NEARLY OPTIMAL BOUNDS FOR DISTRIBUTED WIRELESS SCHEDULING IN THE SINR MODEL

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Abstract. We study the wireless scheduling problem in the physically realistic SINR model. More specifically: we are given a set of \( n \) links, each a sender-receiver pair. We would like to schedule the links using the minimum number of slots, using the SINR model of interference among simultaneously transmitting links. In the basic problem, all senders transmit with the same uniform power.

In this work, we provide a distributed \( O(\log n) \)-approximation for the scheduling problem, matching the best ratio known for centralized algorithms. This is based on an algorithm studied by Kesselheim and Vöcking, improving their analysis by a logarithmic factor. We show this to be best possible for any such distributed algorithm.

Our analysis extends also to linear power assignments, and as well as for more general assignments, modulo assumptions about message acknowledgement mechanisms.

1. Introduction

Given a set of \( n \) wireless links, each a sender-receiver pair, what is the minimum number of slots needed to schedule all the links, given interference constraints? This is the canonical scheduling problem in wireless communication that we study here in a distributed setting.

In a wireless network, communication occurring simultaneously in the same channel interfere with each other. Algorithmic questions for wireless networks depend crucially on the model of interference considered. In this work, we use the physical model or “SINR model” of interference, precisely defined in Section 2. It is known to capture reality more faithfully than the graph-based models most common in the theory literature, as shown theoretically as well as experimentally [9, 15, 16, 18]. Early work on scheduling in the SINR model focused on heuristics and/or non-algorithmic average-case analysis (e.g. [10]). In seminal work, Moscibroda and Wattenhofer [17] proposed the study of the scheduling complexity of arbitrary set of wireless links. Numerous works on various problems in the SINR setting have appeared since (e.g. [6, 8, 13, 15, 12]), to point out just a few.

The scheduling problem has primarily been studied in a centralized setting. In many realistic scenarios, however, it is imperative that a distributed solution be found, since a centralized controller may not exist, and individual nodes in the link may not be aware of the overall topology of the network. For the scheduling problem, the only known rigorous result is due to Kesselheim and Vöcking [15], who show that a simple and natural distributed algorithm provides \( O(\log^2 n) \) approximation to the scheduling problem.

In this work, we adopt the same algorithm as Kesselheim and Vöcking, but provide an improved analysis of \( O(\log n) \)-approximation. Moreover, we show that this is essentially the best result obtainable by a distributed algorithm.

2. Preliminaries and Contributions

Given is a set \( L = \{\ell_1, \ell_2, \ldots, \ell_n\} \) of links, where each link \( \ell_v \) represents a communication request from a sender \( s_v \) to a receiver \( r_v \). The distance between two points \( x \) and \( y \) is denoted \( d(x, y) \). The asymmetric distance from link \( \ell_v \) to link \( \ell_w \) is the distance from \( v \)'s sender to \( w \)'s receiver, denoted \( d_{vw} = d(s_v, r_w) \). The length \( d(s_v, r_v) \) of link \( \ell_v \) is simply denoted by \( \ell_v \).
Let $P_v$ denote the power assigned to link $\ell_v$, or, in other words, $s_v$ transmits with power $P_v$. We adopt the physical model (or SINR model) of interference, in which a receiver $r_v$ successfully receives a message from a sender $s_v$ if and only if the following condition holds:

$$\frac{P_v/\ell_v^\alpha}{\sum_{w \in S \setminus \{v\}} P_w/d_{vw}^\alpha} + N \geq \beta,$$

where $N$ is a universal constant denoting the ambient noise, $\alpha > 0$ denotes the path loss exponent, $\beta \geq 1$ denotes the minimum SINR (signal-to-interference-noise-ratio) required for a message to be successfully received, and $S$ is the set of concurrently scheduled links in the same slot. We say that $S$ is SINR-feasible (or simply feasible) if (1) is satisfied for each link in $S$.

Given a set of links $L$, the **scheduling problem** is to find a partition of $L$ of minimum size such that each subset in the partition is feasible. The size of the partition equals the minimum number of slots required to schedule all links. We will call this number the **scheduling number** of $L$, and denote it by $T(L)$ (or $T$ when clear from context).

The above defines the physical model for uni-directional links. In the bi-directional setting, the asymmetry between senders and receivers disappear. The SINR constraint (1) changes only in that the definition of distance between links changes to $d_{vw} = \min(d(s_w, r_v), d(s_v, r_w), d(s_w, s_v), d(r_w, r_v))$. With this new definition of inter-link distances, all other definitions and conditions remain unchanged. We will focus on the uni-directional model, but our proofs easily extend to the bi-directional case.

A power assignment $P$ is **length-monotone** if $P_v \geq P_w$ whenever $\ell_v \geq \ell_w$ and **sub-linear** if $\frac{P_v}{\ell_v^\alpha} \leq \frac{P_w}{\ell_w^\alpha}$ whenever $\ell_v \geq \ell_w$. Two widely used power assignments in this class are the uniform power assignment, where every link transmits with the same power; and the linear power assignment, where $P_v$ is proportional to $\ell_v^{\alpha}$.

