Construction of the Zero-Energy State of $SU(2)$-Matrix Theory: Near the Origin

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Abstract

We explicitly construct a (unique) $Spin(9) \times SU(2)$ singlet state, $\phi$, involving only the fermionic degrees of freedom of the supersymmetric matrix-model corresponding to reduced 10-dimensional super Yang-Mills theory, resp. supermembranes in 11-dimensional Minkowski space. Any non-singular wavefunction annihilated by the 16 supercharges of $SU(2)$ matrix theory must, at the origin (where it is assumed to be non-vanishing) reduce to $\phi$. 

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1 Introduction

The fermionic degrees of freedom of $SU(2)$-matrix theory (see e.g. [1]) are three $\text{Spin}(9)$ spinors, $\theta_{\dot{\alpha}A}$, ($\dot{\alpha} = 1, \ldots, 16$, $A = 1, 2, 3$, $\theta^\dagger_{\dot{\alpha}A} = \theta_{\dot{\alpha}A}$), satisfying canonical anti-commutation relations

$$\{\theta_{\dot{\alpha}A}, \theta_{\dot{\beta}B}\} = \delta_{\dot{\alpha}\dot{\beta}}\delta_{AB}.$$  

The corresponding $2^{8\cdot3}$-dimensional Hilbert-space $\mathcal{H} = \mathcal{H}_{256} \otimes \mathcal{H}_{256} \otimes \mathcal{H}_{256}$ splits into irreducible $\text{Spin}(9)$ representations built out of the ones occurring in $\mathcal{H}_{256} = 44 \oplus 84 \oplus 128$.

First determining all $\text{Spin}(9)$ singlets occurring in $\mathcal{H}$, in terms of the three representations $44, 84, 128$ (whose elements are denoted $|st\rangle$, $|stu\rangle$ and $|t\dot{\alpha}\rangle$ respectively), the central part of the paper then is the explicit construction, out of these $\text{Spin}(9)$ singlets, of a (unique) $\text{Spin}(9) \times SU(2)$ singlet $\phi$ (whose relevance has been advocated by Wosiek, who was led to the existence of such a $\text{Spin}(9) \times SU(2)$ singlet using symbolic programme [2]).

In the next section we take an independent route to obtain $\phi$, here proving its uniqueness, by listing all possible $\text{Spin}(7) \times SU(2)$ invariant states and then taking their (unique) linear combination such that the result is $\text{Spin}(9) \times SU(2)$ invariant.

While the three representations in $\mathcal{H}_{256}$, forming an 'Euler-triple' (cp. e.g. [3]) - quite likely relevant concerning the existence of a (unique) zero energy state for general $SU(N \geq 2)$ (note the intertwining nature of the two terms $\gamma^t_{\beta\dot{\alpha}} \theta_{\dot{\alpha}A}$ and $\gamma^st_{\beta\dot{\alpha}} \theta_{\dot{\alpha}A}$ in the supercharges of the model) - have quite a long history in supergravity theory (starting with [4]), we could not find a good reference for their Fock space representations and therefore derived them explicitly (see Appendix A) to be sure of the exact intertwining relations.

2 The construction of $\phi$

According to the decomposition of $\mathcal{H}_{256} \otimes \mathcal{H}_{256}$ into irreducible representations of $\text{Spin}(9)$ (cp. e.g. [7], eq. (13)-(18), or [8]) - yielding three 44’s and

\footnote{
  at least not until September 28; our apologies to the authors of [5], [6] (whose results seem to be related to our Fock space representations given in Appendix A)
}
three $84$’s, which are easily seen to be (proportional to)

$$\| \text{st} \rangle := |su\rangle |tu\rangle + |tu\rangle |su\rangle - \frac{2}{9} \delta_{st} |uv\rangle |uv\rangle,$$

$$\| \text{st} \rangle := |suv\rangle |tuv\rangle + |tuv\rangle |suv\rangle - \frac{2}{9} \delta_{st} |uvw\rangle |uvw\rangle,$$

$$\| \text{st} \rangle := |s\hat{a}\rangle |t\hat{a}\rangle + |t\hat{a}\rangle |s\hat{a}\rangle - \frac{2}{9} \delta_{st} |u\hat{a}\rangle |u\hat{a}\rangle,$$

and (for notational convenience we will now write $\alpha$ instead of $\hat{\alpha}$, in this section)

$$\| \text{st} \rangle := \epsilon^{stupqrabc} |pqr\rangle |abc\rangle,$$

$$\| \text{st} \rangle := \gamma^{\alpha\beta} |t\alpha\rangle |u\beta\rangle + \gamma^{u\alpha\beta} |s\beta\rangle + \gamma^{u\alpha\beta} |t\beta\rangle,$$

$$\| \text{st} \rangle := \gamma^{stuv} |v\alpha\rangle |v\beta\rangle,$$

there are 14 $Spin(9)$ singlets in $\mathcal{H}_{256} \otimes \mathcal{H}_{256} \otimes \mathcal{H}_{256}$. Nine of these involve the 128-dimensional spinor-representations while the simplest ones are

$$\|\|1\rangle := |su\rangle_1 |tu\rangle_2 |st\rangle_3, \quad \|\|1\rangle := \epsilon^{stupqrabc} |stu\rangle_1 |pqr\rangle_2 |abc\rangle_3,$$

and the (cyclically invariant) sum of the remaining three,

$$\|\|1\rangle := |suv\rangle_1 |tuv\rangle_2 |st\rangle_3 + |tuv\rangle_1 |st\rangle_2 |suv\rangle_3 + |st\rangle_1 |suv\rangle_2 |tuv\rangle_3.$$

