DENDROIDAL SEGAL SPACES AND $\infty$-OPERADS

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ABSTRACT. We introduce the dendroidal analogs of the notions of complete Segal space and of Segal category, and construct two appropriate model categories for which each of these notions corresponds to the property of being fibrant. We prove that these two model categories are Quillen equivalent to each other, and to the monoidal model category for $\infty$-operads which we constructed in an earlier paper. By slicing over the monoidal unit objects in these model categories, we derive as immediate corollaries the known comparison results between Joyal’s quasi-categories, Rezk’s complete Segal spaces, and Segal categories.

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INTRODUCTION

The category of dendroidal sets is an extension of that of simplicial sets, suitable for constructing nerves, not just of categories but also of (coloured) operads. It was introduced with this purpose, and with the aim of giving an inductive definition of weak higher categories, in [MW07, MW09]. This category $dSet$ of dendroidal sets carries a symmetric monoidal closed structure which is closely related to the Boardman-Vogt tensor product of operads, and the inclusion of the category $sSet$ of simplicial sets into $dSet$ can in fact be identified with the forgetful functor, from the slice (or comma) category of dendroidal sets over the unit $\eta$ of the monoidal structure back to dendroidal sets, via an explicit isomorphism of categories

\[(0.0.1) \quad dSet/\eta = sSet\]
Dendroidal sets carry a very rich homotopical structure, which we began to explore in [CMa]. For example, there is a monoidal Quillen model structure on $d\mathcal{S}et$, whose fibrant objects include all nerves of operads. In fact, these fibrant objects can be thought of as simple combinatorial models of the notion of operad-up-to-homotopy or “$\infty$-operad”. Like any Quillen model structure, this model structure on dendroidal sets induces another model structure on any slice category. Under the identification $d\mathcal{S}et/\eta = s\mathcal{S}et$ this induced model structure can be shown to coincide with the Joyal model structure on simplicial sets, whose fibrant objects are most commonly known under the name “$\infty$-categories” (and are also referred to as quasi-categories, weak Kan complexes, or inner Kan complexes [Joy02, Lur09, BV73]).

These $\infty$-categories model a notion of category-up-to-homotopy. Other ways of modelling such a notion have occurred in the literature, including the theory of Segal categories [Sim, Ber07] and of complete Segal spaces [Rez01]. The latter two concepts are both based on the much older observation that a simplicial set $X$ is the nerve of a category if and only if the canonical map

$$X_n \longrightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

sending a simplex to its one-dimensional ribbons, is an isomorphism. Indeed, Simpson and Rezk both base their theories on bisimplicial sets $X$ for which the map (0.0.2) is a weak equivalence of simplicial sets (and replacing the fibred product on the right hand side by its homotopy version). Building on the work of Simpson and Rezk, the relation between these different ways of modelling categories-up-to-homotopy was recently made precise through the work of Bergner, Joyal and Tierney, and Lurie. Indeed, Simpson’s Segal categories, Rezk’s complete Segal spaces, and Joyal’s $\infty$-categories all arise as the fibrant objects in a specific Quillen model category structure, and these different model category structures have now been related to each other by explicit Quillen equivalences [Ber07, Lur09]. Moreover, they are all Quillen equivalent to the model category of simplicial categories discovered by Bergner [Ber07], thus providing a strictification or rigidification result for each of these notions of category-up-to-homotopy.

The goal of this paper and its sequel [CMb] is to develop analogous theories of Segal operads (rather than categories) and complete dendroidal (rather than simplicial) Segal spaces, to relate these to each other and to dendroidal sets via Quillen equivalences, and to prove a strictification result for each of them by relating them to simplicial operads. By a simple slicing procedure like in (0.0.1), the earlier results just mentioned for categories-up-to-homotopy can all be recovered from our results, which can in this sense be said to be more general.

In more detail, then, we will consider the category $s\mathcal{S}et$ of simplicial objects in dendroidal sets, or what is the same, dendroidal spaces. We will define a Segal type condition on the objects of this category, based on an extension to trees of “the union of 1-dimensional ribbons in an $n$-simplex” to which we will refer as the Segal core of a tree. In Section 5 we will establish a closed model category structure on $s\mathcal{S}et$ whose fibrant objects satisfy a tree-like Segal condition involving these Segal cores, and a completeness condition like the one of Rezk, and prove (Corollary 6.7) that this model category is Quillen equivalent to our earlier model category structure on dendroidal sets [CMa]. The definitions and proofs of these results are based on some elementary observations about these Segal cores presented in Section 2 and on a characterization of the weak equivalences between $\infty$-operads as maps which are “essentially surjective and fully faithful” in a suitable sense (Theorem 5.5). The
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proof also exploits the hybrid nature of the objects of sdSet, which can be viewed alternatively as simplicial objects in one category or as dendroidal objects in another. In fact, the first viewpoint is taken in Section 4 while the second viewpoint underlies the notion of complete dendroidal Segal space and the formulation of the main equivalence [5,7]. The relation between these two viewpoints is most clearly expressed by Theorem 6.6 which equates two seemingly different model category structures.

Again using the Segal cores, we define the notion of a Segal operad in Section 8. These Segal operads will then be shown to be the fibrant objects for a model category structure on a full subcategory of the category sdSet of dendroidal spaces, the category of so-called Segal pre-operads (Theorems 8.13 and 8.17). Using most if not all of the earlier results, we will then be able to show that this model category with Segal operads as fibrant objects is Quillen equivalent to the model category having complete dendroidal Segal spaces as fibrant objects (Theorem 8.15), and hence also Quillen equivalent to the original model category of dendroidal sets.

We believe these results are of interest in themselves, and because they generalize important classical results from the simplicial-categorical context to the dendroidal-operadic one. In addition, they will all be used in our proof of the strictification theorem for ∞-operads presented in [CMb].

1. Preliminaries

We begin by recalling the basic definitions related to dendroidal sets; see [MW07, MW09, CMa]. The starting point is a category Ω of trees. Its objects are finite (non-planar) trees. These trees have internal edges (between vertices) and external ones (attached to just one vertex); the root is one such external edge, the others are called “leaves”, or “input edges”. Each such tree freely generates a coloured operad, and the arrows in Ω are the maps between these operads. Thus, by definition, Ω is a full subcategory of the category of (symmetric coloured) operads.

Each natural number \( n \geq 0 \) defines a linear tree with \( n \) vertices and \( n + 1 \) edges, the input edge being labelled 0, and the output or root edge labelled \( n \). The corresponding coloured operad is the category defined by the linear order \( 0 \leq \cdots \leq n \). Thus the simplicial category \( \Delta \) is a full subcategory of \( \Omega \), and we denote the inclusion by

\[
i : \Delta \hookrightarrow \Omega , \quad [n] \mapsto n = i[n] .
\]

The category \( dSet \) of dendroidal sets is by definition the category of presheaves (i.e. contravariant Set-valued functors) on \( \Omega \), just like the category of simplicial sets is that of presheaves on \( \Delta \). The inclusion functor \( i \) induces a pair of adjoint functors

\[
i : sSet \rightleftarrows dSet : i^* \]

where \( i^* \) is the restriction along \( i \) and \( i_* \) is its fully faithful left adjoint (\( i^* \) also has a fully faithful right adjoint \( i_* \)).

We will write \( \Omega[T] \) for the dendroidal set represented by a tree \( T \). With the similar notation \( \Delta[n] \) for representable simplicial sets, we thus have

\[
i_* \Delta[n] = \Omega[n] ,
\]

and this identification determines \( i_* \) uniquely up to unique isomorphism (as colimit preserving functor).
There is a natural identification of $\Delta$ with the slice category $\Omega/i[0]$, and this leads to an identification

$$sSet = dSet/\eta$$

where $\eta = \Omega[0]$. Under this identification, the functor $i_!$ corresponds to the forgetful functor $dSet/\eta \to dSet$.

The full embedding of $\Omega$ into (coloured) operads gives an adjoint pair

$$\tau_d : dSet \rightleftarrows \text{Operad} : N_d,$$

where the right adjoint $N_d$ is called the dendroidal nerve. These functors restrict to the usual nerve of a small category and its left adjoint.

The category of dendroidal sets carries a symmetric monoidal closed structure, denoted by $\otimes$ and $\text{Hom}$. Its unit object is the representable dendroidal set $\eta = \Omega[0]$.

This structure is compatible with the product of simplicial sets as well as with the Boardman-Vogt tensor product of operads, in the sense that for any simplicial sets $M$ and $N$, and for any dendroidal sets $X$ and $Y$, we have natural identifications

$$i_!(M \times N) = i_!(M) \otimes i_!(N) \quad \text{and} \quad \tau_d(X \otimes Y) = \tau_d(X) \otimes_{BV} \tau_d(Y).$$

We now recall some of the main combinatorial properties in the category of dendroidal sets; see [MW07, MW09, CMa] for more details.

Just like for the simplicial category $\Delta$, the arrows in $\Omega$ are generated by elementary arrows. These are faces and degeneracies like for $\Delta$, together with the isomorphisms (the only isomorphisms in $\Delta$ are the identities). In particular, for a tree $T$, we may define $\partial \Omega[T]$ as the maximal proper subobject of $\Omega[T]$, or, equivalently, as the union of the all the images of the elementary face maps $\Omega[S] \to \Omega[T]$. We refer to $\partial \Omega[T]$ as the boundary of $\Omega[T]$. The saturation of the set of boundary inclusions $\partial \Omega[T] \to \Omega[T]$ (i.e. the closure under transfinite composition, pushout, and retract) gives rise to the class of normal monomorphisms.

The normal monomorphisms can also be characterized as the monomorphisms of dendroidal sets $u : X \to Y$ such that, for any tree $T$ in $\Omega$, the action of $\text{Aut}(T)$ on the set $Y(T) - u(X(T))$ is free. A dendroidal set $X$ is normal if the map $\emptyset \to X$ is a normal monomorphism. We will often use the following property: given any morphism of dendroidal sets $X \to Y$, if $Y$ is normal, then so is $X$.

For an internal edge $e$ in a tree $T$, we denote by $T/e$ the tree obtained from $T$ by contracting the edge $e$. Then there is an elementary face map

$$\partial_e : T/e \to T$$

in $\Omega$. Face maps of this shape are called internal. We write $\Lambda^e[T]$ for the maximal subobject of $\Omega[T]$ which does not contain the image of the internal face $\partial_e : \Omega[T/e] \to \Omega[T]$ (equivalently, $\Lambda^e[T]$ may be described as the union of all the images of the elementary faces $\Omega[S] \to \Omega[T]$ which do not factor through $\partial_e$). We refer to $\Lambda^e[T]$ as the inner horn of $\Omega[T]$ associated to $e$. The class of inner anodyne extensions is defined to be the closure of the set of inner horn inclusions by transfinite composition, pushout, and retract.

A morphism of dendroidal sets $X \to Y$ is called an inner fibration if it has the right lifting property with respect to the set of inner horn inclusions $\Lambda^e[T] \to \Omega[T]$. A dendroidal set $X$ is called an inner Kan complex, or an $\infty$-operad if the map from $X$ to the terminal object of $dSet$ is an inner fibration. Under the identification $dSet/\eta = sSet$, the $\infty$-operads which admit a (necessarily unique) map to $\eta$ are
The category $dSet$ of dendroidal sets carries a cofibrantly generated model category structure, whose cofibrations are the normal monomorphisms, and whose fibrant objects are the $\infty$-operads. This structure is left proper and monoidal (i.e. compatible with tensor product). The induced model category structure on $dSet/\eta = sSet$ corresponds to the Joyal model structure on $sSet$.

The weak equivalences of the model structure above are called the weak operadic equivalences. This model category structure on $dSet$ will be referred to as the model category structure for $\infty$-operads.

