Computing rational points on rank 0 genus 3 hyperelliptic curves

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Abstract

We compute rational points on genus 3 odd degree hyperelliptic curves $C$ over $\mathbb{Q}$ which have Jacobians of Mordell-Weil rank 0. The computation applies the Chabauty-Coleman method to find the zero set of a certain system of $p$-adic integrals, which is known to be finite and include the rational points $C(\mathbb{Q})$. We implemented an algorithm in Sage to carry out the Chabauty-Coleman method on a database of 5,870 curves.

1 Introduction

Given a curve $C$ of genus $g \geq 2$ defined over $\mathbb{Q}$, Faltings’s theorem implies that the set of rational points $C(\mathbb{Q})$ is finite. Our goal is to explicitly compute this finite set of points in the case of a genus 3 hyperelliptic curve with Jacobian having Mordell-Weil rank 0 curve. For a general curve, computing the set of rational points on that curve is an unsolved problem. However, there are techniques available when the Mordell-Weil rank $r$ of the Jacobian is smaller than the genus. In this case it is often possible, for a specific curve, to provably find its rational points. We first give an overview of the history of rational point finding, in Section 2, we give an overview of the background on Coleman integration, in Section 3, we outline our algorithm, and then in Section 4, we discuss our data and examples.

The rational point finding techniques applied here for curves of small Mordell-Weil rank date back to an idea of Chabauty [6], who proved in 1941 that when the rank $r$ is less than the genus $g$, fixing a prime of good reduction $p$, the set $C(\mathbb{Q}_p) \cap \overline{J(\mathbb{Q})}$ is finite, and hence $C(\mathbb{Q})$ is, where the bar denotes the closure in the $p$-adic topology. In the 1980’s, Robert Coleman made Chabauty’s idea effective: he gave an upper bound on the number of rational points [7]. Coleman’s upper bound comes from counting the zeroes of $p$-adic (Coleman) integrals on $\overline{J(\mathbb{Q}_p)}$ which vanish on $\overline{J(\mathbb{Q})}$. This bound was refined by Stoll [12]. We exhibit some curves which reach the Stoll bound. In 2010, Balakrishnan, Bradshaw, and Kedlaya [3] gave a practical algorithm to compute these $p$-adic Coleman integrals on odd degree hyperelliptic curves, and implemented the algorithm in Sage [14].

Recently, Booker, Booker, Platt, Sijssing, and Sutherland [4] constructed a database of genus 3 curves up to discriminant $10^7$. Armed with a decent upper bound on the number of rational points on a curve, a practical algorithm for computing these Coleman integrals that vanish on the rational points, and a large number of curves to experiment on, one might ask: how effective is Coleman’s method for determining rational points?

For example, Balakrishnan, Bianchi, Cantoral-Farfán, Çiperiani, and Etropolski [2] carried out computations to find rational points on 16,977 genus 3 rank 1 odd degree hyperelliptic curves. They found that in most cases, the Chabauty-Coleman method picks up exactly the rational points, or the rational points and the Weierstrass points which are defined over $\mathbb{Q}_p$. In the remaining cases,
they were able to explain the existence of the extra points either by torsion, linearity of the integral, or by extra endomorphisms of the Jacobian.

In order to satisfy the Chabauty-Coleman hypothesis that $r < g$ we need $r = 0, 1, \text{ or } 2$, but in the case of $r = g - 1$, one will likely find many extra $p$-adic but non-rational points when applying the Chabauty-Coleman method, and in general the Chabauty-Coleman set will be larger than the set of rational points. More work would be needed to determine the set of rational points; for example, one could implement a Mordell-Weil sieve. For these reasons, we focused on the case of $g = 3$ and $r = 0$. In this case, we computed the rational points in Magma \cite{5}, up to height $10^5$ and verified these points using Chabauty-Coleman calculations. Most of the time, in 3083 curves, the only extra points in the Chabauty-Coleman set were the Weierstrass points which were defined over $\mathbb{Q}_p$. In 16 cases, we picked up points on the curve giving rise to torsion points on the Jacobian of higher order. These points are defined over degree two number fields where the prime $p$ splits.

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2 Background on Coleman integration

In this section we will define the basics of Coleman integration and necessary background for the rest of the paper. For a more thorough introduction to the rigid analytic geometry and theory behind Coleman integrals, we refer the reader to \cite{8}, whose exposition we have loosely followed herein.

