THE DIRICHLET PROBLEM FOR A CLASS OF DEGENERATE HESSIAN EQUATIONS

HEMING JIAO AND TINGTING WANG

ABSTRACT. In this paper, we study the Dirichlet problem for a class of degenerate Hessian equations. We establish the $C^2$ estimates for an approximating problem and the existence of smooth solution is proved.

Keywords: degenerate Hessian equations, interior second order estimates, smooth solutions.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary $\partial \Omega$. In this paper, we are concerned with the regularity for solutions of the Dirichlet problem

$$
\begin{cases}
    f(\lambda[D^2u + \gamma \Delta u I]) = \psi & \text{in } \Omega, \\
    u = \varphi & \text{on } \partial \Omega,
\end{cases}
$$

where $\gamma \geq 0$ is a constant, $I$ is the unit matrix and $\lambda[D^2u + \gamma \Delta u I] = (\lambda_1, \ldots, \lambda_n)$ denote the eigenvalues of the matrix $\{D^2u + \gamma \Delta u I\}$.

Following [1], $f \in C^2(\Gamma) \cap C(\bar{\Gamma})$ is assumed to be defined in an open convex symmetric cone $\Gamma$, with vertex at the origin and

$$
\Gamma \supseteq \Gamma_n = \{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\},
$$

and to satisfy the following structure conditions:

$$
\begin{align*}
    (1.2) & \quad f_i \equiv \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, 1 \leq i \leq n, \\
    (1.3) & \quad f \text{ is concave in } \Gamma, \\
    (1.4) & \quad f > 0 \text{ in } \Gamma, f = 0 \text{ on } \partial \Gamma.
\end{align*}
$$

A function $u \in C^2(\Omega)$ is called admissible if $\lambda[D^2u + \gamma \Delta u I] \in \bar{\Gamma}$. According to [1], condition (1.2) ensures that equation (1.1) is degenerate elliptic for admissible solutions. While (1.3) implies that the function $F$ defined by $F[A] = f(\lambda[A])$ to be concave for $A \in S^{n\times n}$ with $\lambda[A] \in \Gamma$, where $S^{n\times n}$ is the set of $n$ by $n$ symmetric matrices.

We assume that $\psi \geq 0$ in $\Omega$, so the equation (1.1) is degenerate. In this paper we shall prove the existence of smooth solutions to (1.1) under the basic conditions (1.2)-(1.4).
Theorem 1.1. Let $\gamma > 0$, $\psi \in C^\infty(\bar{\Omega})$ and $\varphi \in C^\infty(\partial \Omega)$. Suppose that (1.2) - (1.4) hold and there exists a strict admissible subsolution $u \in C^2(\bar{\Omega})$ satisfying
\[
\begin{cases} 
F[D^2u + \gamma \Delta u] \geq \psi(x) + \delta_0 \text{ in } \Omega, \\
\varphi = \text{ on } \partial \Omega,
\end{cases}
\]
for some positive constant $\delta_0$. Then there exists a unique admissible solution $u \in C^\infty(\bar{\Omega})$ to (1.1).

Our strategy is to establish the a priori $C^2$ estimates independent of $\varepsilon$ for the approximating problem
\[
\begin{cases} 
F[D^2u_\varepsilon + \gamma \Delta u_\varepsilon] = \psi + \varepsilon \text{ in } \Omega, \\
u_\varepsilon = \varphi \text{ on } \partial \Omega.
\end{cases}
\]
We find that the $C^1$ estimates can be derived without the positivity of $\gamma$ (see Section 3) so that we can prove the following theorem.

Theorem 1.2. Let $\gamma = 0$, $\psi \in C^{0,1}(\bar{\Omega})$ and $\varphi \in C^{0,1}(\partial \Omega)$. Suppose (1.2) - (1.5) hold. Then there exists a unique viscosity solution $u \in C^{0,1}(\bar{\Omega})$ to (1.1).

We refer the readers for the definition of viscosity solutions to [2] and [16] and to [18, 19, 20] for the weak solutions to Hessian equations.

For the non-degenerate case ($\psi \geq \psi_0 > 0$), the existence of smooth solutions to the Dirichlet problem (1.1) with $\gamma = 0$ was established by Caffarelli, Nirenberg and Spruck [1] under additional assumptions on $f$ in a domain $\Omega$ satisfying that there exists a sufficiently large number $R > 0$ such that, at every point $x \in \partial \Omega$,
\[
(\kappa_1, \ldots, \kappa_{n-1}, R) \in \Gamma,
\]
where $\kappa_1, \ldots, \kappa_{n-1}$ are the principal curvatures of $\partial \Omega$ with respect to the interior normal. Their work was further developed and simplified by Trudinger [17].