2. Segal cores

2.1. We recall that, for each $n \geq 0$, the $n$th corolla $C_n$ is defined as the smallest rooted tree with one vertex and $n$ leaves.

\[
C_n = \begin{array}{c}
\text{•} \\
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_n
\end{array}
\]

(2.1.1)

In general, we say that a face map $F \to T$ is a subtree if $F \to T$ is a composition of external faces. In other words, a face map $F \to T$ is a subtree if $F$ is obtained by successively pruning away top vertices, or pruning away root vertices which have only one internal edge attached to them.

\[
T = \begin{array}{c}
\text{α}
\end{array}
\]

(2.1.2)

**Definition 2.2.** Given a tree $T$ with at least one vertex, we define its Segal core $Sc[T]$ as the subobject of $\Omega[T]$ defined as the union of all the images of those maps $\Omega[C_n] \to \Omega[T]$ corresponding to subtrees of shape $C_n \to T$. Remark that, up to isomorphism, such a map $C_n \to T$ is completely determined by the vertex of $T$ in its image, so we can write

\[
Sc[T] = \bigcup_v \Omega[C_{n(v)}],
\]

where the union is over all the vertices of $T$, and $n(v)$ is the number of input edges at $v$.

If $T = [0]$ is the tree with no vertices (so that $\Omega[T] = \Omega[0] = \eta$ is the unit object of the Boardman-Vogt tensor product), it will be convenient to define $Sc[T] = \eta$.

**Definition 2.3.** Let $T$ be a tree. The external boundary of $\Omega[T]$ is the subobject $\partial^\text{ext} \Omega[T]$ of $\Omega[T]$ obtained as the union of all the external faces of $T$.

**Proposition 2.4.** For any tree $T$, the inclusion $Sc[T] \to \Omega[T]$ is inner anodyne.
Proof. Note that, if $T$ has at most one vertex, then this inclusion is an isomorphism, while, if $T$ has exactly two vertices, this is an inner horn. So we may assume that $T$ has at least three vertices.

If $T$ has $N$ vertices, then $\Omega[T]$ has a natural filtration by subobjects

$$\Omega[T]_1 \subset \Omega[T]_2 \subset \ldots \subset \Omega[T]_{N-1} \subset \Omega[T]_N = \Omega[T],$$

where, for $1 \leq n \leq N$,

$$\Omega[T]_n = \bigcup_{\xi} \Omega[F_\xi]$$

is the union over all subtrees $F_\xi$ of $T$ with at most $n$ vertices. Notice that, by definition, we have:

$$\Omega[T]_1 = \text{Sc}[T] \quad \text{and} \quad \Omega[T]_{N-1} = \partial^{\text{ext}} \Omega[T].$$

By virtue of [MW09, Lemma 5.1], the inclusion $\partial^{\text{ext}} \Omega[S] \rightarrow \Omega[S]$ is inner anodyne for any tree $S$ with at least two vertices. We shall use this to prove that the inclusion $\Omega[T]_{n-1} \rightarrow \Omega[T]_n$ is inner anodyne for $2 \leq n \leq N$, which will prove the proposition.

Let $F_0, \ldots, F_k$ be all subtrees of $T$ having $n$ vertices. For $0 \leq j \leq k$, we put

$$S_j = \bigcup_{0 \leq i \leq j} \Omega[F_i] \subset \Omega[T].$$

We shall prove by induction on $j$ that the map

$$S_j \cap \Omega[T]_{n-1} \rightarrow S_j$$

is inner anodyne. The case $j = 0$ follows from the identification $\Omega[F_i] \cap \Omega[T]_{n-1} = \partial^{\text{ext}} \Omega[F_i]$, $0 \leq i \leq k$. Assume that $j > 0$. Note that, since

$$S_{j-1} \cap S_j \cap \Omega[T]_{n-1} = S_{j-1} \cap \Omega[T]_{n-1},$$

the following diagram is a pushout.

Moreover, since $\Omega[F_p] \cap \Omega[F_q] \subset \Omega[T]_{n-1}$ for $p \neq q$, we have

$$\Omega[F_j] \cap (S_{j-1} \cup (\Omega[F_j] \cap \Omega[T]_{n-1})) = \Omega[F_j] \cap \Omega[T]_{n-1},$$

which gives the following pushout square.

Since the top arrows in these two squares are inner anodyne, so are the lower ones, and we obtain that the composite

$$S_j \cap \Omega[T]_{n-1} \rightarrow S_{j-1} \cup (\Omega[F_j] \cap \Omega[T]_{n-1}) \rightarrow S_j$$

is inner anodyne as well. For $j = k$, we conclude that the map

$$\Omega[T]_{n-1} \rightarrow S_k \cup \Omega[T]_{n-1} = \Omega[T]_n$$
is the pushout of an inner anodyne extension.

2.5. We shall say that a class of maps $W$ in $dSet$ is saturated if the class of normal monomorphisms which are in $W$ is closed under pushouts, transfinite compositions, and retracts.

**Proposition 2.6.** Let $W$ the smallest saturated class of maps in $dSet$ which contains the Segal core inclusions and which has the following version of the two-out-of-three property: given any normal monomorphisms between normal dendroidal sets $u : X \to Y$ and $v : Y \to Z$, if two of the maps among $u$, $v$, and $vu$ are in $W$, so is the third. Then any inner anodyne extension belongs to $W$.

**Proof.** As, by definition, the class of inner anodyne extensions is the smallest saturated class which contains the inner horn inclusions, it is sufficient to prove that any inner horn inclusion belongs to $W$. For a tree $T$ with at least two vertices, and any internal edge $e$ in $T$, we have the following natural inclusions.

$$\text{Sc}[T] \to \partial^{ext} \Omega[T] \to \Lambda^e[T] \to \Omega[T]$$

We shall prove by induction on the number $|T|$ of vertices of $T$ that all these inclusions belong to $W$ (note that this is not so if $T$ has only one vertex). In fact, if $A$ is a union of at least two external faces of $T$, then $\text{Sc}[T] \subseteq A$, while, if $B$ is the union of $\partial^{ext} \Omega[T]$ and of a collection of internal faces not including the one contracting $e$, there are interpolating inclusions

$$\text{Sc}[T] \to A \to \partial^{ext} \Omega[T] \to B \to \Lambda^e[T] \to \Omega[T].$$

Our induction on $|T|$ will proceed by showing that all these inclusions belong to $W$. To begin with, if $|T| = 2$, then $\Omega[T]$ has just two external faces, and one internal one (given by the edge $e$), so $\text{Sc}[T] = A = \partial^{ext} \Omega[T] = B = \Lambda^e[T]$, whence, as $W$ contains isomorphisms and Segal core inclusions, all the inclusions in (2.6.2) are in $W$.

Consider now a tree $S$ with $|S| > 2$, and assume that, for any tree $T$ such that $2 \leq |T| < |S|$, all the maps in (2.6.2) are in $W$. We shall first show that, for any set $(R_i)_{0 \leq i \leq j}$ of at least two external faces of $S$, the map

$$\text{Sc}[S] \to A = \bigcup_{0 \leq i \leq j} \Omega[R_i]$$

is in $W$. For the set of all external faces of $S$, the map (2.6.3) is the inclusion $\text{Sc}[S] \to \partial^{ext} \Omega[S]$, so that we shall have shown that this map belongs to $W$ as well. By the two-out-of-three property, it then follows that the map $A \to \partial^{ext} \Omega[S]$ also belongs to $W$.

To prove that (2.6.3) is in $W$, consider the case of just two distinct external faces $R_0$ and $R_1$ in $S$. Then the map

$$\text{Sc}[R_0 \cap R_1] \to \Omega[R_0 \cap R_1]$$

belongs to $W$ (also if $R_0 \cap R_1$ is a tree with just one vertex), as does the map $\text{Sc}[R_1] \to \Omega[R_1]$.

Now, consider the commutative diagram below.
The square is a pushout, so the right hand vertical map belongs to \( W \). As the slanted map also does by assumption, we find that the right hand horizontal map belongs to \( W \). Next, the two pushout diagrams

\[
\begin{align*}
\text{Sc}[R_1] \cup (\Omega[R_0] \cap \Omega[R_1]) & \longrightarrow \Omega[R_1] & \text{Sc}[R_0] & \longrightarrow \Omega[R_0] \\
\text{Sc}[R_1] \cup \Omega[R_0] & \longrightarrow \Omega[R_0] \cup \Omega[R_1] & \text{Sc}[R_0] \cup \text{Sc}[R_1] & \longrightarrow \Omega[R_0] \cup \text{Sc}[R_1]
\end{align*}
\]

and

\[
\begin{align*}
\text{Sc}[R_1] \cup (\Omega[R_0] \cap \Omega[R_1]) & \longrightarrow \Omega[R_1] & \text{Sc}[R_0] & \longrightarrow \Omega[R_0] \\
\text{Sc}[R_1] \cup \Omega[R_0] & \longrightarrow \Omega[R_0] \cup \Omega[R_1] & \text{Sc}[R_0] \cup \text{Sc}[R_1] & \longrightarrow \Omega[R_0] \cup \text{Sc}[R_1]
\end{align*}
\]

show that

\[
\text{Sc}[S] = \text{Sc}[R_0] \cup \text{Sc}[R_1] \longrightarrow \Omega[R_0] \cup \Omega[R_1]
\]

belongs to \( W \). This shows that (2.6.3) belongs to \( W \) for any collection of two distinct external faces. Consider now a collection \( \{R_i\}_{0 \leq i \leq j} \) of distinct external faces of \( S \), with \( j \geq 1 \). We shall prove by induction on \( j \) that (2.6.3) belongs to \( W \). The case \( j = 1 \) just having been dealt with, we may assume that \( j \geq 2 \). By induction, we have that the map

\[
\text{Sc}[R_0] \longrightarrow \bigcup_{0 < i \leq j} \Omega[R_0 \cap R_i]
\]

belongs to \( W \), because \( R_0 \cap R_i \) is an external face of \( R_0 \) for \( 0 < i \leq j \). Also, the map \( \text{Sc}[R_0] \longrightarrow \Omega[R_0] \) is in \( W \), so that, by the two-out-of-three property, the map

\[
\bigcup_{0 < i \leq j} \Omega[R_0 \cap R_i] \longrightarrow \Omega[R_0]
\]

belongs to \( W \). By induction, the map

\[
\text{Sc}[S] \longrightarrow \bigcup_{0 < i \leq j} \Omega[R_i]
\]

is in \( W \), whence we deduce from the pushout

\[
\begin{align*}
\bigcup_{0 < i \leq j} \Omega[R_0 \cap R_i] & \longrightarrow \Omega[R_0] \\
\bigcup_{0 < i \leq j} \Omega[R_i] & \longrightarrow \bigcup_{0 < i \leq j} \Omega[R_i]
\end{align*}
\]

that (2.6.3) is the composite of two maps in \( W \). This completes the proof of the fact that all the maps of type (2.6.3) belong to \( W \).

We now turn to the internal faces of the tree \( S \), and let \( B \) be the union of \( \partial^{\text{ext}} \Omega[S] \) and of a family of internal faces \( \Omega[\partial^{\mu_1}S], \ldots, \Omega[\partial^{\mu_k}S] \), given by internal edges \( a_1, \ldots, a_k \), all distinct from another internal edge \( e \). We shall prove that \( \partial^{\text{ext}} \Omega[S] \longrightarrow B \) belongs to \( W \). Then the composition \( \text{Sc}[S] \longrightarrow \partial^{\text{ext}} \Omega[S] \longrightarrow B \) belongs to \( W \) as well. For the special case where \( B \) contains all internal faces but the one given by \( e \), we find that the inclusion \( \text{Sc}[S] \longrightarrow \Lambda^e[S] \) belongs to \( W \). Since \( \text{Sc}[S] \longrightarrow \Omega[S] \) is in \( W \) by assumption, the two-out-of-three property implies that \( \Lambda^e[S] \longrightarrow \Omega[S] \) is in \( W \). So, to complete the proof of the proposition, it is sufficient to prove that, under our inductive assumption (that the maps in (2.6.2) all belong to \( W \) for smaller trees), the map

\[
(2.6.4) \quad \partial^{\text{ext}} \Omega[S] \longrightarrow B = \partial^{\text{ext}} \Omega[S] \cup \bigcup_{1 \leq i \leq k} \Omega[\partial^{\mu_i}S]
\]
belongs to $W$. We proceed by induction on $k$. If $k = 1$, then $S$ has at least two internal edges, hence $\partial^{ra_1}S$ has at least two vertices, so, by assumption on trees smaller than $S$, the map

$$\partial^{ext}(\partial^{ra_1}S) \rightarrow \Omega(\partial^{ra_1}S)$$

belongs to $W$. If $k > 1$, then we have

$$\left(\partial^{ext}(\partial^{ra_k}S) \cup \bigcup_{1 \leq i < k} \Omega(\partial^{ra_i}S)\right) \cap \Omega(\partial^{ra_k}S) = \partial^{ext}(\partial^{ra_k}S) \cup \bigcup_{1 \leq i < k} \Omega(\partial^{ra_1}\partial^{ra_k}S),$$

so the diagram

$$\begin{array}{ccc}
\partial^{ext}(\partial^{ra_k}S) \cup \bigcup_{1 \leq i < k} \Omega(\partial^{ra_i}S) & \longrightarrow & \Omega(\partial^{ra_k}S) \\
\downarrow & & \downarrow \\
\partial^{ext}(\partial^{ra_k}S) \cup \bigcup_{1 \leq i < k} \Omega(\partial^{ra_i}S) & \longrightarrow & \partial^{ext}(\partial^{ra_k}S) \cup \bigcup_{1 \leq i < k} \Omega(\partial^{ra_i}S)
\end{array}$$

is a pushout. Moreover, the family $\{\partial^{ra_i}\partial^{ra_k}S\}_{1 \leq i < k}$ of internal faces of $T = \partial^{ra_k}S$ misses the edge $e$, so that, by assumption, all the inclusions in (2.6.2) belong to $W$. We conclude that the top arrow in the pushout above belongs to $W$, whence so does the bottom arrow. By induction on $k$, the map

$$\partial^{ext}(\partial^{ra_k}S) \rightarrow \partial^{ext}(\partial^{ra_k}S) \cup \bigcup_{1 \leq i < k} \Omega(\partial^{ra_i}S)$$

is in $W$, and as $W$ is closed under composition, we find that (2.6.4) is in $W$. □

