The Hamiltonian Formalism for the Generalized Rigid Particles.

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Abstract

The Hamiltonian formulation for the mechanical systems with reparametrization-invariant Lagrangians, depending on the worldline external curvatures is given, which is based on the use of moving frame.

A complete sets of constraints are found for the Lagrangians with quadratic dependence on curvatures, for the Lagrangians, proportional to an arbitrary curvature, and for the Lagrangians, linear on the first and second curvatures.

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1 Introduction

As it is known, the curve in a $D$-dimensional space possesses $D-1$ reparametrization invariants (external curvatures) $\tilde{k}_1, \ldots, \tilde{k}_{D-1}$, which are the functions of a natural parameter $\tilde{s}$ (see, e.g. [1]). Therefore, the general reparametrization-invariant mechanical action in $D$-dimensional space can be defined as

$$\mathcal{S} = \int F(\tilde{k}_1, \ldots, \tilde{k}_N) d\tilde{s}, \quad 0 \leq N \leq D - 1. \quad (1.1)$$

Such systems we will call by the models of generalized rigid particles.

The mechanical systems depending on the first and second curvatures became rather intensively studied in the late eighties as toy models of rigid strings and (2+1)-dimensional field theories with the Chern-Simon term [2]. Before long, it became clear, mainly due to the studies of M.Plyushchay that those systems are of independent interest.

For instance, at $D = (2 + 1)$, $F = c_0 + c_1 \tilde{k}_1 + c_2 \tilde{k}_2$, $c_0 \neq 0$ they describe a massive relativistic anyon [3]; at $D = (3 + 1)$, $F = c_0 + c_1 \tilde{k}_1$, $c_0 \neq 0$, a massive relativistic boson [4]; at $D = (3 + 1)$, $F = c \tilde{k}_1$, a massless particle with an arbitrary (both integer and half-integer) helicity [5]. The system with $F = c_0 + c_1 \tilde{k}_1$ corresponds to the effective action of relativistic kink in the field of soliton [6].

Recently, E.Ramos and J.Roca have found that the model with $F = c \tilde{k}_1$ possesses the $W_3$-gauge symmetry [7]. They have also shown in an implicit way that a system with Lagrangian $F = c \tilde{k}_N$ possesses $N + 1$ gauge degrees of freedom, perhaps, forming $W_{N+2}$-algebra [8].

Which (iso)spinning particles are described by the models of generalized rigid particles?

Which gauge $W$-symmetries can be inherent in these models?

To answer on these questions, one should know the dimension and structure of phase spaces of the models under consideration, the generators of their gauge symmetries, and then quantized the models.

First of all, this needs the Hamiltonian formulation of the models with the action (1.1). However, the Lagrangians of that models depend on $(N+1)$-order derivatives, since the external curvatures are determined by the expressions

$$\tilde{k}_I(\tilde{s}) = \frac{\sqrt{\det g_{I+1} \det g_{I-1}}}{\det \hat{g}_I}, \quad (g_I)_{ij} \equiv x_{(i)}x_{(j)}, \quad i, j = 1, \ldots, I,$$

where $x_{(i)} \equiv d^i x(\tilde{s})/(d\tilde{s})^i$. Thus, one should first replace the initial Lagrangian by an equivalent second order one and then pass to the Hamiltonian formalism in $2D(N+1)$-dimensional phase space.

In the latter transition, most authors neglect invariant properties of Lagrangians, which state in their dependence on external curvatures. As a result, even the construction of the complete set of constraints requires tiring structureless calculations. For example, in the referred paper [8] the complete set of constraints was constructed only for $F = c \tilde{k}_2$, the latter being essentially nonlinear.
In this paper, we suggest more geometrical approach for constructing the Hamiltonian formalism for the models of generalized rigid particles, which is based on the use of moving frame.

The resulting system is formulated in terms of the coordinates of the initial space $x$, the components of moving frame $e_i$, and their conjugated momenta $p$ and $\tilde{p}_i$, $i = 1, \ldots, N$. The Lagrangian multipliers in the total Hamiltonian of the system represents the external curvatures of trajectories.