**Distributed algorithms.** In the traditional distributed setting, a communication infrastructure exists which can be used to run distributed algorithms. The current setting is different, where the goal is to construct such an infrastructure. Thus our algorithm will work with very little global knowledge and minimal external input. We assume that the senders and receivers have a rough estimate of the network size $n$, have synchronized clocks, and are given fixed length-monotone, sub-linear power assignments that they must use.

The algorithm works by having the senders repeatedly transmitting until they succeed, necessitating an acknowledgement from receivers to the senders, so that the sender would know when to stop. We assume that these acknowledgements are the only external information that the senders receive. Any algorithm that works under these constraints will be referred to as an **ack-only** algorithm. We study the scheduling problem under two separate assumptions about acknowledgements. In the first model, we require the algorithm to generate explicit acknowledgements. In the second model, we assume that acknowledgements are “free” or much cheaper than data packets, thus do not have to be explicitly realized by the algorithm.

**Affectance.** We will use the notion of affectance, introduced in [8, 13] and refined in [15] to the thresholded form used here, which has a number of technical advantages. The affectance $a^P_w(v)$ on link $\ell_v$ from another link $\ell_w$, with a given power assignment $P$, is the interference of $\ell_w$ on $\ell_v$ relative to the power received, or

$$a^P_w(v) = \min \left\{ 1, c_v \frac{P_w/d_{vw}^\alpha}{P_v/\ell_v^\alpha} \right\} = \min \left\{ 1, c_v \frac{P_w}{P_v} \cdot \left( \frac{\ell_v}{d_{vw}} \right)^\alpha \right\},$$

where $c_v = \beta/(1 - \beta N \ell_v^\alpha/P_v)$ is a constant depending only on the length and power of the link $\ell_v$. We will drop $P$ when it is clear from context. Let $a_v(v) = 0$. For a set $S$ of links and a link $\ell_v$, let $a_v(S) = \sum_{\ell_w \in S} a_v(w)$ and $a_S(v) = \sum_{\ell_w \in S} a_w(v)$. For sets $S$ and $R$, $a_R(S) = \sum_{\ell_v \in R} \sum_{\ell_w \in S} a_v(u)$. Using such notation, Eqn. (1) can be rewritten as $a_S(v) \leq 1$, and **this is the form we will use.**
Signal-strength and robustness. A $\delta$-signal set is one where the affectance on any link is at most $1/\delta$. A set is SINR-feasible iff it is a 1-signal set. We know:

**Lemma 1** ([11]). Let $\ell_u, \ell_v$ be links in a $q^\alpha$-signal set. Then, $d_{uv} \cdot d_{vu} \geq q^2 \cdot \ell_u \ell_v$.

2.1. Our Contributions and Related work. We achieve the following results:

**Theorem 2.** There is a $O(\log n)$-approximate ack-only distributed algorithm for the scheduling problem for uniform and linear power assignments, as well as for all length-monotone sub-linear power assignments for bi-directional links. If we additionally assume free (or cheap) acknowledgements the same result holds for all length-monotone sub-linear power assignments for uni-directional links.

**Theorem 3.** Assuming that all senders use an identical algorithm, no distributed ack-only algorithm can approximate the scheduling problem by a factor better than $\Omega(\log n)$, even with free acknowledgements.

As in [15], our results hold in arbitrary distance metrics (and do not require the common assumption that $\alpha > 2$).

The scheduling problem has been profitably studied in the centralized setting by a number of works. The problem is known to be NP-hard [8]. For length-monotone, sub-linear power assignments, a $O(\log n)$ approximation for general metrics has been achieved recently [12] following up on earlier work [8]. In the bi-directional setting with power control, Fanghanel et al. [6] provided a $O(\log^{3.5+\alpha} n)$ approximation algorithm, recently improved to $O(\log n)$ [11, 12]. For linear power on the plane, [7] provides an additive approximation algorithm of $O(T + \log^2 n)$. On the plane, $O(\log n)$ approximation for power control for unidirectional links has been recently achieved [14]. Chafekar et al. [8] provide a $O(\log^2 \Delta \log^2 \Gamma \log n)$ approximation to the joint multi-hop scheduling and routing problem.

In the distributed setting, the related capacity problem (where one wants to find the maximum subset of $L$ that can be transmitted in a single slot) has been studied a series of papers [11, 4, 2], and have culminated in a $O(1)$-approximation algorithm for uniform power [2]. However, these game-theoretic algorithms take time polynomial in $n$ to converge, thus can be seen more appropriately to determine capacity, instead of realizing it in “real time”.

For distributed scheduling, the only work we are aware of remains the interesting paper by Kesselheim and Vöcking [15], who give a $O(\log^2 n)$ distributed approximation algorithm for the scheduling problem with any fixed length-monotone and sub-linear power assignment. They consider the model with no free acknowledgements, however their results do not improve if free acknowledgements are assumed. Thus in all cases considered, our results constitute a $\Omega(\log n)$ factor improvement.

In [15], the authors introduce a versatile measure, the maximum average affectance $\bar{A}$, defined by

$$\bar{A} = \max_{R \subseteq L} \frac{1}{|R|} \sum_{\ell_u \in R} \sum_{\ell_v \in R} a_{uv}(u).$$

The authors then show two results. On the one hand, they show that $\bar{A} = O(T \log n)$ where $T = T(L)$. On the other hand, they present a natural algorithm (we use the same algorithm in this work) which schedules all links in $O(\bar{A} \log n)$ slots, thus achieving a $O(\log^2 n)$ approximation. We can show (Appendix A) that both of these bounds are tight. Thus it is not possible to obtain improved approximation using the measure $\bar{A}$.

