OPERATOR ESTIMATES FOR HOMOGENIZATION OF THE ROBIN LAPLACIAN IN A PERFORATED DOMAIN

ANDRII KHRABUSTOVSKYI¹,² AND MICHAEL PLUM³

ABSTRACT. Let $\varepsilon > 0$ be a small parameter. We consider the domain $\Omega_\varepsilon := \Omega \setminus D_\varepsilon$, where $\Omega$ is an open domain in $\mathbb{R}^n$, and $D_\varepsilon$ is a family of small balls of the radius $d_\varepsilon = o(\varepsilon)$ distributed periodically with period $\varepsilon$. Let $\Delta_\varepsilon$ be the Laplace operator in $\Omega_\varepsilon$ subject to the Robin condition $\frac{\partial u}{\partial n} + \gamma_\varepsilon u = 0$ on the boundary of the holes and the Dirichlet condition on the exterior boundary. Kaizu (1985, 1989) and Brillard (1988) have shown that, under appropriate assumptions on $d_\varepsilon$ and $\gamma_\varepsilon$, the operator $\Delta_\varepsilon$ converges in the strong resolvent sense to the sum of the Dirichlet Laplacian in $\Omega$ and a constant potential. We improve this result deriving estimates on the rate of convergence in terms of $L^2 \rightarrow L^2$ and $L^2 \rightarrow H^1$ operator norms. As a byproduct we establish the estimate on the distance between the spectra of the associated operators.

1. INTRODUCTION

We revisit one of the classical problems in homogenization theory – homogenization of the Laplacian in a domain with a lot of tiny holes. For the Dirichlet Laplacian it is also known as crushed ice problem. Here we focus on holes with the Robin boundary conditions. In the introduction we recall the setting of the problem along with some important known results, and then sketch the main outcomes of the present work.

1.1. Homogenization of Robin Laplacian in perforated domains. Let $\varepsilon > 0$ be a small parameter. We consider the following perforated domain:

$$\Omega_\varepsilon := \Omega \setminus \left( \bigcup_i D_{i,\varepsilon} \right).$$

Here $\Omega$ is a fixed open domain in $\mathbb{R}^n$ ($n \geq 2$), $D_{i,\varepsilon} \subseteq \Omega$ are identical open balls of the radius $d_\varepsilon$ distributed evenly in $\Omega$ along an $\varepsilon$-periodic lattice. The domain $\Omega_\varepsilon$ is given in Figure 1. A more accurate description of $\Omega_\varepsilon$ is postponed to the next section. In the following, we refer to the sets $D_{i,\varepsilon}$ as holes.

In the present paper we focus on the case of a low concentration of the holes, namely

$$\Lambda_\varepsilon := \frac{d_\varepsilon}{\varepsilon} \to 0 \text{ as } \varepsilon \to 0. \quad (1.1)$$

1 Department of Physics, Faculty of Science, University of Hradec Králové, Czech Republic
2 Department of Theoretical Physics, Nuclear Physics Institute of the Czech Academy of Sciences, Řež, Czech Republic
3 Institute of Analysis, Faculty of Mathematics, Karlsruhe Institute of Technology, Germany
E-mail addresses: andrii.khrabustovskyi@uhk.cz, michael.plum@kit.edu.
2010 Mathematics Subject Classification. 35B27, 35B40, 35P05, 47A55.
Key words and phrases. homogenization; perforated domain; norm resolvent convergence; operator estimates; spectral convergence; varying Hilbert spaces.
We consider the following boundary-value problem in $\Omega_\varepsilon$:

\[
\begin{aligned}
-\Delta u_\varepsilon + u_\varepsilon &= f \quad \text{in } \Omega_\varepsilon, \\
\frac{\partial u_\varepsilon}{\partial n} + \gamma_\varepsilon u_\varepsilon &= 0 \quad \text{on } \cup_i \partial D_{i,\varepsilon}, \\
u_\varepsilon &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]  

(1.2)

Here $f \in L^2(\Omega)$ is a given function, $\frac{\partial}{\partial n}$ stands for the derivative along the exterior unit normal to the boundary $\partial \Omega_\varepsilon$, and the constant $\gamma_\varepsilon$ satisfies $0 \leq \gamma_\varepsilon < \infty$. Homogenization theory is aimed to describe the behavior of the solution $u_\varepsilon$ to this problem as $\varepsilon \to 0$.

In was demonstrated in [11, 33] that the asymptotic behavior of $u_\varepsilon$ is determined by the asymptotic behaviour of two quantities $P_\varepsilon \geq 0$ and $Q_\varepsilon > 0$ given by

\[
P_\varepsilon := \kappa_n \frac{\gamma_\varepsilon d_\varepsilon^{n-1}}{\varepsilon^n}, \quad Q_\varepsilon := \begin{cases} (n-2)\kappa_n \frac{d_\varepsilon^{n-2}}{\varepsilon^n}, & n \geq 3, \\
\frac{2\pi}{\ln d_\varepsilon |\varepsilon|}, & n = 2, \end{cases}
\]

(1.3)

with $\kappa_n$ standing for the surface area of the unit sphere in $\mathbb{R}^n$. We define

\[
P := \lim_{\varepsilon \to 0} P_\varepsilon, \quad Q := \lim_{\varepsilon \to 0} Q_\varepsilon, \quad P, Q \in [0, \infty]
\]

(1.4)

(here we assume that both limits in (1.4), either finite or not, exist). Two cases are possible [11,33]: if $P = \infty$ and $Q = \infty$, then

\[
\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0,
\]

(1.5)

otherwise, if $P < \infty$ or $Q < \infty$, then

\[
\|u_\varepsilon - u\|_{L^2(\Omega_\varepsilon)} \to 0 \quad \text{as } \varepsilon \to 0,
\]

(1.6)

where $u$ is the solution to the problem

\[
\begin{aligned}
-\Delta u + Vu + u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(1.7)
with the constant potential $V$ defined by
\[
V := \begin{cases} 
  P, & P > 0 \quad \text{and} \quad Q = \infty, \\
  Q, & P = \infty \quad \text{and} \quad Q > 0, \\
  PQ(P + Q)^{-1}, & P > 0 \quad \text{and} \quad Q > 0, \\
  0, & P = 0 \quad \text{or} \quad Q = 0.
\end{cases}
\]

It is easy to see that
\[
V = \lim_{\varepsilon \to 0} V_{\varepsilon}, \quad \text{where} \quad V_{\varepsilon} := \frac{P_{\varepsilon}Q_{\varepsilon}}{P_{\varepsilon} + Q_{\varepsilon}}.
\]

Example 1.1. Let $n \geq 3$. Let $d_{\varepsilon} = \varepsilon^s$ with $s > 1$ (this restriction is caused by the assumption (1.1)), $\gamma_{\varepsilon} = \varepsilon^t$ with $t \in \mathbb{R}$. Then $P_{\varepsilon} = \gamma_{\varepsilon} \varepsilon^{s+(n-1)s-n}$ and $Q_{\varepsilon} = (n-2)\gamma_{\varepsilon} \varepsilon^{(n-2)s-n}$. In Figure 2 we sketch five subsets of the set of admissible parameters $\{(s, t) \in \mathbb{R}^2 : s > 1\}$. If $(s, t)$ belongs to the dark gray area (respectively, the light gray area), then (1.5) holds (respectively, (1.6)--(1.7) with $V = 0$ hold). If $(s, t)$ is on the dashed bold open interval (respectively, on the solid bold open ray), one has (1.6)--(1.7) with $V = \gamma_{\varepsilon}$ (respectively, with $V = (n-2)\gamma_{\varepsilon}$). Finally, we have (1.6)--(1.7) with $V = \gamma_{\varepsilon}^{n-2}$ provided $s = -t = \frac{n}{n-2}$. 

