Universal Property of Convolutional Neural Networks

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Abstract

Universal approximation, whether a set of functions can approximate an arbitrary function in a specific function space, has been actively studied in recent years owing to the significant development of neural networks. However, despite its extensive use, research on the universal properties of the convolutional neural network has been limited due to its complex nature. In this regard, we demonstrate the universal approximation theorem for convolutional neural networks. A convolution with padding outputs the data of the same shape as the input data; therefore, it is necessary to prove whether a convolutional neural network composed of convolutions can approximate such a function. We have shown that convolutional neural networks can approximate continuous functions whose input and output values have the same shape. In addition, the minimum depth of the neural network required for approximation was presented, and we proved that it is the optimal value. We also verified that convolutional neural networks with sufficiently deep layers have universality when the number of channels is limited.

**Keywords:** Convolutional Neural Network, Universal Approximation, Deep Narrow Network

1. Introduction

The convolutional neural network (CNN) (O’Shea and Nash, 2015; LeCun et al., 1998), one of the most widely used deep learning modules, has achieved tremendous accomplishment in numerous fields, including object detection (Zaidi et al., 2022), image classification (Elngar et al., 2021), and sound processing (Tan et al., 2021). Starting with the most basic architecture like LeNet5 (LeCun et al., 1998), many well-known deep learning models such as VGGNet (Simonyan and Zisserman, 2014), ResNet (He et al., 2016), and ResNeXt (Xie et al., 2017) have been constructed based on CNN. In this regard, it would be natural to be interested in the universal property of CNN, which justifies using a specific network.

Universal property or universal approximation is the ability of a particular set of functions to approximate the sufficiently wide range of the functions. However, despite its extensive range of applications, research on the universal property of CNN has been barely conducted. One of the rare studies is Zhou (2020). The paper considered the convolutional neural network with a linear layer combined in the last layer and proved the universal property of the network as the function from $\mathbb{R}^d$ to $\mathbb{R}$. However, networks sometimes are expected
to retain the output data in the same shape as the input data. Representative examples include object segmentation (Long et al., 2015), depth estimation (Bhoi, 2019), or image processing such as deblurring (Zhang et al., 2022), inpainting (Suthar and Patel, 2014), and denoising (Fan et al., 2019). Another common usage of CNN is as a feature extractor. The feature extractor extracts information from the data and feeds it to the latter part of the deep learning model. Typically, the feature extracted by CNN is multi-dimensional, and to achieve the purpose of being a module that can be used in common across multiple networks, CNN needs to have universal property. Also, the paper assumed an unrealistic situation in which each convolutional layer expands the dimension of the data, which makes the contribution restrictive.

Some other research papers tackle the universal property of CNN with multi-dimensional output as a translation invariant function. Approaches that tackle the universal property of CNN with multi-dimensional output are investigating the approximation of the translation invariant function with convolutional neural networks (Yarotsky, 2022; Maron et al., 2019). These papers consider the convolutional network as a function from $\mathbb{R}^d$ to $\mathbb{R}^d$. However, the invariance of the network inevitably prevents the use of practically used padding methods like zero padding. In addition, invariance fundamentally contradicts the universal property in the more general continuous function space.

In this regard, we studied the universal property of the convolutional neural network consisting of the convolutional layer with zero padding. Unlike the previous methods that only consider scalar output or the translationally invariant functions, We directly tackle the universal property of CNN as a vector-to-vector function. Despite its dominant use in CNN, zero padding convolution has been outside the interests of the study because it deteriorates the invariance of the network. However, we revealed that zero padding is critical in achieving universal property. More specifically, universal property occurs because zero padding interferes with invariance. We scrutinize the three-kernel convolutional neural network with zero padding and explore the minimal depth and width bound for the universal property. Our contributions are as follows:

- We proved that CNN has the universal property in the continuous function space as a function that preserves the shape of the input data.
- We found the optimal number of convolutional layers for a function with $d$-dimensional input to have universal property.
- We proved that deep CNN with $c_x + c_y + 2$ has the universal property, where $c_x$ and $c_y$ are the number of channels of the input and output data, respectively.

2. Related Works

We will briefly arrange the list of studies investigating the universal approximation theorem of neural networks. Early studies investigated whether two-layered MLPs have universal properties; if they can approximate any continuous function. Cybenko (1989) provided that the two-layered MLP is universal when the activation function is sigmoidal. Several other studies (Hornik et al., 1989; Hornik, 1991) tried to clarify the condition of the activation function. Among them, Leshno et al. (1993) provided a simple and powerful result. They
proved that the necessary and sufficient condition for two-layered MLP to be universal is that the activation function is non-polynomial.

The universal property of deep narrow MLP is another interesting area of research that is widely investigated. Lu et al. (2017) tackled the situation when the number of nodes of MLP is bounded. Hanin and Sellke (2017) expanded the result to the case that the output is a multidimensional vector. Johnson (2019); Kidger and Lyons (2020) expand the result to the more general activation function, confining the lower and upper bound, respectively. Unlike previous research focusing on the continuous function space, Park et al. (2020) investigated the universal approximation in \( L_p \) space.

As the kind of neural network and its range function space of interest increases, the need for the universal approximation theorem adapted to these various structures is raised. Schäfer and Zimmermann (2007); Hanson and Raginsky (2020) studied the universal property of the recurrent neural network. They proved that recurrent neural networks could approximate arbitrary open dynamical systems. Yun et al. (2020) demonstrated the universality of the transformer network. On the other hand, research on the convolutional neural network as a general-purpose function is barely conducted despite its widespread use. One of the research Zhou (2020) studied the universal property of the convolutional neural network as a function from the vector to the scalar value. It tackles the network with a fully connected layer added to the last layer to make the network’s output a scalar. Also, to employ the homomorphism between the composition of convolutional layers and the multiplication of polynomials, the paper assumed the impractical situation that data becomes longer as the data go through the network. On the other hand, we proved the case for a fully convolutional network that retains the shape of the input data to the output data. The authors of Yarotsky (2022) focused on the periodic convolutional network’s universal property as the translation equivariant function. However, the translation equivariance fundamentally contradicts the universal property as the general function from \( d \)-dimensional input data to the \( d \)-dimensional output data. Because the translation equivariance of the convolutional neural network is derived from cyclic padding, we need different padding, such as zero padding.

3. Notion and Definition

When we index the data in \( \mathbb{R}^d \) or \( \mathbb{R}^Z \), we will use the subscript for indexing. For example, we express the components of \( x \in \mathbb{R}^d \) as

\[
x = (x_1, x_2, \ldots, x_d).
\]

(1)

When we index the unique dimension called channel, we will use the superscript for indexing, that is, for \( x \in \mathbb{R}^{c \times d} \),

\[
x = (x^1, x^2, \ldots, x^c),
\]

(2)

where \( x^i \in \mathbb{R}^d \) for \( i \in [1, c] \), and

\[
x^i = (x^i_1, x^i_2, \ldots, x^i_d).
\]

(3)

The channel always comes first compared to other dimensions and is denoted as \( c \) or its variant. We also define the concatenation operation \( \oplus \) along the channel as follows. For
We now define the mathematical contents used in the remaining sections.

- **Infinite-Length Convolution**: Let \( w \) be \( w = (w_{-k}, w_{-k+1}, \ldots, w_k) \in \mathbb{R}^{2k+1} \). Then an infinite-length convolution with kernel \( w \) is a map \( f : \mathbb{R}^Z \to \mathbb{R}^Z \) defined as follows. For \( x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in \mathbb{R}^Z \),

\[
  f_i(x) := \sum_{j=-k}^{k} w_j x_{i+j},
\]

where \( f(x) = (\ldots, f_{-1}(x), f_0(x), f_1(x), \ldots) \in \mathbb{R}^Z \). We say that a convolution has a kernel size of \( 2k+1 \).

- **Zero Padding Convolution**: Let \( \iota : \mathbb{R}^d \to \mathbb{R}^Z \) be a natural inclusion map. Formally, for \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \),

\[
  \iota_{i}(x) := \begin{cases} x_i & \text{if } 1 \leq i \leq d \\ 0 & \text{otherwise} \end{cases},
\]

where \( \iota(x) = (\ldots, \iota_{-1}(x), \iota_0(x), \iota_1(x), \ldots) \in \mathbb{R}^Z \). And let \( p_d : \mathbb{R}^Z \to \mathbb{R}^d \) be a projection map; that is, for \( x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in \mathbb{R}^Z \), \( p_d(x) \) is defined as

\[
  p_d(x) := (x_1, x_2, \ldots, x_d).
\]

Let \( w \in \mathbb{R}^{2k+1} \) be a kernel. Then zero padding convolution with kernel \( w \) is a function \( f : \mathbb{R}^d \to \mathbb{R}^d \) is defined as

\[
  f := p_d \circ g \circ \iota,
\]

where \( g \) is an infinite-length convolution with kernel \( w \). We also define it as operation \( \odot \):

\[
  w \odot x := f(x).
\]

We can interpret the composition as constructing a temporary infinite-length sequence by filling zeros in the remaining components, conducting the convolution with kernel, and cutting off the unnecessary elements.

A zero padding convolution with kernel \( w \) is a linear transformation and hence can be expressed as matrix multiplication; \( w \odot x = Tx \) is satisfied for the following matrix \( T \in \mathbb{R}^{d \times d} \):

\[
  T = \begin{bmatrix}
    w_0 & w_1 & \ldots & w_k \\
    w_{-1} & w_0 & \ldots & w_{k-1} & w_k \\
    \vdots & \ddots & \ddots & \ddots & \ddots \\
    w_{-k} & w_{-k+1} & \ldots & w_0 & w_1 & \ldots & w_{k-1} & w_k \\
    w_{-k} & w_{-1} & w_2 & \ldots & w_{k-2} & w_{k-1} & w_k \\
    \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    w_{-k} & w_{-k+1} & \ldots & w_0
  \end{bmatrix}.
\]
We define the set of Toeplitz matrix as
\[
To_d(s) := \left\{ (x_{i,j})_{1 \leq i, j \leq d} \in \mathbb{R}^{d \times d} \middle| x_{i,j} = \begin{cases} w_{j-i} & \text{if } |i - j| \leq s \\ 0 & \text{else} \end{cases}, w_k \in \mathbb{R}, k \in [-s, s] \right\}.
\]

(11)

Also, define \( U_t = (u_{i,j})_{1 \leq i, j \leq d} \) as
\[
u_{i,j} = \begin{cases} 1 & \text{if } i - j = t \\ 0 & \text{otherwise} \end{cases}.
\]

(12)

By definition, \( U_0 \) is an identity matrix, and \( U_t \) and \( U_{-t} \) have a transpose relationship with each other; \( U_t^T = U_{-t} \). The set \( \{ U_{-s}, U_{-s+1}, \ldots, U_s \} \) is the basis of the set of Toeplitz matrices \( To_d(s) \). Zero padding convolution with kernel \( w = (w_{-s}, w_{-s+1}, \ldots, w_s) \) can be represented as
\[
w \circ x = \sum_{i=-s}^{s} w_i U_{-i}.
\]

(13)

Obviously, \((U_1)^t = U_t\), and \((U_{-1})^t = U_{-t}\) for \( t \geq 0 \). Also, it is convenient to interpret the matrix multiplication in the following way. Let \( A \) be a matrix or a column vector. Then, \( U_t A \) and \( U_{-t} A \) move \( A \) downward \( t \) rows and upward \( t \) rows, respectively. Similarly, \( A U_t \) and \( A U_{-t} \) move \( A \) to the left by \( t \) columns and right by \( t \) columns, respectively. We also define \( E_{n,m} := (e_{i,j})_{1 \leq i, j \leq d} \) as
\[
e_{i,j} = \begin{cases} 1 & \text{if } i = n \text{ and } j = m \\ 0 & \text{otherwise} \end{cases}.
\]

(14)

To deal with the composition of convolutions, we define \( S_N \) as follows.
\[
S_N := \left\{ \sum_{i=1}^{n} \prod_{j=1}^{N} T_{i,j} \middle| T_{i,j} \in To_d(1), n \in \mathbb{N} \right\}.
\]

(15)

\( S_N \) is a vector space of matrix representations of linear transformations that a linear three-kernel \( N \)-layered CNN can express.

