Bounding integral points on the Siegel modular variety $A_2(2)$

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Abstract

We determine an explicit upper bound for the stable Faltings height of principally polarised abelian surfaces over number fields corresponding to $S$-integral points on the Siegel modular variety $A_2(2)\setminus D$ where $D$ is the divisor of products of elliptic curves. This upper bound, using Runge’s method, is uniform in $S$ as long as $|S| < 3$.

1 Introduction

Diophantine methods, commonly used to determine the rational or integral points on algebraic varieties (i.e. to solve diophantine equations), tend to become particularly difficult when the underlying algebraic variety is not a curve; even more so when one desires effective results.

For example, the Siegel varieties $A_g(n)$, which are the moduli spaces of $g$-dimensional principally polarised abelian varieties with full $n$-torsion, are widely studied but still quite mysterious for $g > 1$. Important results include non-effective uniform boundedness of the full torsion under Vojta’s conjecture [1], and effective height bounds for jacobians of hyperelliptic curves with good reduction [20]. We note that the latter result is of a different nature since its proof does not directly use the geometry of $A_g(n)$.

The study of integral points on varieties was given a boost by Aaron Levin, who succeeded in extending Runge’s method [9] (updated in [11]) and Baker’s method [10] – both classical methods for determining integral points on curves – to varieties of any dimension. Recently, the second named author expanded on both methods and applied these to $A_2(2)$ [12], [13]. In this article we push these methods to their limit to find stronger explicit effective results for integral points on $A_2(2)$. In a previous version of the paper, we also applied a variant of Baker’s method on $A_2(2)$, but the result we obtained was in fact less general than what is obtained by [20], so following the referee’s advice we removed this part of the paper. We hope our computations might pave the way for the explicit application of higher-dimensional Runge and Baker to other (modular) varieties.

Our objects of interest are abelian surfaces over number fields with full 2-torsion (so $g = n = 2$). Recall [16] that over any field $k$, a principally polarised abelian surface $A_{2/k}$ is isomorphic over a finite extension to either the jacobian of a (smooth) hyperelliptic curve...
of genus 2, or to a product of elliptic curves (both endowed with the natural associated polarisations).

When we consider an abelian surface $A$ over a number field $K$, its semistable reduction at every finite place $v$ of $K$ can be an abelian surface (potentially good reduction) or not (potentially multiplicative reduction). For our purposes, principally polarised abelian surface will be considered to reduce “nicely” at $v$ if the semistable reduction is not only an abelian surface, but also is isomorphic to the jacobian of a hyperelliptic curve of genus 2 over some finite extension. If we start with $A = \text{Jac}(C)$ for some genus 2 hyperelliptic curve $C$, this is equivalent to saying that $C$ itself has potentially good reduction at $v$.

As we will see below, abelian surfaces which “reduce nicely” in this sense outside of a set of places $S$ correspond to $S$-integral points on $(A_2(2) \setminus D)$ for a certain divisor $D$, in a way which will be made precise later. Furthermore, $A_2(2)$ has explicit (and workable) equations as a subvariety of $\mathbb{P}^9$, which makes it a good example to practise precise computations and methods.

**Theorem 1** Let $(A, \lambda)$ be a principally polarised abelian surface defined over a number field $K$, with full 2-torsion defined over $K$. Consider the set $S$ of places $v$ of $K$ which are infinite or such that the semistable reduction of $A$ modulo $v$ is not isomorphic to the jacobian of a genus 2 curve.

If $|S| < 3$, we have a bound

$$h_F(A) \leq 985,$$

where $h_F$ is the stable Faltings height of $A$.

The proof of Theorem 1 uses Runge’s method in higher dimensions, as introduced by Levin [11] and first applied to $A_2(2)$ in [12]. To be precise, there is an overlap between [12, Theorem 8.2 (a)] and Theorem 1: indeed, they both cover the case where $A$ never reduces to a product of elliptic curves ($s_P = 0$ in the latter theorem, $|S| = 1$ here) and in this case the bound given in [12] is slightly better. Otherwise, [12, Theorem 8.2 (a)] assumes potentially good reduction at all finite places which is why it allows $s_P < 4$, and we do not assume it here. Furthermore, a straightforward application of Levin’s result predicts the condition $|S| < 2$, whereas we allow $|S| < 3$. This means in practice that our set $S$ of “bad” places is allowed to contain a finite place, thus providing a significant improvement to [12, Theorem 8.2], as witnessed by Corollary 2.

Regarding the bound itself, we compute a smaller bound for the Weil height with respect to a model for $A_2(2)$ (see Theorem 23). Choosing $K = \mathbb{Q}$ and $S = \{p, \infty\}$, we can search for $\mathbb{Q}$-rational points on $A_2(2)$ of height up to this bound, and obtain the following consequence (proven in Sect. 3.2). To be clear, the potentially good reduction mentioned here in the Corollary is that of the curve, not that of its jacobian (which can have good reduction as an abelian variety even when the curve does not).

**Corollary 2** There is no genus 2 hyperelliptic curve $C$ over $\mathbb{Q}$ such that all Weierstrass points of $C$ are rational and $C$ has potentially good reduction at all but one of the primes.

**Remark 3** The proof of this Corollary leads us to the curve

$$C: y^2 = x(x - 1)(x^2 - 4)(x - 4),$$
which has potentially good reduction at all primes except for 2 and 3.

For curves (over $\mathbb{Q}$) such that $\text{Jac}(C)$ has CM by the maximal order of a quartic field $K$ above a quadratic field $F$, it is proven in [6] that there are up to isomorphism only finitely such curves with potentially good reduction everywhere (with $F$ fixed and $K$ a quartic cyclic field over $F$ allowed to change). It would thus be interesting to know if integral points techniques could apply in the special case of CM curves and improve the known bounds, although it appears to be far outside the scope of Runge and Baker’s methods at the moment.

The structure of the paper is as follows. In Sect. 2, we start with basic facts and definitions about the variety $A_2(2)$ and its compactification $A_2(2)^S$. The theta coordinates and equations of $A_2(2)^S$ in $\mathbb{P}^9$ and the divisor $D$ are recalled, after which we state precisely the integrality hypothesis and its interpretation in theta coordinates. We prove several useful “transitivity” statements for the natural action of $\text{Sp}_4(\mathbb{F}_2)$ on $A_2(2)^S$. To conclude this section, we compute the “graph of intersection” of the irreducible components of $D$.

In Sect. 3, we make the precise estimates necessary for realising effectively our Runge-type method and regroup them to obtain Theorem 1 and Corollary 2.

We have chosen to verify some simple computations using Magma. The code for these can be found at https://github.com/joshabox/IntegralpointsonA22.

2 Setup of the integrality problem

In this section, we recall the definitions of our objects of interest and establish their basic properties.

2.1 The Siegel modular variety $A_2(2)^S$

We start with $\Gamma = \text{Sp}_4(\mathbb{Z})$ and its action on the Siegel half space

$$\mathcal{H}_2 := \{ \tau \in M_2(\mathbb{C}), i\tau = \tau, \text{Im } \tau > 0 \},$$

where the positivity is as a real symmetric matrix. It acts naturally by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau := (A\tau + B)(C\tau + D)^{-1},$$

where $A, B, C, D$ are $2 \times 2$ matrices. For any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we define $j_M(\tau) := \det(C\tau + D)$, which is a cocycle for this action (see [4, Proposition VII.1.1] for proofs of those claims).

