Binary Non-Tiles

Don Coppersmith    Victor S. Miller

September 2009

Abstract
A subset $V \subseteq \mathbb{F}_2^n$ is a tile if $\mathbb{F}_2^n$ can be covered by disjoint translates of $V$. In other words, $V$ is a tile if and only if there is a subset $A \subseteq \mathbb{F}_2^n$ such that $V + A = \mathbb{F}_2^n$ uniquely (i.e., $v + a = v' + a'$ implies that $v = v'$ and $a = a'$ where $v, v' \in V$ and $a, a' \in A$). In some problems in coding theory and hashing we are given a putative tile $V$, and wish to know whether or not it is a tile. In this paper we give two computational criteria for certifying that $V$ is not a tile. The first involves impossibility of a bin-packing problem, and the second involves infeasibility of a linear program. We apply both criteria to a list of putative tiles given by Gordon, Miller, and Ostapenko in the context of hashing to find close matches, to show that none of them are, in fact, tiles.

1 Tiles

We first define tiles, and make some observations about them. Discrete tiles arise in problems in perfect codes [3, 4], and hashing [5]. Many of their properties have been extensively analyzed in [2].

In [5] the authors give a table of ten putative tiles, which is reproduced in Table 1. In Section 9 of [2] the authors give several criteria for non-tiling. However, none of these apply to the putative tiles given in Table 1. Non-tilings have also been analyzed in [7] and [6].

Definition 1. Let $U$ be a finite-dimensional vector space over $\mathbb{F}_2$. A subset $V \subseteq U$ is a tile (of $U$), if $U$ can be written as the disjoint union of translates of $V$. In other words, there is a subset $A \subseteq U$ such that every element of $U$ can be written uniquely in the form $v + a$, where $v \in V$ and $a \in A$. This may be expressed as

$$V + A = U$$

(1)

$$(V + V) \cap (A + A) = \{0\},$$

(2)

where $X + Y := \{x + y : x \in X, y \in Y\}$. The equation (1) says that every element of $U$ can be written as a sum of an element of $V$ and an element of $A$. The equation (2) says that this representation is unique. Namely, $v + a = v' + a'$ if and only if $v + v' = a + a'$; thus uniqueness is equivalent to both sides being
0. Note that the definition is symmetric in $V$ and $A$, so that $A$ is also a tile, called a complement of $V$.

**Definition 2.** If $X \subseteq U$ denote by $\langle X \rangle$ the linear subspace generated by $X$ (i.e. the smallest linear subspace containing $X$).

Note that $V$ is a tile of $U$ if and only if $V$ is a tile of $\langle V \rangle$. Namely, let $W$ be a linear complement of $\langle V \rangle$ (i.e., $W$ is a linear subspace and $W + \langle V \rangle = U$ and $W \cap \langle V \rangle = \{0\}$), and $A$ be a complement of $V$ in $\langle V \rangle$, then $A + W$ is a complement of $V$. If $B$ is a complement of $V$ in $U$, then $B \cap \langle V \rangle$ is a complement of $V$ in $\langle V \rangle$. Thus we may assume, without loss of generality, that $\langle V \rangle = U$. We say that $V$ is a proper tile of $\langle V \rangle$.

If we also assume that the complementary tile $A$ is proper, we say that the tiling is full-rank.

Suppose that we are given a subset $V \subseteq \mathbb{F}_2^n$. We wish to know whether or not $V$ is a tile, and if it is, to find a complement $A$. In the next two sections we give two methods of showing that $V$ is not a tile, and apply them to the list of putative tiles in Table 1 (taken from [5] using a slightly different notation), to show that none of them are, in fact, tiles. In this table, instead of listing bit vectors, as in [5], we list sets of indices, in which the corresponding bit vector must have a 1. In the column labeled “generators of $V$” we give a list of generators. A set, $S$, is derivable from another set, $T$, if we can obtain it by one or more applications of either replacing an element with one of smaller index, or omitting it entirely. The set $V$ will consist of the generators and any set derivable from them. Each of the putative tiles has cardinality 64. The column labeled $k$ indicates that a putative complement $A$ has cardinality $2^k$.

### Table 1: Putative tiles

| $k$ | $n$ | Generators of $V$         |
|-----|-----|---------------------------|
| 6   | 12  | {11}, {10, 5}, {9, 8}    |
| 7   | 13  | {12}, {10, 4}, {9, 8}    |
| 8   | 14  | {13, 2}, {13, 1, 0}, {3, 2, 0} |
| 9   | 15  | {14, 1, 0}, {10, 2}      |
| 16  | 22  | {21, 1}                   |
| 17  | 23  | {22, 0}, {19, 1}         |
| 18  | 24  | {23, 0}, {17, 1}         |
| 19  | 25  | {24, 0}, {15, 1}         |
| 20  | 26  | {25, 0}, {13, 1}         |
| 21  | 27  | {26, 0}, {11, 1}         |

For example, the first line in Table 1 lists generators

\[ \{11\}, \{10, 5\}, \{9, 8\} \]

Then $V$ consists of the following 64 sets, each derivable from at least one of the generators:
2 Bin Packing

A straightforward combinatorial approach to showing non-tiling is via bin packing. This is the idea: Suppose that we’re given a linear projection \( \pi : U \to W \).

For every \( w \in W \) we have a “bin” \( \pi^{-1}(w) \). We’ll say that two vectors \( v, v' \in V \) are equivalent under \( \pi \) (written \( v \sim v' \)) if \( \pi(v) = \pi(v') \). That is, equivalent vectors are always in the same bin. Since \( \pi \) is linear, if \( v \sim v' \) then \( v + a \sim v' + a \).

Thus we can lump all the vectors in an equivalence class together, and call this aggregate a “piece”. In order for \( V \) to tile \( U \), it is necessary that we be able to pack together \(|U|/|V|\) copies of each piece so as to exactly fill up all the bins.