Guan considered the non-degenerate Hessian equations of the form
\[
f(\lambda|\nabla^2 u + \gamma \Delta u| + sdu \otimes du - \frac{t}{2}|\nabla u|^2g + A)) = \psi(x, u, \nabla u)
\]
on a Riemannian manifold with metric $g$, which is arising from conformal geometry (see [3] and [6]). In these papers Guan also assumed that $f$ is homogenous of degree one which implies that the equation (1.8) is strictly elliptic. It would be interesting to prove Theorem 1.1 for the general form (1.8) on manifolds when $\psi \geq 0$ without any additional conditions on $f$. The case that $\gamma = 0$ seems more complicated. In a recent work [7], Guan proved Theorem 1.1 under (1.2) - (1.6) for $\delta_0 = 0$ in (1.5) when $\gamma = 0$ and $\psi \geq \psi_0 > 0$. Another interesting question would be whether Theorem 1.1 is valid for $\gamma = 0$ when $\psi \geq 0$.

The main difficulty in the degenerate case is from the boundary estimates for pure normal second order derivative. In a series of papers [12, 13, 14, 15], Krylov provided a technique to deal with the Dirichlet problem for the more general Bellman equations. Ivochina, Trudinger and Wang [10] gave an alternative, shorter proof for Hessian equations under various conditions. Roughly speaking, Krylov’s proof consists of two steps. One is the weakly interior estimate (or the interior estimate) for second order derivatives and the other one is the boundary estimate in terms of the interior one. In this paper, the constant $\gamma > 0$ plays a key role in the proof of interior second order estimates and the existence of subsolutions satisfying (1.6) is crucial to the construction of barrier functions.
It was shown in [1] that using (1.7) and the condition that for every $C > 0$ and every compact set $K$ in $\Gamma$ there is a number $R = R(C,K)$ such that

\begin{equation}
 f(R\lambda) \geq C \text{ for all } \lambda \in K
\end{equation}

one can construct admissible strict subsolutions of equation (1.1) with $\gamma = 0$. Obviously $\Gamma \subset \{ \lambda \in \mathbb{R}^n : \sum \lambda_i > 0 \}$ and we have $\Delta u \geq 0$ for any admissible function $u$. So we can construct an admissible strict subsolution of (1.1) when $\gamma \geq 0$ satisfying (1.5) under (1.7) and (1.9) by the same way.

Typical examples are given by $f = \sigma_1^{1/k}$ and $f = (\sigma_k/\sigma_l)^{1/(k-l)}$, $1 \leq l < k \leq n$, defined in the Gårding cone

\[ \Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \ldots, k \}, \]

where $\sigma_k$ are the elementary symmetric functions

\[ \sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \quad k = 1, \ldots, n. \]

Another interesting example is $f = \log P_k$, where

\[ P_k(\lambda) := \prod_{i_1 < \cdots < i_k} (\lambda_{i_1} + \cdots + \lambda_{i_k}), \quad 1 \leq k \leq n \]

defined in the cone

\[ P_k := \{ \lambda \in \mathbb{R}^n : \lambda_{i_1} + \cdots + \lambda_{i_k} > 0 \}. \]

The case when $f = \sigma_n^{1/n}$ (the Monge-Ampèré equation) and $\gamma = 0$ was studied by Guan, Trudinger and Wang [9] and they obtained the $C^{1,1}$ regularity as $\psi^{1/(n-1)} \in C^{1,1}(\bar{\Omega})$. It would be an interesting problem to show whether the result can be improved for the $f = \sigma_k^{1/k}$ (see [10]).

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.1 provided the $C^2$ estimates for (1.6) is established. $C^1$ estimate is treated in Section 3. The interior second order estimate is proved in Section 4. In section 5, the estimates for second derivatives are established.

