Research Article

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Urysohn and Hammerstein operators on Hölder spaces

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Abstract: We present an application-oriented approach to Urysohn and Hammerstein integral operators acting between spaces of Hölder continuous functions over compact metric spaces. These nonlinear mappings are formulated by means of an abstract measure theoretical integral involving a finite measure. This flexible setting creates a common framework to tackle both such operators based on the Lebesgue integral like frequently met in applications, as well as e.g. their spatial discretization using stable quadrature/cubature rules (Nyström methods). Under suitable Carathéodory conditions on the kernel functions, properties like well-definedness, boundedness, (complete) continuity and continuous differentiability are established. Furthermore, the special case of Hammerstein operators is understood as composition of Fredholm and Nemytskii operators. While our differentiability results for Urysohn operators appear to be new, the section on Nemytskii operators has a survey character. Finally, an appendix provides a rather comprehensive account summarizing the required preliminaries for Hölder continuous functions defined on metric spaces.

Keywords: Urysohn integral operator, Hammerstein integral operator, Nemytskii operator, nonlinear operator, Hölder continuity, Lipschitz continuity

MSC: Primary: 47H30, Secondary: 45P05; 45G15

1 Introduction

This treatise is devoted to Urysohn operators, a class of nonlinear integral operators arising in various contexts of nonlinear analysis [4, 30, 38, 40, 41], as right-hand sides of certain integrodifferential (Barbashin) equations [7], as well as in recent applications from control theory [28], mathematical biology [4], economic theory [20] (integral over an unbounded domain) or system identification [15] (sums as integrals). Urysohn operators are traditionally well-studied when acting between spaces of continuous functions over a compact domain [38, pp. 164ff, Sect. V.3], [48, pp. 35–37, Sect. 3.1] or [46, App. B.2], spaces of integrable functions [32] with possibly different exponents, or in mixed form [32, pp. 175ff]. In such a set-up, their well-definedness and continuity is addressed e.g. in [38, pp. 172ff], [44, p. 85], while conditions yielding that they are set contractions w.r.t. ambient spaces of non-compactness can be found in [2, pp. 227ff] (for $L^p$-spaces). Both necessary and sufficient conditions for the complete continuity of Urysohn operators between different function spaces are given in [41]. Furthermore, [32, pp. 162–298] provides an extensive analysis of such mappings between abstract ideal spaces; see also [34]. Properties of Urysohn operators over compact intervals having values in a real Banach space are discussed in [25, pp. 54–90, Sect. 2.1] and differentiability conditions were given in [19] (see also [30, pp. 41ff] or [32, pp. 417ff, Sect. 20] in $L^p$-spaces). Finally, we would like to point out the paper [34] containing complete continuity results for Urysohn operators on the continuous functions over merely locally compact (and possibly unbounded) domains.

A highly relevant special case is given in terms of Hammerstein operators [30, 32, 38, 41]. Our given approach tackles them as composition of (linear) Fredholm integral operators determined by an integral kernel [21, 26, 33] with (nonlinear) Nemytskii operators [8, 18, 23, 39, 40, 41, 44]. Classically Hammerstein

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operators arise in fixed point problems related to nonlinear boundary value problems \cite{38}, pp. 177ff, Sect. V.5] or \cite{38} p. 71, Thm. 5.5], where the kernel is a corresponding Green’s function. A more recent application are integrodifference equations originating in theoretical ecology \cite{31}\cite{39}, where the kernel models the spatial dispersal of species over a habitat, while the Nemyskii operator describes their growth phase. The classical \(L^{p}\)-theory of Hammerstein operators is covered for instance in \cite{11} p. 84 or \cite{38} pp. 68ff, Sect. 5.3.\]

The paper at hand supplements the above contributions. We provide a comprehensive approach to Urysohn operators acting between possibly different spaces of Hölder continuous functions over compact metric spaces. We restrict to Hölder spaces with exponents \(\alpha \leq 1\), i.e. the functions under consideration are not necessarily differentiable with Hölder continuous derivatives of positive order. This endows us with a wide scale of Banach spaces whose elements range from nowhere differentiable to Lipschitz functions, being differentiable almost everywhere. On the one hand, an early contribution to this area is the note \cite{43} addressing well-definedness and complete continuity of general Urysohn operators. On the other hand, Hölder spaces are meanwhile widely used when dealing with linear integral operators having singular kernels \cite{33} pp. 103ff, Ch. 7] or in the field of (quasilinear) elliptic boundary value problems \cite{24}; moreover, \cite{19} investigates nonlinear integral equations in Hölder spaces. Our motivation, nevertheless, is different. It rather comes from the numerical analysis of integral equations \cite{10} and the numerical dynamics of integrodifference equations \cite{31}\cite{36}\cite{35}. In the latter context one aims to show that such infinite-dimensional dynamical systems given by the iterates of integral operators share the long term dynamics with their spatial discretizations. Certain problems in this area require to establish that Fréchet derivatives of Urysohn operators and of their spatial discretization converge to each other in the operator norm. Among the techniques for the numerical solution of integral equations this can be justified for semi-discretizations of projection or degenerate kernel type, cf. \cite{10}\cite{26}\cite{33} and \cite{16}. However, uniform convergence is not feasible when working with full discretizations of Nyström type on the continuous functions (see e.g. \cite{33} p. 225, Thm. 12.8]). In contrast, when working with Hölder spaces appropriate estimates can be established \cite{17}.

Having applications from theoretical ecology to numerical dynamics in mind, it is advantageous to establish a rather flexible setting we are aiming to provide here: First, we consider vector-valued operators (in finite dimensions though), which arise in ecological models describing various interacting species. Second, we allow general measure theoretical integrals induced by a finite measure such that both integral operators based on the Lebesgue integral, as well as their spatial discretization using e.g. Nyström methods fit into a common framework (see Ex. 2.2). For this reason we content ourselves to provide sufficient conditions guaranteeing that an integral operator is well-defined, bounded, (completely) continuous or differentiable. Necessary conditions for the above properties exist for operators on compact intervals, but are beyond the scope of this paper.

Our presentation is subdivided into three parts: In Sect. 2 we provide conditions of Carathéodory type on the kernel functions such that the associated Urysohn operators are well-defined, bounded, (completely and Hölder) continuous, resp. continuously differentiable. We successively study such operators, first having values in the continuous, and second in the Hölder functions. In particular, a subsection is devoted to convolutive Urysohn operators \(\mathbb{U}\), where Hölder continuity of the arguments \(u\) extends to the values \(\mathbb{U}(u)\). The Sect. 3 on Hammerstein operators follows a similar scheme. These mappings are compositions of Fredholm and Nemyskii operators. Since Nemyskii operators between Hölder spaces have rather degenerate mapping and differentiability properties \cite{8} Ch. 7], we retreat to the case that they map into the continuous functions. Hölder continuity of the images is then guaranteed by appropriate assumptions on the kernel of the subsequent Fredholm operator. Addressing well-studied objects, the Sect. 3.2 on Nemyskii operators between Hölder spaces has a survey character. Finally, the App. A provides a broad perspective over the class of Hölder continuous functions defined on a metric space and having values in a normed space.

**Notation and terminology:** Let \(\mathbb{R} := [0, \infty)\) and \(X, Y\) be nonempty sets. We write \(F(X, Y)\) for the set of all functions \(f : X \to Y\). In the setting of metric spaces \(X, Y\), a subset \(\Omega \subseteq X\) is called bounded, if it has finite diameter
\[
\text{diam} \Omega := \sup_{x, \bar{x} \in \Omega} d(x, \bar{x}).
\]

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A function \( f : X \to Y \) is called \textit{bounded}, if it maps bounded sets into bounded sets, i.e. \( f(\Omega) \subseteq Y \) is bounded for every bounded \( \Omega \subseteq X \), and \textit{globally bounded}, if \( f(X) \subseteq Y \) is bounded. A \textit{completely continuous} mapping is continuous and maps bounded sets into relatively compact images.

If \( X, Y \) are normed spaces, then \( L_k(X, Y) \), \( k \in \mathbb{N}_0 \), is the normed space of continuous \( k \)-linear maps from \( X^k \) to \( Y \), where \( L_0(X, Y) := Y \) and \( L(X, Y) := L_1(X, Y) \). We write \( B_r(x_0, X) := \{ x \in X : \| x - x_0 \| < r \} \) for the open and \( B_r(x_0, X) := \{ x \in X : \| x - x_0 \| \leq r \} \) for the closed \( r \)-ball around \( x_0 \in X \) in \( \langle X, \| \cdot \| \rangle \). Norms on finite-dimensional spaces are denoted as \(| | \) and \( B_r(x_0) \), \( B_r(x_0) \) are the corresponding \( r \)-balls.

The remaining introduction anticipates notions from App. A on Hölder spaces: A function \( u : \Omega \to \mathbb{R}^n \) on a metric space \((\Omega, d)\) is called \( \alpha \)-\textit{Hölder} (with \textit{Hölder exponent} \( \alpha \in (0, 1) \)), if it satisfies

\[
[u]_\alpha := \sup_{x, \bar{x} \in \Omega, x \neq \bar{x}} \frac{|u(x) - u(\bar{x})|}{d(x, \bar{x})^\alpha} < \infty;
\]

the finite quantity \([u]_\alpha\) is denoted as \textit{Hölder constant} of \( u \). One speaks of a \textit{Hölder continuous} function \( u \), if it is \( \alpha \)-Hölder for some \( \alpha \in (0, 1) \), in case \( \alpha = 1 \) one denotes \( u \) as \textit{Lipschitz continuous} with \textit{Lipschitz constant} \([u]_1\) and for convenience we denote a continuous function also as \( 0 \)-Hölder. For the linear space of all bounded and \( \alpha \)-Hölder functions we write \( C^\alpha_n(\Omega) := C^\alpha(\Omega, \mathbb{R}^n) \), supplemented by \( C_0^\alpha(\Omega) := C^0(\Omega, \mathbb{R}^n) \) for the bounded, \( \alpha \)-Hölder functions and \( C^\alpha_n(\Omega) := C^\alpha(\Omega, \mathbb{R}) \). Note that \( C^\alpha_n(\Omega) \) is a Banach space w.r.t. the \textit{norm} (cf. Thm. A.11)

\[
\|u\|_\alpha := \begin{cases} \sup_{x \in \Omega} |u(x)|, & \alpha = 0, \\ \max\{\sup_{x \in \Omega} |u(x)|, [u]_\alpha\}, & \alpha \in (0, 1]. \end{cases}
\]

Throughout the remaining paper, our set-up is as follows: Let \( \Omega \) and \( \Omega_1 \) be metric spaces. Suppose additionally that \( \Omega \) is compact and can be interpreted as measure space \((\Omega, \mathcal{A}, \mu)\) with \( \mu(\Omega) < \infty \) whose \( \sigma \)-algebra \( \mathcal{A} \) contains the Borel sets. The notions of measurability and integrability refer to this measure space from now on. In particular, \( f \in L_1(\Omega, d\mu) \) stands for the abstract integral associated to the measure \( \mu \) (e.g. \([15]\)).

Moreover, \( Z \subseteq \mathbb{R}^n \) denotes a nonempty subset throughout. Given a Hölder exponent \( \alpha \in [0, 1] \) we write

\[
U_\alpha := \{ u : \Omega \to Z \mid u \in C^\alpha_n(\Omega) \}
\]

for the \( \alpha \)-Hölder functions over \( \Omega \) having values in \( Z \). If \( 0 \leq \alpha \leq \beta \leq 1 \), then Thm. A.13 guarantees the embedding \( U_\beta \subseteq U_\alpha \) between the continuous and the Lipschitz continuous functions.

## 2 Urysohn integral operators

An \textit{Urysohn operator} is a nonlinear integral operator of the form

\[
\mathcal{U} : U_\alpha \to F(\Omega_1, \mathbb{R}^d), \quad \mathcal{U}(u) := \int_{\Omega} f(\cdot, y, u(y)) \, d\mu(y)
\]  

(2.1)

determined by a \textit{kernel function} \( f : \Omega_1 \times \Omega \times Z \to \mathbb{R}^d \) and a measure \( \mu \) as above. Its overall analysis is based on the following Carathéodory like conditions:

**Hypothesis.** Let \( m \in \mathbb{N}_0 \). With \( 0 \leq k \leq m \) one assumes:

\( (U^k_1) \) The partial derivative \( D^k f(x, y, \cdot) : Z \to L_k(\mathbb{R}^n, \mathbb{R}^d) \) exists and is continuous for all \( x \in \Omega_1 \) and almost all \( y \in \Omega \),

\( (U^m_1) \) for all \( r > 0 \) there exists a function \( h^k : \Omega^2_1 \times \Omega \to \mathbb{R}_+ \), measurable in the third argument and satisfying

\[
\lim_{x \to x_0} \int_{\Omega} h^k_r(x, x_0, y) \, d\mu(y) = 0 \quad \text{for all } x_0 \in \Omega_1,
\]  

(2.2)

\[\text{1 note that } C^1_n(\Omega) = C^1(\Omega, \mathbb{R}^n) \text{ abbreviates the Lipschitz continuous and not the continuously differentiable functions} \]

\[\text{2 also denoted as \textit{nonlinear Fredholm operator}}. \text{ Another transcription is \textit{Uryson operator}}\]
so that for almost all \( y \in \Omega \) the following holds:

\[
D^k_b f(x, y, z) - D^k_b f(x_0, y, z) \leq b^k_b(x, x_0, y) \quad \text{for all } x, x_0 \in \Omega_1, \ z \in Z \cap \overline{B}_r(0),
\]

(2.3)

\((U^k_b)D^k_b f(x, \cdot, z) : \Omega \to L_k(\mathbb{R}^n, \mathbb{R}^d)\) is measurable for all \( x \in \Omega_1, \ z \in Z, \) and suppose that for every \( r > 0 \) there exists a function \( b^k_b : \Omega_1 \times \Omega \to \mathbb{R}_+ \) measurable in the second argument and satisfying

\[
\text{ess sup}_{x \in \Omega_1} \int_{\Omega} b^k_b(x, y) \, d\mu(y) < \infty, \quad \text{so that for almost all } y \in \Omega \text{ the following holds:}
\]

\[
D^k_b f(x, y, z) \leq b^k_b(x, y) \quad \text{for all } x \in \Omega_1, \ z \in Z \cap \overline{B}_r(0).
\]

(2.4)

Because we are working with a general (finite) measure on \( \Omega, \) both spatially continuous and discrete integral operators fit into our framework:

**Example 2.1** (Lebesgue measure). In most applications, e.g. \([3, 20, 31, 36]\), \( \mu \) is the \( \kappa \)-dimensional Lebesgue measure \( \lambda_\kappa \) on compact sets \( \Omega \subset \mathbb{R}^\kappa \) yielding the Lebesgue integral in (2.1) and thus

\[
\mathcal{U}(u) = \int_{\Omega} f(\cdot, y, u(y)) \, d\lambda_\kappa(y) = \int_{\Omega} f(\cdot, y, u(y)) \, dy : \Omega_1 \to \mathbb{R}^d
\]

(2.5)

is a spatially continuous integral operator. One clearly has \( \mu(\Omega) < \infty. \)

**Example 2.2** (Nyström methods). Suppose that \( \Omega \subset \mathbb{R}^\kappa \) is a countable set \( \Omega^{(i)}, \eta \in \Omega^{(i)} \) and \( w_\eta \) denote non-negative reals. Then \( \mu(\Omega^{(i)}) := \sum_{\eta \in \Omega^{(i)}} w_\eta \) defines a measure on the family of countable subsets of \( \mathbb{R}^\kappa \) and precisely the empty set has measure 0. Moreover, the assumption \( \sum_{\eta \in \Omega^{(i)}} w_\eta < \infty \) guarantees that \( \mu(\Omega^{(i)}) \) is finite. The resulting \( \mu \)-integral \( \int_{\Omega} u \, d\mu = \sum_{\eta \in \Omega^{(i)}} w_\eta u(\eta) \) leads to spatially discrete Urysohn operators

\[
\mathcal{U}(u) = \int_{\Omega^{(i)}} f(\cdot, y, u(y)) \, d\mu(y) = \sum_{\eta \in \Omega^{(i)}} w_\eta f(\cdot, \eta, u(\eta)) : \Omega_1 \to \mathbb{R}^d,
\]

which cover Nyström methods with nodes \( \eta \) and weights \( w_\eta \) as used for numerical approximations of spatially continuous integral operators \([2,5], \) cf. \([9\text{ Sect. 3}], \) \([26\text{ pp. 128ff, Sect. 4.7]} \) or \([33\text{ pp. 219ff, Ch. 12]} \) (the latter two references address linear operators only). Alternatively, such mappings arise in theoretical ecology by means of models for populations spreading between finitely many different patches (metapopulation models, see \([31\text{ Example 1]} \]).

**Example 2.3** (evaluation map). In case of singletons \( \Omega = \{ \eta \} \) and the measure from Ex. 2.2 one obtains that \( F(\Omega, \mathbb{R}^n) \cong \mathbb{R}^n \) and the Urysohn operator (2.1) becomes an evaluation map \( \mathcal{U}(u) = w_\eta f(\cdot, \eta, u(\eta)) \), which is simply a mapping from \( \mathbb{R}^n \) into \( F(\Omega_1, \mathbb{R}^d). \)

**Remark 2.1** (differentiability on \( Z \)). We imposed no further conditions of the sets \( Z \subseteq \mathbb{R}^n \) and therefore some remarks on the existence of the partial derivative \( D^k_b f \) for \( k > 0 \) are due:

1. For interior points of \( Z \) the partial derivatives are understood in the Fréchet sense.
2. If \( z_0 \in Z \) is not an interior point of \( Z, \) then we assume that there exists a neighborhood \( V \subseteq \mathbb{R}^n \) of \( z_0 \) and an extension \( \hat{f} : \Omega_1 \times (\mathbb{R}^n \cup \{ V \}) \to \mathbb{R}^d \) such that the partial derivatives \( D^k_b(\hat{f}(x, \cdot)) \) exist in \( z_0 \) as assumed in \((U^k_b)^{(0)}\). Alternatively, there is the notion of cone differentiability \([16\text{ pp. 225–226]} \].

Under continuity the above hypothesis can be simplified as follows:

**Proposition 2.1.** Let \( k \in \mathbb{N}_0, \ \Omega_1 \) be compact and \( Z \subseteq \mathbb{R}^n \) be closed. If the partial derivative \( D^k_b f : \Omega_1 \times \Omega \times Z \to L_k(\mathbb{R}^n, \mathbb{R}^d) \) exists as continuous function, then \((U^k_b, U^k_b, U^k_b)\) are satisfied and the limit relation (2.2) holds uniformly in \( x_1 \in \Omega_1. \)

**Proof.** Since \( Z \) is assumed to be closed, \( Z_r := Z \cap \overline{B}_r(0) \subseteq \mathbb{R}^n \) is compact for all \( r > 0. \) Then the continuous function \( D^k_b f \) is uniformly continuous and globally bounded on each compact product \( \Omega_1 \times \Omega \times Z_r. \) Moreover, since the \( \sigma \)-algebra \( \mathfrak{A} \) contains the Borel sets, continuous functions are measurable. Given this, the assertions
hold with the continuous functions
\[ h^k_r(x, x_0, y) := \sup_{z \in \mathbb{Z}} |D^k f(x, y, z) - D^k f(x_0, y, z)|, \quad b^k_r(x, y) := \sup_{z \in \mathbb{Z}} |D^k f(x, y, z)|. \]
This concludes the proof. \(\square\)

### 2.1 Well-definedness and complete continuity

We begin with basic properties of Urysohn operators [21] and assume \(\alpha \in (0, 1]\):

**Proposition 2.2** (well-definedness of \(\mathbb{U}\)). Assume that \((U^0_0, U^0_1, U^0_2)\) hold. Then an Urysohn operator \(\mathbb{U} : U_\alpha \to C^0_\alpha(\Omega_1)\) is well-defined, bounded and continuous.

**Proof.** W.l.o.g. let \(\mu(\Omega) > 0\) since otherwise \(\mathbb{U}(u) \equiv 0\) on \(U_\alpha\).

(I) Claim: \(\mathbb{U} : U_\alpha \to C^0_\alpha(\Omega_1)\) is well-defined and bounded.