Using the ‘Rarita-Schwinger’ constraints (RSC) $\gamma^{t\alpha\beta} |t\beta\rangle_A = 0$ and the intertwining relations (cp. Appendix A)

$$2 \theta_{\alpha\beta} |st\rangle_A = \gamma^{s\alpha\beta} |t\beta\rangle_A + \gamma^{t\alpha\beta} |s\beta\rangle_A,$$

$$\theta_{\alpha\beta} |stu\rangle_A = \frac{i}{\sqrt{2}} \left( \gamma^{st\alpha\beta} |u\beta\rangle_A + \gamma^{us\alpha\beta} |t\beta\rangle_A + \gamma^{tu\alpha\beta} |s\beta\rangle_A \right),$$

it is straightforward to calculate the action of the $SU(2)$ generators $J_A := \frac{1}{2} \epsilon_{ABC} \theta_{AB} \theta_{AC}$ on the above $Spin(9)$ singlets; e.g.$^2$

$$\theta_{\alpha1} \theta_{\alpha2} \|\|1\rangle = \frac{13}{4} |s\beta\rangle_1 |t\beta\rangle_2 |st\rangle_3,$$

$^2$It may be amusing to speculate about the occurrence of the relatively large prime 13, which played a prominent role in (bosonic) string theory.
and (s.b.)

\[ \theta_{\alpha_1} \theta_{\alpha_2} \ |1\rangle = -9|s\epsilon_1|t\epsilon_2|st\rangle. \quad (3) \]

It follows that

\[ \phi := \ |1\rangle + \frac{13}{36} \ |1\rangle \quad (4) \]

is Spin(9) \times SU(2) invariant.

Eq. (3) easily follows when splitting the calculation into two parts:

\[ -2\theta_{\alpha_1} \theta_{\alpha_2}\langle suv|1|tuv\rangle_2|st\rangle_3 = \]

\[ (\gamma_{su}^{|v\beta_1} + \gamma_{su}^{|s\beta_1} + \gamma_{su}^{|u\beta_1})(\gamma_{\alpha\epsilon}^{|t\epsilon_2} + \gamma_{\alpha\epsilon}^{|s\epsilon_2})|st\rangle_3; \]

using the RSC and the (anti-)commutation relations between \( \gamma \)'s (in particular \( \gamma^{pq} = \gamma^p \gamma^q - \delta^{pq}1 \) and \( [\gamma^{pq}, \gamma^r] = 2\gamma^p \delta^{qr} - 2\gamma^q \delta^{pr} \)), each of the nine terms becomes proportional to \( |s\epsilon_1|t\epsilon_2|st\rangle_3 \), the respective coefficients being: 0 for \( \gamma^{su} \gamma^t \), \( \gamma^{su} \gamma^t \); 72 for \( \gamma^{uv} \gamma^w \); 3 for \( \gamma^{su} \gamma^v \), \( \gamma^{uv} \gamma^t \); -15 for each of the remaining four \( \gamma^{su} \gamma^v \), \( \gamma^{uv} \gamma^t \), \( \gamma^{us} \gamma^v \).

Concerning the second part in (3), one notes that

\[ -\sqrt{8} \theta_{\alpha_1} \theta_{\alpha_2}(\langle tuv|1|st\rangle_2 + |st\rangle_1|tuv\rangle_2)|su\rangle_3 = \]

\[ (\gamma_{\alpha\epsilon}^{|t\epsilon_2} + \gamma_{\alpha\epsilon}^{|s\epsilon_2})|su\rangle_3; \]

which gives rise to 12 terms, 6 of which cancel in pairs \( (\gamma^{uv} \gamma^s, \gamma^{tu} \gamma^t \) and \( \gamma^{vt} \gamma^t \)), due to antisymmetry of \( \gamma^{su} \) (as a matrix) and \( |su\rangle \), while (using again \( \gamma^{pq} = \gamma^p \gamma^q - \delta^{pq}1 \) and the transformation of the \( \gamma^w \)'s as a vector when commuting with \( \gamma^{pq} \)) the 3 remaining terms containing \( |t\epsilon_2 \rangle \), resulting in

\[ 5(\gamma_{\alpha\epsilon}^{|s\beta_1} |ue_2 \rangle - \gamma_{\alpha\epsilon}^{|v\beta_1} |s\beta_1 \rangle |ve_2 \rangle)|su\rangle_3, \]

are cancelled by those arising from the terms containing \( |t\beta_1 \rangle \).

Perhaps it is worth noting that the image (under the action of any of the SU(2) generators, say \( J_3 \)) of \( |1\rangle \) (which is not needed for the Spin(9) \times SU(2) singlet) is, using 

\[ \gamma_{\alpha\epsilon}^{stpq} \propto \gamma^{xyzwv}_w \epsilon_{xyzvwstpq}, \]

proportional to \( \gamma_{\alpha\epsilon}^{|u\beta_1} |ve\rangle |su\rangle \) (and that the apparent 'puzzle' of the Spin(9) singlet \( \gamma_{\alpha\epsilon}^{|t\beta_1} |t\epsilon_2 \rangle |su\rangle \) not entering these considerations is 'resolved' by observing that only the cyclically invariant combination \( |1\rangle \) was considered - in which \( \gamma_{\alpha\epsilon}^{su} \) terms appear symmetrized in \( \beta \) and \( \epsilon \), i.e. cancelling each other).
3 The construction of $\phi$ out of $Spin(7) \times SU(2)$ singlets

