Finite dimensional Hamiltonian formalism for gauge and field theories

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May 11, 2000

Abstract

We discuss in this paper the canonical structure of classical field theory in finite dimensions within the *pataplectic* Hamiltonian formulation, where we put forward the role of Legendre correspondence. We define the generalized Poisson p-brackets which are the analogues of the Poisson bracket on forms. We formulate the equations of motion of forms in terms of p-brackets. As illustration of our formalism we present three examples: the interacting scalar fields, conformal string theory and the electromagnetic field.

1 Introduction

In the standard Hamiltonian formulation of classical point particle mechanics, the phase space of a system with $N$ degrees of freedom is a $2N$ dimensional manifold, which represents the space of possible positions and momenta of the system. In
field theories the objects under consideration typically have an infinite number of degrees of freedom. Thus we need to employ infinite dimensional manifolds to model the dynamical possibilities for such objects. This requires a generalization of the familiar theory of finite dimensional manifolds, and so we are motivated to stipulate that an infinite dimensional manifold is a manifold modeled on an infinite dimensional Banach space.

Infinite dimensional manifolds do, of course, differ in many significant ways from their finite dimensional counterparts. No infinite dimensional manifold is locally compact, for instance, although every finite dimensional is. Furthermore, the fact that the tangent spaces are infinite dimensional leads to some complications which are not present in the finite dimensional case. Two are worth mentioning here:

1) In the finite dimensional case, a linear map \( T : V \to V \) is one to one iff it is onto; in the infinite dimensional case a linear operator on \( V \) can be one to one but not onto.

2) Similarly, in the finite dimensional case, all linear operators are continuous maps from \( V \) to \( V \); in the infinite dimensional case, the continuous linear operators, i.e. the bounded operators, often have as their domain of definition a dense (proper) subset of \( V \).

A crucial step in the formulation of Hamiltonian mechanics is the construction of the Poisson bracket between a pair of physical observables. This is obtained from the natural symplectic structure on \( T^*M \) (where \( M \) is the configuration space of the physical system). In this phase space approach to classical mechanics, the dynamical evolution from an initial point \( x_0 \in T^*M \) is the solution to Hamilton’s first order differential equations. Geometrically, dynamical paths in phase space can be identified with the flow lines of a special vector field \( \xi_H \) on \( T^*M \) associated with the Hamiltonian function \( H \). Those dynamical equations imply the time rate of change of any physical observable \( f \in C^\infty(T^*M, \mathbb{R}) \), precisely through the Poisson bracket of \( f \) with \( H \) which is defined thanks to a Hamiltonian vector field \( \xi_f \) on \( T^*M \) associated with \( f \). There are couple of potential difficulties here. If \( T^*M \) is infinite dimensional, then some perfectly good functions may not have a Hamiltonian vector field (this problem does not arise in the finite dimensional case). Even when \( \xi_f \) exists, its integral curves may be incomplete (i.e., the vector field is only locally defined).

Further, in this viewpoint space and time are treated asymmetrically, therefore we do not have a covariant scheme.

In order to avoid these difficulties, an alternative approach is to construct covariant canonical formulations of (finite dimensional) field theories which treat the space and time in equal footing (symetrically). Remark that there is a whole variety of such theories and interestingly enough they offer a generalization of the Hamilton canonical equations of motion to field theory, see for instance \( [1, 3, 2, 4] \). Further details can be found in \( [5, 7, 8, 9, 24, 28] \), and \( [30, 31, 27, 26] \). One point there is that the observable quantities are not represented by (generalized) functions on a phase space, but rather by \( n - 1 \)-forms, whose integrals on Cauchy hypersurfaces give back the usual observables. But many of those approaches share a characteristic, which is an obstacle to the development of a field quantization,
the lack of an appropriate generalization of the Poisson bracket. And even if a Poisson bracket was proposed, the related construction was too restrictive and not appropriate for representing the generalized Hamiltonian field equations in Poisson bracket formulation.

More recently, a definition of the Poisson brackets on a subclass of forms and the equations of motion of forms from De Donder-Weyl point of view was given for review see [12, 13]. The main point is to derive the Hamiltonian fields equations from the Poincaré-Cartan n-form and its differential, called there polysymplectic form using “vertical multivector fields” (which generalize the Hamiltonian vector fields in mechanics). Constructions of brackets can be done using also the polysymplectic form, but a correct expression of the dynamics of these forms requires a decomposition of forms and multivectors along “vertical” and “horizontal” components. This decomposition, however, essentially implies a triviality of the “extended polymomentum phase space” as a bundle over the space-time manifold. Moreover we notice that in those works although a natural link between Poisson brackets and dynamics exists for n − 1-forms, in the case of forms of arbitrary degrees the link is not clear. This affect the possibility of a precise formulation, for example, (of the dynamics) of Maxwell’s electrodynamics. In addition, we don’t have a representation of the energy-momentum tensor.

In this paper we exhibit a general construction of a universal Hamiltonian formalism (which contains all previously known formalisms, which explains the appellation universal) and define the generalized Poisson p-brackets, the analogues of the Poisson bracket on forms. We formulate the equations of motion of forms in terms of those p-brackets. The main focus in this construction is on the role of Legendre correspondence, and the hypothesis concerning the generalized Legendre condition. We want to emphasize here that in our formalism there is no need to the decomposition “vertical” and “horizontal” parts thanks to the use of Equation (17) which is much more enlightened than Equation (15). This implies Theorems 2 and 3. On the other hand the energy-momentum tensor is clearly represented and the Hamiltonian formulation of Maxwell’s electrodynamics, for instance, is properly given.

In section (2) we establish the Hamiltonian formalism: the Euler-Lagrange equations, Legendre’s correspondance (and the generalized Legendre condition) and Hamilton’s equations. In fact, we recover the Hamilton’s equations using three

\[ L(u^i, \partial_\alpha u^i, x^\alpha) \rightarrow H(u^i, p^\alpha_i, x^\alpha) := p^\alpha_i \partial_\alpha u^i - L \]

So the phase space is replaced by a finite dimensional space.

1 In this approach we associate to the generalized coordinates (the field variables) \( u^i \) a set of \( n \) momentum-like variables (which are defined from the Lagrangian as the conjugate momenta associated with each space-time - here \( \alpha = 1, ..., n \) is the space-time index - derivative of the field), \( p^\alpha_i := \frac{\partial L}{\partial (\partial_\alpha u^i)} \), and we have the Legendre transform: \( \partial_\alpha u^i \rightarrow p^\alpha_i \),

\[ L(u^i, \partial_\alpha u^i, x^\alpha) \rightarrow H(u^i, p^\alpha_i, x^\alpha) := p^\alpha_i \partial_\alpha u^i - L \]

2 This generalization of the Poisson bracket formulation of the equations of motion to forms of arbitrary degree requires a certain extension. Namely, by adding horizontal forms of degree \( n \) and the vertical-vector valued horizontal one forms (objects of formal degree zero) associated with \( n \)-forms. This extension, call for a generalization of Lie, Schouten-Nijenhuis and Frölicher-Nijenhuis brackets.
different approaches: a) as necessary and sufficient conditions for the existence of a critical point \( u : \mathcal{X} \to \mathcal{Y} \), b) by contracting the pataplectic form \( \Omega \) with any \( n \)-vector \( X \in \Lambda^n T_{(q,p)} \mathcal{M} \) where \( \mathcal{M} = \Lambda^n T^*(\mathcal{X} \times \mathcal{Y}) \) and for any \( (q,p) \in \mathcal{M} \) and finally c) by variational formulation i.e. as the Euler-Lagrange equations of some simple functional. In section (3) we review the usual approach to quantum field theory from the standard canonical viewpoint and pataplectic geometry point of view where we express the various brackets using an analogue of the Poisson brackets, the Poisson \( p \)-brackets, defined on \( n-1 \)-forms. In subsection (3.4) we give a dynamical formulation for the \( n-1 \)-forms in terms of \( p \)-brackets with the \( n \)-form \( \mathcal{H} \omega \).

In section (4), and after introducing the internal and external \( p \)-brackets, we generalize the definition of \( p \)-bracket on \( n-1 \)-forms to a class of forms of an arbitrary degrees \( 0 \leq p \leq n \) using anticommuting (Grassmann) variables \( \tau_1 \ldots \tau_n \) which behave under change of coordinates like \( \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \). We should add that those anticommuting variables do not appear in the expression of the dynamics of forms of arbitrary degrees\(^3\). Notice also that the generalized Poisson bracket which is obtained here differs from the one proposed in [12, 13] for forms of degree lower than \( p-1 \). In particular “admissible” \( p \)-forms are composed basically of “position” observables unless we have some gauge symmetry and constraints then we can represent some “momentum” observable by a \( p \)-forms with \( p \leq n \) (in section (5.3) we study the electromagnetic field which is an instance of such a situation). Finally in section (5) we present three examples: the interacting scalar fields, conformal string theory and the electromagnetic field.

## 2 Construction of the Hamiltonian formalism

In this section we show how to build a universal Hamiltonian formalism for a \( \sigma \)-model variational problem involving a Lagrangian functional depending on first derivatives. We derive it through a universal Legendre correspondance.

### 2.1 Notations

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two differentiable manifolds. \( \mathcal{X} \) plays the role of the space-time manifold and \( \mathcal{Y} \) the target manifold. We fix some volume form \( \omega \) on \( \mathcal{X} \), this volume form may be chosen according to the variational problem that we want to study (for instance if we look at the Klein-Gordon functional on some pseudo-Riemannian manifold, we choose \( \omega \) to be the Riemannian volume), but in more general situation, with less symmetries we just choose some arbitrary volume form. We set \( n = \dim \mathcal{X} \) and \( k = \dim \mathcal{Y} \). We denote \( \{x^1, \ldots, x^n\} \) local coordinates on \( \mathcal{X} \) and \( \{y^1, \ldots, y^k\} \) local coordinates on \( \mathcal{Y} \). For simplicity we shall assume that the coordinates \( x^\alpha \) are always chosen such that \( dx^1 \wedge \ldots \wedge dx^n = \omega \), through it is not essential. Then on the product \( \mathcal{X} \times \mathcal{Y} \) we denote \( \{q^1, \ldots, q^{n+k}\} \) local coordinates in such a way that

\(^3\)so the role of these anticommuting variables is similar to the role of ghosts in the quantization of gauge invariant systems
\[ q^\mu = x^\mu = x^\alpha \quad \text{if } 1 \leq \mu = \alpha \leq n \]
\[ q^\mu = y^{\mu-n} = y^i \quad \text{if } 1 \leq \mu - n = i \leq k. \]

Generally we shall denote the indices running from 1 to \( n \) by \( \alpha, \beta, \ldots \), the indices between 1 and \( k \) by \( i, j, \ldots \) and the indices between 1 and \( n + k \) by \( \mu, \nu, \ldots \). To any map \( u : \mathcal{X} \rightarrow \mathcal{Y} \) we may associate the map

\[ U : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{Y} \]
\[ x \mapsto (x, u(x)) \]

whose image is the graph of \( u \), \( \{(x, u(x))/x \in \mathcal{X}\} \). We also associate to \( u \) the bundle \( u^* T \mathcal{Y} \otimes T^* \mathcal{X} \) over \( \mathcal{X} \). This bundle is naturally equipped with the coordinates \((x^\alpha)_{1 \leq \alpha \leq n} \) (for \( \mathcal{X} \)) and \((v^i_{\nu})_{1 \leq i \leq k; 1 \leq \nu \leq n} \), such that a point \((x, v) \in u^* T \mathcal{Y} \otimes T^* \mathcal{X} \) is represented by

\[ v = \sum_{\alpha=1}^{n} \sum_{i=1}^{k} v^i_{\alpha} \frac{\partial}{\partial y^i} \otimes dx^\alpha. \]

We can think \( u^* T \mathcal{Y} \otimes T^* \mathcal{X} \) as a subset of \( T \mathcal{Y} \otimes T^* \mathcal{X} := \{(x, y, v)/(x, y) \in \mathcal{X} \times \mathcal{Y}, v \in T_y \mathcal{Y} \otimes T^*_x \mathcal{X}\} \) by the inclusion map \((x, v) \mapsto (x, u(x), v)\).

The differential of \( u \), \( du \) is a section of the bundle \( u^* T \mathcal{Y} \otimes T^* \mathcal{X} \) over \( \mathcal{X} \). Hence the coordinates for \( du \) are simply \( v^i_{\alpha} = \frac{\partial u^i}{\partial x^\alpha} \). Notice that \( u^* T \mathcal{Y} \otimes T^* \mathcal{X} \) is a kind of analog of of the tangent bundle \( T \mathcal{Y} \) to a configuration space \( \mathcal{Y} \) in classical particle mechanics.

It turns out to be more convenient to consider \( \Lambda^n T(\mathcal{X} \times \mathcal{Y}) \) the analog of \( T(\mathbb{R} \times \mathcal{Y}) \), the tangent bundle to a space-time, or rather \( S \Lambda^n T(\mathcal{X} \times \mathcal{Y}) \), the submanifold of \( \Lambda^n T(\mathcal{X} \times \mathcal{Y}) \), as the analog of the subset \( S T(\mathbb{R} \times \mathcal{Y}) := \{(t, x; \xi^0, \dot{\xi}) \in T(\mathbb{R} \times \mathcal{Y})/dt(\xi^0, \dot{\xi}) = 1\} \), which is diffeomorphic to \( \mathbb{R} \times T \mathcal{Y} \) by the map \((t, x, \xi) \mapsto (t, x, \dot{\xi})\), and where:

\[ S \Lambda^n T(\mathcal{X} \times \mathcal{Y}) := \{(q, z) \in \Lambda^n T(\mathcal{X} \times \mathcal{Y})/z = z_1 \wedge \ldots \wedge z_n, z_1, \ldots, z_n \in T_q(\mathcal{X} \times \mathcal{Y}), \omega(z_1, \ldots, z_n) = 1\}. \]

For any \((x, y) \in \mathcal{X} \times \mathcal{Y}\), the fiber \( S \Lambda^n T_{(x, y)}(\mathcal{X} \times \mathcal{Y}) \) can be identified with \( T_y \mathcal{Y} \otimes T^*_x \mathcal{X} \) by the diffeomorphism

\[ T_y \mathcal{Y} \otimes T^*_x \mathcal{X} \rightarrow S \Lambda^n T_{(x, y)}(\mathcal{X} \times \mathcal{Y}) \]
\[ v = \sum_{\alpha=1}^{n} \sum_{i=1}^{k} v^i_{\alpha} \frac{\partial}{\partial y^i} \otimes dx^\alpha \rightarrow z = z_1 \wedge \ldots \wedge z_n, \quad (1) \]

where for all \( 1 \leq \beta \leq n \), \( z_\beta = \frac{\partial \omega}{\partial x^\alpha} + \sum_{i=1}^{k} v^i_{\alpha} \frac{\partial \omega}{\partial y^i} \). We denote by \((z^\mu_{\alpha})_{1 \leq \mu \leq n+k; 1 \leq \alpha \leq n}\) the coordinates of \( z_\alpha \), so that \( z_\beta = \sum_{\mu=1}^{n+k} z^\mu_{\alpha} \frac{\partial \omega}{\partial x^\mu} \) (or \( z^\mu_{\alpha} = \delta^\beta_{\alpha} \) for \( 1 \leq \beta \leq n \) and \( z^{\mu+i}_{\alpha} = v^i_{\alpha} \) for \( 1 \leq i \leq k \)). This induces an identification \( T \mathcal{Y} \otimes T^* \mathcal{X} \simeq S \Lambda^n T(\mathcal{X} \times \mathcal{Y}) \).

Thus coordinates \((x^\alpha, y^i, v^i_{\nu}) \) (or equivalently \((x^\alpha, y^i, z^\mu_{\alpha})\) can be thought as coordinate on \( T \mathcal{Y} \otimes T^* \mathcal{X} \) or \( S \Lambda^n T(\mathcal{X} \times \mathcal{Y}) \).

Given a Lagrangian function \( L : T \mathcal{Y} \otimes T^* \mathcal{X} \rightarrow \mathbb{R} \), we define the functional

\[ \mathcal{L}[u] := \int_{\mathcal{X}} L(x, u(x), du(x))dx. \]
2.2 The Euler-Lagrange equations

The critical points of the action are the maps \( u : \mathcal{X} \rightarrow \mathcal{Y} \) which are solutions of the system of Euler-Lagrange equations

\[
\frac{\partial}{\partial x^\alpha} \left( \frac{\partial L}{\partial v^i_\alpha}(x, u(x), du(x)) \right) = \frac{\partial L}{\partial y^i}(x, u(x), du(x)). \tag{2}
\]

This equation implies also other equations involving the stress-energy tensor associated to \( u : \mathcal{X} \rightarrow \mathcal{Y} \):

\[
S^\alpha_\beta(x) := \delta^\alpha_\beta L(x, u(x), du(x)) - \frac{\partial L}{\partial v^i_\alpha}(x, u(x), du(x)) \frac{\partial u^i}{\partial x^\beta}(x).
\]

Indeed for any \( u \),

\[
\frac{\partial S^\alpha_\beta}{\partial x^\alpha}(x) = \delta^\alpha_\beta \left( \frac{\partial L}{\partial x^\alpha}(x, u, du) + \frac{\partial L}{\partial y^i}(x, u, du) \frac{\partial u^i}{\partial x^\alpha}(x) + \frac{\partial L}{\partial v^i_\gamma}(x, u, du) \frac{\partial^2 u^i}{\partial x^\alpha \partial x^\gamma}(x) \right) - \frac{\partial}{\partial x^\alpha} \left( \frac{\partial L}{\partial v^i_\alpha}(x, u, du) \frac{\partial u^i}{\partial x^\beta}(x) - \frac{\partial L}{\partial y^i}(x, u(x), du(x)) \frac{\partial u^i}{\partial x^\beta}(x) \right).
\]

Thus we conclude that if \( u \) is a solution of (2), then

\[
\frac{\partial S^\alpha_\beta}{\partial x^\alpha}(x) = \frac{\partial L}{\partial x^\beta}(x, u, du). \tag{3}
\]

It follows that if \( L \) does not depend on \( x \), then \( S^\alpha_\beta \) is divergence-free for all solutions of (2), a property which can be predicted by Noether’s theorem.

2.3 The Legendre correspondance

Let \( \mathcal{M} := \Lambda^n T^*(\mathcal{X} \times \mathcal{Y}) \). Every point \((q, p) \in \mathcal{M}\) has coordinates \( q^\mu \) and \( p_{\mu_1...\mu_n} \) such that \( p_{\mu_1...\mu_n} \) is completely antisymmetric in \((\mu_1, ..., \mu_n)\) and

\[
p = \sum_{\mu_1 < ... < \mu_n} p_{\mu_1...\mu_n} dq^{\mu_1} \wedge ... \wedge dq^{\mu_n}.
\]

We shall define a Legendre correspondance

\[
\begin{align*}
S\Lambda^n T(\mathcal{X} \times \mathcal{Y}) \times \mathbb{R} \quad &\mapsto \quad \mathcal{M} = \Lambda^n T^*(\mathcal{X} \times \mathcal{Y}) \\
(q, v, w) \quad &\mapsto \quad (q, p),
\end{align*}
\]

where \( w \in \mathbb{R} \) is some extra parameter (its signification is not clear for the moment, \( w \) is related to the possibility of fixing arbitrarily the value of some Hamiltonian). Notice that we do not name it a transform, like in the classical theory but a correspondance, since generally there will be many possible values of \((q, p)\) corresponding to a single value of \((q, v, w)\). But we expect that in generic situations, there corresponds a unique \((q, v, w)\) to some \((q, p)\). This correspondance is generated by the function
\[ W : SA^nT(\mathcal{X} \times \mathcal{Y}) \times \mathcal{M} \rightarrow \mathbb{R} \\
(q, v, p) \quad \mapsto \quad \langle p, v \rangle - L(q, v), \]

where

\[ \langle p, v \rangle \simeq \langle p, z \rangle := \sum_{\mu_1, \ldots, \mu_n} p_{\mu_1} \cdots \mu_n z_{\mu_1} \cdots z_{\mu_n} . \]

**Definition 1** We write that \((q, v, w) \leftrightarrow (q, p)\) if and only if

\[ L(q, v) + w = \langle p, v \rangle \quad \text{or} \quad W(q, v, p) = w \] (4)

and

\[ \frac{\partial L}{\partial v^i}(q, v) = \frac{\partial \langle p, v \rangle}{\partial v^i} = \left( p, z_1 \wedge \ldots \wedge z_{\alpha-1} \wedge \frac{\partial}{\partial y^i} \wedge z_{\alpha+1} \wedge \ldots \wedge z_n \right) \quad \text{or} \quad \frac{\partial W}{\partial v^i}(q, v, p) = 0. \] (5)

Notice that for any \((q, v, w) \in SA^nT(\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}\) there exist \((q, p) \in \mathcal{M}\) such that \((q, v, w) \leftrightarrow (q, p)\). This will be proven in Subsection 2.6 below. But \((q, p)\) is not unique in general. In the following we shall need to suppose that the inverse correspondence is well-defined.

