FOURTH-ORDER GRAVITY AS THE INFLATIONARY MODEL REVISITED

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Abstract

We revisit the old (fourth-order or quadratically generated) gravity model of Starobinsky in four space-time dimensions, and derive the (inflaton) scalar potential in the equivalent scalar-tensor gravity model. The inflaton scalar potential is used to compute the (CMB) observables of inflation, associated with curvature perturbations (namely, the scalar and tensor spectral indices, and the tensor-to-scalar ratio), including the new next-to-leading-order terms with respect to the inverse number of e-foldings. The results are compared to the recent (WMAP5) experimental bounds. We confirm both mathematical and physical equivalence between $f(R)$ gravity theories and the corresponding scalar-tensor gravity theories.

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1 Introduction

*Inflation* is a proposal (cosmological paradigm) about the existence of a short but fast (exponential, or de-Sitter-type) accelerated grow of the FLRW scale factor $a(t)$ in the early Universe, after the Big-Bang but before the radiation-dominated epoch [1]. It implies

$$\ddot{a}(t) > 0$$  \hspace{1cm} (1.1)

Though the whole idea of inflation remains to be a speculation, there is the significant (indirect) evidence for it. In the first place, it is the correct prediction of CMB fluctuations and large scale structure, in remarkable agreement with the WMAP observations of CMB — see eg., ref. [2]. Inflation can generate irregularities in the Universe that may lead to the formation of structure. The main discriminators among various inflationary models are the *spectral indices* associated with the primordial power spectrum of curvature perturbations [3]. For instance, the on-going PLANCK satellite mission is going to provide tight constraints on the observable spectral indices with the accuracy of under 0.5 percent [4]. Though the basic formulae for the spectral indices in terms of any inflaton potential are well known [3], their dependence upon the e-foldings number can only be computed in a specific inflationary model. Our motivation here is to reconsider primary candidates among the inflationary models, as to whether they can survive precisional tests in a near future.

The excellent model of chaotic inflation was proposed by Starobinsky in 1980 [5]. It is the simplest version of $f(R)$ gravity theories [6], whose extra term beyond the standard Einstein-Hilbert term is *quadratic* in the scalar curvature. The Starobinsky model is reviewed in Sec. 2, where we also argue why the other (Ricci- and Riemann- curvature) terms in the quadratically generated gravitational action are irrelevant to the FLRW dynamics.

Any $f(R)$ gravity model is known to be *mathematically* equivalent to the certain scalar-tensor gravity via a Legendre-Weyl transform [7]. We review that procedure in Sec. 3. However, even in the current literature on the $f(R)$ gravity (see ref. [6] and references therein), its *physical* equivalence to scalar-tensor gravity is put into doubt. As is known in Field Theory, any two field theories, related by a field redefinition or via duality, have *the same* observables. In other words, the field theories that are mathematically equivalent are also physically equivalent. Of course, in specific cases the full equivalence may be very tricky (cf., for instance, the AdS/CFT correspondence), so it still makes sense to calculate the observables in both equivalent theories. The spectral indices (in the leading approximation) of the Starobinsky model were calculated on the $f(R)$ gravity side a long time ago [8]. In this paper we do a calculation on the corresponding scalar-tensor gravity side. We confirm the leading terms found in ref. [8], and calculate the sub-leading corrections to them in Sec. 4, with respect to the inverse number of e-foldings. Checking the physical equivalence (ie. the same spectral indices) is yet another motivation to our paper.

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There is a priori no reason of restricting the gravitational Lagrangian to the standard Einstein-Hilbert term that is linear in the scalar curvature, as long as it does not contradict an experiment. The first attempt of that kind was made by Weyl as early as 1921. Nowadays, there is no doubt that the extra terms of the higher-order in the curvature should appear in the gravitational effective action of any Quantum Theory of Gravity. For instance, they do appear in String Theory — see eg., ref. [9] for a review. Since the scale of inflation is just a few orders less than the Planck scale [3], it is conceivable that the higher-order gravitational terms may be instrumental for inflation. It is already the case in the simplest modified gravity model having only the terms quadratic in the curvature [7].

