N=2 SuperTime Dependent Oscillator and Spontaneous Breaking of Supersymmetry

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Abstract

Using the nonlinear realizations of the N=2 superVirasoro group we construct the action of the N=2 Superconformal Quantum Mechanics (SCQM) with additional harmonic potential. We show that SU(1,1|1) invariance group of this action is nontrivially embedded in the N=2 Super Virasoro group. The generalization for the (super) time dependent oscillator is constructed. In a particular case when the oscillator frequency depends on the proper-time anticommuting coordinates the unusual effect of spontaneous breaking of the supersymmetry takes place: the Masses of bosons and fermions can have different nonzero values.
1 Introduction

The Time Dependent Oscillator (so called Ermakov system) has so many physical applications (see [2] for list of some references) and was the subject of vigorous research for decades especially due to its elegant mathematical properties and application potential of its invariant.

A vital modification of the Time Dependent Oscillator includes an additional term in the potential proportional to the inverse square of the coordinate - it is often referred to as the anharmonic oscillator. This extra term is conformally invariant. Analogous Conformal Quantum Mechanics (CQM) was investigated in detail by De Alfaro, Fubini and Furlan. It was shown in their paper that the consistent quantum treatment of the model assumes the transition to the new time coordinate which transpires to be equivalent to the introduction of an additional oscillator-like term with constant frequency in the potential. Therefore, the emerging physical Hamiltonian represents the anharmonic oscillator with time independent frequency $\omega$.

The most adequate approach for understanding the geometrical meaning of CQM and SCQM is the method of nonlinear realizations of the symmetry groups underlying both the theories - the group $SL(2, R)$ and its supersymmetrical generalizations $SU(1, 1|1)$, and $SU(1, 1|2)$, respectively [14], [7]. In spite of its power, this method does not allow the oscillator-like potentials introduced in [3] to be included in the Hamiltonian of the theory. As was shown in [2] the explanation for this lies entirely in the fact that in the presence of the oscillator-like term the invariance group of the nonsupersymmetric action, though being the Conformal Group, is realized by more complicated transformations. These transformations for the constant $\omega$, as well as for the time-dependent one (Ermakov system), can naturally be embedded in the reparametrization group of the time variable which is isomorphic to the centerless Virasoro group. This embedding is rather nontrivial in the case of nonvanishing $\omega$.

As will be shown in this paper, the supersymmetrization of the model leads to new possibility of supertime dependence of the oscillator frequency. As a result, the masses of bosons and fermions can have different nonzero values.

The structure of the paper is as follows. In Section 2 we apply the nonlinear realizations method to the Virasoro group and its three dimensional subgroup $SL(2, R)$. Using the Cartan’s invariant Omega-forms we construct the action for Conformal Quantum Mechanics and describe the mechanism for
appearance of oscillator-like terms in the Omega-forms and, correspondingly, in the action. We show how the symmetry group of this action, $SL(2, R)$, is nontrivially embedded in the Virasoro group and generalize these results to the Ermakov systems with time-dependent oscillator frequency.

In Section 3 we supersymmetrize the above constructions and illustrate the mechanism of appearing different nonzero masses for bosons and fermions.

Some further anticipations of the formalism developed are included in the conclusions.

## 2 Nonlinear Realization of the Virasoro Group

The generators of the infinite dimensional reparametrization (diffeomorphisms) group on the line parametrized by some parameter $s$ are $L_m = is^{m+1}d/ds$ and form the Virasoro algebra without central charge

$$[L_n, L_m] = -i(n - m)L_{n+m}. \quad (1)$$

If one restricts oneself to the transformations regular at the origin $s = 0$, it is convenient to parametrize the Virasoro group element as \[15, 16\]

$$G = e^{i\tau L_{-1}} \cdot e^{ix_1 L_1} \cdot e^{ix_2 L_2} \cdot e^{ix_3 L_3} \cdots e^{ix_0 L_0}, \quad (2)$$

where all multipliers, except the last one, are arranged by the conformal weight of the generators in the exponents.