![Figure 2. Example 1.1](image)

For the case ($\gamma_{\varepsilon} = \gamma$ and $P > 0$) the above result was obtained by Kaizu in [30]; in Figure 2 it corresponds to the point ($\frac{1}{n}, 0$). The case ($Q > 0$ and $P > 0$) was studied in [31, 32] for $n = 3$. All other cases were investigated by Brillard in [11] and, using another methods, by Kaizu in [33]. Actually, the articles [11, 32, 33] dealt with the holes of the form $D_{i,\varepsilon} \cong d_{\varepsilon} D$, where the set $D$ may have an arbitrary shape (i.e., it is not necessary a ball). As for the ball-shaped holes, $u_{\varepsilon}$ converges to zero if $P = Q = \infty$; otherwise, if $P < \infty$ or $Q < \infty$, $u_{\varepsilon}$ converges to the solution $u$ of the boundary value problem (1.7) with an appropriately modified constant potential $V$. Namely, if $P > 0$ and $Q = \infty$, then $V = |\partial D| \gamma_n^{-1} P$, where $|\partial D|$ is the surface area of $D$; if $P = \infty$ and $Q > 0$, then $V = \text{cap}(D) \gamma_n^{-1} (n-2)^{-1} Q$ if $n \geq 3$, where $\text{cap}(D)$ is the Newton capacity of $D$ (see, e.g., [51] for its definition), and $V = Q$ if $n = 2$; if $P > 0$ and $Q > 0$, then $V$ is the minimum of some capacity-type functional depending on $P, Q$ and $D$ (cf. [11] Proposition 3.3)]; if either $P = 0$ or $Q = 0$, then $V = 0$ as before. In the current work, in order to make the presentation simpler, we restrict ourselves to the ball-shaped holes.
The same homogenization problem with non-periodically distributed holes or/and with a non-constant $\gamma_\varepsilon$ was studied, e.g., in [3,20,34]. Qualitatively, the main result in these papers remains the same as in [11,33], but the potential $V$ in (1.7) may be non-constant.

Earlier, Marchenko and Khruslov [41], and Cioranescu and Murat [16] considered a similar problem for $\gamma_\varepsilon = \infty$ which corresponds to Dirichlet conditions $u_\varepsilon = 0$ on $\partial D_\varepsilon$. In these articles quite general shapes and distributions of holes are allowed. Notably, the result is similar to the one for Robin conditions, and for identical ball-shaped holes distributed $\varepsilon$-periodically it reads as follows: if $Q = \infty$, then (1.5) holds, otherwise one has (1.6)–(1.7) with $V = Q$. The cases $Q = 0$ and $Q = \infty$ were also considered in [48], moreover, the authors investigated a similar problem for randomly distributed holes under assumptions resembling $Q > 0$.

Despite in the present paper we deal with holes which are asymptotically smaller than the period (see (1.1)), in this introduction we would like to devote some attention to the case when they are of the same order. Namely, let $d_\varepsilon = \varepsilon r$ with $r \in (0, \frac{1}{4})$. We define $P, Q$ as in (1.4), (1.3); clearly, $Q = \infty$. Like in the case (1.1), $u_\varepsilon$ converges to zero (cf. (1.5)) provided $P = \infty$. Otherwise, if $P < \infty$, $u_\varepsilon$ converges in the sense (1.6) to $u \in H^1_0(\Omega)$ satisfying

$$-\text{div}(A\nabla u) + Pu + Bu = Bf.$$  

Here $B > 0$ is the measure of the set $\Box \setminus \overline{B}_{\varepsilon r}$, where $\Box$ is the unit cube, and $B_r$ is the ball of radius $r$, both having the same center; $A \gg 0$ is a constant matrix, whose entries are calculated by solving a certain boundary value problem on $\Box \setminus \overline{B}_{\varepsilon r}$. For $\gamma_\varepsilon = 0$ (Neumann holes) this result was obtained in [17], for $\gamma_\varepsilon > 0$ (Robin holes) – in [11,33]. Similar results for more general shapes and distributions of holes can be found, e.g., in [3,39].

It worth to mention that in [33] (see also the earlier articles [31,32]) not only linear, but also non-linear Robin boundary conditions $\frac{\partial u_\varepsilon}{\partial n} + \gamma_\varepsilon g(x,u_\varepsilon) = 0$ were treated. In this case a nonlinear term $V(u)$ appears in the limiting equation. After [33], a huge number of articles concerning non-linear Robin problems in perforated domains appeared. Not pretending to present an exhaustive overview of these articles here, we refer only to some of them – [19,22,24,25,43], more results and references can be found in the recent monograph [18].

Finally, one can also study a surface distribution of holes, i.e., holes being located near some hypersurface $\Gamma$ intersecting $\Omega$. For the first time this problem was considered in [41] for Dirichlet holes. Robin holes case was treated in [40]; the complete analysis including non-linear case can be found in [22]. The result reads as follows: the solution $u_\varepsilon$ to the problem (1.2) converges in $L^2(\Omega_\varepsilon)$ to the function $u$ (i.e., (1.6) holds), where $u \in H^1_0(\Omega)$ satisfies $-\Delta u + u = f$ in $\Omega \setminus \Gamma$ and certain interface conditions on $\Gamma$.

1.2. Main results. In all mentioned above papers concerning homogenization of the linear boundary value problems in domains with a lot of tiny holes (cf. [3,11,16,20,22,30,33,34,39,41]) the convergence result was established for the fixed right-hand-side $f \in L^2(\Omega)$. In the language of operator theory this means that one has strong resolvent convergence of the differential operators associated with the boundary value problems (1.2) and (1.7).\footnote{Strictly speaking, we are not able to treat the classical resolvent convergence, since the operators act in different Hilbert spaces $L^2(\Omega)$ and $L^2(\Omega)$. Nevertheless, using the operator $I_\varepsilon f := f |_{\Omega_\varepsilon}$, one gets its natural analogue for varying domains $\Omega_\varepsilon \subset \Omega$, see Section 2.} In the current work we improve (1.5)–(1.6), by proving the uniform convergence with respect to $f \in \{g \in L^2(\Omega) : \|g\|_{L^2(\Omega)} = 1\}$. In another words, we establish norm resolvent convergence...
of the underlying operators. Moreover, we estimate the rate of this convergence (the so-called operator estimates).

Recall that \( u_\epsilon \) and \( u \) stand for the solutions to the problems (1.2) and (1.7), respectively. Our first result reads as follows. Let \( P < \infty \) or \( Q < \infty \). Then one has the estimate
\[
\|u_\epsilon - u\|_{L^2(\Omega_\epsilon)} \leq C\eta_\epsilon \|f\|_{L^2(\Omega)}, \quad \eta_\epsilon \to 0 \text{ as } \epsilon \to 0.
\] (1.9)
Here \( C \) is a positive constant independent of \( \epsilon \), and \( \eta_\epsilon \) is given below in (2.8). As a byproduct of (1.9), we obtain the estimate for the distance between the spectra of the underlying operators.

Besides the convergence in the \( L^2(\Omega) \to L^2(\Omega_\epsilon) \) operator norm, in some cases (see (1.12)) we also prove the convergence in the \( L^2(\Omega) \to H^1(\Omega_\epsilon) \) operator norm. Namely, we derive the estimate
\[
\|u_\epsilon - u\|_{H^1(\Omega_\epsilon)} \leq C\eta_\epsilon' \|f\|_{L^2(\Omega)}, \quad \text{where } \eta_\epsilon' := \max\left\{V_\epsilon Q_\epsilon^{-1/2}; \eta_\epsilon\right\}
\] (1.10)
(recall that \( Q_\epsilon \) and \( V_\epsilon \) are given in (1.3) and (1.8), respectively). Evidently, one has
\[
V_\epsilon Q_\epsilon^{-1/2} \leq \min\{Q_\epsilon^{1/2}; P_\epsilon Q_\epsilon^{-1/2}\},
\] (1.11)
whence the right-hand-side of (1.10) converges to zero provided either
\[
(P < \infty \land Q = \infty) \quad \text{or} \quad (P = 0 \land Q > 0) \quad \text{or} \quad Q = 0.
\] (1.12)

To get a reasonable estimate in the remaining case \( (P \neq 0 \land Q > 0) \) we need a special corrector; the corresponding result reads as follows:
\[
\|u_\epsilon - (1 + G_\epsilon)u\|_{H^1(\Omega_\epsilon)} \leq C\tilde{\eta}_\epsilon \|f\|_{L^2(\Omega)}, \quad \tilde{\eta}_\epsilon \to 0.
\]

Here \( G_\epsilon \) is a smooth function given in (2.12), while \( \tilde{\eta}_\epsilon \) is defined in (2.16).

In the last section we also discuss possible improvements of (1.9)–(1.10) for \( P, Q \) satisfying
\[
(P < \infty \land Q = \infty) \quad \text{or} \quad (P = 0 \land Q > 0).
\] (1.13)

Finally, we derive the estimate
\[
\|u_\epsilon\|_{L^2(\Omega_\epsilon)} \leq C\max\left\{P_\epsilon^{-1}; Q_\epsilon^{-1}; \epsilon^2\right\} \|f\|_{L^2(\Omega)},
\] (1.14)
giving the required convergence result if \( P = Q = \infty \).

The main results are collected in Theorems 2.1, 2.2, 2.4, 2.7, 2.8 and 7.1. We formulate them in operator terms. Our proofs (except the one of (1.14)) rely on the abstract results from [2,38,45] concerning the resolvent and spectral convergence in varying Hilbert spaces.

