Open inflation from quantum cosmology with a strong nonminimal coupling

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Abstract

We propose the mechanism of quantum creation of the open Universe in the observable range of values of Ω. This mechanism is based on the no-boundary quantum state with the Hawking-Turok instanton applied to the model with a strong nonminimal coupling of the inflaton field. We develop the slow roll perturbation expansion for the instanton solution and obtain a nontrivial contribution to the classical instanton action. The interplay of this classical contribution with the loop effects due to quantum effective action generates the probability distribution peak with necessary parameters of the inflation stage without invoking any anthropic considerations. In contrast with a similar mechanism for closed models, existing only for the tunneling quantum state of the Universe, the observationally justified open inflation originates from the no-boundary cosmological wavefunction.

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1. Introduction

Hawking and Turok have recently suggested the mechanism of quantum creation of an open Universe from the no-boundary cosmological state [1]. Motivated by the observational evidence for inflationary models with Ω < 1 they constructed a singular gravitational instanton capable of generating expanding universes with open spatially homogeneous sections. The prior quantum probability of such universes weighted by the anthropic probability of galaxy formation was shown to be peaked at Ω ∼ 0.01. This idea, despite its extremely attractive nature, was criticized from various sides. In order to increase the amount of inflation to larger values of Ω and avoid anthropic considerations Linde [2] proposed to replace the no-boundary quantum state [1, 5] by the tunneling one [6, 7]. The singularity of the Hawking-Turok instanton raised a number of objections both in the Euclidean theory...
and from the viewpoint of the resulting timelike singularity in the expanding Universe \[9, 12, 11\]. The criticism of singular instantons was followed by attempts of their justification \[13, 14\] which still leave their issue open.

In any case it seems that the practical goal of quantum cosmology – generating the open Universe with observationally justified modern value of \(\Omega\), not very close to one or zero, – has not yet been reached. The use of anthropic principle, as was recognized by the authors of \[1\], is certainly a retreat in theory, because by and large this principle has such a disadvantage that it can explain practically everything without being able to predict anything. The tunneling state advocated by Linde \[2\] (and strongly criticized in \[15\]) requires special supergravity induced potentials and takes place at energies beyond reliable perturbative domain with the resulting \(\Omega \approx 1\). Other works in the above series discuss conceptual issues of the Hawking-Turok proposal without offering the concrete mechanism of generating the needed \(\Omega\).

On the other hand, in spatially closed context there exists a mechanism of generating the probability peak in the distribution of cosmological models at a low (typically GUT) energy scale. It does not appeal to anthropic considerations. Rather it is based on quantum loop effects \[16, 17, 18\] in the model of chaotic inflation with large negative nonminimal coupling of the inflaton \[19, 20, 21, 22\]. This peak exists for a wide class of particle physics models of GUT type, coupled to the inflaton. Because of its GUT energy scale its mechanism is not essentially affected by the nonrenormalizability problems inherent to quantum gravity, or by modifications due to a more fundamental theory like superstrings. Moreover, big negative nonminimal coupling of the inflaton plays in this model the role of the effective inverse gravitational constant, which improves the loop expansion and suppresses the contributions of higher spins including gravitons. Thus it seems to be robust against nonperturbative issues. Important peculiarity of this result is that it exists only for the tunneling cosmological wavefunction, because a similar peak for the no-boundary state of the Universe, existing in certain domain of coupling constants, can generate only an infinitely long inflationary stage.

In this paper we extend this mechanism to the open Universe and show, what was briefly reported in recent author’s paper \[23\], that the situation qualitatively reverses: the probability peak in the distribution of open inflationary models exists for the no-boundary state based on the Hawking-Turok instanton. Similarly to \[19, 21, 20, 22\] it has GUT scale parameters and the value of \(N\) easily adjustable (without fine tuning of initial conditions and anthropic considerations) for observationally justified values of \(\Omega\). We begin in Sect.2 with a brief overview of the closed model \[19, 21, 20, 22\] with a special emphasis on the interplay between the classical Euclidean action and its quantum effective counterpart, generating the probability peak. Sect.3 is devoted to the construction of the Hawking-Turok instanton and a special algorithm for its classical action, based on the asymptotic behaviour of fields near the singularity. In Sect.4 this construction is extended to the model with a nonminimal inflaton field. Sects.5 and 6 contain a detailed presentation of the slow roll perturbation theory for the Hawking-Turok solution of field equations and the calculation of its action. Remarkably, this tree-level action has a large positive contribution logarithmic in the inflaton field.
It is structurally analogous to loop corrections and already at the tree level is capable of generating the inflation probability peaks. In Sect. 7 we show that these peaks do not satisfy the observational constraints, because they generate either too low, \( \Omega \sim e^{-110} \), or too close to one, \( \Omega \sim 1 - e^{-2 \times 10^6} \), values of the density parameter. For this reason, in Sect. 8 we turn to quantum loop effects on the Hawking-Turok instanton. We discuss the problems with the quantum effective action associated with its singularity and find that the dominant scaling behaviour is robust against this singularity. Finally, in Sect. 9 we calculate the most probable \( H, N \) and \( \Omega \) of the open inflation generated within the no-boundary Hawking-Turok paradigm. In the concluding Sect. 10 we discuss the modifications of the obtained results by loop corrections to the classical equations of motion \[22\] at the stage of the inflationary evolution and show that they do not qualitatively change the conclusions.

2. Quantum cosmological origin of the closed inflation

Quantum cosmology serves as a theory of quantum initial conditions for the evolution of our Universe. One of the main observable cosmological parameters is the density parameter \( \Omega \), and in the model, undergoing the conventional inflationary and standard big-bang stages, its origin can be traced back to the initial conditions for inflation. Matching the inflationary, radiation and then matter dominated stages leads to the following expression for the present day value of \( \Omega \) \[1\],

\[
\Omega \approx \frac{1}{\mp B \exp(-2N)},
\]

(2.1)
in terms of the inflationary e-folding number \( N \) — the logarithmic expansion coefficient for the cosmological scale factor \( a \) during the inflation stage with a Hubble constant \( H = \dot{a}/a \),

\[
N = \int_{t_0}^{t_F} dt H
\]

(2.2)
(with \( t = 0 \) and \( t_F \) denoting the beginning and the end of inflation epoch). The signs \( \mp \) in (2.1) are related respectively to the closed and open models, and \( B \) is the parameter incorporating the details of the reheating and radiation-to-matter transitions in the early Universe. Depending on the model for these transitions, its order of magnitude can vary from \( 10^{25} \) to \( 10^{50} \) (when the reheating temperature varies from the electroweak to GUT scale). In what follows we shall assume the latter as the most probable value of this parameter.

Eq. (2.1) clearly demonstrates rather stringent bounds on \( N \). For the closed model the e-folding number should satisfy the lower bound \( N \geq 60 \) in order to generate the observable Universe of its present size, while for the open model \( N \) should be very close to this bound \( N \approx 60 \) in order to have the present value of \( \Omega \) not very close to zero or one, \( 0 < \Omega < 1 \), the fact intensively discussed on the ground of the recent observational data.

In the chaotic inflation model the effective Hubble constant is generated by the potential of the inflaton scalar field and all the parameters of the inflationary epoch, including its
duration in units of $N$, can be found as functions of the initial value of the inflaton field $\varphi$ at the onset of inflation $t = 0$. This initial condition belongs to the quantum domain, i.e. it has to be considered subject to the quantum distribution following from the cosmological wavefunction. If this distribution function has a sharp probability peak at certain $\varphi$, then, at least within the semiclassical expansion, this value of $\varphi$ serves as the initial condition for the inflationary dynamics.

There are two known quantum states that lead in the semiclassical regime to the closed chaotic inflation Universe. They are given by the no-boundary [3, 4, 5] and tunneling [6] wavefunctions. In the tree-level approximation they generate the distribution functions which are unnormalizable in the high-energy domain $\varphi \to \infty$ and generally devoid of the observationally justified probability peaks. It turned out, however, that the inclusion of quantum loop effects can qualitatively change the situation – make the distributions normalizable [16, 18]. Moreover, in the closed model with a strong nonminimal coupling of the inflaton to curvature, these effects can generate the probability peak at GUT energy scale satisfying the above bound on $N$ [19, 21, 20]. The basic formalism underlying this result is as follows.

In the quantum gravitational domain the conventional expression for the no-boundary and tunneling distributions of the inflaton field $\rho_{\text{NB,T}}(\varphi) \sim \exp[\mp I(\varphi)]$ is replaced by

$$\rho_{\text{NB,T}}(\varphi) \sim \exp[\mp I(\varphi) - \Gamma(\varphi)],$$  

(2.3)

where the classical Euclidean action $I(\varphi)$ on the quasi-DeSitter instanton with the inflaton value $\varphi$ is amended by the loop effective action $\Gamma(\varphi)$ calculated on the same instanton [16, 17, 18, 21]. The contribution of the latter can qualitatively change predictions of the tree-level theory due to the dominant part of the effective action induced by the anomalous scaling behaviour. On the instanton of the size $1/H(\varphi)$ – the inverse of the Hubble constant, it looks like

$$\Gamma(\varphi) \sim Z \ln H(\varphi),$$  

(2.4)

where $Z$ is the total anomalous scaling of all quantum fields in the model. For the model of [19, 20]

$$L(g_{\mu\nu}, \varphi) = g^{1/2} \left\{ \frac{m_P^2}{16\pi} R(g_{\mu\nu}) - \frac{1}{2} \xi \phi^2 R(g_{\mu\nu}) - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4} \phi^4 \right\},$$  

(2.5)

with a big negative constant $-\xi = |\xi| \gg 1$ of nonminimal curvature coupling, and generic GUT sector of Higgs $\chi$, vector gauge $A_\mu$ and spinor fields $\psi$ coupled to the inflaton via the interaction term

$$L_{\text{int}} = \sum_\chi \frac{\lambda_\chi}{4} \chi^2 \phi^2 + \sum_\Delta \frac{1}{2} g_A^2 A_\mu^2 \phi^2 + \sum_\psi f_\psi \phi \bar{\psi} \psi + \text{derivative coupling},$$  

(2.6)

this parameter can be very big, because of the Higgs effect generating large masses of all the corresponding particles directly coupled the inflaton. Due to this effect the parameter
Z (dominated by terms quartic in particle masses) is quadratic in $|\xi|$, 

$$Z = 6\frac{|\xi|^2}{\lambda} A ,$$

$$A = \frac{1}{2\lambda} \left( \sum_{\chi} \lambda_{\chi}^2 + 16 \sum_A g_{\lambda}^A - 16 \sum_{\psi} f_{\psi}^4 \right) ,$$

with a universal combination of the coupling constants above.

