A Drop in the Pond: The Effect of Rapid Mass Loss on the Dynamics and Interaction Rate of Collisionless Particles

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ABSTRACT

In symmetric gravitating systems experiencing rapid mass loss, particle orbits change almost instantaneously, which can lead to the development of a sharply contoured density profile, including singular caustics for collisionless systems. This framework can be used to model a variety of dynamical systems, such as accretion disks following a massive black hole merger and dwarf galaxies following violent early star formation feedback. Particle interactions in the high-density peaks seem a promising source of observable signatures of these mass loss events (i.e. a possible EM counterpart for black hole mergers or strong gamma-ray emission from dark matter annihilation around young galaxies), because the interaction rate depends on the square of the density. We study post-mass-loss density profiles, both analytic and numerical, in idealised cases and present arguments and methods to extend to any general system. An analytic derivation is presented for particles on Keplerian orbits responding to a drop in the central mass. We argue that this case, with initially circular orbits, gives the most sharply contoured profile possible. We find that despite the presence of a set of singular caustics, the total particle interaction rate is reduced compared to the unperturbed system; this is a result of the overall expansion of the system dominating over the steep caustics. Finally we argue that this result holds more generally, and the loss of central mass decreases the particle interaction rate in any physical system.

Key words: galaxies: kinematics and dynamics, dark matter, gamma-rays: galaxies

1 INTRODUCTION

There are a variety of astrophysical systems which experience mass loss on a time-scale much shorter than the dynamical time of the system, leading to a significant shift in the dynamics. One example of this phenomenon, highlighted in the recent literature, is the merger of a binary black hole (BH): the burst of gravitational waves during the last stage of the merger typically carries away a few percent of the binary’s rest-mass. This basic prediction of general relativity (GR) has been confirmed by LIGO observations of stellar-mass BH mergers, which show that a significant fraction of the BHs’ total mass is lost (Abbott et al. 2016, 2017).

Several studies have examined the impact that this mass-loss would imprint on a circumbinary disk, both in the context of super-massive (Schnittman & Krolik 2008; O’Neill et al. 2009; Megevand et al. 2009; Corrales et al. 2010; Rossi et al. 2010; Rosotti et al. 2012) and stellar-mass (de Mink & King 2017) BHs. The key result of these studies is that a sharply contoured density profile quickly emerges, with concentric rings of large under- and over-densities, including shocks. The origin of this morphology is simple: the disk gas, which is initially on circular orbits, instantaneously changes to eccentric orbits. Over time, the orbits at different radii shift out of phase, and in the particle limit, intersect and create caustics (see Lippai et al. 2008 and Shields & Bonning 2008 for a similar effect from BH recoil, demonstrated by test-particle orbits). The concentric density spikes and shocks found in hydrodynamical simulations correspond to these caustics (e.g. Corrales et al. 2010, and the other references above).

In a different context, and on much larger spatial scales, dwarf galaxies are believed to experience a similar rapid mass loss, when early periods of rapid star formation (and associated supernova feedback) blow out a large fraction of the gas from the nucleus. Crucially, this mass ejection also occurs on a time-scale shorter than the dynamical time. Governato et al. (2010) have shown that such rapid supernova feedback can transfer energy to the surrounding dark matter (DM). This model can be extended to repeated mass outflow and infall events (Pontzen & Governato 2012) to

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gradually move DM away from the center of the galaxy and turn a cuspy profile into a core. These simple models have been implemented in hydrodynamical simulations (e.g. Governato et al. 2012; Pontzen & Governato 2014; El-Zant et al. 2016), which confirm this basic result.

In the latter context, the focus has been on the overall expansion of the DM core. However in principle if the outflow is dominated by a single large event rather than repeated energy bursts this collisionless DM particle core could develop shells of overdensities and caustics, analogous to the those in the circumbinary disks. While self-gravity will reduce the effect, these systems are of particular interest for indirect DM detection. As suggested, e.g., in Lake (1990), they are excellent candidates for seeing γ-rays from DM annihilation, due to their abundance in the nearby universe, their high mass-to-light ratio, and their lack of other γ-ray sources. While a detection remains elusive, dwarf galaxies have been used to put strong limits on the mass and interaction cross section of DM particles (e.g. Abazajian et al. 2012; Geringer-Sameth et al. 2015; Gaskins 2016).

While the overall effect of rapid mass loss is a decrease in density of the DM core, the presence of strong density spikes could significantly boost the DM annihilation rate, even if these spikes contain only a small fraction of the mass (note that the annihilation rate is proportional to the square of the density). This would imply a larger γ-ray flux, and strengthen the existing limits on DM properties.

Motivated by the above, in this paper, we compute the density profiles of spherical, collisionless systems, following an instantaneous mass-loss at the center. Our models can include self-gravity, and directly resolve the caustic structures. We emphasize that our results are generic, and are applicable to any quasi-spherical collisionless system on any scale. Our result is negative and completely general: we find that mass-loss always decreases the net interaction rate.1

We start with one of the simplest cases possible: an initially circular orbit of a test particle around a point mass, which at some point instantly loses a fraction of its mass. We choose this case not just for its intuitive behaviour, but because it is relatively simple to extend it to more realistic situations, and it provides an upper limit on the interaction rate in response to rapid mass loss (as we argue in § 3 below).

2 CIRCULAR ORBITS IN A KEPLERIAN POTENTIAL

Particles initially on circular orbits, if they remain bound after the mass loss, will move on ellipses, so we start by recapitulating some basic results for orbits in a Keplerian potential. We will utilize the following results (derivations can be found in textbooks such as Binney & Tremaine 2008).

In a Keplerian potential around a point with mass $M$ at some radius $r$, the radius of a particle’s orbit follows

$$r(\phi) = \sqrt{\frac{a(1-e^2)}{1 + e \cos(\phi - \phi_0)}}$$

where $\phi$ is the phase, $\phi_0$ the initial phase, $a$ the semi-major axis and $e$ the eccentricity. The particle’s specific angular momentum and energy are

$$l = rv = r^2 \dot{\phi} = \sqrt{GMa(1-e^2)}$$

and

$$e = \frac{1}{2}(v^2 + \dot{v}^2) - \Phi(r),$$

both of which are conserved over an orbit. A less easily visualised, but useful parameterisation in terms of the eccentric anomaly $\eta$,

$$\sqrt{1 - e \tan \left(\frac{\phi - \phi_0}{2}\right)} = \sqrt{1 + e \tan \left(\frac{\eta}{2}\right)}$$

allows us to express the radius more simply as

$$r(\eta) = a(1 - e \cos \eta).$$

The expression for the time as a function of $\eta$ can be obtained by integrating $\dot{\phi}$ from equation 3 and using $r$ in the form of equation 6. This yields,

$$t(\eta) - t_0 = \sqrt{\frac{a^3}{GM}}(\eta - e \sin \eta),$$

where $t_0$ is the time of the first pericenter passage (i.e. if the particle is initially at some $\eta_0$, $t_0 = \sqrt{\frac{a^3}{GM}}[\eta_0 - e \sin \eta_0]$). An orbit has been completed when $\eta = 2\pi$, and hence the orbital period is easily confirmed from equation 7 to be

$$T = 2\pi \sqrt{\frac{a^3}{GM}}.$$
Rapid Mass Loss Dynamics

2.2 Response to mass loss

To obtain the evolution of a spherical system, we start with a single particle on a circular orbit, initially at some radius \( r_0 \). Figure 1 shows an illustration and our notation. In the circular case, \( e = 0 \) and hence \( a = r_0 \). (The corresponding solution for non-circular orbits is given in Appendix A.) When the central mass instantly drops from \( M \) to \( m < M \), the particle’s orbit is instantly changed. The particle is now less tightly bound, and has been given a boost in energy (i.e. a less negative gravitational potential) and will continue on an elliptical orbit. The angular momentum is unchanged, as we have given it no tangential impulse, and the position and velocity must be conserved over the instant of mass loss. An interactive demonstration can be found at http://user.astro.columbia.edu/~zpenoyre/causticsWeb.html (a still image of which is shown in Figure 2).

