AVERAGE CASE \((s, t)\)-WEAK TRACTABILITY OF
NON-HOMOGENOUS TENSOR PRODUCT PROBLEMS

JIA CHEN, HEPING WANG, AND JIE ZHANG

Abstract. We study \(d\)-variate problem in the average case setting with re-
spect to a zero-mean Gaussian measure. The covariance kernel of this Gaussian
measure is a product of univariate kernels and satisfies some special proper-
ties. We study \((s, t)\)-weak tractability of this multivariate problem, and obta
in a necessary and sufficient condition for \(s > 0\) and \(t \in (0, 1)\). Our result can ap-
ply to the problems with covariance kernels corresponding to Euler and Wiener
integrated processes, Korobov kernels, and analytic Korobov kernels.

1. Introduction

Recently, there has been an increasing interest in \(d\)-variate problems with large
or even huge \(d\). Examples include problems in computational finance, statistics
and physics. In this paper we investigate multivariate problems \(S = \{S_d\}_{d \in \mathbb{N}}\) in the
average case setting, where \(S_d : F_d \to G_d, F_d\) is a separable Banach space equipped
with a zero-mean Gaussian measure \(\mu_d\), \(G_d\) is a Hilbert space. We only consider
continuous linear functional. We use either the absolute error criterion (ABS) or
the normalized error criterion (NOR). The information complexity \(n^X(\varepsilon, S_d)\) is
defined as the minimal number of continuous linear functionals needed to find an
\(\varepsilon\)-approximation of \(S_d\) for \(X \in \{\text{ABS}, \text{NOR}\}\).

An algorithm \(A : F_d \to G_d\) is said to be an \(\varepsilon\)-approximation of \(S_d\) for \(X \in \{\text{ABS}, \text{NOR}\}\) if

\[
\left( \int_{F_d} \|S_d(f) - A(f)\|^2_{G_d} \mu_d(df) \right)^{1/2} \leq \varepsilon CRI_d,
\]

where

\[
CRI_d = \begin{cases} 
1, & \text{for } X = \text{ABS}, \\
\left( \int_{F_d} \|S_d(f)\|^2_{G_d} \mu_d(df) \right)^{1/2}, & \text{for } X = \text{NOR}.
\end{cases}
\]

Tractability of multivariate problems \(S\) is concerned with the behavior of the
information complexity \(n^X(\varepsilon, S_d)\) for \(X \in \{\text{ABS, NOR}\}\) when the accuracy \(\varepsilon\)
of approximation goes to zero and the number \(d\) of variables goes to infinity. Various
notions of tractability have been studied recently for many multivariate problems.
We briefly recall some of the basic tractability notions (see [9, 10, 11, 12, 15]).

Let \(S = \{S_d\}_{d \in \mathbb{N}}\). For \(X \in \{\text{ABS, NOR}\}\), we say \(S\) is

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• **strongly polynomially tractable (SPT)** iff there exist non-negative numbers $C$ and $p$ such that for all $d \in \mathbb{N}$, $\varepsilon \in (0, 1)$,

$$n^X(\varepsilon, S_d) \leq C(\varepsilon^{-1})^p;$$

The exponent of SPT is defined to be the infimum of all $p$ for which the above inequality holds:

• **polynomially tractable (PT)** iff there exist non-negative numbers $C, p$ and $q$ such that for all $d \in \mathbb{N}$, $\varepsilon \in (0, 1)$,

$$n^X(\varepsilon, S_d) \leq Cd^q(\varepsilon^{-1})^p;$$

• **quasi-polynomially tractable (QPT)** iff there exist two constants $C, t > 0$ such that for all $d \in \mathbb{N}$, $\varepsilon \in (0, 1)$,

$$n^X(\varepsilon, S_d) \leq C \exp(t(1 + \ln \varepsilon^{-1})(1 + \ln d));$$

• **uniformly weakly tractable (UWT)** iff for all $s, t > 0$,

$$\lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n^X(\varepsilon, S_d)}{(\varepsilon^{-1})^s + d^t} = 0;$$

• **weakly tractable (WT)** iff

$$\lim_{\varepsilon^{-1} + d \to \infty} \frac{n^X(\varepsilon, S_d)}{\varepsilon^{-1} + d} = 0;$$

• **($s, t$)-weakly tractable (($s, t$)-WT)** for positive $s$ and $t$ iff

$$\lim_{\varepsilon^{-1} + d \to \infty} \frac{\ln n^X(\varepsilon, S_d)}{(\varepsilon^{-1})^s + d^t} = 0.$$

This paper is devoted to studying average case ($s, t$)-weak tractability of non-homogenous tensor product problems with covariance kernels corresponding to Euler and Wiener integrated processes, Korobov kernels, and analytic Korobov kernels. Such problems were investigated in [14] for Euler and Wiener integrated processes under NOR, and in [8] for analytic Korobov case under NOR and ABS. The authors in [14, 8] obtained that ($s, t$)-WT always holds with $s > 0$ and $t > 1$, and ($s, 1$)-WT with $s > 0$ holds iff WT holds. However, they did not obtain the matching necessary and sufficient conditions on ($s, t$)-WT with $s > 0$ and $t \in (0, 1)$. The matching necessary and sufficient condition on ($s, t$)-WT with $s > 0$ and $t \in (0, 1)$ was first obtained in [1] for average case multivariate approximation with Gaussian covariance kernels.

In this paper, we use a unified method to get a necessary and sufficient condition for ($s, t$)-WT for $t \in (0, 1)$ and $s > 0$. Specially for Euler and Wiener integrated processes, the measures $\mu_{td}$ are defined in terms of the nondecreasing sequence $\{r_k\}_{k \in \mathbb{N}}$ of nonnegative integers

$$0 \leq r_1 \leq r_2 \leq r_3 \leq \ldots.$$  

Roughly speaking, $r_k$ measures the smoothness of the process with respect to the $k$th variable. For the normalized error criterion, we obtain for $t \in (0, 1)$ and $s > 0$,

• for the Euler integrated process,

$$(s, t) - WT \iff \lim_{k \to \infty} k^{1-t}3^{-2r_k}(1 + r_k) = 0;$$

• for the Wiener integrated process,

$$(s, t) - WT \iff \lim_{k \to \infty} k^{1-t}(1 + r_k)^{-2} \ln^+(1 + r_k) = 0,$$
where \( \ln^+ x = \max(\ln x, 1) \).

The paper is organized as follows. In Section 2 we give the preliminaries about non-homogeneous tensor product problems in the average case setting and present the main results, i.e., Theorem 2.1. Section 3 is devoted to proving Theorem 2.1. In Section 4, we give the applications of Theorem 2.1 to the problems with covariance kernels corresponding to Euler and Wiener integrated processes, Korobov kernels, and analytic Korobov kernels.

2. Preliminaries and main results

We recall the concept of non-homogeneous linear multivariate tensor product problems in average case setting, see [4].

Let \( F_d, H_d \) are given by tensor products. That is,

\[
F_d = F_1^{(1)} \otimes F_2^{(1)} \otimes \cdots \otimes F_d^{(1)} \quad \text{and} \quad H_d = H_1^{(1)} \otimes H_2^{(1)} \otimes \cdots \otimes H_d^{(1)},
\]

where Banach spaces \( F_k^{(1)} \) are of univariate real functions equipped with a zero-mean Gaussian measure \( \mu_k^{(1)} \), and \( H_k^{(1)} \) are Hilbert spaces, \( k = 1, 2, \ldots, d \). We set

\[
S_d = S_1^{(1)} \otimes S_2^{(1)} \otimes \cdots \otimes S_d^{(1)}, \quad \mu_d = \mu_1^{(1)} \otimes \mu_2^{(1)} \otimes \cdots \otimes \mu_d^{(1)},
\]

where

\[
S_k^{(1)} = F_k^{(1)} \rightarrow H_k^{(1)}, \quad k = 1, 2, \ldots, d
\]

are continuous linear operators. Then \( \mu_d \) is a zero-mean Gaussian measure on \( F_d \) with covariance operator \( C_{\mu_d} : F_d^* \rightarrow F_d \).

