ON THE INVERSE BRAID AND REFLECTION MONOIDS OF TYPE B

V. V. VERSHININ

Abstract. There are well known relations between braid groups and symmetric groups, between Artin-Briskorn braid groups and Coxeter groups. Inverse braid monoid the same way is related to the inverse symmetric monoid. In the paper we show that similar relations exist between the inverse braid monoid of type B and the inverse reflection monoid of type B. This gives a presentation of the last monoid.

CONTENTS

1. Introduction 1
2. Inverse braid monoid and type B 2
References 7

1. Introduction

Let $V$ be a finite dimensional real vector space ($\dim V = n$) with Euclidean structure. Let $W$ be a finite subgroup of $GL(V)$ generated by reflections. We suppose that $W$ is essential, i.e. that the set of fixed vectors by the action of $W$ consists only of zero: $V^W = 0$. Let $\mathcal{M}$ be the set of hyperplanes such that $W$ is generated by orthogonal reflections with respect to $M \in \mathcal{M}$. We suppose that for every $w \in W$ and every hyperplane $M \in \mathcal{M}$ the hyperplane $w(M)$ belongs to $\mathcal{M}$.

Consider the complexification $V_C$ of the space $V$ and the complexification $M_C$ of $M \in \mathcal{M}$. Let $Y_W = V_C - \bigcup_{M \in \mathcal{M}} M_C$. The group $W$ acts freely on $Y_W$. Let $X_W = Y_W/W$ then $Y_W$ is a covering over $X_W$ corresponding to the group $W$.

This generalized braid group $Br(W)$ corresponding to the Coxeter group $W$ is defined as the fundamental group of the space $X_W$ of regular orbits of the action of $W$ and the corresponding pure braid group $P(W)$ is defined as the fundamental group of the space $Y_W$. So, for the generalized braid groups is $Br(W) = \pi_1(X_W)$, $P(W) = \pi_1(Y_W)$. The groups $Br(W)$ were defined by E. Brieskorn [3], and are called also as Artin–Brieskorn groups. E. Brieskorn [3] and P. Deligne [4] proved that the spaces $X_W$ and $Y_W$ are of the type $K(\pi, 1)$.

The covering which corresponds to the action of $W$ on $Y_W$ gives rise to the exact sequence

$$1 \to \pi_1(Y_W) \xrightarrow{\rho} \pi_1(X_W) \to W \to 1.$$ 

So, there is a naturally defined map $\rho : Br(W) \to W$.

Geometrical braid, as a system of $n$ curves in $\mathbb{R}^3$ lead to the notion of partial braid where several among these $n$ curves can be omitted; partial braids form the inverse braid monoid $IB_n$.

2000 Mathematics Subject Classification. Primary 20F36; Secondary 20F38, 57M.

Key words and phrases. Braid, inverse braid monoid, reflection group of type B, presentation, reflection monoid.
By definition a monoid is inverse if for any element $a$ of it there exists a unique element $b$ (which is called inverse) such that

$$a = aba$$

and

$$b = bab.$$  

This notion was introduced by V. V. Wagner in 1952 [15]. See the books [10] and [9] as general references for inverse semigroups.

The multiplication of partial braids is shown at Figure 1.1. At the last stage it is necessary to remove any arc that does not join the upper or lower planes.

So, the classical braid group (which corresponds to $W = \Sigma_n$, symmetric group) is included into the inverse braid monoid $IB_n$.

The most important example of an inverse monoid is a monoid of partial (defined on a subset) injections of a set into itself. For a finite set this gives us the notion of a symmetric inverse monoid $I_n$ which generalizes and includes the classical symmetric group $\Sigma_n$. A presentation of symmetric inverse monoid was obtained by L. M. Popova [11], see also formulas (2.1-2.3) below.

Now let $W$ be the Coxeter group of type $B_n$. The corresponding inverse braid monoid $IB(B_n)$ was studied in [14] and the reflection monoid $I(B_n)$ in [6].

The aim of the present paper is to show that in the case of type $B$ the situation is quite similar: there exists a map $\rho_B : IB(B_n) \to I(B_n)$ such that the following diagram

$$
\begin{array}{ccc}
Br(B_n) & \longrightarrow & W(B_n) \\
\downarrow & & \downarrow \\
IB(B_n) & \xrightarrow{\rho_B} & I(B_n)
\end{array}
$$

(where the vertical arrows mean inclusion of the group of invertible elements into a monoid) is commutative.