Here we reproduce the result obtained in the previous section using an independent approach. Our strategy is to take advantage of the natural $Spin(7)$ covariance of fermionic creation operators $\lambda_{\alpha A}$, $\alpha = 1, \ldots, 8$, corresponding to the full $Spin(9) \times SU(2)$ model. To do so we first write the fermionic $Spin(9)$ generators $M_{st} = \frac{1}{4} \gamma^{st}_{\alpha \beta} \theta_{\alpha A} \theta_{\beta A}$ as $(i, j = 1, \ldots, 7)$

$$M_{ij} = \frac{1}{2} \Gamma^{ij}_{\alpha \beta} \lambda_{\alpha A} \lambda^{\dagger}_{\beta A}, \quad M_{j8} = \frac{i}{4} \Gamma^{j}_{\alpha \beta} (\lambda_{\alpha A} \lambda_{\beta A} + \lambda^{\dagger}_{\alpha A} \lambda^{\dagger}_{\beta A}),$$

$$M_{89} = -\frac{i}{2} \left( \lambda_{\alpha A} \lambda^{\dagger}_{\alpha A} - 12 \right), \quad M_{j9} = -\frac{1}{4} \Gamma^{j}_{\alpha \beta} (\lambda_{\alpha A} \lambda_{\beta A} - \lambda^{\dagger}_{\alpha A} \lambda^{\dagger}_{\beta A}),$$

where we use the conventions for $\gamma_s$ as given in Appendix A.

The condition $M_{ij} \phi = 0$ is the $Spin(7)$ invariance of $\phi$ while $M_{89} \phi = 0$ tells us that $\phi \in \mathcal{F}_{12}$ (where $\mathcal{F}_{n_F}$ denotes the sector with $n_F$ fermions). Therefore we are led to search for a combination of $Spin(7) \times SU(2)$ invariant states in $\mathcal{F}_{12}$ such that

$$\Gamma^{i}_{\alpha \beta} \lambda_{\alpha A} \lambda_{\beta A} \phi = \Gamma^{i}_{\alpha \beta} \lambda^{\dagger}_{\alpha A} \lambda^{\dagger}_{\beta A} \phi = 0.$$  \hspace{1cm} (5)

3.1 The number of $Spin(7) \times SU(2)$ invariant states

In order to solve Eqn. (5) we attempt to list all $Spin(7) \times SU(2)$ invariant states in $\mathcal{F}_{12}$. In doing so it is helpful to first calculate the number $D_{n_F}$ of such states appearing in $\mathcal{F}_{n_F}$ for arbitrary $n_F$. This is done following the lines of [9] by writing $\mathcal{F}_{n_F}$ as

$$\mathcal{F}_{n_F} = Alt(\bigotimes_{l=1}^{n_F} F^{s=1/2}_l) = \mathcal{F}_{n_F, j=0} \oplus \mathcal{F}_{n_F, j=1/2} \oplus \cdots,$$

where $F^{s=1/2}_l$ is a vector space spanned by $\lambda^{s=1/2}_l |0\rangle$ (operators $\lambda^{s=1/2}_l$ are assumed to carry spin $s = 1/2$ of $SO(7)$) and where $\mathcal{F}_{n_F, j}$ is $\mathcal{F}_{n_F}$ projected into subspaces with given $SO(7)$ angular momentum. Therefore the dimensions of subspaces with angular momentum $j = 0$ are

$$D^{Spin(7) \times SU(2)}_{n_F} = \int d\mu_{SO(7)} \chi^{SO(7)}_{j=0} \int d\mu_{SU(2)} \chi^{\text{Alt}}_{j=1}(R),$$
where \( d\mu_{SO(7)} \) and \( d\mu_{SU(2)} \) are \( SO(7) \) and \( SU(2) \) invariant measures, \( R \) is the adjoint representation of \( SU(2) \) and \( j = 1/2 \) representation of \( SO(7) \), i.e. \( R = R^{SO(7),j=1/2} \otimes R^{SU(2),j=1} \). The characters \( \chi \) can be read off directly from the Weyl character formula while the antisymmetric power of \( \chi(R) \) is given by the Frobenius formula (see e.g [10])

\[
\chi^{[n_F]}(R) = \sum_{\sum_k k_i = n_F} (-1)^{\sum_k i_k} \prod_{k=1}^{n_B} \frac{1}{i_k!} \chi_{i_k}(R^k),
\]

(here \( R \) is considered as a matrix). Taking all into consideration we find that the generating function for the numbers \( D_{n_F} \) is

\[
\sum_{n_F=0}^{24} D_{n_F} b^{n_F} = 1 + 2b^4 + 5b^8 + 7b^{12} + 5b^{16} + 2b^{20} + b^{24}.
\]

Note the duality between \( F_{n_F} \) and \( F_{24-n_F} \), i.e. the particle-hole symmetry.

