A sublinear variance bound for solutions of a random Hamilton-Jacobi equation

Ivan Matic* and James Nolen†

Abstract

We estimate the variance of the value function for a random optimal control problem. The value function is the solution \( w^{\epsilon} \) of a Hamilton-Jacobi equation with random Hamiltonian \( H(p, x, \omega) = K(p) - V(x/\epsilon, \omega) \) in dimension \( d \geq 2 \). It is known that homogenization occurs as \( \epsilon \to 0 \), but little is known about the statistical fluctuations of \( w^{\epsilon} \). Our main result shows that the variance of the solution \( w^{\epsilon} \) is bounded by \( O(\epsilon/|\log \epsilon|) \). The proof relies on a modified Poincaré inequality of Talagrand.

1 Introduction

In this paper we study the random optimal control problem

\[
    u(t, x, \omega) = \sup_{\gamma \in \mathcal{A}_{t,x}} g(\gamma(t)) - \mathcal{L}(\gamma, \omega), \quad x \in \mathbb{R}^d, \quad t > 0
\]

in dimension \( d \geq 2 \), where the supremum is taken over the set of admissible paths

\[ \mathcal{A}_{t,x} = \{ \gamma \in W^{1,\infty}([0,t]; \mathbb{R}^d) \mid \gamma(0) = x \} . \]

The upper-semicontinuous payoff function \( g : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\} \) is given. The cost functional \( \mathcal{L} \) has the form

\[
    \mathcal{L}(\gamma, \omega) = \int_0^t L(\gamma'(s), \gamma(s), \omega) \, ds = \int_0^t K(\gamma'(s)) + V(\gamma(s), \omega) \, ds,
\]

where \( K(p) : \mathbb{R}^d \to [0, \infty) \) is convex and grows super-linearly in \( |p| \). The function \( V(x, \omega) \) is a scalar random field that is statistically stationary and ergodic with respect to certain translations in \( x \). The parameter \( \omega \in \Omega \) denotes a sample from a given probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Thus, the value function \( u(t, x, \omega) \) is random. Our main result shows that the variance of \( u(x, t, \omega) \) grows only sublinearly in \( t \) as \( t \to \infty \).

Under certain conditions on \( g \) and \( L \), \( u(t, x, \omega) \) is uniformly continuous and is a viscosity solution of the random Hamilton-Jacobi equation

\[
    \begin{cases}
        u_t = H(Du, x, \omega), & x \in \mathbb{R}^d, \quad t > 0 \\
        u(0, x) = g(x), & x \in \mathbb{R}^d,
    \end{cases}
\]

\*Mathematics Department, Duke University, Box 90320, Durham, North Carolina, 27708, USA. (matic@math.duke.edu).

†Mathematics Department, Duke University, Box 90320, Durham, North Carolina, 27708, USA. (nolen@math.duke.edu).
Theorem 1.2. The product measure on Ω = \{ω\} depends on the values of ω or at the point k for any k and identically distributed. Now for k all P be the shift-invariant product measure on Ω determined by numbers. Let Ω = V w pertains to this last issue: in terms of about the properties of \(\bar{w}\) not determined by a simple averaging. Beyond this convergence result, relatively little is known of w as kind of law of large numbers for The function \(\bar{w}\) for the case of dimension d = 2 we will make use of an extra non-degeneracy condition: for some \(\eta > 0\). For certain Hamiltonians \(H(p,x,ω)\) which are convex in p, statistically stationary and ergodic with respect to translation in x, it is known [11, 13] that as \(ε \to 0\), homogenization occurs (see also [2, 9] for alternative proofs and [6, 7, 8, 12] for related results). This means that the functions \(w^ε(t,x,ω)\) converge locally uniformly in \([0,∞) × \mathbb{R}^d\), as \(ε \to 0\) to the deterministic function \(\bar{w}(t,x)\) which solves

\[
\begin{align*}
\bar{w}_t &= H(D\bar{w}, x), \quad x ∈ \mathbb{R}^d, \quad t > 0 \\
\bar{w}(0, x) &= g(x).
\end{align*}
\]

(1.4)

The function \(\bar{H}(p) : \mathbb{R}^d → \mathbb{R}\) is called the effective Hamiltonian. We may think of this convergence as kind of law of large numbers for \(w^ε\), although the limit \(\bar{w}\) and the effective Hamiltonian \(\bar{H}\) are not determined by a simple averaging. Beyond this convergence result, relatively little is known about the properties of \(\bar{H}\), about the rate of convergence \(w^ε \to \bar{w}\), or about the statistical behavior of \(w^ε - \mathbb{E}[w]\), where \(\mathbb{E}[\cdot]\) denotes expectation with respect to the probability measure \(P\). Our work pertains to this last issue: in terms of \(w^ε(t,x,ω)\), our estimate on the variance of \(u\) implies that \(\text{var}(w^ε(t,x,ω)) ≤ Cε/\log ε\), as \(ε \to 0\). Before stating our main result, let us make some definitions and assumptions more precise. We will suppose the random field \(V(x,ω)\) has the following special structure. Let \(a < b\) be two real numbers. Let \(Ω = \{a,b\}^{Z^d}\) be the set of all functions \(ω : Z^d → \{a,b\}\). Let the probability measure \(P\) be the shift-invariant product measure on \(Ω\) determined by \(P(ω_k = a) = α\) and \(P(ω_k = b) = \beta\), for all \(k ∈ Z^d\), where \(α ∈ (0,1)\) and \(β = 1 - α\). Thus the random variables \(\{ω_k\}_{k ∈ Z^d}\) are independent and identically distributed. Now for \(k ∈ Z^d\), let \(Q_k = k + [0,1)^d\) denote the unit cube with corner at the point \(k\). Given \(ω ∈ Ω\), define \(V(x,ω) : \mathbb{R}^d × Ω → \{a,b\}\) by

\[
V(x,ω) = ∑_{k ∈ Z^d} ω_k 1_{Q_k}(x),
\]

(1.5)

with \(1_{Q_k}\) is the indicator function for the set \(Q_k\). Thus, \(x → V\) is piecewise constant, taking values \(a\) or \(b\) on the unit cubes. By construction, the law of \(V(x,ω)\) is the same as that of \(V(x+k,ω)\) for any \(k ∈ Z^d\). This precise construction of the field \(V(x,ω)\) is not essential for our result to hold. In particular, the function could be mollified so that it is uniformly continuous, or \(V(x,ω)\) could depend on the values of \(ω_k\) for \(k\) in a bounded neighborhood of \(x\). Nevertheless, the choice of \(P\) as the product measure on \(Ω = \{a,b\}^{Z^d}\) is motivated by the main analytical tool presented below in Theorem 1.2.

We suppose that \(K : \mathbb{R}^d → [0,∞)\) is convex, \(K(0) = 0\), and that

\[
\lim_{|z| → ∞} \frac{K(z)}{|z|} = +∞.
\]

For the case of dimension \(d = 2\) we will make use of an extra non-degeneracy condition: for some \(ν > 1\),

\[
K(z) ≥ |z|^ν \quad ∀ z ∈ B_{1/2}(0).
\]

(1.6)

Given \(V\) and \(K\), let \(L(p,x,ω) = K(x) + V(x,ω)\) and let \(u\) be defined by (1.1). The following estimate of the variance of \(u\) for large \(t\) is our main result:
Theorem 1.1 Let $d \geq 2$. Let $x \in \mathbb{R}^d$ and suppose that $g : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ satisfies
\[
g(y) < g(x) + C_1(1 + |y - x|), \quad \forall y \in \mathbb{R}^d.
\] (1.7)

There is a constant $C > 0$, depending only on $C_1$, $K$, $\alpha$, $\beta$, and $|b - a|$, such that
\[
\text{var}(u(t, x, \omega)) \leq C \frac{t}{\log t}, \quad \forall t \geq 2.
\] (1.8)

The main tool that we use to control the variance of $u(t, x, \omega)$, is the following theorem, which is a slight variation of an inequality of Talagrand (see [14], Theorem 1.5). This result holds for product spaces of the form $\Omega_J = \{a, b\}^J$, $J$ being a finite set, and $\mathbb{P}$ being the product measure on $\Omega_J$ with marginals $\mathbb{P}(\omega_j = a) = \alpha \in (0, 1)$ and $\mathbb{P}(\omega_j = b) = \beta = 1 - \alpha$, for all $j \in J$. Let us define $\phi_j \omega$ to be the element of $\{a, b\}^J$ which is identical to $\omega$ except that the $j$-th component $\omega_j$ is opposite to $\omega_j$.

That is, $\phi_j \omega = \omega'$, where $\omega_k' = \omega_k$ for $k \neq j$, and $\omega_j' \neq \omega_j$. For each random variable $f : \Omega_J \to \mathbb{R}$ define $\sigma_j f(\omega) = f(\phi_j \omega)$ and
\[
\rho_j f(\omega) = \frac{\sigma_j f(\omega) - f(\omega)}{2}.
\]

Theorem 1.2 There is a constant $C > 0$, independent of $|J|$, such that
\[
\text{var}(f) \leq C \sum_{j \in J} \frac{\|\rho_j f\|_2^2}{1 + \log \frac{\|\rho_j f\|_2}{\|\rho_j f\|_1}}
\] (1.9)

holds for all $f \in L^2(\Omega_J)$.

