Projection and Convolution Operations for Integrally Convex Functions

Satoko Moriguchi† and Kazuo Murota‡

October, 2017; Version August 9, 2018

Abstract

This paper considers projection and convolution operations for integrally convex functions, which constitute a fundamental function class in discrete convex analysis. It is shown that the class of integrally convex functions is stable under projection, and this is also the case with the subclasses of integrally convex functions satisfying local or global discrete midpoint convexity. As is known in the literature, the convolution of two integrally convex functions may possibly fail to be integrally convex. We show that the convolution of an integrally convex function with a separable convex function remains integrally convex. We also point out in terms of examples that the similar statement is false for integrally convex functions with local or global discrete midpoint convexity.

Keywords: Discrete convex analysis, Integrally convex function, Minkowski sum, Infimal convolution, Integer programming

1 Introduction

In discrete convex analysis [13, 14, 15], a variety of discrete convex functions are considered. Among others, integrally convex functions, due to Favati–Tardella [1], constitute a common framework for discrete convex functions. A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is called integrally convex if its local convex extension $\tilde{f} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is (globally) convex in the ordinary sense, where $\tilde{f}$ is defined as the collection of convex extensions of $f$ in each unit hypercube $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid a_i \leq x_i \leq a_i + 1 \ (i = 1, \ldots, n)\}$ with $a \in \mathbb{Z}^n$; see Section 2.2. A proximity theorem for integrally convex functions has recently been established in [9, 10], together with a proximity-scaling algorithm for minimizing integrally convex functions.

A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ in $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ is called separable convex if it can be represented as $f(x) = \varphi_1(x_1) + \cdots + \varphi_n(x_n)$ with univariate discrete convex functions $\varphi_i : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ satisfying $\varphi_i(t - 1) + \varphi_i(t + 1) \geq 2\varphi_i(t)$ for all $t \in \mathbb{Z}$. Separable convex functions are an obvious example of integrally convex functions. Moreover, L-convex, $L^\natural$-convex, M-convex, $M^\natural$-convex, $L^2$-convex, and $M^2$-convex functions [13] and BS-convex
and UJ-convex functions \[2\] are integrally convex functions. An integrally convex function is \(L^2\)-convex if and only if it is submodular \[3\].

The concept of integral convexity found applications, e.g., in economics and game theory. It is used in formulating discrete fixed point theorems \[5, 6, 18\] and designing solution algorithms for discrete systems of nonlinear equations \[8, 17\]. In game theory integral concavity guarantees the existence of a pure strategy equilibrium in finite symmetric games \[7\].

Various operations can be defined for discrete functions \(f : \mathbb{Z}^n \to \mathbb{R} \cup \{\pm \infty\}\) through natural adaptations of standard operations in convex analysis, such as

- origin shift \(f(x) \mapsto f(x + b)\) with an integer vector \(b\),
- sign inversion of the variable \(f(x) \mapsto f(-x)\),
- nonnegative multiplication of function values \(f(x) \mapsto af(x)\) with \(a \geq 0\),
- subtraction of a linear function \(f \mapsto f[-p]\), where \(f[-p]\) denotes the function defined by \(f[-p](x) = f(x) - \sum_{i=1}^{n} p_i x_i\) for \(p \in \mathbb{R}^n\).

It is known \[13, 16\] that these basic operations preserve integral convexity as well as \(L^\alpha\), \(L^2\), \(M^\alpha\), \(M^2\), \(M^\alpha\), and \(M^2\)-convexity. For a positive integer \(\alpha\), the \(\alpha\)-scaling of \(f\) means the function \(f^\alpha : \mathbb{Z}^n \to \mathbb{R} \cup \{\pm \infty\}\) defined by \(f^\alpha(x) = f(\alpha x)\) for \(x \in \mathbb{Z}^n\). \(L^\alpha\)- and \(L^2\)-convexity are preserved under scaling, whereas \(M^\alpha\)- and \(M^2\)-convexity are not stable under scaling \[13\]. The scaling operation for integrally convex functions is considered recently in \[9, 10, 11\]. Integral convexity admits the scaling operation only when \(n \leq 2\); when \(n \geq 3\), the scaled function \(f^\alpha\) is not necessarily integrally convex. Within subclasses of integral convex functions with local or global discrete midpoint convexity, the scaling operation can be defined for all \(n\).

In this paper we are concerned with projection and convolution operations. For a set \(S \subseteq \mathbb{Z}^{n+m}\), the projection of \(S\) (to \(\mathbb{Z}^n\)) is the set \(T \subseteq \mathbb{Z}^n\) defined by

\[
T = \{ x \in \mathbb{Z}^n \mid \exists y \in \mathbb{Z}^m : (x, y) \in S \}. \tag{1.1}
\]

For a function \(f : \mathbb{Z}^{n+m} \to \mathbb{R} \cup \{\pm \infty\}\), the projection of \(f\) to \(\mathbb{Z}^n\) is the function \(g : \mathbb{Z}^n \to \mathbb{R} \cup \{\pm \infty\}\) defined by

\[
g(x) = \inf\{f(x, y) \mid y \in \mathbb{Z}^m\} \quad (x \in \mathbb{Z}^n), \tag{1.2}
\]

where it is assumed that \(g(x) > -\infty\) for all \(x\). For sets \(S_1, S_2 \subseteq \mathbb{Z}^n\), their Minkowski sum \(S_1 + S_2\) is defined by

\[
S_1 + S_2 = \{ y + z \mid y \in S_1, z \in S_2 \}. \tag{1.3}
\]

For functions \(f_1, f_2 : \mathbb{Z}^n \to \mathbb{R} \cup \{\pm \infty\}\), their (integer infimal) convolution is the function \(f_1 \circ f_2 : \mathbb{Z}^n \to \mathbb{R} \cup \{\pm \infty\}\) defined by

\[
(f_1 \circ f_2)(x) = \inf\{f_1(y) + f_2(z) \mid x = y + z, y, z \in \mathbb{Z}^n\} \quad (x \in \mathbb{Z}^n), \tag{1.4}
\]

where it is assumed that, for every \(x \in \mathbb{Z}^n\), the infimum on the right-hand side is not equal to \(-\infty\).

The following facts are known about projections.

\[\text{In (ordinary) convex analysis, the projection operation \(\square\) for functions is also referred to as “partial minimization” and the resulting function } g \text{ as “marginal function” [4 Def. 2.4.4]. Note that the epigraph of } g \text{ is the projection of the epigraph of } f \text{ [4 Fig.2.4.1].}\]
• The projection of a separable convex function is separable convex.

• The projection of an $L^\natural$-convex function is $L^\natural$-convex \[13, \text{Theorem 7.11}\]. Similarly, the projection of an $L$-convex function is $L$-convex \[13, \text{Theorem 7.10}\].

• The projection of an $M^\natural$-convex function is $M^\natural$-convex \[13, \text{Theorem 6.15}\]. However, the projection of an $M$-convex function is not necessarily $M$-convex \[3\].

The following facts are found in this paper.

• The projection of an integrally convex function is integrally convex (Theorem \[3, \text{Theorem 3.3}\]).

• The projection of an integrally convex function with global discrete midpoint convexity is an integrally convex function with global discrete midpoint convexity (Theorem \[3, \text{Theorem 3.5}\]).

• The projection of an integrally convex function with local discrete midpoint convexity is an integrally convex function with local discrete midpoint convexity (Theorem \[3, \text{Theorem 3.6}\]).

As for convolutions the following facts are known.

• The convolution of separable convex functions is separable convex \[3\].

• The convolution of $M^\natural$-convex functions is $M^\natural$-convex \[13, \text{Theorem 6.15}\]. Similarly, the convolution of $M$-convex functions is $M$-convex \[13, \text{Theorem 6.13}\].

• The convolution of $L^\natural$-convex functions is not necessarily $L^\natural$-convex, but is integrally convex \[13, \text{Theorem 8.42}\]. Similarly for $L$-convex functions.

• The convolution of an $L^\natural$-convex function and a separable convex function is $L^\natural$-convex \[13, \text{Theorem 7.11}\]. Similarly, the convolution of $L$-convex function and a separable convex function is $L$-convex \[13, \text{Theorem 7.10}\].

• The convolution of integrally convex functions is not necessarily integrally convex \[16, \text{Example 4.12}\], \[13, \text{Example 3.15}\].

The following facts are found in this paper.

• The convolution of an integrally convex function and a separable convex function is integrally convex (Theorem \[4, \text{Theorem 4.2}\]).

• The convolution of an integrally convex function with global discrete midpoint convexity and a separable convex function is not necessarily an integrally convex function with global discrete midpoint convexity (Examples \[4, \text{Examples 4.3 and 4.4}\]).

• The convolution of an integrally convex function with local discrete midpoint convexity and a separable convex function is not necessarily an integrally convex function with local discrete midpoint convexity (Examples \[4, \text{Examples 4.3 and 4.4}\]).

This paper is organized as follows. Section \[2\] is a review of relevant results on integrally convex functions. Section \[3\] deals with projections and Section \[4\] with convolutions. Concluding remarks are given in Section \[5\].
2 Preliminaries

2.1 Basic definition and notation

For integer vectors \( a \in (\mathbb{Z} \cup \{-\infty\})^n \) and \( b \in (\mathbb{Z} \cup \{+\infty\})^n \) with \( a \leq b \), \([a, b]_\mathbb{Z}\) denotes the integer interval (box, discrete rectangle) between \( a \) and \( b \), i.e., \([a, b]_\mathbb{Z} = \{x \in \mathbb{Z}^n \mid a \leq x \leq b\}\). For a function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \), the effective domain of \( f \) means the set \( \{x \in \mathbb{Z}^n \mid f(x) < +\infty\} \) and is denoted by \( \text{dom} f \). The indicator function of a set \( S \subseteq \mathbb{Z}^n \) is a function \( \delta_S : \mathbb{Z}^n \to \{0, +\infty\} \) defined by \( \delta_S(x) = \begin{cases} 0 & (x \in S), \\ +\infty & (x \not\in S). \end{cases} \) For \( x \in \mathbb{R}^n \), \([x]\) and \( \lfloor x \rfloor \) denote the integer vectors obtained by componentwise rounding-up and rounding-down to the nearest integers, respectively.

2.2 Integrally convex functions

For \( x \in \mathbb{R}^n \) the integral neighborhood of \( x \) is defined as

\[
N(x) = \{z \in \mathbb{Z}^n \mid |x_i - z_i| < 1 \ (i = 1, \ldots, n)\}.
\]

In other words, \( N(x) = [[|x|, |x|]]_\mathbb{Z} \). For a function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) the local convex extension \( \tilde{f} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) of \( f \) is defined as the union of all convex envelopes of \( f \) on \( N(x) \). That is,

\[
\tilde{f}(x) = \min\{\sum_{y \in N(x)} \lambda_y f(y) \mid \sum_{y \in N(x)} \lambda_y y = x, (\lambda_y) \in \Lambda(x)\} \quad (x \in \mathbb{R}^n),
\]

where \( \Lambda(x) \) denotes the set of coefficients for convex combinations indexed by \( N(x) \):

\[
\Lambda(x) = \{(\lambda_y \mid y \in N(x)) \mid \sum_{y \in N(x)} \lambda_y = 1, \lambda_y \geq 0 \ (\forall y \in N(x))\}.
\]

If \( \tilde{f} \) is convex on \( \mathbb{R}^n \), then \( f \) is said to be integrally convex [1][13].