More formally, we have

**Proposition 1.** Let \( V \subseteq U \) where \( U \) is a finite dimensional vector space over \( \mathbb{F}_2 \), such that \( |V| \) is a power of 2. Let \( \pi : U \to W \) be a linear projection of vector spaces over \( \mathbb{F}_2 \), and \( c := |U|/|W| \) (the bin size). For each integer \( i, 0 \leq i \leq c \), let \( b_i \) be the number of equivalence classes of cardinality \( i \), of vectors in \( V \), where \( v \sim v' \) if and only if \( \pi(v) = \pi(v') \). Then a necessary condition for \( V \) to tile \( U \) is that there are non-negative integers \( d_{i,j} \) (which are the number of pieces of size \( i \) that go into bin \( j \)), such that

\[
\sum_i d_{i,j} = c, \text{ for all } j = 1, \ldots, |W| \tag{3}
\]

\[
\sum_j d_{i,j} = (|U|/|V|)b_i, \text{ for all } i = 0, \ldots, c. \tag{4}
\]

Although a general bin-packing problem is NP-complete, quite often we can show non-solvability by simple arguments.

This approach easily shows non-tiling for the last eight entries in Table 1. In Table 2 below we give the results of projecting each of the putative tiles in Table 1 onto coordinates \( r, \ldots, n - 1 \). The column labeled “piece census” specifies the multiset of piece sizes obtained. For example, in the row labeled \( k = 8 \), the entry 10*5, 1*6, 1*8 means that there are 10 pieces of size 5, 1 of size 6, and 1 of size 8. We work out this example in detail to show the idea:

There are ten pieces of size 5:

\( \{ \{m, 2\}, \{m, 1, 0\}, \{m, 1\}, \{m, 0\}, \{m\} \} \) for \( m = 4, \ldots, 13 \).

There is one piece of size 6:

\( \{\{3, 2, 0\}, \{3, 1, 0\}, \{3, 2\}, \{3, 1\}, \{3, 0\}, \{3\}\} \).
and one piece of size 8:

\{ \{2, 1, 0\}, \{2, 1\}, \{2, 0\}, \{2\}, \{1, 0\}, \{1\}, \{0\}, \{\}\}.

For the rows with \(k = 8\) and \(k = 9\), the bin size is 8, and the minimum piece size is \(\geq 4\). Placing a piece of size 5 leaves no way to fill up a bin of size 8. Similarly, in the rows for \(k = 16, \ldots, 21\), the bin size is 4, and there are no pieces of size 1. Placing a piece of size 3 leaves no way of filling up the bin in which it is placed.

Thus, none of the last eight rows in Table 1 can be a tile.

| \(k\) | \(r\) | bin size | piece census   |
|------|------|----------|---------------|
| 8    | 3    | 8        | 10*5, 1*6, 1*8 |
| 9    | 3    | 8        | 4*4, 8*5, 1*8  |
| 16   | 2    | 4        | 20*3, 1*4     |
| 17   | 2    | 4        | 3*2, 18*3, 1*4 |
| 18   | 2    | 4        | 6*2, 16*3, 1*4 |
| 19   | 2    | 4        | 9*2, 14*3, 1*4 |
| 20   | 2    | 4        | 12*2, 12*3, 1*4 |
| 21   | 2    | 4        | 15*2, 10*3, 1*4 |

## 3 Linear Programming

In this section we’ll rewrite the defining conditions for a tile in terms of a linear program. We’ll identify subsets \(S \subseteq U\) with their characteristic functions \(\chi_S : U \to \mathbb{R}:\) \(\chi_S(x) = 1\) if \(x \in S\) and 0 otherwise. Denote convolution of functions \(f, g : U \to \mathbb{R}\) by

\[
f \ast g(x) = \sum_{y \in U} f(y)g(x + y),
\]

and the Fourier transform

\[
\hat{f}(y) = \sum_{x \in U} (-1)^{x \cdot y} f(x).
\]

As is well known:

\[
\hat{f \ast g} = \hat{f} \ast \hat{g}.
\]

Note that if \(X, Y \subseteq U\) then

\[
\chi_X \ast \chi_Y(z) := |\{(x, y) : x \in X, y \in Y, x + y = z\}|.
\]
the number of ways of writing \( z \) as the sum of an element in \( X \) and an element of of \( Y \). Thus, we may express the condition for \( V \) to be a tile (with \( A \) as a complement) as
\[
\chi_V \ast \chi_A = 1, \tag{5}
\]
where 1 denotes the constant function with value 1. Although this is a necessary and sufficient condition (along with the condition that \( \chi_A(u) \in \{0, 1\} \)) it proves to be too weak to use as a linear programming criterion to certify non-tiling. We supplement it with the condition derived from (2):
\[
(\chi_V \ast \chi_V)(\chi_A \ast \chi_A) = |U|\delta, \tag{6}
\]
where \( \delta : U \to \mathbb{R} \) is the function \( \delta(0) = 1 \) and \( \delta(x) = 0 \) when \( x \neq 0 \).

Taking the Fourier transform of (5) yields
\[
\hat{\chi}_V \hat{\chi}_A = |U|\delta. \tag{7}
\]
This suggests a linear program in which we use the variables \( \chi_A \ast \chi_A(u) \) instead of \( \chi_A(u) \). We are given \( V \). Since \( |V||A| = 2^n \) we know \( |A| \) (if it exists). We have variables \( b_u \) for \( u \in \mathbb{F}_2^n \) and \( c_x \) for \( x \in \mathbb{F}_2^n \). We’ll want
\[
b_u = \chi_A \ast \chi_A(u),
\]
and
\[
c_x = |\hat{\chi}_A(x)|^2.
\]
We have the conditions
\[
0 \leq b_u \leq |A| \text{ and is an integer}, \tag{8}
\]
\[
0 \leq c_x \leq |A|^2 \text{ and is the square of an integer}, \tag{8a}
\]
\[
b_0 = |A| \tag{8b}
\]
\[
c_0 = |A|^2 \tag{8c}
\]
\[
b_u = 0 \text{ if } u \neq 0 \text{ and } \chi_V \ast \chi_V(u) \neq 0 \tag{8d}
\]
\[
c_x = 0 \text{ if } x \neq 0 \text{ and } \hat{\chi}_V(x) \neq 0 \tag{8e}
\]
\[
c_x = \sum_u (-1)^x \cdot b_u \text{ for all } x \tag{8f}
\]
If we drop the conditions about \( b_u \) being an integer and \( c_x \) being the square of an integer, we get a linear program which must be feasible if \( V \) is a tile.