Fix $K$ a field which is complete with respect to a non-archimedian absolute value and fix some embedding of $K$ into $\mathbb{C}_p$. Let $R$ be the ring of integers of $K$.

Let $C/R$ be a smooth connected affinoid over $K$ and set $C_K = C \times_R K$. Let $F$ be the residue field of $K$, and $\tilde{C}_K$ be $C \times_R F$. There is a map $\text{red} : C_K \to \tilde{C}_K$ which maps $C_K$ to the affine scheme $\tilde{C}_K$. We say $C$ is of good reduction if $\tilde{C}_K$ is smooth. We call the preimage of a point under the reduction map $\text{red}^{-1}(x)$ for $x \in \tilde{C}_K$ the residue disc of $x$. These residue discs partition $C_K$.

Let $C$ have good reduction, $\omega$ be a holomorphic one-form on $C$, and $P, Q \in C(K)$ be points. Coleman defined an integral $\int_P^Q \omega$ in the following way. Fix a Frobenius lift $\phi : C_K \to C_K$, that is, a morphism of rigid analytic varieties which reduces to (relative) Frobenius on $\tilde{C}_K$.

**Theorem 1 (Coleman).** Suppose there is a polynomial $P(T) \in \mathbb{C}_p[T]$ which does not vanish on any root of unity, and that

$$P(\phi^*)\omega$$

is exact. Then there is a function $f_\omega$ on $C(\mathbb{C}_p)$ which is analytic on each residue disc such that $df_\omega = \omega$ and $P(\phi^*)(f_\omega)$ is analytic. The function $f_\omega$ is unique up to a constant and is independent of $\phi$ and $P$.

In this way, Coleman provides a way to compute a locally analytic antiderivative of $\omega$. Coleman proves, in several corollaries, properties of the antiderivative, which we collect here in one theorem:

**Theorem 2 (Coleman).** The integral satisfies the following properties:

1. **Linearity:** $\int_P^Q (\alpha \omega + \beta \omega') = \alpha \int_P^Q \omega + \beta \int_P^Q \omega'$.
2. Additivity: \( \int_P^R \omega = \int_P^Q \omega + \int_Q^R \omega \).

3. Change of variables: if \( C' \) is a curve and \( \phi : C \to C' \) a rigid analytic map between wide opens then \( \int_P^Q \phi^* \omega = \int_{\phi(P)}^{\phi(Q)} \omega \). For example, \( \phi \) can be taken to be a lift of the \( p \)-th power Frobenius.

4. Fundamental theorem of calculus: \( \int_P^Q df = f(Q) - f(P) \).

5. Galois equivariance: if \( \sigma \) is an automorphism of \( \mathbb{C}_p \), then \( \left( \int_P^Q \omega \right)^\sigma = \int_{\sigma(P)}^{\sigma(Q)} \omega^\sigma \).

Then Coleman shows that we can use this integration theory to obtain arithmetic information about points. Let \( J_K \) be the Jacobian of \( C \) over \( K \) and fix \( \iota : C_K \to J_K \) to be an embedding given by fixing a rational basepoint \( b \in C(\mathbb{Q}) \) and mapping \( P \) to \( [P - b] \).

We have functionals
\[
\lambda_\omega(P) = \int_b^{P} \omega, \text{ for } \omega \in H^0(J_K, \Omega^1_{J_K/K}).
\]
Note that \( H^0(J_K, \Omega^1_{J_K/K}) \) is \( g \)-dimensional but the Jacobian has rank \( r < g \). So, if we fix a basis \( \omega_0, \ldots, \omega_{g-1} \) for the space of holomorphic differentials, then we can find \( g - r \) linear relations
\[
\sum_{i=0}^{g-1} \alpha_i \int_b^{z} \omega_i = 0
\]
among the functionals
\[
\lambda_i(z) = \int_b^{z} \omega_i : J_K \to K
\]
where \( \alpha_i \in K \) for \( i = 0, \ldots, g - 1 \).

The embedding \( \iota : C_K \to J_K \) induces an isomorphism of vector spaces \( \iota^* : H^0(J_K, \Omega^1) \to H^0(C_K, \Omega^1) \) which is independent of the chosen basepoint \( b \in C(\mathbb{Q}) \) (Proposition 2.2 of [11]). Thus the linear relations among the functionals of \( J_K \) descend to linear relations among the functionals of \( C_K \).

