Cluster reduction of the four-body Yakubovsky equations in configuration space for bound-state problem and low-energy scattering

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Abstract

A method using an expansion of the four-body Yakubovsky wave function components onto the basis of the Faddeev-equation solutions for the two-cluster sub-Hamiltonian eigenfunctions is proposed. This expansion reduces the Yakubovsky differential equations to a system of coupled-channel equations for functions depending on the relative coordinates between the subsystems of the two-cluster partitions. On the basis of the resulting equations the four-nucleon bound-state problem and the zero-energy $n-\text{\textsuperscript{3}}\text{H}$ scattering problem are solved on the relatively small computer.

1 Introduction

The Faddeev-Yakubovsky approach provides the correct way for a treatment the few-body problem \textsuperscript{1,2}. It is based on the decomposition of the wave function into components. This decomposition leads in the momentum representation to integral equations. The solutions of these equations are uniquely defined by inhomogeneous terms in contrast to the Lippmann-Schwinger equations for which this holds only in the two-body case. In the coordinate representation, the decomposition generates differential equations \textsuperscript{3} which are easier to solve numerically because they involve only interparticle potentials and have simple boundary conditions \textsuperscript{4,5}.

For the three-body case these equations are widely used for practical calculations. Unfortunately, already for the four-body systems, the numerical solution is very difficult and requires the most powerful supercomputer facilities \textsuperscript{10,11}. 
Recently, two new methods were proposed to solve the differential Faddeev-Yakubovsky equations (DYE) in configuration space \[6, 7, 9, 12\]. Both of them reduce the computational difficulties by several orders of magnitude, so that the four-body equations can be solved numerically on relatively small computers. One of the methods [9] elaborated for the bound-state problem consists in the systematic exploitation of the tensor structure of the matrices appearing after discretization of DYE with spline expansion and orthogonal collocation methods. This technique looks very appropriate to calculations of the four-body bound states producing more than five significant digits for the four-nucleon binding energy.

The key ingredient of the other method [12] is the partial diagonalization of four-body DYE by making use of the basis of the eigenfunctions of Faddeev equations corresponding to subsystems of two-cluster fragmentations of the four-body system. Such expansions referred to conventionally as the coupled reaction channel (CRC) decomposition [13] reduce DYE to a system of coupled equations for the functions depending only on the relative coordinates between the subsystems of two-cluster partitions. For this reason we name this procedure the cluster reduction.

The purpose of this paper is to give the systematic description of the latter method and to demonstrate its efficiency for numerical solutions of both the four-nucleon bound-state and scattering-state problems. The bound-state problem of the system \(^4\text{He}\) and the zero-energy \(^n-\text{\(^3\)H}\) scattering problem have been chosen for the actual calculations. The MT I-III and S3 potential models have been used as the \(N-N\) interactions [9, 11, 14, 15]. We would like to point out already here that all the calculations were performed on a personal computer with Intel DX-50 processor and 16 Mb of RAM.

The paper is organized as follows. In the second section we recall those portions of the papers [3, 8] which are required to formulate the four-body bound-state and scattering-state problems on the base of DYE. In Sec. 3 the procedure of the partial diagonalization of DYE is described. There we give the appropriate definition of basis functions and discuss their main properties of completeness and biorthogonality. The resulting CRC equations for the functions depending on the distance between the subsystems of the two-cluster partitions are used in the Sec. 4 for numerical calculations of the four-nucleon bound-state \(^4\text{He}\) and zero-energy \(^n-\text{\(^3\)H}\) scattering problems. The last section summarizes the paper. We have collected some technical details in the Appendix.

## 2 DYE formalism

The four-particle wave function \(\Psi\) should be decomposed into components in one to one correspondence to all chains of partitions. The chains consist of two-cluster partitions \(a_2\) (e.g., \((ijk)l\) or \((ij)(kl)\)) and three-cluster partitions \(a_3\) (e.g., \((ij)kl\)) obeying the relation \(a_3 \subset a_2\). The latter means that the partition
\( a_3 \) can be obtained from the partition \( a_2 \) by splitting up one subsystem. It is easy to see that there exist 18 chains of partitions for the four-particle system.

The Yakubovsky wave-function components can be defined by the formulas

\[
\Psi_{a_3a_2} = R_{a_3}(E)V_{a_3} \sum_{(b_3 \neq a_3) \subset a_2} R_0(E)V_{b_3}\Psi,
\]

(1)

where

\[
R_{a_3}(E) = (H_0 + V_{a_3} - E)^{-1}, \quad R_0(E) = (H_0 - E)^{-1}.
\]

Here \( H_0 \) is the kinetic-energy operator in the c.m. frame and \( V_{a_3} \) stands for the two-particle potential acting inside the two particle subsystem of a partition \( a_3 \) (e.g., \( V_{a_3} = V_{ij} \) with \( a_3 = (ij)kl \)).

If the function \( \Psi \) is the solution of the Schrödinger equation

\[
(H_0 + \sum_{a_3} V_{a_3} - E)\Psi = 0
\]

then the components \( \Psi_{a_3a_2} \) obey the Yakubovsky equations

\[
(H_0 + V_{a_3} - E)\Psi_{a_3a_2} + V_{a_3} \sum_{(c_3 \neq a_3) \subset a_2} \Psi_{c_3a_2} = -V_{a_3} \sum_{d_2 \neq a_2} \sum_{(d_3 \neq a_3) \subset a_2} \Psi_{d_3d_2}.
\]

(2)

The remarkable sum rule

\[
\Psi = \sum_{a_2} \sum_{a_3} \Psi_{a_3a_2}
\]

(3)

allows one to construct the Schrödinger equation solution \( \Psi \).

