Rank-preserving Multidimensional Mechanisms*

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Abstract

We show that the mechanism-design problem for a monopolist selling multiple, heterogeneous objects to a buyer with ex ante symmetric and additive values is equivalent to the mechanism-design problem for a monopolist selling identical objects to a buyer with decreasing marginal values. We derive three new results for the identical-objects model: (i) a sufficient condition on priors, such that prices in optimal deterministic mechanism are not increasing, (ii) a simplification of incentive constraints for deterministic mechanisms, and (iii) a new condition for revenue monotonicity of stochastic mechanisms. We use the equivalence to establish corresponding results in the heterogeneous-objects model.

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1 Introduction

We study two models for selling multiple, indivisible objects to a buyer: an identical-objects model and a heterogeneous-objects model. The seller chooses an incentive compatible (IC) and individually rational (IR) mechanism with the goal of maximizing expected revenue. The buyer’s type is multidimensional, a vector consisting of (marginal) values for each object. With identical objects, the buyer’s value for $k$ units of the object is the sum of the marginal values of these units. With heterogeneous objects, the buyer’s value for a bundle of objects is the sum of the values of objects in the bundle. The seller knows the distribution of buyer values.

We show that any identical-objects model with decreasing marginal values is equivalent to a heterogeneous-objects model in the following sense. There is a bijective mapping between the set of IC and IR mechanisms in the identical-objects model and the set of symmetric, IC and IR mechanisms in the heterogeneous-objects model. If the distribution of buyer values in the heterogeneous-objects model is exchangeable\(^1\) then (i) the expected revenue of a symmetric, IC and IR mechanism in the heterogeneous-objects model is equal to the expected revenue of its equivalent mechanism in the identical-objects model and (ii) in the heterogeneous-objects model, there exists an optimal mechanism that is symmetric. Hence, the optimal revenues in the two models are equal.

While exchangeability is a strong assumption as it entails a presumption of ex ante symmetric buyer values, it is plausible when the seller is somewhat uninformed about buyer preferences. Moreover, exchangeability is a weaker assumption than i.i.d. distribution of buyer values, which is often assumed in the literature.

With $n$ identical objects, allocation rules induce probability distributions with $n + 1$ outcomes (number of objects) whereas with $n$ heterogeneous objects, allocations rules induce probability distributions with $2^n$ outcomes (bundles of objects). Therefore, as it has a smaller allocation space, the identical-objects model is a more tractable setting than the heterogeneous-objects model for the discovery of new results (as we demonstrate in our applications). These results can be adapted to the exchangeable, heterogeneous-objects model via the equivalence. The equivalence is also useful in adapting known results for

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\(^1\)A mechanism is symmetric if a permutation of the allocation probabilities (of objects) at a buyer type is equal to the allocation probabilities at the same permutation of the buyer type.

\(^2\)A distribution is exchangeable if it is symmetric.
heterogeneous objects to identical objects.

We provide three applications to demonstrate the usefulness of the equivalence. In these applications, we establish novel results in the identical-objects model and use the equivalence to establish new results in the heterogeneous-objects model.

First, we show that in the identical-objects model, optimal prices of units cannot be increasing if the marginal distribution of values satisfy the hazard-rate order. The equivalence implies that in the heterogeneous-objects model with i.i.d. distribution of values, optimal prices in the class of symmetric and deterministic mechanisms cannot be supermodular.

In a second application, we show that in the identical-objects model a weaker notion of incentive compatibility implies full incentive compatibility for deterministic mechanisms. This weaker notion, which we call upper-set IC or UIC, requires incentive constraints only for pairs \((v, v')\) such that \(v \geq v'\) or \(v' \geq v\). Under a mild condition, every deterministic UIC mechanism is IC in the identical-objects model. Therefore, every symmetric, deterministic, rank-preserving, UIC mechanism is IC in the heterogeneous-objects model. This leads to a simplification of the design problem for deterministic mechanisms, which we plan to investigate in future research.

In a third application, we obtain new results on revenue monotonicity. Hart and Reny (2015) show that the optimal revenue need not be monotone in the distribution of values. We obtain a new sufficient condition, majorization monotonicity, for a mechanism to be revenue monotone in the identical-objects model and therefore also in the heterogeneous-objects model with an exchangeable distribution. If an optimal mechanism is symmetric and almost deterministic (i.e., at each outcome in the range of the mechanism, there is randomization over at most one item), then it satisfies majorization monotonicity; consequently, the optimal revenue is monotone.

To establish the equivalence of the two models, it is straightforward to show that any IC and IR mechanism in the identical-objects model can be extended to a symmetric mechanism in the heterogeneous-objects model, while preserving IC and IR. In the other direction, a

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3 We show in an example that despite decreasing marginal values, optimal prices may be increasing in the identical-objects model.

4 This condition, which we call object non-bossiness, requires that if the allocation of object \(i\) is the same for types \((v_i, v_{-i})\) and \((v'_i, v_{-i})\), then the entire allocation vector to these two types is the same. We show that object non-bossiness is generically satisfied by any IC mechanism.
complication is that the restriction of an IC and IR mechanism in the heterogeneous-objects model to the domain of identical objects need not yield a mechanism that is feasible in the identical-objects model. This is because in order to allocate the \((i+1)\)st unit in the identical-objects model, the \(i\)th unit must also be allocated; thus, feasibility requires that the \(i\)th unit is allocated with a (weakly) greater probability than the \((i+1)\)st unit. There is no such feasibility restriction in the heterogeneous-objects model.

The property of rank preserving plays a key role in showing that a symmetric mechanism in the heterogeneous-objects model maps into a feasible mechanism in the identical-objects model. A mechanism is rank preserving if whenever the (buyer’s) value for object \(i\) is greater than the value for object \(j\), the probability that object \(i\) is allocated to the buyer is at least as large as the probability that object \(j\) is allocated.

In the identical-objects model, decreasing marginal values implies that any feasible mechanism is rank preserving. In the heterogeneous-objects case, there exist feasible, IC and IR mechanisms that are not rank preserving; however, if a symmetric mechanism is IC, then we show that it must be rank preserving. This is critical in establishing equivalence.

Under exchangeability, in the heterogeneous-objects model there exists an optimal mechanism that is symmetric. Therefore, it must be rank-preserving and has an equivalent mechanism in the identical-objects model; this equivalent mechanism must be optimal in the identical-objects model.

A general solution to the optimal mechanism-design problem for the sale of multiple objects is unknown. Much of the multidimensional screening literature has focused on the sale of heterogeneous objects (see, for instance, McAfee and McMillan (1988), Manelli and Vincent (2006), Hart and Reny (2015), Carroll (2017)). The few papers on multidimensional screening that focus on the sale of identical objects do so in models with two-dimensional private information. In Malakhov and Vohra (2009) and Devanur et al. (2020), a buyer has the same value for each unit of the object up to a capacity, after which the value for additional units is zero. Both the value and the capacity of a buyer are private information. Dobzinski et al. (2012) consider budget-constrained buyers who have the same value for each unit. They obtain impossibility results regarding achieving efficiency when each buyer’s value and budget are private information. Bikhchandani and Mishra (2022) provide sufficient conditions on priors for an optimal mechanism for the sale of two identical objects to be deterministic and almost deterministic. None of these papers consider \(n \geq 3\) dimensional private information,
which is a challenging setting.

The paper is organized as follows. We present the two models in Section 2 and establish the connection between rank preserving and symmetry in Section 3. The equivalence of the two models is obtained in Section 3.2. The three applications are in Section 4. We end with a discussion in Section 5. All the proofs are in an appendix, including some in an online appendix.

## 2 Two Models of Selling Multiple Objects

In both models, the set of objects is denoted by \( N = \{1, \ldots, n\} \) and the type of the buyer is a vector of valuations \( v := (v_1, \ldots, v_n) \), where each \( v_i \in [\underline{v}, \bar{v}] \), \( 0 \leq \underline{v} < \bar{v} < \infty \). The seller is the mechanism designer.\(^5\)

In the **heterogeneous-objects model**, the type space is

\[
\overline{D}^H := [\underline{v}, \bar{v}]^n
\]

For any type \( v \), \( v_i \) is the buyer’s value for object \( i \) and the value for a bundle of objects \( S \subseteq N \) is additive: \( \sum_{i \in S} v_i \). As the \( n \) objects may be distinct, there is no restriction on values across objects, i.e., both \( v_i > v_j \) or \( v_j < v_i \) are possible. The values \( v_1, \ldots, v_n \) are jointly distributed with cumulative distribution function (cdf) \( F^H \) and density function \( f^H \) with support \( \overline{D}^H \). An alternate interpretation is that there is a unit of mass of buyers distributed with density \( f^H \).

In the **identical-objects model**, all objects are identical and \( v_i \) is the (marginal) value of consuming the \( i^{\text{th}} \) unit of the object. We assume that marginal values are decreasing. Thus, the type space is

\[
\overline{D}^I := \{ v \in [\underline{v}, \bar{v}]^n \mid v_1 \geq v_2 \geq \ldots \geq v_n \}
\]

The values \( v_1, \ldots, v_n \) are jointly distributed with cdf \( F^I \) and density function \( f^I \) with support \( \overline{D}^I \).

We refer to the heterogeneous-objects model as \( \mathcal{M}^H := (N, \overline{D}^H, f^H) \). Similarly, the identical-objects model is denoted by \( \mathcal{M}^I := (N, \overline{D}^I, f^I) \).

\(^5\)We assume that the seller’s cost for selling each object is not more than \( \underline{v} \). 
In either model, a mechanism is an allocation probability vector \( q : D^M \to [0,1]^n \) and a payment \( t : D^M \to \mathbb{R} \), \( M = H \) or \( I \).\(^6\) A buyer with (reported) type \( v \) is allocated object \( i \) with probability \( q_i(v) \), \( i = 1, 2, \ldots, n \) and makes a payment of \( t(v) \). The expected utility of a buyer of type \( v \) from mechanism \((q, t)\) is
\[
u(v) := v \cdot q(v) - t(v)
\]
In the identical-objects model, \( q_i(v) \) denotes the probability of getting the \( i \)th unit of the object, which happens whenever at least \( i \) units are allocated. In other words, the \((i+1)\)st unit can be allocated only if the \( i \)th is also allocated. Thus,
\[
q_i(v) \geq q_{i+1}(v) \quad \forall v \in D^i, \forall i \in \{1, \ldots, n-1\}
\]
is a feasibility requirement in the identical-objects model.\(^7\) There are no such restrictions on the allocation probabilities of a mechanism in the heterogeneous-objects model. In either model, a mechanism \((q, t)\) is deterministic if \( q_i(v) \in \{0,1\} \) for all \( v \) and all \( i \).

A mechanism \((q, t)\) is incentive compatible (IC) if for every \( v, v' \in D^M \), we have
\[
u(v) \geq v \cdot q(v') - t(v') = u(v') + (v - v') \cdot q(v')
\]
A mechanism \((q, t)\) is individually rational (IR) if for every \( v \in D^M \), \( u(v) \geq 0 \). If \((q, t)\) is IC, then it is IR if and only if \( u(v, \ldots, v) \geq 0 \).

We assume that every mechanism \((q, t)\) satisfies \( u(v, \ldots, v) = 0 \). This is without loss of generality as the seller is interested in maximizing expected revenue. Then IC implies that \( 0 = u(v, \ldots, v) \geq v' \left[ \sum_i q_i(v) \right] - t(v) \) or \( t(v) \geq v' \left[ \sum_i q_i(v) \right] \). Note that IR implies \( t(v) \leq v \cdot q(v) \). Since the domain of types is bounded, this implies that for any IC and IR mechanism \((q, t)\), both \( u(v) \) and \( t(v) \) are bounded above and below for every \( v \).

### 3 Symmetric and Rank-preserving Mechanisms

We formally define symmetric and rank-preserving IC mechanisms in model \( M^n \), and show that these properties are closely related.

\(^6\)To simplify notation, we do not attach superscript \( H \) or \( I \) to \( q \) and \( t \) when the model is clear from the context.

\(^7\)If we denote the probability of getting exactly \( k \) units by \( Q_k(v) \), then \( q_i(v) = \sum_{k=i}^{n} Q_k(v) \). This immediately shows that \( q_i(v) = Q_i(v) + q_{i+1}(v) \geq q_{i+1}(v) \).
A type vector \( v \) is \textbf{strict} if \( v_i \neq v_j \) for all \( i, j \in \mathbb{N} \). Let \( D^H \) and \( D^I \) denote the \textbf{set of all strict types} in \( D^H \) and \( D^I \), respectively.  

**Lemma 1** Let \( (q, t) \) be an IC and IR mechanism defined on \( D^M \), \( M = H \) or \( I \). There exists an IC and IR mechanism \( (\bar{q}, \bar{t}) \) defined on \( \overline{D}^M \) such that 
\[
(q(v), t(v)) = (\bar{q}(v), \bar{t}(v)) \quad \forall v \in D^M
\]

Throughout, we assume that the probability distribution of types has a density function. Thus, the set of non-strict types has zero probability. Consequently, the expected revenue from \( (\bar{q}, \bar{t}) \) is the same as the expected revenue from \( (q, t) \). Hence, Lemma 1 allows us to define mechanisms on the set of strict types, i.e., on \( D^M \), and then extend them to \( \overline{D}^M \). This results in a simplification of the proofs.

Let \( \sigma \) represent a permutation of the set \( \mathbb{N} \). The identity permutation is \( \sigma^I := (1, \ldots, n) \). This is a slight abuse of notation as \( I \) also refers to the identical-objects model. The set of all permutations of \( \mathbb{N} \) is denoted by \( \Sigma \). We partition the set of strict types in the heterogeneous-objects model, \( D^H \), using permutations in \( \Sigma \). For any permutation \( \sigma \in \Sigma \), let 
\[
D(\sigma) = \{ v \in D^H : v_{\sigma(1)} > v_{\sigma(2)} > \ldots > v_{\sigma(n)} \}\tag{2}
\]
Note that \( D^H \equiv \bigcup_{\sigma \in \Sigma} D(\sigma) \) and \( D(\sigma) \cap D(\sigma') = \emptyset \) if \( \sigma \neq \sigma' \). Also, \( D^I = D(\sigma^I) \).

Every type in \( D^H \) can be mapped to a type in \( D(\sigma^I) \). To see this, take any \( v \in D^H \). There exists a unique \( \sigma \) such that \( v \in D(\sigma) \subset D^H \). Let \( v^\sigma \) denote the permuted type of \( v \), i.e., \( v_j^\sigma = v_{\sigma(j)} \) for all \( j \in \mathbb{N} \). Eq. (2) implies that \( v^\sigma \in D(\sigma^I) \). Thus, for an arbitrary type \( v \in D^H \) and a permutation \( \sigma \), \( v^\sigma \in D(\sigma^I) \) if and only if \( v \in D(\sigma) \).

We start with a mechanism defined on \( \overline{D}^H \) and assume that it satisfies the properties of symmetry and rank preserving, defined below, on the subset \( D^H \). As \( \overline{D}^I \setminus D^H \) has zero measure, these properties are satisfied for almost all \( v \in \overline{D}^H \).

**Definition 1** In model \( \mathcal{M}^H \), a mechanism \( (q, t) \) is \textbf{symmetric} if for every \( v \in D^H \) and for every \( \sigma \in \Sigma \),
\[
q_i(v^\sigma) = q_{\sigma(i)}(v) \quad \forall i \in \mathbb{N}
\]
\[
t(v^\sigma) = t(v)
\]

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8Thus, \( D^I \) is the largest subset of \( \overline{D}^I \) with strictly decreasing types.
In a symmetric mechanism, the allocation probabilities at a permutation of type \( v \) are the permutation of allocation probabilities at \( v \), while the payment function is invariant to permutations of \( v \).\(^9\) Later, we show in Theorem 2 that in an exchangeable environment, it is without loss of generality to consider symmetric mechanisms.