By Coleman’s Theorem 1, the functionals are locally analytic, and we can use bounds on the numbers of zeroes of \( p \)-adic power series to get control over the number of zeroes for \( \lambda_i \) on each residue disc. In particular, there are at most finitely many common zeroes to any non-zero linear combination of functionals. Hence the rational points \( C(\mathbb{Q}) \) are contained in the finite set cut out by the linear relations.

Coleman’s proof suggests an effective method for computing \( C(\mathbb{Q}) \): first, compute the linear relations that the basis differentials satisfy. Then, compute the finite zero set of the functionals on each residue disk. This second step requires an algorithm for computing the Coleman integral to some \( p \)-adic precision.

Balakrishnan, Bradshaw, and Kedlaya give a practical algorithm for computing \( \int_P^Q \omega \), on an odd-degree hyperelliptic curve \( C \), where \( P \) and \( Q \) are in \( C(\mathbb{Q}_p) \) and \( \omega \) is a holomorphic one-form. We give a brief outline here, and for more details refer the reader to [3].

To evaluate Coleman integrals, we work locally in each residue disc, where the Coleman integral is analytic and can be expressed as a power series in a parameter, called a local coordinate. If \( K(C) \) is the function field of \( C \), then our local coordinate is simply a uniformizing parameter \( t \) for \( K(C)_P \), that is, an element \( t \in K(C)_P \) such that the valuation of \( t \) is one.

Our residue discs come in two flavors: those that are the residue disc of a Weierstrass point, the Weierstrass residue discs, and those that are not, the non-Weierstrass residue discs. The infinity
which contains the point at infinity is a Weierstrass residue disc in the odd degree hyperelliptic case.

For a hyperelliptic curve we have a basis of holomorphic differentials $\omega_i$, $i = 0, \ldots, g - 1$ where $\omega_i = (x^i/2y)dx$. For any holomorphic differential $\omega$ we can write $\omega = \sum_{i=0}^{g-1} a_i \omega_i$ for some $a_i \in K$ and use linearity to break up an integral.

To compute a Coleman integral between $P$ and $Q$ in the same residue disc, called a tiny integral, we simply write $P$, $Q$, and $\omega$ in a local coordinate $t$, $P(t)$, $Q(t)$, $\omega(t)$, and integrate formally:

$$\int_Q^P \omega = \sum_{i=0}^{g-1} a_i \int_Q^P \omega_i = \sum_{i=0}^{g-1} a_i \int_{P(t)}^{Q(t)} \frac{x(t)^i}{2y(t)} \frac{dx(t)}{dt} dt.$$

To compute a Coleman integral between $P$ and $Q$ in different non-Weierstrass residue discs we fix Teichmüller points $P'$ in the residue disc of $P$ and $Q'$ in the residue disc of $Q$ such that $\phi(P') = P$ and $\phi(Q') = Q'$. By the previous algorithm we can compute tiny integrals $\int_{P'}^Q \omega$ and $\int_{Q'}^P \omega$ between our desired points and the Teichmüller points. Applying the change of variables property of the integral, $\int_{P'}^Q \phi^* \omega = \int_{\phi(P')}^{\phi(Q')} \omega$. Thus we compute $\phi^* \omega_i$ for each basis vector $\omega_i$, so to represent the action of $\phi^*$ as a matrix, we write

$$\phi^* \omega_i = \sum_j M_{ij} \omega_j + df_i.$$

This gives us a linear system, and we can solve

$$(M - I) \begin{bmatrix} \vdots \\ \int_{P'}^Q \omega_i \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \sum_{i=0}^{g-1} f_i(P') - f_i(Q') \\ \vdots \end{bmatrix}$$

for the value of $\int_{P'}^Q \omega_i$. By linearity, we can compute $\int_{P'}^Q \omega$. The value of the original integral is simply the sum $\int_{P'}^Q \omega + \int_{P'}^P \omega + \int_{Q'}^Q \omega$. Finally, for the case of $P$ or $Q$ in a Weierstrass disc, we use the fact that if $P$ is a Weierstrass point $\int_{P}^Q \omega = 1/2 \int_{i(Q)}^Q \omega$ where $i$ is the hyperelliptic involution (Lemma 16 of [3]).

