Probabilistic bisimilarities between quantum processes

Yuan Feng∗, Runyao Duan†, Zhengfeng Ji‡ and Mingsheng Ying§

State Key Laboratory of Intelligent Technology and Systems, Department of Computer Science and Technology, Tsinghua University, Beijing, China, 100084

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Abstract

Modeling and reasoning about concurrent quantum systems is very important both for distributed quantum computing and for quantum protocol verification. As a consequence, a general framework describing formally the communication and concurrency in complex quantum systems is necessary. For this purpose, we propose a model qCCS which is a natural quantum extension of classical value-passing CCS with the input and output of quantum states, and unitary transformations and measurements on quantum systems. The operational semantics of qCCS is given based on probabilistic labeled transition system. This semantics has many different features compared with the proposals in literature in order to describe input and output of quantum systems which are possibly correlated with other components. Based on this operational semantics, we introduce the notions of strong probabilistic bisimilarity and weak probabilistic bisimilarity between quantum processes and discuss some properties of them, such as congruence under various combinators.

1 Introduction

An important research field in computer science is modeling and reasoning about concurrent systems, especially when computations are performed by physically separated parties. In classical world, various theories of concurrent computation have been proposed, including Petri nets [28], CSP [18, 19], CCS [24, 25] and ACP [5], among which CCS is one of the most mathematically developed models. The simplest version of CCS is so-called pure CCS in which no data are involved when performing actions, and the only form of communication between different components is just pure synchronization. Value-passing CCS is a natural extension of pure CCS where processes can not only synchronize with each other but also transfer data between them during synchronization [15, 16, 17].

Quantum computation and quantum information theory (QCQI) has attracted considerable research efforts in the past twenty years since Feynman [10] proposed the idea that a quantum mechanical system can be used to perform computation. Benefiting from the possibility of superposition of different basis states and the linearity of quantum operations, quantum computing may provide considerable speedup over its classical analogue [35, 13, 14]. However, the design of quantum algorithms is too tricky and there does not yet seem to be a unifying set of principles by which quantum algorithms can be developed. To deal with this difficulty, quantum programming languages are

∗E-mail: feng-y@tsinghua.edu.cn
†E-mail: dry02@mails.tsinghua.edu.cn
‡E-mail: jizhengfeng98@mails.tsinghua.edu.cn
§E-mail: yingmsh@tsinghua.edu.cn
developed which identify and promote useful “high-level” features allowing us to think about a problem in a conceptual way, rather than focusing on the details of its “low-level” implementations [27, 30, 6, 34].

The languages presented so far are, however, mostly designed for sequential quantum computing, where no communications between physically separated parties are considered. Investigation of languages which can describe quantum concurrent systems and the communication behaviors between them has just begun. On the other hand, constructing real quantum computers in which quantum programming can be applied is very difficult; while quantum cryptography [7, 2, 1], which can provide absolutely security in principle even attacked by potential quantum eavesdropper, has grown so rapidly that recently quantum cryptographic systems have been commercially available [29]. So to some extent the need for a language describing concurrent systems is more urgent than that for sequential computation. Furthermore, a framework of modeling and reasoning about quantum concurrent systems will also provide techniques to prove the properties such as correctness and security of quantum cryptographic protocols, just as we see in classical world.

The first steps of constructing such a general framework for modeling quantum concurrent systems were made independently by Jorrand and Lalire [20], and Gay and Nagarajan [11]. In Ref.[20], a process algebra for quantum processes was proposed which can describe both classical and quantum information passing and processing. Later on, Lalire presented for their language a probabilistic branching bisimulation which identifies quantum processes associated with process graphs having the same branching structure [21]. But further properties such as congruence under various combinations of quantum processes were not explored. In Ref.[11], a language called CQP (Communicating Quantum Processes) was defined, which combined the communication primitives of pi-calculus with primitives for measurement and transformation of quantum states. One distinctive feature of CQP is a type system which can guarantee the physical realizability of quantum processes in CQP. However, no equivalence notions between processes were presented there.

The purpose of this paper is to propose a different model for quantum concurrent systems and discuss its properties. This model (we call it qCCS) is a quantum extension of classical value-passing CCS. To avoid no-go operations such as quantum cloning in syntactical level, we explicitly introduce the notion of free quantum variables, which intuitively denote the quantum systems a process can reference. When constructing more complicated processes from simpler ones, this type of variables must be taken into consideration. For example, if q is one of the free quantum variables of E then the expression c!q.E is invalid because we can not reference a quantum system when it has been output. This is in sharp contrast with classical variables, because the value of classical values can be copied arbitrarily so that we can use the variable even after it has been output. As a consequence, the syntax of qCCS is more complicated compared with those in [11] and [20]. But the reward is that a type system as introduced in [11] is not necessary in qCCS. Note also that in [20], there was not such a mechanics to avoid invalid quantum processes.

To present the operational semantics of qCCS, we introduce the notion of configuration which is a pair consisting of a quantum process and an accompanied context instantiating all free quantum variables of the process. Intuitively, the context describes the environment the process is performed in. The operational semantics of qCCS is then given based on probabilistic labeled transition system consisting of configurations. There are some differences between our approach and the previous ones presented in literature. The first one is that in our semantics, transitions are of only one kind, and defined from configurations to distributions over configurations, i.e.

\[ \rightarrow \subseteq \text{Con} \times \text{Act} \times D(\text{Con}) \]

where Con is the set of configurations and \(D(\text{Con})\) is the set of finite-support distributions on Con. Notice that in [20] and [11], in order to resolve probabilistic choice after measurements have been performed, another kind of transitions, i.e. probabilistic transitions, were introduced; while here in the current paper, we do not resolve any probabilistic choices in intermediate steps and instead keep the probability information all the time. To achieve this, we extend the ordinary transitions
defined from configurations to distributions to those defined between configurations. The motivation
for us to make such a design decision is as follows. First, transitions defined in this way make
our operational semantics simple and more CCS-like; second, a single transition in our qCCS can
describe simultaneously different probabilistic execution paths with the same action and the same
starting and ending configurations. As a consequence, the definition of weak transitions, which is the
basis of weak probabilistic bisimulation, becomes possible. Furthermore, by defining transitions in
this way, the notions and techniques introduced in [32] and [33] for classical probabilistic processes
can be extended to investigate the properties of weak probabilistic bisimulations between quantum
processes.

The second difference between our approach of semantics and the previous ones is the way of
dealing with rules of quantum input, quantum output and quantum communication. In Ref.[20],
the rule for quantum input was of the following form (rewritten with our notations)

$$< c?q.E; \bar{q} = \sigma > \rightarrow^{c} < E[\bar{r}/q]; r, \bar{q} = \rho \otimes \sigma >$$

for all $2 \times 2$ density matrix $\rho$ and $r \notin \bar{q}$.

This rule makes sense when the input system (denoted by the quantum variable $r$) is initially not
correlated (neither entangled nor classically correlated) with the quantum systems in $\bar{q}$. However, we
know that one of the essential difference between quantum mechanics and classical correspondence
is that in the former, different systems can lie in an entangled states which can not be determined
by the reduced states of each individual system. Motivated by this observation, we present two rules
considering different situations when quantum states are input. The first rule aims at the case when
the system to be input is not in current quantum context:

$$< c?q.E; \bar{q} = \sigma > \rightarrow^{c} < E[\bar{r}/q]; r, \bar{q} = \rho >$$

where $r \notin \bar{q}$ and $\text{Tr}_r \rho = \sigma$.

In this case, the accompanied context is extended to include the newly input system. Note that in
our rule, any quantum input can be characterized since we put no constraints on the form of the
new state $\rho$ except $\text{Tr}_r \rho = \sigma$ which means that the state of initial systems will not be changed. The
second rule aims at the case when the system to be input is already in current quantum context:

$$< c?q.E; \bar{q} = \sigma > \rightarrow^{c} < E[\bar{r}/q]; \bar{q} = \sigma >$$

where $r \in \bar{q} - (qv(E) - \{q\})$.

Note that the context keeps unchanged after the transition. The intuition behind this rule is that
when the system to be input has already been described in the current context, input action is
nothing but a declaration that the process can reference this system, which of course does not affect
the state of the whole system.

The output rule for quantum system presented in [20] was of the following form

$$< c!q.E; \bar{q} = \sigma > \rightarrow^{c} < E; \bar{q} - \{q\} = \text{Tr}_{\{q\}} \sigma >$$

with the intuition that we do not care about the state of a quantum system when it has been output.
But it is easy to see that the information about how the output system is correlated with the systems
remained in the context is totally lost; problems will arise if we input again the system which was
just output. To deal with this shortcoming, we present the quantum output rule as

$$< c!q.E; \bar{q} = \sigma > \rightarrow^{c} < E; \bar{q} = \sigma >.$$
In Ref. [11], no rules for quantum input and output were introduced because the authors took the viewpoint that any input action is necessarily accompanied with an output action (no matter from another process or the environment). However, we still think it necessary to present rules considering input and output actions, since they give us a compositional way to describe quantum communication between different components.

Another contribution of the current paper is the notions of strong and weak probabilistic bisimilarities between quantum processes. As mentioned above, Lalire [21] has proposed a notion of probabilistic branching bisimulation without any further properties being explored. Our bisimulation, based on different labeled transition system, is motivated by [32] and [33] where classical probabilistic processes are considered. This notion of bisimilarity is different from classical non-probabilistic ones in two ways. First, for two bisimilar configurations, any action performed by one can be simulated by a combined action of the other. That is, different non-deterministic transitions can be chosen simultaneously with different probabilities to simulate a single one. Second, the final states of the quantum contexts when all matching actions have been performed must be the same when we want to check if two configurations are bisimilar. For example, the configurations $<c!0.nil; q = |0\rangle\langle 0|>$ and $<c!0.nil; q = |1\rangle\langle 1|>$ should not be considered bisimilar although both of them can only perform the same action $c!0$ and then terminate.

The notion of weak probabilistic bisimilarity which abstracts from internal actions is a distinctive feature of our qCCS. Note that in qCCS, transition from a configuration generally leads to a finite-support distribution over configurations; and from each resulted configuration, different configurations can again be derived with different probabilities. As a consequence, the execution of a sequence of actions from a quantum configuration typically form a tree rather than a linear path as in classical non-probabilistic case. When weak probabilistic bisimulations are considered, internal actions should be able to be arbitrarily inserted in or deleted from any branch of the tree. This arbitrariness makes the definition of weak transitions difficult. To deal with it, we first introduce the notion of adversary which, depending on the execution history, decides the next transition when nondeterministic choices have to be made. Then a weak transition from a configuration to a distribution is determined by an adversary with all the coinciding execution paths having the same sequence of observable actions. Based on weak transitions, we define weak probabilistic bisimulations between configurations and between quantum processes and discuss some properties of them such as congruence under various combinators in qCCS.

1.1 Overview of this paper

This paper is organized as follows: in Section 2, we give some preliminaries. In particular, we review some basic notions of quantum computing used in this paper, such as quantum state, unitary transformations, measurements, quantum entanglement, and density matrices.

Section 3 is devoted to studying the syntax and operational semantics of qCCS. We first define inductively the syntax of quantum process expressions and at the same time the free quantum variables associated with each process expression. Then the notion of configuration is introduced to make process expressions closed with respect to quantum variables. The operational semantics of qCCS is then given in terms of probabilistic labeled transition system consisting of configurations. We also extend the ordinary transitions defined from configurations to distributions to those defined between distributions. The notion of combined transition, which is the combination of different ordinary transitions with the same action, is also introduced.

