Heat flux and quantum correlations in dissipative cascaded systems

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(Received 19 December 2014; published 25 February 2015)

We study the dynamics of heat flux in the thermalization process of a pair of identical quantum systems that interact dissipatively with a reservoir in a cascaded fashion. Despite that the open dynamics of the bipartite system $S$ is globally Lindbladian, one of the subsystems “sees” the reservoir in a state modified by the interaction with the other subsystem and hence it undergoes a non-Markovian dynamics. As a consequence, the heat flow exhibits a nonexponential time behavior which can greatly deviate from the case where each party is independently coupled to the reservoir. We investigate both thermal and correlated initial states of $S$ and show that the presence of correlations at the beginning can considerably affect the heat-flux rate. We carry out our study in two paradigmatic cases—a pair of harmonic oscillators with a reservoir of bosonic modes and two qubits with a reservoir of fermionic modes—and compare the corresponding behaviors. In the case of qubits and for initial thermal states, we find that the trace distance discord is at any time interpretable as the correlated contribution to the total heat flux.

DOI: 10.1103/PhysRevA.91.022121 PACS number(s): 03.65.Yz, 03.67.–a, 42.50.Lc, 03.65.Ud

I. INTRODUCTION

A fundamental thermodynamic quantity is the amount of energy that can be extracted from nonequilibrium systems. The field of quantum thermodynamics [1–4] is currently experiencing a considerable effort to understand the concepts of work and heat within quantum mechanics [5–10]. While work is commonly analyzed in the presence of external coherent control on the system [3,4,6], heat is associated with energy changes that are due to some system-bath interaction [2,7,11]. Quantum thermodynamics tackles heat transfer by modeling the system-bath interactions as a quantum mechanical process mathematically described, under weak-coupling assumptions, by the Lindblad generator [12]. Scenarios featuring consecutive interactions between individual elements of a quantum multipartite system and their own local environments have recently been investigated [13–16] and the study of these correlated channels has made clear that interesting new features emerge in the presence of correlations.

Given the quantum mechanical nature of such processes, an interesting question concerns if, and how, the heat flux between a multipartite system and its reservoir can be affected by intrasystem quantum correlations (QCs) which are present in the initial state. In particular, one can investigate whether QCs, in the form of either entanglement [17] or quantum discord [18], are fundamental resources for the heat-transfer mechanism. Note that a similar issue was tackled in the completely different framework of quantum biology; see, e.g., [19,20].

It is straightforward to predict that if the various subsystems are not directly coupled and the reservoir is sufficiently large to prevent any cross talking, then correlations do not play any role. In such cases, the heat flux emerging from a composite system is the same for all the initial states admitting the same local representation, regardless of the presence of correlations among its constituents. The scenario, however, changes drastically if we do introduce interactions among the various subsystems or if the reservoir “sees” the compound systems as a unique object (a so-called common bath). For instance, it is well known that a strong coupling between two atoms can inhibit energy dissipation via the formation of dark states effectively decoupled from the reservoir [21]. In all these cases, quantum coherence (at the level of either initial correlations or interactions) plays a major role.

In this paper, we shed light on such issues in the case of a cascade bipartite system where energy flows between its subsystems along a specific direction (say from subsystem 1 to subsystem 2, but not the opposite).

Although thermal equilibrium with the heat bath is always reached after an infinite amount of time, a stronger or weaker heat flux can be obtained by engineering correlations in the initial state of the system, giving rise to very different time scales for the thermalization process. This means that the same amount of energy, stored in different configurations of the system, can be retrieved faster or slower according to the chosen state preparation. In our study, we adopt the master equation approach developed by Gardiner et al. [22,23] in the case of bosonic baths and recently generalized by two of us [24] via a collision-model-based approach. Within this framework, we discuss both the case of continuous-variable (CV) systems (two quantum harmonic oscillators) and the case of two-level systems (a pair of qubits [25]), showing how the presence of initial correlations can influence the system dynamics by speeding up or slowing down the energy flux to or from the reservoir. Interestingly enough, we find that in both scenarios, while entanglement among the subsystems appears not to play an essential role, the extremal performances in terms of heat-flux rate take place in the presence of high values of nonclassical correlations [18] in the initial state of the system. Yet strong QCs are not sufficient to ensure faster or slower energy transfer. This is particularly true in the continuous-variable case, where states featuring the maximum level of nonclassicality do not show any difference in terms of heat fluxes with respect to the completely uncorrelated case.
While our analysis is of a conceptual nature (the systems under study being rather idealized) the effects we describe may find potential applications in designing more efficient energy storage units or energy filters.

The outline of the paper is as follows. In Sec. II, we describe the model under consideration and the master equation describing its open dynamics under a cascade interaction with the reservoir. In Sec. III, we investigate the general form of the total and local heat fluxes and show that the former can be decomposed into three contributions, one of which reflects the interaction between the subsystems mediated by the reservoir. In Sec. IV, we address the general time dependence of heat fluxes for both harmonic oscillators and qubits. In Secs. V (case of harmonic oscillators) and VI (qubits) we analyze extensively the heat-flux dynamics when the initial state of the open system is thermal or correlated (but locally thermal). We furthermore investigate on the role of initial QCs. In Sec. VII, we show that in some cases the correlated heat flux can be directly connected to a discordlike measure of QCs. Finally, in Sec. VIII, we draw our conclusions.

II. MODEL AND MASTER EQUATION

We consider a bipartite open system $S$, consisting of a pair of subsystems $S_1$ and $S_2$, and a thermal reservoir $R$ modeled as a large ensemble of identical ancillas all in the same initial thermal state. The $S$-$R$ interaction occurs in cascade [26], $S_1$ interacts with $R$ through a sequence of system-ancilla collisions under the usual Born-Markov approximation [27]. $S_2$, instead, interacts with $R$ modified by the previous interaction with $S_1$; see Fig. 1(a). No direct mutual coupling between $S_1$ and $S_2$ is present. Yet, $R$ mediates an indirect coupling between them. Such indirect $S_1$-$S_2$ coupling is, however, unidirectional: $S_1$ affects the dynamics of $S_2$, but $S_2$ cannot influence $S_1$ in any way. The master equation in the $S$ state $\rho$ at time $t$ was derived long ago for bosonic baths through the input-output formalism [22,23,28] and, quite recently, generalized to arbitrary baths by means of a collision-model-based approach [24]. To simplify the analysis, in what follows we assume that the delay time between the $S_1$-$R$ and the $S_2$-$R$ collisions is negligible compared to all the other time scales. Still, the causal structure of the process holds: A collision between $S_1$ and a given ancilla of $R$ will anyway occur before the latter collides with $S_2$; see Fig. 1(a). Accordingly, the master equation is of the Kossakowski-Lindblad form [27] and reads

$$\dot{\rho} = -\frac{i}{\hbar}[\hat{H}, \rho] + \mathcal{L}^{(i)}(\rho),$$

where $\hat{H} = \hat{H}_1 + \hat{H}_2$ is the free Hamiltonian of $S$ ($\hat{H}_i$ is the local free Hamiltonian of the $i$th subsystem with $i = 1, 2$) while the cascade Lindbladian superoperator $\mathcal{L}^{(i)}$ is the sum of three terms according to

$$\mathcal{L}^{(i)} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{D}_{12}.$$  

Here $\mathcal{L}_i$ acts locally on $S_i$ only and coincides with the Lindblad superoperator that would be obtained if $S_i$ were in contact with $R$ in the absence of the other subsystem. The superoperator $\mathcal{D}_{12}$, instead, acts on both of the subsystems and accounts for the cascade, i.e., one way, $S_1 \rightarrow S_2$ interaction mediated by $R$. The explicit forms of $\mathcal{L}_i$ and $\mathcal{D}_{12}$ are given below in the cases of concern to this work (for simplicity, we refer to such superoperators as “dissipators” since we focus on purely dissipative reservoirs). The general expressions for $\mathcal{L}_1$ and $\mathcal{D}_{12}$ can be found in [24]. For comparison, we also analyze the case where the cascade link is removed in a way that both systems interact with the reservoir $R$ independently; see Fig. 1(b). Formally, this can be obtained by simply replacing in Eq. (1) $\mathcal{L}^{(i)}$ with $\mathcal{L}^{(ind)} = \mathcal{L}_1 + \mathcal{L}_2$ (i.e., by setting $\mathcal{D}_{12} = 0$).

We next illustrate the explicit form taken by $\mathcal{L}_i$ and $\mathcal{D}_{12}$ for a pair of CV variables (i.e., quantum harmonic oscillators) and qubits (i.e., two-level systems) in contact with a reservoir of harmonic oscillators and qubits, respectively.

A. Harmonic oscillators

In this case, each subsystem $S_i$ is a quantum harmonic oscillator of frequency $\omega$ with associated bosonic annihilation and creation operators $\hat{a}_i$ and $\hat{a}_i^\dagger$, respectively. The free Hamiltonian reads

$$\hat{H} = \hat{H}_1 + \hat{H}_2 = \hbar \omega (\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2).$$

The reservoir $R$ consists of a large collection of bosonic modes. If the interaction Hamiltonian between the system and each reservoir mode does not feature counter-rotating
terms (rotating-wave approximation), the local and nonlocal
dissipators in Eq. (2) are then given by [23,24]
\[
\mathcal{L}_i(\rho) = \frac{\gamma}{2}(N + 1)(\hat{a}_i^\dagger \rho \hat{a}_i - \rho \hat{a}_i^\dagger \hat{a}_i - \hat{a}_i^\dagger \hat{a}_i \rho)
\]
\[+ \frac{\gamma}{2} N(2\hat{a}_i^\dagger \rho \hat{a}_i - \rho \hat{a}_i^\dagger \hat{a}_i - \hat{a}_i^\dagger \hat{a}_i \rho),
\]
\[
\mathcal{D}_{12}(\rho) = \gamma(N + 1)(\hat{a}_1^\dagger \rho \hat{a}_2 - \rho \hat{a}_2^\dagger \hat{a}_1 + [\hat{a}_2^\dagger, \rho] \hat{a}_1^\dagger)
\]
\[+ \frac{\gamma}{2} N(\hat{a}_1^\dagger (\rho \hat{a}_2^\dagger) - [\hat{a}_1^\dagger, \rho] \hat{a}_2^\dagger),
\]
(4)

Here \(\gamma\) coincides with the relaxation rate that would arise
for each subsystem alone (assumed identical for the two
subsystems), \(N = 1/(e^\beta - 1)\) is the thermal excitation
number, \(\beta = 1/(k_B T)\) is the inverse temperature, while \(k_B\)
and \(T\) are the Boltzmann constant and the reservoir’s temperature,
respectively.