**Corollary 2.7.** A dendroidal set $X$ is the nerve of an operad if and only if, for any tree $T$, the map

$$X_T = \text{Hom}_{dSet}(\Omega[T], X) \rightarrow \text{Hom}_{dSet}(\text{Sc}[T], X)$$

is bijective.

**Proof.** Assume that $X = N_d(P)$ for a (coloured) operad $P$. Then, as the functor $\tau_d$ sends inner anodyne inclusions to isomorphisms, it follows from Proposition 2.3 that the map $\text{Hom}_{dSet}(\Omega[T], X) \rightarrow \text{Hom}_{dSet}(\text{Sc}[T], X)$ is bijective. The converse follows easily from Proposition 2.6 and from the characterization of dendroidal nerves given by [MW09, Proposition 5.3 and Theorem 6.1]. □

**Remark 2.8.** The preceding corollary gives in particular the well known characterization of small categories as simplicial sets satisfying the Grothendieck-Segal condition: given a simplicial set $X$, then, for $T = n$ with $n \geq 1$, we have

$$\text{Hom}_{dSet}(\text{Sc}[T], X) = X_1 \times X_0 \times X_0 \times \cdots \times X_0 \times X_1.$$  

$n$ times

3. Equivalences of $\infty$-operads

3.1. For an $\infty$-category $X$, we denote by $k(X)$ the maximal Kan complex contained in $X$; see [Joy02, Corollary 1.5]. Recall that, if $A$ and $X$ are two dendroidal sets, $\mathcal{H}om(A, X)$ denotes their internal Hom (with respect to the Boardman-Vogt tensor product of dendroidal sets).

Given two dendroidal sets $A$ and $X$, we write

$$\text{hom}(A, X) = i^* \mathcal{H}om(A, X).$$
If $X$ is an $\infty$-operad, and if $A$ is normal, then $\mathcal{H}om(A, X)$ is an $\infty$-operad, so that $\mathcal{h}om(A, X)$ is an $\infty$-category; see [MW09, Theorem 9.1].

For an $\infty$-operad $X$ and a simplicial set $K$, we will write $X^{(K)}$ for the subcomplex of $\mathcal{H}om(i_i(K), X)$ which consists of dendrices $a : \Omega[T] \times i_i(K) \to X$ such that, for any edge $u$ in the tree $T$, the induced map $a_u : K \to i^*(X)$ factors through $k(i^*(X))$ (i.e. all the 1-cells in the image of $a_u$ are weakly invertible in $i^*(X)$).

For an $\infty$-operad $X$ and a normal dendroidal set $A$, we write $k(A, X)$ for the subcomplex of $\mathcal{h}om(A, X)$ which consists of maps $u : A \otimes i_i(\Delta[n]) \to X$ such that, for all vertices $a$ of $A$ (i.e. maps $a : \eta \to A$), the induced map $u_a : \Delta[n] \to i^*(X)$ factors through $k(i^*(X))$. So, by definition, for any normal dendroidal set $A$, any simplicial set $K$, and any $\infty$-operad $X$, there is a natural bijection:

\[
\text{Hom}_{sSet}(K, k(A, X)) \simeq \text{Hom}_{dSet}(A, X^{(K)}).
\]

Furthermore, by virtue of [CMa, 6.8], we have the equality:

\[
(3.1.2) \quad k(A, X) = k(\mathcal{h}om(A, X))
\]

(in particular, $k(A, X)$ is a Kan complex).

3.2. Recall that, in any model category $\mathcal{C}$, given a cofibrant object $A$ and a fibrant object $X$, one of the models of the mapping space $Map(A, X)$ is the simplicial set defined by

\[
(3.2.1) \quad Map(A, X)_n = \text{Hom}_{\mathcal{C}}(A, X_n),
\]

where $X_\bullet$ is a Reedy fibrant resolution of $X$ (where $X$ is seen as a simplicially constant object of $\mathcal{C}^\Delta^n$); see [Hov99, 5.4.7 and 5.4.9], for instance. In the case where $\mathcal{C} = dSet$ and $X$ is an $\infty$-operad, then the simplicial dendroidal set $X^{(\Delta[\bullet])}$ is Reedy fibrant, and, for any integer $n \geq 0$, the map $X \to X^{(\Delta[n])}$ is a weak equivalence (this follows immediately from [CMa Corollary 6.9]). In particular, we get:

**Proposition 3.3.** If $A$ is a normal dendroidal set and $X$ an $\infty$-operad, then there is a natural identification

\[
k(A, X) \simeq Map(A, X).
\]

**Proof.** This follows immediately from the identifications (3.1.1) and (3.2.1). \qed

**Lemma 3.4.** Let $X$ be an $\infty$-operad.

(i) For any simplicial set $K$, and for any normal dendroidal set $A$, there is a natural bijection

\[
\text{Hom}_{\mathcal{H}o(dSet)}(K, k(A, X)) \simeq \text{Hom}_{\mathcal{H}o(dSet)}(A, X^{(K)}).
\]
(ii) For any cofibration (resp. trivial cofibration) between normal dendroidal sets $A \to B$, the map $k(B, X) \to k(A, X)$ is a fibration (resp. a trivial fibration) between Kan complexes.

(iii) For any pushout of normal dendroidal sets

$$
\begin{array}{c}
A \\
\downarrow i \\
B
\end{array} 
\quad 
\begin{array}{c}
A' \\
\downarrow i' \\
B'
\end{array}
$$

with $i$ a cofibration, the commutative square

$$
\begin{array}{c}
k(B', X) \\
\downarrow \\
k(B, X)
\end{array} 
\quad 
\begin{array}{c}
k(A', X) \\
\downarrow \\
k(A, X)
\end{array}
$$

is a pullback.

(iv) For any sequence of cofibrations between normal dendroidal sets

$$
A_0 \to A_1 \to \cdots \to A_n \to A_{n+1} \to \cdots,
$$

the map

$$
k(\lim_{\to n} A_n, X) \to \lim_{\to n} k(A_n, X)
$$

is an isomorphism.

Proof. This follows immediately from Proposition 3.3, using the general properties of mapping space functors; see for instance [Hov99, Proposition 5.4.1 and Theorem 5.4.9]. □

Theorem 3.5. Let $f : X \to Y$ be a morphism between $\infty$-operads. The following conditions are equivalent.

(a) For any integer $n \geq 0$, the map $k(\Omega[C_n], X) \to k(\Omega[C_n], Y)$ is a simplicial homotopy equivalence, as is the map $k(\eta, X) \to k(\eta, Y)$.

(b) For any tree $T$, the map $k(\Omega[T], X) \to k(\Omega[T], Y)$ is a simplicial homotopy equivalence.

(c) For any normal dendroidal set $A$, the map $k(A, X) \to k(A, Y)$ is a simplicial homotopy equivalence.

(d) The map $f : X \to Y$ is a weak operadic equivalence.

Proof. Assume condition (a). We claim that, for any tree $T$, the induced map $k(\text{Sc}[T], X) \to k(\text{Sc}[T], Y)$ is a simplicial homotopy equivalence. Note that, for $T = [0]$, this is a special case of (a). Therefore, to prove this, we may assume that $T$ has at least one vertex. Let $v_1, \ldots, v_k$ the vertices of $T$, and, for $1 \leq i \leq k$, write $n_i$ for the number of input edges of $v_i$ in $T$. We then have

$$
\text{Sc}[T] = \bigcup_{1 \leq i \leq k} \Omega[C_{n_i}].
$$

Moreover, for two indices $i \neq j$, the intersection $\Omega[C_{n_i}] \cap \Omega[C_{n_j}]$ is either empty or isomorphic to $\eta$. For $1 \leq j \leq k$, define

$$
A_j = \bigcup_{1 \leq i \leq j} \Omega[C_{n_i}].
$$
For $1 < i \leq k$, there is a pushout square

\[
\begin{array}{ccc}
A_{i-1} \cap \Omega[C_{n_i}] & \longrightarrow & \Omega[C_{n_i}] \\
\downarrow & & \downarrow \\
A_{i-1} & \longrightarrow & A_i
\end{array}
\]

in which the intersection $A_{i-1} \cap \Omega[C_{n_i}]$ is isomorphic to a finite sum of $\eta$’s. By the cube lemma (see the dual version of [Hov99, Lemma 5.2.6]), using properties (ii) and (iii) of Lemma 3.4, we obtain by induction on $i$ that the maps

\[k(A_i, X) \longrightarrow k(A_i, Y)\]

are simplicial homotopy equivalences. In particular, for $i = k$, this means that the map

\[k(\text{Sc}[T], X) \longrightarrow k(\text{Sc}[T], Y)\]

is a simplicial homotopy equivalence. Thus, since the vertical maps in the commutative square

\[
\begin{array}{ccc}
k(\Omega[T], X) & \longrightarrow & k(\Omega[T], Y) \\
\downarrow & & \downarrow \\
k(\text{Sc}[T], X) & \longrightarrow & k(\text{Sc}[T], Y)
\end{array}
\]

are simplicial homotopy equivalences as well (by Proposition 2.4 and Lemma 3.4 (ii)), this proves (b).

The fact that condition (b) implies condition (c) follows by similar arguments from Lemma 3.4 using the skeletal filtration of normal dendroidal sets [MW09, section 4].

A reformulation of Lemma 3.4 (i) in the particular case where $K = \Delta[0]$ is that, for a normal dendroidal set $A$ and an $\infty$-operad $X$, we have a natural bijection

\[\pi_0(k(A, X)) \simeq \operatorname{Hom}_{\mathcal{H}o(dSet)}(A, X).\]

It thus follows from the Yoneda lemma that condition (c) implies condition (d).

Finally, the fact that condition (d) implies condition (c) (and, therefore, condition (a)) is obvious: it follows from Lemma 3.4 (ii) and from Ken Brown’s Lemma [Hov99, Lemma 1.1.12] that the functor $k(A, -)$ sends weak operadic equivalences between $\infty$-operads to simplicial homotopy equivalences.

\[\square\]

3.6. Let $X$ be an $\infty$-operad. Given an $(n+1)$-tuple of 0-cells $(x_1, \ldots, x_n, x)$ in $X$, the space of maps $X(x_1, \ldots, x_n; x)$ is obtained by the pullback below, in which the map $p$ is the map induced by the inclusion $\eta \eta \cdots \eta \longrightarrow \Omega[C_n]$ (with $n+1$ copies

\[\text{Another proof consists to see that, by [CMA 1.7], for a normal dendroidal set $A$, the category $\Omega[A]$ is a regular skeletal category in the sense of [Cis06 8.2.3], from which we deduce that $A$ is the homotopy colimit of the $\Omega[T]$’s over $A$ (see [Cis06 8.2.9]), and we can use Proposition 3.6 to see that the functor $k(-, X)$ turns homotopy colimits into homotopy limits (see for instance [Cis03 6.13]), which implies the result.}\]
of η, corresponding to the \(n + 1\) objects \((a_1, \ldots, a_n, a)\) of \(C_n\): see \((2.1.1)\).

\[
\begin{array}{ccc}
X(x_1, \ldots, x_n; x) & \longrightarrow & \text{Hom}(\Omega[C_n], X) \\
\downarrow \quad \eta & & \downarrow p \\
(x_1, \ldots, x_n, x) & \longrightarrow & X^{n+1}
\end{array}
\]

Using the identification \(sSet = dSet/\eta\), we shall consider \(X(x_1, \ldots, x_n; x)\) as a simplicial set. Observe that \(X(x_1, \ldots, x_n; x)\) is actually a Kan complex (see [CMa, Proposition 6.13]).

