Generalized para-Bose coherent states

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Abstract

In this paper, we construct integrals of motion in a para-Bose formulation for a general time-dependent quadratic Hamiltonian, which, in its turn, commutes with the reflection operator. In this context, we obtain generalizations for the squeezed vacuum states (SS) and coherent states (CS) in terms of the Wigner parameter. Furthermore, we show that there is a positive completeness relation for the SS owing to the Wigner parameter. In the study of the probability transition, we found that the displacement parameter acts as a transition parameter by allowing access to odd states, while the Wigner parameter controls the dispersion of the distribution.

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I. INTRODUCTION

In 1926 Schrödinger obtained quantum states to harmonic oscillator problem, which allowed for a semiclassical description [1]. In the 1960s, with the advent of the laser, these states were rediscovered and widespread in the works of Glauber-Sudarshan-Klauber [2–4] by showing a semiclassical description of electromagnetic radiation, and where the term “coherent states” (CS) was presented for the first time. These states are an overcomplete basis, which for the harmonic oscillator problem can be obtained in three equivalent ways: I - as eigenstates of the annihilation operator; II - by the action of a unitary displacement operator acting in the vacuum state; III - as minimum uncertainty states. These properties guarantee CS a prominent role in modern quantum mechanics, with various applications ranging from quantum optics [5, 6], quantum computing [7], and mathematical physics [8].

There is a wide range of publications in the literature that seeks to generalize CS to be able to describe systems in addition to the harmonic oscillator, such as the time-dependent quadratic systems [9, 10], systems with a given Lie group [11, 12], some non-trivial generalizations [13], and squeezed coherent states (SCS) [14, 15]. On the other hand, CS can be generalized by considering deformations in the canonical commutation relation and thus allow us to study a wide range of relevant problems, such as, for instance, the implications of the gravitational effects in quantum mechanics [16–18], or even in q- and f-deformed oscillators study which is applied, mainly, to quantum optics [19–21]. Thus, our contribution in this field will be through the construction of CS via Wigner-Heisenberg algebra.

The Wigner-Heisenberg algebra originated from the quantization proposed by Wigner [22], which generalizes the canonical commutation relation. In particular, this algebra is generated by the creation-annihilation operators and the reflection operator, which satisfies commutation and anti-commutation relations [23]. In this formulation, it is possible to obtain a self-adjoint momentum operator on the semi-axis [24], which can be applied to study systems with singularity in Calogero-like model [25, 26]. Furthermore, this algebra leads to the parastatistical description of physical systems, which corresponds to generalization of the Bose-Einstein and Fermi-Dirac statistics [27–29].

According to the order of the deformation parameter, also known as the order of statistics, it is possible to describe para-Bose or parafermion particles, see also [30]. Although parastatistics cannot be applied to the particles described by the standard model [31], their
distinctive approach has allowed presenting promising proposals, for instance, in the conjecture of candidate particles to explain dark matter \[32\], in paraquark models description \[33\], in the study of thermodynamics properties of para-Bose systems \[34\], and in optical physics by simulation of para-Fermi oscillators \[35\]. By considering the para-Bose formulation, we will construct integrals of motion, i.e., operators that commute with the Schrödinger’s operator \[36\] in the form of a Bogoliubov transformation \[37\].

In the para-Bose formulation \[38\], the CS were obtained as eigenstates of the annihilation operator that satisfy the characteristic commutation relation of the algebra \[39\] – see \[40–42\]. Recently, some works have paid attention to this topic, for instance, the nonlinear CS problem \[43\], the construction of new types of para-Bose states \[44\], and the study of “Schrödinger cat states” \[45\]. In this context, we will obtain the time-dependent CS via integrals of motion, which, in turn, we can identify as the SCS. Our construction has potential application to describe the recent proposed experimental realization of para-particles \[46, 47\].

This paper is structured as follows. In Sec. \[\text{II}\] the integrals of motion method in the context of Wigner-Heisenberg algebra will be formulated. In Sec. \[\text{III}\] the time-dependent SS in terms of the time-independent para-Bose number states will be constructed. In turn, the completeness relation and probability transition plots will be obtained. In Sec. \[\text{IV}\] a generalization of the time-dependent CS in terms of the Wigner parameter will be constructed. Then, the probability transition graph will be shown, and the mean values and the uncertainty relations will be calculated. In Sec. \[\text{V}\] the generalized CS in the coordinate representation will be considered. The concluding remarks are presented in Sec. \[\text{VI}\].

\section{II. INTEGRALS OF MOTION VIA WIGNER-HEISENBERG ALGEBRA}

The motion integral method consists of building a time-dependent operator, which commutes with the Schrödinger operator. In turn, the eigenstates of these operators are obtained and imposed to satisfy the Schrödinger equation. Here, our contribution will be reformulate the integrals of motion method according to the Wigner-Heisenberg algebra.

The Wigner-Heisenberg algebra is composed by the annihilation $\hat{a}$, creation $\hat{a}^\dagger$ and reflection $\hat{R}$ operators, satisfying commutation $\left[\hat{b}, \hat{c}\right] = \hat{b}\hat{c} - \hat{c}\hat{b}$ and anti-commutation
\[
\{\hat{b}, \hat{c}\} = \hat{b}\hat{c} + \hat{c}\hat{b}, \quad [\hat{a}, \hat{a}^\dagger] = 1 + \nu\hat{R}, \quad \left\{\hat{R}, \hat{a}\right\} = 0 = \left\{\hat{R}, \hat{a}^\dagger\right\}, \quad \hat{R}^2 = 1, \tag{1}
\]

where \(\nu\) is the well-known Wigner parameter associated with the fundamental energy level \(\varepsilon\) of the system defined by

\[
\nu = 2\varepsilon - 1. \tag{2}
\]

For the sake of simplicity, we will consider a Hamiltonian which commutes with the reflection operator, and in that way, the eigenstates of \(H\) can be even or odd. In this sense, the most general form for a one-dimensional time-dependent quadratic Hamiltonian is given by

\[
\hat{H} = \frac{1}{2}\hbar \left(\alpha^* \hat{a}^2 + \alpha \hat{a}^\dagger_2\right) + \frac{1}{2}\hbar\beta \left(\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger\right) + \hbar\delta, \tag{3}
\]

where \(\alpha, \beta\) and \(\delta\) are time-dependent functions and the signs \(\dagger\) and \(*\) denote Hermitian and complex conjugation, respectively. From the hermiticity condition, we have that \(\beta\) and \(\delta\) must be real functions. By assuming the condition \(\beta > |\alpha|\), we get that the Hamiltonian \(\hat{H}\) is positive definite, i.e., it can be written in the form of an oscillator-type Hamiltonian.

