Elements of Fedosov Geometry in Lagrangian BRST Quantization

A.A. Reshetnyak

Tomsk State Pedagogical University, 634041 Tomsk, Russia

Abstract

A Lagrangian BRST quantization for generic gauge theories in general irreducible non-Abelian hypergauges is proposed on a basis of the multilevel Batalin–Tyutin formalism and a special BV–BFV dual description for a reducible gauge model in a symplectic supermanifold $M_0$ locally parameterized by antifields for Lagrangian multipliers and by the fields of the BV method. The quantization rules are based on a set of nilpotent anticommuting operators $\Delta^M, V^M, U^M$ defined using some odd and even symplectic structures in a supersymplectic manifold $M$ whose local representation is an odd (co)tangent bundle over $M_0$ provided by the choice of a flat Fedosov connection and a compatible non-symplectic metric in $M_0$. The generating functional of Green’s functions is constructed in terms of general coordinates in $M$ with the help of contracting homotopy operators with respect to $V^M$ and $U^M$. We prove the gauge independence of the S-matrix and derive the Ward identity.

1. Introduction

The conventional form of Lagrangian \cite{1, 2} and Hamiltonian \cite{3, 4, 5} quantization for general gauge theories, implementing the BRST (BRSTantiBRST) symmetry and maintaining locality and global symmetries, was developed around 15–20 years ago and is largely sufficient for perturbative quantization of gauge models set within the variational principle. Nonetheless, it is of interest to study some additional problems related to geometrically covariant descriptions of the quantization procedure reflecting the global and invariant properties of a manifold of field variables in specific models. This activity was originated by the Lagrangian multilevel formalism \cite{6} and continued by the Hamiltonian coordinate-free formalism \cite{7}, based on the Weyl symbols, as well as by the (modified) triplectic \cite{8, 9} BRSTantiBRST scheme. On the other hand, it is closely related (via the notion of supertime $\chi = (t, \theta)$) with the problem of an equitable description for the dynamics and BRST transformations of a model when treated by the Lagrangian \cite{10, 11, 12, 13} and Hamiltonian \cite{14, 15, 16} superfield quantization.

A solution of the first series of problems with reference to the general aspects of quantization is the construction of a $*$-product by Kontsevich \cite{17} within a deformation quantization for the algebra of functions in an arbitrary Poisson manifold $M_0$ to which one applies \cite{18} a topological Poisson $\sigma$-model defined in $M_0$, whose field-antifield spectrum, following the AKSZ approach \cite{19} in an $N = 2$ superfield (non-spacetime) formulation, coincides with the corresponding fields and antifields of the BV method. One of these problems is a construction of deformation quantization for dynamical systems with second-class constraints and symplectic manifolds \cite{20} on a basis of the BFV–BRST \cite{3} conversion methods \cite{21}, as well as for non-(Lagrangian)-Hamiltonian gauge theories \cite{22}, such as higher-spin gauge fields \cite{23}, where one essentially uses a symmetric connection compatible with a symplectic structure, i.e., the Fedosov connection \cite{24, 25}.

While the Lagrangian BRSTantiBRST (superfield) quantization \cite{26, 27} defined in general coordinates maintains a tensor character of compatible differential operations, i.e., extended antibrackets and odd operators $(\Delta^a, V^a, U^a), a = 1, 2$, only in the case of a flat Fedosov connection in a supermanifold $M_0$ using Darboux coordinates parameterized by the fields $\phi^A$ of the BV method and by the corresponding antifields $\bar{\phi}_A$, being sources for the commutator of BRSTantiBRST transformations \cite{2}, the construction of Lagrangian BRST quantization in irreducible \cite{6} and reducible (originally introduced in \cite{11}) non-Abelian hypergauges \cite{4}, in fact, does not utilize the concept of Fedosov connection. The local superfield

\footnote{E-mail addresses: A-Reshetnyak@yandex.ru, reshet@tspu.edu.ru}

\footnote{The study of the influence of reducibility and non-Abelian properties of hypergauges on specific gauge models in quantum calculations presents an essential part of \cite{28}.}

\footnote{Except for a special connection $F_\rho(\Gamma) = \delta[\rho(\Gamma)]/\delta \Gamma^p$ in \cite{6}, expressed using a density functional $\rho(\Gamma)$ in the “first-level” supermanifold $\mathcal{N} = \{\Gamma^p\}$ determining the (anti)fields $(\phi^A, \bar{\phi}_A)$ of the BV method in the Darboux coordinates $\Gamma^p = (\phi^A, \bar{\phi}_A)$.}
BRST quantization [11, 12] shows that restricting the consideration to the ingredients of the first-level formalism, in view of a special nature of Lagrangian multipliers $\lambda^a$ for hypergauges $G_0(\Gamma)$, is insufficient to introduce a covariant derivative in $N$. For this purpose, one must utilize not only $\lambda^a$, but also the antifields $\lambda^*_a$ that arise in the second-level formalism [4].

This report aims essentially at the following:

1. Description of a gauge algebra for a reducible gauge model by means of a special BV–BFV duality between odd $N_{\text{min}}$ and even $M_{\text{min}}$ symplectic manifolds underlying the quantization procedure and intersecting with a manifold parameterized by the minimal-sector fields of the BV method.

2. Investigation of a supersymplectic structure of the quantization manifold $\mathcal{M}$ compatible with the requirement of anticommutativity for a set of nilpotent operators $\Delta^M, V^M, U^M$.

3. Formulation of quantization rules for gauge models using general coordinates in $\mathcal{M}$ with an essential use of operators $V^+, U^+$ (constructed in terms of even and odd Poisson brackets), whose difference $(U^+ - V^+)$ is a contracting homotopy with respect to $V^M$ for an operator $N^M$ nondegenerate for nonconstant elements of $C_\infty(M)$.

