A new version of a theorem of Kaplansky

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ABSTRACT
A well-known theorem of Kaplansky states that any projective module is a direct sum of countably generated modules. In this paper, we prove the \( w \)-version of this theorem, where \( w \) is a hereditary torsion theory for modules over a commutative ring.

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1. Introduction
A well-known theorem of Kaplansky states that any projective module is a direct sum of countably generated modules (see [7]). This is equivalent to saying that every projective module can be filtered by countably generated and projective modules. In [11], Štovíček and Trlifaj applied Hill’s method [6] to extend Kaplansky’s theorem on projective modules to the setting of cotorsion pairs. Later, Enochs et al. [1] also got the analogous version of Kaplansky’s Theorem for cotorsion pairs for a more general setting on concrete Grothendieck categories. Moreover, several versions of Kaplansky’s theorem have been discussed in the literature. For example, a categorical version of Kaplansky’s theorem on projective modules is proved in [9, Lemma 3.8] by Osofsky. Also, in [2], Estrada et al. prove a version of Kaplansky’s Theorem for quasi-coherent sheaves, by using Drinfeld’s notion of almost projective module and the Hill Lemma.

The purpose of this article is to present a \( w \)-version of Kaplansky’s theorem on projective modules, where \( w \) is a hereditary torsion theory for modules over a commutative ring. Next, we shall review some terminology related to the hereditary torsion theory \( w \), see [14] for details. Throughout, \( R \) denotes a commutative ring with an identity element and all modules are unitary.

Recall from [15] that an ideal \( J \) of \( R \) is called a Glaz-Vasconcelos ideal (a GV-ideal for short) if \( J \) is finitely generated and the natural homomorphism

\[
\varphi : R \rightarrow J^* := \text{Hom}_R(J, R)
\]

is an isomorphism. Notice that the set \( GV(R) \) of GV-ideals of \( R \) is a multiplicative system of ideals of \( R \). Let \( M \) be an \( R \)-module. Define
tor_{GV}(M) := \{x \in M \mid Jx = 0 \text{ for some } J \in GV(R)\}.

Thus tor_{GV}(M) is a submodule of M. Now M is said to be GV-torsion (resp., GV-torsionfree) if tor_{GV}(M) = M (resp., tor_{GV}(M) = 0). A GV-torsionfree module M is called a w-module if Ext^1_R(R/J, M) = 0 for all J \in GV(R). Then projective modules and reflexive modules are both w-modules. In [14, Theorem 6.7.24], it is shown that all flat modules are w-modules. Also it is known that a GV-torsionfree R-module M is a w-module if and only if Ext^1_R(N, M) = 0 for every GV-torsion R-module N (see [14, Theorem 6.2.7]). For any GV-torsionfree module M,

M_w := \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in GV(R)\}

is a w-submodule of E(M) containing M and is called the w-envelope of M, where E(M) denotes the injective envelope of M. It is clear that a GV-torsionfree module M is a w-module if and only if M_w = M.

It is worthwhile to point out that from a torsion-theoretic point of view, the notion of GV-torsionfree modules coincides with that of tor_{GV}-closed (i.e. tor_{GV}-torsionfree and tor_{GV}-injective) modules, where tor_{GV} is the torsion theory whose torsion modules are the GV-torsion modules and torsionfree modules are the GV-torsionfree modules, that is, the pair of classes of R-modules

\text{tor}_{GV} = (\{\text{GV-torsion modules}\}, \{\text{GV-torsionfree modules}\})

is a hereditary torsion theory on the category of R-modules. In the integral domain case, w-modules were called semi-divisorial modules in [3] and (in the ideal case) F_{\infty}-ideals in [5], which have been proved to be useful in the study of multiplicative ideal theory and module theory.

In [13], the first named author and Kim generalized projective modules to the hereditary torsion theory tor_{GV} setting and introduced the notion of w-projective modules. Recall that an R-module M is said to be w-projective if Ext^1_R(L(M), N) is GV-torsion for any torsionfree w-module N, where L(M) = (M/tor_{GV}(M))_w. It is clear that both GV-torsion modules and projective modules are w-projective. Actually, the notion of w-projective modules appeared first in [12] when R is an integral domain. Thus, it is natural to ask if Kaplansky’s theorem on projective modules has a w-module theoretic analog.

To give a w-version of Kaplansky’s theorem, we first introduce and study a class of modules closely related to the w-projective modules called w-split modules (see Section 2). Then we prove, in Section 3, the Kaplansky’s theorem for w-projective w-modules in terms of w-split modules. More precisely, it is shown that every w-projective w-module can be filtered by countably generated and w-split modules (see Theorem 3.5).

Any undefined notions and notations are standard, as in [4, 10, 14].

2. On w-split modules

In this section, we introduce and study w-split modules, which can be used to prove a w-version of Kaplansky’s theorem on projective modules.

Before we give the definition of w-split modules, we first fix the following notation. If M is an R-module and s \in R, then let \eta^m_s : M \rightarrow M denote the multiplication map m \mapsto sm. It is obvious that \eta^M_1 = \text{id}_M is the identity map on M. Moreover, if M, N are R-modules, then for any a \in R and any f \in \text{Hom}_R(M, N), we have the multiplication a \cdot f = f\eta^M_a = \eta^N_a f.

Definition.

(1) A short exact sequence of R-modules

\[ 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0 \]

is said to be w-split if there exist J = \langle d_1, \ldots, d_n \rangle \in GV(R) and h_1, \ldots, h_n \in \text{Hom}_R(C, B) such that \eta^C_{d_k} = gh_k for all k = 1, \ldots, n.
(2) An $R$-module $M$ is said to be $w$-split if there is a $w$-split short exact sequence of $R$-modules 
\[ 0 \to K \to P \to M \to 0 \]
with $P$ projective.

The first part of the following lemma is exactly [14, Exercise 1.60] (without proof). However, for the purposes of showing the second part of the lemma, we still give a proof for it.

**Lemma 2.1.** Consider the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow f & & \downarrow g \\
0 & \longrightarrow & B \\
\downarrow h & & \downarrow \gamma \\
0 & \longrightarrow & C \\
\end{array}
\]

Then there exists a homomorphism $h : B \to A'$ with $hf = \alpha$ if and only if there is a homomorphism $h' : C \to B'$ with $g'h' = \gamma$. In this case, the equality $\beta = f'h + h'g$ holds.

