On Randomized Fictitious Play for Approximating Saddle Points Over Convex Sets

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Abstract

Given two bounded convex sets \(X \subseteq \mathbb{R}^m\) and \(Y \subseteq \mathbb{R}^n\), specified by membership oracles, and a continuous convex-concave function \(F : X \times Y \to \mathbb{R}\), we consider the problem of computing an \(\varepsilon\)-approximate saddle point, that is, a pair \((x^*, y^*) \in X \times Y\) such that
\[
\sup_{y \in Y} F(x^*, y) \leq \inf_{x \in X} F(x, y^*) + \varepsilon.
\]
Grigoriadis and Khachiyan (1995), based on a randomized variant of fictitious play, gave a simple algorithm for computing an \(\varepsilon\)-approximate saddle point for matrix games, that is, when \(F\) is bilinear and the sets \(X\) and \(Y\) are simplices. In this paper, we extend their method to the general case. In particular, we show that, for functions of constant "width", an \(\varepsilon\)-approximate saddle point can be computed using \(O^*(n + m)\) random samples from log-concave distributions over the convex sets \(X\) and \(Y\). As a consequence, we obtain a simple randomized polynomial-time algorithm that computes such an approximation faster than known methods for problems with bounded width and when \(\varepsilon \in (0, 1)\) is a fixed, but arbitrarily small constant. Our main tool for achieving this result is the combination of the randomized fictitious play with the recently developed results on sampling from convex sets.

1 Introduction

Let \(X \subseteq \mathbb{R}^m\) and \(Y \subseteq \mathbb{R}^n\) be two bounded convex sets. We assume that each set is given by a membership oracle, that is an algorithm which given \(x \in \mathbb{R}^m\) (respectively, \(y \in \mathbb{R}^n\)) determines, in polynomial time in \(m\) (respectively, \(n\)), whether or not \(x \in X\) (respectively, \(y \in Y\)). Let \(F : X \times Y \to \mathbb{R}\) be a continuous convex-concave function, that is, \(F(\cdot, y) : X \to \mathbb{R}\) is convex for all \(y \in Y\) and \(F(x, \cdot) : Y \to \mathbb{R}\) is concave for all \(x \in X\). We assume that we can evaluate \(F\) at rational arguments in constant time. The well-known saddle-point theorem (see e.g. [Roc70]) states that
\[
v^* = \inf_{x \in X} \sup_{y \in Y} F(x, y) = \sup_{y \in Y} \inf_{x \in X} F(x, y).
\]
This can be interpreted as a 2-player zero-sum game, with one player, the minimizer, choosing her/his strategy from a convex domain \(X\), while the other player, the maximizer, choosing her/his strategy from a convex domain \(Y\). For a pair of strategies \(x \in X\) and \(y \in Y\), \(F(x, y)\)
denotes the corresponding payoff, which is the amount that the minimizer pays to the maximizer. An equilibrium, when both $X$ and $Y$ are closed, corresponds to a saddle point, which is guaranteed to exist by [1], and the value of the game is the common value $v^*$. When an approximate solution suffices or at least one the sets $X$ or $Y$ is open, the appropriate notion is that of $\varepsilon$-optimal strategies, that a pair of strategies $(x^*, y^*) \in X \times Y$ such that for a given desired absolute accuracy $\varepsilon > 0$, 

$$
\sup_{y \in Y} F(x^*, y) \leq \inf_{x \in X} F(x, y^*) + \varepsilon.
$$

(2)

There is an extensive literature on the existence of saddle points in this class of games and generalization (see e.g. [Dan63, Gro67, KLP90, McLA84, Roc70, Sha58, Ter72, Wal45, Seb90]) and their applications (see e.g. [Bel97, DKR91, Was03]). A particularly important case is when the sets $X$ and $Y$ are polytopes with an exponential number of facets arising as the convex hulls of combinatorial objects (see Appendix A for some applications).

One can easily see that (1) can be reformulated as a convex minimization problem over a convex set given by a membership oracle and hence any algorithm for solving this class of problems, e.g., the Ellipsoid method, can be used to compute a solution to (2), in time polynomial in the input size and polylog(\frac{1}{\varepsilon}). However, there has recently been an increasing interest in finding simpler and faster approximation algorithms for convex optimization problems, sacrificing the dependence on $\varepsilon$ from polylog(\frac{1}{\varepsilon}) to poly(\frac{1}{\varepsilon}), in exchange of efficiency in terms of other input parameters; see e.g. [AHK05, AK07, BBR04, GKL92, GKL95, GKL98, GKL04, Kha04, Kal07, LN93, KY07, You01, PST91] and [GKL94, GKL96, GKPV01, Jan04, DJ07].

In this paper, we show that it is possible to get such an algorithm for computing an $\varepsilon$-saddle point (2). Our algorithm is based on combining a technique developed by Grigoriadis and Khachiyan [GKL93], based on a randomized variant of Brown’s fictitious play [Bro51], with the recent results on random sampling from convex sets [LV06, Vem05]. Our algorithm is superior to known methods when the width parameter $\rho$ (to be defined later) is small and $\varepsilon \in (0, 1)$ is a fixed but arbitrarily small constant.

2 Our Result

We need to make the following technical assumptions:

(A1) We know $\xi^0 \in X$, and $\eta^0 \in Y$, and strictly positive numbers $r_X$, $R_X$, $r_Y$, and $R_Y$ such that $B^m(\xi^0, r_X) \subseteq X \subseteq B^m(0, R_X)$ and $B^n(\eta^0, r_Y) \subseteq Y \subseteq B^n(0, R_Y)$, where $B^k(x^0, r) = \{x \in \mathbb{R}^k : \|x - x^0\|_2 \leq r\}$ is the $k$-dimensional ball for radius $r$ centered at $x^0 \in \mathbb{R}^k$. In particular, both $X$ and $Y$ are full-dimensional in their respective spaces (but maybe open). In what follows we will denote by $R$ the maximum of $\{R_X, R_Y, \frac{1}{r_X}, \frac{1}{r_Y}\}$.

