A NOTE ON DIRECTED POLYMERS IN GAUSSIAN ENVIRONMENTS

YUEYUN HU
Département de Mathématiques, Université Paris XIII, 99 avenue J-B Clément, 93430 Villetaneuse, France
email: yueyun@math.univ-paris13.fr

QI-MAN SHAO
Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China
email: maqmshao@ust.hk

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Abstract
We study the problem of directed polymers in gaussian environments in $\mathbb{Z}^d$ from the viewpoint of a gaussian family indexed by the set of random walk paths. In the zero-temperature case, we give a numerical bound on the maximum of the Hamiltonian, whereas in the finite temperature case, we establish an equivalence between the very strong disorder and the growth rate of the entropy associated to the model.

1 Introduction and main results

1.1 Finite temperature case
Let $(g(i,x))_{i \geq 0, x \in \mathbb{Z}^d}$ be i.i.d. standard real-valued gaussian variables. We denote by $P$ and $E$ the corresponding probability and expectation with respect to $g(\cdot, \cdot)$. Let $\{S_k, k \geq 0\}$ be a simple symmetric random walk on $\mathbb{Z}^d$, independent of $g(\cdot, \cdot)$. We denote by $P_x$ the probability measure of $(S_n)_{n \in \mathbb{N}}$ starting at $x \in \mathbb{Z}^d$ and by $E_x$ the corresponding expectation. We also write $P = P_0$ and $E = E_0$.

The directed polymer measure in a gaussian random environment, denoted by $\langle \cdot \rangle^{(n)}$, is a random probability measure defined as follows: Let $\Omega_n$ be the set of nearest neighbor paths of length $n$: $\Omega_n \overset{\text{def}}{=} \{ \gamma : \{1, \ldots, n\} \rightarrow \mathbb{Z}^d, |\gamma_k - \gamma_{k-1}| = 1, k = 2, \ldots, n, \gamma_0 = 0 \}$. For any function $F : \Omega_n \rightarrow \mathbb{R}_+$,

$$\langle F(S) \rangle^{(n)} = \frac{1}{Z_n} \mathbb{E} \left( F(S) e^{E_n(g, S) - \frac{\beta^2 n}{2}} \right), \quad H_n(g, \gamma) = \sum_{i=1}^n g(i, \gamma_i),$$
where $\beta > 0$ denotes the inverse of temperature and $Z_n$ is the partition function:

$$Z_n = Z_n(\beta, g) = \mathbb{E} \left( e^{\beta H_n(g, \xi)} \right).$$

We refer to Comets, Shiga and Yoshida \cite{3} for a review on directed polymers. It is known (see e.g. \cite{2,3}) that the so-called free energy, the limit of $\frac{1}{n} \log Z_n$ exists almost surely and in $L^1$:

$$p(\beta) := \lim_{n \to \infty} \frac{1}{n} \log Z_n,$$

$p(\beta)$ is some constant and $p(\beta) \leq 0$ by Jensen's inequality since $E Z_n = 1$.

A problem in the study of directed polymer is to determine the region of $\{\beta > 0 : p(\beta) < 0\}$, also called the region of very strong disorder. It is an important problem, for instance, $p(\beta) < 0$ yields interesting information on the localization of the polymer itself.

By using the F-K-G inequality, Comets and Yoshida \cite{4} showed the monotonicity of $p(\beta)$, therefore the problem is to determine

$$\beta_c := \inf \{\beta > 0 : p(\beta) < 0\}.$$

It has been shown by Imbrie and Spencer \cite{8} that for $d \geq 3$, $\beta_c > 0$ (whose exact value remains unknown). Comets and Vargas \cite{5} proved that (for a wide class of random environments)

$$\beta_c = 0, \quad \text{if} \quad d = 1. \quad (1.1)$$

Recently, Lacoin \cite{10}, skilfully used the ideas developed in pinned models and solved the problem in the two-dimensional case:

$$\beta_c = 0, \quad \text{if} \quad d = 2. \quad (1.2)$$

Moreover, Lacoin \cite{10} gave precise bounds on $p(\beta)$ when $\beta \to 0$ both in one-dimensional and two-dimensional cases.

