CLAIRAUT ANTI-ININVARIANT SUBMERSIONS FROM NORMAL ALMOST CONTACT METRIC MANIFOLDS

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Abstract. We investigate new Clairaut conditions for anti-invariant submersions from normal almost contact metric manifolds onto Riemannian manifolds. We prove that there is no Clairaut anti-invariant submersion admitting vertical Reeb vector field when the total manifold is Sasakian. Several illustrative examples are also included.

1. Introduction

In the theory of surfaces, Clairaut’s theorem states that for any geodesic $\alpha$ on a surface $S$, the function $r \sin \theta$ is constant along $\alpha$, where $r$ is the distance from a point on the surface to the rotation axis and $\theta$ is the angle between $\alpha$ and the meridian through $\alpha$. This idea was applied to the Riemannian submersions [16] by Bishop [5] and he gave a necessary and sufficient condition for a Riemannian submersion to be Clairaut. Allison [1] considered Clairaut submersions when the total manifolds were Lorentzian and he also showed that such submersions have interesting applications in static space-times. Lee et al. [15], investigated new conditions for anti-invariant Riemannian submersions [19] to be Clairaut when the total manifolds are Kählerian. A similar study [22] was done by Şahin and the first author of this paper for semi-invariant submersions [20], slant submersions [21] and pointwise slant submersions [14].

In the present paper, we consider anti-invariant Riemannian submersions from normal almost contact metric manifolds onto Riemannian manifolds. After giving a necessary and sufficient condition for a curve on the total manifolds to be geodesic, we focus on investigating new Clairaut conditions for considered submersions. We first give a new necessary and sufficient condition for anti-invariant submersions admitting horizontal Reeb vector field to be Clairaut in the case of the total manifolds are Sasakian. We also give a characterization for such submersions when they satisfy Clairaut condition. Contrary to the case of admitting horizontal Reeb vector field, we prove that there is no anti-invariant submersion satisfying Clairaut condition in the case of admitting vertical Reeb vector field when the total manifold is Sasakian. Finally, we present a new necessary and sufficient condition for anti-invariant submersions to be Clairaut in the case of their total manifolds are Kenmotsu. An illustrative example for each kind of submersion is also given.

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2. Preliminaries

This section consists of four subsections. In subsection 2.1, we present the fundamental definitions and notions some classes of normal almost contact metric manifolds such as Sasakian and Kenmotsu. In subsection 2.2, we give the basic background for Riemannian submersions. In subsection 2.3, we recall the fundamental definitions and notions of anti-invariant Riemannian and Lagrangian submersions. The definition and a characterization of Clairaut submersions are placed in the last subsection.

2.1. Some classes of normal almost contact metric manifolds. Let \((M, g)\) be a \((2m + 1)\)-dimensional Riemannian manifold and denote by \(TM\) the set of vector fields of \(M\). Then, \(M\) is called an almost contact metric manifold \([3]\) if there exists a tensor \(\varphi\) of type \((1, 1)\) and global vector field \(\xi\) which is called the Reeb vector field or the characteristic vector field such that for any \(E, F \in TM\), we have

\[
\varphi \xi = 0, \quad \eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi
\]

and

\[
g(\varphi E, \varphi F) = g(E, F) - \eta(E) \eta,
\]

where \(\eta\) is the dual 1-form of \(\xi\). Also, it can be deduced from the above axioms that

\[
\eta \circ \varphi = 0 \quad \text{and} \quad \eta(E) = g(E, \xi).
\]

In this case, \((\varphi, \xi, \eta, g)\) is called the almost contact metric structure of \(M\). The almost contact metric manifold \((M, \varphi, \xi, \eta, g)\) is called a contact metric manifold if we have

\[
\Phi(E, F) = d\eta(E, F)
\]

for any \(E, F \in TM\), where \(\Phi\) is a 2-form in \(M\) defined by \(g(E, \varphi F)\). The 2-form \(\Phi\) is called the fundamental 2-form of \(M\). A contact metric structure \((\varphi, \xi, \eta, g)\) of \(M\) is said to be normal \([25]\) if we have

\[
[\varphi, \varphi] + 2d\eta \otimes \xi = 0,
\]

where \([\varphi, \varphi]\) is Nijenhuis tensor of \(\varphi\). Any normal contact metric manifold is called a Sasakian manifold. It is not difficult to prove that a contact metric manifold \(M\) is a Sasakian manifold if and only if

\[
(\nabla_E \varphi) F = g(E, F) \xi - \eta(F) E
\]

for any \(E, F \in TM\), where \(\nabla\) denotes the Levi-Civita connection of \(M\). For the further information of Sasakian manifolds, see the classical books \([3, 25]\).

A Kenmotsu manifold \(M\) \([11]\) is a normal almost contact metric manifold satisfying

\[
(\nabla_E \varphi) F = g(\varphi E, F) \xi - \eta(F) \varphi E.
\]

for all \(E, F \in TM\). We refer to the original paper \([11]\) for fundamental definitions and notions of Kenmotsu manifolds.
2.2. **Riemannian submersions.** Let \((M, g)\) and \((N, g_N)\) be Riemannian manifolds, where \(\dim(M) > \dim(N)\). A surjective mapping \(\pi : (M, g) \to (N, g_N)\) is called a **Riemannian submersion** \([10]\) if:

**\((S1)\)** The rank of \(\pi\) is equal to \(\dim(N)\).

In which case, for each \(q \in N\), \(\pi^{-1}(q) = \pi^{-1}(q)\) is a \(k\)-dimensional submanifold of \(M\) and called a **fiber**, where \(k = \dim(M) - \dim(N)\). A vector field on \(M\) is called **vertical** (resp. **horizontal**) if it is always tangent (resp. orthogonal) to fibers. A vector field \(Y\) on \(M\) is called **basic** if \(Y\) is horizontal and \(\pi\)-related to a vector field \(Y_\ast\) on \(N\), i.e., \(\pi_\ast Y_p = Y_\ast \pi(p)\) for all \(p \in M\), where \(\pi_\ast\) is the derivative map of \(\pi\). As usual, we denote by \(\mathcal{V}\) and \(\mathcal{H}\) the projections on the vertical distribution \(\ker \pi_\ast\) and the horizontal distribution \((\ker \pi_\ast)^\perp\), respectively.

