Refuting Tianrong Lin’s arXiv:2110.05942 “Resolution of The Linear-Bounded Automata Question”

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Abstract

In the preprint mentioned in the title Mr. Tianrong claims to prove NSPACE[n] ≠ DSPACE[n], resolving a longstanding open problem in automata theory called the LBA question. He claims to achieve this by showing more generally NSPACE[S(n)] ≠ DSPACE[S(n)] for suitable S(n). We demonstrate that his proof is incomplete, even wrong, and his strategy cannot be repaired.

key words: complexity theory, automata theory
MSC2020: 68Q15, 68Q45

1 Introduction

Mr. Tianrong uploaded his paper first on Monday, 11th Oct. 2021 to arXiv. Over the course of the following two weeks we, Mr. Tianrong and the author of this paper, had a discussion via email about open points in Mr. Tianrong’s work. Since we could not come to agree on the open points, Mr. Tianrong invited me to write a correct paper on my own. I will follow his invitation and refute his claim that he resolved the LBA question.

On Thursday, 21st Oct. 2021, Mr. Tianrong uploaded the latest update of his paper to arXiv as of writing this note. We refer to this second version [3]. We mainly use the notation from there, which is standard, in particular DTM, NTM, DSPACE[S(n)], etc.

The rest of this small note is organized in three main parts. In section 2 we outline the key points of the proof strategy for the claims in [3]. In section 4 we assume that the main argument in [3] was correct and based on that make some harmless modifications which would not invalidate the argument but which lead to an obvious contradiction. By this we show that the approach of Mr. Tianrong is fatally flawed and is beyond repair. In section 5 we make an attempt to locate the source of the error in the main argument of [3]. Finally we sum up our refute of Mr. Tianrong claims in section 5.
2 The strategy of Mr. Tianrong

The heart of \[3\] is Theorem 3 on p. 6. We repeat it here word by word:

**Theorem 3.** Let \(S(n) \geq \log n\) be a space-constructible function. Then there exists a language \(L_d\) accepted by a NTM by using space \(O(S(n))\) but by no DTM of space complexity \(S(n)\). That is, \(L_d \in \text{NSPACE}[S(n)]\) but \(L_d \notin \text{DSPACE}[S(n)]\).

Most of \[3\] is inspired by \[1\] which Mr. Tianrong cites repeatedly. In particular his Theorem 3 parallels \[1, \text{Thm. 11.1.}\] on p. 408. A version with more restrictive assumptions can be found in \[2, \text{Thm. 12.8}\] on p. 297. We repeat this later statement as it seems closer to Theorem 3.

**Theorem 12.8.** If \(S_2(n)\) is a fully space-constructible function,

\[
\inf_{n \to \infty} \frac{S_1(n)}{S_2(n)} = 0,
\]

and \(S_1(n)\) and \(S_2(n)\) are each at least \(\log_2 n\), then there is a language in \(\text{DSPACE}(S_2(n))\) not in \(\text{DSPACE}(S_1(n))\).

**Remark 1.** For a TM \(M\) we denote the language accepted by it by \(L_M\) as is customary. Note that if a DTM \(M\) uses at most \(S(n)\) space, then \(L_M \in \text{DSPACE}(S(n))\). However, even if \(M\) used more space than \(S(n)\), we cannot rule out \(L_M \in \text{DSPACE}(S(n))\): there may be another DTM \(M'\) using at most \(S(n)\) space and accepting the same language \(L_M = L_M\). Below we will focus on the space requirements of a DTM \(M\) and less on whether \(L_M \in \text{DSPACE}(S(n))\).

We give a two page exposition of the proof of \[2, \text{Thm. 12.8}\] along the lines of the proof of \[1, \text{Thm. 11.1.}\]. Details and used terminology can be found in the references.

