1. INTRODUCTION

Vacuum selection is one of the most important issues in string/M theory. The standard cartoon of string vacua (see figure 1.1) showing all the ten dimensional perturbative string asymptopia together with the eleven-dimensional limit, refers to solutions with a large number of supersymmetries, and flat directions (moduli) in the effective action.

![Figure 1.1 Schematic picture of the space of string vacua with extended supersymmetry.](image)

A much richer story arises as more supersymmetries are broken, and such flat directions are lifted or absent altogether. Eventually we will
want to understand vacua which respect no supersymmetries whatsoever. To explore the configuration space, one needs to understand off-shell physics, since the effective potential in such situations will not be flat.

Inflation is an example of such off-shell physics with fully broken supersymmetry; one might consider it as a variant of ‘tachyon condensation’ in string theory, \textit{i.e.} the dynamical decay of an unstable configuration. For example, the dynamics of a D-brane and anti D-brane, filling the noncompact dimensions of space and separated in some compact direction, provides an interesting model of inflation \cite{1}. Here supersymmetry is fully broken, since the half of the supersymmetries preserved on the brane are incompatible with those preserved on the anti-brane. Thus we are interested in understanding tachyon condensation phenomena in string theory, for both closed and open strings.

For open strings, a rather complete picture is developing, largely due to ideas of Sen, Witten, and others \cite{2,3}. The presence of dynamical gravity, while certainly of interest, is however a complication if we wish to isolate the condensation phenomena and their properties.\footnote{Consider for example the \(d = 26\) bosonic string. Here condensation of the closed string tachyon might lead to a change in the dimension of spacetime – the only known stable vacua have \(d \leq 2\) (see \cite{4,5} for reviews).} One can eliminate this complication by studying ‘impurity’ or ‘defect’ dynamics, where the tachyon is confined to the defect, see figure 1.2. The defect could be

- a D-brane system;
- an NS5-brane system;
- an orbifold fixed point;
- \textit{etc.}

If there are sufficiently many noncompact directions transverse to the defect, we expect to be able to largely ignore gravitational back reaction, or at least localize it near the defect (and for open strings, we can tune it to zero by taking the \(g_s \rightarrow 0\) limit). Suppose that the bulk theory is supersymmetric, while the defect breaks supersymmetry. Then typically we will have a tachyon localized at the defect, since the ambient space far away from the defect is locally stable. We wish to understand the process of condensation of this localized tachyon.

There are two common methods used to study this phenomenon. The first is to look for time-dependent solutions of the string equations of motion involving the growing tachyon mode. At asymptotically early times,
Figure 1.2 Decay of a localized defect leads to either a (more) stable relic or vacuum, plus radiation.

The tachyon background corresponds to a modification of the worldsheet action describing string propagation:

\[ S_{WS} \sim S_0 + \int d^2 z e^{E X^0} O_{tach} \]  

(1.1)

where \( O_{tach} \) is a vertex operator for the spatial profile of the tachyon and \( X^0 \) is spacetime time. One then looks to understand the late time behavior of the solution. This procedure is typically quite difficult, and fraught with technical difficulties associated to the fact that little is understood about string theory in time-dependent backgrounds – much of the standard treatment of string perturbation theory rests on the connection to light cone gauge and/or analytic continuation from Euclidean space, neither of which are guaranteed to exist in general time dependent backgrounds.\(^2\)

\(^2\)Although there is recent work of Sen [6, 7] on obtaining exact solutions for open strings via analytic continuation from Euclidean solutions, and of Gutperle and Strominger [8] on time-dependent solutions of the low-energy field equations related to decaying D-branes.
A second approach adopts a worldsheet renormalization group (RG) viewpoint. Conformal invariance of the worldsheet quantum field theory, \textit{i.e.} fixed points of the 2d renormalization group, are classical solutions of the spacetime field equations of string theory (\textit{c.f.} [9] and references therein.). Regarding the couplings \( \{ \lambda \} \) in the worldsheet action as a description of the spacetime background fields, the RG fixed point equations \( \beta(\lambda) = 0 \) characterize background fields satisfying the classical equations of motion, among all possible backgrounds. For example, the usual nonlinear sigma model background parametrized by a metric \( G_{\mu\nu} \), antisymmetric tensor gauge field \( B_{\mu\nu} \), and dilaton \( \Phi \), yields the usual low-energy field equations

\[
\begin{align*}
\beta^{(G)}_{\mu\nu} &= \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{1}{2} \alpha' H_{\mu\lambda \kappa} H^{\lambda \kappa}_{\nu} + O(\alpha'^2) \\
\beta^{(B)}_{\mu\nu} &= -\frac{1}{2} \alpha' \nabla^\lambda H_{\lambda\mu\nu} + \alpha' \nabla^\lambda \Phi H_{\lambda\mu\nu} + O(\alpha'^2) \\
\beta^{(\Phi)} &= \frac{D-10}{4} - \frac{1}{2} \alpha' \nabla^2 \Phi + \frac{1}{2} \alpha' \nabla^\lambda \Phi \nabla^\lambda \Phi - \frac{1}{24} \alpha' H_{\lambda\mu\nu} H^{\lambda \mu \nu} + O(\alpha'^2) (1.2)
\end{align*}
\]

as the condition of conformal invariance; similarly, scale invariance under boundary perturbations yields the open string equations of motion [10, 11, 12, 13].

Nonconformal backgrounds in the worldsheet field theory provide a way of continuing off-shell; RG flows interpolate between classical solutions, and thus provide information about the topology of the effective action and the configuration space.

The mass shell condition \( L_0 - 1/2 = \alpha' (-E^2 + p^2)/2 + h_{\text{int}} = 0 \), where \( h_{\text{int}} \) is the contribution to the conformal dimension of a vertex operator coming from internal structure (\textit{i.e.} the conformal field theory (CFT) describing the defect), gives the correspondence

\[
\alpha' m^2 = (h_{\text{int}} - \frac{1}{2}) \begin{cases} 
> 0 & \text{massive in spacetime} \\
= 0 & \text{massless in spacetime} \\
< 0 & \text{tachyonic in spacetime} 
\end{cases} (1.3)
\]

between spacetime properties and the effect under the renormalization group of perturbing the worldsheet action by the vertex operator.

Tachyon condensation thus corresponds to adding a relevant operator to the worldsheet Lagrangian describing the background in which perturbative strings propagate

\[
S_{WS} = S_0 + \lambda(t) \int d^2 z O_{\text{tach}} . \quad (1.4)
\]
This breaks conformal symmetry; the perturbation grows toward the IR on the worldsheet, and the endpoint of tachyon condensation in this context is the IR fixed point of the worldsheet renormalization group flow. Near the UV fixed point (where the tachyon perturbation vanishes), the coupling scales as

\[ \lambda(t) \sim e^{E^2 t} \]  

where \( t \) is the (logarithmic) worldsheet scale. Note that there is not a direct correspondence between spacetime time dependence (1.1) and worldsheet scale dependence (1.5); in particular the effective spacetime equations of motion are second order differential equations, while the RG equation is first order. Nevertheless, RG flows define interesting paths in the configuration space leading away from unstable extrema toward more stable ones; and there is no known dynamical decay process for which there is no corresponding RG flow.

One can also relate the renormalization group more closely to spacetime dynamics via light front evolution [14]. Given an RG flow in the nonlinear sigma model with \( d \) dimensional target space, one can make a conformally invariant theory in \( D = d + 2 \) dimensional spacetime via

\[
\begin{align*}
\frac{ds^2}{-2dudv + g_{ij}(u, x)dx^i dx^j} \\
\Phi = p\nu + \phi(u, x)
\end{align*}
\]  

with \( p = \text{const.} \); for the full set of conditions on the fields, the reader is referred to [14]. The salient feature is that the equation of motion (1.2) for the transverse metric reduces to

\[
p\frac{\partial g_{ij}}{\partial u} = \beta_{ij}^{(g)} + D_i W_j + 2D_i D_j \phi ,
\]  

so that one can interpret the null coordinate \( u \) as ‘RG time’. The RG flow becomes the profile of a gravitational wave in spacetime, and the geometry interpolates between fixed points of the RG flow of the \( d \)-dimensional transverse geometry as one traverses the wave.

In these lectures, we will explore the renormalization group approach to the condensation of tachyons on localized defects in string theory, following [15, 16]. In section 2 we review a variety of results from the study of open string tachyon condensation on collections of D-branes, which may be simply obtained using the RG approach. The RG trajectories are specified by the gradient of the open string effective action, which therefore monotonically decreases along the flows. We explain the connection of this effective action to the density of states and the intuition that the renormalization group is a process of thinning of degrees of freedom (the ‘dissipated states’ of the title). We then explore in section 3 the
extent to which these ideas can be carried over to closed string tachyon condensation, using the example of nonsupersymmetric orbifolds as the prototype of an unstable closed string defect. The examples studied exhibit rich connections to algebraic geometry, which we exploit to map out the set of RG flows. An attempt is made to characterize the flows by a decrease of the density of states associated to the defect, analogous to the open string case, with inconclusive results.

2. OPEN STRINGS

For open strings, one has boundary RG flow

\[ S_{\text{WS}} = S_0 + \lambda \int_{\partial} ds \mathcal{O}_{\text{tach}}. \]  

The worldsheet bulk theory remains conformal; in this case the RG flow interpolates between two different boundary states of the same target space – in other words, the defect decays.

In this section we survey open string RG flows, restricting our attention to the superstring unless otherwise indicated. The examples we will consider are:

- \( Dp-Dp \rightarrow \text{vacuum} \)
- \( Dp-Dp \rightarrow D(p-2) \)
- \( Dp-D(p-2) \rightarrow Dp \text{ with magnetic flux} \)
- \( N \) \( D0 \) branes on \( SU(2) \rightarrow D2 \)

The first of these examples is the complete decay of an unstable brane system. In the remaining examples, conserved topological charges prevent the complete decay of the initial brane configuration. These topological charges are the subject of K-theory, which is roughly the classification of vector bundles up to isomorphism – appropriate since open string dynamics is that of (generalized) gauge fields. The last decay \( (D0 \rightarrow D2) \) is an example of the Myers effect [17]; in the context of the renormalization group, one finds an amusing application of one of its earliest successes, namely the Kondo model [18] (for a review, see [19]).

A basic property that we will be using is the spectrum of open strings connecting two D-branes (label them \( a \) and \( b \)) oriented at an angle \( \alpha \) relative to one another (see figure 1.3). The boundary conditions on such strings are

\[
\begin{align*}
\text{Re} \partial_\sigma X|_{\sigma=0} &= 0, & \text{Re} e^{i\alpha} \partial_\sigma X|_{\sigma=\pi} &= 0, \\
\text{Im} X|_{\sigma=0} &= 0, & \text{Im} e^{i\alpha} X|_{\sigma=\pi} &= 0
\end{align*}
\]  

(1.9)
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Figure 1.3 A pair of D-branes at relative angles has an open string tachyon localized along their intersection. The tachyon lives in the sector of open strings with mixed $a$–$b$ boundary conditions depicted on the right.

and similarly for the worldsheet superpartners $\psi$ of $X$.

