Lie superalgebra structures in $H^*(g; g^\ast)$

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Received 23 July 2004

Let $g = \text{vect}(M)$ be the Lie (super)algebra of vector fields on any connected (super)manifold $M$; let $\Pi$ be the change of parity functor, $C^*$ and $H^*$ the space of $i$–chains and $i$–cohomology. The Nijenhuis bracket makes $\mathcal{L} = \Pi(C^{-1}(g; g)) = C^{-1}(g; \Pi(g))$ into a Lie superalgebra that can be interpreted as the centralizer of the exterior differential considered as a vector field on the supermanifold $\hat{M} = (M, \Omega(M))$ associated with the de Rham bundle on $M$. A similar bracket introduces structures of DG Lie superalgebra in $L = H^{-1}(g; \Pi(g))$ for any Lie superalgebra $g$. We use a Mathematica–based package SuperLie (already proven useful in various problems) to explicitly describe the algebras $L$ for some simple finite dimensional Lie superalgebras $g$ and their “relatives” — the nontrivial central extensions or derivation algebras of the considered simple ones.

PACS: 02.20.Sv

Key words: Lie superalgebras, cohomology, Nijenhuis bracket

1 Introduction

This paper is a sequel to [10] and [6]. Its aim is to demonstrate usefulness of the Mathematica–based package SuperLie ([3]) already tested on various problems ([5]). Our results — computation of $H^*(g; g)$ (complete, or partial) — are “orthogonal” to those of [12] who computed only $H^1(g; M)$ and $H^2(g; M)$ but for all (finite dimensional) irreducible modules $M$. We also complete Tyutin’s description [13] of deformations of $\mathfrak{po}(0|n)$ who ignored odd parameters; cf. [9].

During the talk at the conference we indicated how Lie algebra cohomology is related to a “new” (designed in late 1960’s) and seemingly never explored method for solving differential equations and for the study of stability of various dynamical systems. For an exposition of this and many other applications of cohomology, see [5]. For basics on Lie superalgebras we consider, see [11]. For main data on Lie algebra cohomology and rudiments of Lie superalgebra cohomology, see Fuchs’s book [2]. It is well-known that if the Lie algebra $g$ is finite dimensional and simple

*) Presented at the 13th International Colloquium on “Integrable Systems and Quantum Groups”, Prague, 17–19 June 2004.
and $M$ a finite dimensional $\mathfrak{g}$-module (both over $\mathbb{C}$), then

$$H^*(\mathfrak{g}; M) = 0 \text{ for any irreducible } M.$$  

(1)

However, there are very few general theorems helping to compute $H^*(\mathfrak{g}; M)$ when both $\mathfrak{g}$ and $M$ are considered over $\mathbb{Z}$ or $\mathbb{F}_q$, even less for Lie superalgebras.

On the other hand, C. Gruson suggested a totally new method (applicable to Lie superalgebras only, not to Lie algebras) [7]. At the moment, Gruson’s method is only applied to the trivial coefficients. Its applicability in other cases is not studied yet. So, for nontrivial modules over Lie superalgebras, we have only the same tool researchers had at their disposal at the birth of the cohomology theory: the definition. To see what phenomena and patterns we might encounter, we use SuperLie ([3]) to get a supply of reliable results to be used in more general analytic study.

The standard proof of (1) uses the (even, quadratic) Casimir operator. Passing to Lie superalgebras $\mathfrak{g}$ we observe that: (A) some Lie superalgebras have no (even) quadratic Casimir operator, (B) for some Lie superalgebras, such an operator exists but vanishes at various $M$’s. The algebras we consider here have these properties (A) and (B), and hence (1) fails sometimes.

In what follows, $C_i := C_i(\mathfrak{g}; \mathfrak{g})$ and $H^i := H^i(\mathfrak{g}; \mathfrak{g})$ for a Lie superalgebra $\mathfrak{g}$; set $C^* = \bigoplus C_i$ and $H^* = \bigoplus H^i$.

