Higher Order Fluctuations of Extremal Eigenvalues of Sparse Random Matrices

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Abstract. We consider extremal eigenvalues of sparse random matrices, a class of random matrices including the adjacency matrices of Erdős-Rényi graphs $G(N, p)$ recently. It was shown that the leading order fluctuations of extremal eigenvalues are given by a single random variable associated with the total degree of the graph (Ann. Probab., 48(2):916–962, 2020; Probab. Theory Related Fields, 180:985–1056, 2021). We construct a sequence of random correction terms to capture higher (sub-leading) order fluctuations of extremal eigenvalues in the regime $N^\epsilon < pN < N^{3/4}$. Using these random correction terms, we prove a local law up to a shifted edge and recover the rigidity of extremal eigenvalues under some corrections for $pN > N^\epsilon$.

1. Introduction

An $N \times N$ random matrix is said to be sparse if the number of nonzero entries per row is much less than $N$ on average as $N \to \infty$. A motivating example is the adjacency matrix of the sparse Erdős-Rényi graph $G(N, p)$, a random graph on $N$ vertices in which each edge independently exists with probability $p \ll 1$. As the canonical model of random graphs, Erdős-Rényi graphs have been used in many areas such as combinatorics, network theory and mathematical physics [2][4][9]. It is important to study the spectral statistics of Erdős-Rényi graphs since their eigenvalues and eigenvectors contain fundamental information, and accordingly, have many applications: combinatorial optimization, spectral partitioning, community detection, etc [1][4][22][24].

The Erdős-Rényi graph $G(N, p)$ exhibits some threshold phenomena with regard to the probability $p$. In the seminal works [12][13], a connectivity transition was shown around $pN = \log N$:

1. If $pN > (1 + \epsilon) \log N$, the Erdős-Rényi graph $G(N, p)$ is almost surely connected.
2. If $pN < (1 - \epsilon) \log N$, the Erdős-Rényi graph $G(N, p)$ almost surely contains isolated vertices, and thus it is disconnected.

In this paper, we consider the regime $pN > N^\epsilon$, which is included in the super-critical regime, $pN > (1 + \epsilon) \log N$. A pioneering work for this case is a series of papers by Erdős, Knowles, Yau and Yin [5][6]. They established many outstanding results such as a local law, eigenvalue rigidity and universality for sparse random matrices.

A so-called local law is an essential ingredient to understand the local behavior of eigenvalues. A three-step strategy based on local laws was developed to solve the well-known Wigner-Dyson-Mehta universality conjecture for Wigner matrices [8][10][11], an open problem for nearly 50 years. A similar approach is believed to be valid for sparse matrices but sparsity introduces some additional technical challenges to overcome, especially near the spectral edge. Recently, bulk universality was established when $pN > N^\epsilon$ [14][16] but progress on edge universality has been slower since extreme eigenvalues fluctuate in a more pronounced way for sparse matrices. In [5][6], both rigidity and universality of the extremal eigenvalues were proved only for $pN > N^{2/3+\epsilon}$. One reason for the restriction on the regime of $pN$, is that large deviation estimates for sparse matrices are not powerful enough: they contain a fixed power of $(pN)^{-1/2}$. (See [6] Lemma 3.8 for more details.)

Later Lee and Schnelli discovered that local law estimates at the edge can be strengthened by introducing a deterministic correction of the semicircle law [21]. Using their improved local law, they established edge rigidity and universality for $pN > N^{1/3+\epsilon}$ with respect to a deterministically shifted edge $L = 2 + O((pN)^{-1})$ where we note that $+2$ is the right edge of the standard semicircle law. (See [21] Theorem 2.9, Theorem 2.10 for precise statements.) They also expected that the order of the fluctuations of the extremal eigenvalues would start to exceed $N^{-2/3}$, the typical order of Tracy-Widom fluctuations, when $pN < N^{1/3-\epsilon}$.

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In recent work by Huang, Landon and Yau [17], the authors indeed confirmed a transition from Tracy-Widom to Gaussian fluctuations for extreme eigenvalues around $pN \sim N^{1/3}$. The main observation is that the random correction term $\mathcal{X}$ defined by

$$\mathcal{X} := \frac{1}{N} \sum_{i,j} \left( h_{ij}^2 - \frac{1}{N} \right),$$

where the entries of an $N \times N$ sparse random matrix are denoted by $(h_{ij})$, becomes the leading order term of the fluctuations as $pN < N^{1/3 - \epsilon}$. In addition, they recovered edge rigidity and Tracy-Widom fluctuations with respect to the randomly shifted edge $L + \mathcal{X}$ under the condition $N^{2/3 + \epsilon} < pN < N^{1/3 - \epsilon}$. One of their novel ideas involved subtracting the main source of Gaussian fluctuations $\mathcal{X}$. However the results in [17] are valid only when $pN > N^{2/9 + \epsilon}$, due to technical constraints.

Finally, He and Knowles resolved some subtle technical issues and showed that the extremal eigenvalues have Gaussian fluctuations in the regime $N^\epsilon < pN < N^{1/3 - \epsilon}$ [15]. Thus, the random quantity $\mathcal{X}$ is the leading order term of fluctuations when $pN < N^{1/3 - \epsilon}$. One can also ask about higher (sub-leading) order fluctuations of extremal eigenvalues. One might anticipate higher order fluctuations to exist and that new random correction terms should capture such fluctuations. Indeed in [17], the authors suspected higher order fluctuations would depend on subgraph counts. In [15], the authors claimed, for new random corrections terms, that there is an infinite hierarchy of random variables which is strongly correlated and asymptotically Gaussian, and the random variable $\mathcal{X}$ would be only the leading order fluctuations of extremal eigenvalues.

In this paper, we investigate such higher order fluctuations of extremal eigenvalues of sparse random matrices in the regime $pN > N^\epsilon$. We construct a series of random correction terms $(Z_n)$ to control higher order fluctuations of extremal eigenvalues. (See (4.55) for more detail.) Using these random correction terms $(Z_n)$, we derive a new (randomly) shifted edge $\mathcal{L} := \mathcal{L}(Z_1, Z_2, \ldots, Z_d)$ which is a polynomial in the variables $(Z_n)_{1 \leq n \leq d}$, and also obtain a local law near this shifted edge $\mathcal{L}$ (Theorem 2.7). As a result, we prove that the extremal eigenvalues are concentrated at the shifted edge $\mathcal{L}$ with scale $N^{-2/3}$, which is precisely the eigenvalue rigidity estimate near the edge (Theorem 2.11).

One of the main ingredients is the so-called recursive moment estimate (RME), Proposition 3.1. The RME is a moment estimate of a self-consistent polynomial. In [15, 17, 21], an approach using the RME was developed and a local law was proved as a consequence of the RME and stability analysis. The RME in [17, 21] has a bound containing powers of $(pN)^{-1/2}$ so the resulting local law bound has powers of $(pN)^{-1/2}$ too. Inspired by [15], we tried to find a way to avoid such powers of $(pN)^{-1/2}$ in the RME bound. As it turns out, by including sufficiently many random correction terms, we are able to obtain the RME without any powers of $(pN)^{-1/2}$, which consequently resulted in a nearly optimal local law bound. From standard arguments using the local law [17], we can then establish the desired rigidity estimate.

2. DEFINITIONS AND MAIN RESULTS

2.1. Basic notions and conventions. To formulate the main results, we introduce some definitions and the notation as preliminaries. Let $N$ be the fundamental positive-integer parameter. Dealing with $N$-dependent quantities, we almost always omit the $N$-dependence to ease notation, unless otherwise stated. We use $c > 0$ for a small universal constant while we denote by $C > 0$ a large universal constant throughout this paper. Their values may change between occurrences.

Definition 2.1 (High probability event). Let $E \equiv E_N$ be an event parametrized by $N$. We say that $E$ holds with very high probability if for any (large) $D > 0$ there exists a constant $C$ such that $\mathbb{P}(\Omega^c) \leq CN^{-D}$ for all $N$.

Definition 2.2 (Stochastic domination). Let $X \equiv X_N$ and $Y \equiv Y_N$ be random (or deterministic) variables depending on $N$. We say that $X$ is stochastically dominated by $Y$ if for any (small) $\epsilon > 0$ and (large) $D > 0$ there exists a constant $C$ such that $\mathbb{P}(|X| > N^\epsilon Y) \leq CN^{-D}$ for all $N$. If $Y$ stochastically dominates $X$, we write $X \prec Y$ or $X = \mathcal{O}_{\prec}(Y)$.

We have some notational conventions for the asymptotics of the limit $N \to \infty$. The symbols $\mathcal{O}(\cdot)$ and $o(\cdot)$ are used for the standard big-O and little-o notation. Let $x \equiv x_N$ and $y \equiv y_N$ be nonnegative $N$-dependent quantities. We write $x \preceq y$ if there exist a constant $C > 0$ such that $x \leq Cy$ for all $N$. We use the notation $x \asymp y$ if $x \preceq y$ and
In this paper, we use the symbol $\ll$ in a less standard way. We write $x \ll y$ if there exist a constant $c > 0$ satisfying $N^c x \ll y$.

For a complex number $w \in \mathbb{C}$, we denote its real part by $\Re w$ and its imaginary part by $\Im w$ throughout this paper.

### 2.2. Main results.

In this paper, we consider the class of sparse random matrices introduced in [5, 6, 17].

**Definition 2.3 (Sparse random matrices).** Let $H = (h_{ij})_{1 \leq i,j \leq N}$ be an $N \times N$ real symmetric random matrix. The entries $(h_{ij})$ are independent up to the symmetry constraint $h_{ij} = h_{ji}$ and have the same moments. We assume that the entries $(h_{ij})$ have zero mean and $(1/N)$-variance, i.e.

$$\mathbb{E}[h_{ij}] = 0, \quad \mathbb{E}[h_{ij}^2] = \frac{1}{N}.$$  

For $p \geq 2$, the $p$-th cumulant $\kappa_p$ of $h_{ij}$ is given by

$$\kappa_p = \frac{(p-1)!C_p}{Nq^{p-2}},$$

where $q = N^b$ is the sparsity parameter with fixed (auxiliary parameter) $b$ satisfying $0 < b < \frac{1}{2}$. We further assume

$$|C_p| \leq C_p$$

for some constant $C_p > 0$, and

$$C_4 \gtrsim 1.$$

Throughout this paper, we denote by $H$ the sparse random matrix as in Definition 2.3. We define the Green function of $H$ by

$$G(z) := (H - z)^{-1}, \quad z \in \mathbb{C}_+,$$

and use the notation

$$m(z) := \frac{1}{N} \text{Tr} G(z), \quad z \in \mathbb{C}_+,$$

for the normalized trace of the Green function. Let $\mu$ be the empirical eigenvalue distribution of $H$ given by

$$\mu := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i},$$

where we denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ the ordered eigenvalues of $H$. Note that the normalized trace $m(z)$ is the Stieltjes transform of the empirical eigenvalue distribution $\mu$. We introduce the random polynomial $P(z, w)$ defined through

$$P(z, w) := 1 + zw + Q(w), \quad z, w \in \mathbb{C}_+,$$

where the random polynomial $Q(w)$ is given by

$$Q(w) = \sum_{n=1}^{\ell} Z_n w^{2n},$$

for a large integer $\ell \geq 1$ to be chosen later, and each random coefficient $Z_n$ is a polynomial in the variables $(h_{ij})_{1 \leq i,j \leq N}$. There are two important assumptions for the random coefficients $(Z_n)_{1 \leq n \leq \ell}$.

**Assumption 2.4.** For the sparsity parameter $q$ as in Definition 2.3, we have

$$Z_1 - 1 \ll \frac{1}{\sqrt{Nq}},$$

and

$$Z_n \ll \frac{1}{q^{n-1}}, \quad 1 \leq n \leq \ell.$$

**Assumption 2.5.** For all $i, j \in \{1, 2, \cdots, N\}$, the following holds:

$$\frac{\partial Z_n}{\partial h_{ij}} \ll \frac{1}{N}, \quad 1 \leq n \leq \ell.$$
If the above two assumptions hold, then we call the random polynomial $P(z, w)$ and the associated equation $P(z, \tilde{m}(z)) = 0$ self-consistent polynomial and self-consistent equation respectively. Some important properties of a solution $\tilde{m}$ of the self-consistent equation are described in the following lemma.

**Lemma 2.6.** Suppose Assumption 2.4 holds. Then, there exists an algebraic function $\tilde{m} : \mathbb{C}_+ \to \mathbb{C}_+$, such that the following properties hold:

1. The function $\tilde{m}$ is a solution of the self-consistent equation $P(z, \tilde{m}(z)) = 0$.
2. The function $\tilde{m}$ is the Stieltjes transform of a random probability measure $\tilde{\rho}$ and the support of $\tilde{\rho}$ is $[-\widetilde{L}, \widetilde{L}]$, where $\widetilde{L}$ and $-\widetilde{L}$ are called (spectral) edges.
3. The probability measure $\tilde{\rho}$ has strictly positive density on $(\widetilde{L}, L)$ and square root behavior at the edges.
4. For any (large) $C > 0$, there exists a polynomial $\widetilde{L} \equiv \widetilde{L}(Z_1, Z_2, \cdots, Z_\ell)$ such that
   \[
   \widetilde{L} = \tilde{L} + O_e(N^{-C}).
   \]

   Thus, an approximate location of the edge is denoted by $\tilde{L}$.
5. Consider $z \in \mathbb{C}_+$ and set $\kappa := \text{dist}(\Re(z), \{-\tilde{L}, \tilde{L}\})$. In a neighborhood of the support $[-\tilde{L}, \tilde{L}]$, we have
   \[
   \Re m(z) \asymp \begin{cases} \sqrt{\kappa + \eta}, & \Re z \in [-\tilde{L}, \tilde{L}], \\ \frac{\eta}{\sqrt{\kappa + \eta}}, & \Re z \notin [-\tilde{L}, \tilde{L}], \end{cases}
   \]
   \[
   |\partial_2 P(z, \tilde{m}(z))| \asymp \sqrt{\kappa + \eta},
   \]
   \[
   \partial_2^2 P(z, \tilde{m}(z)) = 2 + O(q^{-1}),
   \]
   where we write $\partial_2 P(x, y) \equiv \partial_y P(x, y)/\partial y$.

The proof of Lemma 2.6 is essentially same with that of [17, Proposition 2.5, Proposition 2.6] and [21, Lemma 4.1] but we write it up in Appendix B for completeness. Our first main result is the local law up to the edge.

**Theorem 2.7 (Local law near the edge).** Let $H$ be as in Definition 2.2 with any small $b > 0$. For \( \ell \geq 1 \) large enough, we can construct the random coefficients $(Z_n)_{1 \leq n \leq \ell}$ such that the following statements hold:

1. Assumption 2.4 and Assumption 2.5 are satisfied.
2. Let $\tilde{m}$ and $\tilde{\ell}$ be as in Lemma 2.6. We define $\tilde{z}$ by setting $\tilde{z} := \tilde{\ell} + E + i\eta$, and assume $|E| \leq 1$ and $N^{-1} \ll \eta \ll 1$. Then, we have
   \[
   |m(\tilde{z}) - \tilde{m}(\tilde{z})| \ll \left( \frac{\phi}{N\eta} \right)^{1/2} + (\sqrt{\kappa + \eta})^{1/4} \left( \frac{\phi}{N\eta} \right)^{3/8} + \frac{1}{N^{1/4}} \left( \frac{\phi}{N\eta} \right)^{1/8} + (\sqrt{\kappa + \eta})^{1/4} \left( \frac{\phi}{N^2\eta^2} \right)^{1/4}
   \]
   \[
   + \frac{1}{N^{1/2}\eta^{1/4}} + \frac{1}{N\eta} + \frac{(\sqrt{\kappa + \eta})^{2/5} (N\eta)^{3/5}}{(N\eta)^{3/5}} + \frac{1}{N^{2/7}(N\eta)^{1/7}} + \frac{(\sqrt{\kappa + \eta})^{1/3}}{(N\eta)^{2/3}},
   \]
   where $\kappa := \text{dist}(\Re(\tilde{z}), \{-\tilde{L}, \tilde{L}\})$ and the parameter $\phi$ is given by
   \[
   \phi \equiv \phi(\tilde{z}) := \begin{cases} \sqrt{\kappa + \eta}, & \Re \tilde{z} \in [-\tilde{L}, \tilde{L}], \\ \eta/\sqrt{\kappa + \eta}, & \Re \tilde{z} \notin [-\tilde{L}, \tilde{L}], \end{cases}
   \]

   We call these constructed random coefficients $(Z_n)_{n=1}^{\ell}$ random correction terms.

**Remark 2.8.** In contrast to the previous results (e.g. [6, Theorem 2.8], [21, Theorem 2.4], and [17, Theorem 2.1]), the local law estimate (2.5) does not contain any power of $q^{-1}$ since we implicitly assume that $\ell$ is large enough. The condition that $\ell > 2/(3b)$ is enough to guarantee an optimal local law in the vicinity of the edge. Compared with [15, Equations (6.2), (6.4), (9.1)-(9.2)], our results cover a larger spectral domain.

**Remark 2.9.** The estimate (2.5) is perhaps daunting at first glance. The main thing to keep in mind is that the order of the bound is strong enough to be used for spectral analysis near the edge. For example, if $\kappa = \eta = N^{-2/3}$, then by (2.5) it is straightforward to see that
   \[
   |m(\tilde{z}) - \tilde{m}(\tilde{z})| \ll N^{-1/3}.
   \]
   Moreover, we can improve the local law estimate (2.5) outside the spectrum. See (5.5) for more details.
Remark 2.10. We can write a random correction term $Z_n$ explicitly. For example,
\[ Z_1 = \frac{1}{N} \sum_{i \neq j} h_{ij}^2 = 1 + \mathcal{X} + O\left(\frac{1}{N}\right), \]
(See (1.1) for the definition of $\mathcal{X}$.) and
\[ Z_2 = \frac{1}{N} \sum_{i \neq j} h_{ij}^4 - \frac{2}{N^2} \sum_{i \neq j \neq x \neq y} h_{ij}^4 (h_{xy}^2 - \mathbb{E}[h_{xy}^2]) + \frac{1}{N} \sum_{i \neq j} h_{ij}^2 (h_{ij}^2 - \mathbb{E}[h_{ij}^2]) + \frac{1}{N} \sum_{i \neq j} h_{ij}^2 (h_{xj}^2 - \mathbb{E}[h_{xj}^2]), \]
where we denote by $\sum_{i \neq j \neq x \neq y}$ the sum over all distinct indexes, similarly for $\sum_{i \neq j \neq y}$ and $\sum_{i \neq j \neq x}$. Note that $Z_1$ contains the random correction term $\mathcal{X}$ introduced in [17] and the first term of $Z_2$, 
\[ \frac{1}{N} \sum_{i \neq j} h_{ij}^4, \]
has a mean of the order $q^{-2}$ corresponding to the deterministic correction term introduced in [21]. For $n \geq 3$, we can still calculate $Z_n$ but the explicit form is more complicated.

As a result of Theorem 2.7, we can show that the extremal eigenvalues have fluctuations of order $N^{-2/3}$ with respect to the approximate location of the edge, $\tilde{L}$. Recall that the ordered eigenvalues of $H$ is denoted by
\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N. \]

Theorem 2.11 (Recovery of edge rigidity). Let $H$ be as in Definition 2.3 with any small $b > 0$. For $\ell \geq 1$ large enough, let $\{Z_n\}_{n=1}^\ell$ be as in Theorem 2.7 and let $\tilde{L}$ be as in Lemma 2.6. Fix an integer $k \geq 1$. We have for all $1 \leq i \leq k$,
\[ |\lambda_i - \tilde{L}| \ll N^{-2/3}. \]

Remark 2.12. This result extends [17], Theorem 1.4 and reveals what exactly governs the higher order fluctuation of extremal eigenvalues of sparse random matrices when $1 \ll q \ll N^{1/6}$.

Remark 2.13. As an application of Theorem 2.11, we consider the noise sensitivity problem of the top eigenvector, which is a unit eigenvector associated with the largest eigenvalue. Let $v$ be the top eigenvector of the random matrix $H$. We resample $k$ randomly chosen entries of the matrix $H$ and obtain another realization of the random matrix with top eigenvector $v^{[k]}$. According to [3], if $q \gg N^{1/9}$ and $k \gg N^{5/3}$, the top eigenvectors, $v$ and $v^{[k]}$, are “almost orthogonal”, i.e.
\[ \mathbb{E} \left| \langle v, v^{[k]} \rangle \right| = o(1). \]
Applying Theorem 2.11, it might be feasible to extend the noise sensitivity result (2.7) to a larger regime, such as $q \gg 1$, by modifying the argument in [3].

Remark 2.14. Very recently, Huang and Yau improved Theorem 2.11 by including bulk eigenvalues [18]. Let $\tilde{\rho}$ and $\tilde{L}$ as in Lemma 2.6. According to [18], Theorem 1.6], for the classical eigenvalue locations $\gamma_1 > \gamma_2 > \cdots > \gamma_N$ defined by
\[ \frac{i - 1/2}{N} = \int_{\gamma_i}^{\tilde{L}} \tilde{\rho}(x) dx, \quad 1 \leq i \leq N, \]
we have
\[ |\lambda_i - \gamma_i| \ll N^{-2/3} \min(i, N - i + 1)^{-1/3}, \quad 1 \leq i \leq N. \]
In order to get [18], Theorem 1.6, they extended the main argument of this paper, which is the construction of higher order random correction terms $(Z_n)_{1 \leq n \leq \ell}$.
Remark 2.15. By [15, Theorem 1.2], in the regime $1 \ll q \ll N^{1/6}$, all nontrivial eigenvalues away from 0 have Gaussian fluctuations, and the dominating term of the fluctuations is $\mathcal{X} = Z_1 - E[Z_1] + O(\sqrt{N}^{-1})$. It might be possible to give another proof of [15, Theorem 1.2] by showing that
\[ |Z_1 - E[Z_1]| \gg |Z_n - E[Z_n]|, \quad n \geq 2, \]
and using some results in [18].

Since the typical order of Tracy-Widom fluctuation is $N^{-2/3}$, in the previous version of this paper, we conjectured that edge universality can be also recovered for $q \gg 1$. It was indeed confirmed by Huang and Yau very recently.

Theorem 2.16 (Recovery of edge universality, [18 Theorem 1.7]). Let $H$ be as in Definition 2.3 with any small $b > 0$. Let the random correction terms $Z_1, Z_2, \ldots, Z_{\ell-1}, Z_\ell$ be as in Theorem 2.7 and let $\tilde{\mathcal{L}}$ be as in Lemma 2.6. Fix an integer $k \geq 1$. Let $F : \mathbb{R}^k \to \mathbb{R}$ be a bounded test function with bounded derivatives. Then, there exist a constant $c > 0$ such that
\[ \mathbb{E} \left[ F(\lambda_1 - \tilde{\mathcal{L}}), \ldots, N^{2/3}(\lambda_k - \tilde{\mathcal{L}}) \right] = \mathbb{E}_{\text{GOE}} \left[ F(\lambda_1 - 2), \ldots, N^{2/3}(\lambda_k - 2) \right] + O(N^{-c}), \]
where the second expectation is with respect to a GOE matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$.

