Quantum electrodynamics on the 3-torus  
II.- The RG flow

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Abstract

We continue the study of quantum electrodynamics on a three dimensional torus as the limit of a lattice gauge theory. In this paper we give a preliminary treatment of the renormalization group flow. We study the propagators which arise under multiple block spin averaging, both in global and local versions. We also study low order perturbation theory. However we do not control remainders. This is left to the more complete treatment of the following paper.

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## Contents

1. **Global flow**  
   1.1 overview  
   1.2 bosons  
   1.2.1  
   1.2.2  
   1.2.3  
   1.2.4  
   1.3 fermions  
   1.3.1  
   1.3.2  
   1.3.3  
   1.3.4  
   1.4 global flow  
   1.5 perturbation theory  
   1.6 single steps  

2. **Localized flow**  
   2.1 blocking  
   2.2 bosons  
   2.3 fermions  

3. **Propagators**  
   3.1 definitions  
   3.2 fermions  
   3.3 bosons  

A. **Averaging operators**  
   A.1 bosons  
   A.2 fermions  

B. **Perturbation identities**
1 Global flow

1.1 overview

In this paper we continue our analysis of QED on the 3-torus which was begun in paper I [1]. In that paper we explained the first step consisting of a renormalization group (RG) transformation followed by a split into regions with large and small gauge fields. In this paper we study multiple RG transformations as much as possible without the split into regions of large and small fields. The emphasis is on studying the propagators which arise under multiple block spin averaging, both in global and localized versions. We also study low order perturbation theory, but as yet without estimating remainders. The paper is preliminary to paper III [2] in which we also incorporate the split into large and small fields at each step. This gives the full expansion and gives an expression for the effective action on any length scale. The expression exhibits the contribution of low order perturbation theory and controls the remainder.

The analysis starts with a field theory on a toroidal lattice $\mathbb{T}^3_{N} = L^{-N} \mathbb{Z}^3 / L^M \mathbb{Z}^3$ with spacing $L^{-N}$ and volume $L^M$. Our goal is to gain some control over the $N \to \infty$ limit for fixed $M$. We start by scaling the theory up to $\mathbb{T}^3_{N,M}$ which has unit spacing and volume $L^{3(M+1)}$. The density for the theory is then

$$\rho_0(\Psi_0, A_0) = \exp\left(-\frac{1}{2}(A_0, (-\Delta + \mu^2_0)A_0) - (\bar{\Psi}_0, (D_{e_0}(A) + m_0)\Psi_0) - v_0 - \xi_0\right)$$

(1)

Here the gauge fields $A_0 = \{A_0,\mu(x)\}$ are real numbers indexed by $x \in \mathbb{T}^3_{N,M}$ and $\mu = (1, 2, 3)$. The fermi fields $\Psi_0 = \{\Psi_0,\alpha(x)\}$ and $\bar{\Psi}_0 = \{\bar{\Psi}_0,\alpha(x)\}$ are the generators of a Grassmann algebra and are indexed by $x \in \mathbb{T}^3_{N,M}$ and spinor indices $\alpha = (1, \ldots, 4)$. The operator $\Delta$ is the lattice Laplacian and $D_{e_0}(A)$ is the lattice Dirac operator with gauge field $A$. The coupling constant $e_0$, fermion mass $m_0$, and boson mass $\mu_0$ are scaled versions of their bare values $e, m, \mu$:

$$e_0 = L^{-N/2}e, \quad m_0 = L^{-N}m, \quad \mu_0 = L^{-N}\mu$$

(2)

Finally $v_0$ is a fermion mass counterterm and $\xi_0$ is a vacuum energy counterterm. The counterterms play no role in the present paper and are set equal to zero. See paper I for more details.

To analyze this expression we perform a number of renormalization group (RG) transformations consisting of averaging operation followed by scalings back to the unit lattice. We get a sequence of densities $\rho_0, \rho_1, \rho_2, \ldots$. The density $\rho_k$ is defined on fields $\Psi_k, A_k$ on $\mathbb{T}^3_{N,M,k}$. The sequence is generated by

$$\rho_{k+1}(\Psi_{k+1}, A_{k+1}) = \int \mathcal{N}^{-1}_{k+1,a} \mathcal{M}^{-1}_{k+1,b} \exp\left(-\frac{1}{2L^2} |A_{k+1} - Q_k|^2 - \frac{b}{L} \Psi_{k+1} - Q_{e_k}(A_k') \Psi_k|^2\right) \rho_k(\Psi_k, A_k) \, dA_k \, d\Psi_k$$

(3)

Here scaling up by $L$ is defined by

$$A_L(x) = (\sigma_L A)(x) = L^{-1/2} A(L^{-1} x)$$
$$\Psi_L(x) = (\sigma_L \Psi)(x) = L^{-1} \Psi(L^{-1} x)$$

(4)

so that $\Psi_{k+1,L}, A_{k+1,L}$ are fields on $\mathbb{T}^3_{N,M,k}$. The operators $Q, Q_{e_k}(A_k')$ average over blocks of side $L$ and also take us to fields on $\mathbb{T}^3_{N,M,k}$. For fermions this has the gauge covariant form

$$(Q_{e_k}(A_k')\Psi)(y) = L^{-3} \sum_{|x-y| \leq L/2} \exp(i e_k A_k'(G_{yx})) \Psi(x)$$

(5)
Here \( e_k = L^{-(N-k)/2} e \) is a running coupling constant, the gauge field \( A'_k \) is either \( A_k \) or some modification to be specified, and \( \Gamma_{yx} \) is a standard path from \( x \) to \( y \). We are using the notation

\[
|\Psi_{k+1,L} - Q_{e_k}(A'_k)\Psi_k|^2 = (\Psi_{k+1,L} - Q_{e_k}(-A'_k)\Psi_k, \Psi_{k+1,L} - Q_{e_k}(A'_k)\Psi_k)
\]

(6)

The normalization constants are chosen so that

\[
\int \rho_{k+1}(\Psi_{k+1,L}A_{k+1})d\Psi_{k+1}dA_{k+1} = \int \rho_k(\Psi_k,A_k)d\Psi_kdA_k
\]

(7)

That is

\[
N_{k,a}^{-1} = \int e^{-a|A_k|^2/2}dA_k = \left(\frac{2\pi}{a}\right)^{3|T_{N+M-k}^0|/2} \\
M_{k,b}^{-1} = \int e^{-b\langle\Psi_k,\Psi_k\rangle}d\Psi_k = b^4|T_{N+M-k}^3|
\]

(8)

We want to study \( \rho_k \) as \( k \to N \).

Before proceeding we recall our indebtedness to the papers of Balaban and collaborators: 9 - 15.

1.2 bosons

1.2.1

Let us start by dropping the fermions and study the effect of repeated RG transformations for the bosons. If we do the intermediate integrals and scale down we find the following formula

\[
\int \prod_{j=0}^{k-1} dA_j N_{j+1,a}^{-1} \exp\left(-\frac{1}{2}a|A_{j+1,L} - QA_j|^2\right) \exp\left(-\frac{1}{2}(A_0,(-\Delta + \mu^2_0)A_0)\right) F(A_0)
\]

\[= \int N_{k,a_k}^{-1} \exp\left(-\frac{a_k}{2}|A_k - Q_k A|^2 - \frac{1}{2}(A,(-\Delta + \mu^2_k)A)\right) F(A_{L,k})dA
\]

(9)

Here we are integrating over functions \( A \) on \( T_{N+M-k}^0 \) and the inner product is

\[
\langle A, A \rangle = \sum_x L^{-3k} |A(x)|^2 = \int |A(x)|^2 dx
\]

(10)

The operator \( Q_k \) is an averaging operator on this space taking us to functions on the unit lattice \( T_{N+M-k}^0 \) given by

\[
(Q_k A)_\mu(y) = \int_{|x-y| \leq 1/2} A_\mu(x) dx
\]

(11)

With \( \sigma_L = \sigma_L^k \) it satisfies

\[
Q_{k+1} = \sigma^{-1}_{L+1} Q_k \sigma_L
\]

(12)

The numbers \( a_k \) obey the recursion equation \( a_{k+1} = a_k/(a_k + a/L^2) \) which has the solution

\[
a_k = \left(\frac{1 - L^{-2}}{1 - L^{-2k}}\right) a
\]

(13)

The quadratic form \( \langle A, (-\Delta + \mu^2_k)A\rangle \) is the scaled version of \( \langle A_0,(-\Delta + \mu^2_0)A_0\rangle \) and has mass \( \mu_k = L^{-(N-k)} \mu \).
The formula (9) is proved by induction. It comes down to the explicit calculation of the Gaussian integral

\[
\int dA_k \mathcal{N}^{-1}_{k+1,a} \mathcal{N}^{-1}_{k,a} \exp \left( -\frac{a}{2L^2} |A_{k+1,L} - QA_k|^2 \right) \exp \left( -\frac{a}{2} |A_k - Q_kA|^2 \right)
\]

(14)

which we carry out in appendix A.

1.2.2

To evaluate integrals such as those on the right side of (15) we use the identity

\[
\int \mathcal{N}^{-1}_{k,a} \exp \left( -\frac{a}{2} |A_k - Q_kA|^2 - \frac{1}{2} (A, (-\Delta + \mu_k^2)A) \right) f(A) dA
\]

(15)

Here we have defined

\[
\Delta^\#_k = -\Delta + \mu_k^2 + a_k Q_k^T Q_k
\]

(16)

and

\[
G_k = (\Delta^\#_k)^{-1}, \quad \mathcal{H}_k = a_k (\Delta^\#_k)^{-1} Q_k^T, \quad \Delta_k = a_k I - a_k^2 Q_k (\Delta^\#_k)^{-1} Q_k^T
\]

(17)

The operators \( \Delta^\#_k, G_k \) act on functions on \( \mathbb{T}_{N+M-k}^k \), \( \mathcal{H}_k \) maps functions on \( \mathbb{T}_{N+M-k}^0 \) to functions on \( \mathbb{T}_{N+M-k}^{-k} \), and \( \Delta_k \) is an operator on functions on \( \mathbb{T}_{N+M-k}^0 \). We have also defined \( \mu_{G_k} \) to be the Gaussian measure with covariance \( G_k \).

To prove (15) make transformation \( A \to A + \mathcal{H}_k A_k \). This diagonalizes the quadratic form and gives \(-\frac{1}{2} (A_k, \Delta_k A_k) - \frac{1}{2} (A, \Delta^\#_k A)\), whence the result with

\[
Z_k = \mathcal{N}^{-1}_{k,a} \int e^{-\frac{1}{2} (A, (-\Delta^\#_k)A)} dA
\]

(18)

1.2.3

As a special case we take \( f(A) = \exp(i(A, J)) \) and consider the generating function

\[
\Omega_k(A_k, J) = \int \mathcal{N}^{-1}_{k,a} \exp \left( -\frac{a}{2} |A_k - Q_kA|^2 - \frac{1}{2} (A, (-\Delta + \mu_k^2)A) + i(J, A) \right) dA
\]

(19)

Using (15) and \( \int \exp(i(A, J)) d\mu_{G_k}(A) = \exp(-1/2(J, G_k J)) \) this is evaluated as

\[
\Omega_k(A_k, J) = Z_k \exp \left( -\frac{1}{2} (A_k, \Delta_k A_k) - \frac{1}{2} (J, G_k J) + i(J, \mathcal{H}_k A_k) \right)
\]

(20)

We can also take it one step at a time. Using (15) we have

\[
\Omega_{k+1}(A_{k+1}, J) = \int \mathcal{N}^{-1}_{k+1,a} \exp \left( -\frac{a}{2L^2} |A_{k+1,L} - QA_k|^2 \right) \Omega_k(A_k, \sigma_{L-1}^T J) dA_k
\]

(21)
To evaluate we introduce
\[ C_k = (\Delta_k + \frac{a}{L^2} Q T Q)^{-1} \]
\[ H_k = \frac{a}{L^2} C_k Q T \] (22)

Now insert (20) into (21) and make the transformation \( A_k \rightarrow A_k + H_k A_{k+1,L} \) to diagonalize the quadratic form. The integral over \( A_k \) becomes a Gaussian integral with covariance \( C_k \). Carry out the integral and compare the resulting expression with (20) for \( \Omega_{k+1}(A_{k+1,J}) \). We find that
\[ G_{k+1} = \sigma_L^{-1} (G_k + H_k C_k H_k^T) (\sigma_L^{-1})^T \]
\[ \mathcal{H}_{k+1} = \sigma_L^{-1} (H_k \mathcal{H}_k) \sigma_L \]
\[ \Delta_{k+1} = \sigma_L^T \left( \frac{a}{L^2} - \frac{a^2}{L^4} Q C_k Q T \right) \sigma_L \] (23)

(for \( k = 0 \) the conventions are \( G_0 = 0, \mathcal{H}_0 = I, \) and \( \Delta_0 = -\Delta + \mu_0^2 \)) We also have
\[ Z_{k+1} = Z_k N_{k+1,a}^{-1} \int e^{-\frac{1}{2}(A_k, C_k^{-1} A_k)} \, dA_k \] (24)

1.2.4

As an operator on functions on \( T_{N+M-k}^{-1} \), the propagator \( G_k \) has a kernel \( g_k(x, x') \) defined so that
\[ (G_k f)(x) = \int G_k(x, x') f(x') \, dx' \] where again the integral means the weighted sum. Since \( \sigma_L^{-1} = L^{-2} \sigma_L \) for bosons and \( (\sigma_L^{-1} G_k \sigma_L)(x, x') = L^3 G_k(Lx, Lx') \) and we can write (20) as
\[ G_{k+1}(x, x') = L(G_k(Lx, Lx') + (H_k C_k H_k^T)(Lx, Lx')) \] (25)

If we iterate this we find
\[ G_k(x, x') = \sum_{j=0}^{k-1} L^{k-j} \tilde{C}_j(L^{k-j}x, L^{k-j}x') \] (26)

where
\[ \tilde{C}_j = \mathcal{H}_j C_j \mathcal{H}_j^T \] (27)

The operators \( \mathcal{H}_j, C_j, \tilde{C}_j \) have kernels which satisfy for \( x, x' \in T_{N+M-k}^{-1} \) and \( y, y' \in T_{N+M-k}^0 \)
\[ |\mathcal{H}_j(x, y)|, |\partial \mathcal{H}_j(x, y)|, \leq \mathcal{O}(1) \exp(-\mathcal{O}(1)(d(x, y)) \]
\[ |C_j(y, y')| \leq \mathcal{O}(1) \exp(-\mathcal{O}(1)(d(y, y'))) \]
\[ |\tilde{C}_j(x, x')| \leq \mathcal{O}(1) \exp(-\mathcal{O}(1)(d(x, x'))) \] (28)

These are proved by Balaban in [6], but one may prefer to use the methods of [14].

Using this in (20) leads to the estimates
\[ |G_k(x, x')| \leq \begin{cases} \mathcal{O}(1)(d(x, x'))^{-1} e^{-\mathcal{O}(1)(d(x, x'))} & x \neq x' \\ \mathcal{O}(L^k) & x = x' \end{cases} \]
\[ |\partial G_k(x, x')| \leq \begin{cases} \mathcal{O}(1)(d(x, x'))^{-2} e^{-\mathcal{O}(1)(d(x, x'))} & x \neq x' \\ \mathcal{O}(L^{2k}) & x = x' \end{cases} \] (29)

(To see the short distance bound divide the sum into terms satisfying \( L^{k-j} d(x, x') \leq 1 \) and the complement.)

The function \( G_k(x, x') \) is our basic photon propagator after \( k \) steps. This estimate shows the exponential decay whose origin is an effective mass from \( Q_k^T Q_k \), and the characteristic short distant singularity \( d(x, x')^{-1} \).

\[ ^1 \text{In our notation } \mathcal{O}(1) \text{ allows } L \text{ dependence. We do however note that in the second and third bounds the } \mathcal{O}(1) \text{ in the exponent is actually } \mathcal{O}(L^{-1}). \]
1.3 fermions

1.3.1

Next we want to do something similar for fermions. For an arbitrary element \( F(\Psi_0) \) in the Grassmann algebra generated by \( \Psi_0 \) we claim that

\[
\int \prod_{j=0}^{k-1} d\Psi_j M_{j+1,b}^{-1} \exp \left( -\frac{b}{L} |\Psi_{j+1,L} - Q_{e_j}(\tilde{Q}_j A_j)\Psi_j|^2 \right) \exp \left( - (\Psi_0, (D_{e_0}(A_0) + m_0)\Psi_0) \right) F(\Psi_0) = \int d\psi M_{k,b}^{-1} \exp \left( -b_k |\psi - Q_k(A_{k-1,L-1}, \ldots, A_{0,L-k})\psi|^2 \right) \exp \left( - (\tilde{\psi}, (D_{e_k}(A_{0,L-k}) + m_k)\psi) \right) F(\psi_{L,k})
\]

(30)

Here on the left side instead of \( Q_{e_j}(A_j) \) we have taken \( Q_{e_j}(\tilde{Q}_j A_j) \) where \( A_j \) is for the moment an arbitrary function on \( T_{N+M-j}^J \) and \( \tilde{Q}_j \) is an averaging operator which brings the field up to \( T_N^J \). The averaging operator is not the same as \( Q_j \), but given by

\[
(\tilde{Q}_j A)_\mu(y) = \int_{|x-y| \leq 1/2} A(\Gamma_{yx} \cup [x, x + \epsilon]\cup \Gamma_{x+\mu,y+\mu})
\]

(31)

where \( A(\Gamma) \) indicates the integral along the contour \( \Gamma \). For a scalar \( \lambda \) on \( T_{N+M-j}^{-j} \)

\[
(\tilde{Q}_j (A + d\lambda))_\mu(y) = (\tilde{Q}_j A)_\mu(y) + \lambda(y) - \lambda(y + \epsilon)
\]

(32)

so this averaging is gauge covariant.