Choose \(u \in U_\alpha\) and \(r > 0\) such that \(\|u\|_0 \leq r\). Given \(x, x_0 \in \Omega_1\) the Carathéodory conditions \((U^0_0, U^0_2)\) yield that \(f(x, \cdot, u(\cdot)) : \Omega \to \mathbb{R}^d\) is measurable (see [18, p. 62, Lemma 5.1]). From \((U^0_1)\) we conclude
\[
|\mathbb{U}(u)(x) - \mathbb{U}(u)(x_0)| \leq \int_\Omega |f(x, y, u(y)) - f(x_0, y, u(y))| \, d\mu(y) \leq \int_\Omega h^0_r(x, x_0, y) \, d\mu(y) \to 0
\]
for each \(x_0 \in \Omega_1\), which guarantees that \(\mathbb{U}(u)\) is continuous. Furthermore, because \((U^0_2)\) yields
\[
|\mathbb{U}(u)(x)| \leq \int_\Omega |f(x, y, u(y))| \, d\mu(y) \leq \epsilon \sup_{\xi \in \Omega_1} \int_\Omega b^0_\epsilon(\xi, y) \, d\mu(y)
\]
we see that \(\mathbb{U}(u)\) is bounded and thus \(\mathbb{U}(u) \in C^0_\alpha(\Omega_1)\). In addition, \(\mathbb{U}\) maps bounded subsets of \(U_0\) into bounded subsets of \(C^0_\alpha(\Omega_1)\).

(II) Claim: \(\mathbb{U} : U_\alpha \to C^0_\alpha(\Omega_1)\) is continuous.

Let \(u \in U_\alpha\) and \((u_l)_{l \in \mathbb{N}}\) be a sequence in \(U_\alpha\) with \(\lim_{l \to \infty} \|u_l - u\|_0 = 0\) and \(r > 0\) sufficiently large so that \(u, u_l \in B(r, C^0_\alpha(\Omega))\) holds for all \(l \in \mathbb{N}\). Using \((U^0_0)\) this gives \(\lim_{l \to \infty} f(x, y, u_l(y)) = f(x, y, u(y))\) for all \(x \in \Omega_1\) and almost all \(y \in \Omega\). For each \(\epsilon > 0\) there exists a \(\delta > 0\) such that subsets \(\Omega \subseteq \Omega_1\) with \(\mu(\Omega) \leq \delta\) fulfill \(\int_{\Omega} b^0_\epsilon(x, y) \, d\mu(y) \leq \frac{\epsilon}{2}\) and Egoroff’s theorem [13, p. 87, Prop. 3.1.3] guarantees that there exist a \(\Omega' \subseteq \Omega\) with \(\mu(\Omega') \leq \delta\) and an \(L \in \mathbb{N}\) such that \(\int_{\Omega'} |f(x, y, u_l(y)) - f(x, y, u(y))| \leq \frac{\epsilon}{22_m(\Omega)}\) for all \(x \in \Omega_1, y \in \Omega \setminus \Omega'\) and \(l \geq L\).

This implies that we have pointwise convergence due to
\[
\left|\mathbb{U}(u_l)(x) - \mathbb{U}(u)(x)\right| \leq \int_{\Omega \setminus \Omega'} |f(x, y, u_l(y)) - f(x, y, u(y))| \, d\mu(y) + \int_{\Omega'} |f(x, y, u_l(y)) - f(x, y, u(y))| \, d\mu(y)
\]
\[
\leq \int_{\Omega \setminus \Omega'} b^0_\epsilon(\xi, y) \, d\mu(y) + 2 \int_{\Omega'} b^0_\epsilon(\xi, y) \, d\mu(y) \leq \frac{\epsilon}{2} + 2 \frac{\epsilon}{2} = \epsilon
\]
for all \(x \in \Omega_1, l \geq L\).

Passing to the supremum over \(x \in \Omega_1\) yields the limit relation
\[
\lim_{l \to \infty} \|\mathbb{U}(u_l) - \mathbb{U}(u)\|_0 = 0. \tag{2.6}
\]
This shows the continuity of \(\mathbb{U}\). \(\square\)

**Corollary 2.3** (complete continuity of \(\mathbb{U}\)). An Urysohn operator \(\mathbb{U} : U_\alpha \to C^0_\alpha(\Omega_1)\) is completely continuous, provided one of the following holds:
(i) \(\alpha \in (0, 1]\),
(ii) \(\Omega_1\) is compact, \(\alpha = 0\), the limit relation [22] holds uniformly in \(x_0 \in \Omega_1\).
Proof. We write \( u_0 : U_0 \to C^\alpha_d(\Omega_1) \) for the operator defined in (2.1).

(I) For exponents \( \alpha \in (0, 1] \) we observe \( \mathcal{U} = \mathcal{U}_0 \circ J_0^\alpha \) with the compact embedding operator \( J_0^\alpha \) from (A.3) (cf. Thm. A.13). Therefore, \( \mathcal{U} \) inherits the claimed properties from the steps (I) and (II) of the proof to Prop. 2.2. In particular, as composition of the continuous \( u_0 \) with the compact \( J_0^\alpha \), it is completely continuous due to e.g. [58, pp. 25–26, Thm. 2.1(2)].

(II) Claim: If (ii) holds, then \( u_0 : U_0 \to C^\alpha_d(\Omega_1) \) is completely continuous.

In Prop. 2.2 it was shown that \( u_0 : U_0 \to C^\alpha_d(\Omega_1) \) is bounded and continuous. If \( \alpha = 0 \) and (2.2) holds uniformly in \( x_0 \in \Omega_1 \), then the first limit exists in the proof of Prop. 2.2 is true uniformly in \( x_0 \) as well, and each image \( u_0(U_0 \cap \bar{B}_r(0, C^\alpha_d(\Omega_1))) \subseteq C^\alpha_d(\Omega_1) \) is equicontinuous. Therefore, the Arzelà-Ascoli theorem [52, pp. 162–298] provides an extensive analysis of such mappings between different function spaces.

Corollary 2.4. Let \( \Omega_1 \) be compact and \( Z \subseteq \mathbb{R}^n \) be closed. If \( f : \Omega_1 \times \Omega \times Z \to \mathbb{R}^d \) is continuous, then \( u : U_0 \to C^\alpha_d(\Omega_1) \) is completely continuous and uniformly continuous on each set \( U_0 \cap \bar{B}_r(0, C^\alpha_d(\Omega_1)), r > 0 \).

Proof. Due to Prop. 2.1 the assumptions of Prop. 2.2 and Cor. 2.3 are fulfilled. Therefore, \( \mathcal{U} \) is completely continuous. Hence, it remains to show the uniform continuity on bounded sets. For this purpose, let \( \varepsilon > 0 \) and \( r > 0 \). Because \( Z \) is closed, \( Z_r := Z \cap \bar{B}_r(0) \subseteq \mathbb{R}^n \) is compact and since the continuous function \( f \) is uniformly continuous on the compact \( \Omega_1 \times \Omega \times Z_r \), there exists a \( \delta > 0 \) such that for all \( x, y \in \Omega_1 \) and \( z, \varepsilon \in Z_r \) one has \( |z - \varepsilon| < \delta \Rightarrow |f(x, y, z) - f(x, y, \varepsilon)| < \varepsilon \). Let \( u, \bar{u} \in U_0 \cap \bar{B}_r(0, C^\alpha_d(\Omega_1)) \) with \( ||u - \bar{u}||_0 < \delta \). Then the inclusions \( u(y), \bar{u}(y) \in Z_r \) and the estimate \( |u(y) - \bar{u}(y)| < \delta \) hold for all \( y \in \Omega_1 \). They yield
\[
|\mathcal{U}(u)(x) - \mathcal{U}(u_0)(x)| \leq \int \limits_{\Omega} |f(x, y, u(y)) - f(x, y, u_0(y))| \, d\mu(y) \leq \frac{\varepsilon}{2}
\]
for all \( x \in \Omega_1 \).

and passing to the least upper bound over \( x \in \Omega_1 \) results in \( ||\mathcal{U}(u) - \mathcal{U}(u_0)||_0 < \varepsilon \), i.e. \( \mathcal{U} \) is uniformly continuous on each set \( U_0 \cap \bar{B}_r(0, C^\alpha_d(\Omega_1)) \).

The subsequent assumption allows us to infer Hölder continuity of \( \mathcal{U} \).

Hypothesis. Let \( \vartheta \in (0, 1] \).

(\( U_0^\alpha \)) For every \( r > 0 \) there exists a function \( l_r : \Omega_1 \times \Omega \to \mathbb{R}_+ \) measurable in the second argument and satisfying
\[
\text{ess} \sup_{\xi \in \Omega_1} \int \Omega l_r(\xi, y) \, d\mu(y) < \infty,
\]
so that for almost all \( y \in \Omega \) the following holds:
\[
|f(x, y, z) - f(x, y, \varepsilon)| \leq l_r(\xi, y) |z - \varepsilon|^\vartheta
\]
for all \( x \in \Omega_1 \), \( z, \varepsilon \in Z_r \cap \bar{B}_r(0) \).

(2.7)

Obviously, the condition \( (U_0^\alpha) \) is sufficient for \( (U_0^\alpha) \).

Corollary 2.5. If additionally \( (U_0^\alpha) \) holds, then \( u : U_0 \to C^\alpha_d(\Omega_1) \) is Hölder on bounded sets, that is
\[
||\mathcal{U}(u) - \mathcal{U}(\bar{u})||_0 \leq \text{ess} \sup_{\xi \in \Omega_1} \int \Omega l_r(\xi, y) \, d\mu(y) \text{ for all } r > 0.
\]

Proof. Since \( (U_0^\alpha) \) implies \( (U_0^\alpha) \), we obtain from Prop. 2.2 that \( \mathcal{U} : U_0 \to C^\alpha_d(\Omega_1) \) is well-defined. Given \( r > 0 \), for \( u, \bar{u} \in U_0 \cap \bar{B}_r(0, C^\alpha_d(\Omega_1)) \) we derive from \( (U_0^\alpha) \) that
\[
||\mathcal{U}(u) - \mathcal{U}(\bar{u})||_0 \leq \int \Omega |f(x, y, u(y)) - f(x, y, \bar{u}(y))| \, d\mu(y) \leq \int \Omega l_r(x, y) \, d\mu(y) ||u - \bar{u}||_0^\vartheta
\]
and consequently \( ||\mathcal{U}(u) - \mathcal{U}(\bar{u})||_0 \leq \text{ess} \sup_{\xi \in \Omega_1} \int \Omega l_r(\xi, y) \, d\mu(y) ||u - \bar{u}||_0^\vartheta \) after passing to the least upper bound over all \( x \in \Omega_1 \). 


Let us proceed to Urysohn operators having values in Hölder spaces with positive exponent \( \beta \) rather than in \( C_\delta^\alpha(\Omega_1) \). This requires to sharpen our above assumptions beyond \((U_0^k)\):

**Hypothesis.** Let \( m \in \mathbb{N}_0 \) and \( \beta \in (0,1] \). With \( 0 \leq k \leq m \) one assumes that for every \( r > 0 \) there exists \((U_1^k)\) an integral function \( h_1^k : \Omega \to \mathbb{R}_+ \), so that for almost all \( y \in \Omega \) the following holds:

\[
|D_3^k f(x, y, z) - D_3^k f(x, y, \bar{z})| \leq h_1^k(y) d(x, \bar{x})^\beta \quad \text{for all } x, \bar{x} \in \Omega_1, \ z \in Z \cap B_r(0),
\]  

\( (U_1^k) \) a function \( c_r^k : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+ \) measurable in the second argument with \( \lim_{\delta \to 0} \int_{\Omega} c_r^k(\delta, y) \, d\mu(y) = 0 \), so that for almost all \( y \in \Omega \) and all \( \delta > 0 \) the following holds:

\[
|z - \bar{z}| \leq \delta \Rightarrow \left| D_3^k f(x, y, z) - D_3^k f(x, y, \bar{z}) - \left[ D_3^k f(\bar{x}, y, z) - D_3^k f(\bar{x}, y, \bar{z}) \right] \right| \leq c_r^k(\delta, y) d(x, \bar{x})^\beta
\]  

for all \( x, \bar{x} \in \Omega_1, \ z, \bar{z} \in Z \cap B_r(0) \).

**Remark 2.2.** (1) Note that \((U_1^k)\) implies \((U_2^k)\) with the function \( h_2^k(x, x_0, y) := h_1^k(y) d(x, x_0)^\beta \) and in particular the limit relation \((2.2)\) holds uniformly in \( x_0 \in \Omega_1 \).

(2) Since it might be tedious to verify the implication \((2.10)\), we note some sufficient conditions:

- If \( Z \subseteq \mathbb{R}^n \) is convex, then \((U_1^{k+1})\) implies \((U_2^k)\) with \( c_r^k(\delta, y) := \delta h_2^{k+1}(y) \). Indeed, the Mean Value Theorem [35, p. 341, Thm. 4.2] yields for almost all \( y \in \Omega \) that

\[
|D_3^k f(x, y, z) - D_3^k f(x, y, \bar{z})| = \int_0^1 \left| D_3^{k+1} f(x, y, \tilde{z} + \theta(z - \bar{z})) - D_3^{k+1} f(x, y, \bar{z} + \theta(z - \bar{z})) \right| d\theta |z - \bar{z}|
\]  

\[
\leq c_r^{k+1}(y) |z - \bar{z}| d(x, \bar{x})^\beta \quad \text{for all } x, \bar{x} \in \Omega_1, \ z, \bar{z} \in Z \cap B_r(0).
\]

- Let \( \Omega_1 \subset \mathbb{R}^m \) be bounded and convex. Assume for all \( r > 0 \) there is a function \( \gamma_1 \mapsto \mathbb{R}^+ \times \Omega \to \mathbb{R}_+ \) measurable in the second argument satisfying \( \lim_{\delta \to 0} \int_{\Omega} \gamma^k(\delta, y) \, d\mu(y) = 0 \), so that for almost all \( y \in \Omega \) and all \( \delta > 0 \) the following holds: \( D_1 D_3 f(\cdot, y, z) \in \Omega \to L_1(\mathbb{R}^n, L_k(\mathbb{R}^n, \mathbb{R}^d)) \) exists and is continuous for all \( z \in Z \), almost all \( y \in \Omega \),

\[
|z - \bar{z}| < \delta \Rightarrow \left| D_1 D_3^k f(x, y, z) - D_1 D_3^k f(x, y, \bar{z}) \right| \leq \gamma_1^k(\delta, y) \quad \text{for all } x, \bar{x} \in \Omega_1
\]

and \( z, \bar{z} \in Z \cap B_r(0) \). If \( |z - \bar{z}| < \delta \), then the Mean Value Theorem [35, p. 341, Thm. 4.2] yields

\[
|D_3^k f(x, y, z) - D_3^k f(x, y, \bar{z})| = \int_0^1 \left| D_1 D_3^{k+1} f(\bar{x} + \theta(x - \bar{x}), y, z) - D_1 D_3^{k+1} f(\bar{x} + \theta(x - \bar{x}), y, \bar{z}) \right| d\theta |x - \bar{x}|
\]  

\[
\leq (\operatorname{diam} \Omega_1)^{1-\beta} \gamma_1^k(\delta, y) |x - \bar{x}|^\beta \quad \text{for all } x, \bar{x} \in \Omega_1,
\]

which allows us to choose \( c_r^k(\delta, y) := (\operatorname{diam} \Omega_1)^{1-\beta} \gamma_1^k(\delta, y) \).

However, this requires one higher order of continuous partial differentiability for the kernel function \( f \).

(3) Replacing the function \( c_r^k : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+ \) in \((U_2^k)\) with

\[
c_r^k : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+, \quad c_r^k(\delta, y) := \sup_{\rho \leq \delta} c_r^k(\rho, y)
\]

yields a nondecreasing function \( c_r^k(\delta, y) \leq c_r^k(\delta, y) \) inheriting the other relevant properties from \( c_r^k \).

**Theorem 2.6** (well-definedness of \( U \)). Assume that \((U_0^0, U_1^0, U_2^0)\) hold. Then an Urysohn operator \( U : U_\alpha \to C_\delta^\alpha(\Omega_1) \) is well-defined and bounded. If additionally \((U_3^0)\) holds, then \( U \) is continuous.
Proof. Choose \( u \in U_\alpha \) and \( r > 0 \) so large that \( \|u\|_{\beta} \leq r \) holds. Referring to Rem. \([2,2]\)(1) we can apply Prop. \([2.2]\) which guarantees that \( U : U_\alpha \to C^0_d(\Omega_1) \) is well-defined, bounded and continuous.

(I) Claim: \( U : U_\alpha \to C^0_d(\Omega_1) \) is well-defined and bounded.

Given arbitrary \( x, \bar{x} \in \Omega_1 \), using \((U^0_\gamma)\) the estimate

\[
|U(u)(x) - U(u)(\bar{x})| \leq \int_{\Omega} |f(x, y, u(y)) - f(\bar{x}, y, u(y))| \, d\mu(y) + \int_{\Omega} \|h_r(y)\| \, d\mu(y) \, d(x, \bar{x})^\beta
\]

implies \( U(u) \in C^0_d(\Omega_1) \) \((U \) is well-defined) and \( \sup_{\|u\| \leq r} |U(u)|_{\beta} < \infty \) \((U \) is bounded).

(II) Claim: If \((U^0_\gamma)\) holds, then \( U_0 : U_\alpha \to C^0_d(\Omega_1) \) is continuous.

Let \((u_l)_{l \in \mathbb{N}}\) be a sequence in \( U_\alpha \) satisfying \( \lim_{l \to \infty} \|u_l - u_0\| = 0 \). If \( r > 0 \) is chosen sufficiently large that \( u, u_l \in B_r(\bar{0}, C^0_d(\Omega_1)) \) holds for all \( l \in \mathbb{N} \), then \((U^0_\gamma)\) yields

\[
\|U(u_l) - U(u_l)(x) - U(u)(x)\| \leq \int_{\Omega} |f(x, y, u_l(y)) - f(x, y, u_l(y)) - f(\bar{x}, y, u_l(y))| \, d\mu(y)
\]

\[
\leq \int_{\Omega} c^0_r(|u_l(y) - u_l(y), y) \, d\mu(y) \, d(x, \bar{x})^\beta
\]

\[
\leq \int_{\Omega} c^0_r(‖u_l - u‖_\beta, y) \, d\mu(y) \, d(x, \bar{x})^\beta \quad \text{for all } x, \bar{x} \in \Omega_1,
\]

hence, \( \|U(u_l) - U(u_l)\|_{\beta} \leq \int_{\Omega} c^0_r(‖u_l - u‖_0, y) \, d\mu(y) \). This shows \( \lim_{l \to \infty} \|U(u_l) - U(u)\|_{\beta} = 0 \) and combined with \([2.6]\) the claim results.

Completely continuity of \( U \) can be achieved by e.g. slightly increasing the image space:

**Corollary 2.7** (complete continuity of \( U \)). If additionally \((U^0_\gamma)\) holds, then \( U : U_\alpha \to C^0_d(\Omega_1) \) is completely continuous, provided one of the following holds:

(i) \( \alpha \in (0, 1) \), \( \gamma = \beta \) and \( Z \) is closed,

(ii) \( \Omega_1 \) is bounded, \( \alpha \in (0, 1) \), \( \gamma \in (0, \beta) \) and \( Z \) is closed,

(iii) \( \Omega_1 \) is compact, \( \gamma \in (0, \beta) \), \( (U^0_\gamma) \) holds with \( \lim_{x \to x_0} \int_{\Omega} h^0_r(x, x_0, y) \, d\mu(y) = 0 \) uniformly in \( x_0 \in \Omega_1 \),

(iv) \( \Omega_1 \) is compact and \( \gamma \in (0, \beta) \).

Proof. We write \( U_0 : U_\alpha \to C^0_d(\Omega_1) \) and \( U_0^\beta : U_\alpha \to C^0_d(\Omega_1) \) for the operator given in \([2.1]\).

(I) Claim: If \( \alpha \in (0, 1) \), then \( U_0 : U_\alpha \to C^0_d(\Omega_1) \) is completely continuous.

