On the automorphism groups of connected bipartite $vd$-graphs

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Abstract

Let $G = (V, E)$ be a graph with the vertex-set $V$ and the edge set $E$. Let $N(v)$ denotes the set of neighbors of the vertex $v$ of $G$. The graph $G$ is called a $vd$-graph, if for every pare of distinct vertices $v, w \in V$ we have $N(v) \neq N(w)$. In this paper, we present a method for finding automorphism groups of connected bipartite $vd$-graphs. Then, by our method, we determine automorphism groups of some classes of connected bipartite $vd$-graphs, including a class of graphs which are derived from Grassmann graphs. In particular, we show that if $G$ is a connected non-bipartite $vd$-graph such that for a fixed positive integer $a_0$ we have $c(v, w) = |N(v) \cap N(w)| = a_0$, when $v, w$ are adjacent, whereas $c(v, w) \neq a_0$, when $v, w$ are not adjacent, then the automorphism group of the bipartite double of $G$ is isomorphic with the group $\text{Aut}(G) \times \mathbb{Z}_2$. A graph $G$ is called a stable graph, if $\text{Aut}(B(G)) \cong \text{Aut}(G) \times \mathbb{Z}_2$, where $B(G)$ is the bipartite double of $G$. Finally, we show that the Johnson graph $J(n, k)$ is a stable graph.

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1 Introduction

In this paper, a graph $G = (V, E)$ is considered as an undirected simple finite graph, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. For the terminology and notation not defined here, we follow [1, 2, 4, 7].

Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a bipartite graph. It is quite possible that we wish to construct by vertices in $U$ some other graphs which are related to $G$ in some aspects. For instance there are cases in which we can construct a graph $G_1 = (U, E_1)$ such that we have $Aut(G) \cong Aut(G_1)$, where $Aut(X)$ is the automorphism group of the graph $X$. For example note to the following cases.

(i) Let $n \geq 3$ be an integer and $[n] = \{1, 2, ..., n\}$. Let $k$ be an integer such that $1 \leq k < \frac{n}{2}$. The graph $B(n, k)$ which has been introduced in [15] is a graph with the vertex-set $V = \{v \mid v \in [n], |v| \in \{k, k+1\}\}$ and the edge-set $E = \{\{v, w\} \mid v, w \in V, v \subset w \text{ or } w \subset v\}$. It is clear that the graph $B(n, k)$ is a bipartite graph with the vertex-set $V = V_1 \cup V_2$, where $V_1 = \{v \in [n] \mid |v| = k\}$ and $V_2 = \{v \in [n] \mid |v| = k+1\}$. This graph has some interesting properties which are investigated recently [11, 15, 16, 19]. Let $G = B(n, k)$ and let $G_1 = (V_1, E_1)$ be the Johnson graph $J(n, k)$ which can be constructed on the vertex-set $V_1$. It has been proved that if $n \neq 2k+1$, then $Aut(G) \cong Aut(G_1)$, and if $n = 2k+1$, then $Aut(G) \cong Aut(G_1) \times \mathbb{Z}_2$ [15].

(ii) Let $n$ and $k$ be integers with $n > 2k$, $k \geq 1$. Let $[n] = \{1, 2, ..., n\}$ and $V$ be the set of all $k$-subsets and $(n-k)$-subsets of $[n]$. The bipartite Kneser graph $H(n, k)$ has $V$ as its vertex-set, and two vertices $v, w$ are adjacent if and only if $v \subset w$ or $w \subset v$. It is clear that $H(n, k)$ is a bipartite graph. In fact, if $V_1 = \{v \in [n] \mid |v| = k\}$ and $V_2 = \{v \in [n] \mid |v| = n-k\}$, then $\{V_1, V_2\}$ is a partition of $V(H(n, k))$ and every edge of $H(n, k)$ has a vertex in $V_1$ and a vertex in $V_2$ and $|V_1| = |V_2|$. Let $G = H(n, k)$ and let $G_1 = (V_1, E_1)$ be the Johnson graph $J(n, k)$ which can be constructed on the vertex-set $V_1$. It has been proved that $Aut(G) \cong Aut(G_1) \times \mathbb{Z}_2$ [17].

(iii) Let $n, k$ and $l$ be integers with $0 < k < l < n$. The set-inclusion graph $G(n, k, l)$ is the graph whose vertex-set consists of all $k$-subsets and $l$-subsets of $[n] = \{1, 2, ..., n\}$, where two distinct vertices are adjacent if one of them is contained in another. It is clear that the graph $G(n, k, l)$ is a bipartite graph with the vertex-set $V = V_1 \cup V_2$, where $V_1 = \{v \in [n] \mid |v| = k\}$ and $V_2 = \{v \in [n] \mid |v| = l\}$. It is easy to show that $G(n, k, l) \cong G(n, n-k, n-l)$, hence we assume that $k + l \leq n$. It is clear that if $l = k + 1$, then $G(n, k, l) = B(n, k)$, where $B(n, k)$ is the graph which is defined in (i). Also, if $l = n - k$ then $G(n, k, l) = H(n, k)$, where $H(n, k)$ is the graph which is defined in (ii). Let $G = G(n, k, l)$ and let $G_1 = (V_1, E_1)$ be the Johnson graph $J(n, k)$ which can be constructed on the vertex-set $V_1$. It has been proved that if $n \neq k+l$, then $Aut(G) \cong Aut(G_1)$, and if $n = k+l$, then $Aut(G) \cong Aut(G_1) \times \mathbb{Z}_2$ [9].

In this paper we generalize this results to some other classes of bipartite graphs. In fact, we state some accessible conditions such that if for a bipartite graph $G = (V, E) = (U \cup W, E)$ these conditions hold, then we can determine the automorphism group of the graph $G$. Also, we determine the automorphism group of a class of graphs which are derived from Grassmann graphs. In particular, we determine automorphism groups of bipartite double of some classes of graphs. In fact, we show
that if \( G \) is a non-bipartite connected \( vd\)-graph, and \( a_0 \) is a positive integer such that \( c(v, w) = |N(v) \cap N(w)| = a_0 \), when \( v, w \) are adjacent, whereas \( c(v, w) \neq a_0 \) when \( v, w \) are not adjacent, then the automorphism group of the bipartite double of \( G \) is isomorphic with \( Aut(G) \times \mathbb{Z}_2 \). Finally, we show that if \( G = J(n, k) \) is a Johnson graph, then \( Aut(G \times K_2) \) is isomorphic with the group \( Aut(G) \times \mathbb{Z}_2 \). In other words, we show that Johnson graphs are stable graphs.

## 2 Preliminaries

The graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) are called isomorphic, if there is a bijection \( \alpha : V_1 \rightarrow V_2 \) such that \( \{a, b\} \in E_1 \) if and only if \( \{\alpha(a), \alpha(b)\} \in E_2 \) for all \( a, b \in V_1 \). In such a case the bijection \( \alpha \) is called an isomorphism. An automorphism of a graph \( G \) is an isomorphism of \( G \) with itself. The set of automorphisms of \( \Gamma \) with the operation of composition of functions is a group, called the automorphism group of \( G \) and denoted by \( Aut(G) \).