In Section 4, we present an example to show the expressibility of our qCCS. Particularly, we describe the well-known quantum teleportation protocol with the language of qCCS and show that it indeed teleports any qubit from one party to another.

Section 5 and Section 6 are the main part of the present paper. We define the notions of strong and weak probabilistic bisimilarities between quantum processes by first introducing corresponding bisimilarities between configurations. Some properties of these two bisimilarities are also derived.
Particularly, we show that strong/weak probabilistic bisimilarity between quantum configurations is the largest strong/weak probabilistic bisimulation on Con; a weak version of congruence property is also proved in which bisimilarity is preserved by parallel combinator when the common process is classical. The inherent non-commutativity of quantum operations and the potential entanglement between quantum systems make the general case very hard to prove and we put it aside for future investigation.

Section 7 is the concluding section in which we outline the main results of the present paper and point out some problems for further study.

2 Preliminary

In this section, we review some basic notations and concepts in quantum computation used in this paper. For a detailed survey, we refer to [26].

One of the most fundamental notions of quantum computation is Hilbert space. In finite dimensional case, a Hilbert space is just a vector space equipped with an inner product. So there exists a natural isomorphism between a Hilbert space and a complex vector space with the same dimension. According to the basic principles of quantum mechanics, the state of a quantum system can be characterized by a unit vector in some Hilbert space. For example, in a two-level quantum system (i.e., the associated Hilbert space is two-dimensional. Such a system is usually called a ‘qubit’, the quantum analogue of ‘bit’ in classical computation), a general state can be described by a complex vector

\[
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} \in \mathbb{C}^2 \tag{1}
\]

where \(\alpha^* + \beta^* = 1\). An alternative but more frequently used notation for the qubit presented in Eq.(1) is \(\alpha|0\rangle + \beta|1\rangle\), where \(|0\rangle\) and \(|1\rangle\) denote the two basis states respectively, and \(|\cdot\rangle\), called ‘ket’, indicates that it is a quantum state.

The evolution of a closed quantum system is described by a unitary transformation on the associated Hilbert space. To be specific, let a quantum system be in a state \(|\psi_1\rangle\) at time \(t_1\). Then the state \(|\psi_2\rangle\) at time \(t_2\) is related to \(|\psi_1\rangle\) by a unitary operator \(U\) which depends only on the times \(t_1\) and \(t_2\), that is,

\[|\psi_2\rangle = U|\psi_1\rangle.\]

Observation of a quantum system is a quantum measurement represented by a Hermitian operator \(M\) on the associated Hilbert space of the system. Suppose \(M\) has a spectral decomposition

\[M = \sum_m m P_m, \tag{2}\]

where \(P_m\) is the projector onto the eigenspace of \(M\) with eigenvalue \(m\). Then the probability of getting result \(m\) when the system is initially in the state \(|\psi\rangle\) is \(p(m) = \langle \psi | P_m | \psi \rangle\) and the post-measurement state of the system given the outcome \(m\) becomes \(P_m |\psi\rangle / \sqrt{p(m)}\).

The state space of a composite system (for example, a quantum system consisting of many qubits) is the tensor product of the state spaces of its component systems. To be specific, a system consisting of \(n\) qubits has a \(2^n\)-dimensional state space with basis states \(|0\rangle \otimes \cdots \otimes |0\rangle, \ldots, |1\rangle \otimes \cdots \otimes |1\rangle\}. We often simply write \(|0\rangle|0\rangle \ldots |0\rangle\) or even \(|00\ldots0\rangle\) for abbreviations of \(|0\rangle \otimes \cdots \otimes |0\rangle\), and so on. Note that in general, the state of a composite system cannot be decomposed into tensor product of the states of its component systems. A well-known example is the so-called EPR state

\[
\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)
\]

in 2-qubit system. This kind of states are called entangled states and the phenomenon is called entanglement. Mathematically speaking, entanglement illustrates the key difference between tensor
product in quantum systems and Cartesian product in classical systems. To see the power of entanglement, let us consider a measurement \( M = 0|0\rangle\langle 0| + 1|1\rangle\langle 1| \) applied on the first qubit of EPR state. If the result is 0 then we can definitely say that after the measurement, the second qubit collapses into state \( |0\rangle \). In any sequential measurement \( M \) applied on the second state, the result is definitely 0, in contract with the fact that originally it should yield 0 and 1 with respective probabilities one half. In other words, the measurement on the first qubit changes in a way the state of the second qubit, no matter how far away these two qubits are from each other. This is an outstanding property of quantum mechanics which has no correspondence in classical world, and is the source of many strange quantum phenomena such as teleportation [3] and superdense coding [4].

A more powerful notation to describe quantum system is density matrix (or density operator), especially when the state of the system we are concerned with is not completely known. This arises when the system has been interacted with a noisy environment or it is only a part of a larger system. In this setting, the state of a quantum system is represented by a density matrix, i.e., a positive semi-definite matrix with trace 1 on the associated Hilbert space. The unitary transformation \( U \) applied on the system has the effect \( \rho \rightarrow U\rho U^\dagger \), where \( U^\dagger \) denotes the adjoint of \( U \). A quantum measurement in Eq.(2) on an initial state \( \rho \) will get result \( m \) with probability \( \text{Tr}(P_m\rho) \) and when the result \( m \) occurs, the post-measurement state of the system is \( P_m\rho P_m/\text{Tr}(P_m\rho) \).

## 3 Syntax and semantics of qCCS

For the sake of simplicity, we consider only two types of data, the set of real numbers \( \text{Real} \) for classical information and the set of qubits \( \text{Qbt} \) for quantum information. We denote by \( \text{cVar} \) (ranged over by \( x, y, \ldots \)) and \( \text{qVar} \) (ranged over by \( q, r, \ldots \)) the set of classical variables on \( \text{Real} \) and quantum variables on \( \text{Qbt} \) respectively. The set of expressions with the value domain \( \text{Real} \) is denoted by \( \text{Exp} \) and ranged over by \( e \). Let \( \text{cChan} \) be the set of classical channel names, ranged over by \( c, d, \ldots \), and \( \text{qChan} \) the set of quantum channel names, ranged over by \( c, d, \ldots \). Denote \( \text{Chan} = \text{cChan} \cup \text{qChan} \).

A relabeling function \( f \) is a function from \( \text{Chan} \) to \( \text{Chan} \) such that \( f \mid \text{cChan} \subseteq \text{cChan} \) while \( f \mid \text{qChan} \not\subseteq \text{qChan} \).

From these notations, we now propose the syntax of qCCS. For simplicity, we often abbreviate ordered set \( \{q_1, \ldots, q_n\} \) to \( \bar{q} \) when \( q_1, \ldots, q_n \) are distinct quantum variables and the dimension \( n \) is understood.

**Definition 3.1** *(quantum process expression)* The set of quantum process expressions \( q\text{PrE} \) and the free quantum variable function \( qv : q\text{PrE} \rightarrow 2^\text{qVar} \) are defined inductively as follows, where \( E, F \in q\text{PrE}, \; c \in \text{cChan}, \; x, y \in \text{cVar}, \; q \in \text{qChan}, \; q_1, \ldots, q_n \in \text{qVar}, \; e \in \text{Exp}, \; f \) is a relabeling function, \( L \subseteq \text{Chan}, \; b \) is a boolean-valued expression, \( U \) is a unitary matrix, and \( M \) is a Hermitian matrix.

1. \( \text{nil} \in q\text{PrE}, \) and \( qv(\text{nil}) = \emptyset; \)
2. \( \text{c?x}.E \in q\text{PrE}, \) and \( qv(\text{c?x}.E) = qv(E); \)
3. \( \text{c!}.E \in q\text{PrE}, \) and \( qv(\text{c!}.E) = qv(E); \)
4. \( \text{(qbit} \; q).E \in q\text{PrE}, \) and \( qv((\text{qbit} \; q).E) = qv(E) - \{q\}; \)
5. \( \text{c?q}.E \in q\text{PrE}, \) and \( qv(\text{c?q}.E) = qv(E) - \{q\}; \)
6. If \( q \not\in qv(E) \) then \( \text{c?q}.E \in q\text{PrE}, \) and \( qv(\text{c?q}.E) = qv(E) \cup \{q\}; \)
7. \( U[\bar{q}].E \in q\text{PrE}, \) and \( qv(U[\bar{q}].E) = qv(E) \cup \bar{q}; \)
8. \( M[\bar{q};x].E \in q\text{PrE}, \) and \( qv(M[\bar{q};x].E) = qv(E) \cup \bar{q}; \)
9. $E + F \in qPrE$, and $qv(E + F) = qv(E) \cup qv(F)$;
10. If $qv(E) \cap qv(F) = \emptyset$ then $E \parallel F \in qPrE$, and $qv(E \parallel F) = qv(E) \cup qv(F)$;
11. $E[f] \in qPrE$, and $qv(E[f]) = qv(E)$;
12. $E\setminus L \in qPrE$, and $qv(E \setminus L) = qv(E)$;
13. (If $b$ then $E) \in qPrE$, and $qv(if \ b \ then \ E) = qv(E)$;
14. $qPrE$ is the smallest set satisfying 1-13.

Intuitively, for any quantum process expression $E$, $qv(E)$ denotes the set of quantum variables $E$ can reference. Note that in $clq.E$, the assumption $q \notin qv(E)$ guarantees that a quantum system will never be referenced after it has been output. This is a requirement of the quantum no-cloning theorem. For the same reason, we assume $q_{1}, \ldots, q_{n}$ distinct in $U[q].E$ and $M[q;x].E$ (Recall that the denotation $\bar{q}$ implies $q_{1}, \ldots, q_{n}$ are distinct). Furthermore, since we intend to use the parallel combinator $\parallel$ to model separate parties which can perform actions locally on their own systems and communicate with each other through channels, the assumption $qv(E) \cap qv(F) = \emptyset$ guarantees that $E$ and $F$ will never reference a quantum system synchronously.

We do not allow process variables and recursion in our language of quantum process expressions. Allowing them needs careful consideration of free quantum variables and we set it aside for further study.

The notion of free classical variables in quantum process expressions is as usual (only classical input prefix $c?x$ and measurement $M[q;x]$ have binding power on $x$). A quantum process expression $E$ is called a quantum process if it contains no free classical variables, i.e., $f\nu(E) = \emptyset$. The set of quantum processes is denoted by $qProc$ and ranged over by $P, Q, \ldots$.

Note that for any $E \in qPrE$ with $f\nu(E) \subseteq \{x_{1}, \ldots, x_{n}\}$ and indexed set $\nu = \{v_{1}, \ldots, v_{n}\}$ of real values, the expression $E[\nu/x]$ obtained by instantiating classical variables $x$ with $\nu$ is a quantum process. The following definition introduces a corresponding instantiation for free quantum variables. Similar notions were also proposed in [20] and [11] in somewhat different way.

**Definition 3.2 (Configuration)** For any process expression $E \in qPrE$, if $qv(E) \subseteq \bar{q}$ then the pair

$$< E; \bar{q} = \rho >$$

is called a configuration, where $\rho$ is a density matrix in $2^{n}$-dimensional Hilbert space and $n$ is the length of $\bar{q}$. The set of configurations is denoted by $Con$ and ranged over by $C, D, \ldots$. In configuration $C \equiv < E; \bar{q} = \rho >$, $\bar{q} = \rho$ is called the quantum context of $C$ and denoted by $Context(C)$.

