NONCOMMUTATIVE CARTAN C*-SUBALGEBRAS

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Abstract. We characterise Exel’s noncommutative Cartan subalgebras in several ways using uniqueness of conditional expectations, relative commutants, or purely outer inverse semigroup actions. We describe in which sense the crossed product decomposition for a noncommutative Cartan subalgebra is unique. We relate the property of being a noncommutative Cartan subalgebra to aperiodic inclusions and effectivity of dual groupoids.

1. Introduction

Many important C*-algebras may be described as groupoid C*-algebras of Hausdorff, étale, locally compact groupoids. Renault [26] defined a Cartan subalgebra in a C*-algebra B as a maximal abelian, regular C*-subalgebra A of B with a faithful conditional expectation E : B → A. Assuming B to be separable, he proved that B ∼= C*(H, Σ) for a topologically principal, second countable, Hausdorff, étale twisted groupoid (H, Σ). In addition, the isomorphism B → C*(H, Σ) maps A onto C0(H0), and (H, Σ) is unique up to isomorphism. As a result, a Cartan subalgebra allows to reconstruct an underlying dynamical system from a C*-inclusion. Kumjian’s earlier theory of C*-diagonals in [19] covered only the reduced twisted groupoid C*-algebras of principal étale groupoids. The notion of a Cartan subalgebra has become ubiquitous in the study of C*-algebras (see, for instance, [25] and the sources cited there); in particular, it is crucial for the UCT problem [1,2] and rigidity of dynamical systems [8].

The success of Renault’s theory led Exel [14] to generalise Renault’s definition to the case when the subalgebra A ⊆ B need no longer be commutative. In Exel’s definition of a noncommutative Cartan subalgebra, Renault’s requirement that A be maximal Abelian in B is replaced by the requirement that all “virtual commutants” of A in B be trivial. The main result of [14] says that every noncommutative Cartan C*-inclusion A ⊆ B into a separable C*-algebra B is isomorphic to an inclusion A ⊆ A ⊗ S into a reduced crossed product A ⊗ S for some action of a unital inverse semigroup S on A by Hilbert A-bimodules. Actions of inverse semigroups by Hilbert bimodules are equivalent to saturated Fell bundles over inverse semigroups (see [7,23]). And Fell bundles over inverse semigroups are slightly more general than Fell bundles over étale groupoids (see [5,23]). Exel [14] did not identify the inverse semigroup actions for which the inclusion A ⊆ A ⊗ S is Cartan, and he did not study the uniqueness of such crossed product decompositions. Here we answer these questions and push Exel’s theory much further.

Our first main result (Theorem 4.3) characterises noncommutative Cartan inclusions in a number of different ways. We show that an inclusion A ⊆ A ⊗ S is Cartan if and only if the action of the inverse semigroup S on A is “closed” and “purely outer”. The notion of pure outerness is generalised from automorphisms to Hilbert bimodules in [21]. In Renault’s setting, pure outerness corresponds to the effectivity of the underlying groupoid. The condition that the action be closed corresponds to Hausdorffness of the groupoid. We show that an inclusion A ⊆ B has only trivial
virtual commutants if and only if, for each ideal \( I \) in \( A \), the relative commutant \( A' \cap \mathcal{M}(IBI) \) is contained in \( \mathcal{M}(I) \). And a regular \( \mathcal{C}^* \)-inclusion \( A \subseteq B \) with a faithful conditional expectation \( E: B \to A \) is Cartan if and only if, for each ideal \( I \) in \( A \), \( E|_{IBI} \) is the only conditional expectation \( IBI \to I \). When the primitive ideal space of \( A \) is Hausdorff, it is enough to assume that the expectation \( E \) itself is unique, without passing to ideals (see Proposition 7.1). Exel already shows that the conditional expectation \( B \to A \) for a noncommutative Cartan subalgebra is unique. This is a major step in his proof that \( B \) is isomorphic to a reduced inverse semigroup crossed product. Our proof of the converse direction uses ideas of Zarikian [27].

In passing, we extend Exel’s theory, and thus also Renault’s characterisation, to the non-separable case. This extension amounts to replacing frames in Hilbert bimodules by certain special approximate units used in [3]. The separability assumption in the previous works goes back to the prototype result by Feldman and Moore [17] in the von Neumann algebraic setting. These results have recently been extended to the non-separable case in [11] using inverse semigroup methods. In Renault’s approach separability plays only a role in the construction of the dual groupoid. Namely, Renault uses groupoids of germs, while we use transformation groupoids. The two constructions coincide if and only if the transformation groupoid is effective. And a second countable, Hausdorff, étale groupoid is effective if and only if it is topologically principal.

If \( A \) is commutative, then an inverse semigroup action on \( A \) gives rise to a groupoid, the dual groupoid \( \hat{A} \rtimes S \). And \( A \rtimes_{(\mathcal{r})} S \) is isomorphic to the (reduced) section \( \mathcal{C}^* \)-algebra of a Fell line bundle over \( \hat{A} \). Similar results are still available if the primitive ideal space of \( A \) is Hausdorff. Two extreme cases where this happens are when \( A \) is simple or commutative. If \( \hat{A} \) is Hausdorff, then a noncommutative Cartan inclusion \( A \subseteq B \) is isomorphic to the inclusion of \( A \) into the reduced section \( \mathcal{C}^* \)-algebra of a Fell bundle over an étale, Hausdorff groupoid with unit space \( \hat{A} \) (Theorem 7.2). In particular, this result explains how Renault’s theory is a special case of Exel’s theory. As in Renault’s theory, the groupoid and the Fell bundle are unique up to isomorphism.

This leads to our second main result (Theorem 5.6), which settles the uniqueness of the crossed product decomposition. This is more subtle than uniqueness of the underlying twisted groupoid for commutative Cartan subalgebras because many inverse semigroup actions on a space give the same transformation groupoid. Therefore, for an arbitrary action of a unital inverse semigroup \( S \) on a \( \mathcal{C}^* \)-algebra \( A \), we define a refined action that has the same crossed product and the same dual groupoid \( \hat{A} \rtimes S \). Two inverse semigroup actions that model the same noncommutative Cartan subalgebra have isomorphic refinements.

Our characterisations of noncommutative Cartan subalgebras have a number of important consequences when combined with our previous work on aperiodic inclusions in [21,23]. On the one hand, there are noncommutative Cartan \( \mathcal{C}^* \)-subalgebras that do not detect ideals in \( B \). There is a counterexample where \( A \) is an AF-algebra and \( B = A \rtimes_{\alpha} \mathbb{Z} \) for an automorphism \( \alpha \) of \( A \). On the other hand, if \( A \) is prime or contains an essential ideal of Type I, then every noncommutative Cartan inclusion \( A \subseteq B \) is aperiodic in the sense of [23]. Then it follows that \( A \) detects ideals in \( B \) and even supports positive elements in \( B^+ \); these properties are crucial to study the ideal structure and pure infiniteness of \( B \). We prove elegant characterisations of regular aperiodic \( \mathcal{C}^* \)-inclusions \( A \subseteq B \) with a faithful conditional expectation, either as crossed products by aperiodic, closed inverse semigroup actions on \( A \), or as regular inclusions \( A \subseteq B \) whose dual groupoids are effective and have closed space of units (Theorems 6.3 and 5.7).
This article is organised as follows. Section 2 recalls some general results on regular inclusions and inverse semigroup actions and their crossed products. We also define generalised Fourier coefficients for inverse semigroup crossed products, extending a similar definition for Fell bundles over groups. In Section 3, we characterise when a conditional expectation $E: B \to A$ on an $S$-graded $C^*$-algebra induces the canonical expectation on $A \rtimes S$. This gives a sufficient condition, still rather shallow, for an isomorphism $A \rtimes r, S \cong B$. Section 4 introduces noncommutative Cartan subalgebras and characterises them in different ways. The uniqueness of the inverse semigroup action induced by a noncommutative Cartan subalgebra is formulated in Section 5. This replaces the well known fact that Renault’s twisted groupoid is uniquely determined up to isomorphism. Section 6 compares noncommutative Cartan subalgebras with aperiodic inclusions. Section 7 specialises to the case when $A$ is Hausdorff, with Sections 7.1 and 7.2 treating simple and commutative Cartan subalgebras.

2. Preliminaries on inverse semigroup actions and crossed products

In this section, we briefly discuss some basic objects and facts needed later. See Sections 1 and 2 and the sources cited there for more details. The only new tool introduced here are the generalised Fourier coefficients in Proposition 2.1.

2.1. Inverse semigroup actions and regular inclusions. Throughout this article, $S$ stands for a unital inverse semigroup with unit $1 \in S$. It is equipped with the standard partial order, which is defined by $t \leq u$ for $t, u \in S$ if and only if $t = ut^*t$.

**Definition 2.1 ([22, Definition 6.15]).** An $S$-graded $C^*$-algebra is a $C^*$-algebra $B$ with closed subspaces $B_t \subseteq B$ for $t \in S$ such that $\sum_{t \in S} B_t$ is dense in $B$, $B_t B_u \subseteq B_{tu}$ and $B_t^* = B_{t^*}$ for all $t, u \in S$, and $B_t \subseteq B_u$ if $t \leq u$ in $S$. The grading is saturated if $B_t \cdot B_u = B_{tu}$ for all $t, u \in S$. We call $A := B_1 \subseteq B$ the unit fibre of the $S$-grading.

**Remark 2.2.** If the $S$-grading $(B_t)_{t \in S}$ is saturated, then the condition $B_t \subseteq B_u$ if $t \leq u$ in $S$ follows from the other conditions.

An $S$-grading on a $C^*$-algebra $B$ gives a family of Banach spaces $(B_t)_{t \in S}$ with conjugate-linear isometries $B_t \cong B_{t^*}$, $x \mapsto x^*$, multiplication maps $B_t \times B_u \to B_{tu}$ for all $t, u \in S$, and isometric embeddings $B_t \hookrightarrow B_u$ for all $t, u \in S$ with $t \leq u$. This is the data of a Fell bundle over $S$. It is a saturated Fell bundle if the multiplication maps are surjective. The definition of a Fell bundle imposes a long list of conditions on this data (see [14]). These conditions hold if and only if the data is realised by some $S$-graded $C^*$-algebra, which may be taken saturated if the Fell bundle is saturated. The definition of a Fell bundle simplifies in the saturated case. Namely, saturated Fell bundles are equivalent to the following $S$-actions by Hilbert bimodules:

**Definition 2.3 ([7]).** An action of $S$ on a $C^*$-algebra $A$ (by Hilbert bimodules) consists of Hilbert $A$-bimodules $E_t$ for $t \in S$ and Hilbert bimodule isomorphisms $\mu_{t,u}: E_t \otimes_A E_u \xrightarrow{\sim} E_{tu}$ for $t, u \in S$, such that

- (A1) for all $t, u, v \in S$, the following diagram commutes (associativity):
  $$
  \begin{array}{ccc}
  (E_t \otimes_A E_u) \otimes_A E_v & \xrightarrow{\mu_{t,u} \otimes_A \text{Id}_{E_v}} & E_{tu} \otimes_A E_v \\
  \uparrow \text{ass} & & \uparrow \text{ass} \\
  E_t \otimes_A (E_u \otimes_A E_v) & \xrightarrow{\text{Id}_{E_t} \otimes_A \mu_{u,v}} & E_t \otimes_A E_{uv} \\
  \end{array}
  $$

- (A2) $E_t$ is the identity Hilbert $A, A$-bimodule $A$;

- (A3) $\mu_{1,t}: E_t \otimes_A A \xrightarrow{\sim} E_t$ and $\mu_{t,1}: A \otimes_A E_t \xrightarrow{\sim} E_t$ for $t \in S$ are the maps defined by $\mu_{1,t}(a \otimes \xi) = a \cdot \xi$ and $\mu_{t,1}(\xi \otimes a) = \xi \cdot a$ for $a \in A, \xi \in E_t$. 

**Remark 2.4.** The condition (A3) implies that the multiplication maps in the $S$-grading $B_t \subseteq B$ induce inclusions $B_t \subseteq B_{tu}$ if $t \leq u$ in $S$.
Any $S$-action by Hilbert bimodules comes with canonical involutions $\mathcal{E}_t^* \to \mathcal{E}_{te}$, $x \mapsto x^*$, and inclusion maps $j_{u,t}: \mathcal{E}_t \to \mathcal{E}_u$ for $t \leq u$ that satisfy the conditions required for a saturated Fell bundle in $[14]$ (see $[7$, Theorem 4.8]). Thus $S$-actions by Hilbert bimodules are equivalent to saturated Fell bundles over $S$.

**Definition 2.4.** Let $A \subseteq B$ be a $C^*$-subalgebra. We call the elements of

$$N(A,B) := \{ b \in B : bAb^* \subseteq A, b^*Ab \subseteq A \}$$

normalisers of $A$ in $B$ (see $[19]$). We call the inclusion $A \subseteq B$ regular if it is non-degenerate and $N(A,B)$ generates $B$ as a $C^*$-algebra (see $[26]$).

**Proposition 2.5** ($[21$, Proposition 2.11], $[22$, Proposition 6.26]). The following are equivalent for a $C^*$-inclusion $A \subseteq B$:

1. $A$ is a regular subalgebra of $B$;
2. $A$ is the unit fibre for some inverse semigroup grading on $B$;
3. $A$ is the unit fibre for some saturated inverse semigroup grading on $B$.

If these equivalent conditions hold, then

$$S(A,B) := \{ M \subseteq N(A,B) : M \text{ is a closed linear subspace, } AM \subseteq M, MA \subseteq M \}$$

with the operations $M \cdot N := \text{span} \{ mn : m \in M, n \in N \}$ and $M^* := \{ m^* : m \in M \}$ is an inverse semigroup. And the subspaces $M \in S(A,B)$ form a saturated $S(A,B)$-grading on $B$. Let $S$ be a unital inverse semigroup and $(B_t)_{t \in S}$ an $S$-grading on $B$ with $A = B_1$. Then $B_t \in S(A,B)$ for all $t \in S$. The map $t \mapsto B_t$ is a homomorphism $S \to S(A,B)$ if and only if the $S$-grading on $B$ is saturated.

**Definition 2.6.** Elements of $S(A,B)$ are called slices.

### 2.2. Full and reduced crossed products for inverse semigroup actions

Let $\mathcal{E} = ((\mathcal{E}_t)_{t \in S}, (\mu_{t,u})_{t,u \in S})$ be an $S$-action on a $C^*$-algebra $A$. For any $t \in S$, let $r(\mathcal{E}_t)$ and $s(\mathcal{E}_t)$ be the ideals in $A$ generated by the left and right inner products of vectors in $\mathcal{E}_t$, respectively. Thus $\mathcal{E}_t$ is an $r(\mathcal{E}_t)$-$s(\mathcal{E}_t)$-imprimitivity bimodule, and $s(\mathcal{E}_t) = s(\mathcal{E}_{ts_1}) = r(\mathcal{E}_{ts_1}) = r(\mathcal{E}_{s_1})$. If $v \leq t$, then the inclusion map $j_{t,v}$ restricts to a Hilbert bimodule isomorphism $\mathcal{E}_v \cong r(\mathcal{E}_v) \cdot \mathcal{E}_t = s(\mathcal{E}_v)$. For $t,u \in S$ and $v \leq t, u$, this gives Hilbert bimodule isomorphisms $\vartheta_{v,u}^t: \mathcal{E}_v \cdot s(\mathcal{E}_v) \xrightarrow{j_{v,u}} \mathcal{E}_v \xrightarrow{j_{u,v}} \mathcal{E}_u \cdot s(\mathcal{E}_u)$. Let

$$I_{t,u} := \sum_{v \leq t,u} s(\mathcal{E}_v)$$

be the closed ideal generated by $s(\mathcal{E}_v)$ for $v \leq t,u$. This is contained in $s(\mathcal{E}_t) \cap s(\mathcal{E}_u)$, and the inclusion may be strict. There is a unique Hilbert bimodule isomorphism

$$\vartheta_{v,u}^t: \mathcal{E}_t \cdot I_{t,u} \cong \mathcal{E}_v \cdot I_{t,u}$$

that restricts to $\vartheta_{v,u}^t$ on $\mathcal{E}_t \cdot s(\mathcal{E}_v)$ for all $v \leq t,u$ by $[6$, Lemma 2.4].

