ON THE BOUNDEDNESS AND DECAY OF SOLUTIONS FOR A CHEMOTAXIS-HAPTOTAXIS SYSTEM WITH NONLINEAR DIFFUSION

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(Communicated by Eduard Feireisl)

ABSTRACT. This paper deals with a parabolic-parabolic-ODE chemotaxis-haptotaxis system with nonlinear diffusion
\[
\begin{align*}
    u_t &= \nabla \cdot (\varphi(u)\nabla u) - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), \\
    v_t &= \Delta v - v + u, \\
    w_t &= -vw,
\end{align*}
\]
under Neumann boundary conditions in a smooth bounded domain \(\Omega \subset \mathbb{R}^2\), where \(\chi, \xi\) and \(\mu\) are positive parameters and \(\varphi(u)\) is a nonlinear diffusion function. Firstly, under the case of non-degenerate diffusion, it is proved that the corresponding initial boundary value problem possesses a unique global classical solution that is uniformly bounded in \(\Omega \times (0, \infty)\). Moreover, under the case of degenerate diffusion, we prove that the corresponding problem admits at least one nonnegative global bounded-in-time weak solution. Finally, under some additional conditions, we derive the temporal decay estimate of \(w\).

1. Introduction. In this paper, we consider the following chemotaxis-haptotaxis system with nonlinear diffusion
\[
\begin{align*}
    u_t &= \nabla \cdot \big( \varphi(u) \nabla u \big) - \chi \nabla \cdot \big( u \nabla v \big) - \xi \nabla \cdot \big( u \nabla w \big) \\
    &\quad + \mu u(1 - u - w), \\
    v_t &= \Delta v - v + u, \\
    w_t &= -vw, \\
    \varphi(u) \frac{\partial u}{\partial \nu} - \chi u \frac{\partial v}{\partial \nu} - \xi u \frac{\partial w}{\partial \nu} &= 0, \\
    u(x, 0) &= u_0(x), \ v(x, 0) = v_0(x), \ w(x, 0) = w_0(x), \\
    x \in \Omega, \ t > 0,
\end{align*}
\]
2010 Mathematics Subject Classification. Primary: 35K55; Secondary: 35B45, 35B33, 35K57, 92C17.

Key words and phrases. Boundedness, temporal decay, chemotaxis-haptotaxis system, nonlinear diffusion.
where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial \Omega$, $\frac{\partial}{\partial \nu}$ denotes the differentiation with respect to the outward normal derivative on $\partial \Omega$, and $\phi \in C^2([0, \infty))$ is the nonlinear diffusion function. The parameters $\chi$, $\xi$ and $\mu$ are positive and the initial data $(u_0, v_0, w_0)$ is supposed to be satisfied the following conditions

\[
\begin{cases}
  u_0 \in C^0(\bar{\Omega}) \text{ with } u_0 \geq 0 \text{ in } \Omega \text{ and } u_0 \not\equiv 0, \\
v_0 \in W^{1,\infty}(\Omega) \text{ with } v_0 \geq 0 \text{ in } \Omega, \\
w_0 \in C^{2+\alpha}(\Omega) \text{ with } \alpha \in (0, 1) \text{ and } w_0 > 0 \text{ in } \Omega \text{ and } \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial \Omega.
\end{cases}
\]

The oriented movement of biological cells or organisms in response to a chemical gradient is called chemotaxis. The pioneering works of chemotaxis model were introduced by Patlak [22] in 1953 and Keller and Segel [18] in 1970, and we refer the reader to the survey [11, 13, 14] where a comprehensive information of further examples illustrating the outstanding biological relevance of chemotaxis can be found. In recent years, chemotactic mechanisms have also been detected to be crucial in the process of cancer invasion, where they usually interact with haptotaxis, the correspondingly directed cell movement in response to gradients of non-diffusible signals. The combination of these two cell migration mechanisms was initially proposed by Chaplain and Lolas in [4, 5] to describe cancer cell invasion into surrounding healthy tissue. More precisely, their model accounts for both chemotactic migration of cancer cells towards a diffusible matrix-degrading enzyme (MDE) secreted by themselves, and haptotactic migration towards a static tissue, also referred to as extracellular matrix (ECM). In this context, $u(x, t)$ represents the density of cancer cell, $v(x, t)$ denotes the concentration of MDE, and $w(x, t)$ stands for the density of ECM. In addition to random movement, cancer cells are supposed to bias their movement both towards increasing concentrations of urokinase plasminogen activator by chemotaxis (see [3]), and towards increasing densities of the non-diffusible ECM through detecting the macromolecules adhered therein by haptotaxis (see [2]). It is assumed that the cancer cells undergo birth and death in a logistic manner, competing for space with the ECM. The MDE is assumed to be produced by cancer cells, and to diffuse and decay, whereas the ECM is stiff in the sense that it does not diffuse, but it could be degraded upon contact with MDE.

In order to better understand model (1), let us mention some previous contributions in this direction. In recent years, the following initial boundary value problems have been studied by many authors

\[
\begin{cases}
  u_t = \nabla \cdot (\phi(u)\nabla u) - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), & x \in \Omega, t > 0, \\
  \tau v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\
  w_t = -vw, & x \in \Omega, t > 0, \\
  \phi(u) \frac{\partial u}{\partial \nu} - \chi u \frac{\partial v}{\partial \nu} - \xi u \frac{\partial w}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
  u(x, 0) = u_0(x), \quad \tau v(x, 0) = \tau v_0(x), \quad w(x, 0) = w_0(x), \quad x \in \Omega,
\end{cases}
\]

where $\tau \in \{0, 1\}$, $\chi > 0$, $\xi > 0$, $\mu > 0$ and $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded domain with smooth boundary $\partial \Omega$.

When $\tau = 0$, i.e. the diffusion rate of the MDE is much greater than that of cancer cells [5]. Moreover, similar quasi-steady-approximations for corresponding
chemoattractant equations were frequently used to study classical chemotaxis systems (for instance [17, 23]). For the special case \( \varphi(u) = 1 \) in (3), Tao and Wang [27] proved that model (3) possesses a unique global bounded classical solution for any \( \mu > 0 \) in two space dimensions, and for large \( \mu > 0 \) in three space dimensions. Recently, in [31], Tao and Winkler studied global boundedness for model (3) with \( \varphi(u) = 1 \) in (3), Tao and Wang [27] proved that model (3) possesses a unique global bounded classical solution for any \( \mu > 0 \) in two space dimensions, and for large \( \mu > 0 \) in three space dimensions. Recently, in [31], Tao and Winkler studied global boundedness for model (3) with \( \varphi(u) = 1 \) in (3), Tao and Wang [27] proved that model (3) possesses a unique global bounded classical solution for any \( \mu > 0 \) in two space dimensions, and for large \( \mu > 0 \) in three space dimensions. Moreover, if \( \varphi(u) \in C^2([0, \infty)) \), \( \varphi(0) > 0 \) and \( \varphi(u) \geq \delta u^{m-1} \), where \( \delta > 0 \) and \( m > \max\{1, \overline{m}\} \) with

\[
\overline{m} = \begin{cases} 
\frac{2n^2+4n-4}{n(n-4)}, & \text{if } n \leq 8, \\
\frac{2n^2+3n+2-\sqrt{8n(n+1)}}{n(n+2)}, & \text{if } n \geq 9,
\end{cases} \tag{4}
\]

Tao and Winkler [29] proved that model (3) possesses at least one nonnegative global classical solution, however, their boundedness is left as an open problem. In [12], Hillen et al. studied the global boundedness and asymptotic behavior of model (3) with \( \varphi(u) = 1 \) in one space dimension and proposed an open problem about boundedness in the higher-dimensional case. Recently, for the case \( \varphi(u) = 1 \) in (3), Tao [25] showed that under appropriate regularity assumption on the initial data \((u_0, v_0, w_0)\), the corresponding initial-boundary problem possesses a unique classical solution which is global in time and bounded in two space dimensions. To the best of our knowledge, there exist few boundedness results addressing the fully parabolic-parabolic-ODE chemotaxis-haptotaxis model in the higher-dimensional version.

In the present paper, our main purpose is to give a positive answer about the question of global boundedness in [12, 29]. Motivated by the above works, we consider global boundedness and temporal decay estimate of \( w \) for model (1) under some suitable conditions. The crucial assumption in our result is related to the diffusion function \( \varphi(u) \). Here, we suppose that the function \( \varphi \) satisfies

\[
\varphi \in C^2([0, \infty)) \tag{5}
\]

and

\[
\varphi(s) \geq c_0 s^{m-1} \quad \text{for all } s > 0 \tag{6}
\]

with some \( c_0 > 0 \) and \( m > 1 \).

