Regularity results for Choquard equations involving fractional $p$-Laplacian

Reshmi Biswas | Sweta Tiwari

Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati, Assam, India

Correspondence
Sweta Tiwari, Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati, Assam 781039, India.
Email: swetatiwari@iitg.ac.in

Funding information
Indian Institute of Technology Guwahati, Grant/Award Number: Research Grant for Reshmi Biswas

Abstract
In this paper, first we address the regularity of weak solution for a class of $p$-fractional Choquard equations:

\[
(-\Delta)_p^s u = \left( \int_{\Omega} \frac{F(y,u)}{|x-y|^{\mu}} dy \right) f(x,u), \quad x \in \Omega,
\]

\[
u = 0, \quad x \in \mathbb{R}^N \setminus \Omega,
\]

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $1 < p < \infty$ and $0 < s < 1$ such that $sp < N$, $0 < \mu < \min\{N, 2sp\}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function with at most critical growth condition (in the sense of the Hardy–Littlewood–Sobolev inequality), and $F$ is its primitive. Next, for $p \geq 2$, we discuss the Sobolev versus Hölder minimizers of the energy functional $J$ associated with the above problem, and using that we establish the existence of the local minimizer of $J$ in the fractional Sobolev space $W^{s,p}_0(\Omega)$. Moreover, we discuss the aforementioned results by adding a local perturbation term (at most critical in the sense of Sobolev inequality) in the right-hand side in the above equation.

KEYWORDS
a priori bound, Choquard equation, critical exponents, fractional $p$-Laplacian, Sobolev vs. Hölder minimizers

MSC (2020)
35J60, 35R11, 35B33, 35B65, 35B45, 35A15

1 | INTRODUCTION

Our first aim in this paper is to study the following doubly nonlocal $p$-fractional Choquard equation:

\[
(-\Delta)_p^s u = \left( \int_{\Omega} \frac{F(y,u)}{|x-y|^{\mu}} dy \right) f(x,u), \quad x \in \Omega,
\]

\[
u = 0, \quad x \in \mathbb{R}^N \setminus \Omega,
\]

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{1,1}$ boundary, $1 < p < \infty$ and $0 < s < 1$ such that $sp < N$, $0 < \mu < \min\{N, 2sp\}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function with at most critical growth condition (in the sense of the Hardy–Littlewood–Sobolev inequality), described later. Here, $F(x,t) = \int_0^t f(x,\tau)d\tau$ is the primitive of $f$. The nonlocal operator $(-\Delta)_p^s$ is
defined as

$$(\Delta)^p u(x) := 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y))}{|x - y|^{N+sp}} dy, \ x \in \mathbb{R}^N$$

(1.2)

up to a normalized constant. In recent years, significant attention has been given in the study of the problems involving the nonlocal operators, due to its various applications in the real world such as thin obstacle problems, finance, conservation laws, phase transition, crystal dislocation, anomalous diffusion, material science, etc. (see, for example, [2, 8, 27, 44], and references therein for more details). One can refer to the monograph [31] for the study of nonlocal problems driven by the fractional Laplacian and [18, 22] for the detailed discussions on the fractional $p$-Laplacian and problems involving it.

On the other hand, the study of Choquard-type equations was started with the celebrated work of Pekar [39], where the author considered the following nonlinear Schrödinger–Newton equation:

$$-\Delta u + V(x)u = (\mathcal{K}_\mu * u^2)u + \lambda f(x, u),$$

(1.3)

where $\mathcal{K}_\mu$ denotes the Riesz potential. The nonlinearity in the right-hand side of Equation (1.3) is termed as Hartree-type nonlinearity. This type of nonlinearity plays a key role in the study the Bose–Einstein condensation (see [13]) and also describes the self-gravitational collapse of a quantum-mechanical wavefunction (see [40]). For $V(x) = 1, \lambda = 0$, the equations of type (1.3) were extensively studied in [29, 30]. For more results on the existence of solutions of Choquard equations, without attempting to provide a complete list, we refer to [32–35], and references therein. In the fractional Laplacian set up, Wu [49] discussed the existence and stability of solutions for the equations

$$(-\Delta)^s u + \omega u = (\mathcal{K}_\mu * |u|^q)|u|^{q-2}u + \lambda f(x, u) \text{ in } \mathbb{R}^N,$$

(1.4)

where $q = 2, \lambda = 0,$ and $\mu \in (N - 2s, N).$ For the critical case, that is, $q = 2^{*}_{\mu,s} := (2N - 2\mu/2)/(N - 2s)$, Mukherjee and Sreenadh [37] studied existence and multiplicity, and regularity results for the solutions of Equation (1.4) in a smooth bounded domain for $w = 0$ and $f(x, u) = u$. Pucci et al. [41] studied some Schrödinger–Choquard–Kirchhoff equation driven by the fractional $p$-Laplacian with critical Hardy–Littlewood–Sobolev exponent. For more details regarding Choquard-type equations, we refer to the survey paper [38], and references therein.

The regularity of weak solutions has been one of the most interesting topics since years and the literature available on the regularity of weak solutions for both local and non-local problems is quite vast. For the regularity results of the local elliptic problems, we refer to [14, 28, 47]. A systematic study on the regularity results of the non-local elliptic problems involving fractional Laplacian started with the pioneering work of Caffarelli and Silvestre in [9]. Consider the following non-local problem:

$$(-\Delta)^s u = g \text{ in } \Omega, \quad u = h \text{ in } \mathbb{R}^N \setminus \Omega.$$  

(1.5)

When $p = 2$, in [9], Caffarelli and Silvestre established the interior $C^{1+\alpha}, \alpha > 0$, regularity for viscosity solutions to Equation (1.5). The authors also proved interior $C^{2+\alpha}$ regularity for the convex equation (see [10]). For the regularity of weak solutions to the free boundary problem involving the fractional Laplacian ($p = 2$), we refer to [44]. Concerning the boundary regularity for the solution of Equation (1.5), for $p = 2, h = 0$ and $g \in L^\infty(\Omega)$, we refer to the work of Ros-Oton and Serra in [42]. Here, the authors used a barrier function and the interior regularity results for the fractional Laplacian to show that any weak solution $u$ of Equation (1.5) belongs to $C^\alpha(R^N)$ and $\frac{\partial u}{\partial \nu}|_{\partial \Omega} \in C^\alpha$, up to the boundary $\partial \Omega$, for some $\alpha \in (0, 1)$. In [43], the authors discussed the high integrability of these weak solution by using the regularity of Riesz potential established in [45]. Following the approach of De Giorgi in the non-local and nonlinear setting, in [17], the authors established the local Hölder regularity of the viscosity solution of the non-local quasi-linear problem. In [12], Cozzi extended this result to functions belonging to a fractional analog of the so-called De Giorgi classes. The regularity results for the non-local quasi-linear problem are also explored by Squassina et al. in [23], where the authors studied the global Hölder regularity for the weak solutions to Equation (1.5), for $p \in (1, \infty), h = 0,$ and $g \in L^\infty(\Omega)$. In [7], the authors studied the higher Hölder regularity and provided an explicit Hölder exponent for solutions of the non-local quasi-linear problem with non-homogeneous data in $L^q$ for $q > N/(sp)$. Also, regarding the fine boundary regularity results for the problems of type (1.5), for the degenerate case ($p \geq 2$) and $h = 0$, we cite [26]. Here, the authors exhibited a weighted Hölder regularity up to the
boundary, that is, \( \frac{\partial u}{\partial \nu} |_{\partial \Omega} \in C^{\alpha} \) up to the boundary \( \partial \Omega \), for some \( \alpha \in (0, 1) \). We would like to mention that the fine boundary regularity for the singular case \((1 < p < 2)\) is still an open problem. For more works on the regularity estimates of weak solutions to the nonlocal elliptic problems governed by the fractional \( p \)-Laplacian, without attempting of providing the complete list, we refer to [7, 12, 17] and the references there in.

Concerning the regularity of the Choquard equations, we refer to [20], in which Gao and Yang studied the Dirichlet problem involving local Laplacian and the critical Choquard-type nonlinearity (in the sense of the Hardy–Littlewood–Sobolev inequality). Moroz and Schaftingen [33] established the \( W^{2, q}_{\text{loc}}(\mathbb{R}^N) \)-regularity \((q > 1)\) of the weak solutions to the following Choquard problem involving local Laplacian:

\[
-\Delta u + u = (\mathcal{K}_{\mu} * F(u))f(u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.6}
\]

where

\[
|t f(t)| < C(|t|^{N+\mu \over N} + |t|^{N+\mu \over N-2}), \quad \text{for some constant} \quad C > 0.
\]

Although an extensive research is done on the existence of solutions for the doubly non-local problems, there are very few results present in the literature regarding the regularity of weak solutions to such problems. By generalizing the idea of [34], in [15], for the fractional Laplacian framework, the authors established the regularity results for solutions of the following Choquard equation:

\[
(-\Delta)^s u + \omega u = (\mathcal{K}_{\mu} * |u|^r)u^{r-2}u, \quad u \in H^s(\mathbb{R}^N),
\]

where \( \omega > 0, N \geq 3, \mu \in (0, N), s \in (0, 1), \) and \( 2^*_{s, \mu} < r < 2^*_{s, \mu} \). In [46], the authors studied the \( L^\infty(\mathbb{R}^N) \) bound of the non-negative ground state solution to some Kirchhoff–Choquard equation driven by the fractional Laplacian with critical Choquard term (in the sense of the Hardy–Littlewood–Sobolev inequality). Very recently, Giacomoni et al. [21] studied the regularity result for the following generalized doubly non-local problem in a smooth bounded domain \( \Omega \) in \( \mathbb{R}^N \):

\[
(-\Delta)^s u = g(x, u) + \left( \int_{\Omega} \frac{F(u(y))}{|x-y|^\mu} dy \right)f(u(x)) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega, \tag{1.7}
\]

where \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function such that there exists a constant \( C > 0 \),

\[
|t f(t)| \leq C \left( |t|^{2N-\mu \over N} + |t|^{2N-\mu \over N-2} \right)
\]

and \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function satisfying Sobolev-type critical (or singular) growth assumption.

We mention that the techniques used in [21, 33] cannot be implemented straightforward to Equation (1.1) due to the lack of the Hilbert nature of the solution space associated with the problem. The regularity result for the quasilinear Choquard equations involving the local (or fractional) \( p \)-Laplacian are very few. For instance, consider the following equation studied in [3]:

\[
(-\Delta)^s p u + \omega u = \left( \frac{1}{|u|^\mu} * F(u) \right)f(u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.8}
\]

where \( \omega > 0 \) is a real number and \( f \) has sub-critical growth in terms of the Hardy–Littlewood–Sobolev inequality. For the case \( s = 1 \), we cite [1], in which the authors studied Equation (1.8) in the local \( p \)-Laplacian set up. In both the aforementioned works, the authors proved local Hölder regularity of the weak solutions of Equation (1.8) with some restrictive conditions, viz., \( \mu < s \rho \) and \( \mu < p \), respectively.

Inspired by all these works, using a unified boot-strap technique for \( 1 < p < \infty \), first we investigate a priori bound for the weak solutions to Equation (1.1) which covers a large class of nonlinearities (up to the critical level in the sense of Hardy–Littlewood–Sobolev inequality). After achieving \( L^\infty(\Omega) \) estimate on the weak solution to Equation (1.1), we use the result by Squassina et al. [23] along with the Hardy–Littlewood–Sobolev inequality, to infer the Hölder regularity result. To the best of our knowledge, the \( L^\infty(\Omega) \) bound on the weak solutions to the doubly non-local problem of
type (1.1) involving critical Choquard-type nonlinearity is established for the first time in this present work. Next, we discuss the Sobolev versus Hölder minimizers for the energy functional associated with Equation (1.1). We show that local minimizers of the energy functional associated with Equation (1.1) with respect to $C^0$ topology are also local minimizers of the same energy functional with respect to $W^{s,p}_0(\Omega)$ topology. In variational problems, this result plays an important role in establishing the multiplicity of solutions. In the local framework, Brezis and Nirenberg [4] were the first to study this type of result, where the authors showed that the local minima of the associated energy functional in $C^1$ topology and in $H^s$, topology coincides. In the fractional framework ($p = 2$), the analogous result is proved in [24]. In [25], this result is further generalized for the fractional $p$-Laplacian set up for $p \geq 2$. For non-local nonlinearity, Gao and Yang [20] studied such result for the following Brezis–Nirenberg-type critical Choquard problem involving local Laplacian under some appropriate assumptions on $f$:

$$-\Delta u = \lambda f(u) + \left( \int_{\Omega} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} \, dy \right) |u(x)|^{2^*_\mu - 2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \partial \Omega,$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a bounded domain having smooth boundary, $\lambda > 0$, $0 < \mu < N$ and $2^*_\mu = \frac{2N-\mu}{N-2}$ is the critical Choquard exponent in view of the Hardy–Littlewood–Sobolev inequality. In the case of doubly non-local equation, Giacomoni et al. [21] investigated $H^s$ versus $C^0$-weighted minimizers of the functional associated with Equation (1.7). But, to the best of our knowledge, there is no such work regarding Sobolev versus Hölder minimizers for the problems involving the fractional $p$-Laplacian and critical (or sub-critical) Choquard-type nonlinearity. Also, the tools used in [20, 21] to prove this result cannot be adapted for the general case of $1 < p < \infty$.

