Abstract

The Halpern–Shoham logic is a modal logic of time intervals. Some effort has been put in last ten years to classify fragments of this beautiful logic with respect to decidability of its satisfiability problem. We contribute to this effort by showing – what we believe is quite an unexpected result – that the logic of subintervals, the fragment of the Halpern–Shoham where only the operator “during”, or $D$, is allowed, is undecidable over discrete structures. This is surprising as this logic is decidable over dense orders [14] and its reflexive variant is known to be decidable over discrete structures [13]. Our result subsumes a lot of previous results for the discrete case, like the undecidability for $ABE$ [10], $BE$ [11], $BD$ [12], $ADB$, $A\bar{D}$, and so on [2, 6].

1 Introduction

In classical temporal logic structures are defined by assigning properties (propositional variables) to points of time (which is an ordering, discrete or dense). However, not all phenomena can be well described by such logics. Sometimes we need to talk about actions (processes) that take some time and we would like to be able to say that one such action takes place, for example, during or after another.

The Halpern–Shoham logic [10], which is the subject of this paper, is one of the modal logics of time intervals. Judging by the number of papers published, and by the amount of work devoted to the research on it, this logic is probably the most influential of time interval logics. But historically it was not the first one. Actually, the earliest papers about intervals in context of modal logic were written by philosophers, e.g., [9]. In computer science, the earliest attempts to formalize time intervals were process logic [17, 19] and interval temporal logic [15]. Relations between intervals in linear orders from an algebraic point of view were first studied systematically by Allen [1].

Halpern and Shoham introduce six modal operators, acting on intervals. Their operators are “begins” $B$, “during” $D$, “ends” $E$, “meets” $A$, “later” $L$, “overlaps” $O$ and the six inverses of those operators: $\bar{B}$, $\bar{D}$, $\bar{E}$, $\bar{A}$, $\bar{L}$, $\bar{O}$. It is easy to see that the set of operators is redundant. The ,,more expressive” of them, which are $A$, $B$ and $E$ can define $D$ ($B$ and $E$ suffice for that – a prefix of my suffix is my infix) and $L$ (here $A$ is enough – “later” means “meets an interval that meets”). The operator $O$ can be expressed using $E$ and $B$.

In their paper, Halpern and Shoham show that (satisfiability of formulae of) their logic is undecidable. Their proof requires logic with five operators ($B$, $E$ and $A$ are explicitly used in the formulae and, as we mentioned above, once $B$, $E$ and $A$ are allowed, $D$ and $L$ come for free) so they state a question about decidable fragments of their logic.

Considerable effort has been put since this time to settle this question. First, it was shown [11] that the $BE$ fragment is undecidable. Recently, negative results were also given for the classes $BE$, $\bar{B}E$, $\bar{E}B$, $A\bar{D}B$, $\bar{A}\bar{D}B$ [2, 6], and $BD$ [12]. Another elegant negative result was that $OO$ is undecidable over discrete orders [3, 4].

On the positive side, it was shown that some small fragments, like $BB$ or $EE$, are decidable and easy to translate...
into standard, point-based modal logic \[8\]. The fragment using only \(A\) and \(A\) is a bit harder and its decidability was only recently shown \[6, 7\]. Obviously, this last result implies decidability of \(LL\) as \(L\) is expressible by \(A\). Another fragment known to be decidable is \(ABB\) \[16\].

The last interesting fragment of the Halpern and Shoham logic of unknown status was the, apparently very simple, fragment with the single operator \(D\) (,,during''), which we call here the logic of sub-intervals. Since \(D\) does not seem to have much expressive power (an example of a formula would be ,,each morning I spend a while thinking of you'' or ,,each nice period of my life contains an unpleasant fragment'') logic of sub-intervals was widely believed to be decidable. A number of decidability results concerning variants of this logic has been published. For example, it was shown in \[1, 4, 13\] than satisfiability of formulae of logic of subintervals is decidable over dense structures. In \[1, 13\], decidability is proved for (slightly less expressive) ,,reflexive \(D'\). The results in \[21\] imply that \(D\) (as well as some richer fragments of the HS logic) is decidable if we allow models, in which not all the intervals defined by the ordering are elements of the Kripke structure.

