Ghosh, Payel; Roy, Tapan

Article
A goal geometric programming problem (G2P2) with logarithmic deviational variables and its applications on two industrial problems

Journal of Industrial Engineering International

Provided in Cooperation with:
Islamic Azad University (IAU), Tehran

Suggested Citation: Ghosh, Payel; Roy, Tapan (2013) : A goal geometric programming problem (G2P2) with logarithmic deviational variables and its applications on two industrial problems, Journal of Industrial Engineering International, ISSN 2251-712X, Springer, Heidelberg, Vol. 9, pp. 1-9, http://dx.doi.org/10.1186/2251-712X-9-5

This Version is available at:
http://hdl.handle.net/10419/147150

Terms of use:
Documents in EconStor may be saved and copied for your personal and scholarly purposes. You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public. If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

http://creativecommons.org/licenses/by/2.0/
A goal geometric programming problem ($G^2P^2$) with logarithmic deviational variables and its applications on two industrial problems

Payel Ghosh¹* and Tapan Kumar Roy²

Abstract

A very useful multi-objective technique is goal programming. There are many methodologies of goal programming such as weighted goal programming, min-max goal programming, and lexicographic goal programming. In this paper, weighted goal programming is reformulated as goal programming with logarithmic deviation variables. Here, a comparison of the proposed method and goal programming with weighted sum method is presented. A numerical example and applications on two industrial problems have also enriched this paper.

Keywords: Goal programming, Geometric programming, Pareto optimality, Nonlinear programming

Background

The earliest goal programming formulation was introduced by Charnes et al. (1955). Later, Charnes and Cooper (1977), Ijiri (1965), Lee (1972), and Ignizio (1976) are the contributors of goal programming for which goal programming became a useful tool in multi-criteria decision-making (MCDM) problem. The updated presentations of goal programming have been discussed by Tamiz et al. (1998), Lee and Olson (2000), Jones and Tamiz (2002), and Ignizio and Romero (2003). Methodologies of goal programming such as weighted goal programming, min-max goal programming, lexicographic goal programming have been discussed in the study of Romero (2004). Except for these three methods, another method, the logarithmic goal programming, is introduced (Wang et al. 2005).

Our proposed method is goal geometric programming with logarithmic deviational variables. In goal programming formulation with logarithmic deviational variables, we use geometric programming for solving because there are lots of real-life situations and many engineering applications where equations may be nonlinear. For a special type of nonlinear programming problem, geometric programming is a very useful tool. Since we use geometric programming method to solve a nonlinear goal programming problem, therefore, the degree of difficulty has a great role in this context. The degree of difficulty of the proposed method is lesser than that of other methods such as goal geometric programming using weighted sum method.

The concept of taking multiplicative deviational variables as an objective function is not new. Previously, Verma (1990) and a paper entitled ‘Goal geometric programming problem ($G^2P^2$) with product method’ by Ghosh and Roy (2012) used this concept. In this paper, we have started with additive deviational variables as the objective function which were then converted into multiplicative deviational variables as objective function using the logarithmic concept. The method of conversion is given in the form of ‘Result 1’.

The arrangement of the paper is as follows: the background of the study followed by the goal programming model are presented. A result is presented together with its proof (Result 1), and the model of weighted goal programming with logarithmic deviational variables is then presented. The sections for goal geometric programming model with logarithmic deviational variables and its solution procedure are followed by a theorem on the model of weighted goal programming with logarithmic deviational variables and its proof (Result 2). Next, a numerical
example and applications on lightly loaded bearing problem, optimal production, and marketing planning are presented. Finally, the conclusions of the study is presented.

Goal programming
A multi-objective programming can be written as follows:

Find \( X = (x_1, x_2, \ldots, x_n)^T \)

so as to minimize \( f_{10}(X) = \sum_{i=1}^{P_{10}} C_{10i} \prod_{k=1}^{n} x_k^{a_{ki}} \) with target \( C_{10j} \),

minimize \( f_{20}(X) = \sum_{i=1}^{P_{20}} C_{20i} \prod_{k=1}^{n} x_k^{a_{ki}} \) with target \( C_{20}\),

minimize \( f_{n0}(X) = \sum_{i=1}^{P_{n0}} C_{n0i} \prod_{k=1}^{n} x_k^{a_{ki}} \) with target \( C_{n0} \),

subject to \( f_i(X) = \sum_{r=1}^{P_r} C_{ri} \prod_{k=1}^{n} x_k^{a_{ki}} \leq C_{ri} ; \) \( r = 1, 2, \ldots, q \)

\( x_k > 0 ; \) \( k = 1, 2, \ldots, n \).

