Bregman Parallel Direction Method of Multipliers for Distributed Optimization via Mirror Averaging

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Abstract—Distributed optimization aims to optimize a global objective formed by a sum of coupled local convex functions over a graph via only local computation and communication. In this paper, we propose the Bregman parallel direction method of multipliers (PDMM) based on a generalized averaging step named mirror averaging. We establish the global convergence and $O(1/T)$ convergence rate of the Bregman PDMM, along with its $O(n/\ln n)$ improvement over existing PDMM, where $T$ denotes the number of iterations and $n$ the dimension of solution variable. In addition, we can enhance its performance by optimizing the spectral gap of the averaging matrix. We demonstrate our results via a numerical example.

I. INTRODUCTION

Distributed optimization arises in a variety of applications such as distributed tracking and localization [1], estimation in sensor networks [2], and multiagent coordination [3]. In particular, given an undirected connected graph with $m$ vertices, distributed optimization over this graph is defined as

$$\min_{u \in \mathcal{X}} \sum_{i=1}^{m} f_i(u) \quad (1)$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ is a closed convex set, each $f_i$ is a convex function locally known by vertex $i$ only. The optimality is achieved by local optimization on each vertex and efficient communication between neighboring vertices in the graph.

Alternating direction method of multipliers (ADMM) [4] is a primal-dual algorithm that alternatively optimizes a quadratic augmented Lagrangian with respect to splitted primal variables and dual variables. There has been an increasing interest in applying multi-block variants of ADMM to solve problem (1) [5], [6], [7]. One of the main challenges of such methods is to find a separable approximation to the coupled quadratic penalty term in augmented Lagrangian. In particular, a Gauss-Seidel approximation [5], [7] was proposed in [8], which results in sequential updates on the vertices. On the other hand, a Jacobian approximation based variant of ADMM [9], [6] allows simultaneous updates [10], [11]. We call such methods parallel direction method of multipliers (PDMM) since their primal variables are updated parallelly instead of alternatively.

Bregman ADMM [12] is a generalization of ADMM where the quadratic penalty function in ADMM updates is replaced by Bregman divergence, which can potentially exploit the problem structure. There has been attempts to introduce Bregman divergence as proximal term in multi-block variants of ADMM [10], [13], but they are still based on a quadratic augmented Lagrangian. To our best knowledge, all existing ADMM based methods for distributed optimization use quadratic penalty functions.

In this paper, we propose a new solution method, namely Bregman PDMM, for distributed optimization, which combines the advantages of PDMM and Bregman ADMM. We first propose a generalized averaging step named mirror averaging. Based on this, we develop Bregman PDMM which replaces all the quadratic penalty function in PDMM updates with Bregman divergence. We establish the global convergence of the proposed algorithm and its $O(1/T)$ convergence rate, where $T$ is the number of iterations. Furthermore, in some cases, Bregman PDMM can outperform PDMM by a factor of $O(n/\ln n)$, where $n$ is the dimension of solution variable. Finally, we show that by optimizing the spectral gap of the averaging matrix, we can enhance the performance of the Bregman PDMM.

The rest of the paper is organized as follows. §II provides a reformulation of problem (1) using consensus constraints. In §III we develop Bregman PDMM for problem (1), whose convergence properties are established in §IV via Lyapunov analysis. §V presents numerical examples; §VI concludes the paper and comments on the future work.

II. PRELIMINARIES AND BACKGROUND

A. Notation

Let $\mathbb{R}$ (\(\mathbb{R}_+\)) denote the (nonnegative) real numbers, \(\mathbb{R}^n (\mathbb{R}_+^n)\) denote the set of $n$-dimensional (elementwise nonnegative) vectors. Let $\geq (\leq)$ denote elementwise inequality when applied to vectors and matrices. Let $\langle \cdot, \cdot \rangle$ denote the dot product. Let $I_n \in \mathbb{R}^{n \times n}$ denote...
the $n$-dimensional identity matrix, $1_n \in \mathbb{R}^n$ the $n$-dimensional vector of all 1s. Given matrix $A \in \mathbb{R}^{n \times n}$, let $A_{ij}$ denote its $(i, j)$ entry; $A^T$ denotes its transpose. Let $\otimes$ denote the Kronecker product. Given the set $\mathcal{X} \subseteq \mathbb{R}^n$, its indicator function $1_{\mathcal{X}} : \mathbb{R}^n \to \mathbb{R}$ is defined as: $1_{\mathcal{X}}(u) = 0$ if $u \in \mathcal{X}$ and $+\infty$ otherwise.

