Semidefinite tests for quantum network topologies

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Quantum networks play a major role in long-distance communication, quantum cryptography, clock synchronization, and distributed quantum computing. Generally, these protocols involve many independent sources sharing entanglement among distant parties that, upon measuring their systems, generate correlations across the network. The question of which correlations a given quantum network can give rise to, remains almost uncharted. We show that the network topology alone imposes strong constraints on observable covariances, which yield tests that can be cast as semidefinite programs, thus allowing for efficient characterizations of the correlations in arbitrary quantum networks, as well as systematic derivations of device-independent and experimentally testable witnesses. We obtain such semidefinite tests for fixed measurement settings, as well as parties that independently choose among collections of measurement settings. The applicability of the method is demonstrated for various networks, and compared with previous approaches.

A basic scientific goal is the development of causal models explaining observed phenomena. Through the mathematical theory of causality, empirical data can be turned into a causal hypothesis that can be falsified or refined by new observations [1,3]. Not surprisingly, this causal framework has found many applications, ranging from economics [4,5] to biology and medicine [6,7] and also quantum physics [8,14]. Indeed, Bell’s theorem [14] can be regarded as a particular case of a causal inference problem [15,16], and the phenomenon generally known as quantum nonlocality shows that quantum correlations are incompatible with our classical notion of cause and effect [17].

More recently, in view of steady experimental advances [18–21], understanding the role of causality in quantum networks of growing size and complexity, typically composed of many independent sources of entanglement, has become a topic of particular relevance [22]. On the practical side, quantum correlations can be distributed across the whole network via quantum repeaters [23]. Fundamentally, new and stronger notions of nonlocality can emerge in such quantum networks [24] and lead to novel quantum information protocols.

In spite of its clear importance, the characterization of correlations in quantum networks remains in its infancy. Even in the simplest case of two distant laboratories sharing quantum states, this characterization turns out to be extremely demanding [25]. The situation for more complex quantum networks, such as the quantum internet [26,27], and quantum repeaters [23], is yet more intractable. Due to the independence of entanglement sources, correlations compatible with a quantum network form a non-convex set [25,28] that, even in a purely classical case, cannot be easily characterized beyond very small networks [30,32]. To circumvent this problem, novel approaches have recently been proposed to characterize quantum causal structures [19,33,34]. In this context, we consider arbitrary quantum networks where a number of distant parties share quantum states provided by several independent sources, and we introduce a method to characterize the covariances that are compatible with such quantum networks, irrespective of whether the underlying sources are classical or quantum. That is, we determine the constraints on correlations arising from the topology of the network alone. Despite the non-convex nature of the underlying problem, our approach can be implemented via efficient semidefinite programs, which moreover allows for the derivation of device-independent witnesses for the network topology. We demonstrate the applicability of our method, and show that it significantly improves over previous approaches.

We consider typical quantum networks with two types of vertices: i) quantum states distributed among several distant parties, and ii) classical variables standing for the outcomes of measurements performed on such states. These networks can be represented as directed acyclic graphs (DAGs) $G = (V,E)$, consisting of a set of vertices $V$ and directed edges $E$ together with a bipartition of the vertex-set $V$ into the latent set (quantum states) and the observable set (measurement outcomes). Due to the state-distribution scenario, we here exclusively consider the class of DAGs where all edges are directed from latent vertices to observable vertices, but with no edges within these two groups (see Fig. 1).

For this reason, we refer to the elements of the latent set as parents $p_n$, and the elements in the observable set as children $c_m$.

Although the quantum state $\rho_{nm}$, associated to each parent $p_n$, can be entangled internally, the parents have
no correlations between each other, which results in the joint state \( \rho = \rho_{p_1} \otimes \cdots \otimes \rho_{p_N} \). The parents distribute the quantum systems to their children, where child \( c_m \) measures an arbitrary positive-operator valued measure (POVM) \( \{ A^{(m)}_n \}_n \) (see Fig. 1). For a given graph \( G \) with \( N \) parents and \( M \) children, the joint measurement outcome \( x_1, \ldots, x_M \) thus occurs with a probability

\[
P(x_1, \ldots, x_M) = \text{Tr}(A^{(1)}_{x_1} \otimes \cdots \otimes A^{(M)}_{x_M} \rho_{p_1} \otimes \cdots \otimes \rho_{p_N}).
\]

The question is whether a quantum causal structure \( G \) can 'explain' an observed distribution, in the sense that it can be written as in (1), for some choices of states and POVMs that are compatible with \( G \). Proposition 1 below, provides a method to falsify such causal explanations. For this purpose, we consider a collection of orthogonal Hilbert spaces \( V_1, \ldots, V_M \), and let \( Y := \bigoplus_{m=1}^M V_m \). To each possible outcome \( x_m \) of child \( c_m \) we associate a vector \( Y_{x,m}^{(m)} \in V_m \). This type of mappings is often referred to as feature maps [16], and can be chosen freely as part of the analysis. Here, we take the liberty of overloading the notation, and let \( Y^{(m)} \) denote both the feature map, and the random vector that results from applying the feature map to the random measurement outcomes \( x_m \). By combining all the children, we get the global random vector \( Y := \sum_{m=1}^M Y^{(m)} \) with elements \( Y_{x_1,\ldots,x_M} = Y^{(1)}_{x_1} + \cdots + Y^{(M)}_{x_M} \).

Broadly speaking, Proposition 1 says that the covariance matrix \( \text{Cov}(Y) = E(YY^\dagger) - E(Y)E(Y)^\dagger \) necessarily satisfies a specific semidefinite decomposition, Eq. (2), determined by \( G \). This can be viewed as the quantum generalization of a similar result for classical networks in [37] (see Appendix A for a brief summary).

**Proposition 1.** Let the distribution \( P(x_1, \ldots, x_M) \) be compatible, in the sense of (1), with a quantum causal structure \( G \) with parents \( p_1, \ldots, p_N \) and children \( c_1, \ldots, c_M \). Assume that each child \( c_m \) is assigned a feature map \( Y^{(m)} \) into a vector space \( V_m \). Then there exist operators \( R \) and \( \{ C_n \}_{n=1}^N \) on \( V := \bigoplus_{m=1}^M V_m \) such that the covariance matrix of \( Y = \sum_{m=1}^M Y^{(m)} \) satisfies

\[
\text{Cov}(Y) = R + \sum_{n=1}^N C_n, \quad R \geq 0, \quad C_n \geq 0, \quad (2)
\]

\[
p^{(n)} C_n p^{(n)} = C_n, \quad R = \sum_{m=1}^M P_m R P_m, \quad (3)
\]

where the projectors \( p^{(n)} := \sum_{m \in C_n} P_m \) with respect to the given DAG \( G \), and where \( P_m \) is the projector onto \( V_m \).

The proof of Proposition 1 is presented in Appendix B. Since the semidefinite decomposition in Eq. (2) is a necessary condition, it defines a 'semidefinite test' whose failure falsifies \( G \) as an explanation of the observed distribution. This test is identical to the classical counterpart in [37] (see Appendix A), and thus reflects the network topology, irrespective of the classical or quantum nature of the underlying sources of correlations. The semidefinite test can moreover be cast as a semidefinite program, which is efficiently solved via standard convex optimization tools (see Appendix C), in spite of the non-convex nature of the underlying problem. From technical point of view, the semidefi-
FIG. 2. Tests for 4-partite scenarios. Analogous to Fig. 1 we consider the family of four-partite distributions $P_{p,q} = p\delta_{0000} + q\delta_{1111} + (1 - p - q)(1 - \delta_{0000} - \delta_{1111})/14$, with $p, q \geq 0$ and $p + q \leq 1$. We test for values of $p$ and $q$ for which the resulting covariance matrix is compatible with two quantum scenarios: a) Six parents (grey disks), each with two children (white disks). b) Four parents, each with three children. c) For all values of $p, q$ above the solid blue curve, our test rejects scenario a) as an explanation of $P_{p,q}$. Above the dotted dashed curve, the test rejects scenario b) as an explanation of $P_{p,q}$. In scenario a) we moreover compare with the Finner-test [54], which rejects all cases above the dashed red curve.

One can furthermore derive general purpose witnesses. An equivalent (dual) formulation of (2) yields a Hermitian matrix $W$, with the same dimensions as $\text{Cov}(Y)$, such that $\text{Tr}(WC) \leq 0$, for any covariance matrix $C$ that admits the decomposition in Eq. (3). For example, measuring the observable $s'_z$ on each qubit of a GHZ state $|\text{GHZ}\rangle = (\sqrt{1/2})(|000\rangle + |111\rangle)$ generates the distribution $P_{\text{GHZ}} = \delta_{000}/2 + \delta_{111}/2$ with $\delta_{abc}(x_1, x_2, x_3) = \delta_{a,x_1}\delta_{b,x_2}\delta_{c,x_3}$, which has the optimal witness (See Appendix C)

$$W_{\text{GHZ}} := \frac{1}{6} \begin{bmatrix} -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix}.$$

For the corresponding covariance matrix $C_{\text{GHZ}}$, we obtain $\text{Tr}(W C_{\text{GHZ}}) = 0.5$, revealing that the distribution is incompatible with the triangle network (Fig. 1). In Appendix C we show the explicit form of the dual problem, as well as generalized form of the witness for more parties.

To demonstrate the applicability of the semidefinite test, we apply it to the family of distributions

$$P_{p,q} = p\delta_{000} + q\delta_{111} + \frac{1}{6}(1 - p - q)(1 - \delta_{000} - \delta_{111}),$$

where $p, q \geq 0$ and $p + q \leq 1$. As shown in Fig. 1 the semidefinite test rejects the triangular quantum DAG as a causal explanation for a substantially larger range of distributions $P_{p,q}$ than the quantum Finner inequality in [34]. Moreover, while the latter inequality only holds for networks where each source is connected with at most two parties, Fig. 2 illustrates that our method allows for connections to any number of parties.

Next we consider a generalization where each child $c_m$ can choose measurement settings $s_m$ modeled via a collection of POVMs $A^{(m,s_m)} = \{A_x^{(m,s_m)}\}_x$ (see Fig. 3). The distribution of the measurement outcomes $x_1, \ldots, x_M$, conditioned on the choices of measurement settings $s_1, \ldots, s_M$, is

$$P(x_1, \ldots, x_M | s_1, \ldots, s_M) = \text{Tr}([A^{(1,s_1)}_1 \otimes \cdots \otimes A^{(M,s_M)}_M][\rho_{p_1} \otimes \cdots \otimes \rho_{p_M}]).$$

We moreover associate a feature map $\gamma^{(m,s_m)}$, to each child $c_m$ and each choice of measurement $s_m$, which maps into subspaces $\mathcal{Y}^{(m,s_m)}$, each associated with a projector $P_{m,s_m}$. In general, the different choices of POVMs correspond to non-commuting observables, thus implying that covariances of the form $\text{Cov}(\gamma^{(m,s_m)}, \gamma^{(m',s'_m)})$, for $s_m \neq s'_m$, are not empirically observable. Based on the observable covariances only, we define the observable covariance matrix

$$C_{\text{observable}} := \sum_{m \neq m'} \sum_{s_m} \text{Cov}(\gamma^{(m,s_m)}) + \sum_{m \neq m': m \neq m'} \sum_{s_m, s'_m} \text{Cov}(\gamma^{(m,s_m)}, \gamma^{(m',s'_m)}),$$

where the unobservable covariances correspond zero ‘blocks’ $P_{m,s_m}C_{\text{observable}}P_{m',s'_m} = 0$ for $s_m \neq s'_m$. It turns
out that one can complete the zero blocks in $C_{\text{observable}}$ with a matrix $C_{\text{completion}}$, such that $C := C_{\text{observable}} + C_{\text{completion}}$ is positive semidefinite. Moreover, $C_{\text{completion}}$ is non-zero only on the blocks associated to the unobservable covariances, and thus

$$
P_{m,s_n} C_{\text{completion}} P_{m,s_n} = 0, \quad P_{m,s_n} C_{\text{completion}} P_{m',s'_n} = 0, \quad m \neq m'.
$$

The completed covariance matrix can be decomposed according to the DAG $G$, as detailed by the following proposition.

**Proposition 2.** Let the conditional distribution $P(x_1, \ldots, x_M|z_1, \ldots, z_M)$ be compatible, in the sense of [6], with the quantum causal structure $G$ with parents $p_1, \ldots, p_N$ and children $c_1, \ldots, c_M$, with associated inputs $s_1, \ldots, s_M$. Assume that each child $c_m$, and each input $s_m$, is assigned a feature map $Y^{(m,s_m)}$ into a vector space $V_{m,s_m}$. Let the operator $C_{\text{observable}}$ on $V := \bigoplus_{m=1}^M \bigoplus_{s_m} V_{m,s_m}$ be as defined in [7]. Then there exist operators $C_{\text{completion}}$, $R$, and $(C_n)_{n=1}^N$ on $V$, such that $C_{\text{completion}}$ satisfies (8) and

$$
C := C_{\text{observable}} + C_{\text{completion}} \geq 0,
$$

and

$$
C = R + \sum_{n=1}^N C_n, \quad R \geq 0, \quad C_n \geq 0,
$$

$$
p^{(n)} C_n p^{(n)} = C_n, \quad R = \sum_{m,s_m} P_{m,s_n} R P_{m,s_{n'}}
$$

$$
p^{(n)} := \sum_{m \in \mathcal{C}_n} \sum_{s_m} P_{m,s_n} P_{m,s_{n'}}
$$

where $P_{m,s_n}$ is the projector onto $V_{m,s_n}$.

