A Forbidden-Minor Characterization for the Class of graphic Matroids which yield the Co-graphic Element-Splitting Matroids

March 20, 2018

S. B. DHOTRE
Department of Mathematics
University of Pune, Pune-411007 (India)
E-mail: dsantosh2@yahoo.co.in

P. P. MALAVADKAR
MIT College of Engineering, Pune, Pune-411038 (India)
E-mail: pmalavadkar@gmail.com

M. M. SHIKARE
Department of Mathematics
University of Pune, Pune-411007 (India)
E-mail: mmshikare@gmail.com

Abstract

The element splitting operation on a graphic matroid, in general may not yield a cographic matroid. In this paper, we give a necessary and sufficient condition for the graphic matroid to yield cographic matroid under the element splitting operation.

AMS Subject Classification: 05B35
Key words: binary matroids, splitting, element splitting
1 Introduction

Fleischner [5] introduced the idea of splitting a vertex of degree at least three in a connected graph and used the operation to characterize Eulerian graphs. Raghunathan, Shikare and Waphare [10] extended the splitting operation from graphs to binary matroids. $M_{x,y}$ denotes the splitting matroid obtained by applying splitting operation on a binary matroid $M$, by a pair of elements $\{x, y\}$ of $M$.

Slater [13] specified the $n$-point splitting operation on a graph in the following way:

Let $G$ be a graph and $u$ be a vertex of degree at least $2n - 2$ in $G$. Let $H$ be the graph obtained from $G$ by replacing $u$ by two adjacent vertices $u_1$, $u_2$ such that each point formerly joined to $u$ is joined to exactly one of $u_1$ and $u_2$ so that in $H$, $\deg(u_1) \geq n$ and $\deg(u_2) \geq n$. We say that $H$ arises from $G$ by $n$-element splitting operation.

![Figure 1](image)

If $X = \{x_1, x_2, ... x_k\}$ be the set of edges incident at $u_1$ then we denote $H$ by $G'_X$. Tutte [15] characterized 3-connected graphs in terms of edge addition and 3-point splitting. Slater [13] obtained the following two useful results in this regard.

**Theorem 1.1.** The class of 2-connected graphs is the class of graphs obtained from $K_3$ by finite sequence of edge addition and 2-element splitting.

**Theorem 1.2.** If $G$ is $n$-connected and $H$ arise from $G$ by $n$-element splitting, then $H$ is $n$-connected.

Further, he classified 4-connected graphs using $n$-element splitting operation (see [13]).

Shikare and Azadi [12] [11] extended the notion of $n$-point splitting operation on graphs to binary matroids as follows.
Definition 1.3. Let $M$ be a binary matroid on a set $E$ and $A$ be a matrix over $GF(2)$ that represents the matroid $M$. Suppose that $X$ is a subset of $E(M)$. Let $A'_X$ be the matrix that is obtained by adjoining an extra row to $A$ with this row being zero everywhere except in the columns corresponding to the elements of $X$ where it takes the value 1 and then adjoining an extra column (corresponding to $a$) with this column being zero everywhere except in the last row where it takes the value 1. Suppose $M'_X$ be the vector matroid of the matrix $A'_X$. The transition from $M$ to $M'_X$ is called the element splitting operation.

We call matroid $M'_X$ as the element splitting matroid. If $|X| = 2$ and $X = \{x, y\}$. We denote the matroid $M'_X$ by $M'_{x,y}$. Azadi [1], characterized circuits of the element splitting matroid in terms of circuits of the binary matroids as follows.

Proposition 1.4. Let $M(E, C)$ be a binary matroid together with the collection of circuits $C$. Suppose $X \subseteq E$ and $a \notin E$. Then $M'_X = (E \cup \{a\}, C')$ where $C' = C_0 \cup C_1 \cup C_2$ and

$$
C_0 = \{ C \in C \mid C \text{ contains an even number of elements of } X \};
$$

$$
C_1 = \text{The set of minimal members of } \{ C_1 \cup C_2 \mid C_1, C_2 \in C, C_1 \cap C_2 \neq \emptyset \text{ and each of } C_1 \text{ and } C_2 \text{ contains an odd number of elements of } X \text{ such that } C_1 \cup C_2 \text{ contains no member of } C_0 \};
$$

$$
C_2 = \{ C \cup \{a\} \mid C \in C \text{ and } C \text{ contains an odd number of elements of } X \};
$$

Various properties concerning the element splitting matroids have been studied in [1, 6, 12]. The element splitting operation on a graphic (cographic) matroid may not yield a graphic (cographic) matroid.

Dalvi, Borse and Shikare [3, 4] characterized graphic (cographic) matroids whose element splitting matroids are graphic (cographic) when $|X| = 2$. In fact, they proved the following result.

Theorem 1.5. The element splitting operation, by any pair of elements, on a graphic (cographic) matroid yields a graphic (cographic) matroid if and only if it has no minor isomorphic to $M(K_4)$, where $K_4$ is the complete graph on 4 vertices.
In this paper, we obtain a forbidden-minor characterization for graphic matroids whose element splitting matroid is cographic when \(|X| = 2\). The main result in this paper is the following theorem.

**Theorem 1.6.** The element splitting operation, by any pair of elements, on a graphic matroid yields a cographic matroid if and only if it has no minor isomorphic to \(M(K_4)\), where \(K_4\) is the complete graph on 4 vertices.

### 2 Properties of Element Splitting Operation

In this section, we provide necessary Lemmas which are used in the proof of Theorem 1.6. Dalvi, Borse and Shikare proved the following useful Lemma.