As is well known, there exist only three independent quadratic curvature invariants, \(R_{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}, R_{\mu\nu}R_{\mu\nu}\) and \(R^2\). In addition, in four space-time dimensions,

\[
\int d^4x \sqrt{-g} \left( R_{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} - 4R_{\mu\nu}R_{\mu\nu} + R^2 \right)
\]  

(2.2)

is topological for any metric, whereas

\[
\int d^4x \sqrt{-g} \left( 3R_{\mu\nu}R_{\mu\nu} - R^2 \right)
\]  

(2.3)

is topological for any FLRW metric. Those combinations do not contribute to the (Friedmann) equation of motion for the scale factor, indicating that the scalar curvature models play the most important role in cosmological dynamics. Hence, the most general gravitational action of the highest order 2 in the curvature, which may be relevant for inflation, is given by

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( 2\Lambda - R + \alpha R^2 \right)
\]  

(2.4)

where we have introduced the cosmological constant \(\Lambda\) and the dimensional parameter \(\alpha \equiv M^{-2}\) of mass dimension \((-2)\). We use the spacetime signature \((+,-,-,-)\) and the units \(\hbar = c = 1\). The Einstein-Hilbert term in eq. (2.4) has the standard normalization with \(\kappa = M_{\text{Pl}}^{-1}\) in terms of the reduced Planck mass \(M_{\text{Pl}}^2 = 8\pi G_N\). The rest of our notation for space-time (Riemann) geometry is the same as in ref. [10].

The model (2.4) is the simplest representative of the Starobinsky models [5]. As was shown in refs. [5, 7], the equations of motion for the action (2.4) have an inflationary solution with \(\alpha \neq 0\) (even when \(\Lambda = 0\)), which is stable provided that \(\alpha > 0\). The stability is confirmed by our method in Sec. 3.

### 3 f(R) gravity and inflaton

The model (2.4) is the simplest particular case of the \(f(R)\) gravity models characterized by an action

\[
S_f = -\frac{1}{2\kappa^2} \int d^4x f(R)
\]  

(3.5)
with some function \( f(R) \) of the scalar curvature. Those models are quite popular in the current literature — see e.g., the recent reviews [6] and the references therein — due to their theoretical applications to inflation and dark energy.

The gravitational equations of motion derived from the action (3.5) read

\[
f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + g_{\mu\nu} \Box f'(R) - \nabla_\mu \nabla_\nu f'(R) = 0 \quad (3.6)
\]

where the primes denote differentiation. Those equations of motion are the 4th-order differential equations with respect to the metric \( g_{\mu\nu} \) (ie. with the higher derivatives). Taking the trace of eq. (3.6) yields

\[
\Box f'(R) + \frac{1}{3} f'(R) R - \frac{2}{3} f(R) = 0 \quad (3.7)
\]

Hence, in contrast to General Relativity having \( f'(R) = \text{const.} \), in f(R) gravity the field \( A = f'(R) \) is dynamical, ie. it represents the independent propagating (scalar) degree of freedom. In terms of the fields \( (g_{\mu\nu}, A) \) the equations of motion are of the 2nd order in the derivatives of the fields.

In fact, any \( f(R) \) gravity is classically (mathematically) equivalent to a scalar-tensor gravity [7]. The equivalence is established by applying a Legendre-Weyl transform. The action (3.5) is equivalent to

\[
S_A = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ AR - Z(A) \right\} \quad (3.8)
\]

where the real scalar \( A(x) \) is related to the scalar curvature \( R \) by the Legendre transformation

\[
R = Z'(A) \quad \text{and} \quad f(R) = RA(R) - Z(A(R)) \quad (3.9)
\]

A Weyl transformation of the metric

\[
g_{\mu\nu}(x) \to \exp \left[ \frac{2\kappa \phi(x)}{\sqrt{6}} \right] g_{\mu\nu}(x) \quad (3.10)
\]

with the arbitrary field parameter \( \phi(x) \) yields

\[
\sqrt{-g} R \to \sqrt{-g} \exp \left[ \frac{2\kappa \phi(x)}{\sqrt{6}} \right] \left\{ R - \sqrt{\frac{6}{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) \kappa - \kappa^2 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\} \quad (3.11)
\]

Hence, when choosing

\[
A(\kappa \phi) = \exp \left[ -\frac{2\kappa \phi(x)}{\sqrt{6}} \right] \quad (3.12)
\]

and ignoring the total derivative, we can rewrite the action (3.8) to the form

\[
S_\phi = \int d^4x \sqrt{-g} \left\{ -\frac{R}{2\kappa^2} + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2\kappa^2} \exp \left[ \frac{4\kappa \phi(x)}{\sqrt{6}} \right] Z(A(\kappa \phi)) \right\} \quad (3.13)
\]

in terms of the physical (and canonically normalized) scalar field \( \phi(x) \).
Equation (3.13) is the standard action of the real dynamical scalar field $\phi(x)$ minimally coupled to Einstein gravity and having the scalar potential

$$V(\phi) = -\frac{M_{\text{Pl}}^2}{2} \exp\left\{ \frac{4\phi}{M_{\text{Pl}}\sqrt{6}} \right\} Z \left( \exp\left[ \frac{-2\phi}{M_{\text{Pl}}\sqrt{6}} \right] \right)$$  \hspace{1cm} (3.14)