Let \( \nu_d = \mu_d(S_d)^{-1} \) be the induced measure. Then \( \nu_d \) is a zero-mean Gaussian measure on \( H_d \) with covariance operator \( C_{\nu_d} : H_d \rightarrow H_d \) given by

\[
C_{\nu_d} = S_d C_{\mu_d} S_d^*,
\]

where \( S_d^* : H_d \rightarrow F_d^* \) is the operator dual to \( S_d \). Let \( \nu_k^{(1)} = \mu_k^{(1)}(S_k^{(1)})^{-1} \) be the induced zero-mean Gaussian measure on \( H_k^{(1)} \), and let \( C_{\nu_k^{(1)}} : H_k^{(1)} \rightarrow H_k^{(1)} \) be the covariance operator of the measure \( \nu_k^{(1)} \). Then

\[
\nu_d = \nu_1^{(1)} \otimes \nu_2^{(1)} \otimes \cdots \otimes \nu_d^{(1)}, \quad \text{and} \quad C_{\nu_d} = C_{\nu_1^{(1)}} \otimes C_{\nu_2^{(1)}} \otimes \cdots \otimes C_{\nu_d^{(1)}}.
\]

The eigenpairs of \( C_{\nu_k^{(1)}} \) are denoted by \( \{(\lambda(k,j), \eta(k,j))\}_{j \in \mathbb{N}^d} \), and satisfy

\[
C_{\nu_k^{(1)}}(\eta(k,j)) = \lambda(k,j) \eta(k,j), \quad \text{with} \quad \lambda(k,1) \geq \lambda(k,2) \geq \cdots \geq 0.
\]

Then

\[
\text{trace}(C_{\nu_k^{(1)}}) = \int_{H_k^{(1)}} \|f\|^2_{H_k^{(1)}} \nu_k^{(1)}(df) = \sum_{j=1}^{\infty} \lambda(k,j) < \infty.
\]

The eigenpairs of \( C_{\nu_d} \) are given by

\[
\left\{(\lambda_{d,j}, \eta_{d,j})\right\}_{j=(j_1, j_2, \ldots, j_d) \in \mathbb{N}^d},
\]

where

\[
\lambda_{d,j} = \prod_{k=1}^{d} \lambda(k, j_k) \quad \text{and} \quad \eta_{d,j} = \prod_{k=1}^{d} \eta(k, j_k).
\]
Let the sequence \( \{\lambda_{d,j}\}_{j \in \mathbb{N}} \) be the non-increasing rearrangement of \( \{\lambda_{d,j}\}_{j \in \mathbb{N}^d} \). Then we have
\[
\sum_{j \in \mathbb{N}} \lambda_{d,j}^r = \prod_{k=1}^{d} \sum_{j=1}^{\infty} \lambda(k,j)^r, \quad \text{for any} \quad r > 0.
\]

We approximate \( S_d \) by algorithms \( A_{n,d} \) that use only finitely many continuous linear functionals. A function \( f \in F_d \) is approximated by an algorithm (2.1)
\[
A_{n,d}(f) = \Phi_{n,d}(L_1(f), L_2(f), \ldots, L_n(f)),
\]
where \( L_1, L_2, \ldots, L_n \) are continuous linear functionals on \( F_d \), and \( \Phi_{n,d} : \mathbb{R}^n \to H_d \) is an arbitrary measurable mapping. The average case error for \( A_{n,d} \) is defined by
\[
e(A_{n,d}) = \left( \int_{F_d} \|S_d f - A_{n,d} f\|_{H_d}^2 \mu_d(df) \right)^{\frac{1}{2}}.
\]
The \( n \)th minimal average case error, for \( n \geq 1 \), is defined and given by (see [9])
\[
e(n,d) = \inf_{A_{n,d}} e(A_{n,d}) = \left( \sum_{j=n+1}^{\infty} \lambda_{d,j} \right)^{\frac{1}{r}},
\]
where the infimum is taken over all algorithms of the form (2.1). It is achieved by the \( n \)th optimal algorithm
\[
A_{n,d}^*(f) = \sum_{j=1}^{n} \langle f, \eta_{d,j} \rangle_{H_d} \eta_{d,j}.
\]
For \( n = 0 \), we use \( A_{0,d} = 0 \). We remark that the so-called initial error \( e(0,d) \) is defined and given by
\[
e(0,d) = \left( \int_{F_d} \|S_d f\|_{H_d}^2 \mu_d(df) \right)^{\frac{1}{2}} = \left( \sum_{j=1}^{\infty} \lambda_{d,j} \right)^{\frac{1}{r}}.
\]
The information complexity for \( S_d \) can be studied using either the absolute error criterion (ABS), or the normalized error criterion (NOR). Then we define the information complexity \( n_X(\varepsilon, S_d) \) for \( X \in \{\text{ABS, NOR}\} \) as
\[
n_X(\varepsilon, S_d) = \min \{n : e(n,S_d) \leq \varepsilon CRI_d\},
\]
where
\[
CRI_d = \begin{cases} 1, & \text{for } X=\text{ABS}, \\ e(0,S_d), & \text{for } X=\text{NOR}. \end{cases}
\]
In this paper we consider a special class of non-homogeneous tensor product problems \( S = \{S_d\}_{d \in \mathbb{N}} \). Assume that the eigenvalues
\[
\left\{ \prod_{k=1}^{d} \lambda(k,j_k) \right\}_{(j_1,j_2,\ldots,j_d) \in \mathbb{N}^d}
\]
of the covariance operator \( C_{\nu_d} \) of the problem \( S \) satisfy the following three conditions:
1. \( \lambda(k,1) = 1, \quad k \in \mathbb{N}; \)
2. there exist a decreasing positive sequence \( \{f_k\}_{k \in \mathbb{N}} \) and two positive constants \( A_2 \in (0,1), \ A_1 \geq 1 \) such that for all \( k \in \mathbb{N} \), we have
\[
A_2 f_k \leq h_k \leq A_1 f_k,
\]
where \( h_k = \frac{\lambda(k,2)}{\lambda(k,1)} \in (0,1) \):

(3) there exist two constants \( \tau_0 \in (0,1) \) and \( M_{\tau_0} \) for which

\[
\sup_{k \in \mathbb{N}} H(k, \tau_0) \leq M_{\tau_0} < \infty,
\]

where

\[
H(k, x) := \sum_{j=2}^{\infty} \left( \frac{\lambda(k,j)}{\lambda(k,2)} \right)^x.
\]

Then we say that the problem \( S = \{S_d\}_{d \in \mathbb{N}} \) has Property (P).

We make some comments on Property (P). Usually, the sequence \( \{h_k\} \) in Condition (2) is decreasing. In this case, \( A_1 = A_2 = 1 \), and \( f_k = h_k \), \( k \in \mathbb{N} \). For the problem \( S \) with Property (P), we have for \( \varepsilon \in (0,1) \) and \( d \in \mathbb{N} \),

\[
(2.2) \quad n^{\text{ABS}}(\varepsilon, d) \geq n^{\text{NOR}}(\varepsilon, d).
\]

Note that for any \( x \geq \tau_0, \; k \in \mathbb{N} \),

\[
\ln \left( \sum_{k=1}^{d} \lambda_{d,k}^x \right) = \sum_{k=1}^{d} \ln(1 + h_k^x H(k,x)), \quad \text{and} \quad 1 \leq H(k, x) \leq H(k, \tau_0) \leq M_{\tau_0} < \infty.
\]

According to Conditions (2) and (3), we have for any \( x \geq \tau_0 \),

\[
(2.3) \quad \ln 2 \sum_{k=1}^{d} h_{k}^x \leq \sum_{k=1}^{d} \ln(1 + h_k^x) \leq \ln \left( \sum_{k=1}^{d} \lambda_{d,k}^x \right) \leq \sum_{k=1}^{d} \ln(1 + M_{\tau_0} h_k^x) \leq M_{\tau_0} \sum_{k=1}^{d} h_{k}^x,
\]

where in the first inequality we used the inequality \( \ln(1 + x) \geq x \ln 2, \; x \in [0,1] \), and in the last inequality we used the inequality \( \ln(1 + x) \leq x, \; x > 0 \).

We are ready to present the main result of this paper.

