A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model

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Abstract

We provide a new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. The proof applies to infinite range models on arbitrary locally finite transitive infinite graphs.

For Bernoulli percolation, we prove finiteness of the susceptibility in the subcritical regime $\beta < \beta_c$, and the mean-field lower bound $\mathbb{P}_\beta[0 \leftrightarrow \infty] \geq (\beta - \beta_c)/\beta$ for $\beta > \beta_c$. For finite-range models, we also prove that for any $\beta < \beta_c$, the probability of an open path from the origin to distance $n$ decays exponentially fast in $n$.

For the Ising model, we prove finiteness of the susceptibility for $\beta < \beta_c$, and the mean-field lower bound $\langle \sigma_0 \rangle_\beta^2 \geq \sqrt{(\beta^2 - \beta_c^2)}/\beta^2$ for $\beta > \beta_c$. For finite-range models, we also prove that the two-point correlations functions decay exponentially fast in the distance for $\beta < \beta_c$.

The paper is organized in two sections, one devoted to Bernoulli percolation, and one to the Ising model. While both proofs are completely independent, we wish to emphasize the strong analogy between the two strategies.

General notation. Let $G = (V,E)$ be a locally finite (vertex-)transitive infinite graph, together with a fixed origin $0 \in V$. For $n \geq 0$, let

$$\Lambda_n := \{ x \in V : d(x,0) \leq n \},$$

where $d(\cdot,\cdot)$ is the graph distance. Consider a set of coupling constants $(J_{x,y})_{x,y \in V}$ with $J_{x,y} = J_{y,x} \geq 0$ for every $x$ and $y$ in $V$. We assume that the coupling constants are invariant: for every graph automorphism $\gamma$, we have $J_{\gamma(x),\gamma(y)} = J_{x,y}$. We say that $(J_{x,y})_{x,y \in V}$ is finite range if there exists $R > 0$ such that $J_{x,y} = 0$ whenever $d(x,y) > R$. 


1
1 Bernoulli percolation

1.1 The main result

Let $P_{\beta}$ be the bond percolation measure on $G$ defined as follows: for $x, y \in V$, $\{x, y\}$ is open with probability $1 - e^{-\beta J_{x,y}}$, and closed with probability $e^{-\beta J_{x,y}}$. We say that $x$ and $y$ are connected in $S \subset V$ if there exists a path of vertices $(v_k)_{0 \leq k < K}$ in $S$ such that $v_0 = x$, $v_K = y$, and $\{v_k, v_{k+1}\}$ is open for every $0 \leq k < K$. We denote this event by $x \leftrightarrow_S y$. For $A \subset V$, we write $x \leftrightarrow_S A$ for the event that $x$ is connected in $S$ to a vertex in $A$. If $S = V$, we drop it from the notation. Finally, we set $0 \leftrightarrow \infty$ if $0$ is connected to infinity. The critical parameter is defined by

$$\beta_c := \inf \{ \beta \geq 0 : P_{\beta}[0 \leftrightarrow \infty] > 0 \}.$$ 

**Theorem 1.1.** 1. For $\beta > \beta_c$, $P_{\beta}[0 \leftrightarrow \Lambda_n^c] \geq \frac{\beta - \beta_c}{\beta}$. 2. For $\beta < \beta_c$, the susceptibility is finite, i.e.

$$\sum_{x \in V} P_{\beta}[0 \leftrightarrow x] < \infty.$$ 

3. If $(J_{x,y})_{x,y \in V}$ is finite range, then for any $\beta < \beta_c$, there exists $c = c(\beta) > 0$ such that

$$P_{\beta}[0 \leftrightarrow \Lambda_n^c] \leq e^{-cn} \quad \text{for all } n \geq 0.$$ 

Let us describe the proof quickly. For $\beta > 0$ and a finite subset $S$ of $V$, define

$$\varphi_{\beta}(S) := \sum_{x \in S} \sum_{y \not\in S} (1 - e^{-\beta J_{x,y}}) P_{\beta}[0 \leftrightarrow_S x]. \quad (1.1)$$

This quantity can be interpreted as the expected number of open edges on the “external boundary” of $S$ that are connected to 0 by an open path of vertices in $S$. Also introduce

$$\tilde{\beta}_c := \sup \{ \beta \geq 0 : \varphi_{\beta}(S) < 1 \text{ for some finite } S \subset V \text{ containing } 0 \}. \quad (1.2)$$

In order to prove Theorem 1.1, we show that Item 1, 2 and 3 hold with $\tilde{\beta}_c$ in place of $\beta_c$. This directly implies that $\tilde{\beta}_c = \beta_c$, and thus Theorem 1.1.

We proceed in two steps.

The quantity $\varphi_{\beta}(S)$ appears naturally when differentiating the probability $P_{\beta}[0 \leftrightarrow \Lambda_n^c]$ with respect to $\beta$. A simple computation presented in Lemma 1.2 below provides the differential inequality, for every $\beta$.

$$\frac{d}{d\beta} P_{\beta}[0 \leftrightarrow \Lambda_n^c] \geq \frac{1}{\beta} \inf_{S \ni 0} \varphi_{\beta}(S) \cdot (1 - P_{\beta}[0 \leftrightarrow \Lambda_n^c]). \quad (1.3)$$
By integrating (1.3) between $\tilde{\beta}_c$ and $\beta > \beta_c$ and then letting $n$ tends to infinity, we obtain $P_{\beta}[0 \leftrightarrow \infty] \geq \frac{\beta - \tilde{\beta}_c}{\beta}$.

Now consider $\beta < \tilde{\beta}_c$. The existence of a finite set $S$ containing the origin such that $\varphi_\beta(S) < 1$, together with the BK-inequality, imply that the expected size of the cluster the origin is finite.

