ON THE FIRST INFINITESIMAL NEIGHBORHOOD OF A LINEAR CONFIGURATION OF POINTS IN \( \mathbb{P}^2 \)

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Abstract. We consider the following open questions. Fix a Hilbert function \( h \) that occurs for a reduced zero-dimensional subscheme of \( \mathbb{P}^2 \). Among all subschemes, \( X \), with Hilbert function \( h \), what are the possible Hilbert functions and graded Betti numbers for the first infinitesimal neighborhood, \( Z \), of \( X \) (i.e. the double point scheme supported on \( X \))? Is there a minimum \( (h_{\text{min}}) \) and maximum \( (h_{\text{max}}) \) such function?

The numerical information encoded in \( h \) translates to a type vector which allows us to find unions of points on lines, called linear configurations, with Hilbert function \( h \). We give necessary and sufficient conditions for the Hilbert function and graded Betti numbers of the first infinitesimal neighborhoods of all such linear configurations to be the same. Even for those \( h \) for which the Hilbert functions or graded Betti numbers of the resulting double point schemes are not uniquely determined, we give one (depending only on \( h \)) that does occur. We prove the existence of \( h_{\text{max}} \), in general, and discuss \( h_{\text{min}} \).

Our methods include liaison techniques.

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1. Introduction

The classification of all the possible Hilbert functions of reduced zero-dimensional subschemes of the projective space \( \mathbb{P}^n(k) \), \( (k \text{ a field of characteristic zero}) \) is well known (see e.g. [27]). In marked contrast, the analogous classification, even for the important class of non-reduced zero-dimensional schemes which are unions of “2-fat point” schemes (which

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we will refer to as double point schemes; see §2 for the definitions) is wide open. This in spite of the fact that answers to such questions have interesting implications in algebraic geometry, coding theory, computational complexity and statistics.

The one major achievement in this area is the proof of the genericity (apart from some well understood and well known exceptions) of the Hilbert function of a generic double point scheme by J. Alexander and A. Hirschowitz (see §1 and also K. Chandler [14]). Other important contributions to our understanding of the Hilbert function of fat point schemes have been made by Bocci [15], Catalisano, Geramita and Gimigliano [11], Ciliberto [16], Ciliberto and Miranda [17], Laface and Ugaglia [35], [36], Yang [11] (for further references see the survey article of Miranda [41]).

Since one knows the functions which can be the Hilbert function of a reduced zero-dimensional subscheme of \( \mathbb{P}^n \), one may then inquire as to the possible minimal free resolutions for reduced subschemes sharing the same Hilbert function. This is, in general, a very difficult problem which has attracted a great deal of attention. There are: complete results for reduced subschemes of \( \mathbb{P}^2 \) by Campanella [9]; a sharp upper bound for any Hilbert function (in terms of the graded Betti numbers) by Bigatti [14], Hulett [33] and Pardue [44]; complete results in low codimension and under the assumption that the coordinate ring of the reduced scheme is (in some way) special – e.g. for codimension two and codimension 3 Gorenstein see e.g. Diesel [20] and Geramita and Migliore [29], while for Gorenstein rings with the Weak Lefschetz Property see, e.g., Geramita, Harima, and Shin [26], Migliore and Nagel [40].

The results about minimal free resolutions for fat point schemes are much scantier. Notable results (in \( \mathbb{P}^2 \)) are given by: Catalisano (arbitrary fat points schemes supported on a conic) [10]; Harbourne et.al. (for fat point schemes supported on generic sets of points or arbitrary fat point schemes supported on few points on a cubic) [31], [23], [32]. For higher dimensional spaces we have: Catalisano, Ellia and Gimigliano [13] (arbitrary fat points whose support is on a rational normal curve in \( \mathbb{P}^n \)); Fatabbi [21], Francisco [24], Fatabbi and Lorenzini [22] and Valla [45] (arbitrary fat points whose support is on \( \leq n \) points in \( \mathbb{P}^n \)).

In this paper we will deal with the problem of classification of Hilbert functions and resolutions of double point schemes in \( \mathbb{P}^2 \) in the following way. Let \( h \) be the Hilbert function of a reduced zero-dimensional subscheme of \( \mathbb{P}^2 \). What are all the possible Hilbert functions and minimal free resolutions for double point schemes whose support, \( X \), has Hilbert function \( h \)?

Given any such Hilbert function \( h \), there is a well known family of reduced subschemes of \( \mathbb{P}^2 \) whose Hilbert function is \( h \) – the so-called \( k \)-configurations of a specific type (see §2 for the definitions). So, the first natural problem to consider is the following:

\[
\text{if } X \text{ is a } k \text{-configuration in } \mathbb{P}^2 \text{ with Hilbert function } h, \text{ can we describe the Hilbert function of the double point scheme whose support is } X?\]

As is also well known, the \( k \)-configurations in \( \mathbb{P}^2 \) of the same type always have the same graded Betti numbers in their minimal free resolution (see [26] for this and generalizations to \( \mathbb{P}^n \)). This nice result for reduced subschemes of \( \mathbb{P}^2 \) leads naturally to another question:
if $X$ is a $k$-configuration in $\mathbb{P}^2$ with Hilbert function $h$, can we describe the graded Betti numbers in the minimal free resolution of the double point scheme whose support is $X$?

It is also well known that all reduced sets of points which form $k$-configurations with Hilbert function $h$ enjoy many extremal properties (e.g. their minimal free resolutions are the extremal ones described by Bigatti, Hulett and Pardue (see [20]), their conductors are extremal - at least in $\mathbb{P}^2$ (see [3])). It is thus an obvious question to ask if the double point schemes supported on linear configurations have any extremal properties.

Indeed, having raised the problem of extremal properties, one is naturally led to another series of questions:

- **If** $h$ **is the Hilbert function of some reduced zero-dimensional subscheme of** $\mathbb{P}^2$, **do there exist Hilbert functions** $h_{\text{max}}$ **and** $h_{\text{min}}$ **of double point schemes (whose support has Hilbert function** $h$ **) such that if** $h'$ **is the Hilbert function of any double point scheme whose support has Hilbert function** $h$, **then** $h_{\text{min}} \leq h' \leq h_{\text{max}}$.

Moreover, if $h_{\text{min}}$ and $h_{\text{max}}$ exist, what are they?

The questions above are the main ones we will consider in this paper.

We now give a summary description of the results we have obtained below. The first thing we do is restrict our notion of a $k$-configuration to that of a linear configuration (see Definition 2.6). This technical restriction is crucial for our main theorems and yet does not restrict the Hilbert functions we consider for reduced sets of points.

If $h$ is the Hilbert function of a zero-dimensional reduced subscheme of $\mathbb{P}^2$ we first describe an O-sequence (see Definition 3.2), $\text{dbl}(h)$, with the property that: if $X$ is a special linear configuration with Hilbert function $h$ (and $X$ always exists) then the double point scheme with support $X$ has Hilbert function $\text{dbl}(h)$ (see Theorem 5.4).

We then give a complete description of all the Hilbert functions $h$ so that every double point scheme whose support is a linear configuration with Hilbert function $h$ has Hilbert function $\text{dbl}(h)$ (see Theorem 6.1).

Even when it is no longer true that every double point scheme whose support is a linear configuration with Hilbert function $h$ has the same Hilbert function, we prove that all such double point schemes share the same regularity (see Theorem 6.2), which is the maximal possible for double point schemes whose support has Hilbert function $h$. This illustrates one sort of “extremality” property for the Hilbert functions of double point schemes whose support share the same Hilbert function.

We also investigate the minimal free resolutions of double point schemes supported on a linear configuration. We give necessary and sufficient conditions on the Hilbert function $h$ in order that all double point schemes with support on a linear configuration having Hilbert function $h$ have the same graded Betti numbers in their minimal free resolution (see Theorem 6.1). As expected, the results about minimal free resolutions are more subtle and restrictive than those simply about Hilbert functions.

It is worth making some comment here about our method of proof for the results about double point schemes sharing the same Hilbert function for their support and, in particular, when the support is a linear configuration.
Although our principal aim in this paper is the study of the possible Hilbert functions of double point schemes in \( \mathbb{P}^2 \), we spend a great deal of effort (especially in §3 and §4) studying special configurations of reduced point sets in \( \mathbb{P}^2 \) (which we call pseudo linear configurations). Although these reduced point sets are really peripheral to our main concern, there are important reasons for considering them which come out of the strong connections between the numerical information encoded in these reduced schemes and the numerical information we seek about the 2-fat point schemes we are considering. In fact, our approach illustrates (in a very concrete way) how one can get a great deal of mileage out of considering certain collections of 2-fat point schemes in \( \mathbb{P}^2 \) as if they were the union of a collection of triples of reduced points (configured in a special way). This sort of philosophy is evident in J. Alexander and A. Hirschowitz’s “Horace Method (Divide and Conquer)” [1] and also in [12] and [19]. The novelty of our approach is that, for the first time, we use the techniques of Liaison as an additional weapon for Horace’s arsenal.

In §7 we consider the problem of existence for, what we have called, \( h_{\max} \) and \( h_{\min} \). We note that \( h_{\max} \) always exists, even though it is difficult in practice to say exactly what it is. Recall that the results of J. Alexander and A. Hirschowitz (see [1]) give (as a special case in \( \mathbb{P}^2 \)) that: if we denote by \( h_s \) the generic Hilbert function of a set of \( s \) distinct points in \( \mathbb{P}^2 \), then \( h_{3s} = h_{3s} \) except for \( s = 2, 5 \). We have been unable to decide if \( h_{\min} \) exists, in general. Nevertheless, we have found it for \( h_{s} \) when \( s = \binom{12}{2} \). Even though we cannot decide if \( h_{\min} \) actually exists, in general, we can prove something that would be a consequence of that existence: namely the existence of a maximal regularity for all double point schemes whose support has Hilbert function \( h \) (see Remark 6.2).

2. Preliminaries

Let \( k \) be any infinite field of characteristic zero and let \( R = k[x_0, x_1, x_2] \). We denote by \( \mathbb{P}^2(k) \) the scheme \( \text{proj}(R) \). If \( P \) is a point in \( \mathbb{P}^2 \) defined by the prime ideal \( \wp = (L_1, L_2) \) (the \( L_i \) linearly independent linear forms in \( R \)) then any scheme supported on the point \( P \) is defined by a \( \wp \)-primary ideal of \( R \).

Definition 2.1. A scheme supported on the point \( P \) is called a fat point scheme with support \( P \) if it is defined by the primary ideal \( \wp^t \) for some integer \( t > 1 \). If, in particular, \( t = 2 \) then we shall call the scheme defined by \( \wp^2 \) a double point scheme with support on \( P \). This latter is also referred to as the first infinitesimal neighborhood of \( P \).

More generally, if \( X = \{P_1, \ldots, P_t\} \) is any set of distinct points in \( \mathbb{P}^2 \) where \( P_i \) is defined by the prime ideal \( \wp_i \), then the double point scheme with support \( X \) is the scheme defined by the (saturated) ideal \( \wp_1^2 \cap \ldots \cap \wp_t^2 \).

If a scheme supported on \( X \) is defined by an ideal of the type \( \wp_1^{n_1} \cap \ldots \cap \wp_t^{n_t} \) then we sometimes loosely refer to it as a fat point scheme with support \( X \). If, in addition, the \( n_i \) are all the same (and say are equal to \( t \)) then we say that the scheme defined on \( X \) is a \( t \)-fat point scheme on \( X \).

We also recall some terminology that is used in discussing the Hilbert function of zero dimensional subschemes of \( \mathbb{P}^2 \).

Definition 2.2. Let \( h \) be the Hilbert function of a zero dimensional subscheme, say \( X \), of \( \mathbb{P}^2 \). We define:
(i) \( \alpha(h) \) to be the least integer \( t \) for which \( h(t) < \binom{t+2}{2} \);
(ii) \( \Delta h \) to be the first difference of \( h \), i.e.
\[
\Delta h(t) = h(t) - h(t-1);
\]
(iii) \( \sigma(h) \) to be the least integer \( t \) for which \( \Delta h(t) = 0 \).

We also sometimes refer to \( h \) as \( h_X \). In this case, since \( \Delta h_X(t) \neq 0 \) for only finitely many values of \( t \), we refer to the sequence
\[
\Delta h_X(0) = 1 \quad \Delta h_X(1) \quad \cdots \quad \Delta h_X(\sigma(h_X) - 1)
\]
as the \( h \)-vector of \( X \).

Geramita, Harima and Shin defined the notion of an \( n \)-type vector in [25]. Since we only need the case of a 2-type vector, we only recall that definition.

**Definition 2.3.** A 2-type vector is a vector of the form \( T = (d_1, d_2, \ldots, d_r) \), where \( 0 < d_1 < d_2 < \ldots < d_r \) are integers. For such a 2-type vector, we define \( \alpha(T) = r \) and \( \sigma(T) = d_r \).

**Theorem 2.4.** [25, Theorem 2.6] Let \( S_2 \) denote the collection of Hilbert functions of all sets of distinct points in \( \mathbb{P}^2 \). Then there is a 1-1 correspondence \( S_2 \leftrightarrow \{2\text{-type vectors}\} \). Under this correspondence if \( h \in S_2 \) and \( h \) corresponds to \( T \) (we write \( h \leftrightarrow T \)) then \( \alpha(h) = \alpha(T) \) and \( \sigma(h) = \sigma(T) \).

We now give the inductive formula for obtaining the Hilbert function referred to in Theorem 2.4 from its corresponding 2-type vector.

**Theorem 2.5.** [25, Proof of Theorem 2.6] Let \( T = (d_1, d_2, \ldots, d_r) \) be a 2-type vector, and let \( h_i \) denote the Hilbert function of \( d_i \) points on a line. Then \( h : T \leftrightarrow \) where \( h(j) = h_r(j) + h_{r-1}(j-1) + \cdots + h_1(j-(r-1)) \) and \( h(t) = 0 \) for \( t < 0 \).

The notion of an \( n \)-type vector is convenient for defining the notion of a \( k \)-configuration in \( \mathbb{P}^n \). We give here the definition of a \( k \)-configuration in \( \mathbb{P}^2 \).

**Definition 2.6.** a) Let \( T = (d_1, d_2, \ldots, d_r) \) be a 2-type vector. Let \( L_1, L_2, \ldots, L_r \) be distinct lines in \( \mathbb{P}^2 \). Let \( X_i \) consist of \( d_i \) points on \( L_i \) for each \( i \). Suppose, furthermore, that \( L_i \) does not contain any point of \( X_j \) for \( j < i \). Then \( X = \bigcup_{i=1}^r X_i \) is called a \( k \)-configuration of type \( T \).
b) If we assume further that no point of \( X_i \) is on line \( L_j \), for \( j \neq i \), then \( X \) will be called a linear configuration of type \( T \).

**Example 2.7.** In the diagram below, \( X_1 \) consists of the two points of \( L_1 \) that are not on \( L_2 \), \( X_2 \) consists of the five points of \( L_2 \), and \( X_3 \) consists of the 6 points of \( L_3 \). Then \( X = X_1 \cup X_2 \cup X_3 \) is a \( k \)-configuration of type \( T = (2,5,6) \). Notice that \( L_i \) does not contain a point of \( X_j \) for \( j < i \), although \( L_1 \) does contain a point of \( X_2 \). Thus \( X \) is NOT a linear configuration of type \( T = (2,5,6) \).
Notice that $X$ is not a $k$-configuration of type $T = (3,4,6)$ since $X_1$ would have to consist of all 3 points on $L_1$ and this includes a point of $L_2$. □

One can see from Example 2.7 that the fact that the $X_i$ are ordered from smallest to largest is a crucial part of the definition of a $k$-configuration. As well, the example suggests that the same $k$-configuration cannot have two different 2-type vectors associated to it. In fact, more is true: namely, all $k$-configurations of type $T$ have the Hilbert function corresponding to $T$.

**Theorem 2.8** ([25], p. 21). Let $X$ be a $k$-configuration of type $T \leftrightarrow h$. Then $h_X = h$.

**Remark 2.9.** Although some of the results of this paper (and results cited from earlier papers) are true for arbitrary $k$-configurations, our main results are not. For this reason, from now on we will only consider linear configurations (see Definition 2.6).

Recall that the $(i, j)^{th}$ graded Betti number of an ideal $I$ of $R$ is defined to be

$$\beta^I_{i,j} := (\text{Tor}_i(R/I, k))_j.$$ entry

It turns out that the graded Betti numbers of a linear configuration of type $T$ are also completely determined by $T$ ([26], Theorem 3.6). In fact, those Betti numbers are extremal, in a way which we will explain later. We will see in this paper exactly when the Hilbert function (Corollary 6.1 (a)) and graded Betti numbers (Corollary 6.1 (b),(c)) of the double points supported on a linear configuration are determined just from the type of the linear configuration - something that does not always happen!

