I. INTRODUCTION

The magnetic self-effects of the Josephson current in wide superconductor-thin interlayer-superconductor junctions result in the screening inside the junction interlayer of the magnetic field applied at the junction edge, and in the formation of Josephson vortices. Spatial variations of the magnetic field and of the current density along the junction interface are coupled with variations of the phase difference and should be determined jointly.

The corresponding results for standard tunnel junctions with a sufficiently weak Josephson coupling have been obtained within the Ginzburg-Landau (GL) theory since the early days of studying the Josephson effect.1–4 The results imply that the Josephson penetration depth \( \lambda_J \) significantly exceeds the London penetration depth \( \lambda_L \), as is commonly observed. The Meissner screening being significantly stronger than the Josephson one is consistent with the definition of “weak superconductivity”, although it is not generally an integral feature of weak links.

The conventional definition of weak links and, in particular of tunnel junctions, requires the critical current \( j_c \) to be significantly less than the depairing current \( j_{dp} \) deep within the superconducting leads. On the other hand, the condition \( \lambda_J \gg \lambda_L \) is equivalent to \( j_c^{1/2} \ll \left( \frac{j_{dp}}{\kappa} \right)^{1/2} \), where \( \kappa \) is the GL parameter. For junctions involving strongly type-II superconductors, the relation presented can prove to be more restrictive for the critical current than the weak-link requirement \( j_c \ll j_{dp} \). Hence, the standard results only apply to tunnel junctions satisfying the condition \( j_c^{1/2} \ll \left( \frac{j_{dp}}{\kappa} \right)^{1/2} \). In the opposite case \( j_c^{1/2} \gtrsim \left( \frac{j_{dp}}{\kappa} \right)^{1/2} \), \( j_c \ll j_{dp} \), which ensures the relation \( \lambda_J \lesssim \lambda_L \) for weak links, the electrodynamics of tunnel junctions acquires a nonlocal character.5–6 In a strongly nonlocal regime \( \lambda_J \ll \lambda_L \), the characteristic scale of an isolated Josephson vortex along the junction plane is \( \lambda_J^2 / \lambda_L \), which is substantially less than the Josephson penetration depth \( \lambda_J \).

Recently the nonlocality has been experimentally identified in planar junctions with thin superconducting electrodes7, where the conditions for observing the nonlocal effects8–13 are modified, and monitored more easily as compared with the junctions with thick leads.

A distinctive feature of superconducting junctions considered in this paper is the presence of an interfacial pair breaking. An intense interfacial pair breaking can take place, for example, in junctions involving unconventional superconductors and/or magnetic or normal metal interlayers. Since in a small transition region weak links are quite sensitive to local conditions, an interface-induced local weakening of the superconducting condensate density can have a profound influence on the whole of the Josephson effect. As the result, the interplay of the Josephson coupling strength and interfacial pair activity controls the behavior of the supercurrent.

A weak Josephson coupling leads to the sinusoidal (harmonic) current-phase relation, whereas a strongly anharmonic supercurrent emerges at the large values of the coupling constant. In planar junctions with a strong Josephson coupling and vanishing interfacial pair activity the critical current \( j_c \) becomes comparable with the depairing current \( j_{dp} \), and the junctions do not represent weak links.14 Conversely, the critical current of the junctions with an intense interfacial pair breaking is strongly suppressed, as compared to the case of no pair breaking, and can only be substantially less than \( j_{dp} \), irrespective of the Josephson coupling strength.15 Thus the interfacial pair breaking maintains the planar junctions with a pronounced Josephson coupling as weak links \( j_c \ll j_{dp} \) with strongly anharmonic current-phase relations.

This paper addresses effects of the interfacial pair breaking and of the Josephson coupling strength on the magnetic penetration depth \( l_j \) and the Josephson vortex structure in wide planar junctions involving strongly type-II superconductors. For a fixed Josephson coupling, the quantity \( l_j \) is shown to go up with the interfacial...
pair breaking. This substantially extends an applicability domain of the condition \( l_j \gg \lambda_L \) and, hence, of the local Josephson electrodynamics to the junctions with a strong Josephson coupling in the presence of an intense interfacial pair breaking.

The magnetic field dependence of the penetration depth is studied below both for harmonic and anharmonic superconducting junctions. In the junctions with the harmonic supercurrent described by local Josephson electrodynamics, the Josephson penetration depth \( \lambda_J \) is the only characteristic scale of the problem. Along with the critical current, it depends substantially on the strength of the interfacial pair breaking. Under a weak applied field, \( \lambda_J \) exactly coincides with the penetration depth, while the latter is shown to depend on the magnetic flux \( \Phi \) through the junction and to approach the value \( l_{jv} \equiv l_j(\frac{\Phi}{\Phi_0}) = \frac{\pi}{2} \lambda_J \) at half of the flux quantum.

While in harmonic junctions a characteristic size of the Josephson vortex \( l_{jv} \) (a half of its effective width) is of the same order as \( \lambda_J \), in junctions with strongly anharmonic current-phase relations the magnetic field dependence \( l_j(\Phi) \) is demonstrated to become pronounced and to result in a significant difference between \( l_{jv} \) and a weak-field penetration depth \( l_{j0} \). As a specific feature of the strongly anharmonic current-phase relation, a non-monotonic dependence of the Josephson vortex size \( l_{jv} \) and of the lower critical field on the Josephson coupling strength, for a fixed and intense interfacial pair breaking, is identified within the local Josephson electrodynamics. Finally, the spatial structure of an isolated Josephson vortex in the junctions with an intense interfacial pair breaking is studied. In particular, narrow peaks in the current-phase relation of strongly anharmonic junctions are shown to transform into narrow peaks in a spatial profile of the supercurrent density in the vortex.

The paper is organized as follows. The magnetic field dependence of the penetration depth in harmonic junctions is described in Sec. II. In Sec. III the penetration depth in anharmonic junctions is obtained as a function of the magnetic field, of the Josephson coupling constant and of the strength of the interfacial pair breaking. Section IV addresses spatial profiles of the phase difference, of the magnetic field and of the supercurrent density in an isolated Josephson vortex in anharmonic junctions. The lower critical field in such junctions is found in Sec. V. Section VI contains discussions and Sec. VII concludes the paper.

II. \( l_j \) IN HARMONIC JUNCTIONS

Let the static magnetic field \( H = H e_z \) be applied along the \( z \) axis to a symmetric planar junction involving thick leads made of strongly type-II superconductors (see Fig. 1). A homogeneous plane rectangular interlayer at \( x = 0 \) is supposed to be of zero length within the GL approach. The spatially constant widths \( L_y, L_z \) of the junction are considered to significantly exceed the penetration depths: \( L_y, L_z \gg l_j, \lambda_L \). Under such conditions the magnetic field is independent of the \( z \) coordinate inside the interlayer and in the superconductors.

The applied field is assumed to be substantially less than the critical fields of the leads, and to produce a negligibly small influence on the Josephson current as a function of the phase difference \( j(\chi) \). At the same time the self-field effects, generated by the current flowing through wide junctions, interconnect the magnetic field \( H(\chi(y)) \), the supercurrent density \( j(\chi(y)) \) and the spatially dependent phase difference \( \chi(y) \), and can have a profound influence on their spatial distributions.

Within the local Josephson electrodynamics, which presupposes the condition \( l_j \gg \lambda_L \), the spatially dependent static phase difference \( \chi(y) \) in the junctions with a harmonic current-phase relation \( j(\chi) = j_c \sin \chi \) satisfies a well-known one-dimensional sine-Gordon equation\(^1\)–\(^4\)

\[
\frac{d^2 \chi(y)}{dy^2} = \frac{1}{\lambda^2} \sin \chi(y). \tag{1}
\]

Here \( \lambda_J \) is the Josephson penetration depth \( \lambda_J = (c \Phi_0 / 16 \pi^2 \lambda_L j_c) \) and \( \Phi_0 = \pi \hbar c / |e| \) is the superconductor flux quantum.

The self-consistent results of the GL theory for the Josephson current \( j(\chi) \) in planar junctions\(^15\)–\(^17\), is being used below. The order parameters in the two superconducting leads is written as \( f_{1(2)}(x) e^{i \lambda(x)}(x) \), where the moduli \( f_{1(2)}(x) \) are normalized to their values in the bulk in the absence of the supercurrent. In symmetric junctions \( f = f(|x|) \), i.e., \( f_2(x) = f_1(-x) \), and the boundary conditions for \( f \) are

\[
\left( \frac{df}{dx} \right) \pm = \pm (g_6 + 2 g_z \sin^2 \frac{\chi}{2}) f_0, \tag{2}
\]

where \( f_0 \) is an interface value of \( f(x) \), \( \mp = x / \xi(T) \), \( \xi(T) \) is the temperature dependent superconductor coherence length and \( \chi \) is the phase difference \( \chi = \chi_+ - \chi_- \).