**Definition 3.7.** Let \(f : X \longrightarrow Y\) be a morphism of \(\infty\)-operads.

The map \(f\) is **fully faithful** if, for any \((n + 1)\)-tuple of 0-cells \((x_1, \ldots, x_n, x)\) in \(X\), the morphism

\[
X(x_1, \ldots, x_n; x) \longrightarrow Y(f(x_1), \ldots, f(x_n); f(x))
\]

is a simplicial homotopy equivalence.

The map \(f\) is **essentially surjective** if the functor underlying the morphism of operads \(\tau_d(X) \longrightarrow \tau_d(Y)\) is essentially surjective.

**Remark 3.8.** For an \(\infty\)-operad \(X\), the set of isomorphism classes of objects in \(\tau_d(X)\) is in bijection with the set \(\pi_0(k(i^*X))\); see [CMa, 4.1]. The condition of essential surjectivity is thus equivalent to the fact that the map \(k(i^*X) \longrightarrow k(i^*Y)\) induces a surjection on connected components.

Moreover, by virtue of [CMa, Proposition 6.14], we have natural bijections

\[
\pi_0(X(x_1, \ldots, x_n; x)) \simeq \tau_d(X)(x_1, \ldots, x_n; x).
\]

As a consequence, if \(f : X \longrightarrow Y\) is fully faithful, so is the induced morphism of operads \(\tau_d(f) : \tau_d(X) \longrightarrow \tau_d(Y)\). Therefore, if \(f : X \longrightarrow Y\) is fully faithful and essentially surjective, then the map \(k(i^*X)) \longrightarrow k(i^*Y))\) induces a bijection on connected components.

We recall the following well known fact:

**Lemma 3.9.** A commutative square of simplicial sets

\[
\begin{array}{ccc}
X & \longrightarrow & X' \\
\downarrow p & & \downarrow p' \\
Y & \longrightarrow & Y'
\end{array}
\]

in which \(p\) and \(p'\) are Kan fibrations is a homotopy pullback square if and only if, for any 0-simplex \(y\) of \(Y\), the map between the corresponding fibers

\[
p^{-1}(y) = X_y \longrightarrow X'_{v(y)} = p'^{-1}(v(y))
\]

is a simplicial homotopy equivalence.

A direct consequence of the preceding lemma is:
Lemma 3.10. Consider a commutative square of simplicial sets.

\[
\begin{array}{ccc}
X & \xrightarrow{u} & X' \\
\downarrow{p} & & \downarrow{p'} \\
Y & \xrightarrow{v} & Y'
\end{array}
\]

Assume furthermore that \( p \) and \( p' \) are Kan fibrations, and that the map \( v \) is a weak homotopy equivalence. Then the map \( u \) is a weak homotopy equivalence if and only if, for any 0-simplex \( y \) of \( Y \), the map between the corresponding fibers \( X_y \to X'_{v(y)} \) is a simplicial homotopy equivalence.

Theorem 3.11. Let \( f : X \to Y \) be a morphism between two \( \infty \)-operads. Then \( f \) is a weak operadic equivalence if and only if it is fully faithful and essentially surjective.

Proof. Given an \( \infty \)-operad \( X \), and an \((n+1)\)-tuple \((x_1, \ldots, x_n, x)\) of objects of \( X \), i.e. a 0-simplex of \( k(i^* X)^{n+1} = k(\partial \Omega[C_n], X) \), we have the following diagram in which the right hand square is a pullback square (see \[CMa\, Remark 6.2 and Corollary 6.8\]).

\[
\begin{array}{ccc}
X(x_1, \ldots, x_n; x) & \to & k(\Omega[C_n], X) \\
\downarrow{\eta} & & \downarrow{\hom(\Omega[C_n], X)} \\
\to & k(i^* X)^{n+1} & \to i^* X^{n+1}
\end{array}
\]

Hence the left hand square above is a pullback as well. Assume that \( f \) is fully faithful and essentially surjective. We will first prove that the induced morphism \( k(i^* X) \to k(i^* Y) \) is a simplicial homotopy equivalence. As the corresponding map \( \pi_0(k(i^* X)) \to \pi_0(k(i^* Y)) \) is bijective (see \[MM\]), it is sufficient to prove that, for any 0-simplex \( x \) of \( k(i^* X) \), the map of loop spaces

\[
\Omega(k(i^* X), x) \to \Omega(k(i^* Y), f(x))
\]

is a weak homotopy equivalence. For this purpose, it will be sufficient to prove that the commutative square

\[
\begin{array}{ccc}
\Omega(k(i^* X), x) & \to & \Omega(k(i^* Y), f(x)) \\
\downarrow & & \downarrow \\
X(x; x) & \to & Y(f(x); f(x))
\end{array}
\]

(3.11.1)

is homotopy cartesian. In general, the set of connected components of the Kan complex \( X(x_1, \ldots, x_n; x) \) is in bijection with the set \( \tau_d(X)(x_1, \ldots, x_n; x) \); see \[CMa\, Proposition 6.14\]. We can thus describe the loop space \( \Omega(k(i^* X), x) \) as the disjoint union of the connected components of \( X(x; x) \) which correspond to automorphisms of \( x \) in the category underlying the operad \( \tau_d(X) \). In particular, the inclusion \( \Omega(k(i^* X), x) \subset X(x; x) \) is a Kan fibration between Kan complexes. Using the
fact that the functor underlying the map \( \tau_d(X) \to \tau_d(Y) \) is full faithful (whence conservative), we see that the map
\[
\pi_1(k(i^*X), x) = \pi_0(\Omega(k(i^*X), x)) \to \pi_0(\Omega(k(i^*Y), f(x))) = \pi_1(k(i^*Y), f(x))
\]
is bijective. The square (3.11.1) is thus cartesian, and, as its vertical maps are Kan fibrations, it is homotopy cartesian as well. Therefore, the map \( k(i^*X) \to k(i^*Y) \) is a simplicial homotopy equivalence. As a consequence, for any corolla \( C_n \), we also have simplicial homotopy equivalences
\[
k(i^*X)^{n+1} = k(\partial\Omega[C_n], X) \to k(i^*Y)^{n+1} = k(\partial\Omega[C_n], Y).
\]
By applying Lemma 3.10 to the commutative squares
\[
\begin{array}{ccc}
  k(\Omega[C_n], X) & \longrightarrow & k(\Omega[C_n], Y) \\
  \downarrow & & \downarrow \\
  k(\partial\Omega[C_n], X) & \longrightarrow & k(\partial\Omega[C_n], Y)
\end{array}
\]
we conclude that the maps \( k(\Omega[C_n], X) \to k(\Omega[C_n], Y) \) are all simplicial homotopy equivalences. The characterization given by condition (a) of Theorem 3.5 thus implies that \( f \) is a weak operadic equivalence.

For the converse, we just apply again Lemma 3.10 to the commutative squares (3.11.2). □

**Remark 3.12.** As we saw implicitly in the proof above, Theorem 3.5 and Lemma 3.9 lead to a characterization of fully faithful maps: a morphism between \( \infty \)-operads \( X \to Y \) is fully faithful if and only if the commutative squares of shape (3.11.2) are homotopy pullback squares of Kan complexes for any \( n \geq 0 \).

### 4. Locally Constant Simplicial Dendroidal Sets

**4.1.** Let \( sdSet = dSet^{\Delta^o} \simeq \overline{\Delta \times \Omega} \) be the category of simplicial dendroidal sets. The category \( dSet \) (resp. \( sSet \)) of dendroidal (resp. simplicial) sets is naturally embedded in \( sdSet \), by viewing a dendroidal (resp. simplicial) set as a constant simplicial (resp. dendroidal) object in \( dSet \) (resp. in \( sSet \)). For a simplicial dendroidal set \( X \), a tree \( T \), and an integer \( n \geq 0 \), the evaluation of \( X \) at \((T, n)\) will often be denoted by \( X(T)_n \).

Given a simplicial dendroidal set \( X \), we denote by
\[
sSet^{op} \longrightarrow dSet , \quad K \mapsto X^K
\]
the (essentially) unique limit preserving functor which sends \( \Delta[n] \) to \( X^{\Delta[n]} := X_n \).

Starting from the model category structure on \( dSet \), and using that the category of simplices \( \Delta \) is a Reedy category, one obtains the Reedy model structure on \( sdSet \); see [Hov99, Theorem 5.2.5]. We shall call this structure the simplicial Reedy model category structure on \( sdSet \). By definition, the weak equivalences are the termwise weak operadic equivalences (by evaluating at simplices), while the fibrations (resp. the trivial fibrations) are the maps \( X \to Y \) such that, for any integer \( n \geq 0 \), the map
\[
X^{\Delta[n]} \longrightarrow X^{\partial\Delta[n]} \times_{Y^{\partial\Delta[n]}} Y^{\Delta[n]}
\]
is a fibration (resp. a trivial fibration) in \( dSet \). In other words, we have:
Proposition 4.2. The simplicial Reedy model structure on $sdSet$ is a cofibrantly generated model category. A generating set of cofibrations of $sdSet$ is given by the inclusions
\[ \partial \Delta[n] \times \Omega[T] \cup \Delta[n] \times \partial \Omega[T] \longrightarrow \Delta[n] \times \Omega[T], \]
for any integer $n \geq 0$ and any tree $T$, while, if $K$ is a generating set of trivial cofibrations (between normal dendroidal sets) in $dSet$, then a generating set of trivial cofibrations of $sdSet$ is given by the inclusions
\[ \partial \Delta[n] \times \Omega[T] \cup \Delta[n] \times \partial \Omega[T] \rightarrow \Delta[n] \times \Omega[T] \]
for any integer $n \geq 0$ and any map $A \rightarrow B$ in $K$.

Corollary 4.3. The cofibrations of the simplicial Reedy model category structure on $sdSet$ are the termwise normal monomorphisms of dendroidal sets.

Proof. As both $\Omega$ and $\Delta$ are skeletal categories in the sense of [Cis06, 8.1.1], so is there product [Cis06, 8.1.7]. Moreover, for any integer $n \geq 0$ and any tree $T$, the boundary of the representable presheaf $\Delta[n] \times \Omega[T]$ is nothing but the presheaf $\partial \Delta[n] \times \Omega[T] \cup \Delta[n] \times \partial \Omega[T]$. Therefore, the Reedy cofibrations of $sdSet$ are the normal monomorphisms in the absolute sense (see [Cis06, 8.1.30 and 8.1.35] for $A = \Delta \times \Omega$). It is easily seen that those are precisely the termwise normal monomorphisms. □

4.4. In the sequel, we shall simply call normal monomorphisms the cofibrations of the simplicial Reedy model category structure on $sdSet$.

Remark 4.5. Any fibrant object of the simplicial Reedy model category on $sdSet$ is termwise fibrant (i.e. is termwise an $\infty$-operad).

Definition 4.6. We define the locally constant model category structure on $sdSet$ as the left Bousfield localization of the simplicial Reedy model category structure on $sdSet$ by the set of projections $\Delta[n] \times \Omega[T] \longrightarrow \Omega[T]$, for any tree $T$ and any integer $n \geq 0$ (see [Hir03] for the general theory of left Bousfield localization of model categories).