**Lemma 2.1.** Let \(x\) and \(y\) be distinct elements of a binary matroid \(M\) and let \(r(M)\) denote the rank of \(M\). Then, using the notations introduced in Section 1,

1. \(M_{x,y} = M_{x,y}' \setminus \{a\}\);
2. \(M = M'_{x,y}/\{a\}\);
3. \(r(M_{x,y}') = r(M) + 1\);
4. Every cocircuit of \(M\) is a cocircuit of the matroid \(M'_{x,y}\); (v) if \(\{x, y\}\) is a cocircuit of \(M\) then \(\{a\}\) and \(\{x, y\}\) are cocircuits of \(M'_{x,y}\);
5. If \(\{x, y\}\) does not contain a cocircuit, then \(\{x, y, a\}\) is a cocircuit of \(M'_{x,y}\);
6. \(M'_{x,y} \setminus x/y \cong M \setminus x\);
7. If \(M\) is graphic and \(x, y\) are adjacent edges in a corresponding graph, then \(M'_{x,y}\) is graphic;
8. \(M'_{x,y}\) is not eulerian.

The following two results are well known minor based characterizations of graphic and cographic matroids (see [9]).

**Theorem 2.2.** A binary matroid is graphic if and only if it has no minor isomorphic to \(F_7, F_7^*, M^*(K_{3,3})\) or \(M^*(K_5)\).
Theorem 2.3. A binary matroid is cographic if and only if it has no minor isomorphic to $F_7, F_7^*, M(K_{3,3})$ or $M(K_5)$.

Notation. For the sake of convenience, let $\mathcal{F} = \{F_7, F_7^*, M(K_5), M(K_{3,3})\}$.

In the following Lemma, we provide a necessary condition for a graphic matroid whose element splitting matroid is not cographic.

Lemma 2.4. Let $M$ be a graphic matroid and let $x, y \in E(M)$ such that $M'_{x,y}$ is not cographic. Then $M$ has a minor isomorphic to $M(K_4)$ or there is a minor $N$ of $M$ such that no two elements of $N$ are in series and $N_{x,y}^f \setminus \{a\}/\{x\} \cong F$ or $N_{x,y}^f \setminus \{a\}/\{x,y\} \cong F$ or $N_{x,y}^f \setminus \{y\} \cong F$ or $N_{x,y}^f \setminus \{x, y\} \cong F$ for some $F \in \mathcal{F}$.

Proof. Suppose that $M'_{x,y}$ is not cographic and $M$ has no minor isomorphic to $M(K_4)$. Since $M'_{x,y}$ is not cographic $M'_{x,y} \setminus T_1/T_2 \cong F$ for some $T_1, T_2 \subseteq E(M_{x,y})$. Let $T_i' = T_i - \{a, x, y\}$ for $i = 1, 2$. Then $T_i' \subseteq E(M)$ for each $i$. Let $N = M \setminus T_1/T_2$. Then $N_{x,y} = M'_{x,y} \setminus T_1/T_2$. Let $T_i'' = T_i - T_i'$ for $i = 1, 2$. Then $N_{x,y} \setminus T_1''/T_2'' \cong F$. If $a \in T_2''$, then $F$ is a minor of $M'_{x,y}/a$ and hence, by Lemma 2.1(i), $F$ is a minor of $M$. Since $M$ is graphic $F_7, F_7^*$ can not be the minors of $M$. So $F = M(K_5)$ or $F = M(K_{3,3})$, but both $M(K_5)$ and $M(K_{3,3})$ have minor isomorphic to $M(K_4)$. Consequently $M$ has a minor isomorphic to $M(K_4)$. Which is a contradiction. Suppose $a \in T_1''$. By Lemma 2.1(i), $M_{x,y} = M'_{x,y} \setminus a$. Hence $F$ is a minor of $M_{x,y}$. It follows from Theorem 2.3 of [11] that $N$ does not contain a 2-cocircuit and further, $N_{x,y}/x \cong F$ or $N_{x,y}/\{x, y\} \cong F$. This implies that $N_{x,y}^f \setminus \{a\}/\{x\} \cong F$ or $N_{x,y}^f \setminus \{a\}/\{x, y\} \cong F$. Suppose that $a \notin T_1'' \cup T_2''$. Hence $a \notin T_1 \cup T_2$. If $T_1'' \cup T_2'' = \phi$, then $N_{x,y} \cong F$. If $T_2'' = \phi$, then $N_{x,y} \setminus y \cong F$ or $N_{x,y} \setminus y \cong F$ or $N_{x,y} \setminus \{x, y\} \cong F$. In the first case, $a$ forms a 2-cocircuit with $x$ or $y$ which ever is remained, and in the later case, $a$ is a coloop, both are contradictions. Hence $T_2'' \neq \phi$. If $T_1'' \neq \phi$ then, by Lemma 2.1(vi), $F$ is minor of $M$, which is a contradiction. Hence $T_2'' = \phi$ and $N_{x,y}/x \cong F$ or $N_{x,y}/y \cong F$ or $N_{x,y}/x, y \cong F$. Assume that $N$ contains a 2-cocircuit $Q$. By Lemma 2.1(iv), $Q$ is a 2-cocircuit in $N_{x,y}$. Since $F$ is 3-connected, it does not contain a 2-cocircuit. It follows that $N_{x,y}^f$ is not isomorphic to $F$. Hence $N_{x,y}^f \setminus \{a\}/\{x\} \cong F$ or $N_{x,y}^f \setminus \{x, y\} \cong F$ or $N_{x,y}^f \setminus \{y\} \cong F$ or $N_{x,y}^f \setminus \{x, y\} \cong F$. If $Q \cap \{x, y\} \neq \phi$, then it is retained in all these cases and thus $F$ has a 2-cocircuit, which is a contradiction. If $Q = \{x, y\}$, a contradiction follows from Lemma 2.1(v). Hence $Q$ contains exactly one of $x$ and $y$. Suppose that
Let $x \in Q$. Then $N''_{x,y}/y \not\cong F$. Let $x_1$ be the other element of $Q$. Let $L = N/x_1$. Then $L$ is a minor of $M$ in which no pair of elements is in series. Further, $L'_{x,y} = N''_{x,y}/x_1 \cong N''_{x,y}/x$. Thus we have $L'_{x,y} \{a\} \cong F$ or $L'_{x,y} \{a\}/y \cong F$ or $L'_{x,y} \cong F$ or $L'_{x,y}/y \cong F$. Since $L_{x,y} \cong L'_{x,y} \{a\}$, and $x, y$ are in series in $L_{x,y}$, it follows that $L'_{x,y} \{a\} \not\cong F$ and also $L'_{x,y} \{a\}/y \cong L'_{x,y} \{a\}/x$. If $y \in Q$, then $N''_{x,y}/x \not\cong F$. Also, $L'_{x,y} \cong N''_{x,y}/y$. In this case we get $L'_{x,y} \{a\}/x \cong F$ or $L'_{x,y} \cong F$ or $L'_{x,y}/x \cong F$. □