2. Special BV–BFV dual description for a gauge model Recall that an $L$-stage reducible gauge theory based on the variational principle for classical fields $A^i$, $i = 1, \ldots, n = n_+ + n_-$ (with Grassmann parities $\varepsilon(A^i) = \varepsilon_i$ in condensed notation) is defined by a classical bosonic action $S_0(A): C_\infty(M_2) \rightarrow \mathbb{R}$, $M_2 = \{A^i\}$, invariant with respect to gauge transformations, $\delta A^i = R^i_{\alpha_0}(A)x_0^0, \alpha_0 = 1, \ldots, m_0 = m_{0+} + m_{0-}$, $\varepsilon(x_0^0) = \varepsilon_{\alpha_0}$, $S_0, R^i_{\alpha_0}(A) = 0$, for rank $\|S_0, i\| \|S_0, k = 0 \| = \overline{\mu} - \overline{\mu}_1 = (n_+ + n_-) - (m_{1+} - m_{1-}), S_0, \delta \equiv \delta S_0/\delta A^i$, (1) with reducibility relations in condensed notation, for $s = 1, \ldots, L$,

$$
Z_{\alpha_1,1}^{\alpha_1,1}(A)Z_{\alpha_1,1}^{\alpha_1,1}(A) = S_{0, i} I_{\alpha_s}^{-1} (A), \quad \alpha_s = 1, \ldots, m_s = m_{s+} + m_{s-}, \quad m_{s-1} > \sum_{k=0}^{s-1} (-1)^k \overline{\mu}_{k-s+} = 0 = \|Z_{\alpha_s} I_{\alpha_s}^{-1} \| = \|Z_{\alpha_s} I_{\alpha_s}^{-1} \| = (-1)^{i+s}(-1)^{l+s}K_{\alpha_1}^{\alpha_1}.
$$

Definitions (1) and (2) partially determine the first-order structure relations and functions of the gauge algebra, and are encoded, due to the corresponding Koszul–Tate complex resolution [19], when extended (following the BV method) to an odd-Hamiltonian description of the model in $\Pi T^* M_{\text{min}} = \{\Gamma_k^p = (\phi^A, \phi^*_A)|\phi^A = (A, C^\alpha, s = 0, \ldots, L), A^k = 1, \ldots, n = n + \sum_{r=0}^{L} m_r; k = \min\}$ by means of a bosonic functional and a classical master-equation in the minimal sector [11].

$$
S_k(\Gamma_k) = S_0(A) + \sum_{s=0}^{L} \left( C_{\alpha_s,1}^{\alpha_s,1}(A) + o(\phi^A) \right) C_{\alpha_s} + o(C_{\alpha_s}), \quad (\varepsilon, gh)S_k = 0,
$$

$$
S_k(\Gamma_k) = S_0(A) + \sum_{s=0}^{L} \left( C_{\alpha_s,1}^{\alpha_s,1}(A) + o(\phi^A) \right) C_{\alpha_s} + o(C_{\alpha_s}), \quad \delta \Gamma_k \equiv \delta \Gamma_k = 0, \quad \omega^{pq}_{\alpha_s} = \text{antidiag}(-1, m, 1, \ldots, -1).
$$

The quantum action $S^q(\Gamma_k, \hbar)$ of the BV method (in what follows, $k = \text{ext}$) is constructed, first, by an extension of $S_0$, using the pyramids of ghosts and Nakanishi–Lautrup fields, up to $S_k(k, \hbar)$ defined in $\Pi T^* M_k = \{\Gamma_k^p = (\phi^A, \phi^*_A)|\phi^A = (\phi^A_{\text{min}}, C_{\alpha_s}^*, B_{\alpha_s}^*, s^* = 0, \ldots, s = 0, \ldots, L), A^k = 1, \ldots, k = \sum_{r=0}^{L}(2r + 3)m_r; k = \text{ext}\}$, and, second, by imposing an Abelian hypergauges corresponding to a phase anticanonical transformation in $\Pi T^* M_k$ for an $\hbar$-deformed $S_k(k, \hbar)$.

$$
S_k(\Gamma_k) = S_{\text{min}} + \sum_{s=0}^{L} \left( C_{\alpha_s,1}^{\alpha_s,1}(A) + o(\phi^A) \right) C_{\alpha_s} + o(C_{\alpha_s}), \quad W^q(\Gamma_k, \hbar) = \exp \left[ (\psi(\phi^A), k) \right] S_k(k, \hbar).
$$

The functional $\{S_k, W^q\}(\Gamma_k, h)$ obey a quantum master-equation and provide its proper solutions in terms of a nilpotent operator $\Delta^k$ constructed using a nondegenerate antibracket in $\Pi T^* M_k$, as well as using a trivial density function $\rho(\Gamma_k), \rho = 1$, and $\omega^{pq}_{\alpha_s}(\Gamma_k), \|\omega^{pq}_{\alpha_s}\| = \text{antidiag}(1, m, 1, \ldots, -1), \omega^{pq}_{\alpha_s} = \delta^p_q$.

$$
\Delta^k \exp \left[ \frac{i}{\hbar} E(\Gamma_k, \hbar) \right] = 0, \quad E \in \{S^q, S_k\}, \quad \Delta^k = \frac{1}{2} (-1)^{i}(\Gamma^q) \rho^{-1} \omega^{pq}_{\alpha_s} (\Gamma^q, \rho((\Gamma^q, \cdot) k)^k).
$$

3In what follows, we often omit the prefix “super” in “supermanifold”.

4$M$ locally represents a vector bundle over the manifold $M_0, M \rightarrow M_0$, so that $M_0 \supset M_{\text{min}}; N \supset N_{\text{min}} \supset M_0, N \subset M$.

5The coordinates in $\Pi T^* M_k$ possess the following distribution of Grassmann parity and ghost number $[3]$: $(\varepsilon, gh)C_{\alpha_s} = (\varepsilon_{\alpha_s} + s + 1, s + 1), gh(A^i) = 0, \varepsilon(\phi^{A^i}) = \varepsilon_{\alpha_k}; (\varepsilon, gh)\phi^*_A = (\varepsilon_{\alpha_k} + 1, -1 - gh(\phi^{A^i})))$. 