The *algebraic crossed product* $A \rtimes_{\text{alg}} S$ is the quotient vector space of $\bigoplus_{t \in S} \mathcal{E}_t$ by the linear span of $\vartheta_{v,u}^t(\xi)\delta_u - \delta_t \xi$ for all $t,u \in S$ and $\xi \in \mathcal{E}_t \cdot I_{t,u}$. It is a *-algebra with multiplication and involution induced by the maps $\mu_{t,u}$ and the involutions $\mathcal{E}_t^* \to \mathcal{E}_{te}$. There is a maximal $C^*$-norm on $A \rtimes_{\text{alg}} S$.

**Definition 2.7.** The (full) crossed product $A \rtimes S$ of the action $\mathcal{E}$ is the maximal $C^*$-completement of the *-algebra $A \rtimes_{\text{alg}} S$.

**Remark 2.8.** The Hilbert $A$-bimodules $(\mathcal{E}_t)_{t \in S}$ embed naturally into $A \rtimes S$, and then they form a saturated $S$-grading of $A \rtimes S$. In particular, $A \subseteq A \rtimes S$ is a regular $C^*$-inclusion and the subspaces $(\mathcal{E}_t)_{t \in S}$ form an inverse subsemigroup of $S(A, A \rtimes S)$. For every $S$-graded $C^*$-algebra $B$ with grading $(\mathcal{E}_t)_{t \in S}$, there is a surjective *-homomorphism $A \rtimes S \to B$ which is the identity map on the fibres of the grading. This universal property determines $A \rtimes S$ uniquely up to isomorphism.
The reduced section C*-algebra of a Fell bundle over S is defined first in [14]. An equivalent definition appears in [6], where it is called the reduced crossed product A ⊙S of the action E. Here we define A ⊙S in another equivalent way as a quotient of A ⋊ S, using the canonical weak conditional expectation introduced in [6] and studied in [23]. A generalized expectation for a C*-inclusion A ⊆ B a consists of another C*-inclusion A ⊆ A' and a completely positive, contractive map E: B → A' that restricts to the identity map on A. Then E must be an A-bimodule map (see [23, Lemma 3.2]). If A is the bidual von Neumann algebra A'', then E is called a weak conditional expectation. We recall that a generalized expectation E: B → A' ⊆ A is faithful if E(b*b) = 0 for some b ∈ B implies b = 0. It is almost faithful if E((bc)*bc) = 0 for all c ∈ B and some b ∈ B implies b = 0 (equivalently, the largest closed two-sided ideal contained in ker E is zero, see [20] Proposition 2.2 or [23] Proposition 3.5)). There are natural examples of almost faithful conditional expectations that are not faithful (see [4, Example 4.6]). A generalized almost faithful expectation E is faithful if and only if it symmetric, that is, E(b*b) = 0 is equivalent to E(bb*) = 0 for all b ∈ B (see [23], Corollary 3.7).

**Proposition 2.9.** There is a canonical weak conditional expectation E: A ⋊ S → A" defined through the formula

\[
E(ξδ_t) = \text{s-lim}_i \vartheta_{1,i}(ξ \cdot u_i)
\]

for ξ ∈ E_t and t ∈ S, where (u_i) is an approximate unit for I_{1,t} and s-lim denotes the limit in the strict topology on M(I_{1,t}) ⊆ A". The largest ideal contained in ker E is

\[
N_E := \{ b ∈ A ⋊ S : E(b*b) = 0 \} = \{ b ∈ A ⋊ S : E(bb*) = 0 \},
\]

and E factors through to a faithful weak conditional expectation (A ⋊ S)/N_E → A".

**Proof.** The first part follows from [6, Lemma 4.5] (see also [23] Proposition 3.15). The second part follows from [23] Proposition 3.15 and Theorem 3.20.

**Definition 2.10.** The reduced crossed product of the inverse semigroup action E is the quotient A ⋊ S := (A ⋊ S)/N_E. So A ⋊ S is the unique quotient of A ⋊ S for which the weak conditional expectation in (2.3) factors through to a faithful weak conditional expectation E_r: A ⋊ S → A".

**Remark 2.11.** The canonical maps from A ⋊alg S to A ⋊ S and to A ⋊ S are injective by [6, Proposition 4.3]. Both A ⋊ S and A ⋊ S are naturally S-graded with the same Fell bundle (E_t)_{t ∈ S} over S.

**Example 2.12 (Fell bundles over groupoids).** Let A = (A_t)_{t ∈ H} be an upper-semicontinuous Fell bundle over an étale groupoid H with locally compact and Hausdorff unit space X. Let

\[
\text{Bis}(H) := \{ U ⊆ H : U \text{ is open and } s,r: U → X \text{ are injective} \}
\]

be the set of bisections of H. It is a unital inverse semigroup, with multiplication and involution inherited from H and with the unit X. If U ∈ Bis(H), let A_U be the space of C_0-sections of the restriction of (A_t)_{t ∈ H} to U. The spaces A_U for U ∈ Bis(H) form a Fell bundle over Bis(H) in a natural way (see [5,23]). In particular, they are Hilbert bimodules over the C_0(X)-algebra A := A_X corresponding to the bundle of C*-algebras (A_t)_{t ∈ X}. Let S ⊆ Bis(H) be a unital inverse subsemigroup S ⊆ Bis(H) which is wide in the sense that \( \bigcup S = H \) and \( U \cap V \) is a union of bisections in S for all \( U, V \in S \). Then the full and reduced C*-algebras for the Fell bundles (A_U)_{U ∈ S} and (A_t)_{t ∈ H} are naturally isomorphic (see [5, Theorem 2.13], [7] Corollary 5.6 and [6, Theorem 8.11]). The results in [6,7] are formulated for saturated Fell bundles, but they remain true for non-saturated ones, with the same proofs. Thus if (A_U)_{U ∈ S} is saturated, which is always the case when (A_t)_{t ∈ H} is saturated, then
A \times S \cong C^*(H, A) and A \rtimes S \cong C^*_\alpha(H, A). If \((A_U)_{U \in S}\) is not saturated, we extend it to a saturated Fell bundle over an inverse semigroup \(\tilde{S}\) consisting of all Hilbert bimodules of the form \(A_U J\), where \(U \in S\) and \(J\) is an ideal in \(A\). Then \(A \times \tilde{S} \cong C^*_\alpha(H, A)\) and \(A \rtimes \tilde{S} \cong C^*_\beta(H, A)\) (see \[23\] Propositions 7.6 and 7.9).

Fell line bundles over \(H\) correspond to “twists” of \(H\). The resulting section \(C^*\)-algebras are full and reduced twisted groupoid \(C^*\)-algebras.

2.3. Dual groupoids and closed actions. We briefly recall how inverse semigroup actions are related to étale groupoids. Let \(h_t\colon X_t \to X_t\) for \(t \in S\) be partial homeomorphisms forming an action of \(S\) on a topological space \(X\). The arrows of the transformation groupoid \(X \rtimes S\) are equivalence classes of pairs \((t, x)\) for \(x \in X_t \subseteq X\), where two pairs \((t, x)\) and \((t', x')\) are equivalent if \(x = x'\) and there is \(v \in S\) with \(v \leq t, t'\) and \(x \in X_v\). The range and source maps \(r, s\colon X \rtimes S \to X\) and the multiplication are defined by \(r((t, x)) := h_1(t)(x)\), \(s((t, x)) := x\), and \([t, h_u(x)]\) for \(t, u, x\). And the topology on \(X \rtimes S\) is such that \([t, x] \mapsto x\) is a homeomorphism from an open subset of \(X \rtimes S\) onto \(X_t\) for each \(t \in S\). Thus \(X \rtimes S\) is an étale topological groupoid.

Every étale topological groupoid \(G\) arises in this way. Namely, if \(S \subseteq \text{Bis}(H)\) is a unital and wide inverse subsemigroup, then \(H\) is naturally isomorphic to the transformation groupoid \(X \rtimes S\) of the associated action (see \[13\] Propositions 5.3 and 5.4 or \[23\] Proposition 2.2).

Remark 2.13. The transformation groupoid described above differs, in general, from the groupoid of germs considered in \[26\]. By \[26\] Proposition 3.2, a groupoid \(H\) is isomorphic to the groupoid of germs of its bisections if and only if it is effective.

Let \(A\) be a \(C^*\)-algebra with an action \(E\) of a unital inverse semigroup \(S\). Let \(\hat{A}\) and \(\hat{A} = \text{Prim}(A)\) be the space of irreducible representations and the primitive ideal space of \(A\), respectively. Open subsets in \(\hat{A}\) and in \(\hat{A}\) are in natural bijection with ideals in \(A\). The action of \(S\) on \(A\) induces actions \(\hat{E} = (\hat{E}_t)_{t \in S}\) and \(\hat{E} = (\hat{E}_t)_{t \in S}\) of \(S\) by partial homeomorphisms on \(\hat{A}\) and \(\hat{A}\), respectively (see \[7\] Lemma 6.12, \[23\] Section 2.3). The homeomorphisms \(\hat{E}_t\colon s(\hat{E})_t \cong r(\hat{E})_t\) and \(\hat{E}_t\colon s(\hat{E})_t \cong r(\hat{E})_t\) are given by Rieffel’s correspondence and induction of representations, respectively.

Definition 2.14 (\[22\][23]). We call \(\hat{E}\) and \(\hat{E}\) dual actions to the action \(E\) of \(S\) on \(A\). The transformation groupoids \(\hat{A} \rtimes S\) and \(\hat{A} \rtimes S\) are called dual groupoids of \(E\).

Example 2.15. Let \(A\) be a commutative \(C^*\)-algebra. So \(A \cong C_0(X)\) and \(\hat{A} \cong \hat{A} \cong X\). Consider an inverse semigroup action on \(A\). The corresponding saturated Fell bundles are studied in \[5\], where they are called semi-Abelian. As shown in \[5\], they are equivalent to Fell line bundles over étale groupoids and to twisted étale groupoids. The relevant groupoid is the dual groupoid of the action, \(H := X \rtimes S\). There is a unique twist \(\Sigma\) over \(H\) with an isomorphism \(C^*(H, \Sigma) \cong X \rtimes S\), which descends to an isomorphism \(C^*_\alpha(H, \Sigma) \cong A \rtimes S\) (see also Example \[2.12\]).

Definition 2.16. An inverse semigroup action \(E\) on a \(C^*\)-algebra \(A\) is called closed if the weak conditional expectation \(E\colon A \rtimes S \to A^0\) given by \[4, 3\] is \(A\)-valued, that is, it is a genuine conditional expectation \(E\colon A \rtimes S \to A \subseteq A \times S\).

Proposition 2.17. Let \(E\) be an action of \(S\) on \(A\). The following are equivalent:

1. the action \(E\) is closed, that is, \(E(A \rtimes S) \subseteq A\);
2. the subset of units \(\hat{A}\) is closed in the arrow space \(\hat{A} \rtimes S\);
3. the subset of units \(\hat{A}\) is closed in the arrow space \(\hat{A} \rtimes S\);
4. for each \(t \in S\), the ideal \(I_{1,t}\) defined in \[2.1\] is complemented in \(s(\hat{E}_t)\).
Let $t \in S$ and let $I_{1,t} := \{ a \in A : a \cdot I_{1,t} = 0 \}$ be the annihilator of $I_{1,t}$. If hold, then $E|_{E_t}$ is the projection onto the first summand in the decomposition
\begin{equation}
E_t = E_t \cdot I_{1,t} \oplus E_t \cdot I_{t,1} = I_{1,t} \oplus E_t \cdot I_{t,1}.
\end{equation}

**Proof.** Combine [6] Theorem 6.5 and Proposition 6.3 and [23] Proposition 3.18. \(\square\)

2.4. Generalised Fourier coefficients for closed actions. Let $G$ be a group and let $B$ be a C*-algebra with a topological $G$-grading $(B_t)_{t \in G}$. Besides the conditional expectation $E : B \twoheadrightarrow B_1$, there are projections $E_t : B \twoheadrightarrow B_t$ for all $t \in G$, with $E = E_1$ (see [15] Corollary 19.6)). We now define analogous maps for a closed action of an inverse semigroup $S$ by Hilbert bimodules. We cannot expect these to be defined on all of $A \rtimes S$ because this is impossible in very easy examples where $S = \{0,1\}$, $A = B = B_1$, and $B_0$ is an ideal in $A$ that is not complemented.

**Proposition 2.18.** Let $S$ be a unital inverse semigroup and let $(E_t,\mu_{t,u})_{t,u\in S}$ be a closed action of $S$ on a C*-algebra $A$ by Hilbert bimodules. For each $t \in S$ there is a unique contractive linear map
\[ E_t : r(E_t) \cdot (A \rtimes S) \twoheadrightarrow E_t \subseteq r(E_t) \cdot (A \rtimes S) \]
that satisfies $x^* E_t(y) = E(x^* y)$ for all $x \in E_t$, $y \in r(E_t) \cdot (A \rtimes S)$. The map $E_t$ is idempotent and $A$-bilinear. The projection $E_1$ for the unit $1 \in S$ is the canonical conditional expectation $E$. Let $x \in A \rtimes S$. Then $x = 0$ if and only if $E_t(ax) = 0$ for all $t \in S$ and all $a \in r(E_t)$.

**Proof.** The conditional expectation $E$ defines an $A$-valued inner product $\langle x | y \rangle := E(x^* y)$ on $A \rtimes S$. Let $\ell^2(S,A)$ denote the resulting Hilbert $A$-module completion of $A \rtimes S$. If $x \in E_t$, $y \in r(E_t) \cdot (A \rtimes S)$, then $\langle x | E_t(y) \rangle = x^* E_t(y)$ and $\langle x | y \rangle = E(x^* y)$. So the formula defining $E_t$ says exactly that $E_t$ should be the orthogonal projection onto $E_t$ in $B(r(E_t) \cdot \ell^2(S,A))$, restricted to $r(E_t) \cdot (A \rtimes S)$. To construct $E_t$, it remains to show that $E_t$ is complemented as a Hilbert $A$-submodule of $r(E_t) \cdot \ell^2(S,A)$.