The group of all permutations of a set \( V \) is denoted by \( Sym(V) \) or just \( Sym(n) \) when \( |V| = n \). A permutation group \( \Gamma \) on \( V \) is a subgroup of \( Sym(V) \). In this case we say that \( \Gamma \) acts on \( V \). If \( \Gamma \) acts on \( V \), we say that \( \Gamma \) is transitive on \( V \) (or \( \Gamma \) acts transitively on \( V \)) when there is just one orbit. This means that given any two elements \( u \) and \( v \) of \( V \), there is an element \( \beta \) of \( G \) such that \( \beta(u) = v \). If \( X \) is a graph with vertex-set \( V \), then we can view each automorphism of \( X \) as a permutation on \( V \), and so \( Aut(X) = \Gamma \) is a permutation group on \( V \).

A graph \( G \) is called vertex-transitive, if \( Aut(G) \) acts transitively on \( V(\Gamma) \). We say that \( G \) is edge-transitive if the group \( Aut(G) \) acts transitively on the edge set \( E \), namely, for any \( \{x, y\} \), \( \{v, w\} \in E(G) \), there is some \( \pi \) in \( Aut(G) \), such that \( \pi(\{x, y\}) = \{v, w\} \). We say that \( G \) is symmetric (or arc-transitive) if, for all vertices \( u, v, x, y \) of \( G \) such that \( u \) and \( v \) are adjacent, and also, \( x \) and \( y \) are adjacent, there is an automorphism \( \pi \) in \( Aut(G) \) such that \( \pi(u) = x \) and \( \pi(v) = y \). We say that \( G \) is distance-transitive if for all vertices \( u, v, x, y \) of \( G \) such that \( d(u, v) = d(x, y) \), where \( d(u, v) \) denotes the distance between the vertices \( u \) and \( v \) in \( G \), there is an automorphism \( \pi \) in \( Aut(\Gamma) \) such that \( \pi(u) = x \) and \( \pi(v) = y \).

Let \( n, k \in \mathbb{N} \) with \( k < n \), and let \( [n] = \{1, ..., n\} \). The Johnson graph \( J(n, k) \) is defined as the graph whose vertex set is \( V = \{v \mid v \subseteq [n], |v| = k\} \) and two vertices \( v, w \) are adjacent if and only if \( |v \cap w| = k - 1 \). The Johnson graph \( J(n, k) \) is a distance-transitive graph [2]. It is easy to show that the set \( H = \{f_\theta \mid \theta \in Sym([n])\}, f_\theta(\{x_1, ..., x_k\}) = \{\theta(x_1), ..., \theta(x_k)\} \), is a subgroup of \( Aut(J(n, k)) \). It has been shown that \( Aut(J(n, k)) \cong Sym([n]) \), if \( n \neq 2k \), and \( Aut(J(n, k)) \cong Sym([n]) \times \mathbb{Z}_2 \), if \( n = 2k \), where \( \mathbb{Z}_2 \) is the cyclic group of order 2 [10,18].

The group \( \Gamma \) is called a semidirect product of \( N \) by \( Q \), denoted by \( \Gamma = N \rtimes Q \), if \( \Gamma \) contains subgroups \( N \) and \( Q \) such that, (i) \( N \leq \Gamma \) (\( N \) is a normal subgroup of \( \Gamma \)); (ii) \( NQ = \Gamma \); (iii) \( N \cap Q = 1 \). Although, in most situations it is difficult to determine the automorphism group of a graph \( G \), there are various papers in the literature, and some of the recent works include [5,6,10,14,15,17,18,22].
3 Main Results

The proof of the following lemma, however is easy but its result is necessary for proving the results of our work.

**Lemma 3.1.** Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a connected bipartite graph. Let $f$ be an automorphism of $G$. If for a fixed vertex $u_0 \in U$, we have $f(u_0) \in U$, then $f(U) = U$ and $f(W) = W$. Also, if for a fixed vertex $w_0 \in W$, we have $f(w_0) \in W$, then $f(W) = W$ and $f(W) = U$.

**Proof.** It is sufficient to show that if $u \in U$ then $f(u) \in U$. We know that if $u \in U$, then $d_G(u_0, u) = d(u_0, u)$, the distance between $v$ and $w$ in the graph $G$, is an even integer. Assume $d(u_0, u) = 2l$, $0 \leq 2l \leq D$, where $D$ is the diameter of $G$. We prove by induction on $l$, that $f(u) \in U$. If $l = 0$, then $d(u_0, u) = 0$, thus $u = u_0$, and hence $f(u) = f(u_0) \in U$. Suppose that if $u \in U$ and $d(u_0, u) = 2(k - 1)$, then $f(u) \in U$. Assume $u \in U$ and $d(u, u_0) = 2k$. Then, there is a vertex $u_1 \in G$ such that $d(u_0, u_1) = 2k - 2 = 2(k - 1)$ and $d(u, u_1) = 2$. We know (by the induction assumption) that $f(u_1) \in U$, and since $d(f(u), f(u_1)) = 2$, therefore $f(u) \in U$. Now, it follows that $f(U) = U$ and consequently $f(W) = W$.

**Corollary 3.2.** Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a connected bipartite graph. If $f$ is an automorphism of the graph $G$, then $f(U) = U$ and $f(W) = W$, or $f(U) = W$ and $f(W) = U$.

**Definition 3.3.** Let $G = (V, E)$ be a graph with the vertex-set $V$ and the edge-set $E$. Let $N(v)$ denotes the set of neighbors of the vertex $v$ of $G$. We say that $G$ is a $vd$-graph, if for every pare of distinct vertices $x, y \in V$ we have $N(x) \neq N(y)$ ($vd$ is an abbreviation for “vertex-determining”).

From Definition 3.3, it follows that the cycle $C_n$ is a $vd$-graph, but the complete bipartite graph $K_{m,n}$ is not a $vd$-graph, when $m \neq 1$.

**Lemma 3.4.** Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a bipartite $vd$-graph. If $f$ is an automorphism of $G$ such that $f(u) = u$ for every $u \in U$, then $f$ is the identity automorphism of $G$.

**Proof.** Let $w \in W$ be an arbitrary vertex. Since $f$ is an automorphism of the graph $G$, then for the set $N(w) = \{u | u \in U, u \leftrightarrow w\}$, we have $f(N(w)) = \{f(u) | u \in U, u \leftrightarrow w\} = N(f(w))$. On the other hand, since for every $u \in U$, $f(u) = u$, then we have $f(N(w)) = N(w)$, and therefore $N(f(w)) = N(w)$. Now since $G$ is a $vd$-graph we must have $f(w) = w$. Therefore, for every vertex $x$ in $V(G)$ we have $f(x) = x$ and thus $f$ is the identity automorphism of the graph $G$.

Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a bipartite graph. We can construct various graphs on the set $U$. We show that some of these graphs can help us in finding the automorphism group of the graph $G$. 
Definition 3.5. Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a bipartite graph. Let $G_1 = (U, E_1)$ be a graph with the vertex-set $U$ such that the following conditions hold:
(i) every automorphism of the graph $G_1$ can be uniquely extended to an automorphism of the graph $G$. In other words, if $f$ is an automorphism of the graph $G_1$, then there is a unique automorphism $e_f$ in the automorphism group of $G$ such that $(e_f)|_U = f$, where $(e_f)|_U$ is the restriction of the automorphism $e_f$ to the set $U$.
(ii) If $f \in \text{Aut}(G)$ is such that $f(U) = U$, then the restriction of $f$ to $U$ is an automorphism of the graph $G_1$. In other words, if $f \in \text{Aut}(G)$ is such that $f(U) = U$ then $f|_U \in \text{Aut}(G_1)$.

When such a graph $G_1$ exists, then we say that the graph $G_1$ is an \textit{attached} graph to the graph $G$.

Remark 3.6. Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a bipartite $vd$-graph, and $G_1 = (U, E_1)$ be a graph. If $f \in \text{Aut}(G_1)$ can be extended to an automorphism $g$ of the graph $G$, then $g$ is unique. In fact if $g$ and $h$ are extensions of the automorphism $f \in \text{Aut}(G_1)$ to automorphisms of $G$, then $i = gh^{-1}$ is an automorphism of the graph $G$ such that the restriction of $i$ to the set $U$ is the identity automorphism. Hence by Lemma 3.4. the automorphism $i$ is the identity automorphism of the graph $G$, and therefore $g = h$. Hence, according to Definition 3.5. the graph $G_1$ is an attached graph to the graph $G$, if and only if every automorphism of $G_1$ can be extended to an automorphism of $G$ and every automorphism of $G$ which fixes $U$ set-wise is an automorphism of $G_1$.

Example 3.7. Let $G = H(n, k) = (V_1 \cup V_2, E)$ be the bipartite Kneser graph which is defined in (ii) of the introduction of the present paper. Let $G_1 = (V_1, E_1)$ be the Johnson graph which can be constructed on the vertex $V_1$. It can be shown that the graph $G_1$ is an attached graph to the graph $G$ [17].

In the next theorem, we show that if $G = (U \cup W, E)$, $U \cap W = \emptyset$ is a connected bipartite $vd$-graph with $G_1 = (U, E_1)$ as an attached graph to $G$, then we can determine the automorphism group of the graph $G$, provided the automorphism group of the graph $G_1$ has been determined.

Let $G = (U \cup W, E)$, $U \cap W = \emptyset$, be a connected bipartite $vd$-graph such that $G_1 = (U, E_1)$ is an attached graph to $G$. If $f \in \text{Aut}(G_1)$ then we let $e_f$ be its unique extension to $\text{Aut}(G)$. It is easy to see that $E_{G_1} = \{e_f|f \in \text{Aut}(G_1)\}$, with the operation of composition, is a group. Moreover, it is easy to see that $E_{G_1}$ and $\text{Aut}(G_1)$ are isomorphic (as abstract groups).

For the bipartite graph $G = (U \cup W, E)$ we let $S(U) = \{f \in \text{Aut}(G) | f(U) = U\} = \text{Aut}(G)_U$, the stabilizer subgroup of the set $U$ in the group $\text{Aut}(G)$. Next theorem shows that when $G_1 = (U, E_1)$ is an attached graph to $G$, then $S(U)$ is a familiar group.

Proposition 3.8. Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a connected bipartite $vd$-graph such that $G_1 = (U, E_1)$ is an attached graph to $G$. Then $S(U) \cong \text{Aut}(G_1)$, where $S(U) = \{f \in \text{Aut}(G) | f(U) = U\}$. 


Proof. Let $f$ be an automorphism of the graph $G_1$, then by definition of the graph $G_1$ we deduce that $ef$ is an automorphism of the graph $G$ such that $e_f(U) = U$. Hence, we have $E_{G_1} \leq S(U)$, where $E_{G_1}$ is the group which is defined preceding of this theorem.

On the other hand, if $g \in S(U)$, then $g(U) = U$, thus by the definition of the graph $G_1$, the restriction of $g$ to $U$ is an automorphism of the graph $G_1$. In other words, $h = g|_U \in Aut(G_1)$. Therefore by definition 3.5. there is an automorphism $e_h$ of the graph $G$ such that $e_h(u) = g(u)$ for every $u \in U$. Now by Remark 3.6 we deduce that $g = e_h \in E_{G_1}$. Hence we have $S(U) \leq E_{G_1}$. We now deduce that $S(U) = E_{G_1}$. Now, since $E_{G_1} \cong Aut(G_1)$, we conclude that $S(U) \cong Aut(G_1)$.

\hfill \Box

Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a connected bipartite graph. It is quite possible that $G$ be such that $f(U) = U$, for every automorphism of the graph $G$. For example if $|U| \neq |W|$, or $U$ contains a vertex of degree $d$, but $W$ does not contain a vertex of degree $d$, then we have $f(U) = U$ for every automorphism $f$ of the graph $G$. In such a case we have $Aut(G) = S(U)$, and hence by Proposition 3.8. we have the following theorem.

**Theorem 3.9.** Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a connected bipartite vd-graph such that $G_1 = (U, E_1)$ is an attached graph to $G$. If $Aut(G) = S(U)$, then $Aut(G) \cong Aut(G_1)$.

Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a connected bipartite vd-graph. Concerning the automorphism group of $G$, we can more say even if $|U| = |W|$. When $|U| = |W|$ then there is a bijection $\theta : U \to W$. Then $\theta^{-1} \cup \theta = t$ is a permutation on the vertex-set of the graph $G$ such that $t(U) = W$ and $t(W) = U$. In the following theorem, we show that if the graph $G$ has an attached graph $G_1 = (U, E_1)$, and if such a permutation $t$ is an automorphism of the group $G$, then the automorphism group of the graph $G$ is a familiar group.

**Theorem 3.10.** Let $G = (U \cup W, E)$, $U \cap W = \emptyset$ be a connected bipartite vd-graph such that $G_1 = (U, E_1)$ is an attached graph to $G$ and $|U| = |W|$. Suppose that there is an automorphism $t$ of the graph $G$ such that $t(U) = W$. Then $Aut(G) = Aut(G_1) \rtimes H$ where $H = < t >$ is the subgroup generated by $t$ in the group $Aut(G)$.

Proof. Let $S(U) = \{ f \in Aut(G) | f(U) = U \}$. It is clear that $S(U)$ is a subgroup of $Aut(G)$. Let $g \in Aut(G)$ be such that $g(U) \neq U$, then by Lemma 3.1. we have $g(U) = W$, and hence $tg(U) = t(W) = U$. Therefore, $tg \in S(U)$, and hence there is an element $h \in S(U)$ such that $tg = h$. Thus, $g = t^{-1}h \in < t, S(U) >$, where $< t, S(U) > = K$ is the subgroup of $Aut(G)$ which is generated by $t$ and $S(U)$. This follows that $Aut(G) \leq K$. Since $K \leq Aut(G)$, hence we deduce that $K = Aut(G)$. If $f$ is an arbitrary element in the subgroup $S(U)$ of $K$, then we have $(t^{-1}ft)(U) = (t^{-1}f)(W) = t^{-1}(f(W)) = (t^{-1})(W) = U$, hence $t^{-1}ft \in S(U)$. We now deduce that $S(U)$ is a normal subgroup of the group $K$. Therefore $K = < t, S(U) > = S(U) \rtimes < t > = S(U) \rtimes H$, where $H = < t >$. We have seen in Proposition 3.8. that $S(U) \cong Aut(G_1)$, hence we conclude that, $K = Aut(G) \cong Aut(G_1) \rtimes H$.

