Quantisation conditions of the quantum Hitchin system and the real geometric Langlands correspondence

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1. Introduction

1.1 Motivations

– Program of Nekrasov-Shatashvili [1], relations to gauge theory

An important motivation comes from the program initiated by Nekrasov and Shatashvili investigating relations between supersymmetric field theories and quantum integrable models. An interesting family of examples to which this program can be applied is provided by a class of four-dimensional $\mathcal{N}=2$-supersymmetric field theories associated to the choice of a pair $(C, g)$ consisting of a (possibly punctured) Riemann surface $C$ and a Lie-algebra $g$ of ADE-type [2, 3]. The integrable models relevant for this class of theories are known [3] to be the Hitchin systems [4]. Regularising the supersymmetric field theories by means of the so-called Omega-deformation leads to the quantisation of the corresponding integrable models [1, 5, 6].

– Quantum integrable systems

Many quantum integrable models can be solved by the Bethe ansatz method. Whenever the Bethe ansatz is applicable, it is often useful to formulate the Bethe ansatz equations representing the quantisation conditions in terms of a single, model-dependent function $Y(a, t)$ called Yang’s function [7]. This function depends on two types of variables, $a = (a_1, \ldots, a_d)$ and $t = (\tau_1, \ldots, \tau_d)$. The parameters $t$ are parameters of the commuting Hamiltonians, in the context of spin chains often called inhomogeneity parameters. The variables $a$ are auxiliary, allowing
us to represent the Bethe Ansatz equations in the form

$$\frac{\partial}{\partial a_k} Y(a, t) = 2\pi i n_k, \quad k = 1, \ldots, d. \quad (1.1)$$

In a non-degenerate situation equation (1.1) has a unique solution $a = a_{cr}(n)$ for given integers $n = (n_1, \ldots, n_d)$. The eigenvalues $E_r$ of a subset of the commuting conserved quantities $H_r$, $r = 1, \ldots, d'$, can be obtained from $Y(a, t)$ by taking the derivatives

$$E_r = \frac{\partial}{\partial \tau_r} Y(a, t) \bigg|_{a=a_{cr}(n)}. \quad (1.2)$$

Beyond the class of quantum integrable systems soluble by Bethe ansatz techniques, there exists a large class of models where such techniques fail. One important outcome of the Nekrasov-Shatashvili program is strong evidence for the proposal made in [11] that the quantisation conditions in large classes of integrable models which cannot be solved by the Bethe ansatz method can nevertheless be described in terms of suitable Yang’s functions.

However, for the models studied in this paper it will turn out that another type of condition formulated in terms of a single function $Y(a, t)$ is appropriate. In general it is not a priori obvious which type of condition is appropriate for a given model. The scheme of Nekrasov-Shatashvili will be efficient for the solution of quantum integrable systems only if one knows exactly how a given quantisation condition is represented in terms of the Yang’s function $Y(a, t)$. Answering this question for interesting integrable models may lead into fairly profound mathematical problems, as will be illustrated by the examples studied in this paper.

– Geometric Langlands program

The geometric Langlands correspondence is often loosely formulated as a correspondence which assigns $\mathcal{D}$-modules on $\text{Bun}_G$ to $L^G$-local systems on a Riemann surface $C$, see [8] for a review of the aspects relevant here. $L^G$ is the Langlands dual group of a simple complex Lie group $G$. Most interesting for us is the special case considered in the original work of Beilinson and Drinfeld where the $L^G$-local systems are opers, pairs $(E, \nabla')$ in which $\nabla'$ is gauge-equivalent to a certain standard form. The space of opers forms a Lagrangian subspace in the moduli space of all local systems. The corresponding $\mathcal{D}$-modules on $\text{Bun}_G$ can be described more concretely as systems of partial differential equations taking the form of eigenvalue equations $H_r f = E_r f$ for a family of differential operators $H_r$ on $\text{Bun}_G$ quantising the Hamiltonians of Hitchin’s integrable system. The oper corresponding to such a $\mathcal{D}$-module in the geometric Langlands correspondence is the geometric object encoding the eigenvalues $E_r$.

– Relations to conformal field theory

This paper is part of a larger program outlined in [9, 10] on the relations between the quantisation of the Hitchin system, supersymmetric field theories, conformal field theory, and the geometric Langlands program. Some of these relations will be briefly described at the end of this paper.
1.2 Main results

We are going to propose a natural quantisation condition for the Hitchin system, and explain how it can be reformulated in terms of a function $Y(a, t)$. The function $Y(a, t)$ relevant for this task is found to be the generating function for the variety of opers within the space of all local systems as predicted in [6, 9]. However, the condition on $Y$ expressing the quantisation condition turns out to be different from the types of conditions considered in [1]. Our derivation is essentially complete for Hitchin systems associated to the Lie algebra $\text{sl}_2$ in genus 0 and 1, which may be called the Gaudin and elliptic Calogero-Moser models associated to the group $\text{SL}(2, \mathbb{C})$. It reduces to a conjecture of E. Frenkel [11] for $\mathcal{g} > 1$, as will be discussed below.

Reformulating the quantisation conditions in terms of $Y$ can be done using the Separation of Variables (SOV) method pioneered by Sklyanin [12]. This method may be seen as a more concrete procedure to construct the geometric Langlands correspondence relating opers to $D$-modules (eigenvalue equations), as was pointed out in [11]. In our case it will be found that the SOV method relates single-valued solutions of the eigenvalue equations to opers having Fuchsian holonomy. The classification of opers or equivalently projective structures on $\mathcal{C}$ with Fuchsian holonomy has been studied in [13]. Using complex Fenchel-Nielsen coordinates we will reformulate this description in terms of the generating function for the variety of opers.

From the point of view of the geometric Langlands correspondence we obtain a correspondence between a special class of real opers, opers with Fuchsian holonomy which is in particular real, and $D$-modules admitting single-valued solutions. We expect that a generalisation to more general local system with real holonomy will exist. We propose to call such correspondences the real geometric Langlands correspondence.

2. Separation of variables for the classical Hitchin integrable system

2.1 Integrability and special geometry

A complex symplectic manifold $\mathcal{M}$ with holomorphic symplectic form $\Omega$ is called an algebraic integrable system if it can be described as a Lagrangian torus fibration $\pi : \mathcal{M} \to \mathcal{B}$ with fibres being principally polarised abelian varieties. Algebraic integrability is equivalent to the fact that the base $\mathcal{B}$ is a special Kähler manifold satisfying certain integrality conditions [14].

These connections may be reformulated conveniently in terms of a covering of $\mathcal{M}$ with local charts carrying action-angle coordinates consisting of a tuple $a = (a^1, \ldots, a^d)$ of coordinates for the base $\mathcal{B}$, and complex coordinates $z = (z_1, \ldots, z_d)$ for the torus fibres $\Theta_b = \mathbb{C}^d/(\mathbb{Z}^d + \tau_b \cdot \mathbb{Z}^d)$, $b \in \mathcal{B}$, such that

$$\Omega = \sum_{r=1}^{d} da^r \wedge dz_r.$$  

(2.3)
The transformation $z^D := \tau^{-1}_b \cdot z$ gives an equivalent representation of the torus fibres $\Theta_b$. It can be extended to a canonical transformation $(a, z) \rightarrow (a^b, z^D)$ by introducing coordinates $a^D_r$ satisfying $\frac{\partial}{\partial a^b} a^D_r = \tau_{rs}$. As $\tau_{rs} = \tau_{sr}$, there exists a potential $F(a)$ allowing us to represent $a^D_r$ in the form $a^D_r = \frac{\partial}{\partial a^b} F(a)$. It follows that

$$\Omega = \sum_{r=1}^d da^D_r \wedge dz^D_r. \quad (2.4)$$

One may equivalently represent $\Theta_b$ as real torus $\mathbb{R}^{2d}/\mathbb{Z}^{2d}$ using the coordinates $(w, w_D)$, $w = (w_1, \ldots, w_d)$, $w_D = (w_D^1, \ldots, w_D^d)$ such that $z = w + \tau \cdot w_D$. There exists a corresponding set of real action variables $(b, b_D)$ such that

$$\text{Re}(\Omega) = \sum_{r=1}^d (db^r \wedge dw_r + db_D^r \wedge dw_D^r). \quad (2.5)$$

The real action variables $(b, b_D)$ are simply the real parts of $(a, a_b)$. The coordinates above are only locally defined, in general. Different sets of coordinates are related by $\text{Sp}(2d, \mathbb{Z})$-transformations acting in the standard fashion on the vectors $(w, w_D)$.

### 2.2 Integrability of the Hitchin system

The phase space $\mathcal{M}_H(C)$ of the Hitchin system [4] for $G = GL(2)$ on a Riemann surface $C$ with genus $g \geq 1$ is the moduli space of stable pairs $(E, \varphi)$, where $E$ is a holomorphic rank 2 vector bundle, and $\varphi \in H^0(C, \text{End}(E) \otimes K_C)$ is called the Higgs field, modulo gauge transformations. There is a natural stability condition for the pairs $(E, \varphi)$ allowing certain unstable bundles $E$. The open dense subset of $\mathcal{M}_H(C)$ consisting of pairs $(E, \varphi)$ with stable bundles $E$ is isomorphic to the cotangent bundle $T^*\text{Bun}_G(C)$. The moduli space $\mathcal{M}_H(C)$ carries a natural holomorphic symplectic structure restricting to the canonical symplectic structure on the dense open subset $T^*\text{Bun}_G(C)$. Considering bundles $E$ with fixed determinant and Higgs fields $\varphi$ with vanishing trace allows one to describe the Hitchin system for $G = SL(2)$ in a similar way.

The complete integrability of the Hitchin system is demonstrated using the so-called Hitchin map, in our case mapping a pair $(E, \varphi)$ to the coefficients $(\vartheta_1, \vartheta_2)$ of the characteristic polynomial $\det(v \text{id} - \varphi(u)) = v^2 - \vartheta_1 v + \vartheta_2$. The coefficients $(\vartheta_1, \vartheta_2)$ can be identified with elements of the vector space $B = H^0(C, K) \oplus H^0(C, K^2)$. Fixing bases $\{\rho_1, \ldots, \rho_g\}$ and $\{q_1, \ldots, q_{3g-3}\}$ for $H^0(C, K)$ and $H^0(C, K^2)$, respectively, allows us to define the Hamiltonians of the Hitchin system to be the coefficients in the expansions $\text{tr}(\varphi(u)) = \sum_{i=1}^g \rho_i h_i$ and $\text{tr}(\varphi^2(u)) = \sum_{r=1}^{3g-3} q_r H_r$. They form a maximal set of Poisson-commuting globally defined functions on $\mathcal{M}_{H}(C)$. The Hitchin fibres $\Theta_b$ are the subvarieties of $\mathcal{M}_H(C)$ associated to a point $b \in B$. 
In order to see that generic fibres $\Theta_b$ can be represented as abelian varieties (complex tori), one may first define the spectral curve $\Sigma$ as

$$\Sigma = \{ (u, v) \in T^*C ; \det(v \cdot \text{id} - \varphi) = 0 \} .$$

(2.6)

To each pair $(\mathcal{E}, \varphi)$ let us then associate a line bundle $L$ on $\Sigma$, the bundle with fibres being the eigenlines of $\varphi$ for a given eigenvalue $v$, defining a map from $(\mathcal{E}, \varphi)$ to the pair $(\Sigma, L)$. Conversely, given a pair $(\Sigma, L)$, where $\Sigma \subset T^*C$ is a double cover of $C$, and $L$ a holomorphic line bundle on $\Sigma$, one can recover $(\mathcal{E}, \varphi)$ via

$$(\mathcal{E}, \varphi) := (\pi_* (L), \pi_* (v)) ,$$

(2.7)

where $\pi$ is the covering map $\Sigma \to C$, and $\pi_*$ is the direct image. In this way we may identify the Hitchin fibres $\Theta_b$ with the Jacobian of $\Sigma$ parameterising the choices of the line bundles $L$. This is how the space $\mathcal{M}_H(C)$ gets described as torus fibration with fibre over a point $b \in B$ being the Jacobian.