In its turn, the quantum states \(|\Psi\rangle\) which describe the time evolution of the system should satisfy the Schrödinger’s equation

\[
\hat{A} |\Psi\rangle = 0, \quad \hat{A} = \frac{i}{\hbar} \hat{H} + \partial_t, \quad \partial_t = \frac{\partial}{\partial t}, \tag{4}
\]

where \(\hat{A}\) we call of Schrödinger’s operator or equation operator.

Let us consider a time-dependent operator \(\hat{A} = \hat{A}(t)\), as a linear combination of the annihilation \(\hat{a}\) and creation \(\hat{a}^\dagger\) para-Bose operators, in the form

\[
\hat{A} = f\hat{a} + g\hat{a}^\dagger + \varphi, \tag{5}
\]

where \(f = f(t)\), \(g = g(t)\) and \(\varphi = \varphi(t)\) are time-dependent complex functions. The Eq. \([5]\) is the well-known Bogoliubov’s transformation; those coefficients have been studied in Ref. \([37]\).

In order to \(\hat{A}\) to be an integral of motion, it has to commute with the Schrödinger’s operator \([4]\), that is

\[
\hat{A} = \hat{A} = \left[\hat{A}, \hat{A}\right] = 0, \quad \hat{A} = \frac{d\hat{A}}{dt}. \tag{6}
\]
Substituting the Eqs. (3), (4) and (5) into (6), we obtain the following equations for the functions \( f, g \) and \( \varphi \):

\[
\dot{f} = i (\beta f - \alpha^* g), \quad \dot{g} = i (f \alpha - \beta g), \quad \dot{\varphi} = 0,
\]

where \( \varphi(t) = \varphi_0 \) is a constant function for any instant of time.

The commutator between \( \hat{A} \) and \( \hat{A}^\dagger \) reads

\[
[\hat{A}, \hat{A}^\dagger] = \mu \left( 1 + \nu \hat{R} \right), \quad \mu = |f|^2 - |g|^2 = |f_0|^2 - |g_0|^2.
\]

where \( f_0 = f(0) \) and \( g_0 = g(0) \) are initial conditions.

It follows from (5) and (8), that

\[
\dot{a} = \frac{f^* \hat{A} - g \hat{A}^\dagger + u}{\mu}, \quad u = g \varphi_0^* - f^* \varphi_0.
\]

It is worth highlighting which the \( \mu \)-parameter will be useful to study special cases where the Hamiltonian presents linear terms.

### III. TIME-DEPENDENT SS

The SS are pure states known to be one of the most useful nonclassical states, see for example, [48, 49]. In addition, these states make possible to obtain measurements of a physical quantity with a smaller standard deviation than those obtained for the CS. In this section, we will obtain the SS in the para-Bose formulation. Thus, assuming the condition

\[
\varphi_0 = 0 \implies u = 0,
\]

one can obtain the SS in a nonunitary approach [30]. In what follows we will apply this condition.

#### A. Para-Bose number states

In this subsection, we will recall some properties of the para-Bose states, since the SS can be expanded in these states. It is well-known that the para-Bose number states \( |n, \varepsilon\rangle \) form a complete set and are orthonormal, i.e.,

\[
\sum_{n=0}^{\infty} |n, \varepsilon\rangle \langle \varepsilon, n| = 1, \quad \langle \varepsilon, n|m, \varepsilon\rangle = \delta_{nm}.
\]
as can be seen in [39, 40].

Taking into account that the vacuum state has parity even \( \hat{R} |0, \varepsilon\rangle = |0, \varepsilon\rangle \), the action of the generators of the Wigner-Heisenberg algebra in \(|n, \varepsilon\rangle\), reads

\[
\hat{a} |2n, \varepsilon\rangle = \sqrt{2n} |2n-1, \varepsilon\rangle, \quad \hat{a}^\dagger |2n, \varepsilon\rangle = \sqrt{2(n+\varepsilon)} |2n+1, \varepsilon\rangle,
\]

\[
\hat{a} |2n+1, \varepsilon\rangle = \sqrt{2(n+\varepsilon)} |2n, \varepsilon\rangle, \quad \hat{a}^\dagger |2n+1, \varepsilon\rangle = \sqrt{2(n+1)} |2n+2, \varepsilon\rangle,
\]

\[
\hat{n} |n, \varepsilon\rangle = \hat{a}^\dagger \hat{a} |n, \varepsilon\rangle = n |n, \varepsilon\rangle, \quad \hat{R} |n, \varepsilon\rangle = (-1)^n |n, \varepsilon\rangle, \quad n = 0, 1, 2, 3, \ldots
\]

(12)

From a recurrence relation, one can write the number states \(|n, \varepsilon\rangle\) in terms of the vacuum state \(|0, \varepsilon\rangle\), in the form

\[
|2n, \varepsilon\rangle = \sqrt{\frac{\Gamma(\varepsilon)}{2^{2n} n! \Gamma(n+\varepsilon)}} (\hat{a}^\dagger)^{2n} |0, \varepsilon\rangle,
\]

\[
|2n+1, \varepsilon\rangle = \sqrt{\frac{\Gamma(\varepsilon)}{2^{2n+1} (n+\varepsilon+1)! \Gamma(n+\varepsilon+1)}} (\hat{a}^\dagger)^{2n+1} |0, \varepsilon\rangle.
\]

(13)

**B. Time-dependent SS via Para-Bose number states**

In what follows, we aim to apply the nonunitary approach, as in the recent publication [36], to construct the SS. From condition (10), one can write the nonunitary operator \( \hat{S} \), in the form

\[
\hat{S} = \exp \left( \frac{1}{2} \zeta \hat{a}^\dagger \hat{a} \right), \quad \zeta = \frac{g}{f}, \quad \dot{\zeta} = i \alpha^* \zeta^2 - 2i \beta \zeta + i \alpha,
\]

(14)

such that the commutators from Baker-Campbell-Hausdorff relation of the second order onwards become null. Here, \( \zeta = \zeta(t) \) represents the squeeze parameter for the SS.

Applying the Baker–Campbell–Hausdorff theorem, we can express the canonical operator \( \hat{a} \) in terms of the integrals of motion \( \hat{A} \), as follows

\[
\hat{a} = \frac{1}{f} \hat{S} \hat{A} \hat{S}^{-1}.
\]

(15)

The application from (15) on the vacuum state \(|0, \varepsilon\rangle\), which satisfies the annihilation condition \( \hat{a} |0, \varepsilon\rangle = 0 \), yields:

\[
\hat{A} |\zeta\rangle = 0,
\]

(16)

whose general solution is given by

\[
|\zeta\rangle = \Phi \exp \left( -\frac{1}{2} \zeta \hat{a}^\dagger \hat{a} \right) |0, \varepsilon\rangle,
\]

(17)
where \( \Phi = \Phi (t) \) is an arbitrary function, which will be determined such that the states \( |\zeta\rangle \) satisfy the Schrödinger’s equation (4).

