Quantum Langevin equation

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Abstract.
We propose a Langevin equation to describe the quantum Brownian motion of bounded particles based on a distinctive formulation concerning both the fluctuation and dissipation forces. The fluctuation force is similar to that employed in the classical case. It is a white noise with a variance proportional to the temperature. The dissipation force is not restricted to be proportional to the velocity and is determined in a way as to guarantee that the stationary state is given by a density operator of the Gibbs canonical type. To this end we derived an equation that gives the time evolution of the density operator, which turns out to be a quantum Fokker-Planck-Kramers equation. The approach is applied to the harmonic oscillator in which case the dissipation force is found to be non Hermitian and proportional to the velocity and position.
1. Introduction

The Brownian motion of particles in contact with a heat reservoir is well described by the Langevin equation \[1, 2, 3, 4\]. The essential feature of this equation is the presence of two types of forces, one being the dissipation force and the other being a force of a stochastic nature. The interplay between these two forces leads to a mechanism of loss and gain of energy that ensures an unceasing motion of the particles. In addition, a dissipation-fluctuation relation exists or should exist between the two forces to guarantee that in equilibrium the probability distribution of velocities and positions is the equilibrium Gibbs distribution.

The extension of the Langevin equation to describe quantum open system and quantum dissipation \[5, 6, 7, 8, 9\] has been addressed but the quantum description has brought problems that were not present in the classical case, and the approach to quantum open system and quantum dissipation does not seem to be completely solved \[10\]. For instance, an equation that governs the time evolution of the probability density, the Fokker-Planck-Kramers equation, can immediately be obtained from the classical Langevin equation. However, for the quantum case, such a derivation is not a trivial problem \[10\].

The development of an equation that describes quantum dissipation has focused mainly on the harmonic oscillator. This is the case of the paper by Senitzky on quantum dissipation \[11, 12, 13\]. He remarked that in quantum mechanics, dissipation and fluctuations cannot be divorced as in classical mechanics so that these two mechanisms must be considered together. In his treatment of the harmonic oscillator the dissipation force is assumed to be proportional to the velocity, and the fluctuating force has a time correlation proportional to \(E(\omega)\delta(t' - t)\) where \(\omega\) is the oscillator frequency and

\[
E = \frac{\hbar \omega}{2} \coth \frac{\hbar \omega}{2k_B T}
\]

is the Planck function, the average energy of a harmonic oscillator. Another type of time correlation for the fluctuating force was that used by Ford, Kac and Mazur \[14, 15, 16, 17\]. They proposed a symmetrized time correlation proportional to

\[
\int_0^\infty E(\omega) \cos(\omega(t' - t))d\omega.
\]

This type of correlation has been used to study the thermal properties of chains coupled oscillator in equilibrium and out of equilibrium \[18\]. However, it has been shown that the canonical distribution cannot be approached for finite friction but only in the weak friction limit \[19, 20\].

The approach proposed by Bedeaux and Mazur \[21\] for the Langevin equation for a harmonic oscillator has a distinct feature concerning the fluctuation mechanism. In addition to the random force, they introduced a random velocity. The correlation between the random force and the random velocity yields the zero point motion of the oscillator. The correlation of the random force is similar to that employed by Senitzky, \(E(\omega)\delta(t' - t)\), and the correlation of the random velocity is chosen to vanish.
However, the correlation between the random force and random velocity is nonzero and is proportional to $i\hbar \delta(t' - t)$. This approach has recently been used to an axiomatic construction of quantum Langevin equation [10].

Here we propose an approach to the Langevin equation that is distinct from the above approaches in two aspects. In treatments of the Langevin equation, it is usual to consider the dissipative force as proportional to the velocity. In our approach, this restriction is relaxed and we allow for a dissipative force that might depend on velocity and possibly on the position. The second distinct feature concerns the random force. The correlation of the random force is taken to be proportional to the temperature, as in the classical case, and is considered to be a complex scalar. These two features allowed us to derive the following relation between the random dissipative force $F$ and the momentum $p$ and the position $x$ of a particle described by a Hamiltonian $\mathcal{H}(x, p)$,

$$F = \gamma \{ p - \frac{\beta}{2!} [p, \mathcal{H}] + \frac{\beta^2}{3!} [[p, \mathcal{H}], \mathcal{H}] + \ldots \} ,$$

where $\gamma$ is the dissipative constant and $\beta = 1/k_B T$. This expression is non Hermitian in general and has been previously obtained by another procedure namely by the quantization of the classical Fokker-Planck-Kramers equation [22]. Expression (3) gives a dissipation proportional to the form $F = \gamma p$ only for a free Brownian motion in which case $\mathcal{H} = p^2/2m$. In the classical limit, it reduces to the usual form $F = \gamma p$.

