ENUMERATION OF CARLITZ MULTIPERMUTATIONS

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Abstract. A multipermutation with \( k \) copies each of \( 1 \ldots n \) is Carlitz if neighbours are different. We enumerate these objects for \( k = 2, 3, 4 \) and derive recurrences. In particular, we prove and improve a conjectured recurrence for \( k = 3 \), stated in OEIS, the Online Encyclopedia of Integer Sequences.

1. Introduction

Leonard Carlitz [1] enumerated compositions with adjacent parts being different. We will count multipermutations of \( 1^k, 2^k, \ldots, n^k \) with the same condition.

Definition 1.1. A multipermutation is Carlitz if adjacent elements are different.

For \( k = 1 \), these are just the \( n! \) ordinary permutations, but for \( k > 1 \) there are few results. OEIS has entries A114938 for \( k = 2 \), where an expression and a three-term recurrence is given, and A193638 for \( k = 3 \), but with no formula and only a conjectured recurrence.

Let \( A_k(n) \) be the set of Carlitz multipermutations of \( 1^k, 2^k, \ldots, n^k \) and let \( a_k(n) = |A_k(n)| \). The simplest examples are

\[
\begin{align*}
A_2(2) &= \{1212, 2121\}, \quad a_2(2) = 2 \\
A_2(3) &= \{12132, 12312, 12321, 12323, 12332, \ldots\}, \quad a_2(3) = 30
\end{align*}
\]

Table 1. Number of Carlitz multipermutations

| \( a_k(n) \) | \( n = 0 \) | \( n = 1 \) | \( n = 2 \) | \( n = 3 \) | \( n = 4 \) | \( n = 5 \) | \( n = 6 \) |
|-------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| \( k = 1 \) | 1         | 1         | 2         | 6         | 24        | 120       | 720       |
| \( k = 2 \) | 1         | 0         | 2         | 30        | 864       | 39 480    | 2 631 600 |
| \( k = 3 \) | 1         | 0         | 2         | 174       | 41 304    | 19 606 320 | 16 438 575 600 |

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The numbers grow very fast. An upper bound is of course $(kn)!/(k!)^n$, the number of all multipermutations.

We see that $A_2(2)$ has two elements, but only one pattern, $xyxy$. If we identify elements with the same pattern, we get a smaller set $A'_k(n)$. Every pattern may be realized in $n!$ ways as a multipermutation, so $a'_k(n) = a_k(n)/n!$ as the examples show.

$A'_2(2) = \{1212\}$, $a'_2(2) = 1$

$A'_2(3) = \{121323, 123123, 123132, 123213, 123231\}$, $a'_2(3) = 5$

As representative we choose the ordered multipermutation, where the elements appear in order. For any pattern, such as $zyzxyxyxz$, the order condition determines what numeral each letter represents, in this case $121323231$.

Sometimes, it seems more natural to work with $a'_k(n)$, sometimes $a_k(n)$ is more convenient. OEIS has entries A278990 for $a'_2(n)$, with formula and a three-term recurrence, and A190826 for $a'_3(n)$ with no formula and an only conjectured recurrence.

| $a'_k(n)$ | $n = 0$ | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ | $n = 6$ |
|-----------|---------|---------|---------|---------|---------|---------|---------|
| $k = 1$   | 1       | 1       | 1       | 1       | 1       | 1       | 1       |
| $k = 2$   | 1       | 0       | 1       | 5       | 36      | 329     | 3 655   |
| $k = 3$   | 1       | 0       | 1       | 29      | 1 721   | 163 386 | 22 831 355 |

2. Inclusion-exclusion formulas

Computing $a_2(n)$ by inclusion-exclusion is Example 2.2.3 in [3]. We show the method for $a_2(3) = 30$.

To begin with, there are $6!/2^3 = 90$ multipermutations of $1 1 2 2 3 3$. We subtract all containing the subpattern $11$, i.e. multipermutations of the five symbols $11 2 2 3 3$. These are $5!/2^2$. The same goes for $22$ and $33$ so we subtract $\binom{3}{1}5!/2^2 = 90$. Patterns with both $11$ and $22$ were subtracted twice, so we add $4!/2^1$ for every such pair, totalling $\binom{3}{2}4!/2^1 = 36$. Finally, patterns with all three $11$, $22$, $33$ must be subtracted, that is $\binom{3}{3}3!/2^0 = 6$.

The general formula looks like this.

**Proposition 2.1.**

$$a_2(n) = \sum_{s+t=n} \left( \frac{(-1)^t}{s} \binom{n}{s} \frac{(2s + t)!}{(2!)^s} \right)$$
The sum is to be taken over nonnegative $s, t$ that add up to $n$. Here $s$ counts symbols that are separate, like $..x..x..$, and $t$ counts symbols that appear together, like $..xx..$, so there are $2s + t$ blocks to permute and $s$ indistinguishable pairs.

