Primality test for numbers of the form \((2p)^{2^n} + 1\)

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Abstract
We describe a primality test for number \(M = (2p)^{2^n} + 1\) with odd prime \(p\) and positive integer \(n\). And we also give the special primality criteria for all odd primes \(p\) not exceeding 19. All these primality tests run in polynomial time in \(\log_2(M)\). A certain special \(2p\)-th reciprocity law is used to deduce our result.

1 Introduction

Primality testing is an important problem in computational number theory. Although this has been proved to be a \(\textsf{P}\) problem by Agrawal, Kayal and Saxena [1] in 2004, finding more efficient algorithms for specific families of numbers makes yet a lot of sense. Let \(a > 1\) be an integer. Considering the special family of prime numbers of the form \(a^n \pm 1\), when \(a\) is fixed and \(n > 0\) is varied, is then a natural problem. For \(a^n - 1\), it is easy to see that it suffices to consider the case when \(a = 2\) and \(n = p\) is a prime. Numbers of the form \(2^p - 1\) are called Mersenne numbers. For these numbers, there is a famous primality test named Lucas-Lehmer test given by Lucas [6] and Lehmer [5]. Here, we recall it:

**Lucas-Lehmer test.** Let \(M_p = 2^p - 1\) be Mersenne number, where \(p\) is an odd prime. Define a sequence \(\{u_k\}\) as follows: \(u_0 = 4\) and \(u_k = u_{k-1}^2 - 2\) for \(k \geq 1\). Then \(M_p\) is a prime if and only if \(u_{p-2} \equiv 0 \pmod{M_p}\).

For \(a^n + 1\), it is easy to see that it suffices to consider the case when \(a\) is even and \(n\) is a power of 2. When \(a = 2\), numbers of the form \(2^{2^n} + 1\) are called Fermat numbers. For these numbers, there is also a primality test due to Pépin (see [10]):

**Pépin test.** Let \(F_n = 2^{2^n} + 1\) be the \(n\)-th Fermat number, with \(n > 0\). Then \(F_n\) is a prime if and only if \(3^{(F_n-1)/2} \equiv -1 \pmod{F_n}\).
In this paper, we consider the primality of \( M = (2p)^{2^n} + 1 \), where \( p \) is an odd prime. For \( p = 3 \) and \( p = 5 \), Williams obtained primality tests for them using Lucas functions in [9]. However, for a general \( p \), it seems that there is no any known work to deal with the primality of these numbers. Notice that, from the work in [3], it is possible to give primality tests whose seeds of sequences will depend on \( M \), and this is not our desire. We can do better in this paper, that is, using a certain special \( 2p \)-th reciprocity law, we can give primality tests whose seeds of sequences will not depend on \( M \), at least for \( p \leq 19 \).

This paper is organized as follows. In Section 2 we give the definition of power residue symbol and prove a certain special \( 2p \)-th reciprocity law that will be used in later section. In Section 3 we state and prove our main result. In Section 4, we give explicit primality tests for \( M = (2p)^{2^n} + 1 \) for each odd prime \( p \leq 19 \). In Section 5 we give the implementation and computational results for \( p = 3, 5 \).

2 Preliminaries

What we state in this section may be found in [4, Chapter 14].

For a positive integer \( m \), let \( \zeta_m = e^{2\pi\sqrt{-1}/m} \) be the complex primitive \( m \)-th root of unity, and \( D = \mathbb{Z}[\zeta_m] \) the ring of integers of the cyclotomic field \( \mathbb{Q}(\zeta_m) \). Let \( p \) be a prime ideal of \( D \) lying over a rational prime \( p \) with \( \gcd(p, m) = 1 \). For every \( \alpha \in D \), the \( m \)-th power residue symbol \( \left( \frac{\alpha}{p} \right)_m \) is defined by:

1. If \( \alpha \in p \), then \( \left( \frac{\alpha}{p} \right)_m = 0 \).
2. If \( \alpha \notin p \), then \( \left( \frac{\alpha}{p} \right)_m = \zeta_m^i \) with \( i \in \mathbb{Z} \), where \( \zeta_m^i \) is the unique \( m \)-th root of unity in \( D \) such that \( \alpha^{(N(p)-1)/m} \equiv \zeta_m^i \pmod{p} \),

where \( N(p) \) is the absolute norm of the ideal \( p \).

3. If \( \alpha \in D \) is an arbitrary ideal prime to \( m \) and \( \alpha = \prod p_i^{n_i} \) is its factorization as a product of prime ideals, then

\[
\left( \frac{\alpha}{D} \right)_m = \prod \left( \frac{\alpha}{p_i} \right)^{n_i}_m.
\]

We set \( \left( \frac{\alpha}{D} \right)_m = 1 \).

4. If \( \beta \in D \) and \( \beta \) is prime to \( m \) define \( \left( \frac{\alpha}{\beta} \right)_m = \left( \frac{\alpha}{\beta D} \right)_m \).

We will need the following proposition which can be found in [4, Chapter 14, Corollary 2, p. 218].
Proposition 2.1. Suppose $A, B \subset \mathbb{Z}[\zeta_m]$ are ideals prime to $m$ and $A = (\alpha)$ is principal with $\text{gcd}(N(A), N(B)) = 1$. Then

$$
\left(\frac{N(B)}{\alpha}\right)_m = \left(\frac{\varepsilon(\alpha)}{B}\right)_m \left(\frac{\alpha}{N(B)}\right)_m
$$

where $\varepsilon(\alpha) = \pm \zeta_m^i$ for some $i \in \mathbb{Z}$.

From the above proposition, we can obtain a certain special $2p$-th reciprocity law.

Proposition 2.2. Let $M \equiv 1 \pmod{4p^2}$ be an integer with $p$ an odd prime. Let $\pi \in \mathbb{Z}[\zeta_{2p}]$ be coprime with $2pM$. Suppose $M > 1$ is a prime, then we have

$$
\left(\frac{M}{\pi}\right)_{2p} = \left(\frac{\pi}{M}\right)_{2p}.
$$

Proof. Let $\mathfrak{P}$ be a prime ideal of $\mathbb{Z}[\zeta_{2p}]$ lying over $M$. Since $M \equiv 1 \pmod{2p}$, we have $N(\mathfrak{P}) = M$. By Proposition 2.1

$$
\left(\frac{N(\mathfrak{P})}{\pi}\right)_{2p} = \left(\frac{\varepsilon(\pi)}{\mathfrak{P}}\right)_{2p} \left(\frac{\pi}{N(\mathfrak{P})}\right)_{2p}
$$

which implies

$$
\left(\frac{M}{\pi}\right)_{2p} = \left(\frac{\varepsilon(\pi)}{\mathfrak{P}}\right)_{2p} \left(\frac{\pi}{M}\right)_{2p}.
$$

And

$$
\left(\frac{\varepsilon(\pi)}{\mathfrak{P}}\right)_{2p} = \varepsilon(\pi)^{(M-1)/2p} = (\pm \zeta_{2p}^i)^{(M-1)/2p} = 1 \pmod{2p},
$$

as $2p | \frac{M-1}{2p}$. Then

$$
\left(\frac{M}{\pi}\right)_{2p} = \left(\frac{\pi}{M}\right)_{2p}.
$$

3 Primality test for $M = (2p)^2 + 1$

We first define the polynomials $G_n(x)(n \geq 0)$ by $G_0(x) = 1$, $G_1(x) = x$, and for $n \geq 2$ define $G_n(x)$ recursively by the formulas:

$$
G_n(x) = \begin{cases} 
G_{(n-1)/2}(x)G_{(n+1)/2}(x) - x, & \text{if } n \text{ is odd,} \\
G_{n/2}(x)^2 - 2, & \text{if } n \text{ is even.}
\end{cases}
$$

Clearly all the coefficients of $x^i (i \geq 0)$ of $G_n(x)$ can be computed in $O(\log n)$ steps. It is easy to see that $G_n(x)(n \geq 0)$ is a monic polynomial of degree $n$, and that $G_n(x + x^{-1}) = x^n + x^{-n}$ for $n > 0$.