### 3.2 The construction of \( Spin(7) \times SU(2) \) invariant states and \( \phi \)

We now proceed to construct the \( Spin(7) \times SU(2) \) invariant states in \( F_4 \), \( F_8 \) and finally in \( F_{12} \) (note that there are no such states in \( F_{4n+2}, n = 0, \ldots, 5 \) and \( F_{2n+1}, n = 0, \ldots, 11 \)). Let us first consider operators

\[
b_{AB} := \lambda_{\alpha A} \lambda_{\alpha B}, \quad b^i_{AB} := \Gamma^i_{\alpha \beta} \lambda_{\alpha A} \lambda_{\beta B}.
\]

In the \( F_4 \) sector there are only two such states for which we choose

\[
v_1|0\rangle, \quad v_2|0\rangle \quad v_1 := b_{AB} b_{AB} \quad v_2 := b^i_{AB} b^i_{AB}.
\]

They are not orthogonal as the overlap matrix \( G = [\langle 0|v^\dagger_i v_j |0\rangle]_{i,j=1,2} \) is (see Appendix B)

\[
G = \begin{pmatrix} 1344 & 2688 \\ 2688 & 45696 \end{pmatrix}, \quad \det G \neq 0. \tag{6}
\]

We developed a symbolic programme written in Mathematica to confirm that indeed \( G \) is of this form.

In the \( F_8 \) fermion sector there are 5 invariant states. Three of them are simply

\[
v_1 v_1|0\rangle, \quad v_1 v_2|0\rangle, \quad v_2 v_2|0\rangle.
\]
The remaining two are e.g.

\[ w_1|0\rangle, \ w_2|0\rangle, \ w_1 = b_i^{AB} b_i^{BC} b_j^{CD} b_j^{DA}, \ w_1 = b_i^{AB} b_j^{BC} b_j^{CD} b_i^{DA}. \]

We checked in Mathematica that these states are linearly independent.

Finally in the \( F_{12} \) sector we should have 7 states. Considering the previous sectors we can construct already 8. They are

\[ v_1 v_1 |0\rangle, \ v_1 v_2 |0\rangle, \ v_1 v_2 v_2 |0\rangle, \ v_2 v_2 |0\rangle, \]

\[ v_1 w_1 |0\rangle, \ v_2 w_1 |0\rangle, \ v_1 w_2 |0\rangle, \ v_2 w_2 |0\rangle. \]

Accordingly there should be one relation between them. Indeed, we found that

\[
4358 v_1 v_1 |0\rangle + 2652 v_1 v_2 |0\rangle + 984 v_1 v_2 v_2 |0\rangle + 63 v_2 v_2 v_2 |0\rangle - 528 v_1 w_1 |0\rangle
- 88 v_1 w_2 |0\rangle + 24 v_2 w_1 |0\rangle - 152 v_2 w_2 |0\rangle = 0,
\]

and that there are no other identities among these 8 states. Therefore we can choose the \( F_{12} \) basis to be e.g.

\[ r_1 = v_1 v_1 |0\rangle, \ r_2 = v_1 v_2 |0\rangle, \ r_3 = v_1 v_2 v_2 |0\rangle, \ r_4 = v_2 v_2 v_2 |0\rangle, \]

\[ r_5 = v_1 w_1 |0\rangle, \ r_6 = v_2 w_1 |0\rangle, \ r_7 = v_1 w_2 |0\rangle. \]

Finally we checked that there exists a unique combination of \( r_i \) such that Eqn. (5) is satisfied. The result reads

\[ \chi = 326304 r_1 + 488136 r_2 + 72612 r_3 + 1377 r_4 + 114576 r_5 - 176528 r_6 + 10296 r_7, \]

and is proportional to (4).

### 4 Outlook

Having solved a (physically relevant) representation theoretic question, let us now make a comment on the problem of determining the full zero-energy eigenfunction \( \Psi \) of the Hamiltonian

\[ H = -\Delta + \frac{1}{2} (\epsilon_{ABC} x_B x_C)^2 + i x s B \epsilon_{ABC} \gamma^s_{\hat{A} \hat{B}} \theta_{\hat{A} \hat{B}}, \]
which on the physical space of $SU(2)$ invariant states is equal to the square of each of the supercharges
\[ Q_{\beta} = -i\partial_{sA}\gamma_{\beta\dot{\alpha}}^s\theta_{\dot{A}A} + \frac{1}{2}\epsilon_{ABC}\gamma_{\beta\dot{\alpha}}^s\theta_{\dot{A}A} + D_{\beta} + V_{\beta}. \]

Due to elliptic regularity (see e.g. [11]), any solution to $H\Psi = 0$ must be smooth. Accordingly, one can, around the origin, write $\Psi$ in terms of a power series in the coordinates,
\[ \Psi(x) = \sum_{k=0}^{N} \Psi^{(k)}(x) = \psi^{(0)} + x_{tA}\psi^{(1)}_{tA} + \frac{1}{2}x_{tA}x_{uB}\psi^{(2)}_{tA,uB} + \ldots + \Psi^{(N)}(x), \]

with $\psi_{t_{1A_{1}}\ldots t_{kA_{k}}}^{(k)} \in \mathcal{H}$, and $\Psi^{(k)}$ vanishing to order $k$ at $x = 0$.

Examining the equations $Q_{\beta}\Psi = 0$ to each order in the coordinates, we find
\[ D_{\beta}\Psi^{(1)} = 0, \quad D_{\beta}\Psi^{(2)} = 0, \quad D_{\beta}\Psi^{(k+3)} + V_{\beta}\Psi^{(k)} = 0, \quad k = 0, 1, 2, \ldots \]
i.e.
\[ \gamma_{\beta\dot{\alpha}}^{t}\theta_{\dot{A}A}\psi^{(1)}_{tA} = 0, \]
\[ \gamma_{\beta\dot{\alpha}}^{t}\theta_{\dot{A}A}\psi^{(2)}_{tA,uB} = 0, \]
\[ \gamma_{\beta\dot{\alpha}}^{t}\theta_{\dot{A}A}\psi^{(3)}_{tA,uB,vC} + \frac{1}{2}\epsilon_{ABC}\gamma_{\beta\dot{\alpha}}^{uv}\theta_{\dot{A}A}\psi^{(0)} = 0, \]

etc. for all $\hat{\beta}$. Note the three separate towers of equations relating $\Psi^{(k)}$ to $\Psi^{(k+3)}$ via intertwiners.

