Quantization on algebraic curves with Frobenius-projective structure

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Abstract
In the present paper, we study the relationship between deformation quantizations and Frobenius-projective structures defined on an algebraic curve in positive characteristic. A Frobenius-projective structure is an analogue of a complex projective structure on a Riemann surface, which was introduced by Y. Hoshi. Such an additional structure has some equivalent objects, e.g., a dormant PGL$_2$-oper and a projective connection having a full set of solutions. The main result of the present paper provides a canonical construction of a Frobenius-constant quantization on the cotangent space minus the zero section on an algebraic curve by means of a Frobenius-projective structure. It may be thought of as a positive characteristic analogue of a result by D. Ben-Zvi and I. Biswas. Finally, this result generalizes to higher-dimensional varieties, as proved by I. Biswas in the complex case.

Keywords Positive characteristic · Quantization · Projective structure · Indigenous bundle · Oper · Projective connection

Mathematics Subject Classification Primary 53D55 · Secondary 14H99

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Introduction

A (deformation) quantization of a symplectic manifold is a noncommutative deformation of the structure sheaf which is, in a certain sense, compatible with the symplectic structure. We know that every symplectic manifold admits quantizations (cf. [7]; [8]; [9]), but in general the quantization is neither unique nor canonically constructed. Therefore, construction of quantizations will be one of the important subjects in this theory. In [2], D. Ben-Zvi and I. Biswas provided a canonical construction of a quantization on (the total space of) the cotangent bundle minus the zero section on a given Riemann surface \( C \) equipped with a projective structure. Here, recall that a projective structure on \( C \) is an atlas of coordinate charts defining \( C \) whose transition functions may be expressed as Möbius transformations. Projective structures have a very major role to play in understanding the framework of uniformization theorem of Riemann surfaces and have mutually equivalent objects, including PGL\(_2\)-opers and projective connections, etc. Every Riemann surface \( C \) admits a projective structure, and the space of all projective structures on \( C \) forms an affine space for the space \( H^0(C, \Omega^2_C) \) of quadratic differentials on \( C \). The idea behind the construction of D. Ben-Zvi and I. Biswas is that the canonical construction of a quantization on the complex projective line \( \mathbb{P}^1_C \), which is invariant under the action of PGL\(_2(\mathbb{C})\) (= the group of Möbius transformations on \( \mathbb{P}^1_{\mathbb{C}} \)), may extend naturally to any Riemann surface once we choose a projective structure.

On algebraic curves in positive characteristic, there are analogous objects of projective structures, called Frobenius-projective structures. The notion of a Frobenius-projective structure was introduced by Y. Hoshi (cf. [10], § 2, Definition 2.1) as a certain collection of locally defined étale maps on a prescribed curve to the projective line. Just as in the complex case, any smooth curve in positive characteristic admits such a structure. Also, Frobenius-projective structures are equivalent to, for example, dormant indigenous bundles and projective connections having a full set of solutions.
That is to say, given a connected smooth curve $X$ in characteristic $p > 2$ and a theta characteristic $\mathbb{L} := (\mathcal{L}, \psi_{\mathcal{L}} : \mathcal{L} \otimes \mathbb{L} \rightarrow \Omega^1_X)$ (cf. §1.4), we obtain the following diagram consisting of bijective correspondences in parallel with the classical result on Riemann surfaces:

\begin{equation}
\begin{array}{ccc}
\mathfrak{B}_{X} & \cong & \mathfrak{B}_{X}^{zz} \\
\downarrow & & \downarrow \\
\mathfrak{P} \mathfrak{C}_{X, \mathbb{L}} \cong & \mathfrak{P} \mathfrak{C}_{X, \mathbb{L}}^{2, \text{full}} & \mathfrak{P} \mathfrak{C}_{X, \mathbb{L}}^{F}
\end{array}
\end{equation}

where

- $\mathfrak{P} \mathfrak{C}_{X}^{F}$ := the set of Frobenius-projective structures on $X$ (cf. (44));
- $\mathfrak{B}_{X}^{zz}$ := the set of isomorphism classes of dormant indigenous bundles on $X$ (cf. (45));
- $\mathfrak{P} \mathfrak{C}_{X, \mathbb{L}}^{2, \text{full}}$ := the set of projective connections for $\mathbb{L}$ having a full set of solutions (cf. (62)).

(We also discuss, in the present paper, certain intermediate objects equivalent to them, called dormant $(\text{SL}_2, \mathbb{L})$-opers.)

The purpose of the present paper is to prove an analogous assertion of D. Ben-Zvi and I. Biswas, i.e., a canonical construction of a quantization by means of the curve $X$ together with a choice among such additional structures. In [3], [13], and [16], it has been shown that the theory of quantizations can be made to work in the algebraic setting. Also, we can find, in [4] (and [5]), the study of a special class of quantizations on symplectic algebraic varieties in positive characteristic, called Frobenius-constant quantizations. They are quantizations with large center in some suitable sense, and has a cohomological classification given in the point of view of formal geometry.

Let $\mathbb{A}(\Omega_X)^\times$ denote the complement of the zero section in (the total space of) the cotangent bundle of $X$; it admits a symplectic structure $\tilde{\omega}^{\text{can}}$ defined as one half of the Liouville symplectic form. Thus, it makes sense to speak of a (Frobenius-constant) quantization on the symplectic variety $(\mathbb{A}(\Omega_X)^\times, \tilde{\omega}^{\text{can}})$. Denote (cf. (25)) by

\begin{equation}
\Omega^{\text{FC}}_{(\mathbb{A}(\Omega_X)^\times, \tilde{\omega}^{\text{can}})}
\end{equation}

the set of Frobenius-constant quantizations on $(\mathbb{A}(\Omega_X)^\times, \tilde{\omega}^{\text{can}})$. Then, the main result of the present paper (cf. Theorem 4.1) provides a canonical injective assignment from each Frobenius-projective structure (or equivalently, each dormant indigenous bundle, or a projective connection having a full set of solutions) to a Frobenius-constant quantization on $(\mathbb{A}(\Omega_X), \tilde{\omega}^{\text{can}})$, as displayed below:
In particular, we can think of $\mathcal{P}\mathcal{V}_{X}$ as a subset of $\Omega^{\text{FC}}(\mathbb{A}(\Omega_X)^{\times}, \omega_{\text{can}})$ via this assignment; this fact gives a lower bound of the number of Frobenius-constant quantizations on $(\mathbb{A}(\Omega_X)^{\times}, \omega_{\text{can}})$ by applying the result in [14].

The present paper is organized as follows. The first section contains the necessary definitions and conventions used in our discussion, including a symplectic structure, a differential operator, and a theta characteristic. In the second section, we recall the notion of a Frobenius-constant quantization and discuss some related topics. For instance, it will be observed that Frobenius-constant quantizations are functorial with respect to pull-back via étale morphisms (cf. § 2.2) and have a descent property via finite Galois coverings (cf. § 2.3). In the third section, we discuss various bijective correspondences between Frobenius-projective structures on a curve and some equivalent objects, i.e., dormant indigenous bundles (= dormant PGL$_2$-opers), dormant (SL$_2$, $\mathbb{L}$)-opers, and projective connections with a full set of solutions. Some results mentioned in that section have been essentially obtained in other literatures, e.g., [10] (which gives $\mathcal{P}\mathcal{V}_{X} \sim \mathcal{M}^{Zzz}_{X}$) and [1] together with [12] (which gives $\mathcal{O}_{\mathbb{P}(\text{SL}_{n,L}), X} \sim \mathcal{P}\mathcal{C}^{\text{full}}_{n,X,L}$).

Also, in [15], we can find generalizations of these correspondences to a family of pointed stable curves. But, unfortunately, many of such results and related discussions seem not to be familiar (at least in the characteristic $p$ setting) and are unavoidable in the proof of our main theorem, so we decided to review them here and the contents became nearly self-contained. The fourth section is devoted to stating and proving the main theorem. As carried out in [2], we first construct, by means of a Frobenius-projective structure, a Frobenius-constant quantization on the complement of the zero section in the total space of the line bundle $\mathcal{L}$. This quantization turns out to be invariant under the natural involution and hence descends to a Frobenius-constant quantization on $(\mathbb{A}(\Omega_X)^{\times}, \omega_{\text{can}})$. Moreover, the injectivity of this assignment is proved by examining the behavior of the noncommutative multiplication in each quantization. In the final section, we discuss (cf. Theorem 5.3) a higher-dimensional variant of our main theorem, which may be thought of as a positive characteristic analogue of a result in [6].

1 Preliminaries

In this section, we prepare the notation and conventions used in the present paper. Throughout the present paper, let us fix an odd prime $p$ and an algebraically closed field $k$ of characteristic $p$. Unless otherwise stated, all schemes and morphisms of schemes are implicitly assumed to be over $k$, and products of schemes are taken over $k$. We use the word variety (resp., curve) to mean a finite type integral scheme over $k$. 
(resp., a finite type integral scheme over $k$ of dimension 1). For each positive integer $n$, we shall write $\mathbb{A}^n$ (resp., $\mathbb{P}^n$) for the affine space (resp., the projective space) over $k$ of dimension $n$. Also, write $\mathbb{A}^{n \times} := \mathbb{A}^n \setminus \{0\}$, which is an open subscheme of $\mathbb{A}^n$.

1.1 Vector bundles

Let $S$ be a smooth variety of dimension $n > 0$. Given a vector bundle $\mathcal{F}$ on $S$ (i.e., a locally free coherent sheaf on $S$), we denote by $\mathcal{F}^\vee$ its dual sheaf, i.e., $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{O}_S)$. Let $\mathbb{A}(\mathcal{F})$ and $\mathbb{P}(\mathcal{F})$ denote the relative affine and projective spaces, respectively, associated with $\mathcal{F}$, i.e.,

$$\mathbb{A}(\mathcal{F}) := \text{Spec}(S^\bullet(\mathcal{F}^\vee)), \quad \mathbb{P}(\mathcal{F}) := \text{Proj}(S^\bullet(\mathcal{F}^\vee)), \quad (4)$$

where $S^\bullet(\mathcal{F}^\vee) := \bigoplus_{i \geq 0} S^i(\mathcal{F}^\vee)$ denotes the symmetric algebra over $\mathcal{O}_S$ associated with $\mathcal{F}^\vee$. Also, write $\mathbb{A}(\mathcal{F})^{\times}$

$$\quad (5)$$

for the complement of the zero section $S \to \mathbb{A}(\mathcal{F})$ in $\mathbb{A}(\mathcal{F})$, which admits a natural projection

$$\pi_\mathcal{F} : \mathbb{A}(\mathcal{F})^{\times} \to \mathbb{P}(\mathcal{F}) \quad (6)$$

over $S$.

We shall write $\Omega_S$ for the sheaf of 1-forms (in other words, the cotangent bundle) on $S$ relative to $k$ and $T_S$ for its dual. By the smoothness assumption on $S$, both $\Omega_S$ and $T_S$ turn out to be vector bundles of rank $n$. We write $d : \mathcal{O}_S \to \Omega_S$ for the universal derivation. Moreover, denote by $\omega_S$ the canonical line bundle of $S$ (relative to $k$), which is canonically isomorphic to the determinant line bundle $\text{det}(\Omega_S) := \bigwedge^n \Omega_S$ of $\Omega_S$.

1.2 Symplectic structures

Recall that a symplectic structure on $S$ is a nondegenerate closed 2-form $\omega \in \Gamma(S, \bigwedge^2 \Omega_S)$. Here, we say that $\omega$ is nondegenerate if the morphism $(\Omega_S^\vee := T_S \to \Omega_S)$ induced naturally by $\omega$ is an isomorphism. A symplectic variety (over $k$) is a pair $(S, \omega)$ consisting of a smooth variety $S$ and a symplectic structure $\omega$ on it. An isomorphism $(S, \omega) \to (S', \omega')$ between symplectic varieties is an isomorphism $S \to S'$ preserving the symplectic structure.