### 1.2 Comments and consequences

**Bibliographical comments.** Theorem 1.1 was first proved in [AB87] and [Men86] for Bernoulli percolation on the $d$-dimensional hypercubic lattice. The proof was extended to general quasi-transitive graphs in [AV08]. The first item was also proved in [CC87], and a recent proof of Gady Kozma was presented in [Pet13].

**Nearest-neighbour percolation.** We recover the standard results for nearest-neighbor model by setting $J_{x,y} = 1$ if $\{x,y\} \in E$, and $0$ otherwise, and $p = 1 - e^{-\beta}$. In this context, one can obtain the inequality $P_p[0 \leftrightarrow \infty] \geq \frac{p - p_c}{p(1 - p_c)}$ for $p \geq p_c$. This lower bound is slightly better than Item 1 of Theorem 1.1 and is provided by little modifications in our proof (see [DCT15] for a presentation of the proof in this context).

**Finite susceptibility against exponential decay.** Finite susceptibility does not always imply exponential decay of correlations for infinite-range models. Conversely, on graphs with exponential growth, exponential decay does not imply finite susceptibility. Hence, the second condition of Theorem 1.1 is neither weaker nor stronger than the third one.

**Percolation on the square lattice.** This theorem implies that $p_c = 1/2$ on the square lattice. Indeed, it immediately implies that $p_c \leq 1/2$, and the classical Zhang’s argument [Gri99] implies the other inequality $p_c \geq 1/2$. This provides a shorter proof than the original strategy provided by Kesten [Kes80].

**Lower bound on $\beta_c$.** Since $\varphi_\beta(\{0\}) = \sum_{y \in V} 1 - e^{-\beta J_{0,y}}$, we obtain a lower bound on $\beta_c$ by taking the solution of the equation $\sum_{y \in V} 1 - e^{-\beta J_{0,y}} = 1$.

**Behaviour at $\beta_c$.** Under the hypothesis that $\sum_{y \in V} J_{0,y} < \infty$, the set

$\{ \beta \geq 0 : \varphi_\beta(S) < 1 \text{ for some finite } S \subset V \text{ containing } 0 \}$

defining $\tilde{\beta}_c$ in Equation (1.2) is open. In particular, we have that at $\beta = \beta_c$, $\varphi_\beta(\Lambda_n) \geq 1$ for every $n$. This implies the classical result that the susceptibility is infinite at criticality.

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Semi-continuity of $\beta_c$. Consider the nearest-neighbor model. Since $\tilde{\beta}_c$ is defined in terms of finite sets, one can see that $\tilde{\beta}_c$ is lower semi-continuous when seen as a function of the graph in the following sense. Let $G$ be an infinite locally finite transitive graph. Let $(G_n)$ be a sequence of infinite locally finite transitive graphs satisfying that the balls of radius $n$ around the origin in $G_n$ and $G$ are the same. Then,

$$\lim inf \beta_c(G_n) \geq \beta_c(G). \quad (1.4)$$

The equality $\beta_c = \tilde{\beta}_c$ implies that the semi-continuity (1.4) also holds for $\beta_c$. The locality conjecture, due to Schramm and presented in [BNP11], states that a restricted version of $\beta_c$ should be continuous. The discussion above shows that the hard part in the locality conjecture is the upper semi-continuity.

Dependent models. For dependent percolation models, the proof does not extend in a trivial way, mostly due to the fact that the BK inequality is not available in general. Nevertheless, this new strategy may be of some use. For instance, for random-cluster models on the square lattice, a proof (see [DCST13]) based on the strategy of this paper and the parafermionic observable offers an alternative to the standard proof of [BDC12b] based on sharp threshold theorems.

Oriented percolation. The proof applies mutatis mutandis to oriented percolation.

1.3 Proof of Item 1

In this section, we prove that for every $\beta \geq \tilde{\beta}_c$,

$$\mathbb{P}_\beta[0 \leftarrow \infty] \geq \frac{\beta - \tilde{\beta}_c}{\beta}. \quad (1.5)$$

Let us start by the following lemma.

Lemma 1.2. Let $\beta > 0$ and $\Lambda \subset V$ finite,

$$\frac{d}{d\beta} \mathbb{P}_\beta[0 \leftarrow \Lambda^c] \geq \frac{1}{\beta} \inf_{S \subseteq \Lambda} \varphi_\beta(S) \cdot (1 - \mathbb{P}_\beta[0 \leftarrow \Lambda^c]). \quad (1.6)$$

Integrating the differential inequality (1.6) between $\tilde{\beta}_c$ and $\beta$ implies that for every $\Lambda \subset V$, $\mathbb{P}_\beta[0 \leftarrow \Lambda^c] \geq \frac{\beta - \tilde{\beta}_c}{\beta}$. By letting $\Lambda$ tend to $V$, we obtain Equation (1.5).
Proof of Lemma 1.2. Let $\beta > 0$ and $\Lambda$. Define the following random subset of $\Lambda$:

$$S := \{x \in \Lambda \text{ such that } x \leftrightarrow \Lambda^c\}.$$ 

Recall that $\{x, y\}$ is pivotal for the configuration $\omega$ and the event $0 \leftrightarrow \Lambda^c$ if $\omega_{\{x,y\}} \notin \{0 \leftrightarrow \Lambda^c\}$ and $\omega_{\{x,y\}} \in \{0 \leftrightarrow \Lambda^c\}$. (The configuration $\omega_{\{x,y\}}$, resp. $\omega_{\{x,y\}}$, coincides with $\omega$ except that the edge $\{x,y\}$ is closed, resp. open.)