In any case, using that $\Psi$ must be $Spin(9)$ invariant [12], one concludes that $\psi^{(0)}$ must be a scalar multiple of the state we constructed in this paper.

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Appendix A

In this Appendix we work with a single $H_{256}$. A Fock space representation of $H_{256}$ can be obtained by introducing fermionic creation operators $\lambda_\alpha$ and annihilation operators $\frac{\partial}{\partial \lambda_\alpha} = \lambda_\alpha^\dagger$ via

$$\theta_\alpha = \frac{1}{\sqrt{2}}(\lambda_\alpha + \lambda_\alpha^\dagger), \quad \theta_{\alpha+8} = \frac{1}{i\sqrt{2}}(\lambda_\alpha - \lambda_\alpha^\dagger).$$

A basis of the Hilbert space $H_{256}$ is obtained by acting with products of the $\lambda_\alpha$'s on the fermion vacuum state $|0\rangle$ defined by $\lambda_\alpha^\dagger|0\rangle = 0$. The $Spin(9)$ generators $M_{st} = \frac{1}{4}\gamma_{\alpha\beta}^s\theta_\alpha\theta_\beta$, $s, t = 1, \ldots, 9$ where $\gamma_{st} = \frac{1}{2}[\gamma^s, \gamma^t]$ and $\gamma^s$ are $16 \times 16$, real, symmetric matrices satisfying $\{\gamma^s, \gamma^t\} = 2\delta_{st}1_{16	imes16}$, then become

$$M_{ij} = \frac{1}{2}\Gamma_{\alpha\beta}^{ij}\lambda_\alpha\lambda_\beta^\dagger, \quad M_{j8} = \frac{i}{4}\Gamma_{\alpha\beta}^j(\lambda_\alpha\lambda_\beta + \lambda_\alpha^\dagger\lambda_\beta^\dagger),$$

$$M_{89} = -\frac{i}{2}(\lambda_\alpha\lambda_\alpha^\dagger - 4), \quad M_{j9} = -\frac{1}{4}\Gamma_{\alpha\beta}^j(\lambda_\alpha\lambda_\beta - \lambda_\alpha^\dagger\lambda_\beta^\dagger),$$

when choosing

$$\gamma^j = \begin{bmatrix} 0 & i\Gamma^j \\ -i\Gamma^j & 0 \end{bmatrix}, \quad \gamma^8 = \begin{bmatrix} 0 & 1_{8 \times 8} \\ 1_{8 \times 8} & 0 \end{bmatrix}, \quad \gamma^9 = \begin{bmatrix} 1_{8 \times 8} & 0 \\ 0 & -1_{8 \times 8} \end{bmatrix},$$

with $\Gamma^i$ being $8 \times 8$, purely imaginary, antisymmetric matrices satisfying $\{\Gamma^i, \Gamma^j\} = 2\delta_{ij}1_{8 \times 8}$.

As already mentioned, the Hilbert space $H_{256}$ decomposes into three irreducible representations whose elements will be denoted by $|st\rangle$, $|stu\rangle$ and $|s\hat{a}\rangle$ respectively.

An explicit presentation of the $44$ in terms of creation operators $\lambda_\alpha$ was given in [13] as follows:

$$|i \neq j\rangle = b_ib_j|0\rangle, \quad |jj\rangle = \left(b_j^2 - \frac{1}{9}b^2\right)|0\rangle,$$

Furthermore one may choose $i\Gamma_{\alpha8}^j = \delta_{\alpha}^j$, $i\Gamma_{kl}^j = -c_{jkl}$ with totally antisymmetric octonionic structure constants $c_{ijk} = +1$ for $(ijk) = (123), (165), (246), (435), (147), (367), (257)$. 

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\[ |j8\rangle = \frac{1}{2} b_j \left( 1 - \frac{2}{9} b^2 \right) |0\rangle, \quad |j9\rangle = -\frac{i}{2} b_j \left( 1 + \frac{2}{9} b^2 \right) |0\rangle, \]
\[ |88\rangle = \frac{1}{2} \left( |0\rangle - \frac{2}{9} b^2 |0\rangle + |8\rangle \right), \quad |99\rangle = -\frac{1}{2} \left( |0\rangle + \frac{2}{9} b^2 |0\rangle + |8\rangle \right), \]
\[ |89\rangle = -\frac{i}{2} (|0\rangle - |8\rangle), \quad \text{(8)} \]
where
\[ b_j := \frac{i}{4} \Gamma^j_{\alpha\beta} \lambda_\alpha \lambda_\beta, \quad b^2 := \sum_{i=1}^7 b_i b_i, \quad |8\rangle := \lambda_1 \ldots \lambda_8 |0\rangle. \]