\( C_{j0} \) and \( C_{ri} \) are positive real numbers \( \forall j, r, i \), and

\( a_{ki0}, a_{ki} \) are real numbers \( \forall k, i \).

\( P_{j0} \) = Number of terms present in \( j0 \)th objective function,

\( P_r \) = Number of terms present in \( r \)th constraint,

\( C_r \) = Boundary value of \( r \)th constraint.

The multi-objective programming model contains \( m \), the number of minimizing objective functions; \( q \), the number of inequality type constraints; and \( n \), the number of strictly positive decision variables.

Result 1. As mentioned, the goal programming model may be reduced to the following form:

Minimize \( \prod_{j=1}^{m} u_{10}^+ \prod_{r=1}^{q} v_r^+ \)

subject to \( f_{j0}(X)/u_{10}^+ \leq C_{j0} ; j = 1, 2, \ldots, m \),

\( f_r(X)/v_r^+ \leq C_r ; r = 1, 2, \ldots, q \),

\( x_k > 0 ; k = 1, 2, \ldots, n ; u_{10}^+, v_r^+ > 1 \),

with the conditions \( f_{j0}(X) > 0 \), \( C_{j0} > 0 \), \( C_r > 0 \).

Proof. In the multi-objective programming model (1), objective functions are minimized and have target values, e.g., minimize \( f_{j0}(X) \) with target value \( C_{j0} \), i.e., minimize \( \log(f_{j0}(X)) \) with target value \( \log(C_{j0}) \). According to the method of goal formulation, positive deviation should be minimized.

Similarly, in model (1), constraints are of \( \leq \) type. Thus, positive deviations should also be minimized. Therefore, when

\( f_r(X) \leq C_r \),

then

\( \log(f_r(X)) \leq \log(C_r) \).

The goal formulation is as follows:

Minimize \( \sum_{j=1}^{m} d_{j0}^- + \sum_{r=1}^{q} d_r^- \) \hspace{1cm} (2)

subject to

\( \log(f_{j0}(X)) + d_{j0}^+ - d_{j0}^- = \log(C_{j0}) ; j = 1, 2, \ldots, m \),

\( \log(f_r(X)) + d_r^+ - d_r^- = \log(C_r) ; r = 1, 2, \ldots, q \),

\( x_k > 0 ; k = 1, 2, \ldots, n ; d_{j0}^+, d_{j0}^-, d_r^+, d_r^- > 0 \),

\( d_{j0}^+ \times d_{j0}^- = 0 ; d_r^+ \times d_r^- = 0 \).

\( d_r^+ \) = Positive deviation of objective function,

\( d_{j0}^- \) = Negative deviation of objective function,

\( d_r^+ \) = Positive deviation of constraint,

\( d_{j0}^- \) = Negative deviation of constraint.

However, with a logarithmic change of deviational variables \( d_{j0}^+ = \log(u_{j0}^-) \), \( d_{j0}^- = \log(u_{j0}^+) \),

\( d_r^+ = \log(v_r^-) \), \( d_r^- = \log(v_r^+) \), we can turn model (2) into the following problem:

Minimize \( \left( \log \prod_{j=1}^{m} u_{10}^+ \prod_{r=1}^{q} v_r^+ \right) \) \hspace{1cm} (3)

subject to

\( \log(f_{j0}(X) \cdot u_{j0}^- / u_{j0}^+) = \log(C_{j0}) ; j = 1, 2, \ldots, m \),

\( \log(f_r(X) \cdot v_r^- / v_r^+) = \log(C_r) ; r = 1, 2, \ldots, q \),

\( x_k > 0 ; k = 1, 2, \ldots, n ; u_{j0}^+, u_{j0}^-, v_r^+, v_r^- > 1 \),

which is obviously equivalent to the following goal programming form with logarithmic deviational variables:

Minimize \( \prod_{j=1}^{m} u_{j0}^+ \prod_{r=1}^{q} v_r^+ \) \hspace{1cm} (4)

subject to

\( f_{j0}(X) \cdot u_{j0}^- / u_{j0}^+ = C_{j0} ; j = 1, 2, \ldots, m \),

\( f_r(X) \cdot v_r^- / v_r^+ = C_r ; r = 1, 2, \ldots, q \),

\( x_k > 0 ; k = 1, 2, \ldots, n ; u_{j0}^+, u_{j0}^-, v_r^+, v_r^- > 1 \).