**B. Subgradients**

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then $g \in \mathbb{R}^n$ is a subgradient of $f$ at $u \in \mathbb{R}^n$ if and only if for any $v \in \mathbb{R}^n$ one has

$$f(v) - f(u) \geq \langle g, v - u \rangle.$$  
(2)

We denote $\partial f(u)$ the set of subgradients of $f$ at $u$. We will use the following results.

**Lemma 1.** [14, Theorem 27.4] Given a closed convex set $C \subseteq \mathbb{R}^n$ and closed, convex, proper function $f : \mathbb{R}^n \to \mathbb{R}$, then $u^* = \arg\min_u f(u)$ if and only if there exists $g \in N_C(u^*)$ such that $-g \notin \partial f(u^*)$, where

$$N_C(u^*) := \{ g \in \mathbb{R}^n : \langle g, u^* - v \rangle \geq 0, \forall v \in C \}$$  
(3)

is the normal cone of the set $C$ at $u^*$.

**C. Mirror maps and Bregman divergence**

Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a convex open set. We say that $\phi : \mathcal{D} \to \mathbb{R}$ is a mirror map [15, p.298] if it satisfies: 1) $\phi$ is differentiable and strictly convex, 2) $\nabla \phi$ takes all possible values, and 3) $\nabla \phi$ diverges on the boundary of the closure of $\mathcal{D}$, i.e., $\lim_{u \to \partial \mathcal{D}} \|\nabla \phi(u)\| = \infty$, where $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^n$. Bregman divergence $B_\phi : \mathcal{D} \times \mathcal{D} \to \mathbb{R}_+$ induced by $\phi$ is defined as

$$B_\phi(u, v) = \phi(u) - \phi(v) - \langle \nabla \phi(v), u - v \rangle.$$  
(4)

Note that $B_\phi(u, v) \geq 0$ and $B_\phi(u, u) = 0$ only if $u = v$. $\Phi$ and $B_\phi$ also satisfy the following three-point identity,

$$\langle \nabla \phi(u) - \nabla \phi(v), w - u \rangle = B_\phi(w, v) - B_\phi(w, u) - B_\phi(u, v).$$  
(5)

**D. Graphs and distributed optimization**

An undirected connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ contains a vertex set $\mathcal{V} = \{1, 2, \ldots, m\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ such that $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$ for all $i, j \in \mathcal{V}$. Denote $\mathcal{N}(i)$ the set of neighbors of node $i$, where $j \in \mathcal{N}(i)$ if $(i, j) \in \mathcal{E}$.

Consider a symmetric stochastic matrix $P \in \mathbb{R}^{m \times m}$ defined on the graph $\mathcal{G}$ such that $P_{ij} > 0$ implies that $j \in \mathcal{N}(i)$. Such a matrix $P$ can be constructed, for example, by the graph Laplacian [16, Proposition 3.18]. The eigenvalues of $P$ are real and will be ordered non-increasingly in their magnitude, denoted by $|\lambda_1(P)| \geq |\lambda_2(P)| \geq \cdots \geq |\lambda_m(P)|$. From [17, Theorem 8.4.4] we know that $\lambda_1(P) = 1$ is simple with eigenvectors spanned by $1_m$.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote the underlying graph over which the distributed optimization problem (1) is defined. A common approach to solve problem (1) is to create local copies of the design variable $\{x_1, x_2, \ldots, x_m\}$ and impose the consensus constraints: $x_i = x_j$ for all $(i, j) \in \mathcal{E}$ [18], [4]. Many different forms of consensus constraints have been proposed [8], [9], [19], [20]. In this paper, we consider consensus constraints of the form:

$$(P \otimes I_n)x = x,$$  
(6)

where $x = [x_1^\top, x_2^\top, \ldots, x_m^\top]^\top$, $P$ is a symmetric, stochastic and irreducible matrix defined on $\mathcal{G}$. We will focus on solving the following reformulation of (1),

$$\min_{x \in \mathcal{X}^m} \sum_{i \in \mathcal{V}} f_i(x_i)$$  
subject to $(P \otimes I_n)x = x,$  
(7)

where $\mathcal{X}^m$ is the Cartesian product of $m$ copies of $\mathcal{X}$.

