Privacy Amplification, Private States, and the Uncertainty Principle

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We show that three principle means of treating privacy amplification in quantum key distribution, private state distillation, classical privacy amplification, and via the uncertainty principle, are equivalent and interchangeable. By adapting the security proof based on the uncertainty principle, we construct a new protocol for private state distillation which we prove is identical to standard classical privacy amplification. Underlying this approach is a new characterization of private states, related to their standard formulation by the uncertainty principle, which gives a more physical understanding of security in quantum key distribution.

Privacy amplification is the art of extracting a secret key from a string which is partially-known to an eavesdropper \(^1\,2\). In quantum key distribution (QKD) it plays a vital role as the protagonists, Alice and Bob, would like to transform their shared, but not secret, raw key into a verifiably secret key even when the eavesdropper Eve has tampered with the quantum signals.

Heuristically, privacy amplification works by applying a suitable randomly-chosen function to the raw key which scrambles and shortens it so that Eve’s limited knowledge of the input tells her nothing about the output. The canonical example is using a random public string and computing the XOR with the original string. Provided Eve’s information is not too large, Alice and Bob can be confident that the output will be secret.

Broadly speaking, QKD has historically taken three main approaches to privacy amplification. Each is characterized by its treatment of the states held by the various parties to the protocol. The first focuses on Eve’s marginal state conditional on the key string, which she obtains in the course of eavesdropping. Applying a random function to the key string results in new marginal states for Eve which are essentially identical. We term this method classical privacy amplification as it is an adaptation of privacy amplification against classical adversaries. It can be traced through the sequence of papers \(^2\,4\,5\,6\,7\).

The remaining two approaches focus either concretely on the states held by Alice and Bob, including any auxiliary systems, or abstractly on the key itself. In the former, privacy amplification is recast as a virtual form of private state distillation in which Alice and Bob transform their initial shared quantum state into a private state, a state which yields secret keys upon measurement \(^8\). Maximally entangled states are a subset of private states, so this method includes the techniques of applying entanglement distillation to privacy amplification developed in \(^5\,10\,11\) and the subsequent work employing the technique of Shor and Preskill. Means for distilling more general private states were found in \(^12\).

The latter approach of focusing abstractly on the key itself, irrespective of its realization by either honest party and disregarding any auxiliary systems not held by Eve, was employed in the first QKD security proof by Mayers \(^13\), subsequently improved by Koashi and Preskill \(^14\), and finally culminated in a security proof based on the uncertainty principle by Koashi \(^15\). Here privacy amplification is viewed as a means of creating a virtual Pauli X eigenstate and then obtaining the key by measuring the conjugate Z observable.

In this letter we draw these three threads together and show they are equivalent when privacy amplification is based on linear functions. We do so by adapting Koashi’s proof to give a new method of private state distillation and then prove it is identical to classical privacy amplification. The distillation technique follows from a new characterization of private states which is complementary to their standard description in the sense of the uncertainty principle. This unifies various approaches to the security of QKD, allowing the various means of treating privacy amplification to be interchanged. Moreover, it provides a more physical picture of how security arises from quantum mechanics.

The new private state distillation method significantly generalizes that presented in \(^12\), which directly applied entanglement distillation techniques. Correction of phase errors afflicting the key subsystems becomes easier for private states as the shield can store phase error information. Thus, not all phase errors need be corrected, increasing the secret key yield above the entanglement yield. However, the resulting rates still do not always match those of classical privacy amplification as the shared state is not always a classical mixture of states subjected to various phase errors.

Our results are presented as follows. We first show how the uncertainty principle inspires dual descriptions of private states. Then the method of classical privacy amplification is shortly recounted before proceeding to the new approach to private state distillation. The details of the derivation of the secret key rate are presented from which the equivalence of the methods follows. Finally, we conclude with a view to open problems and related issues.

