THE G-STABLE RANK FOR TENSORS

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Abstract. We introduce the $G$-stable rank of a higher order tensors over perfect fields. The $G$-stable rank is related to the Hilbert-Mumford criterion for stability in Geometric Invariant Theory. We will relate the $G$-stable rank to the tensor rank and slice rank. For numerical applications, we express the $G$-stable rank as a solution to an optimization problem. Over the field $F_3$ we discuss an application to the Cap Set Problem.

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1. Introduction

1.1. Ranks of tensors. We will introduce the $G$-stable rank for tensors, describe its properties and relate it to other notions for the rank of a tensor, such as the tensor rank, border rank, slice rank and non-commutative rank. Suppose that $K$ is a field, $V_1, V_2, \ldots, V_d$ are finite dimensional $K$-vector spaces and $V = V_1 \otimes V_2 \otimes \cdots \otimes V_d$ is the tensor product. All tensor products are assumed to be over the field $K$ unless stated otherwise. The definition of tensor rank goes back to Hitchcock [19,20]:

Definition 1.1. The rank $\text{rk}(v)$ of a tensor $v \in V$ is the smallest nonnegative integer $r$ such that we can write $v = \sum_{i=1}^{r} v_{i,1} \otimes v_{i,2} \otimes \cdots \otimes v_{i,d}$ with $v_{i,j} \in V_j$ for all $i$ and $j$.

There are many applications of the tensor rank and the related concept of CP-decomposition (see [2] for a survey). For $d = 2$, tensor rank coincides with matrix rank. Computing the tensor rank is NP-hard [16,17], and tensor rank is ill-behaved. For example, the set $X(\text{rk}, r) \subseteq V$ of all tensors of rank $\leq r$ is not always Zariski closed. The border rank $\text{brk}(v)$ of a tensor $v$ is the smallest positive integer $r$ such that $v \in X(\text{rk}, r)$ (see [6,26]). The slice rank of a tensor was introduced by Terence Tao (see [4,34]).

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Definition 1.2. A non-zero tensor $v \in V$ has slice rank 1 if it is contained in

$$V_1 \otimes \cdots \otimes V_{i-1} \otimes w \otimes V_{i+1} \otimes \cdots \otimes V_d$$

for some $i$ and some $w \in V_i$. The slice rank $\text{srk}(v)$ of an arbitrary tensor $v \in V$ is the smallest nonnegative integer $r$ such that $v$ is the sum of $r$ tensors with slice rank 1.

1.2. The definition of the $G$-stable rank. We will now define the $G$-stable rank. It was noted in [4] that the slice-rank is closely related to the notion of stability in Geometric Invariant Theory (see [28]). The authors also introduce the instability of a tensor and relate it to the slice rank. The instability of a tensor does not behave like a rank function, but it is closely related to the $G$-stable rank. We will define the $G$-stable rank in terms of degenerations and power series. It can also be defined in terms 1-parameter subgroup using the Hilbert-Mumford criterion in Geometric Invariant Theory (see Theorem 2.4). The Hilbert-Mumford criterion is often formulated when working over an algebraically closed field $K$. Kempf showed in [22] that the Hilbert-Mumford criterion still applies when working of a perfect field $K$. For this reason, we will assume that $K$ is a perfect field for the remainder of the paper.

To define the $G$-stable rank, we need to introduce the ring $K[[t]]$ of formal power series in $t$ and its quotient field $K((t))$ of formal Laurent series. The $t$-valuation of a series $a(t) \in K((t))$ is the smallest integer $d$ such that $a(t) = t^db(t)$ with $b(t) \in K[[t]]$. By convention, $\text{val}_t(0) = \infty$. If $W$ is a $K$-vector space and $v(t) \in K((t)) \otimes W$ then we define

$$\text{val}_t(v(t)) = \min\{d \mid v(t) = t^dw(t) \text{ and } w(t) \in K[[t]] \otimes W\}.$$  

We say that $v(t)$ has no poles when $\text{val}_t(v(t)) \geq 0$, which is equivalent to $v(t) \in K[[t]] \otimes W$. In that case we say that $\lim_{t \to 0} v(t)$ exists, and is equal to $v(0) \in W$.

The group $\text{GL}(W,K((t)))$ will denote the group of $K((t))$-linear endomorphisms of the space $K((t)) \otimes_K W$. We may view $\text{GL}(W,K((t)))$ as a subset of $K((t)) \otimes_K \text{End}(W)$. If $W = K^n$ then $K((t)) \otimes_K W \cong K((t))^n$ and we can identify $\text{GL}(W,K((t)))$ with the set of $n \times n$ matrices with entries in the field $K((t))$. If $R \subseteq K((t))$ is a $K$-subalgebra of $K((t))$ (such as $R = K[[t]]$, $R = K[t,t^{-1}]$ or $R = K[t]$), then $\text{GL}(W,R)$ is the intersection of $\text{GL}(W,K((t)))$ with $R \otimes_K \text{End}(W)$ in $K((t)) \otimes_K \text{End}(W)$. Note that the inverse of an element in $\text{GL}(W,R)$ lies in $\text{GL}(W,K((t)))$, but not necessarily in $\text{GL}(W,R)$. If $W = K^n$, then $\text{GL}(W,R)$ is the set of $n \times n$ matrices with entries in $R$ that, viewed as a matrix with entries in $K((t))$, are invertible.

We consider the action of the group $G = \text{GL}(V_1) \times \text{GL}(V_2) \times \cdots \times \text{GL}(V_d)$ on the tensor product space $V = V_1 \otimes V_2 \otimes \cdots \otimes V_d$. For any $K$-subalgebra $R \subseteq K((t))$, we define

$$G(R) = \text{GL}(V_1,R) \times \cdots \times \text{GL}(V_d,R).$$

The group $G(K((t)))$ acts on $K((t)) \otimes V$.

For any weight $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_d) \in \mathbb{R}_{>0}^d$ we will have a notion of $G$-stable rank, but the case $\alpha = (1,1,\ldots,1)$ will be of particular interest. Suppose that $g(t) \in G(K[[t]])$, $v \in V$ and $\text{val}_t(g(t) \cdot v) > 0$. We consider the slope

$$(1) \quad \mu_{\alpha}(g(t),v) = \frac{\sum_{i=1}^d \alpha_i \text{val}_t(\det g_i(t))}{\text{val}_t(g(t) \cdot v)}.$$  

Heuristically, the denominator in the slope measures how fast $g(t) \cdot v$ goes to 0 as $t \to 0$. The numerator measures how fast the eigenvalues of $g_1(t), g_2(t), \ldots, g_d(t)$ go to 0 as $t \to 0$. The
A small slope means that $v$ is very unstable in the sense that $g(t) \cdot v$ goes to 0 quickly, while, on average, the eigenvalues of $g_t(t)$ go to 0 slowly.

**Definition 1.3.** The $G$-stable $\alpha$-rank $\text{rk}^G_\alpha(v)$ of $v$ as the infimum of all $\mu_\alpha(g(t), v)$ where $g(t) \in G(K[[t]])$ and $val_t(g(t) \cdot v) > 0$. If $\alpha = (1, 1, \ldots, 1)$, then we may write $\text{rk}^G$ instead of $\text{rk}^G_\alpha$.

Using a $K$-rational version of the Hilbert-Mumford ([18],[25]) criterion by Kempf [22], we will show that for computing the $G$-stable $\alpha$-rank, one only has to consider $g(t)$ that are 1-parameter subgroups of $G$ without poles (Theorem 2.4). In this context, $g(t) \in G(K[[t]])$ is a 1-parameter subgroup if for every $i$ we can choose a basis of $V_i$ such that the matrix of $g(t)$ is diagonal and each diagonal entry of that matrix is a nonnegative power of $t$.

We denote the standard basis vectors in $K^n$ by $[1],[2],\ldots,[n]$, and we abbreviate a tensor $[\alpha_1] \otimes [\alpha_2] \otimes \cdots \otimes [\alpha_d]$ by $[\alpha_1,\alpha_2,\ldots,\alpha_d]$.

**Example 1.4.** Suppose that $V_1 = V_2 = V_3 = K^2$, and $v = [2,1,1] + [1,2,1] + [1,1,2]$. We take $g(t) = (g_1(t),g_2(t),g_3(t))$ with

$$g_1(t) = g_2(t) = g_3(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.$$  

We have $g(t) \cdot v = t^2 v$, $\det(g_i(t)) = t$, and

$$\mu(g(t), v) = \mu_{(1,1,1)}(g(t), v) = \frac{\text{val}_t(\det(g_1(t))) + \text{val}_t(\det(g_2(t))) + \text{val}_t(\det(g_3(t)))}{\text{val}_t(g(t) \cdot v)} = \frac{1+1+1}{2} = \frac{3}{2}.$$  

This shows that $\text{rk}^G(v) \leq \frac{3}{2}$. One can show that $\text{rk}^G(v) = \frac{3}{2}$ (see Example 1.5 and Example 4.5).

1.3. **Properties of the $G$-stable rank.** If $v$ is a rank 1 tensor, then we have $\text{rk}^G_\alpha(v) = \min\{\alpha_1,\ldots,\alpha_d\}$ and $\text{rk}^G(v) = 1$ (Lemma 3.1). The $G$-stable rank is related to other notions of rank. We have (see Corollary 3.7 and Proposition 4.9)

$$\frac{2\text{srk}(v)}{d} \leq \text{rk}^G(v) \leq \text{srk}(v) \leq \text{brk}(v) \leq \text{rk}(v).$$

This implies that for $d = 2$, the $G$-stable rank, the slice rank and the matrix rank coincide.

The tensor rank depends on the field one is working over. For example, the tensor $[1,1,1] - [1,2,2] - [2,1,2] - [2,2,1]$ has rank 3 as a tensor in $\mathbb{R}^{2 \times 2 \times 2}$ but rank 2 when viewed as a tensor in $\mathbb{C}^{2 \times 2 \times 2}$. Although it is not clear from the definition, the $G$-stable rank does not change when passing to a field extension of $K$ (see Theorem 2.5).

Another nice property of the $G$-stable rank is that the border rank phenomenon does not happen and the set $X(\text{rk}^G_\alpha, r)$ of all tensors $v$ with $\text{rk}^G_\alpha(v) \leq r$ is Zariski closed (Theorem 2.11). Tao proved a similar result for the slice rank [33], and this implies that $\text{srk}(v) \leq \text{brk}(v)$ for all tensors $v$.

Like other rank notions, the $G$-stable rank satisfies the triangle inequality: $\text{rk}^G_\alpha(v + w) \leq \text{rk}^G_\alpha(v) + \text{rk}^G_\alpha(w)$ (see Proposition 3.9). If $v \in V_1 \otimes V_2 \otimes \cdots \otimes V_d$ and $w \in W_1 \otimes W_2 \otimes \cdots \otimes W_d$ then the direct sum of $v$ and $w$, viewed as

$$\begin{pmatrix} v \\ w \end{pmatrix} \in \begin{array}{c} V_1 \otimes V_2 \otimes \cdots \otimes V_d \\ W_1 \otimes W_2 \otimes \cdots \otimes W_d \end{array} \subseteq V \oplus W := \begin{array}{c} V_1 \\ \oplus \\ W_1 \end{array} \otimes \begin{array}{c} V_2 \\ \oplus \\ W_2 \end{array} \otimes \cdots \otimes \begin{array}{c} V_d \\ \oplus \\ W_d \end{array}$$
will be denoted by \( v \square w \). (We will use the notation \( v \oplus w \) and \( V \oplus W \) rather than the more common notation \( v \times w \) and \( V \times W \) to emphasize that this direct sum is a “vertical” operation, i.e., the sum \( V_i \oplus W_i \) is taken within each tensor factor.) The \( G \)-stable rank is additive (Proposition 3.8): \( \text{rk}_G^G(v \square w) = \text{rk}_G^G(v) + \text{rk}_G^G(w) \). In particular, if

\[
v = [1, 1, \ldots, 1] + [2, 2, \ldots, 2] + \cdots + [r, r, \ldots, r] =
\left[1, 1, \ldots, 1 \right] \boxplus \left[1, 1, \ldots, 1 \right] \boxplus \cdots \boxplus \left[1, 1, \ldots, 1 \right] \in K^r \boxtimes K^r \boxtimes \cdots \boxtimes K^r,
\]

then \( \text{rk}_G^G(v) = r \text{rk}_G^G([1, 1, \ldots, 1]) = r \min \{ \alpha_1, \ldots, \alpha_d \} \) and \( \text{rk}_G^G(v) = r \). Strassen conjectured in [32] that tensor rank is additive when \( K \) is infinite, but Shitov recently gave a counterexample to this long standing conjecture (see [29]).

