Quantising on a Category

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Abstract

We review the problem of finding a general framework within which one can construct quantum theories of non-standard models for space, or space-time. The starting point is the observation that entities of this type can typically be regarded as objects in a category whose arrows are structure-preserving maps. This motivates investigating the general problem of quantising a system whose ‘configuration space’ (or history-theory analogue) is the set of objects Ob(Q) in a category Q.

We develop a scheme based on constructing an analogue of the group that is used in the canonical quantisation of a system whose configuration space is a manifold Q \simeq G/H, where G and H are Lie groups. In particular, we choose as the analogue of G the monoid of ‘arrow fields’ on Q. Physically, this means that an arrow between two objects in the category is viewed as an analogue of momentum. After finding the ‘category quantisation monoid’, we show how suitable representations can be constructed using a bundle (or, more precisely, presheaf) of Hilbert spaces over Ob(Q). For the example of a category of finite sets, we construct an explicit representation structure of this type.

\[1\text{Dedicated with respect to the memory of Jim Cushing} \]
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1 Introduction

1.1 Motivational Remarks

Attempts to construct a quantum theory of gravity provide many challenges for quantum theory itself. Some of these involve conceptual issues of the type in which Jim Cushing would certainly have been interested. For example: is it meaningful to talk about a quantum theory in the absence of any background spatio-temporal structure? Or what is meant by a ‘quantum theory of the universe’ in quantum cosmology?

Other issues are of a more mathematical nature and concern the technical challenge of finding a structure that can be understood as a ‘quantum theory of space-time’, or a ‘quantum theory of space’. This is the focus of the present paper, which reviews (hopefully, in a fairly pedagogical way) some recent work that was stimulated by a desire to quantise discrete models of space or space-time [1] [2] [3]. An example of a discrete space-time is a ‘causal set’ in which the fundamental ingredient is the causal relations between space-time points [4]. In mathematical terms, this means modeling space-time with a partially-ordered set (‘poset’) \( C \) where, if \( p, q \in C \) are such that \( p \prec q \), then the space-time point \( q \) lies in the causal future of the space-time point \( p \).

To illustrate some of the main issues, consider the ‘toy’ model in Figure 1, which contains just four causal sets. Can we find a mathematical framework for constructing theories in which the universe could be in a quantum superposition of these four basic space-times? For example, are there quantum states \( |C_i\rangle \), \( i = 1, 2, 3, 4 \), such that the general state can be written as

\[
|\psi\rangle = \sum_{i=1}^{4} \alpha_i |C_i\rangle
\]

for complex coefficients \( \alpha_i \), \( i = 1, 2, 3, 4 \)?

Note that, if there is such a framework, then it must describe a history theory. This is because a causal set represents a space-time, and therefore could not be associated with a quantum state in the standard approaches to quantum theory in which a state is defined at a fixed moment in time. The most fully developed theory of this type is the consistent histories approach to quantum theory [6] [7] [8].

Thus one motivation for the work in the present paper is to construct a framework for discussing quantum theories for a system whose ‘history
configuration space’ is a specific collection of causal sets. A similar problem in non-history quantum theory would be to find a quantum framework for a system in which physical space at each moment in time is represented by one of a collection of topological spaces\(^3\).

There has been much work on quantising a system whose classical (canonical) configuration space is a differentiable manifold \(Q\). One particularly powerful approach is to associate with \(Q\) an algebra whose irreducible representations constitute quantisations of the system. For example, if \(Q\) is the real line \(\mathbb{R}\), the appropriate algebra is generated by the canonical commutation relation

\[
[\hat{x}, \hat{p}] = i\hat{1}.
\]  

(1.2)

This raises the question of whether there is an analogue of such an algebra for a quantum theory of discrete spaces or space-times: for example, a space-time history theory based on a collection of causal sets. As we shall see, the answer is ‘yes’, providing one adopts the approach to consistent history theories developed by the author and collaborators in which propositions about the history of the system are represented by projection operators on a ‘history quantum state space’ \([9] [10]\).

In fact, the quantisation scheme we shall develop applies to far more than just collections of causal sets, or topological spaces. This new scheme

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\(^3\)From a purely mathematical perspective these two examples are related. Indeed, any \(T_0\) topological space gives rise to a partially ordered set in which \(x \leq y\) is defined to mean that \(y \in \overline{\{x\}}\), the closure of the set \(\{x\}\). Conversely, any poset gives rise to a topological space that is generated by the lower sets in the poset \([5]\). For example, moving from left to right in Figure 1, the posets describe a topology on a set with 1, 2, 4 and 3 points respectively. Thus a theory of quantising posets could be construed as a ‘canonical’ theory of quantum topology as well as a history theory of causal sets.
involves constructing a quantisation algebra for a system whose 'configuration space' (or history-theory analogue thereof) is the collection of objects in a category. The arrows in the category are then interpreted as the analogue of momentum variables. For example, Figure 1 illustrates a category with four objects, where the arrows between a pair of causal sets are defined as order-preserving maps.

The invocation of category theory may sound rather abstract, but the idea works well in practice and provides a set of powerful tools for quantising types of physical system that are outside the scope of existing methods.

The plan of the paper is as follows. We start by reviewing the quantisation algebra for a system whose configuration space (or history-theory equivalent) is a manifold. Then, in Section 2.3, we motivate the idea of quantising a system whose configuration space is the collection of objects in a category. A key ingredient is the concept of an arrow field, which is introduced in Section 3. Arrow fields play a central role in constructing the quantisation algebra for a category, and this is explained in Section 4, as are the techniques for constructing representations of this algebra. Then, in Section 5, the general scheme is applied to the special case of a category of sets with structure: for example, causal sets, or topological spaces.

2 Quantisation Algebras

2.1 A Non-Relativistic Particle

Let us consider first the quantum framework for a non-relativistic point particle moving in three spatial dimensions. The canonical commutation relations that specify the quantisation are

\[
[\hat{x}_i, \hat{x}_j] = 0 \\
[\hat{p}_i, \hat{p}_j] = 0 \\
[\hat{x}_i, \hat{p}_j] = i\delta_{ij}\hat{1}
\]

for \(i, j = 1, 2, 3\). On defining \(\hat{U}(\mathbf{a}) := e^{i\mathbf{a} \cdot \hat{p}}\), and \(\hat{V}(\mathbf{b}) := e^{i\mathbf{b} \cdot \hat{x}}\) for three-vectors \(\mathbf{a}, \mathbf{b}\), Eqs. (2.1–2.3) give the exponentiated relations

\[
\hat{U}(\mathbf{a}_1)\hat{U}(\mathbf{a}_2) = \hat{U}(\mathbf{a}_1 + \mathbf{a}_2) \\
\hat{V}(\mathbf{b}_1)\hat{V}(\mathbf{b}_2) = \hat{V}(\mathbf{b}_1 + \mathbf{b}_2)
\]
\[ \hat{U}(a)\hat{V}(b) = e^{ia \cdot b} \hat{V}(b)\hat{U}(a) \] (2.6)

corresponding to a unitary representation of the Weyl-Heisenberg group in three dimensions.

