A polycyclic presentation for the $q$-tensor square of a polycyclic group

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Communicated by Dessislava H. Kochloukova

Abstract. Let $G$ be a group and $q$ a non-negative integer. We denote by $v^q(G)$ a certain extension of the $q$-tensor square $G \otimes^q G$ by $G \times G$. In this paper, we describe an algorithm for deriving a polycyclic presentation for $G \otimes^q G$ when $G$ is polycyclic, via its embedding into $v^q(G)$. Furthermore, we derive polycyclic presentations for the $q$-exterior square $G \wedge^q G$ and for the second homology group $H_2(G, \mathbb{Z}_q)$. Additionally, we establish a criterion for computing the $q$-exterior center $Z^q(G)$ of a polycyclic group $G$, which is helpful for deciding whether or not $G$ is capable modulo $q$. These results extend to all $q \geq 0$ generalizing methods due to Eick and Nickel for the case $q = 0$.

1 Introduction

Let $G$ be a group and $q$ a non-negative integer. The $q$-tensor square $G \otimes^q G$ is a particular case of the $q$-tensor product $G \otimes^q H$ of groups $G$ and $H$ which act compatibly on each other; this construction was defined by Conduché and Rodrigues-Fernandez in [6], in the context of $q$-crossed modules (see also [2, 10, 13]). It reduces to Brown and Loday’s non-abelian tensor product $G \otimes H$ when $q = 0$ (cf. [4]).

For $x, y \in G$, we write the conjugate of $y$ by $x$ as $y^x = x^{-1}yx$; the commutator of $x$ and $y$ is then written as $[x, y] = x^{-1}y^{-1}xy$. Commutators are left normed: $[x, y, z] = [[x, y], z]$, and so on for commutators of higher weights.

For $q \geq 1$, let $\hat{G} := \{\hat{k} \mid k \in G\}$ be a set of symbols, one for each element of $G$. According to Ellis [10], the $q$-tensor square $G \otimes^q G$ is then defined to be the group generated by all symbols $g \otimes h$ and $\hat{k}, g, h, k \in G$, subject to the defining relations

\begin{align}
(g \otimes hh_1) &= (g \otimes h)(g^{h_1} \otimes h^{h_1}), \quad (1.1) \\
(gh_1 \otimes h) &= (g^{h_1} \otimes h^{h_1})(g \otimes h), \quad (1.2) \\
(g \otimes k)^{\hat{k}} &= (g^{k^q} \otimes h^{k^q}), \quad (1.3)
\end{align}

The second author acknowledges partial financial support through grant No. 0193.001.344/2016 from FAPDF, Brazil, during the preparation of this work.
for all \( g, g_1, h, h_1, k, k_1 \in G \). If \( q = 0 \), then we set \( \hat{G} = 0 \) to get the group generated by the symbols \( g \otimes h \), \( g, h \in G \), subject to relations (1.1) and (1.2) only; that is, \( G \otimes^0 G \) is the non-abelian tensor square \( G \otimes G \). By the defining relations (1.1)–(1.6), we see that the diagonal \( \nabla^q(G) = \langle g \otimes g \mid g \in G \rangle \) is a central subgroup of \( G \otimes^q G \). The \( q \)-exterior square \( G \wedge^q G \) is by definition the factor group (see [11])

\[
G \wedge^q G = \frac{G \otimes^q G}{\nabla^q(G)}.
\]

We usually write \( g \wedge h \) for the image of \( g \otimes h \) in \( G \wedge^q G \).

There is a homomorphism

\[
\varrho: G \otimes^q G \to G, \quad g \otimes h \mapsto [g, h], \quad \hat{k} \mapsto k^q
\]

for all \( g, h, k \in G \). Clearly, \( \nabla^q(G) \leq \text{Ker} \varrho \), and we have (see for instance [2, Proposition 18] or [5, Theorem 2.12])

\[
\text{Ker} \varrho/\nabla^q(G) \cong H_2(G, \mathbb{Z}_q),
\]

the second homology group of \( G \) with coefficients in the trivial \( G \)-module \( \mathbb{Z}_q \). The image \( \text{Im} \varrho \) is the subgroup \( G'/G'^q \leq G \), where \( G' \) is the derived subgroup of \( G \), generated by all commutators \([g, h]\) with \( g, h \in G \), and \( G'^q \) is the subgroup of \( G \) generated by all \( q \)-th powers \( g^q \), \( g \in G \). Thus, we get the exact sequence (cf. [2, Proposition 18])

\[
1 \to H_2(G, \mathbb{Z}_q) \to G \wedge^q G \to G'/G'^q \to 1.
\]

A group \( G \) is called \( q \)-perfect in case \( G = G'/G'^q \). If this is the case, then the above sequence shows that \( G \wedge^q G \) is a \( q \)-central extension of \( G \), and in addition it is the unique universal \( q \)-central extension of \( G \) (see [2]). Notice that if \( G \) is \( q \)-perfect, then \( G \otimes^q G \cong G \wedge^q G \).

The \( q \)-exterior square is also helpful in deciding whether or not a group \( G \) is \( q \)-capable; recall that \( G \) is \( q \)-capable if there exists a group \( Q \) such that

\[
Z(Q) = Z_q(Q) \quad \text{and} \quad G \cong Q/Z(Q),
\]
The \( q \)-tensor square of a polycyclic group, \( q \geq 0 \)

where \( Z(Q) \) is the center of \( Q \) and \( Z_q(Q) \) is the \( q \)-center, that is, the elements of the center \( Z(Q) \) of order dividing \( q \). The \( q \)-exterior center of \( G \) is the subgroup of \( G \) defined by

\[
Z^\wedge_q(G) = \{ g \in G \mid g \wedge x = 1 \in G \wedge^q G \text{ for all } x \in G \}.
\]

In [10, Proposition 16], Ellis proved that the group \( G \) is \( q \)-capable if, and only if, \( Z^\wedge_q(G) = 1 \).

So getting a presentation for the \( q \)-tensor square of a group \( G \) and for its subfactors is an interesting task.

It is known that if \( G \) is a polycyclic group, then \( G \otimes^q G \) is polycyclic for all \( q \geq 0 \) (see for instance [5]). In [8], the authors describe algorithms to compute the non-abelian tensor square \( G \otimes G \), the exterior square \( G \wedge G \) and the Schur multiplier \( M(G) \), among others, for a polycyclic group \( G \) given by a consistent polycyclic presentation; the implementation of this algorithm is available in [7]. They manage to find such an algorithm to computing \( G \otimes G \) by finding a presentation of the group \( \nu(G) \), as introduced for instance in [15] (see also [9]), which turns out to be an extension of \( G \otimes G \) by \( G \times G \).

The present paper aims to extend these algorithms to all \( q \geq 0 \). Instead of \( \nu(G) \), we now consider the group \( \nu^q(G) \), defined for instance in [5, Definition 2.1]. To ease reference, we briefly describe this group early in the next section.

The paper is organized as follows. In Section 2, we describe some basic constructs and preliminaries results. In Section 3, we give consistent polycyclic presentations for certain \( q \)-central extensions of \( G \), more specifically, the groups \( E_q(G) \) and \( S^q(G) \). In Section 4, we give polycyclic presentations for the second homology group \( H_2(G, \mathbb{Z}_q) \) and for the \( q \)-exterior square \( G \wedge^q G \); use of the group \( E_q(G) \) is made to exemplify the computation of the \( q \)-exterior center of \( G \).

In Section 5, we provide a consistent polycyclic presentation for the group \( \tau^q(G) \). Finally, in Section 6, we give an algorithm to compute a polycyclic presentation for \( \nu^q(G) \) and the \( q \)-tensor square of a polycyclic group \( G \).

Notation is fairly standard; for basic results on group theory, see for instance [14]. In this article, all group actions are on the right. Basic notation and structural results concerning \( \nu^q(G) \) can be found for instance in [5].

2 Preliminary results

We begin this section by defining the group \( \nu^q(G) \) and giving a brief description of some of its properties.

To this end, let \( G^\varphi \) be an isomorphic copy of \( G \), via an isomorphism \( \varphi \) such that \( \varphi : g \mapsto g^\varphi \) for all \( g \in G \). With these data, we immediately get the group \( \nu(G) \), as
mentioned before, defined as 

\[ v(G) := \langle G \cup G^\varphi \mid [g, h]^k = [g^k, (h^k)^\varphi] = [g, h^\varphi]^k \varphi \text{ for all } g, h, k \in G \rangle. \]

It is well known (see [9, 15]) that the subgroup \([G, G^\varphi]\) of \(v(G)\) is isomorphic to the non-abelian tensor square \(G \otimes G\) so that the strategy of finding an appropriate representation of \(v(G)\) can be useful to compute \(G \otimes G\) and various of its relevant subfactors (see for instance [1, 8, 9, 13, 15]).

