the g-Landau equation (See e.g. Chavanis 2013b)

\[
(\partial_t + v_1 \cdot \nabla_1 + A_1 \cdot \partial_1) f(1, t) = -\partial_1 \cdot \int g(1, 2, t) \tilde{a}_{12} d^2,
\]

\[
(\partial_t + v_1 \cdot \nabla_1 + v_2 \cdot \nabla_2 + A_1 \cdot \partial_1 + A_2 \cdot \partial_2) g(1, 2, t) = -\int \left[ \tilde{a}_{12} \cdot \partial_1 + \tilde{a}_{21} \cdot \partial_2 \right] f(1, t) f(2, t).
\]

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where the unperturbed trajectories of stars 1 and 2 in Langrangian description are

\[ r_i(t) = r_i(t - \tau) + \int_{t-\tau}^{t} v_i(t') \, dt', \quad (i \neq j = 1, 2) \]

(2.9a)

\[ v_i(t) = v_i(t - \tau) + \int_{t-\tau}^{t} A_i(t') \, dt'. \]

(2.9b)

or follows the one-body Hamiltonian

\[ h(r_i, p_i) = \frac{p_i^2}{2m} + \Phi(r_i, t) \quad (i = 1 \text{ or } 2) \]

(2.10)

### 3 Importance of Generalized Landau Kinetic Equation for Evolution of Finite Star Clusters

In the present section, it is shown that the g-Landau equation is a correct approximation of (Gilbert 1968)’s equation from a point of view of statistical mechanics and the other existing associated equations (inhomogeneous- Landau and Balescu-Leandrd equations) are incorrect. This can be discussed based on kinetics in phase space \((r, p)\) before one executes the orbit-averaging of the BBGKY hierarchy.

#### 3.1 Conservation of total-energy and number

Excluding the effect of strong-close encounters, the star-cluster kinetic equation derived in (Gilbert 1968) is the most accurate description at two-body DF truncation level. One needs a fundamental criterion to approximate the equation and the present section relies on the anti-normalization condition. Since the derivation of the (Gilbert 1968)’s equation is based on the first two-equation of the BBGKY hierarchy with three body DF neglecting the ternary correlation function, the definition of the total number of a system is

\[ N(t) = \int \cdots \int \sum_{i=1}^{N} F_N(1 \cdots N, t) \, d1 \cdots dN = N \int \sum_{i=1}^{N} \left( \frac{p_i^2}{2m} + m \sum_{j=1}^{N} \phi_{ij} \right) F_N(1 \cdots N, t) \, d1 \cdots dN = \int \frac{p_i^2}{2m} f_i(1, t) \, dl + U_{m.f.}(t) + U_{cor}(t). \]

(3.1)

and the total energy is

\[ E(t) = \int \cdots \int \sum_{i=1}^{N} \left( \frac{p_i^2}{2m} + m \sum_{j=1}^{N} \phi_{ij} \right) F_N(1 \cdots N, t) \, d1 \cdots dN = \int \frac{p_i^2}{2m} f_i(1, t) \, dl + U_{m.f.}(t) + U_{cor}(t). \]

(3.2a)

\[ U_{m.f.}(t) = \frac{m}{2} \int \Phi(r_1, t) f(1, t) \, dl, \]

(3.2b)

\[ U_{cor}(t) = \frac{m}{2} \int \phi(1, 2) g(1, 2) \, dl + \int \left( f(1, t) g(23, t) + f(2, t) g(31, t) \right) \, dl. \]

(3.2c)

Since the anti-normalization condition can hold if the equation is correctly expanded, one can retrieve the following basic definition for the total number and energy

\[ N(t) = \int f(1, t), \quad \quad E(t) = \int \frac{p_i^2}{2m} f_i(1, t) \, dl + U_{m.f.} + \int d1 \, g(1, 2, t) \phi(r_{12}). \]

(3.3)

The anti-normalization is in importance to correctly define the DF, total energy and total number of stars at order of \(O(1/N)\) following equation (3.3) in other words, if the condition is not satisfied in formulation, one must prove the conservation of ‘total’ number and energy defined by equations (3.1) and (3.2). (In equation (3.2), even the polarization effect contributes to the total energy!)

One must recognize in equation (3.2c) that use of the anti-normalization condition and neglegence of the polarization effect \(\int \left( f(1, t) g(23, t) + f(2, t) g(31, t) \right) \, dl\) are equivalent, meaning as long as the system satisfies the anti-normalization condition, one may neglect the effect of polarization as an approximation, holding the conservation of total energy as well as that of number.
gravitational statistical acceleration

Table 1. The conservation of the total-energy and number under the assumption of the anti-normalization condition for star-cluster kinetic equations in phase spaces \( (\mathbf{r}, \mathbf{p}) \). The letters (s.a., g.p.) below the kinetic equations are added if they have statistical acceleration (s.a.) and/or gravitational polarization (g.p.). \( \Delta N_g \) is defined as \( \Delta N_g = N \int d^2d^2 \mathbf{d} \left[ g(1,2,t) f(3,1,t) + g(2,3,t) f(1,1,t) + g(3,1,t) f(2,1,t) \right] \) and \( U_p \) is \( U_p = \frac{\phi}{\Delta N} \int f(1) d(1,t) g(2,3,t) + f(2) g(3,1,t) d^2 d^2 \). "hold/unhold" in the anti-normalization means the equation is/is not valid after the integral \( \int d1 \) or \( \int d2 \) over each side of the equation is taken while "satisfy" means the explicit form of a kinetic equation satisfies the anti-normalization condition. The latter is applicable only to homogeneous/local FP equation whose explicit form is known in phase spaces \( (\mathbf{r}, \mathbf{p}) \). The references are related to the derivation of the corresponding equations and the orbit-averaging while only the orbit-averaging of Gilbert’s equation has not been done yet.

| kinetic equations | total number | total energy | anti-normalization in \( (\mathbf{r}, \mathbf{p}) \) | Refs |
|-------------------|--------------|--------------|---------------------------------|------|
| Gilbert’s equation | conserved | conserved | hold | Gilbert (1968), Gilbert (1971) |
| (s.a., g.p.) | | | | |
| g-Landau equation | conserved | conserved | hold | Kandrup (1981a), Kandrup (1981b), Chavanis (2013a), the present work |
| (s.a.,) | | | | |
| inhomogeneous Landau | \( \Delta N_g \) | \( \Delta N_g \) | \( \Delta N_g \) | Polyachenko & Shukhman (1982), Chavanis (2013) |
| (s.a., g.p.) | | | | |
| homogeneous Balescu | \( \Delta N_g \) | \( \Delta N_g \) | \( \Delta N_g \) | Heyvaerts (2010), Chavanis (2012) |
| (s.a.,) | | | | |
| local/homogeneous FP | conserved | undefined | satisfy | Rosenbluth et al. (1957), Renon (1961), Cohn (1979) |

3.2 Conservation of total-energy and number of g-Landau equation

It is convenient to employ the (Gilbert 1968)’s equation to show the conservation of energy and total number and the g-Landau equation is a correct approximation of the equation. The first two equations in the BBGKY hierarchy of (Gilbert 1971) are

\[
\begin{align*}
(\dot{\mathbf{v}}_1 + \mathbf{v}_1 \cdot \nabla_1 + A_1 \cdot \partial) f(1,t) &= -\partial_1 \cdot \int g(1,2,t) a_{12} d^2 d^2, \\
(\dot{\mathbf{v}}_2 + \mathbf{v}_2 \cdot \nabla_2 + A_2 \cdot \partial_2) g(1,2,t) &= \int d^3 g(2,3,t) a_{23} \cdot \partial_1 f(1) + \int d^3 g(1,3,t) a_{23} \cdot \partial_2 f(2) = -[\tilde{a}_{12} + \partial_1 + \tilde{a}_{21} + \partial_2] f(1) f(2,t),
\end{align*}
\]

(3.4a)

(3.4b)

To prove the conservation laws, assume the anti-normalization condition is satisfied for equations (3.4a) and (3.4b). It is, of course, not a simple task to prove the anti-normalization condition itself though, one can confirm that equation (3.4b) is not against the anti-normalization condition by taking the integral \( \int d1 \) or \( \int d2 \) over each side of the equation.

