Abstract

The theory of KMS weights is based on a theorem of Combes and a theorem of Kustermans. In applications to KMS states for flows on a unital $C^*$-algebra the relation to KMS weights of the stabilized algebra has proved useful and this relation hinges on a theorem of Laca and Neshveyev. This text presents proofs of these three fundamental results that require a minimum of prerequisites; in particular, they do not depend on the modular theory of von Neumann algebras.
Chapter 1

Weights

Let $A$ be a $C^*$-algebra and let $A^+$ denote the cone of positive elements in $A$. A map $\psi : A^+ \to [0, \infty]$ is a semi-weight on $A$ when

(a) $\psi(a + b) \leq \psi(a) + \psi(b)$ $\forall a, b \in A^+$,
(b) $a \leq b \Rightarrow \psi(a) \leq \psi(b)$ $\forall a, b \in A^+$,
(c) $\psi(ta) = t\psi(a)$ $\forall a \in A^+$, $\forall t \in \mathbb{R}^+$, with the convention that $0 \cdot \infty = 0$,
(d) $\psi$ is lower semi-continuous; i.e. $\{ a \in A^+ : \psi(a) > t \}$ is open in $A^+$ for all $t \in \mathbb{R}$.

It is easy to see that the lower semi-continuity of $\psi$ is equivalent to the following condition which is often useful and easier to verify.

\* $\psi(a) \leq \liminf_n \psi(a_n)$ when $\lim_{n \to \infty} a_n = a$ in $A^+$.

A semi-weight is densely defined when $\{ a \in A^+ : \psi(a) < \infty \}$ is dense in $A^+$. Given a semi-weight $\psi$ on $A$ we set

\[ M^+_\psi = \{ a \in A^+ : \psi(a) < \infty \}, \]
\[ N_\psi = \{ a \in A : \psi(a^*a) < \infty \}, \]

and

\[ M_\psi = \text{Span} M^+_\psi. \]

**Lemma 1.0.1.** Let $\psi$ be a semi-weight on $A$. The sets $M_\psi$, $M^+_\psi$, and $N_\psi$ have the following properties.

(a) $N_\psi$ is a left-ideal in $A$, and it is dense in $A$ when $\psi$ is densely defined.
(b) $M_\psi = \text{Span} N^*_\psi N_\psi$.
(c) $M_\psi \subseteq N_\psi \cap N^*_\psi$. 

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(d) \( M_\psi \cap A^+ = M^+_\psi \).

(e) \( M_\psi \) is a *-subalgebra of \( A \), and it is dense in \( A \) when \( \psi \) is densely defined.

Proof. For (a) note that \((a + \mu b)^*(\lambda a + \mu b) \leq 2|\lambda|^2a^*a + 2|\mu|^2b^*b \) for all \( a, b \in A \) and all \( \lambda, \mu \in \mathbb{C} \). This shows that \( N_\psi \) is a subspace of \( A \). Since \( a^2 \leq \|a\|^2a \) for all \( a \in A^+ \) it follows that \( M^+_\psi \subseteq N_\psi \). In particular, \( N_\psi \) is dense in \( A \) when \( \psi \) is densely defined. \( N_\psi \) is a left ideal because \( b^*a^*ab \leq \|a\|^2b^*b \) for all \( a, b \in A \).

For (b), let \( \psi \) be a semi-weight on \( A \). Since \( b^*a^*ab \subseteq \mathbb{F}^+ \), it follows from the definition that \( \psi \in \mathbb{F}^+ \). Similarly, \( b \in N_\psi \) and it follows from (b) that \( ab \in M_\psi \). Hence \( M_\psi, M_\psi \subseteq M_\psi \) and we conclude that \( M_\psi \) is a *-subalgebra of \( A \). Since every element of \( A \) is a linear combination of elements from \( A^+ \) it follows that \( M_\psi \) is dense in \( A \) when \( \psi \) is densely defined. \( \square \)

1.1 Combes’ theorem

In the following we denote by \( A_1^+ \) the set of bounded positive linear functionals on \( A \). If \( \mathcal{F} \) is a subset of \( A_1^+ \), we can define a semi-weight by the formula \( \sup_{\omega \in \mathcal{F}} \omega(a) \). The fundamental result on semi-weights is the following theorem of Combes, showing that all semi-weights arise like this.

Theorem 1.1.1. Let \( \psi \) be a semi-weight on \( A \). Set
\[
\mathcal{F}_\psi = \{ \omega \in A_1^+ : \omega(a) \leq \psi(a) \ \forall a \in A^+ \}.
\]
Then
\[
\psi(a) = \sup_{\omega \in \mathcal{F}_\psi} \omega(a)
\]
for all \( a \in A^+ \).

The proof of Combes’ theorem requires some preparation. We denote by \( A_1 \) the \( C^* \)-algebra obtained by adjoining a unit to \( A \). In particular, \( A_1 = A \oplus \mathbb{C} \) when \( A \) already has a unit. For \( \alpha > 0 \) and \( a \in A_1^+ \), set
\[
\rho_\alpha(a) = \inf \left\{ \psi(s) + \alpha t : s \in M^+_\psi, \ t \in \mathbb{R}^+, \ a \leq s + t1 \right\}.
\]
The following properties of $\rho_\alpha : A^+_1 \to \mathbb{R}^+$ are straightforward to establish.

\begin{align*}
\rho_\alpha(a + b) & \leq \rho_\alpha(a) + \rho_\alpha(b) \quad \forall a, b \in A^+_1. \quad (1.1.1) \\
\rho_\alpha(\lambda a) & = \lambda \rho_\alpha(a) \quad \forall a \in A^+_1, \ \forall \lambda \in \mathbb{R}^+. \quad (1.1.2) \\
\rho_\alpha(a) & \leq \psi(a) \quad \forall a \in A^+. \quad (1.1.3) \\
a, b \in A^+_1, \ a \leq b & \Rightarrow \rho_\alpha(a) \leq \rho_\alpha(b). \quad (1.1.4) \\
\rho_\alpha(1) & \leq \alpha. \quad (1.1.5)
\end{align*}

It follows from (1.1.4), (1.1.2) and (1.1.5) that

$$\rho_\alpha(a) \leq \alpha \|a\| \quad \forall a \in A^+_1.$$  

(1.1.6)

**Lemma 1.1.2.**

$$\rho_\alpha(a + \lambda 1) = \rho_\alpha(a) + \lambda \alpha \quad \forall a \in A^+_1, \ \forall \lambda \in \mathbb{R}^+.$$

**Proof.** This follows from the equality

\begin{align*}
\{ \psi(s) + t\alpha : s \in M^+_\psi, \ t \in \mathbb{R}^+, \ a + \lambda 1 \leq s + t1 \} & \\
= \{ \psi(s) + t\alpha : s \in M^+_\psi, \ t \in \mathbb{R}^+, \ a \leq s + t1 \} + \lambda \alpha. \quad (1.1.7)
\end{align*}

To establish this equality, let $s \in M^+_\psi, \ t \in \mathbb{R}^+, \ a + \lambda 1 \leq s + t1$. Then $a \leq s + (t - \lambda)1$, and if we let $\chi : A_1 \to \mathbb{C}$ be the canonical character with $\ker \chi = A$, we conclude that $t - \lambda = \chi(s + (t - \lambda)1) \geq \chi(a) \geq 0$. Hence

$$\psi(s) + (t - \lambda)\alpha \in \{ \psi(s) + t'\alpha : s \in M^+_\psi, \ t' \in \mathbb{R}^+, \ a \leq s + t'1 \}.$$  

This proves that the first set in (1.1.7) is contained in the second. The reverse inclusion is trivial.

**Lemma 1.1.3.**

$$\sup_{\alpha > 0} \rho_\alpha(a) = \psi(a) \quad \forall a \in A^+.$$

**Proof.** Let $a \in A^+$. It follows from (1.1.3) that $\sup_{\alpha > 0} \rho_\alpha(a) \leq \psi(a)$. Assume for a contradiction that $\sup_{\alpha > 0} \rho_\alpha(a) < \psi(a)$. There is then a $k > 0$ such that $\rho_\alpha(a) < k < \psi(a)$ for all $\alpha > 0$. Applied with $\alpha = kn$ we get for each $n \in \mathbb{N}$ an element $s_n \in M^+_\psi$ and a $t_n \in \mathbb{R}^+$ such that $a \leq s_n + t_n 1$ and $\psi(s_n) + t_n \alpha < k$.

In particular, $t_n \leq \frac{1}{n}$. Set

$$b_n := a^\frac{1}{n} - a^\frac{1}{n} \left( \frac{1}{n} + s_n \right)^{-1} s_n = \frac{1}{n} a^\frac{1}{n} \left( \frac{1}{n} + s_n \right)^{-1}.$$
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Then
\[ b_n^* b_n = \frac{1}{n^2} \left( \frac{1}{n} + s_n \right)^{-1} a \left( \frac{1}{n} + s_n \right)^{-1} \]
\[ \leq \frac{1}{n^2} \left( \frac{1}{n} + s_n \right)^{-1} \left( s_n + \frac{1}{n} \right) \left( \frac{1}{n} + s_n \right)^{-1} = \frac{1}{n^2} \left( \frac{1}{n} + s_n \right)^{-1} \leq \frac{1}{n}, \]
proving that \( \lim_{n \to \infty} b_n = 0 \). Thus \( a = \lim_{n \to \infty} c_n \) where

\[ c_n := s_n \left( \frac{1}{n} + s_n \right)^{-1} a \left( \frac{1}{n} + s_n \right)^{-1} s_n. \]

Note that

\[ c_n \leq s_n \left( \frac{1}{n} + s_n \right)^{-1} \left( s_n + \frac{1}{n} \right) \left( \frac{1}{n} + s_n \right)^{-1} s_n \]
\[ = s_n \left( \frac{1}{n} + s_n \right)^{-1} s_n \leq s_n, \]

implying that \( \psi(c_n) \leq \psi(s_n) < k \). Since \( \psi \) is lower semi-continuous it follows that \( \psi(a) \leq \lim \inf_n \psi(c_n) \) and we conclude therefore that \( \psi(a) \leq k \) which contradicts that \( k < \psi(a) \). \( \square \)

Let \( A_1^h \) denote the set of self-adjoint elements of \( A_1 \) and define \( \rho'_\alpha : A_1^h \to \mathbb{R}^+ \) by

\[ \rho'_\alpha(a) = \inf \{ \rho_\alpha(b + c) : b, c \in A_1^+, a = b - c \}. \]

**Lemma 1.1.4.** \( \rho'_\alpha \) has the following properties.

- \( \rho'_\alpha(a + b) \leq \rho'_\alpha(a) + \rho'_\alpha(b) \) \( \forall a, b \in A_1^h \),
- \( \rho'_\alpha(\lambda a) = |\lambda| \rho'_\alpha(a) \) \( \forall a \in A_1^h, \forall \lambda \in \mathbb{R} \),
- \( \rho'_\alpha(a) \leq \alpha \|a\| \) \( \forall a \in A_1^h \), and
- \( \rho'_\alpha(a) = \rho_\alpha(a) \) \( \forall a \in A_1^+ \).

**Proof.** The first two items follow easily from (1.1.1) and (1.1.2). Let \( a \in A_1^h \). Recall that there is a (unique) decomposition \( a = a_+ - a_- \) where \( a_+ \in A_1^+ \) and \( a_+ a_- = 0 \). Since \( \|a_+ + a_-\| = \|a\| \) we get the third item from (1.1.3). If \( a \in A_1^+ \) and \( a = b - c \) with \( b, c \in A_1^+ \) we have that \( a \leq b + c \) and hence \( \rho_\alpha(a) \leq \rho_\alpha(b + c) \) thanks to (1.1.3). This shows that \( \rho_\alpha(a) \leq \rho'_\alpha(a) \) in this case. The reverse inequality is trivial. \( \square \)

**Proof of Theorem 1.1.1.** Let \( a \in A^+ \) and \( \alpha > 0 \). By Lemma 1.1.3 it suffices to find \( \omega \in \mathcal{F}_\psi \) such that \( \omega(a) = \rho_\alpha(a) \). Let \( V \) be the real vector subspace of \( A_1^h \) generated by \( a \) and \( 1 \). We can then define a linear map \( \omega : V \to \mathbb{R} \) such that

\[ \omega(sa + t1) = s\rho_\alpha(a) + t\rho_\alpha(1). \]
for all \( s, t \in \mathbb{R} \). We claim that
\[
|\omega(x)| \leq \rho'_\alpha(x) \quad \forall x \in V.
\] (1.1.8)

Assume that \( t \geq 0 \). Then (1.1.5) and Lemma 1.1.2 imply that
\[
|\omega(a + t\alpha)| = \rho_\alpha(a) + t\rho_\alpha(1) \leq \rho_\alpha(a) + t\alpha = \rho_\alpha(a + t\alpha).
\]

Since \( \rho_\alpha(a + t\alpha) = \rho'_\alpha(a + t\alpha) \) by the last item in Lemma 1.1.4 we have established (1.1.8) when \( x = a + t\alpha \). Assume that \( t \leq 0 \). Using the fourth and the second item in Lemma 1.1.4 we find that
\[
\omega(a + t\alpha) = \rho_\alpha(a) + t\rho_\alpha(1) = \rho_\alpha(a) - \rho'_\alpha(t\alpha).
\] (1.1.9)

The first two items in Lemma 1.1.4 imply that
\[
\rho'_\alpha(t\alpha) = \rho'_\alpha(-t\alpha) = \rho'_\alpha(a - (a + t\alpha)) \leq \rho'_\alpha(a) + \rho'_\alpha(a + t\alpha)
\]
and, since \( \rho_\alpha(a) = \rho'_\alpha(a) \),
\[
\rho_\alpha(a) = \rho'_\alpha(a + t\alpha - t\alpha) \leq \rho'_\alpha(a + t\alpha) + \rho'_\alpha(t\alpha).
\]

Thus
\[
-\rho'_\alpha(a + t\alpha) \leq \rho_\alpha(a) - \rho'_\alpha(t\alpha) \leq \rho'_\alpha(a + t\alpha),
\]
which combined with (1.1.9) shows that (1.1.8) holds when \( x \in a + \mathbb{R}1 \). From the second item in Lemma 1.1.4 we see that (1.1.8) holds when \( x \in \mathbb{R}1 \). Finally, when \( s \neq 0 \) we can now conclude that
\[
|\omega(sa + t\alpha)| = |s| \left| \omega(a + \frac{t}{s}\alpha) \right| \leq |s|\rho'_\alpha(a + \frac{t}{s}\alpha) = \rho'_\alpha(sa + t\alpha).
\]
Thus (1.1.8) holds for all \( x \in V \) and it follows from the Hahn-Banach extension theorem that there is a linear map \( \omega : A_1 \rightarrow \mathbb{C} \) extending \( \omega : V \rightarrow \mathbb{R} \) such that \( \omega(x^*) = \overline{\omega(x)} \) for all \( x \in A_1 \), i.e. \( \omega \) is hermitian, and \( |\omega(y)| \leq \rho'_\alpha(y) \) for all \( y \in A_1^h \). Using that \( \omega \) is hermitian together with the third item of Lemma 1.1.4 we find that
\[
\|\omega\| = \sup \{ |\omega(y)| : y \in A_1^h, \|y\| \leq 1 \}
\leq \sup \{ \rho'_\alpha(y) : y \in A_1^h, \|y\| \leq 1 \} \leq \alpha.
\]

Since \( \omega(1) = \rho_\alpha(1) = \alpha \), it follows that \( \omega|_A \in A_1^* \). Furthermore, when \( x \in A_+ \), we find from (1.1.3) that \( \omega(x) \leq \rho'_\alpha(x) = \rho_\alpha(x) \leq \psi(x) \). Thus \( \omega \in \mathcal{F}_\psi \). Since \( \omega(\alpha) = \rho_\alpha(\alpha) \), this completes the proof.

**Notes and remarks 1.1.5.** Theorem 1.1.7 was obtained by Combes in [C]; see Lemme 1.5 and Remarque 1.6 in [C]. An alternative proof was given by Haagerup in [Ha]; see Proposition 2.1 and Corollary 2.3 in [Ha].
1.2 Combes’ theorem in the separable case

Let $A$ be a $C^*$-algebra. A map $\psi : A^+ \to [0, \infty]$ is a weight on $A$ when

- $\psi(a + b) = \psi(a) + \psi(b) \ \forall a, b \in A^+$,
- $\psi(ta) = t\psi(a) \ \forall a \in A^+, \ \forall t \in \mathbb{R}^+$, using the convention $0 \cdot \infty = 0$, and
- $\psi$ is lower semi-continuous; i.e. $\{a \in A_+ : \psi(a) > t\}$ is open in $A^+$ for all $t \in \mathbb{R}$.

Any sequence $\{\omega_n\}_{n=1}^\infty$ in $A_+^*$ defines a weight by the formula

$$\psi(a) = \sum_{n=1}^\infty \omega_n(a) \ \forall a \in A^+ \ (1.2.1)$$

The content of the next theorem is that they all arise this way when $A$ is separable.

**Theorem 1.2.1.** Assume that $A$ is separable and let $\psi$ be a weight on $A$. There is a sequence $\omega_n \in A_+^*$, $n = 1, 2, \cdots$, such that

$$\psi(a) = \sum_{n=1}^\infty \omega_n(a) \ \forall a \in A^+ \ (1.2.1)$$

The proof requires some preparations starting with the following continuation of the proof of Theorem 1.1.1.

**Lemma 1.2.2.** Assume that $A$ is separable and that $\psi$ is a semi-weight on $A$. There is a countable set $C \subseteq F_\psi$ with the following property: For all $a \in A^+$ and all $n \in \mathbb{N}$ there is an element $f \in C$ such that $f(a) > \min\{n, \psi(a) - \frac{1}{n}\}$.

**Proof.** The function $\alpha \mapsto \rho_\alpha(a)$ which was used in the proof of Theorem 1.1.1 is non-decreasing for $a \in A^+$. It follows therefore from Lemma 1.1.3 that $\lim_{k \to \infty} \rho_k(a) = \psi(a)$ for all $a \in A^+$. Let $\{a_i\}_{i=1}^\infty$ be a dense sequence in $A^+$. As shown in the proof of Theorem 1.1.1 there is for each pair $k, i \in \mathbb{N}$ an element $\omega_k,i \in F_\psi$ such that $\omega_k,i(a_i) = \rho_k(a_i)$ and $|\omega_k,i(x)| \leq \|x\|k$ for all $x \in A$. Set

$$C = \{\omega_k,i : k, i \in \mathbb{N}\}.$$ 

To check that $C$ has the required properties, let $a \in A^+$ and $n \in \mathbb{N}$ be given. Since $\lim_{k \to \infty} \rho_k(a) = \psi(a)$ there is a $k \in \mathbb{N}$ such that

$$\rho_k(a) > \min\{n + 1, \psi(a) - \frac{1}{2n}\}.$$ 

Choose $i \in \mathbb{N}$ such that $\|a - a_i\| \leq \frac{1}{4kn}$. Using the properties of $\rho_k$ and $\rho'_k$ we find

$$|\rho_k(a) - \omega_k,i(a)| \leq |\rho_k(a) - \rho_k(a_i)| + |\rho_k(a_i) - \omega_k,i(a)|$$

$$\leq \rho'_k(a - a_i) + |\omega_k,i(a_i) - \omega_k,i(a)| \leq 2k \|a - a_i\| \leq \frac{1}{2n}.$$
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Then
\[ \omega_{k,i}(a) \geq \rho_k(a) - \frac{1}{2n} > \min \left\{ n + 1, \psi(a) - \frac{1}{2n} \right\} - \frac{1}{2n} \]
\[ \geq \min \left\{ n, \psi(a) - \frac{1}{n} \right\}. \]

Note that a weight is also a semi-weight. In the following \( \psi : A \to [0, \infty] \)
will be a weight on the \( C^{*} \)-algebra \( A \).

1.2.1 The GNS construction for weights

Let \( A \) be a \( C^{*} \)-algebra.

**Definition 1.2.3.** A GNS representation \( (H, \Lambda, \pi) \) of \( A \) consists of a Hilbert space \( H \), a linear map \( \Lambda : D(\Lambda) \to H \) defined on a dense left ideal \( D(\Lambda) \subseteq A \) and a representation \( \pi : A \to B(H) \) of \( A \) by bounded operators on \( H \) such that

(a) \( \Lambda(D(\Lambda)) \) is dense in \( H \), and

(b) \( \pi(a) \Lambda(b) = \Lambda(ab) \) for all \( a \in A \) and \( b \in D(\Lambda) \).

A densely defined weight \( \psi : A^{+} \to [0, \infty] \) gives rise to a GNS representation of \( A \) in a way we now describe.

**Lemma 1.2.4.** Let \( \psi : A \to [0, \infty] \) be a weight. There is a unique linear map \( \psi : \mathcal{M}_\psi \to \mathbb{C} \) extending \( \psi : \mathcal{M}_\psi^+ \to \mathbb{R}^{+} \).

**Proof.** An element \( a \in \mathcal{M}_\psi \) can be decomposed as a sum \( a = a_1 - a_2 + i(a_3 - a_4) \) where \( a_i \in \mathcal{M}_\psi^+ \), and we are forced to define the extension such that

\[ \psi(a) = \psi(a_1) - \psi(a_2) + i(\psi(a_3) - \psi(a_4)). \]

This is well-defined and gives a linear map because \( \psi \) is additive on \( \mathcal{M}_\psi^+ \).

It follows from Lemma 1.0.1 and Lemma 1.2.4 that we can define a sesquilinear form

\[ (\cdot, \cdot) : \mathcal{N}_\psi \times \mathcal{N}_\psi \to \mathbb{C} \]

such that

\[ (a, b) = \psi(b^* a). \]

We let \( H_\psi \) denote the Hilbert space obtained by completing the quotient \( \mathcal{N}_\psi / I_\psi \), where \( I_\psi = \{ a \in \mathcal{N}_\psi : \psi(a^* a) = 0 \} \), with respect to the norm on \( \mathcal{N}_\psi / I_\psi \) induced by the Hermitian form \( (\cdot, \cdot) \). Let \( \Lambda_\psi : \mathcal{N}_\psi \to H_\psi \) be the map coming with this construction. We can then define a representation \( \pi_\psi : A \to B(H_\psi) \) of \( A \) by bounded operators such that

\[ \pi_\psi(a) \Lambda_\psi(b) = \Lambda_\psi(ab) \]

for all \( b \in \mathcal{N}_\psi \). When \( \psi \) is densely defined \( \mathcal{N}_\psi \) is a dense left ideal in \( A \) by (a) of Lemma 1.0.1. By construction \( (H_\psi, \Lambda_\psi, \pi_\psi) \) is then a GNS representation of \( A \). We shall refer to \( (H_\psi, \Lambda_\psi, \pi_\psi) \) as the GNS-triple of \( \psi \).
1.2.2 The weight of a GNS representation

Let \((H, \Lambda, \pi)\) be a GNS representation of \(A\). Set

\[ F_A = \{ \omega \in A^*_+ : \omega(a^*a) \leq \langle \Lambda(a), \Lambda(a) \rangle \ \forall a \in D(\Lambda) \} \]

We denote by \(\pi(A)'\) the commutant of \(\pi(A)\) in \(B(H)_\omega\).

**Lemma 1.2.5.** For every \(\omega \in F_A\) there is an operator \(T_\omega \in \pi(A)\)' such that 
\(0 \leq T_\omega \leq 1\) and

\[ \omega(b^*a) = \langle T_\omega \Lambda(a), \Lambda(b) \rangle \ \forall a, b \in D(\Lambda). \tag{1.2.2} \]

**Proof.** Since \(\omega \in F_A\), the Cauchy-Schwarz inequality gives the estimate

\[ \omega(b^*a) \leq \| \Lambda(a) \| \| \Lambda(b) \| \ \forall a, b \in D(\Lambda). \tag{1.2.3} \]

It follows that there is a bounded sesquilinear form \(\langle \cdot, \cdot \rangle_\omega\) on \(H\) such that

\[ \langle \Lambda(a), \Lambda(b) \rangle_\omega = \omega(b^*a) \]

for all \(a, b \in D(\Lambda)\). There is therefore a bounded operator \(T_\omega \in B(H)\) such that (1.2.2) holds. (See for example Theorem 2.4.1 in [KR].) It follows from (1.2.3) that \(\| T_\omega \| \leq 1\) and from the positivity of \(\omega\) that \(T_\omega \geq 0\). Finally, since

\[ \langle \pi(x)T_\omega \Lambda_\omega(a), \Lambda(b) \rangle = \langle T_\omega \Lambda(a), \pi(x)^* \Lambda(b) \rangle = \langle T_\omega \Lambda(a), \Lambda(x^*b) \rangle = \omega(b^*xa) = \langle T_\omega \Lambda(xa), \Lambda(b) \rangle = \langle T_\omega \pi(x) \Lambda(a), \Lambda(b) \rangle \]

for all \(x \in A, a, b \in D(\Lambda)\), it follows that \(T_\omega \in \pi(A)\). \(\square\)

**Lemma 1.2.6.** Let \(B\) be a C*-algebra and \(0 \leq T_i \leq 1, \ i = 1, 2\) elements of \(B\). Let \(\lambda_i \in [0, 1[, i = 1, 2\) and numbers \(\lambda \in [0, 1[\) and \(r > 0\) such that \(\lambda_i T_i \leq \lambda T, \ i = 1, 2,\) and \(T \leq r(T_1 + T_2)\).

