INTEGRAL GENERALIZED EQUIVARIANT COHOMOLOGIES OF GKM ORBIFOLDS AND WEIGHTED GRASSMANN ORBIFOLDS

KOUSHIK BRAHMA AND SOUMEN SARKAR

Abstract. In this paper, we define 'simplicial GKM orbifold complexes' and 'simplicial GKM graph complexes'. We study some of their topological and combinatorial properties. We discuss some necessary conditions to confirm an invariant $q$-cell structure on a simplicial GKM orbifold complex. We prove Thom isomorphism theorem for orbifold $G$-vector bundles for equivariant cohomology and equivariant K-theory with rational coefficients. We use this to extend the main result of Harada-Henriques-Holm (2005) to the category of $G$-spaces equipped with 'singular invariant stratification'. Then we introduce an equivalent definition of weighted Grassmann orbifolds which were introduced by Corti and Reid (2002) and explicitly studied by Abe and Matsumura (2015) to compute their equivariant cohomology with rational coefficients. These orbifolds are a generalization of Grassmann manifolds. We study their different $q$-cell structures and discuss some necessary conditions to determine if they have invariant CW-structures. We employ the techniques from the first part to study some generalized equivariant cohomology theories of weighted Grassmann orbifolds. We introduce 'divisive' weighted Grassmann orbifolds and compute their equivariant cohomology, equivariant K-theory and equivariant cobordism ring with integer coefficients. We discuss Schubert calculus for the equivariant cohomology ring and show how to compute the corresponding structure constants with integer coefficients.

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1. Introduction

Group actions on topological spaces provide rich structures of the spaces. Examples of such actions include torus actions on symplectic manifolds \cite{Aud91, McD11}, locally standard torus actions on even dimensional manifolds \cite{DJ91}, torus action on GKM manifolds \cite{GKM98}. The above references mainly consider manifolds with torus actions. However, one also has ‘nice’ topological spaces with ‘similar type’ of torus actions as follows. Let $M_1, M_2$ be two manifolds in one of the above categories and $f_1: A \hookrightarrow M_1$ and $f_2: A \hookrightarrow M_2$ be two equivariant embeddings. Then one can construct the pushout $(M_1 \cup_A M_2)$ by giving the following commutative diagram.

\[
\begin{array}{ccc}
A & \xrightarrow{f_1} & M_1 \\
\downarrow{f_2} & & \downarrow \\
M_2 & \longrightarrow & (M_1 \cup_A M_2).
\end{array}
\]

This new space $(M_1 \cup_A M_2)$ is not a manifold in general. Observing this construction and the concept of simplicial complexes, we introduce ‘simplicial GKM orbifold complexes’. Of course, one can make this definition in any category. However, we restrict this definition on the category of GKM orbifolds. The word GKM originated after the pioneering work of \cite{GKM98} which describes the $T$-equivariant cohomology ring $H^*_T$ of any smooth projective toric variety in terms of combinatorial data obtained from its 1-dimensional (complex) orbit structure. Subsequently, several papers extended their result to ‘nice’ stratified spaces see for example \cite{GZ01, HHH05} and \cite{DKS20}. We note that simplicial GKM orbifold complexes contain the above categories of manifolds or orbifolds. Also, one can construct plenty of ‘nice’ topological spaces similarly.

Let $X$ be a $G$-space. Then the equivariant map $X \longrightarrow \{pt\}$ induces a graded $E^*_G(pt)$-algebra structure on $E^*_G(X)$ where $E^*_G$ represents the $G$-equivariant generalized cohomology theory. In \cite{HHH05}, the authors have calculated the generalized equivariant cohomology theory ring $E^*_G(X)$ of an equivariant stratified $G$-space $X$. So the natural question arises how to calculate the generalized equivariant cohomology theory $E^*_G$ of a GKM orbifold and more generally of a GKM orbifold simplicial complex. This question is answered in \cite{SS21} for quasitoric orbifolds over simple polytopes. At the beginning of this paper, we study the generalized equivariant cohomology $E^*_G$ (such as $H^*_G, K^*_G, MU^*_G$) of simplicial GKM orbifold complexes.
under some hypothesis arising from the combinatorial structure of its 1-
dimensional (complex) invariant subsets called spindles, see [GZ01].

On the other hand, weighted Grassmann orbifolds were discussed by
Corti and Reid [CR02]. These are a generalization of Grassmann mani-
folds. They are projective varieties with orbifold singularities. Abe and
Matsumura [AM15] defined them explicitly and studied their equivariant
cohomology with rational coefficients. We note that they used the name
‘weighted Grassmannians’. However, keeping other naming in mind like
Milnor manifolds, Seifert manifolds, we prefer to use Grassmann manifolds
and weighted Grassmann orbifolds. In the second part of this paper, we
introduce a different definition of weighted Grassmann orbifolds. We show
that it is equivalent to the definition of [CR02, AM15]. We employ the tech-
niques from the first part to compute the equivariant cohomology ring and
equivariant K-theory ring of weighted Grassmann orbifolds with rational co-
efficients. Kawasaki [Kaw73] proved that weighted projective spaces (which
are simplest weighted Grassmann orbifolds) have no torsion in the integral
cohomology and it is concentrated in even degrees. Then one can ask the
following natural questions.

**Question 1.1.** Is the cohomology of weighted Grassmann orbifolds concen-
trate in even degrees without any torsion?

**Question 1.2.** How to compute the equivariant cohomology ring, equivari-
ant K-theory ring and equivariant cobordism ring of weighted Grassmann
orbifold with integer coefficients?

We have answered Question 1.1 and 1.2 for weighted Grassmann orbifolds
which satisfies some combinatorial hypothesis.

Throughout the paper, we consider the group $G$ is compact abelian, any
$G$-space is compact, $q$-CW structure in the sense of [PS10, Section 4] and
the standard $n$ torus $T^n := (s^1)^n \subset (\mathbb{C}^*)^n$. We consider cohomology with
rational coefficients unless it is mentioned explicitly. The paper is organized
as follows.

In Section 2 we introduce the concept of ‘simplicial GKM orbifold com-
plexes’ (Definition 2.2 and 2.3) extending the concept of simplicial comple-
exes. Then we introduce ‘simplicial GKM graph complexes’ (Definition
2.9) which generalizes the regular graphs to construct a bridge from simplici-
GKM orbifold complexes to some combinatorics associated with them.
We define the equivariant cohomology of simplicial GKM graph complexes
similarly to [GZ01, Section 1.7] and compare it with the equivariant coho-
mology of the corresponding simplicial GKM orbifold complexes equipped
with torus equivariant CW-structure. This generalizes [DKS20, Theorem
2.9]. Then we briefly discuss on orbifold $G$-vector bundles. We prove the
Thom isomorphism for equivariant cohomology and equivariant K-theory
for orbifold $G$-vector bundles (Theorem 2.16) which generalizes the Thom
isomorphism in [SS21, Section 2]. This helps to generalize the main result
[HHH05, Theorem 2.3] to the category of $G$-spaces equipped with a stratification

$$\{pt\} = X_0 \subset X_1 \subset \cdots \subset X_m = X$$

where $X_i - X_{i-1}$ is equivariantly homeomorphic to an orbifold $G$-vector bundle for $1 \leq i \leq m$, see Proposition 2.17. Under some hypothesis on this stratification, we can give a presentation of the equivariant cohomology and the equivariant K-theory rings of $X$, see Proposition 2.18. Then we introduce the concept of filtration of regular graphs and simplicial graph complexes. Using this, we define ‘build-able GKM orbifold complexes’, see Definition 2.23. We note that this contains retractable toric orbifolds (see [SU21, Section 2]), toric varieties over almost simple polytopes (see [SS21, Section 3]), and the weighted Grassmann orbifolds of Section 3. We describe the equivariant cohomology and equivariant K-theory ring of these spaces with rational coefficients, see Theorem 2.26. We also introduce ‘divisible’ simplicial GKM orbifold complexes to compute their equivariant cohomology, equivariant K-theory and equivariant cobordism ring with integer coefficients, see Theorem 2.27. We also note that this contains divisible weighted projective spaces studied in [HHRW16], retractable toric orbifolds (see [SU21, Section 2], divisible toric varieties (see [SS21, Section 5]), and divisible weighted Grassmann orbifolds of Section 4.

In Section 3 we introduce another definition of a weighted Grassmann orbifold $WGr(d, n)$ for $d < n$, a ‘weight vector’ $W = (w_1, \ldots, w_n) \in \mathbb{Z}_{\geq 0}^n$ and $1 \geq a \in \mathbb{Z}$. Interestingly, this definition is equivalent to the previous one in [AM15]. We show that there is a ‘weighted Plücker embedding’ from $WGr(d, n)$ to a weighted projective space $\mathbb{W}P(c_0, c_1, \ldots, c_m)$ for some $c_i \in \mathbb{Z}_{\geq 1}$ for $i = 0, \ldots, k$ see Lemma 3.3. We give a total ordering on the Schubert symbols, see Definition 3.2. We describe a $q$-cell structure of $WGr(d, n)$ in Proposition 3.5 which is independent of [AM15, Section 2.3]. We discuss a filtration (3.20) of $WGr(d, n)$ using the $q$-cell decomposition. We note that one may get different $q$-cell structures depending on the choice of orderings on the Schubert symbols for $d < n$. Hence, one may obtain different filtration of $WGr(d, n)$.

In Section 4 first we recall that there is an equivariant homeomorphism from $\mathbb{W}P(rc_0, rc_1, \ldots, rc_m)$ to $\mathbb{W}P(c_0, c_1, \ldots, c_m)$ for any $r \geq 1$. Using this techniques we show how the orbifold singularity on a $q$-cell of a $q$-cell decomposition of some subcomplexes of $WGr(d, n)$ can be reduced, see Theorem 4.4. Consequently, we get a new $q$-cell structure of these subcomplexes including $WGr(d, n)$ possibly with less singularity on each cell, see Proposition 4.5. Then we show in Lemma 4.6 that two weighted Grassmann orbifolds are weakly equivariantly homeomorphic if their weight vectors differ by a permutation $\sigma \in S_n$. Then we prove our first main result of this section which says that $WGr(d, n)$ have no $p$-torsion in the integral cohomology and it is concentrated in even degree under some combinatorial hypothesis,
Next, we introduce 'divisive' weighted Grassmann orbifolds which generalize the concept of divisive weighted projective space of [HHRW16]. We also show that a divisive weighted Grassmann orbifold has a \((\mathbb{C}^\ast)^n\)-invariant CW-structure see Theorem 4.21. As a corollary, we get that its integral cohomology has no torsion and is concentrated in even degrees. We discuss some non-trivial examples of divisive weighted Grassmann orbifolds.

In Section 5, we show that the weighted Grassmann orbifolds \(WGr(d, n)\) and \(WP(c_0, c_1, \ldots, c_m)\) are GKM orbifolds see Proposition 5.1. We study the GKM graphs of \(WGr(d, n)\) and \(WP(c_0, c_1, \ldots, c_m)\) and calculate the generalized axial functions of the actions of \(T^n\) on \(WGr(d, n)\) and \(WP(c_0, c_1, \ldots, c_m)\), respectively. Then we describe the equivariant cohomology ring of a divisive weighted Grassmann orbifold with integer coefficients, see Theorem 5.3.

Moreover, we deduce a recurrence relation which helps to compute the structure constants corresponding to them with integer coefficients, see Proposition 5.7 and 5.8.

In Section 7, first we show that there is a \(T^n\)-invariant stratification
\[
\{pt\} = X_0 \subset X_1 \subset \cdots \subset X_m = WGr(d, n)
\]
such that \(X_i/X_{i-1}\) is homeomorphic to the Thom space of an orbifold \(T^n\)-bundle
\[
\xi_i: \mathbb{C}^{k_i}/G_i \to \{pt\}
\]
for some positive integers \(k_i\) and finite groups \(G_i\) for \(i = 1, \ldots, m\), see Proposition 7.1. This satisfies conditions \((A_1), (A_2), (A_3)\) in Section 2 and helps to compute the equivariant K-theory and equivariant cohomology ring of any weighted Grassmann orbifolds with rational coefficients see Theorem 7.3. If \(WGr(d, n)\) is divisive then \(G_i\) is trivial in the above filtration for \(i = 1, \ldots, m\). Then we prove the last main result which computes the equivariant cobordism and K-theory ring of a divisive weighted Grassmann orbifold with integer coefficients see Theorem 7.6.

We provide different corresponding examples from Section 2 onward.

2. Generalized equivariant cohomologies of simplicial GKM orbifold complexes

In this section, we introduce the concept of 'simplicial GKM orbifold complexes' and 'simplicial GKM graph complexes' applying the concept of simplicial complexes. We define the equivariant cohomology of simplicial GKM graph complexes and compare it with the equivariant cohomology of simplicial GKM orbifold complexes. We prove the Thom isomorphism for equivariant cohomology and equivariant K-theory for general orbifold \(G\)-bundles which generalizes [SS21 Section 2]. Then we give a presentation of the equivariant cohomology and equivariant K-theory ring of 'build-able simplicial GKM orbifold complexes' which contains the category of GKM orbifolds.
orbifolds. The readers are referred to [ALR07] and [GGKRW18, Section 2] for basic properties on orbifolds.

Let $X$ be a $G$-space. Then the diagonal $G$-action on $EG \times X$ defined by $g(e, x) \mapsto (eg^{-1}, gx)$ is free, and the orbit space $(EG \times X)/G$ has a CW-structure if $X$ has so. The Borel equivariant cohomology of the $G$-space $X$ is defined by the cohomology of $X_G := (EG \times X)/G$. That is

$$H^*_G(X) := H^*(X_G).$$

The collapsing map $X \rightarrow \{pt\}$ is $G$-equivariant and induces a natural map $\pi^* : H^*(BG) \rightarrow H^*(X_G)$. So $H^*(X_G)$ has a $H^*(BG)$-algebra structure via this map. The free action of $G$ on $EG$ and the $G$-equivariant projection $EG \times X \rightarrow EG$ induce the following fibre bundle.

$$X \xrightarrow{i} X_G \xrightarrow{\pi} BG.$$

Then one has

$$H^*(BG) \xrightarrow{\pi^*} H^*(X_G) \xrightarrow{i^*} H^*(X).$$

If $i^*$ is surjective map then $X$ is called equivariantly formal space. Note that if $H^0d(X) = 0$ and $H^*(X)$ is torsion free then $X$ is equivariantly formal. Note that $H^*_{\mathfrak{g}}(pt) = H^*(BT^n; \mathbb{Z}) \cong \mathbb{Z}[y_1, \ldots, y_n]$.

The readers are referred to [May96] for details and results on G-equivariant cohomology theory $H^*_G$, [Seg68] for $G$-equivariant K-theory $K^*_G$, and [D70, Sim01] for $G$-equivariant complex cobordism theory $MU^*_G$.

Next we recall GKM orbifolds explicitly. Let $G$ be a compact abelian Lie group, and it is acting effectively on a $2n$-dimensional orbifold $X$. This $X$ is called a $G$-orbifold. The set $X^G := \{x \in X \mid gx = x \forall g \in G\}$ is called the fixed point set of the $G$-action on $X$. Let $p \in X^G$ be an isolated fixed point. Then there is an orbifold chart $(\tilde{U}, \xi, K)$ over a neighbourhood $U \subset X$ of $p$ and a finite covering $\tilde{G}$ of $G$ such that $\tilde{G}$ acts on $\tilde{U}$ effectively, the map $\xi : \tilde{U} \rightarrow U$ preserves the respective group actions, $\tilde{p}$ is the $\tilde{G}$ fixed point in $\tilde{U}$ with $\tilde{p} = \xi^{-1}(p)$ and $\tilde{G}/K \cong G$, see [GGKRW18, Proposition 2.8]. Then the tangent space of $\tilde{U}$ at $\tilde{p}$ can be decomposed as

$$T_{\tilde{p}}\tilde{U} \cong V(\tilde{\alpha}_1) \oplus \cdots \oplus V_n(\tilde{\alpha}_n)$$

where $V(\tilde{\alpha}_i)$ is a one dimensional irreducible representation of $\tilde{G}$ and $\tilde{\alpha}_i$ is the corresponding character for $i = 1, 2, \ldots, n$. Let $\alpha_i$ is the image of $\tilde{\alpha}_i$ under the Lie algebra map $L(\tilde{G}) \rightarrow L(G)$ induced by the covering homomorphism $\tilde{G} \rightarrow G$. We say that $\alpha_1, \ldots, \alpha_n$ are the characters of the irreducible $G$-representations at $p$ for $T_pX$.

**Definition 2.1.** A $G$-orbifold $X$ is said to be a GKM orbifold if the following holds.

1. $X^G$ is finite and discrete.
2. $X_1 := \{x \in X \mid \dim gx \leq 1\}$ is a finite union of spindles $\bigwedge P(p, q)$.
3. If $\alpha_1, \ldots, \alpha_n$ are the characters of the irreducible $G$-representations at $p \in X^G$ for $T_pX$ then they are pairwise linearly independent.
We remark that if $k$ many ($2 \leq k \leq n$) elements in Definition 2.1 (3) are linearly independent and $G$ is compact torus then $X$ is called a GKM$_k$ orbifold in [GW20]. Next we introduce the concept of simplicial orbifold complexes generalizing the definition of simplicial complexes.

**Definition 2.2.** Let $\mathcal{K}$ be a finite collection of effective orbifolds. We say that $\mathcal{K}$ is a simplicial orbifold complex if $X, Y \in \mathcal{K}$ and $X \cap Y \neq \emptyset$ then $X \cap Y \in \mathcal{K}$ and $X \cap Y$ is a suborbifold of both $X$ and $Y$.

We note that one can think $\mathcal{K}$ as a category where objects are elements in $\mathcal{K}$ and for any two elements $X, Y \in \mathcal{K}$, $\text{Mor}(X, Y)$ is either inclusion or empty. The limit of this category, denoted by $|\mathcal{K}|$, is called its geometric realization. If both $\mathcal{K}$ and $|\mathcal{K}|$ are clear from the context, we may not distinguish this two notations. As usual, the dimension of a simplicial orbifold complex is defined to be the maximum of the dimension of the orbifolds in that simplicial orbifold complex. Note that each simplex has an orbifold structure. If all element in $\mathcal{K}$ are simplexes then $\mathcal{K}$ is called a simplicial complex. Now we introduce another definition which generalizes GKM-orbifolds.

**Definition 2.3.** If each element of a simplicial orbifold complex $\mathcal{K}$ is a GKM orbifold, then we say that $\mathcal{K}$ is a simplicial GKM orbifold complex.

**Example 2.4.** All GKM orbifolds and GKM manifolds are simplicial GKM orbifold complexes.

**Example 2.5.** Let $X$ be a $2n$-dimensional quasitoric orbifold as defined in [PS10] and $\pi: X \to P$ the corresponding orbit map. From [PS10] Subsection 2.1 and Definition 2.1 we get that any quasitoric orbifold has a GKM orbifold structure. [PS10] Subsection 2.3 gives that if $F$ is a $k$-dimensional face then $\pi^{-1}(F)$ is a $2k$-dimensional quasitoric orbifold. Let $Q \subset P$ be such that $Q$ is the union of some faces in $P$. Then $\pi^{-1}(Q) \subset X$ is an example of simplicial GKM orbifold complex. This gives many examples of simplicial GKM orbifold complexes which are not GKM orbifolds.

**Example 2.6.** Let $P_1$ and $P_2$ are weighted projective spaces and $P_0$ is a suborbifold of both $P_1$ and $P_2$. Then the pushout $X$ in the following diagram is a simplicial GKM orbifold complex.

\[
\begin{array}{ccc}
P_0 & \longrightarrow & P_1 \\
\downarrow & & \downarrow \\
P_2 & \longrightarrow & X.
\end{array}
\]

**Remark 2.7.** Note that if $\mathcal{K}$ is a simplicial GKM orbifold complex and $X, Y \in \mathcal{K}$ then the action of $G$ on $X$ and the action of $G$ on $Y$ coincides on $X \cap Y$. Therefore this induce an action of $G$ on the geometric realization $|\mathcal{K}|$ of $\mathcal{K}$.

Next we introduce simplicial GKM graph complex and its equivariant cohomology. We recall that a graph is a regular graph if degree of all vertices are same. For example the edges in a simple polytope form a regular graph.
Definition 2.8. Let $\Gamma$ be a finite collection of regular graphs. We say that $\Gamma$ is a simplicial graph complex if $\Gamma_1, \Gamma_2 \in \Gamma$ and $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ then $\Gamma_1 \cap \Gamma_2 \in \Gamma$ and $\Gamma_1 \cap \Gamma_2$ is a regular subgraph of both $\Gamma_1$ and $\Gamma_2$.

If $\Gamma$ is a simplicial graph complex, then we denote the vertex set by $V(\Gamma) := \bigcup V(\Gamma_i)$ and the set of edges by $E(\Gamma) := \bigcup E(\Gamma_i)$ where the union runs over the regular graphs in $\Gamma$. For simplicity, we may denote the union of $\Gamma_i$’s by $\Gamma$ and the edges in $\Gamma_i$ outgoing from $p \in V(\Gamma_i) \subset V(\Gamma)$ by $E_p(\Gamma_i)$. We also denote the starting vertex of $e$ by $s(e)$, and the terminal vertex by $t(e)$.

The collection of maps $\theta_e : E_p(\Gamma_i) \to E_q(\Gamma_i)$ for $e \in E(\Gamma_i)$ is called a connection on $\Gamma_i$ if $\theta_e(e) = \bar{e}$ and $\theta_e = \theta_{\bar{e}}^{-1}$ for all $e \in E(\Gamma_i)$ for any $\Gamma_i \in \Gamma$. Generalizing [DKS20] Definition 2.2], we introduce the following.

Definition 2.9. For a positive integer $n$, a simplicial GKM graph complex is defined by a triple $(\Gamma, \alpha, \theta)$ such that the following holds.

1. $\Gamma$ is a simplicial graph complex and the edges in $E(\Gamma)$ are orientable.
2. $\alpha : E(\Gamma) \to H^2(BT^n; \mathbb{Q})$ is a map such that
   a. for all $\Gamma_i \in \Gamma$ the following set of vectors $\{\alpha(e) \mid e \in E_p(\Gamma_i)\}$ are pairwise linearly independent for any $p \in V(\Gamma_i)$ as well as
   b. if $e \in E(\Gamma)$ is an oriented edge and $\bar{e}$ its reverse orientation then
      $$r_e \alpha(e) = \pm r_e \alpha(\bar{e}) \in H^2(BT^n; \mathbb{Z})$$
      for some positive integers $r_e$ and $r_{\bar{e}}$.
3. $\theta = \{\theta_{pq} \mid pq \in E(\Gamma)\}$ where $\theta_{pq} : E_p(\Gamma) \to E_q(\Gamma)$ is a map such that the restriction on each $E_p(\Gamma_i) \subseteq E_p(\Gamma)$ defines a connection on $\Gamma_i$ and if $e, e' \in \Gamma_i$ for some $i$ with $s(e) = s(e')$ there exist $c_{e,e'} \in \mathbb{Z} - \{0\}$ such that $c_{e,e'}(\alpha(\theta_e(e'))) - \alpha(e') = 0 \mod r_e \alpha(e)$.