Russo’s formula ([Rus78] or [Gri99, Section 2.4]) implies that

$$\frac{d}{d\beta} P_{\beta} \left[0 \leftrightarrow \Lambda^c\right] = \sum_{\{x,y\}} J_{x,y} P_{\beta} \left[\{x,y\} \text{ pivotal and } 0 \leftrightarrow \Lambda^c\right] \geq \frac{1}{\beta} \sum_{\{x,y\}} \left(1 - e^{-\beta J_{x,y}}\right) P_{\beta} \left[\{x,y\} \text{ pivotal and } 0 \leftrightarrow \Lambda^c\right]$$

In the second line, we used the inequality $t \geq 1 - e^{-t}$ for $t \geq 0$. Observe that the event that $\{x, y\}$ is pivotal and $S = S$ is nonempty only if $x \in S$ and $y \notin S$, or $y \in S$ and $x \notin S$. Furthermore, the vertex in $S$ must be connected to 0 in $S$. We can assume without loss of generality that $x \in S$ and $y \notin S$.

Rewrite the event that $\{x, y\}$ is pivotal and $S = S$ as $\{0 \leftrightarrow x\} \cap \{S = S\}$. Since the event $\{S = S\}$ is measurable with respect to the configuration outside $S$, the two events above are independent. Thus,

$$P_{\beta} \left[\{x,y\} \text{ pivotal and } S = S\right] = P_{\beta} \left[0 \leftrightarrow x\right] P_{\beta} \left[S = S\right].$$

Plugging this equality in the computation above, we obtain

$$\frac{d}{d\beta} P_{\beta} \left[0 \leftrightarrow \Lambda^c\right] \geq \frac{1}{\beta} \sum_{S \ni 0} \varphi_{\beta}(S) P_{\beta} \left[S = S\right]. \quad (1.7)$$

The proof follows readily.

Remark 1.1. In the proof above, we used Russo’s formula in infinite volume, since the model is possibly infinite range. There is no difficulty resolving this technical issue (which does not occur for finite range) by finite volume approximation. The same remark applies below when we use the BK inequality.

1.4 Proof of Items 2 and 3

In this section, we show that Items 2 and 3 in Theorem 1.1 hold with $\beta_c$ in place of $\beta_c$. 
Lemma 1.3. Let $\beta > 0$, and $u \in S \subset A$ and $B \cap S = \emptyset$. We have

$$\mathbb{P}_{\beta}(u \overset{A}{\rightarrow} B) \leq \sum_{x \in S} \sum_{y \in S} \left(1 - e^{-\beta J_x,y}\right) \mathbb{P}_{\beta}(u \overset{S}{\leftrightarrow} x) \mathbb{P}_{\beta}(y \overset{A}{\leftrightarrow} B).$$

Proof of Lemma 1.3. Let $u \in S$ and assume that the event $u \overset{A}{\rightarrow} B$ holds. Consider an open path $(v_k)_{0 \leq k \leq K}$ from $u$ to $B$. Since $B \cap S = \emptyset$, one can define the first $k$ such that $v_{k+1} \notin S$. We obtain that the following events occur disjointly (see [Gri99, Section 2.3] for a definition of disjoint occurrence):

- $u$ is connected to $v_k$ in $S$,
- $\{v_k, v_{k+1}\}$ is open,
- $v_{k+1}$ is connected to $B$ in $A$.

The lemma is then a direct consequence of the BK inequality applied twice ($v_k$ plays the role of $x$, and $v_{k+1}$ of $y$).

Let us now prove the second item of Theorem 1.1. Fix $\beta < \tilde{\beta}_c$ and $S$ such that $\varphi_\beta(S) < 1$. For $\Lambda \subset V$ finite, introduce

$$\chi(\Lambda, \beta) := \max \left\{ \sum_{v \in \Lambda} \mathbb{P}_{\beta}(u \overset{v}{\leftrightarrow} v) : u \in \Lambda \right\}.$$

For every $u$, let $S_u$ be the image of $S$ by a fixed automorphism sending $0$ to $u$. Lemma 1.3 implies that

$$\mathbb{P}_{\beta}(u \overset{\Lambda}{\rightarrow} v) \leq \sum_{x \in S_u} \sum_{y \notin S_u} \mathbb{P}_{\beta}(u \overset{S_u}{\leftrightarrow} x) \left(1 - e^{-\beta J_x,y}\right) \mathbb{P}_{\beta}(y \overset{\Lambda}{\leftrightarrow} v).$$

Summing over all $v \in \Lambda \setminus S_u$, we find

$$\sum_{v \in \Lambda \setminus S_u} \mathbb{P}_{\beta}(u \overset{\Lambda}{\rightarrow} v) \leq \varphi_\beta(S) \chi(\Lambda, \beta).$$

Using the trivial bound $\mathbb{P}_{\beta}(u \overset{}{\leftrightarrow} v) \leq 1$ for $v \in \Lambda \cap S_u$, we obtain

$$\sum_{v \in \Lambda} \mathbb{P}_{\beta}(u \overset{\Lambda}{\rightarrow} v) \leq |S| + \varphi_\beta(S) \chi(\Lambda, \beta).$$

Optimizing over $u$, we deduce that

$$\chi(\Lambda, \beta) \leq \frac{|S|}{1 - \varphi_\beta(S)}$$

which implies in particular that

$$\sum_{x \in \Lambda} \mathbb{P}_{\beta}(0 \overset{\Lambda}{\rightarrow} x) \leq \frac{|S|}{1 - \varphi_\beta(S)}.$$

The result follows by taking the limit as $\Lambda$ tends to $V$. 
We now turn to the proof of the third item of Theorem 1.1. A similar proof was used in [Ham57]. Let $R$ be the range of the $(J_{x,y})_{x,y \in V}$, and let $L$ be such that $S \subset \Lambda_{L-R}$. Lemma 1.3 implies that for $n \geq L$,

$$\mathbb{P}_\beta[0 \leftrightarrow \Lambda^n_\ast] \leq \sum_{x \in S} \sum_{y \notin S} (1 - e^{\beta J_{x,y}}) \mathbb{P}_\beta[0 \leftrightarrow x] \mathbb{P}_\beta[y \leftrightarrow \Lambda^n_\ast] \leq \varphi_\beta(S) \mathbb{P}_\beta[0 \leftrightarrow \Lambda^n_{n-L}] .$$