**Definition 2.5.** Let $M$ be a graphic matroid in which no two elements are in series and let $F \in \mathcal{F}$. We say that $M$ is minimal with respect to $F$ and the element splitting operation if there exist two elements $x$ and $y$ of $M$ such that $M'_{x,y}\{a\}/\{x\} \cong F$ or $M'_{x,y}\{a\}/\{x, y\} \cong F$ or $M'_{x,y} \cong F$ or $M'_{x,y}/\{x\} \cong F$ or $M'_{x,y}/\{x, y\} \cong F$.

**Corollary 2.6.** Let $M$ be a graphic matroid. For $x, y \in E(M)$, the matroid $M'_{x,y}$ is cographic if and only if $M$ has no minor isomorphic to a minimal matroid with respect to any $F \in \mathcal{F}$.

**Proof.** If $M'_{x,y}$ is not cographic for some $x, y$ then, by Lemma 2.4, $M$ has a minor $N$ in which no two elements are in series and $N'_{x,y}\{a\}/\{x\} \cong F$ or $N'_{x,y}\{a\}/\{x, y\} \cong F$ or $N'_{x,y} \cong F$ or $N'_{x,y}/\{x\} \cong F$ or $N'_{x,y}/\{y\} \cong F$ or $N'_{x,y}/\{x, y\} \cong F$ for some $F \in \mathcal{F}$. If $N'_{x,y}/y \cong F$ but $N'_{x,y}/x \not\cong F$, then interchange roles of $x$ and $y$.

Conversely, suppose that $M$ has a minor $N$ isomorphic to a minimal matroid with respect to some $F \in \mathcal{F}$. Then $N'_{x,y}\{a\}$ or $N'_{x,y}/\{x\}$ or $N'_{x,y}/\{x, y\}$ or $N'_{x,y} \cong F$, for some $x, y \in E(M)$. We conclude that $M'_{x,y}$ has a minor isomorphic to $F$ and hence it is not cographic.

In the following Lemma, we prove some basic properties of graphic minimal matroids.

**Lemma 2.7.** Let $M$ be a graphic matroid. If $M$ is minimal with respect to some $F \in \mathcal{F}$, then

(i) $M$ has neither loops nor coloops;

(ii) every pair of parallel elements of $M$ must contain either $x$ or $y$;

(iii) $x$ and $y$ cannot be parallel in $M$;

(iv) if $M'_{x,y} \cong F^*_7$ or $M(K_{3,3})$ then $M$ is simple, and there is no odd circuit of $M$ containing both $x$ and $y$, and also there is no even circuit of $M$ containing precisely one of $x$ and $y$;

(v) if $M'_{x,y}/\{x\} \cong F^*_7$ or $M(K_{3,3})$ then $M$ is simple and there is no
3-circuit of $M$ containing both $x$ and $y$;

(vi) if $M'_{x,y}/\{x\} \cong F_7$ or $M(K_5)$, then $M$ has exactly one pair of parallel elements and there is no 3-circuit of $M$ containing both $x$ and $y$;

(vii) if $M'_{x,y}/\{x,y\} \cong F$ then $M$ is simple and there is no 3 or 4-circuit of $M$ containing both $x$ and $y$; and

(viii) $M'_{x,y}$ is not isomorphic to $F_7$ or $M(K_5)$.

Proof. The proof is straightforward.

Lemma 2.8. Let $F \in \mathcal{F}$ and let $M$ be a binary matroid such that either $M_{x,y}/\{x\} \cong F$ or $M_{x,y}/\{x,y\} \cong F$ for some pair $x, y \in E(M)$. Then the following statements hold.