Thus, the probability peak in this model reduces to the extremum of the function

$$\ln \rho_{NB,T}(\varphi) \simeq \mp I(\varphi) - \frac{3|\xi|^2}{\lambda} A \ln \frac{\varphi^2}{\mu^2} .$$

in which the $\varphi$-dependent part of the classical instanton action

$$I(\varphi) = -\frac{96\pi^2|\xi|^2}{\lambda} - \frac{24\pi(1 + \delta)|\xi| m^2}{\varphi^2_0} + O \left( \frac{m^4}{\varphi^4} \right) ,$$

$$\delta \equiv -\frac{8\pi |\xi| m^2}{\lambda m^2_p} ,$$

should be balanced by the anomalous scaling term, provided the signs of $(1 + \delta)$ and $A$ are properly correlated with the $(\mp)$ signs of the no-boundary (tunneling) proposals. As a result the probability peak exists with parameters – mean values of the inflaton and Hubble constants and relative width

$$\varphi_I^2 = m^2_p \frac{8\pi|1 + \delta|}{L^2 A},$$

$$H^2(\varphi_I) = m^2_p \frac{2\pi|1 + \delta|}{3 L^2 A} ,$$

$$\frac{\Delta \varphi}{\bar{\varphi}_I} \sim \frac{\Delta H}{H} \sim \frac{1}{\sqrt{12 A L} |\xi|} ,$$

which are strongly suppressed by a small ratio $\sqrt{\lambda}/|\xi|$ known from the COBE normalization for $\Delta T/T \sim 10^{-5}$ [24, 27] (because the CMBR anisotropy in this model is proportional to this ratio [28]). This GUT scale peak gives rise to the finite inflationary epoch with the e-folding number

$$N \simeq \frac{48\pi^2}{A} ,$$

only for $1 + \delta > 0$ and, therefore, only for the tunneling quantum state (plus sign in (2.9)). Comparison with $N \geq 60$ necessary for $\Omega > 1$ immediately yields the bound on $A$ [22],

$$A \sim 5.5 ,$$

which can be regarded as a selection criterion for particle physics models [19]. This conclusions on the nature of the inflation dynamics from the initial probability peak remain true
also at the quantum level – with the effective equations replacing the classical equations of motion [22].

For the proponents of the no-boundary quantum states in a long debate on the wave-function discord [29, 15, 24, 25] this situation might seem unacceptable. According to this result the no-boundary proposal does not generate realistic inflationary scenario, while the tunneling state does not satisfy important aesthetic criterion – the universal formulation of the initial conditions and dynamical aspects in one concept – spacetime covariant path integral over geometries. In what follows we show that for the open model this situation qualitatively reverses.

3. The Hawking-Turok instanton in a minimal model

We assume that the reader is familiar with the construction of the Hawking-Turok instanton generating the open inflation. Very briefly, the inflating Lorentzian spacetime originates in both closed and open models by the nucleation from a 3-dimensional section of the gravitational instanton. In the closed model this is the equatorial section – the boundary of the 4-dimensional quasi-hemishpere labelled by the constant value of the latitude angular coordinate. The analytic continuation of this coordinate into the complex plane gives rise to the Lorentzian quasi-DeSitter spacetime modelling the open inflation. In the open case the Hawking-Turok suggestion was to continue the Euclidean solution beyond the equatorial section up to the point where the Euclidean scale factor again vanishes at the point antipoidal to the regular pole on the first hemisphere. The nucleation surface then has to be chosen as the longitudinal section of this quasisherical manifold passing through the regular pole and its antipoidal point. Then the analytic continuation of the corresponding longitudinal angle into the complex plane gives rise to the Lorentzian spacetime. The light cone originating from the regular pole cuts in this spacetime the domain sliced by the open spatially homogeneous sections of constant negative curvature. Their chronological succession serves as the model for the open inflationary Universe.

The difficulty with this construction follows from the fact that the point antipoidal to the regular pole of the Hawking-Turok instanton is not regular – it has a singularity which develops into the time-like singularity in the Lorentzian part of spacetime. The main objections against the Hawking-Turok instanton are targeting this singularity. We shall not, however, dwell on the criticism of this construction, and adopt the treatment of this singularity suggested in [13]. It should be regarded as point punctured from the manifold by choosing a surrounding it boundary surface shrinking to the point as a limiting procedure. The usual Dirichlet boundary conditions for boundary metric coefficients are assumed at this surface,

\footnote{The Lorentzian path integral for the tunneling state of [24] also requires, in this respect, extension beyond minisuperspace level, development of the semiclassical expansion technique, etc.}
so that the Einstein Euclidean action on the Hawking-Turok instanton has the form

\[ I[g_{\mu\nu}, \phi] = \int_M d^4x \, g^{1/2} \left\{ -\frac{m_P^2}{16\pi} R + \frac{1}{2}(\nabla \phi)^2 + V(\phi) \right\} + \frac{m_P^2}{8\pi} \int_{\partial M} d^3x \, (g)^{1/2} K, \]  

(3.1)

with \( K \) – the extrinsic curvature at this boundary. Here we consider the model with minimally coupled inflaton field denoted by \( \phi \) (in contrast with \( \varphi \) reserved for the nonminimally coupled inflaton) and having the inflaton potential \( V(\phi) \).

With the spatially homogeneous ansatz for the Euclidean metric \((d\Omega^2_3)\) is the metric of the 3-dimensional sphere of unit radius, \( a(\sigma) \) is the scale factor and \( \sigma \) is the latitude angular coordinate of the above type),

\[ ds^2 = d\sigma^2 + a^2(\sigma) d\Omega^2_3, \]  

(3.2)

the Euclidean equations of motion take the form

\[ \ddot{\phi} + \frac{3}{a} \dot{\phi} - V' = 0, \]  

(3.3)

\[ a^3 V - \frac{3m_P^2}{8\pi}(a - aa^2) - \frac{1}{2}a^3 \dot{\phi}^2 = 0, \]  

(3.4)

where dots denote the derivatives with respect to the coordinate \( \sigma \) and the prime denotes the derivative with respect to the inflaton scalar field.

The regularity of the instanton solution at \( \sigma = 0 \) implies that \( \dot{\phi}(0) = 0 \) and \( \dot{a}(0) = 1 \) (the absence of the conical singularity). With these initial conditions and when the slope of the inflaton potential is not too steep the solution near \( \sigma = 0 \) can be obtained in the slow roll approximation as an expansion in powers of the dimensionless parameter

\[ \epsilon = \frac{m_P \, V'(\phi_0)}{\sqrt{3\pi} \, V(\phi_0)}. \]  

(3.5)

In the lowest order approximation this solution represents the constant inflaton field \( \phi_0 = \dot{\phi}(0) \) and the scale factor of the Euclidean DeSitter geometry

\[ a = \frac{1}{H_0} \sin \theta + \delta a, \quad \phi = \phi_0 + \delta \phi, \]  

(3.6)

\[ H_0^2 = \frac{8\pi V(\phi_0)}{3m_P^2}, \quad \theta = H_0 \sigma, \]  

(3.7)

\[ \delta \phi(\theta) = m_P O(\epsilon), \quad \delta a(\theta) = \frac{1}{H_0} O(\epsilon^2), \]  

(3.8)

with the effective Hubble constant \( H_0 \) given in terms of the initial value of the potential \( V(\phi_0) \).

The slow roll approximation does not, however, hold for all values of \( \sigma \). As it follows from the dynamical equation for \( a \),

\[ \ddot{a} + \frac{8\pi}{3m_P^2} (V + \dot{\phi}^2) a = 0, \]  

(3.9)
for monotonically growing positive potentials the scale factor has a negative second derivative which converts at certain moment its expansion to a contraction and then makes it vanishing at some $\sigma = \sigma_f$, $a(\sigma_f) = 0$. Near this point, kinetic terms of the equations above start dominating and, thus, allow one to write the approximate solution

\[ a \simeq A(\sigma_f - \sigma)^{1/3}, \]
\[ \phi \simeq -\frac{m_P}{\sqrt{12\pi}} \ln(\sigma_f - \sigma) + \Phi, \quad \sigma \to \sigma_f. \]

It is important that the coefficient of the logarithmic singularity of the scalar field is unambiguously defined from the equations of motion, whereas the coefficients $A$ and $\Phi$ nontrivially depend on the initial condition at $\sigma = 0$, that is on $\phi_0$.

The knowledge of $A(\phi_0)$ and $\Phi(\phi_0)$ allows one to obtain the action of the Hawking-Turok instanton. Its action (3.1) for the metric (ref2.2) has the

\[ I_{HT}(\phi_0) = 2\pi^2 \int_0^{\sigma_f} d\sigma \left[ a^3 V - \frac{3m_P^2}{8\pi} a(1 + \dot{a}^2) + \frac{1}{2} a^3 \dot{\phi}^2 \right], \]

where the surface term was removed by integration by parts (which simultaneously removes the term with the second derivative of the scale factor). It depends on $\phi_0$, and the mechanism of this dependence originates from the behaviour of fields at the singularity. Indeed, differentiating (3.12) with respect to $\phi_0$ one finds that the volume part vanishes in virtue of equations of motion, while integrations by parts give a typical surface term involving the Lagrangian of (3.12) and its derivatives with respect to $(\dot{\phi}, \dot{a})$, which does not vanish at $\sigma_f$.