As is well-known, the variety $\mathbb{A}(\Omega_S)$ (i.e., the total space of the cotangent bundle of $S$) has a canonical symplectic structure

$$\omega_S^{\text{can}} \in \Gamma(\mathbb{A}(\Omega_S), \bigwedge^2 \Omega_{\mathbb{A}(\Omega_S)}) \quad (7)$$

often called the Liouville symplectic form. If there is no fear of causing confusion, we write $\omega_S^{\text{can}}$ instead of $\omega_S^{\text{can}}$ for simplicity. If $q_1, \cdots, q_n$ are local coordinates in $S$
and \( q_1^\vee, \ldots, q_n^\vee \) denote the dual coordinates in \( \mathbb{A}(\Omega_S) \), then \( \omega^{\text{can}} \) may be expressed locally as \( \omega^{\text{can}} = \sum_{i=1}^n dq_i^\vee \wedge d q_i \). By abuse of notation, we also use the notation \( \omega^{\text{can}} \) to denote the restriction of \( \omega^{\text{can}} \) to the open subscheme \( \mathbb{A}(\Omega_S)^\times \subset \mathbb{A}(\Omega_S) \). Also, for each \( c \in k^\times \), \( c \cdot \omega^{\text{can}} \) forms a symplectic structure. In particular, by letting \( \tilde{\omega}^{\text{can}} := \frac{1}{2} \cdot \omega^{\text{can}} \), we have symplectic varieties

\[
(\mathbb{A}(\Omega_S), \tilde{\omega}^{\text{can}}), \quad (\mathbb{A}(\Omega_S)^\times, \tilde{\omega}^{\text{can}}).
\]  

\[ (1) \]

### 1.3 Differential operators

We shall recall the notion of a differential operator. Let \( L_1 \) (\( i = 1, 2 \)) be line bundles on \( S \). By a differential operator from \( L_1 \) to \( L_2 \), we mean a \( k \)-linear morphism \( D : L_1 \to L_2 \) locally expressed, after fixing identifications \( L_1 \cong L_2 \cong \mathcal{O}_S \) and a local coordinate system \( \mathbf{x} := (x_1, \ldots, x_n) \) in \( S \), as

\[
D : v \mapsto D(v) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} a_\alpha \cdot \partial_\mathbf{x}^\alpha(v)
\]  

by means of some local sections \( a_\alpha \in \mathcal{O}_S \) with \( a_\alpha = 0 \) for almost all \( \alpha \), where for each \( \alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n \), we write \( \partial_\mathbf{x}^\alpha(v) := \frac{\partial^{j_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{j_n}}{\partial x_n^{\alpha_n}} \). If \( a_\alpha = 0 \) for any \( \alpha \) with \( |\alpha| \geq p \), then \( j_{\max} := \max \{ |\alpha| : a_\alpha \neq 0 \} (\leq p) \) is well-defined (i.e., depend only on \( D \), not on the choice of the local expression \( (9) \)). In this situation, we say that \( D \) is of order \( j_{\max} \). (We say that \( D \) is of order \( -\infty \) if \( D = 0 \).) Given a nonnegative integer \( j \) with \( j < p \), we denote by

\[
\text{Diff}_{L_1, L_2}^{\leq j}
\]  

the Zariski sheaf on \( S \) consisting of locally defined differential operators from \( L_1 \) to \( L_2 \) of order \( \leq j \); it is a subsheaf of the sheaf \( \text{Hom}_k(L_1, L_2) \) of locally defined \( k \)-linear morphisms \( L_1 \to L_2 \). We shall write

\[
\mathcal{D}_X^{\leq j} := \text{Diff}_{\mathcal{O}_S, \mathcal{O}_X}^{\leq j}.
\]  

\[ (11) \]

Note that \( \text{Diff}_{L_1, L_2}^{\leq j} \) admits two different structures of \( \mathcal{O}_S \)-module — one as given by left multiplication (where we denote the resulting \( \mathcal{O}_S \)-module by \( \text{Diff}_{L_1, L_2}^{\leq j} \)), and the other given by right multiplication (where we denote the resulting \( \mathcal{O}_S \)-module by \( \text{rDiff}_{L_1, L_2}^{\leq j} \)) —. Given an \( \mathcal{O}_X \)-module \( \mathcal{F} \), we equip the tensor product \( \mathcal{F} \otimes \mathcal{D}_X^{\leq j} \) (resp., \( \text{rDiff}_{L_1, L_2}^{\leq j} \otimes \mathcal{F} \) with an \( \mathcal{O}_X \)-module structure arising from \( \text{rDiff}_{L_1, L_2}^{\leq j} \) (resp., \( \text{rDiff}_{L_1, L_2}^{\leq j} \)). The composition with the \( k \)-linear morphism \( L_2 \otimes \mathcal{D}_X^{\leq j} \to L_2 \) given by \( v \otimes D \mapsto v \otimes D(1) \) yields an identification

\[
\text{Hom}_{\mathcal{O}_X}(L_1, L_2 \otimes \mathcal{D}_X^{\leq j}) \cong \text{Diff}_{L_1, L_2}^{\leq j}.
\]  

\[ (12) \]
Moreover, the assignment $D = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} a_\alpha \cdot \partial^\alpha \mapsto \sum_{|\alpha| = j} a_\alpha \cdot \partial^\alpha$ gives a well-defined isomorphism of $O_S$-modules

$$\mathcal{D}iff_{L_1, L_2}^{\leq j} / \mathcal{D}iff_{L_1, L_2}^{\leq (j-1)} \sim \mathcal{H}om_{O_S}(L_1, L_2 \otimes S^j(T_S)),$$

where $S^j(T_S)$ denotes the $j$th component of the symmetric power of $T_S$. Denote by $\Sigma$ the composite

$$\Sigma : \mathcal{D}iff_{L_1, L_2}^{\leq j} / \mathcal{D}iff_{L_1, L_2}^{\leq (j-1)} \sim \mathcal{H}om_{O_S}(L_1, L_2 \otimes S^j(T_S)).$$

For each local section $D \in \mathcal{D}iff_{L_1, L_2}^{\leq j}$, we refer to $\Sigma(D)$ as the principal symbol of $D$.

Next, let us write

$$S^{(1)}$$

for the Frobenius twist of $S$ over $k$ (i.e., the base-change of $S$ via the absolute Frobenius morphism of $k$) and

$$F_{S/k} : S \rightarrow S^{(1)}$$

for the relative Frobenius morphism of $S$ over $k$. To simplify the notation, we regard each $O_{S^{(1)}}$-module (resp., $O_S$-module) as a sheaf on $S$ (resp., $S^{(1)}$) via the underlying homeomorphism of $F_{S/k}$.

Notice that each differential operator $D : L_1 \rightarrow L_2$ of order $j$ (with $0 \leq j < p$) may be considered as an $O_{S^{(1)}}$-linear morphism $F_{S/k*}(L_1) \rightarrow F_{S/k*}(L_2)$ via the underlying homeomorphism of $F_{S/k}$. It follows that the kernel $\text{Ker}(D)$ forms an $O_{S^{(1)}}$-submodule of $F_{S/k*}(L_1)$.

**Definition 1.1** Let $D$ be as above. We shall say that $D$ has a full set of solutions if $\text{Ker}(D)$ is a vector bundle on $S^{(1)}$ of rank $j$.

### 1.4 Theta characteristics

By a *theta characteristic* on $S$, we mean a pair

$$\mathbb{L} := (L, \psi_L)$$

consisting of a line bundle $L$ on $S$ and an isomorphism between line bundles $\psi_L : L^{\otimes (n+1)} \sim \omega_S$. As is well-known, any smooth curve always admits a theta characteristic.

**Example 1.2** We shall prove the claim that there exists a canonical theta characteristic on the projective space $\mathbb{P}^n = \text{Proj}(k[x_0, x_1, \cdots, x_n])$. Let

$$\eta_0 : \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow \mathcal{O}^{\oplus (n+1)}_{\mathbb{P}^n}$$
be the $\mathcal{O}_{\mathbb{P}^n}$-linear injection given by $w \mapsto \sum_{i=0}^{n} w.x_i \cdot e_i$ for each local section $w \in \mathcal{O}_{\mathbb{P}^n}(-1)$, where $(e_0, \cdots, e_n)$ is a canonical basis of $\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}$. The composite

$$\mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\eta_0} \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \xrightarrow{\omega \otimes \mathcal{O}_{\mathbb{P}^n}(n+1)} \Omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} / \text{Im}(\eta_0),$$

which is verified to be $\mathcal{O}_{\mathbb{P}^n}$-linear, induces an isomorphism of $\mathcal{O}_{\mathbb{P}^n}$-modules

$$\mathcal{O}_{\mathbb{P}^n}(-1) \otimes (\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}/\text{Im}(\eta_0))^{\vee} \sim \Omega_{\mathbb{P}^n}. \quad (20)$$

Moreover, we have a composite isomorphism

$$\mathcal{O}_{\mathbb{P}^n} \sim \text{det}(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}) \sim \mathcal{O}_{\mathbb{P}^n}(-1) \otimes \text{det}(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}/\text{Im}(\eta_0)), \quad (21)$$

where the first isomorphism is given by $1 \mapsto e_0 \wedge \cdots \wedge e_n$ and the second isomorphism arises from the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\eta_0} \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}/\text{Im}(\eta_0) \to 0. \quad (22)$$

Denote by $\psi_0$ the composite isomorphism

$$\psi_0 : \mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes(n+1)} (= \mathcal{O}_{\mathbb{P}^n}(-n) \otimes \mathcal{O}_{\mathbb{P}^n}(-1)) \sim \mathcal{O}_{\mathbb{P}^n}(-n) \otimes \text{det}(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}/\text{Im}(\eta_0))^{\vee}$$

$$\sim \text{det}(\mathcal{O}_{\mathbb{P}^n}(-1) \otimes \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}/\text{Im}(\eta_0))^{\vee}$$

$$\sim \text{det}(\Omega_{\mathbb{P}^n}) = \omega_S,$$

where the first isomorphism follows from (21) and the third isomorphism follows from (20). Thus, we have obtained a theta characteristic

$$\mathbb{L}_0 := (\mathcal{O}_{\mathbb{P}^n}(-1), \psi_0) \quad (24)$$

on $\mathbb{P}^n$.

### 2 Frobenius-constant quantizations

In this section, we recall the notion of a Frobenius-constant (= FC) quantization on a given symplectic variety and discuss some related topics.

#### 2.1 Quantizations

Let $(S, \omega_S)$ be a symplectic variety. The nondegenerate pairing $T_S \otimes \mathcal{O}_S \xrightarrow{T_S} \mathcal{O}_S$ given by $\omega_S$ becomes a pairing $\omega_S^{-1} : \Omega_S \otimes \mathcal{O}_S \xrightarrow{\Omega_S} \mathcal{O}_S$ via $\Omega_S \sim T_S$ induced by $\omega_S$. Thus, we obtain a skew-symmetric $k$-bilinear map
\[-, -\]_\omega : \mathcal{O}_S \times \mathcal{O}_S \to \mathcal{O}_S

defined by \(\{ f, g \}_\omega := \omega^{-1}_S(df, dg)\). One verifies from the closedness of \(\omega_S\) that \(\{-, -\}_\omega\) defines a Poisson bracket in the usual sense. Here, let \(k[[h]]\) denote the ring of formal power series in the variable \(h\) over \(k\), and write \(\mathcal{O}^\bullet_S := \varprojlim_{j \geq 1} \mathcal{O}_S[[h]]/(h^j)\).

In the present paper, we shall define a quantization on \((S, \omega_S)\) to be a sheaf of (possibly noncommutative) flat \(k[[\hbar]]\)-algebras \(\mathcal{O}^\bullet_S\) on \(S\) such that \(\mathcal{O}^\bullet_S = \mathcal{O}_S[[\hbar]]\) (as an equality of sheaves of \(k[[\hbar]]\)-modules) and the commutator in \(\mathcal{O}^\bullet_S\) is equal to \(\hbar \cdot \{-, -\}_\omega \mod \hbar^2 \cdot \mathcal{O}^\bullet_S\).

Moreover, a Frobenius-constant quantization (or, an FC quantization, for short) on \((S, \omega_S)\) (cf. [4], Definition 3.3; [5], Definitions 1.1 and 1.4) is a quantization \(\mathcal{O}^\bullet_S\) on \((S, \omega_S)\) such that the image of the natural inclusion \(\mathcal{O}^\bullet_S \to \mathcal{O}_S[[\hbar]]\) coincides with the center \(Z(\mathcal{O}^\bullet_S)\) of \(\mathcal{O}^\bullet_S\).

We shall write \(\mathcal{Q}^{\text{FC}}_{(S, \omega_S)}\) for the set of FC quantizations on \((S, \omega_S)\).

### 2.2 Pull-back of FC quantizations

If we are given an FC quantization on a prescribed symplectic variety, then it induces an FC quantization on each open subvariety via restriction. More generally, we can construct the pull-back of an FC quantization via an étale morphism, as follows.

Let \((T, \omega_T)\) be another symplectic variety and \(f : T \to S\) an étale morphism with \(f^*(\omega_S) = \omega_T\). The étaleness of \(f\) implies that the commutative square diagram

\[
\begin{array}{ccc}
T & \xrightarrow{f} & S \\
\downarrow F_{T/k} & & \downarrow F_{S/k} \\
T^{(1)} & \xrightarrow{f^{(1)}} & S^{(1)}
\end{array}
\]

is Cartesian, where \(f^{(1)}\) denotes the base-change of \(f\) via the absolute Frobenius morphism of \(k\). Given an FC quantization \(\mathcal{O}^\bullet_S\) on \((S, \omega_S)\), we set

\[
f^*(\mathcal{O}^\bullet_S) := \lim_{j > 0} \left( \mathcal{O}_{T^{(1)}} \otimes f^{-1}(\mathcal{O}_{S^{(1)}}) \right) \left( f^{-1}(\mathcal{O}^\bullet_S) \right).
\]

Since \(Z(f^*(\mathcal{O}^\bullet_S)) \cong \mathcal{O}_{T^{(1)}} \otimes f^{-1}(\mathcal{O}_{S^{(1)}}) \lim_{j > 0} (Z(\mathcal{O}^\bullet_S) / (h^j))\), the sheaf \(f^*(\mathcal{O}^\bullet_S)\) specifies an FC quantization on \((T, \omega_T)\). We shall refer to \(f^*(\mathcal{O}^\bullet_S)\) as the pull-back of \(\mathcal{O}^\bullet_S\) (via \(f\)).
2.3 Equivariant FC quantizations

Let \((S, \omega_S)\) be as above and \(G\) a finite group acting freely on \((S, \omega_S)\), i.e., \(S\) is equipped with a free \(G\)-action preserving \(\omega_S\).

A Frobenius \(G\)-constant quantization (or, a \(G\)-FC quantization) on \((S, \omega_S)\) (cf. [4], Definition 5.5) is an FC quantization \(\mathcal{O}_S^\hbar\) on \((S, \omega_S)\) compatible, in the natural sense, with the \(G\)-action on \(S\). Denote by

\[Q^G_{\text{FC}}(S, \omega_S)\tag{28}\]

the set of \(G\)-FC quantizations on \((S, \omega_S)\). We obtain the natural forgetting map

\[Q^G_{\text{FC}}(S, \omega_S) \twoheadrightarrow Q^{\text{FC}}(S, \omega_S).\tag{29}\]

Furthermore, let \((T, \omega_T)\) be the quotient of \((S, \omega_S)\) by the \(G\)-action. The quotient morphism \(f : S \to T\) is a Galois étale covering with Galois group \(G\) and \(f^*(\omega_T) = \omega_S\). Hence, pulling-back via \(\pi\) induces a map of sets

\[Q^{\text{FC}}(T, \omega_T) \twoheadrightarrow Q^{\text{FC}}(S, \omega_S).\tag{30}\]

If \(\mathcal{O}_T^\hbar\) is an FC quantization on \((T, \omega_T)\), then the pull-back \(f^*(\mathcal{O}_T^\hbar)\) has naturally a structure of \(G\)-FC quantization since the \(G\)-actions on \(S\) and \(S^{(1)}\) are compatible via \(F_{S/k}\). Conversely, let \(\mathcal{O}_S^\hbar\) be a \(G\)-FC quantization on \((S, \omega_S)\). Then, the sheaf \(f_*(\mathcal{O}_S^\hbar)^G\) of \(G\)-invariant sections of \(f_*(\mathcal{O}_S^\hbar)\) specifies an FC quantization on \((T, \omega_T)\).