If in addition to (5) and (6), \( \varphi(s) \) satisfies

\[
\varphi(s) > 0 \quad \text{for all } s \geq 0, \tag{7}
\]

then the first equation in (1) becomes uniformly parabolic, thus the solution of (1) may be considered in the classical sense.

Our main results in this paper are stated as follows. Firstly, we consider global boundedness of model (1) in the case of non-degenerate diffusion.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary. Assume that \( \chi > 0 \), \( \xi > 0 \) and \( \mu > 0 \), and the function \( \varphi \) satisfies (5)-(7). Then for any \((u_0, v_0, w_0)\) fulfilling (2), model (1) possesses a unique global classical solution
\( (u,v,w) \) which is uniformly bounded in \( \Omega \times (0,\infty) \) in the sense that there exists \( C > 0 \) such that

\[
\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t > 0.
\]

**Remark 1.** In the two-dimensional case, Theorem 1.1 gives an affirmative answer to the question of global boundedness proposed in [12, 29]. Moreover, when \( \varphi(u) = 1 \), the results in Theorem 1.1 are consistent with those in [25].

In the absence of (7), the first equation of (1) may be degenerate at \( u = 0 \), thus in general there is no classical solution. Hence, we shall study global weak solutions in (1). Before stating our second result, we first give the definition of weak solutions of problem (1).

**Definition 1.2.** Let \( T \in (0,\infty) \). A pair of nonnegative function \( (u,v,w) \) defined in \( \Omega \times (0,T) \) is called a weak solution of problem (1), if

(i) \( u \in L^2((0,T); L^2(\Omega)), \varphi(u)\nabla u \in L^2((0,T); L^2(\Omega)), v \in L^2((0,T); W^{1,2}(\Omega)) \)

\( u\nabla v \in L^2((0,T); L^2(\Omega)), w \in L^2((0,T); W^{1,2}(\Omega)), u\nabla w \in L^2((0,T); L^2(\Omega)) \), \( uv \in L^2((0,T); L^2(\Omega)) \) and \( uw \in L^2((0,T); L^2(\Omega)) \),

(ii) the following integral equalities

\[
-\int_0^T \int_\Omega u\zeta_t \, dx \, dt - \int_\Omega u_0\zeta(x,0) \, dx dt = -\int_0^T \int_\Omega \varphi(u)\nabla u \cdot \nabla \zeta \, dx \, dt + \chi \int_0^T \int_\Omega u\nabla v \cdot \nabla \zeta \, dx \, dt \tag{9}
\]

\[
+ \xi \int_0^T \int_\Omega u\nabla w \cdot \nabla \zeta \, dx \, dt + \mu \int_0^T \int_\Omega u(1 - u - w) \zeta \, dx \, dt
\]

and

\[
-\int_0^T \int_\Omega v\eta_t \, dx \, dt - \int_\Omega v_0\eta(x,0) \, dx dt = -\int_0^T \int_\Omega \nabla v \cdot \nabla \eta \, dx \, dt - \int_0^T \int_\Omega v\eta \, dx \, dt + \int_0^T \int_\Omega u\eta \, dx \, dt \tag{10}
\]

as well as

\[
\int_0^T \int_\Omega w\theta \, dx \, dt = \int_0^T \int_\Omega w_0\theta e^{\int_0^s v(x,s) \, ds} \, dx \, dt \tag{11}
\]

hold for all \( (\zeta, \eta, \theta) \in C_0^\infty(\Omega \times [0,T])^3 \). Furthermore, if \( (u,v,w) \) is a weak solution of problem (1) in \( \Omega \times (0,T) \) for all \( T \in (0,\infty) \), it is said that \( (u,v,w) \) is a global weak solution.

**Theorem 1.3.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary. Assume that \( \chi > 0, \xi > 0 \) and \( \mu > 0 \), and the function \( \varphi \) satisfies (5)-(6). Then for any \( (u_0,v_0,w_0) \) fulfilling (2), model (1) possesses at least one nonnegative global weak solution \( (u,v,w) \).

Finally, under some addition conditions, we consider decay of \( w \).
Theorem 1.4. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume that $\varphi \in C^2((0, \infty))$ satisfies the condition (5)-(7) with $m \in (1, 2)$. Suppose that $\chi > 0$, $\xi > 0$, $\mu > 0$, and the initial data $(u_0, v_0, w_0)$ satisfies (2) and $\|w_0\|_{L^\infty(\Omega)} < 1$. Then there exist positive constants $\kappa$ and $C$ such that the third solution component $w$ satisfies the following decay estimate

$$\|w(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq Ce^{-\kappa t} \quad \text{for all } t > 0. \quad (12)$$

Remark 2. In Theorem 1.1, Theorem 1.3 and Theorem 1.4, we only obtain global boundedness and decay of $w$ for model (1) in the two space dimensions. Unfortunately, as for the higher-dimensions case (i.e $n \geq 3$), we have to leave an open question here. However, we expect that we solve it in future work.

This paper is organized as follows. In Section 2, we shall state some preliminary results which are important for our main proofs. In Section 3, we consider global boundedness of classical solutions for model (1) under some suitable conditions and prove Theorem 1.1. In Section 4, we concern with global weak solutions of problem (1) and prove Theorem 1.3. In Section 5, we give the decay estimate of $w$ under some additional assumptions and prove Theorem 1.4.

2. Preliminaries. We first state one result concerning local-in-time existence of a classical solution to model (1).

Lemma 2.1. Let $\chi > 0$, $\xi > 0$ and $\mu > 0$, and assume that the function $\varphi$ satisfies (5)-(7). Then for any initial data $(u_0, v_0, w_0)$ fulfilling (2), there exists a maximal existence time $T_{\max} \in (0, \infty]$ such that model (1) possesses a unique classical solution

$$u \in C^0(\Omega \times [0, T_{\max}]) \cap C^{2,1}(\Omega \times (0, T_{\max})), \quad \text{and}$$

$$v \in C^0(\Omega \times [0, T_{\max}]) \cap C^{2,1}(\Omega \times (0, T_{\max})).$$

Moreover, the solution $(u, v, w)$ satisfies

$$u \geq 0, \quad v \geq 0 \quad \text{and} \quad 0 < w \leq \|w_0\|_{L^\infty(\Omega)} \quad \text{for all } (x, t) \in \Omega \times [0, T_{\max}). \quad (13)$$

Finally, if $T_{\max} < +\infty$, then

$$\lim_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \quad (14)$$

Proof. The local-in-time existence of classical solution to model (1) is well-established by a fixed point theorem in the context of chemotaxis-haptotaxis systems. By the maximum principle, it is easy to obtain that $u \geq 0$ and $v \geq 0$ for all $(x, t) \in \Omega \times [0, T_{\max})$. Integrating the third equation in (1), it follows from (2) and $v \geq 0$ that $0 < w \leq \|w_0\|_{L^\infty(\Omega)}$ for all $(x, t) \in \Omega \times [0, T_{\max})$. The proof is quite standard, for details, we refer the readers to [6, 29, 32, 37, 15, 34].

Next, we give the basic property on mass for $u$.

Lemma 2.2. Assume that $(u, v, w)$ is a solution for model (1). Then there exists a constant $m^* > 0$ such that the first component $u$ of the solution to (1) satisfies the following estimate

$$\|u(\cdot, t)\|_{L^1(\Omega)} \leq m^* := \max\{|\Omega|, \|u_0\|_{L^1(\Omega)}\} \quad \text{for all } t \in (0, T_{\max}). \quad (15)$$
Proof. Integrating the first equation in (1), we deduce from \( w > 0 \) that
\[
\frac{d}{dt} \int_{\Omega} u(x,t) \, dx = \int_{\Omega} \mu u(1 - u - w) \, dx \leq \mu \int_{\Omega} u \, dx - \mu \int_{\Omega} u^2 \, dx. \tag{16}
\]
According to Hölder’s inequality, we have
\[
\frac{d}{dt} \int_{\Omega} u(x,t) \, dx \leq \mu \int_{\Omega} u \, dx - \frac{\mu}{|\Omega|} \left( \int_{\Omega} u \, dx \right)^2. \tag{17}
\]
By the comparison argument of ODE, we derive
\[
|| u(\cdot,t) ||_{L^1(\Omega)} \leq \max \left\{ ||u_0(x)||_{L^1(\Omega)}, |\Omega| \right\} := m^*. \tag{18}
\]
The proof of Lemma 2.2 is complete.

