Dynamical system analysis of Myrzakulov gravity

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We perform a dynamical system analysis of Myrzakulov or \( F(R, T) \) gravity, which is a subclass of affinely connected metric theories, where ones uses a specific but non-special connection that allows for non-zero curvature and torsion simultaneously. We consider two classes of models, we extract the critical points, and we examine their stability properties alongside their physical features. In the Class 1 models, which possess \( \Lambda \text{CDM} \) cosmology as a limit, we find the sequence of matter and dark energy eras, and we show that the Universe will result in a dark-energy dominated critical point for which dark energy behaves like a cosmological constant. Concerning the dark-energy equation-of-state parameter we find that it lies in the quintessence or phantom regime, according to the value of the model parameter. For the Class 2 models, we again find the dark-energy dominated, de Sitter late-time attractor, although the scenario does not possess \( \Lambda \text{CDM} \) cosmology as a limit. The cosmological behavior is richer, and the dark-energy sector can be quintessence-like, phantom-like, or experience the phantom-divide crossing during the evolution.

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I. INTRODUCTION

An increasing amount of observational data has now led to the establishment of the standard model of cosmology, according to which the universe has passed through two phases of accelerating expansion one at early and one at late times. Although the latter can be explained by a cosmological constant, the possibility of a dynamical nature, some possible tensions, as well as the necessity for an additional description of the former phase, may ask for a kind of modification. In general one has two ways to accomplish this. The first is to maintain general relativity as the underlying gravitational theory but consider extra field contents such as the dark energy sector [1, 2] and/or the inflaton field [3]. The second way is to construct modified gravity theories, which in a particular limit tend to general relativity, but which in general exhibit extra degrees of freedom that can drive the non-standard universe evolution [4–6]. This direction has the additional advantage of bringing gravity closer to a quantum description [7].

Modified gravity can arise from suitable extensions of the Einstein-Hilbert action, such as in \( F(R) \) gravity [8], in theories with non-minimal coupling between matter and curvature [9, 10], in \( F(G) \) gravity [11, 12], in Lovelock gravity [13, 14], in Horndeski [15] and generalized galileon [16, 17] gravities, etc. A different way to construct gravitational modifications is to use as a base the equivalent, teleparallel formulation of gravity [18, 19], and build torsional theories such as \( F(T) \) gravity [20–22], theories with non-minimal coupling between matter and torsion [23, 24], \( F(T, T_G) \) gravity [25], \( F(T, B) \) gravity [26], teleparallel Horndeski [27], etc. One can proceed to other geometrical modifications, thus obtaining novel extended gravity theories, such as using non-metricity [28, 29], or constructing more complex structure such as in Finsler geometry [30–32].

An alternative way to construct gravitational modifications is to alter the connection structure of the theory, namely the extra degrees of freedom will arise from the different connection instead of the different action. This was known in the framework of metric-affine theories [33–36], as well as in Finsler-like theories where the non-linear connection may bring about extra degrees of freedom [37–41]. In Myrzakulov or \( F(R, T) \) gravity [42] (this should not be confused with the \( F(T) \) gravity where \( T \) is the trace of the energy-momentum tensor [43]) ones uses a specific but non-special connection, which allows for non-zero curvature and torsion simultaneously, which then leads to the appearance of extra degrees of freedom that can make the theory phenomenologically viable [44]. As one can show, it can be expressed as a deformation of both general relativity and its teleparallel equivalent. Hence, this theory lies within the class of Riemann-Cartan family of theories, which in turn belong to the general family of affinely connected metric theories [45]. Nevertheless, the theory at hand maintains zero non-metricity.

The cosmological applications of Myrzakulov gravity were investigated in [42, 44, 46–52], while the confrontation with observational data has been performed in [53]. In this work we are interested in investigating the cosmological behavior by applying the powerful method of dynamical system analysis. Such an approach allows
to extract global information on the cosmological evolution, independently of the specific initial conditions or the intermediate-time behavior [54, 55]. In particular, by examining the stable critical points of the autonomously transformed cosmological equations, one can classify the infinite number of possible evolutions into few different classes obtained asymptotically.

