Long memory constitutes a unified mesoscopic mechanism consistent with nonextensive statistical mechanics

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Abstract. - We unify two paradigmatic mesoscopic mechanisms for the emergence of nonextensive statistics, namely the multiplicative noise mechanism leading to a linear Fokker-Planck (FP) equation with inhomogeneous diffusion coefficient, and the non-Markovian process leading to the nonlinear FP equation with homogeneous diffusion coefficient. More precisely, we consider the equation \( \frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x} [F(x)p(x,t)] + \frac{1}{2} D \frac{\partial^2}{\partial x^2} [\phi(x)p(x,t)] \), where \( D \in \mathbb{R} \) and \( F(x) = -\partial V(x)/\partial x \), \( V(x) \) being the potential under which diffusion occurs. Our aim is to find whether \( \phi(x,p) \) exists such that the inhomogeneous linear and the homogeneous nonlinear FP equations become unified in such a way that the (ubiquitously observed) \( q \)-exponentials remain as stationary solutions. It turns out that such solutions indeed exist for a wide class of systems, namely when \( \phi(x,p) = [A + BV(x)]^q p(x,t) \), where \( A, B, \theta \) and \( \eta \) are (real) constants. Our main result can be summarized as follows: For \( \theta \neq 1 \) and arbitrary confining potential \( V(x) \), \( p(x,\infty) \propto \{1 - \beta(1-q)V(x)\}^{\gamma/(1-q)} \equiv \epsilon_q^\beta V(x) \), where \( q = 1 + \eta/(\theta - 1) \). The present approach unifies into a single mechanism, essentially long memory, results currently discussed and applied in the literature.

One of the cornerstones of statistical mechanics is the functional connection of the thermodynamic entropy with the set of probabilities \( \{p_i\} \) of microscopic configurations. For the celebrated Boltzmann-Gibbs (BG) theory, this central functional is given by \( S_{BG} = -k \sum_i W_i \ln p_i \), where \( W \) is the total number of microscopic states which are compatible with the information that we have about the system. This powerful connection is in principle applicable to a vast class of relevant systems, including (classical) dynamical ones whose maximal Lyapunov exponent is positive, thus generically warranting strong chaos, hence mixing in phase space, hence ergodicity (in some sense, Boltzmann embodied all these features in his insightful molecular chaos hypothesis). Within this theory, it ubiquitously emerges the Gaussian distribution \( p_G \propto e^{-\beta x^2} \) (\( \beta > 0 \)). Indeed, this important probabilistic form (i) maximizes the (continuous version of the) entropy \( S_{BG} = -k \int dx p(x) \ln[p(x)] \) under the basic constraints of normalizability and finite width; (ii) constitutes the exact solution, for all values of space and time, of the simplest form of the (linear and homogeneous) Fokker-Planck equation, in turn based on the simplest form of the Langevin equation (which includes additive noise); (iii) is the \( N \to \infty \) attractor of the (appropriately centered and scaled) sum of \( N \) independent (or weakly correlated in an appropriate sense) discrete or continuous random variables whose second moment is finite (Central Limit Theorem, CLT); (iv) is the velocity distribution (Maxwell distribution) of any classical many-body Hamiltonian system whose canonical (thermal equilibrium with a thermostat) partition function is finite, i.e., if the interactions between its elements are sufficiently short-ranged, or inexistant. The simplest probabilistic model which realizes these paradigmatic features is a set of \( N \) independent equal bi-
binary random variables (each of them taking say the values
0 and 1 with probability 1/2). The probability of having, for
fixed $N$, $n$ 1’s is given by $\binom{N}{n} 2^{-N}$. Its
limiting distribution is, after centering and scaling, a Gaussian (as
first proved by de Moivre and Laplace), and its (extensive)
entropy is the $BG$ one, since $S_{BG}(N) = N k \ln 2$.

What happens with the above properties when the cor-
relations between say the elements of a probabilistic model
are strong enough (in the sense that they spread over all el-
ements of the system)? There is in principle no reason for
expecting the relevant limiting distribution to be a Gaussi-
an, and the entropy which is extensive (i.e., $S(N) \propto N$
for $N \gg 1$) to be $S_{BG}$. The purpose of the present paper
is to focus on such and related questions for a class of sys-
tems which are ubiquitous in natural, artificial and even
social systems, namely those which are scale-invariant in
a probabilistic sense which we shall define below. Let us
now discuss the frequent emergence of q-exponentials, de-
defined as

$$P_q(x) = N_q [1 - (1-q)\beta V(x)]^{1/(1-q)} = N_q e^{-\beta V(x)} ,$$

where $N_q$ is a normalization factor; $P_1(x) = N_1 e^{-\beta V(x)}$
is the standard BG case; for $V(x) \propto x^2$, $P_q(x)$ is a q-
Gaussian, which displays asymptotic power-laws and can
be seen as a natural generalization of the Gaussian ($q = 1$).

