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Sharp space-time regularity of the solution to stochastic heat equation driven by fractional-colored noise

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ABSTRACT

In this article, we study the following stochastic heat equation

$$\partial_t u(t,x) = Lu(t,x) + B, \quad u(0,x) = 0, \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^d,$$

where $L$ is the generator of a Lévy process $X$ in $\mathbb{R}^d$, $B$ is a fractional-colored Gaussian noise with Hurst index $H \in \left(\frac{1}{2}, 1\right)$ in the time variable and spatial covariance function $f$ which is the Fourier transform of a tempered measure $\mu$. After establishing the existence of solution for the stochastic heat equation, we study the regularity of the solution $\{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ in both time and space variables. Under mild conditions, we establish the exact uniform modulus of continuity and a Chung-type law of iterated logarithm for the sample function $(t,x) \mapsto u(t,x)$. These results, to our knowledge, are new even for the classical stochastic heat equation (where $L = \Delta$) with space-time white noise and they strengthen the corresponding results of Balan and Tudor (2008) and Tudor and Xiao (2017) where partial regularity results were obtained.

1. Introduction

Stochastic partial differential equations (SPDE) driven by fractional Brownian motion (fBm) or other fractional Gaussian noises have many applications in biology, electrical engineering, finance, physics, among others, see, e.g., [5, 11, 12, 22]. The theoretical studies of SPDEs driven by fBm or other fractional Gaussian noises have been growing rapidly. We refer to, for example, [3, 4, 6–8, 13, 15–18, 24, 27, 28, 30, 31] for recent developments.

In this article, for a fixed constant $T > 0$, we consider the following stochastic heat equation

$$\partial_t u(t,x) = Lu(t,x) + B, \quad u(0,x) = 0, \quad 0 \leq t \leq T, \quad x \in \mathbb{R}^d, \quad (1.1)$$

where $L$ is the generator of a Lévy process taking values in $\mathbb{R}^d$, and $B$ is a fractional-colored Gaussian noise with Hurst index $H \in \left(\frac{1}{2}, 1\right)$ in the time variable and spatial covariance function $f$ as in Balan and Tudor [4]. Namely,
The solution of (1.1) has been studied by Balan and Tudor [4] and the regularity properties of the solution process under mild conditions in Section 3. Finally, we provide a general result for a real-valued centered Gaussian random field with stationary increments to be strongly locally nondeterministic in Appendix A, and we believe this general result is of independent interest.

Throughout this article, for any appropriate measure \( \mu \) on \( \mathbb{R}^d \), we use \( \mathcal{F}\mu(\xi) \) to denote the Fourier transform of \( \mu \), that is

\[
\mathbb{E}\left(B(t,A)B(s,C)\right) = R_H(t,s)\int_{A \cap C} f(z-z')dzdz',
\]

where \( R_H(t,s) := \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \) is the covariance of a fractional Brownian motion with index \( H \in (\frac{1}{2}, 1) \), and \( f \) is the Fourier transform of a tempered measure \( \mu \), which is defined by

\[
\int_{\mathbb{R}^d} f(x)\varphi(x)dx = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\mu(d\xi), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).
\]

Throughout this article, \( \mathcal{B}(\mathbb{R}^d) \) denotes the family of Borel sets in \( \mathbb{R}^d \), \( \mathcal{S}(\mathbb{R}^d) \) denotes the Schwarz space on \( \mathbb{R}^d \), and \( \mathcal{F}\varphi \) denotes the Fourier transform of the function \( \varphi : \mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-i\langle \xi, x \rangle} \varphi(x)dx \). It is known that the mapping \( \mathcal{F} : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d) \) is an isomorphism which extends uniquely to a unitary isomorphism of \( L^2(\mathbb{R}^d) \).

We will make use of the following identity (cf. p.6 of [10]): For any \( \varphi, \psi \in \mathcal{S}(\mathbb{R}^d) \),

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x)f(x-y)\psi(y)dxdy = (2\pi)^{-d} \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi).
\]
\[ \mathcal{F} \mu(\xi) = \int_{\mathbb{R}^d} e^{-i(\xi, x)} \mu(dx), \quad x \in \mathbb{R}^d. \]

2. Existence of the solution

Let \( X = \{ X_t, \ t \geq 0 \} \) be a Lévy process taking values in \( \mathbb{R}^d \), with \( X_0 = 0 \) and characteristic exponent \( \Psi(\xi) \) given by
\[ \mathbb{E}(e^{i(\xi, X_t)}) = e^{-t\Psi(\xi)}, \quad \forall t \geq 0, \ \xi \in \mathbb{R}^d. \]

Let \( \mathcal{L} \) be the generator of \( X \). The domain of \( \mathcal{L} \) is given by
\[ \text{Dom}(\mathcal{L}) = \left\{ \phi \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\mathcal{F}^{-1}\phi(\xi)|^2 |\Psi(\xi)| d\xi < \infty \right\}, \]
where \( \mathcal{F}^{-1} \) denotes the inverse Fourier transform in \( L^2(\mathbb{R}^d) \).

We first recall from [4] some facts about integration of deterministic functions with respect to the fractional-colored noise \( B \). Unless mentioned otherwise, we will use the same notation as in [4].

Let \( \mathcal{D}((0, T) \times \mathbb{R}^d) \) denote the space of all infinitely differentiable functions with compact support contained in \((0, T) \times \mathbb{R}^d\) and let \( \mathcal{H}^2 \) be the completion of \( \mathcal{D}((0, T) \times \mathbb{R}^d) \) with respect to the inner product
\[
\langle \phi, \psi \rangle_{\mathcal{H}^2} = q_H \int_0^T \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(u, x)|u - v|^{2H-2}f(x - y)\psi(v, y) dydxdu dv
\]
\[ = q_H c_H \int_{\mathbb{R}} |\tau|^{1-2H} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x - y)\mathcal{F}_{0,T}\phi(\tau, x)\mathcal{F}_{0,T}\psi(\tau, y) dydx d\tau, \quad (2.1) \]

where \( q_H = H(2H - 1), \ c_H = [2^{2(1-H)}\sqrt{\pi}]^{-1}\Gamma(H - 1/2)/\Gamma(1 - H) \), and \( \mathcal{F}_{0,T}\phi \) is the restricted Fourier transform of \( \phi \) in the variable \( t \in (0, T) \) defined by
\[ \mathcal{F}_{0,T}\phi(\tau) = \int_0^T e^{-it\tau}\phi(t) dt. \]

Note that the inner product in (2.1) is defined under the assumption \( H \in (\frac{1}{2}, 1) \).

Without this assumption, it is not a well-defined inner product. In (2.1), the second equality follows from [4, Lemma A.1.(b)]. It follows from (2.1) and (1.2) that
\[
||\phi||_{\mathcal{H}^2} = q_H c_H \int_{\mathbb{R}} |\tau|^{1-2H} d\tau \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x - y)\mathcal{F}_{0,T}\phi(\tau, x)\mathcal{F}_{0,T}\phi(\tau, y) dydx
\]
\[
= \frac{q_H c_H}{(2\pi)^d} \int_{\mathbb{R}} |\tau|^{1-2H} dt \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{F}_{0,T}\phi(\tau, \cdot)(\xi)\mathcal{F}_{0,T}\phi(\tau, \cdot)(\xi)\mu(d\xi). \]

Let \( B = \left\{ B(\phi) : \phi \in \mathcal{D}((0, T) \times \mathbb{R}^d) \right\} \) be a centered Gaussian process with covariance
\[
\mathbb{E}[B(\phi)B(\psi)] = \langle \phi, \psi \rangle_{\mathcal{H}^2}. \quad (2.2) \]

For any \( t > 0 \) and \( A \in \mathcal{B}(\mathbb{R}^d) \), one can define \( B_t(A) = B(\mathbb{1}[0,t] \times A) \) as the \( L^2(\Omega) \)-limit of the Cauchy sequence \( \{ B(\phi_n) \} \), where \( \{ \phi_n \} \subset \mathcal{D}((0, T) \times \mathbb{R}^d) \) converges to the indicator function \( \mathbb{1}[0,t] \times A \) point wisely. By a routine limiting argument, one can show that
(2.2) remains valid when \( \varphi \) and \( \psi \) are functions of the form \( \mathbb{I}_{[0,t] \times A} \) with \( t > 0 \) and \( A \in \mathcal{B}(\mathbb{R}^d) \).

Let \( \mathcal{E} \) be the space of all linear combinations of indicator functions \( \mathbb{I}_{[0,t] \times A} \), where \( t \in [0, T] \) and \( A \in \mathcal{B}(\mathbb{R}^d) \) which is the class of all bounded Borel sets in \( \mathbb{R}^d \).