We demonstrate efficiency of the presented formulation, constructing the complete sets of constraints and Hamiltonians for models with the following Lagrangians:

i) $F = \frac{1}{2} \sum_i^N b_i \tilde{k}_i^2 + \sum_{i=1}^N c_i \tilde{k}_i + c_0, \quad b_1 b_2 \ldots b_N \neq 0$; This system is characterized by the lowest degeneracy and by absence of the secondary constraints.

ii) $F = c k_N, \quad \forall D, N < D$; The system is specified by the maximal (for a given $N$) degeneracy and by $N + 1$ gauge degree of freedom. All the constraints arising here are quadratic. Surprisingly, this model coincides with the model $N + 1$-pointing discreet string.

We show that systems with the Lagrangians, linear on external curvatures possess the maximally possible set of (quadratic) primary constraints. When the Lagrangian contains the curvatures $k_a, a < N$, the number of secondary constraints and the gauge symmetries of Lagrangian is decreased. To illustrate this phenomena, we present the complete sets of constraints for the thoroughly studied models with Lagrangians linear on first and second curvatures.

Throughout the paper, we assume the signature of the initial space $\mathbb{R}^D$ to be Euclidean, which should not cause misunderstanding when passing to the pseudo-Euclidean signature.

We use the following groups of indices:

$$i, j, k = 1, \ldots N; \quad a, b, c, d = 1, \ldots (N - 1); \quad \alpha, \beta = 1, \ldots, (N - 2);$$

and the notation:

$$F_{,i} \equiv \partial F / \partial \tilde{k}_i, \quad F_{,ij} \equiv \partial^2 F / \partial \tilde{k}_i \partial \tilde{k}_j$$

$$\tilde{\phi}_{0,i} = p e_i, \quad \tilde{\phi}_{i,j} = p_i e_j - p_j e_i, \quad \tilde{\Phi}_{0,0} = p \tilde{L} p, \quad \tilde{\Phi}_{0,i} = p \tilde{L} \tilde{p}_i, \quad \tilde{\Phi}_{i,j} = p_i \tilde{L} p_j, \quad (1.2)$$

where

$$\tilde{L} = \tilde{I} - \sum_{i=1}^N e_i \otimes e_i; \quad \forall a, \quad b : a \equiv a^A, b \equiv a^A, \quad \text{ab} = \sum_{A=1}^D a^A b^A.$$  

2 Frenet Formulae and Legendre Transformation

Consider the Hamiltonian formulation of the models of generalized rigid particle.

Let us rewrite the action (1.1) as

$$S = \int F(k_1/s, \ldots, k_N/s) s d\tau; \quad \text{where} \quad s \equiv \frac{|d\mathbf{x}|}{d\tau}, \quad k_i \equiv \tilde{s} \tilde{k}_i, \quad (2.1)$$
and introduce the moving frame \( \{ e_\mu \} \) for the trajectory of that system
\[
e_\mu e_\nu = \delta_{\mu\nu}, \quad \dot{x} = s e_1, \quad \mu = 1, \ldots, D. \tag{2.2}
\]
In these terms the external curvatures are defined by the Frenet equations
\[
\dot{e}_\mu = k_\mu e_{\mu+1} - k_{\mu-1} e_{\mu-1}, \quad e_0 = e_{D+1} = 0, \tag{2.3}
\]
so
\[
k_{\mu-1} = \dot{e}_{\mu-1} e_\mu, \quad k_\mu^2 = \dot{e}_\mu^2 - k_{\mu-1}^2, \quad \dot{e}_\mu e_\nu = 0, \quad \text{if} \quad |\mu - \nu| > 1. \tag{2.4}
\]
Note that \( k_\mu \geq 0, \) for \( \mu = 1, \ldots, (D - 2), \) whereas \( k_{D-1} \) ("torsion") can assume both positive and negative values. If some \( k_I \neq 0, \) then \( k_{I-1} \neq 0, \) at \( \mu = 1, \ldots, I - 1. \) Vice versa, if \( k_I = 0, \) then \( k_{I-1} = 0, \) at \( \mu = I + 1, \ldots, D - 1 \) (see, e.g. [1]).

With the expressions (2.2), (2.3), (2.4) at hands, we can replace the initial Lagrangian by the following one
\[
\mathcal{L} = F(k_1/s, \ldots, k_N/s) s + p(\dot{x} - s e_1) + \sum_a p_a \dot{e}_a - k_a e_{a+1} + k_{a-1} e_{a-1})
- \sum_{i,j} d^{ij} (e_i e_j - \delta_{ij}) - F_N \left( k_N^2 - (e_N^2 - k_{N-1}^2)^{1/2} \right) \tag{2.5}
\]
where \( s, k_i, d^{ij}, p_a, e_i \) are independent variables.