Our main technical insight is to devise a different measure that avoids bounding the global average, instead looks at the average over a large fraction of the linkset. This allows us to shave off a log $n$ factor. The measure $\Lambda(L)$ (or $\Lambda$ when clear from the context) can be defined as follows:

$$\Lambda(L) = \arg \min_{\ell} \{ \ell \in L : a_R(\ell) \leq 4\ell \} \geq |R|/4.$$
To get an intuitive feel for this measure, consider any given $R$. Since we only insist that a large fraction of $R$ have affectance bounded by $t$, the value of $t$ may be significantly smaller than the average affectance in $R$ (and as a consequence $\Lambda(L)$ may be much smaller than $\bar{A}$). Indeed, we show that $\Lambda = O(T)$ and that the algorithm schedules all links in time $O(\Lambda \log n)$, achieving the claimed approximation factor. We can give instances where $T(L) = \theta(\Lambda(L) \log n)$, and thus the performance of the algorithm is best possible in that respect.

3. $O(\log n)$-Approximate Distributed Scheduling Algorithm

The algorithm from [15] is listed below as Distributed. It is a very natural algorithm, in the same tradition of backoff schemes as ALOHA [19], or more recent algorithms in the SINR model [4] [2].

Algorithm 1 Distributed

1: $k \leftarrow 0$
2: while transmission not successful do
3: \hspace{1cm} $q = \frac{1}{4 \cdot 2^k}$
4: \hspace{1cm} for $\frac{\ln n}{q}$ slots do
5: \hspace{2cm} transmit with i.i.d. probability $q$
6: \hspace{1cm} end for
7: \hspace{1cm} $k \leftarrow k + 1$
8: end while

The algorithm is mostly self-descriptive. One point to note is that Line 2 necessitates some sort of acknowledgement mechanism for the distributed algorithm to stop. For simplicity, we will defer the issue of acknowledgements to Section 3.2 and simply assume their existence for now. Theorem 4 below implies our main positive result.

Theorem 4. If all links of set $L$ run Distributed, then each link will be scheduled by time $O(\Lambda(L) \log n)$ with high probability.

To prove Theorem 4, we claim the following.

Lemma 5. Given is a request set $R$ with measure $\Lambda = \Lambda(R)$. Consider a time slot in which each sender of the requests in $R$ transmits with probability $q \leq \frac{1}{8\Lambda}$. Then at least $\frac{q\cdot|R|}{2}$ transmissions are successful in expectation.

Proof. Let $M = \{\ell_u \in R : a_R(u) \leq 4\Lambda\}$. By definition of $\Lambda$, $M \geq |R|/4$. It suffices to show that at least $q|M|/2$ transmissions are successful in expectation.

For $\ell_u \in R$, let $T_u$ be indicator random variable that link $\ell_u$ transmits, and $S_u$ the indicator r.v. that $\ell_u$ succeeds. Now, $\mathbb{E}(S_u) = \mathbb{P}(S_u = 1) = \mathbb{P}(T_u = 1)\mathbb{P}(S_u = 1|T_u = 1) = q\mathbb{P}(S_u = 1|T_u = 1) = q(1 - \mathbb{P}(S_u = 0|T_u = 1))$. For $\ell_u \in M$,

$$
\mathbb{P}(S_u = 0|T_u = 1) \leq \mathbb{P} \left( \sum_{\ell_v \in R} a_v(u)T_v \geq 1 \right) \leq \mathbb{E} \left( \sum_{\ell_v \in R} a_v(u)T_v \right) = \sum_{\ell_v \in R} a_v(u)\mathbb{E}(T_v) = q \sum_{\ell_v \in R} a_v(u) \leq q \cdot 4\Lambda \leq 1/2 .
$$

for $q \leq \frac{1}{8\Lambda}$. Therefore, $\mathbb{E}(S_u) \geq \frac{q}{2}$. 

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The expected number of successful links is thus

\[ E\left(\sum_{\ell_u \in R} S_u\right) = \sum_{\ell_u \in R} E(S_u) \geq \sum_{\ell_u \in M} E(S_u) \geq |M| \cdot q/2 \geq |R| \cdot q/8, \]

as desired. \(\square\)

of Thm. 4. Given Lemma 5 the proof of the Theorem essentially follows the arguments in Thms. 2 and 3 of [15].

First consider the running time of the algorithm when in Line 3, \(q\) is set to the “right” value \(1/8\) stipulated in Lemma 5. (Note that since the algorithm increases the probabilities by a factor of 2, it will select the right probability within that factor). Let \(n_t\) be the random variable indicating the number of requests that have not been successfully scheduled in the first \(t\) time slots.