The new orbit must also be Keplerian, of the form in equation 6. Let the new eccentricity, semi-major axis and phase be \( e, a \) and \( \psi \) respectively, and let the moment of mass loss be \( t = 0 \). Since the velocity at \( t = 0 \) is purely tangential, the particle must be at its periapsis, and hence \( \psi_0, \eta_0 \) and \( t_0 \) must all be equal to 0. Conserving angular momentum throughout, and energy for \( t > 0 \), we can find the properties of the new orbit.

First let us define the dimensionless constant

\[
\mu = \frac{1}{2 - \frac{M}{m}}. \tag{9}
\]

We have \( r_{\text{min}} = r_0 \) and we find that the apoapsis is at

\[
r_{\text{max}} = \frac{r_0}{2 \frac{M}{m} - 1} = \frac{M}{m} r_0. \tag{10}
\]
where $\alpha = \mu r_0$ and $\epsilon = 1 - \frac{1}{\eta}$. Two consequences of these results are worth noting:

- The physical scale of an orbit depends linearly on the initial radius, and the eccentricity is constant for all orbits. This means the orbit of any two particles with different initial radii are similar, differing only in their period.
- The above solution breaks down for $m \leq \frac{M}{2}$; this corresponds to the particle becoming unbound and elliptical orbits no longer existing.

Thus for a single particle initially at $r_0$,

$$r(r_0, \eta) = \mu r_0 (1 - \epsilon \cos \eta)$$  \hspace{1cm} (11)

and

$$t(r_0, \eta) = \frac{3}{5} \sqrt{\frac{\mu^3}{Gm}} (\eta - \epsilon \sin \eta).$$ \hspace{1cm} (12)

While this equation does not directly yield $r$ as a function of $r_0$ and $t$, we can solve it to find $\eta = \eta(r_0, t)$ and hence find $r$ from equation 6.

The radius of a particle at some time $t$ depends only on the initial radius $r_0$. Hence a family of particles that start at a given $r_0$, regardless of orbital inclination, will always be at the same radius at any moment in time. This is illustrated in Figure 1 by the three particles on the dotted circular arc, which, while they are on different orbits, all coincide at the same radius. As a result, the radial motion of a spherical system, composed of individual particles, can instead be described as that of a series of concentric spherical shells (or cylindrical shells in 2D disks and other axisymmetric systems).

Henceforth we will refer not to individual particles but to spherical shells, with initial and current radii $r_0$ and $r$, which obey the above equations.

### 2.3 Recovering the density profile

With particles suddenly on a range of eccentric orbits, shells can pass through one another and overlap, leading to overdense regions where shells bunch up together and underdense regions where shells are widely spread. Our goal is to compute the time-evolution of the density profile (and use it to compute the particle interaction rate). The density at radius $r$ at time $t$ can be related to the initial density profile and shell positions, using the 1D Jacobian determinant

$$\rho(r, t) = \sum_{r_i(t_i, r_0)} \frac{dV_0}{dV} \rho_0(r_{0i}).$$ \hspace{1cm} (13)

Here the sum is over all individual shells $i$ that are currently at a radius $r$, but may have had different initial radii $r_{0i}$. A similar calculation was used in Schnittman & Krolik (2008) under the approximation of epicyclic orbits, whereas here we make no such approximations. Each shell contributes a density equal to its initial density $\rho_0(r_0)$, multiplied by its change in infinitesimal volume, $dV$. For each individual shell with $dV = 4\pi r^2 dr$, we have

$$\frac{dV_0}{dV} = \frac{r_i^2}{r^2} \frac{dr_0}{dr}.$$ \hspace{1cm} (14)

(Note that all analysis presented here can be easily modified to an axisymmetric system, by replacing densities with

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure3}
\caption{The radius $r$ at which a particle (or spherical shell) resides at time $t$, as a function of its initial radius $r_0$ at the moment $t_0 < t$ of mass loss (red curve). The pericenter of the elliptical orbit of each particle is $r_0$, and its apocenter is taken from equation 10 (black and blue lines, respectively). The outermost six turning points of the function are also marked by vertical black lines. Here we use an initial point mass of $10^3 M_\odot$, which drops by 10%. The particle positions are shown $t = 10$ Myr after the mass loss, although as discussed later the shape of the profile is self-similar and can be expressed by this curve at all times by rescaling using equation 18.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure4}
\caption{Upper panel: The same curve as Figure 3, but with the axes reversed. Lower panel: The corresponding density profile. The two plots have a shared horizontal axis. The profile is analytic though computationally limited to some numerical resolution (detailed in Appendix B) causing truly singular caustics to appear finite. The vertical lines show the $r$ values of the same turning points as shown in Figure 3. Again, the shape of both profiles does not change with time and all time evolution can be captured by rescaling the radial coordinate using equation 18.}
\end{figure}
surface densities and the volume element with \( \text{d}V = 2\pi r \text{d}r \text{d}\theta \).

Finding the density profile then amounts to identifying the set of shells that are at a particular radius \( r \) at a time \( t \).

To simplify operations involving equations 11 and 12, we can rearrange these to make use of the fact that we only want to recover density profiles at fixed values of \( t \). From equation 12, we find \( r_0 = r(t) \) as

\[
\rho(t) = \frac{1}{\mu} \left[ \frac{\text{GM}m^2}{(\eta - \epsilon \sin \eta)^2} \right]^{\frac{1}{2}} \tag{15}
\]

and substituting this into equation 11 we obtain

\[
r = \left[ \frac{\text{GM}m^2 (1 - \epsilon \cos \eta)^3}{(\eta - \epsilon \sin \eta)^2} \right]^{\frac{1}{2}}. \tag{16}
\]

Thus we have a parametric equation for \( r = r(t) \).\(^2\) This solution is shown in Figure 3, for a system of initially circular orbits around a point mass of \( 10^9 M_\odot, t = 10 \) Myrs after the moment of a drop in the central mass by 10%. The same parameters are used throughout the rest of this section.

To interpret the result shown in Figure 3 in an intuitive way, consider the period of each shell, \( T^2 \propto a^3 \propto r_0^3 \). At larger initial radii, the period becomes longer and longer. In the extreme case, there are some particles for whom \( t \ll T \) which have barely moved from their periapsis. Further in, we see particles for whom \( t = \frac{T}{2} \), just reaching apoapsis for the first time. In Figure 3, particles which started at roughly 200 parsecs from the central mass are just completing their first orbit, i.e. \( t = T \). This is the outermost minimum in this figure; each successive minimum, toward smaller radii, corresponds to those particles which have completed another full orbit at time \( t \).

Figure 3 also shows that there are various locations where multiple shells with different initial radii coincide at the same \( r \). To make this clearer, top panel of Figure 4 shows the same plot with the axes reversed, such that it becomes clearer to see the radii at which multiple shells overlap. The bottom panel of Figure 4 shows the corresponding density profile, which we explore next. (A more detailed discussion of how this profile is computed can be found in Appendix B.)

The density profile in Figure 3 has two distinct features, which can be understood from equation 13. First, large step-like over- and underdensities, the cause of which we have already identified as the overlap of shells from various initial radii, i.e. they stem from the summation in equation 13. Second, the sharp density spikes (caustics) arise when the derivative \( \frac{dr}{d\eta} \) in the \( \frac{d\rho}{d\eta} \) term goes to zero. From equations 15 and 16, this derivative can be written as

\[
\frac{dr}{d\eta} = \frac{1 - \epsilon \cos \eta + \frac{3}{2} \sin \eta}{1 - \epsilon \cos \eta}. \tag{17}
\]

In Figure 3, the vertical lines mark the points where \( \frac{dr}{d\eta} = 0 \). The same points are marked by vertical lines in the top panel of Figure 4; their locations clearly coincidence with the caustics in the bottom panel, where \( \rho(r) \propto \infty \). At these turning points we have a shell whose two edges are crossing itself, i.e. where one edge of the shell has passed through a turning point and meets the other, still to halt, travelling in the other direction. Hence all the mass contained within the volume element is now in enclosed in a volume that goes to 0, and the corresponding density is infinite. This can be seen clearly in Figure 4, where the turning points in \( r_0 \) vs \( r \) (which now, with the reversed axes, are vertical with a gradient going to infinity) are again highlighted with vertical bars that correspond perfectly to the caustics in the density profile.