Let \((u_l)_{l \in \mathbb{N}}\) be a bounded sequence in \( U_\alpha \), i.e. there exists a \( r > 0 \) such that

\[
\sup_{x \in \Omega} |u_l(x)| \leq r, \quad \sup_{x, \bar{x} \in \Omega, x \neq \bar{x}} \frac{|u_l(x) - u_l(\bar{x})|}{a(x, \bar{x})^\alpha} \leq r \quad \text{for all } l \in \mathbb{N}.
\]

Thus, the subset \( \{u_l\}_{l \in \mathbb{N}} \subseteq C^0_d(\Omega) \) is bounded and equicontinuous. By the Arzelà-Ascoli theorem [38, Thm. 3.2] there exists a subsequence \((u_{k_l})_{l \in \mathbb{N}}\) and a \( u \in C^0_d(\Omega) \) with \( \lim_{l \to \infty} \|u_{k_l} - u\| = 0 \). Because \( Z \) is closed, we have \( u(x) \in Z \) for all \( x \in \Omega \), i.e. \( u \in U_0 \). Since \([2.12]\) also holds for each \( u_{k_l} \), passing to the limit \( l \to \infty \) shows \( u \in U_\alpha \). The continuity shown in Thm. \([2.6]\) implies \( \lim_{l \to \infty} \|U_0(u_{k_l}) - U_0(u)\|_{\beta} = 0 \).

This establishes that every bounded sequence in \( U_\alpha(U_0) \subseteq C^0_d(\Omega_1) \) has a convergent subsequence, i.e. the image \( U_0(U_\alpha \cap B_r(\bar{0}, C^0_d(\Omega_1))) \) is relatively compact. Therefore, \( U_0 \) maps bounded subsets of \( U_\alpha \) into relatively compact sets. This shows that \( U_0 \) is completely continuous.

(II) Claim: If \( (iii) \) holds, then \( U_0^\beta : U_0 \to C^0_d(\Omega_1) \) is completely continuous.

Let \( B \subseteq C^0_d(\Omega) \) be bounded and choose \( r > 0 \) so large that \( \|u\|_{\beta} \leq r \) holds for all \( u \in B \). We establish that \( U_0^\beta(B) \subseteq C^0_d(\Omega_1) \) fulfills the assumptions of Thm. \([A.14]\). First, \( U_0^\beta(B) \) is bounded due to Thm. \([2.6]\).
Second, given \( \varepsilon > 0 \) by assumption (iii) there exists a \( \delta > 0 \) such that \( d(x, \bar{x}) \leq \delta \) implies the estimate
\[
\frac{\int_{\Omega} h_r(x, \bar{x}, y) \, d\mu(y)}{d(x, \bar{x})^r} \leq \varepsilon
\]
for all \( x, \bar{x} \in \Omega_1, x \neq \bar{x} \). Consequently, we obtain that
\[
[\|\mathcal{U}(u)(x) - \mathcal{U}(u)(\bar{x})\|_\beta] \leq \frac{\int_{\Omega} |f(x, y, u(y)) - f(\bar{x}, y, u(y))| \, d\mu(y)}{d(\bar{x}, x)^\beta} \leq \varepsilon d(\bar{x}, x)^\beta.
\]

Thus, the bounded \( \mathcal{U}^\beta_1(B) \subset C_\alpha^\beta(\Omega_1) \) is relatively compact. Hence, \( \mathcal{U}^\beta_1 \) is completely continuous.

(III) Under (i) the mapping \( \mathcal{U} = \mathcal{U}_0 \) is completely continuous due to step (I), under (ii) the map \( \mathcal{U} = J_\beta^1 \circ \mathcal{U}_0 \) is a composition with a continuous embedding \( J_\beta^1 \) (cf. Thm. A.13) with the completely continuous \( \mathcal{U}_0 \), under assumption (iii) the operator \( \mathcal{U} = J_\beta^1 \mathcal{U}_0^\beta \mathcal{U}_0^0 \) is a composition of bounded embeddings (see Thm. A.13) with the due to step (II) completely continuous \( \mathcal{U}_0^\beta \) and finally under (iv) the embedding \( J_\beta^1 \) in \( \mathcal{U} = J_\beta^1 \mathcal{U}_0^\beta \mathcal{U}_0^0 \) is compact thanks to Thm. A.15. In conclusion, at least one function in the above compositions is completely continuous and the claim results from [38] pp. 25–26, Thm. 2.1(2).

For a Lipschitz condition we have to invest continuous differentiability of the kernel function:

**Corollary 2.8.** If additionally \( (\mathcal{U}_1^1), (\mathcal{U}_0^0) \) with \( \vartheta = 1 \) hold on a convex set \( Z \subseteq \mathbb{R}^n \), then \( \mathcal{U} : U_\alpha \to C_\vartheta^\beta(\Omega_1) \) is Lipschitz on bounded sets, that is
\[
\left[\|\mathcal{U}(u) - \mathcal{U}([u])\|_\infty \leq \max_{\xi \in \Omega_1} \left\{ \text{ess sup } \int_\Omega \mathcal{L}(\xi, y) \, d\mu(y), \int_\Omega \tilde{h}_r(y) \, d\mu(y) \right\} \right]_1 \quad \text{for all } r > 0.
\]

**Proof.** From Rem. 2.2[1] and Cor. 2.5 we obtain that \( \mathcal{U} : U_\alpha \to C_\vartheta^\beta(\Omega_1) \) is Lipschitz on bounded sets. Let \( r > 0 \) and choose \( u, \tilde{u} \in U_\alpha \cap \overline{B}_t(0, C_\vartheta^\beta(\Omega_1)) \). Since \( Z \) is convex, the inclusion \( u(y) + \theta(u(y) - \tilde{u}(y)) \in Z \) holds for all \( y \in \Omega \) and \( \theta \in [0,1] \). Hence, [35] p. 341, Thm. 4.2] applies and implies
\[
[\|\mathcal{U}(u) - \mathcal{U}([u])\|_\infty \leq \max_{\xi \in \Omega_1} \left\{ \text{ess sup } \int_\Omega \mathcal{L}(\xi, y) \, d\mu(y), \int_\Omega \tilde{h}_r(y) \, d\mu(y) \right\} \quad \text{for all } r > 0.
\]
and consequently \( (\mathcal{U}_1^1) \) leads to
\[
[\|\mathcal{U}(u) - \mathcal{U}([u])\|_\infty \leq \max_{\xi \in \Omega_1} \left\{ \text{ess sup } \int_\Omega \mathcal{L}(\xi, y) \, d\mu(y), \int_\Omega \tilde{h}_r(y) \, d\mu(y) \right\} \quad \text{for all } r > 0.
\]

This guarantees the estimate \( [\|\mathcal{U}(u) - \mathcal{U}([u])\|_\infty \leq \int_\Omega \tilde{h}_r(y) \, d\mu(y) \|u - \tilde{u}\|_\vartheta \) and combined with [2.8] it results that \( \mathcal{U} \) is Lipschitz on bounded sets.

### 2.2 Continuous differentiability

In the following, we investigate the smoothness of Urysohn operators (2.1):
Lemma 2.9. Assume that \((U_r^k, U_r^k, U_r^k)\) hold for some \(k \in \mathbb{N}\). Then \(U_r^k : U_\alpha \rightarrow L_k(C^a_n(\Omega), C^\alpha_2(\Omega))\) given by
\[
U_r^k(u)v_1 \cdots v_k := \int_\Omega D_3^k f(\cdot, y, u(y))v_1(y) \cdots v_k(y) \mu(y) \quad \text{for all } v_1, \ldots, v_k \in C^a_n(\Omega)
\] (2.13)
is well-defined and continuous.

Proof. The well-definedness of \(U_r^k\) is shown verbatim to the step (I) of the proof for Prop. 2.2 and we hence focus on continuity. Given \(u \in U_\alpha\) let \((u_t)_{t \in \mathbb{N}}\) be a sequence in \(U_\alpha\) with \(\lim_{t \to \infty} \|u_t - u\|_0 = 0\). Choose \(r > 0\) so large that \(u_t, u_t \in B_r(0, C^a_n(\Omega))\) for all \(k \in \mathbb{N}\). Using \((U_r^k)\) this leads to
\[
\lim_{t \to \infty} D_3^k f(x, y, u_t(y)) = D_3^k f(x, y, u(y)) \quad \text{for all } x \in \Omega_1\text{ and almost all } y \in \Omega.
\]

For \(\varepsilon > 0\) there exists a \(\delta > 0\) such that subsets \(\tilde{\Omega} \subseteq \Omega\) satisfying \(\mu(\tilde{\Omega}) \leq \delta\) fulfill \(\int_{\tilde{\Omega}} b^k_\varepsilon(x, y) \mu(y) \leq \frac{\varepsilon}{4}\) and Egoroff’s theorem [13], Prop. 3.1.3 yields a \(\Omega' \subseteq \Omega\) with \(\mu(\Omega') \leq \delta\) and an \(L_1 \in \mathbb{N}\) such that
\[
\left| D_3^k f(x, y, u_t(y)) - D_3^k f(x, y, u(y)) \right| \leq \frac{\varepsilon}{2^k \mu(\Omega)} \quad \text{for all } x \in \Omega_1, y \in \Omega \setminus \Omega', l \geq L_1.
\]

From this, for any \(x \in \Omega_1\) and integers \(l \geq L_1\) we arrive at
\[
\left\| U_r^k(u_t) - U_r^k(u) \right\|_{0, \Omega_1} \leq \frac{1}{2^L} \int_{\Omega} \left| D_3^k f(x, y, u_t(y)) - D_3^k f(x, y, u(y)) \right| d\mu(y) + \frac{1}{2^L} \int_{\Omega} b^k_\varepsilon(x, y) d\mu(y) \leq \frac{\varepsilon}{4} \leq 2^{L_1} \varepsilon = \varepsilon.
\]

Passing first to the supremum over \(x \in \Omega_1\) therefore implies
\[
\left\| U_r^k(u_t) - U_r^k(u) \right\|_{0, \Omega_1} \leq \varepsilon \quad \text{for all } l \geq L_1, v_1, \ldots, v_k \in \tilde{B}_1(0, C^a_n(\Omega))
\]
and second over the vectors \(v_1, \ldots, v_k\) yields
\[
\left\| U_r^k(u_t) - U_r^k(u) \right\|_{L_k(C^a_n(\Omega), C^\alpha_2(\Omega_1))} \leq \varepsilon \quad \text{for all } l \geq L_1. \quad \text{Hence, since } u \text{ was arbitrary, } U_r^k \text{ is continuous.}
\]

Proposition 2.10 (continuous differentiability of \(U_r^k\)). Let \(m \in \mathbb{N}\). Assume that \((U_r^k, U_r^k, U_r^k)\) hold for all \(0 \leq k \leq m\) on a convex set \(Z \subseteq \mathbb{R}^n\). Then an Urysohn operator \(U : U_\alpha \rightarrow C^m(\Omega)\) is \(m\)-times continuously differentiable with \(D^kU = U_r^k\) for every \(1 \leq k \leq m\).

Proof. (1) Thanks to Lemma 2.9 the mappings \(U_r^k : U_\alpha \rightarrow L_k(C^a_n(\Omega), C^\alpha_2(\Omega_1))\) are well-defined and continuous for \(0 \leq k \leq m\). Let \(u \in U_\alpha\) and \(h \in C^a_n(\Omega)\) such that \(u + h \in U_\alpha\). Due to the convexity of \(Z\), the inclusion \(u(y) + \theta h(y) \in Z\) holds for all \(y \in \Omega\) and \(\theta \in [0, 1]\). Then the remainder functions
\[
r_k(h) := \sup_{\theta \in [0, 1]} \left| U_r^k(u + \theta h) - U_r^{k+1}(u) \right|_{L_k+C^a_n(\Omega), C^\alpha_2(\Omega_1)}
\]
satisfy \(\lim_{h \to 0} r_k(h) = 0\) for all \(0 \leq k < m\). Now we obtain from [35], p. 341, Thm. 4.2] that
\[
\left\| U_r^k(u + h) - U_r^k(u) - U_r^{k+1}(u) \right\|_{0, \Omega} \leq \frac{1}{2^L} \int_\Omega \left| D_3^k f(x, y, u(y) + h(y)) - D_3^k f(x, y, u(y)) - D_3^{k+1} f(x, y, u(y))h(y) \right| d\mu(y)
\]
\[
= \int_\Omega \left[ D_3^{k+1} f(x, y, u(y) + \theta h(y)) - D_3^{k+1} f(x, y, u(y)) \right] h(y) d\theta d\mu(y)
\]
\[
= \frac{1}{2^L} \int_0^1 \left[ (U_r^{k+1}(u + \theta h)) - U_r^k(u) \right] \right|_{0, \Omega} d\theta
\]
by Fubini’s theorem \cite[pp. 155, Thm. 14.1]{17}. Consequently,

\[ \left\| \mathcal{U}^k(u + h) - \mathcal{U}^k(u) - \mathcal{U}^{k+1}(u)h \right\| \leq \frac{1}{0} \left\| \mathcal{U}^{k+1}(u + \theta h) - \mathcal{U}^{k+1}(u) \right\|_{L_{k+1}(C_0^\alpha(\Omega),C_0^\beta(\Omega_1))} d\theta \|h\|_0 \]

\[ \leq r_{k+1}(h) \|h\|_\alpha \text{ for all } x \in \Omega_1 \]

and after passing to the least upper bound over \( x \in \Omega_1 \) it results

\[ \left\| \mathcal{U}^k(u + h) - \mathcal{U}^k(u) - \mathcal{U}^{k+1}(u)h \right\|_{L_k(C_0^\alpha(\Omega),C_0^\beta(\Omega_1))} \leq r_k(h) \|h\|_\alpha. \tag{2.16} \]

This establishes that \( \mathcal{U}^k : U_\alpha \to L_k(C_0^\alpha(\Omega),C_0^\beta(\Omega_1)) \) is differentiable in \( u \) with the derivative \( \mathcal{U}^{k+1}(u) \).

(II) Applying step (I) in case \( k = 0 \) shows that \( \mathcal{U} \) is differentiable on \( U_\alpha \) with the derivative \( \mathcal{U}^1 \). Given this, mathematical induction yields that \( \mathcal{U} : U_\alpha \to C_0^\beta(\Omega_1) \) is actually \( m \)-times differentiable with the derivatives \( D^k \mathcal{U} = \mathcal{U}^k \) for all \( 1 \leq k \leq m \), which in turn are continuous due to Lemma \ref{lem:2.9}.

We proceed to Urysohn operators having values in a Hölder space.

**Lemma 2.11.** Assume that \( (U_0^k, U_1^k, U_2^k) \) hold for some \( k \in \mathbb{N} \). Then \( \mathcal{U}^k : U_\alpha \to L_k(C_0^\alpha(\Omega),C_0^\beta(\Omega_1)) \) given by \( \mathcal{U}_3^k \) is well-defined. If additionally \( \mathcal{U}_3^k \) holds, then \( \mathcal{U}^k \) is continuous.

**Proof.** The well-definedness of \( \mathcal{U}^k \) follows as in step (I) from the proof of Thm. \ref{thm:2.6}. Let \( u \in U_\alpha \) and \((u_\ell)_{\ell \in \mathbb{N}}\) denote a sequence in \( U_\alpha \) fulfilling the limit relation \( \lim_{\ell \to \infty} |u - u_\ell|_0 = 0 \). In addition, choose \( r > 0 \) sufficiently large so that the inclusion \( u, u_\ell \in \bar{B}_r(0, C_0^\alpha(\Omega)) \) for each \( \ell \in \mathbb{N} \) holds. Therefore, \( (2.13) \)

\[ \left( [(\mathcal{U}^k(u_\ell) - \mathcal{U}^k(u)v_1 \cdots v_k)(x) - [(\mathcal{U}^k(u_\ell) - \mathcal{U}^k(u))v_1 \cdots v_k](\bar{x})] \right) \]

\[ \cdot v_1(y) \cdots v_k(y) d\mu(y) \]

and after passing to the norm our assumption \( \mathcal{U}_3^k \) results in

\[ \left( \left| [(\mathcal{U}^k(u_\ell) - \mathcal{U}^k(u)v_1 \cdots v_k)(x) - [(\mathcal{U}^k(u_\ell) - \mathcal{U}^k(u))v_1 \cdots v_k](\bar{x})]\right| \right) \]

\[ \leq \frac{1}{\Omega} \left\| \epsilon^k(|u_\ell(y) - u(y)|, y) d\mu(y)d(x, \bar{x}) \right\|^{2/10} \]

\[ \leq \int_{\Omega} \epsilon^k(|u_\ell - u|_0, y) d\mu(y)d(x, \bar{x})^\beta \text{ for all } x, \bar{x} \in \Omega_1. \]

Consequently, for \( v_1, \ldots, v_k \in \bar{B}_1(0, C_0^\alpha(\Omega)) \) we derive the estimate

\[ \left( [(\mathcal{U}^k(u_\ell) - \mathcal{U}^k(u))v_1 \cdots v_k]_{\beta} \leq \int_{\Omega} \epsilon^k(|u_\ell - u|_0, y) d\mu(y). \tag{2.17} \]

In particular, given \( \varepsilon > 0 \) there exists a \( L_2 \in \mathbb{N} \) such that \( \left( [(\mathcal{U}^k(u_\ell) - \mathcal{U}^k(u))v_1 \cdots v_k]_{\beta} \leq \varepsilon \text{ for all } \ell \geq L_2, v_1, \ldots, v_k \in \bar{B}_1(0, C_0^\alpha(\Omega)) \). In conclusion, with \( \mathcal{U}_3^k \) this implies

\[ \left\| \mathcal{U}^k(u_\ell) - \mathcal{U}^k(u)v_1 \cdots v_k \right\|_{\beta} = \max \left\{ \left\| \mathcal{U}^k(u_\ell) - \mathcal{U}^k(u)v_1 \cdots v_k \right\|_0, \left[ \left\| \mathcal{U}^k(u_\ell) - \mathcal{U}^k(u) \right\|_{\beta} \right] \right\} \]

\[ \leq \varepsilon \text{ for all } v_1, \ldots, v_k \in \bar{B}_1(0, C_0^\alpha(\Omega)) \]

and thus \( \left\| \mathcal{U}^k(u_\ell) - \mathcal{U}^k(u) \right\|_{L_k(C_0^\alpha(\Omega),C_0^\beta(\Omega_1))} \leq \varepsilon \) for all \( \ell \geq \max \{L_1, L_2\} \). Therefore, because the function \( u \) was arbitrarily chosen, \( \mathcal{U}^k \) is continuous on \( U_\alpha \). \( \square \)
In contrast to Prop. 2.10 establishing continuous differentiability now requires to invest one additional order of differentiability on the kernel function:

**Theorem 2.12** (continuous differentiability of $\mathcal{U}$). Let $m \in \mathbb{N}$. Assume that $(U_0^k, U_1^k, U_2^k, U_3^k)$ hold for all $0 \leq k \leq m$ on a convex set $Z \subseteq \mathbb{R}^n$. Then an Urysohn operator $\mathcal{U} : U_\alpha \to C_\alpha^0(\Omega_1)$ is $m$-times continuously differentiable with $D^k \mathcal{U} = \mathcal{U}^k$ for every $1 \leq k \leq m$.

**Proof.** We establish the assertion for $\mathcal{U} : U_\alpha \to C_\alpha^\beta(\Omega_1)$ first. Let $u \in U_\alpha$ and $h \in C_\alpha^\beta(\Omega)$ such that $u + h \in U_\alpha$. Moreover, choose $r > 0$ so large that $u, u + h \in B_r(0, C_\alpha^\beta(\Omega))$ holds.

(I) Let $0 \leq k < m$. Above all, with the function $c_r^k : \mathbb{R}_+ \times \Omega_1 \to \mathbb{R}_+$ in (2.11) we observe that

$$\rho_k(h) := \frac{1}{\Omega} \int_0^1 \int \hat{c}_r^{k+1}(\theta \|h\|_\alpha, y) \, d\mu(y) \, d\theta$$

satisfies $\lim_{h \to 0} \rho_k(h) = 0$. Given arbitrary $x, \bar{x} \in \Omega_1$, again [35] p. 341, Thm. 4.2 yields

$$\left| \mathcal{U}^k(u + h) - \mathcal{U}^k(u) - \mathcal{U}^k(u + h)(\bar{x}) - \mathcal{U}^k(u) - \mathcal{U}^k(u)(\bar{x}) \right| \leq \int_0^1 \int \left[ D^k_{3} f(x, y, u(y) + \theta h(y)) - D^k_{3} f(x, y, u(y)) ight] \, d\theta h(y) \, d\mu(y)$$

due to Fubini’s theorem [17] p. 155, Thm. 14.1 and using the assumption $(U_3^{k+1})$ we obtain

$$\left| \mathcal{U}^k(u + h) - \mathcal{U}^k(u) - \mathcal{U}^k(u + h)(\bar{x}) - \mathcal{U}^k(u) - \mathcal{U}^k(u)(\bar{x}) \right| \leq \int_0^1 \int \hat{c}_r^{k+1}(\theta \|h\|_\alpha, y) \, d\mu(y) \, d\theta d(x, \bar{x})^\beta$$

which in turn implies

$$\left| \mathcal{U}^k(u + h) - \mathcal{U}^k(u) - \mathcal{U}^k(u + h)(\bar{x}) - \mathcal{U}^k(u) - \mathcal{U}^k(u)(\bar{x}) \right| \leq \rho_k(h) \|h\|_\alpha.$$

If we combine this with the inequalities (2.19), then $\|\mathcal{U}^k(u + h) - \mathcal{U}^k(u) - \mathcal{U}_x^k(u + h)\|_\beta \leq R_k(h) \|h\|_\alpha$ with the remainder term $R_k(h) := \max \{r_k(h), \rho_k(h)\}$ satisfying the desired limit relation $\lim_{h \to 0} R_k(h) = 0$.

