Positive entropy using Hecke operators at a single place

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Abstract

We prove the following statement: Let \( X = \text{SL}_n(\mathbb{Z}) \backslash \text{SL}_n(\mathbb{R}) \), and consider the standard action of the diagonal group \( A < \text{SL}_n(\mathbb{R}) \) on it. Let \( \mu \) be an \( A \)-invariant probability measure on \( X \), which is a limit

\[
\mu = \lambda \lim_{i} |\phi_i|^2 \, dx,
\]

where \( \phi_i \) are normalized eigenfunctions of the Hecke algebra at some fixed place \( p \), and \( \lambda > 0 \) is some positive constant. Then any regular element \( a \in A \) acts on \( \mu \) with positive entropy on almost every ergodic component. We also prove a similar result for lattices coming from division algebras over \( \mathbb{Q} \), and derive a quantum unique ergodicity result for the associated locally symmetric spaces. This generalizes a result of Brooks and Lindenstrauss [2].

1 Introduction

Let \( Y \) be a compact manifold of negative sectional curvature, and \( \phi_i \) be a sequence of normalized eigenfunctions of the Laplacian with eigenvalues \( \lambda_i \to \infty \). The Quantum Unique Ergodicity conjecture of Rudnick and Sarnak [10] asserts that the only weak-* limit of the measures \( \mu_i = |\phi_i|^2 \, d\text{vol}_Y \) is \( d\text{vol}_Y \). An important special case of this problem is the case of \( Y \) a compact quotient of the upper half plane with its usual hyperbolic metric. In fact, if \( G \) is a simple Lie group, \( K < G \) is a maximal compact, and \( \Gamma < G \) is any lattice, then QUE is conjectured to hold true for the locally symmetric space \( Y = \Gamma \backslash G/K \) (see e.g. [12, Problem 6.1]). In this case the \( \phi_i \) are assumed to

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be a sequence of normalized eigenfunctions of the ring of $G$-invariant differential operators on $G/K$, with the eigenvalues with respect to the Casimir operator tending to $\infty$ in absolute value. If $\Gamma$ is a congruence lattice, it is natural to consider sequences $\phi_i$ as above which are also eigenfunctions of the Hecke algebra of $Y$. In [8], Lindenstrauss made a significant progress and established QUE for such sequences $\phi_i$ on hyperbolic surfaces $\Gamma \backslash \mathbb{H}$. This is known as Arithmetic QUE. Lindenstrauss’ proof is based on his deep results on the dynamics of diagonal actions on $\Gamma \backslash G$. The link between QUE and dynamics on $X = \Gamma \backslash G$ follows from a general idea due to Shnirel’man. To each of the measures $\mu_i$ on $Y$ Shnirel’man constructed a lifting measure $\tilde{\mu}_i$ on the unit tangent bundle $SY$, called the microlocal lift, with the property that any limit of the $\tilde{\mu}_i$ is invariant under the geodesic flow. In the case of a hyperbolic surface $Y = \Gamma \backslash \mathbb{H}$, the unit tangent bundle is isomorphic to $X = \Gamma \backslash \text{PSL}_2(\mathbb{R})$, and under this isomorphism the geodesic flow is given by multiplying by a diagonal matrix on the right. Thus using an appropriate version of the microlocal lift, the problem is reduced to the following problem concerning invariant measures on $X$: Let $\phi_i$ be a sequence of normalized Hecke eigenfunctions on $X$. Suppose that $\mu$ is a weak-$\ast$ limit of the measures $\mu_i = |\phi_i|^2 dx$ on $X$, invariant under the diagonal action on $X$. Show that $\mu = dx$. In [2], Brooks and Lindenstrauss proved this statement assuming that the $\phi_i$ are eigenfunctions of only one Hecke operator, for certain co-compact lattices in $\text{SL}_2(\mathbb{R})$, coming from a quaternion algebra over $\mathbb{Q}$. Their main innovation is showing that for any such $\mu$ the diagonal group acts with positive entropy on almost every ergodic component. This generalizes the original positive entropy result of Bourgain and Lindenstrauss [1], which uses the full Hecke algebra. Brooks and Lindenstrauss suggested that their results might be generalized to the cases considered by Silberman and Venkatesh in their work on QUE in higher rank [12]. In this paper we generalize the techniques of Brooks and Lindenstrauss to higher rank. Let $G = \text{SL}_n(\mathbb{R})$, $K = \text{SO}(n)$, $\Gamma = \text{SL}_n(\mathbb{Z})$, $A < G$ the diagonal group, and fix a prime number $p$. The following theorem is our main result.

**Theorem 1.1.** Let $\mu$ be an $A$-invariant probability measure on $X$, which is a weak-$\ast$ limit

$$
\mu = \lambda \lim_i |\phi_i|^2 dx,
$$

where $\phi_i$ are normalized eigenfunctions of the Hecke algebra at some fixed place $p$, and $\lambda > 0$ is some positive constant. Then any regular element $a \in A$ acts on $\mu$ with positive entropy on almost every ergodic component.

Using the higher rank microlocal lift constructed by Silberman and
Venkatesh [13, Theorem 1.6], and the measure rigidity results of Einsiedler, Katok, and Lindenstrauss [3, Corollary 1.4], we conclude the following result.

**Theorem 1.2.** Assume $n$ is prime. Let $\phi_i \in L^2(\Gamma \backslash G/K)$ be a non-degenerate (in the sense of [13]) sequence of joint eigenfunctions of the ring of $G$-invariant differential operators on $G/K$ and the Hecke operators at some fixed place $p$. Then any weak-$*$ limit of the measures $|\phi_i|^2 \, dy$ is proportional to the uniform measure $dy$ on $\Gamma \backslash G/K$.

We note that this is a version of the main theorem of [12], except with the weaker hypotheses that the $\phi_i$ are eigenfunctions of the Hecke algebra at only one place. In fact, after some adjustments of our proof of Theorem 1.1, the result holds also when replacing the lattice $\text{SL}_n(\mathbb{Z})$ by lattices coming from division algebras over $\mathbb{Q}$, and the regular element $a$ by any non-trivial element of $A$ (see Theorem 5.1). Since these lattices are co-compact the constant $\lambda$ in Theorem 1.1 must be 1. The precise setting are as follows. Let $D$ be a division algebra of prime degree $n$ over $\mathbb{Q}$, which splits over $\mathbb{R}$ and over $\mathbb{Q}_p$, and Let $\mathcal{O} \subset D$ be a maximal order. Let $\Gamma < G := \text{SL}_n(\mathbb{R})$ be the lattice of all norm 1 elements of $\mathcal{O}$. Using Einsiedler and Katok’s theorem [4, Theorem 4.1] we obtain the following result.

**Theorem 1.3.** Let $\phi_i$ be a sequence of normalized eigenfunctions of the Hecke algebra at $p$ of $\Gamma \backslash G$, and suppose that $\mu$ is an $A$-invariant probability measure on $\Gamma \backslash G$ which is a weak-$*$ limit,

$$\mu = \lim_i |\phi_i|^2 \, dx.$$

Then $\mu = dx$, the normalized Haar measure on $X$.

In the notation above, consider the locally symmetric space $Y = \Gamma \backslash G/K$, where $K$ is the maximal compact subgroup $K = \text{SO}(n)$. Using the higher rank microlocal lift constructed by Silberman and Venkatesh [13, Theorem 1.6], we conclude the following QUE result.

**Theorem 1.4.** Let $\phi_i \in L^2(\Gamma \backslash G/K)$ be a non-degenerate (in the sense of [13]) sequence of joint eigenfunctions of the ring of $G$-invariant differential operators on $G/K$ and the Hecke operators at some fixed place $p$. Then the only weak-$*$ limit of $\phi_i$ is the standard uniform probability measure $dy$ on $Y$.

Note that in Theorem 1.2 we have not ruled out escape of mass. In fact, for sequences of eigenfunctions of the Hecke operators at a single place, escape of mass has not been ruled out even in the case where $n = 2$ (see [2]).
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2 Background on spherical functions on $p$-adic groups

In this section we review what we need from the theory of spherical functions on $p$-adic groups. We mostly follow [9]. Another standard reference is [11].

Notation. For two real functions $f, g$ with the same domain, we write $f \ll g$ if there is a positive constant $C$ so that $f \leq Cg$. We say that $f$ and $g$ are equivalent if $f \ll g$ and $g \ll f$. We write $f \ll_t g$ if the implied constant $C$ above depends on a parameter $t$. Fix a prime number $p$. Let $G = \text{SL}_n(\mathbb{R})$, $G_p = \text{SL}_n(\mathbb{Q}_p)$, and $K_p = \text{SL}_n(\mathbb{Z}_p)$. We denote by $l = n - 1$ the rank of $G$. Throughout, unless otherwise specified, our constants may depend on $G, G_p, K_p$ but not on anything else.

Fix a Haar measure on $G_p$ normalized so that its restriction to $K_p$ is a probability measure. Let $L(G_p, K_p)$ be the $\mathbb{C}$-algebra of all $K_p$-bi-invariant (hence continuous) compactly supported functions on $G_p$, the product in $L(G_p, K_p)$ is being defined by the convolution,

$$f \ast g(x) = \int_{G_p} f(xy)g(y^{-1})dy.$$ 

A complex valued function $\omega$ on $G_p$ is called a zonal spherical function (or just spherical function) on $G_p$ relative to $K_p$ if it is $K_p$-bi-invariant, $\omega(1) = 1$, and for all $f \in L(G_p, K_p)$, $\omega$ is an eigenfunction of the integral operator defined by $f$, i.e. we have

$$f \ast \omega = \lambda_f \omega,$$  

with $\lambda_f \in \mathbb{C}$. For a spherical function $\omega$ we denote the scalar $\lambda_f$ in (1) by $\hat{\omega}(f)$, i.e. we put

$$\hat{\omega}(f) = \int_{G_p} f(g)\omega(g^{-1})dg.$$ 

Then clearly $\hat{\omega}$ is a $\mathbb{C}$-algebra homomorphism from $L(G_p, K_p)$ onto $\mathbb{C}$. Moreover, the correspondence $\omega \mapsto \hat{\omega}$ is a bijection between spherical functions on
$G_p$ relative to $K_p$ and non-zero $\mathbb{C}$-algebra homomorphisms $L(G_p, K_p) \to \mathbb{C}$. Denote by $\Omega$ the set of all spherical functions on $G_p$ relative to $K_p$. For each $f \in L(G_p, K_p)$ its spherical transform $\hat{f} : \Omega \to \mathbb{C}$ is defined by $\hat{f}(\omega) = \omega(f)$.

We say that a spherical function $\omega$ is positive definite if $\omega(g_i g_j^{-1})$ is a positive definite matrix for any $g_1, \ldots, g_m \in G_p$, and we denote by $\Omega^+$ the set of all positive definite spherical functions on $G_p$ relative to $K_p$.

**Theorem 2.1** ([9, Theorem (1.5.1)]). There is a unique positive measure $\mu$ on $\Omega^+$ such that

1. If $f \in L(G_p, K_p)$ then $\hat{f} \in L^2(\Omega^+, \mu)$,
2. $\int_{G_p} |f(x)|^2 dx = \int_{\Omega^+} |\hat{f}|^2 d\mu(\omega)$ for all $f \in L(G_p, K_p)$.