Note that \( r \) and \( r_0 \) both scale with time as \( \propto t^{2/3} \). As a result, the solutions are self-similar, and depend only on \( \eta^2 \). The density profile, in particular, has a fixed shape – any features, such as a caustic at position \( r_c \), correspond to fixed values of \( \eta \) and hence obey

\[
r_c = \frac{g(\eta)}{(GMm^2)^{\frac{1}{2}}}, \tag{18}
\]

where \( g(\eta) \) is a constant function of \( \eta \) and is of order \( \eta^{-2/3} \). Features move outward at the “pattern speed” \( \omega_c \propto t^{-1/3} \). Each caustic, and indeed the whole profile, moves outward initially very fast and then slows at later times.

### 2.4 The nature of caustics

In the caustics shown in Figure 4, the density is formally singular, which suggests a potentially large (if not infinite) interaction rate \( \propto \rho^2 \) in such a system of particles. It can be shown, by expanding the expression for the density close to the caustics, that as \( |r - r_c|/r_c \to 0 \), the density approaches the singularity as

\[
\rho(r) \propto (r - r_c)^{-\frac{1}{2}} \tag{19}
\]

where \( r_c \) is the location of the caustic. This derivation is shown in Appendix C. To understand the profile from a finite distance from the caustics, we can directly compute the gradient of the density from equation 13,

\[
\frac{d\rho(r)}{dr} = \sum_{r_i(r_0) = r} \frac{\rho_i}{dr}, \tag{20}
\]

where

\[
\frac{\rho_i}{dr} = \frac{1}{\mu^3} \left[ \frac{d\rho_0}{dr} \rho_0 + \mu^2 \rho_i(r_0) \frac{d\rho}{d\eta} \eta \right] \left[ \frac{r_c^2}{r_0^2} \right] \frac{dr}{d\eta}, \tag{21}
\]

is the gradient in density for a single shell, with corresponding \( \eta_i \), at radius \( r \). With tedious differentiation which we will not reproduce here, this expression can be evaluated as a function of \( \eta_i \). For completeness, we have included a pseudo-mass-loss gradient \( \frac{d\rho}{dr} \) here, although we for simplicity, we use a flat profile \( \frac{d\rho}{dr} = 0 \) in our calculations. This is justified by the fact that, near the caustic, the right hand term is \( O\left(\frac{dr}{d\eta}\right)^2 \) and dominates over the \( \frac{d\rho}{dr} \approx O\left(\frac{dr}{d\eta}\right) \) term.

Figure 5 shows the profile and its gradient, and includes a zoomed-in view near the outermost caustics. Notice that as we zoom in and the numerical resolution increases, so does the height of the caustics – only numerical resolution keeps them from being truly singular. We also note that the outermost turning point in \( \frac{dr}{d\eta} \) in Figure 3 corresponds to the second-largest-radius density peak in Figure 5. (The order of the caustics differs between Figures 3 and 5: they

\[^2\text{Note that this solution is a generalization of the parametric solution for spherical collapse in cosmology – the latter corresponds to the limiting case of } \epsilon = 1, \text{ i.e. pure radial orbits, in eq. 16.}\]
appear in pairs with the larger $r$ corresponding to the smaller $\rho_c$.)

Let us assume a power-law density profile approaching a caustic from above,

$$\rho_c \propto (r - r_c)^{-n}, \quad (22)$$

where $r_c$ is the radius of the caustic and $n$ is some power greater than 0. Note that the sign of the term in brackets should be reversed for peaks that approach the singular point from below (which is true of every other peak). In either case, we can differentiate equation 22 to give

$$n = -\frac{d \ln \rho}{d \ln |r - r_c|} = \frac{|r_c - r| \, dp}{\rho \, dr}, \quad (23)$$

and comparing this to the calculated density gradient we can find the best-fit value of $n$ as the profile approaches the peak. This expression is true for caustics which approach the singularity either from above or below.

Figure 6 shows the value of this exponent near the peak. Notice that it tends to $n = \frac{1}{2}$ as it reaches the caustic (as expected from equation 19). Immediately away from the peak the profile becomes shallower.

2.5 The particle interaction rate

We now turn to the issue of whether this density profile, with sharp (and formally singular) post-mass-loss density spikes leads to a large boost in the particle interaction rate.

Assuming, for simplicity, a constant velocity dispersion and interaction cross section, the interaction rate is proportional to the integral

$$R \propto \int \rho^2 r^2 \, dr \quad (24)$$

(these assumptions are discussed further in § 4 below). We can calculate the contribution, $R_c$, to the total interaction rate from a thin radial shell stretching from some small distance from the caustic, $\epsilon$, to some macroscopic distance, $\Delta$,

$$R_c = \int_{r_c + \epsilon}^{r_c + \Delta} \rho^2 r^2 \, dr. \quad (25)$$

If the interaction rate over the caustic is finite then the value of the integral should converge as $\epsilon \to 0$.

Using the power-law form of the density near a caustic from equation 22 with $x = r - r_c$ (and swapping all appropriate signs for caustics which approach the singularity from below),

$$R_c = \int_{\epsilon}^{\Delta} k^2 x^{-2n} (r_c + x)^2 \, dx. \quad (26)$$

For small $x$, the integrand approaches $k^2 r_c^2 x^{-2n}$, so that the integral diverges for $n \geq \frac{1}{2}$ and $\epsilon \to 0$, implying an infinite
The contribution to the interaction rate due to the caustics $R_c$ (equation 26), compared to the rate over the whole system before mass loss $R_0$ (integrated to 500 pc, roughly the radius at which the perturbed density profile coincides with the unperturbed). The upper limit of integration, $\Delta$, is fixed at 5pc (and the qualitative result is independent of this value) and the lower limit is varied to demonstrate numerical convergence. We perform the integral on the analytic density profiles of the first six caustics (counting in descending $n_0$). Red curves show the result for caustics which approach the singularity from above, and blue for those which approach from below. Darker curves show the contribution from caustics at larger radii. The integral is performed using Gaussian quadrature and the same parameters are used as in Figure 3.

The total contribution to the interaction rate from the caustics $R_c$ (equation 26), compared to the rate over the whole system before mass loss $R_0$ (integrated to 500 pc, roughly the radius at which the perturbed density profile coincides with the unperturbed). The upper limit of integration, $\Delta$, is fixed at 5pc (and the qualitative result is independent of this value) and the lower limit is varied to demonstrate numerical convergence. We perform the integral on the analytic density profiles of the first six caustics (counting in descending $n_0$). Red curves show the result for caustics which approach the singularity from above, and blue for those which approach from below. Darker curves show the contribution from caustics at larger radii. The integral is performed using Gaussian quadrature and the same parameters are used as in Figure 3.

**Figure 7.** The contribution to the interaction rate due to the caustics $R_c$ (equation 26), compared to the rate over the whole system before mass loss $R_0$ (integrated to 500 pc, roughly the radius at which the perturbed density profile coincides with the unperturbed). The upper limit of integration, $\Delta$, is fixed at 5pc (and the qualitative result is independent of this value) and the lower limit is varied to demonstrate numerical convergence. We perform the integral on the analytic density profiles of the first six caustics (counting in descending $n_0$). Red curves show the result for caustics which approach the singularity from above, and blue for those which approach from below. Darker curves show the contribution from caustics at larger radii. The integral is performed using Gaussian quadrature and the same parameters are used as in Figure 3.

The total contribution to the interaction rate from the caustics, found by summing over the first 100 peaks, is $R_{\text{caustic}}/R_0 = 0.23$, where the contribution from the inner caustics quickly becomes vanishingly small. Thus the interaction rate of particles in caustics, while very large for the small area the reside in, does not lead to a net increase in the total interaction rate.