Proposition 4.7. Let $X$ a simplicial dendroidal set. Assume that $X$ is fibrant for the simplicial Reedy model category structure. Then the following conditions are equivalent:

(i) the map from $X$ to the terminal object has the right lifting property with respect to the inclusions
\[ \Lambda^k[n] \times \Omega[T] \cup \Delta[n] \times \partial \Omega[T] \longrightarrow \Delta[n] \times \Omega[T], \]
for any tree $T$ and for any integers $n \geq 1$ and $0 \leq k \leq n$;

(ii) for any integer $n \geq 0$, the map $X_n \longrightarrow X_0$ is an equivalence of $\infty$-operads;

(iii) $X$ is fibrant for the locally constant model category structure on $sdSet$.

Proof. The equivalence between conditions (ii) and (iii) holds by definition of left Bousfield localizations, as, for any tree $T$, we have a natural identification in $\text{Ho}(sdSet)$
\[ \text{Map}(\Omega[T], X_n) = \text{Map}(\Delta[n] \times \Omega[T], X) \]
for any integer $n \geq 0$. 
Next, it follows easily from [Cis06, Corollary 2.1.20] and [Hov99, Lemma 4.2.4] that the class of monomorphisms $K \to L$ in $sSet$ such that, for any tree $T$, the map

$$K \times \Omega[T] \cup L \times \partial\Omega[T] \to L \times \Omega[T]$$

is a trivial cofibration of the locally constant model category structure on $sdSet$, contains the class of trivial cofibrations of the usual model category structure on $sSet$. Therefore, condition (iii) implies condition (i). Conversely, as the horn inclusions generate the trivial cofibrations of the usual model category structure on $sSet$, it is clear that condition (i) implies condition (iii).

□

**Proposition 4.8.** The inclusion $dSet \subset sdSet$ is a left Quillen equivalence from the model category for $\infty$-operads to the locally constant model category structure.

**Proof.** This follows immediately from [Cis06, Proposition 2.3.27] (which is a very particular case of a much more general result of Rezk, Schwede and Shipley [RSS01, Theorem 3.6]). □

5. **Dendroidal Segal spaces**

5.1. We shall now consider different model category structures on the category of simplicial dendroidal sets.

Given a simplicial dendroidal set $X$, let us denote by

$$dSet^{op} \to sSet, \quad A \mapsto X^A$$

the (essentially) unique limit preserving functor which sends a tree $T$ to $X^{\Omega[T]} := X(T)$.

Starting from the usual model category structure on the category of simplicial sets, we first have:

**Proposition 5.2.** The category $sdSet$ admits a cofibrantly generated and proper model category structure whose weak equivalences are the termwise simplicial weak homotopy equivalences (i.e. the maps $X \to Y$ such that, for any tree $T$, the map $X_T \to Y_T$ is a simplicial weak homotopy equivalence), and whose cofibrations are the normal monomorphisms. Moreover, a morphism of simplicial dendroidal sets $X \to Y$ is a fibration (resp. a trivial fibration) if and only if, for any tree $T$, the map

$$(5.2.1) \quad X^{\Omega[T]} \to X^{\partial\Omega[T]} \times Y^{\partial\Omega[T]} Y^{\Omega[T]}$$

is a trivial Kan fibration (resp. a Kan fibration). In other words, a set of generators for cofibrations is provided by the maps

$$\partial\Delta[n] \times \Omega[T] \cup \Delta[n] \times \partial\Omega[T] \to \Delta[n] \times \Omega[T]$$

for any tree $T$ and any integer $n \geq 0$, while a generating set of trivial cofibrations is given by the maps

$$\Lambda^k[n] \times \Omega[T] \cup \Delta[n] \times \partial\Omega[T] \to \Delta[n] \times \Omega[T]$$

for any tree $T$ and for any integers $n \geq 1$ and $0 \leq k \leq n$.

**Proof.** Any map of $sdSet$ with the right lifting property with respect to normal monomorphisms in $sdSet$ is a termwise trivial fibration of simplicial sets (as it has the right lifting property with respect to the inclusions $\partial\Delta[n] \times \Omega[T] \to \Delta[n] \times \Omega[T]$, for any tree $T$ and any integer $n \geq 0$). The first assertion thus follows from [Cis06].
Theorem 1.6.2. The generating set for cofibrations is provided by Corollary 4.3. Consider now a morphism \( p : X \to Y \) in \( \text{sdSet} \) such that the maps (5.2.1) are Kan fibrations. Using the fact that, for any tree \( T \), \( \Omega/T \) is a Reedy category, we see that, if \( p \) is also a termwise simplicial weak homotopy equivalence, then the maps (5.2.1) are trivial Kan fibrations; see [Hov99, Theorem 5.2.5]. Using [Hov99, Theorem 2.1.19], this completes the proof. □

**Remark 5.3.** As the category \( \Omega \) is a generalized Reedy category (see [BM, Example 2.8]), the model category above can also be obtained directly as special case of [BM, Theorem 1.6]. We shall thus call the model category of Proposition 5.2 the generalized Reedy model category structure on \( \text{sdSet} \).

**Definition 5.4.** We define the model category structure for dendroidal Segal spaces as the left Bousfield localization of the generalized Reedy model category on \( \text{sdSet} \) by the set of maps \( \Lambda^e[T] \to \Omega[T] \), for any tree \( T \) with an inner edge \( e \).

**Proposition 5.5.** The model category structure for dendroidal Segal spaces is the left Bousfield localization of the generalized Reedy model category structure on \( \text{sdSet} \) by the set of maps \( \Lambda^e[T] \to \Omega[T] \), for any tree \( T \) with an inner edge \( e \).

**Proof.** It follows immediately from Proposition 2.6 that, for any tree \( T \) with given inner edge \( e \), the map \( \Lambda^e[T] \to \Omega[T] \) is a weak equivalence of the model category structure for dendroidal Segal spaces. Conversely, let \( W \) be the class of maps of dendroidal sets which are weak equivalences in the left Bousfield localization of the generalized Reedy model category on \( \text{sdSet} \) by the set of maps \( \Lambda^e[T] \to \Omega[T] \), for any tree \( T \) with an inner edge \( e \). It is clear that any inner anodyne extension in \( \text{dSet} \) belongs to \( W \), so that, by virtue of Proposition 2.4, for any tree \( T \), the inclusion \( \text{Sc}[T] \to \Omega[T] \) is in \( W \). □

**Corollary 5.6.** Let \( X \) be a simplicial dendroidal set. Assume that \( X \) is fibrant for the generalized Reedy model category structure. Then the following conditions are equivalent:

(i) \( X \) is a dendroidal Segal space;

(ii) for any tree \( T \), the map \( X^T \to X^{\text{Sc}[T]} \) is a trivial Kan fibration in \( \text{sSet} \);

(iii) for any tree \( T \) with a given inner edge \( e \), the map \( X^T \to X^{\Lambda^e[T]} \) is a trivial Kan fibration in \( \text{sSet} \).

**Proof.** For any normal dendroidal set \( A \), there is a canonical identification

\[
\text{Map}(A, X) = X^A.
\]

The corollary thus follows from the definition of left Bousfield localizations and from Proposition 5.5. □

**Proposition 5.7.** A morphism of dendroidal Segal spaces \( X \to Y \) is a weak equivalence if and only if, its evaluation at \( T \)

\[
X(T) = X^\Omega[T] \to Y^\Omega[T] = Y
\]

is a simplicial homotopy equivalence for \( T = \eta \) as well as for \( T = C_n, n \geq 0 \).
Proof. A morphism of dendroidal Segal spaces is a weak equivalence of the model structure for dendroidal Segal spaces if and only if it is a weak equivalence of the generalized Reedy model category structure. In other words, a morphism of dendroidal Segal spaces \( X \rightarrow Y \) is a weak equivalence if and only if, its evaluation at \( T \)

\[
X^{\Omega[T]} \rightarrow Y^{\Omega[T]}
\]

is a simplicial homotopy equivalence for any tree \( T \). By virtue of condition (ii) of the preceding corollary, we see that evaluating a dendroidal Segal space at a tree gives the same information as evaluating at \( \text{Sc}[T] \). We easily conclude the proof from the fact that \( \text{Sc}[T] \) is a (homotopy) colimit of dendroidal sets of shape \( \eta \) or \( \Omega[C_n], n \geq 0 \).

5.8. If \( X \) is a dendroidal Segal space, and if \( (x_1, \ldots, x_n, x) \) is an \( (n+1) \)-tuple of elements of \( X(\eta)^{n+1} \), we define \( X(x_1, \ldots, x_n; x) \) by the following pullback

\[
\begin{array}{ccc}
X(x_1, \ldots, x_n; x) & \rightarrow & X(C_n) \\
\downarrow & & \downarrow \\
\Delta[0] & \rightarrow & X(\eta)^{n+1}
\end{array}
\]

in which the map \( X(C_n) \rightarrow X(\eta)^{n+1} \) is the map induced by the inclusion \( \coprod_{n+1} \eta = \partial \Omega[C_n] \rightarrow \Omega[C_n] \). As \( X(C_n) \rightarrow X(\eta)^{n+1} \) is a Kan fibration, the pullback square above is homotopy cartesian, and \( X(x_1, \ldots, x_n; x) \) is a Kan complex.

Definition 5.9. A morphism of dendroidal Segal spaces \( f : X \rightarrow Y \) is fully faithful if, for any \( (n+1) \)-tuple of 0-cells \( (x_1, \ldots, x_n, x) \) in \( X(\eta) \), the morphism

\[
X(x_1, \ldots, x_n; x) \rightarrow Y(f(x_1), \ldots, f(x_n); f(x))
\]

is a simplicial homotopy equivalence.

A morphism of dendroidal Segal spaces is a weak equivalence on objects if its evaluation at \( \eta \) is a simplicial weak equivalence.

Corollary 5.10. A morphism of dendroidal Segal spaces is a weak equivalence if and only if it is fully faithful as well as a weak equivalence on objects.

Proof. This follows immediately from Proposition 5.7 and from Lemma 3.10.

6. Complete dendroidal Segal spaces

6.1. Recall the dendroidal interval \( J_d = \iota_I(J) \), where \( J = N\pi_1(\Delta[1]) \) denotes the nerve of the contractible groupoid with two objects 0 and 1.

Definition 6.2. We define the dendroidal Rezk model category as the left Bousfield localization of the model category for dendroidal spaces by the maps

\[
\Omega[T] \otimes J_d \rightarrow \Omega[T], \quad T \in \Omega,
\]

obtained by tensoring with the unique morphism \( J_d \rightarrow \eta \). The fibrant objects of the dendroidal Rezk model category will be called complete dendroidal Segal spaces. The weak equivalences of this model category structure will be called the complete weak equivalences.
Proposition 6.3. For any normal dendroidal set $A$, the map $A \otimes J_d \to A$ induced by the projection $J_d \to \eta$ is a complete weak equivalence.

Proof. It is sufficient to prove that this map is a weak equivalence for the induced model category structure on $sdSet/A$, the latter being equivalent to the category of presheaves on the category $\Delta \times \Omega/A$ (where $\Omega/A$ is the category of elements of $A$). On the other hand, we know that tensoring by $J_d$ preserves colimits as well as normal monomorphisms. As, by virtue of [CMa, Corollary 1.7], $\Omega/A$ is then a regular skeletal category in the sense of [Cis06, 8.2.3], this proposition is a straightforward application of [Cis06, 8.2.14]. □

Corollary 6.4. The inclusion $dSet \subset sdSet$ sends the weak operadic equivalences between normal dendroidal sets to complete weak equivalences.

Proof. It follows immediately from the preceding proposition that this inclusion functor sends $J$-anodyne extensions in the sense of [CMa, 3.2] to complete weak equivalences. This corollary thus follows immediately from [CMa, Proposition 3.16]. □

Corollary 6.5. Let $K$ be a generating set of trivial cofibrations in $dSet$. We assume that all the maps in $K$ are morphisms between normal dendroidal sets (which is a harmless hypothesis, by virtue of [CMa, Remark 3.15]). A simplicial dendroidal space $X$ is a complete dendroidal Segal space if and only if it is a Segal space, and if the map from $X$ to the terminal simplicial dendroidal set has the right lifting property with respect to the inclusions of shape

$$\partial \Delta[n] \times B \cup \Delta[n] \times A \to \Delta[n] \times B$$

for $j: A \to B$ in $K$ and $n \geq 0$.