In this note, we study this problem from the point of view of entropy (see also Birkner \cite{1}). Let

$$e_n(\beta) := E \left( Z_n \log Z_n \right) - (EZ_n) \log (EZ_n) = E \left( Z_n \log Z_n \right)$$

be the entropy associated to $Z_n$ (recalling $E Z_n = 1$).

**Theorem 1.1.** Let $\beta > 0$. The following limit exits

$$\tilde{e}_\infty(\beta) := \lim_{n \to \infty} \frac{e_n(\beta)}{n} = \inf_{n \geq 1} \frac{e_n(\beta)}{n} \geq 0. \quad (1.3)$$

There is some numerical constant $c_d > 0$, only depending on $d$, such that the following assertions are equivalent:

(a) $\tilde{e}_\infty(\beta) > 0$.

(b) $\limsup_{n \to \infty} \frac{e_n(\beta)}{\log n} > c_d$.

(c) $p(\beta) < 0$. 

The proof of the implication (b) \(\implies\) (c) relies on a criterion of \(p(\beta) < 0\) (cf. Fact 3.1 in Section 3) developed by Comets and Vargas [5] in a more general settings.

We can easily check (b) in the one-dimensional case: In fact, we shall show in the sequel (cf. (3.7)) that in any dimension and for any \(\beta > 0\),

\[ e_n(\beta) \geq \frac{\beta^2}{2} \mathbb{E}(L_n(S^1, S^2)), \]

where \(S^1\) and \(S^2\) are two independent copies of \(S\) and \(L_n(\gamma, \gamma') = \sum_{k=1}^{n} 1_{(\gamma_k = \gamma'_k)}\) is the number of common points of two paths \(\gamma\) and \(\gamma'\). It is well known that \(\mathbb{E}(L_n(S^1, S^2))\) is of order \(n^{1/2}\) when \(d = 1\) and of order \(\log n\) when \(d = 2\). Therefore (b) holds in \(d = 1\) and by the implication (b) \(\implies\) (c), we recover Comets and Vargas’ result (1.1) in the one-dimensional gaussian environment.

### 1.2 Zero temperature case

When \(\beta \to 0\), the problem of directed polymers boils down to the problem of first-passage percolation. Let

\[ H_n^* = H_n^*(g) := \max_{\gamma \in \Omega_n} H_n(g, \gamma), \quad H_n(g, \gamma) = \sum_{i=0}^{n} g(\gamma_i - \gamma_{i-1}), \]

where as before \(\Omega_n = \{\gamma : [0, n] \to \mathbb{Z}^d, |\gamma_i - \gamma_{i-1}| = 1, i = 2, \ldots, n, \gamma_0 = 0\}\).

The problem is to characterize these paths \(\gamma\) which maximize \(H_n(g, \gamma)\). See Johansson [9] for the solution of the Poisson points case. We limit here our attention to some explicit bounds on \(H_n^*\).

An easy subadditivity argument (see Lemma 2.2) shows that

\[ \frac{H_n^*}{n} \to \sup_{n \geq 1} \frac{\mathbb{E}H_n^*}{n} \overset{\text{def}}{=} c_d^*, \quad \text{both a.s. and in } L^1. \]

By Slepian’s inequality ([12]),

\[ \mathbb{E}H_n^* \leq \sqrt{n} \mathbb{E} \max_{\gamma \in \Omega_n} Y_\gamma, \]

where \((Y_\gamma)_{\gamma \in \Omega_n}\) is a family of i.i.d. centered gaussian variables of variance 1. Since \(\# \Omega_n = (2d)^n\), it is a standard exercise from extreme value theory that

\[ \frac{1}{\sqrt{n}} \mathbb{E} \max_{\gamma \in \Omega_n} Y_\gamma \to \sqrt{2 \log(2d)}. \]

Hence

\[ c_d^* \leq \sqrt{2 \log(2d)}. \]

It is a natural problem to ask whether this inequality is strict; In fact, a strict inequality means that the gaussian family \(\{H_n(g, \gamma), \gamma \in \Omega_n\}\) is sufficiently correlated to be significantly different from the independent one, exactly as the problem to determine whether \(p(\beta) < 0\).