We consider

$$\Gamma(2) := \{ \gamma \in \Gamma, \gamma = I \mod 2 \},$$

the congruence subgroup of level 2, and denote by $A_2(2)(\mathbb{C})$ the quotient $\Gamma(2) \backslash \mathcal{H}_2$. The following can be found in [5, Theorem V.2.5] and its associated sections. The set $A_2(2)(\mathbb{C})$ is canonically the set of complex points of a quasi-projective normal algebraic variety of dimension 3 over $\mathbb{Q}$ denoted by $A_2(2)$, which admits a Satake compactification (as a projective normal variety) denoted by $A_2(2)^S$. Furthermore, the boundary $\partial A_2(2) := A_2(2)^S \backslash A_2(2)$ is of dimension 1.

Over $\mathbb{Z}$, we can define a scheme $A_2(2)$ which is outside of characteristic 2 the coarse moduli space of principally polarised abelian surfaces in characteristic 2 with full symplectic level structure. We can for example define first $A_2(2)^S$ as the normalisation of $A_2(2)$ in $A_2(1)^S$, and $A_2(2)$ as the complement of the inverse image of the boundary of $A_2(1)$. 
The variety $A_2(2)$ is also the coarse moduli space of principally polarised abelian surfaces in characteristic 0 with full symplectic level 2 structure.

2.2 The integrality question and goal of the paper

As we will be able to prove later (see Theorem 10 or [12, Proposition 7.9 and before]), there are ten irreducible effective divisors $D_1, \ldots, D_{10}$ on $A_2(2)$ (all defined over $\mathbb{Q}$) whose points of the union

$$D = \bigcup_{i=1}^{10} D_i$$

parametrise (outside the boundary) products of elliptic curves with their natural polarisations (and any choice of symplectic basis).

We will thus be interested, for a number field $K$ and a finite set of places $S$ containing the archimedean ones, in

$$(A_2(2)^S \setminus D)(\mathcal{O}_{K,S}),$$

which corresponds to the set of moduli of triples $(A, \lambda, \omega_2)$ defined over $K$ such that the semistable reduction of $(A, \lambda)$ is always outside $S$ an abelian surface not isomorphic (with polarisations) to a product of elliptic curves.

We will see that up to small error for places above 2, this has a natural interpretation in terms of integral points on a model of $A_2(2)$ in $\mathbb{P}^9$.

Our goal is to bound explicitly those integral points in terms of their Faltings height (and projective height), assuming that $|S|$ is small.

In order to apply Runge’s method, it will be convenient to work out precisely the graph of intersection of the divisors, as defined in [13]. For this, it is very worthwhile to closely study the action of $\text{Sp}_4(\mathbb{F}_2)$ on $A_2(2)$, and in particular how it permutes the divisors $D_1, \ldots, D_{10}$.

2.3 Theta constants and equations of the variety

The general definition of theta functions for us, inspired by Igusa, is the following: for any $m = (m', m'') \in \mathbb{Z}^4$ with $m', m'' \in \mathbb{Z}^2$ (all row vectors), we define for all $\tau \in \mathbb{H}_2$

$$\Theta_m(\tau) := \sum_{p \in \mathbb{Z}^2} \exp \left( i\pi (p + m'/2) \tau + (p + m'/2) + i\pi (p + m'/2)^t m'' \right)$$

For any row vector $m \in \mathbb{Z}^4$, it is easily checked that

$$\Theta_{m+2\mathbb{Z}} = (-1)^{m't m''} \Theta_m \quad \text{and} \quad \Theta_{-\cdot m}(\tau) = \Theta_m(\tau).$$

This already proves that when $m't m''$ is odd, the associated theta function is 0 at $\tau = 0$. On another hand, because of this formula, the square of $\Theta_m$ only depends on $m$ modulo $2\mathbb{Z}^4$. We have thus defined 16 functions of $\tau$. Six of those are zero, and the ten remaining ones correspond to the classes of $m$ modulo 2 that are called the even theta characteristics, listed here:

$$E = \{(0000), (0001), (0010), (0011), (0100), (0110), (1000), (1001), (1100), (1111)\}. \quad (1)$$

Recall [19, Theorem 5.2] that the ten even theta functions define an embedding

$$\psi : A_2(2) \longrightarrow \mathbb{P}^9 \\
\tau \longmapsto (x_m = \Theta_m^4(\tau))_{m \in E}$$

(2)
which induces an isomorphism between $A_2(2)^S$ and the subvariety of $\mathbb{P}^9$ (with coordinates indexed by $E$) defined by the linear equations

\begin{align*}
x_{1000} - x_{1100} + x_{1111} - x_{1001} &= 0 \quad (3) \\
x_{0000} - x_{0001} - x_{0110} - x_{1100} &= 0 \quad (4) \\
x_{0110} - x_{0010} - x_{1111} + x_{0011} &= 0 \quad (5) \\
x_{0100} - x_{0000} + x_{1001} + x_{0011} &= 0 \quad (6) \\
x_{0100} - x_{1100} + x_{0001} - x_{0010} &= 0 \quad (7)
\end{align*}

together with the quartic equation

\[ \left( \sum_{m \in E} x_m^2 \right)^2 - 4 \sum_{m \in E} x_m^4 = 0. \]  

(8)

**Remark 4** Following [7, p. 396 and 397], these equations can also be presented in a reduced form as defining a quartic in $\mathbb{P}^4$, for the reader who would prefer doing computations manually (which we do not, with the exception of the proof of Corollary 22). Our choice has been to use Magma throughout to keep pure discussion of the computations to the minimum (and ensure correctness).

Now, the theta functions enjoy a modularity transformation formula [7, p. 227], whose expression is the following:

\[ \Theta_m(M \cdot \tau) = \zeta(M) e(\phi_m(M^{-1})) \sqrt{jM(\tau)} \Theta_{m \odot M}(\tau), \]

where $\zeta(M)$ is an 8-th root of unit depending only on $M$, $\phi_m$ will be defined up to $i\pi/2\mathbb{Z}$ below, and

\[ m \odot \begin{pmatrix} A & B \\ C & D \end{pmatrix} := m \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} - (^tCA)_0, (^tDB)_0, \]  

(9)

where $A_0$ is the row vector formed by the diagonal coefficients of a square matrix $A$. We only care about the fourth powers of our theta functions, and $4\phi_m(M^{-1})$ can be made more explicit: we obtain

\[ x_m(M \cdot \tau) = \zeta(M)^4 (-1)^{^t(A_0)(C_0)}-JM(\tau)^2 x_{m \odot M}(\tau). \]  

(10)

The action of $\Gamma$ on $\mathcal{H}_2$ thus amounts via $\psi$ to an action of $\text{Sp}_4(\mathbb{F}_2)$ on $A_2(2)^S \subset \mathbb{P}^9$ via signed permutations of coordinates, in the shape

\[ (x_m^M)_{m \in E} = \left( (-1)^{^t(\langle B' \rangle_0)(C'D)_0}) x_{m \odot M} \right)_{m \in E}. \]  

(11)

Forgetting about the signs, the permutation of coordinates is induced by the action on $E \subset \mathbb{F}_2^4$ in (9).