For the case of $G = SL(2)$ one needs to impose the condition that the bundle $\mathcal{E}$ has trivial determinant. The Jacobian is then replaced by the so-called Prym variety parameterising line bundles $L$ such that $\det(\pi_* (L)) \simeq \mathcal{O}$.

It can furthermore be shown that the dynamics of the Hitchin system generated by the Hamiltonians with respect to the natural symplectic structure gets linear on the torus fibres [4], completing the proof of the complete integrability of the Hitchin system.

### 2.3 Algebraic integrability of Jacobian fibrations

Algebraic integrability is realised in a canonical fashion in terms of Jacobian or Prym fibrations of spectral curves. Indeed, given a spectral curve $\Sigma$, let us pick a canonical basis for the first homology of $\Sigma$, represented by mutually nonintersecting sets of cycles $\alpha_1, \ldots, \alpha_h$ and $\beta_1, \ldots, \beta_h$ satisfying $\alpha_r \cdot \beta_s = \delta_{r,s}$, where $h = 4g - 3$ is the genus of $\Sigma$. A basic role is played by the periods

$$a^r = \int_{\alpha_r} \lambda, \quad a^D_r = \int_{\beta_r} \lambda .$$

(2.8)

of the canonical differential $\lambda = vdu$ on $\Sigma$. The derivatives $\omega_r = \partial_{a^r} \lambda$ give a basis for the space of abelian differentials normalised as $\delta_{r,s} = \int_{\alpha_r} \omega_s$. The torus fibres may then be represented as $\Theta_{\mathcal{E}} = \mathbb{C}^h / (\mathbb{Z}^h + \tau \cdot \mathbb{Z}^h)$, with period matrix $\tau$ having matrix elements $\tau_{r,s} = \int_{\beta_s} \omega_r$. The Riemann bilinear relations give $\tau_{r,s} = \tau_{sr}$. It follows that there exists a function $\mathcal{F}(a)$ giving the dual periods $a^D_r$ as $a^D_r = \partial_{a^r} \mathcal{F}(a)$.

When the integrable structure is represented in terms of a torus fibration over families of spectral curves which are branched coverings of an underlying curve $C$, one may alternatively represent the integrable structure in terms of a symmetric product $\left(T^*C\right)^{[h]}$ of the cotangent bundle of
This relation is essentially canonical and most easily described when the torus fibres are the Jacobians of \( \Sigma \). The Abel map from divisors \( \hat{D}_u = \sum_{r=1}^{h} \hat{u}_r \) on \( \Sigma \) to the Jacobian,

\[
z_s(a, u) = \sum_{r=1}^{h} \int_{\hat{u}_r} \omega_s.
\]

(2.9)
can be inverted (Jacobi inversion problem), defining a divisor \( D_u = \sum_{r=1}^{h} u_r \) on \( C \) by projection. The locally defined function

\[
\mathcal{X}(a, u) = \sum_{r=1}^{h} \int_{\hat{u}_r} \lambda,
\]

(2.10)
is a generating function for the change of variables from \((a, z)\) to \((v, u)\),

\[
\frac{\partial}{\partial a_r} \mathcal{X}(a, u) = z_r, \quad \frac{\partial}{\partial u_r} \mathcal{X}(a, u) = v_r.
\]

(2.11)
It follows from the existence of the generating function \( \mathcal{X}(a, u) \) that the coordinates \((v, u)\) are Darboux coordinates. Note that the points \((u_k, v_k) \in T^*C\) with \( v_k = v_k(a, u) \) defined in (2.11) automatically satisfy

\[
v_k^2 - \text{tr}(\varphi(u_k)) + \text{tr}(\varphi^2(u_k)) = 0, \quad \Leftrightarrow \quad (u_k, v_k) \in \Sigma,
\]

(2.12)
for \( k \in 1, \ldots, h \). A detailed explanation of the modifications of the Abel map that are necessary in the cases where the torus fibres are Prym varieties can be found in [15]. Only the subspace of \( H_1(\Sigma) \) which is odd under the exchange of sheets is relevant in this case, reducing the number of relevant variables from \( h \) to \( d = 3g - 3 \).

The representation in terms of the symmetric product \((T^*C)^[h]\) will be called Separation of Variables (SOV) representation. We conclude that a SOV representation exists for the classical theory whenever there is a description in terms of pairs \((\Sigma, L)\) as introduced above.

### 2.4 Separation of variables

It may be necessary to describe the passage from the original description in terms of pairs \((E, \varphi)\) to either one of the two descriptions making the integrable structure manifest more explicitly. This requires constructing sections \( \chi \) of the line bundle \( L \) as families of eigenvectors of the Higgs-field \( \varphi \). The divisor \( D_u \) will be identified with the divisor of zeros of \( \chi \) [16, 17].

To begin with, we need to represent the pairs \((E, \varphi)\) more concretely. This can be done by representing the bundles \( E \) as extensions,

\[
0 \rightarrow \mathcal{L}' \rightarrow E \rightarrow \mathcal{L}'' \rightarrow 0.
\]

(2.13)
Describing such extensions by means of a covering \( \mathcal{U}_i \) of \( C \) and transition functions \( \mathcal{E}_{ij} \) between patches \( \mathcal{U}_i \) and \( \mathcal{U}_j \), one may assume that all \( \mathcal{E}_{ij} \) are upper triangular,

\[
\mathcal{E}_{ij} = \begin{pmatrix} L_{ij}' & 0 \\ 0 & L_{ij}'' \end{pmatrix} \begin{pmatrix} 1 & \mathcal{E}_{ij}' \\ 0 & 1 \end{pmatrix}.
\]

(2.14)

This implies that the lower left matrix element \( \varphi_-(y) \) of \( \varphi \) is a section of the line bundle \( \mathcal{L} \otimes K_C \), with \( K_C \) being the canonical line bundle and \( \mathcal{L} = (\mathcal{L}')^{-1} \otimes \mathcal{L}'' \). Without loss of generality one may assume \( \mathcal{L}' = \mathcal{O} \), \( \mathcal{L}'' = \mathcal{L} \), as can always be reached by tensoring \( \mathcal{E} \) with a line bundle. Any holomorphic bundle can be represented as an extension (2.13). At least part of the moduli of the bundle \( \mathcal{E} \) can be represented in terms of extension classes in \( \mathbb{P}H^1(\mathcal{L}^{-1}) \). Since \( \dim H^1(\mathcal{L}^{-1}) = g - 1 + \deg(\mathcal{L}) \) this suffices to represent all moduli of \( \text{Bun}_{SL(2)} \) if \( \deg(\mathcal{L}) > 2g - 2 \). To simplify the discussion we shall assume \( \deg(\mathcal{L}) = 2g - 1 \) in the following.

The matrix elements \( \varphi_- \) of \( \varphi \) represent elements of the vector space \( H^0(C, \mathcal{L} \otimes K_C) \) dual to \( H^1(\mathcal{L}^{-1}) \) by Serre duality. The eigenvectors of \( \varphi = \begin{pmatrix} \varphi_0' & \varphi_0'' \\ \varphi_- & \varphi_- \end{pmatrix} \)

\[
\chi = \begin{pmatrix} v - \varphi_0'' \\ \varphi_- \end{pmatrix}.
\]

vanish at the zeros of \( v - \varphi_0'' \) which project to the \( 4g - 3 \) zeros \( u = (u_1, \ldots, u_h) \) of \( \varphi_- \) on \( C \). The degree \( 4g - 3 \) line bundle \( L = \mathcal{O}(\mathbb{D}_u) \) associated to the divisor \( \mathbb{D}_u = \sum_{r=1}^{h} \hat{u}_r \), represents the point in the Jacobian of \( \Sigma \) associated to \( (\mathcal{E}, \varphi) \). We thereby obtain the relation between pairs \( (\mathcal{E}, \varphi) \), where \( \mathcal{E} \) is represented as extension of the form (2.13), and the tuples of points \( (u, v) \) in \( (T^*C)[h] \) introduced above: \( u = (u_1, \ldots, u_h) \) is the collection of zeros of \( \varphi_- \), while \( v = (v_1, \ldots, v_h) \) is defined by setting \( u_k = \varphi_0'(u_k), k = 1, \ldots, h \).

In order to treat the case of the \( G = SL(2) \) Hitchin system one may consider the line bundle \( \mathcal{L} \simeq \det(\mathcal{E}) \) as fixed, which imposes \( g \) constraints on the positions of the \( u_1, \ldots, u_h \). We furthermore have \( \varphi_0' = -\varphi_0' \equiv \varphi_0 \). Let \( \sigma \) be the sheet involution. The degree zero line bundle \( L = \mathcal{O}(\mathbb{D}) \) associated to the divisor \( \mathbb{D} = \sum_{r=1}^{h} (\hat{u}_r - \sigma(\hat{u}_r)) \) representing the point in the Prym variety of \( \Sigma \) associated to \( (\mathcal{E}, \varphi) \) has lines generated by

\[
\chi = \frac{1}{v - \varphi_0} \begin{pmatrix} v + \varphi_0 \\ \varphi_- \end{pmatrix}.
\]

(2.16)

Variants of this type of representation can be used to parameterise the pairs \( (\mathcal{E}, \varphi) \), and to describe the change of variables defining the tuples \( (u, v) \), much more explicitly [18].

### 2.5 Punctures

It is possible to generalise the set-up by allowing \( n \) marked points on \( C \). In the presence of marked points one may also consider surfaces of genus 0 or 1. The resulting versions of the
Hitchin integrable systems turn out to be related to the integrable models known as Gaudin model \((g = 0)\), or the elliptic Calogero-Moser model \((g = 1)\). We will use the the example of the Gaudin model as guidance for the quantisation of the picture outlined above. The necessary ingredients will have clear analogs in this case, suggesting a path for the treatment of the general case. To this aim let us explain how the separation of variables is realised in this case.