**Hypothesis: Generalized Legendre condition** There exists an open subset \(\mathcal{O} \subset \mathcal{M}\) which is non empty such that for any \((q, p) \in \mathcal{O}\) there exists a unique \(v \in T_q Y \otimes T^*_p \mathcal{X}\) (or equivalently a unique \(z \in SA^nT_q(\mathcal{X} \times \mathcal{Y})\)) which is a critical point of \(v \mapsto W(q, v, p)\). We denote \(v = \mathcal{V}(q, p)\) this unique solution (or \(z = \mathcal{Z}(q, p)\)). We assume further that \(\mathcal{V}\) is a smooth function on \(\mathcal{O}\) (or the same for \(\mathcal{Z}\)).

We now suppose that this hypothesis is true. Then we can define on \(\mathcal{O}\) the following Hamiltonian function

\[ \mathcal{H} : \mathcal{O} \quad \rightarrow \quad \mathbb{R} \\
(q, p) \quad \mapsto \quad \langle p, \mathcal{V}(q, p) \rangle - L(q, \mathcal{V}(q, p)) = W(q, \mathcal{V}(q, p), p). \]

We then remark that (4) is equivalent to \(w = \mathcal{H}(q, p)\).

We now compute the differential of \(\mathcal{H}\). The main point is to exploit the condition

\[ \frac{\partial W}{\partial v^i}(q, \mathcal{V}(q, p), p) = 0 \] (6)

(which defines \(\mathcal{V}\)).
\[ dH = \sum_{\mu} \frac{\partial W}{\partial q^\mu}(q, V(q, p), p) dq^\mu + \sum_{\nu, \alpha} \sum_{\mu_1 < \ldots < \mu_n} \frac{\partial W}{\partial v^\nu}(q, V(q, p), p) \frac{\partial v^\alpha}{\partial q^\mu} dq^\mu + \sum_{\mu_1 < \ldots < \mu_n} \frac{\partial W}{\partial p_{\mu_1 \ldots \mu_n}}(q, V(q, p), p) dp_{\mu_1 \ldots \mu_n}. \]

Now since
\[ \frac{\partial W}{\partial q^\mu}(q, v, p) = - \frac{\partial L}{\partial q^\mu}(q, v, p) \]
and
\[ \frac{\partial W}{\partial p_{\mu_1 \ldots \mu_n}}(q, v, p) = Z_{\mu_1} Z_{\mu_2} \ldots Z_{\mu_n}, \]
we get
\[ dH = - \sum_{\mu} \frac{\partial L}{\partial q^\mu}(q, v, p) dq^\mu + \sum_{\mu_1 < \ldots < \mu_n} Z_{\mu_1 \ldots \mu_n}(q, p) dp_{\mu_1 \ldots \mu_n}, \] (7)

where
\[ Z_{\mu_1 \ldots \mu_n}(q, p) := Z_{\mu_1} Z_{\mu_2} \ldots Z_{\mu_n}, \]
are the components of the \( n \)-vector
\[ Z_1(q, p) \wedge \ldots \wedge Z_n(q, p) = \sum_{\mu_1 < \ldots < \mu_n} Z_{\mu_1 \ldots \mu_n}(q, p) \frac{\partial}{\partial q^{\mu_1}} \wedge \ldots \wedge \frac{\partial}{\partial q^{\mu_n}}. \]

To conclude let us see how the stress-energy tensor appears in this Hamiltonian setting. We define the Hamiltonian tensor on \( \mathcal{O} \) to be
\[ H(q, p) = \sum_{\alpha, \beta} H_{\alpha}^\beta(q, p) \frac{\partial}{\partial x^\alpha} \otimes dx^\beta, \]
with
\[ H_{\beta}^\alpha(q, p) := \frac{\partial L}{\partial v^\alpha}(q, V(q, p)) V_{\beta}^\alpha(q, p) - \delta_{\beta}^\alpha L(q, V(q, p)). \]

It is clear that if \( (x, u(x), du(x), w) \leftrightarrow (q, p) \) then
\[ H_{\beta}^\alpha(q, p) = - S_{\alpha}^\beta(x). \]

Let us now compute \( H_{\beta}^\alpha(q, p) \). We first use \[ \boxed{} \]
\[ \sum_i \frac{\partial L}{\partial v^i_\alpha} (q, \mathcal{V}(q,p)) V^i_\beta(q,p) \]

\[ = \sum_i \frac{\partial(p,v)}{\partial v^i_\alpha} \big|_{v=\mathcal{V}(q,p)} V^i_\beta(q,p) \]

\[ = \sum_i \langle p, Z_1(q,p) \wedge \ldots \wedge Z_{\alpha-1}(q,p) \wedge \frac{\partial}{\partial v^i_\alpha} \wedge Z_{\alpha+1}(q,p) \wedge \ldots \wedge Z_n(q,p) \rangle V^i_\beta(q,p) \]

\[ = \sum_\mu \langle p, Z_1(q,p) \wedge \ldots \wedge Z_{\alpha-1}(q,p) \wedge \frac{\partial}{\partial x^\mu} \wedge Z_{\alpha+1}(q,p) \wedge \ldots \wedge Z_n(q,p) \rangle Z^\mu_\beta(q,p) \]

\[ - \langle p, Z_1(q,p) \wedge \ldots \wedge Z_{\alpha-1}(q,p) \wedge \frac{\partial}{\partial x^\beta} \wedge Z_{\alpha+1}(q,p) \wedge \ldots \wedge Z_n(q,p) \rangle \]

\[ = \langle p, Z_1(q,p) \wedge \ldots \wedge Z_{\alpha-1}(q,p) \wedge Z_\beta(q,p) \wedge Z_{\alpha+1}(q,p) \wedge \ldots \wedge Z_n(q,p) \rangle \]

\[ = \delta^\alpha_\beta \langle p, Z_1(q,p) \wedge \ldots \wedge Z_n(q,p) \rangle \]

\[ - \langle p, Z_1(q,p) \wedge \ldots \wedge Z_{\alpha-1}(q,p) \wedge \frac{\partial}{\partial x^\beta} \wedge Z_{\alpha+1}(q,p) \wedge \ldots \wedge Z_n(q,p) \rangle. \]

Hence since

\[ \langle p, Z_1(q,p) \wedge \ldots \wedge Z_n(q,p) \rangle = \mathcal{H}(q,p) + L(q, \mathcal{V}(q,p)), \]

\[ H^\beta_\beta(q,p) = \delta^\alpha_\beta \mathcal{H}(q,p) - \langle p, Z_1(q,p) \wedge \ldots \wedge Z_{\alpha-1}(q,p) \wedge \frac{\partial}{\partial x^\beta} \wedge Z_{\alpha+1}(q,p) \wedge \ldots \wedge Z_n(q,p) \rangle \]

\[ = \delta^\alpha_\beta \mathcal{H}(q,p) - \frac{\partial(p,z)}{\partial z^\alpha} \big|_{z=Z(q,p)}. \]

(8)

### 2.4 Hamilton equations

Let \( x \mapsto (q(x), p(x)) \) be some map from \( \mathcal{X} \) to \( \mathcal{O} \). To insure that this map is related to a critical point \( u : \mathcal{X} \rightarrow \mathcal{Y} \), we find that the necessary and sufficient conditions split in two parts:

1) **What are the conditions on \( x \mapsto (q(x), p(x)) \) for the existence of a map \( x \mapsto u(x) \) such that \( (x, u(x), du(x)) \leftrightarrow (q(x), p(x)) \)?**

The first obvious condition is \( q(x) = (x, u(x)) = U(x) \). The second condition is that in \( T\mathcal{Y} \otimes T^*\mathcal{X}, (x, y, v^i_\alpha) = (x, y, \frac{\partial U^i}{\partial x^\alpha}) \) coincides with \( (q(x), \mathcal{V}^i_\alpha(q(x), p(x))) \). If we translate that using (11), we obtain that in \( SL^nT(\mathcal{X} \times \mathcal{Y}) \),

\[ \frac{\partial q}{\partial x^1} \wedge \ldots \wedge \frac{\partial q}{\partial x^n} = \frac{\partial U}{\partial x^1} \wedge \ldots \wedge \frac{\partial U}{\partial x^n} = Z_1(q(x), p(x)) \wedge \ldots \wedge Z_n(q(x), p(x)). \]

But we found in (11) that the components in the basis \( \left( \frac{\partial}{\partial q^{\mu_1}} \wedge \ldots \wedge \frac{\partial}{\partial q^{\mu_n}} \right) \) of the right hand side are \( Z^{{\mu_1}^{\mu_2} \ldots} (q(x), p(x)) = \frac{\partial U}{\partial q^{\mu_1} \ldots \mu_n} (q(x), p(x)). \) Hence denoting
\[
\frac{\partial (q^\mu_1, \ldots, q^\mu_n)}{\partial (x^1, \ldots, x^n)} := \begin{vmatrix}
\frac{\partial q^\mu_1}{\partial x^1} & \ldots & \frac{\partial q^\mu_1}{\partial x^n} \\
\vdots & \ddots & \vdots \\
\frac{\partial q^\mu_n}{\partial x^1} & \ldots & \frac{\partial q^\mu_n}{\partial x^n}
\end{vmatrix},
\]

so that
\[
\frac{\partial q}{\partial x^1} \wedge \ldots \wedge \frac{\partial q}{\partial x^n} = \sum_{\mu_1 < \ldots < \mu_n} \frac{\partial (q^\mu_1, \ldots, q^\mu_n)}{\partial (x^1, \ldots, x^n)} \frac{\partial}{\partial q^\mu_1} \wedge \ldots \wedge \frac{\partial}{\partial q^\mu_n},
\]

we obtain the condition
\[
\frac{\partial (q^\mu_1, \ldots, q^\mu_n)}{\partial (x^1, \ldots, x^n)}(x) = - \frac{\partial H}{\partial q^\mu_1 \ldots \mu_n}(q(x), p(x)).
\]

2) Now what are the conditions on \( x \mapsto \langle q(x), p(x) \rangle \) for \( u \) to be a solution of the Euler-Lagrange equations?

It amounts to eliminate \( u \) in \( \mathfrak{2} \) in function of \( (q, p) \). For that purpose we use \( \mathfrak{5} \) to derive
\[
\sum_\alpha \frac{\partial}{\partial x^\alpha} \left( \frac{\partial L}{\partial v^\alpha}(x, u(x), du(x)) \right)
\]
\[
= \sum_\alpha \frac{\partial}{\partial x^\alpha} \left( \langle p, \frac{\partial U}{\partial x^1} \wedge \ldots \wedge \frac{\partial U}{\partial x^{\alpha-1}} \wedge \frac{\partial}{\partial y^1} \wedge \frac{\partial U}{\partial x^{\alpha+1}} \wedge \ldots \wedge \frac{\partial U}{\partial x^n} \rangle \right)
\]
\[
= \sum_\alpha \left( \frac{\partial p}{\partial x^\alpha} \wedge \ldots \wedge \frac{\partial U}{\partial x^1} \wedge \ldots \wedge \frac{\partial U}{\partial x^{\alpha-1}} \wedge \frac{\partial}{\partial y^1} \wedge \frac{\partial U}{\partial x^{\alpha+1}} \wedge \ldots \wedge \frac{\partial U}{\partial x^n} \right)
\]
\[
+ \sum_{\alpha \neq \beta} \left( \frac{\partial p}{\partial x^\alpha} \wedge \ldots \wedge \frac{\partial U}{\partial x^\beta} \wedge \frac{\partial^2 U}{\partial x^\alpha \partial x^\beta} \wedge \ldots \wedge \frac{\partial U}{\partial x^{\alpha-1}} \wedge \frac{\partial}{\partial y^\beta} \wedge \frac{\partial U}{\partial x^{\alpha+1}} \wedge \ldots \wedge \frac{\partial U}{\partial x^n} \right)
\]
\[
= \sum_\alpha \left( \frac{\partial p}{\partial x^\alpha} \wedge \ldots \wedge \frac{\partial U}{\partial x^1} \wedge \ldots \wedge \frac{\partial U}{\partial x^{\alpha-1}} \wedge \frac{\partial}{\partial y^1} \wedge \frac{\partial U}{\partial x^{\alpha+1}} \wedge \ldots \wedge \frac{\partial U}{\partial x^n} \right)
\].

On the other hand we know from \( \mathfrak{6} \) that \( \frac{\partial H}{\partial q}(q, p) = - \frac{\partial H}{\partial q}(q, V(q, p)) \). Thus we obtain
\[
\sum_\alpha \left( \frac{\partial p}{\partial x^\alpha} \wedge \ldots \wedge \frac{\partial q}{\partial x^{\alpha-1}} \wedge \frac{\partial}{\partial y^1} \wedge \frac{\partial q}{\partial x^{\alpha+1}} \wedge \ldots \wedge \frac{\partial q}{\partial x^n} \right) = - \frac{\partial H}{\partial q}(q(x), p(x)).
\]

The latter equation may be transformed using the relation
\[ \sum_{\alpha} \left\langle \frac{\partial p}{\partial x^\alpha}, \frac{\partial q}{\partial x^1} \right\rangle \land \left\langle \ldots \land \frac{\partial q}{\partial x^{\alpha-1}} \land \frac{\partial q}{\partial y^i} \land \left\langle \frac{\partial q}{\partial x^{\alpha+1}} \land \ldots \land \frac{\partial q}{\partial x^n} \right\rangle \right. \]

\[ = \sum_{\alpha} \sum_{\mu_1 < \ldots < \mu_n} \frac{\partial(q^{\mu_1}, \ldots, q^{\mu_{\alpha-1}}, p_{\mu_1 \ldots \mu_n}, q^{\mu_{\alpha+1}}, \ldots, q^{\mu_n})}{\partial(x^1, \ldots, x^n)}. \]

We summarize: the necessary and sufficient conditions we were looking for are

\[ \frac{\partial(q^{\mu_1}, \ldots, q^{\mu_n})}{\partial(x^1, \ldots, x^n)} = \frac{\partial H}{\partial p_{\mu_1 \ldots \mu_n}}(q, p) \]

\[ \sum_{\alpha} \sum_{\mu_1 < \ldots < \mu_n} \frac{\partial(q^{\mu_1}, \ldots, q^{\mu_{\alpha-1}}, p_{\mu_1 \ldots \mu_n}, q^{\mu_{\alpha+1}}, \ldots, q^{\mu_n})}{\partial(x^1, \ldots, x^n)} = -\frac{\partial H}{\partial y^i}(q, p). \]  

(11)

Some further relations

Besides these equations, we have to remark also that equation (8) on the stress-energy tensor has a counterpart in this formalism. For that purpose we use equation (8). Assuming that \((x, u(x), du(x)) \leftrightarrow (q(x), p(x))\), we have

\[ -\frac{\partial S^\beta_{\alpha}}{\partial x^\alpha}(x) = \frac{\partial H^3_{\beta}(q(x), p(x))}{\partial x^\alpha} = \frac{\partial H(q(x), p(x))}{\partial x^\beta} - \frac{\partial}{\partial x^\alpha} \left\langle p(x), \frac{\partial q(x)}{\partial x^1} \land \ldots \land \frac{\partial q(x)}{\partial x^{\alpha-1}} \land \frac{\partial q(x)}{\partial x^{\alpha+1}} \land \ldots \land \frac{\partial q(x)}{\partial x^n} \right\rangle \]

Now assume that \(u\) is a critical point, then because of (8) and (6),

\[ \frac{\partial S_{\beta}}{\partial x^\alpha}(x) = \frac{\partial L}{\partial x^\beta}(x, u(x), du(x)) = -\frac{\partial H}{\partial x^\beta}(q(x), p(x)). \]

And we obtain

\[ \left\langle \frac{\partial p}{\partial x^\alpha}, \frac{\partial q}{\partial x^1} \right\rangle \land \left\langle \ldots \land \frac{\partial q}{\partial x^{\alpha-1}} \land \frac{\partial q}{\partial x^{\alpha+1}} \land \ldots \land \frac{\partial q}{\partial x^n} \right\rangle \right. \]

\[ = -\frac{\partial H}{\partial x^\beta} \langle H(q, p) \rangle = -\frac{\partial H}{\partial x^\beta}(q, p). \]
or equivalently

\[
\sum_\alpha \sum_{\mu_1 < \ldots < \mu_n \atop \mu_\alpha = \beta} \frac{\partial(q^{\mu_1}, \ldots, q^{\mu_{\alpha-1}}, p_{\mu_1 \ldots \mu_\alpha}, q^{\mu_{\alpha+1}}, \ldots, q^{\mu_n})}{\partial(x^1, \ldots, x^n)} - \frac{\partial}{\partial x^\beta} (\mathcal{H}(q, p)) = -\frac{\partial}{\partial q^\nu} (\mathcal{H}(q, p)).
\] (12)

**Conclusion** The Hamilton equations (11) can be completed by adding (12) (which are actually a consequence of (11)). We thus obtain

\[
\sum_\alpha \sum_{\mu_1 < \ldots < \mu_n \atop \mu_\alpha = \nu} \frac{\partial(q^{\mu_1}, \ldots, q^{\mu_{\alpha-1}}, p_{\mu_1 \ldots \mu_\alpha}, q^{\mu_{\alpha+1}}, \ldots, q^{\mu_n})}{\partial(x^1, \ldots, x^n)} - \sum_\alpha \delta_\nu^\alpha \frac{\partial}{\partial x^\alpha} (\mathcal{H}(q, p)) = -\frac{\partial}{\partial q^\nu} (\mathcal{H}(q, p)).
\] (13)

### 2.5 The Cartan-Poincaré and Pataplectic forms on \( \mathcal{M} = \Lambda^n T^*(X \times Y) \)

Motivated by the previous construction, we define the Cartan-Poincaré form on \( \Lambda^n T^*(X \times Y) \) to be

\[
\theta := \sum_{\mu_1 < \ldots < \mu_n} p_{\mu_1 \ldots \mu_n} dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_n}.
\]

Its differential is

\[
\Omega := \sum_{\mu_1 < \ldots < \mu_n} dp_{\mu_1 \ldots \mu_n} \wedge dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_n},
\]

which we will call the *pataplectic form*, a straightforward generalization of the symplectic form.

A first property is that we can express the system of Hamilton’s equations (13) in an elegant way using \( \Omega \). For any \( (q, p) \in \mathcal{M} \) and any \( n \)-vector \( X \in \Lambda^n T_{(q, p)} \mathcal{M} \) we define \( X \cdot \Omega \in T^*_{(q, p)} \mathcal{M} \) as follows. If \( X \) is decomposable, i.e., if there exist \( n \) vectors \( X_1, \ldots, X_n \in T_{(q, p)} \mathcal{M} \) such that \( X = X_1 \wedge \ldots \wedge X_n \), we let

\[
X \cdot \Omega(V) := \Omega(X_1, \ldots, X_n, V), \quad \forall V \in T_{(q, p)} \mathcal{M}.
\]

We extend this definition to non decomposable \( X \) by linearity. Let us analyse \( X \cdot \Omega \) using coordinates. Writing \( X \) as
\[
\sum_{\mu_1 < \ldots < \mu_n} X^{\mu_1 \ldots \mu_n} \partial_{\mu_1} \wedge \ldots \wedge \partial_{\mu_n} + \sum_{\mu_1 < \ldots < \mu_{\alpha-1} < \mu_{\alpha+1} < \ldots < \mu_n} X^{\mu_1 \ldots \mu_{\alpha-1} \mu_{\alpha+1} \ldots \mu_n} \partial_{\mu_1} \wedge \ldots \wedge \partial_{\mu_{\alpha-1}} \wedge \partial_{\mu_{\alpha+1}} \wedge \ldots \wedge \partial_{\mu_n} + \text{ etc} \ldots
\]

with the notations \( \partial_{\mu} := \partial_{q_{\mu}} \), \( \partial_{\nu_1 \ldots \nu_n} := \partial_{p_{\nu_1 \ldots \nu_n}} \), we have

\[
X \oint \Omega = (-1)^n \left[ \sum_{\mu_1 < \ldots < \mu_n} X^{\mu_1 \ldots \mu_n} dp_{\mu_1 \ldots \mu_n} - \sum_{\nu} \sum_{\mu_1 < \ldots < \mu_n} X^{\mu_1 \ldots \mu_{\alpha-1} \mu_{\alpha+1} \ldots \mu_n} \{ \nu_1 \ldots \nu_n \} \partial_{\mu_1} \wedge \ldots \wedge \partial_{\mu_{\alpha-1}} \wedge \partial_{\mu_{\alpha+1}} \wedge \ldots \wedge \partial_{\mu_n} \right].
\]

Algebraic similarities with (13) are evident if we replace \( X \) by \( \partial_{(q,p)} \partial_{x^1, \ldots, x^n} \). In particular we can see easily that the coefficients of \( dy^i \) and \( dp_{\mu_1 \ldots \mu_n} \) in \(( -1)^n \partial_{(q,p)} \partial_{(x^1, \ldots, x^n)} \) \( \oint \Omega \) and \( dH \) coincide if and only if the Hamilton system (11) holds. Thus we are led to define \( \mathcal{I} \) to be the algebraic ideal in \( \Lambda^* M \) spanned by \( \{ dx^1, \ldots, dx^n \} \) and hence (11) is equivalent to

\[
(-1)^n \partial_{(q,p)} \partial_{(x^1, \ldots, x^n)} \oint \Omega = dH \mod \mathcal{I}.
\]

**Definition 2** A \( n \)-vector \( X \in \Lambda^n T_{(q,p)} M \) is \( H \)-Hamiltonian if and only if

\[
(-1)^n X \oint \Omega = dH \mod \mathcal{I}.
\]

For such an \( X \), it is possible to precise the relation between the left and right hand sides of (15) in the case where \( X \) is decomposable, i. e. \( X = X_1 \wedge \ldots \wedge X_n \). Notice that (13) implies in particular \( X^{1 \ldots n} = \frac{\partial H}{\partial \epsilon} = 1 \) (where \( \epsilon := p_{1 \ldots n} \) see (13)), which is equivalent to \( \omega(X_1, \ldots, X_n) = 1 \). Hence we may always assume without loss of generality that the \( X_\alpha \) are chosen so that \( dx^\beta(X_\alpha) = \delta_\alpha^\beta \). Such vectors are unique.

