NEW CHARACTERIZATIONS OF RICCI CURVATURE ON RCD METRIC MEASURE SPACES

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(Communicated by Alessio Figalli)

Abstract. We prove that on a large family of metric measure spaces, if the $L^p$-gradient estimate for heat flows holds for some $p > 2$, then the $L^1$-gradient estimate also holds. This result extends Savaré’s result on metric measure spaces, and provides a new proof to von Renesse-Sturm theorem on smooth metric measure spaces. As a consequence, we propose a new analysis object based on Gigli’s measure-valued Ricci tensor, to characterize the Ricci curvature of RCD space in a local way. In the proof we adopt an iteration technique based on non-smooth Bakry-Émery theory, which is a new method to study the curvature dimension condition of metric measure spaces.

1. Introduction. For any smooth Riemannian manifold $M$ and any $K \in \mathbb{R}$, it is proved by von Renesse and Sturm in [14] that the following properties are equivalent

1) $\text{Ricci}_M \geq K$,

2) there exists $p \in (1, \infty)$ such that for all $f \in C^\infty_c(M)$, all $x \in M$ and $t \geq 0$

\[ |D\text{H}_t f|^p(x) \leq e^{-pKt} \text{H}_t |Df|^p(x), \]

(1)

3) for all $f \in C^\infty_c(M)$, all $x \in M$ and $t \geq 0$

\[ |D\text{H}_t f|(x) \leq e^{-Kt} \text{H}_t |Df|(x), \]

(2)

where $\text{H}_t f$ is the solution of the heat equation with initial datum $f$.

In non-smooth setting, the notions of synthetic Ricci curvature bounds, or non-smooth curvature-dimension conditions, were proposed by Lott-Villani and Sturm (see [13] and [16]) using optimal transport theory. Later on, by assuming the infinitesimally Hilbertianity (i.e. the Sobolev space $W^{1,2}$ is a Hilbert space), RCD condition (or RCD($K, \infty$) condition to emphasize the curvature) which is a refinement of Lott-Sturm-Villani’s curvature-dimension condition, was proposed by Ambrosio-Gigli-Savare (see [4] and [1]). It is known that RCD($K, \infty$) spaces are generalizations of Riemannian manifolds with lower Ricci curvature bound and their limit spaces, as well as Alexandrov spaces with lower curvature bound.

It is known that Lott-Sturm-Villani’s synthetic Ricci bound and 2-gradient estimate (for heat flows) are equivalent in non-smooth setting. Let $(X, d, \text{m})$ be a

2010 Mathematics Subject Classification. Primary: 47D07, 30L99; Secondary: 51F99.

Key words and phrases. Bakry-Émery theory, curvature dimension condition, gradient estimate, heat flow, metric measure space, Ricci curvature.

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RCD\( (K, \infty) \) space, it is proved in [4] that
\[
|DH_t f|^2 \leq e^{-2Kt} |H_t f|^2, \quad m - \text{a.e.} \tag{3}
\]
for any \( f \in W^{1,2} \) and \( t > 0 \), where \( H_t f \) is the heat flow from \( f \) and \( |Df| \) is the minimal weak upper gradient (or weak gradient for simplicity) of \( f \). In particular, by Hölder inequality we know
\[
|DH_t f|^p \leq e^{-pKt} |H_t f|^p, \quad m - \text{a.e.} \tag{4}
\]
for any \( p \geq 2 \). Furthermore, it is proved in [15] that inequality (3) can be improved as
\[
|DH_t f| \leq e^{-Kt} |H_t f|, \quad m - \text{a.e..} \tag{5}
\]
In conclusion, inequality (4) holds for any \( p \in [1, \infty] \).

Conversely, it is shown in [5] that a space satisfying inequality (3) is RCD\( (K, \infty) \). Let \( (X, d, m) \) be an infinitesimally Hilbertian space, we have a well-defined Dirichlet energy:
\[
E(f) := \frac{1}{2} \int |Df|^2 \, dm
\]
for any \( f \in W^{1,2}(X, d, m) \). We denote the \( L^2 \)-gradient flow of \( E(\cdot) \) starting from \( f \) by \( (H_t f)_t \). Assume further that the space \( (X, d, m) \) has Sobolev-to-Lipschitz property: for any function \( f \in W^{1,2} \) with \( |Df| \in L^\infty \), there exists a Lipschitz continuous function \( f \) such that \( f = \tilde{f} \text{ m-a.e. and Lip}(f) = \text{ess sup} \ |Df| \). If
\[
|DH_t f|^2 \leq e^{-2Kt} |H_t f|^2, \quad m - \text{a.e.} \tag{6}
\]
for any \( f \in W^{1,2} \) and \( t > 0 \), then \( (X, d, m) \) is RCD\( (K, \infty) \).

The main goal of this paper is to prove that for any \( p > 2 \), \( p \)-gradient estimate (4) also characterizes the curvature-dimension condition. We prove a non-smooth version of 2) \( \Rightarrow \) 3) in von Renesse-Sturm’s result, thus we complete the circle 1) \( \Leftrightarrow \) 2) \( \Leftrightarrow \) 3) in non-smooth setting.

Now, we introduce our main result in this paper. When \( p = 2 \), it is proved in [15] that there exists a space of test functions TestF\( (X, d, m) \) which is a dense subspace of \( W^{1,2}(X) \) defined as
\[
\text{TestF}(X, d, m) := \left\{ f \in D(\Delta) \cap L^\infty : |Df| \in L^\infty \text{ and } \Delta f \in W^{1,2} \cap L^\infty(X, m) \right\},
\]
such that \( \Delta |Df|^2 \) is a well-defined measure (see Definition 3.1) for any \( f \in \text{TestF} \).

So it is reasonable to the following assumption (Assumption 2, see a similar assumption in [17]): there exists a dense subspace \( \mathcal{A} \) in TestF with respect to the graph norm
\[
f \mapsto \left[ \|(-\Delta)^{\frac{1}{2}} f\|^2_{L^2} + \|f\|^2_{W^{1,2}} \right]^{\frac{1}{2}} = \left[ E(\Delta f) + \|f\|^2_{W^{1,2}} \right]^{\frac{1}{2}}
\]
such that \( |Df|^2 \in M_\infty \) for any \( f \in \mathcal{A} \). We remark that we do not need to assume the density of \( \mathcal{A} \) in \( \mathfrak{W}^{1,2} \).