One problem with this linear program is that it has a large number of nonzero coefficients. Just the condition that the \( c_x \) be the Fourier transform of the \( b_u \) yields \( 2^n(2^n - \max(|\text{supp}(\chi_V \ast \chi_V)||, |\text{supp}(\hat{\chi}_V)|)) \) nonzero coefficients (we can either write the \( c_x \) as the transform of the \( b_u \) or the \( b_u \) as the inverse transform of the \( c_x \), whichever yields a smaller system). We can immediately halve the number of nonzero coefficients by adding the equation
\[
\sum_u b_u = |A|^2
\]
to the remaining equations for the Fourier transform. However, the number
of nonzeros is still quite large. We can greatly reduce this by means of ideas
from the fast Fourier transform. We create new variables corresponding to the
intermediate results of the transform. The usual sort of bookkeeping now yields
3\(n^2\) nonzeros (since each “butterfly” involves 3 variables) and introduces \(n^2\) new variables.

Here are the details: Introduce variables \(t_{i,j}\) with \(0 \leq i \leq n, 0 \leq j < 2^n\),
with \(t_{0,j} = c_j\). For \(0 \leq i < n, 0 \leq k < 2^n\), and \(0 \leq j < 2^n-i-1\),
introduce the equations:

\[
\begin{align*}
t_{i+1,j+2^n-i-k} &= t_{i,j+2^n-i-k} + t_{i,j+2^n-i-k+2^n-i-1} \\
t_{i+1,j+2^n-i-k+2^n-i-1} &= t_{i,j+2^n-i-k} - t_{i,j+2^n-i-k+2^n-i-1}
\end{align*}
\]

The values \(t_{n,j}\) are the values of the Fourier transform of \(c_j\). We can also achieve
a significant savings in our problem by noting that whenever one of the variables
on the right hand sides of (9) is 0 (which is the case for a significant fraction of
the \(c_j\)), then we can “pass through” the remaining variable, or its negation, or a
0 if both are 0, and not create a new variable. We note the effect of this special
case in Table 3, by comparing \(n^2\) to the actual number of variables needed.

One nice feature of the approach using linear programming is that the conditions like full-rank for the complementary tile \(A\) can be described as linear
inequalities.

**Proposition 2.** A subset \(A \subseteq U\) containing 0 generates \(U\) as a linear subspace
if and only if

\[|\chi_A(x)| \leq |A| - 2,\]

for all \(0 \neq x \in U\).

**Proof.** The value \(\chi_A(x)\) is the sum of \(|A|\) terms each of which is \(\pm 1\). Thus
\(\chi_A(x) = |A|\) if and only if \(x \cdot a = 0\) for all \(a \in A\). This can happen if and only
if \(A\) does not generate \(U\). Similarly \(\chi_A(x) = -|A|\) if and only if \(x \cdot a = 1\) for all
\(a \in A\). This is impossible since 0 \(\in A\). Further note that \(\chi_A(x) \equiv |A| \mod 2\),
thus the value of \(|A| - 1\) is impossible for \(|\chi_A(x)|\). \(\square\)

Thus, if we use variables representing \(|\chi(A)|^2\) we can express full rank as
\(|\chi(A)|^2 \leq (|A| - 2)^2\).

For the first four of the ten putative tiles given in Table 1, the resulting linear
programming problem was small enough so that either glpsol \(^1\) or CPLEX \(^2\)
could handle it. The results of this approach applied to the first four cases in
Table 3 is given in Table 3.

The system of linear equations for \(k = 8\) were inconsistent. The linear
programming problems for the last six rows of the table were too large for
CPLEX to handle.

---

\(^1\) glpsol is the standalone solver contained in GLPK – the GNU Linear Programming Kit
http://www.gnu.org/software/glpk

6
### Table 3: Results from CPLEX

| $k$ | $n$ | $n^2$ | time in seconds | rows | variables | nonzeros |
|-----|-----|-------|----------------|------|-----------|----------|
| 6   | 12  | 49152 | 2.13           | 33569| 33414     | 99465    |
| 7   | 13  | 106496| 214.33         | 74349| 74710     | 221693   |
| 8   | 14  | 229376| 1.78           | 140312| 142632   | 419864   |
| 9   | 15  | 491520| 269.71         | 321016| 327828   | 961832   |

#### 3.1 Farkas’s Lemma

Another nice feature of the approach using linear programming is the use of the Farkas lemma, which says that the infeasibility of a linear programming problem can be easily exhibited by means of a vector found from solving the dual problem. Thus, whenever the approach using linear programming is able to certify that a set is not a tile, one can find a vector, by solving the dual program given below, which is a certificate of non-tiling.

**Lemma 1 (Farkas).** Let $G, A, H$ and $B$ be real matrices of dimensions $m \times n$, $s \times n$, $m \times r$ and $s \times r$, and $x, y, h$ and $b$ be real column vectors of dimensions $n, r, m$ and $s$. The system of of linear inequalities and equalities

\[
\begin{align*}
Gx + Hy &\geq h \\
Ax + By &= b
\end{align*}
\]

with $x \geq 0$ and $y$ unconstrained, is infeasible if and only if there exist row vectors $c \geq 0$ and $d$, of dimensions $m$ and $s$ respectively, such that

\[
\begin{align*}
    cG + dA &\leq 0 \\
    cH + dB &= 0 \\
    ch + db &= 1
\end{align*}
\]

The two vectors, $c$ and $d$ produced by solving the dual program (12) constitute a certificate of infeasibility, which is easily checked. Calculating, we find that if (10) is feasible then $c(Gx + Hy) + d(Ax + By) \geq ch + db = 1$. However, we also have $c(Gx + Hy) + d(Ax + By) = (cG + dA)x \leq 0$, which is a contradiction.