Moreover, the mapping $f \mapsto \hat{f}$ extends to an isomorphism of Hilbert spaces $L^2(G_p, K_p) \to L^2(\Omega, \mu)$.

The measure $\mu$ is called the Plancherel measure on $\Omega^+$.

The algebra $L(G_p, K_p)$, as well as the spherical functions on $G_p$ and the Plancherel measure, can be described in terms of an affine root structure on $G_p$, which we describe below.

Let $V$ be the space of all vectors $v = (v_0, \ldots, v_l) \in \mathbb{R}^{l+1}$ with $\sum_{i=0}^l v_i = 0$, endowed with the inner product $\langle u, v \rangle = \sum_i u_i v_i$. Let $V^*$ be the dual of $V$. Then $V$ and $V^*$ are naturally isomorphic through $\langle \cdot, \cdot \rangle$. For each non-zero $a \in V^*$, let $a^\vee \in V$ be the image of $\langle 2a^2, a \rangle$ under this isomorphism. Denote by $e_i$ the $i$th coordinate function on $V$, and consider the root system

$$\Sigma_0 = \{ e_i - e_j \mid i \neq j \} \subset V^*,$$

as well as a fixed set $\Pi_0$ of simple roots $a_i := e_{i-1} - e_i$ ($i = 1, \ldots, l$). For each $a \in \Sigma_0$ and $k \in \mathbb{Z}$ let $a + k$ be the affine function on $V$ given by $v \mapsto a(v) + k$. Let

$$\Sigma = \{ a + k \mid a \in \Sigma_0, k \in \mathbb{Z} \}.$$

The elements of $\Sigma$ are called affine roots. For each $a \in \Sigma$ let $w_\alpha$ be the orthogonal reflection in the (affine) hyperplane on which $\alpha$ vanishes. The group $W$ generated by the $w_\alpha$ is an infinite group of affine transformations from $V$ to itself, called the affine Weyl group of $\Sigma$. Let $W_0$ be the subgroup of $W$ which fixes the point 0; this is the Weyl group of $\Sigma_0$. The set of all
translations in $W$ is a free abelian group $T$ of rank $l$, and $W$ is the semi-direct product of $T$ and $W_0$. For each $a \in \Sigma_0$, let

$$t_a = w_a \circ w_{a+1} \in T.$$  \hfill (2)

The $t_a$ ($a \in \Pi_0$) are a basis of $T$. We have $t_a(0) = a^\vee$ and the mapping $T \to V$ defined by $t \mapsto t(0)$ maps $T$ isomorphically onto the lattice $L$ spanned by $a^\vee$ ($a \in \Sigma_0$). It is easy to see that

$$L = \{(n_0, \ldots, n_l) \in \mathbb{Z}^{l+1} \mid \sum n_i = 0\}.$$  

Let $E_i \in \mathbb{R}^{l+1}$ ($i = 0, \ldots, l$) be the standard basis. We obtain another system of coordinates for $T$ by sending the basis $E_{i-1} - E_i$ ($i = 1, \ldots, l$) of $L$ to the standard basis of $\mathbb{Z}^l$. Namely, $(n_0, \ldots, n_l) \mapsto (m_1, \ldots, m_l)$, where

$$\begin{align*}
(n_0, \ldots, n_l) &= (m_1, m_2 - m_1, m_3 - m_2, \ldots, m_l - m_{l-1}, -m_l) \\
(m_1, \ldots, m_l) &= (n_0, n_0 + n_1, \ldots, n_0 + \cdots + n_{l-1}).
\end{align*}$$  \hfill (3)

Throughout we use both the coordinates $(n_0, \ldots, n_l)$ and the coordinates $(m_1, \ldots, m_l)$ for $T$ interchangeably.

Let $Z$ be the group of all diagonal matrices in $G_p$. We have a surjective homomorphism $$\nu : Z \to T,$$

which maps $\text{diag}(\lambda_0, \ldots, \lambda_l)$ to the translation by the vector $\sum_{i=0}^l v_p(\lambda_i)E_i \in V$. Here $v_p : \mathbb{Q}_p \to \mathbb{Z}$ is the standard $p$-adic valuation. Let

$$S = \text{Hom}(T, \mathbb{C}^\times).$$

Viewing $T$ as a quotient of $Z$ one constructs for each $s \in S$ a spherical function $\omega_s$, using induction of characters. See [9, (3.3)] for the details and the precise definition of $\omega_s$.

**Theorem 2.2 ([9, Theorem 3.3.12]).** Every spherical function is of the form $\omega_s$ for some $s \in S$. Furthermore, $\omega_s = \omega'_s$ if and only if $s = ws'$ for some $w \in W_0$.

Let $L(T)^{W_0}$ be the space of finitely supported, $W_0$-invariant functions on $T$.  

Theorem 2.3 (Satake Isomorphism, [9, Theorem 3.3.6]). Let \( f \in L(G_p, K_p) \). There exists a unique function \( \tilde{f} \in L(T)^{W_0} \) such that for any \( s \in S \),
\[
\tilde{\omega}_s(f) = \sum_{t \in T} \tilde{f}(t)s(t).
\]
The map \( f \mapsto \tilde{f} \) is an isomorphism of \( \mathbb{C} \)-algebras from \( L(G_p, K_p) \) to \( L(T)^{W_0} \).

We refer to \( \tilde{f} \) in Theorem 2.3 as the Fourier transform of \( f \). Let \( U^- < G_p \) be the group of all lower diagonal matrices with 1’s on the diagonal. Then \( U^- \) is nilpotent, and so it is unimodular. The diagonal group \( Z \) normalizes \( U^- \) and we have the following formula for the Jacobian of the action of \( Z \) on \( U^- \) by conjugation [9, Proposition (3.2.4)]. Let \( \Sigma_0^+ \) be the set of positive roots of \( \Sigma_0 \) (i.e. \( e_i - e_j \) with \( i < j \)) and \( r = \frac{1}{2} \sum_{a \in \Sigma_0^+} a \). For any \( t \in T \) and \( z \in \nu^{-1}(t) \), let \( \Delta(z) = \delta(t) = p^{2r(t(0))} \). Then
\[
\frac{dz u^{-1} z^{-1}}{du} = \Delta(z)^{-1}.
\]
In the notation above we have the following formula ([9, (3.3.4)]) for \( \tilde{f} \).
\[
\tilde{f} = \delta^{-\frac{1}{2}}(t) \int_{U^-} f(z u^-) du^-.
\]

Definition 2.4. For \( f \in L(G_p, K_p) \) define the adjoint of \( f \) to be \( f^*(g) = f(g^{-1}) \).

Lemma 2.5. For any \( f \in L(G_p, K_p) \), \( \tilde{f}(t) = (\tilde{f}^*)(-t) \).

Proof. We have by (4) that
\[
(\tilde{f}^*)(t) = \delta^{-\frac{1}{2}}(t) \int_{U^-} f^*(z u^-) du^- = \delta^{-\frac{1}{2}}(t) \int_{U^-} f((u^-)^{-1} z^{-1}) du^- = \delta^{-\frac{1}{2}}(t) \int_{U^-} f((u^-)^{-1} z^{-1}) du^-.
\]
Using the unimodularity of \( U^- \) this yields
\[
\int_{U^-} f((u^-)^{-1} z^{-1}) du^- = \int_{U^-} f(z^{-1} u^- z^{-1}) du^- = \Delta(z) \int_{U^-} f(z^{-1} u^-) du^-.
\]
and so
\[
(f^*)(t) = \delta^{\frac{\tau}{2}}(t) \int_{U^-} f(z^{-1}u^-) du^- = \delta^{-\frac{\tau}{2}}(-t) \int_{U^-} f(z^{-1}u^-) du^- = \tilde{f}(-t),
\]
as needed.

We denote by \( \hat{T} \) the set of all \( s \in S \) with \( |s(t)| = 1 \) \( \forall t \in T \). For \( s \in \hat{T} \), write \( s(t_{e_i-e_j}) = \xi_i^{-1} \xi_j \) \( (\xi_i \in \mathbb{C}) \), and let
\[
c(s) = \prod_{i<j} \frac{\xi_i - p^{-1} \xi_j}{\xi_i - \xi_j},
\]
the Harish-Chandra function, where \( t_{e_i-e_j} \) is as in (2).

**Theorem 2.6 ([9, Theorem (5.1.2)])**. The Plancherel measure \( \mu \) on \( \Omega^+ \) is concentrated on the set \( \{ \omega_s \mid s \in \hat{T} \} \), and up to normalization is given by
\[
d\mu(\omega_s) = \frac{ds}{|c(s)|^2}.
\]
Here \( ds \) is the probability Haar measure on the compact group \( \hat{T} \).

By convention we write integrals over \( \Omega^+ \) with respect to the Plancherel measure as follows: if \( h \) is a function on \( \Omega^+ \) we write
\[
\int_{\hat{T}} h(s)d\mu(s),
\]
meaning
\[
\int_{\hat{T}} h(s) \frac{ds}{|c(s)|^2}.
\]

We denote by \( Z^{++} \) the elements \( \text{diag}(\lambda_0, \ldots, \lambda_l) \in Z \) such that
\[
v_p(\lambda_0) = v_p(\lambda_1) \geq \cdots \geq v_p(\lambda_l),
\]
and its image under \( \nu \) by \( T^{++} \subset T \).

**Theorem 2.7 (Cartan decomposition, [9, Theorem (2.6.11)])**. \( G_p = K_p Z^{++} K_p \), and the mapping \( t \mapsto K_p \nu^{-1}(t) K_p \) is a bijection of \( T^{++} \) onto \( K_p \backslash G_p / K_p \).
We recall Macdonald’s formula for spherical functions. We first treat the case of regular elements of $S$. An element $s \in S$ is called regular if $s(t_a) \neq 1$ for all $a \in \Sigma_0$. Up to normalization we have for any $z_0 \in \mathbb{Z}^{++}$ and any regular $s \in S$,

$$\omega_s(z_0^{-1}) = \delta(t_0)^{-\frac{1}{2}} \sum_{w \in W_0} (ws, t_0)c(ws).$$  \hspace{1cm} (5)

Here $(ws, t_0)$ simply means $ws(t_0)$, and as before $t_0 = \nu(z_0)$. In view of the Cartan decomposition the formula above completely determines $\omega_s$. For a non-regular $s \in S$ a similar formula can be obtained by taking a limit.

Using the Cartan decomposition we obtain for every $g \in G_{\nu}$ a representative $t = t(g) \in T^{++}$. If $t(g)$ is a translation by an element $\sum_{i=0}^{l} n_iE_i \in L$ we say that $p^{-n_l}$ is the denominator of $g$ and we denote it by $d(g)$.

**Lemma 2.8** (Silberman–Venkatesh [12, Lemma 4.3]). $d(gg') \leq d(g)d(g')$ and $d(g^{-1}) \leq d(g)^{\ell}$.

For every $t \in T$ we denote by $\|t\|$ the norm of $t(0)$ coming from the inner product on $V$.