3 Algorithm

In this section we describe our implementation of the Chabauty-Coleman method for our case of interest. For a more general description of this method, see for instance [10]. Our code is available at [9].

We require $C$ to be a hyperelliptic curve of genus 3 given by an odd degree model

$$C : y^2 = F(x),$$

where $F(x) \in \mathbb{Q}[x]$ is a monic polynomial of degree 7. Moreover, we assume that the Jacobian $J$ of $C$ has Mordell-Weil rank 0 over $\mathbb{Q}$. Let $p \geq 7$ be the smallest prime such that $C$ has good reduction mod $p$.  

4
Our goal is to provably compute the set of rational points of $C$. We first use Magma to compute $C(\mathbb{Q})_{\text{known}}$ to be the set of all rational points in $C$ of naive height bounded by $10^5$. Having this set reduces the number of Coleman integrations required by the algorithm.

The input of our algorithm consists on the hyperelliptic curve $C$, the prime $p$ and the set $C(\mathbb{Q})_{\text{known}}$. The output is a finite subset of $C(\mathbb{Q}_p)$ containing $C(\mathbb{Q})$, which is returned as three separate subsets:

- The set $C(\mathbb{Q})$ of rational points of $C$.
- The set of points $Q$ in $C(\mathbb{Q}_p) \setminus C(\mathbb{Q})$ such that $[Q - \infty] \in J(\mathbb{Q}_p)$ is 2-torsion.
- The set of points $Q$ in $C(\mathbb{Q}_p) \setminus C(\mathbb{Q})$ such that $[Q - \infty] \in J(\mathbb{Q}_p)$ is an $n$-torsion point for some $n > 2$.

An outline of our algorithm is given in Algorithm 3.1. Below we explain its main steps.

1: function Chabauty-Coleman($C$, $p$, $C(\mathbb{Q})_{\text{known}}$)
2: Set the $p$-adic precision $N$ to $2p + 4$.
3: Set the $t$-adic precision $M$ to $2p + 1$.
4: Initialize found-points := empty list.
5: for each $P \in C(\mathbb{F}_p)$ up to the standard involution $\iota$ do
6: Compute $f_1, f_2$ and $f_3$ in local coordinates.
7: for each point $Q \in C(\mathbb{Q}_p)$ corresponding to a common zero of the $f_i$ do
8: Add $Q$ and $\iota(Q)$ to found-points.
9: end for
10: end for
11: Classify found-points into three lists: $\mathbb{Q}$-points, non-rational 2-torsion-points, and non-rational higher-torsion-points.
12: return $\mathbb{Q}$-points, non-rational 2-torsion-points, and non-rational higher-torsion-points.
13: end function

Algorithm 3.1: Chabauty-Coleman method for a genus 3, rank 0 hyperelliptic curve

**Step 1 (Required precision.)** We need to choose the $p$-adic precision $N$ and the $t$-adic precision $M$ to guarantee that, in step 3, we will obtain all the roots of $f_i(pt)$ in $\mathbb{Z}_p$. Proposition 3.11 in [2] can be adapted to show that it is enough to set $N = 2p + 4$ and $M = 2p + 1$.

**Step 2 (Annihilator.)** A basis of the space of differentials $H^0(C_{\mathbb{Q}_p}, \Omega^1)$ is given by $\{\omega_1, \omega_2, \omega_3\}$, where $\omega_i = (x^i/2y)dx$. For each $i = 0, 1, 2$, define

$$f_i(z) = \int_\infty^z \omega_i,$$

where $\infty$ denotes the point at infinity. The functions $f_i(z)$ are zero on all rational points of $C$, but not identically zero.