**Proof.** Choose \(\max \{ \lambda_1, \lambda_2 \} < \gamma < 1\). Set

\[ S_i = (1 - \gamma T_i)^{-1} \gamma T_i, \ i = 1, 2, \]

and

\[ T = (1 + S_1 + S_2)^{-1}(S_1 + S_2) \in B. \]

The function \(t \mapsto \frac{t}{1+t} = 1 - (1+t)^{-1}\) is operator monotone by Proposition 1.3.6 in [Pe] and hence

\[ T \geq (1 + S_i)^{-1} S_i = \gamma T_i, \ i = 1, 2. \]

Put \(\lambda = \max \{ \frac{\lambda_1}{\gamma}, \frac{\lambda_2}{\gamma} \} \in [0, 1[\) and note that \(\lambda T \geq \lambda \gamma T_i \geq \lambda_i T_i, \ i = 1, 2\). On the other hand, it follows directly from the definitions that

\[ T \leq S_1 + S_2 \leq \frac{\gamma}{1 - \gamma}(T_1 + T_2). \]

Set \(r = \frac{\gamma}{1 - \gamma} \). \(\square\)
Lemma 1.2.7. Let \((H, \Lambda, \pi)\) be a GNS representation of \(A\). Let \(\omega_1, \omega_2 \in \mathcal{F}_\Lambda\) and \(\lambda_1, \lambda_2 \in [0, 1]\) be given. There is an \(\omega \in \mathcal{F}_\Lambda\) and a \(\lambda \in [0, 1]\) such that \(\lambda_i \omega_i \leq \lambda \omega\) in \(A^*_+\), \(i = 1, 2\).

Proof. By Lemma 1.2.4 there are operators \(T_{\omega_i} \in \pi(A)\) such that \(0 \leq T_{\omega_i} \leq 1\) and \(\omega_i(b^*a) = \langle T_{\omega_i}(\Lambda(a)\Lambda(b))\rangle, \forall a, b \in D(\Lambda), i = 1, 2\). From Lemma 1.2.7 we get an operator \(0 \leq T \leq 1\) in \(\pi(A)\) and numbers \(\lambda \in [0, 1]\) and \(r > 0\) such that \(\lambda_i T_{\omega_i} \leq \lambda \Lambda T, i = 1, 2\), and \(T \leq W\) where \(W = r(T_{\omega_1} + T_{\omega_2})\). We define a sesquilinear form \(s : D(\Lambda) \times D(\Lambda) \to \mathbb{C}\) such that

\[
s(a, b) = \langle T\Lambda(a), \Lambda(b) \rangle.
\]

Note that

\[
|s(a, b)| = \left|\langle T^{1/2}\Lambda(a), T^{1/2}\Lambda(b) \rangle\right| \leq \left\| T^{1/2}\Lambda(a) \right\| \left\| T^{1/2}\Lambda(b) \right\|
\]

\[
= (T\Lambda(a), \Lambda(a))^\frac{1}{2} (T\Lambda(b), \Lambda(b))^\frac{1}{2}
\]

\[
\leq (W\Lambda(a), \Lambda(a))^\frac{1}{2} (W\Lambda(b), \Lambda(b))^\frac{1}{2}
\]

\[
= r\sqrt{(\omega_1 + \omega_2)(a^*a)(\omega_1 + \omega_2)(b^*b)} \leq r \|\omega_1 + \omega_2\| |a||b|.
\]

It follows that \(s\) extends by continuity to a sesquilinear form \(s : A \times A \to \mathbb{C}\) such that the estimate

\[
|s(a, b)| \leq r \|\omega_1 + \omega_2\| |a||b|
\]

holds for all \(a, b \in A\). Using that \(T \in \pi(A)\) we find

\[
s(xa, b) = \langle T\Lambda(xa), \Lambda(b) \rangle = \langle \pi(x)T\Lambda(a), \Lambda(b) \rangle
\]

\[
= \langle \pi(x)\Lambda(a), \Lambda(b) \rangle = (T\Lambda(a), \Lambda(xb)) = s(a, xb)
\]

when \(x \in A, a, b \in D(\Lambda)\). It follows by continuity that \(s(xa, b) = s(a, xb)\) for all \(x, a, b \in A\). Let \(\{v_i\}_{i \in I}\) be a net in \(A\) such that \(0 \leq v_i \leq 1\) in \(A\) when \(i \leq j\) and such that \(\lim_{i \to \infty} v_i a = a\) for all \(a \in A\). It follows from Proposition 2.3.11 of \([BR]\) that \(\lim_{i \to \infty} r(\omega_1 + \omega_2)(v_i) = r \|\omega_1 + \omega_2\|\). Let \(a \in A\). We can then choose a sequence \(\{i_n\}_{n=1}^{\infty} \) in \(I\) such that

(i) \(i_n \leq i_{n+1}\) for all \(n\),

(ii) \(r(\omega_1 + \omega_2)(v_i) \geq r \|\omega_1 + \omega_2\| - \frac{1}{n}\) \(\forall i \geq i_n\), and

(iii) \(\|v_i a - a\| \leq \frac{1}{n}\) \(\forall i \geq i_n\).

The estimate

\[
|s(a, b)| \leq r\sqrt{(\omega_1 + \omega_2)(a^*a)(\omega_1 + \omega_2)(b^*b)}
\]

which was established in (1.2.4) when \(a, b \in D(\Lambda)\) extends by continuity to all \(a, b \in A\). It follows that for \(i \geq j\) in \(I\),

\[
|s(a, v_i) - s(a, v_j)| \leq r\sqrt{(\omega_1 + \omega_2)(a^*a)} \sqrt{(\omega_1 + \omega_2)((v_i - v_j)^2)}
\]

\[
\leq r\sqrt{(\omega_1 + \omega_2)(a^*a)} \sqrt{(\omega_1 + \omega_2)(v_i - v_j)}.
\]
Since \( \lim_{n \to \infty} r(\omega_1 + \omega_2)(v_{i_n}) = r \| \omega_1 + \omega_2 \| \) this estimate implies that \( \{ s(a, v_{i_n}) \} \) is a Cauchy sequence and we set

\[
\omega(a) = \lim_{n \to \infty} s(a, v_{i_n}).
\] (1.2.7)

To see that the limit (1.2.7) is independent of the choice of sequence \( \{ i_n \} \) satisfying (i), (ii) and (iii), let \( \{ j_n \}_{n=1}^{\infty} \) be another sequence in \( I \) with these three properties. Since \( I \) is directed we can then find a sequence \( \{ k_n \}_{n=1}^{\infty} \) in \( I \) such that \( i_n \leq k_n \) and \( j_n \leq k_n \) in \( I \). Inserting \( k_n \) for \( i \) and first \( i_n \) and next \( j_n \) for \( j \) in (1.2.6) we find that

\[
\lim_{n \to \infty} s(a, v_{i_n}) = \lim_{n \to \infty} s(a, v_{k_n}) = \lim_{n \to \infty} s(a, v_{j_n}).
\]

Thus \( \omega(a) \) does not depend on which sequence \( \{ i_n \} \) in \( I \) with the properties (i), (ii) and (iii) we use. It follows therefore that (1.2.7) defines a linear functional \( \omega : A \to \mathbb{C} \) and from (1.2.3) we conclude that \( \| \omega \| \leq r \| \omega_1 + \omega_2 \| \). Note that for any \( a \in A \) we can find a sequence \( \{ i_n \} \) in \( I \) such that

\[
\omega(a^*a) = \lim_{n \to \infty} s(a^*a, v_{i_n}) = \lim_{n \to \infty} s(a, av_{i_n}) = s(a, a).
\]

When \( a \in D(\Lambda) \) we find that

\[
\lambda \omega(a^*a) = \lambda s(a, a) = \lambda \langle T \Lambda(a), \Lambda(a) \rangle \\
\geq \lambda_i \langle T_{\omega_i} \Lambda(a), \Lambda(a) \rangle = \lambda_i \omega_i(a^*a), \ i = 1, 2,
\]

and

\[
\omega(a^*a) = s(a, a) = \langle T \Lambda(a), \Lambda(a) \rangle \leq \langle \Lambda(a), \Lambda(a) \rangle.
\]

The first inequality implies that \( \omega \in A^*_\Lambda \) and \( \lambda \omega \geq \lambda_i \omega_i, \ i = 1, 2 \), and the last inequality implies that \( \omega \in \mathcal{F}_\Lambda \).

Define \( \phi : A^+ \to [0, \infty] \) by

\[
\phi(a) = \sup_{\omega \in \mathcal{F}_\Lambda} \omega(a).
\]

**Lemma 1.2.8.** \( \phi \) is a weight.

**Proof.** It is clear that \( \phi \) is a semi-weight but it remains to show that \( \phi \) is additive. Let \( a, b \in A^+ \). Then \( \phi(a+b) \leq \phi(a) + \phi(b) \) holds by definition of \( \phi \). To conclude that

\[
\phi(a) + \phi(b) \leq \phi(a + b)
\]

we may assume that \( \phi(a + b) < \infty \), and it follows then that \( \phi(a) < \infty \) and \( \phi(b) < \infty \). Let \( \epsilon > 0 \). There are \( \omega_1, \omega_2 \in \mathcal{F}_\Lambda \) such that \( \omega_1(a) > \phi(a) - \epsilon \) and \( \omega_2(b) > \phi(b) - \epsilon \). There is also a number \( \lambda \in [0, 1] \) such that \( \lambda \omega_1(a) > \phi(a) - \epsilon \) and \( \lambda \omega_2(b) > \phi(b) - \epsilon \). By Lemma 1.2.7 there is an element \( \omega \in \mathcal{F}_\Lambda \) such that \( \lambda \omega_1 \leq \omega, \ i = 1, 2 \). It follows that \( \phi(a+b) \geq \omega(a+b) \geq \lambda \omega_1(a) + \lambda \omega_2(b) \geq \phi(a) + \phi(b) - 2\epsilon \). This shows that \( \phi(a+b) \geq \phi(a) + \phi(b) \), and we conclude that \( \phi \) is a weight.
We will refer to \( \phi \) as the weight of the GNS representation. We note that all densely defined weights arise in this way:

**Lemma 1.2.9.** Let \( \psi \) be a densely defined weight on \( A \). Then \( \psi \) is the weight of its GNS representation \( (H_\psi, \Lambda_\psi, \pi_\psi) \).

**Proof.** This is a re-formulation of Combes’ theorem, Theorem 1.1.1, for densely defined weights.

### 1.2.3 Proof of Combes’ theorem in the separable case

**Proof of Theorem 1.2.1** Assume first that \( \psi \) is densely defined. Let \( C \) be the countable subset of \( \mathcal{F}_\psi \) from Lemma 1.2.2 and set

\[
C' = \{ \lambda \omega : \lambda \in [0, 1] \cap \mathbb{Q}, \omega \in C \} \subseteq \mathcal{F}_\psi.
\]

Let \( \mu_i, i \in \mathbb{N} \), be a numbering of the elements of \( C' \). We construct by induction a sequence \( \mu_1' \leq \mu_2' \leq \mu_3' \leq \cdots \) in \( \mathcal{F}_\psi \) such that \( \mu_i \leq \mu_i', i \leq n, \) and \( \mu_n'^i \in \lambda_n \mathcal{F}_\psi \) for some \( \lambda_n \in [0, 1] \). To start the induction set \( \mu_1' = \mu_1 \). When \( \mu_1' \leq \mu_2' \leq \cdots \) \( \mu_n' \) have been constructed we construct \( \mu_{n+1}' \) by using Lemma 1.2.7 to find \( \mu_{n+1}' \in \lambda_{n+1} \mathcal{F}_\psi \) for some \( \lambda_{n+1} \in [0, 1] \) such that \( \mu_{n+1}' \geq \mu_n' \) and \( \mu_{n+1}' \geq \mu_n+1 \). Then \( \{ \mu_n' \}_{n=1}^\infty \) has the stated properties and we claim that

\[
\lim_{n \to \infty} \mu_n'(a) = \psi(a)
\]

for all \( a \in A^+ \). Indeed, when \( a \in A^+ \) and \( N \in \mathbb{N} \) are given we can use Lemma 1.2.2 to get an element \( f \in \mathcal{C} \) such that \( f(a) \geq \min\{ N+1, \psi(a) - \frac{1}{N+1} \} \). Choose \( \lambda \in [0, 1] \cap \mathbb{Q} \) such that \( \lambda f(a) \geq f(a) - \left( \frac{1}{N} - \frac{1}{N+1} \right) \). Then \( \lambda f \in C' \) and

\[
\lambda f(a) \geq \min\{ N, \psi(a) - \frac{1}{N} \}.
\]

There is an \( i \in \mathbb{N} \) such that \( \mu_i = \lambda f \), and then

\[
\psi(a) \geq \mu_k'(a) \geq \mu_i(a) = \lambda f(a) \geq \min\{ N, \psi(a) - \frac{1}{N} \}
\]

for all \( k \geq i \), proving the claim. We define the sequence \( \{ \omega_n \}_{n=1}^\infty \) in \( A_+^* \) such that \( \omega_1 = \mu_1' \) and \( \omega_n = \mu_n' - \mu_{n-1}' \), \( n \geq 2 \). Then (1.2.1) holds for all \( a \in A^+ \).

Consider then the case where \( \psi \) is not densely defined. Set \( B = \overline{\mathcal{M}_\psi} \) which is a \( C^* \)-subalgebra of \( A \) by (c) of Lemma 1.0.1. Let \( b \in B^+ \). Then \( \sqrt{b} \in B \) and we can approximate \( \sqrt{b} \) by an element \( x \in \mathcal{M}_\psi \). Then \( x^*x \) approximates \( b \) and \( x^*x \in \mathcal{M}_\psi^+ \) by (b) and (d) of Lemma 1.0.1. This shows that \( \psi |_{B^+} \) is a densely defined weight on \( B \). It follows then from the first part of the proof that there is a sequence \( \{ \omega_n' \}_{n=1}^\infty \) in \( B_+^* \) such that \( \sum_{n=1}^\infty \omega_n'(b) = \psi(b) \) for all \( b \in B^+ = B \cap A^+ \). Choose \( \omega_n \in A_+^* \) such that \( \omega_n |_{B^+} = \omega_n' \). The closure \( \overline{\mathcal{N}_\psi} \) of \( \mathcal{N}_\psi \) is a closed left ideal in \( A \) by Lemma 1.0.1 and we set \( F = \{ \omega \in A_+^* : \omega(\overline{\mathcal{N}_\psi}) = \{ 0 \}, \| \omega \| \leq 1 \} \), which is a weak* closed face in the quasi-state space of \( A \). It follows from a
result of Effros, reproduced in Theorem 3.10.7 of [Pe], that for every element $a \in A^+ \setminus \mathcal{N}_\psi$ there is $\omega \in F$ such that $\omega(a) > 0$. Indeed, if not the closure of $Aa + \mathcal{N}_\psi$ would be a closed left ideal $\mathcal{I}$ of $A$ strictly larger than $\mathcal{N}_\psi$ with
\[
\{ \omega \in A^*_+ : \omega(\mathcal{I}) = \{0\} \} = F,
\]
contradicting Effros’ theorem. Note that $F$ is a compact metrizable space in the weak* topology because $A$ is separable. There is therefore a countable subset $\mathcal{C}'' \subseteq F$ which is dense in $F$. Therefore
\[
\sum_{n=1}^{\infty} \omega_n(b) = \sum_{n=1}^{\infty} \omega'_n(b) = \psi(b) \quad \forall b \in B \cap A^+.
\]
Consider an element $a \in \mathcal{N}_\psi \cap A^+$. There is a sequence $\{x_n\}$ in $\mathcal{N}_\psi$ such that $\lim_{n \to \infty} x_n x_n = a^2$ so (b) of Lemma 1.0.1 implies that $a^2 \in B$. Since $B$ is a $C^*$-algebra it follows that $a = \sqrt{a^2} \in B$. This shows that $\mathcal{N}_\psi \cap A^+ \subseteq B \cap A^+$, implying that $\psi(a) = \infty$ when $a \in A^+ \setminus B$. For comparison we note that
\[
\sum_{n=1}^{\infty} \omega_n(a) \geq \sum_{n=1}^{\infty} \kappa_n(a) = \infty
\]
when $a \in A^+ \setminus B \subseteq A^+ \setminus \mathcal{N}_\psi$. It follows that the equality in (1.2.1) holds for all $a \in A^+$.

**Notes and remarks 1.2.10.** A version of Theorem 1.2.1, where (1.2.1) is only established for $a \in \mathcal{M}_\psi^+$, was obtained by Combes in Proposition 1.11 of [C1]. A version for non-separable algebras was obtained by Pedersen and Takesaki in Corollary 7.2 of [PT]. The GNS construction and Lemma 1.2.5 was introduced in [CT] while Lemma 1.2.6 and Lemma 1.2.7 are taken from [Ku1] and [QV].
Chapter 2

KMS weights

2.1 Flows and holomorphic extensions

Let $X$ be a complex Banach space. A flow on $X$ is a representation $\sigma = (\sigma_t)_{t \in \mathbb{R}}$ of the group of real numbers $\mathbb{R}$ by linear isometries of $X$ such that

$$\lim_{t \to t_0} \| \sigma_t(a) - \sigma_{t_0}(a) \| = 0$$

for all $t_0 \in \mathbb{R}$ and all $a \in X$.

2.1.1 Entire elements

Let $X$ be a complex Banach space and let $\sigma$ be a flow on $X$. An element $a \in X$ is entire analytic for $\sigma$ if the function $t \mapsto \sigma_t(a)$ is entire analytic in the sense that there is a sequence $\{c_n(a)\}_{n=0}^{\infty}$ in $X$ such that

$$\sum_{n=0}^{\infty} c_n(a) t^n = \sigma_t(a), \quad (2.1.1)$$

with norm-convergence, for all $t \in \mathbb{R}$. Let $A_\sigma$ denote the set of elements of $X$ that are entire analytic for $\sigma$.

Lemma 2.1.1. Let $a \in A_\sigma$. The sequence $\{c_n(a)\}_{n=0}^{\infty}$ of $X$ for which (2.1.1) holds with norm-convergence for all $t \in \mathbb{R}$ is unique, and the series

$$\sum_{n=0}^{\infty} \| c_n(a) \| z^n$$

converges for all $z \in \mathbb{C}$.

Proof. Let $\varphi \in X^*$. Then $\varphi(\sigma_t(a)) = \sum_{n=0}^{\infty} \varphi(c_n(a)) t^n$ for all $t \in \mathbb{R}$, which implies that the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \varphi(c_n(a)) z^n \quad (2.1.2)$$
CHAPTER 2. KMS WEIGHTS

is infinite, and hence the formula (2.1.2) defines an entire holomorphic function $f_{\varphi} : \mathbb{C} \to \mathbb{C}$. If $\{c_n'(a)\}_{n=0}^{\infty}$ is another sequence for which (2.1.1) holds we get in the same way an entire holomorphic function $f_{\varphi}' : \mathbb{C} \to \mathbb{C}$ such that $f_{\varphi}'(z) = \sum_{n=0}^{\infty} \varphi(c_n'(a))z^n$. Since $f_{\varphi}(t) = f_{\varphi}'(t) = \varphi(\sigma_t(a))$ for all $t \in \mathbb{R}$ the two entire holomorphic functions must be identical, implying that $\varphi(c_n(a)) = \varphi(c_n'(a))$ for all $n$. Since $\varphi \in X^*$ was arbitrary we conclude that $c_n(a) = c_n'(a)$ for all $n$.

The Cauchy-estimates, see e.g. Theorem 10.26 of [Ru], applied to the function $f_{\varphi}$ show that

$$|\varphi(c_n(a))| \leq \frac{1}{R^n} \sup_{|z| \leq R} |f_{\varphi}(z)|$$

for all $n \in \mathbb{N}$ and all $R > 0$. The uniform boundedness theorem applied to the family of linear maps $X^* \to \mathbb{C}$ given by

$$X^* \ni \varphi \mapsto f_{\varphi}(z), \quad |z| \leq R,$$

shows that $\sup_{||\varphi|| \leq 1} \sup_{|z| \leq R} |f_{\varphi}(z)| := M_R < \infty$, leading to the conclusion that

$$\|c_n(a)\| \leq \frac{M_R}{R^n}$$

for all $n \in \mathbb{N}$ and all $R > 0$. This implies $\sum_{n=0}^{\infty} \|c_n(a)\| z^n$ converges for all $z \in \mathbb{C}$.

An element $a \in X$ is entire holomorphic for $\sigma$ when there is a function $f : \mathbb{C} \to X$ such that $f(t) = \sigma_t(a)$ for all $t \in \mathbb{R}$ and such that

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

exists in $X$ for all $z \in \mathbb{C}$. Since entire holomorphic functions that agree on $\mathbb{R}$ are identical, the function $f$ is uniquely determined by $a$ when it exists.

**Proposition 2.1.2.** An element $a$ of $X$ is entire analytic for $\sigma$ if and only it is entire holomorphic for $\sigma$.

**Proof.** Assume first that $a$ is entire analytic for $\sigma$. By Lemma 2.1.1 we can define $f : \mathbb{C} \to X$ such that

$$f(z) = \sum_{n=0}^{\infty} c_n(a)z^n.$$

For any $\varphi \in X^*$, $\varphi(f(z)) = \sum_{n=0}^{\infty} \varphi(c_n(a))z^n$ is a analytic and hence holomorphic on $\mathbb{C}$; whence $f$ is entire analytic by Proposition 2.5.21 in [BR].

Assume then that $a$ is entire holomorphic for $\sigma$ and let $f : \mathbb{C} \to X$ be a function such that $f(t) = \sigma_t(a)$ for all $t \in \mathbb{R}$ and such that $\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$ exists in $X$ for all $z \in \mathbb{C}$. Define $f' : \mathbb{C} \to X$ such that

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}.$$
For any $\chi \in X^*$,
\[ \chi(f'(z)) = \frac{d}{dz} \chi(f(z)), \]
and hence $\chi(f'(z))$ is entire holomorphic since $\chi(f(z))$ is. By Proposition 2.5.21 in [BR] this implies that $f' : \mathbb{C} \to X$ is entire holomorphic. Set
\[ \delta(a) := f'(0) = \lim_{s \to 0} \frac{\sigma_s(a) - a}{s}, \]
and note that
\[ f'(t) = \lim_{s \to 0} \frac{\sigma_{t+s}(a) - \sigma_t(a)}{s} = \lim_{s \to 0} \sigma_t \left( \frac{\sigma_s(a) - a}{s} \right) = \sigma_t(\delta(a)) \forall t \in \mathbb{R}. \]
We can therefore continue by induction to get elements $\delta^k(a) \in X$ and entire holomorphic functions $f^k : \mathbb{C} \to X$ such that
\[ f^k(z) = \lim_{h \to 0} \frac{f^{k-1}(z+h) - f^{k-1}(z)}{h} \quad \forall z \in \mathbb{C} \]
when $k \geq 1$. Let $\varphi \in X^*$. Since $z \mapsto \varphi(f(z))$ is entire holomorphic we know that
\[ \varphi(f(z)) = \sum_{k=0}^{\infty} \varphi(\delta^k(a)) \frac{z^k}{k!} \quad \forall z \in \mathbb{C}. \]
and it follows from the preceding that
\[ \frac{d^k}{dz^k} \varphi(f(z))|_{z=0} = \varphi(\delta^k(a)) \]
for all $k$. Hence
\[ \varphi(f(z)) = \sum_{k=0}^{\infty} \varphi(\delta^k(a)) \frac{z^k}{k!} \quad \forall z \in \mathbb{C}. \]
Since this holds for all $\varphi \in X^*$ it follows from the Cauchy estimates and the principle of uniform boundedness, exactly as in the proof of Lemma 2.1.1 that
\[ \sum_{k=0}^{\infty} \frac{\|\delta^k\|}{k!} z^n \text{ converges for all } z \in \mathbb{C}. \]
Since
\[ \sigma_t(a) = f(t) = \sum_{k=0}^{\infty} \frac{\delta^k(a)}{k!} t^n \quad \forall t \in \mathbb{R}, \]
it follows that $a$ is entire analytic for $\sigma$.

