EQUIVARIANT EMBEDDINGS OF COMMUTATIVE LINEAR ALGEBRAIC GROUPS OF CORANK ONE

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Abstract. Let \( K \) be an algebraically closed field of characteristic zero, \( \mathbb{G}_m = (K \setminus \{0\}, \times) \) be its multiplicative group, and \( \mathbb{G}_a = (K, +) \) be its additive group. Consider a commutative linear algebraic group \( G = (\mathbb{G}_m)^r \times \mathbb{G}_a \). We study equivariant \( G \)-embeddings, i.e. normal \( G \)-varieties \( X \) containing \( G \) as an open orbit. We prove that \( X \) is a toric variety and all such actions of \( G \) on \( X \) correspond to Demazure roots of the fan of \( X \). In these terms, the orbit structure of a \( G \)-variety \( X \) is described.

1. Introduction

Let \( K \) be an algebraically closed field of characteristic zero, \( \mathbb{G}_m = (K \setminus \{0\}, \times) \) be its multiplicative group, and \( \mathbb{G}_a = (K, +) \) be its additive group. It is well known that any connected commutative linear algebraic group \( G \) over \( K \) is isomorphic to \( (\mathbb{G}_m)^r \times (\mathbb{G}_a)^s \) with some non-negative integers \( r \) and \( s \), see [19, Theorem 15.5]. We say that \( r \) is the rank of the group \( G \) and \( s \) is the corank of \( G \).

The aim of this paper is to study equivariant embeddings of commutative linear algebraic groups. Let us recall that an equivariant embedding of an algebraic group \( G \) is a pair \( (X, x) \), where \( X \) is an algebraic variety equipped with a regular action \( G \times X \to X \) and \( x \in X \) is a point with the trivial stabilizer such that the orbit \( Gx \) is open and dense in \( X \). We assume that the variety \( X \) is normal. If \( X \) is supposed to be complete, we speak about equivariant compactifications of \( G \). For the study of compactifications of reductive groups, see e.g. [25]. More generally, equivariant embeddings of homogeneous spaces of reductive groups is a popular object starting from early 1970th. Recent survey of results in this field may be found in [26].

Let us return to the case \( G = (\mathbb{G}_m)^r \times (\mathbb{G}_a)^s \). If \( s = 0 \) then \( G \) is a torus and we come to the famous theory of toric varieties, see [13], [22], [17], [12]. Another extreme \( r = 0 \) corresponds to embeddings of a commutative unipotent (=vector) group. This case is also studied actively during last decades, see e.g. [18], [8], [3], [16], [14]. The next natural step is to study the mixed case \( r > 0 \) and \( s > 0 \) and to combine advantages of both torus and additive group actions.

The present paper deals with the case \( s = 1 \), i.e. from now on \( G \) is a connected commutative linear algebraic group of corank one. In other words, \( G = (\mathbb{G}_m)^{n-1} \times \mathbb{G}_a \), where \( n = \dim X \).

Let \( X \) be a toric variety with the acting torus \( T \). Consider an action \( \mathbb{G}_a \times X \to X \) normalized by \( T \). Then \( T \) acts on \( \mathbb{G}_a \) by conjugation with some character \( e \). Such a character is called a Demazure root of \( X \). If \( T = \text{Ker}(e) \), then the group \( G := T \times \mathbb{G}_a \) acts on \( X \) with an open orbit, and \( X \) is a \( G \)-embedding, see Proposition 6. Our main result (Theorem 2) states that all \( G \)-embeddings can be realized this way. To this end we prove...
that for any \( G \)-embedding \( X \) the \((G_a)^{n-1}\)-action on \( X \) can be extended to an action of a bigger torus \( T \) which normalizes the \( G_a \)-action and \( X \) is toric with respect to \( T \).

This result can not be generalized to groups of corank two; examples of non-toric surfaces which are equivariant compactifications of \( \mathbb{G}_a \) can be found in [14]. Similar examples are constructed in [14], [15] for semidirect products \( \mathbb{G}_m \triangleleft \mathbb{G}_a \). Such groups can be considered as non-commutative groups of corank one.

If two toric varieties are isomorphic as abstract varieties, then they are isomorphic as toric varieties [10, Theorem 4.1]. This shows that the structure of a torus embedding on a toric variety is unique up to isomorphism. A structure of a \( G \)-embedding on a given variety may be non-unique, see Examples 2, 4. Such structures are given by Demazure roots and thus the number of structures is finite if \( X \) is complete, and it is at most countable for arbitrary \( X \). At the same time, \( \mathbb{G}_a^6 \)-embeddings into \( \mathbb{P}^6 \) admit a non-trivial moduli space [18, Example 3.6].

The paper is organized as follows. Section 2 contains preliminaries on torus actions on affine varieties. We recall basic facts on affine toric varieties and introduce a description of affine \( T \)-varieties in terms of proper polyhedral divisors due to Altmann and Hausen [1]. A correspondence between \( \mathbb{G}_a \)-actions on \( X \) normalized by \( T \) and homogeneous locally nilpotent derivations (LNDs) of the algebra \( \mathbb{K}[X] \) is explained. We define Demazure roots of a cone and use them to describe homogeneous LNDs on \( \mathbb{K}[X] \), where \( X \) is toric. Also we give a description of homogeneous LNDs of horizontal type on algebras with grading of complexity one obtained by Liendo [21].