The idea of using this inequality to estimate the variance of $f(\omega) = u(t, x, \omega)$ comes from the work of Benjamini, Kalai, and Schramm [4] who used this inequality to estimate the distance variance in first passage percolation, a problem which has some features similar to the control problem (1.1). Specifically, they consider the length of minimal paths between two points in the integer lattice $\mathbb{Z}^d$ under a random metric. Each edge $e$ in the nearest-neighbor graph is assigned an independent random weight $\omega_e \in \{a, b\}$, and the length of a path between two points $x, y \in \mathbb{Z}^d$ is defined as the sum of the edge weights along a path connecting $x$ and $y$. They proved that $\text{var}(d_{\omega_e}(0, v)) \leq C|v|/\log |v|$, where $d_{\omega_e}(0, v)$ is the length of the shortest path connecting $0$ and $v$. See [5] for some extensions of that result. The main difficulty in applying the ideas of [4] to the present setting comes from the different structure of the cost functional $\mathcal{L}(\gamma, \omega)$, which necessitates more control on the optimizing paths.

As we have mentioned, for $d \geq 2$ there are relatively few results about the random fluctuation of $u(t, x, \omega)$ (as $t \to \infty$) or $w^\epsilon(t, x, \omega)$ (as $\epsilon \to 0$). In [10], Rezakhanlou derived conditions under which a central limit theorem holds for $w^\epsilon(t, x)$ where $w^\epsilon$ is the solution of the Hamilton-Jacobi equation (1.3), i.e. whether $\epsilon^{-1/2}(w^\epsilon - \bar{w})$ converges in law to some nontrivial stochastic process as $\epsilon \to 0$. In the case $d = 1$ those conditions can be verified for Hamiltonians having the form $H(p, x, \omega) = K(p) - V(x, \omega)$, and the limit distribution can be computed (see Corollary 2.6 in [10]). For $d \geq 2$, however, it is difficult to verify those conditions. Indeed, our result shows that we may have $\text{var}(w^\epsilon) = o(\epsilon)$, which is less than what a CLT as in [10] would suggest. As this paper was being written, we learned of another work by Armstrong, Cardaliaguet, and Souganidis [11], who study the rate of convergence $w^\epsilon \to \bar{w}$. Our Theorem 1.1 pertains to the variance of $w^\epsilon$, i.e. the statistical error $w^\epsilon - \mathbb{E}[w^\epsilon]$, but does not give an estimate of the bias $\mathbb{E}[w^\epsilon] - \bar{w}$.

The paper is organized as follows. In Section 2 we derive some properties of the paths $\gamma$ which nearly optimize (1.1). Section 3 contains the main argument for the proof of Theorem 1.1. Section 4 and Section 5 contain proofs so some technical estimates needed in Section 3.
Some text from the image...
Lemma 2.2 Suppose that $g: \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$, $g(0) \in \mathbb{R}$, and
\[ g(y) < g(0) + C_1(1 + |y|), \quad \forall \ y \in \mathbb{R}^d. \tag{2.14} \]

There is a constant $R$ depending only on $K$, $C_1$, and $b-a$ such that
\[ |\gamma(t_2) - \gamma(t_1)| \leq R(1 + |t_1 - t_2|), \quad \forall \ t_2, t_1 \in [0, t] \]
holds for all paths $\gamma \in M_1(t, \omega)$ and all $t > 1$.

Proof of Lemma 2.2: We first show there is a constant $R_0$ depending only on $K$, $C_1$, and $b-a$ such that
\[ |\gamma(t) - \gamma(0)| \leq tR_0 \tag{2.15} \]
holds for all $\gamma \in M_1(t, \omega)$ and all $t \geq 1$. Define the path $\hat{\gamma}(s) = \gamma(0) = 0$ for all $s \in [0, t]$. We have
\[ u(t, \omega) \geq g(0) - \mathcal{L}(\hat{\gamma}, \omega) \geq g(0) - tb. \tag{2.16} \]
By (2.12), we also have the lower bound
\[ \mathcal{L}(\gamma, \omega) \geq tK \left( \frac{\gamma(t) - \gamma(0)}{t} \right) + at. \]

Since $\gamma \in M_1(t, \omega)$, we may combine these two estimates with $u(t, \omega) \leq 1 + g(\gamma(t)) - \mathcal{L}(\gamma, \omega)$ to conclude
\[ K \left( \frac{\gamma(t) - \gamma(0)}{t} \right) \leq \frac{1}{t} + (b-a) + \frac{g(\gamma(t)) - g(0)}{t} \leq 1 + (b-a) + C_1 \left( \frac{1 + |\gamma(t)|}{t} \right). \]

Since $K(p)$ grows super-linearly in $|p|$, (2.15) follows.

Next, consider $\gamma$ at integer times $k \in [1, t-1] \cap \mathbb{Z}$. We will show that there is a constant $R_1$, independent of $t > 1$, such that at least one time $k \in [1, t-1] \cap \mathbb{Z}$ must satisfy both
\[ |\gamma(k) - \gamma(k-1)| \leq R_1 \quad \text{and} \quad |\gamma(k+1) - \gamma(k)| \leq R_1. \tag{2.17} \]
Arguing by way of contradiction, let us suppose (2.17) does not hold. Then $|\gamma(j+1) - \gamma(j)| > R_1$ must hold for at least $t/3$ of the times $j \in [1, t-1] \cap \mathbb{Z}$. This implies that
\[ u(t, \omega) \leq 1 + g(\gamma(t)) - \mathcal{L}(\gamma, \omega) \leq 1 + g(\gamma(t)) - at - \frac{t}{3} \min_{|q| \geq R_1} K(q). \]

On the other hand, by Lemma 2.1 and (2.15), we know that
\[ u(t, \omega) \geq g(\gamma(t)) - bt - tK \left( \frac{\gamma(t) - \gamma(0)}{t} \right) - 1 \geq g(\gamma(t)) - bt - t \max_{|q| \leq R_0} K(q) - 1 \]
holds for all $\gamma \in M_1(t, \omega)$. Combining these two bounds we obtain
\[ \frac{1}{3} \min_{|q| \geq R_1} K(q) \leq 1 + (b-a) + \max_{|q| \leq R_0} K(q). \]
If $R_1 > R_0$ is sufficiently large (depending only on $b-a$, $R_0$, and $K$) this forces a contradiction. So, (2.17) must hold.
Now we conclude the proof. Let $R_2 > R_1$, and suppose that for some $t_1, t_2 \in [0, t]$ with $1 \leq |t_2 - t_1| \leq 2$ we have $|\gamma(t_2) - \gamma(t_1)| \geq R_2$. Let $k \in [1, t - 1] \cap \mathbb{Z}$ be such that (2.17) holds. Without loss of generality, we may suppose $k + 1 \leq t_1 < t_2$. Consider the path $\hat{\gamma}$ defined by

$$\hat{\gamma}(s) = \begin{cases} 
\gamma(s), & \text{for } s \in [0, k] \cup [t_2, t], \\
\gamma(k - 1) + (s - k + 1)(\gamma(k + 1) - \gamma(k - 1)), & \text{for } s \in [k - 1, k], \\
\gamma(s + 1), & \text{for } s \in [k, t_1 - 1], 
\end{cases}$$

and for $s \in [t_1 - 1, t_2]$

$$\hat{\gamma}(s) = \gamma(t_1) + (\gamma(t_2) - \gamma(t_1)) \frac{s - t_1 + 1}{t_2 - t_1 + 1}.$$ 

Then we have

$$\mathcal{L}(\hat{\gamma}) - \mathcal{L}(\gamma) \leq 4(b - a) + \int_{k-1}^{k} K(\gamma'(s)) \, ds + \int_{t_1-1}^{t_2} K(\hat{\gamma}'(s)) \, ds$$

$$\quad - \int_{k-1}^{k+1} K(\gamma'(s)) \, ds - \int_{t_1}^{t_2} K(\gamma'(s)) \, ds$$

$$\leq 4(b - a) + K(\gamma(k + 1) - \gamma(k - 1)) + (t_2 - t_1 + 1)K(\frac{\gamma(t_2) - \gamma(t_1)}{t_2 - t_1 + 1})$$

$$\quad - 2K \left( \frac{\gamma(k + 1) - \gamma(k - 1)}{2} \right) - (t_2 - t_1)K \left( \frac{\gamma(t_2) - \gamma(t_1)}{t_2 - t_1} \right)$$

$$\leq M + (t_2 - t_1 + 1)K \left( \frac{\gamma(t_2) - \gamma(t_1)}{t_2 - t_1 + 1} \right) - (t_2 - t_1)K \left( \frac{\gamma(t_2) - \gamma(t_1)}{t_2 - t_1} \right), \quad (2.18)$$

where

$$M = 4(b - a) + \max_{|z| \leq 2R_1} K(z).$$

Let $\Delta t = t_2 - t_1$ and $\sigma = (\Delta t + 1)/(\Delta t)$ and $z = (\gamma(t_2) - \gamma(t_1))/(t_2 - t_1 + 1)$. The inequality (2.18) has the form

$$\mathcal{L}(\hat{\gamma}) - \mathcal{L}(\gamma) \leq M + (\Delta t + 1)K(z) - \Delta tK(\sigma z). \quad (2.19)$$

The properties of $K$ (convexity and super-linear growth) imply that if $R_2$ sufficiently large, then

$$\inf_{|z| \geq R_2/3} K(\sigma z) - \sigma K(z) > M + 1.$$ 

Applying this at (2.19) we conclude $\mathcal{L}(\gamma') - \mathcal{L}(\gamma) < 1$, which contradicts the fact that $\gamma \in M_1(t, \omega)$. Therefore, we must have $|\gamma(t_2) - \gamma(t_1)| \leq R_2$ if $1 \leq |t_1 - t_2| \leq 2$. This and the triangle inequality now imply the desired result for all $t_1, t_2 \in [0, t]$.