**Secret Keys and Private States.**—A perfect secret key shared by Alice and Bob is a uniformly-distributed
random variable about which Eve has zero information. Thus a perfect secret bit is defined as $κ_{ABE} = \left(\frac{1}{2} \sum_{k=0}^{1} \rho^k_A ⊗ P^k_B\right) ⊗ ρ_E$ for any $ρ_E$, where $P^k = |k⟩⟨k|$.

Private states are those quantum states for which independent measurements by Alice and Bob yield a secret key. For secret bits, our focus in the remainder of the paper, these measurements might as well be standard basis measurements on the qubit key registers $A$ and $B$. The overall state can be purified by including additional systems, be they shield systems $S$ under the control of Alice and/or Bob or Eve’s systems $E$. A private state $γ_{ABSE}$ is then a pure state of the form

$$\frac{1}{√2} \sum_k |kk⟩⟨k|_S ⊗ U_{BS} |Φ⟩_{AB} |ξ⟩_S,$$

where the unitaries $U^k$ as well as the state $|ξ⟩$ are arbitrary. The state $|Φ⟩$ is the canonical maximally-entangled state and the unitary $U_{AB} = \sum_j P^j_{AB} ⊗ V^j_S$ is called a twisting operator.

The fact that private states lead to secret keys and secret keys come from private states immediately follows (cf. Theorem 2 of [16]). Measurement of a private state $γ_{ABSE}$ immediately yields $κ_{ABE}$ with $ρ_E = |ξ⟩_E$. Conversely, suppose $γ_{ABSE}$ is a pure state yielding $κ_{ABE}$ under the prescribed measurement. It follows that $|γ⟩_{ABSE} = \frac{1}{√2} \sum_k |kk⟩⟨k|_S ⊗ |ϕ^k⟩_SE$ for some arbitrary normalized states $|ϕ^k⟩$ and furthermore, that $ϕ^k_E = ρ_E$ for all $k$. Calling $|ξ⟩_S$ the purification of $ρ_E$, we must have $|ϕ^k⟩_S = V^k_S |ξ⟩_S$ for some unitaries $V^k_S$ since all purifications of the same state are related by unitaries on the purifying system. We have implicitly proven

**Theorem 1.** A pure state $γ_{ABSE}$ is a private state if and only if (a) $p_{j,k} = Tr[γ_{ABSE} P^j_{AB}^{k}] = \frac{1}{2} δ_{j,k}$, and (b) $γ_E^k = γ_E^k$ for all $j,k$.

This formulation is straightforward: Eve can obtain no information about the key when all her marginal states are identical. The approach of classical privacy amplification is to prove the shared output state has this property.

A different characterization of private states follows from considering a hypothetical measurement by Alice in the $x$-basis. This produces conditional states of the $BS$ subsystem: $κ_{BS} = 2 A⟨\bar{x}|γ_{AB}|\bar{x}⟩_A$, where $|\bar{x}⟩$ is the $x$th basis state. Then one has

**Theorem 2.** A pure state $γ_{ABSE}$ is a private state if and only if (a) $p_{j,k} = Tr[γ_{ABSE} P^j_{AB}^{k}] = \frac{1}{2} δ_{j,k}$, and (b) $κ_{BS} = 0$ for all $j ≠ k$.

Proof. Suppose $γ_{ABSE}$ is a private state, for which condition (a) is satisfied by inspection. The states $BS$ conditional states are $κ_{BS} = Z^k_B U_{BS} (P^k_B ⊗ ρ_E) U_{BS}^† Z^k_B$, where the unitary $U_{BS} = \sum_k P^k_B ⊗ V^k_S$ for $P^k = |\bar{x}⟩⟨\bar{x}|$, and $Z^k$ is the $x$th power of $Z$. In contrast to all other upper indices appearing herein. Since $[Z_B, U_{BS}] = 0$, $κ_{BS} = U_{BS} (P^k_B ⊗ ρ_E) U_{BS}^†$, and (b’) follows immediately.