Proof. By corollary 6.4, the inclusions $\partial \Delta[n] \times B \cup \Delta[n] \times A \to \Delta[n] \times B$, for $A \to B$ in $K$ and $n \geq 0$, are certainly complete weak equivalences. Therefore, the Rezk model category structure can be described as the left Bousfield localization of the model category for dendroidal spaces by this set of maps. The description of fibrant objects in abstract left Bousfield localizations now completes the proof. □

Theorem 6.6. The locally constant model category structure on $sdSet$ (4.6) and the dendroidal Rezk model category structure (6.2) are equal.

Proof. As these two model category structures on $sdSet$ have the same class of cofibrations, it is sufficient to prove that they have the same class of fibrant objects. This follows immediately from Corollary 6.5 and from the characterization of fibrant objects given by 5.2 and 6.2 (iii). □

Corollary 6.7. The inclusion functor $dSet \subset sdSet$ is a left Quillen equivalence from the model category for $\infty$-operads to the dendroidal Rezk model category.

Proof. This follows from the preceding theorem and from Proposition 4.8. □

Corollary 6.8. A morphism of dendroidal sets is a weak operadic equivalence if and only if its image under the inclusion $dSet \subset sdSet$ is a complete weak equivalence.

Corollary 6.9. Let $X \to Y$ be a morphism between complete dendroidal Segal spaces. The following conditions are equivalent.

(a) The map $X \to Y$ is a complete weak equivalence.
(b) For any integer \( n \geq 0 \), the map \( X_n \rightarrow Y_n \) is an equivalence of \( \infty \)-operads.

(c) The map \( X_0 \rightarrow Y_0 \) is an equivalence of \( \infty \)-operads.

(d) For any tree \( T \), the map \( X(T) \rightarrow Y(T) \) is an homotopy equivalence between Kan complexes.

6.10. Consider the cosimplicial dendroidal set

\[
(6.10.1) \quad \Delta_J : \Delta \rightarrow dSet
\]
defined by

\[
(6.10.2) \quad \Delta_J[n] = i_1 N\pi_1(\Delta[n])
\]
(so that \( \Delta_J[1] = J_d \)). This cosimplicial object defines a unique colimit preserving functor

\[
(6.10.3) \quad sdSet \rightarrow dSet , \ X \mapsto |X|_J
\]
such that

\[
(6.10.4) \quad |\Delta[n] \times \Omega[T]|_J = \Delta_J[n] \otimes \Omega[T] .
\]

The functor \( |-|_J \) has a right adjoint

\[
(6.10.5) \quad dSet \rightarrow sdSet , \ X \mapsto Sing_J(X)
\]
defined by

\[
(6.10.6) \quad Sing_J(X)(T)_n = \text{Hom}_{dSet}(\Delta_J[n] \otimes \Omega[T], X) .
\]

**Proposition 6.11.** The functor \( (6.10.3) \) is a left Quillen equivalence from the dendroidal Rezk model category to the model category for \( \infty \)-operads.

**Proof.** Using the fact that \( dSet \) is a monoidal model category, it is easily seen that \( (6.10.3) \) is a left Quillen functor from the generalized Reedy model structure (given by Proposition 5.2) to the model category for \( \infty \)-operads. Therefore, to prove that this is a left Quillen functor for the dendroidal Rezk model structure, it is sufficient to prove that it sends inner horns as well as maps of shape

\[
\Omega[T] \otimes J_d \rightarrow \Omega[T] , \ T \in \Omega,
\]
to weak operadic equivalences. But this latter property follows from the fact that the composition of \( (6.10.3) \) with the inclusion \( dSet \subset sdSet \) is (isomorphic to) the identity. Similarly, to prove that \( (6.10.3) \) is a left Quillen equivalence, by virtue of Corollary 6.7 it is sufficient to prove that its composition with the inclusion \( dSet \subset sdSet \) is a left Quillen equivalence, which is more than obvious. \( \square \)

6.12. Let \( \infty\text{-Operad} \) be the full subcategory of \( dSet \) spanned by \( \infty \)-operads (i.e. fibrant objects). We define a functor

\[
(6.12.1) \quad K : \infty\text{-Operad} \rightarrow sdSet
\]
by the formula below (see 3.1):

\[
(6.12.2) \quad K(X)(T)_n = k(\Omega[T], X)_n .
\]

**Proposition 6.13.** The functor \( (6.12.1) \) takes its values in the full subcategory of \( sdSet \) spanned by complete dendroidal Segal spaces. Moreover, it preserves fibrations as well as weak equivalences between \( \infty \)-operads, and, under the canonical equivalence \( \text{Ho}(\infty\text{-Operad}) \simeq \text{Ho}(dSet) \), the corresponding functor

\[
K : \text{Ho}(dSet) \rightarrow \text{Ho}(sdSet)
\]
is canonically isomorphic to the functor
\[ R \text{Sing}_J : \text{Ho}(d\text{Set}) \longrightarrow \text{Ho}(sd\text{Set}) \]
(which is an equivalence of categories).

**Proof.** In view of the identification of Proposition 3.3, this is a straightforward application of the general properties of mapping spaces; see [Hov99, Propositions 5.4.1, 5.4.3 and 5.4.7] (remark that, for any normal dendroidal set \( A, \Delta_J(\bullet) \otimes A \) provides a canonical cosimplicial frame of \( A \) in the sense of [Hov99, Definition 5.2.7]). □

**Remark 6.14.** The Boardman-Vogt tensor product on \( d\text{Set} \) induces a symmetric monoidal structure on \( sd\text{Set} \): for two simplicial dendroidal sets \( X \) and \( Y \), their tensor product \( X \otimes Y \) is simply defined termwise:
\[
(X \otimes Y)_n = X_n \otimes Y_n, \quad n \geq 0.
\]
Using the fact that \( d\text{Set} \) is a symmetric monoidal model category with respect to the Boardman-Vogt tensor product (Theorem 1.1), it is easily seen that \( sd\text{Set} \), endowed with the dendroidal Rezk model structure, is also a symmetric monoidal model category. Moreover, the functor \( d\text{Set} \subset sd\text{Set} \) is a symmetric monoidal left Quillen equivalence.

### 7. Segal pre-operads

**Definition 7.1.** A Segal pre-operad is a dendroidal space \( A \) such that \( A(\eta) \) is a discrete simplicial set (i.e., all the simplices of positive dimension in \( A(\eta) \) are degenerated). We denote by \( \mathcal{PreOper} \) the full subcategory of \( sd\text{Set} \) spanned by Segal pre-operads.  

**7.2.** The category of Segal pre-operads is in fact the category of presheaves on the category \( \text{S}(\Omega) \), which is obtained as the localization of \( \Delta \times \Omega \) by the arrows of shape \( ([n], \eta) \rightarrow ([m], \eta) \). We denote by
\[
\gamma : \Delta \times \Omega \longrightarrow \text{S}(\Omega)
\]
the localization functor. Under the identification \( \mathcal{PreOper} \simeq \hat{\text{S}(\Omega)} \), the inverse image functor
\[
\gamma^* : \mathcal{PreOper} \longrightarrow sd\text{Set}
\]
is simply the inclusion functor. The inclusion functor \( \gamma^* \) thus has a right adjoint
\[
\gamma_* : sd\text{Set} \longrightarrow \mathcal{PreOper}
\]
as well as a left adjoint
\[
\gamma ! : sd\text{Set} \longrightarrow \mathcal{PreOper}.
\]
The explicit description of these adjoints will be needed later on.

The right adjoint, \( \gamma_* : sd\text{Set} \longrightarrow \mathcal{PreOper} \), is defined as follows. Let \( X \) be a dendroidal space. Then \( \gamma_*(X) \) is the subobject of \( X \) given by all the dendrices whose vertices are degenerated. More explicitly, for a tree \( T \), let write \( E(T) \) for its set of edges (colours), with the evident inclusion (which is natural in \( T \))
\[
\nu_T : \prod_{e \in E(T)} \eta \longrightarrow \Omega[T].
\]
For a simplicial set $K$, we shall identify the set $K_0$ with the corresponding discrete simplicial set, and write $s: K_0 \to K$ for the inclusion. Then $\gamma_s(X)(T)$ is defined as the following pullback of simplicial sets.

$$\gamma_s(X)(T) \to X(T) \to \prod_{e \in E(T)} X(\eta)_0 \leftarrow \prod_{e \in E(T)} X(\eta)$$

(7.2.5)

The left adjoint $\gamma! : sdSet \to PreOper$ can also be made explicit as follows. For a simplicial dendroidal set $X$ as above, consider the set $\pi_0 X(\eta)$ of connected components of the simplicial set $X(\eta)$. We have $\gamma!(X)(T) = X(T)$ for any tree $T$ such that there is no map $T \to \eta$ in $\Omega$. If there is a map $\varepsilon : T \to \eta$ in $\Omega$, then its unique (remember there is a canonical isomorphism $\Omega / \eta = \Delta$), and we can describe $\gamma!(X)(T)$ as the pushout below.

$$X(\eta) \to X(T) \to \pi_0 X(\eta) \leftarrow \gamma!(X)(T)$$

(7.2.6)

7.3. A morphism of Segal pre-operads is a monomorphism if and only if its image by $\gamma^*$ is (because $\gamma^*$ is a fully faithful limit preserving functor). We say that a morphism of Segal pre-operads $X \to Y$ is a normal monomorphism if its image by $\gamma^*$ has the same property (this just means that the map $X_n \to Y_n$ is a normal monomorphism of dendroidal sets for any integer $n \geq 0$).

A Segal pre-operad $X$ is normal if $\emptyset \to X$ is a normal monomorphism.

A morphism of Segal pre-operads is a trivial fibration if it has the right lifting property with respect to the class of normal monomorphisms.

**Lemma 7.4.** If $X \to Y$ is a normal monomorphism of simplicial dendroidal sets and if $\pi_0 X(\eta) \to \pi_0 Y(\eta)$ is injective, then $\gamma!(X) \to \gamma!(Y)$ is a normal monomorphism of Segal pre-operads.

**Proof.** One sees easily from the explicit description of $\gamma!$ given by the pushouts [7.2.6] that, for any tree $T$ above $\eta$, the map $\gamma!(X)(T) \to \gamma!(Y)(T)$ is injective. For any tree $T$ which has a non trivial automorphism in $\Omega$, there is no map from $T$ to $\eta$. As, for such a tree $T$, we have $\gamma!(X)(T) = X(T)$, it is clear that the map $\gamma!(X) \to \gamma!(Y)$ is a normal monomorphism. \qed

**Proposition 7.5.** Let $I$ be the set of maps

$$\gamma!(\partial \Delta[n] \times \Omega[T] \cup \Delta[n] \times \partial \Omega[T]) \to \gamma!(\Delta[n] \times \Omega[T])$$

(7.5.1)

for any tree $T$ with at least one vertex, and for any integer $n \geq 0$, together with the map $\emptyset \to \eta$. Then $I$ generates the class of normal monomorphisms of Segal pre-operads (as a saturated class). In other words, the smallest class of maps in $PreOper$ which is closed under pushout, transfinite composition and retract, and which contains $I$, is the class of normal monomorphisms.

**Proof.** Let us call $I$-cofibrations the elements of the smallest class of maps which contains $I$ and is closed under pushout, transfinite composition, and retract.
If $T$ is a tree with at least one vertex, then, for any integer $n \geq 0$, the evaluation of the map
\begin{equation}
\partial \Delta[n] \times \Omega[T] \cup \Delta[n] \times \partial \Omega[T] \to \Delta[n] \times \Omega[T]
\end{equation}
at $\eta$ is bijective, so that, by virtue of Lemma 7.4, its image by $\gamma!$ is a normal monomorphism. Hence any map in $I$ is a normal monomorphism of Segal pre-operads. Therefore, any $I$-cofibration is a normal monomorphism.

Conversely, consider a normal monomorphism of Segal pre-operads $u : A \to B$. Let $A'$ be the Segal pre-operad obtained from the pushout below.

