CONVERGENCE RATES FOR LINEAR AND NONLINEAR INVERSE PROBLEMS IN HILBERT SPACES VIA HÖLDER STABILITY ESTIMATES

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Abstract. In this article, we look for a new kind of smoothness concept, i.e. Hölder smoothness condition for finding the convergence rates of linear and nonlinear inverse problems in Hilbert spaces. The dependency of variational inequalities on the non-linearity estimates satisfied by \( F \) for obtaining the convergence rates is also shown. For linear problems, the convergence rates are obtained without the use of the spectral theory and results obtained are similar to [3]. The co-action between the variational inequalities and the Hölder stability estimates is also discussed for both the linear and nonlinear problems.

1. Introduction

After many years of the inception of the subject, a string of the rates for convergence for both the linear and nonlinear problems has been developed. The introduction of the novel smoothness conditions, i.e. variational inequalities for obtaining the convergence rates is motivated because of the motive of getting better rates (if possible) or generalizing the existing smoothness conditions satisfied by the exact solution of the problem. The major tool in our analysis is the incorporation of Hölder stability estimates and the inhomogeneous variational inequality separately for obtaining the convergence rates for both the linear as well as nonlinear inverse problems. The convergence rates, which are in particular the norm of the difference between the regularized solution and the exact solution, describe the speed of the approximation. In practice, there are two different approaches for determining the convergence rates. First one is on the basis of source and nonlinearity conditions. See for instance [4, 2, 10, 14] for variational regularization (in particular Tikhonov regularization) and see for instance [8, 10] for Iterative regularization. The other one is exclusively on the basis of stability estimates which have been derived in [6] for iterative regularization (in particular Landweber iteration method) in Banach spaces and in [11] for variational regularization methods. The results of Logarithmic stability results can be found in [12, 13], in particular in [15] for Electrical Impedance Tomography and results of Hölder type stability estimates can be found in [16, 17].

Let \( T : X \to Y \) be a bounded linear operator between the Hilbert spaces \( X \) and \( Y \). Whenever we write \( F : X \to Y \), we mean \( F \) is a nonlinear operator between the Hilbert spaces \( X \) and \( Y \). For the linear inverse problems, our objective is to find out the minimum-norm solution \( x^\dagger \) of

\[
Tx = y, \quad y \in R(T)
\]
where $R(T)$ is the range of $T$. The existence and uniqueness of such a minimum-norm solution for each $y \in R(T)$ is already discussed in [4, Theorem 2.5]. Because of the unavailability of the exact data $y$, we are available with the noisy approximation $y^\delta$ of $y$ satisfying

$$\|y^\delta - y\| \leq \delta.$$  \hfill (1.1)

Because of the ill-posedness of the problem (in general), we look for the regularized approximations of the exact solution $x^\dagger$. There are quite a few regularization techniques for getting the stable approximation of the exact solution (e.g. iterative regularization [8], regularization with sparsity constraints [2], discretization [18]) but one of the facile technique is the Tikhonov regularization. This, in particular, involves the determination of unique minimizers $\{x^\delta_\alpha, \alpha \geq 0\}$ of the Tikhonov functional

$$\| Tx - y^\delta \|^2 + \alpha \| x \|^2.$$ \hfill (1.2)

For the nonlinear inverse problems, we look for the $u_0$-minimum norm solution $u^\dagger$, i.e.

$$\|u^\dagger - u_0\| = \inf \{\|u - u_0\| \mid F(u) = y, \ u \in X\}$$ \hfill (1.3)

where $u_0$ is an a-priori guess of the original solution [4]. In this case too, majority of the problems are ill-posed. Again in place of exact data $y$, we have some approximation $y^\delta$ of $y$ satisfying (1.1) and it is arduous to get the original solution. Thus, as in the case of linear problems, we have to exploit the regularization techniques to get the approximations of the exact solution. For these problems, we find $\{u^\delta_\alpha, \alpha \geq 0\}$, the unique minimizers of the Tikhonov functional

$$\| Fu - y^\delta \|^2 + \alpha \| u - u_0 \|^2.$$ \hfill (1.4)

For both the linear and nonlinear problems in the Hilbert spaces, existence, uniqueness of the Tikhonov minimizers for arbitrary $\alpha > 0$ and their convergence to exact solution when $\alpha$ is properly chosen in accordance with $\delta$ is given in [4, 2]. Coming back to the convergence rates, it is well known that we need to employ some kind of smoothness of the exact solution while obtaining the convergence rates. For solution smoothness, different kind of source conditions and their cross connections are given in [5]. Recently in [3], novel variational inequalities are introduced for obtaining the convergence rates for linear problems. To the best of our knowledge, Hölder stability estimates are introduced here for the first time to find the convergence rates for linear as well as nonlinear problems and the analysis is based on the idea gathered from [6]. The motive behind the introduction of these novel smoothness condition is either to incorporate the more general smoothness condition or to get the better (or atleast same) convergence rates than the existing ones.

This paper is organized in the following manner: Section 2 comprises of some basic Definitions and inequalities which are used in our analysis. In Section 3, we give our main result on the convergence rates using Hölder stability estimate as the smoothness concept. The relation of the Hölder stability estimate satisfied by the exact solution with the already existing smoothness
concepts is also discussed. Results are further supported with the help of an example. Section 4
involves the convergence rate results for nonlinear problems with respect to the inhomogeneous
and homogeneous variational inequality (defined there) with a particular non-linearity estimate.
We also emphasize the cases where we are unable to obtain the convergence rates by our tech-
nique. With respect to the Hölder stability estimates, rate of convergence for nonlinear problems
is also given in this section. In the penultimate section, we discuss the interplay between the
smoothness concepts used for nonlinear problems and then the conclusion is made in the last
section.

2. Preliminaries

**Definition 2.1.** Index function: A function \( \psi : [0, \infty) \to [0, \infty) \) is called an index function if it
is strictly monotonically increasing, continuous and \( \psi(0) = 0 \).

Next, we define the generalized versions of the smoothness conditions defined in [3] for linear
inverse problems in Hilbert spaces.

**Definition 2.2.** Let \( X, Y \) be real Hilbert spaces and \( T : X \to Y \) be a bounded linear operator.
We say that the exact solution \( x^\dagger \) of the problem \( Tx = y \) where \( y \in R(T) \) satisfies
- the inhomogeneous variational inequality with respect to an index function \( \psi \) and for
  \( \mu \in (0,1] \), if there exist constants \( \eta \geq 0 \) and \( \beta \in [0,1) \) such that for every \( x \in X \)
  \[ \langle x^\dagger, x \rangle \leq \eta \| \psi(T^*T)x \|^\mu + \beta \| x \|^2. \]
  (2.1)
- the homogeneous variational inequality with the parameter \( \nu \in (0,1] \), if there exist a
  constant \( \eta \geq 0 \) such that for every \( x \in X \)
  \[ \langle x^\dagger, x \rangle \leq \eta \| \psi(T^*T)x \|^\nu \| x \|^{1-\nu}. \]
  (2.2)
- the generalized source condition with respect to an index function \( \psi \) if for some \( x \in X \)
  \[ x^\dagger = \psi(T^*T)x. \]
  (2.3)

**Remark 2.1.** The respective names given to the inequalities (2.1) and (2.2) is because these ine-
qualities are generalizations of the inhomogeneous and the homogeneous variational inequalities
defined in [3].

**Definition 2.3.** Let \( T : X \to Y \) be a bounded linear operator between the Hilbert spaces \( X \) and
\( Y \). Further, let \( x^\dagger \) denotes the minimum-norm solution of the operator equation \( Tx = y \) for
\( y \in R(T) \). Then, we say that the problem has
- a noise-free convergence rate of order \( \sigma \) if
  \[ \| x_\alpha - x^\dagger \| \leq A \alpha^\sigma, \quad \text{for every } \alpha > 0, \]
  (2.4)
where \( A > 0 \) is some constant and \( x_\alpha \) is the minimizer of the Tikhonov functional (1.2)
with \( y^\delta \) replaced by \( y \).
• a convergence rate of order \( \sigma \) if

\[
\sup \left\{ \inf_{\alpha > 0} \| x_\alpha(y') - x^\dagger \| : y' \in Y, \| y' - y \| \leq \delta \right\} \leq B\delta^\sigma
\]  

(2.5)

where \( B > 0 \) is some constant and \( x_\alpha(y') \) is the minimizer of the Tikhonov functional (1.2) with \( y^\delta \) replaced by \( y' \).