One can extend the definition of \( \mathbb{E}[B(\varphi)B(\psi)] \) to \( \mathcal{E} \) by linearity. Then we have

\[
\mathbb{E}[B(\varphi)B(\psi)] = \langle \varphi, \psi \rangle_{\mathcal{H}^B}, \quad \forall \varphi, \psi \in \mathcal{E},
\]

i.e., \( \varphi \to B(\varphi) \) is an isometry between \( \langle \cdot, \cdot \rangle_{\mathcal{H}^B} \) and \( \mathcal{H}^B \), where \( \mathcal{H}^B \) is the Gaussian space generated by \( \{B(\varphi), \varphi \in \mathcal{D}(0, T) \times \mathbb{R}^d\} \).

Since the space \( \mathcal{H}^P \) is the completion of \( \mathcal{E} \) with respect to \( \langle \cdot, \cdot \rangle_{\mathcal{H}^P} \), the isometry (2.3) can be extended to \( \mathcal{H}^P \), giving us the stochastic integral of \( \varphi \in \mathcal{H}^P \) with respect to \( B \).

We denote this stochastic integral by

\[
B(\varphi) = \int_0^T \int_{\mathbb{R}^d} \varphi(t,x)B(dtdx).
\]

Now we study the existence of solution of (1.1). Before doing this, we prove some preliminary results first.

We assume that the Lévy process \( X = \{X_t\} \) has a transition density which is given by

\[
p_t(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i(x, \xi)} e^{-t\varphi(\xi)} d\xi = (2\pi)^{-d/2} \mathcal{F}e^{-t\varphi}(x), \quad \forall t > 0.
\]

As in [4, (3.25)], we define the solution of the Cauchy problem (1.1) as follows. A random field \( \{u(t,x) : (t,x) \in [0,T] \times \mathbb{R}^d\} \) is said to be a solution of (1.1) if for every \( \eta \in \mathcal{D}(0, T) \times \mathbb{R}^d \),

\[
\int_0^T \int_{\mathbb{R}^d} u(t,x)\eta(t,x)dxdt = \int_0^T \int_{\mathbb{R}^d} \langle \eta \ast \widehat{p}\rangle(t,x)B(dtdx), \quad a.s.,
\]

where \( \widehat{p}_s(y) = p_s(-y) \).

For any \( t \in (0, T), x \in \mathbb{R}^d \), denote the function \( g_{t,x}(s,y) = p_{t-s}(x-y)\mathbb{I}_{\{s < t\}}, \ s \in (0, T), \ y \in \mathbb{R}^d \). We first derive conditions on the characteristic exponent \( \varphi \) such that \( \|g_{t,x}\|_{\mathcal{H}^P} < \infty \), which extends [4, Theorem 3.12].

By noting that

\[
(2\pi)^d \mathcal{F}^{-1} p_{t-s}(\cdot - x)(\xi) = \mathbb{E}[e^{i\xi(t,x,t-s)}] = e^{-i(t,\xi) - (t-s)\varphi(\xi)}
\]

we have

\[
\|g_{t,x}\|_{\mathcal{H}^P} = q_H \int_0^t \int_0^t |s - r|^{2H-2} drds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g_{t,x}(s,y)\mathcal{F}(y-z)g_{t,x}(r,z)dydz
\]

\[
= (2\pi)^d q_H \int_0^t \int_0^t |s - r|^{2H-2} drds \int_{\mathbb{R}^d} \mathcal{F}^{-1} g_{t,x}(s,\cdot)(\xi)\mathcal{F}^{-1} g_{t,x}(r,\cdot)(\xi)\mu(d\xi)
\]

\[
= (2\pi)^d q_H \int_0^t \int_0^t |s - r|^{2H-2} drds \int_{\mathbb{R}^d} \mathcal{F}^{-1} p_{t-s}(\cdot - x)(\xi)\mathcal{F}^{-1} p_{t-r}(\cdot - x)(\xi)\mu(d\xi)
\]

\[
= q_H \int_0^t \int_0^t |s - r|^{2H-2} drds \int_{\mathbb{R}^d} e^{-(t-s)\varphi(\xi) - (t-r)\varphi(-\xi)}\mu(d\xi).
\]
Note that the display above says that \( \|g_{t,x}\|_{\mathcal{F}} \) is independent of \( x \). By Fubini’s theorem, we have

\[
\|g_{t,x}\|_{\mathcal{F}} = q_{H} \int_{\mathbb{R}^{d}} \mu(d\xi) \int_{0}^{t} \int_{0}^{t} |s-r|^{2H-2} e^{-(t-s)\Psi(\xi)-(t-r)\Psi(-\xi)} dsdr.
\]

We consider the inner integral (in \( s \) and \( r \)) first. Define

\[
h_1(s) := \mathbb{I}_{[0,t]}(s)e^{-(t-s)\Psi(\xi)}, \quad h_2(r) := \mathbb{I}_{[0,t]}(r)e^{-(t-r)\Psi(-\xi)}.
\]

It follows from [4, Lemma A.1(b)] that for any constant \( \alpha \in (0, 1) \) and for all \( \varphi, \psi \) from \( L^2(a, b) \), we have

\[
\int_{a}^{b} \int_{a}^{b} \varphi(u)|u-v|^{-(1-\alpha)}\psi(v)dvdu = c_{\alpha+1} \int_{\mathbb{R}} |\tau|^{-\alpha} \mathcal{F}\varphi(\tau)\mathcal{F}\psi(\tau)d\tau,
\]

where \( c_{\alpha+1} = (2^{1-\alpha}\pi)^{-1}\Gamma(\alpha/2)\Gamma((1-\alpha)/2) \). Thus, by choosing \( \alpha = 2H - 1 \), \( \varphi = h_1 \), \( \psi = h_2 \), \( a = 0 \), \( b = t \),

\[
\int_{0}^{t} \int_{0}^{t} |s-r|^{2H-2} e^{-(t-s)\Psi(\xi)-(t-r)\Psi(-\xi)} dsdr = \int_{\mathbb{R}} \int_{\mathbb{R}} |s-r|^{2H-2} h_1(s)h_2(r) dsdr = c_{H} \int_{\mathbb{R}} |\tau|^{-2H} \mathcal{F}h_1(\tau)\mathcal{F}h_2(\tau)d\tau.
\]

By using the change of variables \( t - s = s' \), we get

\[
\mathcal{F}h_1(\tau) = \int_{0}^{t} e^{-its-(t-s)\Psi(\xi)} ds = e^{-itt} \int_{0}^{t} e^{its-t\Psi(\xi)} ds = \frac{e^{-itt}}{it - \Psi(\xi)} \left( 1 - e^{itt-t\Psi(\xi)} \right),
\]

and similarly,

\[
\mathcal{F}h_2(\tau) = \frac{e^{-itt}}{it + \Psi(-\xi)} \left( 1 - e^{itt-t\Psi(-\xi)} \right).
\]

Consequently, letting \( K_H = q_H c_H \), we have

\[
\|g_{t,x}\|_{\mathcal{F}} = K_H \int_{\mathbb{R}^{d}} \mu(d\xi) \int_{\mathbb{R}} |\tau|^{1-2H} \frac{1}{it - \Psi(\xi)} \frac{1}{it + \Psi(-\xi)} \left( 1 - e^{itt-t\Psi(\xi)} \right) \left( 1 - e^{itt+t\Psi(-\xi)} \right) d\tau
\]

\[
= K_H \int_{\mathbb{R}^{d}} \mu(d\xi) \int_{\mathbb{R}} \frac{|\tau|^{1-2H}}{\tau^2 + \Psi(\xi)^2} \left( 1 - e^{itt-t\Psi(\xi)} \right) \left( 1 - e^{-itt-t\Psi(-\xi)} \right) d\tau,
\]

(2.5)

where we have used the fact that \( \Psi(-\xi) = \Psi(\xi) \).