Substituting the states (17) into (4), we obtain the following equation for \( \Phi \):

\[
\dot{\Phi} = \frac{1}{2} \frac{\langle \zeta | \hat{a}^{\dagger 2} | \zeta \rangle}{\langle \zeta | \zeta \rangle} \dot{\zeta} - i \frac{\langle \zeta | \hat{H} | \zeta \rangle}{\hbar} \frac{\langle \zeta | \zeta \rangle}{\langle \zeta | \zeta \rangle}.
\] (18)

Using the representation (9) together with the condition (16), one can easily calculate the mean values in (18), with \( \dot{\zeta} \) given in (14). Thus, the general solution from (18), reads

\[
\Phi = \frac{C f}{\mu^\varepsilon} \exp \left( -i \int \delta dt \right),
\] (19)

where \( C \) is a real normalization constant. Taking into account the normalization condition, we find that:

\[
\langle \zeta | \zeta \rangle = 1 \Rightarrow C = \mu^{\varepsilon/2}.
\] (20)

Then, the normalized states \( |\zeta\rangle \) that satisfy the Schrödinger’s equation are given by

\[
|\zeta\rangle = \frac{\sqrt{\mu^\varepsilon}}{f^\varepsilon} \exp \left( -i \int \delta dt \right) \exp \left( \frac{1}{2} \frac{\zeta^* \hat{a}^{\dagger 2}}{\varepsilon} \right) |0, \varepsilon\rangle
\]

\[
= \frac{\sqrt{\mu^\varepsilon}}{f^\varepsilon} \exp \left( -i \int \delta dt \right) \sum_{n=0}^{\infty} (-\zeta)^n \sqrt{\frac{\Gamma (n + \varepsilon)}{n! \Gamma (\varepsilon)}} |2n, \varepsilon\rangle.
\] (21)

On the other hand, we can express \( f \) and \( \mu \) in terms of squeeze parameter in the form:

\[
f = f_0 \exp \left[ -i \int (\alpha^* \zeta - \beta) dt \right], \quad \mu = |f|^2 \left( 1 - |\zeta|^2 \right).
\] (22)

From here, the states \( |\zeta\rangle \) take the form

\[
|\zeta\rangle = (1 - |\zeta|^2)^{\frac{\varepsilon}{2}} \varepsilon e^{i\vartheta} \sum_{n=0}^{\infty} (-\zeta)^n \sqrt{\frac{\Gamma (n + \varepsilon)}{n! \Gamma (\varepsilon)}} |2n, \varepsilon\rangle,
\] (23)

where the phase \( \vartheta \) is given by

\[
\vartheta = \int [\varepsilon \Re (\alpha \zeta^*) - \varepsilon \beta - \delta] dt.
\] (24)

In what follows, we call the time-dependent states (23) para-Bose SS.

The overlap of two SS with different \( \zeta \), for example \( \langle \zeta_1 | \zeta_2 \rangle \) reads

\[
\langle \zeta_1 | \zeta_2 \rangle = \frac{(1 - |\zeta_1|^2)^{\frac{\varepsilon}{2}} (1 - |\zeta_2|^2)^{\frac{\varepsilon}{2}}}{(1 - \zeta_1^* \zeta_2)^{\varepsilon}} \exp \left\{ i\varepsilon \int \Re [\alpha (\zeta_2^* - \zeta_1^*)] dt \right\}.
\] (25)
In turn, the probability transition $P_{2n}(\zeta, \varepsilon) = |\langle 2n, \varepsilon | \zeta \rangle|^2$ is given by

$$P_{2n}(\zeta, \varepsilon) = \frac{(1 - |\zeta|^2)^\varepsilon \Gamma(n + \varepsilon)}{\Gamma(\varepsilon)} \frac{|\zeta|^{2n}}{n!}.$$  \hspace{1cm} (26)

The probability distribution (26) has been plotted in Fig. 1 by considering some fixing values for the parameters $\zeta$ and $\varepsilon$.

C. Completeness relation on squeeze parameter

Considerations on the completeness relation for the SS and squeezed odd number states, in the context of canonical algebra, can be seen in [50, 51]. Here, our contribution is to
obtain the completeness relation in the context of the Wigner-Heisenberg algebra, and as will be seen, its existence depends on the \( \varepsilon \)-parameter.

Let us consider a weight function \( w(\zeta) \) such that the states \( |\zeta\rangle \) lead to the following closure relation

\[
\int_\mathbb{C} |\zeta\rangle \langle \zeta| w(\zeta) d^2\zeta = 1.
\]  

(27)

This relation ensures that the states \( |\zeta\rangle \) have a P-representation \([52]\), which allows studying classical properties of the fields represented by such states \([53]\).

In turn, substituting the states (23) into Eq. (27), we find

\[
\sum_{n,m=0}^{\infty} |2n,\varepsilon\rangle \langle \varepsilon,2m| \frac{\Gamma(n+\varepsilon)\Gamma(m+\varepsilon)}{n!\Gamma(\varepsilon) \cdot m!\Gamma(\varepsilon)} \int_\mathbb{C} (1-|\zeta|^2)^\varepsilon (-\zeta)^n (-\zeta^*)^m w(\zeta) d^2\zeta = 1.
\]  

(28)

Now, by using polar coordinates in the above relation,

\[
\zeta = r\zeta e^{i\theta}, \quad d^2\zeta = r\zeta dr\zeta d\theta, \quad 0 \leq r \zeta < 1, \quad 0 \leq \theta \zeta \leq 2\pi,
\]  

(29)

we get

\[
\sum_{n,m=0}^{\infty} |2n,\varepsilon\rangle \langle \varepsilon,2m| \frac{\Gamma(n+\varepsilon)\Gamma(m+\varepsilon)}{n!\Gamma(\varepsilon) \cdot m!\Gamma(\varepsilon)} \int_0^1 \int_0^{2\pi} (1-r^2\zeta)\varepsilon (-r\zeta)^n (-r^2 \zeta)^m \times w(\zeta) \exp \left[ i(n-m)\theta \zeta \right] r\zeta dr\zeta d\theta = 1.
\]  

(30)

On the other hand, we may readily show that \( w(\zeta) = w(r\zeta) \), which implies that the integral on \( \theta \zeta \) becomes

\[
\int_0^{2\pi} d\theta \zeta \exp \left[ i(n-m)\theta \zeta \right] = 2\pi \delta_{nm}.
\]  

(31)