**Theorem 2.1.** Let \( S = \{S_d\}_{d \in \mathbb{N}} \) be a non-homogeneous tensor product problem with Property (P). Then for NOR or ABS, \((s, t)\)-WT holds with \( s > 0 \) and \( t \in (0,1) \) iff

\[
(2.4) \quad \lim_{k \to \infty} k^{1-t} f_k \ln + \frac{1}{f_k} = 0.
\]

**Remark 2.2.** Let \( S = \{S_d\}_{d \in \mathbb{N}} \) be a non-homogeneous tensor product problem with Property (P). Using the method of [8, 13], we can obtain that for ABS or NOR, \((s, t)\)-WT always holds with \( s > 0 \) and \( t > 1 \), and \((s, 1)\)-WT holds with \( s > 0 \) iff WT holds iff

\[
\lim_{k \to \infty} f_k = 0.
\]

**Remark 2.3.** Let \( S = \{S_d\}_{d \in \mathbb{N}} \) be a non-homogeneous tensor product problem. If the eigenvalues of the covariance operator \( C_{\nu_d} \) of the problem \( S \) satisfy Conditions (2) and (3), then for NOR, \((s, t)\)-WT holds with \( s > 0 \) and \( t \in (0,1) \) iff (2.4) holds.

Indeed, let \( \tilde{S} = \{\tilde{S}_d\}_{d \in \mathbb{N}} \) be the non-homogeneous tensor product problem which the eigenvalues \( \{ \prod_{k=1}^{d} \tilde{\lambda}(k,j) \}_{(j_1, j_2, \ldots, j_d) \in \mathbb{N}^d} \) of the corresponding covariance operator \( C_{\nu_d} \) of the induced measure of \( \tilde{S} \) satisfy

\[
\tilde{\lambda}(k,j) = \frac{\lambda(k,j)}{\lambda(k,1)} \; \text{for} \; j \in \mathbb{N}, \; k = 1, \ldots, d.
\]
Then \( \tilde{S} \) has Property (P) with the same \( h_k \). Also for NOR, the problems \( S \) and \( \tilde{S} \) have the same tractability. Hence, for NOR, \((s,t)\)-WT holds with \( s > 0 \) and \( t \in (0,1) \) iff (2.4) holds.

In order to prove Theorem 2.1 we need the following lemma.

**Lemma 2.4.** Let \( S = \{S_d\}_{d \in \mathbb{N}} \) be a non-homogeneous tensor product problem. Then for NOR, we have for \( x > 0 \)

\[
n^\text{NOR}(\varepsilon, S_d) \geq (1 - \varepsilon^2)^{\frac{x+1}{\varepsilon_x}} \left( \prod_{k=1}^{d} \frac{1 + h_k}{1 + h_k^{x+1}} \right)^{\frac{1}{x}},
\]

where

\[
h_k = \frac{\lambda(k, 2)}{\lambda(k, 1)} \in (0, 1].
\]

**Proof.** We set

\[
n = n^\text{NOR}(\varepsilon, S_d), \quad \bar{\lambda}_{d,k} = \frac{\lambda_{d,k}}{\sum_{k=1}^{\infty} \lambda_{d,k}}.
\]

It follows from the definition of \( n^\text{NOR}(\varepsilon, S_d) \) that

\[
1 - \sum_{k=1}^{n} \bar{\lambda}_{d,k} = \sum_{k=n+1}^{\infty} \bar{\lambda}_{d,k} \leq \varepsilon^2.
\]

We have

\[
1 - \varepsilon^2 \leq \sum_{k=1}^{n} \bar{\lambda}_{d,k} \leq n^{\frac{1}{\varepsilon_x}} \left( \sum_{k=1}^{n} \bar{\lambda}_{d,k}^{x+1} \right)^{\frac{1}{x+1}} \leq n^{\frac{1}{\varepsilon_x}} \left( \sum_{k=1}^{\infty} \bar{\lambda}_{d,k}^{x+1} \right)^{\frac{1}{x+1}},
\]

which leads to

\[
n^\text{NOR}(\varepsilon, S_d) = n \geq (1 - \varepsilon^2)^{\frac{x+1}{\varepsilon_x}} \left( \sum_{k=1}^{n} \bar{\lambda}_{d,k}^{x+1} \right)^{\frac{1}{x+1}}
\]

\[
= (1 - \varepsilon^2)^{\frac{x+1}{\varepsilon_x}} \left( \prod_{k=1}^{d} \frac{\sum_{j=1}^{\infty} \lambda(k, j)^{x+1}}{\sum_{j=1}^{\infty} (\lambda(k, j))^{x+1}} \right)^{\frac{1}{x}}.
\]

We note that for \( k = 1, 2, \ldots, d, j \geq 3 \) and \( x > 0 \),

\[
\lambda(k, j)(\lambda(k, i))^{x+1} \geq (\lambda(k, j))^{x+1} \lambda(k, i), \quad i = 1, 2.
\]

It follows that

\[
\lambda(k, j)((\lambda(k, 1))^{x+1} + (\lambda(k, 2))^{x+1}) \geq (\lambda(k, j))^{x+1} (\lambda(k, 1) + \lambda(k, 2)),
\]

and so

\[
\sum_{j=1}^{\infty} \lambda(k, j)((\lambda(k, 1))^{x+1} + (\lambda(k, 2))^{x+1}) \geq \sum_{j=1}^{\infty} (\lambda(k, j))^{x+1} (\lambda(k, 1) + \lambda(k, 2)).
\]

This implies that

\[
1 + \frac{\sum_{j=1}^{\infty} \lambda(k, j)}{\lambda(k, 1) + \lambda(k, 2)} \geq 1 + \frac{\sum_{j=1}^{\infty} (\lambda(k, j))^{x+1}}{\prod_{k=1}^{d} (\lambda(k, j))^{x+1} + (\lambda(k, 2))^{x+1}}.
\]

Hence, we have

\[
\left( \frac{\sum_{j=1}^{\infty} \lambda(k, j)}{\lambda(k, 1) + \lambda(k, 2)} \right)^{x+1} \geq \left( \frac{\sum_{j=1}^{\infty} \lambda(k, j)}{\lambda(k, 1) + \lambda(k, 2)} \right)^{x+1} \geq \left( \frac{\sum_{j=1}^{\infty} (\lambda(k, j))^{x+1}}{\prod_{k=1}^{d} (\lambda(k, j))^{x+1} + (\lambda(k, 2))^{x+1}} \right)^{x+1}.
\]
It follows that
\[
\frac{\left( \sum_{j=1}^{\infty} \lambda(k, j) \right)^{x+1}}{\sum_{j=1}^{\infty} (\lambda(k, j))^{x+1}} \geq \frac{\left( \lambda(k, 1) + \lambda(k, 2) \right)^{x+1}}{\lambda(k, 1)^{x+1} + (\lambda(k, 2))^{x+1}} = \frac{(1 + h_k)^{x+1}}{1 + h_k^{x+1}}.
\]

By (2.5) and the above inequality, we get
\[
n^{\text{NOR}}(\varepsilon, S_d) \geq (1 - \varepsilon^2)^{\frac{x+1}{2}} \left( \prod_{k=1}^{d} \frac{1 + h_k}{1 + h_k^{x+1}} \right)^{\frac{x}{2}}.
\]

Lemma 2.4 is proved. \(\square\)

3. Proof of Theorem 2.1

Proof of Theorem 2.1

We first show that (2.4) holds whenever \((s, t)\)-WT holds with \(s > 0\) and \(t \in (0, 1)\) for NOR or ABS. Due to (2.2), it suffices to prove (2.4) under NOR.

Assume that \(s > 0\) and \(t \in (0, 1)\). Suppose that \((s, t)\)-WT holds for NOR. We set
\[
(3.1) \quad u_k := \max(f_k, \frac{1}{2k}), \quad \text{and} \quad s_k := \frac{1}{2} \left( \ln + \frac{1}{u_k} \right)^{-1}, \quad k \in \mathbb{N},
\]
where \(f_k\) is given in Condition (2) of Property (P). Then \(\{u_k\}\) is monotonically decreasing. We want to show that \(\lim_{j \to \infty} u_j = 0\).

