Relativistic constituent model in sector of light mesons

A.F. Krutov1,•, R.G. Polezhaev1,••, and V.E. Troitsky2,•••

1 Samara University, 443086 Samara, Russia
2 D.V. Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow 119991, Russia

Abstract. We present a brief survey of some results on electroweak properties of composite systems that are obtained in the frameworks of our version of the instant form of relativistic quantum mechanics (RQM). Our approach describes well the π- and the ρ- mesons in wide range of momentum transfers $Q^2$. At large $Q^2$ the obtained pion form factor asymptotics coincides with that of QCD predictions. The method permits to perform analytic continuation of pion form factor to complex plane of momentum transfers that is in accordance with predictions of quantum field theory.

1 Introduction

The purpose of this paper is to present a version of relativistic composite quark model developed by the authors (see, e.g., [1]) for the study of composite systems of light quarks. The investigation of electroweak properties of light mesons is an important part of the study of the transition region where the perturbative QCD behavior starts to be valid. In this connection such particles are in the focus of experiments on up-to-date accelerators. The most important results here are the following: measurement of the pion form factor at large momentum transfers in JLab (see, e.g., [2, 3]), the obtaining of the ρ-meson decay constant from the reaction $\tau \to \rho\nu\pi\bar{\nu}$, [4, 5], the magnetic moment measurement of the radiation transition $\rho \to \pi\gamma^*$. The interesting result obtained by BABAR collaboration concerning the deviation of the behavior of the transition form factor $F_{\pi\to\gamma\gamma}(Q^2)$ for large momentum transfer from that predicted by perturbative QCD [6] remains unexplained. One of the main problem in construction of these models is known to be the problem of construction of operators of transition currents. Generally speaking, the complexity of the construction of, for example, the operator of electromagnetic current of the composite system satisfying the Lorentz-covariance and conservation conditions appears in all approaches, including the perturbative quantum field theory. Thus, to ensure the conservation law in the framework of the Bethe-Salpeter equation and quasipotential equations it is necessary to go beyond the framework of the impulse approximation (IA), i.e., it is necessary to add the so called two-particle currents (see, e.g., [9]) to the current operator; these currents are interpreted, for example, in nuclear physics as exchange meson currents. It should be noted that the approach to describing the electroweak structure of two-particle composite systems presented in this survey has the following characteristic features: the matrix element of the electroweak current of the composite system automatically satisfies the relativistic covariance conditions; the matrix element of the electromagnetic current satisfies the conservation law; the IA is formulated in a relativistically invariant way and in the case of the electromagnetic current and with account of the conservation law, the so called modified impulse approximation (MIA) is formulated. This procedure of construction of the current operators actually realizes the Wigner-Eckart theorem on the Poincaré group, i.e., it allows separating from the matrix element of the operator any tensor dimension of the reduced matrix elements (form factors) which are invariants under the Poincaré group transformations. In general, these form factors are not classical but generalized functions.

The method in the relativistic theory of composite systems which will be used here is based on the direct realization of the Poincaré algebra on the set of dynamic observable systems and dates back to the P.Dirac’s paper [10]. This approach is called the theory of direct interac-
tion or the relativistic quantum mechanics with fixed number of particles (RQM) (see, e.g., review [1] and references therein). From the point of view of the principles underlying it, the RQM occupies the intermediate position between the local quantum field theory and nonrelativistic quantum mechanical models.

It should be noted that the field theory and the RQM are formulated as fundamentally different structures and the establishment of a connection between them is a complicated problem that has not been solved yet. Unlike the field theory, in the Poincaré-invariant quantum mechanics, the finite number of degrees of freedom are separated initially, that represents some model. The covariance of description in the RQM is provided by the construction on the Hilbert state space of the composite system with the finite number of degrees of freedom of the unique unitary transformation of the inhomogeneous group $SL(2, C)$, which is the universal covering of the Poincaré group. In this case, the interaction is included in the group generators (operators of observable systems).

The RQM can be realized using different methods and in different forms of dynamics (instant form, point form, light-front form) which differ by methods of inclusion of interaction to the algebra of group generators.

In this survey basic attention was paid to the description of the electroweak structure of composite systems in the framework of the instant form of RQM.

2 Instant form of RQM

Relativistic invariance means that on the Hilbert state space of the system the unitary representation of the Poincaré group (or, more precisely, of the inhomogeneous group $SL(2, C)$ which is the universal covering of the Poincaré group) is realized.