Example 2.10. Figure 1 is obtained by gluing an edge of boundary of two triangles. Note that it is not a regular graph, however it is a simplicial graph complex which is a collection of 3 regular graphs. Let us denote this by $\Gamma$. Then $V(\Gamma) = \{v_0, v_1, v_2, v_3\}$. Take $k = 4$. Let $y_1, \ldots, y_4$ be the standard basis of $H^2(BT^4; \mathbb{Z})$. Now the axial function $\alpha : E(\Gamma) \to H^2(BT^4; \mathbb{Q})$ is defined by

$$\begin{align*}
\alpha(v_1 v_2) &= ((y_1 + y_4) - \frac{k_2}{k_1}(y_1 + y_3), \quad \alpha(v_2 v_1) = ((y_1 + y_3) - \frac{k_1}{k_2}(y_1 + y_4)), \\
\alpha(v_2 v_0) &= ((y_1 + y_2) - \frac{k_0}{k_2}(y_1 + y_3), \quad \alpha(v_0 v_2) = ((y_1 + y_3) - \frac{k_2}{k_0}(y_1 + y_2)), \\
\alpha(v_1 v_0) &= ((y_1 + y_2) - \frac{k_0}{k_1}(y_1 + y_3), \quad \alpha(v_0 v_1) = ((y_1 + y_3) - \frac{k_1}{k_0}(y_1 + y_2)), \\
\alpha(v_3 v_0) &= ((y_1 + y_2) - \frac{k_0}{k_3}(y_2 + y_3), \quad \alpha(v_0 v_3) = ((y_2 + y_3) - \frac{k_3}{k_0}(y_1 + y_2)), \\
\alpha(v_1 v_3) &= ((y_2 + y_3) - \frac{k_3}{k_1}(y_1 + y_3), \quad \alpha(v_3 v_1) = ((y_1 + y_3) - \frac{k_1}{k_3}(y_2 + y_3)), \\
\end{align*}$$

for some non-zero integers $k_0, \ldots, k_5$. 
Let \( r_{v_i v_j} = k_i \) and \( r_{v_j v_i} = k_j \). Then
\[
r_{v_i v_j} \alpha(v_i v_j) = -r_{v_j v_i} \alpha(v_j v_i) \in H^2(BT^4; \mathbb{Z}).
\]

Let \( e = v_0 v_1 \). Define \( \theta_e : E_{v_0}(\Gamma) \to E_{v_1}(\Gamma) \) by \( \theta_e(v_0 v_1) = v_1 v_0 \), \( \theta_e(v_0 v_i) = v_1 v_i \) for \( i = 1, 2 \) and \( \theta_e = (\theta_e)^{-1} \). Let \( e = v_3 v_0 \). Then define \( \theta_e : E_{v_3}(\Gamma) \to E_{v_0}(\Gamma) \) by \( \theta_e(v_3 v_0) = v_0 v_3 \), \( \theta_e(v_3 v_1) = v_0 v_1 \). Let \( e = v_0 v_3 \). Then define \( \theta_e : E_{v_0}(\Gamma) \to E_{v_3}(\Gamma) \) by \( \theta_e(v_0 v_3) = v_3 v_0 \), \( \theta_e(v_0 v_1) = v_3 v_1 \) and \( \theta_e(v_0 v_2) = v_3 v_0 \) or \( v_3 v_1 \). Similarly we can define \( \theta_e \) for the remaining edges so that \( \theta_e \) is a connection on each regular graph in this simplicial graph complex.

Let \( e \) and \( e' \) be two edges such that \( s(e) = s(e') \). Let \( e = v_i v_j \) and \( e' = v_i v_{k} \). Then \( c_{e,e'} = k_j, k_{\ell} \in \mathbb{Z} - \{0\} \) which satisfies the condition
\[
c_{e,e'}(\alpha(\theta_e(e')) - \alpha(e')) = 0 \mod r_e \alpha(e).
\]

Thus \( (\Gamma, \alpha, \theta) \) is a simplicial GKM graph complex.

**Definition 2.11.** Given a simplicial GKM graph complex \( (\Gamma, \alpha, \theta) \), we define the equivariant cohomology ring
\[
H^*_\mathbb{T} \alpha(\Gamma, \alpha, \theta) := \{ f : V(\Gamma) \to H^*(BT^m; \mathbb{Z}) \mid \tilde{r}_e \alpha(e) \text{ divides } (f(i(e)) - f(t(e))) \},
\]
where \( \tilde{r}_e \) is the smallest positive integer satisfying the condition (2)(b) in Definition 2.9.

One can obtain a simplicial GKM graph complex \( (\Gamma, \alpha, \theta) \) from a simplicial GKM orbifold complex \( K \) similarly as in [GZ01]. Let \( K \) be the collection of GKM orbifolds \( \{K_i\} \). So \( \Gamma \) is obtained by considering the fixed points of the action on \( K_i \) as the vertices and the spindles give the edges if they contain two fixed points. Then one can find the associated axial function \( \alpha_i \), connection \( \theta_i \), \( c_{e,e'} \) and \( r_e \) for each GKM orbifold \( K_i \), see for example [DKS20, Section 2]. This collective data gives \( (\Gamma, \alpha, \theta) \), since \( \{K_i\} \) is a finite collection.
Theorem 2.12. Let $\mathcal{K}$ be a compact simplicial GKM orbifold complex with respect to an effective action of the torus $T^n$ such that the geometric realization of $\mathcal{K}$ is homotopic to a $T^n$-CW complex. Assume all the isotropy subgroups of $T^n$ are connected and $H^{\text{odd}}(\mathcal{K};\mathbb{Z}) = 0$, then the equivariant cohomology ring $H^*_T(\mathcal{K};\mathbb{Z})$ is isomorphic to $H^*_T(\Gamma,\alpha,\theta)$ as $H^*_T(BT^n;\mathbb{Z})$ algebra.

Proof. The condition $H^{\text{odd}}(\mathcal{K};\mathbb{Z}) = 0$ implies the Serre spectral sequence of the fibration $\mathcal{K} \xrightarrow{\sim} \mathcal{K}_{T^n} \xrightarrow{\sim} BT^n$ degenerate at $E_2$ page. Thus $H^*(\mathcal{K}_{T^n};\mathbb{Z}) \cong H^*(\mathcal{K};\mathbb{Z}) \otimes H^*(BT^n;\mathbb{Z})$ as $\mathcal{K}$ is homotopic to finite $T^n$-CW complex and $T^n$ is compact. Since all isotropy subgroups of $T^n$ are connected, So we have the exactness of Chang-Skejelbred sequence

$$0 \to H^*_T(\mathcal{K};\mathbb{Z}) \to H^*_T(\mathcal{K}_0;\mathbb{Z}) \to H^*_T(\mathcal{K}_1;\mathbb{Z}).$$

where $\mathcal{K}_0$ denotes the set of all $T^n$-fixed point of $\mathcal{K}$ and $\mathcal{K}_1$ is the union of all zero and one dimensional orbits in $\mathcal{K}$. Thus applying the arguments in the proof of [GKM98] Theorem 7.2 and [DKS20] Theorem 2.9 one can complete the proof for the simplicial GKM orbifold complex $\mathcal{K}$. □

An application of the above theorem for a simplicial GKM orbifold complex which is not a GKM orbifold is given at the end of Section 7.

Rest of this section we study the $G$-equivariant Thom isomorphism for the orbifold $G$-vector bundles and some other generalized cohomology theories (equivariant K-theory and equivariant cobordism theory) of simplicial GKM orbifold complexes. First we give an explicit construction of orbifold $G$-vector bundle.

Let $B$ be an effective orbifold and $\mathcal{A} := \{ (\tilde{V}_i, G_i, \phi_i) \mid i \in \mathcal{I} \}$ an orbifold atlas on $B$. Now assume that $(\tilde{X}_i, \tilde{V}_i, \tilde{P}_i)$ is a $G_i$ invariant $\ell$-dimensional vector bundle for $i \in \mathcal{I}$ such that if there exists an embedding of orbifold chart $\lambda_i : (\tilde{V}_i, G_i, \phi_i) \to (\tilde{V}_j, G_j, \phi_j)$ then $\tilde{X}_i = \lambda^*(\tilde{X}_j)$. Let $\pi_i : X_i \to X_i = \tilde{X}_i / G_i$ be the orbit map. Then $(\tilde{X}_i, G_i, \pi_i)$ is an orbifold chart on $X_i$ for all $i \in \mathcal{I}$.

Now if $\phi_i(\tilde{V}_i) \cap \phi_j(\tilde{V}_j) \neq \emptyset$ for some $i, j \in \mathcal{I}$, then there exists a chart $(\tilde{V}_k, G_k, \phi_k) \in \mathcal{A}$ and two embeddings

$$\lambda_1 : (\tilde{V}_k, G_k, \phi_k) \to (\tilde{V}_i, G_i, \phi_i) \quad \text{and} \quad \lambda_2 : (\tilde{V}_k, G_k, \phi_k) \to (\tilde{V}_j, G_j, \phi_j).$$

Then by the assumption $\tilde{X}_i = \lambda_1^*(\tilde{X}_i) = \lambda_2^*(\tilde{X}_j)$. Therefore, there exist two embeddings

$$\tilde{\lambda}_1 : (\tilde{X}_k, G_k, \pi_k) \to (\tilde{X}_i, G_i, \pi_i) \quad \text{and} \quad \tilde{\lambda}_2 : (\tilde{X}_k, G_k, \pi_k) \to (\tilde{X}_j, G_j, \pi_j).$$

Note that the map $\tilde{\lambda}_2 \tilde{\lambda}_1^{-1} : \tilde{\lambda}_1^*(\tilde{X}_k) \to \tilde{\lambda}_2^*(\tilde{X}_k)$ is an isomorphism. Therefore, using the discussion in [ALR07] Section 1.3, we can construct an orbifold $X := (\bigsqcup \tilde{X}_i / \sim)$ where $\pi_i(x_i) \sim \pi_j(x_j)$ if $x_j = \lambda_2 \lambda_1^{-1}(x_i)$ for $x_i \in \tilde{X}_i$ and $x_j \in \tilde{X}_j$. The collection $\{(\tilde{X}_i, G_i, \pi_i) \mid i \in \mathcal{I} \}$ is an orbifold atlas for $X$. Since $\tilde{P}_i : \tilde{X}_i \to \tilde{V}_i$ is $G_i$-invariant then it induces a map $P_i : \tilde{X}_i / G_i \to \tilde{V}_i / G_i$ for
This gives an \( \ell \)-dimensional orbifold vector bundle \( P: X \to B \). The triple \((X, B, P)\) is said to be an \( \ell \)-dimensional orbifold vector bundle.

Let \( G \) be a Lie group. Then a vector bundle \( \tilde{P}_i: \tilde{X}_i \to \tilde{V}_i \) is called a \( G \)-vector bundle if \( \tilde{X}_i \) and \( \tilde{V}_i \) are \( G \) spaces and \( \tilde{P}_i \) is a \( G \)-equivariant map. For each \( g \in G_i \), the map \( g: \tilde{P}_i^{-1}(x) \to \tilde{P}_i^{-1}(gx) \) is a linear map for \( x \in \tilde{V}_i \).

Note that \( \{V_i = \phi_i(\tilde{V}_i) \mid i \in \mathcal{I}\} \) is an open covering of \( B \).

**Definition 2.13.** Let \( X \) and \( B \) be two \( G \)-spaces such that the orbifold vector bundle \( \tilde{P}_i: \tilde{X}_i \to \tilde{V}_i \) is \( G \)-vector bundle and the action of \( G \) (on \( \tilde{V}_i \) and \( \tilde{X}_i \)) commutes with the action of \( G_i \) for all \( i \in \mathcal{I} \). Then the map \( P: X \to B \) constructed above is called an orbifold \( G \)-vector bundle.

**Remark 2.14.** The action of \( G \) on \( \tilde{V}_i \) commutes with the action of \( G_i \) for all \( i \in \mathcal{I} \). Then this induces a \( G \)-action on \( \tilde{V}_i = V_i \subset B \) and the composition map \( \phi_i \circ \tilde{P}_i: \tilde{X}_i \to \tilde{V}_i \) is also \( G \)-equivariant and this will induce the simple orbifold \( G \)-bundle \( \tilde{P}_i: \tilde{X}_i \to \tilde{V}_i \) which is considered in \([SS21]\) Section 2]. Here we study more general cases.

Since the group \( G_i \) is finite, one can consider the disc bundle \( D(X) \) whose orbifolds chart (as orbifold with boundary) is given by \((D(\tilde{X}_i), G_i, \pi_i)\) and the sphere bundle \( S(X) \) whose orbifold chart is given by \((S(\tilde{X}_i), G_i, \pi_i)\). So, following the above procedure, one can define the Thom space \( Th(X) \) by \( Th(X) = \frac{D(X)}{S(X)} \). Next we prove the Thom isomorphism for general orbifold \( G \)-vector bundles. Let \( X_0 \) be the zero vectors of the bundle \( P: X \to B \).

**Proposition 2.15** (Thom isomorphism for global quotient). Let \( E_G^* \) be one of \( H_G^* \) and \( K_G^* \) and \( P: (X/G_f) \to B \) be an \( \ell \)-dimensional simple orbifold \( G \)-bundle for some finite group \( G_f \). Then \( P \) induces the following isomorphism

\[
P^*: E_G^*(B, \mathbb{Q}) \to E_G^{*+\ell}(X/G_f, (X/G_f)_0; \mathbb{Q}).
\]

*Proof.* This is proved in \([SS21]\) Section 2]. \(\square\)

**Proposition 2.16** (Thom isomorphism for orbifold \( G \)-vector bundle). Let \( E_G^* \) be one of \( H_G^* \) and \( K_G^* \), and \( P: X \to B \) an \( \ell \)-dimensional orbifold \( G \)-vector bundle as in Definition 2.13. Suppose that \( G \)- and \( G_i \)-representations commute on each fiber of \( \tilde{P}_i: \tilde{X}_i \to \tilde{V}_i \) for each \( i \in \mathcal{I} \). If \( B \) is compact, then the map

\[
P^*: E_G^*(B, \mathbb{Q}) \to E_G^{*+\ell}(X, X_0; \mathbb{Q})
\]

is an isomorphism.

*Proof.* Since \( B \) is compact, there is a finite open cover \( V_1, \ldots, V_k \) such that each \( V_i \) has an orbifold chart and each \( V_i \) is \( G \)-invariant. So each restriction \( \tilde{P}_i: \frac{X}{G_i} \to \tilde{V}_i \) satisfies the hypothesis in Proposition 2.15. If \( k = 2 \) (i.e., \( B = V_1 \cup V_2 \) and \( W := V_1 \cap V_2 \subset V_1 \)) then uniformization of \( V_1 \) induces an orbifold chart for \( W \). Thus \( P|_{X|_W}: X|_W \to W \) also satisfies the hypothesis
in Proposition 2.15 where \( X|_W = P^{-1}(W) \). Now using Mayer-Vietoris sequence, we get the following two horizontal long exact sequences such that each square of the following diagram commutes.

\[
\begin{array}{cccccc}
E^*_G(W) & \rightarrow & E^*_G(B) & \rightarrow & E^*_G(V_1) \oplus E^*_G(V_2) & \rightarrow & E^*_G(W) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E^*_G(Th(X|_W)) & \rightarrow & E^*_G(Th(X)) & \rightarrow & E^*_G(Th(X|_V)) \oplus E^*_G(Th(X|_W)) & \rightarrow & E^*_G(Th(X|_W)).
\end{array}
\]

Therefore, using the five-lemma we get that \( P^*: E^*_G(B; \mathbb{Q}) \rightarrow E^*_G(X, X_0; \mathbb{Q}) \) is an isomorphism. Now for \( k > 2 \) one can complete the proof by the inductive arguments. \( \square \)

Note that one can define \( E \)-orientations or Thom classes of a general orbifold \( G \)-vector bundle similarly to the usual \( G \)-vector bundle. Briefly, an element \( u \in E^*_G(Th(X)) \) is called an \( E \)-orientation or Thom class of an orbifold \( G \)-vector bundle \( \xi: X \rightarrow B \) if for each closed subgroup \( H \) of \( G \) and \( x \in B^H \), the restriction of \( u \) to \( X|_{G \cdot x} := \xi^{-1}(G \cdot x) \) is a generator of

\[
E^*_G(Th(X|_{G \cdot x})) \cong E^*_H(D(X|_{G \cdot x}), S(X|_{G \cdot x})).
\]

Note that the Thom class \( u \) is natural under pullback. The restriction of the Thom class \( u \) to the base \( B \) via the zero section \( s: B \rightarrow X \) is called \( G \)-equivariant Euler class \( e_G(\xi) := s^*(u) \in E^*_G(B) \).

Now we consider the following \( G \)-invariant stratification

\[(2.1)\]
\[
\{pt\} = Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y_m
\]

of a \( G \)-space \( Y \) such that \( Y = \bigcup_{j=0}^m Y_j \) and each successive quotient \( Y_j/Y_{j-1} \) is homeomorphic to the Thom space \( Th(X_j) \) of an orbifold \( G \)-vector bundle \( \xi^j: X_j \rightarrow B_j \). Therefore \( Y \) can be built from \( Y_0 \) inductively by attaching \( q \)-disc bundles \( D(X_j) \) to \( Y_{j-1} \) via some \( G \)-equivariant map

\[
\eta_j: S(X_j) \rightarrow Y_{j-1},
\]

for \( j = 1, \ldots, m \). This gives the following cofibration

\[
Y_{j-1} \rightarrow Y_j \rightarrow Th(X_j).
\]

Then, one gets the following proposition about the generalized equivariant cohomologies with rational coefficients by induction on the stratification and by a similar arguments to the proof of [HH05, Theorem 2.3].

**Proposition 2.17.** Let \( Y \) be a \( G \)-space with a stratification as in (2.1) and \( E^*_G \) a generalized \( G \)-equivariant cohomology theory \( (E^*_G = H^*_G \text{ or } K^*_G) \). Assume that each equivariant Euler class \( e_G(\xi^j) \in E^*_G(B_j) \) for the orbifold \( G \)-vector bundle \( \xi^j \) is not a zero divisor. Then the equivariant inclusion \( \iota: \bigsqcup B_j \hookrightarrow Y \) induces an injection

\[(2.2)\]
\[
\iota^*: E^*_G(Y) \rightarrow \bigoplus_{j=0}^m E^*_G(B_j).
\]
An approach to compute the image of $\iota^*$ in (2.2) has been discussed in [HHH05, Section 3] when $\xi^i$’s are $G$-vector bundles. We apply their techniques in the rest of this section according to our setting. Let $Y$ be a $G$-space with the $G$-stratification as in (2.1) which satisfies the following assumptions.

(A1) Each orbifold $G$-vector bundle $\xi^j : X_j \to B_j$ is $E$-orientable and has the following decomposition

\[(\xi^j : X_j \to B_j) \cong \bigoplus_{s < j} (\xi^{js} : X_{js} \to B_j)\]

into $E$-orientable orbifold $G$-vector bundles $\xi^{js}$, (where $X_{js}$ can be trivial).

(A2) The restriction of the attaching map $\eta_j : S(X_j) \to Y_{j-1}$ on $S(X_{js})$ satisfies

\[\eta_j|_{S(X_{js})} = f^{js} \circ \xi^{js}\]

for some $G$-equivariant map $f^{js} : B_j \to B_s \subset Y_{j-1}$, for $s < j$

(A3) The equivariant Euler classes $\{e_G(\xi^{js}) : s < j\}$ are not zero divisors and pairwise relatively prime in $E^*_G(B_j)$.

Note that under the above assumptions on a $G$-space $Y$ with the property as in (2.1), one may obtain the following proposition with rational coefficients.

**Proposition 2.18.** Let $Y$ be a $G$-space with a $G$-stratification as in (2.1) such that assumptions (A1),(A2) and (A3) are satisfied. Then the image of $\iota^* : E^*_G(Y) \to \prod_j E^*_G(B_j)$ is

\[\Gamma_Y := \left\{ (x_j) \in \bigoplus_{j=0}^m E^*_G(B_j) \mid e_G(\xi^{js}) \text{ divides } x_j - f_{js}^*(x_s) \text{ for all } s < j \right\} .\]

**Proof.** The arguments are similar to the proof of [HHH05, Theorem 3.1] and [SS21, Proposition 2.3] \[\blacksquare\]

**Remark 2.19.** If $G$ is non-abelian, one can get the similar conclusion as in Proposition 2.18 whenever the $G$-space has a $G$-stratification as in (2.1) and satisfies the conditions similar to (A1), (A2) and (A3). If all the $G_i$’s are trivial then Proposition 2.18 also holds for $MU^*_G$, see Theorem [HHH05, Theorem 3.1]

Now we study generalized equivariant cohomology of simplicial GKM orbifold complexes. We construct a filtration of a regular graph. Let $\Gamma = (V, E)$ be an $n$ valent graph where $V$ is the vertices and $E$ is the edges of $\Gamma$. Let $b_0 \in V, V_0 = \{b_0\}$ and $\Gamma_0 := (V_0, E_0)$ where $E_0 = \emptyset$. Next we consider $b_1 \in V - V_0$ which is adjacent to $b_0$. Let $V_1 = \{b_0, b_1\}$ and $E_1$ be the edge joining $b_0$ and $b_1$. Define $\Gamma_1 := (V_1, E_1)$. Suppose, inductively, we define $\Gamma_k := (V_k, E_k)$ where $V_k = \{b_0, b_1, \ldots, b_k\}$ and $E_k$ is the edges in $E$ whose
vertices are in \( V_k \). Let
\[
k' := \min\{ \ell \in \{0, 1, 2, \ldots, k\} \mid b_\ell \text{ is adjacent to a vertex in } V - V_k \}.
\]
Now we consider \( b_{k+1} \in V - V_k \) satisfying that \( b_{k+1} \) is adjacent to \( b_{k'} \). Let
\[
V_{k+1} := \{b_0, b_1, \ldots, b_k, b_{k+1}\}
\]
and \( E_{k+1} \) is the edges in \( E \) whose vertices are in \( V_{k+1} \). So
\[
(2.3) \quad E_{k+1} := \{e \in E \mid V(e) \subset V_{k+1}\} = E_k \cup \{e \in E \mid b_{k+1} \in V(e) \subset V_{k+1}\}.
\]
Then define \( \Gamma_{k+1} := (V_{k+1}, E_{k+1}) \). This process stops when there is no remaining vertices. Therefore
\[
(2.4) \quad \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_m = \Gamma
\]
gives a filtration of \( \Gamma \), since \( \Gamma \) is a connected graph, where \( m + 1 = |V| \). Note that \( (2.4) \) induces an ordering on the vertices of the graph \( \Gamma \).

Let \( \Gamma = (V, E) \) be the GKM graph of a GKM manifold \( M \) of dimension \( 2n \). Then \( \Gamma \) is an \( n \) valent graph and \( V \) corresponds bijectively to the set of all fixed points of the \( G \)-action on \( M \) and \( E \) corresponds bijectively to the set of all invariant spheres which connects two fixed points. Thus by the above paragraph we can define the filtration of the GKM graph \( \Gamma \) of a GKM manifold \( M \) and consequently an ordering on the set of all fixed points of \( M \). Consider the orbit map \( h: M_1 \to \Gamma \) where \( M_1 \) is the one skeleton (the union of all invariant spheres) of the \( G \)-action on \( M \) defined as in Definition 2.1. This correspondence maps the fixed points of \( M \) to the corresponding vertices of \( \Gamma \) and the \( G \)-invariant spheres of \( M \) to the corresponding edges of \( \Gamma \).

Let \( (M, R) \) be a Riemannian manifold. For any point \( p \in M \), the tangent space at \( p \) is denoted by \( T_pM \). Let \( B(0_p, r) \) denote the open ball in \( T_pM \) defined by
\[
B(0_p, r) = \{v \in T_pM \mid ||v|| = R_p(0_p, v) < r\}
\]
where \( 0_p \) is the zero vector in \( T_pM \).