(II) Applying step (1) in case $k = 0$ shows that $\mathcal{U}$ is differentiable on $U_\alpha$ with the derivative $\mathcal{U}^1$. Given this, mathematical induction yields that $\mathcal{U} : U_\alpha \to C_\alpha^\beta(\Omega_1)$ is actually $m$-times differentiable with the derivatives $D^k \mathcal{U} = \mathcal{U}^k$ for all $0 \leq k \leq m$. Their continuity is guaranteed by Lemma 2.11.

We close our general analysis of Urysohn operators with several remarks:

**Remark 2.3** (boundedness and continuity of $\mathcal{U}$). (1) The boundedness of Urysohn operators $\mathcal{U}$ stated in Prop. 2.2 and Thm. 2.6 actually means that the $\mathcal{U}$-images of merely $\|\cdot\|_0$-bounded subsets $B \subseteq U_\alpha$ are bounded. This means that the functions in $B$ need not to have uniformly bounded Hölder constants.

(2) The continuity statements for the Urysohn operator $\mathcal{U}$ in Prop. 2.2 and Thm. 2.6 as well as for its derivatives $D^k \mathcal{U}$ in Prop. 2.10 and Thm. 2.12 are to be understood in the following strong form: Already convergence in the domain $U_\alpha$ w.r.t. the norm $\|\cdot\|_0$ is sufficient for convergence of the $\mathcal{U}$-values in the norm $\|\cdot\|_\beta$ resp. $\|\cdot\|_\beta$. A corresponding statement applies to both Cor. 2.13 and 2.8.
Rem. 2.4 (Urysohn operators) Let \( U : U_\alpha \to C^1_d(\Omega_1) \). The above statements extend to Urysohn operators mapping into the \( \gamma \)-Hölder functions over bounded metric spaces \( \Omega_1 \). This is due to the corresponding representation \( T^\alpha_\beta : U_\alpha \to C^\gamma_d(\Omega_1) \), where the embedding \( T^\alpha_\beta \) from Thm. A.13 is continuous.

Rem. 2.5 (Nystrom methods) Let \( \Omega^{(l)} \) be a discrete subset of a compact set \( \Omega_1 \subset \mathbb{R}^k \) and suppose \( w_\eta \geq 0 \) are nonnegative reals, \( \eta \in \Omega^{(l)} \). In Ex. 2.1 resp. 2.2 we pointed out that both Urysohn operators

\[
U_\alpha : U_\alpha \to C^1_d(\Omega_1), \\
U(\alpha) := \int_{\Omega_1} f(\cdot, y, u(y)) \, dy,
\]

\[
U_\alpha^{(l)} : U_\alpha^{(l)} \to C^1_d(\Omega_1), \\
U_\alpha^{(l)}(\alpha) := \sum_{\eta \in \Omega^{(l)}} w_\eta f(\cdot, \eta, u(\eta))
\]

fit well into our abstract setting, where we abbreviated \( U_\alpha^{(l)} := \left\{ \alpha : \Omega^{(l)} \to Z \mid \alpha \in C^0_\alpha(\Omega^{(l)}) \right\} \). However, when dealing with Nystrom methods for or iterating integral operators \( U_\alpha \) it is desirable to work with Urysohn operators defined on \( U_\alpha \) rather than \( U_\alpha^{(l)} \). For this purpose, let us introduce the linear operator \( E_1 : C^\alpha_\omega(\Omega_1) \to C^\alpha(\Omega^{(l)}) \) given by \( E_1 u := u|_{\Omega^{(l)}} \). It satisfies \( E_1 U_\alpha \subseteq U_\alpha^{(l)} \) and is easily seen to be bounded with \( \|E_1\|_{C^\alpha_\omega(\Omega_1), C^\alpha(\Omega^{(l)})} \leq 1 \). Hence, rather than \( U_\alpha \) we consider the composition

\[
U_\alpha^{(l)} : U_\alpha \to C^1_d(\Omega_1), \\
U_\alpha^{(l)}(\alpha) := U_\alpha E_1(\alpha) = \sum_{\eta \in \Omega^{(l)}} w_\eta f(\cdot, \eta, u(\eta)),
\]

which, under appropriate assumptions on the kernel function \( f \), inherits its properties from \( U_\alpha \).

2.3 Convolute operators

In our above analysis the Hölder continuity of an image \( U(\alpha) : \Omega_1 \to \mathbb{R}^d \) of a general Urysohn operator \( U(\alpha) \) was guaranteed and prescribed by the exponent of the kernel function \( f \) in its first variable from assumption \((E_1)\). A higher degree of smoothness cannot be expected, as simple examples like the kernel function \( f(x, y, z) := f_1(x) \) illustrate, where \( U(\alpha)(x) = \int_{\Omega_1} f_1(x) \, d\mu(y) = \mu(\Omega) f_1(x) \) inherits its smoothness from \( f_1 \).

This situation changes for kernel functions of convolution type. Here the smoothness (Hölder continuity, differentiability) of the arguments \( u \) transfers to the images \( U(\alpha) \), i.e., such integral operators possess a smoothing property we are about to analyze over the course of this section. Our results generalize those of [25] pp. 52ff, Sect. 3.4.2] obtained for linear operators.

To be more precise, let us restrict to a compact interval \( \Omega = \Omega_1 = [a, b], a < b \), equipped with the 1-dimensional Lebesgue measure \( \mu = \lambda_1 \) in \((2.1)\). Moreover, the kernel function is of the form

\[
f(x, y, z) = \tilde{f}(x - y, z)
\]

with a function \( \tilde{f} : [a - b, b - a] \times Z \to \mathbb{R}^d \). This yields a convolute Urysohn operator

\[
\tilde{U} : U_\alpha \to F([a, b], \mathbb{R}^d), \\
\tilde{U}(\alpha)(x) := \int_a^b \tilde{f}(x - y, u(y)) \, dy \quad \text{for all} \quad x \in [a, b]. \tag{2.19}
\]

**Hypothesis.** Let \( m \in \mathbb{N}_0 \) and \( \tilde{\Omega} := [a - b, b - a] \). With \( 0 \leq k \leq m \) one assumes:

\( (C_0^k) \) The partial derivative \( D_k^2 \tilde{f}(y, \cdot) : Z \to \mathbb{R}^d \) exists and is continuous for Lebesgue-almost all \( y \in \tilde{\Omega} \).

\( (C_1^k) \) The partial derivative \( D_k^2 \tilde{f}(\cdot, z) : \tilde{\Omega} \to \mathbb{R}^d \) is measurable for all \( z \in Z \) and for every \( r > 0 \) there exists an integrable function \( \tilde{b}_k : \tilde{\Omega} \to \mathbb{R}_+ \) so that for Lebesgue-almost all \( y \in \tilde{\Omega} \) the following holds:

\[
|D_k^2 \tilde{f}(y, z)| \leq \tilde{b}_k(y) \quad \text{for all} \quad z \in \tilde{\Omega} \cap B_r(0), \tag{2.20}
\]

\( (C_2) \) For all \( r > 0 \) there exists an integrable function \( \tilde{L}_r : \tilde{\Omega} \to \mathbb{R}_+ \), so that for Lebesgue-almost all \( y \in \tilde{\Omega} \) the following holds:

\[
|\tilde{f}(y, z) - \tilde{f}(y, \tilde{z})| \leq \tilde{L}_r(y) |z - \tilde{z}| \quad \text{for all} \quad z, \tilde{z} \in \tilde{\Omega} \cap B_r(0). \tag{2.21}
\]
Note that the Lipschitz condition \((C_2)\) implies \((C_0^0)\), but also \((\text{2.20})\) for \(k = 1\) with \(\tilde{b}_r^1 = \tilde{t}_r\).

Under these assumptions the Hölder continuity of \(u \in U_\alpha\) carries over to the values \(\tilde{U}(u)\):

**Theorem 2.13** (Hölder continuity of \(\tilde{U}(u)\)). Assume that \((C_1^0, C_2)\) hold and \(\alpha \in (0,1]\). If \(u \in U_\alpha\), \(r > \|u\|_0\) and there exists a real \(C \geq 0\) satisfying

\[
\int_x^\bar{x} \tilde{b}_r^0(y) \, dy \leq C(x-x)^\alpha \quad \text{for all } a-b \leq x \leq \bar{x} \leq b-a, \quad (2.22)
\]

then the image satisfies \(\tilde{U}(u) \in C_\alpha^0[a,b]\).

**Proof.** Let \(u \in U_\alpha\) and \(r > \|u\|_0\).

(I) Let \(x \in [a,b]\) be given. Above all, \((C_2)\) implies that \(\tilde{f}(y, \cdot) : Z \to \mathbb{R}^d\) is continuous for Lebesgue-almost all \(y \in \tilde{\Omega}\) (i.e. the assumption \((C_0^0)\) holds). Combined with the measurability assumed in \((C_1^0)\) we conclude from [48] p. 62, Lemma 5.1 that \(y \mapsto \tilde{f}(x - y, u(y))\) is measurable. Moreover, due to \((\text{2.20})\) one has the estimate \(|\tilde{f}(x - y, u(y))| \leq \tilde{b}_r^0(x-y)\), where \(\int_a^b \tilde{b}_r^0(x-y) \, dy \leq \int_a^\bar{\tilde{t}}_r^0(\eta) \, d\eta\) for all \(x \in [a,b]\). Consequently the function \(y \mapsto \tilde{f}(x - y, u(y))\) is integrable. Hence \(\tilde{U}(u) : [a,b] \to \mathbb{R}^d\) is well-defined.

(II) For \(x, \bar{x} \in [a,b]\) with \(\bar{x} = x + \Delta > x\) it results

\[
\tilde{U}(u)(\bar{x}) = \tilde{U}(u)(x) + \int_x^{\bar{x}} \tilde{f}(x - y, u(y)) \, dy = \tilde{U}(u)(x) + \int_a^{\bar{x} - \Delta} \tilde{f}(x - \eta, u(\eta + \Delta)) \, d\eta + \int_{\bar{x} - \Delta}^{\bar{x}} \tilde{f}(x - \eta, u(\eta + \Delta)) \, d\eta
\]

via the substitution \(\eta := y - \Delta\) and analogously

\[
\tilde{U}(u)(x) = \int_a^b \tilde{f}(x - \eta, u(\eta)) \, d\eta + \int_{\bar{x} - \Delta}^{\bar{x}} \tilde{f}(x - \eta, u(\eta)) \, d\eta.
\]

Whence, the difference \(\tilde{U}(u)(\bar{x}) - \tilde{U}(u)(x) = \tilde{I}_0 + \tilde{I}_1 + \tilde{I}_2\) can be written as sum of the terms

\[
\tilde{I}_0 := \int_a^{\bar{x} - \Delta} \tilde{f}(x - \eta, u(\eta + \Delta)) - \tilde{f}(x - \eta, u(\eta)) \, d\eta, \quad \tilde{I}_1 := \int_a^{\bar{x} - \Delta} \tilde{f}(\eta - \Delta, u(\eta + \Delta)) \, d\eta,
\]

\[
\tilde{I}_2 := - \int_{\bar{x} - \Delta}^b \tilde{f}(x - \eta, u(\eta)) \, d\eta = \int_{x-b}^{x-b+\Delta} \tilde{f}(y, u(x-y)) \, dy,
\]

which can be estimated separately as

\[
|\tilde{I}_0| \leq \int_a^{\bar{x} - \Delta} |\tilde{f}(x - \eta, u(\eta + \Delta)) - \tilde{f}(x - \eta, u(\eta))| \, d\eta \leq \int_a^{\bar{x} - \Delta} \tilde{t}_r(x - \eta) |u(\eta + \Delta) - u(\eta)| \, d\eta
\]

\[
\leq [u]_\alpha \int_a^{\bar{x} - \Delta} \tilde{t}_r(x - \eta) \, d\eta \Delta^\alpha \leq [u]_\alpha \int_{\tilde{\Omega}} \tilde{t}_r(\eta) \, d\eta \Delta^\alpha
\]

and \((\text{2.22})\) imply that

\[
|\tilde{I}_1| \leq \int_{\bar{x} - \Delta}^a \tilde{b}_r^0(x - \eta) \, d\eta \leq C \Delta^\alpha, \quad |\tilde{I}_2| \leq \int_{x-b}^{x-b+\Delta} \tilde{b}_r^0(y) \, dy \leq C \Delta^\alpha.
\]

Hence, with \(\tilde{I}_0, \tilde{I}_1\) and \(\tilde{I}_2\) also their sum is \(\alpha\)-Hölder due to Thm. A.8 and thus \(\tilde{U}(u) \in C^\alpha[a,b]\).
Theorem 2.14 (Hölder continuity of $\tilde{U}(u)$). Assume that $(C_1^0, C_2)$ hold and $\alpha \in (0, 1]$, the kernel function $\tilde{f}$ and the partial derivative $D_1 \tilde{f}$ exist as a continuous functions on both sets $[a - b, 0) \times Z$ and $(0, b - a] \times Z$. If $u \in U_\alpha$, $r > \|u\|_0$ and there exists $h_0 > 0$ such that $\int_{-h_0}^{+h_0} \tilde{l}_r(y) \, dy < \infty$, then $\tilde{U}(u)$ is $\alpha$-Hölder on every subinterval compact in $[a, b]$.

Proof. Let $u \in U_\alpha$. As in step (I) of the proof to Thm. 2.13, one shows that $\tilde{U}(u)$ is well-defined. Now suppose that $I \subset (a, b)$ is a compact subinterval. For each $x \in I$ we choose $h \in (0, h_0)$ so small that $a \leq x - h$ and $x + h \leq b$. This allows us to represent

$$\tilde{U}(u)(x) = \int_a^b \tilde{f}(x - y, u(y)) \, dy = I_1(x) + I_2(x) + I_3(x)$$

with the functions $I_0, I_1, I_2 : I \to \mathbb{R}^d$ given by

$$I_0(x) := \int_{x-h}^{x+h} \tilde{f}(x - y, u(y)) \, dy,$$

$$I_1(x) := \int_{a}^{x-h} \tilde{f}(x - y, u(y)) \, dy,$$

$$I_2(x) := \int_{x+h}^{b} \tilde{f}(x - y, u(y)) \, dy,$$

where we applied the substitution $\eta = x - y$ in order to rewrite $I_0(x)$. First, we investigate the parameter integral $I_0$. Thereto, using $(C_2)$ we obtain

$$|I_0(x) - I_0(\bar{x})| \leq \int_{-h}^{h} |\tilde{f}(\eta, u(x - \eta)) - \tilde{f}(\eta, u(\bar{x} - \eta))| \, d\eta \leq \int_{-h}^{h} l_r(\eta) |u(x - \eta) - u(\bar{x} - \eta)| \, d\eta \leq |u|_\alpha \int_{-h}^{h} l_r(\eta) \, d\eta |x - \bar{x}|^\alpha$$

for all $x, \bar{x} \in I$. And consequently also $I_0$ is $\alpha$-Hölder on the compact subinterval $I$. Second, the integration variable $y$ in the parameter integrals $I_1(x)$ and $I_2(x)$ satisfies $0 < h \leq x - y$ resp. $x - y \leq -h < 0$. Thus, the functions $I_1, I_2$ are differentiable with the derivatives

$$I_1'(x) = \int_{a}^{x-h} D_1 \tilde{f}(x - y, u(y)) \, dy + \tilde{f}(h, u(x - h)), \quad I_2'(x) = \int_{x+h}^{b} D_1 \tilde{f}(x - y, u(y)) \, dy + \tilde{f}(-h, u(x + h))$$

for all $x \in I$. Because these derivatives are bounded on the interval $I$, we conclude from Ex. A.2 that $I_1, I_2$ are $\alpha$-Hölder on $I$. In conclusion, Thm. A.8 yields that the sum (2.23) is $\alpha$-Hölder. \hfill $\Box$

Theorem 2.15 (continuous differentiability of $\tilde{U}(u)$). Assume that $(C_1^0, C_1^1)$ hold on a convex set $Z \subset \mathbb{R}^n$, $\tilde{f} : \tilde{\Omega} \times Z \to \mathbb{R}^d$ and $D_2 \tilde{f}(\cdot, z)$ are continuous for all $z \in Z$ and $\alpha \in (0, 1]$. If $u : [a, b] \to Z$ is continuously differentiable, then also the image $\tilde{U}(u) : [a, b] \to \mathbb{R}^d$ is continuously differentiable with the derivative

$$\tilde{U}(u)'(x) = \tilde{f}(x - a, u(a)) - \tilde{f}(x - b, u(b)) - \int_{a}^{b} D_2 \tilde{f}(x - y, u(y)) u'(y) \, dy$$

for all $x \in [a, b]$. (2.24)

Since the derivative $\tilde{U}(u)'$ is bounded as a continuous function over the compact interval $[a, b]$, it results from Ex. A.2 that $\tilde{U}(u)$ is $\alpha$-Hölder, $\alpha \in (0, 1]$.

Proof. Let $u : [a, b] \to Z$ be continuously differentiable and choose $r > \|u\|_0$. Since $\tilde{f}$ is continuous, the assumptions $(C_0^0, C_1^0)$ are satisfied and consequently as shown in step (I) of the proof to Thm. 2.13, the expression $\tilde{U}(u)$ is well-defined. Now fix some $x \in [a, b]$.

(I) Claim: One has the limit relation
\[
\lim_{h \to 0} \int_a^b \frac{1}{h} \left( \tilde{f}(x - y, u(y + h)) - \tilde{f}(x - y, u(y)) \right) dy = \int_a^b D_2\tilde{f}(x - y, u(y))u'(y) dy.
\] (2.25)

We define the function
\[
F(h, y) := \begin{cases}
\frac{1}{h} \left( \tilde{f}(x - y, u(y + h)) - \tilde{f}(x - y, u(y)) \right), & h \neq 0,

D_2\tilde{f}(x - y, u(y))u'(y), & h = 0
\end{cases}
\]

having the following properties: First, \(F(h, \cdot) : [a, b] \to \mathbb{R}^d\) is integrable for every fixed \(h\). In order to see this, let us distinguish two cases:

\(h \neq 0\): The continuity of \(\tilde{f}\) and \(u\) yield that \(F(h, \cdot)\) is integrable.

\(h = 0\): Due to (\(C^1\)) the derivative \(D_2\tilde{f}(\cdot, z)\) is measurable for all \(z \in Z\) and the continuity of \(u\) and \(u'\) guarantee that \(F(0, \cdot)\) is measurable due to [35] p. 62, Lemma 5.1. Moreover, for Lebesgue-almost all \(y \in [a, b]\) one has \(|F(0, y)| = |D_2\tilde{f}(x - y, u(y))u'(y)| \leq b_1^1(x - y)\|u'\|_0\) from (2.20) and due to
\[
\int_a^b b_1^1(x - y) dy \leq \int_\Omega b_1^1(\eta) d\eta
\] (2.26)
also the function \(F(0, \cdot)\) is integrable.

Second, \(F(\cdot, y)\) is continuous in \(0\), which readily results from the chain rule [35] p. 337]. Third, applying the Mean Value Theorem [35] p. 341, Thm. 4.2] twice leads to
\[
F(h, y) = \frac{1}{h} D_2\tilde{f}(x - y, u(y) + \theta(u(y) + h) - u(y)) \frac{1}{\theta} u'(y + \theta h) d\theta
\]
and hence \(|F(h, y)| \leq \int_0^1 b_1^1(x - y) d\theta \|u'\|_0 \leq b_1^1(x - y)\|u'\|_0\) for all \(h\). Thanks to (2.26) the right-hand side of this inequality is bounded above by an integrable function independent of \(h\) (cf. \(C^1\)). Combining these three aspects, it is a consequence of the dominated convergence theorem [17] p. 149, Thm. 10.1] that taking the limit and integration in (2.26) can be exchanged, which yields the claim.