The description of \(\mathcal{E}\) as an extension amounts to a description in terms of a cover of \(\mathbb{P}^1\) of the form \(\{\mathbb{P}^1 \setminus \{z_1, \ldots, z_n\}, D_1, \ldots, D_n\}\), where \(D_1, \ldots, D_n\) are small mutually non-intersecting discs around \(z_1, \ldots, z_n\), with transition functions on \(A_r = D_r \setminus \{z_r\}\) being of the form \(\mathcal{E}_r = \begin{pmatrix} 1 & x_r \\ 0 & 1 \end{pmatrix}\). Assuming that \(\varphi\) has a regular singularity of the form \(\frac{1}{y-z_r}(l_r 0 0 p_r -l_r)\) at \(z_r\) it follows that

\[
\varphi(y) = \sum_{r=1}^{n} \frac{\varphi_r}{y-z_r}, \quad \varphi_r = \mathcal{E}_r \cdot \begin{pmatrix} l_r & 0 \\ p_r & -l_r \end{pmatrix}, \quad \mathcal{E}_r^{-1} = \begin{pmatrix} x_r p_r + l_r & x_r^2 p_r + 2l_r x_r \\ p_r & -l_r - x_r p_r \end{pmatrix}.
\]

(2.17)

Regularity of \(\varphi\) at infinity imposes three constraints

\[
\sum_{r=1}^{n} x_r^{k+1} p_r + l_r (k+1) x_r^k = 0, \quad k = -1, 0, 1.
\]

(2.18)

Identifying \(x_r\) with a coordinate on \(\mathbb{P}^1\), and \(p_r\) with a coordinate on the cotangent fibre of \(\mathbb{P}^1\) allows us to describe \(\mathcal{M}_H(C_0, n)\) as symplectic reduction of \((T^*\mathbb{P}^1)^n\) by the constraints (2.18). To this aim one needs to identify points of \((T^*\mathbb{P}^1)^n\) related by the Hamiltonian flows generated by the constraints. These flows generate the group \(G = \text{SL}(2)\) acting on the variables \(x_r\) as Möbius transformations \(x_r \to \frac{ax_r + b}{cx_r + d}\). The quotient \((T^*\mathbb{P}^1)^n/G\) may be represented by fixing a slice \(x_n = \infty, x_{n-1} = 1\) and \(x_{n-2} = 0\) and using (2.18) to express \(p_n, p_{n-1}\) and \(p_{n-2}\) in terms of the remaining variables. This forces us to send \(p_n \to 0\) such that \(x_n p_n + 2l_n = 0\).

The Hamiltonians of this integrable model are defined as the free parameters specifying the quadratic differential \(\text{tr}(\varphi^2)\), which can now be represented explicitly as

\[
\text{tr}(\varphi^2(y)) = \sum_{r=1}^{n} \left( \frac{l_r^2}{(y-z_r)^2} + \frac{H_r}{y-z_r} \right).
\]

(2.19)

The change of variables \((x, p) \to (u, v, u_0)\) defined by

\[
\varphi_-(y) = \sum_{r=1}^{n-1} \frac{p_r}{y-z_r} = u_0 \prod_{k=1}^{n-3} (y-u_k), \quad v_r = \varphi_0(u_r),
\]

(2.20)

gives the isomorphism \(\mathcal{M}_H(C_0, n) \simeq (T^*C_0, n)^{n-3}\) defined by the SOV method.

3. Quantisation of Hitchin’s integrable system

We will now present an overview of known results on the quantisation of the Hitchin system. Starting with the genus zero case we will introduce a variant of the Gaudin model related to the
non-compact group $\text{SL}(2, \mathbb{C})$. Known results on the quantisation of Hitchin’s Hamiltonians in $g > 1$ and their relation to the geometric Langlands correspondence are re-interpreted from the point of view of this paper in the following subsection.

3.1 Genus zero – the $\text{SL}(2, \mathbb{C})$ Gaudin model

The quantisation of the Gaudin model is fairly simple on a purely algebraic level. It starts by turning the algebra of functions on $(T^*\mathbb{P}^1)^n$ with generators $p_r, x_r$, into a non-commutative algebra with generators $p_r, x_r, r = 1, \ldots, n$, satisfying the relations $[p_r, x_s] = \epsilon_1 \delta_{rs}, [p_r, p_s] = 0, [x_r, x_s] = 0$. The matrix elements $\varphi^a_r, a = -, 0, +$ of the residues $\varphi_r$ of $\varphi$ get replaced by the generators of the Lie algebra $\mathfrak{sl}_2$ for all $r = 1, \ldots, n$. The quantised algebra of functions $\mathcal{A}_n$ on $(T^*\mathbb{P}^1)^n$ thereby gets identified with the direct sum of $n$ copies of the Lie algebra $\mathfrak{sl}_2$.

When we are discussing the quantisation of a phase space with complex coordinates it is also natural to consider the conjugate algebra $\bar{\mathcal{A}}_n$ obtained by quantisation of the complex conjugate coordinates $\bar{p}_r, \bar{x}_r$. The generators of $\bar{\mathcal{A}}_n$ will be denoted as $\bar{p}_r, \bar{x}_r, r = 1, \ldots, n$.

Recall that we had represented $\mathcal{M}_H(C_{0,n})$ as symplectic quotient of $(T^*\mathbb{P}^1)^n$ by the three constraints (2.18). The constraints become quantised to the “diagonal” $\mathfrak{sl}_2$ embedded into the direct sum of $n$ copies of $\mathfrak{sl}_2$ in the usual way. It is natural to define the quantised algebra $\mathcal{A}$ of global functions on $\mathcal{M}_H(C_{0,n})$ to be the sub-algebra of $\mathcal{A}_n$ generated by the functions commuting with the diagonal $\mathfrak{sl}_2$. The algebra $\mathcal{A}$ contains the quantised Hamiltonians $H_r$,

$$H_r \equiv \sum_{s \neq r} \frac{J_{rs}}{z_r - z_s},$$

where the differential operator $J_{rs}$ is defined as

$$J_{rs} := \eta_{aa'}J^a_r J^a_s := J^0_{rs} + \frac{1}{2}(J^+_{rs} J^-_{rs} + J^-_{rs} J^+_{rs}).$$

The generators $H_r$ commute, $[H_r, H_s] = 0$ for all $r, s$. Similar statements hold for the conjugate algebra $\bar{\mathcal{A}}$, which commutes with $\mathcal{A}$ and contains the conjugate Hamiltonians $\bar{H}_r, r = 1, \ldots, n$.

A step towards the definition of suitable representations $\mathcal{R}_n$ of $\mathfrak{A}_n$ is to choose a polarisation, a commutative sub-algebra of $\mathfrak{A}_n$ that will be represented by multiplication operators on $\mathcal{R}_n$. In the present case there are are two natural polarisations, defined by choosing either the sub-algebra generated by $x_r, r = 1, \ldots, n$, or the one generated by $p_r, r = 1, \ldots, n$. In both cases one gets an $n$-fold tensor product $\mathcal{R}_n = \bigotimes_{r=1}^N \mathcal{P}_n$ of representations $\mathcal{P}_n$ of the Lie-algebra $\mathfrak{sl}_2$.

In the first case one finds a representation realised by the differential operators $J^\pm_r, J^0_r$,

$$J^-_r = \partial_{x_r}, \quad J^0_r = x_r \partial_{x_r} - j_r, \quad J^+_r = -x_r^2 \partial_{x_r} + 2j_r x_r.$$  

The parameters $j_r$ appearing in (3.23) are related to the parameters $l_r$ of the classical Gaudin model by $l_r = -\epsilon_1 j_r$. In the polarisation generated by $p_r, r = 1, \ldots, n$ we may choose the
operators

\[ \tilde{J}_r^- = p_r, \quad \tilde{J}_r^0 = -p_r \partial_{p_r}, \quad \tilde{J}_r^+ = -p_r \partial_{p_r}^2 + \frac{j_r(j_r+1)}{p_r}, \tag{3.24} \]

as generators for the representation on \( \mathcal{P}_n \). The Casimir operator is in both cases represented as multiplication by \( j_r(j_r+1) \).

In order to fully define the relevant representations of the Lie algebra \( \mathfrak{sl}_2 \), one needs to specify the spaces of functions the differential operators defined in (3.23) and (3.24) should act on.

In the Gaudin model one usually considers finite-dimensional representations, restricting the choice of \( j_r \) to \( j_r = 0, 1/2, 1, \ldots \). The finite-dimensional representations can be realised via (3.23) on polynomial functions of the variables \( x \). We will mostly be interested in infinite-dimensional representations realised by means of the differential operators (3.23) on suitable spaces of non-polynomial functions. One may, for example, consider representations defined by the differential operators \( \tilde{J}_r^a \) together with the conjugate operators \( \tilde{J}_r^b \) obtained by \( x_r \rightarrow \bar{x}_r \), \( \partial_{x_r} \rightarrow \partial_{\bar{x}_r} \) on certain (sub-)spaces of the space of smooth functions on \( \mathbb{C} \). The class of such representations contains the Lie algebra representations associated to principal series representations \( \mathcal{P}_n \equiv \mathcal{P}_{j_n} \) of \( \text{SL}(2, \mathbb{C}) \). The representations \( \mathcal{P}_{j_n} \) are unitary if \( j_n \in -\frac{1}{2} + i\mathbb{R} \).

The symplectic quotient of \( (T^*\mathbb{P}^1)^n \) by the three constraints (2.18) is naturally described by considering the action of \( \mathcal{A} \) and \( \bar{\mathcal{A}} \) on the subspaces \( \mathcal{R}_{n}^{\text{inv}} \subset \mathcal{R}_n \) of invariants under the diagonal \( \mathfrak{sl}_2 \)-action. Representing the tensor product of representations \( \mathcal{R}_n = \bigotimes_{r=1}^N \mathcal{P}_{j_n} \) in terms of functions \( \Psi(x, \bar{x}) \) with \( x = (x_1, \ldots, x_n) \) one may represent the elements of \( \mathcal{R}_{n}^{\text{inv}} \) as functions \( \bar{\Psi}(x, \bar{x}) \) which are invariant under the diagonal action of \( \text{SL}(2, \mathbb{C}) \). We will find it more convenient to represent the elements of \( \mathcal{R}_{n}^{\text{inv}} \) as functions \( \Psi(x, \bar{x}) \) of \( n - 1 \) variables \( x_1, \ldots, x_{n-1} \) which are invariant under translations \( x_r \rightarrow x_r + b \) and behave under dilatations \( x_r \rightarrow a^2 x_r \) as

\[ \Psi(a^2 x, a^2 \bar{x}) = a^{4J} \Psi(x, \bar{x}), \quad J = -j_n + \sum_{r=1}^{n-1} j_r. \tag{3.25} \]

The two representations (3.23) and (3.24) are intertwined by the following slightly modified form of the Fourier-transformation.

\[ \Psi(x, \bar{x}) = \int d^2p_1 \ldots d^2p_{n-1} \Phi(p, \bar{p}) \prod_{r=1}^{n-1} e^{p_r x_r - \bar{p}_r \bar{x}_r} |p_r|^{-2j_r-2}. \tag{3.26} \]

This map establishes an equivalence of the representation defined via (3.23) with a representation of the form (3.24) in which a nilpotent generator is represented as multiplication operator. We will refer to the representations defined on the functions \( \Phi(p, \bar{p}) \) via (3.24) as the Whittaker models for the representations \( \bigotimes_{r=1}^{n-1} \mathcal{P}_{j_r} \). One may note that the conjugate operators \( \tilde{J}_r^\pm, \tilde{J}_r^0 \) get mapped to the complex conjugates of \( \tilde{J}_r^\pm, \tilde{J}_r^0 \).
3.2 Quantisation of Hitchin’s Hamiltonians and the geometric Langlands correspondence

Hitchin’s Hamiltonians have been quantised in the work [19] of Beilinson and Drinfeld on the geometric Langlands correspondence. This means the following: There exist global differential operators \( H_i \) on the line bundle \( K^{1/2} \) on \( \text{Bun}_G \) such that the following holds:

- The differential operators \( H_i \) generate the commutative algebra \( \mathcal{D} \) of global differential operators acting on \( K^{1/2} \), and
- the symbols of the differential operators \( H_i \) coincide with generators of the algebra of functions on the Hitchin base \( B \) defined via Hitchin’s map.

