Quantum de Rham complex with $d^3 = 0$

differential

N. Bazunova *, A. Borowiec †, R. Kerner ‡

To appear in Czechoslovak Journal of Physics v. 51 (2001)

Abstract

In this work, we construct the de Rham complex with differential operator $d$ satisfying the $Q$-Leibniz rule, where $Q$ is a complex number, and the condition $d^3 = 0$ on an associative unital algebra with quadratic relations. Therefore we introduce the second order differentials $d^2 x^i$. In our formalism, besides the usual two-dimensional quantum plane, we observe that the second order differentials $d^2 x$ and $d^2 y$ generate either bosonic or fermionic quantum planes, depending on the choice of the differentiation parameter $Q$.

1 Introduction

Since the discovery of quantum plane by Yu. V. Manin and its possible applications for the description of deformed or more intricate than usual symmetries in mathematical physics by Wess and Zumino, an immense activity followed, especially during the past decade. Quite naturally, after the purely algebraic properties of those newly discovered spaces have been quite deeply investigated, and the related quantum groups and Hopf algebras analyzed and described, the study of analytic properties had followed. This is why the $q$-deformed algebras have become the next object of many excellent studies [1], [2], [3].

Parallelly, novel ternary and $Z_3$-graded algebraic structures have been introduced and investigated [4], then generalized to the $Z_N$-graded case [5], [6], [7]. Thus an important class of $Z_N$-graded differential algebraic structures has been investigated in an exhaustive manner.

It becomes natural now to combine these two novel and important structures in order to see whether they can lead to further generalizations of many useful algebraic and analytic tools such as homology, de Rham complexes, Hecke and braided algebras, and the like. The aim of this article is to show the de Rham complex can be generalized for the case of the non-standard differential satisfying $d^3 = 0$ but with $d^2 \neq 0$.

* University of Tartu, Institute of Pure Mathematics, Vanemuise 46, 51014 Tartu, Estonia, nadegda@ut.ee
† University of Wroclaw, Institute of Theoretical Physics, plac Maksa Borna 9, PL 50-204 Wroclaw, Poland, borow@ift.uni.wroc.pl
‡ L.P.T.L. - Tour 22, 4-ème étage, Boite 142, Université Paris-VI, 4, Place Jussieu, 75005 Paris, rk@ccr.jussieu.fr
The paper is organized as follows. In the first section, we elaborate the general formalism for differential calculus with the differential operator $d$ satisfying the $Q$-Leibniz rule and the condition $d^3 = 0$ on an associative unital algebra with quadratic relations. Supposing $d^2 \neq 0$, we introduce the second order differentials $d^2x^i$ and find the relations connecting the generators $x^i$ and $d^2x^j$, $dx^i$ and $d^2x^j$.

In the second section, we find the values of parameter $Q$ from the commutation relations on the second order differentials $d^2x$ and $d^2y$. For these values of $Q$ we find that $d^2x$ and $d^2y$ generate either bosonic or fermionic quantum plane, depending on the value chosen for $Q$.

In what follows, we shall use the notation $[k]_Q = 1 + Q + Q^2 + \ldots + Q^{k-1}$; we shall denote by $E$ the identity operator (or matrix) acting in linear space defined by the context.

2 General case

Let $A$ be an associative unital algebra generated by variables $x^1$, $x^2$, $\ldots$, $x^n$, which satisfy commutation relations

$$x^i x^j = B_{kl}^{ij} x^k x^l,$$

where $B$ is a matrix with complex number entries.

Our aim is to generalize de Rham complex by assuming that $d^3 = 0$ instead of the usual condition $d^2 = 0$. De Rham complex consists of a first order differential calculus and it’s higher order prolongation. In our construction, the first order differential calculus coincides with Wess-Zumino type differential calculus on the quantum plane \([9]\). It is determined by the commutation relations between the generators $x^i$ and the differential one-forms $dx^j$, $i, j = 1, 2, \ldots, n$,

$$x^i dx^j = C_{kl}^{ij} dx^k x^l,$$

satisfying consistency conditions involving matrices $B$ and $C$:

$$(E_{12} - B_{12})(E_{12} + C_{12}) = 0,$$
$$B_{12}C_{23}C_{12} = C_{23}C_{12}B_{23}.$$  

Where $E_{12}$ denotes the identity operators tensor product, $E_{12} = I_1 \otimes I_2$, operating in the tensor product of the space of 1-forms $dx^i$ with the quantum space generated by $x^k$.