On the right side of (30) we are integrating over \( \psi \) on \( T_{N+M-k}^{-k} \). The multiple averaging operators \( Q_k(a_k - 1, \ldots, a_0) \) depend on fields \( a_k - 1, \ldots, a_0 \) all on \( T_{N+M-k}^{-k} \). They are defined recursively by

\[
Q_{k+1}(a_{k+1}, \ldots, a_0) = \sigma_{L^{-1}} Q_{e_k}(\tilde{Q}_k a_{k,L}) Q_k(a_{k-1,L}, \ldots, a_{0,L}) \sigma_L
\]

(33)

where \( a_k, \ldots, a_0 \) are all on \( T_{N+M-k-1}^{-k} \) and \( \sigma_L = \sigma_L' \). An explicit expression is given in Appendix A.

The numbers \( b_k \) obey

\[
b_k = \left( \frac{1 - L^{-1}}{1 - L^{-k}} \right) b
\]

(34)

The form \( (\tilde{\psi}, (D_{e_k}(A) + m_k)\psi) \) has the Dirac operator on \( T_{N+M-k}^{-k} \) and mass \( m_k = L^{-1}(N-k)m \).

The formula (30) is again proved by induction. It comes down to the Gaussian integral

\[
\int d\Psi_k \mathcal{M}_{k+1,b}^{-1} \mathcal{M}_k^{-1} \exp \left( -\frac{b}{L} |\Psi_{k+1,L} - Q_{e_k}(\tilde{Q}_k A_k)\Psi_k|^2 \right) \exp \left( -b_k |\Psi_k - Q_k(A_{k-1,L-1}, \ldots, A_{0,L-k})\Psi_k|^2 \right) = \mathcal{M}_{k+1,b}^{-1} \exp \left( -b_{k+1} |\Psi_{k+1} - Q_{k+1}(A_{k,L-1}, \ldots, A_{0,L-k-1})\Psi_k|^2 \right)
\]

(35)

which we prove in Appendix A.

1.3.2

We specialize to the case where all the fields are scalings of the last. For any \( A \) on \( T_{N+M-k}^{-k} \), let \( A_j = A_{L-k-j} \). Then on the right side of (30) we identify

\[
Q_k(A) \equiv Q_k(A, \ldots, A)
\]

(36)
From appendix A

\[
(Q_k(A)\psi)(y) = \int_{|x-y|<1/2} \exp(i e_k A(\hat{\Gamma}_{yx}))\psi(x)dx
\]

(37)

Now assume that \(e_k|\partial A|\) is sufficiently small. Then we claim that an integral like the right side of (30) with \(Q_k(A)\) can be evaluated as

\[
\int d\psi \mathcal{M}_{k,b_k}^{-1} \exp(-b_k|\Psi_k - Q_k(A)|^2) \exp\left(-\left(\bar{\psi}, (D_{e_k}(A) + m_k)\psi\right)\right) f(\psi)
\]

(38)

= \(Z_k(A)\) \(\exp(-\bar{\Psi}_k, D_k(A)\Psi_k)\int f(\psi + H_k(A)\Psi_k) \, d\mu_{S_k(A)}(\psi)\)

Here

\[
D_\#^k(A) = D(A) + m_k + b_k Q_k(-A)^T Q_k(A)
\]

(39)

Under the conditions on \(e_k|\partial A|\) this operator is invertible (about which more later) and we define

\[
S_k(A) = D_\#^k(A)^{-1}
\]

\[
H_k(A) = \begin{cases} b_k S_k(A) Q_k(-A)^T & \text{on } \Psi_k \\ b_k S_k(A)^T Q_k(A)^T & \text{on } \bar{\Psi}_k \end{cases}
\]

(40)

Also \(\int \[\ldots\] d\mu_{S_k(A)}(\psi)\) is the fermion Gaussian integral with covariance \(S_k(A)\). To prove (38) make the transformation \(\psi \rightarrow \psi + H_k(A)\Psi_k\) and similarly for \(\bar{\psi}\). This diagonalizes the quadratic form and gives \((\bar{\Psi}_k, D_k(A)\Psi_k) + (\bar{\psi}, D_\#^k(A)\psi)\), whence the result with

\[
Z_k(A) = \mathcal{M}_{k,b_k}^{-1} \int e^{-\bar{\psi}, D_\#^k(A)\psi} \psi
\]

(41)

All these objects are gauge covariant. In particular for a scalar \(\lambda\) on \(T_{N+M-k}\)

\[
S_k(A + d\lambda) = e^{-ie_k\lambda} S_k(A) e^{ie_k\lambda}
\]

(42)

1.3.3

We specifically consider the generating function

\[
\Omega_k(A, \Psi_k, \eta) = \int d\psi \mathcal{M}_{k,b_k}^{-1} \exp(-b_k|\Psi_k - Q_k(A)|^2) \exp\left(-\left(\bar{\psi}, (D_{e_k}(A) + m_k)\psi\right)\right) e^{(\bar{\eta}, \psi) + (\bar{\psi}, \eta)}
\]

(43)

where \(\eta, \bar{\eta}\) are elements of an auxiliary Grassmann algebra indexed by \(T_{N+M-k}\). Using (43) and \(\int \exp((\bar{\eta}, \psi) + (\bar{\psi}, \eta))d\mu_{S_k(A)}(\psi) = \exp((\bar{\eta}, S_k(A)\eta))\) this can be evaluated as

\[
\Omega_k(A, \Psi_k, \eta) = Z_k(A) \exp\left(-\left(\bar{\Psi}_k, D_k(A)\Psi_k\right) + (\bar{\eta}, H_k(A)\Psi_k) + (H_k(A)\bar{\Psi}_k, \eta) + (\bar{\eta}, S_k(A)\eta)\right)
\]

(44)

We can also take it one step at a time. By (43) we have for \(A\) on \(T_{N+M-k-1}\)

\[
\Omega_{k+1}(A, \Psi_{k+1}, \eta) = \int \mathcal{M}_{k+1,b}^{-1} \exp\left(-\frac{b}{L}|\Psi_{k+1,L} - Q_{e_k}(\bar{Q}_k A_L)\Psi_k|^2\right) \Omega_k(A_L, \Psi_k, (\sigma_{L-1})^T \eta)
\]

(45)
where $\sigma$.

For the kernels we can rewrite (47) as

$$
\Gamma_k(A) = (D_k(A) + \frac{b}{L}Q_{e_k}(-\tilde{Q}_kA)^TQ_{e_k}(\tilde{Q}_kA))^{-1}
$$

and making the transformation $\Psi_k \rightarrow \Psi_k + H_k(A_L)\Psi_{k+1,L}$ and similarly for $\tilde{\Psi}_k$. We get an alternative expression for $\Omega_{k+1}$ and by comparing we find

$$
S_{k+1}(A) = \sigma_L^{-1}(S_k(A_L) + \mathcal{H}_k(A_L)\Gamma_k(A_L)\mathcal{H}_k(A_L)^T)(\sigma_L^{-1})^T
$$

$$
\mathcal{H}_{k+1}(A) = \sigma_L^{-1}\mathcal{H}_k(A_L)\mathcal{H}_k(A_L)^T
$$

$$
D_{k+1}(A) = \sigma_L^T\left(\frac{b}{L} - \frac{b^2}{L^2}Q_{e_k}(\tilde{Q}_kA_L)\Gamma_k(A_L)Q_{e_k}(-\tilde{Q}_kA_L)^T\right)\sigma_L
$$

(The convention is that $S_0(A) = 0$, $\mathcal{H}_0(A) = I$, and $D_0(A) = D_{e_0}(A) + m_0$.) We also find that

$$
Z_{k+1}(A) = Z_k(A_L)\mathcal{M}_{k+1,b} \int e^{-\langle \tilde{\Psi}_k, \Gamma_k(A_L)^{-1} \Psi_k \rangle} d\Psi_k
$$

1.3.4

For the kernels we can rewrite (46) as

$$
S_{k+1}(A, x, x') = L^2(S_k(A_L, Lx, Lx') + (\mathcal{H}_k(A_L)\Gamma_k(A_L)\mathcal{H}_k(A_L)^T)(Lx, Lx'))
$$

Here we have used $\sigma_L^{-1}T = L^{-1}\sigma_L$ for fermions. Iterating this yields

$$
S_k(A, x, x') = \sum_{j=0}^{k-1} L^{2(k-j)}\tilde{\Gamma}_j(A_{L^{-j}}; L^{k-j}x, L^{k-j}x')
$$

where

$$
\tilde{\Gamma}_j(A) = \mathcal{H}_j(A)\Gamma_j(A)\mathcal{H}_j(A)^T
$$

provided all the operators exist.

Balaban, O’Carroll, and Schor show for $A$ on $\mathbb{T}_{N+M-k}^k$ that if $e_k|\partial A|$ is sufficiently small then $S_k(A), \mathcal{H}_k(A), \Gamma_k(A)$ all exist and

$$
|\mathcal{H}_k(A, x, y)| \leq O(1) \exp(-O(1)d(x, y))
$$

$$
|\Gamma_k(A, y, y')| \leq O(1) \exp(-O(1)d(y, y'))
$$

$$
|\tilde{\Gamma}_k(A, x, x')| \leq O(1) \exp(-O(1)d(x, x'))
$$

For $A = 0$ this can be found in [13]. For $A \neq 0$ it is a special case of results in [13] and section [12]. If $A$ on $\mathbb{T}_{N+M-k}^k$ has $e_k|\partial A|$ sufficiently small, then $e_j|\partial A_{L^{-j}}|$ on $\mathbb{T}_{N+M-j}^j$ is even smaller by a factor of $L^{-3(k-j)/2}$ so we can use (50) to obtain the bound

$$
|S_k(A, x, x')| \leq \begin{cases} 
O(1)d(x, x')^{-2}e^{-O(1)d(x, x')} & x \neq x' \\
O(L^{2k}) & x = x'
\end{cases}
$$

The function $S_k(A, x, x')$ is our basic fermion propagator with background field $A$. The estimate shows the characteristic short distance singularity $d(x, x')^{-2}$. 

9
1.4 global flow

Now we combine the steps for bosons and fermions and make a first pass at the global flow. Our goal is not yet complete control. We just want to introduce some notation and establish some identities.

We repeatedly apply the basic transformation \( \mathcal{M} \) with fermion averaging operator \( Q_{c_k}(\hat{Q}_kA_k) \) taken to be \( Q_{c_k}(\hat{Q}_kA_k) \) with \( A_k = H_kA_k \). This choice of \( A_k \) is made to match the background field at the \( k \)-th step. Then we have

\[
\rho_k(\Psi_k, A_k) = \int \prod_{i=0}^{k-1} dA_iN_{j+1,a}^{-1} \exp \left( -\frac{1}{2} \frac{a}{L^2} [A_{j+1,L} - QA_j]^2 \right) \exp \left( -\frac{1}{2} (A_0, (-\Delta + \mu_0^2)A_0) \right)
\]

\[
= \frac{1}{L^{\frac{D}{2}}} \int d\Psi_j \mathcal{M}_{j+1,b}^{-1} \exp \left( -\frac{b}{L} |\Psi_j - Q_{c_j}(\hat{Q}_jA_j)|^2 \right) \exp \left( -\frac{1}{2} (\hat{\Psi}_0, (D_{c_0}(A_0) + m_0)\Psi_0) \right)
\]

Integrating out the intermediate fermions by \( \mathcal{M} \) we have

\[
\rho_k(\Psi_k, A_k) = \int \prod_{i=0}^{k-1} dA_iN_{j+1,a}^{-1} \exp \left( -\frac{1}{2} \frac{a}{L^2} [A_{j+1,L} - QA_j]^2 \right) \exp \left( -\frac{1}{2} (A_0, (-\Delta + \mu_0^2)A_0) \right)
\]

\[
\int d\Psi \mathcal{M}_{k,b_k}^{-1} \exp \left( -b_k|\Psi_k - Q_k(A_{k-1,L-i}, \ldots, A_{0,L-i})|^2 \right) \exp \left( -|\hat{\psi}, (D_{c_k}(A_{0,L-i}) + m_k)\psi| \right)
\]

We cannot now integrate out the intermediate boson fields, but we can successively apply the transformations \( A_j \to A_j + H_jA_j+1 \) for \( j = 0, \ldots, k-1 \) and use the identities \( \mathcal{M} \). Under the transformation on \( A_j \) we have \( A_j \to A_j + A_{j+1,L} \). Under all subsequent transformations we have \( A_j \to \sum_{i=j}^{k} \hat{A}_i, L^{-j} \) and thus \( (A_j)_{L^{-j}} \to A_{j,k} \) defined by

\[
A_{j,k} = \sum_{i=j}^{k} A_i
\]

Thus we obtain

\[
\rho_k(\Psi_k, A_k) = Z_k \exp \left( -\frac{1}{2} (A_k, \Delta_k A_k) \right) \rho_k^*(\Psi_k, A_k)
\]

where

\[
\rho_k^*(\Psi_k, A_k) = \int \mathcal{M}_{k,b_k}^{-1} \exp \left( -b_k|\Psi_k - Q_k(A_{k-1,k}, \ldots, A_{0,k})|^2 \right)
\]

\[
\exp \left( -|\hat{\psi}, (D_{c_k}(A_{0,k}) + m_k)\psi| \right) d\Psi \prod_{j=0}^{k-1} d\mu C_j(A_j)
\]

Next for \( a_i \) on \( T_{N+M-k} \) define \( a_{i,j} = \sum_{i=j}^{k} a_i \) and a potential \( V_k(\Psi_k, \psi, a_k, \ldots, a_0) \) by

\[
\begin{align*}
  b_k|\Psi_k - Q_k(a_{k-1,k}, \ldots, a_{0,k})|^2 + (\hat{\psi}, D_{c_k}(a_{0,k})\psi) \\
  = b_k|\Psi_k - Q_k(a_k)|^2 + (\hat{\psi}, D_{c_k}(a_k)\psi) + V_k(\Psi_k, \psi, a_k, \ldots, a_0)
\end{align*}
\]

Note that \( V_k(\Psi_k, \psi, a_k, 0, \ldots, 0) = 0 \). If we evaluate at \( a_i = (A_i)_{L^{-j}} \) for \( i = 0, \ldots, k \) we get the argument of the exponential in \( \mathcal{M} \). Thus

\[
\rho_k^*(\Psi_k, A_k) = \int \mathcal{M}_{k,b_k}^{-1} \exp \left( -b_k|\Psi_k - Q_k(A_k)|^2 - (\hat{\psi}, D_{c_k}(A_k)\psi) \right)
\]

\[
\exp \left( -V_k(\Psi_k, \psi, A_k, A_{k-1,L-i}, \ldots, A_{0,L-i}) \right) d\Psi \prod_{j=0}^{k-1} d\mu C_j(A_j)
\]
Now if we assume \( e_k |\partial \mathcal{A}_k| \) is sufficiently small, we can evaluate the fermion integral by (This assumption would not be suitable for iteration.) Defining \( \psi_k(\mathcal{A}_k) \equiv \mathcal{H}_k(\mathcal{A}_k)\Psi_k \) this yields

\[
\rho_k^\Psi(\Psi_k, \mathcal{A}_k) = Z_k(\mathcal{A}_k) \exp(-\langle \bar{\Psi}_k, D_k(\mathcal{A}_k)\Psi_k \rangle) \rho_k^\Psi(\Psi_k, \psi_k(\mathcal{A}_k), \mathcal{A}_k)
\]

where for \( \psi, \mathcal{A} \) on \( \mathbb{T}_{N-M-k} \)

\[
\rho_k^\Psi(\Psi_k, \psi, \mathcal{A}) = \int \exp\left(-\mathcal{V}_k(\Psi_k, \psi, \mathcal{A}, \mathcal{A}_{k-1,L-1}, \cdots, \mathcal{A}_{0,L-k})\right) d\mu_{S_k(\mathcal{A})}(\psi') \prod_{j=0}^{k-1} d\mu_{C_j}(A_j)
\]

Note that \( \rho_k^\Psi(\Psi_k, \psi_k(\mathcal{A}_k), \mathcal{A}_k) \) is actually just a function of \( \Psi_k, \mathcal{A}_k \) but we find it convenient to keep track of the dependence in the variables \( \Psi_k, \psi_k(\mathcal{A}_k), \mathcal{A}_k \). This will be especially useful for local versions later on.