(A2) $|F(x, y)| \leq 1$ for all $x \in X$ and $y \in Y$.

Assumption (A1) is standard for algorithms that deal with convex sets defined by membership oracles (see, e.g., [GLS93]), and will be required by the sampling algorithms. Assumption (A2) can be made without loss of generality, since the original game can be converted to an equivalent one satisfying (A2) by scaling the function $F$ by $\frac{1}{\rho}$, where the “width” parameter is defined as $\rho = \max_{x \in X, y \in Y} |F(x, y)|$. (For instance, in case of bilinear function, i.e., $F(x, y) = xAy$, where $A$ is given $m \times n$ matrix, we have $\rho = \max_{x \in X, y \in Y} |xAy| \leq \sqrt{mnR_XR_Y} \max_{\{a_{ij} \}} : i \in [m], j \in [n]}$. Replacing $\varepsilon$ by $\frac{\varepsilon}{\rho}$, we get an algorithm that works without assumption (A2) but whose running time is proportional to $\rho^2$. We note that

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1 Minimize $F(x)$, where $F(x) = \max_y F(x, y)$. 

2
such dependence on the width is unavoidable in algorithms that obtain $\varepsilon$-approximate solutions for negative functions and whose running time is proportional to $\text{poly}(\frac{1}{\varepsilon})$ (see e.g. [AHK06, PST91]); otherwise, scaling would yield an exact algorithm for rational inputs. For nonnegative functions $F$, the dependence on the width can be avoided, if we consider relative approximation errors (see [GK98, Kha04, KY07, You01]).

We assume throughout that $\varepsilon$ is a positive constant less than 1.

The main contribution of this paper is to extend the randomized fictitious play result in [GK95] to the more general setting given by (2).

**Theorem 1** Assume $X$ and $Y$ satisfy assumption (A1). Then there is a randomized algorithm that finds a pair of $\varepsilon$-optimal strategies in an expected number of $\mathcal{O}(\frac{\rho^2(n+m)}{\varepsilon^2} \ln \frac{R}{\varepsilon})$ iterations, each computing two approximate samples from log-concave distributions. The algorithm requires $\mathcal{O}^*(\frac{\rho^2(n+m)}{\varepsilon^2} \ln \frac{R}{\varepsilon})$ calls to the membership oracles for $X$ and $Y$.

When the width is bounded and $\varepsilon$ is a fixed constant, our algorithm needs $\mathcal{O}^*((n + m)^6 \ln R)$ oracle calls. This is superior to known methods, e.g., the Ellipsoid method, that compute the $\varepsilon$-saddle point in time polynomial in $\log \frac{1}{\varepsilon}$. In Appendix A we give examples of problems with bounded width arising in combinatorial optimization.

### 3 Relation to Previous Work

**Matrix and polyhedral games.** The special case when each of the sets $X$ and $Y$ is a polytope (or more generally, a polyhedron) and payoff is a bilinear function, is known as polyhedral games (see e.g. [Was03]). When each of these polytopes is just a simplex we obtain the well-known class of matrix games. Even though each polyhedral game can be reduced to a matrix game by using the vertex representation of each polytope (see e.g. [Sch86]), this transformation may be (and is typically) not algorithmically efficient since the number of vertices may be exponential in the number of facets by which each polytope is given.

**Fictitious play.** We assume for the purposes of this subsection that both sets $X$ and $Y$ are closed, and hence the infimum and supremum in (1) are replaced by the minimum and maximum, respectively.

In fictitious play which is a deterministic procedure and originally proposed by Brown [Bro51] for matrix games, each player updates his/her strategy by applying the best response, given the current opponent’s strategy. More precisely, the minimizer and the maximizer initialize, respectively, $x(0) = 0$ and $y(0) = 0$, and for $t = 1, 2, \ldots$, update $x(t)$ and $y(t)$ by

$$x(t + 1) = \frac{t}{t + 1} x(t) + \frac{1}{t + 1} \xi(t), \text{ where } \xi(t) = \arg\min_{\xi \in X} F(\xi, y(t)), \quad (3)$$

$$y(t + 1) = \frac{t}{t + 1} y(t) + \frac{1}{t + 1} \eta(t), \text{ where } \eta(t) = \arg\max_{\eta \in Y} F(x(t), \eta). \quad (4)$$

For matrix games, the convergence to optimal strategies was established by Robinson [Rob51]. In this case, the best response of each player at each step can be chosen from the vertices of the corresponding simplex. A bound of $\left(\frac{2^{m+n}}{\varepsilon}\right)^{m+n-2} \ln R$ on the time needed for convergence to

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2 See Section 5.3 for details.

3 Here, we apply random sampling as a black box for each iteration independently; it might be possible to improve the running time if we utilize the fact that the distributions are only slightly modified from an iteration to the next.

4 $\mathcal{O}^*(\cdot)$ suppresses polylogarithmic factors that depend on $n$, $m$ and $\varepsilon$. 

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an $\varepsilon$-saddle point was obtained by Shapiro [Sha58]. More recently, Hofbauer and Sorin [HS06] showed the convergence of fictitious play for continuous convex-concave functions over compact convex sets; they also gave a dynamic proof for the minmax theorem.

**Randomized fictitious play.** In [GK95], Grigoriadis and Khachiyan introduced a randomized variant of fictitious play for matrix games. Their algorithm replaces the minimum and maximum selections (3)-(4) by a smoothed version, in which, at each time step $t$, the minimizing player selects a strategy $i \in [m]$ with probability proportional to $\exp\{-\varepsilon_2 A y(t)\}$, where $\varepsilon$ denotes the ith unit vector of dimension $m$. Similarly, the maximizing player chooses strategy $j \in [n]$ with probability proportional to $\exp\{\varepsilon x(t) A^T j\}$. Grigoriadis and Khachiyan proved that, if $A \in [-1,1]^{m \times n}$, then this algorithm converges, with high probability, to an $\varepsilon$-saddle point in $O\left(\frac{\log(m+n)}{\varepsilon^2}\right)$ iterations. Our result builds on [GK95].