1.3. Previous works on operator estimates in homogenization. Operator estimates in homogenization theory is a rather young topic. It was initiated by Birman and Suslina [4,5], Griso [26,27], Zhikov and Pastukhova [52,54] (see also the overview [55]) for the classical homogenization problem concerning elliptic operators of the form \( \text{div}(A(\cdot) \nabla) \), where \( A(\cdot) \) is a \( Z^n \)-periodic function.

For Dirichlet holes \( (\gamma_\epsilon = \infty) \) an operator estimate was established by the first author and Post in [37] under the assumption \( Q < \infty \). In a subsequent work Anné and Post [2] derived operator estimates for the Laplace-Beltrami operator on a Riemannian manifold with Dirichlet holes satisfying the conditions resembling either the case \( Q = 0 \) or the case \( Q = \infty \), and for Neumann holes satisfying (a kind of) condition (1.1). For the Neumann Laplacian in periodically perforated domains with holes of the same smallness order as
the period operator estimates were obtained in \[50, 53\]. We also mention the work \[14\], where the norm resolvent convergence without estimates on its rate was demonstrated for three particular cases: Neumann holes \((\gamma_\varepsilon = 0)\) satisfying (1.1), Dirichlet holes satisfying \(Q_\varepsilon > 0\), and Robin holes with \(\gamma_\varepsilon = \gamma, \, d_\varepsilon = \varepsilon^{n/(n-1)}\).

For the surface distribution of holes (see the end of the previous subsection) some operator estimates were derived in \[7, 23\]. In \[23\] operator estimates were obtained for the Robin Laplacian in a bounded domain \(\Omega \subset \mathbb{R}^3\) with a lot of small holes located in a neighborhood of a plane intersecting \(\Omega\); we note that the restrictions on \(\Omega\) (boundedness) and on the dimension \((n = 3)\) played an important role in the proofs in \[23\]. In \[7\] operator estimates were derived for elliptic operators posed in a two-dimensional domain with a lot of small holes located in a neighborhood of a curve; various boundary conditions were treated, in particular, the Robin conditions with \(\varepsilon\)-independent coupling constant \(\gamma\).

Finally, we mention the closely related papers \[6\] and \[8\], where operator estimates were deduced, respectively, for elliptic operators with frequently alternating boundary conditions and for boundary value problems in domains with oscillating boundary.

Let us stress that, in contrast to \[11, 33\], in this paper we do not assume that \(\Omega\) is bounded. We also have no restrictions on the dimension \(n\) as in \[7, 23\].

2. Setting of the problem and main results

Let \(\Omega\) be an open domain in \(\mathbb{R}^n, \, n \geq 2\). We assume that \(\partial \Omega\) is \(C^2\) smooth, moreover, if \(\Omega\) is unbounded, we additionally require it to be uniformly regular of class \(C^2\) in the sense of Browder \[12, \text{Definition 1}\]. The latter is automatically fulfilled, for example, for domains with compact smooth boundaries or for compact smooth perturbations of half-spaces. This requirement is merely technical, and is needed solely to enjoy the global \(H^2\)-regularity of solutions to the homogenized problem. If \(\Omega\) is bounded, our results remain valid under less restrictive assumptions on \(\partial \Omega\), see Remark 5.3 at the end of Section 5.

Let \(\varepsilon\) be a positive parameter. We assume that \(\varepsilon\) is small enough, namely \(\varepsilon \in (0, \varepsilon_0]\), where \(\varepsilon_0 \in (0, 1)\) will be specified later; see (2.4)–(2.6). We denote

\[
\square := (-1/2, 1/2)^n, \quad \square_{i, \varepsilon} := \varepsilon (\square + i), \quad i \in \mathbb{Z}^n, \quad \mathcal{I}_\varepsilon := \{i \in \mathbb{Z}^n : \square_{i, \varepsilon} \subset \Omega\},
\]

i.e., \(\mathcal{I}_\varepsilon\) is the set of those indices \(i \in \mathbb{Z}^n\) for which the corresponding cell \(\square_{i, \varepsilon}\) lies entirely in \(\Omega\). For \(i \in \mathcal{I}_\varepsilon\) we define \(x_{i, \varepsilon} := i \varepsilon\) and

\[
D_{i, \varepsilon} := \{x \in \mathbb{R}^n : |x - x_{i, \varepsilon}| < d_\varepsilon\},
\]

where \(d_\varepsilon \in (0, \varepsilon/2)\). Finally, we set

\[
\Omega_\varepsilon := \Omega \setminus \left( \bigcup_{i \in \mathcal{I}_\varepsilon} D_{i, \varepsilon} \right).
\]

The domain \(\Omega_\varepsilon\) is depicted on Figure 1.

Next we describe the family of operators \(A_\varepsilon\) which will be the main object of our interest. Let \(\gamma_\varepsilon \geq 0\). We introduce the sesquilinear form \(a_\varepsilon\) in the Hilbert space \(L^2(\Omega_\varepsilon)\) by

\[
a_\varepsilon[u, v] = \int_{\Omega_\varepsilon} \nabla u \cdot \overline{\nabla v} \, dx + \sum_{i \in \mathcal{I}_\varepsilon} \gamma_\varepsilon \int_{\partial D_{i, \varepsilon}} u \overline{v} \, ds, \quad \text{dom}(a_\varepsilon) = \left\{ u \in H^1(\Omega_\varepsilon) : u \mid_{\partial \Omega} = 0 \right\}. \quad (2.1)
\]
where \(ds\) is the surface measure on \(\partial \Omega_\varepsilon\). The second term in (2.1) is indeed finite for \(u \in H^1(\Omega_\varepsilon)\) which follows easily from the trace inequality
\[
\|u\|_{L^2(\partial \Omega_\varepsilon)} \leq C_\varepsilon \|u\|_{H^1(\Omega_\varepsilon \setminus \overline{\Omega}_\varepsilon)}, \quad u \in H^1(\Omega_\varepsilon),
\]
where the constant \(C_\varepsilon > 0\) is independent of \(\varepsilon \in I_\varepsilon\). Furthermore, it is also straightforward to check that the form \(a_\varepsilon[\cdot, \cdot]\) is symmetric, densely defined, closed, and positive. Then, by the first representation theorem \([35, \text{Chapter 6, Theorem 2.1}]\), there exists the unique self-adjoint and positive operator \(A_\varepsilon\) associated with \(a_\varepsilon\), i.e., \(\text{dom}(A_\varepsilon) \subset \text{dom}(a_\varepsilon)\) and
\[
(A_\varepsilon u, v)_{L^2(\Omega_\varepsilon)} = a_\varepsilon[u, v], \quad \forall u \in \text{dom}(A_\varepsilon), \forall v \in \text{dom}(a_\varepsilon).
\]

The operator \(A_\varepsilon\) is the Laplace operator in \(\Omega_\varepsilon\) subject to the Dirichlet conditions on \(\partial \Omega_\varepsilon\) and the Robin conditions \(\varepsilon \partial_\nu u + \gamma_\varepsilon u = 0\) on the boundary of the holes \(\Omega_{\varepsilon,i}, i \in I_\varepsilon\). The solution to the boundary value problem (1.2) is given by \(u_\varepsilon = (A_\varepsilon + I)^{-1}f_\varepsilon\), where \(f_\varepsilon := f \mid_{\Omega_\varepsilon}\).

Recall that \(P := \lim_{\varepsilon \to 0} P_\varepsilon, Q := \lim_{\varepsilon \to 0} Q_\varepsilon, V := \lim_{\varepsilon \to 0} V_\varepsilon\), where \(P_\varepsilon, Q_\varepsilon\) are defined in (1.3), and \(V_\varepsilon\) is given in (1.8).

**We start from the case when either** \(P < \infty\) \(\text{or} Q < \infty\). **At first, we introduce the limiting operator** \(A\). Let \(a\) be a sesquilinear form in \(L^2(\Omega)\) given by
\[
a[u, v] = \int_\Omega \nabla u \cdot \nabla v \, dx + V \int_\Omega u \sigma \, dx, \quad \text{dom}(a) = H^1_0(\Omega). \tag{2.2}
\]

We denote by \(A\) the self-adjoint operator associated with this form. Evidently,
\[
A = -\Delta_\Omega + V,
\]
where \(\Delta_\Omega\) is the Dirichlet Laplacian on \(\Omega\). Standard theory of elliptic PDEs (see, e.g., [21]) yields \(\text{dom}(A) \subset H^2_{\text{loc}}(\Omega)\). Furthermore, due to the uniform regularity of \(\Omega\) \([12]\), one has a global \(H^2\)-regularity of functions from \(\text{dom}(A)\), namely, \(\text{dom}(A) = H^2(\Omega) \cap H^1_0(\Omega)\).