The Hilbert submodules $r(E_t) \cdot E_u$ for $u \in S$ are linearly dense in $r(E_t) \cdot (A \rtimes S)$. The orthogonal decomposition $E_t \rtimes u = I_{1,t \rtimes u} \oplus E_t \rtimes u \cdot I_{1,t \rtimes u}$ in (2.4) implies an orthogonal decomposition
\begin{equation}
\langle \xi_t \cdot u | \eta_t \cdot u \rangle = \xi_t \cdot E_u \cdot \eta_t \cdot u = \xi_t \cdot E_u \cdot \eta_t \cdot u = E_t \cdot I_{1,\rtimes u} \oplus E_t \cdot I_{1,\rtimes u}.
\end{equation}

The summand $E_t \cdot I_{1,\rtimes u}$ is contained in $E_t$. In fact, it is equal to $E_t \cdot I_{1,u} = E_t \cap (r(E_t) \cdot E_u)$, where the intersection is taken in $A \rtimes S$. We claim that the other summand is orthogonal to $E_t$. Indeed, if $x_t, y_t \in E_t$, $a \in E_u \subseteq E_t \cdot I_{1,\rtimes u}$, then
\[ \langle y_t | z_{t,u} | x_t \rangle = E(z_{t,u}^* a) \cdot x_t = 0 \]
because $E$ vanishes on $E_u \cdot I_{1,\rtimes u}$ by (2.3). So the orthogonal projection to $E_t$ exists on $r(E_t) \cdot u$, and is the projection onto the first summand in the decomposition in (2.5). Since the submodules $r(E_t) \cdot E_u$ for $u \in S$ are linearly dense in $r(E_t) \cdot \ell^2(S,A)$, it follows that $E_t$ is complemented in $r(E_t) \cdot \ell^2(S,A)$.

The orthogonal projection to $E_t$ is contractive for the Hilbert $A$-module norm on $r(E_t) \cdot \ell^2(S,A)$, which is dominated by the C*-norm on $r(E_t) \cdot (A \rtimes S)$. Hence $E_t$ is contractive as asserted. It is idempotent and right $A$-linear by construction. It is left $A$-linear because the submodules in question are all invariant under left multiplication by $A$. The uniqueness of $E_t$ and the $A$-linearity of $E$ imply $E_1 = E$.

We show the last claim in the proposition. Let $x \in A \rtimes S$ satisfy $E_t(ax) = 0$ for all $t \in S$, $a \in r(E_t)$. Let $y \in E_t$. The Cohen–Hewitt Factorisation Theorem gives $a \in r(E_t)$ and $y' \in E_t$ with $y = ay'$. We compute $\langle x | y' \rangle = \langle a \cdot x | y' \rangle = E_t(a \cdot x) \cdot y' = 0$. Thus $x$ is orthogonal to $E_t$ for all $t \in S$. This implies that $x$ is mapped to 0 in $\ell^2(S,A)$ because the linear span of $E_t$ for $t \in S$ is dense. Then $E_t(x^* x) = \langle x | x \rangle = 0$, and this implies $x = 0$ because $E_t$ is faithful. \(\square\)
3. When the conditional expectation preserves the grading

Before we tackle the main issues in Exel’s theory of noncommutative Cartan subalgebras, we prove a more basic result. Let $S$ be a unital inverse semigroup and let $B$ be a $C^*$-algebra with an $S$-grading $(B_t)_{t \in S}$. Let $A := B_1$ be its unit fibre and view the grading as an action of $S$ on $A$ by Hilbert bimodules. The grading gives a representation of this action, which induces a surjective $*$-homomorphism $\pi: A \rtimes S \to B$. In addition, let $E: B \to A$ be an almost faithful conditional expectation. If $E \circ \pi: A \rtimes S \to A$ is equal to the canonical weak conditional expectation on the crossed product, $E_0: A \rtimes S \to A''$, then $\pi$ descends to a $*$-isomorphism $A \rtimes_t S \tilde{\to} B$ that intertwines the canonical conditional expectations. And then $E_0$ is $A$-valued, that is, the action has to be closed.

So we need conditions that are sufficient for $E \circ \pi = E_0$. We are going to show that a conditional expectation on $B$ satisfies $E \circ \pi = E_0$ if and only if the $S$-grading is “wide” and $E$ “preserves” it. This criterion will be used to characterise noncommutative Cartan subalgebras.

In this article, we assume a genuine conditional expectation $B \to A$. In general, however, an inverse semigroup crossed product only carries an $A^*$-valued conditional expectation. Other interesting targets for conditional expectations are the injective hull or the local multiplier algebra of $A$ (see \[23\]). We prove some technical lemmas for generalised conditional expectations taking values in an arbitrary $C^*$-algebra $\hat{A}$ containing $A$. This extra generality is not going to be used in this article. We hope, however, that it will prove useful for future generalisations of the theory to non-closed actions. The theory of “weak” Cartan subalgebras for non-Hausdorff groupoids is far more difficult than the usual theory of Cartan subalgebras and only in its infancy (see \[10,16\]).

**Definition 3.1** (compare \[14\] Definition 13.7). An $S$-grading $(B_t)_{t \in S}$ is wide if, for all $t \in S$, $\sum_{v \in S_1, t} B_v$ is dense in $B_t \cap A$.

**Example 3.2.** The tautological grading of $B$ over the slice inverse semigroup $S(A, B)$ is always wide. All gradings by groups are wide.

**Definition 3.3.** A conditional expectation $E: B \to A$ on a $C^*$-algebra $B$ with an $S$-grading $(B_t)_{t \in S}$ preserves the grading if $E(B_t) \subseteq B_t$ for all $t \in S$.

**Lemma 3.4.** Let $A$ be a $C^*$-algebra and $\tau \in M(A)$ a multiplier of $A$. If $\tau$ commutes with $A$, it commutes with all of $M(A)$.

**Proof.** Let $m \in M(A)$ and $x \in A$. Then $(\tau \cdot m) \cdot x = \tau(mx) = (mx)\tau = (m \cdot \tau) \cdot x$. This implies $\tau \cdot m = m \cdot \tau$. \hfill \Box

**Lemma 3.5.** Let $\hat{A} \supseteq A \subseteq B$ be $C^*$-inclusions and let $E: B \to \hat{A}$ be a generalised expectation. Let $M \subseteq S(A, B)$ be a slice and let $s(M) = M^*M \subseteq A$ be its source ideal. Then $\hat{A}_M := s(M) \cdot \hat{A} \cdot s(M)$ is a hereditary $C^*$-subalgebra in $\hat{A}$. There is a unique positive element $\tau \in M(\hat{A}_M)$ with $\|\tau\| \leq 1$ and $\tau \cdot m^*n = E(m^*E(n)) = m^*n \cdot \tau$ for all $m, n \in M$. And $\tau$ commutes with $A$ or, briefly, $\tau \in M(\hat{A}_M) \cap A'$.

**Proof.** For a genuine conditional expectation $B \to A$ on a separable $C^*$-algebra, this is \[14\] Lemma 11.5] for the slice $M^*$. We need a different proof. The construction in \[3\] Remark 1.9] gives an approximate unit $(\mu_\lambda)$ in $M^*M$ of the form $\mu_\lambda = \sum_{n=1}^\infty m_{\lambda,n}^*m_{\lambda,n}$, for some $n_\lambda \in \mathbb{N}$ and $m_{\lambda,1}, \ldots, m_{\lambda,n_\lambda} \in M$. Let $\tau_\lambda := \sum_{i=1}^{n_\lambda} E(m_{\lambda,i})^*E(m_{\lambda,i})$.\[
The 2-positivity of $E$ implies $E(m^*)E(m) \leq E(m^*m)$ for all $m \in B$ (see [9, Corollary 2.8]). Hence
\[ 0 \leq \tau_\lambda = \sum E(m^*_{\lambda,i})E(m_{\lambda,i}) \leq \sum E(m^*_{\lambda,i}m_{\lambda,i}) = \mu_\lambda \leq 1. \]
Let $m, n \in M$. Then $m^*n, m_{\lambda,i}m^* \in A$. As in the proof of [14, Lemma 11.5], the $A$-linearity of $E$ implies
\[ \tau_\lambda m^*n = \sum E(m^*_{\lambda,i})E(m_{\lambda,i})m^*n = \sum E(m^*_{\lambda,i})E(m_{\lambda,i}m^*)n = \sum E(m^*_{\lambda,i}m_{\lambda,i}m^*)E(n) = E(\mu_\lambda m^*)E(n). \]
The adjoint of this relation is $m^*n\tau_\lambda = E(m^*)E(n\mu_\lambda)$. Thus $\lim_{\lambda} \tau_\lambda \cdot m^*n \cdot x = E(m^*)E(n) \cdot x$ and $\lim_{\lambda} x \cdot m^*n \cdot \tau_\lambda = x \cdot E(m^*)E(n)$ in the norm topology for all $x \in \hat{A}_M$. Since the net $(\tau_\lambda)$ is bounded and the embedding $M^*M \rightarrow \hat{A}_M$ is non-degenerate, it follows that $(\tau_\lambda)$ converges strictly towards some $\tau \in \mathcal{M}(\hat{A}_M)$, which satisfies $\tau m^*n = E(m^*)E(n)$ for all $m, n \in M$. This determines the multiplier $\tau$ uniquely because $M^*M \cdot \hat{A}_M \cdot M^*M$ is dense in $\hat{A}_M$. The formula above shows that $\tau$ commutes with $M^*M$. Then it commutes with all multipliers of $M^*M$ by Lemma 3.4. Since the map $A \rightarrow \mathcal{M}(\hat{A}_M)$ factors through $\mathcal{M}(M^*M)$, it follows that $\tau$ commutes with $A$.

**Lemma 3.6.** Let $E: B \rightarrow A$ be a conditional expectation, and let $M \in S(A,B)$ be such that $E(M) \subseteq M$. Then $E(M) = M \cap A$, and this is a complemented ideal in $M^*M$.

**Proof.** The inclusion $E(M) \subseteq M$ implies $E(M) \subseteq M \cap A$. Since $M \cap A \subseteq E(M)$ always holds, this implies $E(M) = M \cap A$. Clearly, $M \cap A$ is an ideal in $M^*M$. Let $\tau \in \mathcal{M}(M^*M)$ be as in Lemma 3.5 and let $n, m \in M$. Then
\[ \tau^2 m^*n = \tau E(m^*)E(n) = E^2(m^*)E^2(n) = E(m^*)E(n) = \tau m^*n. \]
Thus $\tau$ is a projection, and $\tau M^*M = E(M)^*E(M) = M \cap A$. Hence $M \cap A$ is complemented in $M^*M$. \hfill \Box

**Proposition 3.7.** Let $B$ be a $C^*$-algebra with a saturated $S$-grading $(B_t)_{t \in S}$ with unit fibre $A := B_1$ and a conditional expectation $E: B \rightarrow A$. Let $\pi: A \rtimes S \rightarrow B$ be the canonical epimorphism and let $E_0: A \rtimes S \rightarrow A^0$ be the canonical weak conditional expectation. The following are equivalent:

1. the $S$-grading is wide and $E$ preserves it;
2. the $S$-grading is wide, $B_t = (B_t \cap A) \oplus B_t \cdot (B_t \cap A)^{\perp}$ and $E|B_t$ is the projection onto the first summand for each $t \in S$;
3. $E \circ \pi = E_0$, and so $(B_t)_{t \in S}$ is a closed action.

Therefore, $\pi$ descends to an isomorphism $A \rtimes_\pi S \cong B$ and $(B_t)_{t \in S}$ is a closed action if and only if the $S$-grading is wide and $B$ admits an almost faithful conditional expectation $E$ that preserves the grading. Then $E$ must be faithful.

**Proof.** Assume first that $E \circ \pi = E_0$. Then $E_0(A \rtimes S) \subseteq A$, that is, the $S$-action on $A$ is closed. And $\ker \pi \subseteq \ker E_0$. Since $\ker(A: A \rtimes S \rightarrow A \rtimes \Lambda)$ is the largest two-sided ideal contained in $\ker E_0$, it follows that $\ker \pi \subseteq \ker \Lambda$. Thus there is a surjective homomorphism $B \rightarrow A \rtimes_\pi S$ such that the composition
\[ A \rtimes_{\text{alg}} S \hookrightarrow A \rtimes S \twoheadrightarrow B \rightarrow A \rtimes_\pi S \]
coincides with $\Lambda$, and $B \rightarrow A \rtimes_\pi S$ is invertible if and only if $E$ is almost faithful (see [23, Lemma 3.10]). If $E$ is almost faithful, it is faithful because the isomorphism $B \cong A \rtimes_\pi S$ identifies it with $E_1$, which is faithful by Proposition 2.9.

The composite map $A \rtimes_{\text{alg}} S \rightarrow A \rtimes_\pi S$ is injective by [6, Proposition 4.3]. Hence so is the map $A \rtimes_{\text{alg}} S \rightarrow B$. If $t, u \in S$, then the intersection of $B_t$ and $B_u$ in $A \rtimes_{\text{alg}} S$
is the closed linear span of $B_v$ for $v \preceq t, u$. Therefore, the $S$-grading on $B$ is wide. So the ideal $I_{1,t} = \sum_{v \preceq t} B_v$ coincides with $B_t \cap A$. Since the action $(B_t)_{t \in S}$ is closed, the last part of Proposition 2.17 implies that $B_t = (B_t \cap A) \oplus (B_t \cdot (B_t \cap A)^\perp)$ and $E_0[B_t]$ is the projection onto the first summand. Since $E \circ \pi = E_0$ and $\pi$ is the identity map on $B_t$ the same holds for $E|_{B_t}$. This shows the last part of the assertion and that [3] implies [2].

It is trivial that [2] implies [1]. We show that [1] implies [3]. Thus assume that the $S$-grading is wide and that the conditional expectation $E : B \to A$ preserves it. Let $t \in S$. Then the ideal $I_{1,t}$ is equal to $B_t \cap A$. Since $B_t \in S(A,B)$ is a slice, Lemma 3.8 implies $E(B_t) = B_t \cap A$ and that the ideal $I_{1,t} = E(B_t)$ is complemented in $s(B_t) = B_t^* B_t$. Therefore, the action $(B_t)_{t \in S}$ is closed by Proposition 2.17. The map $E$ is the identity on $B_t \cdot I_{1,t} = B_t \cap A = I_{1,t} = E(B_t)$. It vanishes on $B_t \cdot I_{1,t}^\perp$ because

$$E(B_t I_{1,t}^\perp) = E(B_t I_{1,t}^\perp (B_t \cap A)) = E(B_t I_{1,t}^\perp (B_t \cap A)) = E(B_t I_{1,t}^\perp I_{1,t}) = 0.$$ 

So $E \circ \pi = E_0$ by the last part of Proposition 2.17.

Proposition 3.7 generalises [12] Theorem 3.3 in the case of group gradings.

**Corollary 3.8.** Let $A \subseteq B$ be a $C^*$-inclusion. There are an inverse semigroup $S$, a closed $S$-action on $A$, and an isomorphism $B \cong A \rtimes S$ if and only if $A$ is the unit fibre of an inverse semigroup grading on $B$ that is preserved by an almost faithful conditional expectation $E : B \to A$.

**Proof.** Compared to Proposition 3.7 only the assumption that the grading is wide and saturated is missing. We remedy this by refining the grading. Let $\tilde{S} := \{B_t J : t \in S, J \in \mathbb{I}(A)\}$. This is an inverse subsemigroup of $S(A,B)$. So $B$ is $\tilde{S}$-graded in a tautological way. If $t, u \in \tilde{S}$, then $B_t \cap B_u = B_t \cdot J$ for some ideal $J \in \mathbb{I}(A)$ because any Hilbert subbimodule of $B_t$ has this form. So $B_t \cap B_u \in \tilde{S}$ and the $\tilde{S}$-grading on $B$ is wide. It is saturated because $\tilde{S} \subseteq S(A,B)$. It is still preserved by $E$ because $E(B_t \cdot J) = E(B_t) \cdot J \subseteq B_t \cdot J \subseteq B_t$. Now Proposition 3.7 finishes the proof.

**Corollary 3.9.** Let $A$ be the unit fibre of an inverse semigroup grading $(B_t)_{t \in S}$ on $B$. At most one conditional expectation $E : B \to A$ preserves this grading.