We prove that the inequality is strict by establishing a numerical bound:

**Theorem 1.2.** For any \(d \geq 1\), we have

\[ c_d^* \leq \sqrt{2 \log(2d) - \frac{(2d - 1)}{2d} \left(1 - \frac{2d - 1}{5d}\right) \left(1 - \Phi(\sqrt{\log(2d)})\right)}, \]

where \(\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du\) is the partition function of a standard gaussian variable.

The proofs of Theorems 1.1 and 1.2 are presented in two separate sections.
2 Proof of Theorem 1.2

We begin with several preliminary results. Recall at first the following concentration of measure property of Gaussian processes (see Ibragimov and al. [7]).

Fact 2.1. Consider a function $F : \mathbb{R}^M \to \mathbb{R}$ and assume that its Lipschitz constant is at most $A$, i.e.

$$|F(x) - F(y)| \leq A ||x - y|| \quad (x, y \in \mathbb{R}^M),$$

where $||x||$ denotes the euclidean norm of $x$. Then if $g = (g_1, \ldots, g_M)$ are i.i.d. $\mathcal{N}(0, 1)$ we have

$$P\left( |F(g) - E(F(g))| \geq u \right) \leq \exp\left(\frac{-u^2}{2A^2}\right) \quad (u > 0).$$

Lemma 2.2. There exists some positive constant $c_d^*$ such that

$$\frac{H_n^*}{n} \to c_d^* = \sup_{n \geq 1} E\left( \frac{H_n^*}{n} \right), \quad \text{a.s. and in } L^1. \quad (2.1)$$

Moreover,

$$P\left( |H_n^* - E(H_n^*)| > \lambda \right) \leq e^{-\frac{\lambda^2}{2n}}, \quad \lambda > 0. \quad (2.2)$$

Proof: We prove at first the concentration inequality. Define a function $F : \mathbb{R}^m \to \mathbb{R}$ by

$$F(z) = \max_{g \in \mathbb{R}^n} \sum_{i=1}^{n} \sum_{|x| \leq n} z_{i,x} 1_{(\gamma_i=x)} \quad z = (z_{i,x})_{1 \leq i \leq n} \in \mathbb{R}^m.$$ 

By the Cauchy-Schwarz inequality,

$$\left| \sum_{i=1}^{n} \sum_{|x| \leq n} z_{i,x} 1_{(\gamma_i=x)} - \sum_{i=1}^{n} \sum_{|x| \leq n} z'_{i,x} 1_{(\gamma_i=x)} \right| \leq n^{1/2} \left( \sum_{i=1}^{n} \sum_{|x| \leq n} (z_{i,x} - z'_{i,x})^2 \right)^{1/2}. $$

Hence $F$ is a Lipschitz function: $|F(z) - F(z')| \leq \sqrt{n} ||z - z'||$. Note that $H_n^* = F((g_i, \gamma_i)_{i \in \Omega_n})$. By the Gaussian concentration inequality Fact [2.1] we get [2.2].

Now we prove that $n \to E H_n^*$ is superadditive: for $n, k \geq 1$, let $\gamma^* \in \Omega_n$ be a path such that $H_n^*(g, \gamma^*) = H_n^*$, then

$$H_{n+k}^* \geq H_n^*(g, \gamma^*) + \max_{g_{i} \in \Omega_k} \sum_{i=1}^{k} g(i + n, \gamma_i + \gamma^*_n),$$

hence by conditioning on $\sigma\{g(i, \cdot), i \leq n\}$, we get that

$$E(H_{n+k}^*) \geq E(H_n^*(g, \gamma^*)) + E(H_k^*) = E(H_n^*) + E(H_k^*),$$

which in view of concentration [2.2] implies [2.1]. \qed

For $x, y \in \mathbb{Z}^d$ and $n \geq 0$, we shall denote by $y - x \leftrightarrow n$ when $\mathbb{P}_x(S_n = y) = \mathbb{P}(S_n = y - x) > 0$. Observe that

$$x \leftrightarrow n \text{ if and only if } n - \sum_{i=1}^{d} x_j \equiv 0 \pmod{2} \text{ and } \sum_{i=1}^{d} |x_i| \leq n,$$

with $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$. We shall also write $\sum_{x \leftrightarrow n}$ or $\max_{x \leftrightarrow n}$ to mean that the sum or maximum is taken over those $x$ such that $x \leftrightarrow n$. 