### B. Qubits

In this case, each subsystem \(S_i\) is a two-level system (qubit)
whose ground and excited states are \(|g_i\rangle\) and \(|e_i\rangle\), respectively.
The corresponding energy gap is \(\hbar \omega_0\). Let \(\{\sigma_{i=1,2}\}\) be the
usual pseudospin operators with \(\sigma_{i=1,2} = \sigma_{i=1,2}^z = |e_i\rangle \langle g_i|\) and
\(\sigma_{i=1,2} = |e_i\rangle \langle e_i| - |g_i\rangle \langle g_i|\). The system’s free Hamiltonian now
reads
\[
\hat{H} = \hat{H}_1 + \hat{H}_2 = \frac{\hbar \omega_0}{2}(\sigma_{1z} + \sigma_{2z}).
\]
If the reservoir consists of a bath of qubits, under the
rotating-wave approximation the local and nonlocal dissipators
in Eq.(2) are given by [24]
\[
\mathcal{L}_i = \frac{\gamma}{4}(1 + \xi)(2\hat{a}_i^\dagger \rho \hat{a}_i + \rho \hat{a}_i^\dagger \hat{a}_i - \sigma_i^z \hat{a}_i^\dagger \hat{a}_i - \hat{a}_i^\dagger \sigma_i^z \hat{a}_i)
\]
\[+ \frac{\gamma}{4} (1 - \xi)(2\hat{a}_i \rho \hat{a}_i^\dagger + \rho \hat{a}_i^\dagger \hat{a}_i - \sigma_i^z \hat{a}_i^\dagger \hat{a}_i - \hat{a}_i^\dagger \sigma_i^z \hat{a}_i),
\]
(7)
\[
\mathcal{D}_{12} = \frac{\gamma}{2}(1 + \xi)(\sigma_{1z} [\rho, \sigma_{2z}^\dagger] + [\sigma_{2z}, \rho] \sigma_{1z})
\]
\[+ \frac{\gamma}{2}(1 - \xi)(\sigma_{1z} [\rho, \sigma_{2z}^\dagger] + [\sigma_{2z}, \rho] \sigma_{1z}),
\]
(8)
with
\[
\xi = \tanh \left[ \frac{\hbar \omega_0/2}{k_B T} \right].
\]
Note that Eqs. (7) and (8) have the same structure as Eqs. (4)
and (5), but differ from these in the statistical nature of
ladder operators (fermionic instead of bosonic) and the rates
associated with the dissipators.

### III. TOTAL AND LOCAL HEAT FLUXES

In the cases of both harmonic oscillators and qubits, any
initial state \(\rho(0)\) of the system asymptotically relaxes towards
the stationary state,
\[
\rho(\infty) = \frac{e^{-\beta \hat{H}_0}}{Z} \otimes \frac{e^{-\beta \hat{H}_2}}{Z},
\]
(10)
with \(Z = \text{Tr}[e^{-\beta \hat{H}_0}]\) (since the subsystems are identical, \(Z\)
does not depend on \(i = 1,2\)). This can be checked by setting
\(\rho = 0\) in Eq. (1) and verifying that the resulting equation
is fulfilled by state (10), as proven in detail in Appendix A
for both harmonic oscillators and qubits. Equation (10) shows
that the system thermalizes to the reservoir temperature. The
asymptotic thermal state coincides with the one that would
be obtained if \(S_1\) and \(S_2\) were in contact with \(R\) independently
[i.e., \(\rho(\infty)\) is also the fixed point associated with the dissipator \(\mathcal{L}^{\text{mil}}]\). Thereby, the presence of the correlated dissipator \(\mathcal{D}_{12}\)
in Eq. (1) has no effect on the steady state, which is indeed
fully factorized and does not feature any \(S_1\)-\(S_2\) correlation,
or on the total amount of energy which is exchanged with
the reservoir, i.e.,
\[
Q(\infty) = \text{Tr}[\{\rho(\infty) - \rho(0)\} \hat{H}].
\]
(11)
However, significant correlations can, in general, arise during
the transient. In turn, these correlations affect the way heat
flows between \(S\)—specifically \(S_2\)—and \(R\). The heat-flux
dynamics during such transient will be the focus of our
analysis.

As in our model, no external work is done on \(S\), the total
heat flux of \(S\)—we call it \(J\)—can be identified with the time
derivative of the system energy \(U = \text{Tr}[\rho \hat{H}]\) [11]. Hence, at
time \(t\), the heat flux is calculated as
\[
J(t) = \dot{U} = \text{Tr}[\rho(\dot{t}) \hat{H}].
\]
In the case of the cascaded system, due to Eqs. (1) and (2), this
yields
\[
J^{(c)}(t) = J_1(t) + J_2(t) + J_{12}(t),
\]
(12)
with
\[
J_1(t) = \text{Tr}[\mathcal{L}_1 \rho(t) \hat{H}^1] \equiv \text{Tr}[\mathcal{L}_1 \rho(t) \hat{H}_1],
\]
(13)
\[
J_2(t) = \text{Tr}[\mathcal{D}_{12} \rho(t) \hat{H}_2] \equiv \text{Tr}[\mathcal{D}_{12} \rho(t) \hat{H}_2].
\]
(14)
The total heat flux can thus be decomposed into three
correlations, two of which stem from the local dissipators
\(\{\mathcal{L}_i\}\), one from the nonlocal dissipator \(\mathcal{D}_{12}\). In Eqs. (13)
and (14), the last identities show that \(\hat{H}_1\) can be replaced
with \(\hat{H}_1(\hat{H}_2)\) in the calculation of \(J_1(D_{12})\). This is due to the
identities
\[
\text{Tr}[\mathcal{L}_1 \rho \hat{H}_2] = \text{Tr}[\mathcal{L}_2 \rho \hat{H}_1] = \text{Tr}[\mathcal{D}_{12} \rho \hat{H}_1] = 0,
\]
(15)
which can be straightforwardly proven upon use of Eqs. (4)
and (7) and the cyclic property of the trace.
As for the local heat fluxes of \(S_1\) and \(S_2\) by using Eqs. (1),
(2), (13)–(15), these are respectively computed as
\[
J_1^{(c)}(t) = \dot{U}_1(t) = \text{Tr}[\rho(\dot{t}) \hat{H}_1^1] \equiv J_1(t),
\]
(16)
\[
J_2^{(c)}(t) = \dot{U}_2(t) = \text{Tr}[\rho(\dot{t}) \hat{H}_2] \equiv J_2(t) + J_{12}(t).
\]
(17)
Upon comparison of these with the total heat flux (12), we find
\(J^{(c)}(t) = J_1^{(c)}(t) + J_2^{(c)}(t)\) as expected. More importantly,
the above equations show that, of the three terms appearing
in Eq. (12), \(J_1(t)\) accounts for the \(S_1\) heat flux while the sum
of the last two, i.e., \(J_2(t) + J_{12}(t)\), is equal to \(J_2^{(c)}(t)\). The
correlated term \(J_{12}(t)\) therefore contributes only to the heat
flux of \(S_2\) (this is reasonable in light of the cascaded nature
of the system dynamics). As anticipated, the reduced dynamics
of $S_1$ fully coincides with that in the absence of $S_2$ since, upon trace over subsystem $S_2$ and using the cyclic property of the partial trace, Eq. (1) yields $\rho_1 = \mathcal{L}_1\rho_1$. Correspondingly, $J_{1}^{(c)}(t)$ is just the same function as in the absence of $S_2$ since in Eq. (16) $\rho(t)$ can be replaced with $\rho_1(t)$.

The heat flux associated with the identical and independent reservoirs model of Fig. 1(b) can be calculated in the same way. Again the total flux is given by the sum of the fluxes from $S_1$ and from $S_2$, i.e., $J_{1}^{(ind)}(t) = J_{1}^{(ind)}(t) + J_{2}^{(ind)}(t)$. Furthermore, the heat flux $J_{1}^{(ind)}(t)$ from $S_1$ coincides with the one we computed for the cascaded system, i.e., $J_{1}^{(ind)}(t) = J_{1}^{(c)}(t)$; hence, the two models give rise to the same reduced local dynamics for $S_1$. On the contrary, the heat flux from $S_2$, $J_{2}^{(ind)}(t)$ is rather different from $J_{1}^{(c)}(t)$. In particular, if we do assume that the initial state $\rho(0)$ is locally indistinguishable for exchange of $S_1$ with $S_2$, we have $J_{2}^{(ind)}(t) = J_{1}^{(ind)}(t) = J_{1}(t)$ (the local dissipative processes being identical). Accordingly, we can write

$$J_{1}^{(ind)}(t) = 2J_{1}(t),$$

with $J_{1}(t)$ being the same function that appears on the right-hand side of Eq. (12). It is finally worth stressing that due to the fact that both the cascade and the independent model yield the same total amount of dissipated energy (11) when integrated over the whole evolution [i.e., $Q(\infty) = \int_{0}^{\infty} J^{(c)}(t)dt = \int_{0}^{\infty} J^{(ind)}(t)dt$], the following identity holds:

$$\int_{0}^{\infty} [J_{1}(t) - J_{2}(t)]dt = \int_{0}^{\infty} J_{12}(t)dt.$$  

IV. TIME DEPENDENCE OF HEAT FLUXES

In this section, we show the explicit procedure to calculate the three contributions to the total heat flux of Eq. (2), for harmonic oscillators and for qubits.

A. Harmonic oscillators

In the case of harmonic oscillators, we focus on initial states $\rho(0)$ of $S$ that are Gaussian [29]. The linearity of the master equation (1) alongside the assumption that the initial state of the ancillas of $R$ (hence Gaussian as well) ensures that the state of $S$ will remain Gaussian at any time $t$. To specify such states, let us introduce the position-momentum quadrature operators $\hat{X}_j = (\hat{a}_j + \hat{a}_j^\dagger)/\sqrt{2}$ and $\hat{Y}_j = i(\hat{a}_j - \hat{a}_j^\dagger)/\sqrt{2}$, with $j = 1, 2$. Correspondingly, let us define the four-dimensional vector operator $\hat{Y} = \{\hat{X}_1, \hat{X}_2, \hat{Y}_1, \hat{Y}_2\}$. By definition, a Gaussian state is fully specified by the expectation value of $\hat{Y}$, i.e., $\{\langle \hat{X}_i \rangle, \langle \hat{Y}_j \rangle \}$, and by the covariance matrix $C_{mn} = \frac{1}{2}\langle [\hat{X}_m, \hat{X}_n] + [\hat{X}_n, \hat{X}_m] \rangle - \langle \hat{X}_m \rangle \langle \hat{X}_n \rangle$, with $m, n = 1, \ldots, 4$. Throughout, we consider states with vanishing first moments, i.e., $\langle \hat{X}_i \rangle = 0$, which amounts to assuming that the energy of $S$ is initially stored solely in the form of fluctuations. Indeed, correlations are entirely described by the fluctuations and our main concern is to highlight the interplay between heat fluxes and correlations. Each initial state we consider, thereby, will be fully specified by the covariance matrix $C_{mn}$ (this has real entries).