Moreover, we show the following one-sided pointwise estimate for \(-\Delta w\), which will be served as a cornerstone for our subsequent analysis.

Lemma 2.3. (see [25]). Assume that \((u, v, w)\) is a solution for model (1). Then we have
\[
-\Delta w(x,t) \leq ||w_0||_{L^\infty(\Omega)} \cdot v(x,t) + M \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad t \in (0, T_{\text{max}}), \tag{19}
\]
where
\[
M := ||\Delta w_0||_{L^\infty(\Omega)} + 4 ||\nabla \sqrt{w_0}||_{L^2(\Omega)}^2 + \frac{||w_0||_{L^\infty(\Omega)}}{e}. \tag{20}
\]
Finally, let us collect some basic statements about the Gagliardo-Nirenberg inequality which will be used in the forthcoming proof of \(L^p\)-boundedness of solutions for model (1). For details, we refer the readers to [7, 21, 37] (see also [28]).

Lemma 2.4. Let \(\Omega \subset \mathbb{R}^n\) be a bounded domain with smooth boundary and assume that \(r \in (0, p)\) and \(\phi \in W^{1,2}(\Omega) \cap L^r(\Omega)\). Then there exists a positive constant \(C_{GN}\) such that
\[
||\phi||_{L^p(\Omega)} \leq C_{GN} (||\nabla \phi||_{L^2(\Omega)} ||\phi||_{H^1(\Omega)}^{1-\lambda} + ||\phi||_{L^r(\Omega)}), \tag{21}
\]
holds with \(\lambda \in (0,1)\) satisfying
\[
\frac{1}{p} = \lambda \left( \frac{1}{2} - \frac{1}{n} \right) + (1 - \lambda) \frac{1}{r}. \tag{22}
\]

3. Global boundedness. In this section, inspired by Liu and Tao in [20] (see also [25]), we first establish an energy-type estimates for \(\int_{\Omega} u \ln u \, dx\) and \(\int_{\Omega} |\nabla v|^2 \, dx\). Next, we further establish a bound for \(\int_{\Omega} u^2 \, dx\) and \(\int_{\Omega} |\nabla v|^4 \, dx\) via coupled estimate techniques. Thirdly, we build a bound for \(\int_{\Omega} u^p \, dx\) for any \(p > 1\). Finally, we derive a \(L^\infty(\Omega)\)-estimate for \(u\) and prove Theorem 1.1. To do this, we need to obtain the following a priori estimates for the solutions of model (1).

Firstly, in the two-dimensional case, by using the properties of the Neumann heat semigroup ([36]) and Lemma 2.2, we give a \(L^p\)-estimate on \(v\).

Lemma 3.1. (see [20]). Let \(\Omega \subset \mathbb{R}^2\) be a bounded domain with smooth boundary and assume that the conditions in Lemma 2.1 hold. Then for any \(p \in [1, \infty)\), there exists \(K(p) > 0\) such that the solution \(v\) of (1) satisfies
\[
\int_{\Omega} v^p \, dx \leq K(p) \quad \text{for all} \quad t \in (0, T_{\text{max}}). \tag{23}
\]
Lemma 3.2. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume that $\varphi$ satisfies the condition \((5)-(7)\). Suppose that $\chi > 0$, $\xi > 0$, $\mu > 0$ and the initial data $(u_0, v_0, w_0)$ satisfies \((2)\). Then there exists $C > 0$ independent of $T_{\text{max}}$ such that the solution of \((1)\) satisfies
\[
\int_{\Omega} u \ln u \, dx \leq C
\] (24)
and
\[
\int_{\Omega} |\nabla v|^2 \, dx \leq C
\] (25)
for all $t \in (0, T_{\text{max}})$.

Proof. The main idea of the proof comes from that in \([20, 25]\). For convenience of the readers, we give the details. Multiplying the first equation in \((1)\) by $(1 + \ln u)$ and integrating by parts, we have
\[
\frac{d}{dt} \int_{\Omega} u \ln u \, dx = -\int_{\Omega} \frac{\varphi(u)}{u} |\nabla u|^2 \, dx + \chi \int_{\Omega} \nabla u \cdot \nabla v \, dx + \xi \int_{\Omega} \nabla u \cdot \nabla w \, dx
\]
\[
+ \mu \int_{\Omega} u (1 + \ln u) (1 - u - w) \, dx
\]
\[
\leq \chi \int_{\Omega} \nabla u \cdot \nabla v \, dx + \xi \int_{\Omega} \nabla u \cdot \nabla w \, dx
\]
\[
+ \mu \int_{\Omega} u (1 + \ln u) (1 - u - w) \, dx
\] (26)
for all $t \in (0, T_{\text{max}})$. With the aid of Lemma 2.2, Lemma 2.3 and Lemma 3.1, it follows from the same way as in \([25]\), we can obtain
\[
\frac{d}{dt} \int_{\Omega} u \ln u \, dx \leq \frac{1}{2} \int_{\Omega} |\Delta u|^2 \, dx + \mu \int_{\Omega} u \ln u \, dx + C_1 \int_{\Omega} u^2 \, dx
\]
\[
- \mu \int_{\Omega} u^2 \ln u \, dx + C_2
\] (27)
for all $t \in (0, T_{\text{max}})$, where $C_1 = \frac{\chi^2 + \xi \|w_0\|_{L^\infty(\Omega)}}{2}$ and $C_2 = \frac{\xi \|w_0\|_{L^\infty(\Omega)} \cdot K(2) + (\xi M + \mu) m^* + \frac{\mu}{2} \|w_0\|_{L^\infty(\Omega)} |\Omega|}{2}$. In order to cancel the first term on the right of \((27)\), we again use the same way as in \([25]\) to derive
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v|^2 \, dx + \int_{\Omega} |\nabla v|^2 \, dx
\]
\[
\leq - \frac{1}{2} \int_{\Omega} |\Delta v|^2 \, dx + \frac{1}{2} \int_{\Omega} u^2 \, dx \quad \text{for all } t \in (0, T_{\text{max}}).
\] (28)
Combining \((27)\) with \((28)\), we have
\[
\frac{d}{dt} \left\{ \int_{\Omega} u \ln u \, dx + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx \right\} + \int_{\Omega} |\nabla v|^2 \, dx
\]
\[
\leq \mu \int_{\Omega} u \ln u \, dx + (C_1 + \frac{1}{2}) \int_{\Omega} u^2 \, dx
\]
\[
- \mu \int_{\Omega} u^2 \ln u \, dx + C_2 \quad \text{for all } t \in (0, T_{\text{max}}).
\] (29)
Let
\[ y(t) := \int_{\Omega} u \ln |v|^2 \, dx + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx \quad \text{for all } t \in (0, T_{\max}), \]
it follows from (29) and Lemma 3.1 in [25] that
\[ \frac{dy}{dt} + y(t) \leq C_3 \quad \text{for all } t \in (0, T_{\max}), \]
where \( C_3 := L|\Omega| + C_2 \). By the comparison argument of ODE, we derive
\[ y(t) \leq \max\{C_3, y(0)\} \quad \text{for all } t \in (0, T_{\max}), \]
which implies that (24) and (25) hold.

**Lemma 3.3.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary. Assume that \( \varphi \) satisfies the condition (5)-(7). Suppose that \( \chi > 0, \xi > 0, \mu > 0 \) and the initial data \((u_0, v_0, w_0)\) satisfies (2). Then there exists \( C > 0 \) independent of \( T_{\max} \) such that the solution of (1) satisfies
\[ \int_{\Omega} u^2 \, dx \leq C \quad \text{and} \]
\[ \int_{\Omega} |\nabla v|^4 \, dx \leq C \]
for all \( t \in (0, T_{\max}) \).

