JACOBI STABILITY ANALYSIS OF THE CLASSICAL RESTRICTED THREE BODY PROBLEM

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Abstract. The circular restricted three body problem, which considers the dynamics of an infinitesimal particle in the presence of the gravitational interaction with two massive bodies moving on circular orbits about their common center of mass, is a very useful model for investigating the behavior of real astronomical objects in the Solar System. In such a system, there are five Lagrangian equilibrium points, and one important characteristic of the motion is the existence of linearly stable equilibria at the two equilibrium points that form equilateral triangles with the primaries, in the plane of the primaries’ orbit. We analyze the stability of motion in the restricted three body problem by using the concept of Jacobi stability, as introduced and developed in the Kosambi-Cartan-Chern (KCC) theory. The KCC theory is a differential geometric approach to the variational equations describing the deviation of the whole trajectory of a dynamical system with respect to the nearby ones. We obtain the general result that, from the point of view of the KCC theory and of Jacobi stability, all five Lagrangian equilibrium points of the restricted three body problem are unstable.

Key words: Celestial Mechanics – Three bodies – Jacobi stability.

1. INTRODUCTION

The three-body problem, the study of the dynamics of three bodies due to their gravitational interactions, is one of the fundamental problems in mathematical physics. One particular case of the general three body problem is the so-called restricted three body problem, in which the mass of one of the bodies can be neglected. Consider two massive bodies that jointly rotate around their common center of mass, with a constant angular velocity \( \omega \). If a third object of negligible mass is placed at rest with respect to a corotating frame of reference, then when viewed from the non-inertial coordinate system it will be initially under the influence of three forces: two due to gravitational interactions with the massive bodies in the system, and a third being the centrifugal force (see Murray (1994), Murray and Dermott (1999)). The small mass particle does not influence the dynamical motion of the two massive bodies, although they influence its motion. In 1772 Joseph Louis Lagrange has shown
that there are five special points in the rotational plane at which the gravitational and the centrifugal forces are in balance. These points are the so-called Lagrangian equilibrium points and are designated by $L_1$, $L_2$, $L_3$, $L_4$ and $L_5$, respectively. The first three points are located on a line passing through the massive bodies, while the remaining two are situated on either side of this line and equidistant from the massive bodies, in the orbital plane of the primaries. The behavior of a particle that is given a small deviation from the equilibrium position can be investigated generally by linearising the equations of motion, and carrying out a linear (Lyapounov) stability analysis. The linear stability analysis of the Lagrangian equilibrium points has been intensively investigated, and it has been shown that stable equilibrium occurs only at $L_4$ and $L_5$, for specific values of the mass parameters (see Murray (1994), Murray and Dermott (1999), Duboshin (1938)). A trapped particle would remain in the neighborhood of these points even if it is slightly perturbed. There is some observational evidence that these theoretical concepts have important astronomical implications. In the case of the Sun-Jupiter system there have been discovered small bodies accumulated near the $L_4$ and $L_5$ points of the system (Trojan asteroids). The two asteroid groups are observed from Earth to lie on either side of Jupiter and to share its synodic period (Van Houten, Van Houten-Groenewald and Gehrels, 1970). It was also speculated that dust may be located in the stable equilibrium points of the Earth-Moon system.

From a purely mathematical point of view, the restricted three body problem is described by a system of two, second order, nonlinear ordinary differential equations. A system of second order differential equations can be investigated by using differential geometric methods, and hence described in geometric terms, by using the general path-space theory of Kosambi-Cartan-Chern (KCC-theory), stimulated and influenced by the geometry of the Finsler spaces (Kosambi (1933), Cartan (1933), Chern (1939), Bucataru and Miron (2007)). The KCC theory describes and interprets the variational equations for the deviation of the whole trajectory with respect to nearby ones in geometric terms. After introducing a nonlinear connection, as well as a Berwald type connection, the differential system of equations can be geometrized, and five geometrical invariants can be constructed.

The second invariant describes the Jacobi stability properties of the dynamical nonlinear system (Sabau, 2005a,b). The KCC theory has been extensively applied for the study of various physical, biological, biochemical, or engineering systems (see Sabau (2005a), Sabau (2005b), Antonelli et al. (1993), Antonelli and Bucataru (2003), Yajima and Nagahama (2007) and references therein) and in astronomy and cosmology (see Harko and Sabau (2008), Boehmer et al. (2012), Abolghasem (2012), Abolghasem (2013), Danila et al. (2016), and Lake and Harko (2016), respectively).