Putting everything together we have

\[
\rho_k(\Psi_k, \mathcal{A}_k) = Z_k Z_{\mathcal{A}_k}(\mathcal{A}_k) \exp\left(-\frac{1}{2}(\mathcal{A}_k, \Delta_k \mathcal{A}_k) - (\bar{\Psi}_k, D_k(\mathcal{A}_k)\Psi_k)\right) \rho_k^\Psi(\Psi_k, \psi_k(\mathcal{A}_k), \mathcal{A}_k)
\]

This separates off a kinematic part, and we now proceed to the study \( \rho_k^\Psi(\Psi_k, \psi_k(\mathcal{A}_k), \mathcal{A}_k) \) which is the interaction part.

### 1.5 Perturbation Theory

For perturbation theory we introduce a parameter \( t \) and define instead of (58)

\[
\rho_k^t(\Psi_k, \mathcal{A}_k) = \int \mathcal{M}_{k,b_k}^{-1} \exp\left(-b_k|\Psi_k - Q_k(\mathcal{A}_{k-1,k})(t), \cdots, \big(A_{0,k}(t)\big)|^2\right) \exp\left(-\langle \bar{\Psi}_k, (D_{e_k} \big(A_{0,k}(t)\big) + m_k)\Psi_k \rangle\right) d\mu_{S_k(\mathcal{A})}(\psi') \prod_{j=0}^{k-1} d\mu_{C_j}(A_j)
\]

where

\[
A_{j,k}(t) = \mathcal{A}_k + t \sum_{i=j}^{k-1} (\mathcal{A}_i)_{L-(k-i)}
\]

These reduce to the previous quantities at \( t=1 \). We continue to assume \( e_k |\partial \mathcal{A}_k| \) is sufficiently small, repeat the steps in the last section, and find

\[
\rho_k^t(\Psi_k, \mathcal{A}_k) = Z_k(\mathcal{A}_k) \exp(-\langle \bar{\Psi}_k, D_k(\mathcal{A}_k)\Psi_k \rangle) \rho_k^t(\Psi_k, \psi_k(\mathcal{A}_k), \mathcal{A}_k)
\]

where

\[
\rho_k^t(\Psi_k, \psi, \mathcal{A}) = \int \exp\left(-\mathcal{V}_k(\Psi_k, \psi, \mathcal{A}, t\mathcal{A}_{k-1,L-1}, \cdots, t\mathcal{A}_{0,L-1})\right) d\mu_{S_k(\mathcal{A})}(\psi') \prod_{j=0}^{k-1} d\mu_{C_j}(A_j)
\]

We study \( \rho_k^t(\Psi_k, \mathcal{A}) \) at \( t=1 \) by expanding it around \( t=0 \): \( \rho_k^t(1) = \rho_k^t(0) + (\rho_k^t)'(0) + (\rho_k^t)''(0)/2 + \cdots \). Write (67) as \( \rho_k^t(1) = <\exp(-\mathcal{V}_k(t)>) \) where \(<\cdots> \) indicates the Gaussian integrals. Then \( \rho_k^0(0) = <\exp(-\mathcal{V}_k(0)>) = 1 \) since \( \mathcal{V}_k(0) = 0 \) and \( (\rho_k^t)'(0) = <\exp(-\mathcal{V}_k'(0)>) = 0 \) since each term in \( \mathcal{V}'(0) \) is odd in some boson field. Thus the first non-trivial term \( (\rho_k^t)''(0)/2 \) which we study further. (This is also the first non-zero term an expansion of the effective potential \( \log(\rho_k^t(1)) \) around \( t=0 \).

Accordingly we define

\[
P_k(\Psi_k, \mathcal{A}) \equiv \frac{1}{2}(\rho_k^t)'(0) = \frac{1}{2} <\mathcal{V}_k'(0)> \quad \frac{1}{2} <\mathcal{V}_k''(0)>
\]
The $t$ derivatives can be evaluated as $a_j$ derivatives and we also use
\[
\int A_{j,L^{-k-j}}(z)A_{j,L^{-k-j}}(w)d\mu_{C_j}(A_j) = L^{k-j}\tilde{C}_j(L^{k-j}z,L^{k-j}w) = \tilde{C}_j,L^{-k-j}(z,w)
\] (69)

Then the boson fluctuation integrals can be evaluated as
\[
P_k(\Psi_k,\psi,A) = \frac{1}{2}\sum_{j=0}^{k-1} \int \frac{\delta^2 V_k}{\delta a_j(z)}(\Psi_k,\psi,\psi',A,0) \frac{\delta^2 V_k}{\delta a_j(w)}(\Psi_k,\psi,\psi',A,0) \tilde{C}_j,L^{-k-j}(z,w)d\mu_{S_k(A)}(\psi')
\] (70)

Now do the fermion Gaussian integral, drop the no fermion part from $P_k$ and call the rest $P_k^+$, and we find
\[
P_k^+(\Psi_k,\psi,A) = \frac{1}{2}\sum_{j=0}^{k-1} \int J_{kj}(z) \tilde{C}_j,L^{-k-j}(z,w) J_{kj}(w) - \int J_{kj}(z,w) \tilde{C}_j,L^{-k-j}(z,w)
\]
(71)

where the vertices are
\[
J_{kj}(z) = \left. \frac{\partial^2 V_k}{\partial a_j(z)}(\Psi_k,\psi,\psi',A,0) \right|_{\psi'=0}
\]
(72)

The terms in (71) can be labeled by Feynman diagrams, see paper I.

We study $J_{kj,\mu}(z) = \partial V_k/\partial a_{j,\mu}(z)$ further. This has two parts $J_{kj,\mu}(z) = J_{kj,\mu}^D(z) + J_{kj,\mu}^Q(z)$. The classical part is
\[
J_{kj,\mu}^D(z) = \left. \frac{\partial}{\partial a_{j,\mu}(z)}(\bar{\psi}, D_{\alpha}(A + \sum_{j=0}^{k-1} a_j)) \right|_{a_j=0} = \left. \frac{\partial}{\partial A_{\mu}(z)}(\bar{\psi}, D_{\alpha}(A)) \right|_{a_j=0}
\]
\[
= ie_k \bar{\psi}(z) \left( \frac{r + \gamma^\mu}{2} \right) \exp(i e_k L^{-k} A_{\mu}(z)) \psi(z + L^{-k} e_\mu)
\]
\[
= ie_k \bar{\psi}(z + L^{-k} e_\mu) \left( \frac{r - \gamma^\mu}{2} \right) \exp(-i e_k L^{-k} A_{\mu}(z)) \psi(z)
\] (73)

The term is independent of $j$. If this were the only contribution we could resum and get the full boson propagator $\sum_{j=0}^{k-1} \tilde{C}_{j,L^{-k-j}} = G_k$ as in (69). Note that as $k = N \to \infty$ it approaches the usual continuum current $ie\bar{\psi}(z)\gamma^\mu \psi(z)$.

The other part is an artifact of our renormalization group procedure. It is
\[
J_{kj,\mu}^Q(z) = \left. \frac{\partial}{\partial a_{j,\mu}(z)} b_k |\Psi_k - Q_k(A + a_{k-1,k-1}^* + \ldots + a_{0,k-1}^*)|^2 \right|_{a_{k-1} = 0, \ldots, a_0 = 0}
\]
\[
= \left. \frac{\partial}{\partial a_{j,\mu}(z)} b_k |\Psi_k - Q_k(A, A + a_j, \ldots, A + a_j)^2 \right|_{a_{j} = 0}
\] (74)

where there are $j + 1$ entries with $A + a_j$. 

12
Similar expressions hold for the other vertices in (72).

The expression $P_k^+$ has no ultraviolet divergences, i.e. it is bounded as $k \rightarrow \infty$. This is true in this global version even without the counterterms. In paper III we prove it for a local version, and in that case the counterterms are needed.

1.6 single steps

We want to get an identity relating $P_k^+$ and $P_{k+1}^+$, and this means investigating how the densities change under a single RG transformation.

Start with the identity

$$
ρ_{k+1}^*(t, \Psi_{k+1}, A_{k+1}) = \int d\Psi_k d\mu_{C_k}(A_k) M_{k+1,b}^{-1} \exp \left( -\frac{b}{L} \Psi_{k+1} - Q_{c_k} (\tilde{Q}_k(A_{k+1} + tA_k)) \Psi_k \right) ρ_k^*(t, \Psi_k, A_{k+1} + tA_k)
$$

(75)

This can be proved by inserting the definition of $ρ_k^*(t)$ on the right and using (55). We also use that replacing $A_k$ by $A_{k+1} + tA_k$ has the effect of replacing $A_{j,k}^*(t)$ by $[A_{j,k}^*(t)]_L$.

We would like to insert into this the expression (61) giving $ρ^*$ in terms of a further reduced density $ρ^*$, and get a recursion relation for $ρ^*$. However this expression for $ρ^*$ is only valid with restrictions on $A_k$ which we cannot assume in the integral. For the moment we proceed formally.

Inserting (61) into (75) we have

$$
ρ_{k+1}^*(t, \Psi_{k+1}, A_{k+1}) = \int d\Psi_k d\mu_{C_k}(A_k) M_{k+1,b}^{-1} \exp \left( -\frac{b}{L} \Psi - Q_{c_k} (\tilde{Q}_k(A + tA_k)) \Psi_k \right) \exp \left( - (\Psi_k, D_k(A + tA_k) \Psi_k) \right) Z_k(A + tA_k) ρ_k^*(t, \Psi_k, A + tA_k, A + tA_k)|_{\Psi = \Psi_{k+1}, L, A = A_{k+1}}
$$

(76)

Now define $V_k, U_k, δH_k$ by

$$
V_k(\Psi, A, A_k) = \frac{b}{L} \Psi - Q_{c_k} (\tilde{Q}_k(A + A_k)) \Psi_k^2 + (\Psi_k, D_k(A + A_k) \Psi_k) - \{A_k = 0\}
$$

$$
Z_k(A + A_k) = Z_k(A) \exp(-U_k(A, A_k))
$$

$$
δH_k(A, A_k) = H_k(A + A_k) - H_k(A)
$$

(77)

We obtain

$$
ρ_{k+1}^*(t, \Psi_{k+1}, A_{k+1}) = Z_k(A) \int d\Psi_k d\mu_{C_k}(A_k) M_{k+1,b}^{-1} \exp \left( -\frac{b}{L} \Psi - Q_{c_k} (\tilde{Q}_k(A)) \Psi_k \right) \exp \left(-V_k(\Psi_k, A, tA_k) - U_k(A, tA_k)\right) ρ_k^*(t, \Psi_k, A + tA_k, A + tA_k)|_{\Psi = \Psi_{k+1}, L, A = A_{k+1}}
$$

(78)

Now make the translation $\Psi_k \rightarrow \Psi_k + H_k(A) \Psi \equiv \Psi_k + \Psi(A)$ This means that $ψ_κ(A_{k+1} + L) = [ψ_κ(A_{k+1})]_L + ψ_k(A_{k+1} + L)$. Using (17) and (48) and identifying a Gaussian measure we find

$$
ρ_{k+1}^*(t, \Psi_{k+1}(A_{k+1}, A_{k+1})) = \int dμ_{T_k(A)}(\Psi_k) dμ_{C_k}(A_k) \exp(-V_k(\Psi, A + \Psi_k, A, tA_k) - U_k(A, tA_k)) ρ_k^*(t, \Psi_k, [ψ_k(A_{k+1})]_L + ψ_k(A) + δH_k(A, tA_k)(ψ_k(A) + ψ_k(A_k)) + tA_k)|_{\Psi = \Psi_{k+1}, L, A = A_{k+1}}
$$

(79)

This is our recursion relation for $ρ_k^*(t)$.

13
Now take two derivatives of the last equation at \( t = 0 \) and obtain

\[
P_{k+1}(\Psi_{k+1},\psi_{k+1}(A_{k+1}),A_{k+1}) = \int P_k(\Psi(A) + \Psi_k, [\psi_{k+1}(A_{k+1})]_L + \psi_k(A), A) d\mu_{r_k}(A)(\Psi_k)
\]

\[
+ \frac{1}{2} \int \frac{\partial V_k}{\partial A_k(z)}(\Psi, \Psi(A) + \Psi_k, A, 0) C_k(z, w) \frac{\partial V_k}{\partial A_k(w)}(\Psi, \Psi(A) + \Psi_k, A, 0) dz dw \ d\mu_{r_k}(A)(\Psi_k)
\]

\[
- \frac{1}{2} \int \frac{\partial^2 V_k}{\partial A_k(z)\partial A_k(w)}(\Psi, \Psi(A) + \Psi_k, A, 0) C_k(z, w) dz d\mu_{r_k}(A)(\Psi_k)|_{\psi = \psi_{k+1}, L} + \ldots
\]

where \ldots indicates no-fermion terms. Next we carry out the fermion integrals and drop the no fermion terms to get the expression

\[
P^+_{k+1}(\Psi_{k+1}, \psi_{k+1}(A_{k+1}), A_{k+1})
\]

\[
= \left[ P^+_k(\Psi(A), [\psi_{k+1}(A_{k+1})]_L, A) - (\Delta_{\psi_k}^1 P_k)^+(\Psi(A), [\psi_{k+1}(A_{k+1})]_L, A) \right]
\]

\[
+ \frac{1}{2} \int_{z,w} J_k(z) C_k(z, w) J_k(w) - J_k(z, w) C_k(z, w)
\]

\[
- \int_{z,w} \sum_{x,y} \tilde{K}_k(z, x) \Gamma_k(A; x, y) K_k(w, y) \tilde{C}_k(z, w)
\]

\[
|_{\psi = \psi_{k+1}, L}^{A = A_{k+1}, L}
\]

where the single step vertices are

\[
J_k(z) = \frac{\partial V_k}{\partial A_k(z)}(\Psi, \Psi(A), A, 0) \quad J_k(z, w) = \frac{\partial^2 V_k}{\partial A_k(z)\partial A_k(w)}(\Psi, \Psi(A), A, 0)
\]

\[
\tilde{K}_k(z, x) = \frac{\partial^2 V_k}{\partial \Psi_k(x)\partial A_k,\mu(z)}(\Psi, \Psi(A), A, 0) \quad K_k(z, x) = \frac{\partial^2 V_k}{\partial \Psi_k(x)\partial A_k,\mu(z)}(\Psi, \Psi(A), A, 0)
\]

and the notation is

\[
(\Delta_{\psi_k}^1 P_k)(\Psi(A), [\psi_{k+1}(A_{k+1})]_L, A)
\]

\[
= \sum_{x,y} \Gamma_k(A, x, y) \left[ \frac{\partial^2}{\partial \Psi_k(x)\partial \Psi_k(y)} P_k(\Psi(A) + \Psi_k, [\psi_{k+1}(A_{k+1})]_L + H_k(A) \Psi_k, A) \right]_{\psi = 0}
\]

The equation [81] is the basic identity we are after. Although the derivation was formal we show in appendix [13] that it is rigorously true provided \( \epsilon_{k+1} | \partial A_{k+1} | \) is sufficiently small.