(i) $M$ has neither loops nor coloops;

(ii) $x$ and $y$ can not be parallel in $M$;

(iii) if $x_1$ and $x_2$ are parallel elements of $M$, then one of them is either $x$ or $y$;

(iv) if $M_{x,y}/\{x,y\} \cong F$, then $M$ has at most one pair of parallel elements;

(v) if $M_{x,y}/\{x\} \cong M(K_{3,3})$ or $M_{x,y}/\{x,y\} \cong M(K_{3,3})$, then every odd circuit of $M$ contains $x$ or $y$; and

(vi) if $M_{x,y}/\{x\} \cong M(K_5)$ or $M_{x,y}/\{x,y\} \cong M(K_5)$, then every odd cocircuit of $M$ contains $x$ or $y$.

A matroid is said to be Eulerian if its ground set can be expressed as a union of circuits [16].

Lemma 2.9. [17] Suppose $x$ and $y$ are non adjacent edges of a graph $G$ and $M = M(G)$. If $M_{x,y}/\{x,y\}$ is Eulerian, then either $G$ is Eulerian or the end vertices of $x$ and $y$ are precisely the vertices of odd degree.

3 A Forbidden-Minor Characterization for the Class of Cographic Matroids which yield the Graphic Element-Splitting Matroids

In this section, we obtain the minimal cographic matroids corresponding to each of the four matroids $F_7, F_7^*, M(K_{3,3})$ and $M(K_5)$ and use them to give a proof of Theorem 1.6.
The minimal graphic matroids corresponding to the matroid $F_7$ and $F_7^*$ are characterized by Shikare and Dalvi \cite{3} in the following Lemma 3.1 and Lemma 3.2

**Lemma 3.1.** Let $M$ be a graphic matroid. Then $M$ is minimal with respect to the matroid $F_7$ if and only if $M$ is isomorphic to one of the cycle matroids $M(G_1)$, $M(G_2)$ or $M(G_3)$, where $G_1$, $G_2$ and $G_3$ are the graphs of figure 2.

![figure 2](image1)

**Lemma 3.2.** Let $M$ be a graphic matroid. Then $M$ is minimal with respect to the matroid $F_7^*$ if and only if $M$ is isomorphic the cycle matroid $M(G_4)$ or $M(G_5)$, where $G_4$ and $G_5$ are the graphs of figure 3.

![figure 3](image2)

In the following Lemma, the minimal matroids corresponding to the matroid $M(K_{3,3})$ are characterized.

**Lemma 3.3.** Let $M$ be a graphic matroid. Then $M$ is minimal with respect to the matroid $M(K_{3,3})$ if and only if $M$ is isomorphic to $M(G_6)$, $M(G_7)$, $M(G_8)$, $M(G_9)$, or $M(G_{10})$, where $G_6$, $G_7$, $G_8$, $G_9$, $G_{10}$ are the graphs of Figure 4.
Proof. We have $M'(G_6)_{x,y\setminus\{a\}/\{x\}} \cong M(K_{3,3})$, $M'(G_7)_{x,y\setminus\{a\}/\{x\}} \cong M(K_{3,3})$, $M'(G_8)_{x,y\setminus\{a\}/\{x\}} \cong M(K_{3,3})$, $M'(G_9)_{x,y\setminus\{a\}/\{x\}} \cong M(K_{3,3})$, $M'(G_{10})_{x,y} \cong M(K_{3,3})$. Therefore $M(G_6)$, $M(G_7)$, $M(G_8)$, $M(G_9)$, $M(G_{10})$ are minimal matroids with respect to the matroid $M(K_{3,3})$.

Conversely, suppose that $M$ is a minimal matroid with respect to the matroid $M(K_{3,3})$. Then there exist elements $x$ and $y$ of $M$ such that $M'_{x,y\setminus\{a\}/\{x\}} \cong M(K_{3,3})$ or $M'_{x,y\setminus\{a\}/\{x,y\}} \cong M(K_{3,3})$ or $M'_{x,y}/\{x\} \cong M(K_{3,3})$ or $M'_{x,y}/\{x,y\} \cong M(K_{3,3})$ and further, no two elements of $M$ are in series.

Case (i) $M'_{x,y\setminus\{a\}/\{x\}} \cong M(K_{3,3})$.

By Lemmas 2.8, 2.9 and 2.7, $M_{x,y}/\{x\} \cong M(K_{3,3})$. Since $r(M_{x,y}/\{x\}) = r(M(K_{3,3})) = 5$. $M_{x,y}$ is a matroid of rank 6 and $|E(M)| = 10$. In the light of the Lemma 2.1(iii), the matroid $M$ has rank 5 and its ground set has 10 elements. Let $G$ be a connected graph corresponding to $M$. Then $G$ has 6 vertices, 10 edges, and has no vertex of degree 2. Hence, by Lemma 2.7, $G$ has minimum degree at least 3 since no two elements are in series. Thus the degree sequence of $G$ is $(5,3,3,3,3,3,3)$ or $(4,4,3,3,3,3)$. By Harary [7], p 223, each simple connected graph with these degree sequences is isomorphic to one of the graphs of Figure 5 below.
By the nature of the circuits of $M(K_{3,3})$ or $M_{x,y}$ and by Lemma 2.7, it follows that $G$ cannot have, (i) two or more edge disjoint triangles, (ii) a circuit of size 3 or 4 or 6 containing both $x$ and $y$. Since each of the graphs (i), (ii) and (iii) of Figure 5 contains two or more edge disjoint triangles, we discard them. The graph (iv) of Figure 5, is isomorphic to the graph $G_6$ in the statement of the Lemma.

Suppose $G$ is a multigraph. Then by Lemma 3.3 of [2], $G$ is isomorphic to $G_7$ or $G_8$ of Figure 4.