Lemma 5 implies that \(E(n_{t+1} | n_t = k) \leq k - \frac{q}{8}k\) and thus

\[ E(n_{t+1}) \leq \sum_{k=0}^{\infty} P(n_t = k) \cdot (1 - q/8)k = (1 - q/8)E(n_t). \]

Setting \(n_0 = n\), this yields \(E(n_t) \leq (1 - q/8)^t n\). Now, after \(8c_3 \log n/q\) time slots for a large enough \(c_3\), the expected number of remaining requests is

\[ E(n_{8c_3 \log n/q}) \leq (1 - q/8)^{8c_3 \log n/q}n \leq \left(\frac{1}{e}\right)^{c_3 \log n} n = n^{1-c_3}. \]

By Markov inequality \(P(n_{8c_3 \log n/q} \neq 0) = P(n_{8c_3 \log n/q} \geq 1) \leq E(n_{8c_3 \log n/q}) \leq n^{1-c_3}\). So we need \(O(\log n/q) = O(\Lambda \log n)\) time, with high probability.

Now we need to show that running the algorithm with the “wrong” values of \(q\) doesn’t cost too much. This holds because the costs of all previous executions form a geometric series increasing the overall time required by no more than a constant factor (see Thm. 3 of [15] for the argument, which is fairly simple). \(\square\)

3.1. **Bounding the Measure.** We will assume that the implicit power assignment is length-monotone and sub-linear. We need two related Lemmas to get a handle on affectances. The first of these Lemmas is known.

**Lemma 6** (Lemma 7, [15]). If \(L\) is a feasible set, and \(\ell_u\) is a link such that \(\ell_u \leq \ell_v\) for all \(\ell_v \in L\), then \(a_L(u) = O(1)\).

Now we prove the following.

**Lemma 7.** If \(L\) is a feasible set such that \(a_u(L) \leq 2\) for all \(\ell_u \in L\), and \(\ell_v\) is a link such that \(\ell_v \leq \ell_u\) for all \(\ell_u \in L\), then \(a_v(L) = O(1)\).

Before we prove this below, note that this Lemma can be contrasted with Lemma 9 of [15], which only gives a \(O(\log n)\) bound (without the extra condition \(a_u(L) \leq 2\)). Intuitively, the power of this Lemma is that while \(a_u(L)\) can be as large as \(O(\log n)\) for feasible \(L\), for a large subset of \(L\) the extra condition \(a_u(L) \leq 2\) holds (as will be easily shown in Lemma 8), for which the stronger \(O(1)\) upper bound will apply.

**Proof.** [of Lemma 7] We use the signal strengthening technique of [13]. For this, we decompose the set \(L\) to \([2 \cdot 3^\alpha/\beta]^2\) sets, each a \(3^\alpha\)-signal set. We prove the claim for one such set; since there are only constantly many such sets, the overall claim holds. Let us reuse the notation \(L\) to be such a \(3^\alpha\)-signal set.
Consider the link $\ell_u = (s_u, r_u) \in L$ such that $d(s_u, s_u)$ is minimum. Also consider the link $\ell_w = (s_w, r_w) \in L$ such that $d(r_w, s_u)$ is minimum. Let $D = d(s_u, s_u)$. We claim that for all links, $\ell_x = (s_x, r_x) \in L$, $\ell_x \not= \ell_w$,

$$d(s_v, r_x) \geq \frac{1}{2} D .$$

To prove this, assume, for contradiction, that $d(s_v, r_x) < \frac{1}{2} D$. Then, $d(s_v, r_w) < \frac{1}{2} D$, by definition of $\ell_w$. Now, again by the definition of $\ell_u$, $d(s_u, s_v) \geq D$ and $d(s_v, s_u) \geq D$. Thus $\ell_w \geq d(s_v, s_w) - d(s_w, r_w) > \frac{D}{2}$ and similarly $\ell_x > \frac{D}{2}$. On the other hand $d(r_w, r_x) < \frac{D}{2} + \frac{D}{2} < D$. Now, $d_{wx} \cdot d_{xw} \leq (\ell_w + d(r_w, r_x))(\ell_x + d(r_x, w_x)) < (\ell_w + D)(\ell_x + D) < 9\ell_w \ell_x$, contradicting Lemma 1.

By the triangle inequality and Eqn. 2, $d_{ux} = d(s_u, r_x) \leq d(s_u, s_v) + d(s_v, r_x) \leq 3d(s_v, r_x) = 3d_{wx}. Now, a_v(x) \leq c_x \frac{P_u \ell_u^a}{P_x} . Since \ell_v \leq \ell_u$, by length-monotonicity $P_v \leq P_u$. Thus,

$$a_v(x) \leq c_x \frac{P_u \ell_u^a}{dx} \leq c_x \frac{3^a P_u \ell_u^a}{dx} \leq 3^a a_u(x)$$

where the last equality holds because $a_u(x) = c_x \frac{P_u \ell_u^a}{dx}$ as $L$ is feasible. Finally, summing over all links in $L$

$$a_v(L) = \sum_{\ell_v \in L} a_v(x) = a_v(w) + \sum_{\ell_v \not= \ell_w} a_v(x) \leq 1 + \sum_{\ell_v \not= \ell_w} a_v(x) \leq 1 + 3^a \sum_{\ell_v \not= \ell_w} a_u(x) \leq 1 + 3^a \cdot 2 = O(1) ,$$

since $\sum_{\ell_v \not= \ell_w} a_u(x) \leq a_u(L) \leq 2$ by assumption. \qed

We can now derive the needed bound on the measure.