If \( v \in V_1 \otimes V_2 \otimes \cdots \otimes V_d \) and \( w \in W_1 \otimes W_2 \otimes \cdots \otimes W_e \), then we can form the “horizontal” tensor product \( v \otimes w \in V_1 \otimes \cdots \otimes V_d \otimes W_1 \otimes \cdots \otimes W_e \). It is clear that \( \text{rk}(v \otimes w) \leq \text{rk}(v) \text{rk}(w) \). It was recently shown in [38] that we do not always have equality. The \( G \)-stable rank behaves quite differently for the horizontal tensor product. We have \( \text{rk}_{G,\beta}^G(v \otimes w) = \min \{ \text{rk}_{G}^G(v), \text{rk}_{G}^G(w) \} \) (see Proposition 3.4). If \( d = e \) then there is another way of forming a tensor product. The tensor product \( v \boxtimes w \) viewed as

\[
v \boxtimes w \subseteq \left( V_1 \right) \otimes \left( W_1 \right) \otimes \cdots \otimes \left( V_d \right) \otimes \left( W_d \right)
\]

will be denoted by \( v \otimes w \). We will refer to this operation as a vertical tensor product or a Kronecker tensor product. It is clear that \( \text{rk}(v \otimes w) \leq \text{rk}(v) \text{rk}(w) \). It has long been known that \( \text{rk}(v \otimes w) \) can be smaller than \( \text{rk}(v) \text{rk}(w) \). For example, if \( v_1 = [1, 1, 1] + [2, 2, 1], v_2 = [1, 1, 1] + [2, 1, 2] \) and \( v_3 = [1, 1, 1] + [2, 2, 1] \) then \( v_1 \otimes v_2 \otimes v_3 \) is the matrix multiplication tensor for \( 2 \times 2 \) matrices which has rank \( 7 \) (see Proposition 3.4), so \( 7 = \text{rk}(v_1 \otimes v_2 \otimes v_3) < \text{rk}(v_1) \text{rk}(v_2) \text{rk}(v_3) = 2^3 \). If \( K \) has characteristic \( 0 \), then we have \( \text{rk}_{G,\beta}^G(v \boxtimes w) = \text{rk}_{G}^G(v) \text{rk}_{G}^G(w) \) (Theorem 5.4). We conjecture that this inequality is also true when \( K \) is a perfect field of positive characteristic. The slice rank does not behave as nicely with respect to vertical tensor product and \( \text{srk}(v \boxtimes w) \) could be larger or smaller than \( \text{srk}(v) \text{srk}(w) \) (see [9] Example 5.2).

1.4. \( G \)-stable rank for complex tensors. If \( K = \mathbb{C} \), then the \( G \)-stable rank can be computed in a different way. For a finite dimensional complex Hilbert space, we will denote the Hermian form by \( \langle \cdot, \cdot \rangle \) and the \( \ell_2 \) norm (or Frobenius norm) by \( \| \cdot \| = \sqrt{\langle v, v \rangle} \). Suppose that \( V_1, V_2, \ldots, V_d \) are finite dimensional Hilbert spaces, which makes \( V \) into a Hilbert space. If \( A \) is a linear map between finite dimensional Hilbert spaces, then its spectral norm \( \| A \|_\sigma \) is the operator norm \( \| A \|_\sigma = \max_{v \neq 0} \frac{\| Av \|}{\| v \|} \), which is also the largest singular value of \( A \).

For a tensor \( v \in V \), let \( \Phi_i(v) : (V_1 \otimes \cdots \otimes \hat{V}_i \otimes \cdots \otimes V_d)^* \rightarrow V_i \) be the \( i \)-th flattening. Then the \( G \)-stable \( \alpha \)-rank of a tensor \( v \in V \) is equal to

\[
(2) \quad \text{rk}_G^G(v) = \sup_{g \in G} \sum_{i} \alpha_i \| g \cdot v \|_G^2
\]

(see Theorem 5.2).
Example 1.5. Consider again the example \( v = [2, 1, 1] + [1, 2, 1] + [1, 1, 2] \in K^{2 \times 2 \times 2} \) as in Example 1.3 but now we will work over \( K = \mathbb{C} \). We have \( \|v\| = \sqrt{3} \). The first flattening of \( v \) is equal to

\[
\Phi_1(v) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

which has singular values 1 and \( \sqrt{2} \). So \( \|\Phi_1(v)\|_\sigma = \sqrt{2} \). By symmetry, we also have \( \|\Phi_2(v)\|_\sigma = \|\Phi_3(v)\|_\sigma = \sqrt{2} \). It follows that

\[
\text{rk}^G(v) = \sup_{g \in G} \min_i \frac{\|g \cdot v\|^2}{\|\Phi_i(g \cdot v)\|^2_\sigma} \geq \min_i \frac{\|v\|^2}{\|\Phi_i(v)\|^2_\sigma} = \frac{3}{2}.
\]

1.5. The Cap Set Problem. We say that a subset \( S \) of an abelian group \( A \) does not contain an arithmetic progression (of length 3) if there are no distinct elements \( x, y, z \in S \) with \( x + z = 2y \). For an abelian group \( A \), let \( r_3(A) \) be the largest cardinality of a subset \( S \subseteq A \) without an arithmetic progression. Finding upper and lower bounds for \( r_3(A) \) has been studied extensively in number theory. For the group \( A = (\mathbb{Z}/3\mathbb{Z})^n \cong \mathbb{F}_3^n \) this is known as the Cap Set Problem. Brown and Buhler [7] showed that \( r_3(\mathbb{F}_3^n) = o(3^n) \) and this was later improved to \( r_3(\mathbb{F}_3^n) = O(3^n/n) \) by Meshulam [27] and to \( o(3^n/n^{1+\varepsilon}) \) by Bateman and Katz [3]. Using the polynomial method of Croot, Lev and Pach [11], who showed that \( r_3((\mathbb{Z}/4\mathbb{Z})^n) = o(c^n) \) for some \( c < 4 \), Ellenberg and Gijswijt showed in [13] that \( r_3(\mathbb{F}_3^n) \leq 3\theta^n = o(2.756^n) \), where \( \theta < 2.756 \). We also have a lower bound \( r_3(\mathbb{F}_3^n) = \omega(2.21^n) \) by Edel. The bound (and the proof) of Ellenberg and Gijswijt is also valid for tri-colored sum-free sets for which an asymptotic lower bound \( \omega(\theta^n) \) was given by Kleinberg, Sawin and Speyer [24]. So for tri-colored sum-free sets, the upper and lower bound have the same exponential growth.

Tao noted that the Ellenberg-Gijswijt proof can be nicely presented using the concept of slice rank. A key idea is to prove the inequality \( r_3(\mathbb{F}_3^n) \leq \text{srk}(u^{\mathbb{F}_3^n}) \) where

\[
u = \sum_{i,j,k \in \mathbb{Z}/3\mathbb{Z}, \ i+j+k=0} [i, j, k] \in \mathbb{F}_3^3 \times 3 \times 3
\]

and to combine this with asymptotic estimates for the slice rank. We will show that \( r_3(\mathbb{F}_3^n) \leq \text{rk}^G(u^{\mathbb{F}_3^n}) \leq \text{srk}(u^{\mathbb{F}_3^n}) \). Using the \( G \)-stable rank, we get better upper bounds for the cardinality of a cap set (or a tri-colored sum-free set). Below is a table of the upper bounds we get for \( n \leq 20 \).

| \( n \) | upper bound | \( n \) | upper bound | \( n \) | upper bound | \( n \) | upper bound |
| --- | --- | --- | --- | --- | --- | --- | --- |
| 1 | 2 | 6 | 274 | 11 | 37477 | 16 | 5235597 |
| 2 | 6 | 7 | 722 | 12 | 100296 | 17 | 14316784 |
| 3 | 15 | 8 | 1957 | 13 | 266997 | 18 | 38685141 |
| 4 | 39 | 9 | 5193 | 14 | 728661 | 19 | 103504935 |
| 5 | 105 | 10 | 13770 | 15 | 1961103 | 20 | 283466139 |

2. The \( G \)-stable rank and the Hilbert-Mumford criterion

2.1. The Hilbert-Mumford criterion. We will discuss the \( K \)-rational version of the Hilbert-Mumford criterion by Kempf [22]. We remind the reader that the base field \( K \) is assumed to be perfect. Suppose that \( G \) is a connected reductive algebraic group over a
field $K$, $X$ is a separated $K$-scheme of finite type and $G \times X \to X$ is a $G$-action that is also a morphism of schemes over $K$. The multiplicative group is defined as $\mathbb{G}_m = \text{Spec } K[t, t^{-1}]$. A 1-parameter subgroup of $G$ is a homomorphism $\lambda : \mathbb{G}_m \to G$ of algebraic groups. We say that this 1-parameter subgroup of $G$ is $K$-rational if the homomorphism is a morphism of algebraic varieties defined over $K$. In the case where $K$ is finite, we caution the reader that the set $G(K)$ of $K$ rational points in $G$ is finite and may not be Zariski dense in the algebraic group $G$. If $x \in X(K)$ is a $K$-rational point of $X$, then $G \cdot x$ denotes a subscheme of $X$ which is not necessarily Zariski closed (even if $G(K)$ is finite). The Zariski closure $\overline{G \cdot x}$ is a closed subscheme of $X$.

**Theorem 2.1** ([22, Corollary 4.3]). Suppose that $x \in X(K)$ is a $K$-rational point, $S \subseteq X$ is a $G$-invariant closed subscheme of $X$ such that $\overline{G \cdot x} \cap S \neq \emptyset$, Then there exists a $K$-rational 1-parameter subgroup $\lambda : \mathbb{G}_m \to G$ such that $\lim_{t \to 0} \lambda(t) \cdot x = y$ for some $y \in S(K)$.

In our situation, $X = V$ is a $K$-vector space which is a representation of $G$, and $S = \{0\}$. A vector $v \in V$ is called $G$-semi-stable if $\overline{G \cdot v}$ does not contain 0. Now Theorem 2.1 implies:

**Corollary 2.2.** If $G$ is a connected reductive algebraic group, $v \in V$ and $0 \in \overline{G \cdot v}$ then there exists a $K$-rational 1-parameter subgroup $\lambda : \mathbb{G}_m \to G$ such that $\lim_{t \to 0} \lambda(t) \cdot v = 0$.

A 1-parameter subgroup of $\text{GL}_n$ is of the form

$$\lambda(t) = C \begin{pmatrix} t^{x(1)} & 0 & \cdots & 0 \\ 0 & t^{x(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^{x(n)} \end{pmatrix} C^{-1}$$

with $C \in \text{GL}_n$ and $x(1), x(2), \ldots, x(n) \in \mathbb{Z}$. In particular, we can view $\lambda$ as an element of $\text{GL}_n(K[t, t^{-1}])$ where $K[t, t^{-1}] \subseteq K((t))$ is the ring of Laurent polynomials. If $v = (v_1 \ v_2 \ \cdots \ v_n)^t \in K^n$ then $\lim_{t \to 0} \lambda(t) \cdot v = 0$ if for all $i$, we have $v_i = 0$ or $x(i) > 0$. We will take $V = V_1 \otimes V_2 \otimes \cdots \otimes V_d$ and $G = \text{GL}(V_1) \times \text{GL}(V_2) \times \cdots \times \text{GL}(V_d)$. A 1-parameter subgroup of $G$ is of the form $(\lambda_1(t), \lambda_2(t), \ldots, \lambda_d(t))$ where $\lambda_i(t) : \mathbb{G}_m \to \text{GL}(V_i)$ is a 1-parameter subgroup for all $i$.

For an integer vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Z}^d$ we define a homomorphism of algebraic groups $\det^\alpha : G \to \mathbb{G}_m$ by $(A_1, \ldots, A_d) \mapsto \prod_{i=1}^d \det(A_i)^{\alpha_i}$. This homomorphism corresponds to a 1-dimensional representation of $G$, which we will also denote by $\det^\alpha$. We will now relate the $G$-stable rank to semi-stability in Geometric Invariant Theory.

**Proposition 2.3.** Suppose that $\beta \in \mathbb{Q}_{>0}^d$, $p$ is a nonnegative integer and $q$ is a positive integer with $q \beta \in \mathbb{Z}^n$. We define a representation $W$ by

$$W = (V^{\otimes p} \otimes \det^{-q \beta}) \oplus V_1^{n_1} \oplus V_2^{n_2} \oplus \cdots \oplus V_d^{n_d}.$$ 

and choose $u_i \in V_i^{n_i} \cong K^{n_i \times n_i}$ of maximal rank $n_i$ for every $i$. Then we have $\text{rk}^G_{\beta}(v) \geq \frac{p}{q}$ if and only if $w = (v^{\otimes p} \otimes 1, u_1, \ldots, u_d)$ is $G$-semi-stable.

**Proof.** Suppose that $\text{rk}^G_{\beta}(v) < \frac{p}{q}$. Then there exists $g(t) = (g_1(t), \ldots, g_d(t)) \in G(K[[t]])$ with

$$\text{val}_1(g(t) \cdot (v^{\otimes p} \otimes 1)) = p \text{val}_1(g(t) \cdot v) - \sum_{i=1}^d q \beta_i \text{val}_1(g_i(t)) > 0.$$
The limit \( \lim_{t \to 0} g(t) \cdot w = (0, g(0) \cdot u) = (0, g(0) \cdot u_1, \ldots, g(0) \cdot u_d) \) lies in the closure of the orbit \( G \cdot w \). Since 0 lies in the orbit closure of \((0, g(0) \cdot u)\), it also lies in the orbit closure of \(w\). We conclude that \(w\) is not \(G\)-semistable.