**Case (ii).** $M_{x,y} \setminus \{a\}/\{x, y\} \cong M(K_{3,3})$.

By Lemma 2.1(i), $M_{x,y}/\{x, y\} \cong M(K_{3,3})$. As $r(M_{x,y}) = 7$, $r(M(K_{3,3})) = 5$. Hence $r(M) = 6$ and $|E(M)| = 11$. Let $G$ be connected graph corresponding to $M$. Then $G$ has 7 vertices, 11 edges and has minimum degree at least 3. Therefore the degree sequence of $G$ is $(4,3,3,3,3,3,3)$. It follows from Lemma 2.7 that $G$ cannot have (i) more than two edge disjoint triangles; (ii) a cycle of size other than 6 which contains both $x$ and $y$; and (iii) a triangle and a 2-circuit which are edge disjoint.

Then, by case (ii) of Lemma 3.3 of [2], $G$ is isomorphic to $G_9$ of Figure 4.

**Case (iii).** $M_{x,y}' \cong M(K_{3,3})$.

Since $r(M(K_{3,3})) = 5$ and $|E(M(K_{3,3}))| = 9$, $r(M) = 4$ and $|E(M)| = 8$. Therefore $M$ cannot have $M(K_{3,3})$ and $M(K_5)$ as a minor. We conclude that $M$ is cographic. Let $G$ be a graph which corresponds to the matroid $M$. Then $G$ has 5 vertices and 8 edges. As $M$ is graphic and cographic, $G$ is
planar. By Lemma 2.7 (iv), \( G \) is simple. Since \( M \) has no coloop and no two elements are in series, minimum degree in \( G \) is at least 3. There is only one non-isomorphic simple graph with 5 vertices and 8 edges [7], see Figure 6.

Thus, \( G \) is isomorphic to this graph, which is nothing but the graph \( G_{10} \) of Figure 4.

**Case (iv).** \( M'_{x,y}/\{x\} \cong M(K_{3,3}) \).

As in the above cases, considering the rank of \( M(K_{3,3}) \) we have \( r(M) = 5 \) and \( |E(M)| = 9 \). If \( M \) is not cographic then \( M \) has minor a isomorphic to \( M(K_{3,3}) \) or \( M(K_5) \). Suppose \( M \) has a minor isomorphic to \( M(K_{3,3}) \) and \( M'_{x,y}/\{x\} \cong M(K_{3,3}) \). Then \( M \cong M(K_{3,3}) \). Now \( x, y \in M \) and since \( M \) is isomorphic to \( M(K_{3,3}) \), \( x, y \) lie in a 4-circuit say \( C \) but then \( C - \{x\} \) is a triangle in \( M'_{x,y}/\{x\} \cong M(K_{3,3}) \), a contradiction to the fact that \( M'_{x,y}/\{x\} \) is bipartite and does not contain any odd circuit. Thus \( M \) does not contain \( M(K_{3,3}) \) as a minor.

If \( M \) has minor a isomorphic to \( M(K_5) \) then \( r(M) = 5 \), \( |E(M)| = 9 \), and \( |E(M(K_5))| = 10 \), so \( M \) does not contain \( M(K_5) \) as a minor. Hence \( M \) is cographic.

Let \( M = M(G) \) be graphic matroid. Then \( G \) has 6 vertices and 9 edges. Further, \( G \) is simple and planar. Also, minimum degree in \( G \) is at least 3. Thus \( G \) is isomorphic to the graph of

**Figure 7**

Figure 7 (see [4]). If a triangle of \( G \) contains neither \( x \) nor \( y \) then it is preserved in \( M'_{x,y}/\{x\} \), a contradiction. Hence \( x \) belongs to a triangle of \( G \). This gives rise to a 4-circuit in \( M'_{x,y} \) containing \( x \) and \( a \). Hence, we get a 3-circuit in \( M'_{x,y}/\{x\} \), a contradiction. Thus \( M \) is not isomorphic to the graph in Figure 7.
Case (v). $M'_{x,y}/\{x,y\} \cong M(K_{3,3})$.

Then $r(M) = 6$ and $|E(M)| = 10$. If $M$ is not cographic then, $M$ has minor isomorphic to $M(K_{3,3})$ or $M(K_5)$.

Suppose $M$ has a minor isomorphic to $M(K_{3,3})$ then, since $M$ is graphic, $M$ must be the graph in Figure 8 (see [7]).

![Figure 8](image)

Any two edges in the graph of Figure 8 are in a 4-cycle or in a 5-cycle so are $x$ and $y$ also. Then these circuits (cycles in $G$) will be preserved in $M'_{x,y}$ and hence a 2-circuit or a triangle is formed in $M'_{x,y}/\{x,y\} \cong M(K_{3,3})$, a contradiction to the fact that $M'_{x,y}/\{x,y\} \cong M(K_{3,3})$ is a simple bipartite matroid. Thus $M$ has no minor isomorphic to $M(K_{3,3})$.

Suppose that $M$ has a minor isomorphic to $M(K_5)$ but then $|E(M)| = 10$, hence $M = M(K_5)$. Consequently $r(M) = 4$ and this is a contradiction to the fact that $r(M) = 6$. Thus $M$ does not have a minor isomorphic to $M(K_5)$.

Thus $M$ is cographic. We conclude that $M$ is graphic as well as cographic. Suppose $G$ is a graph corresponding to $M$, then $G$ has 7 vertices and 10 edges. This implies that $G$ has at least one vertex of degree 2, which is a contradiction to the fact that $M$ has no 2-cocircuit. Therefore, the situation $M'_{x,y}/\{x,y\} \cong M(K_{3,3})$ does not occur.

