On correctness of an $n$ queens program

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Abstract

Thom Frühwirth presented a short, elegant and efficient Prolog program for the $n$ queens problem. However the program may be seen as rather tricky and one may be not convinced about its correctness. This paper explains the program in a declarative way, and provides a proof of its correctness and completeness.

KEYWORDS: logic programming, declarative programming, program completeness, program correctness, specification

1 Introduction

Thom Frühwirth presented a short, elegant and efficient Prolog program for the $n$ queens problem [Frühwirth 1988]. However the program may be seen as rather tricky and one may be not convinced about its correctness. The author’s description is rather operational. So it should be useful to explain the program declaratively, and to provide formal proof that it is correct.

In imperative and functional programming, program correctness implies that the program produces the “right” results. In logic programming, which is nondeterministic, the situation is different. One also needs the program to be complete, i.e. to produce all the results required by the specification. (In particular, the empty program producing no answers is correct whatever the specification is.)

This paper provides proofs of correctness and completeness of the $n$ queens program; the proofs are declarative, i.e. they abstract from any operational semantics.

The paper is organized as follows. After technical preliminaries, the $n$ queens program is presented together with an informal description of its declarative semantics. The next two sections present a formal specification for correctness, and a correctness proof. This is followed by Sections 6 and 7 dealing with the completeness of the program. The last section concludes the paper.
2 Preliminaries

Basics. This paper employs the standard terminology and notation (Apt 1997). We assume a fixed alphabet of function and predicate symbols. The Herbrand universe will be denoted by $\mathcal{HU}$, the Herbrand base by $\mathcal{HB}$, and the set of all terms (atoms) by $\mathcal{TU}$ (respectively $\mathcal{TB}$). A procedure $p$ in a program $P$ is the set of those clauses of $P$ that begin with the predicate symbol $p$. We use the list notation of Prolog. A list (respectively an open list) is a term $[e_1, \ldots, e_n] \in \mathcal{TU}$ ($[e_1, \ldots, e_n|v] \in \mathcal{TU}$), where $n \geq 0$ and $v$ is a variable; the length of the list (the open list) is $n$.

We deal with definite clause programs (in other words, logic programs without negation); $M_P$ stands for the least Herbrand model of a program $P$. By an answer of a program $P$ we mean an instance $Q_0 \theta$ of a query $Q_0$ where $\theta$ is an answer substitution for $Q_0$. (Due to soundness and completeness of SLD-resolution, it does not matter whether correct or computed answer substitutions are considered here.) So an answer of $P$ is any conjunction of atoms $Q$ such that $P \models Q$.

Specifications. In this paper, the treatment of specifications and reasoning about correctness and completeness follows that of (Drabent 2016); missing proofs and further explanations can be found there. For further discussion, examples and references, see (Drabent 2016; 2018; Drabent and Milkowska 2005).

A specification should describe the relation defined by each procedure of the program; each relation is a set of tuples over the Herbrand universe. Such family of relations can conveniently be described as a set of ground atoms. So the relations computed by a program are characterized by its least Herbrand model. And by a specification we mean an Herbrand interpretation $S \subseteq \mathcal{HB}$.

A program $P$ is correct w.r.t. a specification $S$ when $M_P \subseteq S$. This implies that $S \models Q$ for any answer $Q$ of $P$. A program $P$ is complete w.r.t. $S$ when $S \subseteq M_P$. This implies that, for any ground query $Q$, if $S \models Q$ then $Q$ is an answer of $P$.

Often it is inconvenient to specify $M_P$ exactly, i.e. to provide a specification $S$ for which the program is both correct and complete, $S = M_P$. It is useful to use instead an approximate specification, a pair of specifications, for correctness and for completeness. A program $P$ is fully correct w.r.t. an approximate specification $S_{\text{compl}}, S_{\text{corr}}$ when $S_{\text{compl}} \subseteq M_P \subseteq S_{\text{corr}}$.

Note that there are alternative views of what the relations defined by a program are. One may consider relations over nonground terms, and instead of the least Herbrand model $M_P$ use the set of the atomic answers of $P$ (Clark 1979). (The set is a superset of $M_P$; this is sometimes called the c-semantics (Falaschi et al. 1989).) A yet another approach is the s-semantics, where the characterization is the set of answers for the most general atomic queries (Falaschi et al. 1989).