This action is the key to understanding the combinatorics at play, so we first explain all its basic properties.

### 2.4 Properties of the dot action

We identify $E$ with its natural image inside $\mathbb{F}_2^4$.

**Lemma 5** The operation $(m, M) \mapsto m \odot M$ is indeed a group action of $\text{Sp}_4(\mathbb{F}_2)$ on $\mathbb{F}_2^4$, which furthermore preserves the quadratic form $q_2$ on $\mathbb{F}_2^4$ given by

\[ q_2((m', m'')) = (m')^t m''. \]
Consequently, this action stabilises $E$, as it is the set of isotropic vectors of $q_2$.

\textit{Proof} This is the content of [8, Propositions V.6.1 and V.6.3], where $q_2$ is denoted by $e$ and $m'$ and $m''$ are normalised with half-integer values. The curious reader can check it by hand using the definition of $\text{Sp}_4(\mathbb{F}_2)$.

\hfill $\square$

We can now study more finely this action, with definitions borrowed from Igusa again.

**Definition 6** For $x, y, z \in \mathbb{F}_2^4$, we define

$$e(x, y, z) = q_2(x) + q_2(y) + q_2(z) + q_2(x + y + z).$$

A triple of distinct $x, y, z \in E$ is then called \textit{syzygous} if $e(x, y, z) = 0$ (i.e. if $x + y + z \in E$) and \textit{azygous} otherwise.

A quadruple of distinct $x, y, z, t \in E$ is a \textit{Göpel quadruple} if every triple in it is syzygous, and an \textit{azygous quadruple} if every triple in it is azygous.

**Proposition 7** The $\odot$ action of $\text{Sp}_4(\mathbb{F}_2)$ restricted to $E$ has the following properties:

\begin{itemize}
  \item[(a)] It is 2-transitive.
  \item[(b)] It acts transitively on the 60 syzygous triples of $E$ (and also on the 60 azygous triples of $E$).
  \item[(c)] It acts transitively on the 15 Göpel quadruples of $E$ (and hence on their complements in $E$), and the 15 azygous quadruples of $E$.
\end{itemize}

\textit{Proof} First, the counting of triples and quadruples with the required properties can be done by hand or via Magma.

With different notations and wording, [8, Proposition V.6.2] states the following: for any two sequences $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ of $\mathbb{F}_2^4$, there exists $M \in \text{Sp}_4(\mathbb{F}_2)$ such that $x_i \odot M = y_i$ if and only if the subsequences which are affinely dependent have the same indices, and for any triples of distinct indices $i, j, k$, $q_2(x_i) = q_2(y_i)$ (and the same for $j, k$) and $e(x_i, x_j, x_k) = e(y_i, y_j, y_k)$.

For sequences of elements of $E$ (i.e isotropic vectors for $q$), for $k = 2$, it gives part (a) and for $k = 3$, it gives part (b) (affine independence is automatic for distinct triples in $\mathbb{F}_2^3$, as affine lines have only two elements).

Finally, the Göpel quadruples can be checked to be specific translates of maximal totally isotropic subspaces of $\mathbb{F}_2^4$ by elements of $E$; in particular they are automatically not affinely independent and we can use the $k = 4$ case of the property. \hfill $\square$

We now explain how much a given subset of $E$ can be expanded to one of those above.

**Lemma 8** \begin{itemize}
  \item[(a)] Any pair of distinct $x, y \in E$ can be completed into 4 syzygous triples, and 4 azygous triples.
  \item[(b)] Any syzygous triple can be completed into a unique Göpel quadruple, and any azygous triple can be completed into a unique azygous quadruple.
  \item[(c)] A syzygous triple is disjoint with exactly two Göpel quadruples, and an azygous quadruple is disjoint with exactly three Göpel quadruples.
  \item[(d)] No two Göpel quadruples are disjoint.
\end{itemize}

\textit{Proof} Item (a) is simply using that $z \mapsto e(x, y, z)$ is linear on $\mathbb{F}_2^4$ and non-zero when $x \neq y$. The structure of $E$ as set of isotropic vectors then imposes that there are as much
elements of $E$ with image 0 (completing $\{x, y\}$ to a syzygous triple) as there are with image 1 (completing $\{x, y\}$ to an azygous triple).

Items (b), (c) and (d) can be obtained by looking at a fixed triple or quadruple and using transitivity. 

**Remark 9** For a quick exploration of those triples and quadruples, the last section gives a list of all Göpel and azygous quadruples. The Magma code also verifies all the claims above.

### 2.5 Theta coordinates, type of abelian surface, and semistable reduction

First, the vanishing of theta coordinates indicates if the abelian surface is a jacobian or not. More precisely, we have the following.

**Theorem 10** Let $P = (A, \lambda, \alpha_2)$ a principally polarised abelian variety defined over a number field $K$ together with a symplectic 2-torsion basis such that the corresponding point $\psi(P) \in \mathbb{P}^5$ has coordinates in $K$. Then, $A[2](K) = A[2]$ and:

- If no coordinate is 0, there exists a curve $C$ defined over $K$ and of genus 2 such that $\text{Jac}(C)$ is isomorphic to $(A, \lambda)$ over an extension $K'/K$ of degree 2, and its six Weierstrass points are $K$-rational.
- Otherwise, exactly one coordinate is 0 and there exist two elliptic curves $E_1, E_2$ defined over $K$ such that $(A, \lambda)$ is isomorphic to $E_1 \times E_2$ over an extension $K'/K$ of degree 4.

**Proof** The first case relies mainly on Thomae’s formulæ [14, Chapter 6] expressing cross-ratios of fourth powers of theta constants of a jacobian in terms of the roots of the sextic defining a curve. This allows in turn to rebuild normal forms from those cross-ratios, for example with the Rosenhain normal form

$$C : \quad y^2 = x(x - 1)(x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$$

where

$$\lambda_1 = \frac{\Theta^2_{0000} \Theta^2_{0100}}{\Theta^2_{0001} \Theta^2_{0111}}, \quad \lambda_2 = \frac{\Theta^2_{0100} \Theta^2_{1100}}{\Theta^2_{0001} \Theta^2_{1111}}, \quad \lambda_3 = \frac{\Theta^2_{0000} \Theta^2_{1100}}{\Theta^2_{0001} \Theta^2_{1111}}$$

[3, Lemma 2.5 with notations of equation (2.5)]. These only express the parameters in terms of squares of theta constants, but in fact, using classical relations between them (or the equations of $A_2(2)$), one obtains that

$$\lambda_1 = \pm \frac{x_{1000} x_{1001} - x_{0000} x_{0001} - x_{0010} x_{0011}}{2 x_{0001} x_{0011}}.$$

The sign could be determined if necessary by complex analysis, and similar equations hold for $\lambda_2$ and $\lambda_3$. We thus obtain in this case a curve $C$ defined over $K$ such that $\text{Jac}(C) \cong (A, \lambda)$ (over $\overline{K}$) with the implicit principal polarisation canonically associated to $\text{Jac}(C)$, and $\text{Jac}(C)$ is fully defined over $K$ (as the Weierstrass points of $C$, $\{0, 1, \infty, \lambda_1, \lambda_2, \lambda_3\}$ are). After a good permutation of the Weierstrass points (which amounts exactly to the dot action), we ensure that there is a basis $\beta_2$ of the two-torsion on $\text{Jac}(C)$ such that $\psi(P) = \psi((\text{Jac}(C), \beta_2))$, which implies that the two triples are isomorphic over $\overline{K}$.