In this paper we show that satisfiability of formulae from the \(D\) fragment is undecidable over the class of finite orderings as well as over the class of all discrete orderings. Our result subsumes the negative results for the discrete case for \(ABE\) \[10\], \(BE\) \[11\], \(BD\) \[12\] and \(ADB, AAD\) \[2, 6\].

1.1 Main theorems

Our contribution consists of the proofs of the following two theorems:

Theorem 1. The satisfiability problem for the formulae of the logic of subintervals, over models which are suborders of the order \((\mathbb{Z}, \leq)\), is undecidable.

Since truth value of a formula is defined with respect to a model and an initial interval in this model (see Preliminaries), and since the only allowed operator is \(D\), which means that the truth value of a formula in a given interval depends only on the labeling of this interval and its subintervals Theorem 1 can be restated as: The satisfiability problem for the formulae of the logic of subintervals, over finite models is undecidable, and it is this version that will be proved in Section 3.

Theorem 2. The satisfiability problem for the formulae of the logic of sub-intervals, over all discrete models, is undecidable.

2 Preliminaries

Orderings. As in \[10\], we say that a total order \(\langle \mathbb{D}, \leq \rangle\) is discrete if each element is either minimal (maximal) or has a unique predecessor (successor); in other words for all \(a, b \in \mathbb{D}\) if \(a < b\), then there exist points \(a', b'\) such that \(a < a', b' < b\) and there exists no \(c\) with \(a < c < a'\) or \(b' < c < b\).

Semantic of the \(D\) fragment of logic HS (logic of subintervals). Let \(\langle \mathbb{D}, \leq \rangle\) be a discrete ordered set\[1\].

An interval over \(\mathbb{D}\) is a pair \([a, b]\) with \(a, b \in \mathbb{D}\) and \(a \leq b\). A labeling is a function \(\gamma : I(\mathbb{D}) \to \mathcal{P}(\text{Var})\), where \(I(\mathbb{D})\) is a set of all intervals over \(\mathbb{D}\) and \(\text{Var}\) is a finite set of variables. A structure of the form \(M = \langle I(\mathbb{D}), \gamma \rangle\) is called a model.

We say that an interval \([a, b]\) is a leaf iff it has no subintervals (i.e. \(a = b\)).

The truth values of formulae are determined by the following (natural) semantic rules:

1. For all \(v \in \text{Var}\) we have \(M, [a, b] \models v\) iff \(v \in \gamma([a, b])\).
2. \(M, [a, b] \models \neg \varphi\) iff \(M, [a, b] \not\models \varphi\).
3. \(M, [a, b] \models \varphi_1 \land \varphi_2\) iff \(M, [a, b] \models \varphi_1\) and \(M, [a, b] \models \varphi_2\).
4. \(M, [a, b] \models \langle D \rangle \varphi\) iff there exists an interval \([a', b']\) such that \(M, [a', b'] \models \varphi, a \leq a', b' \leq b\), and \([a, b] \neq [a', b']\). In that case we say that \([a, b]\) sees \([a', b']\).

Boolean connectives \(\lor, \Rightarrow, \Leftrightarrow\) are introduced in the standard way. We abbreviate \(\neg \langle D \rangle \varphi\) by \([D] \varphi\) and \(\varphi \land [D] \varphi\) by \([G] \varphi\).

A formula \(\varphi\) is said to be satisfiable in a class of orderings \(\mathcal{D}\) if there exist a structure \(\mathbb{D} \in \mathcal{D}\), a labeling \(\gamma\), and an interval \([a, b]\), called the initial interval, such that \(\langle I(\mathbb{D}), \gamma \rangle, [a, b] \models \varphi\). A formula is satisfiable in a given ordering \(\mathbb{D}\) if it is satisfiable in \(\{\mathbb{D}\}\).

