A NOTE ON VALUES OF NONCOMMUTATIVE POLYNOMIALS

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Abstract. We find a class of algebras \( A \) satisfying the following property: for every nontrivial noncommutative polynomial \( f(X_1, \ldots, X_n) \), the linear span of all its values \( f(a_1, \ldots, a_n), a_i \in A \), equals \( A \). This class includes the algebras of all bounded and all compact operators on an infinite dimensional Hilbert space.

1. Introduction

Starting with Helton’s seminal paper [Hel] there has been considerable interest over the last years in values of noncommuting polynomials on matrix algebras. In one of the papers in this area the second author and Schweighofer [KS] showed that Connes’ embedding conjecture is equivalent to a certain algebraic assertion which involves the trace of polynomial values on matrices. This has motivated us [BK] to consider the linear span of values of a noncommutative polynomial \( f \) on the matrix algebra \( M_d(\mathbb{F}) \); here, \( \mathbb{F} \) is a field with \( \text{char}(\mathbb{F}) = 0 \). It turns out [BK, Theorem 4.5] that this span can be either:

1. \( \{0\} \);
2. the set of all scalar matrices;
3. the set of all trace zero matrices; or
4. the whole algebra \( M_d(\mathbb{F}) \).

From the precise statement of this theorem it also follows that if \( 2d > \deg f \), then (1) and (2) do not occur and (3) occurs only when \( f \) is a sum of commutators.

What to except in infinite dimensional analogues of \( M_d(\mathbb{F}) \)? More specifically, let \( \mathcal{H} \) be an infinite dimensional Hilbert space, and let \( B(\mathcal{H}) \) and \( K(\mathcal{H}) \) denote the algebras of all bounded and compact linear operators on \( \mathcal{H} \), respectively. What is the linear span of polynomial values in \( B(\mathcal{H}) \) and \( K(\mathcal{H}) \)? A very special (but decisive, as we shall see) case of this question was settled by Halmos [Hal] and Pearcy and Topping [PT] (see also Anderson [And]) a long time ago: every operator in \( B(\mathcal{H}) \) and \( K(\mathcal{H}) \), respectively, is a sum of commutators. That is, the linear span of values of the polynomial \( X_1X_2 - X_2X_1 \) on \( B(\mathcal{H}) \) and \( K(\mathcal{H}) \) is all of \( B(\mathcal{H}) \) and \( K(\mathcal{H}) \), respectively. We will prove that the same is true for every nonconstant
polynomial. This result will be derived as a corollary of our main theorem which presents a class of algebras with the property that the span of values of “almost” every polynomial is equal to the whole algebra.

2. Results

By \( \mathbb{F}(\bar{X}) \) we denote the free algebra over a field \( \mathbb{F} \) generated by \( \bar{X} = \{X_1, X_2, \ldots \} \), i.e., the algebra of all noncommutative polynomials in \( X_i \). Let \( f = f(X_1, \ldots, X_n) \in \mathbb{F}(\bar{X}) \). We say that \( f \) is homogeneous in the variable \( X_i \) if all monomials of \( f \) have the same degree in \( X_i \). If this degree is 1, then we say that \( f \) is linear in \( X_i \). If \( f \) is linear in every variable \( X_i, 1 \leq i \leq n \), then we say that \( f \) is multilinear.

Let \( \mathcal{A} \) be an algebra over \( \mathbb{F} \). By \( f(\mathcal{A}) \) we denote the set of all values \( f(a_1, \ldots, a_n) \) with \( a_i \in \mathcal{A}, i = 1, \ldots, n \). Recall that \( f = f(X_1, \ldots, X_n) \in \mathbb{F}(\bar{X}) \) is said to be an identity of \( \mathcal{A} \) if \( f(\mathcal{A}) = \{0\} \). If \( f(\mathcal{A}) \) is contained in the center of \( \mathcal{A} \), but \( f \) is not an identity of \( \mathcal{A} \), then \( f \) is said to be a central polynomial of \( \mathcal{A} \). By \( \text{span} f(\mathcal{A}) \) we denote the linear span of \( f(\mathcal{A}) \). We are interested in the question, when does \( \text{span} f(\mathcal{A}) = \mathcal{A} \) hold?