The coefficient \( g_z \) in (2) is the effective dimensionless Josephson coupling constant, and \( g_6 \) is the effective dimensionless interface parameter. The parameters \( g_6 \) and \( g_z \) are the main characteristics of the interface in the GL theory. They are assumed to be positive and, therefore,
resulting in an interfacial pair breaking in accordance with (2). In the absence of the current, i.e., at \( \chi = 0 \), the suppression of the order parameter at the interface is described solely by \( g_\delta \). When the supercurrent flows, the Josephson coupling contributes to the phase dependent suppression of the order parameter at the interface\(^{17} \).

In macroscopic samples of strongly type-II superconductors, the influence of the interfacial pair breaking on the Meissner effect is small, in the measure of \( \kappa^{-1} \ll 1 \), and will be disregarded below. Thus the local penetration depth of the Meissner effect is considered to be spatially constant, irrespective of the boundary conditions for the order parameter, and equal to \( \lambda_L \) which is related to the bulk condensate density. Contrary to its negligible influence on the Meissner effect, the interfacial pair breaking can have a considerable impact both on the critical current and, in the presence of a pronounced Josephson coupling, on the current-phase relation. For this reason the standard expression and estimates for \( j(\chi) \), which do not take into account effects of the interfacial pair breaking and of the Josephson coupling strength, can fail.

The harmonic current-phase relation \( j(\chi) = j_c \sin \chi \) takes place under the condition \( g \ll \max(1, g_\delta) \), which incorporates not only tunnel junctions, defined as \( g \ll 1 \), but also the junctions with a strong Josephson coupling \( 1 \lesssim g \ll g_\delta \) in the presence of an intense interfacial pair breaking\(^{16,17} \). The junctions satisfying the generalized condition \( g \ll \max(1, g_\delta) \) will be called harmonic junctions. With the corresponding expression for the critical current of harmonic junctions (see (S10) in\(^{17} \) and with those for \( \lambda_L \) and \( \xi \), the quantity \( \lambda_J \) in (1) can be written as

\[
\lambda_J = \left( \frac{c\Phi_0}{16\pi^2 \lambda_L j_c} \right)^{1/2} = \left( \frac{\lambda_L \xi}{g \ell} \right)^{1/2} \frac{1}{\sqrt{2 + g^2 \delta - g_\delta}}. \tag{3}
\]

Hence, the characteristic length scale \( \lambda_J \) substantially depends on the strength of the interfacial pair breaking \( g_\delta \).

In standard tunnel junctions with \( g_\delta \ll 1 \) one gets \( \lambda_J \gg \lambda_L \), since the parameter \( g \) is proportional to the junction transparency and in this case extremely small. The characteristic length (3) decreases \( \propto g^{-1/2} \) with increasing the effective Josephson coupling constant and becomes comparable with \( \lambda_L \) at the characteristic value \( g_{\ell}^{1/2} \sim \kappa^{-1/2} \ll 1 \), which can be still small in the strongly type-II superconductors. At the same time, \( \lambda_J \) increases with the interfacial pair breaking. In junctions with an intense pair breaking \( g_\delta \gg 1 \) the limiting relation \( \lambda_J \approx \left( \frac{\lambda_L \xi}{g_\delta} \right)^{1/2} g_\delta \) follows from (3). One sees that \( \lambda_J \) considerably exceeds \( \lambda_L \) under the condition \( g_\delta^{1/2} \ll g_\delta \kappa^{-1/2} \), which allows the strong coupling constant \( g_\delta \gg 1 \), provided \( g_\delta \gg \kappa^{-1/2} \). Thus, in the presence of an intense interfacial pair breaking the local electrodynamics can be applied to describing the harmonic junctions with a large Josephson coupling.

If a strongly nonlocal regime \( \lambda_J \ll \lambda_L \) takes place, one can combine the results of Ref. 5 with Eq. (3), where the effects of the interfacial pair breaking are taken into account. This leads to the following characteristic scale of an isolated Josephson vortex

\[
\frac{\lambda_J^2}{\Lambda_0} = \frac{\xi}{g_\delta (\sqrt{g_\delta^2 + 2 - g_\delta})^2}. \tag{4}
\]

Further on the condition \( \lambda_J \gg \lambda_L \) will be assumed, which ensures an applicability of the local theory. For the quantitative analysis, let us consider the junction of Ferrell and Prange\(^{1,4} \), i.e., a wide junction occupying the halfspace \( y > 0 \), \( L_y \to \infty \) under the magnetic field applied at the junction edge \( y = 0 \). The magnetic field is assumed to be fully screened far inside the junction plane \( (y \to \infty) \), where the supercurrent density also vanishes. In describing the screening effects, the magnetic flux \( \Phi \) through the junction will be considered not exceeding half of the flux quantum \( |\Phi| \leq \frac{\Phi_0}{2} \). The magnetic field \( H_0 \) at the junction edge at \( \Phi = \frac{\Phi_0}{2} \) is known to be the highest field, for which a solution with no vortex precursors is possible, and, therefore, the magnetic field as well as the current density decay monotonically with increasing the distance \( y \) from the interlayer edge. The screening of such an external field is only metastable, since it exceeds the lower critical field.\(^{2,18} \) At the same time, the spatial distributions of the quantities \( H(y), j(y) \) and \( \chi(y) \), controlled by the screening effect at \( \Phi = \frac{\Phi_0}{2} \), coincide with their spatial profiles in the half of an isolated Josephson vortex involving single flux quantum \( \Phi_0 \). Hence, if \( \Phi = \frac{\Phi_0}{2} \), the magnetic field \( l_1(y) \) at a half of the flux quantum, the penetration depth \( l_1(y) \) represents a characteristic size \( l_1(y) \) of the vortex, a half of its effective width along the \( y \) axis. One also notes, that the magnetic field in the center of the vortex, produced by the vortex Josephson current, coincides with the magnetic field \( H_0 \) at the junction edge at \( \Phi = \frac{\Phi_0}{2} \).

As a weak applied field \( \Phi \ll \Phi_0 \) induces only a small supercurrent in the junction \((|\sin \chi| \ll 1)\), one can consider small phase differences and linearize the sine function in Eq. (1). This results in a simple exponentially decaying solution of (1): \( \chi = \chi_0 \exp(-y/\lambda_J) \), \( H(y) = -\left[\Phi_0 \chi_0 / (4\pi \lambda_L \lambda_J)\right] \exp(-y/\lambda_J) \). The latter expression signifies that the quantity (3) coincides with the weak-field penetration depth exactly: \( l_0 = \lambda_J \).

With the increasing magnetic field the junction, the linearized description fails and one should use the solution of Eq. (1) found in Ref. 1. For the magnetic field at \( x = 0 \) inside the junction \( y > 0 \), with a maximum at the junction edge \( y = 0 \), one has \( H(y) = \mp \Phi_0 / [2\pi \lambda_L \lambda_J \cosh((y + y_0)/\lambda_J)] \), \( y_0 \geq 0 \), and the phase difference is \( \chi(y) = \pm 2 \arcsin \text{sech}((y + y_0)/\lambda_J) \).

Since the spatial profile \( H(y) \) of the magnetic field in the junction interlayer \( (x = 0) \) can substantially differ from the exponential one, the equality

\[
\int_0^{\infty} H(y) dy = l_1 H(0) \tag{5}
\]

will be put to use for a quantitative description of the
juncture penetration depth \( l_j \). Equation (5) is in agreement with the standard definition of magnetic penetration depths in various other circumstances.\(^{18,19}\) Here \( H(0) \) is the magnetic field at the junction edge \( y = 0 \), and (5) defines a characteristic size of an adjacent region, where the magnetic field as well as the d.c. supercurrent are confined within the junction.

Substituting the solution for \( H(y) \) in (5) and taking the integral, one gets \( l_j \) as a function of \( y_0 \). Since \( y_0 \) and \( \Phi \) are implicitly related to each other in accordance with the condition \( \Phi = 2 \Phi_0 \), one obtains eventually the dependence of the Josephson penetration depth on the magnetic flux through the junction

\[
l_j^{-1}(\Phi) = \lambda_j^{-1} \frac{\sin(\pi \Phi/\Phi_0)}{\pi \Phi/\Phi_0}, \quad |\Phi| \leq \frac{1}{2} \Phi_0, \quad (6)
\]

and the relation

\[
H(0) = \frac{\Phi_0}{2 \pi \lambda_j \lambda_L} \sin \left( \frac{\pi \Phi}{\Phi_0} \right). \quad (7)
\]

Thus, \( l_j(\Phi) \) goes up with the increase of the magnetic flux within the given limits. While \( l_j(\Phi) \approx l_0 = \lambda_j \) for \( \pi |\Phi| \ll \Phi_0 \), one gets \( l_j(\frac{\Phi_0}{2}) = l_{jv} = \frac{\pi}{2} \lambda_j \) when half of the flux quantum pierces the junction. Here both the weak-field penetration depth \( l_0 \) and the characteristic size of the Josephson vortex \( l_{jv} \) are associated with one and the same length scale \( \lambda_j \). The difference between them, though quantitatively noticeable, is not significant.