For the sake of completeness, a classical counterpart of this proposition is provided in Appendix [2], while the proof for Proposition [2] is presented in Appendix [2]. The perhaps surprising conclusion is that even though we add inputs, the semidefinite test still only depends on the network topology, without distinguishing the classical and quantum case. As an application of Proposition [2] we tested the nonlocal correlations arising from $\gamma_3$ and $\gamma_1$ measurements on a W-state $|W\rangle = (\sqrt{1/3})(|001\rangle + |010\rangle + |100\rangle)$ with visibility $v = 1/3$, that is, $\psi_v = v|W\rangle\langle W| + (1-v)/8$. We observe numerically that these nonlocal correlations violate our SDP test for visibilities above $v \approx 3/4$, thus witnessing their incompleteness with the quantum triangle with inputs.

In summary, we have presented a general and systematic method to characterize the correlations compatible with quantum networks of any topology, number of inputs and outputs, and involving an arbitrary number of sources. Irrespective of Hilbert space dimensions and the type of quantum measurements being performed, our results show that the topology of the quantum network alone implies constraints on the covariance matrix of distributions compatible with it. Our method can be efficiently implemented via an SDP, even though the original problem is non-convex. Furthermore, it allows for analytical derivations of experimentally testable constraints that can be understood as device-independent witnesses of the topology of the quantum network. In comparison with another recently proposed test [43], our approach not only provides a significantly better description (see Figs. 1 and 2) but can also be applied on a wider range of quantum networks.

Given the ubiquitous role of quantum networks, we believe that our approach, together with other recently proposed alternatives [10, 33, 34, 42, 43], offer a novel suite of tools for network related problems, such as multipartite secure communication [44], distributed computing [45], quantum-repeaters [23], or any other tasks where quantum networks might play a role.

An interesting open problem is whether our approach, based on the covariance of the observed correlations, can be generalized to include higher-order moments of the distribution, thus providing a tighter description of the quantum set of correlations. We hope that our results will trigger further developments in these directions.

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Appendix A: Brief summary: Semidefinite decompositions for classical networks

For convenience, we here restate the core result on the classical semidefinite test obtained in [37]. As explained in the main text, we exclusively consider the class of DAGs where the vertices can be partitioned into a set of parents that is distinct from the set of children, and where the arrows only go from parent vertices to children vertices. This class of DAGs is in [37] referred to as ‘bipartite’. However, here we avoid this terminology, since in the quantum context, this may be misinterpreted as each parent having at most two children, i.e., that each parent only can create bipartite entanglement, while we here allow for scenarios where the parents can create quantum correlations between any number of children.

For this class of DAGs with parents $p_1, \ldots, p_N$ and children $c_1, \ldots, c_M$, the joint distribution of the parents (or latent variables) and children (observables) can be written

$$P(c_1, \ldots, c_M, p_1, \ldots, p_N) = \prod_{m=1}^{M} P(c_m|P_m) \prod_{n=1}^{N} P(p_n),$$

(A1)

where $P_m$ denotes the set of parents of child $c_m$. We say that an observed distribution $P(c_1, \ldots, c_M)$ is compatible with the given classical causal structure $G$ with parents $p_1, \ldots, p_N$ and children $c_1, \ldots, c_M$ (or that $G$ is a causal explanation of the distribution), if it is the margin of some distribution of the form (A1). In [37], it was shown that this structure induces a particular signature on any covariance matrix that can be constructed from the random observables $c_1, \ldots, c_M$. We form covariance matrices by assigning feature maps $Y^{(m)} := Y^{(m)}(c_m)$ into pairwise orthogonal vector spaces $V_m$. On the joint space $V := \bigoplus_{m=1}^{M} V_m$, we let $P_m$ denote the projector onto $V_m$.

The following proposition is taken from [37] (rephrased in order to better fit our discussions).

**Proposition 3** ([37]). Let the distribution $P(c_1, \ldots, c_M)$ be compatible with the classical causal structure $G$ with parents $p_1, \ldots, p_N$ and children $c_1, \ldots, c_M$. Assume that for each child $c_m$ there is assigned a feature map $Y^{(m)}$ into a vector space $V_m$. Then there exist operators $R$ and $(C_n)_{n=1}^{N}$ on $V := \bigoplus_{m=1}^{M} V_m$, such that the covariance matrix of $Y = \sum_{m=1}^{M} Y^{(m)}$ satisfies

$$\text{Cov}(Y) = R + \sum_{n=1}^{N} C_n, \quad R \geq 0, \quad C_n \geq 0,$$

(A2)

where

$$P^{(n)} C_n P^{(n)} = C_n, \quad R = \sum_{m=1}^{M} P_m R P_m,$$

(A3)
where the projectors $P^{(n)} := \sum_{m \in C_n} P_m$ with respect to the given DAG $G$, and where $P_m$ is the projector onto $V_m$.

Remark: From this proposition we can conclude that whenever we find a covariance matrix that cannot be decomposed in this manner, then we can exclude $G$ as an explanation of the observed covariance or underlying distribution. This necessary condition can be used to design an SDP-test that must be satisfied whenever $G$ is a causal explanation, as discussed in [37] (see also the discussion about the quantum counterpart in Section C).

Appendix B: Proof of Proposition 1

Before we turn to the setting and proof of Proposition 1, we first establish a technical lemma that not only will be important for the proof of Proposition 1, but also will play a similar role in Section E when we prove Proposition 2.

1. Technical Lemma

For a Hilbert space $\mathcal{H}$, we let $S(\mathcal{H})$ denote the set of density operators on $\mathcal{H}$. Given a family $\mathcal{H}^1, \ldots, \mathcal{H}^N$ of Hilbert spaces, let

$$\mathcal{H} := \bigotimes_{n=1}^N \mathcal{H}^n. \quad \text{(B1)}$$

In the following we consider linear operators $Q : \mathcal{H} \to \mathcal{H} \otimes \mathcal{V}$. For density operators $\varrho$ on $\mathcal{H}$, we often consider objects such as $\text{Tr}_\mathcal{H}(Q \varrho)$, $\text{Tr}_\mathcal{H}(Q^* \varrho)$ and $\text{Tr}_\mathcal{H}(Q Q^* \varrho)$, where $\text{Tr}_\mathcal{H}$ denotes the partial trace with respect to $\mathcal{H}$. While $\text{Tr}_\mathcal{H}(Q \varrho)$ is an element of $\mathcal{V}$, and $\text{Tr}_\mathcal{H}(Q^* \varrho)$ is an element of the dual $\mathcal{V}^*$, $\text{Tr}_\mathcal{H}(Q Q^* \varrho)$ is an element of the space $L(\mathcal{V})$ of linear operators on $\mathcal{V}$. More specifically, if one chooses $\{|\psi_k\rangle\}_k$ as an orthonormal basis for $\mathcal{H}$ and $\{|a_j\rangle\}_j$ as a basis for $\mathcal{V}$, then any linear operator $Q : \mathcal{H} \to \mathcal{H} \otimes \mathcal{V}$ decomposes as

$$Q = \sum_j \sum_{k,l} Q_{k,l}^j |\psi_k\rangle|a_j\rangle \langle \psi_l|. \quad \text{(B2)}$$

The partial traces thus becomes

$$\text{Tr}_\mathcal{H}(Q \varrho) = \sum_j \sum_{k} Q_{k,j}^j (|\psi_k\rangle|e\rangle|\psi_l\rangle|a_j\rangle) \in \mathcal{V},$$

$$\text{Tr}_\mathcal{H}(Q^* \varrho) = \sum_{j,k,l} (Q^*)_{k,j}^j (|\psi_k\rangle|e\rangle|\psi_l\rangle|a_j\rangle) \in \mathcal{V}^*, \quad \text{(B3)}$$

$$\text{Tr}_\mathcal{H}(Q Q^* \varrho) = \sum_{j,k,l,k',l'} Q_{k,j}^j (Q^*)_{k',j}^{k'} (|\psi_k\rangle|e\rangle|\psi_l\rangle|a_j\rangle) \in L(\mathcal{V}).$$

Lemma 1. Let $Q : \mathcal{H} \to \mathcal{H} \otimes \mathcal{V}$ be a linear operator, and let $\varrho_n \in S(\mathcal{H}^n)$ for each $n = 1, \ldots, N$. Define $\varrho := \varrho_1 \otimes \cdots \otimes \varrho_N$. Then

$$\text{Tr}_\mathcal{H}(Q Q^* \varrho) - \text{Tr}_\mathcal{H}(Q \varrho) \text{Tr}_\mathcal{H}(Q^* \varrho) = \sum_{n=1}^N C_n, \quad \text{(B4)}$$
where
\[
C_1 := \text{Tr}_{P_1, \ldots, P_N} \left[ \left( Q - \hat{1}_{P_1} \otimes \text{Tr}_{P_1}(Qe_{P_1}) \right) \left( Q - \hat{1}_{P_1} \otimes \text{Tr}_{P_1}(Qe_{P_1}) \right)^\dagger [e_{P_1} \otimes \cdots \otimes e_{P_N}] \right],
\]
\[
C_n := \text{Tr}_{P_n, \ldots, P_N} \left[ \left( \text{Tr}_{P_1, \ldots, P_{n-1}}(Q[e_{P_1} \otimes \cdots \otimes e_{P_{n-1}}]) - \hat{1}_{P_n} \otimes \text{Tr}_{P_1, \ldots, P_n}(Q[e_{P_1} \otimes \cdots \otimes e_{P_n}]) \right) \right] (B5)
\]
\[
\left( \text{Tr}_{P_1, \ldots, P_{n-1}}(Q[e_{P_1} \otimes \cdots \otimes e_{P_{n-1}}]) - \hat{1}_{P_n} \otimes \text{Tr}_{P_1, \ldots, P_n}(Q[e_{P_1} \otimes \cdots \otimes e_{P_n}]) \right)^\dagger [e_{P_n} \otimes \cdots \otimes e_{P_N}], \quad 2 \leq n \leq N - 1,
\]
\[
C_N := \text{Tr}_{P_N} \left[ \left( \text{Tr}_{P_1, \ldots, P_{N-1}}(Q[e_{P_1} \otimes \cdots \otimes e_{P_{N-1}}]) - \hat{1}_{P_N} \otimes \text{Tr}_{P_1, \ldots, P_N}(Q[e_{P_1} \otimes \cdots \otimes e_{P_N}]) \right) \right] (B6)
\]
\[
\left( \text{Tr}_{P_1, \ldots, P_{N-1}}(Q[e_{P_1} \otimes \cdots \otimes e_{P_{N-1}}]) - \hat{1}_{P_N} \otimes \text{Tr}_{P_1, \ldots, P_N}(Q[e_{P_1} \otimes \cdots \otimes e_{P_N}]) \right)^\dagger [e_{P_N}],
\]
One can realize that each \( C_n \) is a positive semi-definite operator on \( \mathcal{V} \).

Proof. We break this proof into two main pieces. First of all, one should note that Eq. (B4) can be rewritten as the telescopic sum
\[
\text{Tr}(QQ^\dagger e) - \text{Tr}(Qe)\text{Tr}(Q^\dagger e) = \sum_{n=1}^{N} D_n,
\]
where
\[
D_1 := \text{Tr}_{P_1, \ldots, P_N} \left[ [QQ^\dagger[e_{P_1} \otimes \cdots \otimes e_{P_N}]] - \text{Tr}_{P_2, \ldots, P_N} \left[ \text{Tr}_{P_1}(Qe_{P_1}) \text{Tr}_{P_1}(Q^\dagger e_{P_1}) \otimes [e_{P_2} \otimes \cdots \otimes e_{P_N}] \right] \right],
\]
\[
D_n := \text{Tr}_{P_n, \ldots, P_N} \left[ \text{Tr}_{P_1, \ldots, P_{n-1}}([Q[e_{P_1} \otimes \cdots \otimes e_{P_{n-1}}]] \text{Tr}_{P_1, \ldots, P_{n-1}}(Q^\dagger[e_{P_1} \otimes \cdots \otimes e_{P_{n-1}}]) \otimes [e_{P_n} \otimes \cdots \otimes e_{P_N}] \right] \right] (B7)
\]
\[
- \text{Tr}_{P_{n+1}, \ldots, P_N} \left[ \text{Tr}_{P_1, \ldots, P_n} ([Q[e_{P_1} \otimes \cdots \otimes e_{P_n}]] \text{Tr}_{P_1, \ldots, P_n} (Q^\dagger[e_{P_1} \otimes \cdots \otimes e_{P_n}]) \otimes [e_{P_{n+1}} \otimes \cdots \otimes e_{P_N}] \right],
\]
\[
2 \leq n \leq N - 1,
\]
\[
D_N := \text{Tr}_{P_N} \left[ \text{Tr}_{P_1, \ldots, P_{N-1}}([Q[e_{P_1} \otimes \cdots \otimes e_{P_{N-1}}]] \text{Tr}_{P_1, \ldots, P_{N-1}}(Q^\dagger[e_{P_1} \otimes \cdots \otimes e_{P_{N-1}}]) \otimes e_{P_N} \right] \right] (B6)
\]
\[
- \text{Tr}_{P_1, \ldots, P_N} \left[ [Q[e_{P_1} \otimes \cdots \otimes e_{P_N}]] \text{Tr}_{P_1, \ldots, P_N} (Q^\dagger[e_{P_1} \otimes \cdots \otimes e_{P_N}]) \right] \right],
\]
The second step consists in noticing that for each \( n = 1, \ldots, N \) one has \( D_n = C_n \) with \( C_n \) as in (B5). \( \square \)

2. Semidefinite decompositions for quantum networks

Let \( G \) be a graph with \( N \) parents \( p_1, \ldots, p_N \), and \( M \) children \( c_1, \ldots, c_M \). Recall that \( C_n \) denotes the set of children of parent \( p_n \), and \( P_m \) denotes the set of parents to child \( c_m \). Also recall that we here restrict to the class of graphs \( G \) where there only are arrows directed from the set of parents to the set of children, i.e., no parent is itself a child, and no child itself a parent.