From here, we can write (30) in the form

\[
2\pi \sum_{n=0}^{\infty} |2n,\varepsilon\rangle \langle \varepsilon,2n| \frac{\Gamma(n+\varepsilon)}{n!\Gamma(\varepsilon)} \int_0^1 (1-r^2\zeta)^\varepsilon r^{2n+1}\zeta w(r\zeta) dr\zeta = 1.
\]  

(32)

By using the following relationship among gamma functions

\[
\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2 \int_0^1 (1-t^2)^{y-1} t^{2x-1} dt, \quad \text{Re}(x) > 0, \quad \text{Re}(y) > 0,
\]  

(33)

one can see that the weight function \( w(r\zeta) \) must have the form

\[
w(r\zeta) = \frac{\Gamma(\varepsilon)}{\pi \Gamma(\varepsilon-1) (1-r^2\zeta)^2} = \frac{\varepsilon-1}{\pi (1-r^2\zeta)^2}, \quad \varepsilon > 1,
\]  

(34)
in order to ensure that the completeness relation (27) is satisfied. It is important to highlight that the condition \( \varepsilon > 1 \) leads to a positive weight function \( w(r_\zeta) \). At the same time, this condition shows that the canonical algebra \( (\varepsilon = 1/2) \) does not allow to obtain a completeness relation for the SS. It follows from (25) and (27) that these states form an overcomplete set of states on the Hilbert space. The plot of weight function can be seen in the Fig. 2.

IV. TIME-DEPENDENT GENERALIZED CS

In Ref. [36], the SCS has been built through the nonunitary approach by considering the canonical algebra. In another way, by considering the Wigner-Heisenberg algebra, it is not possible to obtain the SCS since, from Baker–Campbell–Hausdorff theorem, we cannot establish a direct relationship between the para-Bose operators and integral of motion, due the deformed commutation relation. However, instead of following the nonunitary approach, we will construct the eigenstates of the integral of motion \( \hat{A} \), in the form

\[
\hat{A} |z,t\rangle = z |z,t\rangle ,
\]  

where \( z \) is a complex constant. The Eq. (35) allows us to assume

\[
\varphi_0 = 0 \implies u = 0 ,
\]  

without loss of generality.
The states $|z, t\rangle$ can be expanded in terms of the para-Bose number states, as follows

$$|z, t\rangle = \sum_{n=0}^{\infty} c_n |n, \varepsilon\rangle = \sum_{n=0}^{\infty} \left( c_{2n} |2n, \varepsilon\rangle + c_{2n+1} |2n + 1, \varepsilon\rangle \right),$$

where $c_n = c_n(t)$ are time-dependent coefficients and will be determined such that the Eqs. (35) and (4) will be satisfied. Substituting (5), (12) and (37) into (35), we find the following equations for the coefficients

$$c_{2n} = \sqrt{\frac{n! \Gamma(\varepsilon)}{\Gamma(n + \varepsilon)}} \left( -\frac{g}{f} \right)^n \frac{L_n^2}{2 \Gamma(n + \varepsilon + 1)} \left( \frac{z^2}{2gf} \right)^n c_0,$$  
$$c_{2n+1} = \frac{z}{f} \sqrt{\frac{n! \Gamma(\varepsilon)}{\Gamma(n + \varepsilon + 1)}} \left( \frac{g}{f} \right)^n \frac{L_n^2}{2 \Gamma(n + \varepsilon + 1)} \left( \frac{z^2}{2gf} \right)^n c_0,$$

where $L_n^\alpha(x)$ are the associated Laguerre polynomials and $c_0$ is a time-dependent function, which will be determined such that the states $|z, t\rangle$ satisfy the Schrödinger’s equation (4).

Here, it is convenient to introduce the squeeze $\zeta = g/f$ and displacement $\xi = z/f$ parameters to rewrite the states $|z, t\rangle \rightarrow |\zeta, \xi\rangle$, in the form

$$|\zeta, \xi\rangle = \sqrt{\Gamma(\varepsilon)} c_0 \sum_{n=0}^{\infty} (-\zeta)^n \sqrt{n!} \left[ \frac{L_n^{-1} \left( \frac{\xi^2}{2\varepsilon} \right)}{\sqrt{\Gamma(n + \varepsilon)}} |2n, \varepsilon\rangle + \frac{\xi L_n^\xi \left( \frac{\xi^2}{2\varepsilon} \right)}{\sqrt{2\Gamma(n + \varepsilon + 1)}} |2n + 1, \varepsilon\rangle \right],$$

with $\zeta$ and $\xi$ satisfy the following differential equations:

$$\dot{\zeta} = i\alpha^* \xi^2 - 2i\beta \zeta + i\alpha, \quad \dot{\xi} = i(\alpha^* \zeta - \beta) \xi.$$  

From normalization condition

$$\langle\xi, \zeta|\zeta, \xi\rangle = 1,$$

we found the following form for the function $c_0$.

$$c_0 = \left( \frac{\xi}{\sqrt{2}} \right)^{\varepsilon-1} \sqrt{\frac{1 - |\zeta|^2}{\Gamma(\varepsilon)}} \exp \left\{ \frac{\zeta^* \xi^2}{2(1 - |\zeta|^2)} + \frac{i}{\varepsilon} \int [\text{Re}(\alpha \zeta^*) - \beta] \, dt + i\phi \right\},$$

where $I_n(z)$ is the modified Bessel function of the first kind, and $\phi$ is a time-dependent real function. Thus, the normalized states $|\zeta, \xi\rangle$ take the form

$$|\zeta, \xi\rangle = \left( \frac{\xi}{\sqrt{2}} \right)^{\varepsilon-1} \sqrt{\frac{1 - |\zeta|^2}{I_{\varepsilon-1} \left( \frac{|\xi|^2}{1 - |\zeta|^2} \right) + I_{\varepsilon} \left( \frac{|\xi|^2}{1 - |\zeta|^2} \right)}} \exp \left\{ \frac{\zeta^* \xi^2}{2(1 - |\zeta|^2)} + \frac{i}{\varepsilon} \int [\text{Re}(\alpha \zeta^*) - \beta] \, dt + i\phi \right\} \times \sum_{n=0}^{\infty} (-\zeta)^n \sqrt{n!} \left[ \frac{L_n^{-1} \left( \frac{\xi^2}{2\varepsilon} \right)}{\sqrt{\Gamma(n + \varepsilon)}} |2n, \varepsilon\rangle + \frac{\xi L_n^\xi \left( \frac{\xi^2}{2\varepsilon} \right)}{\sqrt{2\Gamma(n + \varepsilon + 1)}} |2n + 1, \varepsilon\rangle \right],$$

$$\ldots$$
Substituting $|\zeta, \xi\rangle$ into Schrödinger’s equation, we find the following expression for $\phi$,

$$\phi = -\int \delta dt.$$ (44)