3. Proof of Theorem 1

In Section 3 only consider finite orderings.

Our representation. We imagine the Kripke structure of intervals of a finite ordering as a directed acyclic graph, where intervals are vertices and each interval \([a, b]\) with the length greater that 0 has two successors: \([a + 1, b]\) and \([a, b - 1]\). Each level of this representation contains intervals of the same length (see Fig. 1).

3.1 The Regular Language \(L_A\)

In this section we will, for a given two-counter finite automaton (Minsky machine) \(A\), define a regular language \(L_A\)

\[1\] To keep the notation light, we will identify the order \(\langle \mathbb{D}, \leq \rangle\) with its set \(\mathbb{D}\).
Figure 1. Our representation of order \( \langle \{ a, a + 1, \ldots, b \}, \leq \rangle \).

whose words will almost encode the computation of \( A \) (beginning from the empty counters).

Let \( Q \) be the set of states of \( A \), and let \( Q' = \{ q' : q \in Q \} \). Define \( B = \{ f, f_1, f_r, f'_1, f'_r, s, s_1, s_r, s'_1, s'_r \} \).

The alphabet \( \Sigma \) of \( L_A \) will consist of all the elements of \( Q \cup Q' \) (jointly called states) and of all the subsets (possibly empty) of \( B \) which consist of at most 4 elements: at most one of them from \( \{ f, f_1, f_r \} \), at most one from \( \{ f'_1, f'_r \} \), at most one from \( \{ s, s_1, s_r \} \), and at most one from \( \{ s'_1, s'_r \} \).

Symbols of \( \Sigma \) containing \( f_1 \) or \( f'_1 \) (or \( s_1 \) or \( s'_1 \)) will be called first (resp. second) counters. Symbols of \( \Sigma \) containing \( f_r \) or \( f'_r \) (or \( s_r \) or \( s'_r \)) will be called first (resp. second) shadows (or shadows of the first/second counter).

The language \( L_A \) will consist of all the words \( w \) over \( \Sigma \) which satisfy all the following six conditions:

- The first symbol of \( w \) is the beginning state \( q_0 \) of \( A \) and the last symbol of \( w \) is either \( q \) or \( q' \), where \( q \) is one of the final states of \( A \).

By a configuration we will mean a maximal sub-word\(^2\) of \( w \), whose first element is a state (called the state of this configuration) and which contains exactly one state (so that \( w \) is split into disjoint configurations). A configuration will be called even if its state is from \( Q \) and odd if it is from \( Q' \).

- Odd and even configurations alternate in \( w \).
- Each configuration, except of the last one (which only consists of the state) contains exactly one first counter and exactly one second counter. If a configuration is even, then its first counter contains \( f_1 \) and its second counter contains \( s_1 \). If a configurations is odd, then its first counter contains \( f'_1 \) and its second counter contains \( s'_1 \). The first non-state symbol of the first configuration is both a first counter and a second counter.
- The are no shadows in the first and the last configuration. Each configuration, except of the first and the last, contains exactly one first shadow and exactly one second shadow. If a configuration is even, then its first shadow contains \( f'_r \) and its second shadow contains \( s'_r \). If a configurations is odd, then its first shadow contains \( f_r \) and its second shadow contains \( s_r \).

It follows, from the conditions above, that if (in a word from the language \( L_A \)) there is a counter containing \( f_1 \) (resp. \( f'_1 \)) then there is its shadow \( f_1 \) (resp. \( f'_1 \)) in the subsequent configuration. Call a sub-word beginning with first (second) counter and ending with its shadow a first (resp. second) shade. Notice, that the above conditions imply in particular that each state (except of the first one and last one) is in exactly one first shade and in exactly one second shade.

- A non-state symbol of \( w \) contains \( f \) (or \( f' \)) if and only if it is inside some shade beginning with \( f_1 \) (resp. \( f'_1 \)).