The construction in [19] uses elements of conformal field theory and the representation-theoretic results of [20]. Our discussion follows the review [8].

Beilinson and Drinfeld put the quantisation of the Hitchin in relation to the geometric Langlands correspondence, schematically represented as

\[
\begin{array}{c}
\text{\( L_g \)-opers} \\
\leftrightarrow
\end{array}
\begin{array}{c}
\mathcal{D} - \text{modules on } \text{Bun}_G
\end{array}
\] (3.27)

as we shall now briefly explain. The relation between the geometric Langlands correspondence and the Gaudin model was described in [21].

3.2.1 Opers

Opers are a special class of holomorphic connections \((\epsilon_1 \partial_y + A(y)) dy \) on \( C \) with \( A(y) \) being gauge equivalent to the form \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). The equation defining horizontal sections \( s \), \((\epsilon_1 \partial_y + A(y)) s = 0 \), reduces to the ODE \((\epsilon_2^2 \partial_u^2 + t(u)) s_2 = 0 \) if \( s = (s_1, s_2) \). Covariance under changes of local coordinates requires that \( t = t(u) \) transforms as

\[
t(u) = (y'(u))^2 \hat{t}_f(y(u)) + \epsilon_y^2 \{y, u\}, \quad \{y, u\} = \left( \frac{y''}{y'} \right)' - \frac{1}{2} \left( \frac{y''}{y'} \right)^2,
\] (3.28)

identifying it as a projective connection. The underlying holomorphic bundle \( \mathcal{E}_{\text{op}} \) must be an extension of the form \( 0 \to K^{1/2} \to \mathcal{E}_{\text{op}} \to K^{-1/2} \to 0 \). As \( \mathcal{E}_{\text{op}} \) is uniquely defined thereby, an oper is completely specified by the choice of the projective connection \( t \).

3.2.2 Geometric Langlands correspondence

One of the main results of Beilinson-Drinfeld is the existence of a canonical isomorphism of algebras

\[
\text{Fun} \mathcal{O}_G(C) \simeq \mathcal{D}.
\] (3.29)
This result implies a special case of the geometric Langlands correspondence. Fixing an oper $\chi$ defines a homomorphism $\text{Fun} \, \text{Op}_{Lg}(\mathbb{C}) \to \mathbb{C}$. Using (3.29) one gets a homomorphism $\tilde{\chi} : \mathcal{D} \to \mathbb{C}$. To each oper $\chi$ one may assign a $\mathcal{D}$-module $\Delta_\chi$ on $\text{Bun}_G$ defined as

$$\mathcal{D}_\chi = \mathcal{D}/\ker \tilde{\chi} \cdot \mathcal{D}.$$  

(3.30)

The correspondence between $Lg$-opers $\chi$ and $\mathcal{D}$-modules $\mathcal{D}_\chi$ on $\text{Bun}_G$ constructed in this way is an important part of what is called geometric Langlands correspondence.

This may be reformulated from the point of view of quantisation of the Hitchin system as follows: To an oper $\chi$ we may associate the following system of differential equations on $\text{Bun}_G$,

$$H_i f = E_i f, \quad E_i = \tilde{\chi}(H_i).$$  

(3.31)

This system of differential equations is regular on the open dense subset of $\text{Bun}_G$ containing the very stable bundles, bundles that do not admit a nilpotent Higgs field. On this locus it defines a vector bundle with flat connection. Conjecturally, the vector bundle has regular singularities along the singular locus. Horizontal sections of the flat connection defined by the equations (3.31) will generically have nontrivial monodromy around the singular loci.

Observing that the differential equations (3.31) are the eigenvalue equations for Hitchin’s Hamiltonians, it seems natural to interpret the results above as the statement that $\text{Op}_{Lg}(\mathbb{C})$ represents the natural geometric “home” for the eigenvalues of the quantised Hitchin Hamiltonians. The space of opers $\text{Op}_{Lg}(\mathbb{C})$ on $\mathbb{C}$ represents the quantum analog $\mathcal{B}_{\epsilon_1}$ of the base $\mathcal{B}$ of the Hitchin fibration.

4. Quantum Separation of Variables

We had noted in Section 3.2 that the geometric Langlands correspondence is related to the eigenvalue problem of the quantised Hitchin Hamiltonians. It characterises the set of eigenvalues for which multi-valued analytic solutions can exist in terms of the opers associated to the Lie algebra $Lg$. In all the cases where the Separation of Variables (SOV) approach has been developed it gives a concrete realisation of a correspondence between opers and eigenfunctions of the quantised Hitchin Hamiltonians. This has been fully realised when the surface $\mathbb{C}$ has genus $g = 0$ [11] or $g = 1$ [22, 23, 24] with any number of punctures. The SOV approach therefore offers an alternative approach to the geometric Langlands correspondence which is similar to the first construction of such a correspondence due to Drinfeld [25], as has been pointed out in [11]. It is natural to expect that the SOV approach can be extended to the cases with $g > 1$, furnishing a more concrete realisation of the geometric Langlands correspondence in all cases.

In this section we will briefly describe how the SOV approach works in the case of genus zero, and then formulate a conjecture about the generalisation of the emerging picture to higher genus.
4.1 Genus zero

The goal is to solve the eigenvalue problem

\[ H_r \Psi_E(x, \bar{x}) = E_r \Psi_E(x, \bar{x}), \quad \bar{H}_r \Psi_E(x, \bar{x}) = \bar{E}_r \Psi_E(x, \bar{x}), \] (4.32)

where \( \Psi_E(x, \bar{x}) \) is a function of the \( n-1 \) variables \( x = (x_1, \ldots, x_{n-1}) \) and their complex conjugates which are invariant under translations \( x_r \to x_r + b \) and behave under dilatations \( x_r \to a^2 x_r \) as in (3.25).

The first step is to pass to the Whittaker model by means of the inverse of the Fourier-transformation (3.25), expressing solutions \( \Psi_E(x, \bar{x}) \) in terms of the eigenfunctions \( \Phi_E(p, \bar{p}) \) in the Whittaker model. Let us then, following Sklyanin [12], perform the change of variables \( p \to (u_0, u) \) defined by the family of equations

\[ \varphi_-(y) = \sum_{r=1}^{n-1} \frac{p_r}{y - z_r} = u_0 \prod_{k=1}^{n-3} \frac{y - u_k}{y - z_r} \quad \Rightarrow \quad p_r(u) = u_0 \prod_{k=1}^{n-3} \frac{z_r - u_k}{z_r - z_s}. \] (4.33)

Abusing notations we will denote \( \Phi_E(p(u_0, u), \bar{p}(u_0, u)) \) by \( \Phi_E(u, \bar{u}) \). Using identities like

\[ \partial u_k = \sum_{r=1}^{n-1} \frac{\partial p_r}{\partial u_k} \partial p_r = \sum_{r=1}^{n-1} \frac{1}{u_k - z_r} p_r \partial p_r, \] (4.34)

it becomes straightforward to show that the eigenvalue equation become equivalent to the set of ordinary differential equations

\[ (\epsilon_1^2 \partial^2 u_k + \ell(u_k)) \Phi_E(u, \bar{u}) = 0, \quad (\epsilon_1^2 \partial^2 \bar{u_k} + \ell(u_k)) \Phi_E(u, \bar{u}) = 0, \] (4.35)

which can be solved in factorised from \( \Phi_E(u, \bar{u}) = \prod_{k=1}^{n-3} \phi_k(u_k, \bar{u}_k) \). Further details can be found in [12][11].

The transformation from eigenfunctions \( \Psi_E(x, \bar{x}) \) to the functions \( \Phi_E(u, \bar{u}) \) can be inverted explicitly [26]. The inverse may be represented as an integral transformation of the form

\[ \Psi_E(x, \bar{x}) = N_J \int d^2 u_1 \ldots d^2 u_{n-3} K^{SOV}(x, u) \Phi_E(u, \bar{u}), \] (4.36)

where the kernel \( K^{SOV}(x, u) \) can be represented explicitly as

\[ K^{SOV}(x, u) = \left| \sum_{r=1}^{n-1} x_r \prod_{k=1}^{n-3} \frac{z_r - u_k}{z_r - z_s} \right|^{2J} \left| \prod_{r=1}^{n-1} \prod_{s \neq r} \frac{z_r - z_s}{z_r - u_k} \right|^{2(j_r+1)} \prod_{k<l} \left| u_k - u_l \right|^2. \] (4.37)

The integral transformation (4.36) with kernel (4.37) is manifestly well-defined for generic \( (x, \bar{x}) \) when the real parts of the parameters \( j_r \) are small enough. It may be defined for more general values of these parameters by analytic continuation. Integrable singularities of a specific type occur at certain loci in the space parameterised by the variables \( x \).
4.2 Higher genus

The SOV approach appears to be less completely understood in the higher genus cases, but there is evidence that the qualitative picture remains essentially unchanged \cite{11}. The first construction of the geometric Langlands correspondence due to Drinfeld \cite{25} starts from a symmetric product $D$-module represented by an oper. This $D$-module can be seen as the result of the canonical quantisation of the coordinates $(u, v)$ introduced in Section 2. Indeed, choosing a polarisation where the coordinates $u_k$ get represented as multiplication operators, and the coordinates $v_k$ as derivatives $\epsilon_1 \partial u_k$, one may identify the differential equations $(\epsilon_1^2 \partial_{u_k}^2 + t(u_k)) \psi(u_k) = 0$ as a quantum counterpart of the equation $v_k^2 + \text{tr}(\varphi^2(u_k)) = 0$ defining the spectral curve $\Sigma$.

The description of Drinfeld’s construction presented in \cite{11} reveals the striking similarity of this construction with a quantum version of the SOV approach. It seems natural to conjecture that the resulting $D$-modules on $\text{Bun}_G(C)$ are isomorphic to the ones furnished by the construction of Beilinson and Drinfeld. The isomorphism of the $D$-modules provided by Drinfeld’s first, and Beilinson and Drinfeld’s second construction of the geometric Langlands correspondence would imply the existence of a quantum version of the SOV for the Hitchin system \cite{11}.