By \(J_\varepsilon\) we denote the operator of restriction to \(\Omega_\varepsilon\), i.e.,
\[
J_\varepsilon : L^2(\Omega) \to L^2(\Omega_\varepsilon), \quad J_\varepsilon f := f \mid_{\Omega_\varepsilon}. \tag{2.3}
\]

Note that \(J_\varepsilon\) can also be regarded as an operator from \(H^1(\Omega)\) to \(H^1(\Omega_\varepsilon)\).

**Let us specify the range for the small parameter** \(\varepsilon\). In view of (1.1), (1.4) and the equality \(|\ln d_\varepsilon|^{-1} = (2\pi)^{-1}Q_\varepsilon\varepsilon^2\), which holds for \(n = 2\), there exists \(\varepsilon_0 \in (0, 1)\) such that
\[
\sup_{\varepsilon \in (0, \varepsilon_0]} \Lambda_\varepsilon \leq 1/4, \tag{2.4}
\]
\[
\begin{align*}
\sup_{\varepsilon \in (0, \varepsilon_0]} P_\varepsilon &< \infty, \quad \text{provided} \ P < \infty, \\
\sup_{\varepsilon \in (0, \varepsilon_0]} Q_\varepsilon &< \infty, \quad \text{provided} \ Q < \infty, \\
\sup_{\varepsilon \in (0, \varepsilon_0]} P_\varepsilon^{-1} &< \infty, \quad \text{provided} \ P = \infty, \\
\sup_{\varepsilon \in (0, \varepsilon_0]} Q_\varepsilon^{-1} &< \infty, \quad \text{provided} \ Q = \infty, \\
\inf_{\varepsilon \in (0, \varepsilon_0]} Q_\varepsilon &> 0, \quad \text{provided} \ Q > 0,
\end{align*}
\tag{2.5}
\]
\[
\sup_{\varepsilon \in (0, \varepsilon_0]} \frac{|\ln \varepsilon|}{|\ln d_\varepsilon|} \leq 1/2, \quad \text{provided} \ n = 2 \text{ and } Q > 0. \tag{2.6}
\]

**In the following we always assume that**
\[
0 < \varepsilon \leq \varepsilon_0.
\]
Note that \( V_\epsilon \leq \min\{P; Q_\epsilon\} \), whence (2.5) implies
\[
\sup_{(0,\epsilon_0)} V_\epsilon < \infty \text{ if } P < \infty \text{ or } Q < \infty. \tag{2.7}
\]

Finally, we define
\[
\eta_\epsilon := \begin{cases} \max \{|V_\epsilon - V|; \epsilon; \Lambda_\epsilon\}, & n \geq 5, \\
\max \{|V_\epsilon - V|; \epsilon; \Lambda_\epsilon \ln \Lambda_\epsilon|\}, & n = 4, \\
\max \{|V_\epsilon - V|; \epsilon; \Lambda_\epsilon^{1/2}\}, & n = 3, \\
\max \{|V_\epsilon - V|; \epsilon \ln \epsilon|; |\ln \Lambda_\epsilon|^{-1/2}\}, & n = 2,
\end{cases} \tag{2.8}
\]
where \( \Lambda_\epsilon \) is given in (1.1). Due to (1.1) and (1.8), \( \eta_\epsilon \to 0 \) as \( \epsilon \to 0 \).

We are now in position to formulate the main results of this work. In the following the notation \( \| \cdot \|_{X \to Y} \) stands for the norm of a bounded linear operator acting between Hilbert spaces \( X \) and \( Y \). By \( C \) we denote generic positive constants being independent of \( \epsilon \) (but it may depend on \( n, \Omega, \epsilon_0 \), and on the values of the suprema and the infimum in (2.5)). We denote by \( I \) the identity operator, either in \( L^2(\Omega) \) or in \( L^2(\Omega_\epsilon) \).

**Theorem 2.1.** Let \( P < \infty \) or \( Q < \infty \). Then one has
\[
\left\| (A_\epsilon + I)^{-1} f_\epsilon - f_\epsilon (A + I)^{-1} \right\|_{L^2(\Omega) \to L^2(\Omega_\epsilon)} \leq C \eta_\epsilon. \tag{2.9}
\]

The next result shows that in some cases (see Remark 2.3 below), the resolvents of \( A_\epsilon \) and \( A \) are close in the \( (H^1 \to L^2) \) operator norm. We define
\[
\eta_\epsilon' := \max \left\{ V_\epsilon Q_\epsilon^{-1/2}; \eta_\epsilon \right\}, \tag{2.10}
\]
where \( \eta_\epsilon \) is given in (2.8), \( Q_\epsilon \) and \( V_\epsilon \) are defined in (1.3) and (1.8), respectively.

**Theorem 2.2.** Let \( P < \infty \) or \( Q < \infty \). Then one has
\[
\left\| (A_\epsilon + I)^{-1} f_\epsilon - f_\epsilon (A + I)^{-1} \right\|_{L^2(\Omega) \to H^1(\Omega_\epsilon)} \leq C \eta_\epsilon'. \tag{2.11}
\]

**Remark 2.3.** For \( P, Q \) satisfying (1.12) the right-hand-side in (2.11) tends to zero which follows immediately from the inequality (1.11). However, in the remaining case \( P \neq 0 \land Q > 0 \) the estimate (2.11) is pointless. To get a reasonable result for this case, one needs a special corrector; see Theorem 2.4 below.

We define the function \( G_\epsilon \in C^\infty(\Omega_\epsilon) \) by
\[
G_\epsilon(x) := \sum_{i \in \mathbb{Z}^n_\epsilon} G_{i,\epsilon}(x) \phi_{i,\epsilon}(x), \quad x \in \Omega_\epsilon. \tag{2.12}
\]
Here \( G_{i,\epsilon} \) is the fundamental solution to the Laplace equation in \( \mathbb{R}^n \) times the constant \( V_\epsilon \epsilon^n \):
\[
G_{i,\epsilon}(x) := V_\epsilon \epsilon^n \begin{cases} \frac{1}{x_\epsilon(n-2)|x-x_\epsilon|^{n-2}}, & n \geq 3, \\
\frac{1}{2\pi} \ln |x-x_\epsilon|, & n = 2,
\end{cases} \tag{2.13}
\]
the cut-off function \( \phi_{i,\epsilon} \) is defined via
\[
\phi_{i,\epsilon}(x) := \phi \left( \frac{4|x-x_\epsilon|}{\epsilon} \right), \tag{2.14}
\]
where $\phi \in C^\infty([0,\infty))$ is such that

$$0 \leq \phi(t) \leq 1, \quad \phi(t) = 1 \text{ as } t \leq 1, \quad \phi(t) = 0 \text{ as } t \geq 2.$$  

(2.15)

We shall use the same notation $G_\varepsilon$ for the operator of multiplication by the function $G_\varepsilon(x)$. Finally, we define

$$\tilde{\eta}_\varepsilon := \begin{cases} \max\{\|V_\varepsilon - V\|; \varepsilon^{2/(n-2)}\}, & n \geq 5, \\
\max\{\|V_\varepsilon - V\|; \varepsilon \ln \varepsilon\}, & n = 4, \\
\max\{\|V_\varepsilon - V\|; \varepsilon\}, & n = 3, \\
\max\{\|V_\varepsilon - V\|; \varepsilon \ln \varepsilon\}, & n = 2. \end{cases}$$  

(2.16)

**Theorem 2.4.** Let $Q > 0$. Then one has

$$\left\|(A_\varepsilon + I)^{-1}I_{\varepsilon} - (I + G_\varepsilon)I_{\varepsilon}(A + I)^{-1}\right\|_{L^2(\Omega) \rightarrow H^1(\Omega)} \leq C\tilde{\eta}_\varepsilon.$$  

(2.17)

**Remark 2.5.** Let $Q > 0$. In this case we have, using (2.3):

$$\exists C_1, C_2 > 0: \quad C_1\varepsilon^n \leq D_\varepsilon \leq C_2\varepsilon^n,$$  

(2.18)

as $\varepsilon \in (0,\varepsilon_0]$, where

$$D_\varepsilon := \begin{cases} d_{\varepsilon}^{n-2}, & n \geq 3, \\
\ln d_{\varepsilon}, & n = 2. \end{cases}$$

It follows easily from (2.18) (taking into account (2.6) for $n = 2$) that

$$\exists C_3, C_4 > 0: \quad C_3\eta_\varepsilon \leq \tilde{\eta}_\varepsilon \leq C_4\eta_\varepsilon.$$  

(2.19)

Our next goal is to estimate the distance between the spectra of $A_\varepsilon$ and $A$ in a suitable metric. Recall that for closed sets $X, Y \subset \mathbb{R}$ the **Hausdorff distance** between them is given by

$$d_H(X, Y) := \max \left\{ \sup_{x \in X} \inf_{y \in Y} |x - y|; \sup_{y \in Y} \inf_{x \in X} |y - x| \right\}.$$  

(2.20)

The notion of convergence provided by this metric is too restrictive for our purposes. Indeed, the closeness of $\sigma(A_\varepsilon)$ and $\sigma(A)$ in the metric $d_H(\cdot, \cdot)$ would mean that these spectra look nearly the same uniformly in the whole of $[0,\infty)$ – a situation which is not guaranteed by norm resolvent convergence. To overcome this difficulty, it is convenient to introduce the new metric $\widetilde{d}_H(\cdot, \cdot)$ which is given by

$$\widetilde{d}_H(X, Y) := d_H((1 + X)^{-1}, (1 + Y)^{-1}), \quad X, Y \subset [0,\infty),$$  

(2.21)

where $(1 + X)^{-1} := \{(1 + x)^{-1}: x \in X\}$, $(1 + Y)^{-1} := \{(1 + y)^{-1}: y \in Y\}$. With respect to this metric two spectra can be close even if they differ significantly at high energies.