Proving program correctness. An obvious way to prove correctness is to use the following sufficient condition. According to Deransart (1993), the condition is due to Clark (1979).

Theorem 1
For a program $P$ and a specification $S$, if $S \models P$ then $P$ is correct w.r.t. $S$.

Proof
As $S$ is a Herbrand model of $P$, the least Herbrand model of $P$ is a subset of $S$. □
As $S$ is an Herbrand interpretation, $S \models P$ means that for each ground instance $H \leftarrow B_1, \ldots, B_n \ (n \geq 0)$ of a clause of $P$, if $B_1, \ldots, B_n \in S$ then $H \in S$.

**Proving program completeness.** Informally, for completeness of a program w.r.t. $S$ it is necessary that each atom $A \in S$ can be produced by some clause of $P$ (out of atoms produced by $P$). The latter can be formalized as follows.

**Definition 2**
A ground atom $H$ is *covered* by a clause $C$ w.r.t. a specification $S$ if $H$ is the head of a ground instance $H \leftarrow B_1, \ldots, B_n \ (n \geq 0)$ of $C$, such that $B_1, \ldots, B_n \in S$ (Shapiro 1983).

A ground atom $H$ is *covered by a program $P$* w.r.t. $S$ if it is covered w.r.t. $S$ by some clause $C \in P$.

Each atom from $S$ being covered by $P$ w.r.t. $S$ does not imply the completeness of $P$ w.r.t. $S$. As an example, consider $S = \{p\}$ and $P = \{p \leftarrow p\}$. What is implied is *seicompleteness*, which means, roughly speaking, completeness for terminating queries (Drabent 2016; 2018).

We will need some additional notions, related to program termination. A *level mapping* is a function $|\cdot| : \mathcal{H} \rightarrow \mathbb{N}$ (where $\mathbb{N}$ is the set of natural numbers). A program $P$ is *recurrent* w.r.t. a level mapping $|\cdot|$ (Bezem 1993) when, for each ground instance $H \leftarrow B_1, \ldots, B_n \ (n \geq 0)$ of $P$ and each $i \in \{1, \ldots, n\}$, we have $|H| > |B_i|$.

The following sufficient condition is an immediate corollary of the results of Drabent 2016 Sections 5.2, 5.2), and is sufficient for the purpose of this paper.

**Lemma 3**
Let $P$ be a program, and $S$ a specification. If each atom $A \in S$ is covered by $P$ w.r.t. $S$, and $P$ is recurrent then $P$ is complete w.r.t. $S$.

### 3 The $n$ queens program

This section presents the $n$ queens program of Frühwirth (1988), and provides its informal declarative description.

The main idea of the program is to describe the placement of the queens by a data structure in which it is impossible that two queens are placed on the same row, column or a diagonal. In this way all the constraints of the problem are treated implicitly and efficiently. Here is the main part of the program; it will be named NQUEENS. (We take from (Frühwirth 1988) the version representing natural numbers as terms in a standard way, and abbreviate predicate names, place_queen as $pq$, and place_queensp as $pq$.)

\[
\begin{align*}
pq(0,_,_,_). \\
pq(s(I),Cs,Us,[_|Ds]):- \\
pq(I,CS,[_|Us],Ds), \\
pq(s(I),Cs,Us,Ds).
\end{align*}
\]

% $pq(Queen,Column,Updiagonal,Downdiagonal)$ places a single queen

\[
\begin{align*}
pq(I,[_|I],[_|I],[_|I]). \\
pq(I,[_|Cs],[_|Us],[_|Ds]):- \\
pq(I,Cs,Us,Ds).
\end{align*}
\]
The answers of the form \( pqs(n, q, t_1, t_2) \), where \( n \) is a number and \( q \) a list of length \( n \) provide solutions to the \( n \) queen problem: a number \( i \) being the \( j \)-th member of list \( q \) means that the queen of row \( i \) is placed in column \( j \). So to obtain the solutions, one can use a query \( pqs(n, q_0, \ldots) \), where \( q_0 \) is a list of \( n \) variables.

