Strong monogamy of multi-party quantum entanglement for partially coherently superposed states

Jeong San Kim

Department of Applied Mathematics and Institute of Natural Sciences, Kyung Hee University, Yongin-si, Gyeonggi-do 446-701, Korea
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We provide an evidence for the validity of strong monogamy inequality of multi-party quantum entanglement using square of convex-roof extended negativity (SCREN). We first consider a large class of multi-qudit mixed state that are in a partially coherent superposition of a generalized W-class state and the vacuum, and provide some useful properties about this class of states. We show that monogamy inequality of multi-qudit entanglement in terms of SCREN holds for this class of states. We further show that SCREN strong monogamy inequality of multi-qudit entanglement also holds for this class of states. Thus SCREN is a good alternative to characterize the monogamous and strongly monogamous properties of multi-qudit entanglement.

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I. INTRODUCTION

Whereas classical correlation can be freely shared in multi-party systems, quantum entanglement is known to have restriction in its shareability. This restricted shareability of entanglement in multi-party quantum systems is known as monogamy of entanglement (MOE) \[1, 2\].

Mathematically, MOE was first characterized as an inequality in three-qubit systems by Coffman-Kundu-Wootters (CKW) \[3\]. Using tangle as bipartite entanglement quantification, CKW inequality shows the mutually exclusive nature of two-qubit entanglement shared in three-qubit systems. Later, three-qubit CKW inequality was generalized for arbitrary multi-qubit systems \[4\] as well as some cases of higher-dimensional quantum systems \[5–8\]. A general monogamy inequality of arbitrary quantum systems was established in terms of the squashed entanglement \[9, 10\].

In three-qubit systems, the residual entanglement from the difference between left and right-hand sides of CKW inequality is interpreted as the genuine three-qubit entanglement, which is referred to as three-tangle \[11\]. Later, the definition of three-tangle was generalized into multiqubit systems, namely n-tangle, and the concept of strong monogamy (SM) inequality of multi-qudit entanglement was proposed by conjecturing the nonnegativity of the n-tangle \[12\].

To support multi-qubit SM inequality, an extensive numerical evidence was presented for four qubit systems as well as an analytical proof for some cases of multi-qubit systems \[12, 13\]. However, it is known that tangle fails in the generalization of CKW inequality for higher dimensional quantum systems due to the existence of counterexamples \[14, 15\]. Because multi-qubit SM inequality in terms of tangle is reduced to the CKW inequality in three-party quantum systems, the existence of counterexamples of CKW inequality also implies the violation of SM monogamy inequality based on tangle in higher-dimensional quantum systems more that qubits.

Recently, square of convex-roof extended negativity (SCREN) was proposed to characterize the strongly monogamous property of multi-party quantum entanglement even in higher-dimensional quantum systems \[16\]. Besides its coincidence with tangle in qubit systems, which can rephrase the multi-qubit SM inequality in terms of SCREN, SCREN SM inequality was shown to be true for the counterexamples of tangle in higher-dimensional systems. It was also analytically shown that SCREN SM inequality is true for a large class of multi-qudit generalized W-class states \[16\]. Thus, SCREN is a good alternative for strong monogamy of multi-party quantum entanglement even in higher-dimensional systems.

Here we provide another evidence for the validity of SCREN SM inequality of multi-qudit entanglement. We first consider a large class of multi-qudit mixed state that are in a partially coherent superposition of a generalized W-class state and the vacuum. After providing some useful properties about the structure of partially coherently superposed states, we show that CKW-type monogamy inequality holds for this class of states in terms of SCREN. We further show that SCREN SM inequality is true for this class of states, which is, we believe, the first result where strong monogamy inequality is studied for multi-qudit mixed states.

The paper is organized as follows. In Sec. II we review the definition of tangle and SCREN, and their relation in monogamy inequality of multi-party quantum entanglement. In Sec. III we recall the multi-qudit SM inequality in terms of tangle, as well as the multi-qudit SCREN SM inequality. In Sec. IV we provide the definition of partially coherent superposition of multi-qudit generalized W-class states and vacuum as well as some useful properties about this class of states. In Sec. V we show that CKW-type monogamy inequality in terms of
SCREN is saturated by partially coherently superposed states. In Sec. VII we show that the SCREN SM inequality of multi-qudit entanglement is saturated by partially coherently superposed states, and we summarize our results in Sec. VII.

II. MONOGAMY OF MULTI-PARTY QUANTUM ENTANGLEMENT

For a two-qubit pure state $|\psi\rangle_{AB}$, its tangle (or one-tangle) is defined as

$$\tau\left(|\psi\rangle_{AB}\right) = 4 \text{det} \rho_A,$$  \hspace{1cm} (1)

with the reduced density matrix $\rho_A = \text{tr}_B|\psi\rangle_{AB}\langle\psi|$. For a two-qubit mixed state $\rho_{AB}$, its tangle (or two-tangle) is defined as

$$\tau\left(\rho_{A|B}\right) = \left[\min_{\{p_h,|\psi_h\rangle\}} \sum_h p_h \sqrt{\text{tr}\left(|\psi_h\rangle_{A|B}\right)}\right]^2,$$  \hspace{1cm} (2)

where the minimization is taken over all possible pure state decompositions

$$\rho_{AB} = \sum_h p_h |\psi_h\rangle_{AB}\langle\psi_h|.$$  \hspace{1cm} (3)

By using one and two-tangles, three-qubit CKW inequality was proposed as

$$\tau\left(|\psi\rangle_{ABC}\right) \geq \tau\left(\rho_{A|B}\right) + \tau\left(\rho_{A|C}\right),$$  \hspace{1cm} (4)

for any three-qubit pure state $|\psi\rangle_{ABC}$ with two-qubit reduced density matrices $\rho_{AB} = \text{tr}_C|\psi\rangle_{ABC}\langle\psi|$ and $\rho_{AC} = \text{tr}_B|\psi\rangle_{ABC}\langle\psi|$. Later CKW inequality in (4) was generalized into multi-qubit systems [13] as

$$\tau\left(|\psi\rangle_{A_1A_2\cdots A_n}\right) \geq \sum_{j=2}^{n} \tau\left(\rho_{A_1|A_j}\right),$$  \hspace{1cm} (5)

for any $n$-qubit state $|\psi\rangle_{A_1A_2\cdots A_n}$ and its two-qubit reduced density matrices $\rho_{A_1A_j}$ on subsystems $A_1A_j$ for each $j = 2, \cdots, n$.

Although tangle is a good bipartite entanglement measure, which also shows the monogamy inequality of multi-qubit entanglement, there exist quantum states in higher dimensions violating CKW inequality, in $3 \otimes 3 \otimes 3$ and even in $3 \otimes 2 \otimes 2$ quantum systems [14, 15]. Thus, tangle itself fails in its generalization of CKW inequality for higher dimensional quantum systems more than qubits.