Relation between the generalized source condition and the homogeneous variational inequality is given in [1, Corollary 4.2]. Relation between standard source conditions and inhomogeneous variational inequality was already discussed in [19]. Next, we give the relation between the homogeneous and the inhomogeneous variational inequality.

**Lemma 2.1.** The homogeneous variational inequality (2.2) with the parameter \( \nu \in (0, 1] \) implies the inhomogeneous variational inequality (2.1) with the parameter \( \mu = \frac{2\nu}{1+\nu} \).

**Proof.** For any \( x \in X \), the homogeneous variational inequality with the parameter \( \nu \in (0, 1] \) is

\[
\langle x^\dagger, x \rangle \leq \eta \| \psi(T^*T)x \|^{\nu} \| x \|^{1-\nu},
\]

where \( \eta > 0 \) is some constant. Now, apply Young’s inequality \( ab \leq a^p/p + b^q/q \) with the parameters \( a = \eta \| \psi(T^*T)x \|^{\nu} \), \( b = \| x \|^{1-\nu} \), \( p = \frac{2}{1+\nu} \), \( q = \frac{2}{1-\nu} \) in the right side of above inequality to obtain

\[
\langle x^\dagger, x \rangle \leq \frac{1+\nu}{2} \eta^{\frac{2}{1+\nu}} \| \psi(T^*T)x \|^{\frac{2\nu}{1+\nu}} + \frac{1-\nu}{2} \| x \|^2
\]

for every \( x \in X \). Thus, the result. \( \square \)

3. Convergence rates for linear problems via Hölder stability estimates

In this section, we derive the convergence rates for linear ill-posed inverse problems by using the Hölder stability estimate as the smoothness condition. The novel thing is the finding of the convergence rates without the use of Spectral theory.

**Theorem 3.1.** Let \( T : X \to Y \) be a bounded linear operator between the real Hilbert spaces \( X \), \( Y \) and \( y \in R(T) \). Further, let \( x^\dagger \) denotes the minimum-norm solution of the problem \( Tx = y \) and it satisfies the Hölder stability estimate

\[
\| x - x^\dagger \| \leq C\| T(x) - T(x^\dagger) \|^k,
\]

(3.1)

for \( 0 < k \leq 1 \), \( C > 0 \) and \( x \in D(F) \). Then, for the noisy data \( y^\delta \in Y \) satisfying (1.1) for \( \delta > 0 \), we have

\[
\| x_\alpha^\delta - x^\dagger \|^2 \leq \frac{(1+k)\delta^2}{\alpha} + C_1\alpha^{\frac{k}{2-\kappa}},
\]

where \( x_\alpha^\delta \) satisfies (1.2) for some \( \alpha > 0 \) and \( C_1 \) is some positive constant.
Proof. By definition of \( x_\alpha^\delta \), we have
\[
\| Tx_\alpha^\delta - y^\delta \|^2 + \alpha \| x_\alpha^\delta \|^2 \leq \| Tx^\dagger - y^\delta \|^2 + \alpha \| x^\dagger \|^2 \leq \delta^2 + \alpha \| x^\dagger \|^2
\]
and hence
\[
\| Tx_\alpha^\delta - y^\delta \|^2 + \alpha \| x_\alpha^\delta - x^\dagger \|^2 \leq \delta^2 + \alpha \| x_\alpha^\delta - x^\dagger \|^2 + \alpha \| x^\dagger \|^2 - \alpha \| x_\alpha^\delta \|^2
\]
\[
= \delta^2 + 2\alpha \langle x^\dagger, x^\dagger - x_\alpha^\delta \rangle.
\]
This with Cauchy-Schwarz inequality and (3.1) leads to
\[
\| Tx_\alpha^\delta - y^\delta \|^2 + \alpha \| x_\alpha^\delta - x^\dagger \|^2 \leq \delta^2 + 2C\alpha \| x^\dagger \| ||T(x_\alpha^\delta) - T(x^\dagger)||^k.
\]
Employing Young’s inequality \( ab \leq a^p/p + b^q/q \) for \( a = 2C\alpha \| x^\dagger \| , b = \| T(x_\alpha^\delta) - T(x^\dagger) \| \), \( p = \frac{2}{2+\alpha}, q = \frac{2}{\alpha} \) in the last term of above inequality to obtain
\[
\| Tx_\alpha^\delta - y^\delta \|^2 + \alpha \| x_\alpha^\delta - x^\dagger \|^2 \leq \delta^2 + C_1 \alpha \frac{2}{2+\alpha} + k \alpha \frac{2}{\alpha} ||T(x_\alpha^\delta) - T(x^\dagger)||^2,
\]
where \( C_1 = \frac{(2-k)(2C\alpha \| x^\dagger \|)^{\frac{2}{2+\alpha}}}{2} \). Above using (1.1) can also be written as On using \( \| x + y \|^2 \leq 2(\| x \|^2 + \| y \|^2) \) and (1.1) in last term of (3.2), we arrive at
\[
(1 - k)||Tx_\alpha^\delta - y^\delta ||^2 + \alpha \| x_\alpha^\delta - x^\dagger \|^2 \leq (1 + k)\delta^2 + C_1 \alpha \frac{2}{2+\alpha}.
\]
Thus, for \( k \leq 1 \), we get
\[
\| x_\alpha^\delta - x^\dagger \|^2 \leq \frac{(1 + k)\delta^2}{\alpha} + C_1 \alpha \frac{2}{2+\alpha}.
\]
\( \Box \)

Remark 3.1. Note that the above estimate is similar to the one obtained when \( x^\dagger \) satisfies the inhomogeneous variational equality (2.1) with same parameters and \( \psi(t) = t^2 \). See [3, Lemma 7].

Corollary 3.1. Let \( T : X \to Y \) be a bounded linear operator between two real Hilbert spaces \( X \), \( Y \) and \( y \in R(T) \). Further, if \( x^\dagger \) denotes the minimum-norm solution of the problem \( Tx = y \) and it satisfies the following Hölder stability estimate
\[
\| x - x^\dagger \| \leq C ||T(x) - T(x^\dagger)||^k,
\]
for \( k = \frac{2\mu}{1+\mu} \), where \( \mu \in (0, 1], C > 0 \) and \( x \in D(T) \). Then, the problem has
1. a convergence rate of the order \( \frac{\mu}{2} \) in the noise free case.
2. a convergence rate of the order \( \frac{\mu}{1+\mu} \) in the case of noisy data.

Proof. First part of the corollary is a direct consequence of (3.3) and \( \delta = 0 \), i.e. (2.4) holds for \( \sigma = \frac{\mu}{2} \). For the second part, by definition of infimum,
\[
\inf_{\alpha>0} \| x_\alpha(y') - x^\dagger \|^2 \leq \| x_{\alpha^\delta-k}(y') - x^\dagger \|^2.
\]
Therefore, (2.5) holds by incorporating (3.3) and \( k = \frac{2\mu}{1+\mu} \). \( \Box \)
Next, we look for the relation between the inhomogeneous variational inequality (2.1) and the Hölder stability estimate (3.1) satisfied by the exact solution.

**Proposition 3.1.** Let $X$ and $Y$ be the real Hilbert spaces, $T : X \to Y$ be a bounded linear operator and $y \in R(T)$. Then, the inhomogeneous variational inequality (2.1) with respect to $\psi(t) = t^{1/2}$, $\mu \in (0, 1]$, i.e.

$$\langle x^\dagger, x^\dagger \rangle \leq \eta \|(T^*T)^{1/2}x\|^\mu + \beta \|x\|^2,$$

with $\beta \in [0, 1)$ and $\eta \geq 0$ implies the Hölder stability estimate

$$\|x' - x^\dagger\| \leq K\|Tx' - Tx^\dagger\|^\nu,$$

under the condition

$$\langle x', x^\dagger - x' \rangle \leq 0,$$

where $x' = x^\dagger - x$ and $K$ is some constant.