Let

$$
f(x_1, \ldots, x_{(p-1)/2}) = \sum_{1 \leq i_1 < \cdots < i_j \leq (p-1)/2} x_{i_1} \cdots x_{i_j}, 1 \leq j \leq (p-1)/2
$$

be the $j$-th elementary symmetric polynomial of $x_1, \ldots, x_{(p-1)/2}$. Let $D = \mathbb{Z}[\zeta_{2p}]$ be the ring of integers of the cyclotomic field $L = \mathbb{Q}(\zeta_{2p})$. Let $G = \text{Gal}(\mathbb{Q}(\zeta_{2p})/\mathbb{Q})$ be the Galois group of $\mathbb{Q}(\zeta_{2p})$ over $\mathbb{Q}$. For every integer $c$ with $\text{gcd}(c, 2p) = 1$ denote by $\sigma_c$ the element of $G$ that sends $\zeta_{2p}$ to $\zeta_{2p}^c$. We know that $\text{Gal}(\mathbb{Q}(\zeta_{2p})/\mathbb{Q}) = \{\sigma_{\pm(2i-1)} | 1 \leq i \leq (p-1)/2\}$. 
For $\tau$ in $\mathbb{Z}[G]$ and $\alpha$ in $L$ with $\alpha \neq 0$ we often denote by $\alpha^\tau$ to the action of the element $\tau$ of $\mathbb{Z}[G]$ on the element $\alpha$ of $L$, that is,

$$\alpha^\tau := \prod_{\sigma \in G} \sigma(\alpha)^k, \text{ if } \tau = \sum_{\sigma \in G} k_\sigma \sigma \text{ where } k_\sigma \in \mathbb{Z}.$$ 

If $\tau \in G$, we will either write $\alpha^\tau$ or $\tau(\alpha)$. We also write $\sigma_1 = 1$ in $\mathbb{Z}[G]$.

Let $K = \mathbb{Q}(\zeta_{2p} + \zeta_{2p}^{-1})$ be the maximal real subfield of $L$. We know that $\text{Gal}(K/\mathbb{Q}) = \{\sigma_{2i-1}|K | 1 \leq i \leq (p-1)/2\}$. Let $\pi \in D$ with $\pi \notin \mathbb{R}$. We denote $\alpha = (\pi/\bar{\pi})^\gamma$ where

$$\gamma = \sum_{i=1}^{(p-1)/2} (2i-1)\sigma_{(2i-1)-1} \in \mathbb{Z}[G]$$

and a bar indicates the complex conjugation, and $(2i-1)^{-1}$ is the inverse of $2i-1$ in the multiplicative group $(\mathbb{Z}/2p\mathbb{Z})^*$. Obviously, we have $\alpha \bar{\alpha} = 1$. Next we define $(p-1)/2$ many sequences: \{$S_k^{(j)} |_{k \geq 0}$, $1 \leq j \leq (p-1)/2$ by $S_k^{(j)} = f^{(j)}(\alpha_1^{(k)}, \ldots, \alpha^{(k)}_{(p-1)/2})$, where $\alpha^{(k)}_i = \sigma_{2i-1}(\alpha^{(2p)^k} + \bar{\alpha}^{(2p)^k})$, $i = 1, \ldots, (p-1)/2$.

We set $F(x) = \sum_{k=0}^{(p-1)/2} G_k(x) := \sum_{j=0}^{(p-1)/2} (-1)^j a_j x^{(p-1)/2-j} \in \mathbb{Z}[x]$, clearly $a_0 = 1$. Since

$$F(\zeta_p + \zeta_p^{-1}) = 1 + \sum_{k=1}^{(p-1)/2} G_k(\zeta_p + \zeta_p^{-1}) = 1 + \sum_{k=1}^{(p-1)/2} (\zeta_p^k + \zeta_p^{-k}) = 0$$

and $F(x)$ is a monic polynomial of degree $(p-1)/2$, so $F(x)$ is the minimal polynomial of $\zeta_p + \zeta_p^{-1}$ over $\mathbb{Q}$.

Our primality test for numbers $M = (2p)^{2^n} + 1$ is described as follows.

**Theorem 3.1.** Let $S_k^{(j)}$ and $a_j$ be as before. Let $M = (2p)^{2^n} + 1$ with $n \geq 1$, $p$ be an odd prime and $r = 2^n$. Let $\pi \in \mathbb{Z}[\zeta_{2p}]$ be coprime with $2pM$ such that $\pi \notin \mathbb{R}$ and $(\frac{M}{\pi})_{2p} \neq \pm 1$. Suppose that if $x^{r-1} \equiv 1 \pmod{p^r}$ and $1 < x < p^r$ then $x$ does not divide $M$. Then $M$ is prime if and only if one of the following holds:

(i) $(\frac{M}{\pi})_{2p} = \zeta_p^l$ for some $l \in \mathbb{Z}$ and $l \not\equiv 0 \pmod{p}$, and $S_{r-1}^{(j)} \equiv a_j \pmod{M}$ for each $1 \leq j \leq (p-1)/2$;

(ii) $(\frac{M}{\pi})_{2p} = -\zeta_p^l$ for some $l \in \mathbb{Z}$ and $l \not\equiv 0 \pmod{p}$, and $S_{r-1}^{(j)} \equiv (-1)^j a_j \pmod{M}$ for each $1 \leq j \leq (p-1)/2$.

**Proof** We first show the necessity for the primality of $M$. Suppose then $M$ is a prime, since $\pi$ is prime to $2pM$, applying Proposition \red{2.2} we get $(\frac{M}{\pi})_{2p} = (\frac{\pi}{M})_{2p}$. From $M \equiv 1 \pmod{2p}$, the ideal $MD$ factors in $D$ as a product of $p-1$ distinct prime ideals. We write