3. OUTLINE OF THE PROOF

The following proposition is the essential tool to prove the local law.

Proposition 3.1 (Recursive moment estimate). Let $H$ be as in Definition 2.3 with any small $b > 0$. For an integer $\ell \geq 1$ large enough, we can construct the random coefficients $(Z_n)_{1 \leq n \leq \ell}$ such that the following statements hold:

1. Assumption 2.3 and Assumption 2.5 are satisfied.
2. Let $\tilde{\mathcal{L}}$ be as in Lemma 2.6. Consider $\tilde{z} = \tilde{\mathcal{L}} + E + i\eta$ such that $|E| \leq 1$ and $N^{-1} \ll \eta \leq 1$. We define the control parameter $\Phi_r$ by
\[ \Phi_r := \mathbb{E} \left[ \left( \frac{3m(\tilde{z})}{N\eta} + \frac{1}{N} \right)^{2r-1} \right] \]
\[ + \max_{1 \leq s \leq 2r-1} \mathbb{E} \left[ \left( \frac{3m(\tilde{z})}{N\eta} + \sqrt{\frac{3m(\tilde{z})}{N\eta}} \right)^{s} \left( \frac{3m(\tilde{z})}{N\eta} + \frac{1}{N} \right)^{2r-s-1} \right], \]
where $\tilde{z} = \tilde{\mathcal{L}} + E + i\eta$ and $\eta \geq 1$. Then, we have
\[ \mathbb{E} \left[ |P(\tilde{z}, m(\tilde{z}))|^{2r} \right] \approx \Phi_r. \]

Remark 3.2. In [18 Proposition 2.6], Huang and Yau improved Proposition 3.1 by removing the term
\[ \max_{1 \leq s \leq 2r-1} \mathbb{E} \left[ \left( \frac{3m(\tilde{z})}{N\eta} + \frac{1}{N} \right)^{s} \left( \frac{3m(\tilde{z})}{N\eta} + \frac{1}{N} \right)^{2r-s-1} \right], \]
in the control parameter $\Phi_r$ so that they can get the rigidity estimates for the bulk eigenvalues in [18 Theorem 1.6].

Let $\phi$ be as in (2.6) and we set $\Lambda := |m(\tilde{z}) - \tilde{m}(\tilde{z})|$. If (3.2) is given, using Young’s inequality and Lemma 2.6 we can show
\[ \mathbb{E} \left[ |P(\tilde{z}, m(\tilde{z}))|^{2r} \right] \approx \mathbb{E} \left[ \left( \frac{\phi}{N\eta} \right)^{2r} + \frac{\Lambda}{(N\eta)^{2r}} + \frac{1}{N} + (\sqrt{\kappa + \eta})^r \left( \frac{\phi}{N\eta} \right)^{3r/2} + (\sqrt{\kappa + \eta})^r \left( \frac{\phi}{N\eta} \right)^{3r/2} \right] \]
\[ + \frac{\Lambda^{r/2}}{N(\eta)^{r/2}} + (\sqrt{\kappa + \eta})^r \left( \frac{\phi}{N\eta} \right)^{r} + (\sqrt{\kappa + \eta})^r \left( \frac{\phi}{N\eta} \right)^{r} + \frac{1}{N^2 \eta^2} \]
Taking Taylor expansion of the self-consistent polynomial $P(\tilde{z}, m(\tilde{z}))$ at $m = \tilde{m}$, we can observe that
\[ \Lambda \approx |P(\tilde{z}, m(\tilde{z}))|. \]
Combining the above two estimates, we can show the local law, Theorem 2.7. The rigidity of extremal eigenvalues, Theorem 2.11 follows from the standard argument using the improved local law outside the spectrum and Helffer-Sjöstrand calculus. We remark that all omitted details can be found in Section 5.
The most technical part is to show the estimate \(3.2\). In order to do that, we shall use the cumulant expansion also known as generalized Stein lemma. The statement is as follows.

**Lemma 3.3** (Cumulant expansion, [15, Lemma 2.1] and [21, Lemma 3.2]). Let \( h \) be a centered random variable with finite moments of all order. We denote by \( \kappa_p(h) \) the \( p \)-th cumulant of \( h \). Let \( f : \mathbb{R} \to \mathbb{C} \) be a smooth function. Then, for every positive integer \( \ell \), we have

\[
\mathbb{E} [h f(h)] = \sum_{p=1}^{\ell} \frac{\kappa_{p+1}(h)}{p!} \mathbb{E} \left[ \frac{d^p f(h)}{dh^p} \right] + R_{\ell+1},
\]

assuming that all expectations in the above equation exist, where \( R_{\ell+1} \) is a remainder term such that for any \( t > 0 \),

\[
R_{\ell+1} = O(1) \cdot \left( \sup_{|x| \leq |h|} \left| \frac{d^{\ell+1} f(h)}{dh^{\ell+1}} \right|^2 \mathbb{E} \left[ |h|^{2\ell+4} \mathbb{1}(|h| > t) \right] \right) + O(1) \cdot \mathbb{E} \left[ |h|^{\ell+2} \right] \sup_{|x| \leq t} \left| \frac{d^{\ell+1} f(h)}{dh^{\ell+1}}(x) \right|.
\]

Since we have the resolvent identity

\[
\sum_k h_{ik} G_{kj} - z G_{ij} = \delta_{ij},
\]

it follows that

\[
1 + zm = \frac{1}{N} \sum_{i,j} h_{ij} G_{ij},
\]

and

\[
\mathbb{E} \left[ |P(z, m(z))|^{2r} \right] = \frac{1}{N} \sum_{i,j} \mathbb{E} [h_{ij} G_{ij} P_r^{-1} \tilde{P}^r] + \mathbb{E} [Q(m) P_r^{-1} \tilde{P}^r].
\]

By the cumulant expansion, Lemma 3.3, we have

\[
\frac{1}{N} \sum_{i,j} \mathbb{E} [h_{ij} G_{ij} P_r^{-1} \tilde{P}^r] = \frac{1}{N} \sum_{i,j} \sum_{p=1}^{\ell} \frac{C_{p+1}}{N^{q^p-1}} \mathbb{E} \left[ \partial_p^p (G_{ij} P_r^{-1} \tilde{P}^r) \right] + O(q^{-\ell}),
\]

where we use \( \partial_{ij} \) to denote the partial derivative with respect to \( h_{ij} \), i.e. \( \partial_{ij} := \partial / \partial h_{ij} \). Thus the problem boils down to showing

\[
\sum_{p=1}^{\ell} \frac{C_{p+1}}{N^{q^p-1}} \sum_{i,j} \mathbb{E} \left[ \partial_p^p (G_{ij} P_r^{-1} \tilde{P}^r) \right] + \mathbb{E} [Q(m) P_r^{-1} \tilde{P}^r] \sim \Phi_r.
\]

Therefore we should find the way to construct \( Q(m) \) so that there is a nice cancellation with leading order terms of

\[
\sum_{p=1}^{\ell} \frac{C_{p+1}}{N^{q^p-1}} \sum_{i,j} \mathbb{E} \left[ \partial_p^p (G_{ij} P_r^{-1} \tilde{P}^r) \right].
\]

In Section 4, we shall describe how to construct the random correction terms \( \{ Z_n \}_{n=1}^{\ell} \) in order to obtain the desired estimates of (3.5).

**4. Recursive moment estimate**

This section is devoted to the proof of Proposition 3.1. For brevity, instead of (3.5), we consider

\[
\sum_{p=1}^{\ell} \frac{C_{p+1}}{N^{q^p-1}} \sum_{i,j} \mathbb{E} \left[ \partial_p^p (G_{ij} P_r^{-1}) \right].
\]

The idea is roughly as follows:

Step 1. Assume there exists a set of random coefficients \( \{ Z_n \}_{n=1}^{\ell} \) satisfying Assumption 2.4 and Assumption 2.5.

Step 2. Observe that the main contribution of (4.1) comes from the terms like

\[
\mathbb{E} \left[ G_{ij}^{\ell+1} \left( \partial_p^p (G_{ij} P_r^{-1}) \right) \right],
\]

where \( s \) is a positive odd integer. (Proposition 4.5)
Step 3. Replace all diagonal entries, \( G_{ii} \) and \( G_{jj} \), in (4.2) with the normalized trace \( m \) to get

\[
\mathbb{E} \left[ m^{s+1} (\partial_{ij}^{p-s} P^{2r-1}) \right],
\]

and keep track of the remaining terms after the replacement. (Proposition 4.10)

Step 4. Repeat Step 2 and Step 3 for the remaining terms until the next remaining terms are negligible. Then, ignoring some negligible errors, we can write (4.4) as a linear combination of the sums of the terms like (4.3). (Proposition 4.13)

Step 5. Show that a sum of the terms like (4.3) can be written as a sum of terms in the following form

\[
\mathbb{E} \left[ F(h_{ij}) m^{s+1} P^{2r-1} \right],
\]

where \( F(h_{ij}) \) is a polynomial in the variable \( h_{ij} \). (Proposition 4.16)

Using this, we can construct \( \{ Z_n \}_{n=1}^\ell \) so that the recursive moment estimate holds. Check Assumption 2.4 and Assumption 2.5 for the constructed set \( \{ Z_n \}_{n=1}^\ell \).

4.1. **Step 1. Assuming the existence of the desired random coefficients.** Suppose there exists a set of random coefficients \( \{ Z_n \}_{n=1}^\ell \) satisfying Assumption 2.4 and Assumption 2.5. Then, we can define \( \tilde{\mathcal{L}} = \tilde{\mathcal{L}}(Z_1, \ldots, Z_\ell) \) as in Lemma 2.6. In this subsection, we shall derive some useful estimates by supposing Assumption 2.4 and Assumption 2.5. Recall that \( \tilde{z} = \tilde{\mathcal{L}} + E + i \eta \) satisfying \( |E| \leq 1 \) and \( N^{-1} \ll \eta \leq 1 \). Let \( \partial_{ij} \) be the partial derivative with respect to \( h_{ij} \). Since \( \tilde{\mathcal{L}} \) is a polynomial in the variables \( \{ Z_n \}_{n=1}^\ell \), it follows from Assumption 2.5 and the product rule that

\[
\partial_{ij} \tilde{z} \prec \frac{1}{N}.
\]

**Remark 4.1.** From Section 4.1 to Section 4.5, we always assume that there exist the random coefficients \( \{ Z_n \}_{1 \leq n \leq \ell} \) satisfying Assumptions 2.4 and 2.5.

**Proposition 4.2.** Suppose Assumption 2.4 and Assumption 2.5 hold. For every \( p \geq 1 \), we define \( D_{ij}^p := (\partial_{ij}^p) (\tilde{z}) \). The following estimates hold:

\[
\partial_{ij}^p G_{ij}(\tilde{z}) = D_{ij}^p G_{ij} + O_{\prec} \left( \frac{3m(\tilde{z})}{N \eta} \right),
\]

(4.4)

\[
\partial_{ij}^p m(\tilde{z}) = O_{\prec} \left( \frac{3m(\tilde{z})}{N \eta} \right),
\]

(4.5)

\[
\partial_{ij}^p P(\tilde{z}, m(\tilde{z})) = \begin{cases} O_{\prec} \left( \frac{|\partial_{ij}^p m(\tilde{z})|}{N \eta} + \frac{1}{N} \right) & p = 1, \\ O_{\prec} \left( \frac{|\partial_{ij}^p m(\tilde{z})|^2}{N \eta} + \frac{3m(\tilde{z})}{N \eta} + \frac{1}{N} \right) & p \geq 2. \end{cases}
\]

(4.6)

**Proof.** We shall prove (4.4) first. Applying the Ward identity

\[
\sum_i \left| G_{il}(\tilde{z}) \right|^2 = \frac{3G_{il}(\tilde{z})}{\eta},
\]

and [17] Proposition A.1], we get

\[
\partial_{ij} G_{ij}(\tilde{z}) = D_{ij} G_{ij}(\tilde{z}) + (\partial_{z} G_{ij})(\tilde{z})\partial_{ij}(\tilde{z}) = D_{ij} G_{ij}(\tilde{z}) + \partial_{ij}(\tilde{z}) \sum_l G_{il}(\tilde{z}) G_{lj}(\tilde{z})
\]

\[
= D_{ij} G_{ij}(\tilde{z}) + O_{\prec} \left( \frac{3m(\tilde{z})}{N \eta} \right).
\]

(4.7)

Next, we will compute \( \partial_{ij}^2 G_{ij}(\tilde{z}) \). Since \( \partial_{ij} \) (or \( D_{ij} \)) and \( \partial_{z} \) commute, the Ward identity (4.7) implies

\[
\partial_{ij} D_{ij} G_{ij}(\tilde{z}) = D_{ij}^2 G_{ij}(\tilde{z}) + (\partial_{z} D_{ij} G_{ij})(\tilde{z})\partial_{ij}(\tilde{z}) = D_{ij}^2 G_{ij}(\tilde{z}) + (D_{ij} \partial_{z} G_{ij})(\tilde{z})\partial_{ij}(\tilde{z})
\]

\[
= D_{ij}^2 G_{ij}(\tilde{z}) + O_{\prec} \left( \frac{3m(\tilde{z})}{N \eta} \right).
\]
We also observe
\[ \partial_{ij} ((\partial_z G_{ij})(\tilde{z})\partial_{ij}(\tilde{z})) = \partial_{ij}((\partial_z G_{ij})(\tilde{z}))\partial_{ij}(\tilde{z}) + (\partial_z G_{ij})(\tilde{z})\partial_{ij}^2(\tilde{z}) = O\left(\frac{3m(\tilde{z})}{N^{1/2}}\right). \]

Repeating the above argument, we conclude (4.3).

Now we take derivative for \(m(\tilde{z})\) to show (4.5). Since \(m = \frac{1}{N} \sum_i G_{ii}\) and \(\partial_{ij} G_{ii}(\tilde{z}) = D_{ij} G_{ii}(\tilde{z}) + O\left(\frac{3m(\tilde{z})}{N^{1/2}}\right)\) by the above argument, it follows from Ward identity (4.7) that
\[ \partial_{ij} m(\tilde{z}) = O\left(\frac{3m(\tilde{z})}{N}\right). \]

We can prove the estimate (4.5) for general \(p \geq 1\) by the same reasoning.

In order to get (4.6), consider the polynomial
\[ P(\tilde{z}, m(\tilde{z})) = 1 + \tilde{z} m(\tilde{z}) + \sum_{n=1}^{\ell} Z_n m^{2n}(\tilde{z}), \]
and take derivative; then we have
\[ \partial_{ij} P(\tilde{z}, m(\tilde{z})) = (\partial_{ij} P) \partial_{ij} m(\tilde{z}) + \left(\partial_{ij}^2 \tilde{z} m(\tilde{z}) + \sum_{n=1}^{\ell} (\partial_{ij} Z_n) m^{2n}(\tilde{z})\right) = O\left(\frac{\partial_{ij} P}{N^{1/2}} + \frac{1}{N}\right), \]
where we denote by \(\partial_{ij} P(x, y)\) the partial derivative \(\partial_y P(x, y)/\partial y\). Using \(\partial_{ij} Z_n < \frac{1}{N}\) and the estimate (4.5), the remaining part of (4.6) immediately follows.

4.2. Step 2. Extracting the main contribution. In this subsection, we shall calculate the main contribution of (4.1).

**Proposition 4.3.** Suppose Assumption (2.1) and Assumption (2.5) hold. Let \(\Phi_r\) be as in (3.1). If \(\ell\) is large enough, we have
\[
\sum_{p=1}^{\ell} \frac{c_{p+1}^{s+p-1}}{N^{2p-1}} \sum_{i,j} E \left[ \partial_{ij}^p (G_{ij} P^{2r-1}) \right] = -\sum_{p=1}^{\ell} \sum_{0 \leq s \leq p} \left(\frac{p}{s}\right) s! c_{p+1}^{s+p-1} \sum_{i \neq j} E \left[ G_{ii}^{s+1} G_{jj}^{s+1} (\partial_{ij}^{p-s} P^{2r-1}) \right] + O\left(\Phi_r\right).
\]

**Proof.** Due to Proposition (4.2) and [17] Proposition A.1, we have
\[
\frac{1}{N^2} \sum_i E \left[ \partial_{ii}^p (G_{ii} P^{2r-1}) \right] = O\left(\Phi_r\right),
\]
which implies
\[
\frac{1}{N^2} \sum_{i,j} E \left[ \partial_{ij}^p (G_{ij} P^{2r-1}) \right] = \frac{1}{N^2} \sum_{i \neq j} E \left[ \partial_{ij}^p (G_{ij} P^{2r-1}) \right] + O\left(\Phi_r\right).
\]

Then, the proof is split into two parts. One is to show that we have for odd integer \(s\) with \(0 \leq s \leq p,\)
\[
\frac{1}{N^2} \sum_{i \neq j} E \left[ (\partial_{ij}^s G_{ij})(\partial_{ij}^{p-s} P^{2r-1}) \right] = -\frac{(s!)}{N^2} \sum_{i \neq j} E \left[ G_{ii}^{s+1} G_{jj}^{s+1} (\partial_{ij}^{p-s} P^{2r-1}) \right] + O\left(\Phi_r\right).
\]

The other is to prove for even integer \(s\) satisfying \(0 \leq s \leq p,\)
\[
\frac{1}{N^2} \sum_{i \neq j} E \left[ (\partial_{ij}^s G_{ij})(\partial_{ij}^{p-s} P^{2r-1}) \right] = O\left(\Phi_r\right).
\]

We first consider the case \(p = 1:\)
\[
\frac{1}{N^2} \sum_{i \neq j} E \left[ \partial_{ij} (G_{ij} P^{2r-1}) \right] = \frac{1}{N^2} \sum_{i \neq j} E \left[ (\partial_{ij} G_{ij}) P^{2r-1} \right] + \frac{(2r-1)}{N^2} \sum_{i \neq j} E \left[ G_{ij} (\partial_{ij} P) P^{2r-2} \right].
\]
Applying (4.4), we get
\[
\frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[ (\partial_{ij} G_{ij}) P^{2r-1} \right] = \frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[ D_{ij} G_{ij} P^{2r-1} \right] + \mathcal{O} \left( \mathbb{E} \left[ \frac{\Im(\tilde{z})}{N\eta} |P|^{2r-1} \right] \right)
\]
\[
= -\frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[ G_{ii} G_{jj} P^{2r-1} \right] + \mathcal{O} \left( \mathbb{E} \left[ \frac{\Im(\tilde{z})}{N\eta} |P|^{2r-1} \right] \right)
\]
We observe that
\[
\left| \sum_{i \neq j} G_{ij} (\partial_{ij} P) \right| \leq \left( \sum_{i,j} |G_{ij}|^2 \right)^{1/2} \left( \sum_{i,j} |\partial_{ij} P|^2 \right)^{1/2} \sqrt{N} \left( \frac{\Im(\tilde{z})}{\eta} \right)^{1/2} \left( \frac{\Im(\tilde{z}) |\partial_2 P|}{\eta} + \frac{1}{N} \right) |P|^{2r-2}.
\]
Then, we obtain
\[
\frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[ G_{ij} (\partial_{ij} P) P^{2r-2} \right] = \mathcal{O}_\prec \left( \mathbb{E} \left[ \frac{\Im(\tilde{z})}{N\eta} |P|^{2r-1} \right] + \mathbb{E} \left[ \frac{\Im(\tilde{z})}{\eta} \frac{\Im(\tilde{z}) |\partial_2 P|}{\eta} + \frac{1}{N} \right] |P|^{2r-2} \right).
\]
We conclude
\[
\frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[ \partial_{ij} \left( G_{ij} P^{2r-1} \right) \right] = -\frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[ G_{ii} G_{jj} P^{2r-1} \right]
\]
\[
+ \mathcal{O}_\prec \left( \mathbb{E} \left[ \frac{\Im(\tilde{z})}{N\eta} |P|^{2r-1} \right] + \mathbb{E} \left[ \frac{\Im(\tilde{z})}{\eta} \frac{\Im(\tilde{z}) |\partial_2 P|}{\eta} + \frac{1}{N} \right] |P|^{2r-2} \right).
\]
Next, we consider the case \( p = 2 \):
\[
\frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[ \partial_{ij}^2 \left( G_{ij} P^{2r-1} \right) \right] .
\]
The term \( \partial_{ij}^2 \left( G_{ij} P^{2r-1} \right) \) is split into \( \partial_{ij} \left( G_{ij} P^{2r-1} \right), \partial_{ij} G_{ij} \partial_{ij} P^{2r-1}, \) and \( G_{ij} \partial_{ij}^2 P^{2r-1} \). Using (4.4), it follows that
\[
\frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[ (\partial_{ij}^2 G_{ij}) P^{2r-1} \right] = \frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[ (D_{ij}^2 G_{ij}) P^{2r-1} \right] + \mathcal{O}_\prec \left( \mathbb{E} \left[ \frac{\Im(\tilde{z})}{N\eta} |P|^{2r-1} \right] \right).
\]
When we estimate
\[
\frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[ (D_{ij}^2 G_{ij}) P^{2r-1} \right] ,
\]
y any term containing at least two off-diagonal Green function entries can be bounded by
\[
\mathcal{O}_\prec \left( \mathbb{E} \left[ \frac{\Im(\tilde{z})}{N\eta} |P|^{2r-1} \right] \right),
\]
using the Ward identity (4.7). Thus, the tricky term is
\[
\frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[ G_{ij} G_{ii} G_{jj} P^{2r-1} \right].
\]
Let \( H^{(i)} \) be the \( N \times N \) matrix defined by
\[
\left( H^{(i)} \right)_{kj} := \mathbbm{1}(k \neq i) \mathbbm{1}(j \neq i) h_{kj}.
\]
We denote by \( G^{(i)} \) the Green function of \( H^{(i)} \). When \( i \neq j \), we have the following resolvent identity:
\[
(4.8) \quad G_{ij} = -G_{ii} \sum_{k} h_{ik} G^{(i)}_{kj},
\]
where we use the notation

\[(4.9) \quad \sum_{k}^{(j)} = \sum_{1 \leq k \leq N, k \neq j}^{(j)} . \]

By the above resolvent identity (4.8), we get

\[ \frac{1}{N^{2}} \sum_{i \neq j} E \left[ G_{ij} G_{ii} G_{jj} P^{2r-1} \right] = - \frac{1}{N^{2}} \sum_{i \neq j} \sum_{k}^{(i)} E \left[ h_{ik} G_{ij}^{(i)} G_{ii}^{2} G_{jj} P^{2r-1} \right] . \]

We apply the cumulant expansion again and obtain

\[ - \sum_{p' = 1}^{\ell} \sum_{i \neq j} \sum_{k}^{(i)} \frac{C_{p'} + 1}{N^{3q} p^{r-1}} E \left[ G_{kj}^{(i)} \partial_{ik}^{p'} (G_{ii}^{2} G_{jj} P^{2r-1}) \right] . \]

For \( i \notin \{k, j\} \), the following identity holds:

\[(4.10) \quad G_{kj}^{(i)} = G_{kj} - G_{ki} G_{ij} . \]