- **Zero Padding Convolutional Layer:** A convolutional layer with \( c_1 \) input channels and \( c_2 \) output channels is a map \( f : \mathbb{R}^{c_1 \times d} \rightarrow \mathbb{R}^{c_2 \times d} \). For each \( 1 \leq i \leq c_2 \) and \( 1 \leq j \leq c_1 \), there exist zero padding convolutions with kernel \( w_{i,j} \in \mathbb{R}^{2k+1} \) and bias \( \delta_i \in \mathbb{R} \) so that for \( x = (x^1, x^2, \ldots, x^{c_1}) \in \mathbb{R}^{c_1 \times d} \),
\[
f^i(x) := \sum_{j=1}^{c_1} w_{i,j} \circ x^j + \delta_{i} 1_d,
\]

(16)

where \( f(x) = (f^1(x), f^2(x), \ldots, f^{c_2}(x)) \). We extend the operation \( \circ \) to the multiplication between the vector-valued matrix. Let \( M_{n,m}(\mathbb{R}^d) \) be the \( n \times m \) matrix whose components are \( d \)-dimensional vectors in \( \mathbb{R}^d \). Then for \( A = (a_{i,j})_{1 \leq i,n \leq j,m} \in \mathbb{R}^{n,m}(\mathbb{R}^d) \)

\[
A \circ f(x) := \sum_{i=1}^{n} \prod_{j=1}^{m} A_{i,j} \circ f^j(x),
\]

(17)
$M_{n,m}(\mathbb{R}^{2k+1})$ and $B = (b_{j,k})_{1 \leq j \leq m,1 \leq k \leq l} \in M_{m,l}(\mathbb{R}^{d})$, we denote matrix multiplication \( \otimes \) between $A$ and $B$ as

$$C := A \otimes B,$$

where $C = (c_{i,k})_{1 \leq i \leq n,1 \leq k \leq l} \in M_{n,l}(\mathbb{R}^{d})$, and $c_{i,k}$ is calculated as

$$c_{i,k} := \sum_{j=1}^{m} a_{i,j} \otimes b_{j,k}.$$  

Zero padding convolutional layer can be interpreted as a matrix multiplication between weight matrix $W = (w_{i,j})_{1 \leq i \leq c_{2,1} \leq j \leq c_{1}} \in M_{c_{2,1}}(\mathbb{R}^{d})$ and input vector $X = (x^{j})_{1 \leq j \leq c_{1}} \in M_{c_{1,1}}(\mathbb{R}^{d})$ and bias summation.

$$
\begin{bmatrix}
  f^{1} \\
  f^{2} \\
  f^{c_{2}} \\
\end{bmatrix} =
\begin{bmatrix}
  w_{1,1} & w_{1,2} & \cdots & w_{1,c_{1}} \\
  w_{2,1} & w_{2,2} & \cdots & w_{2,c_{1}} \\
  \vdots & \vdots & \ddots & \vdots \\
  w_{c_{2},1} & w_{c_{2},2} & \cdots & w_{c_{2},c_{1}} \\
\end{bmatrix}
\oplus
\begin{bmatrix}
  x^{1} \\
  x^{2} \\
  \vdots \\
  x^{c_{1}} \\
\end{bmatrix} +
\begin{bmatrix}
  \delta^{1}_{1} \\
  \delta^{2}_{1} \\
  \vdots \\
  \delta^{c_{1}}_{1} \\
\end{bmatrix}.
\tag{19}
$$

**Activation Function:** An activation function $\sigma$ is a scalar function $\sigma : \mathbb{R} \to \mathbb{R}$. We extend the function component-wise to the multivariate versions $\sigma_{d} : \mathbb{R}^{d} \times \mathbb{R}^{d}$ and $\sigma_{c,d} : \mathbb{R}^{c \times d} \times \mathbb{R}^{c \times d}$. Specifically, for $x \in \mathbb{R}^{d}$,

$$\sigma_{d}(x) := (\sigma(x_{1}), \sigma(x_{2}), \ldots, \sigma(x_{d})), \tag{20}$$

where $x = (x_{1}, x_{2}, \ldots, x_{d})$. And for $x = (x^{1}, x^{2}, \ldots, x^{c}) \in \mathbb{R}^{c \times d}$ and $x^{i} = (x^{i}_{1}, x^{i}_{2}, \ldots, x^{i}_{d}) \in \mathbb{R}^{d}$,

$$\sigma_{c,d}(x)^{j}_{i} = \sigma(x^{i}_{j}) \text{ for } 1 \leq i \leq c, 1 \leq j \leq d. \tag{21}$$

We will slightly abuse notation so that $\sigma$ means $\sigma$, $\sigma_{d}$, and $\sigma_{c,d}$, depending on the context.

We also define a modified version of activation function that selectively applies an activation function to each channel by modifying the activation function as follows. For $I \subset [1,c]$, define $\tilde{\sigma}_{I} : \mathbb{R}^{c \times d} \to \mathbb{R}^{c \times d}$ as follows. If $x = (x^{1}, x^{2}, \ldots, x^{c})$ and $x^{i} \in \mathbb{R}^{d}$,

$$\tilde{\sigma}_{I}^{j}(x) = \begin{cases} 
\sigma(x^{i}) & \text{if } i \in I \\
x^{i} & \text{otherwise}
\end{cases}, \tag{22}$$

where $\tilde{\sigma}_{I} = (\tilde{\sigma}_{I}^{1}, \tilde{\sigma}_{I}^{2}, \ldots, \tilde{\sigma}_{I}^{c})$.

**Convolutional Neural Network:** An $N$-layered convolutional neural network with $c$ input channels and $c'$ output channels is a map $f : \mathbb{R}^{c_{0} \times d} \to \mathbb{R}^{c_{N} \times d}$ that is constructed by following $N$ convolutional layers and the activation function. For the channel sizes $c_{0} = c_{1}, \ldots, c_{N} = c'$, there exist convolutional layers $C_{i} : \mathbb{R}^{c_{i-1} \times d} \to \mathbb{R}^{c_{i} \times d}$, and $f$ is defined as follows.

$$f := C_{N} \circ \sigma \circ C_{N-1} \circ \cdots \circ \sigma \circ C_{1}. \tag{23}$$
We denote the channel sizes of the convolutional layer as \( c_0 - c_1 - \cdots - c_n \). Then, we define \( \Sigma^N_{c,c'} \) as the set of the convolutional neural networks with \( c \) input channels and \( c' \) output channels:

\[
\Sigma^N_{c,c'} := \left\{ C_N \circ \sigma \circ C_{N-1} \circ \cdots \circ \sigma \circ C_1 : \mathbb{R}^{c \times d} \to \mathbb{R}^{c' \times d} \right\}
\]

For some \( c_1, c_2, \ldots, c_{N-1} \in \mathbb{N} \), \( C_i : \mathbb{R}^{c_{i-1} \times d} \to \mathbb{R}^{c_i \times d} \), where \( c = c_0, c' = c_N \). (24)

If we need to indicate the activation function explicitly, we denoted \( \Sigma^N_{c,c'} \) as \( \sigma\Sigma^N_{c,c'} \). Also, define \( \sigma(\Sigma^N_{c,c'}) \) as

\[
\sigma(\Sigma^N_{c,c'}) := \left\{ \sum_{i=1}^{n} a_i(\sigma \circ f_i) : \mathbb{R}^{c \times d} \to \mathbb{R}^{c' \times d} \mid f_i \in \Sigma^N_{c,c'}, a_i \in \mathbb{R}, n \in \mathbb{N}_0 \right\}.
\]

4. Main Theorem

4.1 Problem Formulation

The universal property of CNN, which we will discuss in this paper, is whether a continuous function from \( \mathbb{R}^{c \times d} \) to \( \mathbb{R}^{c' \times d} \) can be uniformly approximated by convolutional neural networks. Let \( C(X,Y) \) be a space of continuous function from \( X \) to \( Y \). Then we define the norm in \( C(\mathbb{R}^{c \times d}, \mathbb{R}^{c' \times d}) \) for each compact subset \( K \subset \mathbb{R}^{c \times d} \) as follows:

\[
||f - g||_{C^\infty(K)} = \sup_{x \in K} ||f(x) - g(x)||_2.
\]

(26)

What we want to show in Section 4.3 is under what condition, the closure with respect to \( C^\infty(K) \) norm satisfy the following statement,

\[
\overline{\Sigma^N_{c,c'}} = C(K, \mathbb{R}^{c' \times d}).
\]

(27)

And in Section 4.4, we will show that convolutional neural networks with bounded width are also dense in \( C(K, \mathbb{R}^{c' \times d}) \) with respect to \( C^\infty(K) \) norm.

4.2 Lemmas

Now we present proofs for theorems. Before we get into the main theorems, we first prove the lemma that will be used for proofs.

**Lemma 1** The following statements are satisfied.

1. \( \Sigma^N_{c,c'} \) is closed under concatenation. In other words, for \( f_1 \in \Sigma^N_{c,c'} \) and \( f_2 \in \Sigma^N_{c,c''} \), the function \( f \) is defined as \( f(x) := f_1(x) \oplus f_2(x) \in \mathbb{R}^{(c+c'') \times d} \). Then, \( f \in \Sigma^N_{c,c''+c''} \).

2. \( \Sigma^N_{c,c'} \) and \( \sigma(\Sigma^N_{c,c'}) \) are vector spaces.

3. For a \( C^\infty \) activation function \( \sigma \), \( \sigma\Sigma^N_{c,c'} \) is closed under partial differentiation; for \( C^\infty \) function \( f(x,\theta) \) and \( f_\theta(x) := f(x,\theta) \), if \( f_\theta(x) \in \Sigma^N_{c,c'} \), then, \( \frac{\partial}{\partial \theta}(f_\theta) \in \Sigma^N_{c,c'} \). Also, \( \sigma(\Sigma^N_{c,c'}) \) is closed under partial differentiation.
4. For \( f \in \sigma^{N}_{c,c'} \) and a convolutional layer \( C \) with \( c' \) input channels and \( c'' \) output channels, \( C \circ f \in \Sigma^{N+1}_{c,c''} \).

