Characterization of Uniform and Hybrid Cellular Automata with Reflecting Boundary

M Rajasekar\(^1\) and R Anbu\(^2\)

\(^1\)Maths Section - FEAT, Annamalai University, Annamalainagar, India
\(^2\) Department of Mathematics, Annamalai University, Annamalainagar, India

E-mail: mdivraj1962@gmail.com, anburaja2291@gmail.com

Abstract. In this paper, we study the characterization of uniform cellular automata with the restricted vertical neighborhood and hybrid cellular automata with both restricted vertical neighborhood and Von Neumann Neighborhood using reflecting boundary condition over the field \(\mathbb{Z}_2\). The transition rule matrix for uniform and hybrid cellular automata with reflecting boundary condition is obtained and also reversibility of the uniform cellular automata is studied.

1. Introduction

Two dimensional cellular automata has a promising approach to solve some problems in human life, such as pseudo random generation, pattern classification, image en-compression and de-compression. The notion of cellular automata (CA) was introduced by John Von Neumann and Stanislaw Ulam in 1950’s. John Von Neumann \([13]\) showed that a cellular automaton (CA) can be universal.

Cellular Automata are also called Cellular Space, Tessellation Automata, Homogeneous structures, Cellular structure, Tessellation structures and Iterative arrays \([14]\).

The study of CA has received remarkable attention in the last few years \([1, 6, 15]\), because CA have been widely investigated in many disciplines (For Example Mathematics, physics, computer science, chemistry, etc.) with different purposes(For Example simulation of natural phenomena, pseudo-random number generation, image processing, analysis of universal model of computations, cryptography). Most of the work for CA\(_{\text{SA}}\) is done for one-dimensional (1-D) case. Recently, two dimensional (2-D) CA\(_{\text{SA}}\) have attracted much of the interest. Some basic and precise mathematical models using matrix algebra built on field with two elements \(\mathbb{Z}_2\) were reported for characterizing the behavior of two dimensional nearest neighborhood linear cellular automata with null or periodic boundary conditions \([5, 6, 15]\).

The 2D finite cellular automata consists of \((m \times n)\) cells arranged. Where each cell takes one of the values of the field \(\mathbb{Z}_2\). The relative positions of the cells is called neighbor of the cell given center cell. The state of these neighbors are used to compute the new state of the center cell.
Type of neighborhood cells in Figure 1. (a) Von Neumann Neighborhood (top, right, bottom and left) (b) Moore Neighborhood (All the nearest surrounding the center cell) (c) Restricted Vertical Neighborhood (The class of CA’s where a cell can have either top or bottom dependency but not both)

The paper is organized as follows. In the 2nd section, the concept used in the paper are formally defined. In 3rd section, the algebraic structure of restricted vertical neighborhood and Von Neumann neighborhood of 2D cellular automata is obtained. In 4th section, the rule of the uniform cellular automata with restricted vertical neighborhood is studied. In 5th section, using both Von Neumann and Restricted Vertical Neighborhood the rule matrix of the hybrid cellular automata is obtained. In 6th section, we compute the rank of rule matrix related to 2D uniform cellular automata. Also study the reversibility of the restricted vertical neighborhood of uniform CA.

2. Preliminaries

Definition 2.1 [9] A periodic Boundary CA is the one in which the extreme cells are adjacent to each other.

Definition 2.2 [9] A null boundary CA is the one in which the extreme cells are connected to logic zero state.

Definition 2.3 [2] Reflecting Boundary: In a reflecting boundary, the state of the opposite neighbor is replicated by the virtual cell.

Definition 2.4 [15] Each state of a CA is called a configuration. In particular, each configuration of a (2-D) CA \( C_{m \times n} \) is a binary information matrix of dimension \((m \times n)\).

Definition 2.5[10] Uniform Cellular Automata: The uniform cellular automata have been presented by Nandi et al. If the same rule is applied to all the cells in a CA, then the CA is said to be uniform or regular CA.

Definition 2.6[3] Hybrid Cellular Automata: The hybrid cellular automata have been explored by Anghelescu et al. If in a CA the different rules are applied to different cells, then the CA is said to be hybrid CA.

Definition 2.7[8] Reversibility of CA: A cellular automata rule is called reversible if there is another rule that makes the automaton retrace its computation steps backwards in time.

Definition 2.8[11] Restricted Vertical Neighborhood: The restricted vertical neighborhood (RVN) cellular automata rule is the class of CA’s where a cell can have either top (or) bottom dependency but not both, this class of CA is referred to as Restricted Vertical Neighborhood CA.