**Lemma 1** Let \( X = X_1 \wedge \ldots \wedge X_n \in \Lambda^n T_{(q,p)} M \) such that \( dx^\beta(X_\alpha) = \delta_\alpha^\beta \). Then \( X \) is \( H \)-Hamiltonian if and only if one of the two following relations are satisfied:

\[
(-1)^n X \oint \Omega = dH - \sum_{\alpha} dH(X_\alpha) dx^\alpha.
\]

or

\[
X \oint (\Omega - d(H\omega)) = 0.
\]
Proof Let us prove first that (15) implies (16). Since for any \( \alpha, \beta, \, dx^\beta \left( X_\alpha - \frac{\partial}{\partial x^\alpha} \right) = 0 \), equation (15) implies that for all \( \alpha \),

\[
(-1)^n X \bigotimes \Omega \left( X_\alpha - \frac{\partial}{\partial x^\alpha} \right) = dH \left( X_\alpha - \frac{\partial}{\partial x^\alpha} \right)
\]

\[
\iff (-1)^n \Omega \left( X_1, \ldots, X_n, X_\alpha - \frac{\partial}{\partial x^\alpha} \right) = dH(X_\alpha) - \frac{\partial H}{\partial x^\alpha}
\]

\[
\iff (-1)^n X \bigotimes \Omega \left( \frac{\partial}{\partial x^\alpha} \right) = \frac{\partial H}{\partial x^\alpha} - dH(X_\alpha).
\]

This implies

\[
(-1)^n \sum_\alpha X \bigotimes \Omega \left( \frac{\partial}{\partial x^\alpha} \right) dx^\alpha = \sum_\alpha \frac{\partial H}{\partial x^\alpha} dx^\alpha - \sum_\alpha dH(X_\alpha) dx^\alpha. \tag{18}
\]

Now if we rewrite (15) as

\[
(-1)^n \left( \sum_i X \bigotimes \Omega \left( \frac{\partial}{\partial y^i} \right) dy^i + \sum_{\mu_1 < \ldots < \mu_n} X \bigotimes \Omega \left( \frac{\partial}{\partial p_{\mu_1 \ldots \mu_n}} \right) dp_{\mu_1 \ldots \mu_n} \right) = \sum_i \frac{\partial H}{\partial y^i} dy^i + \sum_{\mu_1 < \ldots < \mu_n} \frac{\partial H}{\partial p_{\mu_1 \ldots \mu_n}} dp_{\mu_1 \ldots \mu_n},
\]

and sum with (18), we obtain exactly (16).

Now relation (17) is equivalent to (16) because of the following calculation. For all vector \( V \) and for any decomposable \( n \)-vector \( X = X_1 \wedge \ldots \wedge X_n \in \Lambda^n T_{(q,p)} M \) such that \( dx^\beta(X_\alpha) = \delta_\alpha^\beta \) (not necessarily \( H \)-Hamiltonian), we have

\[
X \bigotimes d(H \omega)(V) = dH \wedge \omega(X_1, \ldots, X_n, V)
\]

\[
= \sum_\alpha (-1)^{\alpha - 1} dH(X_\alpha) \omega(X_1, \ldots, X_{\alpha - 1}, X_{\alpha + 1}, \ldots, X_n, V)
\]

\[
+ (-1)^n dH(V) \omega(X_1, \ldots, X_n, V)
\]

\[
= \sum_\alpha (-1)^{n - 1} dH(X_\alpha) dx^\alpha(V) + (-1)^n dH(V)
\]

thus

\[
X \bigotimes d(H \omega) = (-1)^n \left( dH - \sum_\alpha dH(X_\alpha) dx^\alpha \right)
\]

and (16) \iff (17). Conversely it is obvious that (16) and (17) implies (15).

As a Corollary of this result we deduce that a reformulation of (14) is

\[
\frac{\partial (q,p)}{\partial (x^1, \ldots, x^n)} \bigotimes \Omega = \frac{\partial (q,p)}{\partial (x^1, \ldots, x^n)} \bigotimes (dH \wedge \omega). \tag{19}
\]

It is an exercise to check that actually this relation is a direct translation of (13).
2.6 A variational formulation of (13)

We shall now prove that equations (13) are the Euler-Lagrange equations of some simple functional. For that purpose, let $\Gamma$ be an oriented submanifold of dimension $n$ in $\Lambda^n T^*(X \times Y)$ such that $\omega|_{\Gamma} > 0$ everywhere. Then we define the functional

$$A[\Gamma] := \int_{\Gamma} \theta - \lambda \mathcal{H}(q, p) \omega.$$ 

Here $\lambda$ is a (real) scalar function defined over $\Gamma$ which plays the role of a Lagrange multiplier. We now characterise submanifolds $\Gamma$ which are critical points of $A$.

Variations with respect to $p$

Let $\delta p$ be some infinitesimal variation of $\Gamma$ with compact support. We compute

$$\delta A_{\Gamma}(\delta p) = \int_{\Gamma} \delta p_{\mu_1 \ldots \mu_n} \left( dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_n} - \lambda \frac{\partial \mathcal{H}}{\partial p_{\mu_1 \ldots \mu_n}} \omega \right).$$

Assuming that this vanishes for all $\delta p$, we obtain

$$(dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_n})|_{\Gamma} = \lambda \frac{\partial \mathcal{H}}{\partial p_{\mu_1 \ldots \mu_n}} \omega|_{\Gamma}.$$ 

This relation means that for any orientation-preserving parametrization $(t^1, \ldots, t^n) \mapsto (q, p)(t^1, \ldots, t^n)$ of $\Gamma$,

$$\frac{\partial (q^{\mu_1}, \ldots, q^{\mu_n})}{\partial (t^1, \ldots, t^n)} = \lambda \frac{\partial \mathcal{H}}{\partial p_{\mu_1 \ldots \mu_n}} \omega \left( \frac{\partial q^{\mu_1}}{\partial t^1}, \ldots, \frac{\partial q^{\mu_n}}{\partial t^n} \right).$$

But we remark that because $\frac{\partial \mathcal{H}}{\partial p_{\mu_1 \ldots \mu_n}} = 1$, the above relation for $(\mu_1, \ldots, \mu_n) = (1, \ldots, n)$ forces $\lambda = 1$. Hence

$$A[\Gamma] = \int_{\Gamma} \theta - \mathcal{H}(q, p) \omega.$$ 

Moreover the equation obtained here can be written using the natural parametrization $(x^1, \ldots, x^n) \mapsto (x, u(x), p(x))$ (for which $\omega \left( \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right) = 1$) and then we obtain

$$\frac{\partial q}{\partial x^1} \wedge \ldots \wedge \frac{\partial q}{\partial x^n} = \frac{\partial \mathcal{H}}{\partial p}(q, p),$$

i.e. exactly equation (9).

4Note that this relation actually implies $A[\Gamma] = \int_{X} L(x, q, dq) \omega$. Hence, as in the one-dimensional Hamilton formalism, $\theta - \mathcal{H} \omega$ plays the role of the Lagrangian density.
Variations with respect to $q$

Now $\delta q$ is some infinitesimal variation of $\Gamma$ with compact support. And we have

$$
\delta A_{\Gamma}(\delta q) = \int_{\Gamma} \sum_{\mu_1 < \ldots < \mu_n} \sum_{\alpha} p_{\mu_1 \ldots \mu_n} dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_n} - \sum_{\mu} \frac{\partial H}{\partial q^{\mu}} \delta q^{\mu} \omega - \mathcal{H}(q, p) \delta \omega.
$$

We pay special attention to $\delta \omega$:

$$
\delta \omega = d(\delta x^1) \wedge \ldots \wedge dx^n + \ldots + dx^1 \wedge \ldots \wedge d(\delta x^n).
$$

Hence

$$
\int_{\Gamma} \mathcal{H}(q, p) \delta \omega = - \int_{\Gamma} \delta x^1 \left( d(\mathcal{H}(q, p)) \wedge \ldots \wedge dx^n \right) + \ldots + \delta x^n \left( dx^1 \wedge \ldots \wedge d(\mathcal{H}(q, p)) \right)
$$

$$
= - \sum_{\alpha} \delta x^\alpha \frac{\partial}{\partial x^\alpha} (\mathcal{H}(q, p)) \omega.
$$

Thus after integrations by parts, we obtain

$$
\delta A_{\Gamma}(\delta q) = \int_{\Gamma} - \sum_{\mu_1 < \ldots < \mu_n} \sum_{\alpha} \delta q^{\mu_1} dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_n} \wedge dp_{\mu_1 \ldots \mu_n} \wedge dq^{\mu_{n+1}} \wedge \ldots \wedge dq^{\mu_n}
$$

$$
- \sum_{\mu} \frac{\partial H}{\partial q^{\mu}} \delta q^{\mu} \omega + \sum_{\alpha} \delta x^\alpha \frac{\partial}{\partial x^\alpha} (\mathcal{H}(q, p)) \omega.
$$

And this vanishes if and only if

$$
\sum_{\alpha} \sum_{\mu_1 < \ldots < \mu_n} dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_{n+1}} \wedge dp_{\mu_1 \ldots \mu_n} \wedge dq^{\mu_{n+1}} \wedge \ldots \wedge dq^{\mu_n} - \sum_{\alpha} \delta x^\alpha \frac{\partial}{\partial x^\alpha} (\mathcal{H}(q, p)) \omega = - \frac{\partial H}{\partial q^{\nu}} \omega.
$$

Again by choosing the parametrization $(x^1, \ldots, x^n) \mapsto (x, u(x), p(x))$, this relation is easily seen to be equivalent to (10) and (12).

By the same token we have proven that if we look to critical points of the functional $\int_{\Gamma} \theta$ with the constraint $\mathcal{H}(q, p) = h$, for some constant $h$, then the Lagrange multiplier is 1 and they satisfy the same equations.

**Theorem 1** Let $\Gamma$ be an oriented submanifold of dimension $n$ in $\Lambda^n T^* (\mathcal{X} \times \mathcal{Y})$ such that $\Omega_{\Gamma} > 0$ everywhere. Then the three following assertions are equivalent

- $\Gamma$ is the graph of a solution of the generalized Hamilton equations (13)
- $\Gamma$ is a critical point of the functional $\int_{\Gamma} \theta - \mathcal{H}(q, p) \omega$
- $\Gamma$ is a critical point of the functional $\int_{\Gamma} \theta$ under the constraint that $\mathcal{H}(q, p)$ is constant.
2.7 Some particular cases

By restricting the variables \((q, p)\) to lie in some submanifold of \(\mathcal{M} = \Lambda^n T^*(\mathcal{X} \times \mathcal{Y})\), the Legendre correspondence becomes in some situations a true map.

a) We assume that all components \(p_{\mu_1...\mu_n}\) vanishes excepted for

\[
p_{1...n} =: \epsilon \quad \text{and} \quad p_{1...(\alpha-1)(n+1)(\alpha+1)...n} =: p_i^\alpha
\]

and all obvious permutations in the indices. This defines a submanifold \(\mathcal{M}_{\text{Weyl}}\) of \(\mathcal{M}\). It means that

\[
\theta|_{\mathcal{M}_{\text{Weyl}}} = \epsilon \ dx^1 \wedge \ldots \wedge dx^n + \sum_{\alpha} \sum_i p_i^\alpha dx^1 \wedge \ldots \wedge dx^{\alpha-1} \wedge dy^i \wedge dx^{\alpha+1} \wedge \ldots \wedge dx^n.
\]

Then for any \((q, p)\) \(\in \mathcal{M}_{\text{Weyl}}, \langle p, z_1 \wedge \ldots \wedge z_n \rangle = \epsilon + \sum_{\alpha} \sum_i p_i^\alpha v_i^\alpha, W(q, v, p) = \epsilon + \sum_{\alpha} \sum_i p_i^\alpha v_i^\alpha - L(q, v).\) Hence the relation \((\partial W/\partial v^i_\alpha)(q, v) = 0\) is equivalent to

\[
p_i^\alpha = \frac{\partial L}{\partial v^i_\alpha}(q, v) \iff v_i^\alpha = V_i^\alpha(q, p).
\]

The relation \((\partial W/\partial v^i_\alpha)(q, v, p) = w\) gives

\[
\epsilon = w + L(q, v) - \sum_{\alpha} \sum_i \frac{\partial L}{\partial v^i_\alpha}(q, v) v_i^\alpha.
\]

Last we have that \(\mathcal{H}(q, p) = \epsilon + \sum_{\alpha} \sum_i p_i^\alpha V_i^\alpha(q, p) - L(q, V(q, p)).\)

This example shows that for any \((q, v, w)\) \(\in S\Lambda^n T(\mathcal{X} \times \mathcal{Y}) \times \mathbb{R},\) there exist \((q, p)\) \(\in \mathcal{M}\) such that \((q, v, w) \leftrightarrow (q, p)\) and this \((q, p)\) is unique if it is chosen in \(\mathcal{M}_{\text{Weyl}}.\)

To summarize, we recover the Weyl theory (see [4, 23]). As an exercise, the reader can check that in this situation, equations \((11)\) are equivalent to

\[
\frac{\partial y^i}{\partial x^\alpha} = \frac{\partial \mathcal{H}}{\partial p_i^\alpha}, \quad \sum_{\alpha} \frac{\partial p_i^\alpha}{\partial x^\alpha} = -\frac{\partial \mathcal{H}}{\partial y^i}.
\]

(20)

b) We assume that \((q, p)\) are such that there exist coefficients \((\pi^\alpha_\mu)_{\alpha, \mu}\) with

\[
p_{\mu_1...\mu_n} = \begin{vmatrix}
\pi^1_{\mu_1} & \ldots & \pi^1_{\mu_n} \\
\vdots & \ddots & \vdots \\
\pi^n_{\mu_1} & \ldots & \pi^n_{\mu_n}
\end{vmatrix}.
\]

This constraint defines a submanifold \(\mathcal{M}_{\text{Carathéodory}}\) of \(\mathcal{M}\). Then

\[
\theta|_{\mathcal{M}_{\text{Carathéodory}}} = \left(\sum_{\mu_1} \pi^1_{\mu_1} dq^{\mu_1}\right) \wedge \ldots \wedge \left(\sum_{\mu_n} \pi^n_{\mu_n} dq^{\mu_n}\right).
\]
Then it is an exercise to see that, by choosing \( w = 0 \), it leads to the formalism developed in \([4]\) and \([28]\) associated to the Carathéodory theory of equivalent integrals. However it is not clear in general whether it is possible to perform the Legendre transform in this setting by being able to fix arbitrarily the value of \( w \). It is so if we do not impose a condition on \( w \).

3 Comparison with the usual Hamiltonian formalism for quantum fields theory

3.1 Reminder of the usual approach to quantum field theory

Here we compare the preceding construction with the classical approach to quantum field theory by so-called canonical quantization. We shall first explore it in the case where \( X \) is the Minkowski space \( \mathbb{R} \times \mathbb{R}^{n-1} \) and \( y = \phi \) is a real scalar field. Hence \( Y = \mathbb{R} \). Our functional is

\[
\mathcal{L}[\phi] := \int_{\mathbb{R} \times \mathbb{R}^{n-1}} L(x, \phi, d\phi) \, dx.
\]

For simplicity, we may keep in mind the following example of Lagrangian:

\[
\int_{\mathbb{R} \times \mathbb{R}^{n-1}} L(x, \phi, d\phi) \, dx = \int_{\mathbb{R} \times \mathbb{R}^{n-1}} \left( \frac{1}{2} \left( \frac{\partial \phi}{\partial x^0} \right)^2 - \frac{1}{2} \sum_{\alpha=1}^{n-1} \left( \frac{\partial \phi}{\partial x^\alpha} \right)^2 - V(\phi) \right) \, dx^0 \, d\vec{x},
\]

where we denote \( \vec{x} = (x^\alpha)_{1 \leq \alpha \leq n-1} \). We shall also denote \( t = x^0 \).

The usual approach consists in selecting a global time coordinate \( t \) as we already implicitly assumed here. Then for each time the instantaneous state of the field is seen as a point in the infinite dimensional “manifold” \( \mathfrak{F} := \{ \Phi : \mathbb{R}^{n-1} \rightarrow \mathbb{R} \} \). Hence we view the field \( \phi \) rather as a path

\[
\mathbb{R} \rightarrow \mathfrak{F} \quad \mapsto \quad [\vec{x} \mapsto \phi(t, \vec{x}) = \Phi^\vec{x}(t)].
\]

We thus recover the problem of studying the dynamics of a point moving in a configuration space \( \mathfrak{F} \). The prices to pay are 1) \( \mathfrak{F} \) is infinite dimensional 2) we lose relativistic invariance.

In this viewpoint, \( \mathcal{L}[\phi] = \int_{\mathbb{R}} \mathcal{L}[t, \Phi(t), \frac{d\phi}{dt}(t)] \, dt \), where \( \Phi(t) = [\vec{x} \mapsto \phi(t, \vec{x})] \in \mathfrak{F} \), \( \frac{d\phi}{dt}(t) = [\vec{x} \mapsto \frac{\partial \phi}{\partial t}(t, \vec{x})] \in T_{\Phi(t)}\mathfrak{F} \) and \( \mathcal{L}[t, \Phi(t), \frac{d\phi}{dt}(t)] = \int_{\mathbb{R}^{n-1}} L(x, \phi(x), d\phi(x)) \, d\vec{x} \).

Then we consider the “symplectic” manifold which is formally \( T^*\mathfrak{F} \), i. e. we introduce the dual variable
\[
\Pi := \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} \frac{\partial \mathcal{L}}{\partial \Phi} dt,
\]
or equivalently \( \Pi(t) = [\vec{x} \mapsto \pi(t, \vec{x}) = \Pi(\vec{x})] \) with

\[
\Pi(t) = \frac{\partial \mathcal{L}}{\partial \Phi} = \frac{\delta \mathcal{L}}{\delta \Phi(t)}[t, \Phi(t), \frac{d\Phi}{dt}(t)] = \frac{\partial L}{\partial \nu_0}(x, \phi(x), d\phi(x)).
\]

Here \( \frac{\delta}{\delta \phi(\vec{x})} \) is the Fréchet derivative. In our example

\[
\Pi(t) = \frac{\partial \Phi}{\partial t}(t, \vec{x}).
\]

We define the Hamiltonian functional to be

\[
\mathcal{H}[\Phi, \Pi] := \int_{\mathbb{R}^{n-1}} \Pi \Phi \cdot d\vec{x} - \mathcal{L}[t, \Phi, \frac{d\Phi}{dt}] = \int_{\mathbb{R}^{n-1}} \left( \frac{1}{2} \pi(\vec{x})^2 + \frac{1}{2} |\nabla \phi(\vec{x})|^2 + V(\phi(\vec{x})) \right) d\vec{x}.
\]

Now we can write the equations of motion as

\[
\frac{\partial \pi}{\partial t}(t, \vec{x}) = \frac{d\Pi}{dt}(t, \vec{x}) = \frac{\partial \mathcal{H}}{\partial \Phi}(\Phi, \Pi) = \Delta \phi - V'(\phi)
\]

\[
\frac{\partial \Phi}{\partial t}(t, \vec{x}) = \frac{d\Phi}{dt}(t, \vec{x}) = \frac{\partial \mathcal{H}}{\partial \Pi}(\Phi, \Pi) = \pi(t, \vec{x}).
\]

A Poisson bracket can be defined on the set of functionals \( \{ A : T^*\mathfrak{g} \rightarrow \mathbb{R} \} \) by

\[
\{ A, B \} := \int_{\mathbb{R}^{n-1}} \left( \frac{\delta A}{\delta \pi(\vec{x})} \frac{\delta B}{\delta \phi(\vec{x})} - \frac{\delta A}{\delta \phi(\vec{x})} \frac{\delta B}{\delta \pi(\vec{x})} \right) d\vec{x},
\]

where \( \frac{\delta A}{\delta \phi(\vec{x})} \) is the Fréchet derivative with respect to \( \phi(\vec{x}) \), i.e. the distribution such that for any smooth compactly supported deformation \( \delta \phi \) of \( \phi \),

\[
dA_\phi[\delta \phi] = \int_{\mathbb{R}^{n-1}} \delta \phi(\vec{x}) \frac{\delta A}{\delta \phi(\vec{x})} d\vec{x}.
\]

And we may formulate the dynamical equations using the Poisson bracket as

\[
\frac{d\Pi}{dt}(t, \vec{x}) = \{ \mathcal{H}, \Pi \}
\]

\[
\frac{d\Phi}{dt}(t, \vec{x}) = \{ \mathcal{H}, \Phi \},
\]

with

\[
\{ \Phi, \Phi' \} = \{ \Pi, \Pi' \} = 0, \quad \{ \Pi, \Phi' \} = \delta^{n-1}(\vec{x} - \vec{x}') = \delta_{\vec{x}}' = \delta^{n-1}(\vec{x} - \vec{x}').
\]
This singular Poisson bracket means that for any test functions \( f, g \in C_c^\infty(\mathbb{R}^{n-1}, \mathbb{R}) \),
\[
\left\{ \int_{\mathbb{R}^{n-1}} g(\vec{x}) \Pi_{\vec{x}} d\vec{x}, \int_{\mathbb{R}^{n-1}} f(\vec{x}') \Phi_{\vec{x}'} d\vec{x}' \right\} = \int_{\mathbb{R}^{n-1}} f(\vec{x}) g(\vec{x}) d\vec{x}.
\]

This implies in particular
\[
\left\{ \int_{\mathbb{R}^{n-1}} g(\vec{x}) \Pi_{\vec{x}} d\vec{x}, \int_{\mathbb{R}^{n-1}} f(\vec{x}') V(\Phi_{\vec{x}'}) d\vec{x}' \right\} = \int_{\mathbb{R}^{n-1}} V'(\Phi_{\vec{x}}) f(\vec{x}) g(\vec{x}) d\vec{x},
\]
because of the derivation property of the Poisson bracket.