It is the goal of this work to investigate the stability of the dynamical equations of motion in the restricted three body problem by using the Kosambi-Cartan-Chern
theory, and the associated concept of Jacobi stability. In particular, we consider the
Jacobi stability properties of the collinear Lagrangian points $L_1, L_2, L_3,$ as well as of
the triangular Lagrangian points $L_4$ and $L_5$, respectively. As a general result we show
that from the perspective of the KCC theory, all the Lagrange points are unstable with
respect to the perturbations of the orbits.

The present paper is organized as follows. We formulate the restricted three-
body problem in Section 2 where we also obtain the critical points of the system,
and discuss their linear stability properties. We briefly review the KCC theory and
the concept of Jacobi stability in Section 3. The Jacobi stability properties of the
critical points of the restricted three-body problem are analyzed in Section 4. We
concisely discuss and conclude our results in Section 5.

2. THE RESTRICTED THREE-BODY PROBLEM

Let us consider two celestial bodies with masses $m_1$ and $m_2$, respectively, with
$m_1 > m_2$. Let $\mu_1 = m_1/(m_1 + m_2)$ and $\mu_2 = m_2/(m_1 + m_2)$ denote the associated
reduced masses. A third object with mass $m_0 \ll m_1$ has negligible effect on the
motions of $m_1$ and $m_2$, which consequently simply rotates with uniform angular
velocity $\omega$ about their common center of mass. We now establish a coordinate system
whose origin coincides with the center of mass, and which spins so that its $x$ axis
always passes through $m_1$ and $m_2$. Let us measure distances in units of the fixed
separation of $m_1$ and $m_2$, and time in units of $\omega^{-1}$. In these units, the $x$ coordinate of $\mu_1$ is just $-\mu_2$; conversely, $\mu_2$ is located at $x = \mu_1$. When restricted to the $x-y$
plane, the equations of motion of $m_0$ are (see, for example, Murray (1994))

$$\ddot{x} - 2\dot{y} = x - \left[ \frac{\mu_1 (x + \mu_2)}{r_1^2} + \frac{\mu_2 (x - \mu_1)}{r_2^2} \right],$$

(1)

$$\ddot{y} + 2\dot{x} = y - \left[ \frac{\mu_1}{r_1^2} + \frac{\mu_2}{r_2^2} \right] y,$$

(2)

where

$$r_1^2 = (x + \mu_2)^2 + y^2,$$

(3)

$$r_2^2 = (x - \mu_1)^2 + y^2.$$  

(4)

By introducing the function

$$U(x, y) = \frac{\mu_1}{r_1^2} + \frac{\mu_2}{r_2^2} + \frac{1}{2} (x^2 + y^2),$$

(5)

we can represent Eqs. (1) and (2) as

$$\ddot{x} - 2\dot{y} = \frac{\partial U(x, y)}{\partial x}$$

(6)
\[
\ddot{y} + 2\dot{x} = \frac{\partial U(x, y)}{\partial y}
\]  
(7)

From Eqs. (6) and (7) we obtain the Jacobi first integral (see for example, Murray (1994))

\[
\dot{x}^2 + \dot{y}^2 = 2U - C_J,
\]  
(8)

where \(C_J\) is a constant of motion.

The position of the equilibrium points is given by the solution of the system of Eqs. (6) and (7) with \(\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = 0\). There are five different points, denoted by \((x_0, y_0)\), in which these conditions are fulfilled (see for example, Brouwer and Clemence (1961)): the Lagrangian collinear points \(L_1, L_2, L_3\), with \(y_0 = 0\), and \(L_4\) and \(L_5\), the triangular Lagrangian equilibrium points, for which \(r_1 = r_2 = 1\). The position of the triangular Lagrangian equilibrium points \(L_4\) and \(L_5\) is given by \(x_0 = 1/2 - \mu_2\) and \(y_0 = \pm\sqrt{3}/2\). For the Lagrangian collinear points y-coordinate is simple \(y_0 = 0\), but the explicit form of the \(x\)-coordinate of these points is rather lengthy (see, for example, Murray and Dermott (1999) for the explicit value for the equilibrium \(x\)-coordinate for these points and the conditions satisfied by \(r_1, r_2\) and their partial derivatives with respect to \(x\)).