**Proof.** Multiplying the first equation in (1) by \( u \) and integrating by parts, we have
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dx = -\int_{\Omega} \varphi(u)|\nabla u|^2 \, dx + \chi \int_{\Omega} u \nabla u \cdot \nabla v \, dx + \xi \int_{\Omega} u \nabla u \cdot \nabla w \, dx + \mu \int_{\Omega} u^2 (1 - u - w) \, dx \]
\[ := I + II + III + IV \quad \text{for all } t \in (0, T_{\max}). \]

According to (5)-(7), without loss of generality, assume that \( \varphi(u) \geq c_0 (u + 1)^{m-1} \), thus we can obtain
\[ I = -\int_{\Omega} \varphi(u)|\nabla u|^2 \, dx \leq -c_0 \int_{\Omega} (u + 1)^{m-1} |\nabla u|^2 \, dx. \]

By Young’s inequality, we derive from \( m > 1 \) that
\[ II = \chi \int_{\Omega} u \nabla u \cdot \nabla v \, dx \]
\[ \leq c_0 \int_{\Omega} (u + 1)^{m-1} |\nabla u|^2 \, dx + \frac{\chi^2}{2c_0} \int_{\Omega} u^2 (u + 1)^{1-m} |\nabla v|^2 \, dx \]
\[ \leq c_0 \int_{\Omega} (u + 1)^{m-1} |\nabla u|^2 \, dx + \frac{\chi^2}{2c_0} \int_{\Omega} u^2 |\nabla v|^2 \, dx. \]

It follows from Lemma 2.3 that
\[ III = \xi \int_{\Omega} u \nabla u \cdot \nabla w \, dx \]
\[ = -\frac{\xi}{2} \int_{\Omega} u^2 \Delta w \, dx \]
\[ \leq \frac{\xi}{2} ||w_0||_{L^\infty(\Omega)} \int_{\Omega} u^2 v \, dx + \frac{\xi M}{2} \int_{\Omega} u^2 \, dx. \]
Due to the fact that $w > 0$, it is easy to see that

$$ IV = \mu \int_{\Omega} u^2(1 - u - w) dx \leq \mu \int_{\Omega} u^2 dx - \mu \int_{\Omega} u^3 dx. \quad (38) $$

Combining (34)-(38), we have

$$ \frac{d}{dt} \int_{\Omega} u^2 dx \leq -c_0 \int_{\Omega} (u + 1)^{m-1} |\nabla u|^2 dx + \frac{\lambda^2}{c_0} \int_{\Omega} u^2 |\nabla v|^2 dx $$

$$ + \xi ||w_0||_{L^\infty(\Omega)} \int_{\Omega} u^2 vx + (\xi M + 2\mu) \int_{\Omega} u^2 dx - 2\mu \int_{\Omega} u^3 dx. \quad (39) $$

By using Young’s inequality and Lemma 3.1, we derive

$$ \xi ||w_0||_{L^\infty(\Omega)} \int_{\Omega} u^2 vx \leq \frac{\mu}{2} \int_{\Omega} u^3 dx + \frac{16}{27 \mu^2} (\xi ||w_0||_{L^\infty(\Omega)})^3 \int_{\Omega} v^3 dx $$

$$ \leq \frac{\mu}{2} \int_{\Omega} u^3 dx + C_4, \quad (40) $$

where $C_4 := \frac{16}{27 \mu^2} (\xi ||w_0||_{L^\infty(\Omega)})^3 K(3)$. By using Young’s inequality again, we obtain

$$ (\xi M + 2\mu) \int_{\Omega} u^2 dx \leq \frac{\mu}{2} \int_{\Omega} u^3 dx + C_5, \quad (41) $$

where $C_5 := \frac{16}{27 \mu^2} (\xi M + 2\mu)^3 |\Omega|$. Thus, it follows from (39)-(41) that

$$ \frac{d}{dt} \int_{\Omega} u^2 dx \leq -c_0 \int_{\Omega} (u + 1)^{m-1} |\nabla u|^2 dx + \frac{\lambda^2}{c_0} \int_{\Omega} u^2 |\nabla v|^2 dx $$

$$ - \mu \int_{\Omega} u^3 dx + C_6, \quad (42) $$

where $C_6 := C_4 + C_5$. Adding $\int_{\Omega} u^2 dx$ to both sides of (42) and using the inequality

$$ \int_{\Omega} u^2 dx \leq \mu \int_{\Omega} u^3 dx + \frac{4}{27 \mu^2} |\Omega|, \quad (43) $$
	hanks{due to the Young inequality, we have

$$ \frac{d}{dt} \int_{\Omega} u^2 dx + \frac{4c_0}{(m + 1)^2} \int_{\Omega} |\nabla (u + 1)^{m+1}|^2 dx + \int_{\Omega} u^2 dx \leq \frac{\lambda^2}{c_0} \int_{\Omega} u^2 |\nabla v|^2 dx + C_7, \quad (44) $$

where $C_7 := C_6 + \frac{4}{27 \mu^2} |\Omega|$. In order to deal with the first integral term on the right of (44), we need to establish an energy inequality for $\int_{\Omega} |\nabla v|^4 dx$. It follows from the proof of Lemma 3.3 in \cite{30} that

$$ \frac{d}{dt} \int_{\Omega} |\nabla v|^4 dx + \int_{\Omega} |\nabla v|^4 dx + \int_{\Omega} |\nabla |\nabla v||^2 dx $$

$$ \leq 2 \int_{\partial \Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial \nu} dS + 6 \int_{\Omega} u^2 |\nabla v|^2 dx \quad \text{for all } t \in (0, T_{\max}). \quad (45) $$

}
Adding (44) to (45), we derive
\[
\frac{d}{dt} \left( \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla v|^4 dx \right) + \int_{\Omega} u^2 dx + \int_{\Omega} |
abla v|^4 dx \\
+ \frac{4c_0}{(m+1)^2} \int_{\Omega} |\nabla(u+1)^{\frac{m+1}{2} + 1}|^2 dx + \int_{\Omega} |\nabla|\nabla v|^2|^2 dx \\
\leq 2 \int_{\Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial \nu} dS + \left( 6 + \frac{\chi^2}{c_0} \right) \int_{\Omega} u^2 |\nabla v|^2 dx + C_7
\]  
(46)
for all \( t \in (0, T_{\text{max}}) \).

Next, we shall show that the three integrals on the right of (46) can be controlled by \( \frac{4c_0}{(m+1)^2} \int_{\Omega} |\nabla(u+1)^{\frac{m+1}{2} + 1}|^2 dx + \int_{\Omega} |\nabla|\nabla v|^2|^2 dx \). One main contribution of a recent work [16] is to deal with the boundary-related integrals, thus for any \( \varepsilon_1 > 0 \), there exists \( C_b(\varepsilon_1) > 0 \) such that
\[
2 \int_{\Omega} |\nabla v|^2 \frac{\partial |\nabla v|^2}{\partial \nu} dS \leq \varepsilon_1 \int_{\Omega} |\nabla|\nabla v|^2|^2 dx + C_b(\varepsilon_1).
\]  
(47)
By using Young’s inequality with \( \varepsilon_2 > 0 \), we obtain
\[
\left( 6 + \frac{\chi^2}{c_0} \right) \int_{\Omega} u^2 |\nabla v|^2 dx \leq \varepsilon_2 \int_{\Omega} |\nabla v|^6 dx + \frac{\left( 6 + \frac{\chi^2}{c_0} \right)^2}{\varepsilon_2} \int_{\Omega} u^3 dx.
\]  
(48)
It follows from the Gagliardo-Nirenberg inequality in Lemma 2.4 and (25) that
\[
\varepsilon_2 \int_{\Omega} |\nabla v|^6 dx = \varepsilon_2 |||\nabla v|^2||^3_{L^3(\Omega)} \\
\quad \leq \varepsilon_2 C_9 (|||\nabla v|^2||^\frac{2}{L^2(\Omega)} \cdot |||\nabla v|^2||^\frac{1}{L^1(\Omega)} + |||\nabla v|^2||^3_{L^1(\Omega)}) \leq \varepsilon_2 C_{10} |||\nabla v|^2||^2_{L^2(\Omega)} + C_{11},
\]  
(49)
where \( C_9, C_{10}, C_{11} \) are positive constants and we have used the fact that the space dimension is two. By using the Gagliardo-Nirenberg inequality again, we infer from \( m > 1 \), Lemma 2.2 and Young’s inequality that
\[
\left( 6 + \frac{\chi^2}{c_0} \right)^2 \frac{3}{\varepsilon_2} \int_{\Omega} u^3 dx \\
\leq \left( 6 + \frac{\chi^2}{c_0} \right)^{\frac{3}{2}} \left( \|(u+1)^{\frac{m+1}{2}}\|^2_{L^\infty(\Omega)} \cdot \|(u+1)^{\frac{m+1}{2}}\|^\frac{1}{2}_{L^\frac{1}{2}(\Omega)} \right) \\
\leq \left( 6 + \frac{\chi^2}{c_0} \right)^{\frac{3}{2}} C_{12} \left( \|\nabla(u+1)^{\frac{m+1}{2}}\|^2_{L^2(\Omega)} \cdot \|(u+1)^{\frac{m+1}{2}}\|^\frac{1}{2}_{L^\frac{1}{2}(\Omega)} \right) \\
\quad + \|\nabla(u+1)^{\frac{m+1}{2}}\|^\frac{1}{2}_{L^\frac{1}{2}(\Omega)} + 1 \\
\leq \frac{C_{13}}{\varepsilon_2} \left( \|\nabla(u+1)^{\frac{m+1}{2}}\|^2_{L^2(\Omega)} + 1 \right) \\
\leq C_{13} \sqrt{\varepsilon_2} \|\nabla(u+1)^{\frac{m+1}{2}}\|^2_{L^2(\Omega)} + C_{14}(\varepsilon_2),
\]  
(50)
where \( C_{12} > 0, C_{13} > 0 \) and \( C_{14}(\varepsilon_2) > 0 \).
Take $\varepsilon_1 > 0$ sufficiently small fulfilling $\varepsilon_1 \leq \frac{1}{2}$ and $\varepsilon_2 > 0$ sufficiently small fulfilling $\varepsilon_2 \leq \min \left\{ \frac{1}{2 \chi^4}, \frac{16 \varepsilon_0}{C_{13}(m+1)} \right\}$, it follows from (46)-(50) that
\[
\frac{d}{dt} \left\{ \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla v|^4 dx \right\} + \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla v|^4 dx \leq C_{15}
\] for all $t \in (0, T_{\max})$, where $C_{15} := C_7 + C_8(\varepsilon_1) + C_{11} + C_{14}(\varepsilon_2)$.