The present approach allows us to derive the equation that gives the evolution of the density operator, which is a quantum Fokker-Planck-Kramers equation [22]. The stationary solution of this equation shows that the system reaches indeed the appropriate Gibbs state, that is, the system thermalizes properly. We remark that this is valid for any finite dissipative constant and not only for the weak friction limit and for any potential and not for the harmonic potential as long as the expression (3) is used.

Our approach follows as much as possible the usual treatments of the classical Langevin equation. For this reason we will first review briefly the classical approach to the Langevin equation. After that we present our approach to the quantum stochastic equation and derive from it the quantum Fokker-Planck-Kramers equation. In the last section before the conclusion we apply our approach to the harmonic oscillator in which case the expression (3) reduces to the form $F = \gamma (ap + ibx)$.

2. Classical stochastic equation

We consider the Brownian motion of a bounded particle of mass $m$ moving along a straight line and denote by $x$ its position and by $p$ is momentum. The Langevin equation for this particle is

$$\frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial x} - F + \xi,$$

$$\frac{dx}{dt} = \frac{\partial \mathcal{H}}{\partial p},$$

(4) (5)
Quantum Langevin equation

where $H(x, p)$ is the Hamiltonian function

$$H = \frac{p^2}{2m} + V(x),$$

(6)

$F$ is a dissipative force and $\mathcal{F}$ is a fluctuating force holding the properties $\langle \mathcal{F}(t) \rangle = 0$ and

$$\langle \mathcal{F}(t) \mathcal{F}(t') \rangle = \Gamma \delta(t - t').$$

(7)

We consider $\Gamma$ to be constant, independent of $x$ and $p$.

In the stationary regime we require that the particle be in thermodynamic equilibrium with its surroundings, which is assumed to be at at temperature $T$. The requirement is fulfilled by demanding that, in the stationary regime, the probability distribution of the variables $x$ and $p$ is described by the Gibbs canonical distribution

$$\rho_0 = \frac{1}{\zeta} e^{-\beta H},$$

(8)

where $\beta = 1/k_B T$, a condition implying that dissipation and fluctuation are related in some way. One procedure used to find such a relationship amounts to write down the time evolution equation for the probability distribution and impose that, in the stationary regime it is of the form (8).

To find the equation that gives the time evolution of the probability distribution, we start by writing the Langevin equations (4) and (5) in the discretized form

$$\Delta p = -\frac{\partial H}{\partial x} \Delta t - F \Delta t + \xi \sqrt{\Gamma} \Delta t,$$

(9)

$$\Delta x = \frac{\partial H}{\partial p} \Delta t,$$

(10)

where $\xi$ is a random variable with central Gaussian distribution with unit variance.

The variation of an arbitrary function $F(x, p)$ is given by

$$\Delta F = \frac{\partial F}{\partial x} \Delta x + \frac{\partial F}{\partial p} \Delta p + \xi^2 \frac{\Gamma}{2} \frac{\partial^2 F}{\partial p^2} \Delta t.$$

(11)

Replacing $\Delta x$ and $\Delta p$, given by the equations (10) and (9) into this equation, and after integrating in $\xi$ and dividing both sides by $\Delta t$, the following equation is obtained by taking the limit $\Delta t \to 0$,

$$\frac{\partial F}{\partial t} = \{F, H\} - \frac{\partial F}{\partial p} F + \frac{\Gamma}{2} \frac{\partial^2 F}{\partial p^2},$$

(12)

where

$$\{F, H\} = \frac{\partial F}{\partial x} \frac{\partial H}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial H}{\partial x}.$$

(13)

Denoting by $\rho(x, p)$ the probability distribution of variables $x$ and $p$, the time evolution of the average $\langle F \rangle$ becomes

$$\frac{d}{dt} \langle F \rangle = \int \left( \{F, H\} - \frac{\partial F}{\partial p} F + \frac{\Gamma}{2} \frac{\partial^2 F}{\partial p^2} \right) \rho \, dx dp.$$

(14)
Since $F$ is arbitrary, we obtain from this equation the following evolution of the probability distribution, after appropriate integrations by parts, and considering that the density vanishes at the boundaries of the system,

$$\frac{\partial \rho}{\partial t} = \{H, \rho\} + \frac{\partial F \rho}{\partial p} + \frac{\Gamma}{2} \frac{\partial^2 \rho}{\partial p^2},$$

(15)

which is the Fokker-Planck-Kramers equation.