We are now going to employ it as the scalar-tensor gravity model of inflation. In order to explicitly derive the inflaton scalar potential (3.14), one has to solve for $R$ in terms of $\phi$ by inverting the relation

$$f'(R) = A(\phi)$$  \hspace{1cm} (3.15)

that follows from eq. (3.9) by differentiation. In the special case of

$$f(R) = R - \frac{2\Lambda}{M^2} - \frac{1}{M^2} R^2$$  \hspace{1cm} (3.16)

we find

$$V(\phi) = \left( \frac{M_{\text{Pl}}^2 M^2}{8} + \tilde{\Lambda} \right) \exp\left\{ \frac{2\sqrt{2}\phi}{M_{\text{Pl}}\sqrt{3}} \right\} - \frac{M_{\text{Pl}}^2 M^2}{4} \exp\left\{ \frac{\sqrt{2}\phi}{M_{\text{Pl}}\sqrt{3}} \right\} + \frac{M_{\text{Pl}}^2 M^2}{8}$$  \hspace{1cm} (3.17)

where the notation $\tilde{\Lambda} = M_{\text{Pl}}^2 \Lambda$ has been introduced. In terms of the new variable and the parameter,

$$y = \sqrt{\frac{2}{3}} \frac{\phi}{M_{\text{Pl}}} \quad \text{and} \quad V_0 = \frac{1}{8} M_{\text{Pl}}^2 M^2$$  \hspace{1cm} (3.18)

respectively, the potential (3.17) reads

$$v(y) = \frac{V(y)}{V_0} = \left( 1 + \frac{\tilde{\Lambda}}{V_0} \right) e^{2y} - 2e^y + 1$$  \hspace{1cm} (3.19)

The scalar potential appears to be bounded from below with the only minimum at $y = 0$ (stability!). It is also sufficiently steep for a slow-roll inflation. It is the last (third) cosmological term on the right-hand-side of eq. (3.17) that dominates in the potential during the slow-roll inflation (when taken alone, it gives rise to a de-Sitter inflationary solution), the second term represents the 1st-order (leading) correction, and the first term is the 2nd-order (subleading) correction. In what follows we ignore $\tilde{\Lambda}$. Then the scalar potential for the slow-roll inflation gets simplified to

$$V(y) = V_0 \left( e^y - 1 \right)^2$$  \hspace{1cm} (3.20)

A graph of the function $v(y) = e^{2y} - 2e^y + 1$ near its minimum $y = 0$ is given in Fig. 1. After a shift $\phi \to \phi + \phi_0$ with $2 \exp\left[ \frac{\sqrt{2} \phi_0}{3 M_{\text{Pl}}} \right] = 1$, the potential (3.20) for the sufficiently negative values of $y$ can be approximated as

$$V_{\text{eff}}(\phi) \approx V_0 \left[ 1 - \exp\left( \frac{\sqrt{2} \phi}{3 M_{\text{Pl}}} \right) \right]$$  \hspace{1cm} (3.21)

\footnote{As is clear from eq. (3.19), the ‘initial’ cosmological term $\tilde{\Lambda}$ is unimportant during the slow-roll inflation. The ratio $\Lambda/V_0$ is also negligible from physical (scale) arguments.}
where we have ignored the subleading contribution. The scalar potential (3.21) is known in the inflationary model building [3]. In our treatment of Sec. 4 we use the potential (3.20).

The \((R + R^2)\) gravity (or Starobinsky) model is known as the excellent model of chaotic inflation in early Universe, and its spectral indices in the leading approximation are also known [8]. In the next Sec. 4 we derive those indices in the dual (scalar-tensor gravity) picture, and calculate the sub-leading terms.