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The only justification for limiting the integration range as such would be if this was the outer extent of the system, e.g. if a uniform density disk had a sharp cutoff at a finite radius. Similarly if the integration range is taken to be very large the ratio will tend to 1, regardless of the mass-loss, as it will be dominated by mass at large radii which has barely deviated from its original orbit due to its long period.

The integration limits used here are chosen to capture the full region (minus the minor contribution at small radii, which we are limited from resolving numerically) in which the density deviates from its initial state.

As we have shown that despite the presence of a formal singularity, the caustics themselves will provide only a minor contribution only to the total interaction rate. (A similar argument holds for disks and other axisymmetric systems.) In fact the total interaction rate is decreased by rapid mass loss, in direct contribution to the expectation that these sharp density cusps may be an excellent laboratory for observing high interaction rates.

As the shape of the profile is time-independent this result will hold at all times, as long as the integration range is changed to encompass the whole profile.

As an aside, we note that for three-body processes, whose rate is $\propto \rho^2$, the rate near the caustics will diverge, and further study into cases where these are physically relevant may be fruitful.

**2.6 Summary**

In this section, we have presented an analytic derivation of the motions of shells of particles, initially on circular orbits, following an instantaneous mass loss. We then found the corresponding density profile numerically, and found the following properties:

- The profile can be broadly split into two components:
  (i) Step-like over- and underdensities corresponding to regions where multiple shells overlap at one radius
  (ii) Singular caustics at radii where the edges of a single shell cross and hence its volume goes to 0 and its density to $\infty$
- As $r \to 0$, there is an infinite sequence of caustics, coming in pairs and spaced closer together at the edges of the regions where multiple shells overlap
- At large radii, particles have long periods, and well beyond the radius where the orbital time is longer than the time elapsed since the mass loss, the density profile tends to the unperturbed profile
- For a given fractional mass loss, the density profile is...
 Figure 8. Upper panel: The density profile 6 Myrs after instantaneous mass loss, shown for a range of values of $m < M$ for a constant $M = 10^5 M_\odot$. From dark to light, $m/M$ goes from 0.98 to 0.66 in intervals of 0.02. The profile has been computed using the numerical CausticFrog package and sharp peaks of the caustics have been smoothed over 1 pc (more details in Section 3 and Appendix D). Lower panel: The total interaction rate, $R$, compared to the rate before mass loss, $R_0$ (calculated by numerically integrating the above smoothed profiles from 70 to 500 pc using equation 24). Three different smoothing lengths are used: 5 pc (light red), 1 pc (dark red) and $10^{-10}$ pc (black), to show that the results are very weakly dependent on smoothing length.

self-similar, with a shape that expands to larger radii as $r \propto r^2_{s}^{1/3}$.

- The interaction rate in the caustics is large given their small spatial extent. Though the density profile is singular the caustics are shallower than the curve $\rho \propto r^{-2}$ and hence the integral of $r^{2}$ over some small region is finite. However the total interaction rate in the caustics is still small compared to the interaction rate of the profile preceding mass loss.

- Integrating over the whole profile, the interaction rate is less than the unperturbed case, a result that is independent of time.

3 RESPONSE OF LESS IDEALISED SYSTEMS

In § 2, we chose the simplest physical system, consisting of massless particles initially on circular orbits in a Keplerian potential, so as to find a semi-analytic density profile. Here we discuss results from a range of more physical realisations. We show that each amendment leads to a flatter, less sharply contoured, profile than the circular orbit case.

Whenever density profiles are shown, they have been found with our new public 1D Lagrangian simulation code CausticFrog. By evolving the edges of a series of spherical shells, which are able to cross and overlap, we can easily resolve both shell crossing and squeezing (and hence resolve caustics) exploiting the spherical symmetries of the problem to save computation costs. This code can be found at https://github.com/zpenoyre/CausticFrog and is presented in detail in Appendix D.

Numerical discreteness noise, caused by the thousands of interacting shells, makes these profiles difficult to inspect visually and to integrate over numerically, so we smooth the profile. This is done by replacing each spherical shell, extending from radius $r_1$ to $r_2$ with a Gaussian centred at $\frac{1}{2}(r_1 + r_2)$ and with a width

$$\sigma = \sqrt{\frac{1}{4}(r_2 - r_1)^2 + r_s^2},$$

where $r_s$ is a smoothing length. The profile is normalised to conserve mass. Note that the realistic density profiles we consider below consist of convolving a set of discrete caustics with a smooth distribution. We thus expect the caustics structures to be physically smoothed, justifying this approach. Furthermore, in § 8 we show that the choice of smoothing length does not have a large impact on the interaction rate, and hence does not qualitatively affect the results.

3.1 Degree of mass loss

We have shown in Section 2 that the density profile resulting from a specific drop in a point mass potential (10% for all above analysis) leads to a drop in the total interaction rate of particles in the system. Now we extend this to any fractional mass drop.

Figure 8 shows the response of the Keplerian system with initially circular orbits to varying degrees of mass loss. The top panel shows the density profiles at the fixed time 6 Myr following a mass-loss of various degrees. Clearly, more significant mass loss leads to lower overall densities, as the particles are significantly less tightly bound to the smaller remaining point mass. Thus they have more eccentric orbits and move further outwards, giving a lower density. Smaller mass losses lead to flatter and less perturbed profiles.

The bottom panel in Figure 8 shows the ratio of the particle interaction rates 6 Myrs after ($R$) vs immediately before ($R_0$) the moment of mass loss. The interaction rate is reduced, regardless of the degree of mass loss. The larger the mass loss, and thus the more eccentric and larger the orbits of the particles, the lower the density and the lower the interaction rate. Note that the particles become unbound for mass losses of 50%, though the interaction rate will be non-zero for some period while the unbound mass moves outwards.

Hence the drop in interaction rate is true for any fractional mass loss.

3.2 Self-gravity

Depending on the system in question, self-gravity may be safe to ignore (e.g. the inner regions of accretion disks around black holes), or it may be the dominant source of gravitational potential (e.g. the dark matter profile away from the centre of a gas-poor dwarf galaxy).

Equation 18 shows that the speed at which the contours
of the density profile move outwards depends on the enclosed mass. In a self-gravitating system, as a feature moves outwards, the mass enclosed generally increases; hence, the speed increases and the profile spreads out. Equation 18 is no longer exact with the inclusion of self-gravity (and hence unclosed orbits), but in the case of a central point mass it is an increasingly good approximation as the initial density goes to 0.

We next explore the impact of self-gravity with CausticFrog by following a system of particles on initially circular orbits, as before, but including the self-gravity of each shell. We examine the profile for the mass losses and time periods as in the analytic case (see Fig. 3 for details).

The results of this exercise are shown in Figure 9, for initial profiles with increasing density (and hence contribution of self-gravity). The figure shows that denser systems have features that move outward faster, reaching larger radii at a given time, with more dispersed peaks. When the mass of gravitating fluid enclosed, $m_{\text{enc}}$, is of order of the mass of the central object (as is true for the lightest curve) the caustic is almost entirely dispersed. Even when $m_{\text{enc}} \sim 0.1m$ (darker curves) the difference between the point mass and the self-gravitating profiles starts to become apparent.

We conclude that self-gravity will generally disperse caustics and lead to smoother and flatter density profiles.

### 3.3 Non-circular orbits

For most systems, we would not expect the gravitating particles to be on circular orbits. In some cases, such as in a gas disk, viscous dissipation may circularise orbits, but in a dissipationless system such as a dark matter halo or a stellar bulge, we expect a wide distribution of orbital eccentricities.

In this section, we present numerical solutions for non-zero initial eccentricities using CausticFrog, but many of the results below are equally apparent from the analytic derivation presented in Appendix A. In particular, particles with different phases at the moment of mass loss will reach their turning points at different times. We expect that this should cause sharp features of the density profile near caustics to spread out.

Let us start with the simple case of orbits of a fixed initial eccentricity. We would expect the distribution of initial phases to correspond to $P(\phi) \propto \phi^{-1}$ (the probability of a particle being at some phase is inversely proportional to the rate of change of phase), and initialise the initial positions of particles along their elliptical orbits accordingly. We simulate just over a million such orbits.