**Proof.** Let $\tilde{S} := \{B_t J : t \in S, J \in \mathbb{I}(A)\}$ be the wide saturated grading of $B$ as in the proof of Corollary 3.8. Let $\pi : A \rtimes \tilde{S} \to B$ be the canonical epimorphism and let $E_0 : A \rtimes \tilde{S} \to A$ be the canonical conditional expectation. Then $E \circ \pi = E_0$ by Proposition 3.7 and this determines $E$ because $\pi$ is surjective.

**Example 3.10.** An $S$-action need not be closed if there is a conditional expectation $P : A \rtimes S \to A$ that does not preserve the canonical $S$-grading of $A \rtimes S$. By [6] Proposition 5.3, there is a non-Hausdorff étale groupoid $H$ with a compact Hausdorff object space $X$ and a faithful conditional expectation from $B := C^*_r(H)$ onto $A := C(X)$. Let $S$ be the inverse semigroup of bisections of $H$ and let $S$ act on $C(X)$ in the canonical way. Then $C^*_r(H) \cong C(X) \rtimes S$. This $S$-action on $C(X)$ is not closed, although there is a faithful conditional expectation $E : C(X) \rtimes S \to C(X)$.

4. **Exel’s noncommutative Cartan subalgebras**

Noncommutative Cartan subalgebras were introduced in [14], as a generalisation of the (commutative) Cartan subalgebras studied by Renault [26]. Note that Exel considers a subalgebra $B \subseteq A$, so the roles of $A$ and $B$ are exchanged in [14]. And Exel assumes throughout that $B$ is separable.
Definition 4.1. A virtual commutant of $A$ in $B$ is an $A$-bimodule map $J \to B$ defined on an ideal $J \in \mathbb{I}(A)$. An inclusion $A \subseteq B$ is a noncommutative Cartan subalgebra if it is regular, any virtual commutant has range in $A$, and there is an almost faithful conditional expectation $E : B \to A$.

These notions are those in [14 Definition 9.2] and [14 Definition 12.1] modified as in the footnote, replacing a faithful by an almost faithful conditional expectation.

Definition 4.2 ([21]). A Hilbert $A$-bimodule $H$ over a $C^*$-algebra $A$ is purely outer if there is no non-zero ideal $J \in \mathbb{I}(A)$ with $H \cdot J \cong J$ as a Hilbert bimodule. An action $E$ of an inverse semigroup on a $C^*$-algebra $A$ is purely outer if the Hilbert $A$-bimodules $E_t \cdot I_{1,t}^I$ are purely outer for all $t \in S$, where $I_{1,t}^I = \{a \in A : a \cdot I_{1,t} = 0\}$.

Theorem 4.3. Let $A \subseteq B$ be a regular $C^*$-subalgebra with an almost faithful conditional expectation $E : B \to A$. The following conditions are equivalent:

1. there is at most one conditional expectation $IBI \to I$ for all $I \in \mathbb{I}(A)$;
2. there is a unique almost faithful conditional expectation $IBI \to I$ for all $I \in \mathbb{I}(A)$, namely, the restriction of $E$;
3. any virtual commutant in $B$ has range in $A$, that is, $A \subseteq B$ is a noncommutative Cartan subalgebra;
4. $I' \cap \mathcal{M}(IBI) = ZM(I)$ for all $I \in \mathbb{I}(A)$;
5. if a slice $M \subseteq B$ is isomorphic to an ideal $I$ in $A$ as a Hilbert $B$-bimodule, then already $M = I$ as a subset of $B$;
6. if a slice $M \subseteq B$ satisfies $M \otimes_A M \cong M$ as a Hilbert $B$-bimodule, then $M \cdot M = M$;
7. for any unital inverse semigroup $S$ and any saturated, wide grading $(B_t)_{t \in S}$ on $B$ with unit fibre $A$, the action $(B_t)_{t \in S}$ of $S$ on $A$ is closed and purely outer, and the canonical $*-\text{homomorphism} \pi : A \rtimes_S S \to B$ descends to an isomorphism $A \rtimes_x S \cong B$;
8. there are a unital inverse semigroup $S$, a closed and purely outer action of $S$ on $A$, and an isomorphism $A \rtimes_x S \cong B$, mapping $A$ identically to itself.

In particular, if the above equivalent conditions hold, then $E : B \to A$ is the only conditional expectation onto $A$, and it is faithful, not just almost faithful.

Condition [8] without “purely outer” is the conclusion of the main theorem in [14]. So the equivalence of [3] and [8] contains the main result of [14] and, together with [7], characterises precisely to which inverse semigroup actions Exel’s theory applies. Conditions [5] and [6] are slightly different ways of saying that the canonical action of $S(A,B)$ on $A$ is purely outer. This gives a candidate for the inverse semigroup action in [8]. Conditions [1] and [2] concern the uniqueness of conditional expectations studied by Zarikian [27] — but for all inclusions $I \subseteq IBI$ for $I \in \mathbb{I}(A)$ and not just for the inclusion $A \subseteq B$. Condition [4] is equivalent to $I' \cap \mathcal{M}(IBI) \subseteq \mathcal{M}(I)$ by Lemma 3.4. If $A$ is simple, this says that any multiplier of $B$ that commutes with $A$ is already a scalar multiple of 1. If $A$ is commutative, then it is equivalent to $A$ being maximal Abelian in $B$ (compare Corollary 7.6).

The proof of the theorem requires some preparation. Our first goal is Proposition 4.5, which says that [3],[6] in Theorem 4.3 are equivalent for any non-degenerate $C^*$-inclusion, without assuming regularity or a conditional expectation.

Lemma 4.4. Let $A \subseteq B$ be a non-degenerate inclusion and $I \in \mathbb{I}(A)$. If $\tau \in \mathcal{M}(IBI) \cap I'$, then the map $I \to B, x \mapsto \tau \cdot x$, is a virtual commutant. Conversely, any virtual commutant $\varphi : I \to B$ is of this form for a unique $\tau \in \mathcal{M}(IBI) \cap I'$. And $\varphi(I) \subseteq I$ if and only if $\tau \in ZM(I)$.

Proof. If $\tau \in \mathcal{M}(IBI) \cap I'$, then $I \to B, x \mapsto \tau \cdot x = x \cdot \tau$, is a virtual commutant by direct computation. Conversely, let $\varphi : I \to B$ be a virtual commutant. Then
\( \varphi(I) \subseteq IBI \) because \( \varphi \) is \( I \)-bilinear. Represent \( B \) faithfully in \( \mathcal{B}(\mathcal{H}) \) for a Hilbert space \( \mathcal{H} \). There is \( \tau \in \mathcal{B}(\mathcal{H}) \) with \( \varphi(x) = \tau \cdot x = x \cdot \tau \) for all \( x \in I \) by \( [14] \) Theorem 9.5. And \( \tau(I) \subseteq IBI \) by \( [14] \) Proposition 9.3. Hence \( \tau \cdot IBI = \varphi(I)IBI \subseteq IBIBI \subseteq IBI \) and \( IBI \cdot \tau = IB \varphi(I) \subseteq IBIBI \subseteq IBI \). So \( \tau \in \mathcal{M}(IBI) \), and \( \varphi \) has the asserted form. Let \( \tau_1, \tau_2 \in \mathcal{M}(IBI) \) satisfy \( \tau_1 \cdot x = \tau_2 \cdot x \) for all \( x \in I \). Then \( \tau_1 : xy = \tau_2 : xy \) for all \( x \in I, y \in BI \), so that \( \tau_1 \) and \( \tau_2 \) as multipliers of \( IBI \). Thus \( \tau \in \mathcal{M}(IBI) \) above is unique. Clearly, \( \tau \in ZM(I) \) if and only if \( \varphi(I) \subseteq I \). This is equivalent to \( \varphi(I) \subseteq A \) because \( \varphi \) is \( A \)-bilinear and \( I^2 = I \).

**Proposition 4.5.** Let \( A \subseteq B \) be a non-degenerate inclusion and \( I \in \mathcal{I}(A) \). The following are equivalent:

1. any virtual commutant \( \varphi : I \to B \) has range in \( A \);
2. \( \mathcal{M}(IBI) \cap I' = ZM(I) \);
3. if a slice \( M \subseteq B \) is isomorphic to \( I \) as a Hilbert \( A \)-bimodule, then \( M = I \);
4. if a slice \( M \subseteq B \) satisfies \( M^*M = I \) and \( M \otimes_A M \cong M \) as a Hilbert \( A \)-bimodule, then already \( M \cdot M = M \).

**Proof.** Lemma 4.4 shows that (1) and (2) are equivalent. If \( M \subseteq B \) is a slice as in (3), then the isomorphism \( I \cong M \) is a virtual commutant. So (1) implies (3). We are going to show that not (2) implies not (3). This will complete the proof that (1), (2), and (3) are equivalent. A unital \( C^* \)-algebra is spanned by its unitary elements. Therefore, \( \mathcal{M}(IBI) \cap I' \neq ZM(I) \) if and only if there is a unitary \( \tau \in \mathcal{M}(IBI) \cap I' \) with \( \tau \not\in ZM(I) \). Then \( (\tau x)^*(\tau y) = x^*y \) and \( (\tau x)(\tau y)^* = (\tau)(\tau y)^* = xy^* \) for all \( x, y \in I \), and \( a \cdot (\tau x) = ax \tau = \tau(ax) \) and \( (\tau x) \cdot a = \tau(xa) \) for all \( x \in I, a \in A \). So \( \tau \cdot I \subseteq B \) is a slice that is isomorphic to \( I \) as a Hilbert \( A \)-bimodule, but not contained in \( A \).

We prove \((3) \iff (4)\). A slice \( M \) contained in \( A \) is a closed \( A \)-bimodule, that is, a closed two-sided ideal in \( A \). The slices form an inverse semigroup (compare Proposition 2.5). If a slice \( M \) satisfies \( M \cdot M = M \), then \( M \cdot M \cdot M = M \) implies \( M = M^* \). Hence \( M = M \cdot M = M \cdot M^* \subseteq A \). So a slice \( M \) is contained in \( A \) if and only if \( M \cdot M = M \). A Hilbert \( A \)-bimodule \( M \) satisfies \( M \otimes_A M \cong M \) if and only if \( M \cong I \) for some ideal \( I \in \mathcal{I}(A) \) (see [7] Proposition 4.6). Therefore, the two conditions whose equivalence is required in (3) are equivalent to the two conditions whose equivalence is required in (4).

Now we turn to preparatory results about conditional expectations.

**Lemma 4.6.** Let \( \hat{A} \supseteq A \subseteq B \) be non-degenerate \( C^* \)-inclusions and let \( E : B \to \hat{A} \) be a generalised expectation. Then \( E \) is non-degenerate, and it extends uniquely to a strictly continuous generalised expectation \( \hat{E} : \mathcal{M}(B) \to \mathcal{M}(\hat{A}) \) for the induced inclusion \( \mathcal{M}(A) \subseteq \mathcal{M}(B) \). If one of the maps \( E \) and \( \hat{E} \) is faithful, symmetric, or almost faithful, then so is the other.

**Proof.** The non-degeneracy of the inclusions of \( A \) means that an approximate unit for \( \hat{A} \) is also one for \( A \) and \( B \). Since \( E|_A \) is the identity, it maps an approximate unit for \( B \) to one for \( A \). This is the notion of non-degeneracy used in [24] Corollary 5.7 to extend \( E \) to \( \mathcal{M}(B) \). More precisely, it is shown in [24] that the extension \( \hat{E} \) is strictly continuous on the unit ball. Using the Cohen–Hewitt Factorisation Theorem in this argument allows to prove strict continuity everywhere.

Suppose that \( \hat{E} \) is almost faithful. Let \( a \in B \) be such that \( E((ab)^*ab) = 0 \) for all \( b \in B \) and let \( m \in \mathcal{M}(B) \). Let \( (\mu_\lambda) \) be an approximate unit for \( A \). We compute

\[
E((am)^*am) = \lim \mu_\lambda E((am)^*am)\mu_\lambda = \lim E((a(m\mu_\lambda))^*a(m\mu_\lambda)) = 0.
\]

This implies \( a = 0 \). Therefore, \( \hat{E} \) is almost faithful. Conversely, assume \( E \) to be almost faithful. Let \( m \in \mathcal{M}(B) \) satisfy \( \hat{E}((mn)^*mn) = 0 \) for all \( n \in \mathcal{M}(B) \). If
a, b \in A, then \( E(((ma)b)\ast (ma)b)) = \tilde{E}((m(ab))\ast (m(ab))) = 0 \). Thus \( m \cdot a = 0 \) because \( E \) is almost faithful. This implies \( m = 0 \). Hence \( E \) is almost faithful.

Clearly, \( E \) is symmetric if \( \tilde{E} \) is. Conversely, assume \( \tilde{E} \) to be symmetric. Let \((\mu_\lambda)\) be an approximate unit for \( B \). Then \((\mu_\lambda m)\) for \( m \in \mathcal{M}(\tilde{B}) \) converges strictly to \( m \). Assume \( \tilde{E}(m^*m) = 0 \). Then \( E((\mu_\lambda m)^*\mu_\lambda m) = 0 \) because \( E(m^*\mu_\lambda m) \leq \tilde{E}(m^*m) = 0 \). Hence \( E(\mu_\lambda mm^*\mu_\lambda) = 0 \) because \( E \) is symmetric. Since \( \tilde{E} \) is strictly continuous, this implies \( \tilde{E}(mm^*) = \text{s-lim} E(\mu_\lambda mm^*)\mu_\lambda = 0 \). Thus \( \tilde{E} \) is symmetric.

A conditional expectation is faithful if and only if it is almost faithful and symmetric (see [23, Corollary 3.7]). Hence \( E \) is faithful if and only if \( \tilde{E} \) is. \( \square \)

**Lemma 4.7** (compare Proposition 3.1). Let \( \hat{A} \supseteq A \subseteq B \) be non-degenerate \( C^* \)-inclusions and let \( E: B \to A \) be a generalised expectation. Assume that \( E \) is a unique generalised expectation \( B \to \hat{A} \), or that \( E \) is a unique faithful or a unique almost faithful generalised expectation \( B \to \hat{A} \). Let \( E: \mathcal{M}(B) \to \mathcal{M}(\hat{A}) \) be the unique extension of \( E \). Then \( A' \cap \mathcal{M}(B) \) is contained in the multiplicative domain of \( \tilde{E} \) and \( \tilde{E} \) restricts to a \( * \)-homomorphism \( A' \cap \mathcal{M}(B) \to A' \cap \mathcal{M}(\hat{A}) \).