The plan of the work is the following. In Section II we review Myrzakulov gravity and we present the relevant cosmological equations. In Section III we perform a detailed dynamical analysis of various scenarios in this theory, focusing on the stable late-time solutions, and we discuss on the physical behavior. Finally, Section IV is devoted to the Conclusions.

II. COSMOLOGY IN MYRZAKULOV GRAVITY

In this section we provide the cosmological equations in a universe governed by Myrzakulov gravity [42, 44]. The basic feature of the theory is the modification of the connection, maintaining however zero non-metricity. As it is known, choosing a general connection $\omega^a_{bc}$ one can construct the curvature and the torsion tensors through the expressions [25]

$$R^a_{b\mu\nu} = \omega^a_{b\nu,\mu} - \omega^a_{b\mu,\nu} + \omega^a_{c\mu} \omega^c_{b\nu} - \omega^a_{c\nu} \omega^c_{b\mu},$$

$$T^a_{\mu\nu} = \omega^a_{\mu\nu} - \omega^a_{\nu\mu} + \omega^b_{\nu b} \omega^b_{\mu} - \omega^b_{\mu b} \omega^b_{\nu},$$

with $e^a_{\mu} \partial_\mu$ the tetrad field satisfying $g_{\mu\nu} = \eta_{ab} e^a_{\mu} e^b_{\nu}$, with $g_{\mu\nu}$ the metric, $\eta_{ab} = \text{diag}(-1,1,1,1)$, and where Greek and Latin indices run respectively over coordinate and tangent space, and with comma denoting differentiation.

Amongst the infinite connections, the Levi-Civita $\Gamma_{abc}$ is the only one that by construction vanishes to torsion. For clarity we will use the superscript “LC” to denote the curvature tensor calculated using $\Gamma_{abc}$, i.e. $R^{(LC)}_{b\mu\nu} = \Gamma^a_{b\nu,\mu} - \Gamma^a_{b\mu,\nu} + \Gamma^a_{c\mu} \Gamma^c_{b\nu} - \Gamma^a_{c\nu} \Gamma^c_{b\mu}$. Similarly, imposition of the Weitzenböck connection $W_{\mu\nu\lambda} = e^a_{\lambda} e^b_{\mu} \Gamma^c_{b\nu}$ leads to zero curvature, and the corresponding torsion tensor becomes $T^{(W)}_{\mu\nu\lambda} = W^\lambda_{\nu\mu} - W^\lambda_{\mu\nu}$, where we use the superscript “W” to denote quantities calculated using $W^\lambda_{\mu\nu}$. From contractions of the above tensors one can find the Ricci scalar corresponding to the Levi-Civita connection:

$$R^{(LC)} = \eta^{ab} e^a_{\mu} e^b_{\nu} \left[ \Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu,\nu} - \Gamma^\lambda_{\nu,\mu} + \Gamma^\rho_{\mu\nu} \Gamma^\lambda_{\rho\lambda} - \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\rho} \right],$$

as well as the torsion scalar corresponding to the Weitzenböck connection:

$$T^{(W)} = \frac{1}{4} \left( W^\mu_{\nu\lambda} - W^\mu_{\nu\lambda} \right) \left( W^\mu_{\lambda\nu} - W^\mu_{\nu\lambda} \right) + \frac{1}{2} \left( W^\mu_{\nu\lambda} - W^\mu_{\nu\lambda} \right) \left( W^\lambda_{\mu\nu} - W^\lambda_{\nu\mu} \right) - \left( W^\mu_{\nu\lambda} - W^\mu_{\nu\lambda} \right) \left( W^\lambda_{\mu\nu} - W^\lambda_{\nu\mu} \right).$$

In general relativity one uses $R^{(LC)}$ in the Lagrangian, while in teleparallel equivalent of general relativity ones uses $T^{(W)}$. Both these theories possess two propagating degrees of freedom, describing a massless spin-two field, i.e. the graviton. Thus, in their corresponding modifications, namely curvature-based modified gravity or torsion-based modified theories, one can acquire extra degrees of freedom by extending the action, and these extra degrees of freedom are the ones that lead to modified cosmological evolution. Nevertheless, after the above discussion we realize that one can introduce extra degrees of freedom through the consideration of non-special connections, i.e. going beyond the Levi-Civita and Weitzenböck ones. Hence, if ones applies a connection that has both non-zero curvature and torsion, he obtains a theory with more degrees of freedom.