Considering infinitesimal transformations and introducing the generators of translations $\hat{P}^{\mu}$ and space-time rotations $\hat{M}^{\mu\nu}$, we arrive at the Poincaré algebra in the common way,

$$[\hat{M}^{\mu\nu}, \hat{P}^\sigma] = -i(g^{\sigma\mu} \hat{P}^\nu - g^{\sigma\nu} \hat{P}^\mu),$$

$$[\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}] = -i(g^{\mu\rho} \hat{M}^{\nu\sigma} - g^{\mu\sigma} \hat{M}^{\nu\rho}) - (\sigma \leftrightarrow \rho),$$

$$[\hat{P}^\mu, \hat{P}^\nu] = 0. \quad (1)$$

In $g^{\mu\nu}$ is the metric tensor in the Minkowsky space.

The construction of the representation of the Poincaré group in the Hilbert space is reduced to finding the generators $\hat{P}^\mu, \hat{M}^{\mu\nu}$ in terms of dynamic variables of the system. In the case of the system of noninteracting particles generators in (1) have the clear physical meaning, $\hat{P}^\mu \equiv \hat{H}$ is the operator of total energy, $\hat{P} = (\hat{P}^0, \hat{P}^1, \hat{P}^2, \hat{P}^3)$ is the operator of total 3- momentum, $\hat{J} = (\hat{M}^{23}, \hat{M}^{13}, \hat{M}^{12})$ is the operator of total angular momentum, and $\hat{N} = (\hat{M}^{01}, \hat{M}^{02}, \hat{M}^{03})$ are the Lorentz boost generators. However, inclusion of the interaction between particles in this approach involves some problems; the essence of these problems can be illustrated by considering first the quantum nonrelativistic theory and its invariance group, the Galilean group. After the known transition to the central extension of the Galilean group to the covering group $SU(2)$ we obtain the 11-parametric group with the set of generators

$$\hat{H}, \hat{P}, \hat{J}, \hat{K}, \hat{M},$$

where $\hat{K}$ are the generators of the Galilean boosts and $\hat{M}$ is the mass operator. The other generators coincide with the corresponding operators of the Poincaré group.

The following nonzero commutation relations are contained in the Galilean algebra:

$$[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk} \hat{J}_k, \quad [\hat{J}_i, \hat{K}_j] = i\epsilon_{ijk} \hat{K}_k,$$

$$[\hat{J}_i, \hat{P}_j] = i\epsilon_{ijk} \hat{P}_k,$$

$$[\hat{K}_i, \hat{H}] = -i \hat{P}_i, \quad [\hat{K}_i, \hat{P}_j] = -i\delta_{ij} \hat{M}. \quad (2)$$

In nonrelativistic quantum mechanics the operator of interaction is added to the operator of total energy, $\hat{H} \rightarrow \hat{H} + \hat{V}$. In order to preserve the Galilean invariance of the theory, i.e., to preserve algebra (2) under such re-definition of the operator of total energy, the following conditions should be imposed on the operator of interaction:

$$[\hat{P}_i, \hat{V}] = [\hat{J}_i, \hat{V}] = [\hat{V}, \hat{V}] = [\hat{M}, \hat{V}] = 0. \quad (3)$$

Since the generator $\hat{H}$ is absent in the right-hand sides of relations (2), it is not necessary to include the interaction to other generators of the group in order to preserve the Galilean algebra.

The matter is different in the case of the Poincaré group. Let us consider one of the generators of algebra (1) (see, e.g., [8]):

$$[\hat{P}^I \hat{N}^k] = i \delta^{ik} \hat{H}. \quad (4)$$

At inclusion of the interaction to the operator of total energy described above, the right-hand side of (4) depends on the interaction; therefore, either both generators in the left-hand side of (4), or one of them should depend on the interaction. Thus, in order to preserve commutation relations in (1) it is necessary to make other generators in set (1) dependent on the interaction. The generators of the algebra are separated into two types in this case: generators independent of interaction which form the so-called kinematical subgroup and generators depending on interaction, Hamiltonians. The separation of the generators into kinematical generators and Hamiltonians is not unambiguous. Different methods for the separation of the kinematical subgroup result in different forms of dynamics. Usually three basic forms of dynamics are identified: the point form, the instant form, and the light-front form.

Further, the instant form of dynamics will be used in which the kinematical subgroup is comprised of the generators of the group of rotations and shifts of the Euclidean space, $\hat{J}, \hat{P}$, the other generators are the Hamiltonians, i.e., depend on the interaction, $\hat{P}^0, \hat{N}$.