III. \( l_j \) IN JUNCTIONS WITH ANHARMONIC CURRENT-PHASE RELATIONS

In anharmonic junctions the equation for a spatially dependent phase difference takes the form [cf. (1)]

\[
\frac{d^2 \chi([y])}{dy^2} - \frac{16 \pi^2 \lambda_L}{c \Phi_0} j[\chi(y)] = 0, \quad (8)
\]

and the defining relation for the penetration depth \( l_j(\Phi) \) as a function of the magnetic flux is\(^{17}\)

\[
l_j^{-1}(\Phi) = \left[ \frac{8 \lambda L \Phi_0}{c \Phi^2} \int_0^\infty j(\chi) d\chi \right]^{1/2}. \quad (9)
\]

Here \( j(\chi) \), the Josephson current density in the absence of the magnetic field, is assumed to be an odd function of \( \chi \).

Equation (9) describes the junction penetration depth and its magnetic flux dependence, assuming the current-phase relation of the junction to be known. Therefore, making use of the results of the GL theory for the anharmonic phase dependence \( j(\chi) \), allows one to obtain from (9) the quantity \( l_j(\Phi) \). Substituting \( j = j_c \sin \chi \) in (9), one easily reproduces Eq. (6) for harmonic junctions.

The results for harmonic junctions remain applicable to the anharmonic case under sufficiently weak applied magnetic fields, when a spatially dependent current density is small enough throughout the junction plane allowing the linearization of the current-phase relation:

\[
j \approx j_0 \chi, \quad j_0 = \frac{d j(\chi)}{d \chi} \bigg|_{\chi=0}.
\]

Then the integration of the current density in (9) results in the penetration depth

\[
l_j = \left( \frac{c \varphi_0}{16 \pi^2 \lambda_L \chi \Phi_0} \right)^{1/2}. \quad (10)
\]

Thus, \( l_j(\Phi, g_t, g_3) = 2 \left( \frac{\pi |\Phi|/\Phi_0}{\sqrt{(g_3 + g_t) \lambda L \xi}} \right) \ln^{1/2} \left[ 1 + \frac{4 g_t (g_3 + g_t)}{g_3^2 \sin^2 \pi \Phi/\Phi_0} \right]. \quad (11)
\]

Under the condition \( 4 g_t (g_3 + g_t) \sin^2 \frac{\pi \Phi}{\Phi_0} \ll g_3^2 \), which is satisfied in the weak-field and/or in the tunneling limits, Eq. (10) reduces to (6) with \( \lambda_j \) defined in (3) and taken at \( g_3^2 \gg 1 \).

The junction penetration depth (10), as a function of the magnetic flux, monotonically increases from \( l_{j0} = g_3 (\lambda_L \xi / g_t)^{1/2} \) in the weak-field limit to

\[
l_{jv}(g_t, g_3) = \frac{\pi \sqrt{(g_3 + g_t) \lambda L \xi}}{\ln^{1/2} \left[ 1 + \frac{4 g_t (g_3 + g_t)}{g_3^2 \sin^2 \pi \Phi/\Phi_0} \right]}. \quad (11)
\]

when half of the flux quantum pierces the junction.

In strongly anharmonic junctions \( (g_t \gg g_3) \) the relation \( l_{jv} \gg l_{j0} \) takes place, which signifies a pronounced magnetic field dependence \( l_j(\Phi) \). The corresponding quantitative condition follows from (11):

\[
l_{jv} \approx \frac{g_t \pi}{g_3 \sqrt{2 \ln^{1/2} \frac{2 \pi}{g_t} l_{j0} \gg l_{j0}}}. \quad (12)
\]

A significant difference between the characteristic size of the Josephson vortex \( l_j(\frac{\Phi_0}{2}) \) and the weak-field penetration depth \( l_{j0} \) is in striking contrast with the harmonic junctions, where \( l_{jv} \approx \frac{\pi}{2} l_{j0} \approx \frac{\pi}{2} \lambda_j \).

A substantial increase of \( l_{jv} \) as compared to \( l_{j0} \) is associated with the behavior of the quantity \( \int_0^\pi j(\chi) d\chi \), which enters the right hand side of (9) at \( \Phi = \frac{\pi \Phi_0}{2} \). For the harmonic current \( \int_0^\pi j(\chi) d\chi = 2 j_c \) that leads to \( l_{jv} = \frac{\pi}{2} l_{j0} \).

A significant difference between the characteristic size of the Josephson vortex \( l_j(\frac{\Phi_0}{2}) \) and the weak-field penetration depth \( l_{j0} \) as compared to \( l_{j0} \) is in striking contrast with the harmonic junctions, where \( l_{jv} \approx \frac{\pi}{2} l_{j0} \approx \frac{\pi}{2} \lambda_j \).

In the strongly anharmonic regime, when \( g_t \gg g_3 \) and \( g_3^2 \gg 1 \), the Josephson current \( j(\chi) \) is small outside a pronounced narrow peak of the width \( \Gamma \sim \frac{4 \pi}{g_t} \ll 1 \) in a vicinity of \( \chi = \chi_c \) (see (S11), (S12) in Appendix C and also Fig. 2 in Ref. 15). The critical current \( j_c \) is determined...
by the height of the peak: \( j_c = j(\chi_c) \). Therefore, a qualitative estimate is \( \int_0^\chi j(\chi)d\chi \sim \Gamma j_c \ll j_c \). In accordance with Eq. (9), this results in an increase of \( i_{jv} \).

Such an unconventional behavior of \( j(\chi) \) originates from the phase-dependent proximity effect near the interface, which takes place when \( g_s \gg g_\ell \) and \( g_\ell^2 \gg 1 \).

To ensure the applicability of the result (12), the conditions \( g_\ell \gg g_s \), \( g_\ell^2 \gg 1 \), allowing strongly anharmonic effects to manifest themselves in the junctions, have to be restricted further as the consequence of applying the local electrodynamics. This leads to the relation \( l_1(\Phi) \gg \lambda_L \), which is sensitive to the magnetic flux. In weak fields one gets \( l_0 \gg \lambda_L \) and ultimately \( \kappa^{1/2}g_\ell^{1/2} \ll g_s \). Joining the conditions, results in strong inequalities \( \kappa^{1/2}g_\ell^{1/2} \ll g_s \ll g_\ell, g_\ell^2 \gg 1 \). If these are satisfied, the relation \( l_1(\Phi) \gg \lambda_L \) and the results (10)-(12) would take place in the whole region \( |\Phi| \leq \frac{1}{2}\Phi_0 \).

The conditions \( \kappa^{1/2}g_\ell^{1/2} \ll g_s \ll g_\ell, g_\ell^2 \gg 1 \) are quite restrictive and uncommon as they can be satisfied only at huge values of \( g_\ell \). On account of a monotonic increase of \( l_j(\Phi) \) with \( \Phi \), substantially weaker conditions \( l_{jv} \gg \lambda_L \) emerge at \( |\Phi| = \frac{1}{2}\Phi_0 \). Then the local approach is justified in describing the Josephson vortex and, in particular, its size (11), and can fail at smaller values of \( |\Phi| \). This results in \( g_\ell^{1/2} \gg \kappa^{1/2} \), up to a logarithmic factor.

As seen from (11), the dimensionless characteristic size of the Josephson vortex \( \bar{l}_{jv} \) as a function of \( g_\ell \) varies for various \( g_s \gg 1 \): (1) \( g_s = 100 \), (2) \( g_s = 500 \), (3) \( g_s = 1000 \), (4) \( g_s = 2000 \), and (5) \( g_s = 3000 \).

![FIG. 2. The dimensionless characteristic size of the Josephson vortex \( \bar{l}_{jv} \) as a function of \( g_\ell \) for various \( g_s \).](image1)

To ensure the applicability of the result \( i_{jv} \), the conditions \( g_\ell \gg g_s \), \( g_\ell^2 \gg 1 \), allowing strongly anharmonic effects to manifest themselves in the junctions, have to be restricted further as the consequence of applying the local electrodynamics. This leads to the relation \( l_1(\Phi) \gg \lambda_L \), which is sensitive to the magnetic flux. In weak fields one gets \( l_0 \gg \lambda_L \) and ultimately \( \kappa^{1/2}g_\ell^{1/2} \ll g_s \). Joining the conditions, results in strong inequalities \( \kappa^{1/2}g_\ell^{1/2} \ll g_s \ll g_\ell, g_\ell^2 \gg 1 \). If these are satisfied, the relation \( l_1(\Phi) \gg \lambda_L \) and the results (10)-(12) would take place in the whole region \( |\Phi| \leq \frac{1}{2}\Phi_0 \).

The conditions \( \kappa^{1/2}g_\ell^{1/2} \ll g_s \ll g_\ell, g_\ell^2 \gg 1 \) are quite restrictive and uncommon as they can be satisfied only at huge values of \( g_\ell \). On account of a monotonic increase of \( l_j(\Phi) \) with \( \Phi \), substantially weaker conditions \( l_{jv} \gg \lambda_L \) emerge at \( |\Phi| = \frac{1}{2}\Phi_0 \). Then the local approach is justified in describing the Josephson vortex and, in particular, its size (11), and can fail at smaller values of \( |\Phi| \). This results in \( g_\ell^{1/2} \gg \kappa^{1/2} \), up to a logarithmic factor.