Even when the Hilbert function of double points supported on a linear configuration is not determined simply from the type of the linear configuration, we will at least be able to determine the Hilbert function of double points supported on very special linear configurations. We now proceed to the definitions of these two special classes of linear configurations (see [30] before Example 4.1)).
Definition 2.10. A linear configuration of type $T = (d_1, d_2, \ldots, d_r)$ in $\mathbb{P}^2$ is called a standard linear configuration of type $T$ if it consists of:

- $d_r$ points with coordinates $[j : 0 : 1]$ for $0 \leq j \leq d_r - 1, j \in \mathbb{N},$
- $d_2$ points with coordinates $[j : 2 : 1]$ for $0 \leq j \leq d_2 - 2, j \in \mathbb{N},$
- $d_1$ points with coordinates $[j : 1 : 1]$ for $0 \leq j \leq d_1 - 1, j \in \mathbb{N}.$

Definition 2.11. Let $J$ be a homogeneous ideal in $S = k[x_1, \ldots, x_n]$. We say that a radical ideal $I$ of $R = k[x_0, x_1, \ldots, x_n]$ lifts $J$ if there is a linear form $L$ which is a non-zero-divisor on $R/I$ for which $(I, L)/L \simeq J$.

If $I$ is an ideal of $R = k[x_0, \ldots, x_n]$ which lifts the homogeneous ideal $J$ of $S = k[x_1, \ldots, x_n]$, then the minimal free $R$-resolution of $I$ has the same graded Betti numbers as the minimal free $S$-resolution of $J$ (see [8], Proposition 1.1.5).

Note that the ideal of the standard linear configuration of type $T = (d_1, d_2, \ldots, d_r)$ is a lifting of the monomial ideal $\langle x^{d_r}, x^{d_r-1}y, x^{d_r-2}y^2, \ldots, y^r \rangle$ (an ideal is called monomial if it is generated by monomials). We call this the standard lifting.

Note also that the monomial ideal being lifted to obtain the standard linear configuration is by no means random, but rather satisfies the following very special condition: if a monomial $m \in I$, then every larger monomial (using the lexicographic ordering) of the same degree is also in $I$. Such ideals are called lex-segment ideals.

Since the ideal of a standard linear configuration always lifts a lex-segment ideal (by [26], Theorem 4.3), standard linear configurations can be looked at as providing the 1-1 correspondence between Hilbert functions of points and lex-segment ideals.

The special linear configurations for which we will always be able to determine the Hilbert functions of the double points with that support are defined in almost the same way as standard linear configurations.

Definition 2.12. A linear configuration of type $T = (d_1, d_2, \ldots, d_r)$ in $\mathbb{P}^2$ is called a spread out linear configuration of type $T$ if it consists of:

- $d_r$ points with coordinates $[j : 0 = d_r - d_r : 1]$ for $0 \leq j \leq d_r - 1, j \in \mathbb{N},$
- $d_2$ points with coordinates $[j : d_r - d_2 : 1]$ for $0 \leq j \leq d_2 - 1, j \in \mathbb{N},$
- $d_1$ points with coordinates $[j : d_r - d_1 : 1]$ for $0 \leq j \leq d_1 - 1, j \in \mathbb{N}.$
Example 2.13. If $T = (1, 2, 4, 7)$, then the standard linear configuration and the spread out linear configuration of type $T$ are as follows:

| Standard | Spread Out |
|----------|------------|
| ![Standard Configuration](image1.png) | ![Spread Out Configuration](image2.png) |

where the o’s represent “imaginary” points that we are using to properly position the points in which we are interested. Again in this case, we want rows consisting of 1, 2, 4 and 7 points but to obtain the spread out linear configuration of this type we add rows of 3, 5 and 6 “imaginary” points to form an “isosceles right triangle”. $\square$

Remark 2.14. The process of forming a spread out linear configuration ensures that the “diagonal” points are collinear and it is this fact that will enable us to determine the Hilbert function of sets of double points with support on a spread out linear configuration. $\square$

The notion of basic double linkage is extremely useful, both in liaison theory (where it is fundamental) and as a construction tool for building interesting schemes. We use it in this paper to construct sets of double points. Because we are primarily interested in points in $\mathbb{P}^2$, we recall the basic ideas here only in that context, even though they are applicable in far greater generality (cf. [37], [38], [34], [6], [28] for basic properties). We collect the known facts about basic double linkages in $\mathbb{P}^2$ here for the convenience of the reader. If no other reference is given, see [38] for details.

Lemma 2.15 (Basic Double Linkage). Let $X$ be a zero dimensional subscheme of $\mathbb{P}^2$. Let $F \in I_X$ be any polynomial, and let $G$ be any polynomial such that \{F, G\} form a regular sequence. (It makes no difference if $G$ vanishes on a point of $X$ or not.) Form the ideal $I = G \cdot I_X + (F)$. Then

(a) $I = I_Z$ is the saturated ideal of a subscheme $Z$ in $\mathbb{P}^2$.
(b) The support of $Z$ is the union of the support of $X$ and the support of the complete intersection scheme, $V$, defined by $(F, G)$.
(c) If $\deg F = d_1$ and $\deg G = d_2$ then there is an exact sequence

\[
0 \to R(-d_1 - d_2) \to I_X(-d_2) \oplus R(-d_1) \to I_Z \to 0
\]  

(2.1)

(d) We have the Hilbert function formula

\[
h_{R/I_Z}(t) = h_{R/(F,G)}(t) + h_{R/I_X}(t - d_2).
\]

(2.2)

(We will often use the first difference of this formula which, for example, gives that $\deg Z = \deg X + d_1d_2$.)
(e) ([39] Corollary 4.5) Suppose that $I_X$ has a minimal free resolution
\[ 0 \to F_2 \to F_1 \to I_X \to 0. \]

Then $I_Z$ has a free resolution
\[ 0 \to R(-d_1 - d_2) \oplus F_2(-d_2) \to F_1(-d_2) \oplus R(-d_1) \to I_Z \to 0. \]

Furthermore, this resolution is minimal if and only if $F$ is not a minimal generator of $I_X$. If $F$ is a minimal generator of $I_X$ then one term, $R(-d_1 - d_2)$, splits off, yielding a minimal free resolution.

Remark 2.16. It is easy to see that any standard linear configuration or any spread out linear configuration, $X$, can be produced by a sequence of basic double linkages. Simply start at the “top” and choose as the polynomials $F$ suitable unions of “vertical” lines, and choose as $G$ sequentially the “horizontal” lines containing points of $X$, working down from top to bottom. (See Proposition 3.6 below.)

Less obviously, as was observed by one of us (Migliore) several years ago, any linear configuration in $\mathbb{P}^2$ can be viewed as a sequence of basic double links. That fact will be seen later as a consequence of the more general result in Theorem 3.7.

But, the main new idea in this paper comes out of the realization that many double point schemes can also be obtained as the result of a sequence of basic double links. Since this idea is pervasive in this paper, it will be useful to have a simple example that illustrates the point.

Example 2.17. We construct a 2-fat point scheme whose underlying support is a linear configuration of type $(1,2)$. We shall do this example in some detail as it illustrates, in a simple way, some of the key ideas of the proofs in this paper. In particular, it illustrates how: basic double links can be used to “fatten up” points and how one can use basic double links (with the form $F$ progressively growing in degree) to get linear configurations of 2-fat points.

Without loss of generality we may assume that our points are
\[ P_1 = [1 : 0 : 0], \quad P_2 = [0 : 1 : 0], \quad P_3 = [0 : 0 : 1]. \]

We want to “fatten up” $P_1$ by “adding” to it something of length 2 and we then want to fatten up $P_2$ and $P_3$ by adjoining to each a length 2 piece.

So, it is as if we were considering $1 + 2$ points on the first “horizontal” line and then $2 + 4$ points on the second line. We write those numbers down, $1, 2, 2, 4$, and use them as a guide for our construction.

Using the “1” we begin with the point $P_1 = [1 : 0 : 0]$, and the ideal $I_{P_1} = (y, z)$. Now consider the 2’s. We take them both and think of performing a basic double link on $I_{P_1}$ which will, at the same time, “fatten up” $P_1$ (the first 2 in our sequence) and add the two reduced points $P_2$ and $P_3$ on the line $x$ (the second 2 in the sequence).

Let $F = yz$ and $G = (y + z)x$ and form $I = G \cdot I_{P_1} + (F)$. As noted in Lemma 2.15 $I$ is the saturated ideal of a scheme supported on the union of $P_1$ and the support of the scheme defined by $(F,G)$. The latter scheme is supported on $P_1, P_2, P_3$. 

□
Now,
\[
I = ((y + z)xy, (y + z)xz, yz)
\]
\[
= (y^2x, z^2x, yz)
\]
\[
= I_{P_1}^2 \cap I_{P_2} \cap I_{P_3}.
\]
The last equality can be checked locally, since we know that \( I \) is saturated. Now we use the “4” to fatten up \( P_2 \) and \( P_3 \), by letting \( F = yz(x + z)(x + y) \), \( G = x \). Clearly \( F \in I \) and we use \( F \) and \( G \) to form a basic double link on \( I \). We obtain
\[
J = xI + (F)
\]
\[
= (y^2x^2, z^2x^2, xyz, yz(x + z)(x + y))
\]
\[
= I_{P_1}^2 \cap I_{P_2} \cap I_{P_3}^2
\]
as we wanted to show.

\[\square\]

Remark 2.18. Finally, we recall that if \( Z \subset \mathbb{P}^2 \) then \( I_Z \) has regularity \( d \) if
\[
d = \min\{t \mid h^1(I_Z(t - 1)) = 0\}.
\]
If this is the case then \( I_Z \) is generated in degrees \( \leq d \) (12). Furthermore,
\[
d = \min\{t \mid \Delta h_{R/I_Z}(t) = 0\} = \sigma(h_{R/I_Z}).
\]
We will say that \( Z \) has regularity \( d \) if \( I_Z \) does.

\[\square\]

The following elementary result about the regularity of the first infinitesimal neighborhood of a set of distinct points in \( \mathbb{P}^2 \) will be extremely useful.

Lemma 2.19. Let \( \mathbb{X} \) be a reduced set of points in \( \mathbb{P}^2 \) with regularity \( r + 1 \). Let \( Z \) be the first infinitesimal neighborhood of \( \mathbb{X} \). Then \( \text{reg}(I_Z) \leq 2 \cdot \text{reg}(I_{\mathbb{X}}) = 2r + 2 \).

Proof. By hypothesis, \( \mathbb{X} \) imposes independent conditions on forms of degree \( r \). We want to show that \( Z \) imposes independent conditions on forms of degree \( 2r + 1 \). This means that we want to show that if \( P \in \mathbb{X} \) and \( Z' \) is the subscheme of \( Z \) supported on \( \mathbb{X}' = \mathbb{X} \setminus P \) then there is a form of degree \( 2r + 1 \) vanishing on \( Z' \) and also on \( P \) together with any tangent direction at \( P \). But this is clear: let \( F \) be a form of degree \( r \) vanishing on \( \mathbb{X}' \) but not at \( P \). The \( F^2 \) vanishes on \( Z' \) but not at \( P \), and if \( L \) is the line through \( P \) with the desired tangent direction then \( F^2L \) is the desired form.

\[\square\]

3. Pseudo Linear Configurations

Before we can begin to consider configurations of 2-fat points in the plane, it is useful for us to extend the class of configurations of simple points in the plane whose Hilbert function we can control. These will play an important part in our attempt to discover the Hilbert function of all 2-fat point schemes whose underlying supports have the same Hilbert function. The new configurations we consider are inspired by Example 2.17.

Definition 3.1.

i) A pseudo type vector is a sequence \( T' = (m_1, m_2, \ldots, m_p) \), where the \( m_i \) are positive integers for which \( m_1 \leq m_2 \leq \cdots \leq m_p \). Moreover, if \( m_{i-1} = m_i \) then \( m_i < m_{i+1} \).
Given a pseudo type vector $T'$ and lines $L_1, \ldots, L_p$, a pseudo linear configuration of type $T'$ is a set of points $X = X_1 \cup X_2 \cup \cdots \cup X_p$ where $X_i$ is a set of $m_i$ points on $L_i$. We do not allow any point of $X_i$ to lie on line $L_j$ for $j \neq i$.

A pseudo linear configuration of type $T' = (m_1, m_2, \ldots, m_p)$ in $\mathbb{P}^2$ is called standard if it consists of:

- $m_p$ points with coordinates $[j : 0 : 1]$ for $0 \leq j \leq m_p - 1, j \in \mathbb{N}$,
- $m_2$ points with coordinates $[j : p - 2 : 1]$ for $0 \leq j \leq m_2 - 1, j \in \mathbb{N}$,
- $m_1$ points with coordinates $[j : p - 1 : 1]$ for $0 \leq j \leq m_1 - 1, j \in \mathbb{N}$.

We now describe an O-sequence that can be associated to a pseudo type vector which depends only on the numerical information that is contained in the pseudo type vector. We wish to stress, however, that we are not claiming that every pseudo linear configuration with the given pseudo type vector has this O-sequence as the first difference of its Hilbert function. We will see later (see Theorem 3.7) that such a strong statement is true only for certain pseudo-type vectors.

**Definition 3.2.** Let $T' = (m_1, \ldots, m_p)$ be a pseudo type vector. The standard O-sequence associated to $T'$ is given by a “shifted sum” of certain sequences $s_i$ defined as follows: if we formally suppose that $m_0 = 0$ and $m_{p+1} = \infty$, then

- If $m_{i-1} < m_i < m_{i+1}$ then
  \[(s_i)_t = \begin{cases} 
  1 & \text{for } 0 \leq t \leq m_i - 1; \\
  0 & \text{otherwise.} 
  \end{cases} \]

- If $m_{i-1} = m_i < m_{i+1}$ then
  \[(s_i)_t = \begin{cases} 
  1 & \text{for } t = 0; \\
  2 & \text{for } 1 \leq t \leq m_i - 1; \\
  1 & \text{for } t = m_i; \\
  0 & \text{otherwise.} 
  \end{cases} \]

- If $m_{i-1} < m_i = m_{i+1}$ we do not define $s_i$.

Also, for a sequence $s_i$ and non-negative integer $k$, we define the shifted sequence $s_i(-k)$ to be a rightward shift of $s_i$ by $k$ places (so instead of starting in degree 0 it starts in degree $k$).

Then the standard O-sequence associated to $T'$ is:

$$\sum_{i=1}^{p} s_i(i - p)$$

where it is understood that the sum skips any indices for which $s_i$ is not defined.

**Remark 3.3.** The standard Hilbert function associated to $T'$ is the numerical function whose first difference is the standard O-sequence associated to $T'$ (as defined above). □
Example 3.4. Let $T' = (3, 6, 6, 7, 12, 14)$. Then the standard O-sequence associated to $T'$ is:

\[
\begin{array}{cccccccc}
1 & 1 & 1 \\
1 & 2 & 2 & 2 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 & 6 & 5 & 3 & 2 & 2 & 2 & 1
\end{array}
\]

(Note that there is no $s_2$.) This sequence is the first difference of the O-sequence:

\[
1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 33 \ 38 \ 41 \ 43 \ 45 \ 47 \ 48 \ldots
\]

which is the standard Hilbert function associated to $T'$.

□

Remark 3.5. We will see that Definition 3.2 was designed so that the number of sequences $s_i$ correspond to the number of applications of basic double linkage. The sequences $s_i$ containing 2’s will correspond to basic double links of the form $J = QI + (F)$, where $Q = L_1L_2$ is a product of linear forms.

It should be noted that such a basic double link can also be viewed as a sequence of two basic double links $J_1 = L_1I + (F)$ and $J = L_2J_1 + (F)$ (with the same $F$). Because of this, we could also write the O-sequence sum without any 2’s. For instance, the above computation would become

\[
\begin{array}{cccccccc}
1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4 & 5 & 6 & 6 & 5 & 3 & 2 & 2 & 2 & 1
\end{array}
\]

and we would not have to worry about the extra shift. However, for our purposes (constructing 2-fat point schemes) it is important to do the basic double link in one step, so we retain this slightly more complicated notation.

□

We will see in Theorem 3.7 that there is a very precise condition that determines whether or not the Hilbert function of a pseudo linear configuration is uniquely determined by its type. Nevertheless, we now show that the Hilbert function of a standard pseudo linear configuration is uniquely determined, and in fact is equal to the function described in Definition 3.2.

Proposition 3.6. Let $\mathcal{X} \subset \mathbb{P}^2$ be a standard pseudo linear configuration of type $T' = (m_1, m_2, \ldots, m_p)$. Let $\Delta T' = (m_1, m_2 - m_1, \ldots, m_p - m_{p-1})$. Then:

i) $\mathcal{X}$ can be built up by basic double linkage;

ii) the first difference of the Hilbert function of $\mathcal{X}$ is the standard O-sequence associated to $T'$;
iii) assume that between any two zero entries of $\Delta T'$ there is at least one entry $> 1$. If $\Delta T'$ ends with a 0, or with a sequence $(\ldots, 0, 1, \ldots, 1)$ (i.e. a 0 followed by any number of 1's), then the regularity of $X$ is $m_1 + 1$. Otherwise the regularity is $m_p$.

iv) If there are zero entries between which there are no entries $> 1$ then the regularity of $X$ may be arbitrarily larger than $m_p$.

Proof. Let $T' = (m_1, \ldots, m_p)$, $T'' = (m_1, \ldots, m_{p-1})$. Let $X$ be a standard pseudo linear configuration of type $T'$ and let $X_1$ be the obvious subset which is a pseudo linear configuration of type $T''$. We will show that $X$ can be obtained from $X_1$ by basic double linkage. Let

$$F = x(x-z) \ldots (x-(m_{p-1}z)) \ldots (x-(m_p-1)z).$$

Then $F \in I_{X_1}$ (since the configuration $X$ is standard) and $\deg F = m_p$. Let $G = y$.