Imposing that the stationary probability distribution is the Gibbs distribution (8), and using the result $\{H, \rho_0\} = 0$, we reach the condition

$$F \rho_0 + \frac{\Gamma}{2} \frac{\partial \rho_0}{\partial p} = 0,$$

(16)

Replacing the Gibbs distribution (8) into this equation to find

$$F = \frac{\Gamma}{2m} \beta p,$$

(17)

which is the desired relation between dissipation and fluctuation. If we write $F = \gamma p$, where $\gamma$ is the dissipation constant, we reach the usual dissipation-fluctuation relation

$$\Gamma = \frac{2m\gamma}{\beta}.$$

(18)

3. Quantum stochastic equation

In analogy with the classical equations of motion (4) and (5), we assume the same form for the quantum Langevin equation,

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} - F + \tilde{\mathcal{F}},$$

(19)

$$\frac{dx}{dt} = \frac{\partial H}{\partial p},$$

(20)

where $F$ is the dissipative force, that depends on $p$ and possibly on $x$, $\tilde{\mathcal{F}}(t)$ is a fluctuating force of the stochastic nature, and $H$ is the Hamiltonian operator

$$H = \frac{p^2}{2m} + V(x).$$

(21)

The quantities $p$ and $x$ appearing in equations (19) and (20) are quantum operators acting on a Hilbert space, and so are the dissipative force $F$ and the Hamiltonian $H$. However, the noise $\tilde{\mathcal{F}}$ is a scalar and assumed to be complex.

In usual treatments of the Langevin equation, the dissipative force $F$ is considered to be proportional to $p$, but here we relax this condition and let $F$ be a generic function of $p$ and $x$, to be found, which may not be Hermitian. If that is the case we see that the operators $x$ and $p$ in equations (19) and (20) may not be Hermitian either. We also allow the noise to be a complex scalar,

$$\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_1 + i\tilde{\mathcal{F}}_2,$$

(22)

where $\tilde{\mathcal{F}}_1$ and $\tilde{\mathcal{F}}_2$ are real with the properties $\langle \tilde{\mathcal{F}}_n(t) \rangle = 0$ and

$$\langle \tilde{\mathcal{F}}_n(t) \tilde{\mathcal{F}}_{n'}(t') \rangle = \Gamma \delta_{nn'} \delta(t - t').$$

(23)
3.1. Evolution of the density operator

The equations of motion (19) and (20) are understood as equations belonging to the Heisenberg representation, which are characterized by the time dependency of operators. An equivalent representation which will be pursued next is the one in which states rather than operators change in time. However, the states in this representation will not be expressed by state vectors, describing pure states, because the time evolution of \( x \) and \( p \), given by equations (19) and (20), is not unitary. Instead, the states will be expressed by the density operator \( \rho \), describing mixed states.

To find the equation that gives the time evolution of \( \rho \), we proceed in a way similar to the procedure employed in the classical case. We start by writing the discretized version of equations (19) and (20),

\[
\Delta p = -\frac{\partial H}{\partial x} \Delta t - F \Delta t + \xi \sqrt{\Gamma} \Delta t, \tag{24}
\]

\[
\Delta x = \frac{\partial H}{\partial p} \Delta t, \tag{25}
\]

where \( \xi = \xi_1 + i\xi_2 \) and \( \xi_1 \) and \( \xi_2 \) are independent and uncorrelated random variables with central Gaussian distributions with variance equal to 1/2.

Next we consider the time evolution of an operator \( F(x, p) \). The increment of \( F \) during an interval of time \( \Delta t \) is

\[
\Delta F = \left( \frac{\partial F}{\partial x}, \Delta x \right) + \left( \frac{\partial F}{\partial x^\dagger}, \Delta x^\dagger \right) + \left( \frac{\partial F}{\partial p}, \Delta p \right) + \left( \frac{\partial F}{\partial p^\dagger}, \Delta p^\dagger \right) 
+ \frac{\Gamma}{2} \left( \xi^2 \frac{\partial^2 F}{\partial p^2} + 2 \xi^* \xi \frac{\partial^2 F}{\partial p \partial p^\dagger} + (\xi^*)^2 \frac{\partial^2 F}{\partial (p^\dagger)^2} \right) \Delta t. \tag{26}
\]

The notation \( (, ) \) used for a derivative is explained in the Appendix.