As seen from (11), the dimensionless characteristic size of the Josephson vortex \( \bar{l}_{jv} \) as a function of \( g_\ell \) varies for various \( g_s \gg 1 \): (1) \( g_s = 100 \), (2) \( g_s = 500 \), (3) \( g_s = 1000 \), (4) \( g_s = 2000 \), and (5) \( g_s = 3000 \).

![FIG. 3. The dimensionless characteristic size of the Josephson vortex \( \bar{l}_{jv} \) as a function of \( g_\ell \) for various \( g_s \).](image2)

proximated by Eq. (11). As a function of the Josephson coupling strength \( g_\ell \), the quantity \( \bar{l}_{jv} \) shows a nonmonotonic behavior. In the harmonic regime \( g_\ell \ll g_s \) the penetration depth decreases with \( g_\ell \) as \( l_1 \propto g_s^{-1/2} \) (see (3)). When the parameter \( g_s \) increases further and the anharmonic features of the current-phase relation become pronounced, the integral of the supercurrent over the phase difference in Eq. (9) diminishes, as discussed above. As a consequence, the junction penetration depth (9) and, in particular, (10), (11) gradually goes up with increasing \( g_\ell \) in the region \( g_\ell \gg g_s \), \( g_\ell^2 \gg 1 \). As follows from (11), a minimum of \( \bar{l}_{jv}(g_\ell, g_s) \) as a function of \( g_\ell \) at fixed \( g_s \) takes place at \( g_\ell^2 \sim g_s^2 \gg 1 \). Specifically, the minima of \( \bar{l}_{jv}(g_\ell, g_s) \), which correspond to the curves 1 and 2 of Fig. 2, are \( \bar{l}_{jv, \min}(g_s) = 100 \approx 29.7675 \) at \( g_\ell \approx 129.562 \), and \( \bar{l}_{jv, \min}(g_s = 500) \approx 66.5611 \) at \( g_\ell \approx 647.781 \).

After rewriting the condition \( l_{jv}(g_\ell, g_s) \gg \lambda_L \) in the form \( l_{jv}(g_\ell, g_s) \gg \kappa^{1/2} \), one can see that the applicability domain of the results shown in the figures and obtained within the local theory, depends on the GL parameter \( \kappa \gg 1 \). For example, all the curves in Fig. 2 satisfy the condition \( l_{jv}(g_\ell, g_s) \gg 30 \gg 3 \) and, therefore, they are applicable to the case \( \kappa = 10 \). However, for \( \kappa = 100 \) a substantial part of curve 1 does not satisfy the condition \( l_{jv}(g_\ell, g_s) \gg 10 \) in the given region of \( g_\ell \). It is in contrast to other curves, which remain wholly justified. Similar remarks would apply in fact to all subsequent figures of the paper (Figs. 3-8).

With increasing the interface parameter \( g_s \), the critical current \( j_c \) and the integral \( \int_{\Phi_0}^{\Phi} j(\chi)d\chi \) decrease irrespective of the relation between \( g_s \) and \( g_\ell \). For this reason and in accordance with (9) and (10), the junction penetration depth monotonically increases with increasing \( g_s \) at fixed \( \Phi \) and \( g_\ell \). The quantity \( \bar{l}_{jv} \) as a function of \( g_\ell \), taken for various \( g_s \), is depicted in Fig. 3. In the region \( g_\ell^2 \gg 1 \) the exact results are in agreement with those following from (11). As a consequence of the nonmonotonic
experiences abrupt spatial changes in a small region of the vicinity of where varying in space phase difference passes through the harmonic junctions. The greater characteristic length narrow peak as a function of $\chi$, the Josephson current experiences abrupt spatial changes in a small region of $y$, where varying in space phase difference passes through the vicinity of $\chi(y_c) = \chi_c$ with a change of $y$. The quantities $H(y)$ and $\frac{d\chi(y)}{dy}$ change comparatively quickly in that small space region, so that a smaller characteristic length is determined by the particular form of the anharmonic current-phase relation. As a result, the spatial profile of $j(\chi)$ contains narrow peaks, while $\chi(y)$ and $H(y)$ acquire more angular shape as compared to that in the harmonic junctions. The greater characteristic length is the junction penetration depth $l_j(\Phi)$, which can evolve considerably with the varying magnetic field, as shown in the preceding section. Due to a small value of the supercurrent outside the peak region, in strongly anharmonic junctions $\frac{d\chi(y)}{dy}$ is almost constant and $\chi(y)$ is nearly a linear function of $y$ over the scale $\sim l_j(\Phi)$.

Consider here an isolated Josephson vortex deep inside the junction plane, which is known to contain a single flux quantum. The magnetic field is symmetric and the current density is antisymmetric with respect to the vortex center, while $\chi(y)$ changes monotonically overall by $2\pi$. In the presence of a strong interfacial pair breaking $g_3^2 \gg 1$, the equation describing the spatial dependence of the phase difference within the local Josephson electrodynamics can be written as

$$\frac{d\chi}{dy} = \pi \frac{\ln^{1/2} \left[ 1 + \frac{4g_2(g_3 + g_4)}{g_3^2} \sin^2 \chi(\tilde{y}) \right]}{\ln^{1/2} \left[ 1 + \frac{4g_2(g_3 + g_4)}{g_3^2} \right]}.$$  \hspace{1cm} (13)$$

Here the dimensionless coordinate $\tilde{y} = y/l_{Jv}$ is introduced and $H(y) > 0$ assumed.

The spatial profiles of the phase difference $\chi(\tilde{y})$, of the magnetic field $H(\tilde{y})$ and of the supercurrent density $J(\tilde{y})$ in an isolated Josephson vortex are depicted in Figs. 4-6. The vortex center is taken here at $y = 0$, and the asymptotic values of the phase difference are $\chi_{-\infty} = 0$, $\chi_{\infty} = 2\pi$. All the distributions obtained confirm that the overall large scale of the spatial variations is $l_{Jv}$. In this respect, even a comparatively small difference between $\lambda_1$ and $l_{Jv} = \frac{\pi}{2} \lambda_1$ in harmonic junctions can be discerned. In each of these figures curve 1 describes the behavior of the corresponding quantity in harmonic junctions with a strong interfacial pair breaking. It is similar to the analogous profiles in standard tunnel junctions. A particularly interesting case of strongly anharmonic junctions is described by curve 3 in the figures. Curve 3 in Fig. 4 demonstrates a sharp crossover of the gradual behavior of the phase difference and its asymptotic value. It is in contrast to curve 1, which smoothly varies over the only scale $l_{Jv}$. Curve 2 shows an intermediate behavior. Similarly, in contrast to curves 1 and 2, curve 3 in
Fig. 5, which describes the profile of the magnetic field in strongly anharmonic junctions, shows no noticeable tails of the field at distances \(|y| > l_{Jv}\).

Curve 3 in Fig. 6 demonstrates, that the supercurrent flows in strongly anharmonic junctions mostly in a small narrow part of the Josephson vortex. As was noted above, these are the narrow peaks in the current-phase relation of strongly anharmonic junctions, which transform into spatial peaks of the supercurrent density due to a spatial dependence of the phase difference. An effect of similar origin, but with a transformation into the magnetic flux dependence, has been recently predicted in strongly anharmonic junctions, whose widths are much less than the junction penetration depth.\(^\text{20}\) The narrow central Fraunhofer peak of the total critical current was found to possess the following half width at the half of the peak \((\Delta \Phi/\Phi_0) \approx 1.35 g_3/g_t \ll 1\), under the conditions \(g_t \gg g_3\) and \(g_3^2 \gg 1\).

\[ \chi(\tilde{y}) = \text{the solution of Eq. (13) and } \lambda_j \approx (\lambda_L \xi/g_t)^{1/2} g_3 \text{ (see (3)).} \]

For harmonic junctions, when \(g_t \ll g_3\), the logarithmic functions in (14) can be expanded and the integral calculated with the solution \(\chi(\tilde{y}) = -2 \text{arcsin} \text{sech}(\pi \tilde{y}/2)\). This results in \(H_{jc1} = \Phi_0/(\pi^2 \lambda_L \lambda_j)\), in agreement with the conventional expression.\(^{2,18}\)

To single out the dependence of \(H_{jc1}\) on the effective interface parameters \(g_3\) and \(g_t\), it is convenient to introduce the dimensionless lower critical field of the junctions \(\tilde{H}_{jc1} = H_{jc1}/H^*\), taken in units of \(H^* = \Phi_0/(\lambda_L^{3/2} \xi^{1/2})\). Figure 7 displays \(\tilde{H}_{jc1}\) as a function of the strength of the Josephson coupling, for various values of the strong interfacial pair breaking. It is a nonmonotonic function of \(g_t\). In tunnel junctions the field increases with \(g_t\) and decreases with \(g_3\) as \(H_{jc1} = 2 \Phi_0/(\pi^2 \lambda_L^2) \propto g_t^{1/2}/g_3\). A decrease with \(g_t\) under the conditions \(g_t \gg g_3 \gg 1\) takes place for the same reason, for which the penetration depth increases (see Fig. 2 above and its discussion).