The conservation of total number \( \frac{dN(t)}{dt} = 0 \) can be simply proved by taking the integral \( \int d1 \) over each side of equation (3.4a) based on the definition of total number (3.3). To find the conservation of energy one must find the expression for \( f(1,t) \) employing equations (3.4a) and (3.4b) as follows

\[
\begin{align*}
(\dot{\mathbf{v}}_1 + \mathbf{v}_1 \cdot \nabla_1 + \mathbf{v}_2 \cdot \nabla_2 + A_1 \cdot \partial_1 + A_2 \cdot \partial_2) f(1,t) &= \int d^3 g(2,3,t) a_{13} \cdot \partial_1 f(1) + \int d^3 g(1,3,t) a_{23} \cdot \partial_2 f(2) = -[\tilde{a}_{12} + \tilde{a}_{21} + \partial_1 + \partial_2] f(1) f(2,t) + \int d^3 g(1,3,t) a_{23} \cdot \partial_2 f(2) \\
= -[\tilde{a}_{12} + \tilde{a}_{21} + \partial_1 + \partial_2] f(1) f(2,t) - \partial_1 \cdot \int g(1,3,t) a_{13} d^3 - \partial_2 \cdot \int g(2,3,t) a_{23} d^3 \end{align*}
\]

(3.5)

By multiplying equations (3.4a) and (3.5) by the factor \( \frac{\partial f(t)}{\partial t} \) and the phase \( \phi_{12} \) respectively then by adding the two equations up, one can prove that the total energy can conserve \( \frac{\partial E(t)}{\partial t} = 0 \). In the process above to prove the conservation laws, the terms \( \int d^3 g(2,3,t) a_{13} \cdot \partial_1 f(1) + \int d^3 g(1,3,t) a_{23} \cdot \partial_2 f(2) \) associated with the gravitational polarization do not come into play as expected, hence the same result can be true for the g-Landau equation as well.

On one hand the inhomogeneous Balescu-Lenard equation (Heyvaerts 2010) and inhomogeneous Landau equations (Polyachenko & Shukhman 1982; Luciani & Pellat 1987) do not hold the anti-normalization condition as shown in Appendix B2 since one can not correctly employ the condition for the discreteness fluctuation, equation (1.7). Hence, for those kinetic equations, one must prove the conservation of energy and number following equation (3.1) and (3.2), if possible. Table 1 shows the summary of the conservation laws for the Gilbert’s, g-Landau, inhomogeneous Balescu-Lenard, inhomogeneous-Landau equations. As the item goes down, one needs stronger deviation from the correct definition of total-energy and number. As a reference, the most fundamental star-cluster kinetic equation, i.e. homogeneous FP (Landau) kinetic equation is also listed; the total energy, of course, can not be defined correctly while the total number of stars may be correctly defined since the equation satisfies the anti-normalization even at explicit form level (See Appendix B1). The result of section states, only the g-Landau equation among the existing star-cluster kinetic equations is a correct approximation of the Gilbert’s equation under the assumption of the anti-normalization condition in sense that DF, total energy and number of stars are correctly defined at order of \( \Theta(1) \) (compared to \( E \sim N \sim f(1,t) \sim \Theta(N) \)).
4 THE EXPLICIT FORM OF ORBIT-AVERAGED GENERALIZED LANDAU EQUATION IN TERMS OF ACTION VECTOR

The goal of the present section is to find the explicit form of the orbit-averaged generalized Landau equation.

4.1 BBGKY hierarchy in the phase space \((ω,I)\) and orbit-averaging

In secular evolution of a cluster, the DF of stars may be considered at quasi-stationary state on time-scale of dynamical time, which is mathematically described by a time-independent Boltzmann-collisionless (Vlasov) equation

\[ (\mathbf{v}_1 \cdot \nabla_1 + A_1 \cdot \partial) f(1,t) = [f(1,t), h]_{(r,v)} = 0 \quad (t \sim t_{\text{dyn}}, \text{ or, } N \to \infty), \]

where the Poisson bracket is

\[ [A,B]_{(r,v)} = \frac{\partial A}{\partial r} \cdot \frac{\partial B}{\partial v} - \frac{\partial A}{\partial v} \cdot \frac{\partial B}{\partial r}. \]

If the orbits of stars are arguably regular and the m.f. potential is unchanged with time, the strong Jeans theorem (Binney & Tremaine 2011) states that the DF is a function of the action vectors \(I_1 \equiv (I_{1x}, I_{1y}, I_{1z})\)

\[ f(1,t) = f(I_1) \quad (t \sim t_{\text{dyn}}, \text{ or, } N \to \infty), \]

Also, the Hamiltonian is assumed integrable, one has the Hamiltonian equations, under canonical transform to the action-angle variables, read

\[ \frac{d\omega}{dt} = -\frac{\partial h}{\partial I} \equiv \Omega(I) \quad \text{and} \quad \frac{dI}{dt} = \frac{\partial h}{\partial \omega} = 0 \]

that is

\[ \omega = \Omega + \omega_0 \quad \text{and} \quad h = h(I) \]

where the following conservation of the Hamiltonian can hold

\[ \frac{dh}{dt} = 0. \]

On time scale of secular evolution, the discussion above must be modified due to the finite-\(N\) effect of the system, for example, one needs to consider the action vector depends on time and DF depends on the angle too

\[ f(1,t) \approx f(I_1(t),t) + \frac{f(I_1(t),\omega_1,t,t/N)}{N} + \mathcal{O}\left(\frac{1}{N^2}\right) \quad (t \sim t_{\text{rel}}, \text{ or, } N \text{ is finite.}), \]

It is, however, customary to assume the unperturbed orbit of test star is still determined by time-independent regular potential even in relaxation evolution against the correct treatment (e.g. Hénon 1961), then one may consider that the one-body Hamiltonian \(h(I_1)\) is still integrable and time independent. Hence one typically considers equations (4.3) - (4.6) can hold for the secular evolution (e.g. Polyachenko & Shukhman 1982; Heyvaerts 2010). This mathematically profits us to avoid the nonlinearity in the trajectory of stars, equation(2.9) while a ‘cost’ of use of the action-angle approach is to give up a the self-consistency in energy conservation i.e. equation (3.3) due to equation (4.6). Also, the ‘target’ of the action-angle-variables approach is limited to the motion of stars only on scales larger than the ‘encounter radius (Ogorodnikov 1965)’ or ‘Boltzmann-Grad radius (Ito 2018)’ on which the m.f. potential dominate the motions (orbits) of stars so the periodicity of motion can be expected.

