Gibbs Measures For SOS Models On a Cayley Tree

U.A. Rozikov\textsuperscript{1} and Y.M. Suhov \textsuperscript{2}

Abstract. We consider a nearest-neighbor SOS (solid-on-solid) model, with several spin values \(0, 1, \ldots, m, m \geq 2\), and zero external field, on a Cayley tree of order \(k\) (with \(k+1\) neighbors). The SOS model can be treated as a natural generalisation of the Ising model (obtained for \(m = 1\)). We mainly assume that \(m = 2\) (three spin values) and study translation-invariant (TI) and ‘splitting’ (S) Gibbs measures (GMs). [Splitting GMs have a particular Markov-type property specific for a tree.] Furthermore, we focus on symmetric TISGMs, with respect to a ‘mirror’ reflection of the spins. [For the Ising model (where \(m = 1\)), such measures are reduced to the ‘disordered’ phase obtained for free boundary conditions, see [BRZ], [M1,2], [MP].] For \(m = 2\), in the anti-ferromagnetic (AFM) case, a symmetric TISGM (and even a general TISGM) is unique for all temperatures. In the ferromagnetic (FM) case, for \(m = 2\), the number of symmetric TISGMs and (and the number of general TISGMs) varies with the temperature: this gives an interesting example of phase transition. Here we identify a critical inverse temperature, \(\beta_{\text{cr}}^1 (= T_{\text{STISG}}^{\text{cr}}) \in (0, \infty)\) such that \(\forall 0 \leq \beta \leq \beta_{\text{cr}}^1\), there exists a unique symmetric TISGM \(\mu^*\) and \(\forall \beta > \beta_{\text{cr}}^1\), there are exactly three symmetric TISGMs: \(\mu^*_m\) (a ‘bottom’ symmetric TISGM), \(\mu^*_m\) (a ‘middle’ symmetric TISGM) and \(\mu^*_m\) (a ‘top’ symmetric TISGM). For \(\beta > \beta_{\text{cr}}^1\) we also construct a continuum of distinct, symmetric SGMs which are non-TI.

Our second result gives complete description of the set of periodic Gibbs measures for the SOS model on a Cayley tree. A complete description of periodic GMs means a characterisation of such measures with respect to any given normal subgroup of finite index in the representation group of the tree. We show that (i) for an FM SOS model, for any normal subgroup of finite index, each periodic SGM is in fact TI. Further, (ii) for an AFM SOS model, for any normal subgroup of finite index, each periodic SGM is either TI or has period two (i.e., is a chess-board SGM).

KEY WORDS: Gibbs measures, SOS model, Cayley tree

1 Introduction

One of the central problems in the theory of Gibbs measures (GMs) is to describe infinite-volume (or limiting) GMs corresponding to a given Hamiltonian. The existence of such measures for a wide class of Hamiltonians was established in the ground-breaking work of Dobrushin (see, e.g., Ref. [1]). However, a complete analysis of the set of limiting GMs for a specific Hamiltonian is often a difficult problem. On a cubic lattice, for small values of \(\beta = \frac{1}{T}\), where \(T > 0\) is the

\textsuperscript{1}Institute of Mathematics, Uzbek Academy of Sciences, Tashkent 700143, Uzbekistan

\textsuperscript{2}Statistical Laboratory, DPMMS, University of Cambridge, Cambridge CB3 0WB, UK
temperature, a GM is unique (Refs [1-3]) which reflects a physical fact that at high temperatures there is no phase transitions. The analysis for low temperatures requires specific assumptions on the form of the Hamiltonian.

In this paper we consider models with a nearest neighbour interaction on a Cayley tree (CT). Models on a CT were discussed in Refs. [2], [4]–[6]. A classical example of such a model is the Ising model, with two values of spin, ±1. It was considered in Refs. [2], [6] and became a focus of active research in the first half of the 1990’s and afterwards; see Refs [7]–[13]. Models considered in the present paper are generalisations of the Ising model and can be described as SOS (solid-on-solid) models with constraints; see below. In the case of a cubic lattice they were analysed in Ref. [14] where an analogue of the so-called Dinaburg–Mazel–Sinai theory was developed. Besides interesting phase transitions in these models, the attention to them is motivated by applications, in particular in the theory of communication networks; see, e.g., Refs [15].

A CT \( T^k = (V, A) \) of order \( k \geq 1 \) is an infinite homogeneous tree, i.e., a graph without cycles, with exactly \( k+1 \) edges incident to each vertex. Here \( V \) is the set of vertices and \( A \) that of edges (arcs).

We consider models where the spin takes values in the set \( \Phi := \{0, 1, \ldots, m\} \), \( m \geq 2 \), and is assigned to the vertices of the tree. A configuration \( \sigma \) on \( V \) is then defined as a function \( x \in V \mapsto \sigma(x) \in \Phi \); the set of all configurations is \( \Phi^V \). The (formal) Hamiltonian is of an SOS form:

\[
H(\sigma) = -J \sum_{\langle x,y \rangle \in L} |\sigma(x) - \sigma(y)|, \tag{1.1}
\]

where \( J \in \mathbb{R} \) is a coupling constant. As usually, \( \langle x, y \rangle \) stands for nearest neighbor vertices.

The SOS model of this type can be considered as a generalisation of the Ising model (which arises when \( m = 1 \)). Here, \( J < 0 \) gives a ferromagnetic (FM) and \( J > 0 \) an antiferromagnetic (AFM) model. In the FM case the ground states are ‘flat’ configurations, with \( \sigma(x) \equiv j \in \Phi \) (there are \( m+1 \) of them), in the AFM two ‘contrasting’ checker-board configurations where \( |\sigma(x) - \sigma(y)| = m \forall \langle x, y \rangle \). Compared with the Potts model (see, e.g., [16]–[19]), the SOS has ‘less symmetry’ and therefore more diverse structure of phases. For example, in the FM case it is intuitively plausible that the ground states corresponding to ‘middle-level’ surfaces will be ‘dominant’. This observation was made formal in [14] for the model on a cubic lattice.

We consider a standard sigma-algebra \( \mathcal{B} \) of subsets of \( \Phi^V \) generated by cylinder subsets; all probability measures are considered on \( (\Phi^V, \mathcal{B}) \). A probability measure \( \mu \) is called a GM (with Hamiltonian \( H \)) if it satisfies the DLR equation: \( \forall n = 1, 2, \ldots \) and \( \sigma_n \in \Phi^{V_n} \):

\[
\mu \left( \{ \sigma \in \Phi^V : \sigma|_{V_n} = \sigma_n \} \right) = \int_{\Phi^V} \mu(d\omega) \nu_{\omega|W_{n+1}}^{V_n} (\sigma_n), \tag{1.2}
\]

where \( \nu_{\omega|W_{n+1}}^{V_n} \) is the conditional probability:

\[
\nu_{\omega|W_{n+1}}^{V_n} (\sigma_n) = \frac{1}{Z_n (\omega|W_{n+1})} \exp \left( -\beta H (\sigma_n || \omega|W_{n+1}) \right). \tag{1.3}
\]
Here and below, $W_i$ stands for a ‘sphere’ and $V_i$ for a ‘ball’ on the tree, of radius $l = 1, 2, \ldots$, centered at a fixed vertex $x^0$ (an origin):