One of the technical methods for inclusion of interaction to algebra (1) allowing to preserve commutation relations is the additive inclusion of interaction to the mass operator, the so called Bakamajian-Thomas procedure [13]
Here, \( \hat{M}_0 \) is the operator of the invariant mass of the system without interaction, \( \hat{M}_I \) is the mass operator of the system with interaction. In the instant form of dynamics the operator of interaction should satisfy the following conditions:

\[
\hat{M}_I = \hat{M}_I^2, \quad \hat{M}_I > 0 ,
\]

(6)

Conditions (6) represent the spectral conditions for the mass operator. Equalities (7) provide the satisfaction of conditions (6) in the system with an interaction.

\[
\hat{P}_i \hat{\psi} = \hat{J}_i \hat{\psi} = \left[ \hat{\mathbf{J}}, \hat{\psi} \right] = \left\{ \hat{\mathbf{J}}, \hat{\psi} \right\} = 0 .
\]

(7)

Thus, the problem of calculation of the wave function of the system is reduced to the diagonalization of the operator \( \hat{M}_I^2 \) (or \( \hat{M}_I \)).

It should be noted that the eigenvalue problem for the operator \( \hat{M}_I^2 \) has the form of the nonrelativistic Schrödinger equation (see, e.g., review [1]). Thus, the operator can be considered as the phenomenological nonrelativistic potential.

### 3 Construction of the electroweak current matrix elements

Let us describe now a general method of canonical parametrization of local operator matrix elements briefly (see e.g., [1, 15] for details).

The main idea of the parametrization can be formulated as follows. Using the variables entering the state vectors which define the matrix elements one has to construct two types of objects.

1. A set of linearly independent matrices which are Lorentz scalars (scalars or pseudoscalars). This set describes transition matrix elements non-diagonal in spin projections in the initial and finite states, as well as the properties defined by the discrete space–time transformations.

2. A set of linearly independent objects with the same tensor dimension as the operator under consideration (for example, four–vector, or four–tensor of some rank). This set describes the matrix element behavior under the action of Lorentz group transformations.

The operator matrix element is written as a sum of all possible objects of the first type multiplied by all possible objects of the second type. The coefficients in this representation as a sum are just the reduced matrix elements – form factors.

The obtained representation is then modified with the use of additional conditions for the operator, such as the conservation laws, for example. In order to satisfy these additional conditions in some cases some of the coefficients (form factors) occur to be zero.

To demonstrate this let us consider the parameterization of the matrix elements taken between the states of a free particle of mass \( M \) in different simple cases. Let us normalize the state vectors as follows:

\[
\langle \tilde{p}', \tilde{m}' | \tilde{p}, \tilde{m} \rangle = 2p_0 \delta(\tilde{p} - \tilde{p}') \delta_{\tilde{m}\tilde{m}'} ,
\]

(11)

Here \( \tilde{p}, \tilde{p}' \) are three–momenta of particle, \( p_0 = \sqrt{M^2 + \tilde{p}^2} \), \( \tilde{m}, \tilde{m}' \) are spin projections.

Let us consider now the 4-vector operator \( j_\mu(0) \). To parametrize the matrix element one needs a set of quantities of the appropriate tensor dimension. Using the variables entering the particle state vectors one can construct one pseudovector of the covariant spin operator \( \Gamma^\nu(p') \) (see, e.g., [1]):

\[
\Gamma_0(p) = \langle \tilde{p}' | \tilde{J}^\nu | p \rangle,
\]

(12)

\[
\Gamma(p) = M \tilde{J} + \frac{\tilde{p} \tilde{p}' j_\nu}{p_0 + M}.
\]
\[ \Gamma^2 = -M^2 \, j(j+1) \]  
(12)

and three independent vectors:

\[ K_{\mu} = (p - p')_{\mu} = q_{\mu}, \quad K_{\mu} = (p + p')_{\mu}, \]

\[ R_\mu = \epsilon_{\nu\nu'\lambda\rho} p^\nu p'^\lambda \Gamma^\rho(p') \]  
(13)

Here \( \epsilon_{\nu\nu'\lambda\rho} \) is a completely anti-symmetric pseudo-tensor in four dimensional space-time with \( \epsilon_{0123} = -1 \).

The set in question of linearly independent matrices in spin projections of the initial and the final states giving the set of independent Lorentz scalars is presented by \( 2j + 1 \) quantities:

\[ D^j(p, p')(p_n\Gamma^q(p'))^n, \quad n = 0, 1, \ldots, 2j. \]  
(14)

The operator matrix element contains the matrix elements of the listed quantities multiplied by Wigner’s rotation matrix \( D^j(p, p') \) from the left. Each of such products is to be multiplied by the sum of linearly independent scalars \( [14] \):

\[ \langle \vec{p}, M, j, m | j_\mu(0) \rangle \vec{p}', M, j, m' \rangle = \sum_{m''} \langle m | D^j(p, p') | m'' \rangle \langle m'' | F_1 K'_\mu + F_2 K_\mu | m' \rangle, \]

(15)

where

\[ F_i = \sum_{m=0}^{2j} f_m(Q^2)(ip_\mu \Gamma^q(p'))^n. \]  
(16)

Let us impose some additional conditions on the operator: self-adjointness, orthogonality of the vectors in the parametrization \([15]\), parity conservation and the conservation condition: \( q_\mu j_\mu(0) = 0 \).