A set \( S \subseteq \mathbb{Z}^n \) is said to be integrally convex if the convex hull \( \overline{S} \) of \( S \) coincides with the union of the convex hulls of \( S \cap N(x) \) over \( x \in \mathbb{R}^n \), i.e., if, for any \( x \in \mathbb{R}^n \), \( x \in \overline{S} \) implies \( x \in S \cap N(x) \). The effective domain of an integrally convex function is an integrally convex set.

Integral convexity can be characterized by a local condition under the assumption that the effective domain is an integrally convex set.

**Theorem 2.1** ([1][10]). Let \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) be a function with an integrally convex effective domain. Then the following properties are equivalent:

(a) \( f \) is integrally convex.

(b) For every \( x, y \in \mathbb{Z}^n \) with \( \|x - y\|_\infty = 2 \) we have

\[
\tilde{f}\left(\frac{x + y}{2}\right) \leq \frac{1}{2}(f(x) + f(y)).
\]

Integral convexity of a function can also be characterized by integral convexity of the minimizer sets.

**Theorem 2.2** ([13], Theorem 3.29). Suppose a function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) has a nonempty bounded effective domain. Then \( f \) is an integrally convex function if and only if \( \arg\min f[-p] \) is an integrally convex set for every \( p \in \mathbb{R}^n \).
Remark 2.1. The concept of integrally convex functions is introduced in \[1\] for functions defined on integer intervals (discrete rectangles). The extension to functions with general integrally convex effective domains is straightforward, which is found in \[13\]. Theorem 2.1 is proved in \[1, Proposition 3.3\] when the effective domain is an integer interval and in the general case in \[10\].

2.3 Discrete midpoint convexity

A function \(f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}\) is called L\(^5\)-convex \([3, 13]\) if it satisfies discrete midpoint convexity

\[
f(x) + f(y) \geq f\left(\left\lfloor \frac{x+y}{2} \right\rfloor\right) + f\left(\left\lceil \frac{x+y}{2} \right\rceil\right)
\]

for all \(x, y \in \mathbb{Z}^n\). A function is L\(^5\)-convex if and only if it is submodular and integrally convex. L\(^5\)-convex functions form a well-behaved subclass of integrally convex functions.

2.4

A set \(S \subseteq \mathbb{Z}^n\) is called L\(^5\)-convex if

\[
x, y \in S \implies \left\lfloor \frac{x+y}{2} \right\rfloor, \left\lceil \frac{x+y}{2} \right\rceil \in S.
\]

A set \(S\) is L\(^5\)-convex if and only if its indicator function \(\delta_S\) is an L\(^5\)-convex function. The effective domain of an L\(^5\)-convex function is an L\(^5\)-convex set.

A function \(f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}\) is called globally discrete midpoint convex if the discrete midpoint convexity (2.3) is satisfied by every pair \((x, y)\in \mathbb{Z}^n\times \mathbb{Z}^n\) with \(\|x - y\|_\infty \geq 2\). A function \(f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}\) is called locally discrete midpoint convex if \(\text{dom } f\) is a discrete midpoint convex set and the discrete midpoint convexity (2.3) is satisfied by every pair \((x, y)\in \mathbb{Z}^n\times \mathbb{Z}^n\) with \(\|x - y\|_\infty = 2\) (exactly equal to two\(^4\)). The effective domain of a (locally or globally) discrete midpoint convex function is a discrete midpoint convex set. A set \(S\) is discrete midpoint convex if and only if its indicator function \(\delta_S\) is a discrete midpoint convex function.

The inclusion relations among the function classes are summarized as follows:

\[
\{\text{separable convex functions}\} \supseteq \{\text{L}^5\text{-convex functions}\} = \{\text{submodular integrally convex functions}\} \\
\supseteq \{\text{globally discrete midpoint convex functions}\} \\
\supseteq \{\text{locally discrete midpoint convex functions}\} \\
\supseteq \{\text{integrally convex functions}\}.
\]

All the inclusions above are proper; see \[11\].

An inequality, called “parallelogram inequality,” is known for discrete midpoint convex functions. For any pair of distinct vectors \(x, y \in \mathbb{Z}^n\), we can decompose \(y - x\) into \(-1, 0, +1\)-vectors as

\[
y - x = \sum_{k=1}^{m} (\mathbf{1}_{A_k} - \mathbf{1}_{B_k}),
\]

\(\text{Local discrete midpoint convex functions are called “directed integrally convex functions” in [9].}\)}
where $m = \|y-x\|_\infty$, 

$$ A_k = \{ i \mid y_i - x_i \geq m + 1 - k \}, \quad B_k = \{ i \mid y_i - x_i \leq -k \} \quad (k = 1, \ldots, m). \quad (2.8) $$

Note that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m$, $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_m$, $A_m \cap B_1 = \emptyset$, and $A_1 \cup B_m \neq \emptyset$. The following theorem is a reformulation of the parallelogram inequality given in [9,11].

**Theorem 2.3 ([9,11]).** Let $f : \mathbb{Z}^n \to \mathbb{R} \cup \{ \pm\infty \}$ be a (globally or locally) discrete midpoint convex function, and $x, y \in \text{dom } f$. Let $d = \sum_{k \in J} (1_{A_k} - 1_{B_k})$ for any $J \subseteq \{1, 2, \ldots, m\}$ in the decomposition (2.7). Then

$$ f(x) + f(y) \geq f(x+d) + f(y-d). \quad (2.9) $$

## 3 Projection Operation

Recall that the projection $T$ of a set $S \subseteq \mathbb{Z}^{n+m}$ is defined by

$$ T = \{ x \in \mathbb{Z}^n \mid \exists y \in \mathbb{Z}^m : (x, y) \in S \}, \quad (3.1) $$

and the projection $g$ of a function $f : \mathbb{Z}^{n+m} \to \mathbb{R} \cup \{ \pm\infty \}$ is defined by

$$ g(x) = \inf \{ f(x,y) \mid y \in \mathbb{Z}^m \} \quad (x \in \mathbb{Z}^n), \quad (3.2) $$

where it is assumed that $g(x) > -\infty$ for all $x$. 

### 3.1 Projection of integrally convex functions

**Theorem 3.1.** The projection of an integrally convex set is an integrally convex set.

**Proof.** Let $T \subseteq \mathbb{Z}^n$ be the projection of an integrally convex set $S \subseteq \mathbb{Z}^{n+m}$. We will show that $x \in \overline{T}$ implies $x \in \overline{T \cap N(x)}$. Let $x \in \overline{T}$. There exists $y \in \mathbb{R}^m$ such that $(x,y) \in \overline{S}$ (see Lemma 3.2 below). Then, by integral convexity of $S$, we have $(x,y) \in \overline{S \cap N((x,y))}$, i.e., there exist $(u^{(k)}, v^{(k)}) \in S \cap N((x,y))$ and $\lambda_k (k = 1, 2, \ldots, l)$ such that

$$ (x,y) = \sum_{k=1}^l \lambda_k (u^{(k)}, v^{(k)}), \quad \sum_{k=1}^l \lambda_k = 1, \quad \lambda_k \geq 0. $$

We have $x = \sum_{k=1}^l \lambda_k u^{(k)}$ from the first equation, and $u^{(k)} \in T \cap N(x)$ from $(u^{(k)}, v^{(k)}) \in S \cap N((x,y))$. Hence $x \in \overline{T \cap N(x)}$. \hfill \Box

The following fact used in the above proof is stated and proved for completeness, though it is just a basic fact about convexity.

**Lemma 3.2.** $\overline{T} = \{ x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m : (x,y) \in \overline{S} \}$. That is, the convex hull of the projection of $S$ coincides with the projection of the convex hull of $S$. 

6
Proof. We denote \( \{ x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m : (x, y) \in \overline{S} \} \) by proj(\( S \)). To show \( \overline{T} \subseteq \text{proj}(\overline{S}) \), assume \( x \in \overline{T} \). Then there exist \( u^{(k)} \in T \) \( (k = 1, 2, \ldots, l) \) such that
\[
x = \sum_{k=1}^{l} \lambda_k u^{(k)}, \quad \sum_{k=1}^{l} \lambda_k = 1, \quad \lambda_k \geq 0.
\]
Since \( T \) is the projection of \( S \) and \( u^{(k)} \in T \), there exist \( v^{(k)} \) such that \( (u^{(k)}, v^{(k)}) \in S \). Defining \( y = \sum_{k=1}^{l} \lambda_k v^{(k)} \) with the coefficients \( \lambda_1, \lambda_2, \ldots, \lambda_l \) above, we have \( (x, y) = \sum_{k=1}^{l} \lambda_k (u^{(k)}, v^{(k)}) \in \overline{S} \). This shows that \( x \in \text{proj}(\overline{S}) \). Hence \( \overline{T} \subseteq \text{proj}(\overline{S}) \).

To show the converse \( \overline{T} \supseteq \text{proj}(\overline{S}) \), assume \( x \in \text{proj}(\overline{S}) \). Then there exists \( y \) such that \( (x, y) \in \overline{S} \), which in turn implies \( (x, y) = \sum_{k=1}^{l} \lambda_k (u^{(k)}, v^{(k)}) \) for some \( (u^{(k)}, v^{(k)}) \in S \) and \( \lambda_k \) \( (k = 1, 2, \ldots, l) \). Therefore, \( x = \sum_{k=1}^{l} \lambda_k u^{(k)} \) and \( u^{(k)} \in T \) \( (k = 1, 2, \ldots, l) \), which shows \( x \in \overline{T} \).

\[ \square \]

Theorem 3.3. The projection of an integrally convex function is an integrally convex function.