(II) Using the substitution \(\eta = x - y\) we obtain the representation
\[
\tilde{U}(u)(x) = \int_a^b f(x - y, u(y)) dy = \int_{x-a}^{x-b} f(\eta, u(x - \eta)) d\eta,
\]
which in turn yields
\[
\tilde{U}(u)(x + h) - \tilde{U}(u)(x) = \int_{x-a}^{x-b} f(\eta, u(x + h - \eta)) d\eta - \int_{x-a}^{x-b} f(\eta, u(x - \eta)) d\eta
\]
\[
= - \int_{x-a}^{x-b} f(\eta, u(x + h - \eta)) d\eta + \int_{x-b}^{x-h-a} f(\eta, u(x - \eta)) d\eta + \int_{x-a}^{x-h-a} f(\eta, u(x + h - \eta)) d\eta
\]
\[
= - \int_{x-a}^{x-b} f(\eta, u(x + h - \eta)) d\eta - \int_{x-b}^{x-h-a} f(x - y, u(y)) dy + \int_{x-a}^{x-h-a} f(\eta, u(x + h - \eta)) d\eta
\]
for all \(h \in [a - b, b - x]\); note that we re-substituted \(\eta = x - y\) in the center term of the above sum. The Mean Value Theorem from the integral calculus applies to each continuous component function \(f_i\), \(1 \leq i \leq d\), in the first and third term in the above sum. Hence, there exist reals
\[
\xi_1^i(h) \in \{\min \{x - b, x + h - b\}, \max \{x - b, x + h - b\}\},
\]
\[
\xi_2^i(h) \in \{\min \{x - a, x + h - a\}, \max \{x - a, x + h - a\}\},
\]
such that the identities
\[
- \int_{x-a}^{x+h-a} \tilde{f}_i(\eta, u(x + h - \eta)) \, d\eta = h \tilde{f}_i(\xi_1^i(h), u(x + h - \xi_1^i(h)))
\]
\[
- \int_{x-b}^{x+h-b} \tilde{f}_i(\eta, u(x + h - \eta)) \, d\eta = h \tilde{f}_i(\xi_2^i(h), u(x + h - \xi_2^i(h)))
\]
hold, which allow us to conclude
\[
\frac{1}{h} \left( \tilde{U}(u)_i(x + h) - \tilde{U}(u)_i(x) \right) = \frac{1}{h} \left( \int_{x-a}^{x+h-a} \tilde{f}(\eta, u(x + h - \eta)) \, d\eta - \int_{x-b}^{x+h-b} \tilde{f}(\eta, u(x + h - \eta)) \, d\eta \right)
\]
\[
- \int_{a}^{b} \tilde{f}_i(x - y, u(y + h)) - \tilde{f}_i(x - y, u(y)) \, dy = \tilde{f}_i(\xi_2^i(h), u(x + h - \xi_2^i(h))) - \tilde{f}_i(\xi_1^i(h), u(x + h - \xi_1^i(h)))
\]
\[
- \int_{a}^{b} \frac{1}{h} \left( \tilde{f}_i(x - y, u(y + h)) - \tilde{f}_i(x - y, u(y)) \right) \, dy \quad \text{for all} \quad 1 \leq i \leq d.
\]
Thanks to the limit relations \( \lim_{h \to 0} \xi_1^i(h) = x - b \), \( \lim_{h \to 0} \xi_2^i(h) = x - a \) for all \( 1 \leq i \leq d \) and returning to vector notation we consequently arrive at
\[
\lim_{h \to 0} \frac{1}{h} \left( \tilde{U}(u)(x + h) - \tilde{U}(u)(x) \right) = \tilde{f}(x - a, u(a)) - \tilde{f}(x - b, u(b)) - \lim_{h \to 0} \int_{a}^{b} \left( \tilde{f}(x - y, u(y + h)) - \tilde{f}(x - y, u(y)) \right) \, dy
\]
\[
\overset{(2.24)}{=} \tilde{f}(x - a, u(a)) - \tilde{f}(x - b, u(b)) - \int_{a}^{b} D_2 \tilde{f}(x - y, u(y)) u'(y) \, dy.
\]
This establishes that the image \( \tilde{U}(u) \) is differentiable in \( x \in [a, b] \) with the derivative \( (2.24) \). Hence, in order to show that \( U(u)' \) is continuous, it suffices to establish the continuity of the parameter integral \( x \mapsto \int_{a}^{b} \tilde{F}(x, y) \, dy \) with \( \tilde{F}(x, y) := D_2 \tilde{f}(x - y, u(y)) u'(y) \). By assumption follows that \( \tilde{F}(\cdot, y) \) is continuous on \([a, b]\). From [18] p. 62, Lemma 5.1 we conclude that \( \tilde{F}(x, \cdot) \) is continuous and because of
\[
\left| \tilde{F}(x, y) \right| = \left| D_2 \tilde{f}(x - y, u(y)) u'(y) \right| \overset{(2.20)}{\leq} \tilde{b}_1(x - y) \left\| u' \right\|_0 \quad \text{for all} \quad x \in [a, b]
\]
integrable (uniformly in \( x \), cf. \( (2.26) \)). Then the dominated convergence theorem [17] p. 149, Thm. 10.1] shows that \( x \mapsto \int_{a}^{b} \tilde{F}(x, y) \, dy \) is continuous, and thus \( \tilde{U}(u) \) is continuously differentiable.

\[\text{3} \quad \text{Hammerstein integral operators}\]

Let \( \alpha \in [0, 1] \). Hammerstein operators are of the form
\[
\mathcal{H} : U_\alpha \to F(\Omega_1, \mathbb{R}^d), \quad \mathcal{H}(u) := \int_{\Omega} k(\cdot, y)g(y, u(y)) \, d\mu(y)
\]
and represent a relevant special case of the Urysohn operators studied in Sect. 2 having the kernel function \( f(x, y, z) := k(x, y)g(y, z) \). Nevertheless, we investigate them as composition of Fredholm and Nemytskii operators. For this reason, let us study these operator classes independently first.
3.1 Fredholm integral operators

A Fredholm operator is a linear integral operator of the form

\[ \mathcal{K} : C^\alpha_p(\Omega) \to F(\Omega_1, \mathbb{R}^d), \quad \mathcal{K}u := \int_{\Omega} k(\cdot, y)u(y)\,d\mu(y) \quad (3.2) \]

determined by a matrix-valued kernel \( k : \Omega_1 \times \Omega \to \mathbb{R}^{d \times p} \).

Fredholm operators apparently fit in the set-up of Sect. 2 with kernel function \( f(x, y, z) = k(x, y)z \).

Nevertheless, we take the opportunity to formulate our assumptions in terms of integrals over the kernels, rather than over the kernels. Then the corresponding counterparts to \( (U^0_1) \) and \( (U^0_2) \) read as:

Hypothesis. \((K_1)\) \( \lim_{x \to x_0} \int_{\Omega} |k(x, y) - k(x_0, y)|\,d\mu(y) = 0 \) for all \( x_0 \in \Omega_1 \).

\((K_2)\) \( k(x, \cdot) \) is measurable for all \( x \in \Omega_1 \) and \( \sup_{\xi \in \Omega_1} \int_{\Omega} |k(\xi, y)|\,d\mu(y) < \infty \).

Proposition 3.1 (well-definedness of \( \mathcal{K} \)). Assume that \((K_1, K_2)\) hold. Then a Fredholm operator \( \mathcal{K} \) satisfies \( \mathcal{K} \in L(C^\alpha_p(\Omega), C^\alpha_d(\Omega_1)) \) and

\[ \|\mathcal{K}\|_{L(C^\alpha_p(\Omega), C^\alpha_d(\Omega_1))} \leq \max \left\{ 1, (\text{diam } \Omega)^\alpha \right\} \sup_{\xi \in \Omega_1} \int_{\Omega} |k(\xi, y)|\,d\mu(y). \quad (3.2) \]

Proof. Let us abbreviate \( M := \sup_{\xi \in \Omega_1} \int_{\Omega} |k(\xi, y)|\,d\mu(y) \) and write \( \mathcal{K}_0 : C^\alpha_p(\Omega) \to C^\alpha_d(\Omega_1) \) instead of \( \mathcal{K} \).

The inclusion \( \mathcal{K}_0 \in L(C^\alpha_p(\Omega), C^\alpha_d(\Omega_1)) \) with \( \|\mathcal{K}_0\| \leq M \) is shown in [21, p. 244, Satz 1]. In case \( \alpha \in (0, 1] \) we consider the composition \( \mathcal{K} = \mathcal{K}_0 \mathcal{P}_0 \), where the embedding operator \( \mathcal{P}_0 \) from (A.3) satisfies the estimate \( \|\mathcal{P}_0\| \leq \max \{ 1, (\text{diam } \Omega)^\alpha \} \) by Thm. A.13. This implies the remaining assertions.

Corollary 3.2 (compactness of \( \mathcal{K} \)). A Fredholm operator \( \mathcal{K} \in L(C^\alpha_p(\Omega), C^\alpha_d(\Omega_1)) \) is compact, provided one of the following holds:

(i) \( \alpha \in (0, 1] \),

(ii) \( \Omega_1 \) is compact, \( \alpha = 0 \) and \( (K_1) \) holds uniformly in \( x_0 \in \Omega_1 \).

Proof. For compactness of \( \mathcal{K}_0 \in L(C^\alpha_p(\Omega), C^\alpha_d(\Omega_1)) \) we refer to [21, p. 247, Satz 4]. In case \( \alpha \in (0, 1] \) we consider the composition \( \mathcal{K} = \mathcal{K}_0 \mathcal{P}_0 \) of the continuous \( \mathcal{K}_0 \) and the embedding operator \( \mathcal{P}_0 \) introduced in (A.3), which is compact due to Thm. A.15.

Remark 3.1. If \( \Omega_1 \) is compact and \( k : \Omega_1 \times \Omega \to \mathbb{R}^{d \times p} \) is continuous, then \((K_1, K_2)\) are fulfilled. Hence, Prop. 3.1 and Cor. 3.2 guarantee that \( \mathcal{K} \in L(C^\alpha_p(\Omega), C^\alpha_d(\Omega_1)) \) is compact.

In order to handle Fredholm operators which map into the Hölder continuous functions a refinement of assumption \( (U^0_1) \) is due:

Hypothesis. Let \( \beta \in (0, 1] \).

\( (K_1) \) There exists a continuous function \( \tilde{h} : \Omega_1^2 \to \mathbb{R}^+ \) such that

\[ \int |k(x, y) - k(x_0, y)|\,d\mu(y) \leq \tilde{h}(x, x_0)|d(x, x)|^\beta \quad \text{for all } x, x_0 \in \Omega_1. \quad (3.3) \]

Obviously, \((\tilde{K}_1)\) implies \((K_1)\).

Theorem 3.3 (well-definedness of \( \mathcal{K} \)). Assume that \((\tilde{K}_1, K_2)\) hold. Then a Fredholm operator \( \mathcal{K} \) satisfies \( \mathcal{K} \in L(C^\alpha_p(\Omega), C^\alpha_d(\Omega_1)) \) and

\[ \|\mathcal{K}\|_{L(C^\alpha_p(\Omega), C^\alpha_d(\Omega_1))} \leq \max \left\{ 1, (\text{diam } \Omega)^\alpha \right\} \sup_{\xi \in \Omega_1} \int_{\Omega} |k(\xi, y)|\,d\mu(y), \sup_{x, x_0 \in \Omega_1} \tilde{h}(x, x_0). \]
Proof. We abbreviate \( N := \sup_{x,x_0 \in \Omega_1} \overline{h}(x,x_0) \). First, \( \|Ku\|_0 \leq M \max \{1, (\text{diam } \Omega)^{\alpha}\} \|u\|_{\alpha} \) holds due to Prop. 3.1 (\( M \geq 0 \) is defined in its proof). Second, the inequality

\[
|Ku(x) - Ku(\bar{x})| \leq \int_{\Omega} |k(x,y) - k(\bar{x},y)| \|u(y)\|_{\alpha} \ d\mu(y) \leq Nd(x,\bar{x})^\beta \|u\|_{\alpha}
\]

for all \( x, \bar{x} \in \Omega_1 \), consequently implies that \( |\mathcal{H}u|_\beta \leq N \|u\|_{\alpha} \) holds. A combination of these two estimates finally guarantees that \( \|Ku\|_\beta = \max \{\|Ku\|_0, |\mathcal{H}u|_\beta\} \leq M \max \{1, (\text{diam } \Omega)^{\alpha}\}, N \} \|u\|_{\alpha} \) and thus

\[
\|K\|_{L(C^\alpha_\beta(\Omega), C^\gamma_\delta(\Omega_1))} \leq \max \{M \max \{1, (\text{diam } \Omega)^{\alpha}\}, N\}
\]

holds.

Corollary 3.4 (compactness of \( \mathcal{K} \)). A Fredholm operator \( \mathcal{K} \in L(C^\alpha_\beta(\Omega), C^\gamma_\delta(\Omega_1)) \) is compact, provided one of the following holds:

(i) \( \alpha \in (0, 1) \) and \( \gamma = \beta \),

(ii) \( \Omega_1 \) is bounded, \( \alpha \in (0, 1] \) and \( \gamma \in [0, \beta] \),

(iii) \( \Omega_1 \) is compact, \( \gamma \in [0, \beta] \) and \( \lim_{x \to x_0} \overline{h}(x, x_0) = 0 \) uniformly in \( x_0 \in \Omega_1 \),

(iv) \( \Omega_1 \) is compact and \( \gamma \in (0, \beta] \).

Proof. We write \( \mathcal{K}_0 : C^\alpha_\beta(\Omega) \to C^\gamma_\delta(\Omega_1) \) and \( \mathcal{K}_0^\gamma : C^\gamma_\delta(\Omega) \to C^\gamma_\delta(\Omega_1) \) instead of \( \mathcal{K} \).

(I) Claim: If (iii) holds, then \( \mathcal{K}_0^\gamma \in L(C^\alpha_\beta(\Omega), C^\gamma_\delta(\Omega_1)) \) is compact.

Given the unit ball \( B := B_1(0, C^\gamma_\delta(\Omega)) \) we apply the compactness criterion from Thm. A.14 in order to show that \( \mathcal{K}_0^\gamma B \subseteq C^\gamma_\delta(\Omega_1) \) is relatively compact. First, \( \mathcal{K}_0^\gamma B \subseteq C^\gamma_\delta(\Omega_1) \) is relatively compact due to the above Thm. 3.3. Second, given \( \varepsilon > 0 \) we obtain by assumption that there exists a \( \delta > 0 \) such that \( d(x, \bar{x}) < \delta \) yields \( \overline{h}(x, \bar{x}) < \varepsilon \) for all \( x, \bar{x} \in \Omega_1 \). Hence, the assumption (\( \mathcal{K}_1 \)) implies that

\[
|(Ku)(x) - (Ku)(\bar{x})| \leq \int_{\Omega} |k(x,y) - k(\bar{x},y)| \ d\mu(y) \leq \varepsilon d(x,\bar{x})^\beta
\]

and all \( u \in B \), which guarantees relative compactness of \( \mathcal{K}_0^\gamma B \).

(II) Under (i) the operator \( \mathcal{K} = \mathcal{K}_0^\alpha \mathcal{K}_0^\alpha \) is a composition of the continuous \( \mathcal{K}_0^\alpha \) (see Prop. 3.1) with the compact mapping \( \mathcal{T}_0^\alpha \) (see Thm. A.15). Under (ii) one has \( \mathcal{K} = \mathcal{T}_0^\beta \mathcal{K}_0^\beta \) with the bounded embedding \( \mathcal{T}_0^\beta \) (see Thm. A.13) and the compact \( \mathcal{K}_0^\beta \) (due to (i)). Under the assumptions (iii) we have \( \mathcal{K} = \mathcal{T}_0^\beta \mathcal{K}_0^\beta \mathcal{J}_0^\beta \mathcal{K}_0^\beta \) with bounded embeddings and the compact \( \mathcal{K}_0^\beta \) (thanks to step (I)). Finally, in case (iv) one has \( \mathcal{K} = \mathcal{T}_0^\beta \mathcal{K}_0 \), where \( \mathcal{K}_0 \) is continuous and \( \mathcal{T}_0^\beta \) is compact. Since at least one operator in the above compositions is compact, the compactness of \( \mathcal{K} \) results from [35] p. 417, Thm. 1.2.

3.2 Nemytskii operators

A Nemytskii operator is a mapping of the form

\[
S : U_\alpha \to F(\Omega, \mathbb{R}^p), \quad S(u)(x) := g(x, u(x)) \quad \text{for all } x \in \Omega,
\]

which is generated by a function \( g : \Omega \times Z \to \mathbb{R}^p \). Our terminology using the letter ‘g’ comes from growth function met in applications.

Hypothesis. Let \( m \in \mathbb{N}_0 \). With \( 0 \leq k \leq m \) one assumes:

3 also denoted as composition or superposition operator. Further transcriptions are Nemytskii, Nemyzki, Nemytsky, Nemyckij, Nemyckii, Nemitski, Nemitskii, Nemitsky, Nemickij, Nemickii or Niemytski
The partial derivative \( D^k g : \Omega \times Z \rightarrow L_k(\mathbb{R}^n, \mathbb{R}^p) \) exists and can be continuously extended to \( \Omega \times \mathbb{Z} \).

**Proposition 3.5** (well-definedness of \( \mathcal{G} \)). Assume that (\( \mathcal{N}_0^0 \)) holds. Then a Nemytskii operator \( \mathcal{G} : U_\alpha \rightarrow C^0_p(\Omega) \) is well-defined, bounded and continuous. Moreover, \( \mathcal{G} \) is completely continuous, provided \( \alpha \in (0, 1] \).

**Remark 3.2.** For \( \alpha = 0 \) and compact intervals \( \Omega \subset \mathbb{R} \) a Nemytskii operator \( \mathcal{G} : C^0(\Omega) \rightarrow C^0_p(\Omega) \) is well-defined if and only if \( g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^p \) is continuous [5] Thm. 3.1. Indeed, due to [6] Table 1 one has the equivalences:

\[
\text{g is continuous} \quad \Leftrightarrow \quad \mathcal{G} \text{ is bounded} \quad \Leftrightarrow \quad \mathcal{G} \text{ is well-defined} \quad \Leftrightarrow \quad \mathcal{G} \text{ is continuous}
\]

**Proof.** We denote \( \mathcal{G} \) defined on \( U_0 \) as \( \mathcal{G}_0 \). Since every \( u \in U_\alpha \) is continuous, also \( \mathcal{G}(u) : \Omega \rightarrow \mathbb{R}^p \) is continuous and bounded. As a result, \( \mathcal{G} : U_\alpha \rightarrow C^0_p(\Omega) \) is well-defined. Let \( u_0 \in U_\alpha \) and choose \( r := \|u_0\|_0 + 1 \). Now \( g \) can be extended continuously to \( \Omega \times Z \) by assumption (\( \mathcal{N}_0^0 \)) and \( g \) is uniformly continuous on \( \Omega \times (Z \cap B_r(0)) \).

Given \( \varepsilon > 0 \) this means that there exists a \( \delta > 0 \) such that

\[
\begin{cases}
  d(x, \bar{x}) < \delta, \\
  |z - \bar{z}| < \delta
\end{cases} \Rightarrow |g(x, z) - g(x, \bar{z})| < \varepsilon \quad \text{for all } x, \bar{x} \in \Omega, z, \bar{z} \in Z \cap B_r(0).
\]

If \( u \in U_\alpha \cap B_\delta(u_0, C^0_p(\Omega)) \) and \( \delta < 1 \), then \( |u(x)| \leq |u_0(x)| + |u(x) - u_0(x)| \leq r \) and consequently

\[
||\mathcal{G}(u) - \mathcal{G}(u_0)||_0 = |g(x, u(x)) - g(x, u_0(x))| \leq \varepsilon \quad \text{for all } x \in \Omega.
\]

Passing to the supremum over \( x \in \Omega \) yields \( ||\mathcal{G}(u) - \mathcal{G}(u_0)||_0 \leq \varepsilon \), i.e. \( \mathcal{G} \) is continuous. The boundedness of \( \mathcal{G} \) results from the uniform continuity of \( g \) on \( \Omega \times Z \), as well as properties of the norm \( ||\cdot||_0 \). In conclusion, \( \mathcal{G} \) is well-defined, bounded and continuous.