**Lemma 2.9.** There are constants $C, C' > 0$ such that for any $g \in G_{\nu}$ if we let $t = t(g) \in T^{++}$ be the representative of $g$ given by the Cartan decomposition then

$$p^{C'\|t\|} \leq d(g) \leq p^{C\|t\|}.$$

**Proof.** Since everything is defined by taking representatives in $T^{++}$ it suffices to show that there are $C, C'$ such that for every $t = \sum_{i} n_iE_i \in T^{++}$ with $n_0 \geq \cdots \geq n_l$,

$$C\|t\| \leq -n_l \leq C'\|t\|.$$

Consider the function on $V$ given by $v = \sum_{i} v_iE_i \in V \mapsto -\min\{v_i\}$. This function is equivalent to the norm on $V$, given by $v \mapsto \max\{|v_i|\}$. Thus identifying $T^{++}$ with a subset of $V$ we can extend $t = \sum_{i} n_iE_i \in T^{++} \mapsto -\min\{n_i\} = -n_l$ to a function which is equivalent to a norm on $V$. On the other hand $t \in T^{++} \mapsto \|t\|$ extends to a norm on $V$. Since any two norms on $V$ are equivalent, the result follows. \hfill $\Box$

For $t \in T$ with $t(0) = \sum_{i=1}^{l} m_i a_i^{\vee}$ we define the height of $t$ to be

$$ht(t) = \sum_{i} |m_i|.$$

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Lemma 2.10. Let \( t_1, \ldots, t_m \) be a set of generators of the semigroup \( T^{++} \). Then the length of \( t \in T^{++} \) with respect to \( t_1, \ldots, t_m \) is equivalent to \( \text{ht}(t) \). That is, there exist constants \( C, C' > 0 \) such that if \( t = t_{i_1} \cdots t_{i_k} \) (with minimal \( k \)) then

\[
C \text{ht}(t) \leq k \leq C' \text{ht}(t).
\]

Proof. Let \( M > 0 \) be such that the height of any of the \( t_i \)'s is less than \( M \). Then the height of \( t = t_{i_1} \cdots t_{i_k} \) is at most \( kM \). The other direction follows from the fact that the height is additive on \( T^{++} \) and the height of any of the \( t_i \)'s is at least 1. \( \square \)

Lemma 2.11. The height of \( t \in T^{++} \) is equivalent to \( m_1 \), where as before \( t(0) = \sum_{i=1}^{l} m_i a_i^\vee \).

Proof. Write \( t(0) = \sum_{i=0}^{l-1} n_i E_i \) with \( n_0 \geq n_1 \geq \cdots \geq n_l, \sum_i n_i = 0 \). Then

\[
t(0) = \sum_{i=0}^{l-1} (n_0 + \cdots + n_i)(E_i - E_{i+1}) = \sum_{i=1}^{l} (n_0 + \cdots + n_{i-1})a_i^\vee,
\]

so that \( m_i = n_0 + \cdots + n_{i-1} \) and \( \text{ht}(t) = ln_0 + (l - 1)n_1 + \cdots + n_{l-1} \). Thus, since \( n_0 = m_1 \),

\[
m_1 \leq \text{ht}(t) \leq (l + (l - 1) + \cdots + 1)m_1.
\]

\( \square \)

Definition 2.12. Let \( t \in T \) and suppose that \( t(0) = \sum_{i=1}^{l} m_i a_i^\vee \) with \( m_i \in \mathbb{Z} \). Define \( t \geq 0 \) if the first non-zero \( m_i \) is positive, and \( t_1 \geq t_2 \) if \( t_1 t_2^{-1} \geq 0 \).

It is easy to see that \( \leq \) is a total ordering on \( T \).

Lemma 2.13. Let \( t, t' \in T^{++} \). Suppose that \( t \leq t' \) and the lengths of \( t, t' \), with respect to a set of generators \( t_1, \ldots, t_m \) of the semigroup \( T^{++} \), are \( k, k' \). Then \( k \leq Ck' \) for some constant \( C \) (depending only on the set of generators \( t_1, \ldots, t_m \)).

Proof. Suppose that \( t(0) = \sum_{i=1}^{l} m_i a_i^\vee \) and \( t'(0) = \sum_{i=1}^{l} m'_i a_i^\vee \). Then \( m_1 \leq m'_1 \) and so the result follows from Lemma 2.11 and Lemma 2.10. \( \square \)
3 The propagation lemma

We keep the notation of the previous section. Let $\Gamma = \text{SL}_n(\mathbb{Z})$, and $\Gamma_p = \text{SL}_n(\mathbb{Z}^p)$. Identifying $\Gamma_p$ with its diagonal embedding in $G \times G_p$ we have the isomorphism
\[ \Gamma \backslash G \cong \Gamma_p \backslash G \times G_p / K_p, \] (6)
given by $\Gamma x \mapsto \Gamma_p(x, e)K_p$. Through this isomorphism $L(G_p, K_p)$ acts on $L^2(\Gamma \backslash G)$ by convolution on the second coordinate and in this way is viewed as an algebra of operators on $L^2(\Gamma \backslash G)$. In this section, we construct a convolution kernel $K_N \in L(G_p, K_p)$ which spectrally amplifies a given $L(G_p, K_p)$-eigenfunction, but still has small norm.

**Proposition 3.1.** Let $0 < \epsilon < 1$. For any sufficiently large $N \in \mathbb{N}$ (depending on $\epsilon$), and any $L(G_p, K_p)$-eigenfunction $\phi \in L^2(\Gamma \backslash G)$, there exists a self-adjoint kernel $K_N \in L(G_p, K_p)$ satisfying

1. $K_N$ is supported on elements with denominator at most $p^rN$, for some positive constant $r$ (not depending on $\epsilon$ and $\phi$).
2. There exists $\delta > 0$, depending only on $\epsilon$, such that 
   \[ ||K_N||_\infty \ll e^{-N\delta}. \]
3. $\phi$ has $K_N$-eigenvalue $\geq \frac{1}{\epsilon}$.
4. For every $\phi \in L^2(X)$ we have $\langle \phi \ast K_N, \phi \rangle \geq -||\phi||_2$.

The rest of this section is devoted to the proof of Proposition 3.1. For any $L, q \in \mathbb{N}$, let $g_{L,q}(k)$ be the function on $\mathbb{Z}$ defined by
\[ \sum_k g_{L,q}(k)e^{kz} = D_L(qz), \]
where $D_L(z) = \sum_{k=-L}^Le^{kz}$ is the Dirichlet kernel. Explicitly, if $q \neq 0$
\[ g_{L,q}(k) = \begin{cases} 1 & \text{if } q|k \text{ and } |k| \leq qL, \\ 0 & \text{otherwise}, \end{cases} \]
and if $q = 0$
\[ g_{L,0}(k) = \begin{cases} 2L + 1 & \text{if } k = 0, \\ 0 & \text{otherwise}. \end{cases} \]
We extend the definition of $g_{L,q}$ to $l$-tuples. If $q$ is a tuple $q = (q_1, \ldots, q_l) \in \mathbb{Z}^l$ let

$$g_{L,q}(m_1, \ldots, m_l) = \prod_{i=1}^l g_{L,q_i}(m_i),$$

and view it as a function on $T \equiv \mathbb{Z}^l$ (see (3)). We define an element $\tilde{f} \in L(T)^W_0$ by averaging $g_{L,q}$ over $W_0$, and subtracting a multiple of the identity to have $\tilde{f}(0) = 0$. Explicitly,

$$\tilde{f}_{L,q}(t) = \frac{1}{|W_0|} \sum_{w \in W_0} g_{L,q}(wtw^{-1}) - g_{L,q}(0) \delta_0(t). \quad (7)$$

**Lemma 3.2.** The support of $\tilde{f}_{L,q}$ consists of elements $t \in T$ with $\|t\| \ll \|q\|_\infty L$.

**Proof.** We claim that there is a positive constant $C$ such that for any $\|t\| > C\|q\|_\infty L$ and $w \in W_0$, $g_{L,q}(wtw^{-1}) = 0$. Indeed, we start with $w = e$. In this case, $g_{L,q}(t) = \prod_{i=1}^l g_{L,q_i}(m_i)$, where $t(0) = \sum_i m_i a_i^\vee$, and if this does not vanish it means that $|m_i| \leq q_i L$ ($i = 1, 2, \ldots, l$) and so $\|(m_1, m_2, \ldots, m_l)\|_\infty \leq \|q\|_\infty L$. By equivalence of norms $\|t\| \leq C\|q\|_\infty L$ for some $C > 0$. This proves the claim for the case where $w = e$. Since $W_0$ acts by isometries on $T$ the claim follows for any $w \in W_0$, which finishes the proof.

Fix $L \in \mathbb{N}$, and an $L(G_p, K_p)$-eigenfunction $\phi \in L^2(\Gamma \backslash G)$ with corresponding character $s \in S$. For any $N \in \mathbb{N}$ large enough in terms of $L$, we define $k_N \in L(G_p, K_p)$ via its Fourier transform as follows. Select $q_1 = q_1(N) \in \mathbb{N}$ such that $q := (q_1, 0, \ldots, 0) \in \mathbb{N}^l$ has the following properties.

1. $|\hat{\omega}_s(f_{L,q})| \geq L^l$,
2. $N \ll_L q_1 \ll \frac{N}{L}$.

The existence of such $q_1$ is established in Lemma 3.3 below. Finally define

$$\tilde{k}_N = \tilde{k}_{L,N} = \tilde{f}_{L,q} * \tilde{f}^*_{L,q} - \tilde{f}_{L,q} * \tilde{f}^*_{L,q}(0) \delta_0, \quad (8)$$

where $q = q(N)$ is as above.
Lemma 3.3. There exists \( q_1 = q_1(N) \in \mathbb{N} \) which satisfies properties (1) and (2) above.