**Lemma 2.1.3.** Let $\{a_n\}_{n=0}^{\infty}$ be a sequence in $\mathbb{C}$ and $\{b_n\}_{n=0}^{\infty}$ a sequence in $X$ such that $\sum_{n=0}^{\infty} |a_n| |z|^n < \infty$ and $\sum_{n=0}^{\infty} ||b_n|| |z|^n < \infty$ for all $z \in \mathbb{C}$. Set
\[ c_n = \sum_{i=0}^{n} a_i b_{n-i}, \quad n \in \mathbb{N}. \]
Then
\[
\sum_{n=0}^{\infty} c_n z^n = \left( \sum_{n=0}^{\infty} a_n z^n \right) \left( \sum_{n=0}^{\infty} b_n z^n \right)
\]  
with norm-convergence, for all \( z \in \mathbb{C} \).

**Proof.** Let \( z \in \mathbb{C} \). It follows easily from the assumptions on \( \{a_n\}_{n=0}^{\infty} \) and \( \{b_n\}_{n=0}^{\infty} \) that the series \( \sum_{n=0}^{\infty} a_n z^n \) and \( \sum_{n=0}^{\infty} b_n z^n \) converge in \( \mathbb{C} \) and \( X \), respectively. It remains therefore only to establish (2.1.3). Let \( N \in \mathbb{N} \). Then
\[
\left\| \sum_{n=0}^{N} a_n z^n \right\| \left\| \sum_{n=0}^{N} b_n z^n \right\| = \sum_{n=0}^{N} a_n z^n b_n z^n \leq \sum_{n=0}^{\infty} |a_n||z|^n \sum_{n=0}^{N} |b_n||z|^n \leq \left\| a_n \right\| \left\| z \right\|^n \sum_{n=0}^{N} \left\| b_n \right\| \left\| z \right\|^n
\]
where \( \left\lfloor \frac{N}{2} \right\rfloor \) denotes the integer part of \( \frac{N}{2} \). This estimate proves the lemma. \( \square \)

We are going to use some integration theory for Banach space valued function and in Appendix A the reader may find the proofs of the few facts that we shall need.

**Lemma 2.1.4.** Let \( n \in \mathbb{N} \) and \( a \in X \). There is a sequence \( \{b_k\}_{k=0}^{\infty} \) in \( X \) such that \( \sum_{k=0}^{\infty} \|b_k\||z|^k < \infty \) and
\[
\sum_{k=0}^{\infty} b_k z^k = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-n(s-z)^2} \sigma_s(a) \, ds
\]
for all \( z \in \mathbb{C} \).

**Proof.** Write
\[
e^{-n(s-z)^2} = e^{-n s^2} e^{2nsz} e^{-n s^2} = e^{-n s^2} \sum_{k=0}^{\infty} z^k (2n)^k \frac{s^k e^{-n s^2}}{k!}.
\]
Since \( s \mapsto e^{n(2z||s|-s^2)} \) is in \( L^1(\mathbb{R}) \) it follows that \( \int_{\mathbb{R}} e^{-n(s-z)^2} \sigma_s(a) \, ds \) exists, for example by using Proposition 22.5.18 in [BR] or Lemma A.1.2 in Appendix A. Since \( e^{-n(2z||s|-s^2)} \) dominates \( \sum_{k=0}^{\infty} z^k (2n)^k \frac{s^k e^{-n s^2}}{k!} \), for all \( N \) it follows from Lemma A.2.2 in Appendix A that we can interchange integration with summation to get
\[
\sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-n(s-z)^2} \sigma_s(a) \, ds = \sqrt{\frac{n}{\pi}} e^{-n s^2} \sum_{k=0}^{\infty} \frac{z^k (2n)^k}{k!} \int_{\mathbb{R}} s^k e^{-n s^2} \sigma_s(a) \, ds.
\]  
(2.1.4)
Set
\[ b_k = \frac{(2n)^k}{k!} \int_{\mathbb{R}} s^k e^{-ns^2} \sigma_s(a) \, ds \in X. \]

Note that
\[ \left\| \int_{\mathbb{R}} s^k e^{-ns^2} \sigma_s(a) \, ds \right\| \leq 2\|a\| \int_0^\infty t^k e^{-nt^2} \, dt = 2\|a\| n^{\frac{k}{2}} \int_0^\infty t^k e^{-t^2} \, dt. \]

Set \( I_k = \int_0^\infty t^k e^{-t^2} \, dt \) so that
\[ \|b_k\| \leq \lambda_k \tag{2.1.5} \]

where
\[ \lambda_k = 2^{k+1} \|a\| n^{\frac{k+1}{2}} \frac{I_k}{k!} \]

for all \( k \in \mathbb{N} \). By using the relation
\[ I_k = \int_0^\infty t^{k-1} \frac{d}{dt} \left( \frac{1}{2} e^{-t^2} \right) \, dt = \frac{k-1}{2} \int_0^\infty t^{k-2} e^{-t^2} \, dt = \frac{k-1}{2} I_{k-2} \]

for \( k \geq 2 \), it follows that
\[ I_k = 2^{-\frac{k}{2}} (k-1)(k-3)(k-5) \cdots 3I_0 \]

when \( k \) is even and
\[ I_k = 2^{-\frac{k+1}{2}} (k-1)(k-3)(k-5) \cdots 4 \cdot 2I_1 \]

when \( n \) is odd. Hence
\[ \frac{\lambda_{k+1}}{\lambda_k} = 2\sqrt{n} \frac{k(k-2)(k-4) \cdots 4 \cdot 2 \cdot I_1}{(k+1)(k-1)(k-3) \cdots 3 \cdot I_0} \]

when \( k \) is even while
\[ \frac{\lambda_{k+1}}{\lambda_k} = 2\sqrt{n} \frac{k(k-2)(k-4) \cdots 3 \cdot I_0}{(k+1)(k-1)(k-3) \cdots 4 \cdot 2 \cdot I_1} \]

when \( k \) is odd. By using that \( \frac{k}{k+1} \leq e^{-\frac{1}{k+1}} \) we find that
\[ \frac{k(k-2)(k-4) \cdots 4 \cdot 2}{(k+1)(k-1)(k-3) \cdots 3} \leq \exp \left( -\sum_{j=1}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{1}{2j+1} \right) \]

when \( k \) is even and
\[ \frac{k(k-2)(k-4) \cdots 3}{(k+1)(k-1)(k-3) \cdots 4 \cdot 2} \leq \exp \left( -\sum_{j=1}^{\left\lfloor \frac{k+1}{2} \right\rfloor} \frac{1}{2j} \right) \]

when \( k \) is odd. It follows that \( \lim_{k \to \infty} \frac{\lambda_{k+1}}{\lambda_k} = 0 \) and hence that \( \sum_{k=0}^\infty \lambda_k |t|^k < \infty \)

for all \( t \). Combined with (2.1.5) this shows that \( \sum_{k=0}^\infty \|b_k\| |z|^k < \infty \) for all
$z \in \mathbb{C}$. It follows then from Lemma 2.1.3 that there is a sequence $\{c_k\}_{k=0}^{\infty}$ in $X$ such that

$$\sqrt{\frac{n}{\pi}} e^{-nx^2} \sum_{k=0}^{\infty} \frac{z^k (2n)^k}{k!} \int_{\mathbb{R}} s^k e^{-ns^2} \sigma_s(a) \, ds$$

$$= \sqrt{\frac{n}{\pi}} \left( \sum_{k=0}^{\infty} \frac{(-nx^2)^k}{k!} \right) \left( \sum_{k=0}^{\infty} b_k z^k \right) = \sum_{k=0}^{\infty} c_k z^k,$$

with norm-convergence, for all $z \in \mathbb{C}$. \hfill \Box

For $n \in \mathbb{N}$ and $a \in X$, set

$$R_n(a) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} \sigma_s(a) \, ds. \quad (2.1.6)$$

Then $R_n : X \to X$ is a linear operator, and a contraction since $\sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} \, ds = 1$. These operators will be used diligently in the following, and we shall refer to them as smoothing operators.

**Lemma 2.1.5.** $R_n(a) \in \mathcal{A}_\sigma$ for all $n \in \mathbb{N}$, $a \in X$, and $\lim_{n \to \infty} R_n(a) = a$ for all $a \in X$.

**Proof.** Using Lemma 2.1.4 we find that

$$\sigma_t(R_n(a)) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} \sigma_{t+s}(a) \, ds$$

$$= \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-n(s-t)^2} \sigma_s(a) \, ds = \sum_{k=0}^{\infty} b_k t^k$$

for all $t \in \mathbb{R}$, proving that $R_n(a) \in \mathcal{A}_\sigma$. Let $\epsilon > 0$. Note that

$$\|R_n(a) - a\| = \left\| \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} (\sigma_s(a) - a) \, ds \right\|$$

$$\leq \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} \|\sigma_s(a) - a\| \, ds.$$

Choose $\delta > 0$ such that $\|\sigma_s(a) - a\| \leq \frac{\epsilon}{2}$ when $|s| \leq \delta$. Since $\sqrt{n} e^{-ns^2}$ decreases with $n$ when $|s| \geq \delta$ and $n \geq (2\delta^2)^{-1}$, and converges to 0 for $n \to \infty$, it follows from Lebesgue’s theorem that there is an $N \in \mathbb{N}$ such that

$$\sqrt{\frac{n}{\pi}} \int_{|s| \geq \delta} e^{-ns^2} \|\sigma_s(a) - a\| \, ds \leq \frac{\epsilon}{2}.$$
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when \(n \geq N\). Then, for \(n \geq N\),

\[
\| R_n(a) - a \| \leq \frac{\sqrt{\pi}}{n} \int_{|s| \geq \delta} e^{-ns^2} \| \sigma_s(a) - a \| ds + \frac{\sqrt{\pi}}{n} \int_{|s| \leq \delta} e^{-ns^2} \| \sigma_s(a) - a \| ds
\]

\[
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \sqrt{\frac{n}{\pi}} \int_{|s| \leq \delta} e^{-ns^2} ds \leq \epsilon.
\]

\[
\]

For every \(z \in \mathbb{C}\) we define an operator \(\sigma_z : A_\sigma \to X\) such that

\[
\sigma_z(a) = \sum_{n=0}^{\infty} c_n(a) z^n,
\]

where \(\{c_n(a)\}_{n=0}^{\infty}\) is the sequence from (2.1.1) and Lemma 2.1.1. Alternatively, \(\sigma_z(a)\) can be defined as \(f(z)\), where \(f : \mathbb{C} \to X\) is entire holomorphic and has the property that \(f(t) = \sigma_t(a)\) when \(t \in \mathbb{R}\). It follows in particular that the notation is consistent; when \(z \in \mathbb{R}\) the two ways of defining \(\sigma_z(a)\) agree. In the remaining part of the section we prove a series of facts about the operators \(\sigma_z\) that we shall need later on.

Lemma 2.1.6. Let \(a \in A_\sigma\). Then

(a) \(\sigma_z(a) \in A_\sigma\) for all \(z \in \mathbb{C}\), and

(b) \(\sigma_w \circ \sigma_z(a) = \sigma_z \circ \sigma_w(a) = \sigma_{z+w}(a)\) for all \(z, w \in \mathbb{C}\).

Proof. Let \(t \in \mathbb{R}\). We prove first that

(a') \(\sigma_t(a) \in A_\sigma\), and

(b') \(\sigma_t \circ \sigma_z(a) = \sigma_z \circ \sigma_t(a) = \sigma_{t+z}(a)\) for all \(t \in \mathbb{R}\) and all \(z \in \mathbb{C}\).

For all \(s \in \mathbb{R}\),

\[
\sigma_s(\sigma_t(a)) = \sigma_t(\sigma_s(a)) = \sigma_t\left(\sum_{n=0}^{\infty} c_n(a) s^n\right) = \sum_{n=0}^{\infty} \sigma_t(c_n(a)) s^n,
\]

with norm-convergence. This proves (a'). To prove (b') note that the three functions \(\mathbb{C} \to X\) given by \(z \mapsto \sigma_z(\sigma_t(a)), z \mapsto \sigma_t(\sigma_z(a)),\) and \(z \mapsto \sigma_{t+z}(a),\) are all entire holomorphic and agree on \(\mathbb{R}\). They are therefore identical.

Using (a') and (b') we obtain (a) and (b) as follows. By definition there is an entire holomorphic function \(f : \mathbb{C} \to X\) such that \(f(t) = \sigma_t(a)\) for all \(t \in \mathbb{R}\) and \(\sigma_z(a) = f(z)\). It follows from (b') that \(\sigma_t(\sigma_z(a)) = \sigma_{t+z}(a) = f(t+z)\). Since \(\mathbb{C} \ni w \mapsto f(w+z)\) is entire holomorphic it follows that \(\sigma_z(a) \in A_\sigma\) and that \(\sigma_w(\sigma_z(a)) = f(w+z) = \sigma_{w+z}(a)\).

Lemma 2.1.7. The operator \(\sigma_z\) is closable for all \(z \in \mathbb{C}\).
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Proof. When $z \in \mathbb{R}$ there is nothing to prove, so assume that $z \notin \mathbb{R}$. Let $\{a_n\}$ be a sequence in $A_\sigma$ such that $\lim_{n \to \infty} a_n = 0$ and $\lim_{n \to \infty} \sigma_z(a_n) = b$ in $X$. We must show that $b = 0$. It follows from Lemma 2.1.6 that $\lim_{n \to \infty} \sigma_{i\Im z}(a_n) = \sigma_{-\Re z}(b)$. Let $D_{\Im z}$ denote the strip in $\mathbb{C}$ consisting of the elements $u \in \mathbb{C}$ such that $\Im u$ is between 0 and $\Im z$. For $n, m \in \mathbb{N}$ and $u \in D_{\Im z}$, the Phragmen-Lindelöf theorem, Proposition 5.3.5 in [BR], shows that

$$|\varphi(\sigma_u(a_n)) - \varphi(\sigma_u(a_m))| \leq \max \{ ||a_n - a_m||, \|\sigma_{i\Im z}(a_n) - \sigma_{i\Im z}(a_m)\| \} \|\varphi\|$$

for all $\varphi \in X^*$. It follows that the sequence of functions $u \mapsto \varphi(\sigma_u(a_n))$, $n \in \mathbb{N}$, converge uniformly on $D_{\Im z}$ to a continuous function $f$ which is holomorphic in the interior of $D_{\Im z}$. Since $\lim_{n \to \infty} \sigma_t(a_n) = 0$, the function $f$ must vanish on $\mathbb{R}$ and it follows therefore from Proposition 5.3.6 in [BR], applied to $z \mapsto \overline{f(z)}$ if $\Im z < 0$, that $f = 0$. Since

$$f(i\Im z) = \varphi(\sigma_{-\Re z}(b)),$$

it follows that $\varphi(\sigma_{-\Re z}(b)) = 0$, and since $\varphi \in A^*$ is arbitrary, that $b = 0$. □

In the following the symbol $\sigma_z$ will denote the closure of the operator $\sigma_z : A_\sigma \to X$.

Lemma 2.1.8. Let $a \in D(\sigma_z)$. Then $D(\sigma_{t+z}) = D(\sigma_z)$, $\sigma_t(a) \in D(\sigma_z)$ and $\sigma_{t+z}(a) = \sigma_t(\sigma_z(a)) = \sigma_z(\sigma_t(a))$ for all $t \in \mathbb{R}$.

Proof. There is a sequence $\{a_n\}$ in $A_\sigma$ such that $\lim_{n \to \infty} a_n = a$ and

$$\lim_{n \to \infty} \sigma_z(a_n) = \sigma_z(a).$$

Using Lemma 2.1.6 this implies that

$$\lim_{n \to \infty} \sigma_{t+z}(a_n) = \lim_{n \to \infty} \sigma_z(\sigma_t(a_n)) = \lim_{n \to \infty} \sigma_t(\sigma_z(a_n)) = \sigma_t(\sigma_z(a)).$$

Since $\sigma_z$ and $\sigma_{i\Im z}$ are closed operators it follows that $a \in D(\sigma_{t+z})$, $\sigma_t(a) \in D(\sigma_z)$ and $\sigma_{t+z}(a) = \sigma_z(\sigma_t(a)) = \sigma_t(\sigma_z(a))$. The reverse inclusion $D(\sigma_{t+z}) \subseteq D(\sigma_z)$ follows by replacing $z$ with $z + t$ and $t$ with $-t$. □

Lemma 2.1.9. $\sigma_z(D(\sigma_z)) \subseteq D(\sigma_{-z})$ and $\sigma_{-z} \circ \sigma_z(a) = a \; \forall a \in D(\sigma_z)$.

Proof. Let $a \in D(\sigma_z)$. There is a sequence $\{a_n\}$ in $A_\sigma$ such that $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} \sigma_z(a_n) = \sigma_z(a)$. It follows from Lemma 2.1.6 that $\sigma_z(a_n) \in A_\sigma$ and $\sigma_{-z}(\sigma_z(a_n)) = a_n$. Since $\sigma_{-z}$ is closed it follows that $\sigma_{-z}(a) \in D(\sigma_{-z})$ and $\sigma_{-z} \circ \sigma_z(a) = a \; \forall a \in D(\sigma_z)$. □

Lemma 2.1.10. Assume $a \in D(\sigma_z)$. Then

$$\sigma_z(R_n(a)) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} \sigma_z(a) \; ds$$

$$= \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} \sigma_z^2(a) \; ds = R_n(\sigma_z(a)).$$
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Proof. It follows from Lemma 2.1.4 that \( z \mapsto f(z) := \sqrt{2\pi} \int_{\mathbb{R}} e^{-n(s^2)} \sigma_s(a) \, ds \) is entire holomorphic and since

\[
f(t) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} \sigma_{s+t}(a) \, ds = \sigma_t(R_n(a)) \quad \forall t \in \mathbb{R},
\]

the first equality is an immediate consequence of how \( \sigma_z \) is defined. For the second note that \( \sigma_z(\sigma_s(a)) = \sigma_z(\sigma_s(a)) \) depends continuously on \( s \) and that \( \sigma_z \) is closed. It follows therefore from Lemma A.2.1 that \( R_n(a) \in D(\sigma_z) \) and

\[
\sigma_z(R_n(a)) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} \sigma_s(\sigma_z(a)) \, ds.
\]

The third equality follows from how \( R_n \) is defined. \( \square \)

Lemma 2.1.11. \( \bigcap_{z \in \mathbb{C}} D(\sigma_z) = \mathcal{A}_\sigma. \)

Proof. Let \( a \in \bigcap_{z \in \mathbb{C}} D(\sigma_z) \) and let \( \varphi \in X^*. \) We will show that \( \mathbb{C} \ni z \mapsto \varphi(\sigma_z(a)) \) is entire holomorphic. Let \( n \in \mathbb{N} \). It follows from Proposition 5.3.5 in [BR] that

\[
\sup_{|\text{Im } z| \leq n} |\varphi(\sigma_z(R_k(a)) - \varphi(\sigma_z(R_m(a)))| \\
\leq ||\varphi|| \max \{||\sigma_{in}(R_k(a)) - \sigma_{in}(R_m(a))||, ||\sigma_{-in}(R_k(a)) - \sigma_{-in}(R_m(a))||\}.
\]

Since \( \lim_{k \to \infty} \sigma_z(R_k(a)) = \sigma_z(a) \) for all \( z \in \mathbb{C} \) by Lemma 2.1.10 and Lemma 2.1.5 it follows that the sequence of functions \( \varphi(\sigma_z(R_k(a)), k \in \mathbb{N} \), converges uniformly on the strip \( \{ z \in \mathbb{C} : |\text{Im } z| \leq n \} \) to a function \( f \) which is holomorphic in the interior of the strip since \( z \mapsto \varphi(\sigma_z(R_k(a))) \) is for each \( k \). Since \( f(z) = \lim_{k \to \infty} \varphi(\sigma_z(R_k(a))) \) = \( \varphi(\sigma_z(a)) \) for all \( z \) in the strip it follows that \( f \) is holomorphic in the strip. Since \( n \) was arbitrary we conclude that \( \mathbb{C} \ni z \mapsto \varphi(\sigma_z(a)) \) is entire holomorphic. Since \( \varphi \in X^* \) was arbitrary this means that \( a \in \mathcal{A}_\sigma \) by Proposition 2.5.21 in [BR]. This completes the proof because the inclusion \( \mathcal{A}_\sigma \subseteq \bigcap_{z \in \mathbb{C}} D(\sigma_z) \) holds by construction. \( \square \)

2.1.2 Flows on \( C^* \)-algebras

Let \( A \) a \( C^* \)-algebra. A flow on \( A \) is a representation \( \sigma = (\sigma_t)_{t \in \mathbb{R}} \) of \( \mathbb{R} \) by automorphisms of \( A \) such that

\[
\lim_{t \to t_0} \|\sigma_t(a) - \sigma_{t_0}(a)\| = 0
\]

for all \( t_0 \in \mathbb{R} \) and all \( a \in A \). This is a particular case of what we have considered above, but we shall need a few facts that involve the additional structure of a \( C^* \)-algebra. Let therefore now \( \sigma \) be a flow on the \( C^* \)-algebra \( A \).

Lemma 2.1.12. Let \( z \in \mathbb{C} \). Then \( D(\sigma_z)^* = D(\sigma_z) \) and \( \sigma_z(a^*) = \sigma_z(a)^* \) for all \( a \in D(\sigma_z) \).
Proof. Let \( a \in \mathcal{A}_\sigma \). If \( f : \mathbb{C} \to A \) is entire holomorphic and \( f(t) = \sigma_t(a) \) for \( t \in \mathbb{R} \) we have that \( \sigma_z(a) = f(z) \). Note that \( z \mapsto g(z) := f'(z) \) is entire holomorphic and \( g(t) = \sigma_t(a^*) \) for all \( t \in \mathbb{R} \). It follows that \( a^* \in \mathcal{A}_\sigma \) and \( \sigma^*(a^*) = g'(z) = f'(z)^* = \sigma_z(a)^* \). Let then \( a \in D(\sigma_z) \). There is a sequence \( \{a_n\} \) in \( \mathcal{A}_\sigma \) such that \( \lim_{n \to \infty} a_n = a \) and \( \lim_{n \to \infty} \sigma_z(a_n) = \sigma_z(a) \). Since \( \lim_{n \to \infty} a_n^* = a^* \), \( \lim_{n \to \infty} \sigma_z(a_n)^* = \sigma_z(a)^* \) and \( \sigma_z \) is closed, it follows that \( a^* \in D(\sigma_z^*) \) and \( \sigma_z(a^*) = \sigma_z(a)^* \). We have shown that \( D(\sigma_z)^* \subseteq D(\sigma_z^*) \) for all \( z \in \mathbb{C} \). Therefore \( D(\sigma_z) = D(\sigma_z^*) \subseteq D(\sigma_z^*) \).

**Lemma 2.1.13.** Let \( a \in \mathcal{A}_\sigma \) and \( b \in D(\sigma_z) \). Then \( ab \in D(\sigma_z) \) and \( \sigma_z(ab) = \sigma_z(a)\sigma_z(b) \).

**Proof.** Choose a sequence \( \{b_n\} \) in \( \mathcal{A}_\sigma \) such that \( \lim_{n \to \infty} b_n = b \) and \( \lim_{n \to \infty} \sigma_z(b_n) = \sigma_z(b) \). For each \( n \) the functions \( w \mapsto \sigma_w(ab_n) \) are both entire holomorphic and agree on \( \mathbb{R} \). They are therefore the same, and hence \( \sigma_z(ab_n) = \sigma_z(ab) \). The desired conclusion follows because \( \sigma_z \) is closed.

**Lemma 2.1.14.** \( \mathcal{A}_\sigma \) is a dense \( * \)-subalgebra of \( A \).

**Proof.** Let \( a, b \in \mathcal{A}_\sigma \). It follows from Proposition 2.1.2 that there are entire holomorphic functions \( f, g : \mathbb{C} \to A \) such that \( f(t) = \sigma_t(a) \) and \( g(t) = \sigma_t(b) \) for all \( t \in \mathbb{R} \). Then \( z \mapsto f(z)g(z) \) is entire holomorphic and \( f(t)g(t) = \sigma_t(ab) \) for all \( t \in \mathbb{R} \). Thus \( ab \in \mathcal{A}_\sigma \). It follows that \( \mathcal{A}_\sigma \) is a subalgebra of \( A \). It is a right invariant under \( * \) because \( \mathbb{C} \ni z \mapsto f(z)^* \) is holomorphic and \( f(t)^* = \sigma_t(a^*) \) for all \( t \in \mathbb{R} \). It follows from Lemma 2.1.5 that \( \mathcal{A}_\sigma \) is dense in \( A \).