For $\sigma$ close to $\sigma_f$ the functional dependence of the fields in (3.11) on $\phi_0$ enters through the coefficients $A$, $\Phi$ as well as through $\sigma_f$. Since $\sigma_f$ enters the fields in the combination $\sigma - \sigma_f$, the total derivative of the field takes the form

\[ \frac{d\dot{\phi}}{d\phi_0} = \frac{\partial \dot{\phi}}{\partial \phi_0} - \dot{\phi} \frac{\partial \sigma_f}{\partial \phi_0}, \]

where partial derivative with respect to $\phi_0$ acts only on $\Phi$ (and coefficients of higher powers in $(\sigma - \sigma_f)$). Similar relation holds also for the scale factor. Thus, the surface term at $\sigma_f$ equals

\[ \frac{dI_{HT}(\phi_0)}{d\phi_0} = \left( L - \dot{\phi} \frac{\partial L}{\partial \dot{\phi}} - \dot{a} \frac{\partial L}{\partial \dot{a}} \right) \frac{\partial \sigma_f}{\partial \phi_0} \bigg|_{\sigma_f}
\]

\[ + \left( \frac{\partial L}{\partial \dot{a}} \frac{\partial A}{\partial \phi_0} (\sigma_f - \sigma)^{1/3} + \frac{\partial L}{\partial \phi} \frac{\partial \Phi}{\partial \phi_0} \right)_{\sigma \to \sigma_f}, \]

where $L$ is the Lagrangian of the Euclidean action (3.12). The first term here identically vanishes, because it is proportional to the Hamiltonian constraint (in terms of velocities). On using the asymptotic behaviour near the singularity (3.11) one then finds

\[ \frac{dI_{HT}(\phi_0)}{d\phi_0} = \frac{\pi m_P^2}{6} \frac{dA^3}{d\phi_0} + \frac{\pi^{3/2} m_P^3}{\sqrt{3}} \frac{d\Phi}{d\phi_0}, \]
so that the integration of this equation gives \( I_{HT}(\phi_0) \).

To find \( (A(\phi_0), \Phi(\phi_0)) \) we shall develop in Sect.4 the perturbation expansion of the solution near \( \sigma_f \). In contrast with the slow roll expansion near \( \sigma = 0 \) this is the expansion in powers of the potential \( V(\phi) \) itself rather than its gradient, and \( \dot{\phi} \)-derivatives give the dominant contribution at this asymptotics. Then we match the both asymptotic expansions in the domain of \( \sigma \) where they are both valid. From this match one easily finds the unknown parameters \( A, \Phi \) as functions of \( \phi_0 \). As we shall show in Sect.4, in the lowest order of the slow roll expansion they are given by

\[
A = \left( \frac{3\epsilon}{H_0^2} \right)^{1/3} + O(\epsilon), \quad (3.15)
\]

\[
\Phi = \phi_0 - \frac{1}{2} \frac{m_P}{\sqrt{12\pi}} \ln \left[ \frac{9H_0^2}{8\epsilon} \right] + O(\epsilon) = \phi_0 + \frac{1}{2} \frac{m_P}{\sqrt{12\pi}} \ln \frac{V'_0}{V_0^2} + \text{const} + O(\epsilon). \quad (3.16)
\]

A remarkable property of these simple expressions is that, when substituted into (3.14), they yield the equation

\[
\frac{dI_{HT}(\phi_0)}{d\phi_0} \simeq \left( \frac{d}{d\phi_0} + \sqrt{\frac{3}{16\pi}} m_P \frac{d^2}{d\phi_0^2} \right) \left( -\frac{3m_P^4}{8V(\phi_0)} \right) + O(\epsilon), \quad (3.17)
\]

that can be integrated for a generic inflaton potential in a closed form. In the first order of the slow roll expansion inclusive, it reads

\[
I_{HT}(\phi_0) \simeq \left( 1 + \sqrt{\frac{3}{16\pi}} m_P \frac{d}{d\phi_0} \right) \left( -\frac{3m_P^4}{8V(\phi_0)} \right) + O(\epsilon^2). \quad (3.18)
\]

Note that the integration just removed one overall derivative, the integration constant following from the known answer for \( \epsilon = 0 \) – the case of an exact DeSitter solution. One can check that the second term, which represents the first order correction \( O(\epsilon) \), corresponds to the contribution of the extrinsic curvature surface part of the action \( (3.1) \). This expression reproduces the result of ref. \cite{13} obtained by indirect and less rigorous methods\cite{2}. It is interesting that this algorithm is universal for a wide class of inflaton potentials \( V(\phi_0) \) (\( V(\phi) \) should only satisfy typical restrictions imposed by the slow roll approximation) and in a closed form expresses the result in terms of \( V(\phi) \) and its derivative.

Let us now go over to the model with a big nonminimal inflaton coupling and show that the above algorithm is not sufficient in the relevant first order of the slow roll expansion.

### 4. Nonminimal model

In the model \( (2.5) \) the inflaton field \( \varphi \) is nonminimally coupled to curvature via the effective \( \phi \)-dependent Planck “mass” \( 16\pi U(\phi) \)

\[
I = \int_M d^4x g^{1/2} \left\{ V(\varphi) - U(\varphi) R + \frac{1}{2} (\nabla \varphi)^2 \right\} + 2 \int_{\partial M} d^3x \left( g^{1/2} \right) U(\varphi) K. \quad (4.1)
\]

\footnote{The calculations of \cite{13} do not take into account the slow roll corrections to the volume part of the action.}
For completeness we supplied this action with a boundary term necessary in the vicinity of the singularity and also introduced the notation for the inflaton potential $V(\phi)$. It is well known that this action can be reparametrized to the Einstein frame by special conformal transformation and reparametrization of the inflaton field $(g_{\mu\nu}, \phi) \rightarrow (G_{\mu\nu}, \phi)$. These transformations are implicitly given by equations (30)

$$G_{\mu\nu} = \frac{16\pi U(\phi)}{m_P^2} g_{\mu\nu},$$

$$\left(\frac{d\phi}{d\varphi}\right)^2 = \frac{m_P^2 U + 3U'^2}{16\pi U^2}.$$ (4.2)

The action in terms of new fields

$$\bar{I} = \int_M d^4x \sqrt{G} \left\{ \bar{V}(\phi) - \frac{m_P^2}{16\pi} R(G_{\mu\nu}) + \frac{1}{2} (\nabla\phi)^2 \right\} + \frac{m_P^2}{8\pi} \int_{\partial M} d^3x \sqrt{G} \frac{1}{2} \bar{K}$$ (4.4)

has a minimal coupling and the new inflaton potential

$$\bar{V}(\phi) = \left( \frac{m_P^2}{16\pi} \right)^2 \frac{V(\varphi)}{U^2(\varphi)} \bigg|_{\varphi = \varphi(\phi)}.$$ (4.5)

The bar indicates here that the corresponding quantity is calculated in the Einstein frame of fields $(G_{\mu\nu}, \phi)$. In what follows we shall always denote the inflaton field in the Einstein frame (or in the minimal model) by $\phi$ and that of the nonminimal frame by $\varphi$.

The above transition to the Einstein frame allows one to find the Hawking-Turok instanton for the model (4.1) by transforming the results of the previous section to the nonminimal frame. Here we shall do it in the case of a big negative nonminimal coupling $|\xi| \gg 1$ and quartic potential of (2.5):

$$U(\varphi) = \frac{m_P^2}{16\pi} \left[ 1 + \frac{1}{2} |\xi| \varphi^2 \right],$$

$$V(\varphi) = \frac{m_P^2 \varphi^2}{2} + \frac{\lambda \varphi^4}{4}.$$ (4.6)

The integration of eq.(4.3) is rather complicated, but for large values of the inflaton field, $|\xi| \varphi^2/m_P^2 \gg 1$, it expresses $\phi$ in terms of the Einstein frame field $\phi$

$$\phi(\phi) \simeq \frac{m_P}{|\xi|^{1/2}} \exp \left[ \sqrt{\frac{4\pi}{3}} \left( 1 + \frac{1}{6|\xi|} \right)^{-1/2} \frac{\phi}{m_P} \right],$$ (4.8)

where the integration constant is chosen so that the above range of $\phi$ corresponds to $\phi \gg m_P$.

The potential in the Einstein frame equals

$$\bar{V}(\phi) = \frac{\lambda m_P^4}{256\pi^2 |\xi|^2} \left[ 1 - \frac{1 + \delta}{4\pi} \frac{m_P^2}{|\xi| \varphi^2} + \ldots \right]_{\phi = \phi(\phi)},$$ (4.9)

10
where we have retained only the first order term in \( m_p^2/|\xi|^2 \). In view of (3.18), for large \( \phi \) this potential exponentially approaches a constant and satisfies the slow roll approximation with the expansion parameter

\[
\epsilon = \frac{m_p \dot{V}'(\phi_0)}{\sqrt{3\pi} V(\phi_0)} \simeq \frac{1 + \delta}{3\pi} \left( 1 + \frac{1}{6|\xi|} \right)^{-1/2} \frac{m_p^2}{|\xi| \varphi_0^2} \ll 1. \tag{4.10}
\]

This justifies the above choice of range for the values of the inflaton field. In this range the Hawking-Turok instanton is described by the equations of the previous section for the Einstein frame fields \( \bar{a}(\bar{\sigma}) \) and inflaton \( \phi(\sigma) \). Here \( \bar{\sigma} \) is the coordinate in the spacetime interval \( ds^2 = d\sigma^2 + \bar{a}^2(\bar{\sigma}) d\Omega^2(\bar{\sigma}) \) of the Einstein frame metric. In view of (4.2) these intervals are related by the equation \( ds^2 = (16\pi U/m_p^2) d\bar{s}^2 \), so that the coordinates and scale factors of both frames are related by \( d\bar{\sigma} = \sqrt{16\pi U/m_p^2} d\sigma \) and \( \bar{a} = \sqrt{16\pi U/m_p^2} a \). Combining these equations with the asymptotic behaviour of the Einstein frame fields at \( \bar{\sigma} \to \bar{\sigma}_f \) (eqs. (3.11)-(3.14) rewritten for \( \bar{a}(\bar{\sigma}), \phi(\bar{\sigma}) \) with the potential \( \bar{V}(\bar{\phi}) \)) one can easily find the behaviour of fields in the nonminimal frame. We give it in the limit of large \( |\xi| \):

\[
a(\sigma) \simeq \frac{4}{m_p} \left( \frac{m_p}{\phi_0} \right)^{1+2\alpha} \left( \frac{1 + \delta}{4\pi \lambda} \right)^{1/4 + \alpha/2} \left[ m_p(\sigma_f - \sigma) \right]^{1/2 - \alpha}, \tag{4.11}
\]

\[
\varphi(\sigma) \simeq m_p \left( \frac{\phi_0}{m_p} \right)^{1/2 + 3\alpha} \left[ \left( \frac{1 + \delta}{4\pi \lambda} \right)^{1/8 - 3\alpha/4} \right] \left[ m_p(\sigma_f - \sigma) \right]^{-1/4 + 3\alpha/2}, \tag{4.12}
\]

\[
\alpha \equiv \frac{1}{2} \frac{\sqrt{1 + 1/6|\xi|} - 1}{1 + 3\sqrt{1 + 1/6|\xi|}} \simeq \frac{1}{96|\xi|} \ll 1. \tag{4.13}
\]