Conversely, by condition (a) we have $|γ⟩_{ABSE} = \frac{1}{√2} \sum_j |kk⟩⟨k|_S ⊗ |ϕ^j⟩_SE$. From the Schmidt decomposition $|ϕ^j⟩_SE = \frac{1}{√n} \sum_l ν^l_j |ϕ^l⟩_E$ define $Y^k_S = \sqrt{2} |ϕ^k⟩_S$ for unitary $V^k$ so that $|ϕ^k⟩_SE = \sum_l Y^k_S |ℓ⟩_S$. Here $V^k = L^k (R^k)^T$ using the unitaries $L^k |ℓ⟩ = |ϕ^k⟩$ and $R^k |ℓ⟩ = |ν^k⟩$. Now we can write $κ_{BS} = Z^k_B κ_{BS} Z^k_B$ for $κ_{BS} = \frac{1}{2} \sum_j δ_{j,k} |b⟩_S ⊗ Y^k_S (Y^k_S)^†$. Condition (b’) then implies $(Y^k)^† Y^k = (Y^1)^† Y^1$. Defining $|ξ⟩_S = (Y^1)^† Y^1 Σ_1 |ℓ⟩_S |ℓ⟩_E$ we obtain $V^k_S |ξ⟩_S = |ϕ^k⟩_S$ and thus the operator $U_{BS}$ produces the private state: $|γ⟩_{ABSE} = U_{BS} |Φ⟩_{AB} |ξ⟩_S$.

We can understand the relationship between these two characterizations as an instance of the uncertainty principle, which in entropic form requires that the sum of entropies of $x$- and $z$-basis measurements must not be less than unity [17]. Theorem 1 implies that Eve’s entropy of Alice’s $z$-basis measurement (i.e. the key) is itself unity. Complementarily, theorem 2 means Bob’s entropy (actually Bob and shield) of Alice’s $x$-basis measurement is zero, so Eve’s entropy of $z$ must be not less than unity.

**Classical Privacy Amplification.**—An ideal privacy amplification protocol would output a perfectly secret key from the input of only partially secret data. This is too optimistic for practical applications however, and in this section we recapitulate the formulation of protocols which distill an approximately secret key. We say $ρ_{ABE}$ is $ε$-private when $||ρ_{ABE} − κ_{ABE}||_1 ≤ 2ε$. This definition ensures the key can be safely composed with any other cryptographic task and moreover, we can interpret the definition as saying that the actual key $ρ_{ABE}$ is really the ideal key $κ_{ABE}$ with probability at least $1 − ε$ [18, 22].

Here we assume that the input to privacy amplification is $ψ^n_{ABE}$, where $ψ^n_{ABE} = \frac{1}{√n} \sum_{j,k} P^j_{AB}^{k} ⊗ |ϕ^k⟩_E$ describes a shared but not necessarily secret bit. In QKD this product state is the product of a collective attack in which Eve tampers with each signal individually. More general coherent attacks have been dealt with by randomly permuting the quantum signals after receipt and then showing that privacy amplification can extract the same key from the resulting state as from a product state [13, 20].

Now, for $K$ the classical random variable held by Alice and Bob, and $I$ the quantum mutual information, one can show

**Theorem 3** ([11, 12]). There exists a privacy amplification scheme to extract $n(1 − I(K;E))$ secret bits from $ψ^n_{ABE}$ for $n → ∞$. Moreover, this is the maximum possible rate.

The scheme in [3] works by selecting a function at random and applying it to each of the $A$ and $B$ systems; the output size of the function is $n(1 − I(K;E))$ bits. The crux of that proof is a result on measure concentration,
the generic term indicating when a random variable is exponentially likely to be very close to its mean value. The random variable in this case is Eve’s state $\varphi^k_{SE}$, where $k \in \{0,1\}^n$. Initially Eve’s conditional states are not close to the mean, but averaging over some of the $k$ produces a new random variable which is. This partial average comes from regarding the random function as picking a random reversible function on the length-$n$ strings and then discarding (averaging over the last $nI(K:E)$ bits.