**Proof.** Assume that there is a homomorphism $h : B \to A'$ such that $hf = \alpha$. Then for any $c \in C$, since $g$ is epic, $c = g(b)$ for some $b \in B$. Define $h' : C \to B'$ by

\[ h'(c) = \beta(b) - f'h(b). \]

If $c = g(b) = g(b_1)$ for some $b_1 \in B$, then $b - b_1 \in \ker(g) = \im(f)$. Therefore, $b - b_1 = f(a)$ with $a \in A$, and so

\[ \beta(b - b_1) - f'h(b - b_1) = \beta f(a) - f'h f(a) = \beta f(a) - f' \alpha(a) = 0, \]

i.e. $\beta(b) - f'h(b) = \beta(b_1) - f'h(b_1)$. Hence, $h'$ is a well-defined homomorphism with $g'h' = \gamma$. Conversely, let $h' : C \to B'$ be a homomorphism with $g'h' = \gamma$. Then for each $b \in B$, we have

\[ g'h'(b) - g'h'(b) = g'(b) - g'(b) = 0, \]

and so $\beta(b) - h'g(b) \in \ker(g') = \im(f')$. Since $f'$ is monic, there is a unique $a' \in A'$ with $f'(a') = h'(b) - h'g(b)$. Now, define $h : B \to A'$ by $h(b) = a'$. Then it is easy to check that $h$ is a well-defined homomorphism. For any $a \in A$, write $a' = hf(a)$. Then

\[ f'h f(a) = f'(a') = \beta f(a) - h' g f(a) = \beta \alpha(a). \]

Thus, it follows that $hf = \alpha$.

Moreover, the second statement follows immediately by the above proof.

By using the lemma above, it is easy to prove the following proposition.

**Proposition 2.2.** An exact sequence of $R$-modules $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is $w$-split if and only if there are $J = \langle d_1, ..., d_n \rangle \in \text{GV}(R)$ and $q_1, ..., q_n \in \text{Hom}_R(B, A)$ such that $\eta^A_{dk} = q_k f$ for all $k = 1, ..., n$. In this case, for each $k$, the equality $\eta^B_{dk} = q_k h + h_k g$ holds, where $h_k$ is as in the definition.

**Proof.** Assume that the exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is $w$-split. Then there exist $J = \langle d_1, ..., d_n \rangle \in \text{GV}(R)$ and $h_1, ..., h_n \in \text{Hom}_R(C, B)$ such that $\eta^C_{dk} = gh_k$ for all $k = 1, ..., n$. For each $k$, let us consider the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
\downarrow q_k & \downarrow & \downarrow \eta^B_{dk} \\
0 & \longrightarrow & B \\
\downarrow h_k & \downarrow & \downarrow \eta^C_{dk} \\
0 & \longrightarrow & C \\
\end{array}
\]
Thus, by Lemma 2.1, there is a homomorphism \( q_k : B \to A \) with \( \eta_{d_k}^A = q_kf \) and \( \eta_{d_k}^B = f q_k + h_kg \).

In a similar way, we can see that the converse is also true. \( \square \)

The following lemma gives a condition under which a GV-torsion module is \( w \)-split. Also, it will be used to characterize \( w \)-split modules.

**Lemma 2.3.** Let \( M \) be a GV-torsion \( R \)-module. Then \( M \) is \( w \)-split if and only if there exists a \( J \in GV(R) \) with \( J M = 0 \).

**Proof.** If \( M \) is a \( w \)-split module, then there is a \( w \)-split short exact sequence \( 0 \to A \to P \xrightarrow{g} M \to 0 \) with \( P \) a projective module. Then we can pick \( J = \langle d_1, \ldots, d_n \rangle \in GV(R) \) and \( h_k : M \to P \) as in definition. Since \( P \) is GV-torsionfree, we must have \( h_k = 0 \) for all \( k = 1, \ldots, n \). Hence, for each \( x \in M \), \( d_kx = \eta_{d_k}^M(x) = gh_k(x) = 0 \) for all \( k \), and so \( J M = 0 \).

Conversely, suppose that there is a \( J \in GV(R) \) with \( J M = 0 \). Consider a short exact sequence \( 0 \to A \to P \xrightarrow{g} M \to 0 \) of \( R \)-modules with \( P \) projective. Set \( J = \langle d_1, \ldots, d_n \rangle \) and let \( h_k : M \to P \) be the zero maps for all \( k = 1, \ldots, n \). Then for any \( x \in M \), we obtain \( d_kx = 0 = gh_k(x) \), and so \( \eta_{d_k}^M = gh_k \) for each \( k \). Therefore, \( M \) is a \( w \)-split module. \( \square \)

**Proposition 2.4.** The following statements are equivalent for an \( R \)-module \( M \).

1. \( M \) is a \( w \)-split module.
2. \( \text{Ext}_1^R(M, N) \) is GV-torsion for all \( R \)-modules \( N \).
3. \( \text{Ext}_i^R(M, N) \) is GV-torsion for all \( R \)-modules \( N \) and for all integers \( i \geq 1 \).
4. For any \( R \)-epimorphism \( g : B \to C \), the induced map
   \[ g_* : \text{Hom}_R(M, B) \to \text{Hom}_R(M, C) \]
   has a GV-torsion cokernel.
5. For any \( R \)-epimorphism \( g' : F \to M \) with \( F \) projective, the induced map
   \[ g'_* : \text{Hom}_R(M, F) \to \text{Hom}_R(M, M) \]
   has a GV-torsion cokernel.
6. For any \( R \)-epimorphism \( g : B \to C \) and for each homomorphism \( \alpha : M \to C \), there exist \( J = \langle d_1, \ldots, d_n \rangle \in GV(R) \) and homomorphisms \( h_k : M \to B \) such that \( gh_k = d_k \cdot \alpha \) where \( k = 1, \ldots, n \).
7. Every exact sequence \( 0 \to A \xrightarrow{f} B \xrightarrow{g} M \to 0 \) of \( R \)-modules is \( w \)-split.
8. There exist elements \( \{x_i\}_{i \in I} \) of \( M \) and \( J = \langle d_1, \ldots, d_n \rangle \in GV(R) \) such that for all \( k = 1, \ldots, n \), there are homomorphisms \( \{f_k \in M^+\}_{i \in I} \) satisfying that for each \( x \in M \), almost all \( f_k(x) = 0 \) and \( d_kx = \sum_{i \in I} f_k(x_i)x_i \).

**Proof.**  (1) \( \Rightarrow \) (2) Suppose that \( M \) is a \( w \)-split module and let \( N \) be an \( R \)-module. Then there exists a \( w \)-split exact sequence \( 0 \to L \xrightarrow{f} F \xrightarrow{g} M \to 0 \) of \( R \)-modules with \( F \) projective. Since

\[
0 \to \text{Hom}_R(M, N) \to \text{Hom}_R(F, N) \xrightarrow{f^*} \text{Hom}_R(L, N) \to \text{Ext}_1^R(M, N) \to 0
\]

is exact, it suffices to prove that \( \text{coker}(f^*) \) is a GV-torsion module. But this is equivalent to showing that for any \( \alpha \in \text{Hom}_R(L, N) \), there is a \( J \in GV(R) \) with \( J \alpha \subseteq \text{im}(f^*) \). Now let \( \alpha \in \text{Hom}_R(L, N) \) be arbitrary. Pick \( J \) and \( q_k : F \to L \) as in Proposition 2.2, and set \( \beta_k = \alpha q_k \), where \( k = 1, \ldots, n \). Then \( f^*(\beta_k) = \beta_kf = \alpha q_kf = \alpha q_kh_k = d_k \cdot \alpha \) for all \( k \). Thus, it follows that \( J \alpha \subseteq \text{im}(f^*) \), as desired.