In Section 3 we show that if \( X \) is a normal affine \( T \)-variety of complexity one and the algebra \( \mathbb{K}[X] \) admits a homogeneous LND of degree zero, then \( X \) is toric with an acting torus \( T \), \( T \) is a subtorus of \( T \), and \( T \) normalizes the corresponding \( \mathbb{G}_a \)-action. This gives the result for affine \( G \)-embeddings. Moreover, Proposition 3 provides an explicit description of affine \( G \)-embeddings.

Section 4 deals with compactifications of \( G \). Here we use the Cox construction and a lifting of the action of \( G \) to the total coordinate space \( \overline{X} \) of \( X \) to deduce the result from the affine case.

In Section 5 we recall basic facts on toric varieties and introduce the notion of a Demazure root of a fan following Demazure [13]. The action of the corresponding one-parameter subgroup on the toric variety is also described there.

Let \( \Sigma \) be a fan and \( e \) be a Demazure root of \( \Sigma \). In Section 6 we define a \( G \)-embedding associated to the pair \((\Sigma, e)\) and study the \( G \)-orbit structure of \( X \). It turns out that the number of \( G \)-orbits on \( X \) is finite.

Finally, in Section 7 we prove that any \( G \)-embedding is associated with some pair \((\Sigma, e)\). The idea is to reduce the general case to the complete one via equivariant compactification. At the end several explicit examples of \( G \)-embeddings are given.

Some results of this paper appeared in preprint [5]. They form a part of the Ph.D. thesis of the second author [20].

2. \( \mathbb{G}_a \)-actions on affine \( T \)-varieties

Let \( X \) be an irreducible affine variety with an effective action of an algebraic torus \( T \), \( M \) be the character lattice of \( T \), \( N \) be the lattice of one-parameter subgroups of \( T \), and \( A = \mathbb{K}[X] \) be the algebra of regular functions on \( X \). It is well known that there is a bijective correspondence between effective \( T \)-actions on \( X \) and effective \( M \)-gradings on \( A \). In fact, the algebra \( A \) is graded by a semigroup of lattice points in some convex polyhedral cone.
where \( \omega_M = \omega \cap M \) and \( \chi^m \) is the character corresponding to \( m \).

A derivation \( \partial \) on an algebra \( A \) is said to be \textit{locally nilpotent} (LND) if for each \( f \in A \) there exists \( n \in \mathbb{N} \) such that \( \partial^n(f) = 0 \). For any LND \( \partial \) on \( A \) the map \( \varphi_\partial : \mathbb{G}_a \times A \to A \), \( \varphi_\partial(s, f) = \exp(s\partial)(f) \), defines a structure of a rational \( \mathbb{G}_a \)-algebra on \( A \). In fact, any regular \( \mathbb{G}_a \)-action on \( X = \text{Spec} A \) arises this way. A derivation \( \partial \) on \( A \) is said to be \textit{homogeneous} if it respects the \( M \)-grading. If \( f, h \in A \setminus \ker \partial \) are homogeneous, then \( \partial(fh) = f\partial(h) + \partial(f)h \) is homogeneous too and \( \deg \partial(f) - \deg f = \deg \partial(h) - \deg h \). So any homogeneous derivation \( \partial \) has a well defined \textit{degree} given as \( \deg \partial = \deg \partial(f) - \deg f \) for any homogeneous \( f \in A \setminus \ker \partial \). It is easy to see that an LND on \( A \) is homogeneous if and only if the corresponding \( \mathbb{G}_a \)-action on \( X \) is normalized by the torus \( T \) in the automorphism group \( \text{Aut}(X) \).

Any derivation on \( \mathbb{K}[X] \) extends to a derivation on the field of fractions \( \mathbb{K}(X) \) by the Leibniz rule. A homogeneous LND \( \partial \) on \( \mathbb{K}[X] \) is said to be of \textit{fiber type} if \( \partial(\mathbb{K}(X)^T) = 0 \) and of \textit{horizontal type} otherwise. In other words, \( \partial \) is of fiber type if and only if the general orbits of corresponding \( \mathbb{G}_a \)-action on \( X \) are contained in the closures of \( T \)-orbits.

Let \( X \) be an affine toric variety, i.e., a normal affine variety with a generically transitive action of a torus \( T \). In this case

\[
A = \bigoplus_{m \in \omega_M} \mathbb{K} \chi^m = \mathbb{K}[\omega_M]
\]

is the semigroup algebra. Recall that for given cone \( \omega \subset M_Q \), its \textit{dual cone} is defined by

\[
\sigma = \{ n \in N_Q \mid \langle n, p \rangle \geq 0 \; \forall p \in \omega \},
\]

where \( \langle \cdot, \cdot \rangle \) is the pairing between dual lattices \( N \) and \( M \). Let \( \sigma(1) \) be the set of rays of a cone \( \sigma \) and \( n_\rho \) be the primitive lattice vector on the ray \( \rho \). For \( \rho \in \sigma(1) \) we set

\[
S_\rho := \{ e \in M \mid \langle n_\rho, e \rangle = -1 \text{ and } \langle n_\rho', e \rangle \geq 0 \; \forall \rho' \in \sigma(1), \; \rho' \neq \rho \}.
\]

One easily checks that the set \( S_\rho \) is infinite for each \( \rho \in \sigma(1) \). The elements of the set \( \mathcal{R} := \bigsqcup_\rho S_\rho \) are called the \textit{Demazure roots} of \( \sigma \). Let \( e \in S_\rho \). Then \( \rho \) is called the \textit{distinguished ray} of the root \( e \). One can define the homogeneous LND on the algebra \( A \) by the rule

\[
\partial_e(\chi^m) = \langle n_\rho, m \rangle \chi^{m+e}.
\]

In fact, every homogeneous LND on \( A \) has a form \( \alpha \partial_e \) for some \( \alpha \in \mathbb{K}, \; e \in \mathcal{R} \), see [21, Theorem 2.7]. In other words, \( \mathbb{G}_a \)-actions on \( X \) normalized by the acting torus are in bijection with Demazure roots of the cone \( \sigma \).