The privacy amplification function need not be completely random; as shown in [4] any 2-universal family of hash functions suffice. This includes random linear hashing, which we will use for private state distillation.

Private State Distillation.—As in classical privacy amplification, the goal of private state distillation is to distill a state close to a private state, again measured by the trace distance. Since the key measurement is itself a quantum operation, an output state $\epsilon$-close to a private state results in a key in a key at least $\epsilon$-close to $K_{ABE}$.

Koashi’s method is to distill an $X$ eigenstate in a single abstract key register; its immediate application to private states is obscured by the need to respect the form of the twisting operator. But by using a linear hash function for privacy amplification we can neatly avoid this problem. The essential point remains that the honest parties have full information about an observable conjugate to the key.

Initially Alice, Bob, and Eve share $\psi_{ABE}^{\otimes n}$, which can be purified using the shield system $S$ to the state

$$|\Psi\rangle_{ABSE} = |\psi\rangle_{ABSE}^{\otimes n} = \frac{1}{2^n} \sum_{k,x} |\tilde{x}\rangle_A Z^x_B |k\rangle_B |\varphi^k\rangle_{SE}. \quad (2)$$

Generally, Bob cannot perfectly predict the outcome $x'$ of Alice’s hypothetical $x$-basis measurement since his information is limited by the Holevo quantity $\chi$ of the ensemble $E = \{1,\rho_{BS}\}$, where $\rho_{BS} = 2A(\tilde{x}|\psi_{ABS}|\tilde{x})_A$ [21]. But then the distillation strategy suggests itself: have Alice provide Bob the missing information. If she narrows the possible $\rho_{BS}$ to a suitably-random set of size $2^{n\chi(E)}$, then the HSW theorem indicates that with high probability Bob can determine $x'$ [22].

Having sketched the method roughly, we now turn to the details. Alice’s announcement consists of the bits

$$h_i = u_i \cdot x' \mod n \quad \text{for} \quad n[1-\chi(E)] \quad \text{randomly chosen} \quad u_i , \quad i.e. \quad \text{a random linear hash of } \chi.' \quad \text{This can be thought of as the result of measuring the observables } X^u, \quad \text{which define Pauli } X \text{ operators for a set of "encoded" qubits. The complementary subsystem of encoded qubits is associated with the set of } Z^v, \quad \text{where } u_i \cdot v_j = 0 \quad \text{for all } i, j. \quad \text{Thus we can decompose the space of Alice’s (Bob’s) physical qubit systems into virtual systems } A_1, A_2 (B_1, B_2) \text{ corresponding to the observables } Z^v, \text{ and } X^u, \text{ respectively. The post-announcement state is }$$

$$|\Psi'\rangle_{A_1BSE} = A_2(h)|\tilde{Z}^h B_2|\Psi_{ABSE}. \quad (3)$$

where $|\varphi^k\rangle_{BS} = (2^n(1-\chi))^{-\frac{1}{2}} \sum m |m\rangle_{BS} |\varphi^{(m)}\rangle_{SE}$, since Bob can apply $Z^h_{BS}$ after learning $h$ from Alice. He is left to distinguish the states $\varphi_{ABSE} = \varphi^h_{BS} |\psi_{BS}^\prime| Z^h_{BS}$. Note that system $B_2$ has now become part of the shield.