\[2\]
In the second-level formalism \[6\], the presence of antifields \(\lambda^*_a\) for Lagrangian multipliers \(\lambda^a\) introducing non-Abelian first-level hypergaugettes \(G_a(\Gamma_k)\) to the exponent of the path integral \[3\] \(Z^{(1)}\) allows one to construct a special BV–BFV dual description in the cotangent bundle \(T^*\mathcal{M}_k = \{x^*_p = (\phi^A_k, \lambda^*_A_k)\}\), \((\varepsilon, gh)\lambda^*_A_k = (\varepsilon_A, -2 - gh(\phi^A_k))\) for a gauge theory of rank 1 [in general, in a sub-bundle \(\mathcal{N}_{aux} \to \mathcal{M}_k\) of the bundle \(\Pi T^*\mathcal{M}_k \supset \mathcal{N}_{aux} \supset T^*\mathcal{M}_k\) with a fiber over \(\phi^A_k: \mathcal{F}_{\phi^A_{aux}} = \{(\lambda^*_A_k, \phi^A_k)\}\) by means of a BRST-like charge. This object is nilpotent for \(h = 0\) with respect to an even Poisson bracket defined in \(C^\infty(T^*\mathcal{M}_k)\) \([C^\infty(\mathcal{N}_{aux})\)] and determines a formal dynamical system subject to first-class constraints by means of an algorithm which differs from that of \[12, 15, 19\]. To this end, let us consider a functional \(\Omega_k(x_k, \phi^A_k) \in C^\infty(\mathcal{N}_{aux}), (\varepsilon, gh)\Omega_k = (1, -1),\) constructed using nilpotent fermionic quantities \(V^*_k\) and \(\eta = \text{const}, gh(\eta) = -1\),

\[
\Omega_k = V^*_k S_k(\Gamma_k, h) + \eta S_0(A) = \lambda^*_A_k \frac{\delta}{\delta \phi^A_k} + \eta S_0, \tag{6}
\]

as well as using an odd operator \(\Delta^k_\lambda\) dual to \(\Delta^k\) and an even Poisson bracket \{ , \} nondegenerate in \(T^*\mathcal{M}_k\)

\[
\Delta^k_\lambda = \eta(-1)^{\varepsilon\lambda_k} \frac{\delta}{\delta \phi^A_k} \frac{\delta}{\delta \phi^A_k}, \quad \{ , \}^k = \left( \frac{\delta}{\delta x^*_p}, \frac{\delta}{\delta x^*_p} \right) \tag{7}
\]

\[
\varepsilon(\omega^pq_{\phi^A_k}) = \varepsilon(\omega^pq_{\phi^A_k}) + 1 = \varepsilon(x^p_k) + \varepsilon(x^q_k), \quad \left\| \omega^pq_{\phi^A_k} \right\| = \text{antidiag}(1, 1) = 0.
\]

The operator \(V_k\), being a contracting homotopy with respect to a nilpotent operator \(V_k, V_k = \phi^A_k \frac{\delta}{\delta \phi^A_k}\), for the operator \(N_{V_k}^+, N_{V_k} = [V_k, V^*_k]_+, \) nondegenerate in the fibers \(\mathcal{F}^N_{\phi^A_{aux}}\), possesses, along with \(V_k\), the following properties \[\]

\[
[V^*_k, \Delta^k_\lambda]_+ = 0, \quad V^*_k \{F, G\}^k = (V^*_k F, G)^k - (-1)^{\varepsilon(F)} (F, V^*_k G)^k,
\]

\[
[V_k, \Delta^k_\lambda]_+ = 0, \quad V_k \{F, G\}^k = [V_k F, G]^k + (-1)^{\varepsilon(F)} \{F, V_k G\}^k. \tag{8}
\]

The gauge algebra relations \[11–13\] and eqs. \[4, 5\] are equivalently described using a correspondence between Poisson brackets of opposite parities for arbitrary functionals \(F_t(\Gamma_k) \in C^\infty(\Pi T^*\mathcal{M}_k)\), \(F_t(x_k, \phi^A_k) = \{V^*_k F_t(\Gamma_k) + \eta F_{\phi^A}(\phi_k)\}, F_{\phi^A} \equiv F|_{\phi^A_k} \in \mathcal{N}_{aux}\) ker \(N_{V_k}\), where \(t = 1, 2,\)

\[
V_k \{F_1, F_2\}^k = N_{V_k} (F_1, F_2)^k - F_1 \left( \frac{\delta}{\delta \phi^A_k} \frac{\delta}{\delta \phi^A_k} (N_{V_k} - 1) - \left( N_{V_k} - 1 \right) \frac{\delta}{\delta \phi^A_k} \frac{\delta}{\delta \phi^A_k} \right) F_2, \tag{9}
\]

\[
\Delta^k F_t = \eta \Delta^k F_t; \quad \{\Omega_k, \Omega_k\}^k = -V^*_k (S_k, S_k)^k - 2S_k \frac{\delta}{\delta \phi^A_k} \frac{\delta}{\delta \phi^A_k} V^*_k S_k - 2\eta S_0, \quad \frac{\delta}{\delta A^*_k} S_k,
\]

\[
= -2S_k \frac{\delta}{\delta \phi^A_k} \frac{\delta}{\delta \phi^A_k} V^*_k S_k - 2\eta S_0, \quad \frac{\delta}{\delta A^*_k} S_k, \quad \text{for } k = \text{min, ext,}
\]

with \(\hat{N}_{V_k} = \frac{\delta}{\delta \phi^A_k} \phi^A_k + \frac{\delta}{\delta A^*_k} \lambda^*_A_k\) subject to \(N_{V_k} = \mathcal{F} \hat{N}_{V_k}\). From eqs. \[9, 11\], it follows that in rank-1 gauge theories (i.e., \(S_k = S_0 + \phi^A_k H^A(\phi_k) \Leftrightarrow V^*_k S_k \in T^*\mathcal{M}_k, k = \text{ext}\)) to whose class one can always reduce an initial gauge model (modulo the conservation of locality and covariance) the BFV–BRST quantities dual to \(S_k, S^0\) are defined only in \(C^\infty(\Pi T^*\mathcal{M}_k)\). Therefore, only in rank-1 gauge theories does the equation \(\{\Omega_k, \Omega_k\}^k = 0\) hold true, when \(\Delta^k S_k = 0\), with allowance for \[11\] and \((N_{V_k} - 1) S = 0.\) As

---

\[6\] In the case of a fiber bundle \(\Pi T^*\mathcal{M}_k\): \(a = A_k, G_a = G_A(\Gamma_k) = (\phi^A_k - \delta \psi / \delta \phi^A_k)\) and \([\lambda^a, \lambda^*_a] = [\lambda^A_k, \lambda^*_A_k]\), this corresponds to the construction of \(S^0(\Gamma_k, h)\) in \[14\].