Now suppose that \(w\) is not \(G\)-semistable. By the Hilbert-Mumford criterion, there exists a 1-parameter subgroup \(\lambda(t) = (\lambda_1(t), \ldots, \lambda_d(t)) \in G(K[t, t^{-1}])\) of \(G\) such that \(\lim_{t \to 0} \lambda(t) \cdot w = 0\). This implies that \(\lim_{t \to 0} \lambda_i(t) \cdot u_i = 0\). Since \(u_i\) has maximal rank, we get \(\lim_{t \to 0} \lambda_i(t) = 0\) and \(\lambda_i(t) \in \text{GL}(V_i, K[t])\). So we have \(\lambda(t) \in G(K[t]) \subseteq G(K[[t]])\). We also get

\[
0 < \text{val}_t(\lambda(t) \cdot (v^\otimes p \otimes 1)) = p \text{val}_t(\lambda(t) \cdot v) - \sum_{i=1}^d q_i \text{val}_t(\lambda_i(t))
\]

and therefore

\[
\mu_\beta^G(v) = \frac{\sum_{i=1}^d \beta_i \text{val}_t(\lambda_i(t))}{\text{val}_t(\lambda(t) \cdot v)} < \frac{p}{q}.
\]

We conclude that \(\text{rk}_\beta^G(v) < \frac{p}{q}\).

\[\square\]

**Theorem 2.4.** If \(\alpha \in \mathbb{R}^d_{>0}\), then the \(G\)-stable rank \(\text{rk}_\alpha^G(v)\) is the infimum of \(\mu_\alpha(\lambda(t), v)\) where \(\lambda(t) \in G(K[t])\) is a 1-parameter subgroup of \(G\) and \(\text{val}_t(\lambda(t) \cdot v) > 0\).

**Proof.** Assume that \(\text{rk}_\beta^G(v) < r\) for some rational number \(r\). There exists a \(\beta \in \mathbb{Q}^d\) with \(\beta - \alpha \in \mathbb{R}^d_{>0}\) and \(\text{rk}_\beta^G(v) < r\). We can write \(r = \frac{p}{q}\) where \(p\) and \(q\) are positive integers such that \(q \beta \in \mathbb{Z}^d\). By Proposition 2.3, \(w\) is not \(G\)-semistable and from the proof of Proposition 2.3, follow that there exists a 1-parameter subgroup \(\lambda(t) \in G(K[t])\) such that \(\mu_\alpha(\lambda(t), v) \leq \mu_\beta(\lambda(t), v) < r\). This shows that even if \(\lambda(t) \in G(K[t])\) is a 1-parameter subgroup of \(G\), \(\text{rk}_\alpha^G(v)\) can get arbitrarily close to \(\text{rk}_\beta^G(v)\).

\[\square\]

2.2. The relation between \(G\)-stable rank and SL-stability. First we prove that the \(G\)-stable rank does not change when we extend the field.

**Theorem 2.5.** Suppose that \(v \in V = V_1 \otimes_K V_2 \otimes_K \cdots \otimes_K V_d\) where \(V_1, V_2, \ldots, V_d\) are finite dimensional \(K\)-vector spaces, and \(\pi = 1 \otimes v \in \overline{V} = L \otimes_K V \cong \overline{V}_1 \otimes_L \overline{V}_2 \otimes_L \cdots \otimes_L \overline{V}_d\) with \(\overline{V}_i = L \otimes_K V_i\) for all \(i\). Then we have \(\text{rk}_\alpha^G(v) = \text{rk}_\alpha^G(\pi)\). In other words, the \(G\)-stable rank does not change after base field extension.

**Proof.** If \(\beta \in \mathbb{Q}^d\), then we can follow the set up in Proposition 2.3, where \(p, q \in \mathbb{Z}\), \(p > 0\) and \(q \beta \in \mathbb{Z}^d\). We choose \(u_i \in V_i^{n_i}\) invertible for all \(i\), and define

\[
w = (v^\otimes p \otimes 1, u_1, \ldots, u_d) \in W = (V^\otimes p \otimes_K \det^{-q \beta}) \oplus V_1^{n_1} \oplus V_2^{n_2} \oplus \cdots \oplus V_d^{n_d}.
\]

Using the base field extension, we get

\[
\overline{w} = (\overline{v} \otimes 1, \overline{u}_1, \ldots, \overline{u}_d) \in L \otimes_K \overline{W} = (\overline{V}^\otimes p \otimes_L \det^{-q \beta}) \oplus \overline{V}_1^{n_1} \oplus \overline{V}_2^{n_2} \oplus \cdots \oplus \overline{V}_d^{n_d}.
\]

Now \(G\)-semistability does not change after base field extension. So \(w\) is \(G\)-semistable if and only if \(\overline{w}\) is \(G\)-semistable. We have

\[
\text{rk}_\beta^G(w) \geq \frac{p}{q} \iff w \text{ is } G\text{-semistable} \iff \overline{w} \text{ is } G\text{-semistable} \iff \text{rk}_\beta^G(\overline{w}) \geq \frac{p}{q}.
\]

This proves that \(\text{rk}_\beta^G(w) = \text{rk}_\beta^G(\overline{w})\). Since \(\text{rk}_\beta^G(w)\) is the supremum of \(\text{rk}_\beta^G(w)\) over all \(\beta \in \mathbb{Q}^d\) with \(\beta \leq \alpha\), we also get \(\text{rk}_\alpha^G(w) = \text{rk}_\alpha^G(\overline{w})\) for all \(\alpha \in \mathbb{R}^d_{>0}\).
Proposition 2.6. Suppose that $\alpha = (\frac{1}{n_1}, \frac{1}{n_2}, \ldots, \frac{1}{n_d})$ where $n_i = \dim V_i$. For $v \in V = V_1 \otimes V_2 \otimes \cdots \otimes V_d$ we have $\rk(v) \leq 1$. Moreover, $\rk(v) = 1$ if and only if $v$ is semi-stable with respect to the group $H = \text{SL}(V_1) \times \text{SL}(V_2) \times \cdots \times \text{SL}(V_d)$.

Proof. The inequality $\rk(v) \leq 1$ is obvious. Suppose that $v \in V$ is not $H$-semi-stable. Then there exists a 1-parameter subgroup $(\lambda(t)) = (\lambda_1(t), \ldots, \lambda_d(t)) : \mathbb{G}_m \to H$ with $\lim_{t \to 0} \lambda(t) \cdot v = 0$. We can choose $c_1, c_2, \ldots, c_d$ such that $\lambda'(t) = (t^{c_1} \lambda_1(t), \ldots, t^{c_d} \lambda_d(t)) \in G(K[t])$. Note that $\det(t^{c_i} \lambda_i(t)) = \det(t^{c_i} I_{n_i}) \det(\lambda_i(t)) = t^{c_i n_i}$. Now we have have $\text{val}_t(\lambda'(t) \cdot v) = s + c_1 + c_2 + \cdots + c_d$ and

$$\mu(\lambda'(t), v) = \sum_{i=1}^{d} \frac{1}{n_i} \text{val}_t(\det(t^{c_i} \lambda_i(t))) = \frac{\sum_{i=1}^{d} c_i}{s + \sum_{i=1}^{d} c_i} < 1.$$

This proves that $\rk(v) < 1$.

Conversely, suppose that $\rk(v) < 1$. Choose a polynomial 1-parameter subgroup of $G$ such that $\text{val}_t(\lambda(t) \cdot v) = s > 0$ and $\mu_\alpha(\lambda(t), v) < 1$. Let $c_i = \text{val}_t(\det \lambda_i(t))$. Then we have $\mu_\alpha(\lambda(t), v) = \sum_{i=1}^{d} \frac{c_i}{n_i} < s$. After replacing $t$ by $t^k$ for some positive integer $k$ we may assume that $\frac{c_i}{n_i} \in \mathbb{Z}$ for all $i$. Let $\lambda'(t) = (t^{-c_1/n_1} \lambda_1(t), t^{-c_2/n_2} \lambda_2(t), \ldots, t^{-c_d/n_d} \lambda_d(t))$. Then $\lambda'(t)$ is a 1-parameter subgroup of $H$ and $\text{val}_t(\lambda'(t) \cdot v) = s - \sum_{i=1}^{d} \frac{c_i}{n_i} > 0$, so $\lim_{t \to 0} \lambda'(t) \cdot v = 0$. This shows that $v$ is $H$-unstable. \hfill \Box

2.3. The G-stable rank and the non-commutative rank. The non-commutative rank is defined as the rank of $A(t) = t_1 A_1 + t_2 A_2 + \cdots + t_m A_m$ where $t_1, t_2, \ldots, t_m$ are variables in the free skew field $R = K \langle t_1, t_2, \ldots, t_m \rangle$ and $A(t)$ is viewed as a $p \times q$ matrix with entries in $R$ (see [10,14] for more on free skew fields). We will use the following equivalent definition (see [14]):

Definition 2.7. Suppose that $A_1, A_2, \ldots, A_m$ are $p \times q$ matrices. Then the the non-commutative rank $\text{ncrk}(A)$ of $A = (A_1, \ldots, A_m)$ is equal to the maximal value of

$$q + \dim \sum_{i=1}^{m} A_i(W) - \dim W$$

over all subspaces $W \subseteq K^q$.

It was shown in [21] that the non-commutative rank of $A$ is also equal to maximum of

$$\frac{\rk(\sum_{i=1}^{m} T_i \boxtimes A_i)}{d}$$

where $d$ is a positive integer, $T_1, T_2, \ldots, T_m$ are $d \times d$ matrices, and $\boxtimes$ is the Kronecker product of two matrices (so $T_i \boxtimes A_i$ is a $dp \times dq$-matrix).

The non-commutative rank relates to stability. If $A$ is an $m$-tuple of $n \times n$ matrices (i.e., $p = q = n$) then $\text{ncrk}(A) = n$ if and only if $A$ is semi-stable with respect to the simultaneous left-right action of $\text{SL}_n \times \text{SL}_n$ on $m$-tuples of matrices (see [21]).

We can relate the non-commutative and $G$-stable rank as follows. First, we will view the $m$-tuple $A = (A_1, A_2, \ldots, A_m)$ as a tensor. Using a linear isomorphism $K^p \otimes K^q \cong K^{pq}$, we can view $A_1, A_2, \ldots, A_m$ as tensors in $K^p \otimes K^q$. The $m$-tuple $A = (A_1, A_2, \ldots, A_m)$ corresponds to a tensor $T_A = \sum_{i=1}^{m} A_i \otimes [i] \in K^p \otimes K^q \otimes K^m$. 


Lemma 2.8. The non-commutative rank is the smallest value of $r + s$ for which there exist linearly independent vectors $v_1, \ldots, v_r \in K^p$ and linearly independent vectors $w_1, \ldots, w_s \in K^q$ with

$$T_A \in \sum_{i=1}^{r} v_i \otimes K^q \otimes K^m + \sum_{j=1}^{s} K^p \otimes w_j \otimes K^m.$$  

Proof. If (3) holds, then take $W$ to be the $(q - s)$-dimensional space perpendicular to the vectors $w_1, w_2, \ldots, w_s$. The space $A_i(W)$ is contained in the span of $v_1, v_2, \ldots, v_r$. So the non-commutative rank is at most $q + r - (q - s) = r + s$.

We show that $r + s$ can be equal to $\text{ncrk}(A)$. Suppose that $k = \text{ncrk}(A)$. For some $s$ there exists an subspace $V \subseteq K^p$ with $k = q + \dim V - \dim W$, where $V = \sum_{i=1}^{m} A_i(W)$. Choose a basis $w_1, w_2, \ldots, w_s$ of the space orthogonal to $W$. Then we have $s = q - \dim W$. Also choose a basis $v_1, v_2, \ldots, v_r$ of $V$. Now (3) holds and $r + s = q - \dim W + \dim V = k$. □

The following proposition shows that the non-commutative rank can be seen as a special case of the $G$-stable rank.

Proposition 2.9. For $\alpha = (1, 1, \ell)$ and $\ell \geq \min\{p, q\}$ we have $\text{ncrk}(A) = \text{rk}_{\alpha}^G(T_A)$.