4 Spectral indices

The slow-roll inflation parameters are defined by [3]

\[
\varepsilon(\phi) = \frac{1}{2} M_{\text{Pl}}^2 \left( \frac{V'}{V} \right)^2 \tag{4.22}
\]

and

\[
\eta(\phi) = M_{\text{Pl}}^2 \frac{V''}{V} \tag{4.23}
\]

where the primes denote the derivatives with respect to the inflaton field \(\phi\). A necessary condition for the slow-roll approximation is the smallness of the inflation parameters [3],

\[
\varepsilon(\phi) \ll 1 \quad \text{and} \quad |\eta(\phi)| \ll 1 \tag{4.24}
\]

The first condition implies eq. (1.1), whereas the second condition guarantees that inflation lasts long enough, via domination of the friction term in the inflaton equation of motion (in the slow-roll case):

\[
3H \dot{\phi} = -V' \tag{4.25}
\]

Here \(H\) stands for the Hubble ‘constant’ \(H(t) = \dot{a} / a\). Equation (4.25) is to be supplement by the Friedmann equation

\[
H^2 = \frac{V}{3M_{\text{Pl}}^2} \tag{4.26}
\]

\^{3}Though it is irrelevant to the early Universe, the Newtonian (weak field) limit of \(f(R)\) gravity and that of the corresponding scalar-tensor gravity are also \textit{the same}, as can be easily verified by the use of eq. (3.15).
It follows from eqs. (4.25) and (4.26) that
\[ \dot{\phi} = -M_{\text{Pl}} \frac{V'}{\sqrt{3V}} < 0 \tag{4.27} \]
whose solution during the slow-roll inflation \((t_0 < t_{\text{start}} \leq t \leq t_{\text{end}})\) is
\[ \phi(t) = -\sqrt{\frac{3}{2}} M_{\text{Pl}} \ln \left[ \frac{4\sqrt{V_0}}{3\sqrt{3M_{\text{Pl}}}} (t - t_0) \right] \tag{4.28} \]
Substituting it into eq. (4.26) and using the definition \(H = \dot{a}/a\) gives rise to a differential equation on the scale factor \(a(t)\). Its solution is
\[ a(t) = e^{H_0 t} \left[ \frac{t - t_0}{\text{const.}} \right]^{-3/4} \tag{4.29} \]
where we have introduced the notation \(H_0 = M/\sqrt{24}\). The presence of a singularity at \(t = t_0\) in eq. (4.29) is harmless because our inflationary solution is only valid during the slow-roll inflation when \(t \geq t_{\text{start}} > t_0\), so that it does not apply to the Big Bang. A resolution of the Big Bang singularity is supposed to require the higher-order curvature terms in the gravitational effective action (2.4).

The amount of inflation is measured by the e-foldings number
\[ N_e = \int_{t}^{t_{\text{end}}} H dt \approx \frac{1}{M_{\text{Pl}}^2} \int_{\phi_{\text{end}}}^{\phi} \frac{V}{V'} d\phi \tag{4.30} \]
where the \(t_{\text{end}}\) stands for the (time) end of inflation when one of the slow-roll parameters becomes equal to 1. The number of e-foldings between 50 and 100 is usually considered to be acceptable.

In the case of the slow-roll inflation with the scalar potential (3.20), we find that \(\varepsilon(\phi)\) first approaches 1 at \(\phi_{\text{end}} = \sqrt{\frac{3}{2}} M_{\text{Pl}} \ln \left(2\sqrt{3} - 3\right) \approx -0.94 \ M_{\text{Pl}}\), since \(|\eta(\phi)|\) approaches 1 later, at \(\phi_{\text{end}} = -\sqrt{\frac{3}{2}} M_{\text{Pl}} \ln \frac{2}{3} \approx -0.62 \ M_{\text{Pl}}\). Then eq. (4.30) yields
\[ N_e = \frac{3}{4} (e^{-y} + y) - \frac{3}{4} \left( \exp \left[ \sqrt{\frac{2}{3}} \cdot 0.94 \right] - \sqrt{\frac{2}{3}} \cdot 0.94 \right) \approx \frac{3}{4} (e^{-y} + y) - 1.04 \tag{4.31} \]
where we have used the notation (3.18). Similarly, we find
\[ \varepsilon = \frac{4e^{2y}}{3(1 - e^y)^2} \quad \text{and} \quad \eta = \frac{-4e^{y}(1 - 2e^y)}{3(1 - e^y)^2} \tag{4.32} \]
Equation (4.31) can now be used to get \(y\) in terms of \(N_e\), while a substitution of \(y(N_e)\) into eq. (4.32) yields both \(\varepsilon(N_e)\) and \(\eta(N_e)\). The results of our numerical calculations (by using MATHEMATICA) are summarized in Table 1.
An analytic approximation can be obtained by using the expansion with respect to the inverse number of e-foldings. For instance, eq. (4.31) yields

$$e^y = \frac{3}{4N_e} - \frac{9 \ln N_e}{16N_e^2} - \frac{0.94}{N_e^2} + \mathcal{O}\left(\frac{\ln^2 N_e}{N_e^3}\right)$$