**Remark 2.6.** Let $X_\varepsilon, X$ be closed subsets of $[0,\infty)$. One can show (see, e.g., [25] Lemma A.2) that $\widetilde{d}_H(X_\varepsilon, X) \to 0$ if and only if the following two conditions hold simultaneously:

(i) $\forall x \in X$ there exists a family $(x_\varepsilon)_{\varepsilon > 0}$ with $x_\varepsilon \in X_\varepsilon$ such that $x_\varepsilon \to x$ as $\varepsilon \to 0$,

(ii) $\forall x \in \mathbb{R} \setminus X$ there exist $\delta > 0$ such that $X_\varepsilon \cap (x - \delta, x + \delta) = \emptyset$ for small enough $\varepsilon$.

**Theorem 2.7.** Let $P < \infty$ or $Q < \infty$. Then one has

$$\widetilde{d}_H(\sigma(A_\varepsilon), \sigma(A)) \leq C\eta_\varepsilon,$$  

(2.22)

where $\eta_\varepsilon$ is given in (2.8).
Finally, let \( P = Q = \infty \). In this case the solution to the boundary value problem (1.7) converges to zero. The estimate for the rate of this convergence in the operator norm is given in the theorem below.

**Theorem 2.8.** Let \( P = Q = \infty \). One has
\[
\| (A_\varepsilon + I)^{-1} \|_{L^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon)} \leq C \max \left\{ P_\varepsilon^{-1}; Q_\varepsilon^{-1}; \varepsilon^2 \right\}. \tag{2.23}
\]

The remaining part of the work is organized as follows. In Section 3 we present the abstract results for studying convergence of operators in varying Hilbert spaces. In Sections 4 we collect several useful estimates which will be widely used further. In Section 5 we verify the conditions of the above abstract theorems in our concrete setting. The proofs of Theorems 2.1, 2.2, 2.4, 2.7, 2.8 are completed in Section 6. Finally, in Section 7 we revisit the case (1.13), for which we establish an additional \((\mathcal{H}^1 \to L^2)\) operator estimate (Theorem 7.1); in some cases this estimate provides better convergence rate than the estimates (2.9) and (2.11).

### 3. Abstract tools

In [45] Post presented an abstract toolkit for studying convergence of operators in varying Hilbert spaces; further it was elaborated in the monograph [46] and the papers [2, 38, 44, 47]. Originally this abstract framework was developed to study convergence of the Laplace-Beltrami operator on manifolds which shrink to a graph. Recently, it also has shown to be effective for homogenization problems in domains with holes, see [2, 37]. The proofs of our main theorems (except Theorem 2.8) rely on results from [2, 38, 45] which we recall below.

Let \( \mathcal{H}_\varepsilon \) and \( \mathcal{H} \) be Hilbert spaces. Note that within this section \( \mathcal{H}_\varepsilon \) is just a notation for some Hilbert space which (in general) differs from the space \( \mathcal{H} \), i.e., the sub-script \( \varepsilon \) does not mean that this space depends on a small parameter. Of course, later we will apply the results of this section to the \( \varepsilon \)-dependent space \( \mathcal{H}_\varepsilon := L^2(\Omega_\varepsilon) \).

Let \( A_\varepsilon \) and \( A \) be non-negative, self-adjoint, unbounded operators in \( \mathcal{H}_\varepsilon \) and \( \mathcal{H} \), respectively, and \( a_\varepsilon \) and \( a \) be the associated sesquilinear forms. We denote
\[
\mathcal{R}_\varepsilon := (A_\varepsilon + I)^{-1}, \quad \mathcal{R} := (A + I)^{-1}.
\]

Along with \( \mathcal{H}_\varepsilon \) and \( \mathcal{H} \), we also define the Hilbert spaces \( \mathcal{H}_\varepsilon^k \) and \( \mathcal{H}^k \), \( k = 1, 2 \) via
\[
\mathcal{H}_\varepsilon^k := \text{dom}(A_\varepsilon^{k/2}), \quad \| u \|_{\mathcal{H}_\varepsilon^k} := \| (A_\varepsilon + I)^{k/2} u \|_{\mathcal{H}_\varepsilon},
\]
\[
\mathcal{H}^k := \text{dom}(A^{k/2}), \quad \| f \|_{\mathcal{H}^k} := \| (A + I)^{k/2} f \|_{\mathcal{H}}. \tag{3.1}
\]

Note that
\[
\mathcal{H}_\varepsilon^1 = \text{dom}(a_\varepsilon), \quad \| u \|_{\mathcal{H}_\varepsilon^1}^2 = a_\varepsilon[u, u] + \| u \|_{\mathcal{H}_\varepsilon}^2,
\]
\[
\mathcal{H}^1 = \text{dom}(a), \quad \| f \|_{\mathcal{H}^1}^2 = a[f, f] + \| f \|_{\mathcal{H}}^2. \tag{3.2}
\]

Furthermore,
\[
\mathcal{H}^2 \subset \mathcal{H}^1 \subset \mathcal{H} \quad \text{and} \quad \| f \|_{\mathcal{H}} \leq \| f \|_{\mathcal{H}^1} \leq \| f \|_{\mathcal{H}^2},
\]
\[
\mathcal{H}_\varepsilon^2 \subset \mathcal{H}_\varepsilon^1 \subset \mathcal{H}_\varepsilon \quad \text{and} \quad \| u \|_{\mathcal{H}_\varepsilon} \leq \| u \|_{\mathcal{H}_\varepsilon^1} \leq \| u \|_{\mathcal{H}_\varepsilon^2}. \tag{3.3}
\]

It is well-known (see, e.g., [35, Theorem VI.3.6] or [49, Theorem VIII.25]), that convergence of sesquilinear forms with common domain implies norm resolvent convergence of the
Theorem 3.1 ([45] Theorem A.5]). Let \( I_\varepsilon : \mathcal{H} \rightarrow \mathcal{H}_\varepsilon, \tilde{I}_\varepsilon : \mathcal{H}_\varepsilon \rightarrow \mathcal{H} \) be linear operators such that
\[
| (u, I_\varepsilon f)_{\mathcal{H}_\varepsilon} - (\tilde{I}_\varepsilon u, f)_{\mathcal{H}} | \leq \varepsilon \| f \|_{\mathcal{H}} \| u \|_{\mathcal{H}_\varepsilon}, \quad \forall f \in \mathcal{H}, u \in \mathcal{H}_\varepsilon. \tag{3.4}
\]
Also let \( J^1_\varepsilon : \mathcal{H}^1 \rightarrow \mathcal{H}^1_\varepsilon, \tilde{J}^1_\varepsilon : \mathcal{H}^1_\varepsilon \rightarrow \mathcal{H}^1 \) be linear operators satisfying the conditions
\[
\| J^1_\varepsilon f - I_\varepsilon f \|_{\mathcal{H}_\varepsilon} \leq \delta_\varepsilon \| f \|_{\mathcal{H}^1}, \quad \forall f \in \mathcal{H}^1, \tag{3.5}
\]
\[
\| \tilde{J}^1_\varepsilon u - \tilde{I}_\varepsilon u \|_{\mathcal{H}} \leq \delta_\varepsilon \| u \|_{\mathcal{H}_\varepsilon}, \quad \forall u \in \mathcal{H}^1_\varepsilon, \tag{3.6}
\]
\[
| a_\varepsilon [u, J^1_\varepsilon f] - a_\varepsilon [\tilde{J}^1_\varepsilon u, f] | \leq \delta_\varepsilon \| f \|_{\mathcal{H}^2} \| u \|_{\mathcal{H}_\varepsilon^1}, \quad \forall f \in \mathcal{H}^2, u \in \mathcal{H}^1_\varepsilon \tag{3.7}
\]
with some \( \delta_\varepsilon \geq 0 \). Then one has
\[
\| R_\varepsilon J_\varepsilon - J_\varepsilon R \|_{\mathcal{H} \rightarrow \mathcal{H}_\varepsilon} \leq 4\delta_\varepsilon. \tag{3.8}
\]

Remark 3.2. In applications the operators \( J_\varepsilon \) and \( \tilde{J}_\varepsilon \) usually appear in a natural way – in our case \( J_\varepsilon \) is defined in (2.3) and \( \tilde{J}_\varepsilon \) will be defined in (5.4). The other two operators \( J^1_\varepsilon \) and \( \tilde{J}^1_\varepsilon \) should be constructed as “almost” restrictions of \( J_\varepsilon \) and \( \tilde{J}_\varepsilon \) to \( \mathcal{H}^1 \) and \( \mathcal{H}^1_\varepsilon \), respectively.