**III. BREGMAN PARALLEL DIRECTION METHOD OF MULTIPLIERS**

In this section, we first introduce an existing PDMM that contains averaging as an implicit update. Then we generalize averaging to mirror averaging based on the idea of mirror descent, and finally propose our Bregman PDMM based on mirror averaging.

**A. Parallel Direction Method of Multipliers**

PDMM [11] solves (1) with $\mathcal{X} = \mathbb{R}^n$ with parallel and single loop primal updates, and links convergence behavior to graph topology [21], [22]. An adaption of PDMM to formulation (2) is given in Algorithm 1.

Naturally, one will try to generalize the quadratic penalty in Algorithm 1 to Bregman divergence the same way Bregman ADMM generalizes ADMM [12]. However, if we simply replace the quadratic penalty in Algorithm 1 with Bregman divergence induced by a strongly convex function $\phi$, it is challenging to prove its convergence for the following reasons. A crucial step in the proof provided in [11] is to apply the three point identity (5) to a convex function $\Psi : \mathbb{R}^{mn} \to \mathbb{R}$ that satisfies the following differential equation,

$$\nabla \Psi(u) = (P \otimes I_n)\nabla \Phi(u),$$  
(8)

where $\Phi(u) = \sum_{i \in \mathcal{V}} \phi(u_i)$ with $u = [u_1^\top, \ldots, u_m^\top]^\top \in \mathcal{X}^m$. However, it is highly non-trivial to solve (8) for a convex function $\Psi$ unless $\phi$ is quadratic function. Hence we cannot directly utilize the convergence proof in [11].
Algorithm 1 Existing PDMM [11]

Input: Parameter $\rho > 0$; initial point $x(0), \nu(0) \in \mathbb{R}^{mn}$.

for all $t = 0, 1, 2, \ldots$ do

each vertex $i$ updates $x_i$ in parallel
$$x_i^{(t+1)} = \text{argmin}_{x_i} f_i(x_i)$$
$$+ \langle x_i, \nu_i^{(t)} \rangle - \sum_{j \in N(i)} P_{ij} \nu_j^{(t)}$$
$$+ \frac{\rho}{2} \sum_{j \in N(i)} P_{ij} \|x_i - x_j^{(t)}\|^2_2$$

end for

Therefore, we need to take a closer look at the role of the quadratic term in (9).

Consider the following intermediate update,
$$y_i^{(t)} := \text{argmin}_{x_i} \sum_{j \in N(i)} P_{ij} \|x_i - x_j^{(t)}\|^2_2.$$ (11)

The behavior of (11) is characterized by the Markov chain defined by matrix $P$ [16, Proposition 3.21]. In the sequel, we will generalize the quadratic function in (11) to Bregman divergence; then we will introduce Bregman PDMM based on such a generalization.

B. Mirror Averaging

Consider the following update: for all $i \in V$,
$$y_i^{(t)} = \text{argmin}_{x_i \in \mathcal{X}} \sum_{j \in N(i)} P_{ij} B_\phi(x_i, x_j^{(t)}),$$ (12)
where $P$ is symmetric, stochastic and irreducible matrix defined on $G$, and $\phi$ is a mirror map defined on the open set $D$ such that $\mathcal{X}$ is included in the closure of $D$.

Let $\Phi(u) = \sum_{i \in V} \phi(u_i)$ with $u = [u_1, \ldots, u_n]^\top$. Using an argument similar to the one in [15, p. 301], one can obtain the following result.