In the last line, we used that $y$ is connected to distance larger than or equal to $n - L$ since $1 - e^{-\beta J_{x,y}} = 0$ if $x \in S$ and $y$ is not in $\Lambda_L$. By iterating, this immediately implies that

$$\mathbb{P}_\beta[0 \leftrightarrow \Lambda^n_\ast] \leq \varphi_\beta(S)^{n/L} .$$

2 The Ising model

2.1 The main result

For a finite subset $\Lambda$ of $V$, consider a spin configuration $\sigma = (\sigma_x : x \in \Lambda) \in \{-1,1\}^\Lambda$. Introduce the two Hamiltonians

$$H^{+}_\Lambda(\sigma) := - \sum_{\{x,y\} \subset \Lambda} J_{x,y} \sigma_x \sigma_y,$$

$$H^-_\Lambda(\sigma) := H^{\text{free}}_\Lambda(\sigma) - \sum_{x \in \Lambda, y \notin \Lambda} J_{x,y} \sigma_x .$$

For $\beta > 0$, $\# \in \{+, f\}$, define the Gibbs measures on $\Lambda$ with inverse-temperature $\beta$ by the formula

$$\langle f \rangle^\#_{\Lambda, \beta} = \frac{\sum_{\sigma \in \{-1,1\}^\Lambda} f(\sigma) e^{-\beta H^{\#}_\Lambda(\sigma)}}{\sum_{\sigma \in \{-1,1\}^\Lambda} e^{-\beta H^{\#}_\Lambda(\sigma)}}$$

for $f : \{-1,1\}^\Lambda \rightarrow \mathbb{R}$. When $\# = +$ (resp. $\# = f$), one speaks of + boundary condition (resp. free boundary condition). Let the infinite-volume Gibbs measure $\langle \cdot \rangle^+_\beta$ be the weak limit of $\langle \cdot \rangle^+_{\Lambda, \beta}$ as $\Lambda \searrow V$.

Introduce

$$\beta_c := \inf \{ \beta > 0 : \langle \sigma_0 \rangle^+_\beta > 0 \} .$$

Theorem 2.1. 1. For $\beta > \beta_c$, $\langle \sigma_0 \rangle^+_\beta \geq \sqrt{\frac{\beta^2 - \beta_c^2}{\beta^2}}$.

2. For $\beta < \beta_c$, the susceptibility is finite, i.e.

$$\sum_{x \in V} \langle \sigma_0 \sigma_x \rangle^+_\beta < \infty .$$
3. If \((J_{x,y})_{x,y\in V}\) is finite range, then for any \(\beta < \beta_c\), there exists \(c = c(\beta) > 0\) such that

\[
\langle \sigma_0 \sigma_x \rangle_\beta^+ \leq e^{-cd(0,x)} \quad \text{for all } x \in V.
\]

This theorem was first proved in [ABFS78] for the Ising model on the \(d\)-dimensional hypercubic lattice. The proof presented here improves the constant in the mean-field lower bound, and extends to general transitive graphs.

The proof of Theorem 2.1 follows closely the proof for percolation. For \(\beta > 0\) and a finite subset \(S\) of \(V\), define

\[
\varphi_S(\beta) := \sum_{x \in S} \sum_{y \not\in S} \tanh(\beta J_{x,y}) \langle \sigma_0 \sigma_x \rangle_{S,\beta}^F,
\]

which bears a resemblance to (1.1). Similarly to (1.2), set

\[
\tilde{\beta}_c := \sup \{ \beta \geq 0 : \varphi(\beta) < 1 \text{ for some finite } S \subset V \text{ containing } 0 \}.
\]

In order to prove Theorem 2.1, we show that Items 1, 2 and 3 hold with \(\tilde{\beta}_c\) in place of \(\beta_c\). This directly implies that \(\tilde{\beta}_c = \beta_c\), and thus Theorem 2.1.

The proof of Theorem 2.1 proceeds in two steps. As for percolation, the quantity \(\varphi(\beta)\) appears naturally in the derivative of a “finite-volume approximation” of \(\langle \sigma_0 \rangle_\beta^+\). Roughly speaking (see below for a precise statement), one obtains a finite-volume version of the following inequality:

\[
\frac{d}{d\beta} \langle \sigma_0 \rangle_\beta^+ \geq \frac{2}{\beta} \inf_{S \ni 0} \varphi(\beta) \cdot (1 - \langle \sigma_0 \rangle_\beta^+).
\]

This inequality implies, for every \(\beta > \tilde{\beta}_c\),

\[
\langle \sigma_0 \rangle_\beta^+ \geq \sqrt{\frac{\beta^2 - \tilde{\beta}_c^2}{\beta^2}}.
\]

The remaining items follow from an improved Simon’s inequality, proved below.

Remark 2.1. The proof uses the random-current representation. In this context, the derivative of \(\langle \sigma_0 \rangle_\beta^+\) has an interpretation which is very close to the differential inequality (1.6). In some sense, percolation is replaced by the trace of the sum of two independent random sourceless currents. Furthermore, the strong Simon’s inequality plays the role of the BK inequality for percolation.
2.2 Comments and consequences

1. Finite susceptibility does not imply exponential decay of correlations for infinite-range models. Hence, the second condition of Theorem 2.1 is not weaker than the third one.

2. Together with the Kramers-Wannier duality, this theorem implies that $\beta_c = \frac{1}{2} \log(1 + \sqrt{2})$ on the square lattice (see [Ons44, BDC12a] for alternative proofs).