$$W_i = \{ x \in V : d(x, x^0) = l \}, \quad V_i = \{ x \in V : d(x, x^0) \leq l \};$$
distance $d(x, y)$, $x, y \in V$, is the length of (i.e. the number of edges in) the shortest path connecting $x$ with $y$. $\Phi^V_n$ is the set of configurations in $V_n$ (and $\Phi^W_n$ that in $W_n$; see below). Furthermore, $\sigma|_{V_n}$ and $\omega|_{W_{n+1}}$ denote the restrictions of configurations $\sigma, \omega \in \Phi^V$ to $V_n$ and $W_{n+1}$, respectively. Next, $\sigma_n : x \in V_n \mapsto \sigma_n(x)$ is a configuration in $V_n$ and $H (\sigma_n \| \omega|_{W_{n+1}})$ is defined as the sum $H (\sigma_n) + U (\sigma_n, \omega|_{W_{n+1}})$ where

$$H (\sigma_n) = -J \sum_{(x, y) \in L_n} |\sigma_n(x) - \sigma_n(y)|, \quad U (\sigma_n, \omega|_{W_{n+1}}) = -J \sum_{(x, y) : x \in V_n, y \in W_{n+1}} |\sigma_n(x) - \omega(y)|,$$

and

$$L_n = \{ (x, y) \in L : x, y \in V_n \}.$$  

Finally, $Z_n (\omega|_{W_{n+1}})$ stands for the partition function in $V_n$, with the boundary condition $\omega|_{W_{n+1}}$:

$$Z_n (\omega|_{W_{n+1}}) = \sum_{\sigma_n \in \Phi^V_n} \exp \left(-\beta H (\sigma_n \| \omega|_{W_{n+1}}) \right).$$  \hspace{1cm} (1.5)

Because of the nearest-neighbour character of the interaction, the GMs possess a natural Markov property: given a configuration $\omega_n$ on $W_n$, random configurations in $V_{n-1}$ (i.e., ‘inside’ $W_n$) and in $V \setminus V_{n+1}$ (i.e., ‘outside’ $W_n$) are conditionally independent. It is known (see, e.g., [1], [2]) that $\forall \beta > 0$, the GMs form a non-empty convex compact set in the space of probability measures. Extreme measures, i.e., extreme points of this set are associated with pure phases. Furthermore, any GM is an integral of extreme ones (the extreme decomposition). It is true that for any sequence of configurations $\omega(n) \in \Phi^V$, every limiting point of measures $\nu^V_{\omega(n)|_{W_{n+1}}}$ is a GM. Here, for a given $\omega \in \Phi^V$, $\nu^V_{\omega|_{W_{n+1}}}$ is a measure on $\Phi^V$ such that $\forall n' > n$:

$$\nu^V_{\omega|_{W_{n+1}}} \left( \{ \sigma \in \Phi^V : \sigma|_{V_{n'}} = \sigma_n' \} \right) = \begin{cases} \nu^V_{\omega|_{W_{n+1}}} (\sigma_n'|_{V_{n'}}), & \text{if } \sigma_n'|_{V_n \setminus V_{n'}} = \omega|_{V_n \setminus V_{n'}}, \\ 0, & \text{otherwise.} \end{cases}$$  \hspace{1cm} (1.6)

The converse is also true: every GM $\mu$ can be obtained as a limiting point for measures $\nu^V_{\omega(n)|_{W_{n+1}}}$ with a suitable sequence of configurations $\omega(n) \Phi^V$. We call such a sequence $\omega(n)$ the boundary conditions for GM $\mu$.

We use a standard definition of a translation-invariant (TI) measure (see, e.g., [4]). Also, call measure $\mu$ symmetric (S) if it is preserved under the simultaneous change $j \mapsto m - j$ at each vertex $x \in V$. The main object of study in this paper are symmetric TI measures.

An important role is played by a specific monotonicity displayed by the FM model (with $J < 0$). Namely, write $\sigma \leq \sigma'$ if configurations $\sigma$ and $\sigma'$ obey $\sigma(x) \leq \sigma'(x) \forall x \in V$. This partial
order defines a concept of a monotone increasing and monotone decreasing function \( f : \Phi^V \rightarrow \mathbb{R} \). For two probability measures \( \mu_1 \) and \( \mu_2 \) we then write \( \mu_1 \leq \mu_2 \) if \( \int f d \mu_1 \leq \int f d \mu_2 \) for each monotone increasing \( f \). It turns out that for the ‘extreme’ configurations, \( \omega^0 \) with \( \omega^0(x) \equiv 0 \) and \( \omega^2 \) with \( \omega^2(x) \equiv 2 \), there exist the limits \( \nu^0 = \lim_{n \rightarrow \infty} \overline{V}_n^{\omega_0} \mid_{W_{n+1}} \) and \( \nu^2 = \lim_{n \rightarrow \infty} \overline{V}_n^{\omega_2} \mid_{W_{n+1}} \) (both measure sequences are monotone). \( \nu^0, \nu^2 \) are TIGMs and possess the following minimality and maximality properties: \( \nu^1 \leq \mu \leq \nu^2 \) \( \forall \) GM \( \mu \). Because of that, they are both extreme (although not symmetric). The question of whether a GM is non-unique is then reduced to whether \( \nu^1 = \nu^2 \). However, finer properties of GMs require further specifications.

## 2 Construction of splitting GMs

Following Ref. [2] (and subsequent papers [5-13]), we consider a special class of GMs. These measures are called in Ref. [2] Markov chains and in Refs [5], [6] entrance laws. In this paper we call them splitting GMs, to emphasize the property that, in addition to the aforementioned Markov property, they satisfy the following condition: given a configuration \( \sigma_n \) in \( V_n \), the values \( \sigma(y) \) at sites \( y \in W_{n+1} \) are conditionally independent.

Write \( x < y \) if the path from \( x^0 \) to \( y \) goes through \( x \). Call vertex \( y \) a direct successor of \( x \) if \( y > x \) and \( x,y \) are nearest neighbours. Denote by \( S(x) \) the set of direct successors of \( x \). Observe that any vertex \( x \neq x^0 \) has \( k \) direct successors and \( x^0 \) has \( k + 1 \).

Let \( h : x \mapsto h_x = (h_{0,x}, h_{1,x}, ..., h_{m,x}) \in \mathbb{R}^{m+1} \) be a real vector-valued function of \( x \in V \setminus \{x^0\} \). Given \( n = 1, 2, \ldots \), consider the probability distribution \( \mu_n \) on \( \Phi^{V_n} \) defined by

\[
\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp \left( -\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x} \right),
\]

(2.1)

Here, as before, \( \sigma_n : x \in V_n \mapsto \sigma(x) \) and \( Z_n \) is the corresponding partition function:

\[
Z_n = \sum_{\sigma_n \in \Phi^{V_n}} \exp \left( -\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x} \right).
\]

(2.2)

We say that the probability distributions \( \mu^{(n)} \) are compatible if \( \forall \ n \geq 1 \) and \( \sigma_{n-1} \in \Phi^{V_{n-1}} \):

\[
\sum_{\omega_n \in \Phi^{W_n}} \mu^{(n)}(\sigma_{n-1} \lor \omega_n) = \mu^{(n-1)}(\sigma_{n-1}).
\]

(2.3)

Here \( \sigma_{n-1} \lor \omega_n \in \Phi^{V_n} \) is the concatenation of \( \sigma_{n-1} \) and \( \omega_n \). In this case there exists a unique measure \( \mu \) on \( \Phi^V \) such that, \( \forall \ n \) and \( \sigma_n \in \Phi^{V_n} \), \( \mu \left( \left\{ \sigma \big|_{V_n} = \sigma_n \right\} \right) = \mu^{(n)}(\sigma_n) \). Such a measure is called a splitting GM (SGM) corresponding to Hamiltonian \( H \) and function \( x \mapsto h_x, x \neq x^0 \).