(2) \( \Rightarrow \) (3) It follows from standard homological algebra.
(3) ⇒ (4) Suppose that (3) holds and let \( g : B \to C \) be an epimorphism of \( R \)-modules. Then the sequence

\[
\text{Hom}_R(M, B) \xrightarrow{g} \text{Hom}_R(M, C) \to \text{Ext}_R^1(M, \ker(g))
\]

is exact. Thus, by (3), \( \ker(g_*) \) is GV-torsion.

(4) ⇒ (5) is trivial.

(5) ⇒ (1) This follows easily from the definition of w-split modules.

(4) ⇒ (6) Assume that (4) holds. Let \( g : B \to C \) be an epimorphism of \( R \)-modules and \( x : M \to C \) a homomorphism. Then \( \ker(g_*) \) is a GV-torsion module, and there exists a \( J = (d_1, ..., d_n) \in \text{GV}(R) \) with \( Jx \subseteq \text{im}(g) \). Thus, we can find \( h_1, ..., h_n \in \text{Hom}_R(M, B) \) such that for each \( k = 1, ..., n \), \( d_k \cdot x = gh_k \), and so (6) holds.

(6) ⇒ (7) Apply (6) to the identity map \( I_M : M \to M \).

(7) ⇒ (8) Let \( 0 \to A \to F \xrightarrow{g} M \to 0 \) be an exact sequence of \( R \)-modules with \( F \) free. Then by (7), it is w-split, and so there exist \( J = \langle d_1, ..., d_n \rangle \in \text{GV}(R) \) and \( h_1, ..., h_n \in \text{Hom}_R(M, F) \) such that \( h_k = gh_k \) for all \( k = 1, ..., n \). Now, let \( \{ e_i \}_{i \in I} \) be a basis of \( F \) and set \( x_i = g(e_i) \) for each \( i \in I \).

Then for any \( x \in M \), \( h_k(x) = \sum r_k e_i \), where \( r_k \in R \) and only finitely many \( r_k \neq 0 \). Define \( f_k : M \to R \) for all \( k = 1, ..., n \) and for all \( i \in I \), by \( f_k(x) = r_k \). It is clear that all \( f_k \in M^* \) and, for any \( x \in M \), we have \( d_k x = \sum f_k(x)e_i \).

Hence, (8) holds.

(8) ⇒ (1) Assume that (8) holds. Define \( F \) to be the free \( R \)-module with basis \( \{ e_i \}_{i \in I} \), and define an \( R \)-map \( g : F \to M' \) by \( g : e_i \mapsto x_i \), where \( M' \) is the \( R \)-submodule of \( M \) generated by the set \( \{ x_i \}_{i \in I} \). Then we obtain an exact sequence \( 0 \to A \to F \xrightarrow{g} M' \to 0 \) with \( A = \ker(g) \). For each \( k = 1, ..., n \), define \( h_k : M' \to F \) by \( h_k(y) = \sum f_k(y)e_i \), where \( y \in M' \). Since the sum is finite, \( h_k \) is a well-defined homomorphism. Moreover, for any \( y \in M' \),

\[
gh_k(y) = g \left( \sum f_k(y)e_i \right) = \sum f_k(y)x_i = d_k y = \eta_{d_k}^M(y),
\]

i.e. \( \eta_{d_k}^M = gh_k \). Therefore, \( M' \) is a w-split module.

Finally, notice that \( I(M/M') = 0 \) and \( M/M' \) is a GV-torsion \( R \)-module. Then \( M/M' \) is w-split by Lemma 2.3. Thus, the equivalence of (1) and (2) implies that \( M \) is also w-split.

As a corollary of Proposition 2.4, we have:

**Corollary 2.5.** The following statements hold.

1. Every w-split module is w-projective.
2. Let \( 0 \to A \to B \to C \to 0 \) be an exact sequence of \( R \)-modules with \( C \) w-split. Then \( A \) is w-split if and only if so is \( B \).

Next, we will give an example of a w-projective module, which is not w-split.

**Example 2.6.** Let \( R \) be a two dimensional regular local ring with the maximal ideal \( m \) and set

\[
M = \bigoplus \{ R/J \mid J \in \text{GV}(R) \}.
\]

Then \( M \) is a GV-torsion module, and so it is w-projective. Now, we say that \( M \) is not w-split. If not, then there is a \( J_0 \in \text{GV}(R) \) with \( J_0 M = 0 \) by Lemma 2.3. This means that \( J_0 \) is contained in all \( J \in \text{GV}(R) \). We claim next that \( m \in \text{GV}(R) \). Indeed, since \( R \) is a two dimensional regular local ring, \( m \) is generated by an \( R \)-sequence of length two, and so

\[
m^* = \text{Hom}_R(m, R) \cong m^{-1} = R
\]

by [8, p. 102, Exercise 1], where \( m^{-1} = \{ x \in Q \mid mx \subseteq R \} \) and \( Q \) is the quotient field of \( R \). Therefore, \( m \in \text{GV}(R) \), and so is \( m^n \) for any integer \( n \geq 1 \). Thus, it follows that
\[ J_0 \subseteq \bigcap_{n=1}^{\infty} mt^n = 0, \]

whence \( J_0 = 0 \). However, this means that \( R \) as a module over itself is both GV-torsion and GV-torsionfree, hence \( R = 0 \), which is a contradiction.

We close this section with a short discussion of when a \( w \)-projective module is \( w \)-split.

**Proposition 2.7.** Every \( w \)-projective \( w \)-module is \( w \)-split.

**Proof.** Let \( M \) be a \( w \)-projective \( w \)-module over \( R \) and let \( g : P \to M \) be an \( R \)-epimorphism with \( P \) projective. It follows that \( 0 \to K \to P \xrightarrow{g} M \to 0 \) is exact, where \( K = \ker(g) \). Then we have the following exact sequence

\[ \text{Hom}_R(M, P) \xrightarrow{g_*} \text{Hom}_R(M, M) \to \text{Ext}^1_R(M, K) \]

Since \( M \) is GV-torsionfree, \( K \) is a torsionfree \( w \)-module, and so \( \text{Ext}^1_R(M, K) \) is GV-torsion by the \( w \)-projectivity of \( M \). Thus, \( \text{coker}(g_*) \) is also GV-torsion, whence \( M \) is a \( w \)-split module by Proposition 2.4. \( \square \)

**Proposition 2.8.** Let \( M \) be a GV-torsionfree \( w \)-projective \( R \)-module. Then \( M \) is \( w \)-split if and only if there is a \( J \in \text{GV}(R) \) with \( JM_w \subseteq M \).

**Proof.** Write \( T = M_w/M \). Then \( T \) is a GV-torsion module and the sequence \( 0 \to M \xrightarrow{\mu} M_w \xrightarrow{\pi} T \to 0 \) is exact, where \( \mu \) is the inclusion map and \( \pi \) is the natural map. Also, note that \( L(M) = M_w = L(M_w) \). Thus it follows, from the definition of \( w \)-projective modules and the \( w \)-projectivity of \( M \), that \( M_w \) is also \( w \)-projective. Hence, it follows from Proposition 2.7 that \( M_w \) is \( w \)-split.