A slight modification of the HS theorem ensures that with high probability the pretty good measurement $Z^h_{BS}$ can distinguish the $\varphi_{BS}$ with arbitrarily small probability of error. The theorem originally applies to the distinguishability of random subsets of $\rho_{BS}$ and here we have a random subspace. However, in the Appendix we show that the standard proof can be easily adapted to this case, and in fact more generally to the use of 2-universal hashing. Bob’s measurement has elements

$$E^y_{BS} = \sqrt{T_{BS}^{-1}} \left( \Pi_{BS} \Pi^y_{BS} \Pi_{BS} \right) \sqrt{T_{BS}^{-1}} \quad (4)$$

for $T_{BS} = \sum_y \Pi_{BS} \Pi^y_{BS} \Pi_{BS}$, and $\Pi^y_{BS}$ (\Pi_{BS}) the projection onto the typical subspace of $\varphi_{BS}$ ($\varphi'_{BS}$), the subspace spanned by eigenvectors whose eigenvalues are near the likely value. Here $\langle \cdot \rangle$ denotes the average value.

We can now determine the $E^y_{BS}$ explicitly and thereby obtain the twisting operator. Note that $Z^y_{BS} \varphi_{BS} = \sum_{\ell,\ell'} \langle \ell | \Pi_{BS} \Pi^y_{BS} \Pi_{BS} \rangle |\ell\rangle \otimes \mathcal{T}_E [\bar{\varphi}^\ell_{BS} | \varphi_{BS}^\ell ],$ meaning that system $B_2S$ determines typicality in both $\Pi_{BS}$ and $\Pi^y_{BS}$.

Following the proof of Theorem 2 we may then define $Y^y_{BS}$ so that $\Pi_{BS} \Pi^y_{BS} \Pi_{BS}$ becomes

$$Z^y_{BS} \left( \sum_{\ell,\ell'} |\ell\rangle_{BS} |\bar{\varphi}^{\ell}_{BS} \varphi_{BS}^{\ell} \rangle \otimes \bar{\varphi}^\ell_{BS} \right) Z^y_{BS} \quad (5)$$

Direct calculation gives $T_{BS} = 2^{n\chi} \sum_{\ell,\ell'} P_{\ell B_2} |\bar{\varphi}^{\ell}_{BS} \varphi_{BS}^{\ell} \rangle \bar{\varphi}^\ell_{BS} \bar{\varphi}^\ell_{BS}$ and the square root of the (pseudo) inverse follows.

Now consider the unitary $\bar{V}^\ell$ which comes from the polar decomposition $\bar{V}^\ell = (\sqrt{\bar{V}^\ell (\bar{V}^\ell)^\dagger})^\dagger \bar{V}^\ell$; with it we can write

$$E^y_{BS} = Z^y_{BS} \left( \frac{1}{2^{n\chi}} \sum_{\ell,\ell'} |\ell\rangle_{BS} |\bar{\varphi}^{\ell}_{BS} \varphi_{BS}^{\ell} \rangle \otimes \bar{\varphi}^\ell_{BS} \bar{\varphi}^\ell_{BS} \right) Z^y_{BS} \quad (6)$$

Defining the $\bar{U}_{BS} = \sum_{\ell,\ell'} P_{\ell B_2} |\bar{\varphi}^{\ell}_{BS} \varphi_{BS}^{\ell} \rangle \bar{\varphi}^\ell_{BS}$ we can express this in the more appealing form $E^y_{BS} = \bar{U}_{BS} (P_{\ell B_2} \otimes 1_{B_2S}) \bar{U}^\dagger_{BS}$.

Thus, Bob’s strategy is to untwist the shield as best he can and then measure his key system in the $x$-basis. He and Alice obtain the same outcome with probability

$$P_s = \frac{1}{2^{2n\chi}} \sum_{\ell,\ell'} \bar{B}_{BS} |\varphi^{\ell}_{BS} \rangle \langle \bar{\varphi}^\ell_{BS} \rangle \bar{B}_{BS} \langle \varphi^{\ell}_{BS} \rangle \quad (7)$$

If Bob can determine $y$ with high probability, $P_s \approx 1$, and $E^y_{BS}$ functions as an untwisting operator. Defining $|\Psi'_{A_1BSE} = \bar{U}_{BS} |\psi_{A_1BSE} \rangle$, the squared fidelity of $\psi_{A_1B_1}$ with $\Phi_{A_1B_1}^\otimes n$ equals $P_s$. $P_s \geq 1 - e^{-2}$ implies $||\Psi'_{A_1B_1} - \Phi_{A_1B_1}^\otimes n|| \leq 2\epsilon$ [24] and therefore $|\Psi'_{A_1BSE}$ is $\epsilon$-private. Altogether we have sketched a proof of
Theorem 4. There exists a distillation procedure to distill \( n \chi(E) \) private states from \( \psi_{ABS}^\otimes n \) for \( n \to \infty \).