To construct a symmetric mechanism, it is enough to define the mechanism on \( D(\sigma^i) \), say, and then extend it to \( D^H \) symmetrically (as made precise later in Definition 3 and Lemma 2). The following property plays a crucial role in maintaining incentive compatibility in such symmetric extensions.

**Definition 2** In model \( M^H \), a mechanism \((q, t)\) is **rank preserving** if for every \( v \in D^H \) and every \( i, j \), we have \( q_i(v) \geq q_j(v) \) if \( v_i > v_j \).

In the identical-objects model, any feasible mechanism is rank preserving. To see this, note that for any \( v \in D^I \), we have \( v_i > v_{i+1} \). Moreover, by (1) we have \( q_i(v) \geq q_{i+1}(v) \).

In the heterogeneous-objects model, an IC mechanism need not be rank preserving. For example, a mechanism that allocates some fixed object for zero payment to all types is IC and IR, but not rank preserving. This mechanism is not symmetric. Thus, asymmetric, IC, and IR mechanisms need not be rank preserving.

As shown next, if in the heterogeneous-objects model a symmetric mechanism is IC then it must be rank preserving. Conversely, if a symmetric mechanism is rank preserving and IC on \( D(\sigma^i) \), then it is IC on \( D^H \).

**Theorem 1** Suppose that \((q, t)\) is a symmetric mechanism in \( M^H \). Then, the following are equivalent:

(i) \((q, t)\) is IC on \( D^H \).

(ii) \((q, t)\) is rank preserving and \((q, t)\) restricted to \( D(\sigma^i) \) is IC.

As illustrated with an example earlier, asymmetric mechanisms need not be rank preserving. This shows that symmetry is essential for the direction (i) \(\implies\) (ii) in Theorem 1.

\(^9\)We do not impose symmetry on non-strict types as it implies additional restrictions. To see this, consider \( n = 4 \) and suppose that \( v \equiv (1, 0, 0, 1) \) and \( v' \equiv (0, 1, 1, 0) \). The type \( v' \) can be obtained from \( v \) via two permutations: \( \sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 4, \sigma(4) = 3 \) and \( \sigma'(1) = 3, \sigma'(2) = 4, \sigma'(3) = 1, \sigma'(4) = 2 \). Hence, \( v' = v^\sigma = v^\sigma' \). Applying symmetry here would mean \( q_i(v^\sigma) = q_{\sigma(i)}(v) = q_{i}(v^{\sigma'}) = q_{\sigma'(i)}(v) \) for all \( i \) which implies \( q_2(v) = q_3(v) \) and \( q_1(v) = q_4(v) \). Such restrictions limit the set of mechanisms that can be considered.
Symmetry is also essential for the direction \((ii) \implies (i)\). To see this, consider two objects and type space \(D^n\) consisting of strict types in \([0, 1]^2\). A mechanism sells object 1 for free in the lower triangle \(D(\sigma^1) := \{(v_1, v_2) : v_1 > v_2\}\) and does not sell any objects in the upper triangle \(D(\sigma^2) := \{(v_1, v_2) : v_1 < v_2\}\). This is a rank-preserving mechanism which satisfies IC constraints in each \(D(\sigma^i)\). But it is clearly not an IC mechanism for \(D^n\). Notice that this mechanism is not symmetric.

The following example shows that rank preserving is also essential for Theorem 1.

**Example 1** Consider two objects with the buyer’s valuation \(v = (v_1, v_2)\) distributed on the unit square. Let \(t(\cdot) \equiv 0\) and

\[
q(v_1, v_2) = \begin{cases} 
(0, 1), & \text{if } v_1 > v_2 \\
(1, 0), & \text{if } v_1 < v_2 
\end{cases}
\]

This mechanism is symmetric but not rank preserving. Restricted to \(D(\sigma^1)\), this mechanism is IC. It is also IC when restricted to \(D(\sigma^2)\). But the mechanism is not IC on \(D^n\) as any type in \(D(\sigma^1)\) benefits by reporting a type in \(D(\sigma^2)\) and vice versa.

Is every rank preserving and IC mechanism symmetric? The answer is no as the following example illustrates.

![Figure 1: A rank-preserving IC mechanism that is not symmetric](image)

**Example 2** Suppose \(n = 2\) and the type space is \([0, 1]^2\). Figure 1 describes a deterministic mechanism \((q, t)\) for this type space. This mechanism is clearly IC and rank preserving. But it is not symmetric.
We show next that if the distribution of values is exchangeable, then there exists an optimal mechanism which is rank preserving.

3.1 Existence of a Rank-preserving Optimal Mechanism

The expected revenue from an IC and IR mechanism in model $\mathcal{M}^M$, $M = H$ or $I$, is

$$\text{Rev}(q, t; f^M) := \int_{D^M} t(v) f^M(v) dv = \int_{D^M} t(v) f^M(v) dv$$

Further,

$$\text{REV}(q, t; f^H) = \sum_{\sigma \in \Sigma} \text{REV}^\sigma(q, t; f^H)$$

where

$$\text{REV}^\sigma(q, t; f^H) := \int_{D(\sigma)} t(v) f^H(v) dv$$

(3)

We sometimes write $\text{REV}(q, t; F^M)$ instead of $\text{REV}(q, t; f^M)$.

A mechanism $(q^*, t^*)$ is optimal for density function $f^H$ if it is IC and IR and for any other IC and IR mechanism $(q, t)$

$$\text{REV}(q^*, t^*; f^H) \geq \text{REV}(q, t; f^H)$$

In model $\mathcal{M}^H$, the joint density of values, $f^H$, is exchangeable$^{10}$ if

$$f^H(v) = f^H(v^{\sigma}) \quad \forall \ v \in D^H, \ \forall \sigma \in \Sigma$$

Exchangeability is satisfied if $v_1, v_2, \ldots, v_n$ are i.i.d. Exchangeable random variables may be positively correlated such as when $v_1, v_2, \ldots, v_n$ are distributed i.i.d. conditional on an underlying state variable.$^{11}$

As shown next, an exchangeable distribution of buyer types in model $\mathcal{M}^H$ allows one to restrict attention to mechanisms that are symmetric and, therefore, also rank preserving.

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$^{10}$Strictly speaking, the random variables $v_1, v_2, \ldots, v_n$ are exchangeable.

$^{11}$As an example, consider an entity that sells “permits” for operating in $n$ markets that are ex-ante identical. The seller might be a local government that issues licenses for liquor stores or a franchisor introducing its product in a new market via franchises. The buyer is knowledgeable about market conditions in the $n$ markets. The value of market $i$ to the buyer is $v_i = \eta m_i$, where $\eta$ is the buyer’s efficiency level and $m_i$ is the size of market $i$. The buyer knows $\eta$ and $m_i$. The seller has a distribution over $\eta$ and has i.i.d. distributions over $m_i$. The random variables $v_i$ are exchangeable from the seller’s perspective.
Theorem 2 Suppose that $f^H$ in model $\mathcal{M}^H$ is exchangeable. Then, there exists an optimal mechanism which is symmetric and rank preserving.

In an exchangeable environment with two-dimensional types, Pavlov (2011a) notes (in the proof of his Corollary 1) that an optimal mechanism that is symmetric must exist.\textsuperscript{12} For the sake of completeness, we provide a proof for $n$-dimensional types in an online appendix (Appendix B). The proof uses the fact that in the heterogeneous-objects model the simple average of all permutations of an asymmetric, IC and IR mechanism is a symmetric, IC and IR mechanism. Consequently, linearity of the revenue functional and exchangeability imply that for every asymmetric mechanism there exists a symmetric mechanism with the same expected revenue. Thus, in the heterogeneous-objects model, there exists an optimal mechanism that is symmetric and, by Theorem 1(i), rank-preserving; its equivalent mechanism in the identical-objects model is optimal in that model.

3.2 An Equivalence Between the Two Models

Theorem 1 allows us to establish a formal equivalence between the identical objects and the heterogeneous-objects models. A mechanism defined on $D(\sigma^I)$ may be extended symmetrically to $D^H$ using the definition below. Recall that for every $v \in D(\sigma)$, we have $v^\sigma \in D(\sigma^I)$.

Definition 3 Let $(q, t)$ be a mechanism defined on $D(\sigma^I)$ (equivalently on $D^I$). The symmetric extension of $(q, t)$ is a mechanism $(q^s, t^s)$ on $D^H$ such that for every $v \in D(\sigma)$ and for every $\sigma \in \Sigma$

\[ q^s_{\sigma(i)}(v) = q_i(v^\sigma) \quad \forall i \]

\[ t^s(v) = t(v^\sigma) \]

A mechanism defined on $D(\sigma)$, where $\sigma \neq \sigma^I$, may also be extended symmetrically using Definition 3 after first relabeling the axes. A mechanism $(q, t)$ on $D^I$ for model $\mathcal{M}^I$ is rank preserving by definition. But an arbitrary mechanism $(q, t)$ defined on $D(\sigma^I) \equiv D^I$ need not be rank preserving.\textsuperscript{13}

\textsuperscript{12}Maskin and Riley (1984) make a similar observation in a single-object auction setting with ex ante symmetric bidders and one-dimensional types.

\textsuperscript{13}Theorem 1 implies that the symmetric extension of a non-rank-preserving mechanism on $D^I$ will not be IC on $D^H$. Example 1 illustrates this.
Lemma 2 Let \((q, t)\) defined on \(D(\sigma)\) be a rank-preserving, IC and IR mechanism [on \(D(\sigma)\)]. Then the symmetric extension of \((q, t)\) to \(D^H\) is a rank-preserving, IC and IR mechanism.

This leads to an equivalence between identical objects and heterogeneous-objects models:

**Proposition 1**

(i) Any IC and IR mechanism in model \(M^i\) can be extended to a symmetric, IC, and IR mechanism in model \(M^H\).

(ii) The restriction of any symmetric, IC, and IR mechanism in model \(M^H\) to \(D(\sigma^i)\) defines an IC and IR mechanism in model \(M^i\).

(iii) If the density \(f^H\) is exchangeable with \(f^i(v) = n! f^H(v) \forall v \in \mathcal{D}^i\) then optimal mechanisms in models \(M^i\) and \(M^H\) generate the same expected revenue.

In general, the optimization problem for a seller of \(n\) heterogeneous objects is quite different from the optimization problem for a seller of \(n\) identical units of an object. Proposition 1 implies that if the density \(f^H\) in \(M^H\) is exchangeable then the seller’s problem in these two settings is essentially the same.

The equivalence between models \(M^H\) and \(M^i\) relies on the decreasing marginal values assumption in the identical-objects model. Indeed, if marginal values are increasing, then \(v_i \leq v_{i+1}\) for all \(v\) whereas, feasibility of a mechanism \((q, t)\) requires that \(q_i(v) \geq q_{i+1}(v)\). Thus, a feasible mechanism violates the rank-preserving property under increasing marginal values in the identical-objects model. Hence, Theorem 1 does not hold.

### 4 Applications

We provide three applications of the equivalence result for the sale of indivisible objects. These applications establish new results in the identical-objects model, and then extend them to the heterogeneous-objects model using Theorem 1. The applications require a new condition we call object non-bossiness, which is defined next.⁴

⁴Non-bossiness is assumed in the hypothesis of Theorem 4 and is used in the proof of Theorem 5.
Definition 4 A mechanism \( (q, t) \) satisfies object non-bossiness if for all \( i \), for all \( v_{-i} \), and for all \( v_i, v_i' \)

\[
q_i(v_i, v_{-i}) = q_i(v_i', v_{-i}) \implies q_j(v_i, v_{-i}) = q_j(v_i', v_{-i}) \quad \forall \ j \in N
\]

In an object non-bossy mechanism, if the allocation probability of the \( i \)th unit remains the same at types \( (v_i, v_{-i}) \) and \( (v_i', v_{-i}) \) then the allocation of every unit must remain the same at \( (v_i, v_{-i}) \) and \( (v_i', v_{-i}) \).\(^{15}\)

Proposition 2 below shows that for any IC mechanism there exists a non-bossy and IC mechanism which is identical to the original mechanism almost everywhere. Thus, an assumption of object non-bossiness is without loss of generality in our environment. In the applications, we assume that two specific classes of mechanisms are non-bossy: deterministic and almost deterministic. Formally, an allocation rule \( q \) is almost deterministic if for every \( v \) there exists \( k \in \{1, \ldots, n\} \) such that \( q_i(v) \in \{0, 1\} \) for all \( i \neq k \). A mechanism \( (q, t) \) is (almost) deterministic if \( q \) is (almost) deterministic. Proposition 2 shows that if an IC mechanism is (almost) deterministic then its non-bossy version is also (almost) deterministic. We prove the proposition for an arbitrary type space \( D^* \) with decreasing marginal values, which can be finite or infinite. In particular, \( D^* \) can be \( D^I \) or \( \overline{D^I} \).

Proposition 2 Suppose \( (q, t) \) is an IC and IR mechanism in the identical-objects model defined on an arbitrary type space \( D^* \) with decreasing marginal values. Then, there exists an IC, IR, and non-bossy mechanism \( (q^\sharp, t^\sharp) \) such that:

(i) \( t^\sharp(v) \geq t(v) \) and \( u^\sharp(v) = u(v) \) for all \( v \in D^* \).

(ii) If \( D^* \) is convex, then \( (q^\sharp(v), t^\sharp(v)) = (q(v), t(v)) \) for almost all \( v \in D^* \).

(iii) If \( (q, t) \) is deterministic (almost deterministic), then \( (q^\sharp, t^\sharp) \) can be chosen to be deterministic (almost deterministic).

Hence, for any type space \( D^* \), there is an optimal deterministic mechanism which is non-bossy. Note that no assumptions about the distribution of values are made in Proposition 2. While the proposition can be proved for (asymmetric mechanisms in) the heterogeneous-object models, the restriction to identical objects is sufficient for our results.

\(^{15}\)The idea is similar to agent non-bossiness introduced by Satterthwaite and Sonnenschein (1981).
The proof relies on making a non-bossy selection \((q^\ast, t^\ast)\) from the set of seller-favorable mechanisms (as defined in Hart and Reny (2015)) that are equal to \((q, t)\) almost everywhere.\(^{16}\) We illustrate this for a deterministic, and possibly bossy, mechanism next.

Let \((q, t)\) be a deterministic, IC mechanism in the identical-objects model with type space \(D \in \{D^I, \overline{D}^I\}\). The buyer’s utility function from this mechanism, \(u\), is convex and \(\partial u(v)\), the set of subgradients of \(u\) at a type \(v\), is generically a singleton and contains \(q(v)\). For each \(v\), let \(\partial^I u(v)\) be the subset of \(\partial u(v)\) that contains integer vectors only. Then, at any type \(v\), \(\partial^I u(v)\) is finite (generically a singleton) and contains \(q(v)\) (since \(q\) is deterministic).

Note that any \(x \in \partial^I u(v)\) must be of the form \(x = (1, \ldots, 1, \underbrace{0, \ldots, 0}_n)\), where \(0 \leq k \leq n\). Therefore, at each \(v\) there exists a unique largest element in \(\partial^I u(v)\) because either \(\partial^I u(v)\) is a singleton or if \(x, x' \in \partial^I u(v), x \neq x'\), then either \(x < x'\) or \(x' > x\). Let this largest element be \(q^I(v)\). Define \(t^I(v) := v \cdot q^I(v) - u(v)\) for all \(v\).

