Formulation of Matrix Theory at Finite Temperature\(^1\)

Yuri Makeenko

Institute of Theoretical and Experimental Physics
B. Cheremushkinskaya 25, 117218 Moscow, Russia
makeenko@itep.ru

Abstract: The interaction between static D0-branes at finite temperature is considered in the matrix theory and the superstring theory. The results agree in both cases to the leading order in the supersymmetry violation by temperature, where the one-loop approximation is reliable. The effective static potential is short-ranged and attractive.

Talk at the 32nd International Symposium Ahrenshoop on the Theory of Elementary Particles, Buckow, Germany, September 1–5, 1998.

1 Introduction

The matrix theory \[^2\] is the large-\(N\) limit of the 10-dimensional supersymmetric Yang–Mills theory dimensionally reduced to 0 spatial dimensions. When the coupling constant \(g_{YM}^2\) is large, the matrix theory describes 11-dimensional M-theory while the limit of small \(g_{YM}^2\) is associated with 10-dimensional IIA superstring. The matrix theory correctly reproduces properties of D-branes in the superstring theory including their interactions to the leading order in violation of supersymmetry, e.g. at small velocities or large separations between D-branes or weak magnetic fields living on D-branes.

In this talk I consider the formulation of the matrix theory at finite temperature given by an Euclidean path integral with boundary conditions along the compactified “time” which are periodic for the Yang–Mills fields and antiperiodic for fermionic superpartners. I present the result of the computation of the effective potential between static D0-branes in the one-loop approximation and show that it agrees with an analogous computation in superstring theory, where an integration is to be performed over the non-trivial holonomies of the temporal components of Abelian gauge fields living on the D0-branes. This agreement is to the leading order in the supersymmetry violation by temperature, where the one-loop approximation is reliable, thus providing one more argument supporting the validity of the matrix theory. The computed effective static potential which is short-ranged and attractive has consequences for thermal properties of D0-branes.

\(^1\) Based on the paper [3] written in collaboration with Jan Ambjørn and Gordon Semenoff.
2 Matrix theory at finite temperature

The matrix theory [2] is formulated by the reduction of ten dimensional supersymmetric Yang-Mills theory

\[ S_{YM}[A, \theta] = \frac{1}{g_{YM}^2} \int d\tau \, Tr \left( \frac{1}{4} F_{\mu\nu}^2 + \frac{i}{2} \theta \gamma_\mu D_\mu \theta \right) \]  

(1)

to one temporal and zero spatial dimensions: \( A_\mu = A_\mu(\tau), \theta = \theta(\tau) \).

The thermal partition function is given by the Euclidean path integral

\[ Z_{YM} = \int [dA(\tau)][d\theta(\tau)] e^{-S_{YM}[A,\theta]}, \]  

(2)

where the time-coordinate is periodic. The bosonic and fermionic coordinates have, respectively, periodic and antiperiodic boundary conditions

\[ A_\mu(\tau + \beta) = A_\mu(\tau), \quad \theta(\tau + \beta) = -\theta(\tau), \quad \beta = 1/T, \]  

(3)

where \( T \) is the temperature. Gauge fixing involves introducing ghost fields which have periodic boundary conditions.

The representation (2) of the thermal partition function can be derived in the standard way starting from the known Hamiltonian of the matrix theory [2] and representing the thermal partition function

\[ Z_{YM} = \text{Tr} e^{-\beta H} \]  

(4)

via the path integral. The trace is calculated over all states obeying Gauss’s law which is taken care by the integration over \( A_0 \) in (2). This representation of the matrix theory at finite temperature have been discussed in Refs. [3, 4, 5].

The diagonal components of the gauge fields, \( \vec{a}^\alpha \equiv \vec{A}^{\alpha\alpha} \), are interpreted in the matrix theory as the positions of the \( \alpha \)-th D0-brane and should be treated as collective variables. Static configurations play a special role since they satisfy classical equations of motion with the periodic boundary conditions and dominate the path integral as \( g_{YM}^2 \to 0 \). There are no such static zero modes for fermionic components since they would not satisfy the antiperiodic boundary conditions. This is an important difference from the zero temperature case and a manifestation of the fact that supersymmetry is explicitly broken by non-zero temperature.