In contrast with the minimal coupling we now have the power singularities for both fields. For large \( |\xi| \gg 1 \), in particular, they look like \( a \sim (\sigma_f - \sigma)^{1/2} \) and \( \varphi \sim (\sigma_f - \sigma)^{-1/4} \). The inflaton singularity is thus stronger than the logarithmic one in the minimal case, while that of the scale factor is softer \( (1/2 - \alpha \geq 1/3) \). Note, by the way, that the coefficient of strongest singularity of the scalar curvature is also suppressed by \( 1/|\xi|, R \sim (1/|\xi|)(\sigma_f - \sigma)^{-2} \). This property can be qualitatively explained by the fact that the effective gravitational constant \( (m_p^2 + 8\pi|\xi|\varphi^2)^{-1} \) tends to zero at the singularity.

The Hawking-Turok action can now be approximately calculated in the Einstein frame with the aid of eq. (3.18), \( I_{HT}(\varphi_0) = \bar{I}_{HT}(\phi_0) \). Taking into account that

\[
\frac{3m_p^4}{8V(\phi_0)} \simeq \frac{96\pi^2|\xi|^2}{\lambda} + \frac{24\pi(1 + \delta)|\xi| m_p^2}{\lambda \varphi_0^2} + \frac{3}{2} \frac{(1 + 2\delta)^2}{\lambda} \left( \frac{m_p^2}{\varphi_0^2} \right)^2 + \ldots \tag{4.14}
\]

and using the relation

\[
\sqrt{\frac{3}{16\pi}} m_p \frac{d}{d\phi} \simeq \frac{1}{2} \left( 1 + \frac{1}{6|\xi|} \right)^{-1/2} \varphi \frac{d}{d\varphi} \tag{4.15}
\]

\[\text{Note that the relation (4.2) holds in one coordinate system covering the both conformally related spacetimes, while the coordinates } \bar{\sigma} \text{ and } \sigma \text{ are essentially different.}\]
one finds that the term with the derivative in (3.18) almost cancels the first subleading term in the expansion (4.14) and inverts the sign of the second order term

\[
\left(1 + \sqrt{\frac{3}{16\pi}} \frac{m_P}{d\phi_0}\right) \left(-\frac{3m_P^4}{8V(\phi_0)}\right) \approx -\frac{96\pi^2|\xi|^2}{\lambda} \frac{2\pi(1 + \delta)}{\lambda} \frac{m_P^2}{\phi_0^2} + \frac{3}{2} \frac{(1 + 2\delta)^2}{\lambda} \left(\frac{m_P^2}{\phi_0^2}\right)^2 + \ldots
\]

The terms of different powers in \(m_P^2/\phi^2\) here turn out to be of the same order of magnitude in \(1/|\xi|\). Later we shall see that at the probability maximum \(m_P^2/\phi^2 \gg 1\), although \(m_P^2/|\xi|\phi^2 \sim \epsilon \ll 1\), which means that the dominant effect comes from the third term of (4.16). This term is however not reliable because it goes beyond the approximation of eq.(3.18) and belongs to the next order of the slow roll expansion, \(O(\epsilon^2) = O(m_P^2/|\xi|\phi^2)\). Thus we have to construct this expansion to this order inclusive which will be done in the next two sections.

5. Slow roll expansion

Here we develop the slow roll expansion in the minimal inflaton model subject to equations of motion (3.3)-(3.4). The corresponding slow roll corrections to the DeSitter background in (3.6) can be obtained by expanding these equations in perturbations

\[
\delta a = \frac{1}{H_0} \delta \tilde{a}, \quad \delta \phi = \sqrt{\frac{3}{16\pi}} m_P \delta \tilde{\phi}. \quad (5.1)
\]

In the linear order this gives for dimensionless perturbations \((\delta \tilde{a}, \delta \tilde{\phi})\) the following initial value problem

\[
\left(\frac{d^2}{d\theta^2} + 3 \cot \theta \frac{d}{d\theta}\right) \delta \tilde{\phi} = \frac{3}{2} \epsilon + \epsilon_1 \delta \tilde{\phi}, \quad (5.2)
\]

\[
\frac{d}{d\theta} \delta \tilde{a} \cos \theta = -\frac{3}{8} \epsilon \tan^2 \theta \delta \tilde{\phi}, \quad (5.3)
\]

\[
\delta \tilde{\phi}(0) = \frac{d}{d\theta} \delta \tilde{\phi}(0) = 0, \quad \delta \tilde{a}(0) = 0, \quad (5.4)
\]

where \(\epsilon\) and \(\epsilon_1\) are the parameters characterizing the steepness of the inflaton potential at \(\phi_0\)

\[
\epsilon = \frac{m_P}{\sqrt{3\pi}} \frac{V'(\phi_0)}{V(\phi_0)}, \quad (5.5)
\]

\[
\epsilon_1 = \frac{3m_P^2}{8\pi} \frac{V''(\phi_0)}{V(\phi_0)}. \quad (5.6)
\]

When the slope of the inflaton potential is not too steep and \(\epsilon \ll 1\) and \(\epsilon_1 = O(\epsilon) \ll 1\), one can develop the slow roll expansion in powers of the parameter \(\epsilon\) which determines the
rate of change of the potential \( |V/HV| \approx 3\epsilon^2 / 8 \ll 1 \). In the lowest order approximation the second term on the right hand side of (5.2) should be discarded and the solution reads
\[
\delta \tilde{\phi}(\theta) = \epsilon \left( \frac{1}{4} \tan^2 \frac{\theta}{2} - \ln \cos \frac{\theta}{2} \right) + O(\epsilon^2),
\]
\[
\delta \tilde{a}(\theta) = O(\epsilon^2).
\]

The second order approximation is also straightforward. One expands eqs. (3.3) and (3.4) to second order in perturbations and discards the terms \( O(\epsilon^3) \) and \( O(\epsilon^4) \) respectively. As a result one retains the second term on the right hand side of eq.(5.2), while the equation for \( \delta \tilde{a} \) acquires extra \( O(\epsilon^2) \) and \( O(\epsilon^3) \) contributions
\[
\frac{d}{d\theta} \delta \tilde{a} \cos \theta = \frac{1}{8} \tan^2 \theta \left[ -3\epsilon \delta \tilde{\phi} + \left( \frac{d}{d\theta} \delta \tilde{\phi} \right)^2 \right] - \frac{1}{8} \epsilon_1 \tan^2 \theta (\delta \tilde{\phi})^2 + O(\epsilon^4).
\]

Equations (5.2) and (5.9) subject to initial conditions (5.4) are not integrable in elementary functions, for example,
\[
\delta \tilde{\phi}(\theta) = \epsilon \left( \frac{1}{4} \tan^2 \frac{\theta}{2} - \ln \cos \frac{\theta}{2} \right) + \frac{1}{6} \epsilon_1 \left[ \frac{11}{12} \tan^2 \frac{\theta}{2} + \frac{1}{3} \ln \cos \frac{\theta}{2} \right.
\]
\[
- \ln \cos(\theta/2) \right] - \frac{1}{2} + \int_0^{\sin^2(\theta/2)} \frac{dy}{y} \ln(1-y) \right] + O(\epsilon^3).
\]

For our purposes, however, it is sufficient to obtain the leading behaviour in the vicinity of \( \theta = \pi \) – the point where we shall match two perturbation theories. Another simplification that we use here follows from the potential \( V(\phi) = \tilde{V}(\phi) \) given by eq. (4.9)
\[
V(\phi) = \frac{\lambda m_P^4}{256 \pi^2 |\xi|^2} \left[ 1 - \left( 1 + \frac{\delta}{4\pi} \exp \left( - \frac{4\pi}{3} \frac{\phi}{m_P} \right) + \ldots \right) \right], \quad |\xi| \gg 1,
\]

(in this section we omit the bar in \( \tilde{V}(\phi) \), the argument \( \phi \) clearly indicating that the potential belongs to the Einstein frame). For such a potential the two steepness parameters \( \epsilon \) and \( \epsilon_1 \) above are not independent
\[
\epsilon_1 = -\frac{3}{2} \epsilon + O(\epsilon^2).
\]

Thus, putting \( \theta = \pi - \epsilon \) we finally obtain in the vicinity of the Hawking-Turok singularity, \( \epsilon \ll 1 \),
\[
\phi(\pi - \epsilon) = \phi_0 + \sqrt{\frac{3}{16\pi}} m_P \left[ \epsilon \left( \frac{1}{\epsilon^2} - \ln \frac{\epsilon}{2} - \frac{1}{6} + \ldots \right)
\]
\[
+ \epsilon^2 \left( -\frac{11}{12\epsilon^2} + \frac{1}{6} \ln \frac{\epsilon}{2} + \ldots \right) + O(\epsilon^3) \right],
\]
\[
H_0 a(\pi - \epsilon) = \sin \epsilon + \epsilon^2 \left( -\frac{1}{6\epsilon^3} + \ldots \right) + \epsilon^3 \left( \frac{11}{36\epsilon^3} + \ldots \right) + O(\epsilon^4).
\]
where the dots denote terms of higher powers in $\varepsilon$. Perturbation corrections diverge at $\varepsilon \to 0$, and an obvious domain of applicability of this perturbation theory is

$$\frac{\varepsilon}{\varepsilon^2} \ll 1. \quad (5.15)$$

This is the domain of the angular coordinate $\theta = \pi - \varepsilon$ in which we will match the slow roll solution with the perturbative solution near the singularity.