So, finally we have the parametrization of the matrix element which has following form for particle with spin 1/2:

\[ \langle \vec{p}, M, 1/2, m | j_\mu(0) \rangle \vec{p}', M, 1/2, m' \rangle = \sum_{m''} \langle m | D^{1/2}(p, p') | m'' \rangle \langle m'' | f_{10}(Q^2) K'_\mu + i f_{30}(Q^2) R_\mu | m' \rangle, \]  
(17)

The form factors \( f_{10}(Q^2) \) and \( f_{30}(Q^2) \) are the electric and the magnetic form factors of the particle, respectively. These form factors are connected with Sachs form factors \( G_E(Q^2) \) and \( G_M(Q^2) \):

\[ f_{10}(Q^2) = \frac{2M}{\sqrt{4M^2 + Q^2}} G_E(Q^2), \]

\[ f_{30}(Q^2) = -\frac{4}{M \sqrt{4M^2 + Q^2}} G_M(Q^2). \]  
(18)

The developed procedure can be applied to the construction of the electromagnetic current two-quark system. The following integral representation for the pion form factor in the MIA (see, e.g.\([1, 12, 17]\)) holds:

\[ F_\pi(Q^2) = \int d \sqrt{s} d \sqrt{s'} \psi(k) g_0(s, Q^2, s') \psi(k'). \]  
(19)

Here \( \psi(k) \) is pion wave function in the sense of RQM, \( g_0(s, Q^2, s') \) is the free two-particle form factor describing the electromagnetic properties of the two noninteracting quarks without interaction with the quantum numbers of the pion. It may be obtained explicitly by the methods of relativistic kinematics and is a relativistic invariant function.

The wave function in \([19]\) has the following structure:

\[ \psi(k) = \sqrt{2} u(k), \quad s = 4(k^2 + M^2). \]  
(20)

Below for the function \( u(k) \) we use some phenomenological wave functions.

The function \( g_0(s, Q^2, s') \) is written in terms of the quark electromagnetic form factors in the form

\[ g_0(s, Q^2, s') = \frac{(s + s' + Q^2) Q^2}{2Q^2(s - 4M^2)} \times \theta(s, Q^2, s') \times \frac{1}{[1/(s - Q^2) + 1/(s' - 4M^2)]}. \]  
(21)

Here \( \xi = \sqrt{ss'Q^2 - M^2(\alpha - Q^2, s, s')}, \)
\( \omega_1 \) and \( \omega_2 \) are the Wigner rotation parameters:

\[ \omega_1 = \arctan \frac{\xi(s, Q^2, s')}{M(\sqrt{s} + \sqrt{s'})^2 + Q^2 + \sqrt{ss'}(\sqrt{s} + \sqrt{s'})}, \]
\[ \omega_2 = \arctan \frac{\alpha(s, s')(s, Q^2, s')}{M(s + s' + Q^2) + \sqrt{ss'}(\sqrt{4M^2 + Q^2})}. \]

\( \alpha(s, s') = 2M + \sqrt{s} + \sqrt{s'}, \theta(s, Q^2, s') = \theta(s' - s_1) - \theta(s' - s_2), \theta \) is the step function,

\[ s_{1,2} = 2M^2 + \frac{1}{2M^2}(2M^2 + Q^2)(s - 2M^2). \]

Expressions similar to \([19]\) take a place for charge \( G_C(Q^2) \), quadrupole \( G_Q(Q^2) \) and magnetic \( G_M(Q^2) \) form factors of the \( \rho \)-meson:

\[ G_C(Q^2) = \int d \sqrt{s} d \sqrt{s'} \varphi(s) g_{0C}(s, Q^2, s') \varphi(s'), \]
\[ G_Q(Q^2) = \frac{2M^2}{Q} \int d \sqrt{s} d \sqrt{s'} \varphi(s) g_{0Q}(s, Q^2, s') \varphi(s'), \]
\[ G_M(Q^2) = -M \int d \sqrt{s} d \sqrt{s'} \varphi(s) g_{0M}(s, Q^2, s') \varphi(s'), \]  
(22)

where \( \varphi(k) \) is the two-quarks wave function of the \( \rho \)-meson in the sense of RQM, \( g_{0C}, g_{0Q}, g_{0M} \) are the free two-particle form factors describing the electromagnetic properties of the two noninteracting quarks without interaction with the quantum numbers of the \( \rho \)-meson. The explicit
form of free two-particle form factors is cumbersome; it can be found in Ref. [18] which is an extended version of Ref. [19].