It follows from (3.1) and \(\lambda_{d, 1} = 1\) that
\[
1 - \varepsilon^2 \leq \sum_{k=1}^{n^{\text{NOR}}(\varepsilon, S_d)} \lambda_{d,k} \leq n^{\text{NOR}}(\varepsilon,S_d) \lambda_{d,1} = n^{\text{NOR}}(\varepsilon,S_d) \left( \sum_{k=1}^{\infty} \lambda_{d,k} \right)^{-1}.
\]

This implies that
\[
(3.2) \quad \ln n^{\text{NOR}}(\varepsilon,S_d) \geq \ln(1 - \varepsilon^2) + \ln \left( \sum_{k=1}^{\infty} \lambda_{d,k} \right).
\]

By (2.3) and Condition (2) of Property (P), we get
\[
(3.3) \quad \ln \left( \sum_{k=1}^{\infty} \lambda_{d,k} \right) \geq \ln 2 \sum_{k=1}^{d} h_k \geq A_2 \ln 2 \sum_{k=1}^{d} f_k \geq A_2 (\ln 2) d f_d.
\]

Since \((s, t)\)-WT holds for NOR, we obtain by (3.2) and (3.3) that
\[
0 = \lim_{d \to \infty} \frac{\ln(n^{\text{NOR}}(\varepsilon,S_d))}{(\frac{1}{2})^{\frac{\ln(\frac{1}{2}) + A_2 (\ln 2) d f_d}{d^t}}} \geq \lim_{d \to \infty} \frac{\ln \frac{3}{4} + A_2 (\ln 2) d f_d}{d^t} = A_2 \ln 2 \lim_{d \to \infty} d^{1-t} f_d \geq 0,
\]
which implies \(\lim_{d \to \infty} d^{1-t} f_d = 0\) and hence \(\lim_{d \to \infty} u_d = 0\).
Applying Lemma 2.4 with \( x = s_d > 0 \), we obtain
\[
\ln \left( n_{\text{NOR}}^{\text{A}}(1/2, S_d) \right) \geq \frac{s_d + 1}{s_d} \ln \frac{3}{4} + \frac{1}{s_d} \sum_{k=1}^{d} \ln \left( \frac{1 + h_k}{1 + h_k^{s_d+1}} \right)
\]
\[
\geq \frac{1}{s_d} \ln \frac{3}{4} + \frac{1}{s_d} \sum_{k=1}^{d} \left( \frac{h_k - h_k^{s_d+1}}{1 + h_k^{s_d+1}} \right) \ln 2
\]
\[
\geq \frac{1}{s_d} \ln \frac{3}{4} + \frac{2}{2s_d} \sum_{k=1}^{d} (h_k - h_k^{s_d+1}),
\]
(3.4)
where in the second inequality we used the inequality \( \ln(1 + x) \geq x \ln 2, \ x \in [0, 1] \).

We remark that the function \( u(x) = x - x^{1+s_d} \) is monotonically increasing in \((0, e^{-1})\). Since \( \lim_{d \to \infty} u_d = 0 \), there exists a positive integer \( K \) such that \( 0 < u_k < e^{-1} \) holds for any \( k \geq K \). It follows that
\[
\sum_{k=1}^{d} (h_k - h_k^{s_d+1}) \geq \sum_{k=K}^{d} (h_k - h_k^{s_d+1})
\]
\[
\geq \sum_{k=K}^{d} (A_2 f_k - (A_2 f_d)^{s_d+1})
\]
\[
\geq (d - K) (A_2 f_d - (A_2 f_d)^{s_d+1}).
\]
(3.5)

By (3.1) we get
\[
\frac{1}{s_d} = 2 \ln^+ \left( \frac{1}{u_d} \right) \leq 2 \ln^+ (2d), \ \text{and} \ \lim_{d \to \infty} \frac{1}{s_d d^t} = \lim_{d \to \infty} \frac{2 \ln^+ (2d)}{d^t} = 0.
\]
(3.6)

Since \((s, t)\)-WT holds for NOR, we obtain by (3.4), (3.5), and (3.6) that
\[
0 = \lim_{d \to \infty} \frac{\ln \left( n_{\text{NOR}}^{\text{A}}(1/2, S_d) \right)}{2^s + d^t}
\]
\[
\geq \lim_{d \to \infty} \left( \frac{\ln \frac{3}{4}}{s_d d^t} + \frac{(d - K) \ln 2}{2s_d d^t} (A_2 f_d - (A_2 f_d)^{s_d+1}) \right)
\]
\[
= \frac{\ln 2}{2} \lim_{d \to \infty} \frac{d^{1-t} - (A_2 f_d - (A_2 f_d)^{s_d+1})}{s_d}
\]
which yields that
\[
\lim_{d \to \infty} \frac{d^{1-t}}{s_d} (A_2 f_d - (A_2 f_d)^{s_d+1}) = 0.
\]
(3.7)

Applying the mean value theorem to the function \( \phi(x) = a^x, \ a \in (0, 1) \), we obtain for some \( \theta \in (0, 1) \),
\[
a^{s_d} a^{s_d} \ln \left( \frac{1}{a} \right) \leq a - a^{1+s_d} = a^{1+\theta s_d} a^{s_d} \ln \left( \frac{1}{a} \right) \leq a^{s_d} \ln \left( \frac{1}{a} \right)
\]
(3.8)
We get by (3.8) that
\[
0 \leq \lim_{d \to \infty} \frac{d^{1-t}}{s_d} \left( \frac{A_2}{2d} - (\frac{A_2}{2d})^{s_d+1} \right) \leq \lim_{d \to \infty} d^{1-t} \left( \frac{A_2}{2d} \right) \ln \left( \frac{2d}{A_2} \right) = 0
\]
which combining with (3.7), gives that

$$
(3.9) \quad \lim_{d \to \infty} \frac{d^{1-t}}{s_d} \left( A_2 u_d - (A_2 u_d)^{s_d+1} \right) = 0.
$$

Noting that

$$
\lim_{d \to \infty} (A_2 u_d)^{s_d} = \lim_{d \to \infty} \exp \left( -\frac{\ln \frac{1}{A_2} + \ln \frac{1}{u_d}}{2\ln u_d} \right) = e^{-1/2},
$$

by (3.8) we have

$$
0 = \lim_{d \to \infty} \frac{d^{1-t}}{s_d} (A_2 u_d - (A_2 u_d)^{s_d+1})
\geq \lim_{d \to \infty} d^{1-t} (A_2 u_d) (A_2 u_d)^{s_d} \ln \left( \frac{1}{A_2 u_d} \right)
\geq e^{-1/2} A_2 \lim_{d \to \infty} d^{1-t} u_d \ln^+ \left( \frac{1}{u_d} \right) \geq 0,
$$

which implies that

$$
\lim_{d \to \infty} d^{1-t} u_d \ln^+ \left( \frac{1}{u_d} \right) = 0.
$$

Hence, we conclude from the monotonicity of the function \( \varphi(x) = x \ln^+ \left( \frac{1}{x} \right) = x \ln \left( \frac{1}{x} \right), \ x \in (0, 1/e) \) that

$$
0 \leq \lim_{d \to \infty} d^{1-t} f_d \ln^+ \frac{1}{f_d} \leq \lim_{d \to \infty} d^{1-t} u_d \ln^+ \frac{1}{u_d} = 0,
$$

giving (2.4).

Next we show that \((s, t)\)-WT with \(s > 0\) and \(t \in (0, 1)\) holds for NOR or ABS whenever (2.4) holds. Due to (2.2), it suffices to prove \((s, t)\)-WT holds for ABS.