Then the inclusion $A \to A'$ is certainly an $I$-cofibration, and one checks easily that the canonical map $A' \to B$ is still a normal monomorphism. Thus, to prove that $u : A \to B$ is an $I$-cofibration, we may assume, without loss of generality, that the map $A(\eta) \to B(\eta)$ is bijective on 0-simplices. Applying the small object argument to the map $u$ with the set of maps (7.5.1) (for any tree $T$ with at least one vertex, and for any integer $n \geq 0$), we obtain a factorization of $u$ of shape
\begin{equation}
A \to v C \to p B,
\end{equation}
in which $v$ is an $I$-cofibration, while $p$ has the right lifting property with respect to maps of shape (7.5.1) (still for any tree $T$ with at least one vertex, and for any integer $n \geq 0$). Moreover, one checks that $v$ induces a bijection by evaluating at $\eta$, which implies that $p$ has the same property. We claim that $\gamma^*(p)$ has the right lifting property with respect to maps of shape (7.5.2) (for any tree $T$ and any integer $n$). Indeed, in the case $T$ has at least one vertex this follows by a standard adjunction argument. In the case where $T = \eta$, this lifting property means that the map $C(\eta) \to B(\eta)$ is a trivial fibration between discrete simplicial sets, i.e. is a bijective map on the 0-simplices. Hence, since $\gamma^*$ is fully faithful, the map $p$ has the right lifting with respect to $u$. By the retract argument [Hov99, Lemma 1.1.9], this implies that $u$ is a retract of $v$, whence is an $I$-cofibration. □

8. Segal operads

**Definition 8.1.** A *Segal operad* is a Segal pre-operad $X$ such that, for any tree $T$, the map
\begin{equation}
X(T) = X^{\Omega[T]} \to X^{\Sc[T]}
\end{equation}
is a trivial fibration of simplicial sets, where, if $A$ is a dendroidal set, $X^A$ denotes the simplicial set whose $n$-simplices are the maps of dendroidal sets from $A$ to $X_n$ (with the notations of 5.1, we thus have $\gamma^*(X)^A = X^A$). We write $\SegOper$ for the full subcategory of $\PreOper$ spanned by Segal operads.

A *Reedy fibrant Segal operad* is a Segal pre-operad whose image by $\gamma^*$ is fibrant in the model category structure for dendroidal Segal spaces (see Definition 5.4). Note that any Reedy fibrant Segal operad is indeed a Segal operad; see Corollary 5.6.

A morphism of Segal pre-operads is a *Segal weak equivalence* if its image by $\gamma^*$ is a complete weak equivalence (6.2).
A morphism between Reedy fibrant Segal operads is fully faithful if its image by the functor $\gamma^*$ is fully faithful; see §5.9.

**Proposition 8.2.** Let $X$ be a dendroidal Segal space. Then $\gamma_*(X)$ is a Reedy fibrant Segal operad. If $X$ is moreover a complete dendroidal Segal space, then the co-unit map $\gamma^*\gamma_*(X) \to X$ is a complete weak equivalence.

**Proof.** Since, for any tree $T$, the evaluation of the map $\text{Sc}[T] \to \Omega[T]$ at $\eta$ is bijective, the commutative square
\[
\begin{array}{ccc}
\gamma^*\gamma_*(X)^{\Omega[T]} & \to & X^{\Omega[T]} \\
\downarrow & & \downarrow \\
\gamma^*\gamma_*(X)^{\text{Sc}[T]} & \to & X^{\text{Sc}[T]}
\end{array}
\]
is cartesian. Therefore, the functor $\gamma^*\gamma_*$ preserves dendroidal Segal spaces. In other words, the functor $\gamma_*$ sends dendroidal Segal spaces to Reedy fibrant Segal operads.

Assume now that $X$ is a complete dendroidal Segal space. Then $Y = \gamma^*\gamma_*(X)$ is a Segal space. In particular, for any integer $n \geq 0$, the dendroidal sets $Y_n$ and $X_n$ are $\infty$-operads. Essentially by construction, the maps $Y_n \to X_n$ are moreover fully faithful in the sense of definition §3.7 (because isomorphisms of simplicial sets are weak homotopy equivalences). As $X$ is a complete dendroidal Segal space, we know that, for any integer $n \geq 0$, the map $X_n \to X_n$ is a weak operadic equivalence; see Proposition §4.7 and Theorem §6.6. As the map $X(\eta)_0 \to X(\eta)_n$ factors through $Y(\eta)_n \to X(\eta)_n$ (this is easily seen from the pullback square \((7.2.3)\)), the map $Y_n \to X_n$ has to be essentially surjective, whence, by virtue of Theorem §5.11, a weak operadic equivalence for any integer $n \geq 0$. Therefore, the map $Y \to X$ is a complete weak equivalence (using again Theorem §6.6). □

**Lemma 8.3.** Let $X \to Y$ a morphism of dendroidal Segal spaces. Assume that the map $X(\eta)_0 \to Y(\eta)_0$ is bijective, and that, for any tree $T$, the map $X(T) \to Y(T)$ is a weak homotopy equivalence. Then the map $\gamma^*\gamma_*(X) \to \gamma^*\gamma_*(Y)$ is a weak equivalence between fibrant objects in the model category for dendroidal Segal spaces \(5.4\).

**Proof.** It follows immediately from the fact that, for any Segal space $X$, the pullback squares of shape \((7.2.3)\) are homotopy pullback squares of Kan complexes (because the map $X(T) \to \prod_{e \in E(T)} X(\eta)$ is a Kan fibration) that the map $\gamma^*\gamma_*(X)(T) \to \gamma^*\gamma_*(Y)(T)$ is a simplicial weak equivalence for any tree $T$. In particular, the map $\gamma^*\gamma_*(X) \to \gamma^*\gamma_*(Y)$ is a weak equivalence in the model category for dendroidal Segal spaces. □

**Lemma 8.4.** Let $f : X \to Y$ be a morphism of simplicial dendroidal sets. Assume that the map $f$ is injective and induces a bijection $X(\eta)_0 \simeq Y(\eta)_0$. Then the map $\gamma_*(f) : \gamma_*(X) \to \gamma_*(Y)$ is an isomorphism of Segal pre-operads.

**Proof.** Let $T$ be a tree, and $n \geq 0$ an integer. We have then the following identification:
\[
Y(T)_n = (Y(T)_n - X(T)_n) \amalg X(T)_n.
\]
This implies this lemma, by contemplating the explicit description of the functor $\gamma_*$ given by the cartesian squares \((7.2.3)\). □
Lemma 8.5. Let $Y$ be a dendroidal Segal space. Then the evaluation at 0 map $\mathcal{H}om(J_d, Y) \to Y$ has the right lifting property with respect to the maps of shape

$$\partial \Delta[n] \times \Omega[T] \cup \Delta[n] \times \partial \Omega[T] \to \Delta[n] \times \Omega[T],$$

for any tree $T$ with at least one vertex, and any integer $n \geq 0$.

Proof. Notice first that, for any integer $n \geq 0$, the map $Y^n = Y^{\Delta[n]} \to Y^{\partial \Delta[n]}$ is an inner Kan fibration between $\infty$-operads. But, by virtue of [CMa, Theorems 6.4 and 6.10], for any inner Kan fibration between $\infty$-operads $Z \to W$, and for any tree $T$ with at least one vertex, the map

$$\mathcal{H}om(\Omega[T], Z) \to \mathcal{H}om(\Omega[T], W) \times_{\mathcal{H}om(\partial \Omega[T], W)} \mathcal{H}om(\partial \Omega[T], Z)$$

is a fibration of dendroidal sets (for the model category structure for $\infty$-operads), from which we deduce that the map

$$k(\Omega[T], Z) \to k(\Omega[T], W) \times_{k(\partial \Omega[T], W)} k(\partial \Omega[T], Z)$$

is a Kan fibration of simplicial sets; see [CMa, Proposition 6.7 and Corollary 6.8]. In particular, this map has the right lifting property with respect to the inclusion $\{0\} \to J$. Applying this to the inner Kan fibration $Y^{\Delta[n]} \to Y^{\partial \Delta[n]}$, this proves the lemma. $\square$

Remark 8.6. Recall we have the three sets of maps below.

(α) $\Lambda^k[n] \times \Omega[T] \cup \Delta[n] \times \partial \Omega[T] \to \Delta[n] \times \Omega[T]$ for any tree $T$ and integers $0 \leq k \leq n$, $n > 0$,

(β) $\partial \Delta[n] \times \Omega[T] \cup \Delta[n] \times \Lambda^e[T] \to \Delta[n] \times \Omega[T]$ for any tree $T$ with inner edge $e$, and integer $n \geq 0$,

(γ) $\partial \Delta[n] \times (\Omega[T] \otimes J_d) \cup \Delta[n] \times (\partial \Omega[T] \otimes J_d \cup \Omega[T] \otimes \{0\}) \to \Delta[n] \times (\Omega[T] \otimes J_d)$ for any tree $T$ and integer $n \geq 0$.

A simplicial dendroidal set $X$ is a dendroidal Segal space (resp. a complete dendroidal Segal space) if and only if the map, $X \to *$, from $X$ to the terminal dendroidal set $*$, has the right lifting property with respect to the maps of shape (α) and (β) (resp. (α), (β), and (γ)). However, it follows from Lemma 8.5 that a dendroidal Segal space $X$ is complete if and only if the map $X(J) \to X(\eta)$ is a trivial fibration of simplicial sets. Note that the map (β) is a split monomorphism for $T = \eta$ and $n = 0$. In other words, a dendroidal Segal space $X$ is complete if and only if the map $X \to *$ has the right lifting property with respect to the maps of shape

(γ') $\partial \Delta[n] \times J_d \cup \Delta[n] \times \{0\} \to \Delta[n] \times J_d$ for any integer $n > 0$.

Lemma 8.7. Let $X$ be a Segal space. There exists a fibrant replacement $X \to Y$ in the model category for complete Segal spaces such that the induced morphism $\gamma^* \gamma_*(X) \to \gamma^* \gamma_*(Y)$ is a weak equivalence in the model category for dendroidal Segal spaces.
Proof. For a simplicial dendroidal set $X$, we write $X \to L(X)$ (resp. $X \to L'(X)$) for the map obtained by applying the small object argument to the map $X \to *$ with respect to the set of maps of shape $\{\emptyset\}$ and $\{\emptyset\}$ (resp. with respect to the set of maps of shape $\{\emptyset\}$) described in Remark 8.6. It follows from the considerations of Remark 8.6 that the fibrant replacement of $X$ for the Rezk model category structure can be obtained by a transfinite iteration of the endofunctor $L \circ L'$. Notice that $L(X)$ is a dendroidal Segal space for any $X$. Therefore, whenever $X$ is a Segal space, as the map $X \to L(X)$ is a weak equivalence in the model category for Segal spaces, the map $X \to \gamma_*(L(X))$ is a simplicial weak equivalence (between Kan complexes) for any tree $T$. Furthermore, any map of shape $\{\emptyset\}$ and $\{\emptyset\}$ induces a bijection after evaluating at $([0], \eta)$, so that, by virtue of Lemma 8.3, for any Segal space $X$, the canonical map $\gamma_*(L(X)) \to \gamma_*(L(X))$ is a weak equivalence between fibrant objects in the model structure for dendroidal Segal spaces.

On the other hand, by virtue of Lemma 8.4, for any simplicial dendroidal set $X$, the map $\gamma_*(X) \to \gamma_*(L'(X))$ is an isomorphism. As the functor $\gamma_*$ preserves filtered colimits, and as the class of simplicial weak equivalences is closed under filtered colimits, this ends the proof. □

Proposition 8.8. The functor $\gamma_*$ sends complete weak equivalences between dendroidal Segal spaces to Segal weak equivalences, and, for any dendroidal Segal space $X$, the map $\gamma_*(X) \to \gamma_*(L(X))$ is a complete weak equivalence.

Proof. It is clearly sufficient to prove the last assertion, which follows from the fact that, by virtue of Proposition 8.2 and of the preceding lemma, for any dendroidal Segal space $X$, there exists a commutative square

$$
\begin{array}{ccc}
\gamma_*(X) & \to & \gamma_*(Y) \\
\downarrow & & \downarrow \\
X & \to & Y
\end{array}
$$

in which three whence all maps are weak equivalences, with $Y$ a complete Segal space. □

Corollary 8.9. A morphism between dendroidal Segal spaces $X \to Y$ is a complete weak equivalence if and only if $\gamma_*(X) \to \gamma_*(Y)$ is a Segal weak equivalence of Segal operads.