Since \((q^I, t^I)\) assigns an allocation from the subgradient correspondence \(\partial^I u(v)\) and \(u^I \equiv u\), it is a deterministic IC mechanism. Further, \((q^I, t^I)\) coincides with \((q, t)\) almost everywhere, except at \(v\) where \(\partial^I u(v)\) is not a singleton – at these points \(t^I(v) \geq t(v)\) as \(q^I(v) \geq q(v)\). To see that \((q^I, t^I)\) is object non-bossy, pick \((v_i, v_{-i}), (v'_i, v_{-i})\) with \(v_i > v'_i\) such that \(q^I(v_i, v_{-i}) = q^I(v'_i, v_{-i})\). It may be verified that \(q^I(v'_i, v_{-i}) \in \partial^I u(v)\) and \(q^I(v_i) \in \partial^I u(v)\). Since \(q^I(v)\) is the unique largest element of \(\partial^I u(v)\), and \(q^I(v'_i, v_{-i})\) is the unique largest element of \(\partial^I u(v'_i, v_{-i})\), it must be that \(q^I(v'_i, v_{-i}) = q^I(v_i)\), establishing object non-bossiness.

### 4.1 Pricing Mechanisms

We obtain results on optimal mechanisms in the class of deterministic and symmetric mechanisms. Theorem 3 below provides a sufficient condition on the probability distribution of buyer values under which optimal prices in the identical-objects model cannot be increasing. This implies that the optimal prices in the heterogeneous-objects model with i.i.d. buyer values cannot be supermodular (Corollary 1).

A first step in the proof is to show that any deterministic mechanism can be expressed as a pricing mechanism, which is defined next.

\(^{16}\)While \((q^I, t^I)\) in Proposition 2 is seller favorable, in general non-bossy mechanisms need not be seller favorable.
Definition 5 A deterministic mechanism \((q, t)\) for the identical-objects model defined on the type space \(D^i\) is a **pricing mechanism** if there exists prices \(p_0 = 0, p_1, \ldots, p_n \in [0, \overline{v}]\), such that for all \(v \in D^i\),
\[
t(v) = \sum_{k=0}^{k(v)} p_k
\]
where \(k(v) := \sum_{i=1}^{n} q_i(v)\).

Suppose that \((q, t)\) is a deterministic, IC and IR mechanism with **full range**, i.e., for each \(k = 0, 1, \ldots, n\) there exists a type \(v^k\) that is allocated \(k\) units. Then the pricing mechanism implementation of \((q, t)\) has prices \(p_k = t(v^k) - t(v^{k-1})\). That pricing mechanism implementations exist for deterministic mechanisms without full range is shown next.

**Proposition 3** If \((q, t)\) is a deterministic, IC, and IR mechanism in model \(M^i\), then it is a pricing mechanism with prices \(p_0 = 0, p_1, \ldots, p_n \in [0, \overline{v}]\) such that
\[
u(v) \geq \sum_{i=0}^{k} v_i - \sum_{i=0}^{k} p_i \quad \forall \ v \in D^i \ \forall \ k \in \{0, 1, \ldots, n\}
\] (4)
where we use \(v_0 = 0\).

Proposition 3 implies that even if an IC and IR mechanism does not have full range, it can be implemented by defining prices for every unit and letting the buyer choose the payoff maximizing units. Hart and Reny (2015) show a similar result for the heterogeneous-objects model with an unbounded type space. Due to unbounded type space, prices of objects in their model can be infinite. In contrast, the prices in Definition 5 are bounded above by \(\overline{v}\). The proof of Proposition 3 is in Appendix B.

Since marginal values are decreasing, it is tempting to think that prices are decreasing in an optimal pricing mechanism. As shown next, this intuition is incorrect.

**Example 3** To see this, consider an example with two units. The type space consists of the blue line segments \(AB\) and \(CD\) in Figure 2. The values of the buyer are on \(AB\) with probability \(\frac{1}{2}\) and on \(CD\) with probability \(\frac{1}{2}\). Further, conditional on values belonging to \(AB\) or \(CD\), the values are distributed uniformly on each line segment. It may be verified that the optimal prices are \(p_1 = \frac{1}{2}\) and \(p_2 = \frac{3}{4}\); types on \(AB\) buy one unit and types on \(CD\) buy both the units, resulting in an expected revenue of \(\frac{7}{8}\).\(^{17}\)

\(^{17}\)The other two candidates for optimal prices, \((\frac{1}{2}, \frac{1}{2})\) and \((\frac{3}{4}, \frac{3}{4})\), each yield an expected revenue of \(\frac{3}{4}\).
Thus, optimal mechanisms need not have decreasing prices. Next, we give a simple sufficient condition under which the optimal mechanism does not have increasing prices (which ensures decreasing prices for two units).

For probability distribution $F$ over buyer values in the identical-objects model, let $F_k$ denote the marginal distribution of the value for the $k$-th unit. Theorem 3 establishes that if $F_k$ hazard-rate dominates $F_{k+1}$ for each $k$,\(^\text{18}\) then prices cannot be increasing in an optimal pricing mechanism. The proof does not require the existence of a probability density function over values.

**Theorem 3** Suppose $F$ is a probability distribution over buyer values in model $\mathcal{M}^t$ such that $F_k$ hazard-rate dominates $F_{k+1}$ for all $k = 1, \ldots, n - 1$. Then there is no deterministic optimal mechanism with respect to $F$ with prices $p_0, p_1, \ldots, p_n$ such that

$$p_1 \leq p_2 \leq \ldots \leq p_n$$

with at least one strict inequality.

\(^\text{18}\)That is, $\frac{1 - F_k(t)}{1 - F_{k+1}(t)}$ is increasing in $t$.
v^\sigma \in D^I. By symmetry, \( t^H(v) = t^H(v^\sigma) \) and \( q^H(v^\sigma) \) allocates the permuted bundles \( S^\sigma \). But \( q^H(v^\sigma) = q^I(v^\sigma) \) and since \((q^I, t^I)\) is a pricing mechanism, we have

\[
t^H(v^\sigma) = t^I(v^\sigma) = \sum_{j=1}^{\left|S^\sigma\right|} p_j = \sum_{j=1}^{\left|S\right|} p_j = t^H(v)
\]

Hence, the prices of a symmetric, deterministic, IC and IR mechanism in model \( \mathcal{M}^H \) can be described by prices \( p_0, p_1, \ldots, p_n \). For notational convenience, let \( P(S) \equiv \sum_{j=1}^{\left|S\right|} p_j \) be the price of bundle \( S \).

**Definition 6** A deterministic mechanism \((q^H, t^H)\) with prices \( \{P(S)\}_S \) in model \( \mathcal{M}^H \) is supermodular if

\[
P(S \cup \{k\}) - P(S) \leq P(T \cup \{k\}) - P(T), \quad \forall S \subsetneq T \subseteq N, \; \forall k \notin T
\]

A mechanism is strictly supermodular if at least one of the above inequalities is strict.

Supermodularity is well defined for deterministic, symmetric mechanisms as \( P(S) := \sum_{j \in S} p_j \) is defined for every bundle \( S \subseteq N \), even if no type is allocated \( S \). Note that supermodularity is equivalent to \( p_0 = 0 \leq p_1 \leq \ldots \leq p_n \) and strict supermodularity is equivalent to at least requiring one of these inequalities to be strict.

A corollary of Theorem 3 is that if values of objects are distributed independently and identically in the heterogeneous-objects model, then the prices in the optimal mechanism cannot be supermodular.

**Corollary 1** Consider a heterogeneous-objects model \( \mathcal{M}^H \) in which the values of the objects are distributed i.i.d. Then there is no optimal, deterministic, symmetric mechanism which is strictly supermodular.

Babaioff et al. (2018) showed that even if \( F \) is i.i.d. in model \( \mathcal{M}^H \), then the optimal deterministic mechanism need not be symmetric when there are three or more objects for sale. Identifying conditions on i.i.d. \( F \) under which the optimal, deterministic mechanism is symmetric is an open question.
4.2 Upper-set Incentive Compatibility

Next, we ask if a subset of IC constraints imply full incentive compatibility. We establish that only upper-set incentive constraints (i.e., incentive constraints between \( v \) and \( v' \) whenever \( v \geq v' \) or \( v' \geq v \)) are sufficient to imply all IC constraints in deterministic and non-bossy mechanisms in the identical-objects model. Theorem 1 implies that this relaxation of the set of IC constraints applies to symmetric, rank-preserving and deterministic mechanisms in the heterogeneous-objects model (whether or not the distribution of types is exchangeable).

Theorem 4 below, which shows that upper-set IC constraints are sufficient for IC for deterministic mechanisms in model \( M^i \), applies to a broad class of type spaces which include \( D^i \) and \( D^i \). Let \( D \) be an arbitrary type space in the identical-objects model. Denote the IC constraint of a mechanism where type \( v \) does not gain by misreporting type \( v' \) as \( v \rightarrow v' \). Denote \( v \rightarrow v' \) and \( v' \rightarrow v \) by \( v \leftrightarrow v' \). For every \( v \in D \), the upper set of \( v \) is defined as

\[
T(v) = \{ v' \in D : v'_i \geq v_i \ \forall \ i \in N \}
\]

**Definition 7** A mechanism \((q, t)\) for the identical-objects model defined on \( D \) is upper-set incentive compatible (UIC) if for every \( v \in D \) and every \( v' \in T(v) \), the IC constraints \( v \leftrightarrow v' \) hold.\(^{19}\)

Consider the following assumption on the domain of types \( D \):

**Definition 8** A domain of buyer types \( D \) satisfies the strong-lattice property if for every \( v, v' \in D \), for every \( k \in \{1, \ldots, n\} \) the types \( \tilde{v} \) and \( \tilde{v} \), defined below, belong to \( D \):

\[
\tilde{v}_i = \begin{cases} 
v_i & \text{if } i < k \\
\min(v_i, v'_i) & \text{if } i \geq k 
\end{cases}
\]

(5)

\[
\tilde{v}_i = \begin{cases} 
\max(v_i, v'_i) & \text{if } i \leq k \\
v_i & \text{if } i > k 
\end{cases}
\]

(6)

\(^{19}\)Though a deterministic, IC and IR mechanism is a pricing mechanism (by Proposition 3), a deterministic UIC and IR mechanism need not be a pricing mechanism unless it is IC. Example 3 in Appendix B gives a UIC mechanism in which the price for the same allocation is type dependent; such a mechanism cannot be a pricing mechanism.
Note that if in Definition 8, (5) is applied for $k = 1$ only and (6) for $k = n$ only, we get the standard definition of the lattice property.

The strong-lattice property is equivalent to the lattice property under an assumption on the richness of domain of types defined next.

**Definition 9** A domain of buyer types $D$ is **rich** if for every $v, v' \in D$ such that $v_k \geq v'_{k+1}$, the type $v'' := (v_1, \ldots, v_k, v'_{k+1}, \ldots, v'_n) \in D$.

**Lemma 3** A rich domain of types satisfies the strong-lattice property if and only if it satisfies the lattice property.

As $\overline{D}^i$ and $D^i$ are rich and satisfy the lattice property, Lemma 3 implies that these domains satisfy the strong-lattice property.

**Theorem 4** Consider an identical-objects model in which the domain of buyer types satisfies the strong-lattice property. In this setting, every deterministic, object non-bossy, and upper-set incentive compatible mechanism is incentive compatible.

This theorem does not apply to random mechanisms or to bossy deterministic mechanisms or when the domain does not satisfy the strong-lattice property. This is shown in Example 4 in Appendix B.

**Remark 1** We contrast the sufficiency of UIC for IC from previous work that has shown the sufficiency of a reduced set of incentive constraints in multidimensional environments. A mechanism satisfies local IC constraints if for every $v$, there exists an $\epsilon > 0$ such that for all $v'$ in an $\epsilon$-ball around $v$, the IC constraints $v \rightarrow v'$ and $v' \rightarrow v$ hold. Carroll (2012) shows that local IC constraints imply all IC constraints in convex type spaces; this holds for random mechanisms also. This result has been extended to non-convex type spaces by Mishra et al. (2016) and Kumar and Roy (2021). UIC does not correspond to any notion of locality, and hence, the set of redundant IC constraints identified by UIC is quite different.\footnote{In fact, we can use Carroll (2012) to strengthen our result. The upper set of any type $v$ is a convex set if $D$ is convex. Hence, satisfying all IC constraints in the upper set of $v$ is equivalent to satisfying all local IC constraints in the upper set of $v$ provided $D$ is convex. Similarly, if $v \geq v' \geq v''$, and IC constraints $v \leftrightarrow v', v' \leftrightarrow v''$ hold, then a straightforward argument shows that the IC constraint $v \leftrightarrow v''$ holds. Thus, it is possible to define a (weaker) local notion of upper set IC, which implies IC.}
Remark 2 In mechanism-design problems, whenever a subset of IC constraints are dropped, the set of mechanisms satisfying the smaller set of IC constraints is usually larger. This is called a relaxed program. The aim is then to figure out conditions on primitives of the problem when the optimal of the relaxed program is also the optimal of the original problem (with all the IC constraints). Armstrong (2000) uses this approach in an auction with binary types and Bayesian IC constraints.

In contrast, we show that UIC constraints imply all IC constraints for any deterministic and object non-bossy mechanism. That is, the set of deterministic and object non-bossy mechanisms satisfying UIC constraints is not larger in the relaxed program. Thus, for any optimization program in this setting involving IC constraints (be it revenue maximization or some other objective function), it is without loss of generality to consider a strictly smaller set of IC constraints, the set of UIC constraints.

Theorem 4 applies to finite type spaces that satisfy the strong lattice property (for instance, an appropriately defined grid). In such type spaces, the optimal deterministic mechanism is a solution to a finite-dimensional integer program. Theorem 4 reduces the complexity of its constraints, and the number of variables in the dual program. This may be useful for computational purposes, especially since no closed form solution of the optimal deterministic mechanism is known.

Next, we derive an analog of Theorem 4 for model $\mathcal{M}^H$. As we consider only symmetric mechanisms, it is sufficient to define UIC on $D(\sigma^i)$.

Definition 10 A symmetric and rank-preserving mechanism $(q, t)$ for the heterogeneous-objects model defined on type space $D^H$ is upper-set incentive compatible (UIC) if for every $v \in D(\sigma^i)$ and for every $v' \in T(v) \cap D(\sigma^i)$, the IC constraints $v \leftrightarrow v'$ hold.

A symmetric mechanism $(q, t)$ in the heterogeneous-objects model is object non-bossy if $q_i(v_i, v_{-i}) = q_i(v'_i, v_{-i})$ implies $q(v_i, v_{-i}) = q(v'_i, v_{-i})$ for every $(v_i, v_{-i}), (v'_i, v_{-i}) \in D(\sigma^i)$. As we only consider symmetric mechanisms in the heterogeneous-objects model, this definition of object non-bossiness suffices.

Theorem 4 and Theorem 1 imply the following:

Corollary 2 Every symmetric, deterministic, object non-bossy, rank-preserving and upper-set incentive compatible mechanism in a heterogeneous-objects model defined on $D^H$ is incentive compatible.
Since the definition of symmetry only applies to strict types, Corollary 2 does not have a counterpart for $\overline{D}^H$. However, if $(q, t)$ is a symmetric, rank-preserving, deterministic, object non-bossy, UIC mechanism defined on $\overline{D}^H$, then its restriction to $D^H$ is IC by Corollary 2. Using Lemma 1, there exists another IC mechanism $(q', t')$ defined on $\overline{D}^H$ that coincides with $(q, t)$ almost everywhere, and hence, generates the same expected revenue as $(q, t)$.