Proposition 1. Update (12) is equivalent to

$$\nabla \Phi(z^{(t)}) = (P \otimes I_n) \nabla \Phi(x^{(t)}),$$ (13a)
$$y^{(t)} = \text{argmin}_{x \in \mathcal{X}^m} B_\phi(x, z^{(t)}).$$ (13b)

Since (13a) has the same dynamics as averaging step (11), inspired by the idea of mirror descent, we interpret (12) as mirror averaging: to achieve update (12), we first map $x^{(t)}$ to $\nabla \Phi(x^{(t)})$, next run an averaging step via (13a) and obtain $\nabla \Phi(z^{(t)})$, then apply $\left(\nabla \Phi\right)^{-1}$ to it and obtain $z^{(t)}$, and finally get $y^{(t)}$ via the projection (13b).

Remark 1. We provide two special cases [15, p 301] where (12) has a close form solution: 1) If $\mathcal{X} = \mathbb{R}^n$ and $\phi = \|\cdot\|^2$, then (13a) and (13b) reduces to (11). 2) If $\mathcal{X}$ denotes the probability simplex and $\phi$ the negative entropy function, then (13a) reduces to weighted geometric averaging and (13b) to a simple re-normalization.

Algorithm 2 Bregman PDMM

Input: Parameters: $\tau, \rho > 0$, $\delta_1, \ldots, \delta_m \geq 0$; initial point $x(0) \in \mathcal{X}^m \cap D^m$, $\nu(0) \in \mathbb{R}^{mn}$.

for all $t = 0, 1, 2, \ldots$ do

each vertex $i$ updates $y_i$ and $x_i$ in parallel
$$z_i^{(t)} = \text{argmin}_{y_i \in \mathcal{X}} \sum_{j \in N(i)} P_{ij} B_\phi(y_i, x_j^{(t)}).$$ (14a)
$$x_i^{(t+1)} = \text{argmin}_{x_i \in \mathcal{X}} f_i(x_i)$$
$$+ \langle x_i, y_i^{(t)} \rangle - \sum_{j \in N(i)} P_{ij} \nu_j^{(t)}$$
$$+ \rho B_\phi(x_i, y_i^{(t)}) + \delta_i B_{\phi_i}(x_i, x_i^{(t)}).$$ (14b)

end for

We introduce the following useful lemma, whose proof can be found in the Appendix.

Lemma 2. Given update (12), for any $u \in \mathcal{X}$,
$$\sum_{i \in V} B_\phi(u, x_i^{(t)}) - \sum_{i \in V} B_\phi(u, y_i^{(t)}) \geq \sum_{i, j \in V} P_{ij} B_\phi(y_i^{(t)}, x_j^{(t)}).$$ (16)

Remark 2. Lemma 2 turns out to be a key step in our convergence proof. Notice that without the generalization from (11) to (12), we can replace Lemma 2 with Jensen’s inequality for strongly convex function by assuming the Bregman divergence is strongly convex in the second argument, but such an assumption does not hold in general. Hence the generalization from (11) to (12) is necessary.
C. Bregman PDMM via Mirror Averaging

Based on the above observations, we propose Algorithm 2 by generalizing the quadratic penalty term in Algorithm 1 to Bregman divergence. It essentially combines the parallel updates in Algorithm 1 and Bregman penalty term in Bregman ADMM [12]. Notice that Algorithm 1 is a special case of Algorithm 2 with ϕ = \(\frac{1}{2}\|\cdot\|_2^2\), τ = ρ, and δ}_i = 0 for all \(i \in V\).

IV. CONVERGENCE

In this section, we establish the convergence analysis of Algorithm 2. All detailed proofs in this section can be found in the Appendix. We first define the Lagrangian of problem (7) as \(L(x, \nu) = \sum_{i \in V} L_i(x_i, \nu)\) where,

\[
L_i(x_i, \nu) := f_i(x_i) + i_\mathcal{X}(x_i) + \langle x_i, \nu - \sum_{j \in V} P_{ij} \nu_j \rangle, \quad (17)
\]

and \(\nu = [\nu_1^T, \ldots, \nu_m^T]^T\) denote the dual variables.