The following statement describes conditions on \( h_x \) guaranteeing compatibility of distributions \( \mu^{(n)}(\sigma_n) \).
Proposition 2.1. Probability distributions \( \mu^{(n)}(\sigma_n), n = 1, 2, \ldots \), in (2.1) are compatible iff for any \( x \in V \setminus \{x^0\} \) the following equation holds:

\[
h^*_x = \sum_{y \in S(x)} F(h^*_y, m, \theta).
\] (2.4)

Here, and below

\[
\theta = \exp(\beta),
\] (2.5)

\( h^*_x \) is stands for the vector \( (h_{0,x} - h_{m,x}, h_{1,x} - h_{m,x}, \ldots, h_{m-1,x} - h_{m,x}) \) and the vector function

\[
F(\cdot, m, \theta): \mathbb{R}^m \rightarrow \mathbb{R}^m \text{ is } F(h, m, \theta) = (F_0(h, m, \theta), \ldots, F_{m-1}(h, m, \theta)),
\]

with

\[
F_i(h, m, \theta) = \ln \frac{\sum_{j=0}^{m-1} \theta^{i-j} \exp(h_j) + \theta^{m-i}}{\sum_{j=0}^{m-1} \theta^{m-j} \exp(h_j) + 1}, \quad h = (h_0, h_1, \ldots, h_{m-1}), i = 0, \ldots, m-1.
\] (2.6)

Proof. Necessity (cf. [16]). Suppose that (2.3) holds; we want to prove (2.4). Substituting (2.1) in (2.3), obtain, \( \forall \) configurations \( \sigma_{n-1}: x \in V_{n-1} \mapsto \sigma_{n-1}(x) \in \Phi \):

\[
\frac{Z_{n-1}}{Z_n} \sum_{\omega_n \in \Phi_{W_n}} \exp \left( \sum_{x \in W_{n-1}} \sum_{y \in S(x)} (J\beta|\sigma_{n-1}(x) - \omega_n(y)| + h_{\omega_n(y),y}) \right) = \exp \left( \sum_{x \in W_{n-1}} h_{\sigma_{n-1}(x),x} \right),
\] (2.7)

where \( \omega_n: x \in W_n \mapsto \omega_n(x) \).

From (2.7) we get:

\[
\frac{Z_{n-1}}{Z_n} \sum_{\omega_n \in \Phi_{W_n}} \prod_{x \in W_{n-1}} \prod_{y \in S(x)} \exp (J\beta|\sigma_{n-1}(x) - \omega_n(y)| + h_{\omega_n(y),y}) = \prod_{x \in W_{n-1}} \exp (h_{\sigma_{n-1}(x),x}).
\] (2.8)

Consequently, \( \forall i \in \Phi \),

\[
\prod_{y \in S(x)} \frac{\sum_{j \in \Phi} \exp (J\beta|i-j| + h_{j,y})}{\sum_{j \in \Phi} \exp (J\beta|m-j| + h_{j,y})} = \exp (h_{i,x} - h_{m,x}).
\] (2.9)

Introducing \( \theta \) as in (2.5) and denoting \( h^*_i = h_{i,x} - h_{m,x} \), we get (2.4) from (2.9).

Sufficiency. From (2.4) we obtain (2.9), (2.8) and (2.7) i.e. (2.3). The proof is complete.

Proposition 2.2. Any measure \( \mu \) with local distributions \( \mu^{(n)} \) satisfying (2.1), (2.3) is an SGM.

Proof. Straightforward.

Proposition 2.3. An SGM \( \mu \) is TI iff \( h_{j,x} \) does not depend on \( x \): \( h_{j,x} \equiv h_j, \ x \in V, \ j \in \Phi \), and symmetric TI iff \( h_j = h_{m-j}, \ j \in \Phi \).
Proof. Straightforward.

**Proposition 2.4.** Any extreme GM is an SGM.

Proof. See Ref [4], Theorem 12.6.

3 The critical value \( \beta_{cr}^1 \)

From Proposition 2.2 it follows that for any \( h = \{h_x, x \in V\} \) satisfying (2.4) there exists a unique GM \( \mu \) (with restrictions \( \mu^{(m)} \) as in (2.1)) and vice versa. However, the analysis of solutions to (2.4) for an arbitrary \( m \) is not easy. We now suppose that the number of spin values \( m+1 \) is 3 i.e. \( m = 2 \) and \( \Phi = \{0, 1, 2\} \). Throughout the paper we assume that \( h_{2,x} \equiv 0 \) (\( h_{m,x} \equiv 0 \) for general \( m \)).

It is natural to begin with translation-invariant solutions where \( h_x = h \in R^m \) is constant. Unless stated otherwise, we concentrate on the simplest case where \( m = 2 \), i.e. spin values are 0, 1 and 2. In this case we obtain from (2.4), (2.5):

\[
\begin{align*}
  h_{0,x} &= \sum_{y \in S(x)} \ln \frac{\exp(h_{0,y}) + \theta \exp(h_{1,y}) + \theta^2}{\theta^2 \exp(h_{0,y}) + \theta \exp(h_{1,y}) + 1}, \\
  h_{1,x} &= \sum_{y \in S(x)} \ln \frac{\theta \exp(h_{0,y}) + \exp(h_{1,y}) + \theta}{\theta^2 \exp(h_{0,y}) + \theta \exp(h_{1,y}) + 1}.
\end{align*}
\]  

(3.1)

Set \( z_0 = \exp(h_{0,x}) \), \( z_1 = \exp(h_{1,x}) \) (and \( z_2 = 1 \), \( x \in V \). From (3.1) we have

\[
\begin{align*}
  z_0 &= \left( z_0 + \theta z_1 + \theta^2 \right)^k \left( \theta^2 z_0 + \theta z_1 + 1 \right), \\
  z_1 &= \left( \theta z_0 + z_1 + \theta \right)^k \left( \theta^2 z_0 + \theta z_1 + 1 \right). \tag{3.2.a, 3.2.b}
\end{align*}
\]

Observe that \( z_0 = 1 \) satisfies equation (3.2.a) independently of \( k, \theta \) and \( z_1 \). Substituting \( z_0 = 1 \) into (3.2.b), we obtain

\[
z_1 = \left( \frac{2\theta + z_1}{\theta^2 + \theta z_1 + 1} \right)^k. \tag{3.3}
\]

Set:

\[
  a = 2\theta^{k+1}, \quad b = \frac{1 + \theta^2}{2\theta^2}, \quad x = \frac{z_1}{2\theta}. \tag{3.4}
\]

Then from (3.3):

\[
  ax = \left( \frac{1 + x}{b + x} \right)^k. \tag{3.5}
\]

In Proposition 3.1 below we analyse solutions to equation (3.5) with independently varying parameters \( a, b > 0 \). The proof of Proposition 3.1 repeats an argument from [2], Proposition 10.7.
Proposition 3.1. Equation (3.5) with \( x \geq 0, \ k \geq 1, \ a, b > 0 \) has a unique solution if either \( k = 1 \) or \( b \leq \left( \frac{k+1}{k} \right)^2 \). If \( k > 1 \) and \( b > \left( \frac{k+1}{k} \right)^2 \) then there exist \( \nu_1(b, k) \), \( \nu_2(b, k) \), with \( 0 < \nu_1(b, k) < \nu_2(b, k) \), such that the equation has three solutions if \( \nu_1(b, k) < a < \nu_2(b, k) \) and has two if either \( a = \nu_1(b, k) \) or \( a = \nu_2(b, k) \). In fact:

\[
\nu_i(b, k) = \frac{1}{x_i} \left( \frac{1+x_i}{b+x_i} \right)^k,
\]

where \( x_1, x_2 \) are the solutions of

\[
x^2 + [2 - (b-1)(k-1)]x + b = 0.
\]

Now consider \( a \) and \( b \) as functions of \( \beta \) (for a fixed \( J \) as specified in (3.4) and (2.6)).