#### References

[1] ILOG CPLEX 10.1 User’s Manual. CPLEX Optimization, Inc., 2006.

[2] Gérard Cohen, Simon Litsyn, Alexander Vardy, and Gilles Zémor. Tilings of Binary spaces. *SIAM J. Discrete Math.*, 9:393–412, 1996.

[3] Tuvi Etzion and Alexander Vardy. Perfect binary codes, constructions, properties and enumeration. *IEEE Trans. Inf. Thy.*, 40:754–763, 1994.
[4] Tuvi Etzion and Alexander Vardy. On perfect codes and tilings: problems and solutions. *SIAM J. Discrete Math.*, 11:205–223, 1998.

[5] Daniel M. Gordon, Victor S. Miller, and Peter Ostapenko. Optimal hash functions for approximate closest pairs on the $n$-cube. arXiv:0806.3284v1[cs.IT], June 2008. Submitted to IEEE IT.

[6] Patric R. J. Östergård and Alexander Vardy. Resolving the existence of full-rank tiling of binary Hamming spaces. *SIAM J. Discrete Math.*, 18:382–387, 2004.

[7] Ari Trachtenberg and Alexander Vardy. Full-rank tiling of $\mathbb{F}_2^8$ do not exist. *SIAM J. Discrete Math.*, 16:390–392, 2003.
Binary Non-Tiles

Don Coppersmith         Victor S. Miller

August 2, 2011

Abstract

A subset \( V \subseteq \mathbb{F}^n_2 \) is a tile if \( \mathbb{F}^n_2 \) can be covered by disjoint translates of \( V \). In other words, \( V \) is a tile if and only if there is a subset \( A \subseteq \mathbb{F}^n_2 \) such that \( V + A = \mathbb{F}^n_2 \) uniquely (i.e., \( v + a = v' + a' \) implies that \( v = v' \) and \( a = a' \) where \( v, v' \in V \) and \( a, a' \in A \)). In some problems in coding theory and hashing we are given a putative tile \( V \), and wish to know whether or not it is a tile. In this paper we give two computational criteria for certifying that \( V \) is not a tile. The first involves impossibility of a bin-packing problem, and the second involves infeasibility of a linear program. We apply both criteria to a list of putative tiles given by Gordon, Miller, and Ostapenko in the context of hashing to find close matches, to show that none of them are, in fact, tiles.

1 Tiles

We first define tiles, and make some observations about them. Discrete tiles arise in problems in perfect codes [3, 4], and hashing [5]. Many of their properties have been extensively analyzed in [2].

In [5] the authors give a table of ten putative tiles, which is reproduced in Table 1. In Section 9 of [2] the authors give several criteria for non-tiling. However, none of these apply to the putative tiles given in Table 1. Non-tilings have also been analyzed in [8] and [7].

Definition 1. Let \( U \) be a finite-dimensional vector space over \( \mathbb{F}_2 \). A subset \( V \subseteq U \) (which we also refer to as a region) is a tile (of \( U \)), if \( U \) can be written as the disjoint union of translates of \( V \). In other words, there is a subset \( A \subseteq U \) such that every element of \( U \) can be written uniquely in the form \( v + a \), where \( v \in V \) and \( a \in A \).

Note that the definition is symmetric in \( V \) and \( A \), so that \( A \) is also a tile, called a complement of \( V \).

Lemma 1. A subset \( V \subseteq U \) of a vector space over \( \mathbb{F}_2 \) is a tile if and only if

\[
\begin{align*}
V + A & = U \\
(V + V) \cap (A + A) & = \{0\},
\end{align*}
\]

where \( X + Y := \{x + y : x \in X, y \in Y\} \).
Proof. The equation (1) says that every element of $U$ can be written as a sum of an element of $V$ and an element of $A$. The equation (2) says that this representation is unique. Since we’re working in characteristic 2, if $v, v' \in V, a, a' \in A$ then $v + a = v' + a'$ if and only if $v + v' = a + a'$. Uniqueness of the representation is equivalent to both sides being 0.

Definition 2. If $X \subseteq U$ denote by $\langle X \rangle$ the linear subspace generated by $X$ (i.e. the smallest linear subspace containing $X$).

Lemma 2. A subset $V \subseteq U$ is a tile of $U$ if and only if $V$ is a tile of $\langle V \rangle$.

Proof. Let $W$ be a linear complement of $\langle V \rangle$ (i.e., $W$ is a linear subspace and $W + \langle V \rangle = U$ and $W \cap \langle V \rangle = \{0\}$), and $A$ be a complement of $V$ in $\langle V \rangle$, then $A + W$ is a complement of $V$. If $B$ is a complement of $V$ in $U$, then $B \cap \langle V \rangle$ is a complement of $V$ in $\langle V \rangle$.

Thus we may assume, without loss of generality, that $\langle V \rangle = U$.

Definition 3. A subset $V \subseteq U$ is a proper tile of $U$ if $V$ is a tile of $U$ and $\langle V \rangle = U$.

If we also assume that the complementary tile $A$ is proper, we say that the tiling is full-rank.

Suppose that we are given a subset $V \subseteq \mathbb{F}_2^n$. We wish to know whether or not $V$ is a tile, and if it is, to find a complement $A$. In the next two sections we give two methods of showing that $V$ is not a tile, and apply them to the list of putative tiles in Table 1 (taken from [5] using a slightly different notation), to show that none of them are, in fact, tiles.