Let $D(Con)$ be the set of finite-support distributions over $Con$, i.e.

$$D(Con) = \{\mu : Con \rightarrow [0,1] \mid \mu(C) > 0 \text{ for finitely many } C, \text{ and } \sum_{\mu(C) > 0} \mu(C) = 1\}.$$

For any $\mu \in D(Con)$, we denote by $Supp(\mu)$ the support set of $\mu$, i.e. the set of configurations $C$ such that $\mu(C) > 0$. When $\mu$ is a simple distribution such that $Supp(\mu) = \{C\}$ for some $C$, we briefly denote $\mu \equiv C$. Just as in [20] and [11], sometimes we find it convenient to denote a distribution $\mu \in D(Con)$ by an explicit form $\mu \equiv \bigoplus_{i \in I} p_{i} \bullet C_{i}$ (or $\mu \equiv \bigoplus_{i} p_{i} \bullet C_{i}$ when the index set $I$ is understood) where $Supp(\mu) = \{C_{i} \mid i \in I\}$ and $\mu(C_{i}) = p_{i}$. Given $\mu_{1}, \ldots, \mu_{n} \in D(Con)$ and $p_{1}, \ldots, p_{n} \in (0,1]$, $\sum_{i} p_{i} = 1$, we define the combined distribution, denoted by $\sum_{i=1}^{n} p_{i} \mu_{i}$, to be a new $\mu \in D(Con)$ such that

1. $Supp(\mu) = \bigcup_{i=1}^{n} Supp(\mu_{i})$,
2. for any $D \in Supp(\mu)$, $\mu(D) = \sum_{i} p_{i} \mu_{i}(D)$. 

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As usual, the operational semantics of our qCCS is based on probabilistic labeled transition system. Let
\[
\text{Act} = \{c?v, c!v\} \cup \{c?r, c!r\} \cup \{\tau\}
\]
where \(c\) and \(c\) are respectively classical and quantum channels, \(v \in \text{Real}\), \(r \in \text{qVar}\), and \(\tau\) is the silent action. Then the semantics is given by the probabilistic labeled transition system \((\text{Con}, \text{Act}, \rightarrow)\), where \(\rightarrow \subseteq \text{Con} \times \text{Act} \times \mathcal{D}(\text{Con})\) is the smallest relation satisfying the rules defined in the following definitions (For brevity, we write \(C \xrightarrow{\alpha} \mu\) instead of \((C, \alpha, \mu) \in \rightarrow\)).

**Definition 3.3 (Classical rules)**

**C-Inp**:

\[
\begin{align*}
\langle c.x.E; C > & \xrightarrow{c?v} \langle E[v/x]; C > \\
\langle c.e; C > & \xrightarrow{clv} \langle E; C >
\end{align*}
\]

for all \(v \in \text{Real}\)

**C-Outp**:

\[
\begin{align*}
\langle E_1; C > & \xrightarrow{c!v} \langle E'; C >, \quad \langle E_2; C > \xrightarrow{clv} \langle E'_2; C > \\
\langle E_1 || E_2; C > & \xrightarrow{\tau} \langle E'_1 || E'_2; C >
\end{align*}
\]

These three rules describe the passing of classical messages; they are almost the same as in classical process algebra. Note that the input and output of classical messages do not change the context since contexts include only current states of the accompanied quantum systems. In our notion of configurations, classical data are encoded in the structure of quantum processes, just as in classical process algebra; while quantum data are explicitly described in quantum contexts.

**Definition 3.4 Q-New (Quantum-new rule)**

\[
\langle (\text{qbit} \ q).E; \bar{q} = \rho \rangle \xrightarrow{\tau} \langle E[r/q]; r, \bar{q} = |0\rangle \otimes \rho >
\]

where \(r \notin \bar{q}\)

This rule creates a new qubit and initializes it to the standard state \(|0\rangle \otimes \rho >\).

**Definition 3.5 Q-Inp (Quantum-input rule)**

\[
\begin{align*}
\langle c?q.E; \bar{q} = \rho \rangle & \xrightarrow{c!r} \langle E[r/q]; r, \bar{q} = \sigma > \\
\langle c?q.E; \bar{q} = \rho \rangle & \xrightarrow{clr} \langle E[r/q]; \bar{q} = \rho >
\end{align*}
\]

where \(r \notin \bar{q}\) and \(\text{Tr}_r \sigma = \rho\)

The first rule characterizes the situation when a qubit not originally in the context is input; while the second rule deals with the case when the qubit to be input is already in the context. As explained in the Introduction section, this two rules together describe all possible input behaviors for a quantum process.

**Definition 3.6 Q-Outp (Quantum-output rule)**

\[
\langle cl.q.E; \bar{q} = \rho \rangle \xrightarrow{clq} \langle E; \bar{q} = \rho >
\]

Since the definition of configuration guarantees that \(q \in \bar{q}\), we do not specify this in the side-condition.
Definition 3.7 Unit (Unitary transformation rule)

\[ < U[\bar{r}], E; \bar{q} = \rho > \rightarrow < E; \bar{q} = U_r \rho U_r^\dagger > \]

where \( U_r \rho U_r^\dagger \) denotes the application of unitary transformation \( U \) on the system consisting of \( \bar{r} \). To be specific, let length(\( \bar{r} \)) = \( k \) and length(\( \bar{q} \)) = \( n \). Then \( U_r = \Pi_{\bar{r}}^j(U \otimes I^{\otimes(n-k)})\Pi_n \) where \( \Pi_n \) is the permutation transformation which places \( r_1, \ldots, r_k \) at the head of \( \bar{q} \). Similar notations were also introduced in [20].

In our framework of qCCS, unitary transformations are considered internal actions performed by individual systems. As a consequence, they are unobservable from outside and performing a unitary transformation is modeled by a \( \tau \)-action. Similar treatment is applied to measurement on quantum systems.

Definition 3.8 Meas (Measurement rule)

\[ < M[\bar{r}; x], E; \bar{q} = \rho > \rightarrow p_i \cdot < E[\lambda_i/x]; \bar{q} = P_{i,\bar{r}} \rho P_{i,\bar{r}}/p_i > \]

where \( M \) is a Hermitian matrix with spectral decomposition \( M = \sum \lambda_i P_i \). \( P_{i,\bar{r}} \) denotes the projection \( P_i \) performed on the system consisting of \( \bar{r} \), i.e., \( P_{i,\bar{r}} = \Pi_{\bar{r}}^j(P_i \otimes I^{\otimes(n-k)})\Pi_n \), and \( p_i = Tr(P_{i,\bar{r}} \rho) \).

Intuitively, after measurement the configuration will evolve into different configurations with corresponding probabilities depending on different measurement outcomes.

Definition 3.9 Q-Com (Quantum-communication rule)

\[
\begin{align*}
<& E_1; C >_{\bar{r}}^r &<& E_1'; C >_{\bar{r}}^r &<& E_2; C >_{\bar{r}}^r &<& E_2'; C >_{\bar{r}}^r \\
<& E_1||E_2; C > >_{\bar{r}}^r &<& E_1';||E_2; C > >_{\bar{r}}^r \\
<& E_1; C >_{\bar{r}}^r &<& E_2; C >_{\bar{r}}^r &<& E_1';||E_2; C > >_{\bar{r}}^r \\
<& E_1||E_2; C > >_{\bar{r}}^r &<& E_1';||E_2; C > >_{\bar{r}}^r
\end{align*}
\]

It may be surprising at first glance that the quantum context remains unchanged during the process of quantum communication. The reason is that by rules \textbf{Q-Inp} and \textbf{Q-Outp}, contexts here already contain all input/output qubits.

Definition 3.10 (Interleaving rules)

\textbf{Inp-Int} : \[
\begin{align*}
<& E_1; C >_{\bar{r}}^r &<& E_1'; C >_{\bar{r}}^r \\
<& E_1||E_2; C > >_{\bar{r}}^r &<& E_1';||E_2; C > >_{\bar{r}}^r
\end{align*}
\]
where \( r \notin qv(E_2) \)

\textbf{Oth-Int} : \[
\begin{align*}
<& E_1; C >_{\bar{r}}^r &<& E_1'; C >_{\bar{r}}^r \\
<& E_2||E_2; C > >_{\bar{r}}^r &<& E_1';||E_2; C > >_{\bar{r}}^r
\end{align*}
\]
where \( r \notin qv(E_1) \)

\[
\begin{align*}
<& E_1; C >_{\bar{r}}^r &<& E_1'; C >_{\bar{r}}^r \\
<& E_1||E_2; C > >_{\bar{r}}^r &<& E_1';||E_2; C > >_{\bar{r}}^r
\end{align*}
\]
where \( \alpha \notin \{c?r\} \)

\[
\begin{align*}
<& E_2; C >_{\bar{r}}^r &<& E_2'; C >_{\bar{r}}^r \\
<& E_1||E_2; C > >_{\bar{r}}^r &<& E_1';||E_2; C > >_{\bar{r}}^r
\end{align*}
\]
where \( \alpha \notin \{c?r\} \)
It is worth noting that when one component of a composite system attempts to input a quantum system, the system should not be referenced by another component. This is the reason we present the \textbf{Inp-Int} Rule here.

The following rules are similar to their classical correspondences.

**Definition 3.11 Sum** (Summation rule)

\[
\begin{align*}
\langle E; C \rangle &\xrightarrow{\alpha} \mu, \\
\langle E + F; C \rangle &\xrightarrow{\alpha} \mu
\end{align*}
\]

**Definition 3.12 Rel** (Relabeling rule)

\[
\begin{align*}
\langle E; C \rangle &\xrightarrow{\alpha} \exists p_i \cdot \langle E_i; C_i \rangle, \\
\langle E[f]; C \rangle &\xrightarrow{\alpha[f]} \exists p_i \cdot \langle E_i[f]; C_i \rangle
\end{align*}
\]

Here we have extended the definition of relabeling function to quantum process expressions and actions in an obvious way.

**Definition 3.13 Res** (Restriction rule)

\[
\begin{align*}
\langle E; C \rangle &\xrightarrow{\alpha} \exists p_i \cdot \langle E_i; C_i \rangle, \\
\langle E \setminus L; C \rangle &\xrightarrow{\alpha} \exists p_i \cdot \langle E_i \setminus L; C_i \rangle
\end{align*}
\]

Here the function \( cn(\alpha) \) returns the channel name used by an action.

**Definition 3.14 Cho** (Choice rule)

\[
\langle E; C \rangle \xrightarrow{\alpha} \mu, \quad \text{where } [b] = \text{true}
\]

Before concluding this section, we give two extensions of the ordinary transition relation. First we define transitions between distributions on \( \text{Con} \) as follows. Given \( \mu, \nu \in D(\text{Con}) \) and \( \alpha \in \text{Act} \), we write \( \mu \xrightarrow{\alpha} \nu \) if for any \( C \in \text{Supp}(\mu) \), there exists a transition \( C \xrightarrow{\alpha} \mu \) and

\[
\nu = \sum_{C \in \text{Supp}(\mu)} \mu(C)\mu_C.
\]

For an action string \( s = \alpha_1 \ldots \alpha_n \), we write \( \mu \xrightarrow{s} \nu \) or \( \mu \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_n} \nu \) if there exist \( \mu_1, \ldots, \mu_{n-1} \) such that \( \mu \xrightarrow{\alpha_1} \mu_1, \ldots, \mu \xrightarrow{\alpha_n} \nu \).