Now, an automorphism of $\text{Jac}(C)$ fixing the polarisation and the full 2-torsion comes by Torelli’s theorem from an automorphism of $C$ fixing pointwise the Weierstrass
points, and such an automorphism is necessary trivial or the hyperelliptic involution, so by descent \((A, \lambda, \alpha_2)\) and \((\text{Jac}(C), \beta_2)\) are isomorphic over a quadratic extension \(K'\) of \(K\).

- Assume now that at least one coordinate is 0. By [16], one then knows that over \(K\), \((A, \lambda)\) is isomorphic to a product of elliptic curves with the product polarisation. After a permutation by an element of \(\text{Sp}_4(\mathbb{Z})\), one can thus assume that \((A, \lambda, \alpha_2)\) is represented in \(\mathcal{H}_2\) by a diagonal matrix 
\[
\tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}.
\]
Then the coordinates split because 
\[
\Theta_{11}(\tau_1) = \sum_{n \in \mathbb{Z}} \exp(i\pi(n + a/2)^2\tau_1 + i\pi(n + a/2)b),
\]
so we fall back to four possible one-dimensional theta functions. One of them \((\Theta_{11})\) is always 0, and the other three do not vanish on the Poincaré half plane. Apart from the coordinate \((1111)\), we are thus looking (up to permutation of coordinates) at the Segre embedding \(\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8\), where in each \(\mathbb{P}^2\) the coordinates are the three fourth powers of non-zero theta constants for respectively \(\tau_1\) and \(\tau_2\).

We can thus assume (after renormalisation) that each \(\Theta_{ab}(\tau_1)^4\) (\(ab \in \{(00), (01), (10)\}, \ i \in \{1, 2\}\)) belongs to \(K\). Now, the \(j\)-invariant of the elliptic curve associated to \(\tau_1\) is a rational function of the three fourth powers of theta constants with rational coefficients [2, p. 29], so we can find elliptic curves \(E_1\) and \(E_2\) defined over \(K\) and such that \((A, \lambda) \cong E_1 \times E_2\) over \(K\) with the product polarisation. An automorphism of \(E_1 \times E_2\) preserving point wise the 2-torsion necessarily belongs to \(\{\pm 1\}^2\) so by descent again, there exists an extension \(K'/K\) of degree 4 such that \((A, \lambda, \alpha_2)\) is isomorphic over \(K'\) to \(E_1 \times E_2\).

\[\blacksquare\]

Now, this characterisation of the type of abelian surface of theta coordinates extends to every field of characteristic \(\neq 2\), because theta constants can be intrinsically defined as algebraic theta constants (by Mumford’s theory of theta functions), and are compatible with reduction outside of characteristic 2. This leads to the following result; c.f. [12, Proposition 8.4], which also deals with the case of reduction to a product of elliptic curves.

**Proposition 11** Let \(K\) be a number field and \(\mathfrak{P}\) a maximal ideal of \(\mathcal{O}_K\) of residue field \(k(\mathfrak{P})\) with \(\text{char}(k(\mathfrak{P})) \neq 2\). Let \(P = (A, \lambda, \alpha_2) \in A_2(2)(K)\). Then, \(\psi(P) \in \mathbb{P}^9(K)\) and if the semistable reduction of \(A\) modulo \(\mathfrak{P}\) is a jacobian of hyperelliptic curve, the reduction of \(\psi(P)\) modulo \(\mathfrak{P}\) has no zero coordinate, in other words every coordinate of \(\psi(P)\) has the same \(\mathfrak{P}\)-adic norm.

For the places above 2, the situation is a bit more complicated (in part because there is no good theory of algebraic theta constants in characteristic two), but using Igusa invariants, we can say the following.

**Proposition 12** Let \(K\) be a number field and \(\mathfrak{P}\) a maximal ideal of \(\mathcal{O}_K\) above 2, and \(P = (A, \lambda, \alpha_2) \in A_2(2)(K)\) as in the previous proposition.
(a) If the semistable reduction of $A$ modulo $\mathfrak{P}$ is a jacobian of hyperelliptic curve, the coordinates of the reduction of $\psi(P)$ modulo $\mathfrak{P}$ all satisfy
\[ |x_m|_\mathfrak{P} \geq |2|_\mathfrak{P}^6 \cdot \max_{m' \in E} |x_{m'}|_\mathfrak{P} \]
(b) In all cases, the coordinates satisfy
\[ |x_m|_\mathfrak{P} = \max_{m' \in E} |x_{m'}|_\mathfrak{P} \]
with at most 6 exceptions $m \in E$.

**Remark 13** An important point is that, although crude, part (b) only relies on the explicit equations in $\mathbb{P}^9$. Consequently, we can and will use it as a go-to estimate every time one does not have any better option.

**Proof** Part (a) is in [12, Proposition 8.7]. For (b), suppose this is not true. After normalisation to coordinates in $\mathcal{O}_K$, we can assume that at least one of them is invertible and seven have positive valuation. We consider $A_2(2)$ as a scheme over $\mathbb{Z}$ and suppose that 7 coordinates vanish. These 7 coordinates have indices ranging over the complement of a syzygous or an azygous triple, so it suffices to look at one explicit complement of a syzygous triple and one complement of an azygous triple. We first suppose that the variables $x_m$ for $m$ in the complement of the syzygous triple $\{(1001), (0100), (1111)\}$ all vanish. Then the equations between the $x_m$'s directly imply that $x_m = 0$ for all $m \in E$ (and this holds over $\mathbb{Z}$), a contradiction. Next, we consider the complement of the azygous triple $\{(0000), (0100), (0001)\}$. Now the equations imply that the ideal generated by these seven $x_m$'s contains
\[ x_1 + x_5, x_2 + x_5, x_3, x_4, 2x_5, x_6^2, x_7, x_8, x_9, x_{10}. \]
Here $x_i$ is $x_m$ where $m$ is the $i$th element of $E$ as displayed in (1). In particular, the radical of the ideal contains all coordinates, so all of them vanish, also a contradiction. Finally, we base change this argument to the residue field of $\mathcal{O}_K$ at $\mathfrak{P}$. \qed

**Remark 14** When we find 6 simultaneously vanishing coordinates modulo $\mathfrak{P}$, their indices are not random: they will turn out to form the complement subset to a Göpel quadruple (Definition 6), as proven in the next section.

### 2.6 The graph of intersection of the divisors

We can now figure out precisely what is the graph of intersection of our divisors. Recall that it is defined as follows, following [13, Section 5].