We have for \(\tau \in (0, 1)\),

$$
\sum_{k=n+1}^{\infty} \lambda_{d,k} \leq \sum_{k=n+1}^{\infty} \lambda_{d,n+1}^{1-\tau} \leq \sum_{k=1}^{\infty} \lambda_{d,k}^{1-\tau}.
$$

Since

$$
(n+1) \lambda_{d,n+1}^{1-\tau} \leq \sum_{k=1}^{\infty} \lambda_{d,k}^{1-\tau} \leq \sum_{k=1}^{\infty} \lambda_{d,k}^{1-\tau},
$$

we get

$$
\lambda_{d,n+1} \leq (n+1)^{-\frac{1}{1-\tau}} \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1-\tau} \right)^{-\frac{1}{1-\tau}},
$$

which combining with (3.10) yields

$$
\sum_{k=n+1}^{\infty} \lambda_{d,k} \leq (n+1)^{-\frac{1}{1-\tau}} \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1-\tau} \right)^{-\frac{1}{1-\tau}}.
$$

Setting

$$
n = \left\lfloor \varepsilon^{-\frac{2(1-\tau)}{1-\tau}} \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1-\tau} \right)^{\frac{1}{1-\tau}} \right\rfloor
$$

in (3.11), we have

$$
\sum_{k=n+1}^{\infty} \lambda_{d,k} \leq \varepsilon^2.
$$
It follows from the definition of \( n^{\text{ABS}}(\varepsilon, S_d) \) that
\[
(3.12) \quad n^{\text{ABS}}(\varepsilon, S_d) \leq \left\lfloor \frac{\varepsilon}{2} \right\rfloor \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1-\tau} \right)^{\frac{1}{\tau}} \leq \frac{\varepsilon}{2} \left( \sum_{k=1}^{\infty} \lambda_{d,k}^{1-\tau} \right)^{\frac{1}{\tau}}.
\]

We let \( \tau = s_d \), where \( s_k, u_k \) are given in (3.1). By (3.12) and (2.3) we obtain
\[
\ln n^{\text{ABS}}(\varepsilon, S_d) \leq \frac{2(1 - s_d)}{s_d} \ln \varepsilon^{-1} + \frac{1}{s_d} \ln \left( \sum_{k=1}^{d} h_{k}^{1-s_d} \right)
\]
\[
\leq \frac{2}{s_d} \ln \varepsilon^{-1} + \frac{M_{r_0}}{s_d} \sum_{k=1}^{d} h_{k}^{1-s_d}
\]
\[
\leq \frac{2}{s_d} \ln \varepsilon^{-1} + \frac{M_{r_0} A_1^{1-s_d}}{s_d} \sum_{k=1}^{d} u_{k}^{1-s_d}.
\]

Noting that \( A_1 > 1 \) and
\[
u_{k}^{-s_d} = \exp \left( \frac{\ln \frac{1}{u_k}}{2 \ln^2 \frac{1}{u_k}} \right) \leq e^{1/2}, \quad k = 1, 2, \ldots, d,
\]
we continue to get
\[
(3.13) \quad \ln n^{\text{ABS}}(\varepsilon, S_d) \leq \frac{2}{s_d} \ln \varepsilon^{-1} + \frac{e^{1/2} M_{r_0} A_1}{s_d} \sum_{k=1}^{d} u_{k}.
\]

Assume that (2.4) holds. Note that
\[
\lim_{d \to \infty} d^{1-t} \left( \frac{1}{2d} \right) \ln^+(2d) = 0.
\]
It follows from the monotonicity of the function \( \varphi(t) = t \ln + \left( \frac{1}{t} \right), \quad t \in (0, 1/e) \) that
\[
\lim_{d \to \infty} d^{1-t} u_d \ln \frac{1}{u_d} = 0.
\]
It follows from (3.6) that
\[
0 \leq \lim_{\varepsilon \to 0^+, d \to \infty} \frac{2}{s_d} \ln \varepsilon^{-1} \varepsilon^{-s} + d^s \leq \lim_{\varepsilon \to 0^+, d \to \infty} \frac{s_d^{-2} + (\ln \varepsilon^{-1})^2}{\varepsilon^{-s} + d^s} = 0.
\]
In order to show that \((s, t)\)-WT holds for ABS, by (3.13) we only need to prove
\[
(3.14) \quad \lim_{d \to \infty} \frac{1}{d^s s_d} \sum_{k=1}^{d} u_{k} = 0.
\]

We know that \( \varphi(t) = t \ln \left( \frac{1}{t} \right) \) is monotonically increasing in \((0, e^{-\varepsilon})\). So the inverse function \( \varphi^{-1}(t) \) is also monotonically increasing in \( t \in (0, e^{1-e}) \). Let \( y = \varphi(t) = t \ln \left( \frac{1}{t} \right), \quad t \in (0, e^{-\varepsilon}) \). Then we have
\[
\ln \left( \frac{1}{y} \right) = \frac{1}{2} \ln \frac{1}{t} + \left( \frac{1}{2} \ln \frac{1}{t} - \ln \left( \ln \frac{1}{t} \right) \right) \geq \frac{1}{2} \ln \frac{1}{t},
\]
since \( \psi(x) = \frac{x}{2} - \ln x \) is increasing in \([2, \infty)\) and hence
\[
\psi \left( \ln \frac{1}{t} \right) \geq \psi(e) = e/2 - 1 > 0, \quad t \in (0, e^{-\varepsilon}).
\]
We get further
\[(3.15) \quad t = \varphi^{-1}(y) = \frac{y}{\ln y} \leq \frac{2y}{\ln y}.
\]

Since
\[\lim_{d \to \infty} d^{1-t}u_d \ln^+ \frac{1}{u_d} = 0,
\]
for any \(\varepsilon \in (0, 1)\) there exists an integer \(K_1 \geq 4\) such that for all \(k \geq K_1\),
\[0 < u_k < e^{-\varepsilon}, \quad \text{and} \quad k^{1-t}u_k \ln \frac{1}{u_k} \leq \varepsilon.
\]

This yields
\[\varphi(u_k) \leq \varepsilon k^{t-1}.
\]

It follows from (3.15) that
\[u_k \leq \varphi^{-1}(\varepsilon k^{t-1}) \leq \frac{2}\ln \frac{2}{\varepsilon k^{t-1}} = \frac{2\varepsilon k^{t-1}}{\ln \varepsilon^{-1} + (1-t)\ln k} \leq \frac{2\varepsilon k^{t-1}}{(1-t)\ln k}.
\]

We notice that \(v(x) = \frac{t^{1/2}}{\ln x}\) is increasing in \([e^{2/t}, \infty)\) due to the fact that
\[v'(x) = \frac{x^{t/2-1}}{\ln^2 x} \left( \frac{t}{2 \ln x - 1} \right) \geq 0.
\]

It follows from (3.6) that for the above \(\varepsilon \in (0, 1)\) there exists a positive integer \(K_2\) for which
\[(3.16) \quad \frac{1}{sd^t} \leq \varepsilon
\]
holds for any \(k \geq K_2\). We set
\[K = \max(K_1, \lfloor e^{2/t} \rfloor + 1, K_2).
\]

Then for any \(d > K\), we have
\[\sum_{k=1}^{d} u_k \leq \sum_{k=1}^{K} \frac{1}{A_2} \max(h_k, \frac{1}{2k}) + \sum_{k=K+1}^{d} \frac{2\varepsilon k^{t/2-1}k^{t/2}}{(1-t)\ln k}\]
\[\leq \sum_{k=1}^{K} \frac{1}{A_2} + \frac{2\varepsilon d^{1/2}}{(1-t)\ln d} \sum_{k=K+1}^{d} k^{t/2-1}
\]
\[(3.17) \quad \leq \frac{K}{A_2} + \frac{4\varepsilon d^t}{t(1-t)\ln d},
\]
where in the last inequality we used the inequality
\[\sum_{k=K+1}^{d} k^{t/2-1} \leq \sum_{k=1}^{d} k^{t/2-1} \leq \sum_{k=1}^{d} \int_{k-1}^{k} x^{t/2-1}dx \leq \int_{0}^{d} x^{t/2-1}dx = \frac{2}{t} d^{t/2}.
\]
It follows from (3.17), (3.16), and (3.6) that
\[
\frac{1}{d^s s_d} \sum_{k=1}^{d} u_k \leq \frac{K}{A_2} \frac{1}{d^s s_d} + \frac{4\varepsilon}{t(1-t)s_d \ln d} \\
\leq \frac{K \varepsilon}{A_2} + \frac{8\varepsilon \ln(2d)}{t(1-t) \ln d} \\
\leq \varepsilon \left( \frac{K A_2}{A_2} + \frac{16}{t(1-t)} \right).
\]
This gives (3.14). We conclude that \((s, t)\)-WT holds for ABS.
The proof of Theorem 2.1 is completed. \(\square\)

4. Applications of Theorem 2.1

Consider the approximation problem \(S = \{S_d\}_{d \in \mathbb{N}},\)
\(S_d : C([0, 1]^d) \to L_2([0, 1]^d)\) with \(S_d(f) = f.\)
The space \(C([0, 1]^d)\) of continuous real functions is equipped with a zero-mean Gaussian measure \(\mu_d\) whose covariance kernel is given by
\[K_d(x, y) = \int_{C([0, 1]^d)} f(x) f(y) \mu_d(df), \quad x, y \in [0, 1]^d.
\]
The covariance kernels \(K_d(x, y)\) are of tensor product and correspond to Euler and Wiener integrated processes, Korobov kernels, and analytic Korobov kernels. This section is devoted to giving the applications of Theorem 2.1 to these cases.