Replacing \( \Delta x, \Delta p \), given by the equations (24) and (25), and after integrating in \( \xi \) and dividing by \( \Delta t \), we reach the following equation by taking the limit \( \Delta t \to 0 \),

\[
\frac{d}{dt} \langle F \rangle = -\langle \{ F, H \} \rangle - \left( \frac{\partial F}{\partial p}, F \right) 
+ \langle \{ F, H \}^\dagger \rangle - \left( \frac{\partial F}{\partial p^\dagger}, F^\dagger \right) + \Gamma \left( \frac{\partial^2 F}{\partial p \partial p^\dagger} \right), \tag{27}
\]

where

\[
\{ F, H \} = \left( \frac{\partial F}{\partial x}, \frac{\partial H}{\partial p} \right) - \left( \frac{\partial F}{\partial p}, \frac{\partial H}{\partial x} \right), \tag{28}
\]

and we are using the notation \( \langle A \rangle = \text{Tr} A \rho \) for the average of \( A \) with respect to the density operator \( \rho \).

Employing the results

\[
\text{Tr} \{ F, H \} \rho = -\text{Tr} F \{ \rho, H \}, \tag{29}
\]

\[
\text{Tr} \left( \frac{\partial F}{\partial p}, F \right) \rho = -\text{Tr} F \left( \frac{\partial \rho}{\partial p}, F \right) - \text{Tr} F \left( \rho \frac{\partial F}{\partial p} \right), \tag{30}
\]

The notation \( \{, \} \) is explained in the Appendix.
where we have considered boundary conditions such that $\rho$ vanishes, and considering that the function $F$ is arbitrary, we may write the following equation that gives the evolution of the density operator,

$$
\frac{\partial \rho}{\partial t} = -\{\rho, H\} + \left( \frac{\partial \rho}{\partial p}, F \right) + \rho \frac{\partial F}{\partial p} \\
-\{\rho, H\}^\dagger + \left( \frac{\partial \rho}{\partial p^\dagger}, F^\dagger \right) + \frac{\partial F^\dagger}{\partial p^\dagger} \rho + \Gamma \frac{\partial^2 \rho}{\partial p \partial p^\dagger}.
$$

(31)

3.2. Thermodynamic equilibrium

As we have seen above the operators $x$ and $p$, solutions of equations (19) and (20), may not be Hermitian and in fact they remain non-Hermitian even in the stationary state. In this sense they cannot be understood as the position and momentum of a particle, which according to the principles of quantum mechanics are observables and as a consequence should be Hermitian operators. This nuisance is solved by requiring that $p$ and $x$, the solution of equations (19) and (20), be normal operators, which are operators holding the following property

$$
p^\dagger p = pp^\dagger, \quad x^\dagger x = xx^\dagger,
$$

(32)

which allows us to define the Hermitian operators $\overline{x}$ and $\overline{p}$ by

$$
\overline{x} = (x^\dagger x)^{1/2}, \quad \overline{p} = (p^\dagger p)^{1/2}.
$$

(33)

We then postulate that the observables corresponding to the position and momentum of the particle are $\overline{x}$ and $\overline{p}$, respectively, and not properly $x$ and $p$.

In the stationary state we demand that the density operator is of the Gibbs type

$$
\rho_0 = \frac{1}{\zeta} e^{-\beta H}.
$$

(34)

In accordance with our postulate above, the Hamiltonian in equation (34) will be written in terms of $\overline{x}$ and $\overline{p}$, that is,

$$
H = \frac{\overline{p}^2}{2m} + V(\overline{x}).
$$

(35)

To determine $F$, we proceed as follows. First we observe that $\{\rho_0, H\} = 0$, a result that allows us to reach the following equations that determine $F$,

$$
\left( \frac{\partial \rho}{\partial p}, F \right) + \rho \frac{\partial F}{\partial p} + \frac{\Gamma}{2} \frac{\partial^2 \rho}{\partial p \partial p^\dagger} = 0.
$$

(36)

The solution of this equation is

$$
F = -\frac{\Gamma}{2} \rho_0^{-1} \frac{\partial \rho_0}{\partial p} = -\frac{\Gamma}{2} e^{\beta H} \frac{\partial}{\partial p} e^{-\beta H},
$$

(37)

which is the desired dissipation-fluctuation relation.

The result (37) can be expressed in the following form

$$
F = \Gamma \left( \beta \frac{\partial H}{\partial p} - \frac{\beta^2}{2!} \left[ \frac{\partial H}{\partial p}, H \right] + \frac{\beta^3}{3!} \left[ \frac{\partial H}{\partial p}, [H, H] \right] + \ldots \right).
$$

(38)
Quantum Langevin equation

Taking into account equation (35), this expression reduces to
\[ F = \frac{\Gamma \beta}{2m} \left( \mathcal{P} - \frac{\beta}{2} \mathcal{P} \mathcal{H} + \frac{\beta^2}{3!} [\mathcal{P}, \mathcal{H}, \mathcal{H}] + \ldots \right), \] (39)
which is an equivalent form of writing the dissipation-fluctuation relation (37).