**\((S2)\)** For all \(p \in M\) and for any horizontal vectors \(Y\) and \(Z\) at \(p\) and, we have
\[
g(Y_p, Z_p) = g_N(\pi_\ast Y_p, \pi_\ast Z_p),
\]
that is, \(\pi_\ast\) preserves lengths of horizontal vectors.

The geometry of Riemannian submersions is characterized by O’Neill’s tensors \(\mathcal{T}\) and \(\mathcal{A}\), defined as follows:

\[
\mathcal{T}_E F = \mathcal{V} \nabla_{\mathcal{V} E \mathcal{H} F} + \mathcal{H} \nabla_{\mathcal{V} E \mathcal{V} F},
\]
\[
\mathcal{A}_E F = \mathcal{V} \nabla_{\mathcal{H} E \mathcal{H} F} + \mathcal{H} \nabla_{\mathcal{H} E \mathcal{V} F}
\]
for any vector fields \(E\) and \(F\) on \(M\), where \(\nabla\) is the Levi-Civita connection of \(g\). It is easy to see that \(\mathcal{T}_E\) and \(\mathcal{A}_E\) are skew-symmetric operators on the tangent bundle of \(M\) reversing the vertical and the horizontal distributions. We summarize the properties of the tensor fields \(\mathcal{T}\) and \(\mathcal{A}\). Let \(W, U\) be vertical and \(Y, Z\) be horizontal vector fields on \(M\), then we have

\[
\mathcal{T}_W U = \mathcal{T}_U W,
\]
\[
\mathcal{A}_Y Z = -\mathcal{A}_Z Y = \frac{1}{2} \mathcal{V}[Y, Z].
\]
Equation \((2.6)\) says that \(\mathcal{T}\) is symmetric for vertical vector fields, while equation \((2.7)\) says that \(\mathcal{A}\) is skew symmetric for horizontal vector fields. Moreover, from \((2.7)\) it follows the horizontal distribution is integrable if and only if \(\mathcal{A}\) is zero, identically. On the other hand, from \((1)\) and \((2)\), we obtain

\[
\nabla_W U = \mathcal{T}_W U + \hat{\nabla}_W U,
\]
\[
\nabla_W Y = \mathcal{T}_W Y + \mathcal{H} \nabla_W Y,
\]
\[
\nabla_Y W = \mathcal{A}_Y W + \mathcal{V} \nabla_Y W,
\]
\[
\nabla_Y Z = \mathcal{H} \nabla_Y Z + \mathcal{A}_Y Z,
\]
where \(\hat{\nabla}_W U = \mathcal{V} \nabla_W U\). Moreover, if \(Y\) is basic, then we have

\[
\mathcal{H} \nabla_W Y = \mathcal{A}_Y W.
\]
From (2.8), we see that $T$ acts on the fibers as the second fundamental form. We also observe that the horizontal distribution is totally geodesic if and only if $A$ is zero, identically from (2.11). For details on the Riemannian submersions, we refer to the papers, [9, 16] and to the books [8, 18].

2.3. Anti-invariant Riemannian submersions. The notion of anti-invariant Riemannian submersion was first defined by Sahin [18] in almost Hermitian geometry and then this notion was applied to almost contact geometry by Lee [13] as follows.

**Definition 2.1.** ([13]) Let $M$ be a $(2m + 1)$-dimensional almost contact metric manifold with almost contact metric structure $(\varphi, \xi, \eta, g)$ and $N$ be a Riemannian manifold with Riemannian metric $g_N$. Suppose that there exists a Riemannian submersion $\pi: M \to N$ such that the vertical distribution $\ker \pi^*$ is anti-invariant with respect to $\varphi$, i.e., $\varphi \ker \pi^* \subseteq \ker \pi^*$. Then the Riemannian submersion $\pi$ is called an anti-invariant Riemannian submersion. We will briefly call such submersions as anti-invariant submersions.

In this case, the horizontal distribution $\ker \pi^*$ is decomposed as

$$(2.13)\quad \ker \pi^* = \varphi \ker \pi^* \oplus \mu,$$

where $\mu$ is the orthogonal complementary distribution of $\varphi \ker \pi^*$ in $\ker \pi^*$ and it is invariant with respect to $\varphi$.

We say that an anti-invariant Riemannian submersion $\pi: M \to N$ admits vertical Reeb vector field if the Reeb vector field $\xi$ is tangent to $\ker \pi^*$ and it admits horizontal vector Reeb vector field if the Reeb vector field $\xi$ is normal to $\ker \pi^*$. It is easy to see that $\mu$ contains the Reeb vector field $\xi$ in the case of $\pi: M \to N$ admits horizontal vector Reeb vector field $\xi$. For any $Y \in \ker \pi^*$, we write

$$(2.14)\quad \varphi Y = BY + CY,$$

where $BY \in \ker \pi^*$ and $CY \in \ker \pi^*$. For some details and examples of the anti-invariant Riemannian submersions from almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifolds $(N, g_N)$, we refer to the papers [4, 7, 13] and to the book [18].

**Definition 2.2.** ([24]) Let $\pi$ be an anti-invariant Riemannian submersion from an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold $(N, g_N)$. If $\mu = \{0\}$ or $\mu = \text{span}\{\xi\}$, i.e., $\ker \pi^* = \varphi(\ker \pi^*)$ or $\ker \pi^* = \varphi(\ker \pi^*) \oplus < \xi >$, respectively, then we call $\pi$ a Lagrangian submersion.