Clearly, all homogeneous LNDs on a toric variety are of fiber type.

**Example 1.** Consider \( X = \mathbb{A}^k \) with the standard action of the torus \( (\mathbb{K}^*)^k \). It is a toric variety with the cone \( \sigma = Q_{\geq 0}^k \) having rays \( \rho_1 = \langle (1, 0, \ldots, 0) \rangle_{Q_{\geq 0}}, \ldots, \rho_k = \langle (0, 0, \ldots, 0, 1) \rangle_{Q_{\geq 0}} \). The dual cone \( \omega \) is \( Q_{\geq 0}^k \) as well. In this case

\[
S_{\rho_i} = \{ (c_1, \ldots, c_{i-1}, -1, c_{i+1}, \ldots, c_k) \mid c_j \in \mathbb{Z}_{\geq 0} \}.
\]
Theorem 1. Let \( C \) be a smooth curve and \( \mathcal{D} \) a proper \( \sigma \)-polyhedral divisor on \( C \). Then the \( M \)-graded algebra \( A[C, \mathcal{D}] \) is a normal finitely generated effectively graded \((\text{rk } M + 1)\)-dimensional domain. Conversely, for each normal finitely generated domain \( A \) with a grading of complexity one there exist a smooth curve \( C \) and a proper \( \sigma \)-polyhedral divisor \( \mathcal{D} \) on \( C \) such that \( A \) is isomorphic to \( A[C, \mathcal{D}] \).
(2) The $M$-graded domains $\text{Spec} A[C, \mathfrak{D}]$ and $\text{Spec} A[C, \mathfrak{D}']$ are isomorphic if and only if for every $z \in C$ there exists a lattice vector $v_z \in N$ such that

$$\mathfrak{D} = \mathfrak{D}' + \sum_z (v_z + \sigma) \cdot z,$$

and for all $m \in \omega_M$ the divisor $\sum_z \langle v_z, m \rangle \cdot z$ is principal.

The following result is obtained in [1, Section 11].

**Proposition 1.** Let $\mathfrak{D}$ be a proper $\sigma$-polyhedral divisor on a smooth curve $C$, $X = \text{Spec} A[C, \mathfrak{D}]$, and $T \times X \to X$ be the corresponding torus action. Then this action can be realized as a subtorus action on a toric variety if and only if either $C = \mathbb{A}^1$ and $\mathfrak{D}$ can be chosen supported in at most one point, or $C = \mathbb{P}^1$ and $\mathfrak{D}$ can be chosen supported in at most two points.

Also we need a description of homogeneous LNDs of horizontal type for a $T$-variety $X$ of complexity one from [21]. Below we follow the approach given in [7]. We have $\mathbb{K}[X] = A[C, \mathfrak{D}]$ for some $C$ and $\mathfrak{D}$. It turns out that $C$ is isomorphic to $\mathbb{A}^1$ or $\mathbb{P}^1$ whenever there exists a homogeneous LND of horizontal type on $A[C, \mathfrak{D}]$, see [21, Lemma 3.15].

Let $C$ be $\mathbb{A}^1$ or $\mathbb{P}^1$, $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ a $\sigma$-polyhedral divisor on $C$, $z_0 \in C$, $z_\infty \in C \setminus \{z_0\}$, and $v_z$ a vertex of $\Delta_z$ for every $z \in C$. Put $C' = C$ if $C = \mathbb{A}^1$ and $C' = C \setminus \{z_\infty\}$ if $C = \mathbb{P}^1$. A collection $\mathfrak{D} = \{\mathfrak{D}, z_0; v_z, \forall z \in C\}$ if $C = \mathbb{A}^1$ and $\mathfrak{D} = \{\mathfrak{D}, z_0, z_\infty; v_z, \forall z \in C'\}$ if $C = \mathbb{P}^1$ is called a colored $\sigma$-polyhedral divisor on $C$ if the following conditions hold:

**(1)** $\forall z \in C' \exists v_z$ is a vertex of $\deg \mathfrak{D}|_{C'} := \sum_{z \in C'} \Delta_z$;

**(2)** $v_Z \in N$ for all $z \in C'$, $z \neq z_0$.

Let $\mathfrak{D}$ be a colored $\sigma$-polyhedral divisor on $C$ and $\delta \subseteq N_\mathbb{Q}$ be the cone generated by $\deg \mathfrak{D}|_{C'} - v_{\deg}$. Denote by $\mathfrak{D} \subseteq (N \oplus \mathbb{Z})_\mathbb{Q}$ the cone generated by $(\delta, 0)$ and $(v_{z_0}, 1)$ if $C = \mathbb{A}^1$, and by $(\delta, 0)$, $(v_{z_0}, 1)$ and $(\Delta_{z_\infty} + v_{\deg} - v_{z_0} + \delta, -1)$ if $C = \mathbb{P}^1$. By definition, put $d$ the minimal positive integer such that $d \cdot v_{z_0} \in N$. A pair $(\mathfrak{D}, e)$, where $e \in M$, is said to be coherent if

(i) there exists $s \in Z$ such that $\tilde{e} = (e, s) \in M \oplus \mathbb{Z}$ is a Demazure root of the cone $\tilde{\sigma}$ with distinguished ray $\tilde{\rho} = (d \cdot v_{z_0}, d)$;

(ii) $\langle v, e \rangle \geq 1 + \langle v_z, e \rangle$ for all $z \in C' \setminus \{z_0\}$ and all vertices $v \neq v_z$ of the polyhedron $\Delta_{z_\infty}$;

(iii) $d \cdot \langle v, e \rangle \geq 1 + \langle v_{z_0}, e \rangle$ for all vertices $v \neq v_{z_0}$ of the polyhedron $\Delta_{z_\infty}$;

(iv) if $Y = \mathbb{P}^1$, then $d \cdot \langle v, e \rangle \geq -1 - d \cdot \sum_{z \in Y} \langle v_z, e \rangle$ for all vertices $v$ of the polyhedron $\Delta_{z_\infty}$.