The equation (2) can be interpreted as a definition of a left $A$-module, therefore also a bimodule structure on a free right $A$-module generated by the differentials $dx^i$ (see e.g. \([9]\) for more details).

Since we assume $d^2 \neq 0$, we must introduce the second order differentials and replace the classical Leibniz and eventually generalize the classical Leibniz rule. The differential $d$, by means of which we extend the first order differential calculus, is now supposed to satisfy the $Q$-Leibniz rule: $d(\omega \theta) = d\omega \theta + Q^{\deg(\omega)} \omega d\theta$, where $\omega$ and $\theta$ are differential forms, $\deg(\omega)$ is the grade of the element $\omega$, and $Q$ is a complex number. In particular, for $\deg(\omega) = \deg(\theta) = 0$, i.e. for a first order calculus, one recovers the standard (undeformed) Leibniz rule.
The first differentiation of (2) gives rise to the relations between the generators \( x^i \), the first and second order differentials \( dx^j \), \( d^2x^k \):

\[
x^i d^2x^j = C_{k\ell}^{ij} d^2x^k d^2x^\ell + (QC_{k\ell}^{ij} - \delta^i_k \delta^j_\ell) dx^k dx^\ell.
\]

(5)

Next differentiation gives the commutation relations between differentials \( dx^k \) and \( d^2x^l \) only:

\[
([2]Q \delta^i_k \delta^j_\ell - Q^2 C_{k\ell}^{ij}) dx^k d^2x^l = ([2]Q QC_{k\ell}^{ij} - \delta^i_k \delta^j_\ell) d^2x^k dx^l.
\]

(6)

Finally, we obtain extra relations between differentials \( d^2x^i \):

\[
[3]Q d^2x^i d^2x^j = [3]Q^2 C_{k\ell}^{ij} d^2x^k d^2x^\ell.
\]

(7)

When \( Q \) is not a primitive cubic root of unity, i.e. \( Q \neq e^{\frac{2\pi i}{3}} \), we arrive at the following relations

\[
d^2x^i d^2x^j = Q^4 C_{k\ell}^{ij} d^2x^k d^2x^\ell.
\]

(8)

Therefore we refer to the case \( Q = e^{\frac{2\pi i}{3}} \) as specific because in this case there is no need to introduce new relations between the generators \( d^2x^i \) (for more general cases, see [13], [14]).

The relations (1 - 8) define a universal quantum ternary de Rham complex: any other de Rham complex on the quantum plane (2) admitting the first order calculus (2) can be obtained from this one via a standard quotient construction. In particular, we can assume that there exist commutation relations between generators \( x^i \) and second order differentials \( d^2x^i \)

\[
x^i d^2x^j = F_{k\ell}^{ij} d^2x^k d^2x^\ell.
\]

(9)

As a consequence, the second order differentials have to satisfy the following relations

\[
d^2x^i d^2x^j = Q^4 F_{k\ell}^{ij} d^2x^k d^2x^\ell.
\]

(10)

Again, this requirement allows us to introduce a left \( \mathcal{A} \)-module, therefore also a bimodule, structure on a right free module generated by the second order differentials \( d^2x^i \). Because of this, \( F \) should satisfy quadratic consistency conditions analogous to the condition (3):

\[
(B_{12} F_{12} F_{23} - F_{23} F_{12} B_{23}) = 0.
\]

(11)

Substituting now (3) into (3), one finds

\[
(F_{k\ell}^{ij} - C_{k\ell}^{ij}) d^2x^k d^2x^\ell = (QC_{k\ell}^{ij} - \delta^i_k \delta^j_\ell) dx^k dx^\ell,
\]

(12)

and using (3), the consistency condition takes on the form:

\[
E - (Q^2 + Q)C + ((Q^2 + Q)E - Q^3 C) QF = 0.
\]

(13)

The last equation reduces, in the generic case \( Q = e^{\frac{2\pi i}{3}} \) to a linear (cf. 3) Wess-Zumino-like condition on the matrices \( C \) and \( F \):

\[
(E + C)(E - QF) = 0.
\]

(14)
Following the well known Wess-Zumino method, we can now resolve the consistency conditions (11) and (14). To this end let us assume that a Hecke $R$-matrix is given, i.e. the matrix $R$ satisfying the braid relation:

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$$

(15)

together with the second-order minimal polynomial condition

$$(R - \mu E)(R + \lambda E) = 0.$$  

(16)

Rewriting it in the form

$$(E - \frac{1}{\mu} R)(E + \frac{1}{\lambda} R) = 0,$$

(17)

one immediately sees that $B = \frac{1}{\mu} R$, $C = \frac{1}{\lambda} R$ and $F = Q^2 R$ are the solution of the consistency conditions (3), (4) and (11), (14). These can be further generalized for non-Hecke $R$-matrices (cf. [10]).