DYE in configuration space become partial differential equations and require asymptotic boundary conditions to define a unique solution. Let us now introduce the relative coordinates for the four-body problem in order to fix the representation of operators in Eqs. and to describe the asymptotics of the components \( \Psi_{a_3a_2} \). There exist 18 sets of relative Jacobi coordinates in one to one correspondence to 18 chains of partitions \( a_3a_2 \). Note, that among them the only two sets are topologically different. One corresponds to chains with partitions \( a_2 \) of the type 3 + 1 (e.g., (123)4) and the other one to chains with partitions \( a_2 \) of the type 2 + 2 (e.g., (12)(34)). From now on we will assume that all the particles have the same mass, say \( m \). The relative coordinates can be defined through the radius vectors of particles by formulas

\[
\begin{align*}
x_{a_3} &= r_i - r_j \\
y_{a_3a_2} &= (r_i + r_j) / 2 - r_k \\
z_{a_2} &= (r_i + r_j + r_k) / 3 - r_l
\end{align*}
\]

(4)
for \( a_2 = (ijk)l, a_3 = (ij)kl \) and

\[
\begin{align*}
\mathbf{x}_{b_3} &= \mathbf{r}_i - \mathbf{r}_j \\
\mathbf{y}_{b_3,b_2} &= \mathbf{r}_k - \mathbf{r}_l \\
\mathbf{z}_{b_2} &= (\mathbf{r}_i + \mathbf{r}_j)/2 - (\mathbf{r}_k + \mathbf{r}_l)/2
\end{align*}
\]

(5)

for \( b_2 = (ij)kl \), \( b_3 = (ij)kl \). With coordinates (1) and (2) the operators \( H_0 \) and \( V_{a_3} \) have the form

\[
H_0 = -\frac{\hbar^2}{m}(\Delta \mathbf{x}_{a_3} + \alpha_{a_2} \Delta \mathbf{y}_{a_3,a_2} + \beta_{a_2} \Delta \mathbf{z}_{a_2}),
\]

\[
V_{a_3} = V_{a_3}(\mathbf{x}_{a_3}),
\]

where \( \Delta \) stands for Laplacian and \( \alpha_{a_2} = 3/4 \), \( \beta_{a_2} = 2/3 \) for the case of partitions \( a_2 \) of type \( 3 + 1 \) and \( \alpha_{a_2} = 1 \), \( \beta_{a_2} = 1/2 \) for partitions \( a_2 \) of type \( 2 + 2 \).

The asymptotic boundary conditions for solutions of DYE (2) depend on the problem under consideration. For the bound-state problem one should look for quadratically integrable solutions of Eqs. (2) satisfying

\[
\int d\mathbf{x}_{a_3}d\mathbf{y}_{a_3,a_2}d\mathbf{z}_{a_2} |\Psi_{a_3,a_2}(\mathbf{x}_{a_3},\mathbf{y}_{a_3,a_2},\mathbf{z}_{a_2})|^2 < \infty,
\]

(6)

which will give a quadratically integrable wave-function \( \Psi \), according to Eq. (3).

In fact, for the short-range potentials the components \( \Psi_{a_3,a_2} \) have exponentially decreasing asymptotic behavior \( \Psi \)

\[
\Psi_{a_3,a_2}(\mathbf{X}) \sim A_{a_3,a_2}(\dot{X})\frac{\exp\left(-\sqrt{\mathcal{E}}|\mathbf{X}|\right)}{|\mathbf{X}|^{5/2}} \quad (|\mathbf{X}| \to \infty).
\]

(7)

In this equation we have introduced the shorthand notation \( \mathbf{X} \) for the 9-vector \( \{\mathbf{x}_{a_3},\mathbf{y}_{a_3,a_2},\mathbf{z}_{a_2}\} \) and \( \dot{X} \) for the unit vector \( \mathbf{X}/|\mathbf{X}| \). For convenience of notation, let us define six-vectors \( \mathbf{x}_{a_2} = \{\mathbf{x}_{a_3},\mathbf{y}_{a_3,a_2}\} \) and \( \mathbf{y}_{a_3} = \{\mathbf{y}_{a_3,a_2},\mathbf{z}_{a_2}\} \). The vectors \( \mathbf{x}_{a_2} \) and \( \mathbf{y}_{a_3} \) have a simple meaning. The first one describes the relative positions of particles belonging to subsystems of the partition \( a_2 \), and the second one corresponds to the configuration of particles \( k,l \) and the center of mass of the pair \( (ij) \) for \( a_3 = (ij)kl \). Note, that for the four-body position vector \( \mathbf{X} \) we have the representations

\[
\mathbf{X} = \{\mathbf{x}_{a_3},\mathbf{y}_{a_3}\} = \{\mathbf{x}_{a_2},\mathbf{z}_{a_2}\}.
\]

In the case of the scattering problem we will deal with the functions marked by sign \( + \). For these functions the asymptotic form of the components \( \Psi_{a_3,a_2}^{(+)} \) depends on the initial state of the system. We restrict ourselves to the scattering with two clusters in the initial state. Two-cluster channels can be characterized by the bound states of the two-cluster Hamiltonians

\[
H_{a_2} = T_{a_2} + \sum_{a_3 \subset a_2} V_{a_3} \equiv -\frac{\hbar^2}{m}(\Delta \mathbf{x}_{a_3} + \alpha_{a_2} \Delta \mathbf{y}_{a_3,a_2}) + \sum_{a_3 \subset a_2} V_{a_3}(\mathbf{x}_{a_3})
\]

(8)
The first term in (9) is due to the initial state with two clusters being subsystems of the Hamiltonian $H_{a_2}$ obeying the Faddeev equations

$$(T_{a_2} + V_{a_3} - \varepsilon_{a_2})\phi_{a_2}^{a_3} = -V_{a_3} \sum_{(c_3 \neq a_3) \subset a_2} \phi_{a_2}^{c_3},$$

where $\varepsilon_{a_2}$ is the respective binding energy.