This is the original description of the program:

Observing that no two queens can be positioned on the same row, column or diagonals, we place only one queen on each row. Hence we can identify the queen by its row-number. Now imagine that the chess-board is divided into three layers, one that deals with attacks on columns and two for the diagonals going up and down respectively. We indicate that a field is attacked by a queen by putting the number of the queen there.

Now we solve the problem by looking at one row at a time, placing one queen on the column and the two diagonal-layers. For the next row/queen we use the same column layer, to get the new up-diagonals we have to move the layer one field up, for the down-diagonals we move the layer one field down.

Note that the description is rather operational, and does not have much to do with the logic of the program. Let us try to treat the program declaratively, abstracting from the operational semantics. For this we need to understand the relations defined by the procedures (i.e. predicates) of the program.

Assume that columns and rows are numbered from the left/top. Each queen is identified by its row number. Diagonals intersecting a given row are numbered from the left (Figure 1). In contrast to the numbering of rows and columns, this numbering is not fixed, it is specific for the context of the currently considered row. The diagonal number 1 includes the leftmost (column 1) field of the row, and so on (i.e. diagonal \( i \) includes the \( i \)-th field of the row). Thus, in the context of row number \( i \), its queen \( i \) is in the column and in the up and down diagonals of the same number.

The program represents the up (down) diagonals by an open list of numbers, a number \( i \) as the \( j \)-th member of the list means that the \( j \)-th diagonal contains the queen \( i \). If no queen is (yet) placed in the diagonal number \( j \), the \( j \)-th member of the list is a variable, or does not exist. The columns are represented as a (possibly open) list in the same way. This representation guarantees that at most one queen can be placed in each column and diagonal.

Now we try to informally describe the relations defined by the predicates of NQUEENS. Note that the relations are on possibly nonground terms. The tuples of the relation defined by \( pq \) are of the form

\[(i, cs, us, ds)\]

where \( i \) is the number of a row, and \( cs, us, ds \) are (possibly open) lists such that for some \( j > 0 \), the \( j \)-th member of each list is \( i \).

The relation defined by \( pqs \) consists of tuples \((i, cs, us, ds)\), where either \( i = 0 \) (and \( cs, us, ds \) are arbitrary), or \( i > 0 \) and \( ds = [t|ds'] \), \( cs \) describes (as explained above) a placement of queens number 1, \ldots, \( i \) in the columns, and \( us, ds' \) describe their placement respectively on the up and down diagonals (numbered in the context of row \( i \)). Moreover, in the chessboard fragment of rows 1, \ldots, \( i \), each row, each column, and each diagonal contains at most one queen.\[1\]

\[1\] Notice that the last statement follows from the previous one, but only for the columns and diagonals represented by \( cs, us, ds' \) (i.e. those intersecting the row \( i \)). However there are up diagonals of numbers \(-i + 2, \ldots, -1, 0\) that intersect some of the rows 1, \ldots, \( i - 1 \), but not row \( i \).
Such description of the semantics of the predicates of NQUEENS makes it possible to understand the program, in other words, to informally reason about it. In particular, we understand why in the rule for \( pqs \)

\[
pqs(s(I), Cs, Us, [\_|Ds]) \leftarrow pqs(I, Cs, [\_|Us], Ds), \ pq(s(I), Cs, Us, Ds)
\]

the argument representing the up diagonals is \([\_|Us]\) in one place, and \(Us\) in the others. Namely, an up diagonal with number \(j\) in the context of row \(i\) has number \(j-1\) in the context of row \(i+1\). Thus the tail \(Us\) of the (representation of) diagonals in the former context is the (representation of) diagonals in the latter. Similar explanation, with \(j-1\) replaced by \(j+1\), applies to the down diagonals (taking into account that in \(pqs(\ldots)\) the tail of the last argument represents the down diagonals crossing the current row).

The subject of the sections below is precise reasoning about the program. We present a formal specification and detailed proofs of its correctness and completeness.

Note that the presented description is in a sense wrong. This is because the described relations are not closed under substitution (as an instance of an open list may be neither an open list nor a list), while the relations defined by the program are closed under substitution (as an instance of an answer of a program \(P\) is an answer of \(P\)). Actually, the description provides a superset of the s-semantics (Falaschi et al. 1989) of NQUEENS. That is, a superset of the set of its answers for the most general atomic queries.