Proposition 3.2. If \( J \geq 0 \) then the system of equations (3.2.a), (3.2.b) has a unique solution.

Proof. Let \( A = z_0 + \theta z_1 + \theta^2, \ B = \theta^2 z_0 + \theta z_1 + 1 \), then from (3.2.a) we have:

\[
(z_0 - 1)[B^k + (\theta^2 - 1)(A^{k-1} + \ldots + B^{k-1})] = 0
\]

(3.6)

Since \( \theta \geq 1 \ (J \geq 0) \), we deduce from (3.6) that \( z_0 = 1 \) is the only solution. Then \( b = \frac{1+\theta^2}{\theta^2} \leq 1 < \left( \frac{k+1}{k} \right)^2 \). By Proposition 3.1, equation (3.3) has a unique solution. Thus we have proved that system (3.2.a), (3.2.b) has a unique solution.

Proposition 3.3. If \( J < 0 \) then for \( \beta \leq \frac{1}{2J} \ln \frac{(k-1)^2}{k^2 + 6k + 1} \), the system of equations (3.2.a), (3.2.b) has a unique solution of the form \((1, z^*)\) (i.e., a unique solution \((z_0^*, z_1^*)\) with \( z_0^* = 1 \)) and for \( \beta > \frac{1}{2J} \ln \frac{(k-1)^2}{k^2 + 6k + 1} \), precisely three such solutions, \((1, z^-_1), (1, z^+_1), (1, z^*_m)\), with \( 0 < z^-_1 < z^*_m < z^+_1 \) and \( z^*_i = e^{h_1 i}, i = -, m, +, \) (see (3.1)).

Proof. The value \( \frac{1}{2J} \ln \frac{(k-1)^2}{k^2 + 6k + 1} \) is the solution of equation \( b = \left( \frac{k+1}{k-1} \right)^2 \). Other statements of Proposition 3.3 are consequences of Proposition 3.1.

For brevity we say that \( z^* \) and \( z^-_1, z^*_m, z^+_1 \) give symmetric solutions to (3.2.a,b).

Definition. In the FM case, set:

\[
\beta_{cr}^1 = \frac{1}{2J} \ln \frac{(k-1)^2}{k^2 + 6k + 1} > 0.
\]

(3.7)

Going back to (2.5), summarise:
Theorem 1. For the AFM SOS model, with $J > 0$ and $m = 2$, the TISGM exists and is unique $\forall \beta \geq 0$. In fact, it is a symmetric TISGM.

For the FM SOS model, with $J < 0$ and $m = 2$:
1) If $k \geq 2$ and $0 \leq \beta \leq \beta_{\text{cr}}$ then there exists a unique symmetric TISGM, $\mu^*$.
2) If $k \geq 2$ and $\beta > \beta_{\text{cr}}$ then there exist precisely three symmetric TISGMs $\mu_*^-, \mu_*^m, \mu_*^+$ corresponding to $h^*_i = \ln z^*_i$, $i = -, m, +$.

Remark 1. In the AFM case, the phase transition is manifested in the break of the TI property. More precisely, it is expected that for $\beta$ small there exists a unique translation-periodic SGM (which is TI) while for $\beta$ large there are several such measures.

In the FM case, observe that values $z^*_i$, $i = -, m, +$, vary with $\beta$. It is easy to show that as $\beta \to \infty$, $z^-_* \to 0$, $z^*_m \to 1$ and $z^*_+ \to \infty$. Correspondingly, we make a

Conjecture 1. For $m = 2$, $k \geq 2$ and $J < 0$, as $\beta \to \infty$, measure $\mu_*^-$ tends to the half-sum $\frac{1}{2}(\delta_{\omega_0} + \delta_{\omega_2})$, $\mu_*^m$ to the mean $\frac{1}{3}(\delta_{\omega_0} + \delta_{\omega_1} + \delta_{\omega_2})$ and $\mu_*^+$ to $\delta_{\omega_1}$. Here $\delta_\omega$ stands for the Dirac delta-measure sitting on configuration $\omega \in \Phi^V$ and $\omega^i$ has $\omega^i(x) \equiv i$, $i = 0, 1, 2$.

On the other hand, we can say that for $\beta \leq \beta_{\text{cr}}$, all three measures coincide and in the limit $\beta \to 0$ give a Bernoulli measure, with iid and equiprobable values $\sigma(x) = 0, 1, 2$, $x \in V$.

Remark 2. Note that $\beta_{\text{cr}}^1$ may not be the first critical value of the inverse temperature for the FM model. Namely, there exists $\beta_{\text{cr}}^0 (= \beta_{\text{TIGM}}^1) \in (0, \beta_{\text{cr}}^1]$ such that (i) for $0 \leq \beta \leq \beta_{\text{cr}}^0$, a minimal GM, $\mu_-$, and a maximal, $\mu_+$, coincide, and the whole set of GMs is reduced to a unique measure which is therefore extreme (and coincides with symmetric TISGM $\mu^*$), (ii) for $\beta > \beta_{\text{cr}}^0$, $\mu_-$ and $\mu_+$ are distinct (they are always extreme TISGMs, but not symmetric). Thus, for $\beta \geq \beta_{\text{cr}}^1$, there are five TISGMs (in a natural order: $\mu_- \leq \mu_*^* \leq \mu_*^m \leq \mu_*^+ \leq \mu_+$) three of which are symmetric. It is not known whether $\beta_{\text{cr}}^0 = \beta_{\text{cr}}^1$ (it is our Conjecture 2).

The following Proposition 3.5 describes a useful property of general (non-TI) solutions $h_x = (h_{0,x}; h_{1,x})$ to (3.1) with $h_{0,x} \equiv 0$ (or $z_0^* \equiv 1$). As before, $h_{0,x}$ gives a solution to the first equation in (3.1), regardless of $h_{1,x}$ and $\theta$.