Proof. Let \( g \) be the projection of an integrally convex function \( f \). The effective domain \( \text{dom} \, g \) of \( g \) coincides with the projection of \( \text{dom} \, f \), whereas \( \text{dom} \, f \) is an integrally convex set by the integral convexity of \( f \). By Theorem 3.1 \( \text{dom} \, g \) is an integrally convex set. Then by Theorem 2.1 it suffices to show
\[
\frac{1}{2} \left[ g(x^{(1)}) + g(x^{(2)}) \right] \geq \tilde{g} \left( \frac{x^{(1)} + x^{(2)}}{2} \right)
\]  
for any \( x^{(1)}, x^{(2)} \in \text{dom} \, g \) with \( ||x^{(1)} - x^{(2)}||_\infty = 2 \), where \( \tilde{g} \) is the local convex extension of \( g \). Take any \( \varepsilon > 0 \). By the definition (3.2) of projection, there exist \( y^{(1)}, y^{(2)} \in \mathbb{Z}^n \) such that \( g(x^{(i)}) \geq f(x^{(i)}, y^{(i)}) - \varepsilon \) for \( i = 1, 2 \), which implies
\[
\frac{1}{2} \left[ g(x^{(1)}) + g(x^{(2)}) \right] \geq \frac{1}{2} \left[ f(x^{(1)}, y^{(1)}) + f(x^{(2)}, y^{(2)}) \right] - \varepsilon. \tag{3.4}
\]
Consider the local convex extension \( \tilde{f}(z) \) at \( z = [(x^{(1)}, y^{(1)}) + (x^{(2)}, y^{(2)})]/2 \in \mathbb{R}^{n+m} \). There exist \( (u^{(k)}, v^{(k)}) \in N(z) \) and \( \lambda_k \) \( (k = 1, 2, \ldots, l) \) such that
\[
z = \sum_{k=1}^{l} \lambda_k (u^{(k)}, v^{(k)}), \quad \tilde{f}(z) = \sum_{k=1}^{l} \lambda_k f(u^{(k)}, v^{(k)}), \quad \sum_{k=1}^{l} \lambda_k = 1, \quad \lambda_k \geq 0. \tag{3.5}
\]
By Theorem 2.1 for \( f \) and the definition of projection \( g \) we have
\[
\frac{1}{2} \left[ f(x^{(1)}, y^{(1)}) + f(x^{(2)}, y^{(2)}) \right] \geq \tilde{f}(z) = \sum_{k=1}^{l} \lambda_k f(u^{(k)}, v^{(k)}) \geq \sum_{k=1}^{l} \lambda_k g(u^{(k)}). \tag{3.6}
\]
Furthermore, we have
\[
\sum_{k=1}^{l} \lambda_k g(u^{(k)}) \geq \tilde{g} \left( \frac{x^{(1)} + x^{(2)}}{2} \right), \tag{3.7}
\]
since \( (x^{(1)} + x^{(2)})/2 = \sum_{k=1}^{l} \lambda_k u^{(k)} \) by (3.5) and \( u^{(k)} \in N((x^{(1)} + x^{(2)})/2) \). It follows from (3.4), (3.6), and (3.7) that
\[
\frac{1}{2} \left[ g(x^{(1)}) + g(x^{(2)}) \right] \geq \tilde{g} \left( \frac{x^{(1)} + x^{(2)}}{2} \right) - \varepsilon.
\]
This implies (3.3), since \( \varepsilon > 0 \) is arbitrary. \( \square \)
Remark 3.1. In (ordinary) convex analysis, convexity of functions is characterized by convexity of epigraphs. This characterization makes it possible to reduce the proof of convexity for projected functions (marginal functions) to that for projected sets. In discrete convex analysis, however, convexity concepts for functions such as integral convexity, L^∞-convexity, and M^2-convexity, do not admit simple characterizations in terms of the corresponding discrete convexity of epigraphs. Thus we need separate proofs for sets and functions. □

Remark 3.2. Suppose that \( f(x,y) \) is integrally convex in \((x,y)\) and L^2- or M^2-convex in y, and that \( \text{dom} f \) is bounded. We can minimize such a function efficiently on the basis of Theorem 3.3 if the dimension of \( x \) is small.\(^4\) We denote by \( n_x \) and \( n_y \) the dimensions of \( x \) and \( y \), respectively, and by \( K_x \) and \( K_y \) the \( \ell_\infty \)-sizes of \( \text{dom} f \) projected on the spaces of \( x \) and \( y \), respectively. The minimization of \( f \) can be formulated as the minimization of the projected function \( g(x) \) defined by (1.2). Since \( g \) is integrally convex by Theorem 3.3, the algorithm of [10] can be used to find a minimum of \( g \) with \( C(n_x) \log_2 K_x \) evaluations of \( g \), where \( C(n_y) \) is superexponential in \( n_y \). The evaluation of \( g(x) \) itself amounts to minimizing \( f(x,y) \) over \( y \), which can be done in polynomial time in \( n_y \) and \( \log_2 K_y \) using L^2- or M^2-convex function minimization algorithms [13]. Concerning the above-mentioned conditions on \( f(x,y) \), the following facts are known for a quadratic function \( f(x,y) \) represented as

\[
f(x,y) = (x,y)^T \begin{pmatrix} Q_{xx} & Q_{xy} \\ Q_{yx} & Q_{yy} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (c_x, c_y)^T \begin{pmatrix} x \\ y \end{pmatrix}
\]

with a symmetric matrix \( Q = \begin{pmatrix} Q_{xx} & Q_{xy} \\ Q_{yx} & Q_{yy} \end{pmatrix} \). Such \( f \) is integrally convex in \((x,y)\) if \( Q \) is diagonally dominant with nonnegative diagonals (i.e., \( q_{ii} \geq \sum_{j \neq i} |q_{ij}| \) for all \( i \)) [11]; \( f \) is L^2-convex in \( y \) if and only if \( Q_{yy} = (q_{yy}) \) is diagonally dominant with nonnegative diagonals (i.e., \( q_{ij}^y \geq \sum_{j \neq i} |q_{ij}^y| \) for all \( i \) and \( q_{ij}^y \leq 0 \) for all \( i \neq j \) [13]; and \( f \) is M^2-convex in \( y \) if and only if \( Q_{yy} \) satisfies \( q_{ij}^y \geq 0 \) for all \( (i,j) \) and \( q_{ij}^y \geq \min(q_{ii}^y, q_{jj}^y) \) if \( \{i,j\} \cap \{k\} = \emptyset \) [14]. □

3.2 Projection of discrete midpoint convex functions

We begin with sets.

Theorem 3.4. The projection of a discrete midpoint convex set is a discrete midpoint convex set.

Proof. Let \( T \subseteq \mathbb{Z}^n \) be the projection (3.1) of a discrete midpoint convex set \( S \subseteq \mathbb{Z}^{n+m} \). To show discrete midpoint convexity (2.5) for \( T \), take \( x^{(1)}, x^{(2)} \in T \) with \( \|x^{(1)} - x^{(2)}\|_\infty \geq 2 \). By the definition of projection, we have \( (x^{(1)}, y^{(1)}) \in S \) and \( (x^{(2)}, y^{(2)}) \in S \) for some \( y^{(1)}, y^{(2)} \in \mathbb{Z}^m \). Since \( \|(x^{(1)}, y^{(1)}) - (x^{(2)}, y^{(2)})\|_\infty \geq 2 \), discrete midpoint convexity (2.5) for \( S \) shows

\[
\frac{(x^{(1)}, y^{(1)}) + (x^{(2)}, y^{(2)})}{2} \in S, \quad \frac{(x^{(1)}, y^{(1)}) + (x^{(2)}, y^{(2)})}{2} \in S,
\]

in which

\[
\frac{(x^{(1)}, y^{(1)}) + (x^{(2)}, y^{(2)})}{2} = \left( \frac{x^{(1)} + x^{(2)}}{2}, \frac{y^{(1)} + y^{(2)}}{2} \right), \quad (3.8)\]

\[
\frac{(x^{(1)}, y^{(1)}) + (x^{(2)}, y^{(2)})}{2} = \left( \frac{x^{(1)} + x^{(2)}}{2}, \frac{y^{(1)} + y^{(2)}}{2} \right). \quad (3.9)
\]

Therefore, \( \frac{x^{(1)} + x^{(2)}}{2} \in T \) and \( \frac{x^{(1)} + x^{(2)}}{2} \in T \). Hence (2.5) holds for \( T \). □

\(^4\)This fact is pointed out by Fabio Tardella.
For functions we have the following theorems, the first for the global version of discrete midpoint convex functions, and the second for the local version. The proof for the local version relies on the parallelogram inequality in Theorem 2.3.

**Theorem 3.5.** The projection of a globally discrete midpoint convex function is a globally discrete midpoint convex function.

**Proof.** Let \( g \) be the projection \((3.2)\) of a globally discrete midpoint convex function \( f \). To show discrete midpoint convexity \((2.3)\) for \( g \), take \( x^{(1)}, x^{(2)} \in \text{dom} \ g \) with \( \|x^{(1)} - x^{(2)}\|_{\infty} \geq 2 \) and any \( \varepsilon > 0 \). By the definition of projection, there exist \( y^{(1)}, y^{(2)} \in \mathbb{Z}^m \) such that \( g(x^{(i)}) \geq f(x^{(i)}, y^{(i)}) - \varepsilon \) for \( i = 1, 2 \), which implies
\[
g(x^{(1)}) + g(x^{(2)}) \geq f(x^{(1)}, y^{(1)}) + f(x^{(2)}, y^{(2)}) - 2\varepsilon. \tag{3.10}
\]
Noting \( \|(x^{(1)}, y^{(1)}) - (x^{(2)}, y^{(2)})\|_{\infty} \geq 2 \), we use discrete midpoint convexity \((2.3)\) of \( f \), as well as \((3.8)\) and \((3.9)\), to obtain

\[
\begin{align*}
f(x^{(1)}, y^{(1)}) + f(x^{(2)}, y^{(2)}) & \geq f \left( \frac{(x^{(1)}, y^{(1)}) + (x^{(2)}, y^{(2)})}{2} \right) \\
& \geq g \left( \frac{x^{(1)} + x^{(2)}}{2} \right) + g \left( \frac{x^{(1)} + x^{(2)}}{2} \right), \tag{3.11}
\end{align*}
\]

The combination of \((3.10)\) and \((3.11)\) yields
\[
g(x^{(1)}) + g(x^{(2)}) \geq g \left( \frac{x^{(1)} + x^{(2)}}{2} \right) + g \left( \frac{x^{(1)} + x^{(2)}}{2} \right) - 2\varepsilon.
\]
This implies \((2.3)\) for \( g \), since \( \varepsilon > 0 \) is arbitrary. \( \square \)

**Theorem 3.6.** The projection of a locally discrete midpoint convex function is a locally discrete midpoint convex function.

**Proof.** Let \( g(x) \) be the projection \((3.2)\) of a locally discrete midpoint convex function \( f(x, y) \), where \( y \) is \( m \)-dimensional. We may assume \( m = 1 \), since a one-dimensional projection repeated \( m \) times amounts to an \( m \)-dimensional projection. First, \( \text{dom} \ g \) is a discrete midpoint convex set by Theorem 3.4.

To show discrete midpoint convexity \((2.3)\) for \( g \), take \( x^{(1)}, x^{(2)} \in \text{dom} \ g \) with \( \|x^{(1)} - x^{(2)}\|_{\infty} = 2 \) and any \( \varepsilon > 0 \). By the definition of projection, there exist \( y^{(1)}, y^{(2)} \in \mathbb{Z} \) such that \( g(x^{(i)}) \geq f(x^{(i)}, y^{(i)}) - \varepsilon \) for \( i = 1, 2 \), which implies
\[
g(x^{(1)}) + g(x^{(2)}) \geq f(x^{(1)}, y^{(1)}) + f(x^{(2)}, y^{(2)}) - 2\varepsilon. \tag{3.12}
\]
Take \( z^{(1)}, z^{(2)} \in \mathbb{Z} \) that minimize \( |z^{(1)} - z^{(2)}| \) subject to
\[
g(x^{(1)}) + g(x^{(2)}) \geq f(x^{(1)}, z^{(1)}) + f(x^{(2)}, z^{(2)}) - 2\varepsilon. \tag{3.13}
\]
Such \((z^{(1)}, z^{(2)})\) exists by (3.12). We may assume \( z^{(2)} - z^{(1)} \geq 0 \) by interchanging \((x^{(1)}, z^{(1)})\) and \((x^{(2)}, z^{(2)})\) if necessary.
Consider the decomposition of \((x^{(2)}, z^{(2)}) - (x^{(1)}, z^{(1)})\) into vectors of \([-1, 0, +1]^{n+1}\) as in (2.7):
$$
(x^{(2)}, z^{(2)}) - (x^{(1)}, z^{(1)}) = \sum_{k=1}^{m} (1_{A_k} - 1_{B_k}),
$$
(3.14)

where \(m = ||(x^{(1)}, z^{(1)}) - (x^{(2)}, z^{(2)})||_{\infty} = \max(2, z^{(2)} - z^{(1)})\). It should be clear that the components of the vector \((x^{(i)}, z^{(i)})\) are numbered by \(1, 2, \ldots, n\) and \(n + 1\), and accordingly, \(A_k, B_k \subseteq \{1, 2, \ldots, n, n + 1\}\).