**Proof.** First of all, note that as $T^*T$ and $(T^*T)^{1/2}$ are self-adjoint, so for any $x \in X$, we have

$$\|(T^*T)^{1/2}x\|^2 = \langle (T^*T)^{1/2}x, (T^*T)^{1/2}x \rangle = \langle T^*Tx, x \rangle = \|Tx\|^2.$$

So, we can write (3.4) as

$$\langle x^\dagger, x^\dagger \rangle \leq \eta \|Tx\|^\mu + \beta \|x\|^2,$$

$$x \in X$$

Further, we do a bit of manipulation in the inhomogeneous variational inequality. Since for any $x \in X$, $x - x^\dagger \in X$ and hence for each such $x$, there exists some $x' \in X$ such that $x = x^\dagger - x'$. So, put $x = x^\dagger - x'$ in (3.6), we get

$$\langle x^\dagger, x^\dagger - x' \rangle \leq \eta \|Tx' - Tx^\dagger\|^\mu + \beta \|x' - x^\dagger\|^2.$$  

(3.7)

Using (3.7) we have

$$\|x' - x^\dagger\|^2 \leq \langle x', x^\dagger - x' \rangle + \eta \|Tx' - Tx^\dagger\|^\mu + \beta \|x' - x^\dagger\|^2.$$

Above and (3.5) leads to

$$\|x' - x^\dagger\|^2 \leq \frac{\eta}{1 - \beta} \|Tx' - Tx^\dagger\|^\mu.$$

Thus, the assertion follows. \hfill \Box

**Remark 3.2.** Let $\{x_\alpha^\dagger, \alpha \geq 0\}$ be the set of Tikhonov minimizers. Then if (3.4) is satisfied with $x = x^\dagger - x_\alpha^\dagger \in X$, then the Tikhonov minimizers $x_\alpha^\dagger$ satisfy the Hölder stability estimate (3.1) with $k = \frac{\mu}{2}$ under assumption (3.5). So, directly we can not argue that Hölder stability estimates can be obtained from inhomogeneous variational inequality.

Further, in this section we discuss an example which facilitate in the comprehension of results discussed in this section. This example was already discussed in [3] for proving the non equivalency of the different kind of smoothness conditions discussed there. Further, this example also confirms the validity of the inequality (3.5).
Example 3.1. Let $X$ be a separable Hilbert space over field of reals having orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$. Consider $T : X \to X$ given by $T(e_n) = 2^{-n}e_n$ be a compact linear operator. Let the exact data is given to us, i.e.

$$y = \sum_{n=1}^{\infty} 2^{-\frac{1}{2}n}e_n \in R(T).$$

Here, we can easily prove that Moore-Penrose inverse $T^\dagger$ is nothing but equal to $T^{-1}$ and so

$$x^\dagger = T^{-1}y = \sum_{n=1}^{\infty} 2^{-\frac{n}{2}}e_n.$$ 

Now, we show that

(i) given problem is ill-posed.

(ii) Tikhonov minimizers $x_\alpha$ of (1.2) for the non-noisy data satisfies the inequality

$$\langle x_\alpha, x_\alpha - x^\dagger \rangle \leq 0. \quad (3.8)$$

(iii) for noisy data, Tikhonov minimizers $(x_\alpha^\delta)$ of (1.2) satisfies the inequality (3.8) with some additional assumption on $\alpha$.

(iv) for both noise free and noisy data (with additional assumptions) inhomogeneous variational inequality is satisfied with respect to $\psi(t) = t^{1/2}$ and $\mu = \frac{2}{3}$. In particular, Tikhonov minimizers $x_\alpha$ for $\alpha > 0$ satisfies the Hölder stability estimate.

(v) condition (3.5) is not necessary for the Hölder stability estimates to hold.

Proof. For the first part, let $y^{(\delta,n)} = y + \delta e_n$ where $\delta > 0$, be the noisy approximation of the exact data $y$ and $n \in \mathbb{N}$. Clearly $\|y^{(\delta,n)} - y\| \leq \delta$, but

$$\|T^\dagger(y^{(\delta,n)}) - T^\dagger(y)\| = \|T^{-1}(y^{(\delta,n)}) - T^{-1}(y)\|$$

$$= \|T^{-1}(\delta e_n)\| = \delta 2^n \to \infty \text{ as } n \to \infty.$$

So, corresponding to small data error, we get a large error in the solution which clearly implies that the problem is ill-posed.

For the (ii) part, we explicitly calculate the Tikhonov minimizers, i.e.

$$x_\alpha = \arg\min_{x \in X} \{\|Tx - y\|^2 + \alpha\|x\|^2\}$$

for some $\alpha > 0$. But before that, we argue that such a minimizer exists. For $\alpha > 0$, let $g_\alpha(x) = \|Tx - y\|^2 + \alpha\|x\|^2$. As $g_\alpha(x)$ is a strictly convex function, bounded below and $g_\alpha(x) \to \infty$ as $\|x\| \to \infty$, it has a unique minimizer.

Since $X$ has a countable orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$, so any $x \in X$ can be uniquely written as

$$x = \sum_{n=1}^{\infty} b_n e_n, \quad \text{where} \quad b_n = \langle x, e_n \rangle.$$
After substituting the values of $x$ and $y$ in $x_\alpha$, we get

$$ x_\alpha = \arg\min_{x \in X} \left\{ \| T \left( \sum_{n=1}^{\infty} b_n e_n \right) \|_2^2 + \alpha \sum_{n=1}^{\infty} b_n^2 \right\} $$

$$ = \arg\min_{x \in X} \left\{ \sum_{n=1}^{\infty} \left( 2^{-n} b_n - 2^{-\frac{3n}{2}} \right)^2 + \alpha \left( \sum_{n=1}^{\infty} b_n^2 \right) \right\}. $$

Now to find out the minimum value, apply first order necessary criteria, i.e.

$$ \frac{\partial x_\alpha}{\partial b_n} = 0 \text{ for each } n. $$

So, for each $n$

$$ \frac{\partial x_\alpha}{\partial b_n} = 0 \implies 2 \left( b_n 2^{-n} - 2^{-\frac{3n}{2}} \right) 2^{-n} + 2\alpha b_n = 0. $$

Above implies that

$$ x_\alpha = \sum_{n=1}^{\infty} b_n e_n = \sum_{n=1}^{\infty} \frac{2^{-5n/2}}{2^{-2n} + \alpha} e_n. $$

Putting $x_\alpha$ and $x^\dagger$ in (3.8) yields

$$ \langle x_\alpha, x_\alpha - x^\dagger \rangle = \sum_{n=1}^{\infty} \frac{2^{-5n/2}}{2^{-2n} + \alpha} e_n \sum_{n=1}^{\infty} \frac{2^{-5n/2}}{2^{-2n} + \alpha} e_n - \sum_{n=1}^{\infty} 2^{-\frac{5n}{2}} e_n $$

$$ = \sum_{n=1}^{\infty} \left( \frac{2^{-5n/2}}{2^{-2n} + \alpha} \right)^2 - \sum_{n=1}^{\infty} \frac{2^{-3n}}{2^{-2n} + \alpha} $$

$$ = \sum_{n=1}^{\infty} \frac{2^{-5n} - 2^{-3n} (2^{-2n} + \alpha)}{(2^{-2n} + \alpha)^2} < 0. $$

Thus, $\{ x_\alpha : \alpha > 0 \}$ satisfies the estimate (3.8). Now, we check whether the Tikhonov minimizers satisfies (3.7) or not.

$$ \langle x^\dagger, x^\dagger - x_\alpha \rangle = \sum_{n=1}^{\infty} \frac{2^{-n}}{2^{-2n} + \alpha}, \quad \| x_\alpha - x^\dagger \|^2 = \alpha^2 \sum_{n=1}^{\infty} \frac{2^{-n}}{(2^{-2n} + \alpha)^2} $$

and

$$ \| Tx_\alpha - Tx^\dagger \| = \alpha \left( \sum_{n=1}^{\infty} \frac{2^{-3n}}{(2^{-2n} + \alpha)^2} \right)^{1/2}. $$

Let us now consider the noisy case, i.e. if instead of $y$ we are given with some approximation $y^\delta$ of $y$ satisfying (1.1). Since $y^\delta \in X$, we can write

$$ y^\delta = y + \sum_{n=1}^{\infty} 2^{-\frac{3n}{2}} g_n e_n, $$

where we additionally assumed that $g_n$’s are such that

$$ \| \sum_{n=1}^{\infty} 2^{-\frac{3n}{2}} g_n e_n \| \leq \delta, \quad (3.9) $$

which further implies that, for each $n$, we must have (if we look for the individual bound)

$$ |g_n| \leq 2^{\frac{3n}{2}} \delta, \quad (3.10) $$
Now, again we find the minimizer for the Tikhonov functional (1.3). After putting the values of $y^\delta$ and $x$ in above, we arrive at
\[
x^\delta = \arg\min_{x \in X} \left\{ \left\| T \left( \sum_{n=1}^{\infty} b_n e_n \right) \right\|^2 + \alpha \left\| \sum_{n=1}^{\infty} b_n e_n \right\|^2 \right\}
= \arg\min_{x \in X} \left\{ \left( \sum_{n=1}^{\infty} b_n 2^{-n} - 2^{-\frac{5n}{2}} (1 + g_n) \right)^2 + \alpha \left( \sum_{n=1}^{\infty} b_n^2 \right) \right\}.
\]