Therefore, we establish this result for the problem (1.1) considering the degenerate case ($p \geq 2$). Finally, we show that if Equation (1.1) has a weak sub-solution and a weak supersolution, then it attains a solution in between the sub-supersolutions pair, which also appears as a local minimizer of the associated energy functional to the problem (1.1) in $W^{s,p}_0(\Omega)$ topology. In addition, we also study the aforementioned results, established for Equation (1.1), for the local perturbation of the nonlocal Choquard-type nonlinearity, precisely for the nonlinearity $g(x, u) + \left( \int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} \, dy \right) f(x, u)$, where the perturbation term $g(x, u)$ has at most critical growth in sense of Sobolev inequality. In the end, we apply these results to discuss the multiplicity result when $g(x, u)$ is of concave type.

For doubly non-local equations of type (1.1) the main difficulty arises due to the non-Hilbert nature of the solution space and the presence of the nonlinear operator $(-\Delta)^s$, as well as, the non-local nonlinearity of Choquard type. Hence, most of the results and techniques that were used in establishing the similar kind of regularity results in the fractional Laplacian or in the local Laplacian set up (for instance, see [20, 21, 33]) are not applicable to Equation (1.1). Therefore, we need to carry out some extra delicate analysis in our proofs to overcome the stated difficulties. In [38], the regularity of solutions of critical Choquard equations involving the $p$-Laplacian is posed as an open problem and in this work, we come up with the answer to it. In this regard, we would like to remark that the regularity results we establish for Equation (1.1) is also valid in the local $p$-Laplacian framework, which are also new to the literature.

The plan of the paper is described as follows. In Section 2, first we recall some preliminary results regarding fractional Sobolev spaces and state the main results of this paper. In Section 3, we give proofs of the main results of this paper.

2 FUNCTIONAL SETTINGS AND STATEMENTS OF THE MAIN THEOREMS

In this section, first we collect some known results regarding fractional Sobolev spaces. To study $p$-fractional Sobolev spaces in details we refer to [18, 22]. For $0 < s < 1$ and $1 < p < \infty$, the fractional Sobolev space is defined as

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) \bigg| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+p}} \, dx \, dy < \infty \right\}$$

equipped with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N)} + \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}}.$$
We also define

\[ W_{0}^{s,p}(\Omega) = \{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \} \]

with respect to the norm

\[ ||u||_{s,p} = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}}, \]

where \( Q = \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c) \). Then, \( W_{0}^{s,p}(\Omega) \) is a reflexive Banach space. Also, \( W_{0}^{s,p}(\Omega) \hookrightarrow L^q(\mathbb{R}^N) \) continuously for each \( q \in [1, p^*_s) \) and \( W_{0}^{s,p}(\Omega) \hookrightarrow L^q(\Omega) \) compactly for each \( q \in [1, p^*_s) \), where \( p^*_s = \frac{Np}{N-sp} \) is the Sobolev-type critical exponent. The best constant \( S_s \) is given below:

\[ S_s = \inf_{u \in W_{0}^{s,p}(\Omega) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy}{\left( \int_{\Omega} |u|^{p^*_s} \, dx \right)^{p/p^*_s}}. \tag{2.1} \]

The dual of the space \( W_{0}^{s,p}(\Omega) \) is denoted by \( W^{-s,p'}(\Omega) \) with the norm \( || \cdot \||_{-s,p'} \), where \( p' = \frac{p}{p-1} \) is conjugate to \( p \). Also, by \( \langle \cdot, \cdot \rangle \), we denote the dual pairing between \( W_{0}^{s,p}(\Omega) \) and \( W^{-s,p'}(\Omega) \).

Let the distance function \( d : \Omega \to \mathbb{R}_+ \) be defined by

\[ d(x) := \text{dist}(x, \mathbb{R}^N \setminus \Omega), \quad x \in \Omega. \tag{2.2} \]

The weighted Hölder-type spaces are defined as follows:

\[ C^0_d(\Omega) := \left\{ u \in C^0(\Omega) : u/d^s \text{ admits a continuous extension to } \Omega \right\}, \]

\[ C^{0,\alpha}_d(\Omega) := \left\{ u \in C^0(\Omega) : u/d^s \text{ admits an } \alpha\text{-Hölder continuous extension to } \Omega \right\} \]

equipped with the norms

\[ ||u||_{C^0_d(\Omega)} := ||u/d^s||_{L^\infty(\Omega)}, \]

\[ ||u||_{C^{0,\alpha}_d(\Omega)} := ||u||_{C^0_d(\Omega)} + \sup_{x,y \in \Omega, x \neq y} \frac{|u(x)/d^s(x) - u(y)/d^s(y)|}{|x-y|^\alpha}, \]

respectively. The embedding \( C^{0,\alpha}_d(\Omega) \hookrightarrow C^0_d(\Omega) \) is compact for all \( \alpha \in (0, 1) \).

The next lemma states the monotonicity property of the fractional \( p \)-Laplacian for \( p \geq 2 \).

**Lemma 2.1** [25, Lemma 2.3]. Let \( p \geq 2 \). There exists \( C = C(p) > 0 \) such that for all \( u, v \in W_{0}^{s,p}(\Omega) \cap L^\infty(\Omega) \) and all \( q \geq 1 \)

\[ \left\| (u - v)^{\frac{p+1}{p}} \right\|_{s,p}^p \leq C q^{p-1} \langle (-\Delta)_p^s u - (-\Delta)_p^s v, (u - v)^q \rangle. \]

The strong maximum principle for fractional \( p \)-Laplacian is given as follows:

**Lemma 2.2** [36, Lemma 2.3] (Strong maximum principle). Let \( u \in W_{0}^{s,p}(\Omega) \) satisfy

\[ \begin{cases} (-\Delta)_p^s u \geq 0 & \text{weakly in } \Omega, \\ u \geq 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.3} \]

Then, \( u \) has a lower semi-continuous representative in \( \Omega \), which is either identically 0 or positive.
Recalling [26, Theorem 1.1], we have the regularity result for the following problem:

$$(-\Delta)^{s \frac{p}{2}} u = g \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \quad (2.4)$$

**Proposition 2.3.** Let $2 \leq p < \infty$ and $\Omega$ be a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary. Let $g \in L^\infty(\Omega)$. Then, there exist $C$ and $\alpha$, both positive and depending upon $s, p, \Omega, N$ such that any weak solution $u \in W^{s, p}_0(\Omega)$ of Equation (2.4) satisfies

$$\|u\|_{C^0, \alpha(\overline{\Omega})} \leq C \|g\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}. \quad (2.5)$$

Now, we recall the following crucial result to handle the nonlocal Choquard type of nonlinearity:

**Proposition 2.4** (Hardy–Littlewood–Sobolev inequality). Let $q_1, q_2 > 1$ and $0 < \mu < N$ with $1/q_1 + \mu/N + 1/q_2 = 2$, $g_1 \in L^{q_1} (\mathbb{R}^N)$ and $g_2 \in L^{q_2} (\mathbb{R}^N)$. There exists a sharp constant $C(q_1, q_2, N, \mu)$, independent of $g_1, g_2$ such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g_1(x)g_2(y)}{|x-y|^{\mu}} dx dy \leq C(q_1, q_2, N, \mu) \|g_1\|_{L^{q_1}(\mathbb{R}^N)} \|g_2\|_{L^{q_2}(\mathbb{R}^N)}. \quad (2.5)$$

Motivated by the inequality (2.5), we assume the following hypothesis on the continuous function $f : \Omega \times \mathbb{R} \to \mathbb{R}$:

(\textbf{H}) There exists some constant $K_0 > 0$ such that for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$:

$$|f(x, t)| \leq K_0 (1 + |t|^{r-1})$$

with $1 < r \leq p^*_\mu,s$, where $p^*_\mu,s := \frac{(p^N - \mu s / 2)}{(N - ps)}$ denotes the critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality.

Observe that $p^*_\mu,s = \frac{p^*_\mu}{2N - \mu} < p^*_\mu$.

**Definition 2.5** (Weak solution of (1.1)). $u \in W^{s, p}_0(\Omega)$ is said to be weak solution of (1.1), if for all $w \in W^{s, p}_0(\Omega)$

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(w(x) - w(y))}{|x-y|^{N+sp}} dx dy = \int_{\Omega} \int_{\Omega} \frac{F(y, u)f(x, u)}{|x-y|^{\mu}} w(x) dx dy. \quad (2.6)$$

**Definition 2.6.** The energy functional $J : W^{s, p}_0(\Omega) \to \mathbb{R}$ associated with the problem (1.1) is defined as

$$J(u) = \frac{1}{p} \|u\|_{s,p} - \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{F(y, u)f(x, u)}{|x-y|^{\mu}} dx dy. \quad (2.7)$$

Note that Equation (2.5) ensures that the nonlocal terms present in the right-hand side of both Equations (2.6) and (2.7) are well defined. Now, we are in a position to state the main results of this paper.

**Theorem 2.7.** Let $2 \leq p < \infty$, $s \in (0, 1)$ with $sp < N$ and $\Omega$ be a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary. Suppose (\textbf{H}) holds and $0 < \mu < \min\{N, 2sp\}$. Then, there exists $\alpha \in (0, s]$ such that any weak solution $u \in W^{s, p}_0(\Omega)$ of Equation (1.1) belongs to $L^\infty(\mathbb{R}^N) \cap C^{0, \alpha}_d(\overline{\Omega})$.

Next, we use Theorem 2.7 to have the following result:

**Theorem 2.8.** Let $2 \leq p < \infty$, $s \in (0, 1)$ with $sp < N$ and $\Omega$ be a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary. Suppose (\textbf{H}) holds and $0 < \mu < \min\{N, 2sp\}$. Then for any $w_0 \in W^{s, p}_0(\Omega)$ the following assertions are equivalent:
(i) there exists \( \varphi > 0 \) such that \( J(w_0 + w) \geq J(w_0) \) for all \( w \in W^{s,p}_0(\Omega) \cap C^0(\overline{\Omega}), \|w\|_{C^0(\overline{\Omega})} \leq \varphi. \)

(ii) there exists \( \delta > 0 \) such that \( J(w_0 + w) \geq J(w_0) \) for all \( w \in W^{s,p}_0(\Omega), \|w\|_{s,p} \leq \delta. \)

**Definition 2.9.** Let \( u \in W^{s,p}(\mathbb{R}^N) \). Then

(i) \( u \) is a super-solution of Equation (1.1), if we have \( u \geq 0 \) a.e. in \( \Omega^c \) and for all \( v \in W^{s,p}_0(\Omega) \) with \( v \geq 0 \) a.e. in \( \Omega \),

\[
\begin{align*}
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} \, dx \, dy &\geq \int_{\Omega} \int_{\Omega} \frac{F(y,u)f(x,u)}{|x - y|^\mu} v(x) \, dxdy,
\end{align*}
\]

(ii) \( u \) is a sub-solution of Equation (1.1), if we have \( u \leq 0 \) a.e. in \( \Omega^c \) and for all \( v \in W^{s,p}_0(\Omega) \) with \( v \geq 0 \) a.e. in \( \Omega \),

\[
\begin{align*}
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} \, dx \, dy &\leq \int_{\Omega} \int_{\Omega} \frac{F(y,u)f(x,u)}{|x - y|^\mu} v(x) \, dxdy.
\end{align*}
\]

Now, we discuss the following result where Theorem 2.8 plays an important role to ensure that, if problem (1.1) has a weak subsolution and a weak supersolution, then it achieves a solution which is also a local minimizer of \( J \) in \( W^{s,p}_0(\Omega) \).

**Theorem 2.10.** Let \( 2 < p < \infty \), \( s \in (0,1) \) with \( sp < N \) and \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with \( C^{1,1} \) boundary. Let (H) hold, \( 0 < \mu < \min\{N,2sp\} \) and \( f(x,\cdot) \) be a non-decreasing function in \( \mathbb{R} \) for all \( x \in \Omega \). Suppose \( v_0,v \in W^{s,p}_0(\Omega) \) are a weak subsolution and a weak supersolution, respectively, to Equation (1.1), which are not solutions, such that \( v \leq v_0 \). Then, there exists a solution \( v_0 \in W^{s,p}_0(\Omega) \) to Equation (1.1) such that \( v_0 \leq v \leq v_0 \) a.e. in \( \Omega \) and \( v_0 \) is a local minimizer of \( J \) in \( W^{s,p}_0(\Omega) \).

Next, we consider the following perturbation of Equation (1.1):

\[
(-\Delta)_p^s u = g(x,u) + \left( \int_{\Omega} \frac{F(y,u)}{|x-y|^\mu} \, dy \right) f(x,u), \quad x \in \Omega,
\]

\[
u = 0, \quad x \in \mathbb{R}^N \setminus \Omega,
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with \( C^{1,1} \) boundary, \( 1 < p < \infty \) and \( 0 < s < 1 \) such that \( sp < N \). We assume the following hypothesis on the Carathéodory function \( g : \Omega \times \mathbb{R} \to \mathbb{R} \):

\( (H') \) There exists some constant \( C_0 > 0 \) such that for a.e. \( x \in \Omega \) and for all \( t \in \mathbb{R} \):

\[
|g(x,t)| \leq C_0 (1 + |t|^{q-1}),
\]

where \( 1 < q \leq p^*_s \).