Second, we extend the ordinary transitions to combined transitions. Suppose \( C \in \text{Con}, \alpha \in \text{Act}, \) and \( \mu \in D(\text{Con}) \). We say that \( C \) can evolve into \( \mu \) by an \( \alpha \)-combined transition, denoted by \( C \xrightarrow{\alpha} \mu \), if there exist \( \mu_1, \ldots, \mu_n \in D(\text{Con}) \) and \( p_1, \ldots, p_n \in (0,1], \sum_i p_i = 1 \), such that \( C \xrightarrow{\alpha_i} \mu_i \) and \( \mu = \sum_i p_i \mu_i \). The set of combined transitions is denoted by \( \rightarrow_C \), that is

\[
\rightarrow_C := \left\{ (C, \alpha, \mu) : C \xrightarrow{\alpha} \mu \right\}.
\]

Combined transitions can also be extended to those between distributions in a similar way.
4 An example

In this section, we give a simple example to show the expressibility of our qCCS. This example is concerned with quantum teleportation [3], a famous protocol in quantum information theory which teleports an unknown quantum state by sending only classical information, provided that the sender and the receiver shared an entangled state in advance. This example was also considered in [20] and [11].

We first assume some predefined unitary transformations. The transformation CNOT performs on two qubits such that

\[
CNOT |00\rangle = |00\rangle, CNOT |01\rangle = |01\rangle, CNOT |10\rangle = |11\rangle, CNOT |11\rangle = |10\rangle.
\]

The 1-qubit Hadamard transformation \( H \) and Pauli transformations \( \sigma_0, \sigma_1, \sigma_2, \sigma_3 \) are defined respectively as follows:

\[
H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.
\]

We further assume the 2-qubit measurement \( M = \sum_{i=0}^{3} i |i\rangle \langle i| \), where \( i \) is the binary expansion of \( i \).

Then the participating quantum processes are defined as follows:

\[
Alice := c?q_1.CNot[q,q_1].H[q].M[q,q_1;x].clx.nil
\]

\[
Bob := d?q_2.c?x.\sigma_x[q_2].nil
\]

\[
EPR := (qbit q_1).(qbit q_2).H[q_1].CNOT[q_1,q_2].clq_1.dlq_2.nil
\]

\[
Telep := (EPR||Alice||Bob)\{c,d,c\}
\]

where we have adopted the obvious abbreviation

\[
\sigma_x[q_2].nil = \begin{cases} 
\sigma_0[q_2].nil & \text{if } x = 0 \\
\sigma_1[q_2].nil & \text{if } x = 1 \\
\sigma_2[q_2].nil & \text{if } x = 2 \\
\sigma_3[q_2].nil & \text{if } x = 3 
\end{cases}
\]

Now we can show that the process Telep defined above can indeed teleport any qubit from Alice
to Bob.

\[ <\text{Telep}, q = [α|0⟩ + β|1⟩ > \]

\[ \xrightarrow{\sim} \xrightarrow{\sim} <(H[q_1].CNOT[q_1, q_2].cl[q_1, d'l[q_2].nil]|Alice|Bob)\{c, d, c\}; q_2, q_1, q = [00⟩ ⊗ (α|0⟩ + β|1⟩) > \quad \text{Q-New, Res} \]

\[ \xrightarrow{\sim} \xrightarrow{\sim} <(cl[q_1, d'l[q_2].nil]|Alice|Bob)\{c, d, c\}; q_2, q_1, q = [\frac{1}{\sqrt{2}}(|00⟩ + |11⟩) ⊗ (α|0⟩ + β|1⟩) > \quad \text{Unit, Res} \]

\[ \xrightarrow{\sim} <\text{nil}|CNot[q, q_1].H[q].M[q, q_1; x].cl[x].nil|c?x.σ_x[q_2].nil\{c, d, c\}; q_2, q_1, q = [\frac{1}{\sqrt{2}}(|00⟩ + |11⟩) ⊗ (α|0⟩ + β|1⟩) > \quad \text{Q-Com, Res} \]

\[ \xrightarrow{\sim} <\text{nil}|M[q, q_1; x].cl[x].nil|c?x.σ_x[q_2].nil\{c, d, c\}; q_2, q_1, q = [\frac{1}{2}(α(|00⟩ + |01⟩ + |10⟩ + |11⟩) + β(|01⟩ − |01⟩ + |10⟩ − |10⟩) > \quad \text{Unit, Res} \]

\[ \xrightarrow{\sim} \frac{1}{4} <\text{nil}|cl[0].nil|c?x.σ_x[q_2].nil\{c, d, c\}; q_2, q_1, q = [α|00⟩ + β|10⟩ > \]

\[ \xrightarrow{\sim} \frac{1}{4} <\text{nil}|cl[1].nil|c?x.σ_x[q_2].nil\{c, d, c\}; q_2, q_1, q = [α|01⟩ + β|01⟩ > \]

\[ \xrightarrow{\sim} \frac{1}{4} <\text{nil}|cl[2].nil|c?x.σ_x[q_2].nil\{c, d, c\}; q_2, q_1, q = [α|01⟩ - β|10⟩ > \]

\[ \xrightarrow{\sim} \frac{1}{4} <\text{nil}|cl[3].nil|c?x.σ_x[q_2].nil\{c, d, c\}; q_2, q_1, q = [α|11⟩ - β|01⟩ > \quad \text{Meas, Res} \]

\[ \xrightarrow{\sim} \frac{1}{4} <\text{nil}|nil|nil\{c, d, c\}; q_2, q_1, q = [(α|0⟩ + β|1⟩) ⊗ |00⟩ > \]

\[ \xrightarrow{\sim} \frac{1}{4} <\text{nil}|nil|nil\{c, d, c\}; q_2, q_1, q = [(α|0⟩ + β|1⟩) ⊗ |10⟩ > \]

\[ \xrightarrow{\sim} \frac{1}{4} <\text{nil}|nil|nil\{c, d, c\}; q_2, q_1, q = [(α|0⟩ + β|1⟩) ⊗ |01⟩ > \]

\[ \xrightarrow{\sim} \frac{1}{4} <\text{nil}|nil|nil\{c, d, c\}; q_2, q_1, q = [(α|0⟩ + β|1⟩) ⊗ |11⟩ > \quad \text{C-Com, Unit, Res} \]

Here for any pure state |ψ⟩, [|ψ⟩] denotes the abbreviation of |ψ⟩⟨ψ|.

## 5 Strong probabilistic bisimilarity between quantum processes

In this section, we introduce the idea of strong probabilistic bisimulation between quantum processes and prove some properties of strong probabilistic bisimilarity.

Given an equivalence relation \( R \subseteq Con \times Con \), two distributions \( μ \) and \( ν \) on \( Con \) are said to be equivalent under \( R \), denoted by \( μ \equiv_R ν \), if for any equivalence class \( M \in Con/R \) it holds \( μ(M) = ν(M) \). We denote \( C \sim_R \) if there exists no \( μ \in D(Con) \) such that \( C \not\sim_R μ \); and \( C \rightarrow_R \) if \( C \not\sim_R C \) for any \( α \in Act \).

**Definition 5.1** An equivalence relation \( R \subseteq Con \times Con \) is a strong probabilistic bisimulation if for any \( C, D \in Con \), \( (C, D) \in R \) implies that

\[ 12 \]
Lemma 5.1

Theorem 5.2

Example 5.1

Definition 5.2

Example 5.2

Lemma 5.1

Theorem 5.1

Theorem 5.2

Theorem 5.3
(i) \( a = c!x \). Then \( \alpha = c!v \) and \( \mu = < E[\bar{v}/\bar{x},v/x]; C > \) for some \( v \in \text{Real} \). From \textbf{C-Inp} rule, we have \( < a.F[\bar{v}/\bar{x},v/x]; C > \sim \mu \equiv < F[\bar{v}/\bar{x},v/x]; C > \), and furthermore, \( < E[\bar{v}/\bar{x},v/x]; C > \sim < F[\bar{v}/\bar{x},v/x]; C > \) from the assumption that \( E \sim F \).

(ii) \( a = M[\bar{r}/x] \), \( M \) has the spectral decomposition \( M = \sum_i \lambda_i P_i \), and \( C = "\bar{q} = \rho" \). Then \( \alpha = \tau \) and \( \mu = \bigoplus_{i} \bullet < E[\bar{v}/\bar{x},\lambda_i/x]; \bar{q} = P_i,\bar{r} \rho P_i,\bar{r} / p_i > \), where \( p_i = \text{Tr} P_i,\bar{r} \rho \). In this case, we derive from \textbf{Meas} rule that

\[ < a.F[\bar{v}/\bar{x}]; C > \sim \bigoplus_{i} \bullet < F[\bar{v}/\bar{x},\lambda_i/x]; \bar{q} = P_i,\bar{r} \rho P_i,\bar{r} / p_i > = \mu. \]

For any \( M \in \text{Con}/ \sim \),

\[ \mu(M) = \sum_i [ \{ p_i : < E[\bar{v}/\bar{x},\lambda_i/x]; \bar{q} = P_i,\bar{r} \rho P_i,\bar{r} / p_i > \in M \} ] \]

and

\[ \nu(M) = \sum_i [ \{ p_i : < F[\bar{v}/\bar{x},\lambda_i/x]; \bar{q} = P_i,\bar{r} \rho P_i,\bar{r} / p_i > \in M \} ]. \]

Notice that by the assumption \( E \sim F \) we have for any context \( C \), < \( E[\bar{v}/\bar{x},\lambda_i/x]; C > \in M \) if and only if < \( F[\bar{v}/\bar{x},\lambda_i/x]; C > \in M \). Thus \( \mu(M) = \nu(M) \). \( \square \)

For the sake of simplicity, in the following two theorems we only consider quantum processes with no free classical variables. The same results can be proven easily to hold for quantum process expressions.

**Theorem 5.4** If \( P \sim Q \) then \( P + R \sim Q + R \) for any \( R \).

**Proof.** We prove this theorem by proving a stronger result: if \( < P_i; C > \sim < Q_i; D > \) for \( i = 1, 2 \), then \( < P_i + P_2; C > \sim < Q_1 + Q_2; D > \).

Suppose \( < P_i + P_2; C > \sim \mu \) for some \( \alpha \) and \( \mu \). Then from \textbf{Sum} rule, we have \( < P_i; C > \sim \mu \) or \( < P_2; C > \sim \mu \) for some \( \mu \). By the assumption that \( < P_i; C > \sim < Q_i; D > \) we derive \( < Q_1; D > \sim \nu \) or \( < Q_2; D > \sim \nu \) for some \( \nu \) such that \( \mu \equiv \nu \). Using \textbf{Sum} rule again, we know \( < Q_1 + Q_2; C > \sim \nu \).

If \( < P_i + P_2; C > \sim \) then we have \( < P_i; C > \sim \). So we derive that \( C = D \) from the assumption \( < P_i; C > \sim < Q_i; D > \). \( \square \)

Theorems 5.3 and 5.4 imply that strong probabilistic bisimilarity is preserved by prefix and summation combinators. It seems, however, very difficult to show whether it is also preserved by parallel combinator \( \parallel \) because the inherent non-commutativity of quantum operations and the potential entanglement between quantum systems will make the situation very complicated when interleaving is allowed between different components of a composite system. Nevertheless, in the following we prove a weaker result and put aside the general one for further study.