Another generalization of two-qubit tangle into higher-dimensional quantum systems is using negativity. For any bipartite pure state $|\phi\rangle_{AB}$, its negativity is defined as

$$\mathcal{N}(|\phi\rangle_{A|B}) = \left\| |\phi\rangle_{AB}\langle\phi|^{T_B}\right\|_1 - 1,$$  \hspace{1cm} (6)

where $|\phi\rangle_{AB}\langle\phi|^{T_B}$ is the partial transposition of $|\phi\rangle_{AB}$ and $\|\cdot\|_1$ is the trace norm [18–20].

Here, we note that for any two-qubit pure state $|\psi\rangle_{AB}$ with a Schmidt decomposition

$$|\psi\rangle_{AB} = \sqrt{\lambda_1}|e_0\rangle_A \otimes |f_0\rangle_B + \sqrt{\lambda_2}|e_1\rangle_A \otimes |f_1\rangle_B,$$  \hspace{1cm} (7)

the square of negativity coincides with the tangle,

$$\mathcal{N}^2\left(|\psi\rangle_{A|B}\right) = 4\lambda_1\lambda_2 = \tau\left(|\psi\rangle_{A|B}\right).$$  \hspace{1cm} (8)

Thus the two-tangle of any two-qubit state $\rho_{AB}$ in Eq. (2) can be rephrased as

$$\tau\left(\rho_{A|B}\right) = \left[\min_{\{p_h,|\psi_h\rangle\}} \sum_h p_h \mathcal{N}\left(|\psi_h\rangle_{A|B}\right)\right]^2.$$  \hspace{1cm} (9)

Based on this relation, another generalization of two-qubit tangle into higher-dimensional quantum systems was proposed as

$$\mathcal{N}_{sc}(\rho_{A|B}) = \left[\min_{\{p_h,|\psi_h\rangle\}} \sum_h p_h \mathcal{N}\left(|\psi_h\rangle_{A|B}\right)\right]^2,$$  \hspace{1cm} (10)

for any bipartite mixed state $\rho_{AB}$ where the minimization is over all pure-state decompositions of $\rho_{AB}$. The quantity in Eq. (10) is referred to as square of convex-roof extended negativity(SCREN), which is a faithful bipartite entanglement measure [16, 21–23].

Consequently, the multi-qubit CKW inequality in (4) can be rephrased in terms of SCRENs as,

$$\mathcal{N}_{sc}\left(|\psi\rangle_{A_1A_2\cdots A_n}\right) \geq \sum_{j=2}^{n} \mathcal{N}_{sc}\left(\rho_{A_1|A_j}\right).$$  \hspace{1cm} (11)

Moreover, Inequality (11) is still true for the counterexamples violating CKW inequality in terms of tangle [16]. Thus SCREN is a good alternative for monogamy inequality of multi-qubit entanglement without any known counterexamples even in higher-dimensional quantum systems so far.

III. STRONG MONOGAMY OF MULTI-PARTY QUANTUM ENTANGLEMENT

In three-qubit systems, the residual entanglement from the difference between left and right-hand sides of CKW Inequality (5) is referred to as three-tangle,

$$\tau\left(|\psi\rangle_{A|B|C}\right) = \tau\left(|\psi\rangle_{ABC}\right) - \tau\left(\rho_{A|B}\right) - \tau\left(\rho_{A|C}\right),$$  \hspace{1cm} (12)

which is invariant under the permutation of subsystems [16]. This definition of three-tangle was recently generalized for multi-qubit systems [12]; $n$-tangle of an
n-qubit pure state $|\psi\rangle_{A_1A_2...A_n}$ is defined as

$$\tau\left(|\psi\rangle_{A_1A_2...A_n}\right) = \tau\left(|\psi\rangle_{A_1A_2...A_n}\right) - \sum_{m=2}^{n-1} \sum_{j_m} \tau\left(\rho_{A_1|A_j|...|A_{j_m}}\right)^{m/2},$$

(13)

where the index vector $j_m = (j_1^m, ..., j_m^{m-1})$ spans all the ordered subsets of the index set $\{2, ..., n\}$ with $(m-1)$ distinct elements. For each $2 \leq m \leq n-1$, the $m$-tangle of the $m$-qubit reduced density matrix $\rho_{A_1A_j...A_{j_m}}$ is defined as

$$\tau\left(\rho_{A_1|A_j|...|A_{j_m}}\right) = \left[\min_{\{|\psi_h\rangle\}} \sum_r \rho_h|\psi_h\rangle_A|A_j|...|A_{j_m}\rangle \langle |\psi_h\rangle_A|A_j|...|A_{j_m}\rangle\right]^{2},$$

(14)

with the minimization over all possible pure state decompositions

$$\rho_{A_1A_j...A_{j_m}} = \sum_r \rho_h|\psi_h\rangle_A|A_j|...|A_{j_m}\rangle \langle |\psi_h\rangle_A|A_j|...|A_{j_m}\rangle.$$

(15)

For $n = 3$, the definition of $n$-tangle in Eq. (13) reduces to that of three-tangle in Eq. (12). $n$-tangle also reduces to the two-tangle of two-qubit state $\rho_{A_1A_2}$ in Eq. (2) for $n = 2$.

By conjecturing nonnegativity of $n$-tangle, a strong monogamy (SM) inequality of multi-qubit entanglement was proposed as

$$\tau\left(|\psi\rangle_{A_1|A_2|...|A_n}\right) = \sum_{m=2}^{n-1} \sum_{j_m} \tau\left(\rho_{A_1|A_j|...|A_{j_m}}\right)^{m/2}$$

(16)

Inequality (16) encapsulates three-qubit CKW inequality in (14) for $n = 3$, therefore, it is another generalization of three-qubit monogamy inequality into multi-qubit systems in a stronger form (2).

Whereas SM inequality in (16) proposes a stronger monogamous property of multi-qubit entanglement with extensive numerical evidences as well as an analytic proof for some class of of multi-qubit states (16). Inequality (16) is no longer valid for higher-dimensional quantum systems more than qubits due to the existence of counterexamples (14, 15). In other words, $n$-tangle fails in its generalization for SM inequality in higher-dimensional systems.

Based on the coincidence of tangle and SCREEN in two-qubit systems, another generalization of multi-qubit SM inequality into higher-dimensional quantum systems was recently proposed (16); for an $n$-qudit pure state $|\psi\rangle_{A_1A_2...A_n}$, its $n$-SCREEN is defined as

$$\mathcal{N}_{sc}\left(|\psi\rangle_{A_1|A_2|...|A_n}\right) = \mathcal{N}_{sc}\left(|\psi\rangle_{A_1|A_2|...|A_n}\right) - \sum_{m=2}^{n-1} \sum_{j_m} \mathcal{N}_{sc}\left(\rho_{A_1|A_j|...|A_{j_m}}\right)^{m/2},$$

(17)

Moreover, SCREEN SM inequality of multi-party entanglement is then proposed as

$$\mathcal{N}_{sc}\left(|\psi\rangle_{A_1|A_2|...|A_n}\right) \geq \sum_{m=2}^{n-1} \sum_{j_m} \mathcal{N}_{sc}\left(\rho_{A_1|A_j|...|A_{j_m}}\right)^{m/2},$$

(18)

by conjecturing the nonnegativity of $n$-SCREEN in Eq. (17).