For each parent \( p_n \) we assume a collection of Hilbert spaces \( \{ \mathcal{H}_m \}_{m \in C_n} \), and define
\[
\mathcal{H}_{c_n} := \bigotimes_{n \in P_m} \mathcal{H}_m, \quad \mathcal{H}^{p_n} := \bigotimes_{m \in C_n} \mathcal{H}_m, \quad \mathcal{H} := \bigotimes_{m=1}^{M} \bigotimes_{n \in C_m} \mathcal{H}_m = \bigotimes_{m=1}^{M} \mathcal{H}_{c_n} = \bigotimes_{n=1}^{N} \mathcal{H}^{p_n}. \quad (B8)
\]
The interpretation of this structure is that $\mathcal{H}_{m}^n$ is the Hilbert space of the system that parent $p_n$ sends to child $c_m$ (if $p_n$ indeed is a parent of $c_m$).

We let $\varrho_n \in \mathcal{S}(\mathcal{H}^{p_n})$ for $n = 1, \ldots, N$, and the define the joint state
\[ \varrho := \varrho_1 \otimes \cdots \otimes \varrho_N \in \mathcal{S}(\mathcal{H}). \]  
(B9)

Hence, the parents are all uncorrelated.

On each of the Hilbert spaces $\mathcal{H}_{c_m}$ we assign a POVM $\{ A_{x_m}^{(m)} \}_{x_m}$. From the collection of POVMs $\{ A^{(1)}_1 \}, \ldots, \{ A^{(M)}_M \}$, we define the joint POVM
\[ A_{x_1, \ldots, x_M} := A^{(1)}_{x_1} \otimes \cdots \otimes A^{(M)}_{x_M}. \]  
(B10)

By measuring this joint POVM on the state $\varrho$, it follows that the outcome (or observation) $x_1, \ldots, x_M$ occurs with the probability
\[ P(x_1, \ldots, x_M) = \text{Tr}([A^{(1)}_{x_1} \otimes \cdots \otimes A^{(M)}_{x_M}] | \varrho_1 \otimes \cdots \otimes \varrho_N). \]  
(B11)

We say that an observed distribution $P(x_1, \ldots, x_M)$ is compatible with the given quantum causal structure $G$ with parents $p_1, \ldots, p_N$ and children $c_1, \ldots, c_M$, if it can be written as in (B11), for some state as in (B9) and some POVM as in (B10).

Analogous to the classical case in Section A, we assign a feature map $Y^{(m)}$ to each child $c_m$, i.e., to each measurement outcome $x_m$, corresponding to POVM element $A_{x_m}^{(m)}$, we associate an element $Y^{(m)}_{x_m}$ in a vector space $\mathcal{V}_m$. We also define $\mathcal{V} := \bigoplus_{m=1}^M \mathcal{V}_m$. The joint feature map for the joint measurement of all children is
\[ Y_{x_1, \ldots, x_M} := Y^{(1)}_{x_1} + \cdots + Y^{(M)}_{x_M} \in \mathcal{V}. \]

We let $P_m$ denote the projector onto the subspace $\mathcal{V}_m$, and for each $n = 1, \ldots, N$ we define the projector
\[ P^{(n)} := \sum_{m \in C_n} P_m. \]  
(B12)

For the sake of convenience, we here restate Proposition 1 in the main text.

**Proposition 4.** Let the distribution $P(x_1, \ldots, x_M)$ be compatible, in the sense of (B11), with a quantum causal structure $G$ with parents $p_1, \ldots, p_N$ and children $c_1, \ldots, c_M$. Assume that each child $c_m$ is assigned a feature map $Y^{(m)}$ into a vector space $\mathcal{V}_m$. Then there exist operators $R$ and $(C_n)_{n=1}^N$ on $\mathcal{V} := \bigoplus_{m=1}^M \mathcal{V}_m$, such that the covariance matrix of $Y = \sum_{m=1}^M Y^{(m)}$ satisfies
\[ \text{Cov}(Y) = R + \sum_{n=1}^N C_n, \quad R \geq 0, \quad C_n \geq 0, \]  
(B13)

\[ p^{(n)} C_n P^{(n)} = C_n, \quad R = \sum_{m=1}^M P_m R P_m, \]  
(B14)

where the projectors $P^{(n)} := \sum_{m \in C_n} P_m$ with respect to the given DAG $G$, and where $P_m$ is the projector onto $\mathcal{V}_m$.

**Proof.** By the assumption that $P(x_1, \ldots, x_M)$ be compatible with the quantum causal structure $G$ with parents $p_1, \ldots, p_N$ and children $c_1, \ldots, c_M$, there exists a state on the form (B9), and a POVM of the form (B10), such that $P(x_1, \ldots, x_M)$ can be written as in (B11). With the assigned feature maps $Y^{(m)}$ each outcome $x_m$ of the measurement of the POVM $\{ A_{x_m}^{(m)} \}_{x_m}$ corresponds to a vector in $Y^{(m)}_{x_m}$. The measurement of the POVM $A_{x_1, \ldots, x_M}$ thus results in a random vector $Y$ on $\mathcal{V}$. In the following, we will show that the covariance matrix $\text{Cov}(Y)$ can be decomposed as in (B13) for $R \geq 0$ and $C_n \geq 0$. Define $Q \in \mathcal{V} \otimes \mathcal{L}(\mathcal{H})$ by
\[ Q := \sum_{x_1, \ldots, x_M} Y_{x_1, \ldots, x_M} \otimes A^{(1)}_{x_1} \otimes \cdots \otimes A^{(M)}_{x_M} \]
\[ = \sum_{x_1, \ldots, x_M} (Y^{(1)}_{x_1} + \cdots + Y^{(M)}_{x_M}) \otimes A^{(1)}_{x_1} \otimes \cdots \otimes A^{(M)}_{x_M} \]
\[ = \sum_{m=1}^M \sum_{x_m} Y^{(m)}_{x_m} \otimes A^{(m)}_{x_m} \otimes 1_{\mathcal{C}_m}, \]  
(B15)
where $\hat{1}_{/c_m}$ denotes the identity operator on $\mathcal{H}_{c_1} \otimes \cdots \otimes \mathcal{H}_{c_{m-1}} \otimes \mathcal{H}_{c_{m+1}} \otimes \cdots \otimes \mathcal{H}_{c_M}$, and where the last equality follows from $\{A^{(m)}_{x_m}\}_{x_m}$ being a POVM. Next we note that

$$\text{cov}(Y) = E(Y Y^\dagger) - E(Y) E(Y)^\dagger$$

$$= \sum_{x_1, \ldots, x_M} Y_{x_1, \ldots, x_M} Y_{x_1, \ldots, x_M}^\dagger \text{Tr}(A_{x_1, \ldots, x_M} e)$$

$$- \sum_{x_1, \ldots, x_M} Y_{x_1, \ldots, x_M} \text{Tr}(A_{x_1, \ldots, x_M} e) \sum_{x_1', \ldots, x_M'} Y_{x_1', \ldots, x_M'}^\dagger \text{Tr}(A_{x_1', \ldots, x_M'} e)$$

$$= \sum_{x_1, \ldots, x_M} Y_{x_1, \ldots, x_M} Y_{x_1, \ldots, x_M}^\dagger \text{Tr}(A_{x_1, \ldots, x_M} e)$$

$$- \text{Tr}(Q e) \text{Tr}(Q^\dagger e)$$

$$= R + \sum_{n=1}^N C_n,$$

where

$$R := \sum_{x_1, \ldots, x_M} Y_{x_1, \ldots, x_M} Y_{x_1, \ldots, x_M}^\dagger \text{Tr}(A_{x_1, \ldots, x_M} e) - \text{Tr}(Q Q^\dagger e),$$

and where $C_n$ is as in Lemma [1], which also yields $C_n \geq 0$.

Next we shall show that $P(n) C_n P(n) = C_n$, for the projectors $P(n)$ defined with respect to the given DAG $G$. For $2 \leq n \leq N - 1$, recall the definition of $C_n$ in (B5). With the definition of $Q$ in (B15), it follows that

$$C_n = \sum_{m, m' = 1}^M \sum_{x_m, x_m'} Y_{x_m, x_m'}^{(m)} Y_{x_m, x_m'}^{(m')} \text{Tr}_{p_{n-1}} P_n \left[ W_{x_m}^{m, n} W_{x_m'}^{m', n} [e_{p_1} \otimes \cdots \otimes e_{p_N}] \right],$$

where

$$W_{x_m}^{m, n} := \text{Tr}_{p_1, \ldots, p_{n-1}} (\left[ A_{x_m}^{(m)} \otimes \hat{1}_{/c_m} \right] [e_{p_1} \otimes \cdots \otimes e_{p_{n-1}}])$$

$$- \hat{1}_{/c_m} \otimes \text{Tr}_{p_1, \ldots, p_n} (\left[ A_{x_m}^{(m)} \otimes \hat{1}_{/c_m} \right] [e_{p_1} \otimes \cdots \otimes e_{p_n}]).$$

Suppose $m \notin C_n$. This implies that $\hat{1}_{/c_m} \otimes \text{Tr}_{p_n} ([A_{x_m}^{(m)} \otimes \hat{1}_{/c_m}] [e_{p_n}]) = A_{x_m}^{(m)} \otimes \hat{1}_{/c_m}$. Consequently $W_{x_m}^{m, n} = 0$, for all $x_m$. Hence, if $m \notin C_n$ or if $m' \notin C_n$, then

$$W_{x_m}^{m, n} W_{x_m'}^{m', n} = 0, \quad \forall x_m, x_m'.$$

Recall that $Y_{x_m}^{(m)}$ is supported on $V_m$. By comparing (B20) with with (B18) one can note that $P_{m} C_n P_{m'} = 0$ if $m \notin C_n$ or if $m' \notin C_n$. Thus

$$C_n = \sum_{m, m' = 1}^M P_m C_n P_{m'}$$

$$= \sum_{m, m' \in C_n} P_m C_n P_{m'}$$

$$= P(n) C_n P(n).$$

We can conclude that $P(n) C_n P(n) = C_n$, for $2 \leq n \leq N - 1$. The proofs for the cases $n = 1$ and $n = N$ are analogous.

Next we show that $R$, as defined in Eq. (B17), is positive semidefinite. Recall that $R$ is an operator on $V$. In addition, from the definition of $R$, the definition of $Q$ in (B15), and the fact that $\{A_{x_1, \ldots, x_M}\}_{x_1, \ldots, x_M}$ and thus
\[ \sum_{x_1, \ldots, x_M} A_{x_1, \ldots, x_M} = \hat{1}, \] we have
\[
R = \sum_{x_1, \ldots, x_M} \text{Tr}_H \left[ \left( Y_{x_1, \ldots, x_M} \otimes \hat{1} - \sum_{\tau_1, \ldots, \tau_M} Y_{\tau_1, \ldots, \tau_M} \otimes A_{\tau_1, \ldots, \tau_M} \right) \right]
\]
(\text{B22})

and hence, for arbitrary \( c \in \mathcal{V} \) it is the case that
\[
c^\dagger R c = \sum_{x_1, \ldots, x_M} \text{Tr}[Z_{x_1, \ldots, x_M} A_{x_1, \ldots, x_M} Z_{x_1, \ldots, x_M}^\dagger 0],
\]
(\text{B23})

Since \( A_{x_1, \ldots, x_M} \geq 0 \) and \( c \geq 0 \) it follows that \( c^\dagger R c \geq 0 \), and thus \( R \geq 0 \).