**Important cubes**

Our method of estimating the variance of $u$ involves bounding the random variable $|\sigma_j u - u|$. So, we must understand when changing the value of $\omega_j$ leads to a large change in the value of $u(t, \omega)$. Given a path $\gamma \in \mathcal{A}_t$ and an index $j \in \mathbb{Z}^d$, define

$$\pi_j(\gamma) = |\{s \in [0, t] \mid \gamma(s) \in Q_j\}|,$$

which is the total time that the path $\gamma$ occupies the cube $Q_j$. Observe that for any path $\gamma \in \mathcal{A}_t$, we have

$$\mathcal{L}(\gamma, \phi_j \omega) \leq \mathcal{L}(\gamma, \omega) + (b - \omega_j)\pi_j(\gamma). \quad (2.20)$$
In particular, if $\gamma \in M_\delta(t, \omega)$, then
\[
\sigma_j u(t, \omega) = u(t, \phi_j \omega) \geq g(\gamma(t)) - \mathcal{L}(\gamma, \phi_j \omega) \\
\geq g(\gamma(t)) - \mathcal{L}(\gamma, \omega) - (b - \omega_j) \pi_j(\gamma) \\
\geq u(t, \omega) - (b - \omega_j) \pi_j(\gamma) - \delta. \tag{2.21}
\]

From this we deduce that if $\omega_j = b$ or if there is $\gamma \in M_\delta(t, \omega)$ for which $\pi_j(\gamma) = 0$, then it must be true that $u(t, \omega) - \sigma_j u(t, \omega) \leq \delta$. On the other hand, this also shows that if $u(t, \omega) - \sigma_j u(t, \omega) > \delta$, then $\omega_j = a$ and $\pi_j(\gamma) > 0$ must hold for all $\gamma \in M_\delta(t, \omega)$. This motivates the following definition. We say that the cube $Q_j$ is important if $\omega_j = a$ and for some $\delta > 0$ we have
\[
\pi_j(\gamma) > 0, \forall \gamma \in M_\delta(t, \omega). \tag{2.22}
\]

Observe that if (2.22) holds for some $\delta > 0$, then it also holds for all $\delta' \in (0, \delta]$. So, $Q_j$ is important if $\omega_j = a$ and for $\delta$ sufficiently small every $\delta$-approximate optimizer spends time in cube $Q_j$. Let $I_j \subset \Omega$ denote the event that the cube $Q_j$ is important:
\[
I_j = \{ \omega \in \Omega \mid \omega_j = a; \exists \delta > 0 \text{ such that } \pi_j(\gamma) > 0 \forall \gamma \in M_\delta(t, \omega) \} \tag{2.23}
\]
\[
= \bigcup_{n \geq 1} \{ \omega \in \Omega \mid \omega_j = a; \pi_j(\gamma) > 0 \forall \gamma \in M_{1/n}(t, \omega) \}. \tag{2.24}
\]

The above analysis shows that
\[
\{ \omega \in \Omega \mid u(t, \omega) > \sigma_j u(t, \omega) \} \subset I_j, \ \forall j \in \mathbb{Z}^d \tag{2.25}
\]
so we have
\[
\mathbb{P}(\sigma_j u < u) \leq \mathbb{P}(I_j). \tag{2.26}
\]

Observe that $\mathbb{P}(I_j)$ depends on $t$, in addition to $j$.

It will be useful to further classify some cubes as very important. To this end, we define a set of cubes
\[
N(\delta, \omega) = \bigcup_{\gamma \in M_\delta(\omega)} \{ k \in \mathbb{Z}^d \mid \pi_k(\gamma) > 0 \}. \tag{2.27}
\]

This is the set of all cubes visited by some path $\gamma \in M_\delta(t, \omega)$. Next, we define the event $I_j^+ \subset I_j \subset \Omega$ that cube $Q_j$ is very important:
\[
I_j^+ = \{ \omega \in I_j \mid \exists \delta > 0 \text{ such that } \omega_\ell = b \ \forall \ell \in N(\delta, \omega) \setminus \{j\} \}. \tag{2.27}
\]

On this event, $Q_j$ is an important cube, and for any other cube $Q_\ell$ visited by a path $\gamma \in M_\delta$, we have $\omega_\ell = b$, if $\delta$ is sufficiently small. On the event $I_j^- = I_j \setminus I_j^+$, cube $Q_j$ is important but not very important: for any $\delta > 0$ we can find a path $\gamma \in M_\delta(\omega)$ such that $\gamma$ passes through another cube $Q_\ell \neq Q_j$, on which $\omega_\ell = a$. The following lemma shows that the only way for $(u - \sigma_j u)^21_{I_j}$ to be large is if $Q_j$ is very important.

**Lemma 2.3** There is a constant $C_0 > 0$, depending only on $K$ and $|b - a|$, such that
\[
\mathbb{P}\left( \{ \omega \mid (u - \sigma_j u)^21_{I_j^-} < C_0 \} \right) = 1
\]
holds for all $t \geq 1$ and $j \in \mathbb{Z}^d$. 


Proof of Lemma 2.3: If \( \omega_j = b \), then \( \omega \notin I_j^- \), so obviously \( (u - \sigma_j u)^2 I_j^- = 0 \). Hence, we may assume \( \omega_j = a \) and \( \omega \in I_j^- \). When \( \omega_j = a \), we clearly have \( \sigma_j u \leq u \), since \( \mathcal{L}(\gamma, \phi_j \omega) \geq \mathcal{L}(\gamma, \omega) \) in this case. So, we must bound \( u - \sigma_j u \) from above.

Consider an approximate optimizer \( \gamma \in M_\delta(\omega) \) for some \( \delta \leq 1 \). If \( \pi_j(\gamma) \leq C \) then \( u - \sigma_j u \leq (b - a)C + \delta \) according to (2.21). So, we must consider the possibility that \( \pi_j(\gamma) \geq 0 \) is large. Since \( \omega \in I_j^- \), we may assume the path \( \gamma \) also passes through another cube \( Q_\ell \), with \( \ell \neq j \), for which \( \omega_\ell = a \). We will construct a new path \( \hat{\gamma} \) such that \( \pi_j(\hat{\gamma}) \leq 1 \) and \( \mathcal{L}(\hat{\gamma}, \omega) \leq \mathcal{L}(\gamma, \omega) + C \). The two paths \( \hat{\gamma} \) and \( \gamma \) will have the same starting and ending points. This implies that the difference \( u - \sigma_j u \) is bounded by a constant, since by (2.20) we have

\[
\sigma_j u = u(t, \phi_j \omega) \geq g(\hat{\gamma}(t)) - \mathcal{L}(\hat{\gamma}, \phi_j \omega)
\geq g(\hat{\gamma}(t)) - \mathcal{L}(\hat{\gamma}, \omega) - (b - a)\pi_j(\hat{\gamma})
\geq g(\hat{\gamma}(t)) - \mathcal{L}(\gamma, \omega) - C - (b - a)\pi_j(\hat{\gamma}) \geq u(t, \omega) - C - (b - a) - \delta. \tag{2.28}
\]

Suppose that \([t_1, t_2] \) is the smallest interval containing all \( s \) for which \( \gamma(s) \in Q_j \). We may assume \( t_2 - t_1 \geq \pi_j(\gamma) > 1 \). Suppose that \( \gamma(t_3) \in Q_\ell \) where \( \ell \neq j \) and \( \omega_\ell = a \). We may suppose that \( t_3 > t_2 \) (the case \( t_3 < t_1 \) is similar). Define the new path \( \hat{\gamma} \) as follows:

(i) For \( s \in [0, t_1] \), let \( \hat{\gamma}(s) = \gamma(s) \).

(ii) For \( s \in [t_1, t_1 + 1] \), let \( \hat{\gamma}(s) = \gamma(t_1) + (s - t_1)(\gamma(t_2) - \gamma(t_1)) \).

(iii) For \( s \in [t_1 + 1, t_1 + 1 + (t_3 - t_2)] \), let \( \hat{\gamma}(s) = \gamma(s - t_1 - 1 + t_2) \).

(iv) For \( s \in [t_1 + 1 + (t_3 - t_2), t_3] \), let \( \hat{\gamma}(s) = \gamma(t_3) \).

(v) For \( s \in [t_3, t] \), let \( \hat{\gamma}(s) = \gamma(s) \).

Much of \( \hat{\gamma} \) is just a linear reparameterization of \( \gamma \), and we have

\[
\int_0^t L(\gamma'(s), \gamma(s), \omega) ds - \int_0^t L(\hat{\gamma}'(s), \hat{\gamma}(s), \omega) ds \leq \int_{t_1}^{t_1 + 1} K(\gamma'(s)) ds \leq K(\gamma(t_2) - \gamma(t_1)).
\]

Since \( |\gamma(t_2) - \gamma(t_1)| \) is bounded by the diameter of cube \( Q_j \), we have \( \mathcal{L}(\hat{\gamma}', \omega) \leq \mathcal{L}(\gamma, \omega) + C \). \( \Box \)

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. As we have mentioned, the main argument is similar to that of [3]. In particular, it is convenient to average \( u(t, \omega) \) over a random shift of the environment.