\[7\] In general, \(\Pi T^*(\Gamma_k) = \{x^*_p(\phi^A_k, \lambda^*_A_k)\}\) admits the existence of nilpotent operators \((U_k, V^*_k, \Pi \Delta^k_\lambda) = \{(-1)^{\varepsilon A_k} \lambda^*_A_k \frac{\delta}{\delta A^*_k}, (-1)^{\varepsilon A_k} \phi^A_k \frac{\delta}{\delta A^*_k}, \eta(-1)^{\varepsilon A_k+1} \frac{\delta}{\delta A^*_k} \phi^A_k\}\) similar to \((V_k, V^*_k, \Delta^k_\lambda)\), which obey the same properties as in \[8\] with the corresponding exchange \((V_k, V^*_k, \Delta^k_\lambda) \leftrightarrow (U_k, U^*_k, \Pi \Delta^k_\lambda);\) they also anticommute between themselves, \([E, D]_k = 0, E \in \{V_k, V^*_k, \Delta^k_\lambda\}, D \in \{U_k, U^*_k, \Pi \Delta^k_\lambda\}\), and yield the operator \(N_{U_k} = [U_k, U^*_k]_+\) nondegenerate in the fibers \(\mathcal{F} \Pi T^*(\Gamma_k)\).
a consequence,
\[
\frac{1}{2} \{ E_k(x_k), E_k(x_k) \}^k = -\frac{1}{2} V^\ast_k \left( \mathcal{E}_k(\Gamma_k), \mathcal{E}_k(\Gamma_k) \right)^k = 0, \quad i\hbar \Delta^k E_k = i\hbar \Delta^k \mathcal{E}_k = 0, \tag{11}
\]
with \( E_k \in \{ \Omega_k, \Omega^\psi \}, \mathcal{E}_k \in \{ S_k, S^\psi \}, \Omega^\psi = \exp \{ [F, ] \} \Omega_k, \)
\[
\tag{12}
\]
with a gauge boson \( F(\phi), F(\phi) = \eta \psi(\phi_0), \) so that \( \Omega^\psi = \Omega_k + \eta(\psi, S_k) \) for \( k = \text{ext} \).

3. Poisson brackets and triplectic-like algebra of \( \Delta^M, V^M (\Psi^r, U^r) \) Leaving aside the implementation of a gauge model within the BV method, let us consider a Poisson supermanifold \( (M_0, \{ \cdot, \cdot \}_0), M_0 = \{ x^p \}, \dim M_0 = \dim T^\ast M_0 \) equipped with an even Poisson bracket defined by a tensor (bivector) field \( \omega^{pq}(x) \) over \( M_0, \omega^{pq} = -\langle \varepsilon_p, \varepsilon_q \rangle \omega^{pq}, (\varepsilon, gh)\omega^{pq} = \langle \varepsilon_p + \varepsilon_q, gh(x^p) + gh(x^q) + 2 \rangle, \varepsilon(x^p) = \varepsilon_p, \) whose Jacobi identity is
\[
\omega^{pq}(\delta_t \omega^{rs}/\delta x^p) (-1)^{r+s} + \text{cycle}(r, s, \varepsilon) = 0, \tag{13}
\]
and also equipped with a covariant derivative \( \nabla_p \) in \( M_0 \) transforming \( M_0 \) into a Poisson supermanifold with a symmetric connection \( \Gamma^p_{rs}(x), \Gamma^p_{rs} = -(1)^{r+s} \Gamma^p_{sr} \) (for a nondegenerate \( \omega^{pq}(x) \) providing the existence of quantities \( \omega_{pq}(x), \omega_{pq} = -\langle \varepsilon_p, \varepsilon_q \rangle \omega_{pq}, \omega^{pq} \omega_{qr} (-1)^{r+s} = \delta^p_r \), \( M_0 \) transforms into a Fedosov supermanifold \( [30] \))
\[
\nabla_p \omega_{pq} = \frac{1}{2} \left( \frac{\delta \omega^{pq}}{\delta x^p} / (\delta x^p) + 2 \omega^{pq} \Gamma^q_{rs}(x) (-1)^{r+s} (\varepsilon + 1) \right) - (1)^{r+s} (p \leftrightarrow q) = 0. \tag{14}
\]
Let us introduce a manifold \( M = \{ x^p, \eta_p \} \) locally implemented as an odd (co) tangent bundle over \( M_0, M = \Pi^T M_0 \simeq \Pi T M_0, \) whose fibers are parameterized by covariantly constant vectors \( \eta_p, (\varepsilon, gh)\eta_p = (\varepsilon_p + 1, 1 - gh(x^p)), \) being the antifields for \( x^p \) \( [M_0 = T^\ast M_0, (x^p, \eta_p) = (\delta A, \lambda^s, \phi^s_A, \lambda A_k)] \). We next define a functional \( T(x, \eta) \) being a scalar w.r.t. a covariant derivative (extended to act in \( M \) \( \nabla^M_p, \)
\[
T = \frac{1}{2} \eta \omega^{pq}(x) \eta_q, \quad (\varepsilon, gh) T = (0, 0), \quad \nabla^M_p T = \frac{\delta T}{\delta x^p} + \frac{\delta T}{\delta \eta_q} \eta_q \Gamma^q_{p q} = 0, \tag{15}
\]
by virtue of eqs. \( [13] \) and the relations \( \nabla^M_p \eta_q = 0. \) Equipping \( M_0 \) with a bosonic scalar density \( \rho(x) \) is sufficient to determine a set of covariant operations characteristic for the supersymplectic manifold \( M \): an antibracket \( (\cdot, \cdot)^M \) and operators \( \Delta^M, V^M, \)
\[
(\cdot, \cdot)^M = \left( \nabla^M_p \right) \frac{\delta \cdot}{\delta \eta_p} - \left( \nabla^M_p \right) \frac{\delta \cdot}{\delta \eta_q} \nabla^M_q, \quad \Delta^M = -\langle \varepsilon \rangle \delta \cdot \delta \eta_q \left( \nabla^M_q + \frac{1}{2} \delta \varepsilon \rho \right), \quad V^M = \langle T \rangle^M = -\eta_p \omega^{pq} \nabla^M_q, \tag{16}
\]
which (using an explicit verification for scalars in \( M \)) can be shown to obey the relations of a triplectic-like algebra \( [20, E_1, E_2]_+ = 0 \) for \( E_1, E_2 \in \{ \Delta^M, V^M, U^M \}, \) which are consistent with a Leibniz rule similar to \( [33] \) for differentiating an antibracket by any \( E_1, E_2 \) only in the case of a flat Poisson manifold \( M_0 \) \( [14] \).

