Logarithmic modes of critical gravity in de Sitter space-time

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Abstract

In this paper we consider the critical gravity in four dimensional de Sitter space-time. We obtain logarithmic modes in the critical point of the theory. Then we show that these logarithmic modes in de Sitter space-time obey similar properties as the ones in AdS-space-time. Our result in this paper indicate that critical gravity theories in de Sitter space-times could lead to a de Sitter/log CFT correspondence.
1 Introduction

Holography is believed to be one of the fundamental principles of the true quantum theory of gravity [1, 2]. An explicitly calculable example of holography is the much studied Anti de Sitter (AdS)/Conformal Field Theory (CFT) correspondence[3]. We would expect dS/CFT correspondence to proceed along the lines of Anti-de Sitter /Conformal Field Theory (AdS/CFT) correspondence because de Sitter spacetime can be obtained from anti-de Sitter spacetime by analytically continuing the cosmological constant to imaginary values. However, local and global properties of dS spacetime lead to unexpected obstructions. Unlike AdS, the boundary of dS spacetime is spacelike and its dual CFT is Euclidean. Moreover, dS spacetime does not admit a global timelike Killing vector. The time dependence of the spacetime metric precludes a consistent definition of energy and the use of Cardy formula to compute dS entropy. Finally dS/CFT duality leads to boundary operators with complex conformal weights, i.e. to a non-unitary CFT. In spite of these difficulties, some progress towards a consistent definition of dS/CFT correspondence has been achieved [4] (see also [5, 6, 7, 8, 9, 10, 11, 12]).

It is well known that Einstein gravity suffers from the problem that the theory is nonrenormalizable in four and higher dimensions. Adding higher derivative terms such as Ricci and scalar curvature squared terms makes the theory renormalizable at the cost of the loss of unitarity [13]. Usually the theories including the terms given by the square of the curvatures have the massive spin 2 mode and the massive scalar mode in addition to the massless graviton. Also the theory has ghosts due to negative energy excitations of the massive tensor. In a special case that the curvature squared term is given by the square of the Weyl tensor, however, the massive scalar mode does not appear, which can be shown by choosing an appropriate gauge condition. Recently, quadratic-curvature actions with cosmological constant have been introduced in four [14] and D [15] dimensions. It was found that there exist a choice of parameters for which these theories possess one AdS background on which neither massive fields, nor massless scalars propagate. By this special choice of the parameters, which is called as a critical point, there appears a mode which behaves as a logarithmic function of the distance. Logarithmic modes in the framework of the higher-dimensional critical gravity models were recently found in [16, 17]. After that Bergshoeff et. al [18] shown that logarithmic modes are of two types, spin 2 and Proca log modes. The number of independent spin 2 log modes is given by the number of polarization states of a massless spin 2 field, while the number of independent Proca log modes is given by the number of polarization states of a massive
spin 1 field. They have obtained the logarithmic solutions of the linearized 4-dimensional critical gravity. They have considered the linearization of the theory around anti de Sitter space-time. The authors of [18] have not considered the de Sitter case in their work, so it would be interesting to find logarithmic modes, when one consider the linearization of the theory around de Sitter space-time. In fact this is our job in this paper. We solve the equation of motion in 4-dimensional de Sitter background, and show that the logarithmic modes appear as well as anti de Sitter background. Then it would be interesting to see whether the log modes in de Sitter space-time obey similar properties as the ones in AdS-space-time, namely whether the properties of Eq.(64) and especially Eq.(65) in [18] still hold. In the AdS-case, the generator $H_1$ can be identified with the conformal energy operator in the dual field theory. The property of Eq.(65) then indicates that this conformal energy is not diagonalizable, which is the defining property of a logarithmic CFT. Since the generator $H_1$ in dS case is still correspond to the field theory Hamiltonian, by finding an equation similar to the Eq.(65) in [18], we present a strong indication that critical gravity theories in de Sitter space-times could also lead to a de Sitter/log CFT correspondence.