### 4 Specification for correctness

Program NQUEENS employs nonground terms. Open lists are used, and it seems crucial that the not yet assigned columns and diagonals are represented as unbound variables. Also, the informal specification from the previous section is related to the s-semantics, not to the standard semantics of the program. So one may suppose that the standard declarative semantics, based on the notion of logical consequence (and characterized

\[2\] In other words, the tail \(Us\) of the (representation of) diagonals intersecting row \(I\) is the (representation of) diagonals intersecting row \(s(I)\).
by the least Herbrand models) is not suitable here. Hence the notions of specification, correctness, and completeness of Section 2 would have not been suitable. One may expect that the $s$-semantics should be employed.

Actually, this is unnecessary. We specify the program and prove its correctness and completeness in terms of Herbrand interpretations.

Let us informally describe the main idea leading to the specification. Inconvenience is due to the fact that a “correct” data structure may have “bad” instances. For instance, and open (or nonground) list with distinct members may have an instance in which a member appears many times. Roughly speaking, we distinguish a class (say $S_{bad} \subseteq HB$) of atoms that may be such “bad” instances, and we do not bother about them. (The latter means that the specification includes whole $S_{bad}$.) Out of the remaining atoms, $HB \setminus S_{bad}$, the specification selects those that are “correct” answers of the program. Care is taken that each initial query of interest has an instance in $HB \setminus S_{bad}$, so the specification properly deals with all such queries.

The specification for $pq$ is rather obvious. It follows the description from the previous section, representing an open list by all its ground instances:

$$S_{pq} = \{ pq(i, [c_1, \ldots, c_k, i|c], [u_1, \ldots, u_k, i|u], [d_1, \ldots, d_k, i|d]) \in HB \mid k \geq 0 \}.$$ 

In order to formulate the specification for $pq$, we introduce some additional notions.

First we generalize the notion of list membership to terms which are not (open) lists but are instances of open lists.

**Definition 4**

A term $u$ is a **member** of a term $t$ if $t$ is $[t_1, \ldots, t_k, u|t_0]$ (where $k \geq 0$ and $t_0, \ldots, t_k$ are arbitrary terms). A term $u$ is a **single member** of $[t_1, \ldots, t_k|t_0]$ if $u$ is not a member of $[t_1, \ldots, t_k|t_0]$.

We say that $u$ is the $l$-th member of $[t_1, \ldots, t_l-1, u|t_0]$ ($l > 0$).

Let us now formalize the numbering of diagonals. Assume a queen $j$ (i.e. the queen of row $j$) is placed in column $k$ (i.e. $j$ is the $k$-th member of a term $cs$ representing columns).

Then, in the context of row $i$ (say $i \geq j$), the queen $j$ is on the up diagonal of number $k + j - i$. Similarly, the queen $j$ is on the down diagonal of number $k + i - j$, in the context of row $i$. Consider, for instance, the queen $i - 3$ placed in column 2. Then its up (down) diagonal number is, respectively, $-1$ and $5$.

**Definition 5**

Let a number $j$ be the $k$-th member of a term $cs$.  

The **up diagonal number** of $j$, w.r.t. $i$ in $cs$ is $k + j - i$.  

The **down diagonal number** of $j$, w.r.t. $i$ in $cs$ is $k + i - j$.

Now we are ready to introduce the core of our specification.

**Definition 6**

A triple of terms $(cs, us, ds) \in TU^3$ represents a correct placement up to row $m$ in the
context of row i (shortly: is correct up to m w.r.t. i) when 0 ≤ m ≤ i and
numbers 1, . . . , m are members of cs, and
if each 1, . . . , m is a single member of cs then
for each j ∈ {1, . . . , m},
if the up (down) diagonal number of j w.r.t. i in cs is l > 0
then the l-th member of us (respectively ds) is j,
for each j, j′ ∈ {1, . . . , m} with the up diagonal numbers l, l′ w.r.t. i in cs,
if l ≤ 0, l′ ≤ 0, and j ̸= j′ then l ̸= l′. (2)
So we do not bother about those (cs, us, ds) in which some j ∈ {1, . . . , m} is a multiple member of cs (and all 1, . . . , m are its members). Note that for the down diagonal number l we always have l > i − j, hence l > 0 (as i ≥ j). Note also that correctness of (cs, us, ds) implies the required property of cs:

Lemma 7

Assume that (cs, us, ds) is correct up to m (w.r.t. i) and each 1, . . . , m is a single member of cs. Then 1, . . . , m have distinct up diagonal numbers, and distinct down diagonal numbers (w.r.t. any i′).