2 The Lie superalgebra structure on $\Pi(C^{*-1})$ and $\Pi(H^{*-1})$

Let the $e_\alpha$ be elements of the (weight) basis of $\mathfrak{g}$ (for example, the Chevalley basis if $\mathfrak{g}$ is simple), and let the $f^\alpha$ be the elements of the dual basis. The basis of cochains is given by monomials of the form

$$e_\alpha \otimes f^{\alpha_1} \wedge \ldots \wedge f^{\alpha_n}.$$  

For $A = e_A^\alpha \otimes f^{\alpha_1}_A \wedge \ldots \wedge f^{\alpha_n}_A \in C^A_n$ and $B = e_B^\beta \otimes f^{\beta_1}_B \wedge \ldots \wedge f^{\beta_n}_B \in C^B_n$, we set

$$A \cdot B = \sum_{k=1}^{n_A} (-1)^{n_A-k} f^{\alpha_k}_A (e_{\beta}) A \otimes f^{\alpha_1}_A \wedge \ldots \wedge f^{\alpha_k}_A \wedge \ldots \wedge f^{\alpha_n}_A \wedge f^{\beta_1}_B \wedge \ldots \wedge f^{\beta_n}_B.$$  

The Nijenhuis bracket on $\Pi(C^*)$, where $\Pi$ is the shift of parity functor, is given by the formula (hereafter $p(a)$ is the parity of $a$ in $\Pi(C^*)$, not in $C^*$)

$$[A, B] = A \cdot B - (-1)^{p(A)p(B)} B \cdot A \in C^{n_A+n_B-2}.$$  

(2)

Let us not only change parity, but also shift the degree by setting $\deg A = n_A - 1$; we denote this by writing $\Sigma = \Pi(C^{*-1}) := \bigoplus C^{*-1}$. It is subject to a direct verification that (2) defines the Lie superalgebra structure on $\Sigma$, and

$$d[A, B] = [dA, B] + (-1)^{p(A)}[A, dB].$$  

(3)
Therefore $\Pi(Z^{-1})$ is a subalgebra of $\Pi(C^{-1})$, and $\Pi(B'^{-1})$ is an ideal in $\Pi(Z^{-1})$. Hence, we have a DG Lie superalgebras structure on $\mathfrak{L} := \Pi(C^{-1})$ and $\mathfrak{l} := \Pi(H'^{-1})$. In [10], we observed that the differential vanishes on $\mathfrak{L}$, and $\mathfrak{l}$ is often very small, so although researchers mainly study $\mathfrak{L}$, the algebra $\Pi(Z^{-1})$ might be more interesting than $\mathfrak{l}$ in some questions. Observe that representing $\Pi(C' (\mathfrak{g}; \Pi(\mathfrak{g}))$ as $C' (\mathfrak{g}; \Pi(\mathfrak{g}))$ considerably simplifies computations.

### 3 Examples

In what follows, for small $i$, we listed superdimensions of the $H^i$ expressed as $a|\bar{b}$; we write $a$ instead of $a|0$ and $\bar{b}$ instead of $0|\bar{b}$. For $i$ greater than indicated below, we did not calculate (partly due to Mathematica-imposed limitations).

For lists of simple Lie superalgebras, their $Z$-gradings, known central extensions, outer derivations and deformations, see [8], [1] and [11]. On deformations of Poisson superalgebras for various types of functions, see [11], [13], [9] and refs. therein. Here we only consider the $0|n$-dimensional case. Recall that $\text{vect}(m|n)$ is the Lie superalgebra of polynomial vector fields on the $m|n$-dimensional superspace, $\text{svect}(m|n)$ is its divergence-free subalgebra, whereas $\mathfrak{g} = a\mathfrak{g}_2$ and $a\mathfrak{b}_3$ are exceptional finite dimensional Lie superalgebras, most clearly determined by their defining relations [4].

Every $Z$-grading of $\mathfrak{g}$ induces a natural $Z$-grading on the space of all polynomial functions on $\mathfrak{g}$, in particular, on cohomology. Such $Z$-grading will be called the degree. Observe that (1) the parity of a derivation $c : \mathfrak{g} \to \mathfrak{g}$ is opposite to the parity of $c$ considered as a 1-cocycle $c : \Pi(\mathfrak{g}) \to \mathfrak{g}$; (2) to consider $\mathfrak{l}$, one should shift all parities in the tables below. Observe that the deformations with the odd parameter are automatically global.