Note that this is the same rate found by Koashi. Now the associated method of classical privacy amplification is simple. The key is the result of measuring \( Z^{\psi_{i}} \) which commutes with the private state distillation procedure. This key can just as well be reconstructed from individual \( Z \) measurements directly and inherits privacy from the virtual procedure.

Theorems 3 and 4 give the secret key rates \( 1-I(K:E) \) and \( \chi(E) \), corresponding to distillation procedures following from the two descriptions of private states, respectively. Since these descriptions are equivalent, we expect the associated distillation methods to have the same rate. This intuition can be confirmed either by direct calculation or by appealing to upper bounds applicable to either scenario. By the results of 4, \( \chi(E) \leq 1-I(K:E) \). Conversely, \( 1-I(K:E) \leq \chi(E) \) or else by performing the classical privacy amplification coherently, as detailed in 4, Bob would effectively be able to distinguish more of the states \( \rho_{BS}^n \) than possible.

Conclusions.—We have found that the three principal means of treating privacy amplification are essentially identical and interchangeable. The dual descriptions of private states on which the respective distillation methods rest are shown to be elegantly related by the uncertainty principle. This provides an immediate and intuitive understanding of how the quantum information about the key is balanced between the eavesdropper and shield and how the secret information can be extracted.

Care must be taken to incorporate these results into QKD security proofs. Here Alice and Bob begin with a known state \( |\psi\rangle_{ABSE} \), whereas one of the main tasks of a key distribution protocol is to reliably estimate the state shared by the various parties. The presence of a shield system makes this task more difficult, but recent work demonstrates how to estimate the parameters relevant to private state distillation 25.

Reduction of coherent attacks to the case of collective attacks studied here is similarly intricate. This reduction has been accomplished by creating a permutation invariant state \( \psi_{ABE}^{(a)} \) by randomly scrambling the order of the quantum signals and then demonstrating that the chosen key distillation method produces just as many secret bits as from the product input \( \psi_{ABE}^{\otimes n} \). It remains to be shown that when including the shield system this sort of reduction method still applies. In particular, Bob must still be able to distinguish the \( \rho_{BS}^n \) even though the \( x \) are no longer independently and identically distributed. We will report on this in a future publication.

Finally, our result on achieving the Holevo bound using 2-universal hashing can be used for this purpose. A family of functions \( f: X \to Y \) is 2-universal if \( \Pr[|f(x) = f(x')|] \leq 1/|Y|^2 \) for all \( x \neq x' \in X \). Note that random linear hashing, as used in the main text, is 2-universal. Suppose Alice applies a random \( f \) from the hash family to a block (length-\( n \) string) \( x \) of letters, using \( X = \{0, 1, \ldots , d-1\}^n \). Then Bob will be left to distinguish between the elements of \( \{ \rho_{x} = \rho_{x_1} \otimes \cdots \otimes \rho_{x_n}, f(y) = f(x) \} \), for which he uses the measurement \( \{ E_{y} \} \) as defined in Eq. 2 with the slight change that \( E_{y} = 0 \) when \( y \) is nontypical. This rejects nontypical signals, which are in any case exceedingly rare. Adapting the presentation in Appendix B of 26 shows this protocol will have lower error probability. We now specialize to \( d = 2 \), but the argument is essentially the same for the general case.