**Definition 2.11 (Weak solution of Equation (2.8)).** \( u \in W^{s,p}_0(\Omega) \) is said to be weak solution of Equation (2.8), if for all \( w \in W^{s,p}_0(\Omega) \), it holds that

\[
\begin{align*}
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(w(x) - w(y))}{|x - y|^{N+sp}} \, dx \, dy &= \int_{\Omega} g(x,u)w \, dx + \int_{\Omega} \int_{\Omega} \frac{F(y,u)f(x,u)}{|x-y|^\mu} w(x) \, dxdy.
\end{align*}
\]

The weak solution of Equation (2.8) is characterized as the critical point of the associated energy functional \( J_1 : W^{s,p}_0(\Omega) \to \mathbb{R} \), defined as follows:

\[
J_1(u) = \frac{1}{p}\|u\|_{s,p} - \int_{\Omega} G(x,u) \, dx - \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{F(y,u)f(x,u)}{|x-y|^\mu} \, dxdy,
\]
where \( G(x,t) := \int_0^t g(x,\tau)d\tau \) is the primitive of \( g \). Analogous to Equation (1.1), we have the following result for Equation (2.8).

**Theorem 2.12.** Let the assumptions in Theorem 2.7 and \((H')\) hold. Then, there exists \( \alpha \in (0,\gamma] \) such that any weak solution \( u \in W_0^{s,p}(\Omega) \) of Equation (2.8) belongs to \( L^\infty(\mathbb{R}^N) \cap C^{0,\alpha}_{\text{loc}}(\Omega) \). Also, under the assumptions in Theorem 2.8 and \((H')\), the assertions of Theorem 2.8 hold for the functional \( J_1 \).

**Definition 2.13.** Let \( u \in W^{s,p}(\mathbb{R}^N) \). Then

(i) \( u \) is a supersolution of Equation (2.8), if we have \( u \geq 0 \) a.e. in \( \Omega^c \) and for all \( v \in W_0^{s,p}(\Omega) \) with \( v \geq 0 \) a.e. in \( \Omega \)

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} \, dx \, dy \geq \int_{\Omega} g(x,u)v(x) \, dx + \int_{\Omega} \int_{\Omega} \frac{F(y,u)\, f(x,u)}{|x - y|^\mu} v(x) \, dx \, dy;
\]

(ii) \( u \) is a subsolution of Equation (2.8), if we have \( u \leq 0 \) a.e. in \( \Omega^c \) and for all \( v \in W_0^{s,p}(\Omega) \) with \( v \geq 0 \) a.e. in \( \Omega \)

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} \, dx \, dy \leq \int_{\Omega} g(x,u)v(x) \, dx + \int_{\Omega} \int_{\Omega} \frac{F(y,u)\, f(x,u)}{|x - y|^\mu} v(x) \, dx \, dy.
\]

**Theorem 2.14.** Let the assumptions of Theorem 2.10 and \((H')\) hold. Also, let \( g(x,\cdot) \) be a non-decreasing function in \( \mathbb{R} \) for all \( x \in \Omega \). Suppose \( \underline{w}, \overline{w} \in W_0^{s,p}(\Omega) \) are a weak subsolution and a weak supersolution, respectively, to Equation (2.8), which are not solutions, such that \( \underline{w} \leq \overline{w} \). Then, there exists a solution \( w_0 \in W_0^{s,p}(\Omega) \) to Equation (2.8) such that \( \underline{w} \leq w_0 \leq \overline{w} \) a.e in \( \Omega \) and \( w_0 \) is a local minimizer of \( J_1 \) in \( W_0^{s,p}(\Omega) \).

### 3 PROOFS OF THE MAIN RESULTS

In this section, we consider \( C \) to be a generic positive constant which may vary from line to line. In order to prove the main theorems of this paper, we recall some useful inequalities.

**Lemma 3.1** [6, Lemma C.2]. Let \( 1 < p < \infty \) and \( \beta \geq 1 \). For every \( a, b, m \geq 0 \) there holds

\[
|a - b|^{p-2}(a - b)(a_m^{\beta} - b_m^{\beta}) \geq \frac{\beta^{p^\beta} p^{\beta^p}}{(\beta + p-1)^p} \left| a_m^{\beta^p - 1} - b_m^{\beta^p - 1} \right|^p,
\]

where we set \( a_m = \min\{a, m\} \) and \( b_m = \min\{b, m\} \).

**Lemma 3.2** [5, Lemma A.1]. Let \( 1 < p < \infty \) and \( f : \mathbb{R} \to \mathbb{R} \) be a convex function. Then

\[
|a - b|^{p-2}(a - b) \left[ A |f'(a)|^{p-2} f'(a) - B |f'(b)|^{p-2} f'(b) \right] \geq |f(a) - f(b)|^{p-2} (f(a) - f(b))(A - B),
\]

for every \( a, b \in \mathbb{R} \) and every \( A, B \geq 0 \).

In the next lemma, following the approach as in ([5], Theorem 3.1), we derive a priori bound on the weak solution of Equation (1.1).

**Lemma 3.3** (Global \( L^\infty \) - bound). Let the assumptions in Theorem 2.7 hold. Then, any weak solution \( u \in W_0^{s,p}(\Omega) \) of Equation (1.1) belongs to \( L^\infty(\mathbb{R}^N) \).
Proof. From the given assumption on $\mu$, we have $p_{\mu,s}^* > p$. We take $w = \psi[h_\epsilon'(u)]^{p-2}h_\epsilon'(u)$ as the test function in Equation (2.6), where $\psi \in C_c^\infty(\Omega)$, $\psi > 0$ and for every $0 < \epsilon << 1$, we define the smooth convex Lipschitz function

$$h_\epsilon(t) = (\epsilon^2 + t^2)^{1/2}.$$ 

In addition, by choosing

$$a = u(x), b = u(y), A = \psi(x), \text{ and } B = \psi(y)$$

in Lemma 3.2, we obtain

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|h_\epsilon(u(x)) - h_\epsilon(u(y))|^{p-2}(h_\epsilon(u(x)) - h_\epsilon(u(y))) (\psi(x) - \psi(y))}{|x - y|^{N+sp}} dxdy \leq \int_{\Omega} \int_{\Omega} \frac{|F(y, u)| |f(x, u)|}{|x - y|^\mu} |h_\epsilon'(u(x))|^{p-1} \psi(x) dxdy.$$  

(3.1)

Since $h_\epsilon(t)$ converges to $h(t) := |t|$ as $\epsilon \to 0^+$ and $|h_\epsilon'(t)| \leq 1$, passing to the limit and using Fatou's lemma in Equation (3.1), we get

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)| - |u(y)||^{p-2} (|u(x)| - |u(y)|) (\psi(x) - \psi(y))}{|x - y|^{N+sp}} dxdy \leq \int_{\Omega} \int_{\Omega} \frac{|F(y, u)| |f(x, u)|}{|x - y|^\mu} \psi(x) dxdy$$  

(3.2)

for every positive $\psi \in C_c^\infty(\Omega)$. By density, Equation (3.2) holds true for $0 \leq \psi \in W_0^{s,p}(\Omega)$. Next, we define

$$u_l = \min\{l, |u(x)|\}.$$ 

Clearly, $u_l \in W_0^{s,p}(\Omega)$. For $k \geq 1$, let us set

$$\beta := kp - p + 1.$$ 

So $\beta > 1$. In Equation (3.2), choosing $\psi = u_l^\beta$, and using Lemma 3.1, we obtain

$$\frac{\beta}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|(u_l(x))^{\beta+p-1} - (u_l(y))^{\beta+p-1}|^p}{|x - y|^{N+sp}} dxdy \leq \int_{\Omega} \int_{\Omega} \frac{|F(y, u)| |f(x, u)|}{|x - y|^\mu} (u_l(x))^{\beta} dxdy.$$  

(3.3)

By observing that

$$\frac{1}{\beta} \left(\frac{\beta + p - 1}{p}\right)^p \leq \left(\frac{\beta + p - 1}{p}\right)^{p-1}, \text{ for large } \beta$$

and using the continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^{\frac{p'}{s}}(\Omega)$, from Equation (3.3) we get

$$\left\|u_l^k\right\|_{L^{\frac{p'}{s}}(\Omega)}^p \leq \frac{(k)^{p-1}}{S_p} \int_{\Omega} \int_{\Omega} \frac{|F(y, u)| |f(x, u)|}{|x - y|^\mu} (u_l(x))^{\beta} dxdy,$$  

(3.4)

where we have used the relation $k = \frac{\beta+p-1}{p}$ and $S_p$ is as defined in Equation (2.1). Now, we will estimate the right-hand side of Equation (3.4). Using (H), the Hardy–Littlewood–Sobolev inequality and the fact $u_l \leq |u|$ and making use of the
inequalities \( (x_1 + x_2)^\gamma \leq x_1^\gamma + x_2^\gamma, \quad 0 < \gamma < 1, \quad x_1, x_2 \geq 0, \) and \( (x_1 + x_2)^\gamma \leq 2^{\gamma-1}(x_1^\gamma + x_2^\gamma), \quad \gamma > 1, \quad x_1, x_2 \geq 0, \) we deduce

\[
\int_\Omega \int_\Omega \frac{|F(y,u)|}{|x-y|^\mu} |f(x,u)| (u(x))^\beta \, dx \, dy
\]

\[
\leq C \|F(\cdot,u(\cdot))\|_{L^{\frac{2N}{2N-\mu}}(\Omega)} (\int_\Omega (|f(x,u)| |u(x)|^\beta)^{\frac{2N-\mu}{2N}} \, dx)^{\frac{2N-\mu}{2N}}
\]

\[
\leq C \left( \|u\|_{L^{\frac{2N}{2N-\mu}}(\Omega)} + \|u|^{p^*_\mu} \|_{L^{\frac{2N}{2N-\mu}}(\Omega)} \right) \left( \int_\Omega |u|^\beta \frac{p^*_\mu}{p^*_s} \, dx + \int_\Omega \left( |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right)^{\frac{2N-\mu}{2N}}
\]

\[
= C \left( \int \Omega \left( \int |u|^{\beta} \frac{p^*_\mu}{p^*_s} \, dx \right)^{\frac{p^*_\mu}{p^*_s}} \left( \int \Omega \left( \int |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right)^{\frac{p^*_\mu}{p^*_s}} \right)
\]

\[
\leq \mathcal{C} \left( \int \Omega \left( \int |u|^{\beta} \frac{p^*_\mu}{p^*_s} \, dx \right)^{\frac{p^*_\mu}{p^*_s}} \left( \int \Omega \left( \int |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right)^{\frac{p^*_\mu}{p^*_s}} \right)
\]

\[
+ \left( \int_\Omega \left( |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right)^{\frac{p^*_\mu}{p^*_s}} \left( \int \Omega \left( \int |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right)^{\frac{p^*_\mu}{p^*_s}}
\]

\[
\leq \mathcal{C} \left( \frac{\int \Omega |u|^{\beta} \frac{p^*_\mu}{p^*_s} \, dx}{\int \Omega \left( \int |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right) \right)
\]

\[
+ \left( \int_\Omega \left( |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right)^{\frac{p^*_\mu}{p^*_s}} \left( \int \Omega \left( \int |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right)^{\frac{p^*_\mu}{p^*_s}}
\]

\[
\leq \mathcal{C} \left( \frac{\int \Omega |u|^{\beta} \frac{p^*_\mu}{p^*_s} \, dx}{\int \Omega \left( \int |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right) \right)
\]

\[
+ \left( \int_\Omega \left( |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right)^{\frac{p^*_\mu}{p^*_s}} \left( \int \Omega \left( \int |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right)^{\frac{p^*_\mu}{p^*_s}}
\]

\[
\leq \mathcal{C} \left( \frac{\int \Omega |u|^{\beta} \frac{p^*_\mu}{p^*_s} \, dx}{\int \Omega \left( \int |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right) \right)
\]

\[
+ \left( \int_\Omega \left( |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right)^{\frac{p^*_\mu}{p^*_s}} \left( \int \Omega \left( \int |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right)^{\frac{p^*_\mu}{p^*_s}}
\]

\[
\leq \mathcal{C} \left( \frac{\int \Omega |u|^{\beta} \frac{p^*_\mu}{p^*_s} \, dx}{\int \Omega \left( \int |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right) \right)
\]

\[
+ \left( \int_\Omega \left( |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right)^{\frac{p^*_\mu}{p^*_s}} \left( \int \Omega \left( \int |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right)^{\frac{p^*_\mu}{p^*_s}}
\]

\[
\leq \mathcal{C} \left( \frac{\int \Omega |u|^{\beta} \frac{p^*_\mu}{p^*_s} \, dx}{\int \Omega \left( \int |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right) \right)
\]

\[
+ \left( \int_\Omega \left( |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right)^{\frac{p^*_\mu}{p^*_s}} \left( \int \Omega \left( \int |u|^{p^*_\mu} |u u^{\beta-1}| \frac{p^*_\mu}{p^*_s} \, dx \right) \right)^{\frac{p^*_\mu}{p^*_s}}
\]

where \( \Lambda > 1 \) will be chosen later and \( \mathcal{C} = C(\|u\|_{L^{\frac{2N}{2N-\mu}}(\Omega)} + \|u|^{p^*_\mu} \|_{L^{\frac{2N}{2N-\mu}}(\Omega)} ). \) Next, we proceed by adapting the idea of the proof of \[48, \text{Theorem 4.1}\]. Now, two cases arise for the first integration expression in the right-hand side of Equation (3.5).
The above implies

\[
\left( \int_{\Omega \cap |u| < \Lambda} |u|^{\frac{p^*}{p^*_{\mu,s} + \beta - 1}} \right)^{\frac{p}{p^*_{\mu,s} + \beta - 1}} \leq 1 + \int_{\Omega \cap |u| < \Lambda} |u|^{p^*_{\mu,s} + \beta - 1} \right)^{\frac{p}{p^*_{\mu,s} + \beta - 1}} d\mu_s. \tag{3.6}
\]