In that case, for any horizontal vector field $X$, we have

$$(2.15)\quad BX = \varphi X \quad \text{and} \quad CX = 0.$$  

For the general properties of such submersions, see [23, 24].

2.4. Clairaut submersions. Let $S$ be a revolution surface in $\mathbb{R}^3$ with rotation axis $L$. For any $p \in S$, we denote by $r(p)$ the distance from $p$ to $L$. Given a geodesic $\alpha : I \subset \mathbb{R} \to S$ on $S$, let $\theta(t)$ be the angle between $\alpha(t)$ and the meridian curve through $\alpha(t)$, $t \in I$. A well-known Clairaut’s theorem says that for any geodesic $\alpha$ on $S$ the product $r \sin \theta$ is constant along $\alpha$, i.e., it is independent of
t. In the theory of Riemannian submersions, Bishop [5] introduces the notion of Clairaut submersion in the following way.

**Definition 2.3.** ([5]) A Riemannian submersion \( \pi : (M, g) \to (N, g_N) \) is called a Clairaut submersion if there exists a positive function \( r \) on \( M \) such that, for any geodesic \( \alpha \) on \( M \), the function \((r \circ \alpha)\sin \theta\) is constant, where, for any \( t \), \( \theta(t) \) is the angle between \( \dot{\alpha}(t) \) and the horizontal space at \( \alpha(t) \).

He also gave the following necessary and sufficient condition for a Riemannian submersion to be a Clairaut submersion as follows.

**Theorem 2.4.** ([5]) Let \( \pi : (M, g) \to (N, g_N) \) be Riemannian submersion with connected fibers. Then \( \pi \) is a Clairaut submersion with \( r = e^f \) if and only if each fiber is totally umbilical and has the mean curvature vector field \( H = -\nabla f \), where \( \nabla f \) is the gradient of the function \( f \) with respect to \( g \).

### 3. Anti-invariant submersions admitting horizontal Reeb vector field from Sasakian manifolds

In this section, we study anti-invariant submersions from Sasakian manifolds admitting horizontal Reeb vector field. After giving a new necessary and sufficient condition for such submersions to be Clairaut, we prove some characteristic results for this kind of submersions. We also present an illustrative example for such submersions at the end of this section.

As seen from Definition 2.3, the origin of the notion of a Clairaut submersion comes from geodesic on its total space. Therefore, we will investigate a necessary and sufficient condition for a curve on the total space to be geodesic.

**Lemma 3.1.** Let \( \pi \) be an anti-invariant Riemannian submersion from a Sasakian manifold \((M, \varphi, \xi, \eta, g)\) onto a Riemannian manifold \((N, g_N)\) admitting horizontal Reeb vector field. If \( \alpha : I \subset \mathbb{R} \to M \) is a regular curve and \( V(t) \) and \( X(t) \) are the vertical and horizontal components of the tangent vector field \( \dot{\alpha}(t) = E \) of \( \alpha(t) \), respectively, then \( \alpha \) is a geodesic if and only if along \( \alpha \) the following equations hold, where \( \sqrt{\nu} \) is constant speed of \( \alpha \).

\[
\nabla \varphi \varphi + A_X \varphi V + T_V \varphi V + (T_V + A_X)CX + \eta(X)V = 0, \tag{3.1}
\]

\[
\mathcal{H} \nabla \varphi V + \mathcal{H} \nabla \varphi V + (T_V + A_X)BX + \eta(X)X - \nu \xi = 0 \tag{3.2}
\]

where \( \mathcal{H} \) is the mean curvature vector field.

**Proof.** From (2.2), we have

\[
\nabla \varphi \dot{\alpha} = \varphi \nabla \dot{\alpha} + g(\dot{\alpha}, \dot{\alpha})\xi - \eta(\dot{\alpha})\dot{\alpha}. \tag{3.3}
\]

Since \( \dot{\alpha} = V + X \) and \( g(\dot{\alpha}, \dot{\alpha}) = \nu \), we can write

\[
\nabla_{V+X} \varphi (V + X) = \varphi \nabla \dot{\alpha} + \nu \xi - \eta(V)\dot{\alpha} - \eta(X)\dot{\alpha}.
\]

By direct computations, we obtain

\[
\nabla_V \varphi V + \nabla_V \varphi X + \nabla_X \varphi V + \nabla_X \varphi X = \varphi \nabla \dot{\alpha} + \nu \xi - \eta(X)V - \eta(X)X,
\]

since \( \eta(V) = 0 \).
Using (2.8)∼(2.11), we get
\[
\mathcal{H}(\nabla_\alpha \varphi V + \nabla_\alpha CX) + (T_V + A_X)(BX + CX) + \nabla_\alpha BX + A_X \varphi V + T_V \varphi V \\
= \varphi \nabla_\alpha \dot{\alpha} + \nu \xi - \eta(X)V - \eta(X)X.
\]
Taking the vertical and horizontal parts of above equation, we get
\[
\begin{align*}
V \nabla_\dot{\alpha} B X + A X \varphi V &+ T_V \varphi V + (T_V + A_X)BX = \mathcal{V} \varphi \nabla_\alpha \dot{\alpha} - \eta(X)V \\
(3.4)
\end{align*}
\]
and
\[
\begin{align*}
\mathcal{H} \nabla_\alpha CX + \mathcal{H} \nabla_\alpha \varphi V + (T_V + A_X)BX = \mathcal{H} \varphi \nabla_\alpha \dot{\alpha} + \nu \xi - \eta(X)X
\end{align*}
(3.5)
\]
From (3.4) and (3.5), it is easy to see that \( \alpha \) is a geodesic if and only if (3.1) and (3.2) hold.