It follows from [7, Theorem 1.10] that homogeneous LNDs of horizontal type on $A[C, \mathfrak{D}]$ are in bijection with the coherent pairs $(\mathfrak{D}, e)$. Namely, let $(\mathfrak{D}, e)$ be a coherent pair. Without loss of generality we may assume that $z_0 = 0$, $z_\infty = \infty$ if $C = \mathbb{P}^1$, and $v_z = 0 \in N$ for all $z \in C' \setminus \{z_0\}$. Let $\mathbb{K}(C) = \mathbb{K}(t)$. Then the homogeneous LND of horizontal type corresponding to $(\mathfrak{D}, e)$ is given by

$$\partial(\chi^m \cdot t^r) = d(\langle v_0, m \rangle + r)\chi^{m+e} \cdot t^{r+s} \quad \text{for all } m \in M, r \in \mathbb{Z}. \quad (1)$$

In particular, the vector $e$ is the degree of the derivation $\partial$. 

3. The affine case

Let \((X,x)\) be an equivariant embedding of the group \(G = (\mathbb{G}_m)^{n-1} \times \mathbb{G}_a\), where \(n = \dim X\). In this section we assume that \(X\) is normal and affine. Let us denote the subgroup \((\mathbb{G}_m)^{n-1}\) of \(G\) by \(T\). Since the action of \(T\) on \(X\) is effective, it has complexity one and defines an effective grading of the algebra \(\mathbb{K}[X]\) by the lattice \(M\). In particular, the graded algebra \(\mathbb{K}[X]\) has the form \(A[C,D]\) for some smooth curve \(C\) and some proper \(\sigma\)-polyhedral divisor on \(C\), where \(\sigma\) is a cone in \(N_\mathbb{Q}\).

Since the action of the subgroup \(\mathbb{G}_a\) commutes with \(T\)-action on \(X\), the corresponding homogeneous LND on \(\mathbb{K}[X]\) has degree zero. Moreover, the group \(G\) acts on \(X\) with an open orbit. It implies that the \(G_a\)-action on \(X\) is of horizontal type, and hence either \(C = \mathbb{A}^1\) or \(C = \mathbb{P}^1\).

**Proposition 2.** Let \(X = \text{Spec} A[C,D]\) be a \(T\)-variety of complexity one. Suppose that there exists a homogeneous LND of horizontal type and of degree zero on \(A[C,D]\). Then

1. if \(C = \mathbb{A}^1\), then one can assume (via Theorem 1) that \(D\) is a trivial \(\sigma\)-polyhedral divisor;
2. if \(C = \mathbb{P}^1\), then one can choose \(D = \Delta_\infty \cdot [\infty]\), where \(\Delta_\infty \not\subseteq \sigma\) is some \(\sigma\)-tailed polyhedron.

**Proof.** Let \((\mathfrak{D},0)\) be the coherent pair corresponding to the homogeneous LND of horizontal type. Without loss of generality we may assume that \(z_0 = 0\) and \(z_\infty = \infty\) if \(C = \mathbb{P}^1\). By definition of a coherent pair, there exists \(s \in \mathbb{Z}\) such that \((0,s)\) is a Demazure root of the cone \(\mathfrak{D}\) with distinguished ray \((dv_0,d)\). It implies that \(s = -1\), \(d = 1\), and hence \(v_0 \in \mathbb{N}\). Further, the inequality \(\langle v,0 \rangle \geq 1 + \langle v_0,0 \rangle\) should be satisfied for every \(z \in C'\) and every vertex \(v \neq v_z\) of \(\Delta_z\). It means that each polyhedron \(\Delta_z\), where \(z \in C'\), has only one vertex \(v_z\). Replacing \(\sigma\)-polyhedral divisor \(D\) with \(D' = D + \sum_{z \in C'}(-v_z + \sigma) \cdot z\) and using Theorem 1, we obtain the assertion. The condition \(\Delta_\infty \not\subseteq \sigma\) follows from the fact that \(D\) is a proper \(\sigma\)-polyhedral divisor. \(\square\)

**Corollary 1.** Under the conditions of Proposition 2 the variety \(X\) is toric with \(T\) being a subtorus of the action torus \(T\).

**Proof.** It follows immediately from Propositions 1 and 2. \(\square\)

The next proposition is a specification of Corollary 1. In particular, it shows that the \(G_a\)-action on \(X\) is normalized by the acting torus \(T\).

**Proposition 3.** Under the conditions of Proposition 2,

1. if \(C = \mathbb{A}^1\), then \(X \cong Y \times \mathbb{A}^1\), where \(Y\) is the toric variety corresponding to the cone \(\sigma\) and \(G_a\) acts on \(\mathbb{A}^1\) by translations;
2. if \(C = \mathbb{P}^1\), then \(X\) is the toric variety with the cone \(\mathfrak{D} \subseteq N \oplus \mathbb{Z}\) generated by \((\sigma,0), (\Delta_\infty, -1)\) and \((0,1)\). The \(G_a\)-action on \(X\) is given by Demazure root \(\tilde{e} = (0,-1) \in M \oplus \mathbb{Z}\) of the cone \(\mathfrak{D}\).

