Construction of Involutive Monomial Sets for Different Involutive Divisions

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Abstract. We consider computational and implementation issues for the completion of monomial sets to involution using different involutive divisions. Every of these divisions produces its own completion procedure. For the polynomial case it yields an involutive basis which is a special form of a Gröbner basis, generally redundant. We also compare our Mathematica implementation of Janet division to an implementation in C.

1 Introduction and Basic Definitions

In our previous paper [1] we described our first results on implementing in Mathematica 3.0 [2] different involutive divisions introduced in [3,4,5]; the completion of monomial sets to involution for those divisions and application to constructing Hilbert functions and Hilbert polynomials for monomial ideals.

In the present paper we pay more attention to efficient computation and propose some algorithmic improvements. Besides, we shortly describe an implementation of Janet division in C and compare the running times for both implementations. Though in this paper we consider involutivity of monomial ideals, all the underlying operations with involutive divisions and monomials enter in more general completion procedures for polynomial [3,4] and differential systems [9].

Let \(\mathbb{N}\) be a set of non-negative integers, and \(\mathbb{M} = \{x_1^{d_1} \cdots x_n^{d_n} \mid d_i \in \mathbb{N}\}\) be a set of monomials in the polynomial ring \(K[x_1,\ldots,x_n]\) over a field \(K\) of characteristic zero. By \(\text{deg}(u)\) and \(\text{deg}_i(u)\) we denote the total degree of \(u \in \mathbb{M}\) and the degree of variable \(x_i\) in \(u\), respectively.
For the least common multiple of two monomials \( u, v \in \mathbb{M} \) we shall use the conventional notation \( \text{lcm}(u, v) \). An admissible monomial ordering is denoted by \( \succ \), and throughout this paper we shall assume that it is compatible with
\[
x_1 \succ x_2 \succ \cdots \succ x_n.
\]

**Definition 1.** An involutive division \( L \) on \( \mathbb{M} \) is given, if for any finite monomial set \( U \subset \mathbb{M} \) and for any \( u \in U \) there is given a submonoid \( L(u, U) \) of \( \mathbb{M} \) satisfying the conditions:

(a) If \( w \in L(u, U) \) and \( v|w \), then \( v \in L(u, U) \).
(b) If \( u, v \in U \) and \( uL(u, U) \cap vL(v, U) \neq \emptyset \), then \( u \in vL(v, U) \) or \( v \in uL(u, U) \).
(c) If \( v \in U \) and \( v \in uL(u, U) \), then \( L(v, U) \subseteq L(u, U) \).
(d) If \( V \subseteq U \), then \( L(u, U) \subseteq L(u, V) \) for all \( u \in V \).

Elements of \( L(u, U) \) are called multiplicative for \( u \). If \( w \in uL(u, U) \) we shall write \( u|_L w \) and call \( u \) an \( (L-) \) involutive divisor of \( w \). In such an event the monomial \( v = w/u \) is multiplicative for \( u \) and the equality \( w = uv \) will be written as \( w = u \times v \). If \( u \) is a conventional divisor of \( w \) but not an involutive one we shall write, as usual, \( w = u \cdot v \). Then \( v \) is said to be nonmultiplicative for \( u \).

For every monomial \( u \in U \), Definition \( \square \) provides the separation
\[
\{x_1, \ldots, x_n\} = M_L(u, U) \cup NM_L(u, U),
\]  
\( M_L(u, U) \cap NM_L(u, U) = \emptyset \), of the set of variables into two subsets: multiplicative \( M_L(u, U) \subset L(u, U) \) and nonmultiplicative \( NM_L(u, U) \cap L(u, U) = \emptyset \). Conversely, if for any finite set \( U \subset \mathbb{M} \) and any \( u \in U \) the separation \( \square \) is given such that the corresponding submonoid \( L(u, U) \) of monomials in variables in \( M_L(u, U) \) satisfies the conditions (b)-(d), then the partition generates an involutive division.

**Definition 2.** Given an involutive division \( L \), a monomial set \( U \) is involutive with respect to \( L \) or \( L- \) involutive if
\[
(\forall u \in U) \ (\forall w \in \mathbb{M}) \ (\exists v \in U) \ [ uv \in vL(v, U) ].
\]

In this paper as well as in \( \square \) we shall consider the following eight different involutive divisions studied in \( \square \square \):
Example 3. Thomas division \[3\]. Given a finite set $U \subset M$, the variable $x_i$ is considered as multiplicative for $u \in U$ if $\deg_i(u) = \max\{\deg_i(v) \mid v \in U\}$, and nonmultiplicative, otherwise.