Using this identity (4.10), we have for each \( p' \geq 1 \),

\[(4.11) \quad \frac{1}{N^{3q} p^{r-1}} \sum_{i \neq j} \sum_{k}^{(i)} E \left[ G_{kj}^{(i)} \partial_{ik}^{p'} (G_{ii}^{2} G_{jj} P^{2r-1}) \right] = \frac{1}{N^{3q} p^{r-1}} \sum_{i \neq j} \sum_{k}^{(i)} E \left[ \frac{G_{ki} G_{ij}}{G_{ii}} \partial_{ik}^{p'} (G_{ii}^{2} G_{jj} P^{2r-1}) \right] . \]

The second term on the right side of (4.11) contains at least two off-diagonal Green function entries. We deduce from the Ward identity that

\[ \frac{1}{N^{3q} p^{r-1}} \sum_{i \neq j} \sum_{k}^{(i)} E \left[ \frac{G_{ki} G_{ij}}{G_{ii}} \partial_{ik}^{p'} (G_{ii}^{2} G_{jj} P^{2r-1}) \right] = O_{\prec} \left( \max_{0 \leq s \leq 2r-1} E \left[ \frac{3m(\bar{z})}{N \eta} \left( \frac{3m(\bar{z})}{N \eta} \right)^{2} + \frac{1}{N} \right]^{s} \right) . \]

For the first term on the right side of (4.11), if the derivative \( \partial_{ik} \) hits \( G_{jj} \) at least once, there are at least three off-diagonal entries in the summand so we can gain a factor of \( \Im m(\bar{z}) / N \eta \) due to the Ward identity. Similarly, when the derivative \( \partial_{ik} \) hits \( G_{ii} \) odd times, we can find at least two off-diagonal entries and hence an additional factor of \( \Im m(\bar{z}) / N \eta \) follows from the Ward identity. Thus, if the derivative \( \partial_{ik} \) hits \( G_{ii} \), it should hit \( G_{ii} \) even times. If the derivative \( \partial_{ik} \) hits \( P \) at least once, the resulting term is bounded by

\[ O_{\prec} \left( \max_{1 \leq s \leq 2r-1} E \left[ \frac{1}{N \eta} \left( \frac{\partial_{ik} P}{N} \left( \frac{3m(\bar{z})}{N \eta} \right)^{2} + \frac{1}{N} \right]^{s} \right) . \]
since we have (4.6) and
\[
\left| \sum_{k,j} G_{kj} G_{kk} G_{jj} \partial_{ik} P \right| = \left| \sum_k G_{kk}^{\prime} \partial_{ik} P \sum_j G_{kj} G_{jj} \right| \leq \left( \sum_k |G_{kk}^{\prime} \partial_{ik} P|^2 \right)^{1/2} \left( \sum_j \left| \sum_k G_{kj} G_{jj} \right|^2 \right)^{1/2} < \sqrt{N} \left( \frac{\| \partial_2 P \| m(z)}{N \eta} + \left( \frac{\| m(z) \|}{N \eta} \right)^2 + \frac{1}{N} \right) \| \Gamma \| \leq N \left( \frac{\| \partial_2 P \| m(z)}{N \eta} + \left( \frac{\| m(z) \|}{N \eta} \right)^2 + \frac{1}{N} \right) \| \Gamma \|
\]
where \( \Gamma = (G_{11}, \ldots, G_{NN}) \). If the derivative \( \partial_{ik} \) hits \( G_{ii} \) even times, we apply the identities (4.8)-(4.10) and the cumulant expansion again. For example, in the case that \( p' = 2 \), we have the term
\[
\frac{1}{N^3 q} \sum_{i \neq j} \sum_k \mathbb{E} \left[ G_{kj} G_{ki}^{\prime} G_{kk} G_{jj} P^{2r - 1} \right].
\]
Note that we gain at least a factor of \( q^{-1} \) if the derivative \( \partial_{ik} \) hits \( G_{ii} \) even times. Using the identities and the cumulant expansion, we have
\[
- \sum_{p'' = 1}^{\ell} \sum_{i \neq j} \sum_k \sum_s \frac{C_{p'' + 1}}{N^3 q^{p''}} \mathbb{E} \left[ G_{kj}^{(k)} \partial_{kl}^{p''} (G_{ii} G_{kk} G_{jj} P^{2r - 1}) \right].
\]
Following the above argument similarly, a tricky term comes from the case that the derivative \( \partial_{kl} \) hits \( G_{kk} \) even times so at least a factor of \( q^{-1} \) is obtained. Repeating such process until we get these factors of \( q^{-1} \) enough, we conclude
\[
(4.12) \quad \frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[ G_{ij} G_{ii} G_{jj} P^{2r - 1} \right] = \mathcal{O}_s \left( \max_{0 \leq s \leq 2r - 1} \mathbb{E} \left[ \frac{\| m(z) \|}{N \eta} \left( \frac{\| m(z) \|}{N \eta} \right)^2 + \frac{1}{N} \right]^{s} |P|^{2r - s - 1} \right) + \mathcal{O}_s \left( \max_{1 \leq s \leq 2r - 1} \mathbb{E} \left[ \frac{\| \partial_2 P \| m(z)}{N \eta} + \left( \frac{\| m(z) \|}{N \eta} \right)^2 + \frac{1}{N} \right]^{s} |P|^{2r - s - 1} \right).
\]
In addition, we can see that
\[
\left( \begin{array}{c} 2 \\ 1 \end{array} \right) \frac{C_3}{N^2 q} \sum_{i \neq j} \mathbb{E} \left[ (D_{ij} G_{ij}) \partial_{ij} P^{2r - 1} \right] = \left( \begin{array}{c} 2 \\ 1 \end{array} \right) \frac{C_3}{N^2 q} \sum_{i \neq j} \mathbb{E} \left[ (D_{ij} G_{ij}) \partial_{ij} P^{2r - 1} \right] + \mathcal{O}_s \left( \mathbb{E} \left[ \frac{\| m(z) \|}{N \eta} \left( \frac{\| m(z) \|}{N \eta} \right)^2 + \frac{1}{N} \right]^{s} |P|^{r - 2} \right)
\]
\[
= - \left( \begin{array}{c} 2 \\ 1 \end{array} \right) \frac{C_3}{N^2 q} \sum_{i \neq j} \mathbb{E} \left[ G_{ii} G_{jj} \partial_{ij} P^{2r - 1} \right] + \mathcal{O}_s \left( \mathbb{E} \left[ \frac{\| m(z) \|}{N \eta} \left( \frac{\| m(z) \|}{N \eta} \right)^2 + \frac{1}{N} \right]^{s} |P|^{2r - 1} \right). \]
Since we have
\[
\left| \sum_{i \neq j} G_{ij} (\partial_{ij} P^{2r - 1}) \right| < \sqrt{N} \left( \frac{\| m(z) \|}{\eta} \right)^{1/2} N \max_{s = 1, 2} \left( \left( \frac{\| m(z) \|}{N \eta} \right)^2 + \frac{1}{N} \right)^s |P|^{2r - 1},
\]
it follows that
\[
\frac{1}{N^2} \sum_{i \neq j} E \left[ G_{ij} (\partial_{ij}^2 P^{2r-1}) \right] = O_{\prec} \left( \max_{s=1,2} \left[ \left( \frac{3m(\tilde{z})}{N\eta} \right)^{1/2} \left( \frac{\partial_2 P \cdot 3m(\tilde{z})}{N\eta} + \left( \frac{3m(\tilde{z})}{N\eta} \right)^2 + \frac{1}{N} \right)^s |P|^{2r-s-1} \right] \right).
\]
Thus, we deduce that
\[
\frac{C_3}{N^2 q^{p-1}} \sum_{i \neq j} E \left[ \partial_{ij}^2 (G_{ij} P^{r-1}) \right] = -\left( \frac{2}{1} \right) \frac{C_3}{N^2 q^{p-1}} \sum_{i \neq j} E \left[ G_{ij} G_{jj} \partial_{ij} (P^{r-1}) \right] + O_{\prec} (\Phi_r).
\]
Now we consider the case \( p \geq 3 \):
\[
\frac{C_{p+1}}{N^2 q^{p-1}} \sum_{i \neq j} E \left[ \partial_{ij}^p (G_{ij} P^{2r-1}) \right] = \sum_{s=0}^{p} \binom{p}{s} \frac{C_{p+1}}{N^2 q^{p-1}} \sum_{i \neq j} E \left[ (\partial_{ij}^p G_{ij}) (\partial_{ij}^{p-s} P^{2r-1}) \right] + O_{\prec} (\Phi_r), \quad s \equiv 1(\text{mod} 2).
\]
If \( s \) is odd, we can observe that
\[
D_{ij}^s G_{ij} = -(s!)^2 G_{ii}^{s/2} G_{jj}^{s/2} + \text{(the terms having at least two off-diagonal Green function entries)},
\]
which implies due to the Ward identity \[4.7\]
\[
\frac{C_{p+1}}{N^2 q^{p-1}} \sum_{i \neq j} E \left[ (D_{ij}^s G_{ij}) (\partial_{ij}^{p-s} P^{2r-1}) \right] = -\frac{(s!)^2 C_{p+1}}{N^2 q^{p-1}} \sum_{i \neq j} E \left[ (G_{ii}^{s/2} G_{jj}^{s/2}) (\partial_{ij}^{p-s} P^{2r-1}) \right] + O_{\prec} (\Phi_r), \quad s \equiv 0(\text{mod} 2),
\]
Next we consider the case that \( s \) is even. We can see that \( D_{ij}^s G_{ij} \) has the following form:
\[
D_{ij}^s G_{ij} = t_s G_{ij} G_{ii}^{s/2} G_{jj}^{s/2} + \text{(the terms having at least three off-diagonal Green function entries)},
\]
where \( t_s \) is a constant depending on \( s \). Since any term having at least two off-diagonal Green function entries is eventually absorbed into the desired bound \( O_{\prec} (\Phi_r) \) because of the Ward identity \[4.7\], we focus on the terms.

\[
\frac{1}{N^2} \sum_{i \neq j} E \left[ G_{ij} G_{ii}^{s/2} G_{jj}^{s/2} (\partial_{ij}^{p-s} P^{2r-1}) \right], \quad s \equiv 0(\text{mod} 2),
\]
where each term has the only one off-diagonal Green function entry \( G_{ij} \). If \( s = p \) (in this case, \( p \) is even), we should estimate
\[
\frac{1}{N^2} \sum_{i \neq j} E \left[ G_{ij} G_{ii}^{p/2} G_{jj}^{p/2} P^{2r-1} \right].
\]
We shall apply the argument from \[4.8\] to \[4.12\] similarly to get
\[
\frac{1}{N^2} \sum_{i \neq j} E \left[ G_{ij} G_{ii}^{p/2} G_{jj}^{p/2} P^{2r-1} \right] = O_{\prec} (\Phi_r).
\]
Using the identity \[4.8\], we have
\[
\frac{1}{N^2} \sum_{i \neq j} E \left[ G_{ij} G_{ii}^{p/2} G_{jj}^{p/2} P^{2r-1} \right] = -\frac{1}{N^2} \sum_{i \neq j} \sum_{k} (i) E \left[ h_{ik} G_{ki}^{(i)} G_{ii}^{1+p/2} G_{jj}^{p/2} P^{2r-1} \right].
\]
Applying the cumulant expansion (Lemma \[3.3\]), we get
\[
-\sum_{p'=1}^{\ell} \sum_{i \neq j} \sum_{k} (i) C_{p'+1} \frac{N^2 q^{p'-1}}{N^2 q^{p'-1}} E \left[ G_{kj}^{(i)} \partial_{ik}^{p'} (G_{ii}^{1+p/2} G_{jj}^{p/2} P^{2r-1}) \right].
\]
Due to the identity (4.10), we have for each $p' \geq 1$,
\[
\frac{1}{N^3 q^{p'-1}} \sum_{i \neq j} \sum_k^{(i)} \mathbb{E} \left[ G_{ik}^{(i)} \partial_{ik}^p \left( G_{ij}^{1+p/2} G_{jj}^{p/2} P^{2r-1} \right) \right]
\]
\[
= \frac{1}{N^3 q^{p'-1}} \sum_{i \neq j} \sum_k^{(i)} \mathbb{E} \left[ G_{kj} \partial_{ik}^p \left( C_{ii}^{1+p/2} G_{jj}^{p/2} P^{2r-1} \right) \right] - \frac{1}{N^3 q^{p'-1}} \sum_{i \neq j} \sum_k^{(i)} \mathbb{E} \left[ G_{ki} G_{ij} \partial_{ik}^p \left( G_{ii}^{1+p/2} G_{jj}^{p/2} P^{2r-1} \right) \right].
\]

The second term on the right-hand side has at least two off-diagonal entries so it can be absorbed into $\mathcal{O}_< (\Phi_r)$ because of the Ward identity. Next we consider the first term on the right-hand side. If $\partial_{ik}$ hits $G_{jj}$ at least once, then we can find at least three off-diagonal entries so that every term of this case is absorbed into $\mathcal{O}_< (\Phi_r)$ by the Ward identity. If $\partial_{ik}$ hits $P$ at least once, the resulting term is also absorbed into $\mathcal{O}_< (\Phi_r)$ because we have for $l \geq 0$ and $l' \geq 1$
\[
\left| \sum_{k,j} G_{kj} G_{kk} G_{jj}^{p/2} \partial_{ik}^l P \right| = \left| \sum_k G_{kk}^l \partial_{ik}^l P \sum_j G_{kj} G_{jj}^{p/2} \right| \leq \left( \sum_k \left| G_{kk}^l \partial_{ik}^l P \right|^2 \right)^{1/2} \left( \sum_j \left| G_{kj} G_{jj}^{p/2} \right|^2 \right)^{1/2}
\]
\[
\leq N \left( \frac{|\partial_{ik} P| |m(\tilde{z})|}{N \eta} + \left( \frac{|m(\tilde{z})|}{N \eta} \right)^2 + \frac{1}{N} \right) \|G\|
\]
\[
\leq N \eta \left( \frac{|\partial_{ik} P| |m(\tilde{z})|}{N \eta} + \left( \frac{|m(\tilde{z})|}{N \eta} \right)^2 + \frac{1}{N} \right),
\]
where the estimate (4.6) is used. The only remaining case is that $\partial_{ik}$ hits $G_{ii}$ only:
\[
\frac{1}{N^3 q^{p'-1}} \sum_{i \neq j} \sum_k^{(i)} \mathbb{E} \left[ G_{kj} G_{ii}^{1+p/2} G_{jj}^{p/2} P^{2r-1} \right].
\]

If $p'$ is odd, then there are at least two off-diagonal entries so we are done. A tricky term comes from the case that $p'$ is even. Since any term containing at least two off-diagonal entries can be absorbed into $\mathcal{O}_< (\Phi_r)$, it is enough to only consider
\[
\frac{1}{N^3 q^{p'-1}} \sum_{i \neq j} \sum_k^{(i)} \mathbb{E} \left[ G_{kj} G_{ii}^{1+p/2} G_{kk}^{p/2} G_{jj}^{p/2} P^{2r-1} \right].
\]

Since $p'$ is even, we note that $p' \geq 2$ and, as a result, we gain at least a factor of $q^{-1}$ in this case. Let us repeat the similar argument as the above. Using the identities (4.8) and (4.10) with the cumulant expansion, we have
\[
\frac{1}{q^{p'-1}} \sum_{p''=1}^t \sum_{i \neq j} \sum_k^{(i)} \sum_l^{(k)} \frac{C_{p''}}{N^4 q^{p'-1}} \mathbb{E} \left[ G_{ij} \partial_{kl}^{p''} \left( G_{kk}^{1+p/2} G_{ii}^{1+p/2} G_{jj}^{p/2} P^{2r-1} \right) \right] + \mathcal{O}_< (\Phi_r), \quad p' \geq 2.
\]

By the same reasoning, the tricky case is that the derivative $\partial_{kl}$ only hits $G_{kk}$ and $p''$ is even, which give us an additional factor of $q^{-1}$. Repeating similar procedure many times enough, the desired estimate (4.13) follows.

If $s$ is even and $0 \leq s < p$, a tricky part is to bound
\[
\frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[ G_{ij} G_{ii}^{s/2} G_{jj}^{s/2} (\partial_{ij}^{s-2} P^{2r-1}) \right],
\]
where each term has the only one off-diagonal entry. Due to (17) Proposition A.1] and the Cauchy-Schwarz inequality, we have for $0 \leq s < p$,
\[
\left| \sum_{i \neq j} G_{ij} G_{ii}^{s/2} G_{jj}^{s/2} (\partial_{ij}^{s-2} P^{2r-1}) \right| \lesssim \left( \sum_{i,j} |G_{ij}|^2 \right)^{1/2} \left( \sum_{i,j} |\partial_{ij}^{s-2} P^{2r-1}|^2 \right)^{1/2}.
\]
Combining the above with the Ward identity and the estimate (4.6), we get

\[
\frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[ G_{ij} G_{ii}^{s/2} G_{jj}^{s/2} (\partial_{ij}^{p-s} P^{2r-1}) \right] = \mathcal{O}_\prec (\Phi_r).
\]

We complete the proof. \( \square \)

### 4.3. Step 3. Replacing every diagonal entry with the normalized trace.

Changing the order of summations, we can rewrite the conclusion of Proposition 4.3 as follows:

\[
\sum_{\ell=1}^\ell \sum_{p=1}^{\mathcal{C}_{p+1}} \sum_{x,y} \mathbb{E} \left[ \partial_{xy}^p \left( m^d G_{xy} G_{ii}^t \left( \prod_{j=1}^{l} G_{v_{ij}, v_{ij}}^t \right) D(P) \right) \right]
\]

4.3. Step 3. Replacing every diagonal entry with the normalized trace. Changing the order of summations, we can rewrite the conclusion of Proposition 4.3 as follows:

\[
\sum_{\ell=1}^\ell \sum_{p=1}^{\mathcal{C}_{p+1}} \sum_{x,y} \mathbb{E} \left[ \partial_{xy}^p \left( m^d G_{xy} G_{ii}^t \left( \prod_{j=1}^{l} G_{v_{ij}, v_{ij}}^t \right) D(P) \right) \right]
\]

In this subsection, we shall replace all diagonal Green function entries in the right-hand side of (4.16) with the normalized trace. In order to do that, we need a general version of Proposition 4.3.

**Lemma 4.4 (General version of Proposition 4.3).** Let \( d \geq 0 \) be a non-negative integer. Let \( t \geq 1 \) be a positive integer. Fix an integer \( k \geq 1 \). Let \( \{u_j\}_{j=1}^k \) be a finite sequence of non-negative integers. Let \( \{v_j\}_{j=1}^k \) be a finite sequence of positive integers. Let \( D(P) \) be a l-th order derivative of \( P^{2r-1} \) with \( l \geq 0 \), and we use the convention that \( D(P) = P^{2r-1} \) for \( l = 0 \). Then, we have

\[
\sum_{\ell=1}^\ell \sum_{p=1}^{\mathcal{C}_{p+1}} \sum_{x,y} \mathbb{E} \left[ \partial_{xy}^p \left( m^d G_{xy} G_{ii}^t \left( \prod_{j=1}^{l} G_{v_{ij}, v_{ij}}^t \right) D(P) \right) \right]
\]

and

\[
\sum_{\ell=1}^\ell \sum_{p=1}^{\mathcal{C}_{p+1}} \sum_{x} \mathbb{E} \left[ \partial_{xx}^p \left( m^{d+1} G_{xx} G_{ii}^{t-1} \left( \prod_{j=1}^{l} G_{v_{ij}, v_{ij}}^t \right) D(P) \right) \right]
\]

where \( c_s \) is a bounded coefficient for each \( 0 \leq s \leq \ell \). In fact,

\[
c_s = \sum_{0 \leq \tilde{s}_1, \cdots, \tilde{s}_t \leq s} 1[\tilde{s}_1 + \cdots + \tilde{s}_t = s] \times 1[\tilde{s}_1 \equiv 1(\text{mod} \ 2)] \times 1[\tilde{s}_2, \cdots, \tilde{s}_t \equiv 0(\text{mod} \ 2)],
\]

so \( c_1 = 1 \) in particular.

**Proof.** The proof of this lemma is in Appendix C. \( \square \)

Using Lemma 4.4, we start replacing a diagonal entry.
Claim 4.5 (Replacement of a single diagonal entry). Let $s_1$ be a positive odd integer. We have

$$
\sum_{p_1 = s_1}^{\ell} \frac{(p_1)}{s_1} \frac{(s_1)!}{N^2q^{p_1-1}} \sum_{i \neq j} E \left[ G_{ii}^{s_1+1} G_{jj}^{s_1+1} (\partial^{p_1-s_1} f^{2r-1}) \right]
= \sum_{p_1 = s_1}^{\ell} \frac{(p_1)}{s_1} \frac{(s_1)!}{N^2q^{p_1-1}} \sum_{i \neq j} E \left[ mG_{ii}^{s_1+1} G_{jj}^{s_1+1} (\partial^{p_1-s_1} f^{2r-1}) \right]
- \sum_{p_1 = s_1}^{\ell} \sum_{p_2 = 2}^{s_1} \frac{(p_2)}{s_1} \frac{(s_1)!}{N^2q^{p_1-2}} \sum_{i \neq j} E \left[ G_{xy} G_{yy} (\partial^{p_2-s_1} f^{2r-1}) \right]
+ \sum_{1 < s_2 \leq \ell} \sum_{s_2 \equiv 1 (\mod 2)} c_{s_2} \sum_{p_1 = s_1}^{\ell} \frac{(p_1)}{s_1} \frac{(s_1)!}{N^2q^{p_1-2}} \sum_{i \neq j} E \left[ G_{xy} G_{xx} (\partial^{p_2-s_1} f^{2r-1}) \right]
+ O_{\prec} (\Phi_r),
$$

where $c_{s_2}$ is a bounded coefficient for each $0 \leq s_2 \leq \ell$.