**Proof.** Let \(\mathbb{K}(C) = \mathbb{K}(t)\). If \(C = \mathbb{A}^1\) then \(\mathfrak{D}\) is trivial and

\[
A[C,D] = \bigoplus_{m \in \omega_M} \mathbb{K}[t] \cdot x^m = \mathbb{K}[\omega_M] \otimes \mathbb{K}[t] = \mathbb{K}[Y] \otimes \mathbb{K}[t].
\]

Hence \(X \cong Y \times \mathbb{A}^1\). Applying formula (1), we obtain that the homogeneous LND is given by

\[
\partial(\chi^m \cdot t^r) = r \chi^m \cdot t^{r-1}
\]
for all $m \in \omega_M$ and $r \in \mathbb{Z}_{\geq 0}$. Thus $\mathbb{G}_a$ acts on $Y \times \mathbb{A}^1$ as $(y, t) \mapsto (y, t + s)$.

If $C = \mathbb{P}^1$ then $\mathcal{O} = \Delta_\infty \cdot [\infty]$ and we obtain

$$A[C, \mathcal{O}] = \bigoplus_{m \in \omega_M} \bigoplus_{r=0}^{h_{\infty}(m)} \mathbb{K}X^m \cdot t^r = \bigoplus_{(m, r) \in \tilde{\omega}_{\tilde{M}}} \mathbb{K} \chi^m \cdot t^r = \mathbb{K} [\tilde{\omega}_{\tilde{M}}],$$

where $\tilde{M} = M \oplus \mathbb{Z}$ and $\tilde{\omega} \subseteq \tilde{M}$ is the cone dual to $\tilde{\sigma}$. So we see that $A[C, \mathcal{O}]$ is a semigroup algebra and $X$ is a toric variety with the cone $\tilde{\sigma}$. In this case formula (2) gives the LND corresponding to the Demazure root $\tilde{e} = (0, -1)$. \hfill \Box

4. THE COMPLETE CASE

In this section we study equivariant compactifications of the group $G$. First we briefly recall the main ingredients of the Cox construction, see [4, Chapter 1] for more details.

Let $X$ be a normal variety with finitely generated divisor class group $\text{Cl}(X)$ and only constant invertible regular functions.

Suppose that $\text{Cl}(X)$ is free. Denote by $\text{WDiv}(X)$ the group of Weil divisors on $X$ and fix a subgroup $K \subseteq \text{WDiv}(X)$ which maps onto $\text{Cl}(X)$ isomorphically. The Cox ring of the variety $\hat{X}$ is defined as

$$R(X) = \bigoplus_{D \in K} H^0(X, D),$$

where $H^0(X, D) = \{ f \in \mathbb{K}(X) \mid \text{div} f + D \geq 0 \}$ and multiplication on homogeneous components coincides with multiplication in $\mathbb{K}(X)$ and extends to $R(X)$ by linearity.

If $\text{Cl}(X)$ has torsion, we choose finitely generated subgroup $K \subseteq \text{WDiv}(X)$ that projects to $\text{Cl}(X)$ surjectively. Denote by $K_0 \subseteq K$ the kernel of this projection. Take compatible bases $D_1, \ldots, D_s$ and $D_1^0 = d_1 D_1, \ldots, D_s^0 = d_s D_s$ in $K$ and $K_0$ respectively. Let us choose the set of rational functions $\mathcal{F} = \{ F_D \in \mathbb{K}(X)^\times : D \in K_0 \}$ such that $\text{div}(F_D) = D$ and $F_{D+D'} = F_D F_{D'}$. Suppose that $D, D' \in K$ and $D - D' \in K_0$. A map $f \mapsto F_{D-D'} f$ is an isomorphism of the vector spaces $H^0(X, D)$ and $H^0(X, D')$. The linear span of the elements $f - F_{D-D'} f$ over all $D, D'$ with $D - D' \in K_0$ and all $f \in H^0(X, D)$ is an ideal $I(K, \mathcal{F})$ of the graded ring $T_K(X) := \bigoplus_{D \in K} H^0(X, D)$. The Cox ring of the variety $X$ is given by

$$R(X) = T_K(X)/I(K, \mathcal{F}).$$

This construction does not depend on the choice of $K$ and $\mathcal{F}$, see [2, Lemma 3.1 and Proposition 3.2].

Suppose that the Cox ring $R(X)$ is finitely generated. Then $\overline{X} := \text{Spec } R(X)$ is a normal affine variety with an action of the quasitorus $H_X := \text{Spec } \mathbb{K}[\text{Cl}(X)]$. There is an open $H_X$-invariant subset $\hat{X} \subseteq \overline{X}$ such that the complement $\overline{X} \setminus \hat{X}$ is of codimension at least two in $\overline{X}$, there exists a good quotient $p_X : \hat{X} \to \hat{X}/H_X$, and the quotient space $\hat{X}/H_X$ is isomorphic to $X$. So we have the following diagram

$$\hat{X} \xrightarrow{i} \overline{X} = \text{Spec } R(X) \xrightarrow{	ext{proj}} X \xrightarrow{\text{quot}} \hat{X}/H_X = X.$$

Let us return to equivariant compactifications of $G$. 
Proposition 4. Let \( \mathbb{G} = T \times \mathbb{G}_a \) and \( X \) be a normal compactification of \( \mathbb{G} \). Then the \( T \)-action on \( X \) can be extended to an action of a bigger torus \( T' \) such that \( T' \) normalizes \( \mathbb{G}_a \) and \( X \) is a toric variety with the acting torus \( T' \).