The relation \(H_{jc1} \ll H_{c1}\) is to a large extent close to the strong inequality \(l_{Jv} \gg \lambda_j\), which determines the applicability of the local theory to describing the structure of the Josephson vortex. Here \(H_{c1} = \Phi_0/\lambda^2 \kappa (\ln \kappa + 0.08)/(4\pi \lambda_L^2)\) is the lower critical field of the massive strongly type-II superconductor without weak links. After reducing the relation \(H_{jc1} \ll H_{c1}\) to the form \(\tilde{H}_{jc1} \ll (\ln \kappa + 0.08)/(4\pi \sqrt{\kappa})\), one sees that the applicability domain of the condition \(H_{jc1} \ll H_{c1}\), in terms of Fig. 7, depends on the GL parameter \(\kappa \gg 1\). For example, for \(\kappa = 10\) one obtains a strong inequality \(\tilde{H}_{jc1} \ll 0.06\), which applies to all the curves in Fig. 7. For \(\kappa = 100\) one gets the relation \(H_{jc1} \ll 0.037\). It is fully applicable to curves 2-5 and only partially to curve 1. This is very similar to what was
said above regarding Fig. 2 plotted for the dimensionless quantity $l_{xy}$, taken for the same set of parameters.

In tunnel junctions, the field $H_0$ in the center of the Josephson vortex is related to $H_{jc1}$ as $H_{jc1} = \frac{\Phi_0}{2\pi} H_0$. In strongly anharmonic junctions with an intense interfacial pair breaking, the ratio $H_{jc1}/H_0$ depends on $g_\ell$ and $g_\delta$. As shown in Fig. 8, the ratio varies between $\frac{2}{\pi}$ and unity. It monotonically increases with $g_\ell$, while the pair breaking tends to suppress it towards its standard value.

VI. DISCUSSION

A pronounced unconventional behavior of the magnetic properties of the planar junctions emerges, when the parameters $g_\ell$ and $g_\delta$ are large and satisfy the conditions $g_\delta^2 \gg 1$, $g_\ell \gg g_\delta$. A phase dependent suppression of the order parameter at the junction interface, taking place under such conditions due to the proximity effect (see (S13) in (1)), is of key importance here. Qualitatively, the Josephson current (S11) increases and the junction penetration depth (10), (11) decreases with increasing $g_\ell$, when $g_\ell \ll g_\delta$, $g_\delta^2 \gg 1$ and the suppression does not substantially depend on $g_\ell$. However, if $g_\ell \gg g_\delta$, then a phase dependent local decrease of the condensate density at the interface takes place with $g_\ell$ and results in the anharmonic Josephson current, which substantially decreases in the wide region of the phase difference. Though the critical current does not diminish in this regime (see (SI2) in (17)), the integral $\int_0^\pi \tilde{J}(\chi) d\chi$ decreases with $g_\ell$. This induces an increase of the Josephson vortex size $l_{xy}$, demonstrating an important role the anharmonic effects play in the problem in question.

A possibility of achieving large values of $g_\ell$ and $g_\delta$ in experiments has not been established as yet. However, a number of microscopic models persuasively indicate that the strong inequalities $g_\delta^2 \gg 1$, $g_\ell \gg g_\delta$, resulting in a pronounced anharmonic current-phase relation, can be satisfied under certain conditions. More restrictive relations emerge due to the application of local Josephson electrodynamics at large $g_\ell$. In weak fields this results in the uncommon conditions $\kappa^{1/2} g_\ell^{1/2} \ll g_\delta \ll g_\ell$, which are only satisfied at huge values $g_\ell \gtrsim 10^4$, and could be challenging in an experimental realization. For the Josephson vortices the conditions are substantially weaker: $g_\ell^{1/2} \gg \kappa^{1/2}$, $g_\delta^2 \gg 1$, $g_\ell \gg g_\delta$.

There are no fundamental upper bounds to large values of the parameters $g_\ell$ and $g_\delta$. Microscopic model results for $g_\ell$ can be obtained based on the corresponding studies of the Josephson current near $T_c$. or of the boundary conditions for the superconductor order parameter at the interface in the GL theory. As follows from the microscopic results, in dirty junctions with small and moderate transparencies, $g_\ell$ can vary from vanishingly small values in the tunneling limit to those well exceeding $10^2$ and leading to a substantially anharmonic behavior of the Josephson current. The parameter $g_\ell$ goes up, when the interface transparency increases. In highly transparent planar junctions $g_\ell$ can generally take huge values. As $g_\ell \propto \xi(T)$, an additional increase of $g_\ell$ occurs near $T_c$.

Large $g_\delta$ corresponds to a strong suppression of the order parameter at the junction interface. A strong interfacial pair breaking can be induced by proximity to superconductor-normal metal interfaces and to magnetically active boundaries in various superconductors, including isotropic s-wave ones. In unconventional superconductors a significant pair breaking can be present also near superconductor-insulator and superconductor-vacuum interfaces. Under certain conditions, the order parameter can be fully suppressed on the boundary, in particular, for symmetry reasons in unconventional superconductors. This signifies that $g_\delta$ can, in general, take huge values.

A specific microscopic example studied in detail theoretically, in which the parameters $g_\ell$ and $g_\delta$ of the GL theory can satisfy the conditions $g_\delta^2 \gg 1$, $g_\ell \gg g_\delta$, is the dirty SNS junction where the normal conductivity of the leads is significantly less than the conductivity of a thin normal metal interlayer.

VII. CONCLUSIONS

The problem of the magnetic field screening and of the Josephson vortex structure in superconducting planar junctions with anharmonic current-phase relations, has been solved in this paper within the GL theory. Since a strongly anharmonic behavior only appears due to a pronounced Josephson coupling, an intense interfacial pair breaking needs to be present for the planar junctions to be weak links. Another reason for a pronounced interfacial pair breaking to play a crucial role for the theory developed, is that an intense pair breaking significantly
increases the penetration depth and thereby substantially extends an applicability domain of the local Josephson electrodynamics, which in this case applies to the junctions with the strong Josephson coupling.

The magnetic penetration depth \( l_j \) in the junctions is identified theoretically as a function of the magnetic flux, of the Josephson coupling strength and the interfacial pair breaking. Due to a nonexponential spatial profile of the screened magnetic field in the junction plane, a quantitative definition of \( l_j \) is put to use, similar to the standard definitions of magnetic penetration depths in various other circumstances. In harmonic junctions a characteristic size of the Josephson vortex along the junction plane \( jv \) and the weak-field penetration depth \( j0 \) are shown to be related as \( jv = \frac{\theta}{j0} \). A pronounced magnetic field dependence of \( jv \), which induces a significant increase of \( jv \) as compared to \( j0 \), is predicted in strongly anharmonic junctions. A nonmonotonic dependence of \( jv \) on the Josephson coupling strength is obtained in such junctions, as a consequence of the phase-dependent proximity effect, and demonstrated to result in an applicability of the local approach to sufficiently large \( g_y \) at a fixed \( g_s \).

A narrow peak in an anharmonic current-phase relation was found to induce a peak in the spatial profile of the supercurrent density, which is narrow compared to the size of the Josephson vortex. A nonmonotonic dependence on the Josephson coupling as well as a monotonic one on a strength of the interfacial pair breaking, was obtained for the lower critical field of the junctions.

An inclusion of the nonlocal effects into the theory developed above is desirable. Possible restrictions on the results obtained could be also associated with imperfections of the planar geometry of the junctions. Their study would require a significant extension of the theoretical approach used, and lies outside the scope of the paper.

Acknowledgments

The support from Russian Foundation for Basic Research under grants 11-02-00398 and 14-02-00206 is acknowledged.