**The multiplicative weights update method.** Freund and Schapire [FS99] showed how to use the weighted majority algorithm, originally developed by Littlestone and Warmuth [LW94], for computing $\varepsilon$-saddle points for matrix games. Their procedure can be thought of as a derandomization of the randomized fictitious play described above. Similar algorithms have also been developed for approximately solving special optimization problems, such as general linear programs [PST91], multicommodity flow problems [GK98], packing and covering linear programs [PST91, GK98, GK04, KY07, You01], a class of convex programs [Kha04], and semidefinite programs [AIK05, AIK07]. Arora, Hazan, and Kale [AIK06] consider the following scenario: given a finite set $X$ of decisions and a finite set $Y$ of outputs, and a payoff matrix $M \in \mathbb{R}^{X \times Y}$ such that $M(x, y)$ is the penalty that would be paid if decision $x \in X$ was made and output $y \in Y$ was the result, the objective is to develop a decision making strategy that tends to minimize the total payoff over many rounds of such decision making. Arora et al. [AIK06, Kal07] show how to apply this framework to approximately computing $\max_{y \in Y} \min_{i \in [m]} f_i(y)$, given an oracle for finding $\max_{y \in Y} \min_{i \in [m]} f_i(y)$ for any non-negative $\lambda \in \mathbb{R}^m$ such that $\sum_{i=1}^m \lambda_i = 1$, where $Y \subseteq \mathbb{R}^n$ is a given convex set and $f_1, \ldots, f_m : Y \to \mathbb{R}$ are given concave functions (see also [Kha04] for similar results).

There are two reasons why this method cannot be (directly) used to solve our problem. First, the number of decisions $m$ is infinite in our case, and second, we do not assume to have access to an oracle of the type described above; we assume only a (weakest possible) membership oracle on $Y$. Our algorithm extends the multiplicative update method to the computation of approximate saddle points.

**Hazan’s Work.** In his Ph.D. Thesis [Haz06], Hazan gave an algorithm, based on the multiplicative weights updates method, for approximating the minimum of a convex function within an absolute error of $\varepsilon$. Theorem 4.14 in [Haz06] suggests that a similar procedure can be used to approximate a saddle point for convex-concave functions, however, without a running time analysis.

**Sampling algorithms.** Our algorithm makes use of known algorithms for sampling from a given log-concave distribution $f(\cdot)$ over a convex set $X \subseteq \mathbb{R}^m$. The currently best known result achieving this is due to Lovász and Vempala (see, e.g., [LV07] Theorem 2.1): a random walk on $X$ converges in $O^*(m^5)$ steps to a distribution within a total variation distance of $\varepsilon$ from the desired exponential distribution with high probability.

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5 This procedure can be written in the same form as our Algorithm 1 below, except that it chooses the points $x(t) \in X$ and $y(t) \in Y$, at each time step $t = 1, \ldots, T$, as the (approximate) centroids of the corresponding sets with respect to densities $p_k(t) = e^{\sum_{t=1}^{T-k-1} \ln(e^{-F(\xi(t), \eta(t))})}$ and $q_k(t) = e^{\sum_{t=1}^{T-k-1} \ln(e^{+F(\xi(t), \eta(t))})}$ (both of which are log-concave distributions), and outputs $(\frac{1}{T} \sum_{t=1}^{T} x(t), \frac{1}{T} \sum_{t=1}^{T} y(t))$ at the end.

6 that is, log $f(\cdot)$ is concave.
Algorithm 1 Randomized fictitious play

**Input:** Two convex bounded sets $X, Y$ and a function $F(x, y)$ such that $F(\cdot, y) : X \rightarrow \mathbb{R}$ is convex for all $y \in Y$ and $F(x, \cdot) : Y \rightarrow \mathbb{R}$ is concave for all $x \in X$, satisfying assumptions (A1) and (A2)

**Output:** A pair of $\varepsilon$-optimal strategies

1: $t := 0$; choose $x(0) \in X$; $y(0) \in Y$, arbitrarily
2: while $t \leq T$ do
3: Pick $\xi \in X$ and $\eta \in Y$, independently, from $X$ and $Y$ with densities $\frac{p_\xi(t)}{\|p(t)\|_1}$ and $\frac{q_\eta(t)}{\|q(t)\|_1}$, respectively
4: $x(t + 1) := \frac{t}{t+1} x(t) + \frac{1}{t+1} \xi$; $y(t + 1) := \frac{t}{t+1} y(t) + \frac{1}{t+1} \eta$; $t := t + 1$
5: end while
6: return $(x(t), y(t))$

Several algorithms for convex optimization based on sampling have been recently proposed. Bertsimas and Vempala [BV04] showed how to minimize a convex function over a convex set $X \subseteq \mathbb{R}^m$, given by a membership oracle, in time $O^*((m^5C + m^7) \log R)$, where $C$ is the time required by a single oracle call. When the function is linear time $O^*(m^{4.5})$ suffices (Kalai and Vempala [KV06]).

Note that we can write (1) as the convex minimization problem $\inf_{x \in X} F(x)$, where $F(x) = \sup_{y \in Y} F(x, y)$ is a convex function. Thus, it is worth comparing the bounds we obtain in Theorem 1 with the bounds that one could obtain by applying the random sampling techniques of [BV04, KV06] (see Table 1 in [BV04] for a comparison between these techniques and the Ellipsoid method). Since the above program is equivalent to $\inf\{v : x \in X, \text{ and } F(x, y) \leq v \text{ for all } y \in Y\}$, the solution can be obtained by applying the technique of [BV04, KV06], where each membership call involves another application of these techniques (to check if $\sup_{y \in Y} F(x, y) \leq v$). The total time required by this procedure is $O^*(n^{4.5}(m^5C + m^7) \log R)$, which is significantly greater than the bound stated in Theorem 1.