Thanks to Thm. A.15 the embedding \( J^0_\alpha \) is compact and therefore \( \mathcal{G} = J^0_\alpha \mathcal{G}_0 \) is even completely continuous for \( \alpha \in (0, 1] \) (see [13] pp. 25–26, Thm. 2.1(2))).

**Hypothesis.** Let \( \vartheta \in (0, 1] \).

(\( \mathcal{N}_0^0 \)) For every \( r > 0 \) there exists a \( l'_r \geq 0 \) such that

\[
|g(x, z) - g(x, \bar{z})| \leq l'_r |z - \bar{z}|^\vartheta \quad \text{for all } x \in \Omega, z, \bar{z} \in Z \cap B_r(0).
\]

**Corollary 3.6.** Assume that \( g(\cdot, z) : \Omega \rightarrow \mathbb{R}^p \) is continuous for all \( z \in Z \) and (\( \mathcal{N}_0^0 \)) holds. Then a Nemytskii operator \( \mathcal{G} : U_\alpha \rightarrow C^0_p(\Omega) \) is well-defined and Hölder on bounded sets, that is

\[
\left[ \mathcal{G}|_{U_\alpha \cap B_r(0), C^0_p(\Omega)} \right]_\vartheta \leq l'_r \quad \text{for all } r > 0.
\]

The same argument in case \( \sup_{r > 0} l'_r < \infty \) yields a global Hölder condition for \( \mathcal{G} : U_\alpha \rightarrow C^0_p(\Omega) \).

**Remark 3.3.** Note for \( \alpha = 0 \) and a compact interval \( \Omega \subset \mathbb{R} \), Nemytskii operators \( \mathcal{G} : C^0(\Omega) \rightarrow C^0_p(\Omega) \) satisfy a local (resp. global) Lipschitz condition, if and only if \( g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^p \) does in the second variable. In case \( p = 1 \) even the Lipschitz constants (uniformly in \( x \in \Omega \)) are the same, see [5] Thm. 1 and [6] Thm. 3.2.

**Proof.** As a consequence of Thm. A.7 \( \mathcal{G} : \Omega \times Z \rightarrow \mathbb{R}^p \) is continuous and using Lemma A.3 we can show that \( g \) satisfies the assumption (\( \mathcal{N}_0^0 \)). Hence, Prop. 3.5 yields that \( \mathcal{G} \) is well-defined. Moreover, if \( r > 0 \) and \( u, \bar{u} \in U_\alpha \) with \( ||u||_0, ||\bar{u}||_0 \leq r \), then

\[
||\mathcal{G}(u) - \mathcal{G}(\bar{u})||_0 \leq l'_r ||u(x) - \bar{u}(x)||_0 \leq l'_r ||u - \bar{u}||_0 \quad \text{for all } x \in \Omega
\]

and passing to the supremum over \( x \in \Omega \) implies \( ||\mathcal{G}(u) - \mathcal{G}(\bar{u})||_0 \leq l'_r ||u - \bar{u}||_0 \).

We next show that the derivative of a Nemytskii operator is a multiplication operator:
Lemma 3.7. Assume that (N_k^k) holds for some \( k \in \mathbb{N} \). Then \( \mathcal{G}^k : U_\alpha \to L_k(C^\alpha_\alpha(\Omega), C^0_p(\Omega)) \) given by
\[
(\mathcal{G}^k(u)v_1 \cdots v_k)(x) := D^k_2g(x,u(x))v_1(x) \cdots v_k(x) \quad \text{for all } x \in \Omega, \ v_1, \ldots, v_k \in C^\alpha_\alpha(\Omega)
\] (3.6)
is well-defined and continuous.

Proof. Let \( v_1, \ldots, v_k \in C^\alpha_\alpha(\Omega) \) be given. With \( u \in U_\alpha \) also \( x \mapsto D^k_2g(x,u(x))v_1(x) \cdots v_k(x) \) is continuous and consequently \( \mathcal{G}^k(u)v_1 \cdots v_k \in C^\alpha_\alpha(\Omega) \), i.e. \( \mathcal{G}^k \) is well-defined. Let \((u_l)_{l \in \mathbb{N}}\) be a sequence in \( U_\alpha \) with \( \lim_{l \to \infty} \|u_l - u\|_0 = 0 \). Choose \( r > 0 \) so large that \( u, u_l \in B_r(0, C^\alpha_\alpha(\Omega)) \) for all \( l \in \mathbb{N} \). Then \( D^k_2g \) is uniformly continuous on \( \Omega \times (Z \cap B_r(0)) \) and given \( \varepsilon > 0 \), there exists a \( \delta > 0 \) with
\[
|z - z| < \delta \implies \left| D^k_2g(x,z) - D^k_2g(x,z) \right| < \varepsilon \quad \text{for all } x \in \Omega, \ z, z \in Z \cap B_r(0).
\]
Moreover, there exists an \( L \in \mathbb{N} \) such that \( |u_l(x) - u(x)| \leq \delta \) for all \( l \geq L \) and consequently
\[
\left| (\mathcal{G}^k(u_l) - \mathcal{G}^k(u))v_1 \cdots v_k(x) \right| \leq \left| (D^k_2g(x,u_l(x)) - D^k_2g(x,u(x)))v_1(x) \cdots v_k(x) \right| \quad \text{for all } x \in \Omega.
\]
Passing to the supremum over \( x \in \Omega \) yields
\[
\|\mathcal{G}^k(u_l) - \mathcal{G}^k(u)\|_{L_k(C^\alpha_\alpha(\Omega), C^0_p(\Omega))} \leq \varepsilon \quad \text{for } u, u_l \in B_1(0, C^\alpha_\alpha(\Omega)) \text{ and, in turn,}
\]
for all \( l \geq L \). Since \( u \in U_\alpha \) was arbitrary, \( \mathcal{G}^k \) is continuous. \( \square \)

Proposition 3.8 (continuous differentiability of \( \mathcal{G}^k \)). Let \( m \in \mathbb{N} \). Assume that (N_k^k) hold for all \( 0 \leq k \leq m \) on a convex set \( Z \subseteq \mathbb{R}^n \). Then a Nemytskii operator \( \mathcal{G} : U_\alpha \to C^m_p(\Omega) \) is \( m \)-times continuously differentiable with \( D^m \mathcal{G} = \mathcal{G}^m \) for every \( 1 \leq k \leq m \).

Proof. (I) Let \( 0 \leq k < m \). Thanks to Lemma 3.7 the mappings \( \mathcal{G}^k : U_\alpha \to L_k(C^\alpha_\alpha(\Omega), C^0_p(\Omega)) \) are well-defined and continuous. Thus, given \( u \in U_\alpha \) and \( h \in C^\alpha_\alpha(\Omega) \) with \( u + h \in U_\alpha \) the remainders
\[
r_k(h) := \sup_{\theta \in [0,1]} \left\| \mathcal{G}^{k+1}(u + \theta h) - \mathcal{G}^{k+1}(u) \right\|_{L_k(C^\alpha_\alpha(\Omega), C^0_p(\Omega))}
\]
satisfy \( \lim_{h \to 0} r_k(h) = 0 \). Now we obtain from [85, p. 341, Thm. 4.2] that
\[
[D^k_2g(u(x) + h(x)) - D^k_2g(u(x)) - D^{k+1}_2g(u(x))h(x)] = \int_0^1 \left[ D^k_2g(u(x) + \theta h(x)) - D^{k+1}_2g(u(x)) \right] h(x) d\theta,
\]
consequently
\[
\left| [\mathcal{G}^k(u + h) - \mathcal{G}^k(u) - \mathcal{G}^{k+1}(u)h](x) \right| \leq \int_0^1 \left| [\mathcal{G}^{k+1}(u + \theta h) - \mathcal{G}^{k+1}(u)] \right|_{L_k(C^\alpha_\alpha(\Omega), C^0_p(\Omega))} d\theta \|h\|_0
\]
and after passing to the least upper bound over \( x \in \Omega \) it results
\[
\left( \left| [\mathcal{G}^k(u + h) - \mathcal{G}^k(u) - \mathcal{G}^{k+1}(u)h]\right|_{L_k(C^\alpha_\alpha(\Omega), C^0_p(\Omega))} \leq r_k(h) \|h\|_0.
\]
This establishes that the mapping \( \mathcal{G}^k \) is differentiable in \( u \) with the derivative \( \mathcal{G}^{k+1}(u) \) and, in turn, \( \mathcal{G}^{k+1} \) is continuous due to Lemma 3.7.

(II) Applying step (I) in case \( k = 0 \) shows that \( \mathcal{G} \) is continuously differentiable on \( U_\alpha \) with the derivative \( \mathcal{G}^{(1)} \).

Given this, mathematical induction yields that \( \mathcal{G} : U_\alpha \to C^0_p(\Omega) \) is actually \( m \)-times continuously differentiable with the derivatives \( D^m \mathcal{G} = \mathcal{G}^m \) for all \( 0 \leq k \leq m \). \( \square \)
Remark 3.4 (boundedness and continuity of $\mathcal{J}$). (1) The boundedness of Nemytskii operators $\mathcal{J}$ guaranteed in Props. 3.5 means that the $\mathcal{J}$-images of merely $\|\cdot\|_p$-bounded subsets $B \subset U_\alpha$ are bounded. In particular, the functions in $B$ are not required to have uniformly bounded Hölder constants.

(2) The continuity of the Nemytskii operator $\mathcal{U}$ in Prop. 3.5 as well as for the derivatives $D^k \mathcal{U}$ in Prop. 3.8 are to be understood in the following strong form: Already convergence in the domain $U_\alpha$ w.r.t. the norm $\|\cdot\|_0$ is sufficient for convergence of the $\mathcal{J}$-values in the norm $\|\cdot\|_0$. A corresponding statement holds in Cor. 3.6.

In contrast to the above situation, Nemytskii operators $\mathcal{J}$ behave rather differently when mapping into the space of Hölder functions of exponent $\alpha \in (0, 1]$. Indeed, [39] shows that $\mathcal{J}$ fails to map $C^0[0, 1]$ into itself by Rem. 3.2 such that $\mathcal{J}$ maps $C^\alpha[0, 1]$ into itself. Below we survey some properties relating the mappings

$$ g : \Omega \times \mathbb{R} \to \mathbb{R}^p, \quad \mathcal{J} : C^\alpha(\Omega) \to C^\alpha_p(\Omega), $$

when $\Omega \subset \mathbb{R}^n$ is compact; one denotes $g$ as autonomous, if it does not depend on the first argument:

- **Well-definedness and boundedness:** In [13] Thm. 1.1 it is shown that the condition

$$ \forall r > 0 \exists k(r) > 0 : |g(x, z) - g(\bar{x}, \bar{z})| \leq k(r) \left( |x - \bar{x}|^\alpha + |z - \bar{z}|^\beta \right) $$

(3.7)

for all $x, \bar{x} \in \Omega, z, \bar{z} \in \overline{B}_r(0)$ is equivalent to $\mathcal{J}$ being well-defined and bounded (see also [8] Thm. 7.3). In comparison, a necessary and sufficient condition for $\mathcal{J}$ to be merely well-defined is more clumsy, restricted to $\Omega = [a, b]$, and given in terms of (see [6] Thm. 3.8 or [8] Thm. 7.1)

$$ \forall(x_0, z_0) \in \Omega \times \mathbb{R} \forall r > 0 \exists k(r) > 0 \exists \delta > 0 : |g(x, z) - g(\bar{x}, \bar{z})| \leq k(r) \left( |x - \bar{x}|^\alpha + |z - \bar{z}|^\beta \right) $$

(3.8)

for all $x, \bar{x} \in \Omega, z, \bar{z} \in \mathbb{R}$ with $x, \bar{x} \in B_r(x_0), |z - z_0| \leq r |x - x_0|^\alpha$ and $|\bar{z} - \bar{z}_0| \leq r |\bar{x} - \bar{x}_0|^\alpha$.

If $g$ is autonomous, then the Lipschitz condition (3.5) with $\theta = 1$ is even necessary and sufficient for $\mathcal{J}$ being well-defined, see [13] Thm. 1.

Let $g$ be autonomous and $\Omega = [a, b]$. Now every well-defined $\mathcal{J}$ is bounded (see [23] Cor. 2.1) and $g$ is continuous (see [8] Thm. 7.5).

- **Continuity:** If the partial derivative $D_2 g$ exists and satisfies

$$ D_2 g : \Omega \times \mathbb{R} \to \mathbb{R}^p $$

then (3.7) implies that $\mathcal{J}$ is continuous (cf. [22] Thm. 2.2). Conversely, the partial derivative $D_2 g$ exists, if $\mathcal{J}$ is continuous and (3.7) is valid (see [22] Thm. 2.2), or if $\mathcal{J}$ is bounded and $\Omega = [a, b]$ (see [22] Cor. 2.3).

A characterization of $\mathcal{J}$ being uniformly continuous on bounded sets can be found in [13] Thm. 2.1. For intervals $\Omega = [a, b]$ it follows from [6] Table 5] that:

| $\mathcal{J}$ is bounded | $\mathcal{J}$ is well-defined | $\mathcal{J}$ is continuous |
|------------------------|--------------------------|---------------------------|
| $\downarrow$           | $\downarrow$             | $\uparrow$                |
| $g$ satisfies (3.7)    | $g$ satisfies (3.8)      | $g$ is continuously differentiable |

More can be said whenever $g$ is autonomous and $\Omega = [a, b]$: Then reproducing [6] Table 4] the following implications hold:

| $g$ is Lipschitz on bounded sets | $\downarrow$ | $g$ is continuously differentiable |
|---------------------------------|-------------|----------------------------------|
| $\mathcal{J}$ is bounded        | $\downarrow$| $\mathcal{J}$ is well-defined    |
| $\mathcal{J}$ is continuous     |             |                                   |

- **Lipschitz condition:** It is shown in [13] Thm. 3.1 that $\mathcal{J}$ is Lipschitz on bounded sets if and only if both $g$ and $D_2 g$ satisfy an estimate (3.7). The necessity to assume the existence of the partial derivatives also arose for Urysohn operators (see Cor. 2.8). Yet, the assumption of a global Lipschitz condition for $\mathcal{J}$ leads to a degeneracy in $g$. Indeed, [39] shows that $\mathcal{J}$ is globally Lipschitz, if and only if all components $g_1, \ldots, g_p : \Omega \times \mathbb{R} \to \mathbb{R}$ of $g$ are affine linear, i.e. there exist $\tilde{a}_i, \tilde{b}_i \in C^\alpha(\Omega)$ with

$$ g_i(x, z) = z \tilde{a}_i(x) + \tilde{b}_i(x) $$

for all $1 \leq i \leq p, x \in \Omega, z \in \mathbb{R}$. (3.10)
Yet, even uniform continuity of \( \mathcal{J} \) is sufficient for (3.10) to hold (see [50, Thm. 2]). Nonetheless, if \( J^p_α \mathcal{J} \) satisfies a global Lipschitz condition, then \( \sup_{x \in \Omega} |g(x)|_1 < \infty \) (cf. [36, Thm. 2.5] and (3.10) holds in case \( \beta = 1 \) (cf. [36, Thm. 2.6]).

Let \( g \) be autonomous and \( \Omega = [a, b] \): Then a well-defined Nemytskii operator \( \mathcal{J} \) is globally Lipschitz, if and only if \( g : \mathbb{R} \to \mathbb{R}^p \) is affine-linear, i.e. there exist \( \hat{a}, \hat{b} \in \mathbb{R}^p \) such that \( g(z) = \hat{a} + \hat{b} \) (see [23, Thm. 2.3(b)]).

**Continuous differentiability:** If \( g(x, \cdot) \) is twice differentiable such that \( g, D_2 g \) satisfy (3.8) and (3.9) holds with \( D_2^2 g \) (instead of \( D_2 g \)), then \( \mathcal{J} \) is continuously differentiable (cf. [12, Thm. 4.1]). Note that also for the continuous differentiability of Urysohn operators we needed to assume that the second order derivative of the kernel function exists (cf. Thm. 2.12). For a characterization of \( \mathcal{J} \) being continuously differentiable with a derivative being uniformly continuous on bounded sets we refer to [13, Thm. 4.1]. In case \( \Omega = [a, b] \), then differentiability of \( J^p_α \mathcal{J} \) is characterized in [8, Thm. 7.11]. Furthermore, if \( \mathcal{J} \) is differentiable with a globally bounded derivative, then \( g \) is affine linear, i.e. \( g \) satisfies (4.10).

For autonomous \( g \) and \( \Omega = [a, b] \) an elegant characterization holds: The Nemytskii operator \( \mathcal{J} \) is continuously differentiable, if and only if \( g : \mathbb{R} \to \mathbb{R}^p \) is twice continuously differentiable (see [23, Thm. 2.4]).

### 3.3 Hammerstein operators

In the following, we understand *Hammerstein operators* (3.1) as composition

\[
\mathcal{H} = \mathcal{K} \mathcal{J} : C^α_p(\Omega) \to F(\Omega_1, \mathbb{R}^d)
\]

of Fredholm operators \( \mathcal{K} \in L(C^β_d(\Omega_1), \mathbb{R}^d) \) and Nemytskii operators \( \mathcal{J} : C^α_p(\Omega) \to C^β_p(\Omega_1) \) given in (3.2) resp. (3.4). Hence, our above preparations immediately yield properties of \( \mathcal{H} \):

**Theorem 3.9** (well-definedness of \( \mathcal{H} \)). Assume that \( (\bar{K}_1, \bar{K}_2) \) and \( (N^0_0) \) hold. Then a Hammerstein operator \( \mathcal{H} : U_α \to C^β_d(\Omega_1) \) is well-defined, bounded and continuous.

**Proof.** As composition of \( \mathcal{J} : C^α_p(\Omega) \to C^β_p(\Omega) \) and \( \mathcal{K} \in L(C^β_d(\Omega_1), \mathbb{R}^d) \), the claims for \( \mathcal{H} = \mathcal{K} \mathcal{J} \) result directly from Thm. 3.3 (with \( \alpha = 0 \)) and Prop. 3.5.

**Corollary 3.10** (complete continuity of \( \mathcal{H} \)). A Hammerstein operator \( \mathcal{H} : U_α \to C^β_d(\Omega_1) \) is completely continuous, provided one of the following holds:

(i) \( \alpha \in (0, 1) \) and \( \gamma = \beta \),
(ii) \( \Omega_1 \) is bounded, \( \alpha \in (0, 1] \) and \( \gamma \in [0, \beta] \),
(iii) \( \Omega_1 \) is compact, \( \gamma \in [0, \beta] \) and \( \lim_{x \to x_0} \hat{h}(x, x_0) = 0 \) uniformly in \( x_0 \in \Omega_1 \),
(iv) \( \Omega_1 \) is compact and \( \gamma \in [0, \beta] \).

**Proof.** It results from Prop. 3.5 (case (i)) and Cor. 3.4 (cases (ii–iii)) that at least one of the functions in the composition \( \mathcal{H} = \mathcal{K} \mathcal{J} \) is completely continuous.

**Theorem 3.11** (continuous differentiability of \( \mathcal{H} \)). Let \( m \in \mathbb{N} \). Assume that \( (\bar{K}_1, \bar{K}_2) \) and \( (N^k_0) \) hold for all \( 0 \leq k \leq m \) on a convex set \( Z \subseteq \mathbb{R}^n \). Then a Hammerstein operator \( \mathcal{H} : U_α \to C^β_d(\Omega_1) \) is \( m \)-times continuously differentiable with \( D^k \mathcal{H} = \mathcal{K} \mathcal{J}^k \mathcal{H} \) for every \( 1 \leq k \leq m \).