2 Logarithmic modes of critical gravity in de Sitter space-time

We consider the following 4-dimensional gravity action

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [R + \frac{6}{L^2} - \frac{1}{2m^2} (R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2) - \frac{1}{4m^2} E_{GB}], \quad (1)$$

where $\kappa^2 = 8\pi G$, and

$$E_{GB} = R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}, \quad (2)$$

is the the Gauss-Bonnet term. This term is the total derivative in four dimensions. The action (1) can be rewritten as

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g} d^4x [R + \frac{6}{L^2} - \frac{1}{4m^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}], \quad (3)$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. In fact the action (3) is the action of Einstein-Weyl gravity. The linearised fluctuations around the $AdS_4$ vacuum in Einstein-Weyl gravity were investigated in [14]. This theory admits $dS_4$
space-time as the vacuum solution of the field equations as well. Here we want to solve the equation of motion in 4-dimensional de Sitter background with curvature $-\frac{12}{L^2}$. The linearized equation of motion in the following transverse traceless gauge

$$g^{\mu\nu}h_{\mu\nu} = 0, \quad \nabla^\mu h_{\mu\nu} = 0.$$  

becomes

$$\left(\Box + \frac{4}{L^2} - 2m^2\right)\left(\Box + \frac{2}{L^2}\right)h_{\mu\nu} = 0.$$  

The mass of massive mode is

$$M^2 = 2(m^2 - \frac{1}{L^2}).$$  

In the critical case, $m^2 = \frac{1}{L^2}$, in this limit the massive mode becomes massless, and the two quadratic differential operators degenerate. So, in the critical point, we have

$$\left(\Box + \frac{2}{L^2}\right)^2h_{\mu\nu} = 0.$$  

We can write down $\Box$ according to the Killing vectors of de Sitter spacetime and classify the answers of (7) according to the eigenstates of them. We can imagine de Sitter space as a 4-dimensional hypersurface in a 5-dimensional spacetime:

$$\begin{aligned}
ds^2 &= dx^2 + dy^2 + dz^2 + du^2 - dt^2, \\
x^2 + y^2 + z^2 + u^2 - t^2 &= L^2.
\end{aligned}$$  

The isometry group of the hypersurface is SO(1,4). Indeed, there are 6 Killings for rotations and 4 Killings for boosts:

$$\begin{aligned}
L_{XY} &= Y \partial X - X \partial Y, \\
L_{Xt} &= t \partial X + X \partial t, \\
X, Y &= x, y, z, u.
\end{aligned}$$  

We can easily transform coordinates and work with this metric:

$$ds^2 = L^2(d\tau^2 \cosh^2(\rho) + d\rho^2 + \sinh(\rho)^2d\Omega^2),$$  

where $d\Omega^2$ is the metric on $S^2$ with unit radius. So two of the Killing vectors are now the generators of transformation along $\tau$ and $\rho$ axes:

$$H_1 = -L_{ut} = -\partial_\tau, \quad H_2 = iL_{xy} = -i\partial_\varphi,$$

(11) Indeed these are Cartan subalgebra of $so(1,4)$. Now, by choosing the proper linear combination of other Killing vectors, they can satisfy ladder commutation relations:

$$E^\alpha_1 = \frac{1}{2} e^{(\tau+i\varphi)} [ - \sin(\theta) \tanh(\rho) \partial_\tau + \sin(\theta) \partial_\rho + \cos(\theta) \coth(\rho) \partial_\theta 
+i \csc(\theta) \coth(\rho) \partial_\varphi],$$

$$E^\alpha_2 = e^{i\varphi} [(i\partial_\theta - \cot(\theta)\partial_\varphi],$$

$$E^\alpha_3 = \frac{1}{2} e^{(\tau-i\varphi)} [ - \sin(\theta) \tanh(\rho) \partial_\tau + \sin(\theta) \partial_\rho + \cos(\theta) \coth(\rho) \partial_\theta 
-i \csc(\theta) \coth(\rho) \partial_\varphi],$$

$$E^\alpha_4 = e^{\tau} [\cos(\theta) \tanh(\rho) \partial_\tau - \cos(\theta) \partial_\rho + \coth(\rho) \sin(\theta) \partial_\theta],$$