The last condition defining \( L_A \) will depend on the instructions of the automaton \( A \). We say that a configuration has first (second) counter equal zero if the first non-state symbol of this configuration contains \( f_1 \) (resp. \( f'_1 \)). It is good to think, that the number of symbols before the first/second counter is the value of this counter in the given configuration. Notice that the first configuration of a \( w \in L_A \) is indeed the initial configuration of \( A \) – its state is \( q_0 \) and both its counters equal 0.

Since the format of an instruction of \( A \) is:

\[
\begin{align*}
\text{If in state } q & \\
\text{the first counter} & \text{equals/does not equal 0 and} \\
\text{the second counter} & \text{equals/does not equal 0} \\
\text{then change the state to } q_1 & \text{and} \\
\text{decrease/increase/keep unchanged} & \text{the first counter} \\
\text{decrease/increase/keep unchanged} & \text{the second counter}.
\end{align*}
\]

it is clear what we mean, saying that configuration \( C \) matches the assumption of the instruction \( I \).

- If \( C \) and \( C_1 \) are subsequent configurations in \( w \), and \( C \) matches the assumption of an instruction \( I \), then:

\[^{2}\text{See Lemma 2 for an explanation what we mean by "almost".}\]

\[^{3}\text{By a sub-word we mean a sequence of consecutive elements of a word, an infix.}\]
– If \( I \) changes the state into \( q_1 \) then the state of \( C_1 \) is \( q_1 \).
– If \( I \) orders the first (second) counter to remain unchanged, then the first (resp. second) counter in \( C_1 \) coincides with the first (resp. second) shadow in \( C_1 \).
– If \( I \) orders the first (second) counter to be decreased, then the first (resp. second) counter in \( C_1 \) is the direct predecessor of the first (resp. second) shadow in \( C_1 \).
– If \( I \) orders the first (second) counter to be increased, then the first (resp. second) counter in \( C_1 \) is the direct successor of the first (resp. second) shadow in \( C_1 \).

This completes the definition of the language \( L_A \). It is clear, that it is regular. Our main tool will be the following:

**Lemma 1.** The following two conditions are equivalent:

(i) Automaton \( A \), started from the initial state \( q_0 \) and empty counters, accepts.

(ii) There exists a word \( w \in L_A \) and a natural number \( n \) such that:

- each configuration in \( w \) (except of the last one, consisting of a single symbol) has length \( n - 1 \)
- each shade in \( w \) has length \( n \) (this includes the two symbols in the two ends of a shade).

**Proof.** For the \( \Rightarrow \) direction consider an accepting computation of \( A \) and take \( n \) as any number greater than all the numbers that appear on the two counters of \( A \) during this computation. For the \( \Leftarrow \) direction notice that the distance constraint from (ii) imply, that the distance between a state and the subsequent first (second) shadow equals the value of the first (resp. second) counter in the previous configuration. Together with the last of the six conditions defining \( L_A \) this implies that the subsequent configurations in \( w \in L_A \) can indeed be seen as subsequent configurations in the valid computation of \( A \).

\[ \square \]

Since the halting problem for two-counter automata is undecidable, the proof of Theorem 1 will be completed when we write, for a given automaton \( A \), a formula \( \Psi \) of the language of the logic of sub-intervals which is satisfiable (in a finite model) if and only if condition (ii) from Lemma 1 holds. Actually, what the formula \( \Psi \) is going to say is, more or less, that the word written (with the use of the labeling function \( \gamma \)) in the leaves of the model is a word \( w \) as described in Lemma 1(ii).

In the following subsections we are going to write formulae \( \Phi_{\text{orient}}, \Phi_{L_A}, \Phi_{\text{cloud}} \) and \( \Phi_{\text{length}} \), such that \( \Phi_{\text{orient}} \land \Phi_{L_A} \land \Phi_{\text{cloud}} \land \Phi_{\text{length}} \) will be the formula \( \Psi \) we want.