Solving for the string oscillator spectrum, one finds the moding shifted by $\alpha/\pi$. This is just like the twisted sectors of $\mathbb{Z}_N$ orbifolds; thus one deduces that there exists an operator

$$\Sigma_\alpha = \sigma_\alpha \exp[i(\alpha/\pi)H]$$

(1.10)

(where $e^{iH} = \psi_1 + i\psi_2$ bosonizes the worldsheet fermions, and $\sigma_\alpha$ implements the twisting of $X$), which creates the lowest mass $a$–$b$ open string when acting on (say) the $a$–$a$ ground state.\(^3\) The calculation of the conformal dimension of this open string twist field will be exactly as for the corresponding closed string twist operator [20, 21, 22]; one finds

$$h_\Sigma = \frac{1}{2} \frac{\alpha}{\pi} \left(1 - \frac{\alpha}{\pi}\right) + \frac{1}{2} \left(\frac{\alpha}{\pi}\right)^2 = \frac{1}{2} \frac{\alpha}{\pi} \pi$$

(1.11)

($0 < \alpha < \pi$). This operator is GSO odd; one gets a GSO even operator by applying $\psi^* = e^{-iH}$ to get

$$\bar{\Sigma}_\alpha = \sigma_\alpha \exp[-i(1 - \alpha/\pi)H]$$

(1.12)

whose conformal dimension is $h_{\bar{\Sigma}} = \frac{1}{2}(1 - \frac{\alpha}{\pi})$. Thus there is a tachyon whenever the angle between the branes is nonzero.

Among the interesting values of $\alpha$ are

- $\alpha = 0$ describes two BPS $Dp$ branes;
- $\alpha = \pi$ describes $Dp$–$\overline{Dp}$;
- $\alpha = \pi/2$ is related via T-duality to $Dp$–$D(p-2)$.

\(^3\)The conjugate operator $\Sigma^*$ creates the conjugate string when acting on the $b$–$b$ open string.
In the last two cases there is a tachyon in the spectrum related to the decays listed above. Let us discuss their condensation.

2.1 \( DP - \overline{DP} \rightarrow \text{VACUUM} \)

Consider first \( \alpha = \pi \). One expects that the branes can and will decay to vacuum plus radiation, for instance the \( D0 - \overline{D0} \) system is a pair of 11d gravitons with opposite momentum on a circle in M-theory, which can scatter into final states with no momentum on that circle. This expectation of decay has been verified by a variety of complementary approaches:

- String field theory [23];
- Noncommutative effective field theory [24];
- Boundary effective field theory (BEFT)/RG flow [25, 26, 27].

The BEFT/RG approach\(^4\) relies on the following property of the disk worldsheet path integral [29, 30, 25, 26, 27, 31]

\[
\Gamma_{ST}(\lambda_i) = Z_{WS}(\lambda_i) = \int DX D\psi e^{S_0} \text{Tr} \exp \left[ - \int_0^1 dsd\theta \lambda_i O^i \right]. \quad (1.13)
\]

In other words, the conditions of vanishing of the beta functions which are the spacetime equations of motion, can be expressed as

\[
\frac{\partial \Gamma_{ST}}{\partial \lambda_i} = \frac{\partial Z_{WS}}{\partial \lambda_i} = \beta^j(\lambda) G_{ij} \quad (1.14)
\]

where \( G_{ij} \) is the (Zamolodchikov) metric on the coupling space. A further important property is that the variation of the disk partition function with respect to worldsheet scale is

\[
\frac{\partial \Gamma_{ST}}{\partial t} = -\beta^i \frac{\partial}{\partial \lambda_i} \Gamma_{ST} = -\beta^i \beta^j G_{ij}; \quad (1.15)
\]

for unitary worldsheet quantum field theories the RG flow is monotonically in the direction of decreasing spacetime action, since the metric \( G_{ij} \) is positive definite. Thus flows by relevant operators agree with the idea that tachyon condensation is a process of rolling down the effective potential to a more stable extremum [32, 33, 15].

\(^4\)Sometimes called boundary string field theory (BSFT). This is somewhat of a misnomer; since the approach uses continuum Euclidean worldsheet field theory, it is restricted to massless and tachyonic perturbations [28], and so is not a field theory of the entire string spectrum.
The relation $\Gamma_{ST} = Z_{WS}$ is illustrated by the example of open string gauge fields. One finds the Dirac-Born-Infeld action [10, 13] as the space-time effective action for smoothly varying gauge fields

$$\Gamma_{ST} = \mu_p \int d^{p+1}x \ e^{-\Phi} \sqrt{\det[G + B + 2\pi\alpha'F]}$$ \hspace{1cm} (1.16)

from a computation of the disk partition function (here $\mu_p$ is the brane tension). More generally, one has a perturbation

$$\lambda_i O^i = \bar{\Gamma} D \Gamma + \Gamma \cdot \bar{T}(X) + \bar{\Gamma} \cdot \bar{A} \mu DX^\mu + \ldots$$ \hspace{1cm} (1.17)

with $D = \partial_\theta + \theta \partial_s$ is the derivative on boundary superspace, and $\Gamma = \eta + \theta f$ is an auxiliary fermionic superfield living only on the boundary; the latter is needed by the Grassmann parity of the boundary integration measure, so that the tachyon action is Grassmann even. If there are $k$ such superfields, their $2k$ dimensional Hilbert space is a convenient way to realize a Chan-Paton vector space. A calculation of the effective action yields [27, 34, 35]

$$\Gamma_{Dp-\bar{Dp}} = 2\mu_p \int d^{p+1}x \ e^{-\Phi} \ e^{-2\pi\alpha'|T|^2} \left[1 + 8\pi\alpha' \log 2 |D \mu T|^2 + \frac{2\pi\alpha'}{8} (F_{Dp}^2 + F_{\bar{Dp}}^2) + \ldots\right].$$ \hspace{1cm} (1.18)

The overall factor of $\exp[-2\pi\alpha'|T|^2]$ is in hindsight very easy to understand; if one perturbs by constant $T$, this has no effect on the worldsheet dynamics, but the partition function rescales as (upon integrating the boundary action over the boundary supercoordinate, and suppressing the $\eta$ integral which merely constructs the Chan-Paton bundle)

$$Z_{\text{disk}} = \int D\bar{X} D\bar{\psi} D\bar{D} e^{-S_0} \exp\left[\frac{1}{2\pi\alpha'} \int \partial_\theta [f T + \bar{f} \bar{T} + |f|^2]\right] \int D\bar{X} D\bar{\psi} e^{-S_0}.$$

Condensation of $T$ leads to suppression of all disk amplitudes, i.e. open strings disappear from the dynamics as the $D-\bar{D}$ system decays away. The corresponding IR fixed point of the renormalization group is trivial.

The relation $\Gamma_{ST} = Z_{WS}$ is a universal property of open string theory on which the BEFT approach relies; the proof of [26, 36, 31] relies only on general properties of the worldsheet RG such as the Callan-Symanzik equation. This relation is possible because of the nature of the Möbius volume of the disk [29, 30, 37] of the form $Vol(PSL(2, \mathbb{R}) = \frac{c_2}{c_1} + c_2$ with $c_{1,2}$ finite constants, leaving a finite remainder after subtraction; even better, $c_1 = 0$ for the superstring due to Bose-Fermi cancellations. It is believed that a similar approach should work for the closed string, however...
there are some unresolved puzzles. The corresponding Möbius volume
\( \text{Vol}(\text{PSL}(2, \mathbb{C})) = \frac{1}{\kappa} + c_2 \log \epsilon; \) clearly the finite term is not universal.

There is a suggestion [38] that the spacetime action is
\( \frac{\partial Z_{\text{sphere}}}{\partial \log \epsilon}, \) which gives the right answer to one-loop order in the nonlinear sigma model on compact targets. It would be interesting to know if this idea or something similar could define the effective action \( \Gamma_{\text{closed}}^\text{ST} \) in terms of quantities intrinsic to the worldsheet field theory.

2.2 \( \text{DP} - \overline{\text{DP}} \rightarrow D(p - 2) \)

Consider the boundary perturbation
\[
\delta S = \lambda \int ds d\theta (\Gamma X + \bar{\Gamma} \bar{X}) ,
\]
where \( X = X^1 + iX^2 \) are coordinates along the brane; after eliminating the auxiliary field \( f \), this is a generalized mass term. In the Hilbert space of the boundary fermion \( \eta \), the state with fermion number zero (one) is the (anti) brane. The worldsheet bulk remains conformal under such a perturbation, but the string coordinates have a mass on the boundary that breaks worldsheet conformal invariance and also target space translation symmetry. The formerly free boundary value of \( X \) becomes confined to \( X = 0 \) in the worldsheet IR, where \( \lambda(t) \to \infty \) (see figure 1.4). The effective boundary condition passes from Neumann to Dirichlet along two directions parallel to the brane, and \( Dp - \overline{Dp} \rightarrow D(p - 2) \). The winding of the phase of the tachyon at spatial infinity is the \( D(p - 2) \) charge (so that \( T \sim x^n \) makes \( n \) lower branes).

One can generalize this construction [3, 39] to other mass terms by adding indices
\[
\delta S = \lambda \int ds d\theta (\Gamma_i X^i + \bar{\Gamma}_i \bar{X}^i)
\]
to get higher codimension branes at the infrared fixed point (c.f. [27]); \( \Gamma \cdot X \) realizes a coupling of orientation in spacetime and Chan-Paton vector bundle (realized by the Hilbert space of the boundary fermions \( \eta \)). A more elaborate example, that of \( D0 \) branes on \( S^3 \), will be discussed below.

Often, the endpoint of tachyon condensation is the lightest object carrying given topological (RR) charges, indeed there is a close connection between RR charge and K-theory [40]. The coupling of D-branes to RR charge can be seen in the boundary effective field theory formalism by computing \( Z_{\text{disk}} \) with the insertion of a RR vertex operator in the center
Figure 1.4 A boundary potential deepens under RG flow, confining boundary coordinate(s) to a lower dimensional subspace.

of the disk (see for instance [34]). One finds

\[ Z_{RR} = \mu_p \int C \wedge STr \exp[2\pi i \alpha' F] \sqrt{\hat{A}(R)}. \]  

(1.22)

The trace \( STr \) is graded so that branes count with a plus sign and anti-branes with a minus sign (i.e. \( \text{Tr}[-1]^F \cdots \) in the Hilbert space of the boundary fermions). Here \( F \) is the curvature

\[ iF = \begin{pmatrix} iF_{Dp} - \bar{T}T & \frac{DT}{\partial T} \\ \frac{D\bar{T}}{\partial T} & i\bar{F}_{Dp} - \bar{T}T \end{pmatrix} \]  

(1.23)

of a generalized connection

\[ \mathcal{A} = \begin{pmatrix} iA_{Dp} \\ T \\ i\bar{A}_{Dp} \end{pmatrix} \]  

(1.24)

known as the Quillen superconnection. The idea is that the boundary fermion \( \Gamma = \eta + \ldots \) allows us to think of \( T \cdot \Gamma \) as a one form on the Chan-Paton vector space, just as \( A_\mu DX^\mu \) codes one forms in target space via the boundary value of the fermion \( DX = \psi + \ldots \). Expanding the generalized Chern character (1.22), there is a term

\[ \mu_p \int C_{p-2} \wedge e^{-2\pi \alpha' |T|^2} (2\pi \alpha')^2 dT \wedge d\bar{T} \]  

(1.25)
that gives the coupling to the RR \((p-2)\)-form, indicating that the field configuration on the branes indeed properly carries \(D(p-2)\) brane charge.