(3.11)

After applying first order necessary criteria, (3.11) implies
\[
\frac{\partial x^\delta}{\partial b_n} = 0 \implies 2(b_n 2^{-n} - 2^{-\frac{5n}{2}} (1 + g_n)) 2^{-n} + 2\alpha b_n = 0.
\]
On solving this, we get
\[
b_n = \frac{2^{-\frac{5n}{2}} (1 + g_n)}{2^{-2n} + \alpha}.
\]
So, the regularized solution is
\[
x^\delta = \sum_{n=1}^{\infty} b_n e_n = \sum_{n=1}^{\infty} \frac{2^{-\frac{5n}{2}} (1 + g_n)}{2^{-2n} + \alpha} e_n.
\]
Putting $x^\delta$ (in place of $x_\alpha$) and $x^\dagger$ in (3.8) to get
\[
\langle x^\delta, x^\delta - x^\dagger \rangle = \sum_{n=1}^{\infty} \frac{2^{-5n} (1 + g_n)}{2^{-2n} + \alpha} e_n \sum_{n=1}^{\infty} \frac{2^{-\frac{5n}{2}} (1 + g_n)}{2^{-2n} + \alpha} e_n - \sum_{n=1}^{\infty} \frac{2^{-\frac{5n}{2}} e_n}{2^{-2n} + \alpha}
= \sum_{n=1}^{\infty} \left( \frac{2^{-5n} (1 + g_n)^2}{(2^{-2n} + \alpha)^2} - \frac{2^{-3n} (1 + g_n)^2}{2^{-2n} + \alpha} \right)
\sum_{n=1}^{\infty} \frac{(1 + g_n)[2^{-5n} (1 + g_n) - 2^{-3n} - \alpha 2^{-3n}]}{(2^{-2n} + \alpha)^2}
= \sum_{n=1}^{\infty} \frac{(1 + g_n)[g_n - \alpha 2^{2n}]}{2^m (2^{-2n} + \alpha)^2}.
\]

If we select $\alpha$ such that $|g_n| > \alpha 2^n$ and $1 + g_n \geq 0$, then the set of Tikhonov minimizers clearly satisfies (3.5) under the assumption (1.1).

For part $(iv)$, from Example 5 in [3], it is known that the given data fulfills the homogeneous variational inequality with $\psi(t) = t^{1/2}$ and $\nu = \frac{1}{2}$. From Lemma 2.1, we can easily conclude that the given data fulfills inhomogeneous variational inequality with the parameter $\mu = \frac{2}{3}$ in the noise free case. Further, as (3.4) holds good for any $x \in X$, in particular, it is also true for $x^\dagger - x_\alpha \in X$. Thus, by the virtue of $(ii)$, Hölder stability estimate is also satisfied by the Tikhonov minimizers.

For the noisy data, for our convenience we write
\[
x = \sum_{n=1}^{\infty} 2^2 \gamma_n e_n,
\]
where \( x \in X \) is arbitrary and \( \gamma_n \in \mathbb{R} \). Let \( y^\delta \) is same as taken in part \((iii)\). Then, in this case

\[
x^\dagger = T^{-1}y^\delta = T^{-1}(y + \sum_{n=1}^{\infty} 2^{-\frac{n}{2}} g_ne_n) = \sum_{n=1}^{\infty} (1 + g_n)2^{-\frac{n}{2}}e_n
\]

provided \( \|x^\dagger\| \) is finite which further depends on \( g_n \)'s and so we assume them accordingly. Further,

\[
\langle x^\dagger, x \rangle = \sum_{n=1}^{\infty} (1 + g_n)\gamma_n, \quad \|x\|^2 = \sum_{n=1}^{\infty} 2^n|\gamma_n|^2, \quad \|Tx\|^2 = \sum_{n=1}^{\infty} 2^{-n}|\gamma_n|^2. \tag{3.12}
\]

Now, we prove that that the above data fulfills the homogeneous variational inequality with \( \psi(t) = t^{1/2} \) and \( \nu = \frac{1}{2} \), provided the sequence \( \{g_n\} \) is bounded above.

Consider \( G = \sum_{n=1}^{\infty} (1 + g_n)|\gamma_n| \) and choose \( N \in \mathbb{N} \) be such that

\[
\frac{G}{2} \leq G_1 = \sum_{n=1}^{N} (1 + g_n)|\gamma_n| \quad \text{and} \quad \frac{G}{2} \leq G_2 = \sum_{n=N}^{\infty} (1 + g_n)|\gamma_n|. \tag{3.13}
\]

Then by Cauchy-Schwarz inequality,

\[
G_1^2 \leq \sum_{k=1}^{N} 2^k \sum_{n=1}^{N} 2^{-n}(1 + g_n)|\gamma_n|^2 \leq (2^{N+1} - 1) \sum_{n=1}^{N} 2^{-n}(1 + g_n)|\gamma_n|^2.
\]

Since the sequence \( \{g_n\} \) is bounded above, so take \( M_1 = \max \{1 + g_n^2: 1 \leq n \leq N\} \). Thus, we get

\[
G_1^2 \leq M_1(2^{N+1} - 1) \sum_{n=1}^{N} 2^{-n}|\gamma_n|^2 = M_1(2^{N+1} - 1)\|Tx\|^2. \tag{3.14}
\]

Similarly,

\[
G_2^2 \leq \sum_{k=N}^{\infty} 2^{-k} \sum_{n=N}^{\infty} 2^n(1 + g_n)^2|\gamma_n|^2 \leq 2^{-N+1} \sum_{n=N}^{\infty} 2^n(1 + g_n)^2|\gamma_n|^2.
\]

Take \( M_2 = \max \{1 + g_n^2: n \geq N\} \). So, we have

\[
G_2^2 \leq 2^{-N+1}M_2 \sum_{n=N}^{\infty} 2^n|\gamma_n|^2 = 2^{-N+1}M_2\|x\|^2. \tag{3.15}
\]

So, (3.13), (3.14) and (3.15) implies

\[
\langle x^\dagger, x \rangle = \sum_{n=1}^{\infty} (1 + g_n)\gamma_n \leq G \leq 2\sqrt{G_1G_2} \leq C\|Tx\|^\frac{1}{2}\|x\|^\frac{1}{2},
\]

where \( C > 0 \) is some constant. Thus, the homogeneous variational inequality is satisfied with the parameter \( \nu = \frac{1}{2} \). So again on using Lemma 2.1, we get the desired result.

For proving part \((v)\), take \( x = \frac{1}{2}e_m \) where \( m \geq 4 \). Then for this \( x \), left hand side of (3.5) (with \( x' \) replaced by \( x \)) is

\[
\langle x, x - x^\dagger \rangle = \langle \frac{1}{2}e_m, \left( \frac{1}{2}e_m - \sum_{n=1}^{\infty} 2^{-\frac{n}{2}}e_n \right) \rangle = \frac{1}{4} - 2^{-\frac{m}{2} - 1}.
\]

For \( m \geq 4 \), we get \( \langle x, x - x^\dagger \rangle > 0 \). Further,

\[
\|x - x^\dagger\|^2 = \left( \frac{1}{2}e_m - \sum_{n=1}^{\infty} 2^{-\frac{n}{2}}e_n \right) \cdot \left( \frac{1}{2}e_m - \sum_{n=1}^{\infty} 2^{-\frac{n}{2}}e_n \right) = \sum_{n=1, n \neq m}^{\infty} 2^{-n} + (2^{-1} - 2^{-\frac{m}{2}})^2.
\]
\[
\sum_{n=1}^{\infty} 2^{-n} + \frac{1}{4} - 2^{-m} = \frac{5}{4} - 2^{-m}.
\]

So, we have
\[
\|x - x^\dagger\|^2 \leq \frac{5}{4}.
\]
(3.16)

Also,
\[
\|Tx - Tx^\dagger\|^2 = \left(2^{-(m+1)}e_m - \sum_{n=1}^{\infty} 2^{-\frac{3}{2}n}e_n\right), \left(2^{-(m+1)}e_m - \sum_{n=1}^{\infty} 2^{-\frac{3}{2}n}e_n\right)
\]
\[
= \sum_{n=1, n \neq m}^{\infty} 2^{-3n} + 2^{-(m+1)} - 2^{-\frac{3m}{2}}\right)^2 = \sum_{n=1}^{\infty} 2^{-3n} + 2^{-(m+1)} - 2^{-\frac{5m}{2}}\right.
\]
\[
= \frac{1}{7} + 2^{-(m+1)} - 2^{-\frac{5m}{2}}.
\]

For \(m \geq 4\), we have
\[
\frac{1}{7} \leq \|T(x) - T(x^\dagger)\|^2.
\]
(3.17)

So, from the estimates (3.16) and (3.17), we get
\[
\frac{1}{7}\|x - x^\dagger\|^2 \leq \frac{5}{4} \times \frac{1}{7} \leq \frac{5}{4}\|T(x) - T(x^\dagger)\|^2
\]

Thus,
\[
\|x - x^\dagger\|^2 \leq \frac{35}{4}\|T(x) - T(x^\dagger)\|^2.
\]

Thus, Hölder estimate is satisfied with the parameter \(k = 1\) even though the estimate (3.5) is not satisfied.