Claim: \(z^{(2)} - z^{(1)} \leq 4\).

(Proof) To prove the claim by contradiction, suppose that \(z^{(2)} - z^{(1)} \geq 5\). Then we have \(m \geq 5\), \(A_k = \{n + 1\}\) for \(1 \leq k \leq m - 2\), and \(B_k = \emptyset\) for \(3 \leq k \leq m\). Hence \(A_3 = \{n + 1\}\) and \(B_3 = \emptyset\) since \(m \geq 5\). By parallelogram inequality (2.9) for \(f\) with \(x = (x^{(1)}, z^{(1)}), y = (x^{(2)}, z^{(2)})\), and \(d = 1_{A_3} - 1_{B_3} = (0, 1)\), we obtain

$$
f(x^{(1)}, z^{(1)}) + f(x^{(2)}, z^{(2)}) \geq f(x^{(1)}, z^{(1)} + 1) + f(x^{(2)}, z^{(2)} - 1).$$
(3.15)

This is a contradiction to the choice of \((z^{(1)}, z^{(2)})\), since \((z^{(1)} + 1, z^{(2)} - 1)\) satisfies (3.13) by (3.15) and \(|(z^{(1)} + 1) - (z^{(2)} - 1)| = |z^{(1)} - z^{(2)}| - 2\). Thus the claim is proved.

We consider three cases, according to the value of \(z^{(2)} - z^{(1)} \in \{0, 1, 2, 3, 4\}\).

Case 1 \((0 \leq z^{(2)} - z^{(1)} \leq 2)\): In this case we have \(||(x^{(1)}, z^{(1)}) - (x^{(2)}, z^{(2)})||_{\infty} = 2\), which allows us to use discrete midpoint convexity (2.3) to obtain

$$
g(x^{(1)}) + g(x^{(2)}) \geq g\left(\frac{x^{(1)} + x^{(2)}}{2}\right) + 2\varepsilon.
$$
(3.17)

This implies discrete midpoint convexity (2.3) for \(g\), since \(\varepsilon > 0\) is arbitrary.

Case 2 \((z^{(2)} - z^{(1)} = 3)\): In this case we have \(m = 3\). With the notation \(X(p) = \{i \mid 1 \leq i \leq n, x_i^{(2)} - x_i^{(1)} = p\}\) for \(p \in \{-2, -1, 1, 2\}\), we have

\[
A_1 = \{n + 1\}, \quad B_1 = X(-1) \cup X(-2),
\]
\[
A_2 = \{n + 1\} \cup X(1), \quad B_2 = X(-2),
\]
\[
A_3 = \{n + 1\} \cup X(1) \cup X(2), \quad B_3 = \emptyset,
\]

where \(X(2) \cup X(-2) \neq \emptyset\) by \(||x^{(1)} - x^{(2)}||_{\infty} = 2\). Define \(d = 1_{A_1} - 1_{B_1}\) if \(X(2) \neq \emptyset\), and \(d = 1_{A_3} - 1_{B_3}\) otherwise. We denote \(d = (b, 1)\) with \(b \in \mathbb{Z}^n\). By parallelogram inequality (2.9) for \(f\) with \(x = (x^{(1)}, z^{(1)}), y = (x^{(2)}, z^{(2)})\), and \(d\) above, we obtain

$$
f(x^{(1)}, z^{(1)}) + f(x^{(2)}, z^{(2)}) \geq f(x^{(1)} + b, z^{(1)} + 1) + f(x^{(2)} - b, z^{(2)} - 1).
$$
(3.18)
Here we have \(|(x^{(1)} + b, z^{(1)} + 1) - (x^{(2)} - b, z^{(2)} - 1)|_\infty = 2\), since \(|z^{(1)} + 1) - (z^{(2)} - 1)| = 1\) and \(|(x^{(1)} + b) - (x^{(2)} - b)| \leq 2\) with equality for some \(i \in X(2) \cup X(-2)\) by the choice of \(d\).

This allows us to use discrete midpoint convexity (2.3) to obtain
\[
\text{RHS of } (3.18) \geq f \left( \frac{(x^{(1)}, z^{(1)}) + (x^{(2)}, z^{(2)})}{2} \right) + f \left( \frac{(x^{(1)}, z^{(1)}) + (x^{(2)}, z^{(2)})}{2} \right).
\]

The combination of (3.18) and (3.19) yields (3.16). The rest of the proof is the same as in Case 1.

Case 3 \((z^{(2)} - z^{(1)}) = 4\): In this case we have \(m = 4\). With the notation \(X(p)\) introduced in Case 2 we have
\[
A_1 = \{n + 1\}, \quad B_1 = X(-1) \cup X(-2),
\]
\[
A_2 = \{n + 1\}, \quad B_2 = X(-2),
\]
\[
A_3 = \{n + 1\} \cup X(1), \quad B_3 = \emptyset,
\]
\[
A_4 = \{n + 1\} \cup X(1) \cup X(2), \quad B_4 = \emptyset.
\]

Define \(d = \mathbf{1}_{A_1} - \mathbf{1}_{B_1}\) and denote \(d = (b, 1)\) with \(b \in \mathbb{Z}^n\). By parallelogram inequality (2.9) for \(f\) with \(x = (x^{(1)}, z^{(1)}), y = (x^{(2)}, z^{(2)})\), and \(d\) above, we obtain (3.18), in which \(|(x^{(1)} + b, z^{(1)} + 1) - (x^{(2)} - b, z^{(2)} - 1)|_\infty = 2\) since \(|z^{(1)} + 1) - (z^{(2)} - 1)| = 2\). The rest of the proof is the same as in Case 2.

\(\square\)

Remark 3.3. Theorem 3.4 for discrete midpoint convex sets can be proved as a special case of Theorem 3.3 for globally discrete midpoint convex functions, since a set \(S\) is discrete midpoint convex if and only if its indicator function \(\delta_S\) is globally discrete midpoint convex. In this paper, however, we have given a direct proof to Theorem 3.4 which is shorter and more transparent. It is emphasized that Theorem 3.3 for integrally convex sets cannot be proved as a special case of Theorem 3.3 for integrally convex functions, since the proof of the latter depends on the former.

4 Convolution Operation

Recall that the Minkowski sum of \(S_1, S_2 \subseteq \mathbb{Z}^n\) is defined by
\[
S_1 + S_2 = \{y + z \mid y \in S_1, z \in S_2\},
\]
and the convolution of \(f_1, f_2 : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}\) is defined by
\[
(f_1 \square f_2)(x) = \inf \{f_1(y) + f_2(z) \mid x = y + z, \ y, z \in \mathbb{Z}^n\} \quad (x \in \mathbb{Z}^n),
\]
where it is assumed that the infimum on the right-hand side is bounded from below (i.e., \(\neq -\infty\)) for every \(x \in \mathbb{Z}^n\). We have
\[
\text{dom } (f_1 \square f_2) = \text{dom } f_1 + \text{dom } f_2.
\]

Let \(\delta_{S_1}, \delta_{S_2} : \mathbb{Z}^n \to \{0, +\infty\}\) be the indicator functions of \(S_1, S_2 \subseteq \mathbb{Z}^n\), respectively. Then their convolution \(\delta_{S_1} \square \delta_{S_2}\) coincides with the indicator function \(\delta_{S_1 + S_2}\) of the Minkowski sum \(S_1 + S_2\), i.e.,
\[
\delta_{S_1} \square \delta_{S_1} = \delta_{S_1 + S_2}.
\]
4.1 Results for integrally convex functions

It is known [13, 16] that the convolution of two integrally convex functions is not necessarily integrally convex. This is demonstrated by the following example [13, Example 3.15] showing that the Minkowski sum of integrally convex sets is not necessarily integrally convex.

Example 4.1. The Minkowski sum of $S_1 = \{(0, 0), (1, 1)\}$ and $S_2 = \{(1, 0), (0, 1)\}$ is equal to $S_1 + S_2 = \{(1, 0), (0, 1), (2, 1), (1, 2)\}$, which has a “hole” at $(1, 1)$, i.e., $(1, 1) \in S_1 + S_2$ and $(1, 1) \notin S_1 + S_2$. Both $S_1$ and $S_2$ are integrally convex, but $S_1 + S_2$ is not integrally convex. ■

Thus we are motivated to consider the convolution of an integrally convex function and a separable convex function. We denote a separable convex function by $\varphi$, i.e.,

$$\varphi(x) = \sum_{i=1}^{n} \varphi_i(x_i), \quad (4.5)$$

where $\varphi_i : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ is a univariate discrete convex function for $i = 1, 2, \ldots, n$. Also we always use $B$ to denote an integer interval (or box), i.e., $B = [a, b]_\mathbb{Z}$ for some integer vectors $a \in (\mathbb{Z} \cup \{-\infty\})^n$ and $b \in (\mathbb{Z} \cup \{+\infty\})^n$ with $a \leq b$.

The following theorems are the main results of this section, dealing with sets and functions, respectively.

Theorem 4.1. The Minkowski sum $S + B$ of an integrally convex set $S$ and an integer interval $B$ is an integrally convex set.

Proof. The proof is given in Section 4.3. ■

Theorem 4.2. The convolution $f \square \varphi$ of an integrally convex function $f$ and a separable convex function $\varphi$ is an integrally convex function.

Proof. While the details are given in Section 4.4, we mention here that the proof consists of two steps.

1. If the effective domains of $f$ and $\varphi$ are bounded, we can use Theorem 2.2 to reduce the proof of Theorem 4.2 to Theorem 4.1 for integrally convex sets.