Now, for \(q \geq 1\), let \(\hat{G} = \{\hat{k} \mid k \in G\}\) be a set of symbols, one for each element of \(G\) (for \(q = 0\), we set \(\hat{G} = \emptyset\), the empty set), and let \(F(\hat{G})\) be the free group on \(\hat{G}\). Write \(v(G) \ast F(\hat{G})\) for the free product of \(v(G)\) and \(F(\hat{G})\). As \(G\) and \(G^\varphi\) are embedded into \(v(G)\), we shall identify the elements of \(G\) (respectively of \(G^\varphi\)) with their respective images in \(v(G) \ast F(\hat{G})\). Denote by \(J\) the normal closure in \(v(G) \ast F(\hat{G})\) of the following elements for all \(\hat{k}, \hat{k}_1 \in \hat{G}\) and \(g, h \in G\):

\[
\begin{align*}
&g^{-1}k \overline{g(k^g)}, \quad (2.1) \\
&(g^\varphi)^{-1}\hat{k} g^\varphi (k^\varphi)^{-1}, \quad (2.2) \\
&\hat{(k)}^{-1}[g, h^\varphi]\hat{k}[g^{k^q}, (h^{k^q})^\varphi]^{-1}, \quad (2.3) \\
&\hat{(k)}^{-1}\hat{k}_1 (\hat{k})^{-1}
\left(\prod_{i=1}^{q-1} [k, (k^{-i})^\varphi]^{k^q-1-i}\right)^{-1}, \quad (2.4) \\
&[\hat{k}, \hat{k}_1]^{k^q, (k_1^q)^\varphi]^{-1}, \quad (2.5) \\
&[\hat{g}, \hat{h}][g, h^\varphi]^{-q}. \quad (2.6)
\end{align*}
\]

**Definition 2.1.** The group \(v^q(G)\) is defined to be the factor group

\[
v^q(G) := (v(G) \ast F(\hat{G}))/J.
\]

Note that, for \(q = 0\), the sets of relations (2.1) to (2.6) are empty; in this case, we have \(v^0(G) = v(G) \ast F(\hat{G}))/J \cong v(G)\). Let \(R_1, \ldots, R_6\) be the sets of relations corresponding to (2.1)–(2.6), respectively, and let \(R\) be their union, \(R = \bigcup_{i=1}^{6} R_i\). Therefore, \(v^q(G)\) has the presentation

\[
v^q(G) = \langle G, G^\varphi, \hat{G} \mid R, [g, h^\varphi]^k[g^k, (h^k)^\varphi]^{-1}, [g, h^\varphi]^k[g^k, (h^k)^\varphi]^{-1}
\text{ for all } g, h, k \in G \rangle.
\]

The above presentation of \(v^q(G)\) is a variant of the one given by Ellis in [10].

There is an epimorphism \(\rho: v^q(G) \twoheadrightarrow G, g \mapsto g, h^\varphi \mapsto h, \hat{k} \mapsto k^q\). On the other hand, the inclusion of \(G\) into \(v(G)\) induces a homomorphism \(i: G \rightarrow v^q(G)\).
We have \( g^{i\rho} = g \), and thus \( i \) is injective. Similarly, the inclusion of \( G^\varphi \) into \( v(G) \) induces a monomorphism \( j : G^\varphi \rightarrow v^\varphi(G) \). These embeddings allow us to identify the elements \( g \in G \) and \( h^\varphi \in G^\varphi \) with their respective images \( g^i \) and \( (h^\varphi)^j \) in \( v^\varphi(G) \).

Now let \( \mathcal{G} \) denote the subgroup of \( v^\varphi(G) \) generated by the images of \( \hat{G} \). By relations (2.3), \( \mathcal{G} \) normalizes the subgroup \( [G, G^\varphi] \) in \( v^\varphi(G) \), and hence we have that \( \Upsilon^\varphi(G) := [G, G^\varphi] \mathcal{G} \) is a normal subgroup of \( v^\varphi(G) \). Hence we get

\[
v^\varphi(G) = G^\varphi \cdot (G \cdot \Upsilon^\varphi(G)),
\]

where the dots mean internal semidirect products.

By [5, Proposition 2.9], there is an isomorphism \( \varphi : \Upsilon^\varphi(G) \rightarrow G \otimes^q G \) such that \( [g, h^\varphi] \mapsto g \otimes h, \hat{k} \mapsto \hat{k} \) for all \( g, h, k \in G \) and for all \( q \geq 0 \). We then get (see [5, Corollary 2.11])

\[
v^\varphi(G) \cong \varphi \cdot\varphi : G \ltimes (G \ltimes (G \otimes^q G)).
\]

This decomposition of \( v^\varphi(G) \) is analogous to one due to Ellis in [10]; it generalizes a similar result found in [15] for \( q = 0 \).

In view of the above isomorphism, unless otherwise stated, from now on, we will identify \( G \otimes^q G \) with the subgroup \( \Upsilon^\varphi(G) \subseteq v^\varphi(G) \) and write \( [g, h^\varphi] \) in place of \( g \otimes h \) for all \( g, h \in G \). Following [5], we write \( \Delta^\varphi(G) \) for the subgroup \( \langle [g, g^\varphi] \mid g \in G \rangle \leq \Upsilon^\varphi(G) \), which, by Remark 2.2 below, is a central subgroup of \( v^\varphi(G) \). The isomorphism \( \varphi \) restricts to an isomorphism

\[
\Delta^\varphi(G) \cong \Upsilon^\varphi(G)
\]

and, consequently, the factor group \( \Upsilon^\varphi(G) / \Delta^\varphi(G) \) is isomorphic to the \( q \)-exterior square \( G \wedge^q G \). In this case, as usual, we simply write \( g \wedge h \) to denote the coset \( [g, h^\varphi] \Delta^\varphi(G) \) in \( G \wedge^q G \). We shall eventually write \( \mathcal{T} \) to denote the subgroup \( [G, G^\varphi] \) of \( v^\varphi(G) \) in order to distinguish it from the non-abelian tensor square \( G \otimes G \cong [G, G^\varphi] \leq v(G) \) in the case \( q = 0 \). We also write \( \tau^\varphi(G) \) for the factor group \( v^\varphi(G) / \Delta^\varphi(G) \); thus we get

\[
\tau^\varphi(G) \cong ((G \wedge^q G) \ltimes G) \times G.
\]

**Remark 2.2.** It should be noted that the actions of \( G \) and \( G^\varphi \) on \( \Upsilon^\varphi(G) \) are those induced by the defining relations of \( v^\varphi(G) \): for any elements \( g, x \in G \), \( h^\varphi, y^\varphi \in G^\varphi \) and \( \hat{k} \in \hat{G} \), we have \( [g, h^\varphi]^x = [g^x, (h^x)^\varphi] \) and \( \hat{k}^x = \hat{k^x} \). In view of the isomorphism \( \Upsilon^\varphi(G) \cong G \otimes^q G \), these correspond to the action of \( G \) on \( G \otimes^q G \), as given for instance in [10],

\[
\begin{cases}
(g \otimes h)^x = g^x \otimes h^x, \\
(\hat{k})^x = \hat{k^x}.
\end{cases}
\]
Similarly, we get \( [g, h^q]^{y^q} = [g^y, (h^y)^q] \) and \( (\hat{k})^{y^q} = (\hat{k}^y) \). In addition, for any \( \tau \in \Upsilon^q(G) \), \( (g\tau)^{y^q} = g[g, y^q]\tau^{y^q} \in G\Upsilon^q(G) \). Similar actions are naturally induced on the \( q \)-exterior square \( G \wedge^q G \).

It is known that if \( G \) is polycyclic, then \( v^q(G) \) is polycyclic for all \( q \geq 0 \), and thus, as mentioned before, \( G \otimes^q G \) is polycyclic. In [5], the authors proved that, for a polycyclic group \( G \) given by a consistent polycyclic presentation, the defining relations of \( v^q(G) \) can be reduced to relations among the polycyclic generators, with the only exception of relations (2.4) which have a more complicated handling characteristic. Even without reducing relations (2.4), they were able to use the GAP System [18] to compute \( v^q(G), G \otimes^q G \) and \( G \wedge^q G \) for some small groups \( G \) and particular values of \( q \). In addition, in [16], an upper bound to the minimal number of generators of the \( q \)-tensor square of an \( n \)-generator nilpotent group of class 2, \( n > 1 \), for all \( q > 1 \) and \( q \) odd is given.

Our purpose in this article is to overcome in some way the difficulty of dealing with relations (2.4) and describe algorithms for deriving polycyclic presentations for the groups \( v^q(G), G \otimes^q G, G \wedge^q G \) and \( H_2(G, \mathbb{Z}_q) \) for all \( q \geq 0 \), when \( G \) is polycyclic given by a consistent polycyclic presentation. Our approach is based on ideas of Eick and Nickel [8] for the case \( q = 0 \).

The concept of a crossed pairing (biderivation) has been used in order to determine homomorphic images of the non-abelian tensor square \( G \otimes G \) (see [3, Remark 3]). We need to extend this concept in order to the context of the \( q \)-tensor square.