Definition 2.9[4] Cellular Automata: Cellular automata are formally defined as quadruplets \( A = \{d, S, N, f\} \)

* \( d \in \mathbb{Z}_+ \) is the dimension of the cellular space, then each point \( \vec{n} \in \mathbb{Z}^d \) is called a cell.
* \( S = \{0, 1, 2, ..., p - 1\} \) represents the finite state set and the state of any cell at any time must be taken from S.
* \( N = (\vec{n}_1, \vec{n}_2, ..., \vec{n}_m) \) is the neighbor vector, where \( \vec{n}_i \in \mathbb{Z}^d \) and \( \vec{n}_i \neq \vec{n}_j \) when \( i \neq j \) (i, j = 1, 2, ..., m). Thus, the neighbors of the cell \( \vec{n}_i \) are the m cells, i.e. 1, 2, ..., m.
\* f : \mathcal{S}^m \rightarrow \mathcal{S} \text{ is the local rule, which maps the current states of all neighbors of a cell to the next state of this cell.}

A configuration is a mapping \( C : \mathbb{Z}^d \rightarrow \mathcal{S} \) which assigns each cell a state. Make \( C^t \) denote the configuration at time \( t \), then the state of cell \( \vec{n} \) at time is \( C^t(\vec{n}) \) and its state at time \( (t+1) \) goes like this.

\[
C^{t+1}(\vec{n}) = f(C^t(n_1), C^t(n_2), \ldots, C^t(n_m))
\]

now we consider the case in which the local rule \( f \) is a linear function

\[
C^{t+1}(\vec{n}) = f(C^t(n_1) + C^t(n_2) + \ldots + C^t(n_m))
\]

Where \( \lambda_i \in \mathcal{S} \) is the rule co-efficient for neighbor \( n_i \), \( i=1, 2, \ldots, m \).

In this paper, we deal with CA defined by Von neumann rules and Restricted Vertical Neighborhood under reflecting Boundary condition (RB) of Uniform and Hybrid cellular automata. For convenience of analysis, the state of each cell is an element of a finite or an infinite state set. Moreover, the state of the cell \((i,j)\) at time \( t \) is denoted by \( X^{(t)}_{(i,j)} \). The state of the cell \((i,j)\) at time \((t+1)\) is denoted by \( X^{(t+1)}_{(i,j)} = Y^{(t)}_{(i,j)} \).

In [7], they consider the information matrix \( C^{(t)} \)

\[
C^{(t)} = \begin{pmatrix} X^{(t)}_{11} & \cdots & X^{(t)}_{1n} \\ \vdots & \ddots & \vdots \\ X^{(t)}_{m1} & \cdots & X^{(t)}_{mn} \end{pmatrix}
\]

The matrix \( C^{(t)} \) is called the configuration of the 2D finite CA at time \( t \).

We transition the row vectors of \( C^{(t)} \) to \([X]_{1 \times m}^{(t)}\) as follows

\[
([X]_{1 \times m}^{(t)}) = (X^{(t)}_{11}, X^{(t)}_{12}, \ldots, X^{(t)}_{1n}, X^{(t)}_{21}, \ldots, X^{(t)}_{2n}, \ldots, X^{(t)}_{m1}, \ldots, X^{(t)}_{mn}).
\]

where \( X^{(t)}_{(i,j)} \in \mathbb{Z}_2 \).

Hence, we consider the transition matrix \( T_R \) that changes set of states of cellular automata from \( t \) to \((t+1)\) such that

\[
[X]_{1 \times m}^{(t)} (T_R)_{m \times m} = [Y]_{m \times 1}^{(t+1)},
\]

where

\[
([Y]_{m \times 1}^{(t+1)}) = (X^{(t+1)}_{11}, X^{(t+1)}_{12}, \ldots, X^{(t+1)}_{1n}, X^{(t+1)}_{21}, \ldots, X^{(t+1)}_{2n}, \ldots, X^{(t+1)}_{m1}, \ldots, X^{(t+1)}_{mn})
\]

= \((Y_{11}^{(t)}, Y_{12}^{(t)}, \ldots, Y_{1n}^{(t)}, Y_{21}^{(t)}, \ldots, Y_{2n}^{(t)}, \ldots, Y_{m1}^{(t)}, \ldots, Y_{mn}^{(t)}))\).