2. In the general case with possibly unbounded effective domains, we consider sequences $\{f_k\}$ and $\{\varphi_k\}$ with bounded effective domains, which are constructed from $f$ and $\varphi$ as their restrictions to finite intervals. Step 1 shows that $f_k \square \varphi_k$ is integrally convex for each $k$. Then the integral convexity of $f \square \varphi$ is established by a limiting argument. ■

Remark 4.1. It follows from Theorem 4.2 that the $\ell_1$-distance $d^{(1)}$ and the squared $\ell_2$-distance $d^{(2)}$ to an integrally convex set $S$ are both integrally convex, where $d^{(1)}$ and $d^{(2)}$ are defined respectively as

$$d^{(1)}(x) = \inf \{||x - y||_1 \mid y \in S\} \quad (x \in \mathbb{Z}^n), \quad (4.6)$$

$$d^{(2)}(x) = \inf \{||x - y||_2^2 \mid y \in S\} \quad (x \in \mathbb{Z}^n). \quad (4.7)$$

Indeed, the indicator function $\delta_S$ of $S$ is integrally convex, both $\varphi^{(1)}(x) = ||x||_1$ and $\varphi^{(2)}(x) = ||x||_2^2$ are separable convex, and $d^{(k)} = \delta_S \square \varphi^{(k)}$ for $k = 1, 2$ by (4.6) and (4.7). Furthermore, with a parameter $a > 0$, we can define penalty functions $g_a^{(k)}(x) = a d^{(k)}(x)$ ($k = 1, 2$) associated with $S$. Indeed, $g_a = g_a^{(k)}$ with $k \in \{1, 2\}$ satisfies (i) $\text{dom } g_a = \mathbb{Z}^n$, (ii) $g_a(x) \geq 0$ for all $x \in \mathbb{Z}^n$, (iii) $g_a(x) = 0 \Leftrightarrow x \in S$, for each $a > 0$, and (iv) $\lim_{a \to +\infty} g_a(x) = +\infty$ for all $x \notin S$. ■
**Remark 4.2.** It also follows from Theorem 4.2 that any integrally convex function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$, defined effectively on a subset $S$ of $\mathbb{Z}^n$, can be extended to another integrally convex function that takes finite values on the entire integer lattice $\mathbb{Z}^n$. To be specific, with a parameter $a > 0$, we define

$$g_a^{(1)}(x) = \inf \{f(y) + a \|x - y\|_1 \mid y \in \text{dom} f\} \quad (x \in \mathbb{Z}^n),$$

$$g_a^{(2)}(x) = \inf \{f(y) + a \|x - y\|_2^2 \mid y \in \text{dom} f\} \quad (x \in \mathbb{Z}^n).$$

Since $\varphi^{(1)}(x) = \|x\|_1$ and $\varphi^{(2)}(x) = \|x\|_2^2$ are separable convex, both $g_a^{(1)}$ and $g_a^{(2)}$ are integrally convex by Theorem 4.2. Moreover, $g_a = g_a^{(k)}$ with $k \in \{1, 2\}$ satisfies (i) $\text{dom} g_a = \mathbb{Z}^n$, (ii) for each $x \in \text{dom} f$, there exists $a(x) > 0$ such that $g_a(x) = f(x)$ for all $a \geq a(x)$, and (iii) if $\text{dom} f$ is bounded, there exists $\hat{a} > 0$ such that $g_a(x) = f(x)$ for all $x \in \text{dom} f$ and $a \geq \hat{a}$. ■

**Remark 4.3.** Consider the indicator functions $\delta_S, \delta_B : \mathbb{Z}^n \to \{0, +\infty\}$ of $S, B \subseteq \mathbb{Z}^n$. Since their convolution $\delta_S \square \delta_B$ coincides with the indicator function $\delta_{S+B}$ of the Minkowski sum $S + B$ by (4.4), Theorem 4.1 is a special case of Theorem 4.2. It is emphasized, however, that our proof of Theorem 4.2 uses Theorem 3.1 ■

**Remark 4.4.** The projection operation can be regarded as a special case of the convolution operation. Let $g(x) = \inf \{f(x, y) \mid y \in \mathbb{Z}^m\}$ be the projection of $f : \mathbb{Z}^{n+m} \to \mathbb{R} \cup \{+\infty\}$. Consider $B = \{(x, y) \in \mathbb{Z}^{n+m} \mid x = 0\}$, which is an integer interval $[0, -\infty) \times [0, +\infty]$ in $\mathbb{Z}^{n+m}$. Then the projection $g$ coincides with the convolution $f \square \delta_B$ in the sense that $g(x) = (f \square \delta_B)(x, 0)$ for $x \in \mathbb{Z}^n$, which we denote as $g = (f \square \delta_B)|_{\mathbb{Z}^n}$. If $f$ is integrally convex, $f \square \delta_B : \mathbb{Z}^{n+m} \to \mathbb{R} \cup \{+\infty\}$ is also integrally convex by Theorem 4.2, and hence $(f \square \delta_B)|_{\mathbb{Z}^n} = g$ is also integrally convex. This argument serves as an alternative proof of Theorem 5.3 ■

### 4.2 Results for discrete midpoint convex functions

The convolution operation is not amenable to discrete midpoint convexity. In this section we demonstrate this in terms of examples.

We first note the following example ([16 Example 4.11], [13 Note 5.11]) about the Minkowski sum of $L^2$-convex sets.

**Example 4.2.** The Minkowski sum of $S_1 = \{(0, 0, 0), (1, 1, 0)\}$ and $S_2 = \{(0, 0, 0), (0, 1, 1)\}$ is equal to $S_1 + S_2 = \{(0, 0, 0), (0, 1, 1), (1, 1, 0), (1, 2, 1)\}$. For $x = (0, 1, 1)$ and $y = (1, 1, 0)$ in $S_1 + S_2$, we have $\lceil (x + y)/2 \rceil = (1, 1, 1) \notin S_1 + S_2$ and $\lfloor (x + y)/2 \rfloor = (0, 1, 0) \notin S_1 + S_2$. Both $S_1$ and $S_2$ are $L^2$-convex, but $S_1 + S_2$ is not $L^2$-convex. ■

For the Minkowski sum we observe the following.

- The Minkowski sum of two $L^2$-convex sets is not necessarily $L^2$-convex, though it is integrally convex. This is shown by Example 4.2 above and [13 Theorem 8.42].

- The Minkowski sum of two discrete midpoint convex sets is not necessarily discrete midpoint convex (nor integrally convex). This is shown by Example 4.1.

- The Minkowski sum $S + B$ of a discrete midpoint convex set $S$ and an integer interval $B$ is not necessarily discrete midpoint convex, though it is integrally convex. This is shown by Example 4.3 below and Theorem 4.1.
Example 4.3. The Minkowski sum of $S = \{(0, 0, 1), (1, 1, 0)\}$ and $B = \{(0, 0, 0), (1, 0, 0)\}$ is equal to $S + B = \{(0, 0, 1), (1, 1, 0), (1, 0, 1), (2, 1, 0)\}$. $S$ is discrete midpoint convex and $B$ is an integer interval. For $x = (0, 0, 1) \in S + B$ and $y = (2, 1, 0) \in S + B$ we have

$$\|x - y\|_\infty = 2, \quad \left\lfloor \frac{x + y}{2} \right\rfloor = (1, 1, 1) \notin S + B, \quad \left\lfloor \frac{x + y}{2} \right\rfloor = (1, 0, 0) \notin S + B.$$ Therefore, $S + B$ is not discrete midpoint convex.

Next we turn to discrete midpoint convex functions. Recall the relation (4.4) between the Minkowski sums and the convolution of their indicator functions. Our observations above about the Minkowski sums imply the following.

- The convolution $f_1 \Box f_2$ of globally (resp., locally) discrete midpoint convex functions $f_1, f_2$ is not necessarily globally (resp., locally) discrete midpoint convex (nor integrally convex).
- The convolution $f \Box \varphi$ of a globally discrete midpoint convex function $f$ and a separable convex function $\varphi$ is not necessarily globally discrete midpoint convex, though it is integrally convex by (2.6) and Theorem 4.2.
- The convolution $f \Box \varphi$ of a locally discrete midpoint convex function $f$ and a separable convex function $\varphi$ is not necessarily locally discrete midpoint convex, though it is integrally convex by (2.6) and Theorem 4.2.

We show another example of $f$ for the above statements such that the effective domain of $f$ is an integer interval.

Example 4.4. Let $S = \{(0, 0, 1), (1, 1, 0)\}, B = \{(0, 0, 0), (1, 0, 0)\}$, and $\varphi = \delta_B$, and define $f : \mathbb{Z}^3 \to \mathbb{R} \cup \{+\infty\}$ with $\text{dom } f = \{(0, 0, 0), (1, 1, 1)\}_\mathbb{Z}$ by

$$f(x) = \begin{cases} 0 & (x \in S), \\ 1 & (x \in [(0, 0, 0), (1, 1, 1)]_\mathbb{Z} \setminus S). \end{cases}$$

The convolution $f \Box \varphi$, with dom $(f \Box \varphi) = \{(0, 0, 0), (2, 1, 1)\}_\mathbb{Z}$, is given by

$$(f \Box \varphi)(x) = \begin{cases} 0 & (x \in S + B), \\ 1 & (x \in [(0, 0, 0), (2, 1, 1)]_\mathbb{Z} \setminus (S + B)). \end{cases}$$

For $x = (0, 0, 1), y = (2, 1, 0)$ we have $\|x - y\|_\infty = 2$,

$$(f \Box \varphi)(x) = (f \Box \varphi)(y) = 0, \quad (f \Box \varphi)\left(\left\lfloor \frac{x + y}{2} \right\rfloor \right) = (f \Box \varphi)\left(\left\lfloor \frac{x + y}{2} \right\rfloor \right) = 1.$$

Hence $f \Box \varphi$ is not (globally or locally) discrete midpoint convex.

By featuring discrete midpoint convexity (2.3) we can recast our knowledge as follows.

- Discrete midpoint convexity for all $(x, y)$ with $\|x - y\|_\infty \geq 2$ is not preserved in the transformation $f \mapsto f \Box \varphi$.
- Discrete midpoint convexity for all $(x, y)$ with $\|x - y\|_\infty = 2$ is not preserved in the transformation $f \mapsto f \Box \varphi$.
- Discrete midpoint convexity for all \((x, y)\) with \(\|x - y\|_{\infty} \geq 1\) is preserved in the transformation \(f \mapsto f \Box \varphi\).

The last statement is a reformulation of the following (known) fact [13, Theorem 7.11].

**Theorem 4.3.** The convolution \(f \Box \varphi\) of an \(L^1\)-convex function \(f\) and a separable convex function \(\varphi\) is an \(L^1\)-convex function.

In Section 4.3 we give a direct proof of this theorem; the proof in [13] uses a reduction to an \(L\)-convex function, which is defined in terms of submodularity and linearity in the direction of \((1, 1, \ldots, 1)\).

### 4.3 Proof of Theorem 4.1

We prove Theorem 4.1 that the Minkowski sum \(S \oplus B\) of an integrally convex set \(S\) and an integer interval \(B\) is integrally convex.

Let \(S \subseteq \mathbb{Z}^n\) be an integrally convex set and \(B = [a, b]_\mathbb{Z} \subseteq \mathbb{Z}^n\) with \(a \in (\mathbb{Z} \cup \{-\infty\})^n\) and \(b \in (\mathbb{Z} \cup \{+\infty\})^n\). For \(i = 1, 2, \ldots, n\) we denote the \(i\)th unit vector by \(e_i \in \mathbb{Z}^n\) and put \(B_i = \{te_i \mid a_i \leq t \leq b_i, t \in \mathbb{Z}\}\). For \(i = 1\), for example, \(B_1 = \{(t, 0, \ldots, 0) \mid a_1 \leq t \leq b_1, t \in \mathbb{Z}\}\). Then \(B = B_1 + B_2 + \cdots + B_n\) and hence \(S \oplus B = (\cdots ((S \oplus B_1) \oplus B_2) + \cdots) \oplus B_n\). Thus the proof of Theorem 4.1 is reduced to showing the following lemma.

