On the Topology of Kac–Moody groups

Nitu Kitchloo

Abstract We study the topology of spaces related to Kac–Moody groups. Given a Kac–Moody group over $\mathbb{C}$, let $K$ denote the unitary form with maximal torus $T$ having normalizer $N(T)$. In this article we study the cohomology of the flag manifold $K/T$ as a module over the Nil-Hecke algebra, as well as the (co)homology of $K$ as a Hopf algebra. In particular, if $\mathbb{F}$ has positive characteristic, we show that $H_*(K, \mathbb{F})$ is a finitely generated algebra, and that $H^*(K, \mathbb{F})$ is finitely generated only if $K$ is a compact Lie group. We also study the stable homotopy type of the classifying space $BK$ and show that it is a retract of the classifying space $BN(T)$ of $N(T)$. We illustrate our results with the example of rank two Kac–Moody groups.

Keywords Kac–Moody groups · Classifying spaces · Hopf algebras

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1 Introduction

In this paper, we study a family of topological groups known as Kac–Moody groups [9]. By a Kac–Moody group, we shall mean the unitary form of a Kac–Moody group defined over $\mathbb{C}$. We refer the reader to [15] for a beautiful and comprehensive treatment of the subject. These groups form a natural extension of the class of compact Lie groups, and share many of their properties. For example, they admit maximal tori of finite rank, and corresponding Weyl groups (that are Coxeter groups). The family of Kac–Moody groups is known to contain the class of (polynomial) loop groups, which go by the name of affine Kac–Moody groups. With the exception of compact Lie groups, Kac–Moody groups over $\mathbb{C}$ are not even locally compact. Hence geometric arguments used to study the topology of compact Lie groups and their flag varieties no longer extend to general Kac–Moody groups. This led Kac-Peterson to construct of a whole new set of techniques applicable in this context. Underlying these techniques is a collection of (annihilation) operators that generate a ring $\mathcal{O}$ known as the Nil-Hecke ring [8,14] which can be seen as a deformation of the group ring of the Weyl group.

The cohomology rings of the flag varieties admit an action of $\mathcal{O}$ which interacts non-trivially with the action of the Weyl group. This induces a very rich structure that can be exploited to prove various structure theorems about flag varieties [8,12,15], as well as the Kac–Moody group itself [8]. Further techniques were developed by the author in [12] in order to study the classifying spaces of these groups. In [12,13] it was shown that these classifying spaces can be decomposed as a suitable homotopy colimit of classifying spaces of compact Lie groups. This fact turned out to be fundamental in subsequent study of these classifying spaces.

The main purpose of this paper is to prove various new results regarding the topology of Kac–Moody groups. The first set of new results pertain to the (co)homology rings of Kac–Moody groups and their flag varieties with coefficients in a field of positive characteristic. Next, we study the classifying space of a Kac–Moody group and prove various new results. In particular, we show that the stable homotopy type of the classifying space is a retract of the classifying space of the normalizer of the maximal torus.

In addition to the results mentioned above, we will also provide independent proofs of various homological structure theorems, many of which are known to the experts, but whose proofs are either absent in print or exist in weaker generality. Some of these results were stated in [8], but the proofs never made it to print.

2 Background and statement of new results

Kac–Moody groups have been extensively studied and much is known about their general structure, representation theory and topology [8–10,12,15,16,19]. Their construction begins with a finite integral matrix $A = (a_{ij})_{i,j \in I}$ with the properties that $a_{ii} = 2$ and $a_{ij} \leq 0$ for $i \neq j$. Moreover, we demand that $a_{ij} = 0$ if and only if $a_{ji} = 0$. These conditions define a Generalized Cartan Matrix.

Given a generalized Cartan matrix $A$, one may construct a complex Lie algebra $\mathfrak{g}(A)$ using the Harishchandra-Serre relations. This Lie algebra contains a finite dimensional Cartan subalgebra $\mathfrak{h}$ that admits an integral form $\mathfrak{h}_\mathbb{Z}$ and a real form $\mathfrak{h}_\mathbb{R} = \mathfrak{h}_\mathbb{Z} \otimes \mathbb{R}$. The lattice $\mathfrak{h}_\mathbb{Z}$ contains a finite set of primitive elements $h_i, i \in I$ called simple coroots. Similarly, the dual lattice $\mathfrak{h}_\mathbb{Z}^*$ contains a special set of elements called simple roots $\alpha_i, i \in I$. One may decompose $\mathfrak{g}(A)$ under the adjoint action of $\mathfrak{h}$ to obtain a triangular decomposition as in the classical theory of semisimple Lie algebras. Let $\eta_{\pm}$ denote the positive and negative “nilpotent” subalgebras respectively, and let $\mathfrak{b}_{\pm} = \mathfrak{h} \oplus \eta_{\pm}$ denote the corresponding Borel
subalgebras. The structure theory of $\mathfrak{g}(A)$ leads to a construction (in much the same way that Chevalley groups are constructed), of a topological group $G(A)$ called the (minimal, split) Kac–Moody group over the complex numbers. Recall that $G(A)$ is said to be split if it admits a maximal torus which is a product of the multiplicative group over $\mathbb{C}$. The group $G(A)$ supports a canonical anti-linear involution $\omega$, and one defines the unitary form $K(A)$ as the fixed group $G(A)^{\omega}$. The inclusion map $K(A) \subseteq G(A)$ is a homotopy equivalence. In this article we work with the group $K(A)$ for convenience.

Given a subset $J \subseteq I$, one may define a parabolic subalgebra $\mathfrak{g}_J(A) \subseteq \mathfrak{g}(A)$ generated by $b_+$ and the root spaces corresponding to the set $J$. For example, $\mathfrak{g}_\emptyset(A) = b_+$. One may exponentiate these subalgebras to parabolic subgroups $G_J(A) \subseteq G(A)$. We then define the unitary Levi factors $K_J(A)$ to be the groups $K(A) \cap G_J(A)$. In particular, the group $K_\emptyset(A) = T$ is a torus of rank $\vert I \vert - rk(A)$, and is a maximal torus of $K(A)$. The normalizer $N(T)$ of $T$ in $K(A)$, is an extension of a discrete group $W(A)$ by $T$. The Weyl group $W(A)$ has the structure of a crystallographic Coxeter group generated by reflections $r_i, i \in I$. $W(A)$ has a Coxeter presentation given as follows:

$$W(A) = \langle r_i, i \in I \mid r_i^2 = 1, (r_i r_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ij}$ depends on the generalized Cartan matrix $A = (a_{ij})$, and $m_{ij} = 2, 3, 4, 6, \infty$ if $a_{ij} a_{ji} = 0, 1, 2, 3, \geq 4$ respectively. For $J \subseteq I$, let $W_J(A)$ denote the subgroup generated by the corresponding reflections $r_j, j \in J$. The group $W_J(A)$ is a crystallographic Coxeter group in its own right that can be identified with the Weyl group of $K_J(A)$.

**Definition 2.1** Given a generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$, define a category $S(A)$ to be the poset category (under inclusion) of subsets $J \subseteq I$ such that $K_J(A)$ is a compact Lie group. This is equivalent to demanding that $W_J(A)$ is a finite group.

Notice that $S(A)$ contains all subsets of $I$ of cardinality less than two. In particular, $S(A)$ is nonempty and has an initial object given by the empty set. The category $S(A)$ is also known as the poset of spherical subsets. The topology on the group $K(A)$ is the strong topology generated by the compact subgroups $K_J(A)$ for $J \in S(A)$ (See “Appendix”). More precisely, $K(A)$ is the amalgamated product of the compact Lie groups $K_J(A)$, in the category of topological groups. For an arbitrary subset $L \subseteq I$, the topology induced on homogeneous space of the form $K(A)/K_L(A)$ makes it into a CW-complex, with only even cells, indexed by the set of cosets $W(A)/W_L(A)$ (see [15]). More precisely, the cells can be expressed in terms of the Bruhat decomposition:

$$K(A)/K_J(A) = \bigsqcup_{w \in W(A)/W_J(A)} B w G_J/G_J = \bigsqcup_{w \in W^J(A)} B w B/B,$$

where $B = G_\emptyset(A)$ is the Borel subgroup, $G_J(A)$ are the parabolic subgroups, and $W^J(A)$ denotes the set of minimal coset representatives for the cosets in $W(A)/W_J(A)$ (see [5], Chapter 1.10). The space $B w B/B$ is a subspace homeomorphic to $C^{l(w)}$, where $l(w)$ denotes the word length of the element $w$ in terms of the generators $r_i$.

The structure described in the previous paragraph implies that the cohomology groups $H^*(K(A)/K_J(A))$ have a basis $\{\delta^w\}_{w \in W^J(A)}$ where $\delta^w \in H^{2l(w)}(K(A)/K_J(A))$ represents the linear dual of the affine cell $\delta_w = [B w B/B] \in H_{2l(w)}(K(A)/K_J(A))$. The set $\{\delta^w\}$ is called the Schubert basis. If we are given an inclusion $J < L$, then the projection maps $K(A)/K_J(A) \to K(A)/K_L(A)$ are cellular. Hence we may use the same notation $\delta^w$ for the basis of $H^*(K(A)/K_J(A))$ regardless of $J$. 

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Let $\mathcal{K}(A)/T \xrightarrow{\psi} BT$ be the map that classifies the principal $T$-bundle $K(A) \xrightarrow{\pi} \mathcal{K}(A)/T$. Let $\mathbb{F}$ be a field and let $I \subset H^*(BT, \mathbb{F})$ denote the ideal given by the kernel of $\psi^*$. It has been shown in [7] that $I$ is generated by a regular sequence $\langle \sigma_1, \ldots, \sigma_r \rangle$, with $r \leq \text{rank}(T)$. The ideal $I$ is called the ideal of \textit{Generalized Invariants}. Let $S \subseteq H^*(K(A)/T, \mathbb{F})$ denote the subring given by the image of $\psi^*$. It is also shown in [7] that $H^*(K(A)/T, \mathbb{F})$ is a free $S$-module.

Let us define a coproduct (introduced by D. Peterson) on the cohomology of $K(A)/T$:

$$\Delta(\delta^w) = \sum_{uv=w} \delta^u \otimes \delta^v,$$

where the sum is taken over all reduced expressions for $w$ i.e. expressions where the minimal word length of $w$ with respect to the generators $r_i$ equals the sum of the minimal word lengths of $u$ and $v$.

We will prove the following theorem:

\textbf{Theorem 2.2} The image in cohomology of the projection map $\mathcal{K}(A) \xrightarrow{\pi} \mathcal{K}(A)/T$ is isomorphic to $H^*(K(A)/T, \mathbb{F}) \otimes_{S} \mathbb{F}$. Moreover, this image is a Hopf sub-algebra with the coalgebra structure induced via the coproduct $\Delta$ defined above.

\textbf{Remark 2.3} The statement of the above theorem was communicated to the author by D. Peterson. The first part of this theorem has been proved in [7]. The second part is stated there without proof. In [7], V. Kac shows that $H^*(K(A), \mathbb{F})$ is free over the image of $\pi^*$, and there is a short exact sequence of algebras:

$$1 \rightarrow H^*(K(A)/T, \mathbb{F}) \otimes_{\mathbb{F}} \mathbb{F} \xrightarrow{\pi^*} H^*(K(A), \mathbb{F}) \rightarrow \Lambda(x_1, \ldots, x_r) \rightarrow 1.$$

Here $\Lambda(x_1, \ldots, x_r)$ denotes an exterior algebra on classes $x_i$ of degree given by $\text{deg}(\sigma_i) - 1$. The number of exterior generators is bounded by the rank of $T$, and equals it if $\mathbb{F}$ has positive characteristic. In light of the previous theorem, this extension is actually an extension of Hopf algebras. Note that the elements $x_i$ can be choosen to be primitive by [18] (Thm 7.20).

Next, will study the homology ring $H_*(K(A), \mathbb{F})$ and prove the following:

\textbf{Theorem 2.4} Assume that $\mathbb{F}$ has positive characteristic. Then the Pontrjagin ring $H_*(K(A), \mathbb{F})$ is a finitely generated algebra. In addition, $H^*(K(A), \mathbb{F})$ and $H^*(K(A)/T, \mathbb{F})$ are finitely generated if and only if $K(A)$ is a compact Lie group.

\textbf{Remark 2.5} The Pontrjagin ring $H_*(K(A), \mathbb{F})$ will in general be highly non-commutative. The structure of the rational Pontrjagin ring: $H_*(K(A), \mathbb{Q})$ for a general Kac–Moody group remains unclear to the author.

Now let us consider the classifying space of $K(A)$, denoted by $\mathcal{K}(A)$. The study of this space was begun in [12] and continued in [3] and [13]. It was shown in [3, 12, 13] that $\mathcal{K}(A)$ can be described as a homotopy colimit in terms of the classifying space $\mathcal{B}K_{J}(A)$, as $J$ varied through the poset $\mathcal{S}(A)$. A similar result was proved for the spaces $\mathcal{B}N(T)$ and $\mathcal{B}W(A)$. In this paper, we will use this homotopy decomposition to prove the following:

\textbf{Theorem 2.6} Let $\mathcal{B}N(T)_+$ and $\mathcal{B}K(A)_+$ denote the suspension spectra of the spaces $\mathcal{B}N(T)_+$ and $\mathcal{B}K(A)_+$ respectively (each endowed with a disjoint base point). Then the canonical map $\mathcal{B}N(T)_+ \rightarrow \mathcal{B}K(A)_+$ admits a stable retraction $\mathcal{B}K(A)_+ \rightarrow \mathcal{B}N(T)_+$. In addition, this retraction descends to central quotients of $K(A)$ and $N(T)$.
And finally, in the “Appendix” we will study the topology on \( K(A) \). In particular, we will show that \( K(A) \) has the homotopy type of a CW complex, and that it is the colimit of the compact Lie groups \( K_J(A) \) in the category of topological groups.