Proof. To simplify the notation we denote \( f = f_{L,q} \), \( g = g_{L,q} \), and \( g_i = g_{L,q_i} \). We have for \( q = (q_1, 0, \ldots, 0) \),

\[
\hat{\omega}_s(f_{L,q}) = \sum_{t \in T} \hat{f}(t) s(t) = \frac{1}{|W_0|} \sum_{t \in T} \sum_{w \in W_0} g(w t w^{-1}) s(t) - (2L + 1)^{l-1},
\]

and by reversing the order of summation this is equal to

\[
\frac{1}{|W_0|} \sum_{w \in W_0} \sum_{t \in T} g(t) s(w^{-1} t w) - (2L + 1)^{l-1}. \tag{9}
\]

For each \( w \in W_0 \), \( t \mapsto s(w^{-1} t w) \) is given by

\[
(m_1, \ldots, m_l) \mapsto e^{\sum_i m_i z_i^{(w)}},
\]

for some \( z_1^{(w)}, \ldots, z_l^{(w)} \in \mathbb{C} \). In this notation, we have

\[
\sum_{t \in T} g(t) s(w^{-1} t w) = \sum_{m_1, \ldots, m_l \in \mathbb{Z}} g(m_1, \ldots, m_l) e^{\sum_i m_i z_i^{(w)}}
\]

\[
= \prod_{i=1}^l \sum_{m_i \in \mathbb{Z}} g_i(m_i) e^{m_i z_i^{(w)}}
\]

\[
= \prod_{i=1}^l D_L(q_i z_i^{(w)}).
\]

Thus

\[
\hat{\omega}_s(f_{L,q}) = \frac{1}{|W_0|} \sum_{w \in W_0} \prod_{i=1}^l D_L(q_i z_i^{(w)}) - (2L + 1)^{l-1}.
\]

Since \( q_i = 0 \) for \( i = 2, \ldots, l \) we have

\[
\hat{\omega}_s(f_{L,q}) = (2L + 1)^{l-1} \left( \frac{1}{|W_0|} \sum_{w \in W_0} D_L(q_1 z_1^{(w)}) - 1 \right).
\]

Let \( z_1^{(w)} = x_w + iy_w \). Using a quantitative version of Kronecker theorem (see Lemma B.1) we can find \( N \ll L q_1 \ll \frac{N}{L} \) such that

\[
|e^{q_1 i k y_w} - 1| \leq \frac{1}{2},
\]

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for all $|k| \leq L, w \in W_0$. The lemma now follows since for any such $q_1$ we have

$$\frac{1}{|W_0|} \sum_{w \in W_0} (D_L(q_1 z^{(w)}_1) - 1) \geq \frac{L}{2}.$$ 

\[ \square \]

**Lemma 3.4.** The support of $\tilde{k}_N$ consists of elements $t \in T$ with $N \ll_L \|t\| \ll N$. 

**Proof.** For any $t \in T$ we have

$$\tilde{f} * \tilde{f}^*(t) = \sum_{t' \in T} \tilde{f}(t' + t) \overline{\tilde{f}(t')}.$$ 

If this is not zero then there is some $t' \in T$ such that both $\tilde{f}(t') \neq 0$ and $\tilde{f}(t' + t) \neq 0$. Thus if $t = (m_1, \ldots, m_l)$, then each of the $m_i$’s is a multiple of $q_1$. Since at least one of the $m_i$’s is not zero it follows that $\|t\| \gg q_1 \gg L$. For the upper bound, by Lemma 3.2 we have $\|t'\|, \|t + t'\| \ll q_1 L$, and so $\|t\| \ll q_1 L \ll \frac{1}{L} L = N$. 

\[ \square \]

**Lemma 3.5.** Let $q = (q_1, 0, \ldots, 0)$. The support of $\tilde{f}_{L,q}$ is of size at most $O(L)$. 

**Proof.** Let $t \in T$, and assume that $\tilde{f}_{L,q}(t) \neq 0$. We claim that there exists a non-zero multiple of $q_1$, say $y \in \mathbb{Z}$, such that $t$ belongs to the orbit of $(y, 0, \ldots, 0)$ under $W_0$. Indeed, since $\tilde{f}(t) \neq 0$, $g_{L,q}(wtw^{-1}) \neq 0$ for some $w \in W_0$. Let $wtw^{-1} = (m_1, \ldots, m_l)$, so that $g_{L,q}(m_i) \neq 0$ for all $i = 1, \ldots, l$. Since $q_i = 0$ for $i \geq 2$, we have also that $m_i = 0$ for all $i \geq 2$. Hence $wtw^{-1} = (m_1, 0, \ldots, 0)$, $m_1 \neq 0$, and $q_1 | m_1$, as claimed. Write $y = q_1 y'$, then we must have $|y'| q_1 \leq q_1 L$ and so $|y'| \leq L$. Since there are $2L$ such $y'$, the support is of size at most $|W_0| 2L$. 

\[ \square \]

**Lemma 3.6.**

1. $\tilde{f}_{L,q} * \tilde{f}_{L,q}^*(0) \ll L^{2l-1},$

2. For any $L(G_p, K_p)$-eigenfunction in $L^2(\Gamma \backslash G)$ with corresponding $s' \in S$,

$$\hat{\omega}_{s'}(k_N) \gg -L^{2l-1}.$$ 

3. $\hat{\omega}_s(k_N) \gg L^{2l}.$
Proof. We have
\[ \tilde{f} \ast \hat{\tilde{f}}(0) = \sum_{t \in T} |\tilde{f}(t)|^2. \]

If \( t = (m_1, \ldots, m_l) \) is such that \( \tilde{f}(t) \neq 0 \) then at least one of the \( m_i \)'s is not zero and so \( g_{L,q}(t) \leq (2L + 1)^{l-1} \). Similarly \( g_{L,q}(wtw^{-1}) \leq (2L + 1)^{l-1} \) for each \( w \in W_0 \) and so \( \tilde{f}(t) = \frac{1}{|W_0|} \sum_{w \in W_0} g_{L,q}(wtw^{-1}) \leq (2L + 1)^{l-1} \).

Combining this with Lemma 3.5 the first assertion of the lemma follows. The second assertion follows from (1) and from the fact that \( f \ast f^* \) is a positive operator on \( L^2(\Gamma \backslash G) \), and the third assertion follows by combining (1) with \( |\hat{\omega}_s(f_{L,q})| \gg L^l \).

\[ \tag{10} \]

3.1 Geometric properties of \( k_N \)

We start by bounding \( k_N \). We have
\[
k_N(x) = k_N \ast \chi_K(x) \]
\[
= k_N \ast \int_T \omega_s(x) d\mu(s) \]
\[
= \int_T \hat{\omega}_s(k_N) \omega_s(x) d\mu(s) \]
\[
= \int_T \sum_{t \in T} s(t) k_N(t) \omega_s(x) d\mu(s) \]
\[
= \sum_{t \in T} k_N(t) \int_T s(t) \omega_s(x) d\mu(s). \]

Lemma 3.7. There exist \( C > 0 \) and \( \kappa > 0 \) such that for any \( x \in G_p \) we have
\[
| \int_T s(t) \omega_s(x) d\mu(s) | \leq C e^{-\kappa|t|}. \]

Proof. Let \( s \in \hat{T} \) be a non-singular character. Then up to a positive constant we have
\[
\omega_s(x) = \omega_s(z_0^{-1}) = \delta(t_0)^{-\frac{1}{2}} \sum_{w \in W_0} (ws, t_0) c(ws), \]
for every \( z_0 \in Z^{++} \) and \( t_0 = \nu(z_0) \). If \( s_0 \) is singular, then \( \omega_{s_0}(x) \) is just the limit \( \lim_{s \to s_0} \omega_s(x) \) over regular \( s \)'s. Since the limit exists we continue to write the same formula for singular characters as well. By Theorem 2.6 we
have

\[ \int_{\mathcal{T}} s(t) \sum_{w \in W_0} (ws, t_0) c(ws) d\mu(s) = \int_{\mathcal{T}} s(t) \sum_{w \in W_0} (ws, t_0) c(ws) \frac{1}{|c(s)|^2} ds. \]

Recall that $|c(s)|^2$ is $W_0$-invariant, i.e. $|c(s)|^2 = c(ws)c(ws^{-1})$. Thus for each $w \in W_0$ we have

\[ \int_{\mathcal{T}} s(t)(ws, t_0) c(ws) \frac{1}{c(ws^{-1})} ds = \int_{\mathcal{T}} s(t)(ws, t_0) \frac{1}{c(ws^{-1})} ds. \]

The function $s \mapsto (ws, t_0) \frac{1}{c(ws^{-1})}$ is real analytic on $\hat{T}$. That is, viewing it as a function on the torus $(\mathbb{R}/\mathbb{Z})^n$, it can be extended holomorphically to a $\kappa_w$-neighborhood $U$ of $(\mathbb{R}/\mathbb{Z})^n$ in $(\mathbb{C}/\mathbb{Z})^n$ for some positive constant $\kappa_w$, which is clearly independent of $t_0$. Thus by the Paley-Wiener Lemma on exponential decay of Fourier coefficients we have

\[ |\int_{\mathcal{T}} s(t)(ws, t_0) c(ws) \frac{1}{c(ws^{-1})} ds| \leq C_w(t_0) e^{-\kappa_w \|t\|}, \]

where $C_w(t_0) = \sup_{s \in U} (s \mapsto (ws, t_0) \frac{1}{c(ws^{-1})})$. Taking $\kappa_w$ small enough, we can arrange that $\delta(t_0)^{-\frac{1}{2}}C_w(t_0)$ is uniformly bounded in terms of $t_0$, i.e. in terms of $x$, and so

\[ |\delta(t_0)^{-\frac{1}{2}} \int_{\mathcal{T}} s(t)(ws, t_0) \frac{1}{c(ws^{-1})} ds| \ll e^{-\kappa_w \|t\|}. \]

The result now follows since our integral is just a finite sum of such expressions.

**Corollary 3.8.** There exists a positive constant $\delta = \delta(L) > 0$ such that for all $x \in G_p$ and $N$ large enough,

\[ |k_N(x)| \leq e^{-\delta N}. \]

**Proof.** By Lemma 3.4 we have that $\tilde{k}_N(t) \neq 0$ only if $C'N \leq \|t\| \leq CN$ for some constants $C', C > 0$ (depending on $L$). Note that the number of such $t \in T$ is bounded (up to a constant) by $N^l$. By Lemma 3.7 we have that for any such $t \in T$

\[ |\int_{\mathcal{T}} s(t)\omega_s(x) d\mu(s)| \ll e^{-\kappa \|t\|} \leq e^{-\kappa C'N}, \]

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for some constant $\kappa > 0$. Since $\tilde{k}_N(t) \ll L N$ we have by (10) that
\[
|k_N(x)| \ll L N^{t+1} e^{-C'N} ,
\]
and the result follows by absorbing the polynomial coefficient $N^{t+1}$ into the exponent. \qed

Next we estimate the support of $k_N$.

Let $p_t : \hat{T} \to \mathbb{C}$ be defined by
\[
p_t(s) = \sum_{w \in W_0} (ws, t).
\]
We have
\[
k_N(x) = \sum_{t \in T/W_0} \sum_{w \in W_0} \tilde{k}_N(wt) \int_{\hat{T}} (ws, t) \omega_s(x) d\mu(s)
\]
\[
= \sum_{t \in T/W_0} \tilde{k}_N(t) \int_{\hat{T}} p_t(s) \omega_s(x) d\mu(s). \tag{11}
\]

**Lemma 3.9.** Given any $t_0 \in T$ there exists $k_{t_0} \in L(G_p, K_p)$ such that for all $s \in \hat{T}$
\[
k_{t_0} * \omega_s = p_{t_0}(s) \omega_s.
\]

**Proof.** By the Satake isomorphism it is enough to show that there is an element $\tilde{k}_{t_0} \in L(T)^{W_0}$ such that for all $s \in \hat{T}$ we have
\[
p_{t_0}(s) = \sum_t \tilde{k}_{t_0}(t) s(t).
\]
It is clear that $\tilde{k}_{t_0}(t) = \delta_{t \in W_0t_0}$ (here $W_0t_0$ is the orbit of $t_0$ under the action of $W_0$) is such an element. \qed

Following [9], we denote the element $\tilde{k}_t = \delta_{W_0t} \in L(T)^{W_0}$ from Lemma 3.9 by $\langle t \rangle$. We continue to denote by $k_t$ the element of $L(G_p, K_p)$ that corresponds to $\langle t \rangle$. Fix a set of generators $t_1, \ldots, t_m$ of the semigroup $T^{++}$ such that $t_1 \leq \cdots \leq t_m$ (see Definition 2.12) and such that $t_1, \ldots, t_m$ are the first $m$ elements of $T^{++}$.