Define
\[ \phi_n(\lambda) = \log \left( \sum_{x \sim n} \mathbb{E}^e^{\lambda H_{n,x}} \right), \quad \phi_n^*(a) = \sup_{\lambda > 0} (a\lambda - \phi_n(\lambda)). \]

Lemma 2.3. For any \( n \geq 1 \), let \( \varsigma_n \) be \( \inf \{ c > 0 : \phi_n^*(cn) > 0 \} \). We have
\[ c_n^* \leq \varsigma_n \leq c_n^* + 2\sqrt{\frac{d \log(2n + 1)}{n}}. \]

**Proof:** Let \( \tau_{n,x} \) be the time and space shift on the environment:
\[ g \circ \tau_{n,x}(\cdot, \cdot) = g(n + \cdot, x + \cdot). \] (2.3)

We have for any \( n, k, \)
\[ H_{n+k}^* = \max_{x \sim n} \left\{ \max_{\gamma \in \Omega_n: \gamma_n = x} H_n(g, \gamma) + H_k^*(g \circ \tau_{n,x}) \right\}. \]

Write for simplification \( H_{n,x}^* := \max_{\gamma \in \Omega_n: \gamma_n = x} H_n(g, \gamma) \). Then for any \( \lambda \in \mathbb{R}, \)
\[ \mathbb{E} e^{\lambda H_{n+k}} = \mathbb{E} e^{\lambda \max_{x \sim n} (H_{n,x}^* + H_k^*(g \circ \tau_{n,x}))} \leq \mathbb{E} \left( \sum_{x \sim n} e^{\lambda H_{n,x}^*} e^{\lambda H_k^*(g \circ \tau_{n,x})} \right) \leq \mathbb{E} \left( \sum_{x \sim n} e^{\lambda H_{n,x}^*} \right) e^{\lambda H_k^*}. \]

We get
\[ \mathbb{E} e^{\lambda H_{jn}} \leq e^{j \phi_n(\lambda)}, \quad j, n \geq 1, \lambda \in \mathbb{R}. \]

Chebychev’s inequality implies that
\[ P(\lambda H_{jn}^* > cjn) \leq e^{-j \phi_n^*(cn)}, \quad c > 0, \]
where \( \phi_n^*(a) = \sup_{\lambda > 0} (a\lambda - \phi_n(\lambda)) \). Then for any \( n \) and \( c \) such that \( \phi_n^*(cn) > 0 \), \( \limsup_{j \to \infty} \frac{H_{jn}^*}{jn} \leq c \), a.s. It follows that
\[ c_n^* \leq \varsigma_n, \quad \forall n \geq 1. \]

On the other hand, by using the concentration inequality \[ (2.2) \] and the fact that \( \mathbb{E}(H_{jn}^*) \leq nc_n^* \), we get
\[ \mathbb{E} e^{\lambda H_{n,x}^*} \leq e^{\lambda H_n^*} e^{\lambda^2 n/2} \leq e^{\lambda nc_n^* + \lambda^2 n/2}, \quad \lambda > 0. \]

It follows that for any \( \lambda > 0, \)
\[ \phi_n(\lambda) \leq \log \sum_{x \sim n} \mathbb{E}^{\lambda H_{n,x}^*} \leq \lambda nc_n^* + \frac{\lambda^2}{2} n + 2d \log(2n + 1). \]

Choosing \( \lambda = 2\sqrt{\frac{d \log(2n + 1)}{n}} \), we get \( \phi_n^*(\lambda) < \lambda n(c_n^* + a) \) for any \( a > 2\sqrt{\frac{d \log(2n + 1)}{n}} \), which means that \( \phi_n^*((c_n^* + a)n) > 0 \). Hence \( \varsigma_n \leq c_n^* + a \) and the lemma follows. \( \square \)
We conclude from (2.4) that
\[ \phi_2(\lambda) \leq \lambda^2 + 2d(2d - 1)\Phi(\lambda/\sqrt{2}) \]
\[ = \lambda^2 + 2 \log(2d) + \log\left(1 - \frac{(2d - 1)}{2d}(1 - \Phi(\lambda/\sqrt{2}))\right) \]
\[ \leq \lambda^2 + 2 \log(2d) - \frac{(2d - 1)}{2d}(1 - \Phi(\lambda/\sqrt{2})) = h(\lambda) \]