Consider the basic double link

$$I = G \cdot I_{X_1} + (F).$$

Then by Lemma $2.10$, $I$ is a saturated ideal whose support is exactly $X$. To show $I = I_X$, it remains to show that $I$ is reduced. But the degree of the scheme defined by $I$ is $m_p$ more than $\deg X_1$, by Lemma $2.15$, so $I$ and $I_X$ are saturated ideals of zero-dimensional schemes with the same support and same degree, one of which is reduced. Hence they are equal. This proves $i$).

By taking the first difference of (2.2), we obtain

$$(3.1) \quad \Delta h_{R/I_x}(t) = \Delta h_{R/(F,G)}(t) + \Delta h_{R/I_{X_1}}(t-1).$$

Using the fact that $\Delta h_{R/(F,G)}(t)$ is

$$\begin{pmatrix}
1 & 1 & \ldots & 1 & 0 \\
0 & 1 & \ldots & (m_p-1) & (m_p)
\end{pmatrix}$$

and taking Remark $3.10$ into account, it is clear that the first difference of the Hilbert function of $X$ is obtained (inductively) by $3.2$ and so $X$ has the standard O-sequence associated to $T'$. This proves $ii$).

We now verify the conclusions of $iii$) by induction, assuming that it holds for $X_1$. The technical assumption, i.e. that there is at least one entry $> 1$ between any two zero entries, remains true for $X_1$. Note that if $\Delta T''$ ends with a 0 or with a sequence $(\ldots, 0, 1, \ldots, 1)$, then $X_1$ has regularity $m_{p-1} + 1$, by induction. Otherwise $X_1$ has regularity $m_{p-1}$.

**Case 1.** Suppose that $\Delta T''$ ends with a 0 or with a sequence $(\ldots, 0, 1, \ldots, 1)$, and that $m_p = m_{p-1} + 1$. Then the first difference of the Hilbert function of $X_1$ ends in degree $m_{p-1}$ (because of the regularity noted above); but then $3.11$ forces this to be shifted by 1, giving precisely what is required in the O-sequence computation of Definition $3.2$. In particular, the first difference of the Hilbert function of $X$ ends in degree $m_{p-1} + 1 = m_p$, and $X$ has regularity $m_p + 1$ as claimed.

**Case 2.** Suppose that $\Delta T''$ ends with a 0 or with a sequence $(\ldots, 0, 1, \ldots, 1)$, and that $m_p > m_{p-1} + 1$. Then again $3.11$ gives the O-sequence computation of Definition $3.2$. Again the first difference of the Hilbert function of $X_1$, shifted by 1, ends in degree $m_{p-1} + 1 < m_p$, but this time the regularity is determined by the $m_p$ new points, and is equal to $m_p$. 


Case 3. If $\Delta T''$ does not end with a 0 or with a sequence $(\ldots, 0, 1, \ldots, 1)$, and if $m_p > m_{p-1} + 1$ then the same argument as in Case 2 applies.

Case 4. If $m_{p-1} = m_p$ then necessarily we have $m_{p-2} < m_{p-1}$, by the definition of a pseudo linear configuration. Thus, for the pseudo type vector $T''$, $s_{p-1}$ is defined. However, for $T'$, since $m_{p-2} < m_{p-1} = m_p$, we obtain that $s_{p-1}$ is not defined, but $s_p$ is of the second type described in the O-sequence computation of Definition 3.2. Thus, in this case, the induction takes us from $T''' = (m_1, \ldots, m_{p-2})$ to $T' = (m_1, \ldots, m_p)$. But now the technical assumption that between any two zero entries of $\Delta T'$ there is at least one entry $> 1$, together with induction, guarantees that the regularity of the standard pseudo linear configuration $X_2$ determined by $T'''$ is $\leq m_p - 1$. The O-sequence computation of Definition 3.2 indicates that we must shift the O-sequence of $X_2$ by 2, so that it now ends in degree $\leq m_p + 1$. Hence the regularity is computed by the two sets of $m_p$ points, and is equal to $m_p + 1$.

For $iv$) it is enough to realize that in the standard configuration of type

$$(1, 1, 2, 2, 3, 3, \ldots, m, m)$$

there is a set of $2m$ points lying on the “vertical” line $x$, so the regularity is at least $2m$. □

The standard pseudo linear configuration is clearly very special. Nevertheless, we now show that the technical assumption that between any two zero entries there is at least one entry $> 1$ (used to control the regularity of the standard pseudo linear configuration), is enough to guarantee that all pseudo linear configurations of that type have the same Hilbert function.

**Theorem 3.7.** Consider a pseudo type vector $T' = (m_1, m_2, \ldots, m_p)$. Let

$$\Delta T' = (m_1 - 0, m_2 - m_1, \ldots, m_p - m_{p-1})$$

be its first difference (note that $\Delta T'$ has all entries non-negative). Then every pseudo linear configuration of type $T'$ can be realized as the result of a sequence of basic double links if and only if the following condition holds:

(3.2) **Between any two zero entries of $\Delta T'$ there is at least one entry that is $> 1$.**

If this condition holds then the Hilbert function of any pseudo linear configuration of type $T'$ is the same. The first difference of that Hilbert function is the O-sequence given by Definition 3.2.

In particular, if condition (3.2) holds for the pseudo type vector, $T'$, of a given pseudo linear configuration, then the regularity of that pseudo linear configuration is determined as follows: if $\Delta T'$ ends with a 0, or with a sequence $(\ldots, 0, 1, \ldots, 1)$ (i.e. a 0 followed by any number of 1’s), then the regularity is $m_p + 1$. Otherwise the regularity is $m_p$.

If condition (3.2) does not hold then the Hilbert function of a pseudo linear configuration of type $T'$ is not uniquely determined.

**Proof.** Note that by the definition of a pseudo type vector, there must be at least one non-zero entry between any two zeroes in the vector $\Delta T'$. We first prove that the condition (3.2) is sufficient to realize a given pseudo linear configuration as being obtained as a
sequence of basic double linkages, by working from left to right in the pseudo type vector (imagine a pointer moving along the marker and keeping track of our current position).

Having built the subconfiguration corresponding to entries \( m_1, \ldots, m_{i-1}, \) the next step:

- will involve only \( m_i \) if \( m_i < m_{i+1} \),
- will involve \( m_i \) and \( m_{i+1} \) if \( m_i = m_{i+1} \).

We will use the fact that sets consisting of \( m_i \) points on a line, or \( m_i \) points on each of two lines (avoiding the intersection point of the two lines) are both complete intersections in \( \mathbb{P}^2 \). (This is no longer necessarily true for three lines.)

Of course, if \( T' \) satisfies (3.2) then so does every subsequence. To simplify the notation, at each step we will take:

- \( X \) to be the subconfiguration built up so far (by induction);
- \( Y \) to be the set added; and
- \( Z \) will be the new set, \( Z = X \cup Y \).

We will use the fact that sets consisting of \( m_i \) points on a line, or \( m_i \) points on each of two lines (avoiding the intersection point of the two lines) are both complete intersections in \( \mathbb{P}^2 \). (This is no longer necessarily true for three lines.)

Case 1: \( m_i < m_{i+1} \).

This means that we want to add \( m_i \) points on \( L \). We have an exact sequence

\[
0 \to [I_Z : L](-1) \xrightarrow{\times L} I_Z \to \frac{I_Z + (L)}{(L)} \to 0.
\]

We sheafify and take cohomology. Note that \( \left( \frac{I_Z + (L)}{(L)} \right) \) is the ideal sheaf of \( Y \), viewed as a subscheme of \( L = \mathbb{P}^1 \). Its global sections begin in degree \( m_i \). We know that the regularity of \( I_X \) is \( m_{i-1} \) or \( m_{i-1} + 1 \), so \( h^1(I_X(m_{i-1})) = 0 \). Note that \( m_{i-1} \leq m_i - 1 \).

Condition (3.2) does not directly affect this case since we have assumed that it holds for \( X \) and we have \( m_{i-1} < m_i \).

From the exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to I_Y \to I_{Y|L} \to 0
\]
we get \( h^1(I_X(t)) = h^1(I_Y(t)) \) for all \( t \geq -1 \). Sheafifying (3.3), twisting by \( t \geq 0 \) and taking cohomology, we get

\[
(3.4) \quad 0 \to (I_Z)_{t_{-1}} \to (I_Z)_t \to (I_{Y|L})_t \to H^1(I_X(t-1)) \to H^1(I_Z(t)) \to H^1(I_Y(t)) \to \ldots
\]

Taking \( t = m_i \), we have \( H^1(I_X(m_i - 1)) = 0 \) since \( m_{i-1} \leq m_i - 1 \) and \( h^1(I_X(m_{i-1})) = 0 \). Hence the restriction \((I_Z)_{m_i} \to (I_{Y|L})_{m_i}\) is a surjection, and the non-zero element of \((I_{Y|L})_{m_i}\) lifts to a form \( F \in I_Z \) that does not vanish on the line \( L \), so in particular (since the points of \( Y \) are distinct) meets \( L \) transversally in \( Y \). Since \( Z = X \cup Y \) and \( Y \) is the complete intersection of \( F \) and \( L \), \( Z \) is a basic double link: indeed, \( I_Z \) and \( L \cdot I_X + (F) \) are saturated ideals defining the same set of points, so we have \( I_Z = L \cdot I_X + (F) \).

We now verify the Hilbert function calculation. Since we know that \( I_Z = L \cdot I_X + (F) \), we can use the theory of basic double linkage as described in Section 2. Indeed, it follows easily from (2.2). Let \( G = L, d_1 = m_i \) and \( d_2 = 1 (= \deg L) \). We then note that we are in the first case of the O-sequence computation of Definition 3.2 and that in that formula \((s_i)\) is now just the first difference of \( h_{R/(F,L)}(t) \). Then the bottom row of the O-sequence computed in Definition 3.2 (see Example 3.1) corresponds to the first difference of the Hilbert function of \( R/(F, L) \), and the rows above the bottom row all together correspond to (a decomposition of) the first difference of the Hilbert function of \( X \), shifted by 1. The connection is made by (2.2).

As for the regularity, we know that

\[
\reg(I_Z) = 1 + \min \{ t \mid h^1(I_Z(t)) = 0 \}.
\]

The sequence (3.4) shows that \( \reg(I_Z) \) is the larger of \( \reg(I_X) + 1 \) and \( m_i \), depending (respectively) on whether \( m_i = m_{i-1} + 1 \) or \( m_i > m_{i-1} + 1 \). Using induction, this shows that \( \reg(I_Z) = m_i + 1 \) if the first difference vector \((m_1 - 1, m_2 - m_1, \ldots, m_i - m_{i-1})\) ends with a sequence \((0, 1, 1, \ldots, 1)\) (in this Case, it is excluded that this first difference will end with a 0) and \( \reg(I_X) = m_i \) otherwise.

**Case 2.** \( m_i = m_{i+1} \).

In this case we let \( X \) be the set of points corresponding to the pseudo type vector \((m_1, \ldots, m_{i-1})\), and we add \( m_i \) points on each of two lines. In our argument, instead of \( L \) we use \( Q \), the union of the two lines containing \( m_i \) points each. In this Case the first difference vector \((m_1 - 1, m_2 - m_1, \ldots, m_{i+1} - m_i)\) ends with a 0, so condition (3.2) implies that it does not end \((0, 1, 1, \ldots, 1)\) (with only 1’s between the 0’s). The two possibilities are that (a) there is no other 0, or else (b) there is at least one entry that is \( > 1 \) between the 0’s.

We first would like to compute the regularity of \( I_X \). If (a) holds then \( m_1 < m_2 < \cdots < m_{i-1} < m_i = m_{i+1} \). Hence \( X \) is a linear configuration, and its regularity is \( m_{i-1} \). We remark that in this case \( Z \) is in fact a linear configuration minus a point on the “longest” row, so its Hilbert function is just the truncation of the Hilbert function of the linear configuration of type \((m_1, \ldots, m_{i-1}, m_i, m_i + 1)\) (cf. 4.3).

If (b) holds, then there are again two possibilities. First, it could happen that the first difference vector \((m_1 - 1, \ldots, m_{i-1} - m_{i-2})\) ends in a 0. In this case \( m_{i-1} = m_{i-2} \), and by induction the regularity of \( I_X \) is \( m_{i-1} + 1 \). However, condition (b) then means that \( m_i \geq m_{i-1} + 2 \).
The other possibility in (b) is that the first difference vector \((m_1 - 0, \ldots, m_{i-1} - m_i - 2)\) does not end in a 0. If it ends in a string \((\ldots, 0, 1, \ldots, 1)\) (all 1’s after the 0) then again the regularity of \(I_X\) is \(m_i + 1\) by induction, but again (b) forces \(m_i \geq m_{i-1} + 2\). If the first difference vector does not end in such a string, then by induction the regularity of \(I_X\) is \(m_i - 1\).

We conclude from the above analysis that in every case,

\[(3.5) \quad h^1(I_X(m_i - 2)) = 0.\]

By analogy with Case 1, but now using \(Q\) instead of \(L\), the exact sequence \((3.3)\) now becomes

\[
0 \rightarrow [I_Z : Q](-2) \xrightarrow{xQ} I_Z \rightarrow \frac{I_Z + (Q)}{(Q)} \rightarrow 0.
\]

Sheafifying, twisting by \(m_i\) and taking cohomology, we get

\[
0 \rightarrow (I_X)_{m_i - 2} \rightarrow (I_Z)_{m_i} \rightarrow (I_{Y|Q})_{m_i} \rightarrow H^1(I_X(m_i - 2)) \rightarrow H^1(I_Z(m_i)) \rightarrow \ldots
\]

By \((3.3)\), we have \(h^1(I_X(m_i - 2)) = 0\). Since \(Y\) consists of \(m_i\) points on each of the two components of \(Q\), and since by Definition 3.1 we do not allow a point of \(Y\) to lie on both components of \(Q\), we first claim that a non-zero element of \((I_{Y|Q})_{m_i}\) cannot vanish identically on either component of \(Q\). Indeed, if it vanished on either component then it would lift to a homogeneous polynomial \(F \in (I_Z)_{m_i}\) vanishing on a line, which then has as a factor a form \(G\) of degree \(m_i - 1\) that vanishes on \(m_i\) points on the other component of \(Q\), but does not vanish identically on that component (since it is a non-zero element of \(I_{Y|Q}\)). Impossible.

Thus the vanishing of the first cohomology and the fact that \(I_{Y|Q}\) begins in degree \(m_i\) (recall that \(Y\) is a complete intersection of type \((2, m_i)\)) means that there is a form \(F \in I_Z\) of degree \(m_i\) that does not vanish on either component of \(Q\), and cuts out \(Y\) on \(Q\) (in particular it meets \(Q\) transversally). Since \(Z = X \cup Y\), we again recognize \(Z\) as being obtained as a basic double link from \(X\), and we have

\[
I_Z = Q \cdot I_X + (F).
\]

By an argument similar to the one above, we can compute the Hilbert function of \(Z\) using the O-sequence computation in Definition 3.2. In this case the bottom row is given by the second case in that computation, since \(Y\) is a complete intersection of type \((2, m_i)\), and the shift between the bottom row of the computation and the rows above it is now 2 (see Example 3.4).

In fact, starting from \((2.1)\) we can easily compute the minimal free resolution of \(I_Z\), using a mapping cone and using a minimal free resolution

\[
0 \rightarrow F_2 \rightarrow F_1 \rightarrow I_X \rightarrow 0.
\]

We get a free resolution

\[
0 \rightarrow F_2(-2) \oplus R(-m_i - 2) \rightarrow F_1(-2) \oplus R(-m_i) \rightarrow I_Z \rightarrow 0.
\]
However, by (3.5) we have that \( \text{reg}(I_X) \leq m_i - 1 \), so \( F \) (having degree \( m_i \)) cannot be a minimal generator of \( I_X \). But the resolution is minimal if and only if \( F \) is not a minimal generator (Lemma 2.15 (e)). In particular, it follows that \( \text{reg}(I_X) = m_i + 1 \). (Note that in this case the sequence \( \Delta T' \) ends with a 0, so this is the regularity claimed in the statement of the theorem.)

This completes one direction of the theorem. For the converse, we have to show that if \( \Delta T' \) contains a subsequence \( (\ldots, 0, 1, \ldots, 1, 0) \) (all 1’s between the two 0’s) then there exist (at least) two pseudo linear configurations of type \( T' \) with different Hilbert functions. To do this, first we will show that if \( \Delta T' \) ends with such a subsequence, with no such subsequence preceding it, then the conclusion holds. Second, we will show the general statement. Note that we showed in Proposition 3.6 that for any pseudo type vector, there always exists one pseudo linear configuration (the standard one) that can be constructed by basic double linkage, and hence has Hilbert function whose first difference is given by the O-sequence computed in Definition 3.2. So for both the first part and the second part, we have to show that such a subsequence allows for a pseudo linear configuration that can not be constructed entirely by basic double links.