Proof. Let $k = \text{ncrk}(A)$. Then we have

$$T_A \in \sum_{i=1}^{r} v_i \otimes K^q \otimes K^m + \sum_{j=1}^{s} K^p \otimes w_j \otimes K^m.$$  

for some $r$ and $s$ with $r + s = k$ and vectors $v_1, \ldots, v_r, w_1, \ldots, w_s$. We extend $v_1, \ldots, v_r$ to a basis $v_1, \ldots, v_r$ and extend $w_1, \ldots, w_s$ to a basis $w_1, \ldots, w_s$. We define a 1-parameter subgroup $\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t))$ in $G = \text{GL}_p \times \text{GL}_q \times \text{GL}_m$ by $\lambda_i(t) \cdot v_i = t v_i$ for $i = 1, 2, \ldots, r$, $\lambda_1(t) \cdot v_i = v_i$ for $i = r + 1, r + 2, \ldots, p$, $\lambda_2(t) \cdot w_j = t w_j$ for $j = 1, 2, \ldots, s$, and $\lambda_3(t)$ is just the identity. Then we have $\text{val}_\ell(\lambda(t) \cdot T_A) = 1$, $\det(\lambda_1(t)) = t^r$, $\det(\lambda_2(t)) = t^s$, $\det(\lambda_3(t)) = 1$ and

$$\text{rk}_{\alpha}^G(T_A) \leq \mu_\alpha(\lambda(t), T_A) = \frac{1 \cdot r + 1 \cdot s + \ell \cdot 0}{1} = k = \text{ncrk}(A).$$

On the other hand, let $h = \text{rk}_{\alpha}^G(T_A)$ and suppose that $\lambda(t) \in G$ is a 1-parameter subgroup with $\mu_\alpha(\lambda(t), T_A) = h$. If $h = \min\{p, q\}$ then clearly $\text{ncrk}(A) \leq h$, so we assume that $h < \min\{p, q\}$. Suppose $\ell \geq p$ (the case $\ell \geq q$ will go similarly). If $\det(\lambda_3(t)) = t^e$ then we can define another 1-parameter subgroup $\rho(t) = (\rho_1(t), \rho_2(t), \rho_3(t))$ by $\rho_1(t) = t^e \lambda_1(t)$, $\rho_2(t) = \lambda_2(t)$ and $\rho_3(t) = I$. Then $\text{val}_\ell(\rho(t) \cdot T_A) \geq \text{val}_\ell(\lambda(t) \cdot T_A)$, and we get

$$\mu_\alpha(\rho(t), T_A) \leq \frac{\text{val}_\ell(\det \rho_1(t)) + \text{val}_\ell(\det \rho_2(t)) + \ell \text{val}_\ell(\det \rho_3(t))}{\text{val}_\ell(\lambda(t) \cdot T_A)} \leq \frac{p e + \text{val}_\ell(\det \lambda_1(t)) + \text{val}_\ell(\det \lambda_2(t))}{\text{val}_\ell(\lambda(t) \cdot T_A)} \leq \frac{\text{val}_\ell(\det \lambda_1(t)) + \text{val}_\ell(\det \lambda_2(t)) + \ell \text{val}_\ell(\det \lambda_3(t))}{\text{val}_\ell(\lambda(t) \cdot T_A)} = \mu_\alpha(\lambda(t), T_A)$$

because $\ell \geq p$ and $\text{val}_\ell(\det \lambda_3(t)) = e$. We can replace $\lambda(t)$ by $\rho(t)$ and without loss of generality we may assume that $\lambda_3(t) = I$.  

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Let $d := \text{val}_t(\lambda(t) \cdot T_A)$. After base changes, we have

$$
\lambda(t) = \begin{pmatrix}
  t^{x(1)} & \cdots & \cdots & t^{x(p)} \\
  \cdots & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
$$

and $\rho(t) = \begin{pmatrix}
  t^{y(1)} & \cdots & \cdots & t^{y(1)} \\
  \cdots & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots \\
  \cdots & \cdots & \cdots & \cdots
\end{pmatrix}$

From

$$
\frac{\sum_{i=1}^{h+1} (x(i) + y(h + 2 - i))}{d} \leq \frac{\sum_{i=1}^{p} x(i) + \sum_{j=1}^{q} y(j)}{d} = \mu_\alpha(\lambda(t), T_A) = h
$$

follows that $x(r + 1) + y(s + 1) \leq \frac{hd}{k+1} < hd$ for some $r, s$ with $r + s = h$. If a basis vector $[i, j, k] = [i] \otimes [j] \otimes [k]$ appears in $T_A$ then $x(i) + y(j) \geq dk$ and therefore $i \leq r$ or $j \leq s$. This means that

$$
T_A \in \sum_{i=1}^{r} [i] \otimes K^q \otimes K^m + \sum_{j=1}^{s} K^p \otimes [j] \otimes K^m
$$

and $\text{ncrk}(T_A) \leq r + s = h = \text{rk}_G(T_A)$.

2.4. Semi-continuity of the $G$-stable rank. We will show that the $G$-stable rank is semi-continuous, which means that for every $r$, the set of all tensors with $G$-stable rank $\leq r$ is Zariski closed.

Let us for the moment fix a 1-parameter subgroup $\lambda(t)$ of $G$. We can choose bases in the vector spaces $V_i$ for $i = 1, 2, \ldots, d$ such that the matrix of $\lambda_i(t)$ is diagonal, with diagonal entries $t^{x(i,1)}, t^{x(i,2)}, \ldots, t^{x(i,n_i)}$ where $x(i,1) \geq x(i,2) \geq \cdots \geq x(i,n_i) \geq 0$. Define

$$
Z = \{ v \in V \mid \mu_\alpha(\lambda(t), v) < r \}.
$$

The space $Z$ is spanned by all basis vectors $[i_1, i_2, \ldots, i_d] \in V$ with

$$
\sum_{i=1}^{d} \alpha_i \sum_{j=1}^{n_i} x(i, j) < r(x(1, i_1) + x(2, i_2) + \cdots + x(d, i_d)).
$$

Let $B = B_{n_1} \times B_{n_2} \times \cdots \times B_{n_d} \subseteq G$ where $B_k \subseteq \text{GL}_k$ is the Borel group of upper triangular invertible matrices. If $[i_1, i_2, \ldots, i_d]$ lies in $Z$, and $j_k \leq i_k$ for all $k$, then $[j_1, j_2, \ldots, j_d]$ lies in $Z$. This implies that $Z$ is stable under the action of $B$.

Lemma 2.10. The set $G \cdot Z = \bigcup_{g \in G} g \cdot Z$ is Zariski closed.

Proof. Consider the Zariski closed subset $S \subseteq G/B \times V$ defined by

$$
S = \{ (gB, v) \mid g^{-1} \cdot v \in Z \}
$$

and let $\pi : G/B \times V \to V$ be the projection onto $V$. The flag variety $G/B$ is projective, so $\pi$ is a projective morphism which maps closed sets to closed sets. In particular, $G \cdot Z = \pi(S)$ is Zariski closed.

Theorem 2.11. For any weight $\alpha \in \mathbb{R}_{\geq 0}$ and $r \in \mathbb{R}$ the sets $X^\circ(\text{rk}_G^\alpha, r) = \{ v \in V \mid \text{rk}_G^\alpha(v) < r \}$ and $X(\text{rk}_G^\alpha, r) = \{ v \in V \mid \text{rk}_G^\alpha(v) \leq r \}$ are finite unions of sets of the form $G \cdot Z$ where $Z$ is a Borel-fixed subspace. In particular, these sets are Zariski closed.
Proof. If $\text{rk}^G_{\alpha}(v) < r$, then there exists a 1-parameter subgroup $\lambda(t)$ of $G$ such that $\mu_{\alpha}(\lambda(t), v) < r$. If $Z = \{w \in V \mid \mu_{\alpha}(\lambda(t), w) < r\}$ then $X^0(\text{rk}^G_{\alpha}, r)$ contains $Z$ and $G \cdot Z$. Since there are only finite many Borel stable subspaces of $V$, we see that $X^0(\text{rk}^G_{\alpha}, r)$ must be a finite union $G \cdot Z_1 \cup G \cdot Z_2 \cup \cdots \cup G \cdot Z_s$ where $Z_1, Z_2, \ldots, Z_s$ are Borel stable subspaces. Since each $G \cdot Z_i$ is closed, $X^0(\text{rk}^G_{\alpha}, r)$ is closed. Because there are only finitely many Borel stable subspaces, there are only finitely many possibilities for $X^0(\text{rk}^G_{\alpha}, s)$ where $s \in \mathbb{R}_{>0}$. There exists an $\varepsilon > 0$ such that $X^0(\text{rk}^G_{\alpha}, s)$ is the same for all $s \in (r, r + \varepsilon]$. We have $X(\text{rk}^G_{\alpha}, r) = \bigcap_{r < s \leq r + \varepsilon} X^0(\text{rk}^G_{\alpha}, s) = X^0(\text{rk}^G_{\alpha}, r + \varepsilon)$. \hfill $\square$

3. Results on the $G$-stable rank

3.1. Easy observations and a technical lemma.

Lemma 3.1. If $v \neq 0$, then we have $\text{rk}^G_{\alpha}(v) \geq \min\{\alpha_1, \alpha_2, \ldots, \alpha_d\} > 0$. In particular, $\text{rk}^G(v) \geq 1$.

Proof. Choose $g(t) \in G(K[[t]])$ with $\mu_{\alpha}(g(t), v) = \text{rk}^G_{\alpha}(v)$. From $v \neq 0$ follows that $g(t) \cdot v \neq 0$, say $\text{val}(g(t) \cdot v) = s > 0$. Then we get $\sum_{i=1}^{d} \text{val}_i(g_i(t)) \geq s$ and

$$\frac{\sum_{i=1}^{d} \alpha_i \text{val}_i(g_i(t))}{\text{val}(g(t) \cdot v)} \geq \min\{\alpha_1, \ldots, \alpha_d\} \frac{\sum_{i=1}^{s} \text{val}_i(g_i(t))}{s} \geq \min\{\alpha_1, \ldots, \alpha_d\}.$$

It follows that $\text{rk}^G_{\alpha}(v) \geq \min\{\alpha_1, \ldots, \alpha_d\} > 0$. \hfill $\square$

Suppose that $v = u \otimes w$ is nonzero with $u \in V_1$ and $w \in V_2 \otimes \cdots \otimes V_d$. We choose bases in $V_1, \ldots, V_d$ such that $u$ is the first basis vector in $V_1$. We can choose a one parameter subgroup $\lambda(t)$ with

$$\lambda_1(t) = \left( \begin{array}{c} t \\ \vdots \\ 1 \end{array} \right)$$

and $\lambda_k(t) = 1_{n_k}$ for $k = 2, 3, \ldots, d$. Then we have $\lambda(t) \cdot v = tv$ and $\mu_{\alpha}(A(t), v) = \alpha_1$. This shows that $\text{rk}^G_{\alpha}(v) \leq \alpha_1$. From Lemma 3.1 follows that $\text{rk}^G_{\alpha}(v) \leq \alpha_1$. If $v$ has slice rank 1 concentrated in the $i$-th slice, then $\text{rk}^G_{\alpha}(v) \leq \alpha_i \leq \max\{\alpha_1, \alpha_2, \ldots, \alpha_d\}$.

Corollary 3.2. If $v$ has slice rank 1, then $\text{rk}^G(v) = 1$.

Proof. If $v$ has slice rank 1, then $\text{rk}^G(v) = \text{rk}^G_{(1,\ldots,1)}(v) \leq \max\{1, \ldots, 1\} = 1$ and $\text{rk}^G(v) \geq 1$ by Lemma 3.1. \hfill $\square$

Corollary 3.3. If $v$ has rank 1 then $\text{rk}^G_{\alpha}(v) = \min\{\alpha_1, \ldots, \alpha_d\}$.

Proof. If $v$ has rank 1 then $\text{rk}^G_{\alpha} \leq \alpha_i$ for every $i$ and $\text{rk}^G_{\alpha} \geq \min\{\alpha_1, \ldots, \alpha_d\}$ by Lemma 3.1. \hfill $\square$

Proposition 3.4. Suppose that $v \in V_1 \otimes V_2 \otimes \cdots \otimes V_d$ and $w \in W_1 \otimes W_2 \otimes \cdots \otimes W_e$ and $v \otimes w \in V_1 \otimes \cdots \otimes V_d \otimes W_1 \otimes \cdots \otimes W_e$ is the horizontal tensor product. We have $\text{rk}^G_{\alpha,\beta}(v \otimes w) = \min\{\text{rk}^G_{\alpha}(v), \text{rk}^G_{\beta}(w)\}$. 

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Proof. Let $G = \text{GL}(V_1) \times \cdots \times \text{GL}(V_d)$ and $H = \text{GL}(W_1) \times \cdots \times \text{GL}(W_e)$. There exists $g(t) \in G([t])$ with $\mu_\alpha(g(t), v) = \text{rk}_\alpha(v)$. For $(g(t), 1) \in (G \times H)([t])$ we get $\mu_{\alpha, \beta}(g(t), h(t)), v \otimes w) = \text{rk}_\alpha(v)$. This proves that $\text{rk}_G^{\alpha, \beta}(v \otimes w) \leq \text{rk}_\alpha(v)$. Similarly, we have $\text{rk}_G^{\alpha, \beta}(v \otimes w) \leq \text{rk}_G^{\beta}(w)$, so we get $\text{rk}_G^{\alpha, \beta}(v \otimes w) \leq \min\{\text{rk}_G^{\alpha}(v), \text{rk}_G^{\beta}(w)\}$.

