Exact Pointer Properties for Quantum System Projector Measurements with Application to Weak Measurements and Their Accuracy

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Abstract

Exact pointer states are obtained for projection operator measurements performed upon pre-selected (PS) and upon pre- and post-selected (PPS) quantum systems. These states are used to provide simple exact expressions for both the pointer spatial probability distribution profiles and the mean values of arbitrary pointer observables associated with PS and PPS projection operator measurements that are valid for any strength of the interaction which couples a measurement pointer to the quantum system. These profiles and mean values are compared in order to identify the effects of post-selection upon projector measurement pointers. As a special case, these mean value results are applied to the weak measurement regime - yielding PS and PPS mean value expressions which are valid for any operator (projector or non-projector). Measurement sensitivities which are useful for estimating weak measurement accuracies for PS and PPS systems are also obtained and discussed.
I. INTRODUCTION

The pointer of a measurement apparatus is fundamental to quantum measurement theory because the values of measured observables are determined from its properties (e.g. from the mean values for the pointer position and momentum operators $\hat{q}$ and $\hat{p}$, respectively). Understanding these properties has become more important in recent years - in large part due to the increased interest in the theory of weak measurements of pre- and post-selected (PPS) quantum systems and weak value theory. Because of this growing interest the practical value of estimating the associated measurement sensitivities has also become important from both the experimental and device engineering perspectives.

The use of PPS techniques for controlling and manipulating quantum systems was introduced by Schrödinger more than 75 years ago [1, 2]. Since then PPS techniques have found utility in such diverse areas of study as quantum system-environment interactions, e.g. [3]; the quantum eraser, e.g. [4]; and Pancharatnam phase, e.g. [5, 6]. One especially fertile area of application of PPS theory is the time symmetric reformulation of quantum mechanics developed by Aharonov, Bergman and Lebowitz [7] and the closely related notion of the weak value of a quantum mechanical observable, e.g. [8–10].

The weak value $A_w$ of a quantum mechanical observable $A$ is the statistical result of a standard measurement procedure performed upon a PPS ensemble of quantum systems when the interaction between the measurement apparatus and each system is sufficiently weak, i.e. when it is a weak measurement. Unlike a standard strong measurement of $A$ performed upon a prepared, i.e. pre-selected (PS), ensemble which significantly disturbs the measured system and yields the mean value of the associated operator $\hat{A}$ as the measured value of $A$, a weak measurement of $A$ performed upon a PPS system does not appreciably disturb the quantum system and yields $A_w$ as the observable’s measured value. The peculiar nature of the virtually undisturbed quantum reality that exists between the boundaries defined by the PPS states is revealed by the eccentric characteristics of $A_w$, namely that $A_w$ can be complex valued and that $\text{Re} A_w$ can lie far outside the eigenvalue spectral limits of $\hat{A}$. While the interpretation of weak values remains somewhat controversial, experiments have verified several of the interesting unusual properties predicted by weak value theory [11–15]. The theory has also been applied to the theoretical and experimental resolution of such quantum paradoxes as “the quantum box problem” [16–18] and “Hardy’s Paradox” [19–23].
As is well known, projection operators are an important part of the general mathematical formalism of quantum mechanics. There has been a recent increased interest in these operators because the measurement and interpretation of their weak values have played a central role in the theoretical and experimental resolutions of the quantum box problem and Hardy’s paradox. More recently projector weak values have also been exploited in experimental observations of dynamical non-locality induced effects [24, 25].

Projection operators are also interesting because their idempotent property can be used to provide simple exact descriptions of pointers resulting from projection operator measurements. Specifically, when an instantaneous measurement is performed upon a quantum system to determine the value of a projection operator \( \hat{A} \), the associated von Neumann measurement interaction operator \( e^{-i/\hbar \gamma \hat{A}} \) is easily shown (using the series expansion of the measurement interaction operator and the generalized idempotent property \( \hat{A}^n = \hat{A}, n \geq 1 \)) to be given exactly by

\[
e^{-i/\hbar \gamma \hat{A}} = \hat{1} - \hat{A} + \hat{A} \hat{S}.
\]

Here \( \gamma \) is the measurement interaction strength and \( \hat{S} \equiv e^{-i/\hbar \gamma \hat{p}} \) is the pointer position translation operator defined by its action \( \langle q | \hat{S} | \phi \rangle \equiv \phi (q - \gamma) \) upon the pointer state \( | \phi \rangle \) (it is assumed hereinafter that the commutator \( [\hat{A}, \hat{S}] = 0 \)).