We say a quantum process classical if it will never change the context accompanied with it. In other words, it is constructed only by the combinators except 4, 5, 7 and 8 in Definition 3.1.

**Lemma 5.2** If \( R \) is classical and < \( R; C > \sim \alpha \) for some \( C \) and \( \alpha \), then \( < R; D > \sim \alpha < R'_D; D > \) for any context \( D \). Furthermore, \( R'_D \) is also classical.

**Proof.** Direct from definitions. \( \square \)

**Theorem 5.5** If \( P \sim Q \) then

1. \( P[R \sim Q]R \) for any classical \( R \),
2. \( P[f] \sim Q[f] \), for any relabeling function \( f \).
Proof. We only prove (1); (2) are simpler. Define

\[ R' = \{ < P\| R; C >, < Q\| R; D > : < P; C >\sim < Q; D >, \text{ and } R \text{ is classical} \} \]

and \( R = (R' \cup \sim)^* \) the equivalence closure (i.e. the reflexive, symmetric and transitive closure) of \( R' \cup \sim \). We prove in the following that \( R \) is a strong probabilistic bisimulation.

Suppose \( (C, D) \in R \). We may assume that \( (C, D) \in R' \) because the extension to the equivalence closure is straightforward. So we can suppose further that \( C \equiv < P\| R; C > \) and \( D \equiv < Q\| R; D > \) for some \( < P; C >\sim < Q; D > \) and \( R \) classical.

(i) If \( < P\| R; C >\overset{\alpha}{\Rightarrow} \mu \), there are three cases to consider.

Case 1: There exists \( \mu_1 \) such that \( < P; C >\overset{\alpha}{\Rightarrow} \mu_1 \) and

\[ \mu(< T; G >) = \begin{cases} \mu_1(< T'_1; G >), & \text{if } T = T'_1\| R \\ 0, & \text{otherwise.} \end{cases} \]

By the assumption that \( < P; C >\sim < Q; D > \) and Theorem 5.2, we have \( < Q; D >\overset{\alpha}{\Rightarrow} \nu_1 \) and \( \mu_1(M_1) = \nu_1(M_1) \) for any \( M_1 \in Con/ \sim \). Then \( < Q\| R; D >\overset{\alpha}{\Rightarrow} \nu \) for

\[ \nu(< T; G >) = \begin{cases} \nu_1(< T'_1; G >), & \text{if } T = T'_1\| R \\ 0, & \text{otherwise.} \end{cases} \]

For any \( M \in Con/R \), let

\[ M' = \{ < T'_1; G > \mid < T'_1\| R; G > \in M \} \]

From the definition of \( R \), it is not difficult to check that for any configurations \( C \sim D, C \in M' \) if and only if \( D \in M' \). So \( M' \) is the union of some equivalence classes of \( Con/ \sim \), and thus

\[ \mu(M) = \mu_1(M') = \nu_1(M') = \nu(M). \]

Case 2: There exists transition \( < R; C >\overset{\alpha}{\Rightarrow} < R'; C > \) and \( \mu \equiv < P\| R'; C > \). Then from Lemma 5.2, we know \( R' \) is also classical and \( < R; D >\overset{\alpha}{\Rightarrow} < R'; D > \). Thus \( < Q\| R; D >\overset{\alpha}{\Rightarrow} < Q\| R'; D > \) and \( < P\| R'; C >\sim \mu < Q\| R'; D > \) from the definition of \( R \).

Case 3: \( \alpha = \tau \), and the action is caused by a (classical or quantum) communication between \( P \) and \( R \). We assume that

\[ < P; C >\overset{c_{\tau}}{\Rightarrow} < P'; C >, \quad < R; C >\overset{c_{\tau}}{\Rightarrow} < R'; C > \]

and \( \mu = < P\| R'; C > \). Other cases are similar. From the assumption that \( < P; C >\sim < Q; D > \) and Theorem 5.2, we have

\[ < Q; D >\overset{c_{\tau}}{\Rightarrow} C \Xi p_i \Xi < Q_i; D > \text{ and for any } i, < P'; C >\sim < Q_i; D > . \]

On the other hand, from Lemma 5.2 we derive

\[ < R; D >\overset{c_{\tau}}{\Rightarrow} < R'; D > . \]

Thus from \textbf{Q-Com} rule,

\[ < Q\| R; D >\overset{\tau}{\Rightarrow} C \Xi p_i \Xi < Q_i\| R'; D > \equiv \nu. \]

In order to guarantee \( \mu \equiv \nu \), we need only to prove for any \( i, (< P'; R'; C >, < Q_i\| R'; D >) \in R \), which is direct from the fact that \( < P'; C >\sim < Q_i; D > \) and the definition of \( R \).

(ii) If \( < P\| R; C >\Rightarrow \), then we have also \( < P; C >\Rightarrow \). Hence \( Context(C) = Context(D) \) from the assumption \( < P; C >\sim < Q; D > \).
From (i) and (ii) we know that $R$ is a strong probabilistic bisimulation on $\text{Con}$. Now since $P \sim Q$, we have $\langle P; C \rangle \sim \langle Q; C \rangle$ for any context $C$. So $\langle P||R; C \rangle \sim \langle Q||R; C \rangle$ for any $C$ and hence $P||R \sim Q||R$. \hfill \Box

Unfortunately, our notion of strong probabilistic bisimilarity (and also the weak probabilistic bisimilarity) is not preserved by restriction. An example is as follows. Let $U_1, U_2, V_1, V_2$ be unitary transformations such that $U_1 U_2 = V_1 V_2$ but $U_1 \neq V_1$, and

$P \equiv U_1[q].c!0.U_2[q].\text{nil}$, $Q \equiv V_1[q].c!0.V_2[q].\text{nil}$.

It is easy to check that $P \sim Q$ but $P\{c\} \not\sim Q\{c\}$.

To conclude this section, we give a simple property of strong probabilistic bisimilarity.

**Theorem 5.6** For any $E, F \in qPrE$,

1. $E + F \sim F + E$,
2. $E + E \sim E$,
3. $E + (F + G) \sim (E + F) + G$,
4. $E + \text{nil} \sim E$,
5. $E||\text{nil} \sim E$,

**Proof.** Direct from definition.

### 6 Weak probabilistic bisimilarity between quantum processes

As in classical process algebra, the notion of weak probabilistic bisimulation which abstracts from unobservable internal actions is more useful in implementation and verification. In this section, we present weak probabilistic bisimulation in qCCS. Some definitions are motivated by or borrowed directly from [32] and [33] which are designed for classical probabilistic processes.

#### 6.1 Weak transitions

The basis of weak probabilistic bisimulation is the notion of weak transitions, which in turn is a natural extension of combined transitions allowing internal actions before or after external actions in every execution path. To define weak transitions formally, we introduce first the notions of execution fragment and adversary.

**Definition 6.1** An execution fragment $f = C_0 \alpha_1 C_1 \ldots \alpha_n C_n$ is a finite sequence of alternating configurations and actions starting and ending with configurations, such that for each $i = 0, \ldots, n - 1$, there exists a combined transition $C_i \xrightarrow{\alpha_{i+1}} C_{i+1}$ with $\mu_{i+1}(C_{i+1}) > 0$. We call $n$ the length of $f$ and denote by head($f$) and tail($f$) the first and the last configurations of $f$ respectively. The set of all execution fragments is denoted by $\text{frag}$.

For any execution fragment $f$, we denote by $Pre(f)$ the set of execution fragments which are prefixes of $f$.

**Definition 6.2** An adversary $A$ is a function from fragments to the set of finite-support distributions over combined transitions, i.e.

$$A : \text{frag} \rightarrow D(\rightarrow C)$$

such that for any $f \in \text{frag}$, if $A(f) = \oplus_{i \in I} p_i \cdot (C_i, \alpha_i, \mu_i)$ then $\alpha_i$ are distinct, and $C_i = \text{tail}(f)$ for any $i \in I$.
Intuitively, an adversary provides a mechanics to resolve nondeterminism by deciding next transition from the execution history when nondeterministic choices have to be made. The next definition describes the situation where an execution fragment can be obtained according to an adversary.

**Definition 6.3** Suppose \( f = C_0\alpha_1C_1\ldots\alpha_nC_n \) is an execution fragment and \( \mathcal{A} \) an adversary such that \( \mathcal{A}(C_0\alpha_1C_1\ldots\alpha_nC_n) = \bigoplus_{j\in J} \bullet (C_i, \beta_j, \mu_j) \) for \( i = 0, \ldots, n-1 \). We say that \( f \) coincides with \( \mathcal{A} \) if for any \( i = 0, \ldots, n-1 \), there exists \( j_i \in J \) such that \( \beta_{j_i} = \alpha_{i+1} \) and \( \mu_{j_i}(C_{i+1}) > 0 \).

We denote by \( P^j_{\mathcal{A}}(f) = p_{j_i}\mu_{j_i}(C_{i+1}) \) the probability of the \( i \)-th choice in \( f \) according to the adversary \( \mathcal{A} \).

Note that in the above definition, the required \( j_i \) is, if exists, unique from the fact that all \( \beta_j \) are distinct by Definition 6.2. So the notion \( P^i_{\mathcal{A}} \) is well-defined.

For any adversary \( \mathcal{A} \), we denote by \( F_{\mathcal{C}\rightarrow\mathcal{D}}^\mathcal{A} \) the set of execution fragments with head \( \mathcal{C} \) and tail \( \mathcal{D} \) which coincide with \( \mathcal{A} \). If \( f = C_0\alpha_1C_1\ldots\alpha_nC_n \in F_{\mathcal{C}\rightarrow\mathcal{D}}^\mathcal{A} \), we write

\[
P_{\mathcal{A}}(f) = \prod_{i=0}^{n-1} P^i_{\mathcal{A}}(f)
\]

the probability of the execution fragment \( f \) according to \( \mathcal{A} \). When \( f \) does not coincide with \( \mathcal{A} \), we simply let \( P_{\mathcal{A}}(f) = 0 \).

With the above definitions, we are now ready to define the notion of weak transitions.

**Definition 6.4** For any \( \mathcal{C} \in Con \), \( s = \alpha_1\ldots\alpha_n \in Act^* \), and \( \mu \in D(Con) \), we say that \( \mathcal{C} \) can evolve into \( \mu \) by a weak s-transition, denoted by \( \mathcal{C} \xrightarrow{\alpha} \mu \), if there exists an adversary \( \mathcal{A} \) such that for any \( \mathcal{D} \in Supp(\mu) \),

1. \( \sum_{f \in F_{\mathcal{C}\rightarrow\mathcal{D}}^\mathcal{A}} P_{\mathcal{A}}(f) = \mu(\mathcal{D}) \),
2. for any \( f = C_0\beta_1C_1\ldots\beta_mC_m \in F_{\mathcal{C}\rightarrow\mathcal{D}}^\mathcal{A} \), the string \( \beta_1\ldots\beta_m \) has the form \( \tau^*\alpha_1\tau^*\ldots\tau^*\alpha_n\tau^* \).

In the following, we prove some lemmas which are useful for the next subsection. The first lemma shows that any convex-combination of weak s-transitions is also a weak s-transition.

**Lemma 6.1** Suppose \( \mathcal{C} \xrightarrow{s} \mu_1 \) and \( \mathcal{C} \xrightarrow{s} \mu_2 \) for some \( s \in Act^* \). Let \( p \in (0,1] \). Then \( \mathcal{C} \xrightarrow{s} \mu \) where \( \mu = p\mu_1 + (1-p)\mu_2 \).