In the heterogeneous-objects model, IC and symmetric mechanisms are rank preserving (by Theorem 1). However, UIC and symmetric mechanisms need not be rank preserving, and hence we need to assume rank preserving in Corollary 2. Note that the assumption of exchangeability is not required for Corollary 2 as incentive compatibility does not depend on the probability distribution of values.

### 4.3 Revenue Monotonicity

The optimal revenue from the sale of $n$ objects is monotone if the optimal revenue increases when the distribution of the buyer’s values increases in the sense of first-order stochastic dominance. Monotonicity of the optimal revenue is a desirable property as it provides an incentive for the seller to improve her products. It is satisfied in the optimal mechanism for the sale of a single object. However, as Hart and Reny (2015) show, optimal revenue may not be monotone in the heterogeneous objects model. They also show that if the optimal mechanism is symmetric and deterministic or if the optimal payment function is submodular, then the optimal revenue is monotone in the heterogeneous-objects model.\(^{21}\)

We provide other sufficient conditions that guarantee that the expected revenue from the mechanism is monotone in the identical-objects model and the heterogeneous-objects model.

Consider the following definition.

**Definition 11** A mechanism $(q, t)$ is **revenue monotone** if for every cdf $F$ and every cdf $\tilde{F}$, where $\tilde{F}$ first-order stochastic dominates $F$, we have

$$\text{Rev}(q, t; \tilde{F}) \geq \text{Rev}(q, t; F)$$

The definition applies to models $\mathcal{M}^H$ and $\mathcal{M}^I$, where either $F$ and $\tilde{F}$ both have support in $\overline{D}^H$ or both have support in $\overline{D}^I$.

\(^{21}\)In a recent paper, Ben Moshe et al. (2022) show that a restriction to monotone mechanisms can severely reduce expected revenue for some classes of distributions.
If a mechanism \((q, t)\) satisfies\(^{22}\)

\[
t(\hat{v}) \geq t(v) \quad \forall \hat{v} > v
\]

then it satisfies revenue monotonicity, as its expected revenue under a cdf \(\widehat{F}\) is greater than equal to its expected revenue under a first-order stochastically-dominated cdf \(F\).\(^{23}\) Thus, if an optimal mechanism satisfies (7) then it is revenue monotone.

Fix an IC mechanism \((q, t)\) and a pair of types \(v, v'\). Are there sufficient conditions on \(q(v)\) and \(q(v')\) that imply \(t(v) \geq t(v')\)? We show that one such condition takes the form of majorization in model \(\mathcal{M}^I\). We use this to derive new sufficient conditions for revenue monotonicity in both the models.

For any allocation probability vector \(q = (q_1, q_2, \ldots, q_n)\), let \(q_i\) be the \(i\)th highest element of \(q\). That is, \(q_1 \geq q_2 \geq \ldots \geq q_n\).\(^{24}\) If, for two allocation probability vectors \(\hat{q}, q\),

\[
\sum_{i=1}^{j} \hat{q}_i \geq \sum_{i=1}^{j} q_i \quad \forall j \in \{1, \ldots, n\}
\]

then \(\hat{q}\) weakly majorizes \(q\), denoted \(\hat{q} \succ_w q\).\(^{25}\) If each of the inequalities above is satisfied with equality, then \(\hat{q} \succ_w q\) and \(q \succ_w \hat{q}\); in this case, either \(q = \hat{q}\) or \(q\) is a permutation of \(\hat{q}\). The \(\succ_w\) relation is transitive and incomplete.

In \(\mathcal{M}^I\), since \(q_i = q_i\) for each \(i\), a sufficient condition for \(\hat{q} \succ_w q\) is that (the cumulative probability distribution function induced by) \(\hat{q}\) dominates \(q\) by second-order stochastic dominance – for a formal proof see Lemma 4 in Appendix B.

**Proposition 4** Let \((q, t)\) be an IC mechanism which is either (i) in model \(\mathcal{M}^I\) or (ii) in model \(\mathcal{M}^H\) and is symmetric. Then, for almost all \(v, \hat{v}\),

\[
q(\hat{v}) \succ_w q(v) \implies t(\hat{v}) \geq t(v)
\]

The intuition behind Proposition 4 derives from the IC inequality:

\(^{22}\)As we assume the existence of densities, if (7) holds for almost all \(\hat{v} > v\) then revenue monotonicity is satisfied.

\(^{23}\)Note that IC and IR constraints do not involve the distribution of values; therefore, if \((q, t)\) is IC and IR under \(F\) then it is IC and IR under \(\widehat{F}\).

\(^{24}\)In model \(\mathcal{M}^I\), \(q_i = q_i\) for all \(i \in N\).

\(^{25}\)If, in addition, \(\sum_{i=1}^{n} q_i = \sum_{i=1}^{n} \hat{q}_i\) then \(\hat{q}\) majorizes \(q\). The condition \(\sum_{i=1}^{n} q_i = \sum_{i=1}^{n} \hat{q}_i\) is not usually satisfied by mechanisms in our setting.
\( t(\hat{v}) - t(v) \geq v \cdot q(\hat{v}) - v \cdot q(v) \)

For a mechanism \((q, t)\) in model \(M^i\), we have \(q_i(v) \geq q_{i+1}(v)\) and \(v_i \geq v_{i+1}\). Thus, if \(q(\hat{v}) \succ_w q(v)\) then the probabilities of acquiring the most valuable bundles are greater at \(q(\hat{v})\) than at \(q(v)\). Hence, the expected value of the allocation under \(q(\hat{v})\) is at least as high as the expected value of the allocation under \(q(v)\). In consequence, the right-hand expression in the inequality above is non-negative and \(t(\hat{v}) \geq t(v)\).  

Consider the following property for an allocation rule.

**Definition 12** An allocation rule \(q\) in \(M^i\), \(M^u\) satisfies **majorization monotonicity** if for all \((v_i, v_{-i}), (\hat{v}_i, v_{-i}) \in D^i, D^u\),

\[ q_i(\hat{v}_i, v_{-i}) > q_i(v_i, v_{-i}) \implies q(\hat{v}_i, v_{-i}) \succ_w q(v_i, v_{-i}) \]

Note that if \((q, t)\) is IC, then \(q_i(\hat{v}_i, v_{-i}) > q_i(v_i, v_{-i})\) implies \(\hat{v}_i > v_i\). Hence, majorization monotonicity (for an IC mechanism) is weaker than requiring \(\hat{v}_i > v_i\) implies \(q(\hat{v}_i, v_{-i}) \succ_w q(v_i, v_{-i})\). Theorem 5(a) below establishes that majorization monotonicity is sufficient for revenue monotonicity. It also provides a sufficient condition for majorization monotonicity (and, hence, for revenue monotonicity).

**Theorem 5** Suppose that \((q, t)\) is an IC mechanism in model \(M^i\) or a symmetric IC mechanism in model \(M^u\).

(a) If \(q\) satisfies majorization monotonicity then \((q, t)\) is revenue monotone.

(b) If \(q\) is almost deterministic, then it satisfies majorization monotonicity, and hence, \((q, t)\) is revenue monotone.

**Remark 3** By Theorem 2, if the distribution is exchangeable then there exists a symmetric mechanism that is optimal in the heterogeneous-objects model. Hence, such an optimal mechanism is revenue monotone if it is either (a) majorization monotone or (b) almost deterministic. In other words, if \(F^u\) is an exchangeable distribution, then for every \(\bar{F}^u\) that first-order stochastic dominates \(F^u\), the optimal revenue under \(\bar{F}^u\) is no less than the optimal revenue under \(F^u\). Note that \(\bar{F}^u\) need not be an exchangeable distribution.

---

\(^{26}\)Kleiner et al. (2021) study monotone functions which majorize or are majorized by a given monotone function. They characterize the extreme points of such functions and apply their result to several economic problems. Our results do not follow from their characterization.
Theorem 5(b) strengthens the result in Hart and Reny (2015), who showed that the optimal mechanism in model $\mathcal{M}^H$ is revenue monotone if it is symmetric and deterministic. It is difficult to find sufficient conditions on the primitives of the model that guarantee existence of an optimal mechanism which is symmetric and deterministic. On the other hand, when $n = 2$, there is a simple condition on the distribution of values that guarantees that the optimal mechanism is almost deterministic.

Consider the following condition on the density of buyer types, which was introduced by McAfee and McMillan (1988):

$$3f^M(v) + v \cdot \nabla f^M(v) \geq 0 \quad \forall v \in \overline{D}^M, \ M = H \text{ or } I \quad (9)$$

The uniform family of distributions, the truncated exponential distribution, and a family of Beta distributions satisfy condition (9). As shown in Proposition 1 of Pavlov (2011b) for model $\mathcal{M}^H$ and in Proposition 1 of Bikhchandani and Mishra (2022) for model $\mathcal{M}^I$, if there are two objects and $v = 0$, then, in both the models, (9) is sufficient for the existence of an optimal mechanism which is almost deterministic. Thus, we have the following result.

**Corollary 3** Suppose that $n = 2$, $v = 0$, (9) is satisfied, and $f^M$ is continuously differentiable and positive for $M = H$ or $I$. Then

(a) An optimal mechanism in model $\mathcal{M}^I$ is revenue monotone.

(b) Further, if $f^H$ is exchangeable, then an optimal mechanism in model $\mathcal{M}^H$ is revenue monotone.

Note that under the hypothesis of Corollary 3, revenue increases with any distribution $f'$ that dominates $f^M$ by first-order stochastic dominance, whether or not $f'$ satisfies (9).

5 Discussion

We have provided new results in the heterogeneous-objects model by first obtaining new results in the identical-objects model and then using our equivalence result. One can also translate known results in the identical-objects setting to the heterogeneous-objects setting. In a model with two identical objects, Bikhchandani and Mishra (2022) obtain a sufficient condition for the existence of an optimal mechanism that is deterministic. This result can
be directly applied to selling two heterogeneous objects with an exchangeable distribution of values.

It is possible to go in the other direction, i.e., convert results in the heterogeneous-objects model to identical objects. For instance, approximate expected revenue maximization results for i.i.d. priors in Hart and Nisan (2017) imply analogous approximate expected revenue maximization results in model $\mathcal{M}'$ for a class of priors.

Finally, while our presentation is in terms of selling indivisible objects, the results apply to broader settings. For example, consider the following two models:

1. **Model I.** A seller offers one durable product which, depending on its quality level, may be consumed for up to $n$ periods. An object with quality level $i$ lasts $i$ periods. The seller sells (at most) one object to a buyer at the beginning of the first period at one of the $n$ quality levels; no sales take place at any later time period. If a buyer purchases an object of quality $i$ at the beginning of the first period, then she consumes it in each of the periods $1, 2, \ldots, i$. The value of consuming the product in the $n$ periods is $(v_1, \ldots, v_n)$. Owing to discounting, which need not be same across periods, $v_1 \geq v_2 \geq \ldots \geq v_n$. So, $v_i$ is the marginal value of increasing the quality level from $i - 1$ to $i$. Note that the product can be consumed in period $i$ only if it is consumed in period $i - 1$ and hence $q_{i-1}(\cdot) \geq q_i(\cdot)$, where $q_i(\cdot)$ is the probability of consuming the product in period $i$.

2. **Model H.** This is a one-period model in which a seller offers $n$ different products, each of which lasts one period. So, $v_i$ denotes the value for product $i$ to the buyer. The buyer has additive values over any subset of the products.

As long as the products in Model H are *ex-ante symmetric*, the results of Section 3 imply that the two models are equivalent.
Appendix A Omitted Proofs

A.1 Proofs of Section 3

Proof of Lemma 1: Let \((\bar{q}(v), \bar{t}(v)) \equiv (q(v), t(v)), \forall v \in D^m\). For any \(v \in \overline{D^m \setminus D^m}\), take a sequence \(\{v^k\}_k\) in \(D^m\) that converges to \(v\). (As \(D^m\) is dense in \(\overline{D^m}\), for each \(v \in \overline{D^m \setminus D^m}\), there exists such a sequence.) As already noted, for each \(v\), \(t(v)\) is bounded above and below and \(q_i(v^k) \in [0, 1]\). Thus, \(\{q(v^k), t(v^k)\}_k\) is a bounded sequence and hence, it has an accumulation point. Set \((\bar{q}(v), \bar{t}(v))\) equal to an accumulation point of this sequence. For every \(v \in D^m \setminus D^m\), \((\bar{q}(v), \bar{t}(v))\) is an accumulation point of outcomes of a sequence of types in \(D^m\). Therefore, as \((q, t)\) is IC and IR on \(D^m\), and the buyer’s payoff function is continuous in \(v\), it follows that \((\bar{q}, \bar{t})\) is IC and IR on \(\overline{D^m}\). \(\blacksquare\)

Proof of Theorem 1: Let \((q, t)\) be a symmetric mechanism in \(M^H\).

\((i) \Rightarrow (ii)\): Assume that \((q, t)\) is IC on \(D^H\). Fix \(i\) and \(j\). Let \(v \in D^H\) and let \(\sigma\) be the permutation such that \(\sigma(i) = j, \sigma(j) = i\) and \(\sigma(k) = k\) for all \(k \notin \{i, j\}\). We have

\[
0 = t(v) - t(v^\sigma) \quad \text{(by symmetry of \((q, t)\))}
\]

\[
\leq v \cdot (q(v) - q(v^\sigma)) \quad \text{(by IC of \((q, t)\))}
\]

\[
= v_i(q_i(v) - q_i(v^\sigma)) + v_j(q_j(v) - q_j(v^\sigma))
\]

\[
= (v_i - v_j)(q_i(v) - q_j(v))
\]

where the last two equalities follow from symmetry. Thus, if \(v_i > v_j\), then \(q_i(v) \geq q_j(v)\). Hence, \((q, t)\) is rank-preserving.

\((ii) \Rightarrow (i)\): Pick any \(v \in D(\sigma)\) and \(\hat{v} \in D(\hat{\sigma})\). These map to \(v^\sigma, \hat{v}^\hat{\sigma} \in D(\sigma^t)\) such that for every \(i\),

\[
v^\sigma_i = v_{\sigma(i)}, \quad \hat{v}^\hat{\sigma}_i = \hat{v}_{\hat{\sigma}(i)}
\]

(10)

We know that

\[
\sum_{i=1}^n v_i q_i(v) - t(v) = \sum_{i=1}^n v_{\sigma(i)} q_{\sigma(i)}(v) - t(v)
\]

\[
= \sum_{i=1}^n v^\sigma_i q_i(v^\sigma) - t(v^\sigma) \quad \text{(by symmetry of \((q, t)\) and (10))}
\]

\[
\geq \sum_{i=1}^n v^\sigma_i q_i(\hat{v}^\hat{\sigma}) - t(\hat{v}^\hat{\sigma})
\]

25
\[ \sum_{i=1}^{n} v^\sigma_i q_{\bar{\sigma}(i)}(\hat{v}) - t(\hat{v}), \quad \text{(by symmetry of } (q, t)) \]  

(11)

where the inequality follows as \( v^\sigma, \hat{\sigma} \subseteq D(\sigma^I) \) and \( (q, t) \) restricted to \( D(\sigma^I) \) is IC.