**Lemma 3.10.** For each $t \in T^{++}$ there is a polynomial $P_t$ in $m$ variables, of total degree $\ll ||t||$ such that $k_t = P_t(\chi_{t_1}, \ldots, \chi_{t_m})$. 17
Proof. We shall use the following two facts (see e.g. [9, (3.3)]):

1. For any \( t_1, t_2 \in T^+ \) we have
   \[
   \langle t_1 \rangle \ast \langle t_2 \rangle = \langle t_1 t_2 \rangle + \sum_{t' \in T^+, t'< t_1 t_2} c t' \langle t' \rangle.
   \]

2. For any \( t \in T^+ \) we have
   \[
   \tilde{\chi}_t = \delta(t) \tilde{\chi} (t) + \sum_{t' \in T^+, t'< t} \tilde{\chi}(t') \langle t' \rangle.
   \]

Let \( t = t_{i_1} \cdots t_{i_k} \in T^+ \). It follows from the first fact above that
\[
\langle t \rangle = \langle t_{i_1} \rangle \ast \cdots \ast \langle t_{i_k} \rangle - \sum_{t' \in T^+, t'< t} c t' \langle t' \rangle,
\]
and so by induction and by Lemma 2.13 we have that \( \langle t \rangle \) is a polynomial in the \( \langle t_i \rangle \)'s with degree \( \ll k \). Applying the second fact above to \( t = t_i \) we have that each \( \langle t_i \rangle \) is a linear polynomial in \( \tilde{\chi}_{t_{i_1}}, \ldots, \tilde{\chi}_{t_{i_m}} \). Thus \( \langle t \rangle \) is a polynomial of degree \( \ll k \) in \( \tilde{\chi}_{t_{i_1}}, \ldots, \tilde{\chi}_{t_{i_m}} \). Thus by Lemma 2.10 and Lemma 2.13 it follows that if \( t(0) = \sum_{i=1}^l m_i(t) a_i \) then the degree is \( \ll m_1(t) \). But \( t \in T^+ \mapsto m_1(t) \) is equivalent to \( t \mapsto \|t\| \), and so the degree is \( \ll \|t\| \). The result now follows by the Satake isomorphism.

Note that the implied constant in Lemma 3.10 depends on the choice of our set of generators \( t_1, \ldots, t_m \). However we think of this set of generators as being fixed and so we still view the implied constant in this lemma as an absolute constant.

Lemma 3.11. Let \( z_0 \in Z^+ \) and \( t_0 = \nu(z_0) \). If \( \|t\| \ll \|t_0\| \) then
\[
\int_T p_t(s) \omega_s(z_0^{-1}) d\mu(s) = 0,
\]

Proof. We have
\[
\int_T p_t(s) \omega_s(z_0^{-1}) d\mu(s) = \int_T k_t \ast \omega_s(z_0^{-1}) d\mu(s)
= \int_{G_p} k_t(z_0^{-1} g) \int_T \omega_s(g^{-1}) d\mu(s) dg
= k_t \ast \chi_0(z_0^{-1}) = k_t(z_0^{-1}).
\]
By Lemma 3.10 we have that $k_t$ is a polynomial of degree $\ll \|t\|$ in $\chi_t_1, \ldots, \chi_t_m$, where $t_1, \ldots, t_m$ is a fixed choice of generators of the semigroup $T^{++}$. For simplicity write $\chi_{t_i} = \chi_i$. Let $\chi_i \cdots \cdots \chi_{id}$ one of the monomials of our polynomial. We claim that $\chi_i \cdots \cdots \chi_{id}(z_0^{-1}) \neq 0$ only if $z_0^{-1} \in \prod_{j=1}^d K t_i, K$. Indeed, we have

$$
\chi_i \cdots \cdots \chi_{id}(z_0^{-1}) = \int_{G_p} \chi_i \cdots \cdots \chi_{id-1}(g^{-1}) \chi_{id}(z_0^{-1}) dg.
$$

For the integrand to not vanish we need $z_0^{-1} g \in K t_i, K$ and $\chi_i \cdots \cdots \chi_{id-1}(g^{-1}) \neq 0$. Thus by induction $g^{-1} \in \prod_{j=1}^{d-1} K t_i, K$ and so $z_0^{-1} \in \prod_{j=1}^d K t_i, K$ as needed. Thus if $\chi_i \cdots \cdots \chi_{id}(z_0^{-1}) \neq 0$ we have that

$$
p^{C\|t_0\|} \leq d(z_0^{-1}) \leq \prod_{j=1}^d d(t_{ij}) \leq p^{C(\sum_{j=1}^d \|t_{ij}\|)} ,
$$

by Lemma 2.8 and Lemma 2.9, and so

$$
\|t_0\| \ll \sum_{j=1}^d \|t_{ij}\| \ll d \ll \|t\|.
$$

Thus we have that if $\|t\| \ll \|t_0\|$ then $\chi_t_1, \ldots, \chi_t_m(z_0^{-1}) = 0$. Since this holds for each of the monomials we have that $k_t \chi_0(z_0^{-1}) = 0$ whenever $\|t\| \ll \|t_0\|$, as needed.

**Corollary 3.12.** $k_N$ is supported on elements $x \in K_p z_0 K_p$ ($z_0 \in \mathbb{Z}^{++}$) with $\|\nu(z_0)\| \ll N$.

**Proof.** By Lemma 3.4 there exists $C > 0$ such that $\hat{k}_N(t) = 0$ whenever $\|t\| \geq CN$. By Lemma 3.11 there exists $C' > 0$ such that $\int p_t(s) \omega_s(x) d\mu(s) = 0$ whenever $\|\nu(z_0)\| > C' \|t\|$. Thus by (11), we have that $k_N(x) = 0$ whenever $\|\nu(z_0)\| \geq CC'N$.

**Proof of Proposition 3.1.** Let $0 < \epsilon < 1$ and $\phi \in L^2(\Gamma \backslash G)$ an $L(G_p, K_p)$-eigenfunction with character $s \in S$. Let $k_N = k_{L,N}$ be the corresponding kernel as defined in (8). By Lemma 3.6 there exists $L \in \mathbb{N}$ such that

$$
[\tilde{f} \ast \tilde{f}^s(0)]^{-1} \omega_s(k_N) > \frac{1}{\epsilon},
$$

whenever $N$ is large enough in terms of $\epsilon$. Let $K_N = [\tilde{f} \ast \tilde{f}^s(0)]^{-1} k_N$. Then $K_N$ satisfies the last two properties by construction. The first and second properties follow by Corollary 3.12, Corollary 3.8 respectively.
4 Proof of Theorem 1.1

4.1 The partition $\mathcal{P}$

We keep denoting $G = \text{SL}_n(\mathbb{R})$, $\Gamma = \text{SL}_n(\mathbb{Z})$, and $X = \Gamma \backslash G$. We fix a left-invariant Riemannian metric on $G$. If $U \subset G$ is any subset we denote by $\overline{U}$ its image in $X$ under the natural projection. We say that an element $a$ of the diagonal group $A$ is regular if it has distinct entries. We consider measures on $X$ invariant under the action of such $a$. If $\mathcal{P}$ is a partition of $X$, let $\mathcal{P}_N$ be the (symmetric) $N$-th refinement of $\mathcal{P}$ under the action of $a$:

$$\mathcal{P}_N := \bigvee_{i=-N}^{N} a^i \mathcal{P}.$$ 

As before we denote $G_p = \text{SL}_n(\mathbb{Q}_p)$, $K_p = \text{SL}_n(\mathbb{Z}_p)$ and we consider the action of the Hecke algebra $L(G_p, K_p)$ on $L^2(X)$ through the double quotient decomposition (6). In this section we prove the following lemma.

**Lemma 4.1.** Let $a \in A$ be a regular element, and $\mu$ an $a$-invariant probability measure on $X$. For any compact identity neighborhood $\Omega \subset G$ and $r > 0$ there exist a partition $\mathcal{P}$ of $X$, and a sequence of identity neighborhoods $B_N \subset G$ satisfying:

1. There exists $c \in \mathbb{N}$ depending only on $r$ such that the intersection of $\overline{\Omega}$ with any element of $\mathcal{P}_{cN}$ is contained in a translate $xB_N$, $x \in \Omega$.

2. For any $x, y \in \Omega$, the number of cosets $bK_p$ with denominator $\leq p^{rN}$ such that $xB_N \cap yB_N \neq \emptyset$ is at most $N^{O_r(1)}$.

Here the implied constants may depend on $r$ and $\Omega$, but not on $N$.

Following [12], our dynamical balls $B_N$ will be thickened compact pieces of $A$. Given a compact neighborhood of the identity $C \subset A$ let $B(C, \epsilon)$ be an $\epsilon$-neighborhood of $C$ inside $G$. The proof of Lemma 4.1 is based on the following three results.

**Lemma 4.2 (Silberman–Venkatesh [12], Lemma 4.4).** Let $\Omega \subset G$ be a compact neighborhood of the identity. For $c > 0$ sufficiently large, depending only on $n$, and $\epsilon' > 0$ sufficiently small, depending on $\Omega$, for any $g \in \Omega$ the set of $\gamma \in \Gamma_p$ such that

$$\inf \{ \text{dist}(\gamma, t) \mid t \in g(A \cap \Omega^{-1})g^{-1} \} \leq \epsilon, d(\gamma) \leq M$$
is contained in a \( \mathbb{Q} \)-torus \( T < SL_n(\mathbb{Q}) \), provided that \( \epsilon M^c \leq c' \)

**Lemma 4.3** ([5], (7.51)). Let \( a \) be an element of \( A \) and \( \mu \) an \( a \)-invariant probability measure on \( X \). Let \( F \subset X \) a compact subset, and \( \delta > 0 \). There exists a countable partition \( \mathcal{P} \) of \( X \) with finite entropy, containing \( X \setminus F \) as one of its elements, satisfying the following property. For every element \( E \subset F \) in the \( N \)-th refinement \( \mathcal{P}_N \) of \( \mathcal{P} \), there exists \( x \in F \) so that up to a \( \mu \)-null set,

\[
E \subset x \bigcap_{k=-N}^{N} a^{-k}B_G^G a^k.
\]

Here \( B_G^G \subset G \) is the open ball of radius \( \delta \) around the identity in \( G \).

**Lemma 4.4.** Let \( N \in \mathbb{N} \) and \( \delta > 0 \) which is small enough in terms of \( a \). Then there exists some \( \alpha > 0 \), depending only on \( a \) such that

\[
\bigcap_{k=-N}^{N} a^{-k}B_G^G a^k \subset B(C, \kappa e^{-\alpha N}),
\]

for some \( \kappa \ll \delta \), and \( C \subset A \) some compact subset with diameter \( \ll \delta \).