Now we can perform the Legendre transformation for the Lagrangian (2.7). The variables \( p_a \) represent the momenta conjugated to \( e_a, \) whereas momenta, conjugated to \( (s, k_a, d_{ij}), \) lead to the trivial constraints
\[
p^s \approx 0, \quad p^a \approx 0, \quad p^{ij} \approx 0. \tag{2.6}
\]
Setting \( k_N \neq 0, F_N \neq 0 \) we find, that the momentum conjugated to \( e_N, \) is of the form
\[
p_N = F_N \left( e_N^2 - k_{N-1}^2 \right)^{-1/2} \dot{e}_N. \tag{2.7}
\]
So, taking into account (2.4), we get the constraints
\[
\chi_{N,N} \equiv p_N e_N \approx 0, \quad \chi_{N,a} = p_N e_a \approx 0, \tag{2.8}
\]
\[
\Phi_{N,N} \equiv p_N^2 - (p_N e_{N-1})^2 - F_N^2 \approx \Phi_{N,N} - F_{N,N} \approx 0. \tag{2.9}
\]
Thus, after Legendre transformation we obtain the following total Hamiltonian
\[
\mathcal{H}_T = \mathcal{H} + \lambda^{(s)} p_s + \sum_a \lambda^{(k)}_a p^a + \sum_{ij} \lambda^{(d)}_{ij} p^{ij}, \tag{2.10}
\]
where
\[
\mathcal{H} = s \phi_{0,1} + \sum a k_a \phi_{a,a+1} + \lambda \Phi_{N,N} + \sum_{i,j} d^{ij} u_{ij} + \sum_a \lambda_a \chi_{N,a} + \lambda_N \chi_{N,N}, \tag{2.11}
\]
\( \lambda^- \) are the Lagrange multipliers, and
\[
u_{ij} \equiv e_i e_j - \delta_{ij}, \quad \phi_{a,a+1} \equiv \phi_{a,a+1} - F_a, \quad \phi_{0,1} \equiv \phi_{0,1} + \sum_i k_i F_{i} - F, \tag{2.12}
\]
Stabilization of primary constraints \((2.6)\) produces the (secondary) first-stage constraints
\[
u_{ij} \approx 0; \quad s\phi_{0,1} + \sum_a k_a \phi_{a,a+1} \approx 0, \quad \Rightarrow \mathcal{H} \approx 0; (2.13)
\]
\[
s\phi_{a,a+1} = -F_{Na}(k_N - 2\lambda F_N); \quad (k_N - 2\lambda F_N)F_{NN} \approx 0. (2.14)
\]

Now, we can reduce the initial Hamiltonian system by the constraints \((2.6)\), and consider the system with the symplectic structure
\[
\omega_N = dp \wedge dx + \sum_{i=1}^N dp_i \wedge de_i (2.15)
\]
and the Hamiltonian \((2.11)\), where the expressions \((2.8)\) and \((2.13)\) define the primary constraints. The equations \((2.14)\) and \((2.9)\) either determine variables \(k_a, k_N\) as a function of \(\tilde{\phi}_{0,1}, \tilde{\phi}_{a,a+1}\), or define a primary constraints, at which the variables \(k_a, k_N\) represent Lagrange multipliers. The number of primary constraints, arising in that way, is equal to the corank of \(F_{ij}\).

Note that the functions \((1.2)\) form, with respect to \((2.13)\), a closed algebra, and obey the equations
\[
\{\tilde{\phi}_{...}, u_{...}\} \approx \{\tilde{\Phi}_{...}, u_{...}\} \approx 0.
\]
The constraints \(u_{N,N}, u_{N,a}\) and \(\chi_{N,a}, \chi_{N,N}\) are of the second-class,
\[
\{\chi_{N,i}, u_{jk}\} \approx \delta_{Nj} \delta_{ik}, \quad \Rightarrow \lambda_{Na} = \lambda_N = 0;
\]
while the constraints \(u_{N,N-1}, u_{a,b}\) are of the first-class, and their stabilization does not generate secondary constraints; rather, they generate trivial gauge transformations. Consequently, all the secondary constraints are the functions of \((1.2)\).