In order to characterize the three-cluster breakup channels let us introduce the two-body bound-state wave function $\psi_{a_3}(x_{a_3})$ being the solution of the Schrödinger equation

$$(H_{a_3} - \varepsilon_{a_3})\psi_{a_3} = \left\{-\frac{\hbar^2}{2m} \Delta x_{a_3} + V_{a_3}(x_{a_3}) - \varepsilon_{a_3}\right\}\psi_{a_3}(x_{a_3}) = 0.$$

For the sake of simplicity of notation in what follows we will assume that the Hamiltonians $H_{a_2}$ and $H_{a_3}$ possess at most one bound state.

The four-body wave function components $\Psi_{a_3a_2}(X, p_{l_2})$ for scattering with two clusters in the initial state can be arranged as the sum

$$\Psi_{a_3a_2}(X, p_{l_2}) = \delta_{a_2l_2}\phi_{a_2}^{a_3}(x_{a_2}) \exp\{ip_{a_2} \cdot z_{a_2}\} + \phi_{a_3}^{a_3}(x_{a_2})U_{a_2l_2}(z_{a_2}, p_{l_2}) + \psi_{a_3}(x_{a_3})U_{a_3l_2}^{a_2}(y_{a_3}, p_{l_2}) + U_{a_3a_2l_2}(X, p_{l_2}).$$

The first term in (9) is due to the initial state with two clusters being subsystems of the partition $l_2$ and moving with the relative momentum $p_{l_2}$. The total energy $E$ and the momentum $p_{l_2}$ are related by the expression

$$E = \varepsilon_{l_2} + \beta_{l_2} \frac{\hbar^2}{m} (p_{l_2})^2.$$

The second term in (9) describes the elastic ($a_2 = l_2$) and the rearrangement ($a_2 \neq l_2$) processes. For large separations, the functions $U_{a_2l_2}$ become spherical waves:

$$U_{a_2l_2}(z_{a_2}, p_{l_2}) \sim A_{a_2l_2} \exp\{i\sqrt{E - \varepsilon_{a_2}(z_{a_2})}\} \frac{1}{|z_{a_2}|} \quad (|z_{a_2}| \to \infty),$$

with $A_{a_2l_2}$ being the amplitudes of the processes ($2 \to 2$). The remaining terms of (9) correspond to breakup processes ($2 \to 3$) and ($2 \to 4$). The asymptotics of the functions $U$ have the form

$$U_{a_3l_2}^{a_2}(y_{a_3}, p_{l_2}) \sim A_{a_3l_2}^{a_2} \exp\{i\sqrt{E - \varepsilon_{a_3}(y_{a_3})}\} \frac{1}{|y_{a_3}|^{3/2}} \quad (|y_{a_3}| \to \infty),$$

$$U_{a_3a_2l_2}(X, p_{l_2}) \sim A_{a_3a_2l_2} \exp\{i\sqrt{E(|X|)}\} \frac{1}{|X|^4} \quad (|X| \to \infty).$$

Here, the amplitudes $A$ are related to the breakup amplitudes by the formulas

$$f_{a_3l_2}(2 \to 3) = \sum_{a_2 \supset a_3} A_{a_3l_2}^{a_2}.$$
The asymptotic boundary conditions (9-12) make the solution of DYE (2) unique. For the more complicated case of scattering states with three and four clusters in the initial state we refer to the paper [17].

3 Cluster reduction of the DYE

In this section we consider the four-body bound state problem and the low-energy scattering problem when only the two-cluster channels are open. The latter means that the energy of the system obeys the inequalities

\[ E - \varepsilon_{a_2} \geq 0, \ E - \varepsilon_{a_3} < 0, \ E < 0. \]

Under these conditions the two last terms in (9) describing virtual breakup processes vanish exponentially as \( |X| \) approaches \( \infty \) and the asymptotics of \( \Psi_{a_1a_2}^{(+)} \) becomes simpler. Note, that every component \( \Psi_{a_3a_2} \) for both bound and scattering state now is an exponentially decreasing function of the coordinates \( x_{a_2} \) corresponding to the separation of the particles inside the subsystems of the partition \( a_2 \). It means, that the essential part of the component \( \Psi_{a_3a_2} \) is concentrated in a tube-like domain of the configuration space such that \( |x_{a_2}| \leq R_{a_2} \) with sufficiently large parameter \( R_{a_2} \). It is natural to find the approximate solution of Eqs. (2) by imposing the following boundary conditions:

\[ \Psi_{a_3a_2}|_{\Gamma_{a_2}} = 0, \]

where \( \Gamma_{a_2} \) means the boundary of the cylinder: \( |x_{a_2}| = R_{a_2} \). Imposing these boundary conditions is the first step of the reduction. As a second step, we expand the components in the following series:

\[ \Psi_{a_3a_2}(x_{a_2}, z_{a_2}) = \sum_{l=0}^{\infty} \phi_{a_2,l}^{a_3}(x_{a_2}) F_{a_2}^{l}(z_{a_2}). \]  \( (13) \)

Here \( F_{a_2}^{l}(z_{a_2}) \) are new, unknown, functions and \( \phi_{a_2,l}^{a_3}(x_{a_2}) \) are the eigenfunctions of the Faddeev equations for the subsystems of the partition \( a_2 \)

\[ (T_{a_2} + V_{a_3})\phi_{a_2,l}^{a_3} + V_{a_3} \sum_{c_3 \neq a_3} \phi_{a_2,l}^{c_3} = \varepsilon_{a_2,l}^{a_3}\phi_{a_2,l}^{a_3}, \]  \( (14) \)

obeying the boundary conditions

\[ \phi_{a_2,l}^{a_3}(x_{a_2})|_{\Gamma_{a_2}} = 0. \]  \( (15) \)