Thus, we have the following theorem which is an extension of Balan and Tudor [4, Theorem 3.15].
Theorem 2.1. Assume that \( \mathcal{L} \) in (1.1) is the generator of a Lévy process in \( \mathbb{R}^d \) with characteristic exponent \( \Psi(\xi) \) and transition density (2.4). For any fixed \( t > 0 \) and \( x \in \mathbb{R}^d \), \( \|g_{t,x}\|_{H^p} < \infty \) if and only if
\[
\int_{\mathbb{R}^d} \mu(d\xi) \int_{\mathbb{R}} \left| \frac{\tau^{1-2H}}{\tau^2 + \Psi(\xi)^2} \right| \left( 1 - e^{i\tau t - t\Psi(\xi)} \right) \left( 1 - e^{-i\tau t - t\Psi(\xi)} \right) d\tau < \infty. \tag{2.6}
\]
Moreover, if (2.6) holds, then (1.1) has a solution \( \{u(t,x), (t,x) \in [0, T] \times \mathbb{R}^d\} \) and for all \( (t,x) \in [0, T] \times \mathbb{R}^d \),
\[
u(t,x) = \int_0^t \int_{\mathbb{R}^d} g_{t-x}(s,y) B(dy). \tag{2.7}
\]

**Proof.** The first part of the theorem follows directly from (2.5). The proof of the existence of solution and its representation (2.7) is the same as that of [4, Theorem 2.11]. We omit the details here. \( \square \)

In general, the condition (2.6) is rather complicated to verify (even though it is possible to derive a sufficient condition for \( \|g_{t,x}\|_{H^p} < \infty \) when \( \Psi(\xi) \) is complex-valued). In order to derive an easily verifiable equivalent condition to (2.6), we assume from now on that \( X \) is a symmetric Lévy process. This is equivalent to \( \Psi(\xi) = \Psi(-\xi) \geq 0 \) for all \( \xi \in \mathbb{R}^d \). In this case, (2.5) reduces to
\[
\|g_{t,x}\|_{H^p} = K_H \int_{\mathbb{R}^d} \mu(d\xi) \int_{\mathbb{R}} \left| \frac{\tau^{1-2H}}{\tau^2 + \Psi(\xi)^2} \right| \left( 1 - e^{i\tau t - t\Psi(\xi)} \right)^2 d\tau. \tag{2.8}
\]

We now prove the following theorem which provides an easily verifiable condition for \( \|g_{t,x}\|_{H^p} < \infty \) and for the existence of solution of (1.1).

**Theorem 2.2.** Assume that \( \mathcal{L} \) in (1.1) is the generator of a symmetric Lévy process in \( \mathbb{R}^d \) with characteristic exponent \( \Psi(\xi) \) and transition density (2.4). For any fixed \( t > 0 \) and \( x \in \mathbb{R}^d \), \( \|g_{t,x}\|_{H^p} < \infty \) if and only if
\[
\int_{\mathbb{R}^d} \mu(d\xi) \frac{\tau^{1-2H}}{1 + \Psi(\xi)^2} < \infty. \tag{2.9}
\]
Moreover, if (2.9) holds, then (1.1) has a solution \( \{u(t,x), (t,x) \in [0, T] \times \mathbb{R}^d\} \) given by (2.7).

Before proving Theorem 2.2, we prove the following lemma first.

**Lemma 2.3.** We have that
\[
K'^H x^{2H} \leq \int_{\mathbb{R}} |v|^{1-2H} \frac{1}{1 + v^2} \left| 1 - e^{(iv-1)x} \right|^2 dv \leq K_H x^{2H}, \quad \forall x \in (0, 1), \tag{2.10}
\]
where \( K_H, K'_H \) are positive finite constants only depending on \( H \).
Proof. Notice that the integral in (2.10) can be written as
\[
\int_{\mathbb{R}} |v|^{1-2H} \frac{1 - 2e^{-x} \cos (vx) + e^{-2x}}{1 + v^2} dv
\]
\[
= \int_{|v| \leq x^{-1}} |v|^{1-2H} \frac{1 - 2e^{-x} \cos (vx) + e^{-2x}}{1 + v^2} dv
\]
\[
+ \int_{|v| > x^{-1}} |v|^{1-2H} \frac{1 - 2e^{-x} \cos (vx) + e^{-2x}}{1 + v^2} dv
\]
\[
:= I_1 + I_2.
\]

For \( I_1 \), we use the inequality \( \cos (vx) \geq 1 - (vx)^2 \) for \( |vx| \leq 1 \) to get that
\[
I_1 \leq \int_{|v| \leq x^{-1}} |v|^{1-2H} \frac{1 - 2e^{-x}(1 - v^2x^2) + e^{-2x}}{1 + v^2} dv
\]
\[
\leq \int_{|v| \leq x^{-1}} \frac{|v|^{1-2H}}{1 + v^2} (1 - e^{-x})^2 dv + 2 \int_{|v| \leq x^{-1}} |v|^{1-2H} x^2 dv
\]
\[
\leq K_H x^2 + K_H x^{2H} \leq K x^{2H}.
\]  

For \( I_2 \), we have
\[
I_2 \leq 4 \int_{|v| > x^{-1}} |v|^{1-2H} \frac{1}{1 + v^2} dv \leq 8 \int_{x^{-1}}^{\infty} v^{-(1+2H)} dv = K_H x^{2H}.
\]  

Combining (2.11) and (2.12), we get the second inequality in (2.10).

Meanwhile, we use the inequality \( \cos (vx) \leq 1 - (vx)^2/4 \) for \( |vx| \leq 1 \) to get
\[
\int_{\mathbb{R}} |v|^{1-2H} \frac{1 - 2e^{-x} \cos (vx) + e^{-2x}}{1 + v^2} dv
\]
\[
\geq \int_{|v| \leq x^{-1}} |v|^{1-2H} \frac{1 - 2e^{-x} \cos (vx) + e^{-2x}}{1 + v^2} dv
\]
\[
\geq \int_{|v| \leq x^{-1}} |v|^{1-2H} \frac{1 - 2e^{-x}(1 - v^2x^2/4) + e^{-2x}}{1 + v^2} dv
\]
\[
\geq \frac{1}{2e} \int_{|v| \leq x^{-1}} |v|^{3-2H} x^2 dv
\]
\[
\geq K_H x^{2H},
\]

which proves the first inequality in (2.10), and we arrive at the conclusion of Lemma 2.3. □

Remark 2.4. In the proof of Lemma 2.3, we did not use the assumption \( H \in \left( \frac{1}{2}, 1 \right) \), so Lemma 2.3 is true for all \( H \in (0, 1) \).

Now we are ready to prove Theorem 2.2.

Proof. We consider the inner integral with respect to \( \tau \) in (2.8). Let \( \tau = \Psi(\xi)v \), we have
\[
\int_{\mathbb{R}} \frac{1}{\tau^2 + \Psi(\xi)^2} \left| 1 - e^{i\tau - r\Psi(\xi)} \right|^2 d\tau = \frac{1}{\Psi(\xi)^{2H}} \int_{\mathbb{R}} \frac{|v|^{1-2H}}{1 + v^2} \left| 1 - e^{i(r-1)v\Psi(\xi)} \right|^2 dv.
\]  

(2.13)
Clearly, for all $\xi \in \mathbb{R}^d$,
\[
\int_{\mathbb{R}} \frac{|v|^{1-2H}}{1 + v^2} \left| 1 - e^{(iv-1)r\Psi(\xi)} \right|^2 dv \leq 8 \int_0^\infty \frac{v^{1-2H}}{1 + v^2} dv \leq K < \infty \tag{2.14}
\]
for some positive constant $K$ independent of $\xi$.

We prove the sufficiency first. To bound the integral on the right hand side of (2.13) from above, we consider two cases separately:

Case 1. If $t\Psi(\xi) > 1$, then by (2.14), we have
\[
\int_{\mathbb{R}} \frac{|v|^{1-2H}}{1 + v^2} \left| 1 - e^{(iv-1)r\Psi(\xi)} \right|^2 dv \leq \frac{K}{\Psi(\xi)^{2H}}. \tag{2.15}
\]

Case 2. If $t\Psi(\xi) \leq 1$, then by Lemma 2.3, we have
\[
\int_{\mathbb{R}} \frac{|v|^{1-2H}}{1 + v^2} \left| 1 - e^{(iv-1)r\Psi(\xi)} \right|^2 dv \leq Kt^{2H}. \tag{2.16}
\]

Combining (2.14), (2.15) and (2.16), we get
\[
\int_{\mathbb{R}} \frac{|v|^{1-2H}}{1 + v^2} \left| 1 - e^{(iv-1)r\Psi(\xi)} \right|^2 dv \leq \frac{Kt^{2H}}{1 + (t\Psi(\xi))^{2H}}.
\]

Since for any $t > 0$ fixed,
\[
\frac{t^{2H}}{1 + (t\Psi(\xi))^{2H}} < \frac{1}{1 + \Psi(\xi)^{2H}},
\]
we have proved the sufficiency in Theorem 2.2.