The goal programming formulation where the constraints are in inequality form the following:

Minimize \( \prod_{j=1}^{m} u_{j0}^+ \prod_{r=1}^{q} v_r^+ \) \hspace{1cm} (5)

subject to

\( f_{j0}(X) \cdot u_{j0}^- \leq C_{j0} ; j = 1, 2, \ldots, m \),

\( f_r(X) \cdot v_r^- \leq C_r ; r = 1, 2, \ldots, q \),

\( x_k > 0 ; k = 1, 2, \ldots, n ; u_{j0}^+, u_{j0}^-, v_r^+, v_r^- > 1 \),

hence the result. □
Results and discussion

Weighted goal programming with logarithmic deviational variables

According to model (1), all of the objective functions are minimized. If the decision maker wants to get a much more minimized value for any particular objective function or wants to satisfy strictly the constraints, then weight factors (priorities) are introduced. In goal programming formulation with logarithmic deviational variables, weights (priorities) are given with the deviational variable. Hence, the weighted goal programming formulation becomes the following:

\[
\text{Minimize } \prod_{j=1}^{n}(u_j^{+})^{w_j} \prod_{r=1}^{q}(v_r^{+})^{w_r} \tag{6}
\]

subject to

\[
f_{j}(X)/u_j^{+} \leq C_{j0}, j = 1, 2, \ldots, m, \\
f_{r}(X)/v_r^{+} \leq C_{r}, r = 1, 2, \ldots, q, \\
x_k > 0, k = 1, 2, \ldots, n; u_j^{+}, v_r^{+} > 1.
\]

Here, \(W_{j0}\) values are the weights for objective functions and \(W_{r}\) values are the weights for the constraints.

Solutions of goal programming (Romero 1991), even those of weighted goal programming and lexicographic goal programming (Miettinen 1999), are pareto optimal. Here, we prove a result which also shows that goal programming with logarithmic deviation gives pareto optimal solutions.

**Result 2.** The following is the solution of weighted goal programming with logarithmic deviation:

Minimize \(\prod_{i=1}^{k}(u_i^{+})^{w_i}\)

subject to

\[
(\sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} x_l^{a_{rl}}) (u_i^{+})^{-1} \leq C_i, i = 1, 2, \ldots, k, \\
X \in S, u_i^{+} > 1, i = 1, 2, \ldots, k,
\]

which comes from the following goal programming model:

Minimize \(f_i(X) = \sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} x_l^{a_{rl}}\) with target \(C_i\),

\[
i = 1, 2, \ldots, k; X \in S
\]

is pareto optimal if \(u_i^{+}\) for each function \(f_i(X)\) to be minimized has a value greater than 1 at the optimum.

**Proof.** If \(x^{*} \in S\) with a positive deviation vector, then let \((u_i^{+})^{*} (> 1)\) be the solution of the following weighted goal programming problem:

Minimize \(\prod_{i=1}^{k}(u_i^{+})^{w_i}\) \hspace{1cm} \tag{7}

subject to

\[
(\sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} x_l^{a_{rl}}) (u_i^{+})^{-1} \leq C_i, i = 1, 2, \ldots, k, \\
X \in S, u_i^{+} > 1, i = 1, 2, \ldots, k.
\]

If possible, let \(x^{*}\) be not pareto optimal, then there exists a vector \(x^{0}\) with a positive deviational variable \((u_i^{+})^{0}(> 1)\) such that

\[
\left( \sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} x_l^{a_{rl}} \right)^{j} \leq \left( \sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} (x_l^{a_{rl}}) \right)^{j} ; \tag{7.1}
\]

\[
\forall i = 1, 2, \ldots, k
\]

and

\[
\left( \sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} (x_l^{a_{rl}}) \right)^{j} > \left( \sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} (x_l^{a_{rl}}) \right)^{j} ; \tag{7.2}
\]

for at least one \(j\).

Let

\[
\frac{\left( \sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} (x_l^{a_{rl}}) \right)^{j}}{\left( \sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} (x_l^{a_{rl}}) \right)^{j}} = \beta > 1. \tag{7.3}
\]

We set

\[
(u_i^{+})^{0} = (u_i^{+})^{*}( > 1) \text{ for } i = 1, 2, \ldots, k \tag{7.4}
\]

and

\[
(u_i^{+})^{0} = \max(1, (u_i^{+})^{*}/\beta) \geq 1 \text{ and } i \neq j. \tag{7.5}
\]