We also need a variation of [81] in which vertices are localized in a region \( \Theta \subset \mathbb{T}_N^{k+1} \). First define \( \rho^\bullet_{k,\Theta}(t) \) just as in [61] but now with \( A_{1,k}^\bullet(t) \) replaced by

\[
A_{1,k}^\bullet(t, \Theta) = A_k + t \chi_\Theta \sum_{i=j}^{k-1} (A_i)_{L-(k-i)}
\]

where \( \chi_\Theta \) is the characteristic function of \( \Theta \). Then \( \rho^\bullet_{k,\Theta}(t) = Z_k(A_k) \exp(- (\tilde{\Psi}_k, D_k(A_k) \tilde{\Psi}_k)) \rho^\bullet_{k,\Theta}(t) \) where \( \rho^\bullet_{k,\Theta}(t) \) defined as in [67] except that \( tA_{1,k}^\bullet(L-(k-j)) \) is replaced by \( t \chi_\Theta A_{1,k}^\bullet(L-(k-j)) \). Defining \( P_{k,\Theta} = (\rho^\bullet_{k,\Theta})'\rho^\bullet_{k,\Theta}(0)/2 \) we find

\[
P_{k,\Theta}^+(\Psi_k, \psi, A) = \frac{1}{2} \sum_{j=0}^{k-1} \int_{z, w \in \Theta} J_{kj}(z) C_{j, L-(k-j)}(z, w) J_{kj}(w) - J_{kj}(z, w) C_{j, L-(k-j)}(z, w)
\]

\[
- \sum_{j=0}^{k-1} \int_{z, w \in \Theta} K_{kj}(z, x) S_k(A; x, y) K_{kj}(w, y) C_{j, L-(k-j)}(z, w)
\]

\[
(85)
\]
One can also proceed in single steps. One shows for $\Theta \subset T_{N+M-k-1}$ that $\rho_{k+1,\Theta}(t)$ and $\rho_{k,L,\Theta}(t)$ are formally related by an equation like (79) except that under the integral sign $tA_k$ is everywhere replaced by $t\chi_{\Theta}A_k$. Taking two derivatives at $t = 0$ yields the identity

\begin{equation}
P_{k+1,\Theta}^+(\Psi_{k+1}, \psi_{k+1}(A_{k+1}), A_{k+1}) \\
= \left[P_{k,L,\Theta}^+(\Psi(A), [\psi_{k+1}(A_{k+1})]_L, A) - (\Delta_{\Psi_k} P_{k,L,\Theta})^+(\Psi(A), [\psi_{k+1}(A_{k+1})]_L, A) + \frac{1}{2} \int \sum_{z,w \in L \Theta} J_k(z)\tilde{C}_k(z,w)J_k(w) - J_k(z,w)\tilde{C}_k(z,w) \right]_{\Psi=\Psi_{k+1}, L=\Delta_{A_{k+1}, L}}
\end{equation}

Again the derivation is formal, but the result is rigorous by the argument of appendix B.
2 Localized flow

2.1 blocking

We also want to consider a version of our RG transformation in which the averaging is not done on the whole torus, but in a sequence of successively smaller regions. The treatment follows Balaban, O’Carroll, and Schor [14]. A difference is that they do not make the initial scaling to a unit lattice as in [11]. This means they are working up from a finer to a coarser lattice, whereas we are working down from a coarser to a finer lattice.

First we define some blocking and unblocking operations. For \( \Omega \subset T_n^1 \subset T_n^0 \) we defined a blocked set \( B\Omega \subset T_n^0 \) by

\[
B\Omega = \{ x \in T_n^0 : d(x, \Omega) < L/2 \}
\]

Then \( QA \) on \( \Omega \) depends on \( A \) on \( B\Omega \), written \( QA_{\Omega} = Q(A|_{B\Omega}) \). A set \( \Lambda \subset T_n^0 \) has the form \( B\Omega \) iff it is a union of \( L \)-blocks in \( T_n^0 \) centered on points in \( T_n^1 \).

For \( \Lambda \subset T_n^0 \) we also define an unblocked \( \Lambda' \subset T_n^1 \) by

\[
\Lambda' = U\Lambda = \Lambda \cap T_n^1
\]

We have always \( UB\Omega = \Omega \). If \( \Lambda = B\Omega \) then \( BU\Lambda = \Lambda \). Note that if \( \Lambda \subset T_n^0 \) then \( BL\Lambda \subset T_n^0 + 1 \) and \( L^{-1}U\Lambda \subset T_n^0 - 1 \). More generally we define for \( \Lambda \subset T_n^0 \)

\[
B_\ell \Lambda = (BL)^\ell \Lambda \subset T_n^0 + \ell
\]

\[
U_\ell \Lambda = (L^{-1}U)^\ell \Lambda \subset T_n^0 - \ell
\]

We have \( U_\ell B_\ell \Lambda = \Lambda \).

Our regions will be a sequence of the form

\[
\Lambda = (\Lambda_0, \ldots, \Lambda_{k-1})
\]

where \( \Lambda_j \subset T_n^0 \) is a union of \( LM_0 = L^{m_0+1} \) blocks centered on \( T_n^0 \) blocks. We assume the sets are decreasing in the sense that they satisfy one of the equivalent

\[
B_1 \Lambda_{j+1} \subset \Lambda_j \quad \Lambda_j \subset U_1 \Lambda_{j-1} = L^{-1}\Lambda'_{j-1}
\]

We also assume that for some positive integer \( r \)

\[
d((L^{-1}\Lambda'_{j-1})^c, \Lambda_j) \geq rM_0
\]

whenever both subsets are non-empty. This insures that the corridor between successive regions is at least a few \( M_0 \) blocks wide.

We define in \( T_n^{0+M-j} \)

\[
\delta \Lambda_i = L^{-1}\Lambda'_{i-1} - \Lambda_i
\]

Then with the convention \( \Lambda_k = \emptyset \) we have the disjoint union

\[
\Lambda_j = \cup_{i=j+1}^{k} B_{i-j} \delta \Lambda_i
\]

2.2 bosons

We want a generalization of the formula [11] for bosons in which the averaging over \( A_j \) is only done in the region \( \Lambda_j \). The starting point is

\[
\int \prod_{j=0}^{k-1} dA_j, \Lambda_j \exp \left( -\frac{1}{2} \frac{a}{L^2} |A_{j+1,L} - QA_j|_{\Lambda_j}^2 \right) \exp \left( -\frac{1}{2} (A_0, (-\Delta + \mu_0^2) A_0) \right) F(A_0)
\]
where \( N_{\Lambda,a} = (2\pi/a)^{3|\Lambda|/2} \). Break this up by \([\text{II}]\)

\[
N_{L^{-1}\Lambda_j,a}^{-1} \exp \left( -\frac{1}{2} \frac{a}{L^2} |A_{j+1,L} - QA_j|_{L_j}^2 \right) dA_{j},L_j
\]

\[
= \prod_{i=j+1}^{k} N_{L^{-1}((B_{i-j}\Lambda_j),a)^\prime}^{-1} \exp \left( -\frac{1}{2} \frac{a}{L^2} |A_{j+1,L} - QA_j|_{(B_{i-j}\Lambda_j),a}^2 \right) dA_{j,B_{i-j}\Lambda_j} \tag{96}
\]

Change the order of the products \( \prod_{i=j+1}^{k} \prod_{j=0}^{k-j} = \prod_{i=1}^{k-j} \prod_{j=1}^{i-j} \) and evaluate the integral over \( \delta\Lambda_i \) by

\[
\int \prod_{j=0}^{i-1} dA_{j,B_{i-j}\Lambda_j} N_{L^{-1}((B_{i-j}\Lambda_j),a)^\prime}^{-1} \exp \left( -\frac{1}{2} \frac{a}{L^2} |A_{j+1,L} - QA_j|_{(B_{i-j}\Lambda_j),a}^2 \right) \cdots \] \[
= \int dA_{0,B_0\Lambda_0} N_{\delta\Lambda_0,a_0}^{-1} \exp \left( -\frac{a}{2} |A_0 - Q_0 A_{0,L^{-1}}|_{\delta\Lambda_0}^2 \right) \] \[
\int \prod_{i=1}^{k} N_{\delta\Lambda_i,a_i}^{-1} \exp \left( -\frac{a}{2} |A_i - Q_i A_{i,L^{-1}}|_{\delta\Lambda_i}^2 \right) \exp \left( -\frac{1}{2} (A_0,(-\Delta + \mu_0^2)A_0) \right) F(A_0)dA_{0,\Lambda_0} \tag{98}
\]

Next scale \( A_0 = A_{L_k} \) for \( A \) on \( T_{N+k-M+k}^k \) and define \( Q_i^{(k)} = Q_i^k \sigma_{L_k}^{k-i} \), which is a \( i \)-fold averaging operator from \( T_{N+k-M-k}^k \) to \( T_{N+k-M-i}^k \). Then we have that \( \text{[II]} \) is equal to

\[
\int \prod_{i=1}^{k} N_{\delta\Lambda_i,a_i}^{-1} \exp \left( -\frac{a}{2} |A_i - Q_i^{(k)} A_i|_{\delta\Lambda_i}^2 \right) \exp \left( -\frac{1}{2} (A_i,(-\Delta + \mu_i^2)A_i) \right) f(A)dA_{L^{-k}\Lambda_0} \tag{99}
\]

where \( f(A) = F(A_{L_k}) \). In this formula the spectator variables \( A_{0,\Lambda^0} \) appear as \( A_{L^{-k}\Lambda_0^0} \equiv (A_{0,\Lambda^0})_{L^{-k}} \).

We introduce some notation. Let

\[
A' = (A_{1,\delta\Lambda_1}, \ldots, A_{k,\delta\Lambda_k})
\]

\[
Q_{k,\Lambda} A = ((Q_1^{(k)} A_{\delta\Lambda_1}, \ldots, (Q_k^{(k)} A_{\delta\Lambda_k})\tag{100}
\]

Note that since \( \Lambda_k = \emptyset \) we have \( A_{k,\delta\Lambda_k} = A_{L^{-k}L^{-1}\Lambda_{k-1}} \). These are multiscale objects consisting of functions living on subsets \( \delta\Lambda_i \subset T_{N+k-M-k}^k \). A norm on such objects is given by

\[
|A'|^2 = \sum_{i=1}^{k} |A_{i,\delta\Lambda_i}|^2
\tag{101}
\]

We also define \( a = (a_1, \ldots, a_k) \) and \( N_{\Lambda,a} = \prod_{i=1}^{k} N_{\Lambda_i,a_i}^{-1} \). Then \( \text{[99]} \) can be written

\[
\int \prod_{i=1}^{k} N_{\Lambda_i,a_i}^{-1} \exp \left( -\frac{1}{2} a^{1/2} (A' - Q_{k,\Lambda} A)^2 \right) \exp \left( -\frac{1}{2} (A,(-\Delta + \mu_k^2)A) \right) f(A)dA_{L^{-k}\Lambda_0} \tag{102}
\]

To evaluate this integral we again diagonalize the quadratic form, this time separating the variables \( (A_{0,\Lambda'}, A') \) or \( (A_{L^{-k}\Lambda^0}, A') \) from \( A_{L^{-k}\Lambda_0} \). Accordingly we introduce

\[
\Delta_{k,\Lambda}^{\#} = -\Delta + \mu_k^2 + Q_{k,\Lambda} a Q_{k,\Lambda} \tag{103}
\]
and with $A \equiv (A_0, \Lambda^c, A')$

$$G_{k,A}=\left[\Delta^\#\right]_{L^{-k}A_0}^{-1} \quad \mathcal{H}_{k,A}=G_{k,A}(Q_{k,A}^T A')^T + [\Delta|_{\Lambda^c,A^c}]_{\Lambda=L^{-k}A_0}$$

$$(A, \Delta_{k,A}) \equiv (A', \left[ a - aQ_{k,A}G_{k,A}Q_{k,A}^T A' \right] A') - 2(Q_{k,A}^T A') A' - (Q_{k,A}^T A') A'$$

Note that $Q_{k,A}$ vanishes on functions on $L^{-k}A_0$ and thus $Q_{k,A}^T$ maps to functions on $L^{-k}A_0$. Making the change of variables $A_{L^{-k}A_0} \rightarrow A_{L^{-k}A_0} + H_{k,A}$ we find that $102$ can be written

$$Z_{k,A} \exp \left( -\frac{1}{2}(A, \Delta_{k,A}) \right) \int f(A + H_{k,A}) \, d\mu_{G_{k,A}}(A_{L^{-k}A_0})$$

where

$$Z_{k,A} = \int \mathcal{N}_{\Lambda}^{-1} \exp \left( -\frac{1}{2}(A, G_{k,A}^{-1} A) \right) dA_{\Lambda \mid \Lambda = L^{-k}A_0}$$

Our final identity is then $99 = 105$. We are particularly interested in $F = 1$ in which case the identity reads

$$\int \prod_{j=0}^{k-1} dA_{j,\Lambda} \mathcal{N}_{L^{-1}A_j, a}^{-1} \exp \left( -\frac{1}{2} \left[ \frac{a}{L} \right] |A_{j+1,\Lambda} - QA_j |_{\Lambda_j} \right) \exp \left( -\frac{1}{2} (A_0, (-\Delta + \mu_0^2) A_0) \right)$$

$$= Z_{k,A} \exp \left( -\frac{1}{2}(A, \Delta_{k,A}) \right)$$

2.3 fermions

For fermions pick a fixed background $\mathcal{A}$ on $\mathbb{T}_{N+M-k}$ and consider integrals of the form

$$\int \prod_{j=0}^{k-1} d\Psi_{j,\Lambda} \mathcal{M}_{L^{-1}A_j, b}^{-1} \exp \left( -\frac{b}{L} |\Psi_{j+1,\Lambda} - Q(\tilde{Q}_j A_{L^{-k-j}})\Psi_j |_{\Lambda_j}^2 \right)$$

$$\exp \left( -\left( \tilde{\Psi}_0, (D_m(A_{L^k}) + m_0)\Psi_0 \right) \right) F(\Psi_0)$$

Doing the intermediate integrals as for bosons this can be written

$$\int \prod_{i=1}^{k} \mathcal{M}_{\delta\Lambda_i, b_i}^{-1} \exp \left( -b_i |\Psi_i - Q_i(A_{L^{-k-i}})\Psi_i |_{\delta\Lambda_i}^2 \right)$$

$$\exp \left( -\left( \tilde{\Psi}_0, (D_m(A_{L^k}) + m_0)\Psi_0 \right) \right) F(\Psi_0) = \int d\Psi_{0,\Lambda^c}$$

Next scale $\psi_0 = \psi_{L^k}$ for $\psi$ on $\mathbb{T}_{N+M-k}$, and define $Q_i^{(k)}(A) = Q_i(A_{L^{-k-i}})\sigma^{k-i}_L$. Then $109$ is written

$$\int \prod_{i=1}^{k} \mathcal{M}_{\delta\Lambda_i, b_i}^{-1} \exp \left( -b_i |\Psi_i - Q_i^{(k)}(A)\Psi_i |_{\delta\Lambda_i}^2 \right) \exp \left( -\left( \tilde{\psi}_0, (D_m(A) + m_k)\psi \right) \right) f(\psi) \, d\psi_{L^{-k}A_0}$$

where $f(\psi) = F(\psi_{L^k})$. Here $\Psi_{0,\Lambda^c}$ appears as $\psi_{L^{-k}A_0} \equiv (\Psi_{0,\Lambda^c})_{L^{-k}}$. 18
Next introduce
\[
\Psi' = (\Psi_1, \delta \Lambda_1, \cdots, \Psi_k, \delta \Lambda_k) \\
Q_{k, \Lambda}(A) \psi = (Q_1^{(k)}(A) \psi)_{\delta \Lambda_1}, \cdots, (Q_k^{(k)}(A) \psi)_{\delta \Lambda_k}
\]  
(111)

and define
\[
(\overline{\Psi'}, \Psi') = \sum_{i=1}^{k} (\Psi_{i, \delta \Lambda_i}, \Psi_{i, \delta \Lambda_i})
\]  
(112)

We also define \( b = (b_1, \cdots, b_k) \) and \( M_{A, b} = \prod_{i=1}^{k} M_{\delta \Lambda_i, b_i} \). Then (111) is written
\[
\int M_{A, b}^{-1} \exp \left( -\overline{\Psi'} - Q_{k, \Lambda}(-A) \bar{\psi}, b(\Psi' - Q_{k, \Lambda}(A) \psi) \right) \exp \left( -\left( \bar{\psi}, (D_{e_n}(A) + m_k) \psi \right) \right) f(\psi) d\psi_{L^{-k} \Lambda_0}
\]  
(113)

Again we diagonalize the quadratic form. Let
\[
D_{k}(A) = D_{e_n}(A) + m_k + Q_{k, \Lambda}(-A)^T b Q_{k, \Lambda}(A)
\]  
(114)