The primary objectives of this paper are to: (i) use eq. (1) to provide simple exact expressions for pointer spatial probability distribution profiles and for the mean values of arbitrary pointer observables when projection operator measurements of any interaction strength are performed upon PS and upon PPS quantum systems (consequently, these results describe - without approximation - the properties of such pointers over the entire range of measurement interaction strengths - i.e. from the weak measurement regime to the strong measurement regime); (ii) identify some of the effects induced upon the pointer via state post-selection by comparing these PS and PPS pointer results; (iii) use these exact pointer results to obtain pointer mean values for PS and PPS systems which are valid for any operator (i.e., projector or non-projector) in the weak measurement regime, i.e. when \( 0 < \gamma \ll 1 \); and (iv) obtain measurement sensitivity expressions which are useful for estimating the accuracy of weak PS and PPS measurements of any such operator. It is important to note that when direct comparisons between quantities are made it is assumed that the interaction strength is fixed.

This paper is organized as follows: the next section is a development, comparison, and
discussion of the exact theories for pointers resulting from projector operator measurements performed upon PS and PPS systems. Section III is devoted to obtaining from the exact mean values approximate pointer observable mean values that are valid for any operator in the weak measurement regime - as well as to developing measurement sensitivities associated with the weak measurement of any PS and PPS system observable. Concluding remarks comprise the paper’s final section.

II. EXACT POINTER THEORIES FOR PROJECTOR MEASUREMENTS

A. PS Systems

Consider the measurement at time $t$ of a time independent projector $\hat{A}$ performed upon a quantum system prepared in the normalized PS state $|\psi\rangle$. Let the pointer of the measuring apparatus be intitally in the pre-measurement state $|\phi\rangle$. Then - from eq.(1) - the exact normalized state $|\Phi\rangle$ of the pointer immediately after the measurement is

$$|\Phi\rangle = e^{-\frac{i}{\hbar} \gamma \hat{A}} |\psi\rangle |\phi\rangle = (\hat{1} - \hat{A} + \hat{A}\hat{S}) |\psi\rangle |\phi\rangle$$

(2)

(3)

which is simply the weighted sum of the distribution profiles for the pre-measurement state $|\phi\rangle$ and $\hat{S} |\phi\rangle$ - the pre-measurement state translated by $\gamma$. Observe that the idempotency of $\hat{A}$ precludes the existence of an interference cross term proportional to $\text{Re} \langle q | \phi \rangle^* \langle q | \hat{S} | \phi \rangle$ in eq.(3) because the cross terms contain $\langle \psi| \hat{A} (\hat{1} - \hat{A}) |\psi\rangle = \langle \psi| (\hat{A} - \hat{A}^2) |\psi\rangle = \langle \psi| (\hat{A} - \hat{A}) |\psi\rangle = 0$ and $\langle \psi| (\hat{1} - \hat{A}) \hat{A} |\psi\rangle = 0$ as factors.

If $\hat{M}$ is the operator for any pointer observable $M$, then - from eq.(2) - the exact expression for the mean value of $M$ after a PS measurement is readily found to be

$$\langle \Phi | \hat{M} | \Phi \rangle = \left(1 - \langle \psi| \hat{A} |\psi\rangle \right) \langle \phi| \hat{M} |\phi\rangle + \langle \psi| \hat{A} |\psi\rangle \langle \phi| \hat{S}^\dagger \hat{M} \hat{S} |\phi\rangle$$

(4)
it is simply the weighted sum of the pre-measurement mean value of $M$ and its mean value relative to the $\gamma$ translated state $\hat{S} |\phi\rangle$. Note that $\langle\Phi| \hat{M} |\Phi\rangle = \langle\phi| \hat{M} |\phi\rangle$ when: (i) $\gamma = 0$, i.e. no measurement takes place; or (ii) $\langle\psi| \hat{A}|\psi\rangle = 0$. Also note that when $\hat{M} = \hat{p}$, then - since $[\hat{p}, \hat{S}] = 0$ -

$$\langle\phi| \hat{S}^\dagger \hat{p} \hat{S} |\phi\rangle = \langle\phi| \hat{p} |\phi\rangle$$

and (4) becomes

$$\langle\Phi| \hat{p} |\Phi\rangle = \langle\phi| \hat{p} |\phi\rangle.$$  \hspace{1cm} (5)

Thus the mean value of the pointer momentum is not changed by the projector measurement of a PS system, i.e. pointer momentum is a constant of the motion for projector measurements of PS systems (in fact - pointer momentum is a constant of the motion for both projector and non-projector measurements of PS systems since $[\hat{p}, e^{-\frac{i}{\hbar} \hat{A}}] = 0$ so that $\langle\psi| \hat{A} \hat{p} \hat{A}^\dagger e^{-\frac{i}{\hbar} \hat{A}} |\psi\rangle |\phi\rangle = \langle\phi| \hat{p} |\phi\rangle$.

