Topological Density and Instantons on a Lattice

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Abstract: We present an update on the study of topological structure of QCD. Issues addressed include a comparison between the plaquette and the geometric methods of calculating the topological density. We show that the improved gauge action based on $\sqrt{3}$ blocking transformation suppresses the formation of topologically charged dislocations with low action. Using a cooling method we identify the instantons’ location, estimate their size and density, and calculate the renormalization constant $Z_Q$ for the plaquette method.

Issues related to the discretization of the continuum expression for the topological density,

$$q(x) = \frac{1}{2\pi^2} \text{Tr}(\epsilon_{\mu\nu\sigma} F_{\mu\nu} F_{\rho\sigma})$$

depend on which of several methods, of which we consider the plaquette\textsuperscript{2}, and geometric (Lüscher’s\textsuperscript{3}), is used to calculate $q(x)$, and on the lattice action. The geometric method while preserving an integer result for the topological charge $Q = \sum x q(x)$ gives rise to a divergent topological susceptibility $\chi_t$ in the continuum limit due to lattice artifacts called dislocations\textsuperscript{3}. We show that these dislocations are much better controlled by an improved action (IA) derived by approximating the renormalization trajectory(RT) of the $(\sqrt{3})$ blocking transformation \textsuperscript{4}. We also show that cooling\textsuperscript{5} is useful for quantifying instantons’ locations and sizes, and for estimating the large multiplicative renormalization $Z_Q(\beta = 6)$ for the plaquette method\textsuperscript{6}.

Our calculations are performed on the existing ensemble of 34 $(16^3 \times 40)$ lattices at $\beta = 6.0$ using the Wilson action \textsuperscript{7}. For the RGT improved action, we use 55 $(18^3 \times 36)$ lattices at $\beta \approx 6.0$.

1. Improved Action: Geometric Method

On the lattice the action of an instanton goes to zero as the core size $\rho \to 0$, rather than show the expected logarithmic divergence. The charge, measured using the geometric method, jumps from 0 to 1 at some $\rho_c$ where the lattice action $S_{\min}$ is still much smaller than that for the classical instanton. These boundary configurations with $Q = \pm 1$ are called dislocations, and are lattice artifacts because their size $\rho < a$ for all $g$, i.e. $\rho_c$ does not scale as expected. In the case of the Wilson action (WA) the Boltzmann suppression $e^{-\beta S_{\min}}$ is overwhelmed by the entropy factor $e^{2\beta^2}/(6\pi)$, and dislocations dominate the calculation of $\chi_t$ in the continuum limit. To raise $S_{\min}$ we investigate an IA consisting of the plaquette in the fundamental, 8 and 6 representations and the $1 \times 2$ planar loop with couplings \textsuperscript{2}.

$$\frac{K_{1 \times 2}}{K_F} = -0.04; \frac{K_8}{K_F} = -0.12; \frac{K_6}{K_F} = -0.12.$$ (2)

In our normalization, the bare coupling is given by $K_F = c_{(1 \times 1)} 6/g^2$ where $c_{(1 \times 1)}$ is the weight associated with the plaquette. We estimate $S_{\min}^{IA}$ by the following method. We constrain a central plaquette to be $-1$ in a $SU(2)$ projection, a characteristic of most dislocations that have been considered, and then vary the links around the plaquette in a collective fashion until a minimum of the action is found with a topological charge of 1. For such configurations $S_{\min}^{IA} = 18.9$, even larger than the value $8\pi^2/6$ for a classical instanton. A comparison of $S_{\min}^{IA}$ with other actions studied in \textsuperscript{2} is made in Table \textsuperscript{1} along with $c_{(1 \times 1)}$ for each action. We conjecture that dislocations are suppressed for improved actions with large $c_{(1 \times 1)}$ because the central plaquette near the cut locus contributes a significant fraction.
to the total action of the dislocation and a large $c_{(1\times 1)}$ raises the action of this central plaquette.

The results for $\chi_t$ by the geometric method at $K_F = 10.58$ and for the WA [3] at $\beta = 6.0$ are given in Table 1. A rough estimate of the scale for the IA is $a^{-1} = 2.3 \text{ GeV}$, i.e. the same as for the WA at $\beta = 6.0$. $\chi_t$ with the WA lies significantly above that predicted by the Witten-Veneziano formula [8], however, due to the uncertainty in scale for the IA we cannot yet isolate the role of dislocations by comparing Wilson and improved action results or give a reliable estimate for $\chi_t^{IA}$. However, the two advantages of the IA in the study of topology are (i) raising $S_{min}$ to suppress dislocations and (ii) the IA plaquette expectation value $\langle \Box \rangle = 1.871$, is much higher than the WA value $1.782$, consequently the integration routines for calculating $q(x)$ converge about $70\%$ faster as is expected of smoother fields.

2. Identifying instantons via Cooling

In order to correlate hadronic structure with topology we need to identify the shapes and sizes of all instantons on the uncooled lattices. Ultra-violet (UV) noise in $q(x)$ on the uncooled lattices has thwarted our efforts to devise a method to do this. We therefore adiabatically [3] cool a subset of lattices (32 WA lattices and 28 IA lattices) for a total cooling time $\tau = 10$ and measure $q(x)$ using the plaquette and Lüscher’s method for cooling times $\tau = 0, 0.3, 1, 3, 6, 10$. The results are shown in Table 2 using $Z_Q$ as estimated below. Note that for $\tau \geq 1$ the two methods give consistent results. The complete lack of dependence of $\chi_t^{geom}$ on $\tau$ is probably fortuitous.