**Proof.** This results from the chain rule [35, p. 337], Thm. 3.3 (with \( \alpha = 0 \)) and Prop. 3.8.

**Remark 3.5** (convolutive Hammerstein operators). Suppose a growth function \( g : [a, b] \times Z \to \mathbb{R}^p \) generates a Nemytskii operator \( \mathcal{J} \) mapping into \( C^α_p[a, b] \). Then the smoothing properties from Sect. 2.3 extend to convolutive Hammerstein operators \( \mathcal{K}u : \int_a^b \mathcal{K}(x-y)g(y, u(y)) \) dy with an ambient kernel \( \mathcal{K} : [a-b, b-a] \to \mathbb{R}^{d \times p} \).
A H"older continuous functions

The definition of continuity for a function $u$ in e.g. a point $x_0$ is not quantitative in the sense that it provides no information on how fast its values $u(x)$ approach $u(x_0)$ as $x \to x_0$. In consequence, the modulus of continuity $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ of a continuous function $u$ satisfying the estimate $\|u(x) - u(x_0)\| \leq \omega(d(x, x_0))$ may decrease arbitrarily slowly. For this reason, the space of continuous functions is often not suitable for a quantitative analysis in fields such as numerical analysis or partial differential equations. A straightforward and feasible way to strengthen the notion of continuity of $u$ is to assume that its modulus of continuity is proportional to a power of $d(x, x_0)^\alpha$ for some exponent $\alpha \in (0, 1]$. Such functions are denoted as H"older continuous and in the focus of this appendix.

Our overall setting is as follows. Let $(\Omega, d)$ be a metric space and $(Y, \|\cdot\|)$ be a normed space over $\mathbb{K}$, which stands for the real or complex field.

A function $u : \Omega \to Y$ is said to be $\alpha$-H"older (with H"older exponent $\alpha \in (0, 1]$), if it satisfies
\[
[u]_\alpha := \sup_{\substack{x, \bar{x} \in \Omega \\ x \neq \bar{x}}} \frac{\|u(x) - u(\bar{x})\|}{d(x, \bar{x})^\alpha} < \infty;
\]
the finite quantity $[u]_\alpha \geq 0$ is called H"older constant of $u$. One speaks of a H"older continuous function $u$, if it is $\alpha$-H"older for some $\alpha \in (0, 1)$ and in case $\alpha = 1$ one denotes $u$ as Lipschitz continuous with Lipschitz constant $[u]_1$ — a comprehensive approach to this class of functions is given in [14]. For the set of all such $\alpha$-H"older functions $u : \Omega \to Y$ we write $C^\alpha(\Omega, Y)$, supplemented by $C^0(\Omega, Y)$ for the linear space of continuous functions. It is convenient to denote a continuous function as 0-H"older, and unless indicated otherwise, let us assume $\alpha \in [0, 1]$ throughout.

Remark A.1. (1) A function $u : \Omega \to Y$ is constant, if and only if its H"older constant vanishes.

(2) For $\alpha \in (0, 1]$ there is an evident relation between H"older functions and Lipschitz functions: Indeed, $u : (\Omega, d) \to Y$ is $\alpha$-H"older, if and only if $u : (\Omega, d_\alpha) \to Y$ is Lipschitz with the metric $d_\alpha : \Omega \times \Omega \to \mathbb{R}_+$ given by $d_\alpha(x, \bar{x}) := d(x, \bar{x})^\alpha$. Of course the metrics $d$ and $d_\alpha$ on $\Omega$ are not equivalent.

(3) One does restrict to exponents $\alpha \in (0, 1]$ for the following reason. Suppose that a function $u : \Omega \to Y$ on an open subset $\Omega \subseteq \mathbb{R}^n$ satisfies $[u]_\alpha < \infty$ for an exponent $\alpha > 1$. Then
\[
\|u(\bar{x}) - u(\bar{x}) - 0(x - \bar{x})\| = \|u(x) - u(\bar{x})\| \leq [u]_\alpha |x - \bar{x}|^{\alpha - 1}|x - \bar{x}| \quad \text{for all } x, \bar{x} \in \Omega
\]
yields that $u$ is differentiable on $\Omega$ with derivative 0 and thus constant on the components of $\Omega$.

(4) Suppose that $\Omega$ has a finite, positive diameter and that $\phi : [0, 1] \to \mathbb{R}_+$ is a function satisfying $\phi(0) = 0$, $\phi(1) = 1$ such that $t \mapsto \phi(t)$ and $t \mapsto \phi^{-1}(t)$ are positive and increasing on $(0, 1)$. Then $u : \Omega \to Y$ is called generalized H"older, if $\sup_{x, \bar{x} \in \Omega, x \neq \bar{x}} \|u(x) - u(\bar{x})\| \phi\left(\frac{d(x, \bar{x})}{\min\{d(x, \bar{x})\}}\right)^{-1} < \infty$ (see [8] Ch. 7 and [11]). In case $\phi(t) := t^\alpha$, $\alpha \in (0, 1]$ this reduces to the situation studied here.

(5) Differentiable functions on $\Omega \subseteq \mathbb{R}^n$ whose $m$th derivative satisfies an $\alpha$-H"older condition, and associated function spaces $C^{m,\alpha}(\Omega, Y)$, are addressed in [22] pp. 30ff, Sect. 1.5 or [23] pp. 51ff, Sect. 4.1.

(6) The inner structure of H"older spaces from an abstract Banach spaces perspective is studied in [24].

Example A.1. Let $\Omega = Y = \mathbb{R}$ and $\alpha \in (0, 1]$. The function $u_0 : \mathbb{R} \to \mathbb{R}$, $u_0(x) := |x|^\alpha$ is not differentiable in 0 (see Fig. 1) but $\alpha$-H"older. Indeed, given $x, \bar{x} \in \mathbb{R}$ the case $0 < |x| \leq |\bar{x}|$ leads to the inequality $1 - \frac{|x|^\alpha}{|\bar{x}|^{\alpha}} \leq 1 - \frac{|x|}{|\bar{x}|} \leq \left(1 - \frac{|x|}{|\bar{x}|}\right)^\alpha$ and therefore $|u_0(\bar{x}) - u_0(x)| = u_0(\bar{x}) - u_0(x) \leq |\bar{x} - x|^\alpha$. A similar argument in case $0 < |\bar{x}| \leq |x|$ yields the assertion with H"older constant $[u_0]_\alpha \leq 1$.

Theorem A.1 (local H"older continuity). Let $\Omega$ be compact and $\alpha \in (0, 1]$. If $u : \Omega \to Y$ is locally $\alpha$-H"older, i.e. every $x \in \Omega$ has a neighborhood $U \subseteq \Omega$ such that $[u|_U]_\alpha < \infty$ holds, then $u$ is $\alpha$-H"older.

Proof. We proceed indirectly and suppose that for each $c \geq 0$ there exist $x, \bar{x} \in \Omega$ yielding the inequality $\|u(x) - u(\bar{x})\| > cd(x, \bar{x})^\alpha$. In particular, there are sequences $(x_l)_{l \in \mathbb{N}}$, $(\bar{x}_l)_{l \in \mathbb{N}}$ in $\Omega$ so that
\[
\|u(x_l) - u(\bar{x}_l)\| > ld(x_l, \bar{x}_l)^\alpha \quad \text{for all } l \in \mathbb{N} \tag{A.1}
\]
holds. Since $\Omega$ is compact, w.l.o.g. we can assume that these sequences converge to $x^*, \bar{x}^* \in \Omega$, respectively. Passing to the limit $l \to \infty$ in (A.1) therefore enforces $\lim_{l \to \infty} d(x_l, x_i)^\alpha = 0$, i.e. one has $x^* = \bar{x}^*$. Because the function $u$ is assumed to be locally Hölder, there exists a neighborhood $U \subseteq \Omega$ of $x^*$ and a real $C \geq 0$ with $\|u(x) - u(\bar{x})\| \leq Cd(x, \bar{x})^\alpha$ for all $x, \bar{x} \in U$. Thanks to the inclusion $x_l, \bar{x}_i \in U$ for sufficiently large $l \in \mathbb{N}$ this implies $\|u(x_l) - u(\bar{x}_i)\| \leq Cd(x_l, \bar{x}_i)^\alpha$ contradicting (A.1) for $l > C$.

The relationship between differentiable and Hölder continuous functions is addressed in

**Example A.2.** (1) Differentiable functions $u : \Omega \to Y$ on open, bounded and convex sets $\Omega \subset \mathbb{R}^k$ having a bounded derivative $Du : \Omega \to L(\mathbb{R}^k, Y)$ are $\alpha$-Hölder for each $\alpha \in (0, 1]$. This follows from the Mean Value Inequality [33, p. 35, Thm. 4.1], because

$$\|u(x) - u(\bar{x})\| \leq \sup_{\xi \in \Omega} \|Du(\xi)\| |x - \bar{x}| \leq (\text{diam} \Omega)^{1-\alpha} \sup_{\xi \in \Omega} \|Du(\xi)\| |x - \bar{x}|^\alpha$$

and thus $[u]_\alpha \leq (\text{diam} \Omega)^{1-\alpha} \sup_{\xi \in \Omega} \|Du(\xi)\|$. A version of this result on not necessarily convex sets $\Omega$ can be found in [22, p. 11, Prop. 1.1.13]. In the Lipschitz case $\alpha = 1$ boundedness of $\Omega$ is not required.

(2) Rademacher’s theorem [17, p. 414, Thm. 21.2] guarantees that Lipschitz functions $u : \Omega \to \mathbb{R}^d$ on open sets $\Omega \subset \mathbb{R}^k$ are differentiable Lebesgue-almost everywhere in $\Omega$. This situation changes for exponents $\alpha \in (0, 1)$ and [27] shows that the Weierstraß function $u : \mathbb{R} \to \mathbb{R}$ given as Fourier series

$$u(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x)$$

with $a \in (0, 1)$ and integers $b > 1$ satisfying $ab > 1 + \frac{3\pi}{2}$ is $\alpha$-Hölder with exponent $\alpha = -\log_b a$, but nowhere differentiable.

**Theorem A.2.** Hölder continuous functions are uniformly continuous.

**Proof.** Let $\varepsilon > 0$ and $u \in C^\alpha(\Omega, Y)$ (w.l.o.g. $u$ is not constant). Setting $\delta := \left(\frac{\varepsilon}{[u]_\alpha}\right)^{1/\alpha}$ guarantees

$$d(x, \bar{x}) < \delta \implies \|u(x) - u(\bar{x})\| \leq [u]_\alpha d(x, \bar{x})^\alpha \leq \varepsilon$$

for all $x, \bar{x} \in \Omega$ and thus $u$ is uniformly continuous. The converse to Thm. A.2 does not hold.

**Example A.3.** Let $\Omega = [-\frac{1}{2}, \frac{1}{2}]$ and $Y = \mathbb{R}$. The continuous function (see Fig. 2)

$$u : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}, \quad u(x) := \begin{cases} -\frac{1}{\ln|x|}, & x \in [-\frac{1}{2}, \frac{1}{2}] \setminus \{0\}, \\ 0, & x = 0 \end{cases}$$

on the compact interval $[-\frac{1}{2}, \frac{1}{2}]$ is uniformly continuous. However, it is not Hölder continuous, because otherwise there would exist reals $\alpha \in (0, 1], C \geq 0$ such that $-\frac{1}{\ln x} = |u(x) - u(0)| \leq C |x - 0|^{\alpha} = Cx^{\alpha}$ for all $x \in (0, \frac{1}{2}]$ leading to the contradiction $C \geq -\frac{1}{\ln x} x \to \infty$ by the l’Hospital rule.
Lemma A.3. Let \( \alpha \in (0, 1] \) and \( Y \) be a Banach space. Every \( \alpha \)-Hölder function \( u : U \to Y \) on an open set \( U \subseteq \Omega \) allows an \( \alpha \)-Hölder extension to the closure \( \bar{U} \) with the same \( \alpha \)-Hölder constant.

Proof. Let \( u : U \subseteq \Omega \to Y \) be \( \alpha \)-Hölder. By Thm. A.2 it follows that \( u : U \to Y \) is uniformly continuous. Hence, because \( Y \) is complete, there exists a continuous extension \( \bar{u} : \bar{U} \subseteq \Omega \to Y \) of \( u \) to the boundary. We next show that \( \bar{u} \) is \( \alpha \)-Hölder with \( [\bar{u}]_\alpha = [u]_\alpha \). Thereto, given arbitrary \( y, \bar{y} \in \bar{U} \) and the estimate

\[
\|u(x) - u(\bar{x})\| \leq [u]_\alpha d(x, \bar{x})^\alpha
\]

for all \( x, \bar{x} \in U \), we first pass to the limit \( x \to y \), then to \( \bar{x} \to \bar{y} \), and it results from the extension property that \( \|\bar{u}(y) - \bar{u}(\bar{y})\| \leq [\bar{u}]_\alpha d(y, \bar{y})^\alpha \) for all \( y, \bar{y} \in \bar{U} \). This guarantees that \( \bar{u} \) is \( \alpha \)-Hölder with \( [\bar{u}]_\alpha \leq [u]_\alpha \). Moreover, it evidently holds

\[
\|u(x) - u(\bar{x})\| = \|\bar{u}(x) - \bar{u}(\bar{x})\| \leq [\bar{u}]_\alpha d(x, \bar{x})^\alpha \quad \text{for all } x, \bar{x} \in U
\]

and \( [u]_\alpha \leq [\bar{u}]_\alpha \). Thus, the minimum Hölder coefficients of \( u \) and \( \bar{u} \) are equal yielding the claim.

Theorem A.4. Hölder continuous functions are bounded (on bounded sets).

Proof. Let \( u \in C^\alpha(\Omega, Y) \), \( B \subseteq \Omega \) be bounded and choose a fixed \( x_0 \in B \). Then

\[
\|u(x)\| \leq \|u(x_0)\| + \|u(x) - u(x_0)\| \leq \|u(x_0)\| + [u]_\alpha d(x, x_0)^\alpha \leq \|u(x_0)\| + [u]_\alpha (\text{diam } B)^\alpha
\]

for all \( x \in \Omega \) holds and therefore the image \( u(B) \subseteq Y \) is bounded.

For globally bounded \( \alpha \)-Hölder functions \( u : \Omega \to Y \), we define

\[
\|u\|_\alpha := \begin{cases} \sup_{x \in \Omega} \|u(x)\|, & \alpha = 0, \\ \max \left\{ \sup_{x \in \Omega} \|u(x)\|, [u]_\alpha \right\}, & \alpha \in (0, 1]. \end{cases}
\]

On the product \( Y_1 \times Y_2 \) of two normed spaces \( Y_1, Y_2 \) we use the product norm

\[
\|(y_1, y_2)\| = \max \{\|y_1\|, \|y_2\|\} \quad \text{for all } y_1 \in Y_1, y_2 \in Y_2.
\]

Theorem A.5. A function \( u = (u_1, u_2) : \Omega \to Y_1 \times Y_2 \) is \( \alpha \)-Hölder, if and only if both component functions \( u_j : \Omega \to Y_j \) are \( \alpha \)-Hölder for \( j = 1, 2 \). In this case and for \( \alpha \in (0, 1] \) one has \( [u]_\alpha \leq \max \{[u_1]_\alpha, [u_2]_\alpha\} \) and for globally bounded functions \( u \) results \( \|u\|_\alpha \leq \max \{\|u_1\|_\alpha, \|u_2\|_\alpha\} \) for all \( j = 1, 2 \).

Proof. We restrict to the case \( \alpha \in (0, 1] \).

(\(\Rightarrow\)) If \( u : \Omega \to Y_1 \times Y_2 \) is \( \alpha \)-Hölder, then also the components \( u_1, u_2 \) are \( \alpha \)-Hölder due to

\[
\|u_j(x) - u_j(\bar{x})\| \leq \frac{2}{i=1} \|u_i(x) - u_i(\bar{x})\| = \|u(x) - u(\bar{x})\| \leq [u]_\alpha d(x, \bar{x})^\alpha \quad \text{for all } x, \bar{x} \in \Omega, j = 1, 2.
\]

(\(\Leftarrow\)) Conversely, if the component functions \( u_1, u_2 \) are \( \alpha \)-Hölder, then also \( u \) is \( \alpha \)-Hölder because of

\[
\|u(x) - u(\bar{x})\| = \frac{2}{i=1} \|u_i(x) - u_i(\bar{x})\| \leq \frac{2}{i=1} \|u_i\|_\alpha d(x, \bar{x})^\alpha \quad \text{for all } x, \bar{x} \in \Omega.
\]
These inequalities also imply the claimed estimates for the H"{o}lder constants. Combining them with
\[ \| u_j(x) \| \leq \| u(x) \| = \max_{i=1}^{2} \| u_i(x) \| \leq 2 \max_{i=1}^{2} \| u_i \|_\alpha \] for all \( x \in \Omega, j = 1, 2 \)
yields the estimates for \( \| \cdot \|_\alpha \) after passing to the supremum over \( x \in \Omega \).

The product of two metric spaces \( \Omega_1, \Omega_2 \) is equipped with the product metric
\[ d((x_1, x_2), (\bar{x}_1, \bar{x}_2)) := \max \{ d(x_1, \bar{x}_1), d(x_2, \bar{x}_2) \} \] for all \( x_1, \bar{x}_1 \in \Omega_1, x_2, \bar{x}_2 \in \Omega_2 \).

**Theorem A.6.** Let \( \alpha \in (0, 1] \). A function \( u : \Omega_1 \times \Omega_2 \rightarrow Y \) defined on the product of metric spaces \( \Omega_1, \Omega_2 \) is \( \alpha \)-H"{o}lder, if and only if all functions \( u(\cdot, x_2) : \Omega_1 \rightarrow Y \) and \( u(x_1, \cdot) : \Omega_2 \rightarrow Y \) are \( \alpha \)-H"{o}lder uniformly in \( x_2 \in \Omega_2 \) resp. \( x_1 \in \Omega_1 \).

**Proof.** Suppose that \( x_1, \bar{x}_1 \in \Omega_1 \) and \( x_2, \bar{x}_2 \in \Omega_2 \) are arbitrary.
(\( \Rightarrow \)) Let \( u : \Omega_1 \times \Omega_2 \rightarrow Y \) be \( \alpha \)-H"{o}lder. From the estimates
\[ \| u(x_1, x_2) - u(\bar{x}_1, x_2) \| \leq [u]_\alpha d(x_1, \bar{x}_1)\alpha, \quad \| u(x_1, x_2) - u(x_1, \bar{x}_2) \| \leq [u]_\alpha d(x_2, \bar{x}_2)\alpha \]
one deduces that \( u(\cdot, x_2) \) and \( u(x_1, \cdot) \) are \( \alpha \)-H"{o}lder (uniformly in \( x_2 \) resp. \( x_1 \)).

(\( \Leftarrow \)) Conversely, the estimate
\[ \| u(x_1, x_2) - u(\bar{x}_1, \bar{x}_2) \| \leq \| u(x_1, x_2) - u(\bar{x}_1, x_2) \| + \| u(\bar{x}_1, x_2) - u(\bar{x}_1, \bar{x}_2) \| \]
\[ \leq \sup_{x \in \Omega_2} [u(\cdot, x)]_\alpha d(x_1, x_1)\alpha + \sup_{x \in \Omega_1} [u(x, \cdot)]_\alpha d(x_2, x_2)\alpha \]
\[ \leq \left( \sup_{x \in \Omega_2} [u(\cdot, x)]_\alpha + \sup_{x \in \Omega_1} [u(x, \cdot)]_\alpha \right) \max \{ d(x_1, \bar{x}_1), d(x_2, \bar{x}_2) \} \]
implies a H"{o}lder condition for \( u \). \qed

**Theorem A.7.** Let \( \alpha \in (0, 1] \). If a function \( u : \Omega_1 \times \Omega_2 \rightarrow Y \) satisfies
(i) \( \sup_{x_1, \bar{x}_1 \in \Omega_1} [u(x_1, \cdot)]_\alpha < \infty \),
(ii) \( u(\cdot, x_2) : \Omega_1 \rightarrow Y \) is continuous for all \( x_2 \in \Omega_2 \),

then \( u \) is continuous.

**Proof.** Let \( (x_1^i, x_2^i) \in \Omega_1 \times \Omega_2 \) be the limit of sequences \( (x_1^i)_{i \in \mathbb{N}}, (x_2^i)_{i \in \mathbb{N}} \) in the respective metric spaces \( \Omega_1 \) and \( \Omega_2 \). If \( k_2 := \sup_{x \in \Omega_1} [u(x, \cdot)]_\alpha \), then
\[ 0 \leq \left\| u(x_1^1, x_2^1) - u(x_1^2, x_2^2) \right\| \leq \left\| u(x_1^1, x_1^1) - u(x_1^2, x_2^2) \right\| + \left\| u(x_1^2, x_2^2) - u(x_1^2, x_2^2) \right\| \]
\[ \overset{(i)}{=} k_2 d(x_1^1, x_1^2)\alpha + \left\| u(x_1^2, x_2^2) - u(x_1^2, x_2^2) \right\| \overset{(ii)}{\underset{l \to \infty}{\longrightarrow}} 0 \]
establishes the continuity of \( u \), since \( (x_1^1, x_2^1) \) were arbitrary. \( \Box \)

Let us next investigate the algebraic structure of the space of \( \alpha \)-H"{o}lder functions.