Suppose that \( \Delta T' \) ends with the subsequence \( (\ldots, 0, 1, \ldots, 1, 0) \) (all 1’s between the 0’s) and no such subsequence precedes it. This means that if \( T' = (m_1, \ldots, m_{p-2}, m_{p-1}, m_p) \) then \( m_p = m_{p-1} = m_{p-2} + 1 \). In this paper we usually handle the case where \( m_{p-1} = m_p \) by doing a basic double link using a quadric form \( Q \), as described above (because of the application to non-reduced schemes that will be given below). However, just for this step in the current proof, it is convenient to view it as two separate basic double links using linear forms.

Let \( T'' \) be the pseudo type vector \( (m_1, \ldots, m_{p-2}, m_{p-1}) \). Then \( \Delta T' \) ends in a sequence \( (\ldots, 0, 1, \ldots, 1) \) where the end consists of nothing but 1’s. We have assumed that \( T'' \) satisfies (3.2). Let \( X \) be a pseudo linear configuration corresponding to \( T'' \). It can be constructed by basic double linkage, and its Hilbert function is as described. Furthermore, it follows from what we have already proven that the regularity of \( I_X \) is \( m_{p-1} + 1 = m_p + 1 \).

Now consider an additional line \( L_p \), and choose \( Y \) to be a general set of \( m_p \) points on \( L_p \). Let \( Z = X \cup Y \). \( Z \) is a basic double link of \( X \) if and only if there is a form \( F \in (I_X)_{m_p} \) that contains \( Y \) but does not vanish on \( L_p \).

We have an exact sequence

\[
0 \rightarrow I_X(m_p - 1) \xrightarrow{r} I_X(m_p) \rightarrow \mathcal{O}_{L_p}(m_p) \rightarrow 0.
\]

Since the regularity of \( I_X \) is \( m_p + 1 \), we have

\[
0 \rightarrow (I_X)_{m_p-1} \rightarrow (I_X)_{m_p} \xrightarrow{r} H^0(\mathcal{O}_{L_p}(m_p)) \rightarrow H^1(I_X(m_p - 1)) \rightarrow 0
\]

(3.6) where the last cohomology group is not zero. Choosing \( Y \) as above is equivalent to choosing a general element of the vector space \( H^0(\mathcal{O}_{L_p}(m_p)) \). The image of \( r \) is a proper subspace of \( H^0(\mathcal{O}_{L_p}(m_p)) \), so the general section of \( H^0(\mathcal{O}_{L_p}(m_p)) \) defining \( Y \) is not in the image of \( r \). We conclude that any form in \( (I_X)_{m_p} \) that vanishes on \( Y \) must in fact vanish on all of \( L_p \). Hence we cannot express \( Z \) as a basic double link of \( X \).

We claim that the value of the Hilbert function of such a \( Z \) differs, in degree \( m_p \), from the value of the corresponding Hilbert function given by the O-sequence computation in
Definition 3.2 (see Remark 3.3). Indeed, suppose that $Z'$ were a pseudo linear configuration of the same type that was produced by basic double linkage, and hence has the standard Hilbert function for that type. (It is not hard to check that we can even assume that $Z'$ is built up from the same $\mathbb{X}$, choosing the points of $\mathbb{Y}$ in a more careful way.) In degree $m_p$, the forms that vanish on $\mathbb{Z}$ consist entirely of products of $L_p$ with forms of degree $m_p - 1$ vanishing on $\mathbb{X}$ (as discussed above), while $Z'$ has those but also has a form of degree $m_p$ that is not of that form. Hence the Hilbert functions differ in degree $m_p$.

Now we prove the second part. Let $T'$ be a pseudo linear configuration not satisfying (3.2). Its first difference has an initial subsequence with first difference $(\ldots, 0, 1, 1, \ldots, 1, 0)$, where the earlier entries do satisfy (3.2). Let $\mathbb{Z}$ and $\mathbb{Z}'$ be as above, both pseudo linear configurations with type given by this subsequence, and having different Hilbert functions. We claim that term by term we can add $m_i$'s to the subsequence, and correspondingly add points on a line that arise by basic double linkage. (We do not claim that only basic double links are possible if there is another subsequence $(\ldots, 0, 1, 1, \ldots, 1, 0, \ldots)$, but only that in particular a basic double link is possible.) The basic double links at each step are numerically the same, so they add the same amount in each degree to the Hilbert functions. Since we started with $\mathbb{Z}$ and $\mathbb{Z}'$ having different Hilbert functions, this will say that at each step the results have different Hilbert functions, and we will be finished.

We have seen that for any type there exists a standard pseudo linear configuration, so we can assume that $\mathbb{Z}'$ is a standard pseudo linear configuration, and it can continue to be built up by basic double links as claimed. The real assertion is that this is true of $\mathbb{Z}$ as well. First note that if we twist the exact sequence (3.6) by any $t > 0$, we obtain the short exact sequence

$$0 \to (I_\mathbb{X})_{m_p + t - 1} \to (I_\mathbb{X})_{m_p + t} \xrightarrow{r} H^0(\mathcal{O}_{L_p}(m_p + t)) \to 0$$

because of the regularity. Now choose $t$ so that $m_p + t = m_{p+1}$ (recall that both $\mathbb{Z}$ and $\mathbb{Z}'$ ended with $m_p = m_{p-1}$). This says, in particular, that there is a form $F$ in $(I_\mathbb{X})_{m_{p+1}}$ vanishing on $\mathbb{Z}$ but not vanishing identically along $L_p$, because $r$ is surjective and we can choose a section of $H^0(\mathcal{O}_{L_p}(m_{p+1}))$ that vanishes at $\mathbb{Y}$ plus $t$ general points of $L_p$. But then choosing a general line $L_{p+1}$, this meets the same $F$ in $m_{p+1}$ distinct points, forming a basic double link of $\mathbb{Z}$ of type $(m_1, \ldots, m_p, m_{p+1})$. Now it is trivial to build up the rest of the pseudo type by basic double linkage, since we can take products of $F$ with general forms of suitable degree to produce the points.

\begin{proof}
Example 3.8. Theorem 3.7 applies to the pseudo type vectors

$$(2, 2, 3, 4, 5, 7, 7), \ (2, 2, 3, 4, 6, 7, 7), \ (2, 2, 3, 4, 5, 6, 7), \ (5, 5, 7, 7)$$

but not to

$$(2, 2, 3, 4, 5, 6, 7, 7) \text{ or } (6, 6, 7, 7).$$

A simpler example to show that the Hilbert function may vary if (3.2) does not hold is the pseudo type vector $(1, 1, 2, 2)$. If these points form a standard pseudo linear configuration, i.e.

\begin{itemize}
  \item ...
  \item ...
  \item ...
\end{itemize}
then the Hilbert function of the points is \((1, 3, 5, 6, 6, \ldots)\) (note that by considering the “vertical” lines, this set of points is realized as a linear configuration of type \((2, 4)\)). On the other hand, if the points are chosen generically on the four “horizontal” lines then they are in fact 6 generic points in \(\mathbb{P}^2\), so the Hilbert function is \((1, 3, 6, 6, \ldots)\). □

4. The Resolution of the Ideal of a Pseudo Linear Configuration

In the last section we saw the necessary and sufficient condition for the Hilbert function of a pseudo linear configuration to be uniquely determined from the type. This was seen to be equivalent to the condition that every pseudo linear configuration of given type can be built up by basic double linkage in the way prescribed by the type. This is analogous to the situation for linear configurations, where the type uniquely determines the Hilbert function (but with no condition needed).

For linear configurations, in fact, the type uniquely determines the graded Betti numbers (which are maximal among all algebras with the given Hilbert function [26]). We now turn to the question of when the type of a pseudo linear configuration uniquely determines the graded Betti numbers, and how to determine what those graded Betti numbers are. We will use the fact that in \(\mathbb{P}^2\), when the Hilbert function is fixed, the graded Betti numbers depend only on the degrees of the minimal generators.

**Example 4.1.** Consider the pseudo type vector \((1, 2, 2, 3)\). We have seen that any pseudo linear configuration of this type arises from a sequence of two basic double linkages (starting from a single point), so the Hilbert function is uniquely determined. But we will see now that the graded Betti numbers are not uniquely determined. In particular, we will see that the form \(F\) of degree 3 that is used for the last basic double linkage may or may not be a minimal generator of the subconfiguration of type \((1, 2, 2)\).

First suppose that the pseudo linear configuration is standard:

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

\(X\) consists of the five points on the first three “horizontal” lines. In this case \(F\) can be taken to be the product of the three “vertical” lines. Note that the product of the leftmost two vertical lines is an element of the ideal of \(X\) (in fact it is the only generator of \(I_X\) of degree 2), so \(F\) is not a minimal generator of \(I_X\). Hence no splitting occurs, by Lemma 2.15(e). We have a minimal free resolution

\[
0 \to R(-5)^2 \oplus R(-3) \oplus R(-4)^2 \to R(-4) \oplus R(-3) \to I_Z \to 0.
\]

On the other hand, suppose that our pseudo linear configuration of pseudo type \((1, 2, 2, 3)\) is formed by general points on each of the four lines. Note that \(I_X\) still has only one quadric generator, and this quadric meets the fourth “horizontal” line in two points, say \(P_1\) and \(P_2\). Since \(Z\) was chosen with general points on each of the lines, the three points on this fourth “horizontal” line are disjoint from \(P_1\) and \(P_2\). Therefore no \(F\), cutting out the three points on this line, is a multiple of the quadric generator of \(I_X\); hence any such \(F\)
can be chosen as a minimal generator of degree 3. Therefore a copy of $R(-4)$ splits off in the above resolution, for this pseudo linear configuration, and we obtain the minimal free resolution

$$0 \to R(-5)^2 \to R(-3)^2 \oplus R(-4) \to I_Z \to 0.$$  

Therefore, as claimed, the graded Betti numbers are not uniquely determined for the pseudo type vector $(1, 2, 2, 3)$. □

With this example in mind, we consider the graded Betti numbers of a pseudo linear configuration that arises as a result of basic double linkage (e.g. by satisfying condition (3.2) or by being a standard pseudo linear configuration). Suppose that $I_X$ has minimal free resolution

$$0 \to F_2 \to F_1 \to I_X \to 0$$

and that $Z$ arises from $X$ by basic double linkage using $F$ and $L$ as before. Then the diagram

\[
\begin{array}{cccc}
0 & \downarrow & 0 & \downarrow \\
0 & \downarrow & \mathbb{F}_2(-1) \oplus & 0 \\
R(-m_i - 1) & \downarrow & \mathbb{F}_1(-1) \oplus & R(-m_i) \\
0 & \to & R(-m_i - 1) & \to I_X(-1) \oplus R(-m_i) \to I_Z \to 0
\end{array}
\]

yields a resolution (using the mapping cone)

$$0 \to \mathbb{F}_2(-1) \oplus R(-m_i - 1) \to \mathbb{F}_1(-1) \oplus R(-m_i) \to I_Z \to 0.$$ 

As mentioned before, this resolution is minimal if and only if $F$ is not a minimal generator of $I_X$ (Lemma 2.15 (e)).

So we are reduced to the problem of determining whether or not $F$ is a minimal generator of $I_X$. If $m_i > \text{reg}(I_X)$ then clearly $F$ is not a minimal generator of $I_X$. If $m_i = \text{reg}(I_X)$ (the only other possibility) then we know that $m_i = m_{i-1} + 1$, and that the first difference of the partial type vector $(m_1, m_2, \ldots, m_{i-1})$ ends with either a 0 or a sequence $(0, 1, \ldots, 1)$. However, unfortunately in this case the question of whether or not $F$ is a minimal generator of $I_X$ is not merely a numerical one.

**Example 4.2.** One can check that the pseudo type vector $(1, 2, 2, 4, 4, 5)$ does not have the property that all pseudo linear configurations of this type have the same graded Betti numbers. Indeed, letting $X_1$ be a point, with ideal $(A_1, A_2)$ (deg $A_i = 1$), then we successively form

$$I_{X_2} = (QA_1, QA_2, F_1) \quad \text{where } F_1 \in I_{X_1}, \text{ deg } F_1 = 2, \text{ deg } Q = 2$$

$$I_{X_3} = (Q'QA_1, Q'QA_2, Q'F_1, F_2) \quad \text{where } F_2 \in I_{X_2}, \text{ deg } F_2 = 4, \text{ deg } Q' = 2$$

The point is that $I_{X_3}$ does have generators of degree 5, so forming the last basic double link using a form $F \in I_{X_3}$ of degree 5 can be done with $F$ a minimal generator of $I_{X_3}$ or not. □
Theorem 3.7 gave (in particular) a necessary and sufficient condition for the Hilbert function of a pseudo linear configuration, $X$, to be uniquely determined by the pseudo type; namely, (3.2), that between any two zero entries of $\Delta T'$ there is at least one entry that is $> 1$. We would like to do the same thing for the graded Betti numbers. Of course we have to begin by assuming (3.2), since if the Hilbert function can vary then so can the graded Betti numbers. In particular, we can assume that $X$ can be realized as a sequence of basic double links.

The following lemma is trivial, but we will refer to it several times in the next result.

Lemma 4.3. Let $Z$ be a basic double link of $X$, so that $I_Z = A \cdot I_X + (F)$ with $F \in I_X$. Assume that $F$ is not a minimal generator of $I_X = (G_1, \ldots, G_r)$. Assume further that the maximal degree of a minimal generator of $I_X$ is $d$.

(a) If $\deg A = 1$ then the minimal generators of $I_Z$ have degrees $\deg G_1 + 1, \ldots, \deg G_r + 1, \deg F$. If $d \leq \deg F - 1$ then all generators have degree $\leq \deg F$.

(b) If $\deg A = 2$ then the minimal generators of $I_Z$ have degrees $\deg G_1 + 2, \ldots, \deg G_r + 2, \deg F$. If $d \leq \deg F - 2$ then all generators have degree $\leq \deg F$.

Remark 4.4. In the following theorem, we will be constructing a pseudo linear configuration $Z$ inductively from a smaller one, $X$, and studying the question of whether the polynomial $F$ used in the basic double link is a minimal generator of $I_X$ or not. In each case, $F$ will have the largest possible degree allowed by the regularity. If our analysis shows that $I_X$ does have a minimal generator of that degree, then a general element of $I_X$ of that degree can form part of a minimal generating set. Therefore, even though the argument that we use to show that $I_X$ has a minimal generator of that degree will produce $F$ having components in common with other generators, it is just the existence of generators that is important, and then a general choice will have no such common components. □

In the following result, it is helpful to keep in mind Example 4.1 and Example 4.2.

Theorem 4.5. Consider a pseudo type vector $T' = (m_1, m_2, \ldots, m_p)$. Let

$$\Delta T' = (m_1 - 0, m_2 - m_1, \ldots, m_p - m_{p-1})$$

be its first difference, and assume that (3.2) holds, i.e. between any two zero entries of $\Delta T'$ there is at least one entry that is $> 1$. Let $Z$ be a pseudo linear configuration of pseudo type $T'$. Then the following hold.

(a) The graded Betti numbers of $I_Z$ are uniquely determined if and only if $\Delta T'$ contains none of the following as subsequences:

$$(1, 0, 1),$$

$$(1, 0, 2, 0, 1)$$

$$(4.1)$$

$$(1, 0, 2, 0, 2, 0, 1),$$

$$\vdots$$

$$(1, 0, 2, 0, \ldots, 0, 2, 0, 1)$$

(b) If $\Delta T'$ contains none of (4.1) as subsequences then the number of minimal generators of $I_Z$ is $p + 1 - a$ where $a$ is the number of 0's appearing in $\Delta T'$. 

In particular, Z has the maximum number of minimal generators allowed by the Hilbert function if and only if it is a linear configuration (i.e. $\Delta T'$ contains no 0's).

Proof. We know that Z can be obtained by a sequence of basic double links, since (3.2) holds. At each step the ideal has the form $J = AI + (F)$ where $A$ is a form of degree 1 or 2, $F \in I$, and $(A, F)$ is a regular sequence. If $I = (F_1, \ldots, F_r)$ then $J$ is generated by $(F, AF_1, \ldots, AF_r)$. In particular, these are minimal generators if and only if $F$ is not a minimal generator of $I$ (Lemma 2.15 (e)). In this case the graded Betti numbers are uniquely determined.

Hence we have to see when it can happen that the $F$ chosen in any step may (or may not) be a minimal generator. The point is that we are constructing Z inductively. At each step we are adding some set of points on a line, or some set of points on two lines. The graded Betti numbers are uniquely determined if, for regularity or other reasons, the number of points to be added forces $F$ to have a degree such that $F$ has no chance to be a minimal generator of $I$ (e.g. the degree is too large). Alternatively if there is no such prohibition, we have to show that some choices of the points to be added correspond to $F$ a minimal generator of $I$, and other choices of the points to be added correspond to $F$ not a minimal generator of $I$.

By mimicking Example 4.1 (see also Proposition 3.6) we see that the standard pseudo linear configuration always gives an example where $F$ is not a minimal generator of $I$. Therefore, to prove (a) we have to show that the given condition is equivalent to the condition that at each step, $F$ is forced to not be a minimal generator. Notice that if at any step there is a choice between choosing $F$ a minimal generator or not, then not only are the graded Betti numbers at that step not uniquely determined, but neither are the graded Betti numbers for any subsequent step.