### 2.3 DP-D(P-2)

The third example is the tachyon appearing between \(Dp\) and \(D(p-2)\) branes. Because of the coupling \(\int C_{p-2} \wedge F_{Dp}\) in the expansion of (1.22), a \(Dp\) brane can itself carry \(D(p-2)\) brane charge. Let the directions parallel to the \(Dp\) but orthogonal to the \(D(p-2)\) be compactified on a volume \(V_2\). The action of the \(Dp\) brane is

\[
\Gamma_{DBI} = \mu_p \int d^p x e^{-\Phi} \sqrt{\det[G + B + 2\pi \alpha' F]}; \tag{1.26}
\]

with \(n_{p-2}\) units of \(F\) flux on the compact volume \(V_2\), the action of the minimal energy configuration reduces to

\[
\mu_p \sqrt{n_p^2 V_2^2 + n_{p-2}^2 (2\pi \alpha')^2} \int d^{p-2} x (\cdots) \tag{1.27}
\]

which is smaller than the action per unit \((p-2)\) volume, \((n_p V_2 + n_{p-2} 2\pi \alpha') \mu_p\), of the UV fixed point configuration of \(n_p\) \(Dp\)-branes and \(n_{p-2}\) \(D(p-2)\)-branes separately. This is consistent with the idea that the configuration of branes with flux is the endpoint of tachyon condensation; a calculation demonstrating that this is indeed the endpoint was carried out by Gava, Narain and Sarmadi [32].

The fact that a \(Dp\) brane with \(F\) flux is a \(Dp-D(p-2)\) bound state means that another way to make the \(D(p-2)\) brane is to condense the tachyon between this bound state and a \(Dp\); the higher branes will disappear, leaving the lower brane behind.

### 2.4 SYSTEMS WITH LESS SUPERSYMMETRY

So far, we have considered just the simplest properties of branes in situations with a large amount of supersymmetry. One would like to know how much of the above story generalizes to situations with less supersymmetry, say \(\mathcal{N} = 1\) in four dimensions. Douglas and collaborators have made much progress in this direction (c.f. [41] and references therein) in examples with \(\mathcal{N} = 2\) worldsheet supersymmetry. This amount of worldsheet supersymmetry in the absence of D-branes implies \(\mathcal{N} = 2\) spacetime supersymmetry in type II string theory without branes [42, 43]; half-supersymmetric brane configurations then break the symmetry further to \(\mathcal{N} = 1\) in spacetime.
To begin, let us review the connection between worldsheet and target supersymmetry. Consider a worldsheet CFT with $N = (2, 2)$ supersymmetry; one has the supermultiplet of currents containing the stress tensor $T$; two supercurrents $G, G^*$; and the $U(1)$ R-current $J$ that rotates the supersymmetries by opposite phases (and similarly for the right-movers, hence the doubled notation $N = (2, 2)$). A typical example is type II on a Calabi-Yau $n$-fold; there $J$ is related to the $U(1)$ holonomy preserved by the target in the geometrical (large volume) limit. The spacetime $N = 2$ supercharges are related to the integrals of worldsheet supercurrents, whose only dependence on the $N = (2, 2)$ worldsheet theory is through the operator that implements spectral flow between the NS and R sectors. In any worldsheet theory with a $U(1)$ current algebra symmetry, one can decompose all the vertex operators $O$ that are highest weight under the $J$ current algebra into the form

$$O = \Psi \, e^{i\theta H},$$

where $H$ bosonizes the current via $J = iQ_0 \partial H$, and $\Psi$ is an operator that commutes with $J$. In other words, the exponential carries the $U(1)$ charge dependence. Using conventional normalization for the scalar field $H$, $\langle H(z)H(0) \rangle = -\log z$, the two-point function of the current

$$\langle J(z)J(0) \rangle = \left[ \frac{1}{2}(10 - d) \right]^{1/2} \frac{1}{z^2}$$

determines $Q_0 = [(10 - d)/2]^{1/2}$ for the $N = 2$ CFT describing compactification down to $d$ dimensions. One can then represent the spacetime supersymmetry charges as

$$Q = \oint dz (q_{\mathbb{R}^{d-1,1}}) \exp[i(Q_0/2)H]$$
$$\tilde{Q} = \oint d\bar{z} (\bar{q}_{\mathbb{R}^{d-1,1}}) \exp[i(Q_0/2)\bar{H}]$$

where $q_{\mathbb{R}^{d-1,1}}$ is a holomorphic operator of worldsheet dimension $h = \frac{1}{16}(6 + d)$ giving the contribution to the spacetime supersymmetry operator coming from the noncompact directions and the worldsheet BRST ghosts (c.f. [44]).

BPS D-branes will break the spacetime supersymmetry to $N = 1$ via the worldsheet boundary conditions [45]

$$\exp[iQ_0 H] = \exp[i\alpha] \exp[i\theta Q_0 \bar{H}].$$

The phase $\alpha$ determines which of the $S^1$ family of $N = 1$ subalgebras of the spacetime $N = 2$ supersymmetry is preserved by the brane being

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described by the given boundary condition; the key point is that not all branes have the same $\alpha$--supersymmetry is broken for a pair of branes $a, b$ with $\alpha_a \neq \alpha_b$.

The Ramond sector ground states in the $a$--$b$ sector of such a pair will still be chiral fermions in spacetime and therefore massless (the GSO projection acts as a chirality projection on Ramond ground states). Furthermore, spectral flow still relates NS and R boundary conditions, since we still have $\mathcal{N} = 2$ worldsheet supersymmetry. Thus

$$O_{ab}^{NS} = \Psi \exp \left[ i \frac{q}{Q_0} H \right]$$

$$O_{ab}^{R} = \Psi \exp \left[ i \left( \frac{q}{Q_0} - \frac{Q_0}{2} \right) H \right] (q\bar{q})_{R^{d-1,1}} .$$

(1.32)

The dimension $h_{ab}^{R} = 1$ of $O_{ab}^{R}$ is determined by the masslessness of the Ramond ground state; as above, this gets a contribution $\frac{1}{16} (6 + d)$ from the operator $(q\bar{q})_{R^{d-1,1}}$ involving the noncompact directions and worldsheet BRS ghosts, leaving a contribution $Q_0^2/8$ from the compactification CFT. Therefore we can deduce the dimension of $\Psi$ to be

$$h_{\Psi} = \frac{Q_0^2}{8} - \frac{1}{2} \left( \frac{q}{Q_0} - \frac{Q_0}{2} \right)^2 ,$$

(1.33)

and thus the mass of the NS ground state operator $O_{ab}^{NS}$ is

$$m^2 = h_{ab}^{NS} - \frac{1}{2} = \frac{Q_0^2}{8} - \frac{1}{2} \left( \frac{q}{Q_0} - \frac{Q_0}{2} \right)^2 + \frac{1}{2} \left( \frac{q}{Q_0} \right)^2$$

$$= \frac{1}{8} (q - 1) .$$

(1.34)

Here $q = (\alpha_a - \alpha_b)/\pi$.

In fact, we saw a simple example of this sort of calculation before, in the context of two D-branes at relative angles; the mismatch of preserved supersymmetries there gives a direct interpretation to $q$ as the relative angle between the branes. One way of generating the angle $\alpha$ is to wrap two branes on the $a$ and $b$ cycles of a two-torus $T^2$; then the angle between the branes depends on the modulus $\tau$ of the torus, $\alpha = \alpha(\tau)$. This T-dualizes to $\alpha = \alpha(B+iV_2)$ and branes wrapping even-dimensional cycles, c.f. equation (1.27). The generalization to higher dimensions and less supersymmetry is a dependence on the complexified Kähler moduli of the spectrum on such branes $\alpha_{ab} = \alpha_{ab}(M_{\text{Kähler}})$. A property of branes on the torus is $\alpha_{ab} < 0$; this need not be true in the general case (i.e. there need not always be a tachyonic $a$--$b$ string whose condensation generates a bound state of branes $a$ and $b$).

The Quillen superconnection mentioned above in a sense generalizes to a differential on the whole complex of potential brane bound states
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carrying BPS charge vectors. This complex is called the derived category. One has again an equivalence relation between objects whose physical origin is the possibility to add brane-antibrane pairs and condense the resulting tachyons (recall that the tachyon is the ‘off-diagonal’ part of the superconnection connecting different objects); objects modulo equivalence then classifies D-brane charges in a way that generalizes K-theory. For instance, mirror symmetry is automatically incorporated since this operation is simply T-duality \( \bar{J} \rightarrow -\bar{J}, \bar{H} \rightarrow -\bar{H} \) on the \( U(1) \) R-current of the worldsheet \( \mathcal{N} = 2 \). Also, the limit of stringy target spaces, whose geometrical interpretation is more obscure, is incorporated as well.

2.5 D0’S ON SU(2): THE KONDO MODEL

The final example of open string tachyon condensation that we will consider is that of D0 branes in the \( SU(2) \) WZW model, i.e. the three-sphere \( \mathbb{S}^3 \) threaded by \( k \) units of NS three-form flux \( H \) \cite{46, 47}. Suppose there are \( N \) coincident D0 branes on the \( \mathbb{S}^3 \). Then there is a marginally relevant (i.e. logarithmically growing toward the IR) boundary perturbation

\[
\delta S = \int_{\partial} ds j^a(s) S_a ,
\]

where the \( S_a \) are the representation matrices of \( SU(2) \) in the \( N \)-dimensional representation, acting on the Chan-Paton Hilbert space of the branes. The worldsheet theory is none other than the Kondo model – currents interacting with a localized impurity in 2d, taking us back to the very origins of the renormalization group \cite{18}.