4.1. \((s, t)\)-WT of Euler and Wiener integrated processes.

In this subsection we consider multivariate approximation problems \(S = \{S_d\}_{d \in \mathbb{N}}\) defined over the space \(C([0, 1]^d)\) equipped with zero-mean Gaussian measures whose covariance kernels corresponding to Euler and Wiener integrated processes. We briefly recall Wiener and Euler integrated processes.

Let \(W(t), t \in [0, 1],\) be a standard Wiener process, i.e. a Gaussian random process with zero mean and covariance kernel
\[K_{W,0}(s, t) = \min(s, t).
\]

Consider two sequences of integrated random processes \(X^E_r, X^W_r\) on \([0, 1]\) defined inductively on \(r\) by \(X^E_0 = X^W_0 = W\) and for \(r = 0, 1, 2, \ldots\)
\[X^E_{r+1}(t) = \int_{t-r}^{1} X^E_r(s) ds,
\]
\[X^W_{r+1}(t) = \int_{0}^{t} X^W_r(s) ds.
\]
The process \(\{X^E_r\}\) is called the univariate integrated Euler process, while \(\{X^W_r\}\) is called the univariate integrated Wiener process.

Clearly, the corresponding Gaussian measures to \(X^W_r\) and \(X^E_r\) are concentrated on a set of functions which are \(r\) times continuously differentiable but satisfy different boundary conditions.

The covariance kernel of \(X^E_r\) is given by
\[K^E_{1,r}(x, y) = \int_{[0, 1]^r} \min(x, s_1) \min(s_1, s_2) \ldots \min(s_r, y) ds_1 ds_2 \ldots ds_r.
\]
and is called the Euler kernel. The last kernel can be expressed in terms of Euler polynomials. The covariance kernel of \( X^W \) is given by

\[
K^W_{1,r}(x,y) = \int_0^{\min(x,y)} \frac{(x-u)^r}{r!} \frac{(y-u)^r}{r!} du
\]

and is called the Wiener kernel.

The corresponding tensor product kernels on \([0,1]^d\) are given by

\[
K^E_d(x,y) = \prod_{k=1}^d K^E_{1,r_k}(x_k, y_k) \quad \text{and} \quad K^W_d(x,y) = \prod_{k=1}^d K^W_{1,r_k}(x_k, y_k).
\]

Here \( \{r_k\}_{k \in \mathbb{N}} \) is a sequence of nondecreasing nonnegative integers

\[
0 \leq r_1 \leq r_2 \leq r_3 \leq \ldots
\]

They describe the increasing smoothness of a process with respect to the successive coordinates.

For the problems \( S \), the eigenvalues of the covariance operators of the induced measures corresponding to Euler and Wiener integrated processes are known (see [3]):

\[
\{\lambda^Y_{d,j}\}_{j \in \mathbb{N}} = \{\lambda^Y(1,j_1)\lambda^Y(2,j_2)\ldots\lambda^Y(d,j_d)\}_{(j_1, \ldots, j_d) \in \mathbb{N}^d}, \quad Y \in \{E,W\},
\]

where

\[
\lambda^E(k,j) = \left( \frac{1}{\pi(j - \frac{1}{2})} \right)^{2r_k+2},
\]

for all \( j \in \mathbb{N} \), and

\[
\lambda^W(k,j) = \left( \frac{1}{\pi(j - \frac{1}{2})} \right)^{2r_k+2} + O(j^{-2r_k-3}), \quad j \to \infty,
\]

where for two nonnegative sequences \( f, g : \mathbb{N} \to [0, \infty) \),

\[
f(k) = O(g(k)), \quad k \to \infty
\]

means that there exists two constants \( C > 0 \) and \( k_0 \in \mathbb{N} \) for which \( f(k) \leq Cg(k) \) holds for any \( k \geq k_0 \), and

\[
f(k) = \Theta(g(k)), \quad k \to \infty
\]

means that

\[
f(k) = O(g(k)) \quad \text{and} \quad g(k) = O(f(k)), \quad k \to \infty.
\]

Note that for all \( k \in \mathbb{N} \),

\[
f_k^E = h_k^E = \frac{\lambda^E(k,2)}{\lambda^E(k,1)} = \frac{1}{3^{2r_k+2}}.
\]

In this case, we set \( \tau_0 \in (1/2, 1) \). By [42] we have

\[
\sup_{k \in \mathbb{N}} H^E(k, \tau_0) = \sup_{k \in \mathbb{N}} \sum_{j=2}^{\infty} \left( \frac{\lambda^E(k,j)}{\lambda^E(k,2)} \right)^{\tau_0} \leq \sum_{j=1}^{\infty} \left( \frac{3}{2j+1} \right)^{2\tau_0} < \infty.
\]
It is proved in [6] that
\[ \lambda^W(k, 1) = \frac{1}{(r_k!)^2} \left( \frac{1}{(2r_k + 2)(2r_k + 1)} + O(r_k^{-4}) \right), \quad k \to \infty, \]
\[ \lambda^W(k, 2) = \Theta \left( \frac{1}{(r_k!)^2 r_k^2} \right), \quad k \to \infty. \]

Note that
\[ h^W_k = \lambda^W(k, 2) = \Theta \left( \frac{1}{(r_k!)^2 r_k^4} \right), \quad k \to \infty. \]

We conclude that the problem \( S \) corresponding to the Wiener integrated process satisfies Condition (2) with \( \lambda^W(k, 2) = (1 + r_k)^{-2}, k \in \mathbb{N} \).

From [6, Thm. 4.1] it follows that for \( \tau \in (3/5, 1] \),
\[ A_\tau := \sup_{k \in \mathbb{N}} \sum_{j=3}^{\infty} \left( \frac{\lambda^W(k, j)}{\lambda^W(k, 2)} \right)^\tau < \infty. \]

This implies that for \( \tau_0 \in (3/5, 1), x \geq \tau_0 \), we have
\[ \sup_{k \in \mathbb{N}} H^W(k, \tau_0) = 1 + \sup_{k \in \mathbb{N}} \sum_{j=3}^{\infty} \left( \frac{\lambda^W(k, j)}{\lambda^W(k, 2)} \right)^{\tau_0} \leq 1 + A_\tau_0 < \infty. \]

According to (4.3) and (4.4), we know that the problems \( S \) corresponding to the Euler and Wiener integrated processes satisfy Conditions (2) and (3) with \( f^E_k = 3^{-r_k}, f^W_k = (1 + r_k)^{-2} \). By Remark 2.3, we have the following theorem.

**Theorem 4.1.** Consider the problems \( S = \{S_d\} \) in the average case setting with a zero mean Gaussian measure whose covariance kernels corresponding to Euler and Wiener integrated processes with the smoothness \( r_k \) satisfying (4.1). Assume that \( s > 0 \) and \( t \in (0, 1) \). Then for NOR, we have

(1) for the Euler integrated process, \( (s, t) \)-WT holds iff
\[ \lim_{k \to \infty} k^{1-t} (1 + r_k)^{-2r_k} \ln (1 + r_k) = 0. \]

(2) for the Wiener integrated process, \( (s, t) \)-WT holds iff
\[ \lim_{k \to \infty} k^{1-t} (1 + r_k)^{-2} \ln^+ (1 + r_k) = 0. \]

We recall tractability results of the above problems \( S \) corresponding to Euler and Wiener integrated processes under the assumption (4.1) and using NOR. The sufficient and necessary conditions for SPT, PT, QPT and WT were obtained in [6], for UWT in [13], and for \( (s, t) \)-WT with \( s > 0 \) and \( t \geq 1 \) in [14]. In [13], Siedlecki also got the sufficient conditions and the necessary conditions on \( (s, t) \)-WT with \( s > 0 \) and \( t \in (0, 1) \). However, these conditions do not completely match. Combining with our results, we have the following results about the tractability of the above problem \( S \) using NOR.