The case $k = 3$ is trickier as we now have three subpatterns to consider. If $s$ of the symbols appear separated, like $..x..x..$, $t$ of the symbols appear two-plus-one, like $..xx..$, and $u$ of the symbols appear united, like $..xxx..$, inclusion-exclusion will produce a surprisingly simple formula. A more thorough treatment is given in Martin's thesis [7].

**Theorem 2.2.**

$$a_3(n) = \sum_{s + t + u = n} \left[ (-1)^t \binom{n}{s, t, u} \frac{(3s + 2t + u)!}{(3!)^s (2!)^t} \right]$$

**Proof.** A direct application of inclusion-exclusion would be possible if we knew how many multipermutations contain $11$, how many contain $12$ etc. The $t = 2, u = 0$ counts permutations of blocks, some of length 1 and some of length 2, for example $11$ and $22$. This will produce all desired multipermutations, but some of them will be counted twice, for $111$ is the same sequence as $111$. So we must subtract permutations where the ones are united, and this explains the term $t = 1, u = 1$. But now again we must add permutations with both $111$ and $222$ and this explains the term $t = 0, u = 2$. □

Let us try to compute $a_3(3) = 174$ with the formula.

$$1 \cdot \frac{9!}{6^3} - 3 \cdot \frac{8!}{6^2} + 3 \cdot \frac{7!}{6} - 6 \cdot \frac{6!}{6} + 3 \cdot \frac{5!}{6} - 1 \cdot \frac{6!}{1} + 3 \cdot \frac{5!}{1} - 3 \cdot \frac{4!}{1} + 1 \cdot \frac{3!}{1} = 174$$

It is easy to write down similar formulas for $k \geq 4$. We just give $k = 4$ as an example. The proof has no new twists, so we omit it. Just note that $v$ and $w$ count $xx..xx$ resp. $xxxx$.

**Theorem 2.3.**

$$a_4(n) = \sum_{s + t + u + v + w = n} \left[ (-1)^t+w \binom{n}{s, t, u, v, w} \frac{(4s + 3t + 2u + 2v + w)!}{(4!)^s(2!)^{v+t}} \right]$$

We were able to give each term a combinatorial interpretation but the formulas are not new. Ira Gessel [2] used rook polynomials to derive more general expressions than these and Jair Taylor [4] proved the same formulas directly from the generating function. Their elegant version of Th[2,3] is

$$a_4(n) = \Phi((\frac{t^3}{6} - t^2 + t)^n)$$
where \( \Phi(t^n) = n! \), so after expansion each power of \( t \) is replaced with a factorial.

### 3. Recurrences

For many purposes, recurrences are superior to the explicit formulas of last section. We will show how to get recurrences for \( a'_{k}(n) \), the number of ordered Carlitz multipermutations. Recall that \( a'_{k}(n) = a_{k}(n)/n! \).

The OEIS \([5]\) gives conjectural three-term recurrences for \( a'_{2}(n) \) and \( a'_{2}(n) \), a four-term recurrence for \( a_{3}(n) \) and a five-term recurrence for \( a'_{3}(n) \). All these conjectures will be proved below.

**Theorem 3.1.** The sequence \( p_{n} \), recursively defined by

\[
p_{n+1} = (2n + 1)p_{n} + p_{n-1}, \quad p_{0} = 1, \quad p_{1} = 0,
\]

counts ordered Carlitz words of \( 1^2, \ldots, n^2 \).

**Proof.** As \( p_{n} = a'_{2}(n) \), \( p_{2} = 1 \) counts the word 1212 and \( p_{3} = 5 \) counts the words 010212, 012012, 012102, 012120, 012021, using symbols 012. The first four words are of the type \( 0..\hat{0}. \), that is the zero may be removed without violating the Carlitz property, but the fifth word is of the type \( 0..x0x.. \).

Now, we count words in \( 0^2, 1^2, \ldots, n^2 \). according to type.

- \( 0..\hat{0}. \) is counted by \( 2np_{n} \) (insert \( \hat{0} \) anywhere).
- \( 0..x0x \) for \( x > 1 \) is counted by \( p_{n} \) (transform \( 1..1 \rightarrow 0..x0x \)).
- \( 0101.. \) is counted by \( p_{n-1} \) (prefix 0101).

In our example, 1212 \( \rightarrow \) 02x0x2, which is the same pattern as 012021.

**Theorem 3.2.** The sequences \( p_{n}, q_{n} \), recursively defined by

\[
2p_{n+1} = (3n + 3)q_{n} - 2(3n + 1)p_{n} + 2p_{n-1}, \quad p_{0} = 1, \quad p_{1} = 0,
\]

\[
q_{n} = (3n + 2)p_{n} + 2q_{n-1}, \quad q_{0} = 0,
\]

count ordered Carlitz words of \( 1^3, \ldots, n^3 \) resp. of \( 0^2, 1^3, \ldots, n^3 \).