The fact that no interference cross terms appear in eq.(4) is a useful feature of PS system pointers. In particular, when $\hat{M} = \hat{q}$ and $|\phi\rangle$ is such that $\langle\phi| \hat{q} |\phi\rangle = 0$ and $\langle\phi| \hat{S}^\dagger \hat{q} \hat{S} |\phi\rangle = \gamma$ - e.g., when $|\langle q| \phi\rangle|^2$ is Gaussian with 0 mean - then it is found from eq.(4) that

$$\langle\Phi| \hat{q} |\Phi\rangle = \gamma \langle\psi| \hat{A} |\psi\rangle.$$  \hspace{1cm} (6)

Thus - in this case - if $\gamma$ (or $\langle\psi| \hat{A} |\psi\rangle$) is known, then $\langle\psi| \hat{A} |\psi\rangle$ (or $\gamma$) can be determined directly from the measurement pointer’s mean position.

**B. PPS Systems**

Now suppose that a measurement of projector $\hat{A}$ is performed at time $t$ upon a PPS system. If - as above - the pre-measurement pointer state is $|\phi\rangle$, then the exact normalized pointer state $|\Psi\rangle$ immediately after the post-selection measurement is

$$|\Psi\rangle = \frac{1}{N} \left( \frac{\langle\psi_f| \psi_i\rangle}{|\langle\psi_f| \psi_i\rangle|} \right) \left( 1 - A_w + A_w \hat{S} \right) |\phi\rangle,$$  \hspace{1cm} (7)

where $|\psi_i\rangle$ and $|\psi_f\rangle$ are the normalized pre- and post-selected states at $t$, respectively; $A_w$ is the weak value of $A$ at $t$ defined by

$$A_w \equiv \frac{\langle\psi_f| \hat{A} |\psi_i\rangle}{\langle\psi_f| \psi_i\rangle}, \quad \langle\psi_f| \psi_i\rangle \neq 0;$$

and

$$N \equiv \left( |N|^2 \right)^{1/2} = \sqrt{1 - |A_w|^2 + |A_w|^2 + 2 \text{Re} \left[ A_w (1 - A_w^*) \langle\phi| \hat{S} |\phi\rangle \right]}.$$
Using the fact that \[ e^{i\chi} = \frac{\langle \psi_f | \psi_i \rangle}{|\langle \psi_f | \psi_i \rangle|}, \]
where \( \chi \) is the Pancharatnam phase, enables eq. (7) to be more compactly written as

\[ |\Psi\rangle = \frac{e^{i\chi}}{N} \left( 1 - A_w + A_w \hat{S} \right) |\phi\rangle. \quad (8) \]

The associated exact spatial probability distribution profile \( |\langle q | \Psi \rangle|^2 \) of the pointer is

\[ |\langle q | \Psi \rangle|^2 = \left( \frac{1}{N^2} \right) \left\{ |1 - A_w|^2 |\langle q | \phi \rangle|^2 + |A_w|^2 |\langle q | \hat{S} \phi \rangle|^2 + \right. \]
\[ \left. 2 \text{Re} \left[ A_w (1 - A_w^*) \langle q \phi \rangle^* \langle q | \hat{S} \phi \rangle \right] \right\}. \quad (9) \]

The effect of post-selection upon pointer states can be seen by comparing eq. (8) with eq. (2). Even though the measurements are generally not weak measurements, it is interesting that - unlike projector measurement pointer states for PS systems which depend upon \( \hat{A} |\psi\rangle \) - projector measurement pointer states for PPS systems explicitly depend upon the projector’s weak value \( A_w \). This - perhaps - is not surprising in light of the recent discussion in \[26\] concerning von Neumann measurements and the associated ubiquitous nature of weak values. It is also apparent from this comparison that state post-selection is responsible for the presence of the Pancharatnam phase factor \( e^{i\chi} \) in PPS pointer states. This is an expected natural consequence of state post-selection \[27–29\].