### 3.2 Orientation

As we said, we want to write a formula saying that the word written in the leaves of the model is the \( w \) described in Lemma 1(ii).

The first problem we need to overcome is the symmetry of \( D \) – the operator does not see a difference between past and future, or between left and right, so how can we distinguish between the beginning of \( w \) and its end? We deal with this problem by introducing five variables \( L, R, s_0, s_1, s_2 \) and writing a formula \( \Phi_{\text{orient}} \) which will be satisfied by an interval \( [a, b] \) if \( [a, a] \) is the only interval that satisfies \( L \) and \( [b, b] \) is the only interval that satisfies \( R \), or \( [b, b] \) is the only interval that satisfies \( L \) and \( [a, a] \) is the only interval that satisfies \( R \), and if all the following conditions hold:

- any interval that satisfies \( L \) satisfies also \( s_0 \);
- each leaf is labeled either with \( s_0 \) or with \( s_1 \) or with \( s_2 \);
- each interval labeled with \( s_0 \) or with \( s_1 \) or with \( s_2 \) is a leaf;
- if \( c, d, e \) are three consecutive leaves of \( [a, b] \) and if \( s_i \) holds in \( c \), \( s_j \) holds in \( d \) and \( s_k \) holds in \( e \) then \( \{i, j, k\} = \{0, 1, 2\} \).

If \( [a, b] \models \Phi_{\text{orient}} \) then the leaf of \( [a, b] \) where \( L \) holds (resp. where \( R \) holds) will be called the left (resp. the right) end of \( [a, b] \).

Let exactly one of \( X = \bigvee_{x \in X} (x \land \bigwedge_{x' \in X \setminus \{x\}} \neg x') \) be a formula saying (which is not hard to guess) that exactly one variable from the set \( X \) is true in the current interval. \( \Phi_{\text{orient}} \) is a conjunction of the following formulae.

(i) \( \Lbrack D \rbrack ([\Lbrack D \rbrack \models \text{exactly one of} (\{s_0, s_1, s_2\}) \land (s_0 \lor s_1 \lor s_2 \models \Lbrack D \rbrack)]) \)

(ii) \( \Lbrack (D \land \models \text{exactly one of} (\{s_0, s_1, s_2\})) \land (s_0 \lor s_1 \lor s_2 \models (\Lbrack D \rbrack))) \)

(iii) \( (\Lbrack D \rbrack \models s_0) \)

(iv) \( \Lbrack D \rbrack \land \Lbrack D \rbrack \)

(v) \( \Lbrack D \rbrack (\models \neg R) \)

(vi) \( \Lbrack D \rbrack (\models \neg R) \)

(vii) \( \bigvee_{i \in \{0, 1, 2\}} \Lbrack D \rbrack (\models \neg R) \)

Formulae (ii), (iii), and (vii) express the property defined by the conjunction of the four items above (notice, that \( \Lbrack D \rbrack \) means that the current interval is a leaf).

Formula (vi) says that there exists an interval labeled with \( R \) and an interval labeled with \( L \).
Formula (vi) states that intervals labeled with $L$ are also labeled with $s_0$, and intervals labeled with $R$ are labeled with $s_2$, so they are leaves.

Formula (vii) guarantees that no interval containing exactly 2 leaves, which is a super-interval of an interval labeled with $L$, can contain a sub-interval labeled with $s_2$. It implies that an interval labeled with $L$ can only have one super-interval containing exactly 2 leaves — if there were two, then their common super-interval containing 3 leaves would not have a sub-interval labeled with $s_2$, which would contradict (ii).

Finally, formula (viii), finally, works like (vi) but for $R$. We have to use disjunction in this case since we do not know which $s_i$ is satisfied in the interval labeled with $R$.

In the rest of the paper we restrict our attention to models satisfying formula $\Phi_{\text{orient}}$, and treat the leaf labeled with $L$ as the leftmost element of the model.