Let
\[
y(t) := \int_{\Omega} u^2 dx + \int_{\Omega} |\nabla v|^4 dx \quad \text{for all } t \in (0, T_{\max}),
\]
and the initial data $u(0), v(0)$ satisfies (2). Then for any $p > 1$, there exists $C(p) > 0$ independent of $T_{\max}$ such that the solution of (1) satisfies
\[
\int_{\Omega} u^p dx \leq C(p) \quad \text{for all } t \in (0, T_{\max}).
\] (54)

Proof. Multiplying the first equation in (1) by $u^{p-1}$, $(p > 2m - 1)$ and integrating by parts, we have
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p dx = -(p-1) \int_{\Omega} u^{p-1} \varphi(u) |\nabla u|^2 dx + \chi(p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v dx + \xi(p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w dx + \mu \int_{\Omega} u^p (1-u-w) dx
\] (55)

According to (5)-(7), we obtain
\[
I = -(p-1) \int_{\Omega} u^{p-2} \varphi(u) |\nabla u|^2 dx \leq -c_0(p-1) \int_{\Omega} u^{p+m-3} |\nabla u|^2 dx.
\] (56)

By Young’s inequality, we derive
\[
II \leq \frac{c_0(p-1)}{2} \int_{\Omega} u^{p+m-3} |\nabla u|^2 dx + \frac{(p-1)^2 \chi^2}{2c_0} \int_{\Omega} u^{p-m+1} |\nabla v|^2 dx.
\] (57)

It follows from Lemma 2.3 that
\[
III \leq \frac{(p-1)\xi}{p} \int_{\Omega} u^p \Delta w dx \leq \frac{(p-1)\xi}{p} \int_{\Omega} w^p dx + \frac{(p-1)\xi M}{p} \int_{\Omega} u^p dx.
\] (58)
Due to the fact that \( w > 0 \), it is easy to see that
\[
IV = \mu \int_{\Omega} u^p (1 - u - w) dx \leq \mu \int_{\Omega} u^p dx - \mu \int_{\Omega} u^{p+1} dx. \tag{59}
\]
Combining (55)-(59), we have
\[
\begin{align*}
\frac{d}{dt} \int_{\Omega} u^p dx & \leq -\frac{c_0 p(p-1)}{2} \int_{\Omega} u^{p+m-3} |\nabla u|^2 dx + \frac{p(p-1)\chi^2}{2c_0} \int_{\Omega} u^{p-m+1} |\nabla v|^2 dx \\
& \quad + (p-1)\|w_0\|_{L^\infty(\Omega)} \int_{\Omega} u^p dx + ((p-1)\xi M + p\mu) \int_{\Omega} u^p dx - p\mu \int_{\Omega} u^{p+1} dx \\
& \leq \frac{p(p-1)\chi^2}{2c_0} \int_{\Omega} u^{p-m+1} |\nabla v|^2 dx + (p-1)\|w_0\|_{L^\infty(\Omega)} \int_{\Omega} u^p dx \\
& \quad + ((p-1)\xi M + p\mu) \int_{\Omega} u^p dx - p\mu \int_{\Omega} u^{p+1} dx.
\end{align*}
\tag{60}
\]
According to Lemma 3.1 and Lemma 3.3, there exist \( C_{16} > 0 \) and \( K_1(\cdot) \) such that
\[
\|v(\cdot, t)\|_{W^{1,4}(\Omega)} \leq C_{16} \quad \text{for all } t \in (0, T_{\max})
\tag{61}
\]
and
\[
\int_{\Omega} |\nabla v|^q dx \leq K_1(q) \quad \text{for all } q \in [4, \infty) \text{ and } t \in (0, T_{\max}).
\tag{62}
\]
Thus it follows from the Sobolev embedding \( W^{1,4}(\Omega) \hookrightarrow C^0(\Omega) \) thanks to \( 4 > n = 2 \) that
\[
\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{16} \quad \text{for all } t \in (0, T_{\max}).
\tag{63}
\]
Adding \( \int_{\Omega} u^p dx \) to both sides of (60) and using (62), we obtain
\[
\frac{d}{dt} \int_{\Omega} u^p dx + \int_{\Omega} u^p dx
\leq \frac{p(p-1)\chi^2}{2c_0} \int_{\Omega} u^{p-m+1} |\nabla v|^2 dx + C_{17} \int_{\Omega} u^p dx - p\mu \int_{\Omega} u^{p+1} dx, \tag{64}
\]
where \( C_{17} := 1 + C_{16}(p-1)\xi \|w_0\|_{L^\infty(\Omega)} + (p-1)\xi M + p\mu. \)
By using Young’s inequality and (62), we infer from \( p > 2m - 1 \) that
\[
\frac{p(p-1)\chi^2}{2c_0} \int_{\Omega} u^{p-m+1} |\nabla v|^2 dx \leq \frac{p\mu}{2} \int_{\Omega} u^{p+1} dx + C_{18} \int_{\Omega} |\nabla v|^{2(p+1)} dx
\leq \frac{p\mu}{2} \int_{\Omega} u^{p+1} dx + C_{18} K_1 \left( \frac{2(p+1)}{m} \right), \tag{65}
\]
with some \( C_{18} > 0 \) and \( K_1(\cdot) \) is defined in (62).
Similarly, it follows from Young’s inequality again that
\[
C_{17} \int_{\Omega} u^p dx \leq \frac{p\mu}{2} \int_{\Omega} u^{p+1} dx + C_{19}, \tag{66}
\]
with some \( C_{19} > 0. \)
Finally, collecting (64)-(66), we have
\[
\frac{d}{dt} \int_{\Omega} u^p dx + \int_{\Omega} u^p dx \leq C_{20}, \tag{67}
\]
where $C_{20} := C_{19} + C_{18} K_1 \left( \frac{2(p+1)}{m} \right)$. By the comparison argument of ODE, we obtain
\[ \int_{\Omega} u^p \, dx \leq \max \left\{ \int_{\Omega} u_0^p \, dx, C_{20} \right\} \quad \text{for all } t \in (0, T_{\max}). \quad (68) \]
The proof of Lemma 3.4 is complete. \qed

**Lemma 3.5.** Let $(u, v, w)$ be a solution of model (1) and assume that the conditions in Theorem 1.1 hold. Then there exists $C > 0$ such that
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}). \quad (69) \]

**Proof.** By Lemma 3.4 and using parabolic regularity theory (see Lemma 4.1 in [15] or Lemma 1 in [19]) to the second equation in (1), there exists $C_{21} > 0$ such that
\[ \|v(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C_{21} \quad \text{for all } t \in (0, T_{\max}). \quad (70) \]
Combining (60) with (70), we deduce from Young’s inequality that
\[ \frac{d}{dt} \int_{\Omega} u^p \, dx + \frac{c_0 p (p-1)}{2} \int_{\Omega} u^{p+m-3} |\nabla u|^2 \, dx \leq \frac{C_{22} p (p-1) \chi^2}{2 c_0} \int_{\Omega} u^{p-m+1} \, dx + \left[ C_{21} (p-1) \xi \|w_0\|_{L^\infty(\Omega)} \right. \]
\[ + \left. ((p-1) \xi M + pp) \int_{\Omega} u^p \, dx - py \int_{\Omega} u^{p+1} \, dx \right] \leq C_{22} p^2 \int_{\Omega} u^{p-m+1} \, dx + C_{23} \quad \text{for all } t \in (0, T_{\max}) \quad (71) \]
with positive constants $C_{22}$ and $C_{23}$. In view of Lemma 3.4 and using the standard Moser-Alikakos iteration ([1, 30]) (see also Lemma 4.2 in [35]), we derive that $u$ is uniformly bounded in $\Omega \times (0, T_{\max})$. The proof of Lemma 3.5 is complete. \qed

Now we begin with the proof of Theorem 1.1.