This operator is invertible on \( L^{-k} \Lambda_0 \) under certain assumptions on \( A \) and \( \Lambda \) which we explain in the next section. Assuming it is invertible we define with \( \Psi = (\Psi_{0, \Lambda'}, \Psi') \)
\[
\begin{align*}
S_{k, \Lambda}(A) &= [D_{k}(A)]^{-1}_{L^{-k} \Lambda_0} \\
H_{k, \Lambda}(A) \Psi &= S_{k, \Lambda}(A) \left( Q_{k, \Lambda}(-A)^T b \Psi' - [D_{e_n}(A)]_{\Lambda', \Lambda} \psi_{\Lambda'} \right) \mid_{\Lambda = L^{-k} \Lambda_0} \\
H_{k, \Lambda}(A) \overline{\Psi} &= S_{k, \Lambda}(A)^T \left( Q_{k, \Lambda}(A)^T b \overline{\Psi'} - ([D_{e_n}(A)]_{\Lambda', \Lambda})^T \psi_{\Lambda'} \right) \mid_{\Lambda = L^{-k} \Lambda_0} \\
(\overline{\Psi}, D_{k, \Lambda}(A) \Psi) &= (\overline{\Psi} \left[ b - b Q_{k, \Lambda}(A) Q_{k, \Lambda}(-A)^T b \right] \Psi') \\
&+ (\bar{\psi}_{\Lambda'}, [D_{e_n}(A)]_{\Lambda', \Lambda} S_{k, \Lambda}(A) Q_{k, \Lambda}(A)^T b \Psi') \\
&+ (Q_{k, \Lambda}(A)^T b \overline{\Psi'}, S_{k, \Lambda}(A) [D_{e_n}(A)]_{\Lambda', \Lambda} \psi_{\Lambda'}) \\
&+ (\bar{\psi}_{\Lambda'}, \left( [D_{e_n}(A) + m_k - [D_{e_n}(A)]_{\Lambda', \Lambda} S_{k, \Lambda}(A)[D_{e_n}(A)]_{\Lambda', \Lambda} \right] \psi_{\Lambda'}) \mid_{\Lambda = L^{-k} \Lambda_0}
\end{align*}
\]  
(115)

Now make the change of variables \( \psi_{L^{-k} \Lambda_0} \rightarrow \psi_{L^{-k} \Lambda_0} \rightarrow H_{k, \Lambda}(A) \Psi \). Then (113) can be written
\[
Z_{k, \Lambda}(A) \exp \left( -\overline{\Psi}, D_{k, \Lambda}(A) \Psi \right) \int f(\psi + H_{k, \Lambda}(A) \Psi) d\mu_{S_{k, \Lambda}(A)}(\psi_{L^{-k} \Lambda_0})
\]  
(116)

where
\[
Z_{k, \Lambda}(A) = \int M_{A, b}^{-1} \exp \left( -\overline{\psi}_{\Lambda', [S_{k, \Lambda}(A)]_{\Lambda', \Lambda}^{-1} \psi_{\Lambda}} \right) d\psi_{\Lambda} \mid_{\Lambda = L^{-k} \Lambda_0}
\]  
(117)

Our final identity is then (118) = (116). In case \( F = 1 \) it says
\[
\int \prod_{j=0}^{k-1} d\Psi_{j, \Lambda} M_{L^{-k} \Lambda, \delta}^{-1} \exp \left( -\frac{b}{E} \Psi_{j+1, L} - Q(\bar{\Psi}_{j, A_{L, \delta}}) \Psi_j \right) \exp \left( -\left( \psi_0, (D_{e_n}(A_{L, \delta}) + m_0) \psi_0 \right) \right)
\]  
\[
= Z_{k, \Lambda}(A) \exp \left( -\overline{\Psi}, D_{k, \Lambda}(A) \Psi \right)
\]  
(118)
3 Propagators

3.1 Definitions

In this chapter our goal is to show that the propagators on $\mathbb{T}_{N+M-k}$

$$G_{k,A} = (-\Delta + \mu_i^2 + Q_{k,A}^T a Q_{k,A})^{-1} \Lambda_{-k \Lambda_0}$$

$$S_{k,A}(A) = (D_{ei}(A) + m_k + Q_{k,A}(-A)^T b Q_{k,A}(A))^{-1} \Lambda_{-k \Lambda_0}$$  \hspace{1cm} (119)

exist and get estimates on their kernels. The basic tool is a multiscale random walk expansion. The expansion for Dirac operators was developed by Balaban, O’Carroll, and Schor [14] based on earlier work of Balaban [7, 8]. We give expansions both for Dirac operators and Laplacians. For Dirac operators we follow [14] rather closely, but nevertheless have to go into considerable detail to get results in the exact sharp form we want. (In addition the numerous misprints in [14] make it difficult to quote results directly.)

We start with some definitions. To construct the inverses we need to respect the structure of the averaging operators $Q_{k,A} = \{ Q_i^{(k)}(\cdot) |_{\Lambda_{i+1}} \}$. Now $(Q_i^A)_{\Lambda_i}$ depends on $A$ in the set $L^{-1}B_i \delta \Lambda_i$ and hence the scaled version $(Q_i^{(k)}A)_{\delta \Lambda_i}$ depends on $A$ in the set $L^{-1}B_i \delta \Lambda_i$ given by

$$\delta \Lambda_i^{(k)} = L^{-1}B_i \delta \Lambda_i$$  \hspace{1cm} (120)

Here for $\Lambda_i \subset \mathbb{T}_{N+M-k}$ the set $B_i \Lambda_i \subset \mathbb{T}_{N+M}$ is its representative in the original unit lattice, which is then scaled down to $L^{-1}B_i \Lambda_i \subset \mathbb{T}_{N+M-k}$. Since we have assumed that $\Lambda_i$ is a union of $L^0 M_0$ blocks, $B_i \Lambda_i$ is a union of $L^0 M_0$ blocks, $L^{-1}B_i \Lambda_i$ is a union of $L^{-k+1} M_0$ blocks, and $\delta \Lambda_i^{(k)}$ is a union of $L^{-k} M_0$ blocks. The separation condition (122) insures that $\delta \Lambda_i^{(k)}$ is at least a few layers wide.

We have the decomposition of $L^{-k} \Lambda_0 \subset \mathbb{T}_{N+M-k}$ given by the disjoint union

$$L^{-k} \Lambda_0 = \bigcup_{i=1}^k \delta \Lambda_i^{(k)}$$  \hspace{1cm} (121)

Let $\square_i$ be the $L^{-k-i} M_0$ blocks in $\delta \Lambda_i^{(k)}$ denoted $\square$, and let $\square_0 = \bigcup_{i=1}^k \square_i$. Then

$$L^{-k} \Lambda_0 = \bigcup_{i=1}^k \bigcup_{\square \in \square_i} \square = \bigcup_{\square \in \square_0} \square$$  \hspace{1cm} (122)

which is a partition of $L^{-k} \Lambda_0$ into blocks of various sizes. Actually it is convenient to modify this by taking (for $i \geq 2$) $r_0$ layers of $L^{-k-i} M_0$ blocks in $\delta \Lambda_i^{(k)}$ along the boundary of $L^{-k} B_{i-1} \Lambda_{i-1}$ and further subdividing them into $L^{-k+i-1} M_0$ blocks. We assume that $r_0$ is some fraction of $r$ so that we do not exhaust $\delta \Lambda_i^{(k)}$. Let $\square_i$ be the new $L^{-k-i} M_0$ blocks. These are either in $\delta \Lambda_i^{(k)}$ or $\delta \Lambda_i^{(k)}$ and we have

$$L^{-k} \Lambda_0 = \bigcup_{i=1}^k \bigcup_{\square \in \square_i} \square = \bigcup_{\square \in \square_0} \square$$  \hspace{1cm} (123)

We will need partitions of unity concentrated on the sets $\square \in \square$. First take a smooth function $g$ on $\mathbb{R}^3$ so $g$ has support in $\{ x : |x| \leq 2/3 \}$ and $g = 1$ on $\{ x : |x| \leq 1/3 \}$ and $\sum_{n \in \mathbb{Z}^3} g(x - n)^2 = 1$. Then if $\square$ is a $\square_i$ block in the interior of $\bigcup_{\square \in \square_i} \square$ centered on $y \in \mathbb{T}_{N+M-k}^{(k-i)+m_0}$, we define

$$h_{\square}(x) = g \left( \frac{(x - y) L^{-k-i} M_0}{M_0} \right)$$  \hspace{1cm} (124)

20
Then for $x$ well inside $\bigcup_{\square \in \mathcal{D}_i} \square$

$$\sum_{\square \in \mathcal{D}} h_\square(x)^2 = 1 \quad (125)$$

For $\square \in \mathcal{D}_i$ blocks touching $\mathcal{D}_{i-1}$ blocks the scalings do not match. We require in this case that the definition of $h_\square(x)$ be modified on any boundary face by taking the scaling $L^{k-(i-1)}/M_0$ instead of $L^{k-i}/M_0$. (If $\square \subset \delta \Lambda_i^{(k)}$ touches $\Lambda_0^{(k)}$ take $L^k/3M_0$.) Then (124) holds for all $x \in L^{k-i}\Lambda_0$.

Note that $h_\square h_\square' = 0$ unless $\square, \square'$ touch.

We also define some enlargements of each $L^{-(k-i)}M_0$ cube $\square \in \mathcal{D}_i$. We set

$$\square' = 3M_0L^{-(k-i)} \text{ cube centered on } \square \quad (126)$$

$$\square^{(n)} = (1 + 2n)r_1M_0L^{-(k-i)} \text{ cube centered on } \square$$

for some $r_1 < r_0$. We have $\text{supp } h_\square \subset \square'$.

Finally we introduce some modified distances on $T_{N+M-k}^{-k}$. For long distances we use

$$d_A(x,y) = \inf_{\Gamma \ni x \to y} \sum_{i=1}^{k} L^{-(k-i)}|\Gamma \cap \delta \Lambda_i^{(k)}| \quad (127)$$

This is a genuine metric which weighs earlier regions more heavily. For short distances we use

$$d'(x,y) = \begin{cases} 
  d(x,y) & x \neq y \\
  L^{-k} & x = y 
\end{cases} \quad (128)$$

This is not a real metric but it does satisfy the triangle inequality, and it does scale like $d(x,y)$. It is relevant because of its appearance in estimates like (124), (125).

We note the following estimates on integrals in $T_{N+M-k}^{-k}$. As usual $\int dy[\cdots] = \sum_y L^{-3k}[\cdots]$. The estimates refer to $L^{-(k-i)}$ blocks $\Delta$, smaller than the $L^{-(k-i)}M_0$ blocks $\square$.

**Lemma 1** Let $\Delta$ be an $L^{-(k-i)}$ block with $1 \leq i \leq k$ and let $0 \leq \alpha < 3$

$$\int_\Delta d'(x,y)^{-\alpha} \, dy \leq O(L^{-(3-\alpha)(k-i)}) \quad (129)$$

$$\int_\Delta d'(x,y)^{-\alpha} d'(y,z)^{-\alpha} \, dy \leq O(L^{-(3-\alpha)(k-i)}) d'(x,z)^{-\alpha} \quad (130)$$

$$\int_\Delta d'(x,y)^{-1} d'(y,z)^{-2} \, dy \leq O(L^{-(k-i)}) d'(x,z)^{-1} \quad (131)$$

**Proof.** For the first estimate note that if $x$ is well outside $\Delta$ then the integrand is bounded and we get $O(L^{-3(k-i)})$ which suffices. If $x$ is in or near $\Delta$ enlarge $\Delta$ so it is centered on $x$ and still has sides $O(L^{-(k-i)})$. The point with $y = x$ contributes $L^{-(3-\alpha)k}$ which suffices. Points with $y \neq x$ can be dominated by the $\mathbb{R}^3$ integral $\int_{|y| \leq O(L^{-(k-i)})} |y|^{-\alpha} \, dy = O(L^{-(3-\alpha)(k-i)})$.

For the second integral we split into two cases. The first is $d'(x,y) \geq d'(x,z)/2$. Use this in the first factor and then (124) gives the result. The second case is $d'(x,y) < d'(x,z)/2$. In this case we have by the triangle inequality that $d'(y,z) \geq d'(x,z)/2$. Use this in the second factor and again use (124) to obtain the result.

For the last inequality regard the integrand as the product of $d'(x,y)^{-1} d'(y',z)^{-1}$ and $d'(y',z)^{-1}$ and use the Schwarz inequality.
3.2 fermions

The random walk expansion for $S_{k,A}(A, x, y)$ has the form

$$S_{k,A}(A, x, y) = \sum_{\omega:x\rightarrow y} S_{k,A,\omega}(A, x, y)$$  \hspace{1cm} (132)

Here we are summing over paths $\omega$ each of which is a sequence of adjacent cubes (blocks) $\Box_0, \ldots, \Box_n$ from $D$. Equivalently a path $\omega$ is a sequence of links $((\Box_0, \Box_1), (\Box_1, \Box_2), \ldots, (\Box_{n-1}, \Box_n))$ and the adjacency condition is that that $\Box_j, \Box_{j+1}$ should touch, possibly only on corners, and including the possibility $\Box_j = \Box_{j+1}$. The notation $\omega : x \rightarrow y$ means $x \in \Box_0, y \in \Box_n$. We let $|\omega| = n$ be the number of links. If $|\omega| = 0$ then there is just the single square $\Box_0$.

**Theorem 1** Let $A$ satisfy

$$|\partial A| \leq CL^{3(k-i)/2}p(e_i) \text{ on } \delta A_i^{(k)}$$ \hspace{1cm} (133)

for some constant $C$ and $p(e_i) = \log(e_i^{-1})$. Let $M_0$ be sufficiently large and let $e_k$ be sufficiently small. Then $S_{k,A}(A, x, y)$ exists and has the random walk expansion (132). We have the bound for each path

$$|S_{k,A,\omega}(A, x, y)| \leq O(1)(O(1)M_0^{-1})^{|\omega|}d'(x, y)^{-2}\exp(-O(1)d_A(x, y))$$ \hspace{1cm} (134)

and the bound for the full propagator

$$|S_{k,A}(A, x, y)| \leq O(1)d'(x, y)^{-2}\exp(-O(1)d_A(x, y))$$ \hspace{1cm} (135)

In addition $S_{k,A,\omega}(A)$ and $S_{k,A}(A, x, y)$ are gauge covariant, and $S_{k,A,\omega}(A)$ depends on $A$ only in $\bigcup_{\Box \in \omega} \Box^{(5)}$.

**Remarks.**

1. In $\exp(-O(1)d_A(x, y))$ the $O(1)$ is independent of $M_0$.
2. There is no condition on $A$ itself, but only on $\partial A$. This important feature follows from the gauge covariance as we will see. (And actually a condition on the field strength $F = dA$ would suffice.)
3. It is possible that $A$ is a sequence of the full tori, i.e. $A_i = \mathbb{R}^0_{N+M-i}$. In this case $d_A(x, y) = d(x, y)$ and we have the result for $S_k(A, x, y)$.

To prove the theorem we need a result for a single block.

**Lemma 2** Under the same hypotheses for each $\Box_i$ block $\Box$ there is an operator $S_{k,\Box}(A) = S_{k,\Box}^*(A)$ such that for $x \in \Box$

$$\left(D_{e_k}(A) + m_k + Q_{k,A}(-A)^Tb Q_{k,A}(A) \right) S_{k,\Box}^*(A)f(x) = f(x)$$  \hspace{1cm} (136)

and for all $x, y \in \Box$

$$|S_{\Box}^*(A, x, y)| \leq O(1)d'(x, y)^{-2}\exp(-O(1)d_A(x, y))$$ \hspace{1cm} (137)

In addition $S_{\Box}(A)$ is gauge covariant and depends on $A$ only in $\Box^{(5)}$.
Assuming the lemma we have

**Proof of theorem 11**

**Part I.** We define a parametrix on \( L^{-k} \Lambda_0 \)

\[
S^*(A) = \sum_{\square \in \mathcal{D}} h_{\square} S^*_0(A) h_{\square}
\]

Then thanks to (125) and (137) and (138)

\[
\left( D_{ek}(A) + m_k + \mathcal{Q}_{k,\Lambda}((-A)^T b \mathcal{Q}_{k,\Lambda}(A)) \right) S^*(A) = I - \sum_{\square} R_{\square}(A) S_{\square}^*(A) h_{\square} = I - R
\]

where

\[
R_{\square}(A) = - \left[ \left( D_{ek}(A) + m_k + \mathcal{Q}_{k,\Lambda}((-A)^T b \mathcal{Q}_{k,\Lambda}(A)) \right) , h_{\square} \right]
\]

The inverse is now

\[
S_{k,\Lambda}(A) = S^*(A)(I - R)^{-1} = S^*(A) \sum_{n=0}^{\infty} R^n
\]

provided the series converges. This can also be written

\[
S_{k,\Lambda}(A) = \sum_{n=0}^{\infty} \sum_{\square_0,\square_1,\ldots,\square_n} (h_{\square_0} S_{\square_0}(A) h_{\square_0}) (R_{\square_1}(A) S_{\square_1}^*(A) h_{\square_1}) \cdots (R_{\square_n}(A) S_{\square_n}^*(A) h_{\square_n})
\]

The result now follows by that of \( S^*_0(A) \) and \( R_{\square}(A) \), as does the \( A \) dependence.