Specifically, as it was presented in [42, 44] one can construct a theory that is based on a specific but not special connection that leads to both non-zero curvature and non-zero torsion. The action of such a theory would be

$$S = \int d^4 x \left[ \frac{F(R, T)}{2\kappa^2} + L_m \right],$$

with $e = \det(e^a_{\mu}) = \sqrt{-g}$ and $\kappa^2 = 8\pi G$ the gravitational constant, however we mention that $T$ and $R$ are the torsion and curvature scalars of the non-special connection, namely [25]

$$T = \frac{1}{4} T^{\mu\nu\lambda} T_{\mu\nu\lambda} + \frac{1}{2} T^{\mu\nu\lambda} T_{\lambda\nu\mu} - T_{\nu}^{\mu} T_{\lambda\mu}^\lambda,$$

$$R = R^{(LC)} + T - 2T_{\nu}^{\mu} T_{\nu\mu},$$

with ; denoting the covariant differentiation with respect to the Levi-Civita connection. Finally, in the above action we have also added the matter Lagrangian $L_m$.

As one can see from the definitions (1),(2) $T$ depends on the tetrad, its first derivative and the connection, and $R$ depends on the tetrad and its first and second derivatives, and on the connection and its first derivative. These allows one to introduce the parametrization [44]

$$T = T^{(W)} + e,$$

$$R = R^{(LC)} + u,$$

with $u$ being a scalar quantity depending on the tetrad, its first and second derivatives, and the connection and its first derivative, and $e$ a scalar depending on the tetrad, its first derivative and the connection.
The above theory has non-trivial structure and exhibits extra degrees of freedom even in the case where the arbitrary function $F(R, T)$ has a trivial form, since the novel features arise from the non-trivial connection itself, parametrized by the quantities $u$ and $v$. If this connection becomes the Levi-Civita one, we obtain that $u = 0$ and $v = -T(W)$, and thus we recover the standard $F(R)$ gravity (which for $F(R) = R$ becomes general relativity). However, if the connection is the Weitzenböck one, we acquire $v = 0$ and $u = -R^{(LC)}$ and therefore we recover standard $F(T)$ gravity (which for $F(T) = T$ becomes the teleparallel equivalent of general relativity).

In order to proceed to the cosmological applications of the above construction, we follow the mini-super-space procedure \cite{44}. Imposing the flat Friedmann-Robertson-Walker (FRW) metric

\[ ds^2 = dt^2 - a^2(t) \delta_{ij} dx^i dx^j, \]  

namely the tetrad $e^a_{\mu} = \text{diag}[1, a(t), a(t), a(t)]$, with $a(t)$ the scale factor, we find $R^{(LC)} = 6 \left( \dot{a}/a + \dot{a}^2 \right)$ and $T(W) = -6 \left( \dot{a}^2/a^2 \right)$. Taking into account the dependence of $u$ and $v$ on the metric and the connection, we deduce that $u = u(a, \dot{a}, \ddot{a})$ and $v = v(a, \dot{a})$. Furthermore, we take the standard form $L_m = -\rho_m(a)$ \cite{56}. Lastly, in order to explore the dynamics of Myrzakulov gravity arising solely from the non-special connection itself, we make the simple linear choice $F(R, T) = R + \lambda T$, with $\lambda$ the dimensionless coupling parameter.

Inserting the above mini-super-space expressions into (5) we obtain $S = \int L dt$, with

\[ L = \frac{3}{\kappa^2} \left[ \lambda + 1 \right] a \dot{a}^2 - \frac{a^3}{2\kappa^2} \left[ u(a, \dot{a}, \ddot{a}) + \lambda v(a, \dot{a}) \right] + a^3 \rho_m(a). \]  