The final step is to show that \( R \) satisfies the decomposition \( R = \sum_{m=1}^M P_m R P_m \), where \( P_m \) is the projector onto \( \mathcal{V}_m \) in \( \bigoplus_{m=1}^M \mathcal{V}_m \). Recall that \( A_{x_1, \ldots, x_M} = A_{x_1}^{(1)} \otimes \cdots \otimes A_{x_M}^{(M)} \) and \( Y_{x_1, \ldots, x_M} = Y_{x_1}^{(1)} + \cdots + Y_{x_M}^{(M)} \), where \( Y_{x_m}^{(m)} \) has support in \( \mathcal{V}_m \). By combining these observations with Eq. (\text{B22}) for \( m \neq m' \) we get
\[
P_m R P_{m'} = \sum_{x_1, \ldots, x_M} \text{Tr} \left[ \left( Y_{x_m}^{(m)} \otimes \hat{1} - \sum_{\tau_m} Y_{x_m}^{(m)} \otimes A_{\tau_m}^{(m)} \right) \right]
\]
(\text{B24})

\[
A_{x_1, \ldots, x_M} \left( Y_{x_1, \ldots, x_M} \otimes \hat{1} - \sum_{\tau_m} Y_{x_m}^{(m')} \otimes A_{\tau_m}^{(m')} \right) \]

Expanding the last two parentheses in (\text{B24}), and using the fact that \( \{ A_{x_m}^{(m)} \}_{x_m} \) is a POVM and thus \( \sum_{x_m} A_{x_m}^{(m)} = \hat{1} \), yields \( P_m R P_{m'} = 0 \). Hence, \( R \) is block-diagonal with respect to the subspaces \( \mathcal{V}_m \), i.e., \( R = \sum_m P_m R P_m \).

3. Semidefinite test for quantum networks

From Proposition 4, we can conclude that whenever we find a covariance matrix that cannot be decomposed in the manner of (\text{B13}) and (\text{B14}), then we can exclude the quantum network \( G \) as an explanation of the observed covariance. This observation can thus be turned into a test that allows us to falsify hypothetical quantum networks, and we colloquially refer to this as the ‘semidefinite test’. In section 4 we discuss how this semidefinite test can be cast as a semidefinite program.

The semidefinite test provides an outer relaxation to the set of distributions \( P \) that are compatible with \( G \). In other words, the set of distributions (or covariance matrices) that are compatible with \( G \) forms a subset of the distributions (covariance matrices) that are accepted by the semidefinite test. Hence, if a distribution fails the test, we can safely reject \( G \) as an explanation, while if the distribution passes the test, it may still not be compatible with \( G \).

On the level of the covariance matrix \( \text{Cov}(Y) \), the conditions (\text{B13}) and (\text{B14}) in the quantum case are identical to the conditions (\text{A2}) and (\text{A2}) in the classical case. Hence, the semidefinite test in the quantum case is identical to the test in the classical case, which means that it tests the topology of the network, irrespective of whether this network is classical or quantum.

As briefly mentioned in the main text, the semidefinite test does not require us to know the entire distribution \( P(x_1, \ldots, x_M) \) of the measurement outcomes; it suffices that we know the covariance matrix \( \text{Cov}(Y) \) for some suitable choice of feature maps. Since the covariance matrix generally contains less information than the full
distribution, it means that the semidefinite test is correspondingly limited in what it can ‘discern’ about the distribution. However, from the point of view of experimental tests, this may be advantageous, since determining the covariance matrix seems to be less data-intensive than determining the full distribution. The following crude counting of parameters may be illustrative. If we consider a network with $M$ children, where each child has at most $X$ possible measurement outcomes, then the covariance matrix is an operator on a space of dimension $\dim V = \sum_{m=1}^{M} \dim V_m \leq MX$. Hence, the number of parameters of the manifold of covariance matrices cannot be more than quadratic in $M$. The exact number of free parameters may potentially depend on the choice of DAG $G$, and the same can be said about the manifold distributions. However, if we for the latter take the (rather boring) extreme case where we only have one latent system that is parent to all children, then all distributions with $M$ variables that each have $X$ outcomes are compatible with $G$. Consequently, in this case there are $X^M - 1$ free parameters, i.e., an exponential growth in $M$.

### Appendix C: Dual formulation of problem \(^2\) and a general witness for quantum networks with bipartite sources

In this appendix we prove some of the claims made in the main text, regarding the results shown in figures \(^1\) and \(^2\). In the following section, we establish an equivalent dual formulation for the test introduced in Proposition \(^1\), which has the advantage of providing a witness for the particular graph topology imposed by the causal model under consideration—i.e. it provides a linear functional over the covariance matrix $\text{Cov}(Y)$ that determines incompatibility with the proposed causal model for values above a certain threshold.

#### 1. Dual formulations

As established in Proposition \(^1\), the decomposition imposed by a quantum network on the covariance matrix $\text{Cov}(Y)$ (which we denote here as $\mathcal{C}$ for convenience) can be immediately stated in the form of a semidefinite program (SDP) as a feasibility problem \(^46\), namely

\[
\begin{align*}
\text{Given } & \mathcal{C}, & \text{(C1a)} \\
\text{find } & R, \mathcal{C}_n, & \text{(C1b)} \\
\text{subject to } & R \geq 0, \quad \mathcal{C}_n \geq 0, & \text{(C1c)} \\
& R = \sum_m P_m R P_m, \quad \mathcal{C}_n = P(n) \mathcal{C}_n P(n), & \text{(C1d)} \\
& \mathcal{C} = R + \sum_n \mathcal{C}_n. & \text{(C1g)}
\end{align*}
\]

For practical purposes, this test can be slightly simplified into

\[
\begin{align*}
\text{Given } & \mathcal{C}, & \text{(C2a)} \\
\text{find } & R, \mathcal{C}_n, & \text{(C2b)} \\
\text{subject to } & R \geq 0, \quad \mathcal{C}_n \geq 0, & \text{(C2c)} \\
& \mathcal{C} = \sum_m P_m R P_m + \sum_n P(n) \mathcal{C}_n P(n), & \text{(C2e)}
\end{align*}
\]

which subsumes the expected structures for $R$ and $\mathcal{C}_n$ [Eqs. \((C1d)\) \((C1f)\)] within the decomposition for $\mathcal{C}$ [Eq. \((C2e)\)]. There is no loss in this simplification since we may always get a solution for problem \((C2)\) from the solution for \((C1)\) and vice-versa: Any solution for the original problem automatically satisfies the constraints of the simplified problem, while, on the other way, if the $R$ and $\mathcal{C}_n$ obtained from \((C2)\) are not already a solution for the original problem, we may define

\[
\begin{align*}
R' & := \sum_m P_m R P_m, \quad \text{(C3)} \\
\mathcal{C}_n' & := P(n) \mathcal{C}_n P(n), \quad \text{(C4)}
\end{align*}
\]
and the new variables make a viable solution for both (C7) and (C2). In fact, we have $R' \geq 0$ (since $R \geq 0$ from Eq. (C2c)), and $C_0 \geq 0; C = R' + \sum_n C_n$, from Eq. (C2c); $\sum_k P_k R^k P_k = R'$, since
\[
\sum_k P_k R^k P_k = \sum_{k,m} P_k P_m R P_m P_k = \sum_m P_m R P_m = R',
\]
given that $P_m P_k = \delta_{m,k} P_m$. Similarly,
\[
p^{(n)} C'_n p^{(n)} = \sum_{k,k'} P_k P_{k'} C_n P_{k'} P_k = \sum_{k,k'} P_k C_n P_{k'} = p^{(n)} W p^{(n)} = C'_n.
\]

From the SDP (C2), we may obtain an associated dual optimization problem \([45][47]\) that returns a numerical value for the optimal separation between the given covariance matrix $C$ and the set of matrices that satisfy the constraints (C2c,C2e). More specifically, the test described by Eqs. (C2) admits the dual formulation
\[
\begin{aligned}
\text{Maximize} & \quad \Tr[W C], \\
\text{subject to} & \quad \sum_m P_m W P_m \leq 0, \\
& \quad p^{(n)} W p^{(n)} \leq 0,
\end{aligned}
\]
which should detect feasibility or infeasibility in the same way that the other (primal) formulation does. As a byproduct, we obtain the hermitian matrix $W$, which works as a witness for incompatibility with the causal model under test: the dual constraints (C7b,C7c) ensure that $\Tr[W C] \leq 0$ whenever $C$ is decomposable as $R + \sum_n C_n$, since
\[
\Tr[W R] = \sum_m \Tr[W P_m R P_m] = \sum_m \Tr[P_m W P_m R] \leq 0,
\]
and similarly $\Tr[W C_n] \leq 0$. As a consequence, if $W$ is applied over another given covariance matrix $C'$ and $\Tr[W C'] > 0$, we can immediately conclude that $C'$ is not explained by the proposed causal model.

While formulation (C7) is very simple and already useful for obtaining a valid witness for the given causal model, its objective function may be unbounded in some cases, where the primal problem is infeasible. It is interesting then for practical purposes to have the witness $W$ bounded in some manner. To this end, we add a scaling constraint on $\Tr[W]$, rendering the program as
\[
\begin{aligned}
\text{Maximize} & \quad \Tr[W C], \\
\text{subject to} & \quad \sum_m P_m W P_m \leq 0, \\
& \quad p^{(n)} W p^{(n)} \leq 0, \\
& \quad \Tr[W] \geq -1,
\end{aligned}
\]
noting that $\Tr[W]$ is already bounded above due to Eq. (C9b), since $\Tr[W] = \Tr[\sum_m P_m W] = \sum_m \Tr[P_m W P_m] \leq 0$, using that $\sum_m P_m = I$ and $p^2 = p_m$.

2. Applications

As an application, we consider the family of distributions
\[
P_{pq}(x_1, .., x_N) = p \delta_0^{(N)} + q \delta_1^{(N)} + (1 - p - q) \frac{1 - \delta_0^{(N)} - \delta_1^{(N)}}{2^N - 2}
\]
for \( N \) parties with binary outcomes, where \( \delta_{x}^{(N)} = 1 \) if \( x_1 = x_2 = \ldots = x_N = x \), and 0 otherwise, and \( p + q \leq 1 \). These distributions include, in particular, the ones used for the results in Figs. 2c and 2c, which correspond, respectively, to the cases \( N = 3 \) and \( N = 4 \). Figures 2c and 2c also include the results based on the Finner inequality, that in triangle case can be simply written as \( p(x_1, x_2, x_3) \leq \sqrt{p(x_1)p(x_2)p(x_3)} \) (see [34] for more details).

To build the covariance matrix, we use the feature maps \( \chi_{x_m}^{(m)} = \chi_{x_m} \), where \( \{ |x_m\rangle | 1 \leq x_m \leq X_m \} \) forms the canonical basis in the \( X_m \)-dimensional Hilbert space \( V_m \). Let now \( a_d \) be a ladder operator in \( d \) dimensions defined as \( a_d := \sum_{j=1}^{-d-1} |j+1\rangle \langle j| \), \( \Pi_d \) the identity also in \( d \) dimensions, and \( \sigma_x, \sigma_y, \sigma_z \) the usual Pauli matrices. Numerical evidence suggests that the witness

\[
W_{2N} := -\mathbb{I}_{2N} + 2 (\mathbb{I}_N \otimes \sigma_x) + \sum_{j=1}^{2N-1} (-1)^j \left[ \alpha_j^2N \right] (\alpha_j^2N)\dagger
\]

(C11)

is optimal for the family of distributions \( P_{pq}^N \) when the corresponding covariance matrix is incompatible with a network where all possible bipartite sources (i.e., sources connecting 2 children) are present (See, for instance, Figures 1-a and 2-a in the main text for the cases \( N = 3 \) and \( N = 4 \)). This has been verified numerically for \( N = 3, \ldots, 7 \) parties and we conjecture that this witness is always optimal for this family of distributions. Nonetheless, regardless of optimality, it is possible to prove that \( W_{2N} \) is always a valid witness and, in particular, that for any value of \( p \), there is always a value of \( q \) above which the witness detects incompatibility with the causal model.