More generally, having matrices $B$ and $C$ as a solution of the consistency conditions (3, 4), one can set $F = Q^2 B$ provided that $B$ commutes with $C$ and the braid relation (11) is satisfied.

3 Two-dimensional quantum plane

In this section, we shall construct the de Rham complex with $d^3 = 0$ on the two-dimensional quantum plane.

As it is well known, the quantum plane $xy = qyx$, where $q$ being a complex deformation parameter, is determined by an $R$-matrix

$$B = \frac{1}{q} \hat{R}, \quad \text{where} \quad \hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

satisfying the braid relation. There are two infinite and non-equivalent families of covariant first order differential calculus on this plane (parameterized by a complex parameter $r$), which are characterized by means of matrices $C_1$ and $C_2$ which define the commutation relations (3, 4) (cf. [11]).

$$C_1 = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & r - 1 & q & 0 \\ 0 & \frac{q}{r} & 0 & 0 \\ 0 & 0 & 0 & r \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & \frac{1}{q} & r - 1 & 0 \\ 0 & 0 & 0 & r \end{pmatrix}.$$  

As a matter of fact, the two matrices $C_1$ and $C_2$ are not really independent: one can be obtained from another if we substitute $x$ by $y$ and simultaneously $q$ by $q^{-1}$ and vise versa. They define a generalization of the first-order differential calculi obtained by Wess and Zumino [3]. In fact, we get a Wess-Zumino first order differential calculi if $r = q^2$ for the matrix $C_1$ and $r = 1/q^2$ for the matrix $C_2$ (cf. [12]).

Following the general formalism elaborated above, from (3, 4) we get two sorts of commutation relations for generators $d^2x$, $d^2y$. Considering the obtained
relations as equations with respect to the parameter of differentiation $Q$, we get the set of values $Q$: \{ \frac{e\pm i}{\sqrt{2}}, \frac{e\pm i}{\sqrt{3}}, \frac{e\pm i}{\sqrt{4}}, i \}.$

If $Q = e^{\frac{2\pi}{2}}$ or $Q = e^{\frac{2\pi}{4}}$, then $d^2x$ and $d^2y$ can not satisfy any particular binary relations.

If $Q = \pm \frac{1}{\sqrt{3}}$, then we have the commutation relations without the parameter $r$:

\[
\begin{align*}
\frac{d^2x}{d^2x} \frac{d^2y}{d^2y} &= \frac{d^2x}{d^2x} \frac{d^2y}{d^2y}, & d^2x \frac{d^2y}{d^2y} &= q d^2y \frac{d^2x}{d^2x}, \\
\frac{d^2y}{d^2y} &= q^{-1} \frac{d^2x}{d^2y}, & \frac{d^2y}{d^2y} &= \frac{d^2y}{d^2y}
\end{align*}
\]  

(18)

for both choices of matrices $C_1$ and $C_2$. It means that the generators $d^2x$ and $d^2y$ define the quantum plane $d^2x d^2y = q d^2y d^2x$, which is like the given quantum plane $xy = qyx$. Both quantum planes are preserved by the action of quantum group $GL_q(2)$, determined by generators $\alpha, \beta, \gamma, \delta$ satisfying the commutation relations:

\[
\begin{align*}
\alpha \beta &= q \beta \alpha, & \alpha \gamma &= q \gamma \alpha, \\
\gamma \delta &= q \delta \gamma, & \beta \gamma &= q \gamma \beta,
\end{align*}
\]

(19)

The parameter $r$ does not vanish when $Q = i = e^{\frac{2\pi}{4}}$, but in this case the matrices $C_1$ and $C_2$ define two distinct quantum planes. These quantum planes are preserved by the action of two different quantum groups $GL_q(2)$.