**Step 3 (Searching in residue disks.)** For each point $\overline{P}$ in $\overline{C}(\mathbb{F}_p)$, we compute the set of $\mathbb{Q}_p$-rational points $P$ reducing to $\overline{P}$ such that $f_i(P) = 0$ for $i = 0, 1, 2$. To perform this computation, we consider two different cases:

(i) If there is a point $P \in C(\mathbb{Q})_{\text{known}}$ reducing to $\overline{P}$, then we expand each $\omega_i$ in terms of a uniformizer $t$ at $P$ and we formally integrate to obtain three power series $f_i(t)$, which parametrize the integrals of the $\omega_i$ between $P$ and any other point in the residue disk.
(ii) Otherwise, we start by finding a \( \mathbb{Q}_p \)-point \( P \) reducing to \( \overline{P} = (\overline{x}_0, \overline{y}_0) \) (note that \( \overline{P} \) cannot be \( \infty \) in this case). If \( \overline{y}_0 = 0 \) we can take \( P = (x_0, 0) \) where \( x_0 \) is the Hensel lift of \( \overline{x}_0 \) to a root of \( F(x) \). Otherwise, we can take \( P = (x_0, y_0) \) where \( x_0 \) is any lift of \( \overline{x}_0 \) to \( \mathbb{Z}_p \) and \( y_0 \) is obtained from \( \overline{y}_0 \) by applying Hensel’s Lemma to \( y^2 = F(x_0) \). Then we set \( f_i(t) = \tilde{f}_i(t) + \int_{\infty}^t \omega_i \), where each \( \tilde{f}_i(t) \) parametrizes the integral of \( \omega_i \) between \( P \) and any other point in the residue disk.

To provably compute the set of common zeros of \( f_1, f_2 \) and \( f_3 \) to a desired precision, we need to impose the condition that at least one of the \( f_i \) has only simple roots. We check this requirement by computing the discriminant of the truncations of the power series to \( p \)-adic precision \( M \). If all the \( f_i \) have multiple roots, we run the algorithm using a bigger prime \( p \) for which they do not.

Say \( f_{i_0} \) has no double roots. Then we use the PARI/GP function \texttt{polrootsapadic} to compute its roots, truncating it first to \( p \)-adic precision \( M = 2p + 1 \) and computing the coefficients up to \( p \)-adic precision \( N = 2p + 4 \). For each root that lies in \( p\mathbb{Z}_p \), we check whether it is also a root of the other two \( f_i \); if so, it corresponds to a point \( P \in C(\mathbb{Q}_p) \) lying over \( \overline{P} \).

**Step 4 (Identifying the rational points.)** Now, for each of the points \( Q \) found in Step 3, we attempt to reconstruct \( Q \) as a \( \mathbb{Q} \)-rational point, using Sage. If this is not possible, then \( [Q - \infty] \) must be a torsion point in \( J(\mathbb{Q}_p) \), because \( J \) has rank 0. If \( Q \) is a Weierstrass point, it will give rise to a 2-torsion point in the Jacobian; otherwise we classify it as a higher order torsion point.

**Step 5 (Identifying higher order torsion points.)** For each of the points \( Q \) which is not identified as a rational or Weierstrass point in Step 4, we use Sage’s \texttt{algebraic_dependency} function to attempt to find the minimal polynomial of the \( x \)-coordinate and reconstruct the point as a point with coordinates defined over an extension of \( \mathbb{Q} \). We then use Magma to compute the torsion order of \( [Q - \infty] \).

## 4 Analysis and Examples

We ran the Sage implementation of our algorithm on a database of 5870 hyperelliptic curves of genus 3 and rank 0, obtained from the database \([1]\) by scaling the odd-degree models there to monic polynomials. In turn, \([1]\) is a subset of the database of genus 3 curves over \( \mathbb{Q} \) up to discriminant \( 10^7 \) \([2]\).

In 23 of these curves, there was at least one \( \mathbb{Q}_p \)-adic point for which the three power series \( f_1(z), f_2(z) \) and \( f_3(z) \) had double roots when \( p \geq 7 \) was the first prime such that the curve had good reduction. To analyze these curves, we replaced \( p \) by a bigger prime for which at least one of the power series had only simple roots.

Our implementation proves that, for each of the studied curves, the entire set of rational points equals the set of rational points of naive height less than \( 10^5 \). In fact, we proved that, with respect to the odd-degree monic models used in the computation, the maximum among the global heights of rational points in those models is \( 30.7611440827071 \), reached only by points at three curves. Or, if we translate this to the odd degree models in \([1]\), those three curves have points of global height \( 4.39444915467244 \), while the heights of the rational points in the rest of the curves is bounded above by 3. \([1]\)

Next, we show in Figure \([1]\) how many of the curves in our database have a certain number of rational points. We observe that the maximum number of points is six, and that a vast majority of the curves have three or fewer rational points.