For the proof of Lemma 4.4 we use the following notation. Define \( G^- \) to be the set of all elements \( g \in G \) such that \( a^i g a^{-i} \to e \) as \( i \to \infty \) and \( G^+ \) the set of all elements \( g \in G \) such that \( a^{-i} g a^i \to e \). Then \( G^+ \) and \( G^- \) are closed subgroups of \( G \), normalized by \( a, a^{-1} \). If \( g^+ \) and \( g^- \) are the Lie algebras corresponding to \( G^+ \) and \( G^- \) and \( a \) is the Lie algebra corresponding to the centralizer of \( a \) (which is in our case the Lie algebra of \( A \)) then

\[
g = a \oplus g^+ \oplus g^-.
\]

**Proof of Lemma 4.4.** Let \( \theta : G 	o G \) be conjugation by \( a \). Let \( U \subset g \) be a neighborhood of \( 0 \in g \) such that \( \exp|U \) is a diffeomorphism onto its image. Let \( \delta > 0 \) and \( U' \subset U \) such that \( \theta^\pm 1 U' \subset U \) and \( (\exp|_U)^{-1}(B_G^G) = U' \). Let \( g \in \cap_{-N}^{N} \theta^B G \). We claim that \( g = \exp(\theta^N X_N) \) for some \( X_N \in U' \) such that \( \theta^N X_N \in U' \). Indeed, by induction assume that we have

\[
g = \exp(X_0) = \exp(\theta X_1) = \cdots = \exp(\theta^{N-1} X_{N-1}),
\]

for \( X_i \in U' \) with \( \theta^i X_i \in U' \), as above. Since \( g \in \cap_{-N}^{N} \theta^i B_G^G \) there exists \( X_N \in U' \) so that \( g = \exp(\theta^N X_N) \). We have \( g = \exp(\theta^N X_N) = \exp(\theta^{N-1} X_{N-1}) \) and so \( \exp(\theta X_N) = \exp(X_{N-1}) \in B_G^G \). On the other hand we have that
$\theta X_N \in U$, since $X_N \in U'$, and so $\theta X_N \in U'$. Similarly, $\exp(\theta^N X_N) = \exp(\theta^{N-2} X_{N-2})$, and so $\exp(\theta^2 X_N) \in B^G_\delta$. But $\theta X_N \in U'$, so that $\theta^2 X_N \in U$, and so $\theta^2 X_N \in U'$. By applying this argument inductively it follows that $\theta^N X_N \in U'$.

We can write $g$ in the following three forms:

$$g = \exp(H_0 + X_0 + Y_0)$$
$$= \exp(\theta^N(H_N + X_N + Y_N))$$
$$= \exp(\theta^{-N}(H_{-N} + X_{-N} + Y_{-N})),$$

where $H_i \in a, X_i \in g^-, Y_i \in g^+$, and $H_i + X_i + Y_i \in U'$, $\theta^i(H_i + X_i + Y_i) \in U'$ $(i = \pm N, 0)$. Since both $a$ and $a^{-1}$ normalize both $G^+$ and $G^-$ and by the injectivity of $\exp|_U$ we have that $X_0 = \theta^N X_N$ and that $Y_0 = \theta^{-N} Y_{-N}$. Thus there are constants $c, \alpha > 0$ depending only on $a$ such that $||X_0||, ||Y_0|| \leq c\delta e^{-\alpha N}$, and $||H_0|| \leq c\delta$ (see e.g. [5], Lemma 7.29). Thus $H_0 + X_0 + Y_0$ belongs to a $c\delta e^{-\alpha N}$-neighborhood of a $c\delta$-neighborhood of $a$ (where $c'$ is some positive constant depending only on $a$). The claim in the level of the group $G$ follows from the fact that $\exp$ is $c''$-Lipschitz, for some $c'' > 0$, on $U$.

Proof of Lemma 4.1. Let $\delta > 0$. By Lemma 4.3 there exists a partition $P$ containing $X \setminus \Omega$ so that for every $E \in P_N$, $E \subset \Omega$, there exists $x \in \Omega$ such that

$$E \subset x \bigcap_{k=-N}^{N} \theta^k B^G_\delta.$$

Here $\theta$ stands for conjugation by $a$. Taking $\delta$ small enough to satisfy Lemma 4.4 it follows that there exists $\alpha > 0$ depending only on $a$ so that

$$\bigcup_{k=-N}^{N} \theta^k B^G_\delta \subset B(C, \kappa e^{-\alpha N}),$$

where $\kappa \ll \delta$ and $C \subset A$ with diameter $\ll \delta$. Thus we have a partition $P = P(\delta)$, and $\alpha > 0$ depending only on $a$ such that for every $c \in \mathbb{N}$, for every $E \subset \Omega$ in the $cN$-th refinement of $P$,

$$E \subset x B(C, \kappa e^{-\alpha c N})$$

for some $x \in \Omega, C \subset A$ of diameter $\ll \delta$, and $\kappa \ll \delta$. Thus it suffices to show that $\delta$ and $c$ can chosen in a way that if we define $\epsilon_N := \kappa e^{-\alpha c N}$ then

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for any $x, y \in \Omega$, 
\[
xB(C, \epsilon_N)b \cap yB(C, \epsilon_N) \neq \emptyset
\]
for at most $N^{O_r(1)}$ cosets $bK_p$ with $d(b) \leq p^N$. Suppose that the intersection above is non-empty for such $b_1K_p, \ldots, b_kK_p$. Denote $B = B(C, \epsilon_N)$. Then there are $\gamma_i \in \Gamma$ with 
\[
\gamma_ixB_i \cap yB \neq \emptyset
\]
in $G \times G_p/K_p$, and we may assume that $\gamma_ib_i = 1$. So we have $\gamma_i \in yBB^{-1}x^{-1}$ and $d(\gamma_i) \leq p^{O(rN)}$. Let $s_i = \gamma_i^{-1}\gamma_i (i = 2, \ldots, n)$, then $s_i \in yBB^{-1}B^{-1}By^{-1} \cap \Gamma_p$. Taking $c$ large enough it follows from Lemma 4.2 that $s_i$ are all lying on $T \cap \text{SL}_n(\mathbb{Z}[\frac{1}{p}])$, for some torus $T < \text{SL}_n(\mathbb{Q})$.

On the other hand the $s_i$ still have denominator $\leq p^{O(rN)}$. Thus taking $\delta$ sufficiently small (not depending on $T$) the result follows by Lemma 4.5 below.

**Lemma 4.5.** Let $T < \text{SL}_n(\mathbb{Q})$ be a torus (i.e. a commutative subgroup all of its elements are diagonalizable over $\mathbb{C}$), and $U \subset G$ an identity neighborhood, small enough so that the eigenvalues of each element of $UU^{-1}$, lie at distance at most $\frac{1}{4}$ from 1. Then there are at most $N^{O(1)}$ elements $s \in U \cap T \cap \text{SL}_n(\mathbb{Z}[\frac{1}{p}])$ with denominator $d(\gamma) \leq p^N$.

**Proof.** Let $L$ be the splitting field of $T$ over $\mathbb{Q}$ and $\mathcal{O}_L$ its ring of integers. By diagonalizing $T(\mathbb{Z}[\frac{1}{p}]) := T \cap \text{SL}_n(\mathbb{Z}[\frac{1}{p}])$ simultaneously over $L$, we have an embedding of $T(\mathbb{Z}[\frac{1}{p}])$ into $(L^\times)^n$, sending $s$ to its diagonal form $(\lambda_1, \ldots, \lambda_n)$.

Consider the map which takes $s \in U \cap T(\mathbb{Z}[\frac{1}{p}])$ to the tuple $(\lambda_1 \mathcal{O}_L, \ldots, \lambda_n \mathcal{O}_L)$ of fractional ideals of $\mathcal{O}_L$. We claim that if $U$ is small enough this map is injective. Indeed, let $s, s' \in U \cap T(\mathbb{Z}[\frac{1}{p}])$, and let $\theta_i = \lambda_i^{-1}\lambda_i'$. If
\[
\lambda_1 \mathcal{O}_L, \ldots, \lambda_n \mathcal{O}_L = \lambda_1' \mathcal{O}_L, \ldots, \lambda_n' \mathcal{O}_L,
\]
then $\theta_1, \ldots, \theta_n \in \mathcal{O}_L^\times$. Taking $U$ sufficiently small we may assume that $|\theta_i - 1| < \frac{1}{2}$ for all $i$. Since for each $i$ all of the conjugates of $\theta_i$ are contained in \{\theta_1, \ldots, \theta_n\}, and since any element of $\mathcal{O}_L \setminus \{1\}$ has at least one conjugate at distance $\geq \frac{1}{2}$ from 1 (for example by looking at the discriminant), it follows that $\theta_i = 1$. This proves the claim. It thus remains to show that there are at most $N^{O(1)}$ tuples of fractional ideals corresponding to elements $s \in U \cap T(\mathbb{Z}[\frac{1}{p}])$ with denominator at most $p^N$. For this notice first that if we write $\lambda_i \mathcal{O}_L$ as a reduced quotient of integral ideals of $\mathcal{O}_L$, then since $\lambda_i$ belongs to the integral closure of $\mathbb{Z}[\frac{1}{p}]$ in $\mathcal{O}_L$, any prime ideal that appears in
the denominator must divide \( p \). Also since the denominator of \( s \) is at most \( N \), any such prime ideal appears with multiplicity at most \( O(N) \). Since \( \prod_{i=1}^{r} \lambda_i = 1 \), it follows that the ideals appearing in the nominator must satisfy these properties as well. Since there are at most \( O(1) \) prime ideals dividing \( p \) (note that \([ L : \mathbb{Q} ] \) is at most \( O(n(1)) \), we have at most \( N^{O(1)} \) choices for our fractional ideal at each coordinate, hence at most \( N^{O(1)} \) tuples of such ideals.

\[ \square \]

### 4.2 Proof of Theorem 1.1

We keep the notation of the previous section. Instead of working with the definition of entropy we use the following proposition which gives a criterion for positive entropy on almost every ergodic component. The reader who is not familiar with entropy can take this as a definition. See Appendix A (Proposition A.4) for the necessary background and a proof.

**Proposition 4.6.** Let \( a : X \rightarrow X \) be a measurable map. Let \( \mu \) be an \( a \)-invariant probability measure on \( X \). Then \( a \) acts with positive entropy on almost every ergodic component if for any \( \eta > 0 \) there exists a partition \( \mathcal{P} \) of \( X \) satisfying the following condition for some constants \( \delta > 0, c \in \mathbb{N} \): For any \( N \) large enough, if \( J \subset X \) is a union of \( d \) elements of the \( cN \)-th refinement \( \mathcal{P}_{cN} \) of \( \mathcal{P} \) (under \( a \)), of total measure \( \mu(J) > \eta \), then \( d > e^{\delta N} \).

Our proof of Theorem 1.1 is based on analyzing an expression of the form

\[ \langle \phi 1_J * K_N, \phi 1_J \rangle, \]

for a subset \( J \subset X \) as in Proposition 4.6, \( \phi \in L^2(\Gamma \backslash G) \) a suitable \( L(G_p, K_p) \)-eigenfunction, and \( K_N \) the kernel constructed in section 3.

**Lemma 4.7.** Let \( E, E' \subset G \) be two measurable subsets of \( G \), contained in some fixed fundamental domain \( F \) for \( \Gamma \backslash G \). Let \( k \in L(G_p, K_p) \), and let \( v = v(k) \) be the number of cosets \( bK_p \) (\( b \in G_p \)) such that

1. \( K_p b^{-1} K_p \) is in the support of \( k \),
2. \( E' b \cap E \neq \emptyset \) in \( X \).