Therefore, the normalized states $|\zeta, \xi\rangle$ that satisfy the Schrödinger’s equation are given by

$$|\zeta, \xi\rangle = \left(\frac{\xi}{\sqrt{2}}\right)^{\varepsilon-1} \sqrt{I_{\varepsilon-1}(|\zeta|^2) + I_{\varepsilon}(|\zeta|^2)} \exp\left[\frac{\zeta^* \xi^2}{2 (1 - |\zeta|^2)} + i\tilde{\vartheta}\right]$$

$$\times \sum_{n=0}^{\infty} (-\zeta)^n \sqrt{n!} \left[\frac{L_{n}^{\varepsilon-1} \left(\frac{\xi^2}{2}\right)}{\Gamma(n + \varepsilon)} |2n, \varepsilon\rangle + \frac{\xi L_{n}^{\varepsilon} \left(\frac{\xi^2}{2}\right)}{\sqrt{2\Gamma(n + \varepsilon + 1)}} |2n + 1, \varepsilon\rangle\right],$$ (45)

where

$$\tilde{\vartheta} = \int [\text{Re}(\alpha \zeta^*) - \beta - \delta] dt.$$ (46)

In the following, we call the time-dependent states (45) generalized para-Bose CS.

In particular, taking into account the condition $\zeta = 0 \implies \alpha = 0$, the states (45) take the form

$$|\xi\rangle = \left(\frac{\xi}{\sqrt{2}}\right)^{\varepsilon-1} \frac{\exp[-i\int (\beta + \delta) dt]}{\sqrt{I_{\varepsilon-1}(|\xi|^2) + I_{\varepsilon}(|\xi|^2)}} \sum_{n=0}^{\infty} \left(\frac{\xi^2}{2}\right)^n \left[\frac{|2n, \varepsilon\rangle}{\sqrt{n!\Gamma(n + \varepsilon)}} + \frac{\xi |2n + 1, \varepsilon\rangle}{\sqrt{2n!\Gamma(n + \varepsilon + 1)}}\right]$$

$$= \sqrt{\Gamma(\varepsilon)} \exp\left[-i\int (\beta + \delta) dt\right] \left(\frac{\hat{a}^\dagger}{\sqrt{2}}\right)^{1-\varepsilon} \frac{I_{\varepsilon-1}(|\xi\hat{a}\rangle + I_{\varepsilon}(|\xi\hat{a}\rangle)}{\sqrt{I_{\varepsilon-1}(|\xi|^2) + I_{\varepsilon}(|\xi|^2)}} \langle 0, \varepsilon\rangle,$$ (47)

with $\dot{\xi} = -i\beta \xi$ and $|\xi\rangle = |0, \xi\rangle$. These states correspond to the para-Bose CS obtained in [39], which satisfy the Schrödinger equation. In turn, we have that the condition $\varepsilon = 1/2$ reduces the states (47) to the form:

$$|\xi\rangle = \exp\left(\int \frac{\beta + 2\delta}{2i} dt - \frac{|\xi|^2}{2}\right) \exp (\xi \hat{a}^\dagger) |0\rangle, \quad |0\rangle = |0, \varepsilon = 1/2\rangle,$$ (48)

which are the well-known time-dependent canonical CS.

The overlap of the states $|\zeta, \xi\rangle$ for different squeeze $\zeta$ and displacement $\xi$ parameters is given by

$$\langle \xi_1, \zeta_1 | \xi, \zeta \rangle = \left(\frac{\xi \xi_1}{2}\right)^{\varepsilon-1} \sqrt{I_{\varepsilon-1}(|\xi|^2) + I_{\varepsilon}(|\xi|^2)} \left(1 - |\zeta|^2\right) \left(1 - |\xi|^2\right)$$

$$\times \exp\left[\frac{\zeta^* \xi^2}{2} \frac{(1 - |\xi|^2)(1 - |\zeta|^2)}{1 - |\xi|^2} + \zeta \xi \xi_1^2 \frac{(1 - |\xi|^2)(1 - |\zeta|^2)}{1 - |\xi|^2}\right] + i \int \left(\text{Re}(\alpha \zeta^* - \alpha \zeta_1^*) dt\right)$$

$$\times \sum_{n=0}^{\infty} (\zeta_1)^n \sqrt{n!m!} \left[\frac{L_{n}^{\varepsilon-1} \left(\frac{\xi^2}{2}\right)}{\Gamma(n + \varepsilon)} L_{n}^{\varepsilon-1} \left(\frac{\xi_1^2}{2}\right) + \frac{\xi \xi_1 L_{n}^{\varepsilon} \left(\frac{\xi^2}{2}\right) L_{n}^{\varepsilon} \left(\frac{\xi_1^2}{2}\right)}{\Gamma(n + \varepsilon + 1)}\right].$$ (49)
Here, we can analyze the probability transition $P_n(\zeta, \xi, \varepsilon) = |\langle n, \varepsilon | \zeta, \xi \rangle|^2$ of the number states $|n, \varepsilon \rangle$ to the CS $|\zeta, \xi \rangle$:

$$P_n(\zeta, \xi, \varepsilon) = \left( \frac{|\xi|^2}{2} \right)^{\varepsilon^{-1}} \frac{(1 - |\zeta|^2)}{I_{\varepsilon-1} \left( \frac{|\xi|^2}{1-|\zeta|^2} \right) + I_{\varepsilon} \left( \frac{|\xi|^2}{1-|\zeta|^2} \right)} n! |\zeta|^2 \left[ \frac{L_n^{\varepsilon-1} \left( \frac{\xi^2}{2\zeta} \right)}{\Gamma (n + \varepsilon)} + \frac{|\xi|^2 \left| L_n^{\varepsilon} \left( \frac{\xi^2}{2\zeta} \right) \right|^2}{2\Gamma (n + \varepsilon + 1)} \right].$$

(50)

The probability transition $P_n(\zeta, \xi, \varepsilon)$ has been shown in Fig. 3. As we can see, the $\xi$-parameter allows access to odd states while the $\varepsilon$ controls the dispersion of the probability density. Thereby, the displacement parameter enables the conversion of pure squeezed vacuum into mixed states. Furthermore, higher values of the $\varepsilon$-parameter allow access to states with higher principal quantum number $n$.