From this follows, that the dimension of the phase space of the system, \(D_{red}\) satisfy inequality
\[
(2D - 3N - 2)(N + 1) \leq D_{red} \leq (2D - N)(N + 1) - 2,
\]
where the upper limit corresponds to nondegenerate case, \(\det F_{ij} \neq 0\).

Since the gauge transformations of a system are defined by the primary first-class constraints \([4]\), we conclude, that the number of gauge degrees of freedom of the generalized rigid particles does not exceed \(\text{corank } F_{ij} + 1\). For instance, in a maximally nondegenerate case \(\det F_{ij} \neq 0\), the Lagrangian possesses only reparametrization invariance. The system possesses only primary constraints, and the dimension of the phase space of that system is \(D_{\text{max}} = (2D - N)(N + 1) - 2\).

**Example.** The simplest example of nondegenerate system is defined by the Lagrangian
\[
F = \frac{1}{2} \sum_{i=1}^N b_i \tilde{k}_i^2 + \sum_{i=1}^N c_i \tilde{k}_i + c_0, \quad b_1 \cdot b_2 \cdot \ldots \cdot b_N \neq 0.
\]

Solving the constraints \((2.9)\) and \((2.14)\), we find the expressions for curvatures,
\[
\tilde{k}_a = (\tilde{\phi}_{a,a+1} - c_a)/b_a, \quad (b_N \tilde{k}_N + c_N)^2 = \tilde{\Phi}_{N,N}, (2.16)
\]
and the Hamiltonian
\[ H = s\phi_{0.1} + d_{ij}u_{ij}, \quad \phi_{0.1} = \bar{\phi}_{0.1} + \frac{1}{2} \sum_i b_i \bar{k}_i^2 - c_0. \] (2.17)

The system possesses the following complete set of (primary) constraints
\[ \phi_{0.1} \approx 0, \quad u_{ij} \approx 0, \quad \chi_{N,N} \approx 0, \quad \chi_{N,a} \approx 0. \]

3 Lagrangians, Linear on Curvatures

Let consider the models with maximal set of primary constraints, i.e. when \( \text{rank} F_{ij} = 0 \). In this case the Lagrangians are linear functions of the external curvatures,
\[ F = c_0 + \sum_{i=1}^{N} c_i \bar{k}_i, \] (3.1)
and can be considered as a potential candidates on the role of the systems with maximal gauge degrees of freedom.

Such systems possess the following set of primary constraints
\[ \phi_{0.1} = p e_1 - c_0 \approx 0, \]
\[ \phi_{a,a+1} = p_a e_{a+1} - p_{a+1} e_a - c_a \approx 0, \]
\[ \Phi_{N,N} = p_N \hat{L} p_N - c_N^2 \approx 0, \]
\[ \chi_{N,N} = p_N e_N \approx 0, \quad \chi_{N,a} = p_N e_a \approx 0, \]
\[ u_{ij} = e_i e_j - \delta_{ij} \approx 0, \] (3.2)

and the Hamiltonian
\[ H = s\phi_{0.1} + \sum_{a=1}^{N-1} k_a \phi_{a,a+1} + \lambda \Phi_{N,N} + \sum_{i,j=1}^{N} d^{ij} u_{ij}. \] (3.3)

From the equations of motion for \( e_N \) we can see that \( 2c_N \lambda = k_N = s\bar{k}_N \), i.e. all the reparametrization invariants play the role of Lagrange multipliers.

Performing the Legendre transformation we have required the condition \( k_N \neq 0 \). So, stabilizing constraints we should suppose
\[ k_a \neq 0, \quad \lambda \neq 0. \]