Then for any \( f \in L^2(X) \) we have

\[ \langle f 1_E * k, f 1_{E'} \rangle \leq v \| k \|_{\infty} \| f \|_{L^2(E')} \| f \|_{L^2(E')} . \] (12)
Proof. For any function $h \in L^2(X)$ we have

$$h \ast k(x) = \int_{G_p} k(y^{-1})h(xy)dy,$$

where for any $x = (x_{\infty}, 1) \in X$ with $x_{\infty} \in F$, $y \in G_p$, $xy = (x_{\infty}, y)$. The union of the supports of $k$ and $g \mapsto k(g^{-1})$ is a finite union of double cosets $\xi_1, \ldots, \xi_M$, $\xi_i = K_p z_i K_p$, and so we can decompose the integral above as follows.

$$h \ast k(x) = \sum_{i=1}^{M} \int_{\xi_i} k(y^{-1})h(xy)dy = \sum_{i=1}^{M} k(z_i^{-1}) \int_{\xi_i} h(xy)dy.$$

Each of the double cosets $\xi_i$ is a finite union of disjoint right cosets $\xi_i = \bigcup_{b \in B_{\xi_i}} bK_p$,

and so

$$h \ast k(x) = \sum_{i=1}^{M} k(z_i^{-1}) \sum_{b \in B_{\xi_i}} h(xb).$$

Applying this equality to $h = f1_E$ it follows that

$$\langle f1_E \ast k, f1_{E'} \rangle \leq \sum_{i=1}^{M} k(z_i^{-1}) \sum_{b \in B_{\xi_i}} \int_{X} |f(xb)||f(x)| 1_E(xb)1_{E'}(x)dx.$$

Since the last integral vanishes unless $E \cap \bar{E}'b \neq \emptyset$, applying Cauchy-Schwartz yields (12).

Proof of Theorem 1.1. Let $0 < \eta < 1$. Let $\Omega \subset G$ a compact identity neighborhood of measure $\mu(\overline{\Omega}) \geq 1 - \frac{\eta}{2}$. Apply Lemma 4.1 with $r$ as in the first item of Proposition 3.1 to get a partition $\mathcal{P}$ and a sequence of identity neighborhoods $B_N \subset G$ satisfying the two properties of Lemma 4.1. Let $J \subset X$ be a union

$$J = \bigcup_{j=1}^{d} E_j$$

of elements $E_1, \ldots, E_d \in \mathcal{P}_{cN}$ of the $cN$-refinement of $\mathcal{P}$, with total mass $\mu(J) > \eta$. Here $c \in \mathbb{N}$ is the constant from the first item of Lemma 4.1. Put $J' = J \cap \overline{\Omega}$ and $E'_j = E_j \cap \overline{\Omega}$. Since $\mu(\overline{\Omega}) > 1 - \frac{\eta}{2}$, we have $\mu(J') > \frac{\eta}{2}$, and so
there exists $i_0$ such that $\lambda \mu_{i_0}(J') > \frac{\eta}{2}$ as well. Let $K_N \in L(G_p, K_p)$ be the kernel from Proposition 3.1 corresponding to $\epsilon := \frac{\eta}{2N}$ and $\phi_{i_0}$, and consider the inner product

$$\langle \phi_{i_0}1_{J'} * K_N, \phi_{i_0}1_{J'} \rangle = \sum_{j,k} \left\langle \phi_{i_0}1_{E'_j} * K_N, \phi_{i_0}1_{E'_k} \right\rangle. \quad (13)$$

Since each $E'_j$ is contained in a translate $xB_N$ ($x \in \Omega$), we have that for any $1 \leq j, k \leq d$, there exist $x, y \in \Omega$ such that

$$|\langle 1_{E'_j} \phi_{i_0} * K_N, 1_{E'_k} \phi_{i_0} \rangle| \ll \left\langle (1_{E'_j} \cap xB_N) | \phi_{i_0} |, 1_{E'_k} \cap yB_N | \phi_{i_0} | \right\rangle.$$ 

Since $\|K_N\|_\infty \leq e^{-\delta N}$, and since $b \mapsto K_N(b^{-1})$ is supported on elements with denominator at most $p^r N$, it follows by Lemma 4.7 that the last expression is bounded (up to a uniform constant) by

$$ve^{-\delta N} \| \phi_{i_0} \|_{L^2(E'_j)} \| \phi_{i_0} \|_{L^2(E'_k)},$$

where $v$ is the number of cosets $bK_p$ ($b \in G_p$) with denominator at most $p^r N$, such that $xB_N \cap yB_N \neq \emptyset$. But by Lemma 4.1 there are at most $N^{O(1)}$ such cosets, and so

$$|\langle 1_{E'_j} \phi_{i_0} * K_N, 1_{E'_k} \phi_{i_0} \rangle| \ll N^{O(1)} e^{-\delta N} \| \phi_{i_0} \|_{L^2(E'_j)} \| \phi_{i_0} \|_{L^2(E'_k)}.$$ 

Applying Cauchy-Schwartz this yields

$$\langle \phi_{i_0}1_{J'} * K_N, \phi_{i_0}1_{J'} \rangle \ll e^{-N \delta} N^{O(1)} \left( \sum_{j=1}^{d} \| \phi_j \|_{L^2(E_j)} \right)^2 \leq e^{-N \delta} N^{O(1)} d \sum_{j=1}^{d} \| \phi_j \|_{L^2(E_j)}^2 \quad (14)$$

To bound the left-hand side of (13) below, we decompose

$$\phi_{i_0}1_{J'} = \langle \phi_{i_0}1_{J'}, \phi_{i_0} \rangle \phi_{i_0} + R.$$ 

We have

$$\langle \phi_{i_0}1_{J'}, \phi_{i_0} \rangle = \| \phi_{i_0}1_{J'} \|_2^2 \geq \epsilon,$$

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and so
\[ \|R\|_2^2 = \|\phi_{i_0} 1_{J'}\|_2^2 - |\langle \phi_{i_0} 1_{J'}, \phi_{i_0} \rangle|^2 \leq \|\phi_{i_0} 1_{J'}\|_2^2 (1 - \epsilon). \]
Thus by the last two properties of \( K_N \) in Proposition 3.1 we obtain
\[ \langle \phi_{i_0} 1_{J'} * K_N, \phi_{i_0} 1_{J'} \rangle = |\langle \phi_{i_0} 1_{J'}, \phi_{i_0} \rangle|^2 \langle \phi_{i_0} * K_N, \phi_{i_0} \rangle + \langle R * K_N, R \rangle \]
\[ \geq \|\phi_{i_0} 1_{J'}\|_2^4 \epsilon^{-1} - \|R\|_2^2 \]
\[ \geq \epsilon (1 - (1 - \epsilon)) = \epsilon^2. \]
Combining this with (14) yields
\[ d \gg \frac{\epsilon^2}{N^{O(1)}} e^{\delta N}, \]
and the result follows by absorbing the coefficient \( \frac{\epsilon^2}{N^{O(1)}} \) into the exponent. \( \square \)

5 Adjustments for the case of division algebras

Let \( D \) be a division algebra over \( \mathbb{Q} \) of degree \( n \), which splits over \( \mathbb{R} \). Let \( O \) be a maximal order in \( D \), and \( \Gamma \) the group of all norm 1 elements of \( O \). Since \( D \) is \( \mathbb{R} \)-split we can view \( \Gamma \) as a subgroup of \( G := \text{SL}_n(\mathbb{R}) \). It is well-known that \( \Gamma \) is a co-compact lattice in \( G \). Assume that \( D \) splits over \( \mathbb{Q}_p \) as well (this is true for all but finitely many primes). Let \( u_i \) \( (i = 1, \ldots, n^2) \) be a \( \mathbb{Z} \)-basis of \( O \). Using this basis we can embed \( D \) in \( M_{n^2}(\mathbb{Q}) \) by mapping any element \( x \in D \) to the multiplication-by-\( x \) map,
\[ m_{xy} = xy. \]
More precisely, we send \( x \) to the representative matrix of \( m_x \) with respect to the basis \( u_i \). If \( R \) is any subring of \( \mathbb{Q} \), we let \( D_R := m^{-1}(M_{n^2}(R)) \), and \( D_R^1 \) the norm 1 elements of \( D_R \). In particular, \( D_\mathbb{Z} = O \), and \( D_\mathbb{Z}^1 = \Gamma \). Also for any place \( v \) we put \( D_v := D \otimes \mathbb{Q}_v \), and \( D_v^1 \) the norm 1 elements of \( D_v \).
For \( v = \infty, p \), fix isomorphisms \( \varphi_v : D_v^1 \rightarrow G_v := \text{SL}_n(\mathbb{Q}_v) \). It can be shown that \( \varphi_p \) can be chosen so that the completion \( O_p := \Sigma_{i=1}^{n^2} \mathbb{Z}_p u_i \) is mapped to \( K_p := \text{SL}_n(\mathbb{Z}_p) \). Let \( \Gamma_p = D_{\mathbb{Z}(1/p)}^1 \), then similarly to (6), we have the following double quotient decomposition.
\[ \Gamma \setminus G \cong \Gamma_p \setminus G_\infty \times G_p/K_p. \]
Here $\Gamma_p$ is identified with its diagonal copy in $G_\infty \times G_p$ using the isomorphisms $\varphi_\infty, \varphi_p$. As before, through this isomorphism we view $L(G_p, K_p)$ as an algebra of operators acting on $L^2(\Gamma \backslash G)$ by convolution on the right coordinate. In this setting, we have the following theorem.

**Theorem 5.1.** Under the assumptions of Theorem 1.3, any non-trivial element $a \in A$ acts on $\mu$ with positive entropy on almost every ergodic component.

The proof of Theorem 5.1 proceeds along the lines of the proof of Theorem 1.1. The kernel $K_N$ is constructed exactly as in Section 3, and Lemma 4.1 is replaced by the following similar lemma.

**Lemma 5.2.** Let $a \in A$ be any non-trivial element, and $\mu$ an $a$-invariant probability measure on $X$. For any $r > 0$ there exist a partition $\mathcal{P}$ of $X$, and a sequence of identity neighborhoods $B_N \subset G$ satisfying:

1. There exists $c \in \mathbb{N}$ depending only on $r$ such that any element of $\mathcal{P}_{cN}$ is contained in a translate $xB_N$, $x \in G$.

2. For any $x, y \in G$, the number of cosets $bK_p \in G_p/K_p$ with denominator $\leq p^r$ such that $xB_N \cap yB_N = \emptyset$ is at most $N^{O_r(1)}$.

Here the implied constants may depend on $r$, but not on $N$.

Fix a euclidean norm $\| \cdot \|$ on $D \otimes \mathbb{Q} \mathbb{R}$, and use it to define a norm on the Lie algebra of $G$ and thus a left-invariant Riemannian metric on $G$. We have the following three lemmas, similar to Lemma 4.2, Lemma 4.3, and Lemma 4.4.

**Lemma 5.3 (Silberman-Venkatesh [12], Lemma 4.9).** Let $S \subset D_\mathbb{R}$ be a proper subalgebra. For $c > 0$ sufficiently large (in fact depending only on $d$) and for $c' > 0$ sufficiently small (in fact depending only on $D, D_\mathbb{Z}, \| \cdot \|$), the set of $x \in D$ satisfying

$$\|x\| \leq R, \inf_{s \in S} \|x - s\| \leq \epsilon, \tilde{d}(x) \leq M,$$

is contained in a proper subalgebra $F \subset D$ as long as

$$\epsilon R^c M^c < c'.$$  

Here $\tilde{d}(x) = \inf \{ m \in \mathbb{N} \mid mx \in D_\mathbb{Z} \}$.  