To prove that \( W_{2N} \) is a valid witness, first we use that, for the scenario that we consider here, where all parties have binary outcomes, the projector onto \( V_m \) can be written as \( P_m = |m\rangle \langle m| \otimes \mathbb{I}_2 \). We note also that \( a_{2N} = \mathbb{I}_N \otimes a_2 + a_N \otimes a_2^\dagger \). Together these imply that

\[
P_m a_{2N}^k P_n = \delta_{k,1} |m\rangle \langle m| \otimes a_2, \tag{C12}
\]

\[
P_m a_{2N}^{k+1} P_n = \delta_{k,n+k} |m\rangle \langle m| \otimes \mathbb{I}_2, \tag{C13}
\]

\[
P_m a_{2N}^{2k+1} P_n = \delta_{k,n,k+1} |m\rangle \langle m| \otimes a_2 + \delta_{k,n+1,k} |m\rangle \langle m| \otimes a_2^\dagger, \tag{C14}
\]

for \( m, n \in \{1, \ldots, N\} \) and \( m \neq n \). Applying these results on Eq. (C11), we obtain

\[
P_m W_{2N} P_n = (-1)^{\delta_{m,n}} |m\rangle \langle m| \otimes (\mathbb{I}_2 - a_2 - a_2^\dagger) = (-1)^{\delta_{m,n}} |m\rangle \langle m| \otimes |\rangle \langle -| - |\rangle \langle -|, \tag{C16}
\]

where \( |\rangle : = |0\rangle - |1\rangle \). Consequently,

\[
\sum_m P_m W_{2N} P_n = -\mathbb{I}_N \otimes |\rangle \langle -| \leq 0, \tag{C17}
\]

and, given that each latent vertex connects only two different parties,

\[
P^{(n)} W_{2N} P^{(n)} = (P_{i_n} + P_{j_n}) W_{2N} (P_{i_n} + P_{j_n}) = -(|\rangle \langle -| \otimes |\rangle \langle -| - |\rangle \langle -|) \leq 0, \tag{C18}
\]

where \( |\rangle \rangle : = |i\rangle - |j\rangle \) for \( i \neq j \), and \( P_{i_n}, P_{j_n} \) are the projectors that compose \( P^{(n)}, i_n \neq j_n \). Consequently, \( W_{2N} \) satisfy the dual constraints \( \mathbb{C} \) and \( \mathbb{C}_2 \) and, therefore, returns \( \text{Tr}[W_{2N} C] \leq 0 \) for any covariance matrix that is compatible with the proposed causal model.

Now, let \( C_{pq}^N \) be the covariance matrix for \( P_{pq}^N \). It can be written as

\[
C_{pq}^N = (\Delta_{pq}^N \mathbb{I}_N + \chi_{pq}^N |1\rangle \langle 1|_N) \otimes (\mathbb{I}_2 + \sigma_x), \tag{C19}
\]

where \( \Delta_{pq}^N := 2^{N-2}(1 - p - q)/(2^N - 2), \chi_{pq}^N := 1/4 [1 - (p - q)^2 - \Delta_{pq}^N] \), and \( |1\rangle_1 := \sum_{m=1}^{N} |m\rangle \). Evaluating \( \text{Tr}[W_{2N} C_{pq}^N] \), we obtain the expression \( \text{Tr}[W_{2N} C_{pq}^N] = 4N \chi_{pq}^N (N - 2) - \Delta_{pq}^N \). This allows us to solve for \( q \) and establish the analytic formula

\[
q > p + \kappa_N - \sqrt{4 \kappa_N p + (\kappa_N - 1)^2}, \tag{C20}
\]

for the region of incompatibility with the network witnessed by \( W_{2N} \) (i.e., the region where \( \text{Tr}[W_{2N} C_{pq}^N] > 0 \)), where \( \kappa_N := \frac{(N-1)2^{N-2}}{(N-2)(2^{N-1}-1)} \).
It should be remarked that, since $\chi^N_{pq}$ and $\Delta^N_{pq}$ are both symmetric under permutation of $p$ and $q$, the same is true for the region described by (C20), which is expected since $C^N_{pq}$ has this property. Also, if equality is obtained in Eq. (C20) for $q = q_0 < 1$ at $p = 0$, then $C^N_{pq}$ is always incompatible with the network independently of $p$ for $q > q_0$, with the same behavior occurring for $p > q_0$, due to the symmetry. In fact, since $q_0 = \kappa_N - |\kappa_N - 1|$, this always true for $N \geq 4$, where $\kappa_N < 1$. As $N \to +\infty$, $\kappa_N \to 1/2$, implying then that $q_0 \to 0^+$. In fig. 4, we show the analytical curves for different values of $N$ and a comparison with the numerical results.

Appendix D: Semidefinite decompositions for classical networks with inputs

Here we combine the classical test developed in [37] (see Section A) with the notion of ‘inputs’ or ‘measurement settings’. We do this for the sake of comparison with the quantum test with inputs in Proposition 2, which we prove in Section E. As we shall see here, the notion of inputs does in the classical case not introduce anything essentially new compared to [37], in the sense that the classical setting with inputs can be simulated by an ‘extended’ classical setting without inputs.

For a DAG $G$ with parents $p_1, \ldots, p_N$ and children $c_1, \ldots, c_M$ with inputs or (measurement) settings $s = (s_1, \ldots, s_M) \in \{1, \ldots, S_1\} \times \cdots \times \{1, \ldots, S_M\}$, the joint distribution of the parents and children, conditioned on the inputs, can be written

$$P(c_1, \ldots, c_M, p_1, \ldots, p_N|s_1, \ldots, s_M) = \Pi_{m=1}^M P(c_m|s_m, p_m) \Pi_{n=1}^N P(p_n). \quad (D1)$$

One may note that (D1) is nothing but the ordinary causal Markov condition, applied to the scenario with inputs. It expresses that each child $c_m$ may potentially be influenced, not only by its parents $p_m$, but also by the choice of input $s_m$. One should in particular note that each input $s_m$ only affects the output of child $c_m$. We should also remark that we here overload the notion and let $p_1, \ldots, p_N$ and $c_1, \ldots, c_M$ not only denote vertices in the DAG $G$, but also the random variables associated with those vertices, as well as the values that these random variables may attain.

We say that an observed conditional distribution $P(c_1, \ldots, c_M|s_1, \ldots, s_M)$ is compatible with a given classical causal structure $G$ with parents $p_1, \ldots, p_N$ and children $c_1, \ldots, c_M$, with associated inputs $s_1, \ldots, s_M$, if it is the marginal of some distribution of the form (D1), for all inputs $(s_1, \ldots, s_M) \in \{1, \ldots, S_1\} \times \cdots \times \{1, \ldots, S_M\}$.  

FIG. 4. Comparison between numerical results (solid blue lines) obtained as solutions of the SDP [Eq. (C9)] and the analytical curves (dashed red lines) that bound the region defined in (C20). Corresponding numerical and analytic solutions are presented for $N = 3, \ldots, 7$, where it can be observed that both types of solution coincide, implying that the witness $W_{2N}$ [Eq. (C13)] is optimal for the entire family of distributions $P_{pq}^N$ in these cases. The dot-dashed curves correspond to analytical curves for $N = 10$ and $N = 20$. 

\[0.0\ 0.5\ 1.0\n0.0\ 0.5\ 1.0\]
Before turning to the statement and proof of Proposition 5 below, we will in the following give an outline of the main conceptual ideas of our argument, which in essence reduces the proof to a special case of Proposition A1. Suppose that we have a (classical) device that randomly produces an output $X$, but in such a way that this distribution is determined by some input $s = 1, \ldots, S$. In other words, the output distribution $P(X = x|s)$ is conditioned on $s$. Now imagine a second device that simultaneously produces a collection of random variables $X^{(1)}, \ldots, X^{(S)}$, with a joint distribution $P(X^{(1)} = x^{(1)}, \ldots, X^{(S)} = x^{(S)})$, for which the marginal distribution of $X^{(s)}$ is $P(X^{(s)} = x) = P(X = x|s)$. There always exists at least one such distribution, namely the product distribution $P(X^{(1)} = x^{(1)}, \ldots, X^{(S)} = x^{(S)}) = \prod_{s=1}^{S} P(X^{(s)} = x^{(s)}|s)$. Marginalizing over all indices but $s$, the second device can thus simulate the first by presenting only random variable $X^{(s)}$ when the input is $s$. In the proof of Proposition 5 we will apply this ‘extension’ to all the children (that has more that one measurement setting $s_m$).

As a side-remark, one may note that for the corresponding quantum case, in Proposition 2 the parents potentially distribute entangled states to their children. Due to such entangled states, it is difficult to see that one could employ a similar proof strategy of ‘extensions’ in the quantum case. Hence, although the quantum version in Proposition 2 shares an essential structural similarity with Proposition 5, the proof strategy of the former is very different from the latter, and focuses directly on the covariance matrix.

We next make a few observations and definitions concerning the framework of Proposition 5. We consider a scenario where each child $c_m$ at each instance only can choose one single input $s_m$. For each child $c_m$ and for each choice of local input $s_m$ we assign a feature map $Y^{(m,s_m)} := Y^{(m,s_m)}(c_m)$ into a subspace $Y_{m,s_m}$. For each single child, we can observe $\text{Cov}(Y^{(m,s_m)}(c_m)|s_m)$. Moreover, we can observe all kinds of cross covariances $\text{Cov}(Y^{(m,s_m)}(c_m), Y^{(m',s_m')}(c_{m'}|s_m,s_m'))$ for each possible pair of inputs $s_m, s_m'$ of child $m, m'$ with $m \neq m'$. Additionally, one may note that $\text{Cov}(Y^{(m,s_m)}(c_m)|s) = \text{Cov}(Y^{(m,s_m)}(c_m)|s_m)$ and $\text{Cov}(Y^{(m,s_m)}(c_m), Y^{(m',s_m')}(c_{m'}|s_m,s_m')) = \text{Cov}(Y^{(m,s_m)}(c_m), Y^{(m',s_m')}(c_{m'}|s_m,s_m'))$, due to the structure of (D1). However, in this setting, there exist no cross covariances of the type ‘$\text{Cov}(Y^{(m,s_m)}(c_m), Y^{(m',s_m')}(c_{m'}|s_m,s_m'))$’ for $s_m' \neq s_m$, since at each instance, $c_m$ can only choose one input $s_m$. We collect all the observable covariances into the observable covariance matrix

$$C_{\text{observable}} := \sum_{m} \sum_{s_m} \text{Cov}(Y^{(m,s_m)}(c_m)|s_m) + \sum_{m,m', s_m, s_m'} \text{Cov}(Y^{(m,s_m)}(c_m), Y^{(m',s_m')}(c_{m'}|s_m,s_m')).$$

We wish to find a ‘completion’ $C_{\text{completion}}$ of the observable covariances $C_{\text{observable}}$, such that their sum, $C := C_{\text{observable}} + C_{\text{completion}}$, is positive semidefinite, and where $C_{\text{completion}}$ satisfies

$$P_{m,s_m} C_{\text{completion}} P_{m,s_m} = 0, \quad m = 1, \ldots, M, \quad s_m = 1, \ldots, S_m$$
$$P_{m,s_m} C_{\text{completion}} P_{m',s_m'} = 0, \quad m,m' = 1, \ldots, M : m \neq m', \quad s_m = 1, \ldots, S_m, \quad s_m' = 1, \ldots, S_m'. \quad (D3)$$

These conditions in essence say that $C_{\text{completion}}$ regarded as a block matrix with respect to the sub spaces $Y_{m,s_m}$, must be zero on the blocks that constitutes $C_{\text{observable}}$. In the proof of Proposition 5 this completion is constructed via the above described ‘extended’ random variables. The point is that the cross covariances $\text{Cov}(Y^{(m,s_m)}(c_m), Y^{(m',s_m')}(c_{m'}))$ that lack meaning in the conditional setting, become meaningful in the extended setting. Moreover, via the extended collection of random variables, it turns out that Proposition A1 can be applied to the ‘completed’ matrix $C$, which yields Proposition 5.

**Proposition 5.** Let the conditional distribution $P(c_1, \ldots, c_M|s_1, \ldots, s_M)$ be compatible with the classical causal structure $G$ with parents $p_1, \ldots, p_N$ and children $c_1, \ldots, c_M$, with associated inputs $s_1, \ldots, s_M$. Assume that each child $c_m$, and each input $s_m$, is assigned a feature map $Y^{(m,s_m)}$ into a vector space $Y_{m,s_m}$. Let the operator $C_{\text{observable}}$ on $\mathcal{V} := \bigoplus_{m=1}^{M} \bigoplus_{s_m=1}^{S_m} Y_{m,s_m}$ be as defined in (D2). Then there exist operators $C_{\text{completion}}$, $\tilde{C}$ and $\{\tilde{C}_n\}_{n=1}^{N}$ on $\mathcal{V}$, such that $C_{\text{completion}}$ satisfies (D3), and

$$C := C_{\text{observable}} + C_{\text{completion}} \geq 0, \quad (D4)$$
and

\[ C = \bar{C} + \sum_{n=1}^{N} \tilde{c}_n, \quad \bar{C} \geq 0, \quad \tilde{c}_n \geq 0, \]  
(\text{D5})

\[ \tilde{p}(\cdot) \tilde{c}_n \tilde{p}(\cdot) = \tilde{c}_n, \quad \bar{R} = \sum_{m, s_m} P_{m, s_m} \bar{R} P_{m, s_m}, \]  
(\text{D6})

\[ \tilde{p}(\cdot) := \sum_{m \in C_n, s_m = 1} S_m \sum_{m, s_m} P_{m, s_m}, \]  
(\text{D7})

where \( P_{m, s_m} \) is the projector onto \( V_{m, s_m} \).

Remark: One may note that from the requirement that \( C \) is positive semidefinite, it follows that \( C \) also is Hermitian, \( C^\dagger = C \). Since \( C_{\text{observable}} \) necessarily is Hermitian, it follows that \( C_{\text{completion}} = C - C_{\text{observable}} \) is also Hermitian. Hence, in any search for a completion \( C_{\text{completion}} \), one can, without loss of generality, assume that \( C_{\text{completion}} = C \).  