2. Inverse braid monoid and type $B$

Let $N$ be a finite set of cardinality $n$, say $N = \{v_1, \ldots, v_n\}$. The inverse symmetric monoid $I_n$ can be interpreted as a monoid of partial monomorphisms of $N$ into itself. Let us equip
elements of $N$ with the signs, i.e. let $SN = \{\delta_i v_1, \ldots, \delta_n v_n\}$, where $\delta_i = \pm 1$. The Weyl group $W(B_n)$ of type $B$ can be interpreted as a group of signed permutations of the set $SN$:

$$W(B_n) = \{\sigma - \text{bijection of } SN : (-x)\sigma = -(x)\sigma \text{ for } x \in SN\}.$$

The monoid of partial signed permutations $I(B_n)$ is defined as follows:

$$I(B_n) = \{\sigma - \text{partial bijection of } SN : (-x)\sigma = -(x)\sigma \text{ for } x \in SN \text{ and } x \in \text{dom}\sigma \text{ if and only if } -x \in \text{dom}\sigma\},$$

where $\text{dom}\sigma$ means a domain of definition of the monomorphism $\sigma$.

We remind that a monoid $M$ is factorisable if $M = EG$ where $E$ is a set of idempotents of $M$ and $G$ is a subgroup of $M$. Evidently the monoid $I(B_n)$ is factorisable [6] as every partial signed permutation can be extended to an element of the group of units of $I(B_n)$ i.e. a signed permutation with the domain equal to $SN$.

Usually the braid group $Br_n$ is given by the following Artin presentation [1]. It has the generators $\sigma_i, i = 1, \ldots, n - 1$, and two types of relations:

\[
\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i, \text{ if } |i - j| > 1, \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}.
\end{align*}
\]

The following presentation for the inverse braid monoid was obtained in [5]. It has the generators $\sigma_i, \sigma_i^{-1}, i = 1, \ldots, n - 1, \epsilon$, and relations

\[
\begin{align*}
\sigma_i \sigma_i^{-1} &= \sigma_i^{-1} \sigma_i = 1, \text{ for all } i, \\
\epsilon \sigma_i &= \sigma_i \epsilon \text{ for } i \geq 2, \\
\epsilon \sigma_1 \epsilon &= \sigma_1 \epsilon \sigma_1 \epsilon = \epsilon \sigma_1 \epsilon \sigma_1, \\
\epsilon &= \epsilon^2 = \epsilon \sigma_1^2 = \sigma_1^2 \epsilon.
\end{align*}
\]

and the braid relations (2.1).

Geometrically the generator $\epsilon$ means that the first string in the trivial braid is absent.

If we replace the first relation in (2.2) by the following set of relations

\[
\sigma_i^2 = 1, \text{ for all } i,
\]

and delete the superfluous relations

$$\epsilon = \epsilon \sigma_1^2 = \sigma_1^2 \epsilon,$$

we get a presentation of the symmetric inverse monoid $I_n$ [11]. We also can simply add the relations (2.3) if we do not worry about redundant relations. We get a canonical map [5]

$$\rho_n : IB_n \to I_n$$

which is a natural extension of the corresponding map for the braid and symmetric groups.

More balanced relations for the inverse braid monoid were obtained in [7]. Let $\epsilon_i$ denote the trivial braid with $i$th string deleted, formally:

\[
\begin{align*}
\epsilon_1 &= \epsilon, \\
\epsilon_{i+1} &= \sigma_i^{+1} \epsilon_i \sigma_i^{-1}.
\end{align*}
\]

So, the generators are: $\sigma_i, \sigma_i^{-1}, i = 1, \ldots, n - 1, \epsilon_i, i = 1, \ldots, n$, and relations are the following:
\( \sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1 \), for all \( i \), 
\( \epsilon_j \sigma_i = \sigma_i \epsilon_j \) for \( j \neq i, i + 1 \), 
\( \epsilon_i \sigma_i = \sigma_i \epsilon_{i+1} \), 
\( \epsilon_{i+1} \sigma_i = \sigma_i \epsilon_i \), 
\( \epsilon_i = \epsilon_i^2 \), 
\( \epsilon_{i+1} \sigma_i^2 = \sigma_i^2 \epsilon_{i+1} = \epsilon_{i+1} \), 
\( \epsilon_i \epsilon_{i+1} \sigma_i = \sigma_i \epsilon_i \epsilon_{i+1} = \epsilon_i \epsilon_{i+1} \),

(2.4)

plus the braid relations (2.1).