Example 4. Janet division \[4\]. Let the set $U \subset M$ be finite. For each $1 \leq i \leq n$ divide $U$ into groups labeled by non-negative integers $d_1, \ldots, d_i$:

$$[d_1, \ldots, d_i] = \{ u \in U \mid d_j = \deg_j(u), 1 \leq j \leq i \}.$$

A variable $x_i$ is multiplicative for $u \in U$ if $i = 1$ and $\deg_1(u) = \max\{\deg_1(v) \mid v \in U\}$, or if $i > 1$, $u \in [d_1, \ldots, d_{i-1}]$ and $\deg_i(u) = \max\{\deg_i(v) \mid v \in [d_1, \ldots, d_{i-1}]\}$.

Example 5. Pommaret division \[5\]. For a monomial $u = x_1^{d_1} \cdots x_k^{d_k}$ with $d_k > 0$ the variables $x_j, j \geq k$ are considered as multiplicative and the other variables as nonmultiplicative. For $u = 1$ all the variables are multiplicative.

Example 6. Division I \[6\]. Let $U$ be a finite monomial set. The variable $x_i$ is nonmultiplicative for $u \in U$ if there is $v \in U$ such that

$$x_{i_1}^{d_1} \cdots x_{i_m}^{d_m} u = \lcm(u, v), \quad 1 \leq m \leq \lceil n/2 \rceil, \quad d_j > 0 \quad (1 \leq j \leq m),$$

and $x_i \in \{x_{i_1}, \ldots, x_{i_m}\}$.

Example 7. Division II \[6\]. For monomial $u = x_1^{d_1} \cdots x_k^{d_k}$ the variable $x_i$ is multiplicative if $d_i = d_{\max}(u)$ where $d_{\max}(u) = \max\{d_1, \ldots, d_n\}$.

Example 8. Induced division \[7\]. Given an admissible monomial ordering $\succ$ a variable $x_i$ is nonmultiplicative for $u \in U$ if there is $v \in U$ such that $v \prec u$ and $\deg_i(u) < \deg_i(v)$.

To distinguish these divisions we use the abbreviations $T, J, P, I, II, D$. In the implementation described below, three orderings are used to induce division in Example 8: lexicographical, degree-lexicographical and degree-reverse-lexicographical. For these three induced divisions we shall use the subscripts $L, DL, DRL$, respectively.
Every of the above divisions generates its own procedure for completion of a monomial set to involution by means of its enlargement with involutively irreducible nonmultiplicative prolongations. Given a monomial basis and an involutive division, the following algorithm \texttt{MinimalInvolutiveMonomialBasis} produces the uniquely defined minimal involutive basis of the ideal.

Algorithm \texttt{MinimalInvolutiveMonomialBasis}:

Input: $U$, a finite monomial set
Output: $\bar{U}$, the minimal involutive basis of $I_d(U)$

begin

1. $\bar{U} := \text{Autoreduce}(U)$
2. \textbf{choose} any admissible monomial ordering $\prec$
3. \textbf{while} exist $u \in \bar{U}$ and $x \in NM_L(u, \bar{U})$ s.t.
4. \hspace{0.5cm} $u \cdot x$ has no involutive divisors in $\bar{U}$ \textbf{do}
5. \hspace{1cm} \textbf{choose} such $u, x$ with the lowest $u \cdot x$ w.r.t. $\prec$
6. \hspace{1cm} $\bar{U} := \bar{U} \cup \{ u \cdot x \}$
7. \textbf{end}
8. \textbf{end}

end

Here $\text{Autoreduce}(U)$ stands for the conventional (non-involutive) autoreduction.

\section{2 Implementation Issues}

In this section we will describe some observations that allow to speed up the steps of the algorithm \texttt{MinimalInvolutiveMonomialBasis} significantly. Some of them are applicable to different divisions, others are concerned with the completion procedure in general. The basic operations on monomial sets are the same for the computation of involutive bases of polynomial \cite{3,4} and differential systems \cite{9}, so the improvements described here are relevant for these computations.

Our package provides a framework for studying the effect of using different divisions and optimizations. It is implemented using a “generic programming” approach which allows to start with a straightforward implementation of the algorithm and introduce more efficient procedures for special situations later.