2.1. LINEAR STABILITY ANALYSIS OF THE EQUILIBRIUM POINTS OF THE CLASSICAL PROBLEM

By introducing the new variables \((u, v)\) according to the definition \(\dot{x} = u, \dot{y} = v\), the system of equations Eqs. (1) - (2) can be written as a system of first order nonlinear differential equations as

\[
\dot{x} = u, \quad \dot{u} = 2v + \frac{\partial U(x, y)}{\partial x},
\]  
(9)

\[
\dot{y} = v, \quad \dot{v} = -2u + \frac{\partial U(x, y)}{\partial y}.
\]  
(10)

The Jacobian matrix of the above system is given by

\[
J = \begin{pmatrix}
0 & 1 & 0 & 0 \\
\frac{\partial^2 U}{\partial x^2} & 0 & \frac{\partial^2 U}{\partial x \partial y} & 2 \\
0 & 0 & 0 & 1 \\
\frac{\partial^2 U}{\partial x \partial y} & -2 & \frac{\partial^2 U}{\partial y^2} & 0
\end{pmatrix}.
\]  
(11)

The characteristic equation for the eigenvalues \(\lambda\) of the Jacobian is obtained as

\[
\lambda^4 + (4 - U_{xx} - U_{yy})\lambda^2 + U_{xx}U_{yy} - U_{xy}^2 = 0,
\]  
(12)

where a comma denotes the partial derivative with respect to the given independent variable. Equivalently, the characteristic equation can be written as

\[
\lambda^4 + A\lambda^2 - B = 0,
\]  
(13)
where we have denoted $A = (4 - U_{xx} - U_{yy})$ and $B = U_{xy}^2 - U_{xx}U_{yy}$, respectively.

The solution of the characteristic equation is

$$
\lambda^2 = \frac{U_{xx} + U_{yy} - 4 \pm \sqrt{(U_{xx} + U_{yy} - 4)^2 + 4(U_{xy}^2 - U_{xx}U_{yy})}}{2}.
$$

Then for the various derivatives we obtain

$$
U_{xx}(x, y) = 1 + \frac{\mu_1}{r_1^5} \left[\frac{2(x + \mu_2)^2 - y^2}{r_1^3} + \frac{2(x - \mu_1)^2 - y^2}{r_2^3}\right],
$$

$$
U_{yy}(x, y) = 1 - \frac{\mu_1}{r_1^5} \left[\frac{(x + \mu_2)^2 - 2y^2}{r_1^3} - \frac{(x - \mu_1)^2 - 2y^2}{r_2^3}\right],
$$

$$
U_{xy}(x, y) = \frac{3\mu_1 y (x + \mu_2)}{r_1^5} + \frac{3\mu_2 y (x - \mu_1)}{r_2^5}.
$$

For the collinear Lagrange points, using the fact that $y = 0$, we find

$$
U_{xx}(x, 0) = 1 + \frac{2\mu_1}{(x + \mu_2)^3} + \frac{2\mu_2}{(x - \mu_1)^3},
$$

$$
U_{yy}(x, 0) = 1 - \frac{\mu_1}{(x + \mu_2)^3} - \frac{\mu_2}{(x - \mu_1)^3},
$$

$$
U_{xy}(x, 0) = 0.
$$

It can be easily shown that $U_{xx}(x, 0) > 0$ and $U_{yy}(x, 0) < 0$ [Meyer and Hall, 1992]. Hence it follows that the constant term $B$ in Eq. (13) is positive, $B > 0$. Since $B > 0$, the solution of Eq. (13) is

$$
\lambda = -\frac{A}{2} \pm \sqrt{\frac{A^2}{4} + B},
$$

and hence $J$ has two real and two purely imaginary eigenvalues. Therefore, from Lyapunov’s indirect theorem [Swaters, 2000], it follows that the collinear equilibrium points are unstable saddle points.