**Lemma 8.** For any linkset $R$, $\Lambda(R) \leq c_2 T(R)$ for a fixed constant $c_2$. 

**Proof.** It suffices to prove that for every $\hat{R} \subseteq R$, $|\{\ell \in \hat{R} : a_{\hat{R}}(\ell) \leq 4c_2 T\}| \geq \hat{R}/4$. To prove this, the only property we will use of $\hat{R}$ is that $T(\hat{R}) \leq T(R)$, which is obviously true for all $\hat{R}$. Thus, for simplicity we simply prove it for $R$ and the proof will easily carry over to all $\hat{R} \subseteq R$.

Consider a partition of $R$ into $T$ feasible subsets $S_1 \ldots S_T$. For each $i$, define $S'_i = \{\ell_v \in S_i : a_v(S_i) \leq 2\}$.

**Claim 3.1.** For all $i$, $|S_i'| \geq |S_i|/2$.

**Proof.** Since $S_i$ is feasible, the incoming affectance $a_{S_i}(v) \leq 1$ for every link $\ell_v \in S_i$. Let $\hat{S}_i = S_i \setminus S'_i$. Now, $\sum_{v \in \hat{S}_i} a_v(S_i) \leq \sum_{v \in S_i} a_v(S_i) = \sum_{v \in S_i} a_v(S_i) \leq |S_i|$. But, $\sum_{v \in \hat{S}_i} a_v(S_i) \geq 2 |\hat{S}_i|$ by definition of $\hat{S}_i$. Thus, $\hat{S}_i \leq |S_i|/2$, proving the claim. \qed

Let $R' = \cup_i S'_i$. By the above claim, $|R'| \geq |R|/2$. Let $M = \{\ell_u \in R' : \sum_{\ell_v \in R} a_u(\ell_v) \leq 4c_2 T\}$ for some constant $c_2$. We shall show that $|M| \geq |R'|/2 \geq |R|/4$.

We will prove the claim

$$a_R(R') = c_2 |R| \cdot |T| .$$

From this, we get that the average of $a_R(\ell_u)$ over the links $\ell_u \in R'$ is $c_2 |R| \cdot |T| \leq 2c_2 |T|$. At least half of the links in $R'$ have at most double affectance of this average, hence the claim $|M| \geq |R|/4$ and the lemma follow.
Thus our goal becomes to prove Eqn. 3. To prove this note that,

\[ a_R(R') = \sum_{i=1}^{T} \sum_{\ell_u \in S_i^i} \sum_{\ell_v \geq \ell_u} a_v(u) = \sum_{i=1}^{T} \sum_{j=1}^{T} a_{S_j}(S_j^i) . \]

To tackle this sum, we first prove that for any \( i, j \leq T \),

\[ a_{S_j}(S_j^i) \leq O(|S_j| + |S_i|) . \]

This holds because,

\[
\begin{align*}
a_{S_j}(S_j^i) &= \sum_{\ell_u \in S_i^i} \sum_{\ell_v \in S_j, \ell_v \geq \ell_u} a_v(u) + \sum_{\ell_u \in S_i^i} \sum_{\ell_v \in S_j, \ell_v \leq \ell_u} a_v(u) \\
&\leq \sum_{\ell_u \in S_i^i} O(1) + \sum_{\ell_u \in S_i^i} \sum_{\ell_v \in S_j, \ell_v \leq \ell_u} a_v(u) + \sum_{\ell_v \in S_j} \sum_{\ell_v \in S_i^i, \ell_v \leq \ell_u} a_v(u) \\
&\leq O(|S_i|) + \sum_{\ell_v \in S_j} \sum_{\ell_v \in S_i^i, \ell_v \geq \ell_v} a_v(u) + \sum_{\ell_v \in S_j} \sum_{\ell_v \in S_i^i, \ell_v \leq \ell_v} a_v(u) \\
&\leq O(|S_i|) + O(1) \leq O(|S_i| + |S_j|) .
\end{align*}
\]

Explanation for numbered inequalities:

(1) The bound with \( O(1) \) follows from Lemma 5
(2) Noting \( |S_i^i| \leq |S_i| \) for the first term and reorganization of the second term.
(3) The \( O(1) \) bound is from Lemma 7.

Now we can continue with Eqn. 4 to obtain that

\[
a_R(R') = \sum_{i=1}^{T} \sum_{j=1}^{T} a_{S_j}(S_j^i) \leq \sum_{i=1}^{T} \sum_{j=1}^{T} O(|S_i| + |S_j|) \leq \sum_{i=1}^{T} O(|T| \cdot |S_i| + |R|) = O(|T| \cdot |R|) .
\]

Explanations for numbered (in)equalities:

(1) Due to Eqn. 4
(2) As \( \sum_{j=1}^{T} |S_j| = |R| .

This completes the proof by setting \( c_2 \) to be the implicit constant hidden in the Big-O notation. \( \square \)

3.2. **Duality and Acknowledgements.** In the above exposition, we ignored the issue of sending acknowledgements from receivers to senders after the communication has succeeded. It is useful to be able to guarantee that acknowledgements would arrive in time \( O(T \log n) \), with high probability. Note that for bi-directional links, there is no dichotomy between sender and receiver, so we get our result for all length-monotone, sub-linear power assignments automatically. What follows thus concerns the case of uni-directional links.