Now, we prove the necessity. To bound the integral on the right hand side of (2.13) from below, we again consider two cases separately:

Case 1. If $t\Psi(\xi) > 1$, we have
\[
\int_{\mathbb{R}} \frac{|v|^{1-2H}}{1 + v^2} \left| 1 - e^{(iv-1)r\Psi(\xi)} \right|^2 dv \geq \frac{1}{\Psi(\xi)^{2H}} \int_{\mathbb{R}} \frac{|v|^{1-2H}}{1 + v^2} \left( 1 - e^{-r\Psi(\xi)} \right)^2 dv \geq \frac{(1 - e^{-1})^2}{\Psi(\xi)^{2H}} \int_{\mathbb{R}} \frac{|v|^{1-2H}}{1 + v^2} dv = \frac{K'}{\Psi(\xi)^{2H}}, \tag{2.17}
\]

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Case 2. If $t\Psi(\zeta) \leq 1$, then by Lemma 2.3, we have
\[
\frac{1}{\Psi(\zeta)^{2H}} \int_{\mathbb{R}} \frac{|v|^{1-2H}}{1 + v^2} \left| 1 - e^{(iv-1)t\Psi(\zeta)} \right|^2 dv \geq K't^{2H}.
\] (2.18)

Combining (2.17) and (2.18), we get
\[
\frac{1}{\Psi(\zeta)^{2H}} \int_{\mathbb{R}} \frac{|v|^{1-2H}}{1 + v^2} \left| 1 - e^{(iv-1)t\Psi(\zeta)} \right|^2 dv \\
\geq K' \left( \frac{1}{\Psi(\zeta)^{2H}} \mathbb{I}\{t\Psi(\zeta) > 1\} + t^{2H} \mathbb{I}\{t\Psi(\zeta) \leq 1\} \right) \\
\geq \frac{K't^{2H}}{1 + (t\Psi(\zeta))^{2H}}.
\]

Again, since for any $t > 0$ fixed,
\[
\frac{t^{2H}}{1 + (t\Psi(\zeta))^{2H}} \asymp \frac{1}{1 + \Psi(\zeta)^{2H}},
\]
we have proved the necessity in Theorem 2.2.

We end this section with some interesting examples.

**Example 2.5.** If $X$ is a symmetric Lévy process in $\mathbb{R}^d$ whose characteristic exponent $\Psi$ satisfies
\[
\Psi(\zeta) \asymp |\zeta|^\alpha \quad \text{for all} \quad \zeta \in \mathbb{R}^d \quad \text{with} \quad |\zeta| \geq 1,
\] (2.19)
and the measure $\mu$ is given by $\mu(d\zeta) = |\zeta|^{-\beta}d\zeta$ with $0 < \beta < d$, then by Theorem 2.2, $\|g_{t,x}\|_{H^p} < \infty$ is equivalent to
\[
\int_{\mathbb{R}^d} \frac{\mu(d\zeta)}{1 + \Psi(\zeta)^{2H}} \asymp \int_{\mathbb{R}^d} \frac{d\zeta}{|\zeta|^\beta (1 + |\zeta|^{2\alpha H})} < \infty,
\]
which in turn is equivalent to
\[
d - 2xH < \beta < d, \quad \text{or} \quad H > \frac{d - \beta}{2\alpha} > 0.
\]
In the above, the notation $\asymp$ means the ratio of the quantities on the two sides is bounded from above and below by positive and finite constants.

Some concrete examples of symmetric Lévy processes satisfying condition (2.19) are as follows:

(i) Isotropic $\alpha$-stable process, for which $\Psi(\zeta) = |\zeta|^\alpha$;
(ii) relativistic $\alpha$-stable process with mass $m > 0$ ([9, 29]), for which $\Psi(\zeta) \asymp |\zeta|^\alpha$ for all $\zeta \in \mathbb{R}^d$ with $|\zeta| \geq 1$;
(iii) the independent sum of an isotropic $\alpha$-stable process and an isotropic $\gamma$-stable process with $\gamma < \alpha$, for which $\Psi(\zeta) = |\zeta|^\alpha + |\zeta|^\gamma$;
(iv) symmetric $\alpha$-stable process with Lévy density comparable to that of the isotropic $\alpha$-stable process;
truncated $\alpha$-stable process ([21]), for which
\[
\Psi(\xi) = c \int_{|y|<1} \frac{1 - \cos \langle \xi, y \rangle}{|y|^{d+\alpha}} dy
\]
for some constant $c > 0$.

3. Regularity of the solution process in time and space

Throughout this section, we will assume that the conditions of Theorem 2.2 hold. We study the sample path regularities of the solution $u(t, x)$ of (1.1) as a Gaussian random field in variables $(t, x)$. Similar to the case of the random string process in Mueller and Tribe [26] (see also [2, 32]), we consider the Gaussian random fields $\{U(t, x), t \geq 0, x \in \mathbb{R}^d\}$ and $\{Y(t, x), t \geq 0, x \in \mathbb{R}^d\}$ defined, respectively, by
\[
U(t, x) = \int_{-\infty}^{0} \int_{\mathbb{R}^d} \left( p_{t-u}(x - y) - p_{-u}(-y) \right) B(du, dy) + \int_{0}^{t} \int_{\mathbb{R}^d} p_{t-u}(x - y) B(du, dy) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left( p_{(t-u)}(x - y) - p_{-u}(-y) \right) B(du, dy),
\]
and
\[
Y(t, x) = \int_{-\infty}^{0} \int_{\mathbb{R}^d} \left( p_{t-u}(x - y) - p_{-u}(-y) \right) B(du, dy),
\]
where $a_+ = \max\{a, 0\}$, and we use the convention that $p_s(z) = 0$ whenever $s < 0$.

Clearly, $u(t, x) = U(t, x) - Y(t, x)$ for all $t \geq 0$ and $x \in \mathbb{R}^d$. In this section, we will show that the sample function of $Y(t, x)$ is smoother in $(t, x)$ than $U(t, x)$, hence the regularity properties of $u(t, x)$ follow from those of $U(t, x)$. In particular, we can use this decomposition to obtain the exact uniform and local moduli of continuity of the solution $\{u(t, x), t \geq 0, x \in \mathbb{R}^d\}$. This idea is similar to that in [32]. However, we point out that Tudor and Xiao [32] only studied the partial sample path regularities of the solution $u$ in time variable $t$ (with $x$ fixed) and in space variable $x$ (with $t$ fixed), while we provide the corresponding results in time and space variables $(t, x)$ simultaneously here. The key ingredient in our derivation is the strong local nondeterminism of $\{U(t, x), t \geq 0, x \in \mathbb{R}^d\}$, which is proved in Theorem 3.4.

The following theorem shows that $\{U(t, x), t \geq 0, x \in \mathbb{R}^d\}$ has stationary increments and provides information on its spectral measure, which is useful for applying a Fourier-analytic argument to establish the properties of local nondeterminism (see Xiao [35, 36]).

**Theorem 3.1.** The Gaussian random field $U = \{U(t, x), t \geq 0, x \in \mathbb{R}^d\}$ has stationary increments with spectral measure given by
\[
F_U(d\xi, d\tau) = \frac{1}{\tau^{2H-1} \left( \tau^2 + \Psi(\xi)^2 \right)^{\frac{d}{d+\alpha}}} \mu(d\xi) d\tau.
\]
Proof. To prove the stationarity of increments of $U$, we compute $\mathbb{E}(U(t,x) - U(s,y))^2$ for all $0 \leq s < t$ and $x, y \in \mathbb{R}^d$. This will also allow us to identify the spectral measure. Notice that

$$U(t,x) - U(s,y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( p_{(t-u)_+}(x-z) - p_{(s-u)_+}(y-z) \right) B(du,dz).$$

By Parseval’s identity (1.2) we have

$$\mathbb{E}(U(t,x) - U(s,y))^2 = q_H \int_{\mathbb{R}} \int_{\mathbb{R}} |u - v|^{2H-2} dudv \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( p_{(t-u)_+}(x-z) - p_{(s-u)_+}(y-z) \right) \times f(z-z') \left( p_{(t-v)_+}(x-z') - p_{(s-v)_+}(y-z') \right) dzdz'$$

$$= (2\pi)^{-d} q_H \int_{\mathbb{R}} \int_{\mathbb{R}} |u - v|^{2H-2} dudv \int_{\mathbb{R}^d} \mathcal{F} \left( p_{(t-u)_+}(x - \cdot) - p_{(s-u)_+}(y - \cdot) \right)(\xi)$$

$$\times \mathcal{F} \left( p_{(t-v)_+}(x - \cdot) - p_{(s-v)_+}(y - \cdot) \right)(\xi) \mu(d\xi).$$