### 6. Perturbation expansion for the Hawking-Turok instanton

The perturbation theory in powers of the slow roll parameter $\varepsilon$ near the singularity located close to $\theta = \pi$ is qualitatively different from the slow roll expansion near $\theta = 0$. This is the expansion in powers of the potential in the vicinity of this point, rather than only in its gradient. To develop such an expansion we first rewrite the equations of motion as

$$\frac{d}{d\sigma} (a^3 \dot{\phi}) = a^3 V', \quad (6.1)$$

$$\dot{a}^2 = \frac{4\pi}{3m_p^2} (a\dot{\phi})^2 + 1 - H^2(\phi)a^2, \quad (6.2)$$

and then formally integrate the first of equations to convert it to the integral form

$$a\dot{\phi} = \frac{m_p A^3}{\sqrt{12\pi}} \frac{1}{a^2} + \frac{1}{a^2} \int_{\sigma_f}^{\sigma} d\tilde{\sigma} a^3 V'. \quad (6.3)$$

Here the integration constant was chosen in terms of the constant $A$ in the asymptotic behaviour $(3.10)-(3.11)$ at $\sigma_f$, $a(\sigma_f) = 0$. With this choice, this asymptotic behaviour is just the approximate solution of the equations above, corresponding to discarding the right hand side of $(6.1)$ and the last two terms on the right hand side of $(6.2)$.

The key point in finding the perturbation corrections to this approximate solution consists in guessing a special ansatz for the scale factor $a(\sigma)$, which contains a new small parameter $\bar{\varepsilon} = O(\varepsilon)$. The expansion in this parameter will generate the perturbative solution with $A$ and $\Phi$ parametrized by $\bar{\varepsilon}$. Matching the latter with the perturbative solution of the previous section will allow us to establish a concrete relation between the new smallness parameter $\bar{\varepsilon}$ and the old parameter of the slow roll expansion $\varepsilon$, $(4.10)$, and in this way find $A$ and $\Phi$ as functions of $\varepsilon = \varepsilon(\varphi_0)$. This ansatz has the following form

$$a = \frac{\bar{\varepsilon}^{1/2}}{H_0} \bar{g}(x), \quad x = \frac{\theta - \theta_f}{\bar{\varepsilon}^{1/2}}, \quad (6.4)$$

where $\bar{g}(x)$ is the some function of the new argument $x$ replacing the angular coordinate $\theta$ and the parameter $A$ in terms of $\bar{\varepsilon}$ reads as

$$A^3 = \frac{3\bar{\varepsilon}}{H_0^2}. \quad (6.5)$$
Note that the new argument $x$ in (6.4) is zero at the singularity and takes negative values on the Hawking-Turok instanton $0 \leq \theta \leq \theta_f$. With this ansatz $\dot{a} = \bar{g} \dot{\phi}/dx$, $H^2(\phi) a^2 = O(\bar{\epsilon})$ and the second term in the right hand side of (6.3) is also $O(\bar{\epsilon})$. Therefore the equations acquire the form in which all the perturbative terms containing the potential $V(\phi) = 3m_p^2 H^2(\phi)/8\pi$ and its gradient are explicitly multiplied by $\bar{\epsilon}$:

$$\frac{d\phi}{dx} = \sqrt{\frac{3}{4\pi}} \frac{m_p}{g^3} \frac{1}{H_0^2} \int_0^x d\bar{x} \bar{g}^3 V', \quad (6.6)$$

$$\left(\frac{d\bar{g}}{dx}\right)^2 = 1 + \frac{1}{g^4} \left(1 + \sqrt{\frac{4\pi}{3} \frac{\bar{\epsilon}}{m_p H_0^2}} \int_0^x d\bar{x} \bar{g}^3 V''\right)^2 - \bar{\epsilon} \frac{H^2(\phi)}{H_0^2} \bar{g}^2. \quad (6.7)$$

In the next two subsections we consecutively solve these equations in the zeroth and first order approximations sufficient for calculating the Hawking-Turok action up to $\bar{\epsilon}^2$-order inclusive.

### 6.1. Zeroth order approximation

To begin with, note that zeroth order approximation does not exactly coincide with the asymptotic solution (3.10)-(3.11), because the first term on the right hand side of (6.7) is not perturbative in $\bar{\epsilon}$, although it becomes negligible at $\sigma \to \sigma_f$ or $x \to 0$ when $ar{g}(x) \to 0$. This term, however, gives a considerable contribution at the point $\theta = \pi - \varepsilon$, where we match the two solutions, and therefore it should not be treated perturbatively. Therefore, the equation for the function $\bar{g}(x)$, which we denote in the lowest order approximation by $g(x)$ without the bar, reads

$$\bar{g}(x) = g(x) + O(\bar{\epsilon}), \quad (6.8)$$

$$\frac{dg(x)}{dx} = -\sqrt{1 + \frac{1}{g^4(x)}}. \quad (6.9)$$

This function has the asymptotics

$$g(x) \simeq (-3x)^{1/3}, \quad x \to 0, \quad (6.10)$$

obviously matching with the asymptotic behaviour (3.10). We also need its behaviour at large negative $x$ corresponding, as we will shortly see, to the matching point in the domain of validity of the slow roll expansion (5.15). It can be obtained by rewriting (6.9) in the form

$$g(x) = C - x - \int_{g(x)}^\infty dg \left(1 - \frac{1}{1 + 1/g^4}\right), \quad (6.11)$$

$$C \equiv \int_0^\infty dg \left(1 - \frac{1}{\sqrt{1 + 1/g^4}}\right) = \frac{2\pi^{3/2}}{\Gamma^2(1/4)}, \quad (6.12)$$

and solving it for big $g(x)$ by iterations

$$g(x) = C - x - \frac{1}{6(C - x)^3} - \frac{5}{168(C - x)^7} + \ldots, \quad x \to -\infty. \quad (6.13)$$
The zeroth order equation for $\phi$

$$\frac{d\phi}{dx} = \sqrt{\frac{3}{4\pi}} \frac{m_P}{g^3} \frac{1}{\sqrt{1 + g^2}}$$  \hspace{1cm} (6.14)

cannot be integrated in closed form, because the function $g(x)$ is not explicitly known, but the scalar field can be found parametrically in terms of $g(x)$ by integrating the following equation – the corollary of (6.3) and (6.14)

$$\frac{d\phi}{dg} = -\sqrt{\frac{3}{4\pi}} \frac{m_P}{g \sqrt{1 + g^4}}.$$  \hspace{1cm} (6.15)

Its integration gives the result

$$\phi = \Phi + \frac{m_P}{\sqrt{12\pi}} \frac{1}{2} \ln \left( \frac{9H_0^2}{8\bar{\epsilon}} \right) - \sqrt{\frac{3}{16\pi}} m_P \ln \frac{g^2}{1 + \sqrt{1 + g^4}}.$$  \hspace{1cm} (6.16)

where the integration constant follows from comparing the limit of $x = (\theta - \theta_f)/\bar{\epsilon}^{1/2} \to 0$ with the asymptotic behaviour (3.11).

Matching the solution with its slow roll perturbation expansion in $\epsilon$, (5.13)-(5.14), at $\theta = \pi - \epsilon$ with $\epsilon/\bar{\epsilon}^2 \ll 1$ takes place at

$$-x|_{\theta=\pi-\epsilon} = \frac{\epsilon - (\pi - \theta_f)}{\bar{\epsilon}^{1/2}} \gg 1,$$  \hspace{1cm} (6.17)

where the expansion (6.13) for $g(x)$ works. Comparison of the scale factor (5.14) with (6.4)-(6.13), $H_0a = (\bar{\epsilon}^{1/2}C - \pi + \theta_f) + \epsilon + O(\bar{\epsilon}^2)$, shows that the angular coordinate of the Hawking-Turok singularity equals

$$\theta_f = \pi - \bar{\epsilon}^{1/2}C + O(\epsilon).$$  \hspace{1cm} (6.18)

With this value of $\theta_f$, $g(x) = \epsilon/\bar{\epsilon}^{1/2} + O(\bar{\epsilon}^{1/2}) \gg 1$. Therefore, matching the inflaton field (6.10) with (5.13) ($\epsilon^0$ and $\epsilon^{-2}$ parts) gives $\Phi$ and $\bar{\epsilon}$ in the lowest order approximation

$$\Phi = \phi_0 - \frac{m_P}{\sqrt{12\pi}} \frac{1}{2} \ln \left( \frac{9H_0^2}{8\bar{\epsilon}} \right) + O(\epsilon),$$  \hspace{1cm} (6.19)

$$\bar{\epsilon} = \epsilon + O(\epsilon^2).$$  \hspace{1cm} (6.20)

In view of (6.5) these expressions correspond to the relations (3.15)-(3.16) already advocated in Sect.2.

### 6.2. First order approximation

In the first order approximation the second term of eq.(6.6) can be obtained by using the zeroth order results. In view of the approximate expression for the inflaton potential (5.11) one has

$$V'(\phi) = \epsilon \frac{3^{3/2}m_P}{8\sqrt{\pi}} H_0^2 \exp \left[ -\sqrt{\frac{16\pi}{3}} \frac{\phi - \phi_0}{m_P} \right] + O(\epsilon^2),$$  \hspace{1cm} (6.21)
where
\[
\exp \left[ -\sqrt{\frac{16\pi}{3}} \phi - \phi_0 \right] = \frac{g^2}{1 + \sqrt{1 + g^4}} + O(\epsilon)
\]  
(6.22)
due to the results of the zero order approximation. Therefore
\[
\sqrt{\frac{4\pi}{3}} \frac{\bar{\epsilon}}{m_P H_0^2} \int_0^x d\bar{x} \bar{g}^3 V' = -\frac{3}{4} \epsilon^2 \int_0^{\bar{g}(x)} \frac{d\bar{g} g^7}{\sqrt{1 + g^4 (1 + \sqrt{1 + g^4})}} + O(\epsilon^3),
\]  
(6.23)
where the change of integration variable is approximately done with the aid of eq.(6.9).