**Theorem 1.1** (Theorem 3.5, Improved Bakry-Émery theory). Let \( M := (X, d, m) \) be a metric measure space such that there exists an algebra \( \mathcal{A} \) as described above. If for any \( f \in W^{1,2}(X) \cap \text{Lip}(X) \cap L^\infty(X) \) we have the gradient estimate
\[
|DH_t f|^p \leq e^{-pKt} |H_t f|^p, \quad m - \text{a.e.} \tag{7}
\]
for some \( p \in (1, \infty) \). Then (7) holds for \( p = 1 \). In particular, \( M \) is a RCD\( (K, \infty) \) space.
Since we do not have second order differentiation formula for relative entropy along Wasserstein geodesics, or Taylor’s expansion in non-smooth setting, we can not simply use the argument in smooth metric measure space (see the proofs in [14]). The argument we adopt here is the so-called ‘self-improvement’ method in Bakry-Émery’s Γ-calculus, which was used in [15] to deal with the non-smooth problems. We remark that we not only use ‘self-improvement’ technique, but an improved iteration method based on this technique. We believe that this method also has potential application in the future.

It can be seen that Assumption 2 is satisfied in the following cases, where we can apply our main result.

**Example 1.** Smooth metric measure space: obviously, $C^\infty_c(M)$, the space of smooth functions with compact support is a good algebra in Assumption 2. Hence we obtain a new quick proof to von Renesse-Sturm’s theorem, without using Taylor’s expansion method.

**Example 2.** RCD($K,\infty$) metric measure space: it is proved in Lemma 3.2 [15] that $|Df|^2 \in \mathbb{M}_{\infty}$ for any $f \in \text{TestF}$. By Theorem 1.1 we obtain the following proposition which deals with the optimal constant $K$ in the curvature-dimension condition. It is also a complement to Savaré’s result in [15].

**Proposition 1** (Self-improvement of gradient estimate). Let $(X,d,m)$ be a RCD($K,\infty$) metric measure space. If for any $f \in W^{1,2} \cap \text{Lip}(X) \cap L^\infty(X)$ we have the gradient estimate

$$|DH_t f|^p \leq e^{-pK't} H_t |Df|^p, \ m - a.e. \quad (8)$$

for some $p \in [1, \infty)$ and $K' > K$. Then $(X,d,m)$ is a RCD($K',\infty$) space. In particular, we know

$$|DH_t f| \leq e^{-K't} H_t |Df|, \ m - a.e. \quad (9)$$

In [10], Gigli defines measure valued Ricci tensor on RCD metric measure space (see also [12]) as

$$\text{Ricci}(\nabla f, \nabla f) := \Gamma_2(f) - |\text{Hess}[f]|_{\text{HS}}^2 m$$

where $\Gamma_2(f) := \frac{1}{2} \Delta |Df|^2 - \langle \nabla f, \nabla \Delta f \rangle m$ and $|\text{Hess}[f]|_{\text{HS}}$ is the Hilbert-Schmidt norm of the Hessian $\text{Hess}[f]$ as a module (see [10] for details). He shows that $\text{Ricci}(\nabla f, \nabla f) \geq K|Df|^2 m$ if and only if the space is RCD($K,\infty$). However, we do not know if $\text{Ricci}$ has locality in the sense that $\text{Ricci}(\nabla f, \nabla f)|_{|Df|=0} = 0$.

From the proof of Theorem 1.1 we have the following new characterization of curvature bound which extends Gigli’s result:

**Proposition 2** (Proposition 3). Let $(X,d,m)$ be a RCD space. For any $f$ such that $\text{Ricci}(\nabla f, \nabla f)$ is well-defined, we denote the Lebesgue decomposition of $\text{Ricci}(\nabla f, \nabla f)$ with respect to $m$ by

$$\text{Ricci}(\nabla f, \nabla f) = \text{Ricci}_{ac}(\nabla f, \nabla f) m + \text{Ricci}_{sing}(\nabla f, \nabla f).$$

Then the following characterizations are equivalent.

1) $(X,d,m)$ is RCD($K,\infty$),
2) for any test function $f \in \text{TestF}$ we have $\text{Ricci}(\nabla f, \nabla f) \geq K|Df|^2 m$ in the sense that

$$\text{Ricci}_{ac}(\nabla f, \nabla f) \geq K|Df|^2 m - a.e.$$

and $\text{Ricci}_{sing}(\nabla f, \nabla f) \geq 0$. 
3) for any test function \( f \in \text{TestF} \) we have
\[ |Df|^2 \text{Ricci}_{ac}(\nabla f, \nabla f) \geq K|Df|^4 \text{ m-a.e.} \]
and \( \text{Ricci}_{\text{sing}}(\nabla f, \nabla f) \geq 0. \)

We remark that this naive extension is non-trivial, because 2) is not a direct consequence of 3) due to lack of the locality of \( \text{Ricci}(\cdot, \cdot) \). From this proposition, we know that \( \overline{\text{Ricci}}(\nabla f, \nabla f) := |Df|^2 \text{Ricci}_{ac}(\nabla f, \nabla f) \) \( \text{m} \) characterizes the Ricci curvature of \((X, d, m)\) and \( \text{Ricci} \) has locality in the sense that
\[ \overline{\text{Ricci}}(\nabla f, \nabla f)_{\{|\nabla f|=0\}} = 0. \]

2. Preliminaries. First of all, we summarize the basic hypothesis on the metric measure space \((X, d, m)\) below, the notions and concepts in this assumption will be explained later.

**Assumption 1.** We assume that:

1. \((X, d)\) is a complete, separable geodesic space,
2. \( \text{supp} m = X, \ m(B_r(x)) < c_1 \exp(c_2 r^2) \) for every \( r > 0 \),
3. \( W^{1,2}(X) \) is a Hilbert space,
4. \((X, d, m)\) has Sobolev-to-Lipschitz property,
5. there exists a unique heat kernel \( p_t(x, y) \).