Here, \((u_i^{+})^{0}\) is the positive deviational variable corresponding to \(x^{0}\), \(i = 1, 2, \ldots, k\). From (7.1),

\[
\left( \sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} (x_l^{a_{rl}}) \right)^{j} (u_i^{+})^{0} \leq \left( \sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} (x_l^{a_{rl}}) \right)^{j} \times (u_i^{+})^{0} \leq C_i \text{ as } x^{*} \text{ be the solution of (7), i.e.,}
\]

\[
\left( \sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} (x_l^{a_{rl}}) \right)^{j} (u_i^{+})^{0} \leq C_i \text{ for } \tag{7.6}
\]

\[
i = 1, 2, \ldots, k, \text{ but } i \neq j.
\]
From 7.5,

\[(u^+_j)^0 = \max \left(1, \frac{(u^+_j)^*}{\beta}\right),\]

Thus,

\[\frac{(u^+_j)^*}{\beta} > 1, \text{ then using } (7.7),\]

\[\left(\sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} (x^0_{il})^{a_{il}}\right) \left(\left(\frac{(u^+_j)^0}{\beta}\right)^{\beta\gamma} \right) - 1 \leq \frac{C_{j}}{\beta} \leq \left(\sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} (x^0_{il})^{a_{il}}\right) \left(\left(\frac{(u^+_j)^0}{\beta}\right)^{\beta\gamma} \right) - 1 \leq \frac{C_{j}}{\beta}\]

Therefore,

\[\left(\sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} (x^0_{il})^{a_{il}}\right) \left(\frac{(u^+_j)^0}{\beta}\right)^{\beta\gamma} \leq \frac{C_{j}}{\beta}\]

Thus, \(x^0\) satisfies the constraints of (7). From (7.7), \((u^+_j)^0 = \frac{(u^+_j)^*}{\beta} < (u^+_j)^*\).

Since \(\beta > 1\) and \((u^+_j)^* > 1\), using (7.4),
\[(u^+_j)^0 \leq (u^+_j)^*, \forall j = 1, 2, \ldots, k.

Case 2

\[\frac{(u^+_j)^*}{\beta} \leq 1, \text{ then using } (7.8),\]

\[\left(\sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} (x^0_{il})^{a_{il}}\right) \left(\left(\frac{(u^+_j)^0}{\beta}\right)^{\beta\gamma} \right) - 1 \leq \frac{C_{j}}{\beta}\]

Using (7.3),

\[\left(\sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} (x^0_{il})^{a_{il}}\right) \left(\left(\frac{(u^+_j)^0}{\beta}\right)^{\beta\gamma} \right) - 1 \leq \frac{C_{j}}{\beta}\]

from (7.8),

\[\left(\sum_{r=1}^{p} C_{mr} \prod_{l=1}^{n} (x^0_{il})^{a_{il}}\right) \left(\left(\frac{(u^+_j)^0}{\beta}\right)^{\beta\gamma} \right) - 1 \leq \frac{C_{j}}{\beta}\]

Thus, \(x^0\) satisfies the constraints of (7), and from (7.8),
\[(u^+_j)^0 = 1 < (u^+_j)^*\].

Therefore, from (7.4) and (7.11), \((u^+_j)^0 \leq (u^+_j)^* \forall j = 1, 2, \ldots, k\). Thus, \(\forall\) positive weights \(W_i, (i = 1, 2, \ldots, k)\)

\[\left((u^+_j)^0\right)^{W_i} \leq \left((u^+_j)^*\right)^{W_i}\]

(7.12)

Thus, from (7.6), (7.9), (7.10), and (7.12), we have seen that \(x^0\) is a solution of (7), which contradicts the fact that \(x^*\) is a solution of (7). Hence, \(x^*\) is pareto optimal.

Goal geometric programming model with logarithmic deviational variables and its solution procedure

Linear goal programming is a very commonly used tool of the MCDM problem. However, nonlinear goal programming is very rare in this context. In many engineering problems, as well as problems of science, there are nonlinear equations to optimize. To solve that type of nonlinear goal programming problem, the geometric programming method can be used. Hence, we can turn model (6) into a goal geometric programming form as in the following:

Minimize \[\prod_{j=1}^{m} (u^+_{j})^{W_{j0}} \prod_{r=1}^{q} (v^+_{r})^{W_{r}}\] (8)

subject to

\[f_{j0}(X) (u^+_{j0})^{-1}/C_{j0} \leq 1, j = 1, 2, \ldots, m,\]

\[f_{r}(X) (v^+_{r})^{-1}/C_{r} \leq 1, r = 1, 2, \ldots, q,\]

\[x_k > 0, k = 1, 2, \ldots, n; u^+_{j0}, v^+_{r} > 1.\]

The corresponding dual geometric programming of model (8) can be written as follows:

Maximize \[d(\delta) = \left[\frac{1}{\delta_{10}} \prod_{j=1}^{m} \prod_{i=1}^{p} \left(\frac{C_{j0}}{C_{j0}\delta_{ji}}\right) \prod_{r=1}^{n} \left(\frac{P_{r}}{P_{r}\delta_{ri}}\right) \right]\]

such that

\[\delta_{10} = 1, W_{j0}\delta_{10} - \sum_{i=1}^{p} \delta_{ji} = 0, j = 1, 2, \ldots, m;\]

\[W_{r}\delta_{10} - \sum_{i=1}^{p} \delta_{ri} = 0, r = 1, 2, \ldots, q;\]
A multi-objective goal programming problem is:

Minimize $f_1(x_1, x_2) = x_1^{-1}x_2^{-2}$ with target value 4, (9)

Minimize $f_2(x_1, x_2) = 2x_1^{-2}x_2^{-3}$ with target value 50, subject to $x_1 + x_2 \leq 1$, $x_1, x_2 > 0$.

In goal geometric programming model with logarithmic devotional variables, model (8) can be written as follows:

Minimize $u^{W_1}v^{W_2}$

subject to

$$x_1^{-1}x_2^{-2}u^{-1} \leq 4,$$

$$2x_1^{-2}x_2^{-3}v^{-1} \leq 50,$$

$$x_1 + x_2 \leq 1, x_1, x_2 > 0, u, v > 1.$$

From primal dual relation

$$x_1^{-1}x_2^{-2}u^{-1} = \frac{\delta_{11}}{\lambda_1(\delta)} = 1 \text{ or } u = \frac{1}{4x_1x_2^2},$$

$$2x_1^{-2}x_2^{-3}v^{-1} = \frac{\delta_{21}}{\lambda_2(\delta)} = 1 \text{ or } v = \frac{2}{50x_1^2x_2^2},$$

$$x_1 = \frac{W_1 + 2W_2}{3W_1 + 5W_2}, x_2 = \frac{2W_1 + 3W_2}{3W_1 + 5W_2}.$$

Solving from primal dual relation for different values of weights, we get the optimal values of the decision variables which are given in Table 1.

From the table, we see that each deviation $(u_i, v_i)$ has values greater than 1 when minimized. Thus, according to our theorem, the solutions are pareto optimal.

Again, we have solved the mentioned example in goal geometric programming with weighted sum method. Here, we have compared the results of the mentioned example in equal weights solved in two different methods: goal geometric programming with weighted sum method and goal geometric programming with logarithmic devotional variables which are given in Table 2.

From the comparison, it is clear that in both methods, the optimum values of the first and second objectives are almost the same. We have solved the same example in both processes where we have used geometric programming to solve a nonlinear goal programming problem. The advantage of the proposed method lies in the method of solution, i.e., geometric programming where degree of difficulty is less than the degree of difficulty of the previous process (goal geometric programming with weighted sum method). For this reason, the solution procedure of this process becomes easier than that of the previous.

### Application on lightly loaded bearing problem

A lightly loaded bearing is to be designed to minimize the linear combination of frictional moment and angle of twist of the shaft and the temperature rise of the oil while carrying a load of 1,000 lb, and the angular velocity of the shaft is to be greater than 100 rad s$^{-1}$. Assume that 1 in-lb of frictional moment in the bearing is equal to 0.0025 rad of the angle of twist. The following are the goals:

Priority 1: Linear combination of frictional moment, angle of twist of the shaft, and temperature rise of the oil should be minimized and near 10.

Priority 2: Angular velocity of the shaft per 100 rad s$^{-1}$ should be minimized and near 0.2.

In formulating the mentioned goal programming problem and finding the dimension of the bearing that is to be built for this purpose, it should be done in such a way that it can carry the maximum load.
Table 1 Optimal values of decision variables using $G^2p^2$ with logarithmic deviational variables