Assume in secular evolution that the one-body Hamiltonian \(h(I)\) should be autonomous, then the first two equations of BBGKY hierarchy may also be rewritten in term of action-angle variables employing the invariance of Poisson bracket under the canonical transformation

\[ \partial_t f(1,t) + [f(1,t), h(1)]_{(\omega_1,I_1)} = \int [\phi_{12}, g(1,2,t)]_{(\omega_1,I_1)} \, d\Omega_1 \]

\[ \partial_t g(1,2,t) + [g(1,2,t), h(1)]_{(\omega_2,I_1)} + [g(1,2,t), h(1)]_{(\omega_1,I_2)} = \int [\phi_{13}, f(1,t)]_{(\omega_1,I_1)} \left( \delta(2 - 3) - \frac{f(3,t)}{N} \right) \, d\Omega_2 \]

If one assumes that the DF \(f(1,t)\) depends only on the action vector (strong Jeans theorem) and take the orbit-average \(\int d\omega_1\) over the first equation (4.8a), then the corresponding two equation read

\[ \partial_t f(I_1,t) = C(f(I_1,t)) \]

\[ \partial_t g(1,2,t) + \Omega_2 \cdot \frac{\partial g(1,2,t)}{\partial \omega_2} + \Omega_1 \cdot \frac{\partial g(1,2,t)}{\partial \omega_1} + \int \frac{\partial \phi_{12}}{\partial \omega_1} \cdot \frac{\partial f(I_1,t)}{\partial I_1} \left( \delta(2 - 3) - \frac{f(3,t)}{N} \right) \, d\Omega_2 \]

where the conservation of phase space volume elements \((d2 = dI_2 d\omega_2)\) are employed. The collision term \(C(f(I_1,t))\) is defined as

\[ C(f(I_1,t)) = \int \frac{d2d\omega_1}{8\pi^3} \left( \frac{\partial \phi_{12}}{\partial \omega_1} \cdot \frac{\partial g(1,2,t)}{\partial I_1} - \frac{\partial \phi_{12}}{\partial I_1} \cdot \frac{\partial g(1,2,t)}{\partial \omega_1} \right) . \]
4.2 The explicit form of collision terms for orbit-averaged

To find a self-consistent kinetic equation, the second equation of the BBGKY hierarchy, equation (4.9b), may be formally solved by the method of characteristics

\[ g(1, 2, t) = g_a(1, 2, t) + g_\lambda(1, 2, t) \]  

(4.11a)

\[ g_a(1, 2, t) = \int_{t - \tau}^{t} dt' \int d^3 \phi(t', t) \frac{\partial}{\partial t} \delta(2 - 3f(I_2, t)) + (1 \leftrightarrow 2) \]  

(4.11b)

\[ g_\lambda(1, 2, t) = \int_{t - \tau}^{t} dt' \int d^3 \phi(t', t) \left[ -\frac{f(I_3, t)}{N} \right] f(I_2, t) + (1 \leftrightarrow 2) \]  

(4.11c)

Since the explicit form of the correlation function \( g_a(1, 2, t) \) has been shown in (Polyachenko & Shukhman 1982)

\[ g_a(1, 2, t) = \pi \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} \lambda_{n_1, n_2}(I_1, I_2) \delta(n_1 \cdot \Omega_1 - n_2 \cdot \Omega_2)e^{i(n_1 \cdot \omega_1 - n_2 \cdot \omega_2)} \left[ n_1 \cdot \frac{\partial}{\partial I_1} - n_2 \cdot \frac{\partial}{\partial I_2} \right] f(I_1, t)f(I_2, t), \]  

(4.12)

the focus here is the correlation function \( g_\lambda(1, 2, t) \). One may Fourier-transform the potentials \( \phi_{12} \) and \( \phi_{23} \) in terms of the angle variables

\[ \phi_{12} = \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} \lambda_{n_1, n_2}(I_1, I_2) e^{i(n_1 \cdot \omega_1 - n_2 \cdot \omega_2)} \quad (i = 2 \text{ or } 3) \]  

(4.13a)

\[ A_{n_1, n_2}(I_1, I_2) = \frac{1}{(8\pi^3)^2} \int d\omega_1 d\omega_2 \phi_{12}(1, 3)e^{-i(n_1 \cdot \omega_1 - n_2 \cdot \omega_2)} \]  

(4.13b)

where the summation \( \sum_{n = -\infty}^{\infty} \) are taken for all combinations of number vector \( (n_1, n_2, n_1, n_2) \). The conjugate \( A_{n_1, n_2}^{\ast} \) of the Fourier coefficient \( A_{n_1, n_2} \) has the following obvious property

\[ A_{-n_1, -n_2}(I_1, I_2) = [A_{n_1, n_2}(I_1, I_2)]^{\ast} \]  

(4.14)

Employing the inverse Fourier transform (4.13b), the equation (4.11c) reduces to

\[ g_\lambda(1, 2, t) = -\frac{8\pi^3}{N} \int_{t - \tau}^{t} dt' \int d^3 I_3 \sum_{n = -\infty}^{\infty} \left( A_{n_0}(I_1, I_3)e^{in \cdot \omega_1(t')} n \cdot \frac{\partial}{\partial I_1} + A_{n_0}(I_2, I_3)e^{in \cdot \omega_2(t')} n \cdot \frac{\partial}{\partial I_2} \right) f(I_1, t')f(I_2, t')f(I_3, t) \]  

(4.15)

where the Kronecker delta function is employed

\[ \delta(n_0, 0) = \frac{1}{8\pi^3} \int d\omega_1 e^{-i(n_0 \cdot \omega)} \]  

(4.16)

Use of the Hamiltonian equations (4.5)

\[ g_\lambda(1, 2, t) = -\frac{8\pi^3}{N} \int_{0}^{t} dt' \int d^3 I_3 \sum_{n = -\infty}^{\infty} \sum_{m = -\infty}^{\infty} \left( A_{n_0}(I_m, I_3)e^{in \cdot \omega_m - \Omega_m(t')} n \cdot \frac{\partial}{\partial I_m} \right) f(I_1, t - \tau)f(I_2, t - \tau)f(I_3, t) \]  

(4.17)

Assuming the Markovian approximation for the DF and take the limit \( \tau \to \infty \), one obtains

\[ g_\lambda(1, 2, t) = -\frac{8\pi^4}{N} \int_{0}^{\infty} dt' \int d^3 I_3 \left( \int f(I_3, t)A_{n_0}(I_m, I_3) \right) \delta(n \cdot \Omega_m)e^{in \cdot \omega_m} n \cdot \frac{\partial}{\partial I_m} f(I_1, t)f(I_2, t) \]  

(4.18)

where the following identity is employed

\[ \sum_{n = -\infty}^{\infty} \int_{0}^{\infty} dt' e^{-in \cdot \Omega(t')} = \frac{\pi}{\Omega} \sum_{n = -\infty}^{\infty} \delta(n \cdot \Omega) \]  

(4.19)

It is convenient to separate the correlation function \( g_\lambda(1, 2, t) \), according to the angle-vector dependence, into the following functions

\[ g_\lambda(1, 2, t) = g_{A1}(I_1, I_2, \omega_1, t) + g_{A2}(I_1, I_2, \omega_2, t) \]  

(4.20)

Accordingly one may evaluate the collision term \( C(f(I_1, t)) \), equation (4.10), separately to be done in sections 4.2.1 and 4.2.2 respectively.