**For the Euler integrated process under NOR:**

- SPT holds iff PT holds iff
\[ \lim_{k \to \infty} \frac{r_k}{\ln k} > \frac{1}{2 \ln 3}. \]
• QTP holds iff
\[ \sup_{d \in \mathbb{N}} \frac{\sum_{k=1}^{d}(1 + r_k)3^{-2r_k}}{\ln^+ d} < \infty. \]

• UWT holds iff
\[ \lim_{k \to \infty} \frac{r_k}{\ln k} \geq \frac{1}{2 \ln 3}. \]

• \((s, t)\)-WT with \(s > 0\) and \(t > 1\) always holds.

• \((s, 1)\)-WT with \(s > 0\) holds iff WT holds iff
\[ \lim_{k \to \infty} r_k = \infty. \]

• \((s, t)\)-WT with \(s > 0\) and \(t \in (0, 1)\) holds iff
\[ \lim_{k \to \infty} k^{1-t}3^{-2r_k}(1 + r_k) = 0. \]

For the Wiener integrated process under NOR:

• SPT holds iff PT holds iff
\[ (4.5) \lim_{k \to \infty} \frac{\ln r_k}{\ln k} > \frac{1}{2}. \]

• QTP holds iff
\[ \sup_{d \in \mathbb{N}} \frac{\sum_{k=1}^{d}(1 + r_k)^{-2} \ln^+ r_k}{\ln^+ d} < \infty. \]

• UWT holds iff
\[ \lim_{k \to \infty} \frac{\ln r_k}{\ln k} \geq \frac{1}{2}. \]

• \((s, t)\)-WT with \(s > 0\) and \(t > 1\) always holds.

• \((s, 1)\)-WT with \(s > 0\) holds iff WT holds iff
\[ \lim_{k \to \infty} r_k = \infty. \]

• \((s, t)\)-WT with \(s > 0\) and \(t \in (0, 1)\) holds iff
\[ \lim_{k \to \infty} k^{1-t}(1 + r_k)^{-2} \ln^+(1 + r_k) = 0. \]

Remark 4.2. The authors in [6] obtained that the sufficient and necessary condition for SPT or PT under NOR is
\[ \lim_{k \to \infty} \frac{r_k}{k^v} > 0 \text{ for some } v > \frac{1}{2}. \]

However, it is easy to verify that this condition is equivalent to (4.5).
4.2. Average-case \((s,t)\)-WT with Korobov kernels.

In this subsection we consider a multivariate approximation problem \(S = \{S_d\}\) defined over the space \(C([0,1]^d)\) equipped with a zero-mean Gaussian measure whose covariance kernel is given as a Korobov kernel. Assume that the covariance kernel \(K_d\) is of product form,

\[
K_d(x, y) = \prod_{k=1}^{d} R_k(x_k, y_k), \quad x, y \in [0,1]^d,
\]

where \(R_k = R_{r_k, g_k}\) are univariate Korobov kernels,

\[
R_{\alpha, \beta}(x, y) := 1 + 2\beta \sum_{j=1}^{\infty} j^{-2\alpha} \cos(2\pi j(x - y)), \quad x, y \in [0,1].
\]

Here \(\beta \in (0, 1]\) is a scaling parameter, and \(\alpha\) is a smoothness parameter satisfying \(\alpha > \frac{1}{2}\). Note that for \(x = y\) we have

\[
R_{\alpha, \beta}(x, x) = 1 + 2\beta \zeta(2\alpha),
\]

where \(\zeta(x) = \sum_{j=1}^{\infty} j^{-x}\) is the Riemann zeta function which is well-defined only for \(x > 1\). We assume that \(\{r_k\}_{k \in \mathbb{N}}\) and \(\{g_k\}_{k \in \mathbb{N}}\) satisfy

\[
1 \geq g_1 \geq g_2 \geq \cdots \geq g_k \geq \cdots > 0,
\]

and

\[
r_* := \inf_{k \in \mathbb{N}} r_k > \frac{1}{2}.
\]

For the problem \(S = \{S_d\}\), the eigenvalues of the covariance operator \(C_{\nu_d}\) of the induced measure are known, see [5].

\[
\{\lambda_{d,j}\}_{j \in \mathbb{N}} = \{\lambda(1, j_1)\lambda(2, j_2)\cdots\lambda(d, j_d)\}_{(j_1, \ldots, j_d) \in \mathbb{N}^d},
\]

where \(\lambda(k, 1) = 1\) and

\[
\lambda(k, 2j) = \lambda(k, 2j + 1) = \frac{g_k}{j^{2r_k}}, \quad j \in \mathbb{N}.
\]

In this case, we set \(\tau_0 \in \left(\frac{1}{2r_*}, 1\right)\). We have

\[
\sup_{k \in \mathbb{N}} H(k, \tau_0) = \sup_{k \in \mathbb{N}} \sum_{j=2}^{\infty} \left(\frac{\lambda(k, j)}{\lambda(k, 2)}\right)^{\tau_0} = 2 \sup_{k \in \mathbb{N}} \sum_{j=1}^{\infty} j^{-2r_k x}
\]

\[
\leq 2 \sum_{j=1}^{\infty} j^{-2r_* x} = 2 \zeta(2r_* \tau_0) < \infty.
\]

This means that the problem \(S\) has Property (P) with \(f_k = g_k\). By Theorem 2.1, we have the following theorem.

**Theorem 4.3.** Consider the problem \(S = \{S_d\}\) in the average case setting with a zero mean Gaussian measure whose covariance operator is given as the Korobov kernel with the scale \(g_k\) and smoothness \(r_k\) satisfying (4.6) and (4.7), respectively. Assume that \(s > 0\) and \(t \in (0, 1)\). Then \(S\) is \((s,t)\)-WT for \(\text{ABS} or \text{NOR} iff \)

\[
\lim_{k \to \infty} k^{1-t} g_k \ln^+ \frac{1}{g_k} = 0.
\]
Remark 4.4. Using the method of [14, 8], we can get easily that for the above problem $S$ under ABS or NOR, $(s, t)$-WT always holds with $s > 0$ and $t > 1$, and $(s, 1)$-WT with $s > 0$ holds iff WT holds iff $\lim_{k \to \infty} g_k = 0$.

We recall tractability results of the above problem $S$. In [5, 16, 17], the authors considered the problem $S$ under the assumption (4.6) and

\[\frac{1}{2} < r_1 \leq r_2 \leq \cdots \leq r_k \leq \ldots.\]

However, there is no need to assume monotonicity for the smoothness parameters $r_k$, $k \in \mathbb{N}$. Indeed, it suffices to assume (4.7) instead of (4.8). The sufficient and necessary conditions for SPT, PT, WT under NOR were given in [5], for QPT under NOR in [3, 10, 4], and for UWT under ABS or NOR in [17]. Combining with our results, we have the following results about the tractability of the problem $S$ using ABS and NOR.

- For NOR or ABS, SPT holds iff PT holds iff

\[\lim_{j \to \infty} \frac{\ln \frac{1}{g_j}}{\ln j} > 1.\]  

- For NOR, QPT holds iff

\[\sup_{d \in \mathbb{N}} \frac{1}{\ln^+ d} \sum_{k=1}^{d} g_k \ln^+ \frac{1}{g_k} < \infty.\]

- For ABS or NOR, UWT holds iff

\[\lim_{j \to \infty} \frac{\ln \frac{1}{g_j}}{\ln j} \geq 1.\]  

- For ABS or NOR, $(s, t)$-WT with $s > 0$ and $t > 1$ always holds.