**Lemma 4.4.** The Minkowski sum \(S \oplus B\) of an integrally convex set \(S\) and \(B = ((t, 0, \ldots, 0) \mid \hat{a} \leq t \leq \hat{b}, t \in \mathbb{Z})\) is integrally convex, where \(\hat{a} \in \mathbb{Z} \cup \{-\infty\}\) and \(\hat{b} \in \mathbb{Z} \cup \{+\infty\}\).

**Proof.** Let \(x \in \overline{S \oplus B}\). Our goal is to show \(x \in \overline{(S \oplus B) \cap N(x)}\). The proof goes as follows.

Since \(S \oplus B = \overline{S} + \overline{B}\) (see, e.g., [13, Proposition 3.17]) we can represent \(x\) as \(x = y + z\) with \(y \in \overline{S}\) and \(z \in \overline{B}\). Since \(S\) is integrally convex, the vector \(y\) can be represented as \(y = \sum_{k=1}^{l} \lambda_k y^{(k)}\) with some \(y^{(k)} \in S \cap N(y)\) and \(\lambda_k > 0\) \((k = 1, 2, \ldots, l)\), where \(\sum_{k=1}^{l} \lambda_k = 1\). The other vector \(z\) can be represented as \(z = (\zeta + \beta, 0, \ldots, 0)\) with some \(\zeta \in \mathbb{Z}\) and \(\beta \in \mathbb{R}\), where \(0 \leq \beta < 1\). We show \(x \in (S \oplus B) \cap N(x)\) by finding vectors \(v_i \in (S \oplus B) \cap N(x)\) and coefficients \(\mu_i\) of convex combination \((\sum \mu_i = 1, \mu_i \geq 0)\) such that

\[
x = y + z = \sum_{k=1}^{l} \lambda_k y^{(k)} + (\zeta + \beta) e = \sum \mu_i v_i.
\]

By \(x = y + z\) we have \(x_1 - y_1 = z_1 = \zeta + \beta\), which implies \(0 \leq x_1 - y_1 - \zeta < 1\). We divide into two cases: Case 1: \([x_1] - \zeta \leq y_1\), and Case 2: \([x_1] - \zeta > y_1\).

**Case 1** \((|x_1] - \zeta \leq y_1)\). We have \([x_1] - \zeta \leq y_1 \leq x_1 - \zeta \leq [x_1] - \zeta\), and hence \(N(x) - \zeta e \supseteq N(y)\), where \(e = e_1 = (1, 0, \ldots, 0)\). Since \(y^{(k)} \in N(y) \subseteq N(x) - \zeta e\), we have \(y^{(k)} = [x_1] - \zeta\) or \([x_1] - \zeta\) \((k = 1, 2, \ldots, l)\). After renumbering, if necessary, we may assume

\[
y^{(k)}_1 = [x_1] - \zeta \quad (k = 1, 2, \ldots, k_0),
\]

\[
y^{(k)}_1 = [x_1] - \zeta \quad (k = k_0 + 1, k_0 + 2, \ldots, l),
\]

where \(k_0 = 0\) if \(x_1 \in \mathbb{Z}\). Then

\[
y_1 = \sum_{k=1}^{k_0} \lambda_k y^{(k)}_1 + \sum_{k=k_0+1}^{l} \lambda_k y^{(k)}_1 = \sum_{k=1}^{k_0} \lambda_k [x_1] + \sum_{k=k_0+1}^{l} \lambda_k [x_1] - \zeta.
\]
With the use of this expression we obtain
\[
\beta = x_1 - \zeta - y_1
\]
\[
= x_1 - \sum_{k=1}^{k_0} \lambda_k x_1 - \sum_{k=k_0+1}^{l} \lambda_k x_1
\]
\[
= x_1 - \sum_{k=1}^{k_0} \lambda_k (x_1 - 1) - \sum_{k=k_0+1}^{l} \lambda_k x_1 \quad \text{(since } k_0 = 0 \text{ if } x_1 \in \mathbb{Z})
\]
\[
= x_1 - [x_1] + \sum_{k=1}^{k_0} \lambda_k
\]
\[
\leq \sum_{k=1}^{k_0} \lambda_k. \tag{4.12}
\]
Let \( k_1 \) be the minimum \( k' \) satisfying \( \beta \leq \sum_{k=1}^{k'} \lambda_k \). We have \( k_1 \leq k_0 \) by (4.12). It should be clear that \( k_1 = 0 \) if \( \beta = 0 \). Define \( \alpha = \beta - \sum_{k=1}^{k_1-1} \lambda_k \), where \( \alpha = 0 \) if \( \beta = 0 \). We have \( 0 \leq \alpha \leq \lambda_{k_1} \). Using \( \beta = \alpha + \sum_{k=1}^{k_1-1} \lambda_k \) and \( \sum_{k=1}^{l} \lambda_k = 1 \) we obtain
\[
x = y + z
\]
\[
= \sum_{k=1}^{l} \lambda_k y^{(k)} + (\zeta + \beta) e
\]
\[
= \sum_{k=1}^{k_1-1} \lambda_k y^{(k)} + \lambda_{k_1} y^{(k_1)} + \sum_{k=k_1+1}^{l} \lambda_k y^{(k)} + (\zeta + \alpha + \sum_{k=1}^{k_1-1} \lambda_k) e
\]
\[
= \sum_{k=1}^{k_1-1} \lambda_k (y^{(k)} + (\zeta + 1) e) + \alpha (y^{(k_1)} + (\zeta + 1) e)
\]
\[
+ (\lambda_{k_1} - \alpha) (y^{(k_1)} + \zeta e) + \sum_{k=k_1+1}^{l} \lambda_k (y^{(k)} + \zeta e). \tag{4.13}
\]
Since \( 0 \leq \alpha \leq \lambda_{k_1} \) and \( \sum_{k=1}^{k_1-1} \lambda_k + \alpha + (\lambda_{k_1} - \alpha) + \sum_{k=k_1+1}^{l} \lambda_k = 1 \), (4.13) represents \( x \) as a convex combination of
\[
y^{(k)} + (\zeta + 1) e \quad (k = 1, \ldots, k_1 - 1), \quad y^{(k_1)} + (\zeta + 1) e, \tag{4.14}
\]
\[
y^{(k_1)} + \zeta e, \tag{4.15}
\]
\[
y^{(k)} + \zeta e \quad (k = k_1 + 1, \ldots, l), \tag{4.16}
\]
where the vectors in (4.14) and (4.15) are missing when \( \beta = 0 \) (which implies \( k_1 = 0 \)). Then \( x \in (S + B) \cap N(x) \) follows from Claim 1 below.

Claim 1: All vectors in (4.14)–(4.16) belong to \( (S + B) \cap N(x) \).

(Proof) Since \( y^{(k)} \in S \) and \( \zeta e \in B \), the vectors in (4.15) and (4.16) belong to \( S + B \). The vectors in (4.14) exist only when \( \beta > 0 \), in which case \( (\zeta + 1) e \in B \) and hence the vectors in (4.14) belong to \( S + B \). By \( N(x) \supseteq N(y) + \zeta e \), the vectors in (4.15) and (4.16) obviously belong to \( N(x) \). The vectors in (4.14) also belong to \( N(x) \) since \( x = y + (\zeta + \beta, 0, \ldots, 0) \) and
\[
y^{(k_1)} + \zeta + 1 = [x_1] + 1, \quad y^{(k)} = x_i \quad (i \neq 1)
\]
\[
y^{(k)} = x_i \quad (i \neq 1)
\]
Since $0 \leq k \leq s$ satisfies $0 \leq \alpha \leq \lambda$, we may assume $1 \leq l \leq l - 1$. We have $\beta > 0$, since $\beta = x_1 - y_1 - \zeta > x_1 - [x_1] \geq 0$.

It follows from (4.17) and (4.18) that

$$y_1 = \sum_{k=1}^{k_0} \lambda_k y_1^{(k)} + \sum_{k=k_0+1}^{l} \lambda_k y_1^{(k)} = \sum_{k=1}^{k_0} \lambda_k [x_1] + \sum_{k=k_0+1}^{l} \lambda_k [x_1] - (\zeta + 1).$$

With the use of this expression we obtain

$$\beta = x_1 - \zeta - y_1$$

$$= x_1 + 1 - \sum_{k=1}^{k_0} \lambda_k [x_1] - \sum_{k=k_0+1}^{l} \lambda_k [x_1]$$

$$= x_1 + 1 - \sum_{k=1}^{k_0} \lambda_k ([x_1] - 1) - \sum_{k=k_0+1}^{l} \lambda_k [x_1] \quad \text{(since } x_1 \notin \mathbb{Z})$$

$$= x_1 + 1 - [x_1] + \sum_{k=1}^{k_0} \lambda_k$$

$$\geq \sum_{k=1}^{k_0} \lambda_k. \quad (4.19)$$

Let $k_1$ be the maximum $k'$ satisfying $\sum_{k=1}^{k'} \lambda_k \leq \beta$. We have $k_1 \leq l - 1$, since $\sum_{k=1}^{l} \lambda_k = 1$ and $\beta < 1$, and $k_1 \geq k_0$ by (4.19). Therefore, $1 \leq k_1 \leq l - 1$. Define $\alpha = \beta - \sum_{k=1}^{k_1} \lambda_k$, which satisfies $0 \leq \alpha \leq \lambda_{k_1+1}$. Using $\beta = \alpha + \sum_{k=k_1+1}^{l} \lambda_k$ we obtain

$$x = y + z$$

$$= \sum_{k=1}^{l} \lambda_k y^{(k)} + (\zeta + \alpha) e$$

$$= \sum_{k=1}^{k_1} \lambda_k y^{(k)} + (\zeta + 1) e + \alpha (y^{(k_1+1)} + (\zeta + 1) e)$$

$$+ (\lambda_{k_1+1} - \alpha) y^{(k_1+1)} + \zeta e + \sum_{k=k_1+2}^{l} \lambda_k y^{(k)} + \zeta e. \quad (4.20)$$

Since $0 \leq \alpha \leq \lambda_{k_1+1}$ and $\sum_{k=1}^{k_1} \lambda_k + \alpha + (\lambda_{k_1+1} - \alpha) + \sum_{k=k_1+2}^{l} \lambda_k = 1$, (4.20) represents $x$ as a convex
combination of
\[ y^{(k)} + (\zeta + 1)e \quad (k = 1, \ldots, k_1), \quad y^{(k_1+1)} + (\zeta + 1)e, \quad (4.21) \]
\[ y^{(k_1+1)} + \zeta e, \quad (4.22) \]
\[ y^{(k)} + \zeta e \quad (k = k_1 + 2, \ldots, l), \quad (4.23) \]
where the vectors in (4.22) are missing when \( k_1 = l - 1 \). Then \( x \in (S + B) \cap N(x) \) follows from Claim 2 below.