Next, we look for the relation between the Hölder stability estimate and the inhomogeneous variational inequality.

**Proposition 3.2.** Let \(X\) and \(Y\) be real Hilbert spaces, \(T : X \to Y\) be a bounded linear operator and \(y \in R(T)\). Then, for any \(x \in X\), the Hölder stability estimate (3.1) with \(k \in (0, 1]\) implies the inhomogeneous variational inequality (2.1) with respect to \(\psi(t) = t^{1/2}\), \(k \in (0, 1]\) and \(\beta = 0\) for some constant \(C_1\) and \(x = y - x^\dagger\).

**Proof.** Let \(x^\dagger\) satisfies (3.1) for some \(C > 0\), i.e.
\[
\|x - x^\dagger\| \leq C\|T(x) - T(x^\dagger)\|^k.
\]

Further, we have
\[
\langle x^\dagger, x - x^\dagger \rangle \leq \|x^\dagger, x - x^\dagger\| \leq \|x^\dagger\|\|x - x^\dagger\| \leq C_1\|T(x - x^\dagger)\|^k.
\]

where \(C_1 = C\|x^\dagger\|\). Now put \(x - x^\dagger = y\), we get
\[
\langle x^\dagger, y \rangle \leq C_1\|Ty\|^k = C_1\|(T^*T)^{1/2}y\|^k.
\]

Thus, the result holds.
Remark 3.3. Above proposition concludes that for the linear operator $T$, inhomogeneous variational inequality is more general than the Hölder stability estimate which is also clear from the Example 3.1.

In the end, we briefly look for the co-action between the source conditions (2.3), homogeneous variational inequality (2.2) and the Hölder stability estimate (3.1).

Proposition 3.3. Let $X$ and $Y$ be the real Hilbert spaces and $T : X \to Y$ be a bounded linear operator. Then

(i) generalized source condition (2.3) with respect to the index function $\psi(t) = t^\nu$ for $\nu \in (0, 1]$ and the homogeneous variational inequality (2.2) with the index function $\psi(t) = t^{1/2}$, $\nu \in (0, 1)$ implies the Hölder stability estimate (3.1) with $x$ replaced by $x'$ and $k = \frac{\nu}{\nu + 2}$ under the condition (3.5) where $x' = x^\dagger - x$, $x \in X$ is arbitrary.

(ii) The Hölder stability estimate (3.1) implies the source condition (2.3) with the index function $\psi(t) = t^{1/2}$.

Proof. For proving (i), use Lemma 2 in [3] and Proposition 3.1. For (ii), we have

$$\langle x^\dagger, x - x^\dagger \rangle \leq |\langle x^\dagger, x - x^\dagger \rangle| \leq \|x^\dagger\|\|x - x^\dagger\|.$$

Using (3.1) in above and then using $x - x^\dagger = x'$ yields

$$\langle x^\dagger, x' \rangle \leq C_1\|Tx'\|^k$$

where $C_1 = C\|x^\dagger\|$. Put $k = 1$ in above to obtain

$$\langle x^\dagger, x' \rangle \leq C_1\|Tx'\| = C_1\|(T^*T)^{1/2}x'\|.$$

Since, $x$ is arbitrary and hence $x' \in X$ is arbitrary, by Lemma 8.21 in [2], result holds.

4. Convergence rates for NonLinear problems

In this section, we find the convergence rates for nonlinear Inverse problems via different smoothness concepts. First of all, we define the different smoothness conditions which we use later on for obtaining the convergence rates.

Definition 4.1. Let $F$ be a nonlinear operator between the Hilbert spaces $X$ and $Y$ and $u^\dagger$ be the $u_0$-minimum norm solution of the operator equation $F(u) = y$, $y \in R(F)$. Then, the problem fulfills

- the inhomogeneous variational inequality for $\mu \in (0, 1]$, if there exist constants $A \geq 0$ and $\beta \in [0, 1)$ such that

$$\langle u^\dagger, u \rangle \leq A\|F'(u^\dagger)u\|^{\mu} + \beta\|u\|^2, \quad u \in X \quad (4.1)$$

where $F'(u^\dagger)$ is Fréchet derivative of $F$ at $u^\dagger$. 


the homogeneous variational inequality with the parameter \( \nu \in (0, 1] \), if there exist a constant \( A \geq 0 \) such that
\[
\langle u^\dagger, u \rangle \leq A \| F'(u^\dagger)u \| \| u \|^{1-\nu}, \quad u \in X
\] (4.2)
where \( F'(u^\dagger) \) is Fréchet derivative of \( F \) at \( u^\dagger \).

the generalized source condition with respect to an index function \( \psi \), if
\[
x^\dagger = \psi(F'(u^\dagger)F'(u^\dagger))u, \quad \text{for some } u \in X.
\] (4.3)

**Definition 4.2.** We say that the exact solution \( u^\dagger \) of the operator equation \( F(u) = y \) for \( y \in R(F) \) satisfies a Hölder-type stability estimate, if there exist constants \( G > 0 \) and \( 0 < k \leq 1 \) such that
\[
\| u - u^\dagger \| \leq G \| F(u) - F(u^\dagger) \|^{k}, \quad u \in X.
\] (4.4)

**Notation 4.1.** We use the notation \( \sim \), which means that if \( \alpha \sim \delta^s \) for some \( s > 0 \), then there exists constants \( g_1, g_2 > 0 \) such that
\[
g_1 \delta^s \leq \alpha(\delta) \leq g_2 \delta^s.
\]

4.1. **Convergence rates via variational inequalities.** To start this subsection we give the result on convergence rates with respect to the inhomogeneous variational inequality satisfied by the exact solution and a non-linearity estimate satisfied by \( F \).

**Theorem 4.1.** Let \( F : X \to Y \) be a nonlinear, Fréchet differentiable operator between the Hilbert spaces \( X \) and \( Y \). Further, let \( y^\delta \in Y \) satisfies (1.1) and \( u^\dagger \) satisfies (1.3). Moreover, assume that

1. \( F \) satisfies the non-linearity estimate
\[
\| F(u) - F(u^\dagger) - F'(u^\dagger)(u - u^\dagger) \| \leq K \| F(u) - F(u^\dagger) \|, \quad u \in X
\] (4.5)
for some \( K > 0 \) in a sufficiently large ball \( B \) around \( u^\dagger \).

2. \( F'(u^\dagger) \) satisfies the inhomogeneous variational inequality, i.e.
\[
2\langle u^\dagger - u_0, u^\dagger - u \rangle \leq \beta \| F'(u^\dagger)(u - u^\dagger) \|^{\mu} + \gamma \| u - u^\dagger \|^2, \quad u \in X
\] (4.6)
for \( \mu \in (0, 1], \beta \geq 0, \gamma \in [0, 1) \) in a sufficiently large ball around \( u^\dagger \), i.e. \( B_r(u^\dagger) \subseteq B \) for some \( r > 2 \| u^\dagger - u_0 \| \).

3. \( \mu \) and \( K \) satisfies \( 1 - 2\mu(K^2 + 1) \geq 0 \).

Then, for the choice of \( \alpha \sim \delta^\frac{\mu}{\mu+1} \), we have
\[
\| u^\delta_\alpha - u^\dagger \|^2 \leq K_1 \delta^\frac{4\mu}{\mu+1} + K_2 \delta^\frac{2\mu}{(\mu+1)(2-\mu)}
\]
for constants \( K_1, K_2 > 0 \) where \( u^\delta_\alpha \) satisfies (1.4).