The commutation relations Eqs. (2.1–2.3) have the unique\textsuperscript{4} representation on wave functions:

\[ (\hat{x}_i \psi)(x) := x_i \psi(x) \] (2.7)

\[ (\hat{p}_j \psi)(x) := -i \frac{\partial \psi}{\partial x_j}(x) \] (2.8)

or, in exponentiated form,

\[ (\hat{U}(a)\psi)(x) := \psi(x + a) \] (2.9)

\[ (\hat{V}(b)\psi)(x) := e^{ib \cdot x} \psi(x) \] (2.10)

The algebra generated by Eqs. (2.1–2.3) (or, equivalently, Eqs. (2.4–2.6)) constitutes the ‘kinematical’ aspects of constructing the quantum theory of a particle moving in three dimensions: every system of this type has the same algebra and representation theory. However, to this general quantisation structure there must be added the information that specifies any particular such system, which in this case means giving the Hamiltonian (and hence the dynamical evolution). In this context, an important requirement is that the representation of the quantisation algebra in Eqs. (2.1–2.3) (or Eqs. (2.4–2.6)) is irreducible. This means that the Hamiltonian operator will necessarily be expressible as a function of the basic canonical variables \( \hat{x}_i, \hat{p}_j, i, j = 1, 2, 3. \)

In the history theory of a particle moving in three dimensions, the quantisation algebra is [11]

\[ [\hat{x}_it, \hat{x}_{js}] = 0 \] (2.11)

\[ [\hat{p}_it, \hat{p}_{js}] = 0 \] (2.12)

\[ [\hat{x}_it, \hat{p}_{js}] = i\delta_{ij}\delta(t - s)\hat{1} \] (2.13)

for all values of the time parameters \( t, s \in \mathbb{R} \). Note that \( \hat{x}_it \) and \( \hat{p}_{js} \) are Schrödinger-picture operators, not Heisenberg-picture quantities—the labels

\textsuperscript{4}More precisely, according to the Stone-von Neumann theorem all weakly continuous, irreducible, unitary representations of the exponentiated algebra in Eqs. (2.4–2.6) are unitarily equivalent.
\( t, s \) refer to the time at which propositions are made about the system; they are not dynamical variables (these arise in a quite different way \[12\]).

The representation theory of the (infinite-dimensional) algebra in Eqs. (2.11–2.13) is much more complicated than that of the single-time canonical algebra in Eqs. (2.1–2.3). Once again, the representation is required to be irreducible, so that any operator in the theory will be a function of the history variables \( \hat{x}_t, \hat{p}_s \), with \( t, s \in \mathbb{R} \).

### 2.2 The Analogue for a Manifold \( Q \)

Now consider the problem of quantising a system whose configuration space (or history theory analogue) is a general finite-dimensional manifold \( Q \). If \( Q \) is a homogeneous space, \( G/H \), for some finite-dimensional Lie group \( G \) and subgroup \( H \subset G \), the analogue of momentum is played by the generators of \( G \) which, as is expected of momentum, ‘translate’ around the points of \( Q \). Thus \( G \) appears as a finite-dimensional subgroup of the group \( \text{Diff}(Q) \) of all diffeomorphisms of \( Q \). If \( Q \) is not a homogeneous space then the entire group \( \text{Diff}(Q) \) must be used.

Thus, when \( Q \simeq \mathbb{R} \), we anticipate that a quantisation of this system will include a unitary representation \( g \mapsto \hat{U}(g) \) of the group \( G \), so that \( \hat{U}(g_1)\hat{U}(g_2) = \hat{U}(g_1g_2) \) for all \( g_1, g_2 \in G \). In the non-homogeneous case we must use a unitary representation of \( \text{Diff}(Q) \), about which much less is known.

Let us now turn to the configuration variables. When \( Q \) is a vector space, the basic such variables are the linear maps from \( Q \) to \( \mathbb{R} \), i.e., elements of the dual \( Q^* \) of \( Q \). On giving \( Q \) an inner product, these are in one-to-one correspondence with the elements of \( Q \) itself. Essentially, this is what is meant by Eq. (2.7), which can be read as associating to each 3-vector \( a \), an operator \( a \cdot \hat{x} \) defined by \((a \cdot \hat{x}\psi)(x) := a \cdot x\psi(x)\).

For a general manifold \( Q \) there are no such linear maps. However, when \( Q \simeq \mathbb{R} \) it is possible to embed \( Q \) as a \( G \)-orbit in a vector space \( W \), and the real-valued linear maps on the lowest-dimensional such vector space then give a preferred collection of configuration variables \[13\]. A detailed study leads to the following equations (cf. Eqs. (2.4–2.6)):

\[
\hat{U}(g_1)\hat{U}(g_2) = \hat{U}(g_1g_2) \quad (2.14)
\]
\[
\hat{V}(\beta_1)\hat{V}(\beta_2) = \hat{V}(\beta_1 + \beta_2) \quad (2.15)
\]
\[
\hat{U}(g)\hat{V}(\beta) = \hat{V}(\beta \circ \tau_{g^{-1}})\hat{U}(g) \quad (2.16)
\]
for all $g, g_1, g_2 \in G$ and $\beta, \beta_1, \beta_2 \in W^*$. In Eq. (2.16), $\tau_g : Q \to Q$ denotes the left action of $G$ on $G/H$. Thus, if $\beta \in W^* \subset C^\infty(Q)$ then $\beta \circ \tau_g(q) := \beta(gq)$ for all $g \in G$ and $q \in Q$.

The equations (2.14) and (2.16) describe an operator representation of the semi-direct product $G \times \tau W^*$, with the group law

$$(g_1, \beta_1)(g_2, \beta_2) = (g_1g_2, \beta_1 + \beta_2 \circ \tau_{g_1^{-1}})$$

(2.17)

for all $g_1, g_2 \in G$ and $\beta_1, \beta_2 \in W^*$. The various possible quantisations of the system are deemed to be in one-to-one correspondence with the faithful, irreducible representations of this group.

One obvious representation of the 'quantisation group' $G \times \tau W^*$ is given on complex-valued functions on $Q$:

$$\hat{U}(g)\psi(q) := \psi(g^{-1}q)$$

(2.18)

$$\hat{V}(\beta)\psi(q) := e^{i\beta(q)}\psi(q)$$

(2.19)

for all $q \in Q$. However, this not the only such, and Mackey theory [16] shows that the generic irreducible representation of $G \times \tau W^*$ is defined on functions $\psi : Q \simeq G/H \to V$ where the vector space $V$ carries an irreducible representation of the subgroup $H \subset G$. Specifically:

$$\hat{U}(g)\psi(q) := m(g, q)\psi(g^{-1}q)$$

(2.20)

$$\hat{V}(\beta)\psi(q) := e^{i\beta(q)}\psi(q)$$

(2.21)

where the 'multipliers' $m(g, q), g \in G, q \in Q$, are linear maps from $V$ to itself, which—in order to satisfy Eq. (2.14)—must satisfy

$$m(g_1, q)m(g_2, g_1^{-1}q) = m(g_1g_2, q)$$

(2.22)

for all $g_1, g_2 \in G$ and $q \in Q$ [17].

If the manifold $Q$ is not a homogeneous space, the best that can be done is to use the quantisation group $\text{Diff}(Q) \times \tau C^\infty(Q)$ with group law (c.f., Eq. (2.17))

$$(\phi_1, \beta_1)(\phi_2, \beta_2) = (\phi_1 \circ \phi_2, \beta_1 + \beta_2 \circ \tau_{\phi_1^{-1}})$$

(2.23)

for all $\phi_1, \phi_2 \in \text{Diff}(Q), \beta_1, \beta_2 \in C^\infty(Q)$. The representation theory of this group is considerably more complicated than that of $G \times \tau W^*$, although some of the representations are direct analogues of those above.
2.3 Quantising Models for Space or Space-time

Let us turn now to the main topic of interest: namely, to construct a quantum framework for a system whose ‘history configuration space’ \( Q \) is a collection of possible space-times, such as the causal sets in Figure 1; or, in the canonical case, a system whose configuration space is a specific collection of topological spaces.

An obvious first guess for the state vectors is complex-valued functions on \( Q \), with the configuration variables—taken to be real-valued functions \( \beta \) on \( Q \)—being represented by the operators \( (\hat{\beta}\psi)(A) := \beta(A)\psi(A) \) for all \( A \in Q \). In exponentiated form, this is (c.f., Eq. (2.19))

\[
(\hat{V}(\beta)\psi)(A) := e^{i\beta(A)}\psi(A) \tag{2.24}
\]

for all \( A \in Q \).

However, for a manifold \( Q \simeq G/H \), we know that most of the representations involve functions whose values lie in a vector space other than \( \mathbb{C} \). In addition, a causal set (and topological space) has internal structure, and one expects this to be represented in the quantum theory. This also suggests that \( \mathbb{C} \)-valued functions on \( Q \) are not sufficient as state vectors.