Then for each \( v \in B(0_p, r) \) there is a unique geodesic \( \sigma_v(t) \) with \( \sigma_v(0) = p \) and \( \sigma_v'(0) = v \), where \( \sigma_v(t) \) is defined for \( |t| \leq 1 \). The exponential map at \( p \) is the map
\[
exp_p: B(0_p, r) \to M
\]
defined by
\[
exp_p(v) = \sigma_v(1).
\]
[Muk15, Proposition 4.4.4] showed that the exponential map \( exp_p \) is a diffeomorphism on an open neighbourhood of \( 0_p \) in \( T_pM \) to an open neighbourhood of \( p \) in \( M \), and it is defined explicitly by \( \sigma_v(t) = exp_p(tv) \).

Now if \( G \) acts on \( M \) and \( g \in G \) then \( g: M \to M \) is a diffeomorphism which induce an isomorphism \( dg_p: T_pM \to T_pM \) if \( p \) is a fixed point of the
action such that the following diagram commutes.

\[
\begin{array}{ccc}
T_p M & \xrightarrow{d_{pq}} & T_p M \\
\downarrow \exp_p & & \downarrow \exp_p \\
M & \xrightarrow{g} & M.
\end{array}
\]

Let \( M \) be a GKM manifold and \( p \) a fixed point of the action \( G \) on \( M \) with \( h(p) = b_i \in \Gamma \) for some \( 0 \leq i \leq m \) where \( \Gamma \) is the GKM graph of \( M \). Now \( T_p M = V(\alpha_1) \oplus V(\alpha_2) \oplus \cdots \oplus V(\alpha_n) \).

Recall the filtration of \( \Gamma = (V, E) \) as in 2.4. Let \( F_i = E_i - E_{i-1} \) and \( e_1, e_2, \ldots, e_{d_i} \) be the edges in \( F_i \) containing \( b_i \) with weights \( \alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{id_i} \) respectively. By the above discussion, there exist a \( G \)-invariant submanifold \( M_i \) which is the homeomorphic image of a \( G \)-invariant neighbourhood of the origin in \( \bigoplus_{j=1}^{d_i} V(\alpha_{ij}^{(l)}) \) under the exponential map. Then \( M_i \) is equivariantly contractible to \( h^{-1}(b_i) = p \). We consider the subset \( M_i \subset M \) which is maximal with this property. Let

\[
Y_i = \bigcup_{j=1}^{d_i} M_j \subset M
\]

for \( i = 0, 1, \ldots, m = |V| \). The above observation leads the following.

**Definition 2.20.** A GKM manifold \( M \) equipped with the \( G \)-action is called build-able if the filtration constructed above will stop at \( M \). i.e,

\[
\{pt\} = Y_0 \subset Y_1 \subset Y_2 \subset \cdots \subset Y_m = M
\]

is a \( G \)-invariant stratification for some \( m \geq 1 \).

**Remark 2.21.** Note that \( h^{-1}(E_j \setminus E_{j-1}) \) is the one skeleton of \( Y_j \setminus Y_{j-1} \).

We can generalize the concepts of the filtration of regular graphs to the simplicial graph complexes. Let \( \Gamma = (V, E) \) be a simplicial graph complex where \( V \) is the vertices and \( E \) is the edges of \( \Gamma \). In this case \( \Gamma_0 \) and \( \Gamma_1 \) defined similarly as in the paragraph after Remark 2.19. Suppose inductively we can define \( \Gamma_k = (V_k, E_k) \) for \( k \geq 1 \). Let

\[
k' := \min\{\ell \in \{0, 1, 2, \ldots, k\} \mid b_\ell \text{ is adjacent to a vertex in } V - V_k\}.
\]

Now we consider \( b_{k+1} \in V - V_k \) satisfying that \( b_{k+1} \) is adjacent to \( b_{k'} \).

\[
E_{k+1} := E_k \cup \{e \in E \mid b_{k+1} \in V(e) \subset V_{k+1} \text{ and } e \in \Gamma'(k)\}
\]

where \( \Gamma'(k) \) is the GKM graph of a GKM orbifold in \( K \). Therefore \( \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_m \) gives a filtration of \( \Gamma \) if \( \Gamma_m = \Gamma \). Thus it defines an ordering on the vertices of the simplicial graph complex \( \Gamma \).

Note that \( E_{k+1} \) in (2.3) and (2.6) may not be same in general, unless the simplicial graph complex contains only one regular graph.
Example 2.22. In this example we give a filtration of simplicial graph complex $\Gamma$ which is made from a triangle and a rectangle.

Let $K$ be a simplicial GKM orbifold complex and $(\Gamma, \alpha, \theta)$ the corresponding simplicial GKM graph complex. Thus by the above paragraph we can define the filtration of the simplicial graph complex $\Gamma = (V, E)$. This filtration induces an ordering on the vertices of a simplicial GKM graph complex and consequently an ordering on the set of all fixed points of a simplicial GKM orbifold complex. Let $\tilde{p}$ be an isolated fixed point such that $h(\tilde{p}) = b_i$, where $b_i \in V$ (the set of vertices of $\Gamma$) with $i \geq 1$. Let $X(i)$ be the GKM orbifold which corresponds to the GKM graph $\Gamma'(i)$ which is defined in (2.6).

Let $\{\tilde{\alpha}_1^{(i)}, \ldots, \tilde{\alpha}_d_i^{(i)}\} \subset \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\}$ be the weights corresponding to the edges in $F_i = E_i - E_{i-1}$. Then there exists a $\tilde{G}$-invariant submanifold $\tilde{M}_p$ of $\tilde{U}$ containing $\tilde{p}$ and $\tilde{T}_{\tilde{p}}\tilde{M}_p = \oplus_{j=1}^{d_i} V(\tilde{\alpha}_j^{(i)})$. We denote $\xi(\tilde{M}_p)$ by $M_p$. Let $H_i := \{h \in H \mid h\tilde{M}_p = \tilde{M}_p\}$. Define

$$G_i = H/H_i.$$ (2.7)

Then $M_p$ is a $G$-invariant quotient orbifold whose orbifold chart is given by $(\tilde{M}_p, \xi, G_i)$. So there exists a $G$-invariant suborbifold $M_i$ containing $h^{-1}(b_i)$ and its tangential representation is determined by the characters along the the edges of $F_i$. Suppose the subset $M_i$ is $G$-equivariantly homeomorphic to $\mathbb{C}^{d_i}/G_i$ and maximal with this property. Let $K_i := \bigcup_{j=1}^{d_i} M_j \subset K$

for $i = 0, 1, \ldots, m = |V|$. The above observation leads the following.

Definition 2.23. A simplicial GKM orbifold complex $K$ is called build-able if there is a filtration $\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_m = \Gamma$ of its simplicial GKM graph complex and a $G$-invariant stratification

$$K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_m = K$$ (2.8)
such that $h^{-1}(E_j \setminus E_{j-1})$ is the one skeleton of $K_j \setminus K_{j-1}$ and $K_j \setminus K_{j-1}$ is $G$-equivariantly homeomorphic to $C^{d_j}/G_j$ for some finite group $G_j$ for $j = 0, 1, 2, \ldots, m$.

In addition, if the group $G_j$ defined in (2.7) is trivial for any $j$, then $\mathcal{K}$ is called a ‘divisive’ simplicial GKM orbifold complex.

**Remark 2.24.** In Definition 2.23 we always have a filtration on the GKM graph of the GKM manifold or GKM orbifold, whereas this may not be true for any simplicial GKM orbifold complex. For example,

$$\mathcal{K} = \{ [z_1 : z_2 : z_3] \in \mathbb{C}P^2 \mid \text{at least one } z_i \text{ is zero} \}$$

is a simplicial GKM orbifold complex. Consequently this simplicial GKM orbifold complex is not build-able in the sense of Definition 2.23.

**Proposition 2.25.** If $\mathcal{K}$ is a build-able simplicial GKM orbifold complex with filtration as in (2.8), then it satisfies the conditions (A1), (A2) and (A3).

**Proof.** By Definition 2.23 we have a filtration

$$K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_m = \mathcal{K}.$$

Now $K_i \setminus K_{i-1} = M_i$ is $G$-invariant for $i = 0, 1, \ldots, m$. Since $M_i$ is $G$-equivariantly homeomorphic to $\mathbb{C}^{d_i}/G_i$, then $M_i$ is equivariantly homeomorphic to a $G$-invariant vector bundle $\xi: \mathbb{C}^{d_i}/G_i \to h^{-1}(b_i)$ over $h^{-1}(b_i)$ where $d_i$ is same as the number of edges adjacent to $b_i$ in $F_i$. Note that $d_i \leq i$ by the definition of filtration on $\Gamma$. Let $\{ \alpha_j^{(i)} \}_{j=1}^{d_i}$ be the characters along the edges of $F_i$ and $V_i := \bigoplus_{j=1}^{d_i} V(\alpha_j^{(i)})$. Then $V_i$ is the tangent space of $M_i$ at $h^{-1}(b_i)$ and $\xi: \mathbb{C}^{d_i}/G_i \to h^{-1}(b_i)$ is equivariantly homeomorphic to $\xi': V_i \to h^{-1}(b_i)$. Now

$$\xi': V_i \to h^{-1}(b_i) \cong \bigoplus_{j=1}^{d_i} V(\alpha_j^{(i)}) \to h^{-1}(b_i)$$

$$= \bigoplus_{j=1}^{d_i} \xi_j': V(\alpha_j^{(i)}) \to h^{-1}(b_i)$$

Considering $f_{ij}: h^{-1}(v_i) \to h^{-1}(v_j)$ as a map between two fixed points, we conclude the assumption (A2).

The Euler class of $\xi_j': V(\alpha_j^{(i)}) \to h^{-1}(b_i)$ is determined by $e_G(\xi_j') = \alpha_j^{(i)}$. Since $\alpha_1^{(i)}, \ldots, \alpha_d^{(i)}$ are pairwise linearly independent, hence the assumption (A3) follows. \hfill \Box

**Theorem 2.26.** Let $\mathcal{K}$ be a build-able simplicial GKM orbifold complex with the filtration as in (2.8) and $E_G^* = H_G^*$ or $K_G^*$. Then the generalized
G-equivariant cohomology of $K$ is given by

$$E_G^*(K; \mathbb{Q}) = \left\{(x_j) \in \bigoplus_{j=0}^m E_G^*(b_j) \mid e_G(\xi^j) \text{ divides } x_j - f_{js}^*(x_s) \text{ for all } s < j\right\}.$$  

**Proof.** This follows from Proposition 2.18 where $B_j$ are the fixed points of the $G$-action on $K$. $\square$

Next we discuss the generalized $G$-equivariant cohomologies of divisive simplicial GKM orbifold complexes when $E_G^* = H_G^*$, $K_G^*$ or $MU_G^*$. Note that if $K$ is a divisive simplicial GKM orbifold complex then similarly to the proof of Proposition 2.25 one can show that it has an invariant filtration

$$K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_m = K$$

where each $K_i - K_{i-1}$ is an invariant cell which is equivariantly homeomorphic to $\mathbb{C}^d_i$ for $i = 1, 2, \ldots, m$. So a divisive simplicial GKM orbifold complex is integrally equivariantly formal. Moreover, this filtration satisfies conditions (A1), (A2) and (A3). Therefore, using Remark 2.19 we can get the following result with integer coefficient.

**Theorem 2.27.** Let $K$ be a divisive simplicial GKM orbifold complex and $E_G^* = H_G^*$, $K_G^*$ or $MU_G^*$. Then the generalized G-equivariant cohomology of $K$ with integer coefficients is given by

$$E_G^*(K; \mathbb{Z}) = \left\{(x_j) \in \bigoplus_{j=0}^m E_G^*(b_j; \mathbb{Z}) \mid e_G(\xi^j) \text{ divides } x_j - f_{js}^*(x_s) \text{ for all } s < j\right\}.$$  

**Remark 2.28.** Gonzales studies ‘$\mathbb{Q}$-filterable spaces’ in [Gon14]. If these are projective $T$-varieties then they have a stratification similar to (2.8) where $K_j - K_{j-1}$ is a ‘rational cell’ possibly not an orbifold. Under the assumption of ‘$T$-skeletal’, he studies GKM-theory to obtain the Borel equivariant cohomology ring. However we are also interested in other generalized equivariant cohomology theories, like equivariant K-theory and equivariant cobordism ring of these spaces.

3. IN Var iant $q$-cell structure on weighted Grassmann orbifolds

In this section, we introduce another definition of weighted Grassmann orbifold $WGr(d, n)$ where $d < n$ and show that this definition is equivalent to the previous one. We study the orbifold and $q$-cell structure of weighted Grassmann orbifolds generalising the manifolds counter part discussed in [MS74]. We note that this $q$-cell structure is similar to [AM15]. We show that there is an embedding from a weighted Grassmann orbifold to a weighted projective space.

We recall that a Schubert symbol $\lambda$ for $d < n$ is a sequence of $d$ integers $(\lambda_1, \lambda_2, \ldots, \lambda_d)$ such that $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_d \leq n$. For a Schubert symbol $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ the length, denoted by $\ell(\lambda)$, of $\lambda$ is defined by $\ell(\lambda) := (\lambda_1 - 1) + (\lambda_2 - 2) + \cdots + (\lambda_d - d)$. 
Now we recall the definition of Grassmann manifolds following [MS74, Chapter 6]. Let $M_d(n, d)$ be the set of all complex $n \times d$ matrix of rank $d$ and $\text{GL}(d, \mathbb{C})$ the set of all non-singular complex matrix of order $d$. Observe that if $A \in M_d(n, d)$ then the column space of $A$ (i.e., the span of the column vectors of $A$) determines a $d$-dimensional plane in $\mathbb{C}^n$. The column space of $A$ is equal to the column space of $B$ if and only if there exists $T \in \text{GL}(d, \mathbb{C})$ such that $B = AT$.

Define an equivalence relation $\sim$ on $M_d(n, d)$ as follows. For two matrices $A, B \in M_d(n, d)$, $A \sim B$ if and only if $A = BT$ for some $T \in \text{GL}(d, \mathbb{C})$. Then the quotient space

$$\text{Gr}(d, n) := \frac{M_d(n, d)}{\sim}$$

is called a Grassmann manifold. Note that the manifold $\text{Gr}(d, n)$ represents the set of all $d$-dimensional planes in $\mathbb{C}^n$. We denote the quotient map by

$$\pi: M_d(n, d) \to \text{Gr}(d, n).$$

The space $\text{Gr}(d, n)$ is a $d(n - d)$-dimensional smooth manifold. Several basic properties such as manifold and CW structure of $\text{Gr}(d, n)$ can be found in [MS74].

Generalizing the above equivalence relation, we introduce another quotient space of $M_d(n, d)$ in the following. We denote a matrix $A \in M_d(n, d)$ as follows

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nd} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

where $a_i \in \mathbb{C}^d$ for $i = 1, \ldots, n$.

**Definition 3.1.** For $W := (w_1, w_2, \ldots, w_n) \in (\mathbb{Z}_{\geq 0})^n$ and $a \in \mathbb{Z}_{\geq 1}$, we define an equivalence relation $\sim_w$ on $M_d(n, d)$ by

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \sim_w \begin{pmatrix} t^{w_1}a_1 \\ t^{w_2}a_2 \\ \vdots \\ t^{w_n}a_n \end{pmatrix}$$

for $T \in \text{GL}(d, \mathbb{C})$ and $t \in \mathbb{C}^*$ such that $t^a = \det(T) \in \mathbb{C}^*$. We denote the identification space by

$$\text{WGr}(d, n) := \frac{M_d(n, d)}{\sim_w}.$$
Note that if $W = (0, 0, \ldots, 0)$ and $a = 1$ then $\text{WGr}(d, n)$ is the Grassmann manifold $\text{Gr}(d, n)$.

We recall the Plücker map

\[ P: M_d(n, d) \to \Lambda^d(\mathbb{C}^n) \]

deﬁned by $P(A) = v_1 \wedge v_2 \wedge \cdots \wedge v_d$ where $A = (v_1, v_2, \ldots, v_d) \in M_d(n, d)$ considering $v_j$ as the $j$-th column vector of $A$ for $j = 1, \ldots, d$.

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard ordered basis of $\mathbb{C}^n$ over $\mathbb{C}$. For each Schubert symbol $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$, let $A_\lambda$ be the matrix with row vectors $a_{\lambda_1}, a_{\lambda_2}, \ldots, a_{\lambda_d}$. So $A_\lambda$ is a minor of $A$ of order $d$. Then

\[ P(A) = \sum \text{det}(A_\lambda)e_\lambda \]

where $e_\lambda := e_{\lambda_1} \wedge e_{\lambda_2} \wedge \cdots \wedge e_{\lambda_d}$ for the Schubert symbol $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$. Note that the map $P$ is not one-one, in general.

The Schubert symbols form a lattice with respect to a partial order ‘$\preceq$’ on the Schubert symbols defined by $\lambda \preceq \mu$ if $\lambda_i \leq \mu_i$ for all $i = 1, 2, \ldots, d$.

**Definition 3.2.** Let $\alpha$ and $\lambda$ be two Schubert symbols for $d < n$. We say that $\alpha < \lambda$ if $\ell(\alpha) < \ell(\lambda)$, otherwise we use the dictionary order if $\ell(\alpha) = \ell(\lambda)$.

This gives a total order on the set of all Schubert symbols. Note that the total order ‘$<$’ in Definition 3.2 preserve this partial order ‘$\preceq$’. i.e. for two Schubert symbol $\lambda$ and $\mu$, $\lambda \preceq \mu \implies \lambda \leq \mu$, but not conversely. Observe that there may exist several other total orders on the set of all Schubert symbols which preserve the partial order ‘$\preceq$’. For simplicity, by a total order on the set of all Schubert symbols for $d < n$ we mean one of these total orders on it. For $m = \binom{n}{d} - 1$, let

\[ \lambda^0 < \lambda^1 < \lambda^2 < \cdots < \lambda^m \]

be a total order on the Schubert symbols for $d < n$. We remark that the standard ordered basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{C}^n$ induces an ordered basis $\{e_{\lambda^0}, e_{\lambda^1}, \ldots, e_{\lambda^m}\}$ of $\Lambda^d(\mathbb{C}^n)$. Therefore, we can identify $\Lambda^d(\mathbb{C}^n)$ with $\mathbb{C}^{m+1} := \mathbb{C}\{e_{\lambda^0}, e_{\lambda^1}, \ldots, e_{\lambda^m}\}$. Observe that for $A \in M_d(n, d)$, $P(A) \neq 0$ because there always exist a non-singular minor of $A$ of order $d$. Thus we have

\[ P: M_d(n, d) \to \mathbb{C}^{m+1} - \{0\}. \]

Next we recall the definition of weighted projective spaces following [Kaw73].

**Definition 3.3.** Let $c = (c_0, c_1, \ldots, c_m) \in (\mathbb{Z}_{\geq 1})^{m+1}$. There is an weighted action of $\mathbb{C}^*$ on $\mathbb{C}^{m+1} - \{0\}$ defined by

\[ t(\epsilon^{c_0}z_0, \epsilon^{c_1}z_1, \epsilon^{c_2}z_2, \ldots, \epsilon^{c_m}z_m). \]

The quotient space

\[ \mathbb{W}P(c_0, c_1, \ldots, c_m) := \frac{\mathbb{C}^{m+1} - \{0\}}{\text{weighted } \mathbb{C}^*\text{-action}}. \]
is called the weighted projective space with weights \((c_0, \ldots, c_m)\). It is also denoted by \(\mathbb{W}P(c)\), for simplicity.

We denote the orbit map from \(\mathbb{C}^{n+1} - \{0\} \to \mathbb{W}P(c_0, c_1, \ldots, c_m)\) by \(\pi'_c\) and may write \(\pi'_c(z_0, z_1, \ldots, z_m) = [z_0 : z_1 : \cdots : z_m]_{c_0}\). Note that when \(c_0 = c_1 = \cdots = c_m = 1\), then \(\mathbb{W}P(c_0, c_1, \ldots, c_m) = \mathbb{C}P^m\). Then the corresponding orbit map is denoted by \(\pi'\), i.e. \(\pi' : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}P^m\).

Now for \(W = (w_1, w_2, \ldots, w_n) \in (\mathbb{Z}_{\geq 0})^n\) and \(a \in \mathbb{Z}_{\geq 1}\), let

\[
(3.6) \quad w_{\lambda^i} := a + \sum_{j=1}^{d} w_{\lambda^j}
\]

where \(\lambda^i = (\lambda^i_1, \lambda^i_2, \ldots, \lambda^i_d)\) is the \(i\)-th Schubert symbol given in (3.5). There are \(\binom{n}{d}\) many Schubert symbols for \(d < n\). For simplicity, we denote

\[c_i := w_{\lambda^i}\]

for \(i = 0, \ldots, m = \binom{n}{d} - 1\). So \(c_i \geq 1\) for any \(i \in \{0, \ldots, m\}\). Then one can have the weighted projective space \(\mathbb{W}P(c_0, c_1, \ldots, c_m)\).

**Lemma 3.4.** The map in (3.3) induces a weighted Plücker embedding

\[Pl_w : WGr(d, n) \to \mathbb{W}P(c_0, c_1, c_2, \ldots, c_m)\]

**Proof.** If \(A \in M_d(n, d)\), then by (3.4) we have

\[P(A) = \sum_{i=0}^{m} \det(A_{\lambda^i}) e_{\lambda^i}\]

where \(\lambda^i\) is the \(i\)-th Schubert symbol given in (3.5). This implies that

\[P(DAT) = \sum_{i=0}^{m} \det((DAT)_{\lambda^i}) e_{\lambda^i} = \sum_{i=0}^{m} t^{w_{\lambda^i}} \det(A_{\lambda^i}) e_{\lambda^i} = \sum_{i=0}^{m} t^{c_i} \det(A_{\lambda^i}) e_{\lambda^i}\]

where \(T \in \text{GL}(d, \mathbb{C})\) and \(D = \text{diag}(t^{w_1}, t^{w_2}, \ldots, t^{w_n})\) is the diagonal matrix for \(t \in \mathbb{C}^*\) such that \(t^a = \det(T)\). Therefore, this induces a map

\[Pl_w : WGr(d, n) \to \mathbb{W}P(c_0, c_1, \ldots, c_m)\]

defined by

\[P_{\lambda^i}(w) = P_{\lambda^i}(w_1, w_2, \ldots, w_d) = [w_1 \land w_2 \land \cdots \land w_d]_{c_i}\]

where \(v_1, v_2, \ldots, v_d \in \mathbb{C}^n\) are the columns of \(A\) and \(c_i = w_{\lambda^i}\) defined in (3.6) for \(i = 0, 1, 2, \ldots, m\). This map satisfies the following commutative diagram

\[
\begin{array}{ccc}
M_d(n, d) & \xrightarrow{P} & \mathbb{C}^{n+1} - \{0\} \\
\downarrow{\pi_w} & & \downarrow{\pi'_c} \\
WGr(d, n) & \xrightarrow{Pl_w} & \mathbb{W}P(c_0, c_1, \ldots, c_m)
\end{array}
\]

Thus the map \(Pl_w\) is continuous, since \(\pi_w\) and \(\pi'_c\) are quotient maps.

Let \([A]_{\sim_w} \in WGr(d, n)\). Since \(A \in M_d(n, d)\) there exist a Schubert symbol \(\lambda\) such that \(\det(A_\lambda) \neq 0\). Without loss of generality, we can assume that...
$A_{\lambda} = I_d$. If $A_{\lambda} \neq I_d$ calculate $s \in \mathbb{C}^*$ such that $s^{w_{\lambda}} = 1/\det(A_{\lambda})$. Now consider $D = \text{diag}(s^{w_1}, s^{w_2}, \ldots, s^{w_n})$ and $T = (D A_{\lambda})^{-1}$. Then $\det(T) = s^a$ and $(DAT)_{\lambda} = I_d$. Now $[A]_{w} = [DAT]_{w} \in \text{WGr}(d, n)$.