A. Mean values and uncertainty relations

Let us now consider the mean value of the momentum $\hat{P}$ and position $\hat{x}$ operators, which satisfy the commutation relation

$$\left[ \hat{x}, \hat{P} \right] = i\hbar \left[ 1 + (2\varepsilon - 1) \hat{R} \right],$$

(51)

in the generalized CS (45). For this, we will rewrite these operators in terms of the integrals of motion, as seen below

$$\hat{a} = \frac{\hat{A}_f - \zeta \hat{A}^\dagger_f}{1 - |\zeta|^2}, \quad \hat{A}_f \equiv \frac{1}{\tilde{f}} \hat{A},$$

$$\hat{x} = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}} l = \frac{l (1 - \zeta^*) \hat{A}_f + (1 - \zeta) \hat{A}^\dagger_f}{\sqrt{2} (1 - |\zeta|^2)},$$

$$\hat{P} = \frac{\hbar \hat{a} - \hat{a}^\dagger}{i\sqrt{2l}} \frac{l (1 + \zeta^*) \hat{A}_f - (1 + \zeta) \hat{A}^\dagger_f}{i\sqrt{2l} (1 - |\zeta|^2)},$$

(52)

where $l$ is a length-dimensional parameter. The new operator $\hat{A}_f$ acts on states $|\zeta, \xi \rangle$ as follows

$$\hat{A}_f |\zeta, \xi \rangle = \xi |\zeta, \xi \rangle.$$

(53)
Using the relations (52) and (53), we can easily calculate the mean values of the operators \( \hat{x} \) and \( \hat{P} \),

\[
\bar{x} = \bar{x}(t) = \langle \xi, \zeta | \hat{x} | \xi, \zeta \rangle = \sqrt{2l} \frac{\text{Re} \left[ (1 - \zeta^*) \xi \right]}{1 - |\zeta|^2},
\]

\[
P = P(t) = \langle \xi, \zeta | \hat{P} | \xi, \zeta \rangle = \frac{\sqrt{2l} \text{Im} \left[ (1 + \zeta^*) \xi \right]}{1 - |\zeta|^2}.
\]

From here, there is a correspondence between the squeeze \( \zeta \) and displacement \( \xi \) parameters with the mean values of \( \bar{x} \) and \( \bar{P} \),

\[
\xi = \frac{1 + \zeta}{\sqrt{2l}} \bar{x} + \frac{i l}{\hbar \sqrt{2}} \bar{P}.
\]
Taking the square of the operators \( \hat{x} \) and \( \hat{P} \), we have
\[
\hat{x}^2 = \frac{l^2}{2} \left( (1 - \zeta^*)^2 \hat{A}_\ell^2 + (1 - \zeta^2) \hat{A}_\ell^2 + |1 - \zeta|^2 \left( 2\hat{A}_\ell + \hat{R}_{\ell,0} \right) \right),
\]
\[
\hat{P}^2 = -\frac{\hbar^2}{2l^2} \left( (1 + \zeta^*)^2 \hat{A}_\ell^2 + (1 + \zeta^2) \hat{A}_\ell^2 - |1 + \zeta|^2 \left( 2\hat{A}_\ell + \hat{R}_{\ell,1} \right) \right),
\]
\[
\hat{R}_{\ell,1} = \frac{(2\varepsilon - 1) \hat{R} + 1}{|f|^2}. \tag{56}
\]
So, from these results, we can calculate the mean values of \( \bar{x}^2 \) and \( \bar{P}^2 \), as follow
\[
\bar{x}^2 = \bar{x}^2(t) = \langle \xi, \zeta \mid \hat{x}^2 \mid \xi, \zeta \rangle = \frac{2l^2 \text{Re}^2 [(1 - \zeta^*) \xi]}{(1 - |\zeta|^2)^2} + \frac{l^2 |1 - \zeta|^2}{2(1 - |\zeta|^2)^2} \bar{R}_{\ell,1},
\]
\[
\bar{P}^2 = \bar{P}^2(t) = \langle \xi, \zeta \mid \hat{P}^2 \mid \xi, \zeta \rangle = \frac{2\hbar^2 \text{Im}^2 [(1 + \zeta^*) \xi]}{l^2 (1 - |\zeta|^2)^2} + \frac{\hbar^2 |1 + \zeta|^2}{2l^2 (1 - |\zeta|^2)^2} \bar{R}_{\ell,1}, \tag{57}
\]
where
\[
\bar{R}_{\ell,1} = \frac{(2\varepsilon - 1) \bar{R} + 1}{|f|^2}, \quad \bar{R} = \left\langle \xi, \zeta \mid \hat{R} \right| \xi, \zeta \rangle = \frac{I_{\varepsilon-1} \left( |\xi|^2 \right)}{I_{\varepsilon-1} \left( |\xi|^2 \right) + I_\varepsilon \left( |\xi|^2 \right)}. \tag{58}
\]
Note that \( \xi = 0 \) leads to a well-defined even parity of the states \( |45\rangle \), once \( \bar{R} = 1 \).

In what follows, we calculate the standard deviation, i.e.,
\[
\sigma_x = \sigma_x(t) = \sqrt{\bar{x}^2(t) - (\bar{x}(t))^2} = \frac{l}{2} |1 - \zeta| \sqrt{1 + \frac{(2\varepsilon - 1) \bar{R}}{2(1 - |\zeta|^2)}},
\]
\[
\sigma_P = \sigma_P(t) = \sqrt{\bar{P}^2(t) - (\bar{P}(t))^2} = \frac{\hbar}{l} |1 + \zeta| \sqrt{1 + \frac{(2\varepsilon - 1) \bar{R}}{2(1 - |\zeta|^2)}}. \tag{59}
\]
From Eqs. (59), it is easy to see that standard deviations in position and momentum present the squeezing property.

Finally, we aim to find the Heisenberg uncertainty relation in the generalized states \( |45\rangle \) by considering the Wigner-Heisenberg algebra. Therefore, in the hold of previous results, we can explicitly calculate the product \( \sigma_x\sigma_P \), as shown below:
\[
\sigma_x\sigma_P = \hbar \sqrt{\frac{|1 - \zeta| |1 + \zeta| + (2\varepsilon - 1) \bar{R}}{1 - |\zeta|^2}} \sqrt{1 + \frac{4 \text{Im}^2 (\zeta)}{(1 - |\zeta|^2)^2}} \frac{1 + (2\varepsilon - 1) \bar{R}}{2}. \tag{60}
\]
In particular, if we consider that the squeeze parameter assumes real values, this implies that \( \sigma_x\sigma_P = \hbar/2 \left( 1 + (2\varepsilon - 1) \bar{R} \right) \). In turn, for this particular choice, we should note that
the Heisenberg uncertainty assumes the minimum value predicted by the relation
\[
\left\langle \left( \Delta \hat{A} \right)^2 \right\rangle \left\langle \left( \Delta \hat{B} \right)^2 \right\rangle \geq \frac{1}{4} \left| \left\langle [\hat{A}, \hat{B}] \right\rangle \right|^2. \tag{61}
\]

It must be highlighted that in Ref. [40], a detailed analysis has been applied to show how the para-Bose uncertainty relation leads to canonical relation.