Proof

1. Let \( f_1 \) with channel sizes \( c - c_1 - c_2 - \cdots - c_{N-1} - c' \) be
   \[
   f_1 := C_N \circ \sigma \circ C_{N-1} \circ \cdots \circ \sigma \circ C_1,
   \]
   and \( f_2 \) with channel sizes \( c - c'_1 - c'_2 - \cdots - c'_{N-1} - c'' \) be
   \[
   f_2 := C'_N \circ \sigma \circ C'_{N-1} \circ \cdots \circ \sigma \circ C'_1.
   \]
   As in Equation (19), we can express \( C \) as
   \[
   C_i(x) = W_i \sigma x + \delta_i,
   \]
   where \( W_i \) is the \( c_i \times c_{i-1} \) matrix of kernels, and \( \delta_i \) is the vector of length \( c_i \) consisting of \( d \)-dimensional vectors. Similarly, we can denote \( C'_i \) as
   \[
   C'_i(x) = W'_i \sigma x + \delta'_i.
   \]
   Then we can define the concatenation for \( i = 2, 3, \ldots, N \) as
   \[
   C''_i(x \oplus y) := \begin{bmatrix} W_i \\ W'_i \end{bmatrix} \oplus \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \delta_i \\ \delta'_i \end{bmatrix} = C_1(x) \oplus C_2(y).
   \]
   Also, define \( C''_1 \) as
   \[
   C''_1(x) := \begin{bmatrix} W_1 \\ W'_1 \end{bmatrix} \oplus \begin{bmatrix} \delta_1 \\ \delta'_1 \end{bmatrix} = C_1(x) \oplus C'_1(x).
   \]
   Then, we can construct \( f \) with channel sizes \( c - (c_1 + c'_1) - (c_2 + c'_2) - \cdots - (c_{N-1} + c'_{N-1}) - (c' + c'') \) as
   \[
   f := C''_N \circ \sigma \circ C''_{N-1} \circ \cdots \circ \sigma \circ C'_1.
   \]
   which completes the proof.

2. For the arbitrary \( f_1, f_2 \in \Sigma^{N}_{c,c'} \), express \( f_1 \) and \( f_2 \) as
   \[ f_1 := C_N \circ \sigma \circ C_{N-1} \circ \cdots \circ \sigma \circ C_1 \]
   and \( f_2 := C'_N \circ \sigma \circ C'_{N-1} \circ \cdots \circ \sigma \circ C'_1 \). Except for the axiom that \( \Sigma^{N}_{c,c'} \) is closed under addition, other axioms can be shown simply by giving proper operations to the last layer. For example, replacing \( C_N \) with \(-C_N \) gives the inverse element of \( f_1 \). For the axiom that \( \Sigma^{N}_{c,c'} \) is closed under addition, construct \( g \) as the concatenation of
$g_1 := C_{N-1} \circ \cdots \circ \sigma \circ C_1$ and $g_2 := C'_{N-1} \circ \cdots \circ \sigma \circ C'_1$. For the $C_N$ and $C'_N$ expressed as

$$C_N(x) = W \oplus x + \delta,$$

and

$$C'_N(x) = W' \oplus x + \delta',$$

we can construct the convolutional layer $C''_N$, which satisfies

$$C''_N(x \oplus y) = \left[ W \quad W' \right] \oplus \begin{bmatrix} x \\ y \end{bmatrix} + (\delta + \delta') = C_N(x) + C'_N(y).$$

Then,

$$C''_N \circ \sigma \circ g(x) = C''_N \circ \sigma \circ (g_1 \oplus g_2)(x) = C_N \circ \sigma \circ g_1(x) + C'_N \circ \sigma \circ g_2(x) = f_1(x) + f_2(x).$$

Thus, $f_1 + f_2 \in \Sigma^N_{c,c'}$, and $\Sigma^N_{c,c'}$ is a vector space.

For $\sigma(\Sigma^N_{c,c'})$, it is obvious from the definition of $\sigma(\Sigma^N_{c,c'})$.

3. Because $\Sigma^N_{c,c'}$ is a vector space, $\frac{f_{a+}(x) - f_a(x)}{\epsilon} \in \Sigma^N_{c,c'}$. And because $\|\frac{f_{a+}(x) - f_a(x)}{\epsilon}\| < o(\epsilon) \sup_{\Sigma^N_{c,c'}} \|\frac{\partial f}{\partial \theta}(x)\|$, it uniformly converges to zero; thus, $\frac{\partial f}{\partial \theta}(x) \in \Sigma^N_{c,c'}$. Similar argument holds for $\sigma(\Sigma^N_{c,c'})$.

4. For $g \in \sigma(\Sigma^N_{c,c'})$, there exist $g_i \in \sigma(\Sigma^N_{c,c'})$, such that $g_i \xrightarrow{i \to \infty} g$. Then,

$$g_i = \sum_{j=1}^{n_i} a_{i,j}(\sigma \circ g_{i,j}),$$

for $g_{i,j} \in \Sigma^N_{c,c'}$ and $a_{i,j} \in \mathbb{R}$. Decompose $C$ into $C = L + \delta$ where $L$ is the linear transformation and $\delta$ is the bias.

$$C \circ g_i = (L + \delta) \circ \sum_{j=1}^{n_i} a_{i,j}(\sigma \circ g_{i,j}) = \delta + \sum_{j=1}^{n_i} a_{i,j}L \circ \sigma \circ g_{i,j} \in \Sigma^{N+1}_{c,c'},$$

because $\Sigma^{N+1}_{c,c'}$ is a vector space. If $\{g_i\}_{i \in \mathbb{N}}$ uniformly converges to $g$, $\{C \circ g_i\}_{i \in \mathbb{N}}$ uniformly converges to $C \circ g$ for the continuous function $C$ in the compact space, and it completes the proof.

Lemma 2 Consider the activation function $\sigma$, which is the $C^1$-function near $\alpha$ and $\sigma'(-\alpha) \neq 0$. Then, for zero padding convolutional layers $C_1, C_2$ and a positive number $\epsilon \in \mathbb{R}_+$, there exist zero padding convolutional layers $C'_1, C'_2$ with same input and output channels to $C_1, C_2$, respectively, such that

$$||C_2 \circ \bar{\sigma}_1 \circ C_1 - C_2' \circ \sigma \circ C'_1||_{C^\infty(K)} < \epsilon.$$  

where $I \subset [1, c_2]$, and $c_2$ is the number of output channels of $C_1$. 

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Then, define $C'_1$ with kernels $w'_{j,i}$ and biases $\delta'_j$ and $C'_2$ with kernels $w'_{k,j}$ and biases $\delta'_k$ as follows:

$$w'_{j,i} = \begin{cases} w_{j,i} & \text{if } j \in I \\ \frac{w_{j,i}}{N} & \text{otherwise} \end{cases}, \quad \delta'_j = \begin{cases} \delta_j & \text{if } j \in I \\ \alpha + \frac{\delta_j}{N} & \text{otherwise} \end{cases}, \quad (46)$$

and

$$w'_{k,j} = \begin{cases} w_{k,j} \frac{I}{\sigma'(\alpha)} w_{k,j}^{\prime} & \text{if } j \in I \\ \text{otherwise} \end{cases}, \quad \delta'_k = \frac{N\sigma(\alpha)}{\sigma'(\alpha)} + \delta_k^2. \quad (47)$$

Then, $f_k$, the $k$-th component of $C'_2 \circ \sigma \circ C'_1$, becomes

$$f_k(x) := \sum_{j=1}^{c_2} w_{k,j}^{\prime 2} \sigma \left( \sum_{i=1}^{c_1} w_{j,i}^{\prime 1} \odot (x^j) + \delta_j^{1} 1_d \right) + \delta_k^{2} 1_d \quad (48)$$

$$= \sum_{j \in I} w_{k,j}^{\prime 2} \sigma \left( \sum_{i=1}^{c_1} w_{j,i}^{\prime 1} \odot (x^j) + \delta_j^{1} 1_d \right) + \delta_k^{2} 1_d \quad (49)$$

$$+ \sum_{j \notin I} w_{k,j}^{\prime 2} \sigma \left( \sum_{i=1}^{c_1} w_{j,i}^{\prime 1} \odot (x^j) + \delta_j^{1} 1_d \right) + \delta_k^{2} 1_d \quad (50)$$

$$= \sum_{j \in I} w_{k,j}^{\prime 2} \sigma \left( \sum_{i=1}^{c_1} w_{j,i}^{\prime 1} \odot (x^j) + \delta_j^{1} 1_d \right) + \sum_{j \notin I} \frac{N}{\sigma'(\alpha)} w_{k,j}^{\prime 2} \sigma \left( \sum_{i=1}^{c_1} \frac{w_{j,i}^{\prime 1}}{N} \odot (x^j) + \frac{\delta_j}{N} + \alpha \right) + \frac{N\sigma(\alpha)}{\sigma'(\alpha)} + \delta_k^{2} 1_d. \quad (51)$$

And the $k$-th component of $C_2 \circ \bar{\sigma}_I \circ C_1$, $g_k$, is

$$g_k(x) = \sum_{j=1}^{c_2} w_{k,j}^{\prime 2} \bar{\sigma}_I \left( \sum_{i=1}^{c_1} w_{j,i}^{\prime 1} \odot (x^j) + \delta_j^{1} 1_d \right) + \delta_k^{2} 1_d \quad (52)$$

$$= \sum_{j \in I} w_{k,j}^{\prime 2} \bar{\sigma}_I \left( \sum_{i=1}^{c_1} w_{j,i}^{\prime 1} \odot (x^j) + \delta_j^{1} 1_d \right) + \sum_{j \notin I} w_{k,j}^{\prime 2} \bar{\sigma}_I \left( \sum_{i=1}^{c_1} w_{j,i}^{\prime 1} \odot (x^j) + \delta_j^{1} 1_d \right) + \delta_k^{2} 1_d \quad (53)$$

$$= \sum_{j \in I} w_{k,j}^{\prime 2} \sigma \left( \sum_{i=1}^{c_1} w_{j,i}^{\prime 1} \odot (x^j) + \delta_j^{1} 1_d \right) + \sum_{j \notin I} w_{k,j}^{\prime 2} \sigma \left( \sum_{i=1}^{c_1} w_{j,i}^{\prime 1} \odot (x^j) + \delta_j^{1} 1_d \right) + \delta_k^{2} 1_d. \quad (54)$$
Then \( f_k - g_k \) becomes

\[
g_k(x) - f_k(x) = \sum_{j \in I} w_{k,j}^2 \sigma \left( \sum_{i=1}^{c_i} w_{j,i}^1 \odot (x^j) + \delta_j^1 1_d \right) + \sum_{j \notin I} w_{k,j}^2 \left( \sum_{i=1}^{c_i} w_{j,i}^1 \odot (x^j) + \delta_j^1 1_d \right) + \delta_k^2 1_d
\]

(56)

\[
- \sum_{j \notin I} \left( w_{k,j}^2 \sigma \left( \sum_{i=1}^{c_i} w_{j,i}^1 \odot (x^j) + \delta_j^1 1_d \right) \right)
\]

(57)

\[
- \sum_{j \notin I} \left( \frac{N}{\sigma'(\alpha)} w_{k,j}^2 \sigma \left( \sum_{i=1}^{c_i} \frac{w_{j,i}^1}{N} \odot (x^j) + \frac{\delta_j^1 1_d}{N} + \alpha \right) \right) - \frac{N\sigma(\alpha)}{\sigma'(\alpha)} - \delta_k^2 1_d
\]

(58)

\[
= \sum_{j \notin I} w_{k,j}^2 \left( \sum_{i=1}^{c_i} w_{j,i}^1 \odot (x^j) + \delta_j^1 1_d \right) + \delta_k^2 1_d
\]

(59)

\[
- \sum_{j \notin I} \left( \frac{N}{\sigma'(\alpha)} w_{k,j}^2 \sigma \left( \sum_{i=1}^{c_i} \frac{w_{j,i}^1}{N} \odot (x^j) + \frac{\delta_j^1 1_d}{N} + \alpha \right) \right) - \frac{N\sigma(\alpha)}{\sigma'(\alpha)} - \delta_k^2 1_d.
\]