**Notes and remarks 2.1.15.** The main content in this section is a slightly modified version of material from Bratteli and Robinson’s monograph [BR]. The material about \( \sigma_z \) for non-real \( z \) is gleaned from Kustermans, [Ku1]. I have selected facts that will be needed in the following. More can be found in [Ku1] and [Ku2].
Chapter 3

KMS weights

3.1 Flows and invariant weights

In this section we consider a flow \( \sigma \) on the \( C^* \)-algebra \( A \) and a densely defined weight \( \psi \) on \( A \) which we assume is \( \sigma \)-invariant in the sense that \( \psi \circ \sigma_t = \psi \) for all \( t \in \mathbb{R} \). Let \( H_{\psi} \) be the Hilbert space and \( \Lambda_{\psi} : \mathcal{N}_{\psi} \to H_{\psi} \) the linear map from the GNS construction associated to \( \psi \), cf. Section 1.2.1.

**Lemma 3.1.1.** \( \Lambda_{\psi} : \mathcal{N}_{\psi} \to H_{\psi} \) is closed.

**Proof.** Let \( \lim_{n \to \infty} a_n = a \) in \( A \) with \( a_n \in \mathcal{N}_{\psi} \) for all \( n \), and assume that \( \lim_{n \to \infty} \Lambda_{\psi}(a_n) = v \) in \( H_{\psi} \). We must show that \( a \in \mathcal{N}_{\psi} \) and that \( \Lambda_{\psi}(a) = v \). Let \( \omega \in \mathcal{F}_{\psi} \). By Lemma 1.2.5 there is an operator \( T_{\omega} \) on \( H_{\psi} \) such that \( 0 \leq T_{\omega} \leq 1 \) and

\[ \omega(c^* d) = \langle T_{\omega} \Lambda_{\psi}(d), \Lambda_{\psi}(c) \rangle \quad \forall c, d \in \mathcal{N}_{\psi}. \]

Then

\[ \omega(a^* a) = \lim_{k \to \infty} \omega(a_k^* a_k) = \lim_{k \to \infty} \langle T_{\omega} \Lambda_{\psi}(a_k), \Lambda_{\psi}(a_k) \rangle = \langle T_{\omega} v, v \rangle \leq \|v\|^2. \]

It follows then from Combes’ theorem, Theorem 1.1.2, that \( \psi(a^* a) \leq \|v\|^2 \), and hence \( a \in \mathcal{N}_{\psi} \). Let \( \epsilon > 0 \) and let \( b \in \mathcal{N}_{\psi} \). It follows also from Combes’ theorem that there is \( \omega \in \mathcal{F}_{\psi} \) such that

\[ \psi(b^* b) - \epsilon \leq \omega(b^* b) \leq \psi(b^* b). \]

Then

\[
\begin{align*}
\|T_{\omega} \Lambda_{\psi}(b) - \Lambda_{\psi}(b)\|^2 &= \langle T_{\omega}^2 \Lambda_{\psi}(b), \Lambda_{\psi}(b) \rangle + \langle \Lambda_{\psi}(b), \Lambda_{\psi}(b) \rangle - 2 \langle T_{\omega} \Lambda_{\psi}(b), \Lambda_{\psi}(b) \rangle \\
&\leq \omega(b^* b) + \psi(b^* b) - 2\omega(b^* b) \\
&\leq 2\psi(b^* b) - 2(\psi(b^* b) - \epsilon) = 2\epsilon.
\end{align*}
\]
We can choose $k$ so large that $\|v - \Lambda_\psi(a_k)\| \leq \epsilon$ and $|\omega(b^*a_k) - \omega(b^*a)| \leq \epsilon$. Then by the calculation above,

\[ |\langle v, \Lambda_\psi(b) \rangle - \langle \Lambda_\psi(a), \Lambda_\psi(b) \rangle| \]
\[ \leq (\|v\| + \|\Lambda_\psi(a)\|)\sqrt{2\epsilon} + |\langle v, T_\omega \Lambda_\psi(b) \rangle - \langle \Lambda_\psi(a), T_\omega \Lambda_\psi(b) \rangle| \]
\[ \leq (\|v\| + \|\Lambda_\psi(a)\|)\sqrt{2\epsilon} + \|\Lambda_\psi(b)\| \epsilon + |\langle \Lambda_\psi(a_k), T_\omega \Lambda_\psi(b) \rangle - \langle \Lambda_\psi(a), T_\omega \Lambda_\psi(b) \rangle| \]
\[ = (\|v\| + \|\Lambda_\psi(a)\|)\sqrt{2\epsilon} + \|\Lambda_\psi(b)\| \epsilon + |\omega(b^*a_k) - \omega(a^*b)| \]
\[ \leq (\|v\| + \|\Lambda_\psi(a)\|)\sqrt{2\epsilon} + \|\Lambda_\psi(b)\| \epsilon + \epsilon. \]

Since both $\epsilon > 0$ and $b \in \mathcal{N}_\psi$ were arbitrary we conclude that $v = \Lambda_\psi(a)$. \qed

Since $\psi$ is $\sigma$-invariant it follows that $\sigma_t(\mathcal{N}_\psi) = \mathcal{N}_\psi$ and we obtain for each $t \in \mathbb{R}$ a unitary $U^\psi_t \in B(H_\psi)$ such that

\[ U^\psi_t \Lambda_\psi(a) = \Lambda_\psi(\sigma_t(a)) \quad \forall a \in \mathcal{N}_\psi. \]

It is straightforward to check that

\[ U^\psi_t \pi_\psi(a)U^\psi_{-t} = \pi_\psi(\sigma_t(a)) \quad (3.1.1) \]

for all $a \in A$.

**Lemma 3.1.2.** \( \{U^\psi_t\}_{t \in \mathbb{R}} \) is a strongly continuous unitary representation of $\mathbb{R}$; that is, $U^\psi_t U^\psi_s = U^\psi_{t+s}$ for all $t, s \in \mathbb{R}$, $U^\psi_t U^\psi_{-t} = 1$ and $\mathbb{R} \ni t \mapsto U^\psi_t$ is continuous for each $v \in H_\psi$.

**Proof.** The algebraic statements are easily established. By using them it follows that in order to establish continuity it suffices to show that

\[ \lim_{t \to 0} \left\langle U^\psi_t \Lambda_\psi(a), \Lambda_\psi(b) \right\rangle = \langle \Lambda_\psi(a), \Lambda_\psi(b) \rangle \quad (3.1.2) \]

for all $a, b \in \mathcal{N}_\psi$. Let $\epsilon > 0$. It follows from Combes’ theorem, Theorem 1.1.1, that there is $\omega \in \mathcal{F}_\psi$ such that

\[ \psi(a^*a) - \epsilon \leq \omega(a^*a) \leq \psi(a^*a). \]

By Lemma 12.2 there is an operator $T_\omega$ on $H_\psi$ such that $0 \leq T_\omega \leq 1$ and

\[ \omega(c^*d) = \langle T_\omega \Lambda_\psi(d), \Lambda_\psi(c) \rangle \quad \forall c, d \in \mathcal{N}_\psi. \]

The same calculation as in the proof of Lemma 3.1 shows that $\|T_\omega \Lambda_\psi(a) - \Lambda_\psi(a)\|^2 \leq 2\epsilon$ and we find therefore that

\[ \left| \left\langle U^\psi_t \Lambda_\psi(a), \Lambda_\psi(b) \right\rangle - \left\langle \Lambda_\psi(a), \Lambda_\psi(b) \right\rangle \right| = \left| \left\langle \Lambda_\psi(a), U^\psi_{-t} \Lambda_\psi(b) \right\rangle - \left\langle \Lambda_\psi(a), \Lambda_\psi(b) \right\rangle \right| \]
\[ \leq 2 \|\Lambda_\psi(b)\| \sqrt{2\epsilon} + \left| \left\langle T_\omega \Lambda_\psi(a), U^\psi_{-t} \Lambda_\psi(b) \right\rangle - \left\langle T_\omega \Lambda_\psi(a), \Lambda_\psi(b) \right\rangle \right| \]
\[ = 2 \|\Lambda_\psi(b)\| \sqrt{2\epsilon} + |\omega(\sigma_{-t}(b^*)a) - \omega(b^*a)| \]

for all $t \in \mathbb{R}$. Since $\lim_{t \to 0} \omega(\sigma_{-t}(b^*)a) = \omega(b^*a)$ we get (3.1.2). \qed

\[ \square \]
Motivated by the last two lemmas we fix a GNS representation \((H, \Lambda, \pi)\) of \(A\) such that

- \(\sigma_t(D(\Lambda)) = D(\Lambda)\) for all \(t \in \mathbb{R}\),
- there is a continuous unitary representation \((U_t)_{t \in \mathbb{R}}\) of \(\mathbb{R}\) on \(H\) such that \(U_t \Lambda(a) = \Lambda(\sigma_t(a))\) for \(t \in \mathbb{R}, a \in D(\Lambda)\), and
- \(\Lambda : D(\Lambda) \to H\) is closed.

Many of the lemmas we shall need hold for GNS representations with no mention of weights, and it will be crucial that they do, but for some of them it is necessary that the GNS representation arises from a weight. We fix therefore also a densely defined \(\sigma\)-invariant weight on \(A\) and note that its GNS-triple \((H_\psi, \Lambda_\psi, \pi_\psi)\) is a GNS representation with the properties stipulated above. Note that \(D(\Lambda_\psi) = N_\psi\).

**Lemma 3.1.3.** Let \(v \in H\) and \(n \in \mathbb{N}\). There is a sequence \(\{v_k\}_{k=0}^\infty\) in \(H\) such that \(\sum_{k=0}^\infty \|v_k\| |z|^k < \infty\) and

\[
\sum_{k=0}^\infty v_k z^k = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} U_s v \, ds
\]

for all \(z \in \mathbb{C}\).

**Proof.** This follows now from Lemma 2.1.4. \(\square\)

**Lemma 3.1.4.** Let \(a \in D(\Lambda)\). Then \(\sigma_z(R_n(a)) \in D(\Lambda)\) and

\[
\Lambda(\sigma_z(R_n(a))) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-n(s-z)^2} \Lambda(\sigma_s(a)) \, ds
\]

for all \(n \in \mathbb{N}\) and all \(z \in \mathbb{C}\).

**Proof.** It follows from Lemma 2.1.10 that

\[
\sigma_z(R_n(a)) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-n(s-z)^2} \sigma_s(a) \, ds.
\]

The desired equality follows therefore from Lemma 3.2.1 because \(\Lambda\) is closed. \(\square\)

**Lemma 3.1.5.** Let \(a \in D(\Lambda)\). Then \(\lim_{n \to \infty} \Lambda(R_n(a)) = \Lambda(a)\) in \(H\).

**Proof.** As a special case of the formula from Lemma 3.1.4 we get

\[
\Lambda(R_n(a)) = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} e^{-ns^2} U_s \Lambda(a) \, ds.
\]

The conclusion follows therefore from Lemma 2.1.5. \(\square\)
Lemma 3.1.6. Let \( a \in D(\Lambda) \cap D(\sigma_z) \). Assume that \( \sigma_z(a) \in D(\Lambda) \). Then
\[
\lim_{k \to \infty} \Lambda(\sigma_z(R_k(a))) = \Lambda(\sigma_z(a))
\]
in \( H \).

Proof. By Lemma 2.1.10 \( \sigma_z(R_k(a)) = R_k(\sigma_z(a)) \) and hence the conclusion follows from Lemma 3.1.5. \( \square \)

Lemma 3.1.7. \( \psi(R_k(a)) = \psi(a) \) for all \( k \in \mathbb{N} \) and all \( a \in A^+ \).

Proof. The \( C^* \)-subalgebra \( S \) of \( A \) generated by \( \{ \sigma_t(a) : t \in \mathbb{R} \} \) is separable and the restriction \( \psi|_S \) of \( \psi \) to \( S \) is a weight. By Theorem 1.2.1 there is a sequence \( \omega_n \in S^* \) such that \( \psi(b) = \sum_{n=1}^{\infty} \omega_n(b) \) for all \( b \in S^+ \). By using that \( \psi(\sigma_t(a)) = \psi(a) \) and Lebesgue’s theorem on monotone convergence we find
\[
\psi(a) = \frac{k}{\pi} \int_{\mathbb{R}} e^{-kt^2} \psi(\sigma_t(a)) \, dt = \frac{k}{\pi} \int_{\mathbb{R}} e^{-kt^2} \sum_{n=1}^{\infty} \omega_n(a) \, dt
\]
\[
= \sum_{n=1}^{\infty} \frac{k}{\pi} \int_{\mathbb{R}} e^{-kt^2} \omega_n(a) \, dt = \sum_{n=1}^{\infty} \omega_n(R_k(a)) = \psi(R_k(a)).
\]

Set \( M^\sigma_\psi = \{ a \in M_\psi \cap A : \sigma_z(a) \in M_\psi \ \forall z \in \mathbb{C} \} \) \( (3.1.3) \)

Lemma 3.1.8. \( R_k(M_\psi) \subseteq M^\sigma_\psi \) for all \( k \in \mathbb{N} \).

Proof. It follows from Lemma 3.1.7 that \( R_k(M_\psi) \subseteq M_\psi \). Hence \( R_k(M_\psi) \subseteq M_\psi \cap A_\sigma \) by Lemma 2.1.5. Let \( z \in \mathbb{C} \), \( a \in M_\psi \). To show that \( \sigma_z(R_k(a)) \in M_\psi \) we may assume, by linearity, that \( a \in M_\psi^+ \). Since
\[
e^{-k(s-z)^2} = e^{-ks^2} e^{-ks^2} e^{2ksz},
\]
we can write \( e^{-k(s-z)^2} \) as a sum
\[
e^{-k(s-z)^2} = e^{-ks^2} \sum_{j=1}^{4} i^j f_j(s)
\]
where \( f_j : \mathbb{R} \to [0, \infty), j = 1, 2, 3, 4, \) are continuous non-negative functions such that
\[
|f_j(s)| \leq |e^{-ks^2}| e^{2k|z||s|}
\]
for all \( j, s \). It follows from Lemma 2.1.10 that
\[
\sigma_z(R_k(a)) = \frac{k}{\pi} \sum_{j=1}^{4} i^j \int_{\mathbb{R}} e^{-ks^2} f_j(s) \sigma_z(a) \, ds.
\]
3.1. FLOWS AND INVARIANT WEIGHTS

By using Theorem 1.2.1 and Lebesgue’s theorem on monotone convergence in the same way as in the proof of Lemma 3.1.7 it follows that

\[ \psi \left( \int e^{-ks^2} f_j(s) \sigma_s(a) \, ds \right) = \psi(a) \int e^{-ks^2} f_j(s) \, ds < \infty, \]

for each \( j \). Thus \( \sigma_z(R_k(a)) \in M_\psi \).

**Lemma 3.1.9.** \( M_\psi \) is a dense \(*\)-subalgebra of \( A_\sigma \) which is invariant under \( \sigma_z \) for all \( z \in \mathbb{C} \).

**Proof.** It follows from (e) of Lemma 1.0.1 and Lemma 2.1.14 that \( M_\psi \cap A_\sigma \) is a \(*\)-subalgebra and then from Lemma 2.1.6, Lemma 2.1.13 and Lemma 2.1.12 that \( M_\psi \) is a \( \sigma_z \)-invariant \(*\)-subalgebra of \( A_\sigma \). Since \( \psi \) is densely defined it follows from (e) of Lemma 1.0.1 that \( M_\psi \) is dense in \( A_\sigma \) and then from Lemma 3.1.8 and Lemma 2.1.5 that \( M_\psi \) is dense in \( A_\sigma \).

Recall that a representation \( \pi: A \to B(H) \) is non-degenerate when

\[ \{ \pi(a)\eta: a \in A, \eta \in H \} \]

spans a dense subspace of \( H \).

**Lemma 3.1.10.** \( \Lambda_\psi : \mathcal{N}_\psi \to H_\psi \) is closed and \( \pi_\psi \) is non-degenerate.

**Proof.** That \( \Lambda_\psi \) is closed follows from Lemma 3.1.1. To show that \( \pi_\psi \) is non-degenerate, let \( a \in \mathcal{N}_\psi \). There is a sequence \( \{ u_n \} \) in \( A \) such that \( 0 \leq u_n \leq 1 \) for all \( n \) and \( \lim_{n \to \infty} u_n a = a \) in \( A \). Note that

\[
\| \pi_\psi(u_n)\Lambda_\psi(a) - \Lambda_\psi(a) \|^2 = \psi(a^*(1 - u_n)^2 a)
\]

\[
\leq \psi(a^*(1 - u_n)a) = \psi(a^*a) - \psi(a^*u_n a).
\]

Since \( \psi \) is lower semi-continuous, \( \lim_{n \to \infty} \psi(a^*u_n a) = \psi(a^*a) \) and it follows that \( \lim_{n \to \infty} \pi_\psi(u_n)\Lambda_\psi(a) = \Lambda_\psi(a) \) in \( H_\psi \). By definition \( \{ \Lambda_\psi(a): a \in \mathcal{N}_\psi \} \) is dense in \( H_\psi \) and we conclude therefore that \( \pi_\psi \) is non-degenerate.

**Lemma 3.1.11.** Assume that \( \pi: A \to B(H) \) is non-degenerate. There is a net \( \{ e_m \}_{m \in I} \) in \( D(\Lambda) \) which is an approximate unit for \( A \) such that \( 0 \leq e_m \leq 1 \) for all \( m \), \( e_m \leq e_{m'} \) when \( m \leq m' \), and \( \lim_{m \to \infty} \pi(e_m) = 1 \) in the strong operator topology of \( B(H) \).

**Proof.** Let \( I \) be the collection of finite subsets of \( D(\Lambda)^* \) which we consider as a directed set ordered by inclusion. For each \( m \in I \), set

\[ f_m = \sum_{a \in m} aa^*, \]

and

\[ e_m = (\#m)f_m(1 + (\#m)f_m)^{-1} \in D(\Lambda)^*. \]
Note that \( e_m \in D(\Lambda) \) because \( D(\Lambda) \) is a left ideal in \( A \). As shown in the proof of Proposition 2.2.18 in [BR] the net \( \{e_m\}_{m \in I} \) consists of positive contractions, i.e. \( 0 \leq e_m \leq 1 \), it increases with \( m \), i.e. \( m \subseteq m' \Rightarrow e_m \leq e_{m'} \) and it has the property that \( \lim_{m \to \infty} e_m b = b \) for all \( b \in D(\Lambda)^* \). Since \( D(\Lambda)^* \) is dense in \( A \) it follows that \( \lim_{m \to \infty} e_m b = b \) for all \( b \in A \). Since \( \pi \) is non-degenerate,

\[
\lim_{m \to \infty} \langle \pi(e_m)\Lambda(b), \Lambda(b) \rangle = \langle \Lambda \psi(b), \Lambda \psi(b) \rangle
\]

for all \( b \in D(\Lambda) \), implying that \( \lim_{m \to \infty} \pi(e_m) = 1 \) in the weak operator topology, and hence also in the strong operator topology.

**Lemma 3.1.12.** Let \( F \subseteq \mathbb{C} \) be a finite set of complex numbers. Let \( a \in \mathcal{N}_\psi \cap (\bigcap_{z \in F} D(\sigma_z)) \). There is a sequence \( \{a_n\} \) in \( \mathcal{M}_\psi^\sigma \) such that

- \( \lim_{n \to \infty} a_n = a \),
- \( \lim_{n \to \infty} \sigma_z(a_n) = \sigma_z(a) \) for all \( z \in F \),
- \( \lim_{n \to \infty} \Lambda \psi(a_n) = \Lambda \psi(a) \).

If \( \sigma_z(a) \in \mathcal{N}_\psi \) for all \( z \in F \), we can also arrange that

- \( \lim_{n \to \infty} \Lambda \psi(\sigma_z(a_n)) = \Lambda \psi(\sigma_z(a)) \) for all \( z \in F \).

**Proof.** We will construct \( a_n \in \mathcal{M}_\psi^\sigma \) such that

\[
\|a_n - a\| \leq \frac{2}{n}, \quad (3.1.4)
\]

\[
\|\sigma_z(a_n) - \sigma_z(a)\| \leq \frac{2}{n}, \quad (3.1.5)
\]

\[
\|\Lambda \psi(a_n) - \Lambda \psi(a)\| \leq \frac{2}{n}, \quad (3.1.6)
\]

and when \( \sigma_z(a) \in \mathcal{N}_\psi \) for all \( z \in F \) also such that

\[
\|\Lambda \psi(\sigma_z(a_n)) - \Lambda \psi(\sigma_z(a))\| \leq \frac{2}{n}, \quad (3.1.7)
\]

when \( z \in F \). Let \( \{e_m\}_{m \in I} \) be the approximate unit of Lemma 3.1.11. The element \( a_n \) we seek will be

\[
a_n = R_k(e_m a)
\]

for an appropriate choice of \( m \in I \) and \( k \in \mathbb{N} \). Note that since \( a \in \mathcal{N}_\psi \) and \( e_m \in \mathcal{N}_\psi^\sigma \) it follows from (b) of Lemma 1.0.1 that \( e_m a \in \mathcal{M}_\psi \) and hence \( R_k(e_m a) \in \mathcal{M}_\psi^\sigma \) by Lemma 3.1.8. In order to choose \( m \) and \( k \) note first of all that the properties of \( \{e_m\}_{m \in I} \) ensure the existence of \( m_n \in I \) such that

\[
\|e_m a - a\| \leq \frac{1}{n} \text{ when } m_n \subseteq m, \quad (3.1.8)
\]
and
\[ \| \Lambda \psi(e_m a) - \Lambda \psi(a) \| = \| \pi \psi(e_m) \Lambda \psi(a) - \Lambda \psi(a) \| \leq \frac{1}{n} \text{ when } m_n \subseteq m. \quad (3.1.9) \]

It follows from Lemma 3.1.4 that we then also have the estimate
\[ \| \Lambda \psi(R_k(e_m a)) - \Lambda \psi(R_k(a)) \| = \left\| \sqrt{\frac{k}{\pi}} \int_{\mathbb{R}} e^{-k s^2} U_s (\Lambda \psi(e_m a) - \Lambda \psi(a)) \ ds \right\| \leq \| \Lambda \psi(e_m a) - \Lambda \psi(a) \| \leq \frac{1}{n} \forall k \text{ when } m_n \subseteq m. \quad (3.1.10) \]

We will choose \( k \in \mathbb{N} \) first and then subsequently choose \( m \supseteq m_n \). It follows from Lemma 2.1.5 that we can choose \( k \) such that
\[ \| R_k(a) - a \| \leq \frac{1}{n} \quad (3.1.11) \]
and from Lemma 3.1.5 such that also
\[ \| \Lambda \psi(R_k(a)) - \Lambda \psi(a) \| \leq \frac{1}{n} \quad (3.1.12) \]

It follows from Lemma 2.1.10 and Lemma 2.1.5 that
\[ \lim_{k \to \infty} \sigma_z(R_k(a)) = \lim_{k \to \infty} R_k(\sigma_z(a)) = \sigma_z(a), \]
for all \( z \in \mathbb{C} \) and we can therefore arrange that
\[ \| \sigma_z(R_k(a)) - \sigma_z(a) \| \leq \frac{1}{n} \quad (3.1.13) \]
for all \( z \in F \). Similarly, thanks to Lemma 3.1.6 and assuming that \( \sigma_z(a) \in \mathcal{N}_\psi \) for all \( z \in F \), we can also arrange that
\[ \| \Lambda \psi(\sigma_z(R_k(a)) - \Lambda \psi(\sigma_z(a)) \| \leq \frac{1}{n} \quad (3.1.14) \]
for all \( z \in F \). Now we fix \( k \in \mathbb{N} \) such that (3.1.11), (3.1.12), (3.1.13) and (3.1.14) all hold; the latter of course only under the stated additional assumption. We go on to choose \( m \).