The argument in \[1, \text{Thm. 11.1.}\] is essentially based on diagonalization. First a four-tape DTM \(M_0\) with \(L_{M_0} \in \text{DSPACE}(S_2(n))\) is constructed in five steps that on binary input \(x\) of length \(n = \vert x \vert\) attempts to simulate a single-tape DTM \(M = M_x\) which depends on the binary encoding \(x\) (for details cf. \[3\] or \[1\]). There is a twist to this encoding as prepping any number of 1s to \(x\) encodes the same DTM, i.e. “\(M_x = M_{1x} = M_{11x} = \ldots\)”. Thus for any DTM \(M'\) and any bound \(b \in \mathbb{N}\) there is a code word \(x'\) with \(M' = M_{x'}\) and \(\vert x' \vert \geq b\). \(M_0\) uses a binary tape alphabet with an added blank symbol\(^1\). The behavior of \(M_0\) is defined in these five steps:

1. By space-constructibility of \(S_2(n)\) we can detect if \(M_0\) would go beyond that bound and halt without accepting. This enforces \(L_{M_0} \in \text{DSPACE}(S_2(n))\).

2. If \(x\) is not a code word for a DTM, \(M_0\) detects this and halts without accepting. Otherwise we proceed to the next step.

3. Then \(M_0\) prepares to simulate \(M_x\). For this \(M_0\) determines \(s\) the number of states and \(t\) the length of the tape alphabet of \(M_x\). Any tape symbol for \(M_x\) can be represented by \(\lceil \log_2 t \rceil\) binary tape symbols of \(M_0\). We want to ensure that \(L_{M_x} \in \text{DSPACE}(S_1(n))\). For this we limit the space of simulation to \((1 + \lceil \log_2 t \rceil)S_1(n)\) tape cells\(^2\) on \(M_0\) before actually simulating \(M_x\). We need to distinguish two cases:

\(^1\)Technically in \[1\] a forth separator symbol is introduced, but this is not essential.

\(^2\)Instead we could also only use \(\lceil \log_2 t \rceil S_1(n)\) as in \[2\]. Again these are irrelevant details.
(a) If \((1 + \lceil \log_2 t \rceil)S_1(n) \not\in S_2(n)\), then by step 1 \(M_0\) halts without accepting. This can happen despite \(M_x\) not exceeding the \(S_1(n)\) space bound, but this loss will be irrelevant.

(b) If \((1 + \lceil \log_2 t \rceil)S_1(n) \leq S_2(n)\), \(M_0\) proceeds to the next step.

4. On a separate tape \(M_0\) sets up a counter of \(\lceil \log_2 s \rceil + \lceil \log_2 S_1(n) \rceil + \lceil \log_2 t \rceil S_1(n)\) many tape cells. This counter can go up to (possibly slightly more than) \(sS_1(n) t^{S_1(n)}\). It will be used to detect cycles in the configuration space of \(M_x\), i.e. to detect when the simulation of \(M_x\) will not halt. Each of the three factors of \(sS_1(n) t^{S_1(n)}\) is related to an essential component in the configuration space: state, position of read-write head on tape, content of the (space-restricted) tape. Again we distinguish:

(a) If \(\lceil \log_2 s \rceil + \lceil \log_2 S_1(n) \rceil + \lceil \log_2 t \rceil S_1(n) \not\in S_2(n)\), then by step 1 \(M_0\) halts without accepting. This can happen despite \(M_x\) respects the \(S_1(n)\) space bound and properly halts, but this loss will be irrelevant.

(b) If \(\lceil \log_2 s \rceil + \lceil \log_2 S_1(n) \rceil + \lceil \log_2 t \rceil S_1(n) \leq S_2(n)\), \(M_0\) proceeds to the next step.

5. Here is where the heart of the diagonalization is happening. After this setup \(M_0\) simulates \(M_x\) on input \(x\). Its halting behavior is as follows:

(a) If \(M_0\) uses more cells than allowed by step 3, i.e. if \(M_x\) uses more than \(S_1(n)\) cells, then \(M_0\) accepts. Note that both \(L_{M_s} \notin \text{DSPACE}(S_1(n))\) or \(L_{M_y} \notin \text{DSPACE}(S_1(n))\) are possible, but this distinction will be irrelevant.

(b) If the bound from step 3 is respected but the counter from step 4 overflows, i.e. if \(M_x\) does not halt on \(x\), then \(M_0\) accepts.