4 The Algorithm

The algorithm proceeds in steps $t = 0, 1, \ldots$, updating the pair of accumulative strategies $x(t)$ and $y(t)$. Given the current pair $(x(t), y(t))$, define

$$p_\xi(t) = e^{-\frac{tF(\xi, y(t))}{2}} \quad \text{for } \xi \in X,$$

$$q_\eta(t) = e^{\frac{tF(x(t), \eta)}{2}} \quad \text{for } \eta \in Y,$$

and let

$$\|p(t)\|_1 = \int_{\xi \in X} p_\xi(t) d\xi \quad \text{and} \quad \|q(t)\|_1 = \int_{\eta \in Y} q_\eta(t) d\eta.$$

Our algorithm is a direct generalization of the algorithm in [GK95]. The parameter $T$ will be specified later (see Lemma 4).

5 Analysis

Following [GK95], we use a potential function $\Phi(t) = \|p(t)\|_1 \|q(t)\|_1$ to bound the number of iterations required by the algorithm to reach an $\varepsilon$-saddle point. The analysis is composed of
three parts. The first part of the analysis (Section 5.1), is a generalization of the arguments in [GK95] (and [KY07]): we show that the potential function increases, on the average, only by a factor of $e^{O(\varepsilon^2)}$, implying that after $t$ iterations the potential is at most a factor of $e^{O(\varepsilon^2)t}$ of the initial potential. While this was enough to bound the number of iterations by $\varepsilon^{-2} \log(n + m)$ when both $X$ and $Y$ are simplices and the potential is a sum over all vertices of the simplices [GK95], this cannot be directly applied in our case. This is because the fact that a definite integral of a non-negative function over a given region $Q$ is bounded by some $\tau$ does not imply that the function at any point in $Q$ is also bounded by $\tau$. In the second part of the analysis (Section 5.2), we overcome this difficulty by showing that, due to concavity of the exponents in (5) and (6), the change in the function around a given point cannot be too large, and hence, the value at a given point cannot be large unless there is a sufficiently large fraction of the volume of the sets $X$ and $Y$ over which the integral is also too large. In the last part of the analysis (Section 5.3), we show that the same bound on the running time holds when the sampling distributions in line 3 of the algorithm are replaced by sufficiently close approximate distributions.

5.1 Bounding the potential increase

Lemma 1 For $t = 0, 1, 2, \ldots$,

$$\mathbb{E}[\Phi(t + 1)] \leq \mathbb{E}[\Phi(t)](1 + \frac{\varepsilon^2}{6})^2.$$

Proof Conditional on the values of $x(t)$ and $y(t)$, we have

$$\|p(t + 1)\|_1 = \int_{\xi \in X} e^{-\frac{\varepsilon(t+1)F(\xi, y(t+1))}{2}} d\xi \leq \int_{\xi \in X} e^{-\frac{\varepsilon(t+1)F(\xi, y(t))}{2}} e^{-\frac{\varepsilon F(\xi, \eta)}{2}} d\xi = \int_{\xi \in X} p_\xi(t) e^{-\frac{\varepsilon F(\xi, \eta)}{2}} d\xi \leq \int_{\xi \in X} p_\xi(t) \left[1 + \frac{\varepsilon^2}{6} - \frac{\varepsilon}{2} F(\xi, \eta) \right] d\xi = \|p(t)\|_1(1 + \frac{\varepsilon^2}{6} - \frac{\varepsilon}{2} \frac{\int_{\xi \in X} p_\xi(t) F(\xi, \eta) d\xi}{\|p(t)\|_1}),$$

using assumption (A2), concavity of $F(\xi, \cdot) : Y \to \mathbb{R}$ and the inequality $e^\delta \leq 1 + \delta + \frac{3}{2}\delta^2$, valid for all $\delta \in [-\frac{1}{3}, \frac{1}{3}]$. Taking the expectation with respect to $\eta$ (with density proportional to $q_\eta(t)$), we get

$$\mathbb{E}_\eta[\|p(t + 1)\|_1] \leq \|p(t)\|_1 \left[1 + \frac{\varepsilon^2}{6} - \frac{\varepsilon}{2} \frac{\int_{\eta \in Y} q_\eta(t) \int_{\xi \in X} p_\xi(t) F(\xi, \eta) d\xi d\eta}{\|q(t)\|_1 \|p(t)\|_1} \right]. \quad (7)$$

Similarly, by taking the expectation with respect to $\xi$ (with density proportional to $p_\xi(t)$), we can derive

$$\mathbb{E}_\xi[\|q(t + 1)\|_1] \leq \|q(t)\|_1 \left[1 + \frac{\varepsilon^2}{6} + \frac{\varepsilon}{2} \frac{\int_{\xi \in X} p_\xi(t) \int_{\eta \in Y} q_\eta(t) F(\xi, \eta) d\eta d\xi}{\|p(t)\|_1 \|q(t)\|_1} \right]. \quad (8)$$
Now, using independence of \( \xi \) and \( \eta \), we have

\[
\mathbb{E}[\Phi(t+1)|x(t), y(t)] \leq \Phi(t) \left(1 + \frac{\varepsilon^2}{6}\right)^2 \\
+ \varepsilon \left(1 + \frac{\varepsilon^2}{6}\right) \left(\frac{\int_{\xi \in X} p_\xi(t) \int_{\eta \in Y} q_\eta(t) F(\xi, \eta) d\eta d\xi}{\|p(t)\|_1 \|q(t)\|_1} - \frac{\int_{\eta \in Y} q_\eta(t) \int_{\xi \in X} p_\xi(t) F(\xi, \eta) d\xi d\eta}{\|q(t)\|_1 \|p(t)\|_1}\right)
\]

\[
- \frac{\varepsilon^2}{4} \frac{\int_{\xi \in X} p_\xi(t) \int_{\eta \in Y} q_\eta(t) F(\xi, \eta) d\eta d\xi}{\|p(t)\|_1 \|q(t)\|_1} \frac{\int_{\eta \in Y} q_\eta(t) \int_{\xi \in X} p_\xi(t) F(\xi, \eta) d\xi d\eta}{\|q(t)\|_1 - 1 \|p(t)\|_1}.
\]

By interchanging the order of integration, we get that the second part of the sum on the right-hand side is zero, and third part is non-positive. Hence,

\[
\mathbb{E}[\Phi(t+1)|x(t), y(t)] \leq \Phi(t) \left(1 + \frac{\varepsilon^2}{6}\right)^2.
\]

The lemma follows by taking the expectation of (9) with respect to \( x(t) \) and \( y(t) \). \( \square \)

By Markov’s inequality we have the following statement.