An effective action for these coordinates is constructed by integrating the off-diagonal components of the gauge fields, the fermionic variables and the ghosts:

\[ S_{eff}[\vec{a}^\alpha] \equiv -\ln \int \prod_{\beta \neq \alpha} \left[ dq_\alpha^\beta \right] [dA_\mu^\alpha \beta] [d\theta] [d\text{ghost}] e^{-S_{YM} - S_{gf} - S_{gh}}. \]  

(5)

Generally, this integration can only be done in the a simultaneous loop expansion and expansion in the number of derivatives of the coordinates \( \vec{a}^\alpha \). Such an expansion is accurate in the limit where \( |\vec{a}^\alpha - \vec{a}^\beta| \) are large for each pair of D0-branes and where their velocities are small. The remaining dynamical problem then defines the statistical mechanics of a gas of D0-branes:

\[ Z_{YM} = \int \prod_{\tau, \alpha} [d\vec{a}^\alpha(\tau)] e^{-S_{str}[\vec{a}^\alpha]}. \]  

(6)
3 One-loop computation in matrix theory

The computation of the effective action $S_{\text{eff}}$ for the interaction between static D0-branes at one loop is standard for the matrix theory.

The gauge field decomposed into diagonal part which satisfies the classical equation of motion and fluctuating off-diagonal part:

$$A_{\mu}^{\alpha\beta} = a_{\mu}^{\alpha}\delta^{\alpha\beta} + g_{\text{YM}}\bar{A}_{\mu}^{\alpha\beta}. \quad (7)$$

The gauge is fixed by

$$D_{\mu}^{\alpha\beta}\bar{A}_{\mu}^{\alpha\beta} = 0, \quad (8)$$

where

$$D_0^{\alpha\beta} = \partial_0 - i (a_0^{\alpha} - a_0^{\beta}), \quad \bar{D}^{\alpha\beta} = -i \left( \bar{a}^{\alpha} - \bar{a}^{\beta} \right). \quad (9)$$

This adds the Fadeev-Popov ghosts to the action

$$S_{\text{gh}} = \int \sum_{\alpha,\beta} \left\{ \bar{c}_{\alpha\beta} \left( -D^{\alpha\beta}_\mu \right)^2 c_{\beta\alpha} + ig_{\text{YM}}\bar{c}_{\beta\alpha} D^{\alpha\beta}_\mu \left[ \bar{A}_\mu, c \right] \right\}. \quad (10)$$

There is a residual Abelian gauge invariance

$$\bar{A}_{\mu}^{\alpha\beta} \rightarrow \bar{A}_{\mu}^{\alpha\beta} e^{i(\chi^{\alpha} - \chi^{\beta})}, \quad a_{\mu}^{\alpha} \rightarrow a_{\mu}^{\alpha} + \partial_\mu \chi^{\alpha}, \quad (11)$$

which can be used to make $a_0^{\alpha}$ independent on the compactified time-variable ($\partial_0 a_0^{\alpha} = 0$). In contrast to the zero-temperature case, $a_0^{\alpha}$’s can not be completely removed because of the existence of the nontrivial holonomy

$$\text{Pe}^{i \int_0^\beta d\tau A_0(\tau)} = \Omega^\dagger \text{diag} \left( e^{i\beta a_0^1}, \ldots, e^{i\beta a_0^N} \right) \Omega, \quad (12)$$

which is known as the Polyakov loop winding around the compact Euclidean time, whose trace is gauge invariant. Due to periodicity, it is chosen $-\pi/\beta < a_0^{\alpha} \leq \pi/\beta$.