#### 3.1 Simple Lie superalgebras without deformations

**Conjecture.** For $\mathfrak{g} = a\mathfrak{g}_2$ and $a\mathfrak{b}_3$, and also $\mathfrak{g} = \text{vect}(0|n)$, where $n > 1$, we have $\mathfrak{l} = 0$. (Verified to degree 5 and 4, and, for $n = 2, 3$, to degree 7 and 5, respectively.) $\mathfrak{g} = \mathfrak{psl}(2|2) := \mathfrak{sl}(2|2)/\mathbb{C}_{14}$. This algebra is highly symmetric: the group of its outer automorphisms is $\text{Out} \mathfrak{g} \simeq \mathfrak{s}(2)$, so $\dim H^1 = 3$. We also knew that $H^2 = 0$. Let deg be induced by the degree of $\mathfrak{sl}(2|2)$ in the standard format, the even (diagonal) block matrices being of degree 0, the upper/lower ones of degree ±1. Here are known and new results: listed are the superdimensions of the $H^i = l_{i-1}$ for $i \leq 5$ and the degrees of their basis elements:

| $-1$ | deg 6 | deg 4 | deg 2 | deg 0 | deg 2 | deg 4 | deg 6 |
|------|------|------|------|------|------|------|------|
| 0    | 1    | 1    | 1    |      |      |      |      |
| 1    | 2    | 1    | 2    | 2    | 2    |      |      |
| 2    |      |      |      |      |      |      |      |
| 3    |      |      |      |      |      |      |      |
| 4    | 1    | 2    | 2    | 2    | 2    | 2    | 1    |

**Conjecture.** $l_0, l_2$, and $l_3$ generate $l_1$.  

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3.2 Lie superalgebras with deformations

\[ g = \mathfrak{osp}(4|2; \alpha) \] 

(Observe that, for \( \alpha = -1 \) and 0, the algebra is not simple.) The very definition of the algebra as a 1–parameter family indicates that \( \dim H^2 \geq 1 \).

Nothing was known about higher cohomology. Here are new results: for \( \alpha = 1 \) as well as for \( \alpha = \frac{3}{7} \), which served as generic \( \alpha \) (the answer is the same, but for the computer the task is much easier), we have

\[
\begin{array}{c|ccccc}
  i - 1 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
  \dim H^{i-1} & 0 & 0 & 1 & 0 & 0 & 2 & 0 \\
\end{array}
\]

The cocycles depend on \( \alpha \) but the product in \( \mathfrak{l} \) does not (for \( \alpha \neq -1, 0 \)): the product of any two cocycles from the above table vanishes.

**Conjecture.** There are no more cocycles but the above.

The whole pathos of [10] was to express \( \mathfrak{l} \) as an algebra, not as a list of dimensions of the homogeneous components, as classics did (Borel–Weil–Bott–Kostant). Here the product of any two cocycles from the above table vanishes, so the list is as fine as algebra as long as the product is trivial (even if Conjecture fails). In examples that follow, we return, to our regret, to the “list of dimensions” level.

3.2.1 The divergence–free series

We knew ([11]) that \( g = \mathfrak{svect}(0|n) \) for \( n > 2 \) has only one outer derivation, so \( \dim H^1 = 1 \), and \( g \) has only one global deformation, so \( \dim H^2 \geq 1 \), but we knew nothing about other cohomology. New results: indicated are nonzero \( H^i \), in all cases \( H^i = \text{Span}(c_i) \) for some cocycles \( c_i \), i.e., \( \dim H^i = 1 \) in all cases below. The parity of \( c_i \), as an element of \( \mathfrak{l} \), is equal to \( i - 1 + \deg \) (mod 2).

\[ \mathfrak{svect}(0|3): \]

\[
\begin{array}{c|cccc}
  i - 1 & 0 & 1 & 3 & 5 & 6 \\
  \deg & 0 & 3 & -3 & 0 & -6 \\
\end{array}
\]

Obviously, \([c_2, c_4] = 0\), conjecturally \([c_4, c_4] = c_7\). With Mathematica’s inbuilt limitations, we were unable to compute other products nor advance further than indicated (below as well).