Proof. By the assumption that \( P(c_1, \ldots, c_M|s_1, \ldots, s_M) \) it compatible with the classical causal structure \( G \) with parents \( p_1, \ldots, p_N \) and children \( c_1, \ldots, c_M \), with associated inputs \( s_1, \ldots, s_M \), it follows that \( P(c_1, \ldots, c_M|s_1, \ldots, s_M) \) is the margin of a conditional joint distribution \( P(c_1, \ldots, c_M, p_1, \ldots, p_N|s_1, \ldots, s_M) \), as in (\text{D1}), with respect to \( G \).

The first step is to construct a new graph \( \hat{G} \) that contains the same set of parents as \( G \), where each child \( c_m \) in \( G \) is replaced with a collection of children \( c_m^{(1)}, \ldots, c_m^{(s_m)} \), i.e., one new child for each input \( s_m \). We moreover assume that the set of parents of child \( c_m^{(s_m)} \) is \( \mathcal{P}_m \). In other words, child \( c_m^{(s_m)} \) in \( \hat{G} \) has the same set of parents as \( c_m \) has in \( G \).

In the following we construct an extended probability distribution \( \tilde{P} \) of \( \{ c_m^{(s_m)} \}_{m, s_m} \) \( p_1, \ldots, p_N \) regarded as random variables. The distribution \( P(c_1, \ldots, c_M, p_1, \ldots, p_N|s_1, \ldots, s_M) \) is by assumption of the form (\text{D1}), which thus yields a collection of conditional distributions \( P(c_m|s_m, \mathcal{P}_m) \). Define \( \tilde{P}(c_m^{(s_m)}|\mathcal{P}_m) \) by

\[ P(c_m^{(s_m)} = x|\mathcal{P}_m) := P(c_m = x|s_m, \mathcal{P}_m), \quad m = 1, \ldots, M, \quad s_m = 1, \ldots, S_m, \]  
(\text{D8})

and construct the distribution of \( p_1, \ldots, p_N, \{ c_m^{(s_m)} \}_{m, s_m} \) by

\[ \tilde{P}(c_1^{(1)}, \ldots, c_1^{(s_1)}, \ldots, c_M^{(1)}, \ldots, c_M^{(S_M)}, p_1, \ldots, p_N) = \prod_{m=1}^{M} \prod_{s_m=1}^{S_m} P(c_m^{(s_m)}|\mathcal{P}_m) \prod_{n=1}^{N} P(p_n). \]  
(\text{D9})

If we take the marginal distribution of \( c_1^{(1)}, \ldots, c_1^{(s_1)}, \ldots, c_M^{(1)}, \ldots, c_M^{(S_M)} \) in (\text{D9}), it follows by comparison of (\text{D9}) with (\text{A1}) that the resulting distribution \( \tilde{P}(c_1^{(1)}, \ldots, c_1^{(s_1)}, \ldots, c_M^{(1)}, \ldots, c_M^{(S_M)}) \) is compatible with the new DAG \( \hat{G} \). Thus Proposition 3 is applicable to \( \tilde{P} \) with respect to \( \hat{G} \). For child \( c_m^{(s_m)} \) we choose the feature map \( Y_{m, s_m} \). By Proposition 3 there exist operators \( R \) and \( (\mathcal{C}_n)_{n=1}^{N} \) such that the covariance matrix \( \text{Cov}(\hat{Y}) \) of

\[ \hat{Y} := \sum_{m=1}^{M} \sum_{s_m=1}^{S_m} Y_{m, s_m} (c_m^{(s_m)}), \]  
(\text{D10})

satisfies

\[ \text{Cov}(\hat{Y}) = \bar{R} + \sum_{n=1}^{N} \tilde{c}_n, \quad \bar{R} \geq 0, \quad \tilde{c}_n \geq 0, \]  
(\text{D11})

where

\[ \tilde{p}(\cdot) \tilde{c}_n \tilde{p}(\cdot) = \tilde{c}_n, \quad \bar{R} = \sum_{m=1}^{M} \sum_{s_m=1}^{S_m} P_{m, s_m} \bar{R} P_{m, s_m}, \]  
(\text{D12})

and

\[ \tilde{p}(\cdot) := \sum_{m, s_m, \tilde{P}_{m, s_m}} P_{m, s_m} \sum_{m, s_m} P_{m, s_m}, \]  
(\text{D13})
where in the second equality in (D.13) we have used the observation that since \( c_m^{(s_m)} \) in \( G \) has the same parents as \( c_m \) in \( G \), it follows that if \( c_m \) is a child of parent \( p_n \) in \( G \), then all of \( c_m^{(1)}, \ldots, c_m^{(s_m)} \) are children of parent \( p_n \) in \( G \).

In the following, we compare marginal distributions of (D.9) with those of (D.14). The marginal distribution of \( c_m^{(s_m)} = \sum_{m} P(c_m^{(s_m)}|P_m) \Pi_{n=1}^{N} P(p_n) \).

\[ \hat{P}(c_m^{(s_m)}, p_1, \ldots, p_N) = \hat{P}(c_m^{(s_m)}|P_m) \Pi_{n=1}^{N} P(p_n). \]  

(D.14)

Similarly, if we marginalize the conditional distribution (D.14) to \( c_m, p_1, \ldots, p_N \), we obtain

\[ P(c_m, p_1, \ldots, p_N|s_1, \ldots, s_M) = P(c_m|s_m, P_m) \Pi_{n=1}^{N} P(p_n), \]

(D.15)

where we note that only the conditioning on \( s_m \) remains on the right hand side. By recalling that we by construction have \( P(c_m^{(s_m)} = x|P_m) = P(c_m = x|s_m, P_m) \), we can conclude that

\[ \hat{P}(c_m^{(s_m)} = x, p_1, \ldots, p_N) = P(c_m = x, p_1, \ldots, p_N|s_m), \]

(D.16)

where we on the right hand side have dropped all superfluous conditioning.

Analogously, for \( m \neq m' \) we find that the marginalization of (D.9) to \( c_m^{(s_m)} = \sum_{m} P(c_m^{(s_m)}|P_m) \Pi_{n=1}^{N} P(p_n) \)

\[ \hat{P}(c_m^{(s_m)}, c_m^{(s_m)'}, p_1, \ldots, p_N) = \hat{P}(c_m^{(s_m)}|P_m) \Pi_{n=1}^{N} P(p_n). \]

(D.17)

Similarly, the marginalization of the conditional distribution (D.14) to \( c_m, c_m', p_1, \ldots, p_N \) yields

\[ P(c_m, c_m', p_1, \ldots, p_N|s_1, \ldots, s_M) = P(c_m|s_m, P_m) P(c_m'|s_m', P_m) \Pi_{n=1}^{N} P(p_n), \]

(D.18)

and since we again have \( P(c_m^{(s_m)} = x|P_m) = P(c_m = x|s_m, P_m) \), it follows that

\[ \hat{P}(c_m^{(s_m)} = x, c_m' = x', p_1, \ldots, p_N) = P(c_m = x, c_m' = x', p_1, \ldots, p_N|s_m, s_m'). \]

(D.19)

By (D.16) we can conclude that

\[ P_{m,s_m} \text{Cov}(Y) P_{m,s_m} = \text{Cov}(Y^{(m,s_m)}(c_m^{(s_m)})) \]

(D.20)

\[ = \text{Cov}(Y^{(m,s_m)}(c_m)|s_m). \]

By (D.19) it similarly follows that

\[ P_{m,s_m} \text{Cov}(Y) P_{m',s_m'} = \text{Cov}(Y^{(m,s_m)}(c_m^{(s_m)}), Y^{(m',s_m')}(c_m^{(s_m)'}) \]

(D.21)

Equations (D.20) and (D.21) implies that \( C_{\text{observable}} \) defined in (D.2) can be written

\[ C_{\text{observable}} = \sum_{m} \sum_{s_m} P_{m,s_m} \text{Cov}(Y) P_{m,s_m} + \sum_{m \neq m'} \sum_{s_m, s_m'} P_{m,s_m} \text{Cov}(Y) P_{m',s_m'}. \]

(D.22)

Moreover, define

\[ C_{\text{completion}} := \sum_{m} \sum_{s_m} P_{m,s_m} \text{Cov}(Y) P_{m,s_m}. \]

(D.23)

One can confirm that \( C_{\text{completion}} \) so defined satisfies all the conditions in (D.3). Moreover,

\[ C_{\text{observable}} + C_{\text{completion}} = \text{Cov}(Y). \]

(D.24)

Finally, since \( \text{Cov}(Y) \) necessarily is positive semidefinite, we can conclude that \( C_{\text{completion}} \) and \( C := \text{Cov}(Y) \) satisfy (D.4). By combining (D.24) with (D.11), (D.12) and (D.13), it follows that (D.5), (D.6) and (D.7) holds, which proves the proposition.

\[ \square \]
Appendix E: Proof of Proposition 2

Here we consider the classes of quantum networks where we have the freedom to locally choose among collections of measurement settings. In other words, analogous to how Proposition 5 generalizes Proposition 3 to the case of inputs, Proposition 2 generalizes Proposition 1 to the case when we have local measurement settings.

Like for the classical case in Section D, completions of the observable covariance matrix is an important notion also in the quantum setting. In the classical case, we made the completion on the level of probability distributions, where the completion of the covariance matrix followed as a consequence. As already remarked in Section D, it is difficult to see how one could obtain a similar proof-strategy in the quantum case, since the quantum parents may generate entanglement between their children. Locally copying the quantum states, in a manner analogous to how we copy the transition probabilities in the classical case, is thus excluded due to the no-cloning theorem. We circumvent such issues by instead directly completing the covariance matrix itself, and this approach is detailed in Section E.

Let us recall that we here, like in the rest of this investigation, exclusively consider the class of DAGs where the vertices can be partitioned into a class of parents \( p_1, \ldots, p_N \) and a class of children \( c_1, \ldots, c_M \), where there only are arrows from parents to children. For a graph \( G \) with parents \( p_1, \ldots, p_N \) and children \( c_1, \ldots, c_M \), we do, precisely as in Section D, associate a structure of Hilbert spaces as in (B). As in section E we also assume that the parents are initially uncorrelated, with a global state

\[
\varrho := \varrho_{p_1} \otimes \cdots \otimes \varrho_{p_N} \in \mathcal{S}(\mathcal{H}). \tag{E1}
\]

However, as a generalization of the setting in section E, we here consider a collection of POVM \( \{A_{(m,s_m)}^{(s_m)}\}_{s_m} \) for \( s_m = 1, \ldots, S_m \), where the latter is the possible ‘inputs’ or ‘measurement settings’, resulting in the joint POVMs

\[
A_{s_1, \ldots, s_M} := A_{s_1}^{(1,s_1)} \otimes \cdots \otimes A_{s_M}^{(M,s_M)}. \tag{E2}
\]

This construction results in a distribution of measurement outcomes \( x_1, \ldots, x_M \) conditioned on the choice of settings \( s_1, \ldots, s_M \), defined by

\[
P(x_1, \ldots, x_M|s_1, \ldots, s_M) = \text{Tr}([A_{(s_1)}^{(1,s_1)}] \otimes \cdots \otimes A_{(s_M)}^{(M,s_M)}][\varrho_{p_1} \otimes \cdots \otimes \varrho_{p_N}]). \tag{E3}
\]

We say that an observed conditional distribution \( P(x_1, \ldots, x_M|s_1, \ldots, s_M) \) is compatible with the given quantum casual structure \( G \) with inputs, if it can be written as in (E), for some state as in (E1) and some collections of POVMs.

Analogous to the classical case with inputs in Section D, we assign a feature map \( Y_{(m,s_m)}^{(m,s_m)} \) to each child \( c_m \) and each measurement setting \( s_m \), i.e., to each measurement outcome \( x_m, s_m \), corresponding to POVM element \( A_{(m,s_m)}^{(m,s_m)} \), we associate an element \( Y_{(m,s_m)}^{(m,s_m)} \) in a vector space \( V_{m,s_m} \). We also define \( V := \bigoplus_{m=1}^M \bigoplus_{s_m=1}^{S_m} V_{m,s_m} \). To each subspace \( V_{m,s_m} \) we associate the projector \( P_{m,s_m} \).