The resulting picture may be described a bit more concretely as follows. Using extensions to represent the bundles $E$ allows us to introduce $3g-2$ coordinates $x = (x_1, \ldots, x_{3g-2})$ for the vector space $H^1(L^{-1})$ of extension classes. Eigenfunctions of the quantised Hitchin Hamiltonians may then be represented in terms of functions $\Psi_E(x, \bar{x})$ of the coordinates $x$ and their complex conjugates which behave homogeneously of weight $-3g+2$ under dilatations $x_r \rightarrow a^2 x_r$. A Fourier-transformation similar to (3.26) will describe the passage to the Whittaker model in which the quantum counterpart $f(y)$ of $\varphi_-(y)$ is realised as a multiplication operator. The eigenvalue $f(y)$ of $f(y)$ is a section of the line bundle $L \otimes K_C$. It is then natural to introduce the zeros $u_k$ of $f(y)$ as new variables, and to rewrite the eigenvalue equations of the quantised Hitchin Hamiltonians in these variables. The discussion above motivates the following conjecture:

The quantised Hitchin system admits a Whittaker model representing eigenfunctions $\Phi_E(u, \bar{u})$ of the Hitchin Hamiltonians as sections of $(K \otimes \overline{K})^{3g-3}$ which satisfy

$$
(\epsilon_1^2 \partial_{u_k}^2 + t(u_k)) \Phi_E(u, \bar{u}) = 0, \\
(\epsilon_1^2 \partial_{\bar{u}_k}^2 + \bar{t}(\bar{u}_k)) \Phi_E(u, \bar{u}) = 0,
$$

with $t(u)$ representing an oper on $C$.

This would represent a more concrete realisation of the geometric Langlands correspondence proven by Beilinson and Drinfeld. Note that no quantisation condition has been imposed yet.

The existence of such an isomorphism would follow from the uniqueness of the irreducible Hecke eigensheaf associated to an oper via the constructions in \cite{25} and \cite{19}, which has not been established in the literature, as far as we know. According to E. Frenkel, the isomorphism can also be proved directly from the Hecke eigenvalue property. The author thanks E. Frenkel for pointing this out to him.
Inverting the steps outlined above should enable us to construct an integral transformation similar to (4.36) allowing us to construct (generically multi-valued) eigenfunctions of Hitchin’s Hamiltonians from opers on \( C \).

5. Quantisation conditions

The main questions to be addressed in this paper concern exactly what is bypassed in the geometric Langlands correspondence by the use of the \( D \)-module theory: What are interesting spaces of functions or sections of suitable line-bundles in which one can search for the solutions of the eigenvalue equations for Hitchin’s Hamiltonians, and given a particular choice, what are the possible eigenvalues?

This issue will be referred to as the choice of quantisation conditions. After identifying a choice of quantisation condition that appears to be particularly natural for the Hitchin system, we will explain how this choice can be reformulated as a condition on the monodromy of the differential equations

\[
\varepsilon_1^2 \partial_u^2 + t(u) \psi(u) = 0
\]

representing the spectral problem in the SOV representation.

5.1 Natural choices of quantisation conditions

It may be instructive to compare the situation with the case of spectral problems for Schrödinger type operators \( H = -\partial_y^2 + V(y) \), for simplicity restricting attention to functions on the real line. For sufficiently regular potentials there will exist two linearly independent solutions of the eigenvalue equations \( H \psi_E(y) = E \psi_E(y) \) for all real or even complex values of \( E \). Physics usually motivates us to impose additional requirement on the solutions \( \psi_E(y) \) like square-integrability which often can only be satisfied for a discrete set of values of \( E \). In other cases one may be interested in functions \( \psi_E(y) \) which are periodic in \( y \) which may again restrict the possible choices of \( E \) to a discrete set. The supplementary conditions used to define the spectral problem of interest precisely will henceforth be referred to as quantisation conditions. Their mathematical content is to specify the exact class of functions which can be a solution to the spectral problem.

The Gaudin model is usually defined by considering functions of the variables \( x_r \) appearing in the definition of the Gaudin-Hamiltonians which are polynomial in \( x_r \) with degrees being given in terms of the parameters \( j_r \) as \( 2j_r \). The representations defined on such functions via (3.23) correspond to the finite-dimensional representations of \( SU(2) \). This is one possible type of quantisation conditions defining what is called the \( SU(2) \) Gaudin model.

In this paper we are interested in another type of quantisation condition. Following the discussion above we will consider the pair of eigenvalue equations

\[
H_r \Psi(x, \bar{x}) = E_r \Psi(x, \bar{x}), \quad \bar{H}_r \Psi(x, \bar{x}) = \bar{E}_r \Psi(x, \bar{x}),
\]  

(5.39)
where $\bar{H}_r$ is the conjugate of $H_r$ obtained by replacing $x_r \to \bar{x}_r$, $\partial_{x_r} \to \bar{\partial}_{\bar{x}_r}$. We are interested in solutions $\Psi(x, \bar{x})$ which are real-analytic away from possible singularities of the differential operators $H_r$ that are furthermore single-valued. We had seen above that such quantisation conditions are natural if one replaces the representations of SU(2) in the Gaudin model by principal series representations of SL(2, $\mathbb{C}$).

A similar type of quantisation condition can be considered for the quantum Hitchin system on higher genus surfaces $C$. The quantum Hitchin Hamiltonians are conjectured to have regular singularities away from the locus within $\text{Bun}_G(C)$ consisting of the very stable bundles. For generic choice of an oper $\chi$, the solutions $f$ of (3.31) will have nontrivial monodromies around the singular loci. For certain opers $\chi$ there may exist an hermitian form on the space of solutions which is invariant under the monodromy action, allowing us to construct single-valued solutions in the form of linear combinations of products of elements of a basis for the space of solutions to the eigenvalue equations multiplied by elements of the complex-conjugate basis.

Solutions to spectral problems establish generalised duality relations, in the simplest cases relating certain spaces of functions $\mathcal{S}$ to the spaces of functions on the sets of eigenvalues of commuting differential operators acting on functions in $\mathcal{S}$. Imposing additional conditions on one side will be reflected by additional restriction occurring on the other side. From this point of view we may view the geometric Langlands correspondence as the solution to a natural pre-quantisation problem. It characterises the space dual to the multi-valued solutions of the Hitchin eigenvalue equations as the space of opers. This sets the stage for the description of single-valued solutions to the Hitchin eigenvalue equation to be proposed below.

From a physicist’s perspective one might be tempted to look for a natural scalar product, and to look for normalisable solutions within the class of single-valued ones. This would be a natural next step. At the moment we have little to say about it.

The reader may notice that the idea to combine holomorphic with anti-holomorphic functions into single-valued objects is familiar from conformal field theory. What we are proposing here is related to a particular limit of the so-called $H_{3/2}^+$-WZNW model [27, 9].

When this paper was undergoing final revisions, E. Frenkel pointed out to us that ideas similar to the ones presented above have been discussed in the talk he gave at MSRI in Sept. 2014 [28].

### 5.2 Quantisation versus classification of real projective structures

The SOV transformation maps single-valued common eigenfunctions of the Hitchin Hamiltonians to single-value functions having the factorised form

$$\Phi_E(u, \bar{u}) = \prod_{r=1}^{\lambda} \phi_E(u_r, \bar{u}_r).$$

(5.40)

Our goal is therefore to analyse the single-valuedness of the expression (5.40).
As a preparation let us not note that each oper defines a projective structure on $C$, an atlas of local coordinates on $C$ with transition functions all represented as Möbius transformations. Indeed, given two linearly independent solutions $\chi_i$ of $(\epsilon_i^3 \partial_u^2 + t(u))\chi_i = 0$, $i = 1, 2$ one may show that $y(u) = \chi_1/\chi_2$ satisfies $2t(u) = \epsilon_1^2 \{y, u\}$. Using $y$ as a new local coordinate one therefore has $\tilde{t}(y) \equiv 0$. The Möbius transformations are the only allowed transition functions in an atlas formed by a collection of local charts with $\tilde{t}(y) \equiv 0$. The background on projective structures relevant for us is reviewed in \[29\].

### 5.2.1 Single-valuedness

In the form \eqref{5.40} it becomes much easier to analyse the condition that $\Phi_E(u, \bar{u})$ should be single-valued. This will be the case iff the function $\phi_E(u, \bar{u})$ which can be decomposed into linearly independent solutions of the differential equations \eqref{4.38} as

$$
\phi_E(u, \bar{u}) = \sum_{i,j=1}^{2} C_{ij} \chi_i(u) \bar{\chi}_j(\bar{u}), \tag{5.41}
$$

has the property to be single valued. By a change of basis in the space of solutions one may always bring the matrix $C_{ij}$ in \eqref{5.41} into diagonal form. By a rescaling and multiplication of $C_{ij}$ by an overall phase one may assume that the diagonal matrix elements are contained in $\{-1, 0, 1\}$. The matrix $C_{ij} = \delta_{ij}$ and diagonal matrices $C_{ij}$ having a vanishing diagonal matrix element are invariant under representations $\rho : \pi_1(C) \to \text{SL}(2, \mathbb{C})$ which never occur as holonomies of projective structures \[30\]. For the discussion of the remaining case $C = \text{diag}(1, -1)$ we may further transform $C_{ij}$ to the form $C_{ij} = \epsilon_{ij}$, where $\epsilon_{12} = 1$, $\epsilon_{ij} = -\epsilon_{ji}$, having invariance under $\text{SL}(2, \mathbb{R})$. Projective structures having holonomy in $\text{PSL}(2, \mathbb{R})$ are called real projective structures. It follows from the observations above that solutions to the single-valuedness condition correspond to real projective structures.

We may thereby conclude that $\Phi_E(u, \bar{u})$ can be a single-valued solution to the system of equations \eqref{4.38} only if the monodromy of $\epsilon_1^3 \partial_u^2 + t(u)$ is conjugate to a homomorphism of $\pi_1(C)$ into $\text{SL}(2, \mathbb{R})$. The solution $\Phi_E(u, \bar{u})$ can then be represented in the factorised form \eqref{5.40}.

One solution of the conditions above is well-known: For each Riemann surface $C$ there exists a unique metric $d^2s = e^{2\varphi}dyd\bar{y}$ of constant negative curvature. The corresponding projective connection $t(y) = -\frac{1}{4}(\partial_y \varphi)^2 + \frac{1}{2}\partial_y^2 \varphi$ has the Fuchsian group $\Gamma$ uniformising $C \simeq \mathbb{H}/\Gamma$ as its holonomy. We may use it to construct a particular solution $\Phi_{E_0}(u, \bar{u})$ to the quantisation conditions via \eqref{5.40} and \eqref{5.41}. The function $\phi_{E_0}(u, \bar{u})$ appearing in the factorised representation \eqref{5.40} for $\Phi_{E_0}(u, \bar{u})$ is related to the metric of constant negative curvature as $\phi_{E_0}(u, \bar{u}) = e^{-\varphi(u, \bar{u})}$.

There exists a construction called grafting allowing to construct from a given projective structure with Fuchsian holonomy infinitely many other projective structures with the same holonomy \[13\]. It was furthermore shown in \[13\] that all projective structures with holonomy being a
fixed Fuchsian group are obtained by grafting the projective structure furnished by the uniformisation theorem, leading to a classification of the projective structures with Fuchsian holonomy.