Figure 10 shows the density profile for orbits with different initial eccentricities (and a full range of initial phases), between $0 \leq e \leq 0.4$. As we move to higher and higher eccentricities, the density profile near caustics becomes smoother.

Let us now consider the case of random initial eccentricities. We would expect the distribution of initial eccentricities to correspond to $P(e) \propto e^{-n}$, where $n$ controls the width of the distribution, from nearly circular (large $n$) to more eccentric (small $n$). Figure 11 shows the density profile for a range of initial eccentricities, with $e$ ranging from 0 to 0.4. As we move to higher eccentricities, the density profile near caustics becomes smoother.
directly affects another and over time they exchange energy. Mixing, as now the orbit of one particle (or spherical shell) steps, and hence to a continuous mass loss rate. To reducing a single mass loss to any number of distinct bits. Hence, the profile will be flatter than if the mass had strongly peaked features than a family of similar initial or-

§ entropy-distribution. Figure 11 shows density profiles for a system with a variety of initial orbits generally has less spikes, but for broader distributions, those peaks are much smaller. When a large fraction of high-eccentricity orbits are included, a significant amount of mass can again be lost, as particles become unbound.

We conclude that a system with a wider range of initial eccentricities will have smoother features, and a flatter overall density profile, following a mass loss event. For sufficiently large mass loss, there is a net reduction in mass as particles initially near their periapsis can easily become unbound (see Eq. A8).

3.4 Other assumptions and approximations

There are several additional complications that could change the response of a system to rapid mass loss. Here we briefly discuss a few of these complications qualitatively.

3.4.1 Time dependence

The basic premise of this system is that mass loss is almost instantaneous, i.e. occurs on a timescale that is much shorter than the particles’ orbital time. While instantly removing the mass makes our calculation much easier, allowing for the mass to decrease over a finite (if short) period will smooth out peaks and further flatten the density profile.

A simple way to picture this is to imagine the mass dropping not in a single event, but two curiously spaced events. In an initially circular case, the first event sets particles onto elliptical orbits. When the second event occurs, particles are on a range of orbits with different phases. We can think of this a second ’initial’ state, now with particles with different orbital properties at the same radius. As shown in § 3.3, a system with a variety of initial orbits generally has less strongly peaked features than a family of similar initial orbits. Hence, the profile will be flatter than if the mass had dropped in a single event. This argument could be extended to reducing a single mass loss to any number of distinct steps, and hence to a continuous mass loss rate.

The results shown above for a self gravitating fluid (§ 3.2) can also be understood as a time dependant phase mixing, as now the orbit of one particle (or spherical shell) directly affects another and over time they exchange energy. Thus the caustic, a region of high or even infinite phase density, diffuses and flattens over time.

3.4.2 Alternative potentials

We expect most physical potentials to have some degree of asphericity (Pontzen et al. 2015) which will break the spherical symmetry of our solutions. Relativistic effects may also be important; relativistic precession, for example, will also disrupt any simple dependence between an orbit and its period. Furthermore, as shown in § 3.2, the inclusion of self-gravity will also break down sharp features, and thus any self-consistent profile (such as the profile for dark matter halos suggested in (Navarro et al. 1997)) cannot maintain strong features.

3.4.3 Dissipation

We have so far assumed a collisionless system – but, depending on the context, there are several ways for the post-mass loss density waves to dissipate energy. We expect that such dissipation will spread the initially highly coherent waves, and the profile will flatten as a result. For baryonic matter, viscous dissipation due to turbulence and magnetic fields, or due to radiative processes, can all be effective at sapping energy from dense, fast-moving regions. Furthermore, shocks can develop as the overdensities move outwards, heating and transferring energy to the medium they move through. Finally, pressure forces generally smooth the perturbations caused by the mass loss (see e.g. the discussion in Corrales et al. 2010, and references therein).

3.4.4 Unbound mass

We have already seen that in systems with highly eccentric initial orbits, only small changes in the central mass are needed for some particles to become unbound. Any mass loss will of course lead to a lower density, and this will further flatten the profile.

3.5 Summary

We have discussed some effects that should be incorporated quantitatively into a more realistic picture of a dynamical system before and after a period of rapid mass loss. A general trend is clear: compared to the simplest idealised case presented in § 2, a more physical model develops a smoother and flatter density profile. This will generally reduce the particle interaction rate compared to the idealised case.

4 INTERACTION RATES - A GENERAL DISCUSSION

In the one case for which we have calculated the interaction rate (initially circular orbits in a Keplerian potential, § 2) we have shown that there will be a smaller interaction rate than before the mass loss event, as we have observed in § 3.1. Here present a more general heuristic argument: namely, if mass on average moves outwards (as is the case following mass loss), the interaction rate will generically decrease.

First let us more carefully justify our calculation of the particle interaction rate. The rate per unit volume, for a single fluid with a Maxwell-Boltzmann velocity distribution, is
\( \propto n^2 \sigma \sqrt{\langle v^2 \rangle} \) where \( n \) is the number density, \( \sigma \) the interaction cross section and \( \langle v^2 \rangle \) the velocity dispersion.

We will make the simplifying assumptions that (i) the cross section is constant throughout and (ii) the velocity dispersion is unchanged before and after mass loss. We have so far not specified the orientation of the initial orbits. Two limiting cases are isotropic initial velocities for randomly inclined orbits, or zero dispersion if all orbits are co-planar and in the same direction. In the isotropic case, the assumption of constant velocity dispersion before and after mass loss is reasonable, but for anisotropic initial velocity structures, this assumption may break down.

We note that as mass loss induces particles to move outward on average, their velocities are generally lower than the initial velocities. We have now introduced a radial velocity dispersion as shells moving radially, so the assumption that velocity dispersion is unchanged (or reduced) is equivalent to the assumption that the newly introduced radial dispersion is smaller than the original tangential velocity dispersion.

To characterise the change in the total interaction rate of a system, we define the “boost factor” as the ratio of the interaction rate at a given post-merger time to that before the moment of mass loss,

\[
B(t) = \frac{\int \rho_1^2(t) r^2 \, dr}{\int \rho_0^2 r^2 \, dr} \tag{31}
\]

where \( \rho_0 \) and \( \rho_1(t) \) are the density profiles before and after mass loss. For the total interaction rate of the system the integral should be evaluated out to the radius of the system and can be converted to a luminosity (for a given DM particle cross section) to compare to observations. We will assume here that we are interested only in the integrated interaction rate, because the system is unresolved. This is because we are dealing with small objects at extragalactic distances (AGN disks) and/or because the actual signal (e.g. gamma-rays from DM annihilations in dwarf galaxy cores) is spatially unresolved.

We will also assume that mass is conserved as the density profile is modified, i.e.

\[
M = \int 4\pi r^2 \, dr = \text{const.} \tag{32}
\]

This integral will also extend to the outer edge of the system. As we have seen in § 3, mass can become unbound and lost, but this will only reduce the densities and lead to smaller boost factors.

### 4.1 General transformations

#### 4.1.1 Change in volume

Stretching the initial profile (see top panel of Figure 12) such that the outer radius is some factor \( \alpha \) times its initial value yields the new density \( \rho_1 = (1 + \alpha)^{-3} \rho_0 \), and thus the boost\(^3\) \( B = (1 + \alpha)^{-3} \) i.e. \( B < 1 \). The bottom panel of Figure 12 illustrates another volume-expanding operation:

![Figure 12](image)

**Figure 12.** A flat density profile is stretched (top panel), and shifted (bottom panel), both whilst conserving mass. This leads to a drop in the interaction rate.

shifting an initially flat density profile to higher radii. The width of the profile, \( R \), is unchanged, and we use the dimensionless parameters \( \alpha = \frac{R}{R_0} \) and \( \beta = \frac{\delta}{R_0} \) to describe the transformation. Conserving mass leads to the density, and therefore the boost, dropping by a factor

\[
\frac{\rho_1}{\rho_0} = B = \frac{(1 + \alpha)^3 - 1}{(1 + \alpha + \beta)^3 - (1 + \beta)^3} \tag{33}
\]

For \( \beta \) greater than 0 (\( \alpha \) is always \( > 0 \)), this again leads to \( B < 1 \).