Due to the coincidence of tangle and SCREEN in qubit systems, SCREEN SM Inequality in (16) is reduced to tangle-based SM Inequality in (10) for any multi-qubit states. Thus Inequality (18) is valid for the classes of multi-qubit quantum states considered in (12, 13). Moreover, it was recently shown that SCREEN SM inequality is still true for a large class of multi-qudit generalized W-class states as well as the counterexamples of CKW inequality in higher-dimensional quantum systems (16). Thus SCREEN is a good alternative of tangle in characterizing strongly monogamous property of multi-party quantum entanglement.

IV. PARTIALLY COHERENT SUPERPOSITION OF MULTI-QUDIT GENERALIZED W-CLASS STATES AND VACUUM

Let us first recall the definition of multi-qudit generalized W-class state (13); an $n$-qudit generalized W-class state is defined as

$$|W^d_n\rangle_{A_1...A_n} = \sum_{j=1}^{d-1} (a_{ij}|j0\cdots0\rangle + a_{2j}|0j\cdots0\rangle + \cdots + a_{nj}|00\cdots0j\rangle,$$

(19)

for some orthonormal basis $\{|j\rangle^i\}_{j=0}^{d-1}$ of qudit subsystems $A_i$ with $i = 1, \ldots, n$, and the normalization condition $\sum_{i=1}^{n} \sum_{j=0}^{d-1} |a_{ij}|^2 = 1$ (23).

A partially coherent superposition (PCS) of an $n$-qudit generalized W-class state and the vacuum $|0\rangle^\otimes n$ is a two-parameter class of $n$-qudit states,

$$\rho^{(p, \lambda)}_{A_1...A_n} = p|W^d_n\rangle\langle W^d_n| + (1-p)|0\rangle^\otimes n\langle 0|^\otimes n + \lambda \sqrt{p(1-p)} \left(|W^d_n\rangle\langle 0|^\otimes n + |0\rangle^\otimes n\langle W^d_n|\right),$$

(20)
where \(0 \leq p, \lambda \leq 1\) \cite{6}. Here \(\lambda\) is the degree of coherency; \(\rho^{(p, \lambda)}_{A_1 \cdots A_n}\) in Eq. (21) becomes a coherent superposition of a generalized W-class state and vacuum,

\[
|\psi\rangle_{A_1 \cdots A_n} = \sqrt{p} |W^d_{n}| + \sqrt{1-p} |0\rangle^\otimes n, \tag{21}
\]

for the case that \(\lambda = 1\), and it is an incoherent superposition, or a mixture

\[
\rho_{A_1 \cdots A_n} = p |W^d_{n}\langle W^d_{n}| + (1-p) |0\rangle^\otimes n |0\rangle^\otimes n \tag{22}
\]

when \(\lambda = 0\). For the intermediate value of \(\lambda\) between 0 and 1, the superposition coherency of \(\rho^{(p, \lambda)}_{A_1 \cdots A_n}\) is partial, and thus partially coherent superposition.

The PCS state in Eq. (20) can also be interpreted by means of decoherence; \(\rho^{(p, \lambda)}_{A_1 \cdots A_n}\) is the resulting state from a coherent superposition of a generalized W-class state and \(|0\rangle^\otimes n|0\rangle^\otimes n\) after the decoherence process so-called phase damping \cite{20}, which can be represented as

\[
\rho_{A_1 \cdots A_n} = \Lambda (|\psi\rangle \langle \psi|)
= E_0 |\psi\rangle \langle \psi| E_0^\dagger + E_1 |\psi\rangle \langle \psi| E_1^\dagger + E_2 |\psi\rangle \langle \psi| E_2^\dagger, \tag{23}
\]

with Kraus operators

\[
E_0 = \sqrt{I}, E_1 = \sqrt{1-I} |0\rangle \langle 0|, \quad E_2 = \sqrt{1-I} |0\rangle \langle 0| \quad \text{and} \quad I = |0\rangle \langle 0| + |1\rangle \langle 1|.
\]

Thus the PCS state in Eq. (20) naturally arises by the effect of decoherence.

Before we further investigate the monogamous property of PCS states in Eq. (20), we provide some useful properties of the generalized W-class states as well as PCS states.

**Theorem 1.** The \(n\)-qudit generalized W-class state in Eq. (19) can be considered as a \((n-1)\)-party generalized W-class state in higher-dimensional quantum systems.

**Proof.** Let us consider each of the first \((n-2)\)-qudit subsystems \(A_1, A_2, \cdots, A_{n-2}\) as a \(d\)-dimensional quantum system embedded in higher-dimensional system \(B_i\) with dimension \(d^2\) \cite{27},

\[
A_i \subset B_i \cong C^{d^2}, \quad i = 1, \cdots, n-2, \tag{24}
\]

and the last two-qudit systems \(A_{n-1} \otimes A_n\) as a single system \(B_{n-1}\) with dimension \(d^2\),

\[
A_{n-1} \otimes A_n = B_{n-1} \cong C^{d^2}. \tag{25}
\]

For each \(i = 1, 2, \cdots, n-2\), we can extend the orthonormal basis \(|j\rangle_{A_i}\}_{j=0}^{d^2-1}\) of subsystem \(A_i\) to obtain an orthonormal basis \(|j\rangle_{B_i}\}_{j=0}^{d^2-1}\) of subsystem \(B_i\), \(|j\rangle_{A_i}\}_{j=0}^{d^2-1}\) is embedded to \(|j\rangle_{B_i}\) for \(j = 0, \cdots, d - 1\). For the last two-qudit systems \(A_{n-1} \otimes A_n\) with an orthonormal basis \(|j\rangle_{A_{n-1}} \otimes |k\rangle_{A_n}\}_{j,k=0}^{d^2-1}\), we rename these basis element by using the decimal system to obtain an orthonormal basis \(|j\rangle_{B_{n-1}}\}_{j=0}^{d^2-1}\) with the following relation

\[
|j\rangle_{B_{n-1}} = |0\rangle_{A_{n-1}} \otimes |j\rangle_{A_n}, \quad |j\rangle_{B_{n-1}} = |j\rangle_{A_{n-1}} \otimes |0\rangle_{A_n}, \tag{26}
\]

for \(j = 0, \cdots, d - 1\).

