PNJL model for adjoint fermions

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Abstract. Recent work on QCD-like theories has shown that the addition of adjoint fermions obeying periodic boundary conditions to gauge theories on $R^3 \times S^1$ can lead to a restoration of center symmetry and confinement for sufficiently small circumference $L$ of $S^1$. At small $L$, perturbation theory may be used reliably to compute the effective potential for the Polyakov loop $P$ in the compact direction. Periodic adjoint fermions act in opposition to the gauge fields, which by themselves would lead to a deconfined phase at small $L$. In order for the fermionic effects to dominate gauge field effects in the effective potential, the fermion mass must be sufficiently small. This indicates that chiral symmetry breaking effects are potentially important. We develop a Polyakov-Nambu-Jona Lasinio (PNJL) model which combines the known perturbative behavior of adjoint QCD models at small $L$ with chiral symmetry breaking effects to produce an effective potential for the Polyakov loop $P$ and the chiral order parameter $\bar{\psi}\psi$. A rich phase structure emerges from the effective potential. Our results [1] are consistent with the recent lattice simulations of Cossu and D’Elia [2], which found no evidence for a direct connection between the small-$L$ and large-$L$ confining regions. Nevertheless, the two confined regions are connected indirectly if an extended field theory model with an irrelevant four-fermion interaction is considered. Thus the small-$L$ and large-$L$ regions are part of a single confined phase.

1. Introduction
Recent progress in the study of QCD-like gauge theories has revealed that a confined phase can exist under certain conditions when one or more spatial directions are compactified and small [3, 4]. This is surprising, because a small compact direction in Euclidean time gives rise to a deconfined phase for $SU(N)$ gauge theories at high temperatures [5, 6]. It is also intriguing, because one or more small compact directions give rise to a small effective coupling constant if the theory is asymptotically free. Thus we now have four-dimensional field theories in which confinement holds, and holds under circumstances where semiclassical methods may be reliably applied. At present, there are two methods known for achieving this. The first method directly modifies the gauge action with terms nonlocal in the compact direction(s) [4], while the second adds adjoint fermions with periodic boundary conditions in the compact direction(s) [3], which is our subject here.

Confinement in $SU(N)$ gauge theories is associated with an unbroken global center symmetry, which is $Z(N)$ for $SU(N)$. The order parameter for $Z(N)$ breaking in the compact direction is the Polyakov loop, $P(\vec{x}) = \mathcal{P} \exp \left[ i \int_0^L dx_4 A_4(x) \right]$, which is the path-ordered exponential of the gauge field in the compact direction. The trace of $P$ in a representation $R$ represents the insertion of a heavy fermion in that representation into the system. Unbroken $Z(N)$ symmetry implies...
\[ \langle Tr_F P \rangle = 0 \text{ in the confined phase, and correspondingly } \langle Tr_F P \rangle \neq 0 \text{ holds in the deconfined phase where } Z(N) \text{ symmetry is broken.} \]

In the case of adjoint fermions with periodic boundary conditions on \( R^3 \times S^1 \), \( Z(N) \) symmetry is restored if the circumference \( L \) of \( S^1 \) is sufficiently small and the mass \( m \) of the adjoint fermions is sufficiently light \([3, 7]\). If \( mL \) is sufficiently small, the effective potential has a global minimum when the Polyakov loop eigenvalues are uniformly spaced around the unit circle. This is the unique \( Z(N) \)-symmetric solution for \( P \). Experience with phenomenological models \([8, 9]\) suggests that in fact it is the constituent mass which is relevant in determining the size of the fermionic contribution to the effective potential for \( P \).