Now consider the function
\[ h^*(2c) := \sup_{\lambda > 0} (2c\lambda - h(\lambda)), \quad c > 0. \]

Clearly \( h^*(2c) \leq \phi^*(2c) \) for any \( c > 0 \). Let us study the \( h^*(2c) \): The maximal achieves when
\[ 2c = h^*(\lambda) \]

That is
\[ 2c = 2\lambda + \frac{2d - 1}{2d \sqrt{2\pi}} e^{-\lambda^2/2} \]
Now choose $c$ so that
\[ 2c\lambda = h(\lambda) = \lambda^2 + 2\log(2d) - \frac{(2d - 1)}{2d} \left( \frac{1 - \Phi(\lambda/\sqrt{2})}{1 - \Phi(\lambda/\sqrt{2})} \right) \]  
\begin{equation} \tag{2.5} \end{equation}

Let
\[ a = \frac{(2d - 1)}{4d \sqrt{2\pi}} e^{-\lambda^2/4}, \quad b = \frac{(2d - 1)}{2d} (1 - \Phi(\lambda/\sqrt{2})). \]

Then
\[ 2c = 2\lambda + 2a, \quad 2c\lambda = \lambda^2 + 2\log(2d) - b, \]
which gives
\[ \lambda^2 + 2a\lambda = 2\log(2d) - b \]
or
\[ c^2 = (\lambda + a)^2 = 2\log(2d) - b + a^2. \]

It is easy to see that
\[ (1 - \Phi(t)) \geq \frac{5}{16} e^{-t^2}, \quad \text{for } t > 0. \]

Hence
\[ \frac{16}{5} \left( \frac{2d - 1}{4d \sqrt{2\pi}} \right)^2 (1 - \Phi(\lambda/\sqrt{2})) \geq a^2 \]
and
\[ c^2 = 2\log(2d) - \frac{(2d - 1)}{2d} (1 - \Phi(\lambda/\sqrt{2})) + a^2 \]
\[ \leq 2\log(2d) - \left( \frac{2d - 1}{2d} - \left( \frac{2d - 1}{4d \sqrt{2\pi}} \right) \frac{16}{5} \right) (1 - \Phi(\lambda/\sqrt{2})) \]
\[ = 2\log(2d) - \frac{(2d - 1)}{2d} \left( 1 - \frac{2d - 1}{5\pi} \right) (1 - \Phi(\lambda/\sqrt{2})) \]
\[ \leq 2\log(2d) - \frac{(2d - 1)}{2d} \left( 1 - \frac{2d - 1}{5\pi} \right) (1 - \Phi(\sqrt{\log(2d)})) \]
\[ \defeq c_*^2, \]

Here in the last inequality, we used the fact that $\lambda \leq \sqrt{2\log(2d)}$. Recall $(c, \lambda)$ satisfying (2.5). For any $\epsilon > 0$, $\phi^*_{\lambda}(2(\bar{c} + \epsilon)) \geq h^*_{\lambda}(2(\bar{c} + \epsilon)) \geq 2(\bar{c} + \epsilon) - \lambda = 2\epsilon \lambda > 0$. It follows $\zeta_2 \leq \bar{c} + \epsilon$ and hence $c_*^2 \leq \bar{c}$.

\section{Proof of Theorem 1.1}

Let
\[ Z_m(x) := \mathbb{E}\left( 1_{(S_m=x)} e^{\beta H_m(g, x) - \beta^2 m/2} \right), \quad m \geq 1, x \in \mathbb{Z}^d. \]

Fact 3.1 (Comets and Vargas [5]). If there exists some $m \geq 1$ such that
\[ \mathbb{E}\left( \sum_x Z_m(x) \log Z_m(x) \right) \geq 0, \quad \tag{3.1} \]
then $p(\beta) < 0$. 

In fact, the case $E\left(\sum_x Z_m(x) \log Z_m(x)\right) = 0$ follows from their Remark 3.5 in Comets and Vargas (2006), whereas if $E\left(\sum_x Z_m(x) \log Z_m(x)\right) > 0$, which means the derivative of $\theta \to E\sum_x Z_m^\theta(x)$ at 1 is positive, hence for some $\theta < 1$, $E\sum_x Z_m^\theta(x) < 1$ and again by Comets and Vargas (2006) (lines before Remark 3.4), we have $p(\beta) < 0$.