Plugging Equation (3.6) into Equation (3.5), we have

\[
\int\int_{\Omega} \frac{|F(y, u)| \, |f(x, u)|}{|x - y|^{\mu}} (u_s(x))^\beta \, dx \, dy
\]

\[
\leq \bar{C} \left[ C' \left( 1 + \int_{\Omega \cap |u| < \Lambda} |u|^{p^*_{\mu,s} + \beta - 1} \right)^{\frac{p}{p^*_{\mu,s} + \beta - 1}} \right] + \Lambda^{p^*_{\mu,s} - p} \left( \int_{\Omega \cap |u| < \Lambda} |u|^{p + \beta - 1} \right)^{\frac{p}{p^*_{\mu,s} + \beta - 1}} d\mu_s
\]

\[
+ 2 \left( \int_{\Omega \cap |u| < \Lambda} |u|^{p^*_{\mu,s} - p} |u|^{p + \beta - 1} \right)^{\frac{p}{p^*_{\mu,s} + \beta - 1}} d\mu_s \right]
\]

\[
= \bar{C} \left[ C' \left( 1 + \int_{\Omega \cap |u| < \Lambda} |u|^{p^*_{\mu,s} + \beta - 1} \right)^{\frac{p}{p^*_{\mu,s} + \beta - 1}} \right] + \Lambda^{p^*_{\mu,s} - p} \|u\|^{k_p}_{L^{p^*_{\mu,s}}(\Omega)}
\]

\[
+ 2 \left( \int_{\Omega \cap |u| < \Lambda} |u|^{p^*_{\mu,s} - p} |u|^{k_p} \right)^{\frac{p}{p^*_{\mu,s} + \beta - 1}} d\mu_s \right]
\]

\[
\leq \bar{C} \left[ C' \left( 1 + \int_{\Omega \cap |u| < \Lambda} |u|^{p^*_{\mu,s} + \beta - 1} \right)^{\frac{p}{p^*_{\mu,s} + \beta - 1}} \right] + \Lambda^{p^*_{\mu,s} - p} \|u\|^{k_p}_{L^{p^*_{\mu,s}}(\Omega)}
\]

\[
+ 2 \left( \int_{\Omega \cap |u| < \Lambda} |u|^{p^*_{\mu,s} - p} |u|^{k_p} \right)^{\frac{p}{p^*_{\mu,s} + \beta - 1}} d\mu_s \right]
\]

\[
= \bar{C} \left[ C' \left( 1 + \|u\|^{k_p}_{L^{p^*_{\mu,s}}(\Omega)} + \Lambda^{p^*_{\mu,s} - p} \|u\|^{k_p}_{L^{p^*_{\mu,s}}(\Omega)} \right) \right] + \Lambda^{p^*_{\mu,s} - p} \|u\|^{k_p}_{L^{p^*_{\mu,s}}(\Omega)}
\]

\[
+ 2 \left( \int_{\Omega \cap |u| < \Lambda} |u|^{p^*_{\mu,s} - p} |u|^{k_p} \right)^{\frac{p}{p^*_{\mu,s} + \beta - 1}} d\mu_s \right]
\]
\[ \leq C'' \left[ 1 + \Lambda \mu,\beta^{-p} \left\| \frac{u}{\mu,\beta} \right\|_{L^p(\Omega)}^{k_p} + \left( \int_{\Omega \cap \{ |u| \geq \Lambda \}} \left( |u|^{p_{\mu,\beta}^{-p}} |u|^{k_p} \right) \frac{\mu}{\mu,\beta} \, dx \right)^{\frac{p_{\mu,\beta}}{p}} \right], \]  

where \( C', C'' > 1 \) are some positive constants that do not depend on \( k, \beta \). Again by plugging Equation (3.7) into Equation (3.4) and applying Fatou's lemma, we get

\[ \left\| \frac{u}{\mu,\beta} \right\|_{L^p(\Omega)}^{k_p} \leq C'' \left( \int_{\Omega \cap \{ |u| \geq \Lambda \}} \left( |u|^{p_{\mu,\beta}^{-p}} |u|^{k_p} \right) \frac{\mu}{\mu,\beta} \, dx \right)^{\frac{p_{\mu,\beta}}{p}} + C(\Lambda) \left\| \frac{u}{\mu,\beta} \right\|_{L^p(\Omega)}^{k_p}. \]  

Now using the Hölder inequality, we obtain

\[ \left( \int_{\Omega \cap \{ |u| \geq \Lambda \}} \left( |u|^{p_{\mu,\beta}^{-p}} |u|^{k_p} \right) \frac{\mu}{\mu,\beta} \, dx \right)^{\frac{p_{\mu,\beta}}{p}} \leq \left( \int_{\Omega \cap \{ |u| \geq \Lambda \}} |u|^{p_{\mu,\beta}^{-p}} \mu \, dx \right)^{\frac{p_{\mu,\beta}}{p}}. \]  

Combining Equations (3.8) and (3.9), we have

\[ \left\| \frac{u}{\mu,\beta} \right\|_{L^p(\Omega)}^{k_p} \leq C'' \left( \int_{\Omega \cap \{ |u| \geq \Lambda \}} |u| \, dx \right)^{\frac{p_{\mu,\beta}}{p}} + C(\Lambda) \left\| \frac{u}{\mu,\beta} \right\|_{L^p(\Omega)}^{k_p}. \]  

Now by Lebesgue-dominated convergence theorem in Equation (3.9), we choose \( \Lambda > 1 \) large enough so that \( C(\Lambda) \) is appropriately small and consequently \( C(\Lambda) < C'' \frac{1}{(S_3)^p} \). Therefore, by employing the last inequality in Equation (3.10), it follows that

\[ \left\| \frac{u}{\mu,\beta} \right\|_{L^p(\Omega)}^{k_p} \leq \left( \frac{C''}{(S_3)^p} \right) \left[ 1 + \Lambda \mu,\beta^{-p} \left\| \frac{u}{\mu,\beta} \right\|_{L^p(\Omega)}^{k_p} + C(\Lambda) \left\| \frac{u}{\mu,\beta} \right\|_{L^p(\Omega)}^{k_p} \right]. \]  

where \( \check{C} = \frac{C''}{(S_3)^p} > 1 \). Now, we use bootstrap argument on Equation (3.11). For that, we argue as follows.

If there exists a sequence \( k_n \to \infty \) as \( n \to \infty \) such that

\[ \int_{\Omega} |u| \frac{\mu,\beta}{\mu,\beta} \, dx \leq 1, \]
then from Equation (3.11), it immediately follows that

$$\|u\|_{L^\infty(\Omega)} \leq 1.$$ 

If there is no such sequence satisfying the above condition, then there exists $k_0 > 0$ such that

$$\int_\Omega |u|^{k_{P_{\mu,s}}^p} \, dx > 1, \text{ for all } k \geq k_0.$$ 

Then from Equation (3.11), we infer that

$$\|u\|_{L^{k_{P_{\mu,s}}^p}(\Omega)} \leq \left( C_* \right)^{1/p} \frac{1}{k} \frac{1}{k^p} \|u\|_{L^{k_{P_{\mu,s}}^p}(\Omega)} \leq \frac{1}{k^{p-1}} \|u\|_{L^{k_{P_{\mu,s}}^p}(\Omega)}, \text{ for all } k \geq k_0,$$

(3.12)

where $C_* = 2\hat{C} > 1$. Choose $k = k_1 := k_0 \frac{p_{\mu,s}}{p} > 1$ as the first iteration. Thus, Equation (3.12) yields that

$$\|u\|_{L^{k_1P_{\mu,s}^p}(\Omega)} \leq \left( C_* \right)^{1/p} \frac{1}{k_1} \frac{1}{k_1^p} \|u\|_{L^{k_1P_{\mu,s}^p}(\Omega)}.$$

(3.13)

Again by taking $k = k_2 := k_1 \frac{p_{\mu,s}}{p}$ as the second iteration in Equation (3.12) and then employing Equation (3.13) in it, we get

$$\|u\|_{L^{k_2P_{\mu,s}^p}(\Omega)} \leq \left( C_* \right)^{1/p} \frac{1}{k_2} \frac{1}{k_2^p} \|u\|_{L^{k_2P_{\mu,s}^p}(\Omega)} \leq \left( C_* \right)^{1/p} \frac{1}{k_2} \frac{1}{k_2^p} \|u\|_{L^{k_2P_{\mu,s}^p}(\Omega)}.$$

(3.14)

In this fashion, taking $k = k_n := k_{n-1} \frac{p_{\mu,s}}{p}$ as the $n$th iteration and iterating for $n$ times, we obtain

$$\|u\|_{L^{k_nP_{\mu,s}^p}(\Omega)} \leq \left( C_* \right)^{1/p} \frac{1}{k_n} \frac{1}{k_n^p} \|u\|_{L^{k_nP_{\mu,s}^p}(\Omega)} \leq \left( C_* \right)^{1/p} \frac{1}{k_n} \frac{1}{k_n^p} \|u\|_{L^{k_nP_{\mu,s}^p}(\Omega)} \leq \left( C_* \right)^{1/p} \frac{1}{k_n} \frac{1}{k_n^p} \|u\|_{L^{k_nP_{\mu,s}^p}(\Omega)}.$$

(3.15)

where $k_j = \left( \frac{p_{\mu,s}}{p} \right)^{1/j}$. Since $\frac{p_{\mu,s}}{p} > 1$, we have $k_j^{1/k_j} > 1$ for all $j \in \mathbb{N}$ and

$$\lim_{j \to \infty} k_j^{1/k_j} = 1.$$
Hence, it follows that there exists a constant $C^* > 1$, independent of $n$, such that $k_j^{1/k_j} < C^*$ and thus, Equation (3.15) gives

$$\|u\|_{L^k_{\nu_n}(\Omega)} \leq \left( \sum_{j=1}^{n} \frac{1}{k_j} \right)^{\frac{1}{p}} \left( \sum_{j=1}^{n} \sqrt{\frac{1}{k_j}} \right)^{\frac{p-1}{p}} \|u\|_{L^{k_0}p^*_s(\Omega)},$$

(3.16)

As limit $n \to \infty$, the sum of the following geometric series are given as:

$$\sum_{j=1}^{\infty} \frac{1}{k_j} = \sum_{j=1}^{n} \left( \frac{p}{p_{\mu,s}} \right)^j = \frac{p}{p_{\mu,s} - p}$$

and

$$\sum_{j=1}^{\infty} \frac{1}{\sqrt{k_j}} = \sum_{j=1}^{n} \left( \sqrt{\frac{p}{p_{\mu,s}}} \right)^j = \frac{\sqrt{p}}{\sqrt{p_{\mu,s}} - \sqrt{p}}.$$

Thus, from the last two relations and Equation (3.16), we get that

$$\|u\|_{L^{\nu_n}(\Omega)} \leq (C_s)^{p_{\mu,s} - p} (C^*)^{\frac{p-1}{p}} \|u\|_{L^{k_0}p^*_s(\Omega)},$$

(3.17)

where $\nu_n := k_n p^*_s$. Note that, $\nu_n \to \infty$ as $n \to \infty$. Therefore, we claim that

$$u \in L^{\infty}(\Omega).$$

(3.18)

Indeed, if not then there exists $\delta > 0$ and a subset $S$ of $\Omega$ with $|S| > 0$ such that

$$u(x) > C\|u\|_{L^{k_0}p^*_s(\Omega)} + \delta \quad \text{for } x \in S,$$

where

$$C = (C_s)^{p_{\mu,s} - p} (C^*)^{\frac{p-1}{p}} \sqrt{p_{\mu,s} - \sqrt{p}}.$$

The above implies

$$\liminf_{\nu_n \to \infty} \left( \int_{\Omega} |u(x)|^{\nu_n} dx \right)^{\frac{1}{\nu_n}} \geq \liminf_{\nu_n \to \infty} \left( \int_{S} |u(x)|^{\nu_n} dx \right)^{\frac{1}{\nu_n}} \geq \liminf_{\nu_n \to \infty} \left( C\|u\|_{L^{k_0}p^*_s(\Omega)} + \delta \right)(|S|)^{\frac{1}{\nu_n}} = C\|u\|_{L^{k_0}p^*_s(\Omega)} + \delta,$$

a contradiction to Equation (3.17). Therefore, Equation (3.18) holds. Hence, the proof the lemma is complete.
Proof of Theorem 2.7. Now for proving Hölder regularity, we first claim that
\[
\left( \int_{\Omega} \frac{F(y,u)}{|x-y|^\mu} dy \right) f(x,u) \in L^\infty(\Omega).
\] (3.19)