For the proof of our main theorem three rather elementary lemmas will be needed. The first and the simplest one is a slightly simplified version of [BK, Lemma 2.2]. Its proof is based on the standard Vandermonde argument.

**Lemma 2.1.** Let \( \mathcal{V} \) be a vector space over an infinite field \( \mathbb{F} \), and let \( \mathcal{U} \) be a subspace. Suppose that \( c_0, c_1, \ldots, c_n \in \mathcal{V} \) are such that \( \sum_{i=0}^n \lambda^i c_i \in \mathcal{U} \) for all \( \lambda \in \mathbb{F} \). Then each \( c_i \in \mathcal{U} \).

Recall that a vector subspace \( \mathcal{L} \) of \( \mathcal{A} \) is said to be a Lie ideal of \( \mathcal{A} \) if \( [\ell, a] \in \mathcal{L} \) for all \( \ell \in \mathcal{L} \) and \( a \in \mathcal{A} \); here, \( [u, v] = uv - vu \). For a recent treatise on Lie ideals from an algebraic as well as functional analytic viewpoint we refer the reader to [BKS].

Our second lemma is a special case of [BK, Theorem 2.3].

**Lemma 2.2.** Let \( \mathcal{A} \) be an algebra over an infinite field \( \mathbb{F} \), and let \( f \in \mathbb{F}(\bar{X}) \). Then \( \text{span} f(\mathcal{A}) \) is a Lie ideal of \( \mathcal{A} \).

Every vector subspace of the center of \( \mathcal{A} \) is obviously a Lie ideal of \( \mathcal{A} \). Lie ideals that are not contained in the center are called noncentral. The third lemma follows from an old result of Herstein [Her, Theorem 1.2].

**Lemma 2.3.** Let \( \mathcal{S} \) be a simple algebra over a field \( \mathbb{F} \) with \( \text{char}(\mathbb{F}) \neq 2 \). If \( \mathcal{M} \) is both a noncentral Lie ideal of \( \mathcal{S} \) and a subalgebra of \( \mathcal{S} \), then \( \mathcal{M} = \mathcal{S} \).

We are now in a position to prove our main result.

**Theorem 2.4.** Let \( \mathcal{S} \) and \( \mathcal{B} \) be algebras over a field \( \mathbb{F} \) with \( \text{char}(\mathbb{F}) = 0 \), and let \( \mathcal{A} = \mathcal{S} \otimes \mathcal{B} \). Suppose that \( \mathcal{S} \) is simple, and suppose that \( \mathcal{B} \) satisfies

(a) every element in \( \mathcal{B} \) is a sum of commutators; and
(b) for each \( n \geq 1 \), every element in \( \mathcal{B} \) is a linear combination of elements \( b^n \), \( b \in \mathcal{B} \).

If \( f \in \mathbb{F}(\bar{X}) \) is neither an identity nor a central polynomial of \( \mathcal{S} \), then

\[ \text{span} f(\mathcal{A}) = \mathcal{A}. \]

(In case \( \mathcal{A} \) is nonunital, only polynomials \( f \) with zero constant term are considered.)
Proof: Let \( f = f(X_1, \ldots, X_n) \). Let us write \( f = g_i + h_i \) where \( g_i \) is a sum of all monomials of \( f \) in which \( X_i \) appears and \( h_i \) is a sum of all monomials of \( f \) in which \( X_i \) does not appear. Thus, \( h_i = h_i(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) \) and hence

\[
h_i(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n)
\]

for all \( a_i \in A \). Therefore \( \text{span} h_i(A) \subseteq \text{span} f(A) \), which clearly implies \( \text{span} g_i(A) \subseteq \text{span} f(A) \). At least one of \( g_i \) and \( h_i \) is neither an identity nor a central polynomial of \( S \). Therefore there is no loss of generality in assuming that either \( X_i \) appears in every monomial of \( f \) or \( f \) does not involve \( X_i \) at all. Since \( f \) cannot be a constant polynomial and hence it must involve some of the \( X_i \)'s, we may assume, again without loss of generality, that each monomial of \( f \) involves all \( X_i \), \( i = 1, \ldots, n \).