The equations (14) with the spatially restricted conditions (15) are known to have a purely discrete spectrum of real eigenvalues \( \varepsilon_{a_2,l}^{a_3} \). The functions \( \phi_{a_2,l}^{a_3} \) and the eigenvalues \( \varepsilon_{a_2,l}^{a_3} \) are assumed to be enumerated in order of growth the
quantities $\varepsilon_{a_2}^l$. One observes, that for large enough value of $R_{a_2}$, the functions $\phi_{a_2,0}^{a_1}$ and the quantity $\varepsilon_{a_2}^0$ approach the ground-state Faddeev components and the bound-state energy of the Hamiltonian (8), respectively. The remaining part of basis functions extends the function $\phi_{a_2,0}^{a_3}$ up to a complete set [18]. This set of eigenfunctions is not orthogonal due to the nonsymmetry of (14) with respect to the superscript $c_3$. The biorthogonal basis is formed [18] by the solutions of the adjoint to Eqs. (14):

$$ (T_{a_2} + V_{a_3}) \psi_{a_2,k} + \sum_{(c_3 \neq a_3) \subset a_2} V_{c_3} \psi_{a_2,k}^{c_3} = \varepsilon_{a_2}^k \psi_{a_2,k}^{a_3}. $$

(16)

The completeness and biorthogonality properties for the functions $\phi_{a_2,l}^{a_3}$ and $\psi_{a_2,k}^{c_3}$ read

$$ \sum_{k=0}^{\infty} \phi_{a_2,k}^{a_3} (x_{a_2}) \psi_{a_2,k}^{c_3} (x_{a_2}') = \delta_{a_3 c_3} \delta (x_{a_2} - x_{a_2}'), $$

$$ \sum_{a_3 \subset a_2} \int dx_{a_2} \phi_{a_2,l}^{a_3} (x_{a_2}) \psi_{a_2,k}^{c_3} (x_{a_2}) = \delta_{lk}. $$

Using these conditions, we express the coefficients in (13) as the projections

$$ F_{a_2}^l (z_{a_2}) = \sum_{a_3 \subset a_2} \int dx_{a_2} \psi_{a_2,l}^{a_3} (x_{a_2}) \psi_{a_2} (x_{a_2}, z_{a_2}). $$

(17)

Substituting the expansions (13) into the DYE (2) and projecting onto the biorthogonal basis functions $\psi_{a_2,l}^{a_3}$ we get the final equations for the coefficients $F_{a_2}^l (z_{a_2})$:

$$ \left(-\frac{\hbar^2}{m}\beta_{a_2} \Delta_{a_2} - E + \varepsilon_{a_2}^k \right) F_{a_2}^l (z_{a_2}) = $$

$$ - \sum_{a_3 \subset a_2} \langle \psi_{a_2,k}^{a_3} | V_{a_3} \sum_{d_2 \neq a_2 (d_2 \neq a_3)} \sum_{l \geq 0} \phi_{d_2,l}^{d_3} (x_{d_2}) F_{d_2}^l (z_{d_2}) \rangle. $$

(18)

Here the brackets $\langle \cdot | \cdot \rangle$ stand for the integration over $x_{a_2}$ in the domain $|x_{a_2}| \leq R_{a_2}$ and the coordinates $x_{d_2}$ and $z_{d_2}$ are assumed to be expressed through the set $x_{a_3}, y_{a_3 a_2}, z_{a_2}$. The asymptotic boundary conditions for the functions $F_{a_2}^k (z_{a_2})$ can be easily obtained from (3) and (9) [12] by projecting according to formula (17). For the bound state-problem they read

$$ \sum_{k \geq 0} \int dz_{a_2} |F_{a_2}^k (z_{a_2})|^2 < \infty, $$

(19)

and for the scattering states they have the following ‘two-body’ form

$$ F_{a_2}^0 (z_{a_2}) \sim \delta_{a_2 l_2} \exp \{ i p_{a_2} \cdot z_{a_2} \} + A_{a_2 l_2} |z_{a_2}|^{-1} \exp \{ i \sqrt{E - \varepsilon_{a_2}^0} |z_{a_2}| \} $$

(20)
for the open channels and

\[ F_{a_2}^k(z_{a_2}) \sim 0, \quad k \geq 1 \quad (21) \]

for the closed channels. The equations (18) are the desired coupled channel equations for four distinguishable particles.

Let us now turn to the case of identical particles. On the first stage, note that due to the identity of particles, it suffices to define only two Yakubovsky components \( \Psi_{a_3a_2} \), namely, one for the partition \( a_2 \) of the type 3 + 1 and one for the partition \( a_2 \) of the type 2 + 2. We fix these two components choosing 

\[ a_3 = (12)34, \quad a_2 = (123)4 \quad \text{and} \quad b_3 = (12)34, \quad b_2 = (12)(34) \]

and denote \( \Psi_{a_3a_2} = \Psi_1, \quad \Psi_{b_3b_2} = \Psi_2 \).

Let \( P^\pm \) be the cyclic and anticyclic permutation operators for four particles and \( P_i^\pm \) be the same for a three-particle subsystem where the subscript \( i \) corresponds to the fourth particle, that is not participating in the permutation. With these operators the representation (3) can be rewritten in terms of the components \( \Psi_1 \) and \( \Psi_2 \) as follows

\[
\Psi = [I + \sigma P^+ + P^+ P^+ + \sigma P^-][I + P_4^+ + P_4^-] \Psi_1 + [I + P_1^+ + P_1^-][I + P^+ P^+] \Psi_2.
\]

The DYE (2) for the functions \( \Psi_i, \ i = 1, 2 \) read

\[
(H_0 + V - E)\Psi_1 + V(P_4^+ + P_4^-)\Psi_1 = -V[(P_1^+ + \sigma P^+)\Psi_1 + (P_1^+ + P_4^+)\Psi_2],
\]

\[
(H_0 + V - E)\Psi_2 + V(P^+ P^+)\Psi_2 = -V[P^+ + \sigma P_1^+]P^+\Psi_1.
\]

Here \( V \equiv V(x_{12}) \) and \( \sigma = \pm 1(-1) \) for the boson (fermion) case.