Finally, we characterize minimal matroids corresponding to the matroid $M(K_5)$ in the following Lemma.

**Lemma 3.4.** Let $M$ be a cographic matroid. Then $M$ is minimal with respect to the matroid $M(K_5)$ if and only if $M$ is isomorphic to one of the seven matroids $M(G_{11})$, $M(G_{12})$, $M(G_{13})$, $M(G_{14})$, $M(G_{15})$, $M(G_{16})$ and $M(G_{17})$, where $G_{11}$, $G_{12}$, $G_{13}$, $G_{14}$, $G_{15}$, $G_{16}$ and $G_{17}$ are the graphs of Figure 9.
Proof. We have $M'(G_{11})_{x,y} \setminus \{a\}/\{x\} \cong M(K_5), \ M'(G_{12})_{x,y} \setminus \{a\}/\{x,y\} \cong M(K_5), \ M'(G_{13})_{x,y} \setminus \{a\}/\{x\} \cong M(K_5), \ M'(G_{16})_{x,y}/\{x\} \cong M(K_5)$, $M'(G_{14})_{x,y} \setminus \{a\}/\{x,y\} \cong M(K_5), \ M'(G_{15})_{x,y} \setminus \{a\}/\{x,y\} \cong M(K_5), \ M'(G_{17})_{x,y}/\{x,y\} \cong M(K_5)$. Therefore $M(G_{11}), \ M(G_{12}), \ M(G_{13}), \ M(G_{14}), \ M(G_{15}), \ M(G_{16})$ and $M(G_{17})$ are minimal matroids with respect to the matroid $M(K_5)$.

Conversely, suppose that $M$ is a minimal matroid with respect to the matroid $M(K_5)$. Then there exist elements $x$ and $y$ of $M$ such that $M'_{x,y} \setminus \{a\}/\{x\} \cong M(K_5)$ or $M'_{x,y} \setminus \{a\}/\{x,y\} \cong M(K_5)$ or $M'_{x,y} \cong M(K_5)$ or $M'_{x,y}/\{x\} \cong M(K_5)$ or $M'_{x,y}/\{x,y\} \cong M(K_5)$ and also, $M$ does not contain a 2-cocircuit.

Case (i). $M'_{x,y} \setminus \{a\}/\{x\} \cong M(K_5)$.

By Lemma 2.1(i), $M(G)_{x,y}/\{x\} \cong M(K_5)$. Hence, by Lemma 3.4 Chapter 4 of [?], $M$ is isomorphic to the cycle matroid $M(G_{11})$, where $G_{11}$ is the graph of Figure 9.

Case (ii). $M'_{x,y} \setminus \{a\}/\{x,y\} \cong M(K_5)$.

By Lemma 2.1(i), $M(G)_{x,y}/\{x,y\} \cong M(K_5)$. Then $r(M(K_5)) = 4$, $r(M_{x,y}) = 6$ and $|E(M)| = 12$. Let $G$ be a connected graph corresponding to $M$. Then $G$ has 6 vertices, 12 edges and has minimum degree at least 3. Suppose that $G$ is simple. By Lemma 3.4 of [2], there are 5 non isomorphic simple graphs.
each with 6 vertices and 12 edges, out of which, two graphs are discarded in case (ii) of Lemma 3.4 of [2]. So, only three graphs are remaining and these graphs are not planar. These graphs are given in Figure 10.

![Figure 10](image)

In graph (i) of Figure 10, not every odd cocircuit of $M$ contains $x$ or $y$, a contradiction to the fact that if $M(G)_{x,y}/\{x, y\} \cong M(K_5)$, then every odd cocircuit contain $x$ or $y$ otherwise if both of them are absent then that odd cocircuit of $M$ is the odd cocircuit in $M(G)_{x,y}/\{x, y\} \cong M(K_5)$. Consequently $M(G)_{x,y}/\{x, y\}(\cong M(K_5))$ becomes non Eulerian. In each of the graphs (ii) and (iii) of Figure 10, $x$ and $y$ together belong to a 3-cycle or a 4-cycle, a contradiction to Lemma 2.8

Suppose that $G$ is not simple. Then, by Lemma 2.8 (iv), $G$ has exactly one pair of parallel edges. Then $G$ can be obtained from a simple graph on 6 vertices and 11 edges by adding a parallel edge.

There are 8 non isomorphic connected simple graphs, each with 6 vertices and 11 edges ([7] pp. 223) as shown in Figure 11. It follows that by Lemma
Lemma 2.8 and Lemma 2.9 that $G$ cannot be obtained from the graphs (ii), (iii) and (vii) of Figure 11. Suppose that $G$ is obtained from the graphs (i) or (iv). Then $G$ is isomorphic to one of the four graphs of Figure 12. By Lemma 2.8, $G$ is not isomorphic to each of the two graphs (i) and (ii) of Figure 12. Hence $G$ is isomorphic to graphs (iii) and (iv) of Figure 12, which are nothing but the graphs $G_{12}$ and $G_{13}$ of Figure 9.

Figure 12

By Lemma 2.8, $G$ cannot be obtained from graph (v) of Figure 11. Suppose that $G$ is obtained from graph (viii) of Figure 11. Then $G$ is isomorphic to one of the two graphs of Figure 13. By Lemma 2.8 (iv) and Lemma 2.9, $G$ is not isomorphic to graph (i) of Figure 13. By Lemma 2.8 (ii) and (iv) and the fact that $M(K_5)$ does not contain odd cocircuit, $G$ cannot be isomorphic to the graph (ii) of Figure 13.