$$MD = (p\bar{p}) \sum_{i=1}^{(p-1)/2} \sigma_{2i-1}^{-1},$$
thus
\[
\left(\frac{M}{\pi}\right)_{2p} = \left(\frac{\pi}{M}\right)_{2p} = \prod_{i=1}^{(p-1)/2} \left(\frac{\pi}{(\pi^i)^{2i-1}}\right)_{2p}
\]
\[
= \prod_{i=1}^{(p-1)/2} \left(\frac{\pi}{(\pi^i)^{(2i-1)p^{-2}(2i-1)^{-1}}}\right)_{2p} = \left(\frac{\pi}{\pi^{\sum_{k=1}^{p-1} \sigma^k(2i-1)^{-1}}}\right)_{2p}
\]
\[
= \left(\frac{2}{p}\right)_{2p} \equiv \alpha(M^{-1})_{2p} \equiv \alpha(2p)^{r-1} \pmod{p}
\]
Since \(p\) is an arbitrary prime ideal lying over \(M\), we have
\[
\left(\frac{M}{\pi}\right)_{2p} \equiv \alpha(2p)^{r-1} \pmod{M}.
\]
It implies
\[
\alpha^{(r-1)}_1 = \alpha(2p)^{r-1} + \bar{\alpha}(2p)^{r-1} \equiv \left(\frac{M}{\pi}\right)_{2p} + \left(\frac{M}{\pi}\right)^{-1}_{2p} \pmod{M},
\]
and for all \(1 \leq i \leq (p-1)/2\), we obtain
\[
\alpha^{(r-1)}_i = \sigma_{2i-1}(\alpha(2p)^{r-1} + \bar{\alpha}(2p)^{r-1}) \equiv \left(\frac{M}{\pi}\right)^{2i-1}_{2p} + \left(\frac{M}{\pi}\right)^{1-2i}_{2p} \pmod{M}.
\]
Hence for all \(1 \leq j \leq (p-1)/2\),
\[
S^{(j)}_{r-1} = f^{(j)}(\alpha^{(r-1)}_1, \ldots, \alpha^{(r-1)}_{(p-1)/2})
\]
\[
\equiv f^{(j)} \left(\left(\frac{M}{\pi}\right)_{2p} + \left(\frac{M}{\pi}\right)^{-1}_{2p}, \ldots, \left(\frac{M}{\pi}\right)^{p-2}_{2p} + \left(\frac{M}{\pi}\right)^{2-p}_{2p}\right) \pmod{M}.
\]
(i) If \(\left(\frac{M}{\pi}\right)_{2p} = \zeta_l\) for some \(l\) with \(l \equiv 0 \pmod{p}\). Since the minimal polynomial of \(\zeta_p + \zeta_p^{-1}\) is \(F(x)\) defined before, we obtain for all \(1 \leq j \leq (p-1)/2\), \(a_j = f^{(j)}(\zeta_p + \zeta_p^{-1}, \zeta_p^{3}, \zeta_p^{7}, \ldots, \zeta_p^{p-2} + \zeta_p^{2-p})\) because of \(F(x) = \prod_{i=1}^{(p-1)/2} [x - (\zeta_p^{-l} + \zeta_p^{1-2i})]\). Hence from above
\[
S^{(j)}_{r-1} = f^{(j)}(\zeta_p + \zeta_p^{-1}, \zeta_p^{3}, \zeta_p^{7}, \ldots, \zeta_p^{p-2} + \zeta_p^{2-p}) = a_j \pmod{M},
\]
for all \(j = 1, \ldots, (p-1)/2\).
(ii) If \(\left(\frac{M}{\pi}\right)_{2p} = -\zeta_l\) for some \(l\) with \(l \equiv 0 \pmod{p}\). Then by the property of elementary symmetric polynomial, we have
\[
S^{(j)}_{r-1} = f^{(j)} \left(\left(\frac{M}{\pi}\right)_{2p} + \left(\frac{M}{\pi}\right)^{-1}_{2p}, \ldots, \left(\frac{M}{\pi}\right)^{p-2}_{2p} + \left(\frac{M}{\pi}\right)^{2-p}_{2p}\right)
\]
\[
= f^{(j)}(-\zeta_p - \zeta_p^{-1}, -\zeta_p^{3} - \zeta_p^{-3}, \ldots, -\zeta_p^{p-2} - \zeta_p^{2-p})
\]
\[
= (-1)^j f^{(j)}(\zeta_p + \zeta_p^{-1}, \zeta_p^{3}, \zeta_p^{7}, \ldots, \zeta_p^{p-2} + \zeta_p^{2-p}) = (-1)^j a_j \pmod{M},
\]
for all \( j = 1, \ldots, (p - 1)/2 \). This completes the proof of necessity.

Now we turn to the proof of sufficiency. Let \( q \) be an arbitrary prime divisor of \( M \). Let \( \mathfrak{p} \) be a prime ideal in the ring of integers of \( K \) lying over \( q \), and \( \Omega \) be a prime ideal of \( D \) lying over \( q \). Let \( \beta = \alpha^{(2p)^{-1}} + \bar{\alpha}^{(2p)^{-1}} \in K \), then \( S_{r-1}^{(j)} = f^{(j)}(\beta, \sigma_3(\beta), \ldots, \sigma_{p-2}(\beta)) \), \( j = 1, \ldots, (p - 1)/2 \).

(i) If \( S_{r-1}^{(j)} \equiv a_j \pmod{M} \), then \( f^{(j)}(\beta, \sigma_3(\beta), \ldots, \sigma_{p-2}(\beta)) \equiv a_j \pmod{q} \), hence we have

\[
0 = (\beta - \beta)(\beta - \sigma_3(\beta)) \ldots (\beta - \sigma_{p-2}(\beta)) \\
= \beta(p^{-1})/2 + \sum_{j=1}^{(p-1)/2} (-1)^j f^{(j)}(\beta, \sigma_3(\beta), \ldots, \sigma_{p-2}(\beta))\beta(p^{-1})/2-j \\
\equiv \beta(p^{-1})/2 + \sum_{j=1}^{(p-1)/2} (-1)^j a_j \beta(p^{-1})/2-j = F(\beta) \pmod{q},
\]

and

\[
0 \equiv F(\alpha^{(2p)^{-1}} + \bar{\alpha}^{(2p)^{-1}}) \\
= 1 + \sum_{k=1}^{(p-1)/2} \left[ (\alpha^{(2p)^{-1}})k + (\bar{\alpha}^{(2p)^{-1}})k \right] \pmod{\Omega}.
\]

Multiplying both sides of the above congruence by \( \alpha^{(2p)^{-1}(p-1)/2} = \bar{\alpha}^{(2p)^{-1}(p-1)/2} \) gives

\[
\sum_{k=0}^{p-1} \alpha^{(2p)^{-1}k} \equiv 0 \pmod{\Omega}.
\]

It implies that the image of \( \alpha^{(2p)^{-1}} \) has order \( p \) in the multiplicative group \( (D/\Omega)^* \), and so the image of \( \alpha^{2p^{-1}} \) has order \( p^r \). This multiplicative group has order \( N(\Omega) - 1 \) which divides \( q^{p-1} - 1 \), i.e., \( q^{p-1} \equiv 1 \pmod{p^r} \). By the assumption \( M \) is not divisible by all solutions of equation \( x^{p-1} \equiv 1 \pmod{p^r} \) between \( 1 \) and \( \sqrt{p} \). Then \( q > p^r > \sqrt{(2p)^r + 1} = \sqrt{M} \), thus \( q > \sqrt{M} \) for arbitrary prime divisor \( q \) of \( M \), that is to say \( M \) is prime.

(ii) If \( S_{r-1}^{(j)} \equiv (\beta a_j \pmod{M} \), then \( f^{(j)}(\beta, \sigma_3(\beta), \ldots, \sigma_{p-2}(\beta)) \equiv (-1)^j a_j \pmod{q} \), hence we get

\[
0 = \beta(p^{-1})/2 + \sum_{j=1}^{(p-1)/2} (-1)^j f^{(j)}(\beta, \sigma_3(\beta), \ldots, \sigma_{p-2}(\beta))\beta(p^{-1})/2-j \\
\equiv \beta(p^{-1})/2 + \sum_{j=1}^{(p-1)/2} a_j \beta(p^{-1})/2-j = (-1)(p^{-1})/2 F(-\beta) \pmod{q},
\]

and

\[
0 \equiv F(-\alpha^{(2p)^{-1}} - \bar{\alpha}^{(2p)^{-1}}) \\
= 1 + \sum_{k=1}^{(p-1)/2} \left[ (\alpha^{(2p)^{-1}})k + (\bar{\alpha}^{(2p)^{-1}})k \right] \pmod{\Omega}.
\]
Again multiplying both sides of the above congruence by $\alpha^{(2p)^{r-1}.(p-1)/2} = \tilde{\alpha}^{(2p)^{r-1}.(p-1)/2}$ gives
\[
\sum_{k=0}^{p-1} (-1)^{k-(p-1)/2} \alpha^{(2p)^{r-1}k} \equiv 0 \pmod{\Omega}.
\]
That is,
\[
\sum_{k=0}^{p-1} (-1)^{k} \alpha^{(2p)^{r-1}k} \equiv 0 \pmod{\Omega}.
\]
Hence we obtain the image of $\alpha^{(2p)^{r-1}}$ has order $2p$ in the multiplicative group $(D/\Omega)^*$, and so the image of $\alpha$ has order $(2p)^r$. The same as above we have $(2p)^r$ divides the order of group $(D/\Omega)^*$, and $q^{p^{-1}} \equiv 1 \pmod{(2p)^r}$. By the assumption we similarly get $q > (2p)^r + 1 = \sqrt{M}$, and $q > \sqrt{M}$ for arbitrary prime divisor $q$ of $M$, then $M$ is prime. This completes the proof of sufficiency.

**Remark.** (i) The methods on how to find $\pi$ and how to solve the equation $x^{p^{-1}} \equiv 1 \pmod{(2p)^r}$ involved in Theorem 3.1 can be found in [3]. In general, this is not a difficult matter.