There exist real projective structures not having Fuchsian holonomy. The corresponding representations $\rho : \pi_1(C) \rightarrow \text{PSL}(2, \mathbb{R})$ belong to components in the character variety $\mathcal{M}_{\text{char}}(C) = \text{Hom}(\pi_1(C), \text{PSL}(2, \mathbb{R}))/\text{PSL}(2, \mathbb{R})$ which are disconnected from the component containing the Fuchsian representations $\Gamma$. In work in progress [31] we will show that opers with non-Fuchsian real holonomy always have additional singularities. Such opers do not represent solutions to the quantisation problem discussed here. We summarise the discussion as follows:

The quantum Separation of Variables establishes a one-to-one correspondence between single-valued eigenfunctions of the quantised Hamiltonians of the $\text{SL}(2)$ Hitchin system and projective structures with Fuchsian holonomy on $C$.

Projective structures with Fuchsian holonomy not coming from the uniformisation of $C$ are called exotic. The classification of real projective structures from [13] may be used to classify the solutions of the quantisation conditions of the Hitchin system. The functions $\phi_E(u, \bar{u})$ corresponding to exotic projective structures with Fuchsian holonomy define metrics of constant negative curvature via $e^{2\phi(u, \bar{u})} = (\phi_{E_0}(u, \bar{u}))^{-2}$ only away from certain singular loci [32].

5.3 Reformulation in terms of complex Fenchel-Nielsen coordinates

The Riemann-Hilbert correspondence between flat connections $\partial_y + A(y)$ and representations $\rho : \pi_1(C) \rightarrow G$ relates the moduli space $\mathcal{M}_{\text{flat}}(C)$ of flat connections on $C$ to the so-called character variety $\mathcal{M}_{\text{char}}(C) = \text{Hom}(\pi_1(C), \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})$. Useful sets of coordinates for $\mathcal{M}_{\text{flat}}(C)$ are given by the trace functions $L_\gamma := \text{tr} \rho(\gamma)$ associated to simple closed curves $\gamma$ on $C$.

Minimal sets of trace functions that can be used to parameterise $\mathcal{M}_{\text{flat}}(C)$ can be identified using pants decompositions. Cutting a surface $C$ of genus $g$ with $n$ punctures along a maximal set $\{\gamma_1, \ldots, \gamma_{3g-3+n}\}$ on non-intersecting simple closed curves produces a surface having connected components of type $C_{0,3}$ only. Cutting $C$ along all but one of the curves in $\{\gamma_1, \ldots, \gamma_{3g-3+n}\}$ produces a surface containing a single connected component of type $C_{0,4}$ or $C_{1,1}$. This component will be denoted as $C_{\text{Fr}}^r$ if $\gamma_r$ is the curve which was not cut. In order to get a coordinate system for $\mathcal{M}_{\text{char}}(C)$ one needs two independent coordinates for each $C_{\text{Fr}}^r$, $r = 1, \ldots, 3g - 3 + n$. This is what we will define next.
5.3.1 Complex Fenchel-Nielsen coordinates

Conjugacy classes of irreducible representations of $\pi_1(C_{0,4})$ are uniquely specified by seven invariants

\[ L_k = \text{Tr} M_k = 2 \cos 2\pi m_k, \quad k = 1, \ldots, 4, \]
\[ L_s = \text{Tr} M_1 M_2, \quad L_t = \text{Tr} M_1 M_3, \quad L_u = \text{Tr} M_2 M_3, \]

(5.42a)

(5.42b)

generating the algebra of invariant polynomial functions on $\mathcal{M}_{\text{char}}(C_{0,n})$. The monodromies $M_r$ are associated to the curves $\chi_r$ depicted in Figure 1. These trace functions satisfy the quartic equation

\[ L_1 L_2 L_3 L_4 + L_s L_t L_u + L_s^2 + L_t^2 + L_u^2 + L_1^2 + L_2^2 + L_3^2 + L_4^2 = \]
\[ = (L_1 L_2 + L_3 L_4) L_s + (L_1 L_3 + L_2 L_4) L_t + (L_2 L_3 + L_1 L_4) L_u + 4. \]

(5.43)

The affine algebraic variety defined by (5.43) is a concrete representation for the character variety of $C_{0,4}$. For fixed choices of $m_1, \ldots, m_4$ in (5.42a) one may use equation (5.43) to describe the character variety as a cubic surface in $\mathbb{C}^3$. This surface admits a parameterisation in terms of coordinates $(\lambda, \kappa)$ of the form

\[ L_s = 2 \cos (\lambda/2), \]
\[ L_t ((L_s)^2 - 4) = 2(L_2 L_3 + L_1 L_4) + L_s(L_1 L_3 + L_2 L_4) \]
\[ + 2 \cos (\kappa) \sqrt{c_{12}(L_s) c_{34}(L_s)}, \]
\[ L_u ((L_s)^2 - 4) = L_s(L_2 L_3 + L_1 L_4) + 2(L_1 L_3 + L_2 L_4) \]
\[ + 2 \cos \left(\frac{2\kappa - \lambda}{2}\right) \sqrt{c_{12}(L_s) c_{34}(L_s)}, \]

(5.44a)

(5.44b)

(5.44c)

where $L_i = 2 \cos \frac{\lambda_i}{2}$, and $c_{ij}(L_s)$ is defined as $c_{ij}(L_s) = L_s^2 + L_i^2 + L_j^2 + L_s L_i L_j - 4$. 

Figure 1: Basis of loops of $\pi_1(C_{0,4})$ and the decomposition $C_{0,4} = C_{0,3}^L \cup C_{0,3}^R$. 

[Diagram of loops and decomposition]
For $C \simeq C_{1,1}$ one may similarly parameterise the trace functions along the usual $a$- and $b$-cycles on the torus as

$$L_a = 2 \cos(\lambda/2),$$  \hspace{1cm} (5.45)

$$L_b((L_a)^2 - 4)^{3/2} = 2 \cos(\kappa/2) \sqrt{(L_a)^2 + L_0 - 2},$$ \hspace{1cm} (5.46)

with $L_0$ being the trace function associated to the boundary of $C_{1,1}$.

Using pants decompositions as described above one may define a pair of coordinates $(\kappa_r, \lambda_r)$ associated to each cutting curve $\gamma_r$, $r = 1, \ldots, 3g + n - 3$. Taken together, the tuples $\mathbf{k} = (\kappa_1, \ldots, \kappa_{3g+n-3})$ and $\mathbf{l} = (\lambda_1, \ldots, \lambda_{3g+n-3})$ form a system of coordinates for $\mathcal{M}_{\text{flat}}(C)$. The coordinates defined above are Darboux coordinates,

$$\Omega = \frac{1}{4\pi} \sum_{r=1}^{3g-3+n} d\kappa_r \wedge d\lambda_r,$$ \hspace{1cm} (5.47)

where $\Omega$ is the symplectic form on the moduli spaces of flat connections introduced by Goldman and Atiyah-Bott, see [33] and references therein.

### 5.3.2 Quantisation conditions in terms of complex Fenchel-Nielsen coordinates

We had previously reformulated the quantisation conditions as the condition that the monodromy of the differential operator $\epsilon_1^2 \partial_u^2 + t(u)$ defines a representation of $\pi_1(C)$ in $\text{PSL}(2, \mathbb{R})$ with Fuchsian holonomy. We are now going to reformulate this condition in terms of the coordinates $(l, k)$.

To this aim let us recall some basic facts reviewed in [29]. Fixing an oper represented by a projective connection $t_x$ one may represent generic opers via $t = t_x + \varphi$, where $\varphi$ is a quadratic differential. Noting that $d := \dim H^0(C, K^2) = 3g - 3 + n$ we see that the opers represent a half-dimensional subspace $\mathcal{O}_D(C)$ of the moduli space of flat $\text{SL}(2, \mathbb{C})$-connections. The monodromy map from $\mathcal{O}_D(C)$ to the character variety is locally biholomorphic. Fixing coordinates $E = (E_1, \ldots, E_d)$ for the vector space $H^0(C, K^2)$ one may therefore use the monodromy map to define $l = l(E)$ and $k = k(E)$ by analytic continuation in $E$.

Let us next observe that the coordinates $(l, k)$ as introduced above are closely related to the classical Fenchel-Nielsen coordinates for the Teichmüller component in the character variety $\mathcal{M}_{\text{char}}(C)$ parameterising the Fuchsian groups appearing in the uniformization of Riemann surfaces. The Fenchel-Nielsen coordinates $(l_r, k_r)$ are given in terms of the coordinates $(\lambda_r, \kappa_r)$ simply as $l_r = i\lambda_r$, and $k_r = i\kappa_r$. The coordinates $(l_r, k_r)$ are real on Fuchsian groups. Note that the coordinates $(\lambda_r, \kappa_r)$ and $(\lambda_r + 4\pi n_r, \kappa_r + 2\pi \nu_r, m_r)$ give the same values of the trace functions iff $n_r, m_r \in \mathbb{Z}$, and $\nu_r = 1$ if $C^{\dagger_r}$ is of type $C_{0,4}$, while $\nu_r = 2$ if $C^{\dagger_r}$ is of type $C_{1,1}$.

These observations lead us to reformulate the quantisation conditions for the Hitchin system as follows:
To each eigenfunction of the quantised Hamiltonians of the SL(2) Hitchin system on $C$ which is single-valued there exists a projective structure on $C$ having holonomy with complex FN coordinates $(k, l) = (l(E), k(E))$ satisfying

$$\text{Re}(\lambda_r) = 4\pi n_r, \quad \text{Re}(\kappa_r) = \nu_r 2\pi m_r, \quad n_r, m_r \in \mathbb{Z}, \quad r = 1, \ldots, d,$$

(5.48)

where $\nu_r$ was defined above, and $d := 3g - 3 + n$. Conversely, Fuchsian projective structures on $C$ having complex FN coordinates satisfying (5.48) define single-valued eigenfunctions of the quantised Hamiltonians of the SL(2) Hitchin system on $C$ via (5.40) and (5.41).

We observe an interesting point: The definition of the coordinates $(l, k)$ depends on the choice of a pants decomposition. Changing the pants decomposition must relate the integer points defined in (5.48) to each other.

It should be interesting to relate the conditions in (5.48) to Goldman’s classification of projective structures with Fuchsian holonomy in [13]. The data classifying such projective structures are collections $\mu(M)$ of disjoint simple closed curves $\mu_i$ on $C$ with positive integer weights $M_i$ attached to them. We may represent these projective structures in terms of the sum of a reference oper with a quadratic differential $\vartheta = \vartheta(E_{\mu(M)})$, defining a discrete set of points in $B_1 \simeq \text{Op}_{\text{sl}_2}(C)$. In this regard it is suggestive to note that the grafting operation relating different projective structures with Fuchsian holonomy can be represented in terms of the complex Hamiltonian twist flows generated by the complex length functions [33]. What is not clear to us at the moment is how exactly this operation gets represented in terms of the coordinates $l(E)$.

6. Generating functions of varieties of opers

It follows from the observations above that the quantisation conditions are naturally described using the space $\mathcal{P}(S)$ of projective structure on a surface $S$. A projective structure defines in particular a complex structure. The space of projective structures $\mathcal{P}(S)$ on a surface $S$ is therefore fibered over the moduli space $\mathcal{M}(S)$ of complex structures on $S$.