Indeed, by Lemma 3.3, we get \( u \in L^\infty(\Omega) \) and thus, by (H), we have \( f(\cdot,u(\cdot)), F(\cdot,u(\cdot)) \in L^\infty(\Omega) \), which imply that
\[
\left| \int_{\Omega} \frac{F(y,u)}{|x-y|^\mu} dy \right| \leq \|F(\cdot,u(\cdot))\|_{L^\infty(\Omega)} \left[ \int_{\Omega \cap \{|x-y|<1\}} \frac{dy}{|x-y|^\mu} + \int_{\Omega \cap \{|x-y|\geq 1\}} \frac{dy}{|x-y|^\mu} \right]
\]
\[
\leq \|F(\cdot,u(\cdot))\|_{L^\infty(\Omega)} \left[ \int_{\Omega \cap |r|\leq 1} \frac{r^{N-1-\mu}}{d_r} + |\Omega| \right] < \infty,
\]
and since \( 0 < \mu < N \), Equation (3.19) holds. Now by applying Proposition 2.3, we finally can conclude that there exists some \( \alpha \in [0,s) \), depending upon \( s, p, \Omega \) such that \( u \in C^{0,\alpha}_d(\Omega) \). Hence, the proof is complete. \( \square \)

Proof of Theorem 2.8. Here, we follow the approach as in ([25], Theorem 1.1). We consider the two cases separately.
(a) Critical Case: \( r = p^*_\mu,s \) in (H):
First, we show (i) implies (ii). From (i), it follows that \( \langle J'(w_0), \phi \rangle \geq 0 \) for all \( \phi \in W_0^{s,p}(\Omega) \cap C_0^0(\Omega) \). Since \( W_0^{s,p}(\Omega) \cap C_0^0(\Omega) \) is a dense subspace of \( W_0^{s,p}(\Omega) \), we have
\[
\langle J'(w_0), \phi \rangle = 0 \quad \text{for all } \phi \in W_0^{s,p}(\Omega).
\]

Therefore, by Theorem 2.7 we infer that \( w_0 \in C_0^0(\Omega) \cap L^\infty(\Omega) \). Here, we argue by contradiction. Suppose (ii) does not hold. Then, there exists a sequence, say \( \{\tilde{w}_n\} \) in \( W_0^{s,p}(\Omega) \) such that \( \tilde{w}_n \to w_0 \) strongly in \( W_0^{s,p}(\Omega) \) as \( n \to \infty \) and \( J(\tilde{w}_n) < J(w_0) \) for all \( n \in \mathbb{N} \). Next, we introduce a suitable truncation to the nonlinearity \( f \) to handle its critical growth (in the sense of the Hardy–Littlewood–Sobolev inequality). For each \( j \in \mathbb{N} \), we define \( f_j : \Omega \times \mathbb{R} \to \mathbb{R} \) as \( f_j(x,t) := f(x,T_j(t)) \) where
\[
T_j(t) = \begin{cases} 
-j & \text{for } t \leq -j \\
t & \text{for } -j \leq t \leq j \\
j & \text{for } t \geq j.
\end{cases}
\]
We define the corresponding truncated energy functional \( J_j : W_0^{s,p}(\Omega) \to \mathbb{R} \) as
\[
J_j(u) = \frac{\|u\|_{s,p}^p}{p} - \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{F_j(x,u)F_j(y,u)}{|x-y|^\mu} dxdy,
\]
where \( F_j(x,t) = \int_0^t f_j(x,\tau) d\tau \). One can see that \( J_j \in C^1(W_0^{s,p}(\Omega)) \). Note that, by (H),
\[
|f_j(x,t)| \leq \bar{C}_j := K_0(1 + j^{p^*_\mu,s-1}), \quad |F_j(x,t)| \leq K_j(1 + |t|)
\]
are of subcritical growth (in the sense of the Hardy–Littlewood–Sobolev inequality), where \( \bar{C}_j, K_j, j \in \mathbb{N}, \) are positive real numbers. Now by applying the Lebesgue-dominated convergence theorem, for all \( u \in W_0^{s,p}(\Omega) \), we have
\[
\lim_{j \to \infty} F_j(x,u) = \lim_{j \to \infty} \int_0^u f_j(x,t) dt = F(x,u).
\] (3.20)
For fixed \( n \in \mathbb{N} \) and \( 0 < \xi_n < J(w_0) - J(\tilde{w}_n) \), using (3.20), we can find \( j_n > C(\|w_0\|_{L^\infty(\Omega)}) > 1 \) such that

\[
\left| \int_{\Omega} \int_{\Omega} \frac{F_{j_n}(x, \tilde{w}_n)F_{j_n}(y, \tilde{w}_n)}{|x - y|^\mu} \, dx \, dy - \int_{\Omega} \int_{\Omega} \frac{F(x, \tilde{w}_n)F(y, \tilde{w}_n)}{|x - y|^\mu} \, dx \, dy \right| < \xi_n.
\] (3.21)

For all \( n \in \mathbb{N} \), let us set

\[
\sigma_n := \|\tilde{w}_n - w_0\|_{L^{p*}_s(\Omega)}, \quad B_{\sigma_n} := \{ u \in W_0^{s,p}(\Omega) : \|u - w_0\|_{L^{p*}_s(\Omega)} \leq \sigma_n \}.
\]

Using the continuous embedding \( W_0^{s,p}(\Omega) \hookrightarrow L^{p*}(\Omega) \), we have \( \sigma_n \to 0 \) as \( n \to \infty \). Now, \( B_{\sigma_n} \) is a closed convex subset of \( W_0^{s,p}(\Omega) \) and hence weakly closed subset of \( W_0^{s,p}(\Omega) \). Hence by the definition, \( J_{j_n} \) is sequentially weakly lower semi-continuous and coercive in \( B_{\sigma_n} \). Thus, for any \( n \in \mathbb{N} \), there exists \( w_n \in B_{\sigma_n} \) such that

\[
J_{j_n}(w_n) = \inf_{u \in B_{\sigma_n}} J_{j_n}(u).
\] (3.22)

In view of Equation (3.21) and by the choice of \( \xi_n \) and \( j_n \), we get

\[
J_{j_n}(w_n) \leq J_{j_n}(\tilde{w}_n) \leq J(\tilde{w}_n) + \xi_n < J(w_0) = J_{j_n}(w_0).
\] (3.23)

Claim. There exists \( m_n \geq 0 \) such that

\[
(-\Delta)_p^s w_n + m_n (w_n - w_0)^{p_1^* - 1} = \left( \int_{\Omega} \frac{F_{j_n}(y, w_n)}{|x - y|^\mu} \, dy \right) f_{j_n}(x, w_n).
\] (3.24)

Since \( w_n \in B_{\sigma_n} \), in the process of the proof of our claim we encounter with two possible cases:

**Case:** \( \|w_n - w_0\|_{L^{p*}_s(\Omega)} < \sigma_n \). Then, Equation (3.22) yields that \( w_n \) is a local minimizer of \( J_{j_n} \) in \( W_0^{s,p}(\Omega) \) and hence, \( J'_{j_n}(w_n) = 0 \). Thus, Equation (3.24) holds with \( m_n = 0 \).

**Case:** \( \|w_n - w_0\|_{L^{p*}_s(\Omega)} = \sigma_n \). We define the functional \( I : W_0^{s,p}(\Omega) \to \mathbb{R} \) as

\[
I(u) := \frac{\|u - w_0\|_{L^{p*}_s(\Omega)}}{p_1^*}.
\]

One can check that \( I \in C^1(W_0^{s,p}(\Omega), \mathbb{R}) \). Next, we consider the following \( C^1 \)-manifold in \( W_0^{s,p}(\Omega) \):

\[
\mathcal{M}_n := \left\{ u \in W_0^{s,p}(\Omega) : I(u) = \frac{\sigma_n}{p_1^*} \right\}.
\]

Now Equation (3.22) yields that \( w_n \) is a global minimizer of \( J_{j_n} \) on \( \mathcal{M}_n \). Therefore, by applying Lagrange’s multipliers rule, there exists \( m_n \in \mathbb{R} \) such that in \( W^{-s,p}(\Omega) \)

\[
J'_{j_n}(w_n) + m_n I'(w_n) = 0,
\]

the PDE form of which is Equation (3.24). Furthermore, using Equation (3.22) again we can derive

\[
m_n = -\frac{\langle J'_{j_n}(w_n), w_0 - w_n \rangle}{\langle I'(w_n), w_0 - w_n \rangle} \geq 0
\]
such that, possibly \( m_n \to \infty \) as \( n \to \infty \). Hence, our claim is proved. As per the construction, \( w_n \to w_0 \) strongly in \( L^{p*}_s(\Omega) \) as \( n \to \infty \). Moreover, applying Lemma 3.3 for Equation (3.24), we can have \( w_n \in L^\infty(\Omega) \), for all \( n \in \mathbb{N} \). Next, we will show that, up to a subsequence, \( \{w_n\} \) is bounded in \( L^\infty(\Omega) \). Subtracting Equation (1.1) from Equation (3.24), for all \( n \in \mathbb{N} \), we
get

\[ (-Δ)^s_p w_n - (-Δ)^s_p w_0 + m_n(w_n - w_0)^{p_s-1} = \int_{\Omega} \left( \frac{F_{j_n}(y, w_n)}{|x-y|^\mu} \right) f_{j_n}(x, w_n) - \int_{\Omega} \left( \frac{F(y, w_0)}{|x-y|^\mu} \right) f(x, w_0). \]  

(3.25)

We set \( v_n := w_n - w_0 \in W^{s, p}_0(\Omega) \cap L^\infty(\Omega) \). Then for \( \beta := k p - p + 1 \), \( k \geq 1 \), using \( v_n^\beta \in W^{s, p}_0(\Omega) \), as a test function in the weak formulation of Equation (3.25), we deduce

\[
\langle (-Δ)^s_p w_n - (-Δ)^s_p w_0, v^\beta_n \rangle + m_n \int_{\Omega} |v_n|^p^\beta \frac{s}{s-1} dx = \int_{\Omega} \int_{\Omega} \left( \frac{F_{j_n}(y, w_n) f_{j_n}(x, w_n) - F(y, w_0) f(x, w_0)}{|x-y|^\mu} \right) (v_n(x))^\beta dx dy.
\]

(3.26)

By using Lemma 2.1 and the continuous embedding \( W^{s, p}_0(\Omega) \hookrightarrow L^{p^*}(\Omega) \), from the left-hand side of Equation (3.26), we deduce that

\[
\left[ \int_{\Omega} |v_n|^p \frac{s}{s-1} dx \right]^\frac{p}{p^*} \leq C \\| v_n^\beta \\|_{s, p}^{\beta p-1} \langle (-Δ)^s_p w_n - (-Δ)^s_p w_0, v^\beta_n \rangle.
\]

(3.27)

Now, we estimate the right-hand side in Equation (3.26). For that, first observe that by the construction, for \( (x, t) \in \Omega \times \mathbb{R} \), we have \( |F_{j_n}(x, t)| \leq |F(x, t)| \) and \( |f_{j_n}(x, t)| \leq |f(x, t)| \). Therefore, using (H) and the fact \( w_0 \in L^\infty(\Omega) \), together with the inequalities \( (x_1 + x_2)^\gamma \leq x_1^\gamma + x_2^\gamma, \ 0 < \gamma < 1, \ x_1, x_2 \geq 0 \), and \( (x_1 + x_2)^\gamma \leq 2^{\gamma-1}(x_1^\gamma + x_2^\gamma), \ \gamma > 1, \ x_1, x_2 \geq 0 \), we have

\[
|F_{j_n}(y, w_n) f_{j_n}(x, w_n) - F(y, w_0) f(x, w_0)| \leq |F(y, w_n)| |f(x, w_n)| + |F(y, w_0)| |f(x, w_0)| \leq C \left[ 1 + \left( 1 + |v_n + w_0(y)|^{p^*_s} \right)^{p_s} \frac{s}{s-1} \left( 1 + |v_n + w_0(x)|^{p^*_s} \right)^{p_s} \right] \leq 2^{p^*_s} C \left[ 1 + \left( 1 + |v_n(y)|^{p^*_s} + |w_0(y)|^{p^*_s} \right)^{p^*_s} \left( 1 + |v_n(x)|^{p^*_s} + |w_0(x)|^{p^*_s} \right)^{p^*_s} \right] \leq \bar{K} \left[ 1 + |v_n(y)|^{p^*_s} \left( 1 + |v_n(x)|^{p^*_s} \right)^{p^*_s} \right] \]

(3.28)

for some constant \( \bar{K} > 0 \) (independent of \( j, n, w_n, u_n \)). Let us denote \( \hat{g}(x, t) := 1 + |t|^{p^*_s} \) and \( \hat{G}(x, t) := 1 + |t|^{p^*_s} \). Therefore, using Equations (3.26)–(3.28) together with \( m_n \geq 0 \) and the Hardy–Littlewood–Sobolev inequality, for all \( n \in \mathbb{N} \) and \( k, \beta \geq 1 \), we obtain

\[
\| v_n \|^k_p \left\| L^{p^*_s}(\Omega) \right\|
\leq K_2 \beta^{p-1} \left[ \left( \int_{\Omega} |\hat{g}(x, u_n)| |v_n|^{p^*_s} \frac{p^*_s}{p} dx \right)^{\frac{p}{p^*_s}} \left( \int_{\Omega} |\hat{G}(x, u_n)| |v_n|^{p^*_s} \frac{p^*_s}{p} dx \right)^{\frac{p}{p^*_s}} \right]^{\frac{p}{p}}
\leq K_2 \beta^{p-1} \left[ \| v_n \|_{L^{p^*_s}(\Omega)}^{p^*_s} \| L^{p^*_s}(\Omega) \right]\left[ \int_{\Omega \cap \{|v_n| < 1\}} |v_n|^{p^*_s} \frac{p^*_s}{p} dx \right] + \left( \int_{\Omega \cap \{|v_n| \geq 1\}} |v_n|^{p^*_s} \frac{p^*_s}{p} dx \right)
with some constant $K_2 > 0$, independent of $k, \beta, j, n, v_n$. Now, two cases arise for the first integration expression in the right-hand side of Equation (3.29).