(c) If the simulation of \(M_x\) on \(x\) by \(M_0\) respects the bounds from steps 3 and 4 and halts, then, if \(M_x\) rejects, \(M_0\) will accept.

(d) If the simulation of \(M_x\) on \(x\) by \(M_0\) respects the bounds from steps 3 and 4 and halts, then, if \(M_x\) accepts, \(M_0\) will reject.

The four-tape DTM \(M_0\) always halts and by step 1 it uses no more space than \(S_2(n)\), i.e. \(L_{M_0} \in \text{DSPACE}(S_2(n))\). Now \(\text{II}\) argues by contradiction that \(L_{M_0} \not\in \text{DSPACE}(S_1(n))\). If otherwise, by standard lemmas (for details cf. \([3]\) or \(\text{II}\)) there would be a single-tape DTM \(M_y\) using no more than \(S_1(n)\) space with \(L_{M_y} = L_{M_0} \). So far \(M_x\) was variable and with it the code word \(x\), its length \(n = |x|\), the number of states \(s\) and the size of the tape alphabet \(t\). Now \(\text{II}\) fixes \(M_x = M_y\), we will call it \(\vec{M}\) from now on, and with it \(s = s_{\vec{M}}\) and \(t = t_{\vec{M}}\) become constant. However the code word \(z \in \{y, 1y, 11y, \ldots\}\) and \(n = |z|\) can stay variable in a limited sense as \(\vec{M} = M_{z} = M_{y} = M_{1y} = M_{11y} = \ldots\).

Now everything falls into its place (again for details cf. the literature): By step 1 \(L_{M_0} \in \text{DSPACE}(S_2(n))\). Since any of our restricted \(z \in \{y, 1y, 11y, \ldots\}\)

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3This is possible because the set of DTMs respecting the space bound \(S_1(n)\) do not have the full power of the set of all DTMs – i.e. we are not contradicting the unsolvability of the halting problem.

4In \([2]\) detecting non-halting behavior is outsourced to Lemma 12.1

5We actually use a stronger assertion, that the execution of \(M_y\) runs parallel to that of \(M_0\). This is however a direct consequence of the proof of the standard lemmas.
encodes the fixed DTM $\overline{M}$, i.e. $z$ is a valid code word, $\overline{M} = M_z$ executed on $z$ passes step 2. Then \[1\] uses $\inf_{n \to \infty} \frac{S_2(n)}{S_1(n)} = 0$, i.e. that in this limited sense $S_2(n)$ grows asymptotically faster than $S_1(n)$, and they use that $s_{\overline{M}}$ and $t_{\overline{M}}$ are now fixed constants, to pick a $w \in \{y_1, 1y_1, 11y_1, \ldots\}$ of sufficiently large size $n_w = |w|$, s.t. $\overline{M}$ on input $w$ passes steps 3 and 4. By assumption from the last paragraph $\overline{M} = M_w = M_y$ uses at most $S_1(n)$ space, in particular when executed on $w$, thus 5(a) is not relevant. As $\overline{M}$ is a single-tape version of $M_0$ which always halts we also can ignore 5(b). Since only 5(c) and 5(d) are left, $M_y$ accepts $w$ iff $M_0$ rejects $w$, i.e. since $L_{M_w} = L_{\overline{M}} = L_{M_y}$ iff $M_w$ rejects $w$.

Finally \[1\] arrived at a contradiction showing that $L_{M_0} \in \text{DSPACE}(S_2(n)) \setminus \text{DSPACE}(S_1(n))$ and thus finishing the proof.

After this lengthy but relevant digression we come back to Mr. Tianrong’s paper. In his paper he follows the five step construction of $M_0$ in \[1\] but with some modifications:

1. \[3\] replaces $S_2(n)$ by $O(S(n))$ for $M_0$ (called $M$ in \[3\]) and $S_1(n)$ by $S(n)$ for the $M_x$ (called $M_i$ in \[3\]).

2. \[3\] lets $M_0$ be an NTM instead of a DTM.

3. \[3\] reorganizes the five steps into six steps as follows:

   (a) In \[3\] steps 1 and 2 are essentially swapped.