Proposition 3.5. For $J < 0$, $k \geq 2$ and $\beta > \beta_{\text{cr}}^1$, if $h_x = (0; h_{1,x})$ is a solution of (3.1) then, with $h_{1,x} = \ln z_{1,x}$,

$$z^-_* \leq z_{1,x} \leq z^*_+,$$ \hspace{1cm} $x \in V$ \hspace{1cm} (3.8)

where $z^-_* < z^*_+$ are the symmetric solutions of (3.2.a,b) i.e., the solutions of (3.3).
Proof. Denote $z_x = \exp(h_{1,x})$. Then from (3.1) we get
\[
  z_x = \prod_{i=1}^{k} \frac{2\theta + z_{x_i}}{1 + \theta^2 + \theta z_{x_i}}, \quad z_{x_j} > 0, \quad j = 1, \ldots, k,
\]
where $x_j, \quad j = 1, \ldots, k$ are direct successors of $x$. Denote $\varphi(x, \theta) = \frac{2\theta + x}{1 + \theta^2 + \theta x}$. Consider
\[
  G(x_1, \ldots, x_k) = \prod_{i=1}^{k} \varphi(x_i, \theta), \quad x_i > 0, \quad i = 1, \ldots, k.
\]
Set the map $x \mapsto \psi(x, \theta, k) = (\varphi(x, \theta))^k$. Clearly, $\psi(0, \theta, k) \leq G(x_1, \ldots, x_k) \leq \psi(\infty, \theta, k)$. Thus for $z_x$ we get $\psi(0, \theta, k) \leq z_x \leq \psi(\infty, \theta, k)$. Now consider $G(x_1, \ldots, x_k)$ with $\psi(0, \theta, k) \leq x_j \leq \psi(\infty, \theta, k)$. Here we have
\[
  \psi(\psi(0, \theta, k), \theta, k) \leq z_x \leq \psi(\psi(\infty, \theta, k), \theta, k).
\]
Repeating this argument, we see that for the $n$th iteration $\psi^{(n)}$ of $\psi$:
\[
  \psi^{(n)}(0, \theta, k) \leq z_x \leq \psi^{(n)}(\infty, \theta, k),
\]
for all $n \geq 1$ and $x \in V \setminus \{x^0\}$. The sequence $\psi^{(n)}(\infty, \theta, k)$ is decreasing and bounded from below by $z^*_x$. Its limit is a fixed point for $\psi$ and thus equal to $z^*_x$. The lower bound for $z_x$ is similar and gives $z^*_x$.

**Proposition 3.6.** For $J < 0$ and $\beta \leq \beta^1_{cr}$, measure $\mu^*$ is the only splitting GM such that $z_{0,x} = 0, \ x \in V \setminus \{x^0\}$ (regardless whether it is TI or not). Thus, $\mu^*$ is the only symmetric SGM.

**Proof.** In this case equation (3.1) with $h_{0,x} = 0$ has a unique solution $h_x = (0, \ln z^*)$.

**Conjecture 3.** In the case $m = 2, \ J < 0$ and $\beta \leq \beta^1_{cr}$, $\mu^*$ is the unique GM and hence extreme.

**Conjecture 4.** In the case $k \geq 2, \ J < 0$ and $\beta > \beta^1_{cr}$, the boundary condition for the top symmetric TISGM $\mu^*_+\omega$ is $\omega^{(n)}(x) \equiv 1$.

The boundary conditions for the bottom and middle symmetric TISGM, $\mu_-$ and $\mu_m$, are unclear. In the case of a general $m$, we also have two conjectures.

**Conjecture 5.** $\forall m, k \geq 2$ and $J < 0$, there exist symmetric solutions $h = (h_0, h_1, \ldots, h_{m-1})$ to (2.6), with $h_0 = 0$ and $h_i = h_{m-i}, \ i = 1, 2, \ldots, m - 1$.

**Conjecture 6.** $\forall m, k \geq 2$ and $J > 0, \ \forall \beta \geq 0$ the TISGM is unique and is a symmetric TISGM.
4 Periodic SGMs

In this section we study a periodic (see Definition 4.1) solutions of system (3.1).

Note that (see [18]) there exists a one-to-one correspondence between the set $V$ of vertices of the CT of order $k \geq 1$ and the group $G_k$ of the free products of $k + 1$ cyclic groups of the second order with generators $a_1, a_2, ..., a_{k+1}$.

**Definition 4.1.** Let $K$ be a subgroup of $G_k$. We say that a collection (of functions) $h = \{h_x \in \mathbb{R}^2 : x \in G_k\}$ is $K$-periodic if $h_{yx} = h_x$ for all $x \in G_k$ and $y \in K$.

**Definition 4.2.** A Gibbs measure is called $K$-periodic if it corresponds to $K$-periodic collection $h$.

Observe that a TIGM is $G_k$-periodic.

We give a complete description of periodic GMs i.e. a characterisation of such measures with respect to any normal subgroup of finite index in $G_k$.

Let $K$ be a subgroup of index $r$ in $G_k$, and let $G_k/K = \{K_0, K_1, ..., K_{r-1}\}$ be the quotient group, with the coset $K_0 = K$. Let $q_i(x) = \left|S_i(x) \cap K_i\right|$, $i = 0, 1, ..., r - 1$; $N(x) = \left|\{j : q_j(x) \neq 0\}\right|$, where $S_i(x) = \{y \in G_k : \langle x, y \rangle\} \neq 0$, $x \in G_k$ and $\left|\cdot\right|$ is the number of elements in the set. Denote $Q(x) = (q_0(x), q_1(x), ..., q_{r-1}(x))$.

We note (see [21]) that for every $x \in G_k$ there is a permutation $\pi_x$ of the coordinates of the vector $Q(e)$ (where $e$ is the identity of $G_k$) such that

$$\pi_x Q(e) = Q(x).$$

(4.1)

It follows from (4.1) that $N(x) = N(e)$ for all $x \in G_k$.

Each $K$-periodic collection is given by

$$\{h_x = h_i \text{ for } x \in K_i, \ i = 0, 1, ..., r - 1\}.$$

By Proposition 2.1 (for $m = 2$) and (4.1), vector $h_n$, $n = 0, 1, ..., r - 1$, satisfies the system

$$h_n = \sum_{j=1}^{N(e)} q_{i_j}(e) F(h_{\pi_n(i_j)}; \theta) - F(h_{\pi_n(j_0)}; \theta),$$

(4.2)

where $j_0 = 1, ..., N(e)$, and function $h \mapsto F(h, m, \theta)$ defined in Proposition 2.1 takes now the form $h \mapsto F(h) = (F_0(h, \theta), F_1(h, \theta))$ where

$$F_0(h, \theta) = \ln \frac{\exp(h_0) + \theta \exp(h_1) + \theta^2}{\theta^2 \exp(h_0) + \theta \exp(h_1) + 1},$$

$$F_1(h, \theta) = \ln \frac{\theta \exp(h_0) + \exp(h_1) + \theta}{\theta^2 \exp(h_0) + \theta \exp(h_1) + 1}.$$

(4.3)
Recall, $\theta$ has been defined in (2.5).

**Proposition 4.3.** If $\theta \neq 1$, then $F(h) = F(l)$ if and only if $h = l$.

**Proof.** *Necessity.* From $F(h) = F(l)$ we get the system of equations

$$
\begin{cases}
\theta\left(\exp(h_0 + l_1) - \exp(h_1 + l_0)\right) + (1 + \theta^2)\left(\exp(h_0) - \exp(l_0)\right) + \theta\left(\exp(h_1) - \exp(l_1)\right) = 0, \\
\theta\left(\exp(h_0) - \exp(l_0)\right) + \exp(h_1) - \exp(l_1) = 0.
\end{cases}
$$

where $h = (h_0, h_1), \ l = (l_0, l_1)$. Using the fact that

$$
\exp(h_0 + l_1) - \exp(h_1 + l_0) = \exp(l_1)\left(\exp(h_0) - \exp(l_0)\right) - \exp(l_0)\left(\exp(h_1) - \exp(l_1)\right),
$$

we obtain

$$
\begin{cases}
(1 + \theta^2 + \theta \exp(l_1))\left(\exp(h_0) - \exp(l_0)\right) + \theta\left(1 - \exp(l_0)\right)\left(\exp(h_1) - \exp(l_1)\right) = 0, \\
\theta\left(\exp(h_0) - \exp(l_0)\right) + \exp(h_1) - \exp(l_1) = 0.
\end{cases}
$$

From (4.5) we get

$$
(1 + \theta^2 \exp(l_0) + \theta \exp(l_1))\left(\exp(h_0) - \exp(l_0)\right) = 0. \tag{4.6}
$$

It follows from (4.6) that $h_0 = l_0$. Consequently, from second equation in (4.5) we have $h_1 = l_1$.