The method of the parameterization of the current matrix elements has been generalized in the paper [20] to the case of the nondiagonal in the total angular momentum matrix elements. In particular the expression for the lepton decay constant of the $\rho$-meson was obtained in the 4-fermion approximation:

$$f_\rho = \frac{\sqrt{3}}{\sqrt{2}\pi} \int_0^\infty dk k^2 u(k) \frac{\sqrt{k^2 + M^2 + M}}{(k^2 + M^2)^{3/4}} \times \left(1 + \frac{k^2}{3(\sqrt{k^2 + M^2 + M})^2}\right).$$

(23)

4 Parameters of the model

For the calculation of the $\pi$- and $\rho$-meson characteristics basing on the relations [19–23] we use the following model wave functions (see, e.g. [21–23]):

1. The Gaussian or harmonic oscillator wave function

$$u(k) = N_{HO} \exp\left(-k^2/2b_{\pi,\rho}^2\right).$$

(24)

2. The power-law wave function:

$$u(k) = N_{PL} (k^2/b_{\pi,\rho}^2 + 1)^{-n}, \quad n = 2, 3.$$

(25)

In eqs. (24), (25) $b_{\pi,\rho}$ are parameters of wave functions for pion and $\rho$-meson, respectively.

The electromagnetic form factors of constituent quarks in (21), (22) are taken in the form [1,24]:

$$G_{\pi,\rho}^{EM}(Q^2) = e_q f_0(Q^2),$$

$$G_{\pi,\rho}^{EM}(Q^2) = (e_q + \kappa_q) f_0(Q^2),$$

(26)

where $e_q$ is the quark charge and $\kappa_q$ is the quark anomalous magnetic moment,

$$f_0(Q^2) = \frac{1}{1 + \ln(1 + \langle r_q^2\rangle Q^2/6)},$$

(27)

$\langle r_q^2\rangle$ is the MSR of the constituent quark. Values of all parameters used in these expressions are taken from the $\pi$-meson calculation, see e.g. Ref. [25,26].

So, the following parameters enter our calculations:

1) the parameters that describe the constituent quarks per se (the quark mass $M$, the anomalous magnetic moments of the quarks $\kappa_q$, that enter our formulae through the sum $s_q = \kappa_u + \kappa_d$, and the quark mean square radius (MSR) $\langle r_q^2\rangle$);

2) the parameters $b_{\pi,\rho}$ that enter the quark wave functions (24), (25) and is determined by the quark interaction potential.

In the paper [16] on pion we have shown that in our approach all the parameters of the first group are the functions of the quark mass $M$ and are defined by its value. In particular, for the quark MSR we can use the relation (see, also [27]):

$$\langle r_q^2\rangle \approx 0.3M^2.$$

(28)

To calculate electroweak properties of the $\rho$ meson, we use the same values of quark parameters from the first group as that we have used for the pion [16]. So, the wave function parameters $b_{\pi,\rho}$ are the only free parameters in our calculations.

5 Asymptotics of the pion form factor at high momentum transfers

It is worth to consider the pion form factor asymptotic behavior [19] at $Q^2 \to \infty$ especially. In our paper [24] it was shown that in our approach, the pion form-factor asymptotics at

$$Q^2 \to \infty, \quad M(Q^2) \to 0$$

(29)

does not depend on the choice of a wave function but is defined by the relativistic kinematics only. We consider the fact that the asymptotics obtained in our nonperturbative approach does coincide with that predicted by QCD as a very significant one. Our approach occurs to be consistent with the asymptotic freedom, and this feature surely distinguishes it from other nonperturbative approaches.

Let us note that it is obvious that at very high momentum transfers the quark mass decreases as it goes to zero at the infinity. Our approach permits to take into account the dependence $M(Q^2)$ beginning from the range where this becomes necessary to correspond to experimental data. It is possible that this will take place at the values of $Q^2$ lower than 6 GeV$^2$. So, it is obtained in Ref. [24] that

$$F_{\pi}(Q^2) \sim Q^{-2}.$$

(30)

The asymptotics of the pion electromagnetic form factor $F_\pi$ at momenta transfer $Q^2 \to \infty$ has been determined [28–30], in the QCD frameworks, as

$$Q^2 F_{\pi}(Q^2) \to 8\pi a_{\pi}^{1-loop}(Q^2) f_{\pi}^2,$$

(31)

where $a_{\pi}^{1-loop}(Q^2) = 4\pi/(\beta_0 \log (Q^2/\Lambda^2_{QCD}))$ is the one-loop running strong coupling constant, $\beta_0 = 11 - 2N_f/3$ is the first beta-function coefficient, $N_f$ is the number of active quark flavours and $f_\pi \approx 130$ MeV is the pion decay constant. It is important to note that this asymptotic behaviour, consistent with the quark counting rules [31,32], includes the one-loop coupling only and is to be modified whenever the one-loop approximation fails, but not by means of a simple replacing of $a_s$ with its more precise value. Involved QCD calculations have been performed to obtain corrections to Eq. (31), see e.g. [33]. The QCD does not predict the value of $Q^2$ at which this asymptotics should be reached.