Now we prove that $Pl_{w}$ is injective.

Let $[A]_{w}, [B]_{w} \in \text{WGr}(d, n)$ such that $Pl_{w}([A]_{w}) = Pl_{w}([B]_{w})$. Then

$$Pl_{w}([A]_{w}) = Pl_{w}([B]_{w}) \implies \det(A_{\lambda}) = t^j \det(B_{\lambda})$$

for some $t \in \mathbb{C}^*$ and for all $j \in \{0, 1, \ldots, m\}$. Since $A \in M_d(n, d)$ there exists a Schubert symbol $\lambda$ such that $\det(A_{\lambda}) \neq 0$. Thus using $[3.8]$, $\det(B_{\lambda}) \neq 0$.

Thus we can assume $A_{\lambda} = B_{\lambda} = I_d$. Then $t^e_\lambda = 1$. Let $k \notin \lambda^i$. Then $a_{kj} \notin A_{\lambda}$ and $b_{kj} \notin B_{\lambda}$. Let $\lambda' = (\lambda^i \cup \{k\}) \setminus \{k\}$. Then $\det(A_{\lambda'}) = a_{kj}$ and $\det(B_{\lambda'}) = b_{kj}$. Let $D = \text{diag}(t^{w_1}, t^{w_2}, \ldots, t^{w_n})$ and $T = (D_{\lambda'})^{-1}$. Thus using $[3.8]$, we get

$$b_{kj} = \frac{t^{w_k}}{t^{w_{\lambda}} a_{ij}} \implies B = DAT \implies [A]_{w} = [B]_{w}.$$  

In particular, if $W = (0, 0, \ldots, 0)$ and $a = 1$ then the map $Pl_{w}$ is the usual Plücker map

$$Pl : \text{Gr}(d, n) \to \mathbb{P}(\Lambda^d(\mathbb{C}^n)) = \mathbb{CP}^m.$$  

Moreover, we have the following commutative diagram.

$$\begin{array}{ccc}
\text{WGr}(d, n) & \xrightarrow{Pl_{w}} & \mathbb{WP}(c_0, c_1, \ldots, c_m) \\
\pi_{w} \uparrow & & \pi_{c} \uparrow \\
M_d(n, d) & \xrightarrow{P} & \mathbb{C}^{m+1} - \{0\} \\
\pi \downarrow & & \pi' \downarrow \\
\text{Gr}(d, n) & \xrightarrow{Pl} & \mathbb{CP}^m.
\end{array}$$

Note that $\text{Gr}(d, n) = \frac{M_d(n, d)}{\text{Gr}(d, n)}$. Then the map $\pi$ is also an orbit map so it is an open map. Also $\text{Gr}(d, n)$ is compact and $\mathbb{CP}^m$ is Hausdorff. Thus $Pl$ is an injective closed map. Hence $Pl$ is an embedding.

Let $U$ be an open subset of $\text{WGr}(d, n)$. Then $\pi(\pi_{w}^{-1}(U))$ is an open subset of $\text{Gr}(d, n)$. Then $Pl(\pi(\pi_{w}^{-1}(U))) = \pi' \circ P(\pi_{w}^{-1}(U))$ is open subset of $Pl(\text{Gr}(d, n))$. Then $P(\pi_{w}^{-1}(U))$ is an open subset of $P(M_d(n, d))$. Hence $Pl_{w}(U) = \pi'_e(P(\pi_{w}^{-1}(U)))$ is open subset of $Pl_{w}(\text{WGr}(d, n))$.

Thus $Pl_{w}$ is an embedding. $\square$

Let $(t_1, t_2, \ldots, t_n) \in (\mathbb{C}^*)^n$ and $A = (a_1, a_2, \ldots, a_n)^{tr} \in M_d(n, d)$. Then $(\mathbb{C}^*)^n$ acts linearly on $M_d(n, d)$ defined by

$$(t_1, \ldots, t_n)(a_1, a_2, \ldots, a_n)^{tr} := (t_1 a_1, t_2 a_2, \ldots, t_n a_n)^{tr}.$$  

This induces a natural $(\mathbb{C}^*)^n$-action on $\text{WGr}(d, n)$ such that the orbit map $\pi_{w}$ of $[3.2]$ is $(\mathbb{C}^*)^n$-equivariant.
The standard action of \((\mathbb{C}^*)^n\) on \(\mathbb{C}^n\) induces an action of \((\mathbb{C}^*)^n\) on the \((\mathbb{C})^{m+1} - \{0\} = \Lambda^d(\mathbb{C}^n) - \{0\}\) which is defined by

\[
(t_1, t_2, \ldots, t_n)(\sum_{i=0}^m a_i e_{\lambda^i}) = \sum_{i=0}^m a_i t_{\lambda^i} e_{\lambda^i}
\]

where \(t_\lambda = t_{\lambda_1} \cdots t_{\lambda_d}\) and \(e_\lambda = e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_d}\) for any Schubert symbol \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)\). This induces a natural \((\mathbb{C}^*)^n\)-action on the weighted projective space \(\mathbb{P}^\infty(c_0, c_1, \ldots, c_m)\) such that the orbit map \(\pi_c\) is \((\mathbb{C}^*)^n\)-equivariant.

Thus the above actions of \((\mathbb{C}^*)^n\) on \(\text{WGr}(d, n)\) and \(\mathbb{P}^\infty(c_0, c_1, \ldots, c_m)\) implies that the weighted Plücker embedding \(\text{Pl}_w\) in \((3.7)\) is \((\mathbb{C}^*)^n\)-equivariant, and \(\text{Pl}_w(\text{WGr}(d, n))\) is a \((\mathbb{C}^*)^n\)-invariant subset of \(\mathbb{P}^\infty(c_0, c_1, \ldots, c_m)\). Thus all the maps in the diagram \((3.10)\) are \((\mathbb{C}^*)^n\)-equivariant.

Now we show that Definition \(3.1\) is equivalent to the definition of a weighted Grassmann orbifold studied in [AM15]. Note that the algebraic torus \((\mathbb{C}^*)^{n+1}\) acts on \(\Lambda^d(\mathbb{C}^n) - \{0\}\) by

\[
(t_1, t_2, \ldots, t_n, t) \sum a_\lambda e_\lambda = \sum t \cdot t_\lambda a_\lambda e_\lambda
\]

where \(t_\lambda := t_{\lambda_1} \cdots t_{\lambda_d}\) for the Schubert symbol \(\lambda = (\lambda_1, \ldots, \lambda_d)\) for \(d < n\). Consider the subgroup \(WD\) of \((\mathbb{C}^*)^{n+1}\) defined by

\[
WD := \{(t^{w_1}, t^{w_2}, \ldots, t^{w_n}, t^\alpha) \mid t \in \mathbb{C}^*\}.
\]

Then the restricted action of \(WD\) on \(\Lambda^d(\mathbb{C}^n) - \{0\}\) is given by

\[
(t^{w_1}, t^{w_2}, \ldots, t^{w_n}, t^\alpha) \sum a_\lambda e_\lambda = \sum t^{w_\lambda} a_\lambda e_\lambda.
\]

Observe that this action of \(WD\) is same as the weighted \(\mathbb{C}^*\) action as in Definition \(3.3\). Then we have \(\frac{\Lambda^d(\mathbb{C}^n)^*}{WD} = \mathbb{P}^\infty(c_0, \ldots, c_m)\) and by the commutativity of the diagram \((3.10)\) we have

\[
\text{Pl}_w(\text{WGr}(d, n)) = \frac{P(M_d(n, d))}{WD}.
\]

The second one was called a weighted Grassmann orbifold by Abe and Matsumara in [AM15].

Next we show that the quotient space in Definition \(3.1\) has an orbifold structure. Let

\[V_\lambda := \{[A]_{\sim w} \in \text{WGr}(d, n) \mid \det(A_\lambda) \neq 0\}\]

for \(\lambda \in \{\lambda^0, \ldots, \lambda^m\}\), and

\[\widetilde{V}_\lambda := \pi_w^{-1}(V_\lambda) = \{A \in M_d(n, d) \mid \det(A_\lambda) \neq 0\}.
\]

Then \(V_\lambda\) is an open subset of \(\text{WGr}(d, n)\) as \(\widetilde{V}_\lambda\) is open subset of \(M_d(n, d)\). Let

\[U_\lambda := \pi(\pi_w^{-1}(V_\lambda)) = \{[A] \in \text{Gr}(d, n) \mid \det(A_\lambda) \neq 0\}\]
for \( \lambda \in \{\lambda^0, \ldots, \lambda^m\} \). Recall that the standard coordinate chart for the Grassmann manifold \( \text{Gr}(d, n) \) is given by \( \{U_{d, n}^i\}_{i=0}^m \), see [MS74]. The set \( U_{d, n}^i \) is homeomorphic to

\[
Pl(U_{d, n}^i) := \{[z_0 : z_1 : z_2 : \cdots : z_m] \in Pl(\text{Gr}(d, n)) \subseteq \mathbb{C}P^m \mid z_i \neq 0\}.
\]

Indeed, both the spaces \( U_{d, n}^i \) and \( Pl(U_{d, n}^i) \) are homeomorphic to \( \mathbb{C}^d(n-d) \) for all \( i \in \{0, 1, 2, \ldots, m\} \). Lemma 3.4 and the diagram (3.10) give that \( V_\lambda \) is homeomorphic to \( Pl_w(V_\lambda) \), and \( Pl_w(V_\lambda) = \pi'_c(P(\tilde{V}_\lambda)) \). Now

\[
P(\tilde{V}_\lambda) := \{ (z_0, z_1, \ldots, z_m) \in P(M_d(n, d)) \subseteq \mathbb{C}^{m+1} \setminus \{0\} \mid z_i \neq 0\}.
\]

Note that \( (Pl)^{-1} \circ \pi': P(M_d(n, d)) \to \text{Gr}(d, n) \) is a principal \( \mathbb{C}^* \)-bundle with the local trivialization

\[
\phi_{d, n} : P(\tilde{V}_\lambda) \to U_{d, n} \times \mathbb{C}^*
\]

defined by \( \phi_{d, n}(P(A)) = (\pi(A), \det(A_{\lambda_i})) \) where the inverse map is defined by \( (\pi(A), s) \mapsto (s(\det(A_{\lambda_i}))^{-1}P(A)) \).

Let \( t \in \mathbb{C}^* \) and \( A = (a_1, a_2, \ldots, a_n)^{tr} \in M_d(n, d) \). Then \( \pi(A) \in \text{Gr}(d, n) \).

There is a weighted action of \( \mathbb{C}^* \) on \( \text{Gr}(d, n) \) defined by

\[
(3.16) \quad t.\pi(A) = t.\pi((a_1, a_2, \ldots, a_n)^{tr}) := \pi((t^{\lambda_1}a_1, t^{\lambda_2}a_2, \ldots, t^{\lambda_n}a_n)^{tr}).
\]

Then \( \phi_{d, n} \) becomes \( \mathbb{C}^* \)-equivariant with the following weighted action of \( \mathbb{C}^* \) on \( U_{d, n} \times \mathbb{C}^* \) given by \( t.(\pi(A), s) = (t.\pi(A), t^c s) \). Where \( t.\pi(A) \) is defined in (3.16) and \( c_i = w_{\lambda_i} \) is defined in (3.6).

Definition 3.3 implies that

\[
\pi'_c(P(\tilde{V}_\lambda)) = \frac{P(\tilde{V}_\lambda)}{\sim c} = \frac{P(\tilde{V}_\lambda)}{\text{weighted } \mathbb{C}^* \text{ action}} \cong \frac{U_{d, n} \times \mathbb{C}^*}{\text{weighted } \mathbb{C}^* \text{ action}}.
\]

Let \( G(c_i) \) be the group \( c_i \)-th root of unity defined by

\[
(3.17) \quad G(c_i) := \{ t \in \mathbb{C}^* \mid t^{w_{\lambda_i}} = t^{c_i} = 1 \}.
\]

Then the finite subgroup \( G(c_i) \) acts on the second factor of \( U_{d, n} \times \mathbb{C}^* \) trivially. Thus

\[
V_{d, n} \cong \pi'_c(P(\tilde{V}_\lambda)) \cong \frac{U_{d, n}}{G(c_i)}.
\]

Then \( (U_{d, n}, q_i, G(c_i)) \) is an orbifold chart on \( V_{d, n} \in \text{WGr}(d, n) \) where \( q_i : U_{d, n} \to V_{\lambda_i} \) for \( i = 0, 1, \ldots, m \). Now \( \{V_{\lambda_i}\} \) is an open covering of \( \text{WGr}(d, n) \). Therefore the identification space in Definition 3.1 is a weighted Grassmann orbifold. Thus Lemma 3.3 and the above discussion give that Definition 3.1 is equivalent to that of in [AM15].

Next, we recall the Schubert cell decomposition of \( \text{Gr}(d, n) \) following [MS74]. For \( k \leq n \) we identify \( \mathbb{C}^k \) with all vectors of the form

\[
z = (z_1, z_2, \ldots, z_k, 0, \ldots, 0) \in \mathbb{C}^n.
\]
For each Schubert symbol $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$, the Schubert cell $E(\lambda)$ is defined by

$$E(\lambda) := \{X \in \text{Gr}(d, n) \mid \dim(X \cap \mathbb{C}^{\lambda_j}) = j, \ \dim(X \cap \mathbb{C}^{\lambda_j-1}) = j - 1; \ \forall \ j = 1, 2, \ldots, n\}.$$ 

By [MS74, Chapter-6] each element of $E(\lambda)$ can be represented uniquely by the equivalence class of the following $n \times d$ complex matrix $A_{\lambda}$. Where

$$A_{\lambda} := \begin{bmatrix}
* & * & \ldots & * \\
* & * & \ldots & * \\
1 & 0 & \ldots & 0 \\
0 & * & \ldots & * \\
\vdots & \vdots & & \vdots \\
0 & * & \ldots & * \\
0 & 1 & \ldots & 0 \\
0 & 0 & \ldots & * \\
\vdots & \vdots & & \vdots \\
0 & 0 & \ldots & * \\
0 & 0 & \ldots & 1 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \ldots & 0
\end{bmatrix}.$$ (3.18)

Here the $\lambda_j$-th row is $e_j$ and $j$-th column has $\lambda_j$-th entry 1 and all subsequent entries of this column are zero for $j = 1, \ldots, d$. Then $E(\lambda)$ is an open cell of dimension $\ell(\lambda) = (\lambda_1 - 1) + (\lambda_2 - 2) + \cdots + (\lambda_d - d)$.

**Proposition 3.5.** There is a $q$-cell structure on $\text{WGr}(d, n)$ for $0 < d < n$.

**Proof.** Let $\lambda$ be a Schubert symbol for $d < n$ and $\tilde{E}(\lambda) := \pi^{-1}(E(\lambda))$ where $\pi$ is defined in (3.1). Then, by the commutativity of the diagram (3.10), we get $P(\tilde{E}(\lambda)) \cong (\pi')^{-1}(Pl(E(\lambda)))$. The Schubert cell decomposition of $\text{Gr}(d, n)$ gives that $\text{Gr}(d, n) = \sqcup_{i=0}^{m} E(\lambda^i)$. This implies

$$M_d(n, d) = \sqcup_{i=0}^{m} \tilde{E}(\lambda^i) \text{ and } P(M_d(n, d)) = \sqcup_{i=0}^{m} P(\tilde{E}(\lambda^i)),$$ (3.19)

since the map $\pi$ is surjective, $Pl$ is injective and the diagram (3.10) is commutative.

By the commutativity of the diagram (3.10) and injectiveness of $Pl_w$ we get

$$Pl_w(\pi_w(\tilde{E}(\lambda^i))) = \pi'_w(P(\tilde{E}(\lambda^i))) = \frac{P(\tilde{E}(\lambda^i))}{\text{weighted } \mathbb{C}^* \text{ action}} \cong E(\lambda^i) / G(c_i).$$

Therefore we get a $q$-cell decomposition of $\text{WGr}(d, n)$ given by

$$\text{WGr}(d, n) = \pi_w(\tilde{E}(\lambda^0)) \sqcup \pi_w(\tilde{E}(\lambda^1)) \sqcup \cdots \sqcup \pi_w(\tilde{E}(\lambda^i)),$$
or

\[ P_{t_w}(\text{WGr}(d, n)) = \frac{E(\lambda^0)}{G(c_0)} \sqcup \frac{E(\lambda^1)}{G(c_1)} \sqcup \frac{E(\lambda^2)}{G(c_2)} \sqcup \cdots \sqcup \frac{E(\lambda^m)}{G(c_m)}. \]

□

Let \( X^k := \sqcup_{i=0}^k \frac{E(\lambda^i)}{G(c_i)} \subset \text{WGr}(d, n) \). Here \( X^k \) is built inductively by attaching the \( q \)-cells \( \frac{E(\lambda^0)}{G(c_0)}, \ldots, \frac{E(\lambda^k)}{G(c_k)} \). Then we have the following filtration of \( q \)-CW complexes which are invariant under \((\mathbb{C}^*)^n \) action on \( \text{WGr}(d, n) \),

\[
\{pt\} = X^0 \subset X^1 \subset X^2 \subset \cdots \subset X^m.
\]

(3.20)

We note that the paper [AM15] discussed \( q \)-cell structure of \( \text{WGr}(d, n) \). However, our approach is different.

**Remark 3.6.** The set 

\[
\tilde{E}(\lambda^i) = \{ A \in M_d(n, d) \mid \det(A_{\lambda^i}) \neq 0, \det(A_{\lambda^j}) = 0, \text{ for } j > i \}.
\]

Now for each \( k \in \{0, 1, 2, \ldots, m\} \), consider \( Y^k \subset M_d(n, d) \) defined by

\[
Y^k := \{ A \in M_d(n, d) \mid \det(A_{\lambda^j}) = 0, \text{ for } j > k \}.
\]

Then \( Y^k = \sqcup_{j=1}^k \tilde{E}(\lambda^j) \subset M_d(n, d) \) and we have \( X^k = \frac{\cup_{i=0}^k}{\sim_w} = \pi_w(Y^k) \).

4. INTEGRAL COHOMOLOGY OF CERTAIN WEIGHTED GRASSMANN ORBIFOLDS

Kawasaki ([Kaw73]) proved that weighted projective spaces have no torsion in the integral cohomology and it is concentrated in even degrees. In this section we introduce ‘divisive’ weighted Grassmann orbifolds which generalises the concept of devisive weighted projective spaces of [HHRW16]. Then we compute their integral cohomologies. We also show that a divisive weighted Grassmann orbifold has a \((\mathbb{C}^*)^n\)-invariant CW-structure. As a consequence, its integral cohomology has no torsion and concentrated in even degrees.

The following lemma is well-known, but for our purpose we may need its proof.

**Lemma 4.1.** The map \( \pi'_r : \mathbb{C}^{m+1} - \{0\} \rightarrow \mathbb{WP}(c_0, c_1, \ldots, c_m) \) induces an equivariant homeomorphism \( \mathbb{WP}(rc_0, rc_1, \ldots, rc_m) \rightarrow \mathbb{WP}(c_0, c_1, \ldots, c_m) \) for any positive integer \( r \).

**Proof.** The weighted \( \mathbb{C}^* \)-action on \( \mathbb{C}^{m+1} - \{0\} \) for \( \mathbb{WP}(rc_0, rc_1, \ldots, rc_m) \) is given by

\[
t(z_0, z_1, z_2, \ldots, z_m) = (t^{rc_0}z_0, t^{rc_1}z_1, t^{rc_2}z_2, \ldots, t^{rc_m}z_m).
\]

Thus \( (z_0, z_1, z_2, \ldots, z_m) \sim_{rc} (t^{rc_0}z_0, t^{rc_1}z_1, t^{rc_2}z_2, \ldots, t^{rc_m}z_m) \) and

\[
\mathbb{WP}(rc_0, rc_1, \ldots, rc_m) = \sim_{rc} \mathbb{C}^{m+1} - \{0\} (= \mathbb{WP}(rc) \text{ in short}).
\]
Now, for any \( t \in \mathbb{C}^n \)
\[
\pi_c'(t^{rc_0}z_0, t^{rc_1}z_1, t^{rc_2}z_2, \ldots, t^{rc_m}z_m) = [t^{rc_0}z_0 : t^{rc_1}z_1 : t^{rc_2}z_2 : \cdots : t^{rc_m}z_m]_c
\]
\[
= [s^{c_0}z_0 : s^{c_1}z_1 : \cdots : s^{c_m}z_m]_c
\]
\[
= [z_0 : z_1 : \cdots : z_m]_c
\]
\[
= \pi'_c(z_0, z_1, \ldots, z_m)
\]

where \( t^* = s \in \mathbb{C}^n \).

Similarly,
\[
\pi_{rc}'(t^{rc_0}z_0, t^{rc_1}z_1, t^{rc_2}z_2, \ldots, t^{rc_m}z_m) = \pi_{rc}'(z_0, z_1, \ldots, z_m).
\]

So we get a map \( f : \mathbb{W}P(rc) \to \mathbb{W}P(c) \) defined by
\[
f[(z_0, z_1, \ldots, z_m)]_{rc} = [(z_0, z_1, \ldots, z_m)]_c
\]
and a map \( g : \mathbb{W}P(c) \to \mathbb{W}P(rc) \) defined by
\[
g[(z_0, z_1, \ldots, z_m)]_c = [(z_0, z_1, \ldots, z_m)]_{rc}
\]
such that the following diagram commutes
\[
\begin{array}{ccc}
\mathbb{C}^{m+1} - \{0\} & \xrightarrow{\text{Id}} & \mathbb{C}^{m+1} - \{0\} \\
\pi_{rc}' & \downarrow & \pi'_c \\
\mathbb{W}P(rc) & \xrightarrow{f} & \mathbb{W}P(c).
\end{array}
\]

Also we have \( f \circ g = \text{Id}_{\mathbb{W}P(c)} \) and \( g \circ f = \text{Id}_{\mathbb{W}P(rc)} \). Thus \( f \) is a bijective map with the inverse map \( g \).

Let \( U \) be an open subset of \( \mathbb{W}P(c) \) Then \((\pi'_c)^{-1}(U) = (\pi_{rc}')^{-1} \circ f^{-1}(U) \). Since \( \pi_c' \) is a quotient map then \((\pi_{rc}')^{-1}(U) \) is open subset of \( \mathbb{C}^{m+1} - \{0\} \). Thus \( f^{-1}(U) \) is open subset of \( \mathbb{W}P(rc) \) as \( \pi_{rc}' \) is a quotient map. Thus \( f \) is continuous. By the similar logic, we can show that \( g \) is continuous. Hence \( f \) is a homeomorphism and also it is equivariant with respect to the standard torus action on \( \mathbb{W}P(c) \) and \( \mathbb{W}P(rc) \).

\[\Box\]

**Lemma 4.2.** Let \( B \) be a subset of \( \mathbb{C}^{m+1} - \{0\} \). Let \( \pi'_c(B) = B'_c \subset \mathbb{W}P(c) \) and \( \pi_{rc}'(B) = B'_{rc} \subset \mathbb{W}P(rc) \). Then \( f|_{B'_{rc}} : B'_{rc} \to B'_c \) is a homeomorphism.