The total spin \( j^a + S^a \) is conserved by the interaction, so (at least for \( N \ll k \)) one expects the infrared fixed point to be some boundary state in \( SU(2) \) CFT which respects the global \( SU(2) \) symmetry. Boundary states of this sort, having the boundary condition \( j_a = \bar{j}_a \) on the currents, are in one-to-one correspondence with integral conjugacy classes of the group (c.f. \cite{48} for a pedagogical discussion), so there is one for each \( SU(2) \) spin \( j = (N - 1)/2 = 0, \frac{1}{2}, 1, \ldots \). These representations can be seen to describe spherical D2 branes of size

\[
R_{S^2} = \sqrt{k} \sin \left( \frac{\pi N}{k} \right) .
\]

Here tachyon condensation results in the ‘dielectric effect’ \cite{17}, where \( N \) D0 branes in the presence of an \( H \) field puff up into a D2 brane.
2.6 THE $G$-THEOREM

As mentioned above, the relation $\Gamma_{\text{ST}} = Z_{\text{WS}}$ gives a purely worldsheet criterion for allowed decays – namely, the disk partition function must decrease along the corresponding RG flow

$$\Gamma_{\text{IR}} < \Gamma_{\text{UV}}.$$  \hspace{1cm} (1.37)

The existence of a function on open string coupling space that decreases along RG flows is known as the $g$-theorem; it was conjectured by Affleck and Ludwig [49] (who called $\Gamma_{\text{ST}} \equiv g$) and proven by Kutasov, Marino and Moore [26]. As a corollary, we deduce that $\Gamma = g$ is constant along marginal lines of boundary perturbations.\(^5\)

By the magic of worldsheet channel duality, there is another interpretation of $\Gamma = g$. Consider the annulus partition function in the limit of vanishing open string proper time of propagation $\tau_{\text{open}} \to 0$ (see figure 1.5)

$$Z_{\text{annulus}} = \text{Tr}_{\mathcal{H}_{\text{aa}}} e^{2\pi i \tau_{\text{open}} (L_0 - c/24)}$$ \hspace{1cm} (1.38)

By modular invariance, the amplitude factorizes in the closed string channel on the lightest closed string state, namely the identity operator, leaving two copies of the disk partition function.

\(^5\)Note that $\Gamma$ need not be constant under deformations by closed string moduli; for instance, the spacetime energy of a D-brane wrapped on a torus depends on the volume of the torus.
Thus

\[
\lim_{\tau_{\text{open}} \to 0} \text{Tr}_{\mathcal{H}_{aa}} e^{2\pi i \tau_{\text{open}} (L_0 - c/24)} = e^{2\pi i \tau_{\text{closed}} (-c/24)} \left( \frac{\Gamma^{(a)}}{\Gamma_{\text{ST}}} \right)^2 ,
\]

(1.39)

with \( \tau_{\text{closed}} = -1/\tau_{\text{open}} \). On the other hand, the LHS is dominated by the undamped exponential degeneracy of the high energy spectrum of states in the open string Hilbert space; converting the sum to an integral over the density of states, \( \text{Tr}_{\mathcal{H}_{aa}} \to \int dE \rho_{\text{open}}(E) \), where \( E = L_0 - c/24 \), performing an inverse Laplace transform in \( \tau_{\text{open}} \) and making a saddle point approximation to the integral, one finds

\[
\rho_{\text{open}}(E) = \frac{1}{2} \Gamma^{(a)} \left( \frac{c}{6E^3} \right)^{1/4} \exp \left[ 2\pi \sqrt{cE/6} \right] .
\]

(1.40)

Now \( c \) is a property of the bulk of the worldsheet CFT, and cannot change under boundary RG flow (for instance, \( c \) can be measured as the leading coefficient in the stress tensor OPE at points well away from the boundary). Thus another way to state the \( g \)-theorem is that the asymptotic density of open string states monotonically decreases under boundary RG flow, which is to say open string tachyon condensation. This is an embodiment of the usual intuition that the renormalization group implements a ‘thinning’ of field theoretic degrees of freedom.

3. CLOSED STRINGS

We are now interested in asking the analogous questions for closed strings. Adopting the worldsheet renormalization group approach to tachyon condensation, we would like to know

- **Q:** What is the structure of RG flows for localized nonsupersymmetric defects?
  **A:** We will find that, as in the open string case, \( \mathcal{N} = (2,2) \) worldsheet supersymmetry is a powerful tool, enabling us to understand the flows in a variety of examples.

- **Q:** Is there an analogue of the \( g \)-theorem, in the sense of (a) a decrease of the closed string effective action along flows; or (b) a decrease in the asymptotic density of states?
  **A:** There is no obvious analogue of (a) for the closed string case; we will construct the obvious candidate for (b), and show that it does decrease in a wide variety of examples.

- **Q:** Is there an analogue of K-theory or the derived category?
  **A:** Who knows?
A concrete class of examples of localized closed string tachyons arises in noncompact orbifolds of the type $\mathbb{R}^{d-1,1} \times (\mathbb{R}^{10-d}/G)$, where the orbifold group $G \subset SO(10-d)$ but $G \not\subset SU(5-d/2)$. Then supersymmetry is broken by the boundary conditions in twisted sectors of the orbifold, but not in the untwisted sector if we perform the usual GSO projection that gives type IIA/B string theory. The twisted sectors indeed contain closed string tachyons; where do their RG flows take the theory?

The first question to answer is why the central charge does not decrease, leading us to noncritical string theory; in other words, why doesn’t $c$ play the role of $g_{\text{closed}}$, and why doesn’t space just disappear? After all, Zamolodchikov [50] proved that there is an off-shell quantity $c$ that agrees with the Virasoro central charge at fixed points, and which decreases along flows analogous to $g$ for open strings; there have even been attempts to relate $c$ to the closed string effective action [51]. Underlying the proof of the $c$-theorem is a set of assumptions, among which are unitarity and a gapped spectrum of $L_0$. In the context of the non-linear sigma model, this means the target space is compact. However, in infinite volume, the effect of the tachyon condensate is localized to a finite region for any finite point along the flow; since the 2d stress tensor is a continuum normalized operator (it has an implicit factor of the volume of space in its normalization), any localized perturbation that might change $c$ is overwhelmed by the contribution of spatial infinity where $c$ remains unchanged. Of course, this is a bit too trivial; one might ask whether what forms is a bubble of a space of lower $c$, which then grows and takes over the whole target space in the IR. We will see in the examples that this does not occur, but it is an interesting open question to determine whether or not such a scenario is possible in principle.

As mentioned in section 2.1, there is not yet a compelling analogue of the effective action approach here, at least along the lines of the open string relation $\Gamma_{\text{ST}} = Z_{\text{WS}}^{\text{disk}}$. Thus there is no direct way to relate the decay to a decrease in the effective potential. In particular, orbifold spaces are locally flat, so their ADM energy always vanishes. However, in the search for a quantity that characterizes the flows and decreases along them, one might ask how closed string tachyon condensation affects

---

Dabholkar and Vafa [52] proposed an effective action for orbifolds based on solutions to the $tt^*$ equations, in which the value of the effective potential is the mass squared of the least relevant twist operator in the chiral ring. This proposal suffers from the fact that it does not give the same result for $\mathbb{C}^2/\mathbb{Z}_n(p)$ and $\mathbb{C}^2/\mathbb{Z}_n(n-p)$, which as we discuss below are isomorphic as conformal field theories.
the asymptotic density of closed string states
\[
\rho_{cl} \sim \frac{1}{2} g_{cl} \left( \frac{c_{\text{eff}}}{3E^3} \right)^{1/4} \exp \left[ 2\pi \sqrt{c_{\text{eff}}E/3} \right],
\tag{1.41}
\]
where \( E = L_0 + \bar{L}_0 - \frac{1}{12} c_{\text{eff}} \). Typically \( c_{\text{eff}} = c \), the Virasoro central charge; however, in backgrounds with throats, such as the linear dilaton throat of an NS fivebrane source, one can have \( c_{\text{eff}} < c \). Moreover, although as argued above \( c \) is not changing along the RG flows, it can happen that \( c_{\text{eff}} \) changes. In known examples it seems to decrease [53, 16], compatible with the notion of thinning of degrees of freedom. In situations where \( c_{\text{eff}}(\text{IR}) = c_{\text{eff}}(\text{UV}) = c \) such as we have for orbifolds, it is natural to propose that the subleading coefficient \( g_{cl} \) decreases along RG flows (and of course this is only meaningful for constant \( c_{\text{eff}} \)). This is the direct analogue of the open string \( g \)-theorem, where the analogue of \( g_{cl} \) is \( \Gamma_{ST}^2 \), see equation (1.40). Note that tachyon condensation is localized in the target space; \( \rho_{cl} \) in (1.41) should count only the normalizable closed string states. In orbifold examples, this means we only include the twisted sectors in the sum.

The calculation of \( \rho_{cl} \) proceeds much as for the open string case; one calculates the torus partition sum over the Hilbert space of localized closed string states in the limit \( \tau \to 0 \), where the sum is dominated by the high energy spectrum in a saddle point approximation. On the other hand, performing a modular transformation \( \tau \to -1/\tau \) the amplitude factorizes on the lowest energy state (the CFT vacuum) in the dual channel. For orbifold models, one has
\[
Z_{\text{twisted}} = \frac{1}{|G|} \sum_{g \neq 1} Z \left( \begin{array}{c} h \\ g \end{array} \right)
\]
\[
Z \left( \begin{array}{c} h \\ g \end{array} \right)(-1/\tau) = Z \left( \begin{array}{c} g \\ h \end{array} \right)(\tau).
\tag{1.42}
\]

One picks \( h = 1 \) on the RHS of the second line to select the contribution of the identity operator (i.e. the leading contribution) as \( \tau \to i\infty \). The oscillator part is trivial; the important part is the normalization of the zero mode in the path integral
\[
\int dp \delta(\mathcal{R}(g)p - p) = \frac{1}{|\text{det}(1 - \mathcal{R}(g))|^2}
\tag{1.43}
\]

\footnote{Note that the exponential behavior is the usual Hagedorn growth in the string spectrum, since \( E_{WS} = m_{ST}^2 \) via the Virasoro constraints.}
where $R(g)$ is the rotation matrix representing the action of $g$ on the string coordinates, and the integral is only over those directions where the rotation acts nontrivially. Plugging this result into the saddle point evaluation of $\rho_{cl}(E)$, one finds

$$g_{cl} = \frac{1}{|G|} \sum_{g \neq 1} \frac{1}{|\det(1 - R(g))|^2}.$$  \hspace{1cm} (1.44)

One can show that \cite{16} $g_{cl}$ is constant when perturbing along marginal lines of twisted perturbations, much as in the D-brane case.

The examples that have been studied are the orbifolds $\mathbb{C}^m/\mathbb{Z}_n$, acting on the complex coordinates $(X^1, \ldots, X^m)$ of $\mathbb{C}^m$ as

$$(X^1, \ldots, X^m) \mapsto (\exp[2\pi ip_1/n]X^1, \ldots, \exp[2\pi ip_m/n]X^m).$$  \hspace{1cm} (1.45)

Note that by choosing the generator appropriately one can always arrange $e.g. \ p_1 = 1$, and we will do so in what follows. The $g_{cl}$ function (1.44) is then

$$g_{cl}(p_1, \ldots, p_m; n) = \frac{1}{2^{2m} n^2} \sum_{k=1}^{n-1} \frac{1}{\sin^2 \left( \frac{\pi k}{n} \right) \sin^2 \left( \frac{\pi p_1 k}{n} \right) \cdots \sin^2 \left( \frac{\pi p_m k}{n} \right)}; \hspace{1cm} (1.46)$$

which behaves for large $n$ and fixed $p$ as

$$g_{cl} \sim \text{const.} \frac{n^{2m-1}}{p_1^2 p_2^2 \cdots p_m^2}. \hspace{1cm} (1.47)$$

For example, for $\mathbb{C}/\mathbb{Z}_n$ one has $g_{cl} \sim n/12$, and for $\mathbb{C}^2/\mathbb{Z}_n$ one has $g_{cl} \sim n^3/(720p^2)$ (where $p \equiv p_2$). It was argued by Adams, Polchinski, and Silverstein \cite{54} that the generic endpoint of tachyon condensation for nonsupersymmetric orbifolds is flat space. For such flows the $g_{cl}$ conjecture is trivially satisfied, since the endpoint of the flow has no localized closed string states. However, there may be a hierarchy of ‘multicritical points’ of the renormalization group, and one would like to check whether lower multicritical points always have lower $g_{cl}$. For this we must understand what theories the RG flows lead to in the far infrared. Extended $\mathcal{N} = 2$ worldsheet supersymmetry will allow us to make exact statements about the flow structure.