Performing the integration we convert the equation (6.6) to the form
\[
\frac{d\phi}{dx} = \sqrt{\frac{3}{4\pi}} m_P \frac{1}{\bar{g}^3} \left[ 1 - \frac{3}{16} \epsilon^2 \left( \sqrt{1 + g^4} - 1 \right)^2 \right] + O(\epsilon^3).
\]  
(6.24)
This equation can be used in (6.7) along with the expression
\[
\frac{H^2(\phi)}{H_0^2} = 1 + \frac{3}{4} \epsilon g^2 + 1 - \sqrt{1 + g^4} + O(\epsilon^2)
\]  
(6.25)
valid in view of the relation (6.22), whence
\[
\left( \frac{d\bar{g}}{dx} \right)^2 = 1 + \frac{1}{\bar{g}^4} - \bar{\epsilon} \bar{g}^2 - \frac{3}{8} \bar{\epsilon}^2 \left[ 2 (\bar{g}^2 + 1 - \sqrt{1 + \bar{g}^4}) + \frac{(\sqrt{1 + \bar{g}^4} - 1)^2}{\bar{g}^4} \right] + O(\epsilon^3).
\]  
(6.26)
Integration of (6.24) can again be done parametrically in terms of \( \bar{g} \) with \( d\bar{g}(x)/dx \) taken from (6.26). With the integration constant again defined in the limit of \( \bar{g} \to 0 \) one has
\[
\phi = \Phi + \frac{m_P}{\sqrt{12\pi}} \frac{1}{2} \ln \left( \frac{9H_0^2}{8\bar{\epsilon}} \right) - \sqrt{\frac{3}{16\pi}} m_P \ln \frac{\bar{g}^2}{1 + \sqrt{1 + \bar{g}^4}}
\]  
\[
- \bar{\epsilon} \sqrt{\frac{3}{16\pi}} m_P \frac{1}{2} \left[ \ln(\bar{g}^2 + 1 + \bar{g}^4) - \frac{\bar{g}^2}{\sqrt{1 + \bar{g}^4}} \right]
\]  
\[
+ \bar{\epsilon}^2 \sqrt{\frac{3}{16\pi}} m_P \frac{3}{16} \left[ -2 \ln \frac{1 + \sqrt{1 + \bar{g}^4}}{2} + \sqrt{1 + \bar{g}^4} - 1 \right] + O(\epsilon^3).
\]  
(6.27)
The solution of eq.(6.26) with the initial condition \( \bar{g}(0) = 0 \) can be found in several steps. First it can be integrated perturbatively in \( \bar{\epsilon} \). For large \( \bar{g} \gg 1 \) this integration gives the equation
\[
\bar{g} + \frac{1}{6\bar{g}^3} - \frac{3}{56\bar{g}^7} + \ldots + \bar{\epsilon} \left[ \frac{\bar{g}^3}{6} - \frac{5\Gamma^2(1/4)}{48\sqrt{\pi}} + \frac{3}{4\bar{g}} + \ldots \right]
\]  
\[
+ \bar{\epsilon}^2 \left[ \frac{3}{40} \bar{g}^5 - \frac{3}{8} \bar{g} + \left( \frac{63\pi^{3/2}}{40\Gamma^2(1/4)} - \frac{5\Gamma^2(1/4)}{64\sqrt{\pi}} \right) + \frac{9}{16\bar{g}} + \ldots \right] = C - x + O(\epsilon^3),
\]  
(6.28)
which, when solved iteratively in $\bar{\epsilon}$, yields the solution

$$
\bar{g}(x) = C - x - \frac{1}{6(C - x)^3} - \frac{5}{168(C - x)^7} + \ldots
+ \bar{\epsilon} \left[ -\frac{1}{6}(C - x)^3 + \frac{5\pi}{24C} + O(1/(C - x)^5) \right]
+ \epsilon^2 \left[ \frac{1}{120}(C - x)^5 - \frac{5\pi}{48C}(C - x)^2 + O(C - x) \right] + O(\epsilon^3)
$$

(6.29)

expanded in inverse powers of $(C - x)$. Matching this solution with the slow roll perturbation expansion (5.13)-(5.14) at $\theta = \pi - \epsilon$ begins by noting that at the matching point

$$
C - x = \frac{\epsilon - \epsilon_0}{\epsilon^{1/2}},
$$

(6.30)

$$
\epsilon_0 \equiv \pi - \theta_f - C\bar{\epsilon}^{3/2}.
$$

(6.31)

From the zeroth order approximation we know that $\epsilon_0 = O(\bar{\epsilon})$, therefore the scale factor (6.4) with (5.13) at this point takes the form of double series in $\bar{\epsilon}$ and $\epsilon$.

$$
H_0 a(\pi - \epsilon) = \epsilon - \frac{1}{6}\epsilon^3 + \frac{1}{120}\epsilon^5 + \ldots - \epsilon_0 \left( 1 - \frac{3}{6}\epsilon^2 + \ldots \right)
+ \epsilon^{3/2} \left( \frac{5\pi}{24C} - \frac{5\pi}{48C}\epsilon^2 + \ldots \right) + \epsilon^2 \left( -\frac{1}{6}\epsilon^3 + \ldots \right) + O(\epsilon^{5/2})
$$

(6.32)

Comparison with (5.14) then shows that the first group of terms reproduces its unperturbed part, $\sin \epsilon$, while the choice of

$$
\epsilon_0 = \frac{5\pi}{24C}\epsilon^{3/2} + O\left( \epsilon^2 \right)
$$

(6.33)

guarantees the absence of the half integer power of $\bar{\epsilon}$. Eq.(6.32) cannot serve, however, for the determination of $\bar{\epsilon}$ in terms of $\epsilon$, because it lacks the next $-\epsilon^3$ - order of perturbation theory. But the corresponding $-\epsilon^2$ - order is already contained in the expression (6.27) for $\dot{\phi}$. Therefore, its matching with (5.13) can be used for the determination of both unknown quantities $\Phi$ and $\bar{\epsilon}$.

With the above choice of parameters we have $\bar{g} = \sin \epsilon / \bar{\epsilon}^{1/2} + O(\epsilon^{3/2})$ at the matching point and the inflaton (6.27) takes the form

$$
\phi(\pi - \epsilon) = \Phi + \frac{m_P}{\sqrt{12\pi^2}} \ln \left( \frac{9H_0^2}{8\epsilon} \right)
+ \sqrt{\frac{3}{16\pi}} m_P \left[ \bar{\epsilon} \left( \frac{1}{\epsilon^2} + \frac{1}{2}\ln \frac{\bar{\epsilon}}{2\epsilon^2} + \frac{1}{2} + \ldots \right) + \epsilon^2 \left( \frac{3}{8} \ln \frac{2\bar{\epsilon}}{\epsilon^2} - \frac{3}{16} + \ldots \right) + O\left( \epsilon^3 \right) \right]
$$

(6.34)

Now we match this expression with (5.17). The both double pole and logarithmic in $\epsilon$ terms here dutifully coincide with those of (5.13) under the following identification

$$
\bar{\epsilon} = \epsilon - \frac{11}{12} \epsilon^2 + O\left( \epsilon^3 \right)
$$

(6.35)
and \( \Phi \) takes the form

\[
\Phi(\phi_0) = \phi_0 - \frac{1}{2} \frac{m_P}{\sqrt{12\pi}} \ln \left[ \frac{9H_0^2}{8\epsilon} \right] - \frac{1}{2} \sqrt{\frac{3}{16\pi}} m_P \epsilon \left( \ln \frac{\epsilon}{8} + \frac{35}{18} \right) + O\left(\epsilon^2\right).
\] (6.36)

In view of (6.5) and (6.33) the other two parameters of the asymptotic behaviour near the singularity read

\[
\theta_f \equiv H_0\sigma_f \simeq \pi - \frac{2\pi^{3/2}}{\Gamma^2(1/4)} \epsilon^{1/2} - \frac{5\Gamma^2(1/4)}{48\pi^{1/2}} \epsilon^{3/2},
\] (6.37)

\[
A^3(\phi_0) \simeq \frac{3\epsilon}{H_0^2} - \frac{11}{4} \frac{\epsilon^2}{H_0^2}.
\] (6.38)

The expressions (6.36) and (6.38) can now be substituted into the equation (3.14) for \( \bar{I}_{HT}(\phi_0) \). Then the integration of terms belonging to the zeroth order approximation will obviously reproduce the contribution (4.16) which is already written in terms of the inflaton field in the nonminimal frame. The first order corrections will generate the contribution \( \Delta \bar{I}_{HT}(\phi_0) = \Delta I_{HT}(\varphi_0) \) which, unfortunately, can no longer be obtained in a closed form for an arbitrary potential. Rather, the potential (5.11) has to be used in (3.14). The integration and the conversion of the result to the original frame then gives the final result for the Hawking-Turok action to the second order in \( \frac{m_P^2}{|\xi|} \phi^2 \) inclusive

\[
I_{HT}(\varphi) = -\frac{96\pi^2|\xi|^2}{\lambda} - \frac{2\pi(1+\delta)}{\lambda} \frac{m_P^2}{\varphi^2} + 2 \left( \frac{(1+\delta)}{\lambda} \right)^2 \left( \frac{m_P^2}{\varphi^2} \right)^2 \ln \left( \frac{6\pi|\xi|\varphi^2}{m_P^2(1+\delta)\kappa} \right) + O\left(\frac{m_P^6}{|\xi|\varphi^6}\right),
\] (6.39)

where \( \kappa \) absorbs the following combination of numerical parameters

\[
\ln \kappa \equiv \frac{11}{3} - \ln 4 - \frac{3}{4} \left( \frac{1+2\delta}{1+\delta} \right)^2.
\] (6.40)

Due to big \( |\xi| \) it contains a large but slowly varying (in \( \varphi \)) logarithmic coefficient. The positive coefficient of this logarithmic term actually follows from the sign of \( \ln \cos(\theta/2) \) in the equation (5.7) for \( \delta \phi \) above and, thus, it is pretty well fixed. This sign will have important consequences for quantum creation of the open Universe.

