A New Analytical Model for the Estimation of Three-phase Rock-soil Thermal Conductivity for Geothermal Utilization

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Abstract. As a kind of clean energy resource, geothermal energy is widely utilized in many fields, especially for heating and ventilation through heat exchangers buried underground. The use of geothermal heat exchange system will contribute to energy saving as well as building sustainable, therefore it has become more and more popular in recent years. Rock-soil thermal conductivity plays a noticeable role in the performance of ground buried heat exchanger. In the present study, a new analytical model was proposed to describe the spatial structure of the multiphase rock-soil and relative position of solid, liquid and gas phase. Through analyzing the relationship among different phases and coding FORTRAN program, the model structure parameters can be obtained for the calculation of parallel thermal resistances. The expression of the thermal conductivity derived from the model was then applied to obtain the thermal conductivity of the Tripoli sand from North Africa, before the results were compared with previous tests. After comparative analysis, the newly proposed model in this study was proved accurate in predicting the thermal conductivity of the chosen soil with around 20% averaged relative error, which will contribute to the prediction of rock-soil thermal properties as well as the design of ground buried heat exchanger.

1. Introduction

Rock-soil Thermal conductivity plays a decisive role in the prediction of buried heat exchanger performance due to its connection with the heat conduction between the heat exchanger pipe wall and surrounding rock-soil. Rock-soil is a mixture of two or three phases and its thermal conductivity is affected by various factors including the porosity, the volumetric proportion and thermal conductivities of the three components, the radius of solid particles and the water content, etc. Along with the extension in the length of buried heat exchanger, the nearby rocks show disparate thermal properties because of different geological structure, which further makes the theoretical analysis and design procedure more complex compared with shallow buried heat exchanger. Therefore, it is crucial to evaluate the thermal conductivity accurately for either theoretical calculation or simulation with software packages on the geothermal utilization.

Extensive research on more accurate estimation of the rock-soil thermal conductivities was performed in recent years. Dong et al. [1] conducted a review on the existing methods for calculating soil thermal conductivities in terms of the mineralogy, the particle size and shape, the water content, the porosity and the packing geometry, etc. Existing models were categorized into mixing models (regarding the mixtures as an aggregation of series or parallel elements representing the three phases...
(2)], mathematical models (calculated through the thermal conductivity and the volume fraction of the three phases [3]) and empirical models (fit the experimental results with functions for certain type of rock-soil [4]) based on different models’ respective characteristics. These models were verified to be able to predict the thermal conductivity at portions of saturation range for corresponding soil type with relatively small error, but all the above models still lack the consideration of some important factors such as the liquid type and pore size distribution. Haigh et al. [5] derived a simplified two-dimensional analytical model with sectorial solid part for three-phase sands calculation, which was shown to be better than the empirical models in the prediction process involved in the validation. Gori and Corasaniti [6] presented a three-dimensional model which can give good prediction with the soil porosity in the range between 0.0349 and 0.4734 from dryness to saturation without empirical constants. Surrounded by air and liquid water (ice), the solid part was represented by a cubic cell with spherical grain inside it. The porosity and saturation degree were calculated through the structural dimension parameters and the expression of the effective thermal conductivities was obtained by dealing with steady state heat conduction problem of the parallel heat flux through the model. Afterwards, they put forward a unit cell model with a rhomboidal base to obtain the thermal conductivity of two or three-phase media with porosity greater than 0.4764 [7].

The objective of this study is to propose a universal analytical model including the important controlling parameters for estimating the thermal conductivities of various rock-soils. Taking key parameters into consideration including the porosity, the water content, the particle size and the thermal conductivities of different phases, the proposed model aims to maintain the structural features and provide relatively accurate prediction of the soil thermal conductivity for further use in the geothermal utilization.

2. Methodology
The newly proposed analytical model employs a cubic cell (See Figure 1) with two equally sized hemispheres, which represent half the volume of two solid particles, contacted with each other inside the cubic as the elementary unit. Following assumptions were adopted,

(1) The shape of the solid particle is sphere. The distance between the centers of the two equally sized solid spheres $2L$ is determined according to the porosity.
(2) The liquid phase is held by surface tension around the contact segment of the two hemispheres with cylindrical surface and the rest space is filled with gas.
(3) All the phases are homogeneous and each point in the same phase has the same thermal properties.
(4) The heat flux is parallel to the $z$-axis. Heat convection and radiation were ignored during the calculating process.
(5) No chemical reaction occurs among different phases.