If there is a \( J \in \text{GV}(R) \) with \( JM_w \subseteq M \), then \( JT = 0 \), and so \( T \) is \( w \)-split by Lemma 2.3. Therefore, Corollary 2.5(2) says that \( M \) is \( w \)-split.

To prove the converse, it suffices by Lemma 2.3 to show that \( T \) is a \( w \)-split module. This in turn is equivalent to prove that for each \( R \)-module \( N \), \( \text{Ext}^1_R(T, N) \) is GV-torsion. Let \( N \) be an arbitrary \( R \)-module. Thus, consider the following exact sequence

\[ \text{Hom}_R(M_w, N) \xrightarrow{\mu^*} \text{Hom}_R(M, N) \to \text{Ext}^1_R(T, N) \to \text{Ext}^1_R(M_w, N). \]

Then since \( M_w \) is \( w \)-split, \( \text{Ext}^1_R(M_w, N) \) is GV-torsion. Hence, to complete the proof, we need only to show that \( \text{coker}(\mu^*) \) is GV-torsion as well. As \( M \) is a \( w \)-split module, there is a \( w \)-split short exact sequence \( 0 \to K \to P \xrightarrow{g} M \to 0 \) with \( P \) projective, hence there exists \( J = \langle d_1, ..., d_n \rangle \in \text{GV}(R) \) and homomorphisms \( h_1, ..., h_n : M \to P \) such that \( h_k^M = gh_k \), for all \( k = 1, ..., n \). Now, let \( \alpha \in \text{Hom}_R(M, N) \) and consider the following commutative diagram with exact rows

\[
\begin{array}{cccccccc}
0 & \to & K & \xrightarrow{g} & P & \xrightarrow{\beta} & M & \xrightarrow{\alpha} & 0 \\
0 & \to & A & \xrightarrow{\beta} & Q & \xrightarrow{\alpha'} & N & \to & 0.
\end{array}
\]

where \( P \) and \( Q \) are projective modules. Since \( P \) is a \( w \)-module, each \( h_k \) can be extended to a homomorphism \( h_k^P : M_w \to P \). For any \( k \), set \( f_k = g' \beta h_k^P \). Then it is easy to check that \( \mu^*(f_k) = d_k \cdot \alpha \) for all \( k \). Hence, it follows that \( J\alpha \subseteq \text{im}(\mu^*) \), i.e. \( \text{coker}(\mu^*) \) is GV-torsion. \( \square \)
3. Kaplansky’s theorem for w-projective w-modules

To begin with, we fix for all the section the following notation. Let

$$0 \to P \xrightarrow{f} F \xrightarrow{g} M \to 0$$

be a w-split exact sequence of R-modules and $F_1$ a submodule of $F$. Then pick $J \in GV(R)$ and $h_1, \ldots, h_n \in \text{Hom}_R(M, F)$ as in the definition in Section 2 and write

$$g_1 = g|_{F_1}, M_1 = \text{im}(g_1), P_1 = f^{-1}(\ker(g_1)) \text{ and } f_1 = f|_{P_1}.$$ 

Thus, the sequence

$$0 \to P_1 \xrightarrow{f_1} F_1 \xrightarrow{g_1} M_1 \to 0$$

is also exact. Moreover, if for each $k = 1, \ldots, n, h_k(M_1) \subseteq F_1$, then $(\xi_1)$ is also w-split. In this case $(h_k(M_1) \subseteq F_1$ for all $k = 1, \ldots, n)$, we call $(\xi_1)$ a w-split exact sequence induced by $(\xi)$ with respect to the submodule $F_1$.

Let $F$ be as in $(\xi)$ with a direct sum decomposition $F = \bigoplus_{i \in I} F_i$ of projective submodules, where $I$ is an index set. For each subset $H$ of $I$, if $H = \emptyset$, then write

$$F(H) = 0, P(H) = 0 \text{ and } M(H) = 0;$$

otherwise, write

$$F(H) = \bigoplus_{j \in H} F_j, g_H = g|_{F(H)}, M(H) = \text{im}(g_H),$$

$$P(H) = f^{-1}(\ker(g_H)) \text{ and } f_H = f|_{P(H)}.$$ 

It is obvious that $F(I) = F, M(I) = M$ and $P(I) = P$, and that if $H_1, H_2$ are subsets of $I$ with $H_1 \subseteq H_2$, then $F(H_1)$ is a direct summand of $F(H_2), P(H_1) \subseteq P(H_2)$, and $M(H_1) \subseteq M(H_2)$. Now, $M(H)$ is said to be a w-split module induced by $(\xi)$ with respect to the subset $H$ if the sequence

$$0 \to P(H) \xrightarrow{f_H} F(H) \xrightarrow{g_H} M(H) \to 0$$

is a w-split exact sequence induced by $(\xi)$ with respect to the projective submodule $F(H)$.

**Lemma 3.1.** Let $F$ be the module as in the previous paragraph. Suppose that

$$S_1 = \{H_s \mid H_s \subseteq I\}$$

is a set, totally ordered by inclusion, satisfying the sequence

$$0 \to P(H_s) \xrightarrow{f_{H_s}} F(H_s) \xrightarrow{g_{H_s}} M(H_s) \to 0$$

is a w-split exact sequence induced by $(\xi)$ with respect to the submodule $F(H_s)$ for any $H_s \in S_1$. Set $H = \bigcup_{H_s \in S_1} H_s$. Then the following statements hold.

1. $\bigcup_{H_s \in S_1} F(H_s) = F(H)$ and hence it is a projective module. Moreover, $\bigcup_{H_s \in S_1} P(H_s) = P(H)$ and $\bigcup_{H_s \in S_1} M(H_s) = M(H)$.

2. The sequence

$$0 \to \bigcup_{H_s \in S_1} P(H_s) \to \bigcup_{H_s \in S_1} F(H_s) \to \bigcup_{H_s \in S_1} M(H_s) \to 0$$

is a w-split exact sequence induced by $(\xi)$ with respect to the submodule $\bigcup_{H_s \in S_1} F(H_s)$. Hence $\bigcup_{H_s \in S_1} M(H_s)$ is a w-split module induced by $(\xi)$ with respect to the subset $H$. 

Proof. (1) For each \( y \in F(H) \), write \( y = \sum_{j \in H} y_j \), where \( y_j \neq 0 \) for only a finite number of indices \( j \). Since \( S_1 \) is totally ordered, we can choose some \( H_{s_0} \in S_1 \) such that \( y = \sum_{j \in H_{s_0}} y_j \in F(H_{s_0}) \), and so \( F(H) \subseteq \bigcup_{H \in S_1} F(H) \). The other inclusion is clear.

It is obvious that \( \bigcup_{H \in S_1} P(H) \subseteq P(H) \). For the other inclusion, let \( x \in P(H) \). Then we have

\[
    f(x) = \ker(g_H) = \ker(g) \cap F(H).
\]

Since \( F(H) = \bigcup_{H \in S_1} F(H) \), there is some \( H_{s_0} \in S_1 \) such that \( f(x) \in F(H_{s_0}) \), and so \( f(x) \in \ker(g) \cap F(H_{s_0}) = \ker(g_{H_{s_0}}) \). Thus, \( x \in f^{-1}(\ker(g_{H_{s_0}})) = P(H_{s_0}) \), whence \( P(H) \subseteq \bigcup_{H \in S_1} P(H) \).