### 7. Tree-level probability peaks

Note, that the logarithmic structure of the result (6.39) resembles the behaviour of Coleman-Weinberg loop effective potentials (up to inversion of \( \varphi \)), even though this term is entirely of a tree-level origin. Thus, the classical theory somehow feels quantum structures when probing Planckian scales near the singularity. The nontrivial structure of the classical action allows one to expect the existence of the probability peaks already in the tree-level
approximation with the distribution functions $\rho_{NB,T}(\varphi) \simeq \exp(\mp I_{HT}(\varphi)).$ The extremum equation for the exponential

$$
\frac{d}{dx} I_{HT}(\varphi(x)) = \frac{12\pi^2|\xi|}{\lambda \kappa} \frac{1}{x^3} \left( x - \frac{12|\xi|}{\kappa} \ln \frac{x}{\sqrt{e}} \right) = 0,
$$

(7.1)

$$
x \equiv \frac{6\pi|\xi|\varphi^2}{m_P^2(1+\delta)\kappa},
$$

(7.2)

indeed results in two roots $x_{\pm}$ which for big $|\xi|$ read as

$$
x_+ \simeq \frac{12|\xi|}{\kappa} \ln \left( \frac{12|\xi|}{\kappa\sqrt{e}} \right),
$$

(7.3)

$$
x_- \simeq \sqrt{e},
$$

(7.4)

To analyze the inflationary scenario generated by these peaks we have to resort to the classical equations of motion. For the model (4.1) they were considered in much detail in [22]. The slope of the potential (4.9) is positive for $\delta > -1$ which implies the finite inflationary epoch with slowly decreasing inflaton only in this range of $\delta$. The inflationary e-folding number in this case is approximately given by the equation

$$
N \simeq \int_{0}^{\varphi_2} \frac{3H^2(\varphi)}{|F(\varphi)|},
$$

(7.5)

where the Hubble constant

$$
H^2(\varphi) \simeq \frac{\lambda}{12|\xi|} \varphi^2
$$

(7.6)

and $F(\varphi)$ is the rolling force in the inflaton equation of motion for the nonminimal model

$$
\ddot{\varphi} + 3H\dot{\varphi} - F(\varphi) = 0, \quad F(\varphi) \simeq -\lambda m_P^2(1+\delta)\varphi/48\pi \xi^2.
$$

The integration then gives

$$
N \simeq \frac{6\pi|\xi|\varphi^2}{m_P^2(1+\delta)},
$$

(7.7)

so that the parameter $x$ introduced for brevity in (7.2) is related to the e-folding number, $N = \kappa x$. The order of magnitude of the coupling constants, we shall use, is the one that fits the COBE normalization for the microwave background anisotropy generated in this model and good conditions for reheating, $|\xi| \sim 2 \times 10^4$, $\lambda \sim 0.05$, $\sqrt{\lambda/|\xi|} \sim 10^{-5}$. We shall also assume that the mass term in the inflaton potential is small, so that the parameter $\delta$ in (2.11) is negligible in what follows. With this choice the parameter $\kappa$ is approximately given by

$$
\kappa \simeq 4.6.
$$

(7.8)

For the smaller root $x_-$, the second derivative of the action is negative, $d^2I_{HT}/dx^2 = -(12\pi|\xi|/\kappa e)^2/\lambda < 0$, so that it corresponds to the maximum of the tunneling distribution $\rho_T \simeq \exp(\mp I_{HT}).$ This tunneling peak has a a very small relative width $\Delta x_-/x_- \simeq
\[ \kappa \sqrt{\lambda e}/12\pi |\xi| \sim 10^{-6} \text{ and generates a very small e-folding number for inflationary stage} \]

beginning with the GUT scale of the Hubble constant

\[ N_- = \kappa x_- \approx \kappa \sqrt{e} \sim 9, \quad (7.9) \]

\[ H_2^2 \approx m_p^2 \frac{\lambda \kappa (1 + \delta)}{72\pi |\xi|^2} \sim 10^{-11} m_p^2. \quad (7.10) \]

According to (2.1) such an e-folding number generates extremely small value of \( \Omega, \Omega \sim e^{-10^6} \), which makes this peak unacceptable as a candidate for the initial conditions for open inflation.

The second root (7.4) brings us to another extreme. Because of the positive second derivative \[ d^2 I_{HT}(x_+)/dx_+^2 = \frac{\pi^2 \kappa^2}{144\lambda |\xi|^2 \ln(12|\xi|/\kappa \sqrt{e})} \]

this peak exists for the no-boundary case with \( \rho_{NB} \sim \exp(-I_{HT}) \). This peak is rather smeared, for its relative width is \( \Delta x_+/x_+ \sim \sqrt{\lambda} \sim 0.2 \), and its parameters are

\[ N_+ = 12|\xi| \ln \frac{12|\xi|}{\kappa \sqrt{e}} \sim 10^6, \quad (7.11) \]

\[ H_+^2 \approx m_p^2 \frac{\lambda \kappa (1 + \delta)}{72\pi |\xi|^2} \sim 10^{-11} m_p^2. \quad (7.12) \]

The energy scale again belongs to the GUT domain, but the corresponding density parameter is far too close to unity, \( \Omega \sim 1 - e^{-2\times10^6} \), to account for the observable Universe.

### 8. Quantum corrections

The most vulnerable point of the Hawking-Turok instanton is the construction of quantum corrections on its singular background. Although the classical Euclidean action is finite, the quantum part of the effective action involving the higher order curvature invariants is infinite, because their spacetime integrals are not convergent at the singularity. At least naively, this means that the whole amplitude is either suppressed to zero or infinitely diverges indicating strong instability. Clearly, a self-consistent treatment should regularize the arising infinities due to the back reaction of the infinitely growing quantum stress tensor.

The result of such a self-consistent treatment is hardly predictable because we do not yet have for it an exhaustive theoretical framework. This framework might include fundamental stringy structures underlying our local field theory, which are probed by Planckian curvatures near the singularity. However, even without the knowledge of this fundamental framework it is worth considering usual quantum corrections due to local fields on a given singular background. This might help revealing those dominant mechanisms that are robust against the presence of singularities and their regulation due to back reaction and fundamental strings.

These quantum corrections can be divided into two main categories – nonlocal contributions due to massless or light degrees of freedom and local contributions due to heavy massive fields \[ [31] \]. The effects from the first category can be exactly calculable when they
are due to the conformal anomaly of the conformal invariant fields. For the Hawking-Turok instanton this calculation can be based on the (singular) conformal transformation mapping its geometry to the regular metric $d \tilde{s}^2$ of the half-tube $R^+ \times S^3$,

$$ds^2 = a^2(\sigma(X)) d\tilde{s}^2, \quad d\tilde{s}^2 = dX^2 + d\Omega^2,$$

with the conformal coordinate $X = \int_0^{\sigma'} d\sigma'/a(\sigma')$, $0 \leq X < \infty$. With this conformal decomposition of the metric the effective action $\Gamma[g_{\mu\nu}]$ can be represented as a sum of the finite effective action of $\tilde{g}_{\mu\nu}$, $\Gamma[\tilde{g}_{\mu\nu}]$, and $\Delta \Gamma[\tilde{g}_{\mu\nu}, a]$ – the anomalous action obtained by integrating the known conformal anomaly along the orbit of the local conformal group joining $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$. The anomalous action $\Delta \Gamma[\tilde{g}_{\mu\nu}, a]$ is known for problems without boundaries [32, 33, 34]. For a singular conformal factor $a^2(\sigma(X))$ the bulk part of this action is divergent, but the question of its finiteness is still open, because in problems with boundaries the conformal anomaly has surface (simple and double layer) contributions [35] that might lead to finite anomalous action on the Hawking-Turok instanton [36].

Fortunately, the problem with large nonminimal coupling of the inflaton falls into the second category of problems – local effective action of heavy massive fields. Due to the Higgs mechanism for all matter fields interacting with inflaton by (2.6) their particles acquire masses $m^2 \sim \varphi^2$ strongly exceeding the spacetime curvature $R \sim \lambda\varphi^2/|\xi|$ [19, 20, 22]. The renormalized effective action expanded in powers of the curvature to mass squared ratio $R/m^2 \sim \lambda/|\xi| \ll 1$ for generic spacetime background has the following form of the local Schwinger-DeWitt expansion [37, 39, 22]

$$\Gamma^{1\text{-loop}} = \frac{1}{32\pi^2} \int d^4x \, g^{1/2} \text{tr} \left\{ \frac{1}{2} \left( \ln \frac{m^2}{\mu^2} - \frac{3}{2} \right) m^4 \hat{1} + \left( \ln \frac{m^2}{\mu^2} - 1 \right) m^2 \hat{a}_1(x, x) + \ln \frac{m^2}{\mu^2} \hat{a}_2(x, x) - \sum_{n=1}^{\infty} \frac{(n-1)!}{m^{2n}} \hat{a}_{n+2}(x, x) \right\}. \quad (8.2)$$

Here tr denotes the trace over isotopic field indices, hats denote the corresponding matrix structures in vector space of quantum fields and $\hat{a}_n(x, x)$ are the Schwinger-DeWitt coefficients. The latter can be systematically calculated for generic theory as spacetime invariants of growing power in spacetime and fibre bundle curvatures [37, 38, 39, 40].

The situation with this expansion on a singular instanton of Hawking and Turok is also not satisfactory – all integrals of curvature invariants starting with $\hat{a}_2(x, x)$ (quadratic in the curvature and higher) diverge at the singularity. Fortunately, the lessons from the asymptotic theory of semiclassical expansion teach us that the lowest order terms can be trusted as long as they are well defined. In our case this is the term quartic in masses with the logarithm. But this term is exactly responsible for the dominant contribution to the anomalous scaling (2.9) quadratic in $|\xi|$

$$Z = \frac{1}{32\pi^2} \int d^4x \, g^{1/2} \sum_{\text{particles}} m^4 + \ldots. \quad (8.3)$$

This term is dominating the quantum part of effective action, while the others, although being divergent at the singularity, are strongly suppressed by powers of $1/|\xi|$. With the
assumption that the back reaction of quantum stress tensor regulates these divergences, one can conclude that the quantum effective action on the Hawking-Turok instanton in our model is still dominated by the anomalous scaling term of eq.\((2.9)\). In the next section we analyze the consequences of this result.