2. Beginning of Proof

In this Section we explain how to prove Theorem 1.1 when the second order estimates for (1.6) are established. Let $u_\varepsilon \in C^4(\bar{\Omega})$ be the admissible solution of (1.6). For simplicity we shall use the notations $U^\varepsilon = D^2 u_\varepsilon + \gamma \Delta u_\varepsilon I$ and $U = D^2 u + \gamma \Delta u I$. Following the literature, unless otherwise noted, we denote throughout this paper

\[ F^{ij}[U^\varepsilon] = \frac{\partial F}{\partial U_{ij}^\varepsilon}[U^\varepsilon], \quad F^{ij,kl}[U^\varepsilon] = \frac{\partial^2 F}{\partial U_{ij}^\varepsilon \partial U_{kl}^\varepsilon}[U^\varepsilon]. \]

The matrix $\{ F^{ij} \}$ has eigenvalues $f_1, \ldots, f_n$ and is positive definite by assumption (1.2), while (1.3) implies that $F$ is a concave function of $U_{ij}^\varepsilon$ (see [1]). Moreover, when $U^\varepsilon$ is diagonal so is $\{ F^{ij} \}$, and the following identities hold

\[ F^{ij} U_{ij}^\varepsilon = \sum f_i \lambda_i, \quad F^{ij} U_{ik}^\varepsilon U_{kj}^\varepsilon = \sum f_i \lambda_i^2, \quad \lambda[U^\varepsilon] = (\lambda_1, \ldots, \lambda_n). \]

Suppose $\gamma > 0$ and we have proved that there exists a constant independent of $\varepsilon$ such that

\begin{equation}
 |u_\varepsilon|_{C^2(\bar{\Omega})} \leq C.
\end{equation}
Therefore, by the concavity of $F$, 

$$F^{ij}[U_\varepsilon](A\delta_{ij} - U_\varepsilon^{ij}) \geq F[A] - F[U_\varepsilon] \geq c_0 > 0$$

by fixing $A$ sufficiently large. On the other hand, $-F^{ij}U_\varepsilon^{ij} \leq C \sum F^{ii}$ by (2.1). Then we get 

$$\sum F^{ii} \geq \frac{c_0}{A+C} > 0.$$ 

Note that 

$$\{ \partial F/\partial u_i^{ij}[U_\varepsilon]\} = \{F^{ij}[U_\varepsilon]\} + \gamma \sum F^{ii}I \geq \frac{\gamma c_0}{A+C} I.$$ 

Thus, there exists uniform constants $0 < \lambda_0 \leq \Lambda_0 < \infty$ such that 

$$\lambda_0 I \leq \{ \partial F/\partial u_i^{ij}[U_\varepsilon]\} \leq \Lambda_0 I.$$ 

Hence Evans-Krylov theory (see [3] and [11]) assures a bound $M$ independent of $\varepsilon$ such that 

$$|u_\varepsilon|_{C^{2,\alpha}(\bar{\Omega})} \leq M,$$ 

for some constant $\alpha \in (0,1)$. The higher regularity can be derived by the Schauder theory (see [3] for example). Using standard method of continuity, we can obtain the existence of smooth solution to (1.6). By sending $\varepsilon$ to zero (taking a subsequence if necessary), we can prove Theorem 1.1.

In the following sections, we may drop the subscript $\varepsilon$ when there is no possible confusion.

3. The gradient estimates

In this section, we consider the gradient estimates for the admissible solution to (1.6). We first observe that $\lambda[U] \in \Gamma \subset \{ \sum \lambda_i > 0 \}$ and therefore, 

$$tr[U] = (1 + n\gamma)\Delta u > 0.$$ 

Thus we have by the maximum principle that 

$$h \leq u \leq h$$ 

in $\bar{\Omega}$ where $h$ is a superharmonic function in $\Omega$ with $h = \varphi$ on $\partial \Omega$. Then we obtain 

$$\sup_{\bar{\Omega}} |u| + \sup_{\partial \Omega} |Du| \leq C,$$ 

for some positive constant $C$ independent of $\varepsilon$. By (1.5), there exists a sufficiently small constant $\varepsilon_0 > 0$ such that $\lambda[D^2u + \gamma \Delta u I - \varepsilon_0 I] \in \Gamma$ and 

$$F[D^2u + \gamma \Delta u I - \varepsilon_0 I] \geq \psi + \frac{\delta_0}{2},$$ 

for all $x \in \bar{\Omega}$. Without loss of generality, we assume $\varepsilon \leq \frac{\delta_0}{4}$. By the concavity of $F$, 

$$F^{ij}(D_{ij}u + \gamma \Delta u \delta_{ij} - \varepsilon_0 \delta_{ij} - D_{ij}u - \gamma \Delta u \delta_{ij}) \geq F[D^2u + \gamma \Delta u I - \varepsilon_0 I] - F[D^2u + \gamma \Delta u I]$$ 