\(^1\) The global height is the absolute logarithmic height of the point, which is the maximum of the absolute logarithmic heights of its coordinates. For a rational point \( \frac{a}{b} \), this height is computed as \( \max(\log(|n|), \log(|d|)) \).
Figure 1: Number of curves in the database with $n$ rational points

We conclude this section by showing how the algorithm works on a particular curve, giving an example where the algorithm detects a torsion point defined over a number field, and exhibiting an example curve where the Stoll bound is sharp.

**Example 1.** Consider the hyperelliptic curve

$$C : y^2 = x^7 - 37024x^6 + 3134464x^5 - 101220352x^4 + 1613758464x^3 - 13656653824x^2 + 59055800320x - 103079215104,$$

with minimal discriminant 2168084 and conductor 1084042.

Using Magma, we find that the set of rational points of $C$ with height bounded by $10^5$ is

$$C(\mathbb{Q})_{\text{known}} = \{\infty, (32, 0)\}.$$

Since $C$ has good reduction modulo 7, we run our code using this prime. The points of $C(\mathbb{F}_7)$ are:

$$\{\infty, (0, \pm 4), (1, \pm 5), (2, \pm 6), (4, 0), (6, \pm 2)\}.$$

After Hensel lifting each of these points to a point of $C(\mathbb{Q}_7)$, we write $f_1(z)$, $f_2(z)$, and $f_3(z)$ in local coordinates and find their common zeroes. This yields the following results:

| disc     | common roots of $f_1(z)$, $f_2(z)$ and $f_3(z)$ |
|----------|-----------------------------------------------|
| $\infty$ | $\infty$                                      |
| $(0, \pm 4)$ | no common roots                              |
| $(1, \pm 5)$ | no common roots                              |
| $(2, \pm 6)$ | no common roots                              |
| $(4, 0)$   | $(32, 0)$                                     |
| $(6, \pm 2)$ | no common roots                              |
Therefore, we have shown that $C(\mathbb{Q}) = \{\infty, (32, 0)\}$.

**Example 2.** Let

$$C : y^2 = x^7 + \frac{5}{64}x^6 - \frac{51}{256}x^5 - \frac{243}{4096}x^4 - \frac{53}{8192}x^3 - \frac{27}{65536}x^2 - \frac{1}{65536}x - \frac{1}{4194304}$$

be the hyperelliptic curve with minimal discriminant 6856704 and conductor 6856704. Using the algorithm described in the paper, we can determine that this curve has only one rational point, the point at infinity. However, we also detect 7-adic points with the following $x$-coordinates.

$$6 + 6 \cdot 7^2 + 6 \cdot 7^4 + 6 \cdot 7^6 + 6 \cdot 7^8 + 6 \cdot 7^{10} + 6 \cdot 7^{12} + 6 \cdot 7^{14} + O(7^{18})$$

$$6 + 6 \cdot 7^2 + 6 \cdot 7^4 + 6 \cdot 7^6 + 6 \cdot 7^8 + 6 \cdot 7^{10} + 6 \cdot 7^{12} + 6 \cdot 7^{14} + 6 \cdot 7^{16} + O(7^{18})$$

Using sage’s `algebraic_dependency` function, we identified this pair of points as $(-1/8, \pm \sqrt{-3}/2^{11})$. These points have order 18 on the Jacobian over $\mathbb{Q}(\sqrt{-3})$. The Chabauty-Coleman algorithm detects these torsion points at the prime $p = 7$ because 7 splits in this field.

**Example 3.** When $p > 2g$ is a prime of good reduction and $r < g - 1$ then Stoll \[13\] has improved the bound given by Coleman to give $\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2r$. In our case, this shows that $\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p)$. In our database of 5870 curves, 94 achieve equality for Stoll’s bound.

For example, the curve

$$y^2 = 8x^7 - 16x^5 - 7x^4 + 4x^3 + 6x^2 + 4x + 1,$$

which has minimal discriminant 4089600 and conductor 170400, has 6 rational points of height less than $10^5$:

$$\{\infty, (0, 1), (0, -1), (1, 0), (-1, 0), (-1/2, 0)\}$$

as well as 6 points over $\mathbb{F}_7$:

$$\{\infty, (0, 1), (0, 6), (1, 0), (6, 0), (3, 0)\}.$$

By Stoll’s theorem, we know that there cannot be any more rational points.

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