Assume first that $\Delta T'$ contains no subsequence in the list (4.1). Abusing notation slightly, suppose that at some intermediate step we have a pseudo linear configuration Z that has been obtained from the previous step $X$ by a basic double link, using $F \in I_X$ and thus adding a set $Y$ to $X$ to obtain Z. If this basic double link uses a linear form then it corresponds to a single entry in $\Delta T'$; if it uses a quadric then it corresponds to a subsequence $(b, 0)$ in $\Delta T'$. We will assume inductively that the graded Betti numbers of $I_X$ are uniquely determined, and see that then the hypothesis forces that of Z to also be uniquely determined.

If this basic double link corresponds to a single entry in $\Delta T'$ which is 2 or greater, then by Theorem 3.7, $\deg F$ is greater than the regularity of $I_X$ so $F$ cannot be a minimal generator of $I_X$. Suppose that this basic double link corresponds to a single entry in $\Delta T'$ which is 1. If what precedes this 1 is not a sequence $(\ldots, 0, 1, 1, \ldots, 1)$ then again $\deg F$ is greater than the regularity of $I_X$ by Theorem 3.7, so $F$ cannot be a minimal generator.

Next, suppose that this basic double link corresponds to an entry in $\Delta T'$ which is a 1, and that in $\Delta T'$ it is preceded by $(\ldots, b, 0, 1, 1, \ldots, 1)$ (where the number of 1's may be zero). By the definition of a pseudo linear configuration, $b \neq 0$. By hypothesis, $b \neq 1$, and if $b = 2$ then it is not preceded by any sequence $(1, 0), (1, 0, 2, 0)$, etc. We will analyze the cases $b \geq 3$ and $b = 2$ separately, but first we make some general observations.
Corresponding to the subsequence of $\Delta T'$ given by $(\ldots, b, 0, 1, 1, \ldots, 1)$, consider the sequence of configurations

$$\ldots, X_1, X_2, X_3, \ldots, X_r = X, Z.$$ 

Here $X_1$ is the configuration obtained prior to this subsequence, i.e. it corresponds to the initial dots before $b$ in $\Delta T'$. Suppose that the maximum number of collinear points on $X_1$ is $m$. $X_2$ is then obtained from $X_1$ by adding two sets of collinear points, each of which contains $m + b (\geq m + 2)$ points. (This corresponds to the $(b, 0)$ in $\Delta T'$.) Translating to basic double links, $X_2$ is obtained from $X_1$ by a basic double link using a quadric, $Q$, and a form $F_1 \in I_{X_1}$. Each subsequent basic double link uses a linear form.

Note that $F_1$ is not a minimal generator of $I_{X_1}$ (because of the regularity and $b \geq 2$).

Suppose that the minimal generators of $I_{X_1}$ are $G_1, \ldots, G_r$ and the graded Betti numbers of $X_1$ are uniquely determined by the type (by induction). Then

$$I_{X_2} = (QG_1, \ldots, QG_r, F_1)$$

where $F_1 \in I_{X_1}$, not a minimal generator of $I_{X_1}$

$$I_{X_3} = (LQG_1, \ldots, LQG_r, LF_1, F_2)$$

where $F_2 \in I_{X_2}$, deg $F_2 = \deg F_1 + 1$.

Now, if $b \geq 3$ then $\deg F_1 \geq \text{reg} I_{X_1} + 2$. Hence $\deg F_1 \geq \deg QG_i$ for all $i$, and so $F_2$ (having degree $\deg F_1 + 1$) cannot be a minimal generator of $I_{X_2}$ and so the listed generators of $I_{X_3}$ are minimal. The same trickles down to the step from $X$ to $Z$, proving that the graded Betti numbers of $Z$ are uniquely determined.

Now suppose that $b = 2$, but it is not preceeded by any sequence $(1, 0), (1, 0, 2, 0)$, etc.. Again suppose that the subsequence $(b, 0) = (2, 0)$ corresponds to a basic double link $I_{X_2} = QI_{X_1} + (F_1)$ as above, where $I_{X_1} = (G_1, \ldots, G_r)$. Now the pseudo type vector itself has the form $(\ldots, p, q, m, m + 2, m + 2, m + 3, m + 4, \ldots)$, where $X_1$ is a pseudo linear configuration of pseudo type $(\ldots, p, q, m)$. If $q \leq m - 2$ then it follows immediately that $\text{reg} I_{X_1} = m$ and each subsequent step uses an $F$ that is not a minimal generator (not necessarily from a regularity argument, but rather from an analysis of the ideal as above, using Lemma 4.3). If $q = m - 1$ or $q = m$, then the only danger is that $\text{reg} I_{X_1} = m + 1$ and that furthermore $I_{X_1}$ has a minimal generator $G$ of degree $m + 1$, so that $QG \in I_{X_2}$ is a minimal generator of degree $m + 3$ and can be used to construct $X_3$ (thanks to the above analysis). The condition that $\text{reg} I_{X_1} = m + 1$ holds if and only if the first difference of the pseudo type vector for $X_1$ ends either with a 0 or with a sequence $(0, 1, \ldots, 1)$, by Theorem 3.7.

So we are reduced to the two cases

$$\Delta T' = (\ldots, 0, 2, 0, 1, 1, \ldots, 1) \text{ or } \Delta T' = (\ldots, 0, 1, \ldots, 1, 2, 0, 1, 1, \ldots, 1).$$

In these cases, when can it happen that $X_1$ has a minimal generator of degree $m + 1$?

A little thought using Lemma 4.3 shows that in either case it requires that the first 0 be preceeded by a 1, a $(1, 0, 2)$, a $(1, 0, 2, 0, 2)$, etc. But these are eliminated by our hypotheses.

Conversely, suppose that $\Delta T'$ does contain one of the subsequences $(1, 0, 1), (1, 0, 2, 0, 1), (1, 0, 2, 0, 2, 0, 1)$, etc. We know that it is possible to carry out the basic double links using polynomials $F$ at each step that are not minimal generators (mimicking Example 4.1). So to show non-uniqueness of the graded Betti numbers we have to show that at least once it is possible to choose $F$ to be a minimal generator in these cases.
First we consider the case where \( \Delta T' \) contains a subsequence \((1,0,1)\). Hence \( T' \) contains a subsequence \((m,m+1,m+1,m+2)\). Consider a sequence of pseudo linear configurations \( X_1, X_2, X_3 \) where

\[
\begin{align*}
I_{X_2} &= QI_{X_1} + (F_1) \quad \text{where } F_1 \in I_{X_1}, \deg F_1 = m + 1, \\
I_{X_3} &= LI_{X_2} + (F_2) \quad \text{where } F_2 \in I_{X_2}, \deg F_2 = m + 2.
\end{align*}
\]

The construction of basic double linkage guarantees that \( I_{X_1} \) has a minimal generator of degree \( m \). (Notice that it cannot have a minimal generator of degree \( m+1 \) because if it did, the regularity of \( I_{X_1} \) would be \( m+1 \), so \( \Delta T' \) would have a subsequence \((0,1,\ldots,1,0)\), violating \((3.2)\). Hence \( F_1 \) cannot be a minimal generator of \( I_{X_1} \).) But then \( I_{X_2} \) has a minimal generator of degree \( m+2 \). Hence \( F_2 \) can either be chosen to be a minimal generator, or not (as illustrated in Example \( 4.1 \)).

The analysis for the case when \( \Delta T' \) has one of the other subsequences \((1,0,2,0,1)\), \((1,0,2,0,2,0,1)\), etc. is very similar and is left to the reader.

For \((b)\) and \((c)\), the condition that \( \Delta T' \) contains none of these subsequences means (according to the proof of \((a)\)) that each basic double link adds a new generator. An entry of 0 in \( \Delta T' \) corresponds to a repetition in \( T' \), which in turn corresponds to the fact that two entries of \( T' \) come from a single basic double link. The result follows immediately. \( \square \)

A pseudo linear configuration of type \( T' = (\ldots,m,m\ldots) \) satisfying \((3.2)\) can be viewed as being obtained from a pseudo linear configuration of type \((\ldots,m,m+1)\) by removing a point. Indeed, the only danger is that \( T' \) included \((\ldots,m,m,m+1,m+1,\ldots)\) (since putting the point back would give a configuration that has three lines with \( m+1 \) points), but this violates \((3.2)\).

In particular, let \( X \) be a linear configuration in \( \mathbb{P}^2 \) of type \( T \), and remove one point \( P \in X \), giving a pseudo linear configuration (which is possibly still in fact a linear configuration) of type \( T' \) obtained in the obvious way. From our results above we recover the fact (cf. \((4.3)\)) that the Hilbert function of \( X \setminus P \) is determined from the line that \( P \) lies on, but in fact we get more:

**Corollary 4.6.** If \( X \) is a linear configuration in \( \mathbb{P}^2 \) and \( P \in X \) then the graded Betti numbers of \( X \setminus P \) are determined from the line that \( P \) lies on.

**Proof.** When we remove a point of \( X \), we obtain a type \( T' \) that has at most one repetition, i.e. \( \Delta T' \) contains at most one zero. Hence the only danger, according to Theorem \( 4.5 \), is that \( \Delta T' \) contain the subsequence \((1,0,1)\). However, this means that \( T' \) contains the subsequence \((m,m+1,m+1,m+2)\) for some \( m \), and it is clear that adding a point to any line will not give an allowable type vector for a linear configuration. \( \square \)

**Remark 4.7.** Note that this result is not true for arbitrary \( k \)-configurations since even a \( k \)-configuration of type \((2,3)\) that is not a linear configurations provides a counterexample. It would be interesting to determine to what extent the result of Corollary \( 4.6 \) extends to \( \mathbb{P}^n \).
5. First applications to double point schemes

As remarked earlier, linear configurations have the property that their type completely determines their Hilbert function and graded Betti numbers, and these latter are maximal among all zero-dimensional schemes with the same Hilbert function. In this section and the next we are interested in seeing to what extent these properties are preserved for sets of 2-fat points which are supported on a linear configuration, i.e. for the first infinitesimal neighborhood of a linear configuration.

We will use the machinery of pseudo linear configurations as an important component of our study, and indeed the heart of this material is the observation that there are surprisingly few differences between these two situations! In this section our focus will be to find the analog of standard pseudo linear configurations for double point schemes supported on linear configurations. The key point will be that such a double point scheme, , can be constructed (by basic double linkage) starting with an arbitrary type, , by choosing the underlying linear configuration in a suitable way, much as the standard pseudo linear configuration was chosen (for an arbitrary pseudo type vector) in a suitable way. The idea will be to pass to the pseudo type vector associated to (see below). This will lead to the conclusion, analogous to Proposition 3.6, that for any type there is a linear configuration of type whose corresponding double point scheme has Hilbert function computed by the O-sequence computation in Definition 3.2.

Definition 5.1. Let be a linear configuration of type . The associated pseudo type vector of (or of ) is the vector where ( ) means that we list the entries in non-decreasing order. Note that is in fact a pseudo type vector, since for all , so at most two entries of take any particular value (and that happens if and only if we have for some and ).

Example 5.2. We will be using the associated pseudo type vector of the linear configuration to build up a collection of 2-fat points with support . We illustrate the way we will do this with an example.

Let be a linear configuration of type . This gives a pseudo type vector of type . We will build up the 2-fat point scheme with support (using a sequence of basic double links) in 5 steps.

Step 1: Choose the 2 points of on the first line. (This uses the “2” in the pseudo type vector.)

Step 2: Form a basic double link to produce the scheme which consists of the 2 points on the first line of and the 3 points on the second line. (This uses the “3” in the pseudo type vector.)

Step 3: Form a basic double link on the ideal of Step 2 to (simultaneously) fatten up the two points on line 1 and add the four points on line 3 to the previous scheme. (This uses the “4,4” in the pseudo type vector.)

Step 4: Form a basic double link on the ideal of Step 3 to fatten up the three points on the second line of . (This uses the “6” in the pseudo type vector.)

Step 5: Form a basic double link on the ideal of Step 4 to fatten up the four points on the third line of . (This uses the “8” of the pseudo type vector.)
The justification for why these steps are possible will be different in this section and the next. In this section, much as in Proposition 3.6, it will be clear because of the geometry of the configuration. In the next section, as in the preceding section, it will come as a result of showing that numerical conditions force conclusions about the regularity that guarantee the result.

In the next section we will discuss the 2-type vectors, $T$, that have the property that every linear configuration, $X$, of type $T$ has the property that its first infinitesimal neighborhood has the Hilbert function and graded Betti numbers described in Corollary 6.1, that is, for such 2-type vectors, the Hilbert function and graded Betti numbers of any set of double points with such a support are uniquely determined.

In this section, though, we give a construction that gives, for any 2-type vector $T$, an explicit saturated ideal of double points whose support is a (particular) linear configuration of type $T$. We will also see that sometimes there can be more than one Hilbert function for double points whose support is a linear configuration of type $T$. In the next section we will describe exactly when this happens. To illustrate the ideas, we begin with an example.

**Example 5.3.** Let $T = (2, 4, 5)$ be a 2-type vector, so the associated pseudo type vector is $T' = (2, 4, 4, 5, 8, 10)$. Note that we will eventually show (Theorem 6.1) that a double point scheme supported on any linear configuration of type $T$ has the same Hilbert function but not necessarily the same graded Betti numbers. This is not why we have chosen this example. This example was selected because of its simplicity and the fact that it illustrates the method used.

We will form a special linear configuration, namely the “spread out” configuration (placing the points on suitable integer lattice points in the plane – see Definition 2.12), but we will also place “imaginary” points to properly position the points in which we are interested. In this case, we get the following:

![Diagram](image)

We also consider three families of “parallel” lines: $\{L_1, L_2, L_3, \ldots\}$, $\{M_1, M_2, M_3, \ldots\}$, and $\{D_1, D_2, D_3, \ldots\}$, as follows.
Our basic double links will be of the form $L_i \cdot I + (F)$, where $F$ is a suitable product of the $M_i$ and $D_i$. As in the previous section, we will “add rows” (which sometimes means fattening up simple points) according to the dictates of the pseudo type vector. In this case, we begin with two simple points (at the top), which we consider as the complete intersection $I_1 = (M_1D_1, L_2)$. We then simultaneously add two 4’s: one will fatten up $I_1$, while the other adds four simple points on the fourth line. This is done by forming the ideal $I_2 = L_2L_4 \cdot I_1 + (M_1D_1M_2D_2)$. Note that $M_1D_1M_2D_2$ is double at the two points on the second line, and simple at the four points of the fourth line, so basic double linkage does indeed do the required task: $I_2$ is the saturated ideal of the scheme that consists of two double points on $L_2$ and four simple points on $L_4$. We then form

$$
I_3 = L_5 \cdot I_2 + (M_1D_1M_2D_2M_3)
$$

$$
I_4 = L_4 \cdot I_3 + (M_1D_1M_2D_2M_3D_3M_4D_4)
$$

$$
I_5 = L_5 \cdot I_4 + (M_1D_1M_2D_2M_3D_3M_4D_4M_5D_5)
$$

Notice that at each stage, the polynomial playing the role of $F$ (the product of the $M_i$ and $D_i$) is a multiple of the previous one, so it is in the previous ideal. Also, it has no component in common with the polynomial $L_i$ or $L_iL_j$, so basic double linkage applies. Finally, it gives simple points when that is called for, and double points when that is needed, as was argued already in Example 2.17 (and will be formalized below). The end result is the desired configuration of double points.

The Hilbert function of the set of points that we have constructed is again obtained from a simple computation (as are the graded Betti numbers). The Hilbert function is

$$
1 1 \\
1 2 2 2 1 \\
1 1 1 1 1 \\
1 1 1 1 1 1 1 \\
1 1 1 1 1 1 1 1
\rightarrow 1 3 6 10 15 21 27 30 32 33 ... \\
1 2 3 4 5 6 6 3 2 1
$$

and the graded Betti numbers are given by the Macaulay diagram.
This fails to be the graded Betti numbers of the lexsegment ideal with the given Hilbert function only because one step involved a basic double link using a quadric instead of a linear form (to produce $I_2$) — see Theorem 4.5. □

With this example giving the reader our basic ideas, we are now ready to extend Proposition 3.6 to 2-fat points. Notice that part iii) of the following theorem is much cleaner than parts iii) and iv) of Proposition 3.6, because of the extra “compactness” provided by the non-reducedness step. Note also that simply using the standard lifting, without “raising” the rows to fit into the isosceles triangle, is not enough. See Example 5.5.

**Theorem 5.4.** Let $T = (n_1, \ldots, n_r)$ be a 2-type vector, with $n_1 < n_2 < \cdots < n_r$, and let $T' = (m_1, \ldots, m_{2r})$ be the associated pseudo type vector. Let $X$ be the spread out linear configuration with type vector $T$ (see Definition 2.12). Let $Z$ be the set of 2-fat points supported on $X$. Then

i) $Z$ can be built up by basic double linkage.

ii) The first difference of the Hilbert function of $Z$ is the standard $O$-sequence associated to $T'$ (from the $O$-sequence computation in Definition 3.2).

iii) The regularity of $Z$ is $m_{2r} = 2n_r$.