**Part II.** To estimate this expansion we need the following bound. For \( \square \in \mathcal{D}_i \)

\[
|\langle R_{\square}(A) S_{\square}^*(A) \rangle(x,y)| \leq O \left( \frac{L^{k-1}}{M_0} \right) d'(x,y)^{-2} \exp(-O(1) d(A, x,y))
\]

To prove it we write \( R_{\square} = R_{\square}^D + R_{\square}^O \) where \( R_{\square}^D = -[D_{ek}, h_{\square}] \) and \( R_{\square}^O = -[\mathcal{Q}_{k,\Lambda} b \mathcal{Q}_{k,\Lambda}, h_{\square}] \). We have explicitly

\[
(R_{\square}^D(A) S_{\square}^*(A))(x,y) = \sum_{x'} \gamma_{x,x'} L^k e^{i e_k L^{-k} A(x,x')} (h_{\square}(x) - h_{\square}(x')) S_{\square}^*(x', y)
\]

where the sum is over nearest neighbors \( x' \) of \( x \). The result now follows by

\[
L^k |h_{\square}(x) - h_{\square}(x')| \leq \sup_x |\partial h_{\square}(x)| \leq O \left( \frac{L^{k-1}}{M_0} \right)
\]

and \( d'(x',y) \geq d'(x,y)/2 \) and \( d(A,x,y) \geq d(A,x,y) - 1 \).

For the other term we have from (137) on \( \mathcal{T}_{N+M}^{-1} \)

\[
(Q_i(-A)^T b_i \mathcal{Q}_i(A))(x,x') = \exp \left( i e_i A(\tilde{\Gamma}_{x,[x]} \cup \tilde{\Gamma}_{x,x'}) \right) \chi(|x' - [x]| \leq 1/2)
\]
where $[x]$ is the unit lattice point at the center of the block containing $x$. Then for the scaled version on $\mathbb{T}_{N+M-1}^{k}$

$$
(Q_k^{(k)}(-A)^T b_i Q_i^{(k)}(A))(x, x') = \int \int \frac{L^{2(k-1)}(Q_i(-A_{L^k-1})^T b_i Q_i(A_{L^k-1}))(L^{k-1}x, L^{k-1}x')}{L^{2(k-1)} b_i \exp (i e_i A_{L^k-1}(\ldots)) \chi(|x' - [x]| \leq L^{-(k-1)}/2)}
$$

where now $[x]$ is the $L^{-(k-1)}$ lattice point at the center of the $L^{-(k-1)}$ block containing $x$. Now $\Box \subset \delta A_i^{(k)} \cup \delta A_{i+1}^{(k)}$ and the same is true for supp $h \Box \subset \tilde{\Box}$. Hence the only contribution to $R_{\Box}^{Q}$ comes from

$$
(Q_k A (-A)^T b_i Q_k^{(k)}(A)(x, x')) = \left\{ \begin{array}{ll}
(Q_i^{(k)}(-A)^T b_i Q_i^{(k)}(A))(x, x') & x, x' \in \delta A_i^{(k)} \\
(Q_i^{(k)}(-A)^T b_i Q_i^{(k)}(A))(x, x') & x, x' \in \delta A_{i+1}^{(k)}
\end{array} \right.
$$

We concentrate on the first case, the other is similar. Then

$$
(R_{\Box}^{Q}(A) S_{\Box}^{n}(A))(x, y) = b_i \int_{x' - [x] \leq L^{-(k-1)/2}} L^{2(k-1)} b_i \exp (i e_i A_{L^k-1}(\ldots))(h_{\Box}(x) - h_{\Box}(x')) S_{\Box}^{n}(x', y)
$$

Now since $|x - x'| \leq L^{-(k-1)}$

$$
|h_{\Box}(x) - h_{\Box}(x')| \leq L^{-(k-1)} \sup_x |\partial h_{\Box}(x)| \leq O(M_0^{-1})
$$

we obtain

$$
|(R_{\Box}^{Q}(A) S_{\Box}^{n}(A))(x, y)| \leq O\left(\frac{L^{2(k-1)}}{M_0}\int_{|x' - [x]| \leq L^{-(k-1)/2}} d'(x', y)^{-2} \exp(-O(1) d_{\Box}(x, y))\right)
$$

$$
\leq O\left(\frac{L^{2(k-1)}}{M_0}\exp(-O(1) d_{\Box}(x, y))\right)
$$

In the second step we use (124). Since $d'(x, y) \leq d_{\Box}(x, y) + 1$ the result follows.

**Part III.** Now we estimate the expansion. Besides the partition (122) we can also partition $L^{-k} \Lambda_0$ into smaller blocks $\Delta$ of size $L^{-(k-1)}$ in $\delta A_i^{(k)}$. Then for $\omega = (\Box_0, \ldots, \Box_n)$

$$
S_{k, A, \omega}(x, y) = \sum_{\Delta_1, \ldots, \Delta_n} \int_{\Delta_1} \ldots \int_{\Delta_n} d \chi \ldots \chi \left( R_{\Box_0} S_{\Box_0}^{n}(A) h_{\Box_0}(x, x_1, x_2) \ldots (R_{\Box_n} S_{\Box_n}^{n}(A) h_{\Box_n}(x, y) \right)
$$

We can restrict the sum to $\Delta_j$ intersecting both $\tilde{\Box}_{j-1}$ and $\tilde{\Box}_j$. Now use the estimates (137), (148) to obtain

$$
|S_{k, A, \omega}(A, x, y)| \leq (O(1) M_0)^{-n} \sum_{\Delta_1, \ldots, \Delta_n} \int_{\Delta_1} \ldots \int_{\Delta_n} d \chi \ldots \chi \left( \frac{d'(x, x_1)^{-2} e^{-O(1) d_{\Delta}(x, x_1)}}{L^{k-1} d'(x_1, x_2)^{-2} e^{-O(1) d_{\Delta}(x_1, x_2)}} \ldots \left( L^{k-i_n} d'(x_n, y)^{-2} e^{-O(1) d_{\Delta}(x_n, y)} \right) \right)
$$

Here $i_j$ is chosen by $\Box_j \in \mathbb{D}_j$. Now use $d_{\Delta}(x, x_{i+1}) \geq d_{\Delta}(\Delta_i, \Delta_{i+1}) - 2$ where the distance is from the center of the cubes. Then repeatedly use the estimate

$$
L^{k-i_j} \int_{\Delta_j} d'(x_{j-1}, x_{j})^{-2} d'd'(x_{j}, x_{j+1})^{-2} dx_j \leq O(d'(x_{j-1}, x_{j+1})^{-2})
$$
which follow from (130). Here we use that $\Box_j$ is contained in $\delta \Lambda_j \cup \delta \Lambda_j+1$, hence so is $\Delta_j$ and hence it is either a $L^{-(k-i)}$ or a $L^{-(k-i-1)}$ block. We also use the estimate
\[
\sum_{\Delta_1, \ldots, \Delta_n} e^{-O(1)d_A(x,\Delta_1)} e^{-O(1)d_A(\Delta_1,\Delta_2)} \cdots e^{-O(1)d_A(\Delta_n,y)} \leq (O(1))^n \exp\left(-O(1)d_A(x,y)\right)
\]  
(155)

For this see [2], lemma 2.1. These estimates yield the bound on $S_{k,\Lambda}(A,x,y)$. For the bound on $S_{k,\Lambda}(A,x,y)$ we sum over paths. The factor $(O(1)M_0)^{-n}$ is sufficient to control the sum if $M_0$ is sufficiently large.

proof of lemma [2]

part I. We need to invert $\left(D_{c_k}(A) + m_k + Q_{k,\Lambda}(-A)^T bQ_{k,\Lambda}(A)\right)$ on a single block. Inverting with straight Dirichlet boundary conditions makes it awkward to get estimates, so we use a kind of soft Dirichlet conditions, following [14] and the construction of the theorem.

Given $\Lambda$ we take a fixed $\Box_i$ cube $\Box \subset T^{N+M-k}_i$. Then define a decreasing sequence of cubes
\[
\Omega(\Box) = (\Omega_0(\Box), \ldots, \Omega_{k-1}(\Box))
\]  
(156)

with $\Omega_j(\Box) \subset T^{N+M-j}_i$. In reverse order they are specified by
\[
\begin{align*}
\Omega_j &= \emptyset & j > i \\
\Omega_i(\Box) &= U_i L^k \Box^{(2)} \cap \Lambda_i \\
\Omega_{i-1}(\Box) &= U_{i-1} L^k \Box^{(3)}
\end{align*}
\]  
(157)

d$((L^{-1} \Omega^T_{j-1}(\Box))c, \Omega_j(\Box)) = r_i M_0$ \quad $j = i - 1 \ldots 1$

where the cube $L^{-1} \Omega^T_{j-1}(\Box)$ is required to be centered on $\Omega_j(\Box)$. (If $i = k$ then $\Omega_i(\Box) = \emptyset$.)

Now with $\delta \Omega_j(\Box) = L^{-k}(B_{j-1} \Omega^T_{j-1}(\Box) - B_j \Omega_j(\Box))$ we have for $j = 1, \ldots, k$
\[
\begin{align*}
\delta \Omega_j(k) &= \emptyset & j > i + 1 \\
\delta \Omega_{i+1}(k) &= \Box^{(2)} \cap L^{-k} B_i \Lambda_i \\
\delta \Omega_i(k) &= \Box^{(3)} - (\Box^{(2)} \cap L^{-k} B_i \Lambda_i)
\end{align*}
\]  
(158)

d$L^{-k}(B_{j-1} \Omega^T_{j-1}(\Box))c, L^{-k} B_j \Omega_j(\Box)) \leq L^{-(k-j)} L^{-k} M_0$

Here we use that $L^k \Box^{(3)}$ is a $7r_i M_0 L^i$ cube and so has the form $B_i \Box$ for some $7r_i M_0$ cube $\Box$ and hence $B_i U_i = I$ on this set. We have also used $d(B_1 X, B_i Y) \leq L d(X,Y)$ and $(B_1 X)^c = B_1 X^c$. We note that
\[
d(L^{-k} \Omega_0(\Box)^c, \Box^{(3)}) \leq \sum_{j=1}^i L^{-(k-j)} r_i M_0 \leq 2 r_i M_0 L^{-(k-i)}
\]  
(159)

which implies that $L^{-k} \Omega_0(\Box)^c \subset \Box^{(5)}$.

We now define
\[
S^{*}_{\Box}(A) = S_{\Omega(\Box)}(A) = \left[D_{c_k}(A) + m_k + Q_{\Omega(\Box)}(-A)^T bQ_{\Omega(\Box)}(A)\right]^{-1} L^{-k} \Omega_0(\Box)
\]  
(160)

if it exists.

Before considering existence we prove that (160) satisfies (136). It suffices to show that for $x \in \Box$
\[
\left(Q_{\Omega(\Box)}(-A)^T bQ_{\Omega(\Box)}(A)f\right)(x) = \left(Q_{k,\Lambda}(-A)^T bQ_{k,\Lambda}(A)f\right)(x)
\]  
(161)
Recall that \( \hat{\Box} \subset \delta \Lambda^{(k)} \cup \delta \Lambda^{(k+1)} \). If \( x \in \hat{\Box} \cap \delta \Lambda^{(k)} \) then \( x \in \hat{\Box} \cap L^{-k}B_i \Lambda_i \subset \delta \Omega + 1(\Box) \) and the left side of (161) is \( \left( Q_i^{(k)}(-A)^T b_i Q_i^{(k)}(A)f \right)(x) \) which agrees with the right side of (161). If \( x \in \hat{\Box} \cap \delta \Lambda^{(k)} \) then \( x \in \hat{\Box} - L^{-k}B_i \Lambda_i \subset \delta \Omega - (\Box) \) and the left side of (161) is \( \left( Q_i^{(k)}(-A)^T b_i Q_i^{(k)}(A)f \right)(x) \) which agrees with the right side of (161). Thus (161) is true.

**part II.** We are going to treat \( S_{i,\Box}^k(0) \) first and we start with some definitions at \( A = 0 \). Let \( S_j = S_j(0) \) and \( Q_j = Q_j(0) \) and \( S_j = \sigma L_{k-j}, S_j(\sigma L_{k-j})T \) which is the representation of \( S_j \) on \( T^{-k}_{N+M-k} \). We have

\[
S_j = \sigma L_{k-j}^{-1}(D(0) + m_j + Q_j^T b_j Q_j)^{-1}(\sigma L_{k-j}^{-1})T
\]

and the bound from (163)

\[
|\langle S_j(x, y) \rangle| = |L^{2(k-j)}S_j(L^{k-j}x, L^{k-j}y)| \leq O(1)d'(x, y)^{-2}\exp(-O(1)L^{k-j}d(x, y))
\]

We also consider the mixed operator for suitable \( Y \subset T^{-k}_{N+M-k} \)

\[
|\langle S_j(x, y) \rangle| = \left( D(0) + m_k + b_j^T b_j Q_j \right)_{[\Box]} + \left| b_j^T b_j Q_j \right|_{[\Box]}
\]

This has the alternate representation from (14).

\[
S_j = \sigma L_{k-j}^{-1}(S_j + b_j^T b_j Q_j^T [D_j(0) + T Q_j]^{-1}(\sigma L_{k-j})T)
\]

To estimate this we take the bound from (14) for \( z, w \in T^0_{N+M-j} \)

\[
\left| D_j(0) + \left[ D_j(0) + T Q_j \right]^{-1} \right|_{x,y} \leq \exp(-O(1)d(z, w))
\]

Then the expression in parentheses in (165) has a kernel which is \( O(1)d'(x, y)^{-2}\exp(-O(1)d(x, y)) \). (Even without the short distance singularity for the second term.) Thus after scaling we have again

\[
|\langle S_j(x, y) \rangle| \leq O(1)d'(x, y)^{-2}\exp(-O(1)L^{k-j}d(x, y))
\]

**part III.** Now we show \( S_{i,\Box}^k(0) \) exists and get an estimate on the kernel. First define a parametrix on \( L^{-k}\Omega_0(\Box) \)

\[
S^\#(\Box) = \sum_{\Box \subset \Box'} h_{\Box'} S_{\Box'}^\#(\Box) = \Box'
\]

Here \( \Box = \bigcup_j \Box_j(\Box) \) is a partition of \( L^{-k}\Omega_0(\Box) = \bigcup_j \delta \Omega_j^k(\Box) \) defined by dividing \( \delta \Omega_j^k(\Box) \) into cubes of size \( L^{-k-j}M_0 \) and then further subdividing cubes touching \( \delta \Omega_j^k(\Box) \). Then \( \Box_j(\Box) \) is the set of new \( L^{-k-j}M_0 \) cubes. These are contained in \( \delta \Omega_j^k(\Box) \cup \delta \Omega_j^{k+1} \). We define \( S^\#(\Box) \) as follows. If \( \Box \cap \Box_j(\Box) \) and \( (\Box')^2 \) does not intersect \( L^{-k}B_j \Omega_j(\Box) \) then we define \( S^\#(\Box) = S_j \). More generally we define

\[
S_j(\Box) = |\langle S_j \rangle|_{\Box} \quad Y = (\Box')^2 \cap L^{-k}B_j \Omega_j(\Box)
\]
Then $S^\#_a(\square)$ provides a local inverse in the sense that for $x \in \square'$

$$\left(\left(D_{e_k}(0) + m_k + Q_{\square}(0)^T b Q_{\square}(0)\right)S^\#_a(\square)f\right)(x) = f(x)$$  \hspace{1cm} (170)

Check this as in (161). If $\square' \in D_j$ and $x, y \in \widehat{\square}'$ we have from (107)

$$|S^\#_{\square}(x, y)| \leq O(1)d'(x, y)^{-2}\exp(-O(1)d_{\Omega}(\square, x, y))$$  \hspace{1cm} (171)

Now we compute

$$(D_{e_k}(0) + m_k + Q^T_{\square}b Q_{\square})S^\#(\square) = I - \sum_{\square'} R^\#(\square)S^\#(\square)h_{\square'} \equiv I - R^\#(\square)$$  \hspace{1cm} (172)

where

$$R^\#(\square) = -\left[(D_{e_k}(0) + m_k + Q^T_{\square}b Q_{\square}, h_{\square'})\right]$$  \hspace{1cm} (173)

The inverse on $L^{-k}\Omega_0(\square)$ is then

$$S^a_\square(0) = S^\#(\square)(I - R^\#(\square))^{-1} = S^\#(\square)\sum_{n=0}^{\infty} (R^\#(\square))^n$$  \hspace{1cm} (174)

if it converges. This can also be written

$$S^a_\square(0) = \sum_{n=0}^{\infty} \sum_{\square_0, \square_1, \ldots, \square_n} \left(h_{\square_0}, S^\#_{\square_0}(\square)h_{\square_0}, (R^\#_{\square_0}(\square))S^\#_{\square_0}(\square)h_{\square_1}, \ldots, (R^\#_{\square_n}(\square))S^\#_{\square_n}(\square)h_{\square_n}\right)$$  \hspace{1cm} (175)

Convergence is demonstrated just as before and gives

$$|S^\#_{\square}(0, x, y)| \leq O(d'(x, y)^{-2})\exp\left(-O(1)d_{\Omega}(\square, x, y)\right)$$  \hspace{1cm} (176)

Now consider $x, y \in L^{-k}\Omega_0(\square) \subset \square^{(5)}$. Assuming $5r_1 < r_0$ we have $\square^{(5)} \subset \delta\Lambda_i^{(k)} \cup \delta\Lambda_{i+1}^{(k)}$. Then

$$d_{\Omega}(x, y) \geq O(1)L^{(k-i)}d(x, y) \geq O(1)d_{\Lambda}(x, y)$$  \hspace{1cm} (177)

and we have the result we need.