What is needed is a causal-set analogue of the quantisation group \( G \times_{\tau} W^* \), where the generators of the Lie group \( G \) physically represent momentum. Thus we seek an analogue of momentum for a system of the type under consideration; for example, the model in Figure 1.

In the manifold case, each element of \( G \) acts as a diffeomorphism of \( Q \). However causal sets (and topological spaces) differ from each other by virtue of their internal structure, and hence, unlike a diffeomorphism, any ‘momentum’ transformation from one set to another is unlikely to be invertible. This suggests the use of a semi-group of transformations\(^5\).

However, because different members of \( Q \) have different internal structures, it seems unlikely that a single semi-group could act on all of \( Q \). More plausible is a ‘local’ action in which each causal set \( C \in Q \) has its own semi-group that reflects its internal structure. In fact, the admissible transformations from one causal set \( C \) to another \( C' \) will likely depend on \( C' \) as well as \( C \).

\(^5\)A semi-group is a set \( S \) equipped with a combination law that, like a group, is associative. Inverses of elements of \( S \) may not exist however.
We therefore arrive at the idea that to each pair of causal sets \(C, C' \in Q\) there will be a set of transformations from \(C\) to \(C'\) that reflect the internal structures of \(C\) and \(C'\), and which are a local analogue of the global group \(G\) of diffeomorphisms when \(Q\) is a manifold \(G/H\). Clearly, the same argument applies to a collection of topological spaces, or indeed to any other model which involves a collection of spaces, or space-times, with individual internal structure.

Thus, when quantising a system whose configuration space (or history equivalent) is a collection \(Q\) of sets with structure, the starting point will be an association to each pair \(A, A' \in Q\) of a set \(T(A, A')\) of transformations from \(A\) to \(A'\) that, in some suitable sense, ‘respect’, or ‘reflect’, the internal structures of \(A\) and \(A'\). From a physical perspective, these transformations will be related in some way to an analogue of momentum.

In the case of causal sets, the natural choice for the transformations from \(A\) to \(A'\) is order-preserving\(^{6}\) maps from \(A\) to \(A'\); for topological spaces, the natural choice would be continuous maps from \(A\) to \(A'\).

Note that transformation sets of the type postulated above arise also in the case of a group \(G\) acting on a manifold \(Q\), albeit in a somewhat different way since, in this case, there is no internal structure to respect. Specifically, we define an ‘admissible transformation’ from \(q \in Q\) to \(q' \in Q\) to be any group element \(g \in G\) such that \(q' = gq\); thus

\[
T(q, q') = \{g \in G \mid q' = gq\}. \tag{2.25}
\]

The following question arises in general. Namely, if \(A, A', A''\) are three elements in \(Q\), are there any relations between the sets \(T(A, A'), T(A', A'')\) and \(T(A, A'')\)? The obvious requirement is that if \(\alpha \in T(A, A')\), and \(\beta \in T(A', A'')\) then the composition map \(\beta \circ \alpha\) (defined by \(\beta \circ \alpha(a) := \beta(\alpha(a))\) for all \(a \in A\)) should belong to \(T(A, A'')\), i.e., it should respect the internal structure. This works in the example of causal sets, since the combination of two order-preserving maps is itself order preserving. Similarly, in the case of topological spaces, the combination of two continuous maps is continuous.

Note that a natural ‘composition’ also exists in the case of a group \(G\) acting on a manifold \(Q\), with the ‘transformations’ from \(q\) to \(q'\) being defined as in Eq. (2.25). For if \(q' = g_1 q\) and \(q'' = g_2 q'\) then \(q'' = (g_2 g_1)q\), and hence

\(^{6}\)A map \(\alpha : A \rightarrow A'\) is order-preserving if whenever \(a, b \in A\) are such that \(a \leq b\), then \(\alpha(a) \leq \alpha(b)\).
This case fits into the general scheme if the composition \( g_2 \circ g_1 \) is defined to be the group product \( g_2 g_1 \).

This example suggests another property for the sets of transformations \( T(A, A') \), \( A, A' \in Q \). Namely: a group product law is associative, so that
\[
g_1 (g_2 g_3) = (g_1 g_2) g_3 \quad \text{for all } g_1, g_2, g_3 \in G;
\]
and this suggests that we should require the same for the general sets \( T(A, A') \). Thus if \( \alpha \in T(A, A') \), \( \beta \in T(A', A'') \) and \( \gamma \in T(A'', A''') \) then
\[
\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha.
\]
In fact, this is automatically true for the composition of functions, and hence for the examples of causal sets and topological spaces.

Similarly, it is natural to require the existence of a unit element \( \text{id}_A \) in each \( T(A, A) \) so that if \( \alpha \in T(A', A) \) then \( \text{id}_A \circ \alpha = \alpha = \alpha \circ \text{id}_A \). In examples like causal sets, topological spaces, etc, these conditions are satisfied if we choose \( \text{id}_A \) to be the identity map from the set \( A \) to itself. Note that the existence of these unit elements means that each set \( T(A, A), A \in Q \), is a monoid, i.e., a semi-group with a unit element.

In summary, for any collection \( Q \) of entities with internal structure, the quantisation framework will involve an association to each pair \( A, A' \) in \( Q \) of a set of transformations \( T(A, A') \) such that:

i) If \( \alpha \in T(A, A') \) and \( \beta \in T(A', A'') \) then a combination \( \beta \circ \alpha \) can be defined which belongs to \( T(A, A'') \).

ii) To each \( A \in Q \) there is a unit element \( \text{id}_A \in T(A, A) \).

iii) If \( \alpha \in T(A, A') \), \( \beta \in T(A', A'') \) and \( \gamma \in T(A'', A''') \) then
\[
\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha. \quad (2.26)
\]

However, these are nothing but the axioms for a category provided that the ‘transformations’ in \( T(A, A') \) are interpreted as arrows! In fact, the motivating examples of causal sets and topological spaces are special types of category in which the objects are sets with structure, and the arrows are structure-preserving maps between these sets. For example, Figure 1 represents a category with four objects, in which the arrows/morphisms between a pair of causal sets \( C_i, C_j \) are defined to be the order-preserving maps from \( C_i \) to \( C_j \). Similarly, a collection of topological spaces is a category in which the arrows between a pair of topological spaces are defined to be the continuous maps between them.
However, not all categories are collections of structured sets, and we may wish to include this type of category too. Indeed, as shown above, the familiar example of a configuration space manifold $Q$ on which a group $G$ acts, can be realised in this category sense. Namely, the objects in the category are the points of $Q$, and the arrows from $q$ to $q'$ are defined to be the group elements $g$ such that $q' = gq$. Note that the set $T(q, q) = \{ g \in G \mid gq = q \}$ is the ‘little group’, or isotropy group at the point $q \in Q$. Thus, in this case, the monoid $T(q, q)$, $q \in Q$, is actually a group.

Thus we have naturally arrived at the problem of finding the quantum analogue of a system for which (i) the configuration space (or history-theory analogue) is the set, $\text{Ob}(Q)$, of all objects in some category\footnote{Strictly speaking, the category has to be ‘small’, by which is meant that the collection of objects and arrows are genuine sets, not higher-order classes. For example, the category of all sets is not small, since the collection of all sets is not itself a set.} $Q$; and (ii) the sets $T(A, A')$ are identified with the sets, $\text{Hom}(A, A')$, of all arrows (or morphisms) in the category with domain $A$ and range $A'$. We expect that quantising this system will involve representing the elements of $\text{Hom}(A, A')$ with operators on a Hilbert space in some way.

This use of category theory may sound rather abstract but, as I hope is clear from the examples above, it actually arises quite naturally. In particular, if we can find a way of associating a quantum structure with any (small) category $Q$, then this will include the known quantisation scheme for homogeneous manifolds $Q \simeq G/H$, but will extend it to a much wider class of physical system.