In the following we freely use the common practice of identifying a bit vector with the set of indices in which the vector has the value 1. The motivating examples we use here are from calculations performed in [5] which is concerned with finding hash functions which are optimal in a particular sense. One way of constructing an optimal hash function $H : \mathbb{F}_2^n \to \mathbb{F}_2^r$ would be to find a region $V \subseteq \mathbb{F}_2^n$ with $|V| = 2^{n-r}$ which is both a tile and optimal in the following sense:

Definition 4. If $S \subseteq \mathbb{F}_2^n$, and $p \in (0, 1)$, define

$$f_S(p) := \sum_{a,b \in S} (p/(1-p))^{\text{wt}(a+b)},$$

where $\text{wt}(x)$ is the Hamming weight of the bit vector $x$. Say that a region $S \subseteq \mathbb{F}_2^n$ is optimal at $p$ if

$$f_S(p) \geq f_{S'}(p)$$

for all $S' \subseteq \mathbb{F}_2^n$, with $|S'| = |S|$.

We say that $S$ is optimal if there is a $0 < p < 1$ such that $S$ is optimal at $p$.

Definition 5. A set system is a finite collection of finite subsets of the positive integers $\mathbb{N}$. 

2
We find it convenient to identify subsets $S \subseteq \mathbb{F}_2^n$ with the set system given by the non-zero coordinates of the vectors in $S$:

$$S(S) := \{ \{ i \in \mathbb{N} : v_i = 1 \} : v \in S \}.$$ 

So in terms of set systems we have polynomials

$$f_S(p) = \sum_{A,B \in S} (p/(1-p))^{ |A \Delta B|},$$

where $A \Delta B$ denotes the symmetric difference.

In order to state the theorem from [5] we define a partial order on subsets of $\mathbb{N}$:

**Definition 6.** We define a partial order on finite subsets of $\mathbb{N}$, called the right shifted partial order written $S \geq_R T$, as follows: For a finite subset $S \subseteq \mathbb{N}$, denote by $S_i$ the $i$-th largest element of $S$. Then we say that $S \geq_R T$ if $|S| \geq |T|$ and $S_i \geq T_i$ for $1 \leq i \leq |T|$.

**Definition 7.** We say that two set systems, $S$ and $T$ are isomorphic if there is a permutation $\pi$ of $\mathbb{N}$ which fixes all but finitely many elements, and a finite subset $A \subset \mathbb{N}$, such that $T = \{ A \Delta \pi(S) : S \in S \}$, where $\Delta$ denotes the symmetric difference.

Note that if $S$ is isomorphic to $T$ then $f_S(p) = f_T(p)$.

**Theorem 1** (Gordon, Miller, Ostapenko). An optimal region $S$ is isomorphic to an order ideal for the order $\geq_R$.

In the table below we give the putative tiles taken from [5]. These were found by using the above theorem to restrict the search to all order ideals of cardinality 64. Instead of giving them as collections of bit vectors we specify them as set systems. Each is an order ideal for the order $\geq_R$ which we specify by giving its set of maximal elements. Each of the putative tiles has cardinality 64. The column labeled $k$ indicates that a putative complement $A$ has cardinality $2^k$.

For example, the first line in Table 1 lists generators

$$\{11\}, \{10, 5\}, \{9, 8\}.$$ 

Then $V$ consists of the following 64 sets, each $\leq_R$ at least one of the generators:

- $\{m\}, \ 0 \leq m \leq 11 : \ 12$ sets
- $\{10, m\}, \ 0 \leq m \leq 5 : \ 6$ sets
- $\{n, m\}, \ 0 \leq m < n \leq 9 : \ 45$ sets
- $\emptyset : \ 1$ set
Table 1: Putative tiles

| $k$ | $n$ | generators of $V$                                      |
|-----|-----|------------------------------------------------------|
| 6   | 12  | \{11\}, \{10, 5\}, \{9, 8\}                         |
| 7   | 13  | \{12\}, \{10, 4\}, \{9, 8\}                         |
| 8   | 14  | \{13, 2\}, \{13, 1, 0\}, \{3, 2, 0\}               |
| 9   | 15  | \{14, 1, 0\}, \{10, 2\}                            |
| 16  | 22  | \{21, 1\}                                           |
| 17  | 23  | \{22, 0\}, \{19, 1\}                               |
| 18  | 24  | \{23, 0\}, \{17, 1\}                               |
| 19  | 25  | \{24, 0\}, \{15, 1\}                               |
| 20  | 26  | \{25, 0\}, \{13, 1\}                               |
| 21  | 27  | \{26, 0\}, \{11, 1\}                               |

2 Bin Packing

A straightforward combinatorial approach to showing non-tiling is via bin packing. This is the idea: We are given a subset $V \subseteq U$ of an $\mathbb{F}_2$ vector space and we wish to prove that it cannot be a tile. We find an auxiliary linear projection $\pi: U \rightarrow W$, where $W$ is another $\mathbb{F}_2$ vector space. The projection $\pi$ partitions the elements of $U$ into equivalence classes: $v \sim v'$ if and only if $\pi(v) = \pi(v')$. We say that a subset of $U$ of the form

$$P_{a,w} := \{v + a : v \in V, \pi(v + a) = w\},$$

where $a \in U, w \in W$ is a piece, and that a subset of the form $\pi^{-1}(w)$ is a bin. Note that $P_{a,w} = P_{0,w + \pi(a)} + a$. We then have

**Lemma 3.** Let $V \subset U$ be a tile with complement $A \subset U$, and $\pi: U \rightarrow W$ a linear projection of vector spaces. Then for all $w \in W$ the collection $\{P_{a,w} : a \in A, P_{a,w} \neq \emptyset\}$ is a partition of the bin $\pi^{-1}(w)$.