**Part IV.** We continue to consider $\square \in D_i$ and now study $S^a_\square(\mathcal{A})$ for $\mathcal{A} \neq 0$. At first suppose that instead of the bound on $\partial\mathcal{A}$ we have for some $C$

$$|\mathcal{A}| \leq CL^{(k-i)/2}p(e_i)$$  \hspace{1cm} (178)

The same bound holds on $\delta\Lambda_i^{(k)} \cup \delta\Lambda_{i+1}^{(k)}$ since $p(e_{i+1}) < p(e_i)$. Hence the bound holds on $\mathcal{A}^{(5)}$ and hence on $L^{-k}\Omega_0(\square)$, the region we are working in.

Let

$$v_{\square}(\mathcal{A}, \mathcal{A}') = [D_{e_k}(\mathcal{A}) + m_k + Q_{\square}(\mathcal{A})(-\mathcal{A})^T b Q_{\square}(\mathcal{(A)})] - [D_{e_k}(\mathcal{A}') + m_k + Q_{\square}(\mathcal{A}')(-\mathcal{A}')^T b Q_{\square}(\mathcal{(A)')}]$$  \hspace{1cm} (179)

Then $S^a_\square(\mathcal{A})$ exists if the series

$$S^a_\square(\mathcal{A}) = S^a_\square(0)\left(\sum_{n=0}^{\infty} (v_{\square}(\mathcal{A}, 0)S^a_\square(0))^n\right)$$  \hspace{1cm} (180)
converges.

To show convergence we need to estimate the kernel \( (v_{\square}(A,0)S_{\square}^*(0))(x,y) \). Again there are two parts coming from \( v_{\square}(A,0) = v_{\square}^D(A,0) + v_{\square}^Q(A,0) \). For the first term

\[
(v_{\square}^D(A,0)S_{\square}^*(0))(x,y) = \sum_{x'} \gamma_{x,x'} L^k(e^{i \delta_{x,x} k} A(x,x') - 1)(S_{\square}^*(0))(x', y)
\]

which is bounded by

\[
| (v_{\square}^D(A,0)S_{\square}^*(0))(x,y) | \leq O(1) e_k \sup |A| d'(x,y)^{-2} \exp(-O(1)d_{\square}(x,y)) \\
\leq O(1) e_k CL^{(k-i)/2}p(e_i) d'(x,y)^{-2} \exp(-O(1)d_{\square}(x,y))
\]

\[
= O(1) CL^{k-i} e_i p(e_i) d'(x,y)^{-2} \exp(-O(1)d_{\square}(x,y))
\]

The second term has the form for \( x \in \delta\Omega_j^{(k)}(\square) \subset L^{-k}\Omega_0(\square), j \leq i + 1 \):

\[
(v_{\square}^Q(A,0)S_{\square}^*(0))(x,y) = L^{2(k-j)}b_j \int_{|x' - |x|| \leq L^{-(k-j)/2}} \left( \exp(ie_j A_{L^{k-j}}(\Gamma_z,x) \cup \Gamma_{z},x') - 1 \right) |_{z = \Gamma_{L^{k-j}x}} (S_{\square}^*(0))(x', y)
\]

The contour has length bounded by one and so we have bound

\[
| (\exp(ie_j A_{L^{k-j}} \ldots - 1) | \leq e_j \sup |A_{L^{k-j}} | \leq CL^{-(i-j)/2} e_j p(e_i) \leq O(1) Ce_i p(e_i)
\]

and hence

\[
| (v_{\square}^Q(A,0)S_{\square}^*(0))(x,y) | \leq O(1) CL^{2(k-j)} e_i p(e_i) \int_{|x' - |x|| \leq L^{-(k-j)/2}} d'(x,y)^{-2} \exp(-O(1)d_{\square}(x,y)) \\
\leq O(1) CL^{k-j} e_i p(e_i) \exp(-O(1)d_{\square}(x,y))
\]

Combining the two bounds we have for \( x \in \delta\Omega_j^{(k)}(\square) \)

\[
| (v_{\square}(A,0)S_{\square}^*(0))(x,y) | \leq O(1) CL^{k-j} e_i p(e_i) d'(x,y)^{-2} \exp(-O(1)d_{\square}(x,y))
\]

Now we can estimate the expansion. Dividing \( \delta\Omega_j^{(k)}(\square) \) into blocks \( \Delta \) of size \( L^{-(k-j)} \) and hence \( L^{-k}\Omega_0(\square) \) into blocks of various sizes we have

\[
S_{\square}^*(A,x,y) = \sum_{n=0}^{\infty} \sum_{\Delta_1, \ldots, \Delta_n} \int_{\Delta_1} \ldots \int_{\Delta_n} \ldots (v_{\square}(A)S_{\square}^*(0))(x_1, x_2) \ldots (v_{\square}(A)S_{\square}^*(0))(x_n, y)
\]

We use our estimate \( 180 \) as well as \( 182 \) and \( 183 \) and find

\[
|S_{\square}^*(A,x,y)| \leq \left( \sum_{n=0}^{\infty} (O(1) Ce_i p(e_i))^n \right) d'(x,y)^{-2} \exp(-O(1)d_{\square}(x,y))
\]

\[
\leq O(1) d'(x,y)^{-2} \exp(-O(1)d_{\square}(x,y))
\]

Here we use that \( e_i < e_k \) is assumed sufficiently small. By \( 177 \) we can replace \( \exp(-O(1)d_{\square}(x,y)) \) by \( \exp(-O(1)d_{A}(x,y)) \) to complete the proof in this case.
part V. Finally we extend the result to the case $|\partial \mathcal{A}| \leq CL^{3(k-i)/2}p(e_i)$ on $\delta \Lambda_i^{(k)}$, and hence the same bound on $L^{-k} \Omega_0(\Box)$. Instead of (180) we use

$$S^\square_\Box(\mathcal{A}) = S^\square_\Box(\bar{\mathcal{A}}) \left( \sum_{n=0}^{\infty} \left( v^\square_\Box(\mathcal{A}, \bar{\mathcal{A}}) S^\square_\Box(\bar{\mathcal{A}}) \right)^n \right)$$  \hspace{1cm} (189)$$

where $\bar{\mathcal{A}}$ is the average of $\mathcal{A}$ over $L^{-k} \Omega_0(\Box)$. Now $\bar{\mathcal{A}}$ is pure gauge and can be written $\bar{\mathcal{A}} = d\lambda$. and Since the propagator is gauge invariant we have

$$S^\square_\Box(\mathcal{A}, x, y, \lambda) = e^{-ire\lambda(x)} S^\square_\Box(0, x, y) e^{-ire\lambda(y)}$$  \hspace{1cm} (190)$$

which satisfies the same bound (176) as $S^\square_\Box(0, x, y)$. Also

$$|\mathcal{A} - \bar{\mathcal{A}}| \leq 11r_1 M_0 L^{-(k-i)} \sup |\partial \mathcal{A}| \leq C' L^{(k-i)/2} C p(e_i)$$  \hspace{1cm} (191)$$

for a new constant $C'$. This is a bound of the form we assumed on the field in part IV. One can then show that $v^\square_\Box(\mathcal{A}, \bar{\mathcal{A}}) S^\square_\Box(\bar{\mathcal{A}})$ satisfies the same bound (180) used in part IV. Thus we can repeat part IV with the same result. This completes the proof of the lemma and the theorem.

As a corollary we consider perturbation by a complex background field $\mathcal{A}'$. We need a bound on $\mathcal{A}'$ itself, not just $\partial \mathcal{A}'$.

**Corollary 1** Let $\mathcal{A}$ be real and satisfy $|\partial \mathcal{A}| \leq CL^{3(k-i)/2}p(e_i)$ on $\delta \Lambda_i^{(k)}$ and let $\mathcal{A}'$ be complex and satisfy $|\mathcal{A}'| \leq CL^{(k-i)/2}p(e_i)$ on $\delta \Lambda_i^{(k)}$. Then $S^\square_\Box(\mathcal{A} + \mathcal{A}')$ and $S_{k,\Lambda,\omega}(\mathcal{A} + \mathcal{A}')$ all exist, are analytic in $\mathcal{A}'$, and satisfy the $\mathcal{A}' = 0$ bounds (134), (135), (136), now with larger constants.

**Proof.** It suffices to prove the result for $S^\square_\Box(\mathcal{A} + \mathcal{A}')$, the others follow. We make the expansion

$$S^\square_\Box(\mathcal{A} + \mathcal{A}') = S^\square_\Box(\mathcal{A}) \left( \sum_{n=0}^{\infty} \left( v^\square_\Box(\mathcal{A} + \mathcal{A}', \mathcal{A}) S^\square_\Box(\mathcal{A}) \right)^n \right)$$  \hspace{1cm} (192)$$

The bound on $\partial \mathcal{A}$ gives control over $S^\square_\Box(\mathcal{A})$ by the theorem and the bound on complex $\mathcal{A}'$ can be used to show that $v^\square_\Box(\mathcal{A} + \mathcal{A}', \mathcal{A}) S^\square_\Box(\mathcal{A})$ satisfies (180). Here we use also

$$|\exp(ie\lambda L^{-k} \mathcal{A}')| \leq \exp(e\lambda L^{-k} (CL^{(k-i)/2}p(e_i)) \leq \exp(Ce_i p(e_i)) \leq 2$$  \hspace{1cm} (193)$$

Now repeat part IV of the lemma and get the result.

### 3.3 bosons

The treatment for bosons is similar, but easier since there is no background field. The random walk expansion for the boson propagator has the form

$$G_{k,\Lambda}(x, y) = \sum_{\omega : x \rightarrow y} G_{k,\Lambda,\omega}(x, y)$$  \hspace{1cm} (194)$$

29
Theorem 2 Let $M_0$ be sufficiently large. Then $G_{k,\Lambda}$ exists and has the random walk expansion \[^{194}\]. We have

$$
\begin{align*}
|G_{k,\Lambda}(x, y)| & \leq \mathcal{O}(1)(\mathcal{O}(1)M_0^{-1})|\omega|d'(x, y)^{-1}\exp(-\mathcal{O}(1)d_\Lambda(x, y)) \\
|G_{k,\Lambda}(x, y)| & \leq \mathcal{O}(1)d'(x, y)^{-1}\exp(-\mathcal{O}(1)d_\Lambda(x, y))
\end{align*}
$$

(195)

If $x \in \delta\Lambda_i^{(k)}$ then

$$
\begin{align*}
|\partial G_{k,\Lambda}(x, y)| & \leq \mathcal{O}(1)(\mathcal{O}(1)M_0^{-1})|\omega|(L^{k-i}d'(x, y)^{-1} + d'(x, y)^{-2})\exp(-\mathcal{O}(1)d_\Lambda(x, y)) \\
|\partial G_{k,\Lambda}(x, y)| & \leq \mathcal{O}(1)(L^{k-i}d'(x, y)^{-1} + d'(x, y)^{-2})\exp(-\mathcal{O}(1)d_\Lambda(x, y))
\end{align*}
$$

(196)

The proof depends on the lemma:

Lemma 3 Under the same hypotheses for each $D_i$ block $\square$ there is an operator $G_{\square}^*$ such that for $x \in \square$:

$$
(\begin{bmatrix} -\Delta + \mu_k^2 + \mathcal{Q}_{k,\Lambda} T \mathcal{Q}_{k,\Lambda} a \mathcal{Q}_{k,\Lambda} \end{bmatrix} G_{\square}^* f)(x) = f(x)
$$

(197)

and for $x, y \in \square$

$$
\begin{align*}
|G_{\square}^*(x, y)| & \leq \mathcal{O}(1)d'(x, y)^{-1}\exp(-\mathcal{O}(1)d_\Lambda(x, y)) \\
|\partial G_{\square}^*(x, y)| & \leq \mathcal{O}(1)(L^{k-i}d'(x, y)^{-1} + d'(x, y)^{-2})\exp(-\mathcal{O}(1)d_\Lambda(x, y))
\end{align*}
$$

(198)

Assuming the lemma we prove the theorem.

Proof. The random walk expansion has the form

$$
G_{k,\Lambda} = \sum_{n=0}^{\infty} \sum_{\Box_0, \Box_1, \ldots, \Box_n} (h_{\Box_0} G_{\Box_0}^* h_{\Box_0}) (R_{\Box_1} G_{\Box_1}^* h_{\Box_1}) \cdots (R_{\Box_n} G_{\Box_n}^* h_{\Box_n}) \\
= \sum_{\omega} G_{k,\Lambda,\omega}
$$

(199)

where now

$$
R_{\square} = - \left[ (-\Delta + \mu_k^2 + \mathcal{Q}_{k,\Lambda} T \mathcal{Q}_{k,\Lambda} a \mathcal{Q}_{k,\Lambda}) , h_\square \right]
$$

(200)

We write $R_{\square} = R_{\square}^A + R_{\square}^Q$ and estimate for $\square \in D_i$

$$
|\langle R_{\square} G_{\square}^* \rangle(x, y)| = |(-\Delta h_\square)(x)G_{\square}^*(x, y) + (\partial h_\square)(x)\partial G_{\square}^*(x, y) + (\partial^T h_\square)(x)\partial^T G_{\square}^*(x, y)|
\leq \mathcal{O}(1)M_0^{-1} \left( L^{2(k-i)}d'(x, y)^{-1} + L^{k-i}d'(x, y)^{-2} \right)\exp(-\mathcal{O}(1)d_\Lambda(x, y))
$$

(201)

The same bound holds easily for $|\langle R_{\square} G_{\square}^* \rangle(x, y)|$ and hence it holds also for $|\langle R_{\square} G_{\square}^* \rangle(x, y)|$.

Now we follow the proof of theorem \[^{194}]. The only difference is in the short distance estimates which we modify as follows. Instead of \[^{153}\] we have by \[^{124}, 130, 131\]

$$
\begin{align*}
\int_{\Delta_i} \left( L^{2(k-i)}d'(x_{j-1}, x_j)^{-1} + L^{k-i}d'(x_{j-1}, x_j)^{-2} \right)
\left( L^{2(k-i)}d'(x_j, x_{j+1})^{-1} + L^{k-i}d'(x_j, x_{j+1})^{-2} \right) dx_j \\
\leq \mathcal{O}(L^{2(k-i)})d'(x_{j-1}, x_{j+1})^{-1} + \mathcal{O}(L^{k-i})d'(x_{j-1}, x_{j+1})^{-2}
\end{align*}
$$

(202)
As we repeat this estimate we have to adjust the $i$ in the factor $L^{k-i}$ so that neighbors match. But since $|i_j - i_{j+1}| \leq 1$ this costs at most $O(L^{2n})$ which we can afford. In the last step since $|i - i_0| \leq 1$ the inequality is

$$\int_{\Delta_1} d'(x, x_1)^{-1} \left( O(L^{2(k-i)}d'(x, y)^{-1}) + O(L^{k-i})d'(x_1, y)^{-2} \right) dx_1 \leq O(1)d'(x, y)^{-1}$$

(203)

The rest of the proof is as before and gives the result for $G_k, \Lambda, \omega$ and $G_k, \Lambda$. For the bounds on $\partial G_k, \Lambda, \omega$ and $\partial G_k, \Lambda$ the last step is

$$\int_{\Delta_1} \left( L^{k-i}d'(x, x_1)^{-1} + d'(x, x_1)^{-2} \right) \left( L^{2(k-i)}d'(x_1, y)^{-1} + L^{k-i}d'(x_1, y)^{-2} \right) dx_1$$

$$\leq O(1)(L^{k-i}d'(x, y)^{-1} + d'(x, y)^{-2})$$

(204)

to complete the proof.