Expanding the action to the quadratic order in $\bar{A}, c, \bar{c}, \theta$ and doing the Gaussian integration, it is obtained in the standard way

$$S_{\text{eff}} = 8 \sum_{\alpha<\beta} \left\{ \text{Tr}_B \ln \left( -(D^{\alpha\beta}_\mu)^2 \right) - \text{Tr}_F \ln \left( -(D^{\alpha\beta}_\mu)^2 \right) \right\}, \quad (13)$$

where the subscript $B$ denotes contributions from the gauge fields and ghosts, whereas $F$ denotes those from the adjoint fermions. The determinants should be evaluated with periodic boundary conditions for bosons and antiperiodic boundary conditions for fermions.

The boundary conditions are taken into account by proper Matsubara frequencies, so that

$$e^{-S_{\text{eff}}} = \beta^N \int_{-\pi/\beta}^{\pi/\beta} \prod_{\gamma>\alpha} \frac{da_0^{\alpha}}{2\pi} \prod_{n=-\infty}^{\infty} \left( \frac{\left( 2\pi n + \frac{2\beta}{\beta} + a_0^{\alpha} - a_0^{\gamma} \right)^2 + |\bar{a}^{\alpha} - \bar{a}^{\gamma}|^2}{\left( 2\pi n + \frac{2\beta}{\beta} + a_0^{\alpha} - a_0^{\gamma} \right)^2 + |\bar{a}^{\alpha} - \bar{a}^{\gamma}|^2} \right)^8. \quad (14)$$

Using the formula

$$\prod_{n=-\infty}^{\infty} \left( \frac{2\pi n}{\beta} + \omega \right) = \sin \left( \frac{\beta \omega}{2} \right), \quad (15)$$

we obtain finally

$$e^{-S_{\text{eff}}} = \beta^N \int_{-\pi/\beta}^{\pi/\beta} \prod_{\gamma>\alpha} \frac{da_0^{\alpha}}{2\pi} \left( \cosh \beta |\bar{a}^{\alpha} - \bar{a}^{\gamma}| + \cos \beta \left( a_0^{\alpha} - a_0^{\gamma} \right) \right)^8 \cosh \beta |\bar{a}^{\alpha} - \bar{a}^{\gamma}| - \cos \beta \left( a_0^{\alpha} - a_0^{\gamma} \right). \quad (16)$$

3
The integration over the temporal components $a_0^\alpha$ implements the projection onto the gauge invariant eigenstates of the matrix theory Hamiltonian.

If both bosons and fermions had periodic boundary conditions the determinants would cancel because of supersymmetry. This would give the well-known result that the lowest energy state is a BPS state whose energy does not depend on the relative separation of the D0-branes.

4 Comparison with superstring theory

4.1 Open string language

The starting point is the thermal partition function of the single open superstring:

$$Z_{\text{str}} \equiv -\beta F = \text{Tr} e^{-\beta H},$$

where $H$ is the superstring Hamiltonian and the trace is over all physical (GSO projected) superstring states.

Since the ends of the open string end on two D0-branes separated by the distance $L$, the superstring has the Neumann boundary condition along the temporal direction and the Dirichlet boundary conditions along the nine spatial directions. The corresponding superstring spectrum is given by

$$\sqrt{\alpha'} E_N = \sqrt{\frac{L^2}{4\pi^2 \alpha'} + N},$$

where $N$ are eigenvalues of the oscillator number operator.

Knowing the spectrum (18), the thermal partition function of the string gas can be immediately written as

$$Z_{\text{str}}(\beta, L, \nu) = e^{-\beta F} = \prod_{N=0}^{\infty} \frac{1 + e^{-\beta E_N + i\pi \nu}}{1 - e^{-\beta E_N + i\pi \nu}}^{2d_N},$$

where $d_N$ stands for the degeneracy of either bosonic or fermionic superstring states at level $N$:

$$8 \prod_{n=1}^{\infty} \frac{1 + e^{-nl}}{1 - e^{-nl}}^8 = \sum_{N=0}^{\infty} d_N e^{-Nl}. \tag{20}$$

For the lowest levels, $d_0 = 8$ and $E_0 = L/2\pi\alpha'$. The factor of 2 in the exponent $2d_N$ in (19) is the famous one by Polchinski [6] and is due to the interchange of the superstring ends. It is crucial in providing the agreement with the matrix theory computation.