Proof

For a given l > 0, the l-th member of us (or ds) can be at most one of 1, . . . , m, hence by (1) at most one of 1, . . . , m has the up (down) diagonal number l. For up diagonal numbers ≤ 0 their distinctness is required explicitly by (2). Distinct diagonal numbers w.r.t. i mean distinct ones w.r.t. any i′.

Now the specification for pqs is as follows.

\[
S_{pqs} = \{ pqs(0, cs, us, ds) \mid cs, us, ds \in HU \} \cup \\
\left\{ pqs(i, cs, us, [u|ds]) \in HB \mid i > 0, (cs, us, ds) \text{ is correct up to } i \text{ w.r.t. } i \right\}.
\]

And our specification of NQUEENS for correctness is

\[
S = S_{pq} \cup S_{pqs}.
\]

Note that correctness w.r.t. S guarantees the required property of the program: If \( cs = [e_1, . . . , e_n] \) is a list of length n and \( pqs(n, cs, us, ds) \in S \) then \( e_1, . . . , e_n \) is a permutation of 1, . . . , n and 1, . . . , n have distinct up (down) diagonal numbers (w.r.t. any i).

We conclude this section with a property, which will be used in the proofs below.

Lemma 8

Assume m ≤ i. Consider two conditions

\[
(\text{cs, [u|us], ds}) \text{ is correct up to } m \text{ w.r.t. } i \quad (3) \\
(\text{cs, us, [u'|ds]}) \text{ is correct up to } m \text{ w.r.t. } i + 1 \quad (4)
\]

For any u, u′ ∈ HU, (3) implies (4). For any u′ ∈ HU, (4) implies ∃ u ∈ HU (3).
Proof
Assume each 1, . . . , m is a single member of cs. (Otherwise the implications hold vacuously.)

The down diagonal number l w.r.t. i means the down diagonal number l1 = l + 1 w.r.t. i + 1, and the l-th member of ds and the l1-th member of [u′|ds] are the same. Also, [1] implies that for each j ∈ {1, . . . , m} its down diagonal number w.r.t. i + 1 is l1 > 1. So conditions [1] for i, l and ds, and [1] for i + 1, l1 and [u′|ds] are equivalent.

The up diagonal number l w.r.t. i means the up diagonal number l1 = l − 1 w.r.t. i + 1. So we obtain, similarly as above, that [1] for i, l and [u|us] implies [1] for i + 1, l1 and us. The converse implication does not hold, but we obtain, as above, that [1] for i + 1, l1 and us implies that for some u ∈ HU (1) holds for i, l and [u|us]. (If some j has 0 as its up diagonal number w.r.t. i + 1 then u = j, otherwise u is arbitrary.)

Condition [5] implies that 1, . . . , m have distinct up diagonal numbers w.r.t. i, hence w.r.t. i + 1 holds. Similarly, [1] implies [2] w.r.t. i.

Under assumption that 1, . . . , m are single members of cs, we showed that [3] implies that, for any j, j′, conditions [1], and [2] for i + 1 hold, and that [4] implies that there exists u such that, for any j, j′, [1] and [2] hold for i. The two required implications follow immediately. □

5 Correctness proof
Following Theorem[1] to prove correctness of program NQUEENS w.r.t. specification S, one has to show that S is a model of each clause of the program. In other words, to show for each ground instance of a clause of the program that the head is in S provided the body atoms are in S. For the unary clauses of NQUEENS:

\[ pq(I, [I]_, [I]_, [I]_). \]
\[ pqs(0, _, _, _). \]

it is obvious that each ground instance of the clause is in S. Consider the clause

\[ pq(I, [\_|Cs|, [\_|Us|, [\_|Ds|) ← pq(I, Cs, Us, Ds). \]

For any its ground instance

\[ pq(i, \{t\}, cs, \{t\} | us, \{t\} | ds) \]

it immediately follows from the definition of \( S_{pq} \) that if the body atom is in S (thus in \( S_{pq} \)) then its head is in \( S_{pq} \subseteq S \).