**Corollary 1** With probability at least \( \frac{1}{2} \), after \( t \) iterations,

\[
\Phi(t) \leq 2e^{\frac{2}{\varepsilon^2} t} \Phi(0).
\]

At this point one might tend to conclude the proof, as in [GK95, KY07], by implying from Corollary 1 and the non-negativity of the function under the integral

\[
\Phi(t) = \int_{\xi \in X, \eta \in Y} e^{\frac{1}{2} t (F(x(t), \eta) - F(\xi, y(t)))} d\xi d\eta,
\]

that this function is bounded at every point also by \( 2e^{\frac{2}{\varepsilon^2} t} \Phi(0) \) (with high probability). This would then imply that the current strategies are \( \varepsilon \)-optimal. However, this is not necessarily true in general and we have to modify the argument to show that, even though the value of the function at some points can be larger than the bound \( 2e^{\frac{2}{\varepsilon^2} t} \Phi(0) \), the increase in this value cannot be more than an exponential (in the input description), which would be still enough for the bound on the number of iterations to go through.

### 5.2 Bounding the number of iterations

For convenience, define \( Z = X \times Y \), and concave function \( g_t : Z \to \mathbb{R} \) given at any point \( z = (\xi, \eta) \in Z \) by \( g_t(\xi, \eta) := \frac{1}{2} t (F(x(t), \eta) - F(\xi, y(t))) \). Note that, by our assumptions, \( Z \) is a full-dimensional bounded convex set in \( \mathbb{R}^N \) of volume \( \Phi(0) = vol(X) \cdot vol(Y) \), where \( N = n+m \). Furthermore, assumption (A2) implies that \( |g_t(z)| = |\frac{1}{2} t (F(x(t), \eta) - F(\xi, y(t)))| \leq \varepsilon t \) for all \( z \in Z \).

A sufficient condition for the convergence of the algorithm to an \( \varepsilon \)-approximate equilibrium is provided by the following lemma.

**Lemma 2** Suppose that (10) holds and there exists an \( \alpha \) such that

\[
0 < \alpha < 4\varepsilon t,
\]

\[
e^{\frac{1}{2} \alpha} \left(\frac{\alpha}{4\varepsilon t}\right)^N vol(Z) > 1.
\]
Then
\[ e^{g_t(z)} \leq 2e^{\frac{t^2}{3} + \alpha \Phi(0)} \text{ for all } z \in Z. \quad (14) \]

**Proof.** Assume otherwise, i.e., there is \( z^* \in Z \) with \( g_t(z^*) > \frac{t^2}{3} + \alpha + \ln(2\Phi(0)) \). Let \( \lambda^* = \alpha/(4\epsilon t) \),
\[
Z^+ = \{ z \in Z | g_t(z) \geq g_t(z^*) - \alpha/2 \}, \quad \text{and } Z^{++} = \{ z^* + \frac{1}{\lambda^*}(z - z^*) | z \in Z^+ \}.
\]
Concavity of \( g_t \) implies convexity of \( Z^+ \). This implies in particular that \( z^* + \lambda^*(z - z^*) \in Z^{++} \) of all \( 0 \leq \lambda \leq \frac{1}{\lambda^*} \) and \( z \in Z^+ \), since \( z^* + \lambda^*\lambda'(z - z^*) \in Z^+ \). We claim that \( Z \subseteq Z^{++} \). Assume otherwise, and let \( x \in Z \setminus Z^{++} \) (and hence \( x \in Z \setminus Z^+ \)). Let
\[
\lambda^+ = \sup \{ \lambda | z^* + \lambda(x - z^*) \in Z^+ \} \text{ and } z^+ = z^* + \lambda^+(x - z^*).
\]
By continuity of \( g_t \), \( z^+ \in Z^+ \) and \( g_t(z^*) - \alpha/2 = g_t(z^+) \). We have \( x - z^* = \frac{1}{\lambda^+}(z^+ - z^*) \) and hence \( \frac{1}{\lambda^+} > \frac{1}{\lambda^*} \). But \( z^+ = \lambda^+x + (1 - \lambda^+)z^* \) and hence
\[
g_t(z^*) - \alpha/2 = g_t(z^+) = g_t(\lambda^+x + (1 - \lambda^+)z^*) \geq \lambda^+g_t(x) + (1 - \lambda^+)g_t(z^*).
\]
Thus
\[
\frac{\alpha}{2} \leq \lambda^+(g_t(z^*) - g_t(x)) \leq 2\epsilon t \lambda^+,
\]
a contradiction. We have now established \( Z \subseteq Z^{++} \). The containment implies
\[
\text{vol}(Z) \leq \text{vol}(Z^{++}) = \left( \frac{1}{\lambda^*} \right)^N \text{vol}(Z^+)
\]
and further
\[
\Phi(t) = \int_{z \in Z} e^{g_t(z)} \, dz \geq \int_{z \in Z^+} e^{g_t(z)} \, dz
\]
\[
\geq 2\Phi(0)e^{\frac{t^2}{3} + \frac{\alpha}{4\epsilon t}N} \text{vol}(Z^+) \geq 2\Phi(0)e^{\frac{t^2}{3} + \frac{\alpha}{4\epsilon t}}N \text{vol}(Z) > 2\Phi(0)e^{\frac{t^2}{3}t},
\]
a contradiction to \((10)\). \(\square\)

We can now derive an upper-bound on the number of iterations needed to converge to \( \epsilon \)-optimal strategies.