- For ABS or NOR, $(s, 1)$-WT with $s > 0$ holds iff WT holds iff $\lim_{j \to \infty} g_j = 0$.

- For ABS or NOR, $(s, t)$-WT with $s > 0$ and $t \in (0, 1)$ holds iff

\[\lim_{k \to \infty} k^{1-t} g_k \ln^+ \frac{1}{g_k} = 0.\]

Remark 4.5. In [17], Xu obtained that the sufficient and necessary condition for UWT under ABS or NOR is $\lim_{j \to \infty} j^p g_j = 0$ for all $p \in (0, 1)$. This condition is equivalent to (4.10).

Remark 4.6. In [5], the sufficient and necessary condition for SPT or PT only under NOR was given. However, this condition is also true for ABS. Indeed, due to (2.2), it suffices to prove that SPT holds for ABS if (4.9) holds. We assume that (4.9) holds. Then $\sum_{k=1}^{\infty} g_k < \infty$. This means that

\[e(0, d) = \exp \left( \frac{1}{2} \ln \left( \sum_{k=1}^{\infty} \lambda_{d,k} \right) \right) \leq \exp \left( \frac{A_1}{2} \sum_{k=1}^{d} g_k \right) \leq \exp \left( \frac{A_1}{2} \sum_{k=1}^{\infty} g_k \right) =: B < \infty,\]
where in the first inequality we used (2.3), \( A_1 = 2\zeta(2r_0) \). From [5] we know that SPT holds for OR. Using the inequalities
\[
n^{\text{ABS}}(\varepsilon, S_d) = n^{\text{NOR}}(\varepsilon, S_d) = n^{\text{NOR}}(\varepsilon, S_d),
\]
we get that SPT holds for OR if SPT holds for ABS. Hence, SPT for ABS holds.

4.3. \textbf{Average-case} \((s, t)\)-\textbf{WT} with analytic Korobov kernels.

In this subsection we consider a multivariate approximation problem \( S = \{S_d\} \)
defined over the space of \( C([0,1]^d) \) equipped with a zero-mean Gaussian measure
whose covariance kernel is given as an analytic Korobov kernel. Assume that the
 covariance kernel \( K_d \) is of product form,
\[
K_d(x,y) = \prod_{k=1}^{d} K_{1,a_k,b_k}(x_k,y_k), \quad x, y \in [0,1]^d,
\]
where \( K_{1,a_k,b_k} \) are univariate analytic Korobov kernels,
\[
K_{1,a_k,b_k}(x,y) = \sum_{h \in \mathbb{Z}} \omega^{|h|^b} \exp(2\pi i h(x - y)), \quad x, y \in [0,1].
\]
Here \( \omega \in (0,1) \) is a fixed number, \( i = \sqrt{-1} \), \( a, b > 0 \). Hence, we have
\[
K_d(x,y) = \sum_{h \in \mathbb{Z}^d} \omega_h \exp(2\pi i h(x - y)), \quad x, y \in [0,1]^d,
\]
with
\[
\omega_h = \omega^{\sum_{k=1}^d a_k |h_k|^{b_k}}, \quad \forall \ h = (h_1, h_2, \ldots, h_d) \in \mathbb{Z}^d,
\]
for fixed \( \omega \in (0,1) \).

We assume that the sequences \( a = \{a_k\}_{k \in \mathbb{N}} \) and \( b = \{b_k\}_{k \in \mathbb{N}} \) satisfy
\[
0 < a_1 \leq a_2 \leq \cdots \leq a_k \leq \ldots, \quad \text{and} \quad b_* := \inf_{k \in \mathbb{N}} b_k > 0.
\]

For the above problem \( S = \{S_d\} \), the eigenvalues of the covariance operator \( C_{\nu_d} \)
of the induced measure \( \nu_d \) are given by
\[
\{\lambda_{d,j}\}_{j \in \mathbb{N}} = \{\lambda(1,j_1)\lambda(2,j_2)\cdots \lambda(d,j_d)\}_{(j_1,\ldots,j_d) \in \mathbb{N}^d},
\]
where \( \lambda(k,1) = 1 \), and
\[
\lambda(k,2j) = \lambda(k,2j+1) = \omega^{a_k j^{b_k}}, \quad j \in \mathbb{N}.
\]

In this case, we set \( \tau_0 \in (0,1) \). We have
\[
\sup_{k \in \mathbb{N}} H(k,\tau_0) = \sup_{k \in \mathbb{N}} \sum_{j=2}^{\infty} \left( \frac{\lambda(k,j)}{\lambda(k,2)} \right)^{\tau_0} = 2 \sup_{k \in \mathbb{N}} \sum_{j=1}^{\infty} \omega^{\tau_0 a_k (j^{b_k}-1)} \leq 2 \sum_{j=1}^{\infty} \omega^{\tau_0 a_1 (j^{b_*}-1)}. \]

Since
\[
\omega^{\tau_0 a_1 (j^{b_*}-1)} = j^{-\frac{\tau_0 a_1 (j^{b_*}-1) \ln \frac{1}{a}}{\ln j}}, \quad \text{and} \quad \lim_{j \to \infty} \frac{\tau_0 a_1 (j^{b_*}-1) \ln \frac{1}{a}}{\ln j} = \infty,
\]
We get that
\[
M_{\tau_0} := 2 \sum_{j=1}^{\infty} \omega^{\tau_0 a_1 (j^{b_*}-1)} < \infty.
\]
This means that the problem \( S \) has Property (P) with \( f_k = \omega^{a_k} \). By Theorem 2.1,
we have the following theorem.
Theorem 4.7. Consider the problem $S = \{S_d\}$ in the average case setting with a zero mean Gaussian measure whose covariance operator is given as the analytic Korobov kernel with the sequences $a$ and $b$ satisfying (4.11). Assume that $s > 0$ and $t \in (0, 1)$. Then $S$ is $(s, t)$-WT for ABS or NOR iff
\[
\lim_{k \to \infty} k^{1-t} a_k \omega^{a_k} = 0.
\]

We recall tractability results of the above problem $S$ under the assumption (4.11). The sufficient and necessary conditions for SPT, PT, UWT, WT under NOR or ABS, and for QPT under NOR were given in [7], and for $(s, t)$-WT with $s > 0$ and $t \geq 1$ under ABS or NOR in [17]. However, the authors did not find out the matching necessary and sufficient conditions on $(s, t)$-WT with $s > 0$ and $t \in (0, 1)$ under ABS or NOR. Combining with our results, we have the following results about the tractability of the above problem $S$ using ABS and NOR.

- For NOR or ABS, SPT holds iff PT holds iff
\[
\lim_{j \to \infty} a_j \ln j > \frac{1}{\ln (\omega^{-1})}.
\]

- For NOR, QPT holds iff
\[
\sup_{d \in \mathbb{N}} \frac{1}{\ln^d} \sum_{k=1}^d a_k \omega^{a_k} < \infty.
\]

- For ABS or NOR, UWT holds iff
(4.12)
\[
\lim_{j \to \infty} \frac{a_j}{\ln j} \geq \frac{1}{\ln (\omega^{-1})}.
\]

- For ABS or NOR, $(s, t)$-WT with $s > 0$ and $t > 1$ always holds.

- For ABS or NOR, $(s, 1)$-WT with $s > 0$ holds iff WT holds iff $\lim_{j \to \infty} a_j = \infty$.

- For ABS or NOR, $(s, t)$-WT with $s > 0$ and $t \in (0, 1)$ holds iff
\[
\lim_{k \to \infty} k^{1-t} a_k \omega^{a_k} = 0.
\]

Remark 4.8. In [17], Xu obtained that the sufficient and necessary condition for UWT under ABS or NOR is $\lim_{j \to \infty} \omega^{a_j} j^p = 0$ for all $p \in (0, 1)$. This condition is equivalent to (4.12).

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School of Mathematical Sciences, Capital Normal University, Beijing 100048, China.
E-mail address: jiachencd@163.com

School of Mathematical Sciences, Capital Normal University, Beijing 100048, China.
E-mail address: wanghp@cnu.edu.cn.

School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China.
E-mail address: zhangjie91528@163.com.