1. R. A. Ferrell and R. E. Prange, Phys. Rev. Lett. 10, 479 (1963).
2. B. D. Josephson, Adv. Phys. 14, 419 (1965).
3. C. S. Owen and D. J. Scalapino, Phys. Rev. B 64, 538 (1967).
4. A. Barone and G. Paterno, Physics and Applications of the Josephson Effect (John Wiley&Sons, New York, 1982).
5. A. Gurevich, Phys. Rev. B 46, 3187 (1992).
6. Y. M. Aliev, V. P. Silin, and S. A. Uryupin, Sverkhprovidist’ (KIAE) 5, 230 (1992), [Superconductivity 5, 228 (1992)].
7. A. A. Boris, A. Rydh, T. Golod, H. Motzkau, A. M. Khushim, and V. M. Krasnov, Phys. Rev. Lett. 111, 117002 (2013).
8. Y. Ivanchenko and T. Soboleva, Physics Letters A 147, 65 (1990).
9. R. Humphreys and J. Edwards, Physica C: Superconductivity 210, 42 (1993).
10. R. Mints, Journal of Low Temperature Physics 106, 183 (1997).
11. V. G. Kogan, V. V. Dobrovitski, J. R. Clem, Y. Mawatari, and R. G. Mints, Phys. Rev. B 63, 144501 (2001).
12. M. Moshe, V. G. Kogan, and R. G. Mints, Phys. Rev. B 78, 020510 (2008).
13. J. R. Clem, Phys. Rev. B 81, 144515 (2010).
14. A. A. Golubov, M. Y. Kupriyanov, and E. Il’ichev, Rev. Mod. Phys. 76, 411 (2004).
15. Y. S. Barash, Phys. Rev. B 85, 100503 (2012).
16. Y. S. Barash, Phys. Rev. B 85, 174529 (2012).
17. See Supplemental Material for details of the derivations and for results of the GL theory for the Josephson current as a function of \( g_y \) and \( g_s \).
18. M. Tinkham, Introduction to Superconductivity, 2nd ed. (McGraw-Hill, Inc., New York, 1996).
19. L. D. Landau and E. M. Lifshitz, Electrodynamics of Con-


tinuous Media, 2nd ed. (Pergamon Press, Oxford, 1984).
20. Ya. S. Barash, Phys. Rev. B 86, 144502 (2012).
21. Z. G. Ivanov, M. Y. Kupriyanov, K. K. Likharev, S. V. Meriakri, and O. V. Sugiirev, Fiz. Nizk. Temp. 7, 560 (1981), [Sov. J. Low Temp. Phys. 7, 274 (1981)].
22. M. Y. Kupriyanov, Pis’ma Zh. Eksp. Teor. Fiz. 56, 414 (1992), [JETP Lett. 56, 399 (1992)].
23. P. G. de Gennes, Superconductivity of Metals and Alloys (Addison Wesley Publishing Co, Inc., Reading, MA, 1966).
24. V. P. Galaiko, A. V. Svidzinskii, and V. A. Slyusarev, Zh. Eksp. Teor. Fiz. 56, 835 (1969), [Sov. Phys. JETP 29, 454 (1969)].
25. E. N. Bratus’ and A. V. Svidzinskii, Teor. Mat. Fiz. 30, 239 (1977), [Theor. Math. Phys. 30, 153 (1977)].
26. A. V. Svidzinskii, Spatially Inhomogeneous Problems in the Theory of Superconductivity (Nauka, Moscow, 1982).
27. V. B. Geshkenbein, Zh. Eksp. Teor. Fiz. 94, 368 (1988), [Sov. Phys. JETP 67, 2166 (1988)].
28. P. G. de Gennes, Rev. Mod. Phys. 36, 225 (1964).
29. R. O. Zaitsev, Zh. Eksp. Teor. Fiz. 48, 1759 (1965), [Sov. Phys. JETP 21, 1178 (1965)].
30. T. Tokuyasu, J. A. Sauls, and D. Rainer, Phys. Rev. B 38, 8823 (1988).
31. A. Cottet, D. Huertas-Hernando, W. Belzig, and Y. V. Nazarov, Phys. Rev. B 80, 184511 (2009).
32. L. J. Buchholtz and G. Zwicknagl, Phys. Rev. B 23, 5788 (1981).
33. Y. S. Barash, A. V. Galaktionov, and A. D. Zaikin, Phys. Rev. B 52, 665 (1995).
34. M. Matsumoto and H. Shiba, J. Phys. Soc. Jpn. 64, 1703 (1995); 64, 3384 (1995); 64, 4867 (1995); 65, 2194 (1996).
35. Y. Nagato and K. Nagai, Phys. Rev. B 51, 16254 (1995).
36. L. J. Buchholtz, M. Palumbo, D. Rainer, and J. A. Sauls, J. Low Temp. Phys. 101, 1079 (1995); 101, 1099 (1995).
37. M. Alber, B. Bäuml, R. Ernst, D. Kienle, A. Kopf, and
Magnetic penetration depth and vortex structure in anharmonic superconducting junctions with an interfacial pair breaking - Supplemental material

In this supplemental material I present self-consistent results of the GL theory for the Josephson current in planar junctions as a function of the Josephson coupling, of the interfacial pair breaking and of the phase difference. Also the derivations of the defining expressions are given for the magnetic penetration depth and for the lower critical field in anharmonic junctions. The first integral of the basic differential equation for the spatially dependent phase difference is derived for the Josephson vortex in anharmonic junctions.

S1. The Josephson current as a function of $g_{\ell}$, $g_{b}$ and $\chi$

The GL free energy of the junction is represented as a sum of three terms: $F = F_{b1} + F_{b2} + F_{\text{int}}$. For symmetric junctions between two identical s-wave or $d_{xy}$-wave superconductors, the bulk and the interface free energies can be written, in the absence of a magnetic field, as

$$F_{b1(2)} = \int_{V_{b1(2)}} \left( K |\nabla \Psi_{1(2)}|^2 + a |\Psi_{1(2)}|^2 + \frac{b}{2} |\Psi_{1(2)}|^4 \right) dV_{b1(2)}, \quad (S1)$$

$$F_{\text{int}} = \int_{S} \left[ g_{\ell} |\Psi_1 - \Psi_2|^2 + g \left( |\Psi_1|^2 + |\Psi_2|^2 \right) \right] dS. \quad (S2)$$

The interface free energy (S2) contains both the Josephson coupling of the superconducting banks with the coupling constant $g_{\ell}$, and the term with the coupling constant $g$, which describes, for instance in the absence of the current, the interfacial pair breaking ($g_{b} > 0$), or the pair formation ($g_{b} < 0$).

The two independent interface invariants in (S2), which control jointly the relative value of the supercurrent with respect to the depairing current $j_{dp}$, are qualitatively different. In the absence of the current, i.e., at $\chi = 0$, one gets $\Psi_1 = \Psi_2$ in symmetric junctions and, therefore, the first invariant in (S2) vanishes. On the contrary, the invariant $\Psi_1 \Psi_2^* + \Psi_2 \Psi_1^*$ does not vanish at $\chi = 0$. Therefore, if it were used for describing the Josephson coupling instead of the first term in (S2), it would result at $\chi = 0$ in the same interfacial proximity effect as the second invariant in (S2). Such a duplication is microscopically unjustified and would complicate the analysis of the problem. In particular, the free energy (S2), in line with the microscopic theory, satisfies the condition that there should be no influence of a thin interface on the superconductors in symmetric junctions with the vanishing interfacial pair activity ($g_{b} = 0$) and at the zero phase difference ($\Psi_1 = \Psi_2$). The last statement is related, to a certain degree, to Anderson theorem regarding a negligible influence of nonmagnetic impurities on the thermodynamic superconductor properties, in contrast to their profound effect on the supercurrent flow and correlations.

Taking the order parameter in the form $\Psi_{1(2)} = (|a|/b)^{1/2} f_{1(2)}(x) e^{i\lambda (1(2))(x)}$, one can transform the GL equations, which follow from the bulk free energies (S1), to the equations for the normalized order-parameter moduli

$$\frac{d^2 f_{1(2)}}{dx^2} - \frac{4\beta^2}{2\eta f_{1(2)}^3} + f_{1(2)} - f_{1(2)}^3 = 0 \quad (S3)$$

and to the current conservation condition. Here $\eta = x/\xi(T)$, $\xi(T) = (K/|a|)^{1/2}$ is the temperature dependent superconductor coherence length, $j = j_{dp} = -(3\sqrt{3}/2)(d\chi/d\eta)f^2$ is the spatially constant normalized current density, $j_{dp} = (8|e| |a|^{3/2}K^{1/2})/(3\sqrt{3}b)$ is the depairing current deep inside the superconducting leads. As $df/dx = 0$ in the bulk $x \to \pm \infty$, one finds from (S3) the relation $j^2 = (27/4)(1 - f_{\pm \infty}^2)f_{\pm \infty}^4$. Here $\frac{2}{a} < f_{\pm \infty}^2 \leq 1.18$, and, in particular, in the absence of the supercurrent $f_{\pm \infty} = 1$. For symmetric junctions with $f$ continuous through the interface $f = f(|x|)$, i.e., $f_{\pm}(x) = f_{1}(-x)$.

The interface terms in (S2) and the gradient term in (S1) contribute to the boundary conditions for complex order parameters. One can split them into the boundary conditions for $f$ and the expression for the Josephson current via the value $f_0$ at $x = 0$ and the phase difference $\chi = \chi_- - \chi_+$ across the interface:

$$\left( \frac{df}{d\eta} \right)_{\pm} = \pm \left( g_{b} + 2g_{\ell} \sin^2 \frac{\chi}{2} \right) f_0, \quad (S4)$$

$$j = \frac{3\sqrt{3}}{2} g_{b} f_0^2 \sin \chi. \quad (S5)$$

Here $g_{\ell} = g_{\ell} \xi(T)/K$ is the effective dimensionless Josephson coupling constant and $g_{b} = g \xi(T)/K$ is the effective dimensionless interface parameter.