#### 4.1.2 Change in mass distribution

We also examine the effect of a more general transformation to the density profile: changing from one power law to another. Even if the profile has many features, if smoothed, or averaged over time, the profile will be well described by a simple power law. Let us assume an initial and final profiles

\[
\rho \propto \frac{1}{r^{3+\alpha}}
\]

and

\[
\rho' \propto \frac{1}{r'^{3+\beta}}
\]

where the moment of mass loss,

\[
\rho = \rho_0 (r/R_0)^{-\alpha}
\]

and

\[
\rho' = \frac{\rho_0 (r/R_0)^{-\alpha}}{(r/R_0 + \delta)^{-\alpha}}
\]

where \( \delta \) is the radial shift of the profile.

\[
\rho(r) \propto \left( \frac{r}{R_0} \right)^{-\alpha}
\]

\[
\rho'(r) \propto \left( \frac{r}{R_0 + \delta} \right)^{-\alpha}
\]

\[
\rho \propto \frac{1}{r^{3+\alpha}}
\]

and

\[
\rho' \propto \frac{1}{r^{3+\alpha}}
\]

where \( \rho \) is the number density, \( \sigma \) the interaction cross section and \( \langle v^2 \rangle \) the velocity dispersion.

\[
\frac{\rho_1}{\rho_0} = B = \frac{(1 + \alpha)^3 - 1}{(1 + \alpha + \beta)^3 - (1 + \beta)^3} \tag{33}
\]

For \( \beta \) greater than 0 (\( \alpha \) is always \( > 0 \)), this again leads to \( B < 1 \).
of the form
\[ \rho_0 = k r^\nu \quad \text{and} \quad \rho_1 = k r^{\nu_1}, \]
where both extend to the same outer radius, \( R \). Conserving mass gives a boost
\[ B = \left( \frac{3 + 2\nu}{3 + 2\nu_1} \right)^2. \]

Figure 13 shows how the boost varies with the initial and final power law. If both powers are of the same sign, the boost is less than unity if \( |\nu| < |\nu_1| \), i.e. if the resulting power law is shallower than the original, as we might expect for mass becoming less bound and moving outwards.

When the power law changes sign the behaviour is more complex, with the boost going to infinity as the power approaches \(-\frac{1}{2}\). If either power is negative it tends to dominate, unless the other is very large and positive.

An important feature to note is that the largest boosts are seen along the line \( \nu = 0 \), an argument that the case presented in \S 2, with an initially flat density profile, produces the largest boost (though more complex families of solutions with larger boosts may still exist).

Thus if mass moves outward and a density profile flattens, the boost decreases.

### 4.2 Combining a smooth profile with caustics

As we have seen in Figure 3, even when mass in general moves outwards, there can be small regions where the opposite happens: the mass is squeezed into a smaller volume, or the density profile steepens.

This of course is the cause of the caustics, as some finite mass is squeezed into an infinitesimal volume. We have shown numerically, for initially circular orbits around a point mass, in Section 2 (and extended it to more general situations in Section 3) that rapid mass loss does not lead to a boost in the interaction rate in a system.

In other words, the global phenomenon, of mass becoming less bound and moving outwards, dominates over the local phenomenon, of sharp density peaks developing in small regions.

Moving away from a flat initial density profile, as shown in Section 4.1.2, will further decrease the boost. Furthermore, switching to a self-gravitating system, discussed in Section 3.2, flattens out the caustics and reduces the interaction rate. Thus we can generalise this result for other astrophysical potential and density profiles.

### 5 CONCLUSIONS

First, let us briefly summarise the argument presented in this paper:

- A system which develops a large overdensity seems a strong candidate for observing large particle interaction rates (\( \propto r^2 \)).
- Rapid mass loss in a system leads to instantaneously changed orbits and the development of over- and underdensities as orbits cross and overlap.
- After mass loss in a Keplerian potential, the density profile of particles on initially circular orbits is a combination of infinite-density caustics and step-like over- and underdensities.
- The caustics in the circular Keplerian case contribute only a small amount to the interaction rate, significantly less than the total interaction rate before mass loss.
- Away from the caustics, the step-like profile leads to a drop in the interaction rate as mass moves outwards (as particles are less bound after mass loss) and the density drops.
- Overall, rapid mass loss in the circular Keplerian case leads to a smaller interaction rate than the unperturbed case.
- The inclusion of less idealized physical effects smooths and flattens the density profile relative to the circular-orbit Keplerian case. Mass still moves outward and thus the total interaction rate is reduced.
- Hence rapid mass loss will, in any physical case, lead to a drop in the interaction rate, rather than an increase.

Below, we elaborate upon how we arrived at this somewhat surprising conclusion.

In \S 2, we present an analytic derivation of the response of a system of particles, initially on circular orbits, in a Keplerian potential to an instantaneous drop in the central point mass. From this we numerically derive a density profile (Figures 4 and 5) that has a self-similar shape and expands outward with time as \( r \propto t^\frac{2}{3} \). The profile is comprised of step-like over- and underdensities where multiple shells on different orbits overlap, and singular caustics at the boundaries of the multi-shell regions, where a finite mass is squeezed into an infinitesimal volume.

These sharply peaked profiles with singular caustics naively appear promising for a large increase in the total interaction rate. However we show that the rate in this case is still less than in the unperturbed case. The caustics can be shown to contribute only a small amount to the interaction rate, and regardless of degree of mass loss or time (as the shape of the profile is time independent) the interaction rate decreases.

In \S 3, we show that various effects to make the system more realistic (such as self-gravity, non-circular initial orbits, and non-Keplerian potentials) smooth out the sharp density spikes and lead to flatter overall density profiles. Thus, the circular Keplerian case provides the profile that is most sharply peaked.

Thus we have shown that even the best possible candidate environment for observing large interaction rates following rapid mass still has a smaller net interaction rate than the same system before mass loss.

Similar to our results here, the optically thin Bremmsstrahlung luminosity, computed in post-merger binary black hole accretion disk simulations of O’Neill et al. (2009), Megevand et al. (2009) and Corrales et al. (2010) have been found to decrease after the mass-loss caused by the BH merger.\(^4\)

\(^4\) As explained in Corrales et al. (2010), this Bremsstrahlung luminosity is not self-consistent, as it yields an unphysically short cooling time. Nevertheless, this luminosity involves an integral of \( \rho^2 \) over volume, and its post-merger decrease can be traced to
Our results – the absence of a large boost in the particle interaction rate – also justify the simple density profiles used to calculate γ-ray flux from dark matter annihilation in dwarf galaxies (e.g. Geringer-Sameth et al. 2015, and references therein).

We emphasise that the arguments presented here are generalisable and thus applicable to any other system or geometry where we observe mass-loss over a period much shorter than the dynamical time of orbiting particles.

There are extreme cases where the interaction rate may increase, such as if three-body interactions are the main source of the signal, or where the step-like behaviour of the density function is precipitously steep. The large densities in the caustics may also lead to other observable phenomena, such as due to the heating of gas in an AGN disk, but we leave these considerations for future work.

But the overall conclusion of this work is that rapid mass loss in dynamical systems is not the promising laboratory for observing high interaction rates as one may have hoped for.

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APPENDIX A: NON-CIRCULAR ORBITS IN A KEPLERIAN POTENTIAL

In § 2 we derived the response of a system of particles on initially circular orbits to instantaneous mass loss.

The circular case is the simplest and most intuitive but far from the only analytic case. Here we derive the orbital parameters after mass loss for any initial Keplerian orbit.

The general results stated in § 2.1 are as useful here but we’ll also make use of a few more results.