\hfill \Box
In the sequel, we will see how Theorem 3.9. and Theorem 3.10. can help us in determining the automorphism groups of some classes of bipartite graphs.

Some Applications

Let $G = (U \cup W) = G(n, k, l)$ be the bipartite graph which is defined in (iii) of the introduction of the present paper. Then $U = \{v \subseteq [n] \mid |v| = k\}$ and $W = \{v \subseteq [n] \mid |v| = l\}$. It is easy to show that $G$ is a connected $vd$-graph. Let $G_1 = (U, E_1)$ be the Johnson graph which can be constructed on the set $U$. By a proof exactly similar to what is appeared in [15,17] and later [9], it can be shown that $G$ is the Johnson graph which can be constructed on the set $U$. We know that $Aut(G_1) = H = \{f_\theta \mid \theta \in Sym([n])\}$, where $f_\theta(v) = \{\theta(x) \mid x \in v\}$, for every $v \in U$. Because $k < l$ and $k + l \leq n$ implies that $k < \frac{n}{2}$. When $k + l = n$, then the mapping $t : V(G) \to V(G)$, defined by the rule $t(v) = v_c$, where $v_c$ is the complement of the set $v$ in the set $[n] = \{1, 2, 3, \ldots, n\}$, is an automorphism of $G$. It is clear that $t(U) = W$ and $t(W) = U$. Moreover, $t$ is of order 2, hence $t \geq \mathbb{Z}_2$. It is easy to show that if $f \in H$ then $ft = tf$ [15,18].

We now, by Theorem 3.9. and Theorem 3.10. obtain the following theorem which has been given in [9].

Theorem 3.11. Let $n, k$ and $l$ be integers with $1 \leq k < l \leq n - 1$ and $G = G(n, k, l)$. If $n \neq k + l$, then $Aut(G) \cong Sym([n])$, and if $n = k + l$, then $Aut(G) = H \times t > \cong Sym([n]) \times \mathbb{Z}_2$, where $H$ and $t$ are the group and automorphism which are defined preceding of this theorem.

We now consider a class of graphs which are in some combinatorial aspects similar to Johnson graphs.

Definition 3.12. Let $p$ be a positive prime integer and $q = p^m$ where $m$ is a positive integer. Let $n, k$ be positive integers with $k < n$. Let $V(q, n)$ be a vector space of dimension $n$ over the finite field $\mathbb{F}_q$. Let $V_k$ be the family of all subspaces of $V(q, n)$ of dimension $k$. Every element of $V_k$ is also called a $k$-subspace. The Grassmann graph $G(q, n, k)$ is the graph with the vertex-set $V_k$, in which two vertices $u$ and $w$ are adjacent if and only if $dim(u \cap w) = k - 1$.

Note that if $k = 1$, we have a complete graph, so we shall assume that $k > 1$. It is clear that the number of vertices of the Grassmann graph $G(q, n, k)$, that is, $|V_k|$, is the Gaussian binomial coefficient,

$$\binom{n}{k}_q = \frac{(q^n - 1)(q^{n-k} - 1)}{(q^k - 1)(q^k - q^k - 1)} = \frac{(q^n - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1) \cdots (q - 1)}.$$  

Noting that $[\binom{n}{i}_q] = [\binom{n}{n-k}_q]$, it follows that $|V_k| = |V_{n-k}|$. It is easy to show that if $1 \leq i < j \leq \frac{n}{2}$, then $|V_i| < |V_j|$. Let $(, )$ be any nondegenerate symmetric bilinear form on $V(q, n)$. For each $X \subseteq V(q, n)$ we let $X^\perp = \{w \in V(q, n) \mid (x, w) = 0\}$, for every $x \in X$. It can be shown that if $v$ is a subspace of $V(q, n)$ then $v^\perp$ is also a subspace of $V(q, n)$ and $dim(v^\perp) = n - dim(v)$. It can be shown that $G(n, q, k) \cong G(n, q, n - k)$ [2], hence in the sequel we assume that $k \leq \frac{n}{2}$.
It is easy to see that the distance between two vertices \( v \) and \( w \) in this graph is \( k - \text{dim}(v \cap w) \). The Grassmann graph is a distance-regular graph of diameter \( k \) [2]. Let \( K \) be a field, and \( V(n) \) be a vector space of dimension \( n \) over the field \( K \). Let \( \tau : K \rightarrow K \) be a field automorphism. A semilinear operator on \( V(n) \) is a mapping \( f : V(n) \rightarrow V(n) \) such that,

\[
f(c_1v_1 + c_2v_2) = \tau(c_1)f(v_1) + \tau(c_2)f(v_2) \quad (c_1, c_2 \in K, \text{ and } v_1, v_2 \in V(n)).
\]

A semilinear operator \( f : V(n) \rightarrow V(n) \) is a semilinear automorphism if it is a bijection. Let \( PTL_n(K) \) be the group of semilinear automorphisms on \( V(n) \). Note that this group contains \( A(V(n)) \), where \( A(V(n)) \) is the group of non singular linear mappings on the space \( V(n) \). Also, this group contains a normal subgroup isomorphic to \( K^* \), namely, the group \( Z = \{ kI_{V(n)} | k \in K \} \), where \( I_{V(n)} \) is the identity mapping on \( V(n) \). We denote the quotient group \( \frac{PTL_n(K)}{Z} \) by \( PTL_n(K) \).

Note that if \( (a + Z) \in PTL_n(K) \) and \( x \) is an \( m \)-subspace of \( V(n) \), then \( (a + Z)(x) = \{ a(u) | u \in x \} \) is an \( m \)-subspace of \( V(n) \). In the sequel, we also denote \( (a + Z) \in PTL_n(K) \) by \( a \). Now, if \( a \in PTL_n(\mathbb{F}_q) \), it is easy to see that the mapping, \( f_a : V_k \rightarrow V_k \) defined by the rule, \( f_a(v) = a(v) \) is an automorphism of the Grassmann graph \( G = G(q, n, k) \). Therefore if we let,

\[
A = \{ f_a | a \in PTL_n(\mathbb{F}_q) \}
\]

then \( A \) is a group isomorphic to the group \( PTL_n(\mathbb{F}_q) \) (as abstract groups), and we have \( A \leq Aut(G) \).