In particular, we get
\[
\| u^\delta_\alpha - u^\dagger \| = O(\delta^\frac{\mu}{(\mu+1)(2-\mu)}).
\]
Proof. By definition, $u^\delta_\alpha$ is a minimizer of (1.4) which means
\[
\|F(u^\delta_\alpha) - y^\delta\|^2 + \alpha\|u^\delta_\alpha - u_0\|^2 \leq \delta^2 + \alpha\|u^\dagger - u_0\|^2
\]
which further leads to
\[
\|F(u^\delta_\alpha) - y^\delta\|^2 + \alpha\|u^\delta_\alpha - u^\dagger\|^2 \leq \delta^2 + \alpha(\|u^\dagger - u_0\|^2 + \|u^\delta_\alpha - u^\dagger\|^2 - \|u^\delta_\alpha - u_0\|^2)
\]
\[
= \delta^2 + 2\alpha\langle u^\dagger - u_0, u^\dagger - u^\delta_\alpha\rangle.
\]
(4.7)
Now, our claim is that $u^\delta_\alpha$ is in some neighborhood of $u^\dagger$. From above estimate, for $\alpha \sim \frac{2}{\delta^{2+\epsilon}}$ and for $r > 2\|u^\dagger - u_0\|$, we have
\[
\|u^\delta_\alpha - u^\dagger\|^2 \leq K\delta^{\frac{4}{2+\epsilon}} + r\|u^\dagger - u^\delta_\alpha\|
\]
for some constant $K > 0$. If $\delta$ is sufficiently small, then clearly $u^\delta_\alpha \in B_r(u^\dagger)$ and thus, the claim holds. After employing the estimate (4.6) in (4.7), we arrive at
\[
\|F(u^\delta_\alpha) - y^\delta\|^2 + \alpha\|u^\delta_\alpha - u^\dagger\|^2 \leq \delta^2 + \alpha\beta\|F'(u^\dagger)(u^\delta_\alpha - u^\dagger)\|\|u^\dagger\| + \alpha\gamma\|u^\delta_\alpha - u^\dagger\|^2.
\]
Let us assume $A = \|F(u^\delta_\alpha) - y^\delta\|^2$ and $B = \|u^\delta_\alpha - u^\dagger\|^2$. Then, by using the Young’s inequality $ab \leq a^p/p + b^q/q$ where $1/p + 1/q = 1$, in the middle term of the right side of above inequality by taking $a = \alpha\beta, b = \|F'(u^\dagger)(u^\delta_\alpha - u^\dagger)\|\|u^\dagger\|, p = \frac{2}{2-p}$ and $q = \frac{2}{p}$, we get
\[
A + \alpha B \leq \delta^2 + \frac{2-\mu}{2} (\alpha\beta) \frac{\|F'(u^\dagger)(u^\delta_\alpha - u^\dagger)\|^2 + \alpha\gamma\|u^\delta_\alpha - u^\dagger\|^2 + \alpha\gamma B.
\]
This with
\[
\|x_1 + x_2\|^2 \leq 2^{p-1}(\|x_1\|^p + \|x_2\|^p), \quad x_1, x_2 \in X
\]
(4.8)
for $p = 2$, see [2, Lemma 3.20], leads to
\[
A + \alpha(1 - \gamma)B \leq \delta^2 + \frac{2-\mu}{2} (\alpha\beta) \frac{\|F'(u^\dagger) - F(u^\dagger)\|^2 + \alpha\gamma\|F(u^\delta_\alpha) - F(u^\dagger)\|^2}{\alpha\beta}.
\]
(4.9)
Further, (4.5) with above yields
\[
A + \alpha(1 - \gamma)B \leq \delta^2 + \frac{2-\mu}{2} (\alpha\beta) \frac{\|F(u^\delta_\alpha) - F(u^\dagger)\|^2}{\alpha\beta}.
\]
Above with (4.8) for $p = 2$ and the noise estimate (1.1) leads to
\[
A + \alpha(1 - \gamma)B \leq \delta^2 + \frac{2-\mu}{2} (\alpha\beta) \frac{\|F(u^\delta_\alpha) - F(u^\dagger)\|^2}{\alpha\beta} + 2\mu(K^2 + 1)(A + \delta^2).
\]
After some minor rearrangements and using condition (3), we get
\[
\alpha(1 - \gamma)B \leq (2\mu(K^2 + 1) + 1)\delta^2 + \frac{2-\mu}{2} (\alpha\beta) \frac{2^\gamma}{\beta^{2-\gamma}}.
\]
Finally, we have
\[
B = \|u^\delta_\alpha - u^\dagger\|^2 \leq K_1 \frac{\delta^2}{\alpha} + K_2 \alpha^\frac{\beta^2}{\beta^{2-\gamma}}
\]
(4.10)
where $K_1 = \frac{2\mu(K^2+1)+1}{1-\gamma}$ and $K_2 = \frac{2\mu}{2(1-\gamma)} \beta^{2-\gamma}$ are constants and $1 - 2\mu(K^2+1) \geq 0$.
Moreover, if \( \alpha \sim \delta^{\frac{2}{2\mu+1}} \), then we have
\[
\|u_\alpha^\delta - u^\dagger\|^2 \leq K_3 \delta^{\frac{4\mu}{2\mu+1}} + K_4 \delta^{\frac{2\mu}{(2\mu+1)(2-\mu)}},
\]
where \( K_3 \) and \( K_4 \) are constants. Finally, for \( \delta \to 0 \), we have
\[
\|u_\alpha^\delta - u^\dagger\| = O(\delta^{\frac{\mu}{2\mu+1}}).
\]

Remark 4.1. Note that the estimate (4.10) obtained in the Theorem 4.1, i.e.
\[
\|u_\alpha^\delta - u^\dagger\|^2 \leq \frac{\delta^2}{\alpha} + K_2 \alpha^{\frac{\mu}{2-\mu}}
\]
is exactly similar with the expression for the rate of convergence obtained for linear problems when \( x^\dagger \) satisfies the inhomogeneous inequality. See [3, Lemma 7].

Remark 4.2. Employing inhomogeneous variational inequality (4.6) for obtaining the convergence rates depends largely on the non-linearity estimate satisfied by \( F \). For instance, if in Theorem 4.1 instead of using the non-linearity estimate (4.5), we consider the non-linearity estimate
\[
\|F(u) - F(u^\dagger) - F'(u^\dagger)(u - u^\dagger)\| \leq K\|u - u^\dagger\|^2
\]
for some \( K > 0 \) taken in [4, Theorem 10.4], then this technique does not yield convergence rates.

Proof. From equation (4.9) of the Theorem 4.1, we have
\[
A + \alpha(1-\gamma)B \leq \delta^2 + \frac{2-\mu}{2}(\alpha\beta)^{\frac{2}{2-\mu}} + \mu\|F(u_\alpha^\delta) - F(u^\dagger) - F'(u^\dagger)(u_\alpha^\delta - u^\dagger)\|^2 + \mu\|F(u_\alpha^\delta) - F(u^\dagger)\|^2.
\]
This with inequality (4.8) and condition \( 1 - 2\mu \geq 0 \) implies
\[
\alpha(1-\gamma)B \leq (1 + 2\mu)\delta^2 + \frac{2-\mu}{2}(\alpha\beta)^{\frac{2}{2-\mu}} + \mu K^2 B^2 + \mu\|F(u_\alpha^\delta) - F(u^\dagger)\|^2.
\]
From above, it is not possible to find an upper bound on \( B \).

Now, we look for the upper bounds on \( \|u_\alpha^\delta - u^\dagger\| \) with the help of a different non-linearity estimate other than the one used in the Theorem 4.1.

Theorem 4.2. Let \( F : X \to Y \) be a nonlinear, Fréchet differentiable operator between the Hilbert spaces \( X \) and \( Y \). Further, let \( y^\delta \in Y \) satisfies (1.1) and \( u^\dagger \) satisfies (1.3). Moreover, assume that

1. \( F \) satisfies the non-linearity estimate
\[
\|F(u) - F(u^\dagger) - F'(u^\dagger)(u - u^\dagger)\| \leq K\|u - u^\dagger\|, \quad u \in X
\]
for some \( K > 0 \) in a sufficiently large ball \( B \) around \( u^\dagger \).
2. \( F'(u^\dagger) \) satisfies the inequality (4.6) with the same parameters.
With the same notations as in Theorem 4.