Proof. The variety \( X \) is rational with torus action of complexity one. By [4, Theorem 4.3.1.5], the divisor class group \( \text{Cl}(X) \) and the Cox ring \( R(X) \) are finitely generated.

There exists a finite epimorphism \( \epsilon : \mathbb{G}' \to \mathbb{G} \) of connected linear algebraic groups and an action \( \mathbb{G}' \times \hat{X} \to \hat{X} \) which commutes with the quasitorus \( H_X \) and \( p_X(g' \cdot \hat{x}) = \epsilon(g') \cdot p_X(\hat{x}) \) for all \( g' \in \mathbb{G}' \) and \( \hat{x} \in \hat{X} \), see [4, Theorem 4.2.3.1]. The group \( \mathbb{G}' \) has a form \( T' \times \mathbb{G}_a \), where \( \epsilon \) defines a finite epimorphism of tori \( T' \to T' \) and is identical on \( \mathbb{G}_a \).

Since \( \hat{X} = \text{Spec} \mathbb{K}[\hat{X}] \), the action of \( \mathbb{G}' \) extends to the affine variety \( \hat{X} \). This variety is an embedding of the group \( (T' H^0_X) \times \mathbb{G}_a \). By Proposition 3, it is toric with an acting torus \( \overline{T} \) normalizing the \( \mathbb{G}_a \)-action and \( T' H^0_X \) is a subtorus of \( \overline{T} \). Since \( X \) is complete, [24, Corollary 2.5] implies that the subset \( \hat{X} \) is invariant under the torus \( \overline{T} \). By [4, Lemma 4.2.1.3], the action of \( \overline{T} \) descends to an action of the torus \( T := \overline{T} / H^0_X \) on \( X \). Here \( T \) normalizes \( \mathbb{G}_a \), its action extends the action of \( T \) on \( X \), and \( X \) is toric with respect to \( T \). □

5. Toric varieties and Demazure roots

We keep notations of Section 2. Let \( X \) be a toric variety of dimension \( n \) with an acting torus \( T \) and \( \Sigma \) be the corresponding fan of convex polyhedral cones in the space \( N_{\mathbb{Q}} \), see [17] or [12] for details.

As before, let \( \Sigma(1) \) be the set of rays of the fan \( \Sigma \) and \( n_\rho \) be the primitive lattice vector on the ray \( \rho \). For \( \rho \in \Sigma(1) \) we consider the set \( S_\rho \) of all vectors \( e \in M \) such that

1. \( \langle n_\rho, e \rangle = -1 \) and \( \langle n_\rho', e \rangle \geq 0 \) for all \( \rho' \in \sigma(1) \), \( \rho' \neq \rho \);
2. if \( \sigma \) is a cone of \( \Sigma \) and \( \langle v, e \rangle = 0 \) for all \( v \in \sigma \), then the cone generated by \( \sigma \) and \( \rho \) is in \( \Sigma \) as well.

Note that condition (1) implies condition (2) if \( \Sigma \) is a maximal fan with support \( |\Sigma| \). This is the case if \( X \) is affine or complete.

The elements of the set \( \mathcal{R} := \bigsqcup_{\rho} S_\rho \) are called the Demazure roots of the fan \( \Sigma \), cf. [13, Definition 4] and [22, Section 3.4]. Again elements \( e \in \mathcal{R} \) are in bijections with \( \mathbb{G}_a \)-actions on \( X \) normalized by the acting torus. Let us denote the corresponding one-parameter subgroup of \( \text{Aut}(X) \) by \( H_e \).

We recall basic facts from toric geometry. There is a bijection between cones \( \sigma \in \Sigma \) and \( T \)-orbits \( \mathcal{O}_\sigma \) on \( X \) such that \( \sigma_1 \subseteq \sigma_2 \) if and only if \( \mathcal{O}_{\sigma_2} \subseteq \overline{\mathcal{O}_{\sigma_1}} \). Here \( \dim \mathcal{O}_\sigma = n - \dim \langle \sigma \rangle \). Moreover, each cone \( \sigma \in \Sigma \) defines an open affine \( T \)-invariant subset \( U_\sigma \) on \( X \) such that \( \mathcal{O}_\sigma \) is a unique closed \( T \)-orbit on \( U_\sigma \) and \( \sigma_1 \subseteq \sigma_2 \) if and only if \( U_{\sigma_1} \subseteq U_{\sigma_2} \).

Let \( \rho_e \) be the distinguished ray corresponding to a root \( e \), \( n_e \) be the primitive lattice vector on \( \rho_e \), and \( R_e \) be the one-parameter subgroup of \( T \) corresponding to \( n_e \).

Our aim is to describe the action of \( H_e \) on \( X \).

Proposition 5. For every point \( x \in X \setminus X^{H_e} \) the orbit \( H_e x \) meets exactly two \( T \)-orbits \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) on \( X \), where \( \dim \mathcal{O}_1 = \dim \mathcal{O}_2 + 1 \). The intersection \( \mathcal{O}_2 \cap H_e x \) consists of a single point, while

\[ \mathcal{O}_1 \cap H_e x = R_e y \quad \text{for any} \quad y \in \mathcal{O}_1 \cap H_e x. \]

Proof. It follows from the proof of [22, Proposition 3.14] that the affine charts \( U_\sigma \), where \( \sigma \in \Sigma \) is a cone containing \( \rho_e \), are \( H_e \)-invariant, and the complement of their union is
Lemma 1. A pair of $\mathbb{T}$-orbits $(O_{\sigma_1}, O_{\sigma_2})$ is $H_e$-connected if and only if $e|_{\sigma_2} \leq 0$ and $\sigma_1$ is a facet of $\sigma_2$ given by the equation $\langle v, e \rangle = 0$.