An obvious representation of \( M \) as \( \Pi T M_0 \) allows one to lift the nondegenerate Poisson structure \( \{ \cdot, \cdot \}_0 \) to a flat Fedosov manifold \( \{ \cdot, \cdot \} \) by the relation
\[
\{ \cdot, \cdot \} = \left( \nabla^M_p \right) \omega^{pq} \left( \nabla^M_q \right) + \alpha \delta \varepsilon \delta \eta_q \omega_{pq} (-1)^{r+s} \frac{\delta \cdot}{\delta \eta_q}, \quad \alpha = \text{const} \in \mathbb{R}. \tag{18}
\]
A covariant definition for a nilpotent operator \( U^M \) satisfying the relations \( [E_1, E_2]_+ = 0 \) for \( E_1, E_2 \in \{ \Delta^M, V^M, U^M \} \) as in Sec. 2 (see footnote 7), in comparison with the implementation \( [20] \) of a modified triplectic algebra, is impossible in terms of an anti-Hamiltonian vector field, but is provided by equipping \( M_0 \) with an additional Riemann-type nondegenerate even structure \( g_{pq}(x), g_{pq} = (-1)^{r+s} g_{pq}, \) in the form
\[
U^M = -\eta_p \omega^{pq} g_{st} \omega^{ts} (-1)^{r+s} \nabla^M_q, \quad (U^M)^2 = 0 \iff \nabla^M_q g_{pq} = 0. \tag{19}
\]
Since the action of $\nabla$ is defined as a tensor operation in the coordinates $x^p$, an explicit definition for an operator $V^* \mathcal{M} (U^* \mathcal{M})$ of contracting homotopy w.r.t. $V^* \mathcal{M} (U^* \mathcal{M})$ for a nondegenerate (as applied to nonconstant functions in $C^\infty (M)$) operator $N^\mathcal{M} (N^\mathcal{M})$ is possible only for a symplectic $\mathcal{M}_0$, i.e., for $\Gamma^p r_s=0$. To this end, we consider the adjoint action of a fermionic functional $\Omega_T(x, \eta)$, $\Omega_T = -\eta_p x^p$, w.r.t. the non-tensor bracket (15), and then, in order to choose some operators $V^*, U^*$ analogous to $V_k^*, U_k^*$ of Sec. 2, we define an operator $U^* \mathcal{M}$ using the non-tensor bracket (10) with a bosonic $T^* (x, \eta)$, $T^* = -(1/2) x^p g_{pq}(x).x^q (-1)^{\varepsilon_r}$.

$$V^* + \alpha V^{* \mathcal{M}} = \{ \Omega_T, \} = -\eta_p x^p \frac{\delta_i}{\delta x^q} - \alpha (-1)^{\varepsilon_r} x^p \omega_{pr} \frac{\delta_i}{\delta \eta_r}$$

$$U^{* \mathcal{M}} = (T^*, )^{\mathcal{M}} = -x^p \left( g_{pq}(x)(-1)^{\varepsilon_q} + \frac{1}{2} x^p \frac{\delta_r g_{pq}}{\delta x^q} \right) \frac{\delta_i}{\delta \eta_q}$$

Necessary conditions for the quantities $V^{* \mathcal{M}} = 1/\alpha \left( \{ \Omega_T, \} - (T, )^{\mathcal{M}} \right) = -(1)^{\varepsilon_r} x^p \omega_{pr} \left( \delta_i / \delta \eta_r \right)$, $U^{* \mathcal{M}}$ to be nilpotent, as well as to anticommute between themselves and also with $\Delta^\mathcal{M}$, are given by the fulfilment of $(D, D)^{\mathcal{M}} = 0, \Delta^\mathcal{M} D = 0$ for $D \in \{ T, T^* \}$, whereas $N^\mathcal{M}$ is defined according to

$$N^\mathcal{M} = [V^* \mathcal{M}, V^{* \mathcal{M}}] = \left[ \eta_p x^p \frac{\delta_i}{\delta x^q} (-1)^{\varepsilon_r} x^p \omega_{sr} \frac{\delta_i}{\delta \eta_r} \right] = \left[ x^p \frac{\delta_i}{\delta x^q} + \eta_p \left( \delta_q + x^p \frac{\delta_i x^q}{\delta x^q} \right) \frac{\delta_i}{\delta \eta_q} \right]$$

We next introduce some mutually anticommuting nilpotent operators $V^*$ and $U^*$ whose difference coincides with $-V^{* \mathcal{M}}$.

$$(V^*, U^*) = (1/2) \left( \begin{array}{c} (1/\alpha) \{ (T, )^{\mathcal{M}} - \{ \Omega_T, \} \} \end{array} - (T^*, )^{\mathcal{M}} \right), \left( (1/\alpha) \{ (\Omega_T, \} - (T, )^{\mathcal{M}} \} - (T^*, )^{\mathcal{M}} \right).$$

The decomposition (23) and the equations for $T, T^*$ impose a number of restrictions on the choice of $(g_{pq}, \omega_{pq})(x)$ and reduce, by virtue of (20, 21) the quantities $V^*, U^*$ to a simpler representation. In the particular case of Darboux coordinates, also assuming the structure of $g_{pq}(x)$ and the vanishing of the connection $\Gamma^p r_s(x)$ in $\mathcal{M}_0$,

$$(x^p, \eta_p, \Gamma^p r_s(x), \rho(x)) = \left( (\phi^A, \lambda^*_A), (\phi^*_A, \lambda^A), 0, 1 \right), \quad \left[ \omega_{pq}, g_{pq}(x) \right] = \text{antidiag} \left[ (\alpha (-1)^{\varepsilon_A} \delta^A B, (\delta^A B, (-1)^{\varepsilon_A} \delta^A B) \right],$$

we obtain the correspondence

$$(V^* \mathcal{M}, V^*, U^*, \Delta^\mathcal{M}) = \left( V_k - U_k, V_k^*, U_k^*, \Delta^k + (-1)^{\varepsilon_A+1} \frac{\delta_i}{\delta \lambda^A} \frac{\delta_i}{\delta \lambda^*_A} \right),$$

for $k = \text{ext}$, with account taken of $\omega_{pq} = \text{antidiag} ((-1)^{\varepsilon_A} \delta^A B, -\delta^A B)$, thereby providing $\omega_{pq} \omega_{qr} (-1)^{\varepsilon_q} = \delta_r^p$.