As is the case for PS measurements, the pointer state distribution profiles for PPS measurements are also weighted sums of the distribution profiles for \( |\phi\rangle \) and \( \hat{S} |\phi\rangle \). However - unlike the PS case - the profile for PPS measurements contains interference cross terms induced by state post-selection. Interference occurs because post-selection nullifies the idempotency of \( \hat{A} \) by replacing \( \hat{A} |\psi\rangle \) with \( A_w \) - thereby allowing the cross terms to occur. More specifically - unlike a PS measurement where the cross terms contain the vanishing \( \langle \psi | \hat{A} (1 - \hat{A}) |\psi\rangle \) and \( \langle \psi | (1 - \hat{A}) \hat{A} |\psi\rangle \) factors - the cross terms for PPS measurements contain \( A_w (1 - A_w^*) \) and its complex conjugate as non-vanishing factors.

As before, let \( \hat{M} \) be the operator for an arbitrary pointer observable \( M \). Using eq. (8) it is found that the exact expression for the mean value of \( M \) after a PPS measurement has the form anticipated from that of eq. (9) :

\[ \langle \Psi | \hat{M} |\Psi\rangle = \left( \frac{1}{N^2} \right) \left\{ |1 - A_w|^2 \langle \phi | \hat{M} |\phi\rangle + |A_w|^2 \langle \phi | \hat{S}^\dagger \hat{M} \hat{S} |\phi\rangle + \right. \]
\[ \left. 2 \text{Re} \left[ A_w (1 - A_w^*) \langle \phi | \hat{M} \hat{S} |\phi\rangle \right] \right\}. \quad (10) \]
Note that - similar to the PS case - for PPS measurements \( \langle \Psi | \hat{M} | \Psi \rangle = \langle \phi | \hat{M} | \phi \rangle \) when: (i) \( \gamma = 0 \); or (ii) \( A_w = 0 \). Also observe from eq.\( (10) \) that when \( \hat{M} = \hat{p} \) and \( A_w \neq 0 \neq \gamma \), then

\[
\langle \Psi | \hat{p} | \Psi \rangle = \left( \frac{1}{N^2} \right) \left\{ (1 - |A_w|^2 + |A_w|^2) \langle \phi | \hat{p} \phi \rangle + \frac{2 \text{Re} \left[ A_w (1 - A_w^*) \langle \phi | \hat{p} \hat{S} \phi \rangle \right]}{\gamma^2} \right\}
\]

so that - unlike PS systems - measurements of PPS systems generally do change the mean value of \( p \). Thus, pointer momentum is not a constant of the motion for projector measurements of PPS systems (this - in fact - is also the case for non-projector measurements of PPS systems).

Because of the interference term in eq.\( (10) \), pointer positions for PPS systems are not as straightforwardly useful as those for PS systems for measuring \( \hat{A} \) or \( \gamma \). However, for the special case \( \hat{M} = \hat{q} \), \( A_w = 1 \) and \( |\phi\rangle \) is such that \( \langle \phi | \hat{S}^\dagger \hat{q} \hat{S} | \phi \rangle = \gamma \) - e.g., when \( |\langle q | \phi \rangle|^2 \) is Gaussian with 0 mean - the pointer position can be used to determine \( \gamma \) since - from eq.\( (10) \)

\[
\langle \Psi | \hat{q} | \Psi \rangle = \gamma.
\]

This is clearly the PPS analogue of eq.\( (6) \) when \( \langle \psi | \hat{A} | \psi \rangle = 1 \).

### III. THE WEAK MEASUREMENT REGIME

In this section, the exact results given by eqs.\( (4) \) and \( (10) \) of the last section are used to obtain approximate results for these quantities that are valid for PS and PPS systems when the measurements are weak. In this case \( 0 < \gamma \ll 1 \), so that to 1\(^{\text{st}}\) order in \( \gamma \)

\[
\hat{S} \simeq \widehat{1} - \frac{i}{\hbar} \gamma \hat{p} \quad \text{and} \quad \widehat{S}^\dagger \simeq \widehat{1} + \frac{i}{\hbar} \gamma \hat{p}.
\]

Observe that when this approximation to \( \hat{S} \) is applied to eq.\( (11) \), then \( \widehat{1} - \hat{A} + \hat{A} \hat{S} \simeq \widehat{1} - \frac{i}{\hbar} \gamma \hat{A} \hat{p} \) is the 1\(^{\text{st}}\) order approximation to the von Neumann measurement interaction operator. Consequently, the idempotency of \( \hat{A} \) is not relevant to this approximation and - since weak measurement results are expressed only through 1\(^{\text{st}}\) order in \( \gamma \) - all of the following results associated with measurements of PS and PPS systems in the weak measurement regime apply for any operator \( \hat{A} \) (projector or non-projector). In what follows, it is assumed that the weakness conditions (inequalities \((3.5)\) in \[12\]) are satisfied for PPS systems.
A. PS Systems