In order to explore the interrelationship of confinement and chiral symmetry breaking, we use a generalization of Nambu-Jona Lasinio models known as Polyakov-Nambu-Jona Lasinio (PNJL) models \([9]\). In NJL models, a four-fermion interaction induces chiral symmetry breaking. There has been a great deal of work on NJL models, both as phenomenological models for hadrons and as effective theories of QCD \([10, 11]\). NJL models have been used to study hadronic physics at finite temperature, but they include only chiral symmetry restoration, and do not model deconfinement. This omission is rectified by the PNJL models, which include both chiral restoration and deconfinement. The earliest model of this type was derived from strong-coupling lattice gauge theory \([8]\), but later work on continuum models have proven to be extremely powerful in describing the finite-temperature QCD phase transition \([9]\). In PNJL models, fermions with NJL couplings move in a nontrivial Polyakov loop background, and the effects of gluons at finite temperature is modeled in a semiphenomenological way. We will develop a model of this type for both fundamental and adjoint fermions below.

Recent lattice simulations by Cossu and D’Elia \([2]\) have confirmed the existence of the small-\( L \) confined region in \( SU(3) \) lattice gauge theory with two flavors of adjoint fermions, and we will focus on this case in our analysis. Even if the small-\( L \) confined region exists and is accessible in lattice simulations, it is not necessarily the same phase as found for large \( L \). Put slightly differently, we would like to know if the small-\( L \) and large-\( L \) confined regions are smoothly connected, and thus represent the same phase. Our main result will be a phase diagram for adjoint periodic QCD for all values of \( L \), obtained using a PNJL model. On the way to this goal, we will use as tests of our model both standard QCD with fundamental fermions and adjoint QCD with the usual antiperiodic boundary conditions for fermions. Our principal tool will be the effective potential for the chiral symmetry order parameter \( \bar{\psi} \psi \) and the deconfinement order parameter \( P \). For a more detailed discussion, see reference \([1]\).

2. Effective potential

2.1. Fermionic contribution

We take the fermionic part of the Lagrangian of our PNJL model to be \([9, 10, 11]\)

\[
L_F = \bar{\psi} (i \gamma \cdot D - m_0) \psi + \frac{g_S}{2} \left[ (\bar{\psi} \lambda^a \psi)^2 + (\bar{\psi} i \gamma_5 \lambda^a \psi)^2 \right] + g_D [\det \bar{\psi} (1 - \gamma_5) \psi + \text{h.c.}] \tag{1}
\]

where \( \psi \) is associated with \( N_f \) flavors of Dirac fermions in the fundamental or adjoint representation of the gauge group \( SU(N) \). The \( \lambda^a \)'s are the generators of the flavor symmetry group \( U(N_f) \) and \( \lambda_0^a \lambda^a_{kl} = 2 \delta_{ij} \delta_{jk} \); \( g_S \) represents the strength of the four-fermion scalar-pseudoscalar coupling and \( g_D \) fixes the strength of an anomaly induced term. For simplicity, we take the Lagrangian mass matrix \( m_0 \) to be diagonal: \( (m_0)_{jk} = m_{0j} \delta_{jk} \). The covariant derivative \( D_\mu \) couples the fermions to a background Polyakov loop via the component of the gauge field in the compact direction.

It is generally convenient to use the language of finite temperature to describe both the case of finite temperature, \( \beta^{-1} = T > 0 \), with antiperiodic boundary conditions, and the case of a periodic spatial direction, \( L < \infty \). The zero-temperature contribution to the fermionic effective
potential is given by

\[ V_{F0}(m,m_0) = \sum_j g_S \sigma_j^2 + 2g_D (N_f - 1) \prod_j \sigma_j - 2d_R \sum_{j=1}^{N_f} \int_\Lambda \frac{d^3k}{(2\pi)^3} \omega_k^{(j)} \]  

(2)

where \( \omega_k^{(j)} = \sqrt{k^2 + m_j^2} \), \( \sigma_j = \langle \bar{\psi}_j \psi_j \rangle \), \( m_j = m_0j - 2g_S \sigma_j - 2g_D \prod_{k \neq j} \sigma_k \) is a constituent mass, and the constant \( d_R \) is the dimensionality of the color representation, \( N \) for the fundamental and \( N^2 - 1 \) for the adjoint. The last term, representing a sum of one-loop diagrams, is regularized by three-dimensional momentum space cutoff, \( \Lambda \) [10].