**Definition 2.3.** Let \( G \) and \( L \) be arbitrary groups and \( q \) a non-negative integer. A function \( \lambda: G \times G \times G \rightarrow L \) is called a \( q \)-biderivation if the following properties hold:

\[
(gg_1, h, k)\lambda = (g^{g_1}, h^{g_1}, 1)\lambda(g_1, h, k)\lambda,
\]

\[
(g, hh_1, k)\lambda = (g, h_1, 1)\lambda(g^{h_1}, h^{h_1}, k)\lambda,
\]

\[
((1, 1, k)\lambda)^{-1}(g, h, 1)\lambda(1, 1, k)\lambda = (g^{k^q}, h^{k^q}, 1)\lambda,
\]

\[
(1, 1, kk_1)\lambda = (1, 1, k)\lambda \prod_{i=1}^{q-1} \{(k, (k^{-i})^{k^{q-1-i}}, 1)\lambda\}
\times (1, 1, k_1)\lambda,
\]

\[
[(1, 1, k)\lambda, (1, 1, k_1)\lambda] = (k^q, k_1^q, 1)\lambda,
\]

\[
(1, 1, [g, h])\lambda = ((g, h, 1)\lambda)^q
\]

for all \( g, g_1, h, h_1, k_1 \in G \).
By the defining relations in Definition 2.1 of \( v^q(G) \), it is easy to see that a \( q \)-biderivation provides a universal property of the \( q \)-tensor square of a group \( G \).

**Proposition 2.4.** Let \( G \) and \( L \) be arbitrary groups and \( \lambda : G \times G \times G \to L \) a \( q \)-biderivation. Then there exists a unique homomorphism \( \hat{\lambda} : G \otimes^q G \to L \) such that the following hold for all \( g, h, k \in G \):

\[
(g \otimes h)\hat{\lambda} = (g, h, 1)\lambda, \quad (k)\hat{\lambda} = (1, 1, k)\lambda.
\]

To ease reference, we include the next lemma, which relates the \( q \)-exterior square of \( G \) and the second homology group \( H_2(G, \mathbb{Z}_q) \) with an arbitrary free presentation \( F/R \) of \( G \) (see [11, 13]).

**Lemma 2.5.** Let \( F/R \) be a free presentation for the group \( G \). Then

\[
G \wedge^q G \cong \frac{F'F^q}{[R, F]R^q} \quad \text{and} \quad H_2(G, \mathbb{Z}_q) \cong \frac{R \cap F'F^q}{[R, F]R^q}.
\]

Thus, \( H_2(G, \mathbb{Z}_q) \cong (G \wedge^q G) \cap M^q(G) \), where \( M^q(G) = R/[R, F]R^q \) is the \( q \)-multiplier.

Notice that when \( F \) is a free group, then we find that \( F \wedge^q F \cong F'F^q \). A similar result is also valid for projective \( q \)-crossed \( G \)-modules.

**Proposition 2.6** ([13, Proposition 1.3.11]). Given a projective \( q \)-crossed \( G \)-module \( \delta : M \to G \), let \( F/R \) be a free presentation of \( G \), with \( \pi : F \to G \) being the natural epimorphism. Then there exists an isomorphism

\[
M'M^q \cong F'F^q/[R, F]R^q
\]

such that

\[
[m, m']^\delta = [f, f'][R, F]R^q \quad \text{and} \quad (m^q)^\delta = f^q[R, F]R^q,
\]

where \((m)^\delta = (f)^\pi\).

### 3 Consistent polycyclic presentations for the groups \( E_q(G) \) and \( \mathfrak{E}_q(G) \), \( q \)-central extensions of \( G \)

In this section, we describe a method for computing consistent polycyclic presentations for certain \( q \)-central extensions of a polycyclic group \( G \) given by a consistent polycyclic presentation. Our method is a generalization of the one given by Eick and Nickel in [8] for the case \( q = 0 \).
Let $G$ be a polycyclic group defined by a consistent polycyclic presentation $F_n/R$, where $F_n$ is the free group in the generators $g_1, \ldots, g_n$, and let $H$ be a finitely presented group defined by a finite presentation $F_m/S$, where $F_m$ is the free group on the generators $f_1, \ldots, f_m$. For our purposes, we shall assume that $m \leq n$. Suppose that $\xi: H \to G$ is an epimorphism such that $(f_i)\xi = w_i$, $1 \leq i \leq m$, where $w_i$ is a word in the generators $g_1, \ldots, g_n$. Denote by $K/S$ the kernel $\text{Ker}\xi$. Thus, $G \cong F_m/K$.

Define the groups $E_q(G) := \frac{F_n}{R^q[F_n, R]}$ and $\mathfrak{C}^q(G) := \frac{F_m}{K^q[K, F_m]S}$, which, by construction, are $q$-central extensions of $G$.

The following result in the context of crossed modules will be helpful.

**Proposition 3.1** ([13, Lemma 5.2.2]). With the above definition, the natural epimorphism $\pi: E_q(G) \to G$ is a projective $q$-crossed $G$-module.

The relations of a consistent polycyclic presentation $F_n/R$ have the form

\[
g_i^{e_i} = g_{i+1}^{\alpha_{i,i+1}} \cdots g_n^{\alpha_{i,n}} \quad \text{for } i \in I,
\]

\[
g_j^{-1} g_i g_j = g_{j+1}^{\beta_{i,j,j+1}} \cdots g_n^{\beta_{i,j,n}} \quad \text{for } j < i,
\]

\[
g_j g_i g_j^{-1} = g_{j+1}^{\gamma_{i,j,j+1}} \cdots g_n^{\gamma_{i,j,n}} \quad \text{for } j < i \text{ and } j \notin I,
\]

for some set $I \subseteq \{1, \ldots, n\}$, certain exponents $e_i \in \mathbb{N}$ for $i \in I$, and $\alpha_{i,j}, \beta_{i,j,k}, \gamma_{i,j,k} \in \mathbb{Z}$ for all $i$, $j$ and $k$. To ease notation, we shall write the defining relations of $G$ as relators, in the form $r_1, \ldots, r_l$. Thus, each relator $r_j$ is a word in the generators $g_1, \ldots, g_n$; that is, $r_j = r_j(g_1, \ldots, g_n)$.

We now introduce $l$ new generators $t_1, \ldots, t_l$, one for each relator $r_j$, and define a new group $\epsilon(G)$ to be the group generated by $g_1, \ldots, g_n, t_1, \ldots, t_l$, subject to the relators

(a) $r_i(g_1, \ldots, g_n)t_i^{-1}$ for $1 \leq i \leq l$,

(b) $[t_i, g_j]$ for $1 \leq j \leq n$, $1 \leq i \leq l$,

(c) $[t_i, t_j]$ for $1 \leq j < i \leq l$,

(d) $t_i^q$ for $1 \leq i \leq l$.

Denote by $T_q$ the $q$-central subgroup of $\epsilon(G)$ generated by $\{t_1, \ldots, t_l\}$. It follows directly from these relators that $\epsilon(G)$ is a $q$-central extension of $G$ by $T_q$.

The following lemma asserts that the above relations give a polycyclic presentation of $E_q(G)$, possibly inconsistent.
Lemma 3.2. Let \( G \) be a polycyclic group given by a consistent polycyclic presentation \( F_n/R \). Then we have \( \epsilon(G) \cong F_n/R^q[R, F_n] \) and \( \epsilon(G)/T_q \cong G \).

Proof. It follows by relations (a) above that \( \epsilon(G)/T_q \cong G \), while, by relations (b), (c) and (d), we immediately see that \( T_q \) is a \( q \)-central subgroup of \( \epsilon(G) \). Define \( \sigma : F_n \to \epsilon(G) \) given by \( (g_i)^\sigma = g_i \), \( 1 \leq i \leq n \). Relations (a) imply that \( \sigma \) is an epimorphism and, since \( \epsilon(G) \) is a \( q \)-central extension of \( G \), we have \( R^q[R, F_n] \leq \text{Ker}(\sigma) \leq R \). On the other side, there exists a well defined homomorphism from \( \epsilon(G) \) to \( F_n/R^q[F_n, R] \), which is an epimorphism. Consequently, \( \text{Ker}(\sigma) \leq R^q[R, F_n] \leq \text{Ker}(\sigma) \), and thus, we have \( \epsilon(G) \cong F_n/R^q[F_n, R] \). Therefore, \( \epsilon(G) \cong E_q(G) \), where we get \( T_q \cong R/R^q[R, F_n] \).

By using a Smith normal form algorithm in a manner similar to that described by Eick and Nickel in [8] (see also [17, p. 424]), we can determine a consistent polycyclic presentation for \( E_q(G) \) from the (possibly inconsistent) polycyclic presentation given by Lemma 3.2. We then get a consistent polycyclic presentation for \( E_q(G) \) in the generators \( g_1, \ldots, g_n, t_1, \ldots, t_l \) with the following relations:

1. \( r_i(g_1, \ldots, g_n)t_1^{q_{i1}} \cdots t_l^{q_{il}} \) for \( 1 \leq i \leq l \),
2. \( [t_i, g_j] \) for \( 1 \leq i \leq n, 1 \leq j \leq l \),
3. \( [t_i, t_j] \) for \( 1 \leq j < i \leq l \),
4. \( t_i^{d_i} \) for \( 1 \leq i \leq l \), with \( d_i | q \),

where \((q_{ij})_{1 \leq i, j \leq l}\) is an appropriate invertible matrix over \( \mathbb{Z} \). It may happen that \( d_i = 1 \) for some \( i \in \{1, \ldots, l\} \). In this case, the corresponding generator \( t_i \) is redundant and can be removed.