Proof. The proof again reduces to the affine case, where the assertion is [6, Lemma 2.2].

6. The orbit structure

We keep notations of the previous section. Let us begin with a construction mentioned in the Introduction. Let $X$ be a toric variety with the acting torus $\mathbb{T}$. Consider a non-trivial action $\mathbb{G}_a \times X \to X$ normalized by $\mathbb{T}$ and thus represented by a Demazure root $e$ of the fan $\Sigma$ of $X$. Then $\mathbb{T}$ acts on $\mathbb{G}_a$ by conjugation with the character $e$ and the semidirect product $\mathbb{T} \ltimes \mathbb{G}_a$ acts on $X$ as well. Let $T = \text{Ker}(e) \subseteq \mathbb{T}$ and consider the group $\mathbb{G} := T \times \mathbb{G}_a$.

Proposition 6. The variety $X$ is an embedding of $\mathbb{G}$.

Proof. Take a point $x \in X$ whose stabilizers in $\mathbb{T}$ and $\mathbb{G}_a$ are trivial. It suffices to show that the stabilizer of $x$ in $\mathbb{G}$ is trivial. To this end, note that by the Jordan decomposition [19, Theorem 15.3] any subgroup of $T \times \mathbb{G}_a$ is a product of subgroups in $T$ and $\mathbb{G}_a$ respectively.

Remark 1. The $\mathbb{G}$-embedding of Proposition 6 is defined by the pair $(\Sigma, e)$.

Since $\langle n_e, e \rangle = -1$, we have $\mathbb{T} = T \times R_e$.

Lemma 2. Any $(T \times \mathbb{G}_a)$-invariant subset in $X$ is also $\mathbb{T}$-invariant.

Proof. Note that an orbit $Tx$ does not coincide with the orbit $Tx$ if and only if the stabilizer of $x$ in $\mathbb{T}$ is contained in $T$. For $x \in O_\sigma$ this condition is equivalent to $e|_{\sigma} = 0$. It shows that for every $x \in X^{\mathbb{G}_a}$ we have $Tx = Tx$. If $x \in X \setminus X^{\mathbb{G}_a}$, then by Proposition 5 the orbit $\mathbb{G}_a x$ is invariant under $R_e$. This proves that any orbit of $(T \times \mathbb{G}_a)$ is $R_e$- and $\mathbb{T}$-invariant, thus the assertion.

Proposition 7. Let $X$ be a $\mathbb{G}$-embedding given by a pair $(\Sigma, e)$. Then any $\mathbb{G}$-orbit on $X$ is either a union $O_1 \cup O_2$ of two $\mathbb{T}$-orbits on $X$ or a unique $\mathbb{T}$-orbit; the first possibility occurs if and only if the pair $(O_1, O_2)$ is $H_e$-connected. In particular, the number of $\mathbb{G}$-orbits on $X$ is finite.

Proof. The assertion follows directly from Lemma 2 and Proposition 5.

Proposition 8. Let $X$ be a $\mathbb{G}$-embedding given by a pair $(\Sigma, e)$. Then the stabilizer of any point $x \in X$ in $\mathbb{G}$ is connected and the closure of any $\mathbb{G}$-orbit on $X$ is a (normal) toric variety. If $X$ is smooth, then the closure of any $\mathbb{G}$-orbit is smooth.

Proof. The stabilizer of a point $x$ in $\mathbb{G}$ is the direct product of stabilizers in $T$ and in $\mathbb{G}_a$. An algebraic subgroup of $\mathbb{G}_a$ is either $\{0\}$ or $\mathbb{G}_a$ itself, while the stabilizer in $T$ is the kernel of the (primitive) character $e$ restricted to the (connected) stabilizer of $x$ in $\mathbb{T}$. Thus the stabilizer of $x$ in $\mathbb{G}$ is connected.
Proposition 7 shows that any $G$-orbit on $X$ contains an open $T$-orbit, and thus the closure of a $G$-orbit coincides with the closure of some $T$-orbit. Now the last two assertions follow from [17, Section 3.1].

Remark 2. If $X$ contains $l$ torus invariant prime divisors, then the number of $G$-invariant prime divisors on $X$ is $l - 1$. On a toric variety, the closure of any torus orbit is an intersection of torus invariant prime divisors. In contrast, not every $G$-orbit closure on $X$ is an intersection of $G$-invariant prime divisors, see Example 3.

7. The general case

We are going to show that every $G$-embedding can be realized as in Proposition 6.

**Theorem 2.** Let $G = T \times G_a$ and $X$ be a normal equivariant $G$-embedding. Then the $T$-action on $X$ can be extended to an action of a bigger torus $T$ such that $T$ normalizes $G_a$ and $X$ is a toric variety with the acting torus $T$. In particular, every $G$-embedding comes from a pair $(\Sigma, e)$, where $\Sigma$ is a fan and $e$ is a Demazure root of $\Sigma$.

**Proof.** We begin with a classical result of Sumihiro. Let $X$ be a normal variety with a regular action $G \times X \to X$ of a linear algebraic group $G$. By [23, Theorem 3], there exists a normal complete $G$-variety $X$ such that $X$ can be embedded equivariantly as an open subset of $X$. In other words, $X$ is an equivariant compactification of $X$.