The transformation laws of the group parameters $\tau, x_n$ under the left action

$$G' = (1 + ia)G, \quad (3)$$

where the infinitesimal element $a$ belongs to the Virasoro algebra

$$a = a_0 L_{-1} + a_1 L_0 + a_2 L_1 + \ldots + a_m L_m + \ldots = \sum_{n=0}^{\infty} a_n L_{n-1}, \quad (4)$$

are

$$\delta \tau = a(\tau), \quad \delta x_0 = \dot{a}(\tau), \quad \delta x_1 = -\dot{a}(\tau)x_1 + \frac{1}{2}\ddot{a}(\tau), \quad \delta x_2 = -2\dot{a}(\tau)x_2 + \frac{1}{6}\dddot{a}(\tau), \quad (5)$$

where the infinitesimal function $a(\tau)$ is constructed out of the parameters $a_n$

$$a(\tau) = a_0 + a_1 \tau + a_2 \tau^2 + \ldots = \sum_{n=0}^{\infty} a_n \tau^n. \quad (6)$$
One can see from (5) that the parameter $\tau$ transforms precisely as the co-ordinate of the one-dimensional space under the reparametrization. The parameters $x_0$ and $x_1$ transform, correspondingly, as the dilaton and one-dimensional Christoffel symbol. In general, the transformation rule for $x_n$ contains the $(n + 1)$-st derivative of the infinitesimal parameter $a(\tau)$.

All parameters $x_n, n = 0, 1, 2, \ldots$ in (2) in physical models are considered as fields in one-dimensional space parametrized by the coordinate $\tau$.

The conformal group $SL(2, R) \sim SU(1, 1)$ in one dimension is a three-parameter subgroup of (2), namely the one generated by $L_{-1}, L_0$ and $L_1$. Its group element is a product of the first two and last one multipliers in expression (2)

$$G_C = e^{i\tau L_{-1}} \cdot e^{ix_1 L_1} \cdot e^{ix_0 L_0}.$$  

(7)

In other words, the $SL(2, R)$ group is embedded in the Virasoro group in the simplest way by the conditions

$$x_n = 0, \quad n \geq 2$$

(8)

The infinitesimal transformation function $a(\tau)$ (9), which conserves the conditions (8), contains only three parameters

$$a(\tau) = a_0 + a_1 \tau + a_2 \tau^2.$$  

(9)

It is convenient to introduce new variables playing the roles of the coordinate and momentum of the particle

$$x = e^{x_0/2}, \quad p = x_1 x,$$  

(10)

for which the conformal group infinitesimal transformations are

$$\delta \tau = a(\tau), \quad \delta x = \frac{1}{2} \dot{a}(\tau) x, \quad \delta p = -\frac{1}{2} \dot{a}(\tau) p + \frac{1}{2} \ddot{a}(\tau) x,$$  

(11)

with $a(\tau)$ given in this case by (9).

Infinitesimal transformations (11) generate the symmetry group of the Conformal Quantum Mechanics of [3] with the action

$$S = \frac{1}{2} \int d\tau \left( \dot{x}^2 - \frac{\gamma}{x^2} \right).$$  

(12)

As was shown in [14] (see also [12]) this action can be naturally described in terms of the invariant differential Cartan’s form

$$\Omega_C = G_C^{-1} dG_C = \Omega_{-1} L_{-1} + \Omega_0 L_0 + \Omega_1 L_1$$  

(13)
connected with the parametrization (7) of the conformal group. Explicit calculations give
\[\Omega_{-1} = \frac{d\tau}{x^2}, \quad \Omega_0 = \frac{dx - pd\tau}{x}, \quad \Omega_1 = xdp - pdx + p^2d\tau.\]  

(14)

All these differential forms are invariant under transformations (11) and can be used for construction of an invariant action. The simplest one is the combination linear in \(\Omega\)-forms
\[S = -\frac{1}{2} \int \Omega_1 - \frac{\gamma}{2} \int \Omega_{-1} = \frac{1}{2} \int d\tau \left( -x\dot{p} + p\dot{x} - p^2 - \frac{\gamma}{x^2} \right).\]  

(15)

The first term in this expression is appropriately normalized to get the correct kinetic term. The parameter \(\gamma\) plays the role of a cosmological constant in one dimension because \(\Omega_{-1}\), which corresponds to the translation generator \(L_{-1}\), is the differential 1-form einbein.