Let’s consider now the stability of the equilateral Lagrangian points $L_4$ and $L_5$, for which $x_0 = 1/2 - \mu_2$, and $y_0 = \pm \sqrt{3/2}$. Then we obtain first for the eigenvalues of $\lambda$ the expressions $\lambda_{1,2} = \pm \sqrt{-1 + 27\mu_2/4}$ and $\lambda_{3,4} = \sqrt{-27\mu_2/4}$, respectively. This means that the eigenvalues for the triangular Lagrange points consist of two purely imaginary pairs, and therefore these points are linearly stable with respect to small perturbations. In fact it can be shown that $L_4$ and $L_5$ are linearly stable if the condition $\mu_1\mu_2 < 1/27$ is satisfied [Brouwer and Clemence, 1961].
3. BRIEF REVIEW OF KOSAMBI-CARTAN-CHERN (KCC) THEORY, AND OF JACOBI STABILITY

In this Section we summarize the fundamentals of the KCC-theory extensively used in the following Sections. A complete description of the theory can be found in Antonelli et al. (1993); Sabau (2005a); Antonelli and Bucataru (2003).

Let us consider a real, smooth $n$-dimensional manifold $M$, and let $T^*M$ denote its tangent bundle. On an open connected subset $\Omega$ of the Euclidian $(2n+1)$-dimensional space $R^n \times R^n \times R^1$ we introduce a $2n+1$ coordinates system $(x_i, y_i)$, where $y_i$ we have denoted $(y_1, y_2, ..., y_n)$.

Moreover, by $t$ we denote the time coordinate.

On $\Omega$ we consider now an arbitrary system of second order differential equations, given generally as

$$\frac{d^2x^i}{dt^2} + 2G^i(x^i, y^i, t) = 0, \quad i = 1, n,$$

where $G^i(x^i, y^i, t)$, $i = 1, n$ is a $C^\infty$ function in a neighborhood of the initial conditions $(x_0, y_0, t_0) \in \Omega$.

One can prove that the system (23) is generated by a vector field $S$, called semispray,

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x^i, y^i, t) \frac{\partial}{\partial y^i},$$

which introduces a non-linear connection $N^i_j$, defined as

$$N^i_j = \frac{\partial G^i}{\partial y^j}.$$  

For more details see, for example, Miron et al. (2001).

For a vector field $\xi(x)$ one can define the KCC-covariant differential of $\xi(x)$ on an open subset $\Omega \subseteq R^n \times R^n \times R^1$ according to,

$$\frac{D\xi}{dt} = \frac{d\xi}{dt} + N^i_j \xi^j.$$  

For rigorous introductions of the KCC covariant derivative see Antonelli et al. (1993); Antonelli and Bucataru (2003); Sabau (2005a,b).

Let $x^i(t)$ be a trajectory of the system (23), and $\tilde{x}^i(t)$ a small variation into nearby ones, defined by

$$\tilde{x}^i(t) = x^i(t) + \eta \xi^i(t),$$

with $|\eta| << 1$ a small parameter, and $\xi^i(t)$ representing the components of a contravariant vector field, defined along the trajectory $x^i(t)$. Replacing (27) into (23)
and taking the limit \( \eta \to 0 \), we get the variational equations describing the perturbations of the orbit (for more details see, for example [Antonelli et al. (1993); Antonelli and Bucataru (2003); Sabau (2005a,b)],
\[
\frac{d^2 \xi^i}{dt^2} + 2N^j\frac{d\xi^j}{dt} + 2\frac{\partial G^i}{\partial x^j}\xi^j = 0.
\] (28)

Eq. (28) could be written in the covariant form
\[
\frac{D^2 \xi^i}{dt^2} = P^i_j \xi^j,
\] (29)
using the KCC-covariant differential, where
\[
P^i_j = -2\frac{\partial G^i}{\partial x^j} - 2G^i_{jl} + y_j^i \frac{\partial N^j}{\partial x^l} + N^i_j \frac{\partial N^j}{\partial t} + N^i_l \frac{\partial N^j}{\partial x^l} + \frac{\partial N^i}{\partial t},
\] (30)
and \( G^i_{jl} \equiv \partial N^i_j / \partial y^l \) is the Berwald connection (see for example [Antonelli et al. (1993), Miron et al. (2001), Sabau (2005a), Sabau (2005b)]. We recall that Eq. (29) is the Jacobi equation, and \( P^i_j \) is called the second KCC-invariant, or, alternatively, the deviation curvature tensor.