$$\geq \psi(x) + \frac{\delta_0}{2} - \psi(x) - \varepsilon$$ 

$$\geq \frac{\delta_0}{4}$$
Define the linearized operator \( \mathcal{L} \) by
\[
\mathcal{L}v = F^{ij}v_{ij} + \gamma \Delta v \sum F^{ii}
\]
for \( v \in C^2(\Omega) \). We see from (3.3) that
\[
(3.4) \quad \mathcal{L}(u - u) \geq \varepsilon_0 \sum F^{ii} + \frac{\delta_0}{4}
\]
Differentiating the equation (1.4) in direction \( x_l \), we get
\[
\mathcal{L}u_l = \psi_l.
\]
Thus, from (3.4) we see that
\[
\mathcal{L}[a(u - u) \pm u_l] \geq 0 \text{ by choosing } a > 0 \text{ sufficiently large.}
\]
It follows that
\[
a(u - u) \pm u_l \text{ attains its maximum on the boundary } \partial \Omega.
\]
Hence, by (3.2), we get
\[
(3.5) \quad |u|_{C^1(\bar{\Omega})} \leq C.
\]

4. The interior second order estimate

In this section, we prove the interior second order estimate:

**Theorem 4.1.** Let \( \gamma > 0 \) and \( u \in C^4(\Omega) \) be an admissible solution of (1.6). Then for any ball \( B_r \subset \Omega \) of radius \( r > 0 \), there exists a constant \( C \) depending on \( \gamma^{-1}, r^{-1}, |u|_{C^1(B_r)}, |u|_{C^2(B_r)} \) and other known data such that
\[
(4.1) \quad \sup_{B_r} |D^2 u| \leq C.
\]
Proof. Let
\[
W(x, \xi) = \max_{x \in \Omega, |\xi| = 1} e^{\phi} \nabla^2 u
\]
where \( \phi \) is a function to be determined. Assume that \( W \) is achieved at \( x_0 \in \Omega \) and \( \xi_0 = e_1 = (1, 0, \ldots, 0) \). We may also assume that \( D^2 u \) is diagonal at \( x_0 \). We have, at \( x_0 \) where the function \( \log u_{11} + \phi \) attains its maximum,
\[
(4.2) \quad \frac{u_{11i}}{u_{11}} + \phi_i = 0
\]
and
\[
(4.3) \quad \frac{u_{11ii}}{u_{11}} - \left( \frac{u_{11i}}{u_{11}} \right)^2 + \phi_{ii} \leq 0.
\]
Differentiating equation (1.6) twice, by the concavity of \( F \), we obtain at \( x_0 \),
\[
(4.4) \quad F^{ii}u_{i11} + \gamma (\Delta u)_{11} \sum F^{ii} = \psi_{11} \geq -C.
\]
Let
\[
\phi = \frac{\delta |Du|^2}{2} + b (u - u) + \log \eta^3,
\]
where \( b, \delta \) are undetermined constants, \( 0 < \delta < 1 \leq b \) and \( \eta \in C^\infty_0(B_r) \) is a cut-off function satisfying
\[
(4.5) \quad 0 \leq \eta \leq 1, \quad \eta|_{B_r} = 1, \quad |D\eta| \leq \frac{C_r}{\sqrt{\eta}}, \quad |D^2 \eta| \leq C_r
\]
where \( C_r \) is a constant depending on \( r \) (see [8]).
By straightforward calculation, we have
\[ \phi_i = \delta u_i u_{ii} + b(u - u)_i + \frac{3\eta_i}{\eta} \]
and
\[ \phi_{ii} = \delta u_{ii}^2 + \delta u_j u_{jj} + b(u - u)_{ii} + \frac{3\eta_{ii}}{\eta} - \frac{3\eta_i^2}{\eta^2}. \]

Note that
\[ F^{ii} u_j u_{jj} + \gamma u_j \Delta u_j \sum F^{ii} = u_j \psi_j \geq -C \]
and
\[ \phi_i^2 \leq C \delta^2 u_{ii}^2 + b^2 + \frac{C}{\eta^2}. \]

We have
\[ F^{ii} \phi_{ii} + \gamma \Delta \phi \sum F^{ii} \geq \delta F^{ii} u_{ii}^2 + \gamma \delta u_{11}^2 \sum F^{ii} - C \delta \]
\[ + b F^{ii} (u - u)_i + b \gamma \Delta (u - u) \sum F^{ii} - \frac{C}{\eta^2} \sum F^{ii}. \]