Let \( x \in M(H) \). Then \( x = g(y) \) for some \( y \in F(H) \). There is some \( H_{s_0} \in S_1 \) such that \( y \in F(H_{s_0}) \). Therefore, \( x \in g(F(H_{s_0})) = M(H_{s_0}) \), whence \( M(H) \subseteq \bigcup_{H \in S_1} M(H) \). The other inclusion is obvious.

(2) Pick \( h_1, \ldots, h_n \in \text{Hom}_R(M, F) \) as in the definition of \( w \)-split exact sequences. Then we need only show that

\[
    h_k \left( \bigcup_{H \in S_1} M(H) \right) \subseteq \bigcup_{H \in S_1} F(H)
\]

for all \( k = 1, \ldots, n \). Now let \( x \in \bigcup_{H \in S_1} M(H) \) be arbitrary. Then \( x \in M(H_{s_0}) \) for some \( H_{s_0} \in S_1 \). Since \( (\xi_{H_{s_0}}) \) is a \( w \)-split exact sequence induced by \( (\xi) \) with respect to the submodule \( F(H_{s_0}) \), we have \( h_k(M(H_{s_0})) \subseteq F(H_{s_0}) \) for any \( k \), and so \( h_k(x) \in F(H_{s_0}) \subseteq \bigcup_{H \in S_1} F(H) \), as desired.

The second assertion of (2) is clear. \( \square \)

Recall that an ideal \( I \) of \( R \) is said to be a \( w \)-ideal if it is a \( w \)-module as an \( R \)-module, i.e., \( I = I_w \). Let \( w \)-Max(\( R \)) denote the set of \( w \)-ideals of \( R \) maximal among proper integral \( w \)-ideals of \( R \) and we call \( m \in w \)-Max(\( R \)) a maximal \( w \)-ideal of \( R \). Then every proper \( w \)-ideal is contained in a maximal \( w \)-ideal and every maximal \( w \)-ideal is a prime ideal.

Let \( M \) be an \( R \)-module and \( S \) a multiplicatively closed subset of \( R \). Then we denote as usual by \( M_S \) the localization of \( M \) at \( S \). In particular, if \( p \) is a prime ideal of \( R \) and \( S = R - p \), then \( M_S \) is denoted by \( M_p \). Let \( f : M \to N \) be a homomorphism of \( R \)-modules. For \( s \in S \) and \( x \in M \), define

\[
    f_S \left( \frac{x}{s} \right) = f(x) \cdot \frac{1}{s}.
\]

Then \( f_S : M_S \to N_S \) is a well-defined \( R_S \)-homomorphism.

Recall from [14] that an \( R \)-homomorphism \( f : M \to N \) is called a \( w \)-isomorphism if \( f_m : M_m \to N_m \) is an isomorphism over \( R_m \) for any \( m \in w \)-Max(\( R \)). Since an \( R \)-module \( M \) is GV-torsion if and only if \( M_m = 0 \) for any \( m \in w \)-Max(\( R \)) (see [14, Theorem 6.2.15]), it is easy to see that a homomorphism \( f : M \to N \) is a \( w \)-isomorphism if and only if both \( \ker(f) \) and \( \text{coker}(f) \) are GV-torsion.

Now, we call an \( R \)-module \( M \) a \( w \)-countably generated module if there is a \( w \)-isomorphism \( f : M_0 \to M \) with \( M_0 \) a countably generated \( R \)-module. It is easily seen that \( M \) is \( w \)-countably generated if and only if there exists a submodule \( N \) of \( M \) such that for any \( m \in w \)-Max(\( R \)), \( N_m = M_m \).

Lemma 3.2. Let \( F \) be as in \( (\xi) \). Suppose that \( F \) is a \( w \)-module over \( R \) with a direct sum decomposition \( F = \bigoplus_{i \in I} F_i \) of countably generated submodules. For each subset \( H \) of \( I \), let \( F(H) \), \( P(H) \) and \( M(H) \) be as before. If \( H \) is a proper subset of \( I \) satisfying \( (\xi_H) \) is a \( w \)-split exact sequence induced by \( (\xi) \) with respect to the submodule \( F(H) \), then the following statements hold.

(1) There is a subset \( H_1 \) of \( I \) properly containing \( H \) such that

\[
    0 \to P(H_1) \xrightarrow{j_{H_1}} F(H_1) \xrightarrow{g_{H_1}} M(H_1) \to 0 \quad (\xi_{H_1})
\]

is a \( w \)-split exact sequence induced by \( (\xi) \) with respect to the submodule \( F(H_1) \).

(2) \( C := M(H_1)/M(H) \) is a countably generated module.
(3) If \( M \) is GV-torsionfree, then \( C \) is w-isomorphic to \( D := M(H_1)_{\omega}/M(H)_{\omega} \). In this case, \( D \) is a w-countably generated module.

(4) If \( M \) is GV-torsionfree and if each \( F_i \) is projective, then \( M(H) \) and \( M(H_1) \) are w-split modules induced by \( (\xi) \) with respect to the subsets \( H \) and \( H_1 \), respectively, and \( C \) is w-split. In this case, \( D \) is a w-countably generated and w-projective module.

Proof. (1) Since \( (\xi) \) is a w-split exact sequence, we can pick \( J = (d_1, \ldots, d_n) \in GV(R), h_k : M \rightarrow F \) and \( q_k : F \rightarrow P \) as in Proposition 2.2, where \( k = 1, \ldots, n \). Then for any \( j \in I \), both \( f_{q_k}(F_j) \) and \( h_k g(F_j) \) are countably generated modules, and for each \( x \in F_j \), we have \( d_k x = f_{q_k}(x) + h_k g(x) \). Therefore, \( d_k F_j \subseteq f_{q_k}(F_j) + h_k g(F_j) \) for all \( k \).

Choose an \( i_0 \in I \). Then there exists a countable subset \( I_1 \) of \( I \) such that

\[
d_k F_{i_0} \subseteq f_{q_k}(F_{i_0}) + h_k g(F_{i_0}) \subseteq \bigoplus_{i \in I_1} F_i
\]

for all \( k \). Note that \( \bigoplus_{i \in I_1} F_i \) is countably generated as each \( F_i \) is countably generated and \( I_1 \) is a countable set. Thus, we can find another countable subset \( I_2 \) of \( I \) containing \( I_1 \) with

\[
d_k \left( \bigoplus_{j \in I_1} F_j \right) \subseteq f_{q_k} \left( \bigoplus_{j \in I_1} F_j \right) + h_k g \left( \bigoplus_{j \in I_1} F_j \right) \subseteq \bigoplus_{i \in I_2} F_i
\]

for all \( k \). Continuing, we obtain countable subsets \( I_0 = \{i_0\}, I_1, I_2, \ldots, I_s, \ldots \) satisfying

\[
d_k \left( \bigoplus_{j \in I_s} F_j \right) \subseteq f_{q_k} \left( \bigoplus_{j \in I_s} F_j \right) + h_k g \left( \bigoplus_{j \in I_s} F_j \right) \subseteq \bigoplus_{i \in I_{s+1}} F_i
\]

for all \( k \). Hence,

\[
J \left( \bigoplus_{j \in I_s} F_j \right) \subseteq \bigoplus_{i \in I_{s+1}} F_i.
\]

But since \( J = R \) (cf. [15, Proposition 3.5]) and both \( \bigoplus_{j \in I_s} F_j \) and \( \bigoplus_{i \in I_{s+1}} F_i \) are w-modules (cf. [15, Proposition 2.3]), it follows easily from [14, Theorem 6.2.2] that \( \bigoplus_{j \in I_s} F_j \subseteq \bigoplus_{i \in I_{s+1}} F_i \).