The nontrivial part of the proof is to show that S is a model of the clause

\[ pqs(s(I), Cs, Us, [\_|Ds]) ← pqs(I, Cs, [\_|Us|, Ds), pqs(s(I), Cs, Us, Ds). \]

Consider its ground instance, with the head H and the body atoms \( B_1, B_2 \in S \). Thus \( B_2 = pqs(s(i), cs, us, ds) \), where \( s(i) \) is the l-th member of cs, us, ds (for some \( l > 0 \)). Note that \( l \) is the up (down) diagonal number of \( s(i) \) w.r.t. \( s(i) \) in cs.

Consider first the case of \( i = 0 \). Then \( H = pqs(s(0), cs, us, [u|ds]) \) (for some \( u \in HU \)). Notice that \( (cs, us, ds) \) is correct up to \( s(0) \) w.r.t. \( s(0) \). Hence \( H \subseteq S \).

Consider now \( i > 0 \). So \( B_1 = pqs(i, cs, [u|us], [u|ds]) \) is correct up to \( i \) w.r.t. \( i \). Hence by Lemma \( S_\beta = (cs, us, [u|ds]) \) is correct up to \( i \) w.r.t. \( s(i) \). We show that \( \beta \) is correct up to \( s(i) \).

Numbers \( s(0), . . . , s(i) \) are members of cs \( (s(0), . . . , i due to \( B_1 \in S, s(i) due to \( B_2 \in S); \)
assume they are single members of cs. Conditions (1), (2) hold for \( j, j' \in \{s(0), \ldots, i\} \).
As both diagonal numbers of \( s(i) \) (w.r.t. \( s(i) \) in cs) are \( l > 0 \), it remains to show that (1) holds for \( j = s(i) \). Indeed, as stated above, \( s(i) \) is the \( l \)-th member of both \( us \) and of \( ds = [u_2|ds'] \). Thus \( \beta \) is correct up to \( s(i) \) w.r.t. \( s(i) \).

Now the head of the clause instance is \( H = pqs(s(i), cs, us, [u, u_2|ds']) \), and we have \( H \in S \).

### 6 Specification for completeness

The program is not (and should not be) complete w.r.t. specification \( S \). For instance \( S \) includes all atoms \( pqs(s(0), cs, us, ds) \in H \) in which \( s(0) \) is not a single member of \( cs \).

Now we construct a specification for completeness. It describes what has to be computed by the program. As a specification for completeness for procedure \( pq \), we will use \( S_{pq} \), its specification for correctness.

Procedure \( pqs \) should provide all answers to the \( n \) queens problem, so we can restrict the specification \( S_{pqs} \) to atoms in which the \( cs \) argument is a list being a permutation of \( 1, \ldots, n \). This results in a specification for completeness

\[
S^0 = S_{pq} \cup S^0_{pqs},
\]

where

\[
S^0_{pqs} = \big\{ pqs(0, cs, us, ds) \mid cs, us, ds \in H \big\} \cup \bigg\{ pqs(i, cs, us, [u|ds]) \in H \bigg\}
\]

\[
\begin{array}{l}
0 < i \leq n, \\
\quad cs = [e_1, \ldots, e_n] \text{ (for some } n \geq i \text{ where )} \\
\quad e_1, \ldots, e_n \text{ is a permutation of } 1, \ldots, n, \\
\quad (cs, us, ds) \text{ is correct up to } i \text{ w.r.t. } i.
\end{array}
\]

### 7 Completeness proof

Here Lemma 3 is applied. We first show that each atom from specification \( S^0 \) is covered by program NQUEENS. Obviously, each atom

\[
A = pq(i, [c_1, \ldots, c_k, i|c], [u_1, \ldots, u_k, i|u], [d_1, \ldots, d_k, i|d])
\]

from \( S_{pq} \) is covered by NQUEENS w.r.t. \( S^0 \); for \( k = 0 \) by clause \( pq(I, [I|], [I|], [I|]) \), for \( k > 0 \) by \( pq(I, [\ldots|Cs], [\ldots|Us], [\ldots|Ds]) \leftarrow pq(I, Cs, Us, Ds) \). We leave the simple details to the reader.