**Definition 15** The vertices of the graph of intersection are non-empty set-theoretic intersections
\[ Z_I := \bigcap_{i \in I} D_i(\mathbb{K}) \text{ for } I \subset E. \]
A set of indices $I$ is called optimal if there is no set $J \supset I$ such that $Z_I = Z_J$. The depth of a vertex $Z$ is defined to be the size of an optimal set $I$ such that $Z = Z_I$. An edge goes from $Z_I$ to $Z_J$ if $Z_I \subsetneq Z_J$ (equivalently, if $I$ and $J$ are optimal and $I \subset J$) with no intermediary intersection. Finally, the cone of ancestors of a vertex $Z$ is the set of vertices $Z'$ from which starts a path leading to $Z$. 
In our situation, with help of the equations, we obtain the following graph of intersection. The number in each oval is the number of vertices of a given depth (and to which type of optimal subset of $E$ they correspond). The number in each thick arrow corresponds, for each vertex above, to its number of children below.

| Dimension of $Z_I$, irreducible? | Graph of intersection | Depth |
|----------------------------------|------------------------|-------|
| 2, yes                           | Singletons (10)         | 1     |
| 1, no                            | Pairs (45)              | 2     |
| 0, no                            | Syzygous triples (60)   | 3     |
| 1, yes                           | Azygous quadruples (15) | 4     |
| 0, yes                           | Complements of Göpel (15)| 6    |

**Proof of graph of intersection** To build the graph of intersection, we start with singletons and then add elements step by step.

First, note that transitivity of the dot action of $\text{Sp}_4(\mathbb{F}_2)$ on each of the subsets of $E$ displayed in the ovals (and therefore on the corresponding sets of vanishing coordinates because of (11)), detailed in Proposition 7, allows us to reduce to a single singleton, pair, syzygous triple, azygous quadruple or complement of Göpel quadruple and thus saves us a lot of work.

At each step, we determine whether $Z_I$ is optimal. To prove that a set $I$ is not optimal, we formally manipulate the given equations for $A_2(2)$ together with $x_i = 0$ for $i \in I$ to obtain that $x_j = 0$ for some $j \notin I$. Similarly, we extract the dimension and number of irreducible components from the explicit equations.
Conversely, to prove that a set $I$ is optimal, we exhibit for each $j \notin I$ a point $P_j \in Z_I$ such that $P_j$ has non-zero $j$th coordinate. Such points can always be found as deepest points: for each complement $I$ of a Göpel quadruple there is a unique point $P \in A_2(2)$, all of whose coordinates are in $\{0, 1, -1\}$, such that $x_i(P) = 0$ if and only if $i \in I$.

Finally, we use Lemma 8 to determine the number of children displayed in the arrows. This process is rather laborious and error-prone to do by hand, so we have implemented it in Magma.

Remark 16 By formal computations, one can notice that this process would give exactly the same result for these equations over any base field (finite or not) of characteristic unequal to 2 and 3. In particular, reductions of divisors do not intersect more than the divisors over $\mathbb{Q}$ (but to be precise, the scheme-theoretic intersections are sometimes not reduced) except over $\mathbb{F}_2$ and $\mathbb{F}_3$. We will not need this, but have nonetheless worked it out in the Magma file. This phenomenon is implicit in the bounds of Proposition 20 below and its proof.

Remark 17 Even though we have not undertaken the verification of this claim and we do not need it later, it is likely that the 5 types of optimal subsets are closely related to the moduli interpretations of points of $A_2(2)^S$ (as jacobians of stable curves of genus 2). More precisely, following the notations of [15, Proposition 1] and taking into account that their compactification is a blow-up of the Satake compactification, we can expect that singletons are given by products of elliptic curves ($N$, type II), pairs by elliptic curves ($B$, type III), syzygous triples and azygous quadruples by one or two rational curves ($C$, type IV) and complements of Göpel quadruples by two rational curves meeting at three points ($D$, type V).

3 Runge’s method refined

We apply the formalism of the graph of intersection to Runge’s method. Then [13, Proposition 5.5] tells us that if $|S| \leq 2$, one can obtain an explicit bound on the height of points in $(A_2(2)^S\setminus D)(\mathcal{O}_{K,S})$. Let us explain why.

The shape of the graph of intersection tells us the following: no union of two cones of ancestors recovers all the graph in depth 1 (i.e. contains all the divisors). This is due to the fact that any two Göpel quadruples have non-empty intersection, so the union of two complements of Göpel quadruples cannot be the full set $E$.

To exploit this fact, we need to define a consistent notion of $v$-closeness to the intersections of divisors $Z_I$ (for every place $v \in M_K$). For example, if $K$ is a number field and $v$ is a finite place of $K$ not above 2 or 3, then $P \in A_2(2)(K)$ will be $v$-close to $Z_I$ if and only if the reduction of $P$ mod $v$ belongs to the Zariski closure of $Z_I$.

As a consequence, for any integral point $P \in (A_2(2)^S\setminus D)(\mathcal{O}_{K,S})$ where $|S| = 2$, there are at most two places $v \in M_K$ for which $P$ is $v$-close to one of the divisors, and thus generates a cone of ancestors. Taking away those two cones of ancestors, there remains a divisor $D_i$ which is $v$-far from $P$ for all $v \in M_K$, and thus allows to bound the local heights $h_{D_i,v}(P)$ for all $v$, and therefore the global height $h_{D_i}(P)$. This will be particularly easy to do here as the $D_i$ are given as coordinate hyperplanes. This is how we obtain an absolute bound on the height of $\psi(P)$.

To obtain such a bound in practice, more refined estimates are needed for three different reasons:
• Our definition of integral points comes from the moduli space structure (and not the explicit equations), which makes a slight difference in the bounds.

• The graph of intersection of the divisors is different over fields of characteristic 2 or 3 (which tells us that even though $Z_I$s are distinct, they might still be close enough to need a finer definition of closeness to distinguish them).

• We need to evaluate closeness in the archimedean case.

3.1 Estimates on the size of theta functions

In this subsection, we refine the estimates in [12, Proposition 8.5] on sizes of theta functions. For archimedean places and places above 2 and 3, this will provide a quantitative analogue for the part of the graph of intersection that we will need, while at other finite places it is merely a confirmation of what we already knew.

Instead of analysing the Fourier expansions of the theta functions as was done by Streng in [18] and quoted in [12], we only make use of the six equations satisfied by the fourth powers of the theta functions to obtain our estimates.

**Proposition 18** Consider $\tau \in \mathcal{H}_2$, and suppose that $K$ is a number field such that $x_m \in K$ for each $m \in E$. Let $|\cdot|$ be any norm on $K$. The set of $m \in E$ satisfying

$$|x_m| < \begin{cases} \max_{m' \in E} |x_{m'}| & \text{if } |\cdot| \text{ non-archimedean} \\ \frac{1}{27} \max_{m' \in E} |x_{m'}| & \text{if } |\cdot| \text{ archimedean} \end{cases}$$

either has size at most 4, or is contained in one of the 15 complements of Göpel quadruples.

**Remark 19** This constant $1/27 = 0.037\ldots$ is a slight improvement on the constant $0.424 = 0.424\ldots$ found by Le Fourn [12] based on Streng’s estimates [18].