### 3 Cohomology and integration along the fiber

In the following sections we supress reference to the generalized Cartan matrix and let \( K \) denote a Kac–Moody group with compact Levi factors \( K_J \), and maximal torus \( T \). Let \( W \) denote the Weyl group. In this section, we will work with cohomology with coefficients in some arbitrary ring \( R \).

Recall the coroots \( h_i \in h_Z \). Let \( h_i^* \in h_Z^* \) denote the dual characters with the property \( h_i^*(h_j) = \delta_{ij} \), where \( \delta_{ij} \) denotes the Kronecker symbol. The elements \( h_i^* \) need not be unique, but we fix a choice throughout. We shall use the same notation to denote the Euler class \( h_i^* \in H^2(BT) \) of the line bundle \( ET \times T^* \mathbb{C} \) over \( BT \), where the action of \( T \) on \( \mathbb{C} \) is obtained by exponentiating \( h_i^* \). This induces a map from \( T \) to the product of circles indexed by \( i \). By construction, this map is an isomorphism. It follows that the set \( \{ h_i^* \} \) is an \( R \)-basis of \( H^2(BT) \) for any ring \( R \).

We proceed in a similar fashion with the roots \( \alpha_i \in h_Z^* \). We use the same notation to denote the Euler class \( \alpha_i \in H^2(K/T) \) of the line bundles \( K \times T^* \mathbb{C} \), where \( T \) acts on \( \mathbb{C} \) by exponentiating the root \( \alpha_i \). Similarly, let \( \alpha_i \in H^2(BT) \) denote the Euler class of the line bundle \( ET \times T^* \mathbb{C} \) under the same action.

Recall the map \( K/T \xrightarrow{\psi} BT \) classifying the principal bundle \( K \xrightarrow{\pi} K/T \). The homomorphism \( \psi^* : H^*(BT) \to H^*(K/T) \) is called the **Characteristic Homomorphism**.

**Claim 3.1** \( \psi^* : H^2(BT) \to H^2(K/T) \) has the property: \( \psi^*(h_i^*) = \delta_i^* \), where \( \delta_i^* \) denotes elements of the Schubert basis of \( H^*(K/T) \) corresponding to the generating set \( r_i \in W \). In addition, we have the equality \( \alpha_i = \sum_{j \in I} a_{ji} \delta_j^* \), where \( a_{ji} \) denote the entries of the generalized Cartan matrix.

**Proof** The proof follows from the CW decomposition of \( K/T \). By by the construction of Kac–Moody groups, there exist injective homomorphisms: \( \varphi_i : SU_2 \to K_i \subset K \) extending the subgroups \( S^1 \subset T \) obtained by exponentiating the coroot \( h_i \). The induced map on the level of flag varieties: \( SU_2/S^1 \to K/T \) is simply the inclusion of the cell corresponding to the Weyl element \( r_i \). Notice that the map \( \psi \circ \varphi_i : SU_2/S^1 \to BT \) factors through \( BS^1 \), where \( S^1 \subset T \) is the circle generated by the coroot \( h_i \). It follows that the restriction of the line bundle constructed via the \( T \)-representation \( h_j^* : ET \times T^* \mathbb{C} \) to \( SU_2/S^1 \) via \( \psi \circ \varphi_i \) is null if \( i \neq j \), and is the Hopf bundle if \( i = j \). Now, the map \( \psi : K/T \to BT \) is two connected, with a basis of \( H^2(K/T, \mathbb{Z}) \) given by the classes \( \delta_i^* \) dual to the Schubert cells \( \varphi_i : SU_2/S^1 \subset K/T \). Restricting along each individual \( \varphi_i \) shows that: \( \psi^*(h_i^*) = \delta_i^* \). Now under the canonical pairing between \( h_Z^* \) and \( h_Z \), we have the equality: \( \alpha_i(h_j) = a_{ji} \) so we get an expression for \( \alpha_i \) using the previous part of the claim. \( \square \)

We now proceed to construct certain operators \( A_i, i \in I \) acting on \( H^*(K/T) \) and \( H^*(BT) \) that were mentioned in the introduction. Let us first introduce the framework:

**Definition 3.2** Let \( H^* \) denote cohomology with coefficients in a ring \( R \) which will be fixed throughout this section. An oriented fibration is a triple \( (\pi, n, \tau) \) where:
(1) $F \to E \xrightarrow{\pi} B$ is a Serre fibration

(2) $H^i(F) = 0$ for $i > n$ and $\pi_1(B)$ acts trivially on $H^n(F)$.

(3) $\tau : H^n(F) \to R$ is a homomorphism of $R$-modules.

Given an oriented fibration $(\pi, n, \tau)$, we can define a homomorphism of $R$-modules:

$$\int \tau : H^*(E) \to H^{*-n}(B)$$

called Integration along the fiber as follows: Consider the Serre spectral sequence for the fibration $F \to E \xrightarrow{\pi} B$. Then $\int \tau$ is defined as the composite:

$$H^*(E) \to E_{\infty}^{*,n} \to E_2^{*,n} = H^{*-n}(B; H^0(F)) \xrightarrow{\tau^*} H^{*-n}(B).$$

Let us prove an easy claim about this homomorphism:

**Claim 3.3** Given an oriented fibration $(\pi, n, \tau)$, then $\int \tau : H^*(E) \to H^{*-n}(B)$ is a map of $H^*(B)$-modules where $H^*(E)$ is an $H^*(B)$-module via $\pi^*$. In addition, $\int \tau$ is natural with respect to pullbacks of fibrations. Furthermore, we have $\int \tau \circ \pi^* = 0$ if $n > 0$.

**Proof** The first part is obvious using the multiplicative structure of the Serre spectral sequence. Naturality of integration along the fiber is also a consequence of the naturality of the Serre spectral sequence. To show that $\int \tau \circ \pi^* = 0$ if $n > 0$, notice first that the image of $\pi^*$ is detected on $E_{\infty,0}^{*,n}$ via the edge homomorphism. Hence its projection to $E_{\infty,0}^{*,n}$ is trivial. \(\square\)

**Definition 3.4** Given $i \in I$, consider the fiber bundle: $K_i/T \to K/T \xrightarrow{\pi_i} K/K_i$. Note that $K_i/T$ may be canonically identified with $CP^1$ via the homeomorphism $\varphi_i : SU_2/S^1 \to K_i/T$. Let $\tau_i \in \text{Hom}(H^2(K_i/T); R) = \text{Hom}(H^2(CP^1); R) = H_2(CP^1)$ be the fundamental class. Note that $(\pi_i, 2, \tau_i)$ is an oriented fibration. Define operators:

$$A_i : H^*(K/T) \to H^{*-2}(K/T) \quad A_i = \pi_i^* \circ \int \tau_i.$$

Similarly for the bundle $K_i/T \to BT \xrightarrow{\theta_i} BK_i$, we define operators by the same name:

$$A_i : H^*(BT) \to H^{*-2}(BT) \quad A_i = \theta_i^* \circ \int \tau_i,$$

where $\theta_i$ is the inclusion of the maximal torus.

**Remark 3.5** Note that the operators $A_i$ are natural with respect to ring homomorphisms $R \to R'$.

Now we have a commutative diagram:

$$
\begin{array}{ccc}
K_i/T & \xrightarrow{=} & K_i/T \\
\downarrow & & \downarrow \\
K/T & \xrightarrow{\psi} & BT \\
\downarrow & & \downarrow \\
K/K_i & \longrightarrow & BK_i
\end{array}
$$
From Claim 3.3 we observe that the operators \( A_i \) intertwine with \( \psi^* \):

\[
\psi^* \circ A_i = A_i \circ \psi^*.
\]

4 The Schubert basis and the Nil Hecke ring

Recall there is a CW-decomposition of \( K/K_J \), for \( J \subseteq I \):

\[
K/K_J = \bigcup_{w \in W^J} BwB/B.
\]

We define the Schubert subvarieties of \( K/K_J \) as closures of cells:

\[
X_w^J = BwB/B = \bigcup_{w' \in W^J, w' \leq w} Bw'\B/B,
\]

where we define a partial order on \( W^J \) by restricting the Bruhat order [2, Chapter 2].

Notice that \( H_2(K/K_J) \) is a free \( R \)-module generated by the largest cell \( \delta_w = [BwB/B] \).

We define this as the “fundamental class” \( [X_w^J] \), bearing in mind that this is just an algebraic statement as the spaces \( X_w^J \) are not manifolds by any means. Let \( i_w : X_w^J \to K/K_J \) be the inclusion.

**Lemma 4.1** The action of the operators \( A_j \) on the Schubert basis is given by:

\[
A_j(\delta^w) = \begin{cases} 
\delta^{ur_j} & \text{if } l(ur_j) < l(w) \\
0 & \text{otherwise}
\end{cases}
\]

**Proof** Assume \( l(w) = n + 1 \). By the definition of \( A_j \), we known that \( A_j(\delta^w) \) is in the image of \( H^*(K/K_J) \). In particular, we may express \( A_j(\delta^w) \) as:

\[
A_j(\delta^w) = \sum_{v \in W^J, l(v) = n} k_v \delta^v,
\]

for some choice of integers \( k_v \). Given an element \( u \in W^J \) with \( l(u) = n \), we claim that one has a pullback:

\[
\begin{array}{ccc}
X_{ur_j} & \xrightarrow{i_{ur_j}} & K/T \\
\downarrow p \quad & & \quad \downarrow \pi_j \\
X_u^J & \xrightarrow{i_u} & K/K_J
\end{array}
\]

This follows on taking the closure of the Steinberg relation: \((B u B)G_j = (B u B) \cup (B ur_j B)\) (see [15] for a review of Steinberg relations). Now consider the homology Serre spectral sequence for the fibration: \( K_J/T \to X_{ur_j} \xrightarrow{p} X_u^J \) and notice that it collapses at \( E_2 \). Moreover, the class \([X_{ur_j}]\) is represented by \([\tau_j] \otimes [X_u^J] \) at the \( E_2 \) stage, where we recall that \([\tau_j] \in H_2(K_J/T) \) is the fundamental class. Consequently, we have:

\[
\langle x; [X_{ur_j}] \rangle = \left\langle \int_{\tau_j} x; [X_u^J] \right\rangle, \quad x \in H^*(X_{ur_j}).
\]
Taking \( x = i_{ur_j}^*(\delta^w) \), and using the naturality of fiberwise integration, we get equalities:

\[
\left< i_{ur_j}^*(\delta^w); [X_{ur_j}] \right> = \sum_{v \in W^j, l(v) = n} k_v \left< i_u^*(\delta^v); [X_u] \right> = k_u.
\]

But notice that the left hand side is equal to 1 (if \( ur_j = w \)) and is zero otherwise. This is a restatement of the above lemma. \( \square \)

The following theorem is crucial in the study of Kac–Moody groups and their flag varieties. This structure was introduced by Kac-Peterson [8], and studied further by Kostant-Kumar [14]. The theorem below can be shown to follow from the previous lemma, and the Bruhat exchange relations. We refer the reader to [15] for a detailed proof.

**Theorem 4.2** [14] Define the Nil Hecke ring \( O \) to be the ring generated by the operators \( A_i, i \in I \) acting on \( H^*(K/\mathbb{T}) \). Then we have:

(i) \( O \) is generated by \( A_i, i \in I \), and relations \( A_i^2 = 0, A_i A_j A_i \cdots = A_j A_i A_j \cdots \) (mi \( f \) factors).

(ii) \( O \) has an \( \mathbb{R} \)-basis given by \( \{ A_w \}_{w \in W} \) where \( A_w = A_{i_1} A_{i_2} \cdots A_{i_k} \) is well defined whenever \( w = r_{i_1} r_{i_2} \cdots r_{i_k} \) is a reduced (or minimal length) expression. Furthermore,

\[
A_w(\delta^v) = \begin{cases} 
\delta^{vw^{-1}} & \text{if } l(w) + l(vw^{-1}) = l(v) \\
0 & \text{otherwise}.
\end{cases}
\]

**Remark 4.3** Notice that the algebra \( O \) is a deformation of the group algebra of \( W \). It can be shown that it acts on \( H^*(BT) \) through the operators \( A_i \). The operators \( A_i \) therefore generate a (twisted) algebra over \( H^*(BT) \), which acts on \( \mathbb{T} \)-equivariant cohomology. One also has a variant of this algebra acting on the (equivariant) K-theory of the space \( K/\mathbb{T} \) [15], where the relation \( A_i^2 = 0 \) gets replaced by \( A_i^2 = A_i \) (up to a unit).