The following statement returns the minimal involutive basis of a monomial set $U$ with respect to Janet division and with lexicographic selection ordering:
minimalInvolutiveMonomialBasis[Janet][U, lexOrder]

To extend the package for a new involutive division (called, say, newDivision), one would only have to write the specific version of the function separation which computes the multiplicative and nonmultiplicative variables of a monomial $u \in U$ w.r.t. the set $U$:

\[
\text{separation[newDivision][u, U]} := \ldots
\]

All the other steps in the algorithm would then be executed by functions that are generically defined for any involutive division.

On the other hand, an optimized procedure for a specific situation can be introduced later to override the generic version. The pattern matching mechanism in Mathematica dispatches to the specific version wherever it is appropriate.

Monomials are represented as multiindices, i.e. the monomial $x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}$ is represented as the list of its exponents $\{i_1, \ldots, i_n\}$. Thus, the set $U = \{u_1, \ldots, u_m\}$ can be considered as a $m \times n$-matrix of integers. For every monomial $u$, we use two additional lists of length $n$: a list giving the separation of the variables for $u$, and a similar list containing notes about the prolongations that have already been done.

We will now describe observations that can be used to make the basic operations of the algorithm MinimalInvolutiveMonomialBasis faster. Functions like lcm will be applied also to multiindices, with the obvious meaning. The set notation is used for lists, assuming that the order of the elements is given somehow. $U = \{u_1, \ldots, u_n\}$ is a list of monomials, and $u$ is always an element of $U$.

The first step is to compute the separation for each of the input monomials. For globally defined divisions, this is done irrespective of the other monomials in $U$. For Janet division (Example 4), we made use of the following remark:

Remark 1. When the list $U$ is sorted lexicographically in decreasing order, the groups $[d_1, \ldots, d_i]$ mentioned in the definition are grouped together. These groups are sorted lexicographically with respect to their labels of any fixed length $i$. The sorted list starts with the group labeled $[d_{1\text{max}}], d_{1\text{max}} = \max deg_1 u$, the monomials in $[d_{1\text{max}}]$ have $x_1$ as a multiplicative variable. We can split the list into groups given by
labels of length 1 and proceed recursively within each of them, next considering degrees in the second variable $x_2$, and so on.

For a division $D_\triangleright$ (Example 8) that is induced by some ordering $\triangleright$, we can use an auxiliary list:

**Remark 2.** Let the monomials be sorted in descending order: $u_1 \triangleright \ldots \triangleright u_n$. We call the elements of the list $\text{cm}(U) := \{m_1, \ldots, m_n | m_i = \text{lcm}(u_i, \ldots, u_n), i = n, \ldots, 1\}$ the cumulated multiples of $U$. By definition, variable $x_j$ is nonmultiplicative for $u_i$ if and only if it has a higher degree in $m_i$: $\deg_j u_i < \deg_j m_i$. Thus, all we have to do is compute the list $\text{cm}(U)$ of cumulated multiples and then compare each $u \in U$ against its corresponding entry in $\text{cm}(U)$.

For Division I, we are not aware of any property that would allow us to accelerate the computation of separations in a manner similar to Janet or Induced divisions.

The following observation can be used to speed up the process of finding a minimal nonmultiplicative prolongation (line 6 of the algorithm). Let us denote the minimal (w.r.t. the chosen ordering $\triangleright$) nonmultiplicative prolongation by a given variable $x$ with $P_\triangleright(x)$.

**Remark 3.** Let $U$ be sorted w.r.t. the completion ordering: $u_1 \triangleright \ldots \triangleright u_n$. Let $u_i$ and $x$ be fixed such that $u_i \cdot x$ is a minimal nonmultiplicative prolongation w.r.t. $\triangleright$. Then $u_i \cdot x$ is an element of the set $\{P_\triangleright(x_1), \ldots, P_\triangleright(x_n)\}$.

This follows directly from the minimality of $u_i \cdot x$. Furthermore, $u_i$ is the minimal monomial having $x$ as a nonmultiplicative variable, because $v \cdot x \triangleright u \cdot x$ implies $v \triangleright u$.