3.1 $\mathbb{C}/\mathbb{Z}_N$

The simplest example to study is $\mathbb{C}/\mathbb{Z}_n$. The precise value of $g_{cl}$ in this case is

$$g_{cl} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{|\sin \pi k/n|^2} = \frac{1}{12} (n - 1/n); \hspace{1cm} (1.48)$$
thus, if $g_{cl}$ decreases under RG flow, this should have something to do with a decrease of $n$. In fact, one can map out the flows exactly using the fact that while spacetime supersymmetry is broken, one still has the $\mathcal{N} = (2, 2)$ worldsheet supersymmetry generated by the worldsheet stress tensor $T$, supersymmetry currents $G, G^*$, and R-symmetry current $J$ (and their antiholomorphic counterparts). BPS representations of worldsheet supersymmetry are chiral states with $h = q/2$, where $h, q$ are the $L_0$ and $J_0$ eigenvalues. Thus states with $q < 1$ are tachyons. Note the close parallel to the open string example of D-branes at angles; we will return to this analogy below. The vertex operators $\chi_q$ which create the chiral states when acting on the CFT vacuum thus have a nonsingular operator product expansion due to the additivity of their charges/scale dimensions

$$\lim_{z' \to z} \chi_q(z) \chi_{q'}(z') = \chi_{q+q'}(z)$$

(1.49)

which determines a chiral ring structure [55]. Since the operators are BPS, perturbing the worldsheet action by $\int d^2z d^2\theta \chi + c.c$ preserves (the diagonal part of) $\mathcal{N} = (2, 2)$ supersymmetry.

The chiral operators for $\mathbb{C}/\mathbb{Z}_n$ are the operators $\Sigma_{j/n}$ that create the twist ground states

$$\Sigma_{j/n} = \sigma_{j/n} e^{i(j/n)(H - \bar{H})}$$

$$h_j = \frac{1}{2} \frac{j}{n} \left(1 - \frac{j}{n}\right) + \frac{1}{2} \left(\frac{j}{n}\right)^2 = \frac{1}{2} \frac{j}{n} \cdot$$

(1.50)

Here again $\sigma$ is the operator that implements fractional moding on the target coordinates $X$

$$X(e^{2\pi i w}) = e^{2\pi ij/n} X(w) ,$$

(1.51)

and the exponential of $H$ does the same for its worldsheet superpartner $\psi = e^{iH}$. The $\mathcal{N} = 2$ R-current is simply $J = i\partial H$, so the charge of these operators is $q = j/n$; they are all tachyons. The chiral ring is

$$\Sigma_{j/n} \Sigma_{j'/n} = \begin{cases} \Sigma_{(j+j')/n} & j + j' < n \\ \frac{1}{\text{Vol}} e^{i(H - \bar{H})} & j + j' = n \\ 0 & \text{otherwise} \end{cases}$$

(1.52)

In other words, the ring is generated by $W = \Sigma_{1/n}$, with the relation $W^n = \frac{1}{\text{Vol}} \psi \psi^* \equiv \mathcal{V}_X$. Note that we have included a factor of the volume
of the target to remind ourselves of the relative normalizations of the localized twisted states and the delocalized untwisted states. One can render the volume finite by compactifying the plane to a very large $\mathbb{P}^1$ to make these statements precise [56]; the untwisted operators decouple from the twisted sector operators in the infinite volume limit. We may view $W$ as a kind of winding operator around the tip of the orbifold cone, and the chiral ring relation as a description of the ‘location’ of the $n - 1$ twisted vacua in the $W$-plane.

The next question to address is what happens when one condenses the $\Sigma$’s:

$$S_{WS} = S_0 + \left( \lambda_j \int d^2 z d^2 \theta W^j + c.c. \right).$$

With the $\lambda_j$ turned on, the RG flow can deform the chiral ring structure, but cannot destroy it since the operators are BPS and hence protected by supersymmetry. The general deformation is

$$W^n = \mathcal{V}_X \longrightarrow (W - w_1)^{n_1} \cdots (W - w_k)^{n_k} = \mathcal{V}_X .$$

with $\sum_j n_j = n$ and $w_i = w_i(\lambda)$. The $w_i$ grow in the infrared under the renormalization group, i.e. the vacua decouple by becoming widely separated in the $W$-plane. If in the IR we focus on the neighborhood of one of the $w_i$ in the complex $W$-plane we find the chiral ring of one of the lower orbifolds

$$c_i(W - w_i)^{n_i} = \text{Vol. form} ,$$

i.e. in this part of the field space we find $\mathbb{C}/\mathbb{Z}_{n_i}$. But this happens for each of the $w_i$, $i = 1, \ldots, k$, each of which decouples from all the others in the IR. The picture that is suggested is that of a ‘splitting’ of the universe into several disconnected components, see figure 1.6. Although we have not discussed it, only one of these vacua can be type II and therefore not have a bulk tachyon; the rest are type 0, due to an intriguing spontaneous breaking of the discrete wordsheet GSO gauge symmetry under tachyon condensation. See [16] for details.

A possible picture of intermediate stages of the flow is given in figure 1.7; it depicts a kind of ‘delamination’ of spacetime, wherein the separate cones that are forming near the origin join onto the rest of the original cone which is still asymptotically $\mathbb{C}/\mathbb{Z}_n$.

Note that this picture is somewhat of a cartoon; the orbifold fixed points cannot be separating in the spacetime parametrized by $X$, since the twist operators respect the $U(1)$ rotational symmetry of the plane. Rather, as the chiral ring suggests, the fixed points are separating in the ‘T-dual’ coordinate $W = W_1$ which is the field space of the twist operators.

\[8\]
The quantity $g_{cl}$ indeed decreases along the flows; one has

$$g_{cl} = \frac{1}{12} \left( n - \frac{1}{n} \right) \rightarrow \frac{1}{12} \sum_{k=1}^{k} \left( n_j - \frac{1}{n_j} \right)$$

and indeed the IR value of $g_{cl}$ is smaller than the UV value. Thus, $\mathbb{C}/\mathbb{Z}_n$ decays in an interesting way: The cone splits into several ‘smaller’ (i.e. shallower angle) cones, and the asymptotic density of states associated to the defect indeed is dissipated by the flow.

Amusingly, there is actually a rather close analogue of this splitting process in open string condensation, namely the condensation of the tachyon on the intersection of branes at angles of figure 1.3. The effect of the tachyon is to reconnect the branes as shown in figure 1.8.
In the IR the branes move far apart, and the fixed point is two infinitely separated, decoupled branes. Note also that there is an appropriate analogue of $g_{\text{cl}}$ in that what is proper to count here is the density of open string states localized at the intersection and not in the bulk of the brane network; these are the states that are dissipated when the tachyon condenses.

### 3.2 $\mathbb{C}^2/\mathbb{Z}_N$: THE CHIRAL RING

The collection of $\mathbb{C}^2/\mathbb{Z}_N$ orbifolds provides an even richer geometrical structure, which is intimately connected with the resolution of singularities in algebraic geometry. Our main task will be to understand the structure of RG flows, and then to check whether $g_{\text{cl}}$ decreases. The group action is

$$(X \equiv X^1 + iX^2, Y \equiv X^3 + iX^4) \rightarrow (\omega X, \omega^p Y) ,$$

where $p \in \mathbb{Z}$, and $\omega = \exp[2\pi i/n]$. This orbifold group is denoted $\mathbb{Z}_{n(p)}$. Note that for $p = n - 1$, the rotation is in $SU(2)$ rather than $U(2)$ so that spacetime supersymmetry is preserved; these are the well-known $A_{n-1}$ ALE orbifolds (c.f. [57]).

The chiral ring is built out of twist operators for each separate complex plane

$$\Sigma_j = \Sigma_{j/n}^{(X)} \Sigma_{j/p/n}^{(Y)}$$

where $\{\xi\}$ denotes the fractional part of $\xi$, $0 \leq \{\xi\} < 1$.

Before proceeding, a point of notation is in order. Mathematically, a $\mathbb{Z}_{n(p)}$ orbifold is defined by the action (1.57), and $p$ is defined modulo $n$, $p \sim p + n$. However, as conformal field theories, the orbifolds for $p$ and $-p$ are isomorphic, differing only by the field redefinition $Y \rightarrow Y^*$. Correspondingly, these theories contain two sets of ‘BPS protected’
twisted sector operators, namely (1.58) and
\[ \Sigma_j^{(X)}(\Sigma_1^{(Y)})^* \] (1.59)
which is BPS under a different linear combination \( G_X + G_Y^* \) of the supersymmetry currents of the component theories. We can call the ring of these latter operators the \((c_X, a_Y)\) ring, and the ring of operators (1.58) the \((c_X, c_Y)\) ring. The operation \( Y \rightarrow Y^* \) exchanges these two rings, and sends \( p \rightarrow -p \). To reconcile the mathematical \( p = p_{\text{math}} \), defined modulo \( n \), and the conformal field theory \( p = p_{\text{CFT}} \), defined modulo \( 2n \) but with an equivalence \( p_{\text{CFT}} \sim -p_{\text{CFT}} \), we will adopt the convention that \( p_{\text{math}} \) is always positive, so that
\[ p_{\text{math}} = n - |p_{\text{CFT}}| ; \] (1.60)
note that \( p_{\text{math}} \) and \( p_{\text{CFT}} \) are equivalent modulo \( n \) if we restrict ourselves to \( 0 < p_{\text{math}} < n \) and \(-n < p_{\text{CFT}} < 0\).\(^9\) We will do so in the following, and label singularities by their value of \( p_{\text{math}} \).