Combining (4.2), (4.3), (4.4), (4.6) and (4.7), we get
\[ 0 \geq - \frac{C}{u_{11}} - C \delta + (\delta - C \delta^2) F^{ii} u_{ii}^2 + \gamma (\delta - C \delta^2) u_{11}^2 \sum F^{ii} \]
\[ - C b^2 \sum F^{ii} - \frac{C}{\eta^2} \sum F^{ii} + b \mathcal{L}(u - u). \]

Choose \( \delta \) sufficiently small such that
\[ c_1 \equiv \delta - C \delta^2 > 0. \]

Next, by (3.4), we may fix \( b \) sufficiently large such that
\[ \frac{b \delta_0}{4} - \frac{C}{u_{11}} - C \delta > 0. \]

Therefore, it follows from (4.8) that
\[ \left( \gamma c_1 u_{11}^2 - C b^2 - \frac{C}{\eta^2} \right) \sum F^{ii} \leq 0. \]

Then we obtain
\[ u_{11}^2(x_0) \leq \frac{C}{\gamma c_1} \left( b^2 + \frac{1}{\eta^4} \right) \]
and (4.1) is valid.

\[ \square \]

5. Estimates for second order derivatives

In this section, we derive a bound independent of \( \varepsilon \) for second order derivatives of \( u_\varepsilon \). For any unit vector \( \xi \in \mathbb{R}^n \), differentiating the equation (1.4) in the direction \( \xi \), we get
\[ F^{ij} U_{ij \xi \xi} + F^{ij, kl} U_{ij \xi} U_{kl \xi} = \psi_{\xi \xi} \geq -C. \]

It follows that, by the concavity of \( F \),
\[ \mathcal{L} u_{\xi \xi} \geq -C. \]
Hence \( L[a(u - u) + u\xi] \geq 0 \) by choosing \( a \) sufficiently large. Then the maximum principle shows that
\[
\sup_{\Omega} u\xi \leq C + \sup_{\partial\Omega} u\xi.
\]
Therefore we reduce the estimate to the boundary.

It is easy to obtain a bound independent of \( \varepsilon \) for pure tangential second order derivatives on the boundary
\[
|u\xi|_{C^0(\partial\Omega)} \leq C
\]
from the boundary condition in (1.1), where \( \xi \) and \( \eta \) are unit tangential vectors on \( \partial\Omega \). The estimates for the mixed second order derivatives on the boundary
\[
|u\xi\eta|_{C^0(\partial\Omega)} \leq C
\]
can be derived by constructing a sub-barriers using \( u - u \), where \( \xi \) is any unit tangential vector on \( \partial\Omega \) and \( \nu \) is the unit inner normal of \( \partial\Omega \). See e.g. [5], [6]. It suffices to establish an upper bound for the double normal derivative on the boundary \( \partial\Omega \).

As in [10], let \( T = \{ T^j_i \} \) be a skew-symmetric matrix, such that \( e^T \) is orthogonal, where \( T^j_i \) is the entry of \( i^th \) row and \( j^th \) column of \( T \). Let \( \tau = (\tau_1, \ldots, \tau_n) \) be a vector field in \( \Omega \) given by
\[
\tau_i = T^j_i x_j, \quad i = 1, \ldots, n.
\]
Denote \( u_{\tau\tau} = \tau_\tau u_{ij} \) and \( u_{(\tau)(\tau)} = (u_{\tau})_\tau = \tau_\tau u_{ij} + (\tau_i)j \tau_j u_i \). Similar to Lemma 2.1 of [10] we can prove the following lemma.

**Lemma 5.1.** We have
\[
F^{ij}(u_{(\tau)(\tau)})_{ij} + \gamma \sum F^{ii} \Delta(u_{(\tau)(\tau)}) \geq (F[U])_{(\tau)(\tau)}.
\]