At last we recall the notion of Jacobi stability of a dynamical system (see [Antonelli and Bucataru (2003); Sabau (2005a)]).

Definition. Let the system of ordinary differential equations (23) be given. We assume that with respect to the norm ||.||, which is induced by an inner product positive definite, the system satisfies the initial conditions ||\( x^i(0) - \bar{x}^i(0) \)|| = 0, and ||\( \dot{x}^i(0) - \dot{x}^i(0) \)|| \( \neq 0 \), respectively. If the real parts of the eigenvalues of the deviation tensor \( P^i_j \) are strictly negative everywhere, then the trajectories of the system (23) are called Jacobi stable. They are called Jacobi unstable, if this condition does not hold.

For a complete description of the KCC theory see [Antonelli and Bucataru (2003)].

4. JACOBI STABILITY ANALYSIS OF THE CLASSICAL RESTRICTED THREE BODY PROBLEM

By denoting \( x = x^1, y = x^2, \dot{x} = y^1 \), and \( \dot{y} = y^2 \), Eqs. (1) and (2) giving the equations of motion for the classical restricted three body problem can be written in a form similar to Eqs. (23) as
\[
\frac{d^2 x^i}{dt^2} + 2G^i(x^1, x^2, y^1, y^2) = 0, \forall i \in \{1, 2\},
\] (31)
where
\[
G^i(x^1, x^2, y^1, y^2) = \epsilon^i_j y^j - \frac{1}{2} \frac{\partial U(x^1, x^2)}{\partial x^j}, \forall i, j \in \{1, 2\},
\] (32)
with the permutation symbol \( \varepsilon^i_j \) defined as \( \varepsilon^1_2 = -\varepsilon^2_1 = -1 \), and \( \varepsilon^1_1 = \varepsilon^2_2 = 0 \). The nonlinear connection \( N^i_j \) associated to Eqs. (1) and (2) is given by
\[
N^i_j = \frac{\partial G^i}{\partial y^j}, \forall i, j \in \{1, 2\}.\tag{33}
\]
The Berwald connection can be obtained as
\[
G^{i}_{jl} = \frac{\partial N^i_j}{\partial y^l}, \forall i, j, l \in \{1, 2\}.\tag{34}
\]
The deviation tensor \( P^i_j \) is given by
\[
P^i_j = \partial^2 U + \varepsilon^i_l \varepsilon^l_j.\tag{35}
\]
Hence it follows that in the case of the classical restricted three body problem the deviation tensor takes the simple form given by Eq. (35). Moreover, all the other KCC invariants of the system are identically equal to zero. The characteristic equation of the deviation tensor is
\[
\lambda^2 - (P^1_1 + P^2_2)\lambda + P^1_1 P^2_2 - P^1_2 P^2_1 = 0\tag{36}
\]
and its eigenvalues are given by
\[
\lambda_{\pm} = \frac{1}{2} \left[ P^1_1 + P^2_2 \pm \sqrt{(P^1_1 - P^2_2)^2 + 4P^2_2 P^1_2} \right].\tag{37}
\]
The components of the deviation tensor can be explicitly written as
\[
P^1_1 = \frac{\partial^2 U}{\partial x^1 \partial x^1} - 1, P^2_1 = P^2_2 = \frac{\partial^2 U}{\partial x^1 \partial x^2}, P^2_1 = \frac{\partial^2 U}{\partial x^2 \partial x^2}.\tag{38}
\]
Thus, for the eigenvalues of the deviation tensor we find the explicit expression
\[
\lambda_{\pm} = \frac{1}{2} \left[ \Delta_U - 2 \pm \sqrt{\left(\Delta_U\right)^2 + 4 \left(\frac{\partial^2 U}{\partial x^1 \partial x^2}\right)^2} \right],\tag{39}
\]
where we have denoted
\[
\Delta_U = \frac{\partial^2 U}{\partial x^1 \partial x^1} + \frac{\partial^2 U}{\partial x^2 \partial x^2}, \Delta_U = \frac{\partial^2 U}{\partial x^1 \partial x^1} - \frac{\partial^2 U}{\partial x^2 \partial x^2}.\tag{40}
\]
By taking into account the explicit form of the potential \( U \), we obtain after a simple calculation
\[
\Delta_U - 2 = \frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3},\tag{41}
\]
\[
\Delta_U = \frac{3\mu_1 \left( x^1 - x^2 + \mu_2 \right) \left( x^1 + x^2 + \mu_2 \right)}{r_1^3} + \frac{3\mu_2 \left( x^1 + x^2 - \mu_1 \right) \left( x^1 - x^2 - \mu_1 \right)}{r_2^3}.\tag{42}
\]
and
\[
\frac{\partial^2 U}{\partial x^1 \partial x^2} = 3x^2 \left[ \frac{\mu_1 (x^1 + \mu_2)}{r_1^3} + \frac{\mu_2 (x^1 - \mu_1)}{r_2^3} \right],
\]
respectively. In the case of the \(L_4\) and \(L_5\) Lagrangian points we have \(r_1 = r_2 = 1\) and \(x^1 = 1/2 - \mu_2\) and \(x^2 = \pm \sqrt{3}/2\), respectively. Thus, we immediately obtain the eigenvalues of the deviation curvature tensor at these points as
\[
\lambda_{L_4,L_5}^\pm = \frac{1}{2} \pm \frac{3}{4} \sqrt{1 + 3 (1 - 2 \mu_2)^2}.
\]
(44)