(60)

Let \( u_j \) be \( u_j := \sum_{i=1}^{c_i} w_{j,i}^1 \odot (x^j) + \delta_j^1 1_d \). Then,

\[
g_k(x) - f_k(x) = \sum_{j \notin I} w_{k,j}^2 \left( u_j - \frac{N}{\sigma'(\alpha)} \sigma \left( \frac{u_j}{N} + \alpha \right) - \frac{N\sigma(\alpha)}{\sigma'(\alpha)} \right)
\]

(61)

\[
= \sum_{j \notin I} w_{k,j}^2 \frac{u_j^2}{\sigma'(\alpha)} \sigma \left( \frac{1}{N} \right) \xrightarrow{N \to \infty} 0.
\]

(62)

The convergence is uniform because \( x \) is in the compact domain \( K \); thus, \( u_j \) is uniformly bounded for all \( x \). \( \blacksquare \)

**Lemma 3** For the Lipschitz continuous activation function \( \sigma \), \( N \geq 2 \), the channel sizes \( c_0 - c_1 - \cdots - c_N \), indexes \( I_i \subset [1, c_i] \), and the convolutional layers \( C_i \) with \( c_{i-1} \) input channels and \( c_i \) output channels, define the convolutional neural network \( f \) as

\[
f := C_N \circ \bar{\sigma}_{I_N-1} \circ C_{N-1} \circ \cdots \circ \bar{\sigma}_{I_1} \circ C_1.
\]

(63)

Then, there exists \( g \in \sigma \Sigma_{c,c'}^N \) defined as

\[
g := C_N' \circ \sigma \circ C_{N-1}' \circ \cdots \circ \sigma \circ C_1',
\]

(64)

such that

\[
\|f - g\|_{C^\infty(K)} < \epsilon,
\]

(65)

where \( C_i' \) has \( c_{i-1} \) input channels and \( c_i \) output channels.
**Proof** Use the mathematical induction on $N$. By Lemma 2, the induction hypothesis is satisfied for the case $N = 2$. Assume that the induction hypothesis is satisfied for the case $N = N_0$. For the case $N = N_0 + 1$, consider the function $f_{N_0+1}$ defined as

$$f_{N_0+1} = C_{N_0+1} \circ \tilde{\sigma}_{I_{N_0}} \circ C_{N_0} \circ \cdots \circ \tilde{\sigma}_{I_1} \circ C_1. \quad (67)$$

Then, for $f_{N_0} := C_{N_0} \circ \tilde{\sigma}_{I_{N_0}-1} \circ \cdots \circ C_1$,

$$f_{N_0+1} = C_{N_0+1} \circ \tilde{\sigma}_{I_{N_0}} \circ f_{N_0}. \quad (68)$$

By the induction hypothesis, there exists $g \in \sigma \Sigma_{c,c'}^{N_0}$, such that

$$\|f_{N_0} - g\|_{C^\infty(K)} < \frac{\epsilon}{2l}. \quad (69)$$

where $l$ denote the Lipschitz constant of $C_{N_0+1} \circ \tilde{\sigma}_{I_{N_0}}$. Then,

$$\|f_{N_0+1} - C_{N_0+1} \circ \tilde{\sigma}_{I_{N_0}} \circ g\|_{C^\infty(K)} < \frac{\epsilon}{2}. \quad (70)$$

Denote $g$ as

$$g = C'_{N_0} \circ \sigma \circ \cdots \circ \sigma \circ C'_1. \quad (71)$$

By Lemma 2, there exist convolutional layers $C''_{N_0+1}$ and $C''_{N_0}$, such that

$$\|C_{N_0+1} \circ \tilde{\sigma}_{I_{N_0}} \circ C'_{N_0} - C''_{N_0+1} \circ \sigma \circ C''_{N_0}\|_{C^\infty(K')} < \frac{\epsilon}{2}. \quad (72)$$

where $K'$ is the compact space $K' = \sigma \circ C'_{N_0-1} \circ \cdots \circ \sigma \circ C'_1(K)$. Define $h \in \sigma \Sigma_{c,c'}^{N_0}$ as

$$h := C''_{N_0+1} \circ \sigma \circ C''_{N_0} \circ \sigma \circ C'_{N_0-1} \circ \cdots \circ C'_1. \quad (73)$$

Then, the following equation is satisfied:

$$\|C_{N_0+1} \circ \tilde{\sigma}_{I_{N_0}} \circ g - h\|_{C^\infty(K)} < \frac{\epsilon}{2}. \quad (74)$$

To sum up,

$$\|f_{N_0+1} - h\|_{C^\infty(K)} < \|f_{N_0+1} - C_{N_0+1} \circ \tilde{\sigma}_{I_{N_0}} \circ g\|_{C^\infty(K)} + \|C_{N_0+1} \circ \tilde{\sigma}_{I_{N_0}} \circ g - h\|_{C^\infty(K)} < \epsilon. \quad (75)$$

Therefore, the induction hypothesis is satisfied for $N = N_0 + 1$, and it completes the proof.

---

**Corollary 4** For the Lipschitz continuous activation function $\sigma$ and $N \geq 2$, $\text{Id} \Sigma_{c,c'}^{N}$ is the subset of $\sigma \Sigma_{c,c'}^{N}$ as functions defined on the compact set $K$ where $\text{Id}$ is the identity function; that is, $\text{Id} \Sigma_{c,c'}^{N} \subset \sigma \Sigma_{c,c'}^{N}$.

Lemma 3 and Corollary 4 imply that we can freely exchange the activation function to the identity.
Lemma 5 For an arbitrary continuous activation function $\sigma$ and an compact supported $C^\infty$ continuous function $\varphi$, $\overline{\sigma * \varphi}^{N}_{c,e'} \subset \overline{\sigma}^{N}_{c,e'}$.

Proof There exists

$$\sigma * \varphi(x) = \int_{-\alpha}^{\alpha} \sigma(x-y)\varphi(y)dy \quad (76)$$

4.3 The Minimum Depth for the Universal Property of Convolutional Neural Network

In this section, we showed the minimum depth for the three-kernel convolutional neural network to have the universal property. Unlike MLP, which only needs a two-layered network to get universal property, CNN requires a much deeper minimum depth. This is because the receptive field, the range of the input component which affects the specific output component, is restricted by the convolution using the kernel. In the case of a convolutional layer with a kernel size of three, each output receives input from left and right one component. Therefore, when considering the convolutional neural network constructed by composing these $N$ layers of convolutional layers, the input can take values from the left and right $N$ components. Therefore, obviously, in the case of a function with $d$-dimensional input and output, at least $d-1$ layers must be used for the first component of the output to receive the last component of the input. Therefore, for a CNN with kernel size three to have universal property, at least $d-1$ layers are required. The following proposition shows that the minimum depth $d-1$ is insufficient for the case of $d = 3$.

Proposition 6 If a compact domain $K \subset \mathbb{R}^c$ contains an open subset near the origin, three-kernel, two-layered convolutional neural network does not have the universal property in $K$ when $d = 3$; that is, $\overline{\sigma}^{2}_{c,e'} \neq C(K, \mathbb{R}^c)$.

Proof For a 4-dimensional input, consider the case where the numbers of input and output channels are one, and the number of intermediate channels is $n$. Then, for a convolutional layer $C_1$ with kernels $(a^i_{-1}, a_i^0, a_i^1)$ and biases $\delta_i$ and a convolutional layer $C_2$ with kernels $(b^i_{-1}, b_i^0, b_i^1)$ and biases $\delta_0$, the entire CNN $f = (f_1, f_2, f_3) := C_2 \circ \sigma \circ C_1$ satisfies the following equations.

$$f_1(x_1, x_2, x_3) = \sum_{i=1}^{n} b_i^0 \sigma(a_i^0 x_1 + a_i^1 x_2 + \delta_i) + b_i^1 \sigma(a_{-1}^i x_1 + a_0^i x_2 + a_i^1 x_3 + \delta_i) + \delta_0, \quad (77)$$

$$f_2(x_1, x_2, x_3) = \sum_{i=1}^{n} b_{i-1}^i \sigma(a_i^0 x_1 + a_i^1 x_2 + \delta_i) + b_i^0 \sigma(a_{-1}^i x_1 + a_0^i x_2 + a_i^1 x_3 + \delta_i) + b_i^1 \sigma(a_{-1}^i x_2 + a_0^i x_3 + \delta_i) + \delta_0, \quad (78)$$

$$f_3(x_1, x_2, x_3) = \sum_{i=1}^{n} b_{i-1}^i \sigma(a_{-1}^i x_1 + a_0^i x_2 + a_i^1 x_3 + \delta_i) + b_i^0 \sigma(a_{-1}^i x_2 + a_0^i x_3 + \delta_i) + \delta_0. \quad (79)$$
Then, the following equation holds.

\[ f_1(x, y, 0) - f_2(0, x, y) = \left( \sum_{i=1}^{n} b_{i0}^j \sigma(a_{i0}^j x + a_{i1}^j y + \delta_i) + b_{i1}^j \sigma(a_{i-10}^j x + a_{i0}^j y + \delta_i) \right) \]

\[ - \left( \sum_{i=1}^{n} b_{i-1}^j \sigma(a_{i1}^j x + \delta_i) + b_{i0}^j \sigma(a_{i0}^j x + a_{i1}^j y + \delta_i) + b_{i1}^j \sigma(a_{i-1}^j x + a_{i0}^j y + \delta_i) \right) \]

\[ = - \sum_{i=1}^{n} b_{i}^j \sigma(a_{i}^j x + \delta_i). \] (80)

Thus, \( f_1(x, y, 0) - f_2(0, x, y) \) becomes the function of \( x \). Let it

\[ h(x) := f_1(x, y, 0) - f_2(0, x, y). \] (83)

Also, define \( g = (g_1, g_2, g_3) : \mathbb{R}^3 \to \mathbb{R}^3 \) as

\[ g(x_1, x_2, x_3) := (x_2, 0, 0). \] (84)

Let \( K \) contains the open rectangle \( (-\epsilon_0, \epsilon_0)^3 \). Then, the following equation is satisfied for arbitrary \( x, y \in (-\epsilon_0, \epsilon_0) \).

\[ |(f_1 - g_1)(x, y, 0) - (f_2 - g_2)(0, x, y)| = |y - h(x)|. \] (85)

If \( g \in \overline{\Sigma_{c,c'}^2} \), there exists \( f \) such that,

\[ ||f - g||_{C^\infty(K)} < \frac{\epsilon_0}{4}, \] (86)

which implies that \( |(f_1 - g_1)(x, y, 0)| < \frac{\epsilon_0}{4} \) and \( |(f_2 - g_2)(0, x, y)| < \frac{\epsilon_0}{4} \) for arbitrary \( x, y \in (-\epsilon_0, \epsilon_0) \). However,

\[ |y - h(x)| = |(f_1 - g_1)(x, y, 0) - (f_1 - g_2)(0, x, y)| \]

\[ < |(f_1 - g_1)(x, y, 0)| + |(f_1 - g_2)(0, x, y)| < \frac{\epsilon_0}{2}. \] (87)

for arbitrary \( x, y \in [-\epsilon_0, \epsilon_0] \), which becomes a contradiction, and it completes the proof.