The results of the calculation of $F_\pi$ from Ref. [25] are presented in Fig. 1. As can be seen the our asymptotic coincides with QCD prediction at different scenarios for the zero limit of constituent mass [29].

6 The pion form factor in the region of the JLab experiments

The results of the calculation of the charge pion form factor using the wave functions (24), (25) and the value
of constituent-quark mass $M = 0.22$ GeV (this parameter has been fixed as early as in 1998 [16] from the data at $Q^2 \leq 0.26$ (GeV)$^2$ [35]) are shown in Fig. 2.

Let us note that our RQM describes well the experimental data for the pion form factor including the recent data up to $Q^2 = 2.45$ GeV$^2$ [28]. The upper panel of Fig. 3 demonstrates how the QCD asymptotics, Eq. (31) (dashed line) settles down. The thick gray line bounds from below the solutions obtained from the fitting to the experimental points [3]. The upper of our curves corresponds to the prediction of 1998 [16] (full red lines; upper line: wave functions (24), the mean square charge radius, $\langle r_s^2 \rangle$ for the $\rho$-meson lepton decay constant and MSR of the $\rho$-meson

Let us describe the procedure of calculation of the $\rho$-meson MSR in detail, starting from the quark parameter, $M = 0.22$ GeV, used in a successful calculation of the pion parameters [16]. As it has been demonstrated in Ref. [16] (see also Ref. [41]), the actual choice of the wave-function form does not affect the result provided the quark parameters are fixed. In what follows, we illustrate the procedure with the wave function (24) with $n = 3$.

In the lower panel of Fig. 3, the $\rho$-meson lepton decay constant $f_{\rho}$ as a function of the only free parameter of the model, $b_{\rho}$, is presented. The interval on the vertical axis representing the experimental values of $f_{\rho}$, that is $f_{\rho}^{\exp} = (152 \pm 8)$ MeV [4, 5], is shown. It corresponds to the interval of the values of $b_{\rho}$ which give, through our calculation, the correct experimental values of the decay constant. This interval, $b_{\rho} = (0.385 \pm 0.019)$ GeV, is shown on the horizontal axis of Fig. 3.

The calculated MSR of the $\rho$ meson is presented in the upper panel of Fig. 5. The interval of admissible values of $b_{\rho}$ gives now the corresponding interval of MSR predicted in the present study, $\langle r_{\rho}^2 \rangle = (0.56 \pm 0.04)$ fm$^2$.

Table I presents a comparison of our results with the results of calculations of electroweak properties of the $\rho$ meson in other approaches.

The values of $\langle r_{\rho}^2 \rangle$, while not measured directly, are important for testing various conjectures about strongly interacting systems. One of the interesting related predictions was introduced as a consequence of the so-called Wu–Yang hypothesis [53] (see also Refs. [54–57]), though it is remarkable by itself. Namely, one may define the radius of a hadron either in terms of the electroweak interaction (the mean square charge radius, $\langle r_{ch}^2 \rangle$, calculated for the $\rho$ meson in this paper) or in terms of the strong interaction (this radius, $\langle r_{st}^2 \rangle$, is defined by the slope of the cross section of hadron–proton scattering). The conjecture [54], which may be derived from, though not necessary implies, the hypothesis of Ref. [53], is the equality of the two radii, $\langle r_{ch}^2 \rangle = \langle r_{st}^2 \rangle$.

$\langle r_{ch}^2 \rangle = \langle r_{st}^2 \rangle$. (32)
Figure 3. The decay constant \( f_\rho \) and the \( \rho \)-meson MSR, \( \langle r^2_\rho \rangle \), as functions of the only free parameter of the model, \( b_\rho \). The experimental data fix the value of \( f_\rho \), as shown in the lower panel. This fixes the value of \( b_\rho \), which determines in turn the value of \( \langle r^2_\rho \rangle \), as shown in the upper panel. The values of quark parameters are taken from Ref. [16]; the wave function (25) with \( n = 3 \) is used.

Table 1. The lepton decay constant \( f_\rho \) and MSR of the \( \rho \) meson calculated within different approaches.

| Model   | \( f_\rho \), MeV | \( \langle r^2_\rho \rangle \), fm\(^2\) |
|---------|-------------------|-------------------------------|
| This work | 152±8 (fixed)     | 0.56±0.04                     |
| [4]     | 154               | 0.268                         |
| [42]    | 146               | 0.54                          |
| [43]    | 130               | 0.312                         |
| [44]    | —                 | 0.67                          |
| [45]    | —                 | 0.49                          |
| [46]    | 147.4             | —                             |
| [47]    | —                 | 0.67                          |
| [48]    | —                 | 0.655                         |
| [49]    | —                 | 0.33                          |
| [50]    | —                 | 0.35                          |
| [51]    | 134               | 0.296                         |
| [52]    | 133               | —                             |

This remarkable equality between two physical properties of a hadron related to two different interactions of the Standard Model has been verified experimentally with a great degree of accuracy for the proton, \( \pi \) and \( K \) mesons.