Action (15) is the first order representation of that describing Conformal Mechanics of De Alfaro, Fubini and Furlan\(^\text{[3]}\). Indeed, one can find \(p\) by solving its equation of motion, insert it back in the Lagrangian and get the second order action (12).

From the point of view of underlying physics action (12) is not a satisfactory one because the corresponding quantum mechanical Hamiltonian does not have the ground state. The modification of this action with the appropriate energy spectrum was considered in \(\text{[3]}\). It includes the additional harmonic oscillator term
\[S_1 = \frac{1}{2} \int d\tau \left( \dot{x}^2 - \frac{\gamma}{x^2} - \omega^2x^2 \right).\]  

(16)

Though the action (16) contains the dimensional parameter \(\omega\), it is invariant under transformations of the conformal group realized by more complicated expressions, as we will see.

As we have already mentioned in Introduction, action (16) cannot be described in the framework of nonlinear realizations of the \(SL(2, R)\) group, parametrized as in (11). Instead, we will consider the embedding of this group in the Virasoro group \(\text{[2]}\) by conditions different from the simplest ones (8). The structure of the component \(\Omega^V\) in the Cartan’s Omega-form connected with the Virasoro group
\[\Omega_V = G^{-1}dG = \Omega_{-1}L_{-1} + \Omega_0 L_0 + \Omega^V_1 L_1 + \Omega^V_2 L_2 + \ldots\]  

(17)
may serve as a hint in the choice of appropriate conditions. The components \( \Omega_{-1} \) and \( \Omega_0 \) coincide with the corresponding components \( \{14\} \). Though the components \( \Omega_2^V, \Omega_3^V, \ldots \) depend in general on all parameters \( x_n \), the component \( \Omega_1^V \)

\[
\Omega_1^V = x dp - pdx + p^2d\tau - 3x_2 x^2 d\tau
\]  

depends, in addition to the phase space variables \((x, p)\), only on the parameter \( x_2 = x_2(\tau) \). So the last term in expression \( \{18\} \) is the only difference from the corresponding expression \( \{14\} \) calculated for representation \( \{2\} \) of the \( SL(2, R) \) group. Moreover, if we take

\[
x_2(\tau) = -\frac{1}{3}\omega^2, \quad \omega = \text{const},
\]

we obtain exactly an oscillator-like term in the action

\[
S = -\frac{1}{2} \int \Omega_1^V - \frac{\gamma}{2} \int \Omega_{-1} = \frac{1}{2} \int d\tau \left( -x\dot{p} + p\dot{x} - p^2 - \omega^2 x^2 - \frac{\gamma}{x^2} \right) ,
\]

which coincides with the action \( S_1 \) \( \{16\} \) in the second order form.

The component \( \Omega_1^V \) \( \{18\} \) is by construction invariant under the arbitrary infinitesimal transformations \( \{3\} \) of the Virasoro group. The consistency condition of this transformation law with the demand that \( \omega = \text{const} \) can be written in the form

\[
\ddot{a}(\tau) + 4\omega^2 \dot{a}(\tau) = 0.
\]

The solution of this differential equation gives

\[
a(\tau) = a_0 + a_1 \sin(2\omega\tau) + a_2 \cos(2\omega\tau).
\]

So the action of Conformal Mechanics \( \{16\} \) with the additional oscillator-like potential is invariant under the three parameter transformation \( \{22\} \).

In general the variable \( x_2(\tau) \) can be an arbitrary function of time. Nevertheless, it cannot be a dynamical variable because, as one can easily see from expression \( \{18\} \), it plays the role of a Lagrange multiplier leading to the equation of motion \( x^2 = 0 \)\(^1\). So instead of being the constant as in the

\[^1\text{If the variable} \ x \ \text{carries in addition some index} \ I - x \rightarrow x_I \ \text{the situation drastically changes when this index describes the vector representation of the rotation group of the space-time with the signature} \ (D, 2). \ \text{In this case, the action is given by the sum of} \ D+2 \ \text{expressions} \ \{15\} \ \text{(with the corresponding signs) and it describes the massless particle in} \ D \ \text{- dimensional space-time} \ \{17\} \ \text{(or the spinning particle if instead of the Virasoro group one considers the reparametrization group in the superspace} \ (1, N) \ \text{with one bosonic and} \ N \ \text{Grassmann coordinates} \ \{13\} \).