| $W_1$ | $W_2$ | Primal variables | Dual variables | First objective ($f_1^*$) | Second objective ($f_2^*$) |
|---|---|---|---|---|---|
| 0.9 | 0.1 | $x_1^* = 0.34375$, $x_2^* = 0.65625$, $u^* = 1.688724$, $v^* = 1.197751$ | $\Delta_{31} = 1$, $\Delta_{31} = 1.1$ | 6.754896 | 59.88756 |
| 0.8 | 0.2 | $x_1^* = 0.3529412$, $x_2^* = 0.6470588$, $u^* = 1.691804$, $v^* = 1.185288$ | $\Delta_{31} = 1$, $\Delta_{31} = 0.8$, $\Delta_{32} = 1.2$, $\Delta_{32} = 2.2$ | 6.767218 | 59.26442 |
| 0.7 | 0.3 | $x_1^* = 0.36111$, $x_2^* = 0.638889$, $u^* = 1.696088$, $v^* = 1.176257$ | $\Delta_{31} = 1$, $\Delta_{31} = 0.7$, $\Delta_{32} = 1.3$, $\Delta_{32} = 2.3$ | 6.784354 | 58.81286 |
| 0.6 | 0.4 | $x_1^* = 0.3684211$, $x_2^* = 0.6315789$, $u^* = 1.701141$, $v^* = 1.169737$ | $\Delta_{31} = 1$, $\Delta_{31} = 0.6$, $\Delta_{32} = 1.4$, $\Delta_{32} = 2.4$ | 6.804563 | 58.48684 |
| 0.5 | 0.5 | $x_1^* = 0.375$, $x_2^* = 0.625$, $u^* = 1.70667$, $v^* = 1.165084$ | $\Delta_{31} = 1$, $\Delta_{31} = 0.5$, $\Delta_{32} = 1.5$, $\Delta_{32} = 2.5$ | 6.826667 | 58.25422 |
| 0.4 | 0.6 | $x_1^* = 0.3809524$, $x_2^* = 0.6190476$, $u^* = 1.712463$, $v^* = 1.16184$ | $\Delta_{31} = 1$, $\Delta_{31} = 0.4$, $\Delta_{32} = 1.6$, $\Delta_{32} = 2.6$ | 6.849852 | 58.09201 |
| 0.3 | 0.7 | $x_1^* = 0.3836363$, $x_2^* = 0.6136364$, $u^* = 1.718389$, $v^* = 1.159670$ | $\Delta_{31} = 1$, $\Delta_{31} = 0.3$, $\Delta_{32} = 1.7$, $\Delta_{32} = 2.7$ | 6.873558 | 57.98348 |
| 0.2 | 0.8 | $x_1^* = 0.3913043$, $x_2^* = 0.6086957$, $u^* = 1.724348$, $v^* = 1.158324$ | $\Delta_{31} = 1$, $\Delta_{31} = 0.2$, $\Delta_{32} = 1.8$, $\Delta_{32} = 2.8$ | 6.897392 | 57.91620 |
| 0.1 | 0.9 | $x_1^* = 0.3958333$, $x_2^* = 0.6041667$, $u^* = 1.730271$, $v^* = 1.157617$ | $\Delta_{31} = 1$, $\Delta_{31} = 0.1$, $\Delta_{32} = 1.9$, $\Delta_{32} = 2.9$ | 6.921084 | 57.88086 |

Solution Let $R$ (in.) be the radius of the journal and $L$ (in.) be the half length of the bearing, $T$ be the temperature rise of the oil, and frictional moment of the bearing $M = \frac{8\pi \mu \omega \rho^2 L}{\sqrt{1-n^2}c}$ where $\omega$ is the angular velocity of the shaft, $\mu$ is the viscosity of the oil (lubricant), $n$ is the eccentricity ratio, and $c$ is the radial clearance.

The angle of twist of the shaft ($\phi$) = $\frac{\pi L}{Gc}$, where $S_e$ is the shear stress, $l$ is the length between the driving point and rotating mass, and $G$ is the shear modulus. The temperature rise of the oil in the bearing is given by $T = \frac{0.045S_e \rho^2}{c^2\pi\sqrt{1-n^2}}$. For the given data, $\frac{\pi L}{Gc} = 0.0015$, $n = 0.9$, $\mu = 10^{-6}$ lb s in. $^{-2}$, $l = 10$ in., $S_e = 30,000$ psi, and $G = 12 \times 10^6$ psi.
Application on optimal production and marketing planning

Consider a manufacturer who produces a single product where the demand is affected by the selling price. Let $P$ be the selling price per unit, $\alpha$ be the price elasticity to the demand, $M$ be the marketing expenditure per unit, and $\gamma$ be the marketing expenditure elasticity to the demand (Sadjadi et al. 2005). Assume that demand $D = KP^{-\alpha}M^{\gamma}$, where $K$ is the predetermined constant and production cost $C$, which is inversely related to production lot size (units) $Q$, i.e., $C = rQ^{-\beta}$, where $r$ is the predefined constant for unit production cost and $\beta$ is the lot size elasticity of production unit cost. Again, let $\mu$ and $\alpha$ be the production rate and the setup cost of production, respectively. We assume the production rate $\mu$ to vary with the demand $D$ proportionally. Hence, $\mu = uD$ where $u > 1$. There are some restrictions on variables such as $\alpha$, $\gamma$, and $\beta$. The equation $\alpha > 1$ indicates that $D$ increases at a diminishing rate as $P$ decreases. The equation $0 < \beta < 1$ is almost the same as $\alpha$ and $0 < \gamma < 1$.