*Sufficiency.* Straightforward.

Let $G^*_k$ be the subgroup in $G_k$ consisting of all words of even length. Clearly, $G^*_k$ is a subgroup of index 2.

**Theorem 2.** Let $K$ be a normal subgroup of finite index in $G_k$. Then each $K$–periodic GM for SOS model is either TI or $G^*_k$–periodic.

**Proof.** We see from (4.2) that

$$
F(h_{\pi_n(i_1)}) = F(h_{\pi_n(i_2)}) = ... = F(h_{\pi_n(i_{N(z)})}).
$$

Hence from Proposition 4.3 we have

$$
h_{\pi_n(i_1)} = h_{\pi_n(i_2)} = ... = h_{\pi_n(i_{N(z)})}.
$$

Therefore,

$$
h_x = h_y = h, \quad \text{if} \quad x, y \in S_1(z), \quad z \in G^*_k;
$$

11
\( h_x = h_y = l, \) if \( x, y \in S_1(z), \) \( z \in G_k \setminus G_k^* \).

Thus the measures are TI (if \( h = l \)) or \( G_k^* \)-periodic (if \( h \neq l \)). This completes the proof of Theorem 2.

Let \( K \) be a normal subgroup of finite index in \( G_k \). What condition on \( K \) will guarantee that each \( K \)-periodic GM is TI? We put \( I(K) = K \cap \{a_1, ..., a_{k+1}\} \), where \( a_i, \ i = 1, ..., k+1 \) are generators of \( G_k \).

**Theorem 3** If \( I(K) \neq \emptyset \), then each \( K \)-periodic GM for SOS model is TI.

**Proof.** Take \( x \in K \). We note that the inclusion \( xa_i \in K \) holds if and only if \( a_i \in K \). Since \( I(K) \neq \emptyset \), there is an element \( a_i \in K \). Therefore \( K \) contains the subset \( Ka_i = \{xa_i : x \in K\} \). By Theorem 2 we have \( h_x = h \) and \( h_{xa_i} = l \). Since \( x \) and \( xa_i \) belong to \( K \), it follows that \( h_x = h_{xa_i} = h = l \). Thus each \( K \)-periodic GM is TI. This proves Theorem 3.

Theorems 2 and 3 reduce the problem of describing \( K \)-periodic GM with \( I(K) \neq \emptyset \) to describing the fixed points of \( kF(h; \theta) \) (see (3.2,a,b)) which describes TIGM. If \( I(K) = \emptyset \), this problem is reduced to describing the solutions of the system:

\[
\begin{cases}
  h = kF(l; \theta), \\
  l = kF(h; \theta).
\end{cases}
\]

Denote \( z_i = \exp(h_i), \ t_i = \exp(l_i), \ i = 0, 1 \). Then from (4.7) we get

\[
\begin{align*}
  z_0 &= \left( \frac{\theta_0 + \theta_1 + \theta^2}{\theta^2 + \theta_0 + \theta_1 + 1} \right)^k, \\
  z_1 &= \left( \frac{\theta_0 + \theta_1 + \theta^2}{\theta^2 + \theta_0 + \theta_1 + 1} \right)^k, \\
  t_0 &= \left( \frac{\theta z_0 + z_1 + \theta}{\theta^2 z_0 + z_1 + 1} \right)^k, \\
  t_1 &= \left( \frac{\theta z_0 + z_1 + \theta}{\theta^2 z_0 + z_1 + 1} \right)^k.
\end{align*}
\]

**Proposition 4.4.** For a ferromagnetic SOS model, with \( J < 0 \ (\theta < 1) \) (and even for \( J = 0 \)), the system of equations (4.8) has solutions with \( z_0 = t_0 \) and \( z_1 = t_1 \) only.

**Proof.** Denote \( u_i = z_i^{1/k}, \ v_i = t_i^{1/k}, \ i = 0, 1 \). Then from (4.8) we have

\[
u_0 - v_0 = \frac{(1 - \theta^2)[\theta(u_0^k v_0^k - u_0^k v_1^k) + (v_1^k - u_0^k) + \theta(v_1^k - u_0^k)]}{(\theta^2 v_0^k + \theta v_1^k + 1)(\theta^2 u_0^k + \theta u_1^k + 1)}
\]

and

\[
u_1 - v_1 = \frac{(1 - \theta^2)[\theta(u_0^k - u_0^k) + (v_1^k - u_1^k)]}{(\theta^2 v_0^k + \theta v_1^k + 1)(\theta^2 u_0^k + \theta u_1^k + 1)}
\]
Using the fact that
\[ u_1^k v_0^k - u_0^k v_1^k = u_1^k (u_0^k - u_0^k) + u_0^k (u_1^k - v_1^k) \]
we obtain
\[
\begin{cases}
[A + (1 - \theta^2)(\theta u_1^k + \theta^2 + 1)B_0](u_0 - v_0) + \theta(1 - \theta^2)(1 - u_0^k)B_1(u_1 - v_1) = 0, \\
\theta(1 - \theta^2)B_0(u_0 - v_0) + [A + (1 - \theta^2)B_1](u_1 - v_1) = 0,
\end{cases}
\tag{4.11}
\]
where
\[ A = (\theta^2 v_0^k + \theta v_1^k + 1)(\theta^2 u_0^k + \theta u_1^k + 1) > 0, \]
\[ B_i = u_i^{k-1} + u_i^{k-2}v_i + \ldots + v_i^{k-1} > 0, \quad i = 0, 1. \]

From (4.11) we get
\[
[A^2 + (1 - \theta^2)(B_1 + (\theta u_1^k + \theta^2 + 1)B_0)A + (1 - \theta^2)^2(\theta u_1^k + \theta^2 u_0^k + 1)B_0B_1](u_0 - v_0) = 0. \tag{4.12}
\]
Since \( \theta \leq 1 \) (\( J \leq 0 \)), we deduce from (4.12) that \( u_0 = v_0 \). Then from second equation of (4.11) we have \( u_1 = v_1 \). This completes the proof.

Now consider anti-ferromagnetic case, with \( J > 0 \) (\( \theta > 1 \)). By Proposition 3.2 we know that if \( J > 0 \) then the system of equations (4.8) has a unique solution with \( z_0 = t_0, z_1 = t_1 \). Moreover, \( z_0 = 1 \). For \( z_0 = t_0 = 1 \) from (4.8) we have
\[
\begin{cases}
z_1 = \left( \frac{2\theta + z_+}{\theta^2 + \theta z_+ + 1} \right)^k, \\
t_1 = \left( \frac{2\theta + z_1}{\theta^2 + \theta z_1 + 1} \right)^k.
\end{cases}
\tag{4.13}
\]
The following proposition gives a condition under which (4.8) has solutions with \( z_0 = t_0 = 1 \) and \( z_1 \neq t_1 \).