Let \( EF_n \) be a monoid of partial isomorphisms of a free group \( F_n \) defined as follows. Let \( a \) be an element of the symmetric inverse monoid \( I_n \), \( a \in I_n \), \( J_k = \{ j_1, \ldots, j_k \} \) is the image of \( a \), and elements \( i_1, \ldots, i_k \) belong to domain of the definition of \( a \). The monoid \( EF_n \) consists of isomorphisms

\(< x_{i_1}, \ldots, x_{i_k} > \rightarrow < x_{j_1}, \ldots, x_{j_k} > \)

expressed by

\( f_a : x_i \mapsto w_i^{-1} x_{a(i)} w_i \),

if \( i \) is among \( i_1, \ldots, i_k \) and not defined otherwise and \( w_i \) is a word on \( x_{j_1}, \ldots, x_{j_k} \). The composition of \( f_a \) and \( g_b \), \( a, b \in I_n \), is defined for \( x_i \) belonging to the domain of \( a \circ b \). We put \( x_{j_m} = 1 \) in a word \( w_i \) if \( x_{j_m} \) does not belong to the domain of definition of \( g \). If we put \( w_i = 1 \) we get an inclusion of \( I_n \) into \( EF_n \). Sending each \( f_a \in EF_n \) to \( a \in I_n \) we get a homomorphism \( EF_n \rightarrow I_n \).

**Proposition 2.1.** The canonical maps \( I_n \rightarrow EF_n \) and \( EF_n \rightarrow I_n \) give the following splitting

\[ I_n \rightarrow EF_n \rightarrow I_n. \]

We remind that the Artin-Brieskorn braid group of the type \( B \) is isomorphic to the braid group of a punctured disc \([8], [12], [13]\). With respect to the classical braid group it has an extra generator \( \tau \) and the relations of type \( B \):

\[
\begin{align*}
\tau \sigma_1 \tau \sigma_1 &= \sigma_1 \tau \sigma_1 \tau, \\
\tau \sigma_i &= \sigma_i \tau, \quad \text{if } i > 1, \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \\
\sigma_i \sigma_j &= \sigma_j \sigma_i, \quad \text{if } |i - j| > 1.
\end{align*}
\]

(2.5)

The monoid \( IBB_n \) of partial braids of the type \( B \) can be considered also as a submonoid of \( IBB_{n+1} \) consisting of partial braids with the first string fixed. An interpretation as a monoid of isotopy classes of maps is possible as well. As usual consider a disc \( D^2 \) with given \( n + 1 \) points. Denote the set of these points by \( Q_{n+1} \). Consider homeomorphisms of \( D^2 \) onto a copy of the same disc with the condition that the first point is always mapped into itself and among the other \( n \) points only \( k \) points, \( k \leq n \) (say \( i_1, \ldots, i_k \)) are mapped bijectively onto the \( k \) points (say \( j_1, \ldots, j_k \)) of the set \( Q_{n+1} \) (without the first point) of second copy of \( D^2 \). The isotopy classes of such homeomorphisms form the monoid \( IBB_n \).
Theorem 2.1. [14] We get a presentation of the monoid \( IB(B_n) \) if we add to the presentation of the braid group of type \( B \) \((2.5)\) the generator \( \epsilon \) and the following relations

\[
\begin{align*}
\tau \tau^{-1} &= \tau^{-1} \tau = 1, \\
\sigma_i \sigma_i^{-1} &= \sigma_i^{-1} \sigma_i = 1, \text{ for all } i, \\
\epsilon \sigma_i &= \sigma_i \epsilon \text{ for } i \geq 2, \\
\epsilon \sigma_1 \epsilon &= \sigma_1 \epsilon \sigma_1 \epsilon = \epsilon \sigma_1 \epsilon \sigma_1, \\
\epsilon &= \epsilon^2 = \epsilon \sigma_1^2 = \sigma_1^2 \epsilon \\
\epsilon \tau &= \tau \epsilon = \epsilon.
\end{align*}
\]

(2.6)

We get another presentation of the monoid \( IB(B_n) \) if we add to the presentation \((2.4)\) of \( IB_n \) one generator \( \tau \), the type \( B \) relations \((2.5)\) and the following relations

\[
\begin{align*}
\tau \tau^{-1} &= \tau^{-1} \tau = 1, \\
\epsilon_1 \tau &= \tau \epsilon_1 = \epsilon_1.
\end{align*}
\]

It is a factorisable inverse monoid.