**Proof.** Consider the following commutative diagram
\[
\begin{array}{ccc}
B & \xrightarrow{\text{Id}} & B \\
\downarrow{\pi_{rc}'} & & \downarrow{\pi'_c} \\
B'_{rc} & \xrightarrow{f|_{B'_{rc}}} & B'_c.
\end{array}
\]

Since the map \( f \) is well defined and one-one. It follows that \( f|_{B_{rc}} \) is also well defined and one-one. Note that \( f|_{B_{rc}} \) is defined by \( f|_{B_{rc}}(\pi_{rc}'(b)) = \pi'_c(b) \). Thus \( \pi_{rc}'(b) \in B'_{rc} \) is the inverse image of an element \( \pi'_c(b) \in B'_c \). So \( f|_{B_{rc}} \) is
bijective. Also \((f|_{B^r_{\ell}})^{-1} = g|_{B^r_{\ell}}\). Now to conclude, \(f|_{B^r_{\ell}}\) is homeomorphism, recall that restriction of a continuous map is also continuous. \(\square\)

**Remark 4.3.** Let \(A = \{(z_0, z_1, \ldots, z_{m-1}, 0) \in \mathbb{C}^{m+1} - \{0\}\}\) and \(\gcd\{c_0, c_1, \ldots, c_{m-1}\} = r.\) Then \(\pi'_c(A) = \{[(z_0, z_1, \ldots, z_{m-1}, 0)] \in \mathbb{W}P(c_0, c_1, \ldots, c_m)\}\) is homeomorphic to \(\mathbb{W}P(\frac{c_0}{r}, \frac{c_1}{r}, \ldots, \frac{c_{m-1}}{r})\).

Next we apply the previous result onto some subsets of \(P(M_d(n, d)) \subseteq \mathbb{C}^{m+1} - \{0\}.\) Recall the space \(Y^k\) from Remark \ref{remark:cell-structure}. Then

\[
P(Y^k) = \bigcup_{i=0}^{k} P(\tilde{E}(\lambda^i)) \subseteq P(M_d(n, d)) \quad \text{using \ref{remark:cell-structure}.}
\]

Then \(P(Y^k) \subseteq \mathbb{C}^{m+1} - \{0\},\) for \(k \in \{0, 1, \ldots, m\}.\)

Fix \(k \in \{0, 1, 2, \ldots, m\}.\) Now for \(WGr(d, n)\) we have \(c_i = w_{\lambda^i}\) where \(w_{\lambda^i}\) is defined in \ref{definition:restrictions} for \(i \in \{0, 1, \ldots, m\}.\) Let \(r = \gcd\{c_0, c_1, \ldots, c_k\}\) and \(G(r)\) be the group of \(r\)-th roots of unity. Then \(G(r)\) is a subgroup of \(G(c_i)\) and \(G(c_i)/G(r)\) is isomorphic to \(G(c_i/r)\) for \(i \in \{0, 1, 2, \ldots, k\}.\) In the following theorem and proposition \(r\) depends on \(k\) and it is used contextually.

**Theorem 4.4.** The space \(\pi'_c(P(Y^k))\) is homeomorphic to \(\pi'_c(P(Y^k))\). Moreover, \(E(\lambda^k)/G(c_k)\) is homeomorphic to \(E(\lambda^k)/G(c_k/r).\)

**Proof.** Note that the following diagram is commutative.

\[
\begin{array}{ccc}
P(Y^k) & \xrightarrow{\text{Id}} & P(Y^k) \\
\downarrow{\pi'_c} & & \downarrow{\pi'_c} \\
\pi'_c(P(Y^k)) & \xrightarrow{f|_{\pi'_c(P(Y^k))}} & \pi'_c(P(Y^k)).
\end{array}
\]

By Lemma \ref{lemma:homeomorphism} the lower horizontal map is a homeomorphism. The second statement of the theorem follows by the similar arguments with \(P(Y^k)\) is replaced by \(P(\tilde{E}(\lambda^k)).\) \(\square\)

**Proposition 4.5.** The collection \(\{E(\lambda^i)/G(c_i/r)\}_{i=0}^{k}\) gives a \(q\)-cell structure of \(\pi'_c(P(Y^k))\). In particular, if \(k = m,\) this gives a \(q\)-cell structure of \(WGr(d, n).\)

**Proof.** Note that \(P(M_d(n, d)) = \bigcup_{i=0}^{m} P(\tilde{E}(\lambda^i))\) and the set \(P(\tilde{E}(\lambda^i))\) is invariant under the weighted \(\mathbb{C}^*\)-action defined in Definition \ref{definition:weighted-action}. Then we have the following commutative diagram.

\[
\begin{array}{ccc}
\pi'_c(P(Y^k)) & \subseteq & \mathbb{W}P(c_0, c_1, \ldots, c_k) \\
\uparrow{\pi'_c} & & \uparrow{\pi'_c} \\
P(Y^k) & \subseteq & \mathbb{C}^{k+1} - \{0\} \\
\downarrow{\pi'_c} & & \downarrow{\pi'_c} \\
\pi'_c(P(Y^k)) & \subseteq & \mathbb{W}P(\frac{c_0}{r}, \frac{c_1}{r}, \ldots, \frac{c_m}{r}).
\end{array}
\]
Thus

$$
\pi'_\xi(P(Y^k)) = \pi'_\xi(\bigcup_{i=0}^k P(\tilde{E}(\lambda^i)))
$$

$$
= \bigcup_{i=0}^k \pi'_\xi(P(\tilde{E}(\lambda^i)))
$$

$$
= \bigcup_{i=0}^k P(\tilde{E}(\lambda^i)) \cong \bigcup_{i=0}^k \frac{E(\lambda^i)}{G(c_i/r)}.
$$

□

Next, we show that two weighted Grassmann orbifolds are weakly equivariantly homeomorphic if their weights are differed by a permutation $\sigma \in S_n$. Let $X, Y$ be two $G$-spaces. A map $f : X \to Y$ is called a weakly equivariant homeomorphism if $f$ is a homeomorphism and $f(gx) = \eta(g)f(x)$ for some $\eta \in \text{Aut}(G)$ and for all $(g, x) \in G \times X$. If $\eta$ is identity, then $f$ is called an equivariant homeomorphism.

**Lemma 4.6.** Let $W := (w_1, w_2, \ldots, w_n) \in (\mathbb{Z}_{\geq 0})^n$, $0 < a \in \mathbb{Z}$ and $\sigma W := (w_{\sigma 1}, w_{\sigma 2}, \ldots, w_{\sigma n})$ for some $\sigma \in S_n$. If $\text{WGr}(d, n)$ and $\text{WGr}'(d, n)$ are the weighted Grassmann orbifolds corresponding to $(W, a)$ and $(\sigma W, a)$ respectively, then $\text{WGr}(d, n)$ is weakly equivariantly homeomorphic to $\text{WGr}'(d, n)$, and this may induce different $q$-cell structures on $\text{WGr}(d, n)$ for different $\sigma$.

**Proof.** We consider $\mathbb{C}^i = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n \mid x_j = 0 \text{ for } j > i\}$. For $\sigma \in S_n$, define $\sigma \mathbb{C}^n := \{(x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma n})\}$ and

$$
\sigma \mathbb{C}^i := \{(x_{\sigma 1}, x_{\sigma 2}, \ldots, x_{\sigma n}) \in \sigma \mathbb{C}^n \mid x_{\sigma j} = 0 \text{ for } j > i\}.
$$

Note that $A = (a_{ij}) \in M_d(n, d)$ if and only if $\sigma A = (a_{\sigma ij}) \in M_d(n, d)$. Thus $X = \pi(A)$ is a $d$-dimensional subspace of $\mathbb{C}^n$ if and only if $\sigma X = \pi(\sigma A)$ is also so. Thus $\text{Gr}(d, n) \cong \sigma \text{Gr}(d, n) = \text{Gr}(d, n)$, here the equality follows from the definition of a Grassmann manifold. Thus the natural weakly equivariant homeomorphism $\bar{f}_\sigma : \text{M}(d, n,d) \to \text{M}(d, n,d)$ defined by $\bar{f}_\sigma(A) = \sigma A$ induces the following commutative diagram.

$$
\begin{array}{ccc}
\text{M}(d, n,d) & \xrightarrow{\bar{f}_\sigma} & \text{M}(d, n,d) \\
\downarrow \pi_w & & \downarrow \pi_{\sigma w} \\
\text{WGr}(d, n) & \xrightarrow{f_\sigma} & \text{WGr}'(d, n).
\end{array}
$$

Here $\pi_w$ is the quotient map defined in Definition 3.1. Therefore, this induces a weakly equivariant homeomorphism $f_\sigma : \text{WGr}(d, n) \to \text{WGr}'(d, n)$, where $(\mathbb{C}^*)^n$-action is differed by the permutation $\sigma$.

Now we show how the permutation $\sigma$ affects the $q$-cell structure on $\text{WGr}(d, n)$. Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be a Schubert symbol for $d < n$. Then $\sigma E(\lambda) = \{\sigma Y \mid Y \in E(\lambda)\}$

$$
= \{X \in \text{Gr}(d, n) \mid \dim(X \cap \sigma \mathbb{C}^{\lambda}) = i, \dim(X \cap \sigma \mathbb{C}^{\lambda_{i-1}}) = i-1, i \in [n]\}
$$

where $[n] = \{1, 2, \ldots, n\}$. Then $E(\lambda) \cong \sigma E(\lambda)$ and $\dim(\sigma E(\lambda)) = \ell(\lambda)$.
Note that applying the permutation $\sigma$ on the rows of the matrices in $E(\lambda)$ we get the matrices of $\sigma E(\lambda)$. i.e.,

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in E(\lambda) \iff \begin{pmatrix} v_{\sigma 1} \\ v_{\sigma 2} \\ \vdots \\ v_{\sigma n} \end{pmatrix} \in \sigma E(\lambda).$$

So the permutation of the coordinates in $\mathbb{C}^n$ determines a new cell structure for $\text{Gr}(d, n)$ given by $\text{Gr}(d, n) = \bigsqcup_{i=0}^m \sigma E(\lambda^i)$.

Now this new cell structure of $\text{Gr}(d, n)$ induces the following decomposition of $M_d(n, d)$ which is similar to (3.19).

(4.1) \quad M_d(n, d) = \bigsqcup_{i=0}^m \sigma E(\tilde{E}(\lambda^i)) \quad \text{and} \quad P(M_d(n, d)) = \bigsqcup_{i=0}^m P(\sigma(\tilde{E}(\lambda^i))).

Recall that $\lambda^i = (\lambda^i_1, \ldots, \lambda^i_d)$ is a Schubert symbol and $c_i = \lambda^i_1$, for $i = 0, \ldots, m$. Then $\sigma \lambda^i := (\sigma(\lambda^i_1), \ldots, \sigma(\lambda^i_d))$, where $i_1, \ldots, i_d \in \{1, \ldots, d\}$ such that $\sigma(\lambda^i_1) < \sigma(\lambda^i_2) < \ldots < \sigma(\lambda^i_d)$. Let $\sigma c_i := w_{\sigma \lambda^i}$. Now from the commutativity of the diagram (3.10), we have the following

$$\pi_w(\sigma(\tilde{E}(\lambda^i))) \cong P(l_w(\pi_w(\sigma(\tilde{E}(\lambda^i)))) = \frac{P(\sigma(\tilde{E}(\lambda^i))))}{\text{weighted } \mathbb{C}^* \text{ action}} \cong \frac{\sigma(\tilde{E}(\lambda^i))}{G(\sigma c_i)}.$$

Then we get a $q$-cell structure of the weighted Grassmann orbifold $W\text{Gr}(d, n)$ given by

$$W\text{Gr}(d, n) = \frac{\sigma E(\lambda^0)}{G(\sigma c_0)} \bigsqcup \frac{\sigma E(\lambda^1)}{G(\sigma c_1)} \bigsqcup \cdots \bigsqcup \frac{\sigma E(\lambda^m)}{G(\sigma c_m)}.$$

Also, the usual $q$-cell structure on a weighted Grassmann orbifold as in Lemma 3.5 gives the following $q$-cell structure.

$$W\text{Gr}'(d, n) = \frac{E(\lambda^0)}{G(\sigma c_0)} \bigsqcup \frac{E(\lambda^1)}{G(\sigma c_1)} \bigsqcup \cdots \bigsqcup \frac{E(\lambda^m)}{G(\sigma c_m)}.$$

We remark that a permutation $\sigma \in S_n$ induces a permutation on the set $\{\lambda^0, \lambda^1, \ldots, \lambda^m\}$, but not conversely, in general.

**Definition 4.7.** Let $\lambda^j = (\lambda^j_1, \ldots, \lambda^j_d)$ be a Schubert symbol, $\sigma \in S_n$ and $\sigma \lambda^j = \lambda^{\phi_j}$ for $j = 0, \ldots, m$. Then $\phi$ gives a permutation on the set $\{0, 1, 2, \ldots, m\}$, and we call it the permutation corresponding to $\sigma \in S_n$.

**Lemma 4.8.** Let $p$ be a prime and $c_0, c_1, \ldots, c_m$ be positive integers. Then there exists a permutation $\phi$ on the set $\{0, 1, 2, \ldots, m\}$ such that

$$\text{gcd}\{p, \frac{c_{\phi_j}}{r_j}\} = 1 \quad \forall \quad j \in \{1, 2, \ldots, m\}$$

where $r_m = \text{gcd}\{c_i\}_{i=0}^m$ and $r_j = \text{gcd}\{c_i\}_{i=0}^m - \{c_{\phi_j+1}, c_{\phi_j+2}, \ldots, c_{\phi_m}\}$ for $j \in \{1, 2, \ldots, m-1\}$. 

Proof. The existence of \( \phi_0, \phi_1, \phi_2, \ldots, \phi_m \in \{0, 1, 2, \ldots, m\} \) follows by the
inductive computation below. First calculate \( r_m = \gcd\{c_0, c_1, c_2, \ldots, c_m\} \).
Then \( \gcd\{\frac{c_0}{r_m}, \frac{c_1}{r_m}, \ldots, \frac{c_m}{r_m}\} = 1 \). Thus for a prime \( p \), there exist \( \phi_m \in \{0, 1, 2, \ldots, m\} \) which satisfies \( \gcd\{p, \frac{c_m}{r_m}\} = 1 \).

Next we calculate \( r_{m-1} = \gcd\{\{c_0, c_1, c_2, \ldots, c_m\} - \{c_{\phi_m}\}\} \) and it helps
to define \( \phi_{m-1} \) which satisfies (4.2). Thus inductively we can calculate
\( r_{m-2}, r_1 \) which help to define \( \phi_{m-2}, \ldots, \phi_1 \in \{0, 1, 2, \ldots, m\} \) respectively
which satisfy (4.2). We define \( \phi_0 = \{0, 1, 2, \ldots, m\} - \{\phi_j\}_{j=1}^m \).

\[ \Box \]

Remark 4.9. For a prime \( p \) the permutation \( \phi \) on the set \( \{0, 1, 2, \ldots, m\} \)
satisfying (4.2) may not be unique. Also for a prime \( p \) there may not exist
\( \sigma \in S_n \) which induces the permutation \( \phi \) satisfying (4.2).

Proposition 4.10. [BNSS21 Theorem 1.1] Let \( X \) be a \( q \)-CW complex
with no odd dimensional \( q \)-cells and \( p \) a prime number. If \( \{(Y_i, 0, i)\}_{i=0} \) is a
building sequence such that \( \gcd\{p, |G_i|\} = 1 \) for all \( i \) with \( e^{2\pi i}G_i = Y_i/Y_i-1 \),
then \( H^*(X; \mathbb{Z}) \) has no \( p \)-torsion and \( H^{odd}(X; \mathbb{Z}_p) \) is trivial.

Now we prove the first main theorem of this section.

Theorem 4.11. Let \( p \) be a prime and \( \phi \) be a permutation on \( \{0, \ldots, m\} \)
corresponding to a \( \sigma \in S_n \) such that \( \phi \) satisfies (4.2). Then \( H^*(WGr(d, n); \mathbb{Z}) \)
has no \( p \)-torsion and \( H^{odd}(WGr(d, n); \mathbb{Z}_p) \) is trivial.

Proof. Recall that the weights on \( WGr(d, n) \) is determined by the vectors
\( W = (w_1, \ldots, w_n) \) and \( a \geq 1 \). Suppose \( \sigma \in S_n \) induces the permutation
\( \phi \). Then \( \sigma W = (w_{\sigma_1}, w_{\sigma_2}, \ldots, w_{\sigma_n}) \). By Lemma 4.6 we have the following
\( q \)-cell structure

\[ WGr(d, n) \cong \frac{\sigma E(\lambda^0)}{G(\sigma_0)} \sqcup \frac{\sigma E(\lambda^1)}{G(\sigma_1)} \sqcup \cdots \sqcup \frac{\sigma E(\lambda^m)}{G(\sigma_m)} \]

where \( \sigma c_j = \phi_j = w_{\lambda^\phi_j} \). Let \( b_j = c_{\phi_j} \) for simplicity. Then, without loss of
generality, we can assume that the collection \( \{\frac{E(\lambda^0)}{G(\sigma_0)}, \frac{E(\lambda^1)}{G(\sigma_1)}, \ldots, \frac{E(\lambda^m)}{G(\sigma_m)}\} \)
gives a \( q \)-cell structure of \( WGr(d, n) \) which determines a building sequence
for it under some attaching maps.

Then \( \gcd\{p, \frac{b_i}{r_i}\} = 1 \) for all \( i \in \{1, 2, \ldots, m\} \) where \( r_i = \gcd\{b_0, b_1, \ldots, b_i\} \)
by Lemma 4.8. Note that \( r_i \) divides \( r_{i-1} \), so \( r_i \cdot d_i = r_{i-1} \) for some \( d_i > 0 \)
for \( i = 1, \ldots, m \).

Next we construct a \( q \)-cell structure on \( WGr(d, n) \) possibly with less
singularity on each cells. Let \( X^k = \sqcup_{i=0}^{k} \frac{E(\lambda^i)}{G(\sigma_0)} \subseteq WGr(d, n) \) for \( k = 0, \ldots, m \).
Then \( X^0 = \frac{E(\lambda^0)}{G(\sigma_0)} \subseteq WGr(d, n) \) which is a point. So \( X^0 \) is homeomorphic to
\( \pi_1'(P(Y^0)) := \frac{E(\lambda^0)}{G(\sigma_0)} \) by Theorem 4.4 for \( k = 0 \).
Note $\pi'_b(P(Y^1)) \cong \frac{E(\lambda^0)}{G(b_0)} \bigcup \frac{E(\lambda^1)}{G(b_1)} \bigcup \cdots \bigcup \frac{E(\lambda^k)}{G(b_k)} \subseteq WP^1(b_0/r_1, b_1/r_1)$ by Proposition 4.5.

By Theorem 4.1 we have $\pi'_b(P(Y^1)) \cong \pi'_b(P(Y^1)) \cong \pi'_b(P(Y^1)) = X^1$ for $k = 1$, since $d_2 \cdot \frac{b_0}{r_1} = \frac{b_0}{r_2}$ and $d_2 \cdot \frac{b_1}{r_1} = \frac{b_1}{r_2}$.

Now inductively we have

$$\pi'_b(P(Y^i)) = \frac{E(\lambda^0)}{G(b_0)} \bigcup \frac{E(\lambda^1)}{G(b_1)} \bigcup \cdots \bigcup \frac{E(\lambda^i)}{G(b_i)} \cong X^i$$

by Theorem 4.1, since $d_{r+1} \cdot \frac{b_0}{r_1}, \ldots, d_i \cdot \frac{b_i}{r_1}, \frac{b_i}{r_{i+1}}$.

Therefore at the $m$-th step,

$$\pi'_b(P(Y^m)) = \frac{E(\lambda^0)}{G(b_0)} \bigcup \frac{E(\lambda^1)}{G(b_1)} \bigcup \cdots \bigcup \frac{E(\lambda^m)}{G(b_m)} \cong \pi'_b(P(Y^{m-1})) \bigcup \frac{E(\lambda^m)}{G(b_m)}.$$

Now $\pi'_b(P(Y^m))$ is homeomorphic to

$$X^m = \frac{E(\lambda^0)}{G(b_0)} \bigcup \frac{E(\lambda^1)}{G(b_1)} \bigcup \cdots \bigcup \frac{E(\lambda^m)}{G(b_m)}.$$

Since $X^m = WGr(d, n)$, the collection $\{\frac{E(\lambda^0)}{G(b_0)}, \frac{E(\lambda^1)}{G(b_1)}, \ldots, \frac{E(\lambda^m)}{G(b_m)}\}$ gives another $q$-cell structure of $WGr(d, n)$ which determines a building sequence for it under some attaching maps.

Now by our assumption $\gcd\{n, b_i\} = 1$ for all $i \in \{1, 2, \ldots, m\}$. Therefore, by Proposition 4.1.10 $H^*(WGr(d, n); Z)$ has no $p$-torsion and the group $H^{odd}(WGr(d, n); Z_p)$ is trivial. This completes the proof. \hfill $\square$

Corollary 4.12. \textbf{Kaw73} $H^*(WP(b_0, b_1, \ldots, b_m); Z)$ has no torsion.

Example 4.13. Consider the weighted Grassmann orbifold $WGr(2, 4)$ for $W = (1, 0, 2, 0)$ and $a = 1$. Here $n = 4, d = 2, \binom{n}{2} = 6, m = \binom{d}{2} - 1 = 5$.

So in this case, we have 6 Schubert symbols which are

$\lambda^0 = (1, 2), \lambda^1 = (1, 3), \lambda^2 = (1, 4), \lambda^3 = (2, 3), \lambda^4 = (2, 4)$ and $\lambda^5 = (3, 4)$

in the ordering as in Definition 3.2. Then $c_0 = w_{\lambda^0} = 2, c_1 = w_{\lambda_1} = 4, c_2 = w_{\lambda_2} = 2, c_3 = W_{\lambda^3} = 3, c_4 = W_{\lambda^4} = 1$ and $c_5 = W_{\lambda^5} = 3$ using (3.6).

To show that $H^*(WGr(2, 4); Z)$ has no torsion, it is sufficient to prove that it has no 2-torsion and 3-torsion. For the prime $p = 2$, the $q$-cell structure given by Proposition 3.5 and the corresponding cofibrations may not help us to detect all 2-torsions. However, by Lemma 4.8, there exists a permutation $\phi$ on the set $\{0, 1, 2, \ldots, 5\}$ which satisfies (4.2). In this case, one such permutation is $\phi_5 = 4, \phi_4 = 3, \phi_3 = 4, \phi_2 = 0, \phi_1 = 2, \phi_0 = 1$. 
Now one needs to check if it is induced by a $\sigma \in S_4$. Let us consider the permutation $\sigma \in S_4$ defined by
\[ \sigma_1 = 1, \sigma_2 = 3, \sigma_3 = 4, \sigma_4 = 2. \]
Then one can check $\sigma \lambda^j = \lambda^{\phi_j}$ for all $j \in \{0, 1, 2, 3, 4, 5\}$. So this $\phi$ and $\sigma$ satisfy the hypothesis in Theorem 4.11. Thus $H^*(WGr(2, 4); \mathbb{Z})$ has no 2-torsion.

For the prime $p = 3$, consider the permutation $\phi$ on the set $\{0, 1, 2, \ldots, 5\}$ defined by
\[ \phi_5 = 0, \phi_4 = 2, \phi_3 = 4, \phi_2 = 1, \phi_1 = 3, \phi_0 = 5 \]
which satisfies (4.2). Take the permutation $\sigma \in S_4$ defined by
\[ \sigma_1 = 3, \sigma_2 = 4, \sigma_3 = 2, \sigma_4 = 1. \]
Then one can check $\sigma \lambda^j = \lambda^{\phi_j}$ for all $j \in \{0, 1, 2, 3, 4, 5\}$, and they satisfy the hypothesis in Theorem 4.11. Thus $H^*(WGr(2, 4); \mathbb{Z})$ of this example has no 3-torsion.