Proof. According to [15, Lemma 10.1], we have

$$G_{ij} = \delta_{ij} m + \frac{G_{ij}}{N} \sum_{x,y} h_{xy} G_{yx} - m \sum_x h_{ix} G_{xj}.
$$

If $i = j$, it follows that

$$G_{ii} = m + \frac{G_{ii}}{N} \sum_{x,y} h_{xy} G_{yx} - m \sum_x h_{ix} G_{xi}.
$$

Using the identity (4.21), we get

$$
\frac{1}{N^2} \sum_{i \neq j} E \left[ G_{ii}^{s_1+1} G_{jj}^{s_1+1} (\partial^{p_1-s_1} f^{2r-1}) \right]
= \frac{1}{N^2} \sum_{i \neq j} E \left[ mG_{ii}^{s_1+1} G_{jj}^{s_1+1} (\partial^{p_1-s_1} f^{2r-1}) \right]
+ \frac{1}{N^2} \sum_{x \neq y} E \left[ h_{xy} h_{yx} G_{ii} G_{jj} (\partial^{p_1-s_1} f^{2r-1}) \right]
- \frac{1}{N^2} \sum_{x \neq y} E \left[ h_{ix} h_{xi} G_{ii} G_{jj} (\partial^{p_1-s_1} f^{2r-1}) \right].
$$

Applying the cumulant expansion to the last two terms on the right-hand side of (4.22), we obtain

$$
\frac{1}{N^2} \sum_{i \neq j} E \left[ h_{xy} G_{yx} G_{ii} G_{jj} (\partial^{p_1-s_1} f^{2r-1}) \right]
+ \frac{1}{N^2} \sum_{i \neq j} E \left[ h_{ix} G_{xi} G_{ii} G_{jj} (\partial^{p_1-s_1} f^{2r-1}) \right]
= \frac{1}{N^2} \sum_{i \neq j} E \left[ \partial^{p_2}_{xy} \left( G_{xy} G_{ii} G_{jj} (\partial^{p_1-s_1} f^{2r-1}) \right) \right]
- \frac{1}{N^2} \sum_{i \neq j} E \left[ \partial^{p_2}_{ix} \left( G_{xixi} G_{ii} G_{jj} (\partial^{p_1-s_1} f^{2r-1}) \right) \right].
$$
Using Lemma 4.4 to the right-hand side of (4.23) (with setting \(d = 0, t = \frac{v_1 + 1}{2}, k = 1, v_1 = j, u_1 = \frac{v_1 + 1}{2}, D(P) = \partial_{ij}^{p_1 - s_1} P^{2r-1}\)), we have

\[
(4.24) \quad \frac{1}{N^3} \sum_{i,j} \sum_{x,y} \sum_{p_2=1}^{\ell} \frac{C_{p_2+1}}{Nq^{p_2-1}} \mathbb{E} \left[ \partial_{xy}^{2} \left( G_{x,y} G_{ii}^{1/2} G_{jj}^{1/2} (\partial_{ij}^{p_1 - s_1} P^{2r-1}) \right) \right]
= - \sum_{0 \leq s_2 \leq \ell \atop s_2 \equiv 1 \text{ (mod 2)}} \left( \frac{p_2}{s_2} \right) \frac{C_{p_2+1}}{N^3q^{p_2-1}} \sum_{i \neq j} \sum_{x,y} \mathbb{E} \left[ G_{x,x}^{s_2+1} G_{y,y}^{s_2+1} G_{ii}^{s_2+1} G_{jj}^{s_2+1} (\partial_{xy}^{p_1 - s_2} \partial_{ij}^{p_1 - s_1} P^{2r-1}) \right] + \mathcal{O}_\prec (\Phi_r),
\]

and

\[
(4.25) \quad \frac{1}{N^3} \sum_{i,j} \sum_{x,y} \sum_{p_2=1}^{\ell} \frac{C_{p_2+1}}{Nq^{p_2-1}} \mathbb{E} \left[ \partial_{ix}^{2} \left( G_{x,i} G_{ii}^{1/2} G_{jj}^{1/2} (\partial_{ij}^{p_1 - s_1} P^{2r-1}) \right) \right]
= - \sum_{0 \leq s_2 \leq \ell \atop s_2 \equiv 1 \text{ (mod 2)}} \left( \frac{p_2}{s_2} \right) \frac{C_{p_2+1}}{N^3q^{p_2-1}} \sum_{i \neq j} \sum_{x,y} \mathbb{E} \left[ G_{x,x} G_{y,y} G_{ii}^{s_1+1} G_{jj}^{s_1+1} (\partial_{xy}^{p_1 - s_2} \partial_{ij}^{p_1 - s_1} P^{2r-1}) \right] + \mathcal{O}_\prec (\Phi_r),
\]

Combining (4.23), (4.24) and (4.25), it follows that

\[
(4.26) \quad \frac{1}{N^3} \sum_{i,j} \sum_{x,y} \sum_{p_2=1}^{\ell} \frac{C_{p_2+1}}{Nq^{p_2-1}} \mathbb{E} \left[ \partial_{xy}^{2} \left( G_{x,y} G_{ii}^{1/2} G_{jj}^{1/2} (\partial_{ij}^{p_1 - s_1} P^{2r-1}) \right) \right]
= - \frac{1}{N^2} \sum_{i \neq j} \sum_{x} \sum_{p_2=1}^{\ell} \frac{C_{p_2+1}}{Nq^{p_2-1}} \mathbb{E} \left[ \partial_{ix}^{2} \left( G_{x,i} G_{ii}^{s_1+1} G_{jj}^{s_1+1} (\partial_{ij}^{p_1 - s_1} P^{2r-1}) \right) \right]
\]

\[
= - \sum_{0 \leq s_2 \leq \ell \atop s_2 \equiv 1 \text{ (mod 2)}} \left( \frac{p_2}{s_2} \right) \frac{C_{p_2+1}}{N^3q^{p_2-1}} \sum_{i \neq j} \sum_{x,y} \mathbb{E} \left[ G_{x,x} G_{y,y} G_{ii}^{s_1+1} G_{jj}^{s_1+1} (\partial_{xy}^{p_1 - s_2} \partial_{ij}^{p_1 - s_1} P^{2r-1}) \right] + \mathcal{O}_\prec (\Phi_r),
\]

When \(p_2 = s_2 = 1\), there is a cancellation in the right-hand sides of (4.26) as follows: using the fact that \(c_{s_2} = 1\) for \(s_2 = 1\), we have

\[
\frac{1}{N^4} \sum_{i \neq j} \sum_{x \neq y} \mathbb{E} \left[ G_{x,x} G_{y,y} G_{ii}^{s_1+1} G_{jj}^{s_1+1} (\partial_{ij}^{p_1 - s_1} P^{2r-1}) \right] - \frac{1}{N^3} \sum_{i,j} \sum_{x} \mathbb{E} \left[ G_{x,x} G_{ii}^{s_1+1} G_{jj}^{s_1+1} m(\partial_{ij}^{p_1 - s_1} P^{2r-1}) \right]
\]

\[
= \frac{1}{N^4} \sum_{i \neq j} \sum_{x \neq y} \mathbb{E} \left[ G_{x,x} G_{y,y} G_{ii}^{s_1+1} G_{jj}^{s_1+1} (\partial_{ij}^{p_1 - s_1} P^{2r-1}) \right] - \frac{1}{N^2} \sum_{i,j} \sum_{x} \mathbb{E} \left[ G_{ii}^{s_1+1} G_{jj}^{s_1+1} m^2(\partial_{ij}^{p_1 - s_1} P^{2r-1}) \right] + \mathcal{O}_\prec (\Phi_r)
\]

\[
= \frac{1}{N^2} \sum_{i \neq j} \mathbb{E} \left[ m^2 G_{ii}^{s_1+1} G_{jj}^{s_1+1} (\partial_{ij}^{p_1 - s_1} P^{2r-1}) \right] - \frac{1}{N^2} \sum_{i,j} \mathbb{E} \left[ G_{ii}^{s_1+1} G_{jj}^{s_1+1} m^2(\partial_{ij}^{p_1 - s_1} P^{2r-1}) \right] + \mathcal{O}_\prec (\Phi_r)
\]

\[
= \mathcal{O}_\prec (\Phi_r),
\]
where we use Proposition 4.2 and Proposition A.1 to switch over from $\sum_{x\neq y}$ to $\sum_{x,y}$. Then, we get from (4.26)

\begin{equation}
(4.27) \quad \frac{1}{N^3} \sum_{i \neq j} \sum_{x \neq y} \sum_{p_2=1}^{\ell} \frac{C_{p_2+1}}{N^q p_2^{-1}} \mathbb{E} \left[ \partial_{x y} \left( G_{jy} G_{ii}^{\frac{s + 1}{2}} G_{jj}^{\frac{s + 1}{2}} (\partial_{ij}^{p_1-1} p^{2r-1}) \right) \right] \\
- \frac{1}{N^2} \sum_{i \neq j} \sum_{x} \sum_{p_2=1}^{\ell} \frac{C_{p_2+1}}{N^q p_2^{-1}} \mathbb{E} \left[ \partial_{x} \left( G_{jy} G_{ii}^{\frac{s + 1}{2}} G_{jj}^{\frac{s + 1}{2}} m(\partial_{ij}^{p_1-1} p^{2r-1}) \right) \right] \\
= - \sum_{1 < s_2 \leq \ell} \sum_{s_2 \equiv 1 \pmod{2}} \left( \frac{p_2}{s_2} \right) \frac{C_{p_2+1}}{N^q p_2^{-1}} \sum_{i \neq j} \sum_{x \neq y} \mathbb{E} \left[ G_{xx} G_{yy} G_{ii}^{\frac{s + 1}{2}} G_{jj}^{\frac{s + 1}{2}} (\partial_{xy}^{p_2-1} \partial_{ij}^{p_1-1} p^{2r-1}) \right] \\
+ \sum_{s_2 \equiv 1 \pmod{2}} \left( \frac{p_2}{s_2} \right) \frac{C_{p_2+1}}{N^q p_2^{-1}} \sum_{i \neq j} \sum_{x} \mathbb{E} \left[ G_{xx} G_{ii}^{\frac{s + 1}{2}} G_{jj}^{\frac{s + 1}{2}} m(\partial_{xy}^{p_2-1} \partial_{ij}^{p_1-1} p^{2r-1}) \right] + O(\Phi_r) .
\end{equation}

The claim follows from (4.20) and (4.27).

The right-hand side of (4.20) is too complicated so we introduce some new notations for brevity.

**Notation 1:** For a positive integer $k$, we consider a set of index pairs

$$I_k = \{(i_1, j_1), \ldots, (i_k, j_k)\}.$$

All indexes in the set $I_k$ will be used to denote formal indexes in a summation. We allow the case that two formally distinct indexes such as $i_l$ and $i_{l'}$ (or $j_l$ and $j_{l'}$) are **equivalent**, i.e. $i_l \equiv i_{l'}$ (or $j_l \equiv j_{l'}$) where $l \neq l'$. If two formally distinct indexes are not equivalent, these are said to be **non-equivalent**. If $i_l$ and $i_{l'}$ are non-equivalent, we denote this by $i_l \not\equiv i_{l'}$. Similarly, when $j_l$ and $j_{l'}$ are non-equivalent, we write $j_l \not\equiv j_{l'}$.

We denote the maximal number of (pairwise) non-equivalent indexes in $I_k$ by $\ell = \ell(I_k)$, and write

$$\{i_1, j_1, \ldots, i_k, j_k\} = \{i_1, \ldots, i_\ell\},$$

where each $i_l$ ($1 \leq l \leq \ell$) is an element in a set of equivalent indexes, and the set $\{i_1, \ldots, i_\ell\}$ consists of non-equivalent indexes. For example, we consider the case that $k = 2$. We have $I_2 = \{(i_1, j_1), (i_2, j_2)\}$. If $i_1, j_1, j_2$ are non-equivalent but $i_2 \equiv i_1$, then we have $\ell = 3$ and also write $\{i_1, i_2, j_1\} = \{i_1, j_1, j_2\}$.

We denote the summation over all distinct non-equivalent indexes $\{i_1, \ldots, i_\ell\}$ by

\begin{equation}
(4.28) \quad \sum_{I_k} := \sum_{1 \leq i_1, \ldots, i_\ell \leq N} [i_1, \ldots, i_\ell \text{ are all distinct}] \\
\end{equation}

Let $\theta = \theta(I_k)$ be the cardinality of the set

\begin{equation}
(4.29) \quad \{1 \leq l \leq k : i_l \not\equiv i_{l'} \text{ and } j_l \not\equiv j_{l'} \text{ for all } 1 \leq l' < l\}.
\end{equation}

For every positive integer $k \geq 1$, we denote by $J_k$ the collection of all $I_k$ satisfying the following properties:

(P1) For every $1 \leq l, l' \leq k$, the index $i_l$ and $i_{l'}$ are non-equivalent.

(P2) An index $i_l$ can be equivalent to $i_{l'}$ for some $l' < l$ (but not necessarily). Similarly an index $j_l$ can be equivalent to $j_{l'}$ for some $l' < l$ (but not necessarily).

(P3) Suppose $l' < l$. If $i_l \equiv i_{l'}$, then $j_l$ is not equivalent to every index in $\{j_r : i_r = i_l, 1 \leq r \leq l - 1\}$.

Similarly, if $j_l \equiv j_{l'}$, then $i_l$ is not equivalent to every index in $\{i_r : j_r = j_l, 1 \leq r \leq l - 1\}$.

In other words,

\begin{equation}
(4.30) \quad J_k := \{I_k : I_k \text{ satisfies (P1), (P2) and (P3)}\}.
\end{equation}
Notation 2: For an positive odd integer $s$ such that $s \leq p$, we use the notation

$$
\sum_{[p;s]_k} = \begin{cases} 
\sum_{p=2}^{\ell} \left( \frac{p}{1} \right) \frac{(1)\mathcal{C}_{p+1}}{q^{p-1}} & k > 1 \text{ and } s = 1, \\
\sum_{p=s}^{\ell} \left( \frac{s}{s} \right) \frac{(s)\mathcal{C}_{p+1}}{q^{p-1}} & \text{otherwise}.
\end{cases}
$$

(4.31)

Remark 4.6. Throughout this paper, we always consider $\mathcal{I}_k \in \mathcal{J}_k$.

Using the above new notation, we can rewrite the estimate (4.20). First we consider

$$
\sum_{p_1=s_1}^{\ell} \left( \frac{p_1}{s_1} \right) \frac{(s_1)\mathcal{C}_{p_1+1}}{N^2q^{p_1-1}} \sum_{i \neq j} \mathbb{E} \left[ G_{ii}^{\frac{x_1}{i}} G_{jj}^{\frac{x_1}{j}} (\partial_{p_1-s_1} P^{2r-1}) \right],
$$

and

$$
\sum_{p_1=s_1}^{\ell} \left( \frac{p_1}{s_1} \right) \frac{(s_1)\mathcal{C}_{p_1+1}}{N^2q^{p_1-1}} \sum_{i \neq j} \mathbb{E} \left[ mG_{ii}^{\frac{x_1}{i}} G_{jj}^{\frac{x_1}{j}} (\partial_{p_1-s_1} P^{2r-1}) \right].
$$

Simply setting $i_1 = i$, $j_1 = j$ and $k = 1$, we have $\mathcal{I}_1 \in \mathcal{J}_1$ and rewrite the above as follows:

(4.32)

$$
\sum_{[p_1;s_1]_1} \frac{1}{N^{1+\theta(\mathcal{I}_1)}} \sum_{\mathcal{I}_1} \mathbb{E} \left[ G_{i_1 i_1}^{\frac{x_1-1}{i_1} } G_{j_1 j_1}^{\frac{x_1+1}{j_1} } (\partial_{p_1-s_1} P^{2r-1}) \right]
$$

and

(4.33)

$$
\sum_{[p_1;s_1]_1} \frac{1}{N^{1+\theta(\mathcal{I}_1)}} \sum_{\mathcal{I}_1} \mathbb{E} \left[ mG_{i_1 i_1}^{\frac{x_1-1}{i_1} } G_{j_1 j_1}^{\frac{x_1+1}{j_1} } (\partial_{p_1-s_1} P^{2r-1}) \right].
$$

where we refer to (4.29) for the definition of $\theta(\mathcal{I}_k)$ and note that $\theta(\mathcal{I}_1) = 1$. Next we consider

$$
\sum_{p_1=s_1}^{\ell} \sum_{p_2=2}^{p_1} \left( \frac{p_1}{s_1} \right) \left( \frac{p_2}{1} \right) \frac{(s_1)\mathcal{C}_{p_1+1}(1)\mathcal{C}_{p_2+1}}{N^{p_1}q^{p_1-2}} \sum_{i \neq j, x \neq y} \mathbb{E} \left[ G_{xx}^{\frac{x_1}{x}} G_{yy}^{\frac{x_1}{y}} G_{ij}^{\frac{x_1}{ij}} (\partial_{p_2-s_2} \partial_{p_1-s_1} P^{2r-1}) \right]
$$

By setting $i_1 = i$, $j_1 = j$, $i_2 = x$, $j_2 = y$ and $k = 2$, we have $\mathcal{I}_2 \in \mathcal{J}_2$ and rewrite the above as follows:

(4.34)

$$
\sum_{0 \leq s_2 \leq \ell} \sum_{[p_1; s_2]_2} \sum_{[p_1; s_1]_1} \frac{1}{N^{2+\theta(\mathcal{I}_2)}} \sum_{\mathcal{I}_2} \mathbb{E} \left[ G_{i_2 i_2}^{\frac{x_1}{i_2} } G_{j_2 j_2}^{\frac{x_1}{j_2} } G_{i_2 j_2}^{\frac{x_1}{i_2 j_2} } G_{i_1 i_1}^{\frac{x_1}{i_1} } G_{j_1 j_1}^{\frac{x_1}{j_1} } (\partial_{p_2-s_2} \partial_{p_1-s_1} P^{2r-1}) \right] + \mathcal{O}_x (\Phi_r),
$$

where we use Proposition 4.2 and [17] Proposition A.1] to switch over from $\sum_{i \neq j, x \neq y}$ to $\sum_{\mathcal{I}_2}$. We also consider the summations below,

(4.35)

$$
\sum_{c_{s_2}} \sum_{0 \leq s_2 \leq \ell} \sum_{p_1=s_1}^{\ell} \sum_{p_2=2}^{p_1} \left( \frac{p_1}{s_1} \right) \left( \frac{p_2}{1} \right) \frac{(s_1)\mathcal{C}_{p_1+1}(1)\mathcal{C}_{p_2+1}}{N^{p_1}q^{p_1-2}} \sum_{i \neq j, x \neq y} \mathbb{E} \left[ G_{xx}^{\frac{x_1}{x}} G_{ii}^{\frac{x_1}{i}} G_{ij}^{\frac{x_1}{ij}} m(\partial_{p_2-s_2} \partial_{p_1-s_1} P^{2r-1}) \right]
$$

and

$$
\sum_{c_{s_2}} \sum_{0 \leq s_2 \leq \ell} \sum_{p_1=s_1}^{\ell} \sum_{p_2=2}^{p_1} \left( \frac{p_1}{s_1} \right) \left( \frac{p_2}{s_2} \right) \frac{(s_1)\mathcal{C}_{p_1+1}(s_2)\mathcal{C}_{p_2+1}}{N^{p_1}q^{p_1-2}} \sum_{i \neq j, x \neq y} \mathbb{E} \left[ G_{xx}^{\frac{x_1}{x}} G_{ii}^{\frac{x_1}{i}} G_{ij}^{\frac{x_1}{ij}} m(\partial_{p_2-s_2} \partial_{p_1-s_1} P^{2r-1}) \right].
$$
We set \( i_1 \equiv i_2 \equiv i, j_1 \equiv j, j_2 \equiv x \) and \( k = 2 \). Then, we have \( \mathcal{I}_2 \in \mathcal{J}_2 \). Note that \( \theta(\mathcal{I}_2) = 1 \) in this setting. We can replace (4.35) with the following:

\[
(4.36) \quad \sum_{0 \leq s_2 \leq \ell} c_{s_2} \sum_{[p_2:s_2]_2} \sum_{[p_1:s_1]_1} \frac{1}{N^{2+\theta(\mathcal{I}_2)}} \sum_{\mathcal{I}_2} E \left[ G_{i_2i_2}^{s_2+1} G_{j_2j_2}^{s_2+1} G_{i_1i_1}^{s_1+1} G_{j_1j_1}^{s_1+1} m(\partial_{\mathcal{I}_2}^{p_2-s_2} \partial_{\mathcal{I}_1}^{p_1-s_1} P^{2r-1}) \right] + O_{\prec}(\Phi_r),
\]

where we use that \( c_{s_2} = 1 \) when \( s_2 = 1 \), and switch over from \( \sum_{i \neq j} \sum_x \) to \( \sum_{\mathcal{I}_2} \). Now we can restate Claim 4.5 as below.

**Claim 4.7 (Simple form of Claim 4.5).** Let \( s_1 \) be a positive odd integer. Let \( \mathcal{J}_2 \) be as in (4.30) (with \( k = 2 \)). Then, we have

\[
(4.37) \quad \sum_{[p_1:s_1]_1} \frac{1}{N^{1+\theta(\mathcal{I}_1)}} \sum_{\mathcal{I}_1} E \left[ G_{i_1i_1}^{s_1+1} G_{j_1j_1}^{s_1+1} (\partial_{\mathcal{I}_1}^{p_1-s_1} P^{2r-1}) \right] = \sum_{[p_1:s_1]_1} \frac{1}{N^{1+\theta(\mathcal{I}_1)}} \sum_{\mathcal{I}_1} E \left[ mG_{i_1i_1}^{s_1+1} G_{j_1j_1}^{s_1+1} (\partial_{\mathcal{I}_1}^{p_1-s_1} P^{2r-1}) \right]
\]

\[
+ \sum_{\mathcal{I}_2 \in \mathcal{J}_2} \sum_{0 \leq s_2 \leq \ell} \sum_{[p_2:s_2]_2} \sum_{[p_1:s_1]_1} c_{I_2,s_2} \sum_{N^{2+\theta(\mathcal{I}_2)}} \sum_{\mathcal{I}_2} E \left[ G_{i_2i_2}^{s_2+1} G_{j_2j_2}^{s_2+1} G_{i_1i_1}^{s_1+1} G_{j_1j_1}^{s_1+1} m(\partial_{\mathcal{I}_2}^{p_2-s_2} \partial_{\mathcal{I}_1}^{p_1-s_1} P^{2r-1}) \right] + O_{\prec}(\Phi_r),
\]

where each \( c_{I_2,s_2} \) is a constant only depending on \( \mathcal{I}_2 \) and \( s_2 \equiv 1 \pmod{2} \). (Refer to (4.31) for the definition of \( \sum_{[p:s]_s} \).)

**Proof.** We compare (4.20) and (4.37). It immediately follows from (4.32), (4.33), (4.34) and (4.36). \( \square \)

**Remark 4.8.** In fact, \( c_{I_2,s_2} = 0 \) if \( \mathcal{I}_2 \) does not belong to one of the following two cases: (case 1) \( i_1, j_1, i_2, j_2 \) are all non-equivalent; (case 2) \( i_1, j_1, i_2, j_2 \) are all non-equivalent but \( i_2 = i_1 \).

Now we just replace one diagonal entry. We want to replace all diagonal entries with the normalized trace \( m \). We need the following lemma for that.