While it is convenient to work with the overcomplete set of states \(|st\rangle = |ts\rangle\), satisfying \(\sum_{s=1}^9 |ss\rangle = 0\) and transforming according to
\[ M_{st} |uv\rangle = \delta_{ta} |sv\rangle - \delta_{su} |tv\rangle + \delta_{tu} |su\rangle - \delta_{sv} |tu\rangle, \quad \text{(9)} \]
one should be aware of the fact that they are not orthonormal; rather
\[ \langle st|s't'\rangle = \frac{1}{2} (\delta_{ss'} \delta_{tt'} + \delta_{tt'} \delta_{ss'}) (1 - \delta_{st}) (1 - \delta_{s't'}) + \delta_{st} \delta_{s't'} \left( \delta_{ss'} - \frac{1}{9} \right) \]
(in accordance with \(|s \neq t\rangle \cong \frac{1}{2} (||st|| + ||ts||), \langle tt|| + ||tt\rangle - \frac{1}{9} \sum_u ||uu||\),
where \(||st||, ||s't'||\) are unconstrained tensor product-states satisfying \(\langle st|s't'\rangle = \delta_{ss'} \delta_{tt'}\)
\[ \text{(8) follows from (9) when starting with the 27-dimensional traceless symmetric } U(1) \cong M_{99}\text{-invariant } Spin(7) \text{ representation containing } |i \neq j\rangle = b_i b_j |0\rangle:\]
\[ |9j\rangle = M_{9k} |jk\rangle = -i (b_k + b_k^j) b_j b_k |0\rangle = -\frac{i}{2} (b_j + 2b_k^2 b_j) |0\rangle, \]
\[ |8j\rangle = M_{8k} |jk\rangle = \frac{1}{2} (b_j - 2b_k^2 b_j) |0\rangle, \]
\[ M_{9j} |9j\rangle = |99\rangle - |jj\rangle = -\left( \frac{1}{2} + b_j^2 + b_k^2 b_j^2 \right) |0\rangle, \]
\[ M_{8j} |8j\rangle = |88\rangle - |jj\rangle = \left( \frac{1}{2} - b_j^2 + b_j^2 b_k^2 \right) |0\rangle, \]
(with \(j \neq k\), no sum) implying
\[ |88\rangle - |99\rangle = (1 + 2b_j^2 b_k^2) |0\rangle = |0\rangle + |8\rangle, \]
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\[7(|88\rangle + |99\rangle) = -2(|88\rangle + |99\rangle + b^2|0\rangle),\]
i.e.
\[|88\rangle + |99\rangle = -\frac{2}{9}b^2|0\rangle,\]
hence
\[|jj\rangle = b_j^2|0\rangle - \frac{1}{9}b^2|0\rangle.\]

Note that one may use
\[\begin{bmatrix} b_i, b_j^\dagger \end{bmatrix} = \frac{1}{2}M_{ij} + \delta_{ij} \left( 1 - \frac{1}{4}\lambda_\alpha\lambda_\alpha^\dagger \right),\]
\[\begin{bmatrix} M_{ij}, b_k \end{bmatrix} = \delta_{jk}b_i - \delta_{ik}b_j,\]
(10)

which follows from
\[\begin{bmatrix} Tr(\lambda A\lambda), Tr(\lambda^\dagger B\lambda^\dagger) \end{bmatrix} = -4Tr(\lambda B\lambda^\dagger) + 2Tr(AB),\]
\[\begin{bmatrix} Tr(\lambda A\lambda^\dagger), Tr(\lambda^\dagger B\lambda^\dagger) \end{bmatrix} = 2Tr(\lambda^\dagger AB\lambda^\dagger).\]
(11)

Consistency conditions such as \((i \neq j)\)
\[|j8\rangle = M_{8i}|ij\rangle = \frac{1}{2}M_{js}|88\rangle = M_{j9}|98\rangle,\]
lead to useful (Fierz-)identities \((j \neq k)\)
\[2b_jb_k^2|0\rangle = \frac{2}{3}b_j^2|0\rangle = \frac{2}{9}b_jb^2|0\rangle = b_j^2|8\rangle.\]

84

The construction of states \(|stu\rangle\) transforming according to the antisymmetric representation 84 can be done analogously, starting with
\[|ijk\rangle := \sqrt{\frac{2}{9}}(b_ib_{jk} + b_kb_{ij} + b_jb_{ki})|0\rangle,\]
where \(b_{jk} := \frac{1}{4}Tr(\lambda\Gamma^{jk}\lambda).\) In proving
\[\langle ijk|i'j'k'\rangle = \delta_{i'i'}\delta_{jj'}\delta_{kk'},\quad i < j < k, \quad i' < j' < k',\]

it is helpful to use (10) and the commutation rules
\[\begin{bmatrix} M_{ij}, b_{kl} \end{bmatrix} = \delta_{jk}b_{il} + \delta_{il}b_{jk} - \delta_{ik}b_{jl} - \delta_{jl}b_{ik},\]
(10)
\[
[b^\dagger_k, b_{ij}] = \frac{i}{4} Tr(\lambda^i \Gamma^k \lambda^l), \quad \langle 0 | b^\dagger_{ij} b_{kl} | 0 \rangle = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}
\]
which follow from (11).