We want to minimize the equation (Marketing cost + Production cost + Setup cost + Holding cost), which is subject to some constraint that total revenue should be bigger. These are the following goals:

Priority 1: Total revenue should be greater than $0.1386 \times 10^5$,
Priority 2: (Marketing cost + Production cost + Setup cost + Holding cost) should be minimized and near 0.692791.

Thus, the model is as follows:

Minimize $MD + CD + \frac{aD}{Q} + iC \left(1 - \frac{D}{\mu}\right) \frac{Q}{2}$ with target value 0.692791

subject to

$PD \geq 0.1386 \times 10^5$

$P, M, Q > 0$.

Let $\tilde{\alpha} = 1 - \frac{1}{\beta}$, and from assumptions and consideration, the above model becomes the following:

Minimize $KP^{-\tilde{\alpha}}M^{\gamma + 1} + rKP^{-\alpha}M^{\gamma - \beta} + aKP^{-\alpha}M^{\gamma}Q^{-1}$

(12.1)

### Table 2 Comparison of optimal solutions in two different methods

| Method                                           | $W_1$ | $W_2$ | First objective ($f_1^*$) | Second objective ($f_2^*$) |
|--------------------------------------------------|-------|-------|---------------------------|---------------------------|
| Goal geometric programming with weighted sum     | 0.5   | 0.5   | 6.919487                  | 57.88240                  |
| Goal geometric programming with logarithmic     | 0.5   | 0.5   | 6.826667                  | 58.25422                  |
| deviational variables                           |       |       |                           |                           |

Hence, linear combination of frictional moment, angle of twist of the shaft, and temperature rise of the oil equals

$0.038\omega R^2 L + 0.025R^{-1} + 0.592RL^{-3}$ with target value 10

(11.1)

and angular velocity

$\omega \geq 100 \text{ rad s}^{-1}$

(11.2)

From the given data in the chart of ‘Dimensionless performance parameters for full journal bearing’ $\omega R^{-1}L^3 = 11.6$, i.e., $\omega = 11.6R/L^3$.

As per the assumption that 1 in. lb of frictional moment in bearing is equal to 0.0025 rad angle of twist, Equation 11.1 becomes $Z_1 = 0.44R^2L^{-2} + 10R^{-1} + 0.592RL^{-3}$ with the target value of 10.

Equation 11.2 becomes $Z_2 = 8.62R^{-1}L^3$ with the target value of 0.2. Hence, the model of lightly loaded bearing problem in $G^2P^2$ with logarithmic deviational variable is as follows:

Minimize $uW_1vW_2$

subject to

$0.44R^2L^{-2}u^{-1} + 10R^{-1}u^{-1} + 0.592RL^{-3}u^{-1} \leq 10,$

$8.62R^{-1}L^3v^{-1} \leq 0.2,$

$u, v > 1, R, L > 0.$

In solving with the use of geometric programming method where the degree of difficulty is $5 - (4 + 1) = 0$, we get the optimal values of the radius of the journal ($R$) and half length of the bearing ($L$) which are given in Table 3. From the table, we have seen that each deviation ($u, v$) has values greater than 1. Thus, the solution is pareto optimal.

### Table 3 Optimal values of radius of the journal ($R$) and half length of the bearing ($L$)

| $W_1$ | $W_2$ | Dual variables | Primal variables | First objective ($Z_1$) | Second objective ($Z_2$) |
|-------|-------|----------------|------------------|-------------------------|-------------------------|
| 0.6   | 0.4   | $\Delta_{21} = 1,$ | $R^* = 0.97395,$ | 35.2027                 | 0.2485057               |
|       |       | $\Delta_{11} = 0.075,$ | $L^* = 0.3039405,$ |                         |                         |
|       |       | $\Delta_{12} = 0.175, \Delta_{13} = 0.35,$ | $u^* = 3.520266,$ |                         |                         |
|       |       | $\Delta_{21} = 0.4,$ | $v^* = 1.24253,$ |                         |                         |
Consider the following data: $a = 2.5$, $\beta = 0.01$, $\gamma = 0.03$, $r = 5$, $K = 10^6$, $a = 50$, $i = 0.1$, $\hat{u} = 0.7$, and converting the model (12.1) according to the goal geometric programming model, we have the following:

Minimize $(Z) 10^6 P^{-2.5} M^{1.03} + 5 \times 10^6 P^{-2.5} M^{0.03} Q^{-0.01}$

$$+ 50 \times 10^6 P^{-2.5} M^{0.03} Q^{-1} + \frac{0.1 \times 0.7 \times 5}{2}$$

$$\times Q^{0.99} \text{ with target value } 0.692791$$

(12.2)

subject to

$$0.1386 \times 10^5 \times 10^{-6} P^{1.5} M^{-0.03} \leq 1$$

$$P, M, Q > 0.$$}

Transforming the model (12.2) into $G^2 p^2$ with logarithmic deviational variables, we get the following:

Minimize $u^{W_1} v^{W_2}$

(12.3)

subject to

$$\frac{10^6 P^{-2.5} M^{1.03} u^{-1}}{0.692791} + \frac{5 \times 10^6 P^{-2.5} M^{0.03} Q^{-0.01} u^{-1}}{0.692791}$$

$$+ \frac{50 \times 10^6 P^{-2.5} M^{0.03} Q^{-1} u^{-1}}{0.692791}$$

$$+ \frac{0.1 \times 0.7 \times 5 Q^{0.99} u^{-1}}{2} \leq 1$$

$$0.1386 \times 10^5 \times 10^{-6} P^{1.5} M^{-0.03} v^{-1} \leq 1$$

$$P, M, Q > 0, u, v > 1.$$
Acknowledgements
The authors are thankful to the reviewers for their suggestions on improving the quality of this paper as well as to the language editor whose suggestions helped improve the presentation of the paper. The authors are grateful to Mr. Saroj Kumar Rana, assistant professor at the Department of Mechanical Engineering, Adamas Institute of Technology, for the help in the application part of this paper.

Author details
1Department of Mathematics, Adamas Institute of Technology, Barasat, North 24 Parganas 700126, West Bengal, India. 2Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah 711103, West Bengal, India.

Received: 29 September 2012 Accepted: 12 March 2013 Published: 17 April 2013

References
Charnes, A, Cooper WW: Goal programming and multiple objective optimization. European J Oper Res 1977, 1:39–54.
Charnes, A, Cooper W, Ferguson R: Optimal estimation of executive compensation by linear programming. Manag Sci 1955, 1:138–151.
Ghosh, P, Roy TK: Goal geometric programming problem ($G^2P^2$) with product method. In: International Conference of IMBIC on Mathematical Sciences for Advancement of Science and Technology. Kolkata, 2012. 21–23 December 2012.
Ignizio, JP: Goal programming and extensions. Lexington: Lexington Series, DC Heath and Company; 1976.
Ignizio, JP, Romero C: Goal programming. In: Bidgoli H (ed) Encyclopedia of information systems, vol 2. San Diego, Academic Press; 2003. pp 489–500.
Iyri, Y: Management goals and accounting for control. Amsterdam: North Holland; 1965.
Jones, DF, Tamiz M: Goal programming in the period 1990–2000. In: Ehrgott M, Gandibleux X (eds) Multi-criteria optimization: state of the art annotated bibliographic surveys. Dordrecht, Kluwer; 2002. pp 129–170.
Lee, SM: Goal programming for decision analysis. Philadelphia: Auerbach Publishers; 1972.
Lee, SM, Olson D: Goal programming. In: Gal T, Stewart Tj, Hanne, T (eds) Multicriteria decision making: advances in MCDM models, algorithms, theory, and applications. Boston: Kluwer Academic Publishers; 2000.
Miettinen, K: Nonlinear multi-objective optimization. Boston: Kluwer Academic Publishers; 1999.
Romero, C: Handbook of critical issues in goal programming. Oxford: Pergamon Press; 1991.
Romero C: Continuous optimization: a general structure of achievement function for a goal programming model. Eur J Oper Res 2004, 153:675–686.
Sadjadi, SJ, Oroojee M, Anyanrehad MB: Optimal production and marketing planning. Comput Optimization Appl 2005, 30:195–203.
Tamiz, M, Jones DF, Romero C: Goal programming for decision making: an overview of the current state-of-the-art. Eur J Oper Res 1998, 111:569–581.
Verma, RK: Fuzzy geometric programming with several objective functions. Fuzzy Sets Syst 1990, 35(1):115–120.
Wang, YM, Yang JB, Xu DX: A two-stage logarithmic goal programming method for generating weights from interval comparison matrices. Fuzzy Sets Syst 2005, 152:475–498.