For the class of initial states discussed so far, upon use of Eqs. (4), (5), (13), and (14) the three heat fluxes on the right-hand side of Eq. (12) take the form

$$J_{1}(t) = \hbar \omega \left[ \frac{C_{11}(t) + C_{22}(t)}{2} - \left( N + \frac{1}{2} \right) \right],$$  

$$J_{2}(t) = \hbar \omega \left[ \frac{C_{33}(t) + C_{44}(t)}{2} - \left( N + \frac{1}{2} \right) \right],$$  

$$J_{12}(t) = \hbar \omega \left[ C_{13}(t) + C_{24}(t) \right].$$  

To calculate the explicit time evolution of the covariance matrix entries $C_{mn}(t)$ for a given initial state, it is convenient to use the Langevin equations [23] as illustrated in Appendix B.

B. Qubits

In this case, with the help of Eqs. (7), (8), (13), and (14) the contributions to the total heat flux on the right-hand side of Eq. (12) are calculated as

$$J_{1}(t) = \gamma [(1 + \xi)[\rho_{11}(t) + \rho_{22}(t)]$$

$$- (1 - \xi)[\rho_{33}(t) + \rho_{44}(t)]],$$  

$$J_{2}(t) = \gamma [(1 + \xi)[\rho_{11}(t) + \rho_{33}(t)]$$

$$- (1 - \xi)[\rho_{22}(t) + \rho_{44}(t)]],$$  

$$J_{12}(t) = 2\gamma \xi [\rho_{23}(t) + \rho_{32}(t)].$$  

where $\rho_{mn}$, i.e., the matrix elements of $\rho$, are labeled according to the uncoupled basis of the $S$ Hilbert space $\{|ee\rangle_{12}, |eg\rangle_{12}, |ge\rangle_{12}, |gg\rangle_{12}\}$. Equations (23)–(25) hold for an arbitrary initial two-qubit state $\rho(0)$. To calculate the explicit time evolution of the density matrix entries $\rho_{mn}(t)$ for a given $\rho(0)$, it is convenient to use master equation (1) in the Liouville space as shown in Appendix C.

V. HEAT-FLUX DYNAMICS: HARMONIC OSCILLATORS

In this section, we analyze the heat-flux dynamics for a pair of harmonic oscillators. We consider both thermal (hence uncorrelated) and correlated initial states of the reservoir.

A. Thermal initial states

In this case, the pair of harmonic oscillators $S$ is initially in a thermal state $\rho(0) = e^{-\beta_S\hat{H}_S}/Z_S^2$, where $Z_S = Tr[e^{-\beta_S\hat{H}_S}]$, $\beta_S = 1/(k_BT_S)$, and $T_S$ is the system initial temperature. Note that, due to the lack of a direct coupling between $S_1$ and $S_2$, in such a situation the two subsystems are initially fully uncorrelated and identical under mutual exchange. Such initial conditions correspond to a covariance matrix whose only nonzero entries are $C_{ii}(0) = N_S + 1/2$ for any $i = 1, \ldots, 4$. Here $N_S = 1/(e^{\beta_S\omega_S} - 1)$ is the initial average number of excitations in either $S$‘s subsystem, which, in general, differs from $N$ (average number of excitations at the reservoir temperature). With the help of Eqs. (20)–(22) and Appendix B, the explicit time dependencies of $J_1$, $J_2$, and $J_{12}$.
Comparison. So that and hence the heat flux of components is chosen to be comparable with the typical energy scale both above and below the reservoir's temperature. The correlated heat flux in this case is expressed by Eq. (18), i.e.,

$$J^{(\text{ind})}(t) = 2J_1(t) = 2k_B T / (\hbar \omega) e^{-\gamma t}. \quad (30)$$

By a direct comparison with Eq. (29) it follows that the cascading mechanism makes $|J^{(\text{ind})}|$ lower (higher) than $|J^{(\text{ind})}|$ for times shorter (longer) than $\gamma t = 2$ (while maintaining the same sign in any case). In particular, for $T_S > T$ this implies that, when connected in cascade, $S_1$ and $S_2$ tend to retain energy for a longer time.

**B. Correlated initial states**

Next we investigate the effect of initial correlations between $S_1$ and $S_2$ on the heat-flux dynamics. Specifically, we consider initial states $\rho(0)$ such that $\rho_1(0) = \text{tr}_2[\rho(0)] = e^{-\beta_S H_S / Z_S}$ and $\rho_2(0) = \text{tr}_1[\rho(0)] = e^{-\beta_H H_H / Z_S}$, but $\rho(0) \neq \rho_1(0) \otimes \rho_2(0)$. In other words, one such state is locally equivalent to a tensor product of thermal states at the same temperature $T_S$ (like those addressed in Sec. VA), but we allow $S_1$ and $S_2$ to initially share some correlations. For the sake of simplicity, we focus on the case where the reservoir is at zero temperature; i.e., we set $N = 0$ throughout.

In line with Sec. VA, the requirement that the state is locally thermal at the uniform temperature $T_S$ (corresponding to the average excitation number $N_S$) yields that the diagonal entries of the initial-state covariance matrix are $C_{ii}(0) = N_S + 1/2$ for any $i = 1, \ldots, 4$. The energy is then given by $U = \frac{1}{2} \hbar \omega \text{tr}[C(0)] = 2\hbar \omega C_{11}(0)$. The remaining entries of $C(0)$ are set to zero except for $C_{13}(0) = C_{31}(0)$ and $C_{23}(0) = C_{32}(0)$, which can be non-null. This is because, at an arbitrary time $t$, the only off-diagonal entries on which the heat fluxes depend are $C_{13}(t)$ and $C_{24}(t)$ [cf. Eq. (22)]. Moreover, as shown by Eqs. (B8) and (B9) in Appendix B, the initial values of the remaining off-diagonal elements do not affect the heat-flux dynamics since these are fully decoupled from $[C_{13}(t), C_{24}(t)]$. To summarize, we study initial states having the form

$$C(0) = \begin{pmatrix}
C_{11}(0) & 0 & C_{13}(0) & 0 \\
0 & C_{11}(0) & 0 & C_{24}(0) \\
C_{13}(0) & 0 & C_{11}(0) & 0 \\
0 & C_{24}(0) & 0 & C_{11}(0)
\end{pmatrix}. \quad (31)$$

A rigorous parametrization of the family of covariance matrices of the form (31) is presented in Appendix D.

Clearly, the heat flux of $S_1$ is again given by Eq. (26) with $N = 0$. This immediately implies that the total flux $J^{(\text{ind})}(t)$ for the independent system model remains identical to the one computed in Eq. (30) and will not depend upon the presence of

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**FIG. 2.** (Color online) Heat flows $J_1$, $J_2$, $J_{12}$, and total heat flow $J^{(\text{ind})}$ against time in the case of harmonic oscillators (left-column plots) and qubits (right-column plots) for various temperatures $T_S$ (see the color legend in the top left panel). As for the reservoir temperature, we have set it in such a way to have $k_B T / (\hbar \omega) = 1$. Heat flows are expressed in unit of $\hbar \omega \gamma_1$ and time is expressed in units of $\gamma^{-1}$. In the bottom plots, we also report the behavior of $J^{(\text{ind})}$ for comparison.

are shown to be

$$J_1(t) = \hbar \omega \gamma (N_S - N) e^{-\gamma t}, \quad (26)$$

and hence the heat flux of $S_1$ [cf. Eq. (17)] for the cascade model reads

$$J^{(\text{ind})}(t) = (1 - \gamma t)^2 J_1(t), \quad (27)$$

so that

$$J^{(\text{ind})}(t) = [1 + (1 - \gamma t)^2]J_1(t), \quad (28)$$

In Fig. 2 (first column), we plot $J^{(\text{ind})}(t)$ and its the three components $[J_1(t), J_2(t), J_{12}(t)]$ for different values of $T_S$ both above and below the reservoir’s temperature $T$, which is chosen to be comparable with the typical energy scale of the system [specifically, we assume $k_B T / (\hbar \omega) = 1$]. As expected, the heat flux of $S_1$ exponentially decays or increases [depending on the sign of $(N_S - N)$] at the rate $\gamma$. In contrast, both $J_2(t)$ and $J_{12}(t)$ exhibit nonexponential behavior. The correlated heat flux $J_{12}(t)$, in particular, has a nonmonotonic behavior: Its absolute value grows from zero until it reaches a maximum at $\gamma t = 1$ and then decreases. Also, note that the sign of $J_{12}(t)$ is always opposite to that of $J_1(t)$. The nonmonotonic behavior of $J^{(\text{ind})}(t)$ affects the total heat flow $J^{(\text{ind})}(t)$ to a significant extent. To better appreciate this, consider the scenario in which $S_1$ and $S_2$ are fully independent.
we have set harmonic oscillators (qubits). 

J(22) and Appendix B the two contributions to the SALVATORE LORENZO harmonic oscillators for different choices of for the cascade system are calculated as

\[ J_2(t) = (1 + \gamma^2 t^2)J_1(t) - \hbar \omega \gamma t [C_{13}(0) + C_{24}(0)] e^{-\gamma t}. \]  (32)

\[ J_{12}(t) = -2\gamma t J_1(t) + \hbar \omega \gamma [C_{13}(0) + C_{24}(0)] e^{-\gamma t}. \]  (33)

Upon sum of these, we thus obtain

\[ J_{c}^2(t) = (1 - \gamma t^2) J_1(t) + \hbar \omega \gamma (1 - \gamma t) [C_{13}(0) + C_{24}(0)] e^{-\gamma t}. \]  (34)

Equations (32)-(34) generalize Eqs. (27)-(28), featuring additional terms proportional to \( C_{13}(0) + C_{24}(0) \). Importantly, the fact that the heat flux depends on such off-diagonal entries only through their sum entails that for states such that \( C_{13}(0) = -C_{24}(0) \), irrespective of \( |C_{13}(0)| \), the presence of initial correlations has no effect on the heat-flux dynamics.