**Proof.** Let \( x \in A' \cap \mathcal{M}(B) \) satisfy \( x^* = x \) and \( \|x\| < 1 \). Then \( \|\tilde{E}(x)\| < 1 \). So we may define a completely positive map \( E_x: \mathcal{M}(B) \to \mathcal{M}(\hat{A}) \) by

\[
E_x(t) := (1 - \tilde{E}(x))^{-1/2} \cdot \tilde{E}((1 - x)^{1/2}t(1 - x)^{1/2}) \cdot (1 - \tilde{E}(x))^{-1/2}.
\]

Let \( a \in A \). Then \( ax = xa \) and hence \( a\tilde{E}(x) = \tilde{E}(ax) = \tilde{E}(xa) = \tilde{E}(x)a \), and the same holds for \((1 - x)^{\pm 1/2} \) instead of \( x \). This implies \( E_x(a) = a \). Then it follows that \( E_x \) is contractive and a generalised expectation. It maps \( B \to \hat{A} \) and is strictly continuous as well. So \( E_x \) is the unique strictly continuous extension of its restriction to \( E_x|B: B \to \hat{A} \) (see Lemma 4.6). Since \((1 - x)^{1/2} \) is invertible, \( E_x \) is faithful if \( \tilde{E} \) is, and \( E_x \) is almost faithful if \( E \) is. Thus \( E = E_x|B \) by the uniqueness assumption in the lemma. This implies \( \tilde{E} = E_x \) and then \( \tilde{E}(x) = E_x(x) \). So

\[
\tilde{E}(x) - E(x)^2 = (1 - E(x))^{1/2}E_x(x)(1 - E(x))^{1/2}
\]

\[
= (1 - \tilde{E}(x))^{1/2}E_x(x)(1 - \tilde{E}(x))^{1/2} = \tilde{E}((1 - x)^{1/2}x(1 - x)^{1/2}) = \tilde{E}(x - x^2).
\]

This is equivalent to \( \tilde{E}(x^*x)\tilde{E}(x) = \tilde{E}(x^*x) \). Hence \( x \) is in the multiplicative domain of \( \tilde{E} \) (see [9, Theorem 3.1]). It follows that the multiplicative domain contains all self-adjoint elements of \( A' \cap \mathcal{M}(B) \). Thus \( \tilde{E} \) is multiplicative on \( A' \cap \mathcal{M}(B) \). Being \( A \)-bilinear, it must map \( A' \cap \mathcal{M}(B) \) into \( A' \cap \mathcal{M}(\hat{A}) \). \( \square \)

**Proposition 4.8.** Let \( A \subseteq B \) be non-degenerate and assume \( E: B \to A \subseteq B \) is a unique (almost) faithful conditional expectation. Then \( A' \cap \mathcal{M}(B) = Z(\mathcal{M}(A)) \).

**Proof.** Extend \( E \) to multipliers as in Lemma 4.6. Lemma 3.4 implies \( Z(\mathcal{M}(A)) = A' \cap \mathcal{M}(A) \subseteq A' \cap \mathcal{M}(B) \). And \( \tilde{E} \) restricts to a \( * \)-homomorphism \( e: A' \cap \mathcal{M}(B) \to Z(\mathcal{M}(A)) \) by Lemma 4.7. We claim that \( e \) is injective when \( \tilde{E} \) is almost faithful. Otherwise, there is \( a \in A' \cap \mathcal{M}(B) \) with \( a \geq 0 \), \( a \neq 0 \), and \( e(a) = \tilde{E}(a) = 0 \). Then \( \tilde{E}(b^*ab) = 0 \) for all \( b \in B \) because \( a = a^* \) is in the multiplicative domain of \( \tilde{E} \). Since \( \tilde{E} \) is almost faithful, this implies \( a^{1/2} = 0 \) and then \( a = 0 \). So \( e \) is injective. Since \( e^2 = e \), it follows that \( e \) is invertible. Hence \( Z(\mathcal{M}(\hat{A})) = A' \cap \mathcal{M}(B) \). \( \square \)

**Lemma 4.9.** Let \( P: B \to A \) be a conditional expectation and let \( M \subseteq B \) be a slice. Let \( \tau \in \mathcal{M}(\mathcal{M}(M)) \) be as in Lemma 3.5. Then there is a unique isometric virtual commutant \( \varphi \) that maps the closure of \( P(M) \) into \( M \) and that satisfies \( \varphi(P(m)) = m \cdot \tau^{1/2} \) for all \( m \in M \).

**Proof.** Up to a change in conventions, this is [14, Proposition 11.14]. The proof in [14] is literally the same. The closure of \( P(M) \) is a closed ideal. Lemma 3.5
gives \( \tau \in M(M^* M) \) with \( P(m)^* P(n) = \tau^{1/2} m^* n \tau^{1/2} \) for all \( m, n \in M \). So the map \( P(m) \to m \cdot \tau^{1/2} \) is isometric. Then it extends to the closure. The extension is \( A \)-bilinear because \( \tau \) commutes with \( M^* M \) and hence with all multipliers of \( M^* M \) by Lemma 3.4. The values of \( \varphi \) belong to \( M \) because \( M \cdot M(M^* M) \subseteq M \). \( \square 

**Lemma 4.10.** Let \( P: B \to A \) be a conditional expectation. If any virtual commutant in \( B \) has range in \( A \), then \( P \) preserves slices, that is, \( P(M) \subseteq M \) for any slice \( M \subseteq B \).

**Proof.** The proof follows [14]. Lemma 3.5 provides a central, positive multiplier \( \tau \in M(M^* M) \) with \( \|\tau\| \leq 1 \) and \( \tau m^* n = P(m^*) P(n) = m^* n \tau \) for all \( m, n \in M \). Let \( I \) be the closure of \( P(M) \). Lemma 4.9 gives an isometric virtual commutant \( \varphi: I \to B \) with \( \varphi(P(m)) := m \cdot \tau^{1/2} \) for all \( m \in M \). By assumption, \( M \cdot \tau^{1/2} \subseteq I \). The ideal \( I \) is contained in \( M^* M = s(M) \) because an approximate unit for \( s(M) \) is also one for \( I \). Hence \( \tau \) restricts to a multiplier of \( I \). So \( I \cdot \tau^{1/2} \subseteq I \). Then \( M \cdot \tau \subseteq I \cdot \tau^{1/2} \subseteq I \subseteq A \). So \( \tau M^* M = (\tau^*)^* M \subseteq AM = M \). Then \( P(m)^* P(n) = P(m^*) P(n) = \tau m^* n \) for all \( m, n \in M \). Since \( I \) is the closure of \( P(M) \) and a closed two-sided ideal, this implies \( I = P^* I \subseteq M \). That is, \( P(M) \subseteq M \). \( \square 

**Lemma 4.11.** Let \( \mathcal{H} \) be a Hilbert \( A \)-bimodule over a \( C^* \)-algebra \( A \) and let \( J \in \mathcal{I}(A) \) be an ideal. The following conditions are equivalent and imply that \( J \) is \( \mathcal{H} \)-invariant:

1. some subbimodule of \( J \mathcal{H} J \) is isomorphic to the trivial Hilbert \( A \)-bimodule \( J \);
2. \( J \mathcal{H} J \cong J \) as Hilbert \( A \)-bimodules;
3. \( \mathcal{H} J \cong J \) as Hilbert \( A \)-bimodules.

**Proof.** By the Rieffel correspondence, closed subbimodules in \( J \mathcal{H} J \) are naturally in bijection with ideals in \( r(\mathcal{H}) \cap J \) and with ideals in \( s(\mathcal{H}) \cap J \). So the only subbimodule that can be isomorphic to \( J \) is \( J \mathcal{H} J \) itself, and this requires \( J \subseteq s(\mathcal{H}) \cap r(\mathcal{H}) \). Thus \( (1) \) and \( (2) \) are equivalent. If \( \mathcal{H} J \cong J \), then \( J \mathcal{H} J = J^2 = J \) as well. Hence \( (3) \) implies \( (2) \). Conversely, if \( J \mathcal{H} J \cong J \), then \( J \) is a closed subbimodule of the Hilbert bimodule \( \mathcal{H} J \). There is no room for it to be a proper subbimodule because \( s(\mathcal{H} J) \subseteq J \). So \( \mathcal{H} J = J \mathcal{H} J \cong J \). Similarly, \( J \mathcal{H} J \cong J \) implies \( \mathcal{H} J \cong J \mathcal{H} J \), which is the same as \( \mathcal{H} J \). Thus \( J \) is \( \mathcal{H} \)-invariant. \( \square 

**Proof of Theorem 4.3.** It is clear that \( (1) \) implies \( (2) \). Proposition 4.8 applied to the non-degenerate inclusions \( I \to IBI \) for \( I \in \mathcal{I}(A) \) shows that \( (2) \) implies \( (4) \). Conditions \( (3) \) and \( (6) \) are equivalent by Proposition 4.5.

Next we prove that \( (3) \) implies \( (1) \). Then the conditions \( (1) \), \( (5) \), \( (6) \) are equivalent. Let \( I \in \mathcal{I}(A) \) and let \( P: IBI \to I \) be any conditional expectation. The inclusion \( I \subseteq IBI \) is regular as well, and \( (3) \) and Lemma 4.10 imply that \( P \) preserves slices. Thus there is at most one conditional expectation \( IBI \to I \) by Corollary 3.9.

Now assume the equivalent conditions \( (1) \), \( (6) \). Let \( (B_t)_{t \in S} \) be any wide, saturated \( S \)-grading on \( B \). If a Hilbert subbimodule of \( B_t \cdot I_{t,t}^+ \) is isomorphic as an \( A \)-bimodule to an ideal in \( A \), then it is already contained in \( A \) by \( (5) \). Then it is contained in \( B_t \cap A = I_{t,t}^+ \), which is orthogonal to \( B_t \cdot I_{t,t}^- \). Therefore, the action on \( A \) defined by the \( S \)-grading on \( B \) is purely outer. Lemma 4.10 already shows that the given conditional expectation \( E: B \to A \) preserves all slices and hence also the grading \( (B_t)_{t \in S} \). Then Proposition 3.7 implies that the canonical surjection \( A \rtimes S \to B \) descends to a \( * \)-isomorphism \( A \rtimes S \to B \). Thus \( (1) \), \( (6) \) imply \( (7) \). It is trivial that \( (7) \) implies \( (8) \).

The proof of the theorem will be finished by showing that \( (8) \) implies \( (3) \). So let \( S \) be a unital inverse semigroup and let \( E \) be a closed and purely outer \( S \)-action on \( A \) such that \( A \rtimes S \to B \). We may identify \( B \) with \( A \rtimes S \) and assume that the chosen conditional expectation \( E \) is the canonical one on \( A \rtimes S \). Let \( \varphi: I \to A \rtimes S \) be a virtual commutant. We must show that \( \varphi(I) \subseteq A \). Let \( E^\perp = \text{Id} - E \) be the
projection to $\ker E \subseteq B$. The claim $\varphi(I) \subseteq A$ is equivalent to $E^\perp \circ \varphi = 0$. Since $E^\perp \circ \varphi$ is a virtual commutant as well, it suffices to prove $\varphi = 0$ for any virtual commutant $\varphi: I \to \ker E \subseteq B$.

Let $t \in S$. We will use the orthogonal projections $E_t: r(B_t) \cdot (A \rtimes_t S) \to B_t$ for $t \in S$ defined in Proposition 2.18 here $r(B_t) = B_t B_t^\ast$. The map

$$\varphi_t := E_t \circ \varphi|_{r(B_t) \cap I}: r(B_t) \cap I \to B_t \subseteq A \rtimes_t S$$

is a virtual commutant as well. Assume that $\varphi_t \neq 0$ for some $t \in S$. Let $J := r(B_t) \cap I$. Lemma 4.3 provides $\tau_t \in \mathcal{M}(J B_J) \cap J'$ with $\varphi_t(x) = \tau_t \cdot x$ for all $x \in J$. Now $\tau_t \cdot J = \varphi_t(J) \subseteq B_t$ implies $\tau_t^\ast \tau_t \cdot J = J \cdot \tau_t^\ast \tau_t \cdot J \subseteq B_t^\ast B_t \subseteq A$. Thus $\tau_t^\ast \tau_t \in \mathcal{M}(J)$.

Since $\tau_t$ commutes with $J$, the multiplier $\tau_t^\ast \tau_t$ is central by Lemma 3.2. It is positive and non-zero because $\varphi_t \neq 0$ by assumption. So there is $\varepsilon$ with $0 < \varepsilon < ||\tau_t^\ast \tau_t||$. Let $K := (\tau_t^\ast \tau_t - \varepsilon)_+ \cdot J$. This is a two-sided ideal because $\tau_t^\ast \tau_t$ is central, and it is non-zero because $(\tau_t^\ast \tau_t - \varepsilon)_+ \neq 0$. When we restrict $\tau_t^\ast \tau_t$ to a central multiplier of $K$, it becomes strictly positive, hence invertible. So the formula

$$\varphi_t^K(x) := \varphi_t((\tau_t^\ast \tau_t)^{-1/2} x) = \tau_t \cdot (\tau_t^\ast \tau_t)^{-1/2} x$$

well defines a virtual commutant $\varphi_t^K: K \to B_t \subseteq A \rtimes_t S$. It preserves the $K$-valued inner products: $\varphi_t^K(x) \ast \varphi_t^K(y) = x \ast y$, for $x, y \in K$. In particular, $\varphi_t^K$ is isometric.

The range of $\varphi_t$ is contained in ker $E$, $E_t$ is $A$-linear and ker $E \cap B_t = B_t I_{t,t}^\bot$, by (2.4).

Then it follows that the range of $\varphi_t$ and therefore also of $\varphi_t^K$ is contained in $B_t I_{t,t}^\bot$. Thus $\varphi_t^K(K)$ is a Hilbert submodule of $B_t I_{t,t}^\bot \cdot K$ which is isomorphic to $K$. This is equivalent to $B_t I_{t,t}^\bot \cdot K \cong K$ by Lemma 4.11. So $B_t \cdot I_{t,t}^\bot$ is not purely outer, a contradiction. This contradiction shows that the virtual commutants $\varphi_t$ are zero for all $t \in S$. Equivalently, $E_t(r(B_t) \varphi(I)) = E_t(\varphi(I \cap r(B_t))) = 0$ for all $t \in S$. Then $\varphi(I) = 0$ follows by the last part of Proposition 2.18. □

5. Uniqueness of the crossed product decomposition

An important feature of Renault’s theory of Cartan subalgebras is that the twisted groupoid obtained from the Cartan subalgebra $A \subseteq B$ is unique up to isomorphism. So any automorphism $\beta \in \text{Aut}(B)$ with $\beta(A) = A$ lifts to an automorphism of the underlying twisted groupoid. In this article, étale groupoids are replaced by inverse semigroup actions. These are no longer unique up to isomorphism because an étale groupoid $H$ may be written as $H \cong X \rtimes S$ for any wide, unital inverse subsemigroup $S$ of Bis($H$), but only the action of Bis($H$) on $X$ is canonically defined through the groupoid $H$. We are going to define a “refinement” for general inverse semigroup actions on C*-algebras, using either of the dual groupoids $\tilde{A} \rtimes S$ or $\tilde{A} \rtimes S$. Their inverse semigroups of bisections coincide:

**Lemma 5.1.** Let $E$ be an inverse semigroup action of $S$ on $A$. The map $\kappa: \tilde{A} \rtimes S \to \tilde{A} \rtimes S$, $[t, [\pi]] \mapsto [t, \ker \pi]$, is a continuous open epimorphism of groupoids. It induces a lattice isomorphism between open sets $\mathcal{O}(\tilde{A} \rtimes S) \cong \mathcal{O}(\tilde{A} \rtimes S)$ and an isomorphism of inverse semigroups $\text{Bis}(\tilde{A} \rtimes S) \cong \text{Bis}(\tilde{A} \rtimes S)$.