Note that

\[ v^\sigma_1 > v^\sigma_2 > \ldots > v^\sigma_n \quad \text{(since } v^\sigma \subseteq D(\sigma^I)) \]

\[ q_{\bar{\sigma}(1)}(\hat{v}) \geq q_{\bar{\sigma}(2)}(\hat{v}) \geq \ldots \geq q_{\bar{\sigma}(n)}(\hat{v}) \quad \text{(since } (q, t) \text{ is rank-preserving, } \hat{v} \in D(\hat{\sigma}), \text{ and } (2)) \]

As \((v_{\bar{\sigma}(1)}, v_{\bar{\sigma}(2)}, \ldots, v_{\bar{\sigma}(n)})\) is a permutation of \((v^\sigma_1, v^\sigma_2, \ldots, v^\sigma_n)\), these inequalities imply that\(^{27}\)

\[ \sum_{i=1}^{n} v^\sigma_i q_{\bar{\sigma}(i)}(\hat{v}) \geq \sum_{i=1}^{n} v_{\bar{\sigma}(i)} q_{\bar{\sigma}(i)}(\hat{v}) \]  

(12)

Using (11) and (12), we have

\[ \sum_{i=1}^{n} v_i q_i(v) - t(v) \geq \sum_{i=1}^{n} v_{\bar{\sigma}(i)} q_{\bar{\sigma}(i)}(\hat{v}) - t(\hat{v}) \]

\[ = \sum_{i=1}^{n} v_i q_i(\hat{v}) - t(\hat{v}), \]

which is the desired IC constraint. \( \blacksquare \)

**Proof of Lemma 2:** Relabelling the objects if necessary, assume that \((q, t)\) is defined on \( D(\sigma^I) \). Let \((q^s, t^s)\) be the symmetric extension of \((q, t)\). As \((q, t)\) is rank preserving on \( D(\sigma^I) \), \((q^s, t^s)\) is rank preserving on \( D^u \). As \((q^s, t^s) = (q, t)\) on \( D(\sigma^I) \) and \((q, t)\) is IC on \( D(\sigma^I) \), we conclude that \((q^s, t^s)\) is IC on \( D^u \) (by Theorem 1).

For any \( v \in D(\sigma^I) \), \((q^s(v), t^s(v)) = (q(v), t(v)) \). Thus, \((q^s, t^s)\) is IR on \( D(\sigma^I) \). That \((q^s, t^s)\) is IR follows from the fact that the payoff of any type \( v \in D(\sigma) \) is the same as the payoff of type \( v^\sigma \subseteq D(\sigma^I) \).

**Proof of Proposition 1:**

(i) \& (ii) Note that any mechanism in model \( M^I \) is rank preserving due to feasibility restriction (1). Thus, by Lemma 2 the symmetric extension of an IC and IR mechanism in model \( M^I \) is a rank-preserving, IC and IR mechanism on \( D^u \); this symmetric mechanism can be extended to \( \overline{D^u} \), i.e., to model \( M^H \), by Lemma 1. Conversely, if \((q, t)\) is a symmetric, IC, and

\(^{27}\)See also the rearrangement inequality in Theorem 368 of Hardy et al. (1952).
IR mechanism in model $\mathcal{M}^{\text{II}}$, then it is rank preserving by Theorem 1. Hence, the restriction of $(q, t)$ to $D(\sigma^I)$ defines an IC and IR mechanism for model $\mathcal{M}^I$ (since rank preserving implies that the feasibility restriction (1) holds).

(iii) As $f_I$ is exchangeable, Theorem 2 implies that there exists a symmetric and rank-preserving mechanism that is optimal in model $\mathcal{M}^{\text{II}}$. Let $(q^{\text{II}}, t^{\text{II}})$ be this optimal mechanism in model $\mathcal{M}^{\text{II}}$ and let $(q^I, t^I)$ be the corresponding IC and IR mechanism for model $\mathcal{M}^I$ obtained by restricting $(q^{\text{II}}, t^{\text{II}})$ to $D(\sigma^I)$. As $(q^{\text{II}}, t^{\text{II}})$ is symmetric, we have

$$\text{Rev}(q^{\text{II}}, t^{\text{II}}; f^{\text{II}}) = n! \int_{D^I} t^{\text{II}}(v) f^{\text{II}}(v) dv = \int_{D^I} t^I(v) f^I(v) dv = \text{Rev}(q^I, t^I; f^I)$$

(13)

As $(q^{\text{II}}, t^{\text{II}})$ is optimal in $\mathcal{M}^{\text{II}}$, $(q^I, t^I)$ must be optimal in $\mathcal{M}^I$. To see this, suppose that some other mechanism $(q^I', t^I')$ yields a strictly higher revenue than $(q^I, t^I)$ in $\mathcal{M}^I$. Let $(q^{\text{II}}, t^{\text{II}}')$ be the symmetric extension of $(q^I', t^I')$ to $\mathcal{M}^{\text{II}}$. Then we have

$$\text{Rev}(q^{\text{II}}, t^{\text{II}}; f^{\text{II}}) = \text{Rev}(q^I, t^I; f^I) < \text{Rev}(q^I', t^I'; f^I') = \text{Rev}(q^{\text{II}}, t^{\text{II}}'; f^{\text{II}})$$

where the equalities follow from (13). But this contradicts the assumption that $(q^{\text{II}}, t^{\text{II}})$ is optimal in $\mathcal{M}^{\text{II}}$. Thus, optimal mechanisms in the two models yield the same expected revenue.

A.2 Proofs of Section 4

**Proof of Proposition 2:** We provide a proof of parts (i) and (ii) for an arbitrary IC mechanism, and indicate the changes needed for the proof to work for deterministic and almost deterministic mechanisms, i.e., for part (iii).

(i): Let $(q, t)$ be an IC mechanism defined on $D^*$ with corresponding utility function $u$. An $x \in [0, 1]^n$ with $x_1 \geq \ldots \geq x_n$ is a subgradient\(^{28}\) of $u$ at $v \in D^*$ if for every $v' \in D^*$

$$u(v') \geq u(v) + (v' - v) \cdot x \quad \forall \ v' \in D^*$$

(14)

Let $\partial u(v)$ denote the set of all subgradients of $u$ at $v$. By IC, $x = q(v)$ satisfies (14). Hence, $\partial u(v)$ is non-empty for any $v \in D^*$.

\(^{28}\)For the proof of deterministic and almost deterministic mechanisms, only consider subgradients which are deterministic and almost deterministic respectively.
Define a strict linear order $\succ$ on the set of all feasible allocations in the identical-objects model, i.e., on the set $X := \{x \in [0, 1]^n : x_1 \geq \ldots \geq x_n\}$. An example of such an ordering is the following **lexicographic order**: for any $x, y \in X$, $x \succ y$ if either $x_1 > y_1$ or $x_1 = y_1, x_2 > y_2$ or $x_i = y_i$ for $i \in \{1, 2\}$, $x_3 > y_3$, or etc. If $x$ and $y$ are deterministic allocations (i.e., $0 - 1$ vectors) and $\succ$ is the lexicographic order, then $x \succ y$ if and only if $\sum_i x_i > \sum_i y_i$, i.e., $x$ allocates more units than $y$.

For every $v \in D^*$, let

$$X(v) := \{x \in \partial u(v) : v \cdot x \geq v \cdot y \forall y \in \partial u(v)\}$$

**Hart and Reny (2015)** call a mechanism $(q, t)$ seller favorable if $q(v) \in X(v)$ for every $v$ in the domain. If $D^*$ is convex, IC implies that $u$ is convex. For convex $u$ defined on convex $D^*$, Rockafellar (Theorem 23.2) characterizes $v \cdot x$ for each $x \in X(v)$ (note that $v \cdot x$ is the same for all $x \in X(v)$) as the **directional derivative** of $u$ in the direction $v$. Let $x(v)$ denote the maximal vector in $X(v)$ with respect to $\succ$.

Next, we define $(q^*, t^*)$, the **maximal extension** (with respect to $\succ$) of $(q, t)$

$$q^*(v) := x(v), \quad t^*(v) := v \cdot q^*(v) - u(v) \quad \forall v \in D^*$$

Note that $u^*(v) = v \cdot q^* - t^*(v) = u(v)$ for all $v \in D^*$. As $x(v) \in \partial u(v)$, (14) holds. Thus, $(q^*, t^*)$ defines an IC mechanism. Also, $t^*(v) = v \cdot x(v) - u(v) \geq v \cdot q(v) - u(v) = t(v)$ for all $v \in D^*$ as $x(v) \in X(v)$ and $q(v) \in \partial u(v)$.

To complete the proof of (i), we show that $(q^*, t^*)$ is non-bossy. Take $v = (v_i, v_{-i}) \in D^*$ and $(v'_i, v_{-i}) \in D^*$. Suppose that $q^*_i(v) = q_i^*(v'_i, v_{-i})$. IC constraints between $v$ and $(v'_i, v_{-i})$ are

$$u^*(v) - u^*(v'_i, v_{-i}) \geq (v_i - v'_i)q^*_i(v'_i, v_{-i}) = (v_i - v'_i)\hat{q}^*_i(v)$$

$$u^*(v'_i, v_{-i}) - u^*(v) \geq (v'_i - v_i)\hat{q}^*_i(v'_i, v_{-i}) = (v'_i - v_i)q^*_i(v)$$

$$\implies u^*(v) - u^*(v'_i, v_{-i}) = (v_i - v'_i)\hat{q}^*_i(v)$$

We show that $q^*(v) \in \partial u^*(v'_i, v_{-i})$. Pick any $\tilde{v} \in D^*$. The IC constraint from $\tilde{v}$ to $v$ is

$$u^*(\tilde{v}) \geq u^*(v) + (\tilde{v} - v) \cdot q^*_i(v)$$

---

29For the proof of (iii), $x(v)$ will choose a deterministic vector and an almost deterministic vector respectively.

30Note that $(q^*, t^*)$ is a seller-favorable mechanism as in **Hart and Reny (2015)**. In general, there exist non-bossy mechanisms that are not seller favorable as well as seller-favorable mechanisms that are bossy.
\[ u^* (v_i, v_{-i}) = (v_i - v_i^*) q_i^\ast (v) + \sum_{j \neq i} (\dot{v}_j - v_j) q_j^\ast (v) \quad \text{ (by eq. 15)} \]

\[ u^* (v_i, v_{-i}) = (\dot{v}_i - v_i^*) q_i^\ast (v) + \sum_{j \neq i} (\dot{v}_j - v_j) q_j^\ast (v) \]

Thus, \( q^\ast (v) \in \partial u^* (v_i, v_{-i}) \). An identical proof shows \( q^\ast (v_i', v_{-i}) \in \partial u^* (v_i, v_{-i}) \). So, \( q^\ast (v), q^\ast (v_i', v_{-i}) \in \partial u^* (v) \cap \partial u^* (v_i', v_{-i}) \). Since \( (q^\ast, t^\ast) \) is the maximal extension of \( (q, t) \) with respect to the same order \( \succ \), we must have \( q^\ast (v) = q^\ast (v_i', v_{-i}) \). This implies that \( q^\ast \) is non-bossy.

(ii): If \( D^* \) is convex, \( u \) is convex and \( \partial u (v) \) is singleton almost everywhere (Theorems 25.1 and 25.3 in Rockafellar). Consequently, \( (q(v), t(v)) = (q^\ast (v), t^\ast (v)) \) almost everywhere. \( \blacksquare \)

**Proof of Theorem 3:** Suppose that \( (q, t) \) is an optimal deterministic (pricing) mechanism with prices \( p_0 = 0, p_1, \ldots, p_n \). (By Proposition 3, we know that a pricing mechanism representation exists for any deterministic mechanism.)

Suppose that the pricing representation of this optimal mechanism satisfies \( p_1 \leq \ldots \leq p_n \) with at least one one strict inequality. WLOG choose an optimal deterministic mechanism with the least number of strict inequalities in its pricing representation. The types that are allocated exactly \( k \) units must satisfy:

\[ \sum_{i=1}^{k} v_i - \sum_{i=1}^{k} p_i \geq \sum_{i=1}^{j} v_i - \sum_{i=1}^{j} p_i \quad \forall \ j \neq k \]

Hence, the closure of the set of types to which exactly \( k \) units is sold is

\[ R_k (p) = \{ v \in D^i : \sum_{i=k+1}^{j} v_i \leq \sum_{i=k+1}^{j} p_i \text{ for all } j > k, \sum_{i=j+1}^{k} v_i \geq \sum_{i=j+1}^{k} p_i \text{ for all } j < k \} \]

If \( v_k \geq p_k \geq p_{k-1} \geq \ldots \geq p_1 \), then using \( v_1 \geq v_2 \geq \ldots \geq v_k \), we have

\[ \sum_{i=j+1}^{k} v_i \geq \sum_{i=j+1}^{k} p_i \quad \forall j < k \]

Similarly, if \( v_{k+1} \leq p_{k+1} \leq p_{k+2} \leq \ldots \leq p_n \), using \( v_{k+1} \geq v_{k+2} \geq \ldots \geq v_n \), we have

\[ \sum_{i=k+1}^{j} v_i \leq \sum_{i=k+1}^{j} p_i \quad \forall j > k \]

Hence, \( v \in R_k (p) \) if and only if \( v_k \geq p_k \) and \( v_{k+1} \leq p_{k+1} \). Thus, when \( p_1 \leq p_2 \leq \ldots \leq p_n \), we have

\[ R_k (p) = \{ v \in D^i : v_{k+1} \leq p_{k+1}, \ v_k \geq p_k \} \]
Consequently, the closure of the set of types where at least $k$ units are sold is

$$\bigcup_{i=k}^n R_k(p) = \{v \in D^i : v_k \geq p_k\}$$

$$\implies \Pr[v \in \bigcup_{i=k}^n R_k(p)] = 1 - F_k(p_k)$$

where $F_k$ is the marginal distribution of the value of the $k^{th}$ unit. Hence, the expected revenue from the mechanism is

$$\text{Rev}(p) = \sum_{k=1}^n p_k(1 - F_k(p_k))$$

Suppose the inequalities $p_1 \leq p_2 \leq \ldots \leq p_n$ are strict at $j$, i.e., $p_j < p_{j+1}$. Consider two modifications, $\hat{p}$ and $\tilde{p}$, both with increasing prices, of this pricing mechanism and the expected revenue they generate:

$$\hat{p} = (p_0, p_1, \ldots, p_j, p_j, p_{j+2}, \ldots, p_n)$$

$$\text{Rev}(\hat{p}) = \sum_{k \neq j+1} p_k(1 - F_k(p_k)) + p_j(1 - F_{j+1}(p_j))$$

$$\tilde{p} = (p_0, p_1, \ldots, p_j-1, p_{j+1}, p_j, p_{j+2}, \ldots, p_n)$$

$$\text{Rev}(\tilde{p}) = \sum_{k \neq j} p_k(1 - F_k(p_k)) + p_{j+1}(1 - F_j(p_{j+1}))$$

By optimality,

$$\sum_{k=1}^n p_k(1 - F_k(p_k)) > \sum_{k \neq (j+1)} p_k(1 - F_k(p_k)) + p_j(1 - F_{j+1}(p_j))$$

$$\sum_{k=1}^n p_k(1 - F_k(p_k)) > \sum_{k \neq j} p_k(1 - F_k(p_k)) + p_{j+1}(1 - F_j(p_{j+1}))$$

These inequalities are strict since the new mechanisms have one less strict inequality than the original mechanism and by assumption $(q, t)$ is an optimal deterministic mechanism with the least number of strict inequalities in its pricing mechanism. The above inequalities are equivalent to

$$p_{j+1}(1 - F_{j+1}(p_{j+1})) > p_j(1 - F_{j+1}(p_j))$$

$$p_j(1 - F_j(p_j)) > p_{j+1}(1 - F_j(p_{j+1}))$$

$$\implies \frac{1 - F_{j+1}(p_{j+1})}{1 - F_{j+1}(p_j)} > \frac{p_j}{p_{j+1}} > \frac{1 - F_j(p_{j+1})}{1 - F_j(p_j)}$$
\[
1 - F_j(p_j) \over 1 - F_j(p_j) > 1 - F_{j+1}(p_j) \over 1 - F_{j+1}(p_j)
\]

But as \( p_j < p_{j+1} \), this contradicts the assumption that \( F_j \) hazard rate dominates \( F_{j+1} \). □

**Proof of Corollary 1:** Let \( g \) be the (marginal) density of the value of each object in model \( \mathcal{M}^u \). Define

\[
f(v_1, v_2, \ldots, v_n) = n! \cdot g(v_1)g(v_2)\ldots g(v_n) \quad 1 \ge v_1 \ge v_2 \ge \ldots v_n
\]

An identical-objects model, \( \mathcal{M}^i \), with joint density \( f \) is equivalent to the heterogeneous-objects model, \( \mathcal{M}^u \), in the sense of Proposition 1.\(^{31}\) Let \( F_k \) be the marginal distribution of the \( k^{th} \) unit. By Theorem 1.B.26 in Shaked and Shanthikumar (2007), \( F_k \) hazard-rate dominates \( F_{k+1} \). Hence, by Theorem 3, in any optimal pricing mechanism \( p^* \) if \( p^*_1 \le p^*_2 \le \ldots \le p^*_n \) then \( p^*_1 = p^*_n \).