Conversely, suppose that $(g(t), h(t)) \in G \times H(K[[t]])$ satisfies $\mu_{\alpha, \beta}((g(t), h(t)), v \otimes w) = \text{rk}_G^{\alpha, \beta}(v \otimes w)$. Using that

$$\text{val}_t((g(t), h(t)) \cdot (v \otimes w)) = \text{val}_t((g(t) \cdot v) \otimes (h(t) \cdot w)) = \text{val}_t(g(t) \cdot v) + \text{val}_t(h(t) \cdot w)$$

we get

$$\mu_{\alpha, \beta}(v \otimes w) = \frac{\sum_{i=1}^d \text{val}_t(\det g_i(t)) + \sum_{j=1}^e \text{val}_t(\det h_j(t))}{\text{val}_t(g(t) \cdot v) + \text{val}_t(h(t) \cdot w)} = \min \left\{ \frac{\sum_{i=1}^d \text{val}_t(\det g_i(t))}{\text{val}_t(g(t) \cdot v)}, \frac{\sum_{j=1}^e \text{val}_t(\det h_j(t))}{\text{val}_t(h(t) \cdot w)} \right\} = \min\{\text{rk}_G^{\alpha}(v), \text{rk}_G^{\beta}(w)\}.$$ 

$\square$

We will need the following technical lemma to prove Proposition 3.6.

**Lemma 3.5.** If $(g(t), h(t)) \in \text{GL}_n(K[[t]])$ then there exists $u(t), g'(t), h'(t) \in \text{GL}_n(K[[t]])$ such that $u(t) = g'(t)h(t) = h'(t)g(t)$ and $\text{val}_t(\det u(t)) \leq \text{val}_t(\det g(t)) + \text{val}_t(\det h(t))$.

**Proof.** We have

$$\text{val}_t(\det g(t)) = \dim_K \frac{K[[t]]^n}{g(t)K[[t]]^n}.$$ 

The $K[[t]]$-module $g(t)K[[t]]^n \cap h(t)K[[t]]^n$ is a submodule of the free module $K[[t]]^n$, so it is also free of rank $\leq n$. So there exists a matrix $u(t)$ such that $g(t)K[[t]]^n \cap h(t)K[[t]]^n = u(t)K[[t]]^n$. From $u(t)K[[t]]^n \subseteq g(t)K[[t]]^n$ follows that there exists a matrix $h'(t)$ such that $u(t) = h'(t)g(t)$. Similarly, we find a matrix $g'(t)$ with $u(t) = g'(t)h(t)$.

We have

$$\text{val}_t(\det u(t)) \leq \dim \frac{K[[t]]^n}{u(t)K[[t]]^n} = \dim \frac{K[[t]]^n}{g(t)K[[t]]^n \cap h(t)K[[t]]^n} = \dim \frac{K[[t]]^n}{g(t)K[[t]]^n} + \dim \frac{g(t)K[[t]]^n \cap h(t)K[[t]]^n}{h(t)K[[t]]^n} = \text{val}_t(\det g(t)) + \dim \frac{g(t)K[[t]]^n + h(t)K[[t]]^n}{h(t)K[[t]]^n} \leq \text{val}_t(\det g(t)) + \text{val}_t(\det h(t)).$$ 

$\square$

### 3.2. The triangle inequality for the $G$-stable rank.

**Proposition 3.6.** For tensors $v, w \in V$ we have $\text{rk}_G^{\alpha}(v + w) \leq \text{rk}_G^{\alpha}(v) + \text{rk}_G^{\alpha}(w)$.

**Proof.** Suppose that $g(t), h(t) \in G(K[[t]])$. If we replace $t$ by $t^e$, then $\mu_\alpha(g(t), v)$ does not change. Without changing $\mu_\alpha(g(t), v)$ and $\mu_\alpha(h(t), w)$ we may assume that $\text{val}_t(g(t) \cdot v) = \text{val}_t(h(t) \cdot w) = s > 0$. Then there exist $u(t), g'(t), h'(t) \in G(K[[t]])$ such that
u(t) = h'(t)g(t) = g'(t)h(t) and val_i(det u_i(t)) \leq val_i(det g_i(t)) + val_i(det h_i(t)) for all i by Lemma 3.5. We get
\[\text{val}_i(u(t) \cdot (v + w)) = \text{val}_i(h'(t)g(t) \cdot v + g'(t)h(t) \cdot w) \geq \min\{\text{val}_i(h'(t)g(t) \cdot v), \text{val}_i(g'(t)h(t) \cdot w)\} \geq \min\{\text{val}_i(g(t) \cdot v), \text{val}_i(h(t) \cdot w)\} = s\]
and
\[\sum_{i=1}^{d} \alpha_i \text{val}_i(det u_i(t)) \leq \sum_{i=1}^{d} \alpha_i \text{val}_i(det g_i(t)) + \sum_{i=1}^{d} \alpha_i \text{val}_i(det h_i(t)) = s\mu_\alpha(g(t), v) + s\mu_\alpha(h(t), w).
\]
It follows that
\[\mu_\alpha(u(t), v + w) = \frac{\sum_{i=1}^{d} \alpha_i \text{val}_i(det u_i(t))}{\text{val}_i(u(t) \cdot (v + w))} \leq \frac{s\mu_\alpha(g(t), v) + s\mu_\alpha(h(t), w)}{s} = \mu_\alpha(g(t), v) + \mu_\alpha(h(t), w).
\]
Taking the infimum over all g(t) and h(t) gives \(\text{rk}^G_\alpha(v + w) \leq \text{rk}^G_\alpha(v) + \text{rk}^G_\alpha(w).\)

Corollary 3.7. For any tensor \(v \in V\) we have
\[\text{rk}^G_\alpha(v) \leq \text{srk}(v).
\]

Proof. By definition, we can write \(v = v_1 + v_2 + \cdots + v_r\) where \(r = \text{srk}(v)\) and \(v_1, v_2, \ldots, v_r\) are tensors of slice rank 1. Now we have \(\text{rk}^G_\alpha(v) = \text{rk}^G_\alpha(v_1 + \cdots + v_r) \leq \text{rk}^G_\alpha(v_1) + \cdots + \text{rk}^G_\alpha(v_r) = 1 + \cdots + 1 = r = \text{srk}(v).\)

3.3. The additive property of the G-stable rank.

Proposition 3.8. If \(d \geq 2\), the G-stable rank is additive: we have \(\text{rk}^G_\alpha(v \boxplus w) = \text{rk}^G_\alpha(v) + \text{rk}^G_\alpha(w)\).

Proof. From Proposition 3.6 follows that \(\text{rk}^G_\alpha(v \boxplus w) \leq \text{rk}^G_\alpha(v \boxplus 0) + \text{rk}^G_\alpha(0 \boxplus w) \leq \text{rk}^G_\alpha(v) + \text{rk}^G_\alpha(w).\) Suppose that \(g(t) \in G(K[[t]])\) with \(\text{val}_i(g(t) \cdot (v \boxplus w)) = t^s\) for some \(s > 0\). Assume that the block form of \(g_i(t)\) with respect to the decomposition \(V_i \oplus W_i\) is
\[g_i(t) = \begin{pmatrix} a_i(t) & b_i(t) \\ c_i(t) & d_i(t) \end{pmatrix}.
\]
The \(K[[t]]\)-module generated by the rows of \(a_1(t)\) and \(c_1(t)\) is a free submodule of \(K[[t]]^{n_1}\) of rank \(n_1\), where \(n_1 = \dim V_i\). Using the Smith normal form, there exist invertible matrices in \(p(t) \in \text{GL}_{n_1+m_1}(K[[t]])\) and \(q(t) \in \text{GL}_{n_1}(K[[t]])\) such that
\[\begin{pmatrix} a_1(t) \\ c_1(t) \end{pmatrix} = p(t) \begin{pmatrix} r(t) \\ 0 \end{pmatrix} q(t)
\]
where \(r(t)\) is an \(n_1 \times n_1\) diagonal matrix. It follows that
\[p(t)^{-1}g_1(t) = \begin{pmatrix} r(t) * \\ 0 * \end{pmatrix}
\]
So without loss of generality, we may assume that \(c_1(t) = 0\). A similar argument shows that we may assume without loss of generality that \(b_2(t) = b_3(t) = \cdots = b_d(t) = 0\). If we project \(g(t) \cdot v \boxplus w\) onto \(V\), we get \(a(t) \cdot v + b(t) \cdot w = a(t) \cdot v\) because \(b_2(t) = 0\). This
implies that $\text{val}_t(a(t) \cdot v) \geq s$ and $\sum_{i=1}^{d} \alpha_i \text{val}_t(\det a_i(t)) \geq s \text{rk}_G^T(v)$. Similarly, the projection of $g(t) \cdot v \oplus w$ onto $W$ is equal to $c(t) \cdot v + d(t) \cdot w = d(t) \cdot w$ because $c_1(t) = 0$. Therefore, we have $\text{val}_t(d(t) \cdot w) \geq s$ and $\sum_{i=1}^{d} \alpha_i \text{val}_t(\det d_i(t)) \geq s \text{rk}_G^T(w)$. Since $\det g_i(t) = \det a_i(t) \det d_i(t)$ because of the upper triangular or lower triangular form of $g_i(t)$, we get

$$\sum_{i=1}^{s} \alpha_i \text{val}_t(\det g_i(t)) = \sum_{i=1}^{s} \alpha_i \text{val}_t(\det a_i(t)) + \sum_{i=1}^{s} \alpha_i \text{val}_t(\det d_i(t)) \geq s(\text{rk}_G^T(v) + \text{rk}_G^T(w)).$$

This proves that $\text{rk}_G^T(v \oplus w) \geq \text{rk}_G^T(v) + \text{rk}_G^T(w)$. \hfill $\square$

4. The stable $T$-rank

4.1. The $G$-stable rank and the $T$-stable rank. The $G$-stable $\alpha$-rank of a tensor $v$ is the maximum of $\mu_\alpha(\lambda(t), v)$ where $\lambda(t)$ is a 1-parameter subgroup of $G$ with $\text{val}_t(\lambda(t) \cdot v) > 0$. A 1-parameter subgroup is contained in some maximal torus $T$ (which itself is contained in some Borel subgroup $B$ of $G$). We can fix a maximal torus $T$ and consider all 1-parameter subgroups contained in $T$. Choosing a maximal torus of $G$ corresponds to choosing a basis in each vector space $V_i$. So let us choose a basis in each $V_i$ so that we can identify $\text{GL}(V_i)$ with $\text{GL}_{n_i}$. Let $T_k \subseteq \text{GL}_k$ be the subgroup of invertible diagonal $k \times k$ matrices, and $T = T_{n_1} \times T_{n_2} \times \cdots \times T_{n_d} \subseteq G$. Then $T$ is a maximal torus of $G$.

**Definition 4.1.** We define the $\alpha$-stable $T$-rank $\text{rk}_G^T(v)$ as the infimum over all $\mu_\alpha(\lambda(t), v)$ where $\lambda(t) \in T(K[t])$ is a 1-parameter subgroup of $T$ with $\text{val}_t(\lambda(t) \cdot v) > 0$.

Since every 1-parameter subgroup is conjugate to a 1-parameter subgroup in the maximal torus, we get the following corollary.

**Corollary 4.2.** We have

$$\text{rk}_G^T(v) = \inf_{g \in G} \text{rk}_G^T(g \cdot v).$$

4.2. The $T$-stable rank and linear programming. For a tensor $v = (v_{i_1,i_2,\ldots,i_d}) \in V = K^{n_1 \times n_2 \times \cdots \times n_d}$ we define its support by

$$\text{supp}(v) = \{(i_1, \ldots, i_d) \mid v_{i_1,i_2,\ldots,i_d} \neq 0\}.$$ 

As we will see, $\text{rk}_G^T(v)$ only depends on $\text{supp}(v)$ and $\alpha$. For a nonnegative integer $k$, let $k = \{1, 2, \ldots, k\}$. We will fix a support $S \subseteq n_1 \times n_2 \times \cdots \times n_d$ and compute the corresponding $\alpha$-stable $T$-rank.

**Definition 4.3.** Let $x(i, j)$ with $1 \leq i \leq d$ and $1 \leq j \leq n_i$ be real variables and $S \subseteq n_1 \times \cdots \times n_d$ be a support. The linear program $\text{LP}_\alpha(S)$ asks to minimize $\sum_{i=1}^{d} \alpha_i \sum_{j=1}^{n_i} x(i, j)$ under the constraints:

1. $x(i, j) \geq 0$ for $i = 1, 2, \ldots, d$ and $1 \leq j \leq n_i$;
2. $\sum_{i=1}^{d} x(i, s_i) \geq 1$ for all $s \in S$.

**Theorem 4.4.** If $v \in V$ has support $S$, then $\text{rk}_G^T(v)$ is the value of the linear program $\text{LP}_\alpha(S)$.