4. Harmonic oscillator

4.1. Equilibrium

Here we treat the harmonic oscillator for which \( \mathcal{H} = \frac{p^2}{2m} + m\omega^2x^2/2 \). The dissipation force \( F \) turns out to be linear in \( p \) and \( x \) which can be determined from (39) as shown in the Appendix. The result is
\[ F = \gamma (ap - ibx), \quad \gamma = \frac{\Gamma \beta}{2m}, \] (40)
where
\[ a = \frac{1}{\beta \hbar} \sinh \beta \hbar \omega, \quad b = \frac{m}{\beta \hbar} (\cosh \beta \hbar \omega - 1). \] (41)

We use the Langevin equations (19) and (20), which for the harmonic oscillator reads
\[ \frac{dp}{dt} = -m\omega^2x - \gamma (ap - ibx) + \mathcal{F}, \] (42)
\[ \frac{dx}{dt} = \frac{p}{m}. \] (43)

From these equations we obtain
\[ \frac{d}{dt} \langle p\dagger p \rangle = -m\omega^2 \langle x\dagger p + p\dagger x \rangle - 2\gamma a \langle p\dagger p \rangle - i\gamma b \langle x\dagger p - p\dagger x \rangle + \frac{2m\gamma}{\beta}, \] (44)
\[ \frac{d}{dt} \langle x\dagger x \rangle = \frac{1}{m} \langle x\dagger p + p\dagger x \rangle, \] (45)
\[ \frac{d}{dt} \langle x\dagger p \rangle = \frac{1}{m} \langle p\dagger p \rangle - m\omega^2 \langle x\dagger x \rangle - \gamma a \langle x\dagger p \rangle + i\gamma b \langle x\dagger x \rangle, \] (46)
\[ \frac{d}{dt} \langle p\dagger x \rangle = \frac{1}{m} \langle p\dagger p \rangle - m\omega^2 \langle x\dagger x \rangle - \gamma a \langle p\dagger x \rangle - i\gamma b \langle x\dagger x \rangle. \] (47)

Summing and subtracting the last two equations give
\[ \frac{d}{dt} \langle x\dagger p + p\dagger x \rangle = \frac{2}{m} \langle p\dagger p \rangle - 2m\omega^2 \langle x\dagger x \rangle - \gamma a \langle x\dagger p + p\dagger x \rangle, \] (48)
\[ \frac{d}{dt} \langle x\dagger p - p\dagger x \rangle = -\gamma a \langle x\dagger p - p\dagger x \rangle + 2i\gamma b \langle x\dagger x \rangle. \] (49)

The equations (44), (45), (48), and (49) form a set of linear closed equations for the four correlations \( \langle p\dagger p \rangle, \langle x\dagger x \rangle, \langle x\dagger p + p\dagger x \rangle \) and \( \langle x\dagger p - p\dagger x \rangle \), and can be solved as functions of time. Here however we will focus on the steady state solution, which is given by the equations
\[ \langle x\dagger p + p\dagger x \rangle = 0, \] (50)
Quantum Langevin equation

\[
\frac{1}{m} \langle p^\dagger p \rangle = m\omega^2 \langle x^\dagger x \rangle = \frac{a}{\beta(a^2 - b^2/m^2\omega^2)},
\]

\[
\langle x^\dagger p - p^\dagger x \rangle = \frac{2ib}{\beta m\omega^2(a^2 - b^2/m^2\omega^2)}.
\]

Using the values of \(a\) and \(b\) given by (41), we reach the results

\[
\frac{1}{m} \langle p^\dagger p \rangle = m\omega^2 \langle x^\dagger x \rangle = \hbar \omega \left(\frac{1}{e^{\beta\hbar\omega} - 1} + \frac{1}{2}\right),
\]

\[
\langle x^\dagger p - p^\dagger x \rangle = i\hbar,
\]

which are the desired results, if we bear in mind that \(p^\dagger p = \bar{p}^2\) and \(x^\dagger x = \bar{x}^2\).