We group our assumptions in Assumption 1:

Assumption 1. (a) For all \(i \in V\), \(f_i : \mathbb{R} \to \mathbb{R}\{+\infty\}\) is closed, proper and convex.

(b) There exists a saddle point \((x^*, \nu^*)\) that satisfies the KKT conditions of the Lagrangian given in (17), for all \(i \in V\), there exists \(g_i \in \mathcal{N}_\mathcal{X}(x_i^*)\) such that

\[
P_{ij} x_j^* = x_i^* \quad (18a)
\]

\[
-\nu_i^* + \sum_{j \in V} P_{ij} \nu_j^* - g_i \in \partial f_i(x_i^*) \quad (18b)
\]

(c) Functions \(\varphi_1, \varphi_2, \ldots, \varphi_m : \mathcal{D} \to \mathbb{R}\) are strictly convex, where \(\mathcal{D}\) is an open convex set such that \(\mathcal{X}\) is included in its closure. Function \(\phi : \mathcal{D} \to \mathbb{R}\) is a mirror map and is \(\mu\)-strongly convex with respect to \(l_p\)-norm \(\|\cdot\|_p\) over \(\mathcal{X} \cap \mathcal{D}\), i.e., for any \(u, v \in \mathcal{X} \cap \mathcal{D}\),

\[
B_\phi(u, v) \geq \frac{\mu}{2} \|u - v\|_p^2. \quad (19)
\]

(d) The matrix \(P\) is symmetric, stochastic, irreducible and positive semi-definite.

Remark 3. An immediate implication of assumptions in entry (a) is that \(\lambda_2(P) < 1\) due to [17, Corollary 8.4.6].

Notice that we assume a homogeneous mirror map \(\phi\) is used by all vertices in Algorithm 2 but our results can be generalized to the cases of heterogeneous mirror maps as long as they all satisfy (19).

Now we start to construct the convergence proof of Algorithm 2 under Assumption 1. From the definition in (17) we know that the Lagrangian \(L(x, \nu)\) is separable in each \(x_i\); hence the KKT conditions (18b) can be obtained separately for each \(x_i\) using Lemma 1. Similarly one can have the optimality condition of (14b); there exists \(g_i \in \mathcal{N}_\mathcal{X}(x_i^{(t+1)})\) such that

\[
-\nu_i^{(t)} + \sum_{j \in V} P_{ij} \nu_j^{(t)} - \rho \left(\nabla \phi(x_i^{(t+1)}) - \nabla \phi(y_i^{(t)})\right)
\]

\[
-\delta_i \left(\nabla \varphi_i(x_i^{(t+1)}) - \nabla \varphi_i(x_i^{(t)})\right) - g_i \in \partial f_i(x_i^{(t+1)}) \quad (20)
\]

Our goal is to show that as \(t \to \infty\), \(\{x^{(t+1)}, \nu^{(t+1)}\}\) will satisfy (18a) and reduce conditions in (20) to those in (18b). Note that if \(x^{(t+1)} = (P \otimes I_n)x^{(t+1)}\), then \(\nu^{(t+1)} = \nu^{(t)}\). Therefore, KKT conditions (18) are satisfied by \(\{x^{(t+1)}, \nu^{(t+1)}\}\) if the following holds

\[
x^{(t+1)} = (P \otimes I_n)x^{(t+1)}, \quad x^{(t+1)} = x^{(t)} = y^{(t)}. \quad (21)
\]

Define the residuals of optimality conditions (21) at iteration \(t\) as

\[
R(t+1) := \frac{\gamma}{2} \left\|((I_m - P) \otimes I_n)x^{(t+1)}\right\|^2_2 + \sum_{i \in V} B_\phi(x_i^{(t+1)}, y_i^{(t)}) + \sum_{i \in V} \delta_i \frac{1}{\rho} B_{\varphi_i}(x_i^{(t+1)}, x_i^{(t)}), \quad (22)
\]

where \(\gamma > 0\). Notice that \(R(t+1) = 0\) if and only if (21) holds. Hence \(R(t+1)\) is a running distance to KKT conditions in (18). Define the Lyapunov function of Algorithm 2 which measures a running distance to optimal primal-dual pair \((x^*, \nu^*)\), as

\[
V(t) := \frac{1}{2 \tau \rho} \left\|\nu^* - \nu(t)\right\|^2_2 + \sum_{i \in V} B_\phi(x_i^*, y_i^{(t)}) + \sum_{i \in V} \delta_i B_{\varphi_i}(x_i^*, x_i^{(t)}). \quad (23)
\]