The nontrivial part of the proof is to show that each \( A \in S^0_{pqs} \) is covered. If \( A = pqs(0, cs, us, ds) \) then obviously \( A \) is covered by clause \( pqs(0, \ldots, \ldots, \ldots) \). It remains to consider atoms of the form

\[
A = pqs(s(i), cs, us, [u|ds]) \in S^0_{pqs}.
\]

So \( cs = [e_1, \ldots, e_n] \) (\( s(i) \leq n \)), \( e_1, \ldots, e_n \) is a permutation of \( 1, \ldots, n \), and \( (cs, us, ds) \) is correct up to \( s(i) \) w.r.t. \( s(i) \). We show that \( A \) is covered by clause

\[
pqs(s(I), Cs, Us, [\ldots|Ds]) \leftarrow pqs(I, Cs, [\ldots|Us], Ds, pqs(s(I), Cs, Us, Ds)).
\]

Consider its instance with the head \( A \) and body atoms \( B_1, B_2 \). For some \( l, s(i) = e_l \), and
is the up and down diagonal number of \( s(i) \) w.r.t. \( s(i) \). By Def. 6, the \( l \)-th member of \( us \) and of \( ds \) is \( s(i) \). Thus \( B_2 \in S_{pq} \), as \( B_2 = pq(s(i),cs,us,ds) \).

As \( ds \) has a member, it is of the form \( ds = [t_1|ds'] \). The first body atom of the clause instance is \( B_1 = pqs(i,cs,[t_2|us],[t_1|ds']) \), with an arbitrary \( t_2 \in \mathcal{H}U \). As \( (cs,us,[t_1|ds']) \) is correct up to \( s(i) \) (thus also up to \( i \)) w.r.t. \( s(i) \), by Lemma 8 we can choose \( t_2 \) such that \( (cs,[t_2|us],ds') \) is correct up to \( i \) w.r.t. \( i \). Then \( B_1 \in S_{pp}^{0} \), hence \( A \) is covered by NQUEENS w.r.t. \( S^0 \).

It remains to find a level mapping under which NQUEENS is recurrent. Consider the level mapping defined by

\[
|pqs(i,cs,us,ds)| = |i| + |cs|, \\
|pq(i,cs,us,ds)| = |cs|,
\]

where

\[
|h||t| = 1 + |t|, \\
|s(t)| = 1 + |t|, \\
|f(t_1,\ldots,t_n)| = 0,
\]

for any ground terms \( i,cs,us,ds,h,t,t_1,\ldots,t_n \), and any \( n \)-ary function symbol \( f \) distinct from \( s \) and from \([|]\) \((n \geq 0)\). An easy inspection shows that under this level mapping NQUEENS is recurrent. Hence the program is complete w.r.t. \( S^0 \).

8 Conclusions

The paper presents detailed proofs of correctness and completeness of the \( n \) queens program of Frühwirth (1988). This is preceded by informal declarative description of the program (in contrast to the original one, which is rather operational).

Such proofs require a specification for the program. The specification used here is approximate; this means separate specifications for correctness and for completeness. Constructing an exact specification of the program would be too troublesome, and result in more complicated correctness and completeness proofs. This is quite common in logic programming, one often does not need to know the exact semantics of one’s program. It is sufficient to know that its least Herbrand model \( \mathcal{M}_P \) is a superset of a certain set (a specification for completeness) and a subset of another set (a specification for correctness). The latter is equivalent to \( \mathcal{M}_P \) being disjoint with a certain set of “bad” atoms (the complement of the specification for correctness). A basic example is the standard APPEND program, for details and further examples see (Drabent 2016; 2018; Drabent and Milkowska 2005).

The precise proofs presented here may be seen as too impractical due to numerous details. Note however that this is usually the case when proving program properties. Experience with reasoning about programs, among others in imperative programming, provides evidence that the program correctness really does depend on many details. In the author’s opinion, proofs like those presented here can be performed by programmers at an informal level at actual programming, at various levels of precision. A fragment of such informal reasoning, concerning one clause, is shown in Section 3.
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