We can now perform variation and extract the equations of motion for $a$, and we can moreover consider the Hamiltonian constraint $H = \dot{a} \left[ \dot{\dot{a}} - \frac{\dot{a}^2}{a} \right] + \dot{\ddot{a}} \left( \frac{\dot{a}^2}{a} \right) - L = 0$. Hence, we result to the following Friedmann equations \cite{44}

\[ 3\dot{H}^2 = \kappa^2 \left( \rho_m + \rho_{MG} \right) \]  
\[ 2\dot{H} + 3H^2 = -\kappa^2 \left( p_m + p_{MG} \right), \]  

where the dark energy sector that arises effectively from the non-special connection has energy density and press-
sure

\[ \rho_{MG} = \frac{1}{\kappa^2} \left[ \frac{Ha}{2} \left( u_a + v_a \lambda \right) - \frac{1}{2}(u + \lambda v) + \frac{a}{2}(H - 2H^2) - 3\lambda H^2 \right] \]  
\[ p_{MG} = -\frac{1}{\kappa^2} \left[ \frac{Ha}{2} \left( u_a + v_a \lambda \right) - \frac{1}{2}(u + \lambda v) - \frac{a}{6} \left( u_a + \lambda v_a - \dot{u}_a - \lambda \dot{v}_a \right) - \frac{a}{2} \left( H + 3H^2 \right) u_a - Ha \dot{u}_a - \frac{a}{6} \dot{u}_a - \lambda (2H + 3H^2) \right], \]

respectively. In the above expressions $H = \ddot{a}/a$ is the Hubble parameter, $p_m$ is the pressure of the matter sector, and the subscripts $a, \dot{a}, \ddot{a}$ mark partial derivatives with respect to these arguments. Note that the effective dark energy sector is conserved, namely $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$, as it is easily deduced from the above imposing the matter conservation equation $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$ too.

In the following we focus on two classes of the theory at hand, constructed phenomenologically in order to lead to interesting cosmological evolution.

### A. Class 1

As a first example we consider the class where $u = c_1 \dot{a} - c_2$ and $v = c_3 \dot{a} - c_4$, where $c_1, c_2, c_3, c_4$ are constants. For this class equations (12)-(15) lead to

\[ 3\dot{H}^2 = \kappa^2 \left( \rho_m + \rho_{MG} \right) \]  
\[ 2\dot{H} + 3H^2 = -\kappa^2 \left( p_m + p_{MG} \right), \]

with

\[ \rho_{MG} = \frac{1}{\kappa^2} \left[ c - 3\lambda H^2 \right] \]  
\[ p_{MG} = -\frac{1}{\kappa^2} \left[ c - \lambda (2H + 3H^2) \right], \]

where we have defined $c \equiv c_2 + c_4$. Thus, the effective dark-energy equation-of-state parameter reads

\[ w_{DE} = -1 + \frac{2\lambda \dot{H}}{c - 3\lambda H^2}. \]  

It is interesting to mention here that the effective dark energy density (18) falls within particular subclasses of the running vacuum cosmology \cite{57-59}.

### B. Class 2

The second class that we are interested in is the one characterized by $u = c_1 \dot{a} \ln \dot{a}$ and $v = s(a) \dot{a}$, where $s(a)$ is
an arbitrary function. Hence, expressions (12)-(15) give again the Friedmann equations (16)-(17) but now with

\[ \rho_{MG} = \frac{1}{\kappa^2} \left[ \frac{c_1}{2} H - 3\lambda H^2 \right] \]

\[ p_{MG} = -\frac{1}{\kappa^2} \left[ \frac{c_1}{2} H + \frac{c_1}{6} \dot{H} - \lambda(2\dot{H} + 3H^2) \right], \]

and thus we can find

\[ w_{DE} = -1 + \frac{2\lambda \dot{H} - \frac{c_1}{6} \ddot{H}}{2\dot{H} - 3\lambda H^2}. \]

Similarly to the Class 1 above, the effective dark energy density (21) coincides with broader subclasses of the running vacuum cosmology, and as we show below it can lead to very interesting cosmological behavior despite the fact that it does not have ΛCDM scenario as a particular limit.

### III. PHASE SPACE ANALYSIS

In the previous section we presented the cosmological equations of the scenario at hand. As we can see, Myrzakulov gravity leads to the appearance of new terms in the Friedmann equations, that are of geometrical origin and in particular they arise from the non-trivial connection structure through the parametrization in terms of \( u \) and \( v \). In this section we proceed to the full phase-space analysis of these scenarios, by applying the dynamical system method [54, 55]. Hence, we will first introduce suitably the auxiliary variables needed in order to transform the equations into an autonomous dynamical system [54, 55, 61–71], and then we will extract its critical points. Thus, examining the eigenvalues of the perturbation matrix around each of them we can conclude on their stability properties.