We first establish the global convergence of Algorithm 2 by showing that as \(t \to \infty\), \(V(t)\) is monotonically non-increasing and that \(R(t+1) \to 0\) (see [23] for detailed proof).

Theorem 1. Suppose that Assumption 1 holds. Let the sequence \(\{y^{(t)}, x^{(t)}, \nu^{(t)}\}\) be generated by Algorithm 2. Let \(R(k+1)\) and \(V(k)\) be defined as in (22) and (23), respectively. Set

\[
\tau \leq \rho(\mu \sigma - \gamma), \quad 0 < \gamma < \mu \sigma, \quad (24)
\]

where \(\sigma = \min\{1, n^{\frac{2}{n-1}}\}\). Then

\[
V(t) - V(t+1) \geq R(t+1), \quad (25)
\]

As \(t \to \infty\), \(R(t+1)\) converges to zero, and \(\{x^{(t)}, \nu^{(t)}\}\) converges to a point that satisfy KKT conditions (18).

The sketch of the proof is as follows. First apply inequality (2) at \(x^{(t+1)}\) and \(x^*\), which yields a non-negative inner product. Then use identity (5) to break
this inner product into three parts, each of which contributes to $V(t), V(t+1)$, and $R(t+1)$, respectively. Lemma 2 entry (c) and (d) in Assumption 1 together with parameter setting in (24) ensures that intermediate terms cancel each other, and finally we reach (25). Summing up (23) from $t = 0$ to $t = \infty$, we have $\sum_{t=0}^{\infty} R(t+1) = V(0) - V(\infty) \leq V(0)$. Therefore, as $t \to \infty$, we must have $R(t+1) \to 0$, which implies that $\{x(t), \nu(t)\}$ satisfy (18) in the limit.

In general, (24) implies that as $p$ increases, step size $\tau$ needs to decrease. See [12, Remark 1] for details.

The following theorem establishes the $O(1/T)$ convergence rate of Algorithm 2 in an ergodic sense via the Jensen’s inequality.

**Theorem 2.** Suppose that Assumption 1 holds. Let the sequence $\{y(t), x(t), \nu(t)\}$ be generated by Algorithm 2. Let $V(k)$ be defined as in (23), $\mu, \tau, \rho, \gamma$ satisfy (24), $\nu(0) = 0$ and $x(T) = \frac{1}{T} \sum_{t=1}^{T} x(t)$. Then

$$\sum_{i \in V} f_i(\overline{x}_i(T)) - \sum_{i \in V} f_i(x_i^*) \leq \frac{\rho}{1} \sum_{i \in V} B_\phi(x_i^*, y_i(0)) + \sum_{i \in V} \delta_i B_{\phi_i}(x_i^*, x_i(0)),$$

$$\sum_{i \in V} (I_m - P) \otimes I_n \overline{x}(T) \leq \frac{V(0)}{\gamma T}.$$  

(26a)

$$\frac{1}{2} \left\| (I_m - P) \otimes I_n \overline{x}(T) \right\|^2_2 \leq \frac{V(0)}{\gamma T}.$$  

(26b)

Theorem 2 shows that the complexity bound of Algorithm 2 with respect to dimensionality $n$ is determined by the Bregman divergence term. The following corollary gives an example where, with a properly chosen Bregman divergence, Algorithm 2 outperforms Algorithm 1 by a factor of $O(n/\ln n)$ [12, Remark 2].