Then, for the choice of \( \alpha \) assume that \( A \). Proof. From equation (4.7) and (4.6), we get
\[
\| F(u_\alpha^\delta) - y^\delta \|^2 + \alpha \| u_\alpha^\delta - u^\dagger \|^2 \leq \delta^2 + \alpha \beta \| F'(u^\dagger)(u_\alpha^\delta - u^\dagger) \| \mu + \alpha \gamma \| u_\alpha^\delta - u^\dagger \|^2
\]
With the same notations as in Theorem 4.3 and Young’s inequality with the parameters \( a = \alpha^{2/2}, b = \alpha^{2/2} F'(u^\dagger) \| u_\alpha^\delta - u^\dagger \| \mu, p = \frac{2}{2 - \mu}, q = \frac{2}{\mu} \), we get
\[
A + \alpha (1 - \gamma) B \leq \delta^2 + \frac{2 - \mu}{2} \alpha \beta^2 \mu + \mu \alpha \| F(u_\alpha^\delta) - F(u^\dagger) - F'(u^\dagger)(u_\alpha^\delta - u^\dagger) \|^2 + \mu \alpha \| F(u_\alpha^\delta) - F(u^\dagger) \|^2.
\]
On employing the estimate (4.8) for \( p = 2 \) in above, we arrive at
\[
A + \alpha (1 - \gamma) B \leq \delta^2 + \frac{2 - \mu}{2} \alpha \beta^2 \mu + \mu \alpha K^2 B + \mu \alpha \| F(u_\alpha^\delta) - F(u^\dagger) \|^2.
\]
The non-linearity estimate (4.11) with above yields
\[
A + \alpha (1 - \gamma) B \leq \delta^2 + \frac{2 - \mu}{2} \alpha \beta^2 \mu + \mu \alpha K^2 B + \mu \alpha \| F(u_\alpha^\delta) - F(u^\dagger) \|^2
\]
Again using (4.8) for \( p = 2 \), we get
\[
\| u_\alpha^\delta - u^\dagger \|^2 \leq K_1 \frac{\delta^2}{\alpha} + K_2 \delta^2 + K_3,
\]
where \( K_1 = \frac{1}{1 - \gamma - \mu K^2} \) and \( K_2 = \frac{2 \mu}{1 - \gamma - \mu K^2} \) and \( K_3 = \frac{2 - \mu}{2(1 - \gamma - \mu K^2) \beta^2 \mu} \) are constants.

In the next theorem, we find the rates using the homogeneous variational inequality and the non-linearity estimate (4.5).

**Theorem 4.3.** Let \( F : X \to Y \) be a nonlinear, Fréchet differentiable operator between the Hilbert spaces \( X \) and \( Y \). Further, let \( y^\delta \in Y \) satisfies (1.1) and \( u^\dagger \) satisfies (1.3). Moreover, assume that

1. \( F \) satisfies the non-linearity estimate (4.5).
2. \( F'(u^\dagger) \) satisfies the homogeneous variational inequality, i.e.
\[
2 \langle u^\dagger - u_0, u^\dagger - u \rangle \leq \beta \| F'(u^\dagger)(u - u^\dagger) \| \nu \| u - u^\dagger \|^{1-\nu}, \tag{4.12}
\]
for \( \nu \in (0, 1/3), \beta \geq 0 \) in a sufficiently large ball \( B_r(u^\dagger) \subseteq B \) around \( u^\dagger \) for \( r > 2 \| u^\dagger - u_0 \| \).
3. \( K \) and \( \nu \) are such that \( 1 + \nu \geq 4 \nu (K^2 + 1) \).

Then, for the choice of \( \alpha \sim \delta^{\frac{2}{\nu + 1}} \), we get
\[
\| u_\alpha^\delta - u^\dagger \|^2 \leq G \delta^{\frac{4 \nu}{\nu + 1}} + H \delta^{\frac{2 \nu}{\nu + 1}}
\]
for constants \( G, H > 0 \) where \( u_\alpha^\delta \) satisfies (1.4). In particular, we have
\[
\| u_\alpha^\delta - u^\dagger \| = O(\delta^{\frac{\nu}{\nu + 1}}).
\]
Remark 4.3. In the above theorem, if we select \( \nu = \frac{1}{2-\mu} \), for \( \mu \in (0,1/2) \), then the rates obtained in the above theorem are same as that obtained in the Theorem 4.1.

The next remark concludes this particular subsection.

Remark 4.4. Convergence rates for nonlinear problems via inhomogeneous or homogeneous variational inequality depends largely on the type of non-linearity satisfied by \( F \).

4.2. Convergence rates via Hölder stability estimates. In this subsection, we give one of our main result on the determination of convergence rates with the incorporation of a novel smoothness condition, i.e. Hölder stability estimates satisfied by the nonlinear operator and the exact solution of the problem. While using this smoothness concept, we do not require any assumption on the non-linearity of the operator \( F \). Furthermore, unlike the results given in Subsection 4.1, here \( F \) is no longer required to be Fréchet differentiable. Thus, using Hölder stability estimates can be a good alternate to find the convergence rates without the already existing smoothness conditions in the literature.

Theorem 4.4. Let \( F : X \to Y \) be a nonlinear operator between the Hilbert spaces \( X \) and \( Y \). Further, let \( y^\delta \in Y \) satisfies (1.1) and \( u^\dagger \) satisfies (1.3). Moreover, assume \( u^\dagger \) satisfies a Hölder stability estimate, i.e.

\[
\|u - u^\dagger\| \leq G\|F(u) - F(u^\dagger)\|^k, \quad 0 < k \leq 1
\]

(4.13)

where \( G > 0 \) is some constant and \( u \) is in some neighborhood of \( u^\dagger \). Then, for the choice of \( \alpha \sim \delta^\frac{2}{2+k} \), we have

\[
\|u^\delta - u^\dagger\|^2 \leq G_1\delta^\frac{4k}{2+k} + G_2\delta^\frac{2k}{(2+k)(2-k)},
\]

for constants \( G_1, G_2 > 0 \) where \( u^\delta_\alpha \) satisfies (1.4).

In particular, we have

\[
\|u^\delta_\alpha - u^\dagger\| = O(\delta^\frac{k}{2+k}).
\]

Proof. For proving this, we again proceed in the similar manner as done in the Theorem 4.1 upto equation (4.7) and also prove that \( u^\delta_\alpha \) is in some neighborhood of \( u^\dagger \). Further, apply the Cauchy-Schwarz inequality on the right side of (4.7) to get

\[
\|F(u^\alpha^\delta) - y^\delta\|^2 + \alpha\|u^\delta_\alpha - u^\dagger\|^2 \leq \delta^2 + 2\alpha\|u^\dagger - u_0\|\|u^\dagger - u^\delta_\alpha\| \leq \delta^2 + 2\alpha\|u^\dagger - u_0\|\|u^\dagger - u^\delta_\alpha\|. \quad (4.14)
\]

Now, apply Hölder stability estimate (4.13) in (4.14) to get

\[
\|F(u^\alpha^\delta) - y^\delta\|^2 + \alpha\|u^\delta_\alpha - u^\dagger\|^2 \leq \delta^2 + 2\alpha G\|u^\dagger - u_0\|\|F(u^\alpha^\delta) - F(u^\dagger)\|^k.
\]

Let \( A = \|F(u^\alpha^\delta) - y^\delta\|^2 \) and \( B = \|u^\delta_\alpha - u^\dagger\|^2 \). Then, by using Young’s inequality in the middle term of the right side of the inequality by taking \( a = 2\alpha G\|u^\dagger - u_0\| \), \( b = \|F(u^\alpha^\delta) - F(u^\dagger)\|^k \), \( p = \frac{2}{2-k} \) and \( q = \frac{k}{2} \), we get

\[
A + \alpha B \leq \delta^2 + \frac{2-k}{2}(2\alpha G\|u^\dagger - u_0\|)\frac{2}{2-k} + \frac{k}{2}\|F(u^\alpha^\delta) - F(u^\dagger)\|^2.
\]
This with the estimate (4.8) for \( p = 2 \) and \( k \leq 1 \) leads to
\[
B = \| u_\alpha^\delta - u^\dagger \|^2 \leq (1 + k) \frac{\delta^2}{\alpha} + K_1 \alpha^{\frac{k}{2-k}}
\]
where \( K_1 = \frac{2-k}{2} (2G\| u^\dagger - u_0 \|)^{\frac{2}{2-k}} \). Further, if \( \alpha \sim \delta^{\frac{2}{2-k}} \), then
\[
\| u_\alpha^\delta - u^\dagger \|^2 \leq G_1 \delta^{\frac{2k}{2+k}} + G_2 \delta^{\frac{2k}{(2-k)(2-k)}}
\]
where \( G_1, G_2 \) are constants. Hence the assertion follows.