Random shifting of the environment

We now consider an augmented probability space \( \tilde{\Omega} = \Omega \times \Omega_1 \) with product measure \( \tilde{\mathbb{P}} = \mathbb{P} \times P_1 \), and we introduce a random function \( h(\omega_1) : \Omega_1 \rightarrow \mathbb{Z}^d \) to define a random shift of the environment. For \((\omega, \omega_1) \in \tilde{\Omega} \), let us define

\[
\tilde{u}(t, \omega, \omega_1) = u(t, \tau_{h(\omega_1)}\omega) = \sup_{\gamma \in A_t} g(\gamma(t)) - \mathcal{L}(\gamma + h(\omega_1), \omega) \tag{3.29}
\]

where \( \gamma + h(\omega_1) \) denotes the shifted path \( t \mapsto \gamma(t) + h(\omega_1) \). We define \( M_\delta(\omega, \omega_1) = M_\delta(\tau_{h(\omega_1)}\omega) \) to be the set of paths \( \gamma \in A_t \) for which

\[
\tilde{u}(t, \omega, \omega_1) \leq g(\gamma(t)) - \mathcal{L}(\gamma + h(\omega_1), \omega) + \delta.
\]
There exists a constant $P > 0$ independent on $m = \lfloor t^\zeta \rfloor$, and for this reason an estimate of $\text{var}(\tilde{u})$ that is sublinear in $t$ will imply a sublinear bound for $\text{var}(u)$. 

The random shift $h(\omega)$ will lie in the set $[0, m]^d \subset \mathbb{R}^d$ where $m = \lfloor t^\zeta \rfloor$, for some positive $\zeta < 1/2$. For $d \geq 3$, it will suffice to choose $\zeta \in (\frac{1}{7}, \frac{1}{3})$. For $d = 2$, we will require that $\zeta \in \left(\frac{1}{2} - \frac{1}{2d}, \frac{1}{2}\right)$, where $\nu$ was defined by the non-degeneracy condition (1.6). Denote by $P_0$ the product probability measure on the set $\Omega_0 = \{a, b\}^{m^2}$ and having marginal distribution $P_0'(a) = \alpha, P_0'(b) = \beta = 1 - \alpha \in (0, 1)$. The following statement is Lemma 3.2 from [4], so we omit the proof:

**Lemma 3.1** There exists a constant $C > 0$ independent of $m = \lfloor t^\zeta \rfloor$, and a function $\tilde{h} : \Omega_0 \to \{0, 1, \ldots, m - 1\}$ for which the following two conditions hold:

(i) $P_0(\tilde{h} = i) \leq \frac{C}{m}$ for all $i \in [0, m - 1]$, and

(ii) for every $x, y \in \Omega_0$ that differ in at most one coordinate, the difference between $\tilde{h}(x)$ and $\tilde{h}(y)$ satisfies

$$|\tilde{h}(x) - \tilde{h}(y)| \leq 1.$$

Define the set $\Theta = \Theta_1 = \{1, 2, \ldots, d\} \times \{1, 2, \ldots, m^2\}$. Let $\Omega_1 = \{a, b\}^\Theta$, and let $P_1$ be a uniform probability measure on $\Omega_1$. Each $\omega_1 \in \Omega_1$ can be written as $\omega_1 = (\omega_1^1, \omega_1^2, \ldots, \omega_1^m)$, where each $\omega_1^i$ is a binary sequence of length $m^2$. Let $\vec{e}_i$ denote the $i$-th coordinate vector. Define $h(\omega_1) = \sum_{i=1}^{d} \tilde{h}(\omega_1^i) \vec{e}_i$. There exists a constant $C > 0$ independent on $m$ such that for each $x \in \{0, 1, \ldots, m - 1\}^d$ one has $P_1(h = x) \leq \frac{C}{m^d}$. Moreover, if $\omega_1$ and $\omega'_1$ differ in exactly one coordinate then we have $|h(\omega_1) - h(\omega'_1)| \leq 1$. Given the space $\tilde{\Omega} = \Omega \times \Omega_1$ with the product measure $\tilde{\mathbb{P}} = \mathbb{P} \times P_1$ on $\Omega \times \Omega_1$ defined in this way, we now consider the function $\tilde{u}$ defined by (3.29).

**Lemma 3.2** There is a constant $C > 0$ such that

$$|u(t, \omega) - \tilde{u}(t, \omega, \omega_1)| \leq C|h(\omega_1)| \leq Ct^\zeta, \quad \forall t > 1$$

holds $\tilde{\mathbb{P}}$-almost surely, and

$$\text{var} u \leq C \text{var} \tilde{u} + Ct^{2\zeta}, \quad \forall \ t > 1. \quad (3.31)$$

**Proof of Lemma 3.2.** We will prove that $|u(t, \omega) - \tilde{u}(t, \omega, \omega_1)| \leq C|h(\omega_1)|$, $\tilde{\mathbb{P}}$-almost surely, for some constant $C > 0$ independent of $m$ and $t$. Given a path $\gamma \in M_\delta(\omega)$, we can modify it to construct an approximate optimizer for $\tilde{u}(t, \omega, \omega_1)$, thus estimating $\tilde{u}(t, \omega, \omega_1) - u(t, \omega)$ from above. However, we cannot simply shift $\gamma$ by $-h(\omega_1)$, since we must preserve the starting and ending points.

Suppose $|h(\omega_1)| \leq \sqrt{d}\kappa$, with $\kappa \in [1, m] \cap \mathbb{Z}$. Fixing a path $\gamma \in M_\delta(\omega)$, we define the new path $\hat{\gamma}$ in the following way:

(i) For $r \in [0, \kappa]$, set $\hat{\gamma}(r) = \gamma(0) + \left(\frac{r}{\kappa}\right)(\gamma(2\kappa) - h(\omega_1) - \gamma(0))$.

(ii) For $r \in [\kappa, t - \kappa]$, set $\hat{\gamma}(r) = \gamma(r + \kappa) - h(\omega_1)$.

(iii) For $r \in [t - \kappa, t]$, set $\hat{\gamma}(r) = \gamma(t) + (r - t)\frac{h(\omega_1)}{\kappa}$. 


9
We now verify that the path $\hat{\gamma}$ yields the desired bound on $\tilde{u}(t, \omega, \omega_1) - u(t, \omega)$ and $\text{var } u$. Since $\gamma \in M_\delta(\omega)$, we have

$$
\tilde{u}(t, \omega, \omega_1) - u(t, \omega) \geq - \int_0^\kappa L(\hat{\gamma}'(s), \hat{\gamma}(s) + h(\omega_1), \omega) \, ds - \int_t^t L(\hat{\gamma}'(s), \hat{\gamma}(s) + h(\omega_1), \omega) \, ds
$$

$$
+ \int_0^{2\kappa} L(\gamma'(s), \gamma(s), \omega) \, ds - \delta
$$

$$
\geq - C\kappa \left( 1 + \sup_{|z| \leq 2R} |K(z)| \right) - \delta,
$$

where $C$ is a positive real number that depends only on $K, |b - a|$. In a similar way we prove that $\tilde{u}(t, \omega, \omega_1) - u(t, \omega) \leq C\kappa + \delta$. Recalling that $m = \lceil t^\zeta \rceil$, we obtain (3.30). Therefore

$$
\text{var } u = \tilde{E}[(u - \tilde{E}(u))^2]
$$

$$
= \tilde{E} \left[ (\tilde{u} - \tilde{E}(\tilde{u}) + (u - \tilde{u}) - \tilde{E}(u - \tilde{u}))^2 \right]
$$

$$
\leq 3 \text{ var } \tilde{u} + 3\tilde{E} \left[ (u - \tilde{u}) - \tilde{E}(u - \tilde{u}) \right]^2
$$

$$
\leq 3 \text{ var } \tilde{u} + 12C^2m^2,
$$

which is (3.31).

\[ \square \]

**Variance estimate for $\tilde{u}$**

Given Lemma 3.2 and the choice of $\zeta < 1/2$, we now wish to establish a bound of order $t/ \log t$ for the variance of $\tilde{u}(t, \omega, \omega_1)$ under $\tilde{P}$. The augmented probability space was constructed in such a way that $\tilde{u}(t, \omega, \omega_1)$ is amenable to Talagrand’s inequality. The function $u$ depends on $\omega_j$ for only $O(t^d)$ of the indices $j \in \mathbb{Z}^d$.

**Lemma 3.3** There is a constant $R > 0$ such that

$$
\tilde{u}(t, \omega, \omega_1) = \tilde{u}(t, \phi_j \omega, \omega_1), \quad \forall \ j \in \mathbb{Z}^d, \ |j| > Rt, \ t > 0
$$

holds $\tilde{P}$ almost surely.