**Lemma 7** For \( i \in [1, n] \), \( l \in \mathbb{N} \), and a non-polynomial \( C^\infty \) activation function \( \sigma \), if \( A_i \in \overline{\Sigma_{c,1}^l} \), then the following relation holds:

\[ \prod_{i=1}^{n} A_i \in \overline{\sigma(\Sigma_{c,1})}, \] (88)

where the product on the left hand side means the Hadamard product of the vector-valued functions.
Proof Let $a_i \in \mathbb{R}$ for $i \in [1, n]$. Because $\Sigma_{c,1}^l$ is a vector space by Lemma 1, and $\delta 1_d \in \Sigma_{c,1}^l$, the linear summation also in $\Sigma_{c,1}^l$:

$$f := \sum_{i=1}^{n} a_i A_i + \delta 1_d \in \Sigma_{c,1}^l.$$  

(89)

By definition of $\sigma(\Sigma_{c,1}^l)$,

$$\sigma \left( \sum_{i=1}^{n} a_i A_i + \delta 1_d \right) \in \sigma(\Sigma_{c,1}^l).$$  

(90)

By Lemma 1, $\sigma(\Sigma_{c,1}^l)$ is closed under the partial differentiation with respect to the parameters. Therefore, we have

$$\left( \prod_{i=1}^{n} \frac{\partial}{\partial a_i} \right) \left[ \sigma \left( \sum_{i=1}^{n} a_i A_i + \delta 1_d \right) \right] \in \sigma(\Sigma_{c,1}^l).$$  

(91)

And the partial differentiation is calculated as the Hadamard product:

$$\left( \prod_{i=1}^{n} \frac{\partial}{\partial a_i} \right) \left[ \sigma \left( \sum_{i=1}^{n} a_i A_i + \delta 1_d \right) \right] = \prod_{i=1}^{n} A_i \sigma^{(n)}(f) \in \sigma(\Sigma_{c,1}^l).$$  

(92)

Because $\sigma$ is the non-polynomial function, there exist $\delta_0$ such that $\sigma^{(n)}(\delta_0) \neq 0$. By substituting all $a_i$ to zero and $\delta$ to $\delta_0$, we get

$$\prod_{i=1}^{n} A_i \sigma^{(n)}(f) \bigg|_{a_1=\ldots=a_n=0, \delta=\delta_0} = \prod_{i=1}^{n} A_i \sigma^{(n)}(\delta_0) \in \sigma(\Sigma_{c,1}^l).$$  

(93)

Also $\sigma(\Sigma_{c,1}^l)$ is a vector space, and $\prod_{i=1}^{n} A_i \in \sigma(\Sigma_{c,1}^l)$ which completes the proof.

The lemma implies that the sufficiently smooth activation function can transform the input function to the componentwise product.

Now we provide the main proposition which shows that the minimum width $d - 1$ is sufficient for the case of $d \geq 4$.

**Proposition 8** For the continuous activation function $\sigma$ and $d \geq 4$, $(d - 1)$-layered convolutional neural networks have the universal property in the continuous function space; that is, $\Sigma_{c,c'}^{d-1}(K) = C(K, \mathbb{R}^c)$.

**Proof** Before we go any further, we denote that we only have to prove that $\Sigma_{c,1}^{d-1}(K) = C(K, \mathbb{R})$ because the concatenation of the function can be conducted by Lemma 1. The flow of the proof follows the idea of the Leshno et al. (1993). The main idea is that if we can approximate all polynomials, all continuous functions in the compact domain can be approximated by the Stone–Weierstrass theorem (De Branges, 1959). The core difference is to make all multivariate polynomials in all positions of the output vector independently. The complexity made by convolution is the real matter that makes the problem tricky.
The proof is divided into the following steps. First, we will list the functions that can
be approximated by convolution under the assumption that the activation function \( \sigma \) is a
non-polynomial \( C^\infty \) function. Next, we construct the projection, which enables us to split
each component of the output vector and construct an arbitrary polynomial in an arbitrary
position. Finally, we generalize the result for the general non-polynomial activation function
case later.

For the input vector \( x = (x^1, x^2, \ldots, x^c) \in \mathbb{R}^{c \times d} \), define the translation of \( x^i = (x^i_1, x^i_2, \ldots, x^i_d) \)
as follows:

\[
p^i_{-j} := U_j x^i = (0, \ldots, 0, x^i_1, x^i_2, \ldots, x^i_{d-j}),
\]

(94)

\[
p^i_0 := x^i = (x^i_1, x^i_2, \ldots, x^i_d),
\]

(95)

and

\[
p^i_j := U_{-j} x^i = (x^i_{j+1}, \ldots, x^i_d, 0, \ldots, 0),
\]

(96)

**Case 1.** \( d = 4 \): It is obvious by definition that \( p^i_j = U_{-j} x^i \in \Sigma^i_{c,1} \) for \( j \in \{-1, 0, 1\} \).

By Lemma 7, an arbitrary product of \( p^i_j \) is in \( \sigma(\Sigma^i_{c,1}) \). In other words, for some constants
\( \alpha_{i,j} \in \mathbb{N} \) for \( i \in [1, c], j \in \{-1, 0, 1\} \),

\[
\prod_{i=1}^c \prod_{j=-1}^1 (p^i_j)^{\alpha_{i,j}} \in \sigma(\Sigma^i_{c,1}).
\]

(97)

Consider vector-valued functions \( A^1, A^2, \ldots, A^n \in \sigma(\Sigma^1_{c,1}) \), and \( \mathbf{1}_4 \in \sigma(\Sigma^1_{c,1}) \). Also, consider
convolutional layers with kernel \( b^i = (b^i_1, b^i_0, b^i_1) \). By Lemma 1,

\[
b^i \circledast A^i \in \Sigma^2_{c,1},
\]

(98)

for \( i \in [1, n] \), and

\[
b^{n+1} \circledast \mathbf{1}_4 \in \Sigma^2_{c,1}.
\]

(99)

We construct the second convolutional layer \( B \) with \( n \) input channel and one output channel,
which consists of convolutions with kernel and the bias \( \delta \). By Lemma 7, the Hadamard
product of \( b^i \circledast A^i \) is in \( \sigma(\Sigma^2_{c,1}) \):

\[
\prod_{i=1}^n (b^i \circledast A^i) \in \sigma(\Sigma^2_{c,1}).
\]

(100)

Now we construct the projection of the vectors \( \prod_{i=1}^n (b^i \circledast A^i) \) to a certain axis. Because
\( b^{n+1} \circledast \mathbf{1}_4 \in \Sigma^2_{c,1} \) and \( \mathbf{1}_4 \in \Sigma^2_{c,1} \), the linear summation of two functions is also in \( \Sigma^2_{c,1} \):

\[
b^{n+1} \circledast \mathbf{1}_4 + \delta \mathbf{1}_4 \in \Sigma^2_{c,1}.
\]

(101)

Componentwise expression becomes

\[
b^{n+1} \circledast \mathbf{1}_4 + \delta \mathbf{1}_4 = \delta \mathbf{1}_4 + (b^{n+1} + b^{n+1}_0, b^{n+1}_1 + b^{n+1}_0 + b^{n+1}_1, b^{n+1}_0 + b^{n+1}_1, b^{n+1}_1 + b^{n+1}_0 + b^{n+1}).
\]

(102)
With $\delta = -(b_{-1}^{n+1} + b_0^{n+1} + b_1^{n+1})$, $b^{n+1} \otimes 1_4 + \delta 1_4$ becomes

$$b^{n+1} \otimes 1_4 + \delta 1_4 = (-b_{-1}^{n+1}, 0, 0, -b_1^{n+1}). \quad (103)$$

Therefore, $e_1 = (1, 0, 0, 0)$ and $e_4 = (0, 0, 0, 1)$ are in $\overline{\Sigma_{c,1}^2}$. By Lemma 7, the Hadamard product of $b^i \otimes A^i$ and $e_1$ is in $\sigma(\Sigma_{c,1}^2)$:

$$\text{pr}_1 \left( \prod_{i=1}^{n} (b^i \otimes A^i) \right) = e_1 \otimes \left( \prod_{i=1}^{n} (b^i \otimes A^i) \right) \in \sigma(\Sigma_{c,1}^2), \quad (104)$$

where $\text{pr}_i$ means the projection to the $i$-th axis; that is, $\text{pr}_1(\theta_1, \theta_2, \theta_3, \theta_4) = (\theta_1, 0, 0, 0)$, and $\text{pr}_4(\theta_1, \theta_2, \theta_3, \theta_4) = (0, 0, 0, \theta_4)$. Similarly, the Hadamard product of $b^i \otimes A^i$ and $e_4$ is in $\sigma(\Sigma_{c,1}^2)$:

$$\text{pr}_4 \left( \prod_{i=1}^{n} (b^i \otimes A^i) \right) = e_4 \otimes \left( \prod_{i=1}^{n} (b^i \otimes A^i) \right) \in \sigma(\Sigma_{c,1}^2). \quad (105)$$

We also know that

$$\text{pr}_2 \left( \prod_{i=1}^{n} (b^i \otimes A^i) \right) + \text{pr}_3 \left( \prod_{i=1}^{n} (b^i \otimes A^i) \right) = \prod_{i=1}^{n} (b^i \otimes A^i) - \text{pr}_1 \left( \prod_{i=1}^{n} (b^i \otimes A^i) \right) - \text{pr}_4 \left( \prod_{i=1}^{n} (b^i \otimes A^i) \right). \quad (106)$$

Therefore, $\text{pr}_2 \left( \prod_{i=1}^{n} (b^i \otimes A^i) \right) + \text{pr}_3 \left( \prod_{i=1}^{n} (b^i \otimes A^i) \right) \in \sigma(\Sigma_{c,1}^2)$.

Now, we construct the desired polynomials using the ingredients made in the previous steps. First, we will prove that for an monomial $M_1$ consisting of $x_i^1, x_2^i, x_3^i$, except $x_4^i$, $(M_1, 0, 0, 0)$ is the element of $\overline{\sigma(\Sigma_{c,1}^2)}$. More concretely, $M_1$ is defined as

$$M_1 = \prod_{i=1}^{c} \prod_{j=1,2,3} (x_j^i)^{\alpha_{i,j}}, \quad (107)$$

where $\alpha_{i,j} \in \mathbb{N}_0$. Let $A$ be

$$A = \prod_{i}^{c} \prod_{j=-1,0,1} (p_j^i)^{\alpha_{i,j}+2} \in \overline{\sigma(\Sigma_{c,1}^1)} \quad (108)$$

Then with $b = (0, 0, 1)$, $\text{pr}_1 (b \otimes A) = \text{pr}_1 (U_{-1} A) \in \overline{\Sigma_{c,1}^2}$, which means

$$\text{pr}_1 (U_{-1} A) = \text{pr}_1 \left( \prod_{i=1}^{c} \prod_{j=1,2,3} (x_j^i)^{\alpha_{i,j}}, \prod_{i=1}^{c} \prod_{j=1,2,3} (x_j^{i+1})^{\alpha_{i,j}}, \prod_{i=1}^{c} \prod_{j=1,2,3} (x_j^{i+2})^{\alpha_{i,j}}, 0 \right) \quad (109)$$

$$= \left( \prod_{i=1}^{c} \prod_{j=1,2,3} (x_j^i)^{\alpha_{i,j}}, 0, 0, 0 \right) \in \overline{\sigma(\Sigma_{c,1}^2)}, \quad (110)$$
where $x_3^i := 0$. Similarly, for a monomial $M_2$ consisting of $x_2^i, x_3^i, x_4^i$, except $x_1^i$, $(0, 0, 0, M_2)$ is the element of $\sigma(\Sigma^2_{c,1})$; that is, for $\alpha_{i,j} \in \mathbb{N}_0$,

$$M_2 = \prod_{i=1}^{c} \prod_{j=2,3,4} (x_j^i)^{\alpha_{i,j}}. \quad (111)$$

The proof is obvious from symmetry.