In order to perform the dynamical analysis we introduce the quantities

\[ A = \left[ \frac{H a}{2} (u_a + v_a \lambda) - \frac{1}{2} (u + \lambda v) \right. \]
\[ \left. + \frac{a u_a}{2} (\dot{H} - 2H^2) \right] \]
\[ B = \left[ - a (u_a + \lambda v_a - \dot{u_a} - \lambda \dot{v_a}) \right. \]
\[ - 3a \left( \dot{H} + 3H^2 \right) u_a - 6Ha \dot{u_a} \]
\[ \left. - a \ddot{u_a} \right]. \]

Hence, the two Friedman equations can be written as

\[ 3H^2(1 + \lambda) = \kappa^2 \rho_m + A \]

\[ (2\dot{H} + 3H^2)(1 + \lambda) = \kappa^2 \rho_m w_m + A + \frac{B}{6}, \]

where for convenience we have also introduced the matter equation-of-state parameter defined as \( w_m = p_m/\rho_m \).

Let us first examine the limit of the scenario at hand to the ΛCDM cosmology. In order to achieve this we need \( \rho_{MG} = -p_{MG} \), which implies that

\[ \lambda \dot{H} = -\frac{B}{12}. \]

Although this condition can be satisfied in many ways, the simplest one is to consider the case \( \lambda = 0 \), namely to focus on a Lagrangian being just the curvature \( R \) corresponding to the non-special connection. In this case, if we choose a connection with \( u = c_1 \dot{a} - c_2 \), where \( c_1, c_2 \) are constants, we acquire

\[ \rho_{MG} = -p_{MG} = \frac{c_2}{2\kappa^2} \equiv \Lambda. \]

Interestingly enough, we observe that we do obtain ΛCDM cosmology although in the starting action we had not considered an explicit cosmological constant. Thus, the non-trivial structure of the underlying geometry results to an effective cosmological constant, which reveals the capabilities of the theory. Note that even in this simple case where \( \lambda = 0 \), and thus \( T \) disappears from the action, the non-special connection still has a non-zero torsion. In general, such an effective emergence of a cosmological constant due to the richer underlying connection appears in other geometrical modified gravities too [40, 60], and reveals the advantages of the theory.

Having the above discussion in mind we can deduce that Class 1 defined in subsection II A corresponds to a deviation from ΛCDM cosmology, accepting it as a particular limit and thus satisfying the basic requirements to be a viable theory, while still maintaining the possibility to improve ΛCDM behavior. On the other hand, Class 2 defined in subsection II B does not have ΛCDM cosmology as a limit, nevertheless, and interestingly enough, as we will later show it can lead to a cosmological behavior in agreement with observations.

We can now proceed to the dynamical analysis of the above specific classes, keeping a general \( \lambda \neq 0 \).

#### A. Class 1

We start with Class 1 of subsection II A. In this case definitions (24) lead to

\[ A = \frac{1}{2} (c_2 + \lambda c_1) \equiv C \]
\[ B = 0. \]

In order to transform the cosmological equations into an autonomous form we introduce the dark matter and dark energy density parameters as our dimensionless variables,
we depict the corresponding behavior of the system in the $(w_{DE}, \Omega_m)$ space, in the case of dust matter, for various values of $\lambda$. As we can see, the system passes through the saddle point $P_2$ before it results to the stable late-time attractor $P_1$. Additionally, in order to examine the system at both intermediate and late times, in Fig. 2 we present $\Omega_m$ as a function of the redshift $z = -1 + a_0/a$ (setting the current scale factor $a_0 = 1$), since $\dot{H} = - (1 + z)H(z)H'(z)$ with primes denoting derivatives with respect to $z$. We choose different values of $w_m$, and we fix $C$ in order to have $\Omega_m(z = 0) \equiv \Omega_{m0} \approx 0.31$ as required by observations [72]. As we observe, the Universe follows the required evolution, with the sequence of matter and dark energy epochs.