(ii) The testing sequences $\{s_k^{(j)}|k \geq 0\}, 1 \leq j \leq (p - 1)/2$ are sequences of rational numbers. Hence the computation of the sufficient and necessary condition in Theorem 3.1 is taken in residue class ring $\mathbb{Z}/MZ$. And this benefits the explicit realization of the primality test for $M$ in the algorithm.

## 4 Primality tests for $p \leq 19$

We know from [3] Chapter 11 that $\mathbb{Z}[\zeta_{2p}]$ is a PID for $p \leq 19$. In this section we apply Theorem 3.1 to the cases $3 \leq p \leq 19$ with $p$ prime. By the definition of $G_k(x)$ in the previous section, we can compute $G_k(x)$, $0 \leq k \leq 9$, as follows:

- $G_0(x) = 1$, $G_1(x) = x$, $G_2(x) = x^2 - 2$, $G_3(x) = x^3 - 3x$, $G_4(x) = x^4 - 4x^2 + 2$,
- $G_5(x) = x^5 - 5x^3 + 5x$, $G_6(x) = x^6 - 6x^4 + 9x^2 - 2$, $G_7(x) = x^7 - 7x^5 + 14x^3 - 7x$,
- $G_8(x) = x^8 - 8x^6 + 20x^4 - 16x^2 + 2$, $G_9(x) = x^9 - 9x^7 + 27x^5 - 30x^3 + 9x$.

And we denote $F_1(x) = G_0(x) + G_1(x) = x + 1$, $F_2(x) = G_0(x) + G_1(x) + G_2(x) = x^2 + x - 1$, $F_3(x) = \sum_{k=0}^{3} G_k(x) = x^3 + x^2 - 2x - 1$, $F_4(x) = \sum_{k=0}^{5} G_k(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$,
- $F_5(x) = \sum_{k=0}^{6} G_k(x) = x^6 + 5x^5 - 4x^4 - 4x^3 + 6x^2 + 3x - 1$, $F_6(x) = \sum_{k=0}^{8} G_k(x) = x^8 + x^7 - 7x^6 - 6x^5 + 15x^4 + 10x^3 - 10x^2 - 4x + 1$,
- $F_7(x) = \sum_{k=0}^{9} G_k(x) = x^9 + x^8 - 8x^7 - 7x^6 + 21x^5 + 15x^4 - 20x^3 - 10x^2 + 5x + 1$.

Next we can obtain 7 special primality tests.

**Proposition 4.1.** Let $M = 6^{2^n} + 1$, $n \geq 1$ and $r = 2^n$. Let $\pi = 2 + 3 \zeta_3 \in \mathbb{Z}[\zeta_3]$, and $\alpha = \pi/\tilde{\pi}$. We define sequence $\{S_k\}$ with $S_0 = \alpha + \tilde{\alpha}$ and $S_{k+1} = S_k^2 - 6S_k^3 + 9S_k^2 - 2$ for $k \geq 0$. Then $M$ is prime if and only if $S_{r-1} \equiv -1 \pmod{M}$. 


Proof. We let $L = \mathbb{Q}(\zeta_6)$, then $\text{Norm}_{L/\mathbb{Q}}(\pi) = \pi \bar{\pi} = (2 + 3\zeta_3)(-1 - 3\zeta_3) = 7$. Since $M \equiv 2 \pmod{7}$, then we have $(\frac{M}{\pi})_6 \equiv M^{(7-1)/6} = M \equiv 2 \equiv \zeta_3^3 \pmod{\pi}$, and $(\frac{M}{\pi})_6 = \zeta_3^3$.

Let $S_k = \alpha^{6k} + \alpha^{6k}, \ k \geq 0$. We can easily verify that $S_k$ satisfies the recurrent relation in the assumption. We use the same polynomial $F(x)$ as in the above section. Here $F(x) = F_1(x) = x + 1$ implies $a_1 = -1$. Hence by the necessity of Theorem 3.1, we obtain that if $M$ is prime then $S_{r-1} \equiv -1 \pmod{M}$. This completes the proof of necessity.

And by the proof of sufficiency of Theorem 3.1, we get if $S_{r-1} \equiv -1 \pmod{M}$, then $3^r$ divides $q^2 - 1$ for arbitrary prime divisor $q$ of $M$, i.e., $3^r$ divides one of $q + 1$ and $q - 1$ because of $\gcd(q + 1, q - 1) = 2$. So $q \geq 3^r - 1 > \sqrt{6^r + 1} = \sqrt{M}$ and we get $M$ is prime. This completes the proof of sufficiency.

Remark. This primality test is explicit. Compare to the primality test of $G_n = 6^{2^n} + 1$ in [9] they own considerable complexity of running time which is polynomial in $\log_2(M)$. And it seems that our test is more succinct.

Proposition 4.2. Let $M = 10^{2^n} + 1$, $n \geq 1$ and $r = 2^n$. Let $\pi = 1 - \zeta_5 - \zeta_5^3 \in \mathbb{Z}[\zeta_{10}]$, $\alpha = (\pi/\bar{\pi})^{1+3\sigma_3}$. We define sequences $\{S^{(1)}_k\}$ and $\{S^{(2)}_k\}$ with $S^{(1)}_k = \alpha_1(k) + \alpha_2(k)$, $S^{(2)}_k = \alpha_1(k) \cdot \alpha_2(k)$, $k \geq 0$, where $\alpha_1(k) = \alpha^{10k} + \alpha^{10k}$, $\alpha_2(k) = \sigma_3(\alpha_1(k))$. Suppose that if $x^4 \equiv 1 \pmod{5^r}$ and $1 < x < 5^r$ then $x$ does not divide $M$. Then $M$ is prime if and only if $S^{(1)}_{r-1} \equiv 1 \equiv -S^{(2)}_{r-1} \pmod{M}$.\[Q.E.D.\]

Proof. We let $L = \mathbb{Q}(\zeta_{10})$, then $\text{Norm}_{L/\mathbb{Q}}(\pi) = (\pi \bar{\pi})^{1+3\sigma_3} = 11$. Since $M \equiv 2 \pmod{11}$, $n \geq 1$, then $(\frac{M}{\pi})_1 \equiv M^{(11-1)/10} = M \equiv 2 \equiv -\zeta_5 \pmod{\pi}$, and $(\frac{M}{\pi})_1 = -\zeta_5$. We notice that here $F(x) = F_2(x) = x^2 + x - 1$, which implies $a_1 = -1$, $a_2 = -1$. Hence by the necessity of Theorem 3.1 we obtain that if $M$ is prime then $S^{(1)}_{r-1} \equiv -a_1 = 1 \pmod{M}$, $S^{(2)}_{r-1} \equiv a_2 = -1 \pmod{M}$. This completes the proof of necessity.

By the sufficiency of Theorem 3.1, we easily get if $S^{(1)}_{r-1} \equiv 1 \equiv -S^{(2)}_{r-1} \pmod{M}$, i.e., $S^{(1)}_{r-1} \equiv -a_1 \pmod{M}$, $S^{(2)}_{r-1} \equiv a_2 \pmod{M}$, then by assumption we have $M$ is prime. This completes the proof of sufficiency.\[Q.E.D.\]

Proposition 4.3. Let $M = 14^{2^n} + 1$, $n > 1$ and $r = 2^n$. Let $\pi = 1 - \zeta_7 + \zeta_7^4 \in \mathbb{Z}[\zeta_{14}]$, $\alpha = (\pi / \bar{\pi})^{1+3\sigma_3+5\sigma_5}$. We define the sequences $\{S^{(1)}_k\}$, $\{S^{(2)}_k\}$ and $\{S^{(3)}_k\}$ with $S^{(1)}_k = \alpha_1(k) + \alpha_2(k) + \alpha_3(k)$, $S^{(2)}_k = f(2)(\alpha_1(k), \alpha_2(k), \alpha_3(k))$, $S^{(3)}_k = \alpha_1(k) \alpha_2(k) \alpha_3(k)$, $k \geq 0$, where $\alpha_1(k) = \alpha^{14k} + \alpha^{14k}$, $\alpha_2(k) = \sigma_3(\alpha_1(k))$, $\alpha_3(k) = \sigma_5(\alpha_1(k))$. Suppose that if $x^6 \equiv 1 \pmod{7^r}$ and $1 < x < 7^r$ then $x$ does not divide $M$. Then $M$ is prime if and only if one of the following holds:

(i) $M \equiv \pm 8 \pmod{29}$ and $S^{(1)}_{r-1} \equiv 1 \equiv -S^{(3)}_{r-1} \pmod{M}$, $S^{(2)}_{r-1} \equiv -2 \pmod{M}$;

(ii) $M \equiv -5 \pmod{29}$ and $S^{(1)}_{r-1} \equiv -1 \equiv -S^{(3)}_{r-1} \pmod{M}$, $S^{(2)}_{r-1} \equiv -2 \pmod{M}$.