Theorem 3.2. Let \( \pi \) be an anti-invariant Riemannian submersion from a Sasakian manifold \((M, \varphi, \xi, \eta, g)\) onto a Riemannian manifold \((N, g_N)\) admitting horizontal Reeb vector field. Then \( \pi \) is a Clairaut submersion with \( r = e^f \) if and only if along \( \alpha \)
\[
g(\nabla f, X)\|V\|^2 = g(\eta(X)X + \mathcal{H} \nabla_\alpha CX + (T_V + A_X)BX, \varphi V)
\]
holds, where \( V(t) \) and \( X(t) \) are the vertical and horizontal components of the tangent vector field \( \dot{\alpha}(t) \) of the geodesic \( \alpha(t) \) on \( M \), respectively.

Proof. Let \( \alpha(t) \) be a geodesic with speed \( \sqrt{\nu} \) on \( M \), then we have
\[
\nu = \|\dot{\alpha}(t)\|^2,
\]
From this equality, we deduce that
\[
g(V(t), V(t)) = \nu \sin^2 \theta(t) \quad \text{and} \quad g(X(t), X(t)) = \nu \cos^2 \theta(t),
\]
where \( \theta(t) \) is the angle between \( \dot{\alpha}(t) \) and the horizontal space at \( \alpha(t) \). Differentiating the first expression in (3.7), we obtain
\[
\frac{d}{dt}g(V(t), V(t)) = 2g(\nabla_{\dot{\alpha}(t)} V(t), V(t)) = 2\nu \cos \theta(t) \sin \theta(t) \frac{d\theta}{dt}(t).
\]
Hence using the Sasakian structure, we get
\[
g(\varphi \nabla_{\alpha(t)} V(t), \varphi V(t)) = \nu \cos \theta(t) \sin \theta(t) \frac{d\theta}{dt}(t),
\]
(3.8)
At this point, we know
\[
\varphi \nabla_\alpha V = \nabla_\alpha \varphi V - g(\dot{\alpha}, V)\xi
\]
from (2.2). Hence,
\[
g(\varphi \nabla_\alpha V, \varphi V) = g(\nabla_\alpha \varphi V, \varphi V) = g(\mathcal{H} \nabla_\alpha \varphi V, \varphi V),
\]
since \( g(\xi, \varphi V) = 0 \) and \( \varphi V \) is horizontal.

Thus, from (3.8), we obtain
\[
g(\mathcal{H} \nabla_\alpha \varphi V, \varphi V) = \nu \cos \theta(t) \sin \theta(t) \frac{d\theta}{dt}(t)
\]
(3.9)
By (3.2), we find along $\alpha$,
\begin{equation}
-g(\mathcal{H}_{\alpha}C X + (\mathcal{T}_V + A_X)B X + \eta(X)X, \varphi V) = \nu \cos \theta \sin \theta \frac{d\theta}{dt},
\end{equation}
since $g(\xi, \varphi V) = 0$.

On the other hand, $\pi$ is a Clairaut submersion with $r = e^{f}$ if and only if
\begin{equation}
\frac{d}{dt}(e^f \sin \theta) = 0 \Leftrightarrow e^f (\frac{df}{dt} \sin \theta + \cos \theta \frac{d\theta}{dt}) = 0
\end{equation}
Multiplying last equation with non-zero factor $\nu \sin \theta$, we get
\begin{equation}
\frac{df}{dt} \nu \sin^2 \theta + \nu \cos \theta \sin \theta \frac{d\theta}{dt} = 0
\end{equation}
From (3.10) and (3.11), we obtain
\begin{equation}
\frac{df}{dt} (\alpha(t)) \parallel V \parallel^2 = g(\mathcal{H}_{\alpha}C X + (\mathcal{T}_V + A_X)B X - \eta(X)X, \varphi V)
\end{equation}
Since $\frac{df}{dt} (\alpha(t)) = \dot{\alpha}[f] = g(\nabla f, \dot{\alpha}) = g(\nabla f, X)$, the assertion (3.6) follows from (3.12).

Corollary 3.3. Let $\pi$ be a Clairaut anti-invariant Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold $(N, g_N)$ admitting horizontal Reeb vector field. Then we have
\begin{equation}
g(\nabla f, \xi) = 0.
\end{equation}
Next, we give a characterization for Clairaut anti-invariant Riemannian submersion admitting horizontal Reeb vector field.

Theorem 3.4. Let $\pi$ be a Clairaut anti-invariant Riemannian submersion admitting horizontal Reeb vector field from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold $(N, g_N)$ with $r = e^{f}$. Then at least one of the following statements are true:

(a) $f$ is constant on $\varphi \ker \pi$,

(b) the fibers of $\pi$ are one dimensional,

(c) $A_{JW}JX = X(f)W$
for $X \in \mu$ and $W \in \ker \pi$ such that $JW$ is basic.

Proof. Let $\pi$ be a Clairaut anti-invariant Riemannian submersion admitting horizontal Reeb vector field from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold $(N, g_N)$ with $r = e^{f}$. From Bishop’s theorem, we have
\begin{equation}
\mathcal{T}_U V = -g(U, V)\nabla f
\end{equation}
where $U, V \in \ker \pi_*$. If we multiply this equation by $\varphi W$ for $W \in \ker \pi_*$ and using (2.8), we obtain
\[ g(\nabla_U V, \varphi W) = -g(U, V)g(\nabla f, \varphi W). \]

Hence, we get
\[ g(\nabla_U \varphi W, V) = g(U, V)g(\nabla f, \varphi W), \]
since $g(V, \varphi W) = 0$.