Application of approximations (11) to eq.(4) yields
\[
\langle \Phi | \hat{M} | \Phi \rangle \simeq \langle \phi | \hat{M} | \phi \rangle - \frac{i}{\hbar} \gamma \langle \psi | \hat{A} | \psi \rangle \langle \phi | \left[ \hat{M}, \hat{p} \right] | \phi \rangle
\]
as the approximate mean value of the pointer observable \( M \) obtained from the weak measurement of any observable \( A \) when the PS system is in the prepared state \( |\psi\rangle \) at the time of measurement. It is seen from eq.(12) that when \( \hat{M} = \hat{p} \), then since \( [\hat{p}, \hat{p}] = 0 \) eq.(12) is unchanged by a weak measurement process. As required, this is in complete agreement with eq.(5). Also, when \( \hat{M} = \hat{q} \), then - since \( [\hat{q}, \hat{p}] = i\hbar \) eq.(12) becomes
\[
\langle \Phi | \hat{q} | \Phi \rangle \simeq \langle \phi | \hat{q} | \phi \rangle + \gamma \langle \psi | \hat{A} | \psi \rangle
\]
which is in agreement with eq.(6) whenever \( \langle \phi | \hat{q} | \phi \rangle = 0 \).

In general, the calculus of error propagation provides the measurement sensitivity
\[
\delta^2 \langle \psi | \hat{A} | \psi \rangle \equiv \left[ \frac{\Delta^2_{\phi} M}{\partial \langle \Phi | \hat{M} | \Phi \rangle / \partial \langle \psi | \hat{A} | \psi \rangle} \right]^2
\]
as an estimate of the accuracy associated with the determination of \( \langle \psi | \hat{A} | \psi \rangle \) from the measurement of the mean value of pointer observable \( M \). Here
\[
\Delta^2_{\phi} M = \langle \Phi | \hat{M}^2 | \Phi \rangle - \langle \Phi | \hat{M} | \Phi \rangle^2
\]
is the measurement variance of \( M \) relative to the final pointer state \( |\Phi\rangle \). In the weak measurement regime this variance - upon application of eq.(12) - becomes
\[
\Delta^2_{\phi} M \simeq \Delta^2_{\phi} M - \frac{i}{\hbar} \gamma B \left( \hat{M}, \hat{p} \right) \langle \psi | \hat{A} | \psi \rangle,
\]
where
\[
B \left( \hat{M}, \hat{p} \right) \equiv \langle \phi | \left[ \hat{M}^2, \hat{p} \right] | \phi \rangle - 2 \langle \phi | \hat{M} | \phi \rangle \langle \phi | \left[ \hat{M}, \hat{p} \right] | \phi \rangle
\]
and
\[
\frac{\partial \langle \Phi | \hat{M} | \Phi \rangle}{\partial \langle \psi | \hat{A} | \psi \rangle} \simeq - \frac{i}{\hbar} \gamma \langle \phi | \left[ \hat{M}, \hat{p} \right] | \phi \rangle.
\]
Using these results in eq.(13) yields
\[
\delta^2 \langle \psi | \hat{A} | \psi \rangle \simeq \frac{\Delta^2_{\phi} M - \frac{i}{\hbar} \gamma B \left( \hat{M}, \hat{p} \right) \langle \psi | \hat{A} | \psi \rangle}{\left[ \frac{i}{\hbar} \gamma \langle \phi | \left[ \hat{M}, \hat{p} \right] | \phi \rangle \right]^2}, \quad 0 < \gamma \ll 1, \quad \hat{M} \neq \hat{p}, \quad (14)
\]
as the desired sensitivity approximation.

As a useful special case consider the measurement sensitivity when \( \hat{M} = \hat{q} \). Since \([\hat{q}, \hat{p}] = i\hbar\), then \( B(\hat{q}, \hat{p}) = 0 \) and the square root of the last equation becomes

\[
\delta \langle \psi | \hat{A} | \psi \rangle \simeq \frac{\Delta q}{\gamma}, \quad 0 < \gamma \ll 1.
\]