Moreover, in Fig. 3 we depict the corresponding behavior of the dark-energy equation-of-state parameter $w_{DE}$ for various values of $\lambda$. As we see, although for every $\lambda$ at asymptotic late times (i.e. for $z \to -1$) $w_{DE}$ is sta-

In order to show the above feature in a more transparent way, in Fig. 1 we present the behavior of the system in the $(w_{DE}, \Omega_m)$ space, in the case of dust matter, for various values of $\lambda$. As we can see, the system passes through the saddle point $P_2$ before it results to the stable late-time attractor $P_1$. Additionally, in order to examine the system at both intermediate and late times, in Fig. 2 we present $\Omega_m$ as a function of the redshift $z = -1 + a_0/a$ (setting the current scale factor $a_0 = 1$), since $\dot{H} = - (1 + z)H(z)H'(z)$ with primes denoting derivatives with respect to $z$. We choose different values of $w_m$, and we fix $C$ in order to have $\Omega_m(z = 0) \equiv \Omega_{m0} \approx 0.31$ as required by observations [72]. As we observe, the Universe follows the required evolution, with the sequence of matter and dark energy epochs.

![ FIG. 1: The behavior of the system in the $(w_{DE}, \Omega_m)$ space for Class 1 models of (16),(17) with (18),(19), for $w_m = 0$ and with $\lambda = 0.02$ (blue-dashed), $\lambda = 0$ (green-solid) and $\lambda = -0.02$ (orange-dotted).](image)

![ FIG. 2: The evolution of the matter density parameter $\Omega_m(z)$ as a function of the redshift, for Class 1 models of (16),(17) with (18),(19), for $w_m = 0$ (blue-dashed) and $w_m = 0.1$ (orange-dotted).](image)

As we observe, point $P_1$ corresponds to a dark-energy dominated Universe, in which dark energy behaves like a cosmological constant, and the fact that in the usual case of dust matter it is stable implies that it will be the late-time state of the Universe independently of the initial conditions. On the other hand, point $P_2$ is a matter-dominated, non-accelerating solution, and the fact that for dust matter it is saddle implies that this point can describe the necessary intermediate era of the Universe, in which matter structure is formed [70, 71].

| Point $(\Omega_m, \Omega_{MG})$ | Existence $w_{DE}$ Acceleration Stability |
|-------------------------------|------------------------------------------|
| $P_1$ (0, 1)                  | Always $-1$ | yes $w_m > -1$  |
| $P_2$ (1, 0)                  | Always $w_m < -1$ | $w_m < -1$  |

TABLE I: The physically interesting critical points of Class 1, namely of (16),(17) with (18),(19), their features and their stability conditions.

\[
\Omega_m = \frac{k^2 \rho_m}{3H^2(1 + \lambda)} \quad (30)
\]
\[
\Omega_{MG} = \frac{C}{3H^2(1 + \lambda)}, \quad (31)
\]

and therefore the first Friedmann equation (16) becomes

\[
1 = \Omega_m + \Omega_{MG} \quad (note \ that \ the \ case \ \lambda = -1 \ is \ not)
\]

physically interesting since according to (25) leads to $\rho_m = -C/k^2 = \text{const.}$, and hence in the following we focus on the case $\lambda \neq -1)$. Additionally, the second Friedmann equation (17) becomes

\[
\frac{\dot{H}}{H^2} = -\frac{3}{2}(1 + \Omega_m w_m - \Omega_{MG}). \quad (32)
\]

Using this expression, as well as (30),(31), equation (20) can be rewritten as follows:

\[
w_{DE} = -1 - \frac{3\lambda(1 + \Omega_m w_m - \Omega_{MG})}{\Omega_{MG}(1 + \lambda) - 3\lambda}. \quad (33)
\]

In summary, the dynamical system can be straightforwardly written as:

\[
\frac{d\Omega_m}{d\ln a} = -3\Omega_m w_m, \quad (34)
\]
\[
\frac{d\Omega_{MG}}{d\ln a} = 3\Omega_{MG} (\Omega_m w_m + 1 - \Omega_{MG}). \quad (35)
\]