**Lemma 4.9.** Let \( d \geq 0 \) be a non-negative integer. Let \( \ell \geq 1 \) be a positive integer. Fix an integer \( k \geq 1 \). Let \( \{ u_j \}_{j=1}^k \) be a finite sequence of non-negative integers. Let \( \{ v_j \}_{j=1}^k \) be a finite sequence of positive integers. Let \( D(P) \) be a \( l \)-th order derivative of \( P^{2r-1} \) with \( l \geq 0 \), and we use the convention that \( D(P) = P^{2r-1} \) for \( l = 0 \). Then, we have

\[
E \left[ m^d G_{i_1}^d \left( \prod_{j=1}^k G_{v_j,v_j}^{u_j} \right) D(P) \right] = E \left[ m^{d+1} G_{i_1}^{d+1} \left( \prod_{j=1}^k G_{v_j,v_j}^{u_j} \right) D(P) \right] - \sum_{0 \leq s \leq \ell} \sum_{[p:s]_s} \frac{1}{N^2} \sum_{x \neq y} E \left[ m^{d+1} G_{x,x}^{s_1+1} G_{y,y}^{s_1+1} G_{i_1}^{s_1+1} \left( \prod_{j=1}^k G_{v_j,v_j}^{u_j} \right) (\partial_{xy}^s D(P)) \right]
\]

\[
+ \sum_{0 \leq s \leq \ell} c_s \sum_{[p:s]_s} \frac{1}{N} \sum_{x} E \left[ m^{d+1} G_{x,x}^{s_1+1} G_{i_1}^{s_1+1} \left( \prod_{j=1}^k G_{v_j,v_j}^{u_j} \right) \left( \partial_{xx}^s D(P) \right) \right] + O_{\prec}(\Phi_r),
\]

where each coefficient \( c_s \) is as in Lemma 4.4 and,

\[
\sum_{[p:s]_s} = \begin{cases} \sum_{p=2}^\ell \frac{p}{1} \frac{(1)_{s+1}}{q^{s+1}} & s = 1, \\ \sum_{p=s}^\ell \frac{p}{s} \frac{(s)_{s+1}}{q^{s+1}} & s > 1. \end{cases}
\]
Proof. We can prove this following the proof of Claim 4.5.

\[ G_{ii} = m + \frac{G_{ii}}{N} \sum_{x,y} h_{xy} G_{yx} - m \sum_x h_{ix} G_{xi} \]

Using the identity (4.27), we have

\[
(4.38) \quad m^d G_{ii}^{t} \left( \prod_{j=1}^{k} G_{uv_{ij}}^{t_j} \right) D(P) = m^{d+1} G_{ii}^{t-1} \left( \prod_{j=1}^{k} G_{uv_{ij}}^{t_j} \right) D(P)
\]

\[ + \frac{1}{N} \sum_{x,y} h_{xy} m^d G_{yx} G_{ii}^{t} \left( \prod_{j=1}^{k} G_{uv_{ij}}^{t_j} \right) D(P) - \sum_x h_{ix} m^{d+1} G_{xi} G_{ii}^{t-1} \left( \prod_{j=1}^{k} G_{uv_{ij}}^{t_j} \right) D(P). \]

Using Lemma 4.4 directly, we obtain

\[
\frac{1}{N} \sum_{x,y} \mathbb{E} h_{xy} m^d G_{yx} G_{ii}^{t} \left( \prod_{j=1}^{k} G_{uv_{ij}}^{t_j} \right) D(P)
\]

\[ = \sum_{p=1}^{\ell} \frac{c_{p+1}}{N^{q^{p-1}}} \sum_{x,y} \mathbb{E} \partial_{xy}^p \left( m^d G_{yx} G_{ii}^{t} \left( \prod_{j=1}^{k} G_{uv_{ij}}^{t_j} \right) D(P) \right) + \mathcal{O}_< (\Phi_r), \]

and

\[
\sum_x \mathbb{E} h_{ix} m^{d+1} G_{xi} G_{ii}^{t-1} \left( \prod_{j=1}^{k} G_{uv_{ij}}^{t_j} \right) D(P)
\]

\[ = \sum_{p=1}^{\ell} \frac{c_{p+1}}{N^{q^{p-1}}} \sum_x \mathbb{E} \partial_{ix}^p \left( m^{d+1} G_{xi} G_{ii}^{t-1} \left( \prod_{j=1}^{k} G_{uv_{ij}}^{t_j} \right) D(P) \right) + \mathcal{O}_< (\Phi_r). \]

Using Lemma 4.3 directly, we obtain

\[
(4.39) \quad \sum_{p=1}^{\ell} \frac{c_{p+1}}{N^{q^{p-1}}} \sum_{x,y} \mathbb{E} \partial_{xy}^p \left( m^d G_{yx} G_{ii}^{t} \left( \prod_{j=1}^{k} G_{uv_{ij}}^{t_j} \right) D(P) \right)
\]

\[ - \sum_{p=1}^{\ell} \frac{c_{p+1}}{N^{q^{p-1}}} \sum_x \mathbb{E} \partial_{ix}^p \left( m^{d+1} G_{xi} G_{ii}^{t-1} \left( \prod_{j=1}^{k} G_{uv_{ij}}^{t_j} \right) D(P) \right)
\]

\[ = - \sum_{s=1}^{\ell} \sum_{p=s}^{\ell} \frac{(p!)c_{p+1}}{N^{q^{p-1}}} \sum_{x \neq y} \mathbb{E} m^{d+1} G_{xx} G_{iy} G_{ii}^{t-1} \left( \prod_{j=1}^{k} G_{uv_{ij}}^{t_j} \right) \partial_{xy}^{p-s} D(P)
\]

\[ + \sum_{0 \leq s \leq \ell} \sum_{s=1}^{\ell} \frac{(p!)c_{p+1}}{N^{q^{p-1}}} \sum_x \mathbb{E} m^{d+1} G_{xx} G_{iy} G_{ii}^{t-1} \left( \prod_{j=1}^{k} G_{uv_{ij}}^{t_j} \right) \partial_{xx}^{p-s} D(P) + \mathcal{O}_< (\Phi_r), \]
where each $c_s$ is as in Lemma 4.4. For the case that $p = 1$ and $s = 1$, we find a cancellation in the right-hand sides of (4.39) as follows: using the fact that $c_s = 1$ for $s = 1$, we have

\[
\frac{1}{N^2} \sum_{x \neq y} E \left[ m^d G_{x,x} G_{y,y} G_{ii}^t \left( \prod_{j=1}^k G_{v_{i,j}}^u \right) D(P) \right] - \frac{1}{N} \sum_x E \left[ m^{d+1} G_{x,x} G_{ii}^t \left( \prod_{j=1}^k G_{v_{i,j}}^u \right) D(P) \right] \\
= \frac{1}{N^2} \sum_{x,y} E \left[ m^d G_{x,x} G_{y,y} G_{ii}^t \left( \prod_{j=1}^k G_{v_{i,j}}^u \right) D(P) \right] - E \left[ m^{d+2} G_{ii}^t \left( \prod_{j=1}^k G_{v_{i,j}}^u \right) D(P) \right] + \mathcal{O}_\prec (\Phi_r) \\
= E \left[ m^{d+2} G_{ii}^t \left( \prod_{j=1}^k G_{v_{i,j}}^u \right) D(P) \right] - E \left[ m^{d+2} G_{ii}^t \left( \prod_{j=1}^k G_{v_{i,j}}^u \right) D(P) \right] + \mathcal{O}_\prec (\Phi_r) \\
= \mathcal{O}_\prec (\Phi_r),
\]

where we use Proposition 4.2 and [17 Proposition A.1] to switch over from $\sum_{x \neq y}$ to $\sum_{x,y}$ . Due to the above cancellation, we get

\[
(4.40) \quad - \sum_{0 \leq s \leq \ell} \sum_{p=0}^\ell \frac{(p)}{N^2} q^{p-1} \sum_{x \neq y} E \left[ m^d G_{x,x} G_{y,y} G_{ii}^t \left( \prod_{j=1}^k G_{v_{i,j}}^u \right) \left( \partial_{p-x}^{p-s} D(P) \right) \right] \\
+ \sum_{0 \leq s \leq \ell} \sum_{p=0}^\ell \frac{(s)C_p}{N^2} q^{p-1} \sum_{x \neq y} E \left[ m^{d+1} G_{x,x} G_{y,y} G_{ii}^t \left( \prod_{j=1}^k G_{v_{i,j}}^u \right) \left( \partial_{p-x}^{p-s} D(P) \right) \right] \\
= - \sum_{0 \leq s \leq \ell} \sum_{[p]; s} \frac{1}{N^2} \sum_{x \neq y} E \left[ m^d G_{x,x} G_{y,y} G_{ii}^t \left( \prod_{j=1}^k G_{v_{i,j}}^u \right) \left( \partial_{p-x}^{p-s} D(P) \right) \right] \\
+ \sum_{0 \leq s \leq \ell} \sum_{[p]; s} \frac{C_s}{N} \sum_{x} E \left[ m^{d+1} G_{x,x} G_{ii}^t \left( \prod_{j=1}^k G_{v_{i,j}}^u \right) \left( \partial_{p-x}^{p-s} D(P) \right) \right] + \mathcal{O}_\prec (\Phi_r). 
\]

The lemma follows from (4.38), (4.39) and (4.40). \hfill \Box

Using Lemma 4.9 we can replace all diagonal entries in the left-hand side of (4.37) one by one.

**Proposition 4.10.** Suppose Assumption 2.4 and Assumption 2.5 hold. Let $s_1$ be a positive odd integer such that $s_1 \leq \ell$. The difference,

\[
\sum_{[p_1; s_1]} \frac{1}{N^{1+\Theta(2)}} \sum_{x_1} E \left[ G_{i_1,i_1}^{s_1+1} G_{j_1,j_1}^{s_1+1} \left( \partial_{p_1-s_1} D^{2r-1} \right) \right] - \sum_{[p_1; s_1]} \frac{1}{N^{1+\Theta(2)}} \sum_{x_1} E \left[ m^{s_1+1} \left( \partial_{p_1-s_1} D^{2r-1} \right) \right],
\]

is a linear combination of the terms of the following form (with bounded coefficients):

\[
\sum_{[p_2; s_2]} \sum_{[p_1; s_1]} \sum_{x} E[T] + \mathcal{O}_\prec (\Phi_r),
\]

where $s_2$ is a positive odd integer such that $s_2 \leq \ell$, we have $\mathcal{I}_2 \subset J_2$ (refer to (4.30) for the definition of $J_k$), and for some $0 \leq \alpha_1 \leq \frac{s_1-1}{2}$,

\[
T = \frac{1}{N^{2+\Theta(2)}} G_{i_2,i_2}^{s_2+1} G_{j_2,j_2}^{s_2+1} G_{i_1,i_1}^{s_1+1} G_{j_1,j_1}^{s_1+1} \left( \partial_{p_2-s_2} D^{2r-1} \right),
\]

or

\[
T = \frac{1}{N^{2+\Theta(2)}} G_{i_2,i_2}^{s_2+1} G_{j_2,j_2}^{s_2+1} G_{i_1,i_1}^{s_1+1} G_{j_1,j_1}^{s_1+1} \left( \partial_{p_2-s_2} D^{2r-1} \right).
\]

(Refer to (4.31) for the definition of $\sum_{[p]; s_1]$.)

In fact, the length of the linear combination can be bounded by a large constant only depending on $\ell$ but not $N$. 
Proof. This time we replace $G_{j_1,j_1}$ first. By symmetry, it follows from Claim 4.7 that the difference,

$$
\sum_{[p_1;s_1]_1} \frac{1}{N^{1+\delta(z_1)}} \sum_{z_1} \mathbb{E} \left[ G_{i_1 i_1}^{s_1+1} G_{j_1 j_1}^{s_1-1} (\partial \psi_{i_1 i_1}^{p_1-1}s_1 p^{2r-1}) \right] - \sum_{[p_1;s_1]_1} \frac{1}{N^{1+\delta(z_1)}} \sum_{z_1} \mathbb{E} \left[ G_{i_1 i_1}^{s_1+1} G_{j_1 j_1}^{s_1-1} m(\partial \psi_{j_1 j_1}^{p_1-1}s_1 p^{2r-1}) \right],
$$

is a linear combination of the terms of the form (4.41) (with bounded coefficients) where $s_2 \equiv 1(\text{mod } 2)$, $0 \leq s_2 \leq \ell$, $I_2 \in J_2$ and the term $T$ has the form

$$
\frac{1}{N^{2+\delta(z_2)}} G_{i_2 i_2}^{s_2+1} G_{j_2 j_2}^{s_2-1} G_{i_1 i_1}^{s_1+1} G_{j_1 j_1}^{s_1-1} m_{I_2 \equiv j_2}^{1} \left( \partial \psi_{i_2 i_2}^{p_2-2s_2} \partial \psi_{j_1 j_1}^{p_1-1}s_1 p^{2r-1} \right).
$$

According to Claim 4.7, we can find that the length of the linear combination can be bounded by a large constant only depending on $\ell$.

Next, in order to replace one more diagonal entry, we apply Lemma 4.9 to

$$
\sum_{[p_1;s_1]_1} \frac{1}{N^{1+\delta(z_1)}} \sum_{z_1} \mathbb{E} \left[ G_{i_1 i_1}^{s_1+1} G_{j_1 j_1}^{s_1-1} m(\partial \psi_{i_1 i_1}^{p_1-1}s_1 p^{2r-1}) \right].
$$

We assume $\frac{s_1-1}{2} \geq 1$. (Otherwise we will use Lemma 4.9 to replace $G_{i_1,i_1}$.) Setting $d = 1$, $t = \frac{s_1-1}{2}$ and $G_{i_2} = G_{j_2,j_1}$ in Lemma 4.9 we find that the difference,

$$
\sum_{[p_1;s_1]_1} \frac{1}{N^{1+\delta(z_1)}} \sum_{z_1} \mathbb{E} \left[ G_{i_1 i_1}^{s_1+1} G_{j_1 j_1}^{s_1-1} m(\partial \psi_{j_1 j_1}^{p_1-1}s_1 p^{2r-1}) \right] - \sum_{[p_1;s_1]_1} \frac{1}{N^{1+\delta(z_1)}} \sum_{z_1} \mathbb{E} \left[ G_{i_1 i_1}^{s_1+1} G_{j_1 j_1}^{s_1-3} m(\partial \psi_{j_1 j_1}^{p_1-1}s_1 p^{2r-1}) \right],
$$

is, again, a linear combination of the terms of the form (4.41) (with bounded coefficients) where $s_2 \equiv 1(\text{mod } 2)$, $0 \leq s_2 \leq \ell$, $I_2 \in J_2$ and the term $T$ has the form

$$
\frac{1}{N^{2+\delta(z_2)}} G_{i_2 i_2}^{s_2+1} G_{j_2 j_2}^{s_2-1} G_{i_1 i_1}^{s_1+1} G_{j_1 j_1}^{s_1-1} m_{I_2 \equiv j_2}^{1} \left( \partial \psi_{i_2 i_2}^{p_2-2s_2} \partial \psi_{j_1 j_1}^{p_1-1}s_1 p^{2r-1} \right).
$$

The length of the linear combination also can be bounded by a large constant only depending on $\ell$.

We can apply Lemma 4.9 this way repetitively until there is no diagonal entry to replace. Then, the desired result follows. \(\square\)

4.4. Step 4. Iterations. We shall use the approach of [13]. In this subsection, we will show that the main term (4.1) can be approximated by a linear combination of the terms which have a certain form. Let us set

$$
T_1 = T(s_1, p_1) := \frac{1}{N^{1+\delta(z_1)}} G_{i_1 i_1}^{s_1+1} G_{j_1 j_1}^{s_1-1} m(\partial \psi_{i_1 i_1}^{p_1-1}s_1 p^{2r-1}).
$$

According to Proposition 4.10 we obtain

$$
\sum_{[p_1;s_1]_1} \sum_{I_1} \mathbb{E} \left[ T_1 \right] = \sum_{[p_1;s_1]_1} \frac{1}{N^{1+\delta(z_1)}} \sum_{I_1} \mathbb{E} \left[ m_{s_1+1} (\partial \psi_{i_1 i_1}^{p_1-1}s_1 p^{2r-1}) \right] + \sum_{[p_2;s_2]_2} \sum_{[p_1;s_1]_1} \sum_{I_2} \mathbb{E} \left[ T_2^{(f)} \right] + \mathcal{O}(\Phi_r),
$$

where $s_2 \equiv 1(\text{mod } 2)$, $0 \leq s_2 \leq \ell$, $I_2 \in J_2$ (refer to (3.30) for the definition of $J_k$), the summation $\sum_{I_1}$ has a finite length which depends on $\ell$ but not $N$, the coefficient $c_1$ is bounded for every $t$, and $T_2^{(f)}$ in the summation has the following forms, for some $0 \leq \alpha_1 \leq \frac{s_1-1}{2}$,

$$
\frac{1}{N^{2+\delta(z_2)}} G_{i_2 i_2}^{s_2+1} G_{j_2 j_2}^{s_2-1} G_{i_1 i_1}^{s_1+1} G_{i_1 i_1}^{s_1-1} m_{I_2 \equiv j_2}^{1} \left( \partial \psi_{i_2 i_2}^{p_2-2s_2} \partial \psi_{j_1 j_1}^{p_1-1}s_1 p^{2r-1} \right),
$$

or

$$
\frac{1}{N^{2+\delta(z_2)}} G_{i_2 i_2}^{s_2+1} G_{j_2 j_2}^{s_2-1} G_{i_1 i_1}^{s_1+1} G_{i_1 i_1}^{s_1-1} m_{I_2 \equiv j_2}^{1} \left( \partial \psi_{i_2 i_2}^{p_2-2s_2} \partial \psi_{j_1 j_1}^{p_1-1}s_1 p^{2r-1} \right).
$$

Note that

$$
\left| \sum_{t} \sum_{[p_2;s_2]_2} \sum_{[p_1;s_1]_1} \sum_{I_2} \mathbb{E} \left[ T_2^{(f)} \right] \right| < \frac{1}{q} \max_{s \geq 0} \mathbb{E} \left[ \left( \frac{\partial \psi P(\frac{3m(z)}{N\eta}) + (\frac{3m(z)}{N\eta})^2 + \frac{1}{N} \right)^{s} |P|^{2r-s-1} \right],
$$

because in the summation $\sum_{[p_2;s_2]_2}$, we have the factor $q^{p_2-1}$ with $p_2 > 1$, and also we can use (4.6).

Remark 4.11. We will denote by $c_t$ a coefficient of a linear combination. The coefficient $c_t$ may differ whenever it occurs. It is an abuse of notation for convenience.
Pick a single $T_{2}^{(t)}$ and drop the the superscript $(t)$ for brevity. Following the proof of Proposition 4.10, we replace all diagonal entries in $T_{2}$ using Lemma 4.9 and keep track of remaining terms. It follows that

$$\sum_{[p_{2};s_{2}]} \sum_{[p_{1};s_{1}]} \sum_{I_{2}} \mathbb{E} \left[ T_{2} \right] = \sum_{[p_{2};s_{2}]} \sum_{[p_{1};s_{1}]} \frac{1}{N^{2+\theta}(I_{2})} \sum_{I_{2}} \mathbb{E} \left[ m^{s_{1}+s_{2}+2} \left( \partial_{p_{2}}^{s_{2}} \partial_{p_{1}}^{s_{1}} p^{2r-1} \right) \right]$$

$$+ \sum_{t} c_{t} \sum_{[p_{3};s_{3}]} \sum_{[p_{2};s_{2}]} \sum_{[p_{1};s_{1}]} \sum_{I_{3}} \mathbb{E} \left[ T_{3}^{(t)} \right] + O_{\gamma}(\Phi_{r}),$$

where $s_{3} \equiv 1 \pmod{2}$, $0 \leq s_{3} \leq \ell$, $I_{3} \in \mathcal{I}_{3}$, the summation $\sum_{t}$ has a finite length which depends on $\ell$ but not $N$, the coefficient $c_{t}$ is bounded for every $t$, and $T_{3}^{(t)}$ in the summation has the following forms, for some $0 \leq \alpha_{1} \leq \frac{s_{1}-1}{2}$ and $0 \leq \alpha_{2} \leq \frac{s_{2}-1}{2}$,

$$\frac{1}{N^{3+\theta}(I_{3})} G^{s_{2}+1}_{i_{3}j_{3}} G^{s_{3}+1}_{j_{3}i_{3}} G^{s_{1}+1}_{i_{2}j_{2}} G^{s_{1}+1}_{j_{2}i_{2}} G^{s_{1}+1}_{j_{1}i_{1}} \left( \partial_{p_{3}}^{s_{3}} \partial_{p_{2}}^{s_{2}} \partial_{p_{1}}^{s_{1}} p^{2r-1} \right),$$

$$\frac{1}{N^{3+\theta}(I_{3})} G^{s_{2}+1}_{i_{3}j_{3}} G^{s_{3}+1}_{j_{3}i_{3}} G^{s_{1}+1}_{i_{2}j_{2}} G^{s_{1}+1}_{j_{2}i_{2}} G^{s_{1}+1}_{j_{1}i_{1}} \left( \partial_{p_{3}}^{s_{3}} \partial_{p_{2}}^{s_{2}} \partial_{p_{1}}^{s_{1}} p^{2r-1} \right),$$

$$\frac{1}{N^{3+\theta}(I_{3})} G^{s_{2}+1}_{i_{3}j_{3}} G^{s_{3}+1}_{j_{3}i_{3}} G^{s_{1}+1}_{i_{2}j_{2}} G^{s_{1}+1}_{j_{2}i_{2}} \left( \partial_{p_{3}}^{s_{3}} \partial_{p_{2}}^{s_{2}} \partial_{p_{1}}^{s_{1}} p^{2r-1} \right),$$

or

$$\frac{1}{N^{3+\theta}(I_{3})} G^{s_{2}+1}_{i_{3}j_{3}} G^{s_{3}+1}_{j_{3}i_{3}} G^{s_{1}+1}_{i_{2}j_{2}} G^{s_{1}+1}_{j_{2}i_{2}} \left( \partial_{p_{3}}^{s_{3}} \partial_{p_{2}}^{s_{2}} \partial_{p_{1}}^{s_{1}} p^{2r-1} \right).$$

Note that

$$\sum_{t} \sum_{[p_{3};s_{3}]} \sum_{[p_{2};s_{2}]} \sum_{[p_{1};s_{1}]} \sum_{I_{2}} \sum_{i_{3}j_{3}} \mathbb{E} \left[ T_{3}^{(t)} \right] < \frac{1}{q^{2}} \max_{s \geq 0} \mathbb{E} \left[ \left( \frac{\partial_{p}^{s} \mathbb{E} \left[ 3m(\tilde{z}) \right]}{N^{\eta}} + \frac{3m(\tilde{z})}{N^{\eta}} \right)^{2} + \frac{1}{N} \right]^{s} |P|^{2r-s-1},$$

where we use (4.16) and the fact that in the summation $\sum_{[p_{3};s_{3}]}$, if $l \geq 2$, we have the factor $q^{l}-1$ with $p_{l} > 1$.

We shall iterate this procedure until the remaining terms are absorbed into $O_{\gamma}(\Phi_{r})$ by getting many factors of $q^{-1}$. We assume that $I_{k} \in \mathcal{J}_{k}$ and $T_{k}$ has the following forms, for $1 \leq l \leq k - 1$ and $0 \leq \alpha_{l} \leq \frac{s_{l}-1}{2}$.

$$\frac{1}{N^{k+\theta}(I_{k})} G^{s_{k}+1}_{i_{k}j_{k}} G^{s_{k-1}+1}_{j_{k}i_{k}} \ldots G^{s_{1}+1}_{j_{1}i_{1}} G^{s_{1}+1}_{j_{1}i_{1}} \left( \partial_{p_{k}}^{s_{k}} \partial_{p_{k-1}}^{s_{k-1}} \partial_{p_{1}}^{s_{1}} p^{2r-1} \right),$$

or

$$\frac{1}{N^{k+\theta}(I_{k})} G^{s_{k}+1}_{i_{k}j_{k}} G^{s_{k-1}+1}_{j_{k}i_{k}} \ldots G^{s_{1}+1}_{j_{1}i_{1}} G^{s_{1}+1}_{j_{1}i_{1}} \left( \partial_{p_{k}}^{s_{k}} \partial_{p_{k-1}}^{s_{k-1}} \partial_{p_{1}}^{s_{1}} p^{2r-1} \right),$$

where $s_{1}, \ldots, s_{k}$ are positive odd integers such that $s_{l} \leq \ell$ for every $1 \leq l \leq k$. Then, we have the following claim.