4.2. Time dependent solution

The set of equations (42) and (43) are a set of linear differential equations and can thus be solved for any given time dependent function \(\mathbf{f}\). From the solution for \(x(t)\) and \(p(t)\) the covariances are readily obtained. Here, we take a distinct approach and instead we solve the equations for the covariances themselves, which are given by equations (44), (45), (46), and (47). To this end we first observe that these equations can be written in terms of a matrix differential equation. Defining \(X = \langle x^\dagger x \rangle\), \(Y = \langle p^\dagger p \rangle\), \(Z = \langle x^\dagger p \rangle\), then

\[
\frac{dY}{dt} = -m\omega^2(Z + Z^*) - 2\gamma aY - i\gamma b(Z - Z^*) + \frac{2m\gamma}{\beta},
\]

\[
\frac{dX}{dt} = \frac{1}{m}(Z + Z^*),
\]

\[
\frac{dZ}{dt} = \frac{1}{m}Y - m\omega^2X - \gamma aZ + i\gamma bX,
\]

\[
\frac{dZ^*}{dt} = \frac{1}{m}Y - m\omega^2X - \gamma aZ^* - i\gamma bX.
\]

The time evolution of the covariance matrix

\[
Q = \begin{pmatrix} X & Z \\ Z^* & Y \end{pmatrix}
\]

is easily obtained from these equations and is given by

\[
\frac{dQ}{dt} = QM + M^\dagger Q + \Omega,
\]

where the matrices \(M\) are \(\Omega\) are

\[
M = \begin{pmatrix} 0 & -m\omega^2 + i\gamma b \\ 1/m & -\gamma a \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 0 \\ 0 & 2m\gamma/\beta \end{pmatrix},
\]

and \(M^\dagger\) is the transpose conjugate of \(M\).
Next we set up a matrix $B$ whose columns are the right eigenvectors of the matrix $M$. This matrix and its inverse are

$$B = \begin{pmatrix} -\lambda_2 & \lambda_1 \\ 1/m & -1/m \end{pmatrix}, \quad B^{-1} = m \frac{1/m}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 & 1/m \\ 1/m & \lambda_2 \end{pmatrix},$$

(62)

where $\lambda_1$ and $\lambda_2$ are the eigenvalues of $M$ and the roots of

$$m\lambda^2 + \gamma ma\lambda + (m\omega^2 - i\gamma b) = 0.$$  

(63)

The matrix $B$ together with its inverse $B^{-1}$ diagonalizes $M$, that is, $B^{-1}MB = D$, where $D$ is diagonal with elements $\lambda_1$ and $\lambda_2$.

As to the matrix $M^\dagger$, it is not diagonalized by the matrix $B$ but it is diagonalized by the transpose conjugate of $B$. Indeed, taking the transpose conjugate of $B^{-1}MB = D$, we find the result $AM^\dagger A^{-1} = D^\dagger$, where $A = B^\dagger$, and $D^\dagger$ is the diagonal matrix with elements $\lambda_1^*$ and $\lambda_2^*$. The matrix $A$ and its inverse are

$$A = \begin{pmatrix} -\lambda_2^* & 1/m \\ \lambda_1^* & -1/m \end{pmatrix}, \quad A^{-1} = m \frac{1/m}{\lambda_1^* - \lambda_2^*} \begin{pmatrix} \lambda_1^* & 1/m \\ 1/m & \lambda_2^* \end{pmatrix}.$$  

(64)

Multiplying the equation (60) at the left by $A$ and at the right by $B$, we reach the equation

$$\frac{dQ'}{dt} = Q'D + D^\dagger Q' + \Omega',$$

(65)

where $Q' = AQB$ and $\Omega' = A\Omega B$. Denoting by $X'$, $Y'$, $Z'$, and $(Z')^*$ the elements of the matrix $Q'$, the equation (65) is written in an explicit form as

$$\frac{dX'}{dt} = (\lambda_1 + \lambda_1^*)X' + \frac{2\gamma}{m\beta},$$

(66)

$$\frac{dY'}{dt} = (\lambda_2 + \lambda_2^*)Y' + \frac{2\gamma}{m\beta},$$

(67)

$$\frac{dZ'}{dt} = (\lambda_1^* + \lambda_2)Z' - \frac{2\gamma}{m\beta},$$

(68)

$$\frac{d(Z')^*}{dt} = (\lambda_1 + \lambda_2^*)(Z')^* - \frac{2\gamma}{m\beta}.$$  

(69)

The solutions of these equations are readily obtained,

$$X' = X_0 e^{(\lambda_1 + \lambda_1^*)t} - \frac{2\gamma}{m\beta(\lambda_1 + \lambda_1^*)},$$

(70)

$$Y' = Y_0 e^{(\lambda_2 + \lambda_2^*)t} - \frac{2\gamma}{m\beta(\lambda_2 + \lambda_2^*)},$$