**Theorem A.8** (sum rule). With functions \( u_1, u_2 : \Omega \rightarrow Y \) also \( \lambda_1 u_1 + \lambda_2 u_2 \) is \( \alpha \)-H"{o}lder for all \( \lambda_1, \lambda_2 \in \mathbb{K} \).
In case \( \alpha \in (0, 1] \) one has \( |\lambda_1 u_1 + \lambda_2 u_2|_\alpha \leq |\lambda_1||u_1|_\alpha + |\lambda_2||u_2|_\alpha \) and for globally bounded \( u_1, u_2 \) holds
\[ \|\lambda_1 u_1 + \lambda_2 u_2\|_\alpha \leq |\lambda_1|\|u_1\|_\alpha + |\lambda_2|\|u_2\|_\alpha \] for all \( \lambda_1, \lambda_2 \in \mathbb{K} \).

**Proof.** The straightforward proof is left to the reader. \( \Box \)

A mapping \( \cdot : Y_1 \times Y_2 \rightarrow Y \) is called a product, if there exists a constant \( C \geq 0 \) such that
\[ y_1 \cdot (y_2 + \bar{y}_2) = y_1 \cdot y_2 + y_1 \cdot \bar{y}_2, \quad (y_1 + \bar{y}_1) \cdot y_2 = y_1 \cdot y_2 + \bar{y}_1 \cdot y_2, \quad \|y_1 \cdot y_2\| \leq C \|y_1\| \|y_2\| \]
for all \( y_1, \bar{y}_1 \in Y_1, y_2, \bar{y}_2 \in Y_2 \).
Theorem A.9 (product rule). If the functions $u_1 : \Omega \to Y_1$ and $u_2 : \Omega \to Y_2$ are globally bounded and $\alpha$-Hölder, then also their product $u_1 \cdot u_2 : \Omega \to Y$ is $\alpha$-Hölder. In case $\alpha \in (0, 1]$ one has the estimates 
\[
\|u_1 \cdot u_2\|_\alpha \leq C\left(\|u_1\|_0^{\alpha} \|u_2\|_\alpha + \|u_1\|_0 \|u_2\|_\alpha\right) \quad \text{and} \quad \|u_1 \cdot u_2\|_\alpha \leq C\left(\|u_2\|_0 \|u_1\|_\alpha + \|u_1\|_0 \|u_2\|_\alpha\right).
\]

Proof. We restrict to $\alpha \in (0, 1]$. Using properties of a product, we obtain from the triangle inequality
\[
\|(u_1 \cdot u_2)(x) - (u_1 \cdot u_2)(\bar{x})\| \leq \|(u_1(x) - u_1(\bar{x})) \cdot u_2(x)\| + \|(u_1(x) \cdot (u_2(x) - u_2(\bar{x}))\|
\]
\[
\leq C \|u_1(x) - u_1(\bar{x})\| \|u_2(x)\| + C \|u_1(x)\| \|u_2(x) - u_2(\bar{x})\|
\]
\[
\leq C\left(\|u_2\|_0 \|u_1\|_\alpha + \|u_1\|_0 \|u_2\|_\alpha\right) d(x, \bar{x})^\alpha
\]
and this implies that $u_1 \cdot u_2$ is $\alpha$-Hölder. Then the estimate for $\|u_1 \cdot u_2\|_\alpha$ follows easily.

Theorem A.10 (chain rule). Let $\alpha_1, \alpha_2 \in [0, 1]$. If functions $u_1 : \Omega \to Y_1$ is $\alpha_1$-Hölder and $u_2 : u_1(\Omega) \to Y$ is $\alpha_2$-Hölder, then the composition $u_2 \circ u_1 : \Omega \to Y$ is $\alpha_1 \alpha_2$-Hölder. In case $\alpha_1, \alpha_2 \in (0, 1]$ one has the estimate
\[
\|u_2 \circ u_1\|_{\alpha_1 {\alpha}_2} \leq \|u_1\|_{\alpha_1} \|u_2\|_{\alpha_2}
\]
and for globally bounded $u_2$ results
\[
\|u_2 \circ u_1\|_{\alpha_1 \alpha_2} \leq \max\{1, \|u_1\|_{\alpha_1}\} \|u_2\|_{\alpha_2}.
\]

Proof. We focus on the situation $\alpha_1, \alpha_2 \in (0, 1)$. The following holds
\[
\|u_2 \circ u_1(x) - u_2 \circ u_1(\bar{x})\| \leq \|u_2\|_{\alpha_2} \|u_1(x) - u_1(\bar{x})\|^{\alpha_2} \leq \|u_1\|_{\alpha_1} \|u_2\|_{\alpha_2} d(x, \bar{x})^{{\alpha}_1{\alpha}_2}
\]
and so the composition $u_2 \circ u_1$ is $\alpha$-Hölder. The remaining norm estimate is readily derived.

Let $B(\Omega, Y)$ abbreviate the space of globally bounded functions and we define the space of globally bounded $\alpha$-Hölder functions by
\[
C^\alpha(\Omega, Y) := \{u \in B(\Omega, Y) : u \text{ is } \alpha \text{-Hölder}\}.
\]

Due to Thm. A.4 the characterization $C^\alpha(\Omega, Y) = \{u : \Omega \to Y \mid u \text{ is } \alpha \text{-Hölder}\}$ holds on bounded spaces $Y$. By Lemma A.3 it is $C^\alpha(U, Y) = C^\alpha(\overline{U}, Y)$ for subsets $U \subseteq \Omega$ and Banach spaces $Y$.

Theorem A.11. The set $C^\alpha(\Omega, Y)$ is a normed space over $\mathbb{K}$ w.r.t. the norm $\|\cdot\|_\alpha$. Furthermore, with $Y$ also $C^\alpha(\Omega, Y)$ is a Banach space.

Proof. We merely show the completeness of $C^\alpha(\Omega, Y)$ w.r.t. the norm $\|\cdot\|_\alpha$ for $\alpha \in (0, 1)$. Thereto, let $(u_l)_{l \in \mathbb{N}}$ be a Cauchy sequence in $C^\alpha(\Omega, Y)$. Since $C^0(\Omega, Y)$ is complete in the sup-norm, $(u_l)_{l \in \mathbb{N}}$ converges to a continuous function $u : \Omega \to Y$. It remains to show that $\lim_{l \to \infty}[u_l - u]_\alpha = 0$ and that $u$ is $\alpha$-Hölder. Thereto, for $\varepsilon > 0$ first choose $L \in \mathbb{N}$ such that $\|u_l - u_{m}\|_\alpha \leq \frac{\varepsilon}{2}$ for all $l, m \geq L$, and given $x, \bar{x} \in \Omega$, $x \neq \bar{x}$, choose a fixed $l \geq L$ such that $\|u_l(x) - u(x)\| \leq \frac{\varepsilon}{3d(x, \bar{x})^\alpha}$ and $\|u_l(\bar{x}) - u(\bar{x})\| \leq \frac{\varepsilon}{3d(x, \bar{x})^\alpha}$. Now this results in
\[
\frac{\|(u_l - u)(x) - (u_l - u)(\bar{x})\|}{d(x, \bar{x})^\alpha} \leq \left(\frac{\|u_l - u\|_\alpha}{d(x, \bar{x})^\alpha} + \frac{\|u_l(x) - u(x)\|}{d(x, \bar{x})^\alpha} + \frac{\|u_l(\bar{x}) - u(\bar{x})\|}{d(x, \bar{x})^\alpha}\right)
\]
\[
\leq \|u_l - u\|_\alpha + \frac{\varepsilon}{3} \leq \varepsilon
\]
and therefore $\|u_l - u\|_\alpha \leq \varepsilon$. If we set $\varepsilon = 1$ in the above inequality and note that $\{(u_l)_{l \in \mathbb{N}}\}$ is bounded, then the generalized triangle inequality guarantees
\[
\|u(x) - u(\bar{x})\|_\alpha \leq 1 + \frac{\|u_l(x) - u_l(\bar{x})\|_\alpha}{d(x, \bar{x})^\alpha} \leq 1 + \|u_l\|_\alpha \leq 1 + \sup_{l \in \mathbb{N}}\|u_l\|_\alpha
\]
and consequently $u \in C^\alpha(\Omega, Y)$.

Remark A.2. (1) The positive homogeneity $[\lambda u]_\alpha = [\lambda] [u]_\alpha$ for $\lambda \in \mathbb{K}$ and Thm. A.8 guarantee that $[\cdot]_\alpha$ defines a semi-norm on $C^\alpha(\Omega, Y)$. It is not a norm, since $[\cdot]_\alpha$ vanishes on the constant functions. However, if $\alpha \in (0, 1]$ and $x_0 \in \Omega$ is fixed, then $[u]_{\alpha}': := \max\{\|u(x_0)\|, [u]_\alpha\}$ defines an equivalent norm on $C^\alpha(\Omega, Y)$. Indeed, any globally bounded $\alpha$-Hölder function $u : \Omega \to Y$ satisfies (A.2) for all $x \in \Omega$, which implies the inequality $\|u\|_0 \leq \|u(x_0)\| + [u]_{\alpha}(\diam \Omega)^\alpha$ and consequently
\[
\|u\|_\alpha \leq \|u\|_0 + \max\{\|u(x_0)\|, [u]_{\alpha}(\diam \Omega)^\alpha, [u]_\alpha\}
\]
\[
\leq \max\{\|u(x_0)\|, [u]_\alpha\} + \max\{(\diam \Omega)^\alpha, 1\} [u]_\alpha \leq (1 + \max\{(\diam \Omega)^\alpha, 1\}) [u]_\alpha
\]
guarantees that both norms are equivalent.

(2) Let \( \Omega \) be compact. Then \( C^\alpha(\Omega)_+: = \{ u \in C^\alpha(\Omega, \mathbb{R}) : 0 \leq u(x) \text{ for all } x \in \Omega \} \) is an order cone in \( C^\alpha(\Omega) \) with nonempty interior. However, if \( \Omega \) has at least one accumulation point, then \( C^\alpha(\Omega)_+ \) is not normal (see \[3\]).

The following result establishes that \( \alpha \)-Hölder functions on bounded sets form a decreasing scale of spaces between the Lipschitz continuous and the uniformly continuous functions (cf. Thm. \[A.2\]).

**Lemma A.12.** Let \( \Omega \) be bounded. If \( 0 \leq \alpha \leq \beta \leq 1 \), then \( \beta \)-Hölder functions are \( \alpha \)-Hölder and satisfy

\[
|u|_\alpha \leq (\text{diam } \Omega)^{\beta-\alpha}|u|_\beta
\]

**Proof.** Given \( u \in C^\beta(\Omega, Y) \) one has

\[
\|u(x) - u(\bar{x})\| \leq d(x, \bar{x})^{\beta-\alpha}|u|_\beta d(x, \bar{x})^\alpha \leq (\text{diam } \Omega)^{\beta-\alpha}|u|_\beta d(x, \bar{x})^\alpha \quad \text{for all } x, \bar{x} \in \Omega
\]

yielding the assertion. \( \square \)

**Theorem A.13** (continuous embeddings). Let \( \Omega \) be bounded. If \( 0 \leq \alpha \leq \beta \leq 1 \), then \( C^\beta(\Omega, Y) \subseteq C^\alpha(\Omega, Y) \) is a continuous embedding with \( \|u\|_\alpha \leq \max \left\{ 1, (\text{diam } \Omega)^{\beta-\alpha} \right\} \|u\|_\beta \) for all \( u \in C^\beta(\Omega, Y) \).

In order words, one has a bounded embedding operator

\[
\mathcal{I}_\alpha^\beta : C^\beta(\Omega, Y) \rightarrow C^\alpha(\Omega, Y) \quad \text{for all } 0 \leq \alpha \leq \beta \leq 1.
\]

**Proof.** If \( u \in C^\beta(\Omega, Y) \), then Lemma \[A.12\] implies \( |u|_\alpha < \infty \) and thus

\[
\max \left\{ \|u\|_0, |u|_\alpha \right\} \leq \max \left\{ \|u\|_0, (\text{diam } \Omega)^{\beta-\alpha}|u|_\beta \right\} \leq \max \left\{ 1, (\text{diam } \Omega)^{\beta-\alpha} \right\} \|u\|_\beta
\]

yields that \( u \) is also \( \alpha \)-Hölder and satisfies the claimed estimate. \( \square \)

**Remark A.3** (differentiable and Sobolev functions). Let \( \Omega \subseteq \mathbb{R}^n \) be open and bounded.

(1) If \( \Omega \) is convex and \( C^1(\Omega, Y) \) denotes the (canonically normed) space of continuously differentiable functions allowing a continuous extension to \( \bar{\Omega} \), then one has the continuous embedding \[1\] pp. 11–12, 1.34 Thm.]

\[
C^1(\Omega, Y) \subseteq C^\alpha(\bar{\Omega}, Y) \quad \text{for all } 0 \leq \alpha \leq 1.
\]

(2) Let \( \dim Y < \infty \). If \( \Omega \) has a Lipschitz boundary and \( k \in \mathbb{N}, p \geq 1 \) satisfy \( (k - \alpha)p \geq \kappa \), then the Sobolev space \( W^{k,p}(\Omega, Y) \) satisfies the following continuous embedding \[1\]

\[
W^{k,p}(\Omega, Y) \subseteq C^\alpha(\bar{\Omega}, Y) \quad \text{for all } 0 < \alpha \leq 1.
\]

For Hölder exponents \( 0 < \alpha < \beta \leq 1 \) the inclusion from Thm. \[A.15\] will typically be strict.

**Example A.4.** Let \( \Omega = [0, 1] \) and \( Y = \mathbb{R} \). In case \( \alpha \in (0, 1) \) the function \( u : [0, 1] \rightarrow \mathbb{R}, u(x) := x^\alpha \) is contained in \( C^\alpha[0, 1] \). Now if \( u \) would be \( \beta \)-Hölder with exponent \( \beta > \alpha \), then there exists a \( C \geq 0 \) such that \( x^\alpha = |u(x) - u(0)| \leq C|x - 0|^\beta = Cx^\beta \) for \( x \in (0, 1] \) yielding the contradiction \( C \geq x^{\alpha-\beta} \xrightarrow{x \downarrow 0} \infty \).

Concerning the case \( \alpha = 0 \) we refer to Ex. \[A.3\] for a continuous function not being Hölder.

On the space \( C^\alpha(\Omega, Y) \) exist several measures of noncompactness, which even are not necessarily equivalent (cf. \[37\] Sect. 5). Among them, and for finite-dimensional spaces \( Y \), we use

\[
\chi(B) := \lim_{\varepsilon \downarrow 0} \sup_{u \in B} \left\{ \frac{\|u(x) - u(\bar{x})\|}{d(x, \bar{x})^{\beta}} : 0 < d(x, \bar{x}) \leq \varepsilon \right\}
\]

(see \[10\] \[11\] and \[50\]) and obtain a sufficient compactness criterion:
Theorem A.14 (compactness in $C^\alpha(\Omega, Y)$, cf. \cite{RS} Thm. 4). Let $\Omega$ be compact and $\dim Y < \infty$. A subset $B \subseteq C^\alpha(\Omega, Y)$ is relatively compact, provided the following two conditions hold:

(i) $B$ is bounded,

(ii) for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, \tilde{x} \in \Omega$ one has the implication

$$d(x, \tilde{x}) \leq \delta \Rightarrow \|u(x) - u(\tilde{x})\| \leq \varepsilon d(x, \tilde{x})^\alpha$$

for all $u \in B$.

For $\alpha = 0$ this is essentially the sufficiency part of the Arzelà-Ascoli theorem \cite{V} p. 31, Thm. 3.2], which establishes that (i) and (ii) characterize the relatively compact subsets of $C^0(\Omega, Y)$.

Theorem A.15 (compact embeddings). Let $\Omega$ be compact and $\dim Y < \infty$. If $0 \leq \alpha < \beta \leq 1$, then $C^\beta(\Omega, Y) \subseteq C^\alpha(\Omega, Y)$ is a compact embedding. Moreover, the embedding $C^\beta(\Omega, Y) \subseteq C^\alpha(\Omega, Y)$ is even dense, provided one has $\Omega \subset \mathbb{R}^n$.

This means that bounded subsets of $C^\beta(\Omega, Y)$ are relatively compact in $C^\alpha(\Omega, Y)$. In case $\alpha \in (0, 1)$ the embedding (A.4) is compact. Similarly, for $(k-\alpha)p > \kappa$ also (A.5) is compact (see \cite{RS} pp. 11–12, 1.34 Thm.1).

Proof. (I) Let $B \subseteq C^\beta(\Omega, Y)$ be bounded, that is, there exists a $C \geq 0$ such that $\|u\|_\beta \leq C$ for all $u \in B$. This implies $\|u(x)\| \leq C$ and $\|u(x) - u(\tilde{x})\| \leq Cd(x, \tilde{x})^\beta \leq Cd(x, \tilde{x})^{\beta - \alpha}d(x, \tilde{x})^\alpha$ for all $x, \tilde{x} \in \Omega$ and $u \in B$, which guarantees that $B \subseteq C^\beta(\Omega, Y)$ satisfies the assumptions of Thm. A.14. Consequently, $B$ is a relatively compact subset of $C^\alpha(\Omega, Y)$.

(II) Referring to the Stone-Weierstrasz theorem \cite{WS} p. 218, Thm. 16.1] the polynomials over a compact $\Omega \subset \mathbb{R}^n$ form a set of $\beta$-Hölder functions being dense in the continuous functions.

However, note that the embedding $C^\beta(\Omega, Y) \subseteq C^\alpha(\Omega, Y)$ is not dense for $0 < \alpha < \beta \leq 1$.

Example A.5. Let $\Omega = [0, 1]$, $Y = \mathbb{R}$ and $u \in C^\alpha[0, 1]$ be given by $u(x) := x^\alpha$. Choose $v \in C^\beta[0, 1]$ and consider the difference $u - v \in C^0[0, 1]$ satisfying

$$\frac{|u(x) - u(0)|}{|x - 0|^\alpha} \geq \frac{|u(x) - u(0)|}{|x - 0|^\alpha} - \frac{|v(x) - v(0)|}{|x - 0|^\alpha} = 1 - \frac{|x|^\beta}{|x|^\alpha} \frac{|v(x) - v(0)|}{|x - 0|^\beta} \sim 1.$$  

This implies that any function $v \in C^\beta[0, 1]$ has $\alpha$-norm greater or equal to 1 from $u$.

The final example demonstrates that $C^\alpha(\Omega, Y)$ is not separable.

Example A.6. Let $\Omega = [0, 1]$, $Y = \mathbb{R}$ and for reals $c \in (0, 1)$ define the $\alpha$-Hölder functions

$$u_c : [0, 1] \to \mathbb{R}, \quad u_c(x) := \begin{cases} 0, & 0 \leq x \leq c, \\ (x - c)^\alpha, & a < x \leq 1, \end{cases}$$

where $\alpha \in (0, 1]$. This implies the inequality

$$\|u_a - u_b\|_\alpha \geq |u_a - u_b|_\alpha \sup_{x, \tilde{x} \in [0, 1]} \frac{|u_a - u_b|(x) - (u_a - u_b)(\tilde{x})}{|x - \tilde{x}|^\alpha} \geq \frac{|(u_a - u_b)(b) - (u_a - u_b)(a)|}{|b - a|^\alpha} = \frac{(b - a)^\alpha}{|b - a|^\alpha} = 1$$

for all $a, b \in (0, 1)$, with the uncountable family $\{u_c\}_{c \in (0, 1)} \subseteq C^\alpha[0, 1]$.

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Urysohn and Hammerstein operators
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Abstract:
We consider Urysohn and Hammerstein operators, which arise in various contexts of nonlinear analysis.
This treatise is devoted to Urysohn operators, a class of nonlinear integral operators arising in various contexts
of nonlinear analysis. We focus on the differentiability results for Urysohn operators, which appear to be new,
and on the properties of Nemytskii operators. We provide a comprehensive account of the required
preliminaries for Hölder continuous functions defined on metric spaces. Finally, an appendix provides
a comprehensive account summarizing the required conditions.

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