**Proof.** Suppose an adversary corresponding to \( \mathcal{C} \xrightarrow{s} \mu_i \) is \( \mathcal{A}_i \), \( i = 1, 2 \). We construct a new adversary \( \mathcal{A} \), which will be proven to be a corresponding adversary of \( \mathcal{C} \xrightarrow{s} \mu \), as follows. For any \( f \in frag \),

\[
\mathcal{A}(f) = \begin{cases} 
   \frac{pP_{\mathcal{A}_1}(f)}{P_{\mathcal{A}}(f)}A_1(f) + \left(1 - \frac{pP_{\mathcal{A}_1}(f)}{P_{\mathcal{A}}(f)}\right)A_2(f) & \text{if } P_{\mathcal{A}}(f) \neq 0, \\
   pA_1(f) + (1-p)A_2(f) & \text{otherwise}.
\end{cases}
\]

(4)

Note that \( P_{\mathcal{A}}(\mathcal{C}) = 1 \) for any adversary \( \mathcal{A} \) and \( \mathcal{C} \in Con \), and \( P_{\mathcal{A}}(f) \) is dependent only on the set \( \{\mathcal{A}(f') : f' \in Pre(f), f' \neq f\} \). The above definition is meaningful and is an inductive one. Now we show that for any \( f \in frag \) with \( head(f) = \mathcal{C} \),

\[
P_{\mathcal{A}}(f) = pP_{\mathcal{A}_1}(f) + (1-p)P_{\mathcal{A}_2}(f)
\]

(5)

by induction on the structure of \( f \).

When \( f = \mathcal{C} \), we have

\[
P_{\mathcal{A}}(\mathcal{C}) = 1 = p + (1-p)P_{\mathcal{A}_1}(\mathcal{C}) + (1-p)P_{\mathcal{A}_2}(\mathcal{C}).
\]
Now suppose Eq.(5) holds for \( f = C_{o_1}C_1 \ldots C_{o_n} \). Then for \( f' = C_{o_1}C_1 \ldots C_{o_{n+1}}C_{n+1} \), there are two cases to consider.

Case 1. \( P_A(f) = 0 \). Then from Eq.(5) we also find that \( P_{A_1}(f) = P_{A_2}(f) = 0 \). So we have \( P_A(f') = P_{A_1}(f') = P_{A_1}(f') = 0 \), and Eq.(5) holds trivially for \( f' \).

Case 2. \( P_A(f) \neq 0 \). In this case, we derive that

\[
P_A(f') = P_A(f)P_{A_2}(f')
\]

Definition

\[
P_A(f)[P_{A_1}(f)P_{A_2}(f') + (1 - p_{PA_1}(f))P_{A_2}(f')]
\]

Eq.(4)

\[
pP_{A_1}(f)P_{A_2}(f') + (P_A(f) - pP_{A_1}(f))P_{A_2}(f')
\]

\[
pP_{A_1}(f)P_{A_2}(f') + (1 - p)P_A(f)P_{A_2}(f')
\]

Eq.(5)

\[
pP_{A_1}(f') + (1 - p)P_{A_2}(f').
\]

Definition

So for any \( D \in \text{Supp}(\mu) \),

\[
\sum_{f \in F^A_{C \rightarrow D}} P_A(f) = \sum_{f \in F^A_{C \rightarrow D}} \left[ pP_{A_1}(f) + (1 - p)P_{A_2}(f) \right]
\]

\[
= p \sum_{f \in F^A_{C \rightarrow D}} P_{A_1}(f) + (1 - p) \sum_{f \in F^A_{C \rightarrow D}} P_{A_2}(f)
\]

\[
= p\mu_1(D) + (1 - p)\mu_2(D)
\]

\[
= \mu(D).
\]

Here in the second equation, we have used the fact that

\[
F^A_{C \rightarrow D} = F^A_{C \rightarrow D} \cup F^A_{C \rightarrow D}.
\]

which is direct from the definition equation (4), and the assumption that \( P_A(f) = 0 \) when \( f \) does not coincide with \( A \).

Furthermore, it is easy to check that the second condition in Definition 6.4 holds. So we have that \( C \xrightarrow{s} \mu \).

\[ \square \]

**Corollary 6.1** For any \( \mu_1, \ldots, \mu_n \in D(\text{Con}) \) and \( p_1, \ldots, p_n \in (0, 1] \) such that \( C \xrightarrow{s} \mu_i \) and \( \sum_i p_i = 1 \), we have \( C \xrightarrow{s} \mu \) for \( \mu = \sum_i p_i \mu_i \).

**Proof.** Direct from Lemma 6.1. \[ \square \]

**Lemma 6.2** Let \( C \xrightarrow{s} \mu \), \( s = \alpha_1 \ldots \alpha_n \in \text{Act}^* \), and a corresponding adversary be \( A \). Suppose \( A(C) = \bigoplus_{i \in I} \bullet (C, \beta_i, \mu_i) \). Then for any \( i \in I \),

1. \( \beta_i = \tau \) or \( \alpha_1 \),
2. for any \( C_i \in \text{Supp}(\mu_i) \), there exists \( \mu_{C_i} \) such that \( C_i \xrightarrow{s_i} \mu_{C_i} \), and \( \beta_i s_i = s \),
3. \( \mu = \sum_{i \in I} \sum_{C_i \in \text{Supp}(\mu_i)} p_i \mu_i(C_i) \mu_{C_i} \).

**Proof.** (1) is obvious. To prove (2), for any \( C_i \in \text{Supp}(\mu_i) \), let \( \mu_{C_i} \in D(\text{Con}) \) such that for any \( D \in \text{Con} \),

\[
\mu_{C_i}(D) = \frac{1}{p_i \mu_i(C_i)} \sum \{| P_A(f) : f \in F^A_{C \rightarrow D} \text{ and } C \beta_i C_i \in \text{Pre}(f) | \}.
\]

Let \( s_i = s = s \alpha_1 \ldots \alpha_n \) depending on whether \( \beta_i = \tau \) or \( \alpha_1 \). Then \( \beta_i s_i = s \). We now turn to prove that \( C_i \xrightarrow{s_i} \mu_{C_i} \) by constructing a corresponding adversary \( A_{C_i} \) as follows. For any \( f \in \text{frag} \), let

\[
A_{C_i}(f) = \begin{cases} A(C \beta_i f) & \text{if } \text{head}(f) = C_i, \\ A(f) & \text{otherwise.} \end{cases}
\]
It is easy to check that for any $f \in \text{frag}$, if $\text{head}(f) = C_i$ then $p_i \mu_i(C_i) P_{A_{C_i}}(f) = P_A(C_i f)$. So we have for any $D \in \text{Supp}(\mu_{C_i})$,

$$
\sum_{f \in F_{C_i \rightarrow D}} P_{A_{C_i}}(f) = \frac{1}{p_i \mu_i(C_i)} \sum_{f \in F_{C_i \rightarrow D}} P_A(C_i f) \\
= \frac{1}{p_i \mu_i(C_i)} \sum \{| P_A(f') : f' \in F_{C_i \rightarrow D} \text{ and } C_i f_i \in \text{Pre}(f') \} \\
= \mu_{C_i}(D).
$$

Finally, to prove (3) we need only to check that for any $D \in \text{Con}$,

$$
\mu(D) = \sum_{f \in F_{C \rightarrow D}} P_A(f) \\
= \sum_{i \in I} \sum_{C_i \in \text{Supp}(\mu_i)} \sum \{| P_A(f) : f \in F_{C_i \rightarrow D} \text{ and } C_i f_i \in \text{Pre}(f) \} \\
= \sum_{i \in I} \sum_{C_i \in \text{Supp}(\mu_i)} p_i \mu_i(C_i) \mu_{C_i}(D).
$$

To illustrate the definitions and lemmas in this subsection, we present a simple example as follows.

**Example 6.1** Let $M_{0,1} = 0|0\rangle\langle 0| + 1|1\rangle\langle 1|$ and $M_{+,-} = 0|+\rangle\langle +| + 1|-\rangle\langle -|$ be two measurements according to different bases, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Let $I$ be the identity transformation and $U$ be a unitary transformation such that $U|0\rangle = |+\rangle, \ U|1\rangle = |-\rangle$.

Suppose $P = M_{0,1}[q; x].U[q].\text{cl}_q.\text{nil} + M_{+,-}[q; x].I[q].\text{cl}_q.\text{nil}$

is a quantum process which can either perform the measurement according to the $\{+, -\}$ basis on qubit $q$, or apply sequentially the measurement according to computational basis and the unitary transformation $U$ on $q$, and then output $q$. Now consider the configuration $C = <P; q = |+\rangle\langle +|>$. 

The following are two $\text{cl}_q$-weak transitions from configuration $C$:

```
    τ, 1/2
    C1 -- C2
     \ τ, 1/2 /  \\
    C1            C2

    τ, 1
    C3 -- C4
     \ τ, 1 /  \\
    C3            C4
```

Figure 1: The derivation tree of weak transition $C \xrightarrow{\tau, 1/2} C_5 \oplus \frac{1}{2} \cdot C_6$

with a corresponding adversary $A_1$ such that
$$\mathcal{A}_1(C) = (C, \tau, \frac{1}{2} \cdot C_1 \downarrow \frac{1}{2} \cdot C_2), \quad \mathcal{A}_1(C \tau C_1) = (C_1, \tau, C_3), \cdots,$$

and

$$\begin{align*}
\tau, 1 & \quad \tau, 1 & \quad clq, 1 & \quad clq, 1 \\
C & \quad C_7 & \quad C_3 & \quad C_5
\end{align*}$$

Figure 2: The derivation tree of weak transition $C \xrightarrow{clq} C_5$

with a corresponding adversary $A_2$ such that

$$A_2(C) = (C, \tau, C_7), \quad A_2(C \tau C_7) = (C_7, \tau, C_3), \cdots,$$

where

$$C_1 = <U[q], clq.nil; q = |0\rangle \langle 0|>, \quad C_2 = <U[q], clq.nil; q = |1\rangle \langle 1|>,$$

$$C_3 = <clq.nil; q = +\rangle \langle +|>, \quad C_4 = <clq.nil; q = -\rangle \langle -|>,$$

$$C_5 = <nil; q = +\rangle \langle +|>, \quad C_6 = <nil; q = -\rangle \langle -|>,$$

$$C_7 = <I[q], clq.nil; q = +\rangle \langle +|>.$$

Now taking $\frac{1}{2} - \frac{1}{2}$ combination of these two transitions, we derive a new weak transition from $C$ as follows:

$$\begin{align*}
\tau, 1/4 & \quad \tau, 1/4 & \quad \tau, 1/2 & \quad \tau, 1 \\
C & \quad C_1 & \quad C_2 & \quad C_7
\end{align*}$$

$$\begin{align*}
\tau, 1 & \quad clq, 1 & \quad clq, 1 & \quad clq, 1 \\
C_3 & \quad C_5 & \quad C_3 & \quad C_6
\end{align*}$$

Figure 3: The derivation tree of weak transition $C \xrightarrow{clq} \frac{3}{4} \cdot C_5 \downarrow \frac{1}{4} \cdot C_6$

and the corresponding adversary derived using Eq.(4) is $A$ such that

$$\mathcal{A}(C) = \frac{1}{2} \mathcal{A}_1(C) + \frac{1}{2} \mathcal{A}_2(C) = (C, \tau, \frac{1}{4} \cdot C_1 \downarrow \frac{1}{4} \cdot C_2 + \frac{1}{2} \cdot C_7),$$

$$\mathcal{A}(C \tau C_1) = \mathcal{A}_1(C \tau C_1) = (C_1, \tau, C_3), \cdots.$$

$\square$

### 6.2 Weak probabilistic bisimulation

From the notions defined in the previous subsection, we can define our weak probabilistic bisimulation between quantum configurations as follows.