\(^9\)Unfortunately, the notation of [16] is not internally consistent in this regard; the orbifolds labelled by \( p > 0 \) there should actually be shifted by \( p \rightarrow p - n < 0 \) to agree with the above.
The R-charges \((q_1, q_2)\) of the chiral ring under the \(U(1) \times U(1)\) R-symmetry of the orbifold CFT are depicted in figure 1.9 for \(n = 10\) and \(p = 3\). Note that the volume elements \(\mathcal{V}_x, \mathcal{V}_y\) of the two complex planes will always be the maximal charge chiral operators for the R-charges \(q_x, q_y\) (since this is a property of \(\mathbb{C}/\mathbb{Z}_n\)). Note also that, unlike \(\mathbb{C}/\mathbb{Z}_n\), the chiral ring is not in general singly generated – the charge vectors are not all integer multiples of a single basis vector, rather there are typically several generators. For the example in figure 1.9, there are three generators: \(W_1 = \Sigma_1, W_2 = \Sigma_4,\) and \(W_3 = \Sigma_7\). The conformal dimension of operators is just \(\frac{1}{2}(q_x + q_y)\); thus, all operators below the diagonal line in the figure are tachyons, all operators above the line are irrelevant, and operators on the line are marginal.

It will be useful to plot the R-charges \((q_1, q_2)\) of the \(\mathbb{C} \times \mathbb{C}\) twist operators in an integral basis of charges

\[
(q_x, q_y) = (j/n, j'/n = (pj/n)) \to (j, k \equiv \frac{1}{n}(j' - pj)) \quad (1.61)
\]

This new basis is plotted in figure 1.10 for \(n(p) = 10(3)\). Note that the charge vectors \((0, 1)\) of \(\mathcal{V}_y\), and \((1, 0)\) of \(\mathcal{V}_x\), map to \((0, 1)\) and \((n, -p)\) respectively.

![Figure 1.10](image_url) 

**Figure 1.10** The chiral ring states for \(n(p) = 10(3)\) plotted in an integral basis which is closely related to the resolution of the singularity in algebraic geometry. The vectors \(v_i\) correspond to the ring generators.

To give a flavor of what happens as \(p\) varies, all the orbifolds for \(n = 7\) are plotted in figure 1.11.

As we will now show, this way of plotting the chiral ring is directly related to the resolution of the \(\mathbb{C}^2/\mathbb{Z}_{n(p)}\) singularity in algebraic geometry. To see this, we need to embark on a rather lengthy digression into algebraic geometry to prepare the necessary tools. For a more extensive exposition, see [58].
3.3 THE GEOMETRY OF SINGULARITY RESOLUTION

Spaces in algebraic geometry are characterized by their local rings of holomorphic functions; points are characterized by the ideals of holomorphic functions in the ring that vanish at that point. To begin, consider the complex plane \( \mathbb{C} \). Its ring of holomorphic functions has a basis \( \{1, x, x^2, x^3, \ldots\} \). The exponents form a semi-infinite integer lattice which is plotted in figure 1.12a. A less trivial example is \( \mathbb{P}^1 \), which must be covered by two coordinate patches, each of which is isomorphic to \( \mathbb{C} \).

The corresponding functions are those holomorphic outside the origin for one patch, and functions holomorphic away from infinity form the functions on the other patch. This time, the lattice of exponents is infinite in both directions if we superpose the sets coming from each of the two patches (taking into account the map \( X \rightarrow X^{-1} \) relating the coordinates), see figure 1.12b.

In two complex dimensions one has the obvious generalization; the lattice for \( \mathbb{C}^2 \) is depicted in figure 1.13. Note that an isomorphic parametrization of \( \mathbb{C}^2 \) results if we consider any \( SL(2, \mathbb{Z}) \) transformation of the lattice; for instance, a ring of holomorphic functions on \( \mathbb{C}^2 \) can be made from polynomials in say \( X \) and \( X^{-1} Y \), or from \( X Y^{-1} \) and \( Y^{-1} \). So in general, the ring of holomorphic functions on a coordinate patch is equivalent to an integer lattice lying inside a cone, let us denote it as \( \sigma \).

There is now also room for interesting structure. In algebraic geometry, orbifolds are singular spaces; the singularity can be resolved by
Figure 1.12  (a) Semi-infinite 1d lattice characterizing generators for the ring of holomorphic functions on \( \mathbb{C} \). (b) Two such lattices patch together to describe \( \mathbb{P}^1 \).

Figure 1.13  The lattice of generators for the holomorphic function ring for \( \mathbb{C}^2 \). The shaded region is the cone \( \tilde{\sigma} \) containing the lattice of exponents.

excising the singular locus and inserting a smooth space, a procedure known as blowing up. Rather than going directly to the singular orbifold, let us illustrate the technology in a simpler situation by blowing up a point on \( \mathbb{C}^2 \), replacing it by a \( \mathbb{P}^1 \). This operation is defined by the algebraic equation

\[
XT_1 = YT_0
\]  

(1.62)

where \( (X, Y) \) are coordinates for \( \mathbb{C}^2 \) and \( (T_0, T_1) \sim (\lambda T_0, \lambda T_1) \) are homogeneous coordinates for \( \mathbb{P}^1 \). In other words the blown up space is a surface in \( \mathbb{C}^2 \times \mathbb{P}^1 \). Locally the space looks like \( \mathbb{C}^2 \), i.e. \( X, Y \) determine a point on \( \mathbb{P}^1 \) through the equation (1.62), except at \( X = Y = 0 \) where
there is an entire $\mathbb{P}^1$ since (1.62) is automatically satisfied there. Note that the blown up space has a different topology – there is an additional two-cycle in the cohomology coming from the $\mathbb{P}^1$.

To cover the blowup, one needs two coordinate patches isomorphic to $\mathbb{C}^2$: A patch $U_0$, whose ring of holomorphic functions is generated by $(x_0 = X, y_0 = X^{-1}Y = T_1/T_0)$; and a second patch $U_1$, whose ring is generated by $(x_1 = Y, y_1 = XY^{-1} = T_0/T_1)$. These correspond to the two patches needed to cover $\mathbb{P}^1$, as above. On overlaps, one has the relations $x_1 = x_0y_0$ and $y_1 = x_0^{-1}$. The entire structure of coordinate patches and their transition functions is encoded in the relationship between the cones $\tilde{\sigma}_{0,1}$ containing the lattices of exponents, see figure 1.14a.

At this point it is useful to introduce the notion of the cones dual to the cone bounding the lattice of exponents. It will in fact be this dual cone that relates directly to the chiral ring when we come to consider orbifolds and their resolution. The cone $\sigma$ dual to the cone $\tilde{\sigma}$ containing the lattice of monomial exponents is the set of all vectors having a positive inner product with all the exponents in $\tilde{\sigma}$

\[ \mathbf{v} \in \sigma \quad \mathrm{IFF} \quad \langle \mathbf{v}, \mathbf{u} \rangle > 0 \quad \forall \mathbf{u} \in \tilde{\sigma} \, . \quad (1.63) \]

Consider the blowup of $\mathbb{C}^2$; the cones $\tilde{\sigma}_{0,1}$ for the two coordinate patches $U_{0,1}$ are shown in figure 1.14a, and the dual cones $\sigma_{0,1}$ are depicted in figure 1.14b.

Note that in collection of cones $\tilde{\sigma}_{0,1}$ we can directly see the coordinate patches of the $\mathbb{P}^1$ of the blowup (two copies of $\mathbb{C}$ parametrized by $X^{-1}Y$ and $XY^{-1}$). Note also that the collection of dual cones $\sigma_{0,1}$ do not overlap as the $\tilde{\sigma}$’s do, but rather nestle side by side; in fact, they form a subdivision of the dual cone of $\mathbb{C}^2$. It is this feature that generalizes: The resolution of a singularity will consist of a procedure of successive subdivision of the dual cone $\sigma$ to the cone $\tilde{\sigma}$ of the function ring of the singularity, until what remains is a fan of (dual) cones describing the coordinate patches of the resolved space and their relationships. We now turn to orbifold examples.

### 3.4 $\mathbb{C}^2/Z_N$: GEOMETRY

Consider first the supersymmetric orbifold $\mathbb{C}^2/Z_{n(n-1)}$ which is the quotient of $\mathbb{C}^2$ by the action

\[ (X, Y) \rightarrow (\omega X, \omega^{-1}Y) \quad (1.64) \]

with $\omega = \exp[2\pi i/n]$ (note that $G \subset SU(2)$ so that spacetime supersymmetry is preserved). Holomorphic functions on $\mathbb{C}^2$ invariant under this
action are

\[ U = XY \quad , \quad V = Y^n \quad , \quad W = X^n \]  

(1.65)

and in fact these generate the ring of holomorphic functions on the orbifold. Note however that there is a relation

\[ U^n = VW \]  

(1.66)

the surface in \( \mathbb{C}^3 \) defined by this relation is a standard description of this singularity in algebraic geometry. One can make a plot of the cone of basis functions and its dual cone, see figure 1.15. The singularity of the surface is related to the fact that the bounding vectors of the cone, \((1, 0)\) and \((n-1, n)\), define a parallelogram that is not of unit area (i.e. it encloses integral lattice points). This means that the space is not locally like \( \mathbb{C}^2 \) near the origin, i.e. the function ring lattice can’t be mapped to figure 1.13 by an \( SL(2, \mathbb{Z}) \) transformation.

The singularity is resolved by blowing up. The dual cone \( \sigma \) is defined by basis vectors \((0, 1)\) and \((n, -n + 1)\). Choose one of the integral interior points of the the parallelogram (the fundamental cell of the lattice) defined by these two vectors, and subdivide the dual cone into two sub-cones bounded by it and either of the original two bounding vectors. If the sub-cones are still singular (contain integral points in the interior of their fundamental cell), then subdivide further until the full fan of cones...
correspond to nonsingular spaces. This subdivision is depicted in figure 1.16 for \( n = 3 \).

The singularity is now resolved, since the coordinate patches corresponding to this subdivision of the dual cone are each isomorphic to \( \mathbb{C}^2 \), and they are patched together by simple rational functions. For \( n = 3 \), two successive blowups are required; one can see the coordinate patches of the two corresponding \( \mathbb{P}^1 \)'s being formed by the bounding vectors of the coordinate cones in figure 1.16, much as in the example of blowing up \( \mathbb{C}^2 \) considered above. Specifically, the bounding vectors to \( \tilde{\sigma}_i, \tilde{\sigma}_{i+1} \) which are orthogonal to the dual vector \( v_i \) that separates \( \sigma_i, \sigma_{i+1} \) always point in opposite directions, and form the patches of a \( \mathbb{P}^1 \) as in figure 1.12b. If we call \((x_i, y_i)\) the generators of the ring whose corresponding cone is \( \tilde{\sigma}_i \), then one can read off from the figure the (rational) coordinate maps on overlaps from the linear relations among the corresponding exponent vectors.

Now we come to the relation of the resolved singularity and the orbifold CFT. It turns out that the set of vectors that define the (minimal) subdivision of the cone \( \sigma \) which resolves the singularity is in one-to-one correspondence with the generators of the chiral ring of the orbifold CFT. For \( \mathbb{C}^2/\mathbb{Z}_{n(n-1)} \), the chiral ring consists of the twist operators

\[
\Sigma_j = \Sigma_{j/n}^{(X)} \Sigma_{1-j/n}^{(Y)} ; \quad (1.67)
\]
Figure 1.16 (a) A sequence of blowups at the singular point subdivides the dual cone of the orbifold; the example $n(\rho) = 3(2)$ is depicted. (b) The corresponding function ring cones exhibit the $\mathbb{P}^1$'s (c.f. figure 1.12b) of the resolved space as formed by their boundaries – the solid green line and the dashed blue line orthogonal to the subdividing vectors $(1, 0)$ and $(2, -1)$ of $\sigma$.