Next we claim that there is no loss of generality in assuming that \( f \) is homogeneous in \( X_1 \). Write \( f = f_1 + \ldots + f_m \), where \( f_i \) is the sum of all monomials of \( f \) that have degree \( i \) in \( X_1 \). Note that

\[
f(\lambda a_1, a_2, \ldots, a_n) = \sum_{i=1}^{m} \lambda^i f_i(a_1, \ldots, a_n) \in \text{span} f(A)
\]

for all \( \lambda \in F \) and all \( a_i \in A \), so \( f_i(a_1, \ldots, a_n) \in \text{span} f(A) \) by Lemma 2.1. Thus, \( \text{span} f_i(A) \subseteq \text{span} f(A) \). At least one \( f_i \) is neither an identity nor a central polynomial of \( S \). Therefore it suffices to prove the theorem for \( f_i \). This proves our claim.

Let us now show that there is no loss of generality in assuming that \( f \) is linear in \( X_1 \). If \( \deg_{X_1} f > 1 \), we apply the multilinearization process to \( f \); i.e., we introduce a new polynomial \( \Delta_1 f = f'(X_1, \ldots, X_n, X_{n+1}) \):

\[
f'(X_1 + X_{n+1}, X_2, \ldots, X_n) - f(X_1, X_2, \ldots, X_n) - f(X_{n+1}, X_2, \ldots, X_n).
\]

This reduces the degree in \( X_1 \) by one. Clearly, \( \text{span} f'(A) \subseteq \text{span} f(A) \). Observe that \( f \) can be retrieved from \( f' \) by resubstituting \( X_{n+1} \mapsto X_1 \); more exactly

\[
(2^{\deg_{X_1}} f - 2)f = f'(X_1, \ldots, X_n, X_1).
\]

Hence \( f' \) is not an identity nor a central polynomial of \( S \). Note however that \( f' \) is not necessarily homogeneous in \( X_1 \), but for all its homogeneous components \( f'_i \), we have \( \text{span} f'_i(A) \subseteq \text{span} f'(A) \); one can check this by using Lemma 2.1 as in the previous paragraph. At least one of these components, say \( f'_{i*} \), is not an identity nor a central polynomial of \( S \). Thus we restrict our attention to \( f'_{i*} \). If necessary, we continue applying \( \Delta_{i*} \), and after a finite number of steps we obtain a polynomial \( \Delta f \) that is linear in \( X_1 \) which is neither an identity nor a central polynomial of \( S \) and which satisfies \( \text{span} \Delta f(A) \subseteq \text{span} f(A) \). Hence we may assume \( f \) is linear in \( X_1 \).

Repeating the same argument with respect to other variables we finally see that without loss of generality we may assume that \( f \) is multilinear.

Set \( L = \text{span} f(A) \) and \( M = \{ m \in S | m \otimes B \subseteq L \} \). By Lemma 2.2 \( L \) is a Lie ideal of \( A \). Therefore \( [m, s] \otimes b^2 = [m \otimes b, s \otimes b] \in L \) for all \( m \in M, b \in B, s \in S \). Using (b) it follows that \( [m, s] \in M \). Therefore \( M \) is a Lie ideal of \( S \). Pick \( s_1, \ldots, s_n \in S \) such that \( s_0 = f(s_1, \ldots, s_n) \) does not lie in the center of \( S \). For every \( b \in B \) we have

\[
s_0 \otimes b^n = f(s_1 \otimes b, s_2 \otimes b, \ldots, s_n \otimes b) \in L.
\]
In view of (b) this yields $s_0 \in \mathcal{M}$. Accordingly, $\mathcal{M}$ is a noncentral Lie ideal of $\mathcal{S}$. Next, given $m \in \mathcal{M}$ and $b, b' \in \mathcal{B}$, we have

$$m^2 \otimes [b, b'] = [m \otimes b, m \otimes b'] \in \mathcal{L}.$$ 

By (a), this gives $m^2 \in \mathcal{M}$. From $m_1 m_2 = \frac{1}{2} ([m_1, m_2] + (m_1 + m_2)^2 - m_1^2 - m_2^2)$ it now follows that $\mathcal{M}$ is a subalgebra of $\mathcal{S}$. Using Lemma 2.3 we now conclude that $\mathcal{M} = \mathcal{S}$; i.e., $\mathcal{A} = \mathcal{S} \otimes \mathcal{B} \subseteq \mathcal{L} \subseteq \mathcal{A}$. □