3. Exactly as for Bernoulli percolation, the susceptibility is infinite at criticality (see [Sim80] for the original proof).

4. The equality $\beta_c = \tilde{\beta}_c$ implies that $\beta_c$ is lower semi-continuous with respect to the graph (see the discussion for Bernoulli percolation).

2.3 Preliminaries

A useful Ising measure interpolating between free and + boundary conditions. For $S \subset \Lambda$ two finite subsets of $V$, introduce $(\cdot)_S^{\Lambda,\beta}$ obtained from $(\cdot)^{\Lambda,\beta}$ by setting all the coupling constants $J_{x,y}$ with $x$ or $y$ in $\Lambda \setminus S$ to be equal to 0. Note that if $S = \Lambda$, then $(\cdot)_S^{\Lambda,\beta} = (\cdot)^{\Lambda,\beta}$ and for each fixed $S$, $(\cdot)_S^{\Lambda,\beta}$ tends to $(\cdot)_S^{\Lambda,\beta}$ as $\Lambda \nearrow V$.

Griffiths’ inequality. The following is a standard consequence of the second Griffiths’ inequality [Gri67]: for $\beta > 0$ and $S \subset \Lambda$ two finite subsets of $V$,

$$\langle \sigma_0 \rangle_S^{\Lambda,\beta} \leq \langle \sigma_0 \rangle_{S}^{\Lambda,\beta}.$$ (2.1)

Random-current representation. This section presents a few basic facts on the random-current representation. We refer to [Aiz82, ABF87] for details on this representation.

Let $\Lambda$ be a finite subset of $V$ and $S \subset \Lambda$. We consider an additional vertex $g$ not in $\Lambda$, called the ghost vertex, and write $\mathcal{P}_2(S)$ (resp. $\mathcal{P}_2(S \cup \{g\})$) for the set of pairs $\{x,y\}$, $x,y \in S$ (resp. $S \cup \{g\}$). We also define for every $x \in \Lambda$,

$$J_{x,g} = J_{x,g}(\Lambda) := \sum_{y \in \Lambda} J_{x,y}.$$

Definition 2.2. A current $n$ on $S$ (also called a current configuration) is a function from $\mathcal{P}_2(S \cup \{g\})$ to $\{0,1,2,...\}$. A source of $n = (n_{x,y} \cdot \{x,y\} \in \mathcal{P}_2(S \cup \{g\}))$ is a vertex $x$ for which $\sum_{y \in S} n_{x,y}$ is odd. The set of sources of $n$ is denoted by $\partial n$. We say that $x$ and $y$ are connected in $n$ (denoted by $x \leftrightarrow n y$) if there exists $(v_k)_{0 \leq k \leq K}$ such that $v_0 = x$, $v_k = y$ and $n_{v_k,v_{k+1}} > 0$ for every $0 \leq k < K$.
For $S \subset \Lambda$, define

$$w_S(n) = w_S(\Lambda, \beta, n) := \prod_{\{x,y\} \in \mathcal{P}_2(S \cup \{g\})} \frac{(\beta J_{x,y})^{n_{x,y}}}{n_{x,y}!}.$$ 

Introduce

$$Z^\Lambda_{S,\beta} = \sum_{\partial n = \emptyset} w_S(n)$$

so that for every subset $A$ of $V$, we have

$$\prod_{a \in A} \sigma_a^\Lambda_{S,\beta} = \begin{cases} \frac{1}{Z^\Lambda_{S,\beta}} \sum_{\partial n = A} w_S(n) & \text{if } A \text{ is even}, \\ \frac{1}{Z^\Lambda_{S,\beta}} \sum_{\partial n = A \cup \{g\}} w_S(n) & \text{if } A \text{ is odd}. \end{cases}$$

When $S = \Lambda$, we write $w(n)$ instead of $w_\Lambda(n)$, $Z_{\Lambda,\beta}$ instead of $Z^\Lambda_{S,\beta}$, and $\langle \cdot \rangle^\Lambda_{\Lambda,\beta}$ instead of $\langle \cdot \rangle^\Lambda_{S,\beta}$, so that we find

$$\prod_{a \in A} \sigma_a^\Lambda = \begin{cases} \frac{1}{Z_{\Lambda,\beta}} \sum_{\partial n = A} w(n) & \text{if } A \text{ is even}, \\ \frac{1}{Z_{\Lambda,\beta}} \sum_{\partial n = A \cup \{g\}} w(n) & \text{if } A \text{ is odd}. \end{cases}$$ (2.2)

We will use the following standard lemma on random currents.

**Lemma 2.3** (Switching Lemma, [Aiz82, Lemma 3.2]). Let $A \subset S \subset \Lambda$ and $u, v \in S \cup \{g\}$. We have

$$\sum_{\partial n_1 = A \Delta \{u,v\}} \sum_{\partial n_2 = \{u,v\}} w_S(n_1)w_S(n_2) = \sum_{\partial n_1 = A} w_S(n_1)w_S(n_2)\mathbf{1}_{[u \xleftarrow{n_1+n_2} v]},$$

where $\Delta$ is the symmetric difference between sets.