**Proof.** If $P_{a,w} \cap P_{a',w} \neq \emptyset$ then there are $v, v' \in V$ such that $a + v = a' + v'$. By definition of a tile this implies that $a = a', v = v'$. We also have $\cup_{a \in A} P_{a,w} = \pi^{-1}(w)$ by surjectivity of $\pi$. $\square$

This approach easily shows non-tiling for the last eight entries in Table 1 by using arguments about the cardinalities of the pieces $P_{a,w}$. Note that since $P_{a,w} = a + P_{0,w + \pi(a)}$ that the possible cardinalities of $P_{a,w}$ are the same as those for $P_{0,w}$. In addition, by the above lemma, every one of the pieces $P_{0,w} \neq \emptyset$ must be used in the partition of some $\pi^{-1}(w)$. In the arguments below we list the multiset of cardinalities of the non-empty $P_{0,w}$ for all $w$. 
In Table 2 below we give the results of projecting each of the putative tiles in Table 1 onto coordinates $r, \ldots, n - 1$. The column labeled “piece census” specifies the multiset of piece sizes obtained. For example, in the row labeled $k = 8$, the entry $10*5, 4*4, 8*5, 1*8$ means that there are 10 pieces of size 5, 4 of size 6, 8 of size 7, and 1 of size 8. We work out this example in detail to show the idea:

There are ten pieces of size 5:

$$\{\{m, 2\}, \{m, 1, 0\}, \{m, 1\}, \{m, 0\}, \{m\}\} \text{ for } m = 4, \ldots, 13.$$

There is one piece of size 6:

$$\{\{3, 2, 0\}, \{3, 1, 0\}, \{3, 2\}, \{3, 1, 1\}, \{3, 1\}, \{3\}\},$$

and one piece of size 8:

$$\{\{2, 1, 1\}, \{2, 1\}, \{2, 0\}, \{2\}, \{1, 0\}, \{1\}, \{0\}, \{\}\}.$$  

For the rows with $k = 8$ and $k = 9$, the bin size is 8, and the minimum piece size is $\geq 4$. Placing a piece of size 5 leaves no way to fill up a bin of size 8. Similarly, in the rows for $k = 16, \ldots, 21$, the bin size is 4, and there are no pieces of size 1. Placing a piece of size 3 leaves no way of filling up the bin in which it is placed.

Thus, none of the last eight rows in Table 1 can be a tile.

We have not been able to find a projection to make this argument work for the first two rows of the table.

| $k$ | $r$ | bin size | piece census          |
|-----|-----|----------|-----------------------|
| 8   | 3   | 8        | $10*5, 4*4, 8*5, 1*8$ |
| 9   | 3   | 8        | $10*5, 4*4, 8*5, 1*8$ |
| 16  | 2   | 4        | $20*3, 1*4$           |
| 17  | 2   | 4        | $3*2, 18*3, 1*4$      |
| 18  | 2   | 4        | $6*2, 16*3, 1*4$      |
| 19  | 2   | 4        | $9*2, 14*3, 1*4$      |
| 20  | 2   | 4        | $12*2, 12*3, 1*4$     |
| 21  | 2   | 4        | $15*2, 10*3, 1*4$     |

## 3 Linear Programming

In this section we’ll rewrite the defining conditions for a tile in terms of a linear program. We’ll identify subsets $S \subseteq U$ with their characteristic functions $\chi_S : U \to \mathbb{R}$: $\chi_S(x) = 1$ if $x \in S$ and 0 otherwise. Denote convolution of
functions $f, g : U \to \mathbb{R}$ by

$$f \ast g(x) = \sum_{y \in U} f(y)g(x + y),$$

and the Fourier transform

$$\hat{f}(y) = \sum_{x \in U} (-1)^{x \cdot y}f(x).$$

As is well known:

$$\hat{f} \ast \hat{g} = \hat{f} \hat{g}.$$ 

Note that if $X, Y \subseteq U$ then

$$\chi_X \ast \chi_Y (z) := \left| \{(x, y) : x \in X, y \in Y, x + y = z\} \right|,$$

the number of ways of writing $z$ as the sum of an element in $X$ and an element of $Y$. Thus, we may express the condition for $V$ to be a tile (with $A$ as a complement) as

$$\chi_V \ast \chi_A = 1,$$  

where 1 denotes the constant function with value 1. Although this is a necessary and sufficient condition (along with the condition that $\chi_A(u) \in \{0,1\}$) it proves to be too weak to use as a linear programming criterion to certify non-tiling. We supplement it with the condition derived from (2):

$$(\chi_V \ast \chi_V)(\chi_A \ast \chi_A) = |U|\delta,$$  

(4)

where $\delta : U \to \mathbb{R}$ is the function $\delta(0) = 1$ and $\delta(x) = 0$ when $x \neq 0$.

Taking the Fourier transform of (3) yields

$$\hat{\chi}_V \hat{\chi}_A = |U|\delta.$$  

(5)

This suggests a linear program in which we use the variables $\chi_A \ast \chi_A(u)$ instead of $\chi_A(u)$. We are given $V$. Since $|V||A| = 2^n$ we know $|A|$ (if it exists). We have variables $b_u$ for $u \in \mathbb{F}_2^n$ and $c_x$ for $x \in \mathbb{F}_2^n$. We’ll want

$$b_u = \chi_A \ast \chi_A(u),$$

and

$$c_x = |\hat{\chi}_A(x)|^2.$$ 

We have the conditions

$$0 \leq b_u \leq |A|$$

and is an integer,  

(6a)

$$0 \leq c_x \leq |A|^2$$

and is the square of an integer,  

(6b)

$$b_0 = |A|$$  

(6c)

$$c_0 = |A|^2$$  

(6d)

$$b_u = 0 \text{ if } u \neq 0 \text{ and } \chi_V \ast \chi_V (u) \neq 0$$  

(6e)

$$c_x = 0 \text{ if } x \neq 0 \text{ and } \hat{\chi}_V (x) \neq 0$$  

(6f)

$$c_x = \sum_u (-1)^{u \cdot x}b_u \text{ for all } x$$  

(6g)
If we drop the conditions about $b_u$ being an integer and $c_x$ being the square of an integer, we get a linear program which must be feasible if $V$ is a tile.

One problem with this linear program is that it has a large number of nonzero coefficients. Just the condition that the $c_x$ be the Fourier transform of the $b_u$ yields $2^n(2^n - \max(|\text{supp}(\chi_V \star \chi_V^*)|, |\text{supp}(\hat{\chi}_V)|))$ nonzero coefficients (we can either write the $c_x$ as the transform of the $b_u$ or the $b_u$ as the inverse transform of the $c_x$, whichever yields a smaller system). We can immediately halve the number of nonzero coefficients by adding the equation

$$
\sum u b_u = |A|^2
$$

to the remaining equations for the Fourier transform. However, the number of nonzeros is still quite large. We can greatly reduce this by means of ideas from the fast Fourier transform. We create new variables corresponding to the intermediate results of the transform. The usual sort of bookkeeping now yields $3n^2$ nonzeros (since each “butterfly” involves 3 variables) and introduces $n^2n$ new variables.