(12) Basically, these Killing vectors, except an $i$ factor for $\tau$, are the same as in [18]. Indeed, the transformation $\tau \to i\tau$, convert (10) to the AdS metric chosen in [18]. By the above selections for killing vectors, the commutation relations are the same noted in [18]

$$[H_1, E^{-\alpha_x}] = -\alpha_x^i E^{-\alpha_x}, \quad [H_1, E^{\alpha_x}] = \alpha_x^i E^{\alpha_x},$$

$$[H_1, H_2] = 0, \quad [E^{\alpha_x}, E^{-\alpha_x}] = \frac{2}{|\alpha_x|^2} \alpha_x H,$$

(13) which the second one demonstrate $E^{\alpha_x}$ generators as ladder operators with the following values for $\alpha$ vectors:

$$\vec{\alpha}_1 = (-1, 1), \quad \vec{\alpha}_2 = (0, 1), \quad \vec{\alpha}_3 = (-1, -1), \quad \vec{\alpha}_4 = (-1, 0).$$

(14)
Now we can construct Casimir operator by adding proper combinations of squared Killing vectors and then write the equation of motion (5) according to it

\[ \ell = H_1 H_1 + H_2 H_2 + \sum_{x=1}^{4} \frac{|\alpha_x|^2}{2}(E^{\alpha_x} E^{-\alpha_x} + E^{-\alpha_x} E^{\alpha_x}). \] (15)

by acting \( \ell \) on \( h_{\mu\nu} \) we would obtain:

\[ L^2 \Box h_{\mu\nu} = (\ell - 8) h_{\mu\nu}. \] (16)

So, equation (15) converts to:

\[ (\ell - 6 - m^2 L^2) (\ell - 6) h_{\mu\nu} = 0. \] (17)

Now we can solve this equation for those metric perturbations which are the eigenstates of the Cartan subalgebra of \( so(1,4) \):

\[ H_1 \psi_{\mu\nu} = E_0 \psi_{\mu\nu}, \quad H_2 \psi_{\mu\nu} = s \psi_{\mu\nu}. \] (18)

For each root vector, we can construct the \( su(2) \) subalgebra:

\[ E^\pm = |\alpha_x|^{-1} E^{\pm \alpha_x}, \quad E^3 = |\alpha_x|^{-2} \alpha H. \] (19)

then for the highest weight state, we have:

\[ E^{\alpha_1} \psi_{\mu\nu} = 0, \quad E^{\alpha_3} \psi_{\mu\nu} = 0, \quad E^{\alpha_4} \psi_{\mu\nu} = 0. \] (20)

so \( s \) and \( E_0 \) can be interpreted as the eigenvalues of the energy and angular momentum of states. We can choose the below ansatz for \( \psi_{\mu\nu} \):

\[ \psi_{\mu\nu} = f(\tau, \rho, \theta, \phi) \Omega_{\mu\nu}, \] (21)

\[ \Omega_{\mu\nu} = \begin{pmatrix} -1 & ia & ib & -1 \\ ia & -a^2 & -ab & ia \\ ib & -ba & -b^2 & ib \\ -1 & ia & ib & 1 \end{pmatrix}, \]

where \( a \) and \( b \) are only functions of \( \rho \) and \( \theta \). This ansatz ensure the condition \( \nabla^\mu h_{\mu\nu} = 0 \). \( \psi_{\mu\nu} \) is traceless, so:

\[ a = \frac{1}{\sinh(\rho) \cosh(\rho)}, \quad b = \cot(\theta). \] (22)
If we assume \( f(\tau, \rho, \theta, \phi) = T(\tau)R(\rho)\Theta(\theta)\Phi(\phi) \), relations (18) and (22) imply that:

\[
 f(\tau, \rho, \theta, \phi) = e^{E_0\tau} e^{i\phi} \sin(\theta)^s \sinh(\rho)^s \cosh(\rho)^{-E_0} 
\]  

(23)

we can obtain lower states of energy by apply of \( E^{-\alpha_4} \) and lower states of angular momentum by \( E^{\alpha_2} \).