   (b) The determination of $s$ and $t$ moves from step 3 to step 1. Except for this, negligible changes in notation and adjustments for differing space bounds, steps 3 and 4 are literally the same word by word.

   (c) In step 2 of \[3\] the space bound $S_2(n)$ of step 1 in \[1\] is replaced by $1 + [\log_2 s] + [\log_2 t])S(n)$.

   (d) Step 5 is almost literally the same except for a small introduction: “By using nondeterminism in $(1 + [\log_2 s] + [\log_2 t])S(n)$ cells”. This is accompanied by a mysterious footnote, explaining that “$M$ is somewhat deterministic”, but “that $M$ we constructed here is a NTM” (cf. \[3\] for the full footnote).

   (e) Step 1 of \[3\] forks of to a step 6 if $x$ is not a code word for a DTM. This step 6 is original. Instead of simply halting as \[1\] does in this situation, \[3\] sets up another simulation and describes at the end, when this simulation accepts or rejects. Additionally, \[3\] starts with: “Since $x$ is not encoding of some single-tape DTM.” After setting up the simulation, \[3\] goes on: “By using its nondeterministic choices, $M$ moves as per the path given by $x$.”

After the construction, \[3\] follows the proof by contradiction in \[1\]. Of course, the inequalities need adjusting and there seems to be an error in calculating the limit in lines 2-4 of p. 8, however this does not seem to invalidate the overall argument in our point of view. At the end compared to \[1\] a novel argument is made that the NTM $M$ actually can reverse accepting and rejecting states at

\[\overline{M}\] is a single-tape version of $M_0$ and their execution runs in parallel. Thus we may say "$\overline{M}$ on input $z$ passes step X", if $M_0$ on input $z$ does not halt in step X and proceeds further.
the end of its execution: this is indeed less trivial for NTMs than for DTMs, so this remark is rightfully made, but this issue is resolved by standard results from the literature.

With the NTM $M$ of [3] we have $L_M \not\in \text{DSPACE}[S(n)]$ and it is claimed that $M$ uses at most $O(S(n))$ space. By the standard trick of exponentially enlarging the tape alphabet of $M$ if necessary it is not hard to see, that $M$ can be constructed to use at most $S(n)$ space. Thus indeed, if everything was correct before, then there is an $L_d$ with $L_d \in \text{NSPACE}[S(n)]$ but $L_d \not\in \text{DSPACE}[S(n)]$.

This finishes the outline of the core argument for [3, Thm. 3] and its blueprint, the proof of [1, Thm. 11.1.].

3 Contradictory consequences of Mr. Tianrong’s approach

We assume for the time being in this section that the proof strategy of [3] was valid. Based on the modifications of [3] we make additional modifications.

We revert modification 2: In our construction $M_0$ will be constructed as a DTM again. Most things should go through smoothly as the $M_0$ of [1] was a DTM anyways, we only have to pay attention to the parts of [3] where Mr. Tianrong invokes nondeterminism. This happens in exactly two places as far as we were able to observe: modification 3(d) and 3(e).

In 3(d) Mr. Tianrong already admitted that “$M$ is somewhat deterministic” (recall: $M$ is the name in [3] for $M_0$). It seems that nondeterminism is not essential here. At least Mr. Tianrong was not able to explain to the author via emails, why a DTM could not do the same work required to successfully complete step 5. As step 5 in [3] is almost word by word identical to step 5 in [1] except for the introduction “By using nondeterminism in $(1 + \lceil \log_2 s \rceil + \lceil \log_2 t \rceil)S(n)$ cells”, 3(d) cannot be considered an obstacle to switching back to a DTM $M_0$.

In 3(e) something rather strange happens: $x$ is not a valid encoding of a DTM (“Since $x$ is not encoding of some single-tape DTM.”), but nevertheless this garbage code $x$ should be the program governing the execution of $M_0$ a.k.a. $M$ later on (“By using its nondeterministic choices, $M$ moves as per the path given by $x$.”). I did not get an explanation from Mr. Tianrong on this issue, in particular, he was not able to detail to me how essential nondeterminism must enter here. It is also completely unclear to me, how step 6 should be useful and what purpose it serves. Just after explaining the five steps a valid code word $w$ with some additional properties is constructed in both [1] and [3] and nothing that follows the five steps seems to depend on the behavior of non-code words. So this will be our second additional modification: we go back to the original construction of [1], which is simply halting without accepting for any $x$ not a code word.