By (2.2), we arrive at
\[ g(\varphi \nabla_U W, V) = g(U, V)g(\nabla f, \varphi W). \]

Using the Sasakian structure, we find
\[ -g(\nabla_U W, \varphi V) = g(U, W)g(\nabla f, \varphi V). \]

Hence, by (3.14),
\[ g(U, W)g(\nabla f, \varphi V) = g(U, V)g(\nabla f, \varphi W). \]

If take $U = W$ and interchange $U$ with by $V$ in (3.15), we derive
\[ \| V \|^2 g(\nabla f, \varphi U) = g(U, V)g(\nabla f, \varphi V). \]

Using (3.15) with $W = U$ and (3.16), we have
\[ g(\nabla f, \varphi U) = \frac{g^2(U, V)}{\| U \|^2 \| V \|^2} g(\nabla f, \varphi U). \]

On the other hand, using (2.2), we have
\[ g(\nabla_V \varphi W, \varphi X) = g(\varphi \nabla_V W, \varphi X). \]

for $X \in \mu$ and $X \neq \xi$. Hence, using the Sasakian structure, we obtain
\[ g(\nabla_V \varphi W, \varphi X) = g(\nabla_V W, X). \]

Using (2.8) and (3.14), we get
\[ g(\nabla_V \varphi W, \varphi X) = -g(V, W)g(\nabla f, X) \]

Since $\varphi W$ is basic and using the fact that $\mathcal{H}_V \varphi W = A_{\varphi W} V$, we get
\[ g(\nabla_V \varphi W, \varphi X) = g(A_{\varphi W} V, \varphi X) \]

Using (3.18) and (3.19) and the skew-symmetricness of $A$, we find
\[ g(A_{\varphi W} \varphi X, V) = g(\nabla f, X)g(W, V). \]

Since $A_{\varphi W} \varphi X, V$ and $W$ are vertical and $\nabla f$ is horizontal, we deduce that
\[ A_{\varphi W} \varphi X = X(f)W \]
from (3.20).

Now, if $\nabla f \in \varphi \ker \pi_*$, then (3.17) and the equality case of Schwarz inequality imply that either $f$ is constant on $\varphi \ker \pi_*$ or the fibers of $\pi$ are one dimensional.
Thus (a) and (b) follows. If $\nabla f \in \mu \setminus \{\xi\}$, the last assertion follows immediately from (3.21). \hfill \Box

**Corollary 3.5.** Let $\pi$ be a Clairaut anti-invariant Riemannian submersion admitting horizontal Reeb vector field from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold $(N, g_N)$ with $r = e^f$ and $\dim(\ker\pi_\ast) > 1$. Then the fibers of $\pi$ are totally geodesic if and only if $A_JVJX = 0$ for $V \in \Gamma(\ker\pi_\ast)$ such that $JV$ is basic and $X \in \mu$.

Moreover, if the submersion $\pi$ in Theorem 3.4 is Lagrangian, then $A_JVJX$ is always zero, since $\mu = \{0\}$ or $\mu = \text{span}\{\xi\}$. Thus, we have the following result from Theorem 3.4.

**Corollary 3.6.** Let $\pi$ be a Clairaut Lagrangian submersion admitting horizontal Reeb vector field from a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ onto a Riemannian manifold $(N, g_N)$ with $r = e^f$. Then either the fibers of $\pi$ are one dimensional or they are totally geodesic.

We ends this section by giving a (non-trivial) example of a Clairaut anti-invariant submersion from Sasakian manifold admitting horizontal Reeb vector field.

**Example 3.7.** Let $R^3$ be 3-dimensional Euclidean space given by $R^3 = \{(x, y, z) \in R^3 \mid (x, y) \neq (0, 0) \text{ and } z \neq 0\}$. We consider the map $\pi : (R^3, \varphi_0, \xi, \eta, g) \to (R^2, g_2)$ defined by $\pi(x, y, z) = (\sqrt{x^2 + y^2}, z)$ where $(\varphi_0, \xi, \eta, g)$ is the usual Sasakian structure \cite{6} on $R^3$ and $g_2$ is the Euclidean metric on $R^2$. Then the Jacobian matrix of $\pi$ is

$$
\begin{pmatrix}
x/\tau & y/\tau & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

Here, $\tau = \sqrt{x^2 + y^2}$. Since the rank of this matrix is equal to 2, the map $\pi$ is a submersion. Following some computations, we have

$$
\ker\pi_\ast = \text{span}\{U = y/\tau E_1 - x/\tau E_2\}
$$

and

$$
\ker\pi_\ast^\perp = \text{span}\{Z = x/\tau E_1 + y/\tau E_2, E_3 = \xi\},
$$

where $\{E_1, E_2, E_3\}$ is a $\varphi_0$-basis such that $E_1 = 2(\frac{\partial}{\partial x} + y \frac{\partial}{\partial z})$ and $E_2 = 2 \frac{\partial}{\partial y}$.

For this map $\pi$, it is not difficult to satisfy the condition S2). So, $\pi$ is a Riemannian submersion. Also, we have $\varphi_0(U) = -X$. Hence, we see that $\pi$ is an anti-invariant Riemannian submersion admitting horizontal Reeb vector field. In particular, $\pi$ is Lagrangian. Moreover, since the fibers of $\pi$ are one dimensional, they are clearly totally umbilical. Here, we shall show that the fibers are not totally geodesic and find that a function on $R^3$ satisfying $T_UU = -\nabla f$. Indeed, by direct computations, we have