The Sobolev space \( W^{1,2}(X, d, m) \) is defined as in [2]. We say that \( f \in L^2(X, m) \) is a Sobolev function in \( W^{1,2}(X, d, m) \) if there exists a sequence of Lipschitz functions \((f_n) \subset L^2\), such that \( f_n \to f \) and \( \text{lip}(f_n) \to G \) in \( L^2 \) for some \( G \in L^2(X, m) \), where \( \text{lip}(f_n) \) is the local Lipschitz constant of \( f_n \). It is known that there exists a minimal function \( G \) in \( m\text{-a.e.} \) sense. We call the minimal \( G \) the minimal weak upper gradient (or weak gradient for simplicity) of the function \( f \), and denote it by \( \|Df\| \). It is known that the locality holds for \( |Df| \), i.e. \( |Df| = |Dg| \) a.e. on the set \( \{f = g\} \). Furthermore, we have the lower semi-continuity: if \( \{f_n\} \subset W^{1,2}(X, d, m) \) is a sequence converging to some \( f \) in \( m\text{-a.e.} \) sense and \(|Df_n|\) is bounded in \( L^2(X, m) \), then \( f \in W^{1,2}(X, d, m) \) and
\[ \|Df\|_{L^2} \leq \liminf_{n \to \infty} \|Df_n\|_{L^2}. \]

We equip \( W^{1,2}(X, d, m) \) with the norm
\[ \|f\|_{W^{1,2}(X, d, m)} := \|f\|_{L^2(X, m)} + \|Df\|_{L^2(X, m)}. \]

We say that \((X, d, m)\) is an infinitesimally Hilbertian space if \( W^{1,2} \) is a Hilbert space (see [4], [11] for more discussions).

On an infinitesimally Hilbertian space, we have a ‘carré du champ’ operator \( \Gamma(\cdot, \cdot) : [W^{1,2}(X, d, m)]^2 \to L^1(X, d, m) \) defined by
\[ \Gamma(f, g) := \frac{1}{4} \left( |D(f + g)|^2 - |D(f - g)|^2 \right). \]
It can be seen that \( \Gamma(\cdot, \cdot) \) is symmetric, bilinear and continuous. We denote \( \Gamma(f, f) \) by \( \Gamma(f) \). We have the following chain rule and Leibnitz rule (Lemma 4.7 and Proposition 4.17 in [1], see also Corollary 7.1.2 in [8])
\[ \Gamma(\Phi(f), g) = \Phi'(f) \Gamma(f, g) \text{ for every } f, g \in W^{1,2}, \ \Phi \in \text{Lip}(\mathbb{R}), \Phi(0) = 0 \]
and
\[ \Gamma(fg, h) = f \Gamma(g, h) + g \Gamma(f, h) \text{ for every } f, g, h \in W^{1,2} \cap L^\infty. \]
We say that a metric measure space \( M = (X, d, m) \) has Sobolev-to-Lipschitz property if: for any function \( f \in W^{1,2} \) with \( |Df| \in L^\infty \), we can find a Lipschitz continuous function \( \bar{f} \) such that \( f = \bar{f} \) \( m \)-a.e. and \( \text{Lip}(\bar{f}) = \text{ess sup} \ |Df| \).

We define the Dirichlet (energy) form \( E : L^2 \rightarrow [0, \infty] \) by

\[
E(f) := \frac{1}{2} \int \Gamma(f) \, dm.
\]

It is proved (see [2, 3]) that Lipschitz functions are dense in energy: for any \( f \in W^{1,2} \) there is a sequence of Lipschitz functions \( (f_n)_n \subset L^2(X, m) \) such that \( f_n \rightarrow f \) and \( \text{Lip}(f_n) \rightarrow |Df| \) in \( L^2 \). Moreover, if \( W^{1,2} \) is Hilbert we know Lipschitz functions are dense (strongly) in \( W^{1,2} \).

It can be proved that \( E \) is a strongly local, symmetric, quasi-regular Dirichlet form (see [5, 2, 4]). The Markov semigroup \( (H_t)_{t \geq 0} \) generated by \( E \) is called the heat flow. There exists heat kernel which is a family of functions \( p_t(x, y) : X \times X \times \mathbb{R} \rightarrow \mathbb{R} \) such that \( p_t(x, y) \, dm(y) \) is a probability measure for any \( x \in X, t \in \mathbb{R} \), and \( H_t f(x) = \int f(y) p_t(x, y) \, dm(y) \) for any \( f \in L^2(X, m) \).

For any \( f \in L^2(X, m) \) we know that \((0, \infty) \ni t \mapsto H_t f \in L^2 \cap D(\Delta) \) satisfies

\[
\frac{d}{dt} H_t f = \Delta H_t f \forall t \in (0, \infty),
\]

and

\[
\lim_{t \rightarrow 0} H_t f = f \in L^2.
\]

Here the Laplacian is defined in the following way (see [11] for alternative definitions):

**Definition 2.1** (Measure valued Laplacian, [11, 10, 15]). The domain of the Laplacian \( D(\Delta) \subset W^{1,2} \) consists of \( f \in W^{1,2} \) such that there is a measure \( \mu \in \text{Meas}(M) \) satisfying

\[
\int \varphi \, \mu = - \int \Gamma(\varphi, f) \, m, \forall \varphi : M \rightarrow \mathbb{R}, \text{ Lipschitz with bounded support}.
\]

In this case the measure \( \mu \) is unique and we denote it by \( \Delta f \). If \( \Delta f \ll m \), we denote its density with respect to \( m \) by \( \Delta f \).

We define \( \text{TestF}(X, d, m) \subset W^{1,2}(X, d, m) \), the space of test functions as

\[
\text{TestF}(X, d, m) := \left\{ f \in D(\Delta) \cap L^\infty : |Df| \in L^\infty \text{ and } \Delta f \in W^{1,2} \cap L^\infty(X, m) \right\}.
\]

It is known from [15] and [4] that \( \text{TestF}(M) \) is an algebra and it is dense in \( W^{1,2}(X, d, m) \) when \( (X, d, m) \) is a RCD metric measure space. We will see in Lemma 3.4 that \( \text{TestF} \) is dense in \( W^{1,2} \) when \( L^p \)-gradient estimate for heat flow holds for some \( p > 2 \).