5 Properties of \( A_i \) and the action of \( W \)

Note that \( W = N(\mathbb{T})/\mathbb{T} \) acts on \( \mathbb{T} \) and consequently on \( BT \). One also has a well-defined left action of \( W \) on \( K/\mathbb{T} \) given by:

\[
w(k \mathbb{T}) = k w^{-1} \mathbb{T},
\]

where \( k \mathbb{T} \) is a coset in \( K/\mathbb{T} \), and \( w \in W \). Recall that we have a map \( \psi : K/\mathbb{T} \rightarrow BT \) that classified the bundle \( K \rightarrow K/\mathbb{T} \). Up to homotopy, the map \( \psi \) intertwines the actions of \( W \). Consequently, we observe that the characteristic homomorphism \( \psi^* : H^*(BT) \rightarrow H^*(K/\mathbb{T}) \) intertwines with the induced actions of \( W \) in cohomology. In this section, we recall some fundamental results about the structure of the cohomology of Kac–Moody flag varieties as described in [8,14,15].

Recall that we identified the roots \( \alpha_i \), and the dual coroots \( h^*_i \) with elements in cohomology. Using this convention, we have:

**Claim 5.1** The action of \( W \) on \( H^2(BT) \) and \( H^2(K/\mathbb{T}) \) satisfies:

(a) \( r_i(h^*_j) = h^*_j - \delta_{ij} \alpha_i \) where \( \delta_{ij} \) denotes the Kronecker symbol and \( \alpha_i \in H^2(BT) \).

(b) \( r_i(\alpha_j) = \alpha_j - a_{ij} \alpha_i, \) with \( \alpha_i, \alpha_j \in H^2(BT) \).

(c) \( r_i(\delta^r) = \delta^r - \delta_{ij} \alpha_i, \) with \( \alpha_i \in H^2(K/\mathbb{T}) \).
Proof We may canonically identify $H^2(BT, R)$ with $h^*_i \otimes R$. The action of $W$ is therefore given by $r_i(\lambda) = \lambda - \lambda(h_i) \alpha_i$. From this formula, part (a) and (b) follow easily. For part (c) we simply invoke Claim 3.1.

Recall that the operator $A_i$ was defined via integration along the fiber for the fibration:

$$K_i/T \longrightarrow K/T \xrightarrow{\pi_i} K/K_i$$

**Claim 5.2** For any element $x \in H^*(K/T)$, there is a unique pair of elements $y, z \in \text{Im} \pi_i^*$ such that $x = \delta r_i \cup y + z$. Moreover, $y = A_i(x)$ and $z = x - \delta r_i \cup A_i(x)$. In particular, $x \in \text{Im} \pi_i^*$ if and only if $A_i(x) = 0$.

**Proof** The Serre spectral sequence for $K_i/T \to K/T \to K/K_i$ collapses at $E_2$. Since $\delta r_i$ restricts to a generator of $H^2(K_i/T)$, the result follows from the multiplicative structure of the spectral sequence.

**Claim 5.3** One has the relation $\alpha_i \cup A_i(x) = x - r_i(x)$ for $x \in H^*(K/T)$.

**Proof** Write $x$ as $x = \delta r_i \cup A_i(x) + z$ with $z \in \text{Im} \pi_i^*$. Since $r_i$ acts on $K/T$ through right multiplication by an element in $K_i$, it fixes the image of $\pi_i^*$. Thus $r_i(x) = (\delta r_i - \alpha_i) \cup A_i(x) + z$ using 5.1. The difference of the above two equations gives the required result.

**Claim 5.4** If $2 \in R$ is not a zero divisor, then $\alpha_i$ is not a zero divisor on the image of $\pi_i^*$. In particular, under this assumption, $A_i(x)$ is the unique element in the image of $\pi_i^*$ satisfying the equality given in 5.3.

**Proof** Assume $\alpha_i \cup y = 0$ for some $y \in \text{Im} \pi_i^*$. Note that $\alpha_i = 2\delta r_i + z$ for some $z \in \text{Im} \pi_i^*$ using 3.1. Thus we have $\delta r_i \cup 2y + z \cup y = 0$. From 5.2 we get $2y = 0$ which implies $y = 0$ by assumption.

**Remark 5.5** The same proof as above shows that 5.2, 5.3 and 5.4 hold for $H^*(BT)$ once we replace $\delta r_i$ by $h_i^*$, and $\pi_i^*$ by $\theta_i^*$.

**Theorem 5.6** $A_i(u \cup v) = A_i(u) \cup r_i(v) + u \cup A_i(v)$, where $u, v$ are arbitrary homogeneous elements in $H^*(K/T)$ or $H^*(BT)$.

**Proof** Since the operators $A_i$ are natural with respect to ring homomorphisms, it is sufficient to prove the theorem for $R = \mathbb{Z}$. Note that:

$$\alpha_i \cup (A_i(u) \cup r_i(v) + u \cup A_i(v)) = (u - r_i(u)) \cup r_i(v) + u \cup (v - r_i(v))$$

$$= u \cup v - r_i(u \cup v)$$

$$= \alpha_i \cup A_i(u \cup v)$$

Thus, by 5.4, we will be done if we can show that $x = A_i(u) \cup r_i(v) + u \cup A_i(v) \in \text{Im} \pi_i^*$. It suffices to show that $\alpha_i \cup A_i(x) = 0$, but we have:

$$\alpha_i \cup A_i(x) = x - r_i(x) = -\alpha_i \cup A_i(u) \cup A_i(v) + \alpha_i \cup A_i(u) \cup A_i(v) = 0,$$

and so we are done.

**Theorem 5.7** Let $O_j \subseteq O$ be the subalgebra generated by the operators $A_j, j \in J \subseteq I$. Then $H^*(K/K_j) = H^*(K/T)^{O_j}$, where $H^*(K/T)^{O_j}$ denotes all elements annihilated by the augmentation ideal of $O_j$ and $H^*(K/K_j)$ is identified with its image in $H^*(K/T)$ via $\pi_j^*$.
Remark 5.9 Let \( I \) be defined as the subgroup of \( H^* \). The result follows from the formula for the action of \( A_j \) on the Schubert basis 4.1.

\[ \text{Corollary 5.8} \] If \( \mathfrak{g} \) is not a zero divisor, then \( H^*(K/K_J) = H^*(K/T)^{W_J} \) is the submodule of \( W_J \)-invariant elements for \( J \subseteq I \). In particular \( H^*(K/T)^W = H^0(K/T) = R \).

Proof \( H^*(K/K_J) = H^*(K/T)_{\mathfrak{g}J} = H^*(K/T)^{W_J} \) using 5.3 and 5.4.

\[ \text{Remark 5.9} \] Let \( R = \mathbb{F}_2 \) and consider the example of the Kac–Moody group corresponding to the Cartan matrix:

\[ A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \]

Then both the simple roots are zero, and so 5.3 shows that \( H^*(K/T; \mathbb{F}_2)^W = H^*(K/T; \mathbb{F}_2) \). This example demonstrates that 2 not being a zero-divisor is a necessary assumption in the previous corollary.

6 The Cohomology of \( K \)

In this section we recall some results from [7]. Recall the characteristic homomorphism \( \psi^* : H^*(BT) \to H^*(K/T) \). We begin by identifying the kernel of \( \psi^* \).

Let \( H^*(BT)^+ \) denote elements of \( H^*(BT) \) in the kernel of the augmentation to \( H^0(BT) \). Similarly, we denote \( H^*(K/T)^+ \) the elements in the kernel of the augmentation to \( H^0(K/T) \). Let \( \mathcal{I} \subset H^*(BT) \) be defined as the subgroup of \( H^*(BT) \):

\[ \mathcal{I} = \{ u \in H^*(BT) \mid A_i A_j \ldots A_k (u) \in H^*(BT)^+ \forall i_1, \ldots, i_k, k > 0 \}. \]

\[ \text{Theorem 6.1} \] [7] \( \mathcal{I} = \text{Ker} \psi^* \).

Proof Notice first using 4.2 that \( z \in H^*(K/T)^+ \) is nonzero if and only if there exists a sequence \( i_1 \ldots i_k \) such that \( 0 \neq A_{i_1} \ldots A_{i_k} (z) \in H^0(K/T) \). Now since \( \psi^* \) is an isomorphism in degree 0, the statement \( A_{i_1} \ldots A_{i_k} (u) \in H^*(BT)^+ \forall i_1 \ldots i_k \) is equivalent to the statement \( \psi^* A_{i_1} \ldots A_{i_k} (u) \in H^*(K/T)^+ \forall i_1 \ldots i_k \), i.e. \( A_{i_1} \ldots A_{i_k} \psi^* (u) \in H^*(K/T)^+ \forall i_1 \ldots i_k \) which is equivalent to \( \psi^*(u) = 0 \).

\[ \text{Remark 6.2} \] The ideal \( \mathcal{I} \) is known as the ideal of Generalized Invariants of \( W \). It has been studied in detail and appears to be of independent interest [11].

For the rest of this section we work with field coefficients, we call our field \( \mathbb{F} \).

\[ \text{Lemma 6.3} \] [7] The indecomposables: \( (\mathcal{I}/\mathcal{I}^2) \) is a free \( H^*(BT) \)-module.

Proof Let \( y_1, y_2, \ldots \) be a set of homogeneous elements of \( \mathcal{I} \) such that \( y_1, y_2, \ldots \) form a minimal set of generators of \( (\mathcal{I}/\mathcal{I}^2) \) as a \( H^*(BT) \)-module ordered by increasing degree: \( 0 < \deg(y_1) \leq \deg(y_2) \leq \deg(y_3) \ldots \). Now recall that the characteristic homomorphism \( \psi^* \) intertwines the operators \( A_i \) as well as the action of \( W \). Therefore, 5.6 implies that the operators \( A_i \) preserve \( \mathcal{I}^2 \) and that:

\[ A_i(y_k) = \sum_{j < k} r_{ij} y_j + \mathcal{I}^2, \quad (*) \]
with $r_j \in H^*(BT)$. Let $\sum_{j<k} s_jy_j \in I^2$ be some homogeneous relation. We can assume $s_k \notin I$. Choose a sequence $i_1 \ldots i_k$ such that $0 \neq A_{i_1} \ldots A_{i_k}(s_k) \in F$. Applying $A_{i_1} \ldots A_{i_k}$ to the relation and using \((*)\) repeatedly we notice that:

$$y_k = \sum_{j<k} t_jy_j + I^2, \quad t_j \in H^*(BT)$$

which is a contradiction to the minimality of the set of generators. \(\Box\)

**Theorem 6.4** [7] $I$ is generated by a regular sequence $I = \langle \sigma_1, \sigma_2, \ldots, \sigma_r \rangle$, where $r \leq \text{rank}(T)$. In particular, $H^*(BT)$ is a free module over the polynomial algebra generated by the classes $\sigma_i$.

**Proof** First notice that any regular sequence in $H^*(BT)$ must have length $\leq \text{rank}(T)$. So it remains to show that $I$ is generated by a regular sequence. This follows from a theorem of Vasconcelos [20] which says that for a graded algebra $A$ of finite global dimension a homogenous ideal $I \subseteq A$ is generated by a regular sequence if and only if $(I/I^2)$ is a free $(A/I)$-module. \(\Box\)

**Remark 6.5** If $F$ has positive characteristic $p$, the length of this sequence is exactly $\text{rank}(T)$. This can be seen as follows: First notice that if $\lambda \in H^*(BT)^W$, then its $p$th power $\lambda^p$ is annihilated by all the operators $A_i$. Now notice that $H^*(BT)^W$ contains the Dickson invariants [4] defined as the invariants with respect to the action of $GL_n(F)$, where $n$ is the rank of $T$. The Dickson invariants (or their $p$th powers) form a regular sequence of length $n$. Hence $I$ contains a regular sequence of maximal length and must therefore itself be generated by a sequence of maximal length.

Define a subring $S$ of $H^*(K/T)$ via $S := \text{Im}(\psi^* : H^*(BT) \rightarrow H^*(K/T)) \cong (H^*(BT)/I)$. Let us observe three important facts about this ring:

(a) $S$ is the subring of $H^*(K/T)$ generated by $H^2(K/T)$ since the map $K/T \rightarrow BT$ is 2-connected.

(b) If $F$ has characteristic $p > 0$ then $S$ is a finite dimensional vector space over $F$. In fact, since it is a complete intersection ring, it has the structure of of a Poincaré duality algebra, though we will have no use for this fact in this document.

(c) Let $2d_i$ be the degree of $\sigma_i$, and let $n$ be the rank of $T$, then the Poincaré series of $S$ is:

$$P_t(S) = \prod_{i=1}^r \frac{(1 - t^{2d_i})}{(1 - t^2)^n},$$

as can be easily seen using the fact that $H^*(BT)$ is a free module over the polynomial algebra generated by the classes $\sigma_i$.

**Remark 6.6** The facts a) and c) are true in arbitrary characteristic, but fact b) requires the characteristic to be positive. Indeed, examples in rank two (see Sect. 10), show that the regular sequence generating $I$ has length strictly less than the rank, and so $S$ cannot be finite dimensional.