**Remark 4.5.** The convergence rates obtained in the Theorems 4.1 and 4.4 are of the same order. But, we have a huge plus that we get the same convergence rate without assuming \( F \) to be Fréchet differentiable and moreover no non-linearity estimate is required to be satisfied by \( F \).

5. **Interplay between the different smoothness concepts**

In this section, we look for the connections between the different kind of smoothness conditions ((4.1) – (4.4)) defined in the preceding section. In particular, our focus is to find the conditions under which the inhomogeneous variational inequality (4.1) implies the Hölder stability estimate (4.4) (if there exist any such relation) or vice-versa.

**Lemma 5.1.** Let \( F : X \to Y \) be a nonlinear operator between the Hilbert spaces \( X \) and \( Y \). Then, the homogeneous variational inequality (4.2) with the parameter \( \nu \) implies the inhomogeneous variational inequality (4.1) with the parameter \( \mu = \frac{2\nu}{2\nu + 1} \).

**Proof.** See [3, Lemma 2]. The only change is the replacement of \( L \) by \( F'(u^\dagger) \) there. \( \square \)

Next proposition gives us the relation between the inhomogeneous variational inequality and the Hölder stability estimate satisfied by \( u^\dagger \) in terms of \( F'(u^\dagger) \) which can be proved on the similar lines of Proposition 3.1.

**Proposition 5.1.** Let \( F : X \to Y \) be a nonlinear operator between the Hilbert spaces \( X \) and \( Y \). Then, for any \( u \in X \), the inhomogeneous variational inequality (4.1) implies that the exact solution \( u^\dagger \) satisfies the Hölder stability estimate in terms of derivative of \( F \), i.e.
\[
\| u' - u^\dagger \|^2 \leq \frac{A}{1 - \beta} \| F'(u^\dagger)(u^\dagger - u') \|^\mu,
\]
for \( 0 < \mu \leq 1, G > 0 \), under the condition
\[
\langle u', u^\dagger - u \rangle \leq 0,
\]
where \( u' = u^\dagger - u \).

In the next proposition, we find the relation between the inhomogeneous variational inequality (4.1) satisfied by \( u^\dagger \) and the Hölder stability estimate satisfied by the exact solution \( u^\dagger \) in terms of \( F \).
Proposition 5.2. Let $F : X \to Y$ be a nonlinear operator between the Hilbert spaces $X$ and $Y$. Then, for any $u \in X$, the inhomogeneous variational inequality (4.1) (with $u$ replaced by $u^\dagger - u$) and the non-linearity estimate (4.5) implies the estimate

$$
\|u - u^\dagger\|^2 \leq H + G\|F(u) - F(u^\dagger)\|^2
$$

with constants $G$ and $H$ under the condition

$$
\langle u, u - u^\dagger \rangle \leq 0.
$$

Proof. We have

$$
\|u - u^\dagger\|^2 = \langle u - u^\dagger, u - u^\dagger \rangle = \langle u, u - u^\dagger \rangle + \langle u^\dagger, u^\dagger - u \rangle.
$$

Using (4.1) in the above inequality, we get

$$(1 - \beta)\|u - u^\dagger\|^2 \leq \langle u, u - u^\dagger \rangle + A\|F'(u^\dagger)(u^\dagger - u)\|^p.
$$

Now incorporating Young’s inequality $ab \leq a^p/p + b^q/q$ with $a = A, b = \|F'(u^\dagger)(u^\dagger - u)\|^p, p = \frac{\mu}{\mu - 2}$ and $q = \frac{2}{\mu}$ and then (4.8) for $p = 2$ in above, we arrive at

$$(1 - \beta)\|u - u^\dagger\|^2 \leq \langle u, u - u^\dagger \rangle + A_1 + \mu(\|F(u) - F(u^\dagger)\| - F'(u^\dagger)(u - u^\dagger)) + \|F(u) - F(u^\dagger)\|^2 + \|F(u) - F(u^\dagger)\|^2
$$

where $A_1 = \frac{\mu - 2}{\mu}A^\frac{\mu - 2}{\mu}$ and $A_2 = \frac{\mu(K^2 + 1)}{1 - \beta}$. Thus, the assertion follows.

Corollary 5.1. The inhomogeneous variational inequality (4.1) (with $u$ replaced by $u^\dagger - u$) with $A = 0$, i.e.

$$
\langle u^\dagger, u^\dagger - u \rangle \leq \beta\|u^\dagger - u\|^2, \quad u \in X
$$

and the non-linearity estimate (4.5) implies the Hölder stability estimate (4.4) with the parameter $k = 1$ under the condition assumed in the Proposition 5.2.

Proof. If $A = 0$, then from Proposition 5.2, we get

$$
A_1 = \frac{\mu - 2}{\mu}A^\frac{\mu - 2}{\mu} = 0.
$$

Thus, the assertion follows.

Next, we seek for the conditions under which the Hölder stability estimates implies the inhomogeneous variational inequality.

Proposition 5.3. For any $u \in X$, the Hölder stability estimate (4.4) and the non-linearity estimate (4.11) actually implies the estimate

$$
\langle u^\dagger, u - u^\dagger \rangle \leq A + k(K^2\|u - u^\dagger\|^2 + \|F'(u^\dagger)(u - u^\dagger)\|^2),
$$

where $A$ is some constant.
Proof. From (4.4), we get
\[ \langle u^\dagger, u - u^\dagger \rangle \leq \| u^\dagger \| \| u - u^\dagger \| \leq G \| u^\dagger \| \| F(u) - F(u^\dagger) \|^k. \]

Now, using Young’s inequality with parameters \( a = G \| u^\dagger \|, b = \| F(u) - F(u^\dagger) \|^k, p = \frac{2}{2-k}, q = \frac{2}{k} \), we get
\[ \langle u^\dagger, u - u^\dagger \rangle \leq A + \frac{k}{2} \| F(u) - F(u^\dagger) \|^2, \]
where \( A = \frac{2-k}{2}(G \| u^\dagger \|)^{\frac{2}{2-k}} \). Using (4.8) with \( p = 2 \), we get
\[ \langle u^\dagger, u - u^\dagger \rangle \leq A + k(\| F(u) - F(u^\dagger) \| - F'(u^\dagger)(u - u^\dagger) \|^2 + \| F'(u^\dagger)(u - u^\dagger) \|^2). \]

After employing (4.11), we get the result. \( \square \)

**Corollary 5.2.** For any \( u \in X \), the Hölder stability estimate (4.4) and the non-linearity estimate used in the Proposition 5.3 implies the inhomogeneous variational inequality (4.1) with the parameter \( \mu = 2 \) (with \( u \) replaced by \( u - u^\dagger \)) in the case when \( u^\dagger = 0 \).

**Proof.** If \( A = 0 \), then from Proposition 5.3,
\[ A = \frac{2-k}{2}(G \| u^\dagger \|)^{\frac{2}{2-k}} = 0. \]
So, either \( G = 0 \) or \( u^\dagger = 0 \). But as \( G > 0 \), so we must have \( u^\dagger = 0 \). Thus, the result. \( \square \)

6. Conclusion

In this paper, we introduced a new kind of smoothness condition termed as Hölder stability estimate for obtaining the convergence rates of the linear and nonlinear inverse problems. Although the rates obtained via Hölder stability estimates are exactly similar to the one obtained via inhomogeneous variational inequality (2.1), but from Proposition 3.2 it is clear that for the linear problems, inhomogeneous variational inequality is a weaker condition than the Hölder stability estimates for finding the convergence rates. Also the rates are obtained without the use of spectral theory. The interplay between these already existing smoothness concepts and Hölder stability estimate is also explained. For nonlinear problems, rates obtained via Hölder stability estimates are exactly similar to that of inhomogenious variational inequality, but we have a big plus by using Hölder estimates that we do not require two conditions to be satisfied (Fréchet differentiability and the non-linearity estimate). The dependency of the inhomogenous variational inequality on the non-linearity estimates is also shown for getting the convergence rates. Further, Section 5 concludes that no smoothness condition implies the other one directly, but by assuming the stronger conditions we can obtain one from another.
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