The radial and tangential velocities can be found via

\[ v_r = \frac{\dot{l}}{r} \quad \text{and} \quad v_\theta = \frac{le \sin(\phi - \phi_0)}{a(1 - e^2)}. \]  

(A1)

By finding the velocities at peri- and apoapsis (where the radial velocity is 0) we find the Vis-Viva equation:

\[ v^2 = GM \left( \frac{2}{r} - \frac{1}{a} \right). \]  

(A2)
A1 Response to mass loss

Recalling equation 2 let us choose an initial configuration
\[
r_0 = r(t = 0) = \frac{\alpha(1 - e^2)}{1 + \epsilon \cos \phi}
\]  
(A3)

where we’ve set \( \phi_0 \) to be 0 (setting the orientation of our co-ordinate system such that the particle passes through periapsis at \( \phi = 0 \)) and \( \phi \) is the orbital phase at the moment of mass loss.

Now the initial state of the system is expressed in 3 parameters, \( r_0, e \) and \( \phi \), rather than a single parameter in the circular orbit case, \( r_0 \). (Note that using \( r_0 \) is more convenient than \( a \); however, both suffice, and the conversion is trivial.) The problem is still spherically symmetric, and we can still evolve the evolution of shells rather than individual particles, but now the shells correspond to particles with the same \( r_0, e \) and \( \phi \). We have moved from a 1 dimensional parameter space to 3.

The new orbit will be of the similar form,
\[
r = \frac{\alpha(1 - e^2)}{1 + \epsilon \cos(\psi - \phi_0)}
\]
where \( \alpha, \epsilon \) and \( \psi \) are the new semi-major axis, eccentricity and phase respectively.

Figure A1 shows an illustration of this for a single particle. The initial and final orbits will have different orientations, and this difference depends on the phase at the moment of mass loss. Momentarily at \( t = 0 \), \( \psi = \phi \), i.e. both phases have the same orientation; however, \( \phi_0 \) is not in general equal to 0.

The velocity the instant of the mass loss is unchanged, hence at \( t = 0 \) the Vis-Viva equation (equation A2) is satisfied for both the unperturbed and perturbed cases, with the same \( v^2 \). Equating the RHS of both and rearranging gives
\[
a = \frac{m}{M \left( \frac{1}{\epsilon} + \frac{\alpha}{\epsilon} \right)}.
\]
(A5)

The angular momentum of the orbit is constant throughout, and has the same form in both the unperturbed and perturbed case, i.e. equation 3 with the relevant mass, eccentricity and semi-major axis. Setting both expressions for \( l \) equal and rearranging gives
\[
\epsilon = \sqrt{1 - \frac{m a}{\alpha} (1 - e^2)}.
\]
(A6)

The two expressions for radius must match at \( t = 0 \), i.e. when \( \psi = \phi \), setting \( r(\psi = \phi) = r_0(\phi) \) gives
\[
\psi_0 = \phi - \cos^{-1}\left( \frac{1}{\epsilon} \left( \frac{M}{\alpha} \left(1 + \epsilon \cos(\phi)\right) \right) \right).
\]  
(A7)

There are two possible values of the arccos term, with a difference of \( \pi \), but the correct one can be chosen by ensuring \( \sin(\phi - \phi_0) \) and \( \sin(\phi) \) have the same sign, i.e. the direction of the radial velocity is consistent.

Thus for any combination of \( (r_0, e, \phi) \), we can find the parameters of the new orbit.

Setting \( e = 0 \), we can easily recover the relations for circular orbits given in § 2.2.

For the family of orbits with the same \( e \) and \( \phi \) we again find that the new eccentricity is a constant and the semi-major axis is linearly proportional to \( r_0 \). These orbits are again all similar, differing only in period, and any profile will maintain its shape and simply evolve in time as a re-scaling of the \( r \) co-ordinate (using \( \frac{r}{r_0} = \text{const.} \)).

Now the mass loss necessary for a particle to become unbound (\( e > 1 \)) depends on the initial phase and eccentricity. A particle will be unbound for
\[
\frac{m}{M} < 1 - \frac{r_0}{2a} \left( \frac{1}{\epsilon} - \frac{1 - e^2}{2(1 + \epsilon \cos(\phi))} \right).
\]
(A8)
The left hand expression is smallest for particles initially at periapsis, where \( \phi = 0 \), hence for particles of a range of initial phases at least some will be lost if \( \frac{m}{M} < \frac{1 - e^2}{2\epsilon} \). If there is also a distribution of initial eccentricities we may expect it to include particles up to \( e = 1 \) (but not including as these would be unbound) and hence for any finite central mass loss some particles must become unbound.

For particles of a given initial eccentricity and for a specific \( m < \frac{1 - e^2}{2\epsilon} \) all those with
\[
\phi < \cos^{-1}\left( \frac{1}{\epsilon} \left(1 - \frac{2}{1 - e^2} \left(1 - \frac{m}{M}\right) \right) \right)
\]
will be unbound. This means that it is the particles closest to periapsis that are easiest to lose from the system.

The radius, and the time since mass loss, can still be expressed simply via equations 6 and 7, although now \( t_0 \) will not in general be 0 (as in general particles do not start at periapsis). We can find \( t_0 \) using equation 5 using \( e = \epsilon \), \( \phi_0 = \psi_0 \) and the phase at the moment of mass loss, \( \phi \). Hence the same techniques used in § 2.3 (and further detailed in Appendix B) could be used to find the density profile.

To do this for shells with a continuous range of \( r_0, e, \phi \) would, however, require a root-finding in 3 dimensions and is significantly more computationally complex.

APPENDIX B: COMPUTING THE DENSITY PROFILE

In § 2.2 we derived a simple form for the variations in radius of a shell, initially on a circular orbit, that could be solved numerically for a given \( t \) and \( r_0 \). Here we show how to use
these equations, and variations thereof, to find the density profile via a 1D root finding of a well-behaved function.

A similar analysis could be used for the non-circular case, using the results from Appendix A, for a given $t, n_0, \epsilon$ and $\phi$. It is, however, substantially more convoluted and ultimately unnecessary within the scope of this paper, so we will only delve into the circular case.

To find which shells are currently at a given radius we can use equation 16, which we’ll rewrite as the radius of an individual shell,

$$r_i(t, \eta_i) = \left( \frac{G M t^2}{(\eta_i - \epsilon \sin \eta_i)^2} \right)^{\frac{1}{3}},$$

given its corresponding $\eta_i$.

This curve is bounded by

$$r_n(t, \eta) = \left( \frac{G M t^2 (1 \pm \epsilon)^3}{\eta^2} \right)^{\frac{1}{3}}. \tag{B2}$$

All three curves are shown in Figure B1.

The turning points of equation B1 are the same as those of equation 17 (from $\frac{dr}{d\eta} = \frac{dr}{d\eta_0} \frac{d\eta}{d\eta_0}$ and using $\frac{d\eta_0}{d\eta} > 0$ for all $\eta$).

Figure B2 shows $\frac{1}{\mu} \frac{dr}{d\eta}$ and its roots. These roots must be found numerically. This is relatively easy given that the curve has clear periodic behaviour and hence the $n^{th}$ root, $\eta_{r,n}$, must lie between $(2k + n)\pi$ and $(2k + n + 1)\pi$, where $k$ is an integer, dependent on $\epsilon$, which sets the offset of the first root.

For the case shown in Figure B2, the value of $k$ is clearly 1. For a larger mass loss, and hence a larger $\epsilon$, it is possible for $k$ to equal 0, and for smaller mass loss, the first root may be at much higher $\eta$. Given that $r$ is inversely proportional to $\eta$, physically this corresponds to the furthest caustic being at a larger radius. These roots are independent of time and hence need only be calculated once, and though there is an infinite number as $n$ increases, the $n^{th}$ root soon corresponds to vanishingly small radii.

Putting this in simpler terms, there are an infinite number of caustics going down to $r = 0$, and dependent on the fractional mass loss, $\frac{dM}{dt}$, the first caustic can correspond to different values of $\eta$. For minima in Figure B2 with $\frac{dr}{d\eta} > 0$ (if any exist), this corresponds to a smooth bump, rather than a singularity, in the density profile.

With the turning points and the bounds in hand, we can now find all $\eta_i$ for which $r_i = r$, i.e. all values of $\eta_i$ intersecting a horizontal line in Figure B1.