**Proof.** We have $T' = (n_1, 2n_1, n_2, 2n_2, \ldots, n_r, 2n_r)^{ord}$. Note that if an entry of $T'$ is odd then it only occurs once, and if an entry is even then it occurs at most twice. We let $m_i$ denote the (ordered) entries of $T'$, so $m_1 = n_1, \ldots, m_{2r} = 2n_r$.

Consider the spread out linear configuration, $X$, with $r$ rows, each having $n_r$ points, as in Example 5.3. Again as in Example 5.3 we consider three families of lines, $\{L_1, \ldots, L_r\}$, $\{M_1, \ldots, M_{n_r}\}$, $\{D_1, \ldots, D_{n_r}\}$. The $L_i$ are the “horizontal” lines, the $M_i$ are the “vertical” lines and the $D_i$ are the “diagonal” lines (starting at the “hypotenuse”).

A basic double link has the form $I_{i+1} = G \cdot I_i + (F)$, where $F \in I_i$ and $(F, G)$ is a regular sequence. In our case, at each step the role of $F$ will be played by a suitable product $M_1D_1M_2D_2 \cdots$, alternating between them. If we have completed the construction for $m_{i-1}$ in the pseudo type vector $T'$, then to build the ideal corresponding to the entry $m_i$ (which may or may not be equal to $m_{i+1}$) in the pseudo type vector, the number of factors in the polynomial $F$ is equal to $m_i$. Each subsequent $F$ will build on the ones before by
adding factors consisting of the $M_i$ and $D_j$. This guarantees that we will always have $F \in I_t,$ and in fact by checking the regularity we can see that the $F$ we are using is not a minimal generator of $I_t$.

The role of $G$ will always be played by either one $L_j$ (if $m_i < m_{i+1}$), or a product of two $L_j$ (if $m_i = m_{i+1}$), as dictated by $T'$.

We make the following observations:

1. $m_1 = n_1 < m_2$. The construction starts with the ideal $I_1$ that is the complete intersection of $L_1$ and $F$, where $F$ is the product $M_1D_1, \ldots, $ taking $n_1$ factors. This is the ideal of $n_1$ simple points on $L_1$.
2. At any step, if $m_i < m_{i+1}$ then $F$ has $m_i$ factors, and either it cuts out $m_i$ simple points on $L_j$ or else we have $m_i = 2n_k$ for some $k < i$, and $F$ is double at each of $n_k$ points on $L_k$.
3. If $m_i < m_{i+1}$, $I_t$ is the current ideal, and $F$ is chosen as in (2), then the ideal $I_{t+1} = L_j \cdot I_t + (F)$ is a saturated ideal that either adds $n_j$ simple points on $L_j$ or else it “fattens up” ($\text{doubles}$) $n_k$ points on $L_k$, respectively.
4. At any step, if $m_i = m_{i+1}$ then one of them (without loss of generality say it is $m_i$) is the term $2n_j$ for some $j < i$, and the other is equal to $n_k$ for some $k \leq i$. In this case $F$ has $m_i$ factors, and it has $n_j$ singular (double) points along the line $L_j$ ($1 \leq j \leq r$) and $n_k$ simple points along the line $L_k$ ($1 \leq k \leq r$).
5. If $m_i = m_{i+1}$, $I_t$ is the current ideal, and $F$ is chosen as in (1), then the ideal $I_{t+1} = L_j L_k \cdot I_t + (F)$ is a saturated ideal that adds $n_k$ simple points on $L_k$ and “fattens up” $n_j$ already-existing simple points on $L_j$.

The end result, after completing this procedure by reaching $m_{2r}$, is the saturated ideal of double points supported on the spread out linear configuration of type $T$. The computation of the Hilbert function is identical to that in Theorem 3.7. This completes $i)$ and $ii)$.

Now, the numerical information obtained from the basic double linkage is identical to that we saw in the reduced situation – it only depends on the degrees of the polynomials, and not on the geometry of the singularities. In particular, we obtain from Proposition 3.6 that what can play havoc with the regularity here is the existence of certain subsequences in $\Delta T'$. In particular, if $\Delta T'$ has an entry that is $>1$ between any two zero entries then the regularity can only be $m_{2r} + 1$ or $m_{2r}$, depending (respectively) on whether $\Delta T'$ ends with one of the subsequences $0, (0,1), (0,1,1), \ldots, $ or not. If $\Delta T'$ has zero entries between which there are only 1’s then the regularity can (in principle) be arbitrarily bigger than $m_{2r}$. So we have to verify that such things cannot happen for 2-fat points.

First note that $\Delta T'$ can not end with a 0 or a 1. Indeed, we have $m_{2r} = 2n_r$, which is even, and if $m_{2r-1} = 2n_r$ or $2n_r - 1$ then this entry is not the double of a previous one and hence its double is still to come. So if, between any two zero entries of $\Delta T'$, there is at least one entry $>1$, we now know that the regularity of $Z$ is $m_{2r} = 2n_r$.

It is certainly possible for $\Delta T'$ to have a subsequence 0, 1, \ldots, 1, 0. For instance, take $T = (8, 9, 10, 16, 17, 19, 20)$; then $\Delta T' = (8, 1, 1, 6, 0, 1, 1, 1, 1, 0, 12, 2, 4, 2)$. It is clear from the discussion leading to the O-sequence computation in Definition 3.2 that at each step, if we are performing (without loss of generality) a basic double link with $\deg F = m$ (say)
and $\deg G = 1$, building from a zero-dimensional scheme $X$ to $Y$, then
\[
\text{reg}(Y) = \max\{\deg F, \text{reg}(X) + 1\}.
\]

The point that we will make is that (as we have seen) what creates “problems” for the regularity is a double occurrence of an integer in $T'$, say $(\ldots, m, m, \ldots)$, i.e. a 0 in $\Delta T'$. But such an occurrence automatically forces a $2m$ also in $T'$, and this corrects the problems.

Indeed, suppose that the last 0 in $\Delta T'$ occurs in position $d$, and that prior to this 0 there are $k$ zeros. So $T' = (\ldots, m, m, \ldots)$, where the second $m$ occurs in position $d$. Clearly $k \leq \frac{m}{2}$ (a zero in $\Delta T'$ has to correspond to an even number in $T'$). Then the regularity of the subscheme produced up to that point in $T'$ is $\leq m + k \leq m + \frac{m}{2}$. What can happen after this point in $T'$? Either all of the remaining entries are of the form $2n_i$ (so the last one is $2m = 2n_r$), or there are more $n_i > m$ (so the last entry of $T'$ is $2n_r > 2m$).

In the first case, the number of remaining steps is clearly $\leq \frac{m}{2}$, since the number of remaining steps is exactly the number of $n_i$ for which $2n_i > m$. Hence the regularity of the final double point scheme is
\[
\max\left\{2m, \left(m + \frac{m}{2}\right) + \frac{m}{2} \right\} = 2m = 2n_r.
\]

In the second case, when we reach the entry $2m$ in $T'$, we already have regularity being determined by the entry (namely $2m$ in this case), and each subsequent entry preserves this property. Hence again the regularity of the resulting scheme is $2n_r$. \hfill \Box

**Example 5.5.** The construction in this section sometimes has very special properties. For example, suppose that we want to study the Hilbert function of the first infinitesimal neighborhood of a linear configuration of type $T = (4, 5, 8, 9, 10)$. The basic double link prediction for this Hilbert function is
\[
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 10, 10, 7, 4, 3, 3, 3, 2, 1.
\]
But even the standard lifting of the lex-segment ideal (putting the points on the integer lattice points) gives the more general Hilbert function
\[
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 10, 10, 8, 3, 3, 3, 3, 2, 1.
\]
(which is also the Hilbert function for the first infinitesimal neighborhood of a generically chosen linear configuration of this type). But moving the points “upward” as indicated in this section, to add collinearity of the “diagonal” points, is enough to change the value in degree 13 to this more special function. We have verified this on macaulay [2]. Notice that the associated pseudo type vector $T'$ does not satisfy (3.2). \hfill \Box

**Remark 5.6.** It may be noted that a key difference between Theorem 5.4 and Theorem 3.7 is that in the latter we had to use vanishing of first cohomology to guarantee lifting of non-zero elements, which in Theorem 5.4 is not guaranteed simply by the cohomology; rather, we used the simplicity of the geometry to guarantee the existence of suitable curves (the unions of the lines). \hfill \Box

**Remark 5.7.** The construction of Theorem 5.4 would work equally well if the families \{\(L_1, L_2, \ldots\)\}, \{\(M_1, M_2, \ldots\)\} and \{\(D_1, D_2, \ldots\)\} (each of which has a common point at
infinity) were replaced by three different families of lines in \( \mathbb{P}^2 \), each with a common point in \( \mathbb{P}^2 \).

6. When are the Hilbert function and graded Betti numbers uniquely determined?

In this section we will show how to apply the ideas of Theorem 3.7 and Theorem 4.5, and especially their proofs, to the study of double points in \( \mathbb{P}^2 \). We will show that the same ideas in fact produce the (non-reduced) double point scheme by basic double linkage, and the same kind of uniqueness results continue to hold. Some of the important ideas used here were illustrated in Example 2.17.

The following is the main result of this section, and extends to 2-fat points the results on Hilbert functions and graded Betti numbers of pseudo linear configurations.

**Theorem 6.1.** Let \( \bar{X} \) be a linear configuration of type \( T = (n_1, \ldots, n_r) \), and let \( T' = (m_1, \ldots, m_{2r}) \) be the associated pseudo type vector. Let \( \bar{Z} \) be the 2-fat point scheme supported on \( \bar{X} \).

(a) Assume that for each \( i \) we have the property (3.2) of Theorem 3.7, namely that between any two zero entries of \( \Delta T' \) there is at least one entry that is \( > 1 \). Then \( \bar{Z} \) can be constructed as a sequence of basic double links, and its Hilbert function is uniquely determined and can be computed by the O-sequence computation of Definition 3.2.

(b) Conversely, if (3.2) does not hold then there are linear configurations of the given type, \( T \), whose corresponding double points do not arise by basic double linkage. Furthermore there are two different linear configurations of type \( T \) such that the corresponding double points have different Hilbert functions.

(c) Assume again that (3.2) holds. Assume further that \( \Delta T' \) contains no subsequence \( (1,0,1), (1,0,2,0,1), (1,0,2,0,2,0,1), \) etc. Then, in addition, the graded Betti numbers of \( I_{\bar{Z}} \) are uniquely determined, as described in Theorem 4.5.

(d) Conversely, if (3.2) holds, but \( \Delta T' \) does contain a subsequence \( (1,0,1), (1,0,2,0,1), (1,0,2,0,2,0,1), \) etc. then there are two different linear configurations of type \( T \) such that the corresponding double points have the same Hilbert function (by part (a)), but the graded Betti numbers are different.

**Proof.** As usual we assume that \( \bar{X} = \bigcup_{i=1}^r X_i \), where \( X_i \) consists of \( n_i \) points on line \( L_i \), for \( 1 \leq i \leq r \). By the definition of a linear configuration, \( n_{i-1} < n_i \) for all \( i \). If we set \( L = L_i \) then \( I_{\bar{Z}} : L \) is a saturated ideal defining the union of \( X_i \) (the reduced points on \( L \)) and the double points whose supports are not on \( L \). Furthermore, \( \frac{I_{\bar{Z}} : (L)}{(L)} \) is the (non-saturated) ideal of a subscheme of \( L \) that has degree \( 2n_i \) and is supported on \( X_i \) with degree two at each point and tangent direction given by \( L \).

Our strategy will be to consider \( \bar{Z} \) inductively as a “limit” pseudo linear configuration of type \( T' \), and to construct \( \bar{Z} \) in the order dictated by \( T' \), just as in Theorem 3.7 (see Example 2.17 and Example 5.2). Again, if we have reached and completed \( m_{i-1} \) in our construction, then the next step will handle \( m_i \) alone if \( m_i < m_{i+1} \), and it will handle \( m_i \) and \( m_{i+1} \) simultaneously if \( m_i = m_{i+1} \) (which then is necessarily an even number). When we have \( m_i = m_{i+1} = 2m_j \) for some \( i \) and \( j \), this will involve simultaneously “fattening
up” the points corresponding to \( m_j \) and adding the simple points corresponding to \( m_i \). (The next section applies this idea in a more concrete, geometric way.) Note that any intermediate step may or may not produce a scheme consisting entirely of double points. Only the final result will necessarily consist entirely of double points, namely \( \bar{Z} \). Note also that \( L \) does not necessarily progress monotonically through the \( L_i \), since when it “fattens up” a set of points on a line, that line will be a previously considered one (as illustrated in Example 2.17 and Example 5.2).

The “fattening up” process is based on the following observation: if \( P \) is a point in \( \mathbb{P}^2 \) and if \( L_1, L_2, L \in I_P \) are linear forms, then \( L \cdot I_P + (L_1 L_2) \) is the saturated ideal of the double point scheme defined by the (saturated) ideal \( I_P^2 \), as long as \( L \) has no component in common with either \( L_1 \) or \( L_2 \). More generally, let \( P \) be a reduced point of a scheme \( Z \), let \( F \in I_Z \) be a homogeneous polynomial such that \( F \in I_P^2 \) and \( F \notin I_P^3 \), and let \( L \in I_P \) with no component in common with \( F \) even locally (i.e. the intersection of \( F \) and \( L \) is a zero-dimensional scheme that has degree 2 at \( P \)). Then \( L \cdot I_Z + (F) \) is the saturated ideal of a zero-dimensional scheme in \( \mathbb{P}^2 \), and at \( P \) this zero-dimensional scheme is the 2-fat point supported on \( P \). It is worth noting that if we allowed \( F \) to be smooth at \( P \) and \( L \) were tangent to \( F \) at \( P \), then the new zero-dimensional scheme again would have degree \((\geq) 3\) at \( P \), but would be curvilinear, not “fat.”

So, mimicking the approach of Theorem 3.7, suppose that we have reached and completed \( m_{i-1} \). As before, there are two possibilities: either \( m_i < m_{i+1} \) or \( m_i = m_{i+1} \).

We first suppose that \( m_i < m_{i+1} \), and we set \( L \) to be the line containing the \( m_i \) “points” (which is not necessarily \( L_i \)). These will either be

i) \( m_i \) reduced points (which we will add singly), or

ii) \( \frac{m_i}{2} \) length two schemes (not fat) on \( L \), which we will “add” to \( \frac{m_i}{2} \) already-existing single points to obtain \( \frac{m_i}{2} \) double points. Note that then \( \frac{m_i}{2} \) is one of the \( n_j \).

In either case \( X \) will denote this subscheme of \( L \) of degree \( m_i \), and \( Y \) will denote the subscheme of \( Z \) constructed (inductively) up to that point. \( Z \) will denote the “union” of \( X \) and \( Y \), but now this is more delicate to define. If \( Y \) is reduced (case i), we simply take \( Z \) to be the union in the usual sense. If \( Y \) is non-reduced, then \( Z \) will denote the scheme obtained from the scheme of the previous step by replacing the \( \frac{m_i}{2} \) simple points with \( \frac{m_i}{2} \) double points. We have to show that either way, \( Z \) is obtained from \( X \) by basic double linkage.

We again consider the exact sequence

\[
0 \rightarrow [I_Z : L](-1) \xrightarrow{\times L} I_Z \rightarrow \frac{I_Z + (L)}{(L)} \rightarrow 0.
\]

The mechanics of the proof (using regularity to lift elements, and analyzing minimal generators) are identical to those of Theorem 3.7 and Theorem 15, and will not be repeated here. What is new is the justification that it all works even in the non-reduced situation. But in fact, \( Y \) is a divisor on \( L = \mathbb{P}^1 \), and whether it is reduced or not, its ideal in \( L \) begins in degree \( m_i \) just as before. A non-zero element of \( I_{Y \mid L} \) (the saturation of \( \frac{I_Z + (L)}{(L)} \)) in degree \( m_i \) lifts to an element, \( F \), of \( (I_Z)_{m_i} \) just as before, and \( Y \) is the complete intersection of
We then form the ideal $I = L \cdot I_X + (F)$. This is the saturated ideal of a scheme $Z$ that is the same as $X$ for points off $L$, and makes a non-reduced degree three subscheme of $\mathbb{P}^2$ at each point of the support of $Y$. The one remaining subtlety is to ascertain that at each such point in the support of $Y$, the non-reduced scheme that we obtain is really a 2-fat point. This would fail to happen, as noted above, if the polynomial $F$ is smooth at a point of $Y$ and tangent to $L$ there, rather than singular there. (Such an $F$ certainly restricts to a $Y$ that is double at each point, as a subscheme of $L = \mathbb{P}^1$.) But this is resolved by the fact that we know that we are lifting elements of $I_{Y|L}$ to $I_Z$, which we knew in advance to consist of 2-fat points at each of the $\frac{m_i}{2}$ points in the support of $Y$. Hence $F$ is not smooth at any of those points, and must be double there. So now, $L \cdot I_X + (F)$ and $I_Z$ are both saturated ideals defining the same zero-dimensional subscheme, hence they are equal. This completes the proof of (a).

Parts (c) and (d) continue to have (3.2) as a hypothesis, meaning that the configurations of 2-fat points considered there necessarily arise by basic double linkage, but the graded Betti numbers are in question. We consider these parts first, and then turn to (b).