The link set \( L^* \) associated with the required acknowledgement packets is the “dual” of the original link set \( L \), with the sender and receiver of each link swapping places. Let the dual link associated with a link \( \ell_u \in L \) be \( \ell_u^* \). Now, accommodating the acknowledgements in the same algorithm is not difficult. This can be done, for example, by using the odd numbered slots for the original transmissions, and the even numbered slots for the acknowledgement. It is not obvious, however, that \( L^* \) admits a short schedule using an oblivious power assignment.

For uniform power, it was observed in [15] that:

**Claim 3.2** (Observation 12, [15]). Assume both the original and dual link sets use uniform power. Also assume \( \ell_u, \ell_v \in L \) can transmit simultaneously. Then, \( a_u^{\cdot}(v^\cdot) = O(a_u(v)) \), where \( a_u^{\cdot}(v^\cdot) \) is the affectance from \( \ell_u \) on \( \ell_v \).
Thus the feasibility of $S_i$ implies the near-feasibility of $S_i^*$. With this observation in hand, the bound follows from arguments of the previous section in a straightforward manner.

For other power assignments, it is easy to show that one cannot in general use the same assignment for the dual. In [15] an elegant approach to this problem was proposed. The authors introduced the notion of dual power, where dual link $\ell_u^*$ uses power $P_u^* = \gamma \frac{\ell_u^*}{T}$ (the global normalization factor $\gamma$ is chosen so that $c_u^*$ is no worse than $c_u$, for all $\ell_u$). In [15] the following is shown:

**Claim 3.3** (Observation 4, [15]). $a_u^p(v) = \Theta(a_{u^*}^p(u^*))$, for any $\ell_u, \ell_v$ in $L$.

In other words, the incoming affectance of a link $\ell_u$ in $L$ is close to the outgoing affectance of $\ell_u^*$ in the “dual” setting (and vice-versa). Using this claim, it is easy to see that the measure $A$ (maximum average affectance) is invariant (modulo constants) when switching the problem from $L$ to $L^*$. Since the authors claim a bound in terms of $A$, essentially the same bound can be claimed for both $L$ and $L^*$.

The situation for our argument is more delicate, since we seek a bound in terms of $T$, not $A$. Unfortunately, $T^*$ (the scheduling number of $L^*$) cannot be bounded by anything better than $O(T \log n)$, thus a naive application of Thm. 4 only results in a $O(\log^2 n)$ approximation.

However, we can achieve the $O(\log n)$ approximation for linear power, whose dual is uniform power. We define “anti-feasibility” to be: A link set $R$ is anti-feasible if $a_u(R) \subseteq c_4; \forall \ell_u \in R$ for some appropriate constant $c_4$. Consider now a partition of $L$ into feasible sets $S_1, S_2 \ldots S_T$. By the claim above, it is clear that the sets $S_i^*$ are anti-feasible. Define, $S_i^* = \{ \ell_u^* \in S_i^* : a_{S_i^*}(v^*) \leq \alpha \}$, $R^* = \bigcup_{i=1}^T S_i^*$ and $R^* = \bigcup_{i=1}^T S_i^*$. As in the previous section, it is clear that $|R^*| \geq |R^*|/2$.

We claim that a Lemma similar to Lemma 8 holds, after which $O(T \log n)$ algorithm for the dual set follows the same argument as the rest of the proof of Thm. 4.

**Lemma 9.** The following holds for linear power (thus, uniform power for the dual). Let $M^* = \{ \ell_v \in R^* : a_{R^*}(v) \leq 2c_2 T \}$, for some constant $c_2$. Then, $|M^*| \geq |R^*|/8 \geq |R^*|/16$.

The crucial thing to notice is that the bound is in terms of $T$, not the possibly larger $T^*$.

**Sketch.** For uniform power, the following strong result can be proven.

**Lemma 10.** If $L$ is an anti-feasible set using uniform power such that $a_L(u) \leq 4c_4$ for $\ell_u$, and $\ell_v$ some other link, also using uniform power, then $a_v(L) = O(1)$.

This can be proven using techniques we have seen, and has essentially been proven in [2] (Lemma 11). Note that this lemma has no condition on $\ell_v$ (in terms of being smaller/larger than links of $L$, unlike Lemmas 6 and 7). Thus this single Lemma suffices in bounding $a_{S_i^*}(S_i^*)$ à la Eqn. 6 from which the whole argument follows (details omitted).

Finally, if we assume free or cheap acknowledgments, then by Thm. 4 our $O(\log n)$ bound holds for all sub-linear, length-monotone assignments for both uni and bi-directional links. The assumption can be valid in many realistic scenarios, or indeed can guide design decisions. For example, if the size of the acknowledgement packets are smaller than the data packets by a factor of $\Omega(\log n)$, the larger value of $T^*$ is subsumed by having smaller slots for acknowledgement packets. For illustration, a realistic example of data packet sizes and acknowledgement packet sizes is 1500 bytes and 50 bytes, respectively, and in such a case our bound would hold for networks with up to $2^{30}$ nodes. Note here that the analysis of [15] is no better than $O(\log^2 n)$ even assuming free acknowledgments.
We give a construction of $2n$ links on the line that can be scheduled in 2 slots while no distributed algorithm can schedule them in less than $\Omega(\log n)$ slots.