Let $X$ be a normal embedding of $G$ and $X$ be an equivariant compactification of $X$. By Proposition 4, the $T$-action on $X$ can be extended to an action of a bigger torus $T$ such that $T$ normalizes $G_a$ and $X$ is a toric variety with the acting torus $T$. Since the subset $X \subseteq X$ is $(T \times G_a)$-invariant, it is invariant under $T$, see Lemma 2. This provides the desired structure of a toric variety on $X$.

**Proposition 9.** A complete toric variety $X$ admits a structure of a $G$-embedding if and only if $\text{Aut}(X)^0 \neq T$.

**Proof.** The variety $X$ admits a structure of a $G$-embedding if and only if $\text{Aut}(X)^0$ contains at least one root subgroup. It is well known that the group $\text{Aut}(X)^0$ is generated by $T$ and root subgroups [13, Proposition 11], [22, Section 3.4], [11, Corollary 4.7].

Consider two structures of a $G$-embedding on a variety $X$. We say that such structures are equivalent, if there is an automorphism of $X$ sending one structure to the other. Since the structure of a toric variety on $X$ is unique up to automorphism, we may assume that our two structures share the same acting torus $T$ and the same fan $\Sigma$, and are given by two roots $e, e'$ of $\Sigma$. Then the structures are equivalent if and only if $e$ can be sent to $e'$ by an automorphism of the torus $T$. This leads to the following result.

**Proposition 10.** Two structures of a $G$-embedding given by pairs $(\Sigma, e)$ and $(\Sigma, e')$ are equivalent if and only if there is an automorphism $\phi$ of the lattice $N$ which preserves the fan $\Sigma$ and such that the induced automorphism $\phi^*$ of the dual lattice $M$ sends $e$ to $e'$.

Let us finish with explicit examples of $G$-embeddings into a given variety.

**Example 2.** We find all structures of $G$-embeddings on $A^2$. The cone of $A^2$ as a toric variety is $Q^2_{\geq 0}$. The set of Demazure roots of $Q^2_{\geq 0}$ is

$$\mathcal{R} = \{(-1, k) \mid k \in \mathbb{Z}_{\geq 0}\} \cup \{(k, -1) \mid k \in \mathbb{Z}_{\geq 0}\},$$

see Example 1. The $G$-action on $A^2$ corresponding to the root $(-1, k)$ is given by

$$(t, s) \circ (x_1, x_2) = (t^k x_1 + s t^k x_2, tx_2),$$

(3)
where \((x_1, x_2) \in \mathbb{A}^2\), \(s \in \mathbb{G}_a\), and \(t \in \mathbb{K}^\times\). If \(k \neq 0\), then there is a line of \(\mathbb{G}_a\)-fixed points and the stabilizer of a non-zero point on this line is a cyclic group of order \(k\). If \(k = 0\), then there is no \(\mathbb{G}_a\)-fixed point. So formula (3) gives non-equivalent \(\mathbb{G}\)-actions for different \(k\).

With \(k \neq 0\) we have three \(\mathbb{G}\)-orbits on \(\mathbb{A}^2\), while for \(k = 0\) there are two \(\mathbb{G}\)-orbits.

Note that \(\mathbb{G}\)-actions defined by the roots \((k, -1)\) and \((-1, k)\) are equivalent via the automorphism \(x_1 \leftrightarrow x_2\) of \(\mathbb{A}^2\).

**Example 3.** Let \(X = \mathbb{P}^2\). It is a complete toric variety with a fan \(\Sigma\) generated by the vectors \((1, 0)\), \((0, 1)\) and \((-1, -1)\):

\[
\begin{array}{c}
\text{N}_Q \\
\end{array}
\begin{array}{c}
\text{M}_Q \\
\end{array}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\begin{array}{c}
e_5 \leftarrow e_6 \\
e_4 \leftarrow e_3 \\
e_2 \leftarrow e_1 \\
\end{array}
\]

The set of Demazure roots is

\[\mathcal{R} = \{e_1 = (1, 0), e_2 = (1, -1), e_3 = (0, -1), e_4 = (-1, 0), e_5 = (-1, 1), e_6 = (0, 1)\}\]

We see that for any \(i\) and \(j\) there exists isomorphism of the fan \(\Sigma\) sending \(e_i\) to \(e_j\). So any \(\mathbb{G}\)-embedding into \(\mathbb{P}^2\) is equivalent to

\[(t, s) \circ [z_0 : z_1 : z_2] = [tz_0 + stz_1 : tz_1 : z_2].\]

This time seven \(T\)-orbits glue to five \(\mathbb{G}\)-orbits.

**Example 4.** Consider the Hirzebruch surface \(F_1\). The corresponding complete fan \(\Sigma\) is generated by the vectors \((1, 0)\), \((0, 1)\), \((0, -1)\), and \((-1, 1)\):

\[
\begin{array}{c}
\text{N}_Q \\
\end{array}
\begin{array}{c}
\text{M}_Q \\
\end{array}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\begin{array}{c}
e_3 \leftarrow e_4 \\
e_2 \leftarrow e_1 \\
\end{array}
\]

The set of Demazure roots is

\[\mathcal{R} = \{e_1 = (1, 0), e_2 = (-1, 0), e_3 = (0, 1), e_4 = (1, 1)\}\]

By an automorphism, we can send \(e_1\) to \(e_2\) and \(e_3\) to \(e_4\). For the first equivalence class we have six \(\mathbb{G}\)-orbits, while in the second one the number of \(\mathbb{G}\)-orbits in seven.

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