To illustrate the typical behavior of the total heat flux in the general case, in Fig. 3(a) we plot the total flux \( J_{c}^2(t) \) of Eq. (12) for \( N_S = 1 \) and \( C_{13}(0) = C_{24}(0) = -0.7N_S, 0, 0.7N_S \). We point out that, as explained in Appendix D, focusing on states such that \( C_{13}(0) = C_{24}(0) \) does not cause loss of generality. As shown by the plots, in contrast to Fig. 2, a major consequence of the presence of initial correlations is the nonmonotonicity of the heat-flux time. This can be proven in detail through a study of the derivative of \( J_{c}^2(t) \), as resulting from the sum of Eqs. (26) and (34).

The derivative reads

\[ \dot{J}_{c}^2(t) = \hbar \omega (1 - \gamma^2 t^2) e^{-\gamma t} + \left[ \gamma [C_{13}(0) + C_{24}(0) + 4N_S]\right] t e^{-\gamma t} \]

\[ - 2[C_{13}(0) + C_{24}(0) + 2N_S] e^{-\gamma t}. \]

As shown in Appendix D, \( |C_{13}(0) + C_{24}(0)| \leq 2N_S \). Hence, in the above equation, the concave-down parabolic time function between curly brackets is nonpositive at \( t = 0 \). Moreover, this function has the two positive real roots:

\[ t_1 = \frac{2}{\gamma}, \quad t_2 = \frac{2}{\gamma} \left[ 1 + \frac{C_{13}(0) + C_{24}(0)}{2N_S} \right]. \]  (35)

Thereby \( J_{c}^2(t) \) always exhibits a local minimum followed by a local maximum. Specifically, if \( |C_{13}(0) + C_{24}(0)| \leq 0 \) the minimum occurs at \( t_2 \) and the maximum at \( t_1 > t_2 \). Conversely, if \( |C_{13}(0) + C_{24}(0)| > 0 \) the minimum occurs at \( t_1 \) and the maximum at \( t_2 > t_1 \). Such stationary points merge into a single inflection point, thus giving rise to a monotonic \( J_{c}^2(t) \), for \( C_{13}(0) + C_{24}(0) = 0 \).

Remarkably, not only the magnitude but even the sign of \( C_{13}(0) + C_{24}(0) \) affects the heat flux in a significant way. This can be appreciated in Fig. 3(a), which shows that the energy flow of \( S \) into the reservoir proceeds slower when \( C_{13}(0) + C_{24}(0) < 0 \). When the sum is positive, in contrast, most of the energy is released in the early stages of the dynamics. Such different behaviors can be better understood by calculating the value of \( J_{c}^2(t) \) at \( t = 0 \) and at times \( t_{1,2} \) given by (35), which yields

\[ J_{c}^2(0) = \hbar \omega [2N_S + C_{13}(0) + C_{24}(0)]. \]  (36)

\[ J_{c}^2(t_1) = \hbar \omega [2N_S - [C_{13}(0) + C_{24}(0)]] e^{-\gamma t_1}. \]  (37)

\[ J_{c}^2(t_2) = \hbar \omega [2N_S + C_{13}(0) + C_{24}(0)] e^{-\gamma t_2}. \]  (38)

Hence, if \( C_{13}(0) + C_{24}(0) \) is positive, the first minimum always occurs at time \( t_1 \) and equals \( J_{c}^2(t_1) \). As \( e^{-\gamma t} \approx 0.135 \) [cf. Eq. (37)], in this case a drop of the heat flux of at least \( \geq 86\% \) takes place after a time \( 2/\gamma \). The following rise of \( J_{c}^2(t) \) is modest given that also the local maximum \( J_{c}^2(t_2) \) is at most \( \approx 14\% \) of the initial heat flux. Quite differently, if \( C_{13}(0) + C_{24}(0) \) is negative, the minimum occurs at time \( t_2 \); hence, the corresponding drop amounts to the exponential factor in Eq. (38), which does not exceed \( \approx 86\% \), this bound occurring in the limiting case of very small \( C_{13}(0) + C_{24}(0) \). As this grows, the exponential factor rapidly approaches 1 (correspondingly, the drop becomes less and less significant).

To characterize the release time of the system energy in more quantitative terms, in Fig. 4(a) we analyze \( \gamma \tau_p \), namely the time (in units of \( \gamma^{-1} \)) taken by a certain percentage \( p \) of the initial energy of \( S \) to be lost into the reservoir. That is, we compute the energy lost up to some time \( t \) as \( \dot{Q}_{c}^2(t) \equiv \int_0^t \dot{J}_{c}^2(t') dt' \) and we search for the time \( \tau_p \) at which \( \dot{Q}_{c}^2(\tau_p) = p\% \dot{Q}_{c}^2(\infty) \) (i.e., \( p\% \) of the total transferred energy). In Fig. 4(a), we plot \( \gamma \tau_p \) versus \( C_{13}(0) + C_{24}(0) \) for different values of the percentage \( p \) (the outcomes are independent of \( N_S \)). The plots show that positive (negative) values of \( C_{13}(0) + C_{24}(0) \) always speed up (slow down) the energy release compared to the uncorrelated case.

C. Influence of initial quantum correlations

Next we investigate the role played by typical measures of initial QCs possessed by a state of the form (31). Traditionally, QCs have been associated with entanglement [17]. More recently, however, a new paradigm of QCs—associated with the so-called quantum discord—has been put forward [18]. The need for introducing such a new type of QC relies on the observation that, although separable, some bipartite states can feature correlations that are incompatible with classical
physics. Specifically, here we use logarithmic negativity [30] \( E_N \) and Gaussian discord [31] \( D_G \) in order to quantify entanglement and discordlike QCs, respectively. Details on both measures can be found in Appendix F. Figures 5 show density plots of logarithmic negativity (a) and Gaussian discord (b) on the \( C_{13}(0) + C_{24}(0) \) plane for \( N_S = 1 \) and \( N = 0 \) (i.e., the paradigmatic instance addressed in the previous section). Entanglement \( E_N \) arises only in two small regions next to the points \( C_{13}(0) = -C_{24}(0) = \sqrt{N_S(N_S + 1)} \) and \( C_{13}(0) = -C_{24}(0) = -\sqrt{N_S(N_S + 1)} \) [32]. In both cases, the corresponding state is close to an Einstein-Podolsky-Rosen state [33]. Instead, Gaussian discord \( D_G \) is zero only at the point \( C_{13}(0) = C_{24}(0) = 0 \), which corresponds to a fully uncorrelated product state. It grows when the distance from this point increases. The steepest-increase directions are given by \( C_{13}(0) = -C_{24}(0) \) (where also \( E_N \) increases) and \( C_{13}(0) = C_{24}(0) \) (where instead entanglement is fully absent).

As discussed in the previous section (see also Appendix D), for any possible choice of \( C_{13}(0) = C_{24}(0) = c_0 \) there is a class of equivalent states [identified by \( C_{13}(0) + C_{24}(0) = 2c_0 \)] which exhibit the same heat-flux dynamics [cf. Eqs. (32) and (33)]. The union of these classes coincides with the whole set of physical initial states. As shown in Fig. 5, all the states in a given class feature non-null \( D_G \) [except for \( C_{13}(0) = C_{24}(0) = 0 \)], while a relevant fraction of them is not entangled. In Figs. 4(b) and 4(c), for each value of \( C_{13}(0) + C_{24}(0) \), we report all the possible values of \( E_N \) and \( D_G \) in the corresponding equivalence class. We see that the states giving rise to the fastest and slowest energy release [corresponding to the highest and lowest values of \( C_{13}(0) = C_{24}(0) = c_0 \), respectively] are discordant but not entangled. For such states, Gaussian discord lies within a very narrow range (in general, the faster or slower the energy release, the narrower is the interval of possible values of \( D_G \)). Yet, based on Figs. 4 and 5, one can see that a high amount of discord does not necessarily lead to a fast or slow dissipation rate. Moreover, note that the most discordant state gives rise to the same heat-flux time evolution as the completely uncorrelated state [see Fig. 4(c)]. The connection with energy release appears even weaker for entanglement as witnessed by the fact that, for each entangled state, there is always a separable one yielding the same heat-flux dynamics [see Fig. 4(b)].

Overall, the above analysis indicates that it is the peculiar structure of correlations—instead of the featured amount of “quantumness”—that affects the heat-flux dynamics. In particular, the quadratures that are most correlated play the major role. The optimal situation indeed occurs when the pairs \( \{\hat{X}_i, \hat{Y}_j\} \) are equally (anti)correlated by the highest possible amount.

VI. HEAT-FLUX DYNAMICS: QUBITS

A. Thermal initial states

Now consists of a pair of qubits and both subsystems are initially in a local thermal state at temperature \( T_S \), giving a joint (un correlated) initial state \( \rho(0) = \exp\left[-\frac{H_1}{(k_B T_S)}\right]\exp\left[-\frac{H_2}{(k_B T_S)}\right]/Z_S \).

The corresponding density matrix has zero off-diagonal entries, while the diagonal ones read

\[
\rho_{11}(0) = \frac{(1 - \xi_S)^2}{4}, \quad \rho_{22}(0) = \frac{(1 + \xi_S)^2}{4},
\]

\[
\rho_{12}(0) = \rho_{21}(0) = \frac{1 - \xi_S^2}{4},
\]

where \( \xi_S \) is the value taken by Eq. (9) for \( T = T_S \).