Here are the details: Introduce variables $t_{i,j}$ with $0 \leq i \leq n$, $0 \leq j < 2^n$, with $t_{0,j} = c_j$. For $0 \leq i < n$, $0 \leq k < 2^i$, and $0 \leq j < 2^{n-i-1}$, introduce the equations:

$$
t_{i+1,j+2^n-i-k} = t_{i,j+2^n-i-k} + t_{i,j+2^n-i-k+2^{n-i-1}} \quad (7)
$$

$$
t_{i+1,j+2^n-i-k+2^{n-i-1}} = t_{i,j+2^n-i-k} - t_{i,j+2^n-i-k+2^{n-i-1}} \quad (7a)
$$

The values $t_{n,j}$ are the values of the Fourier transform of $c_j$. We can also achieve a significant savings in our problem by noting that whenever one of the variables on the right hand sides of (7) is 0 (which is the case for a significant fraction of the $c_j$), then we can “pass through” the remaining variable, or its negation, or a 0 if both are 0, and not create a new variable. We note the effect of this special case in Table 3 by comparing $n^2n$ to the actual number of variables needed.

One nice feature of the approach using linear programming is that the conditions like full-rank for the complementary tile $A$ can be described as linear inequalities.

**Proposition 1.** A subset $A \subseteq U$ containing 0 generates $U$ as a linear subspace if and only if

$$
|\hat{\chi}_A(x)| \leq |A| - 2,
$$

for all $0 \neq x \in U$.

**Proof.** The value $\hat{\chi}_A(x)$ is the sum of $|A|$ terms each of which is $\pm 1$. Thus $\hat{\chi}_A(x) = |A|$ if and only if $x \cdot a = 0$ for all $a \in A$. This can happen if and only if $A$ does not generate $U$. Similarly $\hat{\chi}_A(x) = -|A|$ if and only if $x \cdot a = 1$ for all $a \in A$. This is impossible since $0 \in A$. Further note that $\hat{\chi}_A(x) \equiv |A| \mod 2$, thus the value of $|A| - 1$ is impossible for $|\hat{\chi}_A(x)|$.

Thus, if we use variables representing $|\hat{\chi}_A(x)|^2$ we can express full rank as $|\hat{\chi}_A(x)|^2 \leq (|A| - 2)^2$. 


For the first four of the ten putative tiles given in Table 1, the resulting linear programming problem was small enough so that either glpsol or CPLEX could handle it. The results of this approach applied to the first four cases in Table 1 is given in Table 3.

Table 3: Results from CPLEX

| $k$ | $n$ | $n^2$ | time in seconds | rows     | variables | nonzeros |
|-----|-----|-------|-----------------|----------|-----------|----------|
| 6   | 12  | 49152 | 0.46            | 33569    | 33414     | 99465    |
| 7   | 13  | 106496| 72.60           | 74349    | 74710     | 221693   |
| 8   | 14  | 229376| 0.54            | 140312   | 142632    | 419864   |
| 9   | 15  | 491520| 39.72           | 321016   | 327828    | 961832   |

The system of linear equations for $k = 8$ were inconsistent. The linear programming problems for the last six rows of the table were too large for CPLEX to handle.

### 3.1 Certificates of non-tiling from Linear Programming

For the use of the linear programming criterion, we use the following well-known criterion to produce a certificate of non-tiling.

**Lemma 4 (Farkas).** Let $A \in \text{Mat}_{m \times n}(\mathbb{R}), b \in \mathbb{R}^n$. Then exactly one of the following statements is true:

1. There is an $y \in \mathbb{R}^n$ such that $Ay = b$ and $y \geq 0$.
2. There is a $z \in \mathbb{R}^m$ such that $A^Tz \geq 0$ and $b^Tz < 0$.

We explicitly describe the matrix $A$ and vector $b$ for our problem. A solution to the second alternative above will constitute a certificate of non-tiling. Note that since the linear programming problem in second alternative is homogeneous, in practice we replace it by

There is a $z \in \mathbb{R}^m$ such that $A^Tz \geq 0$ and $b^Tz = -c$ for some convenient $c > 0$.

For convenience we write $f(x) = \chi_A \ast \chi_A(x)$ The rows and columns of $A$ are divided into regions. The first region (1) of rows is indexed by elements $\hat{x} \in \bar{U}$. The next region of rows, which is indexed by the symbols $f(x)$ for $x \in U$ corresponds to $x \in U$ for which $f(x)$ has a fixed value – either $|A|$ for $x = 0$ (region (2)), or 0 for those $x$ with $\chi_V(x) \ast \chi_V(x) \neq 0$ (region (4)). The next region is similar to the last, is indexed by symbols of the form $f(\hat{x})$, for which $\hat{f}(\hat{x})^2$ has a fixed value – either $|A|^2$ (region (3)) for $\hat{x} = \hat{0}$ or 0 for those $\hat{x}$ for which $\hat{\chi}_V(\hat{x}) \neq 0$ (region (5)).

---

1. glpsol is the standalone solver contained in GLPK – the GNU Linear Programming Kit
2. http://www.gnu.org/software/glpk
The columns are divided into two regions – the first indexed by the elements \( f(x) \) and the second by elements \( \hat{f}(\hat{x}) \).

\[
A = \begin{pmatrix}
  f(x) & \hat{f}(\hat{x}) \\
  \mathcal{F} & \begin{pmatrix}
    -1 & -1 & \ldots \\
    1 & \vdots & 1 \\
    \vdots & 1 & \vdots \\
    1 & 1 & \ldots \\
  \end{pmatrix}
\end{pmatrix}
\]

The vector \( b \) is 0 everywhere, except for \( b_{f(0)} = |A|, b_{\hat{f}(\hat{0})} = |A|^2 \).