(71)

$$Z' = Z_0 e^{(\lambda_1^* + \lambda_2)t} + \frac{2\gamma}{m\beta(\lambda_1^* + \lambda_2)},$$

(72)

$$(Z')^* = Z_0^* e^{(\lambda_1 + \lambda_2^*)t} + \frac{2\gamma}{m\beta(\lambda_1 + \lambda_2^*)}.$$  

(73)
The relation $Q' = AQB$ gives $Q = A^{-1}Q'B^{-1}$ from which follows

$$X = \frac{1}{|\lambda_1 - \lambda_2|^2}(X' + Y' + Z' + (Z')^*),$$  \hspace{1cm} (74)

$$Y = \frac{m^2}{|\lambda_1 - \lambda_2|^2}(|\lambda_1|^2X' + |\lambda_2|^2Y' + \lambda_1^*\lambda_2 Z' + \lambda_2^*\lambda_1 (Z')^*),$$  \hspace{1cm} (75)

$$Z = \frac{m}{|\lambda_1 - \lambda_2|^2}(\lambda_1 X' + \lambda_2 Y' + \lambda_1 (Z')^* + \lambda_2 Z'),$$  \hspace{1cm} (76)

$$Z^* = \frac{m}{|\lambda_1 - \lambda_2|^2}(\lambda_1^*X' + \lambda_2^* Y' + \lambda_1^*Z' + \lambda_2^* (Z')^*),$$  \hspace{1cm} (77)

which are the relations between the covariances $X$, $Y$, $Z$, and $Z^*$, and $X'$, $Y'$, $Z'$, and $(Z')^*$. The replacement of the later variables into the equations (74), (75), (76), and (77), gives the covariances as functions of time.

When $t \to \infty$ its possible to show that these covariances reaches the equilibrium values given by equations (53) and (54). These values are approached exponentially with a correlation time which is the reciprocal of the real part of the eigenvalues.

5. Conclusion

We have proposed a quantum Langevin equation that is very distinct from previous approaches. The distinction concerns both the dissipation and fluctuation forces. The fluctuation force is similar to that employed in the classical case. It is a white noise with a covariance proportional to the temperature and is the same for any system. The fluctuation force is a scalar quantity, instead of an operator, which we found it convenient to consider as a complex quantity. The dissipation force is not restrict to be proportional to the velocity as in the usual approaches to the Langevin equation. In our approach it is determined as to guarantee that the stationary state is given by a density operator of the Gibbs canonical type. The system approaches equilibrium or in other words the system thermalizes. This result is general for any systems as long as one uses the dissipation force given by relation (39). To find the relation between dissipation and fluctuation that leaded to relation (39), we have derived an equation that gives the time evolution of the density operator, which turns out to be a quantum Fokker-Planck-Kramers equation, proposed previously.

The Langevin equation is written in terms of the operators $x$ and $p$ which are not, in general, hermitian operator. To overcome this problem we have postulated that the actual position and momentum are the the operators $\bar{x} = (x^\dagger x)^{1/2}$ and $\bar{p} = (p^\dagger p)^{1/2}$, respectively. In equilibrium the density operator is of the Gibbs canonical type with a Hamiltonian written in terms of $\bar{x}$ and $\bar{p}$.

The position and momentum operator obey, in equilibrium, the commutation relation in the form $x^\dagger p - p^\dagger x = i\hbar$, although this relation is not obeyed out of equilibrium. Although the fulfillment of the commutation relation, at any time, is a basic assumption in other approaches to the Langevin equation, our understanding is
that this demand is only necessary when the dynamics is unitary, which is not the case of the Langevin dynamics. This situation is analogous to the Hamiltonian dynamics in the classical case for which the Poisson brackets of the position and momentum in relation to the initial position and momentum, at any time, is always equal to one. In the Langevin dynamics, this may not be required.

Finally, we remark that the quantum Fokker-Planck-Kramers equation derived here from the Langevin equation is very similar to that obtained by the canonical quantization of the classical Fokker-Planck-Kramers equation [22], which has been applied to the calculation of the transport properties of a chain of coupled harmonic oscillators and of a bosonic chain [23, 24].

Appendix A.

Let $X$ be a matrix that depends on a parameter $s$. The derivative of $X$ with respect to $s$ is defined by

$$
\frac{\partial X}{\partial s} = \lim_{\varepsilon \to 0} \frac{X(s + \varepsilon) - X(s)}{\varepsilon}.
$$

(A.1)

If a matrix $A$ is a function of another matrix $X$, then the derivative of $A$ with respect to $X$ is the matrix defined by

$$
\frac{\partial A}{\partial X} = \lim_{\varepsilon \to 0} \frac{A(X + \varepsilon I) - A(X)}{\varepsilon},
$$

(A.2)

where $I$ is the identity matrix.