$$
(3.22) \quad \nabla_U U = U[\frac{1}{\tau}rU - \frac{x}{\tau} (x E_1 + y E_2) + \left(\frac{x}{\tau}U[E_1] - \frac{y}{\tau}U[E_2]\right)]
$$
Here, one can see that
\[
U[E_1] = \frac{y}{\tau} \nabla E_1 E_1 - \frac{x}{\tau} \nabla E_2 E_1
\]
and
\[
U[E_2] = \frac{y}{\tau} \nabla E_1 E_2 - \frac{x}{\tau} \nabla E_2 E_2.
\]
Using the Sasakian structure, we see that
\[
\nabla E_1 E_1 = \nabla E_2 E_2 = 0
\]
and
\[
\nabla E_1 E_2 = -\nabla E_2 E_1 = -2 \frac{\partial}{\partial z}.
\]
Taking into account these equalities in (3.22), we obtain
\[
\nabla U U = U[\frac{1}{\tau}] \tau V - \frac{2}{\tau^2} (x E_1 + y E_2).
\]
Using (2.8), we get
\[
T U U = -\frac{2}{\tau^2} (x E_1 + y E_2).
\]
After some calculation, we arrive
\[
T U U = -\left\{ \frac{2x}{x^2 + y^2} \frac{\partial}{\partial x} + \frac{2y}{x^2 + y^2} \frac{\partial}{\partial y} + \frac{2xy}{x^2 + y^2} \frac{\partial}{\partial z} \right\}.
\]
For any function \( f \) on \( (\mathbb{R}^3, \varphi_0, \xi, \eta, g) \), the gradient of \( f \) with respect to the metric \( g \) is:
\[
\nabla f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} = 4 \left\{ \left( \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial z} \right) \frac{\partial}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial}{\partial y} + \left( y \frac{\partial f}{\partial x} + (1 + y^2) \frac{\partial f}{\partial z} \right) \frac{\partial}{\partial z} \right\}.
\]
Then, for the function \( f = \frac{1}{4} \ln(x^2 + y^2) \), it is easy to verify that
\[
T_U U = -\nabla f.
\]
Hence, it follows that
\[
T_U V = -\|V\|^2 \nabla f
\]
for any vertical vector field \( V \). Under the given conditions, the tensor \( T \) is never zero. So, the fibers of \( \pi \) are not totally geodesic, but they are totally umbilical with mean curvature field \( H = -\nabla f \). Thus, by Theorem 2.4, we see that this anti-invariant Riemannian submersion is Clairaut with \( r = e^f \), where \( f = \frac{1}{4} \ln(x^2 + y^2) \).

Henceforth, we have alternative theorem, namely Theorem 3.2, to check that whether the submersion is Clairaut or not.

In fact, for any horizontal vector field \( X \) proportional to \( \xi \), we easily verify that the condition (3.6) of Theorem 3.2.

Now, let \( X \) be any horizontal vector field orthogonal to \( \xi \) and \( V \) be any vertical vector field, then using (2.1), (2.15) and the equation (34) of Corollary 6.1 of [24],
we have
\[ g(T_VB, \varphi V) = g(T_V\varphi X, \varphi V) = g(\varphi T_VX, \varphi V) = g(T_VX, V) = -g(T_VV, X). \]

Hence, we obtain

\[ g(T_V\varphi X, \varphi V) = g(\nabla f, X)\|V\|^2, \]

since \( T_VV = -\|V\|^2\nabla f \). Likewise, using (2.1), (2.15) and the equation (35) of Corollary 6.1 of [24], we have
\[ g(A_XB, \varphi V) = g(A_X\varphi X, \varphi V) = -g(\varphi A_XV, X) = -g(A_XV, X). \]

Hence, we obtain

\[ g(A_X\varphi X, \varphi V) = 0, \]

since \( A_XX = 0 \). In addition to, we have

\[ \eta(X) = 0 \quad \text{and} \quad \mathcal{H}\nabla_\alpha CX = 0, \]

since \( \pi \) is Lagrangian and \( X \) is orthogonal to \( \xi \). Using (3.23), (3.24) and (3.25), we easily verify the equation (3.29). Thus, by Theorem 3.2, the considered submersion \( \pi \) is Clairaut.

**Remark 3.8.** We notice that the submersion given in Example 3.7 satisfies one of the conditions of Theorem 3.4 and the condition (3.13) in Corollary 3.3.

### 4. Anti-invariant submersions admitting vertical Reeb vector field from Sasakian manifolds

In this section, we check that the existence of Clairaut anti-invariant submersions from Sasakian manifolds when the Reeb vector field is vertical. First of all, we give a non-trivial example of an anti-invariant submersion from Sasakian manifold admitting vertical Reeb vector field.

**Example 4.1.** Let \( \mathbb{R}^5 \) be a Sasakian manifold with usual Sasakian structure [6]. Consider the map \( \pi: \mathbb{R}^5 \to (\mathbb{R}^2, g_2) \) given by
\[ \pi(x_1, x_2, y_1, y_2, z) = \left( \frac{x_1 + y_1}{\sqrt{2}}, \frac{x_2 + y_2}{\sqrt{2}} \right), \]

where \( g_2 \) is the Euclidean metric on \( \mathbb{R}^2 \). After some calculation, we see that
\[ \ker\pi_* = \text{span}\{V = \frac{1}{\sqrt{2}}(E_4 - E_1), W = \frac{1}{\sqrt{2}}(E_4 - E_2), \xi\} \]
and
\[ \ker\pi^*_+ = \text{span}\{X = \frac{1}{\sqrt{2}}(E_1 + E_3), Y = \frac{1}{\sqrt{2}}(E_2 + E_4)\} \]
It is not difficult to show that π is a Riemannian submersion. Also, we have ϕ₀(V) = −X and ϕ₀(W) = −Y. Hence, π is an anti-invariant submersion admitting vertical Reeb vector field. In particular, π is Lagrangian.

We now assume that there exists an anti-invariant submersion π admitting vertical Reeb vector field from Sasakian manifold satisfying Clairaut condition. Then because of Theorem 2.4, the fibers of π must be totally umbilical. But, the following result forces the fibers to be totally geodesic, since the fibers are submanifolds.

**Theorem 4.2.** (10) Let $\tilde{N}$ be a Sasakian manifold. If $N$ is any totally umbilical submanifold of $\tilde{N}$ tangent to the Reeb vector field $\xi$, then it is totally geodesic.

On the other hand, for any vertical vector field $V$, we have

\[ T_V \xi = -\varphi V \]

from the proof of Theorem 2 of [4]. The equation (4.1) says us the fibers of $\pi$ cannot be totally geodesic. This is a contradiction. Thus, we have the following classification theorem.