Note that it follows from Lemma 2.1.10 that
\[ \| \sigma_z(R_k(e_m a)) - \sigma_z(R_k(a)) \| \]
\[ \leq \sqrt{\frac{k}{\pi}} \int_{\mathbb{R}} e^{-k(s-z)^2} \| \sigma_z(e_m a) - \sigma_z(a) \| \ ds \]
\[ \leq \| e_m a - a \| \sqrt{\frac{k}{\pi}} \int_{\mathbb{R}} e^{-k(s-z)^2} \ ds, \]
and from Lemma 3.1.4 that
\[
\| \Lambda_{\psi}(\sigma_z(R_k(e_m a))) - \Lambda_{\psi}(\sigma_z(R_k(a))) \|
= \left\| \sqrt{\frac{k}{\pi}} \int_{e^{-k(s-z)^2}} U_{\psi}(\Lambda_{\psi}(e_m a) - \Lambda_{\psi}(a)) \, ds \right\|
\leq \| \Lambda_{\psi}(e_m a) - \Lambda_{\psi}(a) \| \sqrt{\frac{k}{\pi}} \int_{e^{-k(s-z)^2}} |e^{-k(s-z)^2}| \, ds
\]
for all \( z \in F \). Thanks to Lemma 3.1.11 we can therefore choose \( m \in \mathcal{I} \) such that \( m_n \subseteq m \) and
\[
\| \sigma_z(R_k(e_m a)) - \sigma_z(R_k(a)) \| \leq \frac{1}{n} \quad (3.1.15)
\]
and
\[
\| \Lambda_{\psi}(\sigma_z(R_k(e_m a))) - \Lambda_{\psi}(\sigma_z(R_k(a))) \| \leq \frac{1}{n} \quad (3.1.16)
\]
for all \( z \in F \). It remains to check that \( a_n = R_k(e_m a) \) satisfies the desired estimates. Using (3.1.11), (3.1.8) and that \( R_k \) is a linear contraction we find that
\[
\|a_n - a\| \leq \| R_k(e_m a) - R_k(a) \| + \| R_k(a) - a \| \leq \| e_m a - a \| + \frac{1}{n} \leq \frac{2}{n}.
\]
Using (3.1.15) and (3.1.13) we find that
\[
\| \sigma_z(a_n) - \sigma_z(a) \|
\leq \| \sigma_z(R_k(e_m a)) - \sigma_z(R_k(a)) \| + \| \sigma_z(R_k(a)) - \sigma_z(a) \| \leq \frac{2}{n},
\]
for all \( z \in F \). It follows from (3.1.10) and (3.1.12) that
\[
\| \Lambda_{\psi}(a_n) - \Lambda_{\psi}(a) \| \leq \frac{2}{n}.
\]
Finally, when \( \sigma_z(a) \in \mathcal{N}_{\psi} \) for all \( z \in F \) it follows from (3.1.14) and (3.1.16) that
\[
\| \Lambda_{\psi}(\sigma_z(a_n)) - \Lambda_{\psi}(\sigma_z(a)) \| \leq \frac{2}{n}
\]
for all \( z \in F \).

Corollary 3.1.13. \( \mathcal{M}_{\psi}^\sigma \) is a core for \( \Lambda_{\psi} \); that is, for all \( a \in \mathcal{N}_{\psi} \) there is a sequence \( \{a_n\} \) in \( \mathcal{M}_{\psi}^\sigma \) such that \( \lim_{n \to \infty} a_n = a \) and \( \lim_{n \to \infty} \Lambda_{\psi}(a_n) = \Lambda_{\psi}(a) \).

Proof. Apply Lemma 3.1.12 with \( F = \{0\} \). \( \qed \)

Lemma 3.1.14. Let \( a, b \in \mathcal{N}_{\psi} \cap \mathcal{A}_\sigma \). Assume that \( \sigma_z(a) \in \mathcal{N}_{\psi} \) and \( \sigma_z(b) \in \mathcal{N}_{\psi} \) for all \( z \in \mathbb{C} \). Then
\[
\psi(\sigma_z(b^* a)) = \psi(b^* a)
\]
for all \( z \in \mathbb{C} \).
3.2. THE DEFINITION OF A KMS WEIGHT

Proof. By using Lemma 2.1.12 and Lemma 2.1.13 we find that
\[ \psi(\sigma_z(b^*a)) = \psi(\sigma_z(b)^*\sigma_z(a)) = \langle \Lambda_\psi(\sigma_z(a)), \Lambda_\psi(\sigma_z(b)) \rangle. \]

It follows from Lemma 3.1.6 that \( \lim_{n \to \infty} \Lambda_\psi(\sigma_z(R_n(a))) = \Lambda_\psi(\sigma_z(a)) \) and \( \lim_{n \to \infty} \Lambda_\psi(\sigma_z(R_n(b))) = \Lambda_\psi(\sigma_z(b)) \) and hence
\[ \langle \Lambda_\psi(\sigma_z(a)), \Lambda_\psi(\sigma_z(b)) \rangle = \lim_{n \to \infty} \langle \Lambda_\psi(\sigma_z(R_n(a))), \Lambda_\psi(\sigma_z(R_n(b))) \rangle. \]

It follows from Lemma 3.1.4 that
\[ \Lambda_\psi(\sigma_z(R_n(a))) = \frac{1}{n} \int \pi \exp(-n(s-z)^2) U_s^\psi \Lambda_\psi(a) \, ds, \]
and then from Lemma 3.1.6 that \( \mathbb{C} \ni z \mapsto \Lambda_\psi(\sigma_z(R_n(a))) \) is entire holomorphic. The same is true for \( a \) replaced by \( b \) and consequently
\[ \mathbb{C} \ni z \mapsto \langle \Lambda_\psi(\sigma_z(R_n(a))), \Lambda_\psi(\sigma_z(R_n(b))) \rangle \]
is entire holomorphic. Since
\[ \langle \Lambda_\psi(\sigma_t(R_n(a))), \Lambda_\psi(\sigma_t(R_n(b))) \rangle = \left\langle U_t^\psi \Lambda_\psi(R_n(a)), U_t^\psi \Lambda_\psi(R_n(b)) \right\rangle \]
\[ = \langle \Lambda_\psi(R_n(a)), \Lambda_\psi(R_n(b)) \rangle, \]
when \( t \in \mathbb{R} \), we conclude that
\[ \langle \Lambda_\psi(\sigma_z(R_n(a))), \Lambda_\psi(\sigma_z(R_n(b))) \rangle = \langle \Lambda_\psi(R_n(a)), \Lambda_\psi(R_n(b)) \rangle \]
for all \( z \in \mathbb{C} \). Using Lemma 3.1.5 again we find therefore that
\[ \psi(\sigma_z(b^*a)) = \lim_{n \to \infty} \langle \Lambda_\psi(R_n(a)), \Lambda_\psi(R_n(b)) \rangle = \langle \Lambda_\psi(a), \Lambda_\psi(b) \rangle = \psi(b^*a). \]

\[ \square \]

3.2 The definition of a KMS weight

Let \( \sigma \) be a flow on the \( C^* \)-algebra \( A \). The definition of a KMS weight for \( \sigma \) will be based on the following theorem. To formulate it, consider a real number \( \beta \in \mathbb{R} \). Set
\[ \mathcal{D}_\beta = \{ z \in \mathbb{C} : \text{ Im } z \in [0, \beta] \} \]
when \( \beta \geq 0 \), and
\[ \mathcal{D}_\beta = \{ z \in \mathbb{C} : \text{ Im } z \in [\beta, 0] \} \]
when \( \beta \leq 0 \), and let \( \mathcal{D}_{\beta}^\circ \) denote the interior of \( \mathcal{D}_\beta \) in \( \mathbb{C} \).

**Theorem 3.2.1.** (Kustermans, [Ku1]) Let \( \psi \) be a non-zero densely defined weight on \( A \) which is \( \sigma \)-invariant in the sense that \( \psi \circ \sigma_t = \psi \) for all \( t \in \mathbb{R} \), and let \( \beta \in \mathbb{R} \) be a real number. The following conditions are equivalent.
(1) $\psi(a^*a) = \psi\left(\sigma_{-i\beta^2}(a)\sigma_{-i\beta^2}(a)^*\right)$ \quad \forall a \in D\left(\sigma_{-i\beta^2}\right)$.

(2) $\psi(a^*a) = \psi\left(\sigma_{-i\beta^2}(a)\sigma_{-i\beta^2}(a)^*\right)$ \quad \forall a \in M_{\psi}^\sigma$.

(3) $\psi(ab) = \psi(b\sigma_i(a))$ \quad \forall a, b \in M_{\psi}^\sigma$.

(4) For all $a, b \in N_{\psi} \cap N_{\psi}^\sigma$ there is a continuous function $f : D_\beta \to \mathbb{C}$ which is holomorphic in the interior $D_0^\beta$ of $D_\beta$ and has the property that

- $f(t) = \psi(b\sigma_i(a))$ \quad $\forall t \in \mathbb{R}$, and
- $f(t + i\beta) = \psi(\sigma_i(a)b)$ \quad $\forall t \in \mathbb{R}$.

We make the following definition.

**Definition 3.2.2.** For $\beta \in \mathbb{R}$ we define a $\beta$-KMS weight for $\sigma$ to be a non-zero densely defined weight on $A$ which is $\sigma$-invariant and satisfies one and hence all of the conditions in Definition 3.2.1.

It is apparent that the case $\beta = 0$ is exceptional, and we make the following definition.

**Definition 3.2.3.** A trace on a $C^*$-algebra $A$ is a non-zero densely defined weight $\psi : A^+ \to [0, \infty]$ with the property that $\psi(a^*a) = \psi(\sigma_{i\beta}^2(a))$ for all $a \in A$.

Thus a 0-KMS weight is a $\sigma$-invariant trace on $A$.

### 3.3 Proof of Kustermans’ theorem

The case $\beta = 0$ is slightly exceptional because the interior $D_0^\beta$ of $D_\beta$ is empty in this case. In the following formulations we focus on the case $\beta \neq 0$ and leave the reader to make the necessary interpretations when $\beta = 0$. It is worthwhile because Theorem 3.2.1 carries non-trivial information also in this case.

The implication (1) $\Rightarrow$ (2) is trivial so it suffices to prove (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

(2) $\Rightarrow$ (3): Let $a, b \in M_{\psi}^\sigma$. Using polarization, Lemma 1.0.1, Lemma 1.2.4, Lemma 2.1.13, Lemma 2.1.12 and Lemma 3.1.14 we find that

$$
\psi(b^*a) = \frac{1}{4} \sum_{k=1}^4 i^k \psi((a + i^k b)^*(a + i^k b))
$$

$$
= \frac{1}{4} \sum_{k=1}^4 i^k \psi\left(\sigma_{-i\beta^2}(a + i^k b)\sigma_{-i\beta^2}(a + i^k b)^*\right)
$$

$$
= \frac{1}{4} \sum_{k=1}^4 i^k \psi\left((\sigma_{-i\beta^2}(a) + i^k \sigma_{-i\beta^2}(b))(\sigma_{-i\beta^2}(a) + i^k \sigma_{-i\beta^2}(b))^*\right)
$$

$$
= \psi(\sigma_{-i\beta^2}(a)\sigma_{-i\beta^2}(b)^*)
$$

$$
= \psi\left(\sigma_{-i\beta^2}(a)\sigma_{i\beta}(b^*)\right)
$$

$$
= \psi(\sigma_{i\beta}(a^*a))
$$

$$
= \psi(\sigma_{i\beta}(a^*a)).
$$

(3.3.1)
\(3 \Rightarrow 4\): Let \(a, b \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^*\). We claim that there are sequences \(\{a_n\}\) and \(\{b_n\}\) in \(\mathcal{M}_\psi^\ast\) such that \(\lim_{n \to \infty} a_n = a\), \(\lim_{n \to \infty} \Lambda_\psi(a_n) = \Lambda_\psi(a)\),
\[
\lim_{n \to \infty} \Lambda_\psi(a_n^*) = \Lambda_\psi(a^*)
\]
and \(\lim_{n \to \infty} b_n = b\), \(\lim_{n \to \infty} \Lambda_\psi(b_n) = \Lambda_\psi(b)\), \(\lim_{n \to \infty} \Lambda_\psi(b_n^*) = \Lambda_\psi(b^*)\). To construct \(\{a_n\}\) note that \(\lim_{k \to \infty} R_k(a) = a\) by Lemma 2.1.5 and
\[
\lim_{k \to \infty} \Lambda_\psi(R_k(a)) = \Lambda_\psi(a)
\]
and \(\lim_{k \to \infty} \Lambda_\psi(R_k(a^*)) = \Lambda_\psi(a^*)\) by Lemma 3.1.5. Note that \(R_k(a) \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^* \cap \mathcal{A}_\sigma\) by Lemma 2.1.5 and Lemma 3.1.4. Furthermore, \(\sigma_z(R_k(a)) \in \mathcal{N}_\psi\) for all \(z \in \mathbb{C}\) by Lemma 3.1.4. We may therefore assume that \(a \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^* \cap \mathcal{A}_\sigma\) and that \(\sigma_z(a) \in \mathcal{N}_\psi\) for all \(z \in \mathbb{C}\). It follows then from Lemma 3.1.12 that there is a sequence \(\{a_n\}\) in \(\mathcal{M}_\psi^\ast\) such that \(\lim_{n \to \infty} a_n = a\), \(\lim_{n \to \infty} \Lambda_\psi(a_n) = \Lambda_\psi(a)\) and \(\lim_{n \to \infty} \Lambda_\psi \left(\sigma_i^\ast(a_n)\right) = \Lambda_\psi \left(\sigma_i^\ast(a)\right)\). Since we assume (3) we have \(\psi(z^\ast y) = \psi(y \sigma_i^\ast(z^\ast))\) for all \(x, y \in \mathcal{M}_\psi^\ast\) so by repeating the last two steps in the calculation 3.3.2 we find that \(\psi(z^\ast y) = \psi(y \sigma_i^\ast(z^\ast))\) for all \(x, y \in \mathcal{M}_\psi^\ast\). In particular,
\[
\|
\Lambda_\psi(a_n^*) - \Lambda_\psi(a_m^*)
\|^2 = \psi((a_n - a_m)(a_n^* - a_m^*))
\]
= \(\psi((a_n^* - a_m^*)(a_n^* - a_m^*)) = \psi(\sigma_i^\ast(a_n^* - a_m^*)\sigma_i^\ast(a_n^* - a_m^*))\)
\[
= \|\Lambda_\psi(\sigma_i^\ast(a_n)) - \Lambda_\psi(\sigma_i^\ast(a_m))\|^2
\]
for all \(m, n\). It follows that \(\{\Lambda_\psi(a_n^*)\}\) converges in \(H_\psi\) and since \(\Lambda_\psi\) is closed by Lemma 3.1.5 it follows that \(\lim_{n \to \infty} \Lambda_\psi(a_n^*) = \Lambda_\psi(a^*)\). The sequence \(\{b_n\}\) is constructed in the same way. We need to arrange that the functions \(z \mapsto \Lambda_\psi(\sigma_z(a_n))\) are entire analytic. To ensure this note that we can choose a sequence \(\{b_n\}\) in \(\mathbb{N}\) such that \(\|R_{b_n}(a_n) - a_n\| \leq \frac{1}{n}\), \(\|\Lambda_\psi(R_{b_n}(a_n)) - \Lambda_\psi(a_n)\| \leq \frac{1}{n}\) and \(\|\Lambda_\psi(b_n^*) - \Lambda_\psi(a_n^*)\| \leq \frac{1}{n}\). This follows from Lemma 2.1.5 and Lemma 3.1.5. Note that \(R_{b_n}(a_n) \in \mathcal{M}_\psi^\ast\) by Lemma 3.1.5. The sequence \(\{R_{b_n}(a_n)\}\) therefore has the same properties as \(\{a_n\}\). Since \(z \mapsto \Lambda_\psi(\sigma_z(R_{b_n}(a_n)))\) is entire analytic by Lemma 3.1.3 and Lemma 3.1.4 we may assume that \(z \mapsto \Lambda_\psi(\sigma_z(a_n))\) is entire analytic for all \(n\).

For each \(n \in \mathbb{N}\) define \(f_n : \mathbb{C} \to \mathbb{C}\) such that
\[
f_n(z) = \psi(b_n \sigma_z(a_n)) = \langle \Lambda_\psi(\sigma_z(a_n)), \Lambda_\psi(b_n^*) \rangle.
\]
Note that \(f_n\) is entire holomorphic since \(z \mapsto \Lambda_\psi(\sigma_z(a_n))\) is entire analytic. For \(z \in \mathcal{D}_\beta\) we get the estimate
\[
|f_n(z) - f_m(z)| \leq \max\left\{ \sup_{t \in \mathbb{R}} |f_n(t) - f_m(t)|, \sup_{t \in \mathbb{R}} |f_n(t + i\beta) - f_m(t + i\beta)| \right\}
\]
(3.3.2)
from Proposition 5.3.5 in [BR] (Phragmen-Lindelöf) for all \( n, m \in \mathbb{N} \). We note that

\[
    f_n(t) = \left\langle U_t \Lambda_\psi(a_n), \Lambda_\psi(b_n^*) \right\rangle,
\]

and that \( U_t \Lambda_\psi(a_n) \) converges to \( U_t \Lambda_\psi(a) \) uniformly in \( t \) since \( \lim_{n \to \infty} \Lambda_\psi(a_n) = \Lambda_\psi(a) \). As \( \lim_{n \to \infty} \Lambda_\psi(b_n^*) = \Lambda_\psi(b^*) \) it follows therefore that

\[
    \lim_{n \to \infty} f_n(t) = \left\langle U_t \Lambda_\psi(a), \Lambda_\psi(b^*) \right\rangle = \psi(b \sigma_t(a))
\]

uniformly in \( t \). It follows from (3) that

\[
    f_n(t + i\beta) = \psi(b \sigma_t(a_n)) = \psi(\sigma_t(a_n)b_n) = \left\langle \Lambda_\psi(b_n), U_t \Lambda_\psi(a_n^*) \right\rangle.
\]

Since \( \lim_{n \to \infty} \Lambda_\psi(b_n) = \Lambda_\psi(b) \) and \( \lim_{n \to \infty} \Lambda_\psi(a_n^*) = \Lambda_\psi(a^*) \), it follows that

\[
    \lim_{n \to \infty} f_n(t + i\beta) = \left\langle \Lambda_\psi(b), U_t \Lambda_\psi(a^*) \right\rangle = \psi(\sigma_t(a)b)
\]

uniformly in \( t \). It follows now from the estimate \([3.3.2]\) that the sequence \( \{f_n\} \) converges uniformly on \( D_\beta \) to a continuous function \( f : D_\beta \to \mathbb{C} \) which is holomorphic in \( D_\beta^0 \) and has the required properties.

\((4) \to (1)\): Let \( a \in \mathcal{N}_\psi \cap D(\sigma_{-i\frac{\beta}{2}}) \). It follows from Lemma \( 3.1.12 \) that there is a sequence \( \{a_n\} \) in \( \mathcal{M}_\psi^* \) such that \( \lim_{n \to \infty} a_n = a \), \( \lim_{n \to \infty} \Lambda_\psi(a_n) = \Lambda_\psi(a) \) and \( \lim_{n \to \infty} \sigma_{-i\frac{\beta}{2}}(a_n) = \sigma_{-i\frac{\beta}{2}}(a) \). Fix \( n, m \in \mathbb{N} \), and set

\[
    H(z) = \psi(a_n \sigma_z(a_m^*)) = \left\langle \Lambda_\psi(\sigma_z(a_m^*)), \Lambda_\psi(a_n^*) \right\rangle\quad \forall z \in \mathbb{C}.
\]

It follows from Lemma \( 3.1.3 \) and Lemma \( 3.1.4 \) that \( H \) is entire holomorphic. By assumption there is a continuous function \( f : D_\beta \to \mathbb{C} \) such that \( f \) is holomorphic on \( D_\beta \),

\[
    f(t) = \psi(a_n \sigma_t(a_m^*))\quad \forall t \in \mathbb{R}, \quad f(t + i\beta) = \psi(\sigma_t(a_m^*)a_n)\quad \forall t \in \mathbb{R}.
\]

Applying Proposition 5.3.6 in [BR] with \( \mathcal{O} = \{ z \in \mathbb{C} : \Im z < \beta \} \) and \( F(z) = H(z) - f(z) \) when \( \beta > 0 \) and with \( \mathcal{O} = \{ z \in \mathbb{C} : \Im z < \beta \} \) and \( F(z) = H(z) - f(z) \) when \( \beta < 0 \), we conclude that \( H(z) = f(z) \) for all \( z \in D_\beta \). Hence

\[
    \psi(\sigma_t(a_m^*)a_n) = f(t + i\beta) = H(t + i\beta) = \psi(a_n \sigma_{t+i\beta}(a_m^*))
\]

for all \( t \in \mathbb{R} \). By combining with Lemma \( 2.1.2 \) Lemma \( 2.1.3 \) and Lemma \( 3.1.4 \) we find

\[
    \psi(a_m^*a_n) = \psi(a_n \sigma_{i\beta}(a_m^*)) = \psi \left( a_n \sigma_{i\beta} \left( \sigma_{i\beta}(a_m^*) \right) \right) = \psi \left( a_m \sigma_{i\beta} \left( \sigma_{-i\beta}(a_n)^* \right) \right) = \psi \left( \sigma_{i\beta}(a_n) \sigma_{-i\beta}(a_m)^* \right).
\]
Thus

\[ \langle \Lambda_\psi(a_n), \Lambda_\psi(a_m) \rangle = \left\langle \Lambda_\psi(\sigma_{-i_\psi}(a_m)^*), \Lambda_\psi(\sigma_{-i_\psi}(a_n)^*) \right\rangle, \] (3.3.4)

and hence

\[
\left\| \Lambda_\psi(\sigma_{-i_\psi}(a_n)^*) - \Lambda_\psi(\sigma_{-i_\psi}(a_m)^*) \right\|^2 \\
= \left\langle \Lambda_\psi(\sigma_{-i_\psi}(a_n)^*), \Lambda_\psi(\sigma_{-i_\psi}(a_n)^*) \right\rangle + \left\langle \Lambda_\psi(\sigma_{-i_\psi}(a_m)^*), \Lambda_\psi(\sigma_{-i_\psi}(a_n)^*) \right\rangle \\
- \left\langle \Lambda_\psi(\sigma_{-i_\psi}(a_n)^*), \Lambda_\psi(\sigma_{-i_\psi}(a_m)^*) \right\rangle - \left\langle \Lambda_\psi(\sigma_{-i_\psi}(a_m)^*), \Lambda_\psi(\sigma_{-i_\psi}(a_n)^*) \right\rangle \\
= \langle \Lambda_\psi(a_n), \Lambda_\psi(a_n) \rangle + \langle \Lambda_\psi(a_m), \Lambda_\psi(a_m) \rangle - \langle \Lambda_\psi(a_n), \Lambda_\psi(a_m) \rangle - \langle \Lambda_\psi(a_m), \Lambda_\psi(a_n) \rangle \\
= \| \Lambda_\psi(a_n) - \Lambda_\psi(a_m) \|^2.
\]

It follows that \( \{ \Lambda_\psi(\sigma_{-i_\psi}(a_n)^*) \} \) converges in \( H_\psi \). Note that \( \Lambda_\psi(\sigma_{-i_\psi}(a_n)^*) = \Lambda_\psi(\sigma_{-i_\psi}(a_n)^*) \) by Lemma 2.1.12. Since \( \lim_{n \to \infty} \sigma_{-i_\psi}(a_n)^* = \sigma_{-i_\psi}(a)^* = \sigma_{i_\psi}(a)^* \) and \( \Lambda_\psi \) is closed by Lemma 3.1.1 it follows that \( \sigma_{i_\psi}(a)^* \in \mathcal{N}_\psi \) and

\[ \lim_{n \to \infty} \Lambda_\psi(\sigma_{-i_\psi}(a_n)^*) = \Lambda_\psi(\sigma_{i_\psi}(a)^*). \]

Combined with (3.3.4) and Lemma 2.1.12 we get

\[
\psi(a^*a) = \langle \Lambda_\psi(a), \Lambda_\psi(a) \rangle \\
= \lim_{n \to \infty} \langle \Lambda_\psi(a_n), \Lambda_\psi(a_n) \rangle \\
= \lim_{n \to \infty} \left\langle \Lambda_\psi(\sigma_{-i_\psi}(a_n)^*), \Lambda_\psi(\sigma_{-i_\psi}(a_n)^*) \right\rangle \\
= \left\langle \Lambda_\psi(\sigma_{i_\psi}(a)^*), \Lambda_\psi(\sigma_{i_\psi}(a)^*) \right\rangle \\
= \psi(\sigma_{i_\psi}(a)^* \sigma_{i_\psi}(a)^*) = \psi(\sigma_{-i_\psi}(a) \sigma_{-i_\psi}(a)^*).
\]

We have established the desired equality in (1) when \( a \in \mathcal{N}_\psi \cap D(\sigma_{-i_\psi}) \). If \( a \in D(\sigma_{-i_\psi}) \cap \mathcal{N}_\psi \) it follows that \( \psi(\sigma_{-i_\psi}(a) \sigma_{-i_\psi}(a)^*) = \infty \). Indeed, if

\[ \psi(\sigma_{-i_\psi}(a) \sigma_{-i_\psi}(a)^*) < \infty \]

we have that \( \sigma_{-i_\psi}(a)^* \in \mathcal{N}_\psi \) and hence \( \sigma_{i_\psi}(a)^* \in \mathcal{N}_\psi \) by Lemma 2.1.12. But \( \sigma_{i_\psi}(a)^* \in D(\sigma_{-i_\psi}) \) by Lemma 2.1.9 and hence, by what we have just established and Lemma 2.1.10

\[ \infty > \psi(\sigma_{i_\psi}(a)^* \sigma_{i_\psi}(a)^*) = \psi(\sigma_{-i_\psi}(\sigma_{i_\psi}(a)^*) \sigma_{-i_\psi}(\sigma_{i_\psi}(a)^*)) = \psi(a^*a); \]

a contradiction. Thus the equality in (1) is valid in all cases where it makes sense, namely for all \( a \in D(\sigma_{-i_\psi}) \). \( \square \)
Some choices were made in the formulation of condition (2) and (3) of Theorem 3.2.1. Specifically, the algebra $\mathcal{M}_\psi^\sigma$ was chosen because it has the nice properties that it is a $\ast$-algebra which is invariant under $\sigma_z$ for all $z \in \mathbb{C}$ on which $\psi$ is an everywhere defined linear functional. However, in specific cases it can be hard to identify $\mathcal{M}_\psi^\sigma$, and in applications it is often important to be able to deduce that a weight is a $\beta$-KMS weight from knowing that it satisfies the identity in (2) or (3) of Theorem 3.2.1 for elements in a set which is as small as possible. The next section is devoted to the proof of a result which seems optimal in this respect.