4.2.1 Collision term due to the correlation function \( g_{A1} \)

The collision term due to the correlation function \( g_{A1}(I_1, I_2, \omega_1, t) \) reads

\[ C_{A1}(f(I_1, t)) = \int d^3 I_1 A_{n_0}(I_1, I_2)f(I_2, t) \left[ \frac{n \cdot \Omega_1}{N} \frac{\partial f(I_1, t)}{\partial I_1} \right] \]  

(4.21)

Employing equations (4.13a), (4.14), (4.16) and (4.11c), one obtains after simple operations

\[ C_{A1}(f(I_1, t)) = -\frac{\pi(8\pi^3)^2}{N} \sum_{n = -\infty}^{\infty} n \cdot \frac{\partial}{\partial I_1} \times \left( \int d^3 I_1 A_{n_0}(I_1, I_2)f(I_2, t) \right) \left[ \frac{n \cdot \Omega_1}{N} \frac{\partial f(I_1, t)}{\partial I_1} \right] \]  

(4.22)
4.2.2 Collision term due to the correlation function $g_{A2}$

Since the correlation function $g_{A2}$ does not depend on the angle vector $\omega_1$, the corresponding collision term due to the correlation function $g_{A2}$ reduces to

$$C_{A2}(f(I_1,t)) \equiv \int \frac{d^2\omega_1}{8\pi^3} \frac{\partial \delta_{12}}{\partial I_1} \cdot \frac{\partial}{\partial I_1} g_{A2}(I_1, I_2, \omega_2, t),$$

(4.23)

Again after employing equations (4.13a), (4.16) and (4.11c), the collision term $C_{A2}(f(I_1,t))$ vanishes as follows

$$C_{A2}(f(I_1,t)) = \frac{\pi (8\pi^3)^2}{N} \sum_{n=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \int dI_2 \int dI_3 A_{n_1 n_2} (I_1, I_2, n_1, n_2) \delta (n_1 - n_1) \cdot \frac{\partial}{\partial I_1} \left( A_{n_0} (I_2, I_3) f(I_3, t) \delta (n_1 - n_2) n \cdot \frac{\partial f(I_1, t)}{\partial I_1} f(I_2, t) \right) = 0$$

(4.24)

4.3 The explicit form of orbit-averaged g-Landau equation

Since the collision term due to the correlation function $g_{a}(1,2, t)$ has the following form as discussed in (Polyachenko & Shukhman 1982)

$$C_{a}(f(I_1,t)) = \frac{\pi (8\pi^3)^2}{N} \sum_{n=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} \int dI_2 n \cdot \frac{\partial}{\partial I_1} \cdot \left( A_{nn} (I_1, I_2)^2 \delta (n \cdot \Omega_1 - n_1 \cdot \Omega_2) \left( n \cdot \frac{\partial}{\partial I_1} - n_1 \cdot \frac{\partial}{\partial I_2} \right) f(I_1, t) f(I_2, t) \right)$$

(4.25)

The final form of the orbit-averaged g-Landau kinetic equation in action vector spaces is

$$\frac{d}{dt} f(I_1, t) = C_{a}(f(I_1,t)) + C_{A1}(f(I_1, t))$$

(4.26)

The newly derived collision term, $C_{A1}(f(I_1,t))$, has a negative sign that brings one into thought of a ‘de-relaxation’ effect. Yet, the fundamental relaxation process should be characterized by the square of the fluctuation in statistical acceleration, equation (1.6). Hence, it is more straightforward to combine the two collision terms in a more convenient form to extract the fluctuation effect. To do so, separate the terms into two associated with finite-scale effect ($n_1 \neq 0$) and infinitely-large-scale one($n_1 \equiv 0$)

$$\frac{d}{dt} f(I_1, t) = \pi (8\pi^3)^2 \sum_{n=-\infty}^{\infty} \int dI_2 n \cdot \frac{\partial}{\partial I_1} \cdot \left( A_{nn} (I_1, I_2)^2 \delta (n \cdot \Omega_1 - n_1 \cdot \Omega_2) \left( n \cdot \frac{\partial}{\partial I_1} - n_1 \cdot \frac{\partial}{\partial I_2} \right) f(I_1, t) f(I_2, t) \right)$$

$$+ \frac{\pi (8\pi^3)^2}{2N} \sum_{n=-\infty}^{\infty} \int dI_2 \int dI_3 \left( A_{n0} (I_1, I_2) - A_{n0} (I_1, I_3) \right) \left( n \cdot \frac{\partial}{\partial I_1} f(I_1, t) f(I_2, t) \right)$$

(4.27)

where, to find the second line of the collision term, the following mathematical identities are employed

$$N = 8\pi^3 \int f(I_1, t) dI_1$$

(4.28a)

$$\int dI_2 \int dI_3 f(I_2, t) f(I_3, t) \left( A_{n0} (I_1, I_2) [A_{n0} (I_1, I_3)]^* - |A_{n0} (I_1, I_2)|^2 \right) = \frac{1}{2} \int dI_2 \int dI_3 f(I_2, t) f(I_3, t) \left( A_{n0} (I_1, I_2) - A_{n0} (I_1, I_3) \right)^2$$

(4.28b)

It is obvious that the orbit-averaged g-Landau equation (4.27) has the Maxwellian DF as an stationary solution

$$f(I_1, t) = \alpha e^{-\beta \Phi (I_1)},$$

(4.29)

where $\alpha$ and $\beta$ are constant. Some basic properties of the orbit-averaged g-Landau equaiton are discussed in section 5.

5 THE BASIC PROPERTIES OF ORBIT-AVERAGED G-LANDAU EQUATION

In order to show the properties of the collision term of the orbit-averaged g-Landau equation (4.27), the basic expression of collision term $C(f(I_1,t))$ may be rewritten as

$$C(f(I_1,t)) = \frac{\partial}{\partial I_1} (\mathcal{F}(f(I_1,t)))$$

(5.1)

where the functional $\mathcal{F}(f(I_1,t))$ means an ‘action flux’ in terms of DF, i.e. a change of DF through a hypersurface of action due to the change in action of star 1 in relaxation process. The boundary conditions of the flux are to be assigned as follows

$$\mathcal{F}(I_{\text{max}}) = 0 = \mathcal{F}(I_{\text{min}})$$

(5.2)

The anti-normalization, the conservation laws and $H$-theorem are discussed in sections 5.1, 5.2 and 5.3 respectively.
5.1 Anti-normalization condition

As explained in section 3, the anti-normalization condition is a fundamental property in defining DF and total-number and energy. The correlation function $g_A(1,2,t)$ in the orbit-averaged Landau kinetic equations in (Polyachenko & Shukhman 1982) does not satisfy the condition as follows

$$\int g_A(1,2,t) d2 = \pi (8\pi^3)^i \sum_{n=0}^\infty \int dI_A \delta(n \cdot \Omega_1) n \frac{\partial}{\partial I_1} f(I_2) \neq 0$$ (5.3)

On one hand, the contribution from the correlation function $g_A(1,2,t)$ can cancel out the residue

$$\int g_A(1,2,t) d2 = - \int g_A(1,2,t) d2.$$ (5.4)

That the orbit-averaged g-Landau equation holds the anti-normalization condition is of importance from the following points of view. First, one still can correctly define the DF of stars in terms of action spaces with orbit-averaging approximation; the failure of the anti-normalization condition prevents one from employing the (one-body) DF itself since the definition of the DF is not correctly made

$$f(1,t) \neq \int \frac{d2}{N-1} f(I_1,t) d1 = \int \left[ f(I_1,t) f(I_2,t) \right] + \frac{1}{N-1} g_A(1,2,t) = f(I_1,t) + \frac{1}{N-1} g_A(1,2,t)$$ (5.5)

Second, the anti-normalization condition allows one to correctly define the effect of ‘discreteness’ fluctuation in acceleration, equation (1.6) in kinetic formulation with the orbit-averaged approximation; without the anti-normalization condition the BBGKY hierarchy only gives one the form of fluctuation $\langle a_{12} \cdot \bar{a}_{12} \rangle$

$$\langle a_{12} \cdot \bar{a}_{12} \rangle \leftrightarrow \int a_{12} \bar{a}_I g(1,2,t; \bar{a}_{12}) d2 \leftrightarrow \int a_{12} \bar{a}_I g(1,2,t; \bar{a}_{12}) d2 \leftrightarrow \langle a_{12} \cdot \bar{a}_{12} \rangle$$ (5.6)

5.2 Conservation of energy and number

Since the anti-normalization condition holds for the orbit-averaged g-Landau equation, as expected, the total number, equation (4.28a) of stars can be conserved

$$\frac{dN(t)}{dt} = \int \frac{\partial}{\partial I_1} \mathcal{F}(f(I_1,t)) \frac{\partial h(I_1)}{\partial I_1} = 0.$$ (5.7)