**Corollary 1.** Suppose that assumption 1 holds. Suppose $\|g_i\|^2 \leq M_0$ for all $g_i \in \partial f_i(x_i^*)$ and $i \in V$, where $M_0 \in \mathbb{R}_+$. Let the sequence $\{y(t), x(t), \nu(t)\}$ be generated by Algorithm 2. Let $\gamma = 1/4$, $\tau = \rho/2$, $\delta_{\max} = \max_i \delta_i$, $\nu(0) = 0$, $x(0) = 1_m \otimes (\frac{1}{n} 1_n)$ and $\overline{x}(T) = \frac{1}{T} \sum_{t=1}^{T} x(t), \ X$ be the probability simplex, $\phi$ and $\phi_i$ be the negative entropy function, then

$$\sum_{i \in V} f_i(\overline{x}_i(T)) - \sum_{i \in V} f_i(x_i^*) \leq \frac{m(\rho + \delta_{\max}) \ln n}{T},$$  

(27a)

$$\frac{1}{2} \left\| (I_m - P) \otimes I_n \overline{x}(T) \right\|^2_2 \leq \frac{4m M_0}{\rho^2 (1 - \lambda_2(P))^2 T} + \frac{4m(m(\rho + \delta_{\max}) \ln n)}{\rho T}. $$  

(27b)

Observe that (27b) implies that the convergence bounds on consensus residual can be tightened by designing $\lambda_2(P)$, which can be achieved efficiently via convex optimization [24].

**V. NUMERICAL EXAMPLES**

In this section, we present numerical examples to demonstrate the performance of Algorithm 2. Consider the following special case of (1) defined over graph $G = (V, E)$:

$$\min_{u \in X} \sum_{i=1}^{m} \langle c_i, u \rangle,$$

(28)

where $X$ is the probability simplex. Such problems have potential applications in, for example, policy design in multi-agent decision making [25], [26].

We use Algorithm 1 as benchmark since it includes other popular variants of distributed ADMM [19], [27], [20], [28] as special cases. Compared to Algorithm 1 which needs efficient Euclidean projection onto probability simplex [29], Algorithm 2 can solve (28) with closed-form updates suited for massive parallelism [12].

We compare the performance of Algorithm 2 with Algorithm 1 on problem (28), where entries in $\{c_1, \ldots, c_m\}$ are sampled from standard normal distribution, graph $G$ is randomly generated with edge probability 0.2 [16, p. 90]. We use the following parameter setting: $\rho = 1$, $\tau = 1/2$, $\delta_i = 0$ for all $i \in V$, $\phi$ is the negative entropy function. We demonstrate the convergence described by (27) in Figure 1. We observe that Algorithm 2 significantly outperforms Algorithm 1 especially for large scale problem, and optimizing $\lambda_2(P)$ further accelerates convergence considerably.

**VI. CONCLUSIONS**

In order to solve distributed optimization over a graph, we generalize PDMM to Bregman PDMM based on mirror averaging. The global convergence and iteration complexity of Bregman PDMM are established, along with its improvement over PDMM. We can further enhance its performance by designing the averaging matrix. Future work directions include the variants of the proposed algorithm for asynchronous and stochastic updates, time-varying graphs, and applications in multi-agent decision making.

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APPENDIX

We will use the following Lemma.

Lemma 3. If $P \in \mathbb{R}^{m \times m}$ is symmetric, stochastic and positive semi-definite, then

$$
\|((I_m - P) \otimes I_n) u\|_2^2 \leq \frac{1}{\sigma} \sum_{i,j \in \mathcal{V}} P_{ij} \|v_i - u_j\|_p^2
$$

where $\|\cdot\|_p$ denote $p$ norm and $\sigma = \min\{1, n^\frac{1}{p} - 1\}$, $u = [u_1^T, \ldots, u_m^T]^T$, $v = [v_1^T, \ldots, v_m^T]^T \in \mathcal{X}^m$, $\mathcal{X}$ is a closed convex set.