One verifies immediately that the assignments \(\mathcal{O}_T^\hbar \mapsto f^*(\mathcal{O}_T^\hbar)\) and \(\mathcal{O}_S^\hbar \mapsto f_*(\mathcal{O}_S^\hbar)^G\) give a bijection correspondence

\[Q^{\text{FC}}(T, \omega_T) \sim Q^G_{\text{FC}}(S, \omega_S)\tag{31}\]

making the following diagram commute:

\[Q^{\text{FC}}(T, \omega_T) \xrightarrow{(31)} Q^G_{\text{FC}}(S, \omega_S) \xrightarrow{(29)} Q^{\text{FC}}(S, \omega_S) \]

\[\xrightarrow{(30)} \]

2.4 Formal Weyl algebras

We here recall a canonical (Frobenius-constant) quantization on the affine space \(A^{2n} := \text{Spec}(k[x_1, \cdots, x_n, y_1, \cdots, y_n])\) \((n > 0)\) equipped with the symplectic structure
\[ \omega^\text{Weyl} := \sum_{i=1}^{n} dy_i \wedge dx_i. \]  

The Poisson bracket \{−, −\}_\omega associated with \( \omega^\text{Weyl} \) is given by \([f, g]_\omega = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i} \cdot \frac{\partial g}{\partial x_i} \) for any local sections \( f, g \in \mathcal{O}_\mathbb{A}^{2n} \).

For each commutative ring \( R \) over \( k \), we define \( W^2_{R} \) to be the (noncommutative) \( R[[h]] \)-algebra \( W^2_{R} := R[x_1, \ldots, x_n, y_1, \ldots, y_n][[h]] \) equipped with the multiplication “∗” given by

\[ f \ast g := \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} \frac{h^{\vert \alpha \vert}}{\alpha!} \cdot \frac{\partial \alpha}{\partial y}(f) \cdot \frac{\partial \alpha}{\partial x}(g) \]  

for any \( f, g \in R[x_1, \ldots, x_n, y_1, \ldots, y_n] \); this multiplication is well-defined by putting \( \frac{1}{a} \cdot \frac{\partial}{\partial x_i}(x_i^a) := x_i^{a-1} \) and \( \frac{1}{a} \cdot \frac{\partial}{\partial y_i}(y_i^a) := y_i^{a-1} \) (\( i = 1, \ldots, n \)) even when \( a \) is divided by \( p \). Hence, the \( R[[h]] \)-algebra \( W^2_{R} \) is generated by the elements \( x_1, \ldots, x_n, y_1, \ldots, y_n \) subject to relations

\[ [x_i, x_j] = [y_i, y_j] = 0, \quad [y_i, x_j] = \delta_{ij} \cdot h \]  

for all \( 0 < i, j \leq n \). Since the center of this algebra coincides with \( R[x_1^p, \ldots, x_n^p, y_1^p, \ldots, y_n^p][[h]] \), \( W^2_{R} \) may be thought of as an \( R[x_1^p, \ldots, x_n^p, y_1^p, \ldots, y_n^p][[h]] \)-algebra.

Here, write \( \text{Sp}_{2n} \) for the symplectic group over \( k \) of rank \( n \), i.e.,

\[ \text{Sp}_{2n}(R) = \{ A \in \text{GL}_{2n}(R) \mid ^t AJ_{2n}A = J_{2n} \} \]

for each commutative ring \( R \) over \( k \), where \( J_{2n} := \begin{pmatrix} O & E \\ -E & O \end{pmatrix} \) (\( E \) denotes the unit matrix of size \( n \)). Each \( A \in \text{Sp}_{2n}(R) \) yields an automorphism \( \eta_A \) of \( W^2_{R} \) given by \((x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_n)^t A \). If \( \text{Aut}(W^2_{R}) \) denotes the automorphism group of the \( R[[h]] \)-algebra \( W^2_{R} \), then the assignment \( A \mapsto \eta_A \) determines an injective homomorphism

\[ \text{Sp}_{2n}(R) \rightarrow \text{Aut}(W^2_{R}). \]  

Notice that the \( \mathcal{O}_{(\mathbb{A}^{2n} \times \{1\})[[h]]} \)-algebra

\[ W^2_{k} \]

corresponding to \( W^2_{R} \) specifies an FC quantization on \( (\mathbb{A}^{2n}, \omega^\text{Weyl}) \), hence also on \( (\mathbb{A}^{2n \times}, \omega^\text{Weyl}) \) via restriction. The variety \( \mathbb{A}^{2n \times} \) admits a free action of \( \mu_2 := \{ \pm 1 \} \) such that the automorphism corresponding to \(-1 \in \mu_2 \) is given by \((x, y) \mapsto (−x, −y) \). This action preserves \( \omega^\text{Weyl} \), and we obtain the quotient symplectic variety
\[(\mathbb{A}^{2n}_{/\mu_2}, \omega_{\text{Weyl}}^{1/\mu_2}). \]  

(39)

One may verify that \(W^2_{n/\mu_2}\) is a \(\mu_2\)-FC quantization, so it descends to a FC quantization

\[W^2_{n/\mu_2}\]  

on \((\mathbb{A}^{2n}_{/\mu_2}, \omega_{\text{Weyl}}^{1/\mu_2})\). (In the subsequent discussion, we only use this quantization in the case \(n = 1\).)

3 Frobenius-projective structures and related objects

In this section, we review a positive characteristic analogue of a complex projective structure, called a Frobenius-projective structure. Also, we discuss various bijective correspondences between Frobenius-projective structures on a curve and some equivalent objects, i.e., dormant indigenous bundles, dormant \((\text{SL}_2, \mathbb{L})\)-opers, and projective connections with a full set of solutions.

3.1 Frobenius-projective structures

Let \(n\) be a positive integer and denote by \(\text{PGL}_{n+1}\) the projective linear group over \(k\) of rank \(n + 1\), which is naturally identified with the automorphism group of \(\mathbb{P}^n\). Given each algebraic group \(G\) over \(k\) and a smooth variety \(S\), we denote by \(G_S\) the sheaf of groups on \(S\) represented by \(G\). Write

\[G^F_S := F^{-1}_{S/k}(G_{S(1)})(\subseteq G_S). \]  

(41)

Also, we shall write

\[\mathcal{P}^\text{ét} \]  

(42)

for the sheaf of sets on \(S\) that assigns, to each open subscheme \(U\) of \(S\), the set of étale \(k\)-morphisms \(\phi : U \to \mathbb{P}^n\) from \(U\) to \(\mathbb{P}^n\). Each local section \(\phi\) of \(\mathcal{P}^\text{ét}\) may be regarded, by taking its graph, as a local section \(\Gamma_\phi : U \to U \times \mathbb{P}^n\) of the trivial \(\mathbb{P}^n\)-bundle \(U \times \mathbb{P}^n \xrightarrow{\text{pr}_1} U\). The sheaf \(\mathcal{P}^\text{ét}\) has a \((\text{PGL}_{n+1})^F_S\)-action described as follows. Let \(U\) be an open subscheme, \(\phi : U \to \mathbb{P}^n\) an element of \(\mathcal{P}^\text{ét}(U)\), and \(A\) an element of \((\text{PGL}_{n+1})^F_S(U)(\subseteq \text{PGL}_{n+1}(U))\). Then, it is immediately verified that the composite

\[A(\phi) : U \xrightarrow{\Gamma_\phi} U \times \mathbb{P}^n \xrightarrow{A} U \times \mathbb{P}^n \xrightarrow{\text{pr}_2} \mathbb{P}^n \]  

(43)

specifies an element of \(\mathcal{P}^\text{ét}(U)\). The assignment \((A, \phi) \mapsto A(\phi)\) defines a \((\text{PGL}_{n+1})^F_S\)-action on \(\mathcal{P}^\text{ét}\).
Definition 3.1 (cf. [10], § 2, Definition 2.1, for the case \( n = 1 \)) We shall say that a subsheaf \( \mathcal{S}^\lozenge \) of \( \mathcal{P}^{\mathcal{E}} \) is a **Frobenius-projective structure (of level 1)** on \( \mathcal{S} \) if \( \mathcal{S}^\lozenge \) is closed under the \((\text{PGL}_{n+1})_S^F\)-action on \( \mathcal{P}_{\text{et}} \), and moreover, forms a \((\text{PGL}_{n+1})_{S^\lozenge}^F\)-torsor on \( S \) with respect to the resulting \((\text{PGL}_{n+1})_{S^\lozenge}^F\)-action on \( S^\lozenge \).

We shall write

\[
P\mathcal{S}^F_S \tag{44}
\]

for the set of Frobenius-projective structures on \( S \).

### 3.2 Dormant indigenous bundles

Let us fix a smooth curve \( X \). Recall from, e.g., [14], § 2, Definition 2.1, (i), that an **indigenous bundle** (or, a \( \text{PGL}_2 \)-oper) on \( X \) is a triple \( \mathcal{E}^\bullet := (\mathcal{E}, \nabla_\mathcal{E}, \sigma_\mathcal{E}) \) consisting of a flat \( \mathbb{P}^1 \)-bundle \( \mathcal{E}, \nabla_\mathcal{E} \) on \( X \) (i.e., a pair of a \( \mathbb{P}^1 \)-bundle \( \mathcal{E} \) on \( X \) and a connection \( \nabla_\mathcal{E} \) on \( \mathcal{E} \)) and a global section \( \sigma_\mathcal{E} : X \to \mathcal{E} \) such that the Kodaira–Spencer map \( \mathcal{T}_X / k \to \sigma_\mathcal{E}^* (\mathcal{T}_X / k) \) associated to \( \sigma_\mathcal{E} \) is nowhere vanishing. (We omit the details of the definition of an indigenous bundle because it will not be necessary for our discussion.) We shall say that an indigenous bundle \( \mathcal{E}^\bullet := (\mathcal{E}, \nabla_\mathcal{E}, \sigma_\mathcal{E}) \) is **dormant** if the connection \( \nabla_\mathcal{E} \) has vanishing \( p \)-curvature (cf. [14], § 3, Definition 3.1). Write

\[
\mathfrak{I} \mathcal{B}^\lozenge_X \quad \text{(resp., } \mathfrak{I} \mathcal{B}^\lozenge_{X^{\text{dorm}}}) \tag{45}
\]

for the set of isomorphism classes of indigenous bundles (resp., dormant indigenous bundles) on \( X \). Then, there exists a canonical bijection of sets

\[
P\mathcal{S}^F_S \sim \mathfrak{I} \mathcal{B}^\lozenge_X \tag{46}
\]

(cf. [10], § 3, Proposition 3.11, in the case where \( X \) is proper). In fact, let \( \mathcal{S}^\lozenge \) be a Frobenius-projective structure on \( X \). The \( \text{PGL}_2 \)-torse over \( X^{(1)} \) corresponding to \( \mathcal{S}^\lozenge \) via the underlying homeomorphism of \( F_{X/k} \) specifies a \( \mathbb{P}^1 \)-bundle over \( X^{(1)} \). The pull-back \( \mathcal{E}_S \) of this \( \mathbb{P}^1 \)-bundle to \( X \) admits naturally a connection \( \nabla_{\mathcal{E}_S}^{\text{can}} \) with vanishing \( p \)-curvature (cf. [11], § 5, Theorem 5.1). Moreover, the local sections \( U \to U \times \mathbb{P}^1 \) (for various open subschemes \( U \) of \( X \)) classified by sections in \( \mathcal{S}^\lozenge \) may be glued together to obtain a well-defined global section \( \sigma_{\mathcal{E}_S} : X \to \mathcal{E}_S \). It follows from the definition of a Frobenius-projective structure that the resulting triple

\[
\mathcal{E}^\bullet_S := (\mathcal{E}_S, \nabla_{\mathcal{E}_S}^{\text{can}}, \sigma_{\mathcal{E}_S}) \tag{47}
\]

specifies a dormant indigenous bundle on \( X \). The resulting assignment \( \mathcal{S}^\lozenge \mapsto \mathcal{E}^\bullet_S \) gives the desired bijection (46).

**Remark 3.2** If \( X \) is proper, then we know an explicit formula for computing the number of dormant indigenous bundles on \( X \), as proved in [14], Theorem A. In particular,
there exists at least one dormant indigenous bundle on any (not necessarily proper) smooth curve because it has a smooth compactification.