For identifying the Josephson current based on (S5), one should know the self-consistent interface value $f_0$ as...
a function of the phase difference $\chi$. The simplest way to obtain the exact result of the GL theory for $f_0$ and, hence, for the Josephson current through the junctions in question, is to make use of the first integral of Eq. (S3), which can be written as

$$\left( \frac{df}{dx} \right)^2 + f^2 - \frac{1}{2} f^4 + \frac{4g^2}{2f^2} = 2f_\infty^2 - \frac{3}{2} f_\infty^4. \quad (S6)$$

The quantity $f_0$ can be found without resorting to a spatially dependent solution of (S6) or (S3). One puts $x = 0$ in (S6), substitutes (S5) for the current and considers the resulting equation as a polynomial one with respect to three unknown quantities $\left( \frac{df}{dx} \right)_0^2$, $f_0^2$ and $f_\infty^2$, for a given phase difference $\chi$. An additional relationship between $f_0^2$ and $f_\infty^2$ is obtained by equating the current (S5) to its expression via $f_\infty^2$ given above. Together with the boundary conditions (S4), one gets three equations for unknown quantities, which are reduced to a single fourth-order polynomial equation

$$2g^2(\chi)\alpha - (1 - \alpha^2)(1 - \alpha(\alpha + 2)g^2 \sin^2 \chi) = 0 \quad (S7)$$

for the quantity $\alpha$. It relates the values of the order parameter taken at the interface and in the bulk to each other: $f_0^2 = \alpha f_\infty^2$. The quantity $g_\delta(\chi)$ in (S7) is defined as $g_\delta(\chi) = (g_\delta + 2g_0 \sin^2 \frac{\chi}{2})$.

Equating the current (S5) to that in the bulk results in $f_\infty^2 = 1 - g_\delta^2 \sin^2 \chi \alpha^2$ and therefore in $f_0^2 = \alpha (1 - g_\delta^2 \sin^2 \chi \alpha^2)$. Substituting the latter formula in (S5), one expresses the Josephson current via $\alpha$:

$$j = \frac{3\sqrt{3}}{2} g_\delta \sin \chi (1 - \alpha^2 g_\delta^2 \sin^2 \chi) j_{dp}. \quad (S8)$$

Simple numerical solution of (S7) allows one to describe, based on (S8), the Josephson current as a function of $\chi$ and of the parameters $g_\delta$ and $g_\ell$.15 Two of the four solutions of (S7) take complex values and, therefore, are of the unphysical character. One of the two remaining solutions satisfies the condition $\alpha < 1$ and corresponds to the pair breaking effects in the presence of the current ($g_\delta(\chi) > 1$). And finally, the fourth solution exceeds the unity $\alpha > 1$ and can be related to the junctions with the pair forming interfaces ($g_\delta(\chi) < 0$). Further on only the junctions with $\chi > 0$ will be considered.

In addition to the exact numerical results, one can also obtain an analytical solution, which describes the Josephson current with a good accuracy. With the pair breaking solution, the following anharmonic current-phase relation has been obtained for the Josephson current in the absence of the magnetic field15

$$j(g_\ell, g_\delta, \chi) = \frac{3\sqrt{3} g_\ell \sin \chi}{2(1 + 2g_\delta^2 \sin^2 \chi)} \left[ 1 + g_\delta^2(\chi) + g_\delta^2 \sin^2 \chi - \sqrt{g_\delta^4(\chi) + 4g_\delta^2(\chi)} \right] j_{dp}. \quad (S9)$$

The exact numerical solution shows that Eq. (S9) describes the current behavior almost perfectly, if $j < 0.7 j_{dp}$. This concerns, in particular, the current at $g_\ell < 1$ for any $g_\delta$, or at $g_\delta > 1$ for any $g_\ell$. For $j > 0.7 j_{dp}$ Eq. (S9) represents a good approximation of the exact numerical solution, with the deviations not exceeding 10%.

There are two basic limiting cases, when the expression (S9) for the supercurrent considerably simplifies. For tunnel junctions $g_\ell < 1$ in the presence of an interfacial pair-breaking, the harmonic current-phase relation is

$$j(g_\ell, g_\delta, \chi) = \frac{3\sqrt{3}}{4} g_\ell \left( \sqrt{g_\delta^2 + 2 - g_\delta} \right)^2 j_{dp} \sin \chi. \quad (S10)$$

On the other hand, in the regime of a pronounced interfacial pair breaking $g_\delta^2 \gg 1$ the anharmonic supercurrent at arbitrary values of $g_\ell$ takes the form

$$j(g_\ell, g_\delta, \chi) = \frac{3\sqrt{3} g_\ell j_{dp} \sin \chi}{4(g_\delta^2 + 4(g_\delta + g_\ell)g_\delta \sin^2 \frac{\chi}{2}). \quad (S11)$$

Strongly anharmonic current-phase relation shows up in (S11) for $g_\ell^2 \gg g_\delta^2 \gg 1$, while in the case $g_\ell \ll g_\delta$, $g_\ell^2 \gg 1$ the first harmonic dominates the current and the result (S11) agrees with (S10).

The critical current $j_c$, following from (S11), is

$$j_c = \frac{3\sqrt{3} g_\ell j_{dp}}{4g_\delta(g_\delta + 2g_\ell)}. \quad (S12)$$

It is small $j_c \ll j_{dp}$ at arbitrary $g_\ell$ due to an intense interfacial pair breaking $g_\delta^2 \gg 1$. At the sufficiently large $g_\ell \gg g_\delta$ the quantity $j_c$ in (S12), taken at $g_\delta^2 \gg 1$, approaches the limiting value $(3\sqrt{3}/8g_\delta) j_{dp}$.

The range of variations of parameters $g_\ell$ and $g_\delta$ is generally quite wide and includes large values that do not allow to confine the analysis to the first order terms in $g_\ell$ or $g_\delta$. Within the GL theory, large positive $g_\delta$ and $g_\ell$ result in a strong local suppression of the order parameter $f_0$ at the interface as compared to its bulk value. As seen from (S5) and (S11), under the condition $g_\delta^2 \gg 1$.

$$f_0^2 = \frac{1}{2(g_\delta^2 + 4(g_\delta + g_\ell)g_\delta \sin^2 \frac{\chi}{2}). \quad (S13)$$

The relation $f_0^2 \approx 1/(2g_\delta^2) \ll 1$ follows from (S13) when $g_\delta^2 \gg g_\ell^2$. In the opposite case $1 \ll g_\delta^2 < g_\ell^2$ a pronounced phase dependence of the quantity $f_0$ shows up. For small phase differences $\chi < g_\ell/g_\delta$ the relation does not change $f_0^2 \approx 1/(2g_\delta^2) \ll 1$, while for $g_\ell/g_\delta \ll |\chi| \lesssim \pi$ a stronger suppression takes place $f_0^2 \approx 1/(8g_\delta^2) \ll 1/(2g_\ell^2) \ll 1$. The phase dependent suppression of $f_0$ is associated not only with the contribution of the Josephson coupling to the boundary conditions (S4), but also with the current depairing, which is locally enhanced near the pair breaking interface. Far inside the superconductors the depairing is small since $j_c \ll j_{dp}$.  

15.}
Substituting (S13) in the boundary conditions (S4), one gets $\frac{df_0}{dx}\bigg|_{\pm} \sim 1$, since the interface value of the order parameter $f_0$ is suppressed to such a degree that the order of magnitude of the right-hand side in (S4) retains unchanged. This signifies that in weak links ($j_c \ll j_{dp}$) the order parameter always varies on the scale $\xi$, even if $g_c^2 \gg 1$.

The phase dependent decrease of $f_0^2$ with $g_c$ at $g_c \gg g_\delta$ results in the corresponding decrease of the Josephson current in an important region of the phase difference, in spite of the factor $g_c$ in (S5). The strong suppression of the order parameter at the junction interface is of key importance in this paper.

**S2. Defining relation for $t_j(\Phi)$ in anharmonic junctions**

Within the local Josephson electrodynamics, which presupposes the condition $t_j \gg \lambda_L$, the equations for spatial profiles of the quantities along the interface at $x = 0$ meet the standard form

$$dH(y) = \frac{4\pi}{c} j(\chi(y)), \quad \frac{d\chi(y)}{dy} = \frac{2\pi d}{\Phi_0} H(y),$$

(S14)

for the given geometry of the junction and the magnetic field. Here $\Phi_0 = \pi \hbar c/e$ is the superconductor flux quantum and $d = 2\lambda_L$, where a small interlayer thickness is neglected.