**Proof.** Similar to Lemma 2.1 of [10], by the skew-symmetry of \( T \), we have
\[
F^{ij}(T^k_i u_{kj} + T^k_j u_{ik}) = -F^{ij, st}(T^s_k u_{kj} + T^s_j u_{ik}) U_{st} \tau
\]
and
\[
F^{ij}(2T^k_i T^j_k u_{kl} + T^k_i T^j_k u_{l}) = -F^{ij, st}(T^s_k u_{kj} + T^s_j u_{ki}) (T^s_k u_{kl} + T^s_k u_{ks}).
\]
Note that
\[
(u_{(\tau)(\tau)})_{ij} = u_{ij(\tau)(\tau)} - 2T^k_i u_{kj} - 2T^k_j u_{ik} + 2T^s_i T^j_i u_{st} + T^s_j T^s_i u_{si} + T^s_i T^s_j u_{sij}.
\]
We find
\[
F^{ij}(u_{(\tau)(\tau)})_{ij} + \gamma \sum F^{ii} \Delta(u_{(\tau)(\tau)})
= F^{ij, u_{ij(\tau)(\tau)}} + \gamma \sum F^{ii} (\Delta u_{(\tau)(\tau)})
+ F^{ij}(2T^s_i T^j_i u_{st} + T^s_i T^s_j u_{si} + T^s_j T^s_i u_{sij} - 2T^k_i u_{kj} - 2T^k_j u_{ik})
+ \gamma \sum F^{ii} (2T^s_i T^j_i u_{st} + 2T^s_i T^s_j u_{si} - 2T^k_i u_{k\tau i})
\]
Next, since \( T \) is skew-symmetric,
\[
2T^s_i T^j_i u_{st} + 2T^s_i T^s_j u_{si} - 4T^k_i u_{k\tau i} = 0.
\]
We have, by (5.3), (5.6) and (5.7),
\begin{equation}
F^{ij}(u_{(\tau)}(\tau))_{ij} + \gamma \sum F^{ii}(u_{(\tau)}(\tau)) = F^{ij}u_{ij(\tau)}(\tau) + \gamma \sum F^{ii}(\Delta u)_{(\tau)}(\tau) \\
- F^{ij, st}(T^k u_{kj} + T^k u_{kt})(T^k u_{kt} + T^k u_{ks}) \\
+ F^{ij, st}(T^k u_{kj} + T^k u_{kt})U_{st\tau} + (T^k u_{kt} + T^k u_{ks})U_{ij\tau}.
\end{equation}

Note that
\begin{equation}
(u_\tau)_{ij} = u_{ij\tau} - T^k u_{kj} - T^k u_{ki}.
\end{equation}

We have
\begin{equation}
F^{ij}(u_{(\tau)}(\tau))_{ij} + \gamma \sum F^{ii}(u_{(\tau)}(\tau)) = (F[U])_{(\tau)}(\tau) - F^{ij, st}(\gamma \delta_{ij}(\Delta u)_\tau + (u_\tau)_{ij})(\gamma \delta_{st}(\Delta u)_\tau + (u_\tau)_{st}) \\
\geq (F[U])_{(\tau)}(\tau).
\end{equation}

\[\square\]

To prove the double normal derivative estimate, we assume the origin is a boundary point such that \(e_n = (0, \ldots, 0, 1)\) is the unit inner normal there. Suppose near the origin, the boundary \(\partial \Omega\) is represented by
\begin{equation}
x_n = \rho(x') = \frac{1}{2} \sum_{\alpha, \beta < n} B_{\alpha\beta} x_\alpha x_\beta + O(|x'|^3)
\end{equation}
for some \(C^\infty\) smooth function \(\rho\), where \(x' = (x_1, \ldots, x_{n-1})\). For any point \(x \in \partial \Omega\), let
\[\tau = \tau(x) = \partial_n + \sum_{\beta < n} B_{\alpha\beta}(x_\beta \partial_n - x_n \partial_\beta), \quad \alpha < n\]
and denote
\[M = \sup_{x \in \partial \Omega} D_{\nu\nu} u(x),\]
where \(\nu\) is the unit inner normal of \(\partial \Omega\) at \(x \in \partial \Omega\). Without loss of generality, we may assume
\[M = \sup_{\partial \Omega} |D^2 u|,
\]
and
\begin{equation}
\sup_{\Omega} |D^2 u| \leq CM,
\end{equation}
for some uniform constant \(C \geq 1\). By Lemma 5.1 we have
\begin{equation}
\mathcal{L}(u_{(\tau)}(\tau)) \geq (F[U])_{(\tau)}(\tau) = \psi_{(\tau)}(\tau) \geq -C.
\end{equation}

According to [10],
\begin{equation}
w(x) \equiv u_{(\tau)}(\tau)(x) - u_{(\tau)}(\tau)(0) \leq C_0(|x'|^2 + M|x'|^4) \equiv h(x')
\end{equation}
for \(x \in \partial \Omega\) with \(|x'| \leq r_0\).