Let \( (q^u, t^u) \) be an optimal deterministic symmetric mechanism for model \( \mathcal{M}^u \) with prices \( p^*_0 = 0, p^*_1, \ldots, p^*_n \), where the price of bundle \( S \) is \( \sum_{j=1}^{|S|} p^*_j \). By Proposition 1, the restriction of \( (q^u, t^u) \) to \( D(\sigma^u) \) defines an IC and IR mechanism for model \( \mathcal{M}^i \); denote this mechanism as \( (q^i, t^i) \). By construction, the expected revenue of \( (q^u, t^u) \) with i.i.d. draws from \( g \) in model \( \mathcal{M}^u \) is equal to the expected revenue of \( (q^u, t^u) \) with i.i.d. draws from \( g \) in model \( \mathcal{M}^u \). By Proposition 1, \( (q^i, t^i) \) is an optimal mechanism for model \( \mathcal{M}^i \). Also, this is a pricing mechanism with prices \( p^*_0 = 0, p^*_1, \ldots, p^*_n \). By Theorem 3, there is no optimal mechanism with prices \( p^*_0 = 0 \le p^*_1 \le \ldots \le p^*_n \) with one inequality strict. Thus, \( (q^u, t^u) \) is not strictly supermodular. □

**Proof of Lemma 3:** Suppose that the domain of types \( D \) satisfies the strong-lattice property. Taking \( k = 1 \) in Definition 8, we see that the lattice property w.r.t. with min and max is implied by the strong-lattice property.

To prove in the other direction, take any \( v, v' \in D \). The two types \( v^\dagger = \min(v, v') \) and \( v^\ddagger = \max(v, v') \) belong to \( D \) by the lattice property. First, we show \( \bar{v} \) as defined in (5) belongs to \( D \). Fix any \( k \). If \( k = 1 \), then \( \bar{v} = v^\dagger \) and we are done. If \( k > 1 \), then \( \bar{v} = (v_1, \ldots, v_{k-1}, v_{k}^\dagger, \ldots, v_n^\dagger) \). By definition, \( v_{k-1} \ge v_k \ge v_k^\dagger \). Hence, richness implies that \( \bar{v} \in D \).

Next we show that \( \hat{v} \) as defined in (6) belongs to \( D \). If \( k = n \), then \( \hat{v} = v^\dagger \), and we are

\[^{31}\text{This identical-objects model is an ordered decreasing values model as defined in Bikhchandani and Mishra (2022).}\]
done. If \( k < n \), then \( \tilde{v} = (v^+_1, \ldots, v^+_k, v^+_{k+1}, \ldots, v_n) \). As \( v^+_k \geq v_k \geq v_{k+1} \), richness implies that \( \tilde{v} \in D \).

**Proof of Theorem 4:** Let \((q, t)\) be a deterministic, object non-bossy, and UIC mechanism defined on type space \( D \) that satisfies the strong-lattice property. We start with some preliminary results. Let \( k(v) := \sum_{i=1}^{n} q_i(v) \), \( k(v') := \sum_{i=1}^{n} q_i(v') \), etc.

**Claim 1** Suppose \((q, t)\) is deterministic and UIC. Then the following are true:

(i) \( \sum_{i=1}^{k(v')} (v_i - v'_i) \geq \sum_{i=1}^{k(v)} (v_i - v'_i) \)

(ii) If \( v \geq v' \), then

(ii-a) either \( k(v) \geq k(v') \) or \( v_i = v'_i \) for all \( i \in \{k(v) + 1, \ldots, k(v')\} \).

(ii-b) either \( k(v) \geq k(v') \) or the IC constraints between \( v \) and \( v' \) bind.

**Proof:** If \( v \geq v' \) or \( v' \geq v \), then UIC between \( v \) and \( v' \) implies:

\[
\sum_{i=1}^{k(v)} v_i - \sum_{i=1}^{k(v')} v_i \geq t(v) - t(v') \geq \sum_{i=1}^{k(v')} v'_i - \sum_{i=1}^{k(v)} v'_i
\]  

(16)

(i) Follows directly from (16).

(ii-a) If \( k(v') > k(v) \), then (i) implies \( \sum_{i=k(v)+1}^{k(v')} (v_i - v'_i) \leq 0 \). Since \( v \geq v' \), this is possible only if \( v_i = v'_i \) for all \( i \in \{k(v) + 1, \ldots, k(v')\} \).

(ii-b) If \( k(v') > k(v) \), the IC constraints in (16) are equivalent to

\[
\sum_{i=k(v)+1}^{k(v')} v'_i \geq t(v') - t(v) \geq \sum_{i=k(v)+1}^{k(v)} v_i
\]

Using (ii-a), the lower bound and upper bound of \( t(v') - t(v) \) is the same. Hence, the IC constraints between \( v \) and \( v' \) bind.

**Claim 2** Suppose that \((q, t)\) is a deterministic, UIC and object non-bossy mechanism. Then, the following hold.

(i) If \( v \geq v' \) and \( v_i = v'_i \) for all \( i \leq \max(k(v), k(v')) \), then \( q(v) = q(v') \) and \( t(v) = t(v') \).

(ii) If \( v \leq v' \) and \( v_i = v'_i \) for all \( i > \min(k(v), k(v')) \), then \( q(v) = q(v') \) and \( t(v) = t(v') \).
Proof: Let \( v = (v_j, v_{-j}) \) and \( v' = (v'_j, v_{-j}) \).\(^{32}\)

(i): Suppose that \( v'_j < v_j \). Since \( v_i = v'_i \) for all \( i \leq \max(k(v'), k(v)) \), we have \( j > \max(k(v'), k(v)) \). Hence, \( q_j(v') = q_j(v) = 0 \). By non-bossiness, \( q(v') = q(v) \). By UIC, we have \( t(v) = t(v') \).

(ii): Suppose that \( v'_j > v_j \). By assumption \( j \leq \min(k(v), k(v')) \). Hence, \( q_j(v) = q_j(v') = 1 \). By non-bossiness, \( q(v) = q(v') \). By UIC, we have \( t(v) = t(v') \). \( \square \)

**Claim 3** Suppose that \((q, t)\) is a deterministic, UIC and object non-bossy mechanism. Then, \( q \) is increasing, i.e., for all \( v \geq v' \), \( \kappa(v) \geq \kappa(v') \).

**Proof:** Let \( v \geq v' \). Without loss of generality assume that \( v' \equiv (v'_j, v_{-j}) \) for some \( j \) with \( v'_j < v_j \). If \( k(v') > k(v) \), then \( j \notin \{k(v) + 1, \ldots, k(v')\} \) since \( v_i = v'_i \) for all \( i \in \{k(v) + 1, \ldots, k(v')\} \) by Claim 1(ii-a). If \( j \leq k(v) \), then \( q_j(v) = q_j(v') = 1 \), and non-bossiness implies \( q(v) = q(v') \).

If \( j > k(v') \), then \( q_j(v) = q_j(v') = 0 \), and non-bossiness implies \( q(v) = q(v') \). This gives us \( k(v) = k(v') \), a contradiction to our assumption that \( k(v') > k(v) \). \( \square \)

The next two claims prove Theorem 4.

**Claim 4** Suppose that \((q, t)\) is a deterministic, UIC and object non-bossy mechanism. Let \( v \) and \( v' \) be such that \( v \nmid v' \), \( v' \nmid v \) and \( k(v) \geq k(v') \) Then, the incentive constraint \( v' \to v \) is satisfied.

**Proof:** We consider two cases.

**Case 1:** \( k(v) = n \). By Claim 3, the type \( v^\uparrow = \max(v, v') \) (by the strong-lattice property, \( v^\uparrow \in D \)) satisfies \( k(v^\uparrow) \geq k(v) = n \). Hence, \( k(v^\uparrow) = n \). As \( n \) units are allocated at \( v \) and at \( v^\uparrow \), and \( v^\uparrow \geq v \), we have \( t(v) = t(v^\uparrow) \) by UIC. As UIC between \( v' \) and \( v^\uparrow \) holds, we have

\[
\sum_{i=1}^{k(v')} v'_i - t(v') \geq \sum_{i=1}^{k(v)} v'_i - t(v^\uparrow) \\
= \sum_{i=1}^{k(v)} v'_i - t(v)
\]

Hence, \( v' \to v \) is satisfied.

**Case 2:** \( k(v) < n \). Define \( \bar{v} \) and \( \bar{v}' \) as follows.

\[
\bar{v}_i = \begin{cases} 
  v_i & \text{if } i \leq k(v) \\
  \min(v_i, v'_i) & \text{if } i > k(v)
\end{cases}
\]

\(^{32}\)If \( v \) and \( v' \) differ in more than one dimension, we can use this argument repeatedly.
\[ \tilde{v}'_i = \begin{cases} v'_i & \text{if } i \leq k(v) \\ \min(v_i, v'_i) & \text{if } i > k(v) \end{cases} \]

By the strong-lattice property, \( \tilde{v}, \tilde{v}' \in D \). Since \( \tilde{v} \leq v \), by Claim 3 we have \( k(\tilde{v}) \leq k(v) \). As \( \tilde{v}_i = v_i \) for all \( i \leq k(v) = \max(k(\tilde{v}), k(v)) \), by Claim 2(i), \( q(\tilde{v}) = q(v) \), i.e., \( k(\tilde{v}) = k(v) \), and \( t(\tilde{v}) = t(v) \).

Similarly, as \( \tilde{v}' \leq v' \), by Claim 3 we have \( k(\tilde{v}') \leq k(v') \leq k(v) \). Since \( \tilde{v}_i = v'_i \) for all \( i \leq k(v) \) and \( k(v') = \max(k(\tilde{v}'), k(v')) \leq k(v) \), by Claim 2(i), \( q(\tilde{v}') = q(v') \), i.e., \( k(\tilde{v}') = k(v') \), and \( t(\tilde{v}') = t(v') \).

Define a new type \( \hat{v} \) as follows:

\[ \hat{v}_i = \max(\tilde{v}_i, \tilde{v}'_i) \quad \forall \ i \in \{1, \ldots, n\} \]

By the strong-lattice property, \( \hat{v} \in D \). Note that for all \( i > k(v) \), \( \hat{v}_i = \tilde{v}_i = \tilde{v}'_i = \min(v_i, v'_i) \). Since \( \hat{v} \geq \tilde{v} \), Claim 3 implies \( k(\hat{v}) \geq k(\tilde{v}) = k(v) \).

Considering UIC from \( \tilde{v}' \) to \( \hat{v} \) and \( \hat{v} \) to \( \tilde{v} \) (and using \( v'_i = \tilde{v}'_i \) for all \( i \leq k(v) \), \( k(v) \geq k(v') \), and \( t(v') = t(\tilde{v}') \), \( t(\hat{v}) = t(v) \)), we get

\[
\begin{align*}
\sum_{i=1}^{k(v')} v'_i - t(v') & = \sum_{i=1}^{k(v')} \tilde{v}'_i - t(\tilde{v}') \geq \sum_{i=1}^{k(\hat{v})} \tilde{v}'_i - t(\hat{v}) \\
\sum_{i=1}^{k(v)} \tilde{v}_i - t(\hat{v}) & \geq \sum_{i=1}^{k(\hat{v})} \hat{v}_i - t(\hat{v}) = \sum_{i=1}^{k(\hat{v})} \hat{v}_i - t(\hat{v}) \\
\end{align*}
\]

Adding these constraints

\[
\begin{align*}
\sum_{i=1}^{k(v')} v'_i - t(v') & \geq \sum_{i=1}^{k(\hat{v})} \hat{v}_i - \sum_{i=1}^{k(\hat{v})} \hat{v}_i + \sum_{i=1}^{k(\hat{v})} \tilde{v}'_i - t(\hat{v}) \\
& = (17)
\end{align*}
\]

If \( k(\hat{v}) = k(v) \), then (17) reduces to

\[
\begin{align*}
\sum_{i=1}^{k(v')} v'_i - t(v') & \geq \sum_{i=1}^{k(\hat{v})} \hat{v}_i - \sum_{i=1}^{k(\hat{v})} \hat{v}_i + \sum_{i=1}^{k(\hat{v})} \tilde{v}'_i - t(\hat{v}) \\
& = \sum_{i=1}^{k(\hat{v})} \hat{v}_i - t(\hat{v}) \\
\end{align*}
\]

where we used the fact that \( v'_i = \tilde{v}'_i \) for all \( i \leq k(v) \). Hence, \( v' \rightarrow v \) holds.

If \( k(\hat{v}) > k(v) \), then (17) reduces to

\[
\begin{align*}
\sum_{i=1}^{k(v')} v'_i - t(v') & \geq \sum_{i=1}^{k(\hat{v})} \hat{v}_i - \sum_{i=k(v)+1}^{k(\hat{v})} \hat{v}_i - t(\hat{v}) \\
& \geq \sum_{i=1}^{k(\hat{v})} \hat{v}_i - t(\hat{v}) \\
\end{align*}
\]

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\[
\begin{align*}
&= \sum_{i=1}^{k(\check{v})} \check{v}_i' - \sum_{i=k(v)+1}^{k(\check{v})} \check{v}_i - t(v) \\
&= \sum_{i=1}^{k(v)} \check{v}_i' - t(v) \\
&= \sum_{i=1}^{k(v)} v_i' - t(v) \\
&\text{ (using } v_i' = \check{v}_i \text{ for all } i \leq k(v))
\end{align*}
\]

Hence, \( v' \rightarrow v \) is satisfied. \( \square \)

**Claim 5** Suppose \((q, t)\) is a deterministic, UIC and object non-bossy mechanism. Let \( v \) and \( v' \) be such that \( v \not\geq v' \), \( v' \not\geq v \) and \( k(v) > k(v') \). Then, the incentive constraint \( v \rightarrow v' \) is satisfied.

**Proof:** We consider two cases.

**Case 1:** \( k(v') = 0 \). As 0 units are allocated at \( v' \), Claim 3 implies that 0 units are allocated at \( v^\dagger = \min(v, v') \) (by the strong-lattice property \( v^\dagger \in D \)). Thus, UIC implies that \( t(v') = t(v^\dagger) \). As UIC between \( v \) and \( v^\dagger \) holds, we have

\[
\sum_{i=1}^{k(v)} v_i - t(v) \geq -t(v^\dagger) = -t(v')
\]

We conclude that \( v \rightarrow v' \).