**Proof.** Suppose $\lambda(t) = \left(\lambda_1(t), \ldots, \lambda_d(t)\right) \in T(K[t])$ is a 1-parameter subgroup, and $\lambda_i(t)$ is diagonal with entries $t^{x(i,1)}, t^{x(i,2)}, \ldots, t^{x(i,n_i)}$ where $x(i, j)$ is a nonnegative integer for all $i, j$. Also, assume that $\text{val}_t(\lambda(t) \cdot v) = q > 0$ where $v$ is a tensor with support $S$. This means that
\[ \sum_{i=1}^{d} \alpha_i x(i, s_i) \geq q \text{ for all } (s_1, s_2, \ldots, s_d) \in S. \] We have \( \mu_\alpha(\lambda(t), v) = \frac{1}{q} \left( \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{n_i} x(i, j) \right) \) and \( \text{rk}^T_\alpha(v) \) is the infimum of all \( \mu_\alpha(\lambda(t), v) \). If we replace \( x(i, j) \) by \( x(i, j)/q \), then we have \( \sum_{i=1}^{d} \alpha_i x(i, s_i) \geq 1 \) for all \((s_1, \ldots, s_d) \in S \) and \( \mu_\alpha(\lambda(t), v) = \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{n_i} x(i, j) \). This shows that \( \text{rk}^T_\alpha(v) \) is the infimum of \( \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{n_i} x(i, j) \) under the constraints \( x(i, j) \geq 0 \) for all \( i, j \), and \( \sum_{i=1}^{d} x(i, s_i) \geq 1 \) for all \( s \in S \) for all \( i, j \). This is the linear program \( \text{LP}_\alpha(S) \), except that the numbers \( x(i, j) \) have to be rational. However, since the constraints are inequalities with coefficients in \( \mathbb{Q} \), there exists an optimal solution over \( \mathbb{Q} \). \( \square \)

**Example 4.5.** Consider the tensor \( v = [2, 1, 1] + [1, 2, 1] + [1, 1, 2] \in K^{2 \times 2 \times 2} = K^2 \otimes K^2 \otimes K^2 \).

with support \( S = \{ (2, 1, 1), (1, 2, 1), (1, 1, 2) \} \). We have to solve the following linear program \( \text{LP}(S) = \text{LP}_{(1,1,1)}(S) \): minimize \( \sum_{i=1}^{3} \sum_{j=1}^{2} x(i, j) \) under the constraints \( x(i, j) \geq 0 \) for \( i = 1, 2, 3 \) and \( j = 1, 2 \) and

\[
\begin{align*}
  x(1, 2) + x(2, 1) + x(3, 1) & \geq 1 \\
  x(1, 1) + x(2, 2) + x(3, 1) & \geq 1 \\
  x(1, 1) + x(2, 1) + x(3, 2) & \geq 1
\end{align*}
\]

An optimal solution is \( x(1, 1) = x(2, 1) = x(3, 1) = \frac{1}{2} \) and \( x(1, 2) = x(2, 2) = x(3, 2) = 0 \). So the optimal value is \( \text{rk}^T(v) = 3 \cdot \frac{1}{2} = \frac{3}{2} \). It follows that \( \text{rk}^G(v) \leq \text{rk}^T(v) \leq \frac{3}{2} \). It is easy to see that \( \text{srk}(v) > 1 \) (and thus equal 2). We will show that \( \text{rk}^G(v) = \frac{3}{2} \).

Suppose that \( \text{rk}^G(v) < \frac{3}{2} \). Then there exists a tensor \( w \in K^{2 \times 2 \times 2} \) in the same \( G \)-orbit as \( v \) such that \( \text{rk}^T(w) < \frac{3}{2} \). Let \( S' = \text{supp}(w) \subseteq 2 \times 2 \times 2 \) be the support of \( w \). Also assume that \( \{ x(i, j) \} \) is an optimal solution for the linear program \( \text{LP}(S') \). By permuting coordinates, we may assume that \( x(i, 1) \geq x(i, 2) \) for \( i = 1, 2, 3 \). The support \( S' \) is not contained in \( \{1\} \times \{1, 2\} \times \{1, 2\} \) because otherwise \( w \) and \( v \) would have slice rank 1. Therefore, \( (2, i, j) \in S' \) for some \( i, j \). Because of the ordering of the variables \( x(i, j), (2, 1, 1) \in S' \). Similarly, \( (1, 2, 1), (1, 1, 2) \in S' \). Now \( \text{supp}(w) = S' \supseteq S = \text{supp}(v) \), so \( \text{rk}^T(w) \geq \text{rk}^T(v) = \frac{3}{2} \). Contradiction.

4.3. **Comparison between the \( G \)-stable rank and the slice rank.** Besides the slice rank, we will also define a slice rank relative to a maximal torus \( T \), or equivalently, relative to bases choices for \( V_1, V_2, \ldots, V_d \).

**Definition 4.6.** We say that a tensor \( v \) has \( T \)-slice rank 1 if \( v \) is contained in a space of the form

\[
V_{i,j} = V_1 \otimes V_2 \otimes \cdots \otimes V_{i-1} \otimes [j] \otimes V_{i+1} \otimes \cdots \otimes V_d.
\]

Now the \( T \)-slice rank \( \text{srk}^T(v) \) of an arbitrary tensor \( v \) is the smallest nonnegative integer \( r \) such that \( v \) is a sum of \( r \) tensors of \( T \)-slice rank 1.

The following result is clear from the definition of slice rank:

**Corollary 4.7.** We have

\[
\text{srk}(v) = \min_{g \in G} \text{srk}^T(g \cdot v).
\]

The \( T \)-slice rank of \( v \) depends only on its support \( S = \text{supp}(v) \) and can be expressed in terms of integer solutions of the linear program \( \text{LP}(S) \).
Proposition 4.8. The T-slice rank $srk_T(v)$ is the smallest possible value of $\sum_{i=1}^d \sum_{j=1}^{n_i} x(i,j)$ where the $x(i,j)$ satisfy the constraints:

1. $x(i,j) \in \{0,1\}$ for $i = 1, 2, \ldots, d$ and $1 \leq j \leq n_i$;
2. $\sum_{i=1}^d x(i,s_i) \geq 1$ for all $s \in S$.

Proof. Suppose that $x(i,j) \in \{0,1\}$ for all $i,j$. Define

$$V(x) = \sum_{i,j, x(i,j)=1} V_{i,j}.$$ 

A vector $[s_1, s_2, \ldots, s_d]$ lies in $V(x)$ if and only if $\sum_{i=1}^d x(i,s_i) \geq 1$. So a tensor $v$ lies in $V(x)$ if and only if $\sum_{i=1}^d x(i,s_i) \geq 1$ for all $s \in \text{supp}(v)$. By definition, $srk_T(v)$ is the smallest possible value of $\sum_{i,j} x(i,j)$ such that $v \in V(x)$. \hfill \square

It is now easy to see that $rk^T(v) \geq \frac{1}{d} \cdot srk_T(v)$ (and this implies $rk^G(v) \geq \frac{1}{d} \cdot srk(v)$): If $x(i,j)$ is a solution to the linear program $\mathbf{LP}(S)$ where $S = \text{supp}(v)$, then we define $x'(i,j) \in \{0,1\}$ such that $x'(i,j) = 1$ if $x(i,j) \geq \frac{1}{d}$ and $x'(i,j) = 0$ otherwise. If $s \in S$ then we have $\sum_{i=1}^d x(i,s_i) \geq 1$. It follows that $x(i,s_i) \geq \frac{1}{d}$ for some $i$ and $x'(i,s_i) = 1$ for some $i$. Therefore, $\sum_{i=1}^d x'(i,s_i) \geq 1$. Now $srk_T(v) \leq \sum_{i,j} x'(i,j) \leq \sum_{i,j} dx(i,j) = d \cdot rk^T(v)$. With a more refined argument, we can improve this bound:

Proposition 4.9. For $d \geq 2$ we have $rk^T(v) \geq \frac{d}{2} \cdot srk_T(v)$ and therefore $rk^G(v) \geq \frac{d}{2} \cdot srk(v)$.

Proof. Suppose that $x(i,j)$ is an optimal solution to the linear program. Note that $0 \leq x(i,j) \leq 1$ for all $i,j$. We define functions $f_1, f_2, \ldots, f_d : [0,1] \to \mathbb{R}$ by

$$f_i(\alpha) = |\{j \mid x(i,j) \geq \alpha\}|.$$ 

We have $\frac{1}{d} \int_0^1 f_i(\alpha) d\alpha = \sum_j x(i,j)$. In particular, $\int_0^1 (f_1(\alpha) + \cdots + f_d(\alpha)) d\alpha = \sum_{i,j} x(i,j)$. Let $s_i = \frac{2i}{d(d-1)}$ for $i = 0, 1, 2, \ldots, d-1$. Note that $s_0 + s_1 + \cdots + s_{d-1} = 1$. We define a closed piecewise linear curve $\gamma = (\gamma_1, \ldots, \gamma_d) : [0,d] \to \mathbb{R}^d$ with $\gamma(d) = \gamma(0) = [s_0, s_1, \ldots, s_{d-1}]$, $\gamma(1) = [s_1, s_2, \ldots, s_{d-1}, s_0], \ldots, \gamma(d-1) = [s_{d-1}, s_0, \ldots, s_{d-2}]$ such that $\gamma$ is linear on each of the intervals $[i,i+1]$, $i = 0,1,\ldots, d-1$. On the intervals $[0,1], [1,2], \ldots, [d-1,d]$, $\gamma_i(t)$ goes through the intervals $[s_0,s_1],[s_1,s_2],\ldots,[s_{d-2},s_{d-1}],[s_{d-1},s_0]$ in order. So $\frac{1}{d} \int_0^d f_i(\gamma_i(t)) dt$ is the average of the averages of $f_i$ of each of these $d$ intervals. This is equal to the average value of $f_i(t)$ on the interval $[0,s_{d-1}] = [0,\frac{2}{d}]$:

$$\frac{1}{d} \int_0^d f_i(\gamma_i(t)) dt = \frac{d}{2} \int_0^{\frac{2}{d}} f_i(t) dt \leq \frac{d}{2} \int_0^1 f_i(t) dt = \frac{d}{2} \sum_{j=1}^{n_i} x(i,j).$$ 

It follows that

$$\frac{1}{d} \int_0^d \left( \sum_{i=1}^d f_i(\gamma_i(t)) \right) dt \leq \frac{d}{2} \sum_{i=1}^d \sum_{j=1}^{n_i} x(i,j) = \frac{d}{2} \cdot rk^T(v).$$

Since the minimal value of $\sum_{i=1}^d f_i(\gamma_i(t))$ is at most the average, there exists a $t \in [0,d]$ such that $\sum_{i=1}^d f_i(\gamma_i(t)) \leq \frac{d}{2} \cdot rk^T(v)$. Now define $x'(i,j) = 1$ if $x(i,j) \geq \gamma_i(t)$ and $x'(i,j) = 0$ if
$x(i, j) < \gamma_i(t)$. If $s = (s_1, s_2, \ldots, s_d) \in \text{supp}(v)$, then $\sum_{i=1}^{d} x(i, s_i) \geq 1$. Since $\sum_{i=1}^{d} \gamma_i(t) = 1$, we have $x(i, s_i) \geq \gamma_i(t)$ for some $i$ and $\sum_{i=1}^{d} x'(i, s_i) \geq 1$. We conclude that

$$\text{srk}^T(v) \leq \sum_{i=1}^{n} \sum_{j=1}^{n_i} x'(i, j) = \sum_{i=1}^{d} f_i(\gamma_i(t)) \leq \frac{d}{2} \text{rk}^T(v).$$

Finally, we get

$$\text{srk}(v) = \inf_{g \in G} \text{srk}^T(g \cdot v) \leq \frac{d}{2} \inf_{g \in G} \text{rk}^T(g \cdot v) = \frac{d}{2} \text{rk}^G(v).$$

\[ \square \]

### 4.4. The dual program and the $T$-stable rank.

**Definition 4.10.** For a support set $S$, the dual program $\text{LP}^\vee(S)$ is to maximize $\sum_{s \in S} y(s)$ under the constraints

1. $y(s) \geq 0$ for all $s \in S$;
2. for all $i, j$ we have

$$\sum_{s \in S \atop s_i = j} y(s) \leq \alpha_i.$$

If $x$ and $y$ are optimal solutions for $\text{LP}^\alpha(S)$ and $\text{LP}^\vee(S)$ respectively, then we have

$$\sum_{s \in S} y(s) = \sum_{i=1}^{d} \alpha_i \sum_{j=1}^{n_i} x(i, j) = \text{rk}^T(v)$$

and

1. for all $i, j$, we have

$$\sum_{s \in S \atop s_i = j} y(s) = \alpha_i \text{ or } x(i, j) = 0;$$

2. for all $s \in S$ we have $\sum_{i=1}^{d} x(i, s_i) = 1$ or $y(s) = 0$.

### 4.5. The super-multiplicative property of the $T$-stable rank.

If $v \in V = V_1 \otimes V_2 \otimes \cdots \otimes V_d$ and $w \in W_1 \otimes W_2 \otimes \cdots \otimes W_d$ then we can consider the “vertical” tensor product $v \boxtimes w \in (V_1 \otimes W_1) \otimes \cdots (V_d \otimes W_d)$.

**Proposition 4.11.** We have $\text{rk}^T_{\alpha\beta}(v \boxtimes w) \geq \text{rk}^T_{\alpha}(v) \text{rk}^T_{\beta}(w)$, where $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\beta = (\beta_1, \ldots, \beta_d)$ and $\alpha \beta = (\alpha_1 \beta_1, \ldots, \alpha_d \beta_d)$.