**Theorem 4.3.** There is no Clairaut anti-invariant submersions admitting vertical Reeb vector field from Sasakian manifolds onto Riemannian manifolds.

### 5. Anti-invariant Submersions from Kenmotsu Manifolds

In this section, we shall give new Clairaut conditions for anti-invariant Riemannian submersions from Kenmotsu manifolds onto Riemannian manifolds. In which case, the Reeb vector field $\xi$ is necessarily horizontal, because Beri et al. [4] showed the non-existence of anti-invariant Riemannian submersions from Kenmotsu manifolds such that the Reeb vector field is vertical.

**Lemma 5.1.** Let $\pi$ be an anti-invariant Riemannian submersion from a Kenmotsu manifold $(M, \varphi, \xi, \eta, g_0)$ onto a Riemannian manifold $(N, g_N)$. If $\alpha : I \subset \mathbb{R} \to M$ is a regular curve and $V(t)$ and $X(t)$ are the vertical and horizontal components of the tangent vector field $\dot{\alpha}(t) = E$ of $\alpha(t)$, respectively, then $\alpha$ is a geodesic if and only if the following two equations

\[ \nabla_{\dot{\alpha}}B X + A_X \varphi V + (T_V + A_X)C X + \eta(X) B X = 0, \]

\[ \mathcal{H} \nabla_{\dot{\alpha}}(\varphi V + CX) + (T_V + A_X)B X + \eta(X) (\varphi V + CX) = 0 \]

hold along $\alpha$.

**Proof.** From (2.15), we have

\[ \nabla_{\dot{\alpha}} \varphi \dot{\alpha} = \varphi \nabla_{\dot{\alpha}} \dot{\alpha} + g(\dot{\alpha}, \dot{\alpha}) \xi - \eta(\dot{\alpha}) \varphi \dot{\alpha}. \]

Since $\dot{\alpha} = V + X$ and $\eta(V) = 0$, we can write

\[ \nabla_V \varphi V + \nabla_X \varphi X + \nabla_X \varphi V + \nabla_X \varphi X = \varphi \nabla_{\dot{\alpha}} \dot{\alpha} - \eta(X) (\varphi V + \varphi X). \]

Using (2.8) ~ (2.11), together with (2.13), we obtain

\[ \mathcal{H} \nabla_{\dot{\alpha}}(\varphi V + CX) + (T_V + A_X)(B X + CX) + \nabla_{\dot{\alpha}}B X + A_X \varphi V = \varphi \nabla_{\dot{\alpha}} \dot{\alpha} - \eta(X) (B X + CX + \varphi V) \]

Taking the vertical and horizontal parts of the last equation, we get

\[ \nabla_{\dot{\alpha}}B X + A_X \varphi V + (T_V + A_X)C X = \nabla_V \nabla_{\dot{\alpha}} \dot{\alpha} - \eta(X) B X, \]
(5.4) \( \mathcal{H}\nabla_\alpha (\varphi V + CX) + (T_V + A_X)BX = \mathcal{H}\varphi \nabla_\alpha \dot{\alpha} - \eta(X)(CX + \varphi V) \),

From (5.3) and (5.4), we see that \( \alpha \) is a geodesic if and only if (5.1) and (5.2) hold along \( \alpha \). \( \square \)

**Theorem 5.2.** Let \( \pi \) be an anti-invariant Riemannian submersion from a Kenmotsu manifold \((M, \varphi, \xi, \eta, g)\) onto a Riemannian manifold \((N, g_N)\). Then, \( \pi \) is a Clairaut submersion with \( r = e^f \) if and only if

\( (5.5) \quad \{g(\nabla f, X) - \eta(X)\} \|V\|^2 = g(\mathcal{H}\nabla_\alpha CX + (T_V + A_X)BX, \varphi V) \)

holds along \( \alpha \), where \( V(t) \) and \( X(t) \) are the vertical and horizontal components of the tangent vector field \( \dot{\alpha}(t) \) of the geodesic \( \alpha(t) \) on \( M \), respectively.

**Proof.** Let \( \alpha \) be a geodesic on \( M \), then we have

\( \|\dot{\alpha}(t)\|^2 = \nu \),

where \( c \) is a constant. Hence, we deduce that

\( (5.6) \quad g(V(t), V(t)) = \nu \sin^2 \theta(t) \quad \text{and} \quad g(X(t), X(t)) = \nu \cos^2 \theta(t) \)

where \( \theta(t) \) is the angle between \( \dot{\alpha}(t) \) and the horizontal space at \( \alpha(t) \). Differentiating the first expression, we obtain

\( (5.7) \quad g(\nabla_{\dot{\alpha}(t)} V(t), V(t)) = \nu \cos \theta(t) \sin \theta(t) \frac{d\theta}{dt}(t) \)

Using the Kenmotsu structure, we get

\( g(\nabla_\alpha V, V) = g(\varphi \nabla_\alpha V, \varphi V) \),

since \( \eta(V) = 0 \). Here, by (2.15), we know

\( \varphi \nabla_\alpha V = \nabla_\alpha \varphi V - g(\varphi \dot{\alpha}, V)\xi. \)

Hence, we obtain

\( (5.8) \quad g(\varphi \nabla_\alpha V, \varphi V) = g(\mathcal{H}\nabla_\alpha \varphi V, \varphi V) \),

since \( \varphi V \) is horizontal. From (5.7) and (5.8), we get

\( (5.9) \quad g(\mathcal{H}\nabla_\alpha \varphi V, \varphi V) = \nu \cos \theta(t) \sin \theta(t) \frac{d\theta}{dt}. \)

Using (5.2), we find

\( (5.10) \quad -g(\mathcal{H}\nabla_\alpha CX + (T_V + A_X)BX, \eta(X)\varphi V, \varphi V) = \nu \cos \theta(t) \sin \theta(t) \frac{d\theta}{dt} \)