**Backbone representation for random currents.** Fix two finite subsets $S \subset \Lambda$ of $V$. Choose an arbitrary order of the oriented edges of the lattice. Consider a current $n$ on $S$ with $\partial n = \{x,y\}$. Let $\omega(n)$ be the edge self-avoiding path from $x$ to $y$ passing only through edges $e$ with $n_e > 0$ which is minimal for the lexicographical order on paths induced by the previous ordering on oriented edges. Such an object is called the backbone of the current configuration. For a backbone $\omega$ with endpoints $\partial \omega = \{x,y\}$, set

$$\rho^\Lambda_S(\omega) = \rho^\Lambda_S(\beta, \omega) := \frac{1}{Z^\Lambda_{S,\beta}} \sum_{\partial n = \{x,y\}} w_S(n)\mathbf{1}_{[\omega(n) = \omega]}.$$ 

The backbone representation has the following properties (see [ABF87, pages 11–12] for a proof of these statements):

1. $\langle \sigma_x \sigma_y \rangle^\Lambda_{S,\beta} = \sum_{\partial \omega = \{x,y\}} \rho^\Lambda_S(\omega).$
2. If the backbone $\omega$ is the concatenation of two backbones $\omega_1$ and $\omega_2$ (this is denoted by $\omega = \omega_1 \circ \omega_2$), then

$$\rho_\Lambda^\Delta(\omega) = \rho_\Lambda^\Delta(\omega_1) \rho_\Lambda^{\Delta,\overline{\omega_1}}(\omega_2),$$

where $\overline{\omega_1}$ is the set of bond whose state is determined by the fact that $\omega_1$ is an admissible backbone (this includes bonds of $\omega_1$ together with some neighboring bonds).

3. For a backbone $\omega$ not using any edge outside $T \subset S$, then

$$\rho_\Lambda^\Delta(\omega) \leq \rho_\Lambda^\Delta(T).$$

### 2.4 Proof of Item 1

In this section, we prove that for every $\beta \geq \tilde{\beta}_c$,

$$\langle \sigma_0 \rangle^\beta_\Lambda \geq \sqrt{\frac{\beta^2 - \tilde{\beta}^2_c}{\beta^2}}. \quad (2.3)$$

**Lemma 2.4.** Let $\beta > 0$ and $\Lambda \subset V$ finite. Then,

$$\frac{d}{d\beta} (\langle \sigma_0 \rangle^\beta_\Lambda)^2 \geq \frac{2}{\beta} \inf_{S \in \Lambda} \varphi_{\beta}(S) \left(1 - (\langle \sigma_0 \rangle^\beta_\Lambda)^2\right). \quad (2.4)$$

We may integrate Equation (2.4) between $\tilde{\beta}_c$ and $\beta$, then let $\Lambda \nearrow V$ to obtain Equation (2.3). In order to prove Lemma 2.4, we use a computation similar to one provided in [ABFS7].

**Proof of Lemma 2.4.** Let $\beta > 0$ and a finite subset $\Lambda$ of $V$. The derivative of $\langle \sigma_0 \rangle^\beta_\Lambda$ is given by the following formula

$$\frac{d}{d\beta} (\langle \sigma_0 \rangle^\beta_\Lambda) = \sum_{\{x,y\} \subset \Lambda \cup \{g\}} J_{x,y}(\langle \sigma_0 \rangle^\beta_{X,Y} - \langle \sigma_0 \rangle^\beta_{X,Y}),$$

where $\sigma_g$ is understood as being +1. Using (2.2) and the switching Lemma, we obtain

$$\frac{d}{d\beta} (\langle \sigma_0 \rangle^\beta_\Lambda) = \frac{1}{Z_{\Lambda,\beta}} \sum_{\{x,y\} \subset \Lambda \cup \{g\}} \sum_{\partial n_1 = \{0,g\} \Delta \{x,y\}} \partial n_2 = \emptyset \ \check{w}(n_1) \check{w}(n_2) \delta^{n_1+n_2}_{x \leftrightarrow g}. \quad (2.5)$$

If $n_1$ and $n_2$ are two currents such that $\partial n_1 = \{0,g\} \Delta \{x,y\}$, $\partial n_2 = \emptyset$ and $0$ and $g$ are not connected in $n_1 + n_2$, then exactly one of these two cases holds: $0 \xleftrightarrow{n_1+n_2} x$ and $y \xleftrightarrow{n_1+n_2} g$, or $0 \xleftrightarrow{n_1+n_2} y$ and $x \xleftrightarrow{n_1+n_2} g$. Since the second case is the same as the first one with $x$ and $y$ permuted, we obtain the following expression,

$$\frac{d}{d\beta} (\langle \sigma_0 \rangle^\beta_\Lambda) = \frac{1}{Z_{\Lambda,\beta}^2} \sum_{\{x,y\} \subset \Lambda \cup \{g\}} \sum_{y \in \Lambda \cup \{g\}} \delta_{x,y}, \quad (2.5)$$

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where

\[ \delta_{x,y} = \sum_{\partial n_1 = \{0,g\} \Delta \{x,y\}} \sum_{\partial n_2 = \emptyset} w(n_1)w(n_2)I[0 \quad n_1+n_2 \quad x \leftrightarrow y, 0 \leftrightarrow g] \]

(see Fig. 1 and notice the analogy with the event involved in Russo’s formula, namely that the edge \( \{x,y\} \) is pivotal, in Bernoulli percolation).

Figure 1: A diagrammatic representation of \( \delta_{x,y} \): the solid lines represent the backbones, and the dotted line the boundary of the cluster of 0 in \( n_1 + n_2 \).

Given two currents \( n_1 \) and \( n_2 \), and \( z \in \{0,g\} \), define \( \mathcal{S}_z \) to be the set of vertices in \( \Lambda \cup \{g\} \) that are NOT connected to \( z \) in \( n_1 + n_2 \). Let us compute \( \delta_{x,y} \) by summing over the different possible values for \( \mathcal{S}_0 \):

\[ \delta_{x,y} = \sum_{S \in \Lambda \cup \{g\}} \sum_{\partial n_1 = \{0,g\} \Delta \{x,y\}} \sum_{\partial n_2 = \emptyset} w(n_1)w(n_2)I[\mathcal{S}_0 = S, 0 \quad n_1+n_2 \quad x \leftrightarrow y, 0 \leftrightarrow g] \]