Figure 13

Suppose that $G$ is obtained from graph (vi) of Figure 11. Then $G$ is isomorphic to one of the two graphs $G_{14}$ and $G_{15}$ of Figure 9.
Case (iii). \( M'_{x,y} \cong M(K_5) \).

Then a contradiction follows from Lemma 2.7 (viii).

Case (iv). \( M'_{x,y}/\{x\} \cong M(K_5) \).

Subcase (i). Suppose that \( M \) is not cographic.

Let \( G \) be a graph that corresponds to the matroid \( M \). Since \( r(M(K_5)) = 4 \), \( r(M'_{x,y}) = 5 \). Further, \( r(M) = 4 \) and \( |E(M)| = 10 \). Therefore, \( G \) has 5 vertices and 10 edges. \( M \) has no minor isomorphic to \( M(K_{3,3}) \) as \( K_{3,3} \) has 6 vertices. Suppose that \( M \) has a minor isomorphic to \( M(K_5) \) then \( M \cong M(K_5) \). By Lemma 2.7, \( x, y \) can not both be in a triangle. Hence \( x \) and \( y \) are not adjacent. Let \( C^* \) ba cocircuit of \( M \) containing \( y \) but not \( x \) such that \( |C^*| = 4 \), since we can always find a set of 4 edges containing \( y \) incident to some vertex in \( K_5 \), that set of edges is a cocircuit of \( M(K_5) \). Then \( C^* \cup \{a\} \) becomes a cocircuit of \( M'_{x,y}/\{x\} \cong M(K_5) \), a contradiction to the fact that \( M(K_5) \) is Eulerian and \( |C^* \cup \{a\}| = 5 \). Thus \( M \) is cographic. Since \( M \) is graphic and cographic, \( G \) is planar. By Harary [7], there is no simple planar graph with 5 vertices and 10 edges. Hence \( G \) must be non-simple. Then, by Lemma 2.7 (vi), \( G \) has exactly one pair of parallel edges. \( G \) can be obtained from a simple planar graph with 5 vertices and 9 edges by adding an edge in parallel. By Harary [7], every simple planar graph with 5 vertices and 9 edges is isomorphic to graph (i) of Figure 14. Therefore, \( G \) is isomorphic to the graph (ii) or (iii) of Figure 14. In graph (iii), there are two edge-disjoint 3-cutsets (i.e. 3-cocircuits in \( M(G) \)). Hence, one of them is preserved in \( M'_{x,y}/\{x\} \), and this is a contradiction. Thus, \( G \) is isomorphic to graph (ii) of Figure 16, which is nothing but the graph \( G_{16} \) of Figure 9.

![Figure 14](image)

Case (v). \( M'_{x,y}/\{x, y\} \cong M(K_5) \).

Subcase (i). Suppose that \( M \) is not cographic.

Let \( G \) be a graph which corresponds to the matroid \( M \). Since \( r(M(K_5)) = 4 \), \( r(M'_{x,y}) = 5 \). Further, \( r(M) = 4 \) and \( |E(M)| = 10 \). Therefore, \( G \) has 5 vertices and 10 edges. \( M \) has no minor isomorphic to \( M(K_{3,3}) \) as \( K_{3,3} \) has 6 vertices. Suppose that \( M \) has a minor isomorphic to \( M(K_5) \) then \( M \cong M(K_5) \). By Lemma 2.7, \( x, y \) can not both be in a triangle. Hence \( x \) and \( y \) are not adjacent. Let \( C^* \) ba cocircuit of \( M \) containing \( y \) but not \( x \) such that \( |C^*| = 4 \), since we can always find a set of 4 edges containing \( y \) incident to some vertex in \( K_5 \), that set of edges is a cocircuit of \( M(K_5) \). Then \( C^* \cup \{a\} \) becomes a cocircuit of \( M'_{x,y}/\{x\} \cong M(K_5) \), a contradiction to the fact that \( M(K_5) \) is Eulerian and \( |C^* \cup \{a\}| = 5 \). Thus \( M \) is cographic. Since \( M \) is graphic and cographic, \( G \) is planar. By Harary [7], there is no simple planar graph with 5 vertices and 10 edges. Hence \( G \) must be non-simple. Then, by Lemma 2.7 (vi), \( G \) has exactly one pair of parallel edges. \( G \) can be obtained from a simple planar graph with 5 vertices and 9 edges by adding an edge in parallel. By Harary [7], every simple planar graph with 5 vertices and 9 edges is isomorphic to graph (i) of Figure 14. Therefore, \( G \) is isomorphic to the graph (ii) or (iii) of Figure 14. In graph (iii), there are two edge-disjoint 3-cutsets (i.e. 3-cocircuits in \( M(G) \)). Hence, one of them is preserved in \( M'_{x,y}/\{x\} \), and this is a contradiction. Thus, \( G \) is isomorphic to graph (ii) of Figure 16, which is nothing but the graph \( G_{16} \) of Figure 9.
4. $r(M_{x,y}') = 6$. So, $r(M) = 5$. Further, $|E(M)| = 11$.

Suppose that $M$ has a minor isomorphic to $M(K_{3,3})$. Let $G$ be the connected graph corresponding to $M$. Then $G$ is the graph with 6 vertices and 11 edges. By Lemma 2.7 (vii), $G$ has to be simple. Consequently, $G$ is isomorphic to one of the following two graphs in Figure 15 (see Harary [7]).