**Proposition 4.5.** Let \((z_*, z_*)\) be the unique solution of (4.13). If
\[
\frac{k z_*(\theta^2 - 1)}{(2\theta + z_*)(1 + \theta^2 + \theta z_*)} > 1, \tag{4.14}
\]
then the system of equations (4.13) has at least three solutions \((z_*, z_+), (z_+, z_*), (z_+, z_*)\), where \( z_* = \psi(z_+, \theta, k) \) and
\[ \psi(x, \theta, k) = \left( \frac{2\theta + x}{1 + \theta^2 + \theta x} \right)^k. \]

**Proof.** Under (4.14) \( z_* \) is unstable fixed point of the map \( z > 0 \to \psi(z, \theta, k) \). For any \( z \geq 1 \), iterates \( \psi^{(2n)}(z, \theta, k) \) remain \( z_* \) monotonically decrease and hence converge to a limit, \( z_+ \geq z_* \) which solves
\[ z = \psi(z, \theta, k). \tag{4.15} \]
However, $z^* > z_*$ as $z_*$ is unstable. Then $z^* = \psi(z^*_i, \theta, k)$ is $< z_*$ and also solves (4.15). This completes the proof.

Summarising, we obtain the following

**Theorem 4.** For the SOS model with respect to any normal subgroup $K \subset G_k$ of finite index the following assertions hold:

(i) In the FM case ($J < 0$), and for $J = 0$ (no interaction), the $K$-periodic GMs coincide with TIGMs.

(ii) In the AFM case ($J > 0$): (a) if $I(K) \neq \emptyset$ then $K$-periodic GMs coincide with TIGMs;
(b) if (4.14) holds and $I(K) = \emptyset$ then there are three $K$-periodic GMs $\mu_{12}$, $\mu_{21}$ and $\mu_*$. Moreover, measure $\mu_*$ is TI and measures $\mu_{12}$ and $\mu_{21}$ are $G^*_k$-periodic.

## 5 Non-periodic SGMs

In this section we consider the case $J < 0$, $m = 2$, $\beta > \beta_{cr}^1$. We use measures $\mu^i_*$, $i = -, m, +$, to show that system (3.1) admits uncountably many non-periodic solutions.

Take an arbitrary infinite path $\pi = \{x_0, x_1, \ldots\}$ on the CT $T^k$ starting at the origin $x^0$.

\[ x_0 = x^0. \]

We will establish a 1-1 correspondence between such paths and real numbers $t \in [0; \frac{k+1}{k}]$ (cf. Ref. [16,17]). In fact, let $\pi_1 = \{x_0, x_1, \ldots\}$ and $\pi_2 = \{y_0, y_1, \ldots\}$ be two such paths, with $x_0 = y_0 = x^0$. We will map the pair $(\pi_1, \pi_2)$ to a vector-function $h^{\pi_1 \pi_2}$: \( x \in V \mapsto h^{\pi_1 \pi_2}_x \) satisfying (3.1). Paths $\pi_1$ and $\pi_2$ split $T^k$ into three components $T_1^k$, $T_2^k$ and $T_3^k$ when $\pi_1$, $\pi_2$ are distinct and into two components $T_1^k$ and $T_3^k$ when $\pi_1$, $\pi_2$ coincide (again cf. Ref. [16,17]).

Vector-function $h^{\pi_1 \pi_2}$ is then defined by

\[
\begin{align*}
  h^{\pi_1 \pi_2}_x & = \begin{cases} 
    h^*_i, & \text{if } x \in T_i^k, \\
    h^*_m, & \text{if } x \in T_2^k, \\
    h^*_+, & \text{if } x \in T_3^k,
  \end{cases} 
\end{align*}
\]  

(5.1)

where vectors $h^*_i = (0, \ln z^*_{i,j})$, $i = -, m, +$, are solutions of (3.1).

Let $h = (h_0, h_1) \in \mathbb{R}^2$. Denote

\[
\|h\| = \max\{|h_0|, |h_1|\}.
\]

Let function $h \mapsto F(h) = F(h, \theta)$ be defined by (4.3).

**Proposition 5.1.** For any $h = (h_0, h_1) \in \mathbb{R}^2$ the following inequalities hold:

a) 

\[
\left| \frac{\partial F_i}{\partial h_j} \right| \leq \frac{\theta^2 - 1}{\theta^2}, \quad i, j = 0, 1,
\]
b) 

\[ \| F(h, \theta) - F(l, \theta) \| \leq 2 \frac{|\theta^2 - 1|}{\theta^2} \| h - l \|, \quad h, l \in \mathbb{R}^2, \]

c) for any \( h = (0, h_1) \) and \( l = (0, l_1) \):

\[ \| F(h) - F(l) \| \leq \frac{|\theta^2 - 1|}{1 + 3\theta^2 + 2\theta \sqrt{2(\theta^2 + 1)}} \| h - l \|, \quad h, l \in \mathbb{R}^2. \]

d) 

\[ |F_0(h)| \leq \frac{|\theta^2 - 1|}{\theta^2 + 1} |h_0|, \quad h = (h_0, h_1) \in \mathbb{R}^2. \]

**Proof.** a) Write:

\[ \frac{\partial F_0}{\partial h_0} = \frac{(1 - \theta^2)e^{h_0}(\theta e^{h_1} + \theta^2 + 1)}{(e^{h_0} + \theta e^{h_1} + \theta^2)(\theta^2 e^{h_0} + \theta e^{h_1} + 1)}. \]

To assess the derivative \( \frac{\partial F_0}{\partial h_0} \), consider two cases:

Case 1: \( h_0 \geq 0 \). Then

\[ \frac{\theta e^{h_1} + \theta^2 + 1}{e^{h_0} + \theta e^{h_1} + \theta^2} \leq 1, \quad \frac{e^{h_0}}{\theta^2 e^{h_0} + \theta e^{h_1} + 1} < \frac{1}{\theta^2}. \]

Case 2: \( h_0 \leq 0 \). Then

\[ \frac{1}{e^{h_0} + \theta e^{h_1} + \theta^2} \leq \frac{1}{\theta^2}, \quad \frac{(\theta e^{h_1} + \theta^2 + 1)e^{h_0}}{\theta^2 e^{h_0} + \theta e^{h_1} + 1} \leq 1. \]

Hence, \( |\frac{\partial F_0}{\partial h_0}| \leq \frac{|1 - \theta^2|}{\theta^2} \).