As in the proof of Theorem 3.2, \( \pi \) is a Clairaut submersion with \( r = e^f \) if and only if (3.11) holds. Thus, from (3.11) and (5.10), we get

\( (5.11) \quad \frac{d(f \circ \alpha)}{dt} \|V\|^2 = g(\mathcal{H}\nabla_\alpha CX + (T_V + A_X)BX, \varphi V) + \eta(X) \|V\|^2 \)

Since \( \frac{d(f \circ \alpha)}{dt} = g(\nabla f, X) \), the assertion immediately follows from (5.11). \( \square \)

From (5.5), we immediately have that:
Corollary 5.3. Let \( \pi \) be a Clairaut anti-invariant Riemannian submersion from a Kenmotsu manifold \((M, \phi, \xi, \eta, g)\) onto a Riemannian manifold \((N, g_N)\). Then, we have

\[
(5.12) \quad g(\nabla f, \xi) = 1.
\]

Example 5.4. Let \( M \) be a 3-dimensional Euclidean space given by

\[
M = \{ (x, y, z) \in \mathbb{R}^3 \mid (x, y) \neq (0, 0) \text{ and } z \neq 0 \}.
\]

Following the Example 1 of [4], we define the Kenmotsu structure \((\phi, \xi, \eta, g)\) on \( M \) given by

\[
\xi = \frac{\partial}{\partial z}, \quad \eta = dz, \quad g = \begin{pmatrix}
e^{2z} & 0 & 0 \\
0 & e^{2z} & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad \phi = \begin{pmatrix}0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

A \( \phi \)-basis for this structure can be given by \( \{E_1 = e^{-z} \frac{\partial}{\partial y}, E_2 = e^{-z} \frac{\partial}{\partial x}, E_3 = \xi\} \).

Let \( N \) be \( \{(u, z) \in \mathbb{R}^2 \mid z \neq 0\} \). We choose the Riemannian metric \( g_N \) on \( N \) in the following form

\[
\begin{pmatrix}
e^{2z} & 0 \\
0 & 1
\end{pmatrix}.
\]

Now, we define the map \( \pi : (M, \phi, \xi, \eta, g) \to (N, g_N) \) by

\[
\pi(x, y, z) = (x + y \sqrt{2}, z).
\]

Then the Jacobian matrix of \( \pi \) is

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Since the rank of this matrix is equal to 2, the map \( \pi \) is a submersion. After simple calculations, we see that

\[
ker \pi_* = span\{U = \frac{E_1 - E_2}{\sqrt{2}}\} \quad \text{and} \quad ker \pi_*^\perp = span\{Z = \frac{E_1 + E_2}{\sqrt{2}}, Y = \xi\}.
\]

By direct calculation, we see that \( \pi \) satisfies the condition \( \text{S2} \) and \( \phi(U) = -X \). Thus, \( \pi \) is an anti-invariant Riemannian submersion. In particular, \( \pi \) is Lagrangian. Moreover, the fibers of \( \pi \) are clearly totally umbilical, since they are one-dimensional. Here, we shall find that a function \( f \) on \( M \) satisfying \( T_U U = -\nabla f \).

Indeed, upon direct computations, we have

\[
\nabla_U U = \frac{1}{2}(\nabla_{E_1} E_1 - \nabla_{E_1} E_2 - \nabla_{E_2} E_1 + \nabla_{E_2} E_2).
\]

Using the given Kenmotsu structure, we find

\[
\nabla_{E_1} E_1 = \nabla_{E_2} E_2 = -\frac{\partial}{\partial z}
\]

and

\[
\nabla_{E_1} E_2 = \nabla_{E_2} E_1 = 0.
\]

Thus, we have

\[
\nabla_U U = -\frac{\partial}{\partial z}.
\]

By \( \text{S3} \), we obtain

\[
T_U U = -\frac{\partial}{\partial z}.
\]
On the other hand, for any function \( f \) on \( M \), the gradient of \( f \) with respect to the metric \( g \) is given by

\[
\nabla f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} = e^{-2z} \frac{\partial f}{\partial x} \frac{\partial}{\partial x} + e^{-2z} \frac{\partial f}{\partial y} \frac{\partial}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial}{\partial z}.
\]

Then, it is easy to see that \( \nabla f = \frac{\partial}{\partial z} \) for the function \( f = z \) and \( T_U U = -\nabla f = -\xi \).

Furthermore, for any vertical vector field \( V \), we conclude that

\[
T_V V = -\|V\|^2 \nabla f
\]

from the last fact. Thus, by Theorem 2.4, the submersion \( \pi \) is Clairaut.

Now, by using our result Theorem 5.2, we show that the submersion \( \pi \) is Clairaut. Indeed, if \( X \) is any horizontal vector field proportional to \( \xi \), then it is easy to see that the condition (5.5) is fulfilled. Next, let \( X \) be any horizontal vector field orthogonal to \( \xi \) and \( V \) be any vertical vector field, then using (2.1), (2.15) and the equation (59) of Corollary 7.2 of [24], we have

\[
g(T_V B X, \varphi V) = g(T_V \varphi X, \varphi V) = g(\varphi T_V X + g(\varphi V, X)\xi - \eta(X)\varphi V, \varphi V) = g(\varphi T_V X, \varphi V) = g(T_V X, V) = -g(T_V V, X) = g(\nabla f, X)\|V\|^2.
\]

Hence, we obtain

(5.13) \[ g(T_V B X, \varphi V) = 0, \]

since \( \nabla f = \xi \). Additionally, by Theorem 7.3 of [24], we have \( A \equiv 0 \), since \( \pi \) is Lagrangian. Then, using this fact, (5.13) and (2.15), the condition (5.5) is fulfilled. Thus, by Theorem 5.2, the given submersion is Clairaut.

Remark 5.5. We notice that the Clairaut Lagrangian submersion given in Example 5.4 satisfies the condition (5.12) of Corollary 5.3.

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