**Proof.** It readily follows from the definitions that $\kappa$ is a groupoid epimorphism. Let $t \in S$ and let $J \in \Pi(A)$ be contained in the source ideal $s(\tilde{E}_t) = \langle \tilde{E}_t | \tilde{E}_t \rangle$. Then $U_t \tilde{J} := \{[t, [p]] : p \in \tilde{J}\}$ is a bisection of $\tilde{A} \rtimes S$ and $W_t \tilde{J} := \{[t, [\pi]] : [\pi] \in \tilde{J}\}$ is a bisection of $\tilde{A} \rtimes S$. Such bisections form bases for the topology of $\tilde{A} \rtimes S$ and $\tilde{A} \rtimes S$, respectively. Moreover, $\kappa^{-1}(U_t \tilde{J}) = W_t \tilde{J}$. This implies that $\kappa$ is continuous open and induces an isomorphism $\mathcal{O}(\tilde{A} \rtimes S) \cong \mathcal{O}(\tilde{A} \rtimes S)$. To see that this isomorphism restricts to Bis($\tilde{A} \rtimes S$) $\cong$ Bis($\tilde{A} \rtimes S$) we describe the corresponding bisections more explicitly.
Any bisection \( u \subseteq \mathcal{A} \times S \) is a union of bisections of the form \( U_t \tilde{J} \) for some \( t \in S \), \( J \subseteq s(\mathcal{E}_t) \). Consider such a union \( u := \bigcup_{t \in I} U_t \tilde{J}_t \). It is a bisection if and only if both \( r \) and \( s \) are injective on \( u \). Recall that \( [t_i, \mathcal{P}_i] = [t_j, \mathcal{P}_j] \) for \( p \in \tilde{J}_i \cap \tilde{J}_j \) holds if and only if there is \( v \in S \) with \( v \leq t_i, t_j \) and \( p \in r(\mathcal{E}_v) = s(U_v) \). And this is further equivalent to \( \mathcal{P} \in \mathcal{I}_{t_i, t_j} \) with \( \mathcal{I}_{t_i, t_j} \), as in \([2, 1]\). Thus \( s \) is injective if and only if \( J_i \cap J_j \subseteq \mathcal{I}_{t_i, t_j} \) for all \( i, j \in I \). The injectivity of \( r \) means that for all \( i, j \in I \) and \( p \in \tilde{E}_t(\tilde{J}_i) \cap \tilde{E}_t(\tilde{J}_j) \), there is \( v \in S \) with \( v \leq t_i, t_j \) and \( p \in r(\mathcal{E}_v) = r(U_v) \). We prefer to rewrite the injectivity of \( r \) on \( u \) through the injectivity of \( s \) on \( u^{-1} := \{ \gamma^{-1} \in \mathcal{A} \times S : \gamma \in u \} \). As a result, a subset \( u \) of \( \mathcal{A} \times S \) is a bisection if and only if both \( u \) and \( u^{-1} \) are of the form \( \bigcup_{t \in I} U_t \tilde{J}_t \) for a set \( I \) and \( t_i \in S \), \( J_i \in \mathcal{I}(A), J_i \subseteq s(\mathcal{E}_{t_i}) \), such that \( J_i \cap J_j \subseteq \mathcal{I}_{t_i, t_j} \) for all \( i, j \in I \).

Similar arguments show that a subset \( w \) of \( \mathcal{A} \times S \) is a bisection if and only if both \( w \) and \( w^{-1} \) are of the form \( \bigcup_{t \in I} W_t \tilde{J}_t \) for a set \( I \) and \( t_i \in S \), \( J_i \in \mathcal{I}(A), J_i \subseteq s(\mathcal{E}_{t_i}) \), such that \( J_i \cap J_j \subseteq \mathcal{I}_{t_i, t_j} \) for all \( i, j \in I \). Since \( \kappa(\bigcup_{t \in I} W_t \tilde{J}_t) = \bigcup_{t \in I} W_t \tilde{J}_t \), we conclude that \( \kappa \) induces a bijection \( \text{Bis}(\mathcal{A} \times S) \cong \text{Bis}(\mathcal{A} \times S) \). It preserves the semigroup product because \( \kappa \) is a groupoid homomorphism. Since we shall need this later, we describe the multiplication in \( \text{Bis}(\mathcal{A} \times S) \) and \( \text{Bis}(\mathcal{A} \times S) \) more explicitly. For each \( t \in S \), the Rieffel correspondence gives a lattice isomorphism \( \tilde{E}_t : (s(\mathcal{E}_t)) \rightarrow (r(\mathcal{E}_t)) \) that restricts to the homeomorphism \( \tilde{E}_t : s(\mathcal{E}_t) \rightarrow r(\mathcal{E}_t) \) which corresponds to the bisection \( U_t := \{ [t, \mathcal{P}] : \mathcal{P} \in s(\mathcal{E}_t) \} \). In particular, for any ideals \( J, J' \) in \( A \) the open set \( U_t^{-1}(J) J' \) is the primitive ideal space of \( \tilde{E}_t^{-1}(J) J' \). Hence for any two open sets \( u = \bigcup_{i} U_t \tilde{J}_t \) and \( w' = \bigcup_{j} W_{s_j} \tilde{J}_{s_j} \), we get

\[
uw' = \bigcup_{i,j} U_t \tilde{J}_t W_{s_j} \tilde{J}_{s_j} = \bigcup_{i,j} U_{t_i,s_j} U_{s_j}^{-1}(J_i) J'_{s_j} = \bigcup_{i,j} U_{t_i,s_j} \tilde{J}_{t_i,s_j,j},
\]

where \( J_{i,j} := \tilde{E}_t^{-1}(J_i) J'_{s_j} \) for all \( i, j \). Similarly, for \( w = \bigcup_{i} W_t \tilde{J}_t \) and \( w' = \bigcup_{j} W_{s_j} \tilde{J}_{s_j} \), we get \( uw' = \bigcup_{i,j} W_{t_i} \tilde{J}_{t_i} W_{s_j} \tilde{J}_{s_j} \). This implies \( \kappa(uw') = \kappa(u) \kappa(w') \). \( \square \)

**Proposition 5.2.** Let \( A \) be a \( C^* \)-algebra, let \( S \) be a unital inverse semigroup, and let \( \mathcal{E} = (\mathcal{E}_t, \mathcal{M}_u, \mathcal{V}_u : u \in S) \) be an \( S \)-action on \( A \). Equip \( \mathcal{A} \) with the dual action and form the transformation groupoid \( \mathcal{A} \times S \). Consider the unital inverse semigroup \( \mathcal{S} := \text{Bis}(\mathcal{A} \times S) \cong \text{Bis}(\mathcal{A} \times S) \).

1. There is a \( \mathcal{S} \)-action \( \tilde{E} \) on \( A \) such that the \( \mathcal{S} \)-action \( \mathcal{E} \) factors through \( \tilde{E} \) via the inverse semigroup homomorphism \( S \ni t \mapsto U_t := \{ [t, \mathcal{P}] : \mathcal{P} \in s(\mathcal{E}_t) \} \in \tilde{S} \) and such that \( s(\tilde{E}_u) = \bigcap_{s(e(u))} \mathcal{P} \) for all \( u \in S \); here \( s(\tilde{E}_u) = s(u) \) if we identify open sets in \( \mathcal{A} \) with ideals in \( \mathcal{I}(A) \). And any two such \( \mathcal{S} \)-actions are isomorphic through a unique isomorphism.

2. The dual groupoids \( \mathcal{A} \times \mathcal{S} \) and \( \mathcal{A} \times \mathcal{S} \) for the action \( \tilde{E} \) are naturally isomorphic to the corresponding dual groupoids \( \mathcal{A} \times S \) and \( \mathcal{A} \times S \) for the action \( \mathcal{E} \).

3. There is a canonical isomorphism \( A \times S \cong A \times \mathcal{S} \), and this isomorphism descends to an isomorphism \( A \times \mathcal{E} \cong A \times \tilde{E} \).

**Proof.** Let \( B \) be an exotic crossed product, that is, a \( C^* \)-algebra between \( A \times S \) and \( A \times S \). View \( \mathcal{E}_t \) for \( i \in I \) as a slice for the inclusion \( A \hookrightarrow B \). Let \( u \in \tilde{S} \). The description of elements in \( \tilde{S} \) in the proof of Lemma [5.1] shows that \( u = \bigcup_{i \in I} U_t \tilde{J}_t \) for some \( t_i \in S \), \( J_i \in \mathcal{I}(A), J_i \subseteq s(\mathcal{E}_{t_i}) \) with \( J_i \cap J_j \subseteq \mathcal{I}_{t_i, t_j} \) for all \( i, j \in I \). We claim
that
\[ \tilde{E}_u := \sum_{i \in I} E_{t_i} \cdot J_i \subseteq B \]
is a slice. As a closed linear span of closed \(A\)-submodules, it is again a closed \(A\)-submodule. We must show that \(E_u, \tilde{E}_u \subseteq A\) and \(E_u^*, \tilde{E}_u \subseteq A\). Since \(J_i \cap J_j \subseteq \mathcal{I}_{t_i, t_j}\) for all \(i, j \in I\), the Hilbert bimodules \(E_{t_i} \cdot \mathcal{I}_{t_i, t_j}\) and \(E_{t_j} \cdot \mathcal{I}_{t_i, t_j}\) are mapped to the same subspace in \(B\). Hence
\[ E_{t_i} \cdot J_i \cdot (E_{t_j} \cdot J_j)^* = E_{t_i} \cdot (J_i \cap J_j) \cdot E_{t_j}^* \subseteq E_{t_i} \cdot \mathcal{I}_{t_i, t_j} \cdot E_{t_j}^* \subseteq E_{t_j} E_{t_j}^* \subseteq A. \]
This implies \(\tilde{E}_u, \tilde{E}_u^* \subseteq A\). The other condition \(\tilde{E}_u^* \cdot \tilde{E}_u \subseteq A\) follows from the same argument applied to \(u^{-1}\), which has the same structure as \(u\).

Hence \((5.2)\) defines a map \(\tilde{S} \ni u \mapsto \tilde{E}_u \in \mathcal{S}(A, B)\). This map preserves the involution because if \(u = \bigcup_{i \in I} U_{t_i, J_i}\) then \(u^{-1} = \bigcup_{i \in I} J_i U_{t_i}^* = \bigcup_{i \in I} U_{t_i}^* \text{Prim}(\tilde{E}_{t_i}(J))\), where \(\tilde{E}_i : I(s(\tilde{E}_i)) \to I(r(\tilde{E}_i))\) is the Rieffel isomorphism, and therefore
\[ \tilde{E}_u^* = \sum_i J_i E_{t_i}^* = \sum_i E_{t_i} \tilde{E}_i(J) = \tilde{E}_{u^{-1}}. \]
This map is a semigroup homomorphism because if \(u' = \bigcup_j U_{s_j, J_j}\) is another element in \(\tilde{S}\), in a canonical form, then \((5.1)\) implies
\[ \tilde{E}_{u' \tilde{E}_u} = \sum_{i, j} E_{t_i} J_i E_{s_j} K_j = \sum_{i, j} E_{t_i s_j} \tilde{E}(J_{i, j}) = \tilde{E}_{u' u}. \]
The multiplication in \(B\) restricts to maps \(\mu_{t, u} : \tilde{E}_t \otimes_A \tilde{E}_u \to \tilde{E}_{t u}\). By construction, \(\tilde{E}_{t u} = \tilde{E}_t\) for all \(t \in S\). Thus the action \(E\) factors through an action \(\tilde{E}\) of \(\tilde{S}\) with the required properties.

Now let \((\tilde{E}_u, \mu_{t, u})_{t, u \in \tilde{S}}\) be any action of \(\tilde{S}\) through which the given action \(E\) factors; this means that \(\tilde{E}_{t u} = \tilde{E}_t\) and \(\mu_{t, u} = \mu_{t, u}\) for all \(t, u \in S\). In addition, we assume that \(s(\tilde{E}_u) = s(u)\) for all \(u \in \tilde{S}\) (we identify the lattice of open subsets in \(A\) with \(I(A)\)). Any action of \(\tilde{S}\) comes with canonical inclusion maps \(\tilde{E}_t \hookrightarrow \tilde{E}_u\) for \(t, u \in \tilde{S}\) with \(t \leq u\). Write \(u \in \tilde{S}\) as \(u = \bigcup_{i \in I} U_{t_i, J_i}\). Then \(U_{t_i} \supseteq U_{t_i, J_i} \subseteq u\) for all \(i \in I\). The inclusion \(\tilde{E}_{U_{t_i}} \hookrightarrow \tilde{E}_{t_i} = \tilde{E}_{t_i} \cdot J_i\) is an isomorphism onto \(E_{t_i} \cdot J_i\). So the inclusion maps above yield canonical inclusion maps \(\tilde{E}_{t_i} \cdot J_i \hookrightarrow \tilde{E}_u\) for all \(i \in I\). The resulting map \(\bigoplus_{i \in I} \tilde{E}_{t_i} \cdot J_i \to \tilde{E}_u\) has dense range because its image is an \(A\)-submodule \(V\) such that \((V : V)\) is dense in \(\langle \tilde{E}_u | \tilde{E}_u \rangle = s(u)\). The Hilbert \(A\)-bimodule \(\tilde{E}_u\) is isomorphic to the Hausdorff completion of \(\bigoplus_{t \in I} \tilde{E}_{t} \cdot J_i\) for the semi-norm given by the inner product \(\langle x_1 | y_1 \rangle = \sum_{i \in I} x_i^* y_i\). The norm in \(\tilde{E}_u\) is given by the same formula. Hence the map \(\bigoplus_{i \in I} \tilde{E}_{t_i} \cdot J_i \to \tilde{E}_u\) descends to an isomorphism \(\tilde{E}_u \cong \tilde{E}_u^*\). These isomorphisms for \(u \in \tilde{S}\) are clearly compatible with the multiplication maps and thus define an isomorphism of \(\tilde{S}\)-actions. Since any isomorphism of \(\tilde{S}\)-actions is compatible with the inclusion maps for \(t \leq u \in \tilde{S}\), this is the only isomorphism of \(\tilde{S}\)-actions that is the identity map on \(\tilde{E}_{U_{t_i}} = \tilde{E}_{t_i}\) for all \(t \in S\). This proves \((1)\).

The dual \(\tilde{S}\)-action of \(\tilde{E}\) on \(A\) turns out to be the standard action of bisections of \(A \times S\) on \(A\). So \(A \times \tilde{S} \cong A \times S\) follows from \(23\) Proposition 2.2, applied to \(\text{Bis}(A \times S)\) itself. The above argument with \(\tilde{S}\) replaced by \(\text{Bis}(A \times S)\) shows \(\tilde{A} \times S \cong A \times S\). This proves \((2)\).

We prove \((3)\). The homomorphism \(\tilde{S} \ni u \mapsto \tilde{E}_u \in \mathcal{S}(A, A \times S)\) given by \((5.2)\) is a representation of \(\tilde{E}\) in \(A \times S\) and thus induces an epimorphism \(\Phi: A \times \tilde{S} \to A \times S\). As \(E\) factors through \(E\), we also have a representation of \(E\) in \(A \times \tilde{S}\) which induces a homomorphism \(\Phi: A \times S \to A \times \tilde{S}\) with \(\Phi \circ \Phi|_{t_i} = \text{Id}_{E_t}\) for \(t \in S\). The images
of \( E \) for \( t \in S \) are linearly dense both in \( A \times S \) and \( A \times \tilde{S} \) because any \( E_u \) for \( u \in \tilde{S} \) is the closed linear span of \( E_t \) for \( t \leq u \). The maps \( \Phi \) and \( \tilde{\Phi} \) restrict to the identity maps between the images of \( E_t \) in \( A \times S \) and \( A \times \tilde{S} \). Hence \( \Phi \) and \( \tilde{\Phi} \) are isomorphisms inverse to each other. And the canonical weak conditional expectation on \( A \times S \) is determined by its restrictions to the slices \( E_{t_u} = E_t \) for \( t \in S \). These restrictions are given by the same formula as the canonical weak conditional expectation on \( A \times S \) (see Proposition 5.4). So the isomorphism \( A \times S \cong A \times S \) intertwines the weak conditional expectations. Then it descends to an isomorphism \( A \times \tilde{S} \cong A \times S \). This proves the statement (3).