The expansion (13) assumes the form

\[
\Psi_i(x_i, y_i) = \sum_{l=0}^{\infty} \phi_i^l(x_i) F_i^l(z_i), \quad i = 1, 2, \quad (22)
\]

with the basis functions \( \phi_i^l \) being the solutions of the symmetrized Faddeev equations

\[
(T_1 + V)\phi_i^1 + V(P_4^+ + P_4^-)\phi_i^1 = \varepsilon_i^l \phi_i^1;
\]

\[
(T_2 + V)\phi_i^2 + VP^+ P^+ \phi_i^2 = \varepsilon_i^l \phi_i^2.
\]

The biorthogonal basis functions \( \psi_i^k \) are the solutions of the respective adjoint equations

\[
(T_1 + V)\psi_i^1 + (P_4^+ + P_4^-)V \psi_i^1 = \varepsilon_i^l \psi_i^1,
\]

\[
(T_2 + V)\psi_i^2 + P^+ P^+ V \psi_i^2 = \varepsilon_i^l \psi_i^2.
\]

The expansion (22) leads to the symmetrized form of (18).
\[
\left\{ -\frac{\hbar^2}{m} \Delta_z - E + \varepsilon_k \right\} F_k^1(z_1) = -\langle \psi^1_k | V[P^+_1 + \sigma P^+] \sum_{l \geq 0} \phi^1_l F_l^1 \rangle \\
- \langle \psi^1_k | V[P^+_1 + P^+_4] \sum_{m \geq 0} \phi^2_m F^m_2 \rangle,
\]
\[
\left\{ -\frac{\hbar^2}{m} \Delta_z - E + \varepsilon_k \right\} F_k^2(z_2) = -\langle \psi^2_k | V[P^+_1 + \sigma P^+] P^+ \sum_{l \geq 0} \phi^1_l F_l^1 \rangle.
\]

The boundary conditions for the equations (23) follow directly from (19), (20), and (21).

4 Application to the four-nucleon system

To apply the equations (23) to the four-nucleon system one needs to perform a partial-wave analysis. Taking into account the conservation of the total angular momentum \(L\) and the total spin \(S\) of the system for chosen \(N - N\) forces S3 and MT I-III we can use the \(L - S\) coupling scheme and obtain for the \(s\)–wave components

\[ F_{S,l}^k(z) = |z| \int d\hat{z} F^l_k(\hat{z}), \quad k = 1, 2 \]

(24)

(here and in what follows the index \(S\) is introduced to distinguish the states with the total spin \(S = 0\) or \(S = 1\)) the following resulting \(s\)–wave equations

\[
\left( -\frac{2}{3} \frac{\partial^2}{\partial z^2} - \varepsilon + \varepsilon_{S,m}^1 \right) F_{S,m}^1(z) = \\
- \frac{1}{2} \langle \Psi_{S,m}^1(x, y) | V^S_1(x) \int_{-1}^{1} du \int_{-1}^{1} dv \frac{x y z}{x_3 y_3 z_3} DS \sum_{l=0}^{N_1} \phi_{S,l}^1(x_3, y_3) F_{S,l}^1(z_3) \rangle \\
- \frac{1}{2} \langle \Psi_{S,m}^1(x, y) | V^S_1(x) \int_{-1}^{1} du \int_{-1}^{1} dv \frac{x y z}{x_4 y_4 z_4} DS \sum_{l=0}^{N_1} \phi_{S,l}^1(x_4, y_4) F_{S,l}^1(z_4) \rangle,
\]

(25)

\[
\left( -\frac{2}{3} \frac{\partial^2}{\partial z^2} - \varepsilon + \varepsilon_{S,n}^2 \right) F_{S,n}^2(z) = \\
- \langle \Psi_{S,n}^2(x, y) | V^S_2(x) \int_{-1}^{1} du \int_{-1}^{1} dv \frac{x y z}{x_5 y_5 z_5} DS \sum_{l=0}^{N_1} \phi_{S,l}^1(x_5, y_5) F_{S,l}^1(z_5) \rangle.
\]

Here the brackets \(\langle \cdot | \cdot \rangle\) mean the integration over \(x\) and \(y\) in the domain \(x^2 + y^2 \leq R^2\) and summation over the components of vector-functions \(\Phi^k_{S,l}\) and \(\Psi^k_{S,m}\) being the solutions of \(s\)–wave Faddeev equations.

1 We have restricted the summation in formula (22) to finite numbers \(N_1\) and \(N_2\), so that (25) should be considered as approximate equations, which become exact in the limit \(N_1, N_2 \to \infty\).
\[ \left\{ -\partial_x^2 - \frac{3}{4} \partial_y^2 + V_1^S(x) \right\} \Phi_{S,l}^1(x, y) + V_1^S(x) \int_{-1}^{1} \frac{xy}{x_1 y_1} B_1^S \Phi_{S,l}^1(x_1, y_1) = \varepsilon_{S,l}^1 \Phi_{S,l}^1(x, y), \]

\[ \left\{ -\partial_x^2 - \partial_y^2 + V_2^S(x) \right\} \Phi_{S,l}^2(x, y) + V_2^S(x) B_2^S \Phi_{S,l}^2(x_2, y_2) = \varepsilon_{S,l}^2 \Phi_{S,l}^2(x, y), \]  