3.3 Dormant \((\text{SL}_n, \mathcal{L})\)-opers

We shall describe indigenous bundles, as well as their higher-rank generalizations, in terms of vector bundles. Let \(n\) be an integer with \(1 < n < p\). Here, recall that, for each vector bundle \(\mathcal{F}\) on \(X\) of rank \(n\), a \textit{connection} on \(\mathcal{F}\) means a \(k\)-linear morphism \(\nabla_{\mathcal{F}} : \mathcal{F} \to \Omega_X \otimes \mathcal{F}\) satisfying that \(\nabla_{\mathcal{F}}(a \cdot v) = da \otimes v + a \cdot \nabla_{\mathcal{F}}(v)\) for any local sections \(a \in \mathcal{O}_X\) and \(v \in \mathcal{F}\). Given such a connection \(\nabla_{\mathcal{F}}\), we have a connection \(\nabla_{\text{det}(\mathcal{F})}\) on the determinant bundle \(\text{det}(\mathcal{F})\) induced by \(\nabla_{\mathcal{F}}\), i.e., given by

\[
\nabla_{\text{det}(\mathcal{F})}(a_1 \wedge \cdots \wedge a_n) = \sum_{i=1}^{n} a_1 \wedge \cdots \wedge \nabla_{\mathcal{F}}(a_i) \wedge \cdots \wedge a_n.
\]

Recall (cf. [1], § 2.1) that a \(\text{GL}_n\)-oper on \(X\) is a collection of data \((\mathcal{F}, \nabla_{\mathcal{F}}, \mathcal{F}^\bullet)\) consisting of a rank \(n\) vector bundle \(\mathcal{F}\) on \(X\), a connection \(\nabla_{\mathcal{F}}\) on \(\mathcal{F}\), and an \(n\)-step decreasing filtration \(\mathcal{F}^\bullet := \{\mathcal{F}_{j}\}_{j=0}^{n}\) on \(\mathcal{F}\) satisfying the following conditions:

- Each \(\mathcal{F}_{j}\) is a subbundle of \(\mathcal{F}\) such that \(\mathcal{F}^0 = \mathcal{F}, \mathcal{F}^n = 0\), and \(\text{gr}_{\mathcal{F}}^j := \mathcal{F}_{j}/\mathcal{F}_{j+1}\) \((0 \leq j \leq n - 1)\) is a line bundle;
- \(\nabla_{\mathcal{F}}(\mathcal{F}_{j}) \subseteq \Omega_X \otimes \mathcal{F}_{j-1}^\wedge\ (1 \leq j \leq n - 1)\) and the morphism \(\text{KS}_{\mathcal{F}, \mathcal{F}^\bullet}^j : \text{gr}_{\mathcal{F}}^j \to \Omega_X \otimes \text{gr}_{\mathcal{F}}^{j-1}\) induced by \(\nabla_{\mathcal{F}}\) (which is verified to be \(\mathcal{O}_X\)-linear) is an isomorphism.

In particular, a \(\text{GL}_2\)-oper on \(X\) is determined by a triple \((\mathcal{F}, \nabla_{\mathcal{F}}, \mathcal{N}_{\mathcal{F}})\) consisting of a pair \((\mathcal{F}, \nabla_{\mathcal{F}})\) as above (with \(n = 2\)) and a line subbundle \(\mathcal{N}_{\mathcal{F}}\) of \(\mathcal{F}\) such that the \(\mathcal{O}_X\)-linear composite

\[
\text{KS}_{\mathcal{N}_{\mathcal{F}}} : \mathcal{N}_{\mathcal{F}} \hookrightarrow \mathcal{F} \xrightarrow{\nabla_{\mathcal{F}}} \Omega_X \otimes \mathcal{F} \to \Omega_X \otimes (\mathcal{F}/\mathcal{N}_{\mathcal{F}}),
\]

called the \textit{Kodaira–Spencer map} associated with \(\mathcal{N}_{\mathcal{F}}\), is an isomorphism.

Next, we shall fix a theta characteristic \(\mathcal{L} := (\mathcal{L}, \psi_{\mathcal{L}})\) (cf. § 1.4) of \(X\). Consider a collection of data

\[
\mathcal{F}^\circ := (\mathcal{F}, \nabla_{\mathcal{F}}, \mathcal{F}^\bullet, \eta_{\mathcal{F}})
\]

consisting of a \(\text{GL}_n\)-oper \((\mathcal{F}, \nabla_{\mathcal{F}}, \mathcal{F}^\bullet)\) on \(X\) and an \(\mathcal{O}_X\)-linear isomorphism \(\eta_{\mathcal{F}} : \mathcal{L}^\otimes(n-1) \sim \mathcal{F}^{n-1}\). For each \(j = 0, \cdots, n - 1\), denote by \(\kappa_j\) the composite isomorphism

\[
\text{gr}_{\mathcal{F}}^j \sim \Omega_X^{(j-1)} \otimes \text{gr}_{\mathcal{F}}^{j+1} \sim \cdots \sim \Omega_X^{(j-n+1)} \otimes \text{gr}_{\mathcal{F}}^{n-1} = \Omega_X^{(j-n+1)} \otimes \mathcal{F}^{n-1}
\]

\(\blacklozenge\) Springer
each of whose constituent arises from $K_{S_{\mathcal{F}}}$. Also, denote by $\delta_{\mathcal{F}}^\diamond$ the composite isomorphism

$$
\delta_{\mathcal{F}}^\diamond : \text{det}(\mathcal{F}) \left( = \bigotimes_{j=0}^{n-1} \text{gr}^j_{\mathcal{F}} \right) \xrightarrow{\otimes j_{\mathcal{F}j}} \bigotimes_{j=0}^{n-1} \Omega_X^{\otimes (j-n+1)} \otimes \mathcal{F}^{n-1}
$$

$$\sim \Omega_{X}^{\otimes -\frac{n(n-1)}{2}} \otimes (\mathcal{F}^{n-1})^{\otimes n}
$$

$$\sim \Omega_{X}^{\otimes -\frac{n(n-1)}{2}} \otimes \mathcal{L}^{\otimes (n-1)}
$$

$$\sim \mathcal{O}_X,
$$

where the third and fourth arrows are the isomorphisms arising from $\eta_{\mathcal{F}}$ and $\psi_{\mathcal{F}}$, respectively.

**Definition 3.3** (cf. [14], § 2, Definition 2.3, (ii), in the case $n = 2$) We shall say that $\mathcal{F}^\diamond$ is an $(\text{SL}_n, \mathbb{L})$-oper on $X$ if the connection $\nabla_{\text{det}(\mathcal{F})}$ on $\text{det}(\mathcal{F})$ coincides with $d : \mathcal{O}_X \to \Omega_X$ via $\delta_{\mathcal{F}}^\diamond$. In a natural manner, we can define the notion of an isomorphism between $(\text{SL}_n, \mathbb{L})$-opers. Also, we shall say that an $(\text{SL}_n, \mathbb{L})$-oper is dormant if it has vanishing $p$-curvature.

We denote by

$$\mathfrak{Op}_{(\text{SL}_n, \mathbb{L}), X} \left( \text{resp., } \mathfrak{Op}_{(\text{SL}_n, \mathbb{L}), X}^{zz...} \right)
$$

the set of isomorphism classes of $(\text{SL}_n, \mathbb{L})$-opers (resp., dormant $(\text{SL}_n, \mathbb{L})$-opers) on $X$. According to [14], § 2, Proposition 2.4, there exists a canonical bijection

$$\mathfrak{B}_X \sim \mathfrak{Op}_{(\text{SL}_2, \mathbb{L}), X},
$$

which restricts to a bijection

$$\mathfrak{B}_X^{zz...} \sim \mathfrak{Op}_{(\text{SL}_2, \mathbb{L}), X}^{zz...}.
$$

Let $\mathcal{E}^\bullet := (\mathcal{E}, \nabla_{\mathcal{E}}, \sigma_{\mathcal{E}})$ be an indigenous bundle on $X$, and let $\mathcal{F}^\diamond := (\mathcal{F}, \nabla_{\mathcal{F}}, N_{\mathcal{F}}, \eta_{\mathcal{F}})$ denote the $(\text{SL}_2, \mathbb{L})$-oper corresponding to $\mathcal{E}^\bullet$ via (54). Then, $(\mathcal{E}, \nabla_{\mathcal{E}})$ may be obtained from $(\mathcal{F}, \nabla_{\mathcal{F}})$ via projectivization, and $\eta_{\mathcal{F}} : L \xrightarrow{\sim} N_{\mathcal{F}} (\subseteq \mathcal{F})$ induces a morphism

$$\eta_{\mathcal{F}}^\Lambda : \mathbb{A}(L)^\times \to \mathbb{A}(\mathcal{F})^\times
$$
over $X$. The following square diagram

\[
\begin{array}{ccc}
\mathbb{A}(\mathcal{L}) \times & \xrightarrow{\eta^L_{\mathcal{F}}} & \mathbb{A}(\mathcal{F}) \times \\
\text{projection} \quad \downarrow & & \quad \downarrow \pi_{\mathcal{F}} \\
X & \xrightarrow{\sigma_{\mathcal{E}}} & \mathcal{E} (= \mathbb{P}(\mathcal{F}))
\end{array}
\]

(57)

is commutative and moreover Cartesian.

**Example 3.4** Let $\mathcal{F}^{\diamond} := (\mathcal{F}, \nabla_{\mathcal{F}}, \mathcal{N}_{\mathcal{F}}, \eta_{\mathcal{F}})$ be an $(\text{SL}_2, \mathbb{L})$-oper on $X$ and $n$ an integer with $1 < n < p$. The $(n-1)$st symmetric power $\mathcal{F}_{\text{SL}_n} := S^{n-1}(\mathcal{F})$ of $\mathcal{F}$ is a rank $n$ vector bundle. For each $j = 0, 1, \ldots, n$, the image $\mathcal{F}_{\text{SL}_n}^j$ of the natural morphism $\mathcal{N}_{\mathcal{F}}^\otimes j \otimes S^{n-1-j}(\mathcal{F}) \to S^{n-1}(\mathcal{F})$ defines a rank $(n-j)$ subbundle of $\mathcal{F}_{\text{SL}_n}$. The collection $\mathcal{F}_{\text{SL}_n}^\bullet := \{\mathcal{F}_{\text{SL}_n}^j\}_{j=0}^n$ forms an $n$-step decreasing filtration on $\mathcal{F}_{\text{SL}_n}$. Let $\eta_{\mathcal{F}, \text{SL}_n} : \mathcal{L}^{\otimes (n-1)} \sim \mathcal{F}_{\text{SL}_n}^{n-1}$ be the composite isomorphism of $\eta_{\mathcal{F}}^{\otimes (n-1)} : \mathcal{L}^{\otimes (n-1)} \sim \mathcal{N}_{\mathcal{F}}^{\otimes (n-1)}$ with the natural isomorphism $\mathcal{N}_{\mathcal{F}}^{\otimes (n-1)} \sim \mathcal{F}_{\text{SL}_n}^{n-1}$. Also, let $\nabla_{\mathcal{F}, \text{SL}_n}$ be the connection on $\mathcal{F}_{\text{SL}_n}$ induced naturally by $\nabla_{\mathcal{F}}$. Then, one may verify immediately that the collection of data

\[
\mathcal{F}_{\text{SL}_n}^{\diamond} := (\mathcal{F}_{\text{SL}_n}, \nabla_{\mathcal{F}, \text{SL}_n}, \mathcal{F}_{\text{SL}_n}^\bullet, \eta_{\mathcal{F}, \text{SL}_n})
\]

(58)

forms an $(\text{SL}_n, \mathbb{L})$-oper on $X$. If, moreover, $\mathcal{F}^{\diamond}$ is dormant, then the resulting $(\text{SL}_n, \mathbb{L})$-oper $\mathcal{F}_{\text{SL}_n}^{\diamond}$ turns out to be dormant. Thus, the assignment $\mathcal{F}^{\diamond} \mapsto \mathcal{F}_{\text{SL}_n}^{\diamond}$ defines a map of sets

\[
\mathcal{O}p_{(\text{SL}_2, \mathbb{L})}, X \to \mathcal{O}p_{(\text{SL}_n, \mathbb{L})}, X,
\]

(59)

and this map restricts to a map

\[
\mathcal{O}p_{Z_{zz\ldots}}^{(\text{SL}_2, \mathbb{L})}, X \to \mathcal{O}p_{Z_{zz\ldots}}^{(\text{SL}_n, \mathbb{L})}, X.
\]

(60)

### 3.4 Projective connections

Next, we discuss higher-order projective connections. Let $\mathbb{L}$ be as above and $n$ an integer with $1 < n < p$. Also, let $D : \mathcal{L}^{\otimes (-n+1)} \to \mathcal{L}^{\otimes (n+1)}$ be an $n$th-order differential operator (i.e., an element of $\mathcal{D}iff_{\leq n} \mathcal{L}^{\otimes (-n+1)}, \mathcal{L}^{\otimes (n+1)}$) satisfying the equality $\Sigma(D) = 1$ (cf. § 1.3) under the identification

\[
\mathcal{H}om_{\mathcal{O}X}(\mathcal{L}^{\otimes (-n+1)}, \mathcal{L}^{\otimes (n+1)} \otimes S^n(T_X)) \sim \mathcal{H}om_{\mathcal{O}X}(\mathcal{L}^{\otimes (-n+1)}, \mathcal{L}^{\otimes (n+1)}) \sim \mathcal{O}_X
\]

(61)
 induced by $\psi_L$. Denote by $^t D$ the transpose of $D$, which is a differential operator $\Omega_X \otimes (L^{\otimes(n+1)})^\vee \to \Omega_X \otimes (L^{\otimes(-n+1)})^\vee$. If $D$ is locally expressed as $D = \sum_{i=0}^n a_i \cdot \partial^i$ (for a local generator $\partial$ of $T_X$), then we have $^t D = \sum_{i=0}^n (-\partial)^i \cdot a_i$. Since $\psi_L$ enables us to identify $\Omega_X \otimes (L^{\otimes(n+1)})^\vee$ and $\Omega_X \otimes (L^{\otimes(-n+1)})^\vee$ with $L^{\otimes(-n+1)}$ and $L^{\otimes(n+1)}$, respectively, $D$ may be thought of as a differential operator in $\mathcal{D}_{\mathcal{L}}^{\leq n} \mathcal{L}^{\otimes(-n+1)} \mathcal{L}^{\otimes(n+1)} \cdot$

In particular, it makes sense to speak of the operator $D' := \frac{1}{2} \cdot (D - (-1)^n \cdot ^t D)$. Moreover, by the equality $\Sigma(D) = 1$, the operator $D'$ turns out to be of order $\leq n - 1$. We refer to the principal symbol $\Sigma_{\text{sub}}(D) := \Sigma(D')$ of $D'$ as the subprincipal symbol of $D$.