When \( n = 2k \), then the Grassmann graph \( G = G(q, n, k) \) has some other automorphisms. In fact if \( n = 2k \), then the mapping \( \theta : V_k \rightarrow V_k \), which is defined by this rule \( \theta(v) = v^1 \), for every \( k \)-subspace of \( V(2k) \), is an automorphism of the graph \( G = G(q, 2k, k) \). Hence \( M = \langle A, \theta \rangle \leq Aut(G) \). It is easy to see that \( A \) is a normal subgroup of the group \( M \). Therefore \( M = A \rtimes \theta \). Note that the order of \( \theta \) is 2 and hence \( \theta \). Concerning the automorphism groups of Grassmann graphs, from a known fact which is appeared in [3], we have the following result [2].

**Theorem 3.13.** Let \( G \) be the Grassmann graph \( G = G(q, n, k) \), where \( n > 3 \) and \( k \leq \frac{n}{2} \). If \( n \neq 2k \), then we have, \( Aut(G) = A \cong PTL_n(\mathbb{F}_q) \), and if \( n = 2k \), then we have \( Aut(G) = \langle A, \theta \rangle \cong A \rtimes \theta \cong PTL_n(\mathbb{F}_q) \rtimes \mathbb{Z}_2 \), where \( A \) is the group which is defined in (1) and \( \theta \) is the mapping which is defined preceding of this theorem.

We now proceed to determine the automorphism group of a class of bipartite graphs which are similar in some aspects to graphs \( B(n, k) \)

**Definition 3.14.** Let \( n, k \) be positive integers such that \( n \geq 3, k \leq n - 1 \). Let \( q \) be a power of a prime and \( \mathbb{F}_q \) be the finite field of order \( q \). Let \( V(q, n) \) be a vector space of dimension \( n \) over \( \mathbb{F}_q \). We define the graph \( S(q, n, k) \) as a graph with the vertex-set \( V = V_k \cup V_{k+1} \), in which two vertices \( v \) and \( w \) are adjacent whenever \( v \) is a subspace of \( w \) or \( w \) is a subspace of \( v \), where \( V_k \) and \( V_{k+1} \) are subspaces in \( V(q, n) \) of dimension \( k \) and \( k + 1 \), respectively.
When $n = 2k + 1$, then the graph $S(q, n, k)$ is known as a doubled Grassmann graph [2]. Noting that $\left[\begin{array}{c}n \\ k\end{array}\right] = \left[\begin{array}{c}n \\ n-k\end{array}\right]$, it is easy to show that $S(n, q, k) \cong S(n, q, n-k-1)$, hence in the sequel we assume $k \leq \frac{n}{2}$. It can be shown that the graph $S(q, n, k)$ is a connected bipartite $vd$-graph. We formally state and prove this fact.

**Proposition 3.15.** The graph $G = S(q, n, k)$ which is defined in Definition 3.14. is a connected bipartite $vd$-graph.

**Proof.** It is clear that the graph $G = S(q, n, k)$ is a bipartite graph with partition $V_k \cup V_{k+1}$. It is easy to show that $G$ is a $vd$-graph. We now, show that $G$ is a connected graph. It is sufficient to show that if $v_1, v_2$ are two vertices in $V_k$, then there is a path in $G$ between $v_1$ and $v_2$. Let $\text{dim}(v_1 \cap v_2) = k - j$, $1 \leq j \leq k$. We prove our assertion by induction on $j$. If $j = 1$, then $u = v_1 + v_2$ is a subspace of $V(n, q)$ of dimension $k + k - (k - 1) = k + 1$, which contains both of $v_1$ and $v_2$. Hence, $u \in V_{k+1}$ is adjacent to both of the vertices $v_1$ and $v_2$. Thus, if $j = 1$, then there is a path between $v_1$ and $v_2$ in the graph $G$. Assume when $j = i, 0 < i < k$, then there is a path in $G$ between $v_1$ and $v_2$. We now assume $j = i + 1$. Let $v_1 \cap v_2 = w$, and let $B = \{b_1, ..., b_{k-i-1}\}$ be a basis for the subspace $w$ in the space $V(q, n)$. We can extend $B$ to bases $B_1$ and $B_2$ for the subspaces $v_1$ and $v_2$, respectively. Let $B_1 = \{b_1, ..., b_{k-i-1}, c_1, ..., c_{i+1}\}$ be a basis for $v_1$ and $B_2 = \{b_1, ..., b_{k-i-1}, d_1, ..., d_{i+1}\}$ be a basis for $v_2$. Consider the subspace $s = \langle b_1, ..., b_{k-i-1}, c_1, d_1, ..., d_{i+1} \rangle$. Then $s$ is a $k$-subspace of the space $V(q, n)$ such that $\text{dim}(s \cap v_2) = k - 1$ and $\text{dim}(s \cap v_1) = k - i$. Hence by the induction assumption, there is a path $P_1$ between vertices $v_2$ and $s$, and a path $P_2$ between vertices $s$ and $v_1$. We now conclude that there is a path in the graph $G$ between vertices $v_1$ and $v_2$. \hfill \Box

**Theorem 3.16.** Let $G = S(q, n, k)$ be the graph which is defined in definition 3.14. If $n \neq 2k + 1$, then we have $\text{Aut}(G) \cong \text{PGL}_n(\mathbb{F}_q)$. If $n = 2k + 1$, then $\text{Aut}(G) \cong \text{PGL}_n(\mathbb{F}_q) \rtimes \mathbb{Z}_2$.

**Proof.** From Proposition 3.15, it follows that the graph $G = S(q, n, k)$ is a connected bipartite $vd$-graph with the vertex-set $V_k \cup V_{k+1}$, $V_k \cap V_{k+1} = \emptyset$. Let $G_1 = G(q, n, k) = (V_k, E)$ be the Grassmann graph with the vertex-set $V_k$. We show that $G_1$ is an attached graph to the graph $G$.

Firstly, the condition (i) of Definition 3.5. holds, because $k < \frac{n}{2}$ and every automorphism of the Grassmann graph $G(q, n, r)$ is of the form $f_a$, $a \in \text{PGL}_n(\mathbb{F}_q)$, is an automorphism of the graph $G(q, n, s)$ when $r, s < \frac{n}{2}$. Also, note that if $X, Y$ are subspaces of $V(q, n)$ such that $X \leq Y$, then $f_a(X) \leq f_a(Y)$.

Now, suppose that $f$ is an automorphism of the graph $G$ such that $f(V_k) = V_k$. We show that the restriction of $f$ to the set $V_k$, namely $g = f|_{V_k}$, is an automorphism of the graph $G$. It is trivial that $g$ is a permutation of the vertex-set $V_k$. Let $v$ and $w$ be adjacent vertices in the graph $G_1$. We show that $g(v)$ and $g(w)$ are adjacent in the graph $G_1$. We assert that there is exactly one vertex $u$ in the graph $G$ such that $u$ is adjacent to the both of the vertices $v$ and $w$. If the vertex $u$ is adjacent to both of the vertices $v$ and $w$, then $v$ and $w$ are $k$-subspaces of the $(k + 1)$-space $u$. Hence $u$ contains the space $v + w$. Since
In fact, if for 

g to see that the group 

\[ \text{Aut} \]

many applications in algebraic graph theory [2].

\[ x \]

and

\[ b \]

bipartite double of

\[ G \]

B and two vertices \((x, y)\) are such that \( \text{dim}(x \cap y) \neq (k - 1) \), then \( x \) and \( y \) have no a common neighbor in the graph \( G \).