One can use these (see Appendix C) to calculate the time evolution of the density matrix elements entering Eqs. (23)–(25), hence the heat fluxes \( J_{12}(t) \), \( J_{12}(t) \), \( J_{12}(t) \), and the total heat flux \( J^{\text{tot}}(t) \). Unfortunately, the resulting analytic expressions are rather involved and uninformative (even in limiting cases). It turns out that no general exact relations as simple as those in Eqs. (27) and (28) can be established. Yet many of the salient features of the heat-flux dynamics are qualitatively quite similar to those emerging for harmonic oscillators. This is shown by the right-column plots of Fig. 2, where we plot \( J_{12}(t) \), \( J_{12}(t) \), \( J_{12}(t) \), and \( J^{\text{tot}}(t) \) against time for different values of \( T/T_S \) (the same considered in Sec. V A). The shape of each curve is quite similar to the corresponding one in the case of harmonic oscillators [a minor difference is that at intermediate times \( J_{12}(t) \) and \( J^{\text{tot}}(t) \) are not as flat as those for continuous-variable systems]. As a distinctive feature, though, saturation appears at growing temperatures for each plotted quantity, which is clearly due to the fermionic nature of each subsystem as well as each reservoir mode.
B. Correlated initial states

In order to select a suitable family of correlated initial states \( \rho(0) \), in full analogy with Sec. VB, we first require the local reduced qubit state to be locally thermal at temperature \( T_S \). This entails that the only possible nonzero off-diagonal entries of \( \rho(0) \) are \( \rho_{23}(0) = \rho_{32}(0)^{\ast} \) and \( \rho_{14}(0) = \rho_{41}(0)^{\ast} \) [the presence of extra off-diagonal entries would be incompatible with the constraint that each reduced state \( \rho(0) \) has a diagonal form]. In a way similar to Sec. VB, to simplify the analysis, we further restrict to states such that \( \rho_{14}(0) = \rho_{41}(0) = 0 \). Indeed, the heat fluxes in Eqs. (23)–(25) depend only on \( \rho_{23}(t) \) and its complex conjugate, which in turn are independent of \( \rho_{14}(0) \) as shown in Appendix C.

Therefore,

\[
\rho(0) = \frac{1}{4} \begin{pmatrix}
1 - \xi_S^2 & 0 & 0 & 0 \\
0 & 1 - \xi_S^2 & \rho_{23}(0) & 0 \\
0 & \rho_{23}(0)^{\ast} & 1 - \xi_S^2 & 0 \\
0 & 0 & 0 & (1 + \xi_S)^2
\end{pmatrix}.
\]

(41)

The allowed values of \( \rho_{23}(0) \) must fulfill the constraint

\[
|\rho_{23}(0)| \leq 1 - \xi_S^2,
\]

which follows from the requirement that density matrix (41) be positive.

As in Sec. VB, we focus on the case of a zero-temperature reservoir (hence \( \xi_N = 1 \)). From Eqs. (23)–(25) and initial state (41)—see also Appendix C—the heat fluxes are calculated as

\[
\begin{align*}
J_1(t) &= \gamma (1 - \xi_S) e^{-\gamma t}, \\
J_2(t) &= \gamma [-2(1 + \xi_S^2)(1 - \xi_S) + 2(1 - \gamma t e^{-\gamma t})(1 - \xi_S)^2 \\
&\quad - \gamma |\rho_{23}(0)| e^{-\gamma t}], \\
J_{12}(t) &= \gamma [2(1 - e^{-\gamma t})(1 - \xi_S)^2 - 2\gamma t (1 - \xi_S) \\
&\quad + |\rho_{23}(0)| e^{-\gamma t}].
\end{align*}
\]

Note that heat fluxes depend on the initial correlations through \( \Re[\rho_{23}(0)] \). In Fig. 3(b), we use these results to plot the total heat flux, as given by Eq. (12), versus time for \( \xi_S = 0.25 \) and three representative values of \( \Re[\rho_{23}(0)] \).

As in the case of initial thermal states (see previous section), again we find a behavior that qualitatively resembles the one observed for harmonic oscillators (a minor difference occurs for the \( \Re[\rho_{23}(0)] = 0.75 \) plot which does not feature stationary points but only concavity changes as time grows). This results from a comparison between Figs. 3(a) and 3(b), which shows that \( \Re[\rho_{23}(0)] \) here behaves similarly to the parameter \( C_{13}(0) + C_{24}(0) \) for harmonic oscillators. Negative (positive) values of \( \Re[\rho_{23}(0)] \) cause a slow (fast) energy release.

In analogy with Fig. 4(a), in Fig. 6(a) we plot \( \gamma t_p \) (time required to dissipate \( p \% \) of the initial energy) for \( \xi_N = 0 \). The plots show that positive (negative) values of \( \Re[\rho_{23}(0)] \) always speed up (slow down) the energy release compared to the uncorrelated case. The relationship between the heat-flux behavior and the initial correlations can be better understood (see Appendix E) by expressing the superoperators (7) and (8) and the initial state (41) in

FIG. 5. (Color online) Gaussian discord \( D_G \) (a) and logarithmic negativity \( E_N \) (b) of a state (31) as functions of \( C_{13}(0) \) and \( C_{24}(0) \) for \( N_S = 1 \) and \( N = 0 \). (c) Quantum discord \( D_Z \) of a state (41) as a function of \( \Re[\rho_{23}(0)] \) and \( \Im[\rho_{23}(0)] \) for \( \xi_S = 0.25 \) and \( \xi = 1 \). The states (41) considered for two qubits are never entangled.

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the collective basis \{ | \psi \rangle , | \psi^+ \rangle , | \psi^- \rangle , | g g \rangle \}, where \(| \psi^\pm \rangle = 1/\sqrt{2} (| g g \rangle_1 \pm | g e \rangle_1)\). Such rearrangement shows that states \(| \psi^+ \rangle \) and \(| \psi^- \rangle \) are coupled to the environment with different strengths. In particular, the singlet \(| \psi^- \rangle \) is fully decoupled from the environment for \( T = 0 \). A positive initial value of \( \rho_{23} \) means a smaller initial population of \(| \psi^- \rangle \) and therefore a faster energy release. A negative initial value of \( \text{Re}[\rho_{23}] \) means a larger initial population of \(| \psi^- \rangle \), hence a slower energy flow. This is shown in more detail in Appendix E.

C. Influence of initial quantum correlations

In line with Sec. V C, we next investigate the connection between heat flux and typical measures of correlations of the initial state (41). These measures, namely the concurrence for entanglement and the quantum discord for general nonclassical correlations, are described in Appendix F. Unlike family (31) for harmonic oscillators, all the qubit states (41) are disentangled (as can be shown by explicitly calculating the concurrence [34]; see Appendix F). They all feature, however, some quantum discord \( D_Z \). To show this, in Fig. 5(c) we set \( \xi_2 = 0.25 \) and plot \( D_Z \) [35,36] as a function of \( \text{Re}[\rho_{23}(0)] \) and \( \text{Im}[\rho_{23}(0)] \). Similarly to the behavior of \( D_C \) in Fig. 5(a), \( D_Z \) is nonzero on the entire plane but the origin \( \text{Im}[\rho_{23}(0)] = 0 \). In the present case, a simpler functional dependence arises since \( D_Z \) depends only on \( \rho_{23}(0) \) and it is thus constant along each circle centered at the origin. As \( | \rho_{23}(0) | \) grows, \( D_Z \) increases.

We see that, similarly to harmonic oscillators, states with different discord can exhibit the same heat-flux dynamics (corresponding to a set value of \( \text{Re}[\rho_{23}(0)] \)). To better highlight this, in Fig. 6(b) for a fixed value of \( \text{Re}[\rho_{23}(0)] \), we report all the possible values of \( D_Z \). Similarly to the harmonic oscillators case, we see that the slowest and fastest heat flows occur only for the maximum value of discord. However, a high amount of discord does not necessarily imply a low or fast energy release, as witnessed by the fact that states with maximum value of \( D_Z \) are compatible with any heat-flux dynamics.

The above indicates that, also in the case of qubits, it is the structure of correlations that decides the speed of heat flux.

VII. INTERPRETATION OF CORRELATED HEAT FLUX FOR QUBITS

The nonlocal nature of the correlated heat \( J_{12} \) [cf. Eqs. (12) and (14)] suggests a possible link between such quantity and some measure of correlations between \( S_1 \) and \( S_2 \). A general formulation of such a connection with some known correlations indicator is not straightforward. Remarkably, however, we next find that, in the case of qubits, this is possible for a relevant class of initial states. Specifically, we show that \( J_{12} \) can be expressed in terms of the so-called trace distance discord (TDD) [37] whenever \( S \) is initially in a product of local thermal states. This is a well-behaved measure of nonclassical correlations exhibited by a bipartite quantum state (not necessarily in the presence of entanglement). Specifically, the one-sided TDD \( D_{-}(\rho) \) from 1 to 2 of a bipartite quantum state \( \rho \) is defined as the minimal trace norm distance [25] between such state and the set of so-called classical-quantum (CQ) states [37]. A CQ state features zero QC with respect to local measurements on A and can be expressed as

\[
\rho_{\text{CQ}} = \sum_j |\alpha_j\rangle\langle\alpha_j| \otimes \varrho_j(\varphi),
\]

with \( \{|\alpha_j\rangle\} \) being a complete set of orthonormal vectors of subsystem 1 and \( \varrho_j(\varphi) \) being a positive (not necessarily normalized) operator of subsystem 2. Specifically, if \( \|\Theta\|_1 = \text{Tr}[\sqrt{\Theta^\dagger\Theta}] \) denotes the trace norm (or Schatten 1-norm) of a generic operator \( \Theta \), then the TDD of state \( \rho \) is defined by

\[
D_{-}(\rho) = \frac{1}{2} \min_{\|\rho\|_1} \|\rho - \rho_{\text{CQ}}\|_1.
\]

where, as shown by the notation, the minimum is over all possible quantum-classical states (43). In other words, the TDD is the minimum distance in the Hilbert space between \( \rho \) and the set of CQ states.