**Lemma 2.2** (Lemma 3.2, [15]). Let \( M = (X, d, m) \) be a metric measure space satisfying Assumptions 1. Assume that the algebra generated by \( \{f_1, \ldots, f_n\} \subset \text{TestF}(M) \) is included in \( \text{TestF}(M) \). Let \( \Phi \in C^\infty(\mathbb{R}^n) \) with \( \Phi(0) = 0 \). Put \( f = (f_1, \ldots, f_n) \), then \( \Phi(f) \in \text{TestF}(M) \).

Let \( f \in \text{TestF}(M) \). We define the Hessian \( \text{Hess}[f](\cdot, \cdot) : \{\text{TestF}(M)\}^2 \rightarrow L^0(M) \) by

\[
2\text{Hess}[f](g, h) = \Gamma(g, \Gamma(f, h)) + \Gamma(h, \Gamma(f, g)) - \Gamma(f, \Gamma(g, h)).
\]

We have the following lemma.
Lemma 2.3 (Chain rules, [7], [15]). Let \( f_1, ..., f_n \in \text{TestF}(M) \) and \( \Phi \in C^\infty(\mathbb{R}^n) \) with \( \Phi(0) = 0 \). Assume that the algebra generated by \( \{f_1, ..., f_n\} \subset \text{TestF}(M) \) is included in \( \text{TestF}(M) \). Put \( f = (f_1, ..., f_n) \), then

\[
|D\Phi(f)|^2 m = \sum_{i,j=1}^{n} \Phi_i \Phi_j (f) \Gamma(f_i, f_j) m,
\]

and

\[
\Delta\Phi(f) = \sum_{i=1}^{n} \Phi_i (f) \Delta f_i + \sum_{i,j=1}^{n} \Phi_{ij} (f) \Gamma(f_i, f_j) m.
\]

The last lemma will be used in the proof of Theorem 3.5.

Lemma 2.4 (Lemma 3.3.6, [10]). Let \( \mu_i = \rho_i m + \mu_i^s \) be measures with \( \mu_i^s \perp m \), \( i = 1, 2, 3 \). We assume that

\[
\lambda^2 \mu_1 + 2\lambda \mu_2 + \mu_3 \geq 0, \ \forall \lambda \in \mathbb{R}.
\]

Then we have

\[
\mu_1^s \geq 0, \ \mu_3^s \geq 0
\]

and

\[
|\rho_2|^2 \leq \rho_1 \rho_3, \ m - a.e.
\]

3. Main results. Firstly, we discuss more about the measure-valued Laplacian. Since \( E \) is quasi-regular, we know (see Remark 1.3.9 (ii), [9]) that every function \( f \in W^{1,2} \) has an quasi-continuous representative \( \tilde{f} \). And \( \tilde{f} \) is unique up to quasi-everywhere equality, i.e. if \( \tilde{f} \) is another quasi-continuous representative, then \( \tilde{f} = \tilde{f} \) holds in a complement of an \( E \)-polar set. For more details, see Definition 2.1 in [15] and the references therein.

Definition 3.1. We define \( \mathcal{M}_\infty \) the space of \( f \in D(\Delta) \cap L^\infty \) such that there exists a measure decomposition \( \Delta f = \mu_+ - \mu_- \) with \( \mu_\pm \) in the positive cone in \( (W^{1,2})' \), such that:

\[
\int \varphi d(\Delta f) = - \int \Gamma(\varphi, f) \, dm
\]

for any \( \varphi \in W^{1,2} \) and the quasi-continuous representative \( \varphi \in L^1(X, \Delta f) \).

In particular, every \( E \)-polar set is \( (\Delta f) \)-negligible and the measure \( \varphi \Delta f \) is well-defined.

In the next lemma we study the measure \( \Delta \Gamma(f)^2 \). Since \( \Gamma(f) \) is not necessarily continuous, and \( \Phi(x) = x^2 \) is not \( C^2(\mathbb{R}) \), we can not use Lemma 2.3 directly.

Lemma 3.2. Let \( (X, d, m) \) be a metric measure space satisfying Assumptions 1. Let \( f \in \text{TestF} \) such that \( \Gamma(f), \Gamma(f)^2 \in \mathcal{M}_\infty, p > 2 \). Then

\[
\frac{1}{p} \Delta \Gamma(f)^2 - \Gamma(f)^{p-1} \Delta \Gamma(f, f) dm \geq K \Gamma(f)^2 dm \tag{10}
\]

if and only if

\[
\frac{1}{2} \Gamma(f) \Delta_{ac} \Gamma(f) + \frac{1}{2} (\frac{p}{2} - 1) \Gamma(f) dm \geq \left( \Gamma(f) \Gamma(\Delta f, f) + K \Gamma(f)^2 \right) dm \tag{11}
\]

and \( \Gamma(f) \Delta_{sing} \Gamma(f) \geq 0 \) as measures, where \( \Delta_{ac} \Gamma(f) \) is the absolutely continuous part in the measure decomposition \( \Delta \Gamma(f) = \Delta_{ac} \Gamma(f) + \Delta_{sing} \Gamma(f) \) with respect to \( m \), and \( \Gamma(f) \) is the quasi-continuous representation of \( \Gamma(f) \).
Proof. Since $p > 2$, it can be seen that (11) is equivalent to
\[
\frac{1}{2} \Gamma(f)^{\frac{p}{2}} - 1 \Delta_{ac} \Gamma(f) + \frac{1}{2} \left(\frac{p}{2} - 1\right) \Gamma(f)^{\frac{p}{2} - 2} \Gamma(\Gamma(f)) \, dm \\
\geq \left(\Gamma(f)^{\frac{p}{2} - 1} \Gamma(\Delta f, f) + K \Gamma(f)^{\frac{p}{2}}\right) \, dm. \tag{12}
\]
Assume that we have the decomposition of the measure $\frac{1}{p} \Delta \Gamma(f)^{\frac{p}{2}}$ with respect to $m$: $\frac{1}{p} \Delta \Gamma(f)^{\frac{p}{2}} = \frac{1}{p} \Delta_{sing} \Gamma(f)^{\frac{p}{2}} + \frac{1}{p} \Delta_{ac} \Gamma(f)^{\frac{p}{2}}$. From (10) we know the singular part $\frac{1}{p} \Delta_{sing} \Gamma(f)^{\frac{p}{2}}$ of the measure $\frac{1}{p} \Delta \Gamma(f)^{\frac{p}{2}}$ is non-negative.