Below, we give a couple of simple examples in order to illustrate these results. The same examples will be used in subsequent sections.

Example 3.3. First we consider the symmetric group \( S_3 \), given by the consistent polycyclic presentation

\[
S_3 = \langle g_1, g_2 \mid g_1^2 = 1, g_1^{-1}g_2g_1 = g_2^2, g_2^3 = 1 \rangle.
\]

According to the definition, we have, say for \( q = 2 \),

\[
E_2(S_3) = \langle g_1, g_2, t_1, t_2, t_3 \mid g_1^2 = t_1, g_1^{-1}g_2g_1 = g_2^2t_2, g_2^3 = t_3, \\
t_1^2 = 1, t_2^2 = 1, t_3^2 = 1 \rangle,
\]

where \( t_1, t_2, t_3 \) are central. Checking for consistency, we find that \( t_2 = 1 \). Thus,
a consistent polycyclic presentation of $E_2(S_3)$ is

$$E_2(S_3) = \langle g_1, g_2, t_1, t_3 \mid g_1^2 = t_1, \, g_1^{-1}g_2g_1 = g_2^2, \, g_2^3 = t_3, \, t_1^2 = 1, \, t_3^2 = 1; \, (t_1, t_3 \text{ central}) \rangle.$$ 

**Example 3.4.** In this second example, we consider the infinite dihedral group, given by the consistent polycyclic presentation

$$D_\infty = \langle g_1, g_2 \mid g_1^2 = 1, \, g_1^{-1}g_2g_1 = g_2^{-1} \rangle.$$ 

From this, for an arbitrary $q \geq 2$, we get

$$E_q(D_\infty) = \langle g_1, g_2, t_1, t_2 \mid g_1^2 = t_1, \, g_1^{-1}g_2g_1 = g_2^{-1}t_2, \, t_1^q = 1, \, t_2^q = 1; \, (t_1, t_2 \text{ central}) \rangle.$$ 

Checking these relations for consistency, we find that this presentation is consistent.

Now, from the polycyclic presentation of $E_q(G)$ given earlier, we can determine a presentation for $\mathcal{C}^q(G)$.

**Lemma 3.5.** Let $\varsigma : F_m \to E_q(G)$, given by $(f_i)_\varsigma = w_i$ for $1 \leq i \leq m$, where, as before, $w_i = w_i(g_1, \ldots, g_n)$ is a word in the generators $g_1, \ldots, g_n$. Then

(i) $\operatorname{Ker}(\varsigma) = [K, F_m]K^q$,

(ii) $\mathcal{C}^q(G) \cong \operatorname{Im}(\varsigma)/(S)_\varsigma$.

**Proof.** (i) Notice that, by definition, $\operatorname{Im}(\varsigma)$ covers $G \cong E_q(G)/T_q$, and hence $F_m/\operatorname{Ker}(\varsigma)$ is a $q$-central extension of $G = F_m/K$. Thus, $[K, F_m]K^q \leq \operatorname{Ker}(\varsigma)$. On the other hand, $F_m/[K, F_m]K^q$ is a polycyclic $q$-central extension of $G$ and, since, by construction, $E_q(G)$ is the largest $q$-central extension of $G$ with this property (by Proposition 2.6 it is a projective $q$-crossed $G$-module; see also [8, Lemma 3]), it follows that $E_q(G)$ contains $F_m/[K, F_m]K^q$ as a sub-factor via $\varsigma$. Thus, $\operatorname{Ker}(\varsigma) = [F_m, K]K^q$.

(ii) Now, by part (i), we get $\operatorname{Im}(\varsigma) \cong \frac{F_m}{[F_m, K]K^q}$ and, by definition,

$$\mathcal{C}^q(G) = \frac{F_m}{S[F_m, K]K^q}.$$ 

But $(S)_\varsigma = \frac{S[K, F_m]K^q}{[K, F_m]K^q}$; consequently, $\frac{\operatorname{Im}(\varsigma)}{(S)_\varsigma} = \mathcal{C}^q(G)$. \hfill \qed

Tuned in this way, in order to determine a presentation of $\mathcal{C}^q(G)$, it suffices to determine generators for the subgroups $\operatorname{Im}(\varsigma)$ and $(S)_\varsigma$ of $E_q(G)$ since standard methods for polycyclic groups can be used in order to construct a consistent polycyclic presentation for the quotient $\operatorname{Im}(\varsigma)/(S)_\varsigma$ (see also [12, Chapter 8]).
Certainly, a set of generators for \( \text{Im}(\zeta) \) is given by \( w_1, \ldots, w_m \). Let \( s_1, \ldots, s_k \) be a set of defining relators for the finitely presented group \( H = F_m/S \). Then \( (S)\zeta \) is generated by \( (s_1)\zeta, \ldots, (s_k)\zeta \) as a subgroup, once \( (S)\zeta \leq T_q \) is central in \( E_q(G) \). Thus, a set of generators for \( (S)\zeta \) can be determined by evaluating the relators \( s_1, \ldots, s_k \) in \( E_q(G) \).

4 Polycyclic presentations for the \( q \)-exterior square \( G \wedge^q G \) and for the second homology group \( H_2(G, \mathbb{Z}_q) \)

According to Lemmas 2.5 and 3.2, we have the following.

**Corollary 4.1.** The following statements hold:

(i) \( G \wedge^q G \cong E_q(G)^q/E_q(G)q \);

(ii) \( H_2(G, \mathbb{Z}_q) \cong (E_q(G)^q/E_q(G)^q) \cap T_q \).

Therefore, in order to obtain a presentation for the groups \( G \wedge^q G, H_2(G, \mathbb{Z}_q) \), for a polycyclic group \( G \) given by a consistent polycyclic presentation, we apply standard methods to determine presentations of subgroups of polycyclic groups (see for instance [12, Chapter 8]).

By the isomorphism given in Corollary 4.1, we obtain generators for \( G \wedge^q G \) via \( E_q(G) \).

**Proposition 4.2.** The subgroup \( (E_q(G))^q/(E_q(G))^q \) of \( E_q(G) \) is generated by the set

\[
\langle [g_i, g_j]^\epsilon, g_k^q \mid 1 \leq i < j \leq n, 1 \leq k \leq n \rangle,
\]

where \( \epsilon = 1 \) if \( G \) is finite and \( \epsilon = \pm1 \) otherwise.

**Proof.** As \( E_q(G) = \langle g_1, \ldots, g_n, t_1, \ldots, t_l \mid r_i = t_i, (t_i \ q\text{-central}) \rangle \) is polycyclic, we immediately get

\[
(E_q(G))^q = \langle [g_i, g_j]^\epsilon \mid 1 \leq i < j \leq n \rangle,
\]

where \( \epsilon \) is as above. Now, by a simple induction on \( q \), we see that each power \( g^q \), \( g \in E_q(G) \), is a word in the commutators \( [g_i, g_j]^\epsilon \) and in the \( q \)-th powers \( g_k^q \) of the generators of \( G \).

Notice that if we consider the natural epimorphism \( E_q(G) \twoheadrightarrow G \) and choose a pre-image \( \widehat{g} \in E_q(G) \) for each \( g \in G \), then, by Proposition 2.6, we can make explicit an isomorphism \( \beta: G \wedge^q G \twoheadrightarrow (E_q(G))^q/(E_q(G))^q \) such that

\[
(g \wedge h)^\beta = [\widehat{g}, \widehat{h}] \quad \text{and} \quad (\widehat{k})^\beta = (\widehat{k})^q.
\]
Remark 4.3. We have the following observations.

(i) As we have seen in Section 1, $G$ acts naturally on $G \wedge^q G$ via

$$(g \wedge h)^k = g^k \wedge h^k, \quad (\hat{k})^g = \hat{k}^g$$

for all $g, h, k \in G$.

In addition, this action is compatible with the isomorphism $\beta$, and $(g \wedge h)^k$ corresponds to

$$[\hat{g}, \hat{h}]^k = [\hat{g}^k, \hat{h}^k],$$

while $(\hat{k})^g$ corresponds to $(\hat{k}^g)^q = \hat{(k^g)^q}$. The image $w^k$ of an arbitrary element $w \in G \wedge^q G$ is obtained by writing $w$ as a product of $q$-th powers and commutators and then computing the action of $k$ upon each factor.

(ii) By construction, the map $\lambda: G \times G \times G \rightarrow G \wedge^q G, (g, h, k) \mapsto [\hat{g}, \hat{h}](\hat{k})^q$ is a $q$-biderivation. Applying $\beta$, it corresponds to the $q$-biderivation

$$\lambda: G \times G \times G \rightarrow (E_q(G))'(E_q(G))^q, \quad (g, h, k) \mapsto (g \wedge h)\hat{k}.$$

Note that we can determine the image of the action of $G$ and of the $q$-biderivation $\lambda$ (see Proposition 2.4) in the polycyclic presentation of $G \wedge^q G$ by using the above remark.