Now the \(n\)-qudit generalized W-class state in Eq. (19) can be rewritten as a \((n-1)\)-party generalized W-class state

\[
|W^d_{n-1}\rangle_{B_1 \cdots B_{n-1}} = \sum_{j=1}^{d^2-1} (b_{ij} |j0\cdots0\rangle + b_{2j} |0j\cdots0\rangle + \cdots + b_{(n-1)j} |00\cdots0j\rangle), \tag{27}
\]

with the coefficients \(b_{ij}\) defined as

\[
b_{ij} = a_{ij} \quad \text{for} \quad i = 1, \cdots, n-2,
\]

\[
b_{(n-1)j} = a_{n(n-1)j}, \quad b_{(n-1)j} = a_{nj}, \tag{28}
\]

for \(j = 0, \cdots, d - 1\), and zero elsewhere.

The proof of Theorem 1 deals with the case when the last two-qudit system \(A_{n-1} \otimes A_n\) is considered as a combined single system. However, we can also analogously show that the choice two-qudit system can be arbitrary among \(A_1, \cdots, A_n\). Moreover, we can iteratively use Theorem 1 to obtain the following corollary.

**Corollary 1.** For a partition \(P = \{P_1, \cdots, P_m\}\), \(m \leq n\) of the set of subsystems \(S = \{A_1, \cdots, A_n\}\), the \(n\)-qudit generalized W-class state in Eq. (19) can be considered as a \(m\)-party generalized W-class state in higher-dimensional quantum systems.

**Proof.** Let us assume that each party \(P_s\) contains \(n_s\) number of qudit subsystems for \(s = 1, \cdots, m\) with \(\sum_{s=1}^{m} n_s = n\). For each \(n_s\)-qudit subsystems of the party \(P_s\), we use the argument in the proof of Theorem 1 iteratively to obtain a single system \(B_s\). After renaming the basis elements and the coefficients analogously as in Eqs. (20) and (28), the \(n\)-qudit generalized W-class state in Eq. (19) can be rewritten as a \(m\)-party generalized W-class state

\[
|W^d_{m}\rangle_{B_1 \cdots B_m} = \sum_{j=1}^{d^{max}-1} (b_{1j} |j0\cdots0\rangle + b_{2j} |0j\cdots0\rangle + \cdots + b_{mj} |00\cdots0j\rangle), \tag{29}
\]

where

\[
n_{max} := \max_{s} n_s, \quad d_{max} := d^{max}. \tag{30}
\]

Corollary 1 shows that the generalized W-class state in Eq. (19) preserves its structure with respect to an arbitrary partition of subsystems. Furthermore, the definition of PCS state in Eq. (20) together with Corollary 1 naturally lead us to the following corollary.

**Corollary 2.** For a partition \(P = \{P_1, \cdots, P_m\}\), \(m \leq n\) of the set of subsystems \(S = \{A_1, \cdots, A_n\}\), the \(n\)-qudit PCS state in Eq. (20) can be considered as a \(m\)-party PCS state in higher-dimensional quantum systems.
Now we provide another useful property about the PCS states.

**Lemma 2.** Let $\rho_{A_1 \cdots A_n}^{(p, \lambda)}$ be a partially coherent superposition of a generalized W-class state and the vacuum in Eq. (20). Then the reduced density matrix of $\rho_{A_1 \cdots A_n}^{(p, \lambda)}$, obtained by tracing out some subsystems, is again a partially coherent superposition of a generalized W-class state and the vacuum in the reduced systems.

Proof. Due to an inductive argument, it is enough to show the case when a single-qudit subsystem is traced out from $\rho_{A_1 \cdots A_n}^{(p, \lambda)}$. Without loss of generality, we consider the case when the last qudit subsystem $A_n$ is traced out.

From a straightforward calculation, we obtain the reduced density matrix $\rho_{A_1 \cdots A_{n-1}} = \text{tr}_{A_n}(\rho_{A_1 \cdots A_n}^{(p, \lambda)})$ as

$$
\rho_{A_1 \cdots A_{n-1}} = p \sum_{j,k=1}^{d-1} (a_{1j} | j \cdots 0 \rangle + \cdots + a_{n-1j} | 0 \cdots j \rangle)_{A_1 \cdots A_{n-1}} (a_{1k}^{*} | k \cdots 0 \rangle + \cdots + a_{n-1k}^{*} | 0 \cdots k \rangle) \\
+ \left( p \sum_{j=1}^{d-1} |a_{nj}|^{2} + 1 - p \right) |0\rangle_{A_1 \cdots A_{n-1}} \langle 0|^{n-1} \\
+ \lambda \sqrt{p(1-p)} \sum_{j=1}^{d-1} (a_{1j} | j \cdots 0 \rangle + \cdots + a_{n-1j} | 0 \cdots j \rangle)_{A_1 \cdots A_{n-1}} (0^{\otimes n-1}) \\
+ \lambda \sqrt{p(1-p)} \left[ | 0 \rangle_{A_1 \cdots A_{n-1}} \sum_{k=1}^{d-1} (a_{1k}^{*} | k \cdots 0 \rangle + \cdots + a_{n-1k}^{*} | 0 \cdots j \rangle) \right],
$$

(31)

where $|0\rangle^{\otimes n-1}_{A_1 \cdots A_{n-1}}$ is the vacuum of subsystems $A_1 \cdots A_{n-1}$. By using the notion $\Omega = \sum_{j=1}^{d-1} |a_{ij}|^{2}$, the normalization condition of $n$-qudit W-class state implies

$$
\sum_{j=1}^{d-1} |a_{nj}|^{2} = 1 - \Omega,
$$

(32)

and Eq. (31) can be rewritten as

$$
\rho_{A_1 \cdots A_{n-1}} = p\Omega \left| W^{d}_{n-1} \rightangle \langle W^{d}_{n-1} | \\
+ (1 - p\Omega) | 0 \rangle^{\otimes n-1} \langle 0|^{\otimes n-1} \\
+ \lambda \sqrt{p(1-p)} \left( | W^{d}_{n-1} \rangle \langle W^{d}_{n-1} |^{\otimes n-1} + | 0 \rangle^{\otimes n-1} \langle W^{d}_{n-1} |^{\otimes n-1} \right),
$$

(33)

where

$$
| W^{d}_{n-1} \rangle = \frac{1}{\sqrt{\Omega}} \sum_{j=1}^{d-1} (a_{1j} | j \cdots 0 \rangle + \cdots + a_{n-1j} | 0 \cdots j \rangle)
$$

(34)

and Eq. (33) becomes

$$
\lambda \sqrt{p(1-p)} = \lambda' \sqrt{p'(1-p')}
$$

(36)

with

$$
\lambda' = \sqrt{\frac{1 - p}{1 - p'}}.
$$

(37)

From Eq. (33) together with Eqs. (35) and (37), the reduced density matrix in Eq. (31) can be rewritten as

$$
\rho_{A_1 \cdots A_{n-1}}^{(p', \lambda')} = p' \left| W^{d}_{n-1} \rightangle \langle W^{d}_{n-1} | + (1 - p') | 0 \rangle^{\otimes n-1} \langle 0|^{\otimes n-1} + \lambda' \sqrt{p'(1-p')} \left( | W^{d}_{n-1} \rangle \langle W^{d}_{n-1} |^{\otimes n-1} + | 0 \rangle^{\otimes n-1} \langle W^{d}_{n-1} |^{\otimes n-1} \right),
$$

(38)
which is a partially coherent superposition of an \((n-1)\)-qudit W-class state \(|W_{n-1}^d\rangle\) and the vacuum with new parameters \(p'\) and \(\lambda'\).