We denote elements of \( C_n^2 \) (the \( n \)-fold product of the group of order 2) by \( x \), and elements of its dual, \( \hat{C}_n^2 \) by \( \hat{x} \). The vector \( z \) giving the Farkas certificate is denoted by \( z \). Note that we’ve eliminated the upper bounds in equation (6).

The part of the matrix denoted by \( \mathcal{F} \) is the matrix of Fourier transform on \( C_n^2 \). We can use the same idea as in (7) to make the resulting set of equations much more sparse at the expense of introducing auxiliary variables.

In terms of equations these are:

For \( b^T z < 0 \):

\[
z_{f(0)}|A| + z_{\hat{f}(\hat{0})}|A|^2 < 0. \tag{9}
\]

As remarked above we set the right hand side of this to any convenient negative constant. Column \( f(0) \):

\[
\sum_{\hat{x} \in \hat{C}_n^2} \hat{x} z_{\hat{x}} + z_{f(0)} \geq 0 \tag{10}
\]

Column \( f(x) \) where \( x \in V + V \):

\[
\sum_{\hat{x} \in \hat{C}_n^2} \hat{x}(x) z_{\hat{x}} + z_{f(x)} \geq 0. \tag{11}
\]

Column \( f(x) \) where \( x \not\in V + V \):

\[
\sum_{\hat{x} \in \hat{C}_n^2} \hat{x}(x) z_{\hat{x}} \geq 0. \tag{12}
\]

Column \( \hat{f}(\hat{0}) \):

\[
- z_{\hat{0}} + z_{\hat{f}(\hat{0})} \geq 0. \tag{13}
\]

Column \( \hat{f}(\hat{x}) \), where \( \hat{\chi}_V(\hat{x}) = 0 \):

\[
- z_{\hat{x}} + z_{\hat{f}(\hat{x})} \geq 0. \tag{14}
\]
Column \( \hat{f}(\vec{x}) \), where \( \vec{\chi}_V(\vec{x}) \neq 0 \):

\[
-z_{\vec{x}} \geq 0.
\]  \hspace{1cm} (15)

In practice, we found that we could require that inequalities (13), (14) and (15) be equalities, and we could still find a certificate of non-tiling. This effectively eliminates the parts of \( z \) indexed by \( \vec{x} \).

For the first and third rows of Table 4 the linear program solver CLP found particular easy to describe certificates. For the second and fourth rows the certificates did not have an apparent structure which allows a compact description.

The certificates below are vectors which have 0.0 everywhere except in the coordinates specified in the tables below. The part above the line corresponds to indices of the form \( \hat{f}(\vec{x}) \) and the part below to indices of the form \( f(x) \). An expression like \((c, l)\) means a segment of all coordinates starting at \( c \) of length \( l \). An expression like \((c, l, s)\) has a similar meaning except that there is a stride of \( s \), meaning the arithmetic progression \( c, c + s, \ldots, c + ms \) where \( m \) is the largest integer such that \( c + ms < c + l \).

| \( \) | \( \) |
|---|---|
| 0 | -1024.0 |
| 320 | 1024.0 |
| 640 | 1024.0 |
| (192, 64) | 1.0 |
| (384, 64) | 1.0 |
| (576, 64) | 1.0 |
| (768, 64) | 1.0 |
| (1216, 64) | 1.0 |
| (1408, 64) | 1.0 |
| (1600, 64) | 1.0 |
| (1792, 64) | 1.0 |
| (2240, 64) | 1.0 |
| (2432, 64) | 1.0 |
| (2624, 64) | 1.0 |
| (2816, 64) | 1.0 |
| (3264, 64) | 1.0 |
| (3456, 64) | 1.0 |
| (3648, 64) | 1.0 |
| (3840, 64) | 1.0 |

Table 4: Certificate of non-tileability for the first putative tile

How widely applicable are these methods? We have tried the linear programming method on all of the examples of non-tiles given in [2], and in all cases it has found a certificate of non-tileability. In [8, 7] the authors show that

---

2The linear program solver included in the COIN-OR package http://www.coin-or.org. See [6]
there are no full-rank tilings in dimension 8 and 9 as a result of the execution of a very long running computer program. It would be interesting to see if the linear programming method could supply certificates of non-tileability for the all the cases examined.

References

[1] ILOG CPLEX 10.1 User’s Manual. CPLEX Optimization, Inc., 2006.

[2] Gérard Cohen, Simon Litsyn, Alexander Vardy, and Gilles Zémor. Tilings of Binary spaces. *SIAM J. Discrete Math.*, 9:393–412, 1996.

[3] Tuvi Etzion and Alexander Vardy. Perfect binary codes, constructions, properties and enumeration. *IEEE Trans. Inf. Thy.*, 40:754–763, 1994.

[4] Tuvi Etzion and Alexander Vardy. On perfect codes and tilings: problems and solutions. *SIAM J. Discrete Math.*, 11:205–223, 1998.

[5] Daniel M. Gordon, Victor S. Miller, and Peter Ostapenko. Optimal hash functions for approximate matches on the n-cube. *IEEE Trans. Inf. Thy.*, 56(3):984–991, mar 2010.

[6] R. Lougee-Heimer. The Common Optimization INterface for Operations Research: Promoting open-source software in the operations research community. *IBM Journal of Research and Development*, 47(1):57–66, January 2003.

[7] Patric R. J. Östergård and Alexander Vardy. Resolving the existence of full-rank tiling of binary Hamming spaces. *SIAM J. Discrete Math.*, 18:382–387, 2004.

[8] Ari Trachtenberg and Alexander Vardy. Full-rank tiling of $\mathbb{F}_2^8$ do not exist. *SIAM J. Discrete Math.*, 16:390–392, 2003.

| x | y |
|---|---|
| 0 | -8192.0 |
| (0,16384,2) | 1.0 |

Table 5: Certificate of non-tileability for the third putative tile