Even more demonstrative is Fig. 4, analogous to a figure from the paper [54], but presenting more recent data. We can see that the value of the \( \rho \)-meson charge radius obtained in this paper fits perfectly the conjecture (32).

Figure 4. Relation between the strong-interaction hadronic radius \( \langle r^2_\rho \rangle \) and the charge radius \( \langle r^2_{\text{ch}} \rangle \) for light hadrons.

8 Analytic continuation of the pion form factor to the complex plain of momentum transfers

As mentioned above, one unsolved theoretical problem of RQM is its relations to the fundamental QFT principles. It is currently unknown whether the basic RQM axioms can be derived from the QFT principles. This paper partially addresses this issue. We compare the corollaries of the QFT principles with the model-independent corollaries of the RQM axioms. In Ref. [58] we compared the predictions for analytic properties of the pion form factor in the complex plane of transferred momenta that follow from QFT general principles with those obtained by the analytic continuation of the form factor integral representation in the spacelike domain derived in the framework of the instant form of RQM [17]. We note that the problem of analytically continuing the form factor in the RQM framework is first formulated in the paper [58].

Technically, the problem of constructing the pion form factor in the complex plane of transferred momenta reduces to continuing expression (19) analytically from the negative part of the real axis to the complex plane of the parameter \( t = -Q^2 \).

We can show that the properties of the analytic continuation of expression (19) depends strongly on the choice of the constituent wave function. We must therefore find which conditions, for example, to impose on the wave functions in order to obtain the experimentally observed resonance behavior of the form factor in the timelike domain of transferred momenta, i.e., on the positive part of the real axis of the parameter \( t \).

It follows from the QFT microcausality condition that the pion form factor is an analytic function in the complex plane of the parameter \( t \) with a cut running from \( 4m_\pi^2 \) to \( \infty \), where \( m_\pi \) is the pion mass (see, e.g., [59, 60] and
the references therein). In this section, we show that our formulation of the composite quark model has analogous properties. As noted above, this fact is interesting from the standpoint of clarifying the relation between the fundamental QFT approach and phenomenological composite quark models.

The obtained analytic properties of the form factor differ from those obtained in the QFT approach only by the position of the branch point on the real axis: it is $4M^2$ in our case and $4m^2$ in the QFT approach.

The main experimentally observed qualitative feature of the form factor behavior in the complex plane is the presence of two resonances, which correspond to $\rho$ and $\omega$ mesons and are located close to each other on the positive part of the real axis (in the timelike domain). Any solution of the problem of constructing the form factor in the complex plane of transferred momenta must ensure the existence of a resonance on the positive part of the real axis for the analytically continued form factor. We can demonstrate that wave functions in the momentum representation without poles in the momentum complex plane do not yield the resonance behavior of the pion form factor for timelike transferred momenta. In particular, this is the case for a wave function of the Gaussian type widely used in composite models (see, e.g., [21, 23]), and this result holds independently of the model parameter values. We therefore encounter the problem of finding a wave function that produces resonances and establishing the relation between the locus of the resonance in the pion form factor and the locus of the wave function poles. In solving this problem, the main idea is to introduce wave functions with poles such that poles of form factor (19) appear on the nonphysical sheet. Below, we show that this ensures the desired resonance behavior of the form factor on the positive part of the real axis.

So, the wave functions $u(k)$ (20), resulting in the resonance behavior of the pion form factor for timelike transferred momenta must have a pole at the point $k_*$: $k_*^2 = \frac{M}{4}(z_* - 2M)$, $z_* = \bar{z}_*'' + i\zeta_*' = \bar{z}_* + i\sqrt{t_r - 4M^2}$, $\bar{z}_*'' < 0$,

where $t_r$ is resonance location.

Let the function $u(k)$ (20) be even and satisfies condition of reality (Im $u(k) = 0$) at real $k$ also.

The simplest rational function satisfying all the above conditions is

$$u_M(k) = N \left[ (k^2 - k_*^2) \left( k^2 - (k_0)^2 \right) \right]^n.$$

The obtained analytic continuation with wave functions of form (34) therefore contains the following parameters: $t_r$ and $z_*''$ from (33), the constituent mass $M$, and the exponent $n$ of the wave function. We fix these data by fitting experimental data for the parameters of the $\rho$ resonance in the pion form factor, namely, its height, width, and location.