**Definition 3.5** An $n$th-order projective connection for $\mathbb{L}$ is an $n$th-order differential operator $D^\bullet : L^{\otimes(-n+1)} \to L^{\otimes(n+1)}$ with $\Sigma(D^\bullet) = 1$ and $\Sigma_{\text{sub}}(D^\bullet) = 0$. For simplicity, we refer to each second order projective connection as a projective connection.

We shall write

$$\mathfrak{P}^n_{X,\mathbb{L}} \quad (\text{resp., } \mathfrak{P}^{n,\text{full}}_{X,\mathbb{L}}) \quad (62)$$

for the set of $n$th-order projective connections for $\mathbb{L}$ (resp., the set of $n$th-order projective connections for $\mathbb{L}$ having a full set of solutions).

**Proposition 3.6** (cf. [1], §2.1 and §2.8; [12], Proposition 6.0.5) There exists a canonical bijection

$$\mathcal{O}p_{(\mathbb{L}, \mathbb{L})}, X \sim \mathfrak{P}^n_{X,\mathbb{L}} \quad (63)$$

Moreover, it restricts to a bijection

$$\mathcal{O}p_{(\mathbb{L}, \mathbb{L})}, X \sim \mathfrak{P}^{n,\text{full}}_{X,\mathbb{L}} \quad (64)$$

**Proof** First, we shall construct the bijection (63). Let $\mathcal{F}^\circ := (\mathcal{F}, \nabla_\mathcal{F}, \mathcal{F}^*, \eta_\mathcal{F})$ be an $(\text{SL}_n, \mathbb{L})$-oper on $X$. The connection $\nabla_\mathcal{F}$ induces, inductively on $i \leq n$, an $\mathcal{O}_X$-linear morphism $\nabla_\mathcal{F}^{D,i} : \mathcal{D}^{\leq i}_X \otimes \mathcal{F} \to \mathcal{F}$ determined by the condition that $\nabla_\mathcal{F}^{D,0} = \text{id}_\mathcal{F}$ and $\nabla_\mathcal{F}^{D,i}(\partial^i \otimes v) = (\partial, \nabla_\mathcal{F}(\nabla_\mathcal{F}^{D,i-1}(\partial^{i-1} \otimes v)))$ for any local generator $\partial \in T_X$ and any local section $v \in \mathcal{F}$. By the definition of a GL$_n$-oper, we see (by induction on $i$) that the morphism $\nabla_\mathcal{F}^{D,i}$ for $i \leq n - 1$ restricts to an isomorphism $\mathcal{D}^{\leq i}_X \otimes \mathcal{F}^{n-1} \to \mathcal{F}^{n-i-1}$ and hence $\nabla_\mathcal{F}^{D,n}$ is surjective. The composite $(\nabla_\mathcal{F}^{D,n})^{-1} \circ \nabla_\mathcal{F}^{D,n}$, regarded as an $\mathcal{O}_X$-linear morphism $\mathcal{D}^{\leq n}_X \otimes \mathcal{L}^{\otimes(n-1)} \to \mathcal{D}^{\leq (n-1)}_X \otimes \mathcal{L}^{\otimes(n-1)}$ via $\eta_\mathcal{F}$, determines a split surjection of the following short exact sequence:

$$0 \to \mathcal{D}^{\leq (n-1)}_X \otimes \mathcal{L}^{\otimes(n-1)} \to \mathcal{D}^{\leq n}_X \otimes \mathcal{L}^{\otimes(n-1)} \to (\mathcal{D}^{\leq n}_X / \mathcal{D}^{\leq (n-1)}_X) \otimes \mathcal{L}^{\otimes(n-1)} \to 0. \quad (65)$$
Let us consider the composite $L^\otimes(-n-1) \rightarrow D_X^{\leq n} \otimes L^\otimes(n-1)$ of the corresponding split injection $(T_X^{\otimes n} \otimes L^\otimes(n-1) = (D_X^{\leq n}/D_X^{\leq n}) \otimes L^\otimes(n-1)) \rightarrow D_X^{\leq n} \otimes L^\otimes(n-1)$ and the isomorphism $L^\otimes(-n-1) \sim T_X^{\otimes n} \otimes L^\otimes(n-1)$ induced naturally by $\psi_L$; it corresponds to an $O_X$-linear morphism $L^\otimes(-n-1) \rightarrow L^\otimes(n+1) \otimes D_X^{\leq n}$, that is to say, an $n$th differential operator $D_X^\bullet : L^\otimes(-n-1) \rightarrow L^\otimes(n+1)$ (cf. (12)). One may immediately verify that $\Sigma(D_X^\bullet) = 1$, and moreover, (by taking account of the fact that $(\det(F), \nabla_{\det(F)}) \cong (O, d)$) that $\Sigma_{\text{sub}}(D_X^\bullet) = 0$. Thus, $D_X^\bullet$ specifies a projective connection on $L$. The resulting assignment $D^\otimes \mapsto D_X^\bullet$ defines a map of sets $\mathcal{D}_\pi(\text{SL}_n, L) \rightarrow \mathfrak{P}^n_{X, L}$.

Conversely, let $D^\bullet$ be a projective connection belonging to $\mathfrak{P}^n_{X, L}$. For each $i = 0, \ldots, n$, we shall write $F^i_D := D_X^{\leq (n-i-1)} \otimes \mathcal{L}^\otimes(n-1)$ and $F^{i*}_D := \{F^i_D\}_{i=0}^n$. The operator $D^\bullet$ may be thought of as an $O_X$-linear morphism $L^\otimes(-n-1) \rightarrow \mathcal{L}^\otimes(n+1) \otimes D_X^{\leq n}$ (via (12)), or equivalently, $\mathcal{L}^\otimes(-n-1) \rightarrow D_X^{\leq n} \otimes \mathcal{L}^\otimes(n-1)$. It specifies a split injection of (65); we shall write $\nabla'$ for the corresponding split surjection $\mathcal{D}_X^{\leq n} \otimes \mathcal{L}^\otimes(n-1) \rightarrow D_X^{\leq(n-1)} \otimes \mathcal{L}^\otimes(n-1) (= F^0_D)$. Then, there exists a unique connection $\nabla_D$ on $F_D^0$ of the form determined by the condition that $\langle \partial, \nabla_D(\partial^i \otimes v) \rangle = \nabla'(\partial^i \otimes v) (i = 0, \ldots, n - 1)$ for any local generator $\partial \in T_X$ and any local section $v \in \mathcal{L}^\otimes(n-1)$. If $\eta_D$ denotes the natural isomorphism $\mathcal{L}^\otimes(n-1) \sim F_D^{n-1}$, then (because of the assumption that $\Sigma(D) = 1$ and $\Sigma_{\text{sub}}(D) = 0$) the collection $F_D^\otimes := (F_D^0, \nabla_D, F^{*}_D, \eta_D)$ forms an $(\text{SL}_n, L)$-oper on $X$. The resulting assignment $D^\bullet \mapsto F_D^\otimes$ turns out to be the inverse of the map $\mathcal{D}_\pi(\text{SL}_n, L) \rightarrow \mathfrak{P}^n_{X, L}$ obtained above. This completes the proof of the former assertion.

Next, we shall consider the latter assertion. Let us take a projective connection $D^\bullet$ belonging to $\mathfrak{P}^n_{X, L}$, and denote by $F^\otimes := (F, \nabla, F^{*}, \eta_F)$ the corresponding $(\text{SL}_n, L)$-oper constructed by the above steps. If $D^\bullet$ may be expressed (after choosing a local identification $L \cong O_X$ locally as $D^\bullet = \partial^n + q_1 \partial^{n-1} + \cdots + q_{n-1} \partial + q_n$ (for a local generator $\partial \in T_X$ and local sections $q_1, \ldots, q_n \in \mathcal{O}_X$), then the dual connection $\nabla_D^\otimes$ of $\nabla_D$ may be expressed locally (with respect to a suitable local basis) as

$$\nabla_D^\otimes = \partial - \left(\begin{array}{cccccc} -q_1 & -q_2 & -q_3 & \cdots & -q_{n-1} & -q_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{array}\right).$$

(66)

Then, $y \mapsto \iota(\partial^{n-1}y, \ldots, \partial y, y)$ gives a bijective correspondence between the solutions of the differential equation $D^\bullet(y) = 0$ and the horizontal local sections of $\mathcal{F}^\otimes$ with respect to $\nabla_D^\otimes$. This implies that $D^\bullet$ has a full set of solutions if and only if the connection $\nabla_D^\otimes$, hence also $\nabla_D$, has vanishing $p$-curvature. Consequently, the bijection $\mathcal{D}_\pi(\text{SL}_n, L) \rightarrow \mathfrak{P}^n_{X, L}$ restricts to a bijection $\mathcal{D}_\pi^{ZL}(\text{SL}_n, L) \rightarrow \mathfrak{P}^n_{X, L}$, as desired.
By various assertions mentioned so far, we have obtained the following commutative diagram of sets:

\[
\begin{array}{cccc}
\mathcal{B}_X^F & \sim & \mathcal{B}_X^{Zzz} & \sim \\
& & & \\
\mathcal{B}_X & \sim & \mathcal{B}_X^{(SL_2, \mathbb{L}), X} & \sim \\
& & & \\
\mathcal{B}_X & \sim & \mathcal{B}_X^{(SL_n, \mathbb{L}), X} & \sim \\
\mathcal{B}_X & \sim & \mathcal{B}_X & \sim \\
\end{array}
\]

inclusion

inclusion

inclusion

Moreover, there is a map from the set of projective connections (resp., projective connections having a full set of solutions) to the set of \(n\)th-order projective connections (resp., \(n\)th-order projective connections having a full set of solutions)

\[
\xi^{2 \rightarrow n} : \mathcal{P}_X^2 \to \mathcal{P}_X^n \quad \text{(resp.,} \xi^{\text{full}}_f \to \mathcal{P}_X^{\text{full}}) \tag{68}
\]

constructed in such a way that the square diagram

is commutative.

### 3.5 Case of the projective line

Let us consider the case where \(X = \mathbb{P}^1 (= \text{Proj}(k[x, y]))\) equipped with the theta characteristic \(\mathbb{L}_0\) (cf. (24)). Hereinafter, we construct a typical example of a Frobenius-projective structure (resp., a dormant indigenous bundle; resp., a dormant \((SL_n, \mathbb{L}_0)\)-oper; resp., a projective connection for \(\mathbb{L}_0\) having a full set of solutions) on \(\mathbb{P}^1\).

First, we shall set

\[
S_0^\vee \tag{70}
\]

to be the subsheaf of \(\mathcal{P}^\text{ét}\) (for \(X = \mathbb{P}^1\)) which, to any open subscheme \(U\) of \(\mathbb{P}^1\), assigns the set

\[
S_0^\vee(U) := \{A(op_U) \in \mathcal{P}^\text{ét}(U) \mid A \in (PGL_2)^F_{\mathbb{P}^1}(U)\}, \tag{71}
\]
where \(\text{op}_{U} \) denotes the natural open immersion \( U \hookrightarrow \mathbb{P}^1 \). Then, \( S_0^\oplus \) forms a trivial \((\text{PGL}_2)^{\text{P}^1}\)-torsor, and hence, specifies a Frobenius-projective structure on \( \mathbb{P}^1 \).

Next, we shall write \( \mathcal{E}_0 := \mathbb{P}^1 \times \mathbb{P}^1 \), which defines the trivial \( \mathbb{P}^1 \)-bundle on \( \mathbb{P}^1 \) by putting the first projection \( \text{pr}_1 : \mathcal{E}_0 \to \mathbb{P}^1 \) as its structure morphism. Write \( \mathcal{V}_0 \) for the trivial connection on this trivial \( \mathbb{P}^1 \)-bundle; it is clear that \( \mathcal{V}_0 \) has vanishing \( p \)-curvature. The Kodaira–Spencer map (with respect to \( \mathcal{V}_0 \)) of the diagonal embedding \( \sigma_0 : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \) (\( = \mathcal{E}_0 \)) is nowhere vanishing. Thus, the triple

\[
\mathcal{E}_0^\bullet := (\mathcal{E}_0, \mathcal{V}_0, \sigma_0)
\]

forms a dormant indigenous bundle on \( \mathbb{P}^1 \).

Moreover, recall the injection \( \eta_0 : \mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \) introduced in Example 1.2, (18) (of the case \( n = 1 \)); we abuse notation to use \( \eta_0 \) for writing the isomorphism \( \mathcal{O}_{\mathbb{P}^1}(-1) \sim \text{Im}(\eta_0) \) induced by \( \eta_0 \). Then, the collection

\[
\mathcal{F}_0^\bullet := (\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}, d^{\oplus 2}, \text{Im}(\eta_0), \eta_0)
\]

forms a dormant \((\text{SL}_2, \mathbb{L}_0)\)-oper on \( \mathbb{P}^1 \).

Finally, let us consider the second order differential operator \( D_{0,u} : \mathcal{O}_{\mathbb{P}^1}(1)|_U \to \mathcal{O}_{\mathbb{P}^1}(-3)|_U \) (resp., \( D_{0,t} : \mathcal{O}_{\mathbb{P}^1}(1)|_T \to \mathcal{O}_{\mathbb{P}^1}(-3)|_T \)) on the open subscheme \( U := \text{Spec}(k[u]) \) (resp., \( T := \text{Spec}(k[t]) \)) of \( \mathbb{P}^1 \), where \( u := x/y \) (resp., \( t := y/x \)), given by \( f(u) \cdot y \mapsto \frac{\partial^2 f}{\partial u \partial y}(u) \cdot y^{-3} \) (resp., \( g(t) \cdot x \mapsto \frac{\partial^2 g}{\partial t \partial x}(t) \cdot x^{-3} \)). Then, \( D_{0,u} \) and \( D_{0,t} \) may be glued together to obtain a globally defined differential operator

\[
D_0^\bullet : \mathcal{O}_{\mathbb{P}^1}(-1)^\vee ( = \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathcal{O}_{\mathbb{P}^1}(-1)^\otimes 3 ( = \mathcal{O}_{\mathbb{P}^1}(-3)).
\]

This operator forms a projective connection for \( \mathbb{L}_0 \).