**Proof** In the non-archimedean case, this can be verified explicitly by considering $A_2(2)$ over $\mathbb{Z}$, as in the proof of Proposition 12 (c). Alternatively one can reason along the lines of the below proof for archimedean norms. We thus assume that $|\cdot|$ is an archimedean norm. We take large rather than small theta functions as our point of view, showing that there are at least four $x \in \{x_m \mid m \in E\}$ of size at least $\frac{1}{27} \max_m |x_m|$ and, whenever there are at most five $x$ of size $\geq \frac{1}{27} \max_m |x_m|$ then these contain a Göpel quadruple.

One of the $x_m$ is the largest, say of size $M := \max_m |x_m|$. By considering a linear equation featuring $x_m$, we find a second $x$-coordinate of size at least $M/3$. By transitivity of $\text{Sp}_4(\mathbb{F}_2)$ on pairs, we may assume this pair of large $x$-coordinates is $\{x_{0000}, x_{0010}\}$. In fact, we may suppose that $|x_{0000}| = M$ and $|x_{0010}| \geq M/3$.

So Eqs. (4)–(7) all contain one $x_m$ of size at least $M/3$, and hence a second $x_m$ of size at least $M/9$. Note that $x_{0110}, x_{0010}, x_{0001}, x_{0011}$ are the four (out of 8 remaining variables) that occur in two of those equations. These are exactly the four variables extending $\{x_{0000}, x_{0010}\}$ to a syzygous triple. By transitivity of the action of $\text{Sp}_4(\mathbb{F}_2)$ on syzygous triples, these choices are thus equivalent. Let us first assume that all four of those are in absolute value $< M/9$. These four together form a Göpel quadruple. Then Eqs. (4)–(7) show that $|x_{1100}|, |x_{1111}|, |x_{1011}|, |x_{1000}| \geq M/9$, thus yielding a total of eight $x_m$ of size $|x_m| \geq M/9$.

We may thus assume that $|x_{0100}| \geq M/9$, giving a syzygous triple of large coordinates. We can use Eqs. (4) and (5) to find more “large” $x_m$. In particular, looking at (5) we have three cases: $|x_{0110}| \geq M/9$ (case (i)), $|x_{0011}| \geq M/9$ (case (ii)) and $|x_{1111}| \geq M/9$ (case
(iii). Consider first case (i). If $|x_{0110}| \geq M/9$, then we have found four coordinates of size $M/9$, namely $\{x_{0000}, x_{0010}, x_{0100}, x_{0110}\}$, and we check that this is a Göpel quadruple: the unique Göpel quadruple extending our syzygous triple.

Next, consider case (ii), so $|x_{0111}| \geq M/9$. Then in Eq. (4), one of $|x_{0001}|, |x_{0110}|, |x_{1100}|$ is $\geq M/9$. The second of these is the case just treated. The first gives us exactly that the variables in the Göpel quadruple $\{x_{0000}, x_{0001}, x_{0010}, x_{0100}\}$ are all at least $M/9$ in absolute value. This is the second (out of two in total) Göpel quadruple containing the pair $\{x_{0000}, x_{0010}\}$. In the third case, if $|x_{1100}| \geq M/9$, then from Eq. (3) we find that a sixth coordinate must be at least $M/27$ in absolute value.

Finally, we consider case (iii), where $|x_{1111}| \geq M/9$. In Eq. (4), we now again have three possibilities, of which $|x_{0110}| \geq M/9$ has already been treated, and the case $|x_{0001}| \geq M/9$ yields six theta functions of size at least $M/27$ by considering Eq. (3).

This leaves $|x_{1100}| \geq M/9$. The 5-set of “large” $x_m$’s we have selected so far does not contain a Göpel quadruple. Moreover, the linear equations appear to be perfectly happy with the sizes of the variables: each equation has two large variables. It is here that we must invoke the power of the degree 4 equation. Assume that all remaining 5 variables have size strictly smaller than $\epsilon M$. Then $|x_{1100} - x_{0000}|, |x_{0100} - x_{0000}| \leq 2\epsilon M$ by Eqs. (4) and (6) and $|x_{1111} - x_{0000}|, |x_{0010} - x_{0000}| \leq 5\epsilon M$ by substituting Eqs. (4) and (6) into Eqs. (3) and (7) respectively. So in the quartic equation (8) we may replace each such $x$ by $x_{0000}$, at the expense of adding a term of size $\leq 5\epsilon M$ or $\leq 2\epsilon M$. This gives

$$5x_{0000}^4 + C = 0,$$

where $|C| \leq (6543\epsilon^4 + 5656\epsilon^3 + \epsilon^2 + 56\epsilon)|x_{0000}|^4$

by the triangle inequality. Now $\epsilon \leq 1/27$ yields $|C| < 5|x_{0000}|^4$, a contradiction. \hfill\Box

**Proposition 20** Let $| \cdot |$ be a non-archimedean norm. Then for each Göpel quadruple $Q$, one $x \in \{x_m \mid m \in Q\}$ must satisfy

$$|x| \geq |2||3| \max_{i \in E} |x_i|.$$

When $| \cdot |$ is archimedean, the same is true with $|2||3|$ replaced by 0.051.

**Remark 21** These factors of $|2|$ and $|3|$ are strictly necessary. Indeed, the subscheme of $A_2(2)/\mathbb{F}_2$ given by the vanishing of the variables indexed by the Göpel quadruple $\{(0000), (0001), (0010), (0011)\}$ is zero-dimensional and contains the point $\{(0 : 0 : 0 : 0 : 1 : 1 : 1 : 1 : 1 : 1)\}$. Similarly, this scheme over $\mathbb{F}_3$ contains $\{(0 : 0 : 0 : 0 : 1 : 1 : 1 : 1 : 1 : 1)\}$.

**Proof** Let $| \cdot |$ be a norm. When $| \cdot |$ is archimedean, suppose that $|x_i| < \epsilon M$ for each $i$ in the Göpel quadruple $G := \{x_{0000}, x_{0001}, x_{0010}, x_{0100}\}$, where $M = \max_{m \in E} |x_m|$ and $\epsilon \leq 1$. We write $o(z)$ for any complex number of size $|o(z)| < \epsilon$. When $| \cdot |$ is non-archimedean, we may assume after scaling that $x_i \in O_K$ for all $i \in E$ and one $x_i$ equals 1. Now suppose for each $i$ in this Göpel quadruple that $x_i \equiv 0 \mod \pi^n$, where $n > 0$ and $\pi$ is a uniformiser. From (4)–(7) we deduce for archimedean norms that

$$-x_{1100} = x_{0110} + o(2\epsilon M), \quad x_{0110} = x_{1111} + o(2\epsilon M)$$

and

$$-x_{1001} = x_{0010} + o(2\epsilon M), \quad x_{0010} = x_{1000} + o(2\epsilon M).$$
Substituting the above into (3) yields

\[ 2x_{0100} + 2x_{0110} = 2o(4\epsilon M). \]  

(14)

This implies that

\[ x_{1000}, x_{1001} = x_{0100} + o(2\epsilon), \quad x_{0110} = x_{0100} + o(4\epsilon), \quad x_{1100}, x_{1111} = x_{0100} + o(6\epsilon). \]

In particular, all six \( x_i \) for \( i \notin G \) are of similar size. When \( \epsilon \) is sufficiently small, this will contradict the quartic equation (8).