Note that it is possible to have two categories with the same set of objects but different arrows. This has immediate physical implications since, if we follow the analogy of $Q \simeq G/H$, we will be seeking an operator representation of configuration functions (i.e., functions on objects) and arrows that is irreducible, and hence such that any other operator in the theory—for example, the Hamiltonian, or (in a history theory) the decoherence operator—can be written as functions of these basic operators. For example, in Figure 1 the arrows from one causal set $C_i$ to another $C_j$ could be chosen to be any order-preserving function from $C_i$ to $C_j$; but we could also restrict ourselves to order-preserving functions that are one-to-one, i.e., which embed $C_i$ in $C_j$. In practice, a lot of physical input will be required to choose the objects and arrows in any particular category.
3 The Theory of Arrow Fields

As emphasised in Section 2.3, when quantising on a category $\mathcal{Q}$ we expect to assign operators to arrows in such a way as to represent the arrow combination law.

However, this runs into an immediate problem. For suppose $\hat{d}(f), f \in \text{Hom}(\mathcal{Q})$, are such operators. Then, representing the arrow combination law presumably means

$$\hat{d}(g)\hat{d}(f) = \hat{d}(g \circ f)$$

(3.1)

whenever the arrows $f, g$ are such that $\text{Ran} \ f = \text{Dom} \ g$, i.e., such that the combination $g \circ f$ is well defined. Then Eq. (3.1) is to be viewed as the category analogue of the representation $\hat{U}(g_1)\hat{U}(g_2) = \hat{U}(g_1g_2)$ of the Lie group $G$ in the quantisation algebra Eqs. (2.4–2.6) for a system whose configuration space is a manifold $Q \simeq G/H$. But the problem with Eq. (3.1) is that the right hand side is only defined when the combination $g \circ f$ exists, whereas the left hand side is defined for all arrows $f, g$.

One possibility would be to define $\hat{d}(g \circ f) := 0$ if $\text{Ran} \ f \neq \text{Dom} \ g$. However, we shall proceed in a somewhat different way that involves introducing the idea of an ‘arrow field’. An arrow field\(^9\) $X$ is defined to be an assignment to each object $A$ of an arrow $X(A)$ whose domain is $A$. For example, consider a category with five objects $A_1, A_2, A_3, B, C$, and with the arrow field shown in Figure 2. Thus\(^10\) $X(A_1) : A_1 \to B$, $X(A_2) : A_2 \to B$, $X(A_3) : A_3 \to B$, $X(B) : B \to C$, and $X(C)$ (not shown) is the identity arrow $\text{id}_C : C \to C$.

The reason for introducing arrow fields is that, unlike arrows, they can always be combined. More precisely, if $X_1$ and $X_2$ are arrow fields, we define

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\(^8\)Equivalently, the collection of arrows in a category is a partial semigroup under the law of arrow combination, whereas the (bounded) operators on a Hilbert space are a (full) semigroup.

\(^9\)The concept of an arrow field has been introduced independently by Gilbert [14] in a different context, where he calls it a ‘flow’ on the category. He refers back in turn to an unpublished preprint of Chase [15]. I am very grateful to Nick Gilbert for drawing my attention to this work.

\(^10\)The notation $f : A \to B$ means that $f$ is an arrow with domain $A$ and range $B$. In categories where the objects are structured sets, such an arrow will be a (structure-preserving) map from the set $A$ to the set $B$.\n
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11
Figure 2: An arrow field in a category with 5 objects

A new arrow field, denoted\(^{11}\) \(X_1 \& X_2\), by

\[(X_1 \& X_2)(A) := X_2(\text{Ran } X_1(A)) \circ X_1(A)\]  \hspace{1cm} (3.2)

for all objects \(A\). More pictorially, if \(A \xrightarrow{X_1(A)} B \xrightarrow{X_2(B)} C\) then \((X_1 \& X_2)(A)\) is the arrow from \(A\) to \(C\) defined by \((X_1 \& X_2)(A) := X_2(B) \circ X_1(A)\).

An important feature of this combination law is that it is \textit{associative}. Thus, for any three arrow fields \(X_1, X_2, X_3\) we have

\[X_1 \&(X_2 \& X_3) = (X_1 \& X_2) \& X_3.\]  \hspace{1cm} (3.3)

Furthermore, if an arrow-field \(\iota\) is defined by

\[\iota(A) := \text{id}_A\]  \hspace{1cm} (3.4)

for all objects \(A\), then \(\iota\) is an identity element for the ‘\&’ composition law, \textit{i.e.}, \(\iota \& X = X \& \iota = X\) for all arrow fields \(X\).

It follows that the set of all arrow fields on the category \(Q\) is a \textit{monoid} when equipped with the combination law in Eq. (3.2) and the identity in Eq. (3.4). We will denote this monoid of arrow fields by \(\text{AF}(Q)\). A key idea in constructing a quantum theory on \(Q\) is that the monoid \(\text{AF}(Q)\) can play a role analogous to that of \(\text{Diff}(Q)\) in the case of a manifold \(Q\).

For a manifold \(Q \simeq G/H\), the group \(G\) acts on \(Q\) as a group of transformations, and we need the analogue of that here. Specifically, we define an action of an arrow field \(X\) on the set \(\text{Ob}(Q)\) by

\[\rho_X(A) := \text{Ran } X(A)\]  \hspace{1cm} (3.5)

Note that the analogous quantity in [1], [2], [3] was denoted \(X_2 \& X_1\), rather than \(X_1 \& X_2\). Of course, this is entirely a matter of convention, but the choice made in the present paper is more convenient for certain purposes.
for all objects $A$; note that this is a right action since
\[ \rho_{X_2} \circ \rho_{X_1} = \rho_{X_1 \& X_2} \] (3.6)
for all arrow fields $X_1, X_2$. An example, of this action is provided by the arrow field shown in Figure 2: the objects $A_1, A_2, A_3$ are mapped to $B$, the object $B$ is mapped to $C$, and $C$ is mapped to itself.

Individual arrows are associated with special arrow fields. Thus, if $f : A \to B$ we define an arrow field $X_f$ by
\[ X_f(C) := \begin{cases} f & \text{if } C = A; \\ \text{id}_C & \text{otherwise.} \end{cases} \] (3.7)
This arrow field acts on $\text{Ob}(Q)$ by $\rho_{X_f}(A) = B$, and $\rho_{X_f}(C) = C$ for all objects $C \neq A$.

Note that if $f$ and $g$ are arrows in the monoid $\text{Hom}(A, A)$, then
\[ X_f \& X_g = X_{g \circ f} \] (3.8)
and hence the set of all arrow fields of the type $X_f, f \in \text{Hom}(A, A)$, gives a (anti)-representation of the monoid $\text{Hom}(A, A)$ in the monoid $\text{AF}(Q)$.

4 Arrow-Field Quantum Theory

4.1 The Basic Quantum Algebra

The next step is to find an analogue for a category $Q$ of the quantisation group Eqs. (2.14–2.16) for a configuration manifold $Q$. The construction of this group involves the cotangent bundle $T^* Q$, which is the classical state space of the system. However, there is no immediate analogue of this for a general category, and therefore we shall proceed in a more heuristic way.

For a general manifold $Q$, physical momenta are associated with the diffeomorphism group $\text{Diff}(Q)$, and—as argued already—the analogue of this for a category $Q$ is the monoid $\text{AF}(Q)$ which acts on the ‘configuration space’ $\text{Ob}(Q)$ according to Eq. (3.5). The configuration variables on a manifold are smooth, real-valued functions on $Q$, and the analogue of this for a category would be some ‘appropriate’ subspace of the vector space $F(\text{Ob}(Q), \mathbb{R})$ of all real-valued functions on $\text{Ob}(Q)$. However, for a general category there is
no obvious preferred such subspace, which obliges us to work with the entire
space $F(\text{Ob}(Q), \mathbb{R})$, albeit with the understanding that for specific categories
$Q$ there may be natural analogues of the subspace $C^\infty(Q) \subset F(Q, \mathbb{R})$ (or
even, perhaps, of the subspace $W \subset C^\infty(Q)$ for the case $Q \simeq G/H$).