**Proof of Theorem 1.1.** By Lemma 3.5, it is easy to see that there exists a positive constant $C_{24} > 0$ such that
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_{24} \quad \text{for all } t \in (0, T_{\max}). \]
With the aid of the blow up criterion (14) in Lemma 2.1, we know $T_{\max} = \infty$. By combining (13) with (63), we can obtain the desired boundedness result. The proof of Theorem 1.1 is complete. \qed

4. **Global weak solutions.** In general, the degenerate diffusion case of (1) might not have classical solutions, thus in order to justify all the formal arguments, we need to introduce the following approximating equation of (1):
\[
\begin{cases}
  u_{et} = \nabla \cdot (\varphi_e(u_e) \nabla u_e) - \chi \nabla \cdot (u_e \nabla v_e) - \xi \nabla \cdot (u_e \nabla w_e) \\
  \quad + \mu u_e (1 - u_e - w_e), & x \in \Omega, t > 0, \\
  v_{et} = \Delta v_e - v_e + u_e, & x \in \Omega, t > 0, \\
  w_{et} = -v_e w_e, & x \in \Omega, t > 0, \\
  \varphi_e \left( \frac{\partial u_e}{\partial \nu} - \chi \frac{\partial v_e}{\partial \nu} - \xi \frac{\partial w_e}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
  u_e(x, 0) = u_{0e}(x), v_e(x, 0) = v_{0e}(x), w_e(x, 0) = w_{0e}(x), & x \in \Omega,
\end{cases}
\quad (72)
\]
where
\[ \varepsilon \in (0, 1), \varphi_\varepsilon(s) = \varphi(s + \varepsilon) \quad \text{for all } s \geq 0, \]
and the initial data \((u_{0\varepsilon}(x), v_{0\varepsilon}(x), w_{0\varepsilon}(x)) \in (W^{1,\infty}(\Omega))^3\) satisfies
\[
\begin{cases}
u_{0\varepsilon} \rightharpoonup u_0 \text{ weakly in } L^\infty(\Omega), \\
v_{0\varepsilon} \rightharpoonup v_0 \text{ weakly in } L^\infty(\Omega) \text{ and } \quad \text{(73)} \\
w_{0\varepsilon} \rightharpoonup w_0 \text{ weakly in } L^\infty(\Omega). 
\end{cases}
\]
Indeed, for each \(\varepsilon \in (0, 1)\), problem (72) possesses a nonnegative classical solution \((u_\varepsilon, v_\varepsilon, w_\varepsilon)\), which is global and bounded by Theorem 1.1.

**Proof of Theorem 1.3.** We first give some estimates for \(u_\varepsilon, v_\varepsilon\) and \(w_\varepsilon\). It follows from Lemma 2.1 and Lemma 3.2-Lemma 3.5 that for each \(T \in (0, \infty)\), there exists a constant \(C(T) > 0\) such that
\[
\int_0^T \|u_\varepsilon(\cdot, t)\|_{L^p(\Omega)} dt \leq C(T) \quad \text{for all } p > 1, T \in (0, \infty),
\]
\[
\int_0^T \int_\Omega \varphi_\varepsilon(u_\varepsilon)u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 dx dt \leq C(T) \quad \text{for all } p > 1, T \in (0, \infty),
\]
\[
\int_0^T \|\nabla v_\varepsilon(\cdot, t)\|_{L^1(\Omega)} dt \leq C(T) \quad \text{for all } T \in (0, \infty) \text{ and }
\|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C(T) \quad \text{for all } t \in (0, T).
\]
In order to achieve a strong precompactness property of \(\{u_\varepsilon\}_{\varepsilon \in (0,1)}\), let us fix \(q \geq \left\{\frac{p+m-1}{2}\right\}\) and multiply the first equation in (72) by \(u_\varepsilon^{q-2} \rho(x)\), where \(\rho(x) \in C_0^\infty(\Omega)\), we infer from integrating it over \(\Omega\) that
\[
\frac{1}{q} \int_\Omega (u_\varepsilon^q) \rho(x) dx
\]
\[\begin{aligned}
= & \int_\Omega \nabla \cdot (\varphi_\varepsilon(u_\varepsilon) \nabla u_\varepsilon) u_\varepsilon^{q-1} \rho(x) dx - \chi \int_\Omega \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) u_\varepsilon^{q-1} \rho(x) dx \\
& - \xi \int_\Omega \nabla \cdot (u_\varepsilon \nabla w_\varepsilon) u_\varepsilon^{q-1} \rho(x) dx + \mu \int_\Omega \nabla \rho(x)(1 - u_\varepsilon - w_\varepsilon) dx \\
= & -(q-1) \int_\Omega \varphi_\varepsilon(u_\varepsilon)u_\varepsilon^{q-2} \rho(x)|\nabla u_\varepsilon|^2 dx - \int_\Omega \varphi_\varepsilon(u_\varepsilon)u_\varepsilon^{q-1} \nabla u_\varepsilon \cdot \nabla \rho dx \\
& + (q-1) \chi \int \Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon dx + \xi \int \Omega \nabla w_\varepsilon \cdot \nabla \rho dx \\
& + (q-1) \xi \int \Omega \nabla w_\varepsilon \cdot \nabla w_\varepsilon dx + \xi \int \Omega \nabla \rho(x)(1 - u_\varepsilon - w_\varepsilon) dx.
\end{aligned}
\]
We choose \(l \in \mathbb{N}\) large enough to satisfy \(l > \frac{n+2}{2}\) and hence \(W_0^{l,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)\). Now collecting (73)-(75), according to our restriction on \(q\), there exists a constant \(C(T) > 0\) (see more details in [32, 34]) such that
\[
\|(u_\varepsilon^q)_t\|_{L^1((0,T);W_0^{l,\infty}(\Omega))^*)} = \sup_{\xi \in C_0^\infty(\Omega), ||\xi||_{W^{l,\infty}(\Omega)} \leq 1} \int_0^T (u_\varepsilon^q)_t \xi(x) dx dt \\
\leq C(T).
\]
Similarly, we have
\[||v_{\varepsilon t}||_{L^2((0,T);(W^{1,2}_{0})(\Omega))} \leq C(T).\]  
(77)

Now in conjunction with (74), (76), (77) and the Aubin-Lions lemma (Theorem III.2.3 in [33]), there exists a subsequence \(\varepsilon = \varepsilon_j \downarrow 0\) as \(j \to \infty\) such that
\[u_{\varepsilon} \to u \quad \text{a.e. in } \Omega \times (0,T),\]
\[\varphi(\varepsilon u_{\varepsilon}) \nabla u_{\varepsilon} \to \varphi(u) \nabla u \quad \text{in } L^2((0,T);L^2(\Omega)),\]
\[v_{\varepsilon} \to v \quad \text{a.e. in } \Omega \times (0,T),\]
\[v_{\varepsilon} \to v \quad \text{weakly in star } L^\infty(\Omega \times (0,T)),\]
\[\nabla v_{\varepsilon} \to \nabla v \quad \text{weakly in star } L^\infty(\Omega \times (0,T)),\]
\[w_{\varepsilon} \to w \quad \text{a.e. in } \Omega \times (0,T),\]
\[w_{\varepsilon} \to w \quad \text{weakly in star } L^\infty(\Omega \times (0,T)) \quad \text{and} \]
\[\nabla w_{\varepsilon} \to \nabla w \quad \text{weakly in star } L^\infty(\Omega \times (0,T)).\]