From hypothesis we know $\Gamma(f), \Gamma(f)^{\frac{p}{2}} \in D(\Delta)$, by chain rule we know
\[
\int \varphi \, d\Delta \Gamma(f)^{\frac{p}{2}} = - \int \Gamma(\varphi, \Gamma(f)^{\frac{p}{2}}) \, dm = - \int \frac{p}{2} \Gamma(f)^{\frac{p}{2} - 1} \Gamma(\varphi, \Gamma(f)) \, dm \tag{13}
\]
for any Lipschitz function $\varphi$ with bounded support.

Denote by $\Gamma(f)$ the quasi-continuous representation of $\Gamma(f)$. From Leibniz rule and chain rule we know $\varphi(\Gamma(f) + \epsilon)^{\frac{p}{2} - 1} \in W^{1,2}$, for any $\epsilon > 0$. According to Definition 3.1 we have
\[
- \int \varphi \, d\Delta \Gamma(f)^{\frac{p}{2}} = - \int \Gamma(\varphi, \Gamma(f)^{\frac{p}{2}}) \, dm = - \int \left[ \varphi(\frac{p}{2} - 1) \Gamma(\Gamma(f)^{\frac{p}{2} - 2} \Gamma(f)) + \Gamma(f)^{\frac{p}{2} - 1} \Gamma(\varphi, \Gamma(f)) \right] \, dm. \tag{14}
\]
Combining (13) and (14) we have
\[
\frac{1}{p} \Delta \Gamma(f)^{\frac{p}{2}} = \frac{1}{2} \Gamma(f)^{\frac{p}{2} - 1} \Delta \Gamma(f) + \frac{1}{2} \left(\frac{p}{2} - 1\right) \Gamma(f)^{\frac{p}{2} - 2} \Gamma(\Gamma(f)) \, dm \tag{15}
\]
as measures. Therefore, we know
\[
\frac{1}{p} \Delta_{ac} \Gamma(f)^{\frac{p}{2}} = \frac{1}{2} \Gamma(f)^{\frac{p}{2} - 1} \Delta_{ac} \Gamma(f) + \frac{1}{2} \left(\frac{p}{2} - 1\right) \Gamma(f)^{\frac{p}{2} - 2} \Gamma(\Gamma(f)) \, dm \\
= \frac{1}{2} \Gamma(f)^{\frac{p}{2} - 1} \Delta_{ac} \Gamma(f) + \frac{1}{2} \left(\frac{p}{2} - 1\right) \Gamma(f)^{\frac{p}{2} - 2} \Gamma(\Gamma(f)) \, dm
\]
and
\[
\frac{1}{p} \Delta_{sing} \Gamma(f)^{\frac{p}{2}} = \frac{1}{2} \Gamma(f)^{\frac{p}{2} - 1} \Delta_{sing} \Gamma(f).
\]
In conclusion, we obtain
\[
- \frac{1}{p} \Delta \Gamma(f)^{\frac{p}{2}} = - \frac{1}{2} \Gamma(f)^{\frac{p}{2} - 1} \Delta_{ac} \Gamma(f) + \frac{1}{2} \left(\frac{p}{2} - 1\right) \Gamma(f)^{\frac{p}{2} - 2} \Gamma(\Gamma(f)) + \frac{1}{2} \Gamma(f)^{\frac{p}{2} - 1} \Delta_{sing} \Gamma(f).
\]
Hence (10) is equivalent to (12), we prove the lemma. \(\square\)

The following lemma will be used in the proof of Theorem 3.5.
Lemma 3.3. Let \( P(r) : [0, \infty) \mapsto [-\frac{1}{4}, \infty) \) be a function defined as
\[
P(r) = r - \frac{1}{4(r+1)},
\]
and \( a_0 \geq 0 \) be an arbitrary initial datum, we define \((a_n)_{n \in \mathbb{N}}\) recursively by the formula
\[
a_{n+1} = P(a_n).
\]
Then there exists an integer \( N_0 \) such that \( 0 \leq a_{N_0} < 1 \) and \(-\frac{1}{4} \leq a_{N_0+1} < 0\).

Conversely, for any \( a \in [0,1) \) and \( b > a \), there exists a sequence \( a_0, ..., a_{N_0} \) defined by the recursive function \( P \) such that \( a_0 > b \) and \( a_{N_0} = a \).

Proof. It can be seen that \( a_{n+1} < a_n \). If \( a_0 \geq 0 \), by monotonicity we know \( a_n - a_{n+1} \in \left[ \frac{1}{4(a_{n+1}+1)}, \frac{1}{4} \right] \) for any \( n \in \mathbb{N} \). So there exists a unique \( N_0 \) such that \( 0 \leq a_{N_0} < 1 \) and \(-\frac{1}{4} \leq a_{N_0+1} < 0\). Conversely, since \( P(r) \) is strictly monotone on \([0, \infty)\), we know \( P^{-1}(r) : [-\frac{1}{4}, \infty) \mapsto [0, \infty) \) is well defined. And \( (P^{-1})^{(n+1)}(a) - (P^{-1})^{(n)}(a) \in \left[ \frac{1}{4((P^{-1})^{(n)}(a)+1)}, \frac{1}{4} \right] \) for any \( n \in \mathbb{N} \). Thus there exists \( N \in \mathbb{N} \) such that \( (P^{-1})^{(N_0)}(a) \geq b \). Finally, we can pick \( a_0 = (P^{-1})^{(N_0)}(a) \), so that \( a_{N_0} = (P)^{(N_0)}(a_0) = a \) fulfills our request.

As we mentioned in the Introduction, the space of test functions is dense in \( W^{1,2}(X,d,m) \) when \( L^p \)-gradient estimate for heat flow holds.

Lemma 3.4 (Density of test functions in \( W^{1,2}(X,d,m) \), Remark 2.5 [5]). Let \( (X,d,m) \) be a metric measure space satisfying Assumption 1. Assume that for any \( f \in W^{1,2} \cap \text{Lip} \cap L^\infty(X,d,m) \) we have the \( L^p \)-gradient estimate
\[
|DH_t f|^p \leq e^{-pKt} |H_t |^p \ m - a.e.
\]
for some \( p \in [1, \infty) \). Then the space of test functions \( \text{TestF}(X,d,m) \) is dense in \( W^{1,2} \).