In the case of the collinear Lagrangian points, since \(\frac{\partial^2 U}{\partial x^1 \partial x^2} \bigg|_{x^2=0} \equiv 0\), we obtain
\[
\lambda_{L_1,L_2,L_3}^\pm = \frac{1}{2} \left( \Delta_+ U - 2 \pm \Delta_- U \right) |_{x^2=0},
\]
and thus
\[
\lambda_{L_1,L_2,L_3}^+ = \frac{1}{2} \frac{\partial^2 U}{\partial x^1 \partial x^2} - 1 \bigg|_{x^2=0} = 2 \left[ \frac{\mu_1}{(x^1 + \mu_2)^3} + \frac{\mu_2}{(x^1 - \mu_1)^3} \right],
\]
\[
\lambda_{L_1,L_2,L_3}^- = \frac{1}{2} \frac{\partial^2 U}{\partial x^2 \partial x^2} - 1 \bigg|_{x^2=0} = -\frac{\mu_1}{(x^1 + \mu_2)^3} - \frac{\mu_2}{(x^1 - \mu_1)^3}.
\]
(47)

To establish the sign of the real part of the eigenvalues of characteristic equation (36), we use the sign of its discriminant and Viète’s formulas. The discriminant of (36) is the term under square root in (39). Taking into account the explicit form of the potential \(U\) from (5), after straightforward calculations, we get
\[
\Delta = \frac{9 \mu_1^2}{r_1^6} + \frac{9 \mu_2^2}{r_2^6} + \frac{18 \mu_1 \mu_2}{r_1^3 r_2^3} \left\{ [(x^1 + \mu_2)(x^1 - \mu_1) + (x^2)^2]^2 - (x^2)^2 \right\}.
\]
(48)

Using Viète’s formulas, we determine the sum and the product of the roots of the characteristic equation (36). The sum, \(S\), is
\[
S = P_1^1 + P_2^1,
\]
(49)
and the product, \(P\), is
\[
P = P_1^1 P_2^2 - P_1^2 P_2^1.
\]
(50)

Taking into account the form of the components of the deviation tensor (38) and the expression of the potential \(U\), given by (5), the sum and the product of the roots of the characteristic equation (36) are
\[
S = \frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}, \quad P = -2 \left( \frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3} \right)^2 + \frac{9 \mu_1 \mu_2}{r_1^3 r_2^3} (x^2)^2.
\]
(51)
In the $L_4$ and $L_5$ Lagrangian points, $x^1 = \frac{1}{2} - \mu_2$, $x^2 = \pm \frac{\sqrt{3}}{2}$ and $r_1 = r_2 = 1$. Replacing these values in (48) and (51) we get

$$\Delta = 9(3\mu_2^2 - 3\mu_2 + 1), \quad P = \frac{1}{4}(-27\mu_2^2 + 27\mu_2 - 8), \quad S = 1.$$  \hspace{1cm} (52)

It is easy to prove that $\Delta$ is always positive and $P$ is negative, for all real values of $\mu_2$. Therefore, the eigenvalues of the characteristic equation (56), in $L_4$ and $L_5$ Lagrangian points (44), are real numbers, with opposite signs. Based on the theory developed in Section 3, we conclude that the triangular Lagrangian equilibrium points of the restricted three body problem are Jacobi unstable.