Example 4.4 (Continuation of Example 3.3). We determine $S_3 \wedge^2 S_3$ by identifying it with the subgroup

$$(E_2(S_3))'(E_2(S_3))^2 = \langle [g_1, g_2], g_1^2, g_2^2 \rangle = \langle w \mid w^6 \rangle \cong C_6,$$

where $w = g_2^2t_1$.

(i) By Remark 4.3, the image of $(g_1 \wedge g_2)^{g_1}$ in the consistent polycyclic presentation of $S_3 \wedge^2 S_3$ corresponds to the element

$$[g_1, g_2]^{g_1} \quad \text{of} \quad (E_2(S_3))'(E_2(S_3))^2.$$

In turn, evaluating this element using the relations of $(E_2(S_3))'(E_2(S_3))^2$, we obtain the element

$$g_1^{-1}[g_1, g_2]g_1 = g_1^{-1}g_2^2g_1 = g_2^2 = w^4.$$

(ii) Analogously, the image of $(g_1, g_2, g_1)\lambda$ in the consistent polycyclic presentation of $S_3 \wedge^2 S_3$ corresponds to the element

$$[g_1, g_2]^{g_1^2} \quad \text{of} \quad (E_2(S_3))'(E_2(S_3))^2,$$

and thus, using the relations of $(E_2(S_3))'(E_2(S_3))^2$, we evaluate this element to get

$$[g_1, g_2]g_1^2 = g_2^2t_1 = w.$$
Example 4.5 (Continuation of Example 3.4). Now we determine $D_\infty \wedge^2 D_\infty$ by identifying it with the subgroup

$$(E_q(D_\infty))'(E_q(D_\infty))^2 = \langle [g_1, g_2], g_1^2, g_2^2 \rangle$$

$$= \langle w_1, w_2, w_3 \mid w_1^2, [w_1, w_2], [w_1, w_3], w_2^2, [w_2, w_3] \rangle$$

$$\cong C_2 \times C_2 \times C_\infty,$$

where $w_1 = g_2^2, w_2 = t_1, w_3 = t_2.$

(i) Analogous to Example 4.4, by Remark 4.3, the image of $(g_1 \wedge g_2)^{g_1}$ in the consistent polycyclic presentation of $D_\infty \wedge^2 D_\infty$ corresponds to the element $[g_1, g_2]^{g_1}$ in $(E_q(D_\infty))'(E_q(D_\infty))^2$, that is, to

$$g_1^{-1}[g_1, g_2]g_1 = g_1^{-1}g_2^2t_2g_1 = g_2^{-1}t_2 = w_1^{-1}w_3.$$  

(ii) Similarly, the image of $(g_1, g_2, g_1)^{\lambda}$ corresponds to the element $[g_1, g_2]g_1^2$ of $(E_q(D_\infty))'(E_q(D_\infty))^2$, which results in

$$[g_1, g_2]g_1^2 = g_2^2t_2t_1 = w_1w_2w_3.$$  

5 The $q$-exterior center of a polycyclic group

Our next step is to show that we can easily determine the $q$-exterior center of a polycyclic group $G$ given by a consistent polycyclic presentation, using a consistent polycyclic presentation for $E_q(G)$ and standard methods for polycyclic groups (see [12, Chapter 8]). These techniques also extend those found in [8] for the exterior center (case $q = 0$).

Theorem 5.1. Let $G$ be a polycyclic group and $\pi : E_q(G) \to G$ the natural epimorphism. Then $Z^*_q(G) = (Z(E_q(G)))^\pi$.

Proof. For each $g \in G$, let $\widetilde{g}$ be a pre-image of $g$ in $E_q(G)$ under the epimorphism $\pi$, i.e., $(\overline{g})\pi = g$. Now, $[\overline{g}, a] = 1$ for all $a \in G$ if, and only if, $[\overline{g}, x] = 1$ for all $x \in E_q(G)$. Indeed, given $x \in E_q(G)$, then $x^\pi \in G$, and by assumption, $[\overline{g}, x^\pi] = 1$. On the other hand, $(\overline{x^\pi})^\pi = x^\pi$. Thus,

$$(\overline{x^\pi})^{-1}x \in \text{Ker}(\pi) \leq Z(E_q(G)).$$

Therefore, we have

$$1 = [\overline{x^\pi}^{-1}x, \overline{g}] = [\overline{x^\pi}^{-1}, \overline{g}][x, \overline{g}] = [\overline{x^\pi}, \overline{g}]^{-1}[x, \overline{g}]$$

$$= [\overline{x^\pi}, \overline{g}][x, \overline{g}] = [x, \overline{g}].$$
Conversely, given \( a \in G \), we have \( \widetilde{a} \in E_q(G) \). By assumption, \([x, \widetilde{g}] = 1\) for all \( x \in E_q(G) \) and, in particular, for \( x = \widetilde{a} \). Thus, \([\widetilde{a}, \widetilde{g}] = 1\) for all \( a \in G \).

Now, remember that the map \( \beta: G \wedge^q G \to (E_q(G))'(E_q(G))^q \) given by
\[
(g \wedge h)\beta = [\widetilde{g}, \widetilde{h}] \quad \text{and} \quad (k)\beta = \widetilde{k}^q
\]
is an isomorphism, and thus we get
\[
Z_{\wedge}^q(G) = \{ g \in G \ | \ 1 = g \wedge a \in G \wedge^q G \ \text{for all} \ a \in G \}
= \{ g \in G \ | \ [\widetilde{g}, \widetilde{a}] = 1 \ \text{for all} \ a \in G \} \quad \text{(by using} \ \beta) \\
= \{ g \in G \ | \ [\widetilde{g}, x] = 1 \ \text{for all} \ x \in E_q(G) \} \\
= \{ g \in G \ | \ \widetilde{g} \in Z(E_q(G)) \} \\
= (Z(E_q(G)))\pi. \hspace{1cm} \square
\]

By Theorem 5.1, the \( q \)-exterior center of a group \( G \) given by a consistent polycyclic presentation can be easily determined: First we determine a polycyclic presentation for \( E_q(G) \) and its corresponding natural epimorphism \( \pi: E_q(G) \to G \). Then we compute the center \( Z(E_q(G)) \) using standard methods for polycyclically presented groups (see [12, Chapter 8]) and, finally, we apply \( \pi \) to obtain \( Z_{\wedge}^q(G) = (Z(E_q(G)))\pi \).

**Example 5.2** (Continuation of Example 3.3). It follows from the consistent polycyclic presentation of \( E_2(S_3) \) that \( Z(E_2(S_3)) = \langle t_1, t_3 \rangle \). Thus, \( Z_{\wedge}^2(S_3) = 1 \) and so, as one should expect, \( S_3 \) is 2-capable. In fact, the group \( Q \) given by
\[
Q = \langle a, b \ | \ a^4 = 1, a^{-1}ba = b^{-1}, b^3 = 1 \rangle
\]
has center \( Z(Q) = \langle a^2 \rangle = Z_2(Q) \), of order 2, and \( S_3 \cong Q/Z(Q) \).

**Example 5.3** (Continuation of Example 3.4). It follows from the polycyclic presentation of \( E_q(D_\infty) \) that \( Z_q(E_q(D_\infty)) = \langle t_1, t_2 \rangle \). Thus, \( Z_{\wedge}^q(D_\infty) = 1 \); hence, \( D_\infty \) is \( q \)-capable for all \( q \geq 0 \). Indeed, the group
\[
Q = \langle a, b \ | \ a^{2q} = 1, a^{-1}ba = b^{-1} \rangle
\]
has center \( Z(Q) = \langle a^2 \rangle = Z_q(Q) \), of order \( q \), and \( D_\infty \cong Q/Z(Q) \).

6 A consistent polycyclic presentation for \( \nu^q(G)/\Delta^q(G) \)

As seen in Section 4, we can determine a consistent polycyclic presentation \( F_r/U \) for the \( q \)-exterior square \( G \wedge^q G \) in the generator \( w_1, \ldots, w_r \) and relators, say
Theorem 6.2. From such a presentation, we will determine a consistent polycyclic presentation for the group \( v^q(G)/\Delta^q(G) \). Remember that
\[
v^q(G)/\Delta^q(G) \cong (G \wedge^q G) \rtimes (G \times G).
\]

According to Remark 4.3, we can determine the image of the \( q \)-biderivation \( \lambda: G \times G \times G \rightarrow G \wedge^q G \): \((g, h, 1) \mapsto (g \wedge h) \) and \((1, 1, k) \mapsto \hat{k} \) in the consistent polycyclic presentation we obtained for \( G \wedge^q G \). Analogously, we can construct the natural action of \( G \) on the presentation found for \( G \wedge^q G \), which is given by \((g \wedge h)^x = g^x \wedge h^x \), \((\hat{k})^x = (\hat{k}^x)\).

Recall that we are given a consistent polycyclic presentation of group \( G \); as before, \( G = \langle g_1, \ldots, g_n \mid r_1, \ldots, r_l \rangle \).