All other states of the 84 can be obtained by application of the \( M_{st} \) operators on \(|ijk\rangle\) using
\[
M_{st}|pqr\rangle = \delta_{tp}|sqr\rangle - \delta_{tq}|spr\rangle + \delta_{tr}|tpq\rangle - \delta_{sr}|tpr\rangle,
\]
from which it follows that for fixed \( i, j \) and \( k \) we have
\[
|i89\rangle = -M_{ij}|ij8\rangle.
\]
Again, using the independence of \( k \) in the above formulas resp. additional relations among various states defined via \( b_i \) and \( b_{jk} \) one can show that
\[
|i89\rangle = \frac{i}{\sqrt{2}} (b_j b_{ij} + b_k b_{ik}) |0\rangle = \frac{i}{3\sqrt{2}} \sum_{l=1}^{7} b_l b_{il} |0\rangle \quad i \neq j \neq k \neq i,
\]
from which it follows that \( \Gamma^{ij}_{\alpha\beta} \Gamma^j_{\rho\epsilon} + \Gamma^{ik}_{\alpha\beta} \Gamma^k_{\rho\epsilon} \) (no sum, \( i \neq j \neq k \neq i \)) must be independent of \( j \) and \( k \) (true even if \( j = k \)).

The 128 representation comprises all odd fermion states in \( \mathcal{H}_{256} \). As a convenient definition one may take
\[
|t\hat{\alpha}\rangle := \frac{2}{11} \gamma^s_{\alpha\beta} \theta_\beta |st\rangle,
\]
which does transform according to
\[ M_{uv}|t\hat{\alpha}\rangle = \delta_{vt}|u\hat{\alpha}\rangle - \delta_{ut}|v\hat{\alpha}\rangle - \frac{1}{2}\gamma_{\alpha\beta}^{uv}|t\hat{\beta}\rangle \]
and explicitly exhibits the crucial RSC \( \gamma_{\alpha\beta}^{t}|t\hat{\alpha}\rangle = 0 \). The intertwining relation
\[ 2\theta_{\alpha}|st\rangle = \gamma_{\alpha\beta}^{s}|t\hat{\beta}\rangle + \gamma_{\alpha\beta}^{t}|s\hat{\beta}\rangle \tag{12} \]
follows when using
\[ \gamma_{\alpha\beta}^{su}\theta_{\beta}|tu\rangle + \gamma_{\alpha\beta}^{tu}\theta_{\beta}|su\rangle = 9\theta_{\alpha}|st\rangle; \]
it is also true that
\[ \theta_{\alpha}|stu\rangle = \frac{i}{\sqrt{2}} \left( \gamma_{\alpha\beta}^{st}|u\hat{\beta}\rangle + \gamma_{\alpha\beta}^{uv}|t\hat{\beta}\rangle + \gamma_{\alpha\beta}^{ut}|s\hat{\beta}\rangle \right), \tag{13} \]
respectively
\[ |t\hat{\alpha}\rangle = \frac{i\sqrt{2}}{42} \gamma_{\alpha\beta}^{sv}\theta_{\beta}|svt\rangle, \]
which of course could have alternatively been used to define \( |t\hat{\alpha}\rangle \).

**Intertwiners**

The above intertwining relations (12) and (13) as well as the ones below (explicitly checked on the computer), we believe to be crucial for the construction of the full zero energy state;
\[ \gamma_{\alpha\beta}^{stu}\theta_{\beta}|stu\rangle = 0, \]
\[ |t\hat{\alpha}\rangle = \gamma_{\alpha\beta}^{t}\theta_{\beta}|tt\rangle, \]
\[ |stu\rangle = \frac{i}{44\sqrt{2}} \left( \theta_{\alpha\beta}^{stu}\theta_{\beta}|uv\rangle + \theta_{\alpha\beta}^{fuv}\theta_{\beta}|sv\rangle + \theta_{\alpha\beta}^{fuv}\theta_{\beta}|sv\rangle \right), \]
\[ |st\rangle = \frac{i}{168\sqrt{2}} \left( \theta_{\alpha\beta}^{suv}\theta_{\beta}|tuv\rangle + \theta_{\alpha\beta}^{fuv}\theta_{\beta}|suv\rangle \right), \]
\[ \gamma_{\alpha\beta}^{su}\theta_{\beta}|tu\rangle - \gamma_{\alpha\beta}^{tu}\theta_{\beta}|su\rangle = \frac{11}{6\sqrt{2t}}\gamma_{\alpha\beta}^{u}\theta_{\beta}|stu\rangle, \]
\[ \gamma_{\alpha\beta}^{su}\theta_{\beta}|tuv\rangle + \gamma_{\alpha\beta}^{ut}\theta_{\beta}|stv\rangle + \gamma_{\alpha\beta}^{tu}\theta_{\beta}|usv\rangle = 9\theta_{\alpha}|stu\rangle, \]
\[ \gamma_{\alpha\beta}^{s}\theta_{\beta}|su\rangle = \frac{11i\sqrt{2}}{84}\gamma_{\alpha\beta}^{st}\theta_{\beta}|stu\rangle. \]
Appendix B

In this Appendix we derive the form of the Gram matrix (6) related to the $F_4$ sector. A similar procedure can be applied for the other sectors.