The physical meaning of Eq. (19) is obvious: the partition function is equal to the ratio of the Fermi and Bose distributions with the power being (twice) the degeneracy of the states.

Equation (19) is derived in Ref. [7] (in Ref. [8] for Dp-branes) by calculating the annulus diagram for the open superstring in compactified Euclidean time of the circumference $\beta$. The parameter $\nu$ has the meaning of the constant $U(1)$ gauge field which enters though the quantized temporal momentum $p^0 = 2\pi(r - \nu)/\beta$ of the open string whose world-sheet winds around the space-time cylinder as is depicted in Fig. 1. In the above formula, $r$ is integer in the NS sector (associated with space-time bosons) and half-integer in the R sector (associated with space-time fermions).
In order to compare with the Yang-Mills computation, we identify the coordinates of D0-branes with $\vec{q}^\alpha = 2\pi\alpha'\vec{a}^\alpha$, so that the separation is $L = 2\pi\alpha' |\vec{a}^1 - \vec{a}^2|$. Then the integrand in (16) coincides for N=2 with (19) truncated to the massless modes ($N = 0$) provided $\pi\nu = \beta(\vec{a}^1_0 - \vec{a}^2_0)$.

The truncation of the stringy modes is justified for $\beta \gg L$ (or $TL \ll 1$) when the energy gap $\Delta$ between the first two levels is finite. From (18) we get

$$\Delta = \sqrt{\left(\frac{L}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}} - \frac{L}{2\pi\alpha'}$$  

and the spectrum can be truncated at the first level when and only when $\beta\Delta \gg 1$. If the temperature is small, this condition is always satisfied unless the length $L$ is not too large.

The integration over $a_0$’s in Eq. (16) corresponds to the integration over $\nu$ in Eq. (19). This integration comes about in the string theory as follows. The open-string gauge field $A_0$ interacts with D0-branes adding the surface term to the action:

$$S_{\text{int}} = \int dq^\mu A_\mu = \int_0^\beta d\tau \left( A_0(\tau, \vec{q}^1) - A_0(\tau, \vec{q}^2) \right) = \pi \nu$$  

The matrix theory automatically takes into account the integration over $A_0$ while in the string theory calculation of Ref. [7] the open-string gauge field is fixed. This integration over $A_0$ is needed to provide Gauss’s law for the charges at the ends of the open string which are induced on D-branes. Therefore, the effective potential between static D0-branes in the superstring theory is given by

$$S_{\text{eff}}[\vec{a}^\alpha] = -\ln \int_{-1}^1 d\nu Z_{\text{str}}(\beta, L, \nu).$$  

This issue will be further discussed in the next Subsection.

### 4.2 Closed string language

At least two issues remain unclear in the open string language. Firstly, why to exponetiate the single string partition function to get the string gas and, secondly, why to integrate over $\nu$ the string gas partition function (19) rather than, say, the single string partition function (17)? This has a natural explanation in the closed string language.

The passage to a closed string is performed by the standard modular transformation which converts the annulus diagram for an open string into a cylinder diagram for a closed
The right-hand side of Eq. (17) can then be represented as
\[ F(L, \beta, \nu) = \frac{8\pi^4}{\sqrt{2\pi \alpha'}} \int_0^\infty \frac{ds}{s^{9/2}} e^{s-L^2/2\alpha'} \Theta_2 \left( \nu \left| i \beta^2 s \right) \frac{2\pi^3 \alpha'}{2\pi \alpha'} \prod_{n=1}^\infty \frac{1-e^{-(2n+1)s}}{1-e^{-2ns}} \right]^8, \tag{24} \]
where
\[ \Theta_2 (\nu | iz) = \sum_{q=-\infty}^{\infty} e^{-\pi z (2q+1)^2 / 4 + i \pi (2q+1) \nu}. \tag{25} \]
The meaning of Eq. (24) is that of the closed-string propagator, which describes the interaction between D0-branes, rather than the thermal partition function as for an open string.