In PNJL models, the finite-temperature contribution from the fermion determinant depends on the background Polyakov loop. It is convenient to work in a gauge where the temporal component of the background gauge field, \( A_t(x,t) \), is constant and diagonal. The covariant derivative then becomes \( \gamma \cdot D = \gamma \cdot \partial - i\gamma^4 A_4 \). The one-loop free energy of fermions in a representation \( R \) of \( SU(N) \) gauge theory with zero chemical potential can be expanded in terms of modified Bessel functions

\[ V_{FL}(P,m) = \sum_j \frac{2m_j^2}{\pi^2 L^2} \sum_{n=1}^{\infty} \frac{(-1)^n Tr_R P_m}{n^2} K_2(n L m_j) \]  

(3)

which is rapidly convergent for all values of the mass [12]. The plus sign is used for periodic boundary conditions and minus for antiperiodic. In what follows, we will take \( N_f = 2 \), and take the masses \( m_0j \) to be equal to a common mass which we also write as \( m_0 \). In this case, the contribution to the effective action from \( g_S \) and \( g_D \) has the same form. It is convenient to take \( g_D = 0 \), and also to write the common constituent mass as \( m = m_0 - 2g_S \sigma \) [10, 11]. There is a possibility of directly modifying the strength of chiral symmetry breaking by adding additional couplings compatible with all symmetries have been added. In the case of adjoint fermions with periodic boundary conditions, the ability to freely vary \( g_S \) allows a clear connection between the large-\( L \) and small-\( L \) confining regions of the phase diagram.

2.2. Gluonic contribution

The boundary conditions for the gauge bosons are periodic in all cases considered here, so \( L \) and \( \beta \) may be used equivalently in the gluonic sector. The one-loop finite-temperature free energy in a background Polyakov loop is given by an expression similar to the one for fermions

\[ V_{g-1\text{loop}}(P) = 2 Tr_A \left[ \frac{1}{L} \int \frac{d^3k}{(2\pi)^3} \ln(1 - Pe^{-L\Omega_k}) \right] \]  

(4)

where we have inserted a mass parameter in \( \Omega_k = \sqrt{k^2 + M^2} \) for purely phenomenological reasons explained below.

The Polyakov loop in the fundamental representation of \( SU(3) \) can be diagonalized by a gauge transformation and written as \( P_{jk} = \exp(i\phi_j) \delta_{jk} \) with two independent angles. With the use of \( Z(3) \) symmetry, it is sufficient to consider the case where \( (Tr_F P) \) is real. Thus we consider only diagonal, special-unitary matrices with real trace, which may be parametrized by taking \( \phi_1 = \phi \), \( \phi_2 = -\phi \), and \( \phi_3 = 0 \), or \( P = \text{diag} \left[ e^{i\phi}, e^{-i\phi}, 1 \right] \) with \( 0 \leq \phi \leq \pi \). The unique set of \( Z(3) \)-invariant eigenvalues are obtained for \( \phi = 2\pi/3 \). For \( SU(3) \), we can write the gluonic effective potential in a high temperature expansion in terms of \( \phi \) [13]

\[ V_g(P) = \left( \frac{3\phi^2}{2\pi^2} - \frac{2\phi}{\pi} + 2 \right) \frac{M^2}{L^2} + \frac{1}{L^4} \left( \frac{135\phi^4 - 300\pi\phi^3 + 180\pi^2\phi^2 - 16\pi^4}{90\pi^2} \right) \]  

(5)
We will set the mass scale $M$ by requiring that $V_g$ yields the correct deconfinement temperature for the pure gauge theory, with a value of $T_d \approx 270 \text{MeV}$. This gives $M = 596 \text{MeV}$ [13]. We stress that the mass parameter $M$ should not be interpreted as a gauge boson mass, nor do we limit ourselves to $ML \ll 1$. The crucial feature of this potential is that for sufficiently large values of the dimensionless parameter $ML$, the potential leads to a $Z(N)$-symmetric, confining minimum for $P$ [7, 14]. On the other hand, for small values of $ML$, the pure gauge theory will be in the deconfined phase. It will be important later that $V_g$ is a good representation of the gauge boson contribution for high temperatures; in other PNJL models, the gauge boson contribution has sometimes been chosen so as to be valid over a more narrow range of temperatures.