Next, we will prove that for a monomial $M_3$ that contains at least one $x_4^i$, $(0, M_3, 0, 0)$ is the element of $\sigma(\Sigma^2_{c,1})$; that is, for $M_3$ defined as

$$M_3 = x_4^i \prod_{i=1}^{c} \prod_{j=1,2,3,4} (x_j^i)^{\alpha_{i,j}}, \quad (112)$$

$(0, M_3, 0, 0) \in \sigma(\Sigma^2_{c,1})$ where $\alpha_{i,j} \in \mathbb{N}_0$. For the proof, define $A_1$ and $A_2$ as

$$A_1 = \prod_{i=1}^{c} \prod_{j=-1,0} (p_j^i)^{\alpha_{i,j+2}} \in \sigma(\Sigma^1_{c,1}), \quad (113)$$

and

$$A_2 = p_1^i \circ \prod_{i=1}^{c} \prod_{j=1,2} (p_{j-1}^i)^{\alpha_{i,j+2}} \in \sigma(\Sigma^1_{c,1}). \quad (114)$$

Also, define $B$ as follows:

$$B := (0, 0, 1) \oplus A_2 = U_1 A_2 \in \Sigma^2_{c,1}. \quad (115)$$

Then, we have

$$(\text{pr}_2 + \text{pr}_3) (A_1 \odot B) \in \sigma(\Sigma^2_{c,1}). \quad (116)$$

Because

$$(\text{pr}_2 + \text{pr}_3) (A_1 \odot B) = (\text{pr}_2 + \text{pr}_3) (A_1) \odot (\text{pr}_2 + \text{pr}_3) (B), \quad (117)$$

and

$$(\text{pr}_2 + \text{pr}_3) (B) = \left( 0, x_4^0 \prod_{i=1}^{c} \prod_{j=3,4} (x_j^i)^{\alpha_{i,j}}, 0, 0 \right), \quad (118)$$

$(\text{pr}_2 + \text{pr}_3) (A_1 \odot B)$ becomes

$$(\text{pr}_2 + \text{pr}_3) (A_1 \odot B) = (\text{pr}_2) (A_1) \odot (\text{pr}_2) (B) \quad (119)$$

$$= \left( 0, \prod_{i=1}^{c} \prod_{j=1,2} (x_j^i)^{\alpha_{i,j}}, 0, 0 \right) \odot \left( 0, x_4^0 \prod_{i=1}^{c} \prod_{j=3,4} (x_j^i)^{\alpha_{i,j}}, 0, 0 \right) \quad (120)$$

$$= \left( 0, x_4^0 \prod_{i=1}^{c} \prod_{j=1}^{4} (x_j^i)^{\alpha_{i,j}}, 0, 0 \right) = (0, M_3, 0, 0) \in \sigma(\Sigma^2_{c,1}). \quad (121)$$

Similarly, the symmetrical argument shows that for a monomial $M_4$ containing at least one $x_1^i$, $(0, 0, M_4, 0)$ is the element of $\sigma(\Sigma^2_{c,1})$. What we have proven in this step is that
• for a monomial \( M_1 \) that does not contain any \( x_i^j \), \( (M_1, 0, 0, 0) \in \sigma(\Sigma_{c,1}^2) \),

• for a monomial \( M_2 \) that does not contain any \( x_i^1 \), \( (0, 0, 0, M_2) \in \sigma(\Sigma_{c,1}^2) \),

• for a monomial \( M_3 \) that contains at least one \( x_i^1 \), \( (0, M_3, 0, 0) \in \sigma(\Sigma_{c,1}^2) \),

• and for a monomial \( M_4 \) that contains at least one \( x_i^1 \), \( (0, 0, M_4, 0) \in \sigma(\Sigma_{c,1}^2) \).

Now we will prove that for arbitrary monomial \( M_0, (M_0, 0, 0, 0), (0, M_0, 0, 0), (0, 0, M_0, 0) \) and \( (0, 0, 0, M_0) \) are in \( \Sigma_{c,1}^2 \). By Lemma 1, for an arbitrary convolutional layer \( C \) and the function \( f \in \sigma(\Sigma_{c,1}^2), C(f) \in \Sigma_{c,1}^3 \). If a monomial \( M \) contains at least one \( x_i^1 \) for some \( i \in [1, c] \), \( (0, M, 0, 0) \in \sigma(\Sigma_{c,1}^2) \). And for \( C(x) = U_0x, C((0, M, 0, 0)) = (0, M, 0, 0) \in \Sigma_{c,1}^3 \), and for \( C(x) = U_{-1}x, C((0, M, 0, 0)) = (M, 0, 0, 0) \in \Sigma_{c,1}^3 \). Otherwise, if a monomial \( M \) does not contain any \( x_i^1 \), \( (M, 0, 0, 0) \in \sigma(\Sigma_{c,1}^2) \). And for \( C(x) = U_0x, C((M, 0, 0, 0)) = (M, 0, 0, 0) \in \Sigma_{c,1}^3 \), and for \( C(x) = U_{-1}x, C((M, 0, 0, 0)) = (0, M, 0, 0) \in \Sigma_{c,1}^3 \). So for an arbitrary monomial \( M, (M, 0, 0, 0) \) and \( (0, M, 0, 0) \) are the elements of \( \Sigma_{c,1}^3 \). And by symmetry, \( (0, 0, M, 0) \) and \( (0, 0, 0, M) \) are also in \( \Sigma_{c,1}^2 \). It completes the proof for the case of \( d = 4 \).

**Case 2.** \( d \geq 5 \): The proof proceeds almost the same to Case 1. The difference is that unlike Case 1, we can construct all the projections \( \text{pr}_k \) for all \( k \in [1, d] \) when \( d \geq 5 \). More concretely, for functions \( A^i \in \sigma(\Sigma_{c,1}^{d-3}) \), \( q_i \), kernels \( b^i \in \mathbb{R}^3 \), and \( q^i \) defined as \( q^i := b^i \oplus A^i \),

the following relation holds for all \( k \in [1, d] \),

\[
\text{pr}_k \left( \prod_{i=1}^n q_i \right) \in \sigma(\Sigma_{c,1}^{d-2}).
\] (123)

The proof is from the following steps.

**Step 1.** In this step, we will show that we can assign different constants to each axis. Let \( e_i \) be the \( i \)-th standard basis in Euclidean space. And define the constant function \( e_i : \mathbb{R}^d \to \mathbb{R}^d \) that has constant value \( e_i \): \( e_i(x) = e_i \) for all \( x \in \mathbb{R}^d \). Then what we will prove is that \( \Sigma_{c,1}^{d-2} \) contains \( e_i \) for \( i \in [1, d-3] \cup [4, d] \). More generally, \( e_i \in \Sigma_{c,1}^{n} \) for \( i \in [1, n-1] \cup [d-n+2, d] \). It can be proved by the following mathematical induction.

1. For the case of \( n = 2 \), constant function \( A(x) := \delta_11_d \in \sigma(\Sigma_{c,1}^{1}) \). Then, for the convolutional layer \( B \) with kernel \( b = (b_{-1}, b_0, b_1) \) and the bias \( \delta_2 \),

\[
B \circ A \in \Sigma_{c,1}^{2}.
\] (124)

More specifically,

\[
B \circ A = \delta_11_d + (\delta_2(b_0 + b_1), \delta_2(b_{-1} + b_0 + b_1), \ldots, \delta_2(b_{-1} + b_0 + b_1), \delta_2(b_{-1} + b_0)).
\] (125)
Then, by substituting $\delta_1$ for $\delta'_l - \delta_2(b_{-1} + b_0 + b_1)$, we get
\[
B \circ A = \delta'_l 1_d + (-\delta_2 b_{-1}, 0, \ldots, 0, -\delta_2 b_1) \in \Sigma_{c,1}^2.
\] (126)
for arbitrary $b_{-1}$ and $b_1$. So $e_1, e_d \in \Sigma_{c,1}^2$, and the induction hypothesis is satisfied for the case of $n = 2$.

2. Assume that for $n = n_0$, the induction hypothesis is satisfied, i.e., $e_i \in \Sigma_{c,1}^{n_0} \subseteq \sigma(\Sigma_{c,1}^{n_0})$ for $i \in [1, n_0 - 1] \bigcup [d - n_0 + 2, d]$. Then, for the convolutional layer $C(x) := U_1x$,
\[
C \circ e_{n_0 - 1} = e_{n_0} \in \Sigma_{c,1}^{n_0+1}.
\] (127)

Similarly, for the convolutional layer $C(x) = U_1x$,
\[
C \circ e_{d-n_0+2} = e_{d-n_0+1} \in \Sigma_{c,1}^{n_0+1}.
\] (128)

Therefore, the induction hypothesis is satisfied for $n = n_0 + 1$, and $\Sigma_{c,1}^{d-3}$ contains $e_i$ for $i \in [1, d - 4] \bigcup [5, d]$.

**Step 2.** In this step, we will similarly construct a polynomial to the case of $d = 4$ and show that its projection can also be constructed. We first prove that for the function $f : \mathbb{R}^{c \times d} \rightarrow \mathbb{R}^d$ defined as $f_j^i(x) = U_j x^i$, $f_j^i \in \Sigma_{c,1}^l$ for $j \in [-l, l]$. We use the mathematical induction. When $l = 1$, it is obviously satisfied. Assume that the induction hypothesis is satisfied for $l$: $f_j^i \in \Sigma_{c,1}^l$ for $j \in [-l, l]$. By Lemma 7, $f_j^i \in \sigma(\Sigma_{c,1}^l)$, and for $C(x) = U_1(x)$, $C \circ f_j^i \in \Sigma_{c,1}^{l+1}$. And because $C \circ f_j^i = f_{j+1}^i$, for $j \in [-l, l]$, $C \circ f_j^{i+1} \in \Sigma_{c,1}^{l+1}$. Similarly, using $C(x) = U_1(x)$, we have $C \circ f_j^i = f_{j-1}^i$, for $j \in [-l, l]$. Therefore, the induction hypothesis is satisfied for $l + 1$.

Consider $l = d - 2$. Then $p_j^i = U_j x^i \in \Sigma_{c,1}^{d-2}$ for $j \in [-d + 2, d - 2]$. By Lemma 7, for $\alpha_{i,j} \in \mathbb{N}_0$, the following relation holds:
\[
\prod_{i=1}^c \prod_{j=-d+2}^{d-2} (p_j^i)^{\alpha_{i,j}} \in \sigma(\Sigma_{c,1}^{d-2}).
\] (129)

Additionally, consider $e_t \in \sigma(\Sigma_{c,1}^{d-2})$. Then by applying Lemma 7 to $p_j^i$ and $e_t$, we get
\[
e_t \odot \left( \prod_{i=1}^c \prod_{j=-d+2}^{d-2} (p_j^i)^{\alpha_{i,j}} \right) \in \sigma(\Sigma_{c,1}^{d-2}).
\] (130)

Because for all $t \in [1, d - 3] \bigcup [4, d]$, $e_t$ is in $\Sigma_{c,1}^{d-2}$, we are able to get the projection $\text{pr}_t$ of $\prod_{i=1}^c \prod_{j=-d+2}^{d-2} (p_j^i)^{\alpha_{i,j}}$ for $t \in [1, d - 3] \bigcup [4, d]$. For $d > 5$, $[1, d - 3] \bigcup [4, d] = [1, d]$; so we have the projection to an arbitrary axis. For $d = 5$, $[1, d - 3] \bigcup [4, d] = \{1, 2, 4, 5\}$, and because $\text{pr}_3 = l_5 - \sum_{t=1,2,4,5} \text{pr}_t$, the projection to an arbitrary axis is also available for $d = 5$. 