Proof. We let $L = \mathbb{Q}(\zeta_{14})$, then $\text{Norm}_{L/\mathbb{Q}}(\pi) = (\pi \bar{\pi})^{1+3\sigma_3+5\sigma_5} = 29$. Since $M \equiv \pm 8, -5 \pmod{29}$, $n > 1$, we have $(\frac{M}{\pi})_{14} \equiv M^{(29-1)/14} = M^2 \equiv 6, -4 \equiv -\zeta_7^2, \zeta_7 \pmod{\pi}$, and
We notice that here $F(x) = F_3(x) = x^3 + x^2 - 2x - 1$, which implies $a_1 = -1$, $a_2 = -2$, $a_3 = 1$. Hence by the necessity of Theorem 3.1 if $M$ is prime and $M \equiv \pm 8 \pmod{29}$, then we obtain $S_{r-1}^{(1)} \equiv -a_1 = 1 \pmod{M}$, $S_{r-1}^{(2)} \equiv a_2 = -2 \pmod{M}$, $S_{r-1}^{(3)} \equiv -a_3 = -1 \pmod{M}$ because of $(\frac{M}{\pi})_{14} = -\zeta_7$. If $M$ is prime and $M \equiv -5 \pmod{29}$, then we similarly get $S_{r-1}^{(1)} \equiv -1 = -S_{r-1}^{(3)} \pmod{M}$, $S_{r-1}^{(2)} \equiv -2 \pmod{M}$ for $(\frac{M}{\pi})_{14} = \zeta_7$. This completes the proof of necessity.

By the sufficiency of Theorem 3.1 whatever (i) or (ii) holds, we can always deduce that $M$ is a prime. This completes the proof of sufficiency.

Proposition 4.4. Let $M = 22^{2n} + 1$, $n \geq 1$ and $r = 2^n$. Let $\pi = 1 + \zeta_{11}^7 + \zeta_{11}^8 \in \mathbb{Z}[\zeta_{22}]$, $\alpha = (\pi/\bar{\pi})^7$, where $\pi = 1 + 3\sigma_7 + 5\sigma_9 + 7\sigma_{11} + 9\sigma_{13} + 11\sigma_{29}$. We define the sequences $\{S_k^{(j)}(\pi)\}$, $1 \leq j \leq 5$ with $S_k^{(j)} = f^{(j)}(\alpha_1^{(k)}, \ldots, \alpha_5^{(k)})$, $k \geq 0$, where $\alpha_1^{(k)} = \alpha^{22k} + \alpha^{-22k}$, and $\alpha_i^{(k)} = \zeta_{29i-1}(\alpha_1^{(k)})$, $2 \leq i \leq 5$. Suppose that if $x^{10} \equiv 1 \pmod{11^r}$ and $1 < x < 11^r$ then $x$ does not divide $M$. Then $M$ is prime if and only if $S_{r-1}^{(1)} \equiv -1 \equiv S_{r-1}^{(5)} \pmod{M}$, $S_{r-1}^{(2)} \equiv -4 \pmod{M}$ and $S_{r-1}^{(3)} \equiv 3 \equiv S_{r-1}^{(4)} \pmod{M}$.

Proof. We let $L = \mathbb{Q}(\zeta_{22})$, then $Norm_{L/\mathbb{Q}}(\pi) = (\pi/\bar{\pi})^{\zeta_{29}} \sum_{i=1}^{5} \zeta_{29i-1} = 23$. Since $M \equiv 2 \pmod{23}$, $n \geq 1$, we have $(\frac{M}{\alpha})_{22} \equiv M^{(23-1)/2} = M \equiv 2 \equiv \zeta_{11}^2 \pmod{29}$, and $(\frac{M}{\alpha})_{22} = \zeta_{11}^2$. Also we notice that here $F(x) = F_4(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$, which implies $a_1 = -1$, $a_2 = -4$, $a_3 = 3$, $a_4 = 3$, $a_5 = -1$. Hence by the necessity of Theorem 3.1 we have if $M$ is prime, then $S_{r-1}^{(1)} \equiv a_1 = -1 \pmod{M}$, $S_{r-1}^{(2)} \equiv a_2 = -4 \pmod{M}$, $S_{r-1}^{(3)} \equiv a_3 = 3 \pmod{M}$, $S_{r-1}^{(4)} \equiv a_4 = 3 \pmod{M}$, $S_{r-1}^{(5)} \equiv a_5 = -1 \pmod{M}$. This completes the proof of necessity.

By the sufficiency of Theorem 3.1 and the assumption, clearly we have $M$ is a prime. This completes the proof of sufficiency.

Proposition 4.5. Let $M = 26^{2n} + 1$, $n > 1$ and $r = 2^n$. Let $\pi = 1 + \zeta_{13}^7 + \zeta_{13}^8 \in \mathbb{Z}[\zeta_{26}]$, $\alpha = (\pi/\bar{\pi})^7$, where $\pi = 1 + 3\sigma_7 + 5\sigma_9 + 7\sigma_{11} + 9\sigma_{13} + 11\sigma_{29}$. We define the sequences $\{S_k^{(j)}(\pi)\}$, $1 \leq j \leq 6$ with $S_k^{(j)} = f^{(j)}(\alpha_1^{(k)}, \ldots, \alpha_6^{(k)})$, $k \geq 0$, where $\alpha_1^{(k)} = \alpha^{26k} + \alpha^{-26k}$, and $\alpha_i^{(k)} = \zeta_{29i-1}(\alpha_1^{(k)})$, $2 \leq i \leq 6$. Suppose that if $x^{12} \equiv 1 \pmod{13^r}$ and $1 < x < 13^r$ then $x$ does not divide $M$. Then $M$ is prime if and only if one of the following holds:

(i) $M \equiv 25, \pm 16, -6, 11, -24, -5, -10, 17 \pmod{53}$ and $S_{r-1}^{(1)} \equiv -1 \equiv S_{r-1}^{(6)} \pmod{M}$, $S_{r-1}^{(2)} \equiv -5 \pmod{M}$, $S_{r-1}^{(3)} \equiv 4 \pmod{M}$, $S_{r-1}^{(4)} \equiv 6 \pmod{M}$ and $S_{r-1}^{(5)} \equiv -3 \pmod{M}$;

(ii) $M \equiv 14, -8, -3 \pmod{53}$ and $S_{r-1}^{(1)} \equiv 1 \equiv -S_{r-1}^{(6)} \pmod{M}$, $S_{r-1}^{(2)} \equiv -5 \pmod{M}$, $S_{r-1}^{(3)} \equiv -4 \pmod{M}$, $S_{r-1}^{(4)} \equiv 6 \pmod{M}$ and $S_{r-1}^{(5)} \equiv 3 \pmod{M}$.