Equation (4.32) now implies

$$\varepsilon = \frac{3}{4N_e^2} + \mathcal{O}\left(\frac{\ln^2 N_e}{N_e^3}\right)$$

and

$$\eta = -\frac{1}{N_e} + \frac{3 \ln N_e}{4N_e^2} + \frac{5}{4N_e^2} + \mathcal{O}\left(\frac{\ln^2 N_e}{N_e^3}\right)$$

We are now ready for a calculation of the CMB observable quantities in our inflationary model, i.e. for its specific physical predictions. The primordial spectrum in the power-law approximation takes the form of $k^{n-1}$ in terms of the comoving wave number $k$ and the spectral index $n$. In particular, the slope $n_s$ of the scalar power spectrum, associated with the density perturbations, is given by [3]

$$n_s = 1 + 2\eta - 6\varepsilon$$

the slope of the tensor primordial spectrum, associated with the gravitational waves, is given by [3]

$$n_t = -2\varepsilon$$

whereas the scalar-to-tensor ratio is given by [3]

$$r = 16\varepsilon$$

Equations (4.34), (4.35) and (4.36) in our model imply

$$n_s = 1 - \frac{2}{N_e} + \frac{3 \ln N_e}{2N_e^2} - \frac{2}{N_e^2} + \mathcal{O}\left(\frac{\ln^2 N_e}{N_e^3}\right)$$

The spectral indices are constrained by cosmological observations — see e.g., the recent WMAP5 data [11] that implies

$$n_s = 0.960 \pm 0.013 \quad \text{and} \quad r < 0.22$$

In addition, the amplitude of the initial perturbations, $\Delta^2_R = M_{Pl}^4 V/(24\pi^2 \varepsilon)$, is yet another physical observable, whose experimental value is [3]

$$\left(\frac{V}{\varepsilon}\right)^{1/4} = 0.027 M_{Pl} = 6.6 \times 10^{16} \text{ GeV}$$

Equation (4.41) determines the normalization of the $R^2$-term in eq. (2.4) as

$$\frac{M}{M_{Pl}} = 4 \cdot \sqrt{\frac{2}{3}} \cdot (2.7)^2 \cdot \frac{e^y}{(1-e^y)^2} \cdot 10^{-4} = (3.5 \pm 1.2) \cdot 10^{-5}$$
Table 1: The slow-roll parameters and spectral indices for some values of $N_e$

| $N_e$ | $\varepsilon \times 10^{-4}$ | $\eta \times 10^{-2}$ | $r \times 10^{-3}$ | $n_t \times 10^{-4}$ | $n_s$ |
|-------|-----------------|-----------------|-----------------|-----------------|-------|
| 35    | 5.13            | -2.56           | 8.20            | -10.3           | 0.946 |
| 40    | 3.99            | -2.27           | 6.39            | -7.98           | 0.952 |
| 45    | 3.20            | -2.03           | 5.12            | -6.40           | 0.957 |
| 50    | 2.62            | -1.84           | 4.19            | -5.24           | 0.962 |
| 55    | 2.19            | -1.69           | 3.50            | -4.37           | 0.965 |
| 60    | 1.85            | -1.55           | 2.96            | -3.71           | 0.968 |
| 65    | 1.59            | -1.44           | 2.54            | -3.18           | 0.970 |
| 70    | 1.38            | -1.34           | 2.21            | -2.76           | 0.972 |
| 75    | 1.21            | -1.26           | 1.93            | -2.42           | 0.974 |

where, in the last step, we have used the value of $N_e = 53.8 \pm 18$, as it follows from eqs. (4.39) and (4.40). The results of our numerical calculations of the spectral indices are collected in Table 1. In particular, we find that the WMAP5 experimental bounds on the scalar spectral index in eq. (4.40) are satisfied in the cosmological model (2.4) provided that the e-foldings number $N_e$ lies between 35.9 and 71.8, with the middle value of $\bar{N}_e = 53.8$. We also find the noticeable suppression of tensor fluctuations as $|r| < 8.2 \cdot 10^{-3}$ and $|n_t| < 10^{-3}$. There is a possibility of further theoretical modification, which would imply more tuning of the spectral indices, when more terms of the higher-order in the curvature are added into the action (2.4).

5 Conclusion

Our main results are given by eqs. (4.29), (4.31), (4.32), (4.39), (4.42) and Table 1. The leading terms agree with the known results [8, 12]. We confirm that the simplest (Starobinsky) model of $(R + R^2)$ gravity with the single new parameter $M$ may theoretically describe inflation and still agree with the experimental (CMB) observations. As regards the possible extensions to the quartic curvature terms, see eg., refs. [13]. The $f(R)$ gravity is extendable to $F(R)$ supergravity [14].

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