We try to check (3.1) in the sequel:

**Lemma 3.2.** There exists some constant $c_d > 0$ such that for any $m \geq 1$,

$$E\left(\sum_x Z_m(x) \log Z_m(x)\right) \geq E\left(Z_m \log Z_m\right) - c_d \log m.$$  

**Proof:** Write

$$u_m(x) := \frac{Z_m(x)}{Z_m} = \{1(S_n=x)\}^{(m)}.$$  

We have

$$E\left(\sum_x Z_m(x) \log Z_m(x)\right) = E\left(Z_m \sum_x u_m(x) \left[ \log Z_m + \log u_m(x) \right]\right) = Q\left(\sum_x u_m(x) \log u_m(x)\right) + Q\left(\log Z_m\right),$$

where the probability measure $Q = Q^{(\beta)}$ is defined by

$$dQ|_{\mathcal{F}^n} = Z_n dP|_{\mathcal{F}^n}, \quad \forall \ n \geq 1,$$

with $\mathcal{F}_n = \sigma\{g(i, \cdot), i \leq n\}$. By convexity,

$$Q\left(\sum_x u_m(x) \log u_m(x)\right) \geq \sum_x Q(u_m(x)) \log Q(u_m(x)).$$

Note that $Q(u_m(x)) = E(Z_m(x)) = \mathbb{P}(S_m = x)$ and $\sum_x \mathbb{P}(S_m = x) \log \mathbb{P}(S_m = x) \sim -c_d' \log m$ for some positive constant $c_d'$. Assembling all the above, we get a constant $c_d > 0$ such that

$$E\left(\sum_x Z_m(x) \log Z_m(x)\right) \geq Q\left(\log Z_m\right) - c_d \log m$$

for all $m \geq 1$, as desired. \hfill \Box

Let $\mu$ be a Gaussian measure on $\mathbb{R}^m$. The logarithmic Sobolev inequality says (cf. Gross [4], Ledoux [11]): for any $f : \mathbb{R}^m \to \mathbb{R}$,

$$\int f^2 \log f^2 d\mu - \left(\int f^2 d\mu\right) \log \left(\int f^2 d\mu\right) \leq 2 \int |\nabla f|^2 d\mu. \quad (3.4)$$

Using the above inequality, we have

**Lemma 3.3.** Let $S^1$ and $S^2$ be two independent copies of $S$. We have

$$\epsilon_n(\beta) \leq \frac{\beta^2}{2} E\left(Z_n \langle L_n(S^1, S^2) \rangle^{(n)}\right), \quad (3.5)$$

$$\frac{d}{d\beta} \epsilon_n(\beta) = \beta E\left(Z_n \langle L_n(S^1, S^2) \rangle^{(n)}\right), \quad (3.6)$$
where \( \langle L_n(S^1, S^2) \rangle^n_2 = \frac{1}{Z_n^2} \mathbb{E}(e^{\beta H_n(g, S^1) + \beta H_n(g, S^2) - \beta^2 L_n(S^1, S^2)}) \). Consequently, \( e_n(\beta) \geq \frac{\beta^2}{2} \mathbb{E}(L_n(S^1, S^2)). \) \( (3.7) \)

**Proof:**
Taking \( f(z) = \sqrt{\mathbb{E} \exp \left( \beta \sum_{i=1}^n \sum_x z(i, x) 1_{(S_i = x)} - \frac{\beta^2}{2} n \right)} \), \( z = (z(i, x), 1 \leq i \leq n, x \leftarrow i) \).

Note that \( Z_n = f^2(g) \) with \( g = (g(i, x), 1 \leq i \leq n, x \leftarrow i) \). Applying the log-Sobolev inequality yields the first estimate \( (3.5) \).