**Lemma 3** If \((14)\) holds and
\[
t \geq \frac{6}{\epsilon^2}(\alpha + \max\{0, \ln(2\text{vol}(Z))\}),
\]
then \((x(t), y(t))\) is an \( \epsilon \)-optimal pair.

**Proof.** By \((14)\) we have \( g_t(z) \leq \frac{e^2}{3}t + \alpha + \ln(2\Phi(0)) = \frac{e^2}{3}t + \alpha + \ln(2\text{vol}(Z)) \) for all \( z \in Z \), or equivalently,
\[
\frac{\epsilon}{2} (F(x(t), \eta) - F(\xi, y(t))) \leq \frac{\epsilon^2}{3}t + \alpha + \ln(2\text{vol}(Z)) \quad \text{for all } \xi \in X \text{ and } \eta \in Y.
\]
Hence,
\[
F(x(t), \eta) \leq F(\xi, y(t)) + \frac{2\epsilon}{3}t + \frac{2}{\epsilon t}(\alpha + \ln(2\text{vol}(Z)) \quad \text{for all } \xi \in X \text{ and } \eta \in Y,
\]
which implies by \((15)\) that
\[
F(x(t), \eta) \leq F(\xi, y(t)) + \epsilon \quad \text{for all } \xi \in X \text{ and } \eta \in Y.
\]
\(\square\)
Lemma 4 For any \( \varepsilon \in (0, 1) \), there exist \( \alpha \) and 
\[
    t = O \left( \frac{N}{\varepsilon^2} \ln \frac{R}{\varepsilon} \right)
\]
satisfying (12), (13) and (15).

Proof Assume \( \text{vol}(Z) \leq \frac{1}{2} \). Let us choose \( t = \frac{6\alpha}{\varepsilon^2} \). Then (13) becomes (after taking logarithms)
\[
    \frac{\alpha}{2} + N \ln \left( \frac{\alpha}{4\varepsilon t} \right) + \ln(\text{vol}(Z)) > 0.
\]
So choosing \( \frac{\alpha}{2} = N \ln \left( \frac{24}{\varepsilon} \right) - \ln(\text{vol}(Z)) \) would satisfy this inequality. Then
\[
    t = O \left( \frac{N}{\varepsilon^2} \ln \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2} \ln \frac{1}{\text{vol}(Z)} \right).
\]
Since \( 1/\text{vol} Z \leq R^N \), the claim follows. Let now \( \text{vol}(Z) > \frac{1}{2} \). Then
\[
    e^{\frac{\alpha}{4\varepsilon t}} (\frac{\alpha}{4\varepsilon t})^N \text{vol}(Z) > \frac{1}{2} e^{\frac{\alpha}{4\varepsilon t}} (\frac{\alpha}{4\varepsilon t})^N,
\]
Thus, in order to satisfy (13), it is enough to find \( \alpha \) and \( t \) satisfying
\[
    \frac{1}{2} e^{\frac{\alpha}{4\varepsilon t}} (\frac{\alpha}{4\varepsilon t})^N > 1.
\]
To satisfy (15), let us simply choose \( t = \frac{6\alpha}{\varepsilon^2} + \frac{6}{\varepsilon^2} \ln(2\text{vol}(Z)) \) and demand that
\[
    \frac{1}{2} e^{\frac{\alpha}{4\varepsilon t}} (\frac{\alpha}{4\varepsilon t})^N = \frac{1}{2} e^{\frac{\alpha}{4\varepsilon t}} (\frac{24\ln(\frac{\alpha}{\varepsilon})}{\varepsilon} + \frac{24}{\varepsilon} \ln(2\text{vol}(Z)))^N > 1,
\]
or equivalently,
\[
    2 \left( \frac{24}{\varepsilon} \right)^N \left( 1 + \frac{\ln(2\text{vol}(Z))}{\alpha} \right)^N < e^{\frac{\alpha}{2}}.
\]
Thus, it is enough to select \( \alpha = \max \left\{ 4(\ln 2 + N \ln(\frac{24}{\varepsilon})), 2\sqrt{N \ln(2\text{vol}(Z))} \right\} \) which satisfies
\[
    2 \left( \frac{24}{\varepsilon} \right)^N \leq e^{\frac{\alpha}{2}} \quad \text{and} \quad \left( 1 + \frac{\ln(2\text{vol}(Z))}{\alpha} \right)^N < e^{\ln(2\text{vol}(Z))} \frac{\alpha}{\alpha} N \leq e^{\frac{\alpha}{2}}.
\]
It follows that
\[
    t = \max \left\{ \frac{24}{\varepsilon^2} (\ln 2 + N \ln(\frac{24}{\varepsilon})), \frac{12}{\varepsilon^2} \sqrt{N \ln(2\text{vol}(Z))} \right\} + \frac{6}{\varepsilon^2} \ln(2\text{vol}(Z)).
\]
Since \( \text{vol}(Z) \leq R^N \), the claim follows. \( \Box \)

Setting \( T \) in Algorithm 1 to the value of \( t \) given by Lemma 4 yields the following result.

Corollary 2 Assume \( X \) and \( Y \) satisfy assumptions (A1) and (A2). Then Algorithm 1 computes a pair of \( \varepsilon \)-optimal strategies in expected \( O(\frac{n+m}{\varepsilon^2} \ln \frac{R}{\varepsilon}) \) iterations.
5.3 Using approximate distributions

We know consider the (realistic) situation when we can only sample approximately from the convex sets. In this case we assume the existence of approximate sampling routines that, upon the call in step 3 of the algorithm, return vectors $\xi \in X$, and (independently) $\eta \in Y$, with densities $\hat{p}_\xi(t)$ and $\hat{q}_\eta(t)$, such that

$$\sup_{X' \subseteq X} \left| \frac{\hat{p}_{X'}(t)}{\hat{p}_X(t)} - \frac{p_{X'}(t)}{p_X(t)} \right| \leq \delta \quad \text{and} \quad \sup_{Y' \subseteq Y} \left| \frac{\hat{q}_{Y'}(t)}{\hat{q}_Y(t)} - \frac{q_{Y'}(t)}{q_Y(t)} \right| \leq \delta,$$

where $\hat{p}_{X'}(t) = \int_{x \in X} \hat{p}_\xi(t) d\xi$ (similarly, define $p_{X'}(t), \hat{q}_{Y'}(t), q_{Y'}(t)$), and $\delta$ is a given desired accuracy. We next prove an approximate version of Lemma 1.