(26)

and adjoint equations

\[ \left\{ -\partial_x^2 - \frac{3}{4} \partial_y^2 + V_1^S(x) \right\} \Psi_{S,l}^1(x, y) + \int_{-1}^{1} \frac{xy}{x_1 y_1} B_1^S \Psi_{S,l}^1(x_1) = \varepsilon_{S,l}^1 \Psi_{S,l}^1(x, y), \]

\[ \left\{ -\partial_x^2 - \partial_y^2 + V_2^S(x) \right\} \Psi_{S,l}^2(x, y) + B_2^S V_2^S(x_2) \Psi_{S,l}^2(x_2, y_2) = \varepsilon_{S,l}^2 \Psi_{S,l}^2(x, y). \]  

(27)

In the above equations we have renormalized the energy of the system according to \( \varepsilon = \frac{mE}{\hbar^2} \). The matrices \( B_k^S, D_k^S \), and \( C_k^S \), \( k = 1, 2 \) realize the representation of operators \( P_4^+ + P_4^-, P_3^+ - P_3^-, P_1^+ + P_1^-, \) and \( (P_3^+ - P_3^-)P_3^+ \) in the spin-isospin space of the four-nucleon system, respectively. We give their values for the systems \( nnpp \) and \( nnnp \) in the Appendix. There we have described the representations of interaction potentials \( V_k^S \) and values of coordinates \( x_i, y_i, z_i \) appearing in the equations.

The components \( F_{S,l}^k(z) \), due to (24), obey the regularity conditions

\[ F_{S,l}^k(0) = 0. \]

The asymptotic boundary conditions for the bound-state problem according to (19) and (24) have the form

\[ \sum_{l \geq 0} \int_0^\infty dz |F_{S,l}^k(z)|^2 < \infty, \quad k = 1, 2. \]  

(28)

For the \( n^-^3^H \) scattering problem the only three-body subsystems have the bound state, so that (21) and (24) lead to the conditions

\[ F_{S,0}^1(z, p) \sim p^{-1} \sin pz + a_S(p) \cos pz, \]

\[ F_{S,l}^1(z, p) \sim 0, \quad l \geq 1, \]

\[ F_{S,l}^2(z, p) \sim 0, \quad l \geq 0. \]  

(29)

For the zero relative \( n^-^3^H \) scattering energy the first condition of (27) should be replaced by

\[ F_{S,0}^1(z, 0) \sim z - A_S, \]  

(30)
where $A_S$ is the $n$-$^3$H scattering length. The relationship between the scattering amplitude $a_S(p)$ and the phase shift $\delta_S(p)$ is of the form

$$a_S(p) = \frac{\tan \delta_S(p)}{p}.$$  

(31)

Alternatively, the scattering length $A_S$ can be obtained from $a_S(p)$ by taking the limit,

$$A_S = -\lim_{p \to 0} a_S(p).$$

(32)

The equations (25) were solved numerically by making use of a finite-difference approximation for the differential operator $\partial_x^2$ and a spline expansion for the integrands on the right hand-side. The mesh parameters were varied extensively to obtain the relative numerical uncertainty less than 1%. The basis functions $\Phi_{S,l}^k$ and $\Psi_{S,l}^k$ were constructed numerically as the solutions of the eigenvalue problems (26) and (27). The maximal nonorthogonality of the basis functions caused by the numerical solutions of (26) and (27) was given by

$$\langle \Psi_{S,l}^k | \Phi_{S,m}^k \rangle \leq 10^{-6}, \ l \neq m.$$

We have used S3 and MT I-III $s$-wave potentials for the bound-state calculations. For $n$-$^3$H scattering we have used only the MT I-III potential. These potentials are known to have a strong infinite (MT I-III) and finite (S3) repulsive core. Such a singular behavior of the potentials requires an appropriate treatment of the problem, so we have chosen these potentials to demonstrate the efficiency of our approach. On the other hand, the data of the direct (except for numerical approximations) solutions of the four-nucleon Yakubovsky equations (YE) for bound states with these potentials are available [9, 11] which facilitates the comparison of the results of calculations.

In Table 1 we show the convergence of our calculations for $^4$He binding energy with respect to the number of terms taken into account in the expansion (22) and in Eqs. (25), respectively. As one can see a converged result within the uncertainty less than 1% can be reached with about 10 basis functions. In Table 2 we have collected our final values for $^4$He binding energy and the data of solutions of YE [3, 14, 19]. The agreement of calculations is near perfect.

The calculations of $n$-$^3$H scattering were performed for the zero neutron laboratory energy. We have used the boundary conditions (31) for zero energy calculations. The scattering length $A_S$ was extracted from the asymptotics of the solutions according to the formula (30). We have collected the results of our calculations for spin-singlet and spin-triplet scattering lengths $A_0$ and $A_1$ together with available data of the other authors and experimental values in Table 3. Comparison of the results is hampered by the fact that different approaches and different models for $N - N$ forces were used. Nevertheless, the agreement of the results is reasonable except for the case of the paper [23] where the Resonating Group Method (RGM) has been used. In our notations, the RGM ansatz corresponds to taking into account only one basis state in [22].
We have performed the calculations within this condition and obtained 3.41 fm for the spin-singlet scattering length $A_0$ which is very close to the result of [23]. Hence, we can suppose that the difference between our converged result and the RGM one is due to the approximations of RGM.

5 Summary

We have described the cluster reduction method to treat the four-body bound-state and scattering-state problem. The method leads to CRC equations for the functions depending on the relative coordinates between the subsystems of the two-cluster partitions. After a suitable partial-wave decomposition, these equations become the one-dimensional integro-differential equations which can be solved numerically on rather small computers in contrast to the original DYE, for which numerical solution even for simple $N-N$ forces requires supercomputer facilities. The results of calculations given in the previous section show the efficiency of reduced DYE (23) and (25) for the numerical solution of both the bound-state and scattering-state problems for the four-nucleon system. The method proposed can be extended both to the case of more realistic nucleon-nucleon potentials in four-nucleon system, and to systems consisting of more than four particles. In the latter case the N-body differential Yakubovsky equations (3) should be taken as a starting point for the reduction procedure.