Rearranging equation B2 we can find, for a given $r$, the maximum and minimum possible values of $\eta_i$,

$$\eta_{\pm}(t, r) = \sqrt{\frac{G M t^2 r^3}{(1 \pm \epsilon)^3}}. \tag{B3}$$

Thus there is some $\eta_i$ for which $r_i(\eta_i) - r = 0$ for each interval from $\eta_{r,m}$ to $\eta_{r,m+1}$ where $m$ runs from the smallest $n$ such that $\eta_{r,n} > \eta_i$ (inclusive) to the largest $n$ such that $\eta_{r,n} < \eta_i$ (exclusive). It is computationally easy to find each of these roots independently. There may also be roots in the two immediately adjacent intervals, but this is dependent on $r$ and these must be checked independently.

With the full range of $\eta_i$ we can then, finally, enumerate equation 13 (using equations 11 and 17) and hence find the density at any given $r$.

As all the functions we have had to explore numerically are well-behaved, the corresponding density profile is truly analytically correct, up to the limits of numerical machine precision.

**APPENDIX C: PERTURBATION ANALYSIS OF CAUSTICS**

Here we reproduce the behaviour of the density profile as it approaches a caustic by looking at how the density profile changes with radius very close to the singularity. This intended as an analytic derivation of the numerical results from Figure 6, where we find that the profile approaches the singularity as an approximate power law with exponent of $-\frac{1}{2}$.

Taking a single shell with density $\rho_i$, we can expand the square of the density,

$$\rho_i^2 = \left( \frac{\rho_i}{\rho} \right)^2 \left( \frac{dr}{d\eta} \right)^2 \rho(\eta_0)^2. \tag{C1}$$

(here we use the square to save worrying about the absolute value). As $\frac{dr}{d\eta}$ is easiest to express in $\eta$, we will expand around $\eta_0$, the value of $\eta$ corresponding to the caustic with $\left( \frac{dr}{d\eta} \right)_{\eta_0} = 0$.

We evaluate equation C1 at some

$$\eta = \eta_0 + \Delta \eta. \tag{C2}$$

We can convert this to the variation around the initial radius of the shells corresponding to this caustic,

$$\Delta \rho_0 = \frac{dr_0}{d\eta} \Delta \eta + O(\Delta \eta^2) = \frac{2\eta_0(1 - \epsilon \cos \eta_0)}{3(\eta_0 - \epsilon \sin \eta_0)} \Delta \eta + O(\Delta \eta^2). \tag{C3}$$

Near the caustic we can Taylor expand $\frac{dr}{d\eta_0}$ to give

$$\left( \frac{dr}{d\eta_0} \right)_{\eta_0} \approx \left( \frac{dr}{d\eta_0} \right)_{\eta_0} + \left( \frac{d}{d\eta} \left( \frac{dr}{d\eta_0} \right) \right) \Delta \eta + O(\Delta \eta^2) \tag{C4}$$

but at the caustic the first term disappears and only the
second term is left, giving

\[
\left( \frac{dr}{dr_0} \right)_r = -\frac{1}{2} \left( \sin \eta \right) + 3 \left( \frac{\eta - \varepsilon \sin \eta}{1 - \varepsilon \cos \eta} \right) \Delta \eta + O(\Delta \eta^2).
\]

where this form is given only to show that the co-efficient of the \( \Delta \eta \) term is non-zero at the caustics.

As \( \frac{d}{dr_0} \) contains only terms linear in \( \Delta \eta \) or higher, and \( \frac{\eta}{r_0} \) does not go to zero at the caustics

\[
\left( \frac{r_0}{\eta} \right)^4 \left( \frac{d^2}{dr \, dr_0} \right)_r \propto \Delta \eta^{-2}(1 + O(\Delta \eta^2))^{-1}.
\]

We could expand the initial density using equation C3 but it is simple to show that any terms beyond the \( \rho_0(r_0) \) term are negligible.

Hence

\[
\rho_i^2 \propto \Delta \eta^{-2}(1 + O(\Delta \eta^2))^{-1}
\]

or

\[
\rho_i^2 \propto \Delta \eta^{-2}(1 + O(\Delta \eta^2))^{-1}.
\]

Finally we can translate this to variation in \( r, \Delta r \), where

\[
\Delta r = \left( \frac{dr}{dr_0} \right)_{r_c} \Delta r_0 + \left( \frac{d}{dr_0} \frac{dr}{dr} \right)_{r_c} \Delta \eta^2 + O(\Delta \eta^3)
\]

where again the first term on the right goes to zero (and it is simple to show the second term is non-zero at \( r_c \)). Thus

\[
\Delta r \propto \Delta \eta^2 + O(\Delta \eta^3).
\]

Putting this back in to equation C7 and taking the square root finally yields

\[
\rho_i \propto \Delta \eta^2
\]

to lowest order. As expected, this fits with Figure 6 as we approach the location of the caustic. This analysis also is completely general to any caustic, regardless of whether it approaches the singularity from above or below.

**APPENDIX D: THE CAUSTICFROG PACKAGE**

Many previous studies of systems undergoing rapid mass loss have modelled the evolution using test-particles, standard N-body or hydrodynamics codes (Lippai et al. 2008; Shields & Bonning 2008). However, these will fail to capture interesting features of these systems, such as the squeezing of finite mass into negligible volume which causes the caustics.

Instead, we present a new code, CausticFrog, designed specifically to resolve this behaviour, while also exploiting the symmetries of the system to simplify computation.

As shown in § 2, the system will always remain spherically (or, in the case of a disk, cylindrically) symmetric and we need not model the motion of individual particles, but can follow the simpler evolution of spherical shells. Our code is effectively Lagrangian, following the evolution of fixed mass shells as they move and stretch radially.

To simplify the terminology we’ll use throughout this section, a shell contains a fixed finite amount of mass, \( m_i \), and is bounded by two edges whose radii, \( r_{a,i} \) and \( r_{b,i} \), we evolve directly. While each shell has a fixed mass enclosed, many shells can overlap, leading to the density at that point being the sum of the densities of all those shells.

We use a leapfrog integration, where at each moment in time the mass enclosed by an edge is calculated as

\[
M_{\text{enc},i} = m_{\text{enc}}(r_i) + \sum_{r_{a,j},r_{b,j}<r_i} m_j \sum_{r_{a,j}<r_i<r_{b,j}} m_j \left( r_j^3 - r_{a,j}^3 \right) \left( r_{b,j}^3 - r_{a,j}^3 \right) + \sum_{r_{b,j}<r_i<r_{a,j}} m_j \left( r_j^3 - r_{a,j}^3 \right) \left( r_{b,j}^3 - r_{a,j}^3 \right)
\]

where \( m_{\text{enc}} \) includes any mass enclosed that is not part of the gravitating fluid (e.g. for the Keplerian potential this would be the central point mass).

The resulting acceleration on the shell therefore is

\[
\ddot{r}_i = \frac{1}{r_i^2} \left( l_i^2 - GM_{\text{enc},i} \right)
\]

where \( l_i \) is the specific angular momentum of the edge and is constant throughout (as there are no tangential impulses).

An edge with a given initial radius, eccentricity, and phase \( (r_{0,i}, \varepsilon_i, \phi_i) \) is initialised via equation 3 for the angular momentum \( l_i \) and equation A1 for the initial radial velocity \( v_{r,i,0} \).

Rather than evolve many separate shells we follow the evolution of a "accordion" of shells, where two consecutive shells share the same edge, i.e. \( r_{b,i} = r_{a,i+1} \). Each accordion has a single initial eccentricity and phase. Grouping shells in this way halves the computation time (as now there is effectively one unique edge per shell).

In this paper, we only show results from simulations in Keplerian potentials or variations thereof, but the code can accept any mass profile for the gravitating particles, and for any external mass before and after mass loss.

The code is written in Python and Cython, and can be found on GitHub at https://github.com/zpenoyre/
CausticFrog. There is an example IPython notebook showing how to initialise and run simulations.

As a simple code test, Figure D1 shows the density profile recovered for a Keplerian potential, compared to the analytic solution shown in § 2.

We encourage anyone who wishes to use the code to contact us so we can provide advice and assistance.

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