Now since, the vertices \( v \) and \( w \) have exactly 1 common neighbor in the graph \( G \), therefore \( f(v) = g(v) \) and \( f(w) = g(w) \) have exactly 1 common neighbor in the graph \( G \). This follows that \( \text{dim}(g(v) \cap g(w)) = k - 1 \), and hence \( g(v) \) and \( g(w) \) are adjacent vertices in the Grassmann graph \( G \).

We now conclude that the graph \( G_1 \) is an attached graph to the graph \( G \).

There are two possible cases, that is (1) \( 2k + 1 \neq n \), or (2) \( 2k + 1 = n \).

(1) Let \( 2k + 1 \neq n \). Noting that \( \left[ \begin{array}{c} n \\ k \\ \end{array} \right]_q < \left[ \begin{array}{c} n \\ k+1 \\ \end{array} \right]_q \), it follows that \( |V_k| \neq |V_{k+1}| \). Therefore by Corollary 3.8. and Theorem 3.14. we have \( \text{Aut}(\Gamma) \cong \text{Aut}(G) \cong \text{PGL}_n(\mathbb{F}_q) \).

(2) If \( 2k + 1 = n \), since \( \left[ \begin{array}{c} n \\ k \\ \end{array} \right]_q = \left[ \begin{array}{c} n \\ k+1 \\ \end{array} \right]_q \), then \( |V_k| = |V_{k+1}| \). Hence, the mapping \( \theta : V(G) \longrightarrow V(G) \) defined by the rule \( \theta(v) = v^\perp \) is an automorphism of the graph \( G \) of order 2 such that \( \theta(V_k) = V_{k+1} \). Hence, by Theorem 3.10. and Theorem 3.14 we have, \( \text{Aut}(G) \cong \text{Aut}(G_1) \times < \theta > \cong \text{PGL}_n(\mathbb{F}_q) \times \mathbb{Z}_2 \). \[ \square \]

We now show another application of Theorem 3.10. in determining the automorphism groups of some classes of graphs which are important in algebraic graph theory.

If \( G_1, G_2 \) are graphs, then their direct product (or tensor product) is the graph \( G_1 \times G_2 \) with vertex set \( \{(v_1, v_2) \mid v_1 \in G_1, v_2 \in G_2\} \), and for which vertices \((v_1, v_2)\) and \((w_1, w_2)\) are adjacent precisely if \( v_1 \) is adjacent to \( w_1 \) in \( G_1 \) and \( v_2 \) is adjacent to \( w_2 \) in \( G_2 \). It can be shown that the direct product is commutative and associative [8].

The following theorem, first proved by Weichsel (1962), characterizes connectedness in direct products of two factors.

**Theorem 3.17.** [8] Suppose \( G_1 \) and \( G_2 \) are connected nontrivial graphs. If at least one of \( G_1 \) or \( G_2 \) has an odd cycle, then \( G_1 \times G_2 \) is connected. If both \( G_1 \) and \( G_2 \) are bipartite, then \( G_1 \times G_2 \) has exactly two components.

Thus, if one of the graphs \( G_1 \) or \( G_2 \) is a connected non-bipartite graph, then the graph \( G_1 \times G_2 \) is a connected graph. If \( K_2 \) is the complete graph on the set \( \{0, 1\} \), then the direct product \( B(G) = G \times K_2 \) is a bipartite graph, and is called the bipartite double of \( G \) (or the bipartite double cover of \( G \)). Then,

\[ V(B(G)) = \{(v, i) \mid v \in V(G), i \in \{0, 1\}\}, \]

and two vertices \((x, a)\) and \((y, b)\) are adjacent in the graph \( B(G) \), if and only if \( a \neq b \) and \( x \) is adjacent to \( y \) in the graph \( G \). The notion of the bipartite double of \( G \) has many applications in algebraic graph theory [2].

Consider the bipartite double of \( G \), namely, the graph \( B(G) = G \times K_2 \). It is easy to see that the group \( \text{Aut}(B(G)) \) contains the group \( \text{Aut}(G) \times \mathbb{Z}_2 \) as a subgroup. In fact, if for \( g \in \text{Aut}(G) \), we define the mapping \( e_g \) by the rule \( e_g(v, i) = (g(v), i) \),
Let $G = (V, E)$ be a connected non-bipartite $vd$-graph. Let $v, w \in V$ be arbitrary. Let $c(v, w)$ be the number of common neighbors of $v$ and $w$ in the graph $G$. Let $a_0 > 0$ be a fixed integer. If $c(v, w) = a_0$, when $v$ and $w$ are adjacent and $c(v, w) \neq a_0$ when $v$ and $w$ are non-adjacent, then we have

$$\text{Aut}(G \times K_2) = \text{Aut}(B(G)) \cong \text{Aut}(G) \times \mathbb{Z}_2,$$

in other words, $G$ is a stable graph.

**Proof.** Note that the graph $G \times K_2$ is a bipartite graph with the vertex set $V = U \cup W$, where $U = \{(v, 0) | v \in V(G)\}$ and $W = \{(v, 1) | v \in V(G)\}$. Since $G$ is a $vd$-graph, then the graph $G \times K_2$ is a $vd$-graph. In fact if the vertices $x, y \in V$ are such that $N(x) = N(y)$, then $x, y \in U$ or $x, y \in W$. Without loss of generality, we can assume that $x, y \in U$. Let $x = (u_1, 0)$ and $y = (u_2, 0)$. Let $N(x) = \{(v_1, 1), (v_2, 1), \ldots, (v_m, 1)\}$ and $N(y) = \{(t_1, 1), (t_2, 1), \ldots, (t_p, 1)\}$, where $v_i$s and $t_j$s are in $V(G)$. Thus $m = p$ and $N(u_1) = \{u_1, \ldots, u_m\} = \{t_1, \ldots, t_m\} = N(u_2)$. Now since $G$ is a $vd$-graph, this follows that $u_1 = u_2$ and therefore $x = y$.

Let $G_1 = (U, E_1)$ be the graph with vertex-set $U$ in which two vertices $(v, 0)$ and $(w, 0)$ are adjacent if and only if $v_1$ and $v_2$ are adjacent in the graph $G$. It is clear that $G_1 \cong G$. Therefore we have $\text{Aut}(G_1) \cong \text{Aut}(G)$. For every $f \in \text{Aut}(G)$ we let,

$$d_f : U \to U, \ d_f(v, 0) = (f(v), 0), \text{ for every } (v, 0) \in U,$$

then $d_f$ is an automorphism of the graph $G_1$. If we let $A = \{d_f | f \in \text{Aut}(G)\}$, then $A$ with the operation of composition is a group, and it is easy to see that $A \cong \text{Aut}(G_1)$ (as abstract groups). We now assert that the graph $G_1$ is a permutation graph to the bipartite graph $B = G \times K_2$. Let $g \in \text{Aut}(B)$ be such that $g(U) = U$. We assert that $h = g|_{U}$, the restriction of $g$ to $U$, is an automorphism of the graph $G_1$. It is clear that $h$ is a permutation of $U$. Let $(v, 0)$ and $(w, 0)$ be adjacent vertices in $G_1$. Then $v, w$ are adjacent in the graph $G$. Hence there are vertices $u_1, \ldots, u_{a_0}$ in the graph $G$ such that the set of common neighbor(s) of $v$ and $w$ in $G$ is $\{u_1, \ldots, u_{a_0}\}$. 