Their R-charges $(q_x, q_y) = (j/n, 1 - j/n)$ translate into the integer basis of equation (1.61) as

$$(j, k) = (j, 1 - j)$$

which are precisely the resolution vectors $v_j$! In this example, each of the $(n - 1)$ twist operators corresponds to a $\mathbb{P}^1$ in the resolution;\(^\text{10}\) but the main point is that the data of the resolution of the singularity is isomorphic to the data of the orbifold chiral ring.

Thus we have the following correspondences:

- The singularity is resolved by subdividing the cone $\sigma$ (dual to the coordinate ring of the singularity) along a set of interior vectors $v_i$. The smallest such subdivision is the minimal resolution of the singularity. There is a $\mathbb{P}^1$, call it $E_i$, blown up in the resolved space for each $v_i$.

\(^{10}\text{This property does not however generalize to other } \mathbb{Z}_{n(\rho)} \text{ orbifolds, where in general there are fewer generators of the ring than there are twist operators. This feature was also encountered in } \mathbb{C}/\mathbb{Z}_n, \text{ where all the twist operators were powers of the lowest one.}
The integer coordinates \((j_i, k_i)\) of each \(v_i\) are the R-charges in the integral basis of a generator \(W_i\) of the chiral ring of the orbifold. The fan vectors \(v_i\) and chiral ring generators are in one-to-one correspondence. The bounding vectors \((0, 1)\) and \((n, -n+1)\) of the cone of the unresolved space correspond to the volume elements \(V_Y, V_X\) of the two complex planes (the only generators of the chiral ring in the untwisted sector).

The linear relations among bounding vectors of sub-cones corresponds to relations among the generators of the chiral ring. For \(\mathbb{C}^2/\mathbb{Z}_{n(n-1)}\), one has

\[
W_j^2 = W_{j-1} W_{j+1} \quad \text{(1.69)}
\]

which also holds for the untwisted generators if we define \(W_0 \equiv V_Y\), \(W_n \equiv V_X\).

We hope the reader is sufficiently convinced of the correspondence between of the orbifold singularity’s resolution in algebraic geometry and the data specifying the chiral ring of the orbifold CFT. To actually prove the relation would take us too far afield; nevertheless it can be done using the gauged linear sigma model construction of [59, 60] (see for example [61, 62]).

While we have illustrated the methodology of singularity resolution on the perhaps more familiar example of \(\mathbb{C}^2/\mathbb{Z}_{n(n-1)}\), the machinery works for all the \(\mathbb{Z}_{n(p)}\) orbifolds. The unresolved dual cone \(\sigma\) of the singularity is bounded by the vectors \((0, 1)\) and \((n, -p)\); note that these are the same as the charges \((j, k)\) of the chiral ring elements \(V_Y\) and \(V_X\) of the orbifold, see equation (1.61). The resolution of the singularity is accomplished by subdividing the cone \(\sigma\) until all of the sub-cones have unit area fundamental cell; each subdividing vector of the minimal such resolution corresponds to a generator of the chiral ring of the orbifold, and the components of the vector determine the R-charges of the ring operator via (1.61). Any three successive vectors \(v_{i-1}, v_i, v_{i+1}\) in the fan obey a linear relation

\[
-a_i v_i + v_{i-1} + v_{i+1} = 0 \quad , \quad i = 1, ..., r \quad \text{(1.70)}
\]

with integer \(a_i\), corresponding to a relation in the chiral ring

\[
W_i^{a_i} = W_{i-1} W_{i+1} \quad \text{(1.71)}
\]

\(\text{Related to this is the fact that the invariant functions can, as in the case of figure 15a, be plotted so as to lie in the cone } \tilde{\sigma} \text{ bounded by } (1, 0) \text{ and } (p, n).\)
generalizing (1.69), again defining $W_{0,r+1}$ as $V_{v',X}$, respectively.

The coefficients in (1.70) code the intersection matrix of the $\mathbb{P}^1$’s of the resolution; namely, the $\mathbb{P}^1$ corresponding to $v_i$ intersects the ones corresponding to the adjacent fan vectors $v_{i-1}$ and $v_{i+1}$ each once, and has self-intersection $-a_i$. Note that these intersection numbers form the Cartan matrix of $A_{n-1}$ in the case of $\mathbb{C}^2/\mathbb{Z}_n(n-1)$, which is one reason why the supersymmetric orbifolds are called $A_n$ singularities. It is convenient to use the notation of Dynkin diagrams even in the more general case of $\mathbb{Z}_n(p)$, since the structure of their resolutions only differs in the self-intersection numbers of the $\mathbb{P}^1$’s; see figure 1.17.

![Figure 1.17](image-url)

**Figure 1.17** The intersection structure of the $\mathbb{P}^1$’s $E_i$ of the resolved space (schematically depicted in the upper figure) is conveniently encoded in a Dynkin-like diagram (the lower figure). The diagonal entries of the intersection matrix – the self-intersection numbers of the $E_i$ – label the nodes of the diagram.

An amusing and somewhat magical fact is that the integers $a_i$ are encapsulated by the continued fraction expansion of $n/p$

$$\frac{n}{p} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \cdots - \frac{1}{a_r}}} \equiv [a_1, a_2, a_3, \ldots, a_r] \quad (1.72)$$

For example, in the supersymmetric orbifold $p = n - 1$ one has

$$\frac{n}{n-1} = [2, 2, \ldots, 2] \quad (e.g. \quad \frac{4}{3} = 2 - \frac{1}{2 - 1/2} = [2, 2, 2]) \quad (1.73)$$

In fact, there is more magic: The ‘dual’ continued fraction

$$\frac{n}{n-p} = [b_2, \ldots, b_{\ell-1}] \quad (1.74)$$

codes the relations among the monomials $\varphi_1, \ldots, \varphi_\ell$ generating the ring of invariant functions on the quotient $\mathbb{C}^2/\mathbb{Z}_n(p)$ as

$$\varphi_i^{b_i} = \varphi_{i-1} \varphi_{i+1} \quad , \quad i = 2, \ldots, \ell - 1 \quad (1.75)$$
These $\ell - 2$ equations embed the singularity as a surface in $\mathbb{C}^\ell$, generalizing equation (1.66). Assigning vectors $u_1 = (1, 0)$ to $\varphi_1$ and $u_2 = (1, 1)$ to $\varphi_2$ as in figure 15a, leads to an increasing set of vectors culminating in $u_\ell = (p, n)$.\footnote{This proves the claim in the previous footnote.} The $G$-invariant monomials are given explicitly as follows: Let

$$
\begin{align*}
t_1 &= (t_1, t'_1) = (n, 0) \\
t_2 &= (t_2, t'_2) = (n - p, 1) \\
t_{i+1} &= b_it_i - t_{i-1}, \quad i = 3, \ldots, \ell .
\end{align*}
$$

Then the ring of invariants is generated by $\varphi_i = X^{t_i} Y^{t'_i}$.

There is an interesting ‘duality’ or ‘mirror symmetry’ of the $\mathbb{Z}_{n(p)}$ and $\mathbb{Z}_{n(n-p)}$ orbifolds. As was noted in section 3.2, in conformal field theory $p$ and $-p$ are related by the exchange $Y \leftrightarrow Y^*$ which induces an exchange of the $(c_X, c_Y)$ and $(c_X, a_Y)$ rings. One can then check that the $(c_X, c_Y)$ ring of the $\mathbb{Z}_{n(p)}$ orbifold is the $(c_X, a_Y)$ ring of the $\mathbb{Z}_{n(n-p)}$ orbifold, and vice versa. A given orbifold singularity can be smoothed in two different ways – by resolving it through a Kähler deformation as above, or by deforming the equations (1.75) which define it as a hypersurface in $\mathbb{C}^\ell$. These two types of smoothings are related by exchanging $p$ and $n - p$, as for instance the data that defines them are given by the dually related continued fractions for $n/p$ and $n/(n - p)$. A given $\mathbb{Z}_{n(p)}$ orbifold has both sets of deformations in it (at least before GSO projection, i.e. in type 0). The GSO projection

$$
H_1 \rightarrow H_1 + p\pi , \quad H_2 \rightarrow H_2 - \pi
$$

keeps some of each ring, namely the $(c_X, c_Y)$ states with $[jp/n] \in 2\mathbb{Z} + 1$ and the $(c_X, a_Y)$ states with $[jp/n] \in 2\mathbb{Z}$ (here $[\xi] = \xi - \{\xi\}$ denotes the integer part of $\xi$). It is only for the supersymmetric orbifold that the entire $(c_X, a_Y)$ ring is projected out and the entire $(c_X, c_Y)$ ring is preserved by the GSO projection. Note that this mirror symmetry is different from the usual one (c.f. [63]), which is T-duality of the $\mathcal{N} = (2, 2)$ $U(1)$ R-current $J$; that operation exchanges chiral and twisted chiral fields (i.e. switching chiral and antichiral for left-movers, keeping right-movers fixed) for all the superfields.

### 3.5 $\mathbb{C}^2/\mathbb{Z}_N$: RG FLOWS

In string theory, one does not actually have to blow up the geometrical size of the $\mathbb{P}^1$’s in order to resolve the singularity, as one must in
algebraic geometry. The orbifold is a completely nonsingular CFT. The size \( V_i \) of the \( i \)th cycle comes complexified as \( V_i + iB_i \) where \( B_i \) is the NS B-flux through the cycle. Aspinwall showed [64, 65] that the supersymmetric orbifolds have \( 1/n \) unit of \( B \)-flux turned on through each \( \mathbb{P}^1 \), thus blowing up the curves in a non-geometrical ‘imaginary’ direction even though the cycles are collapsed to zero size. This \( B \)-field resolves the singularity in a way that strings detect (for instance, it gives finite action to worldsheet instantons wrapping the \( \mathbb{P}^1 \)). One imagines that a similar story transpires for the other \( \mathbb{Z}_n (p) \) orbifolds – that even though the \( \mathbb{P}^1 \)’s that resolve the singularity are collapsed to zero size, a nonzero \( B \)-flux through them ensures the regularity of string propagation on the orbifold.

In the perturbed worldsheet action

\[
S_{\text{WS}} = S_0 + \left( \lambda_j \int d^2z d^2\theta W_j + \text{c.c.} \right), \tag{1.78}
\]

the complex parameter \( \lambda_j \) is some function of the complexified Kähler parameter of the corresponding \( \mathbb{P}^1 \). For the supersymmetric case \( \mathbb{Z}_{n(n-1)} \), all twist operators correspond to different curves, and they are all marginal operators. The resulting family of CFT’s are the Eguchi-Hanson ALE spaces (c.f. [57]). For general \( \mathbb{Z}_{n(p)} \), not all the chiral ring elements correspond to different curves in the resolution; for \( p \neq n - 1 \), one has \( r < n - 1 \). Moreover, the ring elements corresponding to the curves of the algebraic resolution are relevant operators; perturbing the action (1.78) still blows up the curves geometrically, but now the couplings \( \lambda_j \) grow along RG flows and thus the size of the corresponding curve increases as we flow to the IR.