Therefore, we select \(\zeta \in C^\infty_0(\Omega \times [0,T]), \eta \in C^\infty_0(\Omega \times [0,T])\) and \(\theta \in C^\infty_0(\Omega \times [0,T])\) for all \(T \in (0,\infty)\). Multiplying the first, second and third equations of (72) by \(\zeta, \eta\) and \(\theta\), respectively, and then integrating by party, we see that \((u_{\varepsilon}, v_{\varepsilon}, w_{\varepsilon})\) satisfies
\[-\int_0^T \int_\Omega u_{\varepsilon} \zeta_t dxdt - \int_\Omega u_0 \zeta(x,0) dxdt = -\int_0^T \int_\Omega \varphi(\varepsilon u_{\varepsilon}) \nabla u_{\varepsilon} \cdot \nabla \zeta dxdt + \chi \int_0^T \int_\Omega u_{\varepsilon} \nabla v_{\varepsilon} \cdot \nabla \zeta dxdt \]
\[+ \xi \int_0^T \int_\Omega u_{\varepsilon} \nabla w_{\varepsilon} \cdot \nabla \zeta dxdt + \mu \int_0^T \int_\Omega u_{\varepsilon}(1 - u_{\varepsilon} - w_{\varepsilon}) dxdt\]  
(79)

and
\[-\int_0^T \int_\Omega v_{\varepsilon} \eta_t dxdt - \int_\Omega v_0 \eta(x,0) dxdt = -\int_0^T \int_\Omega \nabla v_{\varepsilon} \cdot \nabla \eta dxdt - \int_0^T \int_\Omega v_{\varepsilon} \eta dxdt + \int_0^T \int_\Omega u_{\varepsilon} \eta dxdt\]  
(80)

as well as
\[\int_0^T \int_\Omega w_{\varepsilon} \theta dxdt = \int_0^T \int_\Omega w_0 \theta e^{\int_0^T \varphi(x,t) dt} \eta dxdt.\]  
(81)

By using (73) and (78) in passing to the limit in each term of the identities (79)-(81), we obtain
\[-\int_0^T \int_\Omega u_{\zeta} dxdt - \int_\Omega u_0 \zeta(x,0) dxdt = -\int_0^T \int_\Omega \varphi(u) \nabla u \cdot \nabla \zeta dxdt + \chi \int_0^T \int_\Omega u \nabla v \cdot \nabla \zeta dxdt\]
\[+ \xi \int_0^T \int_\Omega u \nabla w \cdot \nabla \zeta dxdt + \mu \int_0^T \int_\Omega u \zeta(1 - u - w) dxdt\]  
(82)
and
\[
- \int_0^T \int_\Omega v_\eta dxdt - \int_\Omega v_0(x,0) dxdt
- \int_0^T \int_\Omega \nabla v \cdot \nabla \eta dxdt - \int_0^T \int_\Omega v \eta dxdt + \int_0^T \int_\Omega u \eta dxdt
\]
(83)
as well as
\[
\int_0^T \int_\Omega w \theta dxdt = \int_0^T \int_\Omega w_0 \theta e^{\int_0^s v(x,s) ds} dxdt.
\]
(84)
Hence, it is easy to see that \((u, v, w)\) is a global weak solution for (1). Finally, the boundedness statement is a straightforward consequence of the proof of Theorem 1.1. The proof of Theorem 1.3 is complete.  

5. Decay of \(w\). In this section, inspired by Hillen et al. in [12], we consider the decay estimate of \(w\) under a suitable smallness condition on \(w_0\) and prove Theorem 1.4. The proof is mainly based on a lower bound for the mass 
\[
m(t) = \int_\Omega u(x,t) dx.
\]
To do this, we need the following lemmas.

Lemma 5.1. Let \(\Omega \subset \mathbb{R}^2\) be a bounded domain with smooth boundary. Assume that \(\varphi\) satisfies the condition (5)-(7) with \(m \in (1, 2)\). Suppose that \(\chi > 0, \xi > 0, \mu > 0\), and the initial data \((u_0, v_0, w_0)\) satisfies (2) and \(|w_0|_{L^\infty(\Omega)} < 1\). Then there exists \(m_* > 0\) such that
\[
\int_\Omega u(x,s) dx \geq m_* \quad \text{for all } t \in (0, \infty).
\]
(85)

Proof. The proof is mainly based on the arguments in [12]. According to the strong maximum principle and the hypothesis \(u_0 \not\equiv 0\) in (2), then we know that \(u\) is positive in \(\Omega \times (0, \infty)\). Thus, we may multiply the first equation in (1) by \(u^{\gamma-1}, \gamma \in (0, 1)\) and integrate by parts over \(\Omega\) to obtain
\[
\frac{1}{\gamma} \frac{d}{dt} \int_\Omega u^{\gamma} dx
= (1 - \gamma) \int_\Omega u^{\gamma-2} \varphi(u)|\nabla u|^2 dx - \chi(1 - \gamma) \int_\Omega u^{\gamma-1} \nabla u \cdot \nabla v dx
\]
\[
- \xi(1 - \gamma) \int_\Omega u^{\gamma-1} \nabla u \cdot \nabla w dx + \mu \int_\Omega u^{\gamma} dx - \mu \int_\Omega u^{\gamma+1} dx - \mu \int_\Omega u^{\gamma} w dx
\]
\[
:= I - II - III + IV - V - VI \quad \text{for all } t \in (0, \infty).
\]
(86)
According to the conditions (5)-(7), we obtain
\[
I = (1 - \gamma) \int_\Omega u^{\gamma-2} \varphi(u)|\nabla u|^2 dx \geq c_0(1 - \gamma) \int_\Omega u^{\gamma+m-3} |\nabla u|^2 dx.
\]
(87)
By Young’s inequality, we deduce from (70), Hölder’s inequality and \(1 < m < 2\) that
\[
II = \chi(1 - \gamma) \int_\Omega u^{\gamma-1} \nabla u \cdot \nabla v dx
\]
\[
\leq \frac{c_0(1 - \gamma)}{2} \int_\Omega u^{\gamma+m-3} |\nabla u|^2 dx + \frac{(1 - \gamma) \chi^2}{2c_0} \int_\Omega u^{\gamma-m+1} |\nabla v|^2 dx.
\]
\[
\begin{align*}
&\leq \frac{c_0(1-\gamma)}{2} \int_\Omega u^{\gamma+m-3}|\nabla u|^2 dx + \frac{(1-\gamma)C_2^2}{2c_0} \int_\Omega u^{\gamma-m+1} dx \\
&\leq \frac{c_0(1-\gamma)}{2} \int_\Omega u^{\gamma+m-3}|\nabla u|^2 dx + \frac{(1-\gamma)C_2^2}{2c_0} \frac{m-1}{\gamma} \left( \int_\Omega u^\gamma dx \right)^{\frac{\gamma-m+1}{\gamma}}. 
\end{align*}
\]

(88)

It follows from Lemma 2.3 and (63) that

\[
III = \xi (1-\gamma) \int_\Omega u^{\gamma-1} \nabla u \cdot \nabla w dx = -\frac{(1-\gamma)\xi}{\gamma} \int_\Omega u^\gamma \Delta w dx
\]

\[
\leq \frac{(1-\gamma)\xi}{\gamma} \|w_0\|_{L^\infty(\Omega)} \int_\Omega u^\gamma dx + (1-\gamma)\xi M \int_\Omega u^\gamma dx
\]

\[
\leq \left( C_16(1-\gamma)\xi \|w_0\|_{L^\infty(\Omega)} + (1-\gamma)\xi M \right) \int_\Omega u^\gamma dx.
\]

(89)

By the Gagliardo-Nirenberg inequality in Lemma 2.4, we deduce from Lemma 2.2 that there exist \(C_{25} > 0\) and \(C_{26} > 0\) such that

\[
V = \mu \int_\Omega u^{\gamma+1} dx
\]

\[
= \mu \|u^{\frac{\gamma+1}{2}}\|_{L^{2(\gamma+1)}(\Omega)}^{2(\gamma+1)}
\]

\[
\leq C_{25} \left( ||\nabla u^{\frac{\gamma+1}{2}}||_{L^2(\Omega)} \cdot ||u^{\frac{\gamma+1}{2}}||_{L^{2(\gamma+1)}(\Omega)}^{1-\gamma} \right)
\]

\[
= C_{25} \left( ||\nabla u^{\frac{\gamma+1}{2}}||_{L^2(\Omega)} \cdot \|u\|_{L^{2(\gamma+1)}(\Omega)}^{1-\gamma} \right)
\]

\[
\leq C_{26} \left( ||\nabla u^{\frac{\gamma+1}{2}}||_{L^2(\Omega)} + 1 \right),
\]