Let impose the gauge conditions, fixing \( d_{N,N-1} \) and \( d_{a,b} \),
\[ \chi_{N,N-1} \equiv p_N e_{N-1} \approx 0, \quad \chi_{a,a-\kappa} \equiv p_a e_{a-\kappa} \approx 0, \quad \kappa = 0, \ldots, a - 1, \] (3.4)
which turns all the functions \( \bar{\Phi}_{i,j} \) to the quadratic form. These conditions, together with constraints (2.3), satisfy equations,
\[ \{ \chi_{i,j}, \Phi_{N,N} \} \approx 2\delta_{N,i} \delta_{ij} \bar{\Phi}_{N,N}; \quad \{ \chi_{i,j}, \bar{\phi}_{a,a+1} \} \approx \delta_{ij} (\delta_{i,a} - \delta_{i,a+1}) \bar{\phi}_{a,a+1}, \]
which lead to the following gauge fixing

\[ d_{i,j} = \delta_{ij}(k_i c_i - k_{i-1} c_{i-1} - s \delta_{1,i} c_0). \]  

(3.5)

Stabilization of the remaining primary constraints produce the following secondary first-stage constraints

\[ \tilde{\phi}_{0,2} \approx 0, \quad \tilde{\phi}_{0,a+2} \approx 0, \quad \Phi_{N,N-1} \approx 0. \]  

(3.6)

One can easily see from the expressions for the evolution of functions (1.2), that the further realization of the Dirac procedure essentially depends from the values of constants \( c_0, c_a \).

\[
\begin{align*}
\dot{\Phi}_{0,0} &= -4\lambda \Phi_{0,N} \phi_{0,N}, \\
\dot{\Phi}_{0,i} &= -k_{i-1} \phi_{0,i-1} + k_i \phi_{0,i+1} + 2\lambda \Phi_{N,0} \Phi_{0,N} \\
\dot{\phi}_{i,j} &= -s\delta_{i,j} \phi_{0,0} - k_{i-1} \phi_{0,i-1} + k_i \phi_{0,i+1} - 2\lambda \Phi_{N,0} \Phi_{0,N},
\end{align*}
\]

(3.7)

Particularly, if the Lagrangian (3.1) is conformal-invariant, i.e. \( c_0 = 0 \), then stabilization of \( \phi_{0,1} \approx 0 \) leads to the following set of the first-class constraints

\[ \tilde{\phi}_{0,i} \approx 0, \quad \Phi_{0,i} \approx pp_i \approx 0, \quad \Phi_{0,0} \approx p^2 \approx 0, \]  

(3.8)

which corresponds, in the pseudo-Euclidean space, to the massless case \[ 3. \]

3.1 \( F = c k_N \)

Consider the special case, when the Lagrangian is proportional to only one higher curvature, \( F = c k_N \), or, equivalently, \( c_0 = c_1 = ... = c_{N-1} = 0 \), \( c_N \equiv c \neq 0 \).

For this model, the Dirac procedure generates the maximally possible set of constraints, all of which are of the first-class,

\[ \tilde{\phi}_{0,i} \approx 0, \quad \Phi_{0,i} \approx 0, \quad \Phi_{0,0} \approx 0, \quad \Phi_{i,j} \approx 0, \quad \Phi_{i,j} - c^2 \delta_{ij} \approx 0. \]  

(3.9)

So, this system possesses \( (N + 1) \) degrees of gauge freedom. The dimension of its phase space is

\[ D_{min} = (2D - 3N - 2)(N + 1). \]  

(3.10)

3 The total momentum \( P \) and the rotation generators \( M^{(2)} \) of the system are defined by the expressions

\[ P^A = p^A, \quad M^{(2)AB} = g^{[A} e_{B]} + \sum_{i=1}^{N} \tilde{p}_i [A_B]. \]
Let us impose the gauge conditions (3.4) and introduce the complex coordinates
\[ z_i = (p_i + i c e_i)/\sqrt{2}, \quad \omega_2 = d p \wedge d x + \frac{i}{c} \sum_i d z_i \wedge d \bar{z}_i. \] (3.10)

Now the Hamiltonian of a system reads as
\[ \mathcal{H} = \frac{s}{2c} \left[ i \sqrt{2} p (\bar{z}_i - z_i) + i \sum_{a=1}^{N-1} \bar{k}_a (z_a \bar{z}_{a+1} - z_{a+1} \bar{z}_a) + \bar{k}_N (z_N \bar{z}_N - c^2) \right]. \] (3.11)