The efficiency of the cluster-reduction method for the scattering problem demonstrated for the $n-^3\text{H}$ system suggests it will be useful for the direct four-body calculations of various multichannel reactions.

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Appendix

In this Appendix we describe the representation of the potentials $V^S_k$ and the matrices $B^S_k$, $D^S$, and $C^S_k$, $k = 1, 2$ and give the formulas for the coordinates $x_i, y_i, z_i$ appearing in the integrals of Eqs. (25), (26), and (27). For the bound state of the $nnpp$ system the total spin equals to zero, $S = 0$. The operators $V^0_k$ are the diagonal matrices such that

\[ V^0_1 = V^0_2 = \text{diag}\{v^s, v^t\}, \]

where $v^s$ and $v^t$ stand for the singlet and triplet $N - N$ potentials, respectively. The matrices $B^0_1$, $D^0$ and $C^0_1$ have the form

\[ B^0_1 = D^0 = C^0_1 = \begin{pmatrix} \frac{1}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{pmatrix}, \]

\[ B^0_2 = C^0_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

For the system $n$-$^3$H the values of the total spin $S$ equal 0 or 1. The operators $V^S_k$ are the diagonal matrices

\[ V^0_1 = \text{diag}\{v^s, v^t, v^s\}, \]

\[ V^0_2 = v^s, \]

\[ V^1_1 = \text{diag}\{v^s, v^t, v^s, v^t\}, \]

\[ V^1_2 = \text{diag}\{v^s, v^t\}. \]

The explicit form of the matrices $B^S_k$, $D^S$, and $C^S_k$ is as follows

\[ B^0_1 = \begin{pmatrix} \frac{1}{14} & -\frac{3}{4} & 0 \\ -\frac{3}{4} & \frac{1}{14} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \]

\[ B^0_2 = -1 \]

\[ B^1_1 = \begin{pmatrix} \frac{1}{7} & -\frac{3}{7} & 0 & 0 \\ -\frac{3}{7} & \frac{1}{7} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}, \]

\[ B^1_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

\[ D^0 = \begin{pmatrix} \frac{1}{12} & \frac{3}{4} & \frac{1}{2} \sqrt{2} \\ -\frac{3}{4} & \frac{1}{12} & -\frac{1}{2} \sqrt{2} \\ \frac{1}{2} \sqrt{2} & -\frac{1}{2} \sqrt{2} & 0 \end{pmatrix}. \]
\[
D^1 = \begin{pmatrix}
-\frac{1}{12} & -\frac{1}{12} & -\frac{1}{6}\sqrt{2} & -\frac{1}{6}\sqrt{2} \\
\frac{1}{3}\sqrt{2} & \frac{1}{3}\sqrt{2} & 0 & -\frac{1}{6} \\
\frac{1}{3}\sqrt{2} & 0 & -\frac{1}{6} & 0 \\
0 & \frac{1}{3}\sqrt{2} & 0 & -\frac{1}{6}
\end{pmatrix},
\]

\[
C_0^1 = \begin{pmatrix}
\frac{1}{6}\sqrt{2} \\
\frac{1}{6}\sqrt{2}
\end{pmatrix},
\]

\[
C_1^0 = (-\frac{1}{6}\sqrt{2} \ 0 \ -1/\sqrt{3}),
\]

\[
C_1^1 = \begin{pmatrix}
-\frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \\
\frac{1}{3}\sqrt{3} & -\frac{1}{3}\sqrt{3}
\end{pmatrix},
\]

\[
C_2^1 = \begin{pmatrix}
0 & -\frac{1}{3}\sqrt{3} & 0 & \frac{1}{3}\sqrt{6} \\
-\frac{1}{3}\sqrt{3} & 0 & \frac{1}{3}\sqrt{6} & 0
\end{pmatrix}.
\]

The expressions for the coordinates \(x_i, y_i, z_i\) can be given by formulas

\[
x_1 = \left(\frac{1}{4}x^2 + y^2 - xyv\right)^{1/2}, \quad y_1 = \left(\frac{9}{16}x^2 + \frac{1}{4}y^2 + \frac{3}{4}xyv\right)^{1/2},
\]

\[
x_2 = y, \quad y_2 = x, \quad x_3 = x_1, \quad x_4 = x_1, \quad x_5 = y,
\]

\[
y_3 = \left(\frac{1}{9}y_1^2 + z^2 + \frac{2}{3}y_1zu\right)^{1/2}, \quad z_3 = \frac{1}{3}\left(\frac{64}{9}y_1^2 + z^2 - \frac{16}{3}y_1zu\right)^{1/2},
\]

\[
y_4 = \left(\frac{1}{9}y_1^2 + z^2 + \frac{4}{3}y_1zu\right)^{1/2}, \quad z_4 = \left(\frac{4}{9}y_1^2 + \frac{1}{4}z^2 - \frac{2}{3}y_1zu\right)^{1/2},
\]

\[
y_5 = \left(\frac{1}{4}x^2 + z^2 - xzv\right)^{1/2}, \quad z_5 = \frac{2}{3}\left(x^2 + z^2 + 2xzv\right)^{1/2}.
\]
References