![Figure 15](image)

Figure 15

In the graph (i) of Figure 15, any two edges are either in a 3-cycle or a 4-cycle. The elements $x$ and $y$ cannot be in a 3-circuit or a 4-circuit because such circuit becomes a loop or a 2-circuit in $M_{x,y}'/{x, y} \cong M(K_5)$. This is a contradiction to the fact that $M(K_5)$ is simple.

In the graph (ii) of Figure 15, $M_{x,y}'/{x, y} \cong M(K_5)$ for the edges $x, y$ shown in the graph (ii) of Figure 15. Hence the graph (ii) which is $G_{17}$ of Figure 9, is a minimal graph with respect to $M(K_5)$. Suppose that $M$ has a minor isomorphic to $M(K_5)$. Since $r(M(K_5)) = 4$, $r(M_{x,y}') = 6$. So, $r(M) = 5$. Further, $|E(M)| = 11$. Let $G$ be a graph corresponding to the matroid $M$. Thus $G$ is the graph with 6 vertices and 11 edges with $M(K_5)$ as a minor. In fact $G$ is the graph shown in Figure 16.

![Figure 16](image)

Figure 16

Observe that the graph of Figure 16, contains a vertex of degree 2 i.e. there is a 2-cocircuit in $M$. This is a contradiction to the fact that $M$ does not have any pair of elements in series (i.e. 2-cocircuit). Hence $M$ has no minor isomorphic to $M(K_5)$.

Thus $M$ is cographic. Consequently, $G$ is the planar graph with 6 vertices and 11 edges and has minimum degree at least 3. By Lemma 2.7 (vii), $G$ is simple.
There are in all 9 non-isomorphic simple graphs with 6 vertices and 11 edges (see [7]). Out of which, four graphs are non-planar and two graphs contain a degree two vertex, remaining 3 graphs are shown in Figure 17. Here, $G$ cannot have a 3 or a 4-cycle containing both $x$ and $y$. Also, each

![Figure 17](image)

3-cocycle and a 5-cocycle of $G$ must contain $x$ or $y$. These conditions are not satisfied for the graphs (i) and (iii) for any pair of edges $x, y$. The choice for $(x, y)$ in graph (ii) is $(k, l)$. But then $M_{x,y}/\{x, y\}$ is not eulerian. Hence the cycle matroids of the graphs in Figure 17 are not minimal with respect to $M(K_5)$.

Now, we use Lemmas 3.1, 3.2, 3.3, and 3.4 to prove Theorem 1.6.

**Proof of Theorem 1.6** Let $M$ be a graphic matroid. On combining Corollary 2.6 and Lemmas 3.1, 3.2, 3.3, and 3.4 it follows that $M_{x,y}$ is cographic for every pair $\{x, y\}$ of elements of $M$ if and only if $M$ has no minor isomorphic to any of the matroids $M(G_i)$, $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17$, where the graphs $G_i$ are shown in the statements of the Lemmas 3.1, 3.2, 3.3, and 3.4. However, one can check that each of the matroids $M(G_i)$, $i = 1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17$ has matroid $M(G_5)$ of Figure 5 as a minor. Hence the proof.

**References**

[1] Azadi G., *Generalized splitting operation for binary matroids and related results*, Ph.D. Thesis, University of Pune (2001).

[2] Borse Y. M., Shikare M. M. and Dalvi Kiran, Excluded-Minor characterization for the class of cographic splitting matroids, *Ars Combin.* 105(2014), 219-237.
[3] Dalvi K.V., Borse Y.M., Shikare M.M., Forbidden-minors for the class of graphic element-splitting matroids. *Discussiones Mathematicae Graph Theory*, 29 (2009), 629-644.

[4] Dalvi K.V., Borse Y.M., Shikare M.M., Forbidden-minors for the class of cographic element-splitting matroids. *Discussiones Mathematicae Graph Theory*, 31(2011), 601-606.

[5] Fleischner H., *Eulerian Graphs and Related Topics*, Part 1, Vol 1, North Holland, Amsterdam (1990).

[6] Habib Azanchiler, *Some new operations on matroids and related results*, Ph. D. Thesis, University of Pune (2005).

[7] Harary F., *Graph Theory*, Addison-Wesley, Reading, 1969.

[8] Naiyer P., *The Splitting operation for binary matroids and excluded minors for certain classes of splitting Matroids*, Ph.D. Thesis, University of Pune (2011).

[9] Oxley G., *Matroid Theory*, Oxford University Press, Oxford, 1992.

[10] Raghunathan T. T., Shikare M. M. and Waphare B. N., Splitting in a binary matroid, *Discrete Math.* 184 (1998), 267-271.

[11] Shikare M. M. and Waphare B. N., Excluded-Minors for the class of graphic splitting matroids, *Ars Combinatoria* 97 (2010), 111.

[12] Shikare M. M., The Element Splitting Operation for Graphs, Binary Matroids and Its Applications, *The Mathematics Student*, 80(2010), 85-90.

[13] Slater P. J., A Classification of 4-connected graphs, *J. Combin. Theory*, 17 (1974), 282-298.

[14] Tyler M., A minor-based characterization of matroid 3-connectivity, *Advances in Applied Mathematics* 50 (2013), 132-141.

[15] Tutte W. T., A theory of 3-connected graphs, *Indag. Math.* 23 (1961), 441-455.

[16] Welsh D. J. A., *Matroid Theory*, Academic Press, London, 1976.