The another assertion follows from the integration by parts: for a standard gaussian variable \( g \) and any derivable function \( \psi \) such that both \( g \psi(g) \) and \( \psi'(g) \) are integrable, we have
\[
\mathbb{E}(g \psi(g)) = \mathbb{E}(\psi'(g)).
\]

Elementary computations based on the above formula yield \( (3.6) \). The details are omitted. From \( (3.5) \) and \( (3.6) \), we deduce that the function \( \beta \rightarrow e_n(\beta) \) is nondecreasing. On the other hand, it is elementary to check that \( \lim_{\beta \to 0} \frac{e_n(\beta)}{\beta^2} = \frac{1}{2} \mathbb{E}(L_n(S^1, S^2)) \), which gives \( (3.7) \) and completes the proof of the lemma.

\[ \square \]

If \( P \) and \( Q \) are two probability measures on \( (\Omega, \mathcal{F}) \), the relative entropy is defined by
\[
H(Q \| P) \overset{\text{def}}{=} \int \log \frac{dQ}{dP} dQ,
\]
where the expression has to be understood to be infinite if \( Q \) is not absolutely continuous with respect to \( P \) or if the logarithm of the derivative is not integrable with respect to \( Q \). The following entropy inequality is well-known:

**Lemma 3.4.** For any \( A \in \mathcal{F} \), we have
\[
\log \frac{P(A)}{Q(A)} \geq -H(Q \| P) + e^{-1}.
\]

This inequality is useful only if \( Q(A) \sim 1 \). Recall \( (3.3) \) for the definition of \( Q \). Note that for any \( \delta > 0 \), \( Q \left( Z_n \geq \frac{\delta}{1+\delta} \right) \geq \frac{1}{1+\delta} \), it follows that
\[
P \left( Z_n \geq \frac{\delta}{1+\delta} \right) \geq \frac{1}{1+\delta} e^{-(1+\delta)e^{-1}} \exp \left( -(1+\delta) e_n(\beta) \right).
\]

\( (3.8) \)

Now we give the proof of Theorem 1.1

**Proof of Theorem 1.1.** We prove at first \( (1.3) \) by subadditivity argument: Recalling \( (3.2) \) and \( (2.3) \). By markov property of \( S \), we have that for all \( n, m \geq 1 \),
\[
Z_{n+m} = Z_n \sum_x u_n(x) Z_m(x, g \circ \tau_n, x).
\]
Let $\psi(x) = x \log x$. We have

\[ Z_{n+m} \log Z_{n+m} = \psi(Z_n) \sum_x u_n(x) Z_m(x, g \circ \tau_{n,x}) + Z_n \psi \left( \sum_x u_n(x) Z_m(x, g \circ \tau_{n,x}) \right) \leq \psi(Z_n) \sum_x u_n(x) Z_m(x, g \circ \tau_{n,x}) + Z_n \sum_x u_n(x) \psi(Z_m(x, g \circ \tau_{n,x})) \]

by the convexity of $\psi$. Taking expectation gives (1.3).

To show the equivalence between (a), (b), (c), we remark at first that the implication (b) $\Rightarrow$ (c) follows from Lemma 3.2 and (3.1). It remains to show the implication (c) $\Rightarrow$ (a). Assume that $p(\beta) < 0$. The superadditivity says that $p(\beta) = \sup_{n \geq 1} p_n(\beta)$, with $p_n(\beta) = \frac{1}{n} E(\log Z_n)$.

On the other hand, the concentration of measure (cf. [2]) says that

\[ P \left( \left| \frac{1}{n} \log Z_n - p_n(\beta) \right| > u \right) \leq \exp \left( - \frac{n u^2}{2 \beta^2} \right), \quad \forall u > 0. \]

It turns out for $\alpha > 0$,

\[ E \left( Z_n^\alpha \right) = e^{\alpha E(\log Z_n)} e^{\alpha(\log Z_n - E(\log Z_n))} \leq e^{\alpha E(\log Z_n)} e^{\alpha^2 \beta^2 n/2} \leq e^{\alpha p(\beta) n + \alpha^2 \beta^2 n/2}. \]

By choosing $\alpha = -p(\beta)/\beta^2$ (note that $\alpha > 0$), we deduce from the Chebychev’s inequality that

\[ P \left( Z_n > \frac{1}{2} \right) \leq 2^\alpha e^{-p(\beta)n/(2\beta^2)}, \]

which in view of (3.8) with $\delta = 1$ imply that $\liminf_{n \to \infty} \frac{1}{n} e_n(\beta) \geq \frac{p(\beta)^2}{4 \beta^2}$. [square]

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