**Definition 6.1.** Define \( \tau^q(G) \) to be the group generated by \( g_1, \ldots, g_n, g_1^\varphi, \ldots, g_n^\varphi, w_1, \ldots, w_r \), subject to the following defining relations:

1. \( r_i(g_1, \ldots, g_n) = 1 \) for \( 1 \leq i \leq l \),
2. \( r_i(g_1^\varphi, \ldots, g_n^\varphi) = 1 \) for \( 1 \leq i \leq l \),
3. \( u_i(w_1, \ldots, w_r) = 1 \) for \( 1 \leq i \leq s \),
4. \( g_i^{-1}g_j^\varphi g_i = g_j^\varphi((g_i, g_j, 1)\lambda)^{-1} \) for \( 1 \leq i, j \leq n \),
   \[ g_i g_j^{-1} = g_j((g_i^{-1}, g_j, 1)\lambda)^{-1} \] for \( 1 \leq i, j \leq n, i \not\in I \),
5. \( g_j^{-1}w_i g_j = w_j^{g_j} \) for \( 1 \leq i \leq r, 1 \leq j \leq n \),
   \[ g_j w_i g_j^{-1} = w_i^{g_j} \] for \( 1 \leq i \leq r, 1 \leq j \leq n, j \not\in I \),
   \[ g_j^{\varphi} w_i g_j = w_i^{g_j} \] for \( 1 \leq i \leq r, 1 \leq j \leq n \),
   \[ g_j^{\varphi} w_i g_j = w_i^{g_j} \] for \( 1 \leq i \leq r, 1 \leq j \leq n, j \not\in I \).

Notice that we can compute the right-hand side of relations (4) and (5) as words in \( w_1, \ldots, w_r \) (see Remark 4.3).

**Theorem 6.2.** Let \( W \leq \tau^q(G) \) be the subgroup \( \langle w_1, \ldots, w_r \rangle \).

(i) \( W \) is a normal subgroup of \( \tau^q(G) \) and \( \tau^q(G)/W \cong G \times G \).

(ii) The presentation of \( \tau^q(G) \) in Definition 6.1 is a consistent polycyclic presentation.

(iii) \( W \cong G \wedge^q G \).

(iv) The map \( \psi: v^q(G) \rightarrow \tau^q(G) \) defined by
   \[ (g_i)\psi = g_i, \quad (g_i^\varphi)\psi = g_i^\varphi \quad \text{and} \quad (\hat{k})\psi = (1, 1, k)\lambda \]
for all \( 1 \leq i \leq n \) and all \( k \in G \) extends to a well defined homomorphism (also denoted by \( \psi \)) such that \( \text{Ker} \psi = \Delta^q(G) \).
Proof. The proof is mainly based on a careful analysis of the sets of defining relations (1)–(5) of $\tau^q(G)$, as established in Definition 6.1. Relations (5) tell us that $W$ is in fact a normal subgroup of $\tau^q(G)$. Relations (1), (2) and (4) imply that $\tau^q(G)/W \cong G \times G^\varphi$. In addition, relations (3) show that $W$ is a factor of $G \wedge^q G$. Thus, $\tau^q(G)$ satisfies the exact sequence

$$ G \wedge^q G \to \tau^q(G) \to G \times G^\varphi \to 1. $$

Now, relations (5) imply that $G \times G^\varphi$ acts by conjugation on $W$, in the same way as $G \times G$ acts naturally on $G \wedge^q G$. In particular, we get $[w, g] = w^{-1}w^g$ and, analogously, $[w, h^\varphi] = w^{-1}w^{h^\varphi}$ for all words $w$ in $w_1, \ldots, w_r$, all words $g$ in $g_1, \ldots, g_n$ and all words $h^\varphi$ in $g_1^\varphi, \ldots, g_n^\varphi$. Furthermore, the definition of a $q$-biderivation and relations (4) imply $[g, h^\varphi] = (g, h, 1)\lambda$ for all words $g$ in $g_1, \ldots, g_n$ and $h^\varphi$ in $g_1^\varphi, \ldots, g_n^\varphi$. Part (i) then follows directly from the above considerations.

Part (ii). Relations (1)–(5) already have the form of a polycyclic presentation. Thus, it remains to check them for consistency. Well, all consistency relations in the generators $g_1, \ldots, g_n$ are satisfied, once relations (1) come from a consistent polycyclic presentation of $G$. Analogously, for relations (2) and (3); they say that all consistency in the generators $g_1^\varphi, \ldots, g_n^\varphi$ and $w_1, \ldots, w_r$ are also satisfied. Besides that, if a consistency relation involves one generator of the $w_1, \ldots, w_r$, then it is satisfied once $G \times G^\varphi$ acts on $W$ likewise $G \times G$ acts naturally on $G \wedge^q G$. Therefore, the bottom line is really to check the consistency relations in $g_1, \ldots, g_n, g_1^\varphi, \ldots, g_n^\varphi$, involving mixed generators $g_i$ and $g_j^\varphi$. They are

$$ g_k^\varphi(g_j g_i) = (g_k^\varphi g_j) g_i \quad \text{for } j > i, $$
$$ g_k^\varphi(g_j^\varphi g_i) = (g_k^\varphi g_j^\varphi) g_i \quad \text{for } k > j, $$
$$ ((g_j^\varphi)^e_i) g_i = (g_j^\varphi)^{e_j-1}(g_j^\varphi g_i) \quad \text{for } j \in I, $$
$$ g_j^\varphi(g_i^e) = (g_j^\varphi g_i)^{e_i-1} \quad \text{for } i \in I, $$
$$ g_j^\varphi = (g_j^\varphi g_i)^{-1} g_i \quad \text{for } i \notin I. $$

Consider for example the first of these relations. Supposing that

$$ g_i^{-1} g_j g_i = r_{ij}(g_1, \ldots, g_n) = r_{ij} $$

in the defining relations of $G$ and using the fact that $\lambda$ is a $q$-biderivation, we get

$$ g_k^\varphi(g_j g_i) = (g_j g_i)(g_k^\varphi)^{g_j g_i} $$
$$ = (g_i r_{ij})(g_k^\varphi ((g_j g_i, g_k, 1)\lambda)^{-1}). $$
\[
(g_k^q g_j) g_i = (g_j g_k^q ((g_j, g_k, 1)\lambda)^{-1}) g_i \\
= g_j g_k^q g_i ((g_j, g_k, 1)\lambda)^{-1} g_i \\
= g_i r_{ij} g_k^q ((i, g_k, 1)\lambda)^{-1} ((g^q_j, g_k, 1)\lambda)^{-1} \\
= g_i r_{ij} g_k^q ((g^q_j, g_k, 1)\lambda(g_i, g_k, 1)\lambda)^{-1} \\
= g_i r_{ij} g_k^q ((g^q_j g_k, g_k, 1)\lambda)^{-1}.
\]

The other consistency relations can be checked by similar calculations. Thus, we obtain that \[\tau^q(G)\] is given by a consistent polycyclic presentation.

Part (iii) follows from (ii) and from the theory of polycyclic presentations (see for instance [12, Section 8.3]), once \(W\), as a subgroup of \(\tau^q(G)\), has a consistent polycyclic presentation in the generators \(w_1, \ldots, w_r\) and relations \(u_1, \ldots, u_s\).

Therefore, \(W \cong G \wedge^q G\).

(iv) Since \(\lambda\) is a \(q\)-biderivation, all relations of \(\nu^q(G)\) hold in \(\tau^q(G)\). Thus, \(\psi: \nu^q(G) \to \tau^q(G)\) is an epimorphism, once \(g_i, g_j^q \in \text{Im}(\psi)\) for all \(1 \leq i, j \leq n\), and if a word \(w_i \in W \cong G \wedge^q G\) is a product of commutators and \(q\)-th powers, then \(w_i \in \text{Im}(\psi)\) for all \(1 \leq i \leq r\). Consequently, \(\text{Im}(\psi) = \tau^q(G)\). Besides that, \([(g, h^q)]\psi = [g, h^q] = (g, h, 1)\lambda\) and \((k)\psi = (1, 1, k)\lambda\) for all words, \(g\) in the generators \(g_1, \ldots, g_n\), \(h^q\) in \(g_1^q, \ldots, g_n^q\) and all \(k \in G\). Therefore, the map induced by \(\psi\) on the subgroup \(\Upsilon^q(G)\) coincides with the map \(\delta: \Upsilon^q(G) \to G \wedge^q G\) by construction. We then get the commutative diagram

\[
1 \longrightarrow \Upsilon^q(G) \longrightarrow \nu^q(G) \longrightarrow G \times G^q \longrightarrow 1 \\
\phantom{1 \longrightarrow} \downarrow{\delta} \phantom{G \times G^q \longrightarrow 1} \\
1 \longrightarrow W \longrightarrow \tau^q(G) \longrightarrow G \times G^q \longrightarrow 1. 
\]

It follows that \(\text{Ker}(\psi) \leq \Upsilon^q(G)\) and, consequently, \(\text{Ker}(\psi) = \text{Ker}(\delta) = \Delta^q(G)\). This completes the proof.

\[
\text{Example 6.3} \quad \text{(Continuation of Example 3.3). According to the above result, the following polycyclic presentation is a presentation of } \tau^2(S_3) = v^2(S_3)/\Delta^2(S_3) \text{ as the group generated by } g_1, g_2, g_1^q, g_2^q, w \text{ subject to the relations}
\]

(1) \(g_1^2 = 1, g_1^{-1} g_2 g_1 = g_2^{-1}, g_2^3\),

(2) \((g_1^q)^2 = 1, (g_1^q)^{-1} g_2^q g_1^q = (g_2^q)^{-1}, (g_2^q)^3\),

(3) \(w^6 = 1\),
(4) $g_1^{-1}g_1g_1 = g_1^φ, g_1^{-1}g_2g_1 = g_2^φw^2$, $g_2^{-1}g_1g_2 = g_1^φw^4, g_2^{-1}g_2^φg_2 = g_2^φ$.