Since the first Friedman equation acts as a constraint, we finally remain with one-dimensional phase space. The corresponding critical points $P(\Omega_m, \Omega_{MG})$ are summarized in Table I, alongside their features and stability conditions. Note that in this case (33) provides the useful expression

\[
w_{DE} = -1 - \frac{3\Omega_m(1 + w_m)}{(1 - \Omega_m)(1 + \lambda) - 3\lambda}. \quad (36)
\]
bilized at the cosmological constant value $-1$, as it was found in Table I, the behavior at intermediate redshifts and at the present Universe is different. In particular, for $\lambda < 0$ the dark energy sector behaves as quintessence, while for $\lambda > 0$ the $w_{DE}$ lies in the phantom regime. This was expected from the form of (36), and reveals that Class 1 offers a unified description of both quintessence and phantom regimes, without pathologies. Finally, as we see, in the case $\lambda = 0$ the scenario at hand recovers $\Lambda$CDM cosmology.

B. Class 2

Let us now proceed to the investigation of Class 2 of subsection II B. In this case definitions (24) lead to

$$A = \frac{c_1}{2} H \equiv DH. \quad (37)$$

Similarly to the Class 1 case, for $\lambda \neq -1$ we can introduce the dimensionless auxiliary variables

$$\Omega_m \equiv \frac{k^2 \rho_m}{3H^2(1 + \lambda)} \quad (38)$$
$$\Omega_{MG} \equiv \frac{D}{3H(1 + \lambda)}, \quad (39)$$

and thus the first Friedmann equation becomes the constraint $1 = \Omega_m + \Omega_{MG}$. Additionally, for $\lambda \neq -1$ the second Friedmann equation becomes

$$\frac{\dot{H}}{H^2} = \frac{3(1 + \Omega_m w_m - \Omega_{MG})}{2 - \Omega_{MG}}. \quad (40)$$

Hence, using this expression and (38),(39) we can rewrite expression (23) as

$$w_{DE} = -1 - \frac{(1 + \Omega_m w_m - \Omega_{MG})(\lambda(2 - \Omega_{MG}) - \Omega_{MG})}{(2 - \Omega_{MG})(1 + \lambda)\Omega_{MG} - \lambda}. \quad (41)$$

For this class of scenarios, the dynamical system can be straightforwardly written as:

$$\frac{d\Omega_m}{d\ln a} = 3\Omega_m [\Omega_{MG} - w_m(-2 + \Omega_{MG} + 2\Omega_m)] \frac{\Omega_{MG} - 2}{\Omega_{MG} - 2}, \quad (42)$$
$$\frac{d\Omega_{MG}}{d\ln a} = 6\Omega_{MG}(\Omega_{MG} - w_m\Omega_m - 1) \frac{\Omega_{MG} - 2}{\Omega_{MG} - 2}. \quad (43)$$

Due to the constraint first Friedman equation, we result to a one-dimensional phase space. Hence, in this case (41) gives the useful expression

$$w_{DE} = -1 - \frac{\Omega_m(1 + w_m)[\lambda - 1 + \Omega_m(\lambda + 1)]}{(1 + \Omega_m)[1 + \lambda(1 - \Omega_m) - \lambda]}. \quad (44)$$

| Point $(\Omega_m, \Omega_{MG})$ | Existence $w_{DE}$ | Acceleration Stability |
|-----------------------------|-----------------|-----------------------|
| $P_1$                       | (0, 1)          | Always $-1$           | yes $w_m > -1$        |
| $P_2$                       | (1, 0)          | Always $w_m, w_m < -\frac{1}{3}$ | $w_m < -1$ |

TABLE II: The physically interesting critical points of Class 2, namely of (16),(17) with (21),(22), their features and their stability conditions.

The critical points are summarized in Table II. In the same Table we provide their features and their stability conditions. Interestingly enough, Class 2 exhibits the same critical points with Class 1, namely the dark-energy dominated, de Sitter Universe $P_1$, which is stable for dust matter, and the matter-dominated, non-accelerating Universe $P_2$, which is saddle for dust matter. The importance of the current behavior is that it is obtained not only without the consideration of an explicit cosmological constant, but also through the quite rich and complicated dark energy density (21), which does not accept $\Lambda$CDM model as a particular limit.