Proof. As we discussed in the preliminary section, the space
\[
\mathbb{V}^1 := \left\{ \varphi \in W^{1,2} : \Gamma(\varphi) \in L^\infty(X,m) \right\}
\]
is dense in \( W^{1,2} \). We also know that the
\[
\mathbb{V}^1_\infty := \left\{ \varphi \in W^{1,2} \cap L^\infty : \Gamma(\varphi) \in L^\infty(X,m) \right\}
\]
in dense in \( L^2 \), and \( \mathbb{V}^1_\infty \) is invariant under the action \((H_t)_t\) by (16) and Sobolev-to-Lipschitz property. Hence by an approximation argument (see e.g. Lemma 4.9 in [4]), we know \( \mathbb{V}^2_\infty \) is dense in \( W^{1,2} \). Similarly, by a semigroup mollification (see e.g. page 351, [5]) we can prove that
\[
\mathbb{V}^2_\infty := \left\{ \varphi \in \mathbb{V}^1_\infty : \Delta \varphi \in W^{1,2} \cap L^\infty(X,m) \right\}
\]
is dense in \( W^{1,2} \). \( \square \)

We now introduce the following technical assumption, which is important in our proof. It can be proved that Riemannian manifolds and RCD\((K, \infty)\) spaces satisfy this assumption.

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4922 BANG-XIAN HAN
Assumption 2 (Existence of good algebra). We assume the existence of a dense subspace $\mathcal{A}$ in $\text{TestF}(X, d, m)$ with respect to the graph norm

$$f \mapsto \left[\|(-\Delta)^{\frac{3}{2}} f\|_{L^2}^2 + \|f\|_{W^{1,2}}^2\right]^\frac{1}{2} = \left[\|\Gamma(\Delta f)\|_{L^2}^2 + \|f\|_{W^{1,2}}^2\right]^\frac{1}{2}$$

such that $\Gamma(f) \in \mathcal{M}_\infty$ for any $f \in \mathcal{A}$.

It can be seen that $\mathcal{A}$ is an algebra (i.e. $\mathcal{A}$ is closed w.r.t. pointwise multiplication), if it is non-trivial. In particular, by Lemma 3.4 we know that $\mathcal{A}$ is dense in $W^{1,2}$ if $L^p$ gradient estimate holds.

Theorem 3.5 (Improved Bakry-Émery theory). Let $(X, d, m)$ be a metric measure space satisfying Assumption 1 and Assumption 2. If for any $f \in \mathcal{C}_c^{\infty}(X, d, m)$ such that

$$\text{Assumption 2 (Existence of good algebra)}$$

holds $m$-a.e., and

$$E\left[\left(\int_{\mathcal{A}} |\Gamma(f)|^p \, \text{d}m\right)^q\right] \leq \exp\left(\frac{1}{2} \int_{\mathcal{A}} \|\varphi\|_{L^2}^2 \, \text{d}m\right)$$

for some $p \in [1, \infty)$. Then $(X, d, m)$ is a $\text{RCD}(K, \infty)$ space.

Proof. If $p \leq 2$, by the result in [5] we know $(X, d, m)$ is a $\text{RCD}(K, \infty)$. Now we assume $p > 2$.

Part 1. Firstly, we prove

$$\Gamma(f) \Delta_{ac} \Gamma(f) + \epsilon \Gamma(\Gamma(f)) \geq \Gamma(f) \Gamma(\Delta f, f) + K \Gamma(f)^2,$$  

and

$$\Gamma(f) \Delta_{\text{sing}} \Gamma(f) \geq 0, \quad \text{for any } f \in \mathcal{A} \text{ and } \epsilon > 0.$$

For any $f \in \mathcal{A}$, $\varphi \in \text{TestF}(X, d, m)$, $\varphi \geq 0$ and $t > 0$, we define $F : [0, t] \mapsto \mathbb{R}$ by

$$F(s) = \int e^{-pKs} \varphi \Gamma(H_{t-s} f)^\frac{p}{2}.$$

It can be seen that $F$ is a $C^1$ function (see Lemma 2.1, [5]). From (17) we know $F(s) \leq F(t)$ holds for any $s \in [0, t]$. Hence $F'(s)|_{s=t} \geq 0$, and so

$$\int e^{-pKs} \Delta \varphi \Gamma(H_{t-s} f)^\frac{p}{2} \big|_{s=t} - p \int e^{-pKs} \varphi \Gamma(H_{t-s} f)^\frac{p}{2} - \Gamma(\Delta H_{t-s} f, H_{t-s} f) \big|_{s=t}$$

$$\geq pK \int e^{-pKs} \varphi \Gamma(H_{t-s} f)^\frac{p}{2} \big|_{s=t}.$$

Letting $t \to 0$ we obtain

$$\int \Delta \varphi \Gamma(f)^\frac{p}{2} - p \int \varphi \Gamma(f)^\frac{p}{2} - 1 \Gamma(\Delta f, f) \geq pK \int \varphi \Gamma(f)^\frac{p}{2}.$$

In particular, from Lemma 2.6 and Lemma 3.2 in [15] we know $\Gamma(f)^\frac{p}{2} \in \text{D}(\Delta)$ and

$$\frac{1}{p} \Delta \Gamma(f)^\frac{p}{2} - \Gamma(\Delta f, f) \text{dm} \geq K \Gamma(f)^\frac{p}{2} \text{dm}.  \quad (19)$$

By Lemma 3.2, we get that

$$\frac{1}{2} \Gamma(f) \Delta_{ac} \Gamma(f) + \left(\frac{p}{4} - \frac{1}{2}\right) \Gamma(\Gamma(f)) \geq \Gamma(f) \Gamma(\Delta f, f) + K \Gamma(f)^2  \quad (20)$$

holds $m$-a.e., and $\Gamma(f) \Delta_{\text{sing}} \Gamma(f) \geq 0$.