Here we note that \(p' = p\Omega \leq p\) as \(0 \leq \Omega \leq 1\), therefore Eq. (37) implies that \(\lambda' \leq \lambda\). In other words, the parameter of coherency \(\lambda\) is not increasing as we trace out some subsystems from \(\rho_{A_1 \cdots A_n}^{(p, \lambda)}\).

V. SCREN MONOGAMY INEQUALITY FOR PARTIALLY COHERENTLY SUPERPOSED STATES

In this section, we show that multi-party SCREN monogamy inequality in [11] is true for multi-qudit PCS states in Eq. (20). We first recall a useful property about unitary freedom in the ensemble for density matrices provided by Hughston, Jozsa and Wootters (HJW) [25].

Proposition 1. (HJW Theorem) The sets \(|\phi_i \rangle\rangle\) and \(|\psi_j \rangle\rangle\) of (possibly unnormalized) states generate the same density matrix if and only if

\[ |\phi_i \rangle = \sum_j u_{ij} |\psi_j \rangle \]  

(39)

where \((u_{ij})\) is a unitary matrix of complex numbers, with indices \(i\) and \(j\), and we pad whichever set of states \(|\phi_i \rangle\rangle\) or \(|\psi_j \rangle\rangle\) is smaller with additional zero vectors so that the two sets have the same number of elements.

Consequently, Proposition 1 implies that for two pure-state decompositions \(\sum_i p_i |\phi_i \rangle \langle \phi_i |\) and \(\sum_j q_j |\psi_j \rangle \langle \psi_j |\), they represent the same density matrix, that is \(\rho = \sum_i p_i |\phi_i \rangle \langle \phi_i | = \sum_j q_j |\psi_j \rangle \langle \psi_j |\) if and only if \(\prod_i |\phi_i \rangle \langle \phi_i | = \sum_j u_{ij} |\psi_j \rangle \langle \psi_j |\) for some unitary matrix \(u_{ij}\).

Theorem 3. For an \(n\)-qudit PCS state in Eq. (20), we have

\[ N_{sc} (\rho_{A_1 A_2 \cdots A_n}^{(p, \lambda)}) = N_{sc} (\rho_{A_1 A_j}) \]  

(40)

where \(N_{sc} (\rho_{A_1 A_2 \cdots A_n})\) is the 2-SCREN of \(\rho_{A_1 A_2 \cdots A_n}^{(p, \lambda)}\) with respect to bipartition between \(A_1\) and the other qudits, and \(N_{sc} (\rho_{A_1 A_j})\) are 2-SCREN of the two-qudit reduced density matrix \(\rho_{A_1 A_j}\) for \(j = 2, \cdots, n\).

Proof. We use mathematical induction on the number of subsystems \(n\), and first show the saturation of SCREN monogamy inequality for three-qudit systems. For three-qudit PCS states, we have

\[ \rho_{A_1 A_2 A_3}^{(p, \lambda)} = p |W_{3}^d\rangle \langle W_{3}^d| + (1 - p) |000\rangle \langle 000| \]

\[ + \lambda \sqrt{p(1 - p)} \left( |W_{3}^d\rangle \langle 000| + |000\rangle \langle W_{3}^d| \right), \]

(41)

where \(|W_{3}^d\rangle_{A_1 A_2 A_3}\) is a three-qudit W-class state

\[ |W_{3}^d\rangle_{A_1 A_2 A_3} = \sum_{j=1}^{d-1} (a_{1j} |j00\rangle + a_{2j} |0j0\rangle + a_{3j} |00j\rangle) \]

(42)

with normalization \(\sum_{j=1}^{d-1} (|a_{1j}|^2 + |a_{2j}|^2 + |a_{3j}|^2) = 1\).

From the definition of two-SCREN of \(\rho_{A_1 A_2 A_3}^{(p, \lambda)}\) between \(A_1\) and \(A_2 A_3\),

\[ N_{sc} (\rho_{A_1 A_2 A_3}^{(p, \lambda)}) = \left[ \min_{\langle \psi_h |} \sum_{h} p_h \sqrt{N_{sc} (|\psi_h\rangle_{A_1 A_2 A_3})} \right]^2, \]

(43)

we need to consider the minimization over all possible pure-state decompositions of \(\rho_{A_1 A_2 A_3}^{(p, \lambda)}\). By considering two unnormalized states

\[ |\tilde{x}\rangle_{A_1 A_2 A_3} = \sqrt{p} |W_{3}^d\rangle_{A_1 A_2 A_3} + \lambda \sqrt{1 - p} |000\rangle_{A_1 A_2 A_3}, \]

\[ |\tilde{y}\rangle_{A_1 A_2 A_3} = \sqrt{1 - p} |1-\lambda\rangle |000\rangle_{A_1 A_2 A_3}, \]

(44)

we note that the PCS state in Eq. (11) can be rewritten as

\[ \rho_{A_1 A_2 A_3}^{(p, \lambda)} = |\tilde{x}\rangle_{A_1 A_2 A_3} \langle \tilde{x}| + |\tilde{y}\rangle_{A_1 A_2 A_3} \langle \tilde{y}|. \]

(45)

For any pure state decomposition

\[ \rho_{A_1 A_2 A_3}^{(p, \lambda)} = \sum_h |\tilde{w}_{h}\rangle_{A_1 A_2 A_3} \langle \tilde{w}_{h}|, \]

(46)

where \(|\tilde{w}_{h}\rangle_{A_1 A_2 A_3}\) is an unnormalized state in three-qudit subsystem \(A_1 A_2 A_3\), HJW theorem in Proposition 1 assures that there exists an \(r \times r\) unitary matrix \((u_{hkl})\) such that

\[ |\tilde{w}_{h}\rangle_{A_1 A_2 A_3} = u_{h1} |\tilde{x}\rangle_{A_1 A_2 A_3} + u_{h2} |\tilde{y}\rangle_{A_1 A_2 A_3}, \]

(47)

for each \(h\).