The lemma is proved in much the same way. One follows parts I-III of lemma 3 modifying the short distance behavior as indicated above.
A Averaging operators

A.1 bosons

We consider the $k$-step boson averaging operator defined recursively in (33) by $Q_0 = id$ and

$$Q_{k+1} = \sigma_{L^{-1}} Q_k \sigma_L$$

Then we have

**Lemma 4**

$$\int dA_k \mathcal{N}^{-1}_{k+1,a} \mathcal{N}^{-1}_{k,a} \exp \left( -\frac{a}{2L^2} |A_{k+1,L} - QA_k|^2 \right) \exp \left( -\frac{a}{2} |A_k - Q_k A_L|^2 \right)$$

$$= \mathcal{N}^{-1}_{k+1,a_k} \exp \left( -\frac{a}{2L^2} |A_{k+1}|^2 - Q_{k+1} A_L \right)$$

Proof. Shift the integration variable $A_k \to A_k + Q_k A_L$, identify $Q_{k+1}$, and then the left side can be written with $\Phi = A_{k+1} - Q_{k+1} A_L$

$$\int dA_k \mathcal{N}^{-1}_{k+1,a} \mathcal{N}^{-1}_{k,a} \exp \left( -\frac{a}{2L^2} |\Phi_L - QA_k|^2 \right) \exp \left( -\frac{a}{2} |A_k|^2 \right)$$

$$= \text{const} \exp \left( -\frac{a}{2} |\Phi|^2 \right) \int dA_k \exp \left( -\frac{a}{2L^2} (Q^T \Phi_L, A_k) \right) \exp \left( \frac{1}{2} (A_k, (a_k + a L^{-2})^{-1} Q^T \Phi_L) \right)$$

$$= \text{const} \exp \left( -\frac{a}{2} |\Phi|^2 \right) \exp \left( \frac{a^2}{2L^4} (Q^T \Phi_L, (a_k + a L^{-2})^{-1} \Phi) \right)$$

$$= \text{const} \exp \left( -\frac{a}{2} |\Phi|^2 \right)$$

Here we have used $QQ^T = I$ and $a_{k+1} = a a_k / (a_k + a L^{-2})$. Since the left side integrates to one, the right side integrates to one which means the constant must be $\mathcal{N}^{-1}_{k+1,a_k}$ and we have the result.

We also use a local variation of this result. Let $\Lambda \subset \mathbb{T}_{N+M-k-1}^0$ so $\Lambda \subset \mathbb{T}_{N+M-k}^1$ and the blocked set $B_1 \Lambda \subset \mathbb{T}_{N+M-k}^0$. Then with $\mathcal{N}_{\Lambda,a} = (2\pi/a)^{3|\Lambda|/2}$ we have

$$\int dA_k, B_1 \Lambda \mathcal{N}^{-1}_{\Lambda,a} \mathcal{N}^{-1}_{B_1 \Lambda, a_k} \exp \left( -\frac{a}{2L^2} |A_{k+1,L} - QA_k|^2 \right) \exp \left( -\frac{a}{2} |A_k - Q_k A_U|^2 \right)$$

$$= \mathcal{N}^{-1}_{\Lambda,a_k} \exp \left( -\frac{a}{2L^2} |A_{k+1}|^2 - Q_{k+1} A_U \right)$$

**A.2 fermions**

For fermions the multiple averaging $Q_k(a_{k-1}, \ldots, a_0)$ depend on fields $a_{k-1}, \ldots, a_0$ all on $\mathbb{T}^{-k}_{N+M-k}$ and are defined recursively in (33) by $Q_0 = id$ and

$$Q_{k+1}(a_k, \ldots, a_0) = \sigma_{L^{-1}} Q_{e_k} (Q a_{k,L}) Q_k(a_{k-1,L}, \ldots, a_0, L) \sigma_L$$

Then we have
Lemma 5

\[
\int d\Psi_k \mathcal{M}_{k+1,b}^{-1} \mathcal{M}_{k,b}^{-1} \exp \left( -\frac{b}{L} \Psi_{k+1,L} - Q_{c_k}(\tilde{Q}_k a_{k,L}) \Psi_k \right) \exp \left( -b_k |\Psi_k - Q_k(a_{k-1,L}, \ldots, a_{0,L})\psi_L|^2 \right) = \mathcal{M}_{k+1,b}^{-1} \exp \left( -b_k |\Psi_{k+1} - Q_{k+1}(a_k, \ldots, a_0)\psi|^2 \right) \tag{210}
\]

Remark. If we take \( a_{j,L} = (A_j)_L \) for \( A_j \) on \( T_{m+N-j}^- \) then we recover the version quoted in the text.

Proof. First shift \( \Psi_k \to \Psi_k + Q_k(a_{k-1,L}, \ldots, a_{0,L})\psi_L \) and \( \Psi_k \to \bar{\Psi}_k + Q_k(-a_{k-1,L}, \ldots, -a_{0,L})\bar{\psi}_L \) and identify \( Q_{k+1}(a_k, \ldots, a_0) \). Then left side can be written with \( \Phi = \Psi_{k+1} - Q_{k+1}(a_k, \ldots, a_0)\psi \) and \( \Phi = \Psi_{k+1} - Q_{k+1}(-a_k, \ldots, -a_0)\bar{\psi} \) and \( A = Q_k a_{k,L} \):

\[
\int d\Psi_k \mathcal{M}_{k+1,b}^{-1} \mathcal{M}_{k,b}^{-1} \exp \left( -\frac{b}{L} \Phi_L - Q_{c_k}(A) \Psi_k \right) \exp \left( -b_k (\tilde{\Phi}, \Psi_k) \right) = \text{const} \exp \left( -b(\tilde{\Phi}, \Phi) \right) \int d\Psi_k \exp \left( -\frac{b}{L} (Q_{c_k}(A)^T \tilde{\Phi}_L, \Psi_k) \right) \exp \left( -\frac{b}{L} (\tilde{\Psi}_k, Q_{c_k}(-A)^T \Phi_L) \right) = \text{const} \exp \left( -b(\tilde{\Phi}, \Phi) \right) \exp \left( \frac{b^2}{L} (Q_{c_k}(A)^T \tilde{\Phi}_L, [b_k + \frac{b}{L} Q_{c_k}(-A)^T Q_{c_k}(A)]^{-1} Q_{c_k}(-A)^T \Phi_L) \right) = \text{const} \exp \left( -b(\tilde{\Phi}, \Phi) \right) \exp \left( \frac{b^2}{L} (\tilde{\Phi}, (b_k + \frac{b}{L})^{-1} \Phi) \right) = \text{const} \exp \left( -b_{k+1}(\tilde{\Phi}, \Phi) \right) \tag{211}
\]

Here we have used \( Q_{c_k}(A)Q_{c_k}(-A)^T = I \) and \( b_{k+1} = bb_k/(b_k + bL^{-1}) \). Since the left side integrates to one, the right side integrates to one which means the constant must be \( \mathcal{M}_{k+1,b}^{-1} \) and we have the result.

We also want to work out an explicit expression for \( Q_k(a_{k-1}, \ldots, a_0) \). First for \( k = 1 \) we consider \( Q_1(a_0) = \sigma^{-1}_L Q_{c_0}(a_{0,L}) \sigma_L \) and compute it as

\[
(Q_1(a_0)\psi)(y) = L^{-3} \sum_{|x - Ly| < L/2} \exp \left( i e_0 a_{0,L} (\Gamma_{Ly,x}) \right) \psi(x/L) = \int_{|x-y| < L/2} \exp \left( i e_0 a_{0,L} (\Gamma_{Ly,x}) \right) \psi(x) = \int_{|x-y| < L/2} \exp \left( i e_1 a_0 (\Gamma_{y,x}) \right) \psi(x) \tag{212}
\]

Here we have used \( e_1 = L^{1/2} e_0 \) and

\[
A_L(\Gamma_L) = L^{1/2} A(\Gamma) \tag{213}
\]

For the general case we need some definitions. Recall that \( \tilde{Q}_j \) maps from functions on \( T_{N+M-j}^- \) to \( T_{N+M-j}^0 \). For \( j \leq k \) we also consider the scaled version \( \sigma_{(k-j)} \tilde{Q}_j \sigma_{k-j} \) which is a map from functions on \( T_{N+M-k}^- \) to \( T_{N+M-k}^{-(k-j)} \) and is also denoted \( \tilde{Q}_j \). Also given \( y \in T_{N+M-k}^0 \) and \( x \in T_{N+M-k}^k \) with
\(|x - y| \leq 1/2\) we define a sequence of points \(x_0 = x, x_1, \ldots, x_k = y\) where \(x_j\) is the point in \(\mathbb{T}^{-(k-j)}_{N+M-k}\) such that \(|x - x_j| < L^{-(k-j)}/2\). Finally let \(\Gamma_{x_j+1,x_j}\) be the standard contour in \(\mathbb{T}^{-(k-j)}_{N+M-k}\) taking \(x_j\) to \(x_{j+1}\).

Lemma 6

\[
(Q_k(a_{k-1}, \ldots, a_0)\psi)(y) = \int_{|x-y| < 1/2} \exp \left( i e_k \sum_{j=0}^{k-1} (\hat{Q}_j a_j)(\Gamma_{x_{j+1},x_j}) \right) \psi(x)
\]  

(214)

Proof. By induction. We know it is true for \(k = 1\) and assuming it is true for \(k\) we compute

\[
(Q_{k+1}(a_k, \ldots, a_0)\psi)(y') = \int_{|x-Ly'| < L/2} \exp(i e_k(\hat{Q}_k a_k,L)(\Gamma_{Ly',x_k})) \exp \left( i e_k \sum_{j=0}^{k-1} (\hat{Q}_j a_j)(\Gamma_{x_{j+1},x_j}) \right) \psi(x/L)
\]

(215)

Now make the change of variables \(x = Lx'\). Then \(x_j = Lx'_j\) and we obtain

\[
(Q_{k+1}(a_k, \ldots, a_0)\psi)(y') = \int_{|x'-y'| < 1/2} \exp \left( i e_k(\hat{Q}_k a_k,L)(\Gamma_{Lx',x_k}) + i e_k \sum_{j=0}^{k-1} (\hat{Q}_j a_j)(\Gamma_{Lx'_{j+1},Lx'_j}) \right) \psi(x')
\]

(216)

with \(x_{j+1}' = y'\). Now we use (213) and \(e_{k+1} = L^{1/2}e_k\) to write this as

\[
(Q_{k+1}(a_k, \ldots, a_0)\psi)(y') = \int_{|x'-y'| < 1/2} \exp \left( i e_{k+1} \sum_{j=0}^{k} (\hat{Q}_j a_j)(\Gamma_{x'_{j+1},x'_j}) \right) \psi(x')
\]

(217)

which is the statement for \(k + 1\).

B  Perturbation identities

Our goal is to prove the identity (21). We start with the recursion relation (25) for \(\rho_k^*(t, \Psi_k, A_k)\) and introduce under the integral sign the characteristic function

\[
\chi \left( \frac{A_k}{p(t e_k)} \right) = \prod_{x, \mu} \chi \left( \frac{A_k, \mu(x)}{p(t e_k)} \right)
\]

(218)

Here \(\chi\) is a smooth functions satisfying \(\chi = 1\) on \([-1/2, 1/2]\) and \(\chi = 0\) outside \([-1,1]\), and \(p(e) = |\log(e^{-1})|^p\). This is a continuous function for \(t \geq 0\) if we set \(\chi(A_{\mu}(x)/p(t e_k))|_{t=0} = 1\). The new recursion relation defines new functions which we call \(\sigma_k^*(t, \Psi_k, A_k)\). Thus we have

\[
\sigma_{k+1}^*(t, \Psi_{k+1}, A_{k+1}) = \int d\Psi_k d\mu c_{\kappa}(A_k) \chi \left( \frac{A_k}{p(t e_k)} \right) M_{k+1,b}^{-1} \exp \left( -\frac{b}{L} |\Psi_{k+1,L} - Q_{e_k}(\hat{Q}_k(A_{k+1,L} + t A_k))\Psi_k|^2 \right) \sigma_k^*(t, \Psi_k, A_{k+1,L} + t A_k)
\]

(219)
The starting function for $k = 0$ is again $\exp(-(\Psi_0, (D_{c_0} (A_0) + m_0) \Psi_0))$. One can show inductively that $\sigma_k^* (t, \Psi_k, A_k)$ is a bounded analytic function of $A_k$ on a neighborhood of the real axis, and that it is a continuous function of $t \geq 0$, smooth for $t > 0$.

**Lemma 7** The derivatives of $\sigma_k^* (t, \Psi_k, A_k)$ at $t = 0$ exist and are equal to those of $\rho_k^* (t, \Psi_k, A_k)$.

**Proof.** (after [12]) We suppose it is true for $k$ and establish it for $k + 1$. The point is that the derivatives of $\chi(A_k/p(te_k))$ do not contribute, and we focus on this aspect.

Consider the first derivative of $\sigma_{k+1}^* (t)$. Since it is continuous at zero we can find the derivative at zero by taking the limit from positive values. The contribution from the characteristic function has the form

$$
\lim_{t \to 0^+} \int d\Psi_k \ d\mu_{C_k} (A_k) \ \frac{d}{dt} \chi \left( \frac{A_k}{p(te_k)} \right) \mathcal{M}_{k+1,b}^{-1} \exp \left( -\frac{b}{L} |\Psi_{k+1,L} - Q_{c_k} (\tilde{Q}_k(A_{k+1,L} + tA_k)) |\sigma_k^* (t, \Psi_k, A_{k+1,L} + tA_k) \right)
$$

We denote this limit by $f(t, A_k)$. Now we can prove (81) which we repeat:

$$
\sigma_k^* (t, \Psi_k, A_k) = Z_k(A_k) \exp(-(\Psi_k, (\nabla_{\Psi_k} A_k) \Psi_k)) \sigma_k^* (t, \Psi_k, \psi_k(A_k), A_k)
$$

Then $\sigma_k^* (t, \Psi_k, \psi_k(A_k), A_k)$ and $\rho_k^* (t, \Psi_k, \psi_k(A_k), A_k)$ have the same derivatives at $t = 0$. In particular $(\rho_k^*)''(0)/2 = (\rho_k^*)''(0)/2 = P_k$.

Now we can prove [31] which we repeat:

**Lemma 8** Let $\psi_{k+1} | \partial A_{k+1}$ be sufficiently small. Then

$$
P_{k+1}^+ (\Psi_{k+1}, \psi_{k+1}(A_{k+1}), A_{k+1})
$$

$$
=[P_{k+1}^+ (\Psi(A), [\psi_{k+1} (A_{k+1})]_L, A) - (\Delta_{\Psi_k} P_k)^+ (\Psi(A), [\psi_{k+1} (A_{k+1})]_L, A)]
$$

$$
+ \frac{1}{2} \int_{z,w} \mathcal{J}_k(z, w) \mathcal{J}_k(z, w) - \frac{1}{2} \int_{z,w} \mathcal{J}_k(z, w) \tilde{C}_k(z, w)
$$

$$
- \int_{z,w} \sum_{x,y} \tilde{K}_k(z, x) \Gamma_k(A; x, y) K_k(w, y) \tilde{C}_k(z, w) \bigg]_{\Psi = \psi_{k+1,L}, \ A = A_{k+1,L}}
$$

35
Proof. In (219) insert the expression (223) for $\sigma^\ast_k(t, \Psi_k, A_{k+1,L} + tA_k)$. This representation is possible because $e_k |\partial (A_{k+1,L} + tA_k)|$ is small. The first term is small by our assumption and the second term is small by the characteristic function since

$$te_k |\partial A_k| = te_k |\partial H_k A_k| \leq O(te_k) \sup |A_k| \leq O(te_k p(te_k)) \quad (225)$$

Next we carry out the steps in section 1.6. These were formal for $\rho^\ast_k$ but are now rigorous. This yields instead of (29)

$$\sigma^\ast_{k+1}(t, \Psi_{k+1}, \psi_{k+1}(A_{k+1}), A_{k+1})$$

$$= \int d\mu_{\Gamma_k(A)}(\Psi_k) \ d\mu_{G_k}(A_k) \chi \left( \frac{A_k}{p(te_k)} \right) \exp \left( -V_k(\Psi, \Psi(A) + \Psi_k, A, tA_k) - U_k(A, tA_k) \right)$$

$$\sigma^\ast_k(t, \Psi(A) + \Psi_k, [\psi_{k+1}(A_{k+1})]_L + \psi_k(A) + \delta H_k(A, tA_k)(\Psi(A) + \Psi_k), A + tA_k) |_{\Psi = \Psi_{k+1,L}}$$

Now take two derivatives at $t = 0$. The derivatives of $\chi(\frac{A_k}{p(te_k)})$ do not contribute as we have explained. The derivatives of $\sigma^\ast_k, \sigma^\ast_{k+1}$ give $P_k, P_{k+1}$, and the derivatives of $V_k$ give $J_k, K_k$. The details are explained in the text.
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