**Remark 4.14.** Note that considering the total order given in Definition 3.2 on the Schubert symbols, it may not be possible to find $\phi$ and $\sigma$ satisfying the hypothesis in Theorem 4.11. However, if $\ell(\lambda^i) = \ell(\lambda^j)$ for $i \neq j$ then one can interchange their position to obtain another total order on the Schubert symbols, which may help us to find $\phi$ and $\sigma$ satisfying the hypothesis in Theorem 4.11. The following example gives more explanation of this situation.

**Example 4.15.** Consider the weighted Grassmann orbifold $WGr(2, 4)$ for $W = (1, 1, 3, 4)$ and $a = 2$. One can check that the computation similar to Example 4.13 may fail to apply Theorem 4.11. However, we have different total order given by
\[ \lambda^0 = (1, 2), \lambda^1 = (1, 3), \lambda^2 = (2, 3), \lambda^3 = (1, 4), \lambda^4 = (2, 4) \text{ and } \lambda^5 = (3, 4) \]
which is obtained by flipping $(1, 4)$ and $(2, 3)$ in the total order given in Definition 3.2. In this case, $c_0 = w_{\lambda^0} = 4$, $c_1 = w_{\lambda^1} = 6$, $c_2 = w_{\lambda^2} = 6$, $c_3 = w_{\lambda^3} = 7$, $c_4 = w_{\lambda^4} = 7$ and $c_5 = w_{\lambda^5} = 9$ using (3.6).

For the prime $p = 2$, let $\phi$ be the identity permutation on the set $\{0, 1, 2, \ldots, 5\}$. Then it satisfies (4.2).

Note that in this case
\[ \pi'_{Y^2} := \frac{E(\lambda^0)}{G(2)} \sqcup \frac{E(\lambda^1)}{G(3)} \sqcup \frac{E(\lambda^2)}{G(3)} \cong \frac{E(\lambda^0)}{G(4)} \sqcup \frac{E(\lambda^1)}{G(6)} \sqcup \frac{E(\lambda^2)}{G(6)} = X^2. \]
So $H^*(X^2; \mathbb{Z})$ has no 2-torsion. Now we can add the remaining cell by following their order and apply the successive cofibration to conclude that $H^*(WGr(2, 4); \mathbb{Z})$ has no 2-torsion.

For the prime $p = 3$, we choose the order of the Schubert symbols given by
\[ \lambda^0 = (1, 2), \lambda^1 = (1, 3), \lambda^2 = (1, 4), \lambda^3 = (2, 3), \lambda^4 = (2, 4) \text{ and } \lambda^5 = (3, 4). \]
Then \( c_0 = W_{\lambda^0} = 4, \ c_1 = W_{\lambda^1} = 6, \ c_2 = W_{\lambda^2} = 7, \ c_3 = W_{\lambda^3} = 6, \ c_4 = W_{\lambda^4} = 7 \) and \( c_5 = W_{\lambda^5} = 9. \)

For the prime \( p = 3 \), consider the permutation \( \phi \) on the set \( \{0, 1, 2, \ldots, 5\} \) given by
\[
\phi_5 = 0, \ \phi_4 = 4, \ \phi_3 = 2, \ \phi_2 = 3, \ \phi_1 = 1, \ \phi_0 = 5.
\]
This satisfies \((4.2)\). Now take the permutation \( \sigma \in S_4 \) defined by \( \sigma_1 = 3, \ \sigma_2 = 4, \ \sigma_3 = 1, \ \sigma_4 = 2. \)
Then \( \sigma \lambda^j = \lambda^{\phi_j} \). Thus \( H^*(WGr(2, 4); \mathbb{Z}) \) has no 3-torsion by Theorem 4.11.

For the prime \( p = 7 \), consider the permutation \( \phi \) on the set \( \{0, 1, 2, \ldots, 5\} \) given by
\[
\phi_5 = 2, \ \phi_4 = 0, \ \phi_3 = 4, \ \phi_2 = 1, \ \phi_1 = 5, \ \phi_0 = 3.
\]
This satisfies \((4.2)\). Take the permutation \( \sigma \in S_4 \) defined by \( \sigma_1 = 3, \ \sigma_2 = 2, \ \sigma_3 = 4, \ \sigma_4 = 1. \)
Then \( \sigma \lambda^j = \lambda^{\phi_j} \). Thus \( H^*(WGr(d, n); \mathbb{Z}) \) of this example has no 7-torsion by Theorem 4.11.

Since the only primes which divides the orders of the orbifold singularities of this \( WGr(2, 4) \) are \( 2, 3 \) and \( 7 \). Hence in the integral cohomology of \( WGr(2, 4) \) of this example has no torsion.

The \( q \)-cell structure in Theorem 4.11 leads us to introduce the following definition which generalizes the concept of divisive weighted projective spaces of [HHRW16].

**Definition 4.16.** A weighted Grassmann orbifold \( WGr(d, n) \) is called divisive if there exist \( \sigma \in S_n \) such that \( w_{\sigma \lambda^j} \) divides \( w_{\sigma \lambda^j - 1} \) for all \( j = 1, 2, \ldots, m \) where \( w_{\lambda^j} \) is defined in (3.6).

**Example 4.17.** Consider the weighted Grassmann orbifold \( WGr(2, 4) \) for \( w = (1, 6, 1, 1) \) and \( a = 3. \) We have the ordering on the 6 Schubert symbols given by
\[
\lambda^0 = (1, 2), \ \lambda^1 = (1, 3), \ \lambda^2 = (1, 4), \ \lambda^3 = (2, 3), \ \lambda^4 = (2, 4) \text{ and } \lambda^5 = (3, 4).
\]
Now \( w_{\lambda^0} = 10, w_{\lambda^1} = 5, w_{\lambda^2} = 5, w_{\lambda^3} = 10, w_{\lambda^4} = 10, w_{\lambda^5} = 5. \) Now take the permutation \( \sigma \in S_4 \) such that \( \sigma_1 = 2, \ \sigma_2 = 1, \ \sigma_3 = 3, \ \sigma_4 = 4. \)
Then \( w_{\sigma \lambda^0} = 10, w_{\sigma \lambda^1} = 10, w_{\sigma \lambda^2} = 10, w_{\sigma \lambda^3} = 5, w_{\sigma \lambda^4} = 5, w_{\sigma \lambda^5} = 5. \) Thus \( w_{\sigma \lambda^j} \) divides \( w_{\sigma \lambda^j - 1} \) for all \( j = 1, 2, \ldots, 5. \) So \( WGr(2, 4) \) of this example is divisive.

**Example 4.18.** Let \( \alpha \) and \( k \) be any two non-negative integers and \( \beta \) be any positive integer such that \( \beta > d\alpha. \) Consider \( W = (\alpha + k\beta, \alpha, \alpha, \alpha) \in \)
(\mathbb{Z}_{>0})^4$ and $a = \beta - d\alpha > 0$. Let $\text{WGr}(2, 4)$ be the corresponding weighted Grassmann orbifold. Then
\[
w_{\lambda'} = \begin{cases} (k + 1)\beta & \text{if } i = 0, 1, 2 \\ \beta & \text{if } i = 3, 4, 5. \end{cases}
\]
Then $w_{\lambda'}$ divides $w_{\lambda'-1}$ for all $i = 1, 2, 3, 4$ and 5. Thus the above WGr$(2, 4)$ is a divisive weighted Grassmann orbifold.

**Definition 4.19.** Given a Schubert symbol $\lambda$ for $d < n$, a reversal of $\lambda$ is a pair $(k, k')$ such that $k \in \lambda$, $k' \not\in \lambda$ and $k < k'$. Denote $\text{rev}(\lambda)$ be the set of all reversals of $\lambda$. If $(k, k') \in \text{rev}(\lambda)$ then $(k', k)\lambda$ is the Schubert symbol obtained by replacing $k$ by $k'$ in $\lambda$ in the unique position which makes the $d$ elements of $(k', k)\lambda$ a strictly increasing subset of $\{1, 2, \ldots, n\}$.

**Remark 4.20.** Note that if $(k, k') \in \text{rev}(\lambda)$ then $(k', k)\lambda < \lambda$ and $\ell(\lambda)$ is the cardinality of the set $\text{rev}(\lambda)$. In [KT03, AM15] the authors defined an inversion $(k, k')$ of a Schubert symbol $\lambda$ is a pair $k \in \lambda$ and $k' \not\in \lambda$ such that $k < k'$. We note that our definition of reversal is dual to the definition of inversion. If $\text{inv}(\lambda)$ be the set of all inversions of $\lambda$ and $\ell(\lambda)$ be the cardinality of the set $\text{inv}(\lambda)$ then $\ell(\lambda) + \ell(\lambda) = d(n - d)$. Also If $(k, k') \in \text{inv}(\lambda)$ and $(k', k)\lambda = \lambda'$ then $(k', k) \in \text{inv}(\lambda')$ and and $(k', k)\lambda' = \lambda$.

Now we prove the second main theorem of this section. Recall the $(\mathbb{C}^*)^n$-action on WGr$(d, n)$ which is defined after Equation (3.11).

**Theorem 4.21.** If WGr$(d, n)$ is a divisive weighted Grassmann orbifold then it has a $(\mathbb{C}^*)^n$ invariant CW-structure with only even dimensional cells.

**Proof.** Let WGr$(d, n)$ be a divisive weighted Grassmann orbifold with weights $\{w_{\lambda}\}_{i=0}^{m}$ corresponding to the vector $w = (w_1, \ldots, w_n)$ and $a > 0$. Then there exist $\sigma \in S_\lambda$ such that $w_{\sigma\lambda}$ divides $w_{\sigma\lambda'-1}$ for all $j = 1, 2, \ldots, m$. Let us assume $\sigma = \text{Id}$. Then $c_i$ divides $c_{i-1}$ for all $i = 1, 2, \ldots, m$, where $c_i = w_{\lambda'}$ is defined in (3.16). We adhere the notation from Section 3.

Then $\gcd\{c_0, c_1, \ldots, c_\lambda\} = c_i$ for all $i \in \{1, 2, \ldots, m\}$. Thus $\pi_w(E(\lambda')) \cong \frac{E(\lambda')}{G(c_i)} \cong E(\lambda')$ by Theorem 4.4. Thus each element of $\pi_w(E(\lambda'))$ can be represented uniquely by the equivalence class of $n \times d$ matrix $A_\lambda$. Which is defined in (3.18).

Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be a Schubert symbol for $d < n$ and $x \in \mathbb{C}^{\ell(\lambda)}$. Since $\ell(\lambda) = (\lambda_1 - 1) + (\lambda_2 - 2) + \cdots + (\lambda_d - d)$, we can write
\[
x = (x_1, x_2, \ldots, x_d)
\]
where $x_j = (x_1^j, x_2^j, \ldots, \widehat{x_{\lambda_1}}, \ldots, \widehat{x_{\lambda_2}}, \ldots, \widehat{x_{\lambda_{j-1}}}, \ldots, x_{\lambda_j})$ for $j = 1, \ldots, d$.

For $t = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$ define $s \in \mathbb{C}^*$ such that $s^{w_\lambda} = t_{\lambda_1} \cdots t_{\lambda_d}$. Consider $T \in \text{GL}(d, \mathbb{C})$ defined by
\[
T = \text{diag}(\frac{t_{\lambda_1}}{s^{w_{\lambda_1}}}, \frac{t_{\lambda_2}}{s^{w_{\lambda_2}}}, \ldots, \frac{t_{\lambda_d}}{s^{w_{\lambda_d}}}).
\]
Then $\det(T) = s^a$.

Define $g_\lambda: \mathcal{C}^{\ell(\lambda)} \to \pi_{\omega}(\widetilde{E}(\lambda))$ by $g_\lambda(x) = A_\lambda(x).$ where

$$
A_\lambda(x) := \begin{bmatrix}
x_1^1 & x_1^2 & \cdots & x_1^d \\
\vdots & \vdots & \ddots & \vdots \\
x_{\lambda_1-1}^1 & x_{\lambda_1-1}^2 & \cdots & x_{\lambda_1-1}^d \\
1 & 0 & \cdots & 0 \\
0 & x_{\lambda_1+1}^2 & \cdots & x_{\lambda_1+1}^d \\
\vdots & \vdots & \ddots & \vdots \\
0 & x_{\lambda_2-1}^2 & \cdots & x_{\lambda_2-1}^d \\
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & x_{\lambda_2+1}^d \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & x_{\lambda_d-1}^d \\
0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
$$

Then $g_\lambda$ is a homeomorphism. Now we have

$$
(t_1, t_2, \ldots, t_n)A_\lambda(x) = \begin{bmatrix}
t_1x_1^1 & t_1x_1^2 & \cdots & t_1x_1^d \\
\vdots & \vdots & \ddots & \vdots \\
t_{\lambda_1-1}x_{\lambda_1-1}^1 & t_{\lambda_1-1}x_{\lambda_1-1}^2 & \cdots & t_{\lambda_1-1}x_{\lambda_1-1}^d \\
t_{\lambda_1} & 0 & \cdots & 0 \\
0 & t_{\lambda_1+1}x_{\lambda_1+1}^2 & \cdots & t_{\lambda_1+1}x_{\lambda_1+1}^d \\
\vdots & \vdots & \ddots & \vdots \\
0 & t_{\lambda_2-1}x_{\lambda_2-1}^2 & \cdots & t_{\lambda_2-1}x_{\lambda_2-1}^d \\
0 & t_{\lambda_2} & \cdots & 0 \\
0 & 0 & \cdots & t_{\lambda_2+1}x_{\lambda_2+1}^d \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & t_{\lambda_d-1}x_{\lambda_d-1}^d \\
0 & 0 & \cdots & t_{\lambda_d} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix}
$$

Then
\[(t_1, t_2, \ldots, t_n) A_{\lambda}(x) = \left( \begin{array}{cccc}
\frac{s^{w_{\lambda_1}}}{t_{\lambda_1}} t_1 x_1 & \frac{s^{w_{\lambda_2}}}{t_{\lambda_2}} t_1 x_1^2 & \cdots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d}} t_1 x_1^d \\
\vdots & \vdots & \ddots & \vdots \\
\frac{s^{w_{\lambda_1}}}{t_{\lambda_1}} t_{\lambda_1-1} x_{\lambda_1-1} & \frac{s^{w_{\lambda_2}}}{t_{\lambda_2}} t_{\lambda_1-1} x_{\lambda_1-1}^2 & \cdots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d}} t_{\lambda_1-1} x_{\lambda_1-1}^d \\
0 & 0 & \cdots & 0
\end{array} \right) \times T.\]
where $D = \text{diag}(s^{w_1}, \ldots, s^{w_n})$ is a diagonal matrix. So by the equivalence relation $\sim_w$ as in Definition 3.1, we get 
\[(t_1, t_2, \ldots, t_n) A_\lambda(x) = D \times T\]

\[= DMT.\]

where $D = \text{diag}(s^{w_1}, \ldots, s^{w_n})$ is a diagonal matrix. So by the equivalence relation $\sim_w$ as in Definition 3.1, we get 
\[(t_1, t_2, \ldots, t_n) g_\lambda(x) = M \in \pi_w(\overline{E}(\lambda)) \subset \text{WGr}(d, n).\]

Let $a_{ij}$ be the coefficient of $x_i^j$ in the matrix $M$ for $1 \leq j \leq d$, $1 \leq i \leq \lambda_j - 1$, $i \neq \lambda_1, \lambda_2, \ldots, \lambda_{j-1}$. Then 
\[a_{ij} = \frac{t_i s^{w_{\lambda_j}}}{s^{w_i} t_{\lambda_j}}.\]

Now for $1 \leq i \leq \lambda_j - 1$, $i \neq \lambda_1, \lambda_2, \ldots, \lambda_{j-1}$ we have $(\lambda_j, i) \in \text{rev}(\lambda)$ and let $\lambda' = (\lambda_j, i) \lambda$. Recall $w_\lambda$ from (3.6). So 
\[\frac{t_i s^{w_{\lambda_j}}}{s^{w_i} t_{\lambda_j}} = \frac{t_{\lambda'} s^{w_{\lambda_j}}}{t_{\lambda} s^{w_i}} = \frac{t_{\lambda'} s^{w_{\lambda_j} - w_i}}{t_{\lambda} s^{w_i}} = \frac{t_{\lambda'} s^{w_{\lambda_j} - w_i}}{t_{\lambda} s^{w_i}} = t_{\lambda'}(t_\lambda)^{-w_\lambda},\]

since $s^{w_\lambda} = t_{\lambda_1} \cdots t_{\lambda_d} = t_\lambda$ and $t_{\lambda'} = t_{\lambda_1} \cdots t_{\lambda_{j-1}} t_i t_{\lambda_j} t_{\lambda_{j+1}} \cdots t_{\lambda_d}$. Note that 
\[\det(M_{\lambda'}) = \pm a_{ij} x_i^j\]

where $M_{\lambda'}$ is the minor corresponding to the Schubert symbol $\lambda'$ defined just before (3.1). 

Then $\lambda' < \lambda$. As WGr$(d, n)$ is divisive we have $w_\lambda$ divides $w_{\lambda'}$. 
Define a \((\mathbb{C}^*)^n\)-action on \(\mathcal{C}^\ell(\lambda)\) by
\[
(t_1, t_2, \ldots, t_n)(x^j_i) = (t_{\lambda'}(t_{\lambda})^{-\frac{w_{\lambda'}}{w_{\lambda}} x^j_i})
\]
for \(1 \leq j \leq d; 1 \leq i \leq \lambda_j - 1; i \neq \lambda_1, \lambda_2, \ldots, \lambda_{j-1}\). With this action of \((\mathbb{C}^*)^n\) on \(\mathcal{C}^\ell(\lambda)\), the map \(g_\lambda\) becomes \((\mathbb{C}^*)^n\)-equivariant.

If \(\sigma \neq \text{Id}\), then consider the q-cell \(\pi_w(\sigma \tilde{E}(\lambda')) \cong \sigma \tilde{E}(\lambda') \cong \sigma \tilde{E}(\lambda')\). Define \(\sigma g_\lambda: \mathcal{C}^\ell(\lambda) \to \pi_w(\sigma \tilde{E}(\lambda))\) by \(\sigma g_\lambda(x) = \sigma A_\lambda(x)\). Then by similar way as we done above, we get the \((\mathbb{C}^*)^n\)-action on \(\mathcal{C}^\ell(\lambda)\) by
\[
(t_1, t_2, \ldots, t_n)(x^j_i) = (t_{\sigma \lambda'}(t_{\sigma \lambda})^{-\frac{w_{\lambda'}}{w_{\lambda}} x^j_i}).
\]

\[\square\]

**Corollary 4.22.** If \(\text{WGr}(d, n)\) is divisive then \(H^*(\text{WGr}(d, n); \mathbb{Z})\) has no torsion and concentrated in even degrees.

We remark that Corollary 4.22 also follows from the proof of Theorem 4.11 and Definition 4.16. However, Theorem 4.21 says the representation of the \((\mathbb{C}^*)^n\)-action on each invariant cell explicitly. Also a divisive weighted Grassmann orbifold is integrally equivariantly formal.

5. **GKM theory for weighted Grassmann orbifolds**

Abe and Matsumara [AM15] computed the equivariant cohomology ring of weighted Grassmann orbifolds with rational coefficients. In this section, we describe the equivariant cohomology ring for divisive weighted Grassmann orbifolds with integer coefficients.

Recall the \((\mathbb{C}^*)^n\)-action on \(\text{WGr}(d, n)\) and \(\mathbb{WP}(c_0, c_1, \ldots, c_m)\) which is defined after Lemma 3.3. Consider the standard torus \(T^n = (S^1)^n \subset T^n \subset (\mathbb{C}^*)^n\). So we have the restricted \(T^n\)-action on \(\text{WGr}(d, n)\) and \(\mathbb{WP}(c_0, c_1, \ldots, c_m)\).

**Proposition 5.1.** The orbifolds \(\text{WGr}(d, n)\) and \(\mathbb{WP}(c_0, c_1, \ldots, c_m)\) are GKM orbifolds with the above \(T^n\)-actions.

**Proof.** For each Schubert symbol \(\lambda'\) we have a fixed point of the \(T^n\)-action on \(\mathbb{WP}(c_0, c_1, \ldots, c_m)\) which is \([e_{\lambda_i}]\) for \(i = 0, \ldots, m\). Thus the set of all fixed points is \([e_{\lambda_i}]\). To check this, let \((t_1, t_2, \ldots, t_n) \in T^n\). Then there exist \(s \in \mathbb{C}^*\) such that \(s^w = t_1\) then \((t_1, t_2, \ldots, t_n)e_{\lambda} = [t_{\lambda}e_{\lambda}] = [s^w e_{\lambda}] = [e_{\lambda}]\).

The number of all fixed points of the \(T^n\)-action on \(\text{WGr}(d, n)\) is same as the number of all fixed points of the ambient weighted projective space \(\mathbb{WP}(c_0, c_1, \ldots, c_m)\). For each Schubert symbol \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)\), let \(C(\lambda) \in M_d(n, d)\) with column vectors given by \([e_{\lambda_1}, e_{\lambda_2}, \ldots, e_{\lambda_d}]\) where \([e_1, e_2, \ldots, e_n]\) is the standard basis for \(\mathbb{C}^n\). Then \([C(\lambda)] \in \text{WGr}(d, n)\). For \((t_1, t_2, \ldots, t_n) \in T^n\) there exists \(s \in \mathbb{C}^*\) such that \(s^w = t_1\). Then
\[
(t_1, t_2, \ldots, t_n)[C(\lambda)] = [(t_{\lambda_1}e_{\lambda_1}, t_{\lambda_2}e_{\lambda_2}, \ldots, t_{\lambda_d}e_{\lambda_d})]
\]
\[
= [(s^w e_{\lambda_1}, s^w e_{\lambda_2}, \ldots, s^w e_{\lambda_d}) \times T]
\]
\[
= [(e_{\lambda_1}, e_{\lambda_2}, \ldots, e_{\lambda_d})] = [C(\lambda)]
\]
where $T \in \text{GL}(d, \mathbb{C})$ defined by $T = \text{diag}((\frac{t_{\lambda_1}}{s_{w_{\lambda_1}}}), (\frac{t_{\lambda_2}}{s_{w_{\lambda_2}}}), \ldots, (\frac{t_{\lambda_d}}{s_{w_{\lambda_d}}}))$. Then $\det(T) = s^a$. Thus $[C(\lambda)]$ is a fixed point of the $T^n$-action on $\mathbb{WGr}(d, n)$. Note that $P_{lw}([C(\lambda)]) = [e_\lambda]$ for each Schubert symbol $\lambda$.

For any two Schubert symbols $\lambda^i$ and $\lambda^j$, with $i \neq j$ the set $[z_ie_{\lambda^i} + z_je_{\lambda^j}] \subset \mathbb{WPP}(c_0, c_1, \ldots, c_m)$ (where $z_i, z_j \in \mathbb{C}$ and at least one of them is non-zero) is a $T^n$-invariant ‘spindle’ in the weighted projective space containing two fixed points $[e_{\lambda^i}]$ and $[e_{\lambda^j}]$. No other invariant subspace of $\mathbb{WPP}(c)$ is spindle. So this action satisfies Definition 2.1. Let $k, \ell \in \{1, 2, \ldots, m\}$ symbols such that $(\lambda^i, k, \ell) \in \text{rev}(\lambda^i)$ or, $(k, \ell) \in \text{inv}(\lambda^i)$. Let $\lambda^i - \{k\} := \{\lambda_1, \lambda_2, \ldots, \lambda_{d-1}\}$. Consider the matrices

$$C(ij) := (e_{\lambda_1}, e_{\lambda_2}, \ldots, e_{\lambda_{d-1}}, z_ie_k + z_je_\ell) \subset M_d(n, d)$$

where the vectors are columns and at least one of $z_i$ and $z_j$ is non-zero. Then $[C(ij)]$ is a $T^n$-invariant spindle in $\mathbb{WGr}(d, n)$ containing the fixed points $[C(\lambda^i)]$ and $[C(\lambda^j)]$. Recall that $P_{lw}([C(ij)]) = [z_ie_{\lambda^i} + z_je_{\lambda^j}]$. Therefore both the orbifolds satisfy the condition (2) in Definition 2.1.