We assume that the distributed algorithm is an **ack-only** algorithm, and that each sender starts at the same time in the same state and uses an identical (randomized) algorithm. Note that the algorithm presented works within these assumptions.

Our construction uses links of equal length, thus the only possible power assignment is uniform. Let $\alpha > 1$, $\beta = 2$ and noise $N = 0$. For the construction, we start with a **gadget** $g$ with two identical links of length 1. Place $n$ such gadgets $g_i, i = 1 \ldots n$ on the line as follows. The two senders of $g_i$ are placed at point $2ni$ and the two receivers of $g_i$ are placed at $2ni + 1$. Now since $\beta = 2$, it is clear that if the two links in the gadget transmit together, neither succeed. On the other hand, links from other gadgets have negligible affectance on a link. To see this, consider the affectance on a link $\ell_u \in g_i$ from all links of other gadgets, i.e., from all links $\ell_v \in \hat{G} = \cup_{j \neq i}g_j$. There are $2n - 2$ links in $\hat{G}$. The distance $d_{uv} \geq \min\{2n(i+1) - 2ni - 1, 2ni + 1 - 2n(i-1)\} = 2n - 1$. Therefore, $\sum_{\hat{G}}a_v(u) \leq (2n - 2)\frac{1}{(2n-1)^\alpha} < 1$, for large enough $n$ and $\alpha > 1$. Thus, behavior of links in other gadgets is immaterial to the success of a link. This also implies that the scheduling number of all these links is 2.

To prove the lower bound, let us consider a gadget $g_i$ to be “active” at time $t$ if neither link of $g_i$ succeeded by time $t - 1$. Let $T_u(t)$ denote the event that link $\ell_u$ transmits at time $t$, and let $A_i(t)$ denote the event that gadget $g_i$ is active at time $t$.

**Lemma 11.** Let $\ell_u$ and $\ell_v$ be the identical links in gadget $g_i$. Then $\mathbb{P}(T_u(t)|A_i(t)) = \mathbb{P}(T_v(t)|A_i(t))$. Moreover, these two probabilities are independent. In other words, the transmission probabilities of two links in a gadget at time $t$ are identical and independent, conditioned on the gadget being active at time $t$.

**Proof.** Since $g_i$ is active at time $t$, $\ell$ and $\ell'$ have made identical decisions about transmitting at all times $[0, t - 1]$ and got identical information (failure) when they did transmit.

Consider all combinations of paths taken by $\ell$ and $\ell'$ resulting in the event $a_i(t)$. Let $R$ ($R'$) be the set of random strings that $\ell$ ($\ell'$) could have generated, conditioned on $a_i(t)$. By symmetry, $R = R'$ and they clearly have the same distribution as well. Now, crucially, the particular path $\ell$ takes does not affect the distribution on the paths taken by $\ell'$ over $R'$ at all. This is because, conditioned on $a_i(t)$, whatever the particular path may be that $\ell$ chooses, its external behavior is identical to all other paths, and thus the particular choice is not differentiable for any other choice from the point of view of $\ell'$. Thus, the distribution over $R$ and $R'$ are i.i.d., given $a_i(t)$, and the same is clearly the case for the random strings generated by the algorithm in the $t + 1^{st}$ slot. Therefore, the probability of transmission in the $t + 1^{st}$ slot is independent and identical. \hfill $\square$

Let this i.i.d. probability be $p$. Now $\mathbb{P}(A_i(t+1)|A_i(t)) = p^2 + (1 - p)^2$, which is minimized for $p = \frac{1}{2}$ with value $\frac{1}{2}$. Thus, $\mathbb{P}(A_i(t+1)|A_i(t)) \geq \frac{1}{2}$.

**Theorem 12.** $\mathbb{E}(I(n)) = \Omega(\log n)$ where $I(n)$ is the smallest time at which none of the gadgets are active.

**Proof.** Intuitively, the bound $\mathbb{P}(A_i(t+1)|A_i(t)) \geq \frac{1}{2}$ implies that on average, no more than half the active gadgets become inactive in a single round, thus it would take $\Omega(\log n)$ rounds for all gadgets to become inactive.
Consider time $t = \frac{\log n}{10} + 1$. For any gadget $i$, the probability $q_i(t)$ of remaining active at time $t$ is bounded by

\[
q_i(t) = \mathbb{P}(\cap_{k=1}^{t} a_i(k)) = \mathbb{P}(a_i(1)) \prod_{k=2}^{t} \mathbb{P}(a_i(k)|a_i(1) \cap a_i(2) \ldots \cap a_i(k-1))
\]

\[
= \mathbb{P}(a_i(1)) \prod_{k=2}^{t} \mathbb{P}(a_i(k)|a_i(k-1)) \geq \frac{1}{2^{l-1}} = \frac{1}{n^{1/10}}.
\]

Thus the expected number of active gadgets at time $t$, $\mathbb{E}(a(t)) = \sum_{i=1}^{n} q_i(t) \geq \frac{n^{9/10}}{n^{1/10}} = n^{9/10}$.