**Proof:** Since $|h(\omega_1)| \leq m\sqrt{d}$, this is a consequence of Lemma 2.2: no approximate optimizer passes through cube $j$, if $R$ is sufficiently large and $|j| > Rt$. \[ \square \]

In view of Lemma 3.3 we may regard $\tilde{u}$ as a function of no more than $Ct^d + dm^2$ random variables taking values in the set $\{a, b\}$. In this way, Talagrand’s inequality (Theorem 1.2) implies that there is a constant $C > 0$, independent of $t > 0$, such that

$$
\text{var}(\tilde{u}) \leq C \sum_{j \in B_t} \frac{\|\rho^j\tilde{u}\|^2}{1 + \log \frac{\|\rho^j\tilde{u}\|^2}{\|\rho^ju\|^2}} + C \sum_{k \in \Theta_t} \frac{\|\rho^k\tilde{u}\|^2}{1 + \log \frac{\|\rho^k\tilde{u}\|^2}{\|\rho^ku\|^2}}.
$$

(3.32)
where \( B_t \) is the set \( B_t = \{ j \in \mathbb{Z}^d \mid |j| \leq Rt \} \), whose cardinality is bounded by \( Ct^d \). The norms \( \| \cdot \|_2 \) and \( \| \cdot \|_1 \) refer to the \( L^2(\tilde{\Omega}, \tilde{P}) \) and \( L^1(\tilde{\Omega}, \tilde{P}) \) norms, respectively. Observe that if \( k \in \Theta_t \), then \( \rho_k \tilde{u} \) corresponds to translation of the random environment:

\[
\rho_k \tilde{u} = \sigma_k \tilde{u} - \tilde{u} = \frac{\tilde{u}(t, \omega, \phi_k \omega_1) - \tilde{u}(t, \omega, \omega_1)}{2}.
\]

If \( j \in B_t \), then \( \rho_j \tilde{u} \) corresponds to a local change in the random environment over the cube \( Q_j \):

\[
\rho_j \tilde{u} = \frac{\sigma_j \tilde{u} - \tilde{u}}{2} = \frac{\tilde{u}(t, \phi_j \omega, \omega_1) - \tilde{u}(t, \omega, \omega_1)}{2}.
\]

Let us first consider the second sum in (3.32). We will show that this sum is \( O(t^{2\zeta}) \).

**Lemma 3.4** There is a constant \( C > 0 \) such that

\[
\sum_{k \in \Theta_t} \frac{\| \rho_k \tilde{u} \|_2^2}{1 + \log \frac{\| \rho_k \tilde{u} \|_2}{\| \rho_k \tilde{u} \|_1}} \leq Ct^{2\zeta} \tag{3.33}
\]

holds for all \( t > 1 \).

**Proof:** Since there are only \( |\Theta_t| = m^2 \leq t^{2\zeta} \) terms in the sum and since

\[
1 + \log \frac{\| \rho_k \tilde{u} \|_2}{\| \rho_k \tilde{u} \|_1} \geq 1,
\]

the lemma will follow from a uniform bound on \( \| \rho_k \tilde{u} \|_2 \). By definition of \( h(\omega_1) \), we know that \( |h(\phi_k \omega_1) - h(\omega_1)| \leq 1 \). So, by Lemma 3.2 we have \( |\tilde{u}(t, \omega, \omega_1) - \tilde{u}(t, \omega, \phi_k \omega_1)| \leq C|h(\phi_k \omega_1) - h(\omega_1)| \leq C \) holds \( \tilde{P} \) almost surely, for all \( k \in \Theta_t, t \geq 1 \). \( \square \)

Having established (3.33), we now consider the first sum in (3.32).

**Proposition 3.5** There is a constant \( C > 0 \) such that

\[
\sum_{j \in B_t} \frac{\| \rho_j \tilde{u} \|_2^2}{1 + \log \frac{\| \rho_j \tilde{u} \|_2}{\| \rho_j \tilde{u} \|_1}} \leq C \frac{t}{\log t} \tag{3.34}
\]

holds for all \( t > 1 \).

Since we may have \( \alpha \neq \beta \), we will make use of the following fact, proved in the appendix:

**Lemma 3.6** Let \( C' = \min \left\{ \frac{\alpha}{\beta}, \frac{\beta}{\alpha} \right\} \) and \( C'' = \max \left\{ \frac{\alpha}{\beta}, \frac{\beta}{\alpha} \right\} \). For any measurable set \( A \subset \Omega \),

\[
C' \mathbb{P}(A) \leq \mathbb{P}(\phi_j A) \leq C'' \mathbb{P}(A) \tag{3.35}
\]

holds for all \( j \in \mathbb{Z}^d \). Also, for every nonnegative integrable \( \psi \), we have

\[
C' \mathbb{E}(\psi \circ \phi_j) \leq \mathbb{E}(\psi) \leq C'' \mathbb{E}(\psi \circ \phi_j). \tag{3.36}
\]
Proof of Proposition 3.5. Let us begin by estimating $\|\rho_j \tilde{u}\|^2_2$. By Lemma 3.6, we have

$$\|\rho_j \tilde{u}\|^2_2 = \tilde{E} \left[ (\sigma_j \tilde{u} - \tilde{u})^2 1_{\sigma_j \tilde{u} > \tilde{u}} \right] + \tilde{E} \left[ (\sigma_j \tilde{u} - \tilde{u})^2 1_{\sigma_j \tilde{u} < \tilde{u}} \right] \leq C \tilde{E} \left[ (\sigma_j \tilde{u} - \tilde{u})^2 1_{\sigma_j \tilde{u} < \tilde{u}} \right].$$

Recalling the definition (2.23), let $\tilde{I}_j \subset \tilde{\Omega}$ be the event that $Q_j$ is an important cube in the shifted environment:

$$\tilde{I}_j = \{ (\omega, \omega_1) \in \tilde{\Omega} \mid \tau_{\tilde{h}(\omega_1)} \omega \in I_j \}.$$ 

Because of (2.25), the event $\{\sigma_j \tilde{u} < \tilde{u}\}$ is contained in the event $\tilde{I}_j$. So, we have

$$\|\rho_j \tilde{u}\|^2_2 \leq C \tilde{E} \left[ (\sigma_j \tilde{u} - \tilde{u})^2 1_{\tilde{I}_j} \right].$$

(3.37) 

The difference $|\sigma_j \tilde{u} - \tilde{u}|$ could be large in some cases, even on the event $\tilde{I}_j$, so we will distinguish a few possible scenarios. Let $\tilde{I}_j^+ \subset \tilde{I}_j$ denote the event that cube $Q_j$ is very important in the shifted environment:

$$\tilde{I}_j^+ = \{ (\omega, \omega_1) \in \tilde{I}_j \mid \tau_{\tilde{h}(\omega_1)} \omega \in I_j^+ \}.$$ 

Similarly, let $\tilde{I}_j^- = \tilde{I}_j \setminus \tilde{I}_j^+$ be the event that the cube $Q_j$ is important but not very important. Since $\omega \mapsto \tau_{\tilde{h}(\omega_1)} \omega$ is measure preserving on $\Omega$, we have

$$\tilde{P}(\{ (\omega, \omega_1) \in \tilde{\Omega} \mid (\tilde{u} - \sigma_j \tilde{u})^2 1_{\tilde{I}_j^-} > C_0 \}) = P(\{ \omega \in \Omega \mid (u - \sigma_j u)^2 1_{I_j^-} > C_0 \}).$$

Consequently, from Lemma 2.28 and (3.37) we have

$$\|\rho_j \tilde{u}\|^2_2 \leq C \tilde{E} \left[ (\sigma_j \tilde{u} - \tilde{u})^2 1_{\tilde{I}_j^+} \right] + C \tilde{E} \left[ (\sigma_j \tilde{u} - \tilde{u})^2 1_{\tilde{I}_j^-} \right]$$

$$\leq C \tilde{E} \left[ (\sigma_j \tilde{u} - \tilde{u})^2 1_{\tilde{I}_j^+} \right] + C C_0 \tilde{P}(\tilde{I}_j).$$

(3.38) 

Whether the event $\tilde{I}_j^+$ has small probability depends on the function $g(y)$, so we distinguish two cases. Let $\tilde{G} \subset \Omega$ denote the event that

$$|\gamma(t) - \gamma(0)| \geq t^{1/4}, \quad \forall \gamma \in M_1(\omega, \omega_1).$$

On this event, all approximate minimizers must travel a distance at least $O(t^{1/4})$ from their starting point $\gamma(0) = 0$. According to the following lemma, the probability that minimizers travel that far when a cube $Q_j$ is very important must be small.

Lemma 3.7 There are constants $\kappa_1, \kappa_2 > 0$ such that

$$\tilde{P}(\tilde{I}_j^+ \cap \tilde{G}) \leq \kappa_1 e^{-\kappa_2 t^{1/4}}, \quad \forall \ t > 0, \quad j \in \mathbb{Z}^d.$$ 

Therefore, returning to (3.38) and using the fact that $|\sigma_j \tilde{u} - \tilde{u}| \leq O(t)$ and $\tilde{I}_j^+ = (\tilde{I}_j^+ \cap \tilde{G}) \cup (\tilde{I}_j^+ \cap \tilde{G}^c)$, we conclude

$$\|\rho_j \tilde{u}\|^2_2 \leq C t^2 e^{-\kappa_2 t^{1/4}} + C \tilde{P}(\tilde{I}_j) + C \tilde{E} \left[ (\sigma_j \tilde{u} - \tilde{u})^2 1_{\tilde{I}_j^+ \cap \tilde{G}^c} \right].$$

Hence,

$$\sum_{j \in B_t} \|\rho_j \tilde{u}\|^2_2 \leq C |B_t| t^2 e^{-\kappa_2 t} + C \tilde{E} \left[ \sum_{j \in B_t} 1_{\tilde{I}_j} \right] + C \tilde{E} \left[ \sum_{j \in B_t} (\sigma_j \tilde{u} - \tilde{u})^2 1_{\tilde{I}_j^+ \cap \tilde{G}^c} \right].$$

(3.39) 

With probability one, the sum $\sum_{j \in B_t} 1_{\tilde{I}_j}$ is bounded by $O(t)$ because there can be at most $O(t)$ important cubes, as the total number of cubes visited is $O(t)$, by Lemma 2.2.