**Theorem 6.7** [7] $H^*(K/T)$ is a free $S$-module.

**Proof** Proceed as before. Let $y_1, y_2 \ldots$ be a set of homogeneous elements of $H^*(K/T)$ so that $y_1, y_2 \ldots$ form an $F$-basis of $H^*(K/T) \otimes S F$, and $0 = \deg(y_1) \leq \deg(y_2) \leq \deg(y_3) \ldots$
It is clear that the $y_i$ generate $H^*(K/T)$ as an $S$-module. Note that the operators $A_i$ preserve $S$ for all $i$ and

$$A_i(y_k) = \sum_{j<k} r_j y_j; \quad r_j \in S \quad (*)$$

Let $\sum_{j \leq k} s_j y_j = 0$ be some homogeneous relation in $H^*(K/T)$ and assume that $s_k \neq 0$. Choose a sequence $i_1 \ldots i_k$ such that $0 \neq A_{i_1} \ldots A_{i_k}(s_k) \in \mathbb{F}$. Applying $A_{i_1} \ldots A_{i_k}$ to the relation and using $(*)$ repeatedly we get a contradiction. \hfill \Box

We proceed now to study some standard fibrations related to $K$. Recall that given a fibration $F \rightarrow E \rightarrow B$, with $B$ simply connected, the Eilenberg–Moore spectral sequence is a second quadrant cohomological spectral sequence of graded algebras [6]. The $E_2$ term is given by $E_2^{p,q} = \text{Tor}^{-p,q}_{H^*(B)}(\mathbb{F}, H^*(E))$, and it converges to $H^* (F, \mathbb{F})$ for a field $\mathbb{F}$. As an easy consequence of the results we have so far, we recover the result of Kac [7]:

**Theorem 6.8** The Eilenberg–Moore spectral sequence for $K \rightarrow K/T \rightarrow BT$ collapses at $E_2$. Furthermore, $H^*(K, \mathbb{F})$ is free over the image of $\pi^*$, and one has a short exact sequence of algebras:

$$1 \rightarrow H^*(K/T, \mathbb{F}) \otimes_S \mathbb{F} \xrightarrow{\pi^*} H^*(K, \mathbb{F}) \rightarrow \Lambda(x_1, \ldots, x_r) \rightarrow 1.$$

**Proof** We apply the spectral sequence to the fibration $K \rightarrow K/T \rightarrow BT$ in order to compute $H^*(K)$. The $E_2$-term is given by:

$$E_2^{p,q} = \text{Tor}^{-p,q}_{H^*(BT)}(\mathbb{F}, H^*(K/T)).$$

From the previous theorem, we may write $H^*(K/T)$ as the $S$-module $S \otimes_\mathbb{F} H^*(K/T) \otimes_S \mathbb{F}$. It follows that

$$E_2^{p,q} = \text{Tor}^{-p,q}_{H^*(BT)}(\mathbb{F}, S) \otimes_\mathbb{F} H^*(K/T) \otimes_S \mathbb{F} = \Lambda(x_1, x_2, \ldots, x_r) \otimes_\mathbb{F} H^*(K/T) \otimes_S \mathbb{F},$$

where $\Lambda(x_1, x_2, \ldots, x_r)$ denotes an exterior algebra on classes $x_i$ of homogeneous bidegree $(-1, [\sigma_i])$. Due to degree reasons, this spectral sequence collapses. Consider the subring $H^*(K/T) \otimes_S \mathbb{F}$. Since it is in bidegree $(0, *)$, it can be identified via the edge homomorphism with the image of the map $\pi^*: H^*(K/T) \rightarrow H^*(K)$. This proves the above theorem. \hfill \Box

**Remark 6.9** It is natural to ask if $H^*(K/T)$ and $H^*(K)$ are finitely generated algebras. If the characteristic of the field $\mathbb{F}$ is nonzero, then we will use results from the next section to show that both these algebras are not finitely generated, unless $K$ is a compact Lie group. In characteristic zero, this question remains open.

### 7 $H^*(K/T)$ as a module over the Steenrod algebra $A_p$

In this section, we study the structure of the cohomology of $K$ and $K/T$ as modules over the mod $p$ Steenrod algebra $A_p$.

**Theorem 7.1** [8] Let $p$ a prime and let $\mathbb{F} = \mathbb{F}_p$. Let $P = \sum P^i$ be the total Steenrod operation ($P^i = Sq^i$ if $p = 2$). Then $A_i(P(x)) = (1 + \alpha_i^{p-1})P(A_i(x))$ for any element $x \in H^*(K/T)$.
Proof First recall by remark 5.5 that one has a relation in $H^*(BT)$:

$$\alpha_i \cup A_i(x) = x - r_i(x).$$

Now set $x = (h^*_i)^p$, and use 5.1 to obtain the formula over $\mathbb{Z}$:

$$\alpha_i \cup A_i((h^*_i)^p) = (h^*_i)^p - (h^*_i - \alpha_i)^p \equiv \alpha_i^p \mod p.$$  \(\square\)

Reducing mod $p$, and using the fact that $\alpha_i$ is not a zero-divisor, we have the following formula over $\mathbb{F}_p$:

$$A_i((h^*_i)^p) = \alpha_i^{p-1}.$$  

Applying the characteristic homomorphism $\psi^*$, we get the formula: $A_i((\delta^r)^p) = \alpha_i^{p-1}$.

Now given $x \in H^*(K/T)$, we express it as $x = \delta^r \cup A_i(x) + z$. Then using the Cartan formula we have: $P(x) = \delta^r \cup P(A_i(x)) + P(z)$. And consequently, we observe that $A_i(P(x)) = (1 + \alpha_i^{p-1}) \cup P(A_i(x))$ using the fact that $A_i$ is a map of $H^*(K/K_i)$-modules.

Now let $I = \langle \sigma_1, \sigma_2, \ldots, \sigma_r \rangle$ be the ideal of generalized invariants. Let $2d_i$ denote the degree of the element $\sigma_i$. The degree of the top class in $S$ is given by $2m = 2 \sum (d_i - 1)$. We will use the above theorem to show that $H^*(K/T)$ (and therefore $H^*(K)$) is locally finite as a module over $A_p$ (i.e. $A_p(z)$ is a finite dimensional vector space for all $z \in H^*(K/T)$).

For any homogeneous subset $X \subseteq H^*(K/T)$ define $d(X) \leq \infty$ to be the highest degree of any homogeneous element in $X$. For $z \in H^*(K/T)$, let $M(z)$ be the $S$-module given by the span of elements of the form $s \cup a(z)$, $s \in S, a \in A_p$. Note that $M(z)$ is an $A_p$-submodule of $H^*(K/T)$. Let $d(z)$ denote $d(M(z))$.

**Theorem 7.2** If $z$ is an element of positive homogeneous degree $2k$, then $d(z) \leq 2k(m + 1) - 2$.

Proof We work by induction on the degree of $z$. Since $S$ is the subring of $H^*(K/T)$ generated by elements of degree 2, we are done for $k = 1$. Now let $z$ be any element of homogeneous degree $2k + 2$. Let $x = \sum s_\mu \cup P^\mu(z)$ be a homogeneous element of $M(z)$ where $\mu$ ranges over finite sequences of positive integers $i_1 \ldots i_s$ and $P^\mu = P^{i_1} \ldots P^{i_s}$. By repeated application of the previous theorem we notice that $A_i(P^\mu(z)) \in M(A_i(z))$ for any $i \in I$. By induction, $P^\mu(z)$ can have degree at most $2k(m + 1) - 2 + 2 = 2k(m + 1)$. Thus $x$ has degree at most $2k(m + 1) + 2m$. Hence $d(z) = d(M(z)) \leq 2m + 2k(m + 1) = 2(k + 1)(m + 1) - 2$ and we are done. \(\square\)

**Corollary 7.3** $H^*(K/T)$ and $H^*(K)$ are locally finite as modules over $A_p$. In particular, over a field $\mathbb{F}$ of positive characteristic $H^*(K/T)$ is finitely generated if and only if it is finite dimensional (i.e. if and only if $K$ is a compact Lie group).

Proof The first part of the statement follows from the previous theorem. For the second part, notice that for $H^*(K/T)$ to be infinite dimensional and finitely generated, there must exist an element $\lambda$ with arbitrary large nonzero powers. This is impossible since $A_p$ acts locally finitely on $\lambda$. The same argument works for $H^*(K)$.

\(\square\)
The Hopf algebra structure of $H^*(K)$

Recall the extension of algebras:

$$1 \rightarrow H^*(K/T) \otimes_{\mathbb{F}} \mathbb{F} \xrightarrow{\pi^*} H^*(K) \rightarrow \Lambda(x_1, \ldots, x_r) \rightarrow 1.$$  

We will show that this is actually an extension of Hopf algebras. It will be sufficient to show that $H^*(K/T) \otimes_{\mathbb{F}} \mathbb{F}$ is a sub-coalgebra of $H^*(K)$.

We define a coalgebra structure on $H^*(K/T)$ (introduced by D. Peterson) via:

$$\Delta(\delta^w) = \sum_{uv=w} \delta^u \otimes \delta^v$$

where the sum runs over all reduced expressions of $w$. Recall that $H^*(K/T) \otimes_{\mathbb{F}} \mathbb{F}$ maps isomorphically to the image of $\pi^*$ in $H^*(K)$. Our main theorem of this section will state that $\pi^*$ is a map of coalgebras. The first step towards this goal is the construction of equivariant Schubert classes.

Let $H_T^*(K/T) = H^*(ET \times_T (K/T))$ denote the equivariant cohomology of $K/T$. For the moment, we allow coefficients in any ring. Define homomorphisms $E_w$ for $w \in W$ by:

$$E_w : H_T^*(K/T) \xrightarrow{i^*_w} H_T^*(X_w) \xrightarrow{f_{[X_w]}} H_T^*(pt)$$

where $f_{[X_w]}$ denotes integration over the fiber for the oriented fibration:

$$X_w \rightarrow ET \times_T X_w \rightarrow BT$$

Note that these are homomorphisms of $H_T^*(pt)$-modules.

**Claim 8.1** There exists a unique basis $\{\delta^w_T\}_{w \in W}$ of $H_T^*(K/T)$ over $H_T^*(pt)$ with the property:

$$E_v(\delta^w_T) = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

**Proof** Uniqueness will follow easily once we have existence. We proceed by induction on $l(w)$. For $w = 1$, let $\delta^w_T = 1$. Assume we are done defining $\delta^w_T$ for $l(w) < k$. Using the naturality of the pullback:

$$X_w \rightarrow ET \times_T X_w \rightarrow pt \rightarrow BT$$

we see that $\delta^w_T$ restricts to $\delta^w$ under the restriction map $i^*: H_T^*(K/T) \rightarrow H_T^*(K/T)$.

For $w \in W$ such that $l(w) = k$, let $x_w \in H_T^k(K/T)$ be any element that restricts to $\delta^w$ under the (surjective) map $i^*$. For degree reasons, we see that:

$$E_w(x_w) = 1 \quad \text{and} \quad E_v(x_w) = 0 \quad \text{if } l(v) \geq k, w \neq v$$

Now $\delta^w_T$ can be defined as $\delta^w_T = x_w - \sum_{l(v)<k} E_v(x_w) \delta^v_T$, and we are done by induction. \qed
One has a similar fibrations:

\[ \mathbb{K}/T \to \mathbb{K} \times T (\mathbb{K}/T) \xrightarrow{p} \mathbb{K}/T, \quad X_w \to \mathbb{K} \times T X_w \xrightarrow{p} \mathbb{K}/T, \]

and we can define homomorphisms:

\[ \mathcal{F}_w : H^*(\mathbb{K} \times T (\mathbb{K}/T)) \xrightarrow{i_w} H^*(\mathbb{K} \times T X_w) \xrightarrow{\int_{X_w}} H^*(\mathbb{K}/T). \]

Note that \( \mathcal{F}_w \) is a homomorphism of \( H^*(\mathbb{K}/T) \)-modules where \( H^*(\mathbb{K} \times T (\mathbb{K}/T)) \) is viewed as a \( H^*(\mathbb{K}/T) \)-module via \( p^* \).