The constraints (3.8), (2.8), and gauge conditions (3.4), read as
\[ \Phi_{ij}^0 \equiv z_i \bar{z}_j - c^2 \delta_{ij} \approx 0, \quad \Phi_i^+ \equiv p z_i \approx 0, \quad \Phi_0 \equiv p^2 \approx 0, \quad U_{ij}^+ \equiv z_i z_j/2 \approx 0, \] (3.12)
and form the algebra
\[
\begin{align*}
\{ \Phi_{ij}, \Phi_{kj} \} = ic(\delta_{ik} \Phi_{kj} - \delta_{kj} \Phi_{ik}), & \quad \{ \Phi_{ij}, \Phi_{kij} \} = -ic\delta_{kj} \Phi_{ij}, \\
\{ \Phi_{ij}, U_{ki}^+ \} = -ic\left( \delta_{kj} U_{i+}^+ - \delta_{ik} U_{j+}^+ \right), & \quad \{ \Phi_i^+, U_{j+}^+ \} = ic\delta_{ij} \Phi_i^+ / 2, \quad \{ \Phi_i^+, \Phi_j^- \} = ic\delta_{ij} \Phi_0 \\
\{ \Phi, U_{ij}^- \} = \{ \Phi, \Phi_i^+ \} = \{ \Phi_i^+, \Phi_j^+ \} = \{ U_{ij}^+, U_{jk}^- \} = \{ U_{ij}^+, U_{kl}^+ \} = 0, & \\
\{ U_{ij}^+, U_{kl}^- \} = ic\delta_{(i,k} \Phi_{j)l} / 4 + ic^3 \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) / 2
\end{align*}
\]
where \( \Phi_{i^-} = \Phi_i^+, \quad U_{ij}^- = \bar{U}_{ij}^+ \).
So, \( U_{ij}^\pm \) are the second-class constraints, and the remaining ones are of the first-class.

From (3.9) one can see that if \( D \leq 4 \), the dynamics is nontrivial only at \( D = 4, N = 1 \), and the dimension of phase space of the system coincides with that of a (3+1)-dimensional massless particle [3]. In this space, it is possible to ”spinorize” the constraints (3.12) and to carry out covariant quantization of the system [10]. As it can be seen from (3.12), similar trick can be performed also for \( N > 1 \) in (5+1)-, (7+1)- and (9+1)-dimensional spaces, to resolve the part of second-class constraints.

However, it seems most interesting, that the constructed set of constraints coincides with the system of \( N + 1 \)-pointing discreet string [11], [12].

### 3.2 \( N=1, 2 \)

We have constructed above the Hamiltonian systems for generalized rigid particles, which have maximal and minimal possible (for given \( N \) and \( D \)) dimensions of phase spaces.

We have also mentioned, that, even in the case of Lagrangians, linear on curvatures, the presence of curvatures \( k_a \) essentially changes the structure of secondary constraints. Consequently, such systems have the phase spaces of ”intermediate” dimensions and less gauge symmetries.

Below we illustrate this phenomena on the examples of Lagrangians, linear on curvatures, in \( N = 1 \) and \( N = 2 \) cases.

Let us start from \( N = 1, c_0 \neq 0 \) case. There is only one secondary constraint and the condition on the Lagrange multipliers:
\[ \bar{\Phi}_{0,1} \approx 0, \quad s\Phi_{00} + k_1 c_1 c_0 \approx 0, \] (3.13)
Note, that \( \dot{\Phi}_{00} = 0 \), hence \( p^2 = c_0^2 - c_0c_1\dot{k}_1 = \text{const} \), i.e. the trajectory of the system has constant curvature.

In pseudo-Euclidean space the last equations corresponds to the conservation of mass on the given trajectory.

In complex coordinates \((3.10)\), where \( c \equiv c_1 \), the complete set of constraints can be represented by one real and two holomorphic constraints

\[
\Phi \equiv z\bar{z} - c_1^2 \approx 0, \quad U^+ \equiv z^2/2 \approx 0, \quad \Phi^+ \equiv p_0 c_1 / \sqrt{2} \approx 0, \quad (3.14)
\]

forming the algebra

\[
\{\Phi, \Phi^+\} = -ic_1 \Phi_+ + c_1^2 c_0 / \sqrt{2}, \quad \{\Phi, U^+\} = -2ic_1 U^+, \quad \{U^+, U^-\} = ic_1 \Phi + ic_1^3, \\
\{\Phi^+, \Phi^-\} = ic_1 p_2^2, \quad \{\Phi^+, U^-\} = ic_1 \Phi^- + c_0 c_1^2 / \sqrt{2}, \quad \{\Phi^+, U^+\} = 0
\]

where \( \Phi^- \equiv \bar{\Phi}^+ \), \( U^- \equiv \bar{U}^+ \).