[1] Faddeev L.D., Zh. Eksperim. Teor. Fiz. 39 (1960) 1459 [Soviet Phys. - JETP 12 (1961) 1014]
[2] Yakubovsky O.A., Yad. Fiz. 5 (1967) 1312 [Sov. J. Nucl. Phys. 5 (1967) 937]
[3] Merkuriev S.P., Yakovlev S.L., Dans. Akad. Nauk (SSSR) 262 (1982) 591, Teor. Mat. Fiz. 56 (1983) 60 [Theor. Math. Phys. 56 (1983) 673], Yad. Fiz. 39 (1984) 1580 [Sov. J. Nucl. Phys. 39 (1984) 1002]
[4] Merkuriev S.P., Gignoux C., Laverne A., Ann. Phys. [N.Y.] 99 (1976) 30
[5] Friar J.L., Gibson B.F., Payne G.L., Phys. Rev. C39 (1989) 1264
[6] Yakovlev S.L., Filikhin I.N., Yad. Fiz. 56 (1993) 98
[7] Schellingerhout N.W., Kok L.P., Bosveld G.D., Phys. Rev. A40 (1989) 5568
[8] Merkuriev S.P., Yakovlev S.L., Gignoux C., Nucl. Phys. A31 (1984) 125
[9] Schellingerhout N.W., Schut J.J., Kok L.P., Phys. Rev. C46 (1992) 1192
[10] Cerba J., Gignoux C., Merkuriev S.P., Yakovlev S.L., Preprint ISN90.30, Grenoble, 1990
[11] Kamada H., Glöckle W., Nucl. Phys. A548 (1992) 205
[12] Yakovlev S.L., Filikhin I.N., Yad. Fiz. 58 (1995) 817
[13] Bencze Gy., Chandler C., Gibson A.G., Nucl. Phys. A340 (1982) 461; Birse M.C., Redish E.F., Nucl. Phys. A406 (1983) 149; Adhikari S.K., Birse M.C., Kozak R., Levin F.S., Phys. Rev. C30 (1984) 780
[14] Malifiet R.A., Tjon J.A., Ann. Phys. 61 (1970) 425
[15] Afnan I.R., Tang Y.C., Phys.Rev. 175 (1968) 1337
[16] Merkuriev S.P., Faddeev L.D., The scattering theory for the system of several particles (Nauka, Moscow, 1985) (in Russian)
[17] Yakovlev S.L., Teor. Mat. Fiz. 82 (1990) 224 [Theor. Math. Phys. 82 (1990) 157]
[18] Evans J.W., Hoffman D.K., J. Math. Phys. 22 (1981) 2858; Yakovlev S.L., Teor. Mat. Fiz. 102 (1995) 323
[19] Tjon J.A., Phys. Lett. B56 (1975) 217; B63 (1976) 391
[20] Levashev V.P., Yad. Fiz. 38 (1983) 566 [Sov. J. Nucl. Phys. 38 (1983) 336]
[21] Fonseca A.C., Few Body Systems 1 (1986) 69
[22] Belyaev V.B., Pupyshev V.V., Yad. Fiz. 35 (1982) 905 [Sov. J. Nucl. Phys. 35 (1982) 526]
[23] Heiss P., Hackenbroich H.H., Nucl. Phys. A202 (1971) 353
[24] Seagrave J.D., Berman B.L., Phillips T.W., Phys. Lett. B\textbf{91} (1980) 200
[25] Fonseca A.C., Phys. Rev. C\textbf{30} (1984) 35, and private communication
[26] Ciesielski F., Carbonell J., Gignoux C., Preprint ISN96.18, Grenoble, 1996
| $N$ | S3   | MT I-III$^1$ | MT I-III$^2$ |
|-----|------|-------------|-------------|
| 1   | 23.86| 24.6945     | 25.154      |
| 2   | 27.11| 28.3060     | 28.869      |
| 3   | 27.23| 28.4734     | 28.975      |
| 4   | 27.75| 28.7086     | 29.223      |
| 5   | 28.38| 29.3345     | 29.688      |
| 6   | 28.42| 29.4204     | 29.783      |
| 7   | 28.39| 29.9648     | 30.284      |
| 8   | 28.55| 29.5980     | 30.277      |
| 9   | 28.67| 30.0534     | 30.304      |
| 10  | 28.72| 30.0829     | 30.315      |
| 11  | 28.71| 30.0901     | 30.307      |
| 12  | 28.74| 30.1369     |             |
| 13  | 28.73| 30.1392     |             |

**Table 1.** The convergence of the $^4$He ground-state energy $E$(MeV) with respect to the number $N = N_1 = N_2$ of terms in (22) taken into account in the numerical solution of Eqs. (25). Here MT I-III$^1$ is the potential from [14] and MT I-III$^2$ from [9, 11].
Table 2. The results of the calculations of the $^4$He ground-state energy $E$(MeV) for spin-dependent S3 and MT I-III $N-N$ forces. Here MT I-III$^1$ is the potential from [14] and MT I-III$^2$ from [9, 11].

| Refs.   | S3      | MT I-III$^1$ | MT I-III$^2$ |
|---------|---------|--------------|--------------|
| [9]     | 28.7843 | 30.3117      |              |
| [11]    | 28.80   |              | 30.29        |
| [19]    |         | 29.6(2)      |              |
| present work | 28.7(4) | 30.1(3)      | 30.3(1)      |
| Refs.  | $A_0$ fm | $A_1$ fm |
|--------|---------|---------|
| Present work | 4.0     | 3.6     |
| [19]   | 4.09    | 3.61    |
| [20]   | 4.23    | 3.46    |
| [21]   | 3.905   | 3.597   |
| [22]   | 3.8     | 4.9     |
| [23]   | 3.38    | 3.25    |
| [26]   | 4.13    | 3.73    |
| exp. [24] | 3.91 ± 0.12 | 3.6 ± 0.1 |

**Table 3.** The calculated spin-singlet $A_0$ and spin-triplet $A_1$ scattering lengths using the MT I-III\textsuperscript{1} potential of [14], the results of other authors and experimental data for $n$-$^3$H scattering.