Taking into account the covariance \( \sigma_{xP} \),
\[
\sigma_{xP} = \frac{\left\langle \zeta, \xi \right| \hat{P} \hat{x} \left| \xi, \zeta \right\rangle + \left\langle \zeta, \xi \right| \hat{x} \hat{P} \left| \xi, \zeta \right\rangle}{2} - \bar{x}(t) \bar{P}(t) = -\hbar \text{Im} \left( \zeta \right) \frac{1 + (2\varepsilon - 1) \bar{R}}{1 - |\zeta|^2}, \tag{62}
\]
we can calculate the Schrödinger-Robertson uncertainty relation [54],
\[
\sigma_x^2 \sigma_P^2 - \sigma_{xP}^2 = \frac{\hbar^2}{4} \left[ 1 + (2\varepsilon - 1) \bar{R} \right]^2, \tag{63}
\]
which, as we can see, is minimized.

V. COORDINATE REPRESENTATION OF THE GCS

It is well known that the phase space in the context of quantum mechanics encounters difficulties due to the uncertainty principle. Considering the states \( |\zeta, \xi\rangle \) in a coordinate representation allow us to introduce the quasiprobability Wigner distribution, which plays an analogous role to the classical distributions [55–57].

According to Wigner-Heisenberg algebra, \( \hat{P} \) is a self-adjoint operator on semi-axis \( (x \geq 0) \), see [24], and have the following form:
\[
\hat{P} = -i\hbar \partial_x + \frac{i\hbar}{2x} (2\varepsilon - 1) \hat{R}. \tag{64}
\]

In this case, the annihilation operator takes the form
\[
\hat{a} = \frac{1}{\sqrt{2}} \left( \frac{\hat{x}}{l} + \frac{il}{\hbar} \hat{P} \right) = \frac{l}{\sqrt{2}} \left( \partial_x - \frac{2\varepsilon - 1}{2x} \hat{R} + \frac{x}{l^2} \right). \tag{65}
\]

Applying the annihilation condition
\[
\hat{a} \Psi_{0,\varepsilon} (x) = 0, \quad \Psi_{0,\varepsilon} (x) = \langle x | 0, \varepsilon \rangle, \tag{66}
\]
we can obtain a differential equation for the vacuum state, as follows:
\[
\left( \partial_x + \frac{x}{l^2} - \frac{2\varepsilon - 1}{2x} \right) \Psi_{0,\varepsilon} (x) = 0, \quad \hat{R} \Psi_{0,\varepsilon} (x) = \Psi_{0,\varepsilon} (x). \tag{67}
\]
The general solution reads
\[ \Psi_{0,\varepsilon}(x) = C x^{-\frac{1}{2}} \exp \left( -\frac{x^2}{2l^2} \right), \]  
with \( C \) being a real constant, which will be determined through the normalization condition, as seen below,
\[ 2C^2 \int_0^\infty x^{2\varepsilon-1} \exp \left( -\frac{x^2}{l^2} \right) dx = 1 \Rightarrow C = \frac{1}{l^\varepsilon \sqrt{\Gamma(\varepsilon)}}. \]  
It should be noted that the condition (67) leads to the following quantization condition:
\[ \varepsilon = 2\ell + \frac{1}{2}, \quad \ell = 0, 1, 2, \ldots, \]  
where \( \ell \) is analogous with the angular momentum. Such quantization was obtained in [26] to ensure that the eigenfunctions of \( \hat{P} \) are differentiable at the origin.

Thus, the vacuum state with parity even \( \Psi_{0,\varepsilon}(x) \equiv \Psi_{0,\ell}(x) \) takes the form
\[ \Psi_{0,\ell}(x) = \frac{1}{l^{2\ell+\frac{1}{2}} \sqrt{\Gamma(2\ell + \frac{1}{2})}} x^{2\ell} \exp \left( -\frac{x^2}{2l^2} \right). \]  
Taking into account that \( \Psi_{\zeta,\xi}(x,t) = \langle x | \zeta, \xi \rangle \) and replacing the relations (13) in the states (15), we obtain
\[ \Psi_{\zeta,\xi}(x,t) = \zeta^{2\ell-\frac{1}{2}} \sqrt{\frac{(1 - |\zeta|^2)}{\Gamma(2\ell + \frac{1}{2})}} \exp \left[ \frac{\xi \zeta^2}{2(1 - |\zeta|^2)} + i \frac{\bar{\zeta} \bar{\xi}}{2} \right] \]
\[ \times 2^{\frac{\ell-\frac{1}{2}}{2}} \sqrt{I_{2\ell-\frac{1}{2}} \left( \frac{\xi^2}{1 - |\zeta|^2} \right)} + I_{2\ell+\frac{1}{2}} \left( \frac{\xi^2}{1 - |\zeta|^2} \right) \]
\[ \times \sum_{n=0}^{\infty} \left( \frac{-\zeta \hat{a}^2}{2} \right)^n \left[ \frac{L_n^{2\ell-\frac{1}{2}} \left( \frac{\xi^2}{2\ell} \right)}{\Gamma (n + 2\ell + \frac{1}{2})} + \frac{L_n^{2\ell+\frac{1}{2}} \left( \frac{\xi^2}{2\ell} \right)}{\Gamma (n + 2\ell + \frac{3}{2})} \right] \Psi_{0,\ell}(x). \]  
Using the results below
\[ (\hat{a}^\dagger)^{2n} \Psi_{0,\ell}(x) = (-1)^n 2^n n! L_n^{2\ell-\frac{1}{2}} \left( \frac{x^2}{l^2} \right) \Psi_{0,\ell}(x), \]
\[ (\hat{a}^\dagger)^{2n+1} \Psi_{0,\ell}(x) = \frac{\sqrt{\Sigma x}}{l} (-1)^n 2^n n! L_n^{2\ell+\frac{1}{2}} \left( \frac{x^2}{l^2} \right) \Psi_{0,\ell}(x), \]
we can write (72), as follows
\[ \Psi_{\zeta,\xi}(x,t) = \langle x | \zeta, \xi \rangle = \sqrt{\frac{1 - |\zeta|^2}{1 - \zeta^2}} \sqrt{x} \left[ \frac{I_{2\ell-\frac{1}{2}} \left( \frac{\sqrt{2\xi}}{1 - \zeta} \right)}{\Gamma (2\ell - \frac{1}{2})} + I_{2\ell+\frac{1}{2}} \left( \frac{\sqrt{2\xi}}{1 - \zeta} \right) \right] \]
\[ \times \exp \left[ \frac{1 + \zeta x^2}{1 - \zeta 2l^2} - \frac{(1 - \zeta^2) \xi^2}{2 (1 - \zeta) (1 - |\zeta|^2)} + i \frac{\bar{\zeta} \bar{\xi}}{2} \right]. \]
In particular, if \( \ell = 0 \) we obtain the following result:

\[
\Psi_{\zeta, \xi}^0(x, t) = \frac{(1 - |\zeta|^2)^{1/4}}{\sqrt{\pi l (1 - \zeta)}} \exp \left[ -\frac{1}{2l^2} \frac{1 + \zeta}{1 - \zeta} \left( x - l\sqrt{2}\xi \right)^2 + \frac{(1 + \zeta^*)}{(1 - |\zeta|^2)} \frac{\xi^2}{2} - \frac{1}{2} \frac{|\xi|^2}{1 - |\zeta|^2} + i\varrho \right],
\]

\( \varrho = \frac{1}{2} \int [\text{Re}(\alpha \zeta^* - \beta) - 2\delta] \, dt. \)  

Performing the following identifications \( \zeta = g/f, \xi = z/f = -\varphi/f \) and \( \mu = |f|^2 (1 - |\zeta|^2) = 1 \) leads to the results of the recent publication \[36\].