When \( \mathcal{S}_0 = S \), the two currents \( n_1 \) and \( n_2 \) vanish on every \( \{u,v\} \) with \( u \in S \) and \( v \notin S \). Thus, for \( i = 1, 2 \), we can decompose \( n_i \) as

\[ n_i = n_i^S + n_i^{\Lambda \setminus S}, \]

where \( n_i^A \) denotes the current in \( A \subset V \) with source \( \partial n_i^A = A \cap \partial n_i \). Using this observation together with the second identity in (2.2), we obtain

\[ \delta_{x,y} = \sum_{S \in \Lambda \cup \{g\}} \sum_{\partial n_1 = \{0\} \Delta \{x\}} \sum_{\partial n_2 = \emptyset} w(n_1)w(n_2)\{\sigma_y\}^A_{\partial n_2 \cap \partial n_1} I[\mathcal{S}_0 = S]. \]
Multiplying the expression above by \( \langle \sigma_0 \rangle_{\Lambda, \beta} \), and using Inequality (2.1), we find

\[
\langle \sigma_0 \rangle_{\Lambda, \beta} \delta_{x,y} \geq \sum_{S \subseteq \Lambda \setminus \{g\}} \sum_{\partial n_1 = \{0\} \Delta(x) \atop \text{s.t. } y,g \in S \text{ and } 0 \notin S} w(n_1) w(n_2) \left( \langle \sigma_0 \rangle_{\tilde{S}, \beta} \right)^2 \mathbb{I}[\mathcal{F}_0 = S]
\]

\[
= \sum_{S \subseteq \Lambda \setminus \{g\}} \sum_{\partial n_1 = \{0\} \Delta(x) \Delta(y, \beta) \atop \text{s.t. } y,g \in S \text{ and } 0 \notin S} w(n_1) w(n_2) \mathbb{I}[\mathcal{F}_0 = S]
\]

\[
= \sum_{S \subseteq \Lambda \setminus \{g\}} \sum_{\partial n_1 = \{0\} \Delta(x) \atop \text{s.t. } y,g \in S \text{ and } 0 \notin S} w(n_1) w(n_2) \mathbb{I}[\mathcal{F}_0 = S, y \overset{n_1+n_2}{\mapsto} g]
\]

\[
= \sum_{\partial n_1 = \{0\} \Delta(x)} w(n_1) w(n_2) \mathbb{I}[y \overset{n_1+n_2}{\mapsto} g, 0 \leftrightarrow g],
\]

where in the third line we used the switching lemma. We now sum over the possible values of \( \mathcal{F}_g \):\]

\[
\langle \sigma_0 \rangle_{\Lambda, \beta} \delta_{x,y} \geq \sum_{S \subseteq \Lambda} \sum_{\partial n_1 = \{0\} \Delta(x) \atop \text{s.t. } y \in S} w(n_1) w(n_2) \mathbb{I}[\mathcal{F} = S, y \overset{n_1+n_2}{\mapsto} g, 0 \leftrightarrow g]
\]

\[
= \sum_{S \subseteq \Lambda} \sum_{\partial n_1 = \{0\} \Delta(x) \atop \text{s.t. } y \in S} w(n_1) w(n_2) \mathbb{I}[\mathcal{F} = S]
\]

\[
= \sum_{S \subseteq \Lambda} \sum_{\partial n_1 = \emptyset \atop \text{s.t. } y \in S} w(n_1) w(n_2) \langle \sigma_0 \rangle_{\tilde{S}, \beta} \mathbb{I}[\mathcal{F} = S].
\]

The third line follows from the fact that since \( \mathcal{F}_g = S \), the currents \( n_1 \) and \( n_2 \) can be decomposed as \( n_\# = n_\#^S + n_\#^{\Lambda \setminus S} \) as we did before for \( \mathcal{F}_0 = S \). We can therefore use the switching lemma to \( n_\#^S \) and \( n_\#^S \). This corresponds to the measure \( \langle \cdot \rangle_{\tilde{S}, \beta} \) (we may use the switching lemma for free boundary condition simply because the measure can be thought of as the limit of \( \langle \cdot \rangle_{\tilde{S}, \beta} \) as \( \Lambda \nearrow V \) as explained above).
By plugging the inequality above in (2.5), we find
\[
\frac{d}{d\beta} \left( (\sigma_0)_{\Lambda,\beta}^+ \right)^2 = 2(\sigma_0)_{\Lambda,\beta}^+ \frac{d}{d\beta} (\sigma_0)_{\Lambda,\beta}^+
\]
\[
= \frac{2}{Z_{\Lambda,\beta}} \sum_{S \subset \Lambda} \sum_{x \in S} \sum_{y \in S} \sum_{\partial n_1 = \emptyset} \sum_{\partial n_2 = \emptyset} w(n_1) w(n_2) J_{x,y}(\sigma_0 \sigma_x)^f_{S,\beta} \mathbb{I}[S_g = S]
\]
\[
\geq \frac{2}{\beta} \cdot \frac{1}{Z_{\Lambda,\beta}} \cdot \sum_{S \subset \Lambda} \sum_{\partial n_1 = \emptyset} \sum_{\partial n_2 = \emptyset} w(n_1) w(n_2) \mathbb{I}[S_g = S] \geq \frac{2}{\beta} \cdot \inf_{S \subset \Lambda} \varphi_\beta(S) \cdot \sum_{\partial n_1 = \emptyset} \sum_{\partial n_2 = \emptyset} w(n_1) w(n_2)
\]
\[
= \frac{2}{\beta} \cdot \inf_{S \subset \Lambda} \varphi_\beta(S) \cdot (1 - \langle \sigma_0 \rangle_{\Lambda,\beta}^2).
\]