**Lemma 5** Suppose that we use approximate sampling routines with $\delta = \varepsilon/4$ in step 3 of Algorithm 1. Then, for $t = 0, 1, 2, \ldots$, we have

$$\mathbb{E}[\Phi(t + 1)] \leq \mathbb{E}[\Phi(t)](1 + \frac{43}{36}\varepsilon^2).$$

**Proof** The argument up to Equation (17) remains the same. Taking the expectation with respect to $\eta$ (with density proportional to $\hat{q}_\eta(t)$), we get

$$\mathbb{E}_\eta[\|p(t + 1)\|_1] \leq \|p(t)\|_1 \left[ 1 + \frac{\varepsilon^2}{6} - \frac{\varepsilon}{2} \int_{\eta \in Y} \hat{q}_\eta(t) \int_{\xi \in X} p_\xi(t) F(\xi, \eta) d\xi d\eta \right].$$

Similarly,

$$\mathbb{E}_\eta[\|q(t + 1)\|_1] \leq \|q(t)\|_1 \left[ 1 + \frac{\varepsilon^2}{6} + \frac{\varepsilon}{2} \int_{\eta \in Y} \hat{q}_\eta(t) \int_{\eta \in Y} q_\eta(t) F(\xi, \eta) d\eta d\xi \right].$$

Thus, by independence of $\xi$ and $\eta$, we have

$$\mathbb{E}[\Phi(t + 1)|x(t), y(t)] \leq \Phi(t) \left( \left( 1 + \frac{\varepsilon^2}{6} \right)^2 + \frac{\varepsilon}{2} \left( 1 + \frac{\varepsilon^2}{6} \right) \left( \int_{\xi \in X} \hat{p}_\xi(t) \int_{\eta \in Y} q_\eta(t) F(\xi, \eta) d\eta d\xi \right) \right)$$

$$- \frac{\varepsilon^2}{4} \int_{\xi \in X} \hat{p}_\xi(t) \int_{\eta \in Y} q_\eta(t) F(\xi, \eta) d\eta d\xi.$$

We will make use of the following proposition.

**Proposition 1** If we set $\delta = \varepsilon/4$ in (16), then

$$\left| \int_{\xi \in X} \hat{p}_\xi(t) \int_{\eta \in Y} q_\eta(t) F(\xi, \eta) d\eta d\xi \right| \leq \varepsilon.$$  

**Proof** Since

$$\int_{\xi \in X} p_\xi(t) \int_{\eta \in Y} q_\eta(t) F(\xi, \eta) d\eta d\xi = \int_{\eta \in Y} q_\eta(t) \int_{\xi \in X} p_\xi(t) F(\xi, \eta) d\xi d\eta,$$

we have
we can bound the L.H.S. of (19) by
\[
\left| \int_{\xi \in X} p_\xi(t) \int_{\eta \in Y} q_\eta(t) F(\xi, \eta) d\eta d\xi \right| \leq \frac{1}{\|q(t)\|_1} \left| \int_{\xi \in X} \frac{p_\xi(t)}{p_X(t)} - \frac{\hat{p}_\xi(t)}{\hat{p}_X(t)} \right| d\eta d\xi
\]
\[
= \left| \int_{\xi \in X} \frac{p_\xi(t)}{p_X(t)} - \frac{\hat{p}_\xi(t)}{\hat{p}_X(t)} \right| d\xi
\]
\[
= \left| \int_{\xi \in \xi'} \frac{p_\xi(t)}{p_X(t)} - \frac{\hat{p}_\xi(t)}{\hat{p}_X(t)} \right| d\xi + \int_{\xi \in \xi''} \left( \frac{\hat{p}_\xi(t)}{\hat{p}_X(t)} - \frac{p_\xi(t)}{p_X(t)} \right) d\xi
\]
\[
= \left( \frac{p_X'(t)}{p_X(t)} - \frac{\hat{p}_X'(t)}{\hat{p}_X(t)} \right) + \left( \frac{\hat{p}_X''(t)}{\hat{p}_X(t)} - \frac{p_X''(t)}{p_X(t)} \right) \leq \varepsilon \quad \text{(by (16))}.
\]

Proposition 1 implies that
\[
\mathbb{E}[\Phi(t + 1)|x(t), y(t)| \leq \Phi(t) \left[ 1 + \frac{\varepsilon^2}{6} \right]^2 + \frac{\varepsilon^2}{2} \left( 1 + \frac{\varepsilon^2}{6} \right) + \frac{\varepsilon^4}{4} \right] \leq \Phi(t) \left( 1 + \frac{43}{36} \varepsilon^2 \right) .
\]
The rest of the proof is as in Lemma 1.

Combining the currently known bound on the mixing time for sampling (see [LV04, LV06, LV07] and also Section 3) with the bounds on the number of iterations from Corollary 2 gives Theorem 1.

6 Conclusion

We showed that randomized fictitious play can be applied for computing \(\varepsilon\)-saddle points of convex-concave functions over the product of two convex bounded sets. Even though our bounds were stated for general convex sets, one should note that these bounds may be improved for classes of convex sets for which faster sampling procedures could be developed. We believe that the method used in this paper could be useful for developing algorithms for computing approximate equilibria for other classes of games.

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Applications in combinatorial optimization

A.1 Mixed popular matchings

We note that there exist natural examples for which the width parameter $\rho$ is small. For instance, consider the case when $F(x, y) = xAy$, $A \in [-1, 1]^{m \times n}$ and $X$ and $Y$ are convex subsets of the simplices of dimensions $m$ and $n$, respectively.