### 3.3 Encoding a Finite Automaton

In this section we show how to make sure that consecutive leaves of the model, read from $L$ to $R$, are labeled with variables that represent a word of a given regular language.

**Lemma 2.** Let $\mathcal{A} = \langle \Sigma, Q, q^0, \delta, \gamma \rangle$, where $q^0 \in Q$, $F \subseteq Q$, $\delta \subseteq Q \times \Sigma \times Q$ be a finite-state automaton (deterministic or not, it does not matter).

There exists a formula $\psi_\mathcal{A}$ of the $D$ fragment of Halpern–Shoham logic over alphabet $Q \cup \Sigma$ that is satisfiable (with respect to the valuation of the variables from $Q$) if and only if the word, over the alphabet $\Sigma$ written in the leaves of the model, read from $L$ to $R$, belongs to the language accepted by $\mathcal{A}$.

**Proof.** It is enough to write a conjunction of the following properties.

1. In every leaf, exactly one letter from $\Sigma$ is satisfied (so there is indeed a word written in the leaves).
2. Each leaf is labeled with exactly one variable from $Q$.
3. For each interval with the length 1, if this interval contains an interval labeled with $s_1$, with $a \in \Sigma$ and with $q \in Q$ and another interval labeled with $s_{(i+1) \mod 3}$ and with $q' \in Q$, then $\langle q, a, q' \rangle \in \delta$.
4. Interval labeled with $R$ is labeled with such $q \in Q$ and $a \in \Sigma$ that $\langle q, a, q' \rangle \in \delta$ for some $q' \in F$.
5. Interval labeled with $L$ is labeled with $q^0$.

Clearly, a model satisfies properties (i) if and only if its leaves are labeled with an accepting run of $\mathcal{A}$ on the word over $\Sigma$ written in its leaves. The formulae of the $D$ fragment of Halpern–Shoham logic expressing properties (i) are not hard to write:

1. $[G][([D] \perp \Rightarrow \text{exactly one of } (\Sigma)) \land (\forall \Sigma \Rightarrow [D] \perp)]$
2. $[G][([D] \perp \Rightarrow \text{exactly one of } (Q)) \land (\forall Q \Rightarrow [D] \perp)]$
3. $[G][([D][D] \perp \land (D)s_i \land (D)s_{i+1 \mod 3} \Rightarrow \bigvee_{\langle q, a, q' \rangle \in \delta}(D)(s_i \land q \land a) \land (D)(s_{i+1 \mod 3} \land q'))]$

for each $i \in \{0, 1, 2\}$.

4. $[G][R \Rightarrow \bigvee_{\langle q, a, q' \rangle \in \delta}(q \land a)]$
5. $[G][L \Rightarrow q^0]$

Now, let $\mathcal{A}$ be a finite automaton recognizing language $L_\mathcal{A}$ from Section 3.1 and put $\Phi_{L_\mathcal{A}} = \psi_\mathcal{A}$.

### 3.4 A Cloud – how to build it

We still need to make sure, that there exists $n$ such that each configuration (except of the last one) has length $n - 1$ and that each shade has the length exactly $n$. Let us start with:

**Definition 1.** Let $M = \langle I([D]), \gamma \rangle$ be a model and $p$ a variable. We call $p$ a cloud if there exists $k \in \mathbb{N}$ such that $p \in \gamma([a, b])$ if and only if the length of $[a, b]$ is exactly $k$.

So one can view a cloud as a set of all intervals of some fixed length. Notice, that if the current interval has length $k$ then exactly $k + 1$ leaves are reachable from this segment with the operator $D$.

We want to write a formula of the language $D$ fragment of Halpern–Shoham logic saying that $p$ is a cloud. In order to do that we use an additional variable $e$. The idea is that an interval $[a, a + n]$ satisfies $e$ iff $[a + 1, a + n + 1]$ does not.

![Figure 2. An example of a cloud.](image)
Lemma 3. If $M, [a_M, b_M] \models \Phi_{\text{cloud}}$, where $a_M$ and $b_M$ are endpoints of $M$, then $p$ is a cloud.