### 3.2 Translation in pataplectic geometry

We first adapt and modify our notations: the coordinates on \( M = \Lambda^n T^* (\mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}) \) are now written \( (q^\mu, p_{\mu_1...\mu_n}) = (x^\alpha, y, \epsilon, p_\alpha) \) where 0 \( \leq \alpha \leq n-1 \), \( q^0 = x^0 = t \), \( (x^\alpha)_{1 \leq \alpha \leq n-1} = \vec{x} \), \( q^n = y \) and
\[
\epsilon := p_0...n, \quad p_\alpha := p_{0...(\alpha-1)n(\alpha+1)...(n-1)}.
\]

Hence
\[
\theta = \epsilon \ dx^0 \wedge ... \wedge dx^{n-1} + \sum_{\alpha=0}^{n-1} p_\alpha \ dx^0 \wedge ... \wedge dx^{\alpha-1} \wedge dy \wedge dx^{\alpha+1} \wedge ... \wedge dx^{n-1},
\]
or letting \( \omega := dx^0 \wedge ... \wedge dx^{n-1} \) and \( \omega_\alpha := (-1)^\alpha dx^0 \wedge ... \wedge dx^{\alpha-1} \wedge dx^{\alpha+1} \wedge ... \wedge dx^{n-1} \),
\[
\frac{\partial}{\partial x^\alpha} = \omega, \quad \theta = \epsilon \omega + \sum_{\alpha=0}^{n-1} p_\alpha dy \wedge \omega_\alpha \quad \text{and} \quad \Omega = d\epsilon \wedge \omega + \sum_{\alpha=0}^{n-1} dp_\alpha \wedge dy \wedge \omega_\alpha.
\]

Thus we see that in the present case the pataplectic formalism reduces essentially to the Weyl formalism, because the fields are one dimensional.

Let us consider some field \( \phi \) and a map \( x \mapsto p(x) \) such that \( (x, \phi(x), d\phi(x)) \leftrightarrow (x, \phi(x), p(x)) \) \(^5\). This implies the following relations
\[
p_\alpha = \frac{\partial L}{\partial v_\alpha}(x, \phi(x), d\phi(x)) \quad \text{and} \quad \epsilon = w + L(x, \phi(x), d\phi(x)) - \sum_{\alpha=0}^{n-1} p_\alpha \frac{\partial \phi}{\partial x^\alpha}(x).
\]

\(^5\)meaning that for some \( w : \mathbb{R} \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R} \), we have \( (x, \phi(x), d\phi(x), w(x)) \leftrightarrow (x, \phi(x), p(x)) \)
We let \( \Gamma := \{(x, \phi(x), p(x))/x \in \mathbb{R} \times \mathbb{R}^{n-1}\} \subset \mathcal{M} \) and we consider the instantaneous slices \( S_t := \Gamma \cap \{x^0 = t\} \). These slices are oriented by the condition \( \partial_t \omega|_{S_t} > 0 \). Then we can express the observables

\[
\Phi^f(t) := \int_{\mathbb{R}^{n-1}} f(\vec{x}) \Phi^e(t)d\vec{x}, \quad \Pi^g(t) := \int_{\mathbb{R}^{n-1}} g(\vec{x}) \Pi^e(t)d\vec{x}
\]

and

\[
\mathcal{H}[\Phi(t), \Pi(t)] = \int_{\mathbb{R}^{n-1}} \left( \pi(t, \vec{x}) \frac{\partial \phi}{\partial t}(t, \vec{x}) - L(t, \vec{x}, \phi(x), d\phi(x)) \right) d\vec{x} = \int_{\mathbb{R}^{n-1}} H^0_0(t, \vec{x}, \phi) \omega_0
\]
as integrals of \((n-1)\)-forms on \( S_t \). First

\[
\Phi^f(t) = \int_{S_t} f(\vec{x}) \phi(t, \vec{x})d\vec{x}^1 \wedge \ldots \wedge d\vec{x}^{n-1} = \int_{S_t} Q^f, \quad \text{with } Q^f := f(\vec{x}) y \omega_0.
\]

\[
\Pi^g(t) = \int_{S_t} g(\vec{x}) \pi(t, \vec{x})d\vec{x}^1 \wedge \ldots \wedge d\vec{x}^{n-1} = \int_{S_t} P^g, \quad \text{with } P^g := g(\vec{x}) \sum_{\alpha=0}^{n-1} p^\alpha \omega_\alpha,
\]
because \( \pi(t, \vec{x}) = \frac{\partial L}{\partial v_0}(x, \phi(x), d\phi(x)) = p^0 \) and \( \omega_\alpha|_{S_t} = 0 \) if \( \alpha \geq 1 \)

And last

\[
\mathcal{H}[\Phi(t), \Pi(t)] = \int_{\mathbb{R}^{n-1}} \mathcal{H}(q, p) \omega_0 - \int_{\mathbb{R}^{n-1}} \epsilon \omega_0 + \sum_{\alpha=1}^{n-1} p^\alpha dy \wedge \left( \frac{\partial}{\partial x^\alpha} \right) \omega_0 = \int_{S_t} \eta_0,
\]

where

\[
\eta_0 := \mathcal{H}(q, p) \omega_0 - \left( \epsilon \omega_0 + \sum_{\alpha=1}^{n-1} p^\alpha dx^1 \wedge \ldots \wedge dx^{\alpha-1} \wedge dy \wedge dx^{\alpha+1} \wedge \ldots \wedge dx^{n-1} \right),
\]
because \( H^0_0(x, \phi) = \mathcal{H}(q, p) - (p, \frac{\partial}{\partial x^1} \wedge \ldots \wedge dz_{n-1}) = \mathcal{H}(q, p) - (\epsilon + \sum_{\alpha=1}^{n-1} p^\alpha \frac{\partial \phi}{\partial x^\alpha}) \).

We remark \footnote{we observe also that \( P^g = g(\vec{x}) \frac{\partial}{\partial v_0} \left( \epsilon + \sum_{\alpha=1}^{n-1} p^\alpha \frac{\partial \phi}{\partial x^\alpha} \right) \).

\( \eta_0 = -\frac{\partial}{\partial t} \left( \theta - \mathcal{H}(q, p) \omega_0 \right). \)
3.3 Recovering the usual Poisson brackets as a local expression

Our aim is now to express the various Poisson brackets involving the quantities \( \Phi^f(t) \) and \( \Pi^g(t) \) along \( \Gamma \) using some analogue of the Poisson bracket defined on \((n-1)\)-forms. We generalize slightly the definition of \( Q^f \) to be

\[
Q^f = \sum_{\alpha=0}^{n-1} f^\alpha(x) \ y \omega^\alpha,
\]

(21)

where \( f := \sum_{\alpha=0}^{n-1} f^\alpha(x) \frac{\partial}{\partial x^\alpha} \) is some vector field. Hence our observables become

\[
\Phi^f(t) = \int_{S_t} Q^f \quad \text{and} \quad \Pi^g(t) = \int_{S_t} P^g,
\]

(22)

where \( P^g := g(x) \sum_{\alpha=0}^{n-1} p^\alpha \omega^\alpha \) as before [7]. We shall see here that we can define a bracket operation \( \{.,.\} \) between \( Q^f, P^g \) and \( \eta_0 \) such that the usual Poisson bracket of fields actually derives from \( \{.,.\} \) by

\[
\int_{S_t} \{P^g, Q^f\} = \left\{ \int_{S_t} P^g, \int_{S_t} Q^f \right\}, \text{ etc}.
\]

(23)

First we remark that

\[
dQ^f = \sum_{\alpha=0}^{n-1} f^\alpha(x) dy \wedge \omega^\alpha + \sum_{\alpha=0}^{n-1} y \frac{\partial f^\alpha}{\partial x^\alpha} \omega^\alpha = \sum_{\alpha=0}^{n-1} f^\alpha \frac{\partial}{\partial p^\alpha} \omega^\alpha + \sum_{\alpha=0}^{n-1} \frac{\partial f^\alpha}{\partial x^\alpha} \frac{\partial}{\partial \epsilon} \omega^\alpha - \left( \xi_{Q^f} \right) \Omega
\]

and

\[
dP^g = \sum_{\alpha=0}^{n-1} p^\alpha \frac{\partial g}{\partial x^\alpha} \omega^\alpha + \sum_{\alpha=0}^{n-1} g dp^\alpha \wedge \omega^\alpha = \sum_{\alpha=0}^{n-1} p^\alpha \frac{\partial g}{\partial x^\alpha} \frac{\partial}{\partial \epsilon} \omega^\alpha + \sum_{\alpha=0}^{n-1} \frac{\partial g}{\partial x^\alpha} \frac{\partial}{\partial \epsilon} \omega^\alpha - g \frac{\partial}{\partial y} \Omega
\]

where

\[
\xi_{Q^f} := -\sum_{\alpha=0}^{n-1} f^\alpha \frac{\partial}{\partial p^\alpha} - y \sum_{\alpha=0}^{n-1} \frac{\partial f^\alpha}{\partial x^\alpha} \frac{\partial}{\partial \epsilon}
\]

(24)

and

\[
\xi_{P^g} := g \frac{\partial}{\partial y} - \sum_{\alpha=0}^{n-1} p^\alpha \frac{\partial g}{\partial x^\alpha} \frac{\partial}{\partial \epsilon}
\]

(25)

\[\text{notice that actually } \int_{S_t} Q^f = \int_{S_t} f^0(x) y \omega_0.\]
Also notice that
\[ d\eta_0 = (dH - d\epsilon) \wedge \omega_0 - \sum_{\alpha=1}^{n-1} dp^\alpha \wedge dy \wedge \left( \frac{\partial}{\partial x^\alpha} \mathbf{J} \omega_0 \right). \]

**Definition 3** We define the Poisson \( p \)-brackets of these \((n-1)\)-forms to be
\[
\{\eta_0, Q^f\} := -\xi_{Q^f} \lrcorner d\eta_0, \quad \{\eta_0, \mathcal{P}_g\} := -\xi_{\mathcal{P}_g} \lrcorner d\eta_0,
\]
\[
\{\mathcal{P}_g, Q^f\} := -\xi_{Q^f} \lrcorner d\mathcal{P}_g = \xi_{\mathcal{P}_g} \lrcorner dQ^f = \xi_{Q^f} \lrcorner (\xi_{\mathcal{P}_g} \lrcorner \Omega)
\]
and
\[
\{Q^f, Q^{f'}\} := \xi_{Q^{f'}} \lrcorner (\xi_{Q^f} \lrcorner \Omega), \quad \{\mathcal{P}_g, \mathcal{P}_{g'}\} := \xi_{\mathcal{P}_{g'}} \lrcorner (\xi_{\mathcal{P}_g} \lrcorner \Omega).
\]
Let us now compute these \( p \)-brackets. We use in particular the fact that \( \frac{\partial H}{\partial \epsilon} = 1 \).

\[
\{\eta_0, Q^f\} = \left( \sum_{\alpha=0}^{n-1} f^\alpha \frac{\partial}{\partial p^\alpha} + y \sum_{\alpha=0}^{n-1} \frac{\partial f^\alpha}{\partial x^\alpha} \frac{\partial}{\partial \epsilon} \right) \lrcorner \left( (dH - d\epsilon) \wedge \omega_0 - \sum_{\alpha=1}^{n-1} dp^\alpha \wedge dy \wedge \left( \frac{\partial}{\partial x^\alpha} \mathbf{J} \omega_0 \right) \right)
\]
\[
= \sum_{\alpha=0}^{n-1} f^\alpha \frac{\partial H}{\partial p^\alpha} \omega_0 - \sum_{\alpha=1}^{n-1} f^\alpha dy \wedge \left( \frac{\partial}{\partial x^\alpha} \mathbf{J} \omega_0 \right),
\]
\[
\{\eta_0, \mathcal{P}_g\} = \left( \sum_{\alpha=0}^{n-1} p^\alpha \frac{\partial g}{\partial x^\alpha} \frac{\partial}{\partial \epsilon} - g \frac{\partial}{\partial y} \right) \lrcorner \left( (dH - d\epsilon) \wedge \omega_0 - \sum_{\alpha=1}^{n-1} dp^\alpha \wedge dy \wedge \left( \frac{\partial}{\partial x^\alpha} \mathbf{J} \omega_0 \right) \right)
\]
\[
= -g \frac{\partial H}{\partial y} \omega_0 - g \sum_{\alpha=1}^{n-1} dp^\alpha \wedge \left( \frac{\partial}{\partial x^\alpha} \mathbf{J} \omega_0 \right),
\]
\[
\{\mathcal{P}_g, Q^f\} = \left( \sum_{\alpha=0}^{n-1} f^\alpha \frac{\partial}{\partial p^\alpha} + y \sum_{\alpha=0}^{n-1} \frac{\partial f^\alpha}{\partial x^\alpha} \frac{\partial}{\partial \epsilon} \right) \lrcorner \left( \sum_{\alpha=0}^{n-1} p^\alpha \frac{\partial g}{\partial x^\alpha} \omega_0 - g \frac{\partial}{\partial y} \Omega \right)
\]
\[
= g \sum_{\alpha=0}^{n-1} f^\alpha \omega_0,
\]
and \( \{Q^f, Q^{f'}\} = \{\mathcal{P}_g, \mathcal{P}_{g'}\} = 0 \). We now integrate the \( p \)-brackets on a constant time slice \( S_t \subset \Gamma \). We immediately see that
\[
\int_{S_t} \{\mathcal{P}_g, Q^f\} = \int_{S_t} g f^0 \omega_0 = \{\pi_g(t), \Phi^f(t)\} = \left\{ \int_{S_t} \mathcal{P}_g, \int_{S_t} Q^f \right\}
\]
and we recover (23). Second,
\[
\int_{S_t} \{ \eta_0, Q^f \} = \int_{S_t} \sum_{\alpha=0}^{n-1} f^\alpha \frac{\partial H}{\partial p^\alpha} \omega_0 - \sum_{\alpha=1}^{n-1} f^\alpha \frac{\partial \phi}{\partial x^{\alpha}} \omega_0.
\]

Third,
\[
\int_{S_t} \{ \eta_0, P_g \} = \int_{S_t} -g \frac{\partial H}{\partial y} \omega_0 - \sum_{\alpha=1}^{n-1} g \frac{\partial p^\alpha}{\partial x^{\alpha}} \omega_0.
\]

Now let us assume that \( \Gamma \) is the graph of a solution of the Hamilton equations (11) or (20). Since then \( \frac{\partial \phi}{\partial x^{\alpha}} = \frac{\partial H}{\partial p^\alpha} \) along \( \Gamma \),
\[
\int_{S_t} \{ \eta_0, Q^f \} = \int_{S_t} f^0 \frac{\partial \phi}{\partial t} \omega_0,
\]
and because of
\[
-\frac{\partial H}{\partial y} - \sum_{\alpha=1}^{n-1} \frac{\partial p^\alpha}{\partial x^{\alpha}} = \frac{\partial p^0}{\partial t},
\]
\[
\int_{S_t} \{ \eta_0, P_g \} = \int_{S_t} g \frac{\partial p^0}{\partial t} \omega_0.
\]

We conclude that
\[
\frac{d}{dt} \int_{S_t} Q^f = \frac{d}{dt} \Phi^f(t) = \int_{S_t} f^0 \frac{\partial \phi}{\partial t} \omega_0 + \frac{\partial f^0}{\partial t} \phi_0 = \int_{S_t} \{ \eta_0, Q^f \} + \Phi^{\partial f/\partial t}(t)
\]
and
\[
\frac{d}{dt} \int_{S_t} P_g = \frac{d}{dt} \Pi_g(t) = \int_{S_t} g \frac{\partial p^0}{\partial t} \omega_0 + \frac{\partial g}{\partial t} p^0 \omega_0 = \int_{S_t} \{ \eta_0, P_g \} + \Pi^{\partial g/\partial t}(t).
\]

This has to be compared with the usual canonical equations for fields:
\[
\frac{d}{dt} \int_{S_t} Q^f = \left( \int_{S_t} \eta_0, \int_{S_t} Q^f \right) + \Phi^{\partial f/\partial t}(t) \quad \text{and} \quad \frac{d}{dt} \int_{S_t} P_g = \left( \int_{S_t} \eta_0, \int_{S_t} P_g \right) + \Pi^{\partial g/\partial t}(t).
\]

### 3.4 An alternative dynamical formulation using \( p \)-brackets

We can also define the \( p \)-bracket of a \( n \)-form with forms \( Q^f \) or \( P_g \) as given by (21) and (22). If \( \psi \) is such a \( n \)-form,
\[
\{ \psi, Q^f \} := -\xi_Q^f \int d\psi \quad \text{and} \quad \{ \psi, P_g \} := -\xi_{P_g} \int d\psi,
\]
where (24) and (25) have been used. An important instance is for \( \psi = H \omega \):
\[
\{ H \omega, Q^f \} = \sum_{\alpha=0}^{n-1} f^\alpha \frac{\partial H}{\partial p^\alpha} \omega + \sum_{\alpha=0}^{n-1} g \frac{\partial f^\alpha}{\partial x^{\alpha}} \omega.
\]
We shall integrate this \( p \)-bracket on \( \Gamma_{t_1}^{t_2} := \{(q, p) \in \Gamma/t_1 < t < t_2\} \), where we still assume that \( \Gamma \) is the graph of a solution of the Hamilton equations (19). An integration by parts gives

\[
\int_{\Gamma_{t_1}^{t_2}} \{H \omega, Q^f\} = \int_{\partial \Gamma_{t_1}^{t_2}} \phi + \int_{\Gamma_{t_1}^{t_2}} \sum_{\alpha=0}^{n-1} f^\alpha \omega_\alpha + \int_{\Gamma_{t_1}^{t_2}} \sum_{\alpha=0}^{n-1} f^\alpha \left( \frac{\partial H}{\partial p^\alpha} - \frac{\partial \phi}{\partial x^\alpha} \right) \omega
\]

Similarly we find that

\[
\{H \omega, P_g\} = \sum_{\alpha=0}^{n-1} g^\alpha \frac{\partial g}{\partial x^\alpha} \omega - g \frac{\partial H}{\partial y} \omega,
\]

and thus

\[
\int_{\Gamma_{t_1}^{t_2}} \{H \omega, P_g\} = \int_{\partial \Gamma_{t_1}^{t_2}} \sum_{\alpha=0}^{n-1} g^\alpha \omega_\alpha - \int_{\Gamma_{t_1}^{t_2}} \sum_{\alpha=0}^{n-1} g \left( \frac{\partial H}{\partial y} + \sum_{\alpha=0}^{n-1} \frac{\partial p^\alpha}{\partial x^\alpha} \right) \omega
\]

We are tempted to conclude that

\[
dQ^f = \{H \omega, Q^f\} \quad \text{and} \quad dP_g = \{H \omega, P\},
\]

where \( d \) is the differential along a graph \( \Gamma \) of a solution of the Hamilton equations (11). This precisely will be proven in the next section.

4 Poisson \( p \)-brackets for \( (p-1) \)-forms on \( M \)

We have seen on some examples that the Poisson bracket algebra of the classical field theory can actually be derived from brackets on \( (n-1) \)-forms which are integrated on constant time slices. Actually these constructions can be generalized in several ways.