\[ \mathfrak{svect}(0|4): \]

\[
\begin{array}{c|cc}
  i - 1 & 0 & 1 \\
  \deg & 0 & 4 \\
\end{array}
\]

3.2.2 The Poisson series \( \mathfrak{po}(0|m) \)

We knew ([11]) that the infinite dimensional Lie (super)algebra \( g = \mathfrak{h}(2n|m) = \mathfrak{po}(2n|m)/\text{center of Hamiltonian vector fields} \) has only one outer derivation, and we were sure that the same holds for \( g = \mathfrak{po}(0|m) \); we also knew that \( g \) has only one
global deformation, so dim \( H^2 \geq 1 \). We knew nothing about other cohomology nor about the finite dimensional case.

In order not to confuse the elements of \( \mathfrak{po}(0|m) \) with functions, we realize \( \mathfrak{po}(0|m) \) as a subalgebra of contact vector fields \( K_f \) generated by functions \( f \) (for formulas, see [11]). Let deg be the degree of an element of \( \mathfrak{g} \) (and the induced degree of cochains) given by the formula \( \deg(K_f) = \deg_{\text{pol}}(f) - 2 \), where \( \deg_{\text{pol}} \) is the standard grading in the polynomial algebra (the degree of each indeterminate is equal to 1). So, for \( \mathfrak{po}(0|n) \), there are the following obvious cohomology: 0th, given by \( K_f \); 1st, given by the grading by the degree; 2nd, quantization, of deg = -4; and, additionally, the above cocycles wedged (which is possible since \( H^*(\mathfrak{g}; \mathfrak{g}) \) is an \( H^*(\mathfrak{g}) \)–module) by \( (d(K_{\theta_1}, \ldots, \theta_n))^m \), where \( m \) is arbitrary for \( n \) odd and is either 0 or 1 for \( n \) even. (Here \( d(K_{\theta_1}, \ldots, \theta_n) : K_{\theta_1}, \ldots, \theta_n \mapsto 1. \))

\[ g = \mathfrak{po}(0|4) \]

\[
\begin{array}{c|cccccc}
-1 & -8 & -6 & -4 & -2 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[ g = \mathfrak{po}(0|5) \]

\[
\begin{array}{c|cccccc}
-1 & -14 & -12 & -10 & -8 & -6 & -4 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[ g = \mathfrak{po}(0|6) \]

\[
\begin{array}{c|cccccc}
-1 & -14 & -12 & -10 & -8 & -6 & -4 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

3.2.3 The Hamiltonian series \( \mathfrak{h}(0|n) \) and \( \mathfrak{h}'(0|n) = [\mathfrak{h}(0|n), \mathfrak{h}(0|n)] \)

We knew ([11]) that the infinite dimensional \( \mathfrak{g} = \mathfrak{h}(2n|m) \) has only one outer derivation, so dim \( H^1 \geq 1 \), and ([11]) that \( \mathfrak{g} \) has only one global deformation, except for \( (2n, m) = (2, 2) \), so dim \( H^2 \geq 1 \), but we knew nothing about other cohomology nor about the finite dimensional case. Let deg be the degree of an element of \( \mathfrak{g} \) induced by the grading of \( \mathfrak{po} \). Here are new results:

\[ g = \mathfrak{h}(0|4) \]

\[
\begin{array}{c|cccc}
-1 & -8 & -6 & -4 & 0 \\
0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 \\
\end{array}
\]
\( g = h'(0|4). \) This is a special case already considered since \( h'(0|4) \cong \mathfrak{psl}(2|2) \).

\( g = h(0|5) \).

\[
\begin{array}{cccccccccccc}
-1 & -10 & -9 & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 \\
0 & & & & & & & & & & & & \\
1 & & & & & & & & & & & & \\
2 & & & & & & & & & & & & \\
3 & & & & & & & & & & & & \\
\end{array}
\]

\( g = h'(0|5). \)

\[
\begin{array}{cccccccc}
-1 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & & & & & & & & & & & & \\
1 & & & & & & & & & & & & \\
2 & & & & & & & & & & & & \\
3 & & & & & & & & & & & & \\
\end{array}
\]

D.L. was partly supported by l’IHES and MPIMiS.

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