9. Open inflation without anthropic principle

Consider now the distribution functions \((2.9)\) with the Hawking-Turok action \((6.39)\) and the anomalous scaling term of \(\Gamma^{1\text{-loop}}\). A crucial difference from the case of closed cosmology is that the \(\phi\)-dependence of \(I_{HT}(\phi)\) is dominated now by \(|\xi|^0\)-term which contains due to a big slowly varying logarithm a large positive contribution quartic in \(m_P/\phi\). In the case of the no-boundary distribution function the extremum equation reduces to

\[
\frac{d}{dx}(I_{HT} + \Gamma^{1\text{-loop}}) = \frac{3|\xi|^2 \lambda}{x^3} \left( x^2 - \frac{24\pi^2}{\lambda^2 e A} \ln \frac{x^2}{e} + \frac{4\pi^2 x}{\lambda \xi A} \right) = 0
\]

with the same variable \(x\) replacing \(\phi\) as in \((7.2)\). For large \(|\xi| \gg 1\) the last term can be omitted, and the equation again has two roots which equal, in the assumption that \(24\pi^2/\lambda A \gg 1\),

\[
x_-^2 \simeq e,
\]

\[
x_+^2 \simeq \frac{24\pi^2}{\lambda A} \ln \frac{24\pi^2}{\lambda e A}.
\]

The first root \(x_-\) does not yield the probability maximum because \(d^2(I_{HT} + \Gamma)/dx^2 \sim -144\pi^2|\xi|^2/\lambda \ll 0\), while the second root \(x_+\) generates the peak, for

\[
\left. \frac{d^2}{dx^2}(I_{HT} + \Gamma) \right|_{x_+} \simeq \frac{144\pi^2|\xi|^2}{\lambda} \left( \ln \frac{24\pi^2}{\lambda e A} - 1 \right) > 0.
\]

The positivity of this expression and the assumption above will soon be justified from the bound on the e-folding number. The parameters of this peak – the mean value \(\varphi_{I}\), Hubble constant \(H_I\) and quantum dispersion \(\Delta\varphi \equiv [d^2(I_{HT} + \Gamma)/d\varphi^2]^{-1/2}\) – are

\[
\varphi_{I} \simeq \frac{m_P^2}{|\xi|} \left( \frac{2}{3A} \ln \frac{24\pi^2}{\lambda^2 e A} \right)^{1/2},
\]

\[
H_I^2 \simeq m_P^2 \frac{\lambda}{|\xi|^2} \frac{1}{12} \left( \frac{2}{3A} \ln \frac{24\pi^2}{\lambda^2 e A} \right)^{1/2},
\]

\[
\frac{\Delta \varphi}{\varphi_{I}} \sim \frac{\Delta H}{H_I} \sim \frac{\kappa^2 \sqrt{6A}}{72\pi^2} \frac{\sqrt{\lambda}}{|\xi|}.
\]

Similarly to the closed model, these parameters are suppressed relative to the Planck scale by a small dimensionless ratio \(\sqrt{\lambda}/|\xi|\) known from the COBE normalization. As regards the e-folding number \((7.7)\) it is given for this peak entirely in terms of the same universal
combination of coupling constants (2.8) (remember that, apart the negligible dependence of \( \kappa \) on \( \delta \), \( \kappa \sim 4.6 \)).

\[
N \simeq \left( \frac{24\pi^2}{A} \ln \frac{24\pi^2}{\kappa^2 e A} \right)^{1/2}. \tag{9.8}
\]

Comparison of this result with the e-folding number, \( N \sim 60 \), necessary for generating the observable density \( \Omega \), \( 0 < \Omega < 1 \), not very close to one or zero, immediately gives the bound on \( A \)

\[
A \sim \frac{48\pi^2}{N^2} \ln \frac{N}{\kappa e} \sim 0.3. \tag{9.9}
\]

This bound justifies the above assumption on the magnitude of the factor \( 24\pi^2/\kappa^2 e A \sim 10 \gg 1 \), the positivity in (9.4) and the validity of the slow roll approximation throughout the whole paper – the expansion in powers of \( m^2_P/4\pi|\xi|^2 \sim \ln N/N \) (see eqs.(4.9) and (4.16)).

A similar analysis for the case of the tunneling distribution function shows that the corresponding extremum equation has the same form (9.1) but with opposite signs of the last two terms inside the brackets. Therefore, it has only one root \( x \simeq \sqrt{e} \). This is again the maximum of the distribution function, because \( d^2(-I_{HT} + \Gamma)/dx^2 \sim 144\pi^2|\xi|^2/\kappa^2 \lambda > 0 \), but the corresponding e-folding number, \( N \simeq \kappa \sqrt{e} \sim 8 \), and the density parameter \( \Omega \sim e^{-100} \) are far too small to describe the observable Universe. This leaves us with the only candidate for the initial conditions of inflation (9.5)-(9.7) generated by the no-boundary wavefunction of the open Universe.

10. Conclusions

Thus, we have derived the no-boundary and tunneling distribution functions for the open inflationary models originating by the Hawking-Turok mechanism from the singular instanton. By applying the slow roll perturbation expansion we calculated the classical action of this instanton in the model with a strong nonminimal coupling. The resulting distribution functions have in the tree-level approximation sharp probability peaks, which are, however, incompatible with the observational bounds on \( \Omega \). The inclusion of one-loop corrections allows one to get such a probability peak satisfying these bounds for the no-boundary quantum state of the open Universe. The corresponding probability maximum for the tunneling state is far out of the observational range because it predicts absolutely negligible value of \( \Omega \).

These above conclusions are based on classical equations of inflationary dynamics. The latter should certainly be replaced by effective equations for mean fields to have reliable answers within the same accuracy as the calculation of the one-loop distribution function. Since quantum effects qualitatively change the tree-level initial conditions, one should expect that they might strongly influence the dynamics as well. Effective equations of motion for
the model (4.1) were obtained in our recent paper [22], but according to the discussion of the previous section they are not, strictly speaking, applicable here. This is because the Hawking-Turok instanton background does not satisfy the condition of the local Schwinger-DeWitt expansion\(^4\). It does not satisfy this condition globally, in the vicinity of the singularity, but the open inflationary Universe lies inside the light cone originating from the instanton pole antipodal to the singularity. Therefore, if we restrict ourselves with the local part of effective equations, then for this part we can use our old results [22]. The influence of the spatially remote singularity domain is mediated by nonlocal terms. These terms strongly depend on the boundary conditions at the singular boundary and are beyond the control of the local Schwinger-DeWitt approximation. Within the same reservations as those concerning the finiteness of quantum effects on this instanton (and the validity of the Hawking-Turok model as a whole) we can neglect nonlocal terms and use only the local part of effective equations.

There are two arguments in favour of this approximation. Firstly, it is very likely that the nonlocal contribution of the singularity is suppressed by inverse powers of \(|\xi|\gg 1\). At least naively, these effects are inverse to the size of the Universe given by the Hubble constant in (9.7) and also can be damped by \(1/|\xi|\) in the same way as for the singular part of the scalar curvature. Secondly, if we restrict ourselves with a limited spatial domain of the very early open Universe (close to the tip of the light cone originating from the regular pole of the Hawking-Turok instanton), then these nonlocal terms do not contribute at all in view of causality of effective equations, because at early moments of time this domain is causally disconnected from singularity\(^5\).

Local quantum corrections in effective equations depend only on the local geometry of the quasi-DeSitter open Universe. They boil down to the replacement of the classical coefficient functions \((V(\varphi), U(\varphi))\) of the model (4.1) by the effective ones calculated in [22] for a wide class of quantum fields coupled to the inflaton in the limit of big \(|\xi|\gg 1\). It remains to use these functions in classical equations and study the inflation dynamics starting from the initial value of the inflaton (1.5). It turns out that unlike in the closed model (where the quantum terms were of the same order of magnitude as classical ones) the quantum corrections are strongly suppressed by the slow-roll parameter \(\sqrt{A}/4\pi\approx \ln N/\pi\) already at the start of inflation. In particular, the effective rolling force differs from the classical one above by the negligible correction

\[
F_{\text{eff}}(\varphi) \simeq -\frac{\lambda m_{\varphi}^2(1+\delta)}{48\pi \xi^2} \varphi \left\{ 1 + \left[ \frac{A}{96\pi^2} \ln \frac{24\pi^2}{\kappa^2 e A} \right]^{1/2} \frac{\varphi^2}{\varphi_f^2} \right\}. \tag{10.1}
\]

For comparison, in the closed model the second (quantum) term in curly brackets enters with a unit coefficient of \(\varphi^2/\varphi_f^2\), see eqs. (6.7) and (6.10) of [22]. This quantum correction

\(^4\)For closed cosmology with no-boundary or tunneling quantum states the slow roll approximation guarantees the validity of the local Schwinger-DeWitt expansion that was used in [22] for the derivation of the effective equations.

\(^5\)This argument is rigorous but somewhat macabre for the inhabitants of this domain, because after a while they will suffer a fatal influence of fields propagating from the singularity.
gives inessential contribution to the duration of the inflationary stage (9.8) and thus does not qualitatively change the above predictions and bounds. The smallness of quantum corrections (roughly proportional to $\mathcal{A}/32\pi^2$) can be explained by stronger bound on $\mathcal{A} \sim 0.3$ (cf. $\mathcal{A} \leq 5.5$ in the closed model [22]) and another dependence of $\varphi_I$ on $\mathcal{A}$.

Thus, the no-boundary quantum state on the Hawking-Turok instanton in the model with large nonminimal curvature coupling generates open inflationary scenario compatible with observations and, in particular, capable of producing the needed value of $\Omega$. No anthropic considerations or fine tuning of initial conditions is necessary to reach such a final state of the Universe. The only fine tuning we get is the bound on the parameter $\mathcal{A}$ of the matter field sector (9.9) and the estimate of the ratio $\sqrt{\lambda}/|\xi| \sim 10^{-5}$ based on the normalization from COBE which looks as a natural determination of coupling constants of Nature from the experiment. The mechanism of such quantum birth of the Universe is based on quantum effects on the Hawking-Turok instanton, treated within semiclassical loop expansion. The validity of this expansion is in its turn justified by the energy scale of the phenomenon (9.7) which belongs to the GUT domain rather than the Planckian one.

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