Recently, Anné and Post [2] extended the above result to a (suitably sandwiched) resolvent difference regarded as an operator from \( \mathcal{H} \) to \( \mathcal{H}^1_\varepsilon \).

Theorem 3.3 ([2] Proposition 2.5]). Let the operators \( J_\varepsilon, \tilde{J}_\varepsilon, J^1_\varepsilon, \tilde{J}^1_\varepsilon \) satisfy the conditions (3.4)-(3.7) with some \( \delta_\varepsilon \geq 0 \). Then
\[
\| R_\varepsilon J_\varepsilon - J^1_\varepsilon R \|_{\mathcal{H} \rightarrow \mathcal{H}^1_\varepsilon} \leq 6\delta_\varepsilon. \tag{3.9}
\]

Remark 3.4. In fact, Proposition 2.5 in [2] is formulated as follows: the estimate (3.9) holds provided the forms \( a_\varepsilon \) and \( a \) are \( \delta_\varepsilon \)-quasi-unitarily equivalent. The latter means that there exist linear operators \( J_\varepsilon : \mathcal{H} \rightarrow \mathcal{H}_\varepsilon, \tilde{J}_\varepsilon : \mathcal{H}_\varepsilon \rightarrow \mathcal{H}, J^1_\varepsilon : \mathcal{H} \rightarrow \mathcal{H}^1_\varepsilon, \tilde{J}^1_\varepsilon : \mathcal{H}_\varepsilon \rightarrow \mathcal{H}^1 \) such that the conditions (3.4)-(3.7) are fulfilled, and moreover, one has
\[
\| f \|_{\mathcal{H}} \leq (1 + \delta_\varepsilon) \| J_\varepsilon f \|_{\mathcal{H}_\varepsilon}, \quad \| f - \tilde{J}_\varepsilon f \|_{\mathcal{H}_\varepsilon} \leq \delta_\varepsilon \| f \|_{\mathcal{H}^1}, \quad \| u - J^1_\varepsilon \tilde{u} \|_{\mathcal{H}_\varepsilon} \leq \delta_\varepsilon \| u \|_{\mathcal{H}^1_\varepsilon}. \tag{3.10}
\]
However, tracing the proof in [2] Proposition 2.5], one can easily see that the estimates (3.10) are not utilized for the deduction of (3.9).

The assumptions (3.10) are required to deduce other versions of norm resolvent convergence. Namely, if conditions (3.4)-(3.7) are fulfilled, and, moreover, (3.10) are valid, then
\[
\| R_\varepsilon - J_\varepsilon R \|_{\mathcal{H} \rightarrow \mathcal{H}_\varepsilon} \leq C\delta_\varepsilon, \quad \| \tilde{J}_\varepsilon R_\varepsilon - J_\varepsilon \|_{\mathcal{H} \rightarrow \mathcal{H}_\varepsilon} \leq C\delta_\varepsilon,
\]
see [45] Theorem A.10].

Remark 3.5. Tracing the proof of [45] Theorem A.5] one observes that the estimate (3.8) remains valid if (3.7) is substituted by the weaker condition
\[
| a_\varepsilon [u, J^1_\varepsilon f] - a_\varepsilon [\tilde{J}^1_\varepsilon u, f] | \leq \delta_\varepsilon \| f \|_{\mathcal{H}^2} \| u \|_{\mathcal{H}^1_\varepsilon}, \quad \forall f \in \mathcal{H}^2, u \in \mathcal{H}^1_\varepsilon. \tag{3.11}
\]
Nevertheless, in most of the applications one is able to establish stronger estimate (3.7). Note that in Theorem 3.3 condition (3.7) cannot be replaced by (3.11).
The last theorem gives the estimate for the distance between the spectra of $A_\epsilon$ and $A$ in the metrics $d_H(\cdot, \cdot)$. Recall that the distances $d_H(\cdot, \cdot)$ and $\tilde{d}_H(\cdot, \cdot)$ are given in (2.20) and (2.21), respectively. Note that due to spectral mapping theorem one has

$$\tilde{d}_H(\sigma(A_\epsilon), \sigma(A)) = d_H(\sigma(\mathcal{R}_\epsilon), \sigma(\mathcal{R})).$$

(3.12)

It is well-known that the norm convergence of bounded self-adjoint operators in a fixed Hilbert space implies the Hausdorff convergence of their spectra. Indeed, let $\mathcal{R}_\epsilon$ and $\mathcal{R}$ be bounded normal operators in a Hilbert space $\mathcal{H}$, then $\lim_{\epsilon \to 0} \| \mathcal{R}_\epsilon - \mathcal{R} \|_{\mathcal{H}} = 0$. The theorem below is an analogue of this result for the case of operators acting in different Hilbert spaces. In the present form it was established recently by the first author and Post in [38] (see Remark 3.7), a slightly weaker version was proven in [13].

**Theorem 3.6 ([38] Theorem 3.5).** Let $I_\epsilon, \tilde{I}_\epsilon \colon \mathcal{H}_\epsilon \to \mathcal{H}_\epsilon$, $J_\epsilon, \tilde{J}_\epsilon \colon \mathcal{H}_\epsilon \to \mathcal{H}$ be linear bounded operators satisfying

$$\| \mathcal{R}_\epsilon I_\epsilon - I_\epsilon J_\epsilon \|_{\mathcal{H}_\epsilon \to \mathcal{H}_\epsilon} \leq \rho_\epsilon,$$

(3.13)

and, moreover,

$$\| I_\epsilon J_\epsilon \|_{\mathcal{H}_\epsilon} \leq \mu_\epsilon, \quad \| J_\epsilon \|_{\mathcal{H}_\epsilon \to \mathcal{H}_\epsilon} \leq \tilde{\mu}_\epsilon, \quad \| I_\epsilon \|_{\mathcal{H}_\epsilon \to \mathcal{H}_\epsilon} \leq \tilde{\mu}_\epsilon,$$

(3.14)

for some positive constants $\rho_\epsilon$, $\mu_\epsilon$, $\tilde{\mu}_\epsilon$. Then one has

$$\tilde{d}_H(\sigma(A_\epsilon), \sigma(A)) \leq \max \left\{ \frac{\mu_\epsilon}{2} + \sqrt{\frac{\mu_\epsilon^2}{4} + \rho_\epsilon^2 \mu_\epsilon}, \frac{\tilde{\mu}_\epsilon}{2} + \sqrt{\frac{\tilde{\mu}_\epsilon^2}{4} + \tilde{\rho}_\epsilon^2 \tilde{\mu}_\epsilon} \right\}. $$

(3.17)

**Remark 3.7.** In fact, in [38] the obtained estimate reads

$$\tilde{d}_H(\sigma(A_\epsilon), \sigma(A)) \leq \max \left\{ \rho_\epsilon \sqrt{\frac{\mu_\epsilon}{\tau}}, \sqrt{\frac{\mu_\epsilon}{1-\tau}} \right\}, \quad \forall \tau, \bar{\tau} \in (0, 1). $$

(3.18)

Minimizing the right-hand-side of (3.18) over $\bar{\tau}$, one easily arrives at the estimate (3.17).

4. Useful estimates

In this section we collect several estimates which will be used in the proofs of the main theorems. The first result was established in [42]. Here and in what follows the notation $|D|$ stands for the Lebesgue measure of the domain $D \subset \mathbb{R}^n$.

**Lemma 4.1 ([42] Lemma 4.9).** Let $D \subset \mathbb{R}^n$ be a parallelepiped, and let $D_1, D_2 \subset D$ be measurable subsets with $|D_2| \neq 0$. Then

$$\forall g \in H^1(D) : \quad \|g\|_{L^2(D_1)}^2 \leq \frac{2|D_1|}{|D_2|} \|g\|_{L^2(D_2)}^2 + \frac{C(diam(D))^{n+1}|D_1|^{1/n}}{|D_2|} \| \nabla g \|_{L^2(D)}^2,$$

where the constant $C > 0$ depends only on the dimension $n$.