3. 2D cellular automata over the field \( \mathbb{Z}_2 \)

In [12], the 2D finite CA consists of \((m \times n)\) cells arranged in \( m \) rows and \( n \) columns, where each cell takes one of the values of the field \( \mathbb{Z}_2 \). From now on, we denote 2D finite CA order \((m \times n)\) by \( 2D \ CA_{(m \times n)} \). A configuration of the system is an assignment of the states to all cells. Every configuration determines a next configuration via a linear transition rule that is local in the sense that the state of a cell at time \((t+1)\) depends only on the states of some of its neighbor at the time \( t \) using modulo 2 algebra. For 2D CA nearest neighbors, there are \((mn)\) cells arranged in a \((m \times n)\) matrix centering that particular cell.

3.1. Restricted Vertical Neighborhood - 162 Rule

In 2D CA theory, there are some classic types of neighborhood but in this paper we only restrict ourselves to the special neighbors which is called RVN. The RVN comprises the 3 cells surrounding the central cell on two-dimensional.
In Figure 2, we show the RVN which comprises 3 cells which surround the center cell $X_{(p,q)}$. The state $X_{(p,q)}^{(t+1)}$ of the cell $(p,q)^{th}$ at time $(t+1)$ is defined by the local rule function $f: \mathbb{Z}^2 \rightarrow \mathbb{Z}_2$ as follows,

$$X_{(p,q)}^{(t+1)} = aX_{(p,q+1)} + bX_{(p,q-1)} + cX_{(p-1,q)}$$

where $a, b, c \in \mathbb{Z}_2 = \{0, 1\}$.

3.2. Von Neumann Neighborhood - 170 Rule
The Von Neumann neighborhood comprises the four cells surrounding the center cell on a 2D square lattice.

$$X_{(p,q)}^{(t+1)} = f(X_{(p,q+1)}, X_{(p,q-1)}, X_{(p-1,q)}, X_{(p+1,q)})$$

where $a, b, c, d \in \mathbb{Z}_2 = \mathbb{Z}_2 \setminus \{0\}$.

4. Uniform Cellular Automata
4.1. Transition rule matrix of the Restricted Vertical neighborhood for reflecting Boundary
We obtain the rule matrix of 2D finite CA with RVN rule over the field $\mathbb{Z}_2$ under the reflecting boundary condition. The rule matrix which takes the $t^{th}$ finite configuration matrix $C^t$ of order $(m \times n)$ to the $(t+1)^{th}$ state $C^{(t+1)}$.

Theorem 4.1 Let $a, b, c \in \mathbb{Z}_2^* = \mathbb{Z}_2 \setminus \{0\}$, $m \geq 3$ and $n \geq 3$. Then the rule matrix $T_{R-162}$ of finite 2-D Restricted Vertical Neighborhood cellular automata with reflecting boundary condition from the
configuration $c^{(t)}$ to $(t+1)^{th}$ is given by,

$$
T_R = \begin{pmatrix}
A & cI & 0 & 0 & 0 & \cdots & 0 & 0 \\
cI & A & cI & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & A & cI & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & A & cI & \cdots & 0 & 0 \\
& \vdots & & \vdots & & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & A & cI \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & A \\
\end{pmatrix}_{(mn \times mn)}
$$

where,

$$
A = \begin{pmatrix}
0 & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
(a+b) & 0 & b & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & a & 0 & b & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & a & 0 & b & \cdots & 0 & 0 & 0 \\
& \vdots & & \vdots & & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & a & 0 & (a+b) \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & a & 0 \\
\end{pmatrix}_{(n \times n)}
$$

and I is an identity matrix of order $(n \times n)$

**Proof.**

Let $T_R X_{(p,q)} = Y_{(p,q)}$. $Y_{(p,q)} = X_{(p,q)}^{(t+1)}$ is a equal to the linear combination of the neighbors. The co-efficient of $X_{pq} = 0$ if $p \leq 0$ or $q \leq 0$. By using the local rule of the cellular automata we have obtain the following,