### 3.3.1 The GNS triple of a KMS weight

Let $\sigma$ be a flow on the $C^*$-algebra $A$. Two GNS representations, $(H_i, \Lambda_i, \pi_i)$, $i = 1, 2$, of $A$ are isomorphic when there is a unitary $W : H_1 \to H_2$ such that

- $D(\Lambda_1) = D(\Lambda_2)$,
- $WA_1(a) = \Lambda_2(a), \ \forall a \in D(\Lambda_1)$, and
- $W\pi_1(a) = \pi_2(a)W, \ \forall a \in A$.

In order to determine when a GNS representation $(H, \Lambda, \pi)$ of $A$ is isomorphic to the GNS-triple of a KMS weight for $\sigma$, we consider the following conditions:

- (A) $\pi$ is non-degenerate, i.e. $\text{Span} \{\pi(a)\Lambda(b) : a \in A, b \in D(\Lambda)\}$ is dense in $H$,
- (B) $\Lambda : D(\Lambda) \to H$ is closed,
- (C) $\sigma_t(D(\Lambda)) = D(\Lambda)$ for all $t \in \mathbb{R}$,
- (D) there is a continuous unitary group representation $(U_t)_{t \in \mathbb{R}}$ of $\mathbb{R}$ on $H$ such that $U_t\Lambda(a) = \Lambda(\sigma_t(a))$ for all $t \in \mathbb{R}$ and all $a \in D(\Lambda)$, and
- (E) there is a conjugate linear isometry $J : H \to H$ such that $J\Lambda(a) = \Lambda(\sigma_-\frac{\pi}{2}(a)^*)$ for all $a \in M_\Lambda^\sigma$,

where $\mathcal{M}_\Lambda^\sigma := \{a \in D(\Lambda) \cap D(\Lambda)^* \cap \mathcal{A}_\sigma : \sigma_z(a) \in D(\Lambda) \cap D(\Lambda)^* \ \forall z \in \mathbb{C}\}$.

**Lemma 3.3.1.** $\mathcal{M}_\Lambda^\sigma$ is a core for $\Lambda : D(\Lambda) \to H$.

**Proof.** Let $a \in D(\Lambda)$ and let $\{e_j\}_{j \in I}$ be the approximate unit for $A$ in $D(\Lambda)$ from Lemma 3.1.11. Then $e_ja \in D(\Lambda) \cap D(\Lambda)^*$ and $R_k(e_ja) \in D(\Lambda) \cap D(\Lambda)^* \cap \mathcal{A}_\sigma$ by Lemma 3.1.14 and Lemma 2.1.3. Then $\sigma_{-\frac{\pi}{2}}(R_k(e_ja)) \in \mathcal{A}_\sigma$ by Lemma 2.1.1 and $\sigma_{-\frac{\pi}{2}}(R_k(e_ja)) \in D(\Lambda) \cap D(\Lambda)^*$ by Lemma 3.1.3. Hence $R_k(e_ja) \in \mathcal{M}_\Lambda^\sigma$. It follows from Lemma 2.1.5 that $\lim_{k \to \infty} R_k(e_ja) = e_ja$ and from Lemma 3.1.11 that $\lim_{j \to \infty} e_ja$. By Lemma 3.1.15 $\lim_{k \to \infty} \Lambda(R_k(e_ja)) = \Lambda(e_ja) = \pi(e_j)\Lambda(a)$. This completes the proof because $\lim_{j \to \infty} \pi(e_j)\Lambda(a) = \Lambda(a)$ by Lemma 3.1.11. □

**Lemma 3.3.2.** Let $(H, \Lambda, \pi)$ be a GNS representation of $A$ with the properties (A), (B), (C) and (D). Let $\beta \in \mathbb{R}$. There is a net $\{u_j\}_{j \in I}$ in $A$ such that
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(a) \( u_j \in \mathcal{M}_A^\sigma \) for all \( j \in I \),
(b) \( \sup_{j \in I} \|u_j\| < \infty \),
(c) \( \lim_{j \to \infty} u_j a = \lim_{j \to \infty} a u_j = a \) for all \( a \in A \),
(d) \( \left\{ \sigma_{-i \frac{2}{z}}(u_j) \right\}_{j \in I} \) is an increasing approximate unit for \( A \) and \( 0 \leq \sigma_{-i \frac{2}{z}}(u_j) \leq 1 \) for all \( j \),

If (E) also holds there is also a net \( \{\rho_j\}_{j \in I} \) in \( B(H) \), indexed by the same directed set \( I \), such that

(e) \( \|\rho_j\| \leq 1 \) for all \( j \in I \),
(f) \( \rho_j \Lambda(a) = \pi(a) \Lambda(u_j) \) for all \( j \in I \) and all \( a \in D(\Lambda) \), and
(g) \( \lim_{j \to \infty} \rho_j = 1 \) in the strong operator topology.

Proof. By Lemma 3.3.11 there is an increasing approximate unit \( \{e_j\}_{j \in I} \) for \( A \) contained in \( D(\Lambda) \) such that \( 0 \leq e_j \leq 1 \) for all \( j \). In particular, \( e_j \in D(\Lambda) \cap D(\Lambda)^* \). Set

\[ u_j = \sigma_{i \frac{2}{z}}(R_1(e_j)). \]

It follows from Lemma 2.1.3 and Lemma 3.1.4 that \( \sigma_z(u_j) \in D(\Lambda) \cap A_\sigma \) for all \( z \in \mathbb{C} \). Since \( u_j^* = \sigma_{-i \frac{2}{z}}(R_1(e_j)) \) it follows in the same way that \( \sigma_z(u_j) \in D(\Lambda)^* \cap A_\sigma \) for all \( z \in \mathbb{C} \). Thus (a) holds. Note that \( \|u_j\| \leq \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-s^2 + |s|\beta} \, ds \), showing that (b) holds. It follows from the formula for \( \sigma_z(R_1(e_j)) \) from Lemma 2.1.10 and Lemma A.2.2 in Appendix A that

\[ \lim_{j \to \infty} \sigma_z(R_1(e_j)) a = \lim_{j \to \infty} a \sigma_z(R_1(e_j)) = a \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-(s-z)^2} \, ds \quad (3.3.5) \]

for all \( z \in \mathbb{C} \) since \( \{e_j\} \) is an approximate unit for \( A \). The function \( z \mapsto \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-(s-z)^2} \, ds \) is entire analytic\(^1\) and since

\[ \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-(s-t)^2} \, ds = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-s^2} \, ds = 1 \]

for all \( t \in \mathbb{R} \), we conclude that \( \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-(s-z)^2} \, ds = 1 \) for all \( z \in \mathbb{C} \). Applied with \( z = i \frac{2}{\beta} \) we obtain (c). Note that (d) holds by Lemma 2.1.9.

Assume now that also (E) holds. For \( a \in \mathcal{M}_A^\sigma \) we find by using condition (E) that

\[ \|\Lambda(au_j)\| = \|\Lambda \left( \sigma_{-i \frac{2}{z}}(au_j)^* \right) \| = \|\Lambda(R_1(e_j)\sigma_{-i \frac{2}{z}}(a)^*)\| = \|\pi(R_1(e_j))\Lambda \left( \sigma_{-i \frac{2}{z}}(a)^* \right) \| \leq \|\Lambda (\sigma_{-i \frac{2}{z}}(a)^*)\| = \|\Lambda(a)\|. \]

\(^1\)This is a fact from calculus, but it follows also by applying Lemma 2.1.3 to the trivial flow on \( \mathbb{C} \).
It follows from Lemma [3.3.1] that $\Lambda(M)_{\sigma}$ is dense in $H$. Hence the estimate above shows that we can define $\rho_j \in B(H)$ such that

$$\rho_j \Lambda(a) = \Lambda(au_j), \quad \forall a \in M_{\Lambda}^\sigma,$$

and that $\|\rho_j\| \leq 1$. Thus (f) and (e) hold. For $a \in M_{\Lambda}^\sigma$,

$$\|\rho_j \Lambda(a) - \Lambda(a)\|^2 = \|\Lambda(au_j - a)\|^2$$

$$= \|\Lambda\left(\sigma - \frac{i}{\pi}(au_j + \sigma - \frac{i}{\pi}(a))^*\right)\| = \|\Lambda\left(R_1(e_j)\sigma - \frac{i}{\pi}(a)\right)\|$$

$$= \|\pi(R_1(e_j))\Lambda(\sigma - \frac{i}{\pi}(a))\|.$$

Since $\pi$ is non-degenerate and since $\lim_{j \rightarrow \infty} R_1(e_j)b = b$ for all $b \in A$ we conclude first that $\lim_{j \rightarrow \infty} \pi(R_1(e_j)) = 1$ strongly, and then from the calculation above that (g) holds since $\Lambda(M)_{\Lambda}^\sigma$ is dense in $H$. \hfill \Box

**Proposition 3.3.3.** Let $(H, \Lambda, \pi)$ be a GNS representation of $A$ and let $\beta \in \mathbb{R}$. Then $(H, \Lambda, \pi)$ is isomorphic to the GNS-triple of a $\beta$-KMS weight for $\sigma$ if and only if (A), (B), (C), (D) and (E) all hold.

**Proof.** Necessity of the five conditions: (A) and (B) follow from Lemma [3.1.10] and (C) follows because a $\beta$-KMS weight is $\sigma$-invariant. (D) follows from Lemma [3.1.2]. Let $\psi$ be a $\beta$-KMS weight for $\sigma$. Then $\psi(a^*a) = \psi\left(\sigma - \frac{i}{\pi}(a)\sigma - \frac{i}{\pi}(a)^*\right)$ for $a \in A_\sigma$ by (1) of Theorem [3.2.1] and we can therefore define a conjugate linear isometry $J' : \Lambda_\psi(M_{\Lambda}^\sigma) \rightarrow \Lambda_\psi(M_{\Lambda}^\sigma)$ such that $J'\Lambda_\psi(a) = \Lambda_\psi\left(\sigma - \frac{i}{\pi}(a)^*\right)$. Note that $\Lambda_\psi(N_{\psi} \cap A_\sigma)$ is dense in $H_{\psi}$ by Lemma [2.1.3] and Lemma [3.1.3]. It follows therefore that $J'$ extends by continuity to a conjugate linear isometry $J' : H_{\psi} \rightarrow H_{\psi}$. Hence if $(H, \Lambda, \pi)$ is isomorphic to the GNS-triple of $\psi$, and $W : H \rightarrow H_{\psi}$ is the associated unitary, the map $J = W^*J'W$ will have the property required in (E).

Sufficiency: Assume (A) through (E) all hold. Let $\{u_j\}_{j \in I}$ and $\{\rho_j\}_{j \in I}$ be the two nets from Lemma [3.3.4]. Define $\omega_j : A \rightarrow \mathbb{C}$ such that

$$\omega_j(a) = \langle \pi(a), \Lambda(u_j), \Lambda(u_j) \rangle.$$

Then $\omega_j \in A^\sigma_\Lambda$. When $a \in D(\Lambda)$,

$$\omega_j(a^*a) = \langle \pi(a), \Lambda(u_j) \rangle = \langle \Lambda(au_j), \Lambda(au_j) \rangle$$

$$= \langle \rho_j^2 \Lambda(a), \Lambda(a) \rangle \leq \langle \Lambda(a), \Lambda(a) \rangle,$$

showing that $\omega_j \in F_{\Lambda}$. Let $\psi : A^+ \rightarrow [0, \infty]$ be the weight of the GNS-triple $(H, \Lambda, \pi)$, cf. Section [12.2]. Let $a \in D(\Lambda)$. Then

$$\psi(a^*a) = \sup_{\omega \in F_{\Lambda}} \omega(a^*a) \leq \langle \Lambda(a), \Lambda(a) \rangle < \infty,$$
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showing that \( D(\Lambda) \subseteq \mathcal{N}_\psi \). Let \( a \in \mathcal{N}_\psi \). Then \( au_j \in D(\Lambda) \) since \( u_j \in D(\Lambda) \) and \( D(\Lambda) \) is a left ideal. It follows from (c) of Lemma 3.3.2 that \( \lim_{j \to \infty} au_j = a \).

Note that

\[
\|\Lambda(au_j)\|^2 = \omega_j(a^*a) \leq \psi(a^*a)
\]

for all \( j \). Since \( \lim_{j \to \infty} au_j = a \) we can pick out a sequence \( j_1 \leq j_2 \leq \cdots \) in \( \mathbb{N} \) such that the sequence \( a_n = au_{j_n}, n \in \mathbb{N}, \) has the property that \( \|a_n - a\| \leq \frac{1}{k} \) for all \( n \geq k \). Since

\[
\sup_{n \geq k} \|\Lambda(a_n)\| \leq \psi(a^*a)^{\frac{1}{2}},
\]

the weak* closure \( \overline{\text{co}} \{\Lambda(a_n) : n \geq k\} \) of \( \{\Lambda(a_n) : n \geq k\} \) is compact in the weak* topology of \( H^* = H \). In particular, the intersection

\[
\bigcap_{k} \overline{\text{co}} \{\Lambda(a_n) : n \geq k\}
\]

is not empty and we let \( \eta \) be an element of this intersection. By the self-duality of a Hilbert space the weak* topology is the same as the weak topology and hence \( \overline{\text{co}} \{\Lambda(a_n) : n \geq k\} \) is also the norm closure of \( \text{co}\{\Lambda(a_n) : n \geq k\} \). Using the linearity of \( \Lambda \) we can therefore construct \( b_k \in \text{co}\{a_n : n \geq k\} \subseteq D(\Lambda) \) such that \( \|b_k - a\| \leq \frac{1}{k} \) and \( \|\Lambda(b_k) - \eta\| \leq \frac{1}{k} \). Since \( \Lambda \) is closed this implies that \( a \in D(\Lambda) \). We have shown that \( \mathcal{N}_\psi = D(\Lambda) \). For \( a \in D(\Lambda) \) it follows from (g) of Lemma 3.3.2 that

\[
\lim_{j \to \infty} \omega_j(a^*a) = \lim_{j \to \infty} \langle \Lambda(au_j), \Lambda(au_j) \rangle = \lim_{j \to \infty} \langle \rho_j \Lambda(a), \rho_j \Lambda(a) \rangle = \langle \Lambda(a), \Lambda(a) \rangle,
\]

proving that

\[
\|\Lambda_\psi(a)\|^2 = \psi(a^*a) = \langle \Lambda(a), \Lambda(a) \rangle = \|\Lambda(a)\|^2.
\]

It follows that there is a unitary \( W : H \to H_\psi \) such that \( W\Lambda(a) = \Lambda_\psi(a) \) for all \( a \in D(\Lambda) \). Since

\[
W\pi(a)\Lambda(b) = W\Lambda(ab) = \Lambda_\psi(ab) = \pi_\psi(a)\Lambda_\psi(b) = \pi_\psi(a)W\Lambda(b)
\]

for all \( b \in D(\Lambda) \) we conclude that \( W\pi(a) = \pi_\psi(a)W \) for all \( a \in A \), and hence that \( (H, \Lambda, \pi) \) is isomorphic to the GNS-triple \( (H_\psi, \Lambda_\psi, \pi_\psi) \) of \( \psi \). It remains to show that \( \psi \) is a \( \beta \)-KMS weight for \( \sigma \). First of all, \( \psi \) is densely defined because \( D(\Lambda) \) is dense in \( A \) and \( \psi(a^*a) = \langle \Lambda(a), \Lambda(a) \rangle < \infty \) when \( a \in D(\Lambda) \). Let \( t \in \mathbb{R} \) and \( \omega \in \mathcal{F}_\Lambda \). It follows from condition (C) and (D) that

\[
\omega \circ \sigma_t(a^*a) = \omega(\sigma_t(a)^*\sigma_t(a)) \leq \langle \Lambda(\sigma_t(a)), \Lambda(\sigma_t(a)) \rangle
\]

\[
= \langle U_t\Lambda(a), U_t\Lambda(a) \rangle = \langle \Lambda(a), \Lambda(a) \rangle
\]
for all \( a \in D(\Lambda) \), showing that \( \omega \circ \sigma_t \in F_\Lambda \). Since this holds for all \( t \in \mathbb{R} \) and all \( \omega \in F_\Lambda \) we conclude that

\[
\psi(\sigma_t(a)) = \sup_{\omega \in F_\Lambda} \omega \circ \sigma_t(a) = \sup_{\omega \in F_\Lambda} \omega(a) = \psi(a)
\]

for all \( a \in A^+ \) and all \( t \). That is, \( \psi \) is \( \sigma \)-invariant. Let \( a \in M^\sigma_\Lambda = M^\sigma_\Lambda \). Then

\[
\psi(a^*a) = \langle \Lambda_\psi(a), \Lambda_\psi(a) \rangle = \langle \Lambda(a), \Lambda(a) \rangle = \|\Lambda(a)\|^2
\]

\[
= \left\langle \Lambda \left( \sigma_{-i\beta} (a)^* \right), \Lambda \left( \sigma_{-i\beta} (a)^* \right) \right\rangle = \psi \left( \sigma_{-i\beta} (a)^* \right) \left( \sigma_{-i\beta} (a)^* \right).
\]

Since \( M^\sigma_\psi \subseteq M^\sigma_\Lambda \) it follows from Theorem 3.2.1 that \( \psi \) is a \( \beta \)-KMS weight. □

**Lemma 3.3.4.** Let \((H, \Lambda, \pi)\) be a GNS representation of \( A \) with the properties (A), (B), (C) and (D). The map \( \Lambda \circ \sigma_z : M^\Lambda_\Lambda \to H \) is closable for all \( z \in \mathbb{C} \).

**Proof.** Let \( \{a_n\} \) be a sequence in \( M^\Lambda_\Lambda \) such that \( \lim_{n \to \infty} a_n = 0 \) and \( \lim_{n \to \infty} \Lambda (\sigma_z (a_n)) = \eta \in H \).

Let \( k \in \mathbb{N} \). Then \( \lim_{n \to \infty} R_k(a_n) = 0 \) and by using Lemma 2.1.10 and Lemma \( A.2.2 \) we find that

\[
\lim_{n \to \infty} \sigma_z (R_k(a)) = \lim_{n \to \infty} \sqrt{\frac{k}{\pi}} \int_{\mathbb{R}} e^{-ks^2} \sigma_z (a_n) \, ds = 0.
\]

Similarly, by using Lemma 3.1.4 and Lemma \( A.2.2 \) we find that

\[
\lim_{n \to \infty} \Lambda (\sigma_z (R_k(a))) = \lim_{n \to \infty} \sqrt{\frac{k}{\pi}} \int_{\mathbb{R}} e^{-ks^2} U_s \Lambda (\sigma_z (a_n)) \, ds
\]

\[
= \sqrt{\frac{k}{\pi}} \int_{\mathbb{R}} e^{-ks^2} U_s \eta \, ds,
\]

and conclude therefore that \( \sqrt{\frac{k}{\pi}} \int_{\mathbb{R}} e^{-ks^2} U_s \eta \, ds = 0 \) since \( \Lambda \) is closed. Since

\[
\lim_{k \to \infty} \sqrt{\frac{k}{\pi}} \int_{\mathbb{R}} e^{-ks^2} U_s \eta \, ds = \eta
\]

by Lemma 2.1.5 it follows that \( \eta = 0 \). □

**Lemma 3.3.5.** Let \((H, \Lambda, \pi)\) be a GNS representation of \( A \) with the properties (A), (B), (C) and (D). Let \( \beta \in \mathbb{R} \). Condition (E) is equivalent to the following:

\( \text{(F)} \) There is a subspace \( S \subseteq M^\Lambda_\Lambda \) such that
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- for all \(a \in \mathcal{M}^\sigma_\Lambda\) there is a sequence \(\{s_n\}\) in \(S\) such that \(\lim_{n \to \infty} s_n = a\) and \(\lim_{n \to \infty} \Lambda(s_n) = \Lambda(a)\), and

- \(\|\Lambda(s)\| = \|\Lambda \left( \sigma_{-i \frac{\pi}{2}}(s)^* \right) \|\), \(\forall s \in S\).

**Proof.** Assume (F). Let \(a \in \mathcal{M}^\sigma_\Lambda\). By assumption there is a sequence \(\{s_n\}\) in \(S\) such that \(\lim_{n \to \infty} s_n = a\) and \(\lim_{n \to \infty} \Lambda(s_n) = \Lambda(a)\). Since

\[
\left\| \Lambda \left( \sigma_{i \frac{\pi}{2}}(s_n)^* \right) - \Lambda \left( \sigma_{i \frac{\pi}{2}}(s_m)^* \right) \right\| = \left\| \Lambda \left( \sigma_{-i \frac{\pi}{2}}(s_n - s_m) \right) \right\| = \|\Lambda(s_n) - \Lambda(s_m)\|,
\]

it follows that \(\{\Lambda \left( \sigma_{i \frac{\pi}{2}}(s_n)^* \right)\}\) is a Cauchy sequence and hence convergent in \(H\).

Since \(\lim_{n \to \infty} s_n = a^*\) it follows from Lemma 3.3.4 that \(\lim_{n \to \infty} \Lambda \left( \sigma_{i \frac{\pi}{2}}(s_n)^* \right) = \Lambda \left( \sigma_{i \frac{\pi}{2}}(a^*) \right)\). Therefore

\[
\|\Lambda(a)\| = \lim_{n \to \infty} \|\Lambda(s_n)\| = \lim_{n \to \infty} \left\| \Lambda \left( \sigma_{-i \frac{\pi}{2}}(s_n)^* \right) \right\|
= \lim_{n \to \infty} \left\| \Lambda \left( \sigma_{i \frac{\pi}{2}}(s_n)^* \right) \right\| = \left\| \Lambda \left( \sigma_{i \frac{\pi}{2}}(a^*) \right) \right\|.
\]

Since \(\Lambda(\mathcal{M}^\sigma_\Lambda)\) is dense in \(H\) by Lemma 3.3.1, this gives us a conjugate linear isometry \(J : H \to H\) such that \(J\Lambda(a) = \Lambda \left( \sigma_{-i \frac{\pi}{2}}(a^*) \right)\) for all \(a \in \mathcal{M}^\sigma_\Lambda\); that is, (E) holds. That (E) implies (F) is trivial. \(\square\)

**Theorem 3.3.6. (Kustermans, \([Ku]\))** Let \(\psi\) be a non-zero densely defined weight on \(A\) which is \(\sigma\)-invariant in the sense that \(\psi \circ \sigma_t = \psi\) for all \(t \in \mathbb{R}\), and let \(\beta \in \mathbb{R}\) be a real number. Let \(S\) be a subspace of \(\mathcal{M}^\sigma_\Lambda\), with the property that for any element \(a \in \mathcal{M}^\sigma_\Lambda\) there is a sequence \(\{a_n\}\) in \(S\) such that \(\lim_{n \to \infty} a_n = a\) and \(\lim_{n \to \infty} \Lambda_\psi(a_n) = \Lambda_\psi(a)\). The following conditions are equivalent:

1. \(\psi\) is a \(\beta\)-KMS weight for \(\sigma\).
2. \(\psi(a^*a) = \psi \left( \sigma_{-i \frac{\pi}{2}}(a)\sigma_{-i \frac{\pi}{2}}(a)^* \right)\) \(\forall a \in S\).
3. \(\psi(b^*a) = \psi(a\sigma_{i\beta}(b^*))\) \(\forall a, b \in S\).