The thermal conductivity of the model will be acquired by mathematical derivation based on the heat conduction through the cubic and the network of thermal resistance in series and parallel. According to the Fourier's law of heat conduction,

$$Q = -\lambda A \frac{\partial T}{\partial x} = \lambda A \frac{\Delta T}{\delta}$$

where $Q$ is the heat flow, $\lambda$ is the thermal conductivity of the rock-soil, $A$ is the cross-sectional area perpendicular to the heat flow direction, $\Delta T$ is the temperature difference and $\delta$ is the thickness. Then the thermal resistance can be written as,

$$R = \frac{\delta}{A \lambda}$$

In order to derive the expression of the thermal resistance, the porosity $n$, the radius of the solid particle $r$ and the volumetric water content $w$ are applied to obtain the key structure parameters of the cubic model. The height of the cubic $2L$ and the radius of the contact surface of the two hemispheres $r_e$.
can be obtained by establishing the relationship between the porosity \( n \) and the cubic element structure and expressed as,

\[
L = r \sqrt{3\pi - 12 + 12n / \pi} \tag{3}
\]

\[
r_s = \sqrt{r^2 - L^2} \tag{4}
\]

The radius of the liquid column \( r_2 \) is determined according to the water content \( w \). The latter can be expressed through the ratio of the liquid volume and cubic total volume,

\[
w = \frac{V_{\text{liquid}}}{V_0} = \frac{2\left\{ \pi b^3 \left[ r^2 - (b - L)^2 \right] - \int_0^b \pi \left[ r^2 - (z - L)^2 \right] dz \right\}}{8r^2L} \tag{5}
\]

where \( V_{\text{liquid}} \) and \( V_0 \) are the volumes of the liquid phase and the whole cubic respectively. The structure parameter \( b \) can be obtained through solving the following cubic geometric equation,

\[
\frac{2}{3}b^3 - Lb^2 + \frac{4r^2Lw}{\pi} = 0 \tag{6}
\]

It is noted that directly solving the above cubic equation is difficult because the values of \( L, r \) and \( w \) are unknown until specific rock-soil type and parameters are determined. However, after regarding the left side of the equation as a function of \( b \) \( f(b) \), \( 0 < b < L \) and taking the derivative, the function’s monotonicity can be obtained. The value of \( f(0) \) is positive and \( f(L) \) is negative. Therefore, the above cubic equation can be solved using dichotomy by coding the program. Then the structure parameter \( r_2 \) can be calculated as,

\[
r_2 = \sqrt{r^2 - (b - L)^2} \tag{7}
\]

According to Equation (2), the thermal resistance of each part from the inner to the outer regions (as shown in Figure 2) can be calculated directly \( (R_1 \text{ and } R_4) \) or through integration \( (R_2 \text{ and } R_3) \) along the heat flow direction \( (z \text{ direction}) \) and the final expressions are shown respectively as,

\[
\frac{1}{R_i} = \frac{\pi r_s^2 \lambda_s}{2L} \tag{8}
\]
\[ \frac{1}{R_2} = -\frac{\pi \lambda_s r}{\left(1 - \frac{\lambda_s}{\lambda_g}\right)} \left[ \frac{1}{\lambda_s} \left(\cos \alpha_1 - \cos \alpha_2\right) - \cos \alpha_1 \ln \left(1 - \frac{\lambda_s}{\lambda_g} \right) \right] \]  

\[ \frac{1}{R_3} = -\frac{\pi \lambda_l r}{\left(1 - \frac{\lambda_l}{\lambda_g}\right)^2} \left[ \frac{1}{\lambda_l} \cos \alpha_2 + \cos \alpha_1 \ln \left(1 - \frac{\lambda_l}{\lambda_g} \right) \right] \]  

\[ \frac{1}{R_4} = \frac{(4 - \pi) r^2 \lambda_g}{2L} \]  

(9)  

(10)  

(11)  

where, \( \lambda_s, \lambda_l, \lambda_g \) are the thermal conductivities of solid, liquid and gas phase respectively. The intermediate variables \( \alpha_1, \alpha_2 \) are expressed respectively as,

\[ \alpha_1 = \arcsin \frac{r}{r} \]  

\[ \alpha_2 = \arcsin \frac{r}{r} \]  

(12)  

(13)  

Finally, the cubic model effective thermal conductivity can be expressed using total thermal resistance \( R_{total} \) through Equation (2) as,

\[ \lambda = \frac{\delta}{\rho R_{total}} = \frac{2L}{4r^2 R_{total}} \]  

(14)  

Figure 2. Parallel resistance of four parts in the cubic model.