**Proposition 3.7** All the sets \( \mathcal{P}\mathcal{C}^1 \), \( \mathbb{M}_{\mathbb{P}^1} \), \( \mathcal{M}_{\mathbb{P}^1}^{\text{zzz}} \), \( \mathcal{D}_{\mathbb{P}^1} \), \( \mathcal{D}_\mathcal{Z}_{\mathbb{P}^1} \), \( \mathcal{D}_{\mathbb{P}^1,\mathbb{L}_0} \), \( \mathcal{P}\mathcal{C}_{\mathbb{P}^1,\mathbb{L}_0}^2 \), and \( \mathcal{P}\mathcal{C}_{\mathbb{P}^1,\mathbb{L}_0}^{2,\text{full}} \) are singletons. That is to say,

\[
\begin{align*}
\mathcal{P}\mathcal{C}_{\mathbb{P}^1} & = \left\{ S_0^\oplus \right\}, \\
\mathcal{M}_{\mathbb{P}^1} & = \left\{ \mathcal{E}_0^\bullet \right\}, \\
\mathcal{M}_{\mathbb{P}^1}^{\text{zzz}} & = \mathcal{M}_{\mathbb{P}^1} = \left\{ \mathcal{E}_0^\bullet \right\}, \\
\mathcal{D}_{\mathbb{P}^1,\mathbb{L}_0} & = \mathcal{D}_{\text{zzz}}_{\mathbb{P}^1,\mathbb{L}_0} = \left\{ \mathcal{F}_0^\bullet \right\}, \\
\mathcal{P}\mathcal{C}_{\mathbb{P}^1,\mathbb{L}_0}^2 & = \mathcal{P}\mathcal{C}_{\mathbb{P}^1,\mathbb{L}_0}^{2,\text{full}} = \left\{ D_0^\bullet \right\}.
\end{align*}
\]

**Proof** By various bijections in (67), it suffices to prove that \( \mathcal{P}\mathcal{C}_{\mathbb{P}^1,\mathbb{L}_0}^2 \) contains at most one element. As discussed in the proof of Proposition 3.6, each element of \( \mathcal{P}\mathcal{C}_{\mathbb{P}^1,\mathbb{L}_0}^2 \) corresponds to a splitting of the short exact sequence

\( \mathbb{S}^{\text{ Springer}} \)
Let us fix a smooth curve $X$ and a theta characteristic $\mathbb{L} := (\mathcal{L}, \psi_{\mathcal{L}})$ on $X$. The morphism

$$\psi_{\mathcal{L}}^\wedge : \mathbb{A}(\mathcal{L})^\times \to \mathbb{A}(\Omega_X)^\times$$

(78)

over $X$ between algebraic surfaces defined by $\psi_{\mathcal{L}}^\wedge(v) = \psi_{\mathcal{L}}(v \otimes v)$ (for each local section $v \in \mathcal{L}$) is a Galois double covering whose Galois group is isomorphic to $\mu_2 = \{\pm 1\}$. The automorphism of $\mathbb{A}(\mathcal{L})^\times$ determined by $-1 \in \mu_2$ is given by $v \mapsto -v$. Let us write

$$\tilde{\omega}_\mathbb{L} := (\psi_{\mathcal{L}}^\wedge)^* (\tilde{\omega}_{\text{can}}) \in \Gamma(\mathbb{A}(\mathcal{L})^\times, \bigwedge^2 \Omega_{\mathbb{A}(\mathcal{L})^\times})$$

(79)

(cf. (7)), which specifies a symplectic structure on $\mathbb{A}(\mathcal{L})^\times$. In particular, the pair

$$(\mathbb{A}(\mathcal{L})^\times, \tilde{\omega}_\mathbb{L}).$$

forms a symplectic variety equipped with a $\mu_2$-action. Then, the main result of the present paper is as follows.
**Theorem 4.1** There exists a canonical construction of a $\mu_2$-FC quantization on $(A(L)^\times, \tilde{\omega}_L)$ by means of a Frobenius-projective structure (or equivalently, a dormant indigenous bundle, a dormant $(\text{SL}_2, \mathbb{L})$-oper, or a projective connection for $L$ having a full set of solutions) on $X$. The resulting map of sets

$$\star_{X, L} : \Psi_{A}^{F} \rightarrow \Omega_{(A(L)^\times, \tilde{\omega}_L)}^{\mu_2-FC}$$

is injective and the composite injection

$$\star_{X} : \Psi_{A}^{F} \hookrightarrow \Omega_{(A_{\Omega X}^\times, \tilde{\omega}_{can})}^{FC}$$

of this map and the natural bijection $\Omega_{(A(L)^\times, \tilde{\omega}_L)}^{\mu_2-FC} \sim \Omega_{(A_{\Omega X}^\times, \tilde{\omega}_{can})}^{FC}$ (i.e., the inverse of (31)) does not depend on the choice of the theta characteristic $L$.

In the rest of this section, we will prove the above theorem.

**4.2 Step I: Local construction**

Let $S^{\bigodot}$ be a Frobenius-projective structure on $X$, and denote by $\mathcal{E}^{\bigodot} := (\mathcal{E}, \nabla_{\mathcal{E}}, \sigma_{\mathcal{E}})$ and $\mathcal{F}^{\bigodot} := (\mathcal{F}, \nabla_{\mathcal{F}}, N_{\mathcal{F}}, \eta_{\mathcal{F}})$ the corresponding indigenous bundle and $(\text{SL}_2, \mathbb{L})$-oper, respectively. In this first step, we construct FC quantizations on various open subschemes of $(A(L)^\times, \tilde{\omega}_L)$. Since $\nabla_{\mathcal{F}}$ has vanishing $p$-curvature, $(\mathcal{F}, \nabla_{\mathcal{F}})$ is locally trivial. More precisely, there exists a collection

$$\{(U_{\alpha}, \gamma_{\alpha})\}_{\alpha \in I}$$

of pairs $(U_{\alpha}, \gamma_{\alpha})$ (indexed by a set $I$), where $\{U_{\alpha}\}_{\alpha \in I}$ is an open covering of $X$ and $\gamma_{\alpha}$ (for each $\alpha \in I$) denotes an $O_{U_{\alpha}}$-linear isomorphism $\mathcal{F}|_{U_{\alpha}} \sim O_{U_{\alpha}}^{\otimes 2}$ inducing, via taking determinants, the isomorphism $\delta_{\mathcal{F}^{\bigodot}|U_{\alpha}}$ (cf. (52)). Each isomorphism $\gamma_{\alpha}$ induces an isomorphism

$$\gamma_{\alpha} : A_{A}((\mathcal{F}|_{U_{\alpha}})^{\times} \sim \left( A_{A}(O_{U_{\alpha}}^{\otimes 2})^{\times} \sim \right) U_{\alpha} \times \mathbb{A}_{\alpha}^{2\times}$$

over $U_{\alpha}$, where $\mathbb{A}_{\alpha}^{2\times} := \text{Spec}(k[x, y])$ and the isomorphism in the parenthesis is given by $x \mapsto (1, 0), y \mapsto (0, 1)$. Moreover, denote by $\Phi_{\alpha}$ the composite

$$\Phi_{\alpha} : A_{A}(\mathcal{L}|_{U_{\alpha}})^{\times} \xrightarrow{\eta_{\mathcal{F}|U_{\alpha}}} A_{A}(\mathcal{F}|_{U_{\alpha}})^{\times} \xrightarrow{\gamma_{A, \alpha}} U_{\alpha} \times \mathbb{A}_{\alpha}^{2\times} \xrightarrow{pr_2} \mathbb{A}_{\alpha}^{2\times}. $$

This composite is compatible with the respective $\mu_2$-actions on $A_{A}(\mathcal{L}|_{U_{\alpha}})^{\times}$ and $\mathbb{A}_{\alpha}^{2\times}$. The morphism $A_{A}(\Omega_{U_{\alpha}})^{\times} \sim \mathbb{A}_{A/\mu_2}^{2\times}$ induced via quotient does not depend on the choice of $\mathbb{L}$.

**Lemma 4.2** The morphism $\Phi_{\alpha}$ is étale and satisfies the equality $\Phi_{\alpha}^*(\omega_{\text{Weyl}}) = \tilde{\omega}_L|_{U_{\alpha}}$. 

\[ \blacklozenge \] Springer
Proof. To begin with, we shall consider the case where \( X = U_\alpha = \mathbb{P}^1 (= \text{Proj}(k[x, y])) \), \( \mathbb{L} = \mathbb{L}_0 \), \( \mathcal{F}^\phi = \mathcal{F}^\phi_0 \), and \( \gamma_\alpha \) is the identity of \( O_{\mathbb{P}^1}^2 \). Write \( \Phi_0 \) for the morphism "\( \Phi_\alpha \)" in this case. On the open subscheme \( U := \text{Spec}(k[u]) \) of \( \mathbb{P}^1 \), where \( u = x/y \), we have \( \mathbb{A}(O_{\mathbb{P}^1}(-1))^\times \mid U = \text{Spec}(k[u, y, y^{-1}]) \), \( \mathbb{A}(O_{\mathbb{P}^1}(-1))^\times \mid U \) may be given by \( x \mapsto u \cdot y \) and \( y \mapsto y \). Hence,

\[
\Phi_0^*(\omega_{\text{Weyl}}^1) \mid U = \Phi_0^*(dy \wedge dx) = dy \wedge d(u \cdot y) = y \cdot dy \wedge du.
\] (86)

On the other hand, it follows from the definition of \( \psi_0 \) that \( \psi_0^H : \mathbb{A}(O_{\mathbb{P}^1}(-1))^\times \rightarrow \mathbb{A}(\Omega_{\mathbb{P}^1}^1)^\times \) is given by assigning \( f \mapsto (y \cdot f)^2 \cdot du \) for each \( f \in \Gamma(U, O_{\mathbb{P}^1}(-1))^\times \). If \( u^\vee \) denote the dual coordinate of \( u \) in \( \mathbb{A}(\Omega_{\mathbb{P}^1}^1)^\times \), then

\[
\tilde{\omega}_{\mathbb{L}_0} \mid U = (\psi_0^H)^* (\tilde{\omega}_{\mathbb{L}_0}^\text{can} \mid U) = (\psi_0^H)^* \left( \frac{1}{2} \cdot du^\vee \wedge du \right) = \frac{1}{2} \cdot dy^2 \wedge du = y \cdot dy \wedge du.
\] (87)

By (86) and (87), we obtain the desired equality

\[
\Phi_0^*(\omega_{\text{Weyl}}^1) = \tilde{\omega}_{\mathbb{L}_0}^\text{can}
\] (88)

of this case. (This result will be used in (98).)

Now, let us go back to our situation. Denote by \( \gamma_\alpha^E : E \mid U_\alpha \sim \to U_\alpha \times \mathbb{P}^1 \) the isomorphism induced from \( \gamma_\alpha^H \) via projectivization. Denote by \( \phi_\alpha \) the composite

\[
\phi_\alpha : U_\alpha \xrightarrow{\sigma_E \mid U_\alpha} E \mid U_\alpha \xrightarrow{\gamma_\alpha^E} U_\alpha \times \mathbb{P}^1 \xrightarrow{\text{pr}_2} \mathbb{P}^1.
\] (90)

This composite is verified to be \( \text{étale} \) since the Kodaira–Spencer map associated with \( \sigma_E \) is an isomorphism. Under the natural identifications \( U_\alpha \times \mathbb{A}^2 \sim \to \phi_\alpha^* (\mathbb{A}(O_{\mathbb{P}^1}^2)^\times) \) and \( U_\alpha \times \mathbb{P}^1 \sim \to \phi_\alpha^* (E_0) \) (where \( \phi_\alpha^* (-) \) denotes base-change by \( \phi_\alpha \)), we obtain a commutative diagram
Since (57) is Cartesian, the above diagram induces an isomorphism of \( G_m \)-torsors

\[
\gamma^\alpha : \mathbb{A}(\mathcal{L}|_{U_\alpha})^\times \stackrel{\sim}{\longrightarrow} \phi^*_\alpha(\mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(-1))^\times)
\]  

over \( U_\alpha \) such that the following diagram is commutative:

Moreover, the morphism (92) induces an \( \mathcal{O}_{U_\alpha} \)-linear isomorphism

\[
\gamma^\alpha : \mathcal{L}|_{U_\alpha} \stackrel{\sim}{\longrightarrow} \phi^*_\alpha(\mathcal{O}_{\mathbb{P}^1}(-1))
\]  

fitting into the following isomorphism of short exact sequences:

where the upper right-hand horizontal arrow \( \mathcal{F}|_{U_\alpha} \rightarrow \mathcal{L}'|_{U_\alpha} \) is the morphism arising from \( \delta_{\mathcal{F}^\vee} : \det(\mathcal{F}) \sim \mathcal{O}_X \) (cf. (52)). Since \( \gamma_\alpha \) is, moreover, compatible with the connections \( \nabla_{\mathcal{F}} \) and \( d^{\mathbb{P}^2} \), the respective Kodaira–Spencer maps give rise to a commutative square diagram of \( \mathcal{O}_{U_\alpha} \)-modules

\[
\mathcal{L}^{\otimes 2}|_{U_\alpha} \longrightarrow \mathcal{L}|_{U_\alpha} \stackrel{\psi^\alpha}{\longrightarrow} \Omega_{U_\alpha} \longrightarrow \phi^*_\alpha(\mathcal{O}_{\mathbb{P}^1}(-1)^{\otimes 2}) \rightarrow \phi^*_\alpha(\Omega_{\mathbb{P}^1}),
\]
as well as a commutative diagram of $U_\alpha$-schemes

\[
\begin{array}{ccc}
\mathbb{A}(\mathcal{L}|_{U_\alpha})^\times & \xrightarrow{\psi_{\mathcal{L}|_{U_\alpha}}} & \mathbb{A}(\Omega_{U_\alpha})^\times \\
\gamma_{\alpha} & \downarrow & \phi_\alpha \\
\phi_\alpha^*(\mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(-1))^\times) & \xrightarrow{\phi_\alpha^*(\psi_{\mathcal{L}0})} & \phi_\alpha^*(\mathbb{A}(\Omega_{\mathbb{P}^1})).
\end{array}
\]  

(97)

Hence, the following sequence of equalities holds:

\[
\Phi_\alpha^*(\omega_{\text{Weyl}}) (\omega_{\text{can}}_{\mathbb{P}^1})^\times = \frac{1}{2} \cdot \omega_{\mathbb{P}^1}|_{U_\alpha}.
\]  

This completes the proof of the latter assertion.