**Proof.** The implication (1) \(\Rightarrow\) (2) follows immediately from Theorem 3.2.1 and the implication (2) \(\Rightarrow\) (3) results from the following careful inspection of the polarization argument in the proof of the same implication of Theorem 3.2.1 Let \(a, b \in S\). Then \(b^*a\) and \((a + i^k b)^*(a + i^k b) \in \mathcal{M}_\psi\) for all \(k \in \mathbb{N}\) by (b) of Lemma 1.0.1 and 1.0.1 and Lemma 1.2.1 can therefore be applied to conclude that

\[
\psi(b^*a) = \frac{1}{4} \sum_{k=1}^{4} i^k \psi((a + i^k b)^*(a + i^k b)).
\]
Since we assume (2) it follows that
\[ \psi(b^*a) = \frac{1}{4} \sum_{k=1}^{4} i^k \psi \left( \sigma_{-i\frac{\beta}{2}}(a + i^k b) \sigma_{-i\frac{\beta}{2}}(a + i^k b)^* \right). \]

Now, (2) implies also that \( \sigma_{-i\frac{\beta}{2}}(a + i^k b) \sigma_{-i\frac{\beta}{2}}(a + i^k b)^* \in \mathcal{M}_\psi \) and we get
\[ \psi(b^*a) = \psi \left( \frac{1}{4} \sum_{k=1}^{4} i^k \sigma_{-i\frac{\beta}{2}}(a + i^k b) \sigma_{-i\frac{\beta}{2}}(a + i^k b)^* \right). \]

Note that
\[ \frac{1}{4} \sum_{k=1}^{4} i^k \sigma_{-i\frac{\beta}{2}}(a + i^k b) \sigma_{-i\frac{\beta}{2}}(a + i^k b)^* = \sigma_{-i\frac{\beta}{2}}(a) \sigma_{-i\frac{\beta}{2}}(b)^*. \]

It follows from Lemma 2.1.12 and Lemma 2.1.6 that
\[ \sigma_{-i\frac{\beta}{2}}(a) \sigma_{-i\frac{\beta}{2}}(b)^* = \sigma_{-i\frac{\beta}{2}}(a \sigma_{i\beta}(b)^*)), \quad (3.3.8) \]
and we conclude therefore from Lemma 3.1.14 that \( \psi(b^*a) = \psi(a \sigma_{i\beta}(b)^*)) \); that is, (3) holds. The implication (3) \( \Rightarrow \) (2) follows by reversing the arguments just given: Assuming (3), Lemma 3.1.14 and (3.3.8) gives (2) by taking \( b = a \). Finally, (2) \( \Rightarrow \) (1): Let \( (H_\psi, \Lambda_\psi, \pi_\psi) \) be the GNS-triple of \( \psi \). This GNS representation of \( A \) satisfies condition (A) by Lemma 3.1.10, (B) by Lemma 3.1.1, (C) because \( \psi \) is \( \sigma \)-invariant and (D) by Lemma 3.1.2. If \( a \in \mathcal{N}_\psi \) it follows from Corollary 3.1.13 that there is a sequence \( \{a_n\} \) in \( \mathcal{M}_\psi^\sigma \) such that \( \lim_{n \to \infty} a_n = a \) and \( \lim_{n \to \infty} \Lambda_\psi(a_n) = \Lambda_\psi(a) \). By using the assumption about \( S \) we can arrange that \( a_n \in S \) for all \( n \). Since we assume (2) this shows that the triple \( (H_\psi, \Lambda_\psi, \pi_\psi) \) has property (F) of Lemma 3.3.5 and it follows therefore from Lemma 3.3.5 that it also has property (E). By Proposition 3.3.3 this means that it is isomorphic to the GNS-triple of a \( \beta \)-KMS weight for \( \sigma \). Since the weights of isomorphic GNS representations are the same, this means that the weight of \( (H_\psi, \Lambda_\psi, \pi_\psi) \) is a \( \beta \)-KMS weight and hence \( \psi \) is a \( \beta \)-KMS weight by Lemma 1.2.9.

Notes and remarks 3.3.7. The crucial Lemma 3.1.1 appears in the work of Quaegebeur and Verding, [QV], as does some of the arguments in the proof of Proposition 3.3.3. But otherwise most of the material in this section is taken from Kustermans’ paper [Ku1], where he shows that Combes’ original definition of a KMS weight, [C2], which is condition 4) in Theorem 3.2.1, is equivalent to condition 1). The proof here of 4) \( \Rightarrow \) 1) is the same as Kustermans’, but the proof that 1) implies 4) is more direct and it allows us to include the intermediate conditions 2) and 3). Section 3.3.4 here is an interpretation of Section 5.1 in [Ku1].
Chapter 4

Laca-Neshveyev’ theorem

Let $A$ be a $C^*$-algebra and $B \subseteq A$ a $C^*$-subalgebra of $A$. Recall that $B$ is hereditary in $A$ when

$$a \in A, \ b \in B, \ 0 \leq a \leq b \Rightarrow a \in B,$$

and full when $\text{Span} ABA$ is dense in $A$. Given weights $\psi$ on $A$ and $\phi$ on $B$ we say that $\psi$ extends $\phi$ when $\psi(b) = \phi(b)$ for all $b \in B^+$. In this chapter we prove the following theorem due to M. Laca and S. Neshveyev.

**Theorem 4.0.1.** (Theorem 3.2 iv in [LN].) Let $\sigma$ be a flow on the $C^*$-algebra $A$ and let $B \subseteq A$ be a $C^*$-subalgebra of $A$ such that $\sigma_t(B) = B$ for all $t \in \mathbb{R}$. Assume that $B$ is hereditary and full in $A$. Let $\beta \in \mathbb{R}$. For every $\beta$-KMS weight $\phi$ for $\sigma$ on $B$ there is a unique $\beta$-KMS weight for $\sigma$ on $A$ which extends $\phi$.

4.1 Proof of Laca-Neshveyev’ theorem

We fix the setting from Theorem 4.0.1. Since $\beta \in \mathbb{R}$ will be fixed throughout this section and to simplify the notation we set

$$\xi = -\frac{i \beta}{2}.$$ 

Write $A_\sigma M_\phi^*$ for the set of elements of $A$ of the form $ab$, where $a \in A$ is entire analytic for $\sigma$ and $b \in B$ is an element of the algebra $M_\phi^* \subseteq B$ obtained from the restriction of $\sigma$ to $B$. Let $I$ denote the collection of finite subsets $I \subseteq A_\sigma M_\phi^*$ with the property that $\sum_{a \in I} aa^* \leq 1$. When $I \in \mathcal{I}$ we set

$$w_I = \sum_{a \in I} aa^*.$$  

**Lemma 4.1.1.** The collection $\{w_I : I \in \mathcal{I}\}$ is an approximate unit in $A$ in the following sense: For every finite set $S \subseteq A$ and every $\epsilon > 0$ there an element $J \in \mathcal{I}$ such that $\|w_J a - a\| \leq \epsilon$ for all $a \in S$.  

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Proof. For each finite set \( I \subseteq AB \), set
\[
f_I = \sum_{a \in I} aa^*,
\]
and
\[
e_I = (\#I)f_I(1 + (\#I)f_I)^{-1}.
\]
As shown in the proof of Proposition 2.2.18 in [BR] the net \( \{e_I\} \) consists of positive contractions; i.e. \( 0 \leq e_I \leq 1 \), it increases with \( I \); i.e. \( I \subseteq J \Rightarrow e_I \leq e_J \), and it has the property that \( \lim_{I \to \infty} e_Ib = b \) for all \( b \in AB \). Since \( \text{Span} \ ABA \) is dense in \( A \) it follows that \( \lim_{I \to \infty} e_Ib = b \) for all \( b \in A \). Thus, given a finite set \( S \subseteq A \) and an \( \epsilon > 0 \) there is a finite set \( I \subseteq AB \) such that \( \|e_Ib - b\| < \frac{\epsilon}{2} \) for all \( b \in S \). Note that
\[
e_I = \sum_{x \in I} \sqrt{\#I(1 + (\#I)f_I)}^{-\frac{1}{2}} xx^* \sqrt{\#I(1 + (\#I)f_I)}^{-\frac{1}{2}},
\]
where
\[
\sqrt{\#I(1 + (\#I)f_I)}^{-\frac{1}{2}} x \in AB.
\]
Choose \( t \in ]0, 1[ \) such that \( \|te_Ib - b\| \leq \frac{\epsilon}{2} \) for all \( b \in S \). Since \( \mathcal{M}_\sigma^\phi \) is dense in \( B \) and \( A_\sigma \) is dense in \( A \) it follows that \( A_\sigma \mathcal{M}_\sigma^\phi \) is dense in \( AB \) and we can therefore find a finite set \( J \subseteq A_\sigma \mathcal{M}_\sigma^\phi \), such that
\[
\|te_Ib - f_Jb\| \leq \frac{\epsilon}{2}
\]
for all \( b \in S \) and
\[
\|te_I - f_J\| \leq 1 - t.
\]
Then
\[
f_J = f_J - te_I + te_I \leq 1 - t + t = 1,
\]
showing that \( J \in \mathcal{I} \). This completes the proof since \( \|w_jb - b\| = \|f_Jb - b\| \leq \|f_Jb - te_Ib\| + \|te_Ib - b\| \leq \epsilon. \)

Note that \( \sigma_\phi(x)^*a\sigma_\phi(x) \in \mathcal{M}_\sigma^\phi \) for all \( x \in A_\sigma \mathcal{M}_\sigma^\phi \), all \( x \in A^+ \) and all \( \phi \in \mathcal{C}. \)
For each \( I \in \mathcal{I} \) we can therefore define an element \( L_I \in A^*_\sigma \) such that
\[
L_I(a) = \sum_{x \in I} \phi(\sigma_{-\phi}(x)^*a\sigma_{-\phi}(x)) \quad \forall a \in A.
\]
For \( a \in A^+ \) set
\[
\psi(a) = \sup_{I \in \mathcal{I}} L_I(a) = \sup_{I \in \mathcal{I}} \sum_{x \in I} \phi(\sigma_{-\phi}(x)^*a\sigma_{-\phi}(x)). \quad (4.1.1)
\]

Lemma 4.1.2. \( \psi \) is a densely defined weight on \( A \).
4.1. PROOF OF LACA-NESHVEYEV’ THEOREM

Proof. Let \( a, b \in A^+ \). It is clear that \( \psi(a + b) \leq \psi(a) + \psi(b) \). To show that \( \psi(a) + \psi(b) \leq \psi(a+b) \) we may assume that \( \psi(a+b) < \infty \). Since \( \psi(a) \leq \psi(a+b) \) and \( \psi(b) \leq \psi(a+b) \) this implies that \( \psi(a) \) and \( \psi(b) \) are both finite.

Let \( \epsilon > 0 \) and choose \( I_1, I_3 \in \mathcal{I} \) such that \( \psi(a) \leq L_{I_1}(a) + \epsilon \) and \( \psi(b) \leq L_{I_3}(b) + \epsilon \). Let \( k \in \mathbb{N} \) and set \( c = R_k(\sqrt{a}) \) and \( d = R_k(\sqrt{b}) \). Thanks to Lemma 2.1.5 we can choose \( k \) so large that \( L_{I_1}(a) \leq L_{I_1}(c^2) + \epsilon \) and \( L_{I_3}(b) \leq L_{I_3}(d^2) + \epsilon \).

Note that \( c, d \in \mathcal{A}_\sigma \) by Lemma 2.1.5.

Let \( h \in \mathcal{A}_\sigma \) and consider an element \( I_2 \in \mathcal{I} \). Then

\[
\sum_{x \in I_1, y \in I_2} \phi(\sigma_{-\xi}(x)^* h y y^* h \sigma_{-\xi}(x))
= \sum_{x \in I_1, y \in I_2} \phi(\sigma_{\xi}(y^* h \sigma_{-\xi}(x)) \sigma_{\xi}(y^* h \sigma_{-\xi}(x))^*)
= \sum_{x \in I_1, y \in I_2} \phi(\sigma_{\xi}(y^*) \sigma_{\xi}(h) xx^* \sigma_{\xi}(h)^* \sigma_{\xi}(y)^*)
= \sum_{x \in I_1, y \in I_2} \phi(\sigma_{-\xi}(y) \sigma_{\xi}(h) xx^* \sigma_{\xi}(h)^* \sigma_{-\xi}(y)).
\]

Using Lemma 2.1.4 we choose \( I_2 \) such that

\[
L_{I_1}(c^2) \leq \sum_{x \in I_1, y \in I_2} \phi(\sigma_{-\xi}(x)^* c yy^* c \sigma_{-\xi}(x)) + \epsilon
\]

and

\[
L_{I_3}(d^2) \leq \sum_{x \in I_1, y \in I_2} \phi(\sigma_{-\xi}(x)^* d yy^* d \sigma_{-\xi}(x)) + \epsilon.
\]

It follows from calculation (4.1.2) that

\[
\sum_{x \in I_1, y \in I_2} \phi(\sigma_{-\xi}(y) \sigma_{\xi}(c) xx^* \sigma_{\xi}(c)^* \sigma_{-\xi}(y)) = \sum_{x \in I_1, y \in I_2} \phi(\sigma_{-\xi}(y) \sigma_{\xi}(c) \sigma_{\xi}(c) \sigma_{-\xi}(y))
\leq \sum_{y \in I_2} \phi(\sigma_{-\xi}(y) \sigma_{\xi}(c) \sigma_{\xi}(c) \sigma_{-\xi}(y)).
\]

Hence

\[
L_{I_1}(c^2) \leq \sum_{y \in I_2} \phi(\sigma_{-\xi}(y) \sigma_{\xi}(c) \sigma_{\xi}(c) \sigma_{-\xi}(y)) + \epsilon
\]

and similarly,

\[
L_{I_3}(d^2) \leq \sum_{y \in I_2} \phi(\sigma_{-\xi}(y) \sigma_{\xi}(d) \sigma_{\xi}(d) \sigma_{-\xi}(y)) + \epsilon.
\]
Using Lemma 4.1.1 again we choose $I_4 \in \mathcal{I}$ such that
\[
\sum_{y \in I_2} \phi(\sigma_-(y) \sigma(c) \sigma(c)^* \sigma_-(y)^*) \\
\leq \sum_{u \in I_4, y \in I_2} \phi(\sigma_-(y) \sigma(c) uu^* \sigma(c)^* \sigma_-(y)) + \epsilon
\]
and
\[
\sum_{y \in I_2} \phi(\sigma_-(y) \sigma(d) \sigma(d)^* \sigma_-(y)) \\
\leq \sum_{u \in I_4, y \in I_2} \phi(\sigma_-(y) \sigma(d) uu^* \sigma(d)^* \sigma_-(y)) + \epsilon.
\]
The calculation (4.1.2) shows that
\[
\sum_{u \in I_4, y \in I_2} \phi(\sigma_-(y) \sigma(c) uu^* \sigma(c)^* \sigma_-(y)) \\
= \sum_{u \in I_4, y \in I_2} \phi(\sigma_-(y) cyy^* \sigma_-(u)) \\
\leq \sum_{u \in I_4} \phi(\sigma_-(u) c^2 \sigma_-(u))
\]
and
\[
\sum_{u \in I_4, y \in I_2} \phi(\sigma_-(y) \sigma(d) uu^* \sigma(d)^* \sigma_-(y)) \\
= \sum_{u \in I_4, y \in I_2} \phi(\sigma_-(y) dyy^* \sigma_-(u)) \\
\leq \sum_{u \in I_4} \phi(\sigma_-(u) d^2 \sigma_-(u)).
\]

We conclude therefore that
\[
|\psi(a) + \psi(b)| \leq L_{I_4} (c^2 + d^2) + 4\epsilon.
\]

It follows from Kadison’s inequality, Proposition 3.2.4 in [BR], that $c^2 + d^2 = R_k(\sqrt{a})^2 + R_k(\sqrt{b})^2 \leq R_k(a + b)$, and hence
\[
|\psi(a) + \psi(b)| \leq L_{I_4} (R_k(a + b)) + 4\epsilon.
\]

By definition of $R_k$ there are finite sets of numbers, $\lambda_i \in [0, 1], t_i \in \mathbb{R}, i = 1, 2, \ldots, n$, such that $\sum_{i=1}^n \lambda_i = 1$, and the sum
\[
\sum_{i=1}^n \lambda_i \sigma_{t_i} (a + b)
\]
approximates $R_k(a + b)$, cf. Lemma \[\text{A.2.3}\] in Appendix \[A\]. We can therefore choose these sets of numbers such that

$$L_{I_4}(R_k(a + b)) \leq \sum_{i=1}^{n} \lambda_i L_{I_4} \circ \sigma_i(a + b) + \epsilon.$$  

Set

$$I_5 = \left\{ \sqrt{\lambda_i} \sigma_i(u) : u \in I_4, \ i = 1, 2, \cdots, n \right\},$$  

and note that $I_5 \subseteq \mathcal{I}$. Since $\phi$ is $\sigma$-invariant we find that

$$\sum_{i=1}^{n} \lambda_i L_{I_4} \circ \sigma_i(a + b) = \sum_{i=1}^{n} \lambda_i \sum_{y \in I_4} \phi(\sigma_{-\xi}(y) \sigma_i(a + b) \sigma_{-\xi}(y)))$$

$$= \sum_{i=1}^{n} \lambda_i \sum_{y \in I_4} \phi \circ \sigma_i(\sigma_{-\xi}(\sigma_{-\xi}(y))) (a + b) \sigma_{-\xi}(\sigma_{-\xi}(y)))$$

$$= \sum_{i=1}^{n} \lambda_i \sum_{y \in I_4} \phi(\sigma_{-\xi}(\sigma_{-\xi}(y))) (a + b) \sigma_{-\xi}(\sigma_{-\xi}(y))) = L_{I_5}(a + b).$$

We conclude therefore that

$$\psi(a) + \psi(b) \leq L_{I_5}(a + b) + 5\epsilon \leq \psi(a + b) + 5\epsilon.$$  

It follows that $\psi$ is additive on $A^+$. Since $\psi$ is clearly homogeneous and lower semi-continuous, we have shown that $\psi$ is a weight. To see that $\psi$ is densely defined, let $u, v \in \mathcal{A}_\sigma, \ b \in \mathcal{M}_\sigma^\sigma$. For each $I \in \mathcal{I}$ we have that

$$\sum_{x \in I} \phi(\sigma_{-\xi}(x)](uv)\sigma_{-\xi}(x))$$

$$\leq ||u||^2 \sum_{x \in I} \phi(\sigma_{-\xi}(x)](uv)\sigma_{-\xi}(x))$$

$$= ||u||^2 \sum_{x \in I} \phi(\sigma_{-\xi}(x)](uv)\sigma_{-\xi}(x))$$

$$= ||u||^2 \sum_{x \in I} \phi(\sigma_{-\xi}(x)](uv)\sigma_{-\xi}(x))$$

$$\leq ||u||^2 \phi(\sigma_{-\xi}(x)](uv)\sigma_{-\xi}(x))$$

proving that

$$\psi((uv)^* (uv)) \leq ||u||^2 \phi(\sigma_{-\xi}(x)](uv)\sigma_{-\xi}(x))$$

and hence that $\text{Span} \mathcal{A}_\sigma \mathcal{M}_\sigma^\sigma \mathcal{A}_\sigma \subseteq \mathcal{N}_\psi$. Since $\text{Span} \mathcal{A}_\sigma \mathcal{M}_\sigma^\sigma \mathcal{A}_\sigma$ is dense in $A$, every element of $A^+$ can be approximated by elements of the the form $\{x^* x : x \in \mathcal{N}_\psi \} \subseteq \mathcal{M}_\psi^+$. \[\Box\]

**Proposition 4.1.3.** $\psi$ is a $\beta$-KMS weight for $\sigma$.  


Proof. By using that \( \phi \) is \( \sigma \)-invariant we find that

\[ L_I(\sigma_t(a)) = L_I(a), \]

where \( I_t = \sigma_{-t}(I) \), for all \( a \in A^+ \), all \( t \in \mathbb{R} \) and all \( I \in \mathcal{I} \). It follows that \( \psi \) is \( \sigma \)-invariant. Let \( h \in \mathcal{A}_\sigma \) and let \( F_1 \in \mathcal{I} \). Let \( \epsilon > 0 \). It follows from Lemma 4.1.1 that there is an element \( F_2 \in \mathcal{I} \) such that

\[ L_{F_1}(h^*h) - \epsilon \leq \sum_{x \in F_1, y \in F_2} \phi(\sigma_{-\xi}(x)^*h^*yy^*h\sigma_{-\xi}(x)). \]

It follows from the calculation (4.1.2) that

\[ \sum_{x \in F_1, y \in F_2} \phi(\sigma_{-\xi}(x)^*h^*yy^*h\sigma_{-\xi}(x)) \leq L_{F_2}(\sigma_\xi(h)\sigma_\xi(h)^*) \leq \psi(\sigma_\xi(h)\sigma_\xi(h)^*). \]

Since \( \epsilon > 0 \) is arbitrary it follows that \( L_{F_1}(h^*h) \leq \psi(\sigma_\xi(h)\sigma_\xi(h)^*) \), and since \( F_1 \) is arbitrary it follows that \( \psi(h^*h) \leq \psi(\sigma_\xi(h)\sigma_\xi(h)^*) \). The reverse inequality follows by inserting \( \sigma_{-\xi}(h^*) = \sigma_\xi(h)^* \) for \( h \) in the last inequality and we conclude therefore that \( \psi(h^*h) = \psi(\sigma_\xi(h)\sigma_\xi(h)^*) \) for all \( h \in \mathcal{A}_\sigma \). Hence \( \psi \) has property (2) in Kustermans’ theorem, Theorem 3.2.4, implying that \( \psi \) is a \( \beta \)-KMS weight for \( \sigma \). \( \square \)

Lemma 4.1.4. \( \psi(b^*b) = \phi(b^*b) \) for all \( b \in \mathcal{A}_\sigma \cap B \).

Proof. Let \( b \in \mathcal{A}_\sigma \cap B \). For \( F \in \mathcal{I} \) we find that

\[ \sum_{x \in F} \phi(\sigma_{-\xi}(x)^*b^*b\sigma_{-\xi}(x)) = \sum_{x \in F} \phi(\sigma_\xi(b\sigma_{-\xi}(x))\sigma_\xi(b\sigma_{-\xi}(x))^*) \]

\[ = \sum_{x \in F} \phi(\sigma_\xi(b)x^*x^*\sigma_\xi(b)^*) \leq \phi(\sigma_\xi(b)\sigma_\xi(b)^*) = \phi(b^*b). \]

It follows that \( \psi(b^*b) \leq \phi(b^*b) \). In the other direction, let \( \epsilon > 0 \) and \( N > 0 \) be given. It follows from Lemma 4.1.1 and the lower semi-continuity of \( \phi \) that there is \( F \in \mathcal{I} \) such that

\[ \min\{\phi(b^*b) - \epsilon, N\} \leq \sum_{x \in F} \phi(b^*xx^*b) \]

\[ = \sum_{x \in F} \phi(\sigma_\xi(x^*b)\sigma_\xi(x^*b)^*) = L_F(\sigma_\xi(b)\sigma_\xi(b)^*) \]

\[ \leq \psi(\sigma_\xi(b)\sigma_\xi(b)^*) = \psi(b^*b). \]

We conclude that \( \phi(b^*b) \leq \psi(b^*b) \) and hence that \( \psi(b^*b) = \phi(b^*b) \). \( \square \)

Lemma 4.1.5. Let \( \sigma \) be a flow on the \( C^* \)-algebra \( D \). Let \( \rho_i : D^+ \rightarrow [0, \infty] \), \( i = 1, 2 \), be densely defined \( \sigma \)-invariant weights on \( D \). Assume that \( \rho_1(d^*d) = \rho_2(d^*d) \) for all \( d \in \mathcal{A}_\sigma \). Then \( \rho_1 = \rho_2 \).
4.1. PROOF OF LACA-NESHVEYEV’ THEOREM

Proof. Assume first that \( a \in D^+ \) and that \( \rho_i(a) < \infty, i = 1, 2 \). It follows from Lemma 3.1.3 that
\[
\rho_i(a) = \lim_{k \to \infty} \rho_i \left( R_k(\sqrt{a})^2 \right), \quad i = 1, 2.
\]

Since \( R_k(\sqrt{a}) \in A_\sigma \cap D^+ \) we conclude that \( \rho_1(a) = \rho_2(a) \). It remains now only to show that \( N_{\rho_1} = N_{\rho_2} \). Let \( x \in N_{\rho_1} \). It follows from Lemma 3.1.4 and Lemma 3.1.5 and \( x_n := R_n(x) \) gives a sequence \( \{x_n\} \) in \( A_\sigma \cap N_{\rho_1} \) such that \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} \Lambda_{\rho_1}(x_n) = \Lambda_{\rho_1}(x) \). In particular,
\[
\rho_1 ((x_n - x_m)^*(x_n - x_m))
\]
is arbitrarily small if \( n, m \) are both large enough. By assumption, \( \rho_2(x_n^*x_n) = \rho_1(x_n^*x_n) < \infty \), i.e. \( x_n \in N_{\rho_2} \), and
\[
\rho_2 ((x_n - x_m)^*(x_n - x_m)) = \rho_1 ((x_n - x_m)^*(x_n - x_m))
\]
is arbitrarily small if \( n, m \) are both large enough; i.e. \( \{\Lambda_{\rho_2}(x_n)\} \) is a Cauchy sequence in \( H_{\rho_2} \). Since \( \Lambda_{\rho_2} \) is closed by Lemma 3.1.1 it follows that \( x \in N_{\rho_2} \). Thus \( N_{\rho_1} \subseteq N_{\rho_2} \). By symmetry we must have equality; \( N_{\rho_1} = N_{\rho_2} \).