4.1 \( p \)-brackets on \( (n-1) \)-forms

We turn back to \( M = \Lambda^n T^* (\mathcal{X} \times \mathcal{Y}) \) and to the notation of the previous Section. Let \( \Gamma(M, \Lambda^{n-1} T^* M) \) be the set of smooth \( (n-1) \)-forms on \( M \). We consider the subset \( \mathfrak{p}^{n-1} M \) of \( \Gamma(M, \Lambda^{n-1} T^* M) \) of forms \( a \) such that there exists a vector field \( \xi_a = \Xi(a) \) which satisfies the property
Obviously $\Xi(a)$ depends only on $a$ modulo closed forms and the map $a \mapsto \Xi(a)$ from $\mathcal{P}^{n-1}M$ to the set of vector fields induces a map on the quotient $\mathcal{P}^{n-1}M/C^{n-1}(M)$, where $C^{n-1}(M)$ is the set of closed $(n-1)$-forms. A property of vector fields $\Xi(a)$ is that there are infinitesimal symmetries of $\Omega$, for

$$L_{\Xi(a)}\Omega = d(\Xi(a)\lrcorner\Omega) + \Xi(a)\lrcorner d\Omega = -d \circ da = 0.$$  

We shall denote $\text{pp} M$ the set of pataplectic vector fields, i.e. vector fields $X$ such that $X\lrcorner\Omega$ is exact. Clearly $\Xi : \mathcal{P}^{n-1}M/C^{n-1}(M) \rightarrow \text{pp} M$ is a vector space isomorphism.

Then we define the \textit{internal} $p$-bracket on $\mathcal{P}^{n-1}M$ by

$$\{a, b\} := \Xi(b)\lrcorner\Xi(a)\lrcorner\Omega.$$  

\textbf{Lemma 2} For any $a, b \in \mathcal{P}^{n-1}M$,

$$d\{a, b\} = -[\Xi(a), \Xi(b)]\lrcorner\Omega. \quad (26)$$  

\textbf{Proof} Let $\xi_a = \Xi(a)$ and $\xi_b = \Xi(b)$. Then denoting $L_{\xi_a}$ the Lie derivative with respect to $\xi_a$,

$$[\xi_a, \xi_b]\lrcorner\Omega = L_{\xi_a}(\xi_b)\lrcorner\Omega = L_{\xi_a}(\xi_b\lrcorner\Omega) - \xi_b\lrcorner L_{\xi_a}\Omega = d(\xi_a\lrcorner\xi_b\lrcorner\Omega) + \xi_a\lrcorner d(\xi_b\lrcorner\Omega) - d(\xi_a\lrcorner\Omega)\lrcorner d\Omega.$$  

But since $d\Omega = d(\xi_a\lrcorner\Omega) = d(\xi_b\lrcorner\Omega) = 0$, we find that $[\xi_a, \xi_b]\lrcorner\Omega = d(\xi_a\lrcorner\xi_b\lrcorner\Omega) = -d\{a, b\}$. \hfill \blacksquare

\textbf{Lemma 3} $\Xi : \mathcal{P}^{n-1}M/C^{n-1}(M) \rightarrow \text{pp} M$ is a Lie algebra isomorphism. More precisely we have

$$\Xi([a, b]) = [\Xi(a), \Xi(b)]. \quad (27)$$  

This implies the Jacobi identity modulo exact terms in $\mathcal{P}^{n-1}M$:

$$\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = d(\xi_c\lrcorner\xi_b\lrcorner\xi_a\lrcorner\Omega). \quad (28)$$
Proof Relation (27) is a direct consequence of (26) in Lemma 2. The Jacobi identity follows from

\[
\{\{a, b\}, c\} = \{\{a, c\}, b\} + \{\{b, c\}, a\} = \{\{a, b\}, c\},
\]

where we have used (27). ■

We can extend the definition of the p-bracket: for any \(0 \leq p \leq n\) the external p-bracket of a p-form \(a \in \Gamma(\mathcal{M}, \Lambda^pT^*\mathcal{M})\) with a form \(b \in \mathfrak{P}^{n-1}\mathcal{M}\) is

\[
\{a, b\} = -\{b, a\} := -\Xi(b) \downarrow da.
\]

Of course this definition coincides with the previous one when \(a \in \mathfrak{P}^{n-1}\mathcal{M}\).

Examples of external p-brackets For any \(a \in \mathfrak{P}^{n-1}\mathcal{M}\),

\[
\{\theta, a\} = -\Xi(a) \downarrow d\theta = -\Xi(a) \downarrow \Omega = da.
\]

We can add that it is worthwhile to write in the external p-brackets of observable forms like \(q^\mu, q^\mu dq^\nu\), etc ...

\[
\{P_{i,g}, q^\mu\} = \Xi(P_{i,g}) \downarrow dq^\mu = g \delta^\mu_i,
\]

\[
\{Q^{i,f}, q^\mu\} = \Xi(Q^{i,f}) \downarrow dq^\mu = 0
\]

\[
\{P_{i,g}, q^\mu dq^\nu\} = \Xi(P_{i,g}) \downarrow dq^\mu \wedge dq^\nu = g (\delta^\mu_i dq^\nu - \delta^\nu_i dq^\mu).
\]

Theorem 2 Let \(\Gamma\) be the graph in \(\mathcal{M}\) of a solution of the Hamilton equations (11) and write \(\mathcal{U} : x \mapsto \mathcal{U}(x) = (x, u(x), p(x))\) the natural parametrization of \(\Gamma\). Then for any form \(a \in \mathfrak{P}^{n-1}\mathcal{M}\),

\[
da = \{\mathcal{H} \omega, a\},
\]

where \(d\) is the differential along \(\Gamma\) (meaning that \(da|_{\Gamma} = \{\mathcal{H} \omega, a\}|_{\Gamma}\)).

Proof We choose an arbitrary open subset \(D \subset \Gamma\) and denoting \(\xi_a = \Xi(a)\), we compute

\[
\int_D \{\mathcal{H} \omega, a\} = -\int_D \xi_a \downarrow (d\mathcal{H} \wedge \omega)
\]

\[
= -\int_{\mathcal{U}^{-1}(D)} d\mathcal{H} \wedge \omega \left(\xi_a \left.\frac{\partial \mathcal{U}}{\partial x^1}, \ldots, \frac{\partial \mathcal{U}}{\partial x^n}\right)\right) \omega
\]

\[
= -\int_{\mathcal{U}^{-1}(D)} (-1)^n \left.\frac{\partial \mathcal{U}}{\partial x^1} \ldots \frac{\partial \mathcal{U}}{\partial x^n}\right) \downarrow (d\mathcal{H} \wedge \omega)(\xi_a) \omega.
\]
We use equation (19) and obtain
\[
\int_D \{H_\omega, a\} = -\int_{U^{-1}(D)} (-1)^n \frac{\partial U}{\partial x^1 \ldots \partial x^n} \Omega(\xi_a) \omega
\]
\[
= -\int_{U^{-1}(D)} \Omega \left( \xi_a, \frac{\partial U}{\partial x^1}, \ldots, \frac{\partial U}{\partial x^n} \right) \omega
\]
\[
= -\int_D \xi_a \Omega = \int_D da.
\]
And the Theorem follows.

Another way to state this result is that
\[
\int_D \{H_\omega, a\} = \int_{\partial D} a
\]  
along any solution of (11).

4.2 Expression of the standard observable \((n-1)\)-forms

These quantities are integrals of \((n-1)\)-forms on hypersurfaces which are thought as “constant time slices”, the transversal dimension being then considered as a local time. The target coordinates observables \(\hat{\Omega}\) are weighted integrals of the value of the field and are induced by the “position” \(p\)-forms

\[
Q_i^f := y^i \sum_\alpha f^\alpha(x) \omega_\alpha = y^i f \int \omega,
\]
where \(f = \sum_\alpha f^\alpha(x) \frac{\partial}{\partial x^\alpha}\) is a tangent vector field on \(\mathcal{X}\) and \(\omega_\alpha = \frac{\partial}{\partial x^\alpha} \int \omega\).

The “momentum” and “energy” observables are obtained from the momentum form

\[
P^*_\mu, g := g(x) \frac{\partial}{\partial q^\mu} \int (\theta - \mathcal{H}(q, p) \omega),
\]
where \(g\) is a smooth function on \(\mathcal{X}\). Alternatively we may sometimes prefer to use the \(p\)-forms

\[
P_\mu, g := g(x) \frac{\partial}{\partial q^\mu} \int \theta.
\]

For \(1 \leq \mu = \alpha \leq n\), \(P^*_\mu, g := H_{\alpha g}\) generates the components of the Hamiltonian tensor but \(P_{\alpha, g}\) (which is different from \(P^*_\alpha, g\)) does not in general. However the restrictions of \(P^*_\mu, g\) and \(P_{\mu, g}\) on the hypersurface \(\mathcal{H} = 0\) coincide so that if we work on this hypersurface both forms can be used. For \(n + 1 \leq \mu = n + i \leq n + k\), \(P^*_\mu, g = P_{\mu, g} =: P_i, g\) generates the momentum

\(^8\)comparing with the one-dimensional Hamiltonian formalism we can see these target coordinates as generalizations of the position observables.
components.

To check that, we consider a parametrization $U : x \mapsto (x, u(x), p(x))$ of some graph $\Gamma$ and look at the pull-back of these forms by $U$. We write $U^*P_{\mu,g} = \sum_{\beta} s^\beta \omega_\beta$, which implies $s^\beta \omega = dx^\beta \land U^*P_{\mu,g}$ and we compute

$$s^\beta = g(x) \left( p_\alpha \frac{\partial q}{\partial x^\alpha} \land \ldots \land \frac{\partial q}{\partial x^{\beta-1}} \land \frac{\partial}{\partial q^\mu} \land \frac{\partial q}{\partial x^{\beta+1}} \land \ldots \land \frac{\partial q}{\partial x^n} \right) \bigg|_{z=\partial U/\partial x} - g(x) \delta^\beta_\mu \mathcal{H}$$

$$= g(x) \frac{\partial(p, z)}{\partial z^\mu} \bigg|_{z=\partial U/\partial x} - g(x) \delta^\beta_\mu \mathcal{H}.$$

Hence we find that

$$U^*H_{\alpha,g} = -g(x) \sum_{\beta} H^\beta_\alpha (q(x), p(x)) \omega_\beta = g(x) \sum_{\beta} S^\beta_\alpha (x, u(x), du(x)) \omega_\beta,$$

$$U^*P_{i,g} = g(x) \sum_{\beta} \frac{\partial(p, z)}{\partial z^\beta} \omega_\beta = g(x) \sum_{\beta} \frac{\partial L}{\partial v^\beta_\alpha} (x, u(x), du(x)) \omega_\beta.$$

We shall prove below that $P_{\mu,g}$ (and hence $P_{i,g}$) and $Q^{i,f}$ belong to $\mathcal{P}^{n-1}\mathcal{M}$.

### 4.2.1 Larger classes of observable

These forms, which are enough to translate most of the observable studied in the usual field theory, are embedded in two more general classes of observables the definition of which follows.

**Generalised positions** (see the footnote [3]) They are forms $Q^\xi$ in $\Lambda^{n-1} T^*(\mathcal{X} \times \mathcal{Y})$, i. e.

$$Q^\xi := \sum_{\mu_1 < \ldots < \mu_{n-1}} \zeta_{\mu_1\ldots\mu_{n-1}}(q) dq^{\mu_1} \land \ldots \land dq^{\mu_{n-1}}.$$

An example is $Q^{i,f} = y^i f(x) \partial_\alpha \downarrow \omega$. We denote $\mathcal{P}^{n-1}\mathcal{M} = \Lambda^{n-1} T^*(\mathcal{X} \times \mathcal{Y})$.

**Generalised momenta** For each section of $T(\mathcal{X} \times \mathcal{Y})$, i. e. a vector field

$$\xi := \sum_{\mu} \xi^\mu(q) \frac{\partial}{\partial q^\mu},$$

we define the $(n-1)$-form

[9] The advantage of $P_{\mu,g}$ with respect to $P_{\mu,g}^*$ is that $P_{\mu,g}$ belongs to $\mathcal{P}^{n-1}\mathcal{M}$ for all values of $\mu$. 
\[ P_\xi := \xi \mathcal{J} \theta = \sum_\mu \xi^\mu \frac{\partial}{\partial q^\mu} \mathcal{J} \theta. \]

An example is for \( \xi = g(x) \frac{\partial}{\partial q^\nu} \), then we obtain \( P_{g(x) \frac{\partial}{\partial q^\nu}} = P_{\mu, g} \). We denote \( \mathfrak{P}_{p}^{n-1} \mathcal{M} \) the set of such \((n-1)\)-forms.

**Lemma 4** \( \mathfrak{P}_{Q}^{n-1} \mathcal{M} \) ans \( \mathfrak{P}_{p}^{n-1} \mathcal{M} \) are subsets of \( \mathfrak{P}^{n-1} \mathcal{M} \), precisely

\[ \Xi(Q^\xi) = \sum_{\mu_1 < \ldots < \mu_n} \left( -1 \right)^{\alpha} \frac{\partial Q_{\mu_1 \ldots \mu_{n-1} \mu_{n-1} + \ldots + \mu_n}}{\partial q^{\mu_\alpha}} \frac{\partial}{\partial p_{\mu_1 \ldots \mu_n}}, \]

\[ \Xi(P_\xi) = \xi - \sum_\mu \sum_\nu \frac{\partial \xi^\mu}{\partial q^\nu} \Pi_\nu^\mu, \]

where

\[ \Pi_\nu^\mu := \sum_{\mu_1 < \ldots < \mu_n} \sum_{\alpha} p_{\mu_1 \ldots \mu_{n-1} \mu_{n-1} + \ldots + \mu_n} \delta_\mu^\nu \frac{\partial}{\partial p_{\mu_1 \ldots \mu_n}}, \]

so that

\[ dq^\nu \wedge \frac{\partial}{\partial q^\mu} \mathcal{J} \theta = \Pi_\nu^\mu \mathcal{J} \Omega. \]

**Proof** We have

\[ dQ^\xi = \sum_\nu \sum_{\mu_1 < \ldots < \mu_n} \frac{\partial Q_{\mu_1 \ldots \mu_{n-1} \mu_{n-1} + \ldots + \mu_n}}{\partial q^\nu} dq^\nu \wedge dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_{n-1}} \]

\[ = \sum_\alpha \sum_{\mu_1 < \ldots < \mu_n} \frac{\partial Q_{\mu_1 \ldots \mu_{n-1} \mu_{n-1} + \ldots + \mu_n}}{\partial q^{\mu_\alpha}} dq^{\mu_\alpha} \wedge dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_{\alpha-1}} \wedge dq^{\mu_{\alpha+1}} \wedge \ldots \wedge dq^{\mu_n} \]

\[ = \sum_\alpha \left( -1 \right)^{\alpha-1} \sum_{\mu_1 < \ldots < \mu_n} \frac{\partial Q_{\mu_1 \ldots \mu_{n-1} \mu_{n-1} + \ldots + \mu_n}}{\partial q^{\mu_\alpha}} \frac{\partial}{\partial p_{\mu_1 \ldots \mu_n}} \mathcal{J} \Omega. \]

And the expression for \( \Xi(Q^\xi) \) follows.

Next we write

\[ dP_\xi = \sum_\mu \sum_\nu \frac{\partial \xi^\mu}{\partial q^\nu} dq^\nu \wedge \frac{\partial}{\partial q^\mu} \mathcal{J} \theta - \sum_\mu \xi^\mu \frac{\partial}{\partial q^\mu} \mathcal{J} \Omega \]

and we conclude by computing \( dq^\nu \wedge \frac{\partial}{\partial q^\mu} \mathcal{J} \theta \), indeed.
\[ dq^\nu \wedge \frac{\partial}{\partial q^\mu} \mathcal{J} \theta = \sum_{\mu_1<...<\mu_n} \sum_{\alpha} p_{\mu_1...\mu_n} \delta_{\mu}^{\nu} dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_{a-1}} \wedge dq^\nu \wedge dq^{\mu_{a+1}} \wedge \ldots \wedge dq^{\mu_n} \]
\[ = \sum_{\mu_1<...<\mu_n} \sum_{\alpha} p_{\mu_1...\mu_{a-1}\mu_{a+1}...\mu_n} \delta_{\mu_a}^{\nu} dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_n} \]
\[ = \sum_{\mu_1<...<\mu_n} \sum_{\alpha} p_{\mu_1...\mu_{a-1}\mu_{a+1}...\mu_n} \delta_{\mu_a}^{\nu} \frac{\partial}{\partial p_{\mu_1...\mu_n}} \mathcal{J} \Omega. \]

Hence we deduce the result on \(P_\xi\).

\textbf{Proof} These results are all straightforward excepted for \(\{P_\xi, P_\xi\}\). We remark that \(\mathcal{L}(\xi)\mathcal{J} \theta = \xi \mathcal{J} \theta = P_\xi\) and

\[ \mathcal{L}(\xi)\mathcal{J} \theta = \Xi(P_\xi) \mathcal{J} d\theta + d(\Xi(P_\xi) \mathcal{J} \theta) = \Xi(P_\xi) \mathcal{J} \Omega + dP_\xi = 0, \]

so that \(\Xi(P_\xi)\) may be viewed as the extension of \(\xi\) to a vector field leaving \(\theta\) invariant. Now we deduce that

\[ [\xi, \tilde{\xi}] \mathcal{J} \theta = \Xi(P_\xi) \mathcal{J} \theta \]
\[ = \mathcal{L}(\Xi(P_\xi)) \mathcal{J} \theta \]
\[ = \mathcal{L}(\Xi(P_\xi)) (\Xi(P_\xi) \mathcal{J} \theta) - \Xi(P_\xi) \mathcal{J} \mathcal{L}(\Xi(P_\xi)) \theta \]
\[ = \Xi(P_\xi) \mathcal{J} d(\Xi(P_\xi) \mathcal{J} \theta) + d(\Xi(P_\xi) \mathcal{J} \Xi(P_\xi) \mathcal{J} \theta) = \Xi(P_\xi) \mathcal{J} \Omega + d\xi \mathcal{J} \tilde{\xi} \mathcal{J} \theta \]
\[ = \{P_\xi, P_\xi\} - d(\xi \mathcal{J} \xi \mathcal{J} \theta) \]

And the result follows.

\textbf{Poisson p-brackets}

We are now in position to compute the p-brackets of these forms. The results are summarized in the following Proposition.

\textbf{Proposition 1} The p-brackets of forms in \(\mathcal{P}_Q^{n-1} \mathcal{M}\) and \(\mathcal{P}_P^{n-1} \mathcal{M}\) are the following

\[ \{Q^\zeta, Q^{\tilde{\zeta}}\} = 0 \]
\[ \{P_\xi, P_\xi\} = P_{[\xi, \tilde{\xi}]} + d(\tilde{\xi} \mathcal{J} \xi \mathcal{J} \theta) \]
\[ \{P_\xi, Q^\zeta\} = \sum_{\mu_1<...<\mu_n} \sum_{\alpha} \sum_{\mu} (-1)^{\alpha+1} \xi^\mu \frac{\partial Q^\zeta_{\mu_1...\mu_{a-1}\mu_{a+1}...\mu_n}}{\partial q^{\mu_a}} \frac{\partial}{\partial q^\mu} dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_n}. \]

\textbf{Proof} These results are all straightforward excepted for \(\{P_\xi, P_\xi\}\). We remark that \(\mathcal{L}(\xi)\mathcal{J} \theta = \xi \mathcal{J} \theta = P_\xi\) and

\[ \mathcal{L}(\xi)\mathcal{J} \theta = \Xi(P_\xi) \mathcal{J} d\theta + d(\Xi(P_\xi) \mathcal{J} \theta) = \Xi(P_\xi) \mathcal{J} \Omega + dP_\xi = 0, \]

so that \(\Xi(P_\xi)\) may be viewed as the extension of \(\xi\) to a vector field leaving \(\theta\) invariant. Now we deduce that

\[ [\xi, \tilde{\xi}] \mathcal{J} \theta = \Xi(P_\xi) \mathcal{J} \theta \]
\[ = \mathcal{L}(\Xi(P_\xi)) \mathcal{J} \theta \]
\[ = \mathcal{L}(\Xi(P_\xi)) (\Xi(P_\xi) \mathcal{J} \theta) - \Xi(P_\xi) \mathcal{J} \mathcal{L}(\Xi(P_\xi)) \theta \]
\[ = \Xi(P_\xi) \mathcal{J} d(\Xi(P_\xi) \mathcal{J} \theta) + d(\Xi(P_\xi) \mathcal{J} \Xi(P_\xi) \mathcal{J} \theta) = \Xi(P_\xi) \mathcal{J} \Omega + d\xi \mathcal{J} \tilde{\xi} \mathcal{J} \theta \]
\[ = \{P_\xi, P_\xi\} - d(\xi \mathcal{J} \xi \mathcal{J} \theta) \]

And the result follows.
### 4.2.2 Back to the standard observables

As an application of the previous results we can express the pataplectic vector fields associated to $Q^{i,f}$ and $P_{\mu,g}$ and their $p$-brackets. For that purpose, it is useful to introduce other notations:

$$
\epsilon := p_{1...n}
$$

$$
p_{i}^\alpha := p_{1...((\alpha-1)(n+i)(\alpha+1)...n}
$$

$$
p_{i_1i_2}^{\alpha_1\alpha_2} := p_{1...((\alpha_1-1)(n+i_1)+(\alpha_1+1)...(\alpha_2-1)(n+i_2)+(\alpha_2+1)...n}
$$

and

$$
\omega_{i}^{\alpha} := dy_{i} \wedge (\frac{\partial}{\partial x^\alpha}) \Omega =: (dy_{i} \wedge \partial_{\alpha}) \Omega
$$

$$
\omega_{i_1i_2}^{\alpha_1\alpha_2} := (dy_{i_1} \wedge \partial_{\alpha_1}) \Omega (dy_{i_2} \wedge \partial_{\alpha_2}) \Omega
$$

in such a way that

$$
\theta = \epsilon \omega + \sum_{p=1}^{n} \frac{1}{p!^2} \sum_{i_1,...,i_p;\alpha_1,...,\alpha_p} p_{i_1...i_p}^{\alpha_1...\alpha_p} \omega_{\alpha_1...\alpha_p}^{i_1...i_p}.
$$