We are now finally prepared to discuss the RG flows for \( \mathbb{C}^2 / \mathbb{Z}_{n(p)} \). Blowing up the \( i \)th \( \mathbb{P}^1 \) of the minimal resolution to infinite volume turns that \( \mathbb{P}^1 \) into \( \mathbb{C} \). Thus we expect that the corresponding operator \( W_i \) becomes the volume form \( \mathcal{V} \) for a noncompact direction of a new singularity, with the corresponding charge vector \( (j_i, k_i) \) being the bounding vector of its dual cone, see figure 1.18.

The other half of the split fan of cones can be converted into the canonical form with a bounding vector at \( (0, 1) \) by an \( SL(2, \mathbb{Z}) \) transformation of the lattice. For the example \( n(p) = 10(3) \) shown in the figure, the other singularity is \( n(p) = 2(1) \). Note that this is precisely what we would get by splitting apart the Dynkin diagram (figure 1.17) of the singularity by deleting the node corresponding to \( v_2 \), see figure 1.19. This makes sense since this curve is leaving the set of finite volume elements of the middle homology. The limit \( \lambda_i \to \infty \) effectively imposes a Lagrange multiplier constraint \( W_i = 0 \), as can be seen from the action.
Defects, Decay, and Dissipated States

Figure 1.18 The RG flow that blows up to infinite size the middle \( \mathbb{P}^1 \) of the resolved 10(3) singularity, yields a daughter singularity for which the fan vector corresponding to the \( \mathbb{P}^1 \) becomes the bounding vector of the dual cone of the daughter.

(1.78) – just as the strict infinite volume limit removes \( V_{X,Y} \) from the normalizable elements of the chiral ring.

Figure 1.19 The Dynkin diagram of the singularity splits by the deletion of the node corresponding to the curve being blown up to infinite size. As in figure 1.6, spacetime splits into disconnected components.

More generally, one could perturb the action by a chiral ring element that does not correspond to one of the fan vectors \( v_i \) of the minimal resolution. What happens then? In geometry, one can by a succession of further blowups make that operator correspond to a \( \mathbb{P}^1 \) of a non-minimal resolution of the singularity. The additional blowups correspond to adding more vectors to the fan, further subdividing the already non-singular dual cones \( \sigma_i \) (very much like blowing up \( \mathbb{C}^2 \) as we did above). The effect of a single blowing up operation is to modify the data of the continued fraction expansion, which defines the relations among the fan vectors via (1.70), as follows:

\[
[a_1, \ldots, a_k] \rightarrow [a_1, \ldots, (a_{i-1} + 1), 1, (a_i + 1), \ldots, a_r]
\]  
(1.79)
One can check that this continued fraction sequence defines the same fraction \(n/p\), and indeed has the effect of subdividing the \(i\)th cone \(\sigma_i\) while leaving the others intact. The effect of flowing to the IR under the perturbation corresponding to this non-minimal curve should be to split this expanded Dynkin diagram in two by deleting the appropriate node. Similarly, any operator whose curve can be put into the Dynkin diagram by a sequence of blowups ought to work the same way. Any chiral operator that is not a power of another can be treated in this manner.

Finally, what if the chiral operator we perturb by is not a generator of the minimal resolution, or any other resolution that can be obtained from it by a sequence of blowups? Consider for simplicity the case of \(n(p) = 2\ell(1), \ell \in \mathbb{Z}\). Call the generator of the chiral ring \(W\); its charge vector is labelled \(v_1\) in figure 1.20a. Here we understand what the \(W^\ell\) perturbation does, since the orbifold can be thought of as the supersymmetric orbifold \(\mathbb{C}^2/\mathbb{Z}_{2(1)}\), further orbifolded by \(\mathbb{Z}_\ell\); the marginal operator is the one that blows up the \(\mathbb{C}^2/\mathbb{Z}_{2(1)}\) into an \(A_1\) ALE space, whose geometry is that of the cotangent space of the sphere \(T^*\mathbb{P}^1\). Parametrize \(T^*\mathbb{P}^1\) by coordinates \(x_\pm, p_\pm\) related to the standard coordinates \(Z_1, Z_2\) of \(\mathbb{C}^2\) by

\[
\begin{align*}
x_+ &= Z_1/Z_2 \\
x_- &= Z_2/Z_1 \\
p_+ &= Z_2^2 \\
p_- &= -Z_1^2.
\end{align*}
\] (1.80)

Then \(\mathbb{Z}_{2(1)}\) does not act on \(x_\pm\), and acts as \(\mathbb{Z}_\ell\) on \(p_\pm\), so the infinitely blown up theory is \(\mathbb{C}/\mathbb{Z}_\ell \times \mathbb{C}\). But we get exactly this picture in the resolution diagram, see figure 1.20. We recognize along the horizontal axis the toric cone of \(\mathbb{C}/\mathbb{Z}_\ell\); the cone is singular because the cone of \(\mathbb{C}\) is not generated by a primitive vector. Rather, the invariant functions on the space are functions of \(Z_1^\ell\) and \(Z_2\). The picture clearly generalizes to any perturbation not ‘inherited’ from the ALE space, i.e. any of the relevant operators \(W^k, k < \ell\), in the chiral ring of \(\mathbb{C}^2/\mathbb{Z}_{2(1)}\); we expect that in the IR of the RG flow we arrive at the target space \(\mathbb{C}/\mathbb{Z}_k \times \mathbb{C}\).

Thus, a general picture arises for the perturbation of an arbitrary resolution fan by an arbitrary vector \(v_j\) corresponding to a relevant perturbation \(\Sigma_j\) in the chiral ring; one splits the cone \(\sigma\) into two cones \(\sigma_1, \sigma_2\) along the ray generated by the vector \(v_j\), and spacetime splits in the infrared of the RG flow into the orbifolds whose singularities are specified by \(\sigma_1\) and \(\sigma_2\).
3.6 THE $G_{CL}$ CONJECTURE

What about $g_{cl}$? As mentioned above, at large $n$ and fixed $p$ one has $g_{cl} \sim n^3/(720p^2)$. One needs to check whether the conjectured picture of RG flows, obtained from deleting a node on the Dynkin diagram (figure 1.17) of the singularity, leads to daughter singularities whose $n'(p')$ satisfies the conjectured decrease in the asymptotic density of localized closed string states. Assuming the picture of the flows suggested by geometry is valid, it is straightforward to check that the generic flow leads to a decrease in $g_{cl}$. The integer coordinates $(j, k)$ of the perturbing vector become $(n', p')$ of one of the daughter singularities, and in any given example one can find the values of $(n'', p'')$ of the other daughter (this is a question of performing the $SL(2, \mathbb{Z})$ transformation that puts the other sub-cone into the standard position with one of the bounding vectors at $(0, 1)$; then the other bounding vector is at $(n'', -p'')$). Unless the splitting vector is very near the boundaries of the original cone $\sigma$, the daughters will have a much smaller $n'$ and $n''$, and hence $g_{cl}$ will decrease.

There is however a class of candidate counterexamples (first discussed in [54]), where the splitting vector is near the edge of the cone $\sigma$, namely...
Here one has
\[ g_{\text{cl}}(\text{UV}) \sim \frac{1}{720} \frac{(2\ell)^3}{9}, \]
\[ g_{\text{cl}}(\text{IR}) \sim \frac{1}{720} \left( \frac{\ell^3}{9} + \frac{\ell^3}{1} \right), \] (1.81)
i.e. larger by 2/9 in the IR relative to the UV. This flow is marginal, and so \( g_{\text{cl}} \) is constant along the flow, but if this flow works as advertised, one can construct flows by only slightly relevant operators (nearby to the marginal one on the Dynkin diagram) which also violate the conjecture. There is some evidence, however, that this particular conjectured flow is very peculiar, and that perhaps we don’t understand the correspondence between geometry and worldsheet field theory as well as we’d like to think. For instance, to put the perturbing operator into a Dynkin diagram of a (nonminimal) resolution, one must blow up many (of order \( \ell/3 \)) additional \( \mathbb{P}^1 \)’s beyond those needed for the minimal resolution, see figure 1.21.

figure 1.21

\text{Figure 1.21} The flow \( 2\ell(3) \rightarrow \ell(\ell - 3) \oplus \ell(1) \) involves a sequence of blowups of the sort described by equation (1.79), in order to put the curve corresponding to the perturbing operator into the Dynkin diagram of the resolution. Here the vectors \( v_i \) give the subdivision of the cone corresponding to the minimal resolution of the singularity; the \( u_j \) specify the further subdivision that adds the extra curves to the Dynkin diagram (in this example, \( u_2 \) corresponds to the perturbing marginal operator).

\[ ^{13} \text{In fact, one can check using the above rules for splitting fans via infinite blowups, that any flow } n(p) \rightarrow n'(1) \oplus n''(n'' - p), \text{ with } n' = \frac{2n}{p+1} \text{ and } n'' = \frac{p}{p+1}n, \text{ violates the } g_{\text{cl}} \text{ conjecture (here we are assuming that } p+1 \text{ divides } n). \text{ All these flows are however subject to the same caveats discussed below.} \]
In other words, the continued fraction of the UV singularity is \([m+1,3]\) for, say, \(2\ell = 3m + 2\) (the case \(2\ell = 3m + 1\) is similar), while the two daughter singularities have continued fractions \([2,...,2,4]\) and \([\ell]\), where the number of curves with \(a_i = 2\) is \((\ell + 2)/3\). All these curves are associated to relevant perturbations, and it is not immediately obvious that they are not being blown up along the flow as well, leading to a different picture of the IR fixed point that would not contradict the \(g_{\text{cl}}\) conjecture. An argument against this point of view is that, along the flow by the marginal perturbation, the geometry is that of a \(\mathbb{Z}_\ell\) orbifold of \(T^*\mathbb{P}^1\), much as at the end of the previous subsection. Thus it might be that the extra curves are secretly being blown up by the marginal perturbation so that the resulting conformal field theory is nonsingular.

If these orbifolds prove to be valid counterexamples to the \(g_{\text{cl}}\) conjecture, it is actually somewhat more interesting than if they do not. Where are the additional localized closed string states coming from? Naively one expects that in the description of the marginal flow as a \(\mathbb{Z}_\ell\) orbifold of \(T^*\mathbb{P}^1\), there are states localized at the \(\mathbb{Z}_\ell\) fixed points at the north and south poles of the \(\mathbb{P}^1\), as well as normalizable states associated to the \(\mathbb{P}^1\); the latter should disappear from the normalizable spectrum at infinite volume, leaving only the localized orbifold states, and it does not seem as though we have gained any states in the process.

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