The remark obviously extends to the more general situation of the algorithm, where some of the nonmultiplicative prolongations have already been considered. We keep a list $P = \{P_\triangleright(x_1), \ldots, P_\triangleright(x_n)\}$ of nonmultiplicative prolongations, one for each variable $x_1, \ldots, x_n$, sorted by the completion ordering. Let $v = u_i \cdot x_j$ be the minimal prolongation. It is removed from $P$ and checked for involutive divisors. If $v$ is involutively reducible, we have to add another prolongation w.r.t. the same variable $x_j$ to $P$. Otherwise, we add $v$ to the monomial set and recompute the separations and $P$.

The next step in the algorithm is to search for an involutive divisor $w$ of a nonmultiplicative prolongation $v = u \cdot x$. In the polynomial case, the efficiency of this search can be even more important, since we may
want to involutively reduce every term of a prolonged polynomial. Recall that for an involutively reduced set $U$, there can be at most one such $w$. We present now some optimizations that apply to increasingly specialized situations.

The following remark uses a special property of involutive divisions, taking into account that $v$ is a nonmultiplicative prolongation of an element of $U$.

**Remark 4.** Let $U$ be an involutively autoreduced set of monomials and $v = u \cdot x$ a nonmultiplicative prolongation of some $u \in U$. If a monomial $w \in U$ is an involutive divisor of $v$ then $\deg_x w = \deg_x v$.

Since $u \cdot x$ should be involutively reducible by $w$, we can write $u \cdot x = w \times (u \cdot x/w)$. If $w = v = u \cdot x$, we are done. If $w \neq u \cdot x$ and $w | u$, then $u = w \times (u/w)$, which contradicts our assumption that $U$ is involutively autoreduced.

One can gain even more by considering particular divisions. Consider a Janet-autoreduced set $U$. Let us denote the longest common prefix of two monomials $u, v$ by $lcp(u, v)$, where $lcp(u, v) := (u_1, \ldots, u_k)$ with $(u_1, \ldots, u_k) = (v_1, \ldots, v_k)$, and $k$ the maximal index for which $u_k$ and $v_k$ coincide. If $u_1 \neq v_1$, we define $lcp(u, v) := ()$. More generally, we use $lcp(v, U)$ to denote the longest common prefix that $v$ shares with some monomial from the set $U$.

**Remark 5.** Assume that we search for a Janet-involutive divisor $w$ of a monomial $v$. Then, $w$ is in the class $C$ defined by the label $lcp(v, U)$. Let $lcp(v, U) = (v_1, \ldots, v_k)$. Every involutive divisor $w = (w_1, \ldots, w_n)$ is also a conventional divisor, thus $w_i \leq v_i, i = 1, \ldots, k$. We show by contradiction that $w_i = v_i$ for $i = 1, \ldots, k$. Let $s$ be the smallest integer $1 \leq s \leq k$ such that $w_s < v_s$. Then, $x_s$ is nonmultiplicative for $w$ because there exists a monomial in the class $(v_1, \ldots, v_{s-1})$ which has higher degree in $x_s$, and $w$ is not an involutive divisor of $v$.

Note that this remark applies to arbitrary monomials $v$, not only those resulting from a nonmultiplicative prolongation.

Consider a nonmultiplicative prolongation $v = u \cdot x$. For Pommaret division, an involutive divisor $w$ is reverse lexicographically greater than $u$. For a division that is induced by $\succ$, either $u \cdot x = w$ or $u \succ w$ holds.

These properties together with Remark 3.12 in [1] suggest that one should keep the monomials sorted with respect to some suitable order, and use this order as completion order, too.
Finally, when we find no involutive divisor, we have to add the prolongation to the set and adjust separations for all monomials accordingly. Except for globally defined divisions, this step is potentially very time consuming.

**Remark 6.** For all divisions discussed so far, the following holds for a monomial $u \in U$: $\text{NM}(u, U \cup \{v\}) = \text{NM}(u, U) \cup \text{NM}(u, \{u, v\})$.

A detailed discussion of this fact can be found in \cite{5}. After adding a monomial $v$ to $U$, this remark allows us to compute only the “pairwise” separations for every $u \in U$.

Specific divisions give rise to more improvements.

**Remark 7.** Let $v$ be a monomial, and assume that $v$ has no involutive divisor in the Janet-autoreduced set $U$. Then, the separation may only change for monomials in the class $\text{lcp}(v, U) = (v_1, \ldots, v_k)$. The separation of the variables $x_1, \ldots, x_k$ is left unchanged. Furthermore, the separation of the variables $x_1, \ldots, x_k$ for the new monomial $v$ can be copied from the separation of any of the monomials in the class $\text{lcp}(v, U)$.