Since \( \epsilon \leq 1 \), we must have \( |x_i| = M \neq 0 \) for some \( i \in E \setminus G \). In what follows, the two choices \( i = 0100 \) and \( i = 0110 \) will be equivalent, and so will the other four choices \( i \in \{1000, 1001, 1100, 1111\} \).

For non-archimedean norms all error terms have equal size so all choices are equivalent. Hence we may and do assume that \( x_{0100} = 1 \). We obtain that \( x_i \equiv x_{0100} \mod \pi \) for all \( i \notin G \). Substituting equations (12), (13) and (14) into the degree 4 equation (8), we obtain

\[ 12x_{0100}^4 + C = 0, \quad |C| \leq (1712\epsilon^4 + 2880\epsilon^3 + 528\epsilon^2 + 96\epsilon)|x_{0100}^4| \]

for archimedean norms. With \( \epsilon \leq 0.077 \) we obtain a contradiction, unless \( x_{0100} = 0 \). In that case, we can do the same computation with \( x_{1000} \) in place of \( x_{0100} \), and there a contradiction when \( \epsilon \leq 0.051 \).

In the non-archimedean case, the degree 4 equation yields \( 12 \equiv 0 \mod \pi^n \) when \( p \neq 2 \). This is a contradiction unless \( p = 3 \) and \( |\pi^n| \leq |3| \). When \( p = 2 \), we note that \( \epsilon \equiv 0 \mod 2^n \) implies that \( (x_{0100} + \epsilon)^2 \equiv x_{0100}^2 \mod 2^{n+1} \). Hence, \( |\pi^n| \geq |4| \) actually gives \( 12 \equiv 0 \mod 8 \), also a contradiction. \( \square \)

**Corollary 22** Let \( v_1, v_2 \) be two places of a number field \( K \), and \( \psi(P) = (x_m)_{m \in E} \in \mathbb{P}^9(K) \) without a zero coordinate. There exists at least one \( m_0 \in E \) such that \( x = x_{m_0} \) satisfies

\[ |x|_v \geq \max_{m \in E} |x_m| \cdot \begin{cases} 2|3| \text{ if } v \text{ finite and } \\ 1/27 \text{ when } v \text{ infinite} \end{cases} \]  

(15)

for each \( v \in \{v_1, v_2\} \).

**Proof** If (15) is violated by at most four \( x \)-coordinates at each of the two places, then we are free to choose any of the remaining two. Otherwise, there is one \( v \in \{v_1, v_2\} \) such that (15) is violated by the \( x \)-coordinates in a set \( T \subset \{x_m \mid m \in E\} \) of size \( |T| \geq 5 \). In that case, Proposition 18 tells us that \( T \) is contained in the complement \( T_6 \) of a Göpel quadruple. In particular, each \( x \) in the Göpel quadruple \( E \setminus T_6 \) satisfies (15) at \( v \). Now Proposition 20 tells us that one such \( x \) also satisfies (15) at the other place. We note that \( 1/27 \leq 0.051 \). \( \square \)

### 3.2 Proofs of Theorem 1 and Corollary 2

After all this preparatory work, we can finally prove an upper bound on the height of our integral points considered. We prove the following slightly more specific version of Theorem 1 using Runge’s method.
Theorem 23  Let $P = (A, \lambda, \alpha) \in A_2(2)(K)$ representing a triple such that the full 2-torsion $\alpha$ is defined over $K$, and such that the semistable reduction of $A$ is the Jacobian of a smooth curve, except at most at 2 places (including necessarily the archimedean ones). We then have

$$h(\psi(P)) \leq 8.6 \text{ and } h_\mathcal{F}(A) \leq 985,$$

where $h_\mathcal{F}$ is the stable Faltings height of $A$.

Proof  Let $S$ be the set of places including $M_\infty$ and the finite places $v$ such that the semistable reduction of $A$ modulo $v$ is not isomorphic to a Jacobian. By assumption, $|S| \leq 2$. Furthermore, $A$ is necessarily a Jacobian of hyperelliptic curve (after possible extension, see Theorem 10). The coordinates mentioned below refer to the ten coordinates of $\psi(P)$, normalised to belong to $K$.

For the places $v \notin S$ and not dividing 2, all the coordinates have the same valuation by Proposition 11 (a). For the places $v \notin S$ above 2, the smallest possible ratio $|x_i|_v/|x_j|_v$ of coordinates is $|2|_v$ by Proposition 12.

For the (at most two) places of $S$, one can choose by Corollary 22 an index $i \in E$ such that $|x_i|_v \geq C_v \max_{j \in E} |x_j|_v$ with $C_v = |2|_v, |3|_v$ or $1/27$. We keep this choice of $i$. We thus have

$$h(\psi(P)) = \frac{1}{|K : Q|} \sum_{v \in M_K} n_v \log \left( \max_{j \in E} \frac{|x_j|_v}{|x_i|_v} \right)$$

$$\leq \log(27) + 6 \log(2) + \log(3) \leq 8.6.$$

The bound $\log(27)$ comes from the contribution of archimedean places, while $6 \log(2)$ comes from the places above 2 (if $S$ contains a place above 2, the bound obtained is smaller) and $\log(3)$ appears if a place above 3 belongs to $S$. The other places do not contribute. We deduce the bound on the Faltings height by [17, Corollary 1.3], taking into account that $g = r = 2$ here and with his notations, $h_\mathcal{F}(A, L) = \frac{h_\psi(P)}{4}$. $\square$

Remark 24  This proof was conceptualised in the context of Runge’s method for varieties, but an alternative approach was possible to prove a version of this theorem. Indeed, with the same notations and hypotheses for $A$ and $S$, one can exhibit a curve $C$ in Rosenhain normal form such that $\text{Jac}(C) \cong A$ (see the proof of Theorem 10). Its parameters $\lambda_1, \lambda_2$ and $\lambda_3$, belonging to $K$, will have $v$-adic valuation 1 for all $v \notin S$ and not dividing 2, because their squares are cross-ratios of theta constants, using Proposition 11. In fact, the six Weierstrass points $\{0, 1, \infty, \lambda_1, \lambda_2, \lambda_3\}$ will also be distinct modulo $v$ for those $v$ (e.g. because their differences can also be written as Rosenhain parameters and cross-ratios of fourth powers of theta constants), so in particular the $\lambda_i$ are $v$-integral in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ for all $v$ outside $S$ (in other words, solutions of the unit equation). These coefficients thus satisfy the hypotheses of Runge’s method on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and one can bound their height. They then allow to determine back the coordinates of $\psi(P)$ via [3, Lemma 2.5], which finally bounds the height of $\psi(P)$.