3. Validation and analysis

The above-mentioned calculation method is implemented by coding FORTRAN program and validated by the thermal conductivities of Tripoli sand (sandy soil sample from North Africa) measured by Alrtimi et al. [8]. The soil sample is composed of 93.25% silica (Silicon Dioxide) with grain size of 0.2 mm and negligible amounts of inorganic matter. Figure 3 shows the comparison between the experimental and predicted results with the rock-soil thermal conductivities being ordinate and the water content as the abscissa. The porosities are 0.432 and 0.462 respectively for the involved two groups of soil samples. The thermal conductivity values of the solid, liquid and gas phase are determined to be 1.4 W·m⁻¹K⁻¹, 0.6036 W·m⁻¹K⁻¹ and 0.0253 W·m⁻¹K⁻¹ respectively.
according to their thermal properties at room temperature [9]. It can be seen that the model predictions agree well with the measured data. Under circumstance of very low water content (w = 0.006), the predicted thermal conductivities are 0.245 and 0.234 higher than the experimental ones for the chosen two porosities respectively. When \( n = 0.462 \) and \( w = 0.050 \), the maximum difference of thermal conductivity between the experimental and predicted values \( \Delta \lambda \) is 0.390 with the relative error being 22.91%. Meanwhile, with the increase of liquid phase fraction, the thermal conductivities rise gradually associated with the reduced relative errors. The maximum relative error is 66.5% while \( n = 0.462 \) and \( w = 0.014 \) and the minimum relative error is 5.82% while \( n = 0.462 \) and \( w = 0.240 \). The averaged relative error is around 20%, which is much lower than the prediction results using Chen’s empirical equation (63.10% maximum relative error, 13.36% minimum relative error and around 33.94% averaged relative error) [4] and Haigh’s analytical model (81.23% maximum relative error, 44.09% minimum relative error and around 65% averaged relative error) [5] on this sandy soil. In general, the prediction results in this work and test thermal conductivity values show similar tendency with small errors as the water content goes up, which implies that the proposed model is capable to predict the thermal conductivity of selected sandy soil accurately in corresponding ranges of porosity and water content.

![Figure 3. Experimental versus model prediction results.](image)

### 4. Conclusions

To give prediction of the rock-soil thermal conductivity for geothermal application, this study presented a new analytical model to represent the component of three-phase rock-soil. The model aims to reconstruct the relative position of different phases and provide guidance for the thermal resistance calculation. Using physical parameters and thermal conductivities of different phases, the structure parameters of the cubic model can be acquired. Corresponding calculating method based on the relevant physical parameters and parallel theory was also introduced for the estimation of the cubic model thermal conductivity. The comparison with the test results of sandy soil carried out by Alrtimi et al. [8] was demonstrated and analyzed for the verification. The results indicate that the newly proposed cubic model is valid in predicting the thermal conductivity of selected sort of soil with averaged relative error around 20%. There are some points that can be improved and complete in the future work,

1. The present model was put forward and simplified according to the structure characteristics of the actual rock-soil. The thermal resistances of the four parallel parts were calculated to obtain the thermal conductivity without considering the contact resistance. Further research will take the contact thermal resistance into consideration to complete the present model.
2. In the present study, the model was validated with the Tripoli sand from North Africa. More validation with different types of rock-soil needs to be carried out for accuracy improvement of the model in the future work.

The newly proposed model and corresponding calculation method can be used as a tool to predict thermal conductivities of various rock-soils after further perfection. Then the obtained thermal conductivity value can be adopted in predicting the amount of heat conduction between the ground...
buried heat exchanger and surrounding rock-soil using either analytical model (line-source model, cylindrical-source model and so forth) or numerical simulation.

Acknowledgements
This work was supported by the Scientific and Technological Innovation Project in Shaanxi Province (2015KTCQ01-99), the Key Scientific Research Innovation Team Project of Shaanxi Province (2016KCT-16).

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