The total of kinetic energy and (quasi-)static potential energy defined by

$$E_{st}(t) = \int f(I_1,t) h(I_1) d1$$ (5.8)

can also be conserved in relaxation evolution of the cluster

$$\frac{dE(t)_{st}}{dt} = -8\pi^3 \int \mathcal{F}(f(I_1,t)) \frac{\partial h(I_1)}{\partial I_1}$$ (5.9a)

$$= -\pi (8\pi^3)^2 \sum_{n=-\infty}^\infty \sum_{n_1=-\infty}^\infty \int dI_1 \int dI_2 (n \cdot \Omega_1 - n_1 \cdot \Omega_2) \left[ \left| A_{nn}(I_1,I_2) \right|^2 \delta(n \cdot \Omega_1 - n_1 \cdot \Omega_2) \right] n \frac{\partial}{\partial I_1} - n_1 \frac{\partial}{\partial I_2} f(I_1,t) f(I_2,t)$$

$$= -\pi (8\pi^3)^2 \sum_{n=-\infty}^\infty \int dI_1 \int dI_2 \left| A_{nn}(I_1,I_2) \right|^2 f(I_2,t) f(I_1,t) \delta(n \cdot \Omega_1) n \frac{\partial}{\partial I_1} f(I_1,t) = 0$$ (5.9b)

The conservation of total number is an obvious demands from a physical principle even at orbit-averaged-equation level as long as the anti-normalization condition holds while the definition of the total energy $E_{st}$ does not say anything about physically correct arguments since the energy $E_{st}(t)$ merely a result of the assumption that the background potential $\Phi(r_1,t)$ does not change in relaxation (resonance) process, which is rather a necessary assumption to employ the action-angle variable approach for simplification. Yet, the conservation of the energy $E_{st}$ is a fundamental indicator in numerical integration of orbit-averaged FP equations (e.g Cohn 1979; Takahashi 1995).

5.3 H-theorem

Since the DF is correctly defined for the orbit-average g-Landau equation, it is straightforward to examine $H$-theorem for the Boltzmann entropy

$$H(t) = -\int f(I_1,t) \ln[f(I_1,t)] d1$$ (5.10)
One can prove that the $H$-function increases with time as follows
\[
\frac{dH(t)}{dt} = 8\pi^3 \int \mathcal{J}(f(I_1, t)) \cdot \Omega_1 \frac{1}{f(I_1, t)} dI_1
\]
\[
= \frac{\pi(8\pi^3)^2}{2} \sum_{n=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} \int dI_1 \int dI_2 \left| A_{nm}(I_1, I_2) \right|^2 \delta(n \cdot \Omega_1 - n_1 \cdot \Omega_2) \left[ \left( n \cdot \frac{\partial}{\partial I_1} - n_1 \cdot \frac{\partial}{\partial I_2} \right) f(I_1, t) f(I_2, t) \right]^2
\]
\[
+ \frac{\pi(8\pi^3)^3}{2N} \sum_{n=-\infty}^{\infty} \int dI_1 \int dI_2 \int dI_3 \left| A_{nl}(I_1, I_2) - A_{nl}(I_1, I_3) \right|^2 \delta(n \cdot \Omega_1) \left[ n \cdot \frac{\partial}{\partial I_1} f(I_1, t) f(I_2, t) f(I_3, t) \right]^2 \geq 0
\]

6 ANISOTROPIC SPHERICAL SYSTEM

The present section rewrite the orbit-averaged g-Landau equation in action spaces for the evolution of anisotropic spherical systems. One may expect that angular-momentum relaxation due to the statistical acceleration does not occur. This is since, while the orbit-averaged equation does not average the orbit themselves unlike resonance relaxation but the square of fluctuation in acceleration of stars, the m.f. acceleration of stars directed toward radial direction should not contribute to any change in angular momentum of the orbits in making ‘rosette’ trajectory. On one hand, since the ‘centrifugal’ acceleration may contribute to the energy level of orbit, the present section formulates the orbit-averaged equation for anisotropic spherical system based on the formulation of (Polyachenko & Shukhman 1982). Section 6.1 explains the conversion from action spaces to spherical coordinate spaces, section 6.2 rewrites the double coefficients $A_{nl, n_1}(I_1, I_2)$ for spherical coordinates and section 6.3 shows the collision term $C_{A_1}$ for anisotropic spherical system vanishes.

6.1 Action-angle variables and spherical coordinates

To rewrite the orbit-averaged g-Landau equation for anisotropic spherical star clusters, define the following action vectors in spherical coordinates spaces $(r, \theta, \phi)$
\[
I = (I_r, I_\theta, I_\phi) = \left( \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int \int ..
\]
where $L$ is the modulus of the angular momentum of the system, $L_\zeta$ the $\zeta$-component of the angular moment and $v_r$ the radial velocity of test star, i.e.
\[
v_r = \pm \sqrt{2(E(I_r, L) - \Phi(r)) - \frac{L^2}{2r^2}}
\]
and the radial, azimuthal and polar frequencies of orbits are defined as
\[
\Omega = (\Omega_r, \Omega_\theta, \Omega_\phi \text{sgn}(L_z))
\]
where sgn($L_z$) is the sign function of $L_z$. Since the phase space volume element may be rewritten in terms of $(E, L, L_z)$
\[
dI = \frac{\delta(I_r, I_\theta, I_\phi)}{\delta(E, L, L_z)} dEdLdL_z = \frac{1}{\Omega_1} dEdLdL_z
\]
if one assumes the system of concern is a non-rotating anisotropic spherical cluster, one has the total number in terms of the new DF $F(E, L, t)$ as follows
\[
N = \int dI_1 d\omega_1 = \int dE_1 dL_1 F(E, L, t)
\]
\[
F(E, L, t) = \frac{16\pi^3 L}{\Omega_1} f(E, L, t)
\]
and the derivative of any function $\mathcal{J}$ of the arguments $E(= h(I_r, L))$, $L$ and $L_z$ with respect to the action vectors $I$ reads
\[
\frac{\partial}{\partial I} \mathcal{J}(E, L, L_z) = \frac{\partial}{\partial E} \mathcal{J}(E, L, L_z) + \left[ 0, \frac{\partial}{\partial L}, \frac{\partial}{\partial L_z} + \text{sgn}(L_z) \frac{\partial}{\partial L} \right] \mathcal{J}(E, L, L_z)
\]
Hence, the orbit-averaged g-Landau kinetic equation in $(E, L)$ spaces is
\[
\frac{\partial F(E_1, L_1, t)}{\partial t} = C_A(f) + C_1(f)
\]
where the collision term \( C_a(F) \) reads

\[
C_a(f) = \frac{16\pi^3 L_1}{\Omega_1} C_a(F)
\]

\[
= \frac{L_1 \pi}{2\Omega_1^2} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=1}^{\infty} \sum_{l=0}^{\infty} \int dE_2 dL_2 dL_{zz} \delta(\Omega_1 \cdot n_1 - \Omega_2 \cdot n_2) |\lambda_{n_1 n_2}(I_1, I_2)|^2
\]

\[
\times \left[ \Omega_1 \cdot n \frac{\partial}{\partial E_1} + \tilde{\eta}_I \frac{\partial}{\partial L_1} - \Omega_2 \cdot n_2 \frac{\partial}{\partial E_2} - \tilde{\eta}_{I_2} \frac{\partial}{\partial L_2} \right] \frac{\Omega_{r_1} \Omega_{r_2}}{L_{r_1} L_{r_2}} F(E_1, L_1, t) F(E_2, L_2, t)
\]

and refer to (Polyachenko & Shukhman 1982) for further discussion. The term of concern in the present section, the collision term \( C_{A1} \), reduces to

\[
C_{A1}(f(I_1, t)) = \frac{16\pi^3 L_1}{\Omega_1} C_{A1}(F)
\]

\[
= \frac{-2\pi(8\pi^3)^2 L_1}{32\Omega_1^2} \sum_{n=0}^{\infty} n \frac{\partial}{\partial I_1} \left( \int dE_1 A_0(I_1, I_2) f(I_2, t) \right)^2 \delta(n \cdot \Omega_1) \frac{\partial f(I_1, t)}{\partial I_1}.
\]