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Lemma 5.4. Let \( a \) be an element of \( A \) and \( \mu \) an \( a \)-invariant probability measure on \( X \). Let \( \delta > 0 \). There exists a countable partition \( \mathcal{P} \) of \( X \) with finite entropy, such that for every element \( E \) in the \( N \)-th refinement \( \mathcal{P}_N \) of \( \mathcal{P} \), there exists \( x \in X \) so that up to a \( \mu \)-null set,

\[
E \subset x \bigcap_{k=-N}^{N} a^{-k} B^G_\delta a^k.
\]

Here \( B^G_\delta \subset G \) is the open ball of radius \( \delta \) around the identity in \( G \).

Let \( a \) be a non-trivial element of \( A \). We extend the definition of the dynamical balls \( B(C, \epsilon) \) from Section 4.1 to non-regular elements as well. Let \( H := C_G(a) \) be the centralizer of \( a \) in \( G \) (if \( a \) is regular then \( C_G(a) = A \)). For any relatively compact neighborhood of the identity in \( H \), let \( B(C, \epsilon) \) an \( \epsilon \)-neighborhood of \( C \) in \( G \). Define \( G^+, G^- \) as in Section 4.1, and let \( \mathfrak{h} \) be the Lie algebra of \( H \). Then we have \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}^+ \oplus \mathfrak{g}^- \), and the following version of Lemma 4.4.

Lemma 5.5. Let \( N \in \mathbb{N} \) and \( \delta > 0 \) which is small enough in terms of \( a \). Then there exists some \( \alpha > 0 \), depending only on \( a \) such that

\[
\bigcap_{k=-N}^{N} a^{-k} B^G_\delta a^k \subset B(C, \kappa e^{-\alpha N}),
\]

for some \( \kappa \ll \delta \), and \( C \subset H \) some compact subset with diameter \( \ll \delta \).

Using these three results, and noticing that since \( D \) has prime degree any proper subalgebra of \( D \) is a field, Lemma 5.2 can be proved along the lines of the proof of Lemma 4.1.

A Entropy

In this appendix we review what we need from ergodic theory and entropy theory, and we prove a criterion for positive entropy on almost every ergodic component (Proposition A.4). The references are drawn from the book in progress [6].

In what follows we work with locally compact second countable metric spaces. We refer to such a space, equipped with its Borel \( \sigma \)-algebra as standard Borel space. We recall the following theorem regarding existence and uniqueness of conditional measures.
Theorem A.1 (Conditional measure, [6, Theorem 2.2]). Let \((X, \mathcal{B}, \mu)\) be a probability space with \((X, \mathcal{B})\) standard Borel space. Let \(A \subset \mathcal{B}\) be a sub-
\sigma\-algebra. Then there exists a subset \(X' \subset X\) of full measure, belonging to \(A\), and Borel probability measures \(\mu^A_x\) for \(x \in X'\) such that for every \(f \in L^1(X, \mathcal{B}, \mu)\) we have \(E(f | A)(x) = \int f(y) d\mu^A_x(y)\) for almost every \(x\). In particular the right hand side is \(A\)-measurable as a function of \(x\). Moreover, the family of conditional measures \(\mu^A_x\) is almost everywhere uniquely determined by this relationship to the conditional expectation. The map \(x \in X' \to \mu^A_x\) is \(A\)-measurable on \(X'\).

Let \(X\) be a standard Borel space and \(\mu\) a probability measure on \(X\). If \(T : X \to X\) is a measure preserving map and \(\mathcal{E}\) is the \(\sigma\)-algebra of \(T\) invariant sets, then (almost) every \(\mu^\mathcal{E}_x\) is ergodic and we have the ergodic decomposition

\[
\mu = \int \mu^\mathcal{E}_x d\mu(x).
\]

Let \((X, \mathcal{B}, \mu, T)\) an invertible measure preserving system with \(X\) being a standard Borel space and \(\mu\) a probability measure. Let \(\mathcal{P}\) be a finite or countable partition of \(X\). The static entropy of \(\mathcal{P}\) is defined to be

\[
H_\mu(\mathcal{P}) = - \sum_{P \in \mathcal{P}} \mu(P) \log \mu(P),
\]

which in the case where \(\mathcal{P}\) is countable may be finite or infinite. Define \(\mathcal{P}_N = \bigvee_{i=-N}^{N} T^{-i} \mathcal{P}\) the \(N\)-th refinement of \(\mathcal{P}\). The ergodic theoretic entropy \(h_\mu(T)\) is defined using the entropy function \(H_\mu\) as follows:

**Definition A.2.** Let \(\mathcal{P}\) be a partition of \(X\) with \(H_\mu(\mathcal{P}) < \infty\). Define

\[
h_\mu(T, \mathcal{P}) = \lim_{N \to \infty} \frac{1}{2N + 1} H_\mu(\mathcal{P}_N).
\]

The ergodic theoretic entropy is defined to be

\[
h_\mu(T) = \sup_{H_\mu(\mathcal{P}) < \infty} h_\mu(T, \mathcal{P}).
\]

If \(\mathcal{P}\) is a partition of \(X\) denote by \([x]_N\) the atom containing \(x\) of \(\mathcal{P}_N\).

**Theorem A.3** (Relative Shannon–McMillan–Brieman, [6, Theorem 3.2]). Let \(\mathcal{P}\) be a countable partition with \(H_\mu(\mathcal{P}) < \infty\). Then for almost every \(x \in X\) we have

\[
\lim_{N \to \infty} \frac{1}{N} \log \mu([x]_N) = h_\mu^\mathcal{E}(T, \mathcal{P}).
\]
As a corollary we have the following proposition which we use in order to show positive entropy on almost every ergodic component.

**Proposition A.4.** Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system with $X$ being a standard Borel probability space. Suppose that for any $\eta > 0$ there exists a countable partition $\mathcal{P}$ with finite entropy and $\delta > 0$ such that, for any $N$ sufficiently large, if $J$ is a collection of elements of the $N$-th refinement $\mathcal{P}_N$ of $\mathcal{P}$ with total measure $> \eta$, then $J$ has cardinality $\geq e^{N\delta}$. Then for almost every $x \in X$, $h_{\mu_x}(T) > 0$.

**Proof.** Let $B$ be the set of $x \in X$ such that $h_{\mu_x}(T) = 0$ and assume $\mu(B) > \eta > 0$. Then by assumption there exist a countable partition $\mathcal{P}$ with finite entropy, $\delta > 0$, and a positive integer $N_0$, such that for any $N > N_0$ if $J$ is a collection of elements of the $N$-th refinement $\mathcal{P}_N$ of $\mathcal{P}$ with total measure $> \eta$, then $J$ has cardinality $\geq e^{N\delta}$. Let $0 < \epsilon < \delta$, and let $C^N_\epsilon$ be the subset of all $x \in X$ such that $| - \log \frac{1}{N}\mu([x]_N) - h_{\mu_x}(T, \mathcal{P})| > \epsilon$. From Theorem A.3 it follows that $\mu(C^N_\epsilon) \to 0$ as $N \to \infty$. Thus for $N$ large enough $\mu(B \setminus C^N_\epsilon) > \eta$. Take $N$ large enough so that both $\mu(B \setminus C^N_\epsilon) > \eta$, and $N > N_0$. Let $J_N$ be the collection of all the partition elements of $\mathcal{P}_N$ that intersect $B \setminus C^N_\epsilon$ non-trivially. Then $J_N$ has total measure $> \eta$ and so $|J_N| \geq e^{N\delta}$. In other words there are at least $e^{N\delta}$ partition elements of $\mathcal{P}_N$ of the form $[x]_N$ for some $x \in B \setminus C^N_\epsilon$. But for any such partition element we have $| - \log \frac{1}{N}\mu([x]_N)| \leq \epsilon$, and so $\mu([x]_N) \geq e^{-\epsilon N}$. Thus for any $N$ large enough we found a collection $J_N$ of partition elements of $\mathcal{P}_N$ of cardinality at least $e^{N\delta}$ and each of the elements of $J_N$ has measure $\geq e^{-\epsilon N}$. Thus $J_N$ has total measure $\geq e^{(\delta - \epsilon)N}$. Taking $N \to \infty$ we have that $\mu(X) = \infty$ which is a contradiction. 

**B Diophantine approximation**

For any $(\alpha_1, \ldots, \alpha_m) = \alpha \in \mathbb{R}^m$ let $||\alpha|| = \max_i ||\alpha_i||$, where $||\alpha_i||$ is the minimal distance to an integer. In this appendix we prove the following result.

**Lemma B.1.** For any $m \in \mathbb{N}$ and $\epsilon > 0$, there exist constants $C, C' > 0$ so that for every $\alpha \in \mathbb{R}^m$ and $N \in \mathbb{N}$ there exist $q \in \mathbb{N}$ such that $C'N \leq q \leq CN$ and $||q\alpha|| < \epsilon$.

For the proof of the Lemma we use the following version of Kronecker’s theorem:
Theorem B.2 ([7, Proposition 3.1]). For any $0 < \delta \leq \frac{1}{2}$ and $\alpha \in \mathbb{R}^m/\mathbb{Z}^m$ if the sequence $(n\alpha)_{n\in[N]}$ is not $\delta$-equidistributed then there are $k_1, \ldots, k_m \in \mathbb{Z}^m$ with $|k_i| = O_{\delta,m}(1)$ such that $\|\alpha \cdot k\| = O_{\delta,m}(\frac{1}{N})$.

Proof of Lemma B.1. The proof is by induction on $m$. The case where $m = 1$ follows immediately from Dirichlet’s approximation theorem. Assume $m > 1$, and let $\epsilon > 0$. There exists $l = l(\epsilon, m) \in \mathbb{N}$ and $0 < \delta < \frac{1}{2}$ such that for any $N \in \mathbb{N}$, if $(n\alpha)_{n\in[N]}$ is $\delta$-equidistributed then there exists $C_1N \leq q \leq C_2N$ with $\|q\alpha\| < \epsilon$, for some $C_1, C_2$ depending only on $\epsilon$ and $m$. If $(n\alpha)_{n\in[N]}$ is not $\delta$-equidistributed then by Theorem B.2 there are $k_1, \ldots, k_m$ (not all of them are zero), $|k_i| = O_{\epsilon,m}(1)$ such that $\|\sum \alpha_i\| \leq D(\frac{1}{N})$, for some $D = D(\epsilon, m)$. Assume without loss of generality that $k_m \neq 0$. By induction we have $C'(\epsilon', m-1)N \leq q \leq C(\epsilon', m-1)N$ such that $\|q\alpha\| \leq \epsilon'$, for any $\epsilon' > 0$. Since the $k_i$’s are bounded in terms of $m, \epsilon$ we can choose $\epsilon' = \epsilon'(\epsilon, m)$ and $l = l(\epsilon, m)$ so that $\|q\alpha_m\| < \epsilon$ and so we are done. 

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