At this point let us make a few remarks. (i) q-
Gaussians appear as the exact solutions of paradigmatic
non-Markovian Langevin processes and their associated
Fokker-Planck equations. Langevin equations with both
additive and multiplicative noise [1], or Langevin equa-
tions with long-range-memory [2], lead respectively to in-
homogeneous linear [3], or homogeneous nonlinear [4,9]
Fokker-Planck equations (see also [6,10]). (ii) q-CLT at-
tractors are q-Gaussians [11]. (iii) The extremization of
the entropy $S_q$ with norm and finite width constraints
yields q-Gaussians, where $S_q$ is a generalization of BG
entropy, namely

$$S_q = k \frac{1 - f dx [p(x)]^q}{q - 1} \quad (q \in R; S_1 = S_{BG})$$

This entropy is, for $q \neq 1$, nonadditive (see [14] for
the current definition of additivity), i.e., for arbitrary prob-
abilistically independent systems $A$ and $B$, the equality
$S(A + B) = S(A) + S(B)$ is not satisfied. However, for
many systems a value of $q$, denoted by $q_{ent}$, exists for
which $S_{q_{ent}}$ is extensive, i.e., $S_{q_{ent}}(N) \propto N \quad (N \gg 1)$.
As is well known, for all standard short-range-interacting
many-body Hamiltonian systems, we have $q_{ent} = 1$. How-
ever, some systems exist for which $q_{ent} < 1$ [13,18]. (iv)
Numerical indications [13] for the distributions of veloc-
ities in quasistationary states of long-range Hamiltonians
[20] suggest q-Gaussians. Further, experimental and ob-
servational evidence for q-Gaussians exists for the motion of
biological cells [21,22], defect turbulence [23], solar wind
[24,25], cold atoms in dissipative optical lattices [26], dusty
plasma [27], among others (see also [28]). Numerical indi-
cations are also available at the edge of chaos of unimodal
maps [29].

A domain where the nonadditive entropy $S_q$ can be
naturally incorporated is for describing anomalous diffusion
like-phenomena. From modified Langevin equations,
inhomogeneous linear or homogeneous nonlinear Fokker-
Planck equations have been derived and used in order to
obtain the mesoscopic dynamic evolution of systems where
such diffusion occurs.

Let us consider here the nonlinear Fokker-Planck (FP)
equation given by

$$- \frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x} \left[F(x)p(x,t)\right] + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[\phi(x,p)p(x,t)\right]$$

where $D \in R$ is the coefficient of diffusion, $F = \frac{\partial V(x)}{\partial x}$
the drift term, and $V$ a confining potential. We shall further
assume the following wide connection:

$$\phi(x,p) = [g(V)]^p [p(x,t)]^q,$$

with $g(V) = [A + BV]$, $A$, $B$, $\theta$ and $\eta$ being real constants.

It should be noted that the present equation encompasses
also the general inhomogeneous ($\theta \neq 0$) nonlinear
($\eta \neq 0$) case. Our aim is to find whether a q-
exponential $P_q(x)$ exists as a stationary solution (i.e.,
$$\lim_{t \to \infty} p(x,t) = P_q(x))$$
of this FP equation such that the inhomogeneous linear and the homogeneous nonlinear FP
equations become unified.

The condition $\frac{\partial p(x,t)}{\partial t} = 0$ implies

$$\frac{\partial F(x)p(x,\infty)}{\partial x} = \frac{D}{2} \frac{\partial^2}{\partial x^2} \left[\phi(x,p)p(x,\infty)\right].$$

By assuming appropriate boundary conditions (basically
$p(\pm \infty, \infty) = 0$), changing variables and developing, we
obtain

$$\frac{\partial [g(V)^{\theta}p(V)^{1+\eta}]}{\partial V} = -\frac{2p(V)}{D},$$

where $p(V) \equiv p(V(x,\infty))$. It follows that

$$\frac{\partial p(V)}{\partial V} = -\frac{g(V)^{-\theta}p(V)^{1-\eta} + \beta BDg(V)^{-1}p(V)}{D(1+\eta)},$$

Now, we will first consider the $\theta \neq 1$ case. Let us propose
a solution of Eq.(7) satisfying

$$g(V)^{1-\theta}p(V)^{-\eta} = C$$

with $\partial C / \partial V = 0$. By direct substitution in Eq. (7), we
can easily show that $p(V) = P_q(V)$, as given by Eq. (1),
with

$$q = 1 + \frac{\eta}{\theta - 1},$$

and

$$\beta = \frac{2C + \theta BD}{AD(1+\eta)},$$

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where
\[ C = [AN_q^{-1}]^{1-\theta}. \]

These results reproduce, for \( \theta = 0 \), the homogeneous nonlinear case discussed in [2,4,5], i.e., \( q = 1 - \eta \) and \( N_q^{1-\eta} \beta = 2/D(1+\eta) \), which leads to normal diffusion for \( \eta = 0 \).