Noting that \((x, 1)\) is a common neighbor of \((v, 0)\) and \((w, 0)\) in the graph \(B\) if and only if \(x\) is a common neighbor of \(v, w\) in the graph \(G\), we deduce that the set \(\{(u_1, 1), ..., (u_{a_0}, 1)\}\) is the set of common neighbor(s) of \((v, 0)\) and \((w, 0)\) in the graph \(B\). Since \(g\) is an automorphism of the graph \(B\), hence \(g(v, 0)\) and \(g(w, 0)\) have \(a_0\) common neighbor(s) in the graph \(B\). Note that if \(d_{G_1}(g(v, 0), g(w, 0)) > 2\), then these vertices have no common neighbor in the graph \(B\). Also, if \(d_{G_1}(g(v, 0), g(w, 0)) = 2\), then \(d_G(v, w) = 2\), hence \(v, w\) have \(c(v, w) \neq a_0\) common neighbor(s) in the graph \(G\), hence \((v, 0)\) and \((w, 0)\) have \(c(v, w)\) common neighbor(s) in the graph \(B\), and therefore \(g(v, 0), g(w, 0)\) have \(c(v, w) \neq a_0\) common neighbor(s) in the graph \(B\). We now deduce that \(d_{G_1}(g(v, 0), g(w, 0)) = 1\). This follows that \(h = g|_V\) is an automorphism of the graph \(G_1\). Thus, the condition (ii) of Definition 3.5. holds for the graph \(G_1\).

Now, suppose that \(\phi\) is an automorphism of the graph \(G_1\). Then there is an automorphism \(f\) of the graph \(G\) such that \(\phi = df\). Now, we define the mapping \(e_\phi\) on the set \(V(B)\) by the following rule:

\[
(*) \quad e_\phi(v, i) = \begin{cases} (f(v), 0), & \text{if } i = 0 \\ (f(v), 1), & \text{if } i = 1 \end{cases}
\]

It is easy to see that \(e_\phi\) is an extension of the automorphism \(\phi\) to an automorphism of the graph \(B\). We now deduce that the graph \(G_1\) is an attached graph to the graph \(B\).

On the other hand, it is easy to see that the mapping \(t : V(B) \to V(B)\), which is defined by the rule,

\[
(**) \quad t(v, i) = \begin{cases} (v, 0), & \text{if } i = 1 \\ (v, 1), & \text{if } i = 0 \end{cases}
\]

is an automorphism of the graph \(B\) of order 2. Hence, \(< t > \cong \mathbb{Z}_2\). Also, it is easy to see that for every automorphism \(\phi\) of the graph \(G_1\) we have \(te_\phi = e_\phi t\). We now conclude by Theorem 3.10. that,

\[
\text{Aut}(G \times K_2) = \text{Aut}(B) \cong \text{Aut}(G_1) \times < t > \cong \text{Aut}(G) \times < t > \cong \text{Aut}(G) \times \mathbb{Z}_2.
\]

As an application of Theorem 3.18. we show that the Johnson graph \(J(n, k)\) is a stable graph. Since \(J(n, k) \cong J(n, n - k)\), in the sequel we assume that \(k \leq \frac{n}{2}\).

**Theorem 3.19.** Let \(n, k\) be positive integers with \(k \leq \frac{n}{2}\). If \(n \neq 6\), then the Johnson graph \(J(n, k)\) is a stable graph.

**Proof.** We know that the vertex set of the graph \(J(n, k)\) is the set of \(k\)-subsets of \([n] = \{1, 2, 3, ..., n\}\), in which two vertices \(v\) and \(w\) are adjacent if and only if \(|v \cap w| = k - 1\). If \(k = 1\), then \(J(n, k) \cong K_n\), the complete graph on \(n\) vertices. It is easy to see that if \(X = K_n\) then the bipartite double of \(X\), is isomorphic with the
bipartite Kneser graph $H(n, 1)$. From Corollary 3.11, we know that $Aut(H(n, 1)) \cong Sym([n]) \times Z_2 \cong Aut(K_n) \times Z_2$. Hence the Johnson graph $J(n, k)$ is a stable graph when $k = 1$. We now assume that $k \geq 2$. We let $G = J(n, k)$. It is easy to see that $G$ is a vd-graph. It can be shown that if $v, w$ are vertices in $G$, then $d(v, w) = k - |v \cap w|$. Hence, $G$ is a connected graph. It is easy to see that the girth of the Johnson graph $J(n, k)$ is 3. Therefore, $G$ is a non-bipartite graph. It is clear that when $d(v, w) \geq 3$, then $v, w$ have no common vertices. We now consider 2 other possible cases, that is, (i) $d(v, w) = 2$ or (ii) $d(v, w) = 1$. Let $c(v, w)$ denotes the number of common neighbors of $v, w$ in $G$. In the sequel, we show that if $d(v, w) = 2$, then $c(v, w) = 4$, and if $d(v, w) = 1$, then $c(v, w) = n - 2$.

(i) If $d(v, w) = 2$, then $|v \cap w| = k - 2$. Let $v \cap w = u$. Then $v = u \cup \{i_1, i_2\}$, $w = u \cup \{j_1, j_2\}$, where $i_1, i_2, j_1, j_2 \in [n]$, $\{i_1, i_2\} \cap \{j_1, j_2\} = \emptyset$. Let $x \in V(G)$. It is easy to see that if $|x \cap u| < k - 2$, then $x$ can not be a common neighbor of $v, w$. Hence, if $x$ is a common vertex of $v, w$ then $x$ is of the form $x = u \cup \{r, s\}$, where $r \in \{i_1, i_2\}$ and $s \in \{j_1, j_2\}$. We now deduce that the number of common neighbors of $v, w$ in the graph $G$ is 4.

(ii) We now assume that $d(v, w) = 1$. Then $|v \cap w| = k - 1$. Let $v \cap w = u$. Then $v = u \cup \{r\}$, $w = u \cup \{s\}$, where $r, s \in [n]$, $r \neq s$. Let $x \in V(G)$. It is easy to see that if $|x \cap u| < k - 2$, then $x$ can not be a common neighbor of $v, w$. Hence, if $x$ is a common neighbor of $v, w$ then $|x \cap u| = k - 1$. In the first step, we assume that $|x \cap u| = k - 1$. Then $x$ is of the form $x = t \cup \{y\}$, where $y \in [n] - (v \cup u)$. Since, $|v \cup u| = k + 1$, then the number of such $x$'s is $n - k - 1$. We now assume that $|x \cap u| = k - 2$. Hence, $x$ is of the form $x = t \cup \{r, s\}$, where $t$ is a $(k - 2)$-subset of the $(k - 1)$-set $u$. Therefore the number of such $x$'s is $(k-1) = k - 1$. Our argument follows that if $v$ and $w$ are adjacent, then we have $c(v, w) = n - k - 1 + k - 1 = n - 2$.