(Notice that the Weyl theory corresponds to the assumption that $p_{i_1...i_p}^{\alpha_1...\alpha_p} = 0$, $\forall p \geq 2$.) We have

$$
dQ^{i,f} = \sum_{\alpha} f_{\alpha}^\alpha \frac{\partial}{\partial p_i^\alpha} \Omega + y^i \sum_{\alpha} \frac{\partial f_{\alpha}^\alpha}{\partial x^\alpha} \frac{\partial}{\partial \epsilon} \Omega,
$$

$$
dP_{\mu,g} = \sum_{\alpha} \frac{\partial g}{\partial x^\alpha} p_{\alpha}^\mu \Omega - g \frac{\partial}{\partial q^\mu} \Omega,
$$

where

$$
\Pi_{\alpha}^\beta = \delta_{\alpha}^\beta \epsilon \frac{\partial}{\partial \epsilon} + \sum_{p=1}^{n} \frac{1}{p!^2} \sum_{i_1,...,i_p;\alpha_1,...,\alpha_p} \left( p_{i_1...i_p}^{\alpha_1...\alpha_p} \delta_{\alpha}^\alpha - \sum_{j=1}^{p} p_{i_1...j-1i_j...i_p;\alpha_1,...,\alpha_p}^{\alpha_1...\alpha_j...\alpha_j...\alpha_p} \delta_{\alpha}^\alpha \right) \frac{\partial}{\partial p_{i_1...i_p}^{\alpha_1...\alpha_p}}
$$

$$
\Pi_{n+i}^\alpha = \sum_{p=0}^{n-1} \frac{1}{p!^2} \sum_{i_1,...,i_p;\alpha_1,...,\alpha_p} p_{i_1i_2...i_p}^{\alpha_1...\alpha_p} \frac{\partial}{\partial p_{i_1...i_p}^{\alpha_1...\alpha_p}}
$$

The pataplectic vector fields are

$$
\Xi(Q^{i,f}) = - \sum_{\alpha} f_{\alpha}^\alpha \frac{\partial}{\partial p_i^\alpha} - y^i \sum_{\alpha} \frac{\partial f_{\alpha}^\alpha}{\partial x^\alpha} \frac{\partial}{\partial \epsilon}
$$

$$
\Xi(P_{\mu,g}) = g \frac{\partial}{\partial q^\mu} - \sum_{\alpha} \frac{\partial g}{\partial x^\alpha} \Pi_{\alpha}^\mu.
$$

Finally by using Proposition 1, the Poisson $p$-brackets will be
\{Q^{i,j}, Q^{i,j}\} = 0,

\{P_{i,g}, P_{j,\tilde{g}}\} = d \left( \tilde{g} \frac{\partial}{\partial y^j} \frac{\partial}{\partial y^i} \theta \right),

\{P_{i,g}, Q^{i,j}\} = \delta_i^j \sum_\alpha f^\alpha g_{\omega_\alpha}.

Hence if \( g \) and \( \tilde{g} \) have compact support, we obtain that on any submanifold \( S \) of dimension \( n - 1 \) without boundary,

\[ \int_S \{Q^{i,j}, Q^{i,j}\} = \int_S \{P_{i,g}, P_{j,\tilde{g}}\} = 0 \quad \text{and} \quad \int_S \{P_{i,g}, Q^{i,j}\} = \delta_i^j \int_S \sum_\alpha f^\alpha g_{\omega_\alpha} \]

### 4.3 Extension of the \( p \)-bracket to forms of degree less than \( n - 1 \)

The \( p \)-brackets defined above allow us to express the dynamics of an observable which is in \( P^{n-1}M \). We shall extend this bracket to some forms in \( \Gamma(M, \Lambda^{p}T^*M) \), where \( 0 \leq p \leq n - 1 \). Like in the case \( p = n - 1 \), not every \( p \)-form is admissible and, as we shall see, the class of such \( p \)-forms is quite restricted and is basically composed of “position” observables. However when the Hamiltonian system is degenerate, due to some gauge symmetry and constraints, some “momentum” observable can be represented by \( p \)-forms with \( p < n - 1 \). An instance of such a situation is the electromagnetic field studied in Section 5.3.

For \( 1 \leq p \leq n \), we define \( P^{p-1}M \) to be the set of sections \( a \) of \( \Gamma(M, \Lambda^{p-1}T^*M) \), such that, for all \( 1 \leq \alpha_1 < \ldots < \alpha_{n-p} \leq n \),

\[ dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-p}} \wedge a \in P^{n-1}M. \]

We introduce anticommuting (Grassmann) variables \( \tau_1, \ldots, \tau_n \), which behave under change of coordinates like \( \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \). We shall consider functions and forms depending on the variables \( (\tau_\alpha, x^\alpha, y^\alpha, p_{\mu_1}, \ldots, \mu_n) \). Alternatively they can be seen as functions on the bundle \( \Pi T^*X \otimes \pi M \), where \( \Pi T^*X \) is a copy of \( T^*X \) in which the parity of vectors in the fibers \( T_xX \) has been reversed. We consider \( \Gamma(M, \Lambda^{n-1}T^*M) \) to be the set of \((n-1)\)-forms on \( M \) whose coefficients are in the algebra \( \mathbb{R}[\tau_1, \ldots, \tau_n] \). More intrinsically, \( \Gamma(M, \Lambda^{n-1}T^*M) \) can be identified with \( \mathcal{C}^\infty(\Pi T^*X) \otimes \mathcal{C}^\infty(\mathcal{X}) \Gamma(M, \Lambda^{n-1}T^*M) \), meaning that any form \( A \in \Gamma(M, \Lambda^{n-1}T^*M) \) is a finite sum of terms of the form \( \phi(x, \tau) \theta \), where \( \phi \in \mathcal{C}^\infty(\Pi T^*X) \) and \( \theta \in \Gamma(M, \Lambda^{n-1}T^*M) \). Through this identification we can define \( P^{n-1}M \) to be the subset of \( \Gamma(M, \Lambda^{n-1}T^*M) \) linearly spanned by \( \phi(x, \tau) \theta \), where \( \phi \in \mathcal{C}^\infty(\Pi T^*X) \) and \( \theta \in P^{n-1}M \).
Obviously for any \( A \in {^s\mathcal{P}^{n-1}}\mathcal{M} \), there exists some vector field \( \Xi(A) \) on \( \mathcal{M} \) with coefficients in \( \mathbb{R}[\tau_1, \ldots, \tau_n] \) such that \( dA = -\Xi(A) \bigwedge \Omega \). A more geometrical description of \( \Xi(A) \) is that it is a section of the bundle \( \Pi T\mathcal{X} \otimes \mathcal{X} T\mathcal{M} \) over \( \mathcal{M} \).

We embed each \( {^p\mathcal{P}^{n-1}}\mathcal{M} \) in \( {^s\mathcal{P}^{n-1}}\mathcal{M} \) by
\[
{^p\mathcal{P}^{n-1}}\mathcal{M} \rightarrow {^s\mathcal{P}^{n-1}}\mathcal{M}
\]
where the “superform” \( {^s}a \) is defined by
\[
{^s}a := \sum_{\alpha_1 < \ldots < \alpha_{n-p}} \tau_{\alpha_1} \cdots \tau_{\alpha_{n-p}} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{n-p}} \wedge \alpha.
\]
Then
\[
\Xi({^s}a) = \sum_{\alpha_1 < \ldots < \alpha_{n-p}} \tau_{\alpha_1} \cdots \tau_{\alpha_{n-p}} \Xi(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{n-p}} \wedge \alpha).
\]

We endow \( {^s\mathcal{P}^{n-1}}\mathcal{M} \) with the Poisson \( p \)-bracket defined by
\[
\{A, B\}_s := \Xi(A) \bigwedge \Xi(B) \bigwedge \Omega,
\]
where assuming that \( A \) and \( B \) are homogeneous in \( \tau_\alpha \) and are given by
\[
A = \sum_{\alpha_1 < \ldots < \alpha_{n-p}} \tau_{\alpha_1} \cdots \tau_{\alpha_{n-p}} A^{\alpha_1 \ldots \alpha_{n-p}},
\]
\[
B = \sum_{\beta_1 < \ldots < \beta_{n-q}} \tau_{\beta_1} \cdots \tau_{\beta_{n-q}} B^{\beta_1 \ldots \beta_{n-q}},
\]
\[
\{A, B\}_s = \sum_{\alpha_1 < \ldots < \alpha_{n-p}} \sum_{\beta_1 < \ldots < \beta_{n-q}} \tau_{\alpha_1} \cdots \tau_{\alpha_{n-p}} \tau_{\beta_1} \cdots \tau_{\beta_{n-q}} \Xi(B^{\beta_1 \ldots \beta_{n-q}}) \bigwedge \Xi(A^{\alpha_1 \ldots \alpha_{n-p}}) \bigwedge \Omega.
\]

Lemma 2 implies immediately that
\[
d\{A, B\}_s = -[\xi_A, \xi_B]_s \bigwedge \Omega,
\]
where \( \xi_A = \Xi(A) \), \( \xi_B = \Xi(B) \) and the (super)-Lie bracket \([,]_s\) is defined for homogeneous forms \( A \) and \( B \) by \([\xi_A, \xi_B]_s f = \xi_A \bigwedge d(\xi_B \bigwedge df) + (-1)^{|A||B|+1} \xi_B \bigwedge d(\xi_A \bigwedge df)\) (here \(|A|\) and \(|B|\) are the homogeneity degrees in the variables \( \tau_\alpha \)). Furthermore, one can deduce easily from Lemma 3 the following relations for all homogeneous forms \( A, B \) and \( C \) in \( {^s\mathcal{P}^{n-1}}\mathcal{M} \),
\[
\{A, B\}_s = (-1)^{|A||B|+1} \{B, A\}_s.
\]
\[ (-1)^{[A][C]} \{ \{ A, B \}, C \} + (-1)^{[B][A]} \{ \{ B, C \}, A \} + (-1)^{[C][B]} \{ \{ C, A \}, B \} = 0 \]

Hence \( {}^s\mathfrak{p}^{n-1}\mathcal{M} \) has the structure of a graded Lie algebra modulo exact terms.

Now suppose that we can prove that for some forms \( a \in {}^s\mathfrak{p}^{p-1}\mathcal{M} \) and \( b \in {}^s\mathfrak{p}^{q-1}\mathcal{M}, \{ {}^s a, {}^s b \}_s \) is equal to some \( {}^s c \) where \( c \in {}^s\mathfrak{p}^{p+q-n-1}\mathcal{M} \), then we could define the \textit{internal} \( \mathfrak{p} \)-bracket between \( a \) and \( b \) by \( \{ a, b \} := c \). This turns actually to be true for a simple reason: all these brackets vanish by Proposition 2 below \( ^{10} \). However this fact is no longer true in general in the interesting case where we have constraints as shown in Section 5.3.

\textbf{Lemma 5} For \( 1 \leq p < n \), \( {}^s\mathfrak{p}^{p-1}\mathcal{M} \) coincides with \( \Lambda^{p-1}T^*(\mathcal{X} \times \mathcal{Y}) \).

\textbf{Proof} First step: let \( 1 \leq p < n \) and \( a \in {}^s\mathfrak{p}^{p-1}\mathcal{M} \) and choose any \( 1 \leq \alpha_1 < \ldots < \alpha_{n-1} \leq n \), so that \( dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-1}} \wedge a \in {}^s\mathfrak{p}^{n-1}\mathcal{M} \). Let us denote \( \xi := \Xi(dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-1}} \wedge a) \). Decompose \( \xi \):

\[ \xi = \sum_{\mu} \xi^\mu \frac{\partial}{\partial q^\mu} + \sum_{\mu_1 < \ldots < \mu_n} \xi_{\mu_1 \ldots, \mu_n} \frac{\partial}{\partial p_{\mu_1 \ldots, \mu_n}}. \]

Then

\[ -\xi \bigwedge \Omega = - \sum_{\mu_1 < \ldots < \mu_n} \xi_{\mu_1 \ldots, \mu_n} dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_n} \]

\[ - \sum_{\nu} \xi^\nu \sum_{\alpha=1}^{\nu} (-1)^\alpha \sum_{\mu_1 < \ldots < \mu_n} dp_{\mu_1 \ldots, \mu_n} \wedge dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_{\alpha-1}} \wedge dq^{\mu_{\alpha+1}} \wedge \ldots \wedge dq^{\mu_n}. \]

This expression should be equal to \( (-1)^{n-p-1} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-1}} \wedge da \). Note that for any \( 1 \leq \nu \leq n+k \), there exist \( n \) integers \( \mu_1 < \ldots < \mu_n \) such that \( \nu \in \{ \mu_1, \ldots, \mu_n \} \) but \( \alpha_1 \not\in \{ \mu_1, \ldots, \mu_n \} \). This forces \( \xi^\nu = 0 \). Hence we are left with

\[ (-1)^{n-p-1} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-1}} \wedge da = - \sum_{\mu_1 < \ldots < \mu_n} \xi_{\mu_1 \ldots, \mu_n} dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_n} \]

which implies that \( a \) does not depend on the variables \( p_{\mu_1 \ldots, \mu_n} \). Hence \( a \in \Lambda^{p-1}T^*(\mathcal{X} \times \mathcal{Y}) \).

Second step: Conversely let \( a \in \Lambda^{p-1}T^*(\mathcal{X} \times \mathcal{Y}) \). Then, for each \( 1 \leq \alpha_1 < \ldots < \alpha_{n-1} \leq n \), \( dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-1}} \wedge a \) belongs to \( \Lambda^{n-1}T^*(\mathcal{X} \times \mathcal{Y}) \), which is a subset of \( \mathfrak{p}^{n-1}\mathcal{M} \) by Lemma 4. So \( a \in {}^s\mathfrak{p}^{p-1}\mathcal{M} \). \( ^{10} \)
Proposition 2 a) For $1 \leq p, q < n$, $a \in \mathcal{P}^{p-1} \mathcal{M}$ and $b \in \mathcal{P}^{q-1} \mathcal{M}$ the $p$-bracket of $^s a$ with $^s b$ vanishes: $\{ ^s a, ^s b \}_s = 0$ and hence the internal $p$-bracket $\{ a, b \}$ exists and is equal to 0.

b) For $1 \leq p < n$, $a \in \mathcal{P}^{n-1} \mathcal{M}$ and $b \in \mathcal{P}^{p-1} \mathcal{M}$,

$$\{ ^s a, ^s b \}_s = -^s (db) \Xi(a)(\tau) + ^s \{ a, b \}, \tag{31}$$

where $\{ a, b \} = \Xi(a) \bigwedge db$ is the external $p$-bracket and

$$\Xi(a)(\tau) := \sum_{\alpha} dx^\alpha(\Xi(a)) \tau_\alpha.$$

(It is actually a superfunction on $\Pi T \mathcal{X}$.) As a consequence, if $dx^\alpha(\Xi(a)) = 0$, $\forall \alpha$, the internal $p$-bracket $\{ a, b \}$ exists and coincides with the external $p$-bracket.

Proof Case a): by Lemma 5 a and b are in $\Lambda^{p-1} T^* (\mathcal{X} \times \mathcal{Y})$ and $\Lambda^{q-1} T^* (\mathcal{X} \times \mathcal{Y})$ respectively, so $^s a$ and $^s b$ are in $\mathcal{C}^\infty (\Pi T \mathcal{X}) \otimes \mathcal{C}^\infty (\mathcal{Y}) \Lambda^{n-1} T^* (\mathcal{X} \times \mathcal{Y})$. Hence their $p$-bracket vanish by Proposition 1.

Let us consider the case b). Let us denote $\xi_a = \Xi(a)$ and write

$$\xi_a = \sum_{\mu} \xi_{a, \mu} \frac{\partial}{\partial q^\mu} + \sum_{\mu_1 < \ldots < \mu_n} \xi_{a, \mu_1 \ldots \mu_n} \frac{\partial}{\partial p_{\mu_1 \ldots \mu_n}},$$

then

$$\{ a, ^s b \}_s = \sum_{\alpha_1 < \ldots < \alpha_{n-p}} (-1)^{n-p} \tau_{\alpha_1} \ldots \tau_{\alpha_{n-p}} \xi_a \bigwedge (dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-p}} \wedge db)$$

$$= (-1)^{n-p} \sum_{\alpha} \sum_{\alpha_1 < \ldots < \alpha_{n-p}} \sum_{l=1}^{n-p} \delta_{\alpha_1}^\alpha \xi_a \tau_{\alpha_1} \tau_{\alpha_1} \ldots \tau_{\alpha_{l-1}} \tau_{\alpha_{l+1}} \ldots \tau_{\alpha_{n-p}}$$

$$dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{l-1}} \wedge dx^{\alpha_{l+1}} \wedge \ldots \wedge dx^{\alpha_{n-p}} \wedge db$$

$$+ \sum_{\alpha_1 < \ldots < \alpha_{n-p}} \tau_{\alpha_1} \ldots \tau_{\alpha_{n-p}} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-p}} \wedge (\xi_a \bigwedge db)$$

$$= (-1)^{n-p} \sum_{\alpha} \xi_a \tau_{\alpha} \sum_{\alpha_1 < \ldots < \alpha_{n-p-1}} \tau_{\alpha_1} \ldots \tau_{\alpha_{n-p-1}} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-p-1}} \wedge db$$

$$+ ^s (\xi_a \bigwedge db)$$

$$= (-1)^{n-p} \sum_{\alpha} \xi_a \tau_{\alpha} ^s (db) + ^s \{ a, b \}.$$

And the claim is proved. \[\blacksquare\]

One simple example is the 0-form $y^j$. The associated superform is

$$^s y^j = \sum_{\alpha} y^j (\Xi(a)(\tau)) \tau_{\alpha_1} \ldots \tau_{\alpha_{n-p-1}} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-p-1}} \wedge db.$$
Since
\[ d^s y^j = \sum\limits_{\alpha} (-1)^{\alpha-1} \tau_1 \ldots \tau_{\alpha-1} \tau_{\alpha+1} \ldots \tau_n \omega^{\alpha_j}, \]
we have
\[ \Xi( s y^j ) = \sum\limits_{\alpha} (-1)^{\alpha} \tau_1 \ldots \tau_{\alpha-1} \tau_{\alpha+1} \ldots \tau_n \frac{\partial}{\partial p^j_{\alpha}}. \]

Let us compute the \( p \)-sbracket with \( P_i = \frac{\partial}{\partial y^i} \ast \theta \):

\[ \{ P_i, s y^j \}_{s} = \sum\limits_{\alpha} (-1)^{\alpha} \tau_1 \ldots \tau_{\alpha-1} \tau_{\alpha+1} \ldots \tau_n \frac{\partial}{\partial y^i} \ast \partial \frac{\partial}{\partial x^\alpha} \ast \Omega \]
\[ = \sum\limits_{\alpha} (-1)^{\alpha-1} \tau_1 \ldots \tau_{\alpha-1} \tau_{\alpha+1} \ldots \tau_n \delta^i_{\alpha} \frac{\partial}{\partial x^\alpha} \ast \omega \]

Thus
\[ \{ P_i, y^j \} = \delta^i_j. \]

**4.4 Integral of an observable in \( \mathcal{P}^{p-1} \mathcal{M} \) and dynamical equations**

Let \( \Gamma \) be a submanifold of dimension \( n \) on \( \mathcal{M} \) and let \( D \) be be some oriented submanifold with boundary of dimension \( p \) (\( 1 \leq p \leq n \)) included in \( \Gamma \). We consider \( D_\Gamma \), the fiber bundle over \( D \) whose fibers at the point \( m \in D \) is the oriented tangent space to \( \Gamma \) at \( m \).