The former assertion, i.e., the étaleness of $\Phi_\alpha$, follows from the étaleness of $\phi_\alpha$ and the fact that the square diagram

\[
\begin{array}{ccc}
\mathbb{A}(\mathcal{L}|_{U_\alpha})^\times & \xrightarrow{\Phi_\alpha} & \mathbb{A}^2^\times \\
\downarrow & & \downarrow \\
U_\alpha & \xrightarrow{\phi_\alpha} & \mathbb{P}^1
\end{array}
\]  

(99)

is Cartesian. This completes the proof of the lemma.
on the symplectic variety \((\mathbb{A}(\mathcal{L}|_{U_{\alpha}})\times, \omega_{\mathbb{L}}|_{U_{\alpha}})\). The FC quantization on \((\mathbb{A}(\Omega_{U_{\alpha}}), \omega_{\text{can}})\) corresponding to \(\Phi_{\alpha}^{\ast}(\mathcal{W}_{k}^{2})\) via (31) does not depend on the choice of \(\mathbb{L}\).

### 4.3 Step II: Global construction

In this second step, we glue together the locally defined quantizations constructed above to obtain an FC quantization on the entire space \(X\), as follows. After possibly replacing \(\{U_{\alpha}\}_{\alpha \in I}\) with its refinement, we can assume that each \(U_{\alpha}\) is affine. Let us take a pair \((\alpha, \beta) \in I \times I\) with \(U_{\alpha\beta} := U_{\alpha} \cap U_{\beta} \neq \emptyset\). Since \((X, \omega)\) is separated, which implies that \(U_{\alpha\beta}\) is affine, we can write \(U_{\alpha\beta} = \text{Spec}(\mathbb{R}_{\alpha\beta})\) for some \(k\)-algebra \(\mathbb{R}_{\alpha\beta}\). Hereinafter, we shall use the notation \((-)_{(1)}\) to denote the result of base-change by \(F_{k}\). In particular, we obtain the \(k\)-algebra \(\mathbb{R}_{\alpha\beta}^{(1)}\) equipped with a \(k\)-algebra (injective) homomorphism \(\mathbb{R}_{\alpha\beta}^{(1)} \to \mathbb{R}_{\alpha\beta}\). Also, write \(\mathcal{R}_{\alpha\beta} := \Gamma(\mathbb{A}(\mathcal{L}|_{U_{\alpha\beta}})^{\times}, \mathcal{O}_{\mathbb{A}(\mathcal{L}|_{U_{\alpha\beta}})^{\times}})\). If \(\mathcal{I}_{\alpha}\) denotes the ideal of \(\mathbb{R}_{\alpha\beta}[x, y]\) determined by the closed immersion

\[
(\gamma_{\alpha}^{\ast} \circ \eta_{\alpha}^{\ast}|_{U_{\alpha\beta}}) : \mathbb{A}(\mathcal{L}|_{U_{\alpha\beta}})^{\times} = (\text{Spec}(\mathcal{R}_{\alpha\beta})) \\
\hookrightarrow \mathbb{A}^{2x} \times \mathbb{A}^{2y} = (\text{Spec}(\mathbb{R}_{\alpha\beta}[x, y])).
\]

Hence, \(\mathbb{R}_{\alpha\beta}[x, y]/\mathcal{I}_{\alpha} \cong \mathcal{R}_{\alpha\beta}\) and we have a natural isomorphism of \(\mathbb{R}_{\alpha\beta}^{(1)}[x_{p}, y_{p}]/\mathcal{I}_{\alpha}^{(1)} = \Gamma(\mathbb{A}(\mathcal{L}|_{U_{\alpha\beta}})^{\times(1)}, \mathcal{O}_{\mathbb{A}(\mathcal{L}|_{U_{\alpha\beta}})^{\times(1)}})\)-algebras

\[
\Gamma(\mathbb{A}(\mathcal{L}|_{U_{\alpha\beta}})^{\times}, \Phi_{\alpha}^{\ast}(\mathcal{W}_{k}^{2})) \cong \mathbb{W}_{\mathbb{R}_{\alpha\beta}}^{2} \otimes \mathbb{R}_{\alpha\beta}^{(1)}[x_{p}, y_{p}]/\mathcal{I}_{\alpha}^{(1)}.
\]

Next, let \(\gamma_{\alpha\beta}\) be the automorphism of \(\mathbb{R}_{\alpha\beta}[x, y]\) corresponding to \((\gamma_{\alpha}^{\ast}|_{U_{\alpha\beta}}) \circ (\gamma_{\beta}^{\ast}|_{U_{\alpha\beta}})^{-1} \in \text{SL}_{2}(\mathbb{R}_{\alpha\beta}) (= \text{Sp}(\mathbb{R}_{\alpha\beta}))\). This automorphism restricts to an automorphism of \(\mathbb{R}_{\alpha\beta}[x_{p}, y_{p}]\), and hence, we have the following diagram

\[
\begin{array}{ccc}
\mathbb{R}_{\alpha\beta}[x, y] & \sim & \mathbb{R}_{\alpha\beta}[x, y] \\
\gamma_{\alpha\beta} & \sim & \gamma_{\alpha\beta} \\
\mathbb{R}_{\alpha\beta}^{(1)}[x_{p}, y_{p}] & \sim & \mathbb{R}_{\alpha\beta}^{(1)}[x_{p}, y_{p}] \\
\end{array}
\]

The bottom triangle in (103) turns out to be commutative since various other small diagrams are commutative. This implies that \(\gamma_{\alpha\beta}(\mathcal{I}_{\alpha}^{(1)}) = \mathcal{I}_{\beta}^{(1)}\). Hence, the automorphism of the \(\mathbb{R}_{\alpha\beta}^{(1)}\)-algebras \(\mathbb{W}_{\mathbb{R}_{\alpha\beta}}^{2}\) given by \(\gamma_{\alpha\beta}\) via (37) induces an isomorphism
By passing to (102), we obtain an isomorphism
\[ \Phi_{\alpha\beta} : \Phi^*_{\alpha}(W^2_{k}|_{A(\mathcal{L}|U_{\alpha\beta})}) \cong \Phi^*_{\beta}(W^2_{k}|_{\mathcal{A}(\mathcal{L}|U_{\alpha\beta})}) \] (105)
of sheaves of $k[[\hbar]]$-algebras on $\mathcal{A}(\mathcal{L}|U_{\alpha\beta})$. This isomorphism is verified to be compatible with the $\mu_2$-actions, and the induced isomorphism between the respective quotient FC quantizations does not depend on the choice of $\mathbb{L}$.

By means of the isomorphisms $\Phi_{\alpha\beta}$ (for various $\alpha, \beta$), the sheaves $\Phi^*_{\alpha}(W^2_{k})$ may be glued together to obtain a sheaf
\[ W_{S^\circ}, \] (106)
which forms a $\mu_2$-FC quantization on $(\mathcal{A}(\mathcal{L}), \tilde{\omega}_{\mathcal{L}})$. This quantization does not depend on the choice of $(U_{\alpha}, \gamma_{\alpha})_{\alpha \in I}$, and moreover, its quotient by the $\mu_2$-action does not depend on the choice of $\mathbb{L}$. Thus, we have obtained a well-defined map
\[ \star_{X, \mathcal{L}} : \Psi S^F_X \to \Omega^{\mu_2-FC}_{(\mathcal{A}(\mathcal{L}), \tilde{\omega}_{\mathcal{L}})}, \] (107)such that the composite
\[ \star_X : \Psi S^F_X \to \Omega^{FC}_{(\Omega X, \tilde{\omega}_{\mathcal{L}})} \] (108)
of this map and the bijection $\Omega^{\mu_2-FC}_{(\mathcal{A}(\mathcal{L}), \tilde{\omega}_{\mathcal{L}})} \cong \Omega^{FC}_{(\Omega X, \tilde{\omega}_{\mathcal{L}})}$ does not depend on the choice of $\mathbb{L}$.

### 4.4 Step III: Injectivity

The remaining portion of the proof is the injectivity of $\star_{X, \mathcal{L}}$. Here, let $(S, \omega_S)$ be a symplectic variety and $\mathcal{O}_S^k$ an FC quantization on $(S, \omega_S)$. Given two local sections $a, b \in \mathcal{O}_S$, we express $a \ast b$ as
\[ a \ast b = \delta^0_b(a) + \delta^1_b(a) \cdot \hbar + \delta^2_b(a) \cdot \hbar^2 + \cdots \in \mathcal{O}_S[[\hbar]] \] (109)for some local sections $\delta^i_b(a) \in \mathcal{O}_S$ ($i = 0, 1, 2, \cdots$). For each $i$, the assignment $a \mapsto \delta^i_b(a)$ defines a (locally defined) $k$-linear endomorphism $\delta^i_b \in \text{End}_k(\mathcal{O}_S)$. The assignment $b \mapsto \delta^i_b$ gives an $\mathcal{O}_S$-linear morphism
\[ \delta^i : \mathcal{O}_S \to \text{End}_k(\mathcal{O}_S). \] (110)
We equip $\text{End}_k(\mathcal{O}_S)$ with a structure of $\mathcal{O}_S$-module given by multiplication on the left.
Lemma 4.3 If the triple \((S, \omega_S, \mathcal{O}_S^{\hbar})\) is taken to be \((\mathbb{A}(L)^{\times}, \mathcal{O}_L, \mathcal{W}_S^{\natural})\) (as discussed in the previous subsection), then for each \(i < p\) the image of \(\delta^i\) is contained in \(D^{\leq i}_{\mathbb{A}(L)^{\times}}\).

Proof By the local nature of the assertion, it suffices to prove this lemma with \(\mathbb{A}(L)\) replaced by each open subscheme \(\mathbb{A}(L|U_\alpha)\times\). Moreover, since \(\Phi^*_a(D^{\leq i}_{\mathbb{A}(L|U_\alpha)}) \cong D^{\leq i}_{\mathbb{A}(L|U_\alpha)}\wedge\mathcal{W}_k^{\natural}\), it suffices to consider the case where \((S, \omega_S, \mathcal{O}_S^{\hbar})\) is taken to be \((\mathbb{A}^2\times, \omega_{\text{Weyl}}, \mathcal{W}_k^{2})\). Then, it follows from the definition of the multiplication in \(\mathcal{W}_k^{2}\) that for each \(a, b \in k[x, y]\), we have \(\delta^i_b(a) = \frac{1}{i!} \cdot \frac{\partial^i a}{\partial y^i} \cdot \frac{\partial^i b}{\partial x^i}\). This completes the proof of the lemma.

We shall finish the proof of the main theorem. Since \(\pi^*(\mathcal{O}_{\mathbb{A}(L)^{\times}})\) is naturally identified with \(\bigoplus_{j \in \mathbb{Z}} \mathcal{L}^{\otimes j}\), we have the inclusion

\[
\mathcal{L}^{\otimes(-i+1)} \hookrightarrow \pi^*(\mathcal{O}_{\mathbb{A}(L)^{\times}}) \tag{111}
\]

into the \((-i + 1)\)st factor. Next, if \(\pi\) denotes the natural projection \(\mathbb{A}(L)^{\times} \to X\), then the kernel of the surjection \(\mathcal{T}_{\mathbb{A}(L)^{\times}} \to \pi^*(\mathcal{T}_X)\) obtained by differentiating \(\pi\) is isomorphic to \(\pi^*(\mathcal{L})\). The resulting injection \(\pi^*(\mathcal{L}) \hookrightarrow \mathcal{T}_{\mathbb{A}(L)^{\times}}\) induces an injection

\[
\pi^*(\mathcal{L}^{\otimes i}) \hookrightarrow \mathcal{T}^{\otimes i}_{\mathbb{A}(L)^{\times}}. \tag{112}
\]

The projection formula gives the composite isomorphism

\[
\pi^*(\mathcal{L}^{\otimes i}) \cong \pi^*(\mathcal{O}_{\mathbb{A}(L)^{\times}}) \otimes \mathcal{L}^{\otimes i} \cong \left(\bigoplus_{j \in \mathbb{Z}} \mathcal{L}^{\otimes j}\right) \otimes \mathcal{L}^{\otimes i} = \bigoplus_{j \in \mathbb{Z}} \mathcal{L}^{\otimes(j+i)}. \tag{113}
\]

Hence, we have an injection

\[
\mathcal{L}^{\otimes(i+1)} \hookrightarrow \bigoplus_{j \in \mathbb{Z}} \mathcal{L}^{\otimes(j+i)} \overset{(113)}{\rightarrow} \pi^*(\mathcal{L}^{\otimes i}) \overset{(112)}{\rightarrow} \pi^*(\mathcal{T}^{\otimes i}_{\mathbb{A}(L)^{\times}}), \tag{114}
\]

where the first arrow is the inclusion into the first factor.

Proposition 4.4 Let \(i\) be an integer with \(1 < i < p\) and let \(\delta^i : \mathcal{O}_{\mathbb{A}(L)^{\times}} \to D^{\leq i}_{\mathbb{A}(L)^{\times}}\) be the morphism resulting from Lemma 4.3. Denote by \(\mathcal{D}^\bullet\) the projective connection for \(\mathcal{L}\) corresponding to \(S^{\natural}\) via the upper horizontal bijections in the diagram (67). Then, the following diagram is commutative:
In particular; (by considering the case of $i = 2$), the map $\star_X$ is injective.