Definition 8.10. Given a dendroidal Segal space $X$, we associate to it an operad $\mathcal{O}(X)$ whose set of colours is $X(\eta)_0$, and whose sets of maps are given by $\pi_0(X(x_1, \ldots, x_n; x))$.

A morphism of dendroidal Segal spaces $X \to Y$ is essentially surjective if the morphism of operads $\mathcal{O}(X) \to \mathcal{O}(Y)$ is essentially surjective.

Theorem 8.11. Let $f : X \to Y$ be a morphism of dendroidal Segal spaces. The following conditions are equivalent.

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2We leave as an exercise for the reader that this defines an operad. However, we will only need the underlying category of this operad, and the fact that this defines a category is a very standard fact which is easy to check directly. The reader who does not want to make any explicit computation may also prove this by using the explicit description of the operad associated to an $\infty$-operad (see [CMa, Proposition 6.14]), Proposition 8.3 (to reduce to the case of a complete dendroidal Segal space), Proposition 3.3, as well as the Quillen equivalence of Proposition 6.11.
(a) The map $f$ is a complete weak equivalence;
(b) The map $\gamma_*(f) : \gamma_*(X) \rightarrow \gamma_*(Y)$ is a weak equivalence of Segal operads.
(c) The map $\gamma^*\gamma_*(f) : \gamma^*\gamma_*(X) \rightarrow \gamma^*\gamma_*(Y)$ is fully faithful and essentially surjective.
(d) The map $f$ is fully faithful and essentially surjective.

Proof. The equivalence between (a) and (b) follows from Corollary 8.9, while the equivalence between (c) and (d) is a tautology. To prove the remaining equivalences, we will use Theorem 3.11 to deduce the equivalence between conditions (a) and (d). Indeed, we may assume that $X$ and $Y$ are complete, so that the evaluated maps $X(T) \rightarrow Y(T)$ may be identified with the maps of mapping spaces $Map(\Omega[T], X) \rightarrow Map(\Omega[T], Y)$; up to the Quillen equivalence $dSet \subset sSet$ (see Corollary 6.7), and using Proposition 3.3, condition (d) (resp. (a)) may be interpreted by saying that the map between the corresponding $\infty$-operads is fully faithful and essentially surjective (resp. satisfies condition (b) of Theorem 3.5). This completes the proof of the theorem. □

Lemma 8.12. If a morphism of Segal pre-operads has the right lifting property with respect to normal monomorphisms, then it is a Segal weak equivalence.

Proof. Consider first a normal resolution $E_\infty$ of the terminal dendroidal set (i.e. a cofibrant resolution of the terminal dendroidal set for the model category structure of Theorem 1.1), considered as a simplicially constant simplicial dendroidal set. We may see $E_\infty$ as a Segal pre-operad, and it follows immediately from Theorem 6.6 that, for any Segal pre-operad $X$, the projection $E_\infty \times X \rightarrow X$ is a Segal weak equivalence. Moreover, as $E_\infty$ is even a normal Segal pre-operad, $E_\infty \times X$ is always normal. Therefore, it is sufficient to prove that, if $p : X \rightarrow Y$ is a morphism of normal Segal pre-operads which has the right lifting property with respect to normal monomorphisms of pre-operads, then it is a Segal weak equivalence. Note that $J_d$ may be seen as Segal pre-operad, so that, for any normal pre-operad $A$, $J_d \otimes A$ is still a pre-operad, and, whenever $A$ is normal, the map $A \amalg A = (\{0\} \amalg \{1\}) \otimes A \rightarrow J \otimes A$ is a normal monomorphism; see Corollary 4.3 and CMa Proposition 1.9]. We may now finish the proof in the standard way: as $Y$ is normal, the map $p$ admits a section $s : Y \rightarrow X$, and the commutative square

admits a lifting $h$ (where $\pi : J_d \otimes X \rightarrow X$ denotes the map induced by the map $J_d \rightarrow \eta$), which means that the map $p$ is a $J_d$-homotopy equivalence, whence a weak equivalence. □

Theorem 8.13. The category of Segal pre-operads is endowed with a left proper cofibrantly generated model category structure whose weak equivalences are the Segal weak equivalences, and whose cofibrations are the normal monomorphisms.

Proof. The preceding lemma tells us that any trivial fibration of Segal pre-operads is a Segal weak equivalence. On the other hand, by virtue of Proposition 7.5, the class
of normal monomorphism is generated by a small set of maps. The existence of this model category is thus a particular case of J. Smith’s Theorem; see [Bek00, Theorem 1.7 and Proposition 1.18]. The left properness property follows immediately from its counterpart for the Rezk model category structure.

8.14. The model category structure above will be called the Reedy-Segal model category structure on $\mathcal{P}re\mathcal{O}per$. We will always consider the category of simplicial dendroidal sets as a model category with the Rezk model structure (8.2). By construction, the functor $\gamma^* : \mathcal{P}re\mathcal{O}per \rightarrow sd\mathcal{S}et$ is a left Quillen functor. Our purpose is to prove that it is in fact a left Quillen equivalence, and that the fibrant pre-operads are precisely the Reedy fibrant Segal operads.

**Theorem 8.15.** The functor $\gamma^* : \mathcal{P}re\mathcal{O}per \rightarrow sd\mathcal{S}et$ is a left Quillen equivalence from the model category for Segal operads to the model category for complete dendroidal Segal spaces.

**Proof.** As $\gamma^*$ is a fully faithful left Quillen functor which preserves and detects weak equivalences, this follows immediately from Proposition 8.8. □

**Remark 8.16.** Note that Segal pre-operads are closed under tensor product (as defined in Remark 6.14), and that the model category of Theorem 8.13 is symmetric monoidal, in such a way that the left Quillen functor of Theorem 8.15 is symmetric monoidal as well (this is immediate from Remark 6.14).

**Theorem 8.17.** Let $X$ be a Segal pre-operad. The following conditions are equivalent:

(a) $X$ is fibrant in the model structure of Theorem 8.13;
(b) $X$ is a Reedy fibrant Segal operad;
(c) $X$ is a retract of $\gamma_*(Y)$ for some complete dendroidal Segal space $Y$;
(d) $X$ is a retract of $\gamma_*(Y)$ for some dendroidal Segal space $Y$.

**Proof.** Condition (a) implies condition (b) because the inclusions $Sc[T] \rightarrow \Omega[T]$ are trivial cofibrations in the symmetric monoidal model category structure of Theorem 8.13.

Let us prove that condition (b) implies condition (c). If $X$ is Reedy fibrant, then we can choose a trivial cofibration $\gamma^*(X) \rightarrow Y$ with $Y$ a complete dendroidal Segal space. By virtue of Lemma 8.7 and Proposition 8.8 we may assume that the map $\gamma^*X \simeq \gamma_*\gamma^*(X) \rightarrow \gamma^*\gamma_*(Y)$ is a weak equivalence between fibrant objects in the model category for dendroidal Segal spaces. As $\gamma^*$ is fully faithful, to prove that $X$ is a retract of $\gamma_*(Y)$, it is sufficient to prove that the map $\gamma^*X \rightarrow \gamma^*\gamma_*(Y)$ is a cofibration (i.e. a normal monomorphism): this follows from the fact that, by assumption, for any tree $T$, the group $\text{Aut}(T)$ of automorphisms of $T$ in $\Omega$ acts freely on $Y(T) - X(T)$, and that we have an $\text{Aut}(T)$-equivariant inclusion of $\gamma_*(Y)(T) - X(T)$ in $Y(T) - X(T)$; see the cartesian square (7.2.3).

Condition (c) implies condition (a): as $\gamma_*$ is a right Quillen functor (Corollary 8.15), $\gamma_*(Y)$ is fibrant for any complete dendroidal Segal space $Y$, and the class of fibrant objects of any model category is closed under retracts.

It is clear that condition (c) implies condition (d).

Finally, the fact that condition (d) implies condition (b) follows from the fact that $\gamma_*$ sends dendroidal Segal spaces to Reedy fibrant Segal operads (see the first assertion of Proposition 8.2). □
Remark 8.18. Recall from Section 6.12 the canonical functor

\[ K : \infty\text{-Operad} \to sdSet. \]

We know that \( K \) sends \( \infty\text{-operads} \) to complete dendroidal Segal spaces, so that, by virtue of Proposition 8.2, we obtain a functor

\[ \gamma_* K : \infty\text{-Operad} \to SegOper. \]

We also know from Proposition 6.13 and from Theorem 8.15 that \( \gamma_* K \) sends weak equivalences of \( \infty\text{-operads} \) to weak Segal equivalences, and that the induced functor

\[ \gamma_* K : \text{Ho}(\infty\text{-Operad}) \to \text{Ho}(SegOper) \]

is an equivalence of categories.

Remark as well that any dendroidal set is a pre-operad, so that the inclusion \( dSet \subset sdSet \) factors through an inclusion \( dSet \subset PreOper \) which happens to be a left Quillen equivalence (this follows immediately from Corollary 6.4 and from Theorem 8.15). If \( X \) is an \( \infty\text{-operad} \), seen as a Segal pre-operad, then \( \gamma_* K (X) \) is a canonical fibrant replacement of \( X \) in the model category of Theorem 8.13.

Remark 8.19. The identification \( dSet/\eta = sSet \) allows us to deduce the Joyal model category structure for quasi-categories from the model category structure for \( \infty\text{-operads} \); see [CMa, Corollary 2.10]. Similarly, the dendroidal Rezk model structure of Definition 6.2 induces Rezk’s original model structure for complete Segal spaces ([Rez01, Section 12]), while the model category structure for Segal categories can be obtained from the model category structure of Theorem 8.13 by slicing over \( \eta \) as well. The Quillen equivalences relating the homotopy theories of Segal categories and of complete Segal spaces, proved by Joyal and Tierney in [JT07] are deduced immediately from their dendroidal analog, namely Corollary 6.7 and Proposition 8.11, while the Quillen equivalence from the model category for Segal categories to the model category for complete Segal spaces, proved by Bergner in [Ber07], is a direct consequence of Theorem 8.15.

References

[Bek00] T. Beke, Sheafifiable homotopy model categories, Math. Proc. Camb. Phil. Soc. 129 (2000), 447–475.
[BM] C. Berger and I. Moerdijk, On an extension of the notion of Reedy category, arXiv:0809.3311 to appear in Math. Z.
[BV73] J. M. Boardman and R. M. Vogt, Homotopy invariant algebraic structures on topological spaces, Lecture Notes in Math., vol. 347, Springer-Verlag, 1973.
[Cis03] D.-C. Cisinski, Images directes cohomologiques dans les catégories de modèles, Annales Mathématiques Blaise Pascal 10 (2003), 195–244.
[Cis06] Les préfaisceaux comme modèles des types d’homotopie, Astérisque, vol. 308, Soc. Math. France, 2006.
[CMa] D.-C. Cisinski and I. Moerdijk, Dendroidal sets as models for \( \infty\text{-operads} \), preprint arXiv:0902.1954 to appear in Journal of Topology.
[CMB] Dendroidal sets and simplicial operads, in preparation.
[Hir03] P. S. Hirschhorn, Model categories and their localizations, Math. surveys and monographs, vol. 99, Amer. Math. Soc., 2003.
[Hov99] M. Hovey, Model categories, Math. surveys and monographs, vol. 63, Amer. Math. Soc., 1999.
[Joy02] A. Joyal, Quasi-categories and Kan complexes, J. Pure Appl. Algebra 175 (2002), no. 1-3, 207–222.
[JT07] A. Joyal and M. Tierney, *Quasi-categories vs Segal spaces*, Categories in Algebra, Geometry and Physics, Contemp. Math., vol. 431, Amer. Math. Soc., 2007, pp. 277–326.

[Lur09] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, 2009.

[MW07] I. Moerdijk and I. Weiss, *Dendroidal sets*, Algebraic & Geometric Topology 7 (2007), 1441–1470.

[MW09] _, *On inner Kan complexes in the category of dendroidal sets*, Adv. Math. 221 (2009), no. 2, 343–389.

[Rez01] C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. 353 (2001), no. 3, 973–1007.

[RSS01] C. Rezk, S. Schwede, and B. Shipley, *Simplicial structures on model categories and functors*, American Journal of Mathematics 123 (2001), no. 3, 551–575.

[Sim] C. T. Simpson, *Homotopy theory of higher categories*, preprint arXiv:1001.4071.

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