**Definition 5.3.** Let \( A \) be a \( C^* \)-algebra, let \( S \) be a unital inverse semigroup, and let \( E \) be an \( S \)-action on \( A \). The essentially unique action of \( \tilde{E} \) on \( A \) described in Proposition 5.2 is called the refinement of \( E \). The action \( \tilde{E} \) is called fine if the canonical map \( S \to \text{Bis}(A \times S) \) is an isomorphism.

The following example shows that for a general inverse semigroup action \( E \), the inverse semigroup \( S(A, B) \) may be much larger than \( \tilde{S} \). Then the crossed product \( A \rtimes S(A, B) \) and the dual groupoids \( \tilde{A} \rtimes S(A, B) \) and \( \tilde{A} \rtimes S(A, B) \) are much larger than the corresponding objects for \( S \) or \( \tilde{S} \).

**Example 5.4.** Let \( S = \mathbb{Z} \cup \{0\} \) be obtained by adding a zero element to the group \( \mathbb{Z} \) and let \( S \) act trivially on \( C \). This action is fine because \( C \times S \) is isomorphic to the group \( \mathbb{Z} \) and \( S \) is its inverse semigroup of bisections. In contrast, a slice for \( \tilde{A} \rtimes S(A, B) \) is a bisection of \( A \rtimes S(A, B) \) and \( \tilde{A} \rtimes S(A, B) \) is determined by its restrictions to the slices \( \tilde{E}_{t_u} = \tilde{E}_t \) for \( t \in S \). This implies that any \( \tilde{S} \) is much smaller than \( \text{S}(C, C^*(\mathbb{Z})) \). And \( C \times \tilde{S} = C^*(\mathbb{Z}) \) is remarkable because \( C \times S(C, C^*(\mathbb{Z})) \) is not even separable.

**Definition 5.5.** Let \( E_j = (E_{j,t})_{t \in S_j} \) for \( j = 1, 2 \) be actions of unital inverse semigroups \( S_j \) on \( C^* \)-algebras \( A_j \). An isomorphism between these actions is given by isomorphisms \( \varphi: A_1 \sim A_2, \psi: S_1 \sim S_2 \), and \( \varphi_1: E_{1,t} \sim E_{2,\psi(t)} \) for \( t \in S_1 \), such that the isomorphisms \( \varphi_1 \) intertwine the multiplication isomorphisms \( E_{j,t} \otimes A_z E_{j,u} \to E_{j,tu} \) for \( j = 1, 2 \) and \( t, u \in S_j \).

**Theorem 5.6.** Let \( E \) be a closed and purely outer action of a unital inverse semigroup \( S \) on a \( C^* \)-algebra \( A \). Let \( B := A \rtimes S \). The refined action \( \tilde{E} \) is canonically isomorphic to the tautological action of \( S(A, B) \) on \( A \). And

\[ A \rtimes S(A, B) \cong B, \quad \tilde{A} \rtimes S(A, B) \cong \tilde{A} \rtimes S \rtimes S(A, B) \cong \tilde{A} \rtimes S. \]

**Proof.** The proof of Proposition 5.2 shows that we may treat \( \tilde{E}_u \) for \( u \in \tilde{S} \) as elements of \( S(A, B) \) and that the map \( \psi: S \to S(A, B), u \mapsto \tilde{E}_u \), is a homomorphism. Hence the assertion follows once we show that \( \psi \) is an isomorphism (the second part then follows from Proposition 5.2). So let \( X \in S(A, B) \). It suffices to prove that there is a unique \( t \in S \) with \( X = E_t \), where \( E_t \) is defined in (5.2). Let

\[ T = \{ U_t \tilde{J}: J \in I(A), E_t \cdot J \subseteq X \} \subseteq \tilde{S}. \]

We claim that the union \( \bigcup T \) is a bisection of \( \tilde{A} \rtimes S \). Let \( U_t \tilde{J}, U_w \tilde{K} \in T \). The products \( E_J^* \cdot (E_K^* \cdot E_{U_t}) \) and \( (E_J^* \cdot E_{U_w})^* \cdot E_{U_t} \) in \( B \) are contained in \( A \) because \( X^* X \subseteq A \) and \( XX^* \subseteq A \). Since \( (E_J^*)^* \cdot (E_{U_t})^* \cdot (E_K^* \cdot E_{U_w})^* = E_J \cap K \cdot E_{U_w}^* \subseteq E_{U_w}^* \) we get \( E_{U_t} \cap K \cdot E_{U_w}^* \subseteq E_{U_w}^* \cap A = I_{U_w \cdot U_t, 1} \). This implies that any \( p \in r(\tilde{E}_u) \) with \( \tilde{E}_{U_w}(p) \in \tilde{J} \cap \tilde{K} \) already belongs to \( I_{U_t \cdot U_w, 1} \). And this is equivalent to \( \tilde{J} \cap \tilde{K} \subseteq I_{U_t} \) and then to \( s \) being injective on \( U_t \tilde{J} \cup U_w \tilde{K} \subseteq \tilde{A} \rtimes S \). Similarly, \( (E_J \cap K) \cdot (E_{U_w})^* \subseteq A \) is equivalent to \( r \) being injective on \( U_t \tilde{J} \cup U_w \tilde{K} \subseteq \tilde{A} \rtimes S \). This implies that \( \bigcup T \) is a bisection of \( \tilde{A} \rtimes S \).
Let $X_t := \hat{E}_J$. This is the closed linear span in $B$ of $E_t$ for $t \in T$. By construction, $X_0 \subseteq X$. We have described $S$ above through unions of bisections of the form $J_i \hat{J}$ for $J \in \mathcal{J}(s(\hat{E}_t))$, $t \in S$. As a consequence, $\bigcup T$ is the maximal element $u$ in $S$ with the property that $\hat{E}_u \subseteq X$. Since any $u \in S$ with $u < \bigcup T$ satisfies $s(u) \subseteq s(\bigcup T)$, it follows that $\hat{E}_u \neq X$ for $u \neq \bigcup T$. So the proof (injectivity and surjectivity of $\psi$) is finished when we show that $X_1 = X$. Assume the contrary. Then $s(X_1) \subseteq s(X)$. So there is $p \in \hat{s}(X) \setminus \hat{s}(X_1)$. Then there is $x \in X$ with $(x^*x)_p \neq 0$ in $A/p$. Let $\varepsilon := \|x^*x\|_p/2 > 0$. There is $y \in A \times_{alg} S$ with $\|x\| \cdot \|x - y\| < \varepsilon$. Let $E : B \to A$ be the canonical conditional expectation. Then $E(x^*x) = x^*x$ since $X$ is a slice. Then $E(x^*y)_p \neq 0$ in $A/p$ because $\|E(x^*y) - x^*x\|_p = \|E(x^*y) - E(x^*x)\|_p < \varepsilon$. Write $y = \sum_{t \in F} y_t$ for a finite subset $F \subseteq S$ and $y_t \in \hat{E}_t$ for $t \in F$. There must be $t \in F$ with $E(x^*y_t)_p \neq 0$ in $A/p$. We have $x^*y_t \in X^*\hat{E}_t$, and $X^*\hat{E}_t$ is a slice because it is a product of slices. The inclusion $A \subseteq B$ is a noncommutative Cartan subalgebra by Theorem 4.3. Lemma 4.10 shows that $E$ maps the slice $X^*\hat{E}_t$ into itself. Then Proposition 3.7 implies that $X^*\hat{E}_t = J \cap Y$ with $J = X^*\hat{E}_t \cap A$ and $Y \subseteq \ker E$. Since $E(x^*y)_p \neq 0$, it follows that $p \in J$. Now $J \subseteq X^*\hat{E}_t$ implies that $X \cdot J = \hat{E}_t \cdot J$ as slices in $B$. So $\hat{E}_1 \cdot J \subseteq X_1$. Then $p \in \hat{s}(X_1)$, which contradicts our assumption.

**Corollary 5.7.** Let $A \subseteq B$ be a noncommutative Cartan subalgebra and let $(A, S_j, \hat{E}_j)$ for $j = 1, 2$ be two inverse semigroup actions for which there is an isomorphism $A \times \alpha S_j \cong B$ mapping $A \subseteq A \times \alpha S_j$ onto $A \subseteq B$. Then $\hat{E}_1 \cong \hat{E}_2$, $\hat{A} \cong S_1 \cong \hat{A} \times S_2$ and $\hat{A} \times S_1 \cong \hat{A} \times S_2$.

**Proof.** The two actions are closed and purely outer by Theorem 4.3. By Theorem 5.6 they have isomorphic refinements and isomorphic dual groupoids.

The following example shows that the uniqueness result in Corollary 5.7 fails for actions that are not purely outer.

**Example 5.8.** The groups $\mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/4$ are not isomorphic, but their group $C^*$-algebras are isomorphic to the algebra of functions on the discrete set with four points. The inclusion of the unit does not add extra information. So there are two essentially different ways to realise the unital inclusion $\mathbb{C} \subseteq C_1^*$ as the unital inclusion $\mathbb{C} \subseteq C_4^*(H)$ for a discrete group $H$. The resulting dual groupoids are the two groups $\mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/4$.

### 6. Aperiodic inclusions versus Cartan $C^*$-subalgebras

Let $A \subseteq B$ be a $C^*$-inclusion. We say that $A$ detects ideals in $B$ if $J \cap A = 0$ for an ideal $J$ in $B$ implies $J = 0$. Theorem 4.3 implies that a general non-commutative Cartan $C^*$-subalgebra $A \subseteq B$ does not detect ideals in $B$. Indeed, there are counterexamples where $B = A \rtimes \alpha \mathbb{Z}$ is the crossed product by a single automorphism $\alpha$ on a separable $C^*$-algebra. Examples of purely outer automorphisms $\alpha$ where the inclusion $A \to A \rtimes \alpha \mathbb{Z}$ does not detect ideals are described in [21, Examples 2.14 and 2.21]. For such automorphisms, $A$ is a Cartan subalgebra of $A \rtimes \alpha \mathbb{Z}$, but $A$ does not detect ideals in $A \rtimes \alpha \mathbb{Z}$ (see [21, Theorem 9.12]). This means that there is a non-trivial quotient $(A \rtimes \alpha \mathbb{Z})/J$ into which $A$ embeds. This quotient cannot admit an $A$-valued conditional expectation because then the conditional expectation $A \rtimes \alpha \mathbb{Z} \to A$ would not be unique, in contradiction to Theorem 4.3. This is why the lack of “uniqueness” for such counterexamples is not seen in Exel’s theory of noncommutative Cartan subalgebras.

In this section we discuss conditions under which a noncommutative Cartan $C^*$-subalgebra $A \subseteq B$ does detect ideals, and thus is a good tool to study the
structure of $B$. To this end, we compare Cartan inclusions to aperiodic inclusion, which are studied in [23]. If the C*-inclusion $A \subseteq B$ is aperiodic and $E\colon B \to A$ is a faithful conditional expectation, then $A$ detects ideals in $B$ and, even more, $A$ supports $B$ in the sense that for every $b \in B^* \setminus \{0\}$ there is $a \in A^+ \setminus \{0\}$ with $a \preceq b$ (see [23] Theorem 1.1). Here $\preceq$ denotes the Cuntz preorder on the set $B^*$ of positive elements in $B$. We combine our results with some of the results in [23].

**Definition 6.1** ([21],[23]). Let $X$ be a normed $A$-bimodule. An element $x \in X$ satisfies Kishimoto’s condition if for each non-zero hereditary subalgebra $D \subseteq A$ and $\varepsilon > 0$ there is $a \in D^*$ with $\|a\| = 1$ and $\|axa\| < \varepsilon$. We call $X$ aperiodic if all elements $x \in X$ satisfy Kishimoto’s condition. A C*-inclusion $A \subseteq B$ is aperiodic if the Banach $A$-bimodule $B/A$ is aperiodic. An inverse semigroup action $\mathcal{E}$ is aperiodic if the Hilbert $A$-bimodules $\mathcal{E}_t \cdot I_t$ with $I_t = \{a \in A\colon a \cdot I_{t,t} = 0\}$ are aperiodic for all $t \in S$.

**Proposition 6.2.** Let $A \subseteq B$ be a C*-inclusion with a conditional expectation $E\colon B \to A$. The inclusion $A \subseteq B$ is aperiodic if and only if the $A$-bimodule $\ker E$ is aperiodic. In this case, $E$ is the only conditional expectation $B \to A$, and $E|_{IBI}$ is the only conditional expectation $IBI \to I$ for all $I \in \mathbb{I}(A)$.

**Proof.** Since $B = \ker E \otimes A$, there is a contractive $A$-bimodule isomorphism $\ker E \to B/A$ with bounded inverse. This implies that $E$ is aperiodic if and only if $B/A$ is aperiodic. Now let $P\colon B \to A \subseteq B$ be a conditional expectation. Assume $\ker E$ to be aperiodic, and let $x \in \ker E$. Then $P(x)^*x \in \ker E$ and $a := P(P(x)^*x) = P(x)^*P(x) \geq 0$ because $P$ is $A$-bilinear. Since $P$ is a bounded bimodule map, $a$ inherits Kishimoto’s condition from $P(x)^*x \in \ker E$. By [23] Lemma 5.10, 0 is the only positive element of $A$ that satisfies Kishimoto’s condition. So $a = 0$ and then $P(x) = 0$. Both $P$ and $E$ are idempotent maps on $B$ with the same image $A$. We have shown that $\ker E \subseteq \ker P$, and this implies $P = E$. So $E$ is the only conditional expectation $B \to A$. If $I \in \mathbb{I}(A)$, then the inclusion $I \subseteq IBI$ inherits aperiodicity from $A \subseteq B$ by [23] Proposition 5.15. Therefore, the conditional expectation $E|_{IBI}$ is unique as well.

**Theorem 6.3.** Let $A \subseteq B$ be a C*-inclusion and consider the following conditions:

1. $A \subseteq B$ is regular and aperiodic, and there is an almost faithful conditional expectation $E \colon B \to A$;
2. $B \cong A \rtimes S$ for a closed and aperiodic action $\mathcal{E}$ of an inverse semigroup $S$ on $A$, with an isomorphism that restricts to the canonical embedding on $A$;
3. $A \subseteq B$ is a noncommutative Cartan subalgebra.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3), and all conditions (1) (3) are equivalent if $A$ is prime or contains an essential ideal of Type I.

Assume the equivalent conditions (1) and (2). Then $A$ detects ideals in $B$ and $A$ supports $B$. And $B$ is simple if and only if $\mathcal{E}$ is minimal, that is, there are no non-trivial ideals in $A$ that are $\mathcal{E}_t$-invariant for all $t \in S$. If $B$ is simple, then it is purely infinite if and only if all elements of $A^+ \setminus \{0\}$ are infinite in $B$.

**Proof.** By [23] Theorem 6.14, (2) implies (3) and the converse holds when $A$ is prime or contains an essential ideal of Type I. (2) implies (1) by [23] Proposition 6.3. Now assume (1). The last part of Proposition 6.2 verifies (1) in Theorem 4.3. Hence $A \subseteq B$ is a noncommutative Cartan subalgebra, and $B \cong A \rtimes S$ for a closed action $\mathcal{E}$ of an inverse semigroup $S$ on $A$. Under this isomorphism $E\colon B \to A$ is the canonical conditional expectation. Hence $\mathcal{E}$ is aperiodic by [23] Proposition 6.3.

Thus (1) and (2) are equivalent. The last part of the assertion follows from [23] Theorems 6.5, 6.6 and Corollary 6.7. □
Corollary 6.4. Let $A \subseteq B$ be a noncommutative Cartan subalgebra such that $A$ is either prime or contains an essential ideal of Type I. Then $A$ supports $B$ and thus $A$ detects ideals in $B$.