The Hilbert function and regularity of the new scheme are obtained just as in Theorem 3.7. In case (c), the graded Betti numbers are produced just as in Theorem 4.5. If $m_i = m_{i+1}$, instead of $L$ we again use $Q$ which is the product of two linear forms. One of them will contain $m_i$ reduced points and the other will be viewed as containing $\frac{m_i}{2}$ double points, as noted above. Note that these are distinct lines! Again the same proof as in Theorem 3.7 and Theorem 4.5 works, with the same modifications as in the previous paragraph.

For part (c), the point is that we have just shown that the double points are constructed with liaison addition in a manner perfectly analogous to that used for the pseudo linear configurations. The conditions in (c) then guarantee that every step forces us to choose $F$ not a minimal generator of the previous ideal, hence the conclusion that the graded Betti numbers are uniquely determined.

Part (d) is slightly more subtle, however. Each step of the basic double linkage either adds a new set of reduced points, “fattens up” an existing set, or does both simultaneously. Note that there is less freedom if we are constrained to a previously existing support. However, the “fattening up” process can only be done if the corresponding entry in $T'$ is even! A subsequence $(1,0,1)$ in $\Delta T'$ corresponds to a subsequence

$$m, m + 1, m + 1, m + 2$$

in $T'$, and a subsequence $(1,0,2,0,2,\ldots,0,2,0,1)$ in $\Delta T'$ with $k$ 2's corresponds to a subsequence

$$m, m + 1, m + 1, m + 3, m + 3, \ldots, m + 2k + 1, m + 2k + 1, m + 2k + 2$$

in $T'$. In each case, the last entry must be odd $(m + 2$ and $m + 2k + 2$, respectively). Comparing with the proof of Theorem 4.5, it is exactly at this point that there is a choice of choosing $F$ a minimal generator or not, and since the number is odd, this must correspond to adding new reduced points, not “fattening up” already existing points. Hence we have complete freedom with $F$, and (d) follows.

We now turn to (b). The proof is very similar to the last part of Theorem 3.7 with some fine tuning. We have to show that if $\Delta T''$ contains a subsequence $(\ldots, 0, 1, \ldots, 1, 0)$
(all 1’s between the two 0’s) then there exist (at least) two linear configurations of type $T$ whose associated double points (first infinitesimal neighborhoods) have different Hilbert functions. We have already noted that an associated pseudo type vector cannot end with a 0, so the argument will be slightly different from that of Theorem 3.7.

Note that we showed in Theorem 5.4 that for any type vector $T$, there always exists one linear configuration (the spread out one) whose first infinitesimal neighborhood can be constructed by basic double linkage, and hence has Hilbert function whose first difference is given by the O-sequence computation of Definition 3.2. So we have to show that such a subsequence allows for a 2-fat point scheme that can not be constructed entirely by basic double links, and that correspondingly the Hilbert functions are different.

Suppose that we are given the type vector $T = (n_1, \ldots, n_r)$, and a linear configuration $X$ of type $T$. From $T$ we derive the associated pseudo type vector $T' = (n_1, 2n_2, \ldots, n_r, 2n_r)^{ord}$. This information gives the recipe to “fatten up” $X$ to a 2-fat point scheme $Z$ by basic double linkage, if such a process is possible. It is important to note that each entry of $T'$ corresponding to an $n_i$ produces $n_i$ reduced points on a new line, and each entry of $T'$ corresponding to a $2n_i$ “fattens up” $n_i$ previously existing points on a line. The only ambiguity comes when we have two consecutive entries that are equal. Usually we do these simultaneously, by taking $G$ to be the product of the two linear forms. However, for this proof we will consider such a situation as arising from two consecutive basic double links using the same polynomial $F$ and taking $G$ linear, rather than one basic double link using $G$ quadratic.

We will make the convention that the first basic double link corresponds to “fattening up” previously existing points, while the second one corresponds to producing new reduced points.

If basic double linkage is possible at each step, the end result of this process is the desired 2-fat point scheme $Z$ supported on $X$. However, each intermediate step is the saturated ideal of a zero-dimensional scheme that is “2-fat” at some points and reduced at others.

Suppose that $\Delta T'$ contains a subsequence $(\ldots, 0, 1, \ldots, 1, 0)$ (all 1’s between the 0’s) and consider the first occurrence of this subsequence. This means that

$$T' = (m_1, \ldots, m_{p-2}, m_{p-1}, m_p, \ldots)$$

with $2n_i = m_p = m_{p-1} = m_{p-2} + 1$. Let $T''$ be the pseudo type vector $(m_1, \ldots, m_{p-2}, m_{p-1})$. Then $\Delta T''$ ends in a sequence $(\ldots, 0, 1, \ldots, 1)$ where the end consists of nothing but 1’s. We have assumed that $T''$ satisfies (3.2). Let $Z_1$ be the zero-dimensional scheme corresponding to $T''$, following our procedure of basic double linkage; $Z_1$ is supported on some subset of $X$. Its Hilbert function is as described in the O-sequence computation of Definition 3.2. Furthermore, it follows from what we have already proven that the regularity of $I_{Z_1}$ is $m_{p-1} + 1 = m_p + 1 = 2n_i + 1$. The last basic double link in this sequence “fattened up” a previously existing $n_i$ points.

Now consider an additional line $L$, and choose $\Upsilon$ to be a general set of $2n_i$ points on $L$. Let $Z_2 = Z_1 \cup \Upsilon$. $Z_2$ is a basic double link of $Z_1$ if and only if there is a form $F \in (I_{Z_1})_{2n_i}$ that contains $\Upsilon$ but does not vanish on $L$. 
We have an exact sequence
\[ 0 \to \mathcal{I}_{Z_1}(2n_i - 1) \xrightarrow{\times L} \mathcal{I}_{Z_1}(2n_i) \to \mathcal{O}_L(2n_i) \to 0. \]
Since the regularity of \( I_{Z_1} \) is \( 2n_i + 1 \), we have
\[ (6.1) \quad 0 \to (I_{Z_1})^{2n_i - 1} \to (I_{Z_1})^{2n_i} \xrightarrow{r} H^0(\mathcal{O}_L(2n_i)) \to H^1(\mathcal{I}_{Z_1}(2n_i - 1)) \to 0 \]
where the last cohomology group is not zero. Choosing \( Y \) as above is equivalent to choosing a general element of the vector space \( H^0(\mathcal{O}_L(2n_i)) \). The image of \( r \) is a proper subspace of \( H^0(\mathcal{O}_L(2n_i)) \), so the general section of \( H^0(\mathcal{O}_L(2n_i)) \) defining \( Y \) is not in the image of \( r \). We conclude that any form in \( (I_{Z_1})^{2n_i} \) that vanishes on \( Y \) must in fact vanish on all of \( L \). Hence we cannot express \( Z_2 \) as a basic double link of \( Z_1 \).

We claim that the value of the Hilbert function of \( Z_2 \) in degree \( 2n_i \) differs (in fact is larger) from the value of the corresponding Hilbert function given by the O-sequence computation in Definition 3.2 in degree \( 2n_i \) (see Remark 3.3). Indeed, suppose that \( Z' \) were a zero-dimensional scheme that was produced by a sequence of basic double linkages of the same type, and hence has the standard Hilbert function for that type. In degree \( 2n_i \), the forms that vanish on \( Z_2 \) consist entirely of products of \( L \) with forms of degree \( mp - 1 \) vanishing on \( Z_1 \) (as discussed above), while \( Z' \) has those but also has a form of degree \( 2n_i \) that is not of that form. Hence the first Hilbert function is larger than the second in degree \( 2n_i \).

Now we continue along \( T' \). We have reached the entry \( mp = 2n_i \) and constructed a zero-dimensional scheme \( Z_2 \) whose Hilbert function is not the one predicted by the O-sequence computation of Definition 3.2 precisely because at the last step we added a set of points and showed that it could not arise by basic double linkage. At each subsequent step, one of three things can happen: (i) because of regularity arguments like those above, we are guaranteed that that step can be accomplished by basic double linkage; (ii) a step corresponds to “fattening up” an existing set of reduced points, and it happens that it can be accomplished by basic double linkage, or (iii) whether because of the position of the existing points to be “fattened up” or because of the free choice of general reduced points, basic double linkage cannot be performed.

As in Theorem 3.7, if (i) or (ii) hold then the resulting scheme again fails to have the standard O-sequence predicted by the O-sequence computation of Definition 3.2 because we are adding the expected amount to an already larger Hilbert function. In the third case, as in the argument just made, the Hilbert function becomes correspondingly larger than it would have been had basic double linkage been possible, hence gets even farther from the predicted O-sequence.

In the end we obtain a set of 2-fat points supported on a linear configuration whose Hilbert function is different from that of a set of 2-fat points supported on a spread out configuration. This proves (b). □

Remark 6.2. Although, as Theorem 6.1 states, it need not be true that the first infinitesimal neighborhood of two linear configurations of type \( (n_1, \ldots, n_r) \) have the same Hilbert function or the same Betti numbers in their minimal free resolution, there is one thing they will have in common – namely their regularity (which is \( 2n_r \)).

To see why that is so, just observe that by Lemma 2.19 the first infinitesimal neighborhood of any linear configuration of type \( (n_1, \ldots, n_r) \) has regularity \( \leq 2n_r \). However, the
first infinitesimal neighborhood also always has a subscheme of length $2n_r$ on a line and so the regularity is $\geq 2n_r$.

**Example 6.3.** From Theorem 6.1 we see that linear configurations of the same type may have, for their first infinitesimal neighborhoods, the same Hilbert function but not the same graded Betti numbers. It would be interesting to know exactly what the possibilities are for the Betti numbers of these double point schemes in such a case. This example deals with that situation.

Let $T = (2, 3, 4, 5)$. Let $\bar{X}$ be a linear configuration of type $T$ and let $\bar{Z}$ be the first infinitesimal neighborhood of $\bar{X}$. Then the Hilbert function of $\bar{Z}$ is uniquely determined and has first difference

$$1, 2, 3, 4, 5, 6, 7, 8, 5, 1.$$ 

However, the graded Betti numbers are not uniquely determined. The associated pseudo type vector is $(2, 3, 4, 5, 6, 8, 10)$. One can check that there are actually two times, in making the construction of Theorem 6.1 when there is apparently a choice between using a minimal generator or not, namely when we deal with the 5 and when we deal with the 6. However, notice that while there is freedom in choosing where the 5 points are located, there is no such freedom for the 6 since it represents the “fattening up” of three already-existing points.

We have found two examples of linear configurations of type $T$ (above) whose first infinitesimal neighborhoods have the following two Betti diagrams (verified experimentally on macaulay). We are not sure if there are any other Betti diagrams possible.

| total: | 1 | 8 | 7 | total: | 1 | 6 | 5 |
|--------|---|---|---|--------|---|---|---|
|        | 0: 1 | - | - | 0: 1 | - | - | - |
|        | 1: 1 | - | - | 2: 1 | - | - | - |
|        | 2: - | - | - | 3: - | - | - | - |
|        | 3: - | - | - | 4: - | - | - | - |
|        | 4: - | - | - | 5: - | - | - | - |
|        | 5: - | - | - | 6: - | - | - | - |
|        | 6: - | - | - | 7: - | 4 | 2 | - |
|        | 7: - | 3 | 4 | 8: - | 1 | 4 | - |
|        | 8: - | 1 | 1 | 9: - | 1 | 1 | - |

□

It is possible to isolate an important family of type vectors for which all linear configurations of those types have their first infinitesimal neighborhoods sharing both the same Hilbert function and same Betti diagram.

**Corollary 6.4.** Let $T = (n_1, \ldots, n_r)$ be a 2-type vector and let $T' = (m_1, \ldots, m_{2r})$ be the associated pseudo type vector. If $n_i \neq 2n_j$ for all $i, j$, then the pseudo type vector $T'$ is actually a 2-type vector. (This holds, for example, if all the $n_i$ are odd.) In this case the Hilbert function and graded Betti numbers of any set of double points supported on a linear
configuration of type $T$ are uniquely determined, and is that of a linear configuration of type $T'$.

Proof. Immediate. □

7. BEYOND LINEAR CONFIGURATIONS

As indicated in the introduction, this paper is intended as a first step in the study of the following problem: given the Hilbert function $h$ for a reduced, zero-dimensional subscheme of $\mathbb{P}^2$, what are the possible Hilbert functions of double point schemes whose support has Hilbert function $h$? In particular, is there a minimum and maximum such function, $h_{\text{min}}, h_{\text{max}}$ respectively? In this section we address these questions, proving the existence of $h_{\text{max}}$ in general and the existence of $h_{\text{min}}$ at least in a special case. The examples in this section also help to clarify the role of linear configurations, and their limitations, toward an answer to these questions in general.

Example 7.1. It is not hard to find examples of two sets, $X$ and $X'$, of points in $\mathbb{P}^2$ with the same Hilbert function, with the property that the multiplicity two schemes supported on those sets have different Hilbert functions. A consequence of Theorem 6.1 is that $X$ and $X'$ can even have the same graded Betti numbers (e.g. both can be linear configurations of the same type), and yet they can have resulting double point schemes with different Hilbert functions.

A different question is whether there exist unions of double points with the same Hilbert function, but whose supports have different Hilbert functions. The answer is “yes,” and we can use Theorem 5.4 to help produce such an example.

Consider the 2-type vector $(1, 2, 3, 4)$. This corresponds to a Hilbert function whose first difference is $h_4 = (1, 2, 3, 4)$. The construction of Theorem 5.4 gives a set, $Z_1$, of double points whose support, $X_1$, has Hilbert function with first difference $h_1$ and sits on the standard grid, and such that the Hilbert function of $Z_1$ has first difference $(1, 2, 3, 4, 5, 6, 6, 3)$.

Now consider a set $X_2$ of 10 general points on a smooth cubic curve in the plane, and let $Z_2$ be the double points supported on $X_2$. One can check with a computer algebra program (e.g. macaulay [2]) that $Z_2$ has the same Hilbert function as described above, and yet $X_2$ has Hilbert function with first difference $h_5 = (1, 2, 3, 3, 1)$.

Example 7.2. It should be noted that this process of studying the Hilbert function of the first infinitesimal neighborhood of a linear configuration does not give all possible Hilbert functions for double points in $\mathbb{P}^2$. Indeed, a set of seven generally chosen fat points has Hilbert function whose first difference is $(1, 2, 3, 4, 5, 6)$, while any linear configuration of seven points has at least one subset of four points on a line, so the regularity must be at least 8 for the corresponding double points.

We now turn to the question of the existence of $h_{\text{max}}$ and $h_{\text{min}}$. We are grateful to Mike Roth for useful discussions about the following theorem and its proof.

Theorem 7.3. Let $h$ be the Hilbert function of some reduced zero-dimensional subscheme of $\mathbb{P}^2$. Then there is a Hilbert function $h_{\text{max}}$ such that if $h'$ is the Hilbert function of a double point scheme whose support has Hilbert function $h$ then $h' \leq h_{\text{max}}$. □
\textbf{Proof.} Let $\text{Hilb}^s(\mathbb{P}^2) = X(s)$ be the Hilbert scheme which parametrizes all the closed subschemes of $\mathbb{P}^2$ having length $s$. It is well known (e.g. by using the Hilbert-Burch Theorem) that those closed subschemes of $\mathbb{P}^2$ which share the same Hilbert function, $g$ (say), form an irreducible subset of $X(s)$ (which we’ll denote by $X_g(s)$). It is also well known that $X_g(s)$ is locally closed. Thus, for any positive integer $s$ we obtain a (finite) locally closed irreducible partition of $\text{Hilb}^s(\mathbb{P}^2)$.

The partition we described above gives, in the same way, a partition of $\text{Sym}^s(\mathbb{P}^2) = Y(s)$ (the scheme parametrizing families of $s$ distinct points in $\mathbb{P}^2$).

There is also a map from $Y_t = \text{Sym}^t(\mathbb{P}^2)$ into $X(3t) = \text{Hilb}^{3t}(\mathbb{P}^2)$ – we associate to a set of $t$ distinct points in $\mathbb{P}^2$ its first infinitesimal neighborhood. We’ll denote the image of $Y_t$ in $\text{Hilb}^{3t}(\mathbb{P}^2)$ by $D_t$ and the image of $Y_t$ in $X(3t)$ by $D_{X_g}(t)$.

If we restrict the stratification of $\text{Hilb}^{3t}(\mathbb{P}^2)$ to $D_{X_g}(t)$ then exactly one component of this stratification will be dense in $D_{X_g}(t)$, and this stratum will be

$$D_{X_g}(t) \cap X(3t)$$

for some $g$. That $g = h_{\text{max}}$. \hfill \Box

\textbf{Remark 7.4.} Theorem 7.3 is an existence result, valid for any Hilbert function $h$. Unfortunately, we do not know (in general) an explicit formula (or even an algorithm) for computing it. However, in certain special cases we can given an algorithm that easily leads to $h_{\text{max}}$.