$Y_{(1,1)} = aX_{(1,2)} + bX_{(1,2)} + cX_{(2,1)}$

$Y_{(1,2)} = aX_{(1,3)} + bX_{(1,1)} + cX_{(2,2)}$, $2 \leq q \leq n-1$

$Y_{(1,q)} = aX_{(1,q+1)} + bX_{(1,q-1)} + cX_{(2,q)}$

$Y_{(1,n)} = aX_{(1,n-1)} + bX_{(1,n-1)} + cX_{(2,n)}$

For $2 \leq p \leq n-1$, we have

$Y_{(p,1)} = aX_{(p,2)} + bX_{(p,2)} + cX_{(p-1,1)}$

$Y_{(p,q)} = aX_{(p,q+1)} + bX_{(p,q-1)} + cX_{(p-1,q)}$, $2 \leq q \leq n-1$

$Y_{(p,n)} = bX_{(p,n-1)} + bX_{(p,n-1)} + cX_{(p-1,n)}$

Finally, we have

$Y_{(m,1)} = aX_{(m,2)} + bX_{(m,2)} + cX_{(m-1,1)}$

$Y_{(m,q)} = aX_{(m,q+1)} + bX_{(m,q-1)} + cX_{(m-1,q)}$, $2 \leq q \leq n-1$

$Y_{(m,n)} = aX_{(m,n-1)} + bX_{(m,n-1)} + cX_{(m-1,n)}$

Finally we get the rule matrix $T_R - 162$.

**Example 4.1**

If we take $m=5$ and $n=5$, then we get rule matrix $T_R - 162$ of 2D finite CA with RVN rule over the field $\mathbb{Z}_2$ be as follows,

$$
T_{R-162} = \begin{pmatrix}
A & cI & 0 & 0 & 0 \\
cI & A & cI & 0 & 0 \\
0 & 0 & A & cI & 0 \\
0 & 0 & 0 & A & 0 \\
0 & 0 & 0 & 0 & A \\
\end{pmatrix}_{(25 \times 25)}
$$
where, \( A = \begin{pmatrix}
0 & b & 0 & 0 & 0 \\
(a + b) & 0 & b & 0 & 0 \\
0 & a & 0 & b & 0 \\
0 & 0 & a & 0 & (a + b) \\
0 & 0 & 0 & a & 0
\end{pmatrix} \) (5×5)

where \( a, b \in \mathbb{Z}_2^* = \mathbb{Z}_2\backslash\{0\} \).

\( A, cI \) are the sub matrices of order \((5 \times 5)\), \(0\) is the zero matrices.

Similarly, \( I \) is an identity matrix.

5. Hybrid Cellular Automata

In the present study, we work with special 2D CA defined by hybrid rule over the field \( \mathbb{Z}_2 \) under reflecting boundary condition. We will determine the transition matrix rule \( c' \) of matrix of order \((m \times n)\) is considered.

Case(i). \( m \) is even, the rule matrix \( T_R \) is given in the following theorem

**Theorem 5.1**

Let us consider \( a, b, c, d \in \mathbb{Z}_2^* = \mathbb{Z}_2\backslash\{0\}, \) \( m \geq 3, \) \( n \geq 3 \) and \( m \) is even . Then the rule matrix \( T_{R-HYD} \) from \( \mathbb{Z}_2^{mn} \rightarrow \mathbb{Z}_2^{mn} \) which takes \( t^{th} \) state to \((t + 1)^{th} \) state is given by

\[
T_R=
\begin{pmatrix}
A & cI & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
(c + d)I & A & cI & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & A & cI & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & dI & A & cI & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A & cI & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & A & cI & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & A & cI \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & dI & A
\end{pmatrix}_{(mn \times mn)}
\]

where,

\[
A = \begin{pmatrix}
0 & b & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
(a + b) & 0 & b & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & a & 0 & b & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & a & 0 & b & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & A & 0 & (a + b) \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & a & 0
\end{pmatrix}_{(n \times n)}
\]

and \( I \) is an identity matrix of order \((n \times n)\)

**Proof.**
Let \( T_R X_{(p,q)} = Y_{(p,q)} \). \( Y_{(p,q)} = X_{(p,q)}^{(t+1)} \) is equal to the linear combination of the neighbors. The co-efficient of \( X_{pq} = 0 \) if \( p \leq 0 \) or \( q \leq 0 \). The local rule of 2D CA with RB is obtained using rule 162 for even row in \( C(t) \) and rule 170 for odd row in \( C(t) \).

\[
Y_{(1,1)} = aX_{(1,2)} + bX_{(2,1)} + cX_{(2,1)} + dX_{(2,1)}
\]

\[
Y_{(1,q)} = aX_{(1,q+1)} + bX_{(1,q-1)} + dX_{(2,q)} + dX_{(2,q)}, \quad 2 \leq q \leq n - 1
\]

\[
Y_{(1,n)} = aX_{(1,n-1)} + bX_{(1,n-1)} + cX_{(2,n)} + dX_{(2,n)}
\]

\[
Y_{(2,1)} = aX_{(2,2)} + bX_{(2,2)} + cX_{(1,1)} + dX_{(1,1)}
\]

\[
Y_{(2,q)} = aX_{(2,q+1)} + bX_{(2,q-1)} + cX_{(1,q)}, \quad 2 \leq q \leq n - 1
\]

\[
Y_{(2,n)} = aX_{(2,n-1)} + bX_{(2,n-1)} + cX_{(1,n)}
\]