Proof. First, observe that if $P$ is symmetric, stochastic, irreducible and positive semi-definite, $P - P^2$ is positive semi-definite [17, Theorem 8.4.4]. Since $P1_m = P^T 1_m = 1_m$, we can show the following

$$
\sum_{i,j \in \mathcal{V}} P_{ij} \sum_{k \in \mathcal{V}} P_{ik} u_k - u_j \right\|_2^2  \\
\geq \|u\|_2^2 \|P \otimes I_n\|_2 \right\|_2^2  \\
\geq \|u\|_2^2 \|P \otimes I_n\|_2 \right\|_2^2  \\
= \|((I_m - P) \otimes I_n) u\|_2^2
$$

Hence (29) holds due to the fact that

$$
\sum_{k \in \mathcal{V}} P_{ik} u_k = \operatorname{argmin}_{w \in \mathcal{X}} \sum_{j \in \mathcal{V}} P_{ij} \|w - u_j\|_2^2,
$$

for all $i \in \mathcal{V}$, and that $\|w\|_2^2 \leq 1/\sigma \|w\|_p^2$ for all $w \in \mathbb{R}^n$ where $\sigma = \min\{1, n^\frac{1}{p} - 1\}$ [12, Theorem 1]. \hfill \square

A. Lemma 2

Proof. Since $P_{ij} = 0$ if $j \notin \mathcal{N}(i)$, the optimality condition for (27) can be written as follows: for any $u \in \mathcal{X}$, we have

$$
\sum_{j \in \mathcal{V}} P_{ij} \langle \nabla \phi(y^{(t)}) - \nabla \phi(x^{(t)}), u - y^{(t)} \rangle \geq 0
$$

Using three point property (5), we have

$$
\sum_{j \in \mathcal{V}} P_{ij} \beta_\phi(u, x^{(t)}_j) - \sum_{j \in \mathcal{V}} P_{ij} \beta_\phi(u, y^{(t)}_i) \\
\geq \sum_{j \in \mathcal{V}} P_{ij} \beta_\phi(y^{(t)}_i, x^{(t)}_j)
$$

(30)

Summing (30) over all $i$ completes the proof. \hfill \square

B. Theorem 1

Proof. Since $P$ is irreducible, $x^*$ satisfy (18a) if and only if there exists $x^* \in \mathcal{X}$ such that $x^* = 1_m \otimes x^*$.

Substitute (20) into (2) we have: there exists $g_i \in N_\chi(x^{(t+1)}_i)$ for all $i$ such that

$$
\sum_{i \in \mathcal{V}} f_i(x^{(t+1)}_i) - f_i(x^*) \\
\leq \langle -\nu^{(t)}, ((I_m - P) \otimes I_n)x^{(t+1)} \rangle  \\
+ \rho \sum_{i \in \mathcal{V}} \langle \nabla \phi(x^{(t+1)}_i) - \nabla \phi(y^{(t)}_i), x^* - x^{(t+1)}_i \rangle  \\
+ \sum_{i \in \mathcal{V}} \delta_i \langle \nabla \phi_i(x^{(t+1)}_i) - \nabla \phi_i(x^{(t)}_i), x^* - x^{(t+1)}_i \rangle  \\
- \sum_{i \in \mathcal{V}} \langle g_i, x^{(t+1)}_i - x^* \rangle
$$

(31)
Similarly, substitute \((18b)\) into \((2)\) we have
\[
\sum_{i \in V} f_i(x^*) - \sum_{i \in V} f_i(x_i^{(t+1)}) \leq \langle \nu^*, ((I_m - P) \otimes I_n)x^{(t+1)} \rangle 
\]
we obtain
\[
\sum_{i \in V} f_i(x_i^{(t+1)}) - \sum_{i \in V} f_i(x^*) \leq \frac{1}{2\tau} \left| \left| \nu^{(t)} \right| \right|^2 - \frac{1}{2\tau} \left| \left| \nu^{(t+1)} \right| \right|^2 + \frac{\rho}{2} \sum_{i, j \in V} P_{ij} \left| \left| y_i^{(t+1)} - x_j^{(t+1)} \right| \right|^2_p + \frac{\tau/\rho + \gamma}{2} \sum_{i \in V} \left| \left| ((I_m - P) \otimes I_n)x_i^{(t+1)} \right| \right|^2_2
\]