First, suppose that $h$ is the Hilbert function of a complete intersection of type $(a, b)$. Then, in the irreducible family of sets of points with Hilbert function $h$, an open subset corresponds to the complete intersections of type $(a, b)$. But, it is well known that if $I = (F, G)$ is the ideal of such a complete intersection, $X$, then $I^2 = (F^2, G^2) : I$. Since, for a complete intersection $X$, $I^2$ is the defining ideal of the first infinitesimal neighborhood of $X$, easy Liaison techniques give the Hilbert function of $I^2$, which is $h_{\text{max}}$.

Second, suppose that $h$ corresponds to the 2-type vector $(n_1, n_2, \ldots, n_r)$ with $n_i \geq n_{i-1} + 3$ for all $i \geq 2$. Then any reduced set of points whose Hilbert function has this type vector must be a $k$-configuration (using the decomposition techniques of Davis [18]). The general such $k$-configuration is a linear configuration and the Hilbert function of its first infinitesimal neighborhood is uniquely determined by Theorem 6.1. Therefore, this is $h_{\text{max}}$.

We have been unable to prove that $h_{\text{min}}$ exists, in general. However, we will prove its existence in an important special case, and give a conjecture for the general case. In what follows we continue our abuse of notation and refer to a curve and its defining form interchangeably.

\textbf{Lemma 7.5.} Let $F$ be a reduced curve of degree $d$.

(a) If $F$ is a union of $d$ lines, each of which meets the remaining lines in $d - 1$ distinct points, then the number of singular points of $F$ is $\binom{d}{2}$, all double points.

(b) If $F$ is not a union of $d$ lines, each of which meets the remaining lines in $d - 1$ distinct points, then the number of singular points of $F$ is $< \binom{d}{2}$.
Proof. (a) is clear. For (b), suppose first that \( F \) is irreducible. Then the number of double points is \( \leq \frac{(d - 1)(d - 2)}{2} < \binom{d}{2} \). Now suppose that \( F \) is not irreducible, \( F = F_1 \cdot F_2 \). If \( F_1 \) and \( F_2 \) are both unions of lines but at least three lines pass through one point then clearly the number of singular points is \( < \binom{d}{2} \). Finally, suppose that \( F = F_1 \cdot F_2 \) where at least one, say \( F_1 \), is irreducible of degree \( \geq 2 \). Say \( \deg F_i = d_i \), with \( d_1 + d_2 = d \). By induction, then, the number of singular points of \( F_1 \) is \( < \binom{d_1}{2} \) while the number of singular points of \( F_2 \) is \( \leq \binom{d_2}{2} \), with equality if and only if \( F_2 \) is a suitable union of lines. The singular points of \( F \) then come either as singular points of \( F_1 \) or \( F_2 \), or as points of intersection of \( F_1 \) and \( F_2 \). Then the number of singular points of \( F \) is

\[
\# \text{Sing}(F) < \binom{d_1}{2} + \binom{d_2}{2} + d_1 d_2 = \binom{d_1 + d_2}{2} = \binom{d}{2}
\]
as desired. \( \square \)

**Notation 7.6.** Let \( \lambda_1, \ldots, \lambda_t \) be a set of \( t \) distinct lines in \( \mathbb{P}^2 \) such that each \( \lambda_i \) meets the remaining \( t - 1 \) lines in \( t - 1 \) distinct points. We denote by \( C_t \) the configuration consisting of \( \binom{t}{2} \) pairwise intersections of these lines. Let \( 0 \leq r \leq t \). We denote by \( C_{t,r} \) a subconfiguration of \( C_{t+1} \) obtained by removing any \( (t - r) \) points of \( C_{t+1} \) that lie on \( \lambda_{t+1} \). Note that \( C_t \subseteq C_{t,r} \subseteq C_{t+1} \). The first equality holds if \( r = 0 \) and the second holds if \( r = t \). \( \square \)

**Example 7.7.**

The bullets represent \( C_4 \). The bullets together with the squares represent \( C_{4,3} \). The bullets, squares and circle together represent \( C_5 \). Note that \( C_4 \subset C_{4,3} \subset C_5 \). \( \square \)

**Lemma 7.8.**

(a) \( \deg C_t = \binom{t}{2} \)

(b) \( \deg C_{t,r} = \binom{t}{2} + r \)

(c) The first difference of the Hilbert function of \( C_t \) is

\[
1 \ 2 \ 3 \ \ldots \ (t - 1)
\]
and the first difference of the Hilbert function of \( C_{t,r} \) is
\[
1 \quad 2 \quad 3 \quad \ldots \quad (t-1) \quad r
\]

In particular, \( C_{t,r} \) has so-called generic Hilbert function.

Proof. (a) and (b) are clear. For the first part of (c), suppose that \( C_t \) lies on a curve \( F \) of degree \( t-2 \). Each line \( \lambda_i \) contains \( t-1 \) collinear points of \( C_t \), so by Bezout’s theorem \( \lambda_i \) must be a component of \( F \). But there are \( t \) such lines. Contradiction. The second part of (c) comes from the first part together with the inclusions \( C_t \subset C_{t,r} \subset C_{t+1} \), and the fact that consequently the first difference of the Hilbert function of \( C_{t,r} \) must be between those of \( C_t \) and \( C_{t+1} \).

\( \Box \)

Notation 7.9. We denote by \( Z_t \) the first infinitesimal neighborhood of \( C_t \). We denote by \( Z_{t,r} \) the first infinitesimal neighborhood of \( C_{t,r} \). Note that \( Z_{t,0} = Z_t \).

\( \Box \)

Theorem 7.10. (a) The first difference of the Hilbert function of \( Z_t \) is
\[
\begin{array}{c|cccccccc}
\text{degree} & 0 & 1 & 2 & 3 & \ldots & (t-1) & t & (t+1) & \ldots & 2t-3 & 2t-2 \\
\hline
\Delta h_{Z_t} & 1 & 2 & 3 & \ldots & t & t & t & \ldots & t & 0 \\
\end{array}
\]

Note that there are \( t-1 \) occurrences of \( t \) at the end of this function.

(b) Among double point schemes whose support has Hilbert function with first difference \( h = (1, 2, 3, \ldots, t-1) \), \( Z_t \) has minimal Hilbert function.

(c) Up to a different choice of \( \lambda_1, \lambda_2, \ldots, \lambda_t \), \( Z_t \) is the unique double point scheme with this Hilbert function, among double point schemes whose support has Hilbert function with first difference \( h \). In fact, the value of this Hilbert function in degree \( t \) already uniquely determines \( Z_t \).

Proof. For (a), first note that the union of the lines \( \lambda_1, \ldots, \lambda_t \) is a component of any curve of degree \( \leq 2(t-1) - 1 = 2t - 3 \) containing \( Z_t \), by Bezout’s theorem. On the other hand, this union is double at each of the \( \binom{t}{2} \) points of \( C_t \). Hence the ideal has exactly one generator in degree \( t \), and the next generator does not come before degree \( 2t-2 \). So the first difference of the Hilbert function of \( Z_t \) must be as claimed at least up to degree \( 2t-3 \). But
\[
1 + 2 + \cdots + (t-1) + t + t + \cdots + t = \binom{t}{2} + (t-1)t = 3\binom{t}{2} = \deg Z_t,
\]
so this must be the full Hilbert function.

We now prove (b) and (c) at the same time. Let \( \mathcal{X} \) be a reduced set of \( \binom{t}{2} \) points with generic Hilbert function (i.e. the one with Hilbert function with first difference as given in Lemma 7.8 (c) for \( C_t \)) and let \( Z \) be its first infinitesimal neighborhood. Suppose that \( Z \) has Hilbert function that is strictly smaller than that of \( Z_t \) in some degree. We consider the first difference of the Hilbert function of \( Z \), first in degree \( t-1 \). Suppose that \( h_Z(t-1) < h_{Z_t}(t-1) \), i.e. suppose that \( Z \) lies on some curve \( F \) of degree \( t-1 \). Then \( F \) is at least double at all the points of \( \mathcal{X} \). Since \( Z \) has the generic Hilbert function of Lemma 7.8 (c), the initial degree of \( I_{\mathcal{X}} \) is \( t-1 \), and in particular it lies on no curve of degree \( t-2 \). By assumption there is a form \( F \) of degree \( t-1 \) containing \( \mathcal{X} \) that is in fact (at least) double at all the points of \( \mathcal{X} \). We first claim that \( F \) is reduced. If it were not,
then the radical is a form of degree $< t - 1$ containing $X$, contradicting the fact that $t - 1$ is the initial degree of $I_X$. But now Lemma 7.5 says that $F$ has at most $\binom{t - 1}{2}$ singular points. This contradiction shows that $h_Z(t - 1) = h_{Z_t}(t - 1)$.

We now turn to degree $t$. Suppose that the initial degree of $I_Z$ is $t$, so $h_Z(t) \leq h_{Z_t}(t)$. Then there is at least one form, $F$, of degree $t$ that is singular at all the points of $X$.

**Claim:** $F$ is reduced.

To prove this claim, we suppose otherwise. Then $F$ has a factor, $F_1$, that is not reduced. If $\deg F_1 \geq 2$ then the radical of $F$ is a form of degree $\leq t - 2$ that contains $X$, again contradicting the fact that the initial degree of $I_X$ is $t - 1$. So now suppose that there is a linear form, $L$, such that $F = L^2F_2$, with $F_2$ reduced. From the first difference of the Hilbert function of $X$, we see that $X$ contains at most $t - 1$ collinear points. Hence $F_2$ is a reduced form of degree $t - 2$ double at $\binom{t}{2} - (t - 1) = \binom{t - 1}{2}$ points or more. This violates Lemma 7.5 and proves our Claim.

So now we have $h_Z(t) \leq h_{Z_t}(t)$ and $I_Z$ contains a reduced form, $F$, of degree $t$ that is double at $\binom{t}{2}$ points. By Lemma 7.5 then, $F$ is a union of lines and $X$ is the pairwise intersection of these lines. So $Z = Z_t$ (up to the choice of $\lambda_i$).

We may thus assume without loss of generality that the initial degree of $I_Z$ is $\geq t + 1$, so the first difference of the Hilbert function of $Z$ is

| degree | 0 | 1 | 2 | 3 | $\ldots$ | $(t - 2)$ | $(t - 1)$ | $t$ | $(t + 1)$ | $\ldots$
|-------|---|---|---|---|----------|----------|---------|---|---------|------|
| $\Delta h_Z$ | 1 | 2 | 3 | 4 | $\ldots$ | $(t - 1)$ | $t$ | $t$ | $t$ | $\ldots$

Recall that the first difference of the Hilbert function of $Z_t$ is

| degree | 0 | 1 | 2 | 3 | $\ldots$ | $(t - 2)$ | $(t - 1)$ | $t$ | $(t + 1)$ | $\ldots$ | $2(t - 3)$ | $2t - 2$
|-------|---|---|---|---|----------|----------|---------|---|---------|------|----------|--------|
| $\Delta h_{Z_t}$ | 1 | 2 | 3 | 4 | $\ldots$ | $(t - 1)$ | $t$ | $t$ | $t$ | $\ldots$ | $t$ | 0

In particular, we have $h_Z(t) > h_{Z_t}(t)$. We have to show that it cannot happen that later on, the Hilbert function of $Z$ drops below that of $Z_t$. By Lemma 2.19 the regularity of $I_Z$ is $\leq 2 \cdot \reg(I_X) = 2(t - 1) = 2t - 2$, so the first difference of the Hilbert function of $Z$ ends in degree $\leq 2t - 3$ as well.

Suppose that there is a value, $d$, for which $h_Z(d) < h_{Z_t}(d)$. Clearly $t < d < 2t - 3$, since $h_Z(2t - 3) = h_{Z_t}(2t - 3) = 3\binom{t}{2}$. The Hilbert function in any degree is just the sum of the entries of the first difference, up to and including that degree. But since $h_Z(t) > h_{Z_t}(t)$, this means that the first difference of the Hilbert function of $Z$ in some degree $\leq d$ has a value $k < t$. But the first difference of the Hilbert function of a zero-dimensional subscheme of $\mathbb{P}^2$ is non-increasing in degrees $\geq \alpha$ (see Definition 2.29) . Hence

$$\deg Z_t = h_{Z_t}(d) + t(2t - 3 - d)$$
$$> h_Z(d) + t(2t - 3 - d)$$
$$> h_Z(d) + k(2t - 3 - d)$$
$$\geq \deg Z.$$

This contradiction shows that $Z_t$ does in fact have minimal Hilbert function as claimed. □
Remark 7.11. Theorem 7.10 illustrates the necessity of restricting our hypothesis in Theorem 6.1 to linear configurations for the support rather than $k$-configurations. First note that $C_t$ is a $k$-configuration but not a linear configuration. Indeed, every newly added line misses all previous points of the configuration, but the points on the new line do lie on previously existing lines.

We now consider an example. Let $t = 4$. Then the first difference of the Hilbert function of $Z_4$ is

\[
\begin{array}{c|cccccccc}
\text{degree} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\Delta h_{Z_4} & 1 & 2 & 3 & 4 & 4 & 4 & 0 \\
\end{array}
\]

On the other hand, the configuration $C_4$ has Hilbert function with first difference $(1, 2, 3)$, so this is also the type vector (in this case). The associated pseudo type vector is $(1, 2, 3, 4, 6)$. By Theorem 6.1, however, any linear configuration with type vector $(1, 2, 3)$ has first infinitesimal neighborhood whose Hilbert function has first difference $(1, 2, 3, 4, 5, 3)$.

This example also serves as a counterexample to a natural guess, namely that the standard configuration (or in general the spread out configuration) should yield the first infinitesimal neighborhood of minimal Hilbert function among all supports with fixed Hilbert function. Indeed, the problem is that these configurations $C_t$ have even more collinearities than the spread out configurations. □

We now consider generic Hilbert functions $h$ that do not correspond to precisely $\binom{t}{2}$ points. One would like to find the minimal Hilbert function, $h_{\min}$, for the first infinitesimal neighborhoods of point sets with Hilbert function $h$. For example, we now compute the first difference of the Hilbert functions of some low-degree examples.

\[
\begin{array}{c|cccccccccccc}
\text{degree} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\Delta h_{Z_4} & 1 & 2 & 3 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 0 \\
\Delta h_{Z_4,1} & 1 & 2 & 3 & 4 & 5 & 4 & 1 & 1 & 1 & 1 & 1 \\
\Delta h_{Z_4,2} & 1 & 2 & 3 & 4 & 5 & 5 & 2 & 2 & 2 & 2 & 2 \\
\Delta h_{Z_4,3} & 1 & 2 & 3 & 4 & 5 & 5 & 4 & 3 & 3 & 3 & 3 \\
\Delta h_{Z_5} & 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
\Delta h_{Z_5,1} & 1 & 2 & 3 & 4 & 5 & 6 & 5 & 5 & 5 & 5 & 5 \\
\Delta h_{Z_5,2} & 1 & 2 & 3 & 4 & 5 & 6 & 6 & 5 & 2 & 2 & 2 \\
\Delta h_{Z_5,3} & 1 & 2 & 3 & 4 & 5 & 6 & 6 & 6 & 6 & 6 & 6 \\
\Delta h_{Z_5,4} & 1 & 2 & 3 & 4 & 5 & 6 & 6 & 6 & 6 & 6 & 6 \\
\Delta h_{Z_6} & 1 & 2 & 3 & 4 & 5 & 6 & 6 & 6 & 6 & 6 & 6 \\
\end{array}
\]

For instance, why should $h_{Z_4,2}$ be minimal?
We have not been able to find an argument even in this case. However, we have the following:

**Conjecture 7.12.** Among double schemes whose support has a fixed generic Hilbert function \((1, 2, \ldots, t - 1, r)\) (see Lemma 7.8), there is a minimal Hilbert function, and it occurs when the support is \(C_{t, r}\).

Note that when \(0 < r < t\), we do not conjecture that the minimal Hilbert function can only occur when the support is \(C_{t, r}\), as was the case for \(C_t\). For instance, the Hilbert function for the first infinitesimal neighborhood of \(C_{4, 2}\) (illustrated above) can also arise from the first infinitesimal neighborhood of the following configuration (where the oval is a conic):

We generalize the above to any Hilbert function corresponding to type \((n_1, \ldots, n_r)\). We form a configuration \(C_h\) as follows. Chose a set of \(r + 1\) lines (we'll call them \(\lambda_1, \ldots, \lambda_{r+1}\)) and let \(C_{r+1}\) be as above, i.e. the union of all the pairwise intersection points of the \(\lambda_i\)'s. So, each of the \(\lambda_i\) contains \(r\) points of \(C_{r+1}\). Notice, however, that we can view \(C_{r+1}\) as a \(k\)-configuration in the following way: first choose all \(r\) points on \(\lambda_r\); then, on \(\lambda_{r-1}\), choose the remaining \(r - 1\) points (since one was already chosen on \(\lambda_r\)); on \(\lambda_{r-2}\) choose the remaining \(r - 2\) points; \(\ldots\); on \(\lambda_1\) choose the only point remaining. (Note that \(\lambda_{r+1}\)
has become irrelevant in this point of view.) Now we add $n_r - r$ arbitrary points on $\lambda_r$, $n_{r-1} - (r - 1)$ points on $\lambda_{r-1}$, etc. thus forming a $k$-configuration, $C_h$ of type $(n_1, \ldots, n_r)$.

**Conjecture 7.13.** The Hilbert function of $C_h$ is $h_{\text{min}}$.

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