(5) $g_1^{-1}wg_1 = w^5, g_2^{-1}wg_2 = w$,
    $(g_1^φ)^{-1}w_1g_1^φ = w^5, (g_2^φ)^{-1}w_2g_2^φ = w$.

**Example 6.4** (Continuation of Example 3.4). Again, according to Theorem 6.2, we find that $τ^2(D_∞) = v^2(D_∞)/Δ^2(D_∞)$ has the polycyclic presentation in the generators $g_1, g_2, g_1^φ, g_2^φ, w_1, w_2, w_3$ subject to the relations

(1) $g_1^2 = 1, g_1^{-1}g_2g_1 = g_2^{-1}$,
(2) $(g_1^φ)^2 = 1, (g_1^φ)^{-1}g_2^φg_1 = (g_2^φ)^{-1}$,
(3) $w_1^{-1}w_2w_1 = w_2, w_2^{-1}w_3w_1 = w_3, w_2^{-1}w_3w_2 = w_3$,
(4) $g_1^{-1}g_1^φg_1 = g_1^φ, g_1^{-1}g_2^φg_1 = g_2^φw_1^{-1}w_3$,
    $g_2^{-1}g_1^φg_2 = g_1^φw_1w_3, g_2^{-1}g_2^φg_2 = g_2^φ$,
    $g_2g_1^φg_2^{-1} = g_1^φw_1^{-1}w_3, g_2g_2^φg_2^{-1} = g_2^φ$,
(5) $g_1^{-1}w_1g_1 = w_1^{-1}, g_1^{-1}w_2g_1 = w_2, g_1^{-1}w_1g_1 = w_3$,
    $g_2^{-1}w_1g_2 = w_1, g_2^{-1}w_2g_2 = w_2, g_2^{-1}w_3g_2 = w_3$,
    $g_2w_1g_2^{-1} = w_1, g_2w_2g_2^{-1} = w_2, g_2w_3g_2^{-1} = w_3$,
    $(g_1^φ)^{-1}w_1g_1^φ = w_1^{-1}, (g_1^φ)^{-1}w_2g_1^φ = w_2, (g_1^φ)^{-1}w_1g_1^φ = w_3$,
    $(g_2^φ)^{-1}w_1g_2^φ = w_1, (g_2^φ)^{-1}w_2g_2^φ = w_2, (g_2^φ)^{-1}w_3g_2^φ = w_3$,
    $g_2^φw_1(g_2^φ)^{-1} = w_1, g_2^φw_2(g_2^φ)^{-1} = w_2, g_2^φw_3(g_2^φ)^{-1} = w_3$.

**7 A polycyclic presentation for $v^q(G)$**

We can now use the consistent polycyclic presentation of $τ^q(G)$ in place of $F_n/R$ and the finite presentation of $v^q(G)$ in place of $F_m/S$. Note that the epimorphism $ψ: v^q(G) → τ^q(G)$ has the required form.

**Theorem 7.1.** $v^q(G) = π^q(τ^q(G))$.

**Proof.** We have $τ^q(G) = F_n/R, v^q(G) = F_m/S$ and the epimorphism $ψ: v^q(G) → τ^q(G)$ with the kernel $Ker(ψ) = K/S$.

By definition, $π^q(τ^q(G)) ≅ F_m/K^q[K, F_m]S$. By Theorem 6.2 the group $v^q(G)$ is a $q$-central extension of $τ^q(G)$. Thus, $[K, F_m]K^q ≤ S$. Since, by Lemma 3.5, $π^q(τ^q(G)) = F_m/K^q[K, F_m]S$, it follows that $π^q(τ^q(G)) = F_m/S = v^q(G)$, as desired. □

Notice that, in the above proof, we used a finite presentation for $v^q(G)$; however, the presentation coming from Definition 2.1 is finite only if $G$ is finite. On the
other hand, the epimorphism $\psi: v^q(G) \to \tau^q(G)$ does not depend on the finiteness of the presentation of $v^q(G)$, and thus we can consider this epimorphism. If $G$ is an infinite polycyclic group, then, by definition, $v^q(G)$ is given by an infinite presentation, say $F/S$, where $F$ is a free group on the generators of $v^q(G)$, which we denote by $X$, of infinite rank, and where $S$ is the normal closure of relations (2.1)–(2.6). But, being polycyclic, $v^q(G)$ has a finite polycyclic presentation $\langle X_0 | S_0 \rangle$, where $X_0$ is a free group on the generators of $v^q(G)$, which we denote by $X$, of infinite rank, and where $S_0$ is the normal closure of relations (2.1)–(2.6). Thus, we can use the results in Lemma 3.5, and it suffices to prove that the image of $\zeta$ is generated by the elements $g_1, \ldots, g_n, g_1^\varphi, \ldots, g_n^\varphi, \tilde{g}_1, \ldots, \tilde{g}_n$ and that $(S)\zeta$ is generated by the defining relations of $v^q(G)$ evaluated only on the polycyclic generators of $G$.

**Proposition 7.2.** Consider the subgroup $L$ of $E_q(\tau^q(G))$ given by

$$L = \langle g_1, \ldots, g_n, g_1^\varphi, \ldots, g_n^\varphi, (1, 1, g_1)^\lambda, \ldots, (1, 1, g_n)^\lambda \rangle.$$ 

Then, $\text{Im}(\zeta) = L$. In addition, $(S)\zeta$ is generated by the defining relations of $v^q(G)$ in the polycyclic generators of $G$ and $G^\varphi$ in $E_q(\tau^q(G))$.

**Proof.** By definition of $L$, to show that $\text{Im}(\zeta) = L$, it suffices to show that

$$(1, 1, k)^\lambda \in L \quad \text{for all } k \in G.$$ 

We prove this by induction on the number of polycyclic generators of $G$. If $n = 1$, then $G = \langle g_1 \rangle$, so $k = g_1^\alpha$ for some $\alpha \in \mathbb{Z}$. Thus, $(1, 1, k)^\lambda = (1, 1, g_1^\alpha)^\lambda$. For $\alpha \geq 2$, using relation (2.7) in Definition 2.3, we have

$$(1, 1, k)^\lambda = (1, 1, g_1^2)^\lambda$$ 

$$= (1, 1, g_1)^\lambda \prod_{i=1}^{q-1} (((g_1, g_1^{-i}, 1)^\lambda g_1^{q-1-i}) (1, 1, g_1)^\lambda \in L.$$ 

By assuming it for $\alpha - 1$, then, analogously,

$$(1, 1, k)^\lambda = (1, 1, g_1^{\varphi})^\lambda$$ 

$$= (1, 1, g_1)^\lambda \prod_{i=1}^{q-1} (((g_1, g_1^{-i(n-1)}, 1)^\lambda g_1^{q-1-i}) (1, 1, g_1^{n-1})^\lambda \in L.$$ 

In addition, again by the very definition of $\lambda$, as above, we obtain $(1, 1, 1)^\lambda = 1$ and $(1, 1, k^{-1})^\lambda = \left(\prod_{i=1}^{q-1} ((x, x^i, 1)^\lambda)\right)^{-1}((1, 1, x)^\lambda)^{-1}$, which are elements of $L$. This completes the case $n = 1$. 

---

*The $q$-tensor square of a polycyclic group, $q \geq 0*
Suppose \( n \geq 1 \) and that our assertion is true for \( n - 1 \). If \( k = g_1^{\alpha_1} \cdots g_n^{\alpha_n} \), then the same argument used above gives

\[
(1, 1, k)\lambda = (1, 1, g_1^{\alpha_1} \cdots g_n^{\alpha_n})\lambda \\
= (1, 1, g_1^{\alpha_1})\lambda \prod_{i=1}^{q-1} ((g_1^{\alpha_1}, (g_2^{\alpha_2} \cdots g_n^{\alpha_n})^{-i}, 1)\lambda) (g_1^{\alpha_1})^{q-1-i} \\
\times (1, 1, g_2^{\alpha_2} \cdots g_n^{\alpha_n})\lambda \in L.
\]

Thus, \( \text{Im}(\zeta) = L \).

Therefore, having obtained a consistent polycyclic presentation of \( \tau^q(G) \), we can extend it by adding new \((q\text{-central})\) generators \( t_i \), one for each relator \( r_i \) of \( \tau^q(G) \), and changing each relator \( r_i \) by \( r_i t_i^{-1} \). Then, we evaluate the consistency relations among the relators of \( v^q(G) \) in this new presentation and apply Lemma 3.5 (ii).