We next restrict to a system \( S \) made of a pair of qubits and initially in the state \( \rho(0) = \text{exp}(-H_1/\hbar \xi_2) \otimes \text{exp}(-H_2/\hbar \xi_2) \), namely a tensor product of two local thermal states (in general at different temperatures). Note that such a family encompasses the initial state considered in Sec. VI A as a special case. Using the solution of the master equation given in Appendix C, it can be easily shown that the state of \( S \) will maintain the same form at any time (the local temperatures can change with time). A locally thermal state belongs to the family of two-qubit X states (these have nonzero entries on the two main diagonals of the corresponding density matrix). The TDD of such states can be calculated exactly [38]. Using the closed formula of Ref. [38], we find that at any time \( t \) the modulus of the correlated heat flux \( J_{12}(t) \) is proportional to the TDD of state \( \rho(t) \) according to

\[
|J_{12}(t)| = 4\hbar\alpha_\gamma^2 \xi D_{-}(\rho(t)).
\]

It is natural to wonder whether this property holds for more general initial states. This is not the case, as can be seen through the following counterexample: Let us select the initial state, \( \rho(0) = \frac{1}{2} |\psi\rangle_1 \langle\psi| \otimes \begin{pmatrix} 1 - \xi_2 & 0 \\ 0 & 1 + \xi_2 \end{pmatrix} \),

with

\[
|\psi\rangle_1 = \sqrt{\frac{1 + \xi_2}{2}} |0\rangle_1 + \sqrt{\frac{1 - \xi_2}{2}} |1\rangle_1,
\]

where \( \xi_2 \) is the same as in Eq. (9) for \( T = T_2 \). As in the previous case, \( \rho(0) \) is a product state, hence featuring zero correlations, with \( S_2 \) locally in a thermal state. Now, however, despite having the same populations and energy as the thermal state corresponding to \( \xi_1 \), the initial state of \( S_1 \) is fully pure. In other words, \( S_1 \) has the same temperature as \( R \) but features nonzero coherences. In such a case, we can show that \( D_{-}(\rho(t)) \) is, in general, finite but \( J_1(t) = J_{12}(t) = 0 \) identically. In other words, the interaction mediated by the reservoir gives rise to QC between the system’s subparts with no simultaneous development of any correlated heat flux.
VIII. CONCLUSIONS

In this work, we have studied the dynamics of heat flux of a bipartite system interacting with a thermal reservoir in a cascaded way. The cascading makes one of the two subsystems interact with the reservoir modified by the previous interaction with the other subsystem. Because of such circumstance, the local dynamics of the second subsystem is non-Markovian despite the fact that the joint dynamics is Markovian. This affects the heat flux in such a way that it exhibits a nonexponential time behavior. We have carried out a systematic analysis of this after showing that the total heat flux can be decomposed into three components. In particular, one of these—arising from a nonlocal dissipator entering the master equation—can be identified as a correlated heat flux and was shown to play a major role in the nonmonotonic time evolution.

Typical behaviors, occurring in the case of both thermal and correlated initial states, have been scrutinized for two paradigmatic systems: a pair of harmonic oscillators with a reservoir of bosonic modes and two qubits with a reservoir of fermionic modes. While in the case of harmonic oscillators basically all of the observed features can be explained analytically, an analogous analysis is not possible for qubits. Notwithstanding, most of the qualitative features of the heat-flux dynamics are quite similar to those occurring for harmonic oscillators (aside from saturation effects owing to the presence of only two levels for qubits).

In the case of thermal initial states, we have shown that the total heat flux exhibits a monotonic, although nonexponential, time behavior. In particular, an almost flat profile arises at intermediate times which is mostly due to the occurrence of the aforementioned correlated heat flux. To explore the effect of initial correlations in the system state, we have focused on a suitable family of initial states that are locally thermal but additionally feature nonlocal correlations. In general, the effect of these is to cause nonmonotonicity of the total heat flux accompanied by a simultaneous slow down or speed up of the thermalization process. We have investigated the role played by the initial amount of QCs, either in the form of entanglement or discord, on the rate of energy exchange. Our analysis indicates that, although the states featuring the slowest and fastest heat-flux dynamics are characterized by different initial thermal states—the magnitude of the correlated heat flux at any time coincides (up to a proportionality factor) with the trace distance discord of the open system. In particular, this shows the existence of a physical scenario within which such a bona fide measure of QCs acquires a clear physical significance.

It is worth emphasizing that, as already observed, a key feature of our system is that, while the joint dynamics of $S_1$ and $S_2$ is Markovian, the reduced dynamics of system $S_2$ is non-Markovian. Recently, the concept of quantum non-Markovianity has received remarkable attention in the effort of defining on a rigorous basis the distinctive aspects of such phenomenon and, accordingly, ways to quantify it. Within this framework, our work suggests an interesting connection between quantum non-Markovianity and heat-flux dynamics.

In this work, we have focused on initial states—either correlated or not—that are in any case locally thermal at a uniform temperature (i.e., the same for both subsystems). Allowing for a nonuniform temperature makes the heat-flux dynamics as well as its interplay with initial correlations considerably richer, which will be the subject of a future work.

ACKNOWLEDGMENTS

The authors are grateful to R. Fazio for useful discussions. This work is funded by the EU Collaborative Project TherMiQ (Grant Agreement No. 618074) and the Italian PRIN-MIUR 2010/2011.

APPENDIX A: STATIONARY STATE

Here we prove that the thermal state (10) is indeed the asymptotic state reached by $S$ in the cases of both harmonic oscillators and qubits. Let $\rho_h = e^{-\beta H_1} e^{-\beta H_2}$ with $\beta = 1/(k_B T)$ (the tensor product symbol is omitted for simplicity). To demonstrate that this is indeed the system’s steady state, we prove that $\rho_h$ fulfills the master equation under stationary conditions (when all the time derivatives vanish), namely,

$$[\mathcal{L}_1 + \mathcal{L}_2 + D_{12}](\rho_h) = 0.$$  \hspace{1cm} (A1)

1. Harmonic oscillators

Let $\mathcal{U}_k = e^{\pm \beta h_0 a \hat{a}^\dagger}$, where $\hat{a}$ and $\hat{a}^\dagger$ are bosonic annihilation and creation operators. Then, $\mathcal{U}_- \hat{a} \mathcal{U}_+ = e^{\beta h_0 \hat{a}}$ and $\mathcal{U}_- \hat{a}^\dagger \mathcal{U}_+ = e^{-\beta h_0 \hat{a}^\dagger}$. These identities entail

$$[e^{-\beta h_0 \hat{a}^\dagger}, \hat{a}] = (1 - e^{-\beta h_0}) e^{-\beta h_0 \hat{a}^\dagger} \hat{a},$$  \hspace{1cm} (A2)

$$[e^{-\beta h_0 \hat{a}^\dagger}, \hat{a}^\dagger] = (1 - e^{-\beta h_0}) e^{-\beta h_0 \hat{a}^\dagger} \hat{a}^\dagger.$$  \hspace{1cm} (A3)

In the present case, $\rho_{th} = e^{-\beta h_0 \hat{a}^\dagger \hat{a}} e^{-\beta h_0 \hat{a} \hat{a}^\dagger}$. Applying $\mathcal{L}_1$ [cf. Eq. (4)] to such a state, upon use of Eqs. (A2) and (A3), yields

$$\mathcal{L}_1(\rho_{th}) = \gamma(N + 1)(e^{-\beta h_0 \hat{a}^\dagger \hat{a}} e^{-\beta h_0 \hat{a} \hat{a}^\dagger} - e^{-\beta h_0 \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger} - e^{-\beta h_0 \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger}) \gamma N (e^{-\beta h_0 \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger} - e^{-\beta h_0 \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger} + e^{-\beta h_0 \hat{a} \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger} - e^{-\beta h_0 \hat{a} \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger} + e^{-\beta h_0 \hat{a} \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger} - e^{-\beta h_0 \hat{a} \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger}) e^{-\beta h_0 \hat{a}^\dagger \hat{a}^\dagger} \\
= [\gamma N + 1] e^{-\beta h_0} + \gamma(N + 1) e^{-\beta h_0} + \gamma N (e^{-\beta h_0} - 1) \gamma N \rho_{th} \\
= (-\gamma \hat{a}^\dagger \hat{a} + \gamma N + \gamma \hat{a}^\dagger \hat{a} - \gamma N) \rho_{th} = 0.$$
Likewise, the identity $L_2(\rho_{th}) = 0$ is proven by swapping indexes 1 and 2. The last step is thus showing that $\mathcal{D}_{12}(\rho_{th}) = 0$ [cf. Eq. (5)]. Using again Eqs. (A2) and (A3) gives

$$\mathcal{D}_{12}(\rho_{th}) = \gamma(N + 1)(e^{-\beta\hbar} + e^{-\beta\hbar})[e^{\beta\hbar}\hat{a}_1^\dagger \hat{a}_2 - (1 - e^{-\beta\hbar}\hat{a}_1^\dagger \hat{a}_2)] + \gamma N[e^{\beta\hbar}(1 - e^{-\beta\hbar}\hat{a}_1^\dagger \hat{a}_2 - (1 - e^{-\beta\hbar}\hat{a}_1^\dagger \hat{a}_2))\rho_{th} = [\gamma(N + 1)e^{\beta\hbar}(1 - e^{-\beta\hbar})\hat{a}_1^\dagger \hat{a}_2 - \gamma N(1 - e^{-\beta\hbar})\hat{a}_1^\dagger \hat{a}_2 - \gamma(N + 1)(1 - e^{-\beta\hbar})\hat{a}_1^\dagger \hat{a}_2 + \gamma N e^{\beta\hbar}(1 - e^{-\beta\hbar})\hat{a}_1^\dagger \hat{a}_2]\rho_{th} = [\gamma N(1 - e^{-\beta\hbar})\hat{a}_1^\dagger \hat{a}_2 - \gamma N(1 - e^{-\beta\hbar})\hat{a}_1^\dagger \hat{a}_2 - \gamma(N + 1)(1 - e^{-\beta\hbar})\hat{a}_1^\dagger \hat{a}_2 + \gamma(N + 1)(1 - e^{-\beta\hbar})\hat{a}_1^\dagger \hat{a}_2]\rho_{th} = 0.$$

This concludes the proof.