**Definition 6.5** An equivalence relation $R \subseteq Con \times Con$ is a weak probabilistic bisimulation if for any $C, D \in Con$, $(C, D) \in R$ implies that

1. whenever $C \xrightarrow{\alpha} \mu$ for some $\alpha \in Act$, there exists $\nu$ such that $D \xrightarrow{\bar{\alpha}} \nu$ and $\mu \equiv_R \nu$,
2. if $C \leftrightarrow$ and $D \Rightarrow$, then $Contex(C) = Contex(D)$.
The following two lemmas show that the ordinary transition in Clause (1) of the above definition can be strengthened to combined transition and even weak transition.

**Lemma 6.3** Let $\mathcal{R} \subseteq \text{Con} \times \text{Con}$ be a weak probabilistic bisimulation and $(\mathcal{C}, \mathcal{D}) \in \mathcal{R}$. Then for any $\alpha \in \text{Act}$, if $\mathcal{C} \xrightarrow{\alpha} \mu$, then $\mathcal{D} \xrightarrow{\alpha} \nu$ for some $\nu$ such that $\mu \equiv \mathcal{R} \nu$.

**Proof.** Suppose $\mathcal{C} \xrightarrow{\alpha} \mu$. By definition there exist $\mu_1, \ldots, \mu_n \in D(\text{Con})$ and $p_1, \ldots, p_n \in (0, 1]$, $\sum_i p_i = 1$, such that $\mathcal{C} \xrightarrow{\alpha} \mu_i$ and $\mu = \sum_i p_i \mu_i$. From the assumption that $\mathcal{C} \mathcal{R} \mathcal{D}$, we have $\mathcal{D} \xrightarrow{\alpha} \nu_i$ for some $\nu_1, \ldots, \nu_n$ such that $\mu_i \equiv \mathcal{R} \nu_i$ for any $i$. Let $\nu = \sum_i p_i \nu_i$. Then we have $\mu \equiv \mathcal{R} \nu$ and $\mathcal{D} \xrightarrow{\alpha} \nu$ from Corollary 6.1.

**Lemma 6.4** Let $\mathcal{R} \subseteq \text{Con} \times \text{Con}$ be a weak probabilistic bisimulation and $(\mathcal{C}, \mathcal{D}) \in \mathcal{R}$. Then for any $s \in \text{Act}^*$, if $\mathcal{C} \xrightarrow{s} \mu$, then $\mathcal{D} \xrightarrow{s} \nu$ for some $\nu$ such that $\mu \equiv \mathcal{R} \nu$.

**Proof.** Suppose an adversary corresponding to $\mathcal{C} \xrightarrow{s} \mu$ is $\mathcal{A}$. We prove the lemma by induction on the maximal length $h$ of the execution fragments in $\bigcup_{D \in \text{Supp}(\mu)} F_{C \xrightarrow{s} D}$.

If $h = 0$, then we have $s = \epsilon$ and $\mu = \mathcal{C}$. In this case, we need only to take $\nu = \mathcal{D}$.

Suppose the result holds for $h \leq n$. We now prove that it also holds for $h = n + 1$. Let $\mathcal{A}(\mathcal{C}) = \bigoplus_{i \in I} p_i \bullet (\mathcal{C}, \alpha_i, \mu_i)$. Then for each $i \in I$ we have $\mathcal{C} \xrightarrow{\alpha_i} \mu_i$, and from Lemma 6.3 there exists $\nu_i$ such that $\mathcal{D} \xrightarrow{\alpha_i} \nu_i$ and $\mu_i \equiv \mathcal{R} \nu_i$. Furthermore, from Lemma 6.2 we have for any $\mathcal{C}_i \in \text{Supp}(\mu_i)$, there exists $\mu_{\mathcal{C}_i}$ such that $\mathcal{C}_i \xrightarrow{\alpha_i} \mu_{\mathcal{C}_i}$, $\beta_i s_i = s$, and

$$
\mu = \sum_{i \in I} \sum_{\mathcal{C}_i \in \text{Supp}(\mu_i)} p_i \mu_i(\mathcal{C}_i) \mu_{\mathcal{C}_i}.
$$

Now take arbitrarily $\mathcal{D}_i \in \text{Supp}(\nu_i)$. Let $[\mathcal{D}_i]_\mathcal{R}$ denote the equivalence class of $\mathcal{R}$ which includes $\mathcal{D}_i$. For any $\mathcal{C}_i \in \text{Supp}(\mu_i) \cap [\mathcal{D}_i]_\mathcal{R}$, we can choose an adversary $\mathcal{A}_{\mathcal{C}_i}$ corresponding to $\mathcal{C}_i \xrightarrow{\alpha_i} \mu_{\mathcal{C}_i}$ such that the maximal length of the execution fragments in $\bigcup_{D \in \text{Supp}(\mu_i) \cap [\mathcal{D}_i]_\mathcal{R}} F_{\mathcal{C}_i \xrightarrow{s} D}$ is less than $n + 1$. So by induction we have $\mathcal{D}_i \xrightarrow{\alpha_i} \nu_{\mathcal{D}_i}^{\mathcal{C}_i}$ for some $\nu_{\mathcal{D}_i}^{\mathcal{C}_i}$, and $\mu_{\mathcal{C}_i} \equiv \mathcal{R} \nu_{\mathcal{D}_i}^{\mathcal{C}_i}$. Now from Corollary 6.1 we have $\mathcal{D}_i \xrightarrow{\alpha_i} \nu_{\mathcal{D}_i}$, where

$$
\nu_{\mathcal{D}_i} = \sum_{\mathcal{C}_i \in \text{Supp}(\mu_i) \cap [\mathcal{D}_i]_\mathcal{R}} \frac{\mu_i(\mathcal{C}_i)}{\mu_i(\text{Supp}(\mu_i) \cap [\mathcal{D}_i]_\mathcal{R})} \nu_{\mathcal{D}_i}^{\mathcal{C}_i}.
$$

It is now direct to check that $\mathcal{D} \xrightarrow{s} \nu$ for

$$
\nu = \sum_{i \in I} \sum_{\mathcal{D}_i \in \text{Supp}(\nu_i)} p_i \nu_i(\mathcal{D}_i) \nu_{\mathcal{D}_i}.$$
Finally, we show $\mu \equiv R \nu$. For any $M \in Con/R$,

$$\nu(M) = \sum_{i \in I} \sum_{D_i \in Supp(\nu_i)} p_i \nu_i(D_i) \nu_{D_i}(M)$$

$$= \sum_{i \in I} \sum_{D_i \in Supp(\nu_i)} p_i \nu_i(D_i) \left( \sum_{c_i \in Supp(\mu_i) \cap [D_i]_R} \frac{\mu_i(c_i)}{\mu_i(Supp(\mu_i) \cap [D_i]_R)} \nu^{c_i}_{D_i}(M) \right)$$

$$= \sum_{i \in I} \sum_{c_i \in Supp(\mu_i)} p_i \mu_i(c_i) \left( \sum_{D_i \in Supp(\nu_i) \cap [c_i]_R} \frac{\nu_i(D_i)}{\mu_i([c_i]_R)} \right)$$

$$= \sum_{i \in I} \sum_{c_i \in Supp(\mu_i)} p_i \mu_i(c_i) \nu_{c_i}(M)$$

$$= \mu(M).$$

Here the third equation is due to the fact that $\mu_{c_i} \equiv R \nu^{c_i}_{D_i}$ for any $D_i \in Supp(\nu_i)$ and $C_i \in Supp(\mu_i) \cap [D_i]_R$; while the fourth equation holds because $\mu_i \equiv R \nu_i$ for any $i \in I$. □

**Lemma 6.5** Let $R \subseteq Con \times Con$ be a weak probabilistic bisimulation and $(C, D) \in R$.

(1) If $C \not\rightarrow$ then $D \not\rightarrow$ for any $\alpha \in Act - \{\tau\}$.

(2) For any $s \in Act^*$, if $C \not\rightarrow s \nu$ such that $C' \not\rightarrow$ for some $C' \in Supp(\mu)$, then there exists $\nu$ such that $D \not\rightarrow \nu$ and $D' \not\rightarrow$ for some $D' \in Supp(\nu)$. Furthermore, $Contex(C') = Contex(D')$.

**Proof.** (1) is easy. To prove (2), from $C \not\rightarrow s \nu$ we first have some $\nu_1$ such that $D \not\rightarrow \nu_1$ such that $\mu \equiv R \nu_1$. If there exists a $D_1 \in Supp(\nu_1) \cap [C']_R$ such that $D_1 \rightarrow$ then we have done. Otherwise, for any $D_1 \in Supp(\nu_1) \cap [C']_R$, from $C \rightarrow$ and (1) we have $D_1 \not\rightarrow \nu_2$ for some $\nu_2$ and $C'RD_2$ for any $D_2 \in Supp(\nu_2)$. Then we check if there exists a $D_2 \in Supp(\nu_2)$ such that $D_2 \rightarrow$. Note that the quantum processes we consider in this paper are all finitely derivable. It follows that we will finally reach a distribution $\nu$ such that $D \not\rightarrow \nu$ such that $C'RD'$ and $D' \rightarrow$ for some $D' \in Supp(\nu)$. Furthermore, from Definition 6.5 (2) we have $Contex(C') = Contex(D')$. □

Since the union of equivalence relations is not necessarily an equivalence relation, the union of weak probabilistic bisimulations is not necessarily a weak probabilistic bisimulation either. Nevertheless, we can prove the reflexive and transitive closure of the union of weak probabilistic bisimulations is also a weak probabilistic bisimulation. The technique used in the proof is from [12].

**Theorem 6.1** If $R_i, i \in I$, is a collection of weak probabilistic bisimulations on $Con$, then their reflexive and transitive closure $(\cup_i R_i)^*$ is also a weak probabilistic bisimulation.

**Proof.** By definition, $R_i$ is symmetric for any $i \in I$. So $(\cup_i R_i)^*$ is also symmetric and hence an equivalence relation. Now suppose $(C, D) \in (\cup_i R_i)^*$. Then there are integer $n$ and a series of configurations $C_0, \ldots, C_n$ such that $C_0 = C$, $C_n = D$, and $(C_i, C_{i+1}) \in R_{k_i}$ for some $k_i \in I$, $i = 0, \ldots, n - 1$. There are two cases we should consider:

(1) $C \not\rightarrow \mu_0$ for some $\alpha \in Act$. Then from $C\overline{R}_{k_0}C_1$ we know that there exists $\mu_1$ such that $C_1 \not\rightarrow \mu_1$ and $\mu_0(M_0) = \mu_1(M_0)$ for any $M_0 \in Con/R_{k_0}$. Furthermore, from $C_1\overline{R}_{k_1}C_2$ and Lemma 6.4, we have $C_2 \not\rightarrow \mu_2$ for some $\mu_2$, and $\mu_1(M_1) = \mu_2(M_1)$ for any $M_1 \in Con/R_{k_1}$. In this way, we can derive that $C_{i+1} \not\rightarrow \mu_{i+1}$ for some $\mu_{i+1}$, and $\mu_i(M_i) = \mu_{i+1}(M_i)$ for any $M_i \in Con/R_{k_i}$, $i = 0, \ldots, n - 1$.
Now suppose \( M \in \text{Con}/(\cup_i R_i)^* \). Notice that for any \( i = 0, \ldots, n - 1 \), \( M \) is the disjoint union of some equivalence classes of \( \text{Con}/R_{k_i} \). It follows that \( \mu_i(M) = \mu_{i+1}(M) \) for any \( i = 0, \ldots, n - 1 \). Thus we have \( \mu_0(M) = \mu_n(M) \).