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previous Subsection, the parameter $x_2$ in a physical model can be at most some fixed function $x_2(\tau)$. If we look for invariance transformations of action \[20\] with the time dependent frequency $\omega^2(\tau) (x_2(\tau) = -\omega^2(\tau)/3)$, it means that after the time transformation \[3\] $\tau \to \tau' = \tau + a(\tau)$ the functional dependence should remain the same: $x_2(\tau) \to x_2(\tau')$, $\delta x_2(\tau) = a(\tau) \dot{x}_2(\tau)$. The transformation law \[3\] leads then to the equation for the infinitesimal parameter $a(\tau)$

$$\ddot{a}(\tau) + 4\omega^2(\tau) \dot{a}(\tau) + 2 \frac{d}{d\tau}(\omega^2(\tau))a(\tau) = 0.$$ \hspace{1cm} (23)

This differential equation of the third order with the time dependent coefficients has the solution in the form \[19\]

$$a(\tau) = C_1 u_1^2 + C_2 u_1 u_2 + C_3 u_2^2,$$ \hspace{1cm} (24)

where $C_1, C_2, C_3$ are three infinitesimal constants, and the functions $u_1(\tau), u_2(\tau)$ form the fundamental system of solutions to the auxiliary equation

$$\ddot{u}(\tau) + \omega^2(\tau) u(\tau) = 0.$$ \hspace{1cm} (25)

For the time independent $\omega$ this solution reproduces the ones given by \[22\]. For different particular forms of $\omega^2(\tau)$ equation \[23\] becomes, for example, the Lame, Matieu, Hill etc. equations \[19\], each playing very important role in physics.

So solution \[24\] of equation \[23\] describes the invariance transformations of the action for the Time Dependent Oscillator with the frequency $\omega(\tau)$.

3 $N = 2$ SuperTime dependent oscillator

The $N = 2$ SuperVirasoro algebra is formed by the Virasoro generators $L_n$, complex supergenerators $G_r, \bar{G}_s$ and $U(1)$ Kac-Moody algebra generators $T_k$. The indices $n, k$ and $r, s$ are arbitrary integer and halfinteger numbers correspondingly. This algebra has the following form

$$[L_m, L_n] = -i(m-n)L_{m+n}, \quad [L_m, T_k] = ikT_{m+k},$$
$$[L_m, G_s] = -i\left(\frac{m}{2} - s\right)G_{m+s}, \quad [L_m, \bar{G}_s] = -i\left(\frac{m}{2} - s\right)\bar{G}_{m+s}$$ \hspace{1cm} (1)

$$[T_k, G_s] = -\frac{i}{2}G_{k+s}, \quad [T_k, \bar{G}_s] = \frac{i}{2}\bar{G}_{k+s},$$
$$\{G_r, \bar{G}_s\} = -2L_{r+s} - 2(r-s)T_{r+s}.$$
If one restricts to the subalgebra with indices in the region \( m, n \geq -1, k \geq 0, r, s \geq -1/2 \), which we will call in what follows as \( N = 2 \) Superconformal Algebra (SCA), it is convenient to parameterize the coset space of the corresponding group over the \( U(1) \) subgroup generated by \( T_0 \) in the following form \[15, 16\]

\[
G_I = e^{i\Gamma L_{-1}} \cdot e^{\theta G_{1/2} + \theta G_{-1/2}} \cdot e^{\bar{\psi} G_{1/2} + \psi G_{1/2}} \cdot e^{iU^{(1)} L_1} \cdot e^{V^{(1)} T_1} \cdot e^{\Theta (3/2) \bar{G}_{3/2} + \bar{\Theta} (3/2) G_{3/2}} \cdot e^{iU^{(2)} L_2} e^{V^{(2)} T_1} \cdot \ldots \cdot e^{iU^{(0)} L_0}.
\]