Claim 4.12. Let $k$ be a positive integer. Let $s_{1}, \ldots, s_{k}$ be positive odd integers such that $s_{l} \leq \ell$ for every $1 \leq l \leq k$. Assume $T_{k}$ has the form (4.42) or (4.43) for $1 \leq l \leq k - 1$ and $0 \leq \alpha_{l} \leq \frac{s_{l}-1}{2}$. Consider $I_{k} \in \mathcal{J}_{k}$ (refer to (4.50) for the definition of $\mathcal{J}_{k}$). Then, it follows that

$$\sum_{[p_{k};s_{k}]} \sum_{[p_{k-1};s_{k-1}]} \sum_{[p_{1};s_{1}]} \sum_{I_{k}} \mathbb{E} \left[ T_{k} \right]$$

$$= \sum_{[p_{k};s_{k}]} \sum_{[p_{k-1};s_{k-1}]} \sum_{[p_{1};s_{1}]} \sum_{[p_{1};s_{1}]} \sum_{I_{k}} \mathbb{E} \left[ m^{s_{1}+s_{k}+k} \left( \partial_{p_{k}}^{s_{k}} \partial_{p_{k-1}}^{s_{k-1}} \partial_{p_{1}}^{s_{1}} p^{2r-1} \right) \right]$$

$$+ \sum_{t} \sum_{[p_{k+1};s_{k+1}]} \sum_{[p_{k};s_{k}]} \sum_{[p_{1};s_{1}]} \sum_{I_{k+1}} \mathbb{E} \left[ T_{k+1}^{(t)} \right] + O_{\gamma}(\Phi_{r}),$$

where $s_{3} \equiv 1 \pmod{2}$, $0 \leq s_{3} \leq \ell$, $I_{3} \in \mathcal{I}_{3}$, the summation $\sum_{t}$ has a finite length which depends on $\ell$ but not $N$, the coefficient $c_{t}$ is bounded for every $t$, and $T_{3}^{(t)}$ in the summation has the following forms, for some $0 \leq \alpha_{1} \leq \frac{s_{1}-1}{2}$ and $0 \leq \alpha_{2} \leq \frac{s_{2}-1}{2}$,
where \( s_{k+1} \equiv 1 \pmod{2} \), \( 0 \leq s_{k+1} \leq \ell \), \( \mathcal{I}_{k+1} \in \mathcal{J}_{k+1} \), the summation \( \sum_{t} \) has a finite length which depends on \( \ell \) but not \( N \), the coefficient \( c_t \) is bounded for every \( t \), and \( T_{k+1}^{(t)} \) has the form (4.42) or (4.43) (replacing \( k \) with \( k+1 \)) for \( 1 \leq l \leq k \) and \( 0 \leq c_t \leq \frac{s_{k+1}-1}{2} \). (Refer to (4.31) for the definition of \( \sum_{[p; s_1]} \).)

In addition, we have

\[
(4.45) \quad \left| \sum_{t} \sum_{[p_{k+1}; s_{k+1}]} \sum_{[p_k; s_k]} \ldots \sum_{[p_1; s_1]} \sum_{\mathcal{I}_{k+1}} \mathbb{E} \left[ T_{k+1}^{(t)} \right] \right| < \frac{1}{q^k} \max_{s \geq 0} \mathbb{E} \left[ \left( \frac{2N}{N^2} \frac{\Theta m(\hat{\zeta})}{N^2 \eta} + \left( \frac{\Theta m(\hat{\zeta})}{N^2 \eta} \right)^2 + \frac{1}{N} \right)^s \right] |P^{2r-s-1}|.
\]

**Proof.** As in Proposition 4.10, the estimate (4.44) follows from Lemma 4.9. If \( l \geq 2 \), we can get the factor \( q^{p_1-1} \) with \( p_1 > 1 \) in the summation \( \sum_{[p_1; s_1]} \), which implies the other estimate (4.45) due to (4.6).

Now we are ready to show that the main term (4.1) can be written as a linear combination.

**Proposition 4.13.** Suppose Assumption 2.2 and Assumption 2.5 hold. Consider the term (4.1):

\[
\sum_{p=1}^{\ell} \frac{C_{p+1}}{N^2 q^{p-1}} \sum_{i,j} \mathbb{E} \left[ \partial_{ij}^p (G_{ij} P^{2r-1}) \right].
\]

If \( \ell \) is large enough, then the term (4.1) is a linear combination of the terms of the following form (with bounded coefficients):

\[
\sum_{[p_1; s_1]} \sum_{[p_2; s_2]} \ldots \sum_{[p_k; s_k]} \sum_{\mathcal{I}_k} \mathbb{E} \left[ M \right] + O_\omega(\Phi_r), \quad 1 \leq k \leq \ell,
\]

where for \( 1 \leq l \leq k \), each \( s_l \) is a positive odd integer such that \( s_l \leq \ell \), we have \( \mathcal{I}_k \in \mathcal{J}_k \) (refer to (4.30) for the definition of \( \mathcal{J}_k \)), and \( M \) has the form

\[
M = \frac{1}{N^{k+\theta(\mathcal{I}_k)}} \left( \Theta m_{s_1+\ldots+s_k+k} (\partial_{i_1 j_1}^{p_1-s_1} \ldots \partial_{i_{1+j_1}}^{p_1-s_1} P^{2r-1}) \right).
\]

(Refer to (4.31) for the definition of \( \sum_{[p_1; s_1]} \).

In fact, the length of the linear combination can be bounded by a large constant only depending on \( \ell \) but not \( N \).

**Proof.** By Proposition 4.3, the term (4.1) is a linear combination of the terms of the following form:

\[
\sum_{[p_1; s_1]} \sum_{\mathcal{I}_1} \mathbb{E} \left[ T_1 \right] + O_\omega(\Phi_r),
\]

where \( s_1 \equiv 1 \pmod{2} \), \( 0 \leq s_1 \leq \ell \), \( \mathcal{I}_1 \in \mathcal{J}_1 \) and \( T_1 \) has the form

\[
T_1 = \frac{1}{N^{1+\theta(\mathcal{I}_1)}} \left( \Theta m_{s_1+1} (\partial_{i_1 j_1}^{p_1-s_1} P^{2r-1}) \right).
\]

The length of the linear combination can be bounded by \( \ell \).

Applying Claim 4.12, we replace all diagonal entries in \( T_1 \) and observe that the difference,

\[
\sum_{[p_1; s_1]} \sum_{\mathcal{I}_1} \mathbb{E} \left[ T_1 \right] - \sum_{[p_1; s_1]} \frac{1}{N^{1+\theta(\mathcal{I}_1)}} \sum_{\mathcal{I}_1} \sum_{[p_1; s_1]} \mathbb{E} \left[ m_{s_1+1} (\partial_{i_1 j_1}^{p_1-s_1} P^{2r-1}) \right],
\]

is a linear combination of the terms of the form

\[
(4.46) \quad \sum_{[p_2; s_2]} \sum_{[p_1; s_1]} \sum_{\mathcal{I}_2} \mathbb{E} \left[ T_2 \right] + O_\omega(\Phi_r),
\]

where \( s_2 \equiv 1 \pmod{2} \), \( 0 \leq s_2 \leq \ell \), \( \mathcal{I}_2 \in \mathcal{J}_2 \), and \( T_2 \) has the form (4.42) or (4.43) with \( k = 2 \). One important thing is that

\[
\left| \sum_{[p_2; s_2]} \sum_{[p_1; s_1]} \sum_{\mathcal{I}_2} \mathbb{E} \left[ T_2 \right] \right| < \frac{1}{q^{s \geq 0}} \mathbb{E} \left[ \left( \frac{2N}{N^2} \frac{\Theta m(\hat{\zeta})}{N^2 \eta} + \left( \frac{\Theta m(\hat{\zeta})}{N^2 \eta} \right)^2 + \frac{1}{N} \right)^s \right] |P^{2r-s-1}|.
\]
Using Claim 4.12 again, we replace all diagonal entries in $T_2$ and observe that the difference,
\[ \sum_{[p_2:s_2]} \sum_{[p_1:s_1]} \sum_{I_2} \mathbb{E}[T_2] - \sum_{[p_2:s_2]} \sum_{[p_1:s_1]} \frac{1}{N^{2+\theta}(I_2)} \sum_{I_1} \mathbb{E} \left[ m^{s_1+s_2+2}(\partial_{i_2}^{p_2-s_2} \partial_{i_1}^{p_1-s_1} P^{2r-1}) \right], \]
is a linear combination of the terms of the form
\[ (4.47) \quad \sum_{[p_3:s_3]} \sum_{[p_2:s_2]} \sum_{[p_1:s_1]} \sum_{I_3} \mathbb{E}[T_3] + O_{\prec}(\Phi_r), \]
where $s_3 \equiv 1(\text{mod } 2), 0 \leq s_3 \leq \ell, I_3 \in \mathcal{J}_3,$ and $T_3$ has the form \((4.42)\) or \((4.43)\) with $k = 3$. Note that
\[ \left| \sum_{[p_3:s_3]} \sum_{[p_2:s_2]} \sum_{[p_1:s_1]} \sum_{I_3} \mathbb{E}[T_3] \right| \leq \frac{1}{q^2} \max_{s \geq 0} \mathbb{E} \left[ \left( \frac{\partial_2 P^3 m(\tilde{z})}{N \eta} + \left( \frac{3m(\tilde{z})}{N \eta} \right)^2 \right] \frac{1}{N} |P|^{2r-s-1} \right]. \]

In this way, we use Claim 4.12 to replace all diagonal entries in $T_k$ and show that the difference,
\[ \sum_{[p_k:s_k]} \ldots \sum_{[p_1:s_1]} \sum_{I_k} \mathbb{E}[T_k] - \sum_{[p_k:s_k]} \ldots \sum_{[p_1:s_1]} \frac{1}{N^{k+\theta}(I_k)} \sum_{I_k} \mathbb{E} \left[ m^{s_1+\ldots+s_k+k}(\partial_{i_k}^{p_k-s_k} \ldots \partial_{i_1}^{p_1-s_1} P^{2r-1}) \right], \]
is a linear combination of the terms of the form
\[ (4.48) \quad \sum_{[p_k+1:s_{k+1}]} \ldots \sum_{[p_1:s_1]} \sum_{I_{k+1}} \mathbb{E}[T_{k+1}] + O_{\prec}(\Phi_r), \]
where $s_{k+1} \equiv 1(\text{mod } 2), 0 \leq s_{k+1} \leq \ell, I_{k+1} \in \mathcal{J}_{k+1},$ and $T_{k+1}$ has the form \((4.42)\) or \((4.43)\) (replacing $k$ with $k+1$).

After $\ell$-iterations of this argument using Claim 4.12, the term \((4.1)\) can be approximated by a linear combination of terms of the form
\[ \sum_{[p_k:s_k]} \ldots \sum_{[p_1:s_1]} \frac{1}{N^{k+\theta}(I_k)} \sum_{I_k} \mathbb{E}[T_k] = \sum_{[p_k+1:s_{k+1}]} \ldots \sum_{[p_1:s_1]} \sum_{I_{k+1}} \mathbb{E}[T_{k+1}] + O_{\prec}(\Phi_r), \]
and the error is also a linear combination of terms of the form
\[ \sum_{[p_{k+1}:s_{k+1}]} \ldots \sum_{[p_1:s_1]} \sum_{I_{k+1}} \mathbb{E}[T_{k+1}] = O_{\prec}(\Phi_r), \]
where $s_{k+1} \equiv 1(\text{mod } 2), 0 \leq s_{k+1} \leq \ell, I_{k+1} \in \mathcal{J}_{k+1},$ and $T_{k+1}$ has the form \((4.42)\) or \((4.43)\) with $k = \ell + 1$. Since $\ell$ is large enough so that we have
\[ \left| \sum_{[p_{k+1}:s_{k+1}]} \ldots \sum_{[p_1:s_1]} \sum_{I_{k+1}} \right| \leq \frac{1}{q^2} \max_{s \geq 0} \mathbb{E} \left[ \left( \frac{\partial_2 P^3 m(\tilde{z})}{N \eta} + \left( \frac{3m(\tilde{z})}{N \eta} \right)^2 \right] \frac{1}{N} |P|^{2r-s-1} \right] = O_{\prec}(\Phi_r), \]
the desired result follows.

4.5. Step 5. Random correction terms. In this subsection, our first goal is to show
\[ (4.49) \quad \sum_{[p_k:s_k]} \ldots \sum_{[p_1:s_1]} \frac{1}{N^{k+\theta}(I_k)} \sum_{I_k} \mathbb{E} \left[ m^{s_1+\ldots+s_k+k}(\partial_{i_k}^{p_k-s_k} \ldots \partial_{i_1}^{p_1-s_1} P^{2r-1}) \right] = \mathbb{E} \left[ \frac{1}{N^{\theta}(I_k)} \left( \sum_{s_{l+1}}^{k} A_l(s_l) \right)^{m^{s_1+\ldots+s_k+k} P^{2r-1}} + O_{\prec}(\Phi_r) \right], \]
where $s_1, s_2, \ldots, s_k$ are positive odd integers, $I_k \in \mathcal{J}_k,$ and the term $A_l(s_l)$ is defined through
\[ A_l(s_l) := \begin{cases} h_{i_l}^2 - \mathbb{E}\left[h_{i_l}^2\right] & l > 1 \text{ and } s_l = 1, \\ h_{i_l}^{s_{l+1}} & l > 1 \text{ and } s_l \neq 1, \\ h_{i_l}^{s_{l+1}+1} & l > 1 \text{ and } s_l = 1. \end{cases} \]

Then, the term
\[ \frac{1}{N^{\theta}(I_k)} \left( \sum_{s_{l+1}}^{k} A_l(s_l) \right), \]
will be a building block of the random correction term $Z_n$ (as in Theorem 2.7) when $s_1 + \cdots + s_k + k = 2n$. Firstly, we consider the case $k = 1$.

Claim 4.14. Let $s_1$ be a positive odd integer. Consider $I_1 \in J_1$ (refer to (4.30) for the definition of $J_k$). Then, we have

$$\sum_{[p_1:s_1]} \frac{1}{N^{1+\theta(I_2)}} \sum_{I_1} \mathbb{E} \left[ m^{s_1+1} \left( \partial^{p_1-s_1}_{i_1j_1} P^{2r-1} \right) \right] = \mathbb{E} \left[ \frac{1}{N^{\theta(I_2)}} \left( \sum_{I_1} h^{s_1}_{i_1j_1} \right) m^{s_1+1} P^{2r-1} \right] + O_\prec (\Phi_r).$$

(Refer to (4.31) for the definition of $\sum_{[p:s]}$.)

Proof. Consider the first term of the right-hand side

$$\frac{1}{N^{\theta(I_2)}} \sum_{I_1} \mathbb{E} \left[ h^{s_1}_{i_1j_1} m^{s_1+1} P^{2r-1} \right].$$

Referring to (4.29) (for the definition of $\theta(I_k)$), we note that $\theta(I_1) = 1$. Using the cumulant expansion, we have

$$\frac{1}{N} \sum_{i_1 \neq j_1} \sum_{p_1=1}^\ell \frac{C_{p_1+1}}{N^{q^{p_1-1}}} \mathbb{E} \left[ \partial^{p_1}_{i_1j_1} \left( h^{s_1}_{i_1j_1} m^{s_1+1} P^{2r-1} \right) \right] + O_\prec (\Phi_r).$$

One remark here is that, if $\partial_{i_1j_1}$ does not hit $h_{i_1j_1}$ exactly $s_1$ times, then we can apply the cumulant expansion again with a remaining $h_{i_1j_1}$ so that we get an additional factor $1/N$ and the resulting terms are bounded by $O_\prec (\Phi_r)$. For example, we can think of the following case:

$$\frac{1}{N} \sum_{i_1 \neq j_1} \sum_{p_1=1}^\ell \frac{C_{p_1+1}}{N^{q^{p_1-1}}} \mathbb{E} \left[ \partial^{p_1}_{i_1j_1} \partial^{p_1}_{i_1j_1} \left( m^{s_1+1} P^{2r-1} \right) \right]$$

$$= \frac{1}{N} \sum_{i_1 \neq j_1} \sum_{p_1=1}^\ell \frac{C_{p_1+1}}{N^{q^{p_1-1}}} \sum_{p' = 1}^\ell \frac{C_{p'+1}}{N^{q^{p'-1}}} \mathbb{E} \left[ \partial^{p'_{i_1j_1}} \left( h^{s_1}_{i_1j_1} \partial^{p_1}_{i_1j_1} \left( m^{s_1+1} P^{2r-1} \right) \right) \right] + O_\prec (\Phi_r) = O_\prec (\Phi_r).$$

Thus, in order to get a non-negligible contribution, the derivative $\partial_{i_1j_1}$ must hit $h_{i_1j_1}$ exactly $s_1$ times so we have

$$\frac{1}{N} \sum_{i_1 \neq j_1} \mathbb{E} \left[ h^{s_1+1}_{i_1j_1} m^{s_1+1} P^{2r-1} \right] = \sum_{p_1=s_1}^\ell \left( \frac{p_1}{s_1} \right) \frac{C_{p_1+1}}{N^{q^{p_1-1}}} \sum_{i_1 \neq j_1} \mathbb{E} \left[ \partial^{p_1-s_1}_{i_1j_1} \left( m^{s_1+1} P^{2r-1} \right) \right] + O_\prec (\Phi_r).$$

If $\partial_{ij}$ hits the normalized trace $m$ at least once, then the resulting terms are absorbed into $O_\prec (\Phi_r)$ because of (4.5). As a result, we get

$$\sum_{p_1=s_1}^\ell \left( \frac{p_1}{s_1} \right) \frac{C_{p_1+1}}{N^{q^{p_1-1}}} \sum_{i_1 \neq j_1} \mathbb{E} \left[ \partial^{p_1-s_1}_{i_1j_1} \left( m^{s_1+1} P^{2r-1} \right) \right]$$

$$= \sum_{p_1=s_1}^\ell \left( \frac{p_1}{s_1} \right) \frac{C_{p_1+1}}{N^{q^{p_1-1}}} \sum_{i_1 \neq j_1} \mathbb{E} \left[ m^{s_1+1} \partial^{p_1-s_1}_{i_1j_1} \left( P^{2r-1} \right) \right] + O_\prec (\Phi_r)$$

$$= \sum_{[p_1:s_1]} \frac{1}{N^{1+\theta(I_2)}} \sum_{I_1} \mathbb{E} \left[ m^{s_1+1} \partial^{p_1-s_1}_{i_1j_1} \left( P^{2r-1} \right) \right] + O_\prec (\Phi_r),$$

and obtain the claim. (Refer to (4.31) for the definition of $\sum_{[p:s]}$.)

Next, we consider the case $k = 2$. \qed
Claim 4.15. Let $s_1$ and $s_2$ be positive odd integers. Consider $I_2 \in J_2$ (refer to (4.30) for the definition of $J_2$). Let $A_l$ be as in (4.30) for each $l = 1, 2$. Then, we have

$$\sum_{[p_2:s_2] \in [p_1:s_1]} \frac{1}{N^{2+\theta(\xi_2)}} \sum_{I_2} \mathbb{E} \left[ m_1 s_1 + s_2 + 2 \left( \partial_{p_2}^{p_1 - s_1} \partial_{i_1 j_1}^{-1} P_{2r-1}^2 \right) \right]$$

$$= \mathbb{E} \left[ \frac{1}{N^{\theta(\xi_2)}} \left( \sum_{I_2} A_1(s_1) A_2(s_2) \right) m_1 s_1 + s_2 + 2 P_{2r-1}^2 \right] + O_\prec (\Phi_r).$$

(Refer to (4.31) for the definition of $\sum_{[p_1:s_1]}$.)

Proof. If $s_2 = 1$, the first term of the right-hand side is,

$$\frac{1}{N^{\theta(\xi_2)}} \sum_{I_2} \mathbb{E} \left[ h_{1i_1j_1}^{s_1} \left( h_{i_2j_2}^2 - \mathbb{E} \left[ h_{i_2j_2}^2 \right] \right) m_1 s_1 + s_2 + 2 P_{2r-1}^2 \right].$$

Using the cumulant expansion with $h_{i_1j_1}$ and following the proof of Claim 4.14 we have

$$\frac{1}{N^{\theta(\xi_2)}} \sum_{I_2} \mathbb{E} \left[ h_{1i_1j_1}^{s_1} \left( h_{i_2j_2}^2 - \mathbb{E} \left[ h_{i_2j_2}^2 \right] \right) m_1 s_1 + s_2 + 2 P_{2r-1}^2 \right]$$

$$= \sum_{[p_1:s_1]} \frac{1}{N^{1+\theta(\xi_2)}} \sum_{I_2} \mathbb{E} \left[ (h_{i_2j_2}^2 - \mathbb{E} \left[ h_{i_2j_2}^2 \right]) m_1 s_1 + s_2 + 2 \left( \partial_{i_1j_1}^{-1} P_{2r-1}^2 \right) \right] + O_\prec (\Phi_r),$$

where we use that $\partial_{i_1j_1}(h_{i_2j_2}) = 0$ on the summation over distinct indexes $I_2$ (refer to (4.28) for the definition of $\sum_{I_2}$).