Consider the pullback obtained by performing the associated bundle over \( \psi \) for the left \( T \)-action on \( \mathbb{K}/T \):

\[ \begin{array}{ccc}
\mathbb{K} \times T (\mathbb{K}/T) & \xrightarrow{\psi} & ET \times T (\mathbb{K}/T) \\
\downarrow p & & \downarrow p \\
\mathbb{K}/T & \xrightarrow{\psi} & BT
\end{array} \]

**Claim 8.2** Let \( \sigma^w = \Psi^*(\delta^w) \). Then the set \( \{ \sigma^w \}_{w \in W} \) is the unique basis of \( H^*(\mathbb{K} \times T (\mathbb{K}/T)) \) as an \( H^*(\mathbb{K}/T) \)-module with the property:

\[ \mathcal{F}_v(\sigma^w) = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{otherwise} \end{cases} \]

**Proof** The proof follows from the previous claim and naturality with respect to pullback for the above diagram. \( \square \)

We will now need some technical results which will be the content of the next few lemmas. First consider the bundle over \( \mathbb{C}P^1 \) given by \( \mathbb{K}_i \to \mathbb{K}_i/T \). Fix sections \( Y_i \) over the open cell given by the complement of the identity coset \([T]\). Note that we may identify \( Y_i \) with \( \mathbb{C} \subset \mathbb{C}P^1 \) for each \( i \in I \). Consider the following subspace of \( \mathbb{K} \):

\[ Z_w = \bigcup_{v \leq w} \tilde{Y}_v, \quad \text{where} \quad \tilde{Y}_v = Y_v \times T, \quad \text{with} \quad Y_v = Y_{i_1} \times \cdots \times Y_{i_k}, \]

and \( v = r_{i_1} \cdots r_{i_k} \) being some reduced expression. Note that \( Z_w \) is simply the restriction of the fibration \( \mathbb{K} \to \mathbb{K}/T \) to \( X_w \), with \( Y_w \) being a section of this fibration over the top cell of \( X_w \). For \( u, v \in W \), we can form the space \( X_{u,v} = Z_u \times_T X_v \). The space \( X_{u,v} \) has a cellular decomposition:

\[ X_{u,v} = \bigcup_{u' \leq u, v' \leq v} (Y_{u'} \times T) \times_T Y_{v'} = \bigcup_{u' \leq u, v' \leq v} Y_{u'} \times Y_{v'}. \]

The CW decomposition of \( \mathbb{K}/T \) implies that the space \( \mathbb{K} \times_T \mathbb{K}/T \) admits a CW decomposition with even open cells given by the spaces \( \mathbb{Y}_u \times \mathbb{Y}_v \). Let \( \delta^{u,v} \in H^{2l(u)+2l(v)}(X_{u,v}) \) be the class linear dual to the (top) cell: \([Y_u \times Y_v]\).

Let \( \mu : \mathbb{K} \times_T (\mathbb{K}/T) \to \mathbb{K}/T \) denote the left action of \( \mathbb{K} \) on \( \mathbb{K}/T \). One has an induced map:

\[ \mu_{u,v} : X_{u,v} = Z_u \times_T X_v \to \mathbb{K} \times_T (\mathbb{K}/T) \xrightarrow{\mu} \mathbb{K}/T. \]
Lemma 8.3 If \( w \in W \) such that \( l(w) = l(u) + l(v) \) then:

\[
\mu^*_{u,v}(\delta^w) = \begin{cases} 
\delta^u \cup \delta^v & \text{if } uv = w \\
0 & \text{otherwise}
\end{cases}
\]

Proof Recall from the CW-decomposition of \( K/T \) that we have a unique factorization of cells in \( K/T \) [15] given by: \( Y_u \cdot Y_v = Y_{uv} \) if \( uv \) is reduced. Otherwise, \( Y_u \cdot Y_v \) factors through cells of lower dimension. The result follows. \( \square \)

Now consider the oriented fibration

\[
X_u \rightarrow Z_u \times_T X_v \xrightarrow{p_{u,v}} X_u
\]

Lemma 8.4

\[
\int_{[X_v]} \delta^u = \delta^u
\]

Proof The proof is obvious since the Serre spectral sequence for the above filtration collapses at \( E_2 \) and \( \delta^u \) is represented by \( \delta^u \otimes \delta^v \). \( \square \)

Recall that \( H^\ast(K \times_T K/T) \) is a free \( H^\ast(K/T) \)-module via \( p^\ast \), on the basis \( \sigma^v \). We have:

Lemma 8.5 Under the action map \( \mu : K \times_T (K/T) \rightarrow K/T \), we have the equality:

\[
\mu^*(\delta^w) = \sum_{uv = w} \delta^u \cup \delta^v
\]

where the sums runs over all reduced expressions of \( w \).

Proof Let \( l(w) = k \). We can write

\[
\mu^*(\delta^w) = \sum_{l(u) + l(v) = k} a_{u,v} \delta^u \cup \sigma^v
\]

where \( a_{u,v} \) are elements in the coefficient ring. We use the operators \( F_v \) to isolate them:

\[
F_v \mu^*(\delta^w) = \sum_{l(u) = k - l(v)} a_{u,v} \delta^u
\]

and therefore \( i^*_u F_v \mu^*(\delta^u) = a_{u,v} \delta^u \) where \( i^*_u : H^\ast(K/T) \rightarrow H^\ast(X_u) \) is induced by the inclusion. Now recall that \( Z_u \) was the restriction of \( K \rightarrow K/T \) to \( X_u \). Hence, we have the commutative diagram:

\[
\begin{array}{ccc}
Z_u \times_T X_v & \xrightarrow{i_v} & K \times_T X_v \\
\downarrow \mu_{u,v} & & \downarrow \mu \\
X_u & \rightarrow & K/T \\
\end{array}
\]

We have:

\[
i^*_u F_v \mu^*(\delta^w) = \int_{[X_v]} \mu^*_{u,v}(\delta^w) = \begin{cases} 
\delta^u & \text{if } uv = w \\
0 & \text{otherwise}
\end{cases}
\]

using 8.3 and 8.4, so we are done. \( \square \)
Let $m : K \times K \to K$ denote the multiplication map. Consider the commutative diagram:

$$
\begin{array}{ccc}
K \times K & \xrightarrow{\tilde{\pi}} & K \times_T (K/T) \\
\downarrow m & & \downarrow \mu \\
K & \xrightarrow{\pi} & K/T
\end{array}
$$

Let $\pi_1, \pi_2$ be defined as the projections onto the first and second factor respectively:

**Lemma 8.6** In cohomology, $\tilde{\pi}$ is given by

- $a) \quad \tilde{\pi}^*(\sigma^v) = \pi_2^* \circ \pi^*(\delta^v)$
- $b) \quad \tilde{\pi}^*(\delta^u) = \pi_1^* \circ \pi^*(\delta^u)$

**Proof** For a), recall that by definition $\sigma^v = \Psi^*(\delta^v)$. Now notice that we have a commutative diagram:

$$
\begin{array}{ccc}
K \times K & \xrightarrow{f} & ET \times (K/T) \\
\downarrow \tilde{\pi} & & \downarrow g \\
K \times_T (K/T) & \xrightarrow{\Psi} & ET \times_T (K/T)
\end{array}
$$

Note that $ET \times (K/T) \xrightarrow{\sim} K/T$ since ET is contractible. Under this identification we have $f = \pi \circ \pi_2$ and $g$ is the inclusion of $K/T$ in $ET \times_T (K/T)$. Since $g^*(\delta^v_T) = \delta^v$, we get:

$$
\tilde{\pi}^*(\sigma^v) = \tilde{\pi^*} \circ \Psi^*(\delta^v_T) = \pi_2^* \circ \pi^*(\delta^v).
$$

Part b) follows from the commutative diagram:

$$
\begin{array}{ccc}
K \times K & \xrightarrow{\tilde{\pi}} & K \times_T (K/T) \\
\downarrow \pi_1 & & \downarrow p \\
K & \xrightarrow{\pi} & K/T
\end{array}
$$

We now prove the main theorem of this section:

**Theorem 8.7** The map $\pi^* : H^*(K/T) \to H^*(K)$ is a map of coalgebras, where cohomology is taken with coefficient in a field.

**Proof** One recalls the commutative diagram:

$$
\begin{array}{ccc}
K \times K & \xrightarrow{\tilde{\pi}} & K \times_T (K/T) \\
\downarrow m & & \downarrow \mu \\
K & \xrightarrow{\pi} & K/T
\end{array}
$$

Now one invokes 8.5 and 8.6, to get the required equality:

$$
m^* \pi^*(\delta^u) = \sum_{u \cup v = w} \pi_1^* (\pi^*(\delta^u)) \cup \pi_2^* (\pi^*(\delta^v)),
$$

where the sum is begin taken over all reduced expressions. This is exactly the statement of the theorem. 

$\square$
Consequently, we have:

Since $K$ is simply connected, the inclusion of the maximal torus $T \subset H^*(K/T)$ yields a free left $A^*_F$-module 

We begin with some preliminary lemmas:

Lemma 9.1 The left action of $H_*(K)$ on $H_*(K/T)$ factors through the projection $H_*(K) \to A^*_F$, where $A^*_F$ is the dual Hopf algebra of $A_F$. Moreover, $H_*(K/T)$ is a finitely generated free left $A^*_F$-module.

Proof The first part of the above lemma follows by dualizing the result of 8.5. The theory of Hopf-algebras can now be used to show that $H_*(K/T)$ is a free $A^*_F$-module [18](Thm. 4.4). Now since $H^*(K/T)$ is a free module over the finite dimensional algebra $S$, with a basis given by $A_F$, we can compute Poincaré series to see that $H_*(K/T)$ must have a finite basis over $A^*_F$.

For the benefit of the reader, we also provide an alternate proof: Recall that for a topological group $G$, and a principal $G$-bundle $G \to E \to B$, one has a natural homological (Bar) spectral sequence of coalgebras converging to $H_*(B)$ with $E_2$-term given by $E_2^{p,q} = \text{Tor}_{H_*(G)}(F, H_*(E))$ [6]. Consider a pair of pullbacks of principal T-bundles:

This induces an action of $H_*(K)$ on the Bar spectral sequence of converging to $H_*(K/T)$:

$H_*(K) \otimes \text{Tor}_{H_*(T)}(F, H_*(K)) \to \text{Tor}_{H_*(T)}(F, H_*(K)).$

Since $K$ is simply connected, the inclusion of the maximal torus $T \subset K$ is null homotopic. Consequently, we have:

$\text{Tor}_{H_*(T)}(F, H_*(K)) = H_*(K) \otimes F \text{Tor}_{H_*(T)}(F, H_*(K)).$

Differentials in this spectral sequence must annihilate the piece corresponding to the dual exterior algebra: $\Lambda^*(x_1, \ldots, x_r) \subseteq H_*(K)$. This dual algebra is itself an exterior algebra. The generators of this exterior algebra must therefore be targets of differentials originating on elements indecomposable under the $H_*(K)$-action. Now we may write:

$\text{Tor}_{H_*(T)}(F, F) = \Gamma(y_1, \ldots, y_r) = S^* \otimes F \Gamma(\tau_1, \ldots, \tau_r)$

where $\Gamma(y_1, \ldots, y_r)$ denotes the dual of a polynomial algebra. This coalgebra is bigraded by giving $y_i$ bidegree $(1, 1)$. The element $\tau_i$ is an element of bidegree $(1, |x_i|)$, the vector space
$S^*$ is dual to $S$, and is detected in $H_\ast(B\Gamma)$. It follows that the generators $\tau_i$ must hit a set of generators of the dual exterior algebra $\Lambda^\ast(x_1, \ldots, x_r)$ in the spectral sequence. Consequently, the $E_\infty$ term of the spectral sequence is a free left module over $A_F^\ast = H_\ast(K) \otimes \Lambda^\ast(x_1, \ldots, x_r) \mathbb{F}$, with a basis given by the finite dimensional vector space $S^\ast$. The result follows from an easy filtration argument. 

We now need the following general lemma:

**Lemma 9.2**  Let $A$ be a (not necessarily commutative) finitely generated, graded, connected $\mathbb{F}$-algebra. Let $B \subseteq A$ be a graded sub-algebra so that $A$ is finitely generated and free as a left $B$-module. Then $B$ is also a finitely generated algebra.

**Proof**  Let $\{a_1, \ldots, a_n\}$ be a set of algebra generators of $A$ over $\mathbb{F}$, and let $\{c_1, \ldots, c_m\}$ be a basis set of $A$ as a left $B$-module, with $c_1 = 1$. We pick a finite set $\{e_1, \ldots, e_k\} \subseteq B$, so that:

$$a_j \in \sum_{r, s} \mathbb{F} e_r c_s, \quad c_i a_j \in \sum_{r, s} \mathbb{F} e_r c_s, \quad j \leq n, i \leq m.$$ 

Let $b \in B$ be an arbitrary element. Since $A$ is finitely generated, there exists a polynomial $f$ so that $f(a_1, \ldots, a_n) = b$. Using the above properties repeatedly, we may write

$$b = \sum_i g_i(e_1, \ldots, e_k) c_i,$$

for some polynomials $g_i$. But since $A$ is a free left $B$-module, we observe that $g_i = 0$ for all $i > 1$ and that $b = g_1(e_1, \ldots, e_k)$. It follows that $B$ is generated by $\{e_1, \ldots, e_k\}$. 

As an easy consequence of the above lemmas, we have:

**Theorem 9.3**  Let $\mathbb{F}$ be a field of positive characteristic. Then the dual hopf algebra $A_F^\ast$ is a finitely generated $\mathbb{F}$-algebra. It follows that $H_\ast(K, \mathbb{F})$ is also finitely generated $\mathbb{F}$-algebra.

**Proof**  Dualizing the coalgebra structure of $H^\ast(K/\Gamma)$, we observe that $H_\ast(K/\Gamma)$ has the structure of a finitely generated algebra on the set of elements $\delta_{r_i}$ dual to the Schubert basis elements $\delta^\prime$. Working with coefficients in a field $\mathbb{F}$ of positive characteristic, the results of previous sections shows that $A_F^\ast \subseteq H_\ast(K/\Gamma)$ is a sub-algebra. By the first lemma, we see that $H_\ast(K/\Gamma)$ is a finitely generated, free, left $A_F^\ast$-module and so the second lemma implies that $A_F^\ast$ is a finitely generated algebra. The result about $H_\ast(K, \mathbb{F})$ follows easily once we describe it as an extension of $A_F^\ast$ by $\Lambda^\ast(x_1, \ldots, x_r)$. 