**Definition 4** Let \( a \in \mathcal{P}^{p-1} \mathcal{M} \) and \( \psi \in \Gamma(\mathcal{M}, \Lambda^n T^* \mathcal{M}) \). We define the \( p \)-sbracket

\[ \{ \psi, s^a \}_{s} := -\Xi( s^a ) \ast \psi \]
\[ = -\sum_{\alpha_1 < \ldots < \alpha_{n-p}} \tau_{\alpha_1} \ldots \tau_{\alpha_{n-p}} \Xi(dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-p}} \wedge a) \ast \Omega d\psi. \]  

(32)

We define the integral of \( \{ \psi, s^a \}_{s} \) over \( D_\Gamma \) to be

\[ \int_{D_\Gamma} \{ \psi, s^a \}_{s} := \int_{D_\Gamma} -\sum_{\alpha_1 < \ldots < \alpha_{n-p}} (X_{\alpha_1} \wedge \ldots \wedge X_{\alpha_{n-p}} \wedge \Xi(dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-p}} \wedge a)) \ast \Omega d\psi, \]  

(33)

where \( X = X_1 \wedge \ldots \wedge X_n \) is the \( n \)-vector tangent to \( \Gamma \) at \( m \) such that \( dx^{\alpha}(X_\beta) = \delta^\alpha_\beta \). Notice that this definition does not depend on the parametrisation which is used.
Theorem 3 Assume that $\Gamma$ is the graph of some solution of the Hamilton equations (19). Then for any $a \in \mathcal{P}^{p-1} \mathcal{M}$,

$$\int_D da = \int_{D_r} \{ \mathcal{H} \omega, \cdot s a \}_s. \quad (34)$$

Proof We can always assume that $D$ is the image of some parametrisation

$$\Delta \rightarrow D$$

$$t \mapsto U(x(t)) = (x(t), u(t), p(t)),$$

where $\Delta$ is an open subset of $\mathbb{R}^p$. Then

$$\int_{D_r} \{ \mathcal{H} \omega, \cdot s a \}_s$$

$$= \int_{\Delta} \frac{\Omega(\Xi(dx_{\alpha_1} \land \ldots \land dx_{\alpha_{n-p}} \land a), X_{\alpha_1}, \ldots, X_{\alpha_{n-p}}, X_{\beta_1}, \ldots, X_{\beta_p})}{\det \left( \begin{array}{c} \frac{\partial x_{\beta_1}}{\partial t^1}, \ldots, \frac{\partial x_{\beta_p}}{\partial t^p} \end{array} \right) dt^1 \land \ldots \land dt^p}.$$

Now by using (13),

$$\int_{D_r} \{ \mathcal{H} \omega, \cdot s a \}_s$$

$$= \int_{\Delta} \frac{\Omega(\Xi(dx_{\alpha_1} \land \ldots \land dx_{\alpha_{n-p}} \land a), X_{\alpha_1}, \ldots, X_{\alpha_{n-p}}, X_{\beta_1}, \ldots, X_{\beta_p})}{\det \left( \begin{array}{c} \frac{\partial x_{\beta_1}}{\partial t^1}, \ldots, \frac{\partial x_{\beta_p}}{\partial t^p} \end{array} \right) dt^1 \land \ldots \land dt^p}.$$
There exists however a much simpler concept of bracket between $\mathcal{H}\omega$ and observables in $\mathcal{P}^{p-1}\mathcal{M}$ which is also suitable for the dynamical equation in most cases. Namely we call a form $a$ in $\mathcal{P}^{p-1}\mathcal{M}$ an admissible form if

$$dx^\alpha(\Xi(dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-p}} \wedge a)) = 0, \quad \forall \alpha$$

or equivalently

$$dx^\alpha(\Xi(\ast a)) = 0, \quad \forall \alpha.$$  

(35)

The reader may wonder the meaning of this definition since, in view of Lemma 5, all forms in $\mathcal{P}^{p-1}\mathcal{M}$ are admissible for $p < n$. Again the point is that we may encounter variational problems with gauge symmetry and constraints for which non admissible forms exist in $\mathcal{P}^{p-1}\mathcal{M}$.

**Definition 5** Assume that $a \in \mathcal{P}^{p-1}\mathcal{M}$ satisfies (35) and let $\psi \in \Gamma(\mathcal{M}, \Lambda^n\mathcal{T}^*\mathcal{M})$. Then we define the $p$-bracket

$$\{\psi, a\} := - \sum_{\alpha_1 < \ldots < \alpha_{n-p}} \left( -1 \right)^{n-p} \frac{\partial}{\partial x^{\alpha_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{\alpha_{n-p}}} \wedge \Xi(dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-p}} \wedge a) \int d\psi.$$  

**Lemma 6** Let $a \in \mathcal{P}^{p-1}\mathcal{M}$ be an admissible form (i.e. such that (35) holds) and let $\Gamma$ be a $n$-dimensional submanifold of $\mathcal{M}$ which is a graph over $X$. Then for any oriented submanifold $D$ of dimension $p$ included in $\Gamma$,

$$\int_D \{\mathcal{H}\omega, \ast a\} = \int_D \{\mathcal{H}\omega, a\}.$$  

(36)

**Proof** We use the same notations as in the proof of Theorem 3. Because of the condition (35),

$$\sum_{\alpha_1 < \ldots < \alpha_{n-p}} d\mathcal{H} \wedge \omega \left( X_{\alpha_1}, \ldots, X_{\alpha_{n-p}}, \Xi(dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-p}} \wedge a), X_{\beta_1}, \ldots, X_{\beta_{n-p}} \right)$$

$$= \sum_{\alpha_1 < \ldots < \alpha_{n-p}} \left( -1 \right)^{n-p} d\mathcal{H}(\Xi(dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-p}} \wedge a)) \omega \left( X_{\alpha_1}, \ldots, X_{\alpha_{n-p}}, X_{\beta_1}, \ldots, X_{\beta_{n-p}} \right)$$

$$= \sum_{\alpha_1 < \ldots < \alpha_{n-p}} \left( -1 \right)^{n-p} d\mathcal{H}(\Xi(dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-p}} \wedge a)) \omega \left( \frac{\partial}{\partial x^{\alpha_1}}, \ldots, \frac{\partial}{\partial x^{\alpha_{n-p}}}, X_{\beta_1}, \ldots, X_{\beta_{n-p}} \right)$$

$$= \sum_{\alpha_1 < \ldots < \alpha_{n-p}} d\mathcal{H} \wedge \omega \left( \frac{\partial}{\partial x^{\alpha_1}}, \ldots, \frac{\partial}{\partial x^{\alpha_{n-p}}}, \Xi(dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-p}} \wedge a), X_{\beta_1}, \ldots, X_{\beta_{n-p}} \right)$$

$$= - \sum_{\alpha_1 < \ldots < \alpha_{n-p}} \{\mathcal{H}\omega, a\}(X_{\beta_1}, \ldots, X_{\beta_{n-p}}).$$

This implies the result by summation over $\beta_1 < \ldots < \beta_{n-p}$ and integration over $D$. 

■
Corollary 1 Let \( a \in \mathcal{B}^{p-1} \mathcal{M} \) be an admissible form and let \( \Gamma \) be a \( n \)-dimensional submanifold of \( \mathcal{M} \) which is a graph over \( \mathcal{X} \) of a solution of the Hamilton equations \((\ref{Hamilton})\). Then for any oriented submanifold \( D \) of dimension \( p \) included in \( \Gamma \),

\[
\int_D \{ \mathcal{H} \omega, a \} = \int_D da. \tag{37}
\]

Examples The 0-form \( y^i \) and the 1-form \( y^i dy^j \) are admissible and

\[
\{ \mathcal{H} \omega, y^i \} = \sum_\alpha \frac{\partial \mathcal{H}}{\partial p^\alpha_i} dx^\alpha,
\]

\[
\{ \mathcal{H} \omega, y^i dy^j \} = \sum_{\alpha < \beta} \frac{\partial \mathcal{H}}{\partial p^\alpha_\beta} dx^\alpha \wedge dx^\beta.
\]

4.5 Noether theorem

It is natural to relate the Noether theorem to the pataplectic structure.

Let \( \xi \) be a tangent vector field on \( \mathcal{X} \times \mathcal{Y} \), \( \xi \) will be an infinitesimal symmetry of the variational problem if

\[
\mathcal{L}_{\Xi(P^*_\xi)} (\theta - \mathcal{H} \omega) = 0,
\]

since then the integral \( \int_\Gamma \theta - \mathcal{H} \omega \) is invariant under the action of the flow of \( \xi \). Then for any solution \( x \mapsto (U(x), p(x)) \), of the Hamilton equations, the form \( P^* \xi \) is closed along the graph of this solution. This means that if \( \Gamma \) is the graph of \( (U, p) \),

\[
dP^\xi_{\mid \Gamma} = d(\xi \downarrow \mathcal{H} \omega)_{\mid \Gamma} = 0.
\]

This is a direct consequence of Theorem 2 and of the following calculation.

Lemma 7 For any section \( \xi \) of \( \Gamma(\mathcal{X} \times \mathcal{Y}, T(\mathcal{X} \times \mathcal{Y})) \), we have the relation

\[
\{ \mathcal{H} \omega, P^*_\xi \} = \mathcal{L}_{\Xi(P^*_\xi)} (\theta - \mathcal{H} \omega) + d(\xi \downarrow \mathcal{H} \omega). \tag{38}
\]

Proof Using the definition of \( \{ \mathcal{H} \omega, P^*_\xi \} \), we have

\[
\mathcal{L}_{\Xi(P^*_\xi)} (\theta - \mathcal{H} \omega) = \Xi(P^*_\xi) \downarrow (d\theta - d\mathcal{H} \wedge \omega) + d(\Xi(P^*_\xi) \downarrow (\theta - \mathcal{H} \omega))
\]

\[
= \Xi(P^*_\xi) \downarrow \Omega + \Xi(P^*_\xi) \downarrow d\mathcal{H} \wedge \omega + d(\xi \downarrow \mathcal{H} \omega)
\]

\[
= -dP^*_\xi + \{ \mathcal{H} \omega, P^*_\xi \} + d(P^*_\xi - \xi \downarrow \mathcal{H} \omega),
\]

and the result follows. \( \blacksquare \)
Remark 1 It appears that it will be interesting to study solutions of the Hamilton equations with the constraint $H = 0$. This is possible, because of the freedom left in the Legendre correspondance, thanks to the parameter $\epsilon$. The advantage is that then the energy-momentum observables are described by $P_{a,g}$ which belongs to $\mathfrak{p}^{n-1}\mathcal{M}$.

Remark 2 As a consequence of these observations it is clear that on the submanifold $H = 0$, the set of Noether currents can be identified with $\mathfrak{p}^{n-1}\mathcal{M}$. So we can interpret the results of Proposition 1 concerning $\mathfrak{p}^{n-1}\mathcal{M}$ by saying that the set of Noether currents equipped with the $p$-bracket is a representation modulo exact terms of the Lie algebra of vector fields on $\mathcal{X} \times \mathcal{Y}$ with the Lie bracket. We recover thus various constructions of brackets on Noether currents (see for instance [32]).

5 Examples

We present here some examples from the mathematical Physics in order to illustrate our formalism. We shall see that, by allowing variants of the above theory, one can find formalisms which are more adapted to some special situations.

5.1 Interacting scalar fields

As the simplest example, consider a system of interacting scalar fields $\{\phi^1, \ldots, \phi^k\}$ on an oriented (pseudo-)Riemannian manifold $(\mathcal{X}, g)$. One should keep in mind that $\mathcal{X}$ is a four-dimensional space-time and $g_{\alpha\beta}$ is a Minkowski metric. These fields can be seen as a map $\phi$ from $\mathcal{X}$ to $\mathbb{R}^k$ with its standard Euclidian structure. The metric $g$ on $\mathcal{X}$ induces a volume form which reads in local coordinates

$$\omega := g_{\alpha\beta}(x) dx^1 \wedge \ldots \wedge dx^n,$$

where $g := \sqrt{|\det g_{\alpha\beta}(x)|}$.

Let $V : \mathcal{X} \rightarrow \mathbb{R}^k$ be the interaction potential of the fields, then the Lagrangian density is

$$L(x, \phi, d\phi) := \frac{1}{2} g^{\alpha\beta}(x) \frac{\partial \phi^i}{\partial x^\alpha} \frac{\partial \phi_i}{\partial x^\beta} - V(\phi(x)).$$

Here $\phi_i = \phi^i$ and we assume that we sum over all repeated indices. Alternatively one could work with the volume form being $dx^1 \wedge \ldots \wedge dx^n$ and the Lagrangian density being $gL$, in order to apply directly the theory constructed in the previous sections. But we shall not choose this approach here and use a variant which makes clear the covariance of the problem.
We restrict to the Weyl theory, i.e., we work on the submanifold $M_{\text{Weyl}}$, as in subsection 2.7. So we introduce the momentum variables $\epsilon$ and $p_i^\alpha$ and we start from the Cartan form
\[
\theta = \epsilon \omega + p_i^\alpha d\phi^i \wedge \omega_\alpha,
\]
where $\omega_\alpha := \partial_\alpha \int \omega$. But here $\omega_\alpha$ is not closed in general (because $g$ is not constant), so
\[
\Omega = d\theta = d\epsilon \wedge \omega + dp_i^\alpha \wedge d\phi^i \wedge \omega_\alpha - p_i^\alpha \frac{1}{g} \frac{\partial g}{\partial x^\alpha} d\phi^i \wedge \omega.
\]
The Legendre transform is given by
\[
p_i^\alpha = \frac{\partial L}{\partial (\partial_\alpha \phi^i)} = g^{\alpha\beta} \frac{\partial \phi^j}{\partial x^\beta} \iff \frac{\partial \phi^j}{\partial x^\alpha} = g_{\alpha\beta} p_i^\beta,
\]
and the Hamiltonian is
\[
\mathcal{H}(x,\phi, p) = \epsilon + \frac{1}{2} g_{\alpha\beta} p_i^\alpha p_i^\beta + V(\phi).
\]
We use as conjugate variables the 0-forms $\phi^i$ and the $(n-1)$-forms
\[
P_{i,f} := f(x) p_i^\alpha \omega_\alpha = f(x) \frac{\partial}{\partial \phi^i} \int \theta \in \Psi^n M_{\text{Weyl}}.
\]
Taking account of the fact that $\omega_\alpha$ is not closed, one find
\[
\Xi(P_{i,f}) = f \frac{\partial}{\partial \phi^i} - \frac{\partial f}{\partial x^\alpha} p_i^\alpha \frac{\partial}{\partial \epsilon}
\]
and
\[
\{ P_{i,f}, \phi^j \} = \Xi(P_{i,f}) \int d\phi^j = f \delta_i^j.
\]
Also the observables $\phi^i$ are admissible: $\Xi(\tau^i \phi^i) = \sum \frac{\tau^i}{g} \tau_1 \cdots \tau_{\alpha-1} \tau_{\alpha+1} \cdots \tau_n \frac{\partial}{\partial p_j}$, and according to Definition 5,
\[
\{ \mathcal{H}, \phi^j \} = \frac{\partial \mathcal{H}}{\partial P_j^\alpha} dx^\alpha = g_{\alpha\beta} p_j^\beta dx^\alpha.
\]
Moreover
\[
\{ \mathcal{H}, P_{i,f} \} = -\Xi(P_{i,f}) \int d(\mathcal{H}) = \left( -f \frac{\partial V}{\partial \phi^i} + \frac{\partial f}{\partial x^\alpha} p_i^\alpha \right) \omega.
\]
The dynamical equations are that along the graph of a solution,
\[
d\phi^i = \{ \mathcal{H}, \phi^i \} = g_{\alpha\beta} p_i^\beta dx^\alpha
\]
\[
d(f p_i^\alpha \omega_\alpha) = \{ \mathcal{H}, P_{i,f} \} = \left( -f \frac{\partial V}{\partial \phi^i} + \frac{\partial f}{\partial x^\alpha} p_i^\alpha \right) \omega.
\]
The second equation gives
\[
\frac{f}{g} \left( \frac{\partial g}{\partial x^\alpha} p_i^\alpha + g \frac{\partial p_i^\alpha}{\partial x^\alpha} + g \frac{\partial V}{\partial \phi^i} \right) = 0,
\]
while the first relation gives $\frac{\partial \phi^i}{\partial x^\alpha} = g_{\alpha \beta} p_i^\beta$. By substitution in (39) we find
\[
1 \frac{\partial}{\partial x^\alpha} \left( g g_{\alpha \beta} \frac{\partial \phi^i}{\partial x^\beta} \right) + \frac{\partial V}{\partial \phi^i} = 0,
\]
the Euler-Lagrange equations of the problem.

### 5.2 The conformal string theory

We consider maps $u$ from a two-dimensional (pseudo-)Riemannian manifold $(\mathcal{X}, g)$ with values in another (pseudo-)Riemannian manifold $(\mathcal{Y}, h)$ of arbitrary dimension. The most general bosonic action for such maps is $L[u] := \int_\mathcal{X} L(x, u, du) \omega$ with $\omega := g(x) dx^1 \wedge dx^2$ and $g(x) := \sqrt{|\det g_{\alpha \beta}(x)|}$ as before, and

\[
L(x, u, du) := \frac{1}{2} \left( h_{ij}(u(x)) g^{\alpha \beta}(x) + b_{ij}(u(x)) \epsilon^{\alpha \beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} \right),
\]

where $b := \sum_{i<j} b_{ij}(y) dy^i \wedge dy^j$ is a given two-form on $\mathcal{Y}$ and $\epsilon^{12} = -\epsilon^{21} = 1$, $\epsilon^{11} = \epsilon^{22} = 0$. Hence

\[
L[u] = \int_\mathcal{X} \frac{1}{2} h_{ij}(u(x)) g^{\alpha \beta}(x) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} \omega + u^* b.
\]

Setting

\[
G^{\alpha \beta}_{ij}(x, y) := h_{ij}(y) g^{\alpha \beta}(x) + b_{ij}(y) \frac{\epsilon^{\alpha \beta}}{g(x)} = G_{ji}'^{\alpha \beta}(x, y),
\]

we see that $L(x, u, du) = \frac{1}{2} G^{\alpha \beta}_{ij}(x, u) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}$ and the Euler-Lagrange equation for this functional is

\[
1 \frac{\partial}{\partial x^\alpha} \left( g G^{\alpha \beta}_{ij}(x, u(x)) \frac{\partial u^i}{\partial x^\alpha} \right) = \frac{\partial G^{\beta \gamma}_{jk}}{\partial y^i} \frac{\partial u^j}{\partial x^\beta} \frac{\partial u^k}{\partial x^\gamma}.
\]

More covariant formulations exists for the case $b = 0$, which correspond to the harmonic map equation or when the metric on $\mathcal{X}$ is Riemannian using conformal coordinates and complex variables (see [28]). The Cartan-Poincaré form on $\mathcal{M}$ is

\[
\theta := \epsilon \omega + \sum_{\alpha, i} p_i^\alpha \omega_i^\alpha + \sum_{i<j} p_{ij} \omega_{12}^{ij},
\]
(where $\omega^i_1 = g \ dy^i \wedge dx^2$, $\omega^i_2 = g \ dx^1 \wedge dy^i$ and $\omega^{ij}_{12} = g \ dy^i \wedge dy^j$). The pataplectic form is

$$\Omega = d\theta = d\epsilon \wedge + \sum_{\alpha, i} dp^\alpha_i \wedge \omega^i_\alpha + \sum_{i < j} dp^\alpha_{ij} \wedge \omega^{ij}_{12} - \sum_{\alpha, i} \frac{p^\alpha_i}{g} \frac{\partial g}{\partial x^\alpha} dy^i \wedge \omega^i_\alpha + \sum_{i < j} \sum_{\alpha} p_{ij} \frac{\partial g}{\partial x^\alpha} dx^\alpha \wedge dy^i \wedge dy^j.$$

The Legendre correspondance is generated by the function

$$W(x, u, v, p) := \epsilon + p^\alpha_i v^i_{\alpha} + p_{ij} v^i_1 v^j_2 - L(x, u, v) = \epsilon + p^\alpha_i v^i_{\alpha} - \frac{1}{2} M^{\alpha\beta}_{ij}(x, y, p)v^i_{\alpha}v^j_{\beta},$$

where we have denoted

$$M^{\alpha\beta}_{ij}(x, y, p) := h_{ij}(y)g^{1\alpha}(x) + \left(\frac{b_{ij}(y)}{g(x)} - p_{ij}\right)\epsilon^{\alpha\beta} = G^{\alpha\beta}_{ij}(x, y) - p_{ij}\epsilon^{\alpha\beta}.$$

This correspondance is given by the relation $\frac{\partial W}{\partial v^i_{\alpha}} = 0$ which gives

$$M^{\alpha\beta}_{ij}(x, y, p)v^j_{\beta} = p^\alpha_i. \quad (41)$$

Thus, given $(x, y, p)$, finding $(x, y, v, w)$ such that $(x, u, v, w) \leftrightarrow (x, y, p)$ amounts to solving first the linear system $[\mathbf{II}]$ for $v$ and then $w$ is just $W(x, y, v, p)$. This system has a solution in general in the open subset $O$ of $\mathcal{M}$ on which the matrix

$$M = \begin{pmatrix}
    h_{ij}(y)g^{11}(x) & h_{ij}(y)g^{12}(x) + \frac{b_{ij}(y)}{g(x)} - p_{ij} \\
    h_{ij}(y)g^{21}(x) - \frac{b_{ij}(y)}{g(x)} + p_{ij} & h_{ij}(y)g^{22}(x)
\end{pmatrix}$$

is invertible. We remark that $O$ contains actually the submanifold $\mathcal{R} := \{(x, y, p) \in \mathcal{M}/g(x)p_{ij} = b_{ij}(y)\}$, so that the Legendre correspondance induces a diffeomorphism between $TY \otimes T^*\mathcal{X}$ and $\mathcal{R}$.