The quantisation group for a general manifold $Q$ is the semi-direct prod-
uct $\text{Diff}(Q) \times_{\tau} C^\infty(Q)$ with the group law in Eq. (2.23). This suggests
strongly that the analogue for a category $Q$ should be the semi-direct prod-
uct, $\text{AF}(Q) \times_{\rho} F(\text{Ob}(Q), \mathbb{R})$, of the monoid $\text{AF}(Q)$ and the vector space
$F(\text{Ob}(Q), \mathbb{R})$. The combination law is

$$ (X_1, \beta_1)(X_2, \beta_2) = (X_1 \& X_2, \beta_1 + \beta_2 \circ \rho_{X_1}) \quad (4.1) $$

for all $X_1, X_2 \in \text{AF}(Q)$ and $\beta_1, \beta_2 \in F(\text{Ob}(Q), \mathbb{R})$. We shall refer to
$\text{AF}(Q) \times_{\rho} F(\text{Ob}(Q), \mathbb{R})$ as the category quantisation monoid (or just CQM)
for the category $Q$.

4.2 Representations on ‘wave-functions’

Although plausible, the derivation above of the CQM may seem a little ad
hoc, and it is useful therefore to check the overall consistency of these ideas
by looking for the simplest type of representation, which involves taking state
vectors to be functions $\psi : \text{Ob}(Q) \to \mathbb{C}$.

Motivated by Eq. (2.18), we define, for each arrow field $X$, the operator

$$ (\hat{a}(X)\psi)(A) := \psi(\rho_X A) = \psi(\text{Ran } X(A)). \quad (4.2) $$

It can readily be checked that these operators satisfy

$$ \hat{a}(X_1)\hat{a}(X_2) = \hat{a}(X_1 \& X_2) \quad (4.3) $$

for all arrow fields $X_1, X_2$. Thus we have a representation of the monoid of
arrow fields. Equation (4.3) is a precise analogue of the relation Eq. (2.14)
for a manifold.

Turning now to the configuration variables, and guided by Eq. (2.19), we
define, for each $\beta \in F(\text{Ob}(Q), \mathbb{R})$, the operator $\hat{\beta}$:

$$ (\hat{\beta}\psi)(A) := \beta(A)\psi(A) \quad (4.4) $$

and its exponentiated form:

$$ (\hat{V}(\beta)\psi)(A) := e^{i\beta(A)}\psi(A) \quad (4.5) $$
for all objects $A$ in the category.

These operators satisfy the relations (c.f., Eqs. (2.14–2.16))

\begin{align*}
\hat{a}(X_1)\hat{a}(X_2) &= \hat{a}(X_1\&X_2) \quad (4.6) \\
\hat{V}(\beta_1)\hat{V}(\beta_2) &= \hat{V}(\beta_1 + \beta_2) \quad (4.7) \\
\hat{a}(X)\hat{V}(\beta) &= \hat{V}(\beta \circ \rho_X)\hat{a}(X) \quad (4.8)
\end{align*}

for all arrow fields $X, X_1, X_2$ and real-valued functions $\beta, \beta_1, \beta_2$. The relations in Eqs. (4.6–4.8) constitute a representation of the CQM, $AF(Q) \times_\rho F(\text{Ob}(Q), \mathbb{R})$, defined in Eq. (4.1). This helps to justify the claim that the CQM is the category analogue of the group $\text{Diff}(Q) \times_\tau C^\infty(Q)$ used when the configuration space is a manifold $Q$.

Thus I shall define the quantisations on the category $Q$ to be in one-to-one correspondence with faithful, irreducible representations of the category quantisation monoid $AF(Q) \times_\rho F(\text{Ob}(Q), \mathbb{R})$. However, as we shall see, the representation on wave-functions given by Eq. (4.2) and Eq. (4.5) is not faithful, and therefore we need some more complicated representations of the CQM.

In the case of a configuration manifold $Q$, the operators $\hat{U}(g), g \in G$, and $\hat{V}(\beta), \beta \in C^\infty(Q)$, are required to be unitary, and we may wonder what the analogue of that would/should be for the CQM. This requires placing an inner product on the quantum states and, in the simple example of the ‘wave-function’ representation given by Eq. (4.2) and Eq. (4.5), this would be of the form

$$\langle \psi | \phi \rangle := \int_{\text{Ob}(Q)} d\mu(A) \psi(A)^*\phi(A) \quad (4.9)$$

where $\mu$ is some suitable measure on the space $\text{Ob}(Q)$. Of course, at this stage it is not clear what ‘suitable’ means since representations of a monoid (as opposed to a group) are not unitary (because of the absence of inverse elements) whereas, in the group case, it is the desired unitarity of the representations that essentially determines the measure.

However, if the category has only a finite number of objects (such as the toy model in Figure 1) it is natural to use the simple ‘counting’ measure

$$\langle \psi | \phi \rangle := \sum_{A \in \text{Ob}(Q)} \psi(A)^*\phi(A). \quad (4.10)$$

With due care, this can also be used if the category has a countably infinite number of objects.
4.3 Introducing a Multiplier

An important requirement for a representation of the CQM is that it separates arrows. Namely: if \( f, g \) are different arrows with the same domain and the same range, then \( \hat{a}(X_f) \neq \hat{a}(X_g) \). In particular, this applies if the domain and range of the arrows \( f, g \) are a single object \( A \), in which case the requirement is that the (anti-) representation of each monoid \( \text{Hom}(A, A) \), \( A \in \text{Ob}(Q) \) given by the operators \( \hat{a}(X_f), f \in \text{Hom}(A, A) \), (using Eq. (3.8)) is faithful. Indeed, the monoid \( \text{Hom}(A, A) \) is one measure of the internal structure of the object \( A \), and we want the quantum theory to reflect this structure in a faithful way.

However, the representation given by the operators \( \hat{a}(X) \) defined in Eq. (4.2) is clearly far from faithful. Indeed, since the action of \( \hat{a}(X) \) on a state function at an object \( A \) depends only on the object \( \text{Ran} X(A) \), arrows with the same domain and range are never separated. In particular, the monoid \( \text{Hom}(A, A) \) is represented trivially for all objects \( A \).

Turning for guidance to the case of a configuration manifold \( Q \simeq G/H \), the theory of induced representations shows that the most general irreducible representation of the quantisation group \( G \times \tau W^* \) is specified by (i) an orbit of \( G \) on \( W \); and (ii) an irreducible representation of \( H \) on a Hilbert space \( V \). This representation of \( G \times \tau W^* \) is defined on cross-sections of a vector bundle over the orbit with fibre \( V \). The generic orbits on \( W \) are diffeomorphic to the configuration manifold \( Q \).

This suggests that in the category case, we should introduce some bundle \( K \) of Hilbert spaces over the set of objects \( \text{Ob}(Q) \), with an associated inner product on the sections of this bundle given by

\[
\langle \psi|\phi \rangle := \int_{\text{Ob}(Q)} d\mu(A) \langle \psi(A), \phi(A) \rangle_{K(A)}
\]

(4.11)

where \( \langle \psi(A), \phi(A) \rangle_{K(A)} \) denotes the inner product in the Hilbert space \( K(A) \) of the vectors \( \psi(A), \phi(A) \in K(A) \), where \( K(A) \) is the fibre over the object \( A \).

A necessary requirement for this Hilbert bundle to give an arrow-separating representation is that, for each object \( A \), the vector space \( K(A) \) should carry a faithful, irreducible, representation of the monoid \( \text{Hom}(A, A) \) of arrows from \( A \) to itself. However, the internal structure of an object \( A \), and hence of \( \text{Hom}(A, A) \), generally depends on the object \( A \) (for example, consider the monoid of order-preserving maps from each object in Figure 1 to itself), and
this means that the Hilbert space $\mathcal{K}(A)$ should depend on $A$. But in a normal
vector bundle the fibres are all isomorphic to each other, and hence we are
looking for something more general: as we shall see shortly, this is a presheaf
of Hilbert spaces.