6.2 Fourier coefficients in terms of spherical coordinate variables

The explicit expression for the Fourier coefficients are given as

\[
A_{n_1 n_2}(I_1, I_2) = \sum_{l=0}^{\infty} \int_0^{2\pi} d\phi_1 \int_0^\pi d\delta_1 \cos(\delta_1) \frac{\partial}{\partial I_1} \left( \int_0^\pi d\delta_2 \right) 2 \eta_{I_2}^2 \delta(\eta_{I_2}) \delta(\eta_{I_1}) \delta(n_2 \cdot \Omega_2) \delta(n_1 \cdot \Omega_1) \frac{1}{L_{r_1} L_{r_2}} F_0(I_1, r_1, r_2)
\]

\[
= \frac{1}{\sqrt{\pi^2}} \int d\omega_1 \int d\omega_2 e^{-i\omega_1 n_1 \cdot \Omega_1} e^{i\omega_2 n_2 \cdot \Omega_2} F_0(I_1, r_1, r_2)
\]

\[
F_0(I_1, r_1, r_2) = (2l + 1) \int_0^\pi J_{l+1/2}(\omega_1) J_{l+1/2}(\omega_2) \sqrt{\omega_1} \sqrt{\omega_2}
\]

where the function \( p_{l,m,n}(z) \) is the special function defined by the following Rodrigues formula

\[
p_{l,m,n}(z) = \frac{(-1)^l n^l}{2^l(2m)!} \frac{(l+m)!}{(l-n)(l+n)!} \frac{(1-z)^{l-m}}{(1+z)^{l+m}}
\]

and the function is associated with the Legendre- and associated-Legendre- functions as follows

\[
P_{l,m,n}(z) = \frac{p_{l,m,n}(z)}{p_{l,0,0}(z)}
\]

\[
P_{l,m,n}(z) = \frac{p_{l,m,n}(z)}{p_{l,0,0}(z)}
\]

6.3 The disappearance of the collision term \( C_A(F) \)

The summation with respect to the number vector \( n_2 \) is zero in the collision term \( C_A(F) \), one obtains

\[
A_{n_1 n_0}(I_1, I_2) = \sum_{l=0}^{\infty} \int d\omega_1 \int d\omega_2 e^{-i\omega_1 n_1 \cdot \Omega_1} e^{i\omega_2 n_0 \cdot \Omega_2} \delta(\eta_{I_1}) \delta(\eta_{I_2}) \frac{1}{L_{r_1} L_{r_2}} F_0(I_1, r_1, r_2)
\]

\[
= \int dL_2 \int dL_{zz} \frac{F(E_2, L_2, t)}{16\pi^2 L_2} A_{n_0}(I_1, I_2) \delta(n_2 \cdot \Omega_2)
\]

Since the coefficients \( A_{n_0}(I_1, I_2) \) is proportional to the Legendre function \( P_l(\sin\gamma_2) \), one can extract the factors associated with the momentum \( L_{zz} \) and the resonance conditions \( \delta(n_1 \cdot \Omega_1) \)

\[
K \propto \int dL_{zz} P_l(\sin\gamma_2) = L \int_0^1 d(\sin\gamma_2) P_l(\sin\gamma_2) = 2\lambda_{I_0} \delta(n_1 \cdot \Omega_1)
\]

Since the number \( \tilde{\eta}_I \) is defined on the interval \(-I \leq \tilde{\eta}_I \leq 1\), for \( l = 0 \) one always obtains \( \tilde{\eta}_I = 0 \). This implies the orbit must take ‘ground state’ i.e. circular orbit to satisfy the comensurability condition holding \( \tilde{\eta}_I = 0 \). Yet, even the circular orbit is realized due to the boundary condition of the DF, the collision term \( C_{A1} \) vanishes in sense of distribution.
The mathematical structure of the double Fourier coefficients \( A_{n0}(I_1, I_2) \) obviously shows that if the system has \( z \)-component of angular momentum, the value of the number \( l \) does not have to be 0. This implies that the effect of statistical acceleration may be of importance for (non-)spherical star clusters with internal rotational motion. In stellar systems, internal rotational motion of stars has been ubiquitously observed in globular cluster, young cluster and nuclear star clusters as observation method has been improved. It is of significance to establish a mathematical model to correctly capture the effect of rotation on the evolution. Yet one can not analytically formulate the evolution of rotational systems systematically at the same level done in the present paper for non-rotating system; the discussion for rotating systems is not made here.

7 CONCLUSION

The most fundamental kinetic equation for finite star clusters is (Gilbert 1968)'s kinetic equation. While there exist some approximated forms of the kinetic equation, only the g-Landau kinetic equation correctly approximates the (Gilbert 1968)'s kinetic equation since the g-Landau equation satisfies basic demands from fundamental statistical mechanics (Section 3). The present paper derived the orbit-averaged g-Landau equation in action spaces and spherical coordinates by extending (Polyachenko & Shukhman 1982)'s work. The equations can correctly model the effect of ‘discreteness’ fluctuation on the relaxation process in secular evolution of weakly-coupled dense star clusters.

In section 4, the orbit-averaged g-Landau equation in action spaces was derived. In section 5, it was proved that the equation can conserve the total number of energy and total of kinetic energy of stars and quasi static energy. Also, the \( H \)-theorem and anti-normalization condition were shown. In section 6, the orbit-averaged g-Landau kinetic equation is rewritten in terms of spherical coordinates. It is strictly shown that the statistical acceleration can be replaced by a typical acceleration of star if the system of concern is anisotropic spherical star cluster, on one hand the mathematical structure of the Fourier coefficients implies that the statistical acceleration may be of importance in rotating star clusters.

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APPENDIX A: KLIMONTOVICH'S FORMULATION

APPENDIX B: ANTI-NORMALIZATIONS FOR STAR-CLUSTER KINETIC EQUATIONS

The present Appendix shows that the classical star-cluster kinetic equations (homogeneous- and local- Landau kinetic equations) can weakly satisfy the anti-normalization condition while inhomogeneous Balescu-Lenard equation and its orbit-averaged one can not. One can discuss if condition only from the second equation of the BBGKY hierarchy

\[
\dot{\mathbf{r}}_1 + \mathbf{v}_1 \cdot \nabla_1 + \mathbf{v}_2 \cdot \nabla_2 + \mathbf{A}_1 \cdot \partial_1 + \mathbf{A}_2 \cdot \partial_2 \mathbf{g}(1,2,t) + \int a_{23} g(1,3) d^3 \partial_2 f(2,t) + \int a_{13} g(2,3) d^3 \partial_1 f(1,t) = -[\mathbf{a}_{12} \cdot \partial_1 + \mathbf{a}_{21} \cdot \partial_2] f(1,t)/f(2,t),
\]

(B.1)

Assume that correlation function satisfies the anti-normalization condition and take the integral over the left side of equation (B.1), which results in vanishing the side. The discussion separates for the right side of equation (B.1) if the system is local or homogeneous (Appendix B1) or inhomogenous (Appendix B2)

B1 homogeneous- and local- Landau kinetic equations

Assume the DF for star are local or homogenous

\[
f(1,t)f(2,t) = \begin{cases} \rho^2 \chi(p_1,t) \chi(p_2,t) & \text{(homogeneous)} \\ f(r_1,p_1,t)f(r_2,p_2,t) & \text{(local)} \end{cases}
\]

(B.2)

where \( \chi \) is any function of momentum vector and time. Such system corresponds with the fundamental approximation for the encounter of stars in two-body relaxation evolution (Hénon 1965; Chavanis 2013b). One can easily confirm that the integral of the right-hand side of equation (B.1) over phase-space volume for star 2 vanishes

\[
\int a_{12} dr_2 \int f(1,t)f(2,t) dp_2 = 0
\]

(B.3)

since for a local-encounter approximation one can take the approximation \( dr_2 = dr_{12} \). As a matter of fact the explicit form of the correlation function for the Landau collision term with homgeous or local approximation satisfies the anti-normalization condition:

\[
\int g(1,2,t) d^2 \propto \int_0^\infty dr \int dr_2 a_{12}(t-\tau) = \int dr_{12} \int_0^\infty dk \int_0^\infty d\tau \frac{Gmkk}{2\pi^2} e^{-ik\cdot(r_{12}-r_{12}\tau)} = 0
\]

(B.4)

where the trajectory of the relative motion of stars follows the rectilinea motion and the acceleration of star 1 due to the pair-wise Newtonian potential force from star 2 is Fourier-transformed.