We will use the following notation

$$ (\lambda_A A_\lambda B) := \lambda_A a^\alpha A_\lambda B_\beta, \quad (\lambda_A A_\lambda B^\dagger) := \lambda_A a^\alpha A_\lambda B_\beta^\dagger, $$

$$ b_{AB} := (\lambda_A 1\lambda_B), \quad b_{AB}^i := (\lambda_A 1\Gamma^i \lambda_B), \quad b_{AB}^{ij} := (\lambda_A 1\Gamma^i \lambda_B), $$

$$ M_{AB} := (\lambda_A 1\lambda_B^\dagger), \quad M_{AB}^i := (\lambda_A 1\Gamma^i \lambda_B^\dagger), \quad M_{AB}^{ij} := (\lambda_A 1\Gamma^i \lambda_B^\dagger). $$

It is now useful to write down the generalization of commutation relations (11) for operators involving $\lambda_A$ and $\lambda_A^\dagger$ with color indices. We have

$$ [(\lambda_A A_\lambda B), (\lambda_A^\dagger B^{\lambda})] = - (\lambda_C B A_\lambda B^\dagger) \delta_A - (\lambda_D B^T A^T \lambda_A^\dagger) \delta_B $$

$$ + (\lambda_D B^T A_\lambda B^\dagger) \delta_A + (\lambda_C B A^T \lambda_A^\dagger) \delta_B - (AB)^T \delta_A \delta_B + (AB) \delta_A \delta_B, $$

$$ [(\lambda_A A_\lambda B), (\lambda_C B A)] = (\lambda_A AB \lambda_D) \delta_B - (\lambda_A AB^T \lambda_C) \delta_B. $$

The commutators of the $b - b$ type are now

$$ [b_{AB}^i, b_{CD}^i] = M_{CB}^i \delta_{DA} + M_{DA}^i \delta_{CB} - M_{DB}^i \delta_{CA} - M_{CA}^i \delta_{BD} + 8(\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}), $$

$$ [b_{AB}^i, b_{CD}^j] = M_{CB}^i \delta_{AD} - M_{DA}^i \delta_{BC} + M_{DB}^i \delta_{AC} - M_{CA}^i \delta_{BD}, $$

$$ [b_{AB}^i, b_{CD}^{ij}] = -M_{CB}^i \delta_{AD} + M_{DA}^i \delta_{BC} - M_{DB}^i \delta_{AC} + M_{CA}^i \delta_{BD}, $$

$$ [b_{AB}^i, b_{CD}^{ij}] = M_{CB}^i \delta_{AD} - M_{DA}^i \delta_{BC} + M_{DB}^i \delta_{AC} - M_{CA}^i \delta_{BD} $$

$$ - \delta^{ij} (M_{CB} \delta_{AD} + M_{DA} \delta_{CB} + M_{DB} \delta_{CA} + M_{CA} \delta_{DB}) + 8 \delta^{ij} (\delta_{AC} \delta_{BD} + \delta_{AD} \delta_{BC}), $$

while the commutators of the $M - b$ type are

$$ [M_{AB}, b_{CD}] = b_{AD} \delta_{BC} - b_{AC} \delta_{BD}, $$

$$ [M_{AB}, b_{CD}^i] = b_{AD}^i \delta_{BC} + b_{AC}^i \delta_{BD}, $$

$$ [M_{AB}, b_{CD}^{ij}] = b_{AD}^{ij} \delta_{BC} - b_{AC}^{ij} \delta_{BD}, $$

$$ [M_{AB}^i, b_{CD}^{ij}] = b_{AD}^{ij} \delta_{BC} + b_{AC}^{ij} \delta_{BD} + b_{BD} \delta_{BC} + b_{AC} \delta_{BD}. $$

Now it is straightforward to evaluate the scalar product $\langle 0 | v_1^\dagger v_1 | 0 \rangle$ with $v_1 := b_{AB} b_{AB}^\dagger$, we have

$$ [b_{AB}^i, b_{CD} b_{CD}^i] = -4 b_{AD} b_{DB} + 4 b_{BD} b_{DA} + 28 b_{AB}, $$

$$ [b_{AB}^{ij}, b_{CD}^{ij} b_{CD}^{ij}] = -4 b_{AD} b_{DB} + 4 b_{BD} b_{DA} + 28 b_{AB}. $$
which implies

\[ [v_1^\dagger, v_1] = 8M_{AB}M_{AB} - 128M_{AA} + 16b_{AB}M_{BC}b_{CA}^\dagger + 40b_{AB}b_{AB}^\dagger + 1344, \]

hence \( \langle 0|v_1^\dagger v_1|0 \rangle = 1344. \)

To calculate the scalar product \( \langle 0|v_2^\dagger v_1|0 \rangle \) with \( v_2 := b_{AA}^i b_{BB}^i \) we need

\[ [b_{AB}^i, b_{CD}^j b_{CD}] = -4b_{AC}M_{CB}^i + 4b_{BC}M_{CA}^i - 4b_{AB}^i + 4\delta_{AB}b_{CC}^i, \]

which gives

\[ [b_{AA}^i, b_{CD}b_{CD}] = 8b_{AA}^i, \quad [v_2^\dagger, v_1] = 16b_{AB}^i b_{AB}^i - 224M_{AA} + 2688, \]

hence \( \langle 0|v_2^\dagger v_1|0 \rangle = 2688 \)

Finally the scalar product \( \langle 0|v_2^\dagger v_2|0 \rangle \) is obtained with use of

\[ [b_{AA}^i, b_{BB}^j b_{CC}^j] = 8(b_{AA}^i M_{BB}^{ij} - b_{AA}^i M_{BB}) + 136b_{AA}^i, \]

hence

\[
\begin{align*}
[v_2^\dagger, v_2] &= 16b_{AA}^i (M_{BB}^{ij} - \delta^{ij} M_{BB}) b_{CC}^j \\
&+ 32M_{AA}^j M_{BB}^{ij} + 224M_{AA}M_{BB} + 1120M_{AA} + 352b_{AA}^i b_{BB}^i + 45696,
\end{align*}
\]

therefore \( \langle 0|v_2^\dagger v_2|0 \rangle = 45696. \)

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