Overall, this modifications on top of Mr. Tianrong’s modifications remove the nondeterminism and we demonstrated that, if the argument before our modifications was correct, it is so after our modifications.

With this, we could establish

**Theorem 3’**. Let $S(n) \geq \log n$ be a space-constructible function. Then there exists a language $L_d$ accepted by a DTM by using space $O(S(n))$ but by no DTM of space complexity $S(n)$. That is, $L_d \in \text{DSPACE}[S(n)]$ but $L_d \not\in \text{DSPACE}[S(n)]$. 

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Clearly, the last part is an obvious contradiction. This shows that the assumption that Mr. Tianrong’s argument for Theorem 3. was correct must be wrong. Even worse, the whole approach of simply modifying the proof of [1 Thm. 11.1.] by replacing faster growth (in form of the pair $S_1(n)$ and $S_2(n)$) by nondeterminism does not work (at least not without major new ideas, which are absent in the work of Mr. Tianrong), as it is easily possible to go back from NTMs to DTMs as demonstrated above arriving at a fatal contradiction.

4 The reason for the failure of Mr. Tianrong’s strategy

We believe that the source of this contradiction lies in modification 3(c). Mr. Tianrong changes the space bound for $M_0$ a.k.a. $M$ from $S_2(n)$ to $(1 + \lceil \log_2 s \rceil + \lceil \log_2 t \rceil)S(n)$. At this stage the input $x$ is still allowed to vary freely over all binary words. With changing $x$ the length $n = |x|$ varies and also the associated DTM $M_x$ (assuming the code word is valid) and with it $s$ and $t$. For what follows let $s(n)$ be the maximum of all cardinalities of states $s$ for some DTM $M_x$ associated to a code word $x$ of length $n = |x|$ and introduce mutatis mutandis $t(n)$ for the lengths of encoded tape alphabets. Let $a(n)$ be the analog maximum for the expression $\lceil \log_2 s \rceil + \lceil \log_2 t \rceil$.

Let’s assume $a(n)$ was bounded. Then $s(n)$ and $t(n)$ were both bounded and we could only encode a finite number of single-tape DTMs; but we need to encode all DTMs which use space no more than $S(n)$ tape cells. However, if we had $S(n) = 0$, corresponding to DFAs, we would already have infinitely many different languages (regular languages) and therefore infinitely many different automata. The more for $S(n) \geq \log n$. Thus $s(n)$ and $t(n)$ cannot both be bounded, if we want to prove something about the infinite set of languages $\text{DSPACE}[S(n)]$. But then $(1 + a(n))S(n) \not\in O(S(n))$ or informally “$(1 + \lceil \log_2 s \rceil + \lceil \log_2 t \rceil)S(n) \not\in O(S(n))$” and this is, where the error is hidden.

How does [1] deal with this issue? They work with $S_2(n)$ instead of $(1 + \lceil \log_2 s \rceil + \lceil \log_2 t \rceil)S(n)$, but [1] does not require $S_2(n) \in O(S_1(n))$. Rather to the contrary $S_2(n)$ is supposed to grow faster than $S_1(n)$ in the sense of their infimum condition, thus there is not even an issue for [1] to deal with.

5 Summary

We outlined the proof strategy of [3] and gave a teleological argument why it cannot work and is beyond repair. Precisely, if assumed to be correct, slight and logically harmless modifications would yield contradicting results. We pinpointed the main source of this issues in a simple-minded modification of a correct proof of [1 Thm. 11.1.] trading faster growth by insinuating to nondeterminism to get a flawed attempt at a proof of [3 Thm. 3.]. Since any new claims made by Mr. Tianrong all built on his [3 Thm. 3.], all his claims on proving any new result (Thms. 1., 2. and 3. and Corollaries 1. and 2.) are unjustified and void.
References

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