**10 Examples of rank two**

In this section, we describe the structure of the (co)homology of rank two Kac–Moody groups and their flag varieties. By a rank two Kac–Moody group, we shall mean a Kac–Moody group for which the set $I$ has cardinality two.

Generalized Cartan matrices representing Kac–Moody groups of rank two are given by:

$$A(a, b) = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}.$$
Throughout this section let $K = K(a, b)$ denote the semisimple factor inside the corresponding unitary form. If $ab < 4$, then $K$ is a compact Lie group. In particular

$$(a, b) = (0, 0) \quad K = SU(2) \times SU(2)$$

$$(a, b) = (1, 1) \quad K = SU(3)$$

$$(a, b) = (1, 2) \quad K = Spin(5) = Sp(2)$$

$$(a, b) = (1, 3) \quad K = G_2$$

Henceforth, we only work with a generalized Cartan matrix $A = A(a, b)$ with $ab \geq 4$. Let $T \subset K$ denote the maximal torus. Then the Weyl group has a presentation given by:

$W = \langle r_1, r_2 | r_1^2 = 1 \rangle$. Thus the Poincaré series for $H^*(K/T; \mathbb{Z})$ is

$$P_*H^*(K/T; \mathbb{Z}) = 1 + 2t^2 + 2t^4 + \cdots = \frac{1 + t^2}{1 - t^2}$$

hence $H^*(K/T, \mathbb{Z})$ contains two elements from the Schubert basis in every positive even degree. Let $\delta_n$ be the element $\delta^w$ where $l(w) = n$, and $l(wr_1) < l(w)$. Thus $w = \ldots r_1 r_2 r_1$ ($n$ terms). Let $\tau_n$ be the other element from the Schubert basis in the same degree. Denote $\delta_1$ and $\tau_1$ by $\delta$ and $\tau$ respectively. The action of the Weyl group on $\tau$ and $\delta$ can be easily deduced from Claim 5.1.

Given a generalized Cartan matrix $A = A(a, b)$, define integers $c_1, d_i$ recursively via:

$$c_0 = d_0 = 0; \quad c_1 = d_1 = 1; \quad c_{j+1} = ad_j - c_{j-1}; \quad d_{j+1} = bc_j - d_{j-1}.$$  

**Theorem 10.1** In $H^*(K/T; \mathbb{Z})$ we have the relations:

$$\delta \cup \delta_n = d_{n+1} \delta_{n+1}; \quad \delta \cup \tau_n = \delta_{n+1} + d_n \tau_{n+1}$$

$$\tau \cup \tau_n = c_{n+1} \tau_{n+1}; \quad \tau \cup \delta_n = \tau_{n+1} + c_n \delta_{n+1}$$

**Proof** We proceed by induction. Since $\delta_0 = \tau_0 = 1$, the result is true for $n = 0$. Now write $\delta \cup \delta_n = A \delta_{n+1} + B \tau_{n+1}$ where $A, B \in \mathbb{Z}$. We recall the annihilation operators $A_1$ and $A_2$ acting diagonally with respect to the Schubert basis. Therefore, we get the equality: $A_1(\delta \cup \delta_n) = A_1(A \delta_{n+1} + B \tau_{n+1}) = A \tau_n$. On the other hand we have the twisted derivation property given by Theorem 5.6 and claim 5.1:

$$A_1(\delta \cup \delta_n) = r_1(\delta) \cup A_1(\delta_n) + \delta_n \cup A_1(\delta)$$

$$= (\delta - (2\delta - b \tau)) \cup \tau_{n-1} + \delta_n$$

$$= b \tau \cup \tau_{n-1} - \delta \cup \tau_{n-1} + \delta_n$$

$$= bc_n \tau_n - \delta_n - d_{n-1} \tau_n + \delta_n$$

$$= d_{n+1} \tau_n$$

where we used induction and the recursive definition of $d_i$. Thus $A = d_{n+1}$. Now we apply $A_2$ and observe that $B = 0$. The other equalities follow similarly. \hfill \square

**Remark 10.2** The proof of the above theorem shows that these formulas for the multiplicative structure of $H^*(K/T; \mathbb{Z})$ are also valid for rank 2 Kac–Moody groups $K$ of finite type, provided one works below the top degree so that $\delta_{n+1}$ and $\tau_{n+1}$ are distinct generators.
**Definition 10.3** Given a generalized cartan matrix $A(a, b)$, we define the generalized binomial coefficients$^1$:

\[
\begin{align*}
D(n,m) &= \frac{d_{n+m}d_{n+m-1} \cdots 1}{d_{n}d_{n-1} \cdots 1 \; d_{m}d_{m-1} \cdots 1} \\
C(n,m) &= \frac{c_{n+m}c_{n+m-1} \cdots 1}{c_{n}c_{n-1} \cdots 1 \; c_{m}c_{m-1} \cdots 1}
\end{align*}
\]

Note that if $a = b = 2$, then $c_n = d_n = n$ and thus $C(n,m) = D(n,m) = \binom{n+m}{n}$.

The previous theorem on the ring structure of $H^*(K/T, \mathbb{Z})$ immediately implies the following theorem about the cohomology of the partial flag varieties:

**Theorem 10.4** Let $K_1, K_2$ be the maximal compact subgroups of the standard parabolic subgroups corresponding to $[1], [2] \subset [1, 2]$ respectively. Then

\[
\begin{align*}
H^{2n}(K/K_1; \mathbb{Z}) &= \mathbb{Z}[\tau_n]; \quad \tau_n \cup \tau_m = C(n,m) \tau_{n+m} \\
H^{2n}(K/K_2; \mathbb{Z}) &= \mathbb{Z}[\delta_n]; \quad \delta_n \cup \delta_m = D(n,m) \delta_{n+m}
\end{align*}
\]

In particular we see that the generalized binomial coefficients $C(n,m)$ and $D(n,m)$ are integers.$^2$

**Remark 10.5** The above theorems completely determine the ring structure of $H^*(K/T; \mathbb{Z})$, and $H^*(K/K_f, \mathbb{Z})$.

**Claim 10.6** $H^*(K/T; \mathbb{Q}) = \mathbb{Q}[\delta, \tau]/\mathcal{I}$, where $\mathcal{I}$ is the ideal (of generalized invariants) given by the quadratic relation: $a\delta^2 + b\tau^2 - ab\delta \tau = 0$. In particular $H^*(K/T; \mathbb{Q})$ is generated by $H^2(K/T; \mathbb{Q})$.

**Proof** In $H^4(K/T; \mathbb{Z})$ we have a relation $a\delta^2 + b\tau^2 - ab\delta \tau = 0$. This yields a map:

\[
\mathbb{Q}[\delta, \tau]/\mathcal{I} \rightarrow H^*(K/T; \mathbb{Q}).
\]

Using the ring structure of $H^*(K/T, \mathbb{Q})$, this map is surjective. To see that it is an isomorphism, one simply compares the Poincaré series. We leave this as an exercise. \hfill \Box

10.1 The additive structure of $H^*(K, \mathbb{Z})$, and the Hopf algebras $A_\mathbb{Z}, A_{\mathbb{F}_p}$:

Now consider the Serre spectral sequence in integral cohomology for the fibration:

\[
T \rightarrow K \rightarrow K/T.
\]

Let $H^1(T, \mathbb{Z}) = \mathbb{Z} \alpha \oplus \mathbb{Z} \beta$, with $d_2(\alpha) = \delta$ and $d_2(\beta) = \tau$. The ring structure of $H^*(K/T, \mathbb{Z})$ now allows us to compute the structure of the differential $d_2$. It is easy to see that $d_2$ is injective on $E_2^{2,2}$, and that $E_3 = E_\infty$. Let $g_n = \text{gcd}(c_n, d_n)$ denote the g.c.d of the pair $c_n, d_n$. The following results are easy consequences of the explicit formulas given in Theorem 10.1:

\[
E_3^{2n,0} \cong E_3^{2n+2,1} \cong \mathbb{Z}/g_n\mathbb{Z}.
\]

**Corollary 10.7** The additive structure of $H^*(K, \mathbb{Z})$ is given by:

\[
H^{2n+3}(K, \mathbb{Z}) = H^{2n}(K, \mathbb{Z}) = \mathbb{Z}/g_n\mathbb{Z}.
\]

1 This terminology is due to Haynes Miller.

2 We thank Kasper Andersen for showing us a nice algebraic proof of integrality.
From the above Serre spectral sequence, we notice that $E_3^{*,0}$ is given by the Hopf algebra $A_Z = \text{Im}(H^*(K/T, \mathbb{Z}) \to H^*(K, \mathbb{Z}))$, and it can be identified with $H^*(K/T, \mathbb{Z}) \otimes S \mathbb{Z}$. Recall that in degree $2n$, $A_Z$ is a cyclic group of order $g_n$ generated by $\delta_n$ or $\tau_n$:

$$A_Z^{2n} = \mathbb{Z}/g_n\mathbb{Z}; \quad g_n = \gcd(c_n, d_n)$$

The coalgebra structure on $A_Z$ was induced by:

$$\Delta(\delta^w) = \sum_{u \cdot v = w} \delta^u \otimes \delta^v$$

where the sum is over all reduced expressions, thus

$$\Delta(\delta_n) = \sum_{i=0}^{n} \delta_i \otimes \delta'_{n-i}; \quad \delta'_{n-i} = \begin{cases} \delta_{n-i} & \text{if } i \text{ even} \\ \tau_{n-i} & \text{if } i \text{ odd} \end{cases}$$

$$\Delta(\tau_n) = \sum_{i=0}^{n} \tau_i \otimes \tau'_{n-i}; \quad \tau'_{n-i} = \begin{cases} \tau_{n-i} & \text{if } i \text{ even} \\ \delta_{n-i} & \text{if } i \text{ odd} \end{cases}$$

Now fix a prime $p$. Recall that $A_{F_p} = F_p \otimes \mathbb{Z} A_Z$. Hence to understand $A_{F_p}$, we need to know when $p$ divides $g_n$. We have the following theorem on the arithmetic properties of the integers $c_n$ and $d_n$:

**Theorem 10.8** Let $g_n = \gcd(c_n, d_n)$. Given a prime $p$, there is a smallest positive integer $k$ with the property that $p$ divides $g_k$. Further, $p$ divides $g_n$ if and only if $k$ divides $n$. More precisely, $k$ is given by:

1) $k = 2p$ if $p$ divides $a$ or $b$ but not both
2) $k = p$ if $ab = 4 \pmod{p}$ and the conditions of 1) do not hold. In all other cases, we have:
3) $k = r$ where $r$ is the multiplicative order of any root of the quadratic polynomial given by $x^2 - (ab - 2)x + 1$ defined over $F_p[x]$.

**Proof** Note that values of $c_i$, $d_i$ and $g_i$ for small values of $i$ are given by the table:

| $i$ | $c_i$ | $d_i$ | $g_i$ |
|-----|-------|-------|-------|
| 0   | 0     | 0     | 0     |
| 1   | 1     | 1     | 1     |
| 2   | $a$   | $b$   | $(a, b)$ |
| 3   | $ab - 1$ | $ab - 1$ | $ab - 1$ |
| 4   | $a(ab - 2)$ | $b(ab - 2)$ | $(a, b)(ab - 2)$ |
| 5   | $ab(ab - 3) + 1$ | $ab(ab - 3) + 1$ | $ab(ab - 3) + 1$ |
| 6   | $a(ab - 1)(ab - 3)$ | $b(ab - 1)(ab - 3)$ | $(a, b)(ab - 1)(ab - 3)$ |

Let us consider the four cases given by the respective parities of the integers $a$ and $b$. Working mod two, the above table, along with the recursion, shows that the sequence $g_i$ is periodic with period being $k$ as defined above. This establishes the theorem for $p = 2$. 

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Assuming \( p \) is odd, consider the generating function:

\[
F(x) = \sum_{i=0}^{\infty} \left( \frac{c_i}{d_i} \right) x^i
\]

This generating function is to be thought of as a formal power series with coefficients in the two dimensional vector space over the field \( \mathbb{F}_p \). Thus we are interested in when the coefficient of \( x^n \) is zero. Now the recursion is equivalent to the functional equation:

\[
\left( x^2 - \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} x + I \right) F(x) = x \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)
\]

i.e.

\[
(x - M)(x - M^{-1}) F(x) = x \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)
\]

where \( M = \frac{1}{2} \begin{pmatrix} \mu & a \\ b & \mu \end{pmatrix} \) and \( \mu = \sqrt{ab - 4} \in \mathbb{F}_p^2 \). Thus

\[
F(x) = \frac{x}{(x - M)(x - M^{-1})} \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)
\]

(1)

First consider the case \( ab \equiv 4 \pmod{p} \). In this case \( M = M^{-1}, \) so (1) says

\[
F(x) = \sum_{i=0}^{\infty} i x^i M^{i+1} \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)
\]

Since \( M \) is invertible, the coefficients of \( x^n \) are zero if and only if \( n \) is a multiple of \( p \), and that is what we wanted to show. For all other cases \( M - M^{-1} \) is invertible, so (1) becomes:

\[
F(x) = \sum_{i=0}^{\infty} x^i \left( \frac{M^i - M^{-i}}{M - M^{-1}} \right) \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)
\]

Thus we are interested in \( n \) where

\[
\left( \frac{M^n - M^{-n}}{M - M^{-1}} \right) \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)
\]

i.e.

\[
M^{2n} \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)
\]

Now consider the case when \( p \) divides \( a \) or \( b \) but not both. Assume without loss of generality \( p \nmid b \). Then \( M = \begin{pmatrix} \eta & \frac{a}{2} \\ 0 & \eta \end{pmatrix} \), and \( \eta = \sqrt{-1} \in \mathbb{F}_p^2 \). Hence

\[
M^{2n} = \begin{pmatrix} (-1)^n & (-1)^{n+1} na \eta \\ 0 & (-1)^n \end{pmatrix}
\]

and \( M^{2n} \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \) if and only if \( n \) is a multiple of \( 2p \).
In all remaining cases, \( M \) is diagonalizable over \( \mathbb{F}_p^2 \). Since \( M^2 \) has determinant 1, the only way the equality

\[
M^{2n} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

can hold is that \( M^{2n} \) is the identity matrix. This is equivalent to the condition that the eigenvalues of \( M^2 \) have multiplicative order dividing \( n \). These eigenvalues are exactly the roots of the characteristic polynomial of \( M^2 \), which is given by \( x^2 - (ab - 2)x + 1 \).