For \( m \) is odd row following equation

\[
Y_{(p,1)} = aX_{(p,2)} + bX_{(p,2)} + cX_{(p-1,1)} + dX_{(p+1,1)}
\]

\[
Y_{(p,q)} = aX_{(p,q+1)} + bX_{(p,q-1)} + cX_{(p-1,q)} + dX_{(p+1,q)}, \quad 2 \leq p \leq m - 1 \text{ and } 2 \leq q \leq n - 1
\]

\[
Y_{(p,n)} = aX_{(p,n-1)} + bX_{(p,n-1)} + cX_{(p-1,n)} + dX_{(p+1,n)}
\]

For \( m \) is even row following equation

\[
Y_{(p,1)} = aX_{(p,2)} + bX_{(p,2)} + cX_{(p-1,1)} \]

\[
Y_{(p,q)} = aX_{(p,q+1)} + bX_{(p,q-1)} + cX_{(p-1,q)}, \quad 2 \leq p \leq m - 1 \text{ and } 2 \leq q \leq n - 1
\]

\[
Y_{(p,n)} = aX_{(p,n-1)} + bX_{(p,n-1)} + cX_{(p-1,n)}
\]

Finally we get the rule matrix.

**Case(ii).** \( m \) is odd, the rule matrix \( T_R \) is given in the following theorem

![Rule Matrix Diagram](image)

Figure 5. Hybrid rule means that it is applied 170RB and 162RB respectively for each rows when \( m \) is odd on \((m \times n)\) CA.

**Theorem 5.2**

Let us consider \( a, b, c, d \in \mathbb{Z}_2^* = \mathbb{Z}_2 \backslash \{0\}, m \geq 3, n \geq 3 \) and \( m \) is odd. Then the rule matrix \( T_{R-HYD} \) from \( \mathbb{Z}_2^{mn} \rightarrow \mathbb{Z}_2^{mn} \) which takes \( t^{th} \) state to \((t + 1)^{th} \) state is given by

\[
T_R = \begin{pmatrix}
A & cI & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
(c + d)I & A & cI & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & A & cI & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & dI & A & cI & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A & cI & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & A & cI & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & dI & A & (c + d)I \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & A & \end{pmatrix}_{(mn \times mn)}
\]

Where,
For \( n = 4 \),
\[
A = \begin{pmatrix}
0 & b & 0 & 0 & \ldots & 0 & 0 & 0 \\
(a + b) & 0 & b & 0 & \ldots & 0 & 0 & 0 \\
0 & a & 0 & b & \ldots & 0 & 0 & 0 \\
0 & 0 & a & 0 & b & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & a & 0 & (a + b) \\
0 & 0 & 0 & 0 & \ldots & 0 & a & 0
\end{pmatrix}_{(n \times n)}
\]

and I is an identity matrix of order \((n \times n)\).

**Proof.**
Let \( T_R X_{(p,q)} = Y_{(p,q)} \), \( Y_{(p,q)} = X^{(t+1)}_{(p,q)} \) is a equal to the linear combination of the neighbors. The co-efficient of \( X_{pq} = 0 \) if \( p \leq 0 \) or \( q \leq 0 \). The local rule of 2D CA with RB is obtained using rule 162 for even row in \( C^{(t)} \) and rule 170 for odd row in \( C^{(t)} \).

\[
Y_{(1,1)} = aX_{(1,2)} + bX_{(1,2)} + cX_{(2,1)} + dX_{(2,1)} \\
Y_{(1,q)} = aX_{(1,q+1)} + bX_{(1,q-1)} + dX_{(2,q)} + dX_{(2,q)}, \quad 2 \leq q \leq n - 1 \\
Y_{(1,n)} = aX_{(1,n-1)} + bX_{(1,n-1)} + cX_{(2,n)} + dX_{(2,n)} \\
Y_{(2,1)} = aX_{(2,2)} + bX_{(2,2)} + cX_{(1,1)} \\
Y_{(2,q)} = aX_{(2,q+1)} + bX_{(2,q-1)} + cX_{(1,q)}, \quad 2 \leq q \leq n - 1 \\
Y_{(2,n)} = aX_{(2,n-1)} + bX_{(2,n-1)} + cX_{(1,n)}
\]