Eq. (9) shows that, for \( \eta \neq 0 \), the limit \( \theta \to 1 \) corresponds to a singularity in the \(|q|\) value. However, in this limit, we still find \( q \)-exponentials as stationary solutions of the above FP equation, in two different situations:

a) For \( \eta \) considered as a function of \( \theta \) with the leading term given by \( \eta \sim c_2 \theta \), the analytical extension of the solution given by Eqs. (8-11) results in \( q \)-exponentials presenting the indices \( q = 1 \) for \( \delta > 1 \) and \( q = 1+\alpha \) for \( \delta = 1 \), with \( \beta = (2+BD)/AD \) in both cases.

b) An isolated solution (which is not a particular case of the previous ones) can also be found by setting \( \eta = 0 \) and \( \theta = 1 \) in Eq. (7). By so doing we obtain
\[ \frac{\partial p(V)}{\partial V} = -\frac{2+BD}{D}g(V)^{-1}p(V). \]

Now, imposing the relation
\[ g(V)p(V)^{q-1} = \bar{C}, \]
where \( \bar{C} \) is independent of the potential \( V \). Following the same procedure used to obtain Eqs. (9) and (10), we finally have \( p(V) = P_q(V) \), with
\[ q = \frac{(1+BD)}{(1+BD/2)}, \]
and
\[ \beta = \frac{(2+BD)}{AD}. \]

It should be noticed that this case ((\( \eta, \theta \)) = (0,1)) recovers previous results already obtained in [1,3].

As final remarks, we note that:

(i) For \( A \neq 0 \), we may take \( A = 1 \) without loss of generality, if the \((B,D)\) parameters are properly rescaled;

(ii) It is known [13], that the \( q \)-Gaussians, which emerge for \( V(x) \propto x^2 \) are normalized only for \( q < 3 \), which implies the following restrictions to the values of the parameters of the FP equation: (a) For \( \theta \neq 1 \), \( \frac{\alpha}{\eta-1} < 2 \), (b) For \( \theta \to 1 \), \( \alpha < 2 \) if \( \delta = 1 \) and \( BD > -2 \) or \( BD < -4 \). For each \( V(x) \), an analysis of integrability of \( P_q \) must be performed, in order to establish the accepted range of values of the parameters \((\eta, \theta, B, D, q)\) (e.g., if \( V(x) \propto |x|^\rho \), in the limit \( |x| \to \infty \), then it must be \( q < 1 + \rho \));

(iii) We stress that the Eq. (3) used in this paper corresponds to the Itô form of a generalized FP equation (a Stratonovich approach has already been examined [10], and the existence of \( q \)-exponential distributions as stationary solutions has also been proved).

Let us emphasize that Eq. (9), here presented for the first time, enables us to analyze in an unique way the present general inhomogeneous nonlinear case, which contains, as particular cases, several FP equations used to describe a large class of nongaussian natural systems. This unification constitutes the main result of this paper.

To summarize, the nonlinear inhomogeneous FP process given by Eq. (3) has, as a stationary solution for any confining potential \( V(x) \), a probability distribution given by the \( q \)-exponential \( P_q(V) \), with finite values of \( q \) and \( \beta \) (see Fig. 1 and Table 1). These results exhibit that we may retain long memory (basically, the probability distribution longistically maintains memory of its form at \( t = 0 \)) as a unified mesoscopic mechanism consistent with nonextensive statistical mechanics.

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| FOKKER-PLANCK | Linear \((\eta \rightarrow 0)\) | Nonlinear \((\eta \neq 0)\) |
|---------------|-----------------|-----------------|
| **Homogeneous** \((\theta = 0)\) | \(q = 1\) | \(q = 1 - \eta\) |
| **Inhomogeneous** \((\theta \neq 0)\) | If \(\eta \sim (\theta - 1)^{\delta}\) and \(\theta \rightarrow 1\), \(q = 1\) for \(\delta > 1\), \(q = 1 + \alpha\) for \(\delta = 1\) | \(q = 1 + \frac{\eta}{\theta - 1}\) |
| | If \(\eta\) is strictly 0 and \(\theta \rightarrow 1\), \(q = \frac{2(1+BD)}{2+BD}\) | |

Table 1: Nonlinear inhomogeneous Fokker-Planck equation with \(q\)-exponential stationary-state distribution. The present \(\theta = 0\) result recovers that of Eqs. (2.8) and (2.9) of [4] with the notation correspondence \((F, Q, D, K)\) in [4] \(\leftrightarrow (P, D, N_q, F)\) here. It also recovers that of [5] by using there \(\mu = 1\) and \(\nu = 1 + \eta\). The result presented by the isolated solution at \((\theta, \eta) = (1, 0)\) recovers that cited (for the Itô approach) in [1] with the notation correspondence \((\frac{\eta}{\theta})\) in [1] \(\leftrightarrow \frac{\eta}{\theta}\) here. A comparison is also possible through Eq. (11) of [3] by using the notation correspondence \((K, D(U(x)))\) in [3] \(\leftrightarrow (F, Dg(V)^{\theta}, V(x))\) here.