Applying the cumulant expansion with $h_{i_2j_2}$, we get

$$\sum_{[p_1:s_1]} \frac{1}{N^{1+\theta(\xi_2)}} \sum_{I_2} \mathbb{E} \left[ (h_{i_2j_2}^2 - \mathbb{E} \left[ h_{i_2j_2}^2 \right]) m_1 s_1 + s_2 + 2 \left( \partial_{i_1j_1}^{-1} P_{2r-1}^2 \right) \right]$$

$$= \sum_{[p_1:s_1]} \frac{1}{N^{1+\theta(\xi_2)}} \sum_{I_2} \sum_{p_2=1}^{\ell} \left\lfloor \frac{p_2}{2} \right\rfloor \frac{C_{p_2+1}}{N^{q_{p_2-1}}-1} \mathbb{E} \left[ \partial_{i_2j_2}^{p_2+1} \left( h_{i_2j_2}^2 m_1 s_1 + s_2 + 2 \left( \partial_{i_1j_1}^{-1} P_{2r-1}^2 \right) \right) \right]$$

$$- \sum_{[p_1:s_1]} \frac{1}{N^{1+\theta(\xi_2)}} \sum_{I_2} \mathbb{E} \left[ h_{i_2j_2}^2 m_1 s_1 + s_2 + 2 \left( \partial_{i_1j_1}^{-1} P_{2r-1}^2 \right) \right] + O_\prec (\Phi_r).$$

The derivative $\partial_{i_2j_2}$ have to hit $h_{i_2j_2}$ otherwise the resulting terms are absorbed into $O_\prec (\Phi_r)$, which implies

$$\sum_{[p_1:s_1]} \frac{1}{N^{1+\theta(\xi_2)}} \sum_{I_2} \sum_{p_2=1}^{\ell} \left\lfloor \frac{p_2}{2} \right\rfloor \frac{C_{p_2+1}}{N^{q_{p_2-1}}-1} \mathbb{E} \left[ \partial_{i_2j_2}^{p_2+1} \left( m_1 s_1 + s_2 + 2 \left( \partial_{i_1j_1}^{-1} P_{2r-1}^2 \right) \right) \right]$$

$$= \sum_{[p_1:s_1]} \frac{1}{N^{1+\theta(\xi_2)}} \sum_{I_2} \sum_{p_2=1}^{\ell} \left\lfloor \frac{p_2}{2} \right\rfloor \frac{C_{p_2+1}}{N^{q_{p_2-1}}-1} \mathbb{E} \left[ \partial_{i_2j_2}^{p_2+1} \left( m_1 s_1 + s_2 + 2 \left( \partial_{i_1j_1}^{-1} P_{2r-1}^2 \right) \right) \right] + O_\prec (\Phi_r).$$

We find the cancellation for $p_2 = 1$ as follows:

$$\sum_{[p_1:s_1]} \frac{1}{N^{1+\theta(\xi_2)}} \sum_{I_2} \sum_{p_2=1}^{\ell} \left\lfloor \frac{p_2}{2} \right\rfloor \frac{C_{p_2+1}}{N^{q_{p_2-1}}-1} \mathbb{E} \left[ \partial_{i_2j_2}^{p_2+1} \left( m_1 s_1 + s_2 + 2 \left( \partial_{i_1j_1}^{-1} P_{2r-1}^2 \right) \right) \right]$$

$$- \sum_{[p_1:s_1]} \frac{1}{N^{1+\theta(\xi_2)}} \sum_{I_2} \mathbb{E} \left[ h_{i_2j_2}^2 m_1 s_1 + s_2 + 2 \left( \partial_{i_1j_1}^{-1} P_{2r-1}^2 \right) \right]$$

$$= \sum_{[p_1:s_1]} \frac{1}{N^{1+\theta(\xi_2)}} \sum_{I_2} \sum_{p_2=2}^{\ell} \left\lfloor \frac{p_2}{2} \right\rfloor \frac{C_{p_2+1}}{N^{q_{p_2-1}}-1} \mathbb{E} \left[ \partial_{i_2j_2}^{p_2-1} \left( m_1 s_1 + s_2 + 2 \left( \partial_{i_1j_1}^{-1} P_{2r-1}^2 \right) \right) \right],$$

where we use that $C_2 = 1$ and $\mathbb{E} \left[ h_{i_2j_2}^2 \right] = N^{-1}$ by Definition 2.3.
If $\partial_{i_2j_2}$ hits the normalized trace $m$ at least once, then the resulting terms are absorbed into $O_\prec (\Phi_r)$ due to (4.5), which means

$$
\sum_{[p_1:s_1]} \frac{1}{N^{1+h/O_2}} \sum_{i_2} \sum_{p_2=2}^\ell \left( \begin{array}{c} p_2 \\ 1 \end{array} \right) \frac{C_{p_2+1}}{Nq^{p_2-1}} E \left[ \partial_{i_2j_2}^{p_2-1} \left( m^{s_1+s_2+2}(\partial_{i_1j_1}^{p_1-s_1} P^{2r-1}) \right) \right] = \sum_{[p_1:s_1]} \frac{1}{N^{1+O_2}} \sum_{i_2} \sum_{p_2=2}^\ell \left( \begin{array}{c} p_2 \\ 1 \end{array} \right) \frac{C_{p_2+1}}{Nq^{p_2-1}} E \left[ m^{s_1+s_2+2}(\partial_{i_2j_2}^{p_2-s_2}(\partial_{i_1j_1}^{p_1-s_1} P^{2r-1})) \right] + O_\prec (\Phi_r) \quad s_2 = 1,
$$

which concludes the proof for the case $s_2 = 1$. (Refer to (4.31) for the definition of $\sum_{[p,s]}$.)

If $s_2 > 1$, we consider

$$
\frac{1}{N^{O(2)}} \sum_{i_2} E \left[ h_{i_1j_1}^{s_1+1} h_{i_2j_2}^{s_2+1} m^{s_1+s_2+2} P^{2r-1} \right] = \sum_{[p_1:s_1]} \frac{1}{N^{1+O(2)}} \sum_{i_2} E \left[ h_{i_2j_2}^{s_2+1} m^{s_1+s_2+2}(\partial_{j_1j_1}^{p_1-s_1} P^{2r-1}) \right] + O_\prec (\Phi_r),
$$

where we use that $\partial_{j_1j_1} (h_{i_2j_2}) = 0$ on the summation over distinct indexes $\sum_{i_2}$.

By the cumulant expansion with $h_{i_2j_2}$, we have

$$
\sum_{[p_1:s_1]} \frac{1}{N^{1+O(2)}} \sum_{i_2} E \left[ h_{i_2j_2}^{s_2+1} m^{s_1+s_2+2}(\partial_{i_1j_1}^{p_1-s_1} P^{2r-1}) \right] = \sum_{[p_1:s_1]} \frac{1}{N^{1+O(2)}} \sum_{i_2} \sum_{p_2=1}^\ell \left( \begin{array}{c} p_2 \\ 1 \end{array} \right) E \left[ \partial_{i_2j_2}^{p_2} h_{i_2j_2}^{s_2+1} m^{s_1+s_2+2}(\partial_{i_1j_1}^{p_1-s_1} P^{2r-1}) \right] + O_\prec (\Phi_r).
$$

The derivative $\partial_{i_2j_2}$ have to hit $h_{i_2j_2}$ exactly $s_2$ times so it follows that

$$
\sum_{[p_1:s_1]} \frac{1}{N^{1+O(2)}} \sum_{i_2} \sum_{p_2=1}^\ell \frac{C_{p_2+1}}{Nq^{p_2-1}} E \left[ \partial_{i_2j_2}^{p_2} h_{i_2j_2}^{s_2+1} m^{s_1+s_2+2}(\partial_{i_1j_1}^{p_1-s_1} P^{2r-1}) \right] = \sum_{[p_1:s_1]} \frac{1}{N^{1+O(2)}} \sum_{i_2} \sum_{p_2=s_2}^\ell \left( \begin{array}{c} p_2 \\ s_2 \end{array} \right) \frac{C_{p_2+1}}{Nq^{p_2-1}} E \left[ \partial_{i_2j_2}^{p_2-s_2} m^{s_1+s_2+2}(\partial_{i_1j_1}^{p_1-s_1} P^{2r-1}) \right] + O_\prec (\Phi_r).
$$

If $\partial_{i_2j_2}$ hits the normalized trace $m$ at least once, then the resulting terms are absorbed into $O_\prec (\Phi_r)$ due to (4.5). Therefore we obtain the desired result:

$$
\sum_{[p_1:s_1]} \frac{1}{N^{1+O(2)}} \sum_{i_2} \sum_{p_2=s_2}^\ell \left( \begin{array}{c} p_2 \\ s_2 \end{array} \right) \frac{C_{p_2+1}}{Nq^{p_2-1}} E \left[ \partial_{i_2j_2}^{p_2-s_2} m^{s_1+s_2+2}(\partial_{i_1j_1}^{p_1-s_1} P^{2r-1}) \right] = \sum_{[p_1:s_1]} \frac{1}{N^{1+O(2)}} \sum_{i_2} \sum_{p_2=s_2}^\ell \left( \begin{array}{c} p_2 \\ s_2 \end{array} \right) \frac{C_{p_2+1}}{Nq^{p_2-1}} E \left[ m^{s_1+s_2+2}(\partial_{i_2j_2}^{p_2-s_2}(\partial_{i_1j_1}^{p_1-s_1} P^{2r-1})) \right] + O_\prec (\Phi_r) \quad s_2 > 1.
$$

□
Let us consider a general case.

**Proposition 4.16.** Suppose Assumption 2.4 and Assumption 2.5 hold. Let \( s_1, s_2, \ldots, s_k \) are positive odd integers. Consider \( I_k \in J_k \) (refer to (4.30) for the definition of \( J_k \)). Let \( A_l \) be as in (4.50) for each \( l = 1, 2, \ldots, k \). Then, (4.49) holds. (Refer to (4.31) for the definition of \( \sum_{|p|=s_k} \).)

**Proof.** Since the case \( k = 1 \) or \( k = 2 \) is already proved in Claim 4.14 and Claim 4.15 it is enough to consider the case \( k \geq 3 \). We first do the cumulant expansion with \( h_{i_1 j_1} \). Following the proof of Claim 4.14 we get

\[
\frac{1}{N^{\theta}} \sum_{I_k} \mathbb{E} \left[ A_1(s_1) A_2(s_2) \cdots A_k(s_k) m^{s_1 + \cdots + s_k + k} p^{2r-1} \right] = \sum_{[p_1:s_1]} \frac{1}{N^{1+\theta(I_k)}} \sum_{I_k} \mathbb{E} \left[ A_2(s_2) A_3(s_3) \cdots A_k(s_k) m^{s_1 + \cdots + s_k + k} \left( \partial_{i_1 j_1}^{p_1-s_1} p^{2r-1} \right) \right] + O_{\prec} (\Phi_r),
\]

where we use that \( \partial_{i_1 j_1} \left( h_{i_1 j_1} \right) = 0 \) for any \( 1 < l \leq k \) on the summation over distinct indexes \( \sum_{I_k} \) (refer to (4.28) for the definition of \( \sum_{I_k} \)).

Next we use the cumulant expansion with \( h_{i_2 j_2} \) and follow the proof of Claim 4.15. Since \( \partial_{i_2 j_2} \left( h_{i_1 j_1} \right) = 0 \) for any \( 2 < l \leq k \) on the summation over distinct indexes \( \sum_{I_k} \), we observe that

\[
\sum_{[p_1:s_1]} \frac{1}{N^{1+\theta(I_k)}} \sum_{I_k} \mathbb{E} \left[ A_2(s_2) \cdots A_k(s_k) m^{s_1 + \cdots + s_k + k} \left( \partial_{i_1 j_1}^{p_1-s_1} p^{2r-1} \right) \right] = \sum_{[p_2:s_2]} \sum_{[p_1:s_1]} \frac{1}{N^{2+\theta(I_k)}} \sum_{I_k} \mathbb{E} \left[ A_3(s_3) \cdots A_k(s_k) m^{s_1 + \cdots + s_k + k} \left( \partial_{i_1 j_1}^{p_1-s_1} p^{2r-1} \right) \right] + O_{\prec} (\Phi_r),
\]

Repeating the cumulant expansion and following the proof of Claim 4.15 we have

\[
\sum_{[p_2:s_2]} \sum_{[p_1:s_1]} \frac{1}{N^{2+\theta(I_k)}} \sum_{I_k} \mathbb{E} \left[ A_3(s_3) \cdots A_k(s_k) m^{s_1 + \cdots + s_k + k} \left( \partial_{i_1 j_1}^{p_1-s_1} p^{2r-1} \right) \right] = \sum_{[p_3:s_3]} \cdots \sum_{[p_1:s_1]} \frac{1}{N^{k+\theta(I_k)}} \sum_{I_k} \mathbb{E} \left[ m^{s_1 + \cdots + s_k + k} \left( \partial_{i_1 j_1}^{p_1-s_1} \cdots \partial_{i_{k-j} j_k}^{p_{k-j}-s_k} p^{2r-1} \right) \right] + O_{\prec} (\Phi_r),
\]

where we use that \( \partial_{i_1 j_1} \left( h_{i_l j_l} \right) = 0 \) for any \( l \neq l' \) on the summation over distinct indexes \( \sum_{I_k} \). The desired result follows.

Therefore, the random correction term \( Z_n \) should be a linear combination of terms of the form

\[
1 \frac{1}{N^{\theta}} \sum_{I_k} \prod_{l=1}^{k} A_l(s_l),
\]

where \( s_1, s_2, \ldots, s_k \) are positive odd integers such that \( s_1 + \cdots + s_k + k = 2n \).

4.6. **Proof of Proposition 3.1**

Recall (3.3) and (3.4). Then, we have to construct \( Q(m) = \sum_{n=1}^{\ell} \mathbb{E} \left[ \partial_{ij}^{p} \left( G_{ij} \right) p^{r-1} \right] + \mathbb{E} \left[ Q(m) p^{r-1} \right] = O_{\prec} (\Phi_r) \).

For brevity, instead of (4.51), we shall show

\[
\sum_{p=1}^{\ell} \frac{C_p^{p+1}}{N^{2q^{p-1}}} \sum_{i,j} \mathbb{E} \left[ \partial_{ij}^{p} \left( G_{ij} \right) p^{2r-1} \right] + \mathbb{E} \left[ Q(m) p^{2r-1} \right] = O_{\prec} (\Phi_r).
\]
Assume that a set of random coefficients \( \{ Z_n \}_{n=1}^\ell \) satisfies Assumptions 2.4 and Assumption 2.5. Consider the first term of the left-hand side of (4.52):

\[
\sum_{p=1}^\ell \frac{C_p+1}{N^{2q-1}} \sum_{i,j} \mathbb{E} \left[ \partial_{ij}^P(G_{ij} P^{2r-1}) \right].
\]

By Proposition 4.13 the term (4.53) is a linear combination of the terms of the following form (with bounded coefficients):

\[
\sum_{[p_1,s_1]} \ldots \sum_{[p_k,s_k]} \frac{1}{N^{k+\theta(\mathcal{I}_k)}} \sum_{\mathcal{I}_k} \mathbb{E} \left[ m^{s_1+\ldots+s_k+k} \left( \partial_{i_1,j_1}^{p_1-s_1} \ldots \partial_{i_k,j_k}^{p_k-s_k} P^{2r-1} \right) \right] + O(\Phi_r), \quad 1 \leq k \leq \ell,
\]

where for \( 1 \leq l \leq k \), each \( s_l \) is a positive odd integer such that \( s_l \leq \ell \), and \( \mathcal{I}_k \in \mathcal{J}_k \). Note that the length of the linear combination is bounded by a large constant only depending on \( \ell \) but not \( N \).

In addition, due to Proposition 4.16, the term (4.53) is also a linear combination of the terms of the following form (with bounded coefficients):

\[
\mathbb{E} \left[ \frac{1}{N^{\theta(\mathcal{I}_k)}} \left( \prod_{i=1}^{k} A_i(s_i) \right) m^{s_1+\ldots+s_k+k} P^{2r-1} \right] + O(\Phi_r), \quad 1 \leq k \leq \ell,
\]

where for \( 1 \leq l \leq k \), each \( s_l \) is a positive odd integer such that \( s_l \leq \ell \), \( \mathcal{I}_k \in \mathcal{J}_k \), and \( A_l \) has the form (4.50) for each \( l = 1, 2, \ldots, k \).

Let us consider the form

\[
\mathbb{E} \left[ \frac{1}{N^{\theta(\mathcal{I}_k)}} \left( \prod_{i=1}^{k} A_i(s_i) \right) m^{s_1+\ldots+s_k+k} P^{2r-1} \right], \quad 1 \leq k \leq \ell,
\]

where for \( 1 \leq l \leq k \), each \( s_l \) is a positive odd integer such that \( s_l \leq \ell \), \( \mathcal{I}_k \in \mathcal{J}_k \). Formally we can write

\[
\sum_{p=1}^\ell \frac{C_p+1}{N^{2q-1}} \sum_{i,j} \mathbb{E} \left[ \partial_{ij}^P(G_{ij} P^{2r-1}) \right] \approx \left( \text{linear combination of terms of the form (4.54)} \right) + O(\Phi_r).
\]

Then, we can construct \( \{ Z_n \}_{n=1}^\ell \) such that

\[
\left( \text{linear combination of terms of the form (4.54)} \right) + \mathbb{E} \left[ Q(m) P^{2r-1} \right] = 0,
\]

by setting \( Z_n \) as a linear combination of terms of the form

\[
\frac{1}{N^{\theta(\mathcal{I}_k)}} \prod_{i=1}^{k} A_i(s_i),
\]

where \( s_1, s_2, \ldots, s_k \) are positive odd integers such that \( s_1 + \ldots + s_k + k = 2n \). By the construction, the estimate (4.52) holds. Moreover, Assumptions 2.4 and Assumption 2.5 hold for the constructed \( \{ Z_n \}_{n=1}^\ell \) according to Appendix A.

Therefore we complete the proof. \( \Box \)

**Figure 1.** Interpretation of random correction terms on the graph.
Remark 4.17. The term (4.55) can be associated with the number of bipartite subgraphs of a certain form; see Figure 4.17 in [18]. Huang and Yau describe the term (4.55) using a weighted forest. We refer to [18, Definition 2.2, Eq (2.7)] for more detail.

5. LOCAL LAW AND EDGE RIGIDITY

Proof of Theorem 2.7. We start with a standard argument using stability analysis (e.g. [3, Lemma 6.1]). Let us set

\[ g(\tilde{z}) = m(\tilde{z}) - \bar{m}(\tilde{z}), \quad \Lambda(\tilde{z}) = |g(\tilde{z})|. \]

From Taylor expansion, it follows that

\[ P(\tilde{z}, m(\tilde{z})) = \partial_2 P(\tilde{z}, \bar{m}(\tilde{z})) g(\tilde{z}) + \frac{1}{2} \partial_2^2 P(\tilde{z}, \bar{m}(\tilde{z})) g^2(\tilde{z}) + R(g(\tilde{z})), \]

where \( R(y) = b_1 y^3 + b_2 y^4 + \cdots \) is a polynomial whose coefficients are random and stochastically dominated by \( q^{-1} \).

We write

\[
\begin{align*}
 f(\tilde{z}) &= P(\tilde{z}, m(\tilde{z})) - R(g(\tilde{z})), \\
 b(\tilde{z}) &= \partial_2 P(\tilde{z}, \bar{m}(\tilde{z})), \\
 a(\tilde{z}) &= \partial_2^2 P(\tilde{z}, \bar{m}(\tilde{z})).
\end{align*}
\]

Then, we have the

\[ a(\tilde{z})g^2(\tilde{z}) + 2b(\tilde{z})g(\tilde{z}) - 2f(\tilde{z}) = 0. \]

Solving the quadratic equation with respect to \( g(\tilde{z}) \), we obtain

\[ a(\tilde{z})g(\tilde{z}) = -b(\tilde{z}) \pm \sqrt{b^2(\tilde{z}) + 2f(\tilde{z})a(\tilde{z})}. \]

Since \( m \) and \( \bar{m} \) has a trivial bound \( \eta^{-1} \), we find \( g(\tilde{z}) \to 0 \) as \( \eta \to \infty \). Note that

\[
\begin{align*}
 b(\tilde{z}) &= \tilde{z} + \sum_{n=1}^\ell 2n Z_n \bar{m}^{2n-1}(\tilde{z}), \\
 a(\tilde{z}) &= 2 + 2(Z_n - 1) + \sum_{n=2}^\ell 2n(2n-1)Z_n \bar{m}^{2n-2}(\tilde{z}).
\end{align*}
\]

If \( E \) is bounded and \( \eta > 0 \) is large, we get \( b(\tilde{z}) \approx \eta \) for large \( N \). We also have \( a(\tilde{z}) \approx 2 \) for any \( \eta > 0 \) if \( N \) is large. Combining these estimates, we conclude that \( g(\tilde{z}) \to 0 \) and \( b^2(\tilde{z}) \gg f(\tilde{z})a(\tilde{z}) \) as \( \eta \to \infty \). Thus we should take + sign in the equation (5.1):

\[ a(\tilde{z})g(\tilde{z}) = -b(\tilde{z}) + \sqrt{b^2(\tilde{z}) + 2f(\tilde{z})a(\tilde{z})}. \]

We claim that

\[ \Lambda \leq C \sqrt{|f(\tilde{z})|}. \]

If \( b^2(\tilde{z}) \leq 2|f(\tilde{z})a(\tilde{z})| \), then \( |g(\tilde{z})| \leq C(|b(\tilde{z})| + \sqrt{4|f(\tilde{z})a(\tilde{z})|}) \leq C \sqrt{|f(\tilde{z})|} \). Otherwise, we can observe

\[
\sqrt{b^2(\tilde{z}) + 2f(\tilde{z})a(\tilde{z})} = b(\tilde{z}) \sqrt{1 + \frac{2f(\tilde{z})a(\tilde{z})}{b^2(\tilde{z})}} = b(\tilde{z}) \left( 1 + O \left( \frac{|f(\tilde{z})a(\tilde{z})|}{b^2(\tilde{z})} \right) \right) \leq b(\tilde{z}) + C \sqrt{|f(\tilde{z})|},
\]

so the above claim follows. Furthermore, since \( |R(g(\tilde{z}))| \leq \Lambda^3(\tilde{z})/q < \Lambda^2(\tilde{z})/q \) by the a priori bound \( \Lambda \ll N^{-c} \) for large \( N \), we can show \( \Lambda \leq C \sqrt{|P(\tilde{z}, m(\tilde{z}))| + CA(\tilde{z})/\sqrt{q}} \) so it follows

\[ \Lambda \leq C \sqrt{|P(\tilde{z}, m(\tilde{z}))|}. \]

We are now prepared to derive the local law from the recursive moment estimate, Proposition 3.1. Using Young’s inequality, we have

\[
\begin{align*}
 E \left[ |P(\tilde{z}, m(\tilde{z}))|^{2r} \right] &< E \left[ \left( \frac{3m(\tilde{z})}{N\eta} \right)^{2r} + \frac{1}{N^{2r}} \right] \\
 &+ |\partial_2 P|^r \left( \frac{3m(\tilde{z})}{N\eta} \right)^{3r/2} + \frac{1}{N^r} \left( \frac{3m(\tilde{z})}{N\eta} \right)^{r/2} + \left( |\partial_2 P| \frac{3m(\tilde{z})}{N^2\eta^2} \right)^r + \frac{1}{N^{2r}\eta^r}.
\end{align*}
\]
Let $\phi$ be as in (2.6). It follows from Lemma 2.6 that

\begin{equation}
\mathbb{E} \left[ |P(\tilde{z}, m(\tilde{z}))|^{2r} \right] \leq \mathbb{E} \left[ \left( \frac{\phi}{N\eta} \right)^{2r} + \frac{\Lambda^{2r}}{(N\eta)^{2r}} + \frac{1}{N^{2r}} + (\sqrt{\kappa + \eta} + \Lambda r/ \sqrt{N}\eta) \right]^{3r/2} + (\sqrt{\kappa + \eta})^{r} \left( \frac{\phi}{N\eta} \right)^{r} + \frac{\Lambda^r}{(N\eta)^{r}} + \frac{1}{N^{2r} \eta^r} \right].
\end{equation}

Since the estimate (5.2) implies

$$\mathbb{E} A^{4r} \leq C \mathbb{E} \left[ |P(\tilde{z}, m(\tilde{z}))|^{2r} \right],$$

one of the following estimates should hold:

$$\mathbb{E} A^{4r} \leq \frac{1}{(N\eta)^{2r}}; \quad \mathbb{E} A^{2r} \leq \frac{1}{(N\eta)^{3r/5}}; \quad \mathbb{E} A^{r/2} \leq \frac{1}{(N\eta)^{r/7}}; \quad \mathbb{E} A^r \leq \frac{(\sqrt{\kappa + \eta} + \Lambda)^{r/3}}{(N\eta)^{2r/3}}.$$