To assess \( \frac{\partial F_0}{\partial h_1} \), we again consider two cases:

Case 3: \( h_0 \geq 0 \). Then

\[ |\frac{dF_0}{dh_1}| = |\theta^2 - 1| \theta \frac{e^{h_1}}{e^{h_0} + \theta e^{h_1} + \theta^2} \frac{e^{h_0} - 1}{\theta^2 e^{h_0} + \theta e^{h_1} + 1} \leq \frac{|1 - \theta^2|}{\theta^2}. \]

Case 4: \( h_0 < 0 \). Then

\[ |\frac{dF_0}{dh_1}| = |\theta^2 - 1| \frac{e^{h_1}}{\theta^2 e^{h_0} + \theta e^{h_1} + 1} \frac{1 - e^{h_0}}{e^{h_0} + \theta e^{h_1} + \theta^2} \leq \frac{|1 - \theta^2|}{\theta^2}. \]

Finally, to assess the derivatives of \( F_1 \), write:

\[ |\frac{\partial F_1}{\partial h_0}| = \left| \theta(\theta^2 - 1) \frac{1}{\theta e^{h_0} + e^{h_1} + \theta} \frac{e^{h_0}}{\theta^2 e^{h_0} + \theta e^{h_1} + 1} \right| \leq \frac{|1 - \theta^2|}{\theta^2}. \]
and
\[ \left| \frac{\partial F_1}{\partial h_1} \right| = \left| (\theta^2 - 1) \frac{1}{\theta e^{h_0} + e^{h_1} + \theta} \right| \leq \frac{|1 - \theta^2|}{\theta^2}. \]

b) Write:
\[ \| F(h) - F(l) \| = \max \{ |F_0(h) - F_0(l)|, |F_1(h) - F_1(l)| \} \leq \max_{i=0,1} \{ |(F_i)'_{h_0}| h_0 - l_0 | + |(F_i)'_{h_1}| h_1 - l_1 | \} \leq 2 \frac{|\theta^2 - 1|}{\theta^2} \| h - l \|. \]

In cases c) and d) the inequalities are straightforward. This completes the proof of Proposition 5.1.

With the help of Proposition 5.1 it is easy to prove the following Theorem 5, similar to Theorem 3 of [17]:

**Theorem 5.** For any two infinite paths \( \pi_1, \pi_2 \), there exists a unique vector-function \( h_{\pi_1,\pi_2} \) satisfying (3.1) and (5.1).

Next, we map \((\pi_1, \pi_2)\) to a pair \((t, s)\) \(\in [0, \frac{k+1}{k}] \times [0, \frac{k+1}{k}]\). In the standard way (see [5, 16-18]) one can prove that functions \( h_{\pi_1(t)\pi_2(s)} \) are different for different pairs \((t, s)\) \(\in D\) where \( D = \{(u, v) \in [0, \frac{k+1}{k}]^2 : u \leq v \} \).

Now let \( \mu(t, s) \) denote the SGM corresponding to function \( h_{\pi_1(t)\pi_2(s)} \), \((t, s) \in D\). We obtain the following

**Theorem 6.** For any pair \((t, s) \in D\), there exists a unique SGM \( \mu(t, s) \). Moreover, the above GMs \( \mu_i^* \), \( i = -, m, + \), are specified as \( \mu(0, 0) = \mu_m^* \), \( \mu(0, \frac{k+1}{k}) = \mu_m^* \), \( \mu(\frac{k+1}{k}, \frac{k+1}{k}) = \mu_m^* \).

Because measures \( \mu(t, s) \) are different for different \((t, s) \in D\) we obtain a continuum of distinct extreme SGMs.

Concluding the paper, we state our final (an perhaps most ambitious) conjecture:

**Conjecture 7.** For a general ferromagnetic SOS model \( J < 0 \), for temperature \( T > 0 \) small enough, there exists at least three translation-invariant SGMs for \( m > 1 \) even and at least four for \( m > 1 \) odd. The precise numbers may depend on \( m \).

**Acknowledgements.** UAR thanks Cambridge Colleges Hospitality Scheme for supporting.
the visit to Cambridge in July, 2002. YMS worked in association with the ESF/RSDES Programme “Phase Transitions and Fluctuation Phenomena for Random Dynamics in Spatially Extended Systems” and was supported by the INTAS Grant 0265 Mathematics of Stochastic Networks. UAR and YMS thank IHES, Bures-sur-Yvette, and the IGS programme at the Isaac Newton Institute, University of Cambridge, for support and hospitality.

References

1. Ya.G. Sinai, Theory of phase transitions: Rigorous Results (Pergamon, Oxford, 1982).
2. C. Preston, Gibbs states on countable sets (Cambridge University Press, London 1974);
   F. Spitzer, Markov random fields on an infinite tree, Ann. Prob. 3: 387–398 (1975).
3. V.A. Malyshev, R.A. Minlos. Gibbs random fields (Nauka, Moskow 1985).
4. H.O. Georgii, Gibbs states and phase transitions (Walter de Gruyter, Berlin, 1988).
5. S. Zachary, Countable state space Markov random fields and Markov chains on trees. Ann. Prob. 11: 894–903 (1983).
6. S. Zachary, Bounded, attractive and repulsive Markov specifications on trees and on the one-dimensional lattice. Stochastic Process. Appl. 20: 247–256 (1985).
7. P.M. Bleher, N.N. Ganikhodjaev, On pure phases of the Ising model on the Bethe lattice, Theor. Probab. Appl. 35: 216–227 (1990).
8. P.M. Bleher, Extremity of the disordered phase in the Ising model on the Bethe lattice, Comm. Math. Phys. 128: 411-419 (1990).
9. P.M. Bleher, J. Ruiz, V.A. Zagrebnov, On the purity of the limiting Gibbs state for the Ising model on the Bethe lattice, Journ. Statist. Phys. 79: 473-482 (1995).
10. P.M. Bleher, J. Ruiz, V.A. Zagrebnov, On the phase diagram of the random field Ising model on the Bethe lattice, Journ. Statist. Phys. 93: 33-78 (1998).
11. D. Ioffe, On the extremality of the disordered state for the Ising model on the Bethe lattice, Lett. Math. Phys. 37: 137-143 (1996).
12. D. Ioffe. Extremality of the disordered state for the Ising model on general trees, Trees, Versailles, 1995. Progr. Probab. 40: 3-14 (Progr. Probab., 40, Birkhauser, Basel, 1996).
13. P.M. Bleher, J. Ruiz, R.H. Schonmann, S. Shlosman, V.A. Zagrebnov, Rigidity of the critical phases on a Cayley tree, Moscow Mathematical Journ. 3: 345-363 (2001).
14. A.E. Mazel, Yu.M. Suhov. Random surfaces with two-sided constraints: an application of the theory of dominant ground states, Journ. Statist. Phys. 64: 111-134 (1991).
15. F.P. Kelly, Stochastic models of computer communication systems. With discussion. Journ. Roy. Statist. Soc. Ser B. 47: 379–395, 415–428 (1985); K. Ramanan, A. Sengupta, I. Ziedins and P. Mitra, Markov random field models of multicasting in tree networks, Adv. Appl. Probab. 34: 58–84 (2002).
16. U.A. Rozikov, Description of limiting Gibbs measures for λ— models on the Bethe lattice, Siberian Math. Journ. 39: 427-435 (1998).
17. U.A. Rozikov, Description of uncountable number of Gibbs measures for inhomogeneous Ising model, *Theor. Math. Phys.* **118**:95-104 (1999).

18. N.N. Ganikhodjaev, U.A. Rozikov, Description of periodic extreme Gibbs measures of some lattice models on the Cayley tree, *Theor. Math. Phys.* **111**: 480-486 (1997).

19. N.N. Ganikhodjaev, U.A. Rozikov, On disordered phase in the ferromagnetic Potts model on the Bethe lattice, *Osaka Journ. Math.* **37**:373-383 (2000).

20. N.N. Ganikhodjaev, On pure phases of the ferromagnet Potts with three states on the Bethe lattice of order two. *Theor. Math. Phys.* **85**:163–175 (1990).

21. U.A. Rozikov, Partition structures of the group representation of the Cayley tree into cosets by finite-index normal subgroups and their applications to the description of periodic Gibbs distributions. *Theor. Math. Phys.* **112**: 929-933 (1997).