The last term in (3.39) is bounded as follows. First,
Lemma 3.8 There are $\kappa_1, \kappa_2 > 0$ such that
\[
\tilde{\mathbb{P}} \left( \{ (\omega, \omega_1) \mid (\sigma_j \tilde{u} - \tilde{u})^2 \mathbf{1}_{\tilde{G}^c} > C_0 t^{1/2} \} \right) \leq \kappa_1 e^{-\kappa_2 t^{1/4}}
\]
holds for all $t > 1$ and all $j \in \mathbb{Z}^d$.

Furthermore, if $\omega \in \mathcal{I}^+_j$, then $\omega \notin \mathcal{I}^+_k$ for any $k \neq j$, since $Q_j$ must be the only important cube. Therefore, since $|\sigma_j \tilde{u} - \tilde{u}| \leq |b - a| t$ always holds (by (2.21)), we must have
\[
\tilde{\mathbb{E}} \left[ \sum_{j \in B_t} (\sigma_j \tilde{u} - \tilde{u})^2 1_{\tilde{I}_j \cap \tilde{G}^c} \right] \leq C_0 t^{1/2} + |b - a|^2 \kappa_1 e^{-\kappa_2 t^{1/4}}.
\]
Considering (3.39), we have now shown that there is a constant $C' > 0$ for which
\[
\sum_{j \in B_t} \|\rho_j \tilde{u}\|_2^2 \leq C't
\]
holds for all $t > 1$.

Next we consider the denominator in (3.34). We show that there is a constant $C'' > 0$ such that
\[
\log \frac{\|\rho_j \tilde{u}\|_2}{\|\rho_j \tilde{u}\|_1} \geq C'' \log t,
\]
for all $t > 1$. By the Cauchy-Schwarz inequality we see that
\[
\|\rho_j \tilde{u}\|_1 = \|\rho_j \tilde{u} \cdot 1_{\sigma_j \tilde{u} \neq \tilde{u}}\|_1 \leq \|\rho_j \tilde{u}\|_2 \cdot \sqrt{\tilde{\mathbb{P}}(\sigma_j \tilde{u} \neq \tilde{u})}.
\]
Since $\sigma_j \sigma_j u = u$, Lemma 3.6 implies
\[
\tilde{\mathbb{P}}(\sigma_j \tilde{u} \neq \tilde{u}) = \tilde{\mathbb{P}}(\sigma_j \tilde{u} > \tilde{u}) + \tilde{\mathbb{P}}(\sigma_j \tilde{u} < \tilde{u}) \leq (1 + C'') \tilde{\mathbb{P}}(\sigma_j \tilde{u} < \tilde{u}).
\]
Hence,
\[
\frac{\|\rho_j \tilde{u}\|_2}{\|\rho_j \tilde{u}\|_1} \geq \frac{1}{\sqrt{(1 + C'') \tilde{\mathbb{P}}(\sigma_j \tilde{u} < \tilde{u})}}.
\]
Therefore, to bound $\log(\|\rho_j \tilde{u}\|_2/\|\rho_j \tilde{u}\|_1)$ from below, we should find an upper bound for $\tilde{\mathbb{P}}(\sigma_j \tilde{u} < \tilde{u})$. Because of (2.26), we know that
\[
\tilde{\mathbb{P}}(\sigma_j \tilde{u} < \tilde{u}) \leq \tilde{\mathbb{P}}(\tilde{I}_j).
\]

To estimate $\tilde{\mathbb{P}}(\tilde{I}_j)$ we average in $\omega_1$, as was done in [4]:
\[
\tilde{\mathbb{P}}(\tilde{I}_j) = \tilde{\mathbb{E}} \left[ 1_{\tilde{I}_j} \right] = \tilde{\mathbb{E}} \left[ \tilde{\mathbb{E}} \left[ 1_{\tilde{I}_j} \mid \omega \right] \right] = \mathbb{E} \left[ \sum_{z \in [0, m-1]^d} \tilde{\mathbb{E}} \left[ 1_{\tilde{I}_j} \mid \omega, h(\omega_1) = z \right] P_1(h(\omega_1) = z) \right].
\]
Observe that $(\omega, \omega_1) \in \tilde{I}_j$ if and only if there is $\delta > 0$ such that
\[
\pi_j(\gamma) > 0 \quad \text{for all} \quad \gamma \in \hat{M}(\omega, \omega_1),
\]
which holds if and only if for some \( \delta > 0 \)
\[
\pi_{j-z}(\gamma) > 0 \quad \text{for all} \quad \gamma \in M_\delta(t, \tau_z \omega), \quad z = h(\omega_1).
\]

So, for \( I_j \) defined by (2.24), we have
\[
\tilde{E} \left[ 1_{I_j} \mid \omega, h(\omega_1) = z \right] = 1_{I_{j-z}}(\tau_z \omega).
\]

By Lemma 3.1, we also know that \( P_1(h(\omega_1) = z) \leq C m^{-d} \) for any \( z \in [0, m - 1]^d \). Therefore,
\[
\tilde{P}(\tilde{I}_j) \leq C m^{-d} E \left[ \sum_z \tilde{E} \left[ 1_{I_j} \mid \omega, h(\omega_1) = z \right] \right] = C m^{-d} E \left[ \sum_z 1_{I_{j-z}}(\tau_z \omega) \right]
\]
\[
= C m^{-d} E \left[ \sum_z 1_{I_{j-z}}(\omega) \right].
\]

The last equality follows from the stationarity of \( \mathbb{P} \) with respect to \( \tau_z \). Now, given \( \omega \in \Omega \), the sum
\[
\sum_{z \in [0, m - 1]^d} 1_{I_{j-z}}(\omega)
\]
counts the number of important cubes within the box \( j - [0, m - 1]^d \). These cubes are visited by all paths \( \gamma \in M_\delta(\omega) \) for some \( \delta > 0 \) sufficiently small. Hence, \( \tilde{P}(\tilde{I}_j) \leq C m^{-d} E[\# \Lambda_j] \) where
\[
\Lambda_j = \bigcup_{n \geq 1} \left\{ k \in \mathbb{Z}^d \mid j - k \in [0, m - 1]^d, \quad \pi_k(\gamma) > 0 \quad \forall \gamma \in M_{1/n}(\omega) \right\}.
\]

We may interpret the random variable \( \# \Lambda_j \) as the number of important cubes in the box \( j - [0, m - 1]^d \).

Obviously we have the trivial bound \( \# \Lambda_j \leq O(t) \). This is because each path \( \gamma \in M_\delta(\omega) \) has length \( O(t) \), by Lemma 2.2, so \( \pi_k(\gamma) > 0 \) for at most \( O(t) \) indices \( k \). Therefore,
\[
E[\# \Lambda_j] \leq t \leq (m + 1)^{1/\zeta}.
\]

If \( d \geq 3 \), we may choose \( \zeta \in (1/d, 1/2) \), so that
\[
\tilde{P}(\sigma_j \tilde{u} < \tilde{u}) \leq \tilde{P}(\tilde{I}_j) \leq C m^{-d} E[\# \Lambda_j] \leq C m^{-d+1/\zeta} \leq C t^{1-d\zeta}
\]
with \( 1 - d\zeta < 0 \). If \( d = 2 \), we need the following:

**Lemma 3.9** Let \( d = 2 \) and assume the non-degeneracy condition (1.6) holds. Then for each \( p \in \left( \frac{\nu - 1 + \zeta}{\zeta \nu}, 2 \right) \) there exists a constant \( C > 0 \) such that \( \# \Lambda_j \leq C m^p \) holds with probability one, for all \( j \in \mathbb{Z}^d \).

So, for \( d = 2 \) we still have
\[
\tilde{P}(\sigma_j \tilde{u} < \tilde{u}) \leq C m^{-d} E[\# \Lambda_j] \leq C n^{(p-2)/\zeta}, \quad \text{with} \quad (p-2)/\zeta < 0.
\]

Therefore, returning to (3.42), we conclude that there is a constant \( C > 0 \) such that
\[
\log \frac{\|\rho_j \tilde{u}\|_2}{\|\rho_j \tilde{u}\|_1} \geq C \log t
\]
holds for all \( t \) sufficiently large. Therefore, the proof of Proposition 3.5 is reduced to a proof of Lemma 3.7, Lemma 3.8 and, in case \( d = 2 \), Lemma 3.9. These are proved in the next section. \( \square \)

Theorem 1.1 now follows immediately from (3.32), Lemma 3.4 and Proposition 3.5.
4 Proofs of the technical estimates

Proof of Lemma 3.7.

Observe that
\[ \bar{P}(\overline{I}_j^+ \cap \tilde{G}) = P(\overline{I}_j^+ \cap G) \]
where \( G \subset \Omega \) is the event for which
\[ |\gamma(t) - \gamma(0)| \geq t^{1/4}, \quad \forall \gamma \in M_b(t, \omega). \]  
(4.46)

We will show that on the event \( \overline{I}_j^+ \cap G \), any approximate optimizer \( \gamma \in M_b(\omega) \) must touch a set of \( O(t^{1/4}) \) cubes which are almost uniformly spaced on a straight line segment of length \( O(t) \) and on each of those cubes we have \( V(x, \omega) = b \). Such an event can occur only with small probability.