Claim 2: All vectors in (4.21)–(4.23) belong to \((S + B) \cap N(x)\).

(Proof) Since \( y^{(k)} \in S \) and \( \{\zeta e, (\zeta + 1)e\} \subseteq B \) as a consequence of \( \beta > 0 \), all the vectors in (4.21)–(4.23) belong to \( S + B \). By \( N(x) \supseteq N(y) + (\zeta + 1)e \), the vectors in (4.21) obviously belong to \( N(x) \). The vectors in (4.22) and (4.23) also belong to \( N(x) \) since \( x = y + (\zeta + \beta, 0, \ldots, 0) \) and
\[ y^{(k_1)} + \zeta = [x_1] - 1, \quad y^{(k_i)}_i = x_i \quad (i \neq 1) \]
for \( k = k_0 + 1, k_0 + 2, \ldots, l \) by (4.18) and \( k_i \geq k_0 \). Thus Claim 2 is proved.

In both cases, Case 1 and Case 2, we have arrived at \( x \in (S + B) \cap N(x) \). This completes the proof of Lemma 4.4. \( \square \)

### 4.4 Proof of Theorem 4.2

In this section we prove Theorem 4.2 that the convolution \( f \circ \varphi \) of an integrally convex function \( f \) and a separable convex function \( \varphi \) is integrally convex.

The proof consists of two steps. In Step 1, we prove integral convexity of \( g = f \circ \varphi \) when \( \text{dom } f \) and \( \text{dom } \varphi \) are bounded. In Step 2, we cope with the general case by considering sequences \( \{f_k\} \) and \( \{\varphi_k\} \), with \( \text{dom } f_k \) and \( \text{dom } \varphi_k \) bounded, that converge to \( f \) and \( \varphi \), respectively. We first note that \( \text{dom } g \) is an integrally convex set by Theorem 4.1 since \( \text{dom } g = \text{dom } f + \text{dom } \varphi \), in which \( \text{dom } f \) is an integrally convex set and \( \text{dom } \varphi \) is an integer interval.

Step 1: We assume that \( \text{dom } f \) and \( \text{dom } \varphi \) are bounded. Then \( \text{dom } g = \text{dom } f + \text{dom } \varphi \) is also bounded. Let \( p \in \mathbb{R}^n \) and note
\[ g[-p] = f[-p] \circ \varphi[-p], \]
\[ \text{argmin } g[-p] = \text{argmin } f[-p] + \text{argmin } \varphi[-p]. \]
Since \( \text{argmin } f[-p] \) is an integrally convex set by Theorem 2.2 (“only if”) and \( \text{argmin } \varphi[-p] \) is an integer interval, \( \text{argmin } g[-p] \) is an integrally convex set by Theorem 4.1. Then Theorem 2.2 (“if”) shows that \( g \) is an integrally convex function.

Step 2: For \( k = 1, 2, \ldots \), let \( f_k \) denote the function obtained from \( f \) by restricting the effective domain to the integer interval \( \{x \in \mathbb{Z}^n \mid \|x\|_\infty \leq k\} \); define \( \varphi_k \) from \( \varphi \) in a similar manner. For each \( k \), \( f_k \) is an integrally convex function by Theorem 2.1 and \( \varphi_k \) is a separable convex function, both with bounded effective domains, and therefore \( g_k = f_k \circ \varphi_k \) is integrally convex by Step 1. By Lemma 4.5 below, \( \{g_k(x)\}_k \) converges to \( g(x) \) for each \( x \in \text{dom } g \). Therefore \( g \) is integrally convex since the limit of integrally convex functions is integrally convex by Lemma 4.6 below.

**Lemma 4.5.** For each \( x \in \text{dom } g \), the sequence \( \{g_k(x)\}_k \) is nonincreasing and converges to \( g(x) \).

*We only need to consider sufficiently large \( k \) for which \( \text{dom } f_k \) and \( \text{dom } \varphi_k \) are nonempty.*
Proof. Since
\[ g_k(x) = \inf \{ f_k(y) + \varphi_k(z) \mid x = y + z \} = \inf \{ f(y) + \varphi(z) \mid x = y + z, \| y \|_\infty \leq k, \| z \|_\infty \leq k \}, \]
the sequence \( \{ g_k(x) \}_k \) is obviously nonincreasing, while it is bounded from below by our standing assumption stated at the beginning of Section 4. Therefore, the limit \( \lim_{k \to \infty} g_k(x) \) exists. The limit is obviously equal to \( g(x) \), but we give a formal proof for completeness. Let \( \varepsilon > 0 \) be any positive number. By the definition of \( g = f \square \varphi \), there exist \( y_x \) and \( z_x \) in \( \mathbb{Z}^n \) for which \( f(y_x) + \varphi(z_x) \leq g(x) + \varepsilon \). Let \( k_0 = \max(||y_x||_\infty, ||z_x||_\infty) \). For any \( k \geq k_0 \), we have \( g_k(x) \leq f(y_x) + \varphi(z_x) \). Therefore, \( g_k(x) \leq g(x) + \varepsilon \). This shows that \( \lim_{k \to \infty} g_k(x) = g(x) \). \( \square \)

**Lemma 4.6.** If a sequence of integrally convex functions \( g_k : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{ +\infty \} \) converges pointwise to a function \( g : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{ +\infty \} \) with an integrally convex effective domain, then \( g \) is also integrally convex.

Proof. By Theorem 2.1, it suffices to show
\[ \tilde{g}(u) \leq \frac{1}{2}(g(x^{(1)}) + g(x^{(2)})) \] (4.24)
for any \( x^{(1)}, x^{(2)} \in \text{dom } g \) with \( \| x^{(1)} - x^{(2)} \|_\infty = 2 \), where \( u = (x^{(1)} + x^{(2)})/2 \). For each \( k \) we have
\[ \tilde{g}_k(u) \leq \frac{1}{2}(g_k(x^{(1)}) + g_k(x^{(2)})) \] (4.25)
by Theorem 2.1. Recall that the local convex extension \( \tilde{g}_k(u) \) is defined as
\[ \tilde{g}_k(u) = \min \left\{ \sum_{y \in N(u)} \lambda_y g_k(y) \mid \sum_{y \in N(u)} \lambda_y y = u, \ (\lambda_y) \in \Lambda(u) \right\}, \]
where the integral neighborhood \( N(u) \) is independent of \( k \). For each simplex \( \Delta \) with vertices taken from \( N(u) \), consider the linear interpolation of \( g_k \) on \( \Delta \) and denote its value at \( u \) by \( \tilde{g}_k^\Delta(u) \), where \( \tilde{g}_k^\Delta(u) = +\infty \) if \( u \notin \Delta \). Then \( \tilde{g}_k(u) \) is equal to the minimum of \( \tilde{g}_k^\Delta(u) \) over all \( \Delta \), which are finite in number. By defining \( \tilde{g}^\Delta(u) \) from \( g \) in a similar manner we have \( \tilde{g}(u) = \min_{\Delta} \tilde{g}^\Delta(u) \).

The pointwise convergence of \( g_k \) to \( g \) implies \( \lim_{k \to \infty} \tilde{g}_k^\Delta(u) = \tilde{g}^\Delta(u) \) for each \( \Delta \), and hence
\[ \lim_{k \to \infty} \tilde{g}_k(u) = \lim_{k \to \infty} \min_{\Delta} \tilde{g}_k^\Delta(u) = \min_{\Delta} \lim_{k \to \infty} \tilde{g}_k^\Delta(u) = \min_{\Delta} \tilde{g}^\Delta(u) = \tilde{g}(u). \]
Therefore, (4.24) follows from (4.25) by letting \( k \to \infty \). \( \square \)

### 4.5 Proof of Theorem 4.3 by discrete midpoint convexity

In this section we prove the following proposition. This serves as an alternative proof for Theorem 4.3.

**Proposition 4.7.** If \( f \) satisfies discrete midpoint convexity (2.3) for all \( x, y \in \mathbb{Z}^n \), so does its convolution \( f \square \varphi \) with a separable convex function \( \varphi \).
Proof. Let \( x^{(1)}, x^{(2)} \in \text{dom } g \), and take any \( \varepsilon > 0 \). There exist \( y^{(i)}, z^{(i)} \) for \( i = 1, 2 \) such that

\[
g(x^{(i)}) \geq f(y^{(i)}) + \varphi(z^{(i)}) - \varepsilon, \quad x^{(i)} = y^{(i)} + z^{(i)} \quad (i = 1, 2).
\] (4.26)

By discrete midpoint convexity \((2.3)\) of \( f \) we have

\[
f(y^{(1)}) + f(y^{(2)}) \geq f \left( \frac{y^{(1)} + y^{(2)}}{2} \right).
\] (4.27)

Define

\[
z' = \left\lfloor \frac{x^{(1)} + x^{(2)}}{2} \right\rfloor - \left\lfloor \frac{y^{(1)} + y^{(2)}}{2} \right\rfloor, \quad z'' = \left\lfloor \frac{x^{(1)} + x^{(2)}}{2} \right\rfloor - \left\lfloor \frac{y^{(1)} + y^{(2)}}{2} \right\rfloor.
\] (4.28)

By the definition of \( f \circ \varphi = g \) we have

\[
f \left( \frac{y^{(1)} + y^{(2)}}{2} \right) + \varphi(z') \geq g \left( \frac{x^{(1)} + x^{(2)}}{2} \right),
\] (4.29)

\[
f \left( \frac{y^{(1)} + y^{(2)}}{2} \right) + \varphi(z'') \geq g \left( \frac{x^{(1)} + x^{(2)}}{2} \right).
\] (4.30)

For \( \varphi(z) = \sum_{i=1}^{n} \varphi_i(z_i) \), on the other hand, we can show (see below)

\[
\varphi(z^{(1)}) + \varphi(z^{(2)}) \geq \varphi(z') + \varphi(z'').
\] (4.31)

for \( i = 1, 2, \ldots, n \), from which follows

\[
\varphi(z^{(1)}) + \varphi(z^{(2)}) \geq \varphi(z') + \varphi(z'').
\] (4.32)

By adding \((4.26), (4.27), (4.29), (4.30), \text{ and } (4.32)\), we obtain

\[
g(x^{(1)}) + g(x^{(2)}) \geq g \left( \frac{x^{(1)} + x^{(2)}}{2} \right) + 2 \varepsilon.
\] (4.33)

This implies discrete midpoint convexity \((2.3)\) for \( g \), since \( \varepsilon > 0 \) is arbitrary.