With this caveat, we now consider the category analogue of the multiplier
group representations defined in Eqs. (2.20–2.21). Thus, to each arrow field
$X$ and object $A$, we require a linear map (a ‘multiplier’) $m(X, A) : \mathcal{K}(\rho_X A) \to
\mathcal{K}(A)$ from the Hilbert space $\mathcal{K}(\rho_X A)$ to the Hilbert space $\mathcal{K}(A)$. Then, in
direct analogy with Eqs. (2.20–2.21), we define the operators:

\begin{align}
(\hat{a}(X)\psi)(A) &:= m(X, A)\psi(\rho_X A) \\
(\hat{V}(\beta)\psi)(A) &:= e^{i\beta(A)}\psi(A).
\end{align}

It is easy to check that Eq. (4.12) gives a representation of the monoid
$AF(Q)$ provided the multipliers satisfy the conditions

\begin{equation}
m(X_1, A)m(X_2, \rho_{X_1} A) = m(X_1 \& X_2, A)
\end{equation}

for all arrow fields $X_1, X_2$ and objects $A$. This is the precise analogue for
$AF(Q)$ of the well-known condition Eq. (2.22) on the multipliers in a repre-
sentation of a group $G$ that acts on a configuration space $Q$.

### 4.4 The Presheaf Perspective

As things stand, the multiplier $m(X, A) : \mathcal{K}(\rho_X A) \to \mathcal{K}(A)$ could depend on
the value of the arrow field $X$ on objects other than $A$. However, it seems
natural to try imposing the condition that $m(X, A)$ depends only on the
arrow $X(A)$. Thus we suppose that, for each arrow $f : A \to B$, there is a
linear map $\kappa(f) : \mathcal{K}(B) \to \mathcal{K}(A)$ such that $m(X, A) = \kappa(X(A))$.

The requirement on the multipliers in Eq. (4.14) then becomes the condition

\begin{equation}
\kappa(f)\kappa(g) = \kappa(g \circ f)
\end{equation}

for all arrows $f, g$ such that $\text{Ran } f = \text{Dom } g$. This is illustrated in the

diagram

\begin{equation}
\begin{array}{ccc}
\mathcal{K}(A) & \xleftarrow{\kappa(f)} & \mathcal{K}(B) \\
A & \xrightarrow{f} & B \\
\mathcal{K}(C) & \xrightarrow{\kappa(g)} & C
\end{array}
\end{equation}
However, these are precisely the conditions for a \textit{presheaf} of Hilbert spaces over the category $\mathcal{Q}$. Thus a representation of the CQM can be obtained from each such presheaf of Hilbert spaces over $\mathcal{Q}$.\footnote{Something rather similar happens mathematically in the case of topological quantum field theory, and the category quantisation methods may have important applications to that area. This is currently being investigated.}

\section{The adjoints of the operators $\hat{a}(X)$}

As mentioned earlier, because of the existence of non-invertible elements, a monoid representation will not be unitary. This lends interest to the question of what the adjoints of the operators $\hat{a}(X)$ look like.

The main features can be illustrated with the simple example of a category whose number of objects is finite, and with state functions $\psi : \text{Ob}(\mathcal{Q}) \to \mathbb{C}$ (\textit{i.e.}, no multipliers), as in Eq. (4.2) and Eq. (4.5). The simplest inner product is Eq. (4.10), and using standard Dirac notation, with $\psi(A)$ denoted $\langle A|\psi \rangle$, we have

$$\hat{a}(X)^\dagger |B\rangle = |\rho_X B\rangle = |\text{Ran } X(B)\rangle$$

$$\hat{a}(X) |B\rangle = \sum_{A \in \rho_X^{-1}\{B\}} |A\rangle$$

where $\rho_X^{-1}\{B\}$ denotes the set of all objects $A$ such that $\rho_X(A) = B$, \textit{i.e.}, such that the range of the arrow $X(A)$ is $B$.

For example, for the arrow field illustrated in Figure 2 in a simple 5-object category, we have (remembering that $X(C) = \text{id}_C$)

$$\hat{a}(X)^\dagger |A_1\rangle = \hat{a}(X)^\dagger |A_2\rangle = \hat{a}(X)^\dagger |A_3\rangle = |B\rangle$$

$$\hat{a}(X)^\dagger |B\rangle = |C\rangle$$

$$\hat{a}(X)^\dagger |C\rangle = |C\rangle$$

and

$$\hat{a}(X) |A_1\rangle = \hat{a}(X) |A_2\rangle = \hat{a}(X) |A_3\rangle = 0$$

$$\hat{a}(X) |B\rangle = |A_1\rangle + |A_2\rangle + |A_3\rangle$$

$$\hat{a}(X) |C\rangle = |B\rangle + |C\rangle$$
Note that the right hand side of Eq. (4.22) is 0 since no objects in the category are mapped to $A_1$, $A_2$, or $A_3$ by the arrow field $X$. In that sense, the operator $\hat{a}(X)$ resembles an *annihilation* operator.

On the other hand, we see from Eq. (4.17) that there are no states $|B\rangle$ such that $\hat{a}(X)^\dagger |B\rangle = 0$ (since every arrow has a range), and in that limited sense $\hat{a}(X)^\dagger$ resembles a *creation* operator. Of course, Eq. (4.17) does not mean that there are *no* states $|\psi\rangle$ such that $\hat{a}(X)^\dagger |\psi\rangle = 0$. For example, for the arrow field represented by Figure 2 we have $\hat{a}(X)^\dagger |A_1\rangle = |B\rangle$ and $\hat{a}(X)^\dagger |A_2\rangle = |B\rangle$, and hence $\hat{a}(X)^\dagger (|A_1\rangle - |A_2\rangle) = 0$.

In a similar way, we find

\[
\hat{a}(X)^\dagger \hat{a}(X) |A\rangle = \sum_{C \in \rho^{-1}_X \{A\}} |C\rangle
\]

\[
\hat{a}(X)^\dagger \hat{a}(X) |A\rangle = |\rho^{-1}_X \{A\}||A\rangle
\]

where $|\rho^{-1}_X \{A\}|$ denotes the number of elements in the set $\rho^{-1}_X \{A\}$. For example, for the arrow field in Figure 2 we have

\[
\hat{a}(X)^\dagger \hat{a}(X) |B\rangle = 3 |B\rangle.
\]

The results for multiplier representations are similar to the above; for details see [1].

### 4.6 The analogue of momentum

If $Q$ is a manifold $G/H$, the analogue of momentum is played by the elements of the Lie algebra, $L(G)$, of the Lie group $G$. Thus, to each $k \in L(G)$ there is associated the one-parameter subgroup $t \mapsto \exp itk$ of $G$, and this is represented on the quantum Hilbert space by a one-parameter group of unitary operators. Then, by Stone’s theorem, there exists a unique self-adjoint operator $\hat{k}$ such that $\hat{U}(\exp itk) = e^{it\hat{k}}$.

The operators $\hat{k}$, $k \in L(G)$, give a Hilbert space representation of the Lie algebra $L(G)$. From a physical perspective, these operators are the analogue for this system of momentum variables.

The situation for a general category $Q$ is different. The analogue of the operators $\hat{U}(g)$, $g \in G$, is the operators $\hat{a}(X)$, $X \in \operatorname{AF}(Q)$. But there are some important differences: (i) there is no analogue of the Lie algebra of $G$; and (ii) the operators $\hat{a}(X)$ are not unitary. However, quantum physical
variables are associated with self-adjoint operators, and we have to construct these in some way from what we have: namely the operators $\hat{a}(X)$, $X \in \text{AF}(Q)$. To this end we define
\begin{align*}
\hat{\alpha}(X) & := \frac{1}{2} \left( \hat{a}(X) + \hat{a}(X)^\dagger \right) \\
\hat{\beta}(X) & := \frac{1}{2i} \left( \hat{a}(X) - \hat{a}(X)^\dagger \right)
\end{align*}
so that $\hat{a}(X) = \hat{\alpha}(X) + i\hat{\beta}(X)$, where $\hat{\alpha}(X)$ and $\hat{\beta}(X)$ are self-adjoint.

For a manifold $Q \simeq G/H$, we can write $\exp it\hat{k} = \cos t\hat{k} + i \sin t\hat{k}$, and hence $\hat{\alpha}(X)$ and $\hat{\beta}(X)$ are analogous to $\cos t\hat{k}$ and $\sin t\hat{k}$ respectively. The expansion $\sin r \simeq r + O(r^3)$ then suggests that $\hat{\beta}(X)$ is an analogue of a momentum operator.