For the normalized state \(|\tilde{w}_{h}\rangle_{A_1 A_2 A_3} = |\tilde{w}_{h}\rangle_{A_1 A_2 A_3} / \|\tilde{w}_{h}\|\) with \(p_h = |\langle \tilde{w}_{h}| \tilde{w}_{h}\rangle|\), the two-SCREN of \(|\tilde{w}_{h}\rangle_{A_1 A_2 A_3}\) between \(A_1\) and \(A_2 A_3\) can be obtained as

\[ N_{sc} (|\tilde{w}_{h}\rangle_{A_1 A_2 A_3}) = \frac{4}{p_h^2} |u_{h1}|^2 |u_{h2}|^2 \left( \sum_{j=1}^{d-1} (|a_{2j}|^2 + |a_{3j}|^2) \right) \sum_{k=1}^{d-1} |a_{1k}|^2, \]

(48)

for each \(h\). Thus the average of the square-root of pure state two-SCREN for the pure state decomposition in Eq. (46) is

\[ \sum_h p_h \sum_{k=1}^{d-1} \frac{4}{p_h^2} |u_{h1}|^2 |u_{h2}|^2 \left( \sum_{j=1}^{d-1} (|a_{2j}|^2 + |a_{3j}|^2) \right) \sum_{k=1}^{d-1} |a_{1k}|^2 \]

\[ = 2p \sum_{k=1}^{d-1} (|a_{2j}|^2 + |a_{3j}|^2) \sum_{k=1}^{d-1} |a_{1k}|^2 \]

(49)
where the last equality is due to the unitary matrix \((u_{hl})\).

Eq. (39) implies that the average of the square-root of pure state two-SCREN does not depend on the choice of pure state decompositions in Eq. (40). Thus the two-SCREN of \(\rho_{A_1A_2A_3}^{(p, \lambda)}\) with respect to the bipartition between \(A_1\) and \(A_2A_3\) is

\[
\mathcal{N}_{sc} \left( \rho_{A_1A_2A_3}^{(p, \lambda)} \right) = 4p^2 \sum_{j=1}^{d-1} (|a_{2j}|^2 + |a_{3j}|^2) \sum_{k=1}^{d-1} |a_{1k}|^2.
\]

(50)

Now we consider the two-qudit reduced density matrices \(\rho_{A_1A_2A_3}^{(p, \lambda)}\) and their two-SCREN. By tracing out the subsystem \(A_3\), we have

\[
\rho_{A_1A_2} = \text{Tr}_{A_3} \rho_{A_1A_2A_3}
\]

\[
= p \sum_{j,k=1}^{d-1} \left[ |a_{1j}a_{1k}^*|j0\rangle_{A_1A_2}\langle j0| + a_{1j}a_{2k}^*|j0\rangle_{A_1A_2}\langle k0| + a_{2j}a_{1k}^*|0j\rangle_{A_1A_2}\langle k0| + a_{2j}a_{2k}^*|0j\rangle_{A_1A_2}\langle 0k| \right] + \left( p \sum_{j=1}^{d-1} |a_{3j}|^2 + 1 - p \right) |00\rangle_{A_1A_2}\langle 00|
\]

\[
+ \lambda \sqrt{p(1-p)} \sum_{j=1}^{d-1} \left[ (a_{1j}|k0\rangle + a_{2j}|0k\rangle)\langle A_1A_2(00) + a_{1k}^*|00\rangle_{A_1A_2}\langle k0| + a_{2k}^*|0k\rangle \right].
\]

(51)

By considering two unnormalized states

\[
|\tilde{h}\rangle_{A_1A_2} = \sqrt{p} \sum_{j=1}^{d-1} (a_{1j}|j0\rangle + a_{2j}|0j\rangle)_{A_1A_2}
\]

\[
+ \lambda \sqrt{1-p(00)_{A_1A_2}},
\]

\[
|\tilde{\xi}\rangle_{A_1A_2} = \sqrt{d-1} \sum_{j=1}^{d-1} |a_{3j}|^2 + (1-p)(1-\lambda^2)(00)_{A_1A_2},
\]

(52)

the two-qudit reduced density matrix \(\rho_{A_1A_2}\) in Eq. (51) can be rewritten as

\[
\rho_{A_1A_2} = |\tilde{h}\rangle_{A_1A_2}\langle \tilde{h}| + |\tilde{\xi}\rangle_{A_1A_2}\langle \tilde{\xi}|.
\]

(53)

For any pure state decomposition of \(\rho_{A_1A_2}\)

\[
\rho_{A_1A_2} = \sum_{h} |\tilde{\psi}_h\rangle_{A_1A_2}\langle \tilde{\psi}_h|
\]

\[
= \sum_{h} q_h |\tilde{\psi}_h\rangle_{A_1A_2}\langle \tilde{\psi}_h|
\]

(54)

with \(q_h = \langle \tilde{\psi}_h|\tilde{\psi}_h\rangle\) for each \(h\), HJW theorem in Proposition 1 assures that there exists an \(r \times r\) unitary matrix \((v_{hl})\) such that

\[
|\tilde{\psi}_h\rangle_{A_1A_2} = v_{h1}|\tilde{h}\rangle_{A_1A_2} + v_{h2}|\tilde{\xi}\rangle_{A_1A_2},
\]

for each \(h\).

From a straightforward calculation, the two-SCREN of \(|\psi_h\rangle_{A_1A_2}\) in Eq. (52) is obtained as

\[
\mathcal{N}_{sc} \left( |\psi_h\rangle_{A_1A_2} \right) = \frac{4}{q_h^2} p^2 |v_{h2}|^2 \sum_{j=1}^{d-1} |a_{2j}|^2 \sum_{k=1}^{d-1} |a_{1k}|^2,
\]

(56)

therefore the average of the square-root of two-SCRENs for the decomposition in Eq. (54) is

\[
\sum_{h} q_h \sqrt{\mathcal{N}_{sc} \left( |\psi_h\rangle_{A_1A_2} \right)}
\]

\[
= 2p \sum_{h} |v_{h2}|^2 \sqrt{\sum_{j=1}^{d-1} |a_{2j}|^2 \sum_{k=1}^{d-1} |a_{1k}|^2}
\]

\[
= 2p \sqrt{\frac{1}{d-1} \sum_{j=1}^{d-1} |a_{2j}|^2 \sum_{k=1}^{d-1} |a_{1k}|^2},
\]

(57)

where the last equality is due to the unitary matrix \((u_{hl})\).