We define an action of \( IB(B_n) \) on \( SN \) by partial isomorphisms as follows

\[
\begin{align*}
\sigma_i(\delta_j v_j) &= \begin{cases} 
\delta_i v_{i+1}, & \text{if } j = i, \\
\delta_{i+1} v_i, & \text{if } j = i + 1, \\
\delta_j v_j, & \text{if } j \neq i, i + 1,
\end{cases} \\
\tau(\delta_j v_j) &= \begin{cases} 
-\delta_i v_1, & \text{if } j = 1, \\
\delta_j v_j, & \text{if } j \neq 1,
\end{cases} \\
dom \epsilon &= \{\delta_2 v_2, \ldots, \delta_n v_n\}, \\
\epsilon(\delta_j v_j) &= \delta_j v_j, \text{ if } j = 2, \ldots, n, \\
dom \epsilon_i &= \{\delta_i v_1, \ldots, \hat{\delta_i} v_i, \ldots, \delta_n v_n\}, \\
\epsilon_i(\delta_j v_j) &= \delta_j v_j, \text{ if } j = 1, \ldots, i, \ldots, n.
\end{align*}
\]

Direct checking shows that the relations of the inverse braid monoid of type \( B \) are satisfied by the compositions of partial isomorphisms defined by \( \sigma_i, \tau \) and \( \epsilon_i \).

Theorem 2.2. The action given by the formulas \((2.7) - (2.12)\) defines a homomorphism of inverse monoids \( \rho_B : IB(B_n) \rightarrow I(B_n) \) such that the diagram \((1.1)\) is commutative.

\[\square\]

Theorem 2.3. The homomorphism \( \rho_B : IB(B_n) \rightarrow I(B_n) \) is an epimorphism. We get a presentation of the monoid \( I(B_n) \) if in a presentation of \( IB(B_n) \) we replace the first relation in \((2.2)\) by the following set of relations

\[
\sigma_i^2 = 1, \text{ for all } i,
\]

and delete the superfluous relations

\[
\epsilon = \epsilon \sigma_1^2 = \sigma_1^2 \epsilon,
\]
and we replace the first relation in (2.6) by the following relation

\[ \tau^2 = 1. \]

**Proof.** Let us temporarily denote by \( IB_n \) the monoid with the presentation given in the statement of Theorem. To see that the homomorphism \( \rho_B \) is an epimorphism we use the fact that the monoid \( I(B_n) \) is factorisable, so its every element can be written in the form \( \epsilon g \) where \( \epsilon \) belong to the set of idempotents and \( g \) is an element of the the Weyl group of type \( B \), \( W(B_n) \). For the Weyl group the map \( \rho_B \) is an epimorphism \( W(B_k) = Br(B_k)/P(B_k) \) and the sets of idempotents for the monoids \( IB(B_n) \) and \( I(B_n) \) coincide and the map \( \rho_B \) restricted to \( E(IB(B_n)) \) is identity.

It follows from the definition of the action that \( \tau^2 \) and \( \sigma_i^2 \) are mapped to the unit by the map \( \rho_B \). So the homomorphism \( \rho_B \) is factorised by the homomorphism \( \tilde{\rho}_B : IB(B_n) \rightarrow IB_n \rightarrow I(B_n) \).

To show that \( \tilde{\rho}_B \) is an isomorphism we compare the cardinalities of \( IB_n \) and \( I(B_n) \). It is easy to calculate that the cardinality of \( I(B_n) \) is equal to \( \sum_{k=0}^{n} 2^k \binom{n}{k} k! \).

Let \( \epsilon_{k+1,n} \) denote the partial braid with the trivial first \( k \) strings and absent of the rest \( n-k \) strings. It can be expressed using the generator \( \epsilon \) or the generators \( \epsilon_i \) as follows

\[ \epsilon_{k+1,n} = \epsilon \sigma_{n-1} \cdots \sigma_{k+1} \epsilon \sigma_{n-1} \cdots \sigma_{k+2} \epsilon \cdots \epsilon \sigma_{n-1} \sigma_{n-2} \epsilon \sigma_{n-1} \epsilon, \]

\[ \epsilon_{k+1,n} = \epsilon_{k+1} \epsilon_{k+2} \cdots \epsilon_n. \]