In the case of the collinear Lagrangian points, $L_1$, $L_2$, $L_3$, the discriminant of the characteristic equation (56) reduces to a nonzero perfect square

$$\Delta = 9 \left( \frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3} \right)^2,$$  \hspace{1cm} (53)

meaning that the roots of the characteristic equation are always real numbers. The product of the eigenvalues is

$$P = -2 \left( \frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3} \right)^2,$$  \hspace{1cm} (54)

a negative, nonzero, real number. Therefore, we reach the conclusion that the collinear Lagrangian equilibrium points in the restricted three body problem are Jacobi unstable.

5. DISCUSSIONS AND FINAL REMARKS

As we have mentioned in introduction, the linear analysis stability of the Lagrangian equilibrium points was intensively investigated. The studies demonstrated that the collinear points $L_1$, $L_2$, $L_3$ are unstable for all $\mu_2 < 1/2$, but in special initial conditions, the third body could describe a stable, periodic orbit (Szebehely, 1967). For example, the spacecraft of SOHO mission describes a periodic orbit around the Lagrangian point $L_1$ of the Earth-Sun system (Domingo, Fleck and Poland, 1995). The Jacobi stability analysis revealed that the collinear Lagrangian equilibrium points of the restricted three body problem are unstable for all values of $\mu_2$.

In the case of the triangular Lagrangian equilibrium points $L_4$ and $L_5$, the stability was proven for $\mu_2 \leq \frac{27 - \sqrt{621}}{34} \approx 0.0385$ (Brouwer and Clemence, 1961), excepting few specific values, where there is a possibility for resonances to appear (Deprit and Deprit-Bartholome, 1972). Performing the Jacobi stability analysis we obtained that the triangular Lagrangian equilibrium points of the restricted three body problem are unstable for all values of $\mu_2$. 


The concepts and methods of the KCC theory are very well suited for the description of the geometric and stability properties of the dynamical systems. There is a fundamental difference between the classic linear stability analysis, and the Jacobi one. The Lyapunov, or linear, stability analysis is based on the linearization near the critical points of a dynamical system via the Jacobian matrix of the nonlinear system. On the other hand, the KCC theory considers the stability of an entire bunch of trajectories in a tubular region around the considered path (Sabau, 2005a).

The Jacobi analysis revealed that the Lagrangian equilibriums are unstable for all values of the mass parameter \( \mu_2 \). This instability can be interpreted in the sense that the trajectories of the particles in the restricted three body problem in the given coordinate system will scatter when approaching the initial point. The Jacobi stability of the trajectories of the restricted three body problem can also be regarded as characterizing the robustness of the system with respect to the small perturbations of the entire trajectory. We can also regard the Jacobi stability as describing the resistance of the entire trajectory to the emergence of the chaotic dynamics due to small perturbations of the system. In this context it is worth to point out again that all Lagrangian points of the restricted three body problem are Jacobi unstable.

It is also interesting to note that our results show a complex relation between the linear (Lyapunov) stability analysis of the Lagrangian critical points of the restricted three body problem, and the robustness of the whole system trajectories with respect to small perturbations, as described by the Jacobi stability via the deviation curvature tensor. If the Lyapunov stability analysis introduces regions of stability/instability, determined by the numerical values of the parameter \( \mu_2 \), the Jacobi instability is independent on the numerical values of \( \mu_1 \) and \( \mu_2 \).

In the present paper we have performed a full stability analysis of the circular restricted three body problem, in which we have considered a description of the deviations of the full bunch of trajectories of the complex nonlinear differential system describing the motion of the three gravitationally interacting particles. Moreover, we have developed some basic theoretical and computational tools for the in depth study of this type of stability, and its applications to astronomical problems. Further investigations of the Jacobi stability properties of the three body and other celestial mechanical problems may provide some powerful analytical methods for describing the stability and evolutionary properties of the gravitationally interacting many body systems, and help the better understanding of the dynamical characteristics of the particle motion.

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