The definition of the generating function $\mathcal{W}$ given below will use the results of [34, 35, 36] comparing two natural holomorphic symplectic structures on $\mathcal{P}(S)$. The first is defined using the symplectic structure (5.47) on the character variety via the holonomy map. The second comes from the non-canonical isomorphisms of $\mathcal{P}(S)$ to the cotangent bundle $T^*\mathcal{M}(S)$ briefly reviewed below, referring to [34, 35, 36] for a more detailed review of the relevant background and further references.
6.1 Generating function $W$

Let us first recall that the space of quadratic differentials on a Riemann surface $C$ is canonically isomorphic to the cotangent fiber $T^*\mathcal{M}(S)$ at the point in the moduli space $\mathcal{M}(S)$ represented by the Riemann surface $C$. This means that $\mathcal{P}(S)$ is non-canonically isomorphic to $T^*\mathcal{M}(S)$. A set of local coordinates $q = (q_1, \ldots, q_d)$ for $\mathcal{M}(S)$ canonically defines a set $(q, E)$ of coordinates for $T^*\mathcal{M}(S)$, with $E = (E_1, \ldots, E_d)$ being the coefficients in the expansion of $\vartheta \in H^0(C, K^2)$ with respect to the dual of the basis for the tangent space $T\mathcal{M}(S)$ generated by the vector fields $\partial_{q_r}$, $r = 1, \ldots, d$. The isomorphisms $\mathcal{P}(S) \simeq T^*\mathcal{M}(S)$ defined by the choice of a reference projective connection allow us to use $(q, E)$ as coordinates for $\mathcal{P}(S)$.

Associating to the projective connection $t(y)$ the holonomy of the connection $(\partial_y + (0, t))dy$ defines a map from $\mathcal{P}(S)$ to the character variety. It follows from the theorems proven in [34, 35, 36] that this map relates the natural symplectic structures,

$$\Omega = \frac{1}{i} \sum_{r=1}^{h} dE_r \wedge dq_r = \frac{1}{4\pi} \sum_{r=1}^{h} d\kappa_r \wedge d\lambda_r. \quad (6.49)$$

The change of Darboux-coordinates from $(q, E)$ to $(l, k)$ can be described by a generating function $W(l, q)$ satisfying

$$\kappa_r(l, q) = -4\pi i \frac{\partial}{\partial \lambda_r} W(l, q), \quad \frac{\partial}{\partial q_r} W(l, q) = E_r. \quad (6.50)$$

It follows in particular that the subspaces of opers in the moduli spaces of flat connections are Lagrangian, and that the functions $W$ are the generating function of this Lagrangian subspace. Note that the functions $W(l, q)$ satisfying (6.50) depend on the choices of coordinates $(q, E)$ and $(l, k)$. Equation (6.50) furthermore determines $W(l, q)$ only up to a constant. We will discuss the resulting issues in Section 7 below.

6.2 Quantisation conditions in terms of the generating function $W$

It remains to observe that the quantisation conditions can be reformulated in terms of the function $W(l, q)$ in a way which resembles the use of the function introduced by Yang and Yang [7] for the description of the quantisation conditions in quantum integrable models soluble by the Bethe ansatz method. We immediately find from (5.48) and (6.50) that

For each single-valued eigenfunction of the quantised Hamiltonians of the SL(2) Hitchin system on $C$ there exist tuples $n = (n_1, \ldots, n_d)$, $m = (m_1, \ldots, m_d)$ of integers and a solution $l = l(n, m)$ to the equations

$$\text{Re}(\lambda_r(n, m)) = 4\pi n_r, \quad \text{Re}\left(\frac{\partial}{\partial \lambda_r} \mathcal{V}(l, q)\right)_{l=l(n, m)} = \nu_r 2\pi m_r. \quad (6.51)$$
where \( r = 1, \ldots, d \), and the function \( \mathcal{Y}(l, q) \) is given as
\[
\mathcal{Y}(l, q) = 4\pi i \mathcal{W}(l, q).
\]

For a given tuple \((n, m)\) of integers characterising a single-valued eigenfunction one gets the corresponding eigenvalues \( E = (E_1, \ldots, E_d) \) as
\[
E_r(n, m) = \frac{\partial}{\partial q_r} \mathcal{W}(l(n, m), q). \quad (6.52)
\]

This had previously been observed in [9] for the \( SL(2, \mathbb{C}) \)-Gaudin model. In a parallel development it has been proposed in [6] that quantisation conditions for the Hitchin system are naturally formulated in terms of the function \( \mathcal{W} \). The proposal above completes the proposal from [6] by establishing the precise relation between a particular quantisation condition for the quantised Hitchin system to a specific condition formulated in terms of the function \( \mathcal{W} \).

7. Global definition of the Yang’s function

The goal of this section is to clarify which global geometric object is locally represented by the function \( \mathcal{W} \). One may note that the local definition for the functions \( \mathcal{W} \) given above is sufficient for the goal to formulate the quantisation conditions for the Hitchin integrable system. Readers only interested in this aspect can safely skip this section. However, from a mathematical point of view it seems desirable to clarify if there is a globally defined geometric object on \( P(S) \) locally represented by the functions \( \mathcal{W}(l, q) \). We are now going to propose that one can define line-bundles on \( P(S) \) having \( \mathcal{W}(l, q) \) as their local sections. The proposal can be motivated using the observation that \( \mathcal{W}(l, q) \) represents the leading asymptotics of the Virasoro conformal blocks for large central charge [37]. It then follows from known facts on the conformal blocks.

7.1 Global issues in the definition of the functions \( \mathcal{W} \)

In order to define the coordinates \((q, E)\) one needs coordinates \( q_i \) defined in open sets \( U_i \subset \mathcal{M}(S) \) such that \( \{U_i; i \in \mathcal{J}\} \) forms a cover of \( \mathcal{M}(S) \), and a family of reference opers \( t_i = t_i(q_i) \) holomorphic on \( U_i \). Coordinates \((q_i, E_i)\) and \((q_j, E_j)\) defined on sets \( U_i \) and \( U_j \) with nontrivial intersection \( U_{ij} = U_i \cap U_j \) will transform as
\[
\sum_{r=1}^{d} E_{r}^i \, dq_r^i = \sum_{r=1}^{d} E_{r}^j \, dq_r^j + df_{ij}(q_i), \quad (7.53)
\]
with \( f_{ij} \) being locally defined functions on \( U_{ij} \). The functions \( f_{ij} \), being defined in this way only up to a constant, must satisfy the condition
\[
f_{i1j2} + f_{i2j3} = f_{i1j3} + \phi_{i12j3}, \quad (7.54)
\]
with $\phi_{123}$ constant on triple overlaps $U_{12} \cap U_2 \cap U_3$. A collection of functions $f_{ij}$ defined on the overlaps of a cover defines what was called a projective line bundle in \[38\].

The dependence of the coordinates $(l, k)$ on the choice of a pants decomposition $\sigma$ will be made explicit by using the notation $(l_\sigma, k_\sigma)$. The generating function for the change of coordinates from $(l_\sigma, k_\sigma)$ to $(l, E)$ will be denoted as $W_{\sigma}(l_\sigma, q_\sigma)$.

Changes of the defining coordinate systems are described as follows. A change from coordinates $(l_\sigma^1, k_\sigma^1)$ to $(l_\sigma^2, k_\sigma^2)$ is described by a generating function $F_{\sigma^1\sigma^2}(l_\sigma^1, l_\sigma^2)$ such that

$$
\sum_{r=1}^d \left( \kappa_{r}^{\sigma^1} d\lambda_{r}^{\sigma^1} - \kappa_{r}^{\sigma^2} d\lambda_{r}^{\sigma^2} \right) = dF_{\sigma^1\sigma^2}(l_\sigma^1, l_\sigma^2).
$$

(7.55)

The generating functions must satisfy a condition of the form

$$
F_{\sigma^1\sigma^2}(l_\sigma^1, l_\sigma^2(l_\sigma^1, l_\sigma^3)) + F_{\sigma^2\sigma^3}(l_\sigma^2(l_\sigma^1, l_\sigma^3), l_\sigma^3) = F_{\sigma^1\sigma^3}(l_\sigma^1, l_\sigma^3) + \Phi_{\sigma^1\sigma^2\sigma^3},
$$

(7.56)

with $\Phi_{\sigma^1\sigma^2\sigma^3}$ being constant.

The generating functions transform under changes of coordinates for $P(S)$ as

$$
W_{\sigma^1}(l_\sigma, q_\sigma) = W_{\sigma^2}(l_\sigma, q_\sigma(q_\sigma)) + f_{\sigma^1\sigma^2}(q_\sigma),
$$

(7.57a)

and

$$
W_{\sigma^1}(l_\sigma, q_\sigma) = W_{\sigma^2}(l_\sigma(q_\sigma, q_\sigma), q_\sigma) + F_{\sigma^1\sigma^2}(l_\sigma, l_\sigma(q_\sigma, q_\sigma)).
$$

(7.57b)

It is known that the $\phi_{123}$ define a non-trivial cohomology class \[38\]. It is therefore not yet clear if there can be any globally defined object locally represented by the functions $W$ on $P(S)$.

### 7.2 Use of the gluing construction

However, there is a way out. There is a well-known construction defining Riemann surfaces of arbitrary topology by gluing three-punctured spheres. The gluing construction identifies parameterised annular neighbourhoods of two punctures on a possibly disconnected Riemann surface $\hat{C}$ to produce a new surface $C'$. For each cutting curve $\gamma_r$, $r = 1, \ldots, 3g-3+n$, defining a pants decomposition one may introduce a parameter $q_r$ specifying the identification in the gluing construction in such a way that $q_r \to 0$ corresponds to the nodal degeneration where the length of $\gamma_r$ vanishes. In this way one gets families of coordinates $q_\sigma$ for a neighbourhood $U_\sigma$ of the boundary component in the moduli space $\overline{M}(S)$ associated to a pants decomposition $\sigma$.

By varying the pants decompositions $\sigma$ one gets a cover of $\overline{M}(S)$.

It is possible to choose the identification maps in such a way that all transition functions in the Riemann surface produced by the gluing construction are Möbius transformations. This means that the Riemann surfaces defined in this way come equipped with a natural projective structure. We may use this projective structure to define the coordinates $E_r$. In this way we get a family of coordinate systems $(q_\sigma, E_\sigma)$ covering $P(S)$. 
We may then consider the functions \( W_\sigma(I_\sigma, q_\sigma) \) defined as the generating function for the change of coordinates from \((I_\sigma, k_\sigma)\) to \((q_\sigma, E_\sigma)\). The functions \( W_\sigma(I_\sigma, q_\sigma) \) transform under the changes of the coordinates \((I_\sigma, k_\sigma)\) and \((q_\sigma, E_\sigma)\) as

\[
W_{\sigma_1}(I_{\sigma_1}, q_{\sigma_1}) = f_{\sigma_1\sigma_2}(q_{\sigma_1}) + F_{\sigma_1\sigma_2}(I_{\sigma_1}, I_{\sigma_2}, q_{\sigma_1}) + W_{\sigma_1}(I_{\sigma_1}, q_{\sigma_1}), (7.58)
\]

We are now going to argue that there exists a line-bundle over \( \mathcal{P}(S) \) having the functions \( W_\sigma(I_\sigma, q_\sigma) \) as its local sections. Indeed, our claim must hold if we are able to give an unambiguous definition of the function \( W_\sigma(I_\sigma, q_\sigma) \) satisfying \( (6.50) \) for the coordinates \((q_\sigma, E_\sigma)\) and \((I_\sigma, k_\sigma)\) associated to any pants decomposition \( \sigma \). In overlaps of the respective domains of definition there exist relations between \((q_{\sigma_1}, E_{\sigma_1})\) and \((q_{\sigma_2}, E_{\sigma_2})\), as well as \((I_{\sigma_1}, k_{\sigma_1})\) and \((I_{\sigma_2}, k_{\sigma_2})\). It follows that there must exist relations of the form \( (7.58) \). The consistency of these relations on triple overlaps then implies a cancellation between the constants \( \Phi_{i_1i_2i_3} \) and \( \Phi_{i_1i_2i_3} \) appearing in the consistency conditions \( (7.54) \) and \( (7.56) \) for the generating functions \( f_{\sigma_1\sigma_2} \) and \( F_{\sigma_1\sigma_2} \), respectively.