Either

\[
\left( \int_{\Omega \cap \{|v_n| < 1\}} |v_n|^{p_{\mu,s}^{-1} (p_{\mu,s} + \beta - 1)} dx \right)^{\frac{\beta}{p_{\mu,s}^{+\beta-1}}} \leq 1 \text{ or } > 1,
\]

that is,

\[
either \left( \int_{\Omega \cap \{|v_n| < 1\}} |v_n|^{p_{\mu,s}^{-1} (p_{\mu,s} + \beta - 1)} dx \right)^{\frac{\beta}{p_{\mu,s}^{+\beta-1}}} \leq 1 \text{ or } \left( \int_{\Omega \cap \{|v_n| < 1\}} |v_n|^{p_{\mu,s}^{-1} (p_{\mu,s} + \beta - 1)} dx \right)^{\frac{\beta}{p_{\mu,s}^{+\beta-1}}} < \int_{\Omega \cap \{|v_n| < 1\}} |v_n|^{p_{\mu,s}^{-1} (p_{\mu,s} + \beta - 1)} dx.
\]

The above implies

\[
\left\{ \left( \int_{\Omega \cap \{|v_n| < 1\}} |v_n|^{p_{\mu,s}^{-1} (p_{\mu,s} + \beta - 1)} dx \right)^{\frac{\beta}{p_{\mu,s}^{+\beta-1}}} \right\} \leq 1 + \left\{ \int_{\Omega \cap \{|v_n| < 1\}} \left( |v_n|^{p_{\mu,s}^{-1} (p_{\mu,s} + \beta - 1)} \right)^{\frac{\beta}{p_{\mu,s}^{+\beta-1}}} dx \right\}
\]

\[
\leq 1 + \left( \int_{\Omega \cap \{|v_n| < 1\}} \left( |v_n|^{p+\beta-1} \right)^{\frac{p_{\mu,s}^{+\beta-1}}{p_{\mu,s}} \frac{p_{\mu,s}}{p_{\mu,s}} dx \right)
\]

\[
\leq 1 + \|v_n\|^{k \frac{k p_{\mu,s}^{+\beta-1}}{p_{\mu,s}} \frac{p_{\mu,s}}{p_{\mu,s}}}_{L^{p_{\mu,s}} (\Omega)}
\]

Now using the fact that $v_n \to 0$ strongly in $L^{p_{\mu,s}} (\Omega)$ as $n \to \infty$, we eventually have $v_n \to 0$ strongly in $L^{p_{\mu,s}} (\Omega)$ as $n \to \infty$. So, for sufficiently large $n \in \mathbb{N}$, we can have $\left( \|v_n\|^{p_{\mu,s}^{-1} (p_{\mu,s} + \beta - 1)}_{L^{p_{\mu,s}} (\Omega)} + \|v_n\|^{p_{\mu,s}^{-1} (p_{\mu,s} + \beta - 1)}_{L^{p_{\mu,s}} (\Omega)} \right) = \varepsilon_n \to 0$. By plugging Equation (3.30) into
Equation (3.29), we obtain

\[
\|v_n\|_{L^{k\beta_p}\leq k3 \beta^{p-1} \varepsilon_n \left[ \left( \int_{\Omega} |v_n|^{\frac{\beta_p}{k\beta_p}} \frac{p^*_{\mu,s}}{p} dx \right)^{\frac{p}{p^*_{\mu,s}}} + \left( \int_{\Omega} |v_n|^{(p^*_{\mu,s}+p-1)} \frac{p^*_{\mu,s}}{p} \right) dx \right] \]

\leq K_3 \beta^{p-1} \varepsilon_n \left[ 1 + \|v_n\|_{L^{k\beta_p}(\Omega)} + \|v_n\|_{L^{p^*_{\mu,s}}(\Omega)} \right] \]

where the constant $K_3 > 0$ is independent of $j, n, k, \beta, v_n$. Now, choosing $n \in \mathbb{N}$ sufficiently large in the above inequality such that $K_3 \|v_n\|_{L^{p^*_{\mu,s}-1}(\Omega)} (\beta^{p-1} \varepsilon_n < \frac{1}{2}$, we get

\[
\|v_n\|_{L^{k\beta_p}(\Omega)} \leq 2K_3 \beta^{p-1} \varepsilon_n \left[ 1 + \|v_n\|_{L^{k\beta_p}(\Omega)} \right] \]

\[
\leq 2K_3 \beta^{p-1} \varepsilon_n \left[ 1 + \|v_n\|_{L^{k\beta_p}(\Omega)} \right].
\]

Again, by taking $\varepsilon_n \leq \frac{1}{4K_3 \beta^{p-1}}$ for sufficiently large $n \in \mathbb{N}$ in the above inequality, we get

\[
\|v_n\|_{L^{k\beta_p}(\Omega)} \leq 4K_3 \beta^{p-1} \varepsilon_n < 1.
\]

This implies that $\|v_n\|_{L^{k\beta_p}(\Omega)} \leq 1^{1/kp}$. Now, letting $k \to \infty$, we infer that $\|v_n\|_{L^{\infty}(\Omega)} \leq 1$ for sufficiently large $n \in \mathbb{N}$. Thus, \{v_n\} is bounded in $L^{\infty}(\Omega)$ and hence the boundedness of \{w_n\} in $L^{\infty}(\Omega)$ follows. Now for sufficiently large $n \in \mathbb{N}$, Equation (3.24) can be rewritten as

\[
(-\Delta)^{\frac{p}{p^*}} w_n = \left( \int_{\Omega} \frac{F(y, w_n)}{|x-y|^\mu} dy \right) f(x, w_n) - m_n (w_n - w_0)^{\beta - 1} \text{ in } W^{-s, p'}(\Omega).
\]

(3.31)

Since \{w_n\} is bounded in $L^{\infty}(\Omega)$ and hence, by (H), it follows that the sequence $\left\{ \left( \int_{\Omega} \frac{F(y, w_n)}{|x-y|^\mu} dy \right) f(\cdot, w_n) \right\}$ is also uniformly bounded. Therefore, using this fact and again using $\psi^\beta_n$ where $\psi^\beta_n = w_n - w_0$, $\beta = kp + p - 1, k \geq 1$ as a test function in the weak formulation of Equation (3.25) and applying Lemma 2.1, for all sufficiently large $n \in \mathbb{N}$, we achieve

\[
m_n \int |v_n|^{\beta + 1} dx \leq K_4 \int |v_n|^{\beta} dx
\]

\[
\leq K_4 \left[ \int |v_n|^{\beta + 1} dx \right] |\Omega|^{\frac{1}{p^*_{\mu,s}}} \frac{p^*_{\mu,s}-1}{p^*_{\mu,s}+\beta-1}
\]

where $K_4 > 0$ is a constant that is independent of $j, n, k, \beta, v_n$. The above implies

\[
m_n \|v_n\|_{L^{p^*_{\mu,s}+\beta-1}(\Omega)} \leq K_5 |\Omega|^{\frac{1}{p^*_{\mu,s}+\beta-1}},
\]

with some constant $K_5 > 0$ (independent of $j, n, k, \beta, v_n$). Letting $k \to \infty$, we have $\beta \to \infty$ and hence, from the last relation, we deduce that

\[
m_n \|v_n\|_{L^{\infty}(\Omega)} \leq K_5
\]
that is, \(\{m_n (w_n - w_0)^{p_s-1}\}\) is a bounded sequence in \(L^\infty(\Omega)\). Hence, combining this fact along with Equation (3.31) and Theorem 2.7, we see that \(\{w_n\}\) is bounded in \(C^0_d(\Omega)\). By the compact embedding \(C^0_d(\Omega) \hookrightarrow C^0_d(\Omega)\), passing to a subsequence, still denoted by \(\{w_n\}\), we have \(w_n \to w_0\) strongly in \(C^0_d(\Omega)\) as \(n \to \infty\). So, for all \(n \in \mathbb{N}\) large enough, we obtain that \(\|w_n - w_0\|_{C^0_d(\Omega)} \leq \varepsilon\).

On the other hand, since \(\{w_n\}\) is bounded in \(L^\infty(\Omega)\), we get \(J_{j_n}(w_n) = J(w_n)\) for sufficiently large \(n \in \mathbb{N}\). Hence, from Equation (3.23), it follows that \(J(w_n) < J(w_0)\). Thus, we reach at a contradiction to (i) and hence, (ii) is proved.

Next, we show that (ii) implies (i). By (ii), we have \(J'(w_0) = 0\), for all \(v \in W_s^p(\Omega)\). Therefore, Lemma 3.3 and Proposition 2.3 imply that \(w_0 \in C^0(\Omega)\). Supposing the contrary, let there exist a sequence \(\{w_n\}\) in \(W_s^p(\Omega) \cap C^0(\Omega)\) such that \(w_n \to w_0\) in \(C^0_d(\Omega)\) as \(n \to \infty\) and \(J(w_n) < J(w_0)\), for all \(n \in \mathbb{N}\). Thus, we have \(w_n \to w_0\) strongly in \(L^\infty(\Omega)\) as \(n \to \infty\). Hence, by the continuity and (H), the sequence \(\{F(\cdot, w_n)\}\) is bounded in \(L^\infty(\Omega)\) and

\[
F(\cdot, w_n) \to F(\cdot, w_0) \quad \text{strongly in } L^\infty(\Omega) \quad \text{as } n \to \infty.
\]

Observe that

\[
I^{(n)}_0 := \int_\Omega \int_\Omega \frac{F(x, w_n)F(\mu, w_n)}{|x-y|^\mu} \, dx \, dy - \int_\Omega \int_\Omega \frac{F(x, w_0)F(\mu, w_0)}{|x-y|^\mu} \, dx \, dy
\]

\[
= I^{(n)}_1 + I^{(n)}_2,
\]

where

\[
I^{(n)}_1 := \int_\Omega \int_\Omega \frac{F(x, w_n)[F(\mu, w_n) - F(\mu, w_0)]}{|x-y|^\mu} \, dx \, dy,
\]

\[
I^{(n)}_2 := \int_\Omega \int_\Omega \frac{[F(x, w_n) - F(x, w_0)]F(\mu, w_0)}{|x-y|^\mu} \, dx \, dy.
\]

By the Hardy–Littlewood–Sobolev inequality and Equation (3.32), we get

\[
|I^{(n)}_1| \leq \frac{\|F(\cdot, w_n)\|_{L^{2N/2N-\mu}(\Omega)}}{L^{2N/2N-\mu}(\Omega)} \frac{\|F(\cdot, w_n) - F(\cdot, w_0)\|_{L^{2N}(\Omega)}}{L^{2N}(\Omega)} \to 0 \quad \text{as } n \to \infty.
\]

Arguing similarly, we obtain \(|I^{(n)}_2| \to 0\) as \(n \to \infty\), which together with Equations (3.34) and (3.33) implies that \(I^{(n)}_0 \to 0\), that is,

\[
\int_\Omega \int_\Omega \frac{F(x, w_n)F(\mu, w_n)}{|x-y|^\mu} \, dx \, dy \to \int_\Omega \int_\Omega \frac{F(x, w_0)F(\mu, w_0)}{|x-y|^\mu} \, dx \, dy \quad \text{as } n \to \infty.
\]

Using Equation (3.35), we have

\[
\limsup_{n \to \infty} \frac{\|w_n\|_{L^p_s}}{p} = \limsup_{n \to \infty} \left[ J(w_n) + \frac{1}{2} \int_\Omega \int_\Omega \frac{F(x, w_n)F(\mu, w_n)}{|x-y|^\mu} \, dx \, dy \right]
\]

\[
\leq J(w_0) + \frac{1}{2} \int_\Omega \int_\Omega \frac{F(x, w_0)F(\mu, w_0)}{|x-y|^\mu} \, dx \, dy
\]

\[
= \frac{\|w_0\|_{L^p_s}}{p},
\]

that is, \(\{w_n\}\) is bonded in \(W^{s, p}_0(\Omega)\). Hence, passing to a subsequence, still denoted by \(\{w_n\}\), we have \(w_n \to w_0\) weakly in \(W^{s, p}_0(\Omega)\) as \(n \to \infty\). Therefore, using the lower semicontinuity of norm,

\[
\liminf_{n \to \infty} \|w_n\|_{L^p_s} \geq \|w_0\|_{L^p_s}.
\]
Since $W^{s,p}_0(\Omega)$ is uniformly convex, we get $\|w_n\|_{s,p} \to \|w_0\|_{s,p}$ and consequently Brezis–Lieb lemma gives that $\|w_n - w_0\|_{s,p} \to 0$ as $n \to \infty$. So, we have, for large $n \in \mathbb{N}$, $\|w_n - w_0\|_{s,p} \leq \delta$ with $J(w_n) < J(w_0)$, which is a contradiction. Therefore, (i) holds.