(90)

where we have used the fact that the space dimension is two, and

\[
\lambda = \frac{\gamma}{\gamma+1} \in (0,1).
\]

(91)

Due to the fact that \(m > 1\), it is easy to see that

\[
\frac{2\lambda(\gamma+1)}{\gamma+m-1} = \frac{2\gamma}{\gamma+m-1} < 2.
\]

(92)

Thus, using Young’s inequality, it follows from (90) and (92) that there exists a positive constant \(C_{27}\) such that

\[
V = \mu \int_\Omega u^{\gamma+1} dx
\]

\[
\leq \frac{c_0(1-\gamma)}{2} \int_\Omega u^{\gamma+m-3}|\nabla u|^2 dx + C_{27}.
\]

(93)

Finally, we denote

\[
\eta := \mu - \mu \|w_0\|_{L^\infty(\Omega)} > 0,
\]
due to the condition \( \| w_0 \|_{L^\infty(\Omega)} < 1 \). By using the fact that \( w_i \leq 0 \), we have

\[
VI = \mu \int_{\Omega} u^\gamma w dx \leq \mu \int_{\Omega} u^\gamma w_0 dx \leq (\mu - \eta) \int_{\Omega} u^\gamma dx.
\]

(94)

Hence, collecting (86)-(89), (93) and (94), we derive

\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u^\gamma dx \geq \left\{ \mu - \left( C_{16} (1 - \gamma) \xi \| w_0 \|_{L^\infty(\Omega)} + \frac{(1 - \gamma) \xi M}{\gamma} \right) - (\mu - \eta) \right\} \int_{\Omega} u^\gamma dx \\
- \frac{(1 - \gamma) C_{21}^2 x^2 |\Omega|^{\frac{m-1}{2}}}{2c_0} \left( \int_{\Omega} u^\gamma dx \right)^{\frac{2-m}{m+1}} - C_{27}.
\]

(95)

Now, picking \( \gamma \in (0, 1) \) sufficiently close to one such that

\[
(\xi - \gamma) C_{16} \| w_0 \|_{L^\infty(\Omega)} + M \left( \frac{1 - \gamma}{\gamma} \right) < \eta
\]

(96)

and

\[
\frac{\gamma - m + 1}{\gamma} \in (0, 1),
\]

(97)

due to the condition \( 1 < m < 2 \). Therefore, we infer from Young’s inequality that

\[
\frac{d}{dt} \int_{\Omega} u^\gamma dx \geq \frac{\eta \gamma}{2} \int_{\Omega} u^\gamma dx - C_{28} \gamma \left( \int_{\Omega} u^\gamma dx \right)^{\frac{2-m}{m+1}} - C_{27} \gamma \\
\geq \frac{\eta \gamma}{2} \int_{\Omega} u^\gamma dx - \frac{\eta \gamma}{4} \int_{\Omega} u^\gamma dx - C_{29} \\
= \frac{\eta \gamma}{4} \int_{\Omega} u^\gamma dx - C_{29},
\]

(98)

where \( C_{28} := \frac{(1 - \gamma) C_{21}^2 x^2 |\Omega|^{\frac{m-1}{2}}}{2c_0} > 0 \) and \( C_{29} > 0 \). By the ODE comparison argument, we have

\[
\int_{\Omega} u^\gamma dx \geq C_{30} := \min \left\{ \int_{\Omega} u_0^\gamma dx, \frac{4C_{29}}{\eta \gamma} \right\} \text{ for all } t > 0.
\]

(99)

Since \( \gamma \in (0, 1) \), it follows from H"{o}lder’s inequality that

\[
\int_{\Omega} u^\gamma dx \leq |\Omega|^{1 - \gamma} \left( \int_{\Omega} u dx \right)^{\gamma}.
\]

(100)

Thus, we deduce from (99) that

\[
\int_{\Omega} u dx \geq \frac{4C_{29}}{\eta \gamma} \left( \int_{\Omega} u^\gamma dx \right) \geq |\Omega|^{\frac{\gamma}{\gamma - 1}} C_{30}^\frac{\gamma}{\gamma - 1} := m_*.
\]

(101)

The proof of Lemma 5.1 is complete.

Next, we shall give the following lower bound estimate for \( v \). For the details of the proof, please see Corollary 3.3 in [12] (see also Lemma 4.5 in [31] or Lemma 2.1 in [8, 9, 10]).

**Lemma 5.2.** (see Corollary 3.3 in [12]). Under the conditions of Theorem 1.4, there exist \( \Gamma > 0 \) and \( C_{31} > 0 \) such that the solution \( v \) of (1) fulfills the inequality

\[
\int_{0}^{t} v(x,s)ds \geq \Gamma \int_{0}^{t} \int_{\Omega} u(y,s)dyds - C_{31} \text{ for all } x \in \Omega \text{ and } t > 0.
\]

(102)
Proof of Theorem 1.4. By combining Lemma 5.1 and Lemma 5.2, we have

$$\int_0^t v(x,s)ds \geq \Gamma \int_{\Omega} \int_0^t u(y,s)dyds - C_{31} \geq m_\star \Gamma t - C_{31} \quad \text{for all } t > 0. \quad (103)$$

Since the third equation in (1) is a ODE, then

$$w(x,t) = w_0(x)e^{-f_0^s v(x,s)ds} \quad \text{for all } x \in \Omega \text{ and } t \in (0, \infty), \quad (104)$$

so we obtain

$$\nabla w(x,t) = \nabla w_0(x)e^{-f_0^s v(x,s)ds} - w_0(x)e^{-f_0^s v(x,s)ds} \int_0^t \nabla v(x,s)ds \quad (105)$$

for all $x \in \Omega$ and $t \in (0, \infty)$.

Thus it follows from (104) that

$$||w(\cdot,t)||_{L^\infty(\Omega)} \leq ||w_0||_{L^\infty(\Omega)}e^{-m_\star \Gamma t + C_{31}} = e^{C_{31}} ||w_0||_{L^\infty(\Omega)}e^{-m_\star \Gamma t} \quad \text{for all } t > 0. \quad (106)$$

On the other hand, collecting (70), (103) and (105), we have

$$||\nabla w(\cdot,t)||_{L^\infty(\Omega)} \leq ||\nabla w_0||_{L^\infty(\Omega)}e^{-m_\star \Gamma t + C_{31}} + ||w_0||_{L^\infty(\Omega)}e^{-m_\star \Gamma t + C_{31}} C_{21} t$$

$$= e^{C_{31}} ||\nabla w_0||_{L^\infty(\Omega)}e^{-m_\star \Gamma t} + \frac{2C_{21}}{m_\star \Gamma} e^{C_{31}} ||w_0||_{L^\infty(\Omega)}e^{-\frac{m_\star \Gamma t}{2}}e^{-\frac{m_\star \Gamma t}{2}} \quad (107)$$

$$\leq C_{32} e^{-\frac{m_\star \Gamma t}{2}} \quad \text{for all } t > 0,$$

where $C_{32} := e^{C_{31}} \left( ||\nabla w_0||_{L^\infty(\Omega)} + \frac{2C_{21}}{em_\star \Gamma} ||w_0||_{L^\infty(\Omega)} \right)$ and we have used the facts that $ze^{-z} \leq \frac{1}{2}$ for all $z \in \mathbb{R}$ and $e^{-m_\star \Gamma t} \leq e^{-\frac{m_\star \Gamma t}{2}}$. Thus, combining (106) with (107), we know that (12) holds. The proof of Theorem 1.4 is complete. \qed

Acknowledgments. The authors would like to thank the editors and anonymous reviewers for their valuable comments and suggestions for the original manuscript. The first author is partially supported by the Doctor Start-up Funding and the Natural Science Foundation of Chongqing University of Posts and Telecommunications (Grant No. A2014-25 and A2014-106), partially supported Scientific and Technological Research Program of Chongqing Municipal Education Commission (Grant No. KJ1500403) and the Basic and Advanced Research Project of CQC-STC (Grant No. cstc2015jcyjA0008); the second author is partially supported by NSFC (Grant No. 11371384); and the third author is partially supported by the Scientific Research Found of Sichuan Provincial Education Department (Grant No.12ZA288). The authors would like to appreciate Professor Michael Winkler who raised this interesting problem.

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Received December 2014; revised May 2015.

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