**Case 2:** \( k(v') > 0 \). Define \( \check{v} \) and \( \check{v}' \) as follows.

\[
\check{v}_i = \begin{cases} 
\max(v_i, v_i') & \text{if } i \leq k(v') \\
\check{v}_i & \text{if } i > k(v')
\end{cases}
\]

\[
\check{v}_i' = \begin{cases} 
\max(v_i, v_i') & \text{if } i \leq k(v') \\
v_i' & \text{if } i > k(v')
\end{cases}
\]

By the strong-lattice property, \( \check{v}, \check{v}' \in D \). Since \( \check{v} \geq v \), by Claim 3, \( k(\check{v}) \geq k(v)[> k(v')] \).

Hence, \( \check{v}_i = v_i \) for all \( i > k(v') \) implies \( \check{v}_i = v_i \) for all \( i > \min(k(\check{v}), k(v)) \). By Claim 2(ii), we have \( q(v) = q(\check{v}) \), i.e., \( k(v) = k(\check{v}) \), and \( t(v) = t(\check{v}) \).

Similarly, \( \check{v}' \geq v' \), and Claim 3 imply \( k(\check{v}') \geq k(v') \). Hence, \( \check{v}_i' = v_i' \) for all \( i > k(v') \) implies \( \check{v}_i' = v_i' \) for all \( i > \min(k(\check{v}'), k(v')) \). By Claim 2(ii), we have \( q(v') = q(\check{v}') \), i.e., \( k(v') = k(\check{v}') \), and \( t(v') = t(\check{v}') \).
Now, define \( \hat{v} \) as follows:

\[
\hat{v}_i = \min(\tilde{v}_i, \hat{v}'_i) \quad \forall \; i \in \{1, \ldots, n\}
\]

By the strong-lattice property, \( \hat{v} \in D \). By Claim 3, \( q(\hat{v}) \leq q(v) \) and \( q(\hat{v}) \leq q(v') \). Hence, \( k(\hat{v}) \leq k(v')[< k(v)] \). Applying UIC constraints \( \tilde{v} \rightarrow \hat{v} \) and \( \hat{v} \rightarrow \tilde{v}' \), and recalling that \( k(v) = k(\tilde{v}) \), \( t(v) = t(\tilde{v}) \), and \( k(v') = k(\tilde{v}') \), \( t(v') = t(\tilde{v}') \), we have

\[
\sum_{i=1}^{k(\tilde{v})} \tilde{v}_i - t(\tilde{v}) = \sum_{i=1}^{k(v)} \tilde{v}_i - t(v) \geq \sum_{i=1}^{k(\tilde{v})} \hat{v}_i - t(\tilde{v}) \]

Adding the two constraints, we get

\[
\sum_{i=1}^{k(v)} \tilde{v}_i - t(v) \geq \sum_{i=1}^{k(\tilde{v})} \hat{v}_i - t(\tilde{v}) = \sum_{i=1}^{k(v')} \tilde{v}_i - t(v') \tag{18}
\]

If \( k(\hat{v}) = k(v') \), then (18) reduces to

\[
\sum_{i=1}^{k(v)} \tilde{v}_i - t(v) \geq \sum_{i=1}^{k(v')} \tilde{v}_i - t(v') \tag{19}
\]

If, instead, \( k(\hat{v}) < k(v') \), then (18) reduces to

\[
\sum_{i=1}^{k(v)} \tilde{v}_i - t(v) \geq \sum_{i=1}^{k(\tilde{v})} \hat{v}_i - t(\tilde{v}) + \sum_{i=k(\tilde{v})+1}^{k(v')} \hat{v}_i - t(v') \]

\[
= \sum_{i=1}^{k(\tilde{v})} \tilde{v}_i + \sum_{i=k(\tilde{v})+1}^{k(v')} \hat{v}_i - t(v') \quad \text{(since } \tilde{v}_i = \hat{v}_i \text{ for all } i \leq k(v'))
\]

\[
= \sum_{i=1}^{k(v')} \tilde{v}_i - t(v')
\]

This is the same equation as (19).

Next, we show that (19) implies \( v \rightarrow v' \). Since \( k(v') < k(v) \), (19) reduces to

\[
\sum_{i=k(v')+1}^{k(v)} \tilde{v}_i - t(v) \geq -t(v')
\]
Using $\tilde{v}_i = v_i$ for all $i > k(v')$ and adding $\sum_{i=1}^{k(v')} v_i$ on both sides, we get

$$\sum_{i=1}^{k(v)} v_i - t(v) \geq \sum_{i=1}^{k(v')} v_i - t(v')$$

Hence, $v \rightarrow v'$.

Claims 4 and 5 prove Theorem 4.

**Proof of Corollary 2**: Let $(q, t)$ be a symmetric, rank-preserving, deterministic, object non-bossy, and UIC mechanism defined on $D^u$. Since $(q, t)$ is rank preserving, $(q, t)$ restricted to $D(\sigma^l)$ defines a feasible, object non-bossy, and UIC mechanism for the identical-objects model. By Theorem 4, such a mechanism is IC. Hence, $(q, t)$ restricted to $D(\sigma^l)$ is an IC and rank-preserving mechanism. By Theorem 1, $(q, t)$ is an IC mechanism.

**Proof of Proposition 4**: We provide a proof (i) for all $v, \hat{v} \in D^l$ and (ii) for all $v, \hat{v} \in D^u$. Thus, (8) is satisfied for all $v, \hat{v} \in M^l$ and almost all $v, \hat{v} \in M^u$.

Take $v, \hat{v} \in D^l$. By IC,

$$t(\hat{v}) - t(v) \geq \sum_{j=1}^{n} v_j q_j(\hat{v}) - \sum_{j=1}^{n} v_j q_j(v) \quad (20)$$

Let $\Delta_j(v) := v_j - v_{j+1}$ for all $j \in \{1, \ldots, n\}$, where $v_{n+1} := 0$. As $\hat{v}, v \in D^l$, $\Delta_j(v) \geq 0$ for all $j$. So,

$$\sum_{j=1}^{n} v_j q_j(\hat{v}) = \sum_{j=1}^{n} q_j(\hat{v}) \left( \sum_{k=j}^{n} \Delta_k(v) \right) = \sum_{k=1}^{n} \Delta_k(v) \left( \sum_{j=1}^{k} q_j(\hat{v}) \right) \quad (21)$$

Using (20) and (21), we have

$$t(\hat{v}) - t(v) \geq \sum_{k=1}^{n} \Delta_k(v) \sum_{j=1}^{k} (q_j(\hat{v}) - q_j(v)) \quad (22)$$

If $q(\hat{v}) \succ_w q(v)$, then the RHS of (22) is non-negative. As a result, $t(\hat{v}) \geq t(v)$. This completes the proof for $(q, t)$ defined on $D^l$.

Next, consider $(q, t)$ defined on domain $D^u$. As $(q, t)$ is symmetric, IC and IR, Theorem 1 implies that it is rank preserving. Thus, $(q, t)$ on $D^u$ is the symmetric extension of $(q, t)$ on $D(\sigma^l) = D^l$. For any $\hat{v} \in D(\hat{\sigma})$, $\hat{\sigma} \in D(\sigma^l)$. By symmetry,
\[ t(\hat{v}) = t(\hat{v}'), \quad t(\hat{v}) = t(\hat{v}'), \quad q^\#(\hat{v}) = q(\hat{v}'), \quad \text{and } q^\#(\hat{v}) = q(\hat{v}'). \] As weak majorization is invariant to permutations of vectors, and \((q, t)\) is symmetric,

\[ q(\hat{v}) \succ_w q(\hat{v}) \iff q^\#(\hat{v}) \succ_w q^\#(\hat{v}) \iff q(\hat{v}') \succ_w q(\hat{v}'). \]

Thus, the fact that (8) for holds for types in \(D(\sigma^i)\) implies that (8) holds for types in \(D^i\).

**Proof of Theorem 5:**

(a) Let \((q, t)\) be an IC and IR mechanism that satisfies majorization monotonicity in model \(\mathcal{M}^i\). By Proposition 2, there exists a non-bossy mechanism \((q^\#(v), t^\#(v))\) such that \((q(v), t(v)) = (q^\#(v), t^\#(v))\) almost everywhere. To be precise, there exists a set \(\hat{D}^i \subseteq D^i\), where \(D^i \setminus \hat{D}^i\) has zero measure and \((q(v), t(v)) = (q^\#(v), t^\#(v))\) for all \(v \in \hat{D}^i\). Thus, \(\text{Rev}(q, t; f^i) = \text{Rev}(q^\#, t^\#; f^i)\). Moreover, \((q^\#, t^\#)\) satisfies majorization monotonicity on the set \(\hat{D}^i\) and is rank preserving.

Let \((v_i, v_{-i}), (\hat{v}_i, v_{-i}) \in \hat{D}^i\), with \(\hat{v}_i > v_i\). By IC, \(q^\#_i(\hat{v}_i, v_{-i}) \geq q^\#_i(v_i, v_{-i})\). If \(q^\#_i(\hat{v}_i, v_{-i}) = q^\#_i(v_i, v_{-i})\), then object non-bossiness implies \(q^\#(v_i, v_{-i}) = q^\#(\hat{v}_i, v_{-i})\), and IC implies \(t^\#(v_i, v_{-i}) = t^\#(\hat{v}_i, v_{-i})\). If, instead, \(q^\#_i(\hat{v}_i, v_{-i}) > q^\#_i(v_i, v_{-i})\), then majorization monotonicity of \((q^\#, t^\#)\) on \(\hat{D}^i\) implies that \(q^\#(\hat{v}_i, v_{-i}) \succ_w q^\#(v_i, v_{-i})\). By Proposition 4, we have \(t^\#(\hat{v}_i, v_{-i}) \geq t^\#(v_i, v_{-i})\). Thus, for all \((v_i, v_{-i}), (\hat{v}_i, v_{-i}) \in \hat{D}^i\), we have \(\hat{v}_i > v_i\) implies \(t^\#(\hat{v}_i, v_{-i}) \geq t^\#(v_i, v_{-i})\). This in turn implies that for all \(\hat{v}, v \in \hat{D}^i\), if \(\hat{v} > v\) then \(t^\#(\hat{v}) \geq t^\#(v)\). Since \(D^i \setminus \hat{D}^i\) has zero measure, for any pair of distributions \(F^i\) with density \(f^i\) and \(\hat{F}^i\) with density \(\hat{f}^i\) such that \(\hat{F}^i\) first-order stochastically dominates \(F^i\), we have

\[
\text{Rev}(q, t; F^i) = \text{Rev}(q^\#, t^\#; F^i) = \int_{D^i} t^\#(v)f^i(v)dv \leq \int_{D^i} t^\#(v)\hat{f}^i(v)dv
\]

\[
= \text{Rev}(q^\#, t^\#; \hat{F}^i) = \text{Rev}(q, t; \hat{F}^i)
\]

This establishes revenue monotonicity of \((q, t)\) in model \(\mathcal{M}^i\).

Next, let \((q, t)\) be a symmetric, IC, and IR mechanism that satisfies majorization monotonicity in model \(\mathcal{M}^i\). By Theorem 1, its restriction to \(D(\sigma^i)\) defines an IC, IR, and rank-preserving mechanism; clearly, it satisfies majorization monotonicity. Hence, this is a mechanism for model \(\mathcal{M}^i\) on type space \(D^i\). By the argument above, there exists another IC and IR mechanism \((q^\#, t^\#)\) which coincides with the restriction of \((q, t)\) to \(D(\sigma^i)\) almost everywhere and is object non-bossy. By our earlier argument, majorization monotonicity implies that if \(\hat{v} \geq v\) and \(\hat{v}, v \in D(\sigma^i)\), then \(t^\#(\hat{v}) \geq t^\#(v)\). Consequently, \(t(\hat{v}) \geq t(v)\) for
almost all $\hat{v}, v \in D(\sigma^i)$. Since $(q, t)$ is a symmetric mechanism, for almost all $v, \hat{v} \in D^u$, with $\hat{v} \geq v$, we have $t(\hat{v}) \geq t(v)$. This in turn implies revenue monotonicity of $(q, t)$ in model $\mathcal{M}^u$.

(b) In model $\mathcal{M}^i$, let $(q, t)$ be an IC and IR mechanism which is almost deterministic. WLOG, we assume that $q$ is non-bossy.\(^{33}\) Let $(\hat{v}_i, v_{-i})$ and $(v_i, v_{-i})$ be two type profiles with $\hat{v}_i > v_i$. By IC, $q_i(\hat{v}_i, v_{-i}) \geq q_i(v_i, v_{-i})$. By non-bossiness, if $q_i(\hat{v}_i, v_{-i}) = q_i(v_i, v_{-i})$, we have $q(\hat{v}_i, v_{-i}) = q(v_i, v_{-i})$. Suppose, instead, that $q_i(\hat{v}_i, v_{-i}) > q_i(v_i, v_{-i})$. Since $q$ is almost deterministic, for all $k < i$, $q_k(\hat{v}_i, v_{-i}) = 1 \geq q_k(v_i, v_{-i})$. Further, for all $k > i$, $q_k(v_i, v_{-i}) = 0 \leq q_k(\hat{v}_i, v_{-i})$. Hence, $q(\hat{v}_i, v_{-i}) \succ_w q(v_i, v_{-i})$. Thus, $q$ satisfies majorization monotonicity. From part (a), $(q, t)$ is revenue monotone.

The proof for model $\mathcal{M}^u$ is similar to the proof in part (a). ■

\(^{33}\)If $(q, t)$ is bossy then by Proposition 2, there exists another mechanism $(q^\sharp, t^\sharp)$ which is non-bossy, almost deterministic, and rank preserving which agrees with $(q, t)$ except on a set of measure zero. Hence, the expected revenue from $(q^\sharp, t^\sharp)$ and $(q, t)$ is equal, and $(q, t)$ is revenue monotone if and only if $(q^\sharp, t^\sharp)$ is revenue monotone.
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Appendix B  Online Appendix

This appendix, which contains some proofs and an example, is not for publication.

Proof of Theorem 2: Suppose \((q, t)\) is an asymmetric, IC, and IR mechanism in model \(\mathcal{M}^h\). From \((q, t)\) we construct another IC and IR mechanism \((q^*, t^*)\) which is symmetric and has the same expected revenue as \((q, t)\). Consequently, there exists an optimal mechanism which is symmetric.

For any \(\sigma \in \Sigma\), let \(\sigma^{-1} \in \Sigma\) be such that \(\sigma \sigma^{-1} = \sigma^1 = \sigma^1\). For all \(v \in D^h\), define

\[
q(v; \sigma) := q^{\sigma^{-1}}(v^\sigma) = (q_{\sigma^{-1}(1)}(v^\sigma), \ldots, q_{\sigma^{-1}(n)}(v^\sigma))
\]

\[
t(v; \sigma) := t(v^\sigma)
\]

Then for any \(v, \tilde{v} \in D^h\)

\[
v \cdot q(v; \sigma) - t(v; \sigma) = v \cdot q^{\sigma^{-1}}(v^\sigma) - t(v^\sigma)
\]

\[
= v^\sigma \cdot q(v^\sigma) - t(v^\sigma)
\]

\[
\geq v^\sigma \cdot q(\tilde{v}^\sigma) - t(\tilde{v}^\sigma) \quad \text{(since } (q, t) \text{ is IC)}
\]

\[
= v \cdot q^{\sigma^{-1}}(\tilde{v}^\sigma) - t(\tilde{v}^\sigma)
\]

\[
= v \cdot \hat{q}(\tilde{v}; \sigma) - t(\tilde{v}; \sigma)
\]

Hence \((\hat{q}(\cdot; \sigma), \hat{t}(\cdot; \sigma))\) is IC. That \((\hat{q}(\cdot; \sigma), \hat{t}(\cdot; \sigma))\) is IR follows from IR of \((q, t)\).