To describe the experimental data pertaining to the pion form factor, we use the fitting by the celebrated Breit-Wigner formula (see [61]):

$$F_{\pi\text{BW}}(t) = \frac{4k^2}{4k^2 - t - 2\kappa^2},$$

where $\kappa = 0.375 \text{ GeV}$, $\Gamma = 0.1 \text{ GeV}$. (we take the parameters from [61]).

We numerically tuned the parameters of our model such that the absolute value of the form factor best agrees with the Breit-Wigner approximation of the experimental data, which gives

$$t_r = 0.57 \text{ GeV}^2, \quad z_*'' = -0.01, \quad M = 0.32 \text{ GeV}, \quad n = 0.86.$$

We note that the parameter $t_r$ is the resonance mass, and its value is very close to the value $0.5625 \text{ GeV}^2$ provided by formula (35); the value of the constituent quark mass is typical for different formulations of the composite quark model in which constituent masses are in the interval $(0.2-0.33) \text{ GeV}$ (see, e.g., [16, 21, 23, 27]). The parameter $z_*''$ determining the position of the form factor singularity on the nonphysical sheet and the wave function parameter $n$ are purely model parameters determined by the tuning procedure.

The results of numerically integrating expression (19) with wave function (34) and of calculating with the Breit-Wigner formula are shown in Fig. 5. We can see that our analytic continuation of pion form factor (19) from the spacelike domain to the complex plane provides a good qualitative description of the form factor behavior in the timelike domain already for the simplest wave function choice given by (34). The deviation from the Breit-Wigner formula (or from the experimental data) in the domain of the squared transferred momentum above the maximum point is presumably because our choice (34) of the wave function is too robust. We note that the choice of the wave function only weakly affects the form factor description in the spacelike domain; for a more precise description of the pion form factor in the timelike domain of transferred momenta, we need a more refined expression for the wave function, for instance, with a larger number of poles.

By the parameters:

$$t_r = 0.57 \text{ GeV}^2, \quad z_*'' = -0.01, \quad M = 0.32 \text{ GeV}, \quad n = 0.86.$$
analytic continuation against the condition that the dispersion relations with good accuracy.

We can demonstrate the correctness of our constructed form factor satisfies the known dispersion relation in the spacelike domain

\[ F_{\pi}(t) = \frac{1}{4\pi} \int_{4M^2}^{\infty} \frac{Im [F_{\pi}(t')]}{t - t'} dt', \]

and also the dispersion relation with one subtraction (with the weakened condition of decreasing at infinity)

\[ F_{\pi}(t) = 1 + \frac{t}{4\pi} \int_{4M^2}^{\infty} \frac{Im [F_{\pi}(t')]}{t'(t' - t)} dt'. \]

In Fig. 6 we compare our result with formulas (37) and (38). We use parameter values (36). It can be seen that the constructed form factor satisfies the known dispersion relation (38).

In Fig. 7 we compare our result with formulas (37) and (38). We use parameter values (36). It can be seen that the constructed form factor satisfies the known dispersion relation (38).

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We also verified the correctness of the proposed analytic continuation against the condition that the \( \pi\pi \)-scattering S-matrix is unitary, which can be written in terms of the elastic pion form factor in the form (see, e.g., [62, 63]):

\[ Im F_{\pi}(t) = F_{\pi}(t) \exp \left( -i\delta_{1}(t) \right) \sin \delta_{1}(t), \]

where \( \delta_{1}(t) \) is \( \pi\pi \)-scattering phase at the state \( I = J = 1 \).

We present calculation results for the \( \pi\pi \)-scattering phase in Fig. 7. It can be seen that our constructed form factor provides a scattering phase description close to the results of calculations based on the Breit-Wigner formula.

As already noted, the fitting based on the Breit-Wigner formula satisfactorily describes experimental data for the pion form factor. The problem of a more detailed description of the phase and character of the resonance behavior of the pion form factor in the framework of our approach is related to the problem of choosing a wave function of interacting quarks that is more realistic than (34). Calculation results depend strongly on this choice, which ensures the possibility of a more precise description of the experimental data.

9 Conclusions

The approach developed in IF RQM has following main features: predictivity: parameters of model are fixed from experimental data at small momentum transfers. The experimental data for the pion form factor at large momentum transfers obtained later are described without tuning of parameters; robustness: the behavior of the pion and \( \rho \)-meson electromagnetic form factors does not depend on the choice of wave functions and are determined by the mass of the constituent quarks; the approach gives the asymptotics that agrees with QCD asymptotic behavior at large momentum transfers; the approach gives the self-consistent description of the electroweak properties of the pion and \( \rho \)-meson; the approach gives the right analytical properties of the pion form factor in the complex plane of momentum transfers.

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