It was proved in [3] that every partial braid has a representative of the form

\[ \sigma_{i_1} \cdots \sigma_{i_k} \cdots \sigma_{k+1,n} \epsilon \sigma_{k+1,n} \sigma_{k+1} \cdots \sigma_{j_k} \cdots \sigma_{j_1}, \]

\[ k \in \{0, \ldots, n\}, 0 \leq i_1 < \cdots < i_k \leq n-1 \text{ and } 0 \leq j_1 < \cdots < j_k \leq n-1, \]

where \( x \in Br_k \). The same is true for \( IB(B_n) \), where \( x \in Br(B_n) \). The elements \( \tau^2 \) and \( \sigma_i^2 \) are mapped to 1 by \( \rho_B \), so each equivalence class modulo pure braid group of the type \( B_k \) is mapped to the same element in \( I(B_n) \). These equivalence classes form the Weyl group \( W(B_k) \). The order of the Weyl group of type \( B_k \) is equal to \( 2^k k! \). We see that the set of cardinality less or equal than \( \sum_{k=0}^{n} 2^k \binom{n}{k} k! \) is mapped epimorphically onto the set of exactly this cardinality. It means that the epimorphism \( \tilde{\rho}_B \) is an isomorphism. \( \square \)

Let \( E \) be the monoid generated by one idempotent generator \( \epsilon \).

**Proposition 2.2.** The abelianization \( Ab(IBB_n) \) of the monoid \( IBB_n \) is isomorphic to \( E \oplus \mathbb{Z}^2 \), factorised by the relations

\[ \begin{align*}
\epsilon + \tau &= \epsilon, \\
\epsilon + \sigma &= \epsilon,
\end{align*} \]

where \( \tau \) and \( \sigma \) are generators of \( \mathbb{Z}^2 \). The canonical map

\[ a : IBB_n \rightarrow Ab(IBB_n) \]

is given by the formulas:

\[ \begin{align*}
a(\epsilon_i) &= \epsilon, \\
a(\tau) &= \tau, \\
a(\sigma_i) &= \sigma.
\end{align*} \]
The canonical map from \( Ab(IBB_n) \) to \( Ab(I(B_n)) \) consists of factorising \( \mathbb{Z}^2 \) modulo 2.

\[ \square \]

References

[1] E. Artin, Theorie der Zöpfe. Abh. Math. Semin. Univ. Hamburg, 1925, v. 4, 47–72.
[2] N. Bourbaki, Groupes et algèbres de Lie, Chaps. 4–6, Masson, Paris, 1981.
[3] E. Brieskorn, Sur les groupes de tresses [d’après V. I. Arnol’d]. (French) Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401, pp. 21–44. Lecture Notes in Math., Vol. 317, Springer, Berlin, 1973.
[4] P. Deligne, Les immeubles des groupes de tresses généralisés. (French) Invent. Math. 17 (1972), 273–302.
[5] D. Easdown; T. G. Lavers, The inverse braid monoid. Adv. Math. 186 (2004), no. 2, 438–455.
[6] B. Everitt, J. Fountain Partial mirror symmetry I: reflection monoids. 22 pages. \texttt{arXiv:math/0701313}
[7] N. D. Gilbert, Presentations of the inverse braid monoid. J. Knot Theory Ramifications 15 (2006), no. 5, 571–588.
[8] S. Lambropoulou, Solid torus links and Hecke algebras of B-type. Proceedings of the Conference on Quantum Topology (Manhattan, KS, 1993), 225–245, World Sci. Publishing, River Edge, NJ, 1994.
[9] M. V. Lawson, Inverse semigroups. The theory of partial symmetries. World Scientific Publishing Co., Inc., River Edge, NJ, 1998. xiv+411 pp.
[10] M. Petrich, Inverse semigroups. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1984. x+674 pp.
[11] L. M. Popova, Defining relations of a semigroup of partial endomorphisms of a finite linearly ordered set. (Russian) Leningrad. Gos. Ped. Inst. Učen. Zap. 238 1962 78–88.
[12] V. V. Vershinin, On braid groups in handlebodies. Sib. Math. J. 39, No.4, 645-654 (1998); translation from Sib. Mat. Zh. 39, No.4, 755-764 (1998).
[13] V. V. Vershinin, Braid groups and loop spaces. Russ. Math. Surv. 54, No.2, 273-350 (1999); translation from Usp. Mat. Nauk 54, No.2, 3-84 (1999).
[14] V. V. Vershinin, On the inverse braid monoid. Topology and Appl. 156 (2009) 1153-1166.
[15] V. V. Wagner, Generalized groups. (Russian) Doklady Akad. Nauk SSSR (N.S.) 84, (1952). 1119–1122.

Département des Sciences Mathématiques, Université Montpellier II, Place Eugène Bataillon, 34095 Montpellier cedex 5, France
E-mail address: vershini@math.univ-montp2.fr

Sobolev Institute of Mathematics, Novosibirsk 630090, Russia
E-mail address: versh@math.nsc.ru