It remains to prove \((4.31)\). First we note a simple consequence of the convexity of \( \varphi_i \). Let \( a \) and \( b \) be integers with \( a \leq b \), and \( p, q \in \mathbb{Z} \). If (i) \( a + b = p + q \), (ii) \( a \leq p \leq b \), and (iii) \( a \leq q \leq b \), then \( \varphi_i(a) + \varphi_i(b) \geq \varphi_i(p) + \varphi_i(q) \). Therefore the proof of \((4.31)\) is reduced to showing the following:

\[
z_i^{(1)} + z_i^{(2)} = z_i' + z_i'',
\] (4.34)

\[
\min(z_i^{(1)}, z_i^{(2)}) \leq z_i' \leq \max(z_i^{(1)}, z_i^{(2)}),
\] (4.35)

\[
\min(z_i^{(1)}, z_i^{(2)}) \leq z_i'' \leq \max(z_i^{(1)}, z_i^{(2)}).
\] (4.36)

The first equation \((4.34)\) is a consequence of the identity \([\xi/2] + [\xi/2] = \xi\) valid for any \( \xi \in \mathbb{Z} \) as follows:

\[
z_i' + z_i'' = \left( \left\lfloor \frac{x_i^{(1)} + x_i^{(2)}}{2} \right\rfloor - \left\lfloor \frac{y_i^{(1)} + y_i^{(2)}}{2} \right\rfloor \right) + \left( \left\lfloor \frac{y_i^{(1)} + y_i^{(2)}}{2} \right\rfloor - \left\lfloor \frac{y_i^{(1)} + y_i^{(2)}}{2} \right\rfloor \right)
\]
\[
= \left( \left\lfloor \frac{x_i^{(1)} + x_i^{(2)}}{2} \right\rfloor + \left\lfloor \frac{y_i^{(1)} + y_i^{(2)}}{2} \right\rfloor \right) - \left( \left\lfloor \frac{y_i^{(1)} + y_i^{(2)}}{2} \right\rfloor + \left\lfloor \frac{y_i^{(1)} + y_i^{(2)}}{2} \right\rfloor \right)
\]
\[
= (x_i^{(1)} + x_i^{(2)}) - (y_i^{(1)} + y_i^{(2)}) = (x_i^{(1)} - y_i^{(1)}) + (y_i^{(2)} - y_i^{(2)}) = z_i^{(1)} + z_i^{(2)}.
\]
To show (4.35) and (4.36) we substitute \( z^{(1)}_i + z^{(2)}_i = (x^{(1)}_i + x^{(2)}_i) - (y^{(1)}_i + y^{(2)}_i) \) into
\[
\min(z^{(1)}_i, z^{(2)}_i) \leq \frac{z^{(1)}_i + z^{(2)}_i}{2} \leq \max(z^{(1)}_i, z^{(2)}_i),
\]
to obtain
\[
\min(z^{(1)}_i, z^{(2)}_i) + \frac{y^{(1)}_i + y^{(2)}_i}{2} \leq \frac{x^{(1)}_i + x^{(2)}_i}{2} \leq \max(z^{(1)}_i, z^{(2)}_i) + \frac{y^{(1)}_i + y^{(2)}_i}{2}.
\]

By applying \([ \cdot ]\) and \([ \cdot , \cdot ]\), we obtain
\[
\min(z^{(1)}_i, z^{(2)}_i) + \frac{y^{(1)}_i + y^{(2)}_i}{2} \leq \frac{x^{(1)}_i + x^{(2)}_i}{2} \leq \max(z^{(1)}_i, z^{(2)}_i) + \frac{y^{(1)}_i + y^{(2)}_i}{2},
\]
\[
\min(z^{(1)}_i, z^{(2)}_i) + \frac{y^{(1)}_i + y^{(2)}_i}{2} \leq \frac{x^{(1)}_i + x^{(2)}_i}{2} \leq \max(z^{(1)}_i, z^{(2)}_i) + \frac{y^{(1)}_i + y^{(2)}_i}{2},
\]
which are equivalent to (4.35) and (4.36), respectively. \( \square \)

5 Concluding Remarks

Besides projection and convolution, there are a number of fundamental operations for discrete convex functions. Here we touch upon conjugation, restriction, and addition operations for integrally convex functions.

For a function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) with \( \text{dom} f \neq \emptyset \), the (integer) conjugate of \( f \) is the function \( f^* : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) defined by
\[
f^*(p) = \sup\{\sum_{i=1}^n p_i x_i - f(x) \mid x \in \mathbb{Z}^n\} \quad (p \in \mathbb{Z}^n),
\]
which is a discrete version of the Fenchel–Legendre transformation. The conjugate of an integrally convex (resp., globally or locally discrete midpoint convex) function \( f \) is not necessarily integrally convex (resp., globally or locally discrete midpoint convex). This is shown by the following example.

**Example 5.1** ([16 Example 4.15]). \( S = \{(1, 1, 0, 0), (0, 1, 1, 0), (1, 0, 1, 0), (0, 0, 0, 1)\} \) is obviously an integrally convex set, as it is a subset of \( \{0, 1\}^4 \). Accordingly, its indicator function \( \delta_S : \mathbb{Z}^4 \to \{0, +\infty\} \) is integrally convex. The conjugate function \( g = \delta_S^* \) is given by
\[
g(p_1, p_2, p_3, p_4) = \max\{p_1 + p_2, p_2 + p_3, p_1 + p_3, p_4\} \quad (p \in \mathbb{Z}^4).
\]

For \( p = (0, 0, 0, 0) \) and \( q = (1, 1, 1, 2) \) we have \( \tilde{g}(p + q)/2 > (g(p) + g(q))/2 \), a violation of the inequality (2.2) in Theorem 2.1 where \( \tilde{g} \) denotes the local convex closure of \( g \). Indeed, \( (p + q)/2 = (1/2, 1/2, 1/2, 1) \) and \( \tilde{g}((p + q)/2) = 3/2 \), whereas \( (g(p) + g(q))/2 = (0 + 2)/2 = 1 \). Hence \( g \) is not integrally convex. Moreover, \( S \) is a discrete midpoint convex set, and the indicator function \( \delta_S \) is globally (and hence locally) discrete midpoint convex. Its conjugate function \( g \) is not globally or locally discrete midpoint convex, as it is not integrally convex. \( \blacksquare \)
Table 1: Convolution and projection operations for discrete convex functions ($\varphi$: separable convex)

| Function class | Convolution | Convol. with sep. conv. | Projection |
|----------------|-------------|-------------------------|------------|
| $C \boxconv C \subseteq C$ | true (obvious) | true (obvious) | true (obvious) |
| Separable convex | $\text{[13, Th. 6.15]}$ | $\text{[13, Th. 6.15]}$ | $\text{[13, Th. 6.15]}$ |
| $M^2$-convex | true | true | true |
| $L^2$-convex | false | true | true |
| Integrally convex | false | true | true |
| Globally discrete midpoint convex | false | false | true |
| Locally discrete midpoint convex | false | false | true |

As is well known in convex analysis, the operations that are conjugate to projection and convolution are restriction and addition, respectively. For a function $f : \mathbb{Z}^{n+m} \to \mathbb{R} \cup \{+\infty\}$, the restriction of $f$ to $\mathbb{Z}^n$ is the function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$g(x) = f(x, \mathbf{0}_m) \quad (x \in \mathbb{Z}^n),$$

where $\mathbf{0}_m$ means the zero vector in $\mathbb{Z}^m$. It is easy to see that the restriction of an integrally convex (resp., globally or locally discrete midpoint convex) function $f$ is integrally convex (resp., globally or locally discrete midpoint convex).

The sum of integrally convex functions is not necessarily integrally convex, as the following example shows. On the other hand, it is known [9, 11] (and easy to see) that the sum of globally (resp. locally) discrete midpoint convex functions is globally (resp. locally) discrete midpoint convex.

**Example 5.2 ([16, Example 4.4]).** Consider $D_1 = \{(0, 0, 0), (0, 1, 1), (1, 1, 0), (1, 2, 1)\}$ and $D_2 = \{(0, 0, 0), (0, 1, 0), (1, 1, 1), (1, 2, 1)\}$, which are both integrally convex sets. Their intersection $D_1 \cap D_2 = \{(0, 0, 0), (1, 2, 1)\}$ is not an integrally convex set. Therefore, the indicator functions $\delta_{D_1}, \delta_{D_2} : \mathbb{Z}^3 \to \{0, +\infty\}$ are integrally convex, and their sum $\delta_{D_1} + \delta_{D_2}$ is not integrally convex.

The authors thank Fabio Tardella for helpful comments, especially for suggesting the use of projection operation described in Remark 3.2.
References

[1] Favati, P., Tardella, F.: Convexity in nonlinear integer programming. Ricerca Operativa 53, 3–44 (1990)

[2] Fujishige, S.: Bisubmodular polyhedra, simplicial divisions, and discrete convexity. Discrete Optimization 12, 115–120 (2014)

[3] Fujishige, S., Murota, K.: Notes on L-/M-convex functions and the separation theorems. Mathematical Programming 88, 129–146 (2000)

[4] Hiriart-Urruty, J.-B., Lemaréchal, C.: Fundamentals of Convex Analysis. Springer, Berlin (2001)

[5] Iimura, T.: Discrete modeling of economic equilibrium problems. Pacific Journal of Optimization 6, 57–64 (2010)

[6] Iimura, T., Murota, K., Tamura, A.: Discrete fixed point theorem reconsidered. Journal of Mathematical Economics 41, 1030–1036 (2005)

[7] Iimura, T., Watanabe, T.: Existence of a pure strategy equilibrium in finite symmetric games where payoff functions are integrally concave. Discrete Applied Mathematics 166, 26–33 (2014)

[8] van der Laan, G., Talman, D., Yang, Z.: Solving discrete systems of nonlinear equations. European Journal of Operational Research 214, 493–500 (2011)

[9] Moriguchi, S., Murota, K., Tamura, A., Tardella, F.: Scaling and proximity properties of integrally convex functions. In: Seok-Hee Hong (ed.) ISAAC2016, Leibniz International Proceedings in Informatics (LIPIcs), 64, Article No. 57, 57:1–57:12 (2016)

[10] Moriguchi, S., Murota, K., Tamura, A., Tardella, F.: Scaling, proximity, and optimization of integrally convex functions. To appear in Mathematical Programming, arXiv 1703.10705, March 2017

[11] Moriguchi, S., Murota, K., Tamura, A., Tardella, F.: Discrete midpoint convexity. arXiv 1708.04579, August 2017

[12] Murota, K.: Discrete convex analysis. Mathematical Programming 83, 313–371 (1998)

[13] Murota, K.: Discrete Convex Analysis. Society for Industrial and Applied Mathematics, Philadelphia (2003)

[14] Murota, K.: Recent developments in discrete convex analysis. In: Cook, W., Lovász, L., Vygen, J. (eds.) Research Trends in Combinatorial Optimization, Chapter 11, pp. 219–260. Springer, Berlin (2009)

[15] Murota, K.: Discrete convex analysis: A tool for economics and game theory. Journal of Mechanism and Institution Design 1, 151–273 (2016)

[16] Murota, K., Shioura, A.: Relationship of M-/L-convex functions with discrete convex functions by Miller and by Favati–Tardella. Discrete Applied Mathematics 115, 151–176 (2001)
[17] Yang, Z.: On the solutions of discrete nonlinear complementarity and related problems. Mathematics of Operations Research 33, 976–990 (2008)

[18] Yang, Z.: Discrete fixed point analysis and its applications. Journal of Fixed Point Theory and Applications 6, 351–371 (2009)