**Remark 8.** Consider adding a nonmultiplicative prolongation $v = u \cdot x_j$ to an autoreduced set w.r.t. some induced division $D_\succ$.

Only the variable $x_j$ can change from multiplicative to nonmultiplicative, and it can do so only for monomials $s \succ v$ satisfying $\text{deg}_j s = \text{deg}_j v - 1$.

Not all of the improvements mentioned here were actually implemented in the package. Our experience suggests that sometimes the practical performance in *Mathematica* differs from what one expects from looking at the algorithm. This is due to the interpreted nature of *Mathematica* and its flexible evaluation mechanism. Operations which are performed in the kernel are usually much faster than their equivalent expressed in a user defined function, and it was often a matter of trial and error to decide which variant of an operation one should use for a given division.

In practice, the size of the resulting involutive basis is certainly the dominating factor for the overall running time of the algorithm. It was thus worthwhile to invest more programming work in improvements for those divisions which yield relatively small involutive bases (see below).
The improvements for Janet division resulted in the biggest gain in speed compared to the generic implementation. When the completion ordering is lexicographic, all optimizations described above are applied. For induced divisions $D_r$, we always use $\succ$ as completion ordering and Remark 3 to recompute the separations. Only for Division I, the time for changing the separations dominates the time for the other basic operations. Division I is also the only division for which the property mentioned in Remark 6 is used. The optimizations for finding an involutive divisor described above have a positive effect for all divisions.

We have applied the package to examples taken from various sources. For each polynomial system, we computed the degree-reverse-lexicographical Gröbner basis and took the resulting set of leading monomials as input to the algorithm MinimalInvolutiveMonomialBasis. As we described in [1] the output can then be used to compute the Hilbert function, the Hilbert polynomial and the index of regularity of the corresponding polynomial ideal.

**Example 9.** [1] Consider a $n \times n$ matrix $A = (\alpha_{ij})_{n,n}$ with unspecified entries. The condition $A^2 = 0$ leads to a system of $n^2$ polynomial equations in the variables $\alpha_{11}, \ldots, \alpha_{1n}, \alpha_{21}, \ldots, \alpha_{nn}$. We treated the leading monomials of the degree reverse lexicographic Gröbner basis, where the variables are ordered according to $\alpha_{11} \succ \ldots \succ \alpha_{1n} \succ \alpha_{21} \succ \ldots \succ \alpha_{nn}$.

**Example 10.** The system of “$n$-th cyclic roots” is a well known example. For $n = 4$, it is given by:

\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 &= 0, \\
x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 &= 0, \\
x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2 &= 0, \\
x_1x_2x_3x_4 - 1 &= 0.
\end{align*}
\]

**Example 11.** The Reimer system in 5 variables:

\[
\begin{align*}
1 - 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 - 2x_5^2 &= 0, \\
1 - 2x_1^3 + 2x_2^3 + 2x_3^3 + 2x_4^3 - 2x_5^3 &= 0, \\
1 - 2x_1^4 + 2x_2^4 + 2x_3^4 + 2x_4^4 - 2x_5^4 &= 0, \\
\end{align*}
\]
Example 12. The Katsura system in 7 variables:

\[
\begin{align*}
1 - 2x_1^5 + 2x_2^5 + 2x_3^5 + 2 \cdot x_4^5 - 2x_5^5 &= 0, \\
1 - 2x_1^6 + 2x_2^6 + 2x_3^6 + 2 \cdot x_4^6 - 2x_5^6 &= 0.
\end{align*}
\]

The following table shows the results of applying the algorithm \texttt{MinimalInvolutiveMonomialBasis} to our examples. In the first three columns, the size of the input is given where \(m\) is the number of monomials, \(n\) is the number of variables, and \(d\) is the maximum total degree of the input monomials. The divisions are indicated by the abbreviations used above. For each division, we give the length of the minimal involutive monomial basis, the number of prolongations considered during completion, the portion of reducible prolongations, and the computation time. Thus, 100\% reducible prolongations means that the input is already an involutive basis. An empty entry in the column for Pommaret division means that we did not compute a minimal Pommaret basis because the ideal is not zero dimensional. For the other divisions, it means that the timing is larger than 10000 seconds at our computer\footnote{a 200 MHz 586 running Linux}. 