Note that the Fourier transform of $p_t(x)$ is given by

$$\mathcal{F} p_t(x - \cdot)(\xi) = e^{i(x,\xi) - t\Psi(\xi)} \mathbb{1}_{\{t > 0\}}, \quad \xi \in \mathbb{R}^d,$$

thus

$$\mathcal{F} \left( p_{(t-u)_+}(x - \cdot) - p_{(s-u)_+}(y - \cdot) \right)(\xi)$$

$$= \mathcal{F} p_{(t-u)_+}(x - \cdot)(\xi) - \mathcal{F} p_{(s-u)_+}(y - \cdot)(\xi)$$

$$= e^{-i(x,\xi)} e^{-(t-u)\Psi(\xi)} \mathbb{1}_{\{t > u\}} - e^{-i(y,\xi)} e^{-(s-u)\Psi(\xi)} \mathbb{1}_{\{s > u\}}$$

$$:= \phi_{t,s}(u,\xi).$$

By applying Fubini’s theorem and Parseval’s identity (in $x$), we have

$$\mathbb{E}(U(t,x) - U(s,y))^2$$

$$= c_H \int_{\mathbb{R}} \int_{\mathbb{R}} |u - v|^{2H-2} dudv \int_{\mathbb{R}^d} \phi_{t,s}(u,\xi) \overline{\phi_{t,s}(v,\xi)} \mu(d\xi)$$

$$= c_H \int_{\mathbb{R}^d} \mu(d\xi) \int_{\mathbb{R}} \left| \widehat{\phi_{t,s}}(\cdot,\xi)(\tau) \overline{\widehat{\phi_{t,s}}(\cdot,\xi)(\tau)} |\tau|^{1-2H} d\tau \right|^2$$

$$= c_H \int_{\mathbb{R}^d} \mu(d\xi) \int_{\mathbb{R}} |\widehat{\phi_{t,s}}(\cdot,\xi)(\tau)|^2 |\tau|^{1-2H} d\tau,$$

where

$$\widehat{\phi_{t,s}}(\cdot,\xi)(\tau) = \int_{\mathbb{R}} e^{i\tau r} \phi_{t,s}(r,\xi) dr.$$
By change of variables, we have
\[\phi_{t,s}(\cdot, \xi)(\tau) = e^{-i(x, \xi)} \int_{-\infty}^{\tau} e^{itr} e^{-(t-r)^2 \Psi(\xi)} dr - e^{-i(y, \xi)} \int_{-\infty}^{\tau} e^{itr} e^{-(s-r)^2 \Psi(\xi)} dr\]
\[= e^{-i(x, \xi) - it} \int_{0}^{\infty} e^{-itr} e^{r^2 \Psi(\xi)} dr - e^{-i(y, \xi) - it} \int_{0}^{\infty} e^{-itr} e^{r^2 \Psi(\xi)} dr\]
\[= \left( e^{-i((x, \xi) + t\tau)} - e^{-i((y, \xi) + t\tau)}\right) \frac{1}{it + \Psi(\xi)},\]
thus
\[\mathbb{E}(U(t,x) - U(s,y))^2 = c_H \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-i((x-y, \xi) + (t-s)\tau)} \left| \tau \right|^{1-2H} \left| \tau^2 + \Psi(\xi)^2 \right| \mu(d\xi) d\tau\]
\[= 2c_H \int_{\mathbb{R}^{1+d}} \left[ 1 - \cos((x - y, \xi) + (t - s)\tau) \right] \frac{1}{\left| \tau \right|^{2H-1} \left( \tau^2 + \Psi(\xi)^2 \right)} \mu(d\xi) d\tau,\]
which only depends on \(t - s\) and \(x - y\). Therefore \(U\) has stationary increments and its spectral measure is given by (3.1). This completes the proof of Theorem 3.1.

To state a corollary of Theorem 3.1, we introduce a notation similar to Tudor and Xiao [32, (19)]. For two measures \(\mu\) and \(\nu\) on \(\mathbb{R}^d\), \(\mu \prec \nu\) means that, for every non-negative function \(h\) on \(\mathbb{R}^d\), there exists a finite constant \(K \geq 1\) (which may depend on \(h\)) such that
\[K^{-1} \int_{\mathbb{R}^d} h(\xi) \nu(d\xi) \leq \int_{\mathbb{R}^d} h(\xi) \mu(d\xi) \leq K \int_{\mathbb{R}^d} h(\xi) \nu(d\xi).\]

**Corollary 3.2.** If \(\mu\) satisfies the condition that
\[\mu(d\xi) \asymp |\xi|^{-\beta} d\xi, \quad \text{with} \quad 0 < \beta < d,\]
then the spectral measure of \(U\) satisfies
\[F_U(d\xi, d\tau) \asymp \frac{1}{\tau^{2H-1} \left| \tau^2 + \Psi(\xi)^2 \right| |\xi|^{\beta}} d\xi d\tau.\]

If, in addition, we assume that
\[\Psi(\xi) \asymp |\xi|^\alpha L(\xi),\]
where \(0 < \alpha \leq 2\) and \(L(\cdot)\) is a slowly varying function at \(\infty\), then
\[F_U(d\xi, d\tau) \asymp \frac{1}{\tau^{2H-1} \left( \tau^2 + |\xi|^{2\alpha} L(\xi)^2 \right) |\xi|^{\beta}} d\xi d\tau.\]

**Remark 3.3.** Under (3.3) and (3.4), the stochastic heat equation (1.1) has a solution iff
\[d - 2\alpha H < \beta < d, \quad \text{or} \quad H > \frac{d - \beta}{2\alpha} \sqrt{\frac{1}{2}}.\]
In the following, we assume that $\mu$ and $\Phi$ satisfy (3.3) and (3.4) with $L(\cdot) \equiv 1$. Let
\[ \theta_1 = H - \frac{d - \beta}{2x} \quad \text{and} \quad \theta_2 = x\theta_1. \]
We will characterize the regularity properties of the solution $\{u(t,x), t \geq 0, x \in \mathbb{R}^d\}$ in terms of $\theta_1$ and $\theta_2$. Since $\theta_1 \in (0, 1)$, the solution is rough in $t$. However, $\theta_2 = x\theta_1$ may be bigger than 1 and, in this case, we will show that $x \mapsto u(t,x)$ is differentiable.

The following theorem plays an important role for studying regularity and other sample path properties of $\{U(t,x), t \geq 0, x \in \mathbb{R}^d\}$.

**Theorem 3.4.** Assume that $\mu$ and $\Psi$ satisfy (3.3) and (3.4) with $L(\cdot) \equiv 1$, respectively. For any constant $M > 0$, there exists a finite constant $K_1 > 0$ such that
\[
E\left( (U(t,x) - U(s,y))^2 \right) \leq K_1 \left| t - s \right|^{2\theta_1} + \sigma\left( |x - y| \right),
\] (3.5)
for all $(t,x), (s,y) \in [0, \infty) \times [-M,M]^d$, where the function $\sigma$ is defined as
\[
\sigma(r) = \begin{cases} 
  r^{2\theta_2} & \text{if } 0 < \theta_2 < 1, \\
  r^{2\theta_2} \log r & \text{if } \theta_2 = 1, \\
  r^{2\theta_2} & \text{if } \theta_2 > 1.
\end{cases}
\] (3.6)

Furthermore, if $\theta_2 \leq 1$, then the Gaussian random field $U$ is strongly locally nondeter-
ministic (SLND) in the following sense: there exists a positive constant $K_2$ such that for any positive integer $n$ and any $(t,x) \in (0, \infty) \times \mathbb{R}^d \setminus \{0\}$ and $(s^1, x^1), \ldots, (s^n, x^n) \in \mathbb{R}_+ \times \mathbb{R}^d$, we have
\[
\text{Var}\left( U(t,x) | U(s^1, x^1), \ldots, U(s^n, x^n) \right) \geq K_2 \min_{k=0, \ldots, n} \left( |t - s^k|^\theta_1 + |x - x^k|^\theta_2 \right)^2.
\] (3.7)