Similar to the case of \(\mathcal{N}_{sc} \left( \rho_{A_1A_2A_3}^{(p, \lambda)} \right)\), we note that the average in Eq. (57) does not depend on the choice of pure state decomposition of \(\rho_{A_1A_2}\). Thus the two-SCREN of
\( \rho_{A_1A_2} \) is

\[
N_{sc} (\rho_{A_1|A_2}) = \left[ \min_{\{q_h, |\psi_h\rangle\}} \sum_h q_h \sqrt{N_{sc} (|\psi_h\rangle_{A_1|A_2})} \right]^2
\]

\[= 4p^2 \sum_{j=1}^{d-1} |a_{2j}|^2 \sum_{k=1}^{d-1} |a_{1k}|^2. \tag{58} \]

Moreover, we can analogously obtain the two-SCREN of the two-qudit reduced density matrix \( \rho_{A_1A_3} = \text{tr}_{A_2} \rho_{A_1A_2A_3} \) as

\[
N_{sc} (\rho_{A_1|A_2}) = 4p^2 \sum_{j=1}^{d-1} |a_{2j}|^2 \sum_{k=1}^{d-1} |a_{1k}|^2. \tag{59} \]

From Eq. (50) together with Eqs. (58) and (59), we have

\[
N_{sc} (\rho_{A_1A_2|A_3}) = N_{sc} (\rho_{A_1|A_2}) + N_{sc} (\rho_{A_1A_3|A_2}). \tag{60} \]

for any three-qudit PCS state \( \rho_{A_1A_2A_3} \).

Now we assume the induction hypothesis, that is, Eq. (60) is true for any \((n-1)\)-qudit PCS state, and show the validity of Eq. (60) for \(n\)-qudit PCS states. From Corollary 2 we note that the \(n\)-qudit PCS state in Eq. (20) can be considered as a \((n-1)\)-party PCS state in higher-dimensional quantum system where the last two-qudit subsystem \(A_{n-1} \otimes A_n\) is considered as a single subsystem.

Due to the induction hypothesis, we have

\[
N_{sc} (\rho_{A_1|A_2\ldots A_n}) = \sum_{j=2}^{n-2} N_{sc} (\rho_{A_1|A_j}) + N_{sc} (\rho_{A_1A_{n-1}|A_n}), \tag{61} \]

where \(N_{sc} (\rho_{A_1A_{n-1}|A_n})\) is the two-SCREN of the three-qudit reduced density matrix \(\rho_{A_1A_{n-1}A_n}\) with respect to the bipartition between \(A_1\) and \(A_{n-1}A_n\). Moreover, Lemma 2 implies that the three-qudit reduced density matrix \(\rho_{A_1A_{n-1}A_n}\) is a three-qudit PCS state. Thus our induction hypothesis assures that

\[
N_{sc} (\rho_{A_1A_{n-1}|A_n}) = N_{sc} (\rho_{A_1A_{n-1}}) + N_{sc} (\rho_{A_1A_n}). \tag{62} \]

Now Eq. (61) together with Eq. (62) complete the proof. \(\square\)

For the case when \(\lambda = 1\), the PCS state \(\rho_{A_1A_2\ldots A_n}^{(p, \lambda)}\) in Eq. (20) is reduced to a coherently superposed state \(|\psi\rangle_{A_1\ldots A_n}\) in Eq. (21) where its saturation of the monogamy inequality in terms of SCREN was provided as a result in 16. Thus Theorem 3 encapsulates the result of 16.

\section{VI. SCREN STRONG MONOGAMY INEQUALITY FOR PARTIALLY COHERENTLY SUPERPOSED STATES}

Now we show the validity of SCREN SM inequality in (15) for PCS states in Eq. (20). We first note that for the case when \(\lambda = 1\), \(\rho_{A_1A_2\ldots A_n}^{(p, \lambda)}\) becomes the coherent superposition of an \(n\)-qudit generalized W-class state and vacuum in Eq. (21) where the saturation of SCREN SM inequality for this case was already provided in 16.

**Proposition 2.** For the class of \(n\)-qudit states that is a coherent superposition of an \(n\)-qudit generalized W-class state and the vacuum,

\[
|\psi\rangle_{A_1\ldots A_n} = \sqrt{p} |W_n^d\rangle + \sqrt{1-p} |0\rangle^\otimes n,
\]

SCREN SM inequality of entanglement is saturated;

\[
N_{sc} (|\psi\rangle_{A_1|A_2\ldots A_n}) = \sum_{m=2}^{n-1} \sum_{m} N_{sc} (\rho_{A_1|A_2^m\ldots A_n^{m-1}})^{m/2}. \tag{63} \]

Thus our new result derived below encapsulates the result in 16 as a special case. To generalize Proposition 2 for arbitrary PCS states, we first provide the following theorem.

**Theorem 4.** For the PCS state in Eq. (20), its n-SCREN is zero,

\[
N_{sc} (\rho_{A_1|A_2\ldots A_n}) = 0. \tag{64} \]

**Proof.** From the definition of the mixed state n-SCREN, we have

\[
N_{sc} (\rho_{A_1|A_2\ldots A_n}) = \left[ \min_{\{q_h, |\psi_h\rangle\}} \sum_h q_h \sqrt{N_{sc} (|\psi_h\rangle_{A_1|A_2\ldots A_n})} \right]^2, \tag{65} \]

where the minimization is over all possible pure-state decompositions of

\[
\rho_{A_1|A_2\ldots A_n} = \sum_h |\psi_h\rangle_{A_1A_2\ldots A_n} \langle \psi_h|.
\]

Let us consider two unnormalized states in an \(n\)-qudit system

\[
|\tilde{\eta}\rangle = \sqrt{p} |W_n^d\rangle + \lambda \sqrt{1-p} |0\rangle^\otimes n, \quad |\xi\rangle = \sqrt{(1-p)(1-\lambda)} |0\rangle^\otimes n, \tag{67} \]

where \(|W_n^d\rangle\) is the \(n\)-qudit W-class state and \(|0\rangle^\otimes n\) is the vacuum. Then the PCS state in Eq. (20) can be represented as

\[
\rho_{A_1A_2\ldots A_n}^{(p, \lambda)} = |\tilde{\eta}\rangle \langle \tilde{\eta}| + |\xi\rangle \langle \xi|. \tag{68} \]

From the HJW theorem in Proposition 1 any pure state decomposition of \(\rho_{A_1A_2\ldots A_n}^{(p, \lambda)} = \sum_{h=1}^r |\psi_h\rangle \langle \psi_h| \) of

\[\text{Inequality for Partially Coherent Monogamy} \]
size $r$ can be realized by some choice of an $r \times r$ unitary matrix $(u_{ij})$ such that
\[
|\tilde{\psi}_h\rangle = u_{11}|\tilde{\eta}\rangle + u_{12}|\tilde{\xi}\rangle = u_{11}\sqrt{p} |W^d_n\rangle + \left(u_{h1}\lambda \sqrt{1-p} + u_{h2}\sqrt{(1-p)(1-\lambda^2)}\right)|0\rangle^\otimes n. \tag{69}
\]

By considering the normalization $|\tilde{\psi}_h\rangle = \sqrt{p_h} |\psi_h\rangle$ with $p_h = \langle \tilde{\psi}_h | \tilde{\psi}_h \rangle$ for each $h$, we have
\[
\rho^{(p,\lambda)}_{A_1A_2\cdots A_n} = \sum_{h=1}^r p_h |\psi_h\rangle \langle \psi_h|. \tag{70}
\]