Proof. We let $L = \mathbb{Q}(\zeta_{26})$, then $Norm_{L/\mathbb{Q}}(\pi) = (\pi/\bar{\pi})^{\zeta_{29}} \sum_{i=1}^{6} \zeta_{29i-1} = 53$. Since $M \equiv 25, \pm 16, -6, 11, -24, -5, -10, 17, 14, -8, -3 \pmod{53}$, $n > 1$, we have $(\frac{M}{\pi})_{26} \equiv M^{(53-1)/2} = M^2 \equiv -11, -9, -17, 15, -7, 25, -6, 24, -16, 11, 9 \equiv \zeta_{13}^3, \zeta_{13}^5, \zeta_{13}^7, \zeta_{13}^8, \zeta_{13}^9, -\zeta_{13}^2, -\zeta_{13}^3, -\zeta_{13}^4, -\zeta_{13}^5, -\zeta_{13}^7, -\zeta_{13}^8, \zeta_{13}^9, \zeta_{13}^{10}, -\zeta_{13}^2, -\zeta_{13}^3, -\zeta_{13}^4, -\zeta_{13}^5, -\zeta_{13}^7, -\zeta_{13}^8, \zeta_{13}^9, \zeta_{13}^{10}$.
we notice that \( F(x) = F_5(x) = x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 3x - 1 \), which implies \( a_1 = -1 \), \( a_2 = -5 \), \( a_3 = 4 \), \( a_4 = 6 \), \( a_5 = -3 \), \( a_6 = -1 \). Hence by the necessity of Theorem 3.1 if \( M \) is prime and \( M \equiv 25, \pm 16, -6, 11, -24, -5, -10 \pmod{53} \) then we obtain \( S_{r-1}^{(1)} \equiv a_1 = -1 \pmod{M}, S_{r-1}^{(2)} \equiv a_2 = -5 \pmod{M}, S_{r-1}^{(3)} \equiv a_3 = 4 \pmod{M}, S_{r-1}^{(4)} \equiv a_4 = 6 \pmod{M}, S_{r-1}^{(5)} \equiv a_5 = -3 \pmod{M}, S_{r-1}^{(6)} \equiv a_6 = -1 \pmod{M} \), due to here \( (\frac{M}{\pi})_{26} \) is a primitive 13-th root of unity. If \( M \) is prime and \( M \equiv 14, -8, -3 \pmod{53} \), then we also can get \( S_{r-1}^{(1)} \equiv 1 \equiv -S_{r-1}^{(6)} \pmod{M}, S_{r-1}^{(2)} \equiv -5 \pmod{M}, S_{r-1}^{(3)} \equiv -4 \pmod{M}, S_{r-1}^{(4)} \equiv 6 \pmod{M}, S_{r-1}^{(5)} \equiv 3 \pmod{M} \) because \( (\frac{M}{\pi})_{26} \) is a primitive 26-th root of unity. This completes the proof of necessity.

Next by the sufficiency of Theorem 3.1 and the assumption, whatever (i) or (ii) holds, we can easily deduce that \( M \) is a prime. This completes the proof of sufficiency.

\[ \square \]

**Proposition 4.6.** Let \( M = 342^n + 1, n \geq 1 \) and \( r = 2^n \). Let \( \pi = 1 + \zeta_{17}^2 + \zeta_{17}^9 \in \mathbb{Z}[\zeta_{34}] \), \( \alpha = (\pi/\bar{\pi})^{\tau} \), where \( \tau = 1 + 3\sigma_{11} + 5\sigma_7 + 7\sigma_5 + 9\sigma_{13} + 11\sigma_9 + 13\sigma_{15} + 17\sigma_9 \). We define the sequences \( \{S_k^{(j)}\} \), \( 1 \leq j \leq 8 \) with \( S_k^{(j)} = f_j(\alpha_1^{(k)}, \ldots, \alpha_8^{(k)}) \), \( k \geq 0 \), where \( \alpha_1^{(k)} = \alpha^{34k} + \bar{\alpha}^{34k} \), and \( \alpha_2^{(k)} = \sigma_{2i-1}(\alpha_1^{(k)}) \), \( 2 \leq i \leq 8 \). Suppose that if \( x^{16} = 1 \pmod{17^r} \) and \( 1 < x < 17^r \) then \( x \) does not divide \( M \). Then \( M \) is prime if and only if one of the following holds:

(i) \( M \equiv -21, 15 \pmod{103} \) and \( S_{r-1}^{(1)} \equiv -1 \equiv -S_{r-1}^{(8)} \pmod{M}, S_{r-1}^{(2)} \equiv -7 \pmod{M}, S_{r-1}^{(3)} \equiv 6 \pmod{M}, S_{r-1}^{(4)} \equiv 15 \pmod{M}, S_{r-1}^{(5)} \equiv -10 \equiv S_{r-1}^{(6)} \pmod{M} \) and \( S_{r-1}^{(7)} \equiv 4 \pmod{M} \);

(ii) \( M \equiv 35, 24, -2, -9, 10, -30 \pmod{103} \) and \( S_{r-1}^{(1)} \equiv 1 \equiv S_{r-1}^{(6)} \pmod{M}, S_{r-1}^{(2)} \equiv -7 \pmod{M}, S_{r-1}^{(3)} \equiv -6 \pmod{M}, S_{r-1}^{(4)} \equiv 15 \pmod{M}, S_{r-1}^{(5)} \equiv 10 \equiv -S_{r-1}^{(6)} \pmod{M} \) and \( S_{r-1}^{(7)} \equiv 4 \pmod{M} \).

**Proof** We let \( L = \mathbb{Q}(\zeta_{34}) \), then \( \text{Norm}_{\mathbb{Q}(\zeta_{34})/\mathbb{Q}((\pi))} = (\pi \bar{\pi})^{\sum_{i=1}^{8} \sigma_{2i-1}} = 103. \) Since \( M \equiv -21, 15, 35, 24, -2, -9, 10, -30 \pmod{103}, n \geq 1 \), we have \( (\frac{M}{\pi})_{34} \equiv M^{(103-1)/34} = M^3 \equiv 9, -24, 27, 22, -8, -30, -14 \equiv \zeta_1^{7}, -\zeta_1^{7}, -\zeta_1^{7}, -\zeta_1^{10}, -\zeta_1^{10}, -\zeta_1^{10}, -\zeta_1^{10}, -\zeta_1^{10} \pmod{\pi} \), where \( (-2)^3 \equiv (-9)^3 \equiv -8 \pmod{103} \), thus \( (\frac{M}{\pi})_{34} = \zeta_1^{7}, -\zeta_1^{7}, -\zeta_1^{7}, -\zeta_1^{10}, -\zeta_1^{10}, -\zeta_1^{10}, -\zeta_1^{10}, -\zeta_1^{10} \). Again we notice that \( F(x) = F_6(x) = x^8 + x^7 - 7x^6 - 6x^5 + 15x^4 + 10x^3 - 10x^2 - 4x + 1 \), which implies \( a_1 = -1, a_2 = -7, a_3 = 6, a_4 = 15, a_5 = -10, a_6 = -10, a_7 = 4, a_8 = 1 \). Hence by the necessity of Theorem 3.1 if \( M \) is prime and \( M \equiv -21, 15 \pmod{103} \) then we obtain \( S_{r-1}^{(1)} \equiv a_1 = -1 \pmod{M}, S_{r-1}^{(2)} \equiv a_2 = -7 \pmod{M}, S_{r-1}^{(3)} \equiv a_3 = 6 \pmod{M}, S_{r-1}^{(4)} \equiv a_4 = 15 \pmod{M}, S_{r-1}^{(5)} \equiv a_5 = -10 \pmod{M}, S_{r-1}^{(6)} \equiv a_6 = -10 \pmod{M}, S_{r-1}^{(7)} \equiv a_7 = 4 \pmod{M}, S_{r-1}^{(8)} \equiv a_8 = 1 \pmod{M} \), since here \( (\frac{M}{\pi})_{34} \) is a primitive 17-th root of unity. If \( M \) is prime and \( M \equiv 35, 24, -2, -9, 10, -30 \pmod{103} \), then we also obtain \( S_{r-1}^{(1)} \equiv 1 \equiv S_{r-1}^{(6)} \pmod{M}, S_{r-1}^{(2)} \equiv -7 \pmod{M}, S_{r-1}^{(3)} \equiv -6 \pmod{M}, S_{r-1}^{(4)} \equiv 15 \pmod{M}, S_{r-1}^{(5)} \equiv 10 \equiv -S_{r-1}^{(6)} \pmod{M}, S_{r-1}^{(7)} \equiv -4 \pmod{M} \), because \( (\frac{M}{\pi})_{34} \) is a primitive...
34-th root of unity. This completes the proof of necessity.