Given a function \( f \), the average probability of error is given by \( P_{E|f} = \sum_{x} p_{x} \text{Tr}[\rho_{x}(1 - E_{f}^{x})] \). Lemma 2 of 27 states that \( 1 - (S+T)^{-1/2} S(S+T)^{-1/2} \leq 2(1 - S) + 4T \) for \( 0 \leq S \leq 1 \) and \( T \geq 0 \), which we can apply to \( E_{x}^{f} \) using \( \Lambda_{x} = E \Pi_{x} E \) as \( S \) and \( \sum_{x' \neq x} \Lambda_{x'} \) as \( T \) to obtain

\[
P_{E|f} \leq 2 - 2 \sum_{x} p_{x} \left( \text{Tr}[\rho_{x}\Lambda_{x}] - 2 \sum_{x' \neq x} \text{Tr}[\rho_{x}\Lambda_{x'}] \right),
\]

where \( p_{x} = p_{x_1} \cdots p_{x_n} \). When \( x \) is typical, \( \text{Tr}[\rho_{x}\Lambda_{x}] \geq 1 - 3\epsilon \) and by construction \( \Lambda_{x} = 0 \) when \( x \) is not typical. Moreover, the total probability of typical strings exceeds \( 1 - \epsilon \), so we obtain

\[
P_{E|f} \leq 8 \epsilon + 4 \sum_{x} p_{x} \sum_{x' \neq x} \text{Tr}[\rho_{x}\Lambda_{x'}].
\]

Now average over the possible \( f \):

\[
P_{E} \leq 8 \epsilon + 4 \sum_{x} p_{x} \sum_{x' \neq x} \Pr[|f(x') = f(x)| \text{Tr}[\rho_{x}\Lambda_{x'}]]
\]
\[
\leq 8 \epsilon + \frac{4}{|Y|} \sum_{x} p_{x} \sum_{x' \neq x} \text{Tr}[\rho_{x}\Lambda_{x'}]
\]
\[
\leq 8 \epsilon + \frac{4}{|Y|} \sum_{x, x'} p_{x} \text{Tr}[\rho_{x}\Lambda_{x'}]
\]
\[
= 8 \epsilon + \frac{4}{|Y|} \sum_{x'} \text{Tr}[\rho_{x}^{\otimes n}\Lambda_{x'}]
\]

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To evaluate the trace, note that $\text{Tr}[\rho^{\otimes n} \Lambda_{x'}] = \text{Tr}[\Pi \rho^{\otimes n} \Pi \Lambda_{x'}]$. Since $\Pi \rho^{\otimes n} \Pi \leq 2^{-n[S(\rho) - \delta]} \Pi$ (Eq. 19 of [26]) we have

$$P_E \leq 8\epsilon + 4^{2 - n[S(\rho) - \delta]} |\mathcal{Y}| \sum_{x'} \text{Tr}[\Lambda_{x'}].$$

But again $\Lambda_{x'} = 0$ for nontypical $x'$ while $\text{Tr}[\Lambda_{x'}] \leq 2^{n[\sum p_i S(p_i) + \delta]}$ otherwise (Eq. 18), leading to

$$P_E \leq 8\epsilon + 4^{2 - n[\chi(\mathcal{E}) - 2\delta]} |\mathcal{Y}| \sum_{x \in \text{TYP}} 1.$$  

Finally, the size of the typical set is less than $2^{n[H(p_i) + \delta]}$, so putting it all together we have

$$P_E \leq 8\epsilon + 4^{2^{n[H(p_i) - \chi(\mathcal{E}) + 3\delta]} |\mathcal{Y}|^{-1}}.$$  

By choosing $\log_2 |\mathcal{Y}| = n[H(p_i) - \chi(\mathcal{E}) + 4\delta]$, the probability of error can be made arbitrarily small.

Since Bob ultimately learns $x$, an information gain of $H(p_i)$ bits, but Alice only provides $H(p_i) - \chi(\mathcal{E})$, the quantum states themselves provide on average $\chi(\mathcal{E})$ bits, in accordance with the Holevo bound.

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