**Remark 3.5.** It follows from (3.7) and (3.5) that
\[
K_2 \left( |t - s|^{2\theta_1} + |x - y|^{2\theta_2} \right) \leq E\left( (U(t,x) - U(s,y))^2 \right) \leq K_1 \left( |t - s|^{2\theta_1} + \sigma(|x - y|) \right)
\] (3.8)
for all $(t,x), (s,y) \in [0, \infty) \times [-M,M]^d$. As in Xue and Xiao [37], it can be proved that (3.8) holds if $|x - y|^{2\theta_2}$ in the lower bound is replaced by $\sigma(|x - y|)$. Namely, the upper bound in (3.8) is optimal. On the other hand, (3.5) implies that
\[
\text{Var}\left( U(t,x) | U(s^1, x^1), \ldots, U(s^n, x^n) \right) \leq K_1 \min_{k=0, \ldots, n} \left( |t - s^k|^{2\theta_1} + \sigma(|x - x^k|) \right).
\] (3.9)
Observe that, when $\theta_2 = 1$, the lower bound in (3.7) is smaller than the upper bound in (3.9), and (3.8) shows that (3.7) can be improved if $n = 1$. However, when $\theta_2 \geq 1$ and $n \geq 2$, it is an open problem to establish optimal bounds for the conditional variance $\text{Var}\left( U(t,x) | U(s^1, x^1), \ldots, U(s^n, x^n) \right)$.
Proof of Theorem 3.4. It follows from (3.2) and (3.13) that for all \( (t,x), (s,y) \in \mathbb{R}^{1+d} \),
\[
\mathbb{E}
\left[
(U(t,x) - U(s,y))^2
\right]
\leq K \int_{\mathbb{R}^{1+d}} 1 - \cos \left( (x-y, \xi) + (t-s) \tau \right) \frac{d\tau}{|\tau|^{2H-1}(\tau^2 + |\xi|^{2x})|\xi|^{\beta}}.
\]
To prove (3.5), we write
\[
\mathbb{E}
\left[
(U(t,x) - U(s,y))^2
\right]
\leq 2 \left\{ \mathbb{E}
\left[
(U(t,x) - U(s,x))^2
\right] + \mathbb{E}
\left[
(U(s,x) - U(s,y))^2
\right] \right\}
\leq K \int_{\mathbb{R}^{1+d}} 1 - \cos \left( (t-s) \tau \right) \frac{d\tau}{|\tau|^{2H-1}(\tau^2 + |\xi|^{2x})|\xi|^{\beta}}
+ \int_{\mathbb{R}^{1+d}} 1 - \cos \left( (x-y, \xi) \right) \frac{d\tau}{|\tau|^{2H-1}(\tau^2 + |\xi|^{2x})|\xi|^{\beta}}
:= K \{ I_3 + I_4 \}.
\]
By the change of variables \( \zeta = |\tau|^{1/\gamma}, \) we see that
\[
I_3 = \int_{\mathbb{R}} [1 - \cos \left( (t-s) \tau \right)] \frac{dt}{|\tau|^{1+2\theta_1}} \int_{\mathbb{R}^d} \frac{d\eta}{(1 + |\eta|^{2x})|\eta|^{\beta}},
\]
where \( \theta_1 = H - (d - \beta)/(2x) \). Notice that the last equality in (3.11) is a well-known fact for fractional Brownian motion with index \( \theta_1 \). Similarly, we have
\[
I_4 = \int_{\mathbb{R}^d} [1 - \cos \left( (x-y, \xi) \right)] \frac{d\xi}{|\xi|^{d+2\theta_1}} \int_{\mathbb{R}^d} \frac{d\zeta d\tau}{|\tau|^{2H-1}(\tau^2 + 1)}.
\]
where \( \theta_2 = \alpha \theta_1 \). If \( \theta_2 < 1 \), then \( I_4 = K|x-y|^{2\theta_1} \) (which is related to a fractional Brownian motion of index \( \theta_2 \)). If \( \theta_2 \geq 1 \), then it can be verified as in the proof of Tudor and Xiao [32, Theorem 4] that \( I_4 \leq K\sigma(|x-y|) \) for all \( x, y \in [-M, M]^d \), where the function \( \sigma \) is defined in (3.6). By combining (3.10), (3.11) and (3.12), we obtain (3.5).

Now we prove (3.7). By Theorem 4.1 in the Appendix A, it is sufficient to check that
\[
f_U(\tau, \xi) := \frac{1}{\tau^{2H-1}(\tau^2 + |\xi|^{2x})|\xi|^{\beta}}
\]
satisfies (4.1) for some \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{d+1}) \in (0, 1)^{d+1} \). In fact, by taking \( \gamma_1 = \theta_1, \gamma_2 = \ldots = \gamma_{d+1} = \theta_2 = \alpha \theta_1 \leq 1 \), we see that for any \( c > 0 \),
\[
f_U(c^{\gamma_1} \tau, c^{\gamma_2} \xi) = e^{-\left( \frac{2H+1+\beta}{c^2}\right)} f_U(\tau, \xi) = e^{-\left( 2+Q \right) f_U(\tau, \xi)},
\]
where \( Q = \frac{1}{\theta_1} + \frac{d}{\theta_2} \). Therefore, (3.7) follows from Theorem 4.1. This completes the proof of Theorem 3.4.

We now work on the regularity properties of the Gaussian random field \( \{ Y(t,x), t \geq 0, x \in \mathbb{R}^d \} \).
Theorem 3.6. Assume that the conditions of Theorem 3.4 hold. Let \( I := [a, b] \times [-M, M]^d \), where \( b > a > 0 \) and \( M > 0 \) are fixed constants. There is a modification of \( \{ Y(t, x) \} \) such that its sample function is almost surely continuously (partially) differentiable on \([a, b] \times [-M, M]^d\).

Proof. The method for proving this theorem is similar to those in [32, Theorem 2] and [37, Theorem 4.8]. It has three steps: (a) consider the mean square partial derivatives of \( Y(t, x) \) in \( t, x_j \) (\( j = 1, \ldots, d \)), which are Gaussian random fields; (b) show that they all have continuous modifications by using Kolmogorov’s continuity theorem (or metric entropy condition, cf. [1]); (c) construct a modification of \( Y(t, x) \) such that its sample function has continuous partial derivatives a.s. For the convenience of the reader, we provide a proof for Steps (a) and (b) here. Step (c) is exactly the same as in the proof of part 2 of [37, Theorem 4.8] and is omitted.

The mean square partial derivative of \( Y(t, x) \) in \( t \) can be written as

\[
\frac{\partial}{\partial t} Y(t, x) = \int_0^t \int_{\mathbb{R}^d} \frac{\partial}{\partial t} p_{t-u}(x-y) B(du,dy). \tag{3.15}
\]

This can be checked by verifying the covariance functions of both sides. The mean square derivatives of \( Y(t, x) \) in \( x_j \) (\( j = 1, \ldots, d \)) can be represented similarly. For ease of notation, we write \( \frac{\partial}{\partial t} Y(t, x) \) as \( Y'(t, x) \), and \( \frac{\partial}{\partial t} p_{t-u}(x-y) \) as \( p_{t-u}'(x-y) \). As in the proof of Theorem 3.1, we use Parseval’s identity (1.2) to write that for all \( t > s > 0 \) and \( x, y \in \mathbb{R}^d \),

\[
\mathbb{E}\left[ (Y'(t, x) - Y'(s, y))^2 \right] = q_H \int_{-\infty}^0 \int_{-\infty}^0 |u-v|^{2H-2} dv \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (p_{t-u}'(x-z) - p_{t-u}'(y-z)) \cdot f(z-z') (p_{t-u}'(x-z) - p_{t-u}'(y-z)) \, dz \, dz' \frac{f(z-z')}{(\pi)^{-d}} \, dz \, dz' = (2\pi)^{-d} q_H \int_{-\infty}^0 \int_{-\infty}^0 |u-v|^{2H-2} dv \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{F}(p_{t-u}'(x-\cdot) - p_{t-u}'(y-\cdot))(\xi) \, d\mu(\xi) \cdot d\mu(\xi)
\]

Since

\[
\mathcal{F}(p_{t-u}'(x-\cdot)(\xi) = -\Psi(\xi) e^{-i(\xi, x)-(t-u)\Psi(\xi)},
\]

we have

\[
\mathbb{E}\left[ (Y'(t, x) - Y'(s, y))^2 \right] = (2\pi)^{-d} q_H \int_{-\infty}^0 \int_{-\infty}^0 |u-v|^{2H-2} dv \int_{\mathbb{R}^d} \Psi(\xi)^2 \left[ e^{-(t-u)\Psi(\xi)} - e^{-(s-u)\Psi(\xi)} \right] \cdot e^{-i(\xi, x-y)} \cdot e^{-(t-u)\Psi(\xi)} \cdot e^{-i(\xi, x-y)} e^{-(s-u)\Psi(\xi)} \, d\mu(\xi) 
\]

\[= K \int_{\mathbb{R}^d} \Psi(\xi) \, d\mu(\xi) \left[ |(t-u)\Psi(\xi) - (s-u)\Psi(\xi)| \right] \right. \cdot \left. d\mu(\xi) \right] \frac{1}{|\tau|^{1-2H}} |p_{t-u}^{\sim}(\cdot, \xi)(\tau)|^2 \, d\tau,
\]