**Proof.** \( p_{2} = 1 \) counts the word 121212 and \( q_{2} = 8 \) counts the words 01\( \hat{0} \)21212, \ldots, 0121212\( \hat{0} \), 01202121, 01212021. The first six words of the type \( 0..\hat{0}. \) are counted by \( 3np_{n} \), the last two \( 0..x0x..x. \) and \( 0..x..x0x. \) with \( x > 1 \) by \( 2p_{n} \). Finally, \( 0101..1. \) and \( 01..101. \) are counted by \( 2q_{n-1} \). This proves the recurrence for \( q_{n} \).

We now count \( p_{n+1} \) by cases according to type of \( 0. \) As there are two noninitial zeros, the cases will sum to \( 2p_{n+1} \).

- \( 0..0..\hat{0}. \) is counted by \( (3n - 1)q_{n} \) (insert \( \hat{0} \) in empty slot).
0..0..x..x0x. for \( x > 1 \) is counted by \( 2(q_n - p_n - q_{n-1}) \), for our transformation 1..0..1..1. \( \rightarrow 0..0..x..x0x. \) does not work for 101..1. (counted by \( p_n \)) or for 10..1..1. (counted by \( q_{n-1} \)).

0101..1.0 and the equinumerous 011..011..0 split into subcases depending on the position of the other zero.

010101.. is counted by \( p_{n-1} \).

0101..1., 0101..01. and 0101..10. are counted by \( 3q_{n-1} \).

0101..0..1. and 0101..1..0. are counted by \( 2p_n \).

Collecting terms and replacing \( 2q_{n-1} \) with \( q_n - (3n + 2)p_n \) we get the recurrence for \( p_{n+1} \).

\[ p_{n+1} = \lambda p_n + \mu p_{n-1} + \nu p_{n-2}, \quad p_0 = 1, \quad p_1 = 0, \quad p_2 = 1 \]

where \( \lambda = (9n^2 + 9n + 8)/2 + 2/n, \mu = (6n + 3) - 4/n, \nu = -2 - 2/n \) counts ordered Carlitz words of \( 1^3, \ldots, n^3 \)

Proof. Lowering indices in Th 3.2 we get

\[ 2p_n = 3nq_{n-1} - 2(3n - 2)p_{n-1} + 2p_{n-2} \]

Adding \(-2 - \frac{2}{n}\) times this to the \( 2p_{n+1}\)-recurrence and then using the \( q_n\)-recurrence, we get the desired four-term recurrence. \( \square \)

The five-term recurrence in OEIS entry A190826 was found by Richard J. Mathar using an ansatz with twenty unknown coefficients [6]. It is of course easily derived by adding two versions of our four-term recurrence, one of them with lowered indices.

The four-term recurrence in OEIS entry A193638 was found by Alois P. Heintz. It is now a corollary obtained by multiplication with \((n+1)!\). Recurrences for \( a'_k(n) \) with \( k > 3 \) may be derived in exactly the same way. We state the result for \( k = 4 \) here and leave the details to the reader.

**Theorem 3.4.** The sequences \( p_n, q_n, r_n \), recursively defined by

\[
egin{align*}
3p_{n+1} &= (4n + 1)q_n + 3(10q_{n-1} - r_n + 4r_{n-1} + (6n + 7)p_n + p_{n-1}) \\
2q_n &= (4n + 6)r_n + 6r_{n-1} - (16n + 6)p_n \\
r_n &= (4n + 3)p_n + 3q_{n-1}, \quad p_0 = 1, \quad p_1 = 0, \quad q_0 = 0, \quad r_0 = 0
\end{align*}
\]

count ordered Carlitz words of \( 1^4, \ldots, n^4 \) resp. of \( 0^3, 1^4, \ldots, n^4 \), and of \( 0^2, 1^4, \ldots, n^4 \).
References

[1] L. Carlitz, Restricted compositions, *Fibonacci Quart.* **14** (1976), 254–264.
[2] Ira M. Gessel, Generalized rook polynomials and orthogonal polynomials. In D. Stanton, editor, *q-Series and Partitions*, pages 159-176. Springer-Verlag, New York, 1989.
[3] R. Stanley, *Enumerative Combinatorics*, vol. 1, Cambridge University Press, Cambridge, 1997.
[4] Jair Taylor, Counting words with Laguerre series, *Electron. J. Comb.* **21(2)**, 2014
[5] The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org
[6] R. J. Mathar, personal communication, 2015-10-30.
[7] Alexis Martin, Sequences without equal adjacent elements, Bachelor thesis, 2015.

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