Recall from the discussion above that if \( \langle \psi | \hat{A} | \psi \rangle \) is known and \( \langle \phi | \hat{q} | \phi \rangle = 0 \), then \( \gamma \) can also be determined from the measurement of \( \langle \Phi | \hat{q} | \Phi \rangle \). The sensitivity \( \delta \gamma \) associated with this measurement is

\[
\delta \gamma \simeq \frac{\Delta q}{\langle \psi | \hat{A} | \psi \rangle}, \quad 0 < \gamma \ll 1,
\]

which follows from the application of

\[
\frac{\partial \langle \Phi | \hat{q} | \Phi \rangle}{\partial \gamma} \simeq \langle \psi | \hat{A} | \psi \rangle
\]

to the appropriate analogue of eq. (13). These intuitively pleasing results clearly show the accuracy trade-offs associated with weak measurements of PS systems when they are used to obtain \( \langle \psi | \hat{A} | \psi \rangle \) or \( \gamma \) from measurements of the mean pointer position. In particular, the accuracy of the determination of \( \langle \psi | \hat{A} | \psi \rangle (\gamma) \) from a measurement of \( q \) can be arbitrarily increased only when \( \Delta q \) can be made arbitrarily small relative to \( \gamma (\langle \psi | \hat{A} | \psi \rangle) \).

**B. PPS Systems**

The weak measurement approximation for eq. (10) is given by

\[
\langle \Psi | \hat{M} | \Psi \rangle \simeq \langle \phi | \hat{M} | \phi \rangle - \frac{i}{\hbar} \gamma A^{*} \langle \phi | \{ \hat{M}, \hat{p} \} | \phi \rangle + 2 \frac{\gamma}{\hbar} \text{Im} A \cdot ccv(\hat{M}, \hat{p}),
\]

where \( ccv(\hat{M}, \hat{p}) \) is the ”complex covariance” of \( \hat{M} \) and \( \hat{p} \) relative to the initial pointer state \( | \phi \rangle \) defined as

\[
ccv(\hat{M}, \hat{p}) \equiv \langle \phi | \hat{M} \hat{p} | \phi \rangle - \langle \phi | \hat{M} | \phi \rangle \langle \phi | \hat{p} | \phi \rangle.
\]

Since this quantity is related to the associated real valued covariance \( cov(\hat{M}, \hat{p}) \) defined by

\[
cov(\hat{M}, \hat{p}) \equiv \frac{1}{2} \langle \phi | \{ \hat{M}, \hat{p} \} | \phi \rangle - \langle \phi | \hat{M} | \phi \rangle \langle \phi | \hat{p} | \phi \rangle
\]

according to

\[
2cov(\hat{M}, \hat{p}) = ccv(\hat{M}, \hat{p}) + ccv(\hat{p}, \hat{M}),
\]

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then the ”complex covariance” measures - in some sense - the strength of the ”correlation” between \( \hat{M} \) and \( \hat{p} \) relative to \( |\phi\rangle \). Here \( \{\hat{M}, \hat{p}\} \equiv \hat{M}\hat{p} + \hat{p}\hat{M} \) is the anti-commutator of \( \hat{M} \) and \( \hat{p} \).

Although written in a somewhat different form, it should be noted that eq.(16) agrees precisely with that of eq.(17) in [31]. Consequently, the results also agree for the special cases \( \hat{M} = \hat{p} \) and \( \hat{M} = \hat{q} \). In particular, since \( ccv(\hat{p}, \hat{p}) = \Delta_{\phi}^2 p \) and \( [\hat{p}, \hat{p}] = 0 \), then - in complete agreement with [31] -

\[
\langle \Psi | \hat{p} | \Psi \rangle \simeq \langle \phi | \hat{p} | \phi \rangle + 2\frac{\gamma}{\hbar} \text{Im} A_w \cdot \Delta_{\phi}^2 p.
\] (17)

Also, it follows that for a pointer of mass \( m \)

\[
2ccv(\hat{q}, \hat{p}) = [\hat{q}, \hat{p}] + m \frac{d}{dt} (\Delta_{\phi}^2 q).
\] (18)

Using this and \( [\hat{q}, \hat{p}] = i\hbar \) in eq.(16) gives

\[
\langle \Psi | \hat{q} | \Psi \rangle \simeq \langle \phi | \hat{q} | \phi \rangle + \gamma \text{Re} A_w + m \frac{\gamma}{\hbar} \text{Im} A_w \cdot \frac{d}{dt} (\Delta_{\phi}^2 q)
\] (19)

which is also in complete agreement with [31].