Noting that $n - 2 \neq 4$, we conclude from Theorem 3.18 that the Johnson graph $J(n, k)$ is a stable graph, when $n \neq 6$.

Although, Theorem 3.19. does not say anything about the stability of the Johnson graph $J(6, k)$, we show by the next result that this graph is a stable graph.

**Proposition 3.20.** The Johnson graph $J(6, k)$ is a stable graph.

**Proof.** When $k = 1$ the assertion is true, hence we assume that $k \in \{2, 3\}$. In the first step we show that the Johnson graph $J(6, 2)$ is a stable graph. Let $B = J(6, 2) \times K_2$. We show that $Aut(B) \cong Sym([6]) \times Z_2$, where $[6] = \{1, 2, \ldots, 6\}$. It is clear that $B$ is a bipartite vd-graph. Let $V = V(B)$ be the vertex-set of the graph $B$. Then $V = V_0 \cup V_1$, where $V_i = \{(v, i)|v \in [6], |v| = 2\}, i \in \{0, 1\}$. Let $G_1 = (V_0, E_{1})$ be the graph with the vertex-set $V_0$, in which two vertices $(v, 0), (w, 0)$ are adjacent whenever $|v \cap w| = 1$. It is clear that, $G_1$ is isomorphic with the Johnson graph $J(6, 2)$. Hence, we have $Aut(G_1) \cong Sym([6])$. We show that $G_1$ is an attached graph to the graph $B$. By what is seen in (*) of the proof of Theorem 3.18. it is clear that if $h$ is an automorphism of the graph $G_1$, then $h$ can be extended to an automorphism $e_h$ of the graph $B$. Thus, the condition (i) of Definition 3.5. holds for the graph $G_1$. 


Let \( a = (v, 0) \) and \( b = (w, 0) \) be two adjacent vertices in the graph \( G_1 \), that is, \( |v \cap w| = 1 \). Let \( N(a, b) \) denotes the set of common neighbors of \( a \) and \( b \) in the graph \( B \). Let \( X(a, b) = \{ a, b \} \cup N(a, b) \cup t(N(a, b)) \), where \( t \) is the automorphism of the graph \( B \), defined by the rule \( t(v, i) = (v, i^c) \), \( i^c \in \{0, 1\}, i^c \neq i \). Let \( < X(a, b) > \) be the subgraph induced by the set \( X(a, b) \) in the graph \( B \). It can be shown that if \( a, b \) are adjacent vertices in \( G_1 \), that is \( |v \cap w| = 1 \), then \(< X(a, b) > \) has a vertex of degree 0. On the other hand, when \( a, b \) are not adjacent vertices in \( G_1 \), that is \( |v \cap w| = 0 \), then \( < X(a, b) > \) has no vertices of degree 0. In the sequel of the proof, we let \( \{x, y\} = xy \). For example, let \( r = (12, 0) \) and \( s = (13, 0) \) be two adjacent vertices of \( G_1 \). Then \( X(r, s) = \{(12, 0), (13, 0), (14, 1), (15, 1), (16, 1), (23, 1), (14, 0), (15, 0), (16, 0), (23, 0)\} \). Now, in the graph \( < X(r, s) > \) the vertex \((23, 0)\) is a vertex of degree 0. Whereas, if we let \( r = (12, 0) \), \( u = (34, 0) \), then \( r, u \) are not adjacent in the graph \( G_1 \). Then \( X(r, u) = \{(12, 0), (34, 0), (13, 1), (14, 1), (23, 1), (24, 1), (13, 0), (14, 0), (23, 0), (24, 0)\} \). Now, it is clear that the graph \( < X(r, u) > \) has no vertices of degree 0.

Note that the graph \( G_1 \) is isomorphic with the Johnson graph \( J(6, 2) \), hence \( G_1 \) is a distance-transitive graph. Now if \( c, d \) be two adjacent vertices in the graph \( G_1 \) then there is an automorphism \( f \) in \( Aut(G_1) \) such that \( f(r) = c \) and \( f(s) = d \). Let \( e_f \) be the extension of \( f \) to an automorphism of the graph \( B \). Therefore, \(< X(c, d) >= < X(e_f(r), e_f(s)) >= e_f(< X(r, s) >) \) has a vertex of degree 0. This argument also shows that if \( p, q \) are non-adjacent vertices in the graph \( G_1 \), then \(< X(p, q) > \) has no vertices of degree 0.

Now, let \( g \) be an automorphism of the graph \( B \) such that \( g(V_0) = V_0 \). We show that \( g|_{V_0} \) is an automorphism of the graph \( G_1 \). Let \( a = (v, 0) \) and \( b = (w, 0) \) be two vertices of the graph \( G_1 \), that is, \( |v \cap w| = 1 \). Then \( < X(a, b) > \) has a vertex of degree 0. Hence, \( g(< X(a, b) >) = < X(g(a), g(b)) > \) has a vertex of degree 0. Then \( g(a) \) and \( g(b) \) are adjacent in the graph \( G_1 \). We now deduce that if \( g \) is an automorphism of the graph \( B \) such that \( g(V_0) = V_0 \), then \( g|_{V_0} \) is an automorphism of the graph \( G_1 \). Therefore, the condition (ii) of Definition 3.5. holds for the graph \( G_1 \). Therefore, \( G_1 \) is an attached graph to graph \( B \). Note that \( t \) is an automorphism of the graph \( B \) of order 2 such that \( t(V_0) = V_1 \) and \( t(V_1) = V_0 \). Also, we have \( tf = ft, f \in Aut(G_1) \). We now, conclude by Theorem 3.10. that:

\[
Aut(B) \cong Aut(G_1) \times t \cong Aut(G_1) \times t \cong Aut(G) \times \mathbb{Z}_2 \cong Sym([6]) \times \mathbb{Z}_2
\]

Therefore, the graph \( G = J(n, 6) \) is a stable graph.

By a similar argument, we can show that the graph \( J(6, 3) \) is a stable graph. \( \Box \)

Combining Theorem 3.19. and Proposition 3.20. we obtain the following result.

**Theorem 3.21.** The Johnson graph \( J(n, k) \) is a stable graph.

## 4 Conclusion

In this paper, we gave a method for finding the automorphism groups of connected bipartite \( vd \)-graphs (Theorem 3.9. and Theorem 3.10). Then by our method, we explicitly determined the automorphism groups of some classes of bipartite \( vd \)-graphs,
including the graph $S(q, n, k)$ which is a derived graph from the Grassman graph $G(q, n, k)$ (Theorem 3.16). Also, we provided a sufficient ascertainable condition such that, when a connected non-bipartite $vd$-graph $G$ satisfies this condition, then $G$ is a stable graph (Theorem 3.18). Finally, we showed that the Johnson graph $J(n, k)$ is a stable graph (Theorem 3.21).

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