3. Fundamental Fermions
As a test of all the components of the effective potential we have assembled, we consider the case of two flavors of fundamental fermions at finite temperature. A very common choice of zero-temperature parameters for two degenerate light flavors is $m_0 = 5.5 \text{MeV}$, $\Lambda = 631.4 \text{MeV}$, and $g_S = 2 \times 5.496 \text{GeV}^{-2}$ [9, 11]. We will use these parameters, augmented by the gluonic sector parameter $M = 596 \text{MeV}$ discussed in the previous section. In Figure 1, we show the constituent mass $m$ and Polyakov expectation value $\langle Tr_P P \rangle$ as a function of temperature, normalized by dividing by their values at $T = 0$ and $T = \infty$, respectively. The behavior in the crossover region is very similar to the results of Fukushima [9], and shows the explanatory power of PNJL models. The constituent mass $m$ is heavy at low temperatures, due to chiral symmetry breaking. The larger the constituent mass, the smaller the $Z(3)$ breaking effect of the fermions. On the other hand, a small value for $\langle Tr_P P \rangle$ reduces the effectiveness of finite-temperature effects in restoring chiral symmetry. These synergistic effects combine in the case of fundamental representation fermions to give a single crossover temperature at which both order parameters are changing rapidly, in agreement with lattice simulations.

![Figure 1. The constituent mass $m$ and $\langle Tr_P P \rangle$ for two-flavor QCD with fundamental representation fermions with antiperiodic boundary conditions as a function of temperature.](image1)

![Figure 2. The constituent mass $m$ and $\langle Tr_P P \rangle$ for two-flavor QCD with adjoint representation fermions with antiperiodic boundary conditions as a function of temperature.](image2)

4. Adjoint Fermions with antiperiodic boundary conditions
Adjoint $SU(3)$ fermions at finite temperature show a completely different behavior in lattice simulations from fundamental fermions. Because the adjoint fermions respect the $Z(3)$ center symmetry, there is a true deconfinement transition where $Z(3)$ spontaneously breaks. Lattice simulations have shown that chiral symmetry is restored at a substantially higher temperature
than the deconfinement temperature \[15, 16\]. The \( T = 0 \) parameters needed are \( g_S \) and \( \Lambda \). Rather than work directly with \( g_S \), we will consider the dimensionless coupling \( \kappa = g_S \Lambda^2 \). A given ratio of \( m(T=0)/\Lambda \) determines the value of \( \kappa \), and vice versa. The value of \( \Lambda \) is determined by the requirement that \( T_c/T_d \) is near 7.8 \[16\]. This in turn determines the value of the constituent mass for all \( T \).

The ratio \( m(T=0)/\Lambda \) should be less than one in order for the cutoff theory to be meaningful. In the case of fundamental fermions, this ratio is relatively large, on the order of 0.5. We have generally found that for adjoint fermions a larger ratio of \( m(T=0)/\Lambda \) with \( T_c/T_d \) fixed implies a larger value of \( m(T=0) \). We will work with the representative case of \( m_0 = 0 \) and \( m(T=0)/\Lambda = 0.1 \). This gives \( \Lambda = 23.22 \text{ GeV} \) and thus \( m(T=0) = 2.322 \text{ GeV} \), with \( \kappa = 1.2653 \). For comparison, the critical value of \( \kappa \), \( \kappa_c \), below which \( m(T=0) = 0 \), is \( \pi^2/8 \approx 1.234 \). In Figure 2, we show the constituent mass \( m \) and Polyakov expectation value \( \langle TFP \rangle \) as a function of temperature, normalized by dividing by their values at \( T = 0 \) and \( T = \infty \), respectively. We see that the deconfinement temperature \( T_d \) is very close to its value in the pure gauge theory, due to the large adjoint fermion constituent mass. The transition is first order. The constituent mass \( m \) has a slow decline to a second-order transition at a substantially higher temperature, as indicated by lattice simulations \[15, 16\].