(b) Subcritical Case: $r < p^*_\mu$ in (H):
In this case, the proof follows using the similar arguments as in the Critical Case, discussed above, by taking $f_j(x,t) = f(x,t)$ for each $j \in \mathbb{N}$ and using the compact embedding $W^{s,p}_0(\Omega) \hookrightarrow L^q(\Omega), 1 < q < p^*_\mu$. Hence, the proof of the theorem is complete.

Proof of Theorem 2.10. To prove this theorem, we follow the approach as in [19, Lemma 3.2]. First, we define the following truncated function $\hat{f}(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ as

$$\hat{f}(x,t) := \begin{cases} f(x,\underline{v}(x)) & \text{if } t \leq \underline{v}(x) \\ f(x,t) & \text{if } \underline{v}(x) < t < \overline{v}(x) \\ f(x,\overline{v}(x)) & \text{if } t \geq \overline{v}(x). \end{cases}$$

Clearly by (H), we have that $\hat{f}$ is continuous such that for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$|\hat{f}(x,t)| \leq C_1(1 + |v|^{r-1} + |\overline{v}|^{r-1}),$$

$$|\hat{F}(x,t)| = \left| \int_0^t \hat{f}(x,\tau)d\tau \right| \leq C_1(1 + |v|^{r-1} + |\overline{v}|^{r-1})|t|,$$ for some constant $C_1 > 0$. We define the operator $T : W^{s,p}_0(\Omega) \to W^{-s,p'}(\Omega)$ as

$$\langle T(u), v \rangle = -\int_\Omega \int_\Omega \hat{F}(y,u) \hat{f}(x,u)v(x) \frac{1}{|x-y|^{\mu}} dxdy, \text{ for all } u, v \in W^{s,p}_0(\Omega).$$

In view of the Hardy–Littlewood–Sobolev inequality and Equation (3.38), $T$ is well posed. We will show that $T$ is strongly continuous (see [11, Definition 2.95 (iv)]). Indeed, let $\{u_n\}$ be a sequence in $W^{s,p}_0(\Omega)$ such that $u_n \to u_*$ weakly in $W^{s,p}_0(\Omega)$ as $n \to \infty$. Now using compact embedding $W^{s,p}_0(\Omega) \hookrightarrow L^q(\Omega), 1 < q < p^*_\mu$, passing to a subsequence, still denoted by $\{u_n\}$, we have $u_n \to u_*$ strongly in $L^q(\Omega), u_n(x) \to u_*(x)$ as $n \to \infty$ and hence, there exists some $\tilde{h} \in L^{2N/\mu}(\Omega)$ with $|u_n(x)| \leq \tilde{h}(x)$ a.e. in $\Omega$. Moreover, for a.e. $x \in \Omega$, we have $\hat{f}(x,u_n(x)) \to \hat{f}(x,u_*(x))$ and $\hat{F}(x,u_n(x)) \to \hat{F}(x,u_*(x))$ as $n \to \infty$ and $\{F(\cdot,u_n)\}$ is bounded in $L^{2N/\mu}(\Omega)$ and thus, $F(\cdot,u_n) \to F(\cdot,u_*)$ weakly in $L^{2N/\mu}(\Omega)$ as $n \to \infty$. For $v \in W^{s,p}_0(\Omega)$, consider the linear continuous map $\Sigma : L^{2N/\mu}(\Omega) \to \mathbb{R}$, defined as

$$\Sigma(w) = \int_\Omega \int_\Omega \hat{w}(y) \hat{f}(x,u_*)v(x) \frac{1}{|x-y|^{\mu}} dxdy.$$ Then by letting $n \to \infty$, we get

$$\int_\Omega \int_\Omega \hat{F}(y,u_n) \hat{f}(x,u_*)v(x) \frac{1}{|x-y|^{\mu}} dxdy \to \int_\Omega \int_\Omega \hat{F}(y,u_*) \hat{f}(x,u_*)v(x) \frac{1}{|x-y|^{\mu}} dxdy. \quad (3.39)$$

Therefore, using the Hardy–Littlewood–Sobolev inequality, Equations (3.38), (3.39), and Lebesgue-dominated convergence theorem, we obtain

$$|\langle T(u_n) - T(u_*), v \rangle| \leq \int_\Omega \int_\Omega \hat{F}(y,u_n) \left| \hat{f}(x,u_n) - \hat{f}(x,u_*) \right|v(x) \frac{1}{|x-y|^{\mu}} dxdy + \int_\Omega \int_\Omega \left[ \hat{F}(y,u_n) - \hat{F}(y,u_*) \right] f(x,u_*) v(x) \frac{1}{|x-y|^{\mu}} dxdy.$$
\[
\begin{align*}
&\leq \|\tilde{F}(\cdot, u_n)\|_{L^{\frac{2N}{2N-\mu}}(\Omega)} \|\tilde{F}(\cdot, u_n) - \tilde{F}(\cdot, u_\star)\|_{L^{\frac{2N}{2N-\mu}}(\Omega)} + \left| \int_\Omega \int_\Omega \frac{\tilde{F}(y, u_n) - \tilde{F}(y, u_\star)}{|x-y|^\mu} \tilde{f}(x, u_\star)v \, dx \, dy \right| \\
&\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\end{align*}
\]

That is, \(T(u_n) \rightarrow T(u_\star)\) in \(W^{-s,p}(\Omega)\) as \(n \rightarrow \infty\) and thus, \(T\) is strongly continuous. Hence, \([11, \text{Lemma 2.98 (ii)}]\) yields that \(T\) is pseudomonotone.

Next, using Lemma 2.1 and arguing as in \([19, \text{Lemma 3.2}]\) we get that \((-\Delta)_p^s : W^{s,p}_0(\Omega) \rightarrow W^{-s,p}(\Omega)\) is pseudomonotone. Therefore, \((-\Delta)_p^s + T\) is a pseudomonotone operator. On the other hand, using the Hardy–Littlewood–Sobolev inequality, the Hörder inequality, Equation (3.38) and the continuous embedding \(W^{s,p}_0(\Omega) \hookrightarrow L^q(\Omega), 1 < q < p^*_s\), for any \(u \in W^{s,p}_0(\Omega)\), we deduce

\[
\|T(u)\|_{-s,p'} = \sup_{\|v\|_{-s,p'} \leq 1} \left| \int_\Omega \int_\Omega \frac{\tilde{F}(y, u) \tilde{f}(x, u)v(x)}{|x-y|^\mu} \, dx \, dy \right|
\]

\[
\leq C_2 \|\tilde{f}(\cdot, u)\|_{L^{\frac{2N}{2N-\mu}}(\Omega)} \|\tilde{F}(\cdot, u)\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}
\]

\[
\leq C_3 \left[ \|v\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}^{\frac{2N}{2N-\mu}} + \left( \int_\Omega |v|^{(r-1)\frac{2N}{2N-\mu}} |v|^{\frac{2N}{2N-\mu}} \, dx \right)^{\frac{1}{r-1}} \left( \int_\Omega |v|^{\frac{2N}{2N-\mu}} \, dx \right)^{\frac{r-1}{r}} \right]
\]

\[
\leq C_4 \left[ \|v\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}^{\frac{2N}{2N-\mu}} + \left( \int_\Omega |v|^{(r-1)\frac{2N}{2N-\mu}} |v|^{\frac{2N}{2N-\mu}} \, dx \right)^{\frac{1}{r-1}} \left( \int_\Omega |v|^{\frac{2N}{2N-\mu}} \, dx \right)^{\frac{r-1}{r}} \right]
\]

\[
\leq C_5 \|u\|_{s,p} \|u\|_{s,p} (1 + \|v\|_{s,p}^{-1} + \|\overline{v}\|_{s,p}^{-1})^2
\]

\[
\leq C_6 \|u\|_{s,p}
\]
where $C_2, C_3, C_4, C_5, C_6$ are some positive constants which do not depend on $u, v$. This implies that $T$ is bounded. Again arguing as in [19, Lemma 3.2], it follows that $(-\Delta)_p^r + T$ is bounded. Finally, we show that $(-\Delta)_p^r + T$ is coercive. Indeed, again using the Hardy–Littlewood–Sobolev inequality, the Hölder inequality, Equation (3.38) and the continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$, $1 < q \leq p^*$, for all $u \in W_0^{s,p}(\Omega) \setminus \{0\}$, and estimating as above, we have

\[
\langle (-\Delta)_p^r (u) + T(u), u \rangle = \frac{1}{\|u\|_{s,p}^p} \int \int \frac{\hat{F}(y,u)\hat{F}(x,u)u(x)}{|x-y|\mu} \, dx \, dy
\]

\[
\geq \|u\|_{s,p}^{r-1} - \frac{C_7}{\|u\|_{s,p}^{2N}} \|u\|_{L^{2N}}^{2N} \leq \|u\|_{L^{2N}}^{2N} \|u\|_{L^{2N}}^{2N}
\]

\[
\geq \|u\|_{s,p}^{r-1} - \frac{C_8}{\|u\|_{s,p}^{2N}} \left( \|u\|_{L^{2N}}^{2N} + \|\tilde{u}\|_{L^{2N}}^{2N} \right) \]

\[
\geq \|u\|_{s,p}^{r-1} - \frac{C_9}{\|u\|_{s,p}^{2N}} \left( \|u\|_{s,p}^{2N} + \|\tilde{u}\|_{s,p}^{2N} \right) \]

\[
= \|u\|_{s,p}^{r-1} - C_{10} \|u\|_{s,p} \to \infty \quad \text{as} \quad \|u\|_{s,p} \to \infty ,
\]

where $C_7, C_8, C_9, C_{10} > 0$ are some constants, independent of $u$. Applying [11, Theorem 2.99], we get that there exists a solution, say $v_0 \in W_0^{s,p}(\Omega)$ to the following equation:

\[
(-\Delta)_p^r u + T(u) = 0 \quad \text{in} \quad W^{-s,p'}(\Omega),
\]

that is,

\[
\hat{J}(v_0) = \min_{u \in W_0^{s,p}(\Omega)} \hat{J}(u),
\]

where $\hat{J} \in C^1(W_0^{s,p}(\Omega), \mathbb{R})$ is the energy functional associated with Equation (3.41) and is defined as

\[
\hat{J}(u) = \frac{\|u\|_{s,p}^p}{p} - \int \int \frac{\hat{F}(y,u)\hat{F}(x,u)}{|x-y|\mu} \, dx \, dy.
\]

Now, we claim that

\[
v \leq v_0 \leq \tilde{u} \quad \text{in} \quad \Omega.
\]

Observe that, Equation (3.42) holds in $\mathbb{R}^N \setminus \Omega$. Using $(v_0 - \tilde{u})^+ \in W_0^{s,p}(\Omega)$ as a test function in the weak formulation of Equation (3.41) and using the fact that $\tilde{u}$ is a supersolution of Equation (1.1), we deduce

\[
\langle (-\Delta)_p^r (v_0 - \tilde{u})^+, v_0 - \tilde{u}^+ \rangle = \int \int \frac{\hat{F}(y,v_0)\hat{F}(x,v_0)}{|x-y|\mu} \, dx \, dy
\]

\[
= \int \int \frac{F(y,\tilde{u})}{|x-y|\mu} \, dx \, dy
\]

\[
\leq \langle (-\Delta)_p^r \tilde{u}, (v_0 - \tilde{u})^+ \rangle .
\]
Therefore, we get
\[
\langle (-\Delta)^s_p v_0 - (-\Delta)^s_p \overline{v}, (v_0 - \overline{v})^+ \rangle \leq 0.
\] (3.43)

From [5, Lemma A.2] and [25, Lemma 2.3] (with \(g(t) = t^+\)), we recall the following standard inequalities: for all \(a, b \in \mathbb{R}\),
\[
|a^+ - b^+|^p \leq (a - b)^{p-1}(a^+ - b^+), \quad (a - b)^{p-1} \leq C_p(a^p - b^p).
\]

Using the last two inequalities along with Equation (3.43), we obtain
\[
\| (v_0 - \overline{v})^+ \|^p_{s,p} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|(v_0(x) - \overline{v}(x))^+ - (v_0(y) - \overline{v}(y))^+|^p}{|x - y|^{N+sp}} \, dx \, dy
\]
\[
\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|(v_0 - \overline{v})(x) - (v_0 - \overline{v})(y)|^{p-1}[|(v_0 - \overline{v})^+(x) - (v_0 - \overline{v})^+(y)|]}{|x - y|^{N+sp}} \, dx \, dy
\]
\[
\leq C_p \left[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v_0(x) - v_0(y))^{p-1}[|(v_0 - \overline{v})^+(x) - (v_0 - \overline{v})^+(y)|]}{|x - y|^{N+sp}} \, dx \, dy \right]
\]
\[
- \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\overline{v}(x) - \overline{v}(y))^{p-1}[|(v_0 - \overline{v})^+(x) - (v_0 - \overline{v})^+(y)|]}{|x - y|^{N+sp}} \, dx \, dy
\]
\[
= C_p \langle (-\Delta)^s_p v_0 - (-\Delta)^s_p \overline{v}, (v_0 - \overline{v})^+ \rangle \leq 0,
\]
and thus, we have \((v_0 - \overline{v})^+ = 0\) which infers that \(v_0 \leq \overline{v}\) in \(\Omega\). Similarly, testing Equation (3.41) with \((v_0 - v)^- \in W^{s,p}_0(\Omega)\), we can prove that \(v_0 \geq v\) and hence, Equation (3.42) holds true.