\[\begin{array}{|c|c|c|c|c|}
\hline
\text{Division} & \text{Length} & \text{Prolongations} & \text{Reducible} & \text{Time (s)} \\
\hline
\end{array}\]
| Input | Size | Division |
|-------|------|----------|
|       | m n d| J T P I  | D_L D_DRL D_DL |
| Ex. [1] | 38 5 8 | 55 4392 55 | 151 242 894 594 |
|        | 190 17406 190 | - - | 503 798 3994 2639 |
|        | 91% 75% 91% | - - | 77% 74% 79% 79% |
|        | 3.7 s 4484 s 3.4 s | 11 s 48 s 556 s 267 s |
|         |                                               |
| Ex. [12] | 41 7 7 | 43 43 | 201 201 1337 1346 |
|        | 211 211 | - - | 861 892 7600 7663 |
|        | 99% 99% | - - | 81% 82% 83% 83% |
|        | 3.5 s 3.7 s | 20 s 44 s 1500 s 1539 s |
| cyc 4 | 7 4 6 | 7 98 | 98 25 41 9 7 |
|        | 14 242 | - - | 242 55 92 20 14 |
|        | 100% 62% | - - | 62% 67% 63% 90% 100% |
|        | 0.19 s 5.4 s | 18 s 0.87 s 2.3 s 0.33 s 0.21 s |
| cyc 5 | 20 5 8 | 23 1010 23 1010 | 93 154 135 106 |
|        | 76 3544 76 3544 | 297 488 548 419 |
|        | 96% 72% 96% 72% | 75% 72% 79% 79% |
|        | 1.1 s 266 s 1.1 s 1656 s | 5.5 s 21 s 21 s 14 s |
| cyc 6 | 45 6 9 | 46 46 | 201 385 841 972 |
|        | 194 194 | - - | 807 1527 4230 4899 |
|        | 99% 99% | - - | 81% 78% 81% 81% |
|        | 3.2 s 3.1 s | 19 s 123 s 586 s 754 s |
| Ex. [9] | n = 3 | 56 239 | 612 531 1711 1479 |
|         | 87% | - - | 2972 2920 9362 8044 |
|         | 4.5 s | - - | 80% 83% 82% 82% |
|         |                                               |
| Ex. [9] | n = 4 | 1324 11836 90% | - - - - - - |
|         | 161 166 | - - - - | - - - - - - |
|         | 923 s | - - - - | - - - - - - |

For some examples, bases for two different divisions may coincide. For the fourth cyclic roots (Example [10]), the bases for Thomas division and Division I, as well as those for Janet division and the induced division $D_{DL}$ coincide, respectively.

The computations with monomial sets should give at least some hint to the performance of different divisions in the polynomial and
differential cases. From our experience, Janet division, generally, and Induced divisions, sometimes, seem to be the most promising in terms of prolongations that have to be considered. Pommaret division – even though it is not noetherian – deserves further investigation, because it is globally defined and rather “compact”, too.

3 Conclusion

In addition to the above described implementation of different involutive divisions in Mathematica we implemented the completion algorithm for Janet division (Example 4) in C. In this case an input monomial set is represented as an array of lexicographically ordered multiindices and its completion is done with respect to the same order. This choice of completion ordering was motivated by the monotonicity of Janet division with respect to the lexicographical order. The partial involutivity of an intermediate monomial set is preserved in the course of completion and the time for recomputation of the separations is minimized.

The set of nonmultiplicative prolongations to be treated is also represented as a lexicographically sorted array of multiindices, that provides the simplest way to choose a minimal prolongation. Every time an irreducible nonmultiplicative prolongation occurs it is inserted in the intermediate monomial set and its nonmultiplicative prolongations are inserted in the prolongation set. The determination of their position in the sorted arrays is performed using the binary search algorithm. In so doing, the check of Janet reducibility of the prolongation under consideration is done in the course of the position determination. This is a rather straightforward procedure that makes use of the partition into prefix-groups as defined in Example.

The C implementation was done in GNU C/C++ version 2.81 on a 100 MHz Pentium computer running Windows 95. The running times for examples in the above table are less than 0.01 seconds, except Example for \( n = 4 \) which took about 5 seconds.

We plan to extend both Mathematica and C codes to polynomial and then to linear differential systems. Whereas the highly flexible and easily extensible Mathematica code allows one to experiment with different involutive divisions, in the further development of the C code we are going to restrict ourselves to Janet, Pommaret and may be Induced divisions which are more preferable from the computational efficiency point of view.
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