Now using (3.7) we can see $\mathbb{WGr}(d, n)$ as a suborbifold of $\mathbb{WPP}(c_0, c_1, \ldots, c_m)$ and the action of $T^n$ on $\mathbb{WGr}(d, n)$ is the restriction of action of $T^n$ on $\mathbb{WPP}(c_0, c_1, \ldots, c_m)$. So at a fixed point $[C(\lambda^i)] \in \mathbb{WGr}(d, n)$, the tangent space $T_{[C(\lambda^i)]} \mathbb{WGr}(d, n)$ is a subspace of $T_{[e_{\lambda^i}]} \mathbb{WPP}(c_0, c_1, \ldots, c_m)$ and the isotropy representation of $T^n$ on $T_{[C(\lambda^i)]} \mathbb{WGr}(d, n)$ is the sub representation of the isotropy representation of $T^n$ on $T_{[e_{\lambda^i}]} \mathbb{WPP}(c_0, c_1, \ldots, c_m)$. This representation satisfies the pairwise linear independent condition for GKM orbifold. Since the action of the finite group $G(c_i)$ on this tangent space is linear, the orbifold tangent space at $[C(\lambda^i)] \in \mathbb{WGr}(d, n)$ satisfies condition (3) in Definition 2.1. In particular this is true when $d = 1$.

We remark that $\mathbb{WGr}(d, n)$ and $\mathbb{WPP}(c_0, c_1, \ldots, c_m) \cong \mathbb{WGr}(1, m)$ have $q$-CW structures with even dimensional $q$-cells, see Proposition 3.5. Using this, one can show that they are equivariantly formal with rational coefficients.

Next we describe the GKM graphs and the corresponding weights on each edges of these graphs for $\mathbb{WPP}(c_0, c_1, \ldots, c_m)$ and $\mathbb{WGr}(d, n)$ respectively.

Let $V$ consists the fixed points $\{e_{\lambda^i}\}_{i=0}^m$ of the $T^n$-action on $\mathbb{WPP}(c_0, c_1, \ldots, c_m)$ where $m = \frac{n}{d} - 1$. We have shown in the proof of Proposition 5.1 that there is an equivariant spindle containing any two fixed points $[e_{\lambda^i}]$ and $[e_{\lambda^j}]$. We define an edge between any two fixed points $[e_{\lambda^i}]$ and $[e_{\lambda^j}]$ for $i \neq j$. We denote these edges by $E$. Thus we get a graph $(V, E)$. This is the complete graph on $m + 1$ vertices, and we can think it as the one skeleton of an $m$-simplex.

Now we calculate the axial function for the GKM graph of $\mathbb{WPP}(c_0, c_1, \ldots, c_m)$. We fix $j \in \{0, \ldots, m\}$. The set

$$\{\sum_{i=0}^{m} a_i e_{\lambda^i} \in \mathbb{WPP}(c_0, c_1, \ldots, c_m) \mid a_j = 1\}$$
is an invariant open neighborhood of the fixed point \([e_\lambda]\). Then,

\[
(t_1, t_2, \ldots, t_n)[(a_0, a_1, \ldots, \underbrace{1}_{\text{j-th place}}, \ldots, a_m)]
\]

\[
= [(t_\lambda a_0, t_\lambda a_1, \ldots, \underbrace{t_\lambda}_{\text{j-th place}} a_1, \ldots, t_\lambda a_m)]
\]

\[
= [((t_\lambda) c_0 t_\lambda a_0, (t_\lambda) c_1 t_\lambda a_1, \ldots, \underbrace{1}_{\text{j-th place}}, \ldots, (t_\lambda) c_m t_\lambda a_m)]
\]

where \([\cdots]\) represents an equivalence class in \(\mathbb{W}P(c_0, c_1, \ldots, c_m)\). The corresponding weight on the edge joining \((e_\lambda)\) and \((e_\lambda)\) going outside from \((e_\lambda)\) is determined by the character

\[
(5.1) \quad (t_1, \ldots, t_n) \rightarrow (t_\lambda) c_j t_\lambda i.
\]

Thus the weight on the edge \([e_\lambda, e_\lambda]\) is given by \(-\frac{c_0}{c_j}(Y_\lambda) + Y_\lambda\), where

\[
Y_\lambda := \sum_{i=1}^{d} y_{\lambda i}
\]

for a Schubert symbol \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)\). So the above calculation gives the GKM-graph of \(\mathbb{W}P(c_0, c_1, \ldots, c_m)\) where the weights are in rational coefficients.

Similarly, one can describe the GKM graph of \(WGr(d, n)\). However, for the purpose of this paper we give the details. Note that the fixed points of the \(T^n\)-action on \(WGr(d, n)\) corresponds bijectively to the Schubert symbols. We also denote these fixed points by \(V\). Next we find the complex one-dimensional orbits of this \(T^n\)-action. We have shown in the proof of Proposition 5.1 that there is an equivariant spindle containing any two fixed points \([C(\lambda^i)]\) and \([C(\lambda^j)]\) if \((k, \ell)\lambda^i = \lambda^j\) for either \((k, \ell) \in \text{rev}(\lambda^i)\) or, \((k, \ell) \in \text{inv}(\lambda^i)\).

Thus there is an edge between two points in \(V\) if the corresponding Schubert symbols are differed by exactly one coordinate in \(WGr(d, n)\). We denote these edges by \(E\) also. Since the dimension of \(WGr(d, n)\) is \(d(n - d)\), hence \((V, E)\) is a \((d(n - d))\) valent graph on \(m + 1\) vertices in this case, and it is a subgraph of the GKM graph of \(\mathbb{W}P(c_0, c_1, \ldots, c_m)\).

Similarly to the \(T^n\)-action on the GKM orbifold \(\mathbb{W}P(c_0, c_1, \ldots, c_m)\), the corresponding weight on the edge joining \([C(\lambda^i)]\) and \([C(\lambda^j)]\) outgoing from \([C(\lambda^j)]\) is \(-\frac{c_0}{c_j}(Y_\lambda) + Y_\lambda\) if \(\lambda^i\) and \(\lambda^j\) are differed by exactly one coordinate.

Thus the axial function corresponding to the graph of \(WGr(d, n)\) is the restriction of the axial function corresponding to the graph of \(\mathbb{W}P(c_0, c_1, \ldots, c_m)\). This describes the GKM graph of \(WGr(d, n)\).

Note that this is similar as the GKM graph of \(Gr(d, n)\) (weights are different) and also it is a subgraph of the GKM graph corresponding to \(\mathbb{W}P(c_0, c_1, \ldots, c_m)\).

In particular if \((w_1, \ldots, w_n) = (0, \ldots, 0)\) and \(a = 1\), then \((t_\lambda)^{-1}t_\lambda = t_k/t_\ell\). Thus the corresponding weight on the edge joining \((e_\lambda)\) and \((e_\lambda)\) outgoing from \((e_\lambda)\) is \((Y_\lambda - Y_\lambda) = y_k - y_\ell\).
**Remark 5.2.** Let $\lambda^i$ and $\lambda^j$ be two Schubert symbol with $i < j$. If \( W_{\text{Gr}}(d, n) \) be a divisive weighted Grassmann orbifold then \( W_{\lambda^i} \) divides \( W_{\lambda^j} \). Then \( d_{ij} := \frac{W_{\lambda^i}}{W_{\lambda^j}} \in \mathbb{Z} \) and the corresponding axial function joining \([C(\lambda^i)]\) and \([C(\lambda^j)]\) outgoing from \([C(\lambda^j)]\) is \((Y_{\lambda^i} - d_{ij} Y_{\lambda^j})\) if \(|\lambda^j \cap \lambda^i| = d - 1\).

**Theorem 5.3.** The integral equivariant cohomology ring of a divisive weighted Grassmann orbifold \( W_{\text{Gr}}(d, n) \) corresponding to the order \( \lambda^0 < \cdots < \lambda^m \) is given by

\[
H^*_T(W_{\text{Gr}}(d, n); \mathbb{Z}) = \left\{ (x_j) \in \bigoplus_{j=0}^m \mathbb{Z}[y_1, y_2, \ldots, y_n] \mid (Y_{\lambda^i} - d_{ij} Y_{\lambda^j}) \text{ divides } (x_j - x_i) \right\}.
\]

if \(|\lambda^j \cap \lambda^i| = d - 1, i < j\).

**Proof.** This follows from Theorem 2.26 and Corollary 4.22 and Remark 5.2. \(\square\)

We have assumed that \( \sigma = \text{Id} \) in Remark 5.2 and Theorem 5.3. In general if \( W_{\text{Gr}}(d, n) \) is divisive for an arbitrary \( \sigma \in S_n \) as in Definition 4.16 then we can calculate \( \sigma d_{ij} := \frac{W_{\sigma \lambda^i}}{W_{\sigma \lambda^j}} \in \mathbb{Z} \) and the corresponding axial function joining \([C(\sigma \lambda^i)]\) and \([C(\sigma \lambda^j)]\) outgoing from \([C(\sigma \lambda^j)]\) is \((Y_{\sigma \lambda^i} - \sigma d_{ij} Y_{\sigma \lambda^j})\) if \(|\lambda^j \cap \lambda^i| = d - 1\).

**Theorem 5.4.** The integral equivariant cohomology ring of a divisive weighted Grassmann orbifold \( W_{\text{Gr}}(d, n) \) corresponding to the order \( \lambda_0 < \cdots < \lambda^m \) and \( \sigma \in S_n \) is given by

\[
H^*_T(W_{\text{Gr}}(d, n); \mathbb{Z}) = \left\{ (x_j) \in \bigoplus_{j=0}^m \mathbb{Z}[y_1, y_2, \ldots, y_n] \mid (Y_{\sigma \lambda^i} - \sigma d_{ij} Y_{\sigma \lambda^j}) \text{ divides } (x_j - x_i) \right\}.
\]

if \(|\lambda^j \cap \lambda^i| = d - 1, i < j\).

**Remark 5.5.** We can consider the weighted Grassmann orbifolds \( W_{\text{Gr}}'(d, n) \) corresponding to \((\sigma W, a)\), where \( \sigma W = (w_{\sigma(1)}, w_{\sigma(2)}, \ldots, w_{\sigma(n)}) \in (\mathbb{Z}_{\geq 0})^n \). Then \( W_{\text{Gr}}(d, n) \) is weakly equivariantly homeomorphic to \( W_{\text{Gr}}'(d, n) \) (with the permuted \( T_n \)-action for the permutation \( \sigma \)). Then \( W_{\text{Gr}}'(d, n) \) is divisive corresponding to the natural ordering on the Schubert symbols. Then we can apply Theorem 5.3 to calculate \( H^*_T(W_{\text{Gr}}'(d, n); \mathbb{Z}) \) which is isomorphic to \( H^*_T(W_{\text{Gr}}(d, n); \mathbb{Z}) \).
6. Schubert Calculus for divisive weighted Grassmann orbifolds

In this section, we show that there exist equivariant Schubert classes which form a basis for the cohomology ring of divisive weighted Grassmann orbifold with integer coefficients. Moreover, we compute the structure constants corresponding to this Schubert basis with integer coefficients.

For \( x \in H^*_T(WGr(d, n); \mathbb{Z}) \) the support of \( 'x' \) denoted by \( \text{supp}(x) \) is the set of all Schubert symbols \( \lambda \) such that \( x|_\lambda \neq 0 \). Now the Schubert symbols form a lattice with respect to a partial order ‘\( \preceq \)’ on the Schubert symbols defined by \( \lambda \preceq \mu \) if \( \lambda_i \leq \mu_i \) for all \( i = 1, 2, \ldots, d \). Note that the total order ‘\( < \)’ in Definition 3.2 preserve this partial order ‘\( \preceq \)’. That is for two Schubert symbol \( \lambda \) and \( \mu \), \( \lambda \preceq \mu \implies \lambda \leq \mu \), but not conversely. In this section we follow this partial order and we call an element \( x \in H^*_T(WGr(d, n); \mathbb{Z}) \) is supported above by \( \lambda \) if \( \lambda \preceq \lambda' \) for all \( \lambda' \in \text{supp}(x) \).

Let \( WGr(d, n) \) be a divisive weighted Grassmann orbifold. Then there exist \( \sigma \in S_n \) such that

\[
(6.1) \quad w_{\sigma \lambda'} \text{ divides } w_{\sigma \lambda} \text{ for } i=1,2,\ldots,m.
\]

For any Schubert symbol \( \lambda \), suppose the Schubert cycle \( X_{\lambda} \) is defined to be the closure of the Schubert cell \( E(\lambda) \). The only fixed points contained in \( X_{\lambda} \) are \( \{C(\mu)\} \in WGr(d, n) \), such that \( \mu \preceq \lambda \). Let \( X_{\lambda} \) induces the equivariant cohomology class \( [X_{\lambda}] \in H^*_T(WGr(d, n); \mathbb{Z}) \). Then \( [X_{\lambda}]|_\mu \neq 0 \implies \mu \preceq \lambda \). At the fixed point \( [C(\lambda)] \) the weights of the \( T^n \)-action on the normal bundle of \( X_{\lambda} \) are

\[
\{Y_{(k,k')_{\lambda}} - \frac{w_{(k,k')_{\lambda}}}{w_{\lambda}} Y_{\lambda} \mid (k, k') \in inv(\lambda)\}.
\]

So we have \( [X_{\lambda}]|_{\lambda} = \prod_{(k,k')\in inv(\lambda)} \left(Y_{(k,k')_{\lambda}} - \frac{w_{(k,k')_{\lambda}}}{w_{\lambda}} Y_{\lambda}\right) \), which is of degree \( \ell' (\lambda) \).

Now \( Y_{(k,k')_{\lambda}} - \frac{w_{(k,k')_{\lambda}}}{w_{\lambda}} Y_{\lambda} \in \mathbb{Z}[y_1, y_2, \ldots, y_n] \) for all \( (k, k') \in inv(\lambda) \) and for all Schubert symbol \( \lambda \in \{\lambda^0, \lambda^1, \ldots, \lambda^m\} \) if

\[
(6.2) \quad w_{\lambda_{i}} \text{ divides } w_{\lambda_{i+1}} \text{ for } i = 0, 1, \ldots, m - 1.
\]

Note that applying the permutation \( \sigma_r \) on the set \( \{\lambda^0, \lambda^1, \ldots, \lambda^m\} \) we get \( \sigma_r \lambda^i = \lambda^{m-i} \) for all \( i = 0, 1, \ldots, m \), where

\[
\sigma_r := \begin{pmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ n & n-1 & n-2 & \cdots & 2 & 1 \end{pmatrix}.
\]

So condition in (6.2) is true if we take \( \sigma = \sigma_r \) in (6.1).

Now for \( \sigma = \text{Id} \) in (6.1) we have

\[
w_{\lambda_{i}} \text{ divides } w_{\lambda_{i-1}} \text{ for } i = 1, 2, \ldots, m.
\]

which is same as \( w_{\sigma_r \lambda} \) divides \( w_{\sigma_r \lambda+1} \) for \( i = 0, 1, \ldots, m - 1 \).

In this case for every Schubert symbol \( \lambda \) let the Schubert cycle \( \sigma_r X_{\lambda} \) is defined to be the closure of \( \sigma_r E(\lambda) \). Then \( \sigma_r X_{\lambda} \) contains the fixed points
\[
\{[C(\sigma, \mu)] \in WGr(d, n) \mid \mu \leq \lambda\} = \{[C(\mu)] \in WGr(d, n) \mid \mu \geq \sigma, \lambda\}. \]

Now it induces equivariant cohomology class \([\sigma, X_{\lambda}] \in H^*_T(WGr(d, n); \mathbb{Z})\). Then
\[
[\sigma, X_{\lambda}]|_{\mu} \neq 0 \implies \mu \geq \sigma, \lambda.
\]

At the fixed point \([C(\sigma, \lambda)]\) the weights of the \(T^n\)-action on the normal bundle of \(\sigma, X_{\lambda}\) are \(\{Y_{\sigma, \mu} - \frac{w_{\lambda}}{w_{\sigma, \lambda}} Y_{\sigma, \lambda} \mid \mu = (k, k') \lambda\) for \((k, k') \in \text{inv}(\lambda)\). Now \((k, k') \in \text{inv}(\lambda)\) implies that \((n - k, n - k') \sigma, \lambda\). So we have
\[
[\sigma, X_{\lambda}]|_{\sigma, \lambda} = \prod_{(k, k') \in \text{rev}(\sigma, \lambda)} (Y_{(k, k'), \sigma, \lambda} - \frac{w_{(k, k'), \sigma, \lambda}}{w_{\sigma, \lambda}} Y_{\sigma, \lambda})
\]
which is of degree \(\ell(\sigma, \lambda) = \ell(\lambda)\).

Although we get the above expression corresponding to the Schubert cell \(\sigma, E(\lambda)\) but it may not contain \([C(\lambda)]\). Note that \([C(\lambda)] \in \sigma, E(\sigma, \lambda)\). So we denote the Schubert basis corresponding to \(\lambda\) is the equivariant cohomology class corresponding to the closure of \(\sigma, E(\sigma, \lambda)\). Which leads us to define the following.

**Definition 6.1.** An element \(\alpha \in H^*_T(WGr(d, n); \mathbb{Z})\) is said to be an equivariant Schubert class corresponding to a Schubert symbol \(\lambda\) if the following conditions satisfied
\begin{enumerate}
\item \(\alpha|_{\mu} \neq 0 \implies \mu \geq \lambda\) (say that \(\alpha\) is supported above \(\lambda\)).
\item \(\alpha|_{\lambda} = \prod_{(k, k') \in \text{rev}(\lambda)} (Y_{(k, k'), \lambda} - \frac{w_{(k, k'), \lambda}}{w_{\lambda}} Y_{\lambda})\)
\item \(\alpha|_{\mu}\) is a homogeneous polynomial of \(y_1, y_2, \ldots, y_n\) of degree \(\ell(\lambda)\)
\end{enumerate}

**Proposition 6.2 (Uniqueness).** For each Schubert symbol \(\lambda\) there is at most one equivariant Schubert class \(\alpha\) corresponding to \(\lambda\).

**Proof.** Suppose for contradiction that there were two distinct equivariant Schubert symbols \(\alpha, \alpha'\) corresponding to \(\lambda\). Let \(\mu\) be the minimal Schubert symbol such that \(\alpha - \alpha' \neq 0\). By Definition 6.1 (1) and (2), we get \(\lambda < \mu\). Then from the condition in the expression of the equivariant cohomology ring in Theorem 5.3, we have \(\alpha - \alpha' \mid_{\mu}\) is a multiple of \(\prod_{(k, k') \in \text{rev}(\mu)} (Y_{(k, k'), \mu} - \frac{w_{(k, k'), \mu}}{w_{\mu}} Y_{\mu})\) which is of degree \(\ell(\mu)\). This contradicts the fact that \(\alpha - \alpha'\) is homogeneous of degree \(\ell(\lambda) < \ell(\mu)\). \(\square\)

Let us denote the equivariant Schubert class corresponding to a Schubert symbol \(\lambda\) by \(w_{\lambda}\). Using the arguments in the proof of [KT03, Proposition 1], one gets the following.

**Proposition 6.3.** The set of equivariant Schubert classes \(\{w_{\lambda}\}_{i=0}^{m}\) form a \(H^*_T(WGr(d, n); \mathbb{Z})\)-module basis for \(H^*_T(WGr(d, n); \mathbb{Z})\). In particular, any element \(x \in H^*_T(WGr(d, n); \mathbb{Z})\) can be written uniquely as an \(H^*_T(WGr(d, n); \mathbb{Z})\) linear combination of \(w_{\lambda}\) using only those \(\lambda\) such that \(\lambda \geq \mu\) for some \(\mu \in \text{supp}(\lambda)\).
Example 6.4. In Figure 3 we compute the equivariant Schubert class \( w\tilde{S}_{23} \in H_T^*(WGr(2, 4); \mathbb{Z}) \) where \( WGr(2, 4) \) is a divisive weighted Grassmann orbifold for some \( w = (w_1, w_2, w_3, w_4) \) and \( 0 < a \in \mathbb{Z}_{\geq 1} \).

In the rest of this section, we compute the structure constant for the equivariant cohomology of a divisive weighted Grassmann orbifold. Since the set \( \{ w\tilde{S}_\lambda \}_{i=0}^m \) form a \( H_T^n(\{pt\}; \mathbb{Z}) \)-basis for \( H_T^n(WGr(d, n); \mathbb{Z}) \), for any two \( \lambda \) and \( \mu \), one has the following

\[
(w\tilde{S}_\lambda w\tilde{S}_\mu) = \sum_{\nu} wc_{\lambda\mu}^\nu w\tilde{S}_\nu
\]

where \( \nu \) runs over the Schubert symbols. The constant \( wc_{\lambda\mu}^\nu \in H_T^n(\{pt\}; \mathbb{Z}) \) in the formula is called ‘weighted structure constant’. We note that similar to the Grassmann manifold case ([K103 Lemma 2]), we can have the following.

**Lemma 6.5.** The weighted structure constant \( wc_{\lambda\mu}^\nu \) have the following properties.

1. The weighted structure constant \( wc_{\lambda\mu}^\nu \) has degree \( \ell(\lambda) + \ell(\mu) - \ell(\nu) \).
2. \( wc_{\lambda\mu}^\nu = 0 \) unless \( \ell(\nu) \leq \ell(\lambda) + \ell(\mu) \) and \( \nu \succeq \lambda, \mu \).
3. When \( \lambda = \nu \) we have \( wc_{\lambda\mu}^\lambda = w\tilde{S}_\mu |_\lambda \).

**Proof.** (1). The degree of \( w\tilde{S}_\lambda \) is \( \ell(\lambda) \). So the degree of the weighted structure constant \( wc_{\lambda\mu}^\nu \) is given by degree\( (wc_{\lambda\mu}^\nu) = \) degree\( (w\tilde{S}_\lambda) + \) degree\( (w\tilde{S}_\mu) - \) degree\( (w\tilde{S}_\nu) = \ell(\lambda) + \ell(\mu) - \ell(\nu) \).

(2). The weighted structure constant \( wc_{\lambda\mu}^\nu = 0 \) if \( \ell(\lambda) + \ell(\mu) - \ell(\nu) < 0 \).

Also

\[
(w\tilde{S}_\lambda w\tilde{S}_\mu)|_{\lambda'} \neq 0 \iff \lambda' \succeq \lambda, \mu
\]

Thus by Proposition 6.3 \( wc_{\lambda\mu}^\nu \neq 0 \iff \nu \succeq \lambda, \mu \).
(3) Compare the $\lambda$-th component of both sides in Equation (6.3) we get
\[ wS_{\lambda|\lambda}wS_{\mu|\lambda} = wc_{\lambda|\mu} wS_{\lambda|\lambda} + \sum_{\nu \neq \lambda} wc_{\mu|\lambda} wS_{\nu|\lambda} \]
Since, $wc_{\mu|\lambda} = 0$ unless $\ell(\nu) > \ell(\lambda)$, or, $\nu = \lambda$. But $wS_{\nu|\lambda} = 0$ for $\ell(\nu) > \ell(\lambda)$, and $\nu \neq \lambda$. Thus all the terms in the summation vanish. Now the claim follows since $wS_{\lambda|\lambda} \neq 0$. □

The unique Schubert symbol $\lambda$ for which $\ell(\lambda) = 0$ is denoted by $\text{int}$ and the unique Schubert symbol $\lambda$ for which $\ell(\lambda) = 1$ is denoted by $\text{gen}$.