\tag{3.16}
where the function \( \varphi_{t,s}(\cdot, \xi) \) is defined by

\[
\varphi_{t,s}(u, \xi) = (e^{-i(\xi,x-y)}e^{-(s+u)\Psi(\xi)})I_{\{u \geq 0\}}
\]

and, in deriving the last equality of (3.16), we have used again Parseval’s identity (in \( u \)). Simple computation shows that

\[
\varphi_{t,s}(\cdot, \xi)(\tau) = \frac{-1}{i\tau - \Psi(\xi)}(e^{-i\Psi(\xi)} - e^{i(\xi,x-y) - s\Psi(\xi)}).
\]

(3.17)

As in the proof of Theorem 3.4, we write

\[
\mathbb{E}\left[ (Y'(t,x) - Y'(s,y))^2 \right] \leq 2 \left\{ \mathbb{E}\left[ (Y'(t,x) - Y'(s,x))^2 \right] + \mathbb{E}\left[ (Y'(s,x) - Y'(s,y))^2 \right] \right\}
\]

and estimate the last two terms separately. It follows from (3.16), (3.17) and conditions (3.3) and (3.4) on \( \mu \) and \( \Psi \) that for all \( s, t \in [a, b] \) and \( x \in \mathbb{R}^d \),

\[
\mathbb{E}\left[ (Y'(t,x) - Y'(s,x))^2 \right] \\
\leq K \int_{\mathbb{R}^d} \Psi(\xi)^2 |e^{-i\Psi(\xi)} - e^{-s\Psi(\xi)}|^2 \mu(d\xi) \int_{\mathbb{R}} |\tau|^{1-2H} \frac{d\tau}{\tau^2 + \Psi(\xi)^2} \\
= K \int_{\mathbb{R}^d} \Psi(\xi)^2 |e^{-i\Psi(\xi)} - e^{-s\Psi(\xi)}|^2 \mu(d\xi) \int_{\mathbb{R}} |\tau|^{1-2H} \frac{d\tau}{\tau^2 + 1} \\
\leq K|t - s|^2.
\]

Similarly, we can verify that \( s \in [a, b] \) and \( x, y \in \mathbb{R}^d \),

\[
\mathbb{E}\left[ (Y'(s,x) - Y'(s,y))^2 \right] \\
\leq K \int_{\mathbb{R}^d} \Psi(\xi)^2 e^{-s\Psi(\xi)}|1 - e^{-i(\xi,x-y)}|^2 \mu(d\xi) \int_{\mathbb{R}} |\tau|^{1-2H} \frac{d\tau}{\tau^2 + \Psi(\xi)^2} \\
= K \int_{\mathbb{R}^d} \Psi(\xi)^2 e^{-s\Psi(\xi)}(1 - \cos\langle \xi, x - y \rangle) \mu(d\xi) \\
\leq K|x - y|^2.
\]

Therefore we have

\[
\mathbb{E}\left[ (Y'(t,x) - Y'(s,y))^2 \right] \leq K(|t - s|^2 + |x - y|^2)
\]

for all \( s, t \in [a, b] \) and \( x, y \in \mathbb{R}^d \). Then by using Kolmogorov’s continuity theorem, we can find a modification of \( \{Y'(t,x), t \in [a, b], x \in [-M,M]^d\} \) such that its sample function is continuous in \((t, x) \in [a, b] \times [-M,M]^d\). By using the same method, we can show that for each \( j = 1, \ldots, d \), the mean square partial derivative \( Y_j(t, x) \) in \( x_j \) also has a continuous modification on \([a, b] \times [-M,M]^d\). This proves (a) and (b). With these ingredients, we can construct a modification of the Gaussian field \( \{Y(t,x), t \in [a, b], x \in [-M,M]^d\} \) such that its sample function is a.s. continuously differentiable. This proves the theorem. \( \square \)
Because of Theorem 3.6, the regularity properties of \( \{u(t,x)\} \) on \( I = [a,b] \times [-M,M]^d \) are the same as those of \( \{U(t,x)\} \). Since \( \theta_1 \in (0, 1) \), both \( \{u(t,x)\} \) and \( \{U(t,x)\} \) are rough in \( t \). However, \( \theta_2 = \pi \theta_1 \) may be bigger than 1, and in this case the sample functions of \( \{u(t,x)\} \) and \( \{U(t,x)\} \) are continuously differentiable in \( x \). In the following, we distinguish three cases: (i) \( \theta_2 < 1 \), (ii) \( \theta_2 = 1 \) and (iii) \( \theta_2 > 1 \). We will study case (i) and case (iii) in this article, case (ii) is more subtle and we have not been able to solve it completely.

We consider case (i) at first. By applying the results on the exact uniform and local moduli of continuity for Gaussian processes/fields in Meerschaert, Wang and Xiao [25], we have the corresponding regularity results on the solution \( \{u(t,x), t \geq 0, x \in \mathbb{R}^d\} \).

**Proposition 3.7.** Assume that the conditions of Theorem 3.4 hold and \( \theta_2 < 1 \). Then the following results hold:

(i) **(Uniform modulus of continuity)** For any \( I = [a, b] \times [-M, M]^d \subset \mathbb{R}_+ \times \mathbb{R}^d \) with \( 0 < a < b < \infty \) and \( M > 0 \), there is a constant \( \kappa_1 \in (0, \infty) \) such that

\[
\lim_{\varepsilon \to 0^+} \sup_{(t,s),(y,z) \in I: |t-s| < \varepsilon} \frac{|u(t,x) - u(s,y)|}{\rho(t,x; s,y) \sqrt{\log (1 + \rho(t,x; s,y)^{-1})}} = \kappa_1, \quad \text{a.s.}
\]

where \( \rho(t,x; s,y) = |t-s|^{\theta_1} + |x-y|^{\theta_2} \).

(ii) **(Local modulus of continuity)** There is a constant \( \kappa_2 \in (0, \infty) \) such that for any \( (t,x) \in I \),

\[
\lim_{\varepsilon \to 0^+} \sup_{(s,y) \in \bar{I}: |t-s| \leq \varepsilon} \frac{|u(t+s,x+y) - u(t,x)|}{\tilde{\rho}(s,y) \sqrt{\log (1 + \tilde{\rho}(s,y)^{-1})}} = \kappa_2, \quad \text{a.s.}
\]

where \( \tilde{\rho}(s,y) = |s|^{\theta_1} + |y|^{\theta_2} \).

As corollaries of Proposition 3.7, we have the corresponding results that generalize those in Tudor and Xiao [32] for fixed \( x \) and \( t \), respectively.

**Corollary 3.8.** Assume that the conditions of Proposition 3.7 hold. Then for every \( x \in \mathbb{R}^d \) fixed, we have the following modulus of continuity results in time:

(i) **(Uniform modulus of continuity)** For any \( b > 0 \), there is a constant \( \kappa_3 \in (0, \infty) \) such that

\[
\lim_{\varepsilon \to 0^+} \sup_{t,s \in [0,b]: |t-s| \leq \varepsilon} \frac{|u(t,x) - u(s,x)|}{|t-s|^{\theta_1} \sqrt{\log (1 + |t-s|^{-1})}} = \kappa_3, \quad \text{a.s.}
\]

(ii) **(Local modulus of continuity)** There is a constant \( \kappa_4 \in (0, \infty) \) such that for any \( t \in (0, \infty) \),

\[
\lim_{\varepsilon \to 0^+} \sup_{|h| \leq \varepsilon} \frac{|u(t+h,x) - u(t,x)|}{|h|^{\theta_1} \sqrt{\log (1 + |h|^{-1})}} = \kappa_4, \quad \text{a.s.}
\]
Corollary 3.9. Assume that the conditions of Proposition 3.7 hold. Then for every $t > 0$ fixed, we have the following modulus of continuity results in space:

(i) (Uniform modulus of continuity) For any $M > 0$, there is a constant $\kappa_5 \in (0, \infty)$ such that

$$\lim_{\varepsilon \to 0^+} \sup_{x, y \in [-M, M]^d; |x - y| \leq \varepsilon} \frac{|u(t, x) - u(t, y)|}{|x - y|^{\theta_2} \sqrt{\log (1 + |x - y|^{-1})}} = \kappa_5, \quad a.s.$$

(ii) (Local modulus of continuity) There is a constant $\kappa_6 \in (0, \infty)$ such that for all $x \in \mathbb{R}^d$

$$\lim_{\varepsilon \to 0^+} \sup_{y \in [0, |y|] \leq \varepsilon} \frac{|u(t, x + y) - u(t, x)|}{|y|^{\theta_2} \sqrt{\log (1 + |y|^{-1})}} = \kappa_6, \quad a.s.$$

Further properties on the local time and fractal behavior of the solution $\{u(t, x)\}$ can also be derived from [34–36].