From the above theorem, we observe that the Poincaré series if \( A_{\mathbb{F}_p} \) is:

\[
P_t(A_{\mathbb{F}_p}) = 1 + t^{2k} + t^{4k} + \cdots = \frac{1}{1 - t^{2k}},
\]

and both \( \delta_{mk}, \tau_{mk} \) are generators in degree \( 2mk \). It is easier to understand the dual \( A^*_p \mathbb{F} \), which turns out to be a polynomial algebra:

**Claim 10.9** \( A^*_p \mathbb{F} = \mathbb{F}_p[x_{2k}] \) where \( x_{2k} \) is a primitive class in degree \( 2k \).

**Proof** Let \( x \in A^*_p \mathbb{F} \) be any generator in degree \( 2k \). We prove by induction that \( x^n \) is a generator in degree \( 2nk \). Let \( \tau_{nk} \in A_{\mathbb{F}_p} \) be the Schubert basis generator in degree \( 2nk \). We have:

\[
\Delta(\tau_{nk}) = \sum_{i=0}^{n} \lambda_i \tau_{ik} \otimes \tau_{(n-i)k}, \quad \lambda_i \neq 0
\]

Now using the induction hypothesis, we get:

\[
\langle x \cdot x^{n-1}, \tau_{nk} \rangle = \langle x \otimes x^{n-1}, \Delta \tau_{nk} \rangle = \lambda_1 \langle x \otimes x^{n-1}, \tau_k \otimes \tau_{(n-1)k} \rangle \neq 0
\]

Thus \( x^n \) generates \( A^*_p \mathbb{F} \) in degree \( 2nk \). \( \square \)

**Theorem 10.10** There is an isomorphism of algebras:

\[
H_*(K; \mathbb{F}_p) = \Lambda(y_3, y_{2k-1}) \otimes \mathbb{F}_p[x_{2k}]
\]

where the subscripts denote the homogeneous degree of the generators. These generators are related via a higher Bockstein homomorphism: \( \beta^{(m)} x_{2k} = y_{2k-1} \), \( m \) being the exponent of \( p \) in \( g_k \). Moreover, the generators \( y_3, y_{2k-1} \) are primitive, and if \( p \) is odd, then so is the generator \( x_{2k} \).

**Proof** Recall the short exact sequence of Hopf algebras:

\[
1 \rightarrow A_{\mathbb{F}_p} \rightarrow H^*(K; \mathbb{F}_p) \rightarrow \Lambda(x_1, x_2) \rightarrow 1
\]

On dualizing, we get:

\[
1 \rightarrow \Lambda(z_1, z_2) \rightarrow H_*(K; \mathbb{F}_p) \rightarrow \mathbb{F}_p[x_{2k}] \rightarrow 1
\]

Now recall that \( H_*(K; \mathbb{Q}) = \Lambda(z), \) \( |z| = 3 \). This forces \( |z_1| = 3 \), where \( z_1 \) is a permanent cycle in the Bockstein spectral sequence for \( H_*(K; \mathbb{F}_p) \) and \( |z_2| = 2k - 1 \), where \( z_2 \) is the target of a higher Bockstein of height \( m \) supported on \( x_{2k} \). Let us relabel these classes by their subscript and call them \( y_3 \) and \( y_{2k-1} \) respectively. It is clear for dimensional reasons that these classes are primitive. Now since \( \mathbb{F}_p[x_{2k}] \) is a free algebra, we may fix a section to the above short exact sequence. Again, for dimensional reasons, the class \( x_{2k} \) is primitive,

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with the possible exception of the case when \( p = 2, k = 3 \) and the coproduct on \( x_{2k} \) is given by:

\[
\Delta(x_6) = 1 \otimes x_6 + x_6 \otimes 1 + x_3 \otimes x_3.
\]

Finally, to show that \( H^*(K; \mathbb{F}_p) \) is a tensor product of \( \mathbb{F}_p[x_{2k}] \) and \( \Lambda(x_1, x_2) \), it is sufficient to show that \([x_{2k}, y_3] = [x_{2k}, y_{2k-1}] = 0\). This is easy to establish since both the elements: \([x_{2k}, y_3]\) and \([x_{2k}, y_{2k-1}]\) are primitive, but on the other hand, there are no non-zero primitive elements in those degrees.

For the sake of completeness, we include the following theorem whose proof can be found in [1].

**Theorem 10.11** Given an odd prime \( p \), let \( v_p(s) \) denote the exponent of the prime \( p \) dividing \( s \). Let \( g_k \) denote the first integer in the sequence \( \{g_n\} \), so that \( p \) divides \( g_k \). Then we have:

\[
v_p(g_{sk}) = v_p(s) + v_p(g_k).
\]

## 11 The stable transfer from \( BK \) to \( BN(T) \)

It is well known for a compact Lie group \( G \), with maximal torus \( T \) and normalizer \( N \), that the suspension spectrum of \( BG \) is a stable retract of \( B N \). The retraction is constructed as a transfer map, and uses the essential fact that \( G/N \) is a finite complex (with Euler characteristic equal to one). In the case of Kac–Moody groups \( K \), the space \( K/N(T) \) is not even homologically finite, and so there is no (apriori) obvious transfer map. Nevertheless, in this section we will construct a transfer using the homotopy decomposition of the spaces \( BK \) and \( BN(T) \) given in [3,12,13]. The author would like to thank Bill Dwyer for motivating the argument used in this section. The construction of the stable transfer proceeds as follows:

Let \( BK_+ \) and \( BN(T)_+ \) denote the suspension spectra of \( BK_+ \) and \( BN(T)_+ \), respectively each endowed with a disjoint base point. In order to construct a stable transfer map from \( BK_+ \) to \( BN(T)_+ \), first recall [3,12,13] that the following canonical maps are homotopy equivalences:

\[
\hocolim_{J \in S(A)} BN_J(T) \longrightarrow BN(T), \quad \hocolim_{J \in S(A)} BK_J \longrightarrow BK
\]

Let us fix a representation \( \mathcal{V} \) of the Kac–Moody group \( K \) with a countable basis, and the property that given \( J \in S(A) \), every representation of \( K_J \) appears in \( \mathcal{V} \) with infinite multiplicity. Such a representation \( \mathcal{V} \) is easy to construct:

**Lemma 11.1** Let \( \mathcal{V} \) be the \( K \)-representation given by countable sums of all representations of \( K \) of the form \( L_{\mu} \otimes L_\tau \), where \( \mu \) is a dominant weight, \( \tau \) is an anti-dominant weight, and \( L_{\mu} \) (resp. \( L_\tau \)) denote the highest (resp. lowest) weight irreducible representations of \( K \). Then every representation of \( K_J \) appears in \( \mathcal{V} \) for \( J \in S(A) \).

**Proof** Let \( h_i^* \in \mathfrak{h}_Z^* \) denote the dual co-roots. The weight lattice \( \mathfrak{h}_Z^* \) may be written as:

\[
\mathfrak{h}_Z^* = \sum_{i \in I} \mathbb{Z} h_i^* \oplus M,
\]

where \( M \) is the lattice of central weights (i.e. weights that annihilate all co-roots), and has rank given by the co-rank of the generalized Cartan matrix. Let \( C_J \) denote the Weyl chamber.
of $K_J$ in $h^\ast$. The $K_J$-dominant weights: $h^\ast_2 \cap C_J$, classifying all irreducible representations of $K_J$. We have:

$$h^\ast_2 \cap C_J = \sum_{i \in I} \mathbb{N} h^\ast_i \oplus \sum_{k \in I - J} -\mathbb{N} h^\ast_k \oplus M.$$  

Notice that the weights of $K_J$ given by the lattice $\sum_{k \in I - J} \mathbb{Z} h^\ast_k \oplus M$ are both dominant and anti-dominant. Let $L^J_\mu$ denote the irreducible representation of $K_J$ with highest weight $\mu$. Given two dominant weights $\nu, \mu$, the representation $L^J_\mu \otimes L^J_\nu$ is a summand in $L^J_\mu \otimes L^J_\nu$ generated by the tensor product of individual highest weight vectors. Now given a weight $\mu$ in $\sum_{i \in I} \mathbb{N} h^\ast_i$, the highest-weight $K$-representation $L_\mu$ clearly contains $L^J_\mu$ as a summand. Similarly for $\nu$ in $\sum_{k \in I - J} -\mathbb{N} h^\ast_k \oplus M$, the lowest-weight representation of $K$ given by $L_\nu$ contains the $K_J$-representation $L^J_\nu$ (which we recall is an irreducible $K_J$-representation of highest-weight $\nu$ and also lowest-weight $\nu$). Therefore we may tensor representations of the form $L_\mu$ and $L_\nu$ together to obtain representations of $K$ that belong to $\mathcal{V}$, and contain arbitrary representations of $K_J$. This is what we wanted to show. 

Let $\text{Met}$ denote contravariant functor on the category $S(A)$, taking values in spaces given by $\text{Met}(J) = \text{Met}_{K_J}(\mathcal{V})$: the contractible space of $K_J$-invariant metrics on $\mathcal{V}$. The assignment that sends $J$ to $\text{Met}(J)$ is a contravariant functor on $S(A)$. Consider the homotopy inverse limit $\text{lim}_{S(A)} \text{Met}$, of this functor. A standard spectral sequence computes the homotopy groups of $\text{lim}_{S(A)} \text{Met}$ and is given by:

$$E^2_{2,i} = \text{lim}_{S(A)}^{i} \pi_j \text{Met}(J) \Rightarrow \pi_{j-i} \text{lim}_{S(A)} \text{Met}.$$  

Since $\text{Met}(J)$ is contractible for all $J$, we conclude that $\text{lim}_{S(A)} \text{Met}$ is weakly contractible. In particular, it is non-empty. An element in this homotopy inverse limit may be interpreted as a family of metrics parametrized over the simplicial complex $|S(A)|$ (given by the geometric nerve of $S(A)$) and with the property that the metrics over the face corresponding to the sequence of inclusions: $J_0 < \cdots < J_k$ are $K_{J_0}$-invariant. Let us fix one such family, and identify a metric with the corresponding point in $|S(A)|$. This describes $\mathcal{V}$ as a complete universe parametrized over $|S(A)|$. Working with this parametrized universe, let $S^0$ denote the equivariant sphere spectrum. In addition, let $K_J/N_J(T)_+$ denote the suspension spectrum of the $K_J$-space $K_J/N_J(T)_+$, endowed with a disjoint base point.

The subtle part in the construction of the transfer will be to construct a zig–zag of spectra:

$$T : \text{hocolim}_{j \in S(A)} S^0 \leftarrow X \rightarrow \text{hocolim}_{j \in S(A)} K_J/N_J(T)_+.$$  

Each spectrum above will parametrized over the simplicial complex $|S(A)|$. The maps in $T$ will fiber over self maps of $|S(A)|$ which preserves the faces (though not point-wise). By construction, each spectrum above will admit the fiberwise action of the group $K_{J_0}$ over the face of $|S(A)|$ corresponding to the sequence of inclusions $J_0 < \cdots < J_k$. Moreover, $T$ will be equivariant with respect to $K_{J_0}$, over this face. If we let $T(J)$ denote the map over the vertex of $|S(A)|$ given by the object $J \in S(A)$, then our construction will also show that $T(J)$ is equivalent to the standard equivariant splitting of $S^0$ from $K_J/N_J(T)_+$.

Taking homotopy orbits of $T$, and inverting the equivalence in the zig–zag, we get a map which we will define as the stable transfer $\overline{T} : BK_+ \rightarrow BN(T)_+$:

$$\overline{T} : \text{hocolim}_j EK_+ \wedge_{K_J} S^0 \rightarrow \text{hocolim}_j EK_+ \wedge_{K_J} (K_J/N_J(T)_+).$$  

So it remains to actually construct a map $T$ with all the required properties. We begin with some auxiliary constructions. Define $X$ as the parametrized spectrum given by the co-end
construction induced by obvious restrictions:

\[
X = \coprod_{J_0 < \cdots < J_k} \Delta^k \times (\Emb(K_{J_k}/N_{J_k}(T))_+ \wedge S^0)/\sim,
\]

where \( \Emb(K_{J}/N_{J}(T)) \) is the space of \( K_J \)-equivariant embeddings of \( K_J/N_J(T) \) in \( \mathcal{V} \). The space \( \Emb(K_{J}/N_{J}(T)) \) is contractible, and so it is clear that the parametrized projection map from \( X \) to the parametrized spectrum: \( \hocolim_{J \in S(A)} S^0, \) is an equivalence.