Lemma 4.1.6. \( \psi|_B = \phi \).

Proof. This follows from Lemma 4.1.4 and Lemma 4.1.5.

Lemma 4.1.7. Let \( \psi' \) be a \( \beta \)-KMS weight for \( \sigma \) such that \( \psi'|_B = \phi \). Then \( \psi' = \psi \).

Proof. Assume \( a \in A_\sigma \). For each \( F \in \mathcal{I} \),
\[
\sum_{x \in F} \phi(\sigma_{-\xi}(x)^*a^*a\sigma_{-\xi}(x)) = \sum_{x \in F} \psi'(\sigma_{-\xi}(x)^*a^*a\sigma_{-\xi}(x))
\]
\[
= \sum_{x \in F} \psi'(\sigma_{\xi}(a)xx^*\sigma_{\xi}(a)^*) \leq \psi'(\sigma_{\xi}(a)\sigma_{\xi}(a)^*) = \psi'(a^*a)
\]
proving that \( \psi(a^*a) \leq \psi'(a^*a) \). In the other direction the lower semi-continuity of \( \psi' \) combined with Lemma 4.1.1 shows that for arbitrary \( N, \epsilon > 0 \) there is \( F \in \mathcal{I} \) such that
\[
\min \{\psi'(a^*a) - \epsilon, N\} \leq \sum_{x \in F} \psi'(\sigma_{\xi}(a)xx^*\sigma_{\xi}(a)^*)
\]
\[
= \sum_{x \in F} \psi'(\sigma_{-\xi}(x)^*a^*a\sigma_{-\xi}(x)) = \sum_{x \in F} \phi(\sigma_{-\xi}(x)^*a^*a\sigma_{-\xi}(x)) \leq \psi(a^*a).
\]
We conclude therefore now that \( \psi(a^*a) = \psi'(a^*a) \). It follows from Lemma 4.1.6 that \( \psi = \psi' \).

Laca-Neshveyev’ theorem follows by combining Proposition 4.1.3, Lemma 4.1.6 and Lemma 4.1.7.
4.2 Restriction to corners

A weight \( \psi : A^+ \to [0, \infty] \) on \( A \) is bounded when

\[
\sup \{ \psi(a) : a \in A^+, 0 \leq a \leq 1 \} < \infty.
\]

When this holds the algebra \( \mathcal{M}_\psi \) is all of \( A \) and by Lemma 1.2.13 \( \psi \) is the restriction to \( A^+ \) of an element from \( A^*_+ \) which we also denote by \( \psi \). A bounded weight \( \psi \) is a state when \( \| \psi \| = 1 \). When \( A \) unital a weight \( \psi \) on \( A \) is bounded if and only if \( \psi(1) < \infty \), and a state if and only if \( \psi(1) = 1 \).

Let \( \beta \in \mathbb{R} \). A ray of \( \beta \)-KMS weights for \( \sigma \) is a set of \( \beta \)-KMS weights for \( \sigma \) of the form

\[
\{ t\psi : t \in \mathbb{R}, t > 0 \}
\]

for some \( \beta \)-KMS weight \( \psi \) for \( \sigma \). We say that this is the ray generated by \( \psi \).

In this section we use Laca-Neshveyev’s theorem to prove the following:

**Theorem 4.2.1.** Let \( \sigma \) be a flow on the \( C^* \)-algebra \( A \) and \( p = p^* = p^2 \) a projection in the fixed point algebra \( A^\sigma \) of \( \sigma \). Then \( \psi(p) < \infty \) for all KMS-weights \( \psi \) for \( \sigma \), and for each \( \beta \in \mathbb{R} \) the map

\[
\psi \mapsto \psi|_{pAp}
\]

is a bijection from the set of \( \beta \)-KMS weights \( \psi \) for \( \sigma \) with the property that \( \psi(p) = 1 \) onto the set of \( \beta \)-KMS states for the restriction of \( \sigma \) to \( pAp \).

Assume in addition that \( pAp \) is a full corner in \( A \). Then \( \psi(p) > 0 \) for all KMS weights \( \psi \) for \( \sigma \), and the map

\[
\psi \mapsto \psi(p)^{-1}\psi|_{pAp}
\]

is a bijection from the set of rays of \( \beta \)-KMS weights for \( \sigma \) onto the set of \( \beta \)-KMS states for the restriction of \( \sigma \) to \( pAp \).

Besides the theorem of Laca and Neshveyev the main ingredient in the proof of Theorem 4.2.1 is the following result by Christensen, [Ch].

**Proposition 4.2.2.** Let \( \sigma \) be a flow on \( A \) and \( \psi \) a \( \beta \)-KMS weight for \( \sigma \). If \( a \in A^+ \cap D(\sigma_{-\frac{c}{b^2}}) \) and \( b \in A^+ \) are such that \( ab = a \), then \( a \in \mathcal{N}_\psi \).

**Proof.** We assume, as we can, that \( 0 \leq a \leq 1 \). Then limit \( q = \lim_{n \to \infty} \pi_\psi(a)^{\frac{1}{n}} \) exists in the strong operator topology of \( B(H_\psi) \) and \( q \) is the range projection of \( \pi_\psi(a) \). Since \( \pi_\psi(a)^{\frac{1}{n}} \pi_\psi(b) = \pi_\psi(b) \pi_\psi(a)^{\frac{1}{n}} = \pi_\psi(a)^{\frac{1}{n}} \) for all \( n \), we find that \( q \pi_\psi(b) = \pi_\psi(b) q = q \). Since \( \psi \) is densely defined there is a \( c \in A^+ \) such that \( \psi(c^2) < \infty \) and \( \| b^2 - c^2 \| < \frac{1}{2} \). Note that \( \lim_{k \to \infty} R_k(b) = c \) and \( R_k(b) \in \mathcal{A}_\sigma \) by Lemma 2.1.3. Furthermore, \( \psi(R_k(c)^2) \leq \psi(R_k(c^2)) \leq \psi(c^2) \) by Proposition 3.2.4 in [BR] and Lemma 3.1.7 above. By substituting \( R_k(c) \) for \( c \) for some large \( k \) we can therefore arrange that \( c \in \mathcal{A}_\sigma \cap A^+ \). Note that

\[
\| q \pi_\psi(c^2) q - q \| = \| q (\pi_\psi(c^2) - \pi_\psi(b^2)) q \| \leq \| c^2 - b^2 \| < \frac{1}{2}.
\]
4.2. RESTRICTION TO CORNERS

It follows that the spectrum of \( q\pi_\psi(c^2)q \) in \( qB(H_\psi)q \) is contained in \( ]\frac{1}{2}, \frac{3}{2}[ \), implying that \( q \leq 2q\pi_\psi(c^2)q \). Thus

\[
\pi_\psi(a^2) = \pi_\psi(a)q_\psi(a) \leq 2\pi_\psi(a)q\pi_\psi(c^2)q\pi_\psi(a) = 2\pi_\psi(ac^2a). \quad (4.2.2)
\]

Let \( \omega \in F_\psi \), cf. Theorem \[1.1.1\] and let \( T_\omega \in \pi_\psi(A)' \) be the operator from Lemma \[1.2.3\]. It follows from (4.2.2) that for any \( x \in N_\psi \) we have

\[
\omega(x^*a^2x) = \langle T_\omega \Lambda_\psi(ax), \Lambda_\psi(ax) \rangle = \langle T_\omega \pi_\psi(a^2)\Lambda_\psi(x), \Lambda_\psi(x) \rangle 
\leq 2 \langle T_\omega \pi_\psi(ac^2a)\Lambda_\psi(x), \Lambda_\psi(x) \rangle = 2\omega(x^*ac^2ax).
\]

Since \( N_\psi \) is dense in \( A \) and \( \omega \) is continuous it follows that \( \omega(a^2) \leq 2\omega(ac^2a) \). By Combes’ theorem, Theorem \[1.1.1\] it follows that \( \psi(a^2) \leq 2\psi(ac^2a) \). By using Lemma \[2.1.13\] and that \( \psi \) is a \( \beta \)-KMS weight it follows that

\[
\psi(ac^2a) = \psi(\sigma_{-\frac{i}{2}}(c)a\sigma_{-\frac{i}{2}}(c)^*) = \psi(\sigma_{-\frac{i}{2}}(c)\sigma_{-\frac{i}{2}}(a)\sigma_{-\frac{i}{2}}(c)^*) 
\leq \|\sigma_{-\frac{i}{2}}(a)\|^2 \psi(\sigma_{-\frac{i}{2}}(c)\sigma_{-\frac{i}{2}}(c)^*) = \|\sigma_{-\frac{i}{2}}(a)\|^2 \psi(c^2).
\]

Hence \( \psi(a^2) \leq 2\|\sigma_{-\frac{i}{2}}(a)\|^2 \psi(c^2) < \infty \). \qed

**Proof of Theorem 4.2.1**. It follows from Proposition \[1.2.2\] that \( \psi(p) < \infty \) for every KMS weight \( \psi \) and then from Theorem \[4.0.1\] that the restriction map \( \psi \to \psi|_{pAp} \) is a bijection from the set of \( \beta \)-KMS weights \( \psi \) for \( \sigma \) with the property that \( \psi(p) = 1 \) onto the set of \( \beta \)-KMS states for the restriction of \( \sigma \) to \( pAp \). To handle the case where \( pAp \) is a full corner in \( A \) it suffices, again thanks to Theorem \[4.0.1\] only to show that \( \psi(p) > 0 \) for every \( \beta \)-KMS weight \( \psi \) for \( \sigma \). To this end, assume for a contradiction that \( \psi(p) = 0 \). Let \( a, b \in A_\sigma \). Then

\[
\psi(a^*pb^*pa) \leq \|b\|^2 \psi(a^*pa) = \|b\|^2 \psi(\sigma_{-\frac{i}{2}}(pa)\sigma_{-\frac{i}{2}}(pa)^*) 
= \|b\|^2 \psi(\sigma_{-\frac{i}{2}}(a)\sigma_{-\frac{i}{2}}(a)^*p) \leq \|b\|^2 \|\sigma_{-\frac{i}{2}}(a)\|^2 \psi(p) = 0.
\]

Thus \( b^*pa \in N_\psi \) and \( \Lambda_\psi(b^*pa) = 0 \). It follows that \( x \in N_\psi \) and \( \Lambda_\psi(x) = 0 \) for all \( x \in \text{Span} \, A_{\sigma}pA_{\sigma} \). Since \( p \) is full, \( \text{Span} \, A_{\sigma}pA_{\sigma} \) is dense in \( A \) and it follows therefore from the lower semi-continuity of \( \psi \) that \( \psi(x) = 0 \) for all \( x \in A^+ \); a contradiction. \qed

**Notes and remarks 4.2.3.** Laca-Neshveyev’s theorem, Theorem \[4.0.1\] appears in \[LN\] in a slightly different context. The formula \[4.1.1\] for the extension is taken from \[LN\], but otherwise the proof given here is different from that in \[LN\]. When \( \beta = 0 \) the theorem is about traces and it appears in work by Cuntz and Pedersen, \[CP\], albeit without a flow. Proposition \[4.2.2\] here is Proposition 3.1 in \[Ch\] while Theorem \[4.2.1\] is Theorem 2.4 in \[TH1\].
Appendix A

Riemann type integration in Banach spaces

Let $X$ be a Banach space, $M$ a locally compact Hausdorff space and $\mu$ a regular measure on $M$. Let $f : M \to X$ be continuous function such that

$$\int_M \|f(x)\| \, d\mu(x) < \infty. \quad \text{(A.0.1)}$$

In this appendix we describe a way to define the integral

$$\int_M f(x) \, d\mu(x),$$

and we deduce various results about it that are used in the main text.

A.1 The definition

We consider pairs $(\mathcal{U}, \epsilon)$ where $\mathcal{U}$ is a finite collection of mutually disjoint Borel sets $U$ of $M$ and $\epsilon > 0$ is a positive number such that

- $\bigcup_{U \in \mathcal{U}}$ is pre-compact; that is, its closure is compact,
- $\int_{M \setminus (\bigcup_{U \in \mathcal{U}} U)} \|f(x)\| \, d\mu(x) \leq \epsilon$, and
- $\sup_{x,y \in U} \|f(x) - f(y)\| (\mu(\bigcup_{U \in \mathcal{U}} U) + 1) \leq \epsilon$ for all $U \in \mathcal{U}$.

The collection of such pairs will be denoted by $\mathcal{I}$. We consider $\mathcal{I}$ as a pre-ordered set where $(\mathcal{U}, \epsilon) \leq (\mathcal{V}, \delta)$ means that

- $\delta \leq \epsilon$,
- $\bigcup_{U \in \mathcal{U}} \subseteq \bigcup_{V \in \mathcal{V}} V$, and
- for every $V \in \mathcal{V}$ there is an element $U' \in \mathcal{U}$ such that $V \cap (\bigcup_{U \in \mathcal{U}} U) \subseteq U'$.
Lemma A.1.1. I is a directed set.

Proof. Let \((U, \epsilon), (V, \delta) \in I\). Set \(\epsilon' = \min\{\epsilon, \delta\}\). We choose a compact set \(K \subseteq M\) such that 

\[
\bigcup_{U \in U} U \cup \bigcup_{V \in V} V \subseteq K
\]

and

\[
\int_{M \setminus K} \|f(x)\| \, d\mu(x) \leq \epsilon'.
\]

The sets

\[
K \setminus \left( \bigcup_{U \in U} U \cup \bigcup_{V \in V} V \right),
\]

\[
U \cap V, \quad U \in U, V \in V,
\]

\[
U \setminus \bigcup_{V \in V} V, \quad U \in U
\]

and

\[
V \setminus \bigcup_{U \in U} U, \quad V \in V,
\]

constitute a partition \(P\) of \(K\) into Borel sets. Since \(f\) is continuous and \(K\) is compact there is a finite partition

\[
K = \bigcup_{W \in W} W
\]

of \(K\) subordinate to \(P\) such that

\[
\sup_{x,y \in W} \|f(x) - f(y)\| (\mu(K) + 1) \leq \epsilon'
\]

for all \(W \in W\). Then \((U, \epsilon) \leq (W, \epsilon')\) and \((V, \delta) \leq (W, \epsilon')\). \(\square\)

For \((U, \epsilon) \in I\), we denote by

\[
S(U, \epsilon)
\]

the set of elements \(S\) in \(X\) of the form

\[
S = \sum_{U \in U} f(i(U))\mu(U)
\]

for some function \(i : U \to M\) with the property that \(i(U) \subseteq U\) for all \(U \in U\).

Lemma A.1.2. There is an element \(I \in X\) with the property that for every \(\delta > 0\) there is a \((U, \epsilon) \in I\) such that

\[
\|I - S\| \leq \delta
\]

for all \(S \in S(V, \epsilon')\) when \((U, \epsilon) \leq (V, \epsilon')\).
Proof. It follows from (A.0.1) that for each \( n \in \mathbb{N} \) there is a compact subset \( K_n \subseteq M \) such that

\[
\int_{M \setminus K_n} \|f(x)\| \, d\mu(x) \leq \frac{1}{n}.
\]

Since \( f \) is continuous and \( K_n \) compact there is a finite partition \( U_n \) of \( K_n \) into mutually disjoint Borel subsets \( U \) of \( K_n \) such that \( \|f(x) - f(y)\| (\mu(K_n) + 1) \leq \frac{1}{n} \) for all \( x, y \in U \) and all \( U \in U_n \). Then \( (U_n, \frac{1}{n}) \in I \). We proceed inductively, as we can, to arrange that \( (U_n, \frac{1}{n}) \leq (U_{n+1}, \frac{1}{n+1}) \) in \( I \) for all \( n \). For each \( n \) choose \( S_n \in S(U_n, \frac{1}{n}) \). Let \( (V, \delta) \geq (U_n, \epsilon_n) \) and consider an element

\[
S = \sum_{V \in V} f(j(V))\mu(V) \in S(V, \delta).
\]

Then

\[
S = \sum_{V \in V} f(j(V))\mu \left( V \setminus \bigcup_{U \in U_n} U \right) + \sum_{V \in V, U \in U_n} f(j(V))\mu(V \cap U).
\]

Note that

\[
\begin{align*}
\left\| \sum_{V \in V} f(j(V))\mu \left( V \setminus \bigcup_{U \in U_n} U \right) \right\| & \leq \sum_{V \in V} \|f(j(V))\| \mu \left( V \setminus \bigcup_{U \in U_n} U \right) \\
& = \sum_{V \in V} \int_{V \setminus \bigcup_{U \in U_n} U} \|f(j(V))\| \, d\mu(x) \\
& \leq \sum_{V \in V} \int_{V \setminus \bigcup_{U \in U_n} U} \left( \|f(x)\| + \frac{\delta}{\mu(\bigcup_{W \in V} W) + 1} \right) \, d\mu(x) \\
& \leq \delta + \int_{M \setminus \bigcup_{U \in U_n} U} \|f(x)\| \, d\mu(x) \leq \delta + \frac{1}{n} \leq \frac{2}{n}.
\end{align*}
\]

Let \( i_n : U_n \to X \) be the function defining \( S_n \); that is,

\[
S_n = \sum_{U \in U_n} f(i_n(U))\mu(U).
\]
Then
\[
\left\| S_n - \sum_{V \in V, U \in U_n} f(j(V)) \mu(V \cap U) \right\|
\]
\[
= \left\| \sum_{V \in V, U \in U_n} f(i_n(U)) \mu(V \cap U) - \sum_{V \in V, U \in U_n} f(j(V)) \mu(V \cap U) \right\|
\]
\[
\leq \sum_{V \in V, U \in U_n} \|f(i_n(U)) - f(j(V))\| \mu(V \cap U)
\]
\[
\leq \sum_{V \in V, U \in U_n} \left( \frac{1}{n \mu(K_n) + 1} + \frac{\delta}{\mu(\bigcup_{V \in V} V) + 1} \right) \mu(V \cap U)
\]
\[
\leq \frac{1}{n} + \delta \leq \frac{2}{n}.
\]

It follows that \(\|S - S_n\| \leq \frac{1}{n}\). In particular, the sequence \(\{S_n\}\) is Cauchy in \(X\) and we set \(I = \lim_{n \to \infty} S_n\). Then \(I\) has stated property: Let \(\delta > 0\) be given. There is an \(N \in \mathbb{N}\) such that \(\frac{1}{n} \leq \frac{\delta}{2}\) for all \(n \geq N\) and \(\|S_n - I\| \leq \frac{\delta}{2}\) for all \(n \geq N\). If \((V, \epsilon') \geq (\mathcal{U}_N, \frac{1}{N})\) the calculations above imply that \(\|S - I\| \leq \frac{\|S - S_N\|}{n} + \frac{\delta}{n} \leq \frac{1}{n} + \frac{\delta}{2} \leq \frac{\delta}{2}\) for all \(S \in S(V, \epsilon')\).

The element \(I\) of Lemma [A.1.2] will be denoted by
\[
\int_M f(x) \mu(x).
\]

By construction there is a sequence \(\{S_n\}\) in \(X\) such that
\[
\lim_{n \to \infty} S_n = \int_M f(x) \mu(x)
\]
and for each \(n\) there is a finite collection \(\{t_i : i = 1, 2, \cdots, N_n\}\) of non-negative real numbers and mutually disjoint Borel sets \(\{U_i : i = 1, 2, \cdots, N_n\}\) in \(M\) such that
\[
I_n = \sum_{i=1}^{N_n} f(x_i) \mu(U_i)
\]
and
\[
\|I_n\| \leq \int_M \|f(x)\| \, d\mu(x) + \frac{1}{n}.
\]

In particular, we find that
\[
\left\| \int_M f(x) \, d\mu(x) \right\| \leq \int_M \|f(x)\| \, d\mu(x).
\]
A.2. MISCELLANEOUS

Lemma A.2.1. Let $X$ and $Y$ be Banach spaces, $D \subseteq X$ a subspace and $L : D \to Y$ a linear map. Let $M$ be a locally compact Hausdorff space and $\mu$ a regular Borel measure on $M$. Let $f : M \to X$ be a continuous function. Assume that

- $L : D \to Y$ is closed,
- $f(x) \in D$ for all $x \in M$,
- $\int_M \|f(x)\| \, dx < \infty$, and
- $\int_M \|L(f(x))\| \, dx < \infty$.

It follows that $\int_M f(x) \, \mu(x) \in D$ and $L(\int_M f(x) \, d\mu(x)) = \int_M L(f(x)) \, d\mu(x)$.

Proof. Consider the function $M \to X \oplus Y$ defined by

$$M \ni x \mapsto (f(x), L(f(x))).$$

By applying Lemma A.1.2 to this function we obtain sequences $I_n$ in $D$ and $J_n$ in $Y$ such that $L(I_n) = J_n$ and

$$\lim_{n \to \infty} (I_n, J_n) = \left( \int_M f(x) \, \mu(x), \int_M L(f(x)) \, d\mu(x) \right)$$

in $X \oplus Y$. This gives the conclusion because $L$ is closed.

Lemma A.2.2. Let $X$ be a Banach space, $M$ a locally compact Hausdorff space and $\mu$ a regular measure on $M$. Let $\{f_n\}$ be a sequence of continuous functions $f_n : M \to X$ and $f : M \to X$ a continuous function such that $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in M$. Assume that there is function $h \in L^1(M, \mu)$ such that

$$\|f_n(x)\| \leq h(x) \quad \forall x \in M.$$

Then $\lim_{n \to \infty} \int_M f_n(x) \, d\mu(x) = \int_M f(x) \, d\mu(x)$ in $X$.

Proof. Note that all the integrals $\int_M f_n(x) \, d\mu(x)$ and $\int_M f(x) \, d\mu(x)$ are defined since $\int_M h(x) \, d\mu(x) < \infty$. It follows from (A.1.1) that

$$\left\| \int_M f_n(x) \, d\mu(x) - \int_M f(x) \, d\mu(x) \right\| \leq \int_M \|f_n(x) - f(x)\| \, d\mu(x).$$

An application of Lebesgue’s theorem on dominated convergence shows that $\lim_{n \to \infty} \int_M \|f_n(x) - f(x)\| \, d\mu(x) = 0$.

The setting of the next lemma is that from Section 2.1.1. In particular, it deals with the smoothing operators defined in (2.1.0).
**Lemma A.2.3.** Let \( k \in \mathbb{N}, a \in A \). For every \( \epsilon > 0 \) there are numbers \( \lambda_i \in [0, 1] \) and \( t_i \in \mathbb{R}, i = 1, 2, \cdots, n, \) such that \( \sum_{i=1}^{n} \lambda_i = 1 \) and

\[
\left\| R_k(a) - \sum_{i=1}^{n} \lambda_i \sigma_{t_i}(a) \right\| \leq \epsilon.
\]

**Proof.** Define the continuous function \( f : \mathbb{R} \to A \oplus \mathbb{C} \) such that

\[
f(t) = \left( \sqrt{\frac{k}{\pi} e^{-kt^2}} \sigma_t(a), \sqrt{\frac{k}{\pi} e^{-kt^2}} \right).
\]

Let \( \delta \in ]0, 1[ \) be so small that

\[
\frac{\delta}{1 - \delta} (\|a\| + 1) \leq \epsilon.
\]

By applying Lemma A.1.2 to \( f \) we get numbers \( s_i \geq 0 \) and \( t_i \in \mathbb{R}, i = 1, 2, \cdots, n, \) such that

\[
\left\| \sum_{i=1}^{n} \sqrt{\frac{k}{\pi} e^{-kt_i^2}} \sigma_{s_i}(a)s_i - R_k(a) \right\| \leq \delta
\]

and

\[
\left| \sum_{i=1}^{n} \sqrt{\frac{k}{\pi} e^{-kt_i^2}} s_i - 1 \right| \leq \delta.
\]

Set \( x = \sum_{i=1}^{n} \sqrt{\frac{k}{\pi} e^{-kt_i^2}} s_i \) and

\[
\lambda_i = x^{-1} \sqrt{\frac{k}{\pi} e^{-kt_i^2}} s_i.
\]

Then \( \sum_{i=1}^{n} \lambda_i = 1 \) and

\[
\left\| R_k(a) - \sum_{i=1}^{n} \lambda_i \sigma_{t_i}(a) \right\|
\leq \left\| x^{-1} R_k(a) - R_k(a) \right\|
+ \left\| x^{-1} R_k(a) - x^{-1} \sum_{i=1}^{n} \sqrt{\frac{k}{\pi} e^{-kt_i^2}} \sigma_{s_i}(a)s_i \right\|
\leq \frac{\delta}{1 - \delta} \|a\| + \frac{\delta}{1 - \delta} \leq \epsilon.
\]

\( \Box \)
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