Next, by combining Theorem 3.4 with Theorem 1.1 in Luan and Xiao [23], we derive the following Chung-type law of the iterated logarithm for the solution $\{u(t, x)\}$.

Proposition 3.10. Assume that the conditions of Theorem 3.4 hold and $\theta_2 < 1$. Then there is a constant $\kappa_7 \in (0, \infty)$ such that for any $(t, x) \in I$,

$$\liminf_{\varepsilon \to 0^+} \sup_{(s, y) \in [0, \varepsilon]^{\mathbb{R}^d}; |(s, y)| \leq \varepsilon} \frac{|u(t + s, x + y) - u(t, x)|}{\varepsilon (\log \log 1/\varepsilon)^{-1/Q}} = \kappa_7, \quad a.s.$$

where $Q = \frac{1}{\theta_1} + \frac{d}{\theta_2}$.

Proof. Thanks to Theorem 3.6, we only need to prove that

$$\liminf_{\varepsilon \to 0^+} \sup_{(s, y) \in [0, \varepsilon]^{\mathbb{R}^d}; |(s, y)| \leq \varepsilon} \frac{|U(t + s, x + y) - U(t, x)|}{\varepsilon (\log \log 1/\varepsilon)^{-1/Q}} = \kappa_7, \quad a.s. \quad (3.18)$$

Because $\theta_2 < 1$, we know $\sigma(r) = r^{\theta_2}$. By Theorem 3.4, we see that $U(t, x)$ satisfies Condition (C) in Luan and Xiao [23], and thus (3.18) follows directly from their Theorem 1.1.

When $\theta_2 > 1$, the solution process $\{u(t, x)\}$ has a version $\tilde{u}(t, x)$ such that $x \mapsto \tilde{u}(t, x)$ is continuously differentiable. More precisely, we now prove the following result.

Proposition 3.11. Assume that the conditions of Theorem 3.4 hold and $\theta_2 > 1$. Then the solution process $\{u(t, x)\}$ has a version $\tilde{u}(t, x)$ with continuous sample functions such that the partial derivatives $\frac{\partial u(t, x)}{\partial x_j}$ ($j = 1, \ldots, d$) is continuous almost surely. Moreover, for any $M > 0$, there exists a positive positive random variable $K$ with all moments such that for every $j = 1, \ldots, d$, $\frac{\partial}{\partial x_j} \tilde{u}(t, x)$ has the following modulus of continuity on $[-M, M]^d$: 
The proof of Proposition 3.11 is similar to that of Theorem 3.6. We can carry out the steps (a)–(c), and then apply the metric entropy bound for the uniform modulus of continuity (cf. [1]) to derive (3.19). See also the proofs of Xue and Xiao [37, Theorem 4.8] and Tudor and Xiao [32, Theorem 5] for related arguments. Therefore, we omit the details.

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Appendix A

Strong local nondeterminism of a family of Gaussian random fields

In this appendix, we prove the following general result on strong local nondeterminism for a class of Gaussian random fields with stationary increments, which generalizes [36, Theorem 3.2] and may be of independent interest.

**Theorem 4.1.** Let \( \{X(t), t \in \mathbb{R}^N\} \) be a real-valued, centered Gaussian random field with stationary increments and spectral density \( f(\lambda) \). If there exists a vector \( \gamma = (\gamma_1, ..., \gamma_N) \in (0, 1]^N \) such that for all \( a > 0 \)

\[
f(a^2 \lambda) \geq a^{-(2+Q)} f(\lambda) \quad \forall \lambda \in \mathbb{R}^N \setminus \{0\},
\]

where \( E \) is an \( N \times N \) diagonal matrix with diagonal entries given by \( \gamma_1^{-1}, ..., \gamma_N^{-1} \) and \( Q = \sum_{j=1}^{N} \gamma_j^{-1} \). Then, there exists a positive constant \( c \) such that for any positive integer \( n \), and all \( u, t_1, ..., t_n \in \mathbb{R}^N \),

\[
\text{Var}(X(u)|X(t_1), ..., X(t_n)) \geq c \min_{k=1, ..., n} \rho(u, t^k)^2,
\]

where \( t^0 = 0 \) and \( \rho(u, t) := \sum_{j=1}^{n} |u_j - t_j|^\gamma \).

**Proof.** Denote \( r \equiv \min_{0 \leq k \leq n} \rho(u, t^k) \). Since the conditional variance in (4.2) is the square of the \( L^2(\mathbb{P}) \)-distance of \( X(u) \) from the subspace generated by \( \{X(t^1), ..., X(t^n)\} \), it is sufficient to prove that for all \( a_k \in \mathbb{R} \) (\( 1 \leq k \leq n \)),

\[
\mathbb{E} \left( X(u) - \sum_{k=1}^{n} a_k X(t^k) \right)^2 \geq c \, r^2,
\]

where \( c > 0 \) is a constant which depends only on \( \gamma \) and \( N \).

By the stochastic integral representation (cf. [36, (2.9)]) of \( X \), and thanks to the fact that \( X \) has stationary increments, the left hand side of (4.3) can be written as

\[
\mathbb{E} \left( X(u) - \sum_{k=1}^{n} a_k X(t^k) \right)^2 = \int_{\mathbb{R}^N} \left| e^{i(u, \lambda)} - 1 - \sum_{k=1}^{n} a_k \left( e^{i(t^k, \lambda)} - 1 \right) \right|^2 f(\lambda) \, d\lambda.
\]

Hence, we only need to show

\[
\int_{\mathbb{R}^N} \left| e^{i(u, \lambda)} - \sum_{k=0}^{n} a_k e^{i(t^k, \lambda)} \right|^2 f(\lambda) \, d\lambda \geq c_{4,1} \, r^2,
\]

where \( t^0 = 0 \) and \( a_0 = -1 + \sum_{k=1}^{n} a_k \).
Let $\delta(\cdot) : \mathbb{R}^N \to [0, 1]$ be a function in $C^\infty(\mathbb{R}^N)$ such that $\delta(0) = 1$ and $\delta$ vanishes outside the open ball $B_{\rho}(0, 1)$ in the metric $\rho$. Denote by $\hat{\delta}$ the Fourier transform of $\delta$. Then $\hat{\delta}(\cdot) \in C^\infty(\mathbb{R}^N)$ as well and $\hat{\delta}(\lambda)$ decays rapidly as $|\lambda| \to \infty$.

Let $\delta_r(t) = r^{-Q}\delta(r^{-E}t)$. Then the inverse Fourier transform and a change of variables yield

$$\delta_r(t) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{-i\langle t, \lambda \rangle} \hat{\delta}(r^E\lambda) \, d\lambda. \quad (4.6)$$

Since $\min\{\rho(u, t^k) : 0 \leq k \leq n\} \geq r$, we have $\delta_r(u - t^k) = 0$ for $k = 0, 1, \ldots, n$. This and (4.6) together imply that

$$J := \int_{\mathbb{R}^N} \left( e^{i\langle u, \lambda \rangle} - \sum_{k=0}^{n} a_k e^{i\langle t^k, \lambda \rangle} \right) e^{-i\langle t, \lambda \rangle} \hat{\delta}(r^E\lambda) \, d\lambda = (2\pi)^N \left( \delta_r(0) - \sum_{k=0}^{n} a_k \delta_r(u - t^k) \right) = (2\pi)^N r^{-Q}. \quad (4.7)$$

On the other hand, by the Cauchy-Schwarz inequality, (4.1) and (4.4), we have

$$f^2 \leq \int_{\mathbb{R}^N} \left| e^{i\langle u, \lambda \rangle} - \sum_{k=0}^{n} a_k e^{i\langle t^k, \lambda \rangle} \right|^2 f(\lambda) \, d\lambda \cdot \int_{\mathbb{R}^N} \left| \hat{\delta}(r^E\lambda) \right|^2 \, d\lambda$$

$$\leq \mathbb{E} \left( X(u) - \sum_{k=1}^{n} a_k X(t^k) \right)^2 \cdot r^{-Q} \int_{\mathbb{R}^N} \left| \hat{\delta}(r^E\lambda) \right|^2 \, d\lambda$$

$$\leq c \mathbb{E} \left( X(u) - \sum_{k=1}^{n} a_k X(t^k) \right)^2 \cdot r^{-2Q-2}, \quad (4.8)$$

where $c > 0$ is a constant which only depends on $H$ and $N$.

We square both sides of (4.7) and use (4.8) to obtain

$$(2\pi)^{2N} r^{-2Q} \leq c r^{-2Q-2} \mathbb{E} \left( X(u) - \sum_{k=1}^{n} a_k X(t^k) \right)^2.$$ 

Hence (4.5) holds. This finishes the proof of the theorem. \qed