From now on, all the inequalities are considered in $m$-a.e. sense. We denote

$$\frac{1}{2} \Delta_{ac} \Gamma(f) - \Gamma(\Delta f, f) \text{ by } \Gamma_2(f), \quad \text{and} \quad \frac{1}{2} \Delta_{ac} \Gamma(f) - \Gamma(\Delta f, f) = K \Gamma(f) \text{ by } \Gamma_2(f, K),$$

then (20) becomes

$$\Gamma_2(f, K) \Gamma(f)^{\frac{p}{4} - \frac{1}{2}} \Gamma(\Gamma(f)) \geq 0.$$
For any real number \( r \geq 0 \), we say that the property \( B(r) \) holds if
\[
\Gamma_{2,K,r}(f) := \Gamma_{2,K}(f) + r\Gamma(\Gamma(f)) \geq 0
\]
for any \( f \in \text{TestF} \). For example, (20) means \( B(\frac{3}{4} - \frac{1}{2}) \).

Now we define
\[
P(r) = r - \frac{1}{4(r+1)}.
\]
Then we will prove that \( B(r) \) implies \( B(P(r)) \). We choose the smooth function \( \Phi : \mathbb{R}^3 \mapsto \mathbb{R} \) defined by
\[
\Phi(f) := \lambda f_1 + (f_2 - a)(f_3 - b) - ab, \ a,b, \lambda \in \mathbb{R}.
\]
Then we know
\[
\Phi_{23}(f) = \Phi_{32} = a, \ \Phi_{ij}(f) = 0, \text{ if } (i,j) \notin \{(2,3), (3,2)\}
\]
\[
\Phi_1(f) = \lambda, \ \Phi_{2}(f) = f_3 - b, \ \Phi_3(f) = f_2 - a.
\]
If \( f := (f, g, h) \in A^3 \), we know \( \Phi(f) \in A \) by Lemma 2.2. Hence we know
\[
\Gamma_{2,K}(\Phi(f)) \Gamma(\Phi(f)) + r\Gamma(\Gamma(\Phi(f))) \geq 0.
\]
(21)

By direct computation using Lemma 2.3 (see also Theorem 3.4, [15]), we have
\[
\Gamma(\Phi(f)) = g^{ij}\Phi_i\Phi_j(f)
\]
\[
= \lambda^2 \Gamma(f) + (g-a)A_1 + (h-b)B_1
\]
where \( g^{ij} = \Gamma(f, f, f) \), \( A_1, A_2 \) are some additional terms.

Similarly, we have
\[
\Gamma(\Gamma(\Phi(f)))
\]
\[
= \Gamma(g^{ij}\Phi_i\Phi_j(f))
\]
\[
= (g^{ij})^2 \Gamma(\Phi_i\Phi_j) + (\Phi_i\Phi_j)^2 \Gamma(g^{ij}) + 2g^{ij}\Phi_i\Phi_j \Gamma(\Gamma(\Phi(f)))
\]
\[
= (g^{ij})^2 \left[ \Phi_i^2 \Gamma(\Phi_i) + \Phi_j^2 \Gamma(\Phi_j) + 2\Phi_i\Phi_j \Gamma(\Phi_i) \Gamma(\Phi_j) \right]
\]
\[
+ (\Phi_i\Phi_j)^2 \Gamma(\Gamma(\Phi(f))) + 2g^{ij}\Phi_i\Phi_j \Gamma(\Gamma(\Phi(f)))
\]
\[
= 2(g^{ij})^2 \lambda^2 \Gamma(h) + 2(g^{ij})^2 \lambda^2 \Gamma(g) + \lambda^4 \Gamma(\Gamma(f)) + (g-a)A_2 + (h-b)B_2.
\]
We also know (see Theorem 3.4, [15] or Lemma 3.3.7, [10]) that
\[
\Gamma_2(f) - K\Gamma(\Phi(f)) = \lambda^2 \Gamma_2(f) + 4\lambda \text{Hess}[f](g, h) + 2\left( \Gamma(g) \Gamma(h) + \Gamma(g, h)^2 \right)
\]
\[
+ (g-a)A_3 + (h-b)B_3 - K\lambda^2 \Gamma(f).
\]

Combining the computations above, (21) becomes an inequality with parameters \( a, b, \lambda \). By locality of weak gradients and density of simple functions, we can replace \( b \) by \( h \) and replace \( a \) by \( g \) (similar arguments are used in Theorem 3.4 [15] and Lemma 3.3.7 [10]). Then we obtain the following inequality from (21)
\[
\lambda^2 \Gamma(f) \left[ \lambda^2 \Gamma_2(f) + 4\lambda \text{Hess}[f](g, h) + 2\left( \Gamma(g) \Gamma(h) + \Gamma(g, h)^2 \right) - K\lambda^2 \Gamma(f) \right]
\]
\[
+ r \left[ 2\Gamma(f, g)^2 \lambda^2 \Gamma(h) + 2\Gamma(f, h)^2 \lambda^2 \Gamma(g) + \lambda^4 \Gamma(\Gamma(f)) \right]
\]
\[
\geq 0.
\]
Since \( r \geq 0 \) and
\[
\Gamma(g) \Gamma(h) \geq \Gamma(g, h)^2,
\]
Then we have
\[ \Gamma(f) \left[ \lambda^2 \Gamma_2(f) + 4 \lambda \text{Hess}[f](g, h) + 4 \left( \Gamma(g) \Gamma(h) - K \lambda^2 \Gamma(f) \right) \right] + r \left[ 4 \Gamma(f) \Gamma(g) \Gamma(h) + \lambda^2 \Gamma(f) \right] \geq 0. \]

Therefore, we can see that
\[ B \Gamma(f) \Gamma(g) \Gamma(h) \geq \Gamma(f)^2 \text{Hess}[f](g, h). \]

Applying Lemma 2.4 we obtain
\[ (1 + r) \Gamma_{2,K,r} \Gamma(f) \Gamma(g) \Gamma(h) \geq \Gamma(f)^2 \text{Hess}[f](g, h). \]

Since \( B(r) \) means \( \Gamma_{2,K,r} \geq 0 \), this inequality is equivalent to
\[ (1 + r) \Gamma_{2,K,r} \Gamma(f) \Gamma(g) \Gamma(h) \geq \Gamma(f)^2 \text{Hess}[f](g, h). \]

Recall that \( 2 \text{Hess}[f](g, h) = \Gamma(g, \Gamma(f, h)) + \Gamma(h, \Gamma(f, g)) - \Gamma(f, \Gamma(g, h)) \), we know
\[ \text{Hess}[f](g, h) + \text{Hess}[g](f, h) = \Gamma(f, g, h). \]