From the identity

$$n! b = \sum_{i=0}^{n-1} (-1)^{n-1-i} \binom{n-1}{i} ((b + i)^n - i^n)$$

it is immediate that (b) is fulfilled if $\mathcal{B}$ has a unity. In this case the proof can be actually slightly simplified by avoiding use of powers of elements in $\mathcal{B}$. Further, every $C^*$-algebra $\mathcal{B}$ satisfies (b). Indeed, every element in $\mathcal{B}$ is a linear combination of positive elements, and for positive elements we can define $n$th roots.

**Corollary 2.5.** Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then

$$\text{span } f(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H})$$

for every nonconstant polynomial $f \in \mathbb{C}(\bar{X})$.

**Proof.** It is well known that there does not exist a nonzero polynomial that is an identity of $M_d(\mathbb{C})$ for every $d \geq 1$, cf. [Row, Lemma 1.4.3]. Therefore there exists $d \geq 1$ such that $[f, X_{n+1}]$ is not an identity of $M_d(\mathbb{C})$. This means that $f$ is neither an identity nor a central polynomial of $M_d(\mathbb{C})$. Since $\mathcal{H}$ is infinite dimensional, we have $\mathcal{B}(\mathcal{H}) \cong M_d(\mathcal{B}(\mathcal{H})) \cong M_d(\mathbb{C}) \otimes \mathcal{B}(\mathcal{H})$. Now we are in a position to use Theorem 2.4. Indeed, $M_d(\mathbb{C})$ is a simple algebra, and the algebra $\mathcal{B}(\mathcal{H})$ satisfies (a) by [Hal], and satisfies (b) since it is unital. □

**Corollary 2.6.** Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Then

$$\text{span } f(\mathcal{K}(\mathcal{H})) = \mathcal{K}(\mathcal{H})$$

for every nonzero polynomial $f \in \mathbb{C}(\bar{X})$ with zero constant term.

**Proof.** The proof is essentially the same as that of Corollary 2.5. The only difference occurs in verifying whether $\mathcal{K}(\mathcal{H})$ satisfies the conditions of Theorem 2.4. For this we just note that (a) is shown in [PT], and (b) follows by the remark preceding the statement of Corollary 2.5. □

**References**

[And] J. Anderson, Commutators of compact operators, *J. Reine Angew. Math.* 291 (1977) 128-132. MR0442742 (56:1122)

[BKS] M. Brešar, E. Kissin, V. Shulman, Lie ideals: from pure algebra to $C^*$-algebras, *J. Reine Angew. Math.* 623 (2008) 73-121. MR2458041 (2009i:47168)

[BK] M. Brešar, I. Klep, Values of noncommutative polynomials, Lie skew-ideals and tracial Nullstellensätze, *Math. Res. Lett.* 16 (2009) 605-626. MR2525028

[Hal] P.R. Halmos, Commutators of operators II, *Amer. J. Math.* 76 (1954) 191-198. MR0059484 (15:538a)

[Hel] J.W. Helton, “Positive” noncommutative polynomials are sums of squares, *Ann. of Math.* (2) 156 (2002) 675-694. MR1933721 (2003k:12002)
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[Her] I.N. Herstein, *Topics in ring theory*, The University of Chicago Press, 1969. MR0271135 (42:6018)

[KS] I. Klep, M. Schweighofer, Connes’ embedding conjecture and sums of Hermitian squares, *Adv. Math.* 217 (2008) 1816-1837. MR2382741 (2009g:46109)

[PT] C. Pearcy, D. Topping, On commutators in ideals of compact operators, *Michigan J. Math.* 18 (1971) 247-252. MR0284853 (44:2077)

[Row] L.H. Rowen, *Polynomial identities in ring theory*, Academic Press, 1980. MR576061 (82a:16021)

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