Denote \(\Omega_\delta \equiv \{ x \in \Omega : |x| < \delta \}\). By taking \(A_1 \gg A_2 \gg 1\) and fixing \(\delta > 0\) small enough, it follows from (5.10) and (5.4) that
\begin{equation}
\mathcal{L} \left( w + \frac{A_1}{\delta^2} M(u - u) - \frac{A_2}{\delta^4} M|x|^4 - h(x') \right) \geq 0 \quad \text{in} \ \Omega_\delta
\end{equation}
And for fixed $\delta$, we see from (5.11) and (5.9) that
\[
 w + \frac{A_1}{\delta^2} M(u - u) - \frac{A_2}{\delta^4} M|x|^4 - h(x') \leq 0 \quad \text{on } \partial \Omega_\delta.
\]
Thus the maximum principle yields
\[
(5.12) \quad w + \frac{A_1}{\delta^2} M(u - u) - \frac{A_2}{\delta^4} M|x|^4 - h(x') \leq 0 \quad \text{on } \Omega_\delta.
\]
Then for each small $\sigma > 0$, we can find a positive constant $\delta_1 = C\sigma < \delta_1$ such that
\[
(5.13) \quad w \leq CM(u - u) + \frac{\sigma}{2} M + C \quad \text{on } \Omega_{\delta_1}
\]
and
\[
(5.14) \quad Lh \leq (\sqrt{\sigma} M + C) \sum F^{ij} \quad \text{on } \Omega_{\delta_1}.
\]
Next, there exists a positive constant $\delta_2 < \delta_1$ such that
\[
C|u - u| \leq \sigma \quad \text{on } \Omega - \Omega_{\delta_2},
\]
where $\Omega_{\delta_2} \equiv \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta_2 \}$, since $|D(u - u)| \leq C$ independent of $\varepsilon$. Hence we obtain by (5.13)
\[
(5.15) \quad w \leq \sigma M + C \quad \text{on } \Omega_{\delta_1} \cap (\Omega - \Omega_{\delta_2}).
\]
On the other hand, by (4.1), there exists a positive constant $C$ depending on $\delta_2$ and $\text{diam}(\Omega)$ such that
\[
(5.16) \quad |w| \leq C \quad \text{in } \Omega_{\delta_2}.
\]
It follows from (5.10), (3.4) and (5.14) that there exists constants $A'_1 \gg A'_2 \gg 1$ such that
\[
L \left( w + A'_1 \left( \frac{\sigma M + C_\sigma}{\delta_1^2} + \sqrt{\sigma} M + C \right)(u - u) - \frac{A'_2}{\delta_1^4} (\sigma M + C_\sigma)|x|^2 - h(x') \right) \geq 0
\]
in $\Omega_{\delta_1}$ and
\[
\left( w + A'_1 \left( \frac{\sigma M + C_\sigma}{\delta_1^2} + \sqrt{\sigma} M + C \right)(u - u) - \frac{A'_2}{\delta_1^4} (\sigma M + C_\sigma)|x|^2 - h(x') \right) \leq 0
\]
on $\partial \Omega_{\delta_1}$. Thus, by the maximum principle again, we have
\[
(5.17) \quad (u_{(\tau)}(x))_n(0) \leq C\sqrt{\sigma} M + C_\sigma.
\]
For any tangential unit vector field $\xi$ on $\partial \Omega$ near the origin, we see that at 0,
\[
u_{n(\xi)} = u_{n(\tau)(\tau)} = (u_{(\tau)}(x))_n - (\tau_i(\tau)_j)_{n} u_{ij} - (\tau_i \tau_j)_{n} u_{ij}.
\]
Then we have
\[
u_{n(\xi)} \leq C\sqrt{\sigma} M + C_\sigma \quad \text{on } \partial \Omega
\]
for any tangential unit vector field $\xi$ on $\partial \Omega$.