Combining with inequality (22) we have
\[ \sqrt{\frac{1}{1 + r} \Gamma(f, g, h)} \sqrt{\Gamma(f, g, h)} \leq \sqrt{\Gamma_{2,K,r}(f) \Gamma(g, h) + \Gamma_{2,K,r}(g) \Gamma(f) \Gamma(h)} = \left( \sqrt{\Gamma_{2,K,r}(f) \Gamma(g, h)} + \sqrt{\Gamma_{2,K,r}(g) \Gamma(f) \Gamma(h)} \right) \sqrt{\Gamma(h)}. \]

Then we fix \( f, g \in A \), and approximate any \( h \in W^{1,2} \cap L^\infty \) with a sequence \( (h_n) \subset A \) converging to \( h \) strongly in \( W^{1,2} \) such that
\[ \Gamma(h_n) \to \Gamma(h), \quad \Gamma(h_n, \Gamma(f, g)) \to \Gamma(h, \Gamma(f, g)) \]
pointwise and in \( L^1(X, m) \). Thus we can replace \( h \) by \( \Gamma(f, g) \) in the last inequality and obtain
\[ \sqrt{\frac{1}{1 + r} \Gamma(f, g, h)} \Gamma(f, g, h) = \left( \sqrt{\Gamma_{2,K,r}(f) \Gamma(g, h)} + \sqrt{\Gamma_{2,K,r}(g) \Gamma(f) \Gamma(h)} \right) \Gamma(h). \]

Let \( g = f \) in (23), we obtain
\[ \frac{1}{1 + r} \Gamma(f) \Gamma(f) \leq 4 \Gamma_{2,K,r}(f) \Gamma(f). \]

Therefore,
\[ \left( \frac{1}{1 + r} - r \right) \Gamma(f) \Gamma(f) \leq 4 \Gamma_{2,K}(f) \Gamma(f). \]

In other words, we have \( B(P(r)) \).

From Lemma 3.3 we know there exists \( a_0 \geq \frac{e}{4} - \frac{e}{2} \) and \( N_0 \in \mathbb{N} \) such that \( a_{N_0} = \epsilon \), where \( a_{n+1} = P(a_n), \quad n = 0, ..., N_0 - 1 \). Then we know \( B(a_0) \) from (20). From the result above, we can see that \( B(a_{N_0}) \) holds by induction. So we prove (18).

**Part 2.** From (18) and Lemma 3.2 we know
\[ \frac{1}{p_n} \Delta \Gamma(f)^{\frac{p_n}{2}} - \Gamma(f)^{\frac{p_n}{2}} \Gamma(\Delta f, f) \geq K \Gamma(f)^{\frac{p_n}{2}} \]
for any \( p_n = 2 + \frac{2}{2n}, \quad n \in \mathbb{N} \).

Let \( f \in A, \varphi \in \text{Test} F \) and \( \varphi \geq 0 \). From (24) we know
\[ \int \frac{1}{p_n} \Delta \varphi \Gamma(f)^{\frac{p_n}{2}} \text{d} m - \int \varphi \Gamma(f)^{\frac{p_n}{2}} \Gamma(\Delta f, f) \text{d} m \geq K \int \varphi \Gamma(f)^{\frac{p_n}{2}} \text{d} m. \]
Letting $n \to \infty$, by dominated convergence theorem and monotone convergence theorem we know
\begin{equation}
\frac{1}{2} \int \Delta \varphi \Gamma(f) \, dm - \int \varphi \Gamma(\Delta f, f) \, dm \geq K \int \varphi \Gamma(f) \, dm.
\end{equation}

Combining with the density of $\mathcal{A}$ in TestF, we know (25) holds for all $f \in \text{TestF}$. Finally, by Theorem 4.17 [5] we know that $(X,d,m)$ is a RCD$(K,\infty)$ space. \hfill \Box

As a corollary, we have the following proposition. We recall (see [10]) that the measure-valued Ricci tensor on RCD metric measure space is defined as
\begin{equation*}
\Gamma(f) := \Gamma_2(f) - |\text{Hess}[f]|_{\text{HS}}^2 m,
\end{equation*}
where $\Gamma_2(f) := \frac{1}{2} |\Delta \Gamma(f) - \Gamma(f,\Delta f)| m$ and $|\text{Hess}[f]|_{\text{HS}}$ is the minimal $L^2$ function $G$ such that $|\sum_{i,j} \text{Hess}[f](g_i, h_j)| \leq G \sqrt{\sum_{i,j} \Gamma^2(g_i, h_j)}$ for any $(g_i, h_j) \in \text{TestF}$. It is proved that $\text{Ricci}$ is well defined for any $f \in \text{TestF}(X,d,m)$ when $(X,d,m)$ is RCD.

**Proposition 3.** Let $(X,d,m)$ be a RCD space. Then the following characterizations are equivalent.

1) $(X,d,m)$ is RCD$(K,\infty)$,

2) for any test function $f \in \text{TestF}$ we have $\text{Ricci}(\nabla f, \nabla f) \geq K|Df|^2 m$ in the sense that
\begin{equation*}
\text{Ricci}_{\text{ac}}(\nabla f, \nabla f) \geq K|Df|^2 m - \text{a.e.}
\end{equation*}

and $\text{Ricci}_{\text{sing}}(\nabla f, \nabla f) \geq 0$.

3) for any test function $f \in \text{TestF}$ we have
\begin{equation*}
|Df|^2 \text{Ricci}_{\text{ac}}(\nabla f, \nabla f) \geq K|Df|^4 m - \text{a.e.}
\end{equation*}

and $\text{Ricci}_{\text{sing}}(\nabla f, \nabla f) \geq 0$.

**Proof.** 1) $\Rightarrow$ 2) is Lemma 3.6.2 [10], 2) $\Rightarrow$ 3) is trivial. So we just need to prove 3) $\Rightarrow$ 1).

From 3) we know $\Gamma_{2,K,0}(f) \geq 0$, m-a.e. for any $f \in \text{TestF}$. Therefore $\Gamma_{2,K,r}(f) \geq 0$ for any $r > 0$. Using the same argument as in the proof of Theorem 3.5, we know $(X,d,m)$ is RCD$(K,\infty)$.

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Received September 2017; revised April 2018.

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