The equation for a spatially dependent phase difference follows directly from (S14):

$$\frac{d^2\chi(y)}{dy^2} - \frac{16\pi^2 \lambda_L}{c \Phi_0} j(\chi(y)) = 0.$$

(S15)

Substituting $j(\chi(y)) = j_c \sin(\chi(y))$ in the right hand side of Eq. (S15), one gets a well-known one-dimensional sine-Gordon equation (1) describing the static magnetic properties of the harmonic junctions.1-4

Under the condition $t_j \ll \lambda_L$ an interplay of the Meissner and the Josephson screenings takes place, and Eq. (S15) should be modified to incorporate the corresponding nonlocal effects. The generalized equation has been obtained for the phase difference in an isolated Josephson vortex far inside the junctions in question.5-6

It takes the form

$$\frac{c \Phi_0}{16\pi^3 \lambda_L^2} \int_{-\infty}^{\infty} du K_0 \left( \frac{|y - u|}{\lambda_L} \right) \frac{d^2\chi(u)}{du^2} = j(\chi).$$

(S16)

While the characteristic scale of the phase difference is $t_j$, the Macdonald function $K_0$ in the integrand in (S16) varies on the scale $\lambda_\Omega$. On account of the relation $2 \int_0^{\infty} K_0(t)dt = \pi$, equation (S16) reduces to (S15) provided $\lambda_\Omega \gg \lambda_L$. In a strongly nonlocal regime $\lambda_\Omega \ll \lambda_L$ the solution of (S16) has been obtained in Ref. 5 for the junctions with a harmonic current-phase relation. In particular, the supercurrent in the Josephson vortex was shown to be localized mostly over a characteristic length $\frac{\lambda_\Omega^2}{\lambda_L} \ll \lambda_\Omega$, with a decaying power law tail on this scale.

In the absence of the magnetic field, the phase dependent thermodynamic potential per unit area $\Omega_{S0}(\chi)$ and the supercurrent density $j(\chi)$ are related to each other as

$$j(\chi) = \frac{2|\epsilon|}{\hbar} \frac{d}{d\chi} \Omega_{S0}(\chi).$$

(S17)

According to the Josephson electrodynamics, the spatial dependence of the supercurrent arises due to variations of $\chi(y)$ with no change to $j(\chi)$. A macroscopic scale of the junction penetration depth allows one to consider, within the local theory, the spatially dependent quantities $\Omega_{S0}[\chi(y)]$ and $j(\chi(y))$ to be locally related by (S17), at a given $y$, as if the phase difference were constant along the interface.

The first integral of Eq. (S15) for the anharmonic Josephson junctions can be obtained after a substitution of the phase-dependent thermodynamic potential (S17) for the current in Eq. (S15), and a multiplication of all terms by $d\chi(y)/dy$. Since the magnetic field is assumed to be fully screened deep inside the junction plane, one obtains from here making use of the second equation in (S14)

$$H^2(y) = \frac{4\pi}{\lambda_L} \left( \Omega_{S0}[\chi(y)] - \Omega_{S0}(0) \right) = \frac{2\Phi_0}{c\lambda_L} \int_0^{\chi(y)} j(\chi) d\chi,$$

(S18)

where $\Omega_{S0}[\chi_{\infty}] = 0 \equiv \Omega_{S0}(0)$.

After integrating Eqs. (S14) along the y-axis, under the given conditions one also gets

$$H(0) = -\frac{4\pi}{cL_z} I, \quad \chi(0) = \frac{2\pi}{\Phi_0} \Phi,$$

(S19)

where $I$ and $\Phi = 2\lambda_L \int_{-\infty}^{\infty} H(y)dy = 2\lambda_L J_z H(0)$ are the total current and the magnetic flux through the junction.

The first relation in (S19) and (S7) allow one to identify the total current through the harmonic junction of Ferrell and Prange, as a function of the magnetic flux:

$$I = -\frac{cL_z \Phi_0}{8\pi^2 \lambda_1 \lambda_L} \sin \left( \frac{\pi \Phi}{\Phi_0} \right).$$

(S20)

Having in mind the current-phase relations (S8) - (S11), one considers 0-junctions ($g_\epsilon > 0$) with $2\pi$-periodic anharmonic phase dependent thermodynamic potential, which has only one minimum and one maximum per period. One notes that this condition excludes from the consideration the $\varphi$-junctions with several different solutions for an isolated Josephson vortex.39-41 The $2\pi$-periodicity of $\Omega_{S0}(\chi)$ and $j(\chi)$ allows a global shift of $\chi$ by $2\pi n$ ($n = \pm 1, \pm 2, \ldots$). The values $\chi = 2\pi n$ (and, in particular, $\chi = 0$) correspond to the minima of $\Omega_{S0}(\chi)$ and to the vanishing supercurrent. As seen from (S18) and (S19), by fixing the phase difference $\chi(0)$ at the junction edge, one simultaneously specifies the total magnetic...
flux \( \Phi \) penetrating through the junction, the magnetic field \( H(0) \) at the junction edge and the total Josephson current \( I \). Thus the thermodynamic potential \( \Omega_{SO}[\chi(0)] \) can also be considered, for example, as a function of the magnetic field \( H(0) \), or the magnetic flux \( \Phi \). Eqs. (S18) and (S19) also ensure, for given current-phase relations, that the maxima of \( |H(0)|, \Omega_{SO}[\chi(0)] \) and \( |I| \) take place at the phase differences \( \chi(0) = 2\pi (n+ \frac{1}{2}) \). The magnetic flux at these phase differences takes half-integral values of the flux quantum, and the Josephson current density \( j[\chi(0)] \) vanishes at the junction edge.

The magnetic penetration depth \( l_j(\Phi) \) as a function of the magnetic flux through the junction with an anharmonic current-phase relation, follows from the relation \( \Phi = 2\lambda_1 l_j H(0) \), where one should take into account (S18) and (S19):

\[
l_j^{-1}(\Phi) = \left\{ \frac{16\pi \lambda_L}{\Phi^2} \left[ \frac{2\pi \Phi}{\Phi_0} - \Omega_{SO}(0) \right] \right\}^{1/2}.
\] (S21)

Here \( \Omega_{SO}(\chi) \) is assumed to be an even function of \( \chi \).

The defining relation (9) for the junction penetration depth follows from (S21) and (S18).

For the junctions with a pronounced interfacial pair breaking \( g_s^2 \gg 1 \) one gets from (S17) and (S11)

\[
\Omega_{SO}[\chi(y)] - \Omega_{SO}(0) = \frac{\hbar j_d p}{2|e|} \int_0^{\chi(y)} j(\chi) d\chi = \\
= \frac{3\sqrt{3}\hbar j_d p}{16|e|(g_s + g_t)} \ln \left[ 1 + \frac{4g_s(g_s + g_t)}{g_s^2} \sin^2 \frac{\chi(y)}{2} \right].
\] (S22)

Eq. (10) follows from (S21), (S22) and from the expressions for \( j_d p, \lambda_L \) and \( \xi \).

### S3. Basic equation for \( \chi(y) \) in the vortex in anharmonic junctions

For describing the spatial dependence of the phase difference in the Josephson vortex, it is convenient to base it on the equation, which follows from Eq. (S18) after the derivative of the phase difference is substituted for the magnetic field using (S14). Introducing the dimensionless coordinate \( \tilde{y} = \frac{y}{\lambda_v} \), where \( \lambda_v = l_j(\frac{\Phi_0}{2}) \) and \( \lambda_j \) is defined in (S21), one gets

\[
\left( \frac{d\chi}{d\tilde{y}} \right)^2 = \frac{\pi^2}{l_j} \left[ \frac{\Omega_{SO}[\chi(y)] - \Omega_{SO}(0)}{\Omega_{SO}(\pi) - \Omega_{SO}(0)} \right].
\] (S23)

Eq. (13) follows from (S23) and (S22).

### S4. Basic expression for \( H_{jc1} \) in anharmonic junctions

The lower critical field of the junction is known to satisfy the relation \( H_{jc1} = 4\pi \Omega_l / \Phi_0 \), where \( \Omega_l \) is the thermodynamic potential of the Josephson vortex per unit length. The quantity \( \Omega_l \) can be written as

\[
\Omega_l = \int_0^\infty \left\{ \frac{H^2(y)}{8\pi} 2\lambda_L + \left[ \Omega_{SO}[\chi(y)] - \Omega_{SO}(0) \right] \right\} dy.
\] (S24)

It includes both the magnetic field energy and the junction term \( \Omega_{SO}[\chi(y)] \) in the absence of the magnetic field, taken for the phase difference \( \chi(y) \). The spatial dependence of \( \chi(y) \) is described by a solution of Eq. (S23) for an isolated Josephson vortex and thereby it takes into account the self-field effects. The relation (S18) shows that the two terms in (S24) are equal to each other. Hence,

\[
H_{jc1} = \frac{8\pi l_j}{\Phi_0} \int_0^\infty \left[ \Omega_{SO}[\chi(y)] - \Omega_{SO}(0) \right] d\tilde{y}.
\] (S25)

Eq. (14) for \( H_{jc1} \) follows from (S25), (S22) and (11) after taking into account the expressions for \( j_d p, \lambda_L, \xi \) and \( \lambda_j \).