Nevertheless, although Class 2 has the same critical points with Class 1, the behavior of the system at intermediate times is radically different. In Fig. 4 we
we present \( \Omega_z^2 \).

\[ \text{we present the evolution of the dark-energy sector during the evolution.} \]

\[ \text{quintessence regime, in the phantom regime, or experience the phantom-divide crossing during the evolution.} \]

\[ \text{Additionally, the behavior at intermediate redshifts is even more different, and the dark-energy sector can lie in the quintessence regime, in the phantom regime, or experience the phantom-divide crossing during the evolution.} \]

\[ \text{This was expected from the form of (44), and reveals the capabilities of this Class of models.} \]

\[ \text{IV. DISCUSSION} \]

\[ \text{We performed a dynamical system analysis of Myrzakulov gravity. The latter is a subclass of affinely connected metric theories, in which ones uses a specific but non-special connection, which allows for non-zero curvature and torsion simultaneously. Thus, one obtains extra degrees of freedom which in turn lead to extra terms in the Friedman equations that can lead to interesting phenomenology. Hence, by applying the dynamical system approach and performing a phase-space analysis, one is able to bypass the non-linearities of the equations and investigate the global behavior of the system, independently of the specific initial conditions or the intermediate-time behavior evolution.} \]

\[ \text{We considered two classes of models and for each case we transformed the equations into an autonomous dynamical system. We extracted the critical points, and we examined their stability properties alongside their physical features. In the Class 1 models, which possess } \Lambda \text{CDM cosmology as a limit, we found that independently of the initial conditions the Universe will result in a dark-energy dominated critical point in which dark energy behaves like a cosmological constant. Moreover, we found a matter-dominated, non-accelerating solution, which is saddle and thus it can describe the necessary corresponding intermediate matter era of the Universe. Hence, the Universe follows the required evolution, with the sequence of matter and dark energy eras. Concerning the dark-energy equation-of-state parameter } w_{DE}, \text{ we showed that although at asymptotic late times it is stabilized at the cosmological constant value } -1 \text{ for every value of the model parameter } \lambda, \text{ the behavior at intermediate redshifts and at the present Universe is different, since for } \lambda < 0 \text{ the dark energy sector behaves as quintessence, while for } \lambda > 0 \text{ the } w_{DE} \text{ lies in the phantom regime.} \]

\[ \text{For the Class 2 models, we again found the dark-energy dominated, de Sitter late-time attractor, and the saddle critical point corresponding to matter-dominated, non-accelerating Universe. Furthermore, at asymptotically late times } w_{DE} \text{ tends to the cosmological constant value } -1. \text{ However, the interesting feature is that this was obtained without the scenario possessing } \Lambda \text{CDM cosmology as a particular limit. This Class can also describe the sequence of of matter and dark energy epochs, nevertheless the at intermediate times the behavior is radically different than the previous Class, since the dark-energy sector can lie in the quintessence regime, in the phantom regime, or experience the phantom-divide crossing during the evolution.} \]

\[ \text{Let us stress here that, as we mentioned above, the two examined classes of theories, at a cosmological framework, fall within the class of generalized running vac-} \]
uum theories [57–59]. Hence, one can perform the Big-Bang Nucleosynthesis analysis in the same way [73] and deduce that the early-universe evolution is not spoiled in the present models, too.

In summary, the phase-space analysis revealed the interesting features of Myrzakulov gravity, and in particular the ability to possess a stable de Sitter solution as a late-time attractor even without the explicit consideration of a cosmological constant. It would be interesting to apply the Noether symmetry approach [74] in order to extract exact analytic solutions at intermediate times too. Furthermore, since the resulting cosmological equations are similar to subclasses of the running vacuum cosmology, it is necessary to further investigate their possible connection, and examine whether the current framework offers the way to provide a Lagrangian for running vacuum models, a well-known open issue in the corresponding literature. Finally, it would be interesting to investigate the relation and differences of the present theory with theories with Weyl connection (not to be confused with Weyl gravity, that uses the standard Levi-Civita connection), which have an altered connection but non-zero non-metricity [75]. These studies will be performed in separate works.

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