**Lemma 6.6.** The equivariant Schubert divisor class $w\tilde{S}_{\text{gen}} \in H^*_T(WGr(d, n); \mathbb{Z})$ is given by
\[ w\tilde{S}_{\text{gen}}|_{\mu} = Y_{\text{int}} - \frac{w_{\text{int}}}{w_{\mu}} Y_{\mu}. \]

**Proof.** Note that $w\tilde{S}_{\text{gen}}|_{\text{gen}} = Y_{\text{int}} - \frac{w_{\text{int}}}{w_{\text{gen}}} Y_{\text{gen}}$. For other Schubert symbol $\mu$, it follows from Definition 6.1. □

Let $\lambda$ and $\mu$ be two Schubert symbols such that $\lambda \leq \mu$. Then Lemma 6.6 gives
\[ w\tilde{S}_{\text{gen}}w\tilde{S}_{\lambda} = (w\tilde{S}_{\text{gen}}|_{\lambda})w\tilde{S}_{\lambda} + \sum_{\lambda' \rightarrow \lambda} \frac{w_{\text{int}}}{w_{\lambda}} w\tilde{S}_{\lambda'}. \]

(6.4)

For any two Schubert symbol $\lambda$ and $\lambda'$ we denote $\lambda' \rightarrow \lambda$ if $\ell(\lambda') = \ell(\lambda) + 1$ and $\lambda' \succeq \lambda$.

**Proposition 6.7** (Weighted Peiri rule).
\[ w\tilde{S}_{\text{gen}}w\tilde{S}_{\lambda} = (w\tilde{S}_{\text{gen}}|_{\lambda})w\tilde{S}_{\lambda} + \sum_{\lambda' \rightarrow \lambda} \frac{w_{\text{int}}}{w_{\lambda}} w\tilde{S}_{\lambda'}. \]

**Proof.** Using the fact that $\deg(w\tilde{S}_{\text{gen}}) = 1$, we have
\[ w\tilde{S}_{\text{gen}}w\tilde{S}_{\lambda} = (wc_{\lambda|\text{gen}}) w\tilde{S}_{\lambda} + \sum_{\lambda' \rightarrow \lambda} (wc_{\lambda'|\text{gen}}) w\tilde{S}_{\lambda'}. \]

From Lemma 6.5, we get $wc_{\lambda|\text{gen}} = w\tilde{S}_{\text{gen}}|_{\lambda}$. Now fix $\lambda'$ such that $\lambda' \rightarrow \lambda$ and compare $\lambda'$-th component of both side we get
\[ w\tilde{S}_{\text{gen}}|_{\lambda'} w\tilde{S}_{\lambda}|_{\lambda'} = (wc_{\lambda'|\text{gen}}) w\tilde{S}_{\lambda}|_{\lambda'} + (wc_{\lambda|\text{gen}}) w\tilde{S}_{\lambda}|_{\lambda'} \]
\[ \implies (wc_{\lambda'|\text{gen}}) w\tilde{S}_{\lambda}|_{\lambda'} = (w\tilde{S}_{\text{gen}}|_{\lambda'}) w\tilde{S}_{\lambda}|_{\lambda'} \]
\[ \implies (wc_{\lambda'|\text{gen}}) w\tilde{S}_{\lambda}|_{\lambda'} = \frac{w_{\text{int}}}{w_{\lambda}} (Y_{\lambda} - \frac{w_{\lambda'}}{w_{\lambda'}} Y_{\lambda'}) w\tilde{S}_{\lambda}|_{\lambda'} \]

Thus $wc_{\lambda|\text{gen}} = \frac{w_{\text{int}}}{w_{\lambda}}$. Hence the proof. □
By applying Proposition 6.7 repeatedly we can compute the following as well as the higher products.
\[(w\tilde{S}_{\text{gen}})^2w\tilde{S}_{\lambda} = w\tilde{S}_{\text{gen}}((w\tilde{S}_{\text{gen}}|_{\lambda})w\tilde{S}_{\lambda} + \sum_{\lambda' \to \lambda} \frac{w_{\text{int}}}{w_{\lambda}} w\tilde{S}_{\lambda'})\]
\[= (w\tilde{S}_{\text{gen}}|_{\lambda})^2w\tilde{S}_{\lambda} + \sum_{\lambda' \to \lambda} (w\tilde{S}_{\text{gen}}|_{\lambda}) \frac{w_{\text{int}}}{w_{\lambda}} w\tilde{S}_{\lambda'} + \sum_{\lambda'' \to \lambda' \to \lambda} \frac{w_{\text{int}}}{w_{\lambda}} \frac{w_{\text{int}}}{w_{\lambda'}} w\tilde{S}_{\lambda''}\]

**Proposition 6.8.** For any three Schubert symbols \(\lambda, \mu, \nu\), we have the following recurrence relation
\[(w\tilde{S}_{\text{gen}}|_{\nu} - w\tilde{S}_{\text{gen}}|_{\lambda})w\xi_{\lambda\mu}^{\nu} = (\sum_{\lambda' \to \lambda} \frac{w_{\text{int}}}{w_{\lambda}} w\xi_{\lambda'\mu}^{\nu} - \sum_{\nu' \to \nu} \frac{w_{\text{int}}}{w_{\nu'}} w\xi_{\lambda\mu}^{\nu'}).\]

**Proof.** We use the associativity of multiplication in \(H_{\text{Gr}}(\text{WGr}(d, n); \mathbb{Z})\) and equivariant pieri rule to expand \(w\tilde{S}_{\text{gen}}w\tilde{S}_{\lambda}w\tilde{S}_{\mu}\) in two different way. Then
\[(w\tilde{S}_{\text{gen}} w\tilde{S}_{\lambda}) w\tilde{S}_{\mu} = ((w\tilde{S}_{\text{gen}}|_{\lambda}) w\tilde{S}_{\lambda} + \sum_{\lambda' \to \lambda} \frac{w_{\text{int}}}{w_{\lambda}} w\tilde{S}_{\lambda'}) w\tilde{S}_{\mu}\]
\[= (w\tilde{S}_{\text{gen}}|_{\lambda}) \sum_{\rho \to \lambda} w\xi_{\lambda\mu}^{\rho} w\tilde{S}_{\rho} + \sum_{\lambda' \to \lambda} \frac{w_{\text{int}}}{w_{\lambda}} \sum_{\rho \to \lambda} w\xi_{\lambda'\mu}^{\rho} w\tilde{S}_{\rho}.\]

Also
\[w\tilde{S}_{\text{gen}}(w\tilde{S}_{\lambda} w\tilde{S}_{\mu}) = w\tilde{S}_{\text{gen}} \sum_{\rho} w\xi_{\lambda\mu}^{\rho} w\tilde{S}_{\rho}\]
\[= \sum_{\rho} w\xi_{\lambda\mu}^{\rho} ((w\tilde{S}_{\text{gen}}|_{\rho}) w\tilde{S}_{\rho} + \sum_{\rho' \to \rho} \frac{w_{\text{int}}}{w_{\rho'}} w\tilde{S}_{\rho'}).\]

Comparing the coefficient of \(w\tilde{S}_{\nu}\) i.e. putting \(\rho = \nu\) in both the expression we get
\[(w\tilde{S}_{\text{gen}}|_{\nu}) w\xi_{\lambda\mu}^{\nu} + \sum_{\lambda' \to \lambda} \frac{w_{\text{int}}}{w_{\lambda}} w\xi_{\lambda'\mu}^{\nu} = w\xi_{\lambda\mu}^{\nu} (w\tilde{S}_{\text{gen}}|_{\nu}) + \sum_{\nu' \to \nu} \frac{w_{\text{int}}}{w_{\nu'}} w\xi_{\lambda\mu}^{\nu'} w\xi_{\lambda'\mu}.\]

\[\square\]

**Remark 6.9.** The numbers \(\ell(\lambda)\) and \(\ell'(\lambda)\) have geometric interpretation. If we look at the GKM graph of \(\text{WGr}(d, n)\) and at a vertex \(V_{\lambda}\) corresponding to a Schubert symbol \(\lambda\), then
\[\ell(\lambda) = \# \{V_{\lambda'} \mid j < i ; \ V_{\lambda'} \text{ is adjacent to } V_{\lambda} \}\]
and
\[\ell'(\lambda) = \# \{V_{\lambda'} \mid j > i ; \ V_{\lambda'} \text{ is adjacent to } V_{\lambda} \}.\]

**Remark 6.10.** For each Schubert symbol \(\lambda = (\lambda_1, \ldots, \lambda_d)\), one can consider a string consisting of \(d\) ones and \(n - d\) zeroes in the following order. One for each \(\lambda_i\)-th position and zero for the remaining \(n - d\) positions. In this
way we can identify the set of all Schubert symbols $\lambda = (\lambda_1, \ldots, \lambda_d)$ and the set of strings consisting of $d$ ones and $n - d$ zeroes in arbitrary order. With this consideration, the Schubert cycle $X_\lambda$ defined in [KT03] is same as the closure of $E(\lambda)$.

7. Equivariant cobordism and K-theory of weighted Grassmann orbifolds

In this section, first we compute the equivarant K-theory ring of any weighted Grassmann orbifold with rational coefficients. Then we compute the equivariant cobordism and K-theory ring of divisive weighted Grassmann orbifolds with integer coefficients. We adhere the notation of previous sections.

Proposition 7.1. Let $\text{WGr}(d, n)$ be a weighted Grassmann orbifold corresponding to $W = (w_1, w_2, \ldots, w_n) \in (\mathbb{Z}_{\geq 0})^n$ and $a \geq 1$. Then there is a $T^n$-equivariant stratification

$\{pt\} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m = \text{WGr}(d, n)$

such that the quotient $X_j/X_{j-1}$ is homeomorphic to the Thom space $Th(\xi^j)$ of the orbifold $T^n$-vector bundle

$\xi^j : C^\ell(\lambda_j)/G(c_j) \to [C(\lambda_j)]$,

where $G(c_j)$ is the cyclic group of $c_j$-th roots of unity, for $j = 1, \ldots, m$.

Proof. Recall the $T^n$-invariant stratification

$\{pt\} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m = \text{WGr}(d, n)$

from (3.20) which is obtained from the $q$-cell structure of $\text{WGr}(d, n)$ as in Lemma 3.5. Note that $X_j/X_{j-1}$ is the one point compactification of $\frac{E(\lambda_j)}{G(c_j)}$ which is the Thom space of the orbifold $T^n$-vector bundle

$\frac{E(\lambda_j)}{G(c_j)} \to [C(\lambda_j)]$

for $j = 1, \ldots, m$. Where $[C(\lambda_j)]$ is the $T^n$-fixed point corresponding to the Schubert symbol $\lambda_j$ for $j = 1, \ldots, m$ (see the proof of Proposition 5.1). Since $E(\lambda_j)$ is $T^n$-equivariantly homeomorphic to $C^\ell(\lambda_j)$, we get the proposition. □

Proposition 7.2. The $T^n$-equivariant stratification in (7.1) satisfies the assumptions (A1), (A2) and (A3) in Section 2.

Proof. Let $V_j$ be the fixed points of the $T^n$-action on $X_j$ and $X_j^1$ the union of all spindles in $X_j$ for $j = 1, \ldots, m$. Define

$E_j := \{e \in \Gamma | \text{ if } V(e) \subseteq V_j\}$.
Then $\Gamma_j = (V_j, E_j)$ is a subgraph of $\Gamma = (V, E)$ where $\Gamma$ is the GKM-graph of $WGr(d, n)$ and we get the following filtration

$$\{pt\} = \Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_m = \Gamma.$$  

Recall $\Gamma$ is a $d(n - d)$ valent graph. This filtration induces an ordering on the vertices of $\Gamma$ which is same as the total ordering on the corresponding Schubert symbols to obtain the $q$-cell structure in (3.20). Since (3.20) is built inductively by attaching an invariant $q$-cell at each step $WGr(d, n)$ is build-able. Therefore the proof follows from Proposition 2.25.

**Theorem 7.3.** Let $WGr(d, n)$ be the weighted Grassmann orbifold for $d < n$ corresponding to $W = (w_1, w_2, \ldots, w_n) \in (\mathbb{Z}_{\geq 0})^n$ and $a \geq 1$. Then the generalized $T^n$-equivariant cohomology $E^*_{T^n}(WGr(d, n))$ can be given by

$$\left\{(x_j) \in \bigoplus_{j=0}^m E^*_{T^n}(pt) \mid e_{T^n}(\xi_j) \text{ divides } x_j - x_i \text{ for all } i < j \text{ and } |\lambda^j \cap \lambda^i| = d-1\right\}$$

for $E^*_{T^n} = K^*_{T^n}, H^*_{T^n}$, where $e_{T^n}(\xi)$ represents the equivariant Thom class.

**Proof.** This follows from Theorem 2.20 and Proposition 7.2.

We note that equivariant cohomology of $WGr(d, n)$ with rational coefficients is discussed in [AM15]. In the rest we give a description of the equivariant $K$-theory and equivariant cobordism rings of a divisive weighted Grassmann orbifold with integer coefficients.

**Proposition 7.4.** Let $WGr(d, n)$ be a divisive weighted Grassmann orbifold for $d < n$ corresponding to $W = (w_1, w_2, \ldots, w_n) \in (\mathbb{Z}_{\geq 0})^n$ and $a \geq 1$. Then there is a $T^n$-equivariant stratification

$$(7.3) \quad \{pt\} = X_0 \subset X_1 \subset \cdots \subset X_m = WGr(d, n)$$

such that the quotient $X_j/X_{j-1}$ is homeomorphic to the Thom space $Th(\xi_j)$ of the orbifold $T^n$-vector bundle

$$(7.4) \quad \xi_j : \mathbb{C}^{l(\lambda_j)} \to [C(\lambda_j)],$$

for $j = 1, \ldots, m$.

**Proof.** Since $WGr(d, n)$ is divisive, then $r_j = \gcd\{c_0, c_1, \ldots, c_j\} = c_j$ for all $j$. So by Theorem 4.4 the space $E(\lambda^j)/G(c_j)$ is equivariantly homeomorphic to $E(\lambda^j)/G(c_j/c_j) \cong \mathbb{C}^{l(\lambda)}$ for $j = 1, \ldots, m$. So this follows from the proof of Proposition 7.1.

**Proposition 7.5.** The $T^n$-equivariant stratification in (7.3) satisfies the assumptions (A1), (A2) and (A3) in Section 2.

**Proof.** This follows from Proposition 7.2 and 7.4.
Theorem 7.6. Let $WGr(d, n)$ be a divisive weighted Grassmann orbifold for $d < n$. Then the generalized $T^n$-equivariant cohomology $E^*_T(WGr(d, n); \mathbb{Z})$ can be given by

$$\left\{ (x_j) \in \bigoplus_{j=0}^m E^*_T(pt) \mid e^*_T(\xi^{ji}) \text{ divides } x_j - x_i \text{ for all } i < j \text{ and } |\lambda^j \cap \lambda^i| = d-1 \right\}$$

for $E^*_T = H^*_T, K^*_T$ and $MU^*_T$.

Proof. This follows from Theorem 2.27 and Proposition 7.5.

Next we give the computation of $e^*_T(\xi^{ji})$. We recall that

$$K^*_T(pt) \cong R(T^n)[z, z^{-1}]$$

where $R(T^n)$ is the complex representation ring of $T^n$ and $z$ is the Bott element in $K^{-2}(pt)$. Note that $R(T^n)$ is isomorphic to the ring of Laurent polynomials with $n$-variables, that is $R(T^n) \cong \mathbb{Z}[\alpha_1, \ldots, \alpha_n]_{(\alpha_1 \cdots \alpha_n)}$, where $\alpha_i$ is the irreducible representation corresponding to the projection on the $i$-th factor, see [Hus94]. Then (4.3) gives that the one-dimensional representation for the edge joining $e_{\lambda^j}$ to $e_{\lambda^i}$ is given by

$$\xi^{ji} \cong \alpha_{\lambda^j} \alpha_{\lambda^i}^{-d_{ij}}$$

where $\alpha_\lambda = \alpha_{\lambda_1} \cdots \alpha_{\lambda_d}$ for the Schubert symbol $\lambda = (\lambda_1, \ldots, \lambda_d)$ and $i < j$ and $|\lambda^j \cap \lambda^i| = d - 1$. Therefore, one has the following.

$$e^*_T(\xi^{ji}) = \begin{cases} 1 - \alpha_{\lambda^j} \alpha_{\lambda^i}^{-d_{ij}} & \text{in } K^0_T \\ e^*_T(\alpha_{\lambda^j} \alpha_{\lambda^i}^{-d_{ij}}) & \text{in } MU^2_T \\ Y_{\lambda^j} - d_{ij} Y_{\lambda^i} & \text{in } H^2_T. \end{cases}$$

We remark that the structure of $MU^*_T(pt)$ is unknown, however it is referred as the ring of $T^n$-cobordism forms in [HHHRW16].

Example 7.7. Consider the weighted Grassmann orbifold $WGr(2, 4)$ for $w = (12, 2, 2, 2)$ and $a = 6$. We have the ordering on the 6 Schubert symbols given by $\lambda^0 = (1, 2), \lambda^1 = (1, 3), \lambda^2 = (1, 4), \lambda^3 = (2, 3), \lambda^4 = (2, 4)$ and $\lambda^5 = (3, 4)$. Now $W_{\lambda^0} = 20, W_{\lambda^1} = 20, W_{\lambda^2} = 20, W_{\lambda^3} = 10, W_{\lambda^4} = 10, W_{\lambda^5} = 10$. Then $W_{\lambda^j}$ divides $W_{\lambda^j-1}$ for all $j = 1, 2, 3, 4, 5$. Then

$$d_{ij} = \begin{cases} 1 & \text{if both } i, j \in \{0, 1, 2\} \text{ or } \{3, 4, 5\} \\ 2 & \text{if } i \in \{0, 1, 2\} \text{ and } j \in \{3, 4, 5\}. \end{cases}$$

The integral equivariant cohomology ring of this divisive weighted Grassmann orbifold $WGr(2, 4)$ is given by

$$H^*_T(WGr(2, 4); \mathbb{Z}) = \{ (x_j) \in \bigoplus_{i=0}^5 \mathbb{Z}[y_1, y_2, y_3, y_4] \mid e^*_T(\xi^{ji}) \text{ divides } (x_j - x_i) \text{ if } |\lambda^j \cap \lambda^i| = 1, i < j \}. $$
Then by Theorem 2.12, $K$ by Theorem 4.21, are all GKM orbifolds with the above complex $\Gamma$ in Example 2.10. Now if $c_i$ in (3.12). This induces $T^4$-actions on $O_1, O_2$ and $O_3$. Now $O_1, O_2$ and $O_3$ are all GKM orbifolds with the above $T^4$-action and hence $K$ is a simplicial GKM orbifold complex. The GKM graphs corresponding to $O_1$ and $O_2$ are triangles and they have a common edge $(v_1v_3)$ which is the GKM graph of $O_3$. Let $v_i$ be the vertex corresponding to $[e_i]$ (a fixed point of the $T^4$-action) for $i = 0, 1, 2, 3$ where $\{e_0, e_1, \ldots, e_3\}$ is the standard basis of $\mathbb{C}^6$. Recall the order on the Schubert symbols from Definition 3.2. Now from (5.1) it follows that the axial function corresponding to the edge $v_i v_j$ (the edge joining $v_i$ and $v_j$ outgoing from $v_i$) is $\omega_{e_i} \omega_{e_j}$. Thus simplicial GKM graph complex associated to $K$ is same as the simplicial GKM graph complex $\Gamma$ in Example 2.10 Now if $c_i$ devides $c_{i-1}$ for all $i \in \{1, 2, 3\}$ then by Theorem 4.21, $K$ has a $T^4$-invariant cell structure and $H^{odd}(K; \mathbb{Z}) = 0$. Then by Theorem 2.12

$$H^{74}_K(K, \mathbb{Z})$$

$$= \{(f_i) \in \bigoplus_{i=0}^{5} \mathbb{Z}[y_1, y_2, y_3, y_4] \mid (Y_{\lambda^j} - d_{ij} Y_{\lambda^i}) \text{ divides } (f_j - f_i) \}
$$

if $|\lambda^j \cap \lambda^i| = 1, i < j$.

Let $X^3 = \bigcup_{i=0}^{3} \frac{E(\lambda^i)}{C(c_i)} \subset WGr(2, 4)$ be the 3rd skeleton of $WGr(2, 4)$ same as Section 3. Recall the action of $T^4$ on $WGr(2, 4)$ given in (3.11) which restricts an action of $T^4$ on $X^3$. Then $X^3$ is also simplicial GKM orbifold complex and simplicial GKM graph complex associated to $X^3$ is same as the simplicial GKM graph complex $\Gamma$ in Example 2.10. Note that $P_{1p}$ is $T^4$-equivariant homeomorphism between the geometric realization of $K$ and $X^3$. Thus if $c_3 | c_2 | c_1 | c_0$ is divisive then the equivariant cohomology ring of $X^3$ with integer coefficients is given by (7.5).

Remark 7.9. For $0 \leq k \leq m$, let $X^k = \bigcup_{i=0}^{k} \frac{E(\lambda^i)}{C(c_i)} \subset WGr(d,n)$ be the $k$-th skeleton of $WGr(d,n)$ same as in Section 3. Now we can calculate the equivariant cohomology ring and equivariant K theory ring of $X^k$ with rational coefficients by Theorem 7.3. Now if $WGr(d,n)$ is divisive then we can calculate the equivariant cohomology ring, equivariant K theory ring and equivariant cobordism ring of $X^k$ with integer coefficients by Theorem 7.6.
Next we give equivariant cohomology ring of some weighted Grassmann orbifold with integer coefficients which are not divisive.

**Theorem 7.10.** Let \(WGr(d, n)\) be a weighted Grassmann orbifold corresponding to the order \(\lambda^0 < \cdots < \lambda^m\) such that \(W_{\lambda^i}|W_{\lambda^k}\) for \(k \leq i\) and \(i \geq 2\) but \(W_{\lambda^1}\) does not divide \(W_{\lambda^0}\). Then the integral equivariant cohomology ring of \(WGr(d, n)\) is given by

\[
H^*_T(WGr(d, n); \mathbb{Z}) = \left\{ (f_i) \in \bigoplus_{i=0}^m \mathbb{Z}[y_1, y_2, \ldots, y_n] : (Y_{\lambda^j} - d_{ij} Y_{\lambda^i}) \text{ divides } (f_j - f_i) \text{ if } |\lambda^j \cap \lambda^i| = d - 1; (i, j) \neq (0, 1) \text{ and } c_1 Y_{\lambda^0} - c_0 Y_{\lambda^1} \text{ divides } (f_0 - f_1) \right\}.
\]

**Proof.** By given condition \(r_i = \text{gcd}\{c_0, c_1, \ldots, c_i\} = c_i\) for \(i \geq 2\). So by Theorem 4.4 we have \(E(\lambda^i)/G(c_i)\) is homeomorphic to \(E(\lambda^i)/G(c_i/c_i) \cong \mathbb{C}^{\ell(\lambda^i)}\) for \(i = 1, \ldots, m\). When \(i = 1\), we have \(\{pt\} \sqcup E(\lambda^i)/G(c_i/r_i) = WP(p, q)\), for some \(p, q \in \mathbb{Z}_{\geq 1}\). Therefore it has \(T^n\)-invariant CW-structure. Thus by Theorem 2.12 we get the result. \(\square\)

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