To clarify this, consider a model quantum theory with basic states $|n\rangle$ where $n = 0, 1, 2, \ldots$, and with the translation operator $\hat{T}^\dagger$ defined by
\begin{equation}
\hat{T}^\dagger |n\rangle := |n+1\rangle
\end{equation}
for all $n$. This is the analogue of Eq. (4.17), and the analogue of Eq. (4.18) is
\begin{equation}
\hat{T} |n\rangle := \begin{cases} |n-1\rangle & \text{if } n > 0; \\ 0 & \text{if } n = 0. \end{cases}
\end{equation}
Then, the operator $\hat{\beta} := (\hat{T} - \hat{T}^\dagger)/2i$ acts on the states $|n\rangle$ as the difference operator
\begin{equation}
\hat{\beta} |n\rangle = \frac{1}{2i} \left( |n+1\rangle - |n-1\rangle \right)
\end{equation}
for all $n > 0$ (and with $\hat{\beta} |0\rangle = \frac{i}{2} |1\rangle$). This can be viewed as a discrete analogue of the familiar momentum differential operator.

In the category case, Eqs. (4.17–4.18) give
\begin{equation}
\hat{\beta}(X) |B\rangle = \frac{i}{2} \left( |\text{Ran } X(B)\rangle - \sum_{A \in \rho_{X}^{-1}(B)} |A\rangle \right)
\end{equation}
\begin{footnote}
13 This model can be viewed as a special case of the category scheme in which the objects are the integers $0, 1, 2, \ldots$, and there is a single arrow from $n$ to $m$ if $m \geq n$. The operator $\hat{T}$ then corresponds to the special arrow field that associates to each $n$ the arrow $n \rightarrow n+1$.
\end{footnote}
which is also a type of difference operator. So, in that sense, $\hat{\beta}(X)$ is the category equivalent of momentum.

Similarly, if we define $\hat{\alpha} := (\hat{T} + \hat{T}^\dagger)/2$, then (if $n > 0$)

$$\hat{\alpha}|n\rangle = \frac{1}{2}(|n - 1\rangle + |n + 1\rangle)$$

which is a type of ‘average position’ operator. In the category case we have

$$\hat{\alpha}(X)|B\rangle = \frac{1}{2} \left( \sum_{A \in \rho_X \setminus \{B\}} |A\rangle + |\text{Ran } X(B)\rangle \right)$$

which is therefore a type of ‘average configuration point’ operator.

We note that the special arrow fields of the type $X_f$ defined in Eq. (3.7) satisfy $X_f \& X_f = X_f$. As a result, if we define $\hat{a}(f) := \hat{a}(X_f)$ we find that $\hat{a}(f)\hat{a}(f) = \hat{a}(f)$, so that $\hat{a}(f)$ is a non-hermitian projection operator. It is easy to check that, in this case, we have, for all arrows $f$,

$$\hat{\alpha}(f)^2 - \hat{\alpha}(f) = \hat{\beta}(f)$$

and hence $\hat{\beta}(f)$ is not an independent variable from $\hat{\alpha}(f)$.

### 4.7 Irreducibility of the representations

Let us now say something about the irreducibility, or otherwise, of the multiplier representations of the category quantisation monoid.

It seems unlikely that one could develop a complete representation theory of the CQM for an arbitrary small category $\mathcal{Q}$. However, in the manifold analogy with $\mathcal{Q} \simeq G/H$, it is important that $G$ acts transitively on $\mathcal{Q}$. If this is not the case—so that $\mathcal{Q}$ can be decomposed into more than one $G$-orbit—then there is a corresponding decomposition of the group representation of $G \times \tau W$ into a direct sum or direct integral. It seems plausible that this has a category analogue, and so a natural question is whether $\text{Ob}(\mathcal{Q})$ is a single orbit under the action of $\text{AF}(\mathcal{Q})$.

Because of the absence of inverse elements, the concept of an ‘orbit’ is more subtle for an action of a monoid on a set than it is for a group. However, on looking at the operators $\hat{a}(X)$ and $\hat{a}(X)^\dagger$—as, for example, in the simple expressions in Eq. (4.18) and Eq. (4.17)—it seems natural to define a subset
$O$ of $\text{Ob}(\mathcal{Q})$ to be ‘connected’ if for any pair of objects $A, B \in O$ there exists a finite collection of objects $\{A_1, A_2, \ldots, A_N\} \subset O$, with $A_1 = A$, $A_N = B$ and such that, for all $i = 1, 2, \ldots, N - 1$, there exists an arrow with domain $A_i$ and range $A_{i+1}$, or an arrow with range $A_i$ and domain $A_{i+1}$.

Clearly, if $\text{Ob}(\mathcal{Q})$ decomposes into a disjoint union of connected subsets, then the representation of the CQM will decompose in a corresponding way. Thus a necessary condition for the representation to be irreducible is that $\text{Ob}(\mathcal{Q})$ is connected. However, connectedness alone is not sufficient to guarantee irreducibility. Another necessary requirement is that, for all objects $A$, the representations of $\text{Hom}(A, A)$ on the Hilbert spaces $\mathcal{K}(A)$ should be irreducible (analogous to the requirement for an induced group representation that the representation of the little group $H$ on $V$ is irreducible). However, there is no reason to expect that these conditions are sufficient to guarantee irreducibility for a general category $\mathcal{Q}$, and it seems likely that this issue has to be settled with a case-by-case study for categories of physical importance.

5 When $\mathcal{Q}$ is a Category of Sets

5.1 Finite or countably infinite sets

In practice, many of the categories encountered in physics have objects that are sets with internal structure, such as causal sets or topological spaces. It is important, therefore, to see how the general quantisation scheme outlined above can be explicitly implemented in this case. The key questions to be addressed are:

i) Is there a natural choice for the Hilbert spaces $\mathcal{K}(A)$, $A \in \text{Ob}(\mathcal{Q})$?

ii) Given such a choice, can we find linear maps $\kappa(f) : \mathcal{K}(B) \to \mathcal{K}(A)$, where $f : A \to B$, that satisfy the (presheaf) conditions in Eq. (4.15)?

To make life simpler, we start with the case in which $\mathcal{Q}$ is a category of finite sets; for example, as in Figure 1.

A natural vector space associated with any set $A$ is the space $F(A, \mathbb{C})$ of all complex-valued functions on $A$. If $A = \{a_1, a_2, \ldots, a_n\}$ is a finite set, then each function $f : A \to \mathbb{C}$ leads to an $n$-tuple of complex numbers, namely $(f(a_1), f(a_2), \ldots, f(a_n))$. Conversely, to each $n$-tuple of complex numbers
(c_1, c_2, \ldots, c_n) there is associated a function \( f : A \rightarrow \mathbb{C} \), defined by \( f(a_i) := c_i \) for all \( i = 1, 2, \ldots, n \). Thus we have an isomorphism \( F(A, \mathbb{C}) \simeq \mathbb{C}^{|A|} \), where \(|A|\) denotes the number of elements in the finite set \( A \). Furthermore, since \( \mathbb{C}^n \) is a Hilbert space for any \( n < \infty \), we can make \( F(A, \mathbb{C}) \) into a Hilbert space by defining the inner product:

\[
\langle f, g \rangle := \sum_{a \in A} f^*(a) g(a).
\] (5.1)

If \( A \) is countably infinite, the inner product in Eq. (5.1) can still be used provided we restrict our attention to square-summable functions \( f \); i.e., functions for which the sum

\[
\| f \| := \left\{ \sum_{a \in A} |f(a)|^2 \right\}^{1/2}
\] (5.2)

converges. We shall denote this Hilbert space by \( \ell^2(A) \).

Thus if \( A \) is finite or countably infinite, there is a natural choice for the Hilbert space \( \mathcal{K}(A) \): namely, \( \mathbb{C}^{|A|} \) or \( \ell^2(A) \) respectively. The next thing to consider is if \( f : A \rightarrow B \) is a function between sets \( A \) and \( B \), is there some natural induced map \( \kappa(f) : \mathcal{K}(B) \rightarrow \mathcal{K}(A) \)?