In the first inequality, we used that $J_{x,y} \geq \frac{1}{\beta} \tanh(\beta J_{x,y})$. In the last line, we used the switching lemma and (2.2) one more time. Recall that we are working in $\Lambda$ and therefore $y \notin S$ really means $y \in \Lambda \cup \{g\} \setminus S$. Nevertheless in this context we have
\[
\sum_{y \in S} J_{x,y} := \sum_{y \in \Lambda \setminus S} J_{x,y} + J_{x,g} = \sum_{y \in V \setminus S} J_{x,y},
\]
which enables us to claim that
\[
\sum_{x \in S} \sum_{y \in S} J_{x,y}(\sigma_0 \sigma_x)^f_{S,\beta} = \varphi_\beta(S).
\]

**Remark 2.2.** Inequality (2.6) is reminiscent of (1.7). Indeed, one can consider a measure $P_\beta$ on sourceless currents attributing a probability proportional to $w_\beta(n)$ to the current $n$. Then, (2.6) can be rewritten as
\[
\frac{d}{d\beta} \left( (\sigma_0)_{\Lambda,\beta}^+ \right)^2 \geq \frac{2}{\beta} \sum_{S \subset \Lambda} \varphi_\beta(S) P_\beta \otimes P_\beta(S_g = S).
\]
Interpreted like that, the sum of two independent sourceless currents plays the role of the percolation configuration.

### 2.5 Proof of Items 2 and 3

In this section, we show that Items 2 and 3 in Theorem 2.1 hold with $\tilde{\beta}_c$ in place of $\beta_c$. 
We need a replacement for the BK inequality used in the case of Bernoulli percolation. The relevant tool for the Ising model will be an improved version of Simon's inequality, which we state as follows (see [Sim80] for the original inequality and [Lie80] for an improvement which is not quite sufficiently strong for the application that we have in mind).

**Lemma 2.5** (Improved Simon’s inequality). Let $S$ be a subset of $V$ containing 0. For every $z \not\in S$,

$$\langle \sigma_0 \sigma_z \rangle_\beta^+ \leq \sum_{x \in S} \sum_{y \not\in S} \tanh(\beta J_{x,y}) \langle \sigma_0 \sigma_x \rangle_{S,\beta}^+ \langle \sigma_y \sigma_z \rangle_{\beta}^+.$$ 

**Proof.** Fix $\Lambda$ a finite subset of $V$ containing $S$. We consider the backbone representation of the Ising model on $\Lambda$ defined in the previous section. Let $\omega = (v_k)_{0 \leq k < K}$ be a backbone from 0 to $z$. Since $z \not\in S$, one can define the first $k$ such that $v_k+1 \not\in S$. We obtain that the following occur:

- $\omega$ goes from 0 to $v_k$ staying in $S$,
- then $\omega$ goes through $\{v_k, v_k+1\}$,
- finally $\omega$ goes from $v_k+1$ to $z$ in $\Lambda$.

We find

$$\langle \sigma_0 \sigma_z \rangle_{\Lambda,\beta}^+ = \sum_{\omega=\{0,z\}} \rho_{\Lambda}^\beta(\omega)$$

$$\leq \sum_{x \in S} \sum_{y \not\in S} \sum_{\omega_1=\{0,x\}} \sum_{\omega_2=\{x,y\}} \sum_{\omega_3=\{y,z\}} \rho_{S}^\beta(\omega_1) \rho_{\{x,y\}}^\beta(\omega_2) \rho_{S,\beta}^\Lambda(\omega_3)$$

$$\leq \sum_{x \in S} \sum_{y \not\in S} \sum_{\omega_1=\{0,x\}} \sum_{\omega_2=\{x,y\}} \sum_{\omega_3=\{y,z\}} \langle \sigma_0 \sigma_x \rangle_{S,\beta}^+ \rho_{\{x,y\}}^\Lambda(\{x,y\}) \langle \sigma_y \sigma_z \rangle_{\Lambda,\beta}^+.$$ 

The first and third lines are based on the first property of backbones. The second line follows from the second and third properties of the backbone as well as Griffiths’ inequality. The proof follows by taking $\Lambda$ to infinity and by observing that

$$\rho_{\{x,y\}}^\Lambda(\{x,y\}) \to \frac{\sinh(\beta J_{x,y})}{\cosh(\beta J_{x,y})} = \tanh(\beta J_{x,y}).$$

We are now in a position to conclude the proof. Let $\beta < \tilde{\beta}_c$. Fix a finite set $S$ such that $\varphi_\beta(S) < 1$. Define

$$\chi_n(\beta) := \max \left\{ \sum_{y \in \Lambda_n} \langle \sigma_x \sigma_y \rangle_{\beta}^+ : x \in \Lambda_n \right\}.$$ 

Using the same reasoning as for percolation, we find

$$\chi_n(\beta) \leq |S| + \left( \sum_{x \in S} \sum_{y \not\in S} \tanh(\beta J_{x,y}) \langle \sigma_0 \sigma_x \rangle_{S,\beta}^f \right) \chi_n(\beta) \leq |S| + \varphi_\beta(S) \chi_n(\beta).$$

Letting $n \to \infty$, we obtain the second item.
We finish by the proof of the third item. Let $R$ be the range of the $(J_{x,y})_{x,y \in V}$, and let $L$ be such that $S \subset \Lambda_{L-R}$. Lemma 2.5 implies that for any $z$ with $d(0,z) \geq n > L$,

$$\langle \sigma_0 \sigma_z \rangle^+_\beta \leq \sum_{x \in S} \sum_{y \in S} \tanh(\beta J_{x,y}) \langle \sigma_0 \sigma_x \rangle^+_S \langle \sigma_y \sigma_z \rangle^+_\beta \leq \varphi_\beta(S) \max_{y \in \Lambda_L} \langle \sigma_y \sigma_z \rangle^+_\beta.$$ 

The proof follows by letting $\Lambda \nearrow V$ and by iterating $\lfloor n/L \rfloor$ times.

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