The next lemma is the Poincaré inequality for the cube $[i, i+\epsilon]$. 

Lemma 4.2. One has

\[ \forall g \in H^1(\Box_{i,\varepsilon}) \text{ with } \int_{\Box_{i,\varepsilon}} g(x) \, dx = 0 : \|g\|_{L^2(\Box_{i,\varepsilon})}^2 \leq \pi^{-2}\varepsilon^2 \|\nabla g\|_{L^2(\Box_{i,\varepsilon})}^2. \]  \hspace{1cm} (4.1)

Proof. Let \( \lambda_\varepsilon \) be the second (first non-zero) eigenvalue of the Neumann Laplacian on \( \Box_{i,\varepsilon} \). It is easy to compute that \( \lambda_\varepsilon = \left( \frac{2}{\varepsilon^2} \right)^2 \). Moreover, by the min-max principle

\[ \lambda_\varepsilon = \min \left\{ \frac{\|\nabla g\|_{L^2(\Box_{i,\varepsilon})}^2}{\|g\|_{L^2(\Box_{i,\varepsilon})}^2} : g \in H^1(\Box_{i,\varepsilon}) \setminus \{0\}, \int_{\Box_{i,\varepsilon}} g(x) \, dx = 0 \right\}, \]

whence we immediately obtain the inequality (4.1). \( \square \)

Recall that \( n \geq 2 \) stands for the dimension, \( \Box \) is a unit cube in \( \mathbb{R}^n \). By the Sobolev embedding theorem (see, e.g., [1, Theorem 5.4 and Remark 5.5(6)]) the space \( H^2(\Box) \) is embedded continuously into the space \( L^p(\Box) \) provided \( p \) satisfies

\[ 1 \leq p \leq \frac{2n}{n-4} \quad \text{as} \quad n \geq 5, \quad 1 \leq p < \infty \quad \text{as} \quad n = 4, \quad 1 \leq p \leq \infty \quad \text{as} \quad n = 2, 3. \]  \hspace{1cm} (4.2)

Furthermore, the space \( H^2(\Box) \) is embedded continuously into \( W^{1,p}(\Box) \) provided \( p \) satisfies

\[ 1 \leq p \leq \frac{2n}{n-2} \quad \text{as} \quad n \geq 3, \quad 1 \leq p < \infty \quad \text{as} \quad n = 2. \]  \hspace{1cm} (4.3)

In the following, the constants \( C_{n,p} \) (for \( p \) satisfying (4.2)) and \( \tilde{C}_{n,p} \) (for \( p \) satisfying (4.3)) stand for the norms of these embeddings, respectively. In the lemma below we give two Sobolev-type inequalities for the re-scaled cubes \( \Box_{i,\varepsilon} \cong \varepsilon \Box \).

Lemma 4.3. One has

\[ \forall g \in H^2(\Box_{i,\varepsilon}) \text{ with } \int_{\Box_{i,\varepsilon}} g(x) \, dx = 0 : \|g\|_{L^p(\Box_{i,\varepsilon})} \leq C \cdot C_{n,p} \cdot \varepsilon^{n/p+(2-n)/2} \|g\|_{H^2(\Box_{i,\varepsilon})} \]  \hspace{1cm} (4.4)

provided \( p \) satisfies (4.2) (for \( p = \infty \) one has the convention \( 1/p = 0 \)). Moreover,

\[ \forall f \in H^2(\Box_{i,\varepsilon}) : \|\nabla f\|_{L^p(\Box_{i,\varepsilon})} \leq C \cdot \tilde{C}_{n,p} \cdot \varepsilon^{n/p-n/2} \|f\|_{H^2(\Box_{i,\varepsilon})} \]  \hspace{1cm} (4.5)

provided \( p \) satisfies (4.3). The constant \( C \) in (4.4)–(4.5) equals \( (1 + \pi^{-2})^{1/2} \).

Proof. The estimate (4.4) is proven in [37, Lemma 4.3], therefore we present the proof only for the estimate (4.5) (in fact, both proofs are based on similar arguments).

The Sobolev embedding theorem yields

\[ \forall g \in H^2(\Box) : \|\nabla g\|_{L^p(\Box)} \leq \tilde{C}_{n,p} \|g\|_{W^{1,p}(\Box)} \]

\[ = \tilde{C}_{n,p} \left( \|g\|_{L^2(\Box)}^2 + \|\nabla g\|_{L^2(\Box)}^2 + \sum_{k,l=1}^n \left\| \frac{\partial^2 g}{\partial x_k \partial x_l} \right\|_{L^2(\Box)}^2 \right)^{1/2} \]  \hspace{1cm} (4.6)
provided \((4.3)\) holds. Making the change of variables \(\Box \ni y = x\epsilon^{-1} - i\) with \(x \in \Box_{i,\epsilon}\) we reduce \((4.6)\) to the following estimate:

\[
\forall \mathcal{g} \in H^2(\Box_{i,\epsilon}) : \quad \epsilon^{(p-n)/p} \|\nabla \mathcal{g}\|_{L^p(\Box_{i,\epsilon})} \leq \tilde{C}_{n,p} \left( \epsilon^{-n} \|\mathcal{g}\|_{L^2(\Box_{i,\epsilon})}^2 + \epsilon^{2-n} \|\nabla \mathcal{g}\|_{L^2(\Box_{i,\epsilon})}^2 \right)^{1/2}.
\]

(4.7)

Finally, let \(f \in H^2(\Box_{i,\epsilon})\). We denote \(f_{i,\epsilon} := \epsilon^{-n} \int_{\Box_{i,\epsilon}} f(x) \, dx\). One has

\[
\nabla(f - f_{i,\epsilon}) = \nabla f, \quad \frac{\partial^2 (f - f_{i,\epsilon})}{\partial x_k \partial x_l} = \frac{\partial^2 f}{\partial x_k \partial x_l}, \quad \|f - f_{i,\epsilon}\|_{L^2(\Box_{i,\epsilon})} \leq \pi^{-2} \epsilon^2 \|\nabla f\|_{L^2(\Box_{i,\epsilon})}^2
\]

(4.8)

(the last property follows from Lemma 4.2). Using \((4.7)\) with \(g := f - f_{i,\epsilon}\) and taking into account \((4.8)\) and that \(\epsilon < 1\), we arrive at the required estimate \((4.5)\). \(\Box\)

In the next section (see the estimate \((5.37)\)) we apply the inequality \((4.4)\) for the largest \(p\) satisfying \((4.2)\). For \(n = 4\) we are not able to choose a largest \(p\) (since in the dimension 4 the embedding \(H^2 \hookrightarrow L^p\) holds for any \(p < \infty\), but not for \(p = \infty\)), and in this case we need more information on the constant \(C_{4,p}\) in the right-hand-side of \((4.4)\) – see the lemma below.

**Lemma 4.4.** For any \(p \in [1, \infty)\) one has the estimate

\[
C_{4,p} \leq Cp,
\]

(4.9)

where the constant \(C > 0\) is independent of \(p\).

**Proof.** By the Sobolev embedding theorem, the space \(W^{1,4}(\mathbb{R}^4)\) is embedded continuously into \(L^p(\mathbb{R}^4)\) provided \(p \in [4, \infty)\); we denote by \(C'_p\) the norm of this embedding. Also, the space \(H^2(\mathbb{R}^4)\) is embedded into \(W^{1,4}(\mathbb{R}^4)\); the norm of this embedding is denoted by \(C''\). Furthermore, by the Calderon extension theorem \([1, \text{Theorem 4.32}]\), there exists a linear bounded operator \(E : H^2(\Box) \to H^2(\mathbb{R}^4)\) such that \((E f)(x) = f(x)\) a.e. in \(\Box = (-1/2, 1/2)^4\). Consequently, for any \(f \in H^2(\Box)\) and \(p \geq 4\) one has the following chain of inequalities:

\[
\|f\|_{L^p(\Box)} \leq \|Ef\|_{L^p(\mathbb{R}^4)} \leq C'_p \|Ef\|_{W^{1,4}(\mathbb{R}^4)} \leq C'_p C'' \|Ef\|_{H^2(\mathbb{R}^4)} \leq C'_p C'' \|E\| \cdot \|f\|_{H^2(\Box)}.
\]

(4.10)

We will demonstrate below that

\[
C'_p \leq \tilde{C}' := 4^{1 - \frac{p}{4}} \left( \frac{3p}{4} - j \right) \frac{4^j}{\prod_{j=0}^{v-1} \left( \frac{3p}{4} - j \right)},
\]

(4.11)

where \(v\) is the largest integer \(\leq \frac{3p}{4} - 3\). It is easy to see, using Stirling’s formula, that

\[
\tilde{C}' \leq \bar{C}' p
\]