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**Step 3.** For an arbitrary monomial \( M = \prod_{i=1}^{c} \prod_{j=1}^{d} (x_{ij}^{i})^{\alpha_{i,j}} \), we will show that the vector \( M_{e_{t}} \) is in \( \sigma(\Sigma_{c,1}^{d-2}) \):

\[
M_{e_{t}} = (0, \ldots, 0, M, 0, \ldots, 0) \in \sigma(\Sigma_{c,1}^{d-2}),
\]

for \( t \in [2, d-1] \). We know that

\[
e_{t} \odot \left( \prod_{i=1}^{c} \prod_{j=1}^{d} (p_{j}^{i})^{\alpha_{i,j}} \right) \in \sigma(\Sigma_{c,1}^{d-2}).
\]

By proper calculation, we get

\[
e_{t} \odot \left( \prod_{i=1}^{c} \prod_{j=1}^{d} (p_{j}^{i})^{\alpha_{i,j}} \right) = \prod_{i=1}^{c} \prod_{j=1}^{d} (e_{t} \odot p_{j}^{i})^{\alpha_{i,j}} = \prod_{i=1}^{c} \prod_{j=1}^{d} (x_{ij}^{i} e_{t})^{\alpha_{i,j}}
\]

\[
= \prod_{i=1}^{c} \prod_{j=1}^{d} (x_{ij}^{i})^{\alpha_{i,j}} e_{t} = M_{e_{t}}.
\]

Therefore, \( M_{e_{t}} \in \sigma(\Sigma_{c,1}^{d-2}) \) for \( t \in [2, d-1] \).

Finally, by using proper \( U_{1}, U_{0}, U_{-1} \) for the last convolutional layer, we can get \( M_{e_{t}} \in \Sigma_{c,1}^{d-1} \) for all \( i \in [1, d] \), and it completes the proof for the non-polynomial \( C_{\infty} \) activation function \( \sigma \).

Now remaining is to generalize the result of the non-polynomial \( C_{\infty} \) activation function for the general non-polynomial function. It comes from the Section 6 of the Leshno et al. (1993). For any non-polynomial activation function \( \sigma \), there exists the compact supported \( C_{\infty} \) function \( \phi \) such that \( \sigma \ast \phi \) is smooth and not a polynomial function(Step 5 and Step 6 of Section 6 Leshno et al. (1993)). And because \( \sigma \ast \phi \) can be uniformly approximated by \( \sigma \) (Step 4 of Section 6 Leshno et al. (1993)), any convolutional neural network with the activation function \( \sigma \ast \phi \) can be uniformly approximated by the convolutional neural networks with the activation function \( \sigma \). And because CNN with the activation function \( \sigma \ast \phi \) has the universal property, CNN with the activation function \( \sigma \) also has the universal property, and it completes the entire proof.

**Remark 9** Translation equivariance is often referred to as the basis of the advantages of CNN models:

\[
f_{s}(x_{t}) = f_{s+i}(x_{t+i}).
\]

In fact, infinite-length convolution without padding is translation equivariant. However, this property contradicts the universal property because of the relation between the output vector and the input vector. Actually, as shown in the proof process, padding plays an important role. The asymmetry that starts at the boundary gradually propagates toward the center, making it possible to achieve the universal property.

**Lemma 10** For the continuous activation function \( \sigma \) and \( d = 2, 3, \) \( d \)-layered convolutional neural networks has the universal property in the continuous function space; that is, \( \Sigma_{c,\nu}^{d}(K) = C(K, \mathbb{R}^{c}) \).
The proof is almost same to Proposition 8. Divide the case into $d = 2$ and $d = 3$.

**Case 1 $d = 2$:** For the vectors $p^i_{-1} = (0, x^i_1), p^0_0 = (x^i_1, x^i_2)$, and $p^i_1 = (x^i_2, 0)$, Lemma 7 gives the following equation: for some constants $\alpha_{i,j} \in \mathbb{N}_0$ for $i \in [1, c], j \in \{-1, 0, 1\}$,

$$\prod_{i,j}(p^i_j)^{\alpha_{i,j}} \in \sigma(\Sigma_{c,1}^1).$$

(136)

For a monomial $M$ that contains at least one $x^i_1$, $(0, M)$ is the element of $\sigma(\Sigma_{c,1}^1)$; that is, for $M = x^0_1 \prod_{i=1}^c \prod_{j=1,2}(x^i_j)^{\alpha_{i,j}}$, $(0, M) \in \sigma(\Sigma_{c,1}^1)$. It is obvious from the following equation.

$$p^i_{-1} \prod_{i,j}(p^i_j)^{\alpha_{i,j}} = (0, M) \in \sigma(\Sigma_{c,1}^1).$$

(137)

Then, for $C(x) = U_{-1}x$, $C((0, M)) = (M, 0) \in \Sigma_{c,1}^2$, and for $C(x) = U_0x$, $C((0, M)) = (0, M) \in \Sigma_{c,1}^2$. By symmetric process, for a monomial $M$ that contains at least one $x^i_1$, $(M, 0), (0, M) \in \Sigma_{c,1}^2$. Now remaining is to prove that the constant functions $e_1$ and $e_2$ are in $\Sigma_{c,1}^2$. Because $(1, 1) \in \sigma(\Sigma_{c,1}^1)$, $U_{-1}((1, 1)) = (1, 0) = e_1 \in \Sigma_{c,1}^2$, and $U_1((1, 1)) = (0, 1) = e_2 \in \Sigma_{c,1}^2$. It completes the proof for the case $d = 2$.

**Case 2 $d = 3$:** In the proof for Proposition 8, the following relation holds:

$$e_i \in \Sigma_{c,1}^2,$$

(138)

for $i \in [1, n-1] \cup [d] = \{1, 3\}$. Because $p^i_j = U_{-j}x^i \in \Sigma_{c,1}^2$ for $j \in [-2, 2]$, by Lemma 7, we have

$$\prod_{i=1}^c \prod_{j=-2}^2 (p^i_j)^{\alpha_{i,j}} \in \sigma(\Sigma_{c,1}^2),$$

(139)

and

$$e_t \odot \left( \prod_{i=1}^c \prod_{j=-2}^2 (p^i_j)^{\alpha_{i,j}} \right) \in \sigma(\Sigma_{c,1}^2),$$

(140)

for $\alpha_{i,j} \in \mathbb{N}_0$ and $t \in \{1, 3\}$ Because $pr_2 = I_3 - pr_1 - pr_3$, above equation is also satisfied for $t = 2$.

For an arbitrary monomial $M = \prod_{i=1}^c \prod_{j=1}^3(x^i_j)^{\alpha_{i,j}}$,

$$pr_2 \left( \prod_{i}^c \prod_{j=1}^3 (p^i_j)^{\alpha_{i,j}} \right) = (0, M, 0) \in \sigma(\Sigma_{c,1}^2).$$

(141)

Thus, using the convolutional layers $U_{-1}, U_0$, and $U_1$ as the last layer, $(M, 0, 0), (0, M, 0), (0, 0, M) \in \Sigma_{c,1}^3$. And it completes the proof.

Combining Lemma 10, Lemma 6, and Proposition 8 altogether, we get the following theorem:
\textbf{Theorem 11} For the continuous activation function \(\sigma\), the minimal depth \(N_d\) for convolutional neural network to have the universal property is

\[
N_d = \begin{cases} 
2 & \text{if } d = 1, 2 \\
3 & \text{else if } d = 3 \\
d - 1 & \text{else if } d \geq 4 
\end{cases}
\]

(142)

In other words, for a compact set \(K \subset \mathbb{R}^c, \overline{\Sigma_{c,c}} = C(K, \mathbb{R}^c), \) and \(\overline{\Sigma_{c,c}}^{-1} \neq C(K, \mathbb{R}^c)\).

\subsection*{4.4 The Minimum Width for the Universal Property of Convolutional Neural Network}

In this section, we prove the universal property of deep narrow convolutional neural networks. The proof process is as follows. First, construct the convolutional neural networks, which can compute arbitrary linear summation of the input in Lemma 14. Second, in Lemma 15, compose the linear summation and the activation function to get the convolutional neural network which can approximate the arbitrary continuous function using only one activation function layer. Finally, construct the deep narrow neural network that can approximate the network mentioned above.

\textbf{Lemma 12} \(S_{d-1}\) contains the following elements.

- If \(n + m \leq d - 1\), \(E_{n,m} \in S_{d-1}\).
- If \(n + m \geq d + 3\), \(E_{n,m} \in S_{d-1}\).
- If \(n + m = d + 1\), \(E_{n,m} \in S_{d-1}\).
- If \(n + m = d\), \(E_{n,m} + E_{n+1,m+1} \in S_{d-1}\).

\textbf{Proof}

- By simple operation, we can know that \(U_0 - U_1U_{-1} = E_{1,1}\). And \(U_1^{m-1}(U_0 - U_1U_{-1})U_{-1}^{m-1} = U_1^{m-1}U_{-1}^{m-1} - U_1^{m}U_{-1}^{m} = E_{n,m}\). So if \(n + m \leq d - 1\), \(E_{n,m} \in S_{d-1}\).
- Similarly, \(U_0 - U_1U_1 = E(d, d)\). And \(U_1^{m-1}(U_0 - U_1U_{-1})U_{1}^{m-1} = U_1^{m-1}U_{1}^{m-1} - U_1^{m}U_{1}^{m} = E(d - n + 1, d - m + 1)\). So if \((d - n + 1) + (d - m + 1) \geq d + 3\), then \(n + m \leq d - 1\), and thus \(E_{d-n+1,d-m+1} \in S_{d-1}\).
- Divide the case into two cases again. First, consider the case of \(n \geq m\). Then, We can easily observe that \((U_1)^{n-m} = \sum_{i=-n+1}^{d-m} E_{n+i,m+i}\). Because \(E_{n+i,m+i} \in S_{d-1}\) for all \(i < 0\), \((n+i) + (m+i) = d+1+2i \leq d+1\) and \(i \geq 0\), \((n+i) + (m+i) = d+1+2i \geq d+3\), and \((U_1)^{n-m} \in S_{d-1}\), \(E_{n,m} = (U_1)^{n-m} - \sum_{i\neq 0} E_{n+i,m+i} \in S_{d-1}\). Similarly, if \(n < m\), \((U_1)^{m-n} = \sum_{i=-n+1}^{d-m} E_{n+i,m+i}\), and thus \(E_{n,m} = (U_1)^{m-n} - \sum_{i\neq 0} E_{n+i,m+i} \in S_{d-1}\).
- Similar to the above case, if \(n \geq m\), then \((U_1)^{n-m} = \sum_{i=-n+1}^{d-m} E_{n+i,m+i}\). \(E_{n,m} + E_{n+1,m+1} = (U_1)^{n-m} - \sum_{i\neq 0,1} E_{n+i,m+i} \in S_{d-1}\). If \(n < m\), \(E_{n,m} + E_{n+1,m+1} = (U_1)^{m-n} - \sum_{i\neq 0,1} E_{n+i,m+i} \in S_{d-1}\).
Corollary 13 For arbitrary $1 \leq n, m \leq d$, $E_{n,m} \in S_d$.