### 2. Qubits

In this case $\rho_{th} = e^{-\beta\hbar}\hat{a}_1^\dagger e^{-\beta\hbar}/Z^2$, which we rearrange as

$$\rho_{th} = \frac{1}{Z} \left( \begin{array}{cc} \gamma & -\xi \\ -\xi & \frac{1}{2} \end{array} \right).$$

(A4)

Using $\hat{\sigma}_1 \hat{\sigma}_2, \hat{\sigma}_1 \hat{\sigma}_3 = \mp \hat{\sigma}_2 \hat{\sigma}_3$, and $\hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_3 = \pm \hat{\sigma}_2 \hat{\sigma}_3 \hat{\sigma}_1$, it is immediate to see that $L_1(\rho_{th}) = 0$ [cf. (7)] since $L_1(\rho_{th}) = \frac{1}{Z} L_1(\hat{\sigma}_z)$. On the other hand, from Eq. (8) follows

$$\mathcal{D}_{12}(\rho_{th}) = \frac{\gamma}{2} \left[ \hat{\sigma}_1 \hat{\sigma}_3, \rho_{th}^\dagger \hat{\sigma}_2 + \gamma \left( \hat{\sigma}_1 \hat{\sigma}_3, \rho_{th}^\dagger \hat{\sigma}_2 + \hat{\sigma}_1 \hat{\sigma}_3 \right) \right]$$

which upon use of $[\hat{\sigma}_1 \hat{\sigma}_3, \rho_{th}^\dagger] = \pm \hat{\sigma}_1 \hat{\sigma}_3$ and $[\hat{\sigma}_1 \hat{\sigma}_3, \rho_{th}^\dagger] = \hat{\sigma}_1 \hat{\sigma}_3$ yields

$$\mathcal{D}_{12}(\rho_{th}) = \frac{\gamma}{2} \left( \hat{\sigma}_1 \hat{\sigma}_3 - \xi \hat{\sigma}_1 \hat{\sigma}_3 \right) + \frac{\gamma}{2} \left( \hat{\sigma}_1 \hat{\sigma}_3 + \xi \hat{\sigma}_1 \hat{\sigma}_3 \right)$$

which is equivalent to the master equation (1) and read

$$\frac{d}{dt} \begin{pmatrix} \hat{X}_1 \\ \hat{Y}_1 \\ \hat{X}_2 \\ \hat{Y}_2 \end{pmatrix} = -\gamma \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \hat{X}_1 \\ \hat{Y}_1 \\ \hat{X}_2 \\ \hat{Y}_2 \end{pmatrix} - \sqrt{\gamma} \begin{pmatrix} \hat{X}_m \\ \hat{Y}_m \\ \hat{X}_m \\ \hat{Y}_m \end{pmatrix},$$

(B1)

where $\hat{X}_m$ and $\hat{Y}_m$ are zero-mean Gaussian noises characterized by the correlations $\langle \hat{X}_m \hat{X}_m \rangle = 0, \langle \hat{X}_m \hat{Y}_m \rangle = \langle \hat{Y}_m \hat{X}_m \rangle = \langle \hat{Y}_m \hat{Y}_m \rangle = N + \frac{1}{2}$. Correspondingly, the covariance matrix evolves in time as

$$\frac{d}{dt} C = A C + C A^T + M,$$

(B2)

where $A$ is the matrix appearing in Eq. (B1) and

$$M = \gamma \left( N + \frac{1}{2} \right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$
defined by \( \mathcal{K}_{kj, mn} = \langle \langle k | \mathcal{L} | mn \rangle \rangle = \text{Tr}[\langle k | \mathcal{L} | m \rangle \langle n |] \). In our case, such matrix is explicitly given by

\[
\mathcal{K} = \begin{pmatrix}
-2(1 + \xi) & 0 & 0 & 0 & 1 - \xi & 1 - \xi & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 - \xi & \xi - 1 & 1 - \xi & 0 & 0 & 0 & 0 & 1 - \xi & 0 & 0 & 0 \\
0 & 1 - \xi & -2 - \xi & 0 & 0 & 0 & 0 & 1 - \xi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 + \xi & 0 & 0 & 0 & -2 & \xi - 1 & 0 & 0 & \xi - 1 & 0 & 0 & 0 \\
1 + \xi & 0 & 0 & 0 & -1 - \xi & -2 & 0 & 0 & 0 & \xi - 1 & 0 & 0 \\
0 & 1 + \xi & 1 + \xi & 0 & 0 & 0 & -2 + \xi & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 - \xi & 0 & 0 & -2 & \xi - 1 & 0 & 0 & 0 \\
1 + \xi & 0 & 0 & 0 & 0 & 0 & -1 - \xi & 0 & 0 & -2 + \xi & 0 & 0 \\
0 & 0 & 0 & 0 & 1 + \xi & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 + \xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 + \xi & 0 & 0 & 1 + \xi & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 + \xi & 0 & 0 & 0 & 1 + \xi & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 + \xi & 1 + \xi & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where we have used the ordering

\[
\begin{pmatrix}
\mathcal{K}_{11,11} & \mathcal{K}_{11,12} & \mathcal{K}_{11,13} & \mathcal{K}_{11,14} & \mathcal{K}_{11,21} & \cdots \\
\mathcal{K}_{21,11} & \mathcal{K}_{21,12} & \mathcal{K}_{21,13} & \mathcal{K}_{21,14} & \mathcal{K}_{21,21} & \cdots \\
\mathcal{K}_{31,11} & \mathcal{K}_{31,12} & \mathcal{K}_{31,13} & \mathcal{K}_{31,14} & \mathcal{K}_{31,21} & \cdots \\
\mathcal{K}_{41,11} & \mathcal{K}_{41,12} & \mathcal{K}_{41,13} & \mathcal{K}_{41,14} & \mathcal{K}_{41,21} & \cdots \\
\mathcal{K}_{12,11} & \mathcal{K}_{12,12} & \mathcal{K}_{12,13} & \mathcal{K}_{12,14} & \mathcal{K}_{12,21} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

\tag{C2}
\]

The solution of the linear first-order differential system \( \dot{\rho} = \mathcal{K} \rho \) is found in an exponential form as

\[
\rho_{\text{max}}(t) = \sum_{k,j} \langle \exp(\mathcal{K} t) \rangle_{mn,kj} \rho_{kj}(0).
\]

In particular, it turns out that

\[
\rho_{\text{1d}}(t) = \exp(-\nu t) \rho_{\text{1d}}(0),
\]

which shows that the off-diagonal terms \( \rho_{1d}(t) = \rho_{1d}(t)^* \) are decoupled from other elements of the density matrix regardless of the system initial state.

**APPENDIX D: PARAMETRIZATION OF INITIAL CORRELATED STATES FOR HARMONIC OSCILLATORS**

As discussed in the main text (Sec. VI), in the case of harmonic oscillators we focus on the family of initial states whose associated covariance matrix reads

\[
C(0) = \begin{pmatrix}
C_{11}(0) & 0 & C_{13}(0) & 0 \\
0 & C_{11}(0) & 0 & C_{22}(0) \\
C_{13}(0) & 0 & C_{11}(0) & 0 \\
0 & C_{22}(0) & 0 & C_{11}(0)
\end{pmatrix},
\]

\tag{D1}
\]

where \( C_{11}(0) = N_S + \frac{1}{2} \) and the total energy is fixed to \( U = \frac{1}{2} \text{Tr}[C(0)] = 2C_{11}(0) \). This choice is motivated by the fact that the heat flux depends only on \( C_{11}(t), C_{13}(t), C_{22}(t) \) [see Eqs. (20)–(22)] and these instantaneous values are completely determined by the initial conditions \( C_{11}(0), C_{13}(0), C_{22}(0) \) [see Eqs. (B4)–(B9)]. We could then choose any value for the remaining off-diagonal terms without affecting the heat flux, but the optimal choice is zero, as argued at the end of the section. Our essential task is to derive the conditions on the off-diagonal elements \( C_{13}(0) \) and \( C_{22}(0) \), in order for \( C(0) \) to describe a physical state once the total energy is fixed. In general, a covariance matrix of a physically admissible Gaussian state must be such that all the second moments fulfill the Heisenberg uncertainty relations. This requirement can be expressed compactly as the semipositivity condition

\[
C(0) + \frac{1}{2} \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix} \geq 0.
\]

\tag{D2}
\]

This is equivalent to two necessary and sufficient conditions [43]. First, the covariance matrix needs to be positive, i.e., \( C(0) > 0 \), which is, in turn, equivalent to the two inequalities

\[ |C_{13}(0)| < C_{11}(0) = N_S + \frac{1}{2}, \quad |C_{22}(0)| < N_S + \frac{1}{2}. \]

\tag{D3}
\]

Second, the symplectic eigenvalues \( v_{\pm} \) must fulfill

\[
v_{\pm} = \sqrt{I_A + I_B + 2IC \pm \sqrt{(I_A + I_B + 2IC)^2 - 4I_E}} \geq \frac{1}{2}.
\]

\tag{D4}
\]

where we introduced the symplectic invariants [29] \( I_A = I_B = C_{13}(0)^2, I_C = C_{13}(0)C_{22}(0), \) and \( I_E = C_{11}(0)^2 + C_{13}(0)^2C_{22}(0)^2 - C_{11}(0)^2C_{13}(0)^2 + C_{22}(0)^2 \). Note that if the pair \( C_{13}(0), C_{22}(0) = [c,d] \) satisfies the two conditions, so do the pairs \( [C_{13}(0), C_{22}(0)] = [d,c] \) and \( [C_{13}(0), C_{22}(0)] = [c,-d] \). Hence, the region of physicality is symmetric across the two diagonals of the \( C_{13}(0) - C_{22}(0) \) plane. In Fig. 7, we plot this region for different values of \( N_S \). One can see that the area of the physicality region grows with \( N_S \). Indeed, if \( N_S = 0 \) each local state [i.e., \( \rho_{1d}(0) \) and \( \rho_{2d}(0) \)] is pure since \( S \) is in the vacuum state; hence, no correlations are present. Moreover, note that the line where \( C_{13}(0) = C_{22}(0) \) (red line in the figure) spans all the allowed values of \( C_{13}(0) + C_{22}(0) \) (this is constant along each black dashed line in the plots). As heat fluxes depend on \( C_{13}(0) \) and \( C_{22}(0) \) through their
the set of points such that $\langle 1 - \xi_S \rangle^2$ is constant along each black dashed line. The red dashed line is instead state for different values of $\text{Re}\left[ 1/\Gamma_1 \right]$.

Clearly, a positive (negative) value of $\text{Re}[\rho_{23}]$ means a smaller (larger) initial population of $|\Psi^-\rangle$ compared to the case where $\text{Re}[\rho_{23}] = 0$. On the other hand, master equation (1) can be reexpressed as [44]

$$\dot{\rho} = -i[\hat{H}, \rho] + \hat{L}(\rho),$$

where we have defined

$$\hat{L}(\rho) = \Gamma^+ L[|\Psi^+\rangle\langle gg| + |ee\rangle\langle \Psi^+ |](\rho)$$

$$+ \Gamma^- L[|\Psi^-\rangle\langle gg| - |ee\rangle\langle \Psi^- |](\rho),$$

$$\hat{H} = H - i\frac{\gamma}{2}(|\Psi^+\rangle\langle \Psi^- | - |\Psi^-\rangle\langle \Psi^+ |),$$

with $L[\delta](\rho) = \delta\rho\delta^\dagger - \frac{1}{2}(\delta\rho^\dagger, \rho)$ for a generic operator $\delta$ and $\Gamma^\pm = (1 \pm \xi)/2$. It is clear that for $N = 0$, i.e., $\xi = 1$, as in the plot in Fig. 6, $|\Psi^-\rangle$ is not directly affected by dissipation, which yields a slow down of energy releasing if $\text{Re}[\rho_{23}] < 0$.