Theorem 6.3 indicates that regular aperiodic inclusions $A \subseteq B$ with a faithful conditional expectation $E : B \to A$ form an important subclass of noncommutative Cartan inclusions. The example mentioned in the beginning of this section shows that this subclass is strictly smaller (unless $A$ is prime or contains an essential ideal of Type I). The condition of pure outerness of the action suffices to recover the dynamics from the inclusion $A \subseteq A \rtimes_{\varepsilon} S$ in an essentially unique way, but not to relate the ideal structure or the Cuntz semigroups of $A$ and $A \rtimes_{\varepsilon} S$. So some applications will only work for aperiodic noncommutative Cartan inclusions.

Aperiodic noncommutative Cartan inclusions can often be characterised by the topological freeness (effectivity) of the dual groupoid:

Definition 6.5. An étale groupoid $H$, possibly with non-Hausdorff unit space $X$, is effective if any open bisection $U \subseteq H$ with $r|_U = s|_U$ is contained in $X$.

Remark 6.6. In [23, Definition 2.19], an étale groupoid $H$ is called topologically free if there is no non-empty open bisection $U \subseteq H \times X$ with $r|_U = s|_U$. Topological freeness is weaker than effectivity in general, but the two notions coincide if $X$ is closed in $H$.

Theorem 6.7. Let $A \subseteq B$ be a C*-inclusion with a faithful conditional expectation. Assume that $A$ contains an essential ideal which is separable or of Type I. The conditions (1) and (2) in Theorem 6.3 are equivalent to each of the following:

1. $B \cong A \rtimes_{\varepsilon} S$, by an $A$-preserving isomorphism, for a closed action $E$ of an inverse semigroup $S$ on $A$ whose dual groupoid $\hat{A} \rtimes S$ is effective;
2. $A \subseteq B$ is regular and the dual groupoid $\hat{A} \rtimes S(A, B)$ is effective and has a closed space of units;
3. there is a saturated, wide grading $(B_t)_{t \in S}$ on $B$ with unit fibre $A$, whose dual groupoid $\hat{A} \rtimes S$ is effective and has a closed space of units.

If these conditions hold, then the dual groupoids $\hat{A} \rtimes S(A, B)$ and $\hat{A} \rtimes S$ above are canonically isomorphic to each other.

Proof. By [23, Theorem 6.14], the conditions in Theorem 6.3 are equivalent to (1) in the present assertion. In particular, they imply that $A \subseteq B$ is a noncommutative Cartan algebra. Together with Theorem 4.3(7) and Corollary 5.7, it follows that the dual groupoid in (2) is canonically isomorphic to the dual groupoid for the action $E$. These groupoids have closed space of units by Proposition 2.17. Thus (1) implies (2). Condition (2) obviously implies (3).

Assume (3) and let $E$ be the action coming from the grading $(B_t)_{t \in S}$. By definition, $\hat{A} \rtimes S$ is the dual groupoid for the action $E$. Hence this action is closed and aperiodic (see Proposition 2.17 and [23, Theorem 6.14]). Thus the canonical conditional expectation $E_0 : A \rtimes S \to A$ is the unique conditional expectation from $A \rtimes S$ onto $A$ by Proposition 6.2. This implies that the canonical epimorphism $\pi : A \rtimes S \to B$ intertwines $E_0$ and $E : B \to A$. Then Proposition 3.7 shows that $\pi$ descends to an isomorphism $B \cong A \rtimes_{\varepsilon} S$ as in (1). □

7. Cartan C*-subalgebras with Hausdorff primitive ideal space

In this section, we assume that $A$ is a C*-algebra with Hausdorff primitive ideal space $X := \text{Prim}(A) = \hat{A}$. We are going to show that a regular inclusion with a conditional expectation is Cartan if and only if the conditional expectation is unique, and we shall rewrite the crossed product description in Theorem 4.3 through a Fell
bundle over a Hausdorff groupoid. We also consider the two extreme cases where \( A \) is either simple or commutative.

The Dauns–Hofmann isomorphism \( C_0(X) \cong Z(\mathcal{M}(A)) \) shows that \( A \) is a \( C_0(X) \)-algebra in such a way that \( (fa)(x) = f(x)a(x) \in A(x) := A/\pi_x \) for all \( x \in X \), \( f \in C_0(X) \) and \( a \in A \). An ideal \( I \) in \( A \) corresponds to an open subset \( \overline{I} \subseteq \overline{A} = X \).

**Proposition 7.1.** Let \( A \subseteq B \) be a non-degenerate \( C^* \)-inclusion where \( \overline{A} \) is Hausdorff. If there is a unique conditional expectation \( E : B \to A \), then there is a unique conditional expectation \( IBI \to I \) for each \( I \in \mathbb{I}(A) \). A similar implication holds for unique (almost) faithful conditional expectations.

**Proof.** Suppose that there are an ideal \( I \in \mathbb{I}(A) \) and a conditional expectation \( P : IBI \to I \) that differs from the restriction of \( E \). Then there is \( b_0 \in IBI \) with \( P(b_0) \neq E(b_0) \). Then there is \( x \in \overline{I} \subseteq \overline{A} \) with \( P(b_0)(x) \neq E(b_0)(x) \). Pick a function \( f \in C_0(\overline{I}) \) with \( f(x) \neq 0 \) and \( 0 \leq f \leq 1 \). If \( b \in B = ABA \), then \( f \cdot b \cdot f \in IBI \), so that \( P(fbf) \) is defined. We define

\[
E : B \to A, \quad b \mapsto P(fbf) + (1 - f^2 \cdot E(b)).
\]

Since \( P \) and \( E \) are completely positive, \( 0 \leq f \leq 1 \), and \( f \in C_0(\overline{I}) \subseteq C_0(X) \subseteq Z(\mathcal{M}(A)) \), the map \( E \) is positive and \( E(a) = f(a) + (1 - f^2) a = a \) for all \( a \in A \). It follows that \( |E| = 1 \) and so \( E \) is a conditional expectation \( B \to A \). And

\[
E(b_0)(x) - E(b_0)(x) = f^2(x)P(b_0)(x) - f^2(x)E(b_0)(x) \neq 0
\]

by construction. So the conditional expectation \( E \) is not unique. If, in addition, \( E \) and \( P \) are (almost) faithful, then so is \( E \). \( \square \)

**Theorem 7.2.** Let \( A \subseteq B \) be a regular \( C^* \)-subalgebra with an almost faithful conditional expectation \( E : B \to A \). Assume \( \overline{A} \) to be Hausdorff. The conditions in Theorem 4.3 which characterise Cartan subalgebras in the sense of Exel, are equivalent to the following conditions:

1. there is no other conditional expectation \( B \to A \) besides \( E \);
2. there is no other almost faithful conditional expectation \( B \to A \) besides \( E \).

If these conditions hold, then there is a continuous Fell bundle \( A = (A_\gamma)_{\gamma \in H} \) over a Hausdorff, étale, locally compact groupoid \( H \) with unit space \( \overline{A} \) such that \( B \cong C^*_r(H, A) \) with an isomorphism restricting to the canonical isomorphism \( A \cong C_0(\overline{A}, A) \). This Fell bundle is unique up to isomorphism, and \( H \) is isomorphic to the dual groupoid \( \overline{A} \rtimes S \) for any saturated, wide grading \( (B_t)_{t \in S} \) on \( B \) with unit fibre \( A \).

**Proof.** Conditions (1) and (2) in Theorem 4.3 are equivalent to (1) and (2) in this theorem by Proposition 7.1. Assume these conditions. By Theorem 4.3 for any saturated, wide grading \( (B_t)_{t \in S} \) on \( B \) with unit fibre \( A \), the action \( (B_t)_{t \in S} \) of \( S \) on \( A \) is closed and purely outer, and there is a canonical isomorphism \( A \rtimes S \cong B \).

Theorem 5.6 shows that the dual groupoid \( \overline{A} \rtimes S \) is canonically isomorphic to the dual groupoid \( H = \overline{A} \rtimes S(A, B) \), and \( \text{Bis}(H) \cong S(A, B) \) as inverse semigroups. Hence [7, Theorem 6.1] applied to the tautological \( S(A, B) \)-grading on \( B \) gives both existence and uniqueness of the desired Fell bundle \( A = (A_\gamma)_{\gamma \in H} \). The Fell bundle is a continuous field of Banach spaces because \( A \) is a continuous field of \( C^* \)-algebras over \( \overline{A} \) (see [23, Remark 7.17]). \( \square \)

### 7.1. Simple Cartan subalgebras.

Assume \( A \) to be simple. The assumptions in Theorem 4.3 involving ideals become empty for \( I = \{0\} \), so that only the case \( I = A \) remains. Any non-zero slice \( M \subseteq B \) satisfies \( M^* M = A = MM^* \). Hence \( S(A, B) \setminus \{0\} \) is a group, and any grading on \( B \) by an inverse semigroup \( S \) with unit
fibre $A$ may be simplified to a saturated grading by a group by taking the image of $S$ in $S(A,B)$ and discarding 0. Pure outerness simplifies to outerness:

**Definition 7.3** ([21]). A Hilbert $A$-bimodule $H$ is outer if it is not isomorphic to $A$ as a Hilbert bimodule. A saturated Fell bundle $(B_t)_{t \in G}$ over a group $G$ with unit fibre $A$ is outer if, for each $t \in G \setminus \{1\}$, the Hilbert $A$-bimodule $B_t$ is outer.

**Corollary 7.4.** Let $A \subseteq B$ be a regular $C^*$-subalgebra with an almost faithful conditional expectation $E : B \to A$. Let $A$ be simple. The following are equivalent:

1. $A \subseteq B$ is a noncommutative Cartan subalgebra;
2. $A \subseteq B$ is an aperiodic inclusion;
3. there is at most one conditional expectation $B \to A$;
4. $E : B \to A$ is the only almost faithful conditional expectation onto $A$;
5. $A' \cap \mathcal{M}(B) = \mathbb{C} \cdot 1$;
6. any $A$-bimodule map $\varphi : A \to B$ has range in $A$;
7. if a slice $M \subseteq B$ is isomorphic to $A$, then $M = A$;
8. for any saturated group grading $(B_t)_{t \in G}$ of $B$ with unit fibre $A$, the Fell bundle $B = (B_t)_{t \in G}$ is outer and saturated, and the canonical $*-$homomorphism $\pi : C^*_r(B) \to B$ descends to an isomorphism $C^*_t(B) \cong B$;
9. there are a discrete group $G$, a saturated outer Fell bundle $B = (B_t)_{t \in G}$ over $G$, and a $*-$isomorphism $C^*_t(B) \cong B$.

If the above conditions hold then $B$ is simple and the group $G$ and the Fell bundle in (9) are unique up to isomorphism.

**Proof.** Each of the conditions in the assertion, except (2), is equivalent to one of the conditions in Theorem 4.3. Theorem 6.3 implies that (1) and (2) are equivalent because simple algebras are prime. The last part of the assertion follows from Theorem 6.3 as well as Corollary 5.7 or Theorem 7.2. \qed

**Remark 7.5.** The above corollary implies that, for any purely outer action of a discrete group $G$ on a simple $C^*$-algebra $A$, the inclusion $A \subseteq A \rtimes_G G$ remembers the group and the crossed product $A \rtimes_G G$ is simple (simplicity is a classical result of Kishimoto [18]). Both claims fail without outerness assumption (see Example 5.8). Outerness is only sufficient for simplicity. For instance, the inclusion $\mathbb{C} \subseteq \mathbb{C} \rtimes_\varepsilon F_n = C^*_r(F_n)$ for the free group $F_n$ on $n \geq 2$ generators is not Cartan although $C^*_r(F_n)$ is simple.

When $B := A \rtimes_G G$ is an ordinary reduced crossed product by a group action $\alpha : G \to \text{Aut}(A)$ and $A$ is unital, then Zarikian showed in [27, Theorem 3.2] that (3) and (4) in Corollary 7.4 are equivalent to each other and to $\alpha$ acting freely, that is, if $t \in G \setminus \{1\}$ and $d \in A$ are such that $da = \alpha_t(a)d$ for all $a \in A$, then $d = 0$. The case of a crossed product for an outer group action is special because each slice is contained in a global slice defined on all of $A$. Hence we do not expect such equivalences for inverse semigroup actions on non-simple $C^*$-algebras.

### 7.2. Commutative Cartan subalgebras.

Now let $A \subseteq B$ be a $C^*$-inclusion where $A$ is commutative. Renault defined (commutative) Cartan subalgebras as regular $C^*$-inclusions $A \subseteq B$ where $A$ is a maximal Abelian subalgebra of $B$ and there is a faithful conditional expectation onto $A$ (see [20, Definition 5.1]). He shows that Cartan subalgebras $A \subseteq B$ with separable $B$ are equivalent to twists of topologically principal, Hausdorff, étale, locally compact, second countable groupoids (see [20, Theorems 5.2 and 5.9]). We now use Theorem 4.3 to extend Renault’s characterisation to the non-separable case. Then topologically principal, second countable groupoids are replaced by effective groupoids.
Corollary 7.6. Let $A \subseteq B$ be a regular $C^*$-subalgebra with an almost faithful conditional expectation $E : B \to A$. Assume $A$ to be commutative. The conditions in Theorems 4.3 and 7.2, which characterise Cartan subalgebras in the sense of Exel, are equivalent to the following conditions:

1. the inclusion $A \subseteq B$ is aperiodic;
2. $A$ is maximal Abelian in $B$;
3. there is a twist $\Sigma$ of an effective, Hausdorff, étale, locally compact groupoid $H$ such that $B \cong C^*_\Sigma(H, A)$ with an isomorphism mapping $A$ onto $C_0(H^0)$.

The twisted groupoid $(H, \Sigma)$ in (3) is unique up to isomorphism.

Proof. Condition (1) holds if and only if $A \subseteq B$ is a Cartan inclusion by Theorem 6.3. By Theorem 7.2 being Cartan implies existence of an (essentially) unique Fell bundle $A = (A_\tau)_{\tau \in H}$ over a Hausdorff, étale, locally compact groupoid $H$ with unit space $\tilde{A}$ such that $B \cong C^*_\Sigma(H, A)$ with an isomorphism restricting to $A \cong C_0(\tilde{A}, A)$. Since $A = C_0(\tilde{A})$, this Fell bundle is necessarily a line bundle and Fell line bundles over groupoids are equivalent to twisted groupoids. Since the inclusion $A \subseteq B$ is aperiodic this groupoid is effective by Theorem 6.7. Hence (1) implies (3). The converse follows from Theorem 6.7. Thus (3) is equivalent to $A \subseteq B$ being Cartan in the sense of Exel.

Next we prove that condition (2) in the present theorem is equivalent to condition (4) in Theorem 4.3, that is, to $I' \cap M(IBI) = ZM(I)$ for each $I \in \mathfrak{I}(A)$. This condition for $I = A$ implies that $A$ is maximal abelian. Conversely, assume that there are an ideal $I \subseteq I(\tilde{A})$ and $\tau \in I' \cap M(IBI)$ with $\tau \notin ZM(I) = M(I)$. Then there is $f \in I$ with $\tau f = f \tau = I$. And $\tau f \in B$. Since $A$ is commutative, $\tau f \in A'$; if $a \in A$, then $a \cdot (\tau f) = (a \tau) f = \tau (a f) = (\tau f) a$, that is, $f \tau = \tau f$ commutes with $A$. Hence $A$ is not maximal abelian in $B$. Thus all conditions in this theorem are equivalent to those in Theorem 4.3. \hfill \qed 

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