4. Quantization rules Let us define a vacuum functional and a generating functional of Green’s functions in general coordinates $z^p = (x^p, \eta_p)$,

$$Z^\mathcal{M}_X = \int d\bar{z} D_0(\bar{z}) q^\mathcal{M}(\bar{z}) \exp \left\{ \frac{i}{\hbar} \left( W + X \right) - H \right\}(\bar{z}, \hbar)$$

$$Z^\mathcal{M} \left[J_V, J^\mathcal{M} \eta \right] = \int d\bar{z} D_0(\bar{z}) q^\mathcal{M}(\bar{z}) \exp \left\{ \frac{i}{\hbar} \left[ W(\bar{z}, \hbar) + [X + i\hbar H] (\bar{z}, \eta - \hbar) + J^\mathcal{M}_\eta \eta + J^\mathcal{M}_p \bar{z} \right] \right\}$$

where $J^\mathcal{M}_V, J^\mathcal{M}_p$ form a redundant set of sources for the corresponding variables $x^p, \eta_p$ with the properties $(\varepsilon, gh) J^\mathcal{M}_p = (\varepsilon_p, -gh(x^p)) = (\varepsilon, -gh) J^\mathcal{M}_p + (1, -1)$; $H, W, X$ denote a set of bosonic functionals providing a correct reduction, e.g. for $Z^\mathcal{M}_X$, to the BV partition function $\Pi$, as well as a respective quantum action and a gauge-fixing bosonic action for irreducible hypergauges $G_a(z)$ satisfying generalized master-equations and one more restriction on $W$,

$$\Delta^\mathcal{M} \exp [(i/\hbar) E(z, \hbar)] = 0, \quad E \in \{ W, X + i\hbar H \}, \forall W(z, \hbar) = 0,$$
for the first of which a proper solution in $\mathcal{M}$ is provided only by $X$, while $W(z, h)$ is subject to a boundary condition $W(z, h)|_{y=h=0} = S_0(x)$, with the classical action $S_0$, and also subject to $U^*W(z, h) = 0$, for a non-Fedosov (merely symplectic) $\mathcal{M}_0$. At the same time, the density function $D_0(x)$, which determines an invariant measure in $\mathcal{M}$ (corresponding for $\alpha = 1$ to the even bracket $[\mathfrak{g}]$, so that the functional $\rho$ is defined as $\rho = \ln s\det^{-1}\|\omega_{pq}(x)\|$, following (20), and the weight functional $q^M[z]$ are given by

$$D_0(x) = s\det^{-1}\|\omega_{pq}(x)\|, \quad q^M(z) = \delta \left( G_{a_1}^V(z) \right), \quad a_1 = 1, \ldots, 1/2 (\dim \mathcal{M}_0 + \dim - \mathcal{M}_0). \quad (29)$$

In (27), (29), according to the decomposition of the operator $V^*M$ in (23), we have used a polarization of $V^*M$ into the corresponding difference of operators,

$$V^*M = V - U : (V, U) = \frac{1}{\hbar} \eta_p \left( -\omega_{pq} + \omega_{pq} g_{st} \omega^{sg}(-1)^{sz}, \omega_{pq} + \omega_{pq} g_{st} \omega^{sg}(-1)^{sz} \right) \vec{\nabla}_q^M, \quad (30)$$

with a nilpotent anticommuting $V, U$. In turn, the independent functions $G_{a_1}^V(z) = 0$, playing the role of second-level hypergauge conditions in the formalism [6], are necessary to retain the explicitly covariant form of both functionals $Z_X^M, Z^M$. The independent functions $G_{a_1}^V(z)$ are equivalent to a (known explicitly only for a symplectic $\mathcal{M}_0$) set of functions $V^* \eta_p, G_{a_1}^V(z) = [Y_{a_1}^p(z) V^* \eta_p]$ with certain $Y_{a_1}^p(z)$, so that

$$\text{rank} \left\| \vec{\nabla}_p^M E_t(z) \right\| \vec{\nabla}_q^M w = \delta h/\delta q = \vec{\nabla}_q^M x = \delta x/\delta q = \omega_{V^*M}^* = 0 = \frac{1}{2} \dim \mathcal{M}_0, \quad (E_1, E_2, (G_{a_1}^V, V^* \eta_p), \quad (31)$$

and define the functional $H(z, h)$ in (20)–(25) in an explicitly covariant form [6]:

$$H(z, h) = - \frac{1}{2} \ln \left\{ J(z) D_0^{-1}(x) s\det M \right\}, \quad M = \left( \begin{array}{c} (F_{a_1}^V(z), C_{V_1}^V(z)) \quad (G_{a_1}^V(z)) \quad (F_{a_1}^V(z), F_{a_1}^V(z)) \quad (G_{a_1}^V(z)) \quad (F_{a_1}^V(z), F_{a_1}^V(z)) \end{array} \right)^M, \quad (32)$$

where the functions $(F_{a_1}^V, F_{a_1}^V, C_{V_1}^V) = (Z_{a_1}^V U x_p, Z_{a_1}^V U \eta_p, Y_{a_1}^V V z_p)$ with certain $(Z_{a_1}^V, Z_{a_1}^V, Y_{a_1}^V)(z)$ determine an invertible change of variables, $z^p \rightarrow \mathbf{z}^p = (F_{a_1}^V, C_{V_1}^V, G_{a_1}^V, F_{a_1}^V)$, with the Jacobian $J = s\det \left| \vec{\nabla}_q^M \right| \delta z^p/\delta \mathbf{z}^p \right| |

The basic properties of the functionals $Z_X^M, Z^M$ are encoded by a generalized generator $s^M$ of BRST-like transformations with an arbitrary bosonic functional $R^M(z)$,

$$s^M = (h/\i) T^{-1}(z) \left( \nabla T(z) R^M(z) \right)^M, \quad T(z) = \exp \left( (i/\hbar) (W - X - i h H) \right)(z). \quad (33)$$