Similarly by the sufficiency of Theorem \ref{thm:34-root-of-unity} and the assumption, whatever (i) or (ii) holds, we always deduce that \( M \) is a prime. This completes the proof of sufficiency. \( \square \)

**Proposition 4.7.** Let \( M = 38^{2n} + 1 \), \( n > 1 \) and \( r = 2^n \). Let \( \pi = -\zeta_2^2 + \zeta_9^{15} \in \mathbb{Z}[\zeta_{38}] \), \( \alpha = (\pi/\bar{\pi})^r \), where \( \tau = 1 + 3\sigma_{13} + 5\sigma_{-15} + 7\sigma_{11} + 9\sigma_{17} + 11\sigma_{7} + 13\sigma_{3} + 15\sigma_{-5} + 17\sigma_{9} \). We define the sequences \( \{S^{(j)}_k\} \), \( 1 \leq j \leq 9 \) with \( S^{(j)}_k = f^{(j)}(\alpha^{(k)}_1, \ldots, \alpha^{(k)}_9) \), \( k \geq 0 \), where \( \alpha^{(k)}_1 = \alpha^{38k} + \bar{\alpha}^{38k} \), and \( \alpha^{(k)}_i = \sigma_{2i-1}(\alpha^{(k)}_1) \), \( 2 \leq i \leq 9 \). Suppose that if \( x^{18} \equiv 1 \pmod{19r} \) and \( 1 < x < 19r \) then \( x \) does not divide \( M \). Then \( M \) is prime if and only if one of the following holds:

(i) \( M \equiv -48, -44, 15, -4, 56, -55, -45, -61, 26, 49 \pmod{229} \) and \( S^{(1)}_{r-1} \equiv -1 \pmod{4} \), \( S^{(2)}_{r-1} \equiv -8 \pmod{4} \), \( S^{(3)}_{r-1} \equiv 7 \pmod{4} \), \( S^{(4)}_{r-1} \equiv 21 \pmod{4} \), \( S^{(5)}_{r-1} \equiv -15 \pmod{4} \), \( S^{(6)}_{r-1} \equiv -20 \pmod{4} \), \( S^{(7)}_{r-1} \equiv 10 \pmod{4} \) and \( S^{(8)}_{r-1} \equiv 5 \pmod{4} \);

(ii) \( M \equiv -98, 38, 92, -69, 112, -35, -77, -32 \pmod{229} \) and \( S^{(1)}_{r-1} \equiv 1 \pmod{4} \), \( S^{(2)}_{r-1} \equiv -8 \pmod{4} \), \( S^{(3)}_{r-1} \equiv 7 \pmod{4} \), \( S^{(4)}_{r-1} \equiv 21 \pmod{4} \), \( S^{(5)}_{r-1} \equiv 15 \pmod{4} \), \( S^{(6)}_{r-1} \equiv -20 \pmod{4} \), \( S^{(7)}_{r-1} \equiv -10 \pmod{4} \) and \( S^{(8)}_{r-1} \equiv 5 \pmod{4} \).

**Proof.** We let \( L = \mathbb{Q}(\zeta_{38}) \), then \( \text{Norm}_{L/\mathbb{Q}}(\pi) = (\pi/\bar{\pi})^{\sum_{i=1}^{9} \sigma_{2i-1}} = 229 \). Since \( M \equiv -48, -44, 15, -4, 56, -55, -45, -61, 26, 49 \pmod{229} \), \( n > 1 \), we have \( (\frac{M}{\pi})_{38} \equiv M^{(229-1)/38} = M^6 \equiv -4, 16, -64, -26, 42, -15, 60, -68 \).

43, 4, -42, 15, -60, -44, -53, -17 \( \equiv \zeta_{19}, \zeta_{2}^{19}, \zeta_{3}^{19}, \zeta_{6}^{19}, \zeta_{8}^{19}, \zeta_{10}^{19}, \zeta_{11}^{19}, \zeta_{16}^{19}, \zeta_{17}^{19}, -\zeta_{19}, -\zeta_{8}^{19}, -\zeta_{6}^{19}, -\zeta_{6}^{19}, -\zeta_{13}^{19}, -\zeta_{14}^{19}, -\zeta_{15}^{19} \pmod{\pi} \), where \( 26^6 \equiv 49^6 \equiv 43 \pmod{229} \) and \( 38^6 \equiv 92^6 \equiv -42 \pmod{229} \), thus we get \( (\frac{M}{\pi})_{38} = \zeta_{19}, \zeta_{2}^{19}, \zeta_{3}^{19}, \zeta_{6}^{19}, \zeta_{8}^{19}, \zeta_{10}^{19}, \zeta_{11}^{19}, \zeta_{16}^{19}, \zeta_{17}^{19}, -\zeta_{19}, -\zeta_{8}^{19}, -\zeta_{6}^{19}, -\zeta_{6}^{19}, -\zeta_{13}^{19}, -\zeta_{14}^{19}, -\zeta_{15}^{19} \). Here we notice that \( P(x) = F_{7}(x) = x^7 + x^6 - 8x^7 - 7x^5 + 21x^5 + 15x^4 - 20x^3 - 10x^2 + 5x + 1 \), which implies \( a_1 = -1, a_2 = -8, a_3 = 7, a_4 = 21, a_5 = -15, a_6 = -20, a_7 = 10, a_8 = 5, a_9 = -1 \). Hence by the necessity of Theorem \ref{thm:34-root-of-unity} if \( M \) is prime and \( M \equiv -48, -44, 15, -4, 56, -55, -45, -61, 26, 49 \pmod{229} \), we have \( S^{(1)}_{r-1} \equiv a_1 = -1 \pmod{4} \), \( S^{(2)}_{r-1} \equiv a_2 = -8 \pmod{4} \), \( S^{(3)}_{r-1} \equiv a_3 = 7 \pmod{4} \), \( S^{(4)}_{r-1} \equiv a_4 = 21 \pmod{4} \), \( S^{(5)}_{r-1} \equiv a_5 = -15 \pmod{4} \), \( S^{(6)}_{r-1} \equiv a_6 = -20 \pmod{4} \), \( S^{(7)}_{r-1} \equiv a_7 = 10 \pmod{4} \), \( S^{(8)}_{r-1} \equiv a_8 = 5 \pmod{4} \), \( S^{(9)}_{r-1} \equiv a_9 = -1 \pmod{4} \), since every \( (\frac{M}{\pi})_{38} \) is a primitive 19-th root of unity. If \( M \) is prime and \( M \equiv -98, 38, 92, -69, 112, -35, -77, -32 \pmod{229} \), then we can obtain \( S^{(1)}_{r-1} \equiv 1 \pmod{4} \), \( S^{(2)}_{r-1} \equiv -8 \pmod{4} \), \( S^{(3)}_{r-1} \equiv 7 \pmod{4} \), \( S^{(4)}_{r-1} \equiv 21 \pmod{4} \), \( S^{(5)}_{r-1} \equiv 15 \pmod{4} \), \( S^{(6)}_{r-1} \equiv -20 \pmod{4} \), \( S^{(7)}_{r-1} \equiv -10 \pmod{4} \), \( S^{(8)}_{r-1} \equiv 5 \pmod{4} \), because \( (\frac{M}{\pi})_{38} \) is a primitive 38-th root of unity. This completes the proof of necessity.