Suppose \( \omega \in \overline{I}_j^+ \cap G \), and let \( \gamma \in M_b(\omega) \) be such that (4.46) holds. Let \([t_1, t_2] \subset [0, t] \) be the smallest interval containing all \( s \) for which \( \gamma(s) \in Q_j \). Hence, \( \omega(\gamma(s)) = b \) for all \( s \notin [t_1, t_2] \), since \( \omega \in \overline{I}_j^+ \). Since \( \omega \in G \), we know that \( |\gamma(t) - \gamma(0)| \geq t^{1/4} \), which means that either
\[ |\gamma(t_1) - \gamma(0)| > \frac{t^{1/4}}{3} \quad \text{or} \quad |\gamma(t) - \gamma(t_2)| > \frac{t^{1/4}}{3} \]
must hold, because \( \gamma(t_1), \gamma(t_2) \in \overline{Q}_j \). Let us assume that \( |\gamma(t) - \gamma(t_2)| > (t^{1/4})/3 \) holds; the other case is treated in a similar manner.

First, since \( \omega(\gamma(s)) = b \) for all \( s \in (t_2, t] \), we may assume that \( \gamma \) is a straight line between \( \gamma(t_2) \) and \( \gamma(t) \). Specifically, by redefining \( \gamma \) slightly, we may assume that
\[ \gamma(s) = \gamma(t_2) + \frac{\gamma(t) - \gamma(t_2)}{t - t_2}(s - t_2), \quad \forall \ s \in [t_2, t], \]
for otherwise, \( \gamma \) would not be an optimal path. This follows from (2.12).

Next, given points \( \gamma(t_2) \) and \( \gamma(t) \), there is a unique pair \( x_{t_2}, x_t \in \mathbb{Z}^d \) such that
\[ \gamma(t_2) = x_{t_2} + y_{t_2}, \quad \gamma(t) = x_t + y_t \]
for some \( y_{t_2}, y_t \in [0, 1]^d \). Therefore, if we define the linear path
\[ \hat{\gamma}(s) = x_{t_2} + \frac{x_t - x_{t_2}}{t - t_2}(s - t_2), \quad s \in [t_2, t] \]
we have \( |\gamma(s) - \hat{\gamma}(s)| \leq 2\sqrt{d} \) for \( s \in [t_2, t] \). Therefore, for each \( s \in [t_2, t] \) there must be a cube \( Q_t \) such that \( \text{dist}(\hat{\gamma}(s), Q_t) \leq 2\sqrt{d} \) and \( \omega(Q_t) = b \). For \( y \in \mathbb{R}^d \), let \( B_y \) denote the event that there is at least one cube \( Q \) such that \( \text{dist}(Q, y) \leq 2\sqrt{d} \) and \( \omega(Q) = b \). Then \( \mathbb{P}(B_y) = 1 - \mathbb{P}(B'_y) \leq 1 - \alpha C_3 < 1 \), for a constant \( C_3 > 0 \) that depends only on the dimension \( d \). Moreover, if \( |y - z| > 5\sqrt{d} \), then \( B_y \) and \( B_z \) are independent events. Therefore, for fixed times \( t_2 < t \) and a fixed pair of points \( x_{t_2}, x_t \in \mathbb{Z}^d \) satisfying \( |x_{t_2} - x_t| \geq t^{1/4}/2 \) we have
\[ \mathbb{P}\left( \bigcap_{s \in [t_2, t]} B_{\hat{\gamma}(s)} \right) \leq (1 - \alpha C_3)C_4 t^{1/4} \]
for some constant \( C_4 > 0 \) independent of \( \epsilon \).
By Lemma 2.2, we know there is a constant $R > 0$ such that $|\gamma(s) - \gamma(0)| \leq tR$ for all $s \in [0,t]$. There are at most $O(t^{2d})$ possible pairs $x_t, x_s \in \mathbb{Z}^d$ satisfying $|x_t - \gamma(0)| \leq tR$ and $|x_s - \gamma(0)| \leq tR$ and $|x_t - x_s| \geq t^{1/4}$. Therefore, we conclude that

$$\mathbb{P} \left( \mathcal{I}_j^+ \cap G \right) \leq O(t^{2d})(1 - \alpha^{C_3})C_4t^{1/4}. \quad (4.47)$$

The last inequality immediately implies the lemma. \hfill \Box

Proof of Lemma 3.8

Because $\omega \mapsto \tau_{h(\omega)}\omega$ is measure preserving on $\Omega$, we have

$$\tilde{\mathbb{P}} \left( \{ (\omega, \omega_1) \mid (\sigma_j\tilde{u} - \tilde{u})^21_{G^C} > C_4t^{1/2} \} \right) = \mathbb{P} \left( \{ \omega \mid (\sigma_ju - u)^21_{G^C} > C_4t^{1/2} \} \right)$$

where the event $G \subset \Omega$ is defined by (4.46). So, on the event $G^C$ we know there is $\gamma \in M_\delta(\omega)$ such that

$$|\gamma(t) - \gamma(0)| < t^{1/4}. \quad (4.48)$$

Let $B_r(x)$ denote the ball of radius $r > 0$ centered at $x$. We may assume that there are at least two indices $j, k \in \mathbb{Z}^d \cap B_{t^{1/4}}(0)$ such that $\omega_j = a$ and $\omega_k = a$. This is because the event that $\omega_k = a$ for at most one of the cubes contained in $B_{t^{1/4}}(0)$ has probability less than $O(\beta^{N_t})$ where $N_t \geq Ct^{1/4}$ is the number of cubes contained in $B_{t^{1/4}}(0)$.

Let $\gamma \in M_\delta(\omega)$ with $|\gamma(t) - \gamma(0)| \leq t^{1/4}$. Then

$$u(t, \omega) \leq g(\gamma(t)) - at + \delta.$$

Suppose $\omega_k = a$ for some $k \neq j$ and $k \in B_{t^{1/4}}(0)$. Let $x_k \in Q_k$, so that $V(x_k, \omega) = a$. Define the path $\hat{\gamma}$ by

$$\hat{\gamma}(s) = \begin{cases} 
\gamma(0) + s^t_0 - \gamma(0), & s \in [0, t^{1/4}] \\
x_k, & s \in [t^{1/4}, t - t^{1/4}] \\
x_k + (s - t + t^{1/4})\gamma(t) - x_k, & s \in [t - t^{1/4}, t].
\end{cases} \quad (4.49)$$

Then

$$\sigma_ju(t, \omega) \geq g(\hat{\gamma}(t)) - \mathcal{L}(\hat{\gamma}, \omega) \geq g(\gamma(t)) - a(t - 2t^{1/4} - b2t^{1/4} - 2t^{1/4} \max_{|z| \leq 1} K(z)).$$

Therefore,

$$u(t, \omega) - \sigma_ju(t, \omega) \leq (a - b)2t^{1/4} + 2t^{1/4} \max_{|z| \leq 1} K(z).$$

Hence $(u - \sigma_ju)^2 \leq C_4t^{1/2}$ except possibly on a set of probability less than $O(\beta^{N_t})$. \hfill \Box

Proof of Lemma 3.9 for $d = 2$

Assuming the non-degeneracy condition (1.6), we may choose real numbers $\nu > 1$ and $\varepsilon_0 > 0$ such that $K(q) \geq |q|^\nu$ for all $q$ that satisfy $|q| < \varepsilon_0$. Having fixed $\zeta \in (\frac{\nu - 1}{2\nu - 1}, 1)$, we see that $\frac{\nu - 1 + \zeta}{\zeta \rho} < 2$. So, we may choose $p \in (\frac{\nu - 1 + \zeta}{\zeta \rho}, 2)$.

Arguing by contradiction, we assume that $\# \Lambda_j > m^p$: there are more than $m^p$ important cubes within the box $B_j = j - [0, m - 1]^d$. Fix $\delta > 0$ small. Consider a path $\gamma \in M_\delta(n, \omega)$. Let $[t_1, t_2]$ be
the smallest interval containing all \( s \) for which \( \gamma(s) \in B_j \). Hence \( |t_2 - t_1| \geq Cm^p \). Choose any one
of the important cubes in \( B_j \) and let \( x_c \) denote its center point. Let us define a modified path \( \hat{\gamma} \) as follows:

(i) For \( s \in [0, t_1] \cup [t_2, t] \), let \( \hat{\gamma}(s) = \gamma(s) \).

(ii) For \( s \in [t_1, t_2] \) let

\[
\hat{\gamma}(s) = \begin{cases} 
\gamma(t_1) + (s - t_1) \cdot \frac{x_c - \gamma(t_1)}{m}, & s \in [t_1, t_1 + m], \\
x_c, & s \in [t_1 + m, t_2 - m], \\
x_c + (s - t_2 + m) \cdot \frac{\gamma(t_2) - x_c}{m}, & s \in [t_2 - m, t_2].
\end{cases}
\]

We have the bound

\[
\mathcal{L}(\gamma) - \mathcal{L}(\hat{\gamma}) \geq -2m \cdot \max_{|q| \leq 1} K(q) - 2m(b - a) + \int_{t_1}^{t_2} K(\gamma'(s)) \, ds. 
\] (4.50)

We will prove that for sufficiently large \( m \) the right side of (4.50) is larger than \( \delta > 0 \), contradicting the fact that \( \gamma \in M_\delta(n, \omega) \).