For instance, the BRST transformations with a constant $\mu, \delta_{\mu} s^p = s^M z^p \mu$, for $Z_X^M$ and $Z^M [0, 0, \eta]$, are derived using (33), with $R^M = 1$, and also using additional equations providing the BRST invariance of $q^M$, namely,

$$\left( G_{a_1}^V(z), T(z) \right)^M = 0 \iff \delta_{\mu} G_{a_1}^V(z) = 0. \quad (34)$$

Thus the constant (being covariantly constant for a flat Fedosov $\mathcal{M}_0$) functions $Y_{a_1}^p(x)$ belong to solutions of eqs. (33), imposing strong restrictions on the geometry of $\mathcal{M}$. The Ward identity for the functional $Z^M$ and the gauge-independence of the S-matrix follow from the master-equations (28), the transformation (33), and also from some additional equations for $G_{a_1}^V(z)$ (34), along the lines of [11] [12]. The transformation (33) in $Z_{X+\delta X}^M$, with the choice of $R^M$ as $R^M_{\mu} = \delta Y^M$, where the fermionic functional $\delta Y(z)$ is found from the equality $\delta X(z) = Q(X, H) \delta Y(z)$ [with a nilpotent $Q(X, H), Q(X, H) = (X + i h H)^M - i h \Delta^M$,] being a general non-vanishing solution for the linearized equation $Q(X, H) \delta X(z) = 0$, establishes that $Z_{X+\delta X}^M = Z_X^M$. In turn, after exponentiating the quantity $q^M$ in the functional integral (27),

$$q^M(z) = \int d\lambda_{(2)} \exp \left\{ (i/\hbar) G_{a_1}^V(z) \lambda_{a_1}^{(2)} \right\}, \quad \varepsilon(\lambda_{a_1}^{(2)}) = \varepsilon(G_{a_1}^V) = \varepsilon_{a_1}, \quad \text{for } \delta_{\mu} \lambda_{a_1}^{(2)} = 0,$n

the corresponding Ward identity (with a loss of covariance, $\vec{\nabla}_q^M \rightarrow (\delta_{\mu}/\delta x^p)$) is given by

$$\left[ J_{V^*} + \left( \delta_x W + \left( -1 \right)^{e_{a_1}} \lambda_{a_1}^{(2)} \delta_x G_{a_1}^V \right) \right] \frac{\delta g}{\delta \lambda_{a_1}^{(2)}} \left\{ \frac{h}{i} \frac{\delta_i}{\delta J_{V^*}} \frac{h}{i} \frac{\delta_i}{\delta J_Y} \right\} \delta_{\mu} \mathcal{D}^M \left\{ J_Y, J_{V^*}, \eta \right\} = 0. \quad (35)$$

To obtain eq. (35), we start with a functional averaging of the master-equation (28) for $(X + i h H)(\tilde{x}, \tilde{\eta} - \eta)$ w.r.t. the weight functional exp\left\{ (i/\hbar) [W(\tilde{x}) + G_{a_1}^V(\tilde{x}) \lambda_{a_1}^{(2)} + J_{V^*} \eta_p + J_{V^*} \tilde{z}^p] \right\}, and then use a
non-tensor character of $x^p$ to integrate by parts with allowance for $(\delta_l/\delta \eta_p + \delta_l/\delta \eta_p)(X + i\hbar H)(\tilde{\eta} - \eta) = 0$. Note that the symbol $\langle F(\tilde{z}, \lambda(2)) \rangle$ denotes a source-dependent average expectation value for a quantity $F(\tilde{z}, \lambda(2))$ corresponding to a gauge-fixing $X$ w.r.t. $Z^M [J_V, J^{\nu*}, \eta]$ in the presence of an external $\eta$:

$$
\langle F \rangle_{X, J_V, J^{\nu*}} = Z^{M-1} (J_V, J^{\nu*}) \int d\tilde{z} d\lambda(2) D_0(\tilde{x}) \exp \left\{ \frac{i}{\hbar} [W(\tilde{z}, h) + [X + i\hbar H] (\tilde{\eta}, \eta, h)] \right\} + G^{\nu*}(\tilde{z}) \lambda_{(2)} + J_V \tilde{\eta} + J^{\nu*} \tilde{x} \right\} \right\}, \text{ with } \langle 1 \rangle_{X, J_V, J^{\nu*}} = 1.
$$

Having chosen the Darboux coordinates for $z^P$, along with the conditions [24], supplemented by the relations $(Z^p_{(1)}, Y^p_{(1)}, \tilde{Z}^p_{(1)}, \tilde{Y}^p_{(1)}) = (\delta^p_A, \delta^p_{\tilde{A}}, \delta^A_{\tilde{p}}, \delta_{\tilde{A}p})$, for which the sources can be written as $(J^p; J^{\nu*}) = (-I^A, I^A; J^A, J^{A*})$ and $(D_0, H) = (1, 0)$, we obtain a coincidence between the (independent of $J^{A*}$) functional $Z^M [I_A, J^A, \phi^*; \lambda^A = 0]$ and the generating functional of Green’s functions $Z(\theta)_{\theta=0}$ of the superfield quantization [12] under the evident designations $(\phi^A, \phi^*_A, \lambda^A) = (\varphi^a, \varphi^*_a, \lambda^a)$.

In conclusion, we note that the suggested quantization procedure allows one to introduce an effective action and can be elaborated both under reducible hypergauges [11] and in the multilevel formalism. Besides, it suggests a possibility for an explicit correspondence between the BRST and BRSTantiBRST quantization approaches in the $N = 2$ superfield formalism.

Acknowledgments The author is grateful to P.M. Lavrov and P.Yu. Moshin for useful discussions.

References

[1] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. B102 (1981) 27; Phys. Rev. D28 (1983) 2567; J. Math. Phys. 26 (1985) 172.

[2] I.A. Batalin, P.M. Lavrov and I.V. Tyutin, J. Math. Phys. 31 (1990) 1487; ibid. 32 (1991) 532; ibid. 32 (1991) 2513; C.M. Hull, Mod. Phys. Lett. A5 (1990) 1871.