The compact form of eq.(16) is convenient for identifying the effect of post-selection upon the mean value of \( M \) when the measurements are weak. Comparison with eq.(12) reveals that - in addition to replacing \( \langle \psi | \hat{A} | \psi \rangle \) with \( A_w^\ast \) in the second term - post-selection also induces the peculiar third term containing the ”complex covariance” factor that ”correlates” the observable \( M \) with the pointer momentum \( p \) - even when \( [\hat{M}, \hat{p}] = 0 \). Since this term depends upon \( \text{Im} A_w \), this ”correlation” only exists when \( \text{Im} A_w \neq 0 \). Thus, if \( \text{Im} A_w \neq 0 \), this ”correlation” is manifested as: (i) pointer momentum variance when \( \hat{M} = \hat{p} \); and (ii) the sum of the quantum dynamical term \( [\hat{q}, \hat{p}] \) and the rate of change of pointer position variance just prior to measurement time (for massive pointers) when \( \hat{M} = \hat{q} \). This is consistent with the observation in [31] that the second term in eq.(17) is ”an artifact of post-selection rather than a quantum dynamical effect” whereas the translation of the mean pointer position in eq.(19) is quantum dynamical in nature. Note from eq.(18) - however - that post-selection ”correlates” \( q \) and \( p \) through the quantum dynamical term \( [\hat{q}, \hat{p}] \) in \( ccv(\hat{q}, \hat{p}) \) even when \( \frac{d}{dt} (\Delta_{\phi}^2 q) = 0 \). Clearly, if \( A_w \) is real valued, then the third ”complex covariance” term in eq.(16) vanishes and eqs.(12) and (16) share a similar two term form.
The measurement sensitivity $\delta \text{Re} A_w$ follows from the ratio

$$
\delta^2 \text{Re} A_w \equiv \frac{\Delta^2 \text{Re} A_w}{\left| \frac{\partial}{\partial \text{Re} A_w} \right|^2}
$$

where - from eq.(16) -

$$
\Delta^2 \text{Re} A_w \simeq \Delta^2 \text{Re} A_w - \frac{i}{\hbar} \gamma B \left( \hat{M}, \hat{p} \right) \text{Re} A_w + \frac{\gamma}{\hbar} \left[ C \left( \hat{M}, \hat{p} \right) - 2 \langle \phi | \hat{p} | \phi \rangle \left( \Delta^2 \text{Re} A_w - \langle \phi | \hat{M} | \phi \rangle^2 \right) \right] \text{Im} A_w,
$$

with

$$
C \left( \hat{M}, \hat{p} \right) \equiv \langle \phi | \left\{ \hat{M}^2, \hat{p} \right\} | \phi \rangle - 2 \langle \phi | \hat{M} | \phi \rangle \langle \phi | \left\{ \hat{M}, \hat{p} \right\} | \phi \rangle
$$

and

$$
\frac{\partial \langle \Psi | \hat{M} | \Psi \rangle}{\partial \text{Re} A_w} \simeq - \frac{i}{\hbar} \gamma \langle \phi | \left[ \hat{M}, \hat{p} \right] | \phi \rangle.
$$

It is interesting to note that $C \left( \hat{M}, \hat{p} \right) \rightarrow B \left( \hat{M}, \hat{p} \right)$ when the anti-commutators in $C \left( \hat{M}, \hat{p} \right)$ are replaced with commutators.

Comparison of $\delta^2 \text{Re} A_w$ with eq.(14) shows that if $A_w$ is real valued, then the accuracy does not depend upon $C \left( \hat{M}, \hat{p} \right)$ and the difference $\left( \Delta^2 \text{Re} A_w - \langle \phi | \hat{M} | \phi \rangle^2 \right)$. Thus, in this case - except for the $\langle \psi | \hat{A} | \psi \rangle$ and $\text{Re} A_w$ factors - $\delta^2 \langle \psi | \hat{A} | \psi \rangle$ and $\delta^2 \text{Re} A_w$ have the same form (in fact, if $| \psi_i \rangle = | \psi_f \rangle = | \psi \rangle$, then $\delta \langle \psi | \hat{A} | \psi \rangle = \delta \text{Re} A_w$). It is also clear from this that if $A_w$ is complex valued, then $\text{Im} A_w$ effects the accuracy of $\text{Re} A_w$ when it is determined from a measurement of the mean value of $M$.