We will construct the required face-preserving map from \( X \) to \( \hocolim_{J \in S(A)} K_J/N_J(T)_+ \) as a parametrized family of Pontrjagin–Thom collapse maps induced by equivariant embeddings of \( K_J/N_J(T) \) in \( \mathcal{V} \). The subtlety here is to ensure coherence between the individual collapse maps. To address the coherence problem, we will take advantage of the following two general facts (which are easy to prove, and are left to the reader):

1. Given \( J \in S(A) \), let \( \mathfrak{h}_J \) denote the Lie algebra: \( \mathfrak{h}_J = \{ h \in \mathfrak{h}, \ |\alpha_i(h) = 0, \ j \in J \}. \) Then the infinitesimal action of \( \mathfrak{h}_J \) on \( K_S/N_S(T) \) is \( K_J \)-invariant for any \( J < S \). Moreover, the \( \mathfrak{h}_J \)-fixed set of \( K_S/N_S(T) \) is given by \( K_J/N_J(T) \).

2. Fix elements \( \rho_J \in \mathfrak{h}_J \) with the property \( \alpha_i(\rho_J) = 1 \) for \( i \in J \). Then the simplicial complex \( |S(A)| \) given by the geometric nerve of \( S(A) \) can be canonically identified with an affine subspace of \( \mathfrak{h} \), determined by the property that the vertex of \( |S(A)| \) corresponding to \( J \in S(A) \), maps to the element \( \rho_J \).

Given \( G \subseteq K_J \), let \( x \) be a \( G \) invariant metric on \( \mathcal{V} \). Now given a \( G \)-equivariant embedding \( e : K_J/N_J(T) \to \mathcal{V} \), one constructs the Pontrjagin–Thom map by collapsing the complement of a small enough tubular neighbourhood of \( e \). By using the exponential map to identify this tubular neighborhood with the normal bundle of \( e \), we obtain a \( G \)-equivariant stable map \( S^0 \to K_J/N_J(T)^\eta \), where \( K_J/N_J(T)^\eta \) is the Thom-spectrum of the stable normal bundle. Let \( \tau \) denote the stable tangent bundle of \( K_J/N_J(T) \). Including \( \eta \) into the trivial bundle, we get the \( G \)-equivariant transfer map given by the composite:

\[
T_x(J, e) : S^0 \longrightarrow K_J/N_J(T)^\eta \longrightarrow K_J/N_J(T)^{\eta \oplus \tau} = K_J/N_J(T)_+.
\]

The collection of maps \( T(J, e) \) yield a map from the restriction of \( X \) over the zero-skeleton of \( |S(A)| \) to the corresponding restriction of \( \hocolim_{J \in S(A)} K_J/N_J(T)_+ \). It remains to extend this map to the whole simplicial complex \( |S(A)| \). This is the point where the two properties stated above are crucial. By property (2), we may identify the simplicial complex \( |S(A)| \) with a piecewise-affine subspace of \( \mathfrak{h} \), with the property that the vertex corresponding to \( J_i \) is identified with the element \( \rho_J \). Let \( B|S(A)| \) denote the barycentric subdivision of \( |S(A)| \). Working inductively with the faces, we may define a simplicial map: \( \pi : B|S(A)| \to |S(A)| \), with the property \( \pi(b(\Delta)) = J_k \), where \( b(\Delta) \) denotes the vertex given by the barycenter of a k-dimensional face \( \Delta \) of \( |S(A)| \) corresponding to a sequence of inclusions: \( J_0 < \cdots < J_k \). Similarly, one has a simplicial map \( \lambda : B|S(A)| \to |S(A)| \) with the property \( \lambda(b(\Delta)) = J_0 \).

Given \( \mu \in \Delta \), define \( \iota(\mu) \) as the section of the bundle \( \eta \oplus \tau \to \eta \) generated by the vector field given by the infinitesimal action of \( \mu \in \mathfrak{h} \). Now consider the following face-preserving map \( T \) over the k-simplex \( \Delta \):

\[
T : \Delta \times (\Emb(K_{J_k}/N_{J_k}(T))_+ \wedge S^0) \longrightarrow \hocolim_{J \in S(A)} K_J/N_J(T)_+,
\]

\[
T(x, e) = (\pi(x), \iota(\lambda(x)) \circ T_x(J_k, e)),
\]

where given \( \mu \in \Delta \), the composite \( \iota(\mu) \circ T_x(J_k, e) \) is defined as:

\[
S^0 \longrightarrow K_{J_k}/N_{J_k}(T)^\eta \overset{\iota(\mu)}{\longrightarrow} K_{J_k}/N_{J_k}(T)^{\eta \oplus \tau} = K_{J_k}/N_{J_k}(T)_+.
\]
Notice that if \( J_i \) represents a vertex of \( |S(A)| \), then property (1) above, implies that the vector field on \( K_{J_i}/N_{J_i}(T) \), generated by \( \rho_{J_i} \), vanishes exactly on \( K_{J_i}/N_{J_i}(T) \), and hence the map \( (\rho_{J_i}) \) collapses the complement of a tubular neighborhood of \( K_{J_i}/N_{J_i}(T) \) to the basepoint, thereby showing that the above definition of \( T \) extends \( T(J_i, e) \). Moreover, the above definition of \( T \) is compatible with overlaps of faces, hence \( T \) yields a face-preserving map of equivariant spectra from \( X \) to \( \text{hocolim}_{J \in S(A)} K_{J}/N_{J}(T)_{+} \) over \( |S(A)| \). The upshot of this argument is that we have a face-preserving transfer map of equivariant spectra over \( |S(A)| \) given by a zig–zag:

\[
T : \text{hocolim}_{J \in S(A)} S^0 \xleftarrow{\sim} X \xrightarrow{} \text{hocolim}_{J \in S(A)} K_{J}/N_{J}(T)_{+}.
\]

It is now easy to see that the induced transfer map:

\[
\overline{T} : \text{hocolim}_{J \in S(A)} E_{K_{+}} \wedge K_{J} S^0 \longrightarrow \text{hocolim}_{J \in S(A)} E_{K_{+}} \wedge K_{J} (K_{J}/N_{J}(T)_{+}).
\]

is indeed a stable retraction. This can be established by observing that \( \overline{T} \) induces a map of the respective Bousfield-Kan spectral sequences that compute the stable homotopy of the spectra \( BK_{+} \) and \( BN(T)_{+} \) respectively. The properties of \( \overline{T} \) ensure that this map is a retraction on the \( E_2 \)-term, and hence is a retraction. As a consequence, we have:

**Theorem 11.2** The above map \( \overline{T} \) is a stable retraction of \( BK_{+} \) from \( BN(T)_{+} \). In addition, \( \overline{T} \) is compatible with the stable retractions of each spectra \( BK_{J+} \) from \( BN_{J}(T)_{+} \).

**Remark 11.3** Let \( Z(K) \subseteq T \) denote the center of \( K \). Then the above construction shows that \( \overline{T} \) is equivariant with respect to the action of \( BZ(K) \) on \( BK_{+} \) and \( BN(T)_{+} \). In particular, \( \overline{T} \) descends to stable transfers for central quotients of \( K \).

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**12 Appendix**

In this section we establish some basic facts about the topology of the Kac–Moody groups \( K \), and their classifying spaces \( BK \).

Recall that a subset \( J \subseteq I \) is called spherical if the subgroup \( K_{J} \subseteq K \) is a compact Lie group. The poset of spherical subsets of \( I \) is denoted by \( S(A) \). In [9] (Theorem A) it is shown that as an abstract group, \( K \) is an amalgamated product of subgroups of the form \( K_{J} \), where \( J \in S(A) \) has cardinality at most two. In other words, the following canonical map is an isomorphism:

\[
\text{colim}_{J \in S(A), |J| \leq 2} K_{J} \longrightarrow K,
\]

where the colimit is taken in the category of groups. Now given \( J \in S(A) \) it is easy to see that \( K_{J} \) is generated by the groups \( K_{j} \) for \( j \in J \). Hence, the map above factors through a sequence of two surjective maps:

\[
\text{colim}_{J \in S(A), |J| \leq 2} K_{J} \longrightarrow \text{colim}_{J \in S(A)} K_{J} \xrightarrow{\varphi} K.
\]
As a consequence we see that the map $\varphi$ above is an isomorphism of (abstract) groups.

The next step is to understand the topology on $K$. Let us begin by recalling some constructions from Sect. 8. The reader is referred to [8,15] for details regarding the Bruhat decomposition of $K$ that is used in the arguments that follow. Let $N(T) \subset K$ denote the normalizer of $T$. Given $w \in W$, let $\tilde{w} \in N(T)$ denote any lift of $w$ in $N(T)$. We will denote the space $\mathbb{B} \tilde{w} B \cap K$ by $\tilde{Y}_w$. This is a well defined subspace of $K$ homeomorphic, as a right $T$-space, to $\mathbb{C} \setminus \{0\} \times T$. Now for a generating reflection $r_i$, let $Y_i \subset \tilde{Y}_{r_i}$ be the subspace $\mathbb{C} \times \{1\} \subset \mathbb{C} \times T$ under the above identification. Then the group product in $K$ induces a homeomorphism:

$$\tilde{Y}_w = Y_{i_1} \times \cdots \times Y_{i_s} \times T,$$

where $w = r_{i_1} \cdots r_{i_s}$ is a reduced expression. Furthermore, the closure of $\tilde{Y}_w$ is given by:

$$Z_w := \bigcup_{v \leq w} \tilde{Y}_v.$$

With this structure, $K$ becomes a T-CW complex, constructed by successively attaching $T$-cells. The topology is generated by the closed subspaces $Z_w$. Hence a subspace $Z \subset K$ is closed if and only if $Z \cap Z_w$ is closed for all $w \in W$. Now given $J \in \mathcal{S}(A)$, let $w_0 \in W_J$ denote the longest element. It follows from the closure relation that $Z_{w_0} = K_J$ as compact subspaces of $K$.

Assume now that $H$ is any topological group and that we are given a homomorphism $\varphi : K \to H$ that restricts to a continuous map on each $K_J$ for $J \in \mathcal{S}(A)$. Given an element $w \in W$, let $w = r_{i_1} \cdots r_{i_s}$ be a reduced expression. Notice that $\varphi$ extends to a canonical continuous map $\tilde{\varphi}$ from the product $K_{i_1} \times \cdots \times K_{i_s}$ to $H$ given by the product of the individual restriction maps. Moreover, $\tilde{\varphi}$ factors through the projection map from $K_{i_1} \times \cdots \times K_{i_s}$ onto the subspace $Z_w$. It follows that $\varphi$ restricts to a continuous map on $Z_w$. By the definition of the topology on $K$, we see and that $\varphi$ is in fact a continuous homomorphism. The upshot of the argument given above is that $K$ is in fact the colimit of the groups $K_J$ indexed over the poset $\mathcal{S}(A)$ in the category of Topological Groups. We conclude:

**Theorem 12.1** The topological group $K$ has the following properties:

(a) $K$ is a free T-CW complex of finite type under the right action of $T$. This structure is compatible with the CW structure on the homogeneous space $K/T$.

(b) $K$ is equivalent to the colimit, in the category of topological groups, of the compact Lie groups $K_J$ indexed over the poset $\mathcal{S}(A)$.

**Remark 12.2** Since $K$ is a T-CW complex, it is built by successively attaching $T$-cells. Decomposing $T$ as a CW-complex, we see that $K$ may be constructed by successively attaching (standard) cells. However, it fails to be a CW complex by virtue of the fact that the boundary of cells being attached may glue to cells of higher dimension. We will call a space built by attaching cells in a possibly non-sequential order a Cell Complex (there is some conflict in the literature on the terminology for such an object). Working inductively with the stages, we see that a cell complex is homotopy equivalent to a CW complex.

**Remark 12.3** Principal $K$-bundles that appear in physical applications tend to be defined over base spaces (like solutions of differential equations) that may not have an obvious homotopy type of a CW complex. It is therefore desirable to study the homotopy type of the space $\mathbb{B}K$ that classifies numerable $K$-bundles. This is Milnor’s model [17] of the classifying space of $K$, which is defined as a colimit of certain spaces $\mathbb{B}_nK$ under inclusions $\mathbb{B}_nK \subset \mathbb{B}_{n+1}K$. 

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The space $B_n K$ is defined as the quotient $E_n K/K$, where $E_n K$ is the $n$-fold join of $K$ with itself given the quotient (weak) topology, and the diagonal action of $K$. Hence $E_n K$ can be seen as a quotient of $\Delta^{n-1} \times K \times K$. Using the fact that $K$ is a cell complex of finite type, we see that $B_n K$ has the structure of a cell complex of finite type. It is clear that the inclusions $B_n K \subset B_{n+1} K$ are cellular, and therefore, by the previous remark, $B K$ has the homotopy type of a CW complex.

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