The probability density that corresponds to the generalized CS is given by

\[
\rho_{\zeta, \xi}(x, t) = \left| \Psi_{\zeta, \xi}(x, t) \right|^2 = \frac{1 - |\zeta|^2}{|1 - \zeta|^2} \frac{x^2}{l^2} \left| I_{2l - \frac{1}{2}} \left( \frac{\sqrt{2}\xi}{1 - \zeta} \right) + I_{2l + \frac{1}{2}} \left( \frac{\sqrt{2}\xi}{1 - \zeta} \right) \right|^2 
\times \exp \left\{ -\frac{1}{2} \frac{1 - |\zeta|^2}{(1 - \zeta)(1 - \zeta^*)} x^2 - \text{Re} \left[ \frac{(1 - \zeta^*) \xi^2}{(1 - \zeta)(1 - |\zeta|^2)} \right] \right\}. \tag{76}
\]

In Fig. 4 we have obtained some plots of the probability density. As we can see, the generalized CS has a Gaussian probability density which moves in space as \( \ell \) increases. On the other hand, the displacement parameter \( \xi \) is responsible for concentrating the distribution around a maximum point of the density function. The length dimension parameter \( l \) has direct relation with the initial standard deviation \( l \geq \sqrt{2}\sigma_{x_0} \), see \[36\].

Finally, substituting \( \{52\} \) into \( \{3\} \), the Hamiltonian takes the form

\[
\dot{H} = \frac{1}{2} \hbar \left( \alpha^* \dot{a}^2 + \alpha \dot{a}^2 \right) + \frac{1}{2} \hbar \beta \left( \dot{a}^\dagger \dot{a} + \dot{a} \dot{a}^\dagger \right) + \hbar \delta 
= \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \dot{x}^2 + \frac{1}{2} \Omega \left( \hat{P} \dot{x} + \dot{x} \hat{P} \right) + \mathcal{E},
\]

\[
= -\frac{\hbar^2}{2m} \partial_x^2 - i\hbar \Omega x \partial_x + \frac{\hbar^2}{2m \omega^2} 2\ell \left( 2\ell - \hat{R} \right) + \frac{1}{2} m\omega^2 x^2 + \mathcal{E} - \frac{i\hbar \Omega}{2}, \tag{77}
\]

where the time-dependent quantities \( m, \omega, \Omega \) and \( \mathcal{E} \) reads

\[
\frac{1}{m} = \frac{l^2}{\hbar} \text{Re} \left( \beta - \alpha \right), \quad m\omega^2 = \frac{\hbar}{l^2} \text{Re} \left( \beta + \alpha \right), \quad \Omega = \text{Im} \alpha, \quad \mathcal{E} = \hbar \delta.
\]

\[
\omega^2 = \beta^2 - \text{Re}^2 \alpha. \tag{78}
\]

From Hamiltonian \( \{77\} \), we can describe the physical systems such as harmonic oscillators, which are described by confining potential \( V_{HO} \sim x^2 \). We also can establish an analogy of the potential \( V_C = \frac{\hbar^2}{2m \omega^2} 2\ell \left( 2\ell - \hat{R} \right) \) with the centrifugal potential. On the other hand, one
FIG. 4: Displayed is the probability density by assuming fixed value \( l = 3 \). In figures (a) and (b) we have kept fixed the squeezed parameter \( \zeta = 0.45 \), and the displacement parameter assumes the values \( 0.45i \) and \( 0.9i \), respectively. The figures (b) and (d) have been obtained assuming \( \zeta = 0.9 \), while the \( \xi = 0.45i \) and \( \xi = 0.9i \).

may identify the potential \( V_C \) with the conformal sector of Quantum conformal mechanics, except by the \( \hat{R} \) operator [58]. Since the reflection operator acting on even parity states leads to \( \hat{R}\psi_e = \psi_e \), it must be noticed that the potential \( V_C \) has a negative correction term owing to the reflection operator. Furthermore, the repulsiveness of the \( V_C \) potential is weakened by the correction arising from the reflection operator.

Finally, we can relate the action of the reflection operator on a particular state to the reduction of the dimensionality of space. Let us analyze the centrifugal term of the \( d \)-dimensional Laplacian, which can be written in the form [26]:

\[
\Delta^{(d)} = \frac{\partial^2}{\partial r^2} + \frac{(d-1)}{r} \frac{\partial}{\partial r} + \frac{L(L+d-2)}{r^2}.
\]  

(79)
At this point, we can establish a direct relationship between the action of the reflection operator on even parity states and the reduction of the space from three to one dimension \((d = 1)\), from the perspective of the centrifugal term by considering that \(L = 2\ell\).

VI. CONCLUDING REMARKS

In this article, we study the integrals of motion method in a para-Bose formulation. This approach generalizes the usual canonical commutation relation. In turn, we obtain a generalization of the usual SS, which admits a completeness relation in terms of the Wigner parameter. This relation depends on a range of values for the Wigner parameter, which does not include that corresponding to canonical algebra. We also obtain a generalization of the CS in terms of the even and odd time-independent para-Bose number states. These states are thoroughly determined in terms of the time-dependent squeeze and displacement parameters, as well as the Wigner parameter. In the study of the probability transition, we saw that the displacement parameter has an additional role, which is a kind of transition parameter by allowing access to the odd states of the system. Meanwhile, the Wigner parameter has the role of controlling the “dispersion,” i.e., it allows the system to access different excitation levels. We show that the minimization of the Heisenberg uncertainty relation is easily obtained by taking the real value of the squeeze parameter and that the squeezing properties can be seen from the standard deviation of the position and momentum. Taking the coordinate representation of the generalized CS, we found a quantization condition on the Wigner parameter, analogous to the quantization of the angular momentum, which arises by imposing that the parity of the vacuum state is even. This quantization condition also ensures that the eigenstates of the momentum operator are differentiable at the origin.

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