In fact, if the first order differential calculus is determined by the matrix $C_1$, then we have the following commutation relations between $d^2x$, $d^2y$:

\[
\begin{align*}
\frac{d^2x}{d^2x} \frac{d^2x}{d^2x} &= -r \frac{d^2x}{d^2x} \frac{d^2x}{d^2x}, & \frac{d^2x}{d^2x} \frac{d^2y}{d^2y} &= \frac{1}{r} \frac{d^2y}{d^2x} \frac{d^2x}{d^2x}, \\
\frac{d^2y}{d^2y} &= \frac{d^2y}{d^2y}, & \frac{d^2y}{d^2y} &= -r \frac{d^2y}{d^2y}
\end{align*}
\]  

(20)

From first and fourth relations, it follows that $(d^2x)^2 = (d^2y)^2 = 0$. The quantum plane determined by relations $d^2x d^2y = \frac{1}{r} d^2y d^2x$, $(d^2x)^2 = (d^2y)^2 = 0$ is preserved by action of quantum group $GL_{r,q}(2)$ whose generators $\alpha, \beta, \gamma, \delta$ satisfy the commutation relations:

\[
\begin{align*}
\alpha \beta &= \frac{1}{q} \beta \alpha, & \alpha \gamma &= q \gamma \alpha, \\
\beta \gamma &= \gamma \beta, & \alpha \delta &= q \delta \alpha, & \beta \delta &= q \delta \beta,
\end{align*}
\]

(21)

In the case of the first order differential calculus determined by matrix $C_2$, we get the commutation relations:

\[
\begin{align*}
\frac{d^2x}{d^2x} \frac{d^2x}{d^2x} &= -r \frac{d^2x}{d^2x} \frac{d^2x}{d^2x}, & \frac{d^2x}{d^2x} \frac{d^2y}{d^2y} &= -\frac{q}{r} \frac{d^2y}{d^2x} \frac{d^2x}{d^2x}, \\
\frac{d^2y}{d^2y} &= -\frac{q}{r} \frac{d^2x}{d^2y}, & \frac{d^2y}{d^2y} &= -r \frac{d^2y}{d^2y}
\end{align*}
\]  

(22)

which define the quantum plane $d^2x d^2y = -qr d^2y d^2x$, $(d^2x)^2 = (d^2y)^2 = 0$. This quantum plane is preserved by action of quantum group $GL_{r,q}(2)$, generated by $\alpha, \beta, \gamma, \delta$ satisfying a different set of commutation relations:

\[
\begin{align*}
\alpha \beta &= \frac{1}{q} \beta \alpha, & \alpha \gamma &= q \gamma \alpha, \\
\gamma \delta &= \frac{1}{q} \delta \gamma, & \beta \gamma &= q \gamma \beta,
\end{align*}
\]

(23)

N.B. and A.B. wish to thank for hospitality the Laboratoire LPTL where this paper has been written. N.B. is very grateful to V. Abramov and to the organizers of the present Colloquium for financial support, and acknowledges the financial support of Estonian Science Foundation under the grants No.1134 and No.4515.
References

[1] U. Carow-Watamura, S. Watamura: Int. J. Mod. Phys. A 13, No.19, 1998, 3235-3243.
[2] B.L. Cherchiai, R. Henterding, J. Madore, J. Wess: Eur. Phys. J. 8, No.3, 1998, p.547-558.
[3] R. Kerner: Z₃-graded exterior differential calculus and gauge theories of higher order
Lett. Math. Phys. 36, No.4, 1996, p.441-454; math-ph/0004033.
[4] M. Dubois-Violette, R. Kerner: Acta Math. Univ. Comen., New Ser. 65, No.2, 1996, p.175-188; q-alg/9608023.
[5] R. Kerner: in the proceedings of the Conference ICGTMP "Group-23", Dubna, Russia, July 30 - August 6, 2000; math-ph/0011023.
[6] M. Dubois-Violette and I.T. Todorov: Lett. Math. Phys. 48, No.4, 1999, p.323-338; hep-th/9704063.
[7] M. Dubois-Violette: Czech. J. Phys. 46, No.12, 1996, p.1227-1233; q-alg/9609012.
[8] J. Wess, B. Zumino: Nucl. Phys. B, Proc. Suppl. 18B 1990, p.302-312.
[9] V. K. Kharchenko, A. Borowiec, Questions of algebra and logic, Izdatel'stvo Instituta Matematiki SO RAN, Tr. Inst. Mat. Im. S. L. Soboleva SO RAN, 30, Novosibirsk, 1996, p.164-185 (in Russian).
[10] L. Hlavaty: J. Phys. A, Math. Gen. 25 1992, p.485-494.
[11] W. Pusz, S. Woronowicz: Rep. Math. Phys. 27, No.2, 1989, p.231-257.
[12] S.K. Soni: J. Phys. A, Math. Gen. 24, 1991, p.169-174.
[13] Dubois-Violette, M., Kerner, R.: Acta Math. Univ. Comen., New Ser. 65, No.2, 1996, p.175-188.
[14] Dubois-Violette, M.: K-Theory 14, No.4, 1998, p.371-404.