(2) \( C \leftrightarrow \). Then from \( CR_{k_i}C_1 \) and Lemma 6.5 we have \( C_1 \xrightarrow{\cdot} \mu_1, D_1 \xrightarrow{\cdot} \) for some \( D_1 \in \text{Supp}(\mu_1) \), and \( \text{Context}(C) = \text{Context}(D_1) \). Similarly, we can derive that for any \( i = 2, \ldots, n \), we have \( C_i \xrightarrow{\cdot} \mu_i, D_i \xrightarrow{\cdot} \) and \( \text{Context}(D_{i-1}) = \text{Context}(D_i) \) for some \( D_i \in \text{Supp}(\mu_i) \). Finally, since \( D \xrightarrow{\cdot} \), we find it is the only case that \( D_n = D \) and so \( \text{Context}(D) = \text{Context}(D_{n-1}) = \cdots = \text{Context}(C) \).

From (1) and (2), we know that \( (\cup_i R_i)^* \) is also a weak probabilistic bisimulation. \[ \Box \]

Definition 6.6 Two configurations \( C \) and \( D \) are weakly bisimilar, denoted by \( C \approx_c D \), if there is a weak probabilistic bisimulation relation \( R \) such that \( (C, D) \in R \). For any \( P, Q \in q\text{Proc} \), \( P \) and \( Q \) are weakly bisimilar, denoted by \( P \approx_p Q \), if \( < P; C > \approx_c < Q; C > \) for any context \( C \). Suppose process expressions \( E \) and \( F \) contain classical variables \( \bar{x} \) at most. Then \( E \) and \( F \) are weakly bisimilar, denoted by \( E \approx_c F \), if for all indexed set \( \bar{v} \) of values, \( E[\bar{v}/\bar{x}] \approx_p F[\bar{v}/\bar{x}] \).

From Theorem 6.1, we find that the weak bisimilarity relation \( \approx_c \) is also a weak probabilistic bisimulation; it is in fact the largest weak probabilistic bisimulation on \( \text{Con} \).

Corollary 6.2 \( \approx_c \) is a weak probabilistic bisimulation on \( \text{Con} \).

Proof. By definition, we have
\[
\approx_c = \bigcup \{ R \, : \, R \text{ is a weak probabilistic bisimulation on } \text{Con}\}.
\]

From Theorem 6.1, the reflexive and transitive closure \( \approx_c^* \) is also a weak probabilistic bisimulation and hence \( \approx_c^* \subseteq \approx_c \). On the other hand, we have obviously \( \approx_c \subseteq \approx_c^* \). So \( \approx_c = \approx_c^* \) is also a weak probabilistic bisimulation on \( \text{Con} \). \[ \Box \]

In the following, we usually omit the subscripts of \( \approx_c, \approx_p \), and \( \approx_c \) where no confusion arises.

Definition 6.7 A relation \( \approx' \) on \( \text{Con} \) is defined such that \( C \approx' D \) if for any \( s \in \text{Act}^* \),
\begin{enumerate}
\item whenever \( C \xrightarrow{s} \mu \) then there exists \( \nu \) such that \( D \xrightarrow{s} \nu \) and \( \mu \equiv \approx \nu \),
\item whenever \( D \xrightarrow{s} \nu \) then there exists \( \mu \) such that \( C \xrightarrow{s} \mu \) and \( \mu \equiv \approx \nu \),
\item if \( C \leftrightarrow \) and \( D \leftrightarrow \), then \( \text{Context}(C) = \text{Context}(D) \).
\end{enumerate}

Lemma 6.6 \( \approx' \) is a weak probabilistic bisimulation on \( \text{Con} \).

Proof. First, it is direct that \( \approx' \) is an equivalence relation. Furthermore, from Lemma 6.4 and Theorem 6.1, we have \( \approx \subseteq \approx' \). Thus for any \( M \in \text{Con}/\approx' \), \( M \) is the disjoint union of some equivalence classes of \( \text{Con}/\approx \). From these fact, we can easily prove that \( \approx' \) is also a weak probabilistic bisimulation on \( \text{Con} \). \[ \Box \]

From Lemma 6.6, we have \( \approx' \subseteq \approx \) and then \( \approx = \approx' \). We reexpress this result explicitly as the following theorem.

Theorem 6.2 For any \( C, D \in \text{Con} \), \( C \approx D \) if and only if for any \( s \in \text{Act}^* \),
\begin{enumerate}
\item whenever \( C \xrightarrow{s} \mu \) then there exists \( \nu \) such that \( D \xrightarrow{s} \nu \) and \( \mu \equiv \approx \nu \),
\item whenever \( D \xrightarrow{s} \nu \) then there exists \( \mu \) such that \( C \xrightarrow{s} \mu \) and \( \mu \equiv \approx \nu \),
\item if \( C \leftrightarrow \) and \( D \leftrightarrow \), then \( \text{Context}(C) = \text{Context}(D) \).
\end{enumerate}

Now we turn to prove some congruence properties of weak probabilistic bisimilarity.
Theorem 6.3  (1) If $E \approx F$ then $a.E \approx a.F$, for any $a \in \{c?x, c!e, c?q, c!q, U[q], M[q;x]\}$.

(2) If $E \approx F$ and $G$ is classical, then $E\|G \approx F\|G$.

(3) If $E \approx F$ and $f$ is a relabeling function, then $E[f] \approx F[f]$.

Proof. Similar to the case for strong probabilistic bisimilarity. □

As in classical process algebra, $\approx$ is not preserved by summation $\|$. To deal with it, we introduce the notion of equality.

Definition 6.8 Two configurations $C$ and $D$ are equal, denoted by $C \simeq D$, if for any $\alpha \in \text{Act}$,

(1) whenever $C \overset{\alpha}{\rightarrow} \mu$ then there exists $\nu$ such that $D \overset{\alpha}{\rightarrow} \nu$ and $\mu \equiv \nu$,

(2) whenever $D \overset{\alpha}{\rightarrow} \nu$ then there exists $\mu$ such that $C \overset{\alpha}{\rightarrow} \mu$ and $\mu \equiv \nu$,

(3) if $C \rightarrow$ and $D \rightarrow$, then $\text{Contex}(C) = \text{Contex}(D)$.

The only difference between the definitions of $\approx$ and $\simeq$ is that in the latter $D \overset{\alpha}{\rightarrow} \nu$ is replaced by $D \overset{\hat{\alpha}}{\rightarrow} \nu$, i.e., the matching action for a $\tau$-move has to be a real $\tau$-move.

Further, we extend the definition of equality to the case of quantum process and process expressions as follows. For any $P,Q \in q\text{Proc}$, $P \simeq Q$ if $P;C \triangleright= Q;C >\triangleright Q;C >$ for any context $C$; while for any $E, F \in q\text{Pr}E$ which contain free classical variables $\bar{x}$ at most, $E \simeq F$ if $E[\bar{v}/\bar{x}] \simeq F[\bar{v}/\bar{x}]$ for any indexed set $\bar{v}$ of values.

Lemma 6.7 For any $E,F \in q\text{Pr}E$, $E \simeq F$ if and only if $E + G \simeq F + G$ for all $G \in q\text{Pr}E$.

Proof. Similar to Milner [25] Proposition 7.3. □

The following theorem shows that the quality relation is preserved by all the operators in our language except restriction.

Theorem 6.4 If $E \simeq F$ then

(1) $a.E \simeq a.F$, for any $a \in \{c?x, c!e, c?q, c!q, U[q], M[q;x]\}$,

(2) $E + G \simeq F + G$, for any $G \in q\text{Pr}E$,

(3) $E\|G \simeq F\|G$, for any classical $G$,

(4) $E[f] \simeq F[f]$, for any relabeling function $f$.

Proof. (2) is from Lemma 6.7. Others are similar to the proofs of corresponding results about $\simeq$.

7 Conclusions and further work

In this paper, we propose a framework qCCS for modeling and reasoning about the behavior of quantum concurrent systems. This framework is a natural extension of classical value-passing CCS with the input and output of quantum states, and unitary transformations and measurements on quantum systems. To make qCCS consistent with the laws of quantum mechanics, many syntactical restrictions on valid quantum process expressions are introduced, which make our syntax of qCCS more complicated than the ones proposed in literature. The operational semantics of qCCS is introduced based on probabilistic labeled transition system. This semantics has many different features compared with the proposals in literature in order to describe input and output of quantum systems which are potentially correlated with other components. We also adopt the way to express the probabilities raised by quantum measurements by the resulted distributions instead of by the transition relation itself (by introducing probabilistic transitions, as done in [20] and [11]). Based on this operational semantics, we define the notions of strong/weak probabilistic bisimilarity and equality between quantum processes and discuss some properties such as congruence under various
combinators introduced in this paper. The congruence property we proved for parallel combinator is, however, a weak one in which the common process is classical and never changes the accompanied quantum context. So a direction for further research is to investigate the general case when fully quantum processes are considered.

Recursive definitions are very useful in modeling infinite behavior of processes, and the uniqueness of recursion equations in the sense of strong bisimilarity and equality provides us with a powerful tool for reasoning about correctness of implementations with respect to specifications in various process algebras, including CCS [25], π-calculus [23] and higher-order process calculi [31, 36, 37, 44]. However, there are some technical difficulties to introduce the constructor of recursion in the quantum setting. The main reason is that it is difficult to define the notion of free quantum variables for recursively defined process expressions. For example, if we allow the process expression defined by

$$A \triangleq \text{c}?q.A$$

(6)

to be valid, then some problem will occur when we attempt to assign free quantum variables to $A$: on one hand, from Definition 3.1 6, to make $\text{c}?q.A$ meaningful we must have $q \notin qv(A)$; on the other hand, also from Definition 3.1 6, we know $q \in qv(\text{c}?q.A)$. This is a contradiction because we will naturally require $qv(\text{c}?q.A) \subseteq qv(A)$ in definition equation (6). However, the difficulty will not exist in the following recursively defined quantum process

$$A \triangleq \text{c}?q.U[q],\text{c}?q.A$$

(7)

which consequently inputs a qubit through quantum channel $c$, applies a predefined unitary transformation on it, and outputs it through $c$. Here we can freely let $qv(A) = \emptyset$.

In order to provide some useful mathematical tools for describing approximate correctness and evolution of concurrent systems, one of the authors [39] has tried to develop topology in process algebras. In particular, he and Wirsing [45] introduced the notions of λ-bisimulation and approximate bisimilarity in CCS equipped with a metric on its set of action names, and further applied to probabilistic processes [40]. So an interesting question is to see whether these notions can be also extended to the quantum setting.

Another direction worthy of future investigations is the relation between the expressibility of our qCCS and that of quantum λ-calculus presented in [38]. Recall that Milner has proven that classical CCS is equivalent to λ-calculus in the sense that any one of them can be induced to another. It is interesting to examine whether qCCS presented in this paper and quantum λ-calculus also enjoy a similar property.

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