### 1. Semidefinite completion of the observable covariance matrix

Here we show that the observable covariance matrix \( C_{\text{observable}} \) always possesses a positive semidefinite completion \( C_{\text{completion}} \), by explicitly constructing one such completion. We first make a few observations. As discussed in the main text, we define the observable covariance matrix as

\[
C_{\text{observable}} := \sum_{m} \sum_{s_m} \text{Cov}(Y_{(m,s_m)}^{(m,s_m)}) + \sum_{m \neq m'} \sum_{s_m \neq s_m'} \sum_{s_m,s_m'} \text{Cov}(Y_{(m,s_m)}^{(m,s_m)}, Y_{(m',s_m')}^{(m',s_m')}). \tag{E4}
\]

With the conditional distribution defined by (E3), it follows that the covariances \( \text{Cov}(Y_{(m,s_m)}^{(m,s_m)}) \) for each single child \( c_m \) can be written

\[
\text{Cov}(Y_{(m,s_m)}^{(m,s_m)}) = \sum_{x_m,s_m} Y_{(m,s_m)}^{(m,s_m)} Y_{(m,s_m)}^{(m,s_m)*} \text{Tr}([A_{(m,s_m)}^{(m,s_m)}] \otimes \mathbb{I}/c_m)\varrho

- \sum_{x_m,s_m} \sum_{x_m',s_m'} Y_{(m,s_m)}^{(m,s_m)} Y_{(m,s_m')}^{(m,s_m')} \text{Tr}([A_{(m,s_m)}^{(m,s_m)}] \otimes \mathbb{I}/c_m)\varrho \text{Tr}([A_{(m,s_m')}^{(m,s_m')} \otimes \mathbb{I}/c_m])\varrho, \tag{E5}
\]

where we have defined

\[
\text{Cov}(Y_{(m,s_m)}^{(m,s_m)}) := \sum_{x_m,s_m} Y_{(m,s_m)}^{(m,s_m)} Y_{(m,s_m)}^{(m,s_m)*} \text{Tr}([A_{(m,s_m)}^{(m,s_m)}] \otimes \mathbb{I}/c_m)\varrho

- \sum_{x_m,s_m} \sum_{x_m',s_m'} Y_{(m,s_m)}^{(m,s_m)} Y_{(m,s_m')}^{(m,s_m')} \text{Tr}([A_{(m,s_m)}^{(m,s_m)}] \otimes \mathbb{I}/c_m)\varrho \text{Tr}([A_{(m,s_m')}^{(m,s_m')} \otimes \mathbb{I}/c_m])\varrho.
\]
where \( \hat{1}_{/c_{m}} \) denotes the identity operator on \( \mathcal{H}_{c_1} \otimes \cdots \otimes \mathcal{H}_{c_{m-1}} \otimes \mathcal{H}_{c_{m+1}} \otimes \cdots \otimes \mathcal{H}_{c_M} \). For the ‘cross-covariances’ between each pair of children \( c_{m}, c_{m'} \) with \( m \neq m' \), we analogously get

\[
\text{Cov}(Y^{(m,s_m)}, Y^{(m',s_{m'})}) = \sum_{x_{m',s_{m'}}, x_{m,s_m}} Y^{(m,s_m)}(x_{m',s_{m'}})^\dagger \times \left[ \text{Tr}(A^{(m,s_m)}_{x_{m,s_m}} \otimes A^{(m',s_{m'})}_{x_{m',s_{m'}}} \otimes \hat{1}_{/c_{m}} \rho) \right. \\
- \text{Tr}(A^{(m,s_m)}_{x_{m,s_m}} \otimes \hat{1}_{/c_{m}} \rho) \left. \times \text{Tr}(A^{(m',s_{m'})}_{x_{m',s_{m'}}} \otimes \hat{1}_{/c_{m}} \rho) \right].
\] (E6)

It is convenient to define the operator

\[
Q := \sum_{m} \sum_{s_m} \sum_{x_{m,s_m}} Y^{(m,s_m)}_{x_{m,s_m}} \otimes A^{(m,s_m)}_{x_{m,s_m}} \otimes \hat{1}_{/c_{m}}.
\] (E7)

We moreover define

\[
R := \sum_{m} \sum_{s_m} \sum_{x_{m,s_m}} Y^{(m,s_m)}_{x_{m,s_m}} \otimes \text{Tr}(A^{(m,s_m)}_{x_{m,s_m}} \otimes \hat{1}_{/c_{m}} \rho) \\
- \sum_{m} \sum_{s_m} \sum_{x_{m,s_m}} \sum_{x_{m',s_{m'}}} Y^{(m,s_m)}_{x_{m,s_m}} Y^{(m',s_{m'})}_{x_{m',s_{m'}}} \times \text{Tr}(A^{(m,s_m)}_{x_{m,s_m}} A^{(m',s_{m'})}_{x_{m',s_{m'}}} \otimes \hat{1}_{/c_{m}} \rho).
\] (E8)

The completion, of the observable covariance matrix stemming from a given state and given collections of POVMs, can be stated as follows.

**Lemma 2.** Let \( \rho \) be any density operator on \( \mathcal{H} = \mathcal{H}_{c_1} \otimes \cdots \otimes \mathcal{H}_{c_M} \). Let \( Q \) be defined as in (E7), and \( R \) as defined in (E8), and \( C_{\text{observable}} \) as defined by (E12), (E13) and (E14). Then it is the case that

\[
\text{Tr}_{\mathcal{H}} \left[ \left( Q - \text{Tr}_{\mathcal{H}}(Q \rho) \right) \left( Q - \text{Tr}_{\mathcal{H}}(Q \rho) \right)^\dagger \right] + R = C_{\text{observable}} + C_{\text{completion}},
\] (E9)

where

\[
C_{\text{completion}} := \sum_{m} \sum_{s_m} \sum_{x_{m,s_m}} \sum_{x_{m',s_{m'}}} Y^{(m,s_m)}_{x_{m,s_m}} Y^{(m',s_{m'})}_{x_{m',s_{m'}}} \times \left[ \text{Tr}(A^{(m,s_m)}_{x_{m,s_m}} A^{(m',s_{m'})}_{x_{m',s_{m'}}} \otimes \hat{1}_{/c_{m}} \rho) \left( \text{Tr}(A^{(m,s_m)}_{x_{m,s_m}} \otimes \hat{1}_{/c_{m}} \rho) \right) \right].
\] (E10)

Moreover

\[
C_{\text{observable}} + C_{\text{completion}} \geq 0, \quad R \geq 0,
\] (E11)

\[
P_{m,s_m} C_{\text{completion}} P_{m,s_m} = 0, \quad m = 1, \ldots, M, \quad s_m = 1, \ldots, S_m,
\] (E12)

\[
P_{m,s_m} C_{\text{completion}} P_{M',s_{m'}} = 0, \quad m, m' = 1, \ldots, M, \quad m \neq m', \quad s_m = 1, \ldots, S_m, \quad s_{m'} = 1, \ldots, S_{m'},
\] (E13)

\[
C_{\text{completion}}^\dagger = C_{\text{completion}}.
\] (E13)

**Remarks:** One may note that this Lemma holds for any density operator \( \rho \), i.e., we are not restricted to density operators on the form \( \rho = \rho_1 \otimes \cdots \otimes \rho_{P}, \) which however is needed in the setting of Proposition 2.

One may also note that while \( C_{\text{observable}} \) in (E14) has a direct physical interpretation in terms of (strong) measurements of the POVMs \( A^{(m,s_m)}_{x_{m,s_m}} \), it seems difficult to provide \( C_{\text{completion}} \) with such an interpretation, due to the terms \( \text{Tr}(A^{(m,s_m)}_{x_{m,s_m}} A^{(m',s_{m'})}_{x_{m',s_{m'}}} \otimes \hat{1}_{/c_{m}} \rho) \) in (E10). (One could potentially find interpretations in terms of weak measurements.) However, for our needs it suffices that \( C_{\text{completion}} \), on a purely mathematical level, completes \( C_{\text{observable}} \) into a positive semidefinite matrix.
Proof. We first use (E7) to see that
\[
\text{Tr}_H \left[ \left( Q - \text{Tr}_H(Q)e \right) \left( Q - \text{Tr}_H(Q)e \right)^\dagger \right] e
\]
\[
= \text{Tr}_H(QQ^\dagger e) - \text{Tr}_H(Q^\dagger e)\text{Tr}_H(Q)e^\dagger
\]
\[
= \sum_{m,m' s_{m,s_{m'}}, x_{m,s_{m'}}, x'_{m',s_{m'}}} \gamma_{x_{m,s_{m'}}, x'_{m',s_{m'}}}^{(m,s_{m})} \gamma_{x'_{m',s_{m'}}, x_{m,s_{m'}}}^{(m',s_{m'})} \dagger
\times \left[ \text{Tr} \left( A_{x_{m,s_{m}}} A_{x'_{m',s_{m'}}}^{\dagger} \otimes \mathbb{I}_{/c_{m}} \right) e \right] - \text{Tr} \left( [A_{x_{m,s_{m}}} \otimes \mathbb{I}_{/c_{m}}] e \right) \text{Tr} \left( [A_{x'_{m',s_{m'}}}^{\dagger} \otimes \mathbb{I}_{/c_{m'}}] e \right)
\]
\[
= \sum_{m} \sum_{s_{m} x_{m,s_{m}} x'_{m,s_{m'}}} \gamma_{x_{m,s_{m}} x'_{m',s_{m'}}}^{(m,s_{m})} \gamma_{x'_{m',s_{m'}} x_{m,s_{m}}}^{(m',s_{m'})} \dagger
\times \left[ \text{Tr} \left( A_{x_{m,s_{m}}} A_{x'_{m',s_{m'}}}^{\dagger} \otimes \mathbb{I}_{/c_{m}} \right) e \right] - \text{Tr} \left( [A_{x_{m,s_{m}}} \otimes \mathbb{I}_{/c_{m}}] e \right) \text{Tr} \left( [A_{x'_{m',s_{m'}}}^{\dagger} \otimes \mathbb{I}_{/c_{m'}}] e \right)
\]
\[
+ \sum_{m,m' s_{m,s_{m'}}, x_{m,s_{m'}}, x'_{m',s_{m'}}} \gamma_{x_{m,s_{m}} x'_{m',s_{m'}}}^{(m,s_{m})} \gamma_{x'_{m',s_{m'}} x_{m,s_{m}}}^{(m',s_{m'})} \dagger
\times \left[ \text{Tr} \left( A_{x_{m,s_{m}}} A_{x'_{m',s_{m'}}}^{\dagger} \otimes \mathbb{I}_{/c_{m}} \right) e \right] - \text{Tr} \left( [A_{x_{m,s_{m}}} \otimes \mathbb{I}_{/c_{m}}] e \right) \text{Tr} \left( [A_{x'_{m',s_{m'}}}^{\dagger} \otimes \mathbb{I}_{/c_{m'}}] e \right)
\]
\[
+ \sum_{m,m' s_{m,s_{m'}}, x_{m,s_{m'}}, x'_{m',s_{m'}}} \gamma_{x_{m,s_{m}} x'_{m',s_{m'}}}^{(m,s_{m})} \gamma_{x'_{m',s_{m'}} x_{m,s_{m}}}^{(m',s_{m'})} \dagger
\times \left[ \text{Tr} \left( A_{x_{m,s_{m}}} A_{x'_{m',s_{m'}}}^{\dagger} \otimes \mathbb{I}_{/c_{m}} \right) e \right] - \text{Tr} \left( [A_{x_{m,s_{m}}} \otimes \mathbb{I}_{/c_{m}}] e \right) \text{Tr} \left( [A_{x'_{m',s_{m'}}}^{\dagger} \otimes \mathbb{I}_{/c_{m'}}] e \right)
\]
\[
+ \sum_{m,m' s_{m,s_{m'}}, x_{m,s_{m'}}, x'_{m',s_{m'}}} \gamma_{x_{m,s_{m}} x'_{m',s_{m'}}}^{(m,s_{m})} \gamma_{x'_{m',s_{m'}} x_{m,s_{m}}}^{(m',s_{m'})} \dagger
\times \left[ \text{Tr} \left( A_{x_{m,s_{m}}} A_{x'_{m',s_{m'}}}^{\dagger} \otimes \mathbb{I}_{/c_{m}} \right) e \right] - \text{Tr} \left( [A_{x_{m,s_{m}}} \otimes \mathbb{I}_{/c_{m}}] e \right) \text{Tr} \left( [A_{x'_{m',s_{m'}}}^{\dagger} \otimes \mathbb{I}_{/c_{m'}}] e \right)
\]
\[
+ \text{Cov}(Y^{(m,s_{m})}, Y^{(m',s_{m'})})
\]
\[
Hence,
\text{Tr}_H \left[ \left( Q - \text{Tr}_H(Q)e \right) \left( Q - \text{Tr}_H(Q)e \right)^\dagger \right] e
\]
\[
= \text{Cov}(Y^{(m,s_{m})}, Y^{(m',s_{m'})})
\]
\[
+ \sum_{m,m' s_{m,s_{m'}}, x_{m,s_{m'}}, x'_{m',s_{m'}}} \text{Cov}(Y^{(m,s_{m})}, Y^{(m',s_{m'})})
\]
\[
= \text{Cov}(Y^{(m,s_{m})}) + \text{Cov}(Y^{(m',s_{m'})})
\]
\[
Hence, we have confirmed (E9).