Let us denote by \( J_0 \subset [t_1, t_2] \) the set of times for which \( |\gamma'(s)| \geq \varepsilon_0 \). Therefore, we may assume

\[
\int_{J_0} K(\gamma'(s)) \, ds \leq 2m \left( \max_{|q| \leq 1} K(q) + (b - a) \right) + \delta, 
\] (4.51)

for otherwise the right side of (4.50) would be larger than \( \delta \). Let \( J_1 = [t_1, t_2] \setminus J_0 \); for these times \( s \in J_1 \) we have \( |\gamma'(s)| \leq \varepsilon_0 \). From (4.50) we also obtain:

\[
\mathcal{L}(\gamma) - \mathcal{L}(\hat{\gamma}) \geq -2m \left( \max_{|q| \leq 1} K(q) + b - a \right) + \int_{J_1} K(\gamma'(r)) \, dr \\
\geq -2m \left( \max_{|q| \leq 1} K(q) + b - a \right) + \int_{J_1} |\gamma'(r)| \, dr \\
\geq -2m \left( \max_{|q| \leq 1} K(q) + b - a \right) + |J_1| \cdot \left( \frac{1}{|J_1|} \int_{J_1} |\gamma'(r)| \, dr \right) \nu. 
\] (4.52)

In the last line we applied Jensen’s inequality. We will now prove that there exists a real number \( \varepsilon_1 > 0 \) such that

\[
\int_{J_1} |\gamma'(r)| \, dr \geq \varepsilon_1 m^p. 
\] (4.53)

By our assumption, the number of important cubes within \( B_j \) is more than \( m^p \). Let us now paint
all these cubes in \( 2^d \) colors so that no two cubes share the same color. By the pigeon-hole principle there are at least \( m^p 2^{-d} \) important cubes having the same color. The distance between two cubes of the same color is at least 1, hence we have \( \int_{J_1} |\gamma'(r)| \, dr \geq m^p 2^{-d} \). Therefore, since there is \( C > 0 \) such that \( |\gamma'(s)| \leq CK(\gamma'(s)) \) for all \( s \in J_0 \), we have

\[
m^p 2^{-d} \leq \int_{J_1} |\gamma'(r)| \, dr + \int_{J_0} |\gamma'(r)| \, dr \\
\leq \int_{J_1} |\gamma'(r)| \, dr + C \int_{J_0} K(\gamma'(r)) \, dr \\
\leq \int_{J_1} |\gamma'(r)| \, dr + C(m + \delta). 
\] (4.54)
In the last step we have applied (4.51). This last inequality implies (4.53), since \( p > 1 \).

Now the inequalities (4.52) and (4.53) imply:

\[
L(\gamma) - L(\hat{\gamma}) \geq -2m \left( \max_{|q| \leq 1} K(q) + b - a \right) + |J_1| \cdot \left( \frac{\epsilon_1 m^p}{|J_1|} \right)^
u
\]

\[
= -2m \left( \max_{|q| \leq 1} K(q) + b - a \right) + (\epsilon_1)\nu m^{p\nu} \cdot |J_1|^{1-\nu}
\]

\[
\geq -2m \left( \max_{|q| \leq 1} K(q) + b - a \right) + (\epsilon_1)\nu m^{p\nu} \cdot \left( m^{1/\zeta} \right)^{1-\nu}.
\]

In the last inequality we have used \( |J_1| \leq n = m^{1/\zeta} \). If we have

\[
p > \frac{\nu + \zeta - 1}{\zeta \nu},
\]

then \( p \nu + (1 - \nu)/\zeta > 1 \). In this case, the right side of (4.55) is positive, and larger than \( \delta \), for \( t \) sufficiently large. Since this contradicts the approximate optimality of \( \gamma \in M_\delta \), we must have

\[
\#\Lambda_j \leq m^p.
\]

5 Appendix

Proof of Lemma 3.6: The bounds in (3.35) follow from the fact that \( \mathbb{P} \) is the product measure on \( \Omega = \{a, b\}^{2d} \), with \( \mathbb{P}(\omega(j) = a) = \alpha \) and \( \mathbb{P}(\omega(j) = b) = \beta \). For every nonnegative integrable \( \psi \), (3.35) implies

\[
E(\psi) = \int \psi \, d\mathbb{P} \leq C'' \int \psi \, d\mathbb{P} \circ \phi_j = \int \psi(\phi_j \omega) \, d\mathbb{P} = E(\psi \circ \phi_j).
\]

Similarly \( E(\psi) \geq C' E(\psi \circ \phi_j) \) for all such \( \psi \).

Proof of Theorem 1.2: Let us define

\[
\Delta_j f(\omega) = \begin{cases} \beta(f(\phi_j \omega) - f(\omega)), & \text{if } \omega_j = a \\ \alpha(f(\phi_j \omega) - f(\omega)), & \text{if } \omega_j = b. \end{cases}
\]

Then Theorem 1.2 is a slight modification of the following

Theorem 5.1 ([14], Theorem 1.5) There is a constant \( C > 0 \), such that

\[
\text{var}(f) \leq C \cdot \sum_{j \in J} \frac{||\Delta_j f||_2^2}{1 + \log \frac{||\Delta_j f||_2}{||\Delta_j f||_1}}.
\]

holds for all \( f \in L^2(\Omega_J) \).

To derive Theorem 1.2 from this, we start with elementary observation

\[
C' \cdot |\rho_j f(\omega)| \leq |\Delta_j f(\omega)| \leq C'' |\rho_j f(\omega)|
\]

for \( C' = \min\{2\alpha, 2\beta\} \) and \( C'' = \max\{2\alpha, 2\beta\} \). Let \( \kappa = \log(C''/C') \geq 0 \). If \( \log \frac{||\rho_j f||_1}{||\rho_j f||_1} \geq \kappa \), then

\[
\log \frac{||\Delta_j f||_2}{||\Delta_j f||_1} \geq \log \frac{C'}{C''} + \log \frac{||\rho_j f||_2}{||\rho_j f||_1} \geq \frac{1}{2} \log \frac{||\rho_j f||_2}{||\rho_j f||_1}.
\]
Consequently, Theorem 5.1 implies

\[ \text{var}(f) \leq C \cdot \sum_{j \in J} \| \Delta_j f \|_2^2 \leq 2(C^m)^2 C \sum_{j \in J} \frac{\| \rho_j f \|_2^2}{1 + \log \frac{\| \rho_j f \|_2}{\| \rho_j f \|_1}}. \]

On the other hand, if \( \log \frac{\| \rho_j f \|_2}{\| \rho_j f \|_1} \in [0, 2\kappa) \), then Theorem 5.1 implies

\[ \text{var}(f) \leq C \cdot \sum_{j \in J} \| \Delta_j f \|_2^2 \leq (1 + 2\kappa)(C^m)^2 C \sum_{j \in J} \frac{\| \rho_j f \|_2^2}{1 + \log \frac{\| \rho_j f \|_2}{\| \rho_j f \|_1}}. \]

\[ \square \]

References

[1] S. Armstrong, P. Cardaliaguet, P.E. Souganidis, Error estimates and convergence rates for the stochastic homogenization of Hamilton-Jacobi equations, preprint 2012, [arXiv:1206.2601].

[2] S. Armstrong and P.E. Souganidis, Stochastic homogenization of level-set convex Hamilton-Jacobi equations, preprint 2012, [arXiv:1203.6303].

[3] M. Bardi and I. Capuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations, Birkhäuser, 1997.

[4] I. Benjamini, G. Kalai; O. Schramm, First passage percolation has sublinear distance variance, Ann. Probab. 31 (2003), pp. 1970-1978.

[5] M. Benaim and R. Rossignol, Exponential concentration for first passage percolation through modified Poincaré inequalities, Ann. Inst. Henri Poincaré - Prob. Stat. 44 (2008), pp. 544-573.

[6] E. Kosygina and S.R.S. Varadhan, Homogenization of Hamilton-Jacobi-Bellman equations with respect to time-space shifts in a stationary ergodic medium, Comm. Pure Appl. Math, 61 (2008) pp. 816-847.

[7] E. Kosygina, F. Rezakhanlou, S.R.S. Varadhan, Stochastic homogenization of Hamilton-Jacobi-Bellman equations, Comm. Pure Appl. Math 59 (2006), pp. 1489-1521.

[8] P.-L. Lions and P.E. Souganidis, Homogenization of “viscous” Hamilton-Jacobi equations in stationary ergodic media, Comm. PDE 30 (2005), pp. 335375.

[9] P.-L. Lions and P.E. Souganidis, Stochastic homogenization of Hamilton-Jacobi and “viscous”-Hamilton-Jacobi equations with convex nonlinearities– revisited, Comm. Math. Sci. 8 (2010), pp. 672-637.

[10] F. Rezakhanlou, Central limit theorem for stochastic Hamilton-Jacobi equations, Comm. Math. Phys. 211 (2000), pp. 413–438.

[11] F. Rezakhanlou and J. E. Tarver. Homogenization for stochastic Hamilton-Jacobi equations, Arch. Ration. Mech. Anal., 151 (2000), pp. 277-309.

[12] R. Schwab, Stochastic homogenization of Hamilton-Jacobi equations in stationary ergodic spatio-temporal media, Indiana Univ. Math. J. 58 (2009), pp. 527-581.
[13] P.E. Souganidis, *Stochastic homogenization of Hamilton-Jacobi equations and some applications*, Asymptotic Analysis, 20 (1999), pp. 1-11.

[14] M. Talagrand, *On Russo’s approximate zero-one law*, Ann. of Probab. 22 (1994), pp. 1576-1587.