In the case of a category of finite sets, there is indeed such a map. For if \( v : B \rightarrow \mathbb{C} \) belongs to \( \mathcal{K}(B) \simeq \mathbb{C}^{|B|} \), then, using the diagram \( A \xrightarrow{f} B \xrightarrow{v} \mathbb{C} \), we see that the natural 'pull-back' of \( v : B \rightarrow \mathbb{C} \) is the complex-valued function \( v \circ f \) on \( A \). Thus we try defining \( \kappa(f) : \mathbb{C}^{|B|} \rightarrow \mathbb{C}^{|A|} \) by

\[
(\kappa(f)v)(a) := v(f(a))
\] (5.3)

for all \( a \in A \). Then, if \( A \xrightarrow{f} B \xrightarrow{g} C \), and if \( v : C \rightarrow \mathbb{C} \), we have

\[
(\kappa(g \circ f)v)(a) = v(g \circ f(a)) = v(g(f(a))
\] (5.4)

whereas

\[
(\kappa(f)\kappa(g)v)(a) = (\kappa(g)v)(f(a)) = v(g(f(a))),
\] (5.5)

and hence \( \kappa(f)\kappa(g) = \kappa(g \circ f) \), as required.

One could try to do the same thing in the case where there are a countably infinite number of objects, with the Hilbert spaces \( \mathcal{K}(A) \), \( A \in \text{Ob}(\mathcal{Q}) \), chosen to be \( \ell^2(A) \). However, a more careful analysis of the situation is required now, since a square-summable sequence (i.e., an element of \( \ell^2(B) \)) may not be taken into another such sequence by the map \( \kappa(f) \) defined in Eq. (5.3).
5.2 A very simple example

We will now illustrate the general scheme with the aid of a very simple example. This is a category with just two objects: the causal sets $A := \{a\}$, and $B := \{b_1, b_2\}$ with $b_1 \leq b_2$. This is shown in Figure 3.

\[ \begin{array}{c}
\bullet a \\
A \\
\bullet b_1 \\
B \\
\bullet b_2
\end{array} \]

Figure 3: A collection of two causal sets

The arrows are order-preserving maps, and there are two such, $f_1$, $f_2$, from $A$ to $B$ defined by

\[ f_1(a) = b_1 \tag{5.6} \]

and

\[ f_2(a) := b_2 \tag{5.7} \]

respectively, and one arrow $g$ from $B$ to $A$ defined by

\[ g(b_1) := a, \ g(b_2) := a. \tag{5.8} \]

A non-trivial arrow $r : B \to B$ is

\[ r(b_1) := b_1, \ r(b_2) := b_1 \tag{5.9} \]

as is $s : B \to B$ defined by

\[ s(b_1) := b_2, \ s(b_2) := b_2. \tag{5.10} \]

If we forget the causal structure on $B$, an additional arrow $p : B \to B$ is

\[ p(b_1) := b_2, \ p(b_2) := b_1 \tag{5.11} \]

which is the non-trivial element of the permutation group $\mathbb{Z}_2$ of the set $B$. It is not, however, an arrow in the category $\mathcal{Q}$ since it reverses the ordering of the elements $b_1, b_2 \in B$. 

24
In summary, we have the following sets of arrows:

\[
\text{Hom}(A, B) = \{f_1, f_2\} \quad (5.12)
\]
\[
\text{Hom}(B, A) = \{g\} \quad (5.13)
\]
\[
\text{Hom}(A, A) = \{\text{id}_A\} \quad (5.14)
\]
\[
\text{Hom}(B, B) = \{\text{id}_B, r, s\} \quad (5.15)
\]
to which should be added the map \( p : B \to B \) in Eq. (5.11) if we forget the causal structure on \( B \).

The discussion in Section 5.1 suggests that the appropriate choices for the Hilbert space fibres are \( \mathcal{K}[A] = \mathbb{C}^1 \cong \mathbb{C} \), and \( \mathcal{K}[B] = \mathbb{C}^2 \). Thus the quantum state space of the system is \( \mathbb{C} \oplus \mathbb{C}^2 \cong \mathbb{C}^3 \); a vector \( \psi \) in this space will be denoted by \( (\psi_A; \psi_{B_1}, \psi_{B_2}) \in \mathbb{C}^3 \).

The expression Eq. (5.3) for the multiplier, gives the following representations Eq. (4.12) of the non-trivial arrow fields \( X_f, f \in \text{Hom}(Q) \), where we denote \( \hat{a}(X_f) \) by \( \hat{a}(f) := \hat{a}(X_f) \):

\[
\hat{a}(f_1) : (\psi_A; \psi_{B_1}, \psi_{B_2}) \mapsto (\psi_{B_1}; \psi_{B_1}, \psi_{B_2}) \quad (5.16)
\]
\[
\hat{a}(f_2) : (\psi_A; \psi_{B_1}, \psi_{B_2}) \mapsto (\psi_{B_2}; \psi_{B_1}, \psi_{B_2}) \quad (5.17)
\]
\[
\hat{a}(g) : (\psi_A; \psi_{B_1}, \psi_{B_2}) \mapsto (\psi_A; \psi_A, \psi_A) \quad (5.18)
\]
\[
\hat{a}(r) : (\psi_A; \psi_{B_1}, \psi_{B_2}) \mapsto (\psi_A; \psi_{B_1}, \psi_{B_1}) \quad (5.19)
\]
\[
\hat{a}(s) : (\psi_A; \psi_{B_1}, \psi_{B_2}) \mapsto (\psi_A; \psi_{B_2}, \psi_{B_2}). \quad (5.20)
\]

Note that arrows with the same domain and range are indeed separated.

Of course, it is easy to write these operators as \( 3 \times 3 \) matrices. For example,

\[
\hat{a}(f_1) : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} b \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}; \quad (5.21)
\]

so that

\[
\hat{a}(f_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (5.22)
\]

The complete set of operators is

\[
\hat{a}(f_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{a}(f_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (5.23)
\]
\[ \hat{a}(g) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \]

\[ \hat{a}(r) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{a}(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ \hat{\beta} = \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_2 \end{pmatrix} \]

where \( \beta_1 := \beta(A) \) and \( \beta_2 := \beta(B) \).

If we forget the causal structure on the set \( B \), then there is the additional arrow \( p : B \to B \) defined in Eq. (5.11). The corresponding arrow field \( X_p \) is represented in the quantum theory by the matrix

\[ \hat{a}(p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \] (5.24)

### 6 Conclusions

We have seen how to construct a quantum scheme for a system whose configuration space (or history analogue) is the set of object \( \text{Ob}(\mathcal{Q}) \) in a small category \( \mathcal{Q} \). A key ingredient is the monoid \( \text{AF}(\mathcal{Q}) \) of arrow fields and its action on \( \text{Ob}(\mathcal{Q}) \). Multiplier representations are needed to distinguish quantum theoretically between arrows with the same range and domain. Each such representation can be expressed in terms of a presheaf of Hilbert spaces over \( \text{Ob}(\mathcal{Q}) \). For the example of a category of finite sets we have shown how to construct an explicit example of such a presheaf.

This scheme includes as a special case the situation in which the configuration space is a manifold \( Q \) on which a Lie group \( G \) acts transitively, so that \( Q \simeq G/H \) for some subgroup \( H \) of \( G \). In this sense, the scheme can be viewed as a big extension of standard quantisation methods to include types of system whose configuration spaces (or history analogue) are far from being points in a smooth manifold.

It is an exciting challenge to use these techniques for constructing novel theories of quantised space or space-time. However, it should be emphasised
that what is described in the present paper is only a ‘tool-kit’ for constructing such theories: it needs a creative leap to use these tools to construct a physically realistic model of, for example, quantum causal sets. The key step would be to choose a decoherence functional for the quantum history theory. This decoherence functional would be constructed from the operators described in this paper, but new physical principles are needed to decide its precise form. This is an important topic for future research.

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