Next, we wish to show that $R \geq 0$ and $C_{\text{completion}} + C_{\text{observable}} \geq 0$. To this end, define
\begin{equation}
R := \sum_{m} \sum_{s_n} \sum_{x_{m,n}} \sum_{m'} \sum_{s'_n} \sum_{x'_{m,n}} \left[ Y_{x_{m,n}}^{(m,s_n)} \otimes i - \sum_{x_{m,n}} Y_{x_{m,n}}^{(m,s_n)} \otimes A_{x_{m,n}}^{(m,s_n)} \right] A_{x_{m,n}}^{(m,s_n)} \left[ Y_{x_{m,n}}^{(m,s_n)} \otimes i - \sum_{x_{m,n}} Y_{x_{m,n}}^{(m,s_n)} \otimes A_{x_{m,n}}^{(m,s_n)} \right]^\dagger.
\end{equation}

One can confirm that $R = \text{Tr}_H(\mathcal{R} \varrho)$. Moreover, by the manner in which $R$ is constructed in (E14), one can see that $R \geq 0$. Consequently, $R = \text{Tr}_H(\mathcal{R} \varrho) \geq 0$. One can also realize that
\[\text{Tr}_H \left[ \left( Q - \text{Tr}_H(Q \varrho) \right) \left( Q - \text{Tr}_H(Q \varrho) \right)^\dagger \right] \geq 0.\]

Hence, the left hand side of (E10) is positive semidefinite, and thus is also the right hand side. We can thus conclude that $C_{\text{completion}} + C_{\text{observable}} \geq 0$.

Next, we wish to prove (E12). First note that
\[P_{m'',m'} C_{\text{completion}} P_{m''',m''} = \sum_{m} \sum_{s_n} \sum_{x_{m,n}} P_{m''',m''} Y_{x_{m,n}}^{(m,s_n)} Y_{x_{m,n}}^{(m',s_n')} \sum_{x_{m,n}} P_{m'',m'} Y_{x_{m,n}}^{(m,s_n)} Y_{x_{m,n}}^{(m',s_n')} = 0.\]

Similarly, for $m'' \neq m'''$, it is the case that
\[P_{m'',m'} C_{\text{completion}} P_{m''',m''} = \sum_{m} \sum_{s_n} \sum_{x_{m,n}} P_{m''',m''} Y_{x_{m,n}}^{(m,s_n)} Y_{x_{m,n}}^{(m',s_n')} \sum_{x_{m,n}} P_{m'',m'} Y_{x_{m,n}}^{(m,s_n)} Y_{x_{m,n}}^{(m',s_n')} = 0.\]

Hence, we have shown (E12).

Finally, for the proof of (E13), one should keep in mind that $\text{Tr}(A_{x_{m,n}}^{(m,s_n)} A_{x'_{m,n}}^{(m',s_n')} \varrho)$ in general can be complex, and that
\[\text{Tr}(A_{x_{m,n}}^{(m,s_n)} A_{x'_{m,n}}^{(m',s_n')} \varrho)^* = \text{Tr}(A_{x'_{m,n}}^{(m',s_n')} A_{x_{m,n}}^{(m,s_n)} \varrho).\]

Finally, we wish to show that $R \geq 0$ and $C_{\text{completion}} + C_{\text{observable}} \geq 0$. To this end, define
\begin{equation}
C := \sum_{m} \sum_{s_n} \sum_{x_{m,n}} \sum_{m'} \sum_{s'_n} \sum_{x'_{m,n}} \left[ Y_{x_{m,n}}^{(m,s_n)} \otimes i - \sum_{x_{m,n}} Y_{x_{m,n}}^{(m,s_n)} \otimes A_{x_{m,n}}^{(m,s_n)} \right] A_{x_{m,n}}^{(m,s_n)} \left[ Y_{x_{m,n}}^{(m,s_n)} \otimes i - \sum_{x_{m,n}} Y_{x_{m,n}}^{(m,s_n)} \otimes A_{x_{m,n}}^{(m,s_n)} \right]^\dagger.
\end{equation}

One can confirm that $C = \text{Tr}_H(\mathcal{C} \varrho)$. Moreover, by the manner in which $C$ is constructed in (E14), one can see that $C \geq 0$. Consequently, $C = \text{Tr}_H(\mathcal{C} \varrho) \geq 0$. One can also realize that
\[\text{Tr}_H \left[ \left( Q - \text{Tr}_H(Q \varrho) \right) \left( Q - \text{Tr}_H(Q \varrho) \right)^\dagger \right] \geq 0.\]

Hence, the left hand side of (E10) is positive semidefinite, and thus is also the right hand side. We can thus conclude that $C_{\text{completion}} + C_{\text{observable}} \geq 0$.
which demonstrates (E15).

2. Semidefinite decompositions for quantum networks with inputs

For convenience we here restate Proposition 5 in the main text.

**Proposition 6.** Let the conditional distribution $P(x_1, \ldots, x_M|s_1, \ldots, s_M)$ be compatible, in the sense of (E5), with the quantum causal structure $G$ with parents $p_1, \ldots, p_N$ and children $c_1, \ldots, c_M$, with associated inputs $s_1, \ldots, s_M$. Assume that each child $c_m$, and each input $s_m$, is assigned a feature map $Y(m|s_m)$ into a vector space $V_{m,s_m}$. Let the operator $C_{\text{observable}}$ on $\mathcal{V} := \bigoplus_{m=1}^M \bigoplus_{s_m} V_{m,s_m}$ be as defined in (E4). Then there exist operators $C_{\text{completion}}$ and $(C_n)_{n=1}^N$ on $\mathcal{V}$, such that $C_{\text{completion}}$ satisfies (E12) and

$$C := C_{\text{observable}} + C_{\text{completion}} \geq 0,$$

where $C$ is Hermitian. Thus, in any search for a completion, one can assume that $C$ is positive semidefinite, that $C_{\text{completion}}$ is Hermitian. As stated in Lemma 2, the specific choice of completion in (E16) is indeed also Hermitian.

**Proof.** By the assumption that the conditional distribution $P(x_1, \ldots, x_M|s_1, \ldots, s_M)$ is compatible with the quantum causal structure $G$ with parents $p_1, \ldots, p_N$ and children $c_1, \ldots, c_M$, with associated inputs $s_1, \ldots, s_M$, there exists a state as in (E3) and POVMs on the form (E2), such that $P(x_1, \ldots, x_M|s_1, \ldots, s_M)$ can be written as in (E3). With the assigned feature maps $Y(m|s_m)$ to the outcomes $x_{m,s_m}$ corresponding to POVM $A_n^{(m|s_m)}$, the observable covariance $C_{\text{observable}}$ is defined as in (E4). By Lemma 2, we know that $C_{\text{completion}}$, defined in (E12), is such that $C := C_{\text{observable}} + C_{\text{completion}} \geq 0$ and thus (E15) holds. By Lemma 2, we moreover know that $C_{\text{completion}}$ satisfies (E12).

Next, we consider $Q$, defined in (E7), and note that

$$\text{Tr}_H \left[ \left( Q - \text{Tr}_H (Qe) \right) \left( Q - \text{Tr}_H (Qe) \right)^\dagger \right] = \text{Tr}_H (QQ^\dagger e) - 2 \text{Tr}_H (Qe) \text{Tr}_H (Q^\dagger e)$$

with the last equality follows by Lemma 1, where $C_n$ are defined as in (B5).

By equation (E26) in Lemma 2, we can conclude that $C_{\text{observable}} + C_{\text{completion}} = C = R + \sum_{n=1}^N C_n$, with $R$, defined by (E5). Furthermore, Lemma 2 yields that $R$ is positive semidefinite, and Lemma 1 yields that each $C_n$ is positive semidefinite. Hence we can conclude that $C_n$ holds.

Next we shall show that $P^{(n)} C_n P^{(n)} = C_n$, for the projectors $P^{(n)}$ defined in (E18) with respect to the given DAG $G$. For $2 \leq n \leq N - 1$, recall the definition of $C_n$ in (B5). With the definition of $Q$ in (E7), it follows that

$$C_n := \sum_{m,n'} \sum_{s_m,s_{n'}} \sum_{x_{mn},x_{n'n'}} Y^{(m,s_m)}(x_{mn}) Y^{(n',s_{n'})}(x_{n'n'})^\dagger \text{Tr}_{p_1,\ldots,p_N} \left[ W^{m,s_m,n,n'}_{x_{mn},x_{n'n'}} \sum_{p_0} \cdots \sum_{p_{n-1}} e_{p_0} \cdots e_{p_{n-1}} \right],$$

where

$$W^{m,s_m,n,n'}_{x_{mn},x_{n'n'}} := \text{Tr}_{p_1,\ldots,p_{n-1}} \left[ (A^{(m|s_m)}_{x_{mn}} \otimes I_{E_{n-m}}) e_{p_1} \cdots e_{p_{n-1}} \right] - \sum_{p_n} \text{Tr}_{p_1,\ldots,p_{n-1}} \left[ A^{(m|s_m)}_{x_{mn}} (I_{E_{n-m}}) e_{p_1} \cdots e_{p_{n-1}} \right].$$
Suppose that \( m \notin C_n \). This implies that \( \mathbf{1}_{p_a} \otimes \text{Tr}_{p_a} [A^{(m, s_m)}] = \mathbf{1}_{p_a} \otimes \hat{1}_{p_a} \). Consequently \( W_{s_m, \hat{m}^\prime, n}^{m, s_m, n} = 0 \), for all \( s_m \) and all \( x_m, s_n \). Hence, if \( m \notin C_n \) or if \( m' \notin C_n \), then

\[
W_{s_m, \hat{m}^\prime, n}^{m, s_m, n} W_{\hat{m}^\prime, m', n}^{\hat{m}, s_n, n} = 0, \quad \forall s_m, s_n, x_m, x_{m'}, x_{m'}, x_{m'}.
\]

(E22)

Recall that \( Y_{x_m, s_m}^{(m, s_m)} \) is supported on \( \mathcal{V}_{m, s_m} \). By comparing (E20) with (E22) one can realize that \( P_{m, s_m} C_n P_{m', s'_n} = 0 \) if \( m \notin C_n \) or if \( m' \notin C_n \), for all \( s_m \) and \( s'_n \). Hence

\[
C_n = \sum_{m=1}^{M} \sum_{s_m=1}^{S_m} P_{m, s_m} C_n \sum_{m'=1}^{M} \sum_{s'_m=1}^{S_{m'}} P_{m', s'_m}
= \sum_{m \in C_n, s_m=1}^{S_m} P_{m, s_m} C_n \sum_{m' \in C_n, s'_m=1}^{S_{m'}} P_{m', s'_m}
= p(n) C_n p(n)
\]

We can thus conclude that \( p(n) C_n p(n) = C_n \), for \( 2 \leq n \leq N - 1 \). The proofs for the cases \( n = 1 \) and \( n = N \) are analogous.

As the final step we need to show that \( R \), defined in (E8), satisfies the decomposition \( R = \sum_{m, s_m} P_{m, s_m} R_P_{m, s_m} \). Since \( Y_{x_m, s_m}^{(m, s_m)} \) is supported on \( \mathcal{V}_{m, s_m} \), it follows that \( P_{m', s'_m} Y_{x_m, s_m}^{(m, s_m)} = \delta_{m, m'} \delta_{s_m, s'_m} Y_{x_m, s_m}^{(m, s_m)} \) for all \( x_m, s_m \). By comparison with (E8) it follows that \( P_{m, s_m} R_P_{m', s'_m} = \delta_{m, m'} \delta_{s_m, s'_m} P_{m, s_m} R_P_{m, s_m} \), and thus \( R = \sum_{m, s_m} P_{m, s_m} R_P_{m, s_m} \).

3. Semidefinite test for quantum networks with inputs

Analogous to how the case of quantum networks without inputs in Proposition 5 gives rise to a semidefinite test, Proposition 6 can also be turned into a semidefinite test that allows us to rule out hypothetical causal structures as explanations of a given conditional distribution \( P \). Proposition 6 can be phrased schematically as follows: If \( P \) is compatible with \( G \), then for every choice of feature maps there exist \( C_{\text{completion}} \), \( R \), and \( (C_n)_n \) such that the ‘conditions’ \([\text{E15}] - \text{E18}\) are satisfied. This can equivalently be rephrased as follows: If there exists a choice of feature maps for which the ‘conditions’ fail for all choices of \( C_{\text{completion}} \), \( R \), and \( (C_n)_n \), then \( P \) is not compatible with \( G \). This alternative formulation is the basis of the semidefinite test. It is worth stressing that one has to test all completions \( C_{\text{completion}} \) before a given quantum causal structure \( G \) can be excluded as an explanation of an observed conditional distribution \( P \). In particular, when applying the semidefinite test, one cannot in general fix the completion \( C_{\text{completion}} \) to be \( C_{\text{E10}} \). For example, if the conditional distribution is obtained via some quantum state and families of POVMs (not necessarily compatible with \( G \)), then one should not be tempted to use the completion in \( \text{E10} \) for the semidefinite test, since there may exist some other states and collections of POVMs that would be compatible with \( G \), and yield the same distribution.

By comparison of Proposition 6 and Proposition 5 one can see that the conditions of the completion and semidefinite decomposition is identical in both cases. We can thus conclude that even though we allow for measurement settings, the semidefinite test does not distinguish classical and quantum networks. This may seem surprising, since including measurement setting in many cases yield such distinctions. In view of this, it is worth pointing out that although the semidefinite test has identical structure in both the classical and quantum case, this does not exclude the possibility that the observable covariance matrix per se may contain information that would allow for distinctions in some cases. More precisely, it may be the case that the set of observable covariance matrices that are compatible with a quantum network is larger than the set compatible with the corresponding classical network, even though the semidefinite test is oblivious to such differences.