B2 inhomogeneous Balescu-Lenard equation and its orbit-averaged one

Approximating the statistical accleration \( \mathbf{a}_{12} \) to typical one \( \mathbf{a}_{12} \) in (B.1) and take the integral \( \int d2 \) over the right hand side of the equation

\[
\int d2 g(1,2,t) = \int A_1 \cdot \partial_1 f(1,t) \neq 0
\]

(B.5)

which can be cancelled out if there exists the term \( A_1 \cdot \partial_1 f(1,t)/N \) in the correlation function. To show the explicit form of the correlation function is slightly complicated as follows.

APPENDIX C: THE PROPERTIES OF THE FOURIER COEFFICIENTS

In the g-Landau kinetic equation, the roles of factor are relatively easy relation in its derivation thanks to the linearity in the angle variables

\[
\int dr \leftrightarrow \delta(n \cdot \Omega_t) \quad \text{Resonance condition}
\]

\[
\frac{\partial}{\partial \omega} \leftrightarrow n \quad \text{angular change}
\]

(C.1a)

(C.1b)

The resonance condition shows the perturbation process due to the two-body relaxation and the angular change measures the 'number of orbits'. Yet, the double-Fourier coefficient is more complicated to understand than the rest of quantities. According to its definition, equation (4.13b), the fundamental mathematical property of the coefficient holds the conditions

\[
A_{nm}(I_1; I_2) = [A_{nm}(I_2; I_1)]^* \quad \text{(self-adjoint)}
\]

\[
A_{-n-m}(I_1; I_2) = [A_{nm}(I_1; I_2)]^* \quad \text{(anti-adjoint)}
\]

(C.2a)

(C.2b)

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The special case of the coefficients are for the m.f. potential
\[
\Phi(I_1, \omega) = 8\pi^3 \sum_{n_1=-\infty}^{\infty} \int dI_2 e^{i n_1 \cdot \omega_1} A_{n_0}(I_1, I_2) f(I_2, t)
\] (C.3a)
\[
A_{n_0}(I_1, I_2) = \frac{1}{8\pi} \int d\omega_1 \phi_{12} e^{-i n_1 \cdot \omega_1}
\] (C.3b)
where the second subscript '0' in the coefficient takes stands for the effect of the m.f. potential and the orbit-averaged pair-wise potential is defined as
\[
\phi_{12} = \frac{1}{8\pi^2} \int \phi_{12} d\omega_1
\] (C.4)

To further understand the structure of the Fourier coefficients, one may consider the effect of the 'discreteness' fluctuation in potential
\[
\hat{h} = \phi_{12}(1, 2) - \frac{1}{N} \Phi(1, t)
\] (C.5)
and assume test star follows the Hamiltonian equations
\[
\frac{dI_1}{dt} = \frac{\partial (h + \hat{h})}{\partial \omega_1} = -\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} A_{n_1 n_2}(I_1, I_2) i n_1 \cdot \omega_1 e^{i (n_1 \cdot \omega_1 - n_2 \cdot \omega_1)}
\] (C.6a)
\[
\frac{d\omega_1}{dt} = \frac{\partial (h + \hat{h})}{\partial I_1} = \Omega_1 + \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \frac{\partial}{\partial I_1} A_{n_1 n_2}(I_1, I_2) e^{i (n_1 \cdot \omega_1 - n_2 \cdot \omega_1)}
\] (C.6b)

The Hamiltonian equations imply that the change in integrals \(I_1\) depends on the 'number' of orbits \(n_1\) while that in the angles on the Fourier coefficients. This may encourage one to take a constraint on the relaxation process to understand the properties of the Fourier coefficient at kinetic-equation level. The following two cases are considered: (i) the discreteness fluctuation \(\hat{h}\) vanishes in section C1 and (ii) either or both of action vectors and angle variables does not change in relaxation process in section C2.

C1 'discreteness fluctuation' and fine DF

The orbit-averaged squares of m.f.- and pair-wise- potentials may be expressed as
\[
\frac{1}{8\pi^3} \int d\omega_1 |\Phi(1, t)|^2 = 8\pi^3 \sum_{n_1=-\infty}^{\infty} \left| \int dI_2 A_{n_0}(I_1, I_2) \right|^2
\] (C.7a)
\[
\frac{1}{8\pi^3} \int d\omega_1 |\phi(1, 2)|^2 = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \left| A_{n_1 n_2}(I_1, I_2) \right|^2
\] (C.7b)

Employing the squares, one can simply shows the dependence of the collision term \(C(f(I_1, t))\) on the Fourier coefficients. To show the relation between the dispersion of the discreteness fluctuation and the g-Landau collision term, one may define the mean square discrete fluctuation in pair-wise Newtonian potential
\[
\langle \delta_{12}^2 \rangle = \frac{1}{N} \left( \int |f(1, t)| |\phi_{12}|^2 d^2 - \frac{1}{N} |\Phi(1, t)|^2 \right)
\] (C.8)

The orbit-averaged potential dispersion can reproduce the basic mathematical structure except for the resonance condition and the 'angular forces'
\[
\frac{1}{8\pi^3} \int d\omega_1 \langle \delta_{12}^2 \rangle = \left(8\pi^3\right) \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \int dI_2 \left| A_{n_1 n_2}(I_1, I_2) \right|^2 f(I_2, t)
\]
\[
+ \frac{(8\pi^3)^2}{2N} \sum_{n_1=-\infty}^{\infty} \int dI_2 \int dI_3 \left| A_{n_0 n_1 n_2}(I_1, I_2, I_3) - A_{n_0 n_1}(I_1, I_3) \right|^2 f(I_2, t) f(I_3, t)
\] (C.9)

This result merely re-stresses that the resonant relaxation itself does not occur if no discreteness fluctuation occurs i.e. the stars has the fine (discrete) DF
\[
f(1, t) = \pi^3 \delta(I - I(t)) \quad I_1 = \oint p(q(t), t) \cdot dq(t)
\] (C.10)

The condition for the relation of the fine structure is more explicitly derived in terms of the Fourier coefficients. One may consider that the effect of m.f. potential does not directly appear i.e. the second line in equation (C.9) vanishes, which implies the condition
\[
A_{n_0}(I_1, I_2) = A_{n_0}(I_1, I_3) \equiv A_{n_0}(I_1) \quad \text{for any } I_2 \text{ and } I_3
\] (C.11)
meaning from this condition one retrieve the m.f. potential due to the orbit-averaged potentials from the fine DF

\[
\Phi(I_1, \omega) = N \sum_{n_1 = -\infty}^{\infty} e^{i n_1 \cdot \omega_1} A_{n_1,0}(I_1) = N \phi_{12}(I_1, \omega_1)
\]

\[
A_{n_1,0}(I_1) = \frac{1}{8\pi^2} \int d\omega_1 \phi_{12}(I_1, \omega_1) e^{-i n_1 \cdot \omega_1}
\]