**Proof.** Let $S = \text{supp}(v)$, $S' = \text{supp}(w)$, $y(s), s \in S$ be an optimal solution for the $\text{LP}^\vee(v)$ and $y'(s), s \in S'$ be an optimal solution for $\text{LP}^\beta(w)$. The tensor $v \boxtimes w$ has support $S \times S'$. For the dual program for $v \boxtimes w$ we have to maximize $\sum_{s \in S, s' \in S'} Y(s, s')$ under the constraints $Y(s, s') \geq 0$ for all $s \in S, s' \in S'$ and

$$\sum_{s \in S, s' \in S' \atop s_i = j, s'_i = j'} Y(s, s') \leq \alpha_j \beta_{j'}.$$
for all \(i, j, j'\). One solution for this linear program is \(Y(s, s') = y(s)y'(s')\). We get
\[
\text{rk}^T_{\alpha\beta}(v \boxtimes w) \geq \sum_{s \in S} \sum_{s' \in S'} Y(s, s') = \sum_{s \in S} y(s) \sum_{s' \in S'} y(s') = \text{rk}^T_{\alpha}(v) \text{rk}^T_{\beta}(w).
\]

\[
\square
\]

5. G-stable rank over \(\mathbb{C}\)

5.1. Kempf-Ness theory. We recall some of the main results from Kempf-Ness theory \([23, 38]\). Suppose that \(G\) is an complex reductive algebraic group with a maximal compact subgroup \(C\) and \(V\) is a representation of \(G\). We fix a Hermitian inner product \(\langle \cdot, \cdot \rangle\) on \(V\) that is invariant under \(C\), i.e., \(\langle g \cdot v, g \cdot w \rangle = \langle v, w \rangle\) for all \(v, w \in V\) and \(g \in C\). Let \(\mathfrak{c}\) and \(\mathfrak{g}\) be the Lie algebras of \(C\) and \(G\) respectively, and let \(\mathfrak{c}^*\) be the dual space of \(\mathfrak{c}\). We have \(\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{ic}\). For \(v \in V\), we define a morphism \(\psi_v : G \to \mathbb{R}\) by \(g \mapsto \| g \cdot v \|^2 = \langle g \cdot v, g \cdot v \rangle\). The differential \((d\psi_v)_I : \mathfrak{g} \to \mathbb{R}\) of \(\psi_v\) at the identity \(I \in G\) is given by
\[
(d\psi_v)_I : \xi \mapsto \langle \xi v, v \rangle + \langle v, \xi v \rangle \in \mathbb{R}
\]
Because \(\| g \cdot v \|^2\) is constant on \(C\), \((d\psi_v)_I\) vanishes on \(\mathfrak{c}\). So \(\langle v, \xi v \rangle = -\langle \xi v, v \rangle\) for \(\xi \in \mathfrak{c}\). If \(\xi \in \mathfrak{c}\) then we have \((d\psi_v)_I(i\xi) = (i\xi v, v) + \langle v, i\xi v \rangle = i\langle \xi v, v \rangle - i\langle v, \xi v \rangle = 2i\langle \xi v, v \rangle\). For the following result, see \([38\), Corollary 5.2.5.\].

**Theorem 5.1** (Kempf-Ness). An orbit \(G \cdot v\) is closed if and only there exists \(w \in G \cdot v\) with \((d\psi_w)_I = 0\).

Let \(V = V_1 \otimes V_2 \otimes \cdots \otimes V_d\) with \(V_i = \mathbb{C}^{n_i}\). For \(v \in V\), let \(\Phi_i(v) \in (V_1 \otimes \cdots \hat{V}_i \otimes \cdots \otimes V_d)^* \to V_i\) be the \(i\)-th flattening of \(v\).

5.2. A formula for the \(G\)-stable rank over \(\mathbb{C}\). We will use Kempf-Ness theory to prove the following theorem:

**Theorem 5.2.** For \(\alpha \in \mathbb{R}_{>0}\) we have
\[
\text{rk}^G_{\alpha}(v) = \sup_{g \in G} \min_i \frac{\alpha_i \| g \cdot v \|^2}{\| \Phi_i(g \cdot v) \|^2_{\alpha}}
\]

For the proof of the theorem, we need the following lemma:

**Lemma 5.3.** Suppose that \(\beta \in \mathbb{Q}^d_{>0}, r = \frac{\beta}{q}\) with \(p, q\) positive integers, \(q \beta \in \mathbb{Z}^d\) and \(v \in V = V_1 \otimes V_2 \otimes \cdots \otimes V_d\). As in Proposition 2.3, let
\[
W = (V^\otimes p \otimes \det^{-q\beta}) \oplus V_1^{n_1} \oplus V_2^{n_2} \oplus \cdots \oplus V_d^{n_d}.
\]
and \(w = (v^\otimes p \otimes 1, u_1, \ldots, u_d)\). Define \(\psi_w : G \to W\) by \(\psi_w(g) = g \cdot w\). Then we have \((d\psi_w)_I = 0\) if and only if
\[
p\| v \|^2 - 2p\Phi_i(v)\Phi^*_i(v) - q\beta_i\| v \|^2 I_{n_i} + u_i u^*_i = 0
\]
for all \(i\).

**Proof.** The Hermitian scalar products on \(V_1, V_2, \ldots, V_d\) induce Hermitian scalar products on \(V_1^{n_1}, \ldots, V_d^{n_d}, V, V^\otimes p, V^\otimes p \otimes \det^{-q\beta}\) and \(W\) in a natural way. We have
\[
\| w \|^2 = \| v \|^2 + \sum_{i=1}^d \| u_i \|^2
\]
and
\[ \psi_w(g) = \|g \cdot w\|^2 = \|g \cdot v\|^{2p} \det^{-2p\beta}(g) + \sum_{i=1}^d \|g_i u_i\|^2. \]

The Lie algebra of \( G \) can be identified with
\[ \mathfrak{g} = \text{End}(V_1) \oplus \text{End}(V_2) \oplus \cdots \oplus \text{End}(V_d). \]

The Lie algebra \( \mathfrak{c} \) consists of all \( d \)-tuples \((\xi_1, \ldots, \xi_d)\) of skew-Hermitian matrices, and \( i\mathfrak{c} \) consists of \( d \)-tuples of Hermitian matrices. We compute the differential \((d\psi_w)_I\). Note that \( \text{GL}(V_i) \) acts on the \( i \)-th mode. If we view \( v \) as the flattened tensor \( \Phi_i(v) \), then \( g_i \) acts just by left multiplication: \( \Phi_i(g_i \cdot v) = g_i \Phi_i(v) \). Let \( \text{Tr}(\cdot) \) denote the trace. The differential of \( g_i \mapsto \|g_i \cdot v\|^2 = \text{Tr}(g_i \Phi_i(v) \Phi_i^*(v) g_i^*) \) at the identity is given by \( \xi_i \in \text{End}(V_i) \mapsto \text{Tr}(\xi_i \Phi_i(v) \Phi_i^*(v)) + \text{Tr}(\Phi_i(v) \Phi_i^*(v) \xi_i^*) \). If we restrict to Hermitian \( \xi_i \), then this is equal to \( 2 \text{Tr}(\xi_i \Phi_i(v) \Phi_i^*(v)) \).

The differential of \( g_i \mapsto \text{det}(g_i) \) at the identity is \( \xi_i \mapsto \text{Tr}(\xi_i) \). Combining these results with the product rule of differentiation, we get for \( \xi \in i\mathfrak{c} \) that
\[ (d\phi_w)_I(\xi) = \sum_{i=1}^d \left( 2p\|v\|^{2p-2} \text{Tr}(\xi_i \Phi_i(v) \Phi_i^*(v)) - 2q_i \beta_i \|v\|^{2p} \text{Tr}(\xi_i) + 2 \text{Tr}(\xi_i u_i u_i^*) \right) = \sum_{i=1}^d \langle \xi_i, \|v\|^{2p-2} \Phi_i(v) \Phi_i^*(v) - 2q_i \beta_i \|v\|^{2p} I_{n_i} + 2u_i u_i^* \rangle \]
We have \((d\phi_w)_I = 0\) if and only if
\[ 2p\|v\|^{2p-2} \Phi_i(v) \Phi_i^*(v) - 2q_i \beta_i \|v\|^{2p} I_{n_i} + 2u_i u_i^* = 0 \]
for all \( i \).

**Proof of Theorem 5.2.** Let us define
\[ f_\alpha(v) = \sup_{g \in G} \min_i \frac{\alpha_i \|g \cdot v\|^2}{\|\Phi_i(g \cdot v)\|^2}. \]
Suppose that \( r \in \mathbb{Q} \) and \( f_\alpha(v) \leq r \). Assume that \( \beta \in \mathbb{Q}^d_+ \) with \( \beta_i > \alpha_i \) for all \( i \). We can write \( r = p/q \) such that \( p, q \in \mathbb{Z} \) are positive and \( q \beta_i \in \mathbb{Z} \) for all \( i \). From \( f_\alpha(v) \leq r \) follows that
\[ \alpha_i \|g \cdot v\|^2 I_{n_i} - r \Phi_i(g \cdot v) \Phi_i^*(g \cdot v) \]
is nonnegative definite for all \( i \). This implies that
\[ \beta_i \|g \cdot v\|^2 I_{n_i} - r \Phi_i(g \cdot v) \Phi_i^*(g \cdot v) \]
is positive definite for all \( i \). Multiplying with \( p\|g \cdot v\|^{2p-2} \) we get that
\[ p\beta_i \|g \cdot v\|^{2p} I_{n_i} - q \|g \cdot v\|^{2p-2} \Phi_i(g \cdot v) \Phi_i^*(g \cdot v) \]
is positive definite and equal to \( u_i u_i^* \) for some \( u_i \in V_i^{n_i} \). This shows that \((d\psi_g)_w = 0\) if \( \beta > \alpha \), we get \( \text{rk}_G^\beta(v) \geq r \). Because this is true for every rational \( \beta > \alpha \), we get \( \text{rk}_G^\alpha(v) \geq r \). Since this is true for any \( r \in \mathbb{Q} \) with \( r \geq f_\alpha(v) \), we can conclude that \( \text{rk}_G^\alpha(v) \geq f_\alpha(v) \).
Suppose that $\beta \in \mathbb{Q}^d_{>0}$ and $\beta_i < \alpha_i$ for all $i$. Let $r = \text{rk}_G^G(v) < \text{rk}_\alpha^G(v)$. We can write $r = \frac{p}{q}$ such that $p$, $q$ are positive integers, and $q\beta \in \mathbb{Z}^d$. We can choose an invertible $u_i \in V_i^{n_i}$ for all $i$. Now

$$w = (v^\otimes p \otimes 1, u_1, u_2, \ldots, u_d) \in (V^\otimes p \otimes \det^{-q\beta}) \oplus V_1^{n_1} \oplus V_2^{n_2} \oplus \cdots \oplus V_d^{n_d}$$

is $G$-semi-stable by Proposition 2.3. So there exists a nonzero $w' \in G \cdot w$ with $(d\psi_w)_I = 0$. We can write $w' = ((v')^\otimes d, u'_1, \ldots, u'_d)$. Using Lemma 5.3, we get

$$p\|v'\|^{2p-2}\Phi_i(v')\Phi_i^*(v') - q\beta_i\|v'\|^{2p}\text{rk}_i u'_i(u'_i)^* = 0.$$ 

So

$$q\beta_i\|v'\|^{2p}\text{rk}_i u'_i - p\|v'\|^{2p-2}\Phi_i(v')\Phi_i^*(v')$$

is nonnegative definite for all $i$. Therefore,

$$q\alpha_i\|v'\|^{2p}\text{rk}_i u'_i - p\|v'\|^{2p-2}\Phi_i(g \cdot v)\Phi_i^*(g \cdot v)$$

is positive definite for all $i$. Since $w'$ lies in $G \cdot w$, there exists a $g \in G$ such that

$$q\alpha_i\|g \cdot v\|^{2p}\text{rk}_i u'_i - p\|g \cdot v\|^{2p-2}\Phi_i(g \cdot v)\Phi_i^*(g \cdot v)$$

is positive definite for all $i$. It follows that

$$\|\Phi_i(g \cdot v)\|_\sigma^2 = \|\Phi_i(g \cdot v)\Phi_i^*(g \cdot v)\|_\sigma \leq \frac{q\alpha_i\|g \cdot v\|^{2p}}{p\|g \cdot v\|^{2p-2}} \frac{\alpha_i\|g \cdot v\|^2}{r}$$

for all $i$ and

$$\min_i \frac{\alpha_i\|g \cdot v\|^2}{\|\Phi_i(g \cdot v)\|_\sigma^2} \geq r.$$

This shows that $f_\alpha(v) \geq r = \text{rk}_G^G(v)$. Since $\beta \in \mathbb{Q}^d_{>0}$ was arbitrary with $\beta < \alpha$, we obtain $f_\alpha(v) \geq \text{rk}_\alpha^G(v)$. We conclude that $f_\alpha(v) = \text{rk}_\alpha^G(v)$. \hfill \Box

### 5.3. The super-multiplicative property of the $G$-stable rank in characteristic 0.

**Theorem 5.4.** If $v \in V_1 \otimes V_2 \otimes \cdots \otimes V_d$ and $w \in W_1 \otimes W_2 \otimes \cdots \otimes W_d$ where $V_1, \ldots, V_d, W_1, \ldots, W_d$ are $\mathbb{C}$-vector spaces and $\alpha, \beta \in \mathbb{R}_{>0}$, then we have

$$\text{rk}_{G\alpha}^G(v \boxtimes w) \geq \text{rk}_\alpha^G(v) \text{rk}_\beta^G(w).$$