Proof. We will prove that if an interval $[x, y]$ is labeled with $p$, then also $[x + 1, y + 1]$ is labeled with $p$. A symmetric proof shows that the same holds for $[x - 1, y - 1]$, so all the intervals of length equal to $m$, where $m$ is the length of $[x, y]$, are labeled with $p$.

This will imply that no other intervals can be labeled with $p$ and $p$ is indeed a cloud. This is because each such interval either has a length greater than $m$, and thus contains an interval of length $m$, and as such labeled with $p$, or has a length smaller than $m$, and is contained in an interval labeled by $p$, in both cases contradicting (ii).

Consider an interval $[x, y]$ labeled with $p$. Interval $[x, y + 1]$ contains an interval labeled with $p$, so it has to contain two different intervals labeled with $p$ – one labeled with $e$ and the other one with $\neg e$. Suppose without loss of generality that $[x, y]$ is the one labeled with $e$, and let us call the second one $[u, t]$. If $t < y + 1$, then $[u, t]$ is a subinterval of $[x, y]$ and is labeled with $p$, a contradiction. So $t = y + 1$.

Let us assume that $u > x + 1$. The interval $[u - 1, y + 1]$ must contain two different intervals labeled with $p$. One of them is $[x, y + 1]$, and it cannot contain another interval labeled with $p$, so the other one must be a subinterval of $[u - 1, y]$. But then it is a subinterval of $[x, y]$ (because $u - 1 > x + 1 - 1 = x$) which also is labeled with $p$ — a contradiction. So $u = x + 1$.

3.5 A Cloud – how to use it.

Let us now concentrate on models which satisfy $\Phi_{\text{orient}} \land \Phi_{\text{LA}} \land \Phi_{\text{cloud}}$. Since $\Phi_{\text{cloud}}$ is satisfied then $p$ is a cloud. Let $n - 1$ denote number of leaves contained in the intervals that form the cloud. Our goal is to write a formula $\Phi_{\text{length}}$ that would guarantee the following properties:

1. Configurations and shades are not too short. If you see two states (i.e. more than an entire configuration) or an entire shade, then you must see a lot, at least $n$ leaves. So you must be high enough. Higher than the cloud.

2. Configurations and shades are not too long. If you only see an interior of a configuration (i.e. you do not see states) or an interior of some shade, then you do not see much, at most $n - 2$ leaves. So you must be under the cloud.

Once we do that, the formula $\Psi = \Phi_{\text{orient}} \land \Phi_{\text{LA}} \land \Phi_{\text{cloud}} \land \Phi_{\text{length}}$ will be satisfiable if and only if there exists a word satisfying the conditions from Lemma 1(ii) – it is straightforward how to translate such a word into a model of $\Psi$ and vice versa.

So put $\Phi_{\text{length}} = \Phi_{1c} \land \Phi_{1s} \land \Phi_{2c} \land \Phi_{2s}$ where:

$\Phi_{1c} = \Phi_{\text{length}}[G](\bigwedge_{q \in Q} (\langle D \rangle q \land \langle D \rangle q' \Rightarrow \langle D \rangle p))$

$\Phi_{1s} = \Phi_{\text{length}}[G](\bigwedge_{q \in Q} [D] \neg q \Rightarrow \neg p \land [D] \neg p)$

Formulas for shades are a little bit more complex. Let $F_1, F_2, F_r, F_s, F', F''$ be a set of symbols that contain $f_1$ (as a second one $u, t$). Suppose without loss of generality that $f_1$ contains two different intervals labeled with $e$, $e'$. Suppose $u = (x, y)$ and $t = (u, t)$. Then $\Phi_{1s}$ is the formula $\Phi_{\text{length}}[G](\bigwedge_{l \in L} (\langle D \rangle l \land \langle D \rangle l' \Rightarrow \langle D \rangle p))$.

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