So, for \( c_0 \neq 0, N = 1 \) the dimension of phase space is equal to \( D_{\text{red}} = 2(2D - 3) \).

The system possesses one gauge degree of freedom given by the Hamiltonian

\[
\mathcal{H}_{1, c_0 \neq 0} = \frac{s}{2c_1^2 c_0} \left[ 2ic_1 p_0 (z - \bar{z}) + (c_0^2 - p_2^2) (z\bar{z} - c_1^2) \right]. \quad (3.15)
\]

At \( c_0 = 0 \) (see Subsection 3.1) the dimension of system phase space equals to \( D_{\text{red}} = 2(2D - 5) \) and it has two gauge degrees of freedom.

Now let us consider the case \( N = 2 \) with arbitrary constants \( c_0, c_1, c_2 \).

When \( c_0 \neq 0 \), we have two secondary second-class constraints defined by expressions \((3.6)\). Their stabilization results in the conditions

\[
c_2 c_0 k_1 = k_2 \bar{\Phi}_{0,2}, \quad sc_0 c_2 \bar{\Phi}_{0,2} = k_2 \left( \Phi_{1,1} \bar{\Phi}_{0,2} - c_2^2 c_1 c_0 \right), \quad (3.16)
\]

where \( \Phi_{1,1} \equiv \bar{\Phi}_{1,1} - c_2^2 \), and \( \bar{\Phi}_{0,2} \neq 0 \).

At \( c_1 = 0 \) the second condition takes the form \( sc_0 c_1 + k_2 \Phi_{1,1} = 0 \).

The system possesses one gauge degree of freedom. The dimension of its phase space equals to \( D_{\text{red}} = 6(D - 2) \). Like to the case of \( N = 1 \), there is the motion constant, \( \bar{\Phi}_{0,0} \), which corresponds in the pseudo-Euclidean space, to the conservation of the mass on a given trajectory.

When \( c_0 = 0, c_1 \neq 0 \), the secondary constraints are defined by expressions \((3.7)\) and \( \bar{\Phi}_{1,2} \approx 0 \). There is the condition

\[
k_2 c_2 c_1 + k_1 \Phi_{1,1} = 0. \quad (3.17)
\]

Notice, that \( \Phi_{1,1} \) is the motion constant \( \Phi_{1,1} = 0 \), so \( k_2 / k_1 = \text{const} \).

This system possesses two gauge degrees of freedom. The dimension of its phase space is \( D_{\text{red}} = 2(3D - 10) \).

At \( c_0 = c_1 = 0 \) (see Subsection 3.1), the dimension of phase space of the system equals \( D_{\text{red}} = 6(D - 4) \), and there are three gauge degrees of freedom.
4 Conclusion.

We presented the Hamiltonian formulation for the models of generalized rigid particles, based on the use of the moving frame. This strongly simplify formulation of the system and its subsequent analyses. In particular, we found that the dimension of phase space of the system with the Lagrangian depending on the first $N$ external curvatures satisfy the inequality

$$ (2D - 3N - 2)(N + 1) \leq D_{\text{red}} \leq (2D - N)(N + 1) - 2, $$

where the upper limit is corresponds to the Lagrangian, quadratic on first $N$ curvatures, while the lower limit corresponds to the Lagrangian proportional to $N$-th curvature.

In the first case, the Lagrangian possess only reparametrization degree of freedom, while in the last case, it has $(N + 1)$ gauge degrees of freedom. Moreover, in the last case, in appropriate gauge fixing, the complete set of constraints and gauge-fixing conditions become quadratic, and coincides with the $N + 1$-particle discreet string [11], [12], which was quantized recently in BRST approach both for $N = 1$ [13] as well as for arbitrary $N$ [14]. We think, that this surprising parallel deserve to be studied separately.

In the case of Lagrangian with arbitrary linear dependence from curvatures, the set of primary constraints turns out to be quadratic too. However, the full set of secondary constraints is essentially depending by the constants $c_i$, although the algorithm of constructing the secondary constraints, and the generators of gauge symmetries, is the sequence of algebraic operations.

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