**Proof.** If $g \in \text{GL}(V_1) \times \cdots \times \text{GL}(V_d)$ and $h \in \text{GL}(W_1) \times \cdots \times \text{GL}(W_d)$ then we consider $g \boxtimes h \in \text{GL}(V_1 \otimes W_1) \times \cdots \times \text{GL}(V_d \otimes W_d)$. We have

$$\frac{\alpha_i\beta_i\|g \boxtimes h \cdot (v \boxtimes w)\|_\sigma^2}{\|\Phi_i((g \boxtimes h) \cdot (v \boxtimes w))\|_\sigma} = \alpha_i\beta_i\|g \cdot v \boxtimes h \cdot w\|_\sigma^2 \leq \frac{\alpha_i\|g \cdot v\|^{2p}\beta_i\|h \cdot w\|^{2p}}{\|\Phi_i(g \cdot v)\|_\sigma \cdot \|\Phi_i(h \cdot w)\|_\sigma}.$$ 

Therefore, we get

$$\min_i \frac{\alpha_i\beta_i\|g \boxtimes h \cdot (v \boxtimes w)\|_\sigma^2}{\|\Phi_i((g \boxtimes h) \cdot (v \boxtimes w))\|_\sigma} \geq \min_i \frac{\alpha_i\|g \cdot v\|^2}{\|\Phi_i(g \cdot v)\|_\sigma} \cdot \min_j \frac{\beta_j\|h \cdot w\|^2}{\|\Phi_j(h \cdot w)\|_\sigma}.$$ 

Taking the supremum over all $g$ and $h$ now gives $\text{rk}_{G\alpha\beta}^G(v \boxtimes w) \geq \text{rk}_\alpha^G(v) \text{rk}_\beta^G(w)$. \hfill \Box
6. Application of the $G$-stable rank to the Cap Set Problem

The Cap Set Problem asks for a largest possible subset $S \subseteq \mathbb{F}_3^n$ without an arithmetic progression. Let $c(n)$ be the largest possible cardinality of such a set. It was recently proved by Ellenberg and Gijswijt that $c(n) = O(\theta^n)$, where $\theta = \frac{3}{2}(207 + 33\sqrt{33})^{\frac{1}{3}} < 2.756$. Tao gave an elegant formulation of the proof of this bound using the notion of slice rank. Here we will use a similar approach, using the $G$-stable rank instead of the slice rank to get an explicit bound for all $n$ which has the same asymptotic behavior. We view $K^3$ as the vector space with basis $[0], [1], [2]$ where we view $0, 1, 2$ as elements in $\mathbb{F}_3$. More generally, we view $K^{3^n}$ as the vector space with basis $[a], a \in \mathbb{F}_3^n$. Note that $a, b, c$ form an arithmetic progression in $\mathbb{F}_3^n$ if and only if $a + b + c = 0$. Consider the tensor

$$v_n = \sum_{(a,b,c) \in \mathbb{F}_3^n} [a] \otimes [b] \otimes [c] = \sum_{(a,b,c) \in \mathbb{F}_3^n} [a,b,c] \in K^{3^n} \otimes K^{3^n} \otimes K^{3^n}.$$ 

Suppose that $S \subseteq \mathbb{F}_3^n$ is a set without arithmetic progression. Then we have

$$w = \sum_{(a,b,c) \in S^3} [a,b,c] \in K^3 \otimes K^3 \otimes K^3 = \sum_{a \in S} [a,a,a]$$

The tensor $w$ is a projection of $v$ and lies in the orbit closure of $v$. In particular, we have $\text{rk}^G(w) \leq \text{rk}^G(v)$. Since $w$ is a direct sum of $|S|$ rank 1 tensors, we get $\text{rk}^G(w) \geq |S|$ by Proposition 3.8. So we have $\text{rk}^G(v) \geq \text{rk}^G(w) \geq |S|$.

We will work over the field $K = \mathbb{F}_3$. For a function $f : \mathbb{F}_3^n \to \mathbb{F}_3$ we define

$$\langle f \rangle = \sum_{a \in \mathbb{F}_3^n} f(a)[a] \in K^{3^n}.$$ 

In particular, we have $\langle 1 \rangle = [0] + [1] + [2]$, $\langle x \rangle = [1] + 2[2] = [1] - [2]$ and $\langle x^2 \rangle = [1] + [2]$. A basis of $K^{3^n}$ is formed by taking all $\langle p(x) \rangle$ where $p(x) = p(x_1, \ldots, x_n)$ is a polynomial of degree $\leq 2$ in each of the variables $x_1, x_2, \ldots, x_n$. With respect to the basis $\langle 1 \rangle, \langle x \rangle, \langle x^2 \rangle$, we have $v_n = \langle f \rangle$ where $f : \mathbb{F}_3^n \times \mathbb{F}_3^n \times \mathbb{F}_3^n \to \mathbb{F}_3$ is given by

$$f(x, y, z) = \begin{cases} 1 & \text{if } x + y + z = 0; \\ 0 & \text{otherwise}. \end{cases}$$

For $n = 1$ we have $v_1 = \langle f \rangle$ where $f : \mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{F}_3 \to \mathbb{F}_3$ is given by $f(x, y, z) = 1 - (x + y + z)^2 = 1 - x^2 - y^2 - z^2 + x + y + z$. So we have

$$v_1 = \langle 1, 1, 1 \rangle = \langle x^2, 1, 1 \rangle - \langle 1, x^2, 1 \rangle - \langle 1, 1, x^2 \rangle = \langle 1, x, x \rangle + \langle 1, x, 1 \rangle + \langle x, x, 1 \rangle.$$ 

The support of $S$ with respect to the basis $\langle 1 \rangle, \langle x \rangle, \langle x^2 \rangle$ is

$$\{(0,0,0), (2,0,0), (0,2,0), (0,0,2), (0,1,1), (1,0,1), (1,1,0)\}$$

An optimal solution to the linear program is $x(1,0) = x(2,0) = x(3,0) = \frac{1}{2}$, $x(1,1) = x(2,1) = x(3,1) = \frac{1}{4}$ and $x(1,2) = x(2,2) = x(3,2) = 0$, which gives $\text{rk}^G(v) \geq \text{rk}^T(v) = \sum_{i,j} x(i,j) = 0 \frac{9}{4} = 2.25$. An optimal solution for the dual program is $y(2,0,0) = y(0,2,0) = y(0,0,2) = \frac{1}{4}$ and $y(0,1,1) = y(1,0,1) = y(1,1,0) = \frac{1}{2}$ and $y(0,0,0) = 0$.

The support of the tensor $v^{3^n} = v \boxtimes v \boxtimes \cdots \boxtimes v$ is contained in the set

$$T_n = \{(\lambda, \mu, \nu) \in (\{0,1,2\}^n)^3 \mid |\lambda| \leq 2n, |\mu| \leq 2n, |\nu| \leq 2n\}.$$
We will give a solution to the linear program $\mathbf{LP}(S^n)$ that we conjecture to be optimal. Whether optimal or not, it will give an upper bound for the $G$-stable rank of $v^{2n}$. Suppose that $t_0, t_1, t_2, \ldots, t_{2n} \geq 0$ are numbers such that $t_i + t_j + t_k \geq 1$ whenever $i + j + k \leq 2n$. If we define $x(i, \lambda) = t_{|\lambda|}$ for all $\lambda \in \{0, 1, 2\}^n$, and $i = 1, 2, 3$ then we have $x(1, \lambda) + x(2, \mu) + x(3, \nu) = t_{|\lambda|} + t_{|\mu|} + t_{|\nu|} \geq 1$, so we have a solution to the linear program. So we get

$$\text{rk}^G(v) \leq 3 \sum_{i=1}^{3n} x(i, \lambda) = 3 \sum_{\lambda} t_{|\lambda|} = 3 \sum_{i=0}^{2n} f_{n,i} t_i$$

where $f_{n,i}$ is the number of solutions to $a_1 + a_2 + \cdots + a_n = d$ with $a_1, a_2, \ldots, a_n \in \{0, 1, 2\}$. So $f_{n,i}$ is the coefficient of $x^i$ in $(1 + x + x^2)^n$. To choose the $t$’s optimally, we have to solve a linear program by minimizing $3 \sum_{i=0}^{2n} f_{n,i} t_i$ under the constraints:

1. $t_i + t_j + t_k \geq 1$ if $i + j + k \leq 2n$;
2. $t_i \geq 0$ for all $i$.

We get the following optimal solutions for the $t_i$:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | UB | EG | EG | best cap set |
|-----|---|---|---|---|---|---|---|----|----|----|--------------|
| 1   | $f_{1,i}$ | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 2 | 3 | 3 | 2 |
| $t_i$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | 0 | 0 | | | | |
| 2   | $f_{2,i}$ | 1 | 2 | 3 | 2 | 1 | 0 | 0 | 6 | 7 | 9 | 4 |
| $t_i$ | $\frac{3}{5}$ | $\frac{2}{5}$ | $\frac{1}{5}$ | 0 | 0 | 0 | 0 | | | | |
| 3   | $f_{3,i}$ | 1 | 3 | 6 | 7 | 6 | 3 | 1 | 15 | 18 | 30 | 9 |
| $t_i$ | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 | 0 | 0 | 0 | 0 | | | | |
| 4   | $f_{4,i}$ | 1 | 4 | 10 | 16 | 19 | 16 | 10 | 39 | 45 | 45 | 20 |
| $t_i$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 0 | | | | |
| 5   | $f_{5,i}$ | 1 | 5 | 15 | 30 | 45 | 51 | 45 | 105 | 123 | 153 | 45 |
| $t_i$ | $\frac{3}{5}$ | $\frac{3}{5}$ | $\frac{2}{5}$ | $\frac{2}{5}$ | $\frac{1}{5}$ | 0 | 0 | | | | |
| 6   | $f_{6,i}$ | 1 | 6 | 21 | 50 | 90 | 126 | 141 | 274 | 324 | 504 | 112 |
| $t_i$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 0 | 0 | | | | |

In the table, the column UB gives the value of $3 \sum_{i=0}^{2n} f_{n,i} t_i$, which is an upper bound for the $G$-stable rank and the cardinality of a cap set in $\mathbb{F}_3^n$. The column labeled “best cap set” gives the cardinality of the largest known cap set in $\mathbb{F}_3^n$. The column EG gives the Ellenberg–Gijswijt upper bound, which is $3 \sum_{i=0}^{\lfloor \frac{4n}{3} \rfloor} f_{n,i}$. This estimate relies on the fact that if $i, j, k$ are nonnegative integers with $i + j + k \leq 2n$, then it follows that $\min\{i, j, k\} \leq \lfloor \frac{2n}{3} \rfloor$. But one can say something stronger, namely $i \leq \lfloor \frac{2n}{3} \rfloor$, $j \leq \lfloor \frac{2n-1}{3} \rfloor$ or $k \leq \lfloor \frac{2n-2}{3} \rfloor$. This observation gives a better bound that is still based on the slice rank in the column labeled EG’.

In the table of Section 1.3 we have computed the optimal value of $3 \sum_{i=0}^{2n} f_{n,i} t_i$ rounded down to the nearest integer for $n \leq 20$. This bound is an upper bound for the cardinality of a cap set in $\mathbb{F}_3^n$.

Looking at optimal solutions for small $n$, we make the following conjecture:
Conjecture 6.1. The optimal solution of the linear program for \( t_0, t_1, t_2, \ldots, t_{2n} \) is as follows:

\[
\begin{aligned}
&1, 1, \ldots, 1, \frac{2}{3}, \frac{1}{3}, 0, 0, \ldots & \text{if } n \equiv 0 \mod 3 \\
&1, 1, \ldots, 1, \frac{3}{4}, \frac{1}{4}, 0, 0, \ldots & \text{if } n \equiv 1 \mod 3 \\
&1, 1, \ldots, 1, \frac{4}{5}, \frac{2}{5}, \frac{1}{5}, 0, 0, \ldots & \text{if } n \equiv 2 \mod 3
\end{aligned}
\]

7. Conclusion and further directions

The \( G \)-stable rank is a new notion of rank for tensors. Up to a constant it is equal to the slice rank, but it is more refined in the sense that it can take non-integer values, and unlike the slice rank it is supermultiplicative with respect to vertical tensor products. As an illustration, we showed that the \( G \)-stable rank can be used to improve upper bounds for the cardinality of cap sets. A proof of Conjecture 6.1 may lead to stronger asymptotic upper bounds for the cap set problem. Numerical experiments suggest an upper bound of the form \( C\theta^n/\sqrt{n} \) for some constant \( C \).

Besides algebraic applications of tensor decompositions there are also many numerical applications such as psychometrics [7, 15, 35–37] and chemometrics [1]. For more details and references, see the survey article [2] or the books [25, 26]. The formula (2) allows us to compute or approximate the \( G \)-stable rank for real or complex tensors using optimization. Future directions of research include algorithms for approximating the \( G \)-stable rank of a tensor, or to approximate a given tensors by tensors of low \( G \)-stable rank and apply these to such tasks as denoising, dimension reduction and tensor completion.

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