Let us give a more concrete application. Let $S, T$ be two families (say, of combinatorial objects), and $A \in [-1, 1]^{S \times T}$ be a given matrix. We assume that these families have exponential size (in some input parameter) and hence, the matrix is given by an oracle that specifies for each $S \in S$ and $T \in T$ the value of $A(S, T)$. The objective is to find a saddle point for the matrix game defined by $A$ on the set of mixed strategies $\Delta_S = \{p \in \mathbb{R}_+^S : \sum_{S \in S} p_S = 1\}$ and $\Delta_T = \{q \in \mathbb{R}_+^T : \sum_{T \in T} q_T = 1\}$.

In general the optimal strategies might have exponential support (i.e., exponential number of non-zero entries). However, if the families arise from combinatorial objects in a natural way, then the supports of optimal strategies may be polynomially bounded. More precisely, let $E$
and $F$ be two sets of sizes $m$ and $n$ respectively, such that each element $S \in S$ (respectively, $T \in T$), is characterized by a vector $x^S \in \{0,1\}^E$ (respectively, $y^T \in \{0,1\}^F$). We assume that $X = \text{conv}\{x^S : S \in S\}$ and $Y = \text{conv}\{y^T : T \in T\}$ have explicit linear descriptions, and furthermore that there exists an $m \times n$ matrix $A$ such that $\mathcal{A}(S,T) = x^S Ay^T$, for all $S \in S$ and $T \in T$. Then it follows from Von Neumann's Saddle point theorem [Dan63] (which is a special case of [KMN09]) that

$$
\min_{p \in \Delta_S} \max_{q \in \Delta_T} pAq = \min_{u \in X} \max_{y \in Y} xAy.
$$

(21)

Indeed,

$$
\begin{align*}
\min_{p \in \Delta_S} & \max_{q \in \Delta_T} pAq \\
&= \min_{p \in \Delta_S} \max_{q \in \Delta_T} \sum_{S \in \delta, T \in T} pSQT \mathcal{A}(S,T) \\
&= \min_{p \in \Delta_S} \max_{q \in \Delta_T} \sum_{S \in \delta} pSx^S \mathcal{A} \sum_{T \in T} qTy^T \\
&= \max_{y \in Y} \min_{x \in X} xAy.
\end{align*}
$$

(see [KMN09]). Thus the original matrix game corresponds to a problem of the form (1).

A special case of this framework was considered in [KMN09] under the name of Mixed popular matchings. Let $G = (U \cup V, E)$ be a bipartite graph, and $r : E \to \mathbb{Z}$ be a given rank function. A U-matching $M : U \to V$ is an injective mapping such that $\{(u, M(u)) : u \in U\} \subseteq E$. Let $\mathcal{S} = T = \{\{(u, M(u)) : u \in U\} : M \text{ is a U-matching of } G\} \subseteq 2^E$. Given $S, T \in \mathcal{S}$, define $\phi(S,T) = |\{u \in U : r(u, S(u)) \leq r(u, T(u))\}|/|U|$ to be the fraction of the vertices of $U$ that “prefer” $S$ to $T$, and $\mathcal{A}(S,T) = \phi(S,T) - \phi(T,S)$.

It is well-known (see e.g. [GLS93]) that the convex hull of U-matchings has the linear description $X = Y = \{x \in \mathbb{R}_+^E : \sum_{\{u,v\} \in E} x_{u,v} = 1 \ \forall u \in U, \ \sum_{\{u,v\} \in E} x_{u,v} \leq 1 \ \forall v \in V\}$. Furthermore, if we define $A \in \mathbb{R}^{E \times E}$ to be the matrix with entries

$$
a_{\{u,v\},\{u',v'\}} = \begin{cases} 
\frac{1}{|U|} & \text{if } u = u' \text{ and } r(u,v) < r(u',v'), \\
-\frac{1}{|U|} & \text{if } u = u' \text{ and } r(u,v) > r(u',v'), \\
0 & \text{otherwise},
\end{cases}
$$

then for any $S,T \in \mathcal{S}$, we can write $\mathcal{A}(S,T) = x^SAy^T$, where $x^S, y^T \in \{0,1\}^E$ are the characteristic vectors of $S$ and $T$ respectively. Note that in this case $\rho \leq 1$ and $\text{vol}(X) = \text{vol}(Y) \leq 1$.

Note that in the above example, the problem can be written as a linear program of polynomially bounded size [KMN09]. However, this is not the case when the known linear descriptions of $X$ and $Y$ are not polynomially bounded, e.g., when in the above example $G$ is a general non-bipartite graph. In this case the solution requires using the Ellipsoid method, or the sampling techniques of [BV03] [KV06], or the use of our algorithm.

### A.2 Linear relaxation for submodular set cover

Here is another example. Let $f : 2^{[n]} \to [0,1]$ be a monotone submodular set-function. Consider the problem of minimizing $f(X)$ subject to the constraint that the characteristic vector $e_X \in \{0,1\}^n$ belongs to a polytope $P \subseteq \mathbb{R}^n$. For instance, in the submodular set covering problem, the polytope $P = \{x \in [0,1]^n : \sum_{i \in S} x_i \geq 1 \text{ for all } e \in E\}$, where $S_1, \ldots, S_n \subseteq E$ are given subsets of a finite set $E$. Let $P_f = \{y \in \mathbb{R}_+^n : \sum_{i \in X} y_i \leq f(X) \text{ for all } X \subseteq [n]\}$ be the polymatroid associated with $f$. Then it is well-known that $f(X) = \max_{y \in P_f} e_Xy$. Thus we
arrive at the following saddle point computation which provides a lower bound on the optimum submodular set cover: \( \min_{x \in P} \max_{y \in P_j} xy \), where \( \rho \leq 1 \). For other applications of polyhedral games, we refer the reader to [Was03].