[3] E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. B55 (1975) 224; I.A. Batalin and G.A. Vilkovisky, Phys. Lett. B69 (1977) 309; E.S. Fradkin and T.E. Fradkina, Phys. Lett. B72 (1978) 343; I.A. Batalin and E.S. Fradkin, Phys. Lett. B122 (1983) 157; For a review see: M. Henneaux, Phys. Rept. 126 (1985) 1.

[4] I.A. Batalin, P.M. Lavrov and I.V. Tyutin, J. Math. Phys. 31 (1990) 6; ibid. 31 (1990) 2708; Int. J. Mod. Phys. A6 (1991) 3599.

[5] M. Henneaux, Phys. Lett. B282 (1992) 372; P. Grigoire and M. Henneaux, Phys. Lett. B277 (1992) 459; Comm. Math. Phys. 157 (1993) 279.

[6] I.A. Batalin and I.V. Tyutin, Int. J. Mod. Phys. A8 (1993) 2333; Mod. Phys. Lett. A8 (1993) 3673; Mod. Phys. Lett. A9 (1994) 1707, hep-th/9403180.

[7] I.A. Batalin and I.V. Tyutin, Nucl. Phys. B345 (1990) 645.

[8] I.A. Batalin and R. Marnelius, Phys. Lett. B350 (1995) 44; Nucl. Phys. B465 (1996) 521; I.A. Batalin, R. Marnelius and A.M. Semikhatov, Nucl. Phys. B446 (1995) 249.

[9] B. Geyer, D.M. Gitman and P.M. Lavrov, Mod. Phys. Lett. A14 (1999) 661; Theor. Math. Phys. 123 (2000) 813.

[10] A.A. Reshetnyak, Basic features of general superfield quantization method for gauge theories in Lagrangian formalism, Proceedings of the International Seminar on Supersymmetries and Quantum Symmetries SQS 03, Dubna, Russia, July 24–29, 2003 (Eds. E. Ivanov and A. Pashnev, JINR, Dubna, 2004, 345); hep-th/0312118.

[11] A.A. Reshetnyak, Russ. Phys. J. 47 (2004) 1026, hep-th/0512327.

[12] D.M. Gitman, P.Yu. Moshin and A.A. Reshetnyak, J. Math. Phys. 46 (2005) 072302, hep-th/0507160; Phys. Lett. B621 (2005) 295, hep-th/0507046.
[13] P.M. Lavrov, P.Yu. Moshin and A.A. Reshetnyak, Mod. Phys. Lett. \textbf{A10} (1995) 2687, [hep-th/9507104]; JETP Lett. \textbf{62} (1995) 780; B. Geyer, P.M. Lavrov and P.Yu. Moshin, Phys. Lett. \textbf{B463} (1999) 188.

[14] I.A. Batalin, K. Bering, P.H. Damgaard, Nucl. Phys. \textbf{B515} (1998) 455; Phys. Lett. \textbf{B446} (1999) 175.

[15] M. Grigoriev and P.H. Damgaard, Phys. Lett. \textbf{B474} (2000) 323.

[16] I.A. Batalin and P.H. Damgaard, Phys. Lett. \textbf{B578} (2004) 223; I.A. Batalin and K. Bering, Nucl. Phys. \textbf{B700} (2004) 439.

[17] M. Kontsevich, Lett. Math. Phys. \textbf{66} (2003) 157, [q-alg/9709040].

[18] A.S. Cattaneo and G. Felder, Comm. Math. Phys. \textbf{212} (2000) 591; Mod. Phys. Lett. \textbf{A16} (2001) 179; Lett. Math. Phys. \textbf{56} (2001) 163.

[19] M. Alexandrov, M. Kontsevich, A. Schwarz and O. Zaboronsky, Int. J. Mod. Phys. \textbf{A12} (1997) 1405.

[20] M.A. Grigoriev and S.L. Lyakhovich, Comm. Math. Phys. \textbf{218} (2001) 437; I.A. Batalin, M.A. Grigoriev and S.L. Lyakhovich, Theor. Math. Phys. \textbf{128} (2001) 1109; J. Math. Phys. \textbf{46} (2005) 072301.

[21] L.D. Faddeev and S.L. Shatashvili, Phys. Lett. \textbf{B167} (1986) 225; I.A. Batalin and E.S. Fradkin, Phys. Lett. \textbf{B180} (1987) 156; Nucl. Phys. \textbf{B279} (1987) 514; I.A. Batalin and I.V. Tyutin, Int. J. Mod. Phys. \textbf{A6} (1991) 3255.

[22] S.L. Lyakhovich and A.A. Sharapov, JHEP \textbf{03} (2005) 011; P.O. Kazinski, S.L. Lyakhovich and A.A. Sharapov, JHEP \textbf{07} (2005) 076.

[23] M.A. Vasiliev, Phys. Lett. \textbf{B243} (1990) 378; \textit{ibid}. \textbf{B285} (1992) 225; \textit{ibid}. \textbf{B567} (2003) 139; Fortsch. Phys. \textbf{52} (2004) 702.

[24] B.V. Fedosov, J. Diff. Geom. \textbf{40} (1994) 369; \textit{Deformation Quantization and Index Theory} (Akademie Verlag, Berlin, 1996);

[25] I. Gelfand, V. Retakh and M. Shubin, Advan. Math. \textbf{136} (1998) 104; [dg-ga/9707024].

[26] B. Geyer, P. Lavrov and A. Nersessian, Phys. Lett. \textbf{B512} (2001) 211; Int. J. Mod. Phys. \textbf{A17} (2002) 1183; B. Geyer and P.M. Lavrov, Int. J. Mod. Phys. \textbf{A19} (2004) 1639.

[27] B. Geyer, D.M. Gitman, P.M. Lavrov and P.Yu. Moshin, Int. J. Mod. Phys. \textbf{A19} (2004) 737.

[28] P.Yu. Moshin and A.A. Reshetnyak, \textit{in preparation}.

[29] M. Henneaux and C. Teitelboim, \textit{Quantization of Gauge Systems} (Princeton U.P., NJ 1992); J.M.L. Fisch and M. Henneaux, Comm. Math. Phys. \textbf{128} (1990) 627.

[30] B. Geyer and P. Lavrov, Int. J. Mod. Phys. \textbf{A19} (2004) 3195; \textit{ibid}. \textbf{A20} (2005).