This implies the local law:

\begin{equation}
\Lambda \leq \left( \frac{\phi}{N\eta} \right)^{1/2} + (\sqrt{\kappa + \eta})^{1/4} \left( \frac{\phi}{N\eta} \right)^{3/4} + \frac{1}{N^{1/4}} + \frac{\Lambda^{1/4}}{N^{1/2}(N\eta)^{1/4}} + (\sqrt{\kappa + \eta})^{1/2} \left( \frac{\phi}{N^{2\eta^2}} \right)^{1/2} + (\sqrt{\kappa + \eta})^{1/2} \left( \frac{\phi}{N^{2\eta^2}} \right)^{1/2} + \frac{\Lambda^{1/2}}{N\eta} + \frac{1}{N^{1/2} \eta^{1/2}}.
\end{equation}

**Proof of Theorem 2.11** We shall follow the proof of [17] Theorem 1.4. Let $c > 0$ be a small positive real. Take $E \geq N^{-2/3+c}$ and $\eta = N^{-2/3}$. It follows from (5.3) that

$$|P(\tilde{z}, m(\tilde{z}))| \leq \frac{\phi}{N\eta} + \frac{\Lambda}{N\eta \sqrt{\kappa + \eta}} + (\sqrt{\kappa + \eta})^{1/2} \left( \frac{\phi}{N\eta} \right)^{3/4} + (\sqrt{\kappa + \eta})^{1/2} \frac{\Lambda^{3/4}}{(N\eta)^{3/4}} + \frac{1}{N^{1/2}} \left( \frac{\phi}{N\eta} \right)^{1/4}$$

and hence conclude with very high probability

\begin{equation}
|m(\tilde{z})| \ll \frac{1}{N\eta}.
\end{equation}

Applying [17] Lemma 2.13, we get

\begin{equation}
\Lambda \leq \frac{\phi}{N\eta \sqrt{\kappa + \eta}} + \frac{\Lambda}{N^{1/2}(N\eta)^{1/4} \sqrt{\kappa + \eta}} + (\sqrt{\kappa + \eta})^{-1/2} \left( \frac{\phi}{N\eta} \right)^{3/4} + (\sqrt{\kappa + \eta})^{-1/2} \frac{\Lambda^{3/4}}{(N\eta)^{3/4}} + \frac{1}{N^{1/2} \sqrt{\kappa + \eta}} \left( \frac{\phi}{N\eta} \right)^{1/4} + (\sqrt{\kappa + \eta})^{-1/2} \left( \frac{\phi}{N^{2\eta^2}} \right)^{1/2} + (\sqrt{\kappa + \eta})^{-1/2} \frac{\Lambda^{1/2}}{N\eta} + \frac{1}{N^{1/2} \sqrt{\kappa + \eta}}.
\end{equation}

Using the estimate (5.5), we can show that there is no eigenvalue in the interval $[\tilde{\mathcal{L}} + E - \eta, \tilde{\mathcal{L}} + E + \eta]$ with very high probability. If there is an eigenvalue $\lambda_i$ in the small interval $[\tilde{\mathcal{L}} + E - \eta, \tilde{\mathcal{L}} + E + \eta]$, then it follows that

$$\Im(m(\tilde{z})) \geq \frac{1}{N} \Im \left( \frac{1}{\lambda_i - (\tilde{\mathcal{L}} + E + \eta i)} \right) \geq \frac{1}{2N\eta},$$

which is in contradiction to the bound (5.5). Following the lattice argument and using a priori bound (e.g. [6] Lemma 4.4 or [21] Theorem 2.9), we have with very high probability

\begin{equation}
\lambda_1 - \tilde{\mathcal{L}} \leq N^{-2/3+c}.
\end{equation}
In addition, using the standard argument of Helffer-Sjöstrand calculus (e.g. [7, Lemma B.1]), the following estimate holds with very high probability:

$$\mu \left( \tilde{L} - N^{-2/3+c}, \infty \right) \asymp N^{-1+2c/3},$$

where $\mu$ is the empirical eigenvalue distribution of $H$ given by (2.1). The desired result follows if $c > 0$ is small enough.

□

**APPENDIX A. ASYMPTOTICS OF THE RANDOM CORRECTION TERMS**

We should check Assumption 2.4 and Assumption 2.5 for $\{Z_n\}_{n=1}^\ell$ constructed in Section 4.6. By the construction, it follows that

$$Z_1 = \frac{1}{N} \sum_{i,j} h_{ij}^2.$$  

We have

$$Z_1 - 1 = \frac{1}{N} \sum_{i,j} (h_{ij}^2 - E[h_{ij}^2]) + O_\prec \left( \frac{1}{N} \right).$$

According to [17], we get

$$\frac{1}{N} \sum_{i,j} (h_{ij}^2 - E[h_{ij}^2]) = X \prec \frac{1}{\sqrt{Nq}},$$

where recall (1.1) for the definition of $X$. Thus the estimate (2.2) follows.

Now we want to check (2.3) and (2.4). Since $Z_n$ is a linear combination of terms of the form (4.55), it is enough to check that for

(A.1)  

$$\frac{1}{N^q} \sum_{Z_k} \prod_{l=1}^k A_l(s_l) \prec \frac{1}{q^{n+1}},$$

and

(A.2)  

$$\partial_{ij} \left( \frac{1}{N^q} \sum_{Z_k} \prod_{l=1}^k A_l(s_l) \right) \prec \frac{1}{N},$$

where $s_1, s_2, \ldots, s_k$ are positive odd integers such that $s_1 + \cdots + s_k + k = 2n$.

Let us consider

$$E \left[ \left( \frac{1}{N} \sum_{i,j} h_{ij}^{2m} \right)^r \right].$$

It follows that

$$\frac{1}{N^r} E \left[ \left( \sum_{i,j} h_{ij}^{2m} \right)^r \right] = \frac{1}{N^r} \sum_{l=1}^r \sum_{r_1, r_2, \ldots, r_l} \sum_{(i_1, j_1) \neq \cdots \neq (i_l, j_l)} E \left[ (h_{i_1 j_1}^{2m})^{r_1} \cdots (h_{i_l j_l}^{2m})^{r_l} \right]$$

$$\leq C \sum_{l=1}^r \left( \frac{N^2}{l} \right)^{r-1} \left( \frac{1}{N^{l+1} q^{2mr-2l}} \right)$$

$$\leq \frac{C}{q^{2r(m-1)}},$$

where in the first line, the notation

$$\sum_{(i_1, j_1) \neq \cdots \neq (i_l, j_l)}$$

means the summation over all $l$ distinct index pairs $\{(i_1, j_1), \cdots, (i_l, j_l)\}$. By Markov’s inequality, we have for $m \geq 1$

$$\frac{1}{N} \sum_{i,j} h_{ij}^{2m} \prec \frac{1}{q^{2(m-1)}}.$$
Similarly, for $m \geq 1$, we get

\[(A.3)\]

\[
\sum_j h_{ij}^{2m} \prec \frac{1}{q^{2(m-1)}},
\]

We also have

\[
\frac{1}{N} \sum_{i,j} (h_{ij}^2 - \mathbb{E}[h_{ij}^2]) \prec \frac{1}{\sqrt{N} q},
\]

and

\[
\sum_j (h_{ij}^2 - \mathbb{E}[h_{ij}^2]) \prec \frac{1}{q}.
\]

Thus we can see that

\[
\frac{1}{N} \sum_{i_1, j_1} h_{i_1 j_1}^{s_1+1} \sum_{i_2, j_2} h_{i_2 j_2}^{s_2+1} \sum_{i_3, j_3} h_{i_3 j_3}^{s_3+1} \prec \frac{1}{q^{s_1+s_2+s_3+3}},
\]

where $s_1, s_2, s_3$ are positive odd integers. Extending the above argument, we can obtain $(A.1)$ using the condition that $s_1, s_2, \ldots, s_k$ are positive odd integers such that $s_1 + \cdots + s_k + k = 2n$.

By the product rule, we have

\[
\partial_{ij} \left( \frac{1}{N^q} \sum_{I_k} \prod_{l=1}^k A_l(s_l) \right)
\]

\[= \frac{1}{N^q} \sum_{I_k} (\partial_{ij}(A_1(s_1))A_2(s_2) \cdots A_k(s_k) + A_1(s_1)\partial_{ij}(A_2(s_2))A_3(s_3) \cdots A_k(s_k)
\]

\[+ \cdots + A_1(s_1)A_2(s_2) \cdots A_{k-1}(s_{k-1})\partial_{ij}(A_k(s_k))).
\]

Thus, it is enough to show for $1 \leq l \leq k$

\[
\frac{1}{N^q} \sum_{I_k} A_1(s_1) \cdots A_{l-1}(s_{l-1})\partial_{ij}(A_l(s_l))A_{l+1}(s_{l+1}) \cdots A_k(s_k) \prec \frac{1}{N}
\]

We claim that

\[
\frac{1}{N} \sum_{i_1, j_1} \partial_{ij}(A_l(s_l)) \prec \frac{1}{N}.
\]

We first observe that

\[
\partial_{ij}(A_l(s_l)) = \begin{cases} (s_l + 1)h_{ij}^{s_l} & \{i, j\} = \{i, j\}, \\ 0 & \text{otherwise}. \end{cases}
\]

Then, it follows that

\[
\frac{1}{N} \sum_{i_1, j_1} \partial_{ij}(A_l(s_l)) = \frac{2(s_l + 1)}{N} h_{ij}^{s_l}.
\]

Since we have $h_{ij} \prec q^{-1}$ due to the given condition in Definition 2.3, the claim is proved. Consider an index pair $(i_{l'}, j_{l'})$ such that $1 \leq l' \leq k$ and $l' \neq l$. Due to $(A.3)$, if $i_{l'} \equiv i_l$, then we have

\[
\sum_{j_{l'}} h_{i_{l'} j_{l'}}^{s_{l'}-1} \prec \frac{1}{q^{s_{l'}-1}}.
\]

Similarly, if $j_{l'} \equiv j_l$, we get

\[
\sum_{i_{l'}} h_{i_{l'} j_{l'}}^{s_{l'}-1} \prec \frac{1}{q^{s_{l'}-1}}.
\]

Combining the above observations with $(A.1)$, we can deduce that

\[
\frac{1}{N^q} \sum_{I_k} A_1(s_1) \cdots A_{l-1}(s_{l-1})\partial_{ij}(A_l(s_l))A_{l+1}(s_{l+1}) \cdots A_k(s_k) \prec \frac{1}{N}.
\]

Therefore $(A.2)$ is also proved.
APPENDIX B. PROPERTIES OF A SOLUTION OF THE SELF-CONSISTENT EQUATION

Proof of Lemma 2.6 We shall use the argument in [17] Proposition 2.5, Proposition 2.6] and [21] Lemma 4.1. Define

\[
R(w) = -\frac{1}{w} - w - \frac{Q(w) - w^2}{w} = -\frac{1}{w} - w - (Z_1 - 1)w - \sum_{n=2}^{\ell} Z_n w^{2n-1}.
\]

Observe that \(P(z, w) = 0\) if and only if \(R(w) = z\). Taking derivative, we have

\[
R'(w) = \frac{1}{w^2} - 1 - (Z_1 - 1) - \sum_{n=2}^{\ell} (2n - 1)Z_n w^{2n-2},
\]

and

\[
R''(w) = -\frac{2}{w^3} - \sum_{n=2}^{\ell} (2n - 1)(2n - 2)Z_n w^{2n-3}.
\]

Note that

\[
Z_1 - 1 \asymp \frac{1}{\sqrt{Nq}},
\]

and for \(n \geq 2\)

\[
Z_n \ll \frac{1}{q^{n-1}}.
\]

Following the argument in [17] Appendix B], we notice that \(R'(w)\) has two solutions on \((-C, C)\) if \(N\) is large enough. Denote these solutions by \(\pm \tau\). From now on, we shall construct \(\tilde{L}\). We write \(w = 1 - \epsilon\). Then we can see that

\[
R'(w) = (1 + \epsilon + \epsilon^2 + \cdots)^2 - 1 - (Z_1 - 1) - \sum_{n=2}^{\ell} (2n - 1)Z_n (1 - \epsilon)^{2n-2}.
\]

We set

\[
\epsilon_0 := \frac{1}{2} \left( (Z_1 - 1) + \sum_{n=2}^{\ell} (2n - 1)Z_n \right).
\]

Next we write \(w = 1 - (\epsilon_0 + \epsilon)\). It follows that

\[
R'(w) = (1 + (\epsilon_0 + \epsilon) + (\epsilon_0 + \epsilon)^2 + \cdots)^2 - 1 - (Z_1 - 1) - \sum_{n=2}^{\ell} (2n - 1)Z_n (1 - (\epsilon_0 + \epsilon))^{2n-2}
\]

\[
= (1 + (\epsilon_0 + \epsilon) + (\epsilon_0 + \epsilon)^2 + \cdots)^2 - 1 - 2\epsilon_0 - E((Z_n), \epsilon_0) - \text{(remainder)}.
\]

where in the second line \(E((Z_n), \epsilon_0)\) is given by

\[
E((Z_n), \epsilon_0) := \sum_{n=2}^{\ell} (2n - 1)Z_n \left( (1 - \epsilon_0)^{2n-2} - 1 \right).
\]

To make cancellation, we set

\[
\epsilon_1 := \frac{1}{2} E((Z_n), \epsilon_0) - (\epsilon_0^2 + \epsilon_0^3 + \cdots).
\]

Repeating the above argument similarly, we can make a sequence \(\{\epsilon_m\}\) such that \(\tau \asymp 1 - \sum_m \epsilon_m + O_<(N^{-C})\), \(\epsilon_m \gg \epsilon_{m+1}\) and each \(\epsilon_m\) is a polynomial in variables \((Z_n)\). Let us define \(\tilde{L}\) by setting

\[
\tilde{L} := R(-\tau).
\]

Using \(\tau \asymp 1 - \sum_m \epsilon_m + O_<(N^{-C})\), we obtain

\[
\tilde{L} = \tilde{L} + O_<(N^{-C}).
\]
where \( \tilde{L} \) is a polynomial in variables \( (Z_n) \). We define

\[
D_w := \{ w \in \mathbb{C} : |w| < 5 \},
\]

and

\[
D_z := \{ z = E + i\eta : |E|, |\eta| \leq 2\sqrt{2} \}.
\]

According to [17, Appendix B], by Rouché’s theorem, we find that \( R(w) \) has exactly two critical points \( \pm \tau \) on \( D_w \).

Again, by Rouché’s theorem, if \( z \in D_z \), the equation \( P(z, w) = 0 \) has exactly two solutions on \( D_w \). Furthermore, if \( z \in (-\tilde{L}, \tilde{L}) \), one solution of \( P(z, w) = 0 \) is on \( \mathbb{C}_+ \) and the other is on \( \mathbb{C}_- \). Considering \( z \in (-\tilde{L}, \tilde{L}) \), the solution \( w(z) \in \mathbb{C}_+ \) of \( P(z, w) = 0 \) forms a curve on \( \mathbb{C}_+ \), joining \( \tau \) and \( -\tau \). We denote this curve by \( \Gamma \).

Consider the region \( D_\Gamma \), bounded by the curve \( \Gamma \) and the interval \( [-\tilde{L}, \tilde{L}] \). We can see that \( R(w) \) is biholomorphic from the region \( D_\Gamma \) to the upper half-plane by using maximum principle and considering one-point compactification of the complex plane. This let us take \( \tilde{m} \) to be the inverse of \( R(w) \), i.e.,

\[
\tilde{m}(z) := R^{-1}(z), \quad z \in \mathbb{C}_+.
\]

Let \( \tilde{\rho} \) be the probability measure obtained by Stieltjes inversion of \( \tilde{m}(z) \). The probability measure \( \tilde{\rho} \) is supported on \( [-\tilde{L}, \tilde{L}] \) and has strictly positive density on \( (-\tilde{L}, \tilde{L}) \). Considering Taylor expansion of \( R(w) \) in a small neighborhood of \( \pm \tau \), we can show \( \tilde{\rho} \) has square root behavior at the edges \( \pm \tilde{L} \). The asymptotics of derivatives of \( P(z, \tilde{m}(z)) \) easily can be checked as in [17, Appendix B].

\[\tag{C.1}\]

**APPENDIX C. PROOF OF LEMMA 4.4**

**Proof of Lemma 4.4** Consider the left-hand side of (4.17)

\[
\sum_{p=1}^{\ell} \sum_{x,y} \frac{C_{p+1}}{N^2q^{p-1}} E \left[ \frac{\partial^p}{\partial x^p} \left( m^d G_{yx} G_{xi} \left( \prod_{j=1}^{k} G_{v_j v_j} \right) D(P) \right) \right].
\]

Due to Proposition 4.2 and [17] Proposition A.1, we can replace \( \sum_{x,y} \) with \( \sum_{x \neq y} \) as follows:

\[
\sum_{p=1}^{\ell} \frac{C_{p+1}}{N^2q^{p-1}} \sum_{x \neq y} E \left[ \frac{\partial^p}{\partial x^p} \left( m^d G_{yx} G_{xi} \left( \prod_{j=1}^{k} G_{v_j v_j} \right) D(P) \right) \right] = \sum_{p=1}^{\ell} \frac{C_{p+1}}{N^2q^{p-1}} \sum_{x \neq y} E \left[ \frac{\partial^p}{\partial x^p} \left( m^d G_{yx} G_{xi} \left( \prod_{j=1}^{k} G_{v_j v_j} \right) D(P) \right) \right] + O_\prec (\Phi_r).
\]

Note that

\[
(C.1) \quad \sum_{x \neq y} E \left[ \frac{\partial^p}{\partial x^p} \left( m^d G_{yx} G_{xi} \left( \prod_{j=1}^{k} G_{v_j v_j} \right) D(P) \right) \right] = \sum_{s=0}^{p} \binom{p}{s} \frac{C_{p+1}}{N^2q^{p-1}} \sum_{x \neq y} E \left[ \frac{\partial^s}{\partial x^s} G_{yx} \frac{\partial^{p-s}}{\partial y^{p-s}} \left( m^d G_{xi} \left( \prod_{j=1}^{k} G_{v_j v_j} \right) D(P) \right) \right] = \sum_{s=0}^{p} \binom{p}{s} \frac{C_{p+1}}{N^2q^{p-1}} \sum_{x \neq y} E \left[ \frac{\partial^s}{\partial x^s} G_{yx} \frac{\partial^{p-s}}{\partial y^{p-s}} \left( m^d G_{xi} \left( \prod_{j=1}^{k} G_{v_j v_j} \right) D(P) \right) \right] + O_\prec (\Phi_r),
\]

...
where we use (4.4) for the second equality. If the derivative $\partial_{xy}$ hits $m$, $G_{ii}$ or $G_{v_j v_j}$, then the resulting terms are absorbed into $O_{\prec}(\Phi_r)$ due to (4.5) and (4.7), which implies

$$
(C.2) \quad \sum_{s=0}^{p} \left( \frac{p}{s} \right) \frac{C_{p+1}}{N^2 q^{p-1}} \sum_{x \neq y} \sum_{x_{xy}} \left[ (D_{xy}^s G_{xy}) \partial_{xy}^{p-s} \left( m^d G_{ii}^t \left( \prod_{j=1}^{k} G_{v_j v_j}^{u_j} \right) D(P) \right) \right]
$$

$$
= \sum_{s=0}^{p} \left( \frac{p}{s} \right) \frac{C_{p+1}}{N^2 q^{p-1}} \sum_{x \neq y} \sum_{x_{xy}} \left[ (D_{xy}^s G_{xy}) m^d G_{ii}^t \left( \prod_{j=1}^{k} G_{v_j v_j}^{u_j} \right) \left( \partial_{xy}^{p-s} D(P) \right) \right]
$$

If $s$ is odd, we observe that

$$
D_{xy}^s G_{xy} = - (s!) G_{xx}^{s+1} G_{yy}^{s+1} + \text{(the terms having at least two off-diagonal entries)}.
$$

Thus, due to (4.7), we have

$$
\sum_{s=0}^{p} \left( \frac{p}{s} \right) \frac{C_{p+1}}{N^2 q^{p-1}} \sum_{x \neq y} \sum_{x_{xy}} \left[ G_{xx}^{s+2} G_{yy}^{s+2} m^d G_{ii}^t \left( \prod_{j=1}^{k} G_{v_j v_j}^{u_j} \right) \left( \partial_{xy}^{p-s} D(P) \right) \right], \quad s \equiv 1(\text{mod } 2).
$$

If $s$ is even, we can see that

$$
D_{xy}^s G_{xy} = t_s G_{xx} G_{xx}^{s/2} G_{yy}^{s/2} + \text{(the terms having at least three off-diagonal entries)},
$$

where $t_s$ is a constant depending on $s$. Since any term having at least two off-diagonal entries is absorbed into $O_{\prec}(\Phi_r)$, it is enough to consider

$$
\sum_{s=0}^{p} \left( \frac{p}{s} \right) \frac{C_{p+1}}{N^2 q^{p-1}} \sum_{x \neq y} \sum_{x_{xy}} \left[ G_{xx} G_{xx}^{s/2} G_{yy}^{s/2} m^d G_{ii}^t \left( \prod_{j=1}^{k} G_{v_j v_j}^{u_j} \right) \left( \partial_{xy}^{p-s} D(P) \right) \right].
$$

Following the argument from (4.14) and (4.15), we get

$$
\sum_{s=0}^{p} \left( \frac{p}{s} \right) \frac{C_{p+1}}{N^2 q^{p-1}} \sum_{x \neq y} \sum_{x_{xy}} \left[ G_{xx} G_{xx}^{s/2} G_{yy}^{s/2} m^d G_{ii}^t \left( \prod_{j=1}^{k} G_{v_j v_j}^{u_j} \right) \left( \partial_{xy}^{p-s} D(P) \right) \right] = O_{\prec}(\Phi_r).
$$

Thus, we have (4.17).

Next we consider the left-hand side of (4.18)

$$
\sum_{p=1}^{\ell} \frac{C_{p+1}}{N^2 q^{p-1}} \sum_{x} \sum_{x_{xy}} \left[ \partial_{ix}^p \left( m^d G_{x_i}^{t-1} \left( \prod_{j=1}^{k} G_{v_j v_j}^{u_j} \right) D(P) \right) \right].
$$

As in (C.1) and (C.2), using (4.4), (4.5) and (4.7), we have

$$
\frac{C_{p+1}}{N^2 q^{p-1}} \sum_{x} \sum_{x_{xy}} \left[ \partial_{ix}^p \left( m^d G_{x_i}^{t-1} \left( \prod_{j=1}^{k} G_{v_j v_j}^{u_j} \right) D(P) \right) \right] = \sum_{s=0}^{p} \left( \frac{p}{s} \right) \frac{C_{p+1}}{N^2 q^{p-1}} \sum_{x \neq y} \sum_{x_{xy}} \left[ D_{ix}^s (G_{x_i}^{t-1}) m^d \left( \prod_{j=1}^{k} G_{v_j v_j}^{u_j} \right) \left( \partial_{ix}^{p-s} D(P) \right) \right].
$$

As in the previous case, if $s$ is even, it is absorbed into $O_{\prec}(\Phi_r)$ by the same argument from (4.14) and (4.15). Thus, we focus on the case $s$ is odd. If $s$ is odd, we observe that

$$
D_{ix}^s (G_{x_i}^{t-1}) = -c_s \times (s!) G_{xx}^{s+1} G_{ii}^{t+1} + \text{(the terms having at least two off-diagonal entries)},
$$

where $c_s$ is a constant depending on $s$. This implies (4.18) and, in fact, $c_s$ satisfies (4.19).
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