This approach would work, but we would need to deal with similar complications (such as what happens modulo 2, where no standard Rosenhain form exists), which ultimately boils down to using theta constants again, and it is not clear it would give better bounds than the one we found, so we decided to present the results via the graph of intersection.
Proof of Corollary 2. Suppose that \( P = (x_1 : \ldots : x_{10}) \in A_2(2)(\mathbb{Q}) \) corresponds to the jacobian of a hyperelliptic curve with potentially bad reduction at a single prime \( p > 2 \). We will apply Theorem 23 with \( K = \mathbb{Q} \) and \( S = \{\infty, p\} \). First, by Proposition 11, we can (and do) scale the \( x_i \) such that \( x_1, \ldots, x_{10} \in \mathbb{Z} \) and at most one of them satisfies \( v_p(x_i) > 0 \). By transitivity of the \( \text{Sp}_4(\mathbb{F}_2) \)-action, we may suppose this is \( x_5 \). Similarly, after scaling each \( x_i \) by a power of \( \ell \), we have \( v_\ell(x_i) = 0 \) for all primes \( \ell \not\in \{2, p\} \). We also scale by a power of 2 such that one \( x_i \) satisfies \( v_2(x_i) = 0 \). By Proposition 12 (b), at least four \( x_i \) now satisfy \( v_2(x_i) = 0 \), and we may suppose by 2-transitivity of the \( \text{Sp}_4(\mathbb{F}_2) \)-action that one of them is \( x_6 \). Now \( x_6 = \pm 1 \). The height bound \( \log(2^6 \cdot 3 \cdot 27) \) from Theorem 23 now implies that \( x_5 \) is divisible by a power of \( p \) of size at most \( 2^6 \cdot 3 \cdot 27 \). Finally, by Proposition 12 (a), we find (after possibly scaling further by a minus sign) that

(i) \( x_6 = 1 \),
(ii) \( x_5 = q \cdot x_6^2 \), where \( 1 \leq q \leq 2^6 \cdot 3 \cdot 27 \) is a prime power and
(iii) \( x_1, x_2, x_3, x_4, x_5^2, x_7, x_8, x_9, x_{10} \in \{\pm 1, \pm 2, \pm 2^2, \ldots, \pm 2^6\} \).

Now a priori there appear to be too many possibilities to check by computer, but recall that the \( x_i \) satisfy a bunch of linear relations. In fact, following Igusa [7, p 396, 397], we define the rational map

\[
\phi : \mathbb{P}^9 \rightarrow \mathbb{P}^4, \quad (x_1 : \ldots : x_{10}) \mapsto (x_6 : x_5 : x_1 : -x_6 - x_7 : -x_6 - x_9),
\]

mapping \( A_2(2) \) isomorphically onto the threefold \( Y \subset \mathbb{P}^4 \) defined by

\[
y_1^2y_2^2 - 2y_1y_2y_3 + 3 + 2y_3y_4^2 - 2y_1y_2y_3^2 + y_2^2y_3^2 - 4y_1y_2y_3y_4 - 4y_1y_2y_3y_5 - 2y_1y_3y_4y_5 - 2y_2y_3y_4y_5 + y_3^2y_5^2 = 0.
\]

We now search for solutions \( (y_1 : \ldots : y_5) \in Y(\mathbb{Q}) \) satisfying \( y_1 = 1, y_2 = q \cdot y_6^2 \) where \( 1 \leq q \leq 2^6 \cdot 3 \cdot 27 \) is a prime power and \( y_2, y_3, y_4 + 1, y_5 + 1 \in \{\pm 1, \pm 2, \ldots, \pm 2^6\} \). This amounts to evaluating the quartic polynomial defining \( Y \) at a total of 27,736,352 values \( (y_1, \ldots, y_5) \), which is sufficiently small to do on a computer in a matter of minutes. The inverse of \( \phi \) is given by \( \psi : (y_1 : \ldots : y_5) \mapsto (y_3 : y_3 + y_5 : y_1 + y_2 + y_3 + y_4 + y_5 : y_3 + y_4 : y_2 : y_1 : -y_1 - y_4 : -y_2 - y_4 : -y_1 - y_5 : -y_2 - y_5) \). We apply \( \psi \) to each solution, remove those \( (x_1 : \ldots : x_{10}) \) with a zero coordinate (they correspond to boundary points) and check whether (iii) is satisfied. This leaves only two options, which are \( \text{Sp}_4(\mathbb{F}_2) \)-equivalent. We thus find only one possible hyperelliptic curve, corresponding to the point

\[
P = (-4 : 1 : -4 : 1 : -9 : 1 : -4 : 4 : -4 : 4).
\]

By Thomae’s formulae used in the proof of Theorem 10, we find that \( P \) corresponds to a hyperelliptic curve \( C : y^2 = (x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3) \) satisfying \( \lambda_1^2 = 16, \lambda_2^2 = 4 \) and \( \lambda_3^2 = 4 \). This yields 2 non-isomorphic hyperelliptic curves. Using the genus2reduction function in \text{sage} for genus 2 hyperelliptic curves, we find that one of the two has potentially bad reduction at 5, so it cannot correspond to \( P \). The other curve,

\[
C : y^2 = x(x-1)(x^2-4)(x-4),
\]

must therefore correspond to the point \( P \). Indeed, as predicted by Proposition 11, \( C \) has potentially good reduction at all primes \( p > 3 \) and does not have potentially good reduction at 3. However, we find that \( C \) also does not have potentially good reduction at 2, leaving us with no solutions.
Finally, we need to consider \( p = 2 \). Now we need to drop the assumption that the valuation at 2 of the coordinates is at most 6. In return, we obtain from the proof of the above theorem a smaller height bound: \( \log(2 \cdot 3 \cdot 27) \). Since \( 2^8 > 2 \cdot 3 \cdot 27 \), we may now assume that \((x_1: \ldots : x_{10}) \in A_2(2)(\mathbb{Q})\) satisfies

(i) \( x_6 = 1 \)
(ii) \( x_1, x_2, x_3, x_4, x_7, x_8, x_9, x_{10} \in \{ \pm 1, \pm 2, \ldots, \pm 2^7 \} \).

This is an even faster computation, and we find no solutions. \( \square \)

4 List of Göpel and azygous quadruples

Göpel quadruples:

\[
\begin{align*}
\{(0011), (0010), (1001), (1000)\}, \\
\{(0011), (0001), (0110), (0100)\}, \\
\{(0000), (0110), (1111), (1001)\}, \\
\{(1100), (0010), (0110), (1000)\}, \\
\{(1100), (0011), (0000), (1111)\}, \\
\{(0001), (0110), (1111), (1000)\}, \\
\{(0010), (0000), (0110), (0100)\}, \\
\{(0011), (1111), (1000), (0100)\}.
\end{align*}
\]

Azygous quadruples:

\[
\begin{align*}
\{(0011), (0010), (0110), (1111)\}, \\
\{(1100), (0001), (0000), (0110)\}, \\
\{(0110), (1001), (0000), (0100)\}, \\
\{(1100), (0110), (1111), (0100)\}, \\
\{(0001), (0010), (0100), (1000)\}, \\
\{(0011), (0000), (0100), (0000)\}, \\
\{(1100), (1111), (1000), (0100)\}, \\
\{(1100), (0010), (0000), (1000)\}.
\end{align*}
\]

Supplementary Information
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Data availability
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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