APPENDIX F: COMPUTATION OF QUANTUM CORRELATIONS

1. Discordlike measures

Given a pair of quantum systems $A$ and $B$, quantum discord [35] is the gap between two classically equivalent expressions

$$\sum C_{13}(0) + C_{24}(0) \geq 2N_S.$$

If we had other nonzero off-diagonal terms, the constraints (D3) and (D4) would be more restrictive on $C_{13}(0)$ and $C_{24}(0)$. In other words, we would get $|C_{13}(0) + C_{24}(0)| \leq C_{\text{MAX}} < 2N_S$ and some possible evolutions of the heat flux would remain unexplored. Starting with a state of the form (D1) allows instead for a complete analysis of the problem.

APPENDIX E: ROLE OF INITIAL CORRELATIONS

Introducing the collective basis $\{|ee\rangle, |\Psi^+\rangle, |\Psi^-\rangle, |gg\rangle\}$, where $|\Psi^\pm\rangle \equiv 1/\sqrt{2}(|eg\rangle_{12} \pm |ge\rangle_{12})$, the initial state (41) becomes

$$\rho(0) = \frac{1}{4} \begin{pmatrix}
1 - \xi_S^2 & 0 & 0 & 0 \\
0 & 1 - \xi_S^2 + \text{Re}[\rho_{23}(0)] & 0 & 0 \\
0 & 0 & 0 & 1 - \xi_S^2 - \text{Re}[\rho_{23}(0)] \\
0 & 0 & 0 & (1 + \xi_S^2)
\end{pmatrix}.$$

Clearly, a positive (negative) value of $\text{Re}[\rho_{23}]$ means a smaller (larger) initial population of $|\Psi^-\rangle$ compared to the case where $\text{Re}[\rho_{23}] = 0$. On the other hand, master equation (1) can be reexpressed as [44]

$$\dot{\rho} = -i[\hat{H}, \rho] + \hat{L}(\rho),$$

where we have defined

$$\hat{L}(\rho) = \Gamma^+ L[|\Psi^+\rangle\langle gg| + |ee\rangle\langle \Psi^+ |](\rho)$$

$$+ \Gamma^- L[|\Psi^-\rangle\langle gg| - |ee\rangle\langle \Psi^- |](\rho),$$

$$\hat{H} = H - i\frac{\gamma}{2}(|\Psi^+\rangle\langle \Psi^- | - |\Psi^-\rangle\langle \Psi^+ |),$$

with $L[\delta](\rho) = \delta\rho\delta^\dagger - \frac{1}{2}(\delta\rho^\dagger, \rho)$ for a generic operator $\delta$ and $\Gamma^\pm = (1 \pm \xi)/2$. It is clear that for $N = 0$, i.e., $\xi = 1$, as in the plot in Fig. 6, $|\Psi^-\rangle$ is not directly affected by dissipation, which yields a slow down of energy releasing if $\text{Re}[\rho_{23}] < 0$.

of the mutual information content given by

$$\mathcal{D}(B|A) = \mathcal{I}(AB) - C(B|A),$$

where

$$\mathcal{I}(AB) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

is quantum mutual information [45], while

$$C(B|A) = \max_{\{a_i\}} \left\{ S(\rho_B) - \sum_a p_a S \left( \frac{\text{Tr}[\rho_{AB} E_a]}{p_a} \right) \right\}$$

is interpreted as the total amount of classical correlations in the above expression. Here $S(\rho)$ is the von Neumann entropy, $\sum_a E_a = 1$ is a positive-operator valued measure (POVM) on $A$, and $p_a = \text{Tr}[\rho_{AB} E_a]$ is the probability of outcome $a$.

a. Gaussian discord for harmonic oscillators

Originally proposed for qubits, the above definition of quantum discord has been generalized to Gaussian states for
continuous-variable systems [30,46] under the name of Gaussian discord \( D_G \). This is obtained by restricting the optimization in Eq. (F2) to Gaussian POVM. As a consequence \( D_G \) provides, in general, only a lower bound for \( D \) (namely, states with non-zero values of \( D_G \) will certainly exhibit a certain amount of discord). For Gaussian states, yet, it is conjectured to be optimal, i.e., \( D_G = D \) [30,46–48]. Gaussian discord is analytically computable for all two-mode Gaussian states (notably, all such states, except product states, have non-zero Gaussian discord).

The correlation matrix (see Sec. IV A) can be arranged in a \((2 \times 2)\)-block form as

\[
C = \begin{pmatrix}
C_1 & C_2 \\
C_3 & C_2
\end{pmatrix}.
\]

(F4)

From the correlation matrix \( C \), five symplectic invariants [29] can be constructed,

\[
\begin{align*}
I_1 &= 4 \text{ Det}(C_1), \quad I_2 = 4 \text{ Det}(C_2), \quad I_3 = 4 \text{ Det}(C_3), \\
I_4 &= 16 \text{ Det}(C), \quad I_\Delta = I_1 + I_2 + 2I_3,
\end{align*}
\]

The analogous quantity \( D_G(A|B) \) can be computed by exchanging \( I_1 \) with \( I_2 \) in the above formulas and describes the correlations retrieved by measuring subsystem \( B \) first (instead of subsystem \( A \)). For the initial states considered in Sec. V, exchanging the role of the two subsystems has no effect, so that the two quantities coincide and we simply call them \( D_G \).

b. Qubits

For a two-qubit system, the local measurement on system \( A \) is written as \( \Pi_{\lambda}(\theta, \phi) = |l⟩_A \otimes 1_B \) \((l = 1,2)\), with

\[
\begin{align*}
|1⟩ &= \cos \left( \frac{\theta}{2} \right) |e⟩ + e^{i\phi} \sin \left( \frac{\theta}{2} \right) |g⟩, \\
|2⟩ &= \sin \left( \frac{\theta}{2} \right) |e⟩ - e^{i\phi} \cos \left( \frac{\theta}{2} \right) |g⟩,
\end{align*}
\]

being orthogonal single-qubit states. The total amount of classical correlations [cf. (F3)] reads

\[
C(B|A) = \max_{\theta, \phi} \left[ S(\rho_B) - \sum_l p_l S \left( \frac{\text{Tr}_{A}[\Pi_{\lambda}(\theta, \phi)\rho_A\Pi_{\lambda}(\theta, \phi)]}{p_l} \right) \right].
\]

2. Entanglement

a. Harmonic oscillators

In Sec. V, we use logarithmic negativity for measuring entanglement of harmonic oscillators. It directly stems from the positive partial transpose (PPT) criterion [49] for discriminating entangled and separable states. A bipartite separable state can be written, by definition, as \( \rho_{SEP} = \sum_l p_l \rho_A^{(l)} \otimes \rho_B^{(l)} \), with \( \rho_A^{(l)} \), \( \rho_B^{(l)} \) being states of the subsystems \( A \) and \( B \), and two symplectic eigenvalues,

\[
\lambda_\pm = \sqrt{\frac{I_\Delta \pm \sqrt{I_\Delta^2 - 4I_4}}{2}}.
\]

(F5)

Gaussian discord can be defined in terms of these quantities (which are invariant under local unitary operations) as

\[
D_G(B|A) = f(\sqrt{I_\lambda}) - f(\lambda_-) - f(\lambda_+) + f(\sqrt{W}),
\]

where

\[
f(x) \equiv \left( \frac{x + 1}{2} \right) \log_2 \left( \frac{x + 1}{2} \right) - \left( \frac{x - 1}{2} \right) \log_2 \left( \frac{x - 1}{2} \right)
\]

and

\[
\text{if } (I_4 - I_2 I_1)^2 \leq (1 + I_1)I_2^2(I_2 + I_4),
\]

otherwise.

respectively, and \( p_l \) being probabilities. It is easy to see that its partial transpose with respect to one subsystem (say \( A \)) \( \rho_{SEP}^T_A = \sum_l p_l \rho_A^{(l)} \otimes \rho_B^{(l)} \) is still a valid density matrix and hence is positive definite. Conversely, a non-PPT always indicates the presence of entanglement. The logarithmic negativity quantities how negative the partial transpose is.

For 1 \( \otimes \) 1-mode Gaussian states the PPT criterion is both necessary and sufficient [50]. This also implies that the logarithmic negativity is a faithful measure of entanglement. In terms of correlation matrix \( C \), partial transposition is equivalent to changing the sign of momenta for a subsystem (say \( A \)). The partial transpose \( C_A^T \) is positive if and only if its symplectic eigenvalue \( \tilde{\lambda}_- \) is greater than 1/2 [29]. The symplectic eigenvalue \( \tilde{\lambda}_- \) can be found, analogously to Eq. (F5), as

\[
\tilde{\lambda}_- = \sqrt{\frac{I_\Delta - \sqrt{I_\lambda^2 - 4I_4}}{2}},
\]

(F11)

where now \( I_\Delta = I_1 + I_2 - 2I_3 \) (note the change of sign due to partial transposition). The logarithmic negativity \( E_N \) is then defined as

\[
E_N = \max(0, -\log_2(2\tilde{\lambda}_-)).
\]

(F12)

Consistently, \( E_N > 0 \) when \( \tilde{\lambda}_- < 1/2 \).

b. Qubits

The concurrence is a measure of entanglement of two-qubit states, which is given by

\[
C(\rho) = \max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0),
\]

(F13)
where \(\{\lambda_i\}\) are the square roots of the eigenvalues of matrix 
\(M(\rho) = \rho (\sigma_1 \sigma_2 + \sigma_2 \sigma_1) \rho^* (\sigma_1 \sigma_2 + \sigma_2 \sigma_1)\) sorted in decreasing order, 
while \(\rho^*\) is the complex conjugate of density matrix \(\rho\). For two-qubit \(X\) states, 
\[
\rho = \begin{pmatrix}
a & 0 & 0 & w^* \\
0 & b & z^* & 0 \\
0 & z & c & 0 \\
w & 0 & 0 & d
\end{pmatrix}.
\] (F14)

Equation (F13) in this case becomes 
\[
C(\rho) = \max[2(|w| - \sqrt{bc}), 2(|z| - \sqrt{ad})].
\] (F15)

For the initial states addressed in Sec. V IC, we thus find 
\[
C[\rho(0)] = \max \left\{ 1 \left[ |\rho_{23}(0)| + \xi_3^2 - 1 \right] \right\} = 0,
\] (F16)

where we have taken into account Eq. (42) in the main text.

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