Next, we show that \(v_0\) is a local minimizer of \(J\) in \(W^{s,p}_0(\Omega)\). By exploiting the monotonicity and the definition of \(\hat{f}\) along with Equation (3.42), we obtain, in the weak sense,
\[
(-\Delta)^s_p (\overline{v} - v_0) \geq \left( \int_{\Omega} \frac{F(y,\overline{v})}{|x - y|^d} \, dy \right) f(x,\overline{v}) - \left( \int_{\Omega} \frac{\hat{f}(y,v_0)}{|x - y|^d} \, dy \right) \hat{f}(x,v_0)
\]
\[
\geq 0
\]
in \(\Omega\) and by definition, \(\overline{v} - v_0 \geq 0\ in \ \mathbb{R}^N \setminus \Omega\). In view of the fact that \(\overline{v}\) is not a solution to Equation (1.1), we have \(v_0 \neq \overline{v}\). Therefore, by the strong maximum principle for fractional \(p\)-Laplacian (Lemma 2.2), it follows that \(\overline{v} - v_0 > 0\ in \ \Omega\ and similarly \(v_0 - v > 0\ in \ \Omega\). Thus, it follows that \(v_0 \in W^{s,p}_0(\Omega)\) is a weak solution to Equation (1.1). Now, again using Lemma 2.2 and Hopf’s lemma for fractional \(p\)-Laplacian (see [16, Theorem 1.5]), we can have \(\overline{v} - v_0 \geq Rd^s\ in \ \Omega, for some \(R > 0\), where \(d\) is the distance function, defined in Equation (2.2). Likewise, \(v_0 - v \geq Rd^s\ in \ \Omega, for some \(R > 0\). Also, from Lemma 3.3 and Proposition 2.3, we get \(v_0 \in C^{0,\alpha}_d(\overline{\Omega})\). Let us denote
\[
B^d_{R/2}(v_0) := \{ u \in W^{s,p}_0(\Omega) : \| u - v_0 \|_{0,d} \leq R/2 \}.
\]

For each \(w \in B^d_{R/2}(v_0)\), we have
\[
\frac{\overline{v} - w}{d^{s}} - \frac{v_0 - w}{d^{s}} \geq R - \frac{R}{2} = \frac{R}{2} \quad \text{in} \ \overline{\Omega}.
\]
The last relation implies that \(\overline{v} - w > 0\ in \ \Omega\). On a similar note, we have \(w - v > 0\ in \ \Omega\). Therefore, in \(W^{s,p}_0(\Omega) \cap \overline{B}^d_{R/2}(v_0)\), \(\hat{f}\) agrees with \(J\) and thus, \(v_0\) emerges as a local minimizer of \(J\) in \(W^{s,p}_0(\Omega) \cap \overline{B}^d_{R/2}(v_0)\). Finally, from Theorem 2.8, we infer that \(v_0\) is a local minimizer of \(J\) in \(W^{s,p}_0(\Omega)\) as well and hence, the theorem is proved. \(\square\)
Proof of Theorem 2.12 and 2.14. The proofs of Theorems 2.12 and 2.14 follow using the similar arguments as in the proofs of Theorems 2.7–2.10, by incorporating appropriate modifications in the calculations using Sobolev embedding $W^{s, p}_0(\Omega) \hookrightarrow L^q(\Omega)$, $1 < q \leq p^*_s$, for the additional perturbation term $g(x, u)$ and defining the required truncation for $g$ in a similar fashion as defined for $f$ in the previous proofs.

As discussed in [21], Theorem 2.14 finds application in proving the multiplicity of the solutions of the problem (2.8) with concave-type perturbation $g(x, u)$. In particular, consider the following problem:

$$
\left\{ \begin{array}{ll}
(-\Delta)^s_p u &= \lambda |u|^{q-2} u + \left( \int_{\Omega} \frac{F(y, u)}{|x-y|^{\mu}} dy \right) f(x, u), & u > 0 \text{ in } \Omega, \\
u &= 0 \text{ in } \mathbb{R}^N \setminus \Omega,
\end{array} \right.
$$

(3.44)

where $\lambda > 0$, $1 < q < p$ and $f$ is a non-decreasing function and satisfies (H) with $1 < r < p^*_s, \mu$. One can check that a solution $w$ to

$$
(-\Delta)^s_p u = \lambda |u|^{q-2} u, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,
$$

is subsolution to Equation (3.44) for all $\lambda > 0$ whereas for $\lambda > 0$ small enough, a solution $\overline{w}$ to

$$
(-\Delta)^s_p u = 1, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega.
$$

is a supersolution to Equation (3.44) such that $w \leq \overline{w}$. Now using Theorem 2.14, we infer that there exists a solution, say $w_0$, to Equation (3.44), which is a local minimizer in $W^{s, p}_0(\Omega)$. Then, the existence of the second solution is guaranteed by showing that the energy functional associated with Equation (3.44) satisfies the Mountain pass geometry and Palais–Smale condition.

REFERENCES

[1] C. O. Alves and M. Yang, Investigating the multiplicity and concentration behaviour of solutions for a quasi-linear Choquard equation via the penalization method, Proc. R. Soc. Edinb. 146A (2016), 23–58.

[2] O. G. Bakunin, Turbulence and diffusion: scaling versus equations, Springer Science & Business Media, 2008.

[3] P. Belchiner, H. Bueno, O. H. Miyagaki, and G. A. Pereria, Remarks about fractional Choquard equation: groundstate, regularity and polynomial decay, Nonlinear Anal. 164 (2017), 38–53.

[4] H. Brezis and L. Nirenberg, $H^1$ versus $C^1$ local minimizers, C. R. Acad. Sci. Paris Sér. I Math. 317 (1993), 465–472.

[5] L. Brasco and E. Parini, The second eigenvalue of the fractional $p$-Laplacian, Adv. Calc. Var. 9 (2016), 323–355.

[6] L. Brasco, E. Lindgren, and E. Parini, The fractional Cheeger problem, Interfaces Free Bound. 16 (2014), 419–458.

[7] L. Brasco, E. Lindgren, and A. Schikorra, Higher Hölder regularity for the fractional $p$-Laplacian in the superquadratic case, Adv. Math. 338 (2018), 782–846.

[8] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, Commun. Partial Dif. Equ. 32 (2007), 1245–1260.

[9] L. Caffarelli and L. Silvestre, Regularity theory for fully nonlinear integro–differential equations, Commun. Pure Appl. Math. 62 (2009), 597–638.

[10] L. Caffarelli and L. Silvestre, Regularity results for nonlocal equations by approximation, Arch. Rat. Mech. Anal. 200 (2011), 59–88.

[11] S. Carl, V. K. Le, and D. Motreanu, Nonsmooth variational problems and their inequalities, Springer, New York, 2007.

[12] M. Cozzi, Regularity results and Harnack inequalities for minimizers and solutions of nonlocal problems: a unified approach via fractional De Giorgi classes, J. Funct. Anal. 272 (2017), 47624837.

[13] R. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Theory of Bose–Einstein condensation in trapped gases, Rev. Mod. Phys. 71 (1999), 463.

[14] E. DiBenedetto, $C^{1+\alpha}$ regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal. 7 (1983), no. 8, 827–850.

[15] P. d’Avenia, G. Siciliano, and M. Squassina, On fractional Choquard equations, Math. Models Methods Appl. Sci. 25 (2015), 1447–1476.

[16] L. M. Del Pezzo and A. Quaas, A Hopf’s lemma and a strong minimum principle for the fractional $p$-Laplacian, J. Diff. Equ. 263 (2017), 765–778.

[17] A. Di Castro, T. Kuusi, and G. Palatucci, Local behavior of fractional $p$-minimizers, Ann. Inst. Henri Poincaré, Anal. Non Linéaire. 33 (2016), 1279–1299.

[18] E. Di Nezza, G. Palatucci, and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012), 521–573.

[19] S. Frassu and A. Iannizzotto, Extremal constant sign solutions and nodal solutions for fractional $p$-Laplacian, J. Math. Anal. Appl. (2020), 124205. DOI: http://doi.org/10.1016/j.jmaa.2020.124205
[20] F. Gao and M. Yang, *On nonlocal Choquard equations with Hardy–Littlewood–Sobolev critical exponents*, J. Math. Anal. Appl. **448** (2017), 1006–1041.

[21] J. Giacomoni, D. Goel, and K. Sreenadh, *Regularity results on a class of doubly nonlocal problems*, J. Diff. Equ. **268** (2020), 5301–5328.

[22] S. Goyal and K. Sreenadh, *Existence of multiple solutions of p-fractional Laplace operator with sign changing weight function*, Adv. Nonlinear Anal. **4** (2015), 37–58.

[23] A. Iannizzotto, S. Mosconi, and M. Squassina, *Global Hölder regularity for the fractional p-Laplacian*, Rev. Mat. Iberoam. **32** (2016), 1353–1392.

[24] A. Iannizzotto, S. Mosconi, and M. Squassina, *H^s versus C^0-weighted minimizers*, Nonlinear Diff. Equ. Appl. **22** (2015), 477–497.

[25] A. Iannizzotto, S. Mosconi, and M. Squassina, *Sobolev versus Hölder minimizers for the degenerate fractional p-Laplacian*, Nonlinear Anal. **191** (2020), 108635. DOI: https://doi.org/10.1016/j.na.2019.111635.

[26] A. Iannizzotto, S. Mosconi, and M. Squassina, *Fine boundary regularity for the degenerate fractional p-Laplacian*, J. Funct. Anal. **279** (2020), 108659. DOI: https://doi.org/10.1016/j.jfa.2020.108659.

[27] S. Levendorski, *Pricing of the American put under Lévy processes*, Int. J. Theor. Appl. Finance **7** (2004), 303–335.

[28] M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), no. 11, 1203–1219.

[29] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Stud. Appl. Math. **57** (1977), 93–105.

[30] P. Lions, *The Choquard equation and related questions*, Nonlinear Anal. **4** (1980), 1063–1072.

[31] G. Molica Bisci, V. D. Radulescu, and R. Servadei, *Variational methods for nonlocal fractional problems*, Vol. 162, Cambridge University Press, 2016.

[32] V. Moroz and J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: Hardy–Littlewood–Sobolev critical exponent*, Commun. Contemp. Math. **17** (2015), 1550005.

[33] V. Moroz and J. Van Schaftingen, *Existence of groundstates for a class of nonlinear Choquard equations*, Trans. Amer. Math. Soc. **367** (2015), 6557–6579.

[34] V. Moroz and J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties, decay asymptotics*, J. Funct. Anal. **265** (2013), 153–184.

[35] V. Moroz and J. Van Schaftingen, *A guide to the Choquard equation*, J. Fixed Point Theory Appl. **19** (2017), 773–813.

[36] S. Mosconi and M. Squassina, *Nonlocal problems at nearly critical growth*, Nonlinear Anal. **191** (2016), 84–101.

[37] T. Mukherjee and K. Sreenadh, *Fractional Choquard equation with critical nonlinearities*, Nonlinear Differ. Equ. Appl. **24** (2017), 63.

[38] T. Mukherjee and K. Sreenadh, *Critical growth elliptic problems with Choquard type nonlinearity: a survey*, Mathematical modelling, optimization, analytic and numerical solutions, Springer, Singapore, 2020.

[39] S. Pekar, *Untersuchung über die Elektronentheorie der Kristalle*, Akademie Verlag, Berlin, 1954.

[40] R. Penrose, *Quantum computation, entanglement and state reduction*, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. **356** (1998), 1927–1939.

[41] P. Pucci, M. Xiang, and B. Zhang, *Existence results for Schrödinger–Choquard–Kirchhoff equations involving the fractional p-Laplacian*, Adv. Calc. Var. **12** (2019), 253–275.

[42] X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, J. Math. Pures Appl. **101** (2014), 275–302.

[43] X. Ros-Oton and J. Serra, *The extremal solution for the fractional Laplacian*, Calc. Var. Partial Diff. Equ. **50** (2014), 723–750.

[44] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Commun. Pure Appl. Math. **60** (2007), 67–112.

[45] E. M. Stein, *Singular integrals and differentiability properties of functions*, Vol. 2, Princeton University Press, 1970.

[46] Y. Su and H. Chen, *Fractional Kirchhoff-type equation with Hardy–Littlewood–Sobolev critical exponent*, Comput. Math. Appl. **78** (2019), 2063–2082.

[47] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Diff. Equ. **51** (1984), 126–150.

[48] P. Winkert, *L^∞-Estimates for nonlinear elliptic Neumann boundary value problems*, Nonlinear Differ. Equ. Appl. **17** (2010), 289–302.

[49] D. Wu, *Existence and stability of standing waves for nonlinear fractional Schrödinger equations with Hartree type nonlinearity*, J. Math. Anal. Appl. **411** (2014), 530–542.

How to cite this article: R. Biswas and S. Tiwari, *Regularity results for Choquard equations involving fractional p-Laplacian*, Math. Nachr. **296** (2023), 4060–4085. https://doi.org/10.1002/mana.202100469