Proof First, we shall consider the former assertion. Just as in the proof of Lemma 4.3, it suffices to consider the case where the triple $(X, L, S)$ is taken to be $(P^1, L_0, S^\circ)$. Let us identify $A(O_{P^1}(-1)) \times \mathbb{A}^{2n}$ with $A^2 \times \Phi_1$ via $\Phi_0$ (cf. the proof in Lemma 4.2 for the definition of $\Phi_0$). Over the open subsheme $U := \text{Spec}(k[u])$ of $P^1$, the composite $\Sigma \circ \delta_i$ sends each element $f(u) \cdot y^{i-1} \in \Gamma(U, \mathcal{O}_{P^1}(i-1))$ to

$$1 \cdot \frac{\partial}{\partial y} (f(u) \cdot y^{i-1}) \cdot \left( \frac{\partial}{\partial y} \right)^{\otimes i} = \frac{1}{y \cdot i!} \cdot \frac{\partial f}{\partial u^i}(u) \cdot \left( \frac{\partial}{\partial y} \right)^{\otimes i} \in \Gamma(\pi^{-1}(U), T^{\otimes i}_{\mathcal{L}}).$$

(116)

On the other hand, since the injection $\mathcal{O}_{P^1}(-i - 1) \to \pi_*(T^{\otimes i}_{\mathcal{L}})$ of (114) is given by $\frac{1}{y^{i+1}} \mapsto \frac{1}{y} \left( \frac{\partial}{\partial y} \right)^{\otimes i}$, the image of $\frac{1}{y^i} \cdot \xi^{2-i} u(D^{\bullet})_0(f(u) \cdot y^{i-1})$ via this injection coincides with $\frac{1}{y^i} \cdot \frac{\partial}{\partial u^i}(u) \cdot \left( \frac{\partial}{\partial y} \right)^{\otimes i}$ (cf. Remark 3.8), which is identical to (116). This implies the commutativity of (115), as desired. The latter assertion follows from the former assertion.

By the above results, we complete the proof of the main theorem.

5 A higher-dimensional variant of the main theorem

In this final section, we shall prove a higher dimensional variant of our main theorem (cf. Theorem 5.3 described later). That result may be thought of as a positive characteristic analogue of [6], Proposition 4.3.

5.1 Frobenius-Sp structures

Let $n$ be an integer with $n > 1$ and $S$ a smooth variety of dimension $2n - 1$.

Definition 5.1 A Frobenius-Sp structure (of level 1) on $S$ is a triple

$$S^{\otimes \circ} := (S^\circ, S^\otimes, \kappa)$$

consisting of a Frobenius-projective structure (of level 1) $S^\circ$ on $S$, an $(\text{Sp}_{2n})^F$-torsor $S^\otimes$ on $S$, and an isomorphism $\kappa : S^\circ \times \text{Sp}_{2n} \xrightarrow{\sim} S^\otimes$, where $S^\circ \times \text{Sp}_{2n} \xrightarrow{\sim} S^\otimes$. Springer
denotes the \((\text{PGL}_{2n})_S^F\) torsor induced from \(S^\otimes\) via change of structure group by the composite \(\text{Sp}_{2n} \hookrightarrow \text{GL}_{2n} \twoheadrightarrow \text{PGL}_{2n}\).

Hereinafter, we will observe that each Frobenius-Sp structure gives rise to a theta characteristic and an FC quantization on a certain symplectic variety.

First, let us consider a procedure for constructing a theta characteristic by means of a Frobenius-Sp structure. Let \(S^\otimes := (S^\otimes, \delta, \kappa)\) be a Frobenius-Sp structure on \(S\). The \((\text{SL}_{2n})_S^F\) torsor corresponding to \(S^\otimes\) via change of structure group by the natural inclusion \(\text{Sp}_{2n} \hookrightarrow \text{SL}_{2n}\) determines a vector bundle on \(S^{(1)}\) with trivial determinant. If \(F_S\) denotes its pull-back via \(F_S/k\), then \(F_S\) admits a canonical connection \(\nabla\) with vanishing \(p\)-curvature (cf. [11], § 5, Theorem 5.1). In particular, we have \((\text{det}(F_S), \nabla_{\text{det}(F_S)}) \cong (\mathcal{O}_S, d)\). Denote by \(E_S\) the \(\mathbb{P}^{2n-1}\)-bundle on \(S\) defined to be the pull-back of the \(\mathbb{P}^{2n-1}\)-bundle on \(S^{(1)}\) corresponding to \(S^\otimes\). The local sections \(U \to U \times \mathbb{P}^n\) (for various open subschemes \(U\) of \(S\)) classified by sections in \(S^\otimes\) may be glued together to obtain a well-defined global section \(\sigma_S : S \to E_S\). Then, there exists a unique line subbundle \(L_S\) of \(F_S\) such that the square diagram

\[
\begin{array}{c}
\mathbb{A}(L_S)^\times \\
\downarrow \text{projection} \\
S \\
\downarrow \sigma_S \\
E_S
\end{array}
\begin{array}{c}
\mathbb{A}(F_S)^\times \\
\downarrow \text{projection} \\
\end{array}
\]

is commutative and Cartesian, where the upper horizontal arrow denotes the morphism arising from the inclusion \(L_F \hookrightarrow F_S\). By the definitions of \(\sigma_S\) and \(L_S\), the \(\mathcal{O}_S\)-linear morphism

\[
\Omega_S^\vee \otimes L_S \sim \to F_S/L_S
\]

induced naturally by the composite

\[
L_S \hookrightarrow F_S \xrightarrow{\nabla_S} \Omega_S \otimes F_S \twoheadrightarrow \Omega_S \otimes (F_S/L_S)
\]

turns out to be an isomorphism. This isomorphism yields, via taking determinants, an isomorphism

\[
\omega_S^\vee \otimes \mathcal{L}_S^\otimes(2n-1) \ (= \text{det}(\Omega_S^\vee \otimes L_S)) \sim (\text{det}(F_S/L_S) =) \mathcal{L}_S^\vee,
\]

hence also, an isomorphism

\[
\psi_S : \mathcal{L}_S^\otimes2n \sim \to \omega_S.
\]
Thus, we obtain a theta characteristic

$$L := (L_S, \psi_S)$$

(123)
on S.

**Remark 5.2** Let us mention the case where $n = 1$, i.e., $S$ is a smooth curve. As discussed above, each Frobenius-Sp structure yields a theta characteristic. Conversely, suppose that we are given a theta characteristic $L$ and a projective structure $S\overset{\sim}{\rightarrow} S\overset{\sim}{\rightarrow}$. Denote by $F := (F, \nabla_F, N_F, \eta_F)$ the (SL$_2$, $L$)-oper corresponding to $S\overset{\sim}{\rightarrow}$. Since it has vanishing $p$-curvature, there exists a unique (SL$_2$, $L$)-torsor $S\overset{\sim}{\rightarrow} S\overset{\sim}{\rightarrow}$ such that the induced SL$_2$-torsor on $S$ equipped with a canonical connection is isomorphic to $(F, \nabla_F)$. Then, (since SL$_2 = Sp_2$) the triple consisting of $S\overset{\sim}{\rightarrow}$, $S\overset{\sim}{\rightarrow}$, and the natural isomorphism $\kappa : S\overset{\sim}{\rightarrow} \times SL_2 PGL_2 \sim \rightarrow S\overset{\sim}{\rightarrow}$, specifies a Frobenius-Sp structure on $S$. By this construction, we see that giving a Frobenius-Sp structure on a smooth curve $S$ is equivalent to giving a pair of a theta characteristic $L$ and a Frobenius-projective structure on $S$.

### 5.2 An FC quantizations arising from a Frobenius-Sp structure

Next, let us construct an FC quantization on a certain symplectic variety associated to $S\overset{\sim}{\rightarrow}$. We shall keep the above notation. By the definition of a Frobenius-Sp structure, there exists a collection

$$\{(U_\alpha, \gamma_\alpha)_{\alpha \in I}\},$$

(124)
of pairs $(U_\alpha, \gamma_\alpha)$ (indexed by a set $I$), where $(U_\alpha)_{\alpha \in I}$ is an open covering of $S$ and $\gamma_\alpha$ (for each $\alpha \in I$) denotes an $O_{U_\alpha}$-linear isomorphism $F_S|_{U_\alpha} \sim \rightarrow O_{U_\alpha}^{\otimes 2n}$ inducing, via taking determinants, the fixed isomorphism $det(F_S) \sim \rightarrow O_S$ (restricted to $U_\alpha$). Moreover, we can assume that for any pair $(\alpha, \beta) \in I \times I$ with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, the automorphism $\gamma_{\alpha\beta} := (\gamma_\alpha|_{U_{\alpha\beta}})^{-1} \circ (\gamma_\beta|_{U_{\alpha\beta}})$ of $O_{U_{\alpha\beta}}^{\otimes 2n}$ corresponds to a $U_{\alpha\beta}$-rational point of Sp$_{2n}$ ($\subseteq$ SL$_{2n}$). Let

$$\gamma_\alpha^A : \mathbb{A}(F_S|_{U_\alpha})^{\times} \sim \rightarrow \left(\mathbb{A}(O_{U_\alpha}^{\otimes 2n})^{\times} \sim \rightarrow \right) U_\alpha \times \mathbb{A}^{2n},$$

(125)

be the isomorphism induced by $\gamma_\alpha$, and let $\gamma_\alpha^P$ be the isomorphism $E_S|_{U_\alpha} \sim \rightarrow U_\alpha \times \mathbb{P}^{2n-1}$ obtained from $\gamma_\alpha^A$ via projectivization. Then, we have two composites

$$\Phi_\alpha : A(L_S|_{U_\alpha})^{\times} \xrightarrow{\text{inclusion}} A(F_S|_{U_\alpha})^{\times} \xrightarrow{\gamma_\alpha^A} U_\alpha \times \mathbb{A}^{2n} \xrightarrow{\text{pr}_1} \mathbb{A}^{2n},$$

$$\phi_\alpha : U_\alpha \xrightarrow{\sigma_S|_{U_\alpha}} E_S|_{U_\alpha} \xrightarrow{\gamma_\alpha^P} U_\alpha \times \mathbb{P}^{2n-1} \xrightarrow{\text{pr}_1} \mathbb{P}^{2n-1}.$$

(126)
Since $\phi_\alpha$ is étale and the square diagram

\[
\begin{array}{ccc}
\mathbb{A}(\mathcal{L}_S|U_\alpha)^x & \xrightarrow{\Phi_\alpha} & \mathbb{A}^{2n}\times \\
projection & & projection \\
U_\alpha & \xrightarrow{\phi_\alpha} & \mathbb{P}^{2n-1}
\end{array}
\]

is commutative and Cartesian, $\Phi_\alpha$ turns out to be étale. The pull-back $\Phi_\alpha^*(\omega^{\text{Weyl}})$ by $\Phi_\alpha$ specifies a symplectic structure on $\mathbb{A}(\mathcal{L}_S|U_\alpha)^x$. For each pair $(\alpha, \beta) \in I \times I$ with $U_{\alpha\beta} \neq \emptyset$, we shall denote by $\gamma_{\alpha\beta}^A$ the automorphism of $U_{\alpha\beta} \times \mathbb{A}^{2n}\times$ corresponding to $\gamma_{\alpha\beta}$. Since $\gamma_{\alpha\beta} \in \text{Sp}_2(U_{\alpha\beta})$, the equality $\gamma_{\alpha\beta}^A(\text{pr}_2^*(\omega^{\text{Weyl}})) = \text{pr}_2^*(\omega^{\text{Weyl}})$ holds, which implies that $\Phi_\alpha^*(\omega^{\text{Weyl}})|_{U_{\alpha\beta}} = \Phi_\beta^*(\omega^{\text{Weyl}})|_{U_{\alpha\beta}}$. Thus, the $\Phi_\alpha^*(\omega^{\text{Weyl}})$’s may be glued together to obtain a symplectic structure $\omega_S$ on $\mathbb{A}(\mathcal{L}_S)^x$. In particular, we have a symplectic variety

\[
(\mathbb{A}(\mathcal{L}_S)^x, \omega_S).
\]

Moreover, for each $\alpha \in I$, the pull-back $\Phi_\alpha^*(\mathcal{W}^{2n}_k)$ of $\mathcal{W}^{2n}_k$ by $\Phi_\alpha$ specifies an FC quantization on $(\mathbb{A}(\mathcal{L}_S|U_\alpha)^x, \Phi_\alpha^*(\omega^{\text{Weyl}}))$. It follows from an argument similar to the argument in § 4.3 (together with the homomorphism (37)) that $\Phi_\alpha^*(\mathcal{W}^{2n}_k)$ may be glued together to obtain an FC quantization $\mathcal{W}_S$

\[
\mathcal{W}_S
\]

on $(\mathbb{A}(\mathcal{L}_S)^x, \omega_S)$. Consequently, we have obtained the following assertion. (In the case $n = 1$, the symplectic structure $\omega_S$ coincides with $\omega_L$ and the asserted construction of FC quantizations is consistent with $\star_X, L$ mentioned in our main theorem.)

**Theorem 5.3** Let $n$ be a positive integer and $S$ a smooth variety of dimension $2n$. Then, by means of a Frobenius-$\text{Sp}$ structure $S^{2\infty}$ on $S$, we can construct canonically a theta characteristic $\mathbb{L} := (\mathcal{L}_S, \psi_S)$ on $S$, a symplectic structure $\omega_S$ on $\mathbb{A}(\mathcal{L}_S)^x$, and an FC quantization $\mathcal{W}_S$ on the resulting symplectic variety $(\mathbb{A}(\mathcal{L}_S)^x, \omega_S)$.

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