Set \( L = \bigcup_{s=0}^{\infty} I_s \). Then it is a countable set, and so \( L_1 := L \cap H \) is countable too. Write \( H_1 = H \cup L \). Then \( H_1 = H \cup L_1 \). Thus \( V := \bigoplus_{i \in I_1} F_i \) is countably generated and \( F(H_1) = F(H) \oplus V \). For each \( j \in H_1 \), if \( j \in H \), then since \( (\xi_H) \) is a w-split exact sequence induced by \( (\xi) \) with respect to the submodule \( F(H) \), we obtain

\[
h_k g(F_j) \subseteq h_k (M(H)) \subseteq F(H) \subseteq F(H_1).
\]

Otherwise, \( j \in L_1 \), and so \( j \in I_s \) for some \( s \). Hence, by \((\dagger)\), we also have \( h_k g(F_j) \subseteq F(H_1) \). So it follows that \( h_k (M(H_1)) \subseteq F(H_1) \), whence (1) holds.

(2) Consider the following commutative diagram with exact rows and columns

\[
\begin{array}{cccccc}
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow 0 \\
0 & \rightarrow & F(H) & \rightarrow & F(H_1) & \rightarrow A & \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & F(H) & \rightarrow & F(H_1) & \rightarrow V & \rightarrow 0 \\
& & \downarrow \gamma' & & \downarrow & & \downarrow \\
0 & \rightarrow & M(H) & \rightarrow & M(H_1) & \rightarrow C & \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow 0
\end{array}
\]

where \( V \) is as in the proof of (1). Then since \( V \) is countably generated, so is \( C \).
(3) Consider the following commutative diagram having exact rows.

\[
\begin{array}{c}
0 \longrightarrow M(H) \longrightarrow M(H_1) \longrightarrow C \longrightarrow 0 \\
\downarrow h^{(k)} \downarrow k^{(k)} \downarrow 1_{\alpha_k} \downarrow \downarrow V \\
0 \longrightarrow F(H) \longrightarrow F(H_1) \longrightarrow V \longrightarrow 0
\end{array}
\]

Then the Snake Lemma implies that the sequence

\[0 \rightarrow \ker(h) \rightarrow M(H) \rightarrow M(H_1) \rightarrow \ker(h) \rightarrow 0\]

is exact. Therefore, both \(\ker(h)\) and \(\coker(h)\) are GV-torsion, i.e., \(h\) is a \(w\)-isomorphism. Hence, by (2), \(D\) is \(w\)-countably generated.

(4) If each \(F_i\) is a projective module, then it is clear that \(M(H)\) and \(M(H_1)\) are \(w\)-split modules induced by \((\xi)\) with respect to the subsets \(H\) and \(H_1\), respectively. Let \(J = (d_1, \ldots, d_n)\) and \(h_k\) \((k = 1, \ldots, n)\) as in the proof of (1). Then to show that \(C\) is \(w\)-split, let us consider, for each \(k\), the following diagram having exact rows

\[
\begin{array}{c}
0 \longrightarrow M(H) \longrightarrow M(H_1) \longrightarrow C \longrightarrow 0 \\
\downarrow h^{(k)} \downarrow k^{(k)} \downarrow 1_{\alpha_k} \downarrow \downarrow V \\
0 \longrightarrow F(H) \longrightarrow F(H_1) \longrightarrow V \longrightarrow 0
\end{array}
\]

where \(h^{(k)} = h_{k|_{M(H)}}\), \(k^{(k)} = h_{k|_{M(H_1)}}\), and \(V\) is as in the proof of (1). Clearly, the left square commutes, and so there is a homomorphism \(z_k : C \rightarrow V\) such that the right square commutes as well. Let \(g'\) be as in the proof of (2). Then it is not difficult to see that \(g'z_k = n_k^{\alpha_k}\) for all \(k\). Thus, \(0 \rightarrow A \rightarrow V \xrightarrow{g'} C \rightarrow 0\) is a \(w\)-split exact sequence with \(V\) projective, and consequently \(C\) is \(w\)-split. \(\square\)

Let \(x\) be an ordinal and \(A = (A_{\lambda} \mid \lambda \leq x)\) a sequence of modules. Then \(A\) is called a continuous chain of modules (see [4]) if \(A_0 = 0, A_\lambda \subseteq A_{\lambda+1}\) for all \(\lambda < x\) and \(A_\lambda = \bigcup_{\mu \leq \lambda} A_\mu\) for all limit ordinals \(\lambda \leq x\).

Let \(M\) be a \(w\)-split \(R\)-module. Then there exists a \(w\)-split exact sequence \((\xi)\) with \(F\) projective. Write \(F = \bigoplus_{i \in I} F_i\), where each \(F_i\) is a countably generated projective module. Now, a submodule \(N\) of \(M\) is said to be filtered by countably generated \(w\)-split modules if, for some ordinal \(x\), there exist a subset \(H\) of \(I\) and a sequence

\[\mathcal{H} = (H_\lambda \mid \lambda \leq x)\]

of subsets of \(H\) such that \(H_0 = \emptyset, H_x = H\) and

(i) the sequence of modules

\[N = (N_\lambda := M(H_\lambda) \mid \lambda \leq x)\] \(\quad (\dagger)\)

is a continuous chain of \(N\) with \(N_x = N\);  

(ii) each \(N_\lambda\) is a \(w\)-split module induced by \((\xi)\) with respect to the subset \(H_\lambda\);  

(iii) for each \(\lambda < x, N_{\lambda+1}/N_\lambda\) is a countably generated \(w\)-split module.

The chain \(N\) is called a countably generated \(w\)-split filtration of \(N\).