5. Adjoint Fermions with periodic boundary conditions

We consider the behavior of \( m \) and \( TFP \) with periodic fermions using the same parameters we used for the antiperiodic case. Figure 3 shows the behavior of \( m \) and \( \langle TFP \rangle \) as a function of \( L^{-1} \) for the \( m(L=\infty)/\Lambda = 0.1 \) parameter set, with \( m_0 = 0 \). We see that chiral symmetry breaking persists at \( L^{-1} = 10 \text{ GeV} \), which is much higher than the chiral restoration temperature for antiperiodic fermions. The constituent mass \( m \) does fall eventually as \( L^{-1} \) increases, and chiral symmetry is ultimately restored, but at a temperature on the order of \( \Lambda \). In Figure 4, we show the phase diagram in the \( L^{-1} - \kappa \) plane, obtained by numerically minimizing \( V_{eff} \). For most values of \( \kappa \) larger than \( \kappa_c \), the confined large-\( L \) phase and the reconfined phase at small \( L \) are separated by three phase transitions as in Figure 3. All of these transitions are characterized by abrupt changes in \( TFP \), while the chiral order parameter shows only a slow decrease with increasing temperature. However, there is a narrow range of \( \kappa \) between approximately 1.250 and \( \kappa_c \approx 1.234 \) where confinement holds at all temperatures, and chiral symmetry remains broken. In this extended phase diagram, the confined and reconfined regions are smoothly connected. Although this connection appears only for small range of values, the corresponding range of constituent mass values is not necessarily small \[1\]. Our results bear directly on the recent work by Cossu and D’Elia \[2\], in which they performed lattice simulations of two-flavor \( SU(3) \) gauge theory with periodic adjoint fermions.

6. Conclusions

We have extended the PNJL treatment of \( SU(3) \) gauge theories to the case of two adjoint fermions with periodic boundary conditions on \( R^3 \times S^1 \). Our simple model reproduces the known successes of PNJL models for fundamental fermions while at the same time reproducing the expected behavior at high temperatures needed with adjoint fermions. The large separation between the deconfinement transition and the chiral symmetry restoration transition for adjoint fermions is near 7.8, which is consistent with lattice simulations. As \( \kappa \) increases, there is a narrow range of values where confinement holds at all temperatures, and chiral symmetry remains broken. This extended phase diagram allows for a slow change in the constituent mass as a function of temperature, as indicated by lattice simulations.
The constituent mass $m$ and $\langle \text{Tr}_F P \rangle$ for two-flavor QCD with adjoint representation fermions with periodic boundary conditions as a function of $L^{-1}$. C, D, S and R refer to the confined, deconfined, skewed, and reconfined phase, respectively.

The phase diagram for two-flavor QCD with adjoint representation fermions with periodic boundary conditions in the $L^{-1}$-$\kappa$ plane. $\kappa = g_S A^2$. C, D, and S refer to the confined, deconfined, and skewed phase, respectively.

Fermion theories with antiperiodic boundary conditions requires a PNJL model which reproduces the behavior of the pure gauge theory to much smaller values of $L$ than have been considered before.

The results for our $SU(3)$ PNJL model with two flavors of periodic adjoint Dirac fermions can be summarized in the phase diagram in Figure 4. They are completely compatible with the lattice simulations of Cossu and D’Elia [2]. If $m_0$ is set to zero, there is a small region in the $L^{-1} - \kappa$ plane, lying above $\kappa_c$, that connects the large-$L$ and small-$L$ confined regions. Because the largest contribution to the constituent mass $m$ is from chiral symmetry breaking, this behavior will persist for some small range of nonzero $m_0$. Thus there is a single confining region, accessible in principle in lattice simulations.

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