Because $|\tilde{\psi}_h\rangle$ in Eq. (69) is a coherent superposition of an $n$-qudit W-class state and vacuum, so is its normalized state $|\psi_h\rangle$ in Eq. (70). In other words, any pure state appears in any pure-state decomposition of $\rho^{(p,\lambda)}_{A_1A_2\cdots A_n}$ is a coherent superposition an $n$-qudit W-class state and vacuum. Thus Proposition 2 assures that $\rho^{(p,\lambda)}_{A_1A_2\cdots A_n}$ is a PCS state of a $m$-qudit W-class state and vacuum. Thus we have
\[
N_{sc} \left(\rho^{(p,\lambda)}_{A_1|A_j^m\cdots A_{j=m-1}}\right) = 0 \tag{75}
\]

by Theorem 1. Now, Eqs. (71) and (72) together with Theorem 3 complete the proof.

VII. CONCLUSIONS

We have considered a large class of multi-qudit mixed state that are in a partially coherent superposition of a generalized W-class state and the vacuum, and have provided various useful properties about the structure of partially coherently superposed states. We have shown that CKW-type monogamy inequality of multi-qudit entanglement holds for this class of PCS states in terms of SCREN. We have further shown that SCREN SM inequality of multi-qudit entanglement is saturated for PCS states.

Our result proposes the use of SCREN over tangle in characterizing strongly monogamous property of multi-party quantum entanglement by providing analytic proofs of the SCREN monogamy inequalities for some class of multi-party PCS states. However, it is also interesting and important to investigate how the class of PCS states behaves with respect to the corresponding tangle based monogamy inequalities. We believe this investigation will provide a better clarification of the usefulness of PCS states in understanding the constraints on description of entanglement distribution using different kinds of entanglement measures.

Our result presented here is the first case where strong monogamy inequality of multi-qudit mixed states is studied. Noting the importance of the study on multi-party quantum entanglement, our result can provide a rich reference for future work on the study of multi-party quantum entanglement.

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The definition of negativity for a quantum state $\rho$ was first proposed by Vidal and Werner [18] as $\|\rho^{AB}\|_1 - 1$, and another definition of the negativity with a normalizing factor, $\frac{\|\rho^{AB}\|_1 - 1}{d-1}$, was also used for two-qudit states $\rho$. To avoid this inconsistency, here we only use $\|\rho^{AB}\|_1 - 1$ for our definition of the negativity.

Equivalently, negativity of $|\phi\rangle_{AB}$ can be defined by using the reduced density matrix

$$\mathcal{N}(|\phi\rangle_{A|B}) = (\text{tr}\sqrt{\rho_A})^2 - 1,$$

where $\rho_A = \text{tr}_B|\phi\rangle_A\langle B|$.

From the property of convex-roof extension, SCREN clearly has the separability criterion: $\mathcal{N}_{sc}(\rho_{A|B}) = 0$ if and only if $\rho_{AB}$ is separable. SCREN is also an entanglement monotone, which does not increase under local quantum operations and classical communications.

For the same terminology, we also refer $\mathcal{N}_{sc}(\rho_{A|B})$ as one-SCREN and two-SCREN, respectively.

The lower term in Inequality (16) appears in between the both sides of the $n$-qubit CKW inequality in [5] as

$$\tau \left( |\psi\rangle_{A_1|A_2\cdots|A_n} \right) \geq \sum_{j=2}^{n} \tau \left( \rho_{A_1|A_j} \right) + \sum_{m=3}^{n-1} \sum_{j=m}^{n} \tau \left( \rho_{A_1|A_{j_1}\cdots|A_{j_{m-1}}} \right)^{m/2} \geq \sum_{j=2}^{n} \tau \left( \rho_{A_1|A_j} \right),$$

thus it is a stronger inequality.

The term “generalized” naturally arises because Eq. (16) reduces to three-qubit W state for the case when $n = 3$ and $d = 2$. Eq. (19) also includes multi-qubit generalized W-class states when $d = 2$.

When we deal with pure states, we sometimes consider each qudit subsystem $A_i$ as a $d$-dimensional Hilbert space, that is, $A_i \cong \mathbb{C}^d$ for $i = 1, \ldots, n$ if there is no confusion.

[1] B. M. Terhal, IBM J. Research and Development 48, 71 (2004).
[2] J. S. Kim, G. Gour and B. C. Sanders, Contemp. Phys. 53, 5 p. 417-432 (2012).
[3] V. Coffman, J. Kundu and W. K. Wootters, Phys. Rev. A 61, 052306 (2000).
[4] T. Osborne and F. Verstraete, Phys. Rev. Lett. 96, 220503 (2006).
[5] J. S. Kim, A. Das and B. C. Sanders, Phys. Rev. A 79, 012329 (2009).
[6] J. S. Kim and B. C. Sanders, J. Phys. A: Math. and Theor. 43, 445305 (2010).
[7] J. S. Kim, Phys. Rev. A 90, 062328 (2010).
[8] J. S. Kim and B. C. Sanders, J. Phys. A: Math. and Theor. 44, 295303 (2011).
[9] M. Koashi and A. Winter, Phys. Rev. A 69, 022309 (2004).
[10] F. G. S. L. Brandao, M. Christandl and J. Yard, Commun. Math. Phys. 306, 805 (2011).
[11] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A 62, 062314 (2000).
[12] B. Regula, S. D. Martino, S. Lee and G. Adesso, Phys. Rev. Lett. 113, 110501 (2014).
[13] J. S. Kim, Phys. Rev. A 90, 062306 (2014).
[14] Y. Ou, Phys. Rev. A 75, 034305 (2007).
[15] J. S. Kim and B. C. Sanders, J. Phys. A 41, 495301 (2008).
[16] J-H Choi and J. Kim, Phys. Rev. A 92, 042307 (2015).
[17] This definition is also valid for any bipartite pure state with Schmidt-rank 2.
[18] G. Vidal and R. F. Werner, Phys. Rev. A 65, 032314 (2002).
[19] The definition of negativity for a quantum state $\rho$ was first proposed by Vidal and Werner [18] as $\|\rho^{AB}\|_1 - 1$, and another definition of the negativity with a normalizing factor, $\frac{\|\rho^{AB}\|_1 - 1}{d-1}$, was also used for two-qudit states $\rho$. To avoid this inconsistency, here we only use $\|\rho^{AB}\|_1 - 1$ for our definition of the negativity.
[20] Equivalently, negativity of $|\phi\rangle_{AB}$ can be defined by using the reduced density matrix

$$\mathcal{N}(|\phi\rangle_{A|B}) = (\text{tr}\sqrt{\rho_A})^2 - 1,$$