In this case, if \(B\) is also a submodule of \(M\) and if, for some ordinal \(\beta \leq x\), \(B\) is filtered by countably generated \(w\)-split modules having a continuous chain

\[B = (B_\mu := M(H_\mu) \mid \mu \leq \beta)\]

of submodules such that \(B_\mu = N_\mu\) for each \(\mu \leq \beta\), then we call \(N\) a filtered extension of \(B\) by countably generated \(w\)-split modules.
Lemma 3.3. Let, as in the \((\xi)\) above, \(M\) be a w-split module and \(F\) a projective module with a direct sum decomposition \(F = \bigoplus_{i \in I} F_i\) of countably generated projective submodules. Suppose that \(\{H_j\}_{j \in \Gamma}\) is a totally ordered family of subsets of \(I\) satisfying that for each \(j \in \Gamma\), \(A_j := M(H_j)\) can be filtered by countably generated w-split modules, and that for \(j, k \in \Gamma\), if \(H_j \subseteq H_k\), then \(A_k\) is a filtered extension of \(A_j\) by countably generated w-split modules. Then \(N := \bigcup_{j \in \Gamma} A_j\) can also be filtered by countably generated w-split modules and for each \(j \in \Gamma\), it is a filtered extension of \(A_j\).

Proof. Notice that, if \(N = A_j\) for some \(j \in \Gamma\), then we have nothing to prove. So we assume that for any \(j \in \Gamma, N \neq A_j\). For all \(j \in \Gamma\), since \(A_j\) is filtered by countably generated w-split modules, there are an ordinal \(\alpha_j\) and \(H_j\) a subset of \(I\) such that

\[
\left( A_j^{(j)} := M(H_j) \mid \lambda_j \leq \alpha_j \right)
\]

is a countably generated w-split filtration of \(A_j\), where \(\left( A_j^{(j)} \mid \lambda_j \leq \alpha_j \right)\) is a sequence of subsets of \(H_j\) with \(A^{(0)}_j = \emptyset\) and \(A^{(j)}_j = H_j\).

Set \(\alpha = \bigcup_{j \in \Gamma} \alpha_j\). Then we claim that \(\alpha \notin \{\alpha_j\}_{j \in \Gamma}\), and so it is a limit ordinal. Otherwise, \(\alpha = \alpha_j\) for some \(j \in \Gamma\). Since \(N \neq A_j\), there exists a \(k \in \Gamma\) with \(A_k \nsubseteq A_j\), whence \(A_j \subset A_k\). Hence, it follows, from the definition of filtered extensions above, that

\[
\alpha_j \leq \alpha_k(\leq \alpha = \alpha_j) \text{ and } A_j = A_j^{(j)} = A_j^{(k)} = A_j^{(k)} = A_k,
\]

which is a contradiction.

Set \(H = \bigcup_{j \in \Gamma} H_j\). Then we construct a sequence

\[
\mathcal{H} = (H_\lambda \mid \lambda \leq \alpha)
\]

defines a sequence of subsets of \(H\) as follows.

(1) Write \(H_0 = \emptyset\) and \(H_\alpha = H\).
(2) Let \(0 < \lambda < \alpha\). Then we must have some \(j \in \Gamma\) with \(\lambda < \alpha_j\). Define \(H_\lambda = H^{(j)}_\lambda\).

Next, we prove that the sequence of submodules

\[
\mathcal{N} = (N_\lambda := M(H_\lambda) \mid \lambda \leq \alpha)
\]

is a countably generated w-split filtration of \(N\).

(i) The sequence \(\mathcal{N}\) is a continuous chain of \(\mathcal{N}\) with \(N_\alpha = N\).
   (a) \(N_0 = M(H_0) = 0\) is clear.
   (b) \(N_\alpha = N\).

By Lemma 3.1, we have

\[
N_\alpha = M(H_\alpha) = M(H) = \bigcup_{j \in \Gamma} M(H_j) = \bigcup_{j \in \Gamma} A_j = N.
\]

(c) \(N_\lambda \subseteq N_{\lambda + 1}\) for all \(\lambda < \alpha\).
If \(\lambda < \alpha\), then \(\lambda + 1 < \alpha\) as \(\alpha\) is a limit ordinal. Consequently, \(\lambda < \lambda + 1 < \alpha_j\) for some \(j \in \Gamma\), and hence

\[
N_\lambda = M(H_\lambda) = M(H^{(j)}_\lambda) \subseteq M(H^{(j)}_{\lambda + 1}) = M(H_{\lambda + 1}) = N_{\lambda + 1}.
\]

(d) \(N_\lambda = \bigcup_{\beta < \lambda} N_\beta\) for all limit ordinals \(\lambda \leq \alpha\).
Note that we need only consider the case \(\lambda = \alpha\). For each \(j \in \Gamma\), we have some \(k\) with \(\alpha_j < \alpha_k\), and so \(A_j \subset A_k\). It follows that
This means that $\bigcup_{\beta<\alpha} N_{\beta}$ contains all $A_j$’s, whence $\bigcup_{\beta<\alpha} N_{\beta} = N = N_{\alpha}$.

(ii) Each $N_j$ is a $w$-split module induced by $(\xi)$ with respect to the subset $H_j$.

Clearly, this is true for any $\lambda < \alpha$. For the case $\lambda = \alpha$, it is proved in Lemma 3.1 (3).

(iii) For each $\lambda < \alpha, N_{\lambda+1}/N_{\lambda}$ is a countably generated $w$-split module.

The proof is similar to that of (c) of (i).

Thus, we see that $N$ is also filtered by countably generated $w$-split modules. While the second assertion follows immediately from the constructions of $\mathcal{H}$ and $\mathcal{N}$.

**Remark 3.4.** For the construction of $\mathcal{H}$ in the proof of Lemma 3.3, if $0 < \lambda < \alpha$ and if there are $j_1, j_2 \in \Gamma$ such that $\lambda < \alpha_{j_1}$ and $\lambda < \alpha_{j_2}$, then either $H_{j_1} \subseteq H_{j_2}$ or $H_{j_2} \subseteq H_{j_1}$ as $\{H_j\}_{j \in \Gamma}$ is totally ordered, and so either $A_{j_1} \subseteq A_{j_2}$ or $A_{j_2} \subseteq A_{j_1}$. However, in both cases, from the definition of filtered extensions, it is easy to see that

$$M(H_{\lambda}^{(j_1)}) = M(H_{\lambda}^{(j_2)}) .$$

Thus, $H_{\lambda}$ can be defined to be any of $H_{\lambda}^{(j_1)}$ and $H_{\lambda}^{(j_2)}$.

Similarly, if $M$ is a $w$-projective $w$-module over $R$ and if for some ordinal $\alpha$, there is a continuous chain

$$\mathcal{M} = (M'_{\lambda} \mid \lambda \leq \alpha)$$

of $w$-projective $w$-submodules such that $M'_0 = M$ and $M'_{\lambda+1}/M'_\lambda$ is a $w$-countably generated $w$-projective module for each $\lambda < \alpha$, then $\mathcal{M}$ is said to be filtered by $w$-countably generated $w$-projective modules.

Now, we can prove the Kaplansky’s theorem for $w$-projective $w$-modules.

**Theorem 3.5.** Let $M$ be a $w$-projective $w$-module. Then

1. $M$ can be filtered by countably generated $w$-split modules.
2. $M$ can be filtered by $w$-countably generated $w$-projective modules.