Now choose a new coordinate system and suppose the maximum $M$ is attained at the origin $0 \in \partial \Omega$, and near the origin $\partial \Omega$ is given by (5.8). By the Taylor expansion, we have
\[
u(x) \leq \nu_n(0) + \sum_{\alpha < n} u_{n\alpha}(0)|x_\alpha + (C\sqrt{\sigma} M + C_\sigma)|x'|^2
\]
for \( x \in \partial \Omega \) near the origin, where \( u_{nn}(0) \) is bounded by (5.4). Denote

\[
g \equiv u_n(x) - u_n(0) - \sum_{\alpha<n} u_{n\alpha}(0)x_{\alpha} - (C\sqrt{\sigma}M + C_\sigma)|x'|^2.
\]

We may choose positive constants \( A''_1 \gg A''_2 \gg 1 \) and \( \delta \) sufficiently small such that

\[
L\left(g + A''_1(\sqrt{\sigma}M + C_\sigma)(u - u) - A''_2(\sqrt{\sigma}M + C_\sigma)|x|^2\right) \geq 0 \text{ in } \Omega_\delta
\]

and

\[
g + A''_1(\sqrt{\sigma}M + C_\sigma)(u - u) - A''_2(\sqrt{\sigma}M + C_\sigma)|x|^2 \leq 0 \text{ on } \partial \Omega_\delta.
\]

Applying the maximum principle again we obtain

\[
M = u_{nn}(0) \leq C\sqrt{\sigma}M + C_\sigma.
\]

Choosing \( \sqrt{\sigma} < 1/2C \), we get a bound \( M \leq C \) and (2.1) is proved.

References

[1] L. A. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations III: Functions of eigenvalues of the Hessians, Acta Math. 155 (1985), 261-301.

[2] M. Crandall, H. Ishii and P. Lions, User’s guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1992) 1-67.

[3] L. C. Evans, Classical solutions of fully nonlinear, convex, second order elliptic equations, Comm. Pure Applied Math. 35 (1982), 333-363.

[4] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, 2nd Edition; Springer-Verlag: Berlin, 1983.

[5] B. Guan, Conformal metrics with prescribed curvature function on manifolds with boundary, Amer. J. Math. 129 (2007), 915-942.

[6] B. Guan, Complete conformal metrics of negative Ricci curvature on compact manifolds with boundary, Int. Math. Res. Not. 2008, rnn105, 25pp. Addendum, IMRN 2009, 4354-4355, rnp166.

[7] B. Guan, The Dirichlet problem for fully nonlinear elliptic equations on Riemannian manifolds, arXiv:1403.2133

[8] P.-F. Guan and G.-F. Wang, Local estimates for a class of fully nonlinear equations arising from conformal geometry, Int. Math. Res. Not. 2003, no.26, 1413-1432.

[9] P.-F. Guan, N. S. Trudinger and X.-J. Wang, Dirichlet problems for degenerate Monge-Ampère equations, Acta. Math. 182 (1999), 87-104.

[10] N. M. Ivochkina, N. S. Trudinger and X.-J. Wang The Dirichlet problem for degenerate Hessian equations, Comm. Partial Diff. Eqns. 29 (2004), 219-235.

[11] N. V. Krylov, Boundedly nonhomogeneous elliptic and parabolic equations in a domain, Izvestia Math. Ser. 47 (1983), 75-108.

[12] N. V. Krylov, Barriers for derivatives of solutions of nonlinear elliptic equations on a surface in Euclidean space, Comm. Partial Diff. Eqns. 19 (1984), 1909-1944.

[13] N. V. Krylov, Weak interior second order derivative estimates for degenerate nonlinear elliptic equations, Diff. Int. Eqns 7 (1994), 133-156.

[14] N. V. Krylov, A theorem on the degenerate elliptic Bellman equations in bounded domains, Diff. Int. Eqns. 8 (1995), 961-980.

[15] N. V. Krylov, On the general notion of fully nonlinear second-order elliptic equations, Trans. Amer. Math. Soc. 347 (1995), 857-895.

[16] N.S. Trudinger, The Dirichlet problem for the prescribed curvature equations, Arch. Rational Mech. Anal. 111 (1990) 153-179.

[17] N. S. Trudinger, On the Dirichlet problem for Hessian equations, Acta Math. 175 (1995), 151-164.

[18] N. S. Trudinger, Weak solutions of Hessian equations, Comm. Partial Diff. Eqns 22 (1997), 1251-1261.

[19] N. S. Trudinger and X.-J. Wang, Hessian measures. II., Ann. of Math. 150 (1999), 579604.
[20] X.-J. Wang, *The k-Hessian Equation*, Lecture Notes, Australian National University, 2007.

Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, China
E-mail address: jiaoheming@163.com

Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, China
E-mail address: twanghit@gmail.com