For \( m \) is odd row following equation
\[
Y_{(p,1)} = aX_{(p,2)} + bX_{(p,2)} + cX_{(p-1,1)} + dX_{(p+1,1)} \\
Y_{(p,q)} = aX_{(p,q+1)} + bX_{(p,q-1)} + cX_{(p-1,q)} + dX_{(p+1,q)}, \quad 2 \leq p \leq m - 1 \text{ and } 2 \leq q \leq n - 1 \\
Y_{(p,n)} = aX_{(p,n-1)} + bX_{(p,n-1)} + cX_{(p-1,n)} + dX_{(p+1,n)}
\]

For \( m \) is even row following equation
\[
Y_{(p,1)} = aX_{(p,2)} + bX_{(p,2)} + cX_{(p-1,1)} \\
Y_{(p,q)} = aX_{(p,q+1)} + bX_{(p,q-1)} + cX_{(p-1,q)}, \quad 2 \leq p \leq m - 1 \text{ and } 2 \leq q \leq n - 1 \\
Y_{(p,n)} = aX_{(p,n-1)} + bX_{(p,n-1)} + cX_{(p-1,n)}
\]

Finally we get the rule matrix.

### 6. Reversibility of Uniform Cellular Automata

In this section our aim is to study whether the rule matrix is invertible or not. It is known that the 2D finite cellular automata is reversible if and only if its rule matrix is non singular. If the rule matrix has full rank, then it is invertible, so the 2D finite cellular automata is reversible, otherwise it is irreversible.

To determine the invertibility of the rule matrix associated to the Uniform Cellular Automata, we first study the submatrix \( A \) for all case.

**Lemma 6.1**
For all \( n \geq 3 \), \( rank(A) \neq n \). Then \( A \) is not invertible.

**Proof.**
We prove the lemma by induction on \( n \).

For \( n = 3 \),
\[
A = \begin{pmatrix}
0 & b & 0 \\
(a + b) & 0 & (a + b) \\
0 & a & 0
\end{pmatrix}
\]
\[
det(A_{(3 \times 3)}) = -b(0) \\
det(A_{(3 \times 3)}) = 0
\]

For \( n = 4 \),
\[ A = \begin{pmatrix} a + b & 0 & 0 & 0 \\ 0 & a & 0 & (a + b) \\ 0 & 0 & a & 0 \\ 0 & 0 & a & 0 \end{pmatrix} \]

\[ \det(A_{4 \times 4}) = -b \begin{pmatrix} 0 & b & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & a \end{pmatrix} = -b(A_{3 \times 3}) \]

\[ \det(A_{4 \times 4}) = 0 \]

For \( n = k \),
\[ A = \begin{pmatrix} 0 & b & 0 & 0 & \cdots & 0 & 0 & 0 \\ (a + b) & 0 & b & 0 & \cdots & 0 & 0 & 0 \\ 0 & a & 0 & b & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & 0 & b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & 0 & (a + b) \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & 0 \end{pmatrix} \]

\[ = b \begin{pmatrix} 0 & b & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & b & \cdots & 0 & 0 & 0 \\ 0 & a & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & a & 0 \end{pmatrix} \]

\[ = -b(A_{(k-1) \times (k-1)}) \]

\[ = -b(0) \]

\[ \det(A_{(k \times k)}) = 0 \]

If the result is true for \( n = k \), we prove that is also true for \( n = k + 1 \)
\[ A = \begin{pmatrix} 0 & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ (a + b) & 0 & b & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a & 0 & b & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a & 0 & b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & 0 & (a + b) \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & 0 \end{pmatrix} \]
\[
\begin{pmatrix}
0 & b & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & a & 0 & b & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & a & 0 & b & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & a & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & a & 0 \\
\end{pmatrix}
\]
\[= -b(A(k_{x,k}))
\]
\[= -b(0)
\]
\[\text{det}(A((k+1)\times(k+1))) = 0
\]
Therefore,
A is not invertible.

**Theorem 6.1**
Prove that 2D uniform cellular automata with restricted vertical neighborhood is not reversible.

**Proof.**
The proof of above lemma can be obtained by following the theorem.

7. **Conclusions**
In this paper, 2D cellular automata of Restricted Vertical Neighborhood and Von Neumann Neighborhood are studied over the field \(\mathbb{Z}_2\). The rule matrix of the transition \(T_R\) matrix of 2D uniform cellular automata is computed. We prove that the RVN for RB rule matrix is not reversible.

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