(2)

where all multipliers, except the last one, are arranged by the conformal weight of the generators in the exponents. The transformation laws of the group parameters \( \tau, \theta, \psi, U^{(0)}, U^{(1)}, V^{(1)}, \ldots \) in (2) under the left action

\[ G' = (1 + ia)G, \]

(3)

where infinitesimal element \( a \) belongs to the \( N = 2 \) Virasoro algebra, are incoded in the infinitesimal real superfunction

\[ \Lambda = a(\tau) + 2i\theta \bar{\epsilon}(\tau) + 2i\bar{\theta} \epsilon(\tau) + \theta \bar{\theta} b(\tau) \]

(4)

and are, in particular,

\[
\begin{align*}
\delta \tau & = \Lambda - \frac{1}{2}(\theta D + \bar{\theta} \bar{D}) \Lambda, \\
\delta \theta & = -\frac{i}{2} \bar{D} \Lambda, \\
\delta \bar{\theta} & = -\frac{i}{2} D \Lambda, \\
\delta V^{(1)} & = -\Lambda V^{(1)} - \frac{1}{2}(D \bar{D} - \bar{D} D) \Lambda.
\end{align*}
\]

(5) (6) (7) (8)

where \( D \) and \( \bar{D} \) are supercovariant derivatives \( D = \partial/\partial \theta + i\bar{\theta} \partial/\partial \tau, \bar{D} = \partial/\partial \bar{\theta} + i\theta \partial/\partial \tau \).

Parameters \( \tau, \theta \) and \( \bar{\theta} \) transform as coordinates of \( N = 2 \) superspace. All other parameters can be viewed as superfunctions in this superspace.

As was shown in \[17\], the superspace action constructed with the help of Cartan’s invariant forms (we omit here details, as well, as supersymmetrization of the conformal potential \( 1/x^2 \)) has the form

\[
S = \int \! d\tau \, d\theta \, d\bar{\theta} \left( \frac{1}{2} DX \bar{D} X + \frac{1}{4} V^1 X^2 \right),
\]

(10)
where $X = e^{U(0)} = x(\tau) + i\theta\bar{\gamma}(\tau) + i\bar{\theta}\gamma(\tau) + \bar{\theta}\theta F(\tau)$ is the $N = 2$ superfield coordinate.

The last term in (10) after superspace integration reproduces the mass terms for bosonic $x(\tau)$ and fermionic $\gamma(\tau), \bar{\gamma}(\tau)$ coordinates. If the function $V^1$ is a constant, both masses are equal. On the other hand, $V^1$ can depend on all coordinates of the superspace. The consistency equation analogous to (23) have now the form

$$\Lambda \dot{V}^1 - i/2D\Lambda \bar{D}V^1 - i/2\bar{D}\Lambda DV^1 + \dot{\Lambda}V^1 + 1/2(D\bar{D} - \bar{D}D)\dot{\Lambda} = 0.$$  \hspace{1cm} (11)

The solutions of this equation define the embedding of the $SU(1,1|1)$ invariance group of the action in the $N = 2$ Super Virasoro group.

As an example, one can take $V^1 = 2\omega_1 + 2\bar{\theta}\theta \Delta$. Then masses of boson and fermions will be different

$$M_f = \omega_1, \quad M_b^2 = \omega_1^2 + \Delta.$$

4 Conclusions

In this paper we applied the methods of nonlinear realizations approach for construction of the actions of SuperConformal Quantum Mechanics, as well, as the action of the SuperTime Dependent Oscillator. Actions are invariant under the transformations of $SU(1,1|1)$, which is nontrivially embedded in the $N = 2$ Super Virasoro group. It would be interesting to carry up the analogous considerations in the more complicated theories, such $N = 4$ SuperConformal Quantum Mechanics. The mechanism of spontaneous breakdown of the supersymmetry established in this paper gives the possibility to regulate boson and fermion masses without destroying the symmetry properties of the model. The question whether this mechanism is specific only to one dimensional models or the same mechanism will also be work for $D= 4$ dimensional models -is still an open.

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