For all \(v \in D^h\), define,

\[
q^*(v) := \frac{1}{n!} \sum_{\sigma \in \Sigma} \hat{q}(v; \sigma)
\]

\[
t^*(v) := \frac{1}{n!} \sum_{\sigma \in \Sigma} \hat{t}(v; \sigma)
\]

The mechanism \((q^*, t^*)\) is IC and IR as it is a convex combination of IC and IR mechanisms. Extend \((q^*, t^*)\) to \(\overline{D}^h\) as in Lemma 1. To see that \((q^*, t^*)\) is a symmetric mechanism, note that for any fixed permutation \(\bar{\sigma}\) and any \(v \in D^h\),

\[
t^*(v^\sigma) = \frac{1}{n!} \sum_{\sigma \in \Sigma} \hat{t}(v^\sigma; \sigma) = \frac{1}{n!} \sum_{\sigma \in \Sigma} t(v^{\sigma \sigma}) = \frac{1}{n!} \sum_{\sigma' \in \Sigma} t(v^{\sigma'})
\]

\[34\]Note that even if the mechanism \((q, t)\) is deterministic (and asymmetric), the mechanism \((q^*, t^*)\) may be random.
\[
q^*(v^\sigma) = \frac{1}{n!} \sum_{\sigma' \in \Sigma} \hat{q}(v^\sigma; \sigma') = \frac{1}{n!} \sum_{\sigma' \in \Sigma} q^{\sigma^{-1}}(v^\sigma) = \frac{1}{n!} \sum_{\sigma' \in \Sigma} q^{\sigma^{-1} \sigma^{-1}}(v^\sigma)
\]

where \(\sigma' = \tilde{\sigma}\sigma\).

Finally, the expected revenue from \((q^*, t^*)\) is

\[
\text{REV}(q^*, t^*; f^H) = \int_{\mathcal{D}^H} t^*(v)f^H(v)dv = \int_{\mathcal{D}^H} \frac{1}{n!} \left( \sum_{\sigma} t(v; \sigma) \right) f^H(v)dv = \int_{\mathcal{D}^H} \frac{1}{n!} \left( \sum_{\sigma} t(v^\sigma) f^H(v^\sigma) \right)dv
\]

where we used exchangeability of \(f^H\) in the third and fifth equalities. Hence, \((q^*, t^*)\) is a symmetric, IC and IR mechanism. By Theorem 1, \((q^*, t^*)\) is rank-preserving. Thus, for every IC and IR mechanism, there exists a symmetric and rank-preserving IC and IR mechanism that generates the same expected revenue. Hence, there exists a symmetric and rank-preserving optimal mechanism.

\[\text{PROOF OF PROPOSITION 3:}\] Let \((q, t)\) be a deterministic, IC, and IR mechanism. Let the range of \(q\) be \(R(q) := \{k(v) : v \in D^i\}\). By IC, if \(k(v) = k(v')\), then \(t(v) = t(v')\). For every \(k \in R(q)\), define \(P(k) := t(v)\) for some \(v \in D^i\) with \(k(v) = k\).

We now define \(P\) for every \(k \notin R(q)\). If \(k = 0 \notin R(q)\), let \(P(0) := 0\). Consider \(k \notin R(q)\) for some \(k \geq 1\). For every \(k' \in R(q)\), define\(^{35}\)

\[
d(k', k) := \sup_{v \in D : k(v) = k'} \left[ \sum_{i=0}^{k} v_i - \sum_{i=0}^{k'} v_i \right] \quad (23)
\]

\[
P(k) := \max_{k' \in R(q)} \left[ P(k') + d(k', k) \right] \quad (24)
\]

This completes the definition of \(P\). We now proceed in several steps.

\[\text{STEP 1.}\] We show that \(P(0) = 0\). If \(0 \notin R(q)\), we have \(P(0) = 0\) by definition. If \(0 \in R(q)\), then IC implies that \(q(v) = 0\). Thus, \(0 = u(v) = -P(0)\) implies that \(P(0) = 0\).

\(^{35}\)We use the convention that \(v_0 = 0\).
Step 2. We show the IC constraints hold with respect to transfers $P$, i.e.,

$$u(v) \geq \sum_{i=0}^{k'} v_i - P(k') \quad \forall v \in D^i, \forall k' \in \{0, 1, \ldots, n\}$$

(25)

If $k' = 0$, then the above inequality holds as $P(0) = 0$ and $(q, t)$ is IR. Consider $k' \geq 1$. If $k' \in R(q)$, then IC constraint (25) follows from IC of $(q, t)$ and the definition of $P$. Suppose, instead, that $k' \notin R(q)$. Take any $v \in D^i$ and suppose that $k(v) = k \in R(q)$. Then, by the definition of $P(k')$, we see that

$$P(k') \geq P(k) + d(k, k') \geq P(k) + \sum_{i=1}^{k'} v_i - \sum_{i=1}^{k} v_i$$

$$\implies u(v) = \sum_{i=1}^{k} v_i - P(k) \geq \sum_{i=1}^{k'} v_i - P(k')$$

Step 3. Next, we show that $P$ is monotone. Pick $k' > k$. If $k \in R(q)$, by (25) for some $v \in D^i$ with $k(v) = k$, we have

$$P(k') - P(k) \geq \sum_{i=0}^{k'} v_i - \sum_{i=0}^{k} v_i \geq 0$$

If $k \notin R(q)$, we consider two cases. If $k = 0$, then by (25) for type $v$, $u(v) = 0 \geq \sum_{i=0}^{k'} v_i - P(k')$. Hence, $P(k') \geq 0 = P(0)$, where $P(0) = 0$ follows from Step 1.

If $k \notin R(q)$ and $k \geq 1$, from (24) we know that there exists a $k'' \in R(q)$ such that

$$P(k) = P(k'') + d(k'', k)$$

(26)

Pick any type $v$ such that $k(v) = k''$. Then (25) implies

$$\sum_{i=0}^{k''} v_i - P(k'') \geq \sum_{i=0}^{k'} v_i - P(k')$$

$$\iff P(k') - P(k'') \geq \sum_{i=0}^{k'} v_i - \sum_{i=0}^{k''} v_i = \sum_{i=0}^{k'} v_i - \sum_{i=0}^{k} v_i + \sum_{i=0}^{k} v_i - \sum_{i=0}^{k''} v_i \geq \sum_{i=0}^{k} v_i - \sum_{i=0}^{k''} v_i$$

where the last inequality is implied by $k' > k$. As this holds for all $v$ with $k(v) = k''$, we get

$$P(k') - P(k'') \geq \sup_{v: k(v) = k''} \left[ \sum_{i=0}^{k} v_i - \sum_{i=0}^{k''} v_i \right] = d(k'', k) = P(k) - P(k'')$$
where the last equality follows from (26). Hence, we get \( P(k') \geq P(k) \). Thus, \( p_0 := 0 \) and
\[
p_k := P(k) - P(k - 1) \geq 0 \quad \forall k \geq 1
\]
constitute a pricing mechanism that implements \((q, t)\), provided that \( p_k \leq \bar{v} \) for all \( k \geq 1 \). This is shown next.

**Step 4.** If \( k \in R(q) \), then by (25), for some \( v \in D \) with \( k(v) = k \), we have
\[
p_k = P(k) - P(k - 1) \leq \sum_{i=1}^{k} v_i - \sum_{i=1}^{k-1} v_i = v_k \leq \bar{v}
\]
If \( k \not\in R(q) \), by the definition of \( P(k) \), there is \( k' \in R(q) \) such that
\[
P(k) = P(k') + d(k', k) \tag{27}
\]
By (25), for any \( v \) with \( k(v) = k' \), we have
\[
\sum_{i=1}^{k'} v_i - P(k') \geq \sum_{i=1}^{k-1} v_i - P(k - 1)
\]
\[\iff P(k') - P(k - 1) \leq \sum_{i=1}^{k'} v_i - \sum_{i=1}^{k-1} v_i \]
\[= \sum_{i=1}^{k'} v_i - \sum_{i=1}^{k} v_i + v_k \]
\[\leq \sum_{i=1}^{k'} v_i - \sum_{i=1}^{k} v_i + \bar{v}
\]
Thus,
\[
\bar{v} + P(k - 1) \geq P(k') + \sup_{v: k(v) = k'} \left[ \sum_{i=1}^{k} v_i - \sum_{i=1}^{k'} v_i \right] = P(k') + d(k', k) = P(k)
\]
where the last equality is from (27). Hence, we get \( p_k = P(k) - P(k - 1) \leq \bar{v} \).

**Example 4** We present examples to show that none of the three sufficient conditions in Theorem 4 can be dropped. In these examples, there are two identical objects with decreasing marginal values.

The type space is \( D^I \) for \( n = 2 \). The mechanism is specified as follows, with buyer types identified by the number of units they receive:

\[
(q(v), t(v)) = \begin{cases} 
((0, 0), 0), & \text{if } v_1 \leq 0.5 \text{ and } v_1 + v_2 < 0.75, \quad \text{Type 0} \\
((1, 0), 0.5), & \text{if } v_1 > 0.5 \text{ and } v_2 < 0.25, \quad \text{Type 1} \\
((1, 1), 0.75), & \text{if } v_1 + v_2 > 0.75 \text{ and } v_2 \geq 0.25, \quad \text{Type 2} \\
((\bar{q}, 0), \bar{q}v_1), & \text{if } v_1 + v_2 = 0.75 \text{ and } v_1 \leq 0.5, \text{ or } v_2 \geq 0.25, \quad \text{Type } \bar{q}
\end{cases}
\]
Type $\tilde{q}$ buyers are on the thick blue line-segment in Figure 3. Although each type $\tilde{q}$ buyer receives the same allocation, $\tilde{q} \in (0, 1]$, the transfer $\tilde{q}v_1$ is a function of the buyer’s value for the first unit. Thus, each type $\tilde{q}$ buyer has a payoff of zero.

We verify that this mechanism satisfies UIC. First, note that a type 0 cannot get a positive payoff by reporting their type as $\tilde{q}$, 1, or 2. By UIC, type 1 buyers can misreport their type as 0 and 2 buyers only; it is easily verified that a misreport to one of these two types is not profitable.

Next, the upper (or lower) set of any type $\tilde{q}$ buyer does not include any another type $\tilde{q}$ buyer or a type 1 buyer. Under truthful reporting, a type $\tilde{q}$ gets payoff zero which is the same payoff it would obtain with a misreport to a type 2 buyer, as $v_1 + v_2 = 0.75$ for any type $\tilde{q}$ buyer. Its payoff remains zero by misreporting as type 0. Hence, type $\tilde{q}$ buyers also satisfy UIC.

Finally, consider a type 2 buyer with value, say, $(\hat{v}_1, \hat{v}_2)$. If it misreports a type $\tilde{q}$ within its UIC constraint set, it gets $\tilde{q} \in (0, 1]$ units. The lowest possible price it pays with such a misreport is $\tilde{q}\max(0.375, 0.75 - \hat{v}_2)$ (see Figure 3). This deviation is not profitable as

$$\tilde{q}\hat{v}_1 - \tilde{q}\max(0.375, 0.75 - \hat{v}_2) \leq \tilde{q}[\hat{v}_1 + \hat{v}_2 - 0.75] \leq \hat{v}_1 + \hat{v}_2 - 0.75$$

The other UIC constraints (unprofitability of misrepresentations to a type 0 or 1) hold for a type 2 buyer.

However, any type $\tilde{q}$ buyer with $v_1 > 0.375$ profits from misreporting their type as $(0.375, 0.375)$. Thus IC, is not satisfied. In addition, IC is not satisfied for a positive measure of type 2 buyers. Take any type 2 buyer whose values satisfy $0.75 < v_1 + v_2 \leq 0.75 + \epsilon_1$, 46
\(v_1 \geq 0.5 - \epsilon_2\). Under truthful reporting, this buyer has a payoff less than \(\epsilon_1\) in the mechanism. If this type misreports its type as \((0.375, 0.375)\) it obtains a payoff of at least \(\tilde{q}(0.125 - \epsilon_2)\). This deviation is profitable when \(\epsilon_1 < \tilde{q}(0.125 - \epsilon_2)\).

In this example, for any \(\tilde{q} \in [0, 1)\), the mechanism is UIC but not IC. If \(\tilde{q} \in (0, 1)\), this mechanism is non-bossy. As the domain is \(D^i\), the strong-lattice property is satisfied. Hence, if \(\tilde{q} \in (0, 1)\), this example shows that a stochastic, non-bossy, and UIC mechanism in a strong lattice type space need not be IC.

If, instead, \(\tilde{q} = 1\) then the mechanism is deterministic but bossy. To see this, note that the allocation to a type \(\tilde{q}\) buyer with valuation \((v_1, v_2)\) is \(q(v_1, v_2) = (1, 0)\) and the allocation to a type 2 buyer with value \((v_1 + \epsilon, v_2)\) is \(q(v_1, v_2) = (1, 1)\). Thus, in going from \((v_1, v_2)\) to \((v_1 + \epsilon, v_2)\), only \(v_1\) changes and only \(q_2\) changes which is bossiness.

Finally, consider another example in which the domain is the set of \(\tilde{q}\) types in the previous example (the blue types in Figure 3). The domain does not satisfy the lattice property and the upper or lower set of each type is empty. Every mechanism defined on this type space is UIC. Clearly, not every mechanism is IC.

Thus, none of the assumptions of Theorem 4 can be dropped. ■

Weak Majorization and Second-order Stochastic Dominance

In Lemma 4 below, we show the equivalence between weak majorization and second-order stochastic dominance of allocation rules in model \(M^i\).

Let \(x \equiv (x_0, x_1, x_2, \ldots, x_n = 1)\) and \(y \equiv (y_0, y_1, \ldots, y_n = 1)\) be two cumulative distribution functions (cdfs) over \(\{0, 1, \ldots, n\}\). Since \(x\) and \(y\) are cdfs, \(1 = x_n \geq x_{n-1} \geq \ldots \geq x_0 \geq 0\) and \(1 = y_n \geq y_{n-1} \geq \ldots \geq y_0 \geq 0\).

**Definition 13** The cdf \(x\) second-order stochastically dominates (SOSD) cdf \(y\), denoted \(x \succ_{SOSD} y\), if

\[
\sum_{i=0}^{k} x_i \leq \sum_{i=0}^{k} y_i \quad \forall \ k \in \{0, 1, \ldots, n\}
\]

Take two allocation probability vectors \(q, q' \in M^i\). The pdf over \(\{0, 1, \ldots, n\}\) induced by \(q\) is

\[
(1 - q_1, q_1 - q_2, q_2 - q_3, \ldots, q_n)
\]
and the cdf over \(\{0, 1, \ldots, n\}\)

\[ F(q) := (1 - q_1, 1 - q_2, 1 - q_3, \ldots, 1) \]

**Lemma 4** The following are equivalent for any pair of allocation probability vectors \(q, q' \in \mathcal{M}'\).

\[ [q \succ_w q'] \iff [F(q) \succ_{SOSD} F(q')] \]

**Proof:**

\[ q \succ_w q' \iff \sum_{i=1}^{k} q_k \geq \sum_{i=1}^{k} q'_i \quad \forall \ k \in \{1, \ldots, n\} \]

\[ \iff \sum_{i=0}^{k} (1 - F_i(q)) \geq \sum_{i=0}^{k} (1 - F_i(q')) \quad \forall \ k \in \{0, \ldots, n\} \]

\[ \iff \sum_{i=0}^{k} F_i(q) \leq \sum_{i=0}^{k} F_i(q') \quad \forall \ k \in \{0, \ldots, n\} \]

\[ \iff F(q) \succ_{SOSD} F(q') \]

\[ \blacksquare \]