The following result can be used in order to reduce the number of new generators added and the number of relators evaluated in this process.

**Lemma 7.3.** It is redundant to add new generators corresponding to relations (1) and (2) in the definition of \( \tau^q(G) \). If these generators are not introduced, then it is redundant to evaluate the relators (1) and (2) in the definition of \( v^q(G) \).

**Proof.** The relators (1) and (2) in the definition of \( v^q(G) \) coincide with the relators (1) and (2) in the definition of \( \tau^q(G) \). Therefore, if we add new generators corresponding to those relators in (1) and (2) of Definition 6.1 and then we evaluate the relators (1) and (2) in the definition of \( v^q(G) \), then, as a result, we obtain the corresponding generators. This means that the corresponding generators are eliminated in the process of the constructing the factor group as described in Lemma 3.5 (ii). This proves the result.

**Example 7.4** (Continuation of Example 3.3). We compute a polycyclic presentation of \( v^2(S_3) \) as a central extension \( \mathcal{E}^2(\tau^2(S_3)) \) of \( \tau^2(S_3) \). There are a lot of calculations to get such a presentation (by hand), so we omit the details. We obtain a polycyclic presentation for \( v^2(S_3) \) in the generators \( g_1, g_2, g_1^\varphi, g_2^\varphi, w, t \) and defining relations given by

1. \( g_1^2 = 1, g_1^{-1} g_2 g_1 = g_2^{-1}, g_2^3 \),
2. \( (g_1^\varphi)^2 = 1, (g_1^\varphi)^{-1} g_2^\varphi g_1^\varphi = (g_2^\varphi)^{-1}, (g_2^\varphi)^3 \),
3. \( w^6 = t, t^2, t\text{-central} \),
(4) \(g_1^{-1}g_1 g_1 = g_1^w w^6, g_1^{-1}g_1 g_1 = g_2^w w^8, g_2^{-1}g_1 g_2 = g_1^w w^4, g_2^{-1}g_2 g_2 = g_2^w,\)

(5) \(g_1^{-1}w g_1 = w^5, g_2^{-1}w g_2 = w, g_1^{-1}w g_1 = w^5, g_2^{-1}w g_2 = w.\)

From this, we get the 2-tensor square
\[
S_3 \otimes^2 S_3 \cong \langle w \rangle \leq \nu^2(S_3),
\]
that is, \(S_3 \otimes^2 S_3 \cong \mathbb{Z}_{12}.\) In addition, we immediately find that \(\Delta^2(S_3) \cong \mathbb{Z}_2.\)

**Example 7.5.** Here we compute a polycyclic presentation for \(\nu^3(D_{\infty})\) as the group generated by \(g_1, g_2, g_1^\varphi, g_2^\varphi, w_1, w_2\) subject to the relations

1. \(g_1^2 = 1, g_1^{-1}g_2 g_1 = g_2^{-1},\)
2. \((g_1^\varphi)^2 = 1, (g_1^\varphi)^{-1}g_2^\varphi g_1 = (g_2^\varphi)^{-1},\)
3. \(w_1^{-1}w_2 w_1 = w_2^{-1},\)
4. \(g_1^{-1}g_1 g_1 = g_1^w, g_1^{-1}g_2 g_1 = g_2^w w^2,\)
\[g_2^{-1}g_1 g_2 = g_1^w w^2, g_2^{-1}g_2 g_2 = g_2^w,\]
\[g_2 g_1^{-1} g_2^\varphi = g_1^w w^2, g_2 g_2^{-1} = g_2^w,\]
5. \(g_1^{-1}w g_1 = w_1, g_1^{-1}w g_1 = w_2^{-1},\)
\[g_2^{-1}w g_2 = g_1 w_2, g_2^{-1}w g_2 = w_2,\]
\[g_2 w_1 g_2^{-1} = g_1 w_2^{-2}, g_2 w_2 g_2^{-1} = w_2,\]
\[(g_1^\varphi)^{-1}w_1 g_1^\varphi = w_1, (g_1^\varphi)^{-1}w_2 g_1^\varphi = w_2^{-1},\]
\[(g_2^\varphi)^{-1}w_1 g_2^\varphi = w_1 w_2, (g_2^\varphi)^{-1}w_2(g_2^\varphi) = w_2,\]
\[g_2^\varphi w_1(g_2^\varphi)^{-1} = w_1 w_2^{-2}, g_2^\varphi w_2(g_2^\varphi)^{-1} = w_2.\]

According to this presentation, we find that \(D_{\infty} \otimes^3 D_{\infty} \cong D_{\infty}.\)

Notice that the computation of a presentation of \(\nu^q(G)\) becomes relatively simple if the group \(G\) is \(q\)-perfect, according to Theorem 7.6 below. We shall continue using the same notation as before. More specifically, let \(F_n/R\) be a consistent polycyclic presentation for the polycyclic group \(G\) in the generators \(g_1, \ldots, g_n,\) relators \(r_1, \ldots, r_t\) and index set \(I.\) Let \(F_r/U\) be a consistent polycyclic presentation for \(G \wedge^q G\) in the generators \(w_1, \ldots, w_r\) and relators \(u_1, \ldots, u_s,\) as found in Section 4. We determine the image of the \(q\)-biderivation \(\lambda: G \times G \times G \to G \wedge^q G:\) \((g, h, 1) \mapsto (g \wedge h)\) and \((1, 1, k) \mapsto \hat{k}\) in the consistent polycyclic presentation obtained for \(G \wedge^q G\) and construct the natural action of \(G\) on that presentation found for \(G \wedge^q G\) (as defined before: \((g \wedge h)^x = g^x \wedge h^x, \hat{k}^x = \hat{k}^x,\) according to Remark 4.3).
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Theorem 7.6. Let $G$ be a polycyclic group given as above. If $G$ is $q$-perfect, then the group $v^q(G)$ is the group generated by $g_1, \ldots, g_n, g^q_1, \ldots, g^q_n, w_1, \ldots, w_r$, subject to the defining relations

1. $r_i(g_1, \ldots, g_n) = 1$ for $1 \leq i \leq l$,
2. $r_i(g^q_1, \ldots, g^q_n) = 1$ for $1 \leq i \leq l$,
3. $u_i(w_1, \ldots, w_r) = 1$ for $1 \leq i \leq s$,

4. $g_i^{-1}g_j g_i = g_j^{(g_i, g_j, 1)\lambda^{-1}}$ for $1 \leq i, j \leq n$,
   \[ g_i g_j^{-1} g_i = g_j^{(g_i^{-1}, g_j, 1)\lambda^{-1}} \text{ for } 1 \leq i, j \leq n, i \notin I \]

5. $g_j^{-1}w_i g_j = w_i^{g_j}$ for $1 \leq i \leq r, 1 \leq j \leq n$,
   \[ g_j w_i g_j^{-1} = w_i^{g_j^{-1}} \text{ for } 1 \leq i \leq r, 1 \leq j \leq n, j \notin I, \]
   \[ g^q_j w_i g^q_j = w_i^{g_j^q} \text{ for } 1 \leq i \leq r, 1 \leq j \leq n, \]
   \[ g^q_j w_i g^q_j = w_i^{g_j^q} \text{ for } 1 \leq i \leq r, 1 \leq j \leq n, j \notin I. \]

Proof. In effect, according to Definition 6.1, the above presentation is the same as that of $\tau^q(G)$. By Theorem 6.2, $\tau^q(G) \cong v^q(G)/\Delta^q(G)$. If $G$ is $q$-perfect, then we have $\Delta^q(G) = 1$ and so $\tau^q(G) \cong v^q(G)$. Consequently, the given presentation is a presentation of $v^q(G)$.

7.1 A polycyclic presentation for the $q$-tensor square of a polycyclic group

By all we have seen, a method for determining a consistent polycyclic presentation for the $q$-tensor square $G \otimes^q G$ from a given consistent polycyclic presentation of $G$ consists of the following steps.

Algorithm 7.7. Determine a consistent polycyclic presentation for

(a) $G \otimes^q G$,
(b) $\tau^q(G)$,
(c) $v^q(G)$,
(d) the subgroup $\Upsilon^q(G)$ of $v^q(G)$.

Step (a) is a direct application of the method for computing a central extension in Section 3. If $G = F_n/R$ is a consistent polycyclic presentation of $G$, then we can determine a consistent polycyclic presentation for $E_q(G) = F_n/([F_n, R^q] R^q)$, and we get $G \otimes^q G$ as the subgroup $(E_q(G))^q (E_q(G))^q$. Step (b) is thus a direct application of the method developed in Section 4. Step (c) is obtained by another application of the method for computing a central extension in Section 3, in order
to compute $C^q(\tau^q(G))$ which, by Theorem 7.1, is isomorphic to $\nu^q(G)$. Finally, step (d) is an application of the standard method for computing presentations of subgroups of polycyclically presented groups.

Acknowledgments. The authors are grateful to a referee for the valuable comments and suggestions.

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Received February 16, 2019; revised May 12, 2019.

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