One way to define the functions \( W_\sigma(I_\sigma, q_\sigma) \) unambiguously is to specify the asymptotic behaviour they have at the maximal nodal degeneration of \( C \) in the Deligne-Mumford compactification of \( \mathcal{M}(S) \). We claim that the following choice does the job:

\[
W_\sigma(I_\sigma, q_\sigma) \sim \sum_{r \in I_{0,4}} (\delta(l_r) - \delta(l_{r,1}) - \delta(l_{r,2})) \log q_r + \sum_{r \in I_{1,1}} \delta(l_r) \log q_r (7.59)
\]

\[
+ \sum_{v \in \mathbb{P}_\sigma} N(l_{v,1}, l_{v,2}, l_{v,3}) + \mathcal{O}(q_r),
\]

where \( I_{0,4} \) and \( I_{1,1} \) are the subsets of \( \{1, \ldots, 3g - 3 + n\} \) for which \( C^r \simeq C_{0,4} \) and \( C^r \simeq C_{1,1} \), respectively, \( \mathbb{P}_\sigma \) is the set of pairs of pants appearing in the pants decomposition \( \sigma \) of \( C \), \( l_{v,i} \), \( i = 1, 2, 3 \), are the complex length coordinates associated to the boundary curves of the pair of pants labelled by \( v \in \mathbb{P}_\sigma \), and \( N(I_{l_3, l_2, l_1}) \) is defined as

\[
N(I_{l_3, l_2, l_1}) = \frac{1}{2} \sum_{s_1, s_2 = \pm} \text{Y}_{\text{cl}} (1 + \frac{s_1}{4\pi} (s_1 l_1 + s_2 l_2 + l_3)) - \frac{1}{2} \sum_{i=1}^{3} \text{Re} (\text{Y}_{\text{cl}} (1 + \frac{i}{2\pi} l_1)) (7.60)
\]

assuming \( l_i \in \mathbb{R} \) to simplify the expression and using the notation \( \text{Y}_{\text{cl}}(x) = \int_{1/2}^{x} du \log \frac{\Gamma(u)}{\Gamma(1-u)} \). The proof of this claim is outlined in Section 10.3 and Appendix E of [37]. It is interesting to note that the function \( N(I_{l_3, l_2, l_1}) \) appears in the semiclassical limit of the Liouville three point functions [39].
8. Concluding remarks

8.1 Semiclassical limit

The semiclassical limit of the quantisation conditions formulated above is closely related to the geometric picture presented in Section 2.

In order to see this, we need to observe that the leading behaviour of the solutions to the differential equations

\[ \epsilon_1^2 \partial_u^2 + t(u) \chi(u) = 0 \]

for \( \epsilon_1 \to 0 \) may be represented as

\[ \chi(u) \sim e^{i \epsilon_1 \int u \, du' \, v(u')} , \]

where \( v(u) \) satisfies \( (v(u))^2 = t(u) \). The asymptotic behaviour of the monodromies of the differential equations

\[ \epsilon_1^2 \partial_u^2 + t(u) \chi(u) = 0 \]

can be expressed in terms of the periods of the canonical one-form \( \lambda = v \, du \) on the double cover \( \Sigma = \{ (u, v); v^2 = t(u) \} \subset T^*C \),

\[ L_{r,s} \sim \exp \left( \frac{i}{\epsilon_1} a^r \right), \quad L_{r,u} \sim \exp \left( \frac{i}{\epsilon_1} a^D_r \right) , \]

where \( a^r \) and \( a^D_r \) are the periods defined in (2.8) by integrating along a suitable canonical basis for the first odd homology of \( \Sigma \). It follows immediately from (6.50) that

\[ \mathcal{W} \left( \frac{i}{\epsilon_1} a, q \right) \sim \frac{1}{\epsilon_1} \mathcal{F}(a, q) + \text{regular}, \]

where \( \mathcal{F}(a, q) \) is the potential appearing in the discussion of the algebraic integrability of the Hitchin system in Section 2.1. The coordinates \( (a, a^D) \) form yet another set of Darboux coordinates for \( T^*M(S) \) called homological coordinates in \([36]\), and \( \mathcal{F}(a, q) \) is the generating function for the change of coordinates from \( (a, a^D) \) to \( (q, E) \).

The quantisation conditions (6.51) therefore have the following leading asymptotics

\[ \text{Re}(a^r) = \epsilon_1 \, 2\pi \, n_r, \quad \text{Re}(a^D_r) = \epsilon_1 \, 2\pi \, m_r. \]

These are the natural Bohr-Sommerfeld quantisation conditions for the real action-angle variables introduced in Section 2.1 indicating that the quantisation conditions studied in this paper are indeed very natural.

8.2 Real versus complex integrable systems

Somewhat different types of conditions expressed in terms of generalised Yang’s functions have previously been found in other cases admitting such a formulation \([1]\). Rather than the pair of real equations (6.51) it was shown in \([1]\) that the quantisation conditions for the Toda chain and for the elliptic Calogero-Moser models can be represented as a single complex equation of the form

\[ \partial_{a^r} \mathcal{Y}(a, q) = 2\pi n_r, \quad r = 1, \ldots, d. \]
It seems that the quantisation conditions of the type \((8.65)\) are natural in algebraically integrable systems which are complexifications of real integrable systems like the Toda chain. In this paper we have been considering integrable systems which are genuinely complex. The two types of quantisation conditions, \((8.65)\) and \((6.51)\) are naturally associated to these two cases, respectively, as is also supported by the semiclassical considerations in Section 8.1 above. Comparing the results of a semiclassical analysis of the quantisation conditions for XXX-type spin-chains with \(\text{SL}(2, \mathbb{C})\)-symmetry carried out in [40] with \((8.64)\) indicates that quantisation conditions of such spin chains are of the same type as found for the Hitchin system in this paper.

### 8.3 Relation to conformal field theory

WZW-type conformal field theories can be defined mathematically using the representation theory of the affine Lie algebra \(\hat{g}_k\) at level \(k\) extending a semisimple finite-dimensional Lie algebra \(g\). The conformal blocks of WZW-type conformal field theories are defined as elements \(f\) in the dual of the vacuum representation \(V_0\) of \(\hat{g}_k\) invariant under the natural action of the Lie-algebra of meromorphic functions allowed to have poles only at a single point \(P \in \mathbb{C}\). The defining invariance condition may be twisted by families of holomorphic \(G\)-bundles \(E_x\), introducing a dependence on a collection of parameters \(x = (x_1, \ldots, x_d)\) representing coordinates on \(\text{Bun}_G\).

The conformal blocks can be characterised in terms of the solutions \(Z(x, q)\) to the KZB-equations, taking the form

\[
(k + h^\vee) \frac{\partial}{\partial q_r} Z(x, q) = H_r Z(x, q),
\]

where \(q\) are complex coordinates for the Teichmüller space \(\mathcal{T}(S)\), \(H_r\) are the quantised Hitchin-Hamiltonians and \(h^\vee\) is the dual Coxeter number of \(g\). In the critical level limit \(\epsilon_2 \to 0\), \(\epsilon_2 : = -(k + h^\vee)\epsilon_1\) one may solve \((8.66)\) with the ansatz [41] (see [42] for related results)

\[
Z(x, q) \sim e^{-\frac{\epsilon_1}{\epsilon_2} S(x, q; \epsilon_1)} \Psi(x, q; \epsilon_1)(1 + O(q)),
\]

where \(S(x, q; \epsilon_1)\) and \(\Psi(x, q; \epsilon_1)\) satisfy

\[
H_r \Psi(x, q; \epsilon_1) = E_r \Psi(x, q; \epsilon_1), \quad \frac{\partial}{\partial q_r} S(x, q; \epsilon_1) = E_r.
\]

This result can be made more precise by using the gluing construction to construct bases of conformal blocks associated to pants decompositions. In the case \(g = sl_2\) one gets solutions \(Z(p; x, q)\) to the the KZB-equations depending on additional parameters \(l = (\lambda_1, \ldots, \lambda_d)\) parameterising the intermediate representations used in the gluing construction, one complex

\footnote{The parameters \(\lambda_r\) parameterise the weights \(j_r\) of the intermediate representations as \(j_r = -\frac{1}{2} + i\lambda_r\).}
number \( \lambda_r \) for each cutting curve \( \gamma_r \). The analysis above can then be refined by using the Verlinde loop operators in a similar way as it was done in [26, 43], giving

\[
\kappa_r(l, q) = -4\pi i \frac{\partial}{\partial \lambda_r} S(l, q).
\]  

(8.69)

It follows that the function \( S(l, q) \) representing the leading term in the asymptotics (8.67) coincides with the function \( \mathcal{W}(l, q) \) studied in this paper.

### 8.4 Real geometric Langlands

The results of this note can be re-interpreted as a variant of the geometric Langlands program. One aspect of the ordinary Langlands program is the classification of the (cuspidal) spectrum of the Laplacian on certain locally symmetric spaces. From the point of view of integrable systems one may view the Laplacian as the “local” observable one is interested in. Possible degeneracies can be resolved by using additional “non-local” observables called Hecke operators.

From this point of view one may interpret the geometric Langlands program as a conjectural answer to a “pre-spectral” problem. It describes the natural geometric home for the eigenvalues of the Hitchin Hamiltonians - the variety of opers within the moduli space of local systems.

The natural next step is to define natural quantisation conditions defining what might be called cuspidal eigenfunctions of the Hitchin Hamiltonians. In this paper we propose such a quantisation condition. It selects a discrete subset within the variety of opers - a particular subset of the intersections between the variety of opers and the real slice. We propose to view correspondences between real opers and single-valued eigenfunctions of the Hitchin Hamiltonians as a natural variant of the geometric Langlands correspondence.

\[
\text{Real } \mathbf{L}^g\text{-opers with Fuchsian holonomy} \longleftrightarrow \text{Single-valued eigenfunctions of the Hitchin-Hamiltonians}
\]  

(8.70)

As opposed to the versions of the geometric Langlands correspondence intensively studied in the literature, this version is not of algebro-geometric nature: It is based on the relation between the two natural algebraic structures on the moduli of flat connections furnished by the non-algebraic Riemann-Hilbert correspondence. However, the version of the geometric Langlands correspondence proposed above has the virtue to be somewhat closer analogous to the original Langlands program, in the sense that the single-valued eigenfunctions of the Hitchin-Hamiltonians can be viewed as analogs of the automorphic forms.

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