(C.12a)

(C.12b)

C2 Conservation of action-angle variables and homogeneous approximation

C2.1 Conservation of integrals

Beginning with the conservation of the total integral, standard local Landau collision term can be derived. The collision term in action-angle variables takes equation (4.10) and the total integrals of the collision term are

\[
\int dI f(\mathbf{I}, t) = \int dI_1 \int dI_2 \left( \frac{\partial \phi_{12}}{\partial I_1} \frac{\partial g(1,2,t)}{\partial I_1} - \frac{\partial \phi_{12}}{\partial I_2} \frac{\partial g(1,2,t)}{\partial I_2} \right)
\]

By employing equation (4.13a), one can rewrite (C.13) as follows

\[
\int dI f(\mathbf{I}, t) = -\frac{1}{2} \int dI_1 \int dI_2 \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} g(1,2,t) \delta(n_1 - n_2) A_{n_1,n_2}(I_1, I_2) e^{i n_1 \cdot \omega_1 - n_2 \cdot \omega_2)
\]

An obvious condition to hold the total integrals is

\[
n_1 = n_2
\]

(C.15)

This condition merely assign an extra condition on equation (4.13a) that the potential \( \phi_{12} \) is a function of \( \omega_1 - \omega_2 \) and so one may Fourier-transform the potential as follows

\[
\phi_{1i} = \sum_{n_1 = -\infty}^{\infty} A_{n_1,n_1}(I_1, I_i) e^{i n_1 \cdot (\omega_1 - \omega_i)} \quad (i = 2 \text{ or } 3)
\]

(C.16a)

\[
A_{n_1,n_1}(I_1, I_i) = \frac{1}{8\pi^2} \int d\omega_1 \phi_{11}(1,1) e^{-i n_1 \cdot (\omega_1 - \omega_i)}
\]

(C.16b)

\[
\Phi(I_1, \omega_1) = 0
\]

(C.16c)

Now, one can apply the new Fourier expansion to the condition to the orbit-averaged g-Landau kinetic equation, then one obtains

\[
\partial_t f(I_1, t) = \pi (8\pi^3) \sum_{n_1 = -\infty}^{\infty} \int dI_2 n \cdot \frac{\partial}{\partial I_1} \left( |A_{n_1,n_2}(I_1, I_2)|^2 \delta(n \cdot [\Omega_1 - \Omega_2]) n \cdot \left( \frac{\partial}{\partial I_1} - \frac{\partial}{\partial I_2} \right) f(I_1, t) f(I_2, t) \right)
\]

(C.17)

C2.2 Conservation of angles

In a similar way to the total integrals, one may consider the conservation of the total angles

\[
\int d\omega_1 f(\mathbf{I}, t) = \int dI_1 \frac{\partial \phi_{12}}{\partial I_1} g(1,2,t) = \int dI_1 \frac{\partial \phi_{12}}{\partial I_2} g(1,2,t)
\]

Hence, if imposing the conservation of the total angles, one can have the conditions

\[
\frac{\partial}{\partial I_1} A_{n_1,n_2}(I_1, I_2) = 0 \quad \text{or} \quad A_{n_1,n_2}(I_1, I_2) = A_{n_1,n_2}
\]

(C.19)

The corresponding pair-wise- and m.f.- potentials are

\[
\phi_{1i} = \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} A_{n_1,n_1} e^{i n_1 \cdot (\omega_1 - \omega_i)} \quad (i = 2 \text{ or } 3)
\]

(C.20a)

\[
A_{n_1,n_1} = \frac{1}{(8\pi^3)\pi} \int d\omega_1 \omega_1 \phi_{11}(1,1) e^{-i n_1 \cdot (\omega_1 - \omega_i)}
\]

(C.20b)

\[
\Phi(\omega_1) = N \sum_{n_1 = -\infty}^{\infty} e^{i n_1 \cdot \omega_1} A_{n_1,0} = N \phi_{12}(\omega_1)
\]

(C.20c)

and the g-Landau equation reduces to

\[
\partial_t f(I_1, t) = 8\pi^3 \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} |A_{n_1}|^2 \int dI_2 n \cdot \left( \frac{\partial}{\partial I_1} - \frac{\partial}{\partial I_2} \right) f(I_1, t) f(I_2, t)
\]

(C.21)
C2.3 the conventional Landau kinetic equation for star in homogeneous background approximation

Holding the both of conservations of the integrals and angles corresponds with typical two-body relaxation in homogeneous background. To understand the statement employ for the corresponding conservation conditions, equations (C.15) and (C.19), to the Hamiltonian equations (C.6a) and (C.6b)

\[
\frac{d I_1}{dt} = - \sum_{n_1 = -\infty}^{\infty} N_{n_1 n_1} i n_1 \cdot \omega_1 e^{i n_1 \cdot (\omega_1 - \omega_2)} \tag{C.22a}
\]

\[
\frac{d \omega_1}{dt} = \Omega_1 \tag{C.22b}
\]

and the corresponding potentials

\[
\phi_{1i} = \sum_{n_1 = -\infty}^{\infty} N_{n_1 n_1} e^{i n_1 \cdot (\omega_1 - \omega_2)} \quad (i = 2 \text{ or } 3) \tag{C.23a}
\]

\[
A_{n_1 n_1} = \frac{1}{(8\pi^3)} \int \delta n_1 \phi_{11}(1,i) e^{-i n_1 \cdot (\omega_1 - \omega_2)} \tag{C.23b}
\]

\[
\Phi(\omega_1) = 0 \tag{C.23c}
\]

Then the potential can be rewritten as

\[
\phi_{12} = - \frac{G m}{|r_1(\omega_1) - r_2(\omega_2)|} = - \frac{G m}{|r_{12}(\omega_1)|} \tag{C.24}
\]

implying one needs only one variable the modulus of the relative displacement vector \(r_{12} = |r_{12}|\) and the corresponding Fourier variable, wavenumber, \(\omega_{12} = |\omega_1|\). Also, due to the absence of m.f. potential the original Hamiltonian \(h\) reduces to

\[
h = \frac{p_1^2}{2m} \tag{C.25}
\]

Hence the corresponding actions and angles read the rectilinear motion

\[
I_1 = p_1^2 = \text{const.} \tag{C.26a}
\]

\[
\Omega_1 = v_1 \tag{C.26b}
\]

\[
\omega_1 = r_1 \tag{C.26c}
\]

As a result one obtains the following equation

\[
\frac{\partial f(p_1,t)}{\partial t} = \pi (8\pi^{-3}) \sum_{n_1 = -\infty}^{\infty} |A_{n_1 n_1}|^2 \int dp_2 n_1 \cdot \frac{\partial}{\partial p_1} \left( \delta(n_1 \cdot [v_1 - v_2]) \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] f(p_1,t) f(p_2,t) \right) \tag{C.27}
\]

Since in physical spaces the background m.f. potential dominates stars’ motions only on scales larger than ’encounter radius’ (Ogorodnikov 1965) or ’Boltzmann-Grad radius’, in order to count the non-periodicity on small scales, one must apply continuous Fourier transform in place of discreteness Fourier series expansion of the potential

\[
\frac{\partial f(p_1,t)}{\partial t} \approx \pi (8\pi^{-3}) \int d n_1 |A_{n_1 n_2}|^2 \int dp_2 n_1 \cdot \frac{\partial}{\partial p_1} \left( \delta(n_1 \cdot [v_1 - v_2]) \left[ \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2} \right] f(p_1,t) f(p_2,t) \right) \tag{C.28}
\]

The explicit form can be found in (e.g. Kandrup 1981a; Chavanis 2013b).

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