The sum over \( q \) in Eq. (24) represents the sum over all possible winding numbers \( w = 2q + 1 \) of the closed string around the compact dimension \( X_0 \). Only odd winding numbers survive since the contribution of the even ones vanishes due to supersymmetry. The vanishing of the term with zero winding number is analogous to that at zero temperature and is due to the cancellation between the NS-NS and RR sectors.

When two D0-branes interact, they can exchange several closed strings, not necessarily one. All such exchanges are of the same order of magnitude in the string coupling constant and exponentiate since the closed strings are identical. This is analogous to the exponentiation of the single-gluon exchange when the interaction between static quark and antiquark in the Yang–Mills theory is calculated via the correlator of two Polyakov loops. Therefore, Eq. (19) naturally emerges in the closed string language. It is also clear why there is only one \( \nu \) for each multi-string term: we have just two interacting D0-branes rather than a gas of D0-branes. This results in Eq. (23).

Each of the closed strings mediating the interaction between D0-branes has its own winding number \( w_i \). In the open string language, this induces on the D-brane the charge \( \sum_i w_i \) with respect to the open-string gauge field. Such charged states look suspicious since they are missing at zero temperature where \( X_0 \) is not compact and there are no windings along the \( X_0 \) direction, so that the total charge equals zero at each value of the time \( \tau \). But the integration over \( \nu \) picks up exactly the state with \( \sum_i w_i = 0 \), i.e. which is not charged! In particular, all states with a single closed string vanish after the integration over \( \nu \). The leading order contribution to the D0-brane interaction comes from the state with two closed strings having unit winding numbers of opposite signs.

It is worth noting that in the closed string language \( w_i \) is associated with the NS-NS charge of the closed string. Therefore, the condition \( \sum_i w_i = 0 \) implies the vanishing of the total NS-NS charge.

5 Discussion

The effective static potential between two D0-branes emerges because supersymmetry is broken by finite temperature. This effect of breaking supersymmetry is somewhat analogous to the velocity effects at zero temperature where the matrix theory and superstring computations agree to the leading order of the velocity expansion [9]. It is thus shown that the leading term in a low temperature expansion is correctly reproduced by the matrix theory.

The effective static potential between D0-branes at one loop is logarithmic and attractive at short distances. The singularity occurs when the distance between the D0-branes vanishes and the SU(N) symmetry which is broken by finite distances is restored. The
integration over the off-diagonal components of the gauge field can no longer be treated in the one-loop approximation! (This issue has been further discussed recently in Ref. [10].) In the superstring theory, the singularity is exactly the same as in the matrix theory since it is determined only by the massless bosonic modes in the NS sector.

The computed partition functions take into account only thermal fluctuations of superstring stretched between D0-branes but not the fluctuations of D0-branes themselves. These separation of the degrees of freedom is justified by the fact that D0-branes have a mass $1/g_s \sqrt{\alpha'}$ and are very heavy as $g_s \to 0$. To calculate the thermal partition function of D0-branes, a further path integration over their periodic trajectories $\vec{a}(\tau)$ is to be performed as in (6). Classical statistics is not applicable to this problem due to the singularity of the one-loop effective static potential at small distances in spite of the fact that D0-branes are very heavy. However, this singularity is only in the classical partition function. The path integral over the periodic trajectories $\vec{a}(\tau)$ can not diverge since the two-body quantum mechanical problem has a well-defined spectrum.

Acknowledgements
This work is supported in part by the grants INTAS 96–0524 and RFFI 97–02–17927.

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