We shall need to define on $O$ the inverse of $M$, i.e. $K^{ij}_{\alpha\beta}(x, y, p)$ such that

$$K^{ij}_{\alpha\beta}(x, y, p)M^{\beta\gamma}_{jk}(x, y, p) = \delta^i_k \delta^j_\gamma. \quad (42)$$

Now we can express the solution of $[\mathbf{II}]$ by

$$v^i_{\alpha} = K^{ij}_{\alpha\beta}(x, y, p)p^\beta_j \quad (43)$$

and the Hamiltonian function is

$$\mathcal{H}(x, y, p) := \epsilon + \frac{1}{2} K^{ij}_{\alpha\beta}(x, y, p)p^\alpha_i p^\beta_j.$$
We use as conjugate variables the position functions $y^i$ and the momentum 1-forms

$$P_i := \frac{\partial}{\partial y^i} \theta = p_i^\alpha \omega_\alpha + g \ p_{ij} dy^j.$$ 

The Poisson brackets are obtained as follows. First concerning $\{H\omega, y^i\}$ we compute $\Xi(y^i) = \frac{1}{g} \left( \tau_1 \frac{\partial}{\partial p_i^1} - \tau_2 \frac{\partial}{\partial p_i^2} \right)$. Since $dx^\alpha(\Xi(y^i)) = 0, \forall \alpha$,

$$\{H\omega, y^i\} = -\frac{1}{g} \left( \frac{\partial}{\partial p_i^1} \wedge \frac{\partial}{\partial x^1} - \frac{\partial}{\partial p_i^2} \wedge \frac{\partial}{\partial x^2} \right) \wedge dH \wedge \omega = \frac{\partial H}{\partial p_i^2} dx^\alpha = K_{ij} p_j^\beta dx^\alpha.$$ 

Next we compute $dP_i$:

$$dP_i = dp_i^\alpha \wedge \omega_\alpha + g \ dp_{ij} \wedge dy^j + p_i^\alpha \frac{\partial g}{\partial x^\alpha} \omega + p_{ij} \frac{\partial g}{\partial x^\alpha} dx^\alpha \wedge dy^j = -\frac{\partial}{\partial y^i} \Omega.$$ 

Hence

$$\Xi(P_i) = \frac{\partial}{\partial y^i}.$$ 

Because of $dx^\alpha(\Xi(P_i)) = 0, \forall \alpha$ and of Proposition 2 we deduce that

$$\{P_i, y^j\} = \Xi(P_i) \wedge dy^j = \delta_i^j$$

$$\{H\omega, P_i\} = -\Xi(P_i) \wedge d(H\omega) = -\frac{\partial K_{ij}^k}{\partial y^i} p_j^\beta p_k^\omega.$$ 

Notice that, because of (12),

$$\frac{\partial K_{ij}^k}{\partial y^i} = -K_{ij}^l \frac{\partial M_{lm}^\gamma}{\partial y^i} K_{\gamma k}^{\delta \beta},$$

and thus

$$\{H\omega, P_i\} = \frac{\partial M_{im}^\delta}{\partial y^i} K_{ij}^l K_{\delta \beta}^{\gamma \alpha} p_j^\beta p_k^\omega.$$ 

The equations of motion are

$$dy^i = \{H\omega, y^i\} = K_{ij}^i p_j^\beta dx^\alpha$$

$$dP_i = \{H\omega, P_i\} = \frac{\partial M_{lm}^\delta}{\partial y^i} K_{ij}^l K_{\delta \beta}^{\gamma \alpha} p_j^\beta p_k^\omega,$$

(44)
along the graph $\Gamma$ of any solution of the Hamilton equations. From the first equation we deduce that
\[
\frac{\partial y^i}{\partial x^\alpha} = K^i_{\alpha\beta} p^\beta_j \iff p^\alpha_i = M^\alpha_{\beta j} \frac{\partial y^j}{\partial x^\beta}.
\] (45)

Now using (45) we see that along $\Gamma$,
\[
P^i|_\Gamma = (p^\alpha_i \omega_\alpha + g p^\alpha_i \partial y^j) |_{\Gamma} = \left( M^\alpha_{\beta j} \frac{\partial y^j}{\partial x^\beta} + p^\alpha_i \epsilon^\alpha_\beta \frac{\partial y^j}{\partial x^\beta} \right) \omega_\alpha,
\]
and so the left hand side of the second equation of (44) is
\[
dP^i|_\Gamma = \frac{1}{g} \frac{\partial}{\partial x^\alpha} \left[ g G^\alpha_{\beta j} \frac{\partial y^j}{\partial x^\beta} \right] \omega.
\]

And still using (45) the right hand side of the second equation of (44) along $\Gamma$ is
\[
\frac{\partial M^\alpha_{\beta j}}{\partial y^j} \frac{\partial y^j}{\partial x^\alpha} \frac{\partial y^k}{\partial x^\beta} \omega.
\]

Hence we recover the Euler-Lagrange equation (40).

5.3 The electromagnetic field

Here the field is a 1-form $A = A_\alpha dx^\alpha$ defined on the (pseudo-)Riemannian manifold $X$ (which can also be thought as a connection 1-form on a $U(1)$-bundle). Its differential $dA := F$ is the electromagnetic field. We still denote $g_{\alpha\beta}$ the metric on $X$ and $\omega = gdx^1 \wedge \ldots \wedge dx^n$ the volume form. We are given some vector field $\vec{j} = j^\alpha \partial_\alpha$ on $X$ (the electric current field) and we define the Lagrangian density by
\[
L(x, A, dA) := -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - j^\alpha A_\alpha,
\]
where $F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha$ and $F^{\alpha\beta} := g^{\alpha\gamma} g^{\beta\delta} F_{\gamma\delta}$.

The Euler-Lagrange equation could be written
\[
\frac{1}{g} \frac{\partial}{\partial x^\alpha} \left( g F^{\alpha\beta} \right) \frac{\partial y^\beta}{\partial x^\gamma} = j^\gamma \omega_\beta,
\] (46)
or using the notations
\[
*F := \sum_{\alpha<\beta} F^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} \omega \quad \text{and} \quad j := j^\alpha \omega_\alpha.
\]
We remark that in order to have solutions it is necessary to suppose that \(dj = 0\), which is the electric charge conservation law\(^{11}\).

In our framework the fact that the field is a 1-form means that we replace the configuration space \(X \times Y\) by \(T^*X\). Thus the pataplectic manifold is \(\mathcal{M} := \Lambda^n T^*(T^*X)\). We shall here restrict to the “Weyl” submanifold of \(\mathcal{M}\) (see Subsection 2.7) which is described by

\[
\mathcal{M}_{\text{Weyl}} := \{(x, A, p) / x \in X, A \in T^*_x X, p \in \Lambda^n T^*_{(x, A)} (T^*X), \partial A_\alpha \wedge \partial A_\beta \downarrow p = 0, \forall \alpha, \beta\}.
\]

The latter condition on \(p\) means that in local coordinates, \(p = \epsilon \omega + \sum_{\alpha, \beta} p^{A_\alpha A_\beta} dA_\alpha \wedge \omega_\beta\). The Cartan form is

\[
\theta := \epsilon \omega + \sum_{\alpha, \beta} p^{A_\alpha A_\beta} dA_\alpha \wedge \omega_\beta,
\]

with still \(\omega_\beta := \partial_\beta \downarrow \omega\) and the pataplectic form is

\[
\Omega := d\theta = d\epsilon \wedge \omega + \sum_{\alpha, \beta} dp^{A_\alpha A_\beta} \wedge dA_\alpha \wedge \omega_\beta - \sum_{\alpha, \beta} p^{A_\alpha A_\beta} \frac{1}{g} \frac{\partial g}{\partial x^\beta} dA_\alpha \wedge \omega.
\]

Computing the Legendre transform in \(\mathcal{M}_{\text{Weyl}}\), using \(W(x, A, dA, p) = \epsilon + \sum_{\alpha, \beta} p^{A_\alpha A_\beta} \partial_\beta A_\alpha - L(x, A, dA)\) gives the momenta

\[
p^{A_\alpha A_\beta} := \frac{\partial L}{\partial (\partial_\beta A_\alpha)} = F^{\alpha \beta}.
\]

We see that the Legendre transform works only provided the compatibility condition

\[
p^{A_\alpha A_\beta} + p^{A_\beta A_\alpha} = 0 \quad (48)
\]

is satisfied. It is an example of a Dirac primary constraint. Henceforth we shall be restricted to the submanifold

\[
\mathcal{M}_{\text{Maxwell}} := \{(x, A, p) \in \mathcal{M}_{\text{Weyl}} / p^{A_\alpha A_\beta} + p^{A_\beta A_\alpha} = 0\},
\]

along which we are able to obtain an expression for the Hamiltonian

\[
\mathcal{H}(x, A, p) = \epsilon - \frac{1}{4} g_{\alpha \gamma} g_{\beta \delta} p^{A_\alpha A_\beta} p^{A_\gamma A_\delta} + j^\alpha A_\alpha.
\]

\(^{11}\)note also that by writing the electromagnetic functional \(\int \frac{1}{2} F \wedge \star F - A \wedge j\), one sees immediately that the condition \(dj = 0\) ensures that this functional is invariant by gauge transformations \(A \mapsto A + df\) up to the addition of \(\int df\).
A naive use of this Hamiltonian function leads to incorrect dynamical equations. Another possibility, which was already proposed in [12] is to use as dynamical variable the 1-form on $\mathcal{M}_{\text{Maxwell}}$

$$A := A_\alpha dx^\alpha.$$  

Note that here $A_\alpha$ is not a local function of $x$ but an independant variable. Then, as it will be proved below, the momentum variable canonically conjugate to $A$ may be chosen to be the $(n-2)$-form

$$\pi := \frac{1}{2} \sum_{\alpha, \beta} p^{A_\alpha \beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} \omega.$$  

Its associated superform is

$$s_\pi = \frac{1}{2} \sum_{\alpha, \beta} p^{A_\alpha \beta} \left( \tau_\alpha \frac{\partial}{\partial x^\beta} - \tau_\beta \frac{\partial}{\partial x^\alpha} \right) \omega$$

$$= \sum_{\alpha, \beta} p^{A_\alpha \beta} \tau_\alpha \frac{\partial}{\partial x^\beta} \omega.$$  

Hence using (48)

$$d^{s_\pi} = \sum_{\alpha, \beta} \tau_\alpha dp^{A_\alpha \beta} \wedge \left( \frac{\partial}{\partial x^\beta} \omega \right) + \sum_{\alpha, \beta} \frac{p^{A_\alpha \beta}}{g} \tau_\alpha \frac{\partial g}{\partial x^\beta} \omega$$

$$= -\sum_{\alpha} \tau_\alpha \frac{\partial}{\partial A_\alpha} \omega$$

and

$$\Xi(s_\pi) = \sum_{\alpha} \tau_\alpha \frac{\partial}{\partial A_\alpha}.$$  

We also have (denoting $\omega_\beta := \frac{\partial}{\partial x^\beta} \omega$ and $\tau_1^{\alpha_1} \ldots^{\alpha_n} := \tau_1 \ldots \tau_{\alpha_1-1} \tau_{\alpha_1+1} \ldots \tau_{\beta_1-1} \tau_{\beta_1+1} \ldots \tau_n$)

$$s_A = \sum_{\alpha < \beta} (-1)^{n+\alpha+\beta} \frac{A_\alpha}{g} \tau_1^{\alpha} \ldots^{\alpha} \omega_\beta - \sum_{\beta < \alpha} (-1)^{n+\alpha+\beta} \frac{A_\alpha}{g} \tau_1^{\alpha} \ldots^{\alpha} \omega_\beta$$

and

$$\Xi(s_A) = \sum_{\alpha < \beta} (-1)^{n+\alpha+\beta+1} \frac{A_\alpha}{g} \tau_1^{\alpha} \ldots^{\alpha} \frac{\partial}{\partial p^{A_\alpha \beta}} - \sum_{\beta < \alpha} (-1)^{n+\alpha+\beta+1} \frac{A_\alpha}{g} \tau_1^{\beta} \ldots^{\beta} \frac{\partial}{\partial p^{A_\alpha \beta}}.$$  

A computation using (48) gives
\[
\{ s\pi, sA \}_s = \sum_{\alpha, \beta, \gamma} (-1)^{n+\alpha+\beta+1} \tau_\gamma \left( \sum_{\alpha<\beta} \frac{A_\alpha}{g_{\tau_1, \ldots, \tau_\alpha}} - \sum_{\beta<\alpha} \frac{A_\beta}{g_{\tau_1, \ldots, \tau_\beta}} \right) \frac{\partial}{\partial p^{A_\alpha}} \Omega
\]

Thus
\[
\{ \pi, A \} = 2(-1)^n(n-1).
\]

As it may be anticipated by Corollary 1, the dynamical equations are described by the following identities to be true along the graph of a solution of the Hamilton equations

\[
dA = \{ H\omega, A \}
= (-1)^{n+\alpha+\beta} \left[ \sum_{\alpha<\beta} \frac{1}{g} \frac{\partial}{\partial x^\alpha} \wedge \ldots \wedge \frac{\partial}{\partial x^{\alpha+\beta}} \wedge \ldots \wedge \frac{\partial}{\partial x^{n-1}} \wedge \ldots \wedge \frac{\partial}{\partial x^n} \wedge \frac{\partial}{\partial p^{A_\beta}} \right] \wedge \Omega
\]

\[
d\pi = \{ H\omega, \pi \} = -\sum_{\alpha} \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial A_\alpha} \Omega
= \frac{\partial H}{\partial x^\alpha} \omega_\alpha = j^\alpha \omega_\alpha = j.
\]

The first equation leads to \( \frac{\partial A_\alpha}{\partial x^\alpha} - \frac{\partial A_\beta}{\partial x^\beta} = g_{\alpha\gamma} g_{\beta\delta} p^{A_\gamma A_\delta} \) which implies \( F_{\alpha\beta} = g_{\alpha\gamma} g_{\beta\delta} p^{A_\gamma A_\delta} \) or equivalently \( p^{A_\alpha A_\beta} = F_{\alpha\beta} \). This can be translated into the relation

\[
\pi = *F, \quad \text{along } \Gamma.
\]

By substitution in the second equation, \( d\pi = j \), it gives immediately (47).

A last observation is that infinitesimal gauge transformations \( \delta A = df \) (for \( f \in C^\infty(\mathcal{X}) \)) are generated by the Poisson bracket with \( df \wedge \pi \). We have indeed

\[
d(df \wedge \pi) = -df \wedge d\pi = -\sum_{\alpha} \frac{\partial f}{\partial x^\alpha} \frac{\partial}{\partial A_\alpha} \Omega,
\]

so that \( df \wedge \pi \in \mathfrak{P}^{n-1}\mathcal{M} \) and \( \Xi(df \wedge \pi) = \sum_{\alpha} \frac{\partial f}{\partial x^\alpha} \frac{\partial}{\partial A_\alpha} \). We deduce that
\[ \{ df \wedge \pi, \pi \} = \Xi (df \wedge \pi) \int d\pi = 0 \]
\[ \{ df \wedge \pi, A \} = \Xi (df \wedge \pi) \int dA = df. \]

Notice that we could replace \( df \wedge \pi \) by \( -f d\pi \) or \( f (x - d\pi) \) without changing the brackets with \( \pi \) and \( A \).

6 Conclusion

We obtained an Hamiltonian formulation for variational problems with an arbitrary number of variables. This could be the starting point for building a fully relativistic quantum field theory without requiring the space-time to be Minkowskian. This will be the subject of a forthcoming paper. Notice also that we may enlarge the concept of pataplectic manifolds as manifolds equipped with a closed \( n + 1 \)-form and extend to this context notions like the \( p \)-bracket.

References

[1] Th. De Donder, Théorie Invariante du Calcul des Variations, Nuov. éd. (Gauthier-Villars, Paris, 1935)

[2] H. Weyl, Geodesic fields in the calculus of variations, Ann. Math. (2) 36 (1935) 607-629

[3] C. Carathéodory, Über die Extremalen und geodätischen Felder in der Variationsrechnung der mehrfachen Integrale, Acta Sci. Math. (Szeged) 4 (1929) 193-216

[4] H. Rund, The Hamilton-Jacobi Theory in the Calculus of Variations, (D. van Nostrand Co. Ltd., Toronto, etc. 1966) (Revised and augmented reprint, Krieger Publ., New York, 1973)

[5] H. Kastrup, Canonical theories of Lagrangian dynamical systems in physics, Phys. Rep. 101 (1983) 1-167

[6] E. Binz, J. Šniatycki and H. Fisher, Geometry of Classical Fields, (North-Holland, Amsterdam, 1989)

[7] M.J. Gotay, An exterior differential systems approach to the Cartan form, in Symplectic Geometry and Mathematical Physics, eds. P. Donato, C. Duval, e.a. (Birkhäuser, Boston, 1991) p. 160-188

[8] M.J. Gotay, A multisymplectic framework for classical field theory and the calculus of variations I. Covariant Hamiltonian formalism, in Mechanics.
Analysis and Geometry: 200 Years after Lagrange, ed. M. Francaviglia (North Holland, Amsterdam, 1991) p. 203-235

[9] M.J. Gotay, A multisymplectic framework for classical field theory and the calculus of variations II. Space + time decomposition, Diff. Geom. and its Appl. 1 (1991) 375-390

[10] S. P. Hrabak On a multisymplectic formulation of the classical BRST symmetry for first order field theories PartII: Geometric Structure, arXiv: math-ph/9901013

[11] S. P. Hrabak On a multisymplectic formulation of the classical BRST symmetry for first order field theories PartII: Geometric Structure, arXiv: math-ph/9901012

[12] I. V. Kanatchikov Canonical structure of classical field theory in the polymomentum phase space, arXiv:hep-th/9709229

[13] I. V. Kanatchikov, On the canonical structure of the De Donder-Weyl covariant Hamiltonian formulation of field theory I. Graded Poisson brackets and the equation of motion, hep-th/9312162

[14] M.J. Gotay, J. Isenberg, J. Marsden and R. Montgomery, Momentum Maps and Classical Relativistic Fields, arXiv:physics/9801019

[15] A.Echeverria-Enriquez, M.Munoz-Lecanda, N.Roman-Roy, Geometry of multisymplectic Hamiltonian first-order field theories, math-ph/0004007

[16] C.Pauller, A vertical exterior derivative in multisymplectic geometry and a graded Poisson bracket for nontrivial geometries, math-ph/0002032

[17] A.Echeverria-Enriquez, M.Munoz-Lecanda, N.Roman-Roy, On the multimomentum bundles and the Legendre maps in field theories, Rep. Math. Phys., v. 45 (2000) 85, math-ph/9904007

[18] A.Echeverria-Enriquez, M.Munoz-Lecanda, N.Roman-Roy, Multivector field formulation of Hamiltonian field theories, J.Phys.A, v.32 (1999) 8461, math-ph/9907007

[19] G.Sardanashvily, SUSY-extended field theory, hep-th/9911108 (appear in Int.J.Mod.Phys. A (2000))

[20] G.Giachetta, L.Mangiarotti, G.Sardanashvily, Covariant Hamilton equations for field theory, J.Phys.A, v.32 (1999) 6629, hep-th/9904062

[21] G.Sardanashvily, Generalized Hamiltonian Formalism for Field Theory, ed. World Scientific, Singapore, 1995

[22] L.Mangiarotti and G.Sardanashvily, Gauge Mechanics, ed. World Scientific, Singapore, 1998

[23] G.Giachetta, L.Mangiarotti and G.Sardanashvily, New Lagrangian and Hamiltonian Methods in Field Theory, ed. World Scientific, Singapore, 1997
[24] H. Rund, *A Cartan form for the field theory of Carathéodory in the calculus of variations of multiple integrals*, in Differential Geometry, Calculus of Variations and Their Applications, Lect. Notes Pure and Appl. Math. vol. 100, ed. G.M. Rassias and T.M. Rassias, (Marcel Dekker etc., 1985) p. 455-469

[25] D. H. Martin, *Canonical variables and geodesic fields for the calculus of variations of multiple integrals in parametric form*, Math. Z. 104 (1968), 16-27.

[26] H. Goldschmidt and S. Sternberg, *The Hamilton-Cartan formalism in the calculus of variations*, Ann. Inst. Fourier (Grenoble)23, fasc. 1 (1973) 203-267.

[27] J. Kijowski, *A finite dimensional canonical formalism in the classical field theory*, Comm. Math. Phys. 30 (1973) 99-128; J. Kijowski and W. Szczyrba, *A Canonical Structure for Classical Field Theories*, Comm. Math. Phys. 46 (1976) 183-206

[28] F. Hélein, *Problèmes variationnels invariants par transformation conforme en dimension 2*, to appear in *Partial differential equations and variational calculus in Physics*, ed. Joseph Kouneiher.

[29] K. Gawędzki, *On the generalization of the canonical formalism in the classical field theory*, Rep. Math. Phys. 3 (1972) 307-326;

[30] P. Dedecker, *Calcul des variations, formes différentielles et champs géodésiques*, in *Géometrie Differentielle*, Colloq. Intern. du CNRS LII, Strasbourg 1953, (Publ. du CNRS, Paris, 1953) p. 17-34

[31] P. Dedecker, *On the generalization of symplectic geometry to multiple integrals in the calculus of variations*, in *Differential Geometrical Methods in Mathematical Physics*, eds. K. Bleuler and A. Reetz, Lect. Notes Math. vol. 570 (Springer-Verlag, Berlin etc., 1977) p. 395-456

[32] P. Deligne, D. Freed, *Classical field theory*, in *Quantum fields and strings: a course for mathematicians, Volume 1*, P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, D.R. Morrison and E. Witten, editors, American Mathematical Society, 1999.