On the $\tau = 10$ cooled lattices we identify hyperspherical volumes of radius $r^2 \leq r_{max}^2$ containing a total net geometric charge of magnitude greater than $q_{min}$ (we generally set $q_{min} = 1$) and consider these volumes to contain an instanton or anti-instanton. Our search routine first isolates small instantons and excludes their volumes from future search. As a result, if there are overlapping instantons, larger instantons ($r_{max} \geq 30$) lose the spherical shape we normally associate with classical instantons, therefore for each instanton volume containing $v$ points we define an effective radius $r_{eff} = (2v/\pi^{\frac{3}{2}})^{1/4}$. Figure 1 shows the distribution of instanton volumes for IA in terms of $r_{eff}^2$, with $r_{max}^2 = 35$. Most are in the range $4a \leq r_{eff} \leq 5.5a$, or about 0.35 to 0.5 fm. Taking the mean size to be $r_{eff} = 5$ and assuming a uniform distribution, we estimate the core size of classical instantons to be $\rho \sim 0.3$ fm. On WA lattices we find a total of $N_+ = 180$ instantons and $N_- = 200$ anti-instantons, corresponding to an instanton density of $1.3 \text{fm}^{-4}$. Similarly, for the IA we find $N_+ = 440, N_- = 421$, which gives a factor of two larger density of $2.7 \text{fm}^{-4}$.

The average charge $\langle q_{v} \rangle_{\tau}$ in volumes identified with instantons at $\tau = 10$ is shown in Fig. 2 versus the cooling time. For the IA action we find that $\langle q_v^{IA} \rangle_{\tau}$ decreases monotonically with $\tau$ which we interpret as (pair) annihilation of small instantons during cooling. The behavior with the WA at small $\tau$ is not clear, but for $\tau > 0.3$ $\langle q_v^{WA} \rangle_{\tau}$ shows a similar decrease.
We use the correlation between the average charge inside the instanton volume as measured by the plaquette and geometric methods to define the renormalization constant to be $Z_Q(\tau) = \langle q \rangle^\text{plaq.}(\tau)/\langle q \rangle^\text{geom.}(\tau = 10)$. For uncooled lattices we find $Z_Q^W(\beta = 6.0) = 0.158(13)$, and $Z_Q^I(\beta \approx 6.0) = 0.230(9)$. On varying the search parameters between $0.8 \leq q_{min} \leq 1.0$ and $24 \leq r_{max}^2 \leq 35$, the $Z_Q$ ranges roughly between 0.14 and 0.17 (WA), and 0.21 to 0.24 (IA), i.e. it has a rather weak dependence on the definition of instanton volumes. Including this dependence as a systematic uncertainty, we estimate the renormalization constants to be

$$Z_Q^W = 0.16(2), \quad Z_Q^I = 0.22(2).$$

Our estimate of $Z_Q^W$ is consistent with the value 0.18(2) obtained by Alles et al. [8] using a heating method. The larger value of $Z_Q^I$ is consistent with the expectation that the IA is a better approximation of the continuum theory (where $Z_Q = 1$). We use this estimate of $Z_Q(\tau)$ to compile the data presented in Table 3.

Our cooling procedure does not guarantee that all physical instantons survive. We assume that instantons that survive are physical and their properties, as a function of cooling time, can be studied once their location is known. The data show that instantons identified on cooled lattices can be tracked back to the uncooled lattices, where UV noise prevented their identification. Systematic errors due to the uncertainty in the scale and the non-uniqueness of the search algorithm have not been fully included in the rough estimates of $\rho$, density, and $Z_Q(\tau)$ given above.

We are refining this analysis in several ways. We are calculating the string tension for the IA

Table 3

| $\tau$ | Geometric $\chi_t^{1/4}$ (MeV) | Plaquette $Z_Q^{1/2}/\chi_t^{1/4}$ | $\chi_t^{1/4}$ |
|--------|---------------------------------|---------------------------------|----------------|
| 0      | 1.871                           | 238(14)                         | 170(9)         |
| 1      | 2.687                           | 238(14)                         | 172(10)        |
| 2      | 2.922                           | 238(15)                         | 196(13)        |
| 3      | 2.970                           | 238(16)                         | 211(13)        |
| 6      | 2.984                           | 238(16)                         | 218(17)        |
| 10     | 2.984                           | 238(16)                         | 254(20)        |

Figure 2. Average charge $\langle q \rangle_v(\tau)$ contained within instanton volume versus $\tau$ for geometric (crosses) and plaquette (diamonds) methods.

lattices to get a reliable estimate of the scale. We are redoing the IA analysis with $\tau = 30$ as the starting point because the data in Fig. 3 show that $\langle q \rangle_v(\tau = 10)$ with the geometric method has not stabilized. We are investigating alternate schemes for identifying instanton volumes, and are generating quark propagators to correlate hadronic properties with instantons for the IA configurations. Once the signal in all these quantities is under control we plan to do the calculation at weaker coupling to check scaling.

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