**Proof.** (1) Since $M$ is a $w$-projective $w$-module, it is $w$-split by Proposition 2.7. Let as before $(\xi)$ be the $w$-split exact sequence with $F = \bigoplus_{i \in I} F_i$ a projective module, where each $F_i$ is a countably generated projective module. With the same notation as in Lemma 3.1, let $S$ be a collection of subsets $H$ of $I$ satisfying:

(a) $M(H)$ is a $w$-split module induced by $(\xi)$ with respect to the subset $H$;
(b) $M(H)$ can be filtered by countably generated $w$-split modules.

Clearly, $S$ is non-empty as it contains $\emptyset$. Define a partial order $\leq$ on $S$ by $H_1 \leq H_2$ if $H_1 \subseteq H_2$ and $M(H_2)$ is a filtered extension of $M(H_1)$ by countably generated $w$-split modules. Let $S_1 = \{H_i\}$ be a totally ordered subset of $S$ and $H = \bigcup_{H_i \in S_1} H_i$. Then Lemma 3.1 says that $M(H) = \bigcup_{H_i \in S_1} M(H_i)$ is a $w$-split module induced by $(\xi)$ with respect to the subset $H$.

Now, we claim that the second condition of Lemma 3.3 is also satisfied. Indeed, as $S_1$ is totally ordered by $\leq$, it is also totally ordered by inclusion. Moreover, by the choice of $S$, for each $H_i \in S_1 \subseteq S$, $M(H_i)$ can be filtered by countably generated $w$-split modules. Also, if $H_{i_1}, H_{i_2} \in S_1$ with $H_{i_1} \subseteq H_{i_2}$, then either $H_{i_1} \leq H_{i_2}$ or $H_{i_2} \leq H_{i_1}$. In the first case, the definition of $\leq$ implies that $M(H_{i_1})$ is a filtered extension of $M(H_{i_2})$ by countably generated $w$-split modules. In the
second case, \( H_{s_2} \leq H_{s_1} \), we have \( H_{s_2} \subseteq H_{s_1} \), whence \( H_{s_1} = H_{s_2} \), and \( M(H_{s_1}) = M(H_{s_2}) \). But it is obvious that \( M(H_{s_1}) \) is a filtered extension of itself \( (M(H_{s_1})) \) by countably generated \( w \)-split modules.

Thus, it follows from Lemma 3.3 that \( M(H) \) can be filtered by countably generated \( w \)-split modules and that for each \( H_i \in S_1 \), \( M(H) \) is a filtered extension of \( M(H_i) \) by countably generated \( w \)-split modules. Thus, \( H \in S \) and it is an upper bound of \( S_1 \). By Zorn’s Lemma, \( S \) has a maximal element, say, \( H \).

If \( H \neq I \), then by Lemma 3.2, there is a subset \( H_1 \) of \( I \) properly containing \( H \) such that \( M(H_1) \) is a \( w \)-split module induced by \( (\xi) \) with respect to the subset \( H_1 \) and that \( C = M(H_1)/M(H) \) is a countably generated \( w \)-split module. Therefore, if \( (M_\lambda(H) \mid \lambda \leq \alpha) \) is a countably generated \( w \)-split filtration of \( M(H) \), then by setting \( M_{\lambda+1}(H_1) = M(H_1) \) and \( M_\lambda(H_1) = M_\lambda(H) \) for \( \lambda \leq \alpha \), we see that \( (M_\lambda(H_1) \mid \lambda \leq \alpha + 1) \) is a countably generated \( w \)-split filtration of \( M(H_1) \). Hence, \( H_1 \in S \) and \( H \leq H_1 \), which contradicts the maximality of \( H \). Therefore, \( H = I \) and \( M(H) = M \), whence \( M \) can be filtered by countably generated \( w \)-split modules.

By (1), \( M \) can be filtered by countably generated \( w \)-split modules with a continuous chain \( (M_\lambda \mid \lambda \leq \alpha) \). For each ordinal \( \lambda \), set \( M'_\lambda = (M_\lambda)_w \). Then by Corollary 2.5(1), \( M_\lambda \) is \( w \)-projective, and hence \( M'_\lambda \) is a \( w \)-projective \( w \)-submodule of \( M \). By the similar proof of Lemma 3.2(3), we have that \( M_{\lambda+1}/M_\lambda \) is \( w \)-isomorphic to \( M'_{\lambda+1}/M'_\lambda \), and so \( M'_{\lambda+1}/M'_\lambda \) is \( w \)-countably generated \( w \)-projective module. Thus, it follows that \( M \) can be filtered by \( w \)-countably generated \( w \)-projective modules. 

\( \square \)

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References

[1] Enochs, E., Estrada, S., Iacob, A. (2014). Cotorsion pairs, model structures and homotopy categories. 
Houston J. Math. 40:43–61.

[2] Estrada, S., Asensio, P. A. G., Odabaş, S. (2013). A lazer-like theorem for quasi-coherent sheaves. 
Algebr. Represent. Theor. 16(4):1193–1205. DOI: 10.1007/s10468-012-9353-3.

[3] Glaz, S., Vasconcelos, W. V. (1977). Flat ideals II. Manuscripta Math. 22(4):325–341. DOI: 10.1007/BF01168220.

[4] Göbel, R., Trlifaj, J. (2006). Approximations and Endomorphism Algebras of Modules, De Gruyter Expositions in Mathematics, Vol. 41. Berlin: Walter de Gruyter GmbH & Co, KG.

[5] Hedstrom, J. R., Houston, E. G. (1980). Some remarks on star-operations. J. Pure Appl. Algebra 18:37–44. 
DOI: 10.1016/0022-4049(80)90114-0.

[6] Hill, P. (1981). The third axiom of countability for abelian groups. Proc. Amer. Math. Soc. 82(3):347–350. 
DOI: 10.1090/S0002-9939-1981-0612716-0.

[7] Kaplansky, I. (1958). Projective modules. Ann. of Math. 68(2):372–377. DOI: 10.2307/1970252.

[8] Kaplansky, I. (1974). Commutative Rings. Chicago: The University of Chicago Press.

[9] Osofsky, B. L. (1978). Projective dimension of “nice” directed unions. J. Pure Appl. Algebra 13(2):179–219. 
DOI: 10.1016/0022-4049(78)90008-7.

[10] Rotman, J. J. (1979). An Introduction to Homological Algebra. New York: Academic Press.

[11] Šťovíček, J., Trlifaj, J. (2009). Generalized Hill lemma, Kaplansky theorem for cotorsion pairs and some applications. Rocky Mountain J. Math. 39(1):305–324. DOI: 10.1216/RMJ-2009-39-1-305.
[12] Wang, F. (1997). On w-projective modules and w-flat modules. Algebra Colloq. 4:111–120.
[13] Wang, F., Kim, H. (2015). Two generalizations of projective modules and their applications. J. Pure Appl. Algebra 219(6):2099–2123. DOI: 10.1016/j.jpaa.2014.07.025.
[14] Wang, F., Kim, H. (2016). Foundations of Commutative Rings and Their Modules, Algebra and Applications, Vol. 22. Singapore: Springer.
[15] Yin, H., Wang, F., Zhu, X., Chen, Y. (2011). w-modules over commutative rings. J. Korean Math. Soc. 48(1):207–222. DOI: 10.4134/JKMS.2011.48.1.207.