Proof Obviously, $S_{d-1} \subset S_d$. And $E_{n,m} \in S_d$, except for the cases of $n + m = d$ and $n + m = d + 2$. If $n + m = d$, $E_{n,m} = E_{n+1,m}U_1$. Because $n + 1 + m = d + 1$, $E_{n+1,m} \in S_{d-1}$, and thus $E_{n+1,m}U_1 \in S_d$. If $n + m = d + 2$, $E_{n,m} = E_{n-1,m}U_{-1}$. Because $n - 1 + m = d + 1$, $E_{n-1,m} \in S_{d-1}$, and thus $E_{n-1,m}U_{-1} \in S_d$.

Corollary 14 For arbitrary matrix $L \in \mathbb{R}^{d \times d}$, $L \in S_d$.

In the following lemma, we prove that the convolutional neural networks with only one activation function layer can approximate the arbitrary continuous function.

Lemma 15 Define the set of functions as follows. For $x = (x_1, x_2, \ldots, x_c) \in \mathbb{R}^{c \times d}$ and $x^i \in \mathbb{R}^d$,

$$T := \left\{ \sum_{j=1}^{n} a_j \sigma \left( \sum_{i=1}^{c} L_{j,i} x^i + \delta_j \right) : \mathbb{R}^{c \times d} \to \mathbb{R}^{1 \times d} \left| L_{j,i} \in \mathbb{R}^{d \times d}, \delta_j \in \mathbb{R}^d, a_j \in \mathbb{R} \right. \right\}. \quad (143)$$

Then, $\overline{T} = C(K, \mathbb{R}^{1 \times d})$ for the compact set $K \in \mathbb{R}^{c \times d}$.

Proof Let $x^i$ be $x^i = (x_1^i, x_2^i, \ldots, x_d^i) \in \mathbb{R}^d$. Define the arbitrary monomial of $x_j^i$ as follows:

$$M = \prod_{i=1}^{c} \prod_{j=1}^{d} (x_j^i)^{\alpha_{i,j}}, \quad (144)$$

for some degrees $\alpha_{i,j} \in \mathbb{R}$. We will show that for $k \in [1, d]$,

$$Me_k = (0, 0, \ldots, 0, M, 0, \ldots, 0) \in \overline{T}. \quad (145)$$

Then, it is sufficient by Stone–Weierstrass theorem (De Branges, 1959). As in Lemma 1, $\overline{T}$, the closure of $T$, is a vector space and is closed under partial differentiation with respect to the parameters. For $\delta = (\delta_1, \delta_2, \ldots, \delta_d)$ and $b_{i,t} \in \mathbb{R}$,

$$\sigma \circ f(x) = \sigma \left( \sum_{i=1}^{c} b_{i,j} E_{k,j} x^i + \delta \right) \in \overline{T}. \quad (146)$$

Then, partial differentiation with respect to $\delta_j$ and $b_{i,t}$ gives the following equation.

$$\left( \frac{\partial}{\partial \delta_j} \prod_{i=1}^{c} \prod_{j=1}^{d} \left( \frac{\partial}{\partial b_{i,j}} \right)^{\alpha_{i,j}} \right) \sigma(f) = \left( \prod_{i=1}^{c} \prod_{j=1}^{d} (x_j^i)^{\alpha_{i,j}} \right) e_k \circ \sigma^{(n)}(f), \quad (147)$$

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where \( n = \sum_{i=1}^{c} \sum_{j=1}^{d} \alpha_{i,j} + 1 \). Then, with \( \delta_j \) such that \( \sigma^{(n)}(\delta_j) \neq 0 \) and \( b_{i,j} = 0 \), we get,

\[
M e_k \in T .
\]

Therefore, all polynomials are in \( T \), and by Stone–Weierstrass theorem, \( T = C(K, \mathbb{R}^{1 \times d}) \).

We demonstrate the universal property of the deep narrow convolutional neural network in the next theorem.

**Theorem 16** Any function \( f : \mathbb{R}^{c_x \times d} \to \mathbb{R}^{c_y \times d} \) can be approximated by convolutional neural networks with at most \( c_x + c_y + 2 \) channels; for any \( \epsilon > 0 \), there exists convolutional neural network \( g \) with \( c_x + c_y + 2 \) channels such that,

\[
\| f - g \|_{C^\infty(K)} < \epsilon .
\]

**Proof** First, consider the function \( f \) with \( c \) input channels and one output channel:

\[
f : \mathbb{R}^{c \times d} \to \mathbb{R}^{1 \times d}.
\]

We denote the input as \( x \) and each channel of input as \( x = (x^1, x^2, \ldots, x^c) \). By Lemma 15, there exist \( g : \mathbb{R}^{c \times d} \to \mathbb{R}^{1 \times d} \) such that defined as follows:

\[
g(x) := \sum_{j=1}^{n} a_j \sigma \left( \sum_{i=1}^{c} L_{j,i} x^i + \delta_j \right),
\]

which can approximate \( f \) with an arbitrarily small error. Now construct the convolutional neural network with channel size \( c + 3 \) which approximates \( g \). By Lemma 13, for arbitrary \( L_{j,i} \in \mathbb{R}^{d \times d} \), there exists \( C_{i,j}^{k,l} \in T_{ol}(1) \) such that

\[
L_{j,i} = \sum_{l=1}^{m_{i,j}} \prod_{k=1}^{d} C_{i,j}^{k,l}.
\]

Also, there exist \( \tilde{C}_{j}^{k,l} \in T_{ol}(1) \) such that

\[
\delta_j = \sum_{l=1}^{\tilde{m}_{j}} \prod_{k=1}^{d} \tilde{C}_{j}^{k,l} 1_d.
\]

Then, \( g \) becomes

\[
g(x) = \sum_{j=1}^{n} a_j \sigma \left( \sum_{i=1}^{c} L_{j,i} x^i + \delta_j x^i \right) = \sum_{j=1}^{n} a_j \sigma \left( \sum_{i=1}^{c} \sum_{l=1}^{m_{i,j}} \prod_{k=1}^{d} C_{i,j}^{k,l} x^i + \sum_{l=1}^{\tilde{m}_{j}} \prod_{k=1}^{d} \tilde{C}_{j}^{k,l} 1_d \right).
\]

Then we define the convolutional neural network with \( c + 3 \) channels that calculate the aforementioned equation. By Lemma 2, if we can approximate the function with the convolutional neural network with the partial activation function, we can approximate the function with the original convolutional neural network. Therefore, we can preserve \( c \) channels from the input and process the \((c+1)\)-th, \((c+2)\)-th, and \((c+3)\)-th channels. We get the desired output according to the following process of function compositions.
1. Repeat the following for \( j = 1, 2, \ldots, n \).

2. Calculate \( \sigma \left( \sum_{i=1}^{c} \sum_{l=1}^{m_{i,j}} \prod_{k=1}^{d} C_{i,j}^{k,l} x^i + \delta_j \right) \) in the \((c+2)\)-th channel, not using the \((c+3)\)-th channel.

2.1. Repeat the following for \( i = 1, 2, \ldots, c \) and \( l = 1, 2, \ldots, m_{i,j} \).

2.2. Calculate \( \prod_{k=1}^{d} C_{i,j}^{k,l} x^i \) in the \((c+1)\)-th channel, not using the \((c+2)\)-th and the \((c+3)\)-th channels.

2.2.1. Copy \( x^i \) from the \( i \)-th channel to the \((c+1)\)-th channel.

2.2.2. Conduct convolution with kernel \( C_{i,j}^{k,l} \) and the bias 0 on the \((c+1)\)-th channel for \( k = 1, 2, \ldots, d \).

2.3. Add \( \prod_{k=1}^{d} C_{i,j}^{k,l} x^i \) to the \((c+2)\)-th channel and set the \((c+1)\)-th channel to 0.

2.4. Add \( \delta_j = \sum_{l=1}^{\bar{m}_j} \prod_{k=1}^{d} \tilde{C}_{\bar{j}}^{k,l} 1_d \) to the \((c+2)\)-th channel.

2.4.1. Repeat the following for \( l = 1, 2, \ldots, \bar{m}_j \).

2.4.2. Conduct the convolution with kernel \((0, 0, 0)\) and the bias 1 on the \((c+1)\)-th channel and get \( 1_d \) on the \((c+1)\)-th channel.

2.4.3. Conduct the convolution with kernel \( \tilde{C}_{\bar{j}}^{k,l} \) and the bias 0 on the \((c+1)\)-th channel for \( k = 1, 2, \ldots, d \) and get \( \prod_{k=1}^{d} \tilde{C}_{\bar{j}}^{k,l} 1_d \) in the \((c+1)\)-th channel.

2.4.4. Add \( \prod_{k=1}^{d} \tilde{C}_{\bar{j}}^{k,l} 1_d \) to the \((c+2)\)-th channel and set the \((c+1)\)-th channel to 0.

2.5. Apply the activation function on the \((c+2)\)-th channel and get \( \sigma \left( \sum_{i=1}^{c} \sum_{l=1}^{m_{i,j}} \prod_{k=1}^{d} C_{i,j}^{k,l} x^i + \delta_j \right) \) in the \((c+2)\)-th channel.

3. Add \( \sigma \left( \sum_{i=1}^{c} \sum_{l=1}^{m_{i,j}} \prod_{k=1}^{d} C_{i,j}^{k,l} x^i + \delta_j \right) \) to the \((c+3)\)-th channel and set the \((c+2)\)-th channel to 0.

4. Get \( \sum_{j=1}^{n} a_j \sigma \left( \sum_{i=1}^{c} \sum_{l=1}^{m_{i,j}} \prod_{k=1}^{d} C_{i,j}^{k,l} x^i + \delta_j \right) \) in the \((c+3)\)-th channel.

5. Set the final convolutional layer with one output channel, which takes the value from the \((c+3)\)-th channel.

In this process, the \((c+1)\)-th channel is used to calculate the product \( \prod_{k=1}^{d} C_{i,j}^{k,l} \). And the \((c+2)\)-th channel is used to accumulate the summation \( \sum_{i=1}^{c} \sum_{l=1}^{m_{i,j}} \prod_{k=1}^{d} C_{i,j}^{k,l} x^i \) calculated in the \((c+1)\)-th channel. The \((c+3)\)-th channel is used to accumulate the final summation \( \sum_{j=1}^{n} a_j \sigma \left( \sum_{i=1}^{c} \sum_{l=1}^{m_{i,j}} \prod_{k=1}^{d} C_{i,j}^{k,l} x^i + \delta_j \right) \) after the activation function is applied to the \((c+2)\)-th channel. For the general case, when the output channel size is \( c_y \), we can repeat the above process while preserving the output components already processed, and using \( c_x + c_y + 2 \) channels is enough to generate \( c_y \) output vectors. It completes the proof.
5. Conclusion

In this paper, we deal with the universal property of convolutional neural networks with both limited depth and unlimited width and with limited width and unlimited depth. Although we have only dealt with the universal property of three-kernel convolutions, we expect that the same idea can be simply generalized to networks of other kernel sizes. We think that convolution using striding and dilation and the convolutional layer mixed with pooling are also interesting research topics for the universal property of convolutional neural networks. We hope that our research will serve as a basis for active research in this field.
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