When $\hat{M} = \hat{q}$ it has already been noted elsewhere that $\text{Im} A_w$ has an impact upon the pointer spatial distribution profile and consequently can change the size of the associated PPS ensemble. In addition to this - the above results also show the effect of $\text{Im} A_w$ upon the accuracy associated with mean pointer position measurements of PPS systems. Specifically, when $\hat{M} = \hat{q}$, then $B \left( \hat{q}, \hat{p} \right) = 0$ and $C \left( \hat{q}, \hat{p} \right) \neq 0$ so that

$$
\delta^2 \text{Re} A_w \simeq \delta^2 \langle \psi | \hat{A} | \psi \rangle + \left( \frac{1}{\gamma \hbar} \right) \left[ C \left( \hat{q}, \hat{p} \right) - 2 \langle \phi | \hat{p} | \phi \rangle \left( \Delta^2 \phi - \langle \phi | \hat{q} | \phi \rangle^2 \right) \right] \text{Im} A_w, \quad 0 < \gamma \ll 1.
$$

Thus, the effect of a complex valued $A_w$ is to increase (decrease) the accuracy of $\text{Re} A_w$ if it is determined from a measurement of the mean value of $q$ whenever

$$
\left[ C \left( \hat{q}, \hat{p} \right) - 2 \langle \phi | \hat{p} | \phi \rangle \left( \Delta^2 \phi - \langle \phi | \hat{q} | \phi \rangle^2 \right) \right] \text{Im} A_w < 0 \ (> 0).
$$

It is important to note that it may be possible to exploit this effect to increase the accuracy of the determination of $\text{Re} A_w$. If $A_w$ is real valued, then

$$
\delta A_w = \delta \langle \psi | \hat{A} | \psi \rangle.
$$
For the sake of completeness, consider the case mentioned above where it was noted that
\( \gamma \) can be determined from a measurement of \( \langle \Psi | \hat{q} | \Psi \rangle \) when \( \langle \phi | \hat{q} | \phi \rangle = 0 \) and \( A_w = 1 \). Since
\[
\Delta^2_{\Psi \hat{q}} \simeq \Delta^2_{\phi \hat{q}}
\]
and
\[
\frac{\partial \langle \Psi | \hat{q} | \Psi \rangle}{\partial \gamma} \simeq 1,
\]
then
\[
\delta \gamma \simeq \Delta_{\phi \hat{q}}
\]
which is clearly the analogue of eq. (15) when \( \langle \psi | \hat{A} | \psi \rangle = 1 \).

IV. CONCLUDING REMARKS

The idempotent property of projection operators has been used to provide for any measurement interaction strength exact simple expressions for both state vectors and arbitrary pointer observable mean values that are associated with projector measurements of PS and PPS quantum systems. These results demonstrate that: (i) the idempotency of the projector precludes the existence of interference cross terms in the distribution profiles and in the exact expressions for pointer observable mean values for PS systems; (ii) post-selection nullifies the effect of projector idempotency in projector measurements of PPS systems so that interference cross-terms appear in the distribution profiles (thereby providing an observable distinction between PS and PPS systems) and in the exact expressions for pointer observable mean values; (iii) post-selection induces a Pancharatnam phase into the exact states for PPS systems; (iv) regardless of the strength of the interaction both the exact state and the exact expression for the mean value of a pointer observable for PPS systems depends upon the weak value of the projector; (v) whereas pointer momentum is a constant of the motion for projector measurements of PS systems - it is not a constant of the motion for projector measurements of PPS systems; (vi) measurement interaction strengths and mean values for projectors can both be straightforwardly determined from mean pointer position measurements of PS systems when the mean pre-measurement position of the pointer is zero; and (vii) only measurement interaction strengths can be straightforwardly determined from mean pointer position measurements of PPS systems when the weak value of the projector is unity.
When applied to the weak measurement regime these results demonstrate that: (i) the exact pointer observable mean values yield approximate expressions which are valid for any operator (projector or non-projector); (ii) the approximate expression for the mean value of an arbitrary pointer observable for PPS systems agrees exactly with \[31\]; (iii) complex valued weak values "correlate" the pointer observable for a PPS system with the pointer momentum - even when the pointer observable and momentum commute; (iv) the accuracy associated with determining the mean value of an operator from a measurement of the mean pointer position for a PS system can be made arbitrarily small only when the pointer’s pre-measurement position uncertainty can be made arbitrarily small relative to the strength of the measurement interaction; (v) the accuracy associated with determining the real part of the weak value of an operator from a measurement of the mean pointer position for PPS systems is affected by the imaginary part of the weak value (it may be possible to exploit this to increase the measurement accuracy of the real part of a complex weak value); and (vi) when the weak value of an operator is real valued the accuracy associated with determining its weak value from a mean pointer position measurement of a PPS system is precisely the same as the accuracy associated with determining its mean value from a mean pointer position measurement of a PS system.

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