Next by the sufficiency of Theorem \ref{thm:34-root-of-unity} and the assumption, we have whatever (i) or (ii) holds, \( M \) is prime. This completes the proof of sufficiency. \( \square \)

**Remark.** (i) For \( 5 \leq p \leq 19 \), every element \( \alpha^{(k+1)}_i \), \( 1 \leq i \leq (p - 1)/2 \) exists a
polynomial recurrent relation about the corresponding $a_i^{(k)}$. By the definition of $S_k^{(j)}$, $1 \leq j \leq (p-1)/2$, we may get a recurrent relation for every $S_k^{(j)}$ on all elements $S_k^{(i)}$, $1 \leq i \leq (p-1)/2$. Hence there are $(p-1)/2$ many recurrent relations theoretically.

(ii) Once the recurrent relations are given, we will acquire the explicit primality tests for $p \geq 5$. And we will show the definite recurrent relations of $S_k^{(j)}$, $j = 1, 2$, involved in Proposition 4.2 for $p = 5$ later. Then we could deduce the explicit primality test for $p = 5$. As to others $p \geq 7$ the recurrent relations may be very long and complicated, we don’t write down them in the paper.

5 Implementation and Computational results

In this section we will verify the correctness of the algorithms related to Propositions 4.1 and 4.2. Let $G_n = 6^{2^n} + 1$ and $H_n = 10^{2^n} + 1$. First we make some preparations for the case $p = 5$. When $k \geq 0$, the recurrent relations of $S_k^{(j)}$, $j = 1, 2$, involved in Proposition 4.2 can be obtained as follows.

By the definition of $a_1^{(k)}$ and $a_2^{(k)}$, we have

$$a_1^{(k+1)} = (a_1^{(k)})^5 - 10(a_1^{(k)})^4 + 35(a_1^{(k)})^3 - 50(a_1^{(k)})^2 + 25(a_1^{(k)})^2 - 2.$$ 

Hence

$$a_2^{(k+1)} = \sigma_3(a_1^{(k+1)}) = (a_2^{(k)})^5 - 10(a_2^{(k)})^4 + 35(a_2^{(k)})^3 - 50(a_2^{(k)})^2 + 25(a_2^{(k)})^2 - 2.$$ 

From the expression of $S_k^{(1)}$ and $S_k^{(2)}$ in Proposition 4.2 and under some computations, we get

$$S_{k+1}^{(1)} = (S_k^{(1)})^5 - 10(S_k^{(1)})^4 S_k^{(2)} + 35(S_k^{(1)})^3 (S_k^{(2)})^2 - 50(S_k^{(1)})^2 (S_k^{(2)})^3$$

$$+ 25(S_k^{(1)}) (S_k^{(2)})^4 - 10(S_k^{(1)})^8 + 80(S_k^{(1)})^6 S_k^{(2)} - 200(S_k^{(1)})^4 (S_k^{(2)})^2$$

$$+ 160(S_k^{(1)})^2 (S_k^{(2)})^3 - 20(S_k^{(2)})^4 + 35(S_k^{(1)})^6 - 210(S_k^{(1)})^4 S_k^{(2)}$$

$$+ 315(S_k^{(1)})^2 (S_k^{(2)})^2 - 70(S_k^{(2)})^3 - 50(S_k^{(1)})^4 + 200(S_k^{(1)})^2 S_k^{(2)}$$

$$- 100(S_k^{(2)})^2 - 2(S_k^{(2)})^5 + 25(S_k^{(1)})^2 - 50S_k^{(2)} - 4,$$
and

\[ S_{k+1}^{(2)} = (S_k^{(2)})^{10} + 20(S_k^{(2)})^9 - 10(S_k^{(1)})^2(S_k^{(2)})^8 + 170(S_k^{(2)})^8 - 140(S_k^{(1)})^2(S_k^{(2)})^7 + 800(S_k^{(2)})^7 + 35(S_k^{(1)})^4(S_k^{(2)})^6 - 800(S_k^{(1)})^2(S_k^{(2)})^6 + 2275(S_k^{(2)})^6 + 300(S_k^{(1)})^4(S_k^{(2)})^5 - 2400(S_k^{(1)})^2(S_k^{(2)})^5 + 4004(S_k^{(2)})^5 - 50(S_k^{(1)})^6(S_k^{(2)})^4 + 1000(S_k^{(1)})^4(S_k^{(2)})^4 - 4050(S_k^{(1)})^2(S_k^{(2)})^4 + 4290(S_k^{(2)})^4 - 200(S_k^{(1)})^6(S_k^{(2)})^3 + 1600(S_k^{(1)})^4(S_k^{(2)})^3 - 3820(S_k^{(1)})^2(S_k^{(2)})^3 + 2640(S_k^{(1)})^3 + 25(S_k^{(1)})^8(S_k^{(2)})^2 - 320(S_k^{(1)})^6(S_k^{(2)})^2 + 1275(S_k^{(1)})^4(S_k^{(2)})^2 - 1880(S_k^{(1)})^2(S_k^{(2)})^2 + 825(S_k^{(2)})^2 + 20(S_k^{(1)})^8S_k^{(2)} - 160(S_k^{(1)})^6S_k^{(2)} + 420(S_k^{(1)})^4S_k^{(2)} - 400(S_k^{(1)})^2S_k^{(2)} - 2(S_k^{(1)})^{10} + 20(S_k^{(1)})^8 - 70(S_k^{(1)})^6 + 100(S_k^{(1)})^4 - 50(S_k^{(1)})^2 + 100S_k^{(2)} + 4. \]

With the above two recurrent formulas, we can easily reach the explicit primality test for numbers \( H_n \).

**Remark.** Though the expression of the above two recurrent relations is a bit long. Its corresponding algorithm has the same complexity of running time compared with the primality test of \( H_n \) in [9], which runs in polynomial time in \( \log_2(M) \).

We implement our two algorithms of \( p = 3 \) and \( p = 5 \) in Magma [2]. And our program is run on a personal computer with Intel Core i5-3470 3.20GHz CPU and 4GB memory.

We verified the correctness of our program by comparing with the results in [7]. Since the growth of numbers \( G_n \) and \( H_n \) is very fast on index \( n \). When \( n \geq 15 \) the computation involved in our program is out of memory in our personal computer. If we deal with better and more efficient representation of larger integers, we may test the primality of bigger \( G_n \) or \( H_n \). But this is not the stress of our paper. And we will not mention it any more here. We verified all numbers for \( p = 3, 5 \) in the range \( 1 \leq n < 15 \) and found no mistakes. Note that the assumption about equation \( x^4 \equiv 1 \pmod{5^r} \) in Proposition 4.2 holds for \( H_n, 1 \leq n < 15 \), by applying the algorithm in [3]. The prime numbers are rare on such \( G_n \) and \( H_n \). We list the following two tables to show all the cases and their corresponding cost time.
Table 1: Primality of $G_n = 6^{2^n} + 1(p = 3)$

| n  | $G_n$ | Primality | Time(s) |
|----|-------|-----------|---------|
| 1  | 37    | yes       | 0.011   |
| 2  | 1297  | yes       | 0.015   |
| 3 to 10 | -     | no        | 0.921   |
| 11 | -     | no        | 3.931   |
| 12 | -     | no        | 23.228  |
| 13 | -     | no        | 139.293 |
| 14 | -     | no        | 738.805 |

Table 2: Primality of $H_n = 10^{2^n} + 1(p = 5)$

| n  | $H_n$ | Primality | Time(s) |
|----|-------|-----------|---------|
| 1  | 101   | yes       | 0.015   |
| 2 to 10 | -     | no        | 7.909   |
| 11 | -     | no        | 37.004  |
| 12 | -     | no        | 204.579 |
| 13 | -     | no        | 1180.226|
| 14 | -     | no        | 6576.924|

Acknowledgments

The authors are very grateful to Yupeng Jiang and Chang Lv for their useful discussion. The work of this paper was supported by the NNSF of China (Grants Nos. 11071285, 61121062), 973 Project (2011CB302401) and the National Center for Mathematics and Interdisciplinary Sciences, CAS.

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