Now, we can consider the effect of the Casimir operator on \( \psi_{\mu\nu} \):

\[
 \ell \psi_{\mu\nu} = (E_0(E_0 - 3) + s(s + 1))\psi_{\mu\nu}. 
\]  

(24)

So, (17) for \( \psi_{\mu\nu} \) leads to:

\[
 (E_0(E_0 - 3) + s(s + 1) - 6) = 0, 
\]  

or

\[
 (E_0(E_0 - 3) + s(s + 1) - 6 - m^2L^2) = 0. 
\]  

(25)

We can obtain two \( E_0 \) from these equations which lead to two answers for (23). Any linear combination of these two modes still is an answer. In the massless limit, there would be a degeneracy and both equations imply \( s \leq 2 \).

We can find a log mode by the below combination [19]:

\[
 \psi_{\mu\nu}^{\log} = \left( \frac{3}{m} \right)^2 (\psi_{\mu\nu}^{E_0(1)} - \psi_{\mu\nu}^{E_0(2)}). 
\]  

(26)

For \( s = 2 \), in the limit of \( m \to 0 \) we would have:

\[
 \psi_{\mu\nu}^{\log} = \pm (\tau - \ln(\cosh(\rho)))\psi_{\mu\nu}^{E_0(1)}, 
\]  

(27)

the positive sign is for \( E_0^{(1)} = 3 \) and the minus sign is for \( E_0^{(1)} = 0 \). These log modes correspond to eigenstates of \( H_2 \)

\[
 H_2 \psi_{\mu\nu}^{\log} = s\psi_{\mu\nu}^{\log}, 
\]  

(28)

The effect of \( H_1 \) on this state is:

\[
 H_1 \psi_{\mu\nu}^{\log} = \pm \psi_{\mu\nu}^{E_0(1)} + E_0^{(1)} \psi_{\mu\nu}^{\log}, 
\]  

(29)

so \( H_1 \) is not diagonal and the log modes in de Sitter space-time obey similar properties as the ones in AdS-space-time [18]. Since the generator \( H_1 \) can be identified with the conformal energy operator in the dual field theory, then Eq. (29) indicates that this conformal energy is not diagonalizable, which is the defining property of a logarithmic CFT. Therefore this result would be a strong indication that critical gravity theories in de Sitter space-times could lead to a de Sitter/log CFT correspondence.
3 Conclusion

In the present paper we have considered the critical gravity in 4-dimensional de Sitter space-time. The Lagrangian consists of the Einstein-Hilbert term with a cosmological constant \( \Lambda \) and an additional higher-order term proportional to the square of the Weyl tensor, with a coupling constant \( [14] \). It was shown that there is a critical relation between coupling constant and \( \Lambda \) for which the generically-present massive spin-2 modes disappear, and are instead replaced by modes with a logarithmic fall-off \([16, 18]\). These log modes are ghostlike, but since they fall off more slowly than do the massless spin-2 modes, they can be truncated out by imposing an appropriate AdS boundary condition \([20, 21]\). The resulting theory is then somewhat trivial, in the sense that the remaining massless graviton has zero on-shell energy. Furthermore, the mass and the entropy of standard Schwarzschild-AdS black holes are both zero in the critical theory. Now may be one ask why the study of these models are important, and why the existence of these logarithmic solutions is interesting? In answer to these question we can say that these models could provide gravity descriptions for logarithmic CFTs. Logarithmic CFT arise in various contexts in condensed matter physics \([22]\). A defining feature of logarithmic CFTs is that the Hamiltonian does not diagonalise, but rather contains Jordan cells of rank two or higher. In the AdS-case, where the generator \( H_1 \) can be identified with the conformal energy operator in the dual field theory, Bergshoeff et. al \([18]\) have shown that this Hamiltonian is not diagonalizable, which is the defining property of a logarithmic CFT. In this paper we extended this work to the de Sitter space. We have obtained the logarithmic mode for \( s = 2 \) case by Eq. (27). Then we have shown that these logarithmic modes in de Sitter space-time obey similar properties as the ones in AdS-space-time. Namely the properties of Eq. (64) and especially Eq. (65) in paper \([18]\) still hold. These equations would be a strong indication that critical gravity theories in de Sitter space-times could also lead to a de Sitter/log CFT correspondence. Although de Sitter/CFT and the subject is certainly not as well established as the usual AdS/CFT correspondence, but a de Sitter/log CFT conjecture could certainly be interesting to speculate about (since log CFT’s have found applications in condensed matter physics and possible gravity duals might thus be interesting).
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