We will use the following form of the Chernoff Bound ([5], Sec. 1.6).

**Theorem 13.** Let $X = \sum_{i \in [n]} X_i$ where the $X_i$ are independently distributed in $[0, 1]$. Let $\mathbb{E}(X) \geq \mu_1$. Then, for all $0 < \epsilon < 1$, $\mathbb{P}(X < (1-\epsilon)\mu_1) \leq \exp\left(-\frac{\epsilon^2 \mu_1}{2}\right)$.

Plugging in $\epsilon = 0.5$, $\mu_1 = n^{9/10}$, and $X = a(t)$, we get,

\[
\mathbb{P}\left(a(t) < \frac{n^{9/10}}{2}\right) \leq \exp\left(-\frac{n^{9/10}}{8}\right) < 0.01,
\]

for large enough $n$. Now for contradiction, assume $\mathbb{E}(I(n)) \leq t/4$. Then, using the Markov inequality, $\mathbb{P}(I(n) \geq t/2) \leq \frac{1}{t}$ or $\mathbb{P}(I(n) < t/2) \geq \frac{1}{t}$. Now, at time $I(n)$, no gadgets are active. Thus if $I(n) < t/2 < t$, then $a(t) = 0$. Thus, $\mathbb{P}(I(n) < t/2) \geq \frac{1}{t}$ implies $\mathbb{P}(a(t) = 0) \geq \frac{1}{t}$. But this contradicts Eqn. [5]. Therefore, $\mathbb{E}(I(n)) \geq t/4 = \Omega(\log n)$, completing the proof of the Theorem. □

Note that bounding $\mathbb{E}(I(n))$ suffices to lower bound the expected time before all links successfully transmit, since a link cannot succeed as long as the corresponding gadget is active, by definition.

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verify that analysis is required. The tightness of the second bound follows from Section 4, since it is easy to verify that both these bounds are tight, thus going through $O(\ell) = \Omega(\ell)$. For the first bound, we show a construction for which the bound is tight.

First we construct a feasible set $L = \{\ell_0, \ell_1, \ell_2 \ldots \ell_n\}$ of size $n + 1$ such that $a_0(L) = \Omega(\log n)$. We give the construction in a tree metric (that can realized on the plane as well).

Let the length of $\ell_i$ be $\ell_i = (i + 1)^{1/\alpha}$. Assume $d_{0i} = c(i + 1)^{\frac{2}{\alpha}}$ for all $i > 0$, where $c$ is a large constant (to be chosen later). Assume that all other distances are defined by transitivity. We claim that $L$ is feasible, and $a_0(L) = \Omega(\log n)$. We will assume that links use uniform power.

Now, $a_0(L) = \sum_{i > 0} a_0(i) = \sum_{1 \leq i \leq n} \frac{(i + 1)^{1/\alpha}}{c(i + 1)^{\frac{1}{\alpha}}} = 1/c^\alpha \sum_{i \leq n} \frac{1}{i} = \Omega(\log n)$, since the term $\sum_{i \leq n} \frac{1}{i}$ is essentially the harmonic series. We now show that $a_L(\ell_i) \leq 1$ for all $\ell_i$ (which proves that $L$ is feasible). For any $\ell_i$, we write $a_L(\ell_i) = \sum_{k < i} a_k(i) + \sum_{k > i} a_k(i)$. The first term can be bounded by \[ \sum_{k < i} a_k(i) \leq \sum_{k < i} \left( \frac{(i + 1)^{1/\alpha}}{c(i + 1)^{\frac{1}{\alpha}}} \right)^\alpha \leq \frac{1}{c^\alpha} \frac{1}{i + 1} \leq \frac{1}{c^\alpha}, \]

and the second term by \[ \sum_{k > i} a_k(i) = \sum_{k > i} \left( \frac{(i + 1)^{1/\alpha}}{c(k + 1)^{\frac{1}{\alpha}}} \right)^\alpha \leq \frac{1}{c^\alpha} (i + 1) \sum_{k > i} \frac{1}{(k + 1)^2} = \frac{1}{c^\alpha} (i + 1) O(1/i) = \frac{1}{c^\alpha} O(1). \]

The expression $\sum_{k > i} \frac{1}{(k + 1)^2}$ is the tail of the $p$-series with $p = 2$ and can be seen to be $O(1/i)$. Thus $a_L(\ell_i) \leq \frac{1}{c^\alpha} (1 + O(1)) \leq 1$ by choosing a large enough $c$.

Now we place near $L_0$ a very small copy of $L$ (call it $L'$). Making the lengths in $L'$ small enough, we have for any $\ell' \in L'$ and any $\ell_i \in L$ ($i > 0$), $d_{\ell' \ell_i} = O(d_{0i})$. Since we use uniform power, for all $\ell' \in L'$, $a_L'(\ell') = \Omega(\log n)$ and $\hat{A} \geq \frac{1}{2n+2} a_{L \cup L'}(L \cup L') \geq \frac{1}{2n+2} a_L'(L) = \frac{1}{2n+2} n \times \Omega(\log n) = \Omega(\log n)$. On the other hand, the set $L \cup L'$ clearly has a scheduling number of 2.

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