If $I$ is replaced by a matrix $B$, the resulting derivative will depend on $B$. In this case we use a distinct notation,

$$
\left( \frac{\partial A}{\partial X}, Y \right) = \lim_{\varepsilon \to 0} \frac{A(X + \varepsilon Y) - A(X)}{\varepsilon}.
$$

(A.3)

If a series expansion of $A(X + Y)$ in powers of $Y$ is carried out, the derivatives to be used are derivatives of this type. Up to second order in $Y$,

$$
A(X + B) = A(X) + \left( \frac{\partial A}{\partial X}, Y \right) + \frac{1}{2} \left( \frac{\partial}{\partial X} \left( \frac{\partial A}{\partial X}, Y \right), Y \right).
$$

(A.4)

Notice that this derivative is not in general equal to $(\partial A/\partial X)$ multiplied by $Y$. The derivative given by (A.2) and that given by (A.3) are related by

$$
\frac{\partial A}{\partial X} = \left( \frac{\partial A}{\partial X}, I \right).
$$

(A.5)

As a simple example of the derivatives defined above, let us consider $A = X^2$. Then the usual derivative given by (A.2) is $2X$ whereas that given by (A.3) is $YX + XY$.

If $A(Y)$ depends on $Y(X)$ that depends on $X$, the derivative of $A$ with respect to $X$ is given by the chain rule is

$$
\frac{\partial A}{\partial X} = \left( \frac{\partial A}{\partial Y}, \frac{\partial Y}{\partial X} \right).
$$

(A.6)
We calculate here the expression
\[ g = p - \beta \frac{\beta^2}{2!} [p, \mathcal{H}] + \beta \frac{\beta^2}{3!} [[[p, \mathcal{H}], \mathcal{H}], \mathcal{H}] + \ldots, \] (B.1)
for the harmonic oscillator, for which
\[ \mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2. \] (B.2)
To this end we define the quantities \( A_n \) by
\[ A_{n+1} = [A_n, \mathcal{H}] \quad A_0 = p, \] (B.3)
so that
\[ g = \sum_{n=0}^{\infty} \frac{(-\beta)^n}{(n+1)!} A_n. \] (B.4)

Now, using the commutation relation \([x, p] = i\hbar\), we find
\[ [p, \mathcal{H}] = -i\hbar m\omega^2 x, \] (B.5)
and
\[ [x, \mathcal{H}] = i\hbar \frac{p}{m}, \] (B.6)
which allows us to conclude draw the following conclusion: if \( n \) is even then \( A_n = a_n p \), and if \( n \) is odd then \( A_n = b_n x \). Replacing these two results in the recursive definition of \( A_n \), we find \( b_{n+1} = -i\hbar m\omega^2 a_n \) for \( n \) even, and \( a_{n+1} = (i\hbar/m)b_n \) for \( n \) odd. These results give \( a_{n+2} = (\hbar \omega)^2 a_n \) and \( b_{n+2} = (\hbar \omega)^2 b_n \) from which follows that
\[ a_n = (\hbar \omega)^n a_0, \] (B.7)
\[ b_{n+1} = (\hbar \omega)^n b_1, \] (B.8)
and expression (B.1) becomes
\[ g = \sum_{n \text{ even}} \frac{(\beta \hbar \omega)^n}{(n+1)!} a_0 p - \sum_{n \text{ odd}} \frac{\beta^n (\hbar \omega)^{n-1}}{(n+1)!} b_1 x, \] (B.9)
or
\[ g = \frac{\sinh \beta \hbar \omega}{\beta \hbar \omega} a_0 p - \frac{1}{\beta (\hbar \omega)^2} (\cosh \beta \hbar \omega - 1) b_1 x. \] (B.10)
Considering that \( a_0 = 1 \) and \( b_1 = -i\hbar m\omega^2 \), we reach the result
\[ g = \frac{\sinh \beta \hbar \omega}{\beta \hbar \omega} p + \frac{im}{\beta \hbar} (\cosh \beta \hbar \omega - 1) x, \] (B.11)
which we write as
\[ g = ap + ibx, \] (B.12)
where
\[ a = \frac{\sinh \beta \hbar \omega}{\beta \hbar \omega}, \quad b = \frac{m}{\beta \hbar} (\cosh \beta \hbar \omega - 1). \] (B.13)
Quantum Langevin equation

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