(4.12)

with a constant \(\bar{C}' > 0\) being independent of \(p\). It follows from \((4.10) -(4.12)\) that the estimate \((4.9)\) holds with \(C := \bar{C}' C'' \|E\|\) provided \(p \in [4, \infty)\). Moreover, the Hölder inequality yields \(\|f\|_{L^{p_1}(\Box)} \leq \|f\|_{L^{p_2}(\Box)}\) as \(1 \leq p_1 \leq p_2\), hence \((4.9)\) is also valid for \(p \in [1, 4)\).
It remains to prove (4.11). Let \( f \in C_0^\infty(\mathbb{R}^4) \). One has for all \( x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \):

\[
|f(x)|_2^\frac{3p}{2} \leq \frac{3p}{4} \int_{-\infty}^{x_1} |f(t, x_2, x_3, x_4)|_2^\frac{3p}{2} \left| \frac{\partial f}{\partial x_1}(t, x_2, x_3, x_4) \right| \, dt, \tag{4.13}
\]

\[
|f(x)|_2^\frac{3p}{2} \leq \frac{3p}{4} \int_{x_1}^{\infty} |f(t, x_2, x_3, x_4)|_2^\frac{3p}{2} \left| \frac{\partial f}{\partial x_1}(t, x_2, x_3, x_4) \right| \, dt. \tag{4.14}
\]

Adding the inequalities (4.13) and (4.14) and dividing by 2, we obtain

\[
|f(x)|_2^\frac{3p}{2} \leq \frac{3p}{8} \int_{-\infty}^{\infty} |f(t, x_2, x_3, x_4)|_2^\frac{3p}{2} \left| \frac{\partial f}{\partial x_1}(t, x_2, x_3, x_4) \right| \, dt =: F_1(x_2, x_3, x_4). \tag{4.15}
\]

In the same way, replacing the variable \( x_1 \) by any other of the remaining variables, we get

\[
|f(x)|_2^\frac{3p}{2} \leq F_2(x_1, x_2, x_4), \quad |f(x)|_2^\frac{3p}{2} \leq F_3(x_1, x_2, x_4), \quad |f(x)|_2^\frac{3p}{2} \leq F_4(x_1, x_2, x_3), \tag{4.16}
\]

with obvious definitions of \( F_2, F_3, F_4 \). Multiplying the inequalities in (4.15)-(4.16) and then taking the \( \frac{1}{2} \)-power, one has

\[
|f(x)|_2 \leq (F_1(x_2, x_3, x_4) F_2(x_1, x_2, x_4) F_3(x_1, x_2, x_4) F_4(x_1, x_2, x_3))^{\frac{1}{2}}. \tag{4.17}
\]

Integrating (4.17) with respect to \( x_1 \) and then applying the Hölder inequality, we obtain

\[
\int_{\mathbb{R}} |f(x)|_2 \, dx_1 = (F_1(x_2, x_3, x_4))^{\frac{1}{2}} \int_{\mathbb{R}} (F_2(x_1, x_3, x_4) F_3(x_1, x_2, x_4) F_4(x_1, x_2, x_3))^{\frac{1}{2}} \, dx_1
\]

\[
\leq (F_1(x_2, x_3, x_4))^{\frac{1}{2}} \left( \int_{\mathbb{R}} F_2(x_1, x_3, x_4) \, dx_1 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} F_3(x_1, x_2, x_4) \, dx_1 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} F_4(x_1, x_2, x_3) \, dx_1 \right)^{\frac{1}{2}}.
\]

Successively repeating the last steps (i.e., the integration with respect to one of the directions and the subsequent application of the Hölder inequality with respect to this direction), now for the variables \( x_2, x_3 \) and \( x_4 \), we arrive at

\[
\int_{\mathbb{R}^4} |f(x)|_2 \, dx \leq \left( \int_{\mathbb{R}^3} F_1 \, dx_2 \, dx_3 \, dx_4 \cdot \int_{\mathbb{R}^3} F_2 \, dx_1 \, dx_3 \, dx_4 \cdot \int_{\mathbb{R}^3} F_3 \, dx_1 \, dx_2 \, dx_4 \cdot \int_{\mathbb{R}^3} F_4 \, dx_1 \, dx_2 \, dx_3 \right)^{\frac{1}{2}},
\]

whence, using the definitions of the functions \( F_k \) and again the Hölder inequality, we derive

\[
\int_{\mathbb{R}^4} |f(x)|_2 \, dx \leq \left( \frac{3p}{8} \right)^{\frac{1}{2}} \prod_{k=1}^{4} \left( \int_{\mathbb{R}^4} |f(x)|_2 \left| \frac{\partial f}{\partial x_k} \right| \, dx \right)^{\frac{1}{2}}
\]

\[
\leq \left( \frac{3p}{8} \right)^{\frac{1}{2}} \prod_{k=1}^{4} \left( \int_{\mathbb{R}^4} |f(x)|_2^{-\frac{4}{p-2}} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^4} \left| \frac{\partial f}{\partial x_k} \right|^4 \, dx \right)^{\frac{1}{2}}
\]

\[
\leq \left( \frac{3p}{8} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^4} |f(x)|_2^{-\frac{4}{p-2}} \, dx \right) \left( \frac{1}{4} \sum_{k=1}^{4} \int_{\mathbb{R}^4} \left| \frac{\partial f}{\partial x_k} \right|^4 \, dx \right)^{\frac{1}{2}} \leq A p^{\frac{1}{p}} \int_{\mathbb{R}^4} |f(x)|_2^{-\frac{4}{p-2}} \, dx, \tag{4.18}
\]

where \( A := (\frac{3}{8})^{\frac{1}{2}} (\frac{1}{4})^{\frac{1}{2}} \|f\|_{W^{1,4}(\mathbb{R}^4)}^{\frac{1}{2}} \). Iterating this inequality \( \nu - 1 \) times we get:

\[
\int_{\mathbb{R}^4} |f(x)|_2 \, dx \leq A^\nu \prod_{j=0}^{\nu-1} \left( p - \frac{4j}{3} \right)^{\frac{1}{2}} \int_{\mathbb{R}^4} |f(x)|_2^{-\frac{4}{p-2}} \, dx \tag{4.19}
\]
Lemma 4.5. The estimate (4.21) is proven in [36, Lemma 2.1]. Let us prove (4.22). We denote by $C$ where the constant $C$

$$\int_{\mathbb{R}^4} |f|^p \, dx \leq A \frac{4^{\frac{3}{4}}}{3} \left( \frac{16}{3} \right)^{\frac{p-4}{3}} \int_{\mathbb{R}^4} |f|^4 \, dx.$$

(4.20)

Inserting (4.20) into (4.19) and taking into account $\frac{3q}{4} = \frac{3p}{4} - \nu$, we get

$$\int_{\mathbb{R}^4} |f|^p \, dx \leq A \frac{4^{\frac{3}{4}}}{3} \left( \frac{16}{3} \right)^{\frac{p-4}{3}} \prod_{j=0}^{\nu-1} \left( \frac{p-4j}{3} \right) \int_{\mathbb{R}^4} |f|^4 \, dx \leq (\tilde{C}_p)^p \|f\|^{p}_{W^{1,4}(\mathbb{R}^4)},$$

where $\tilde{C}_p$ is defined in (4.11). By density arguments this estimate holds for all $f \in W^{1,4}(\mathbb{R}^4)$, which implies the desired inequality (4.11). The lemma is proven.

The following lemma is a simple consequence of the fact that the set $\Omega \setminus \bigcup_{i \in \mathcal{I}_\varepsilon} \square_{i,\varepsilon}$ belongs to the $\sqrt{n}\varepsilon$-neighborhood of $\partial \Omega$ (this follows easily from the definition of $\mathcal{I}_\varepsilon$; note that $\sqrt{n}$ is the length of the diagonal of the cube $\square$).

**Lemma 4.5** (37 Lemma 4.7). One has

$$\forall g \in H^0_0(\Omega) : \|g\|_{L^2(\Omega) \setminus \bigcup_{i \in \mathcal{I}_\varepsilon} \square_{i,\varepsilon}) \leq C \varepsilon \|\nabla g\|_{L^2(\Omega)},$$

where the constant $C > 0$ depends only on $\Omega$.

The lemma below is used in the proof of Theorem 7.1.

**Lemma 4.6.** One has

$$\forall g \in H^1(\square_{i,\varepsilon}) : |g_{i,\varepsilon} - \bar{g}_{i,\varepsilon}|^2 \leq C \varepsilon^2 \left( \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon^2} \right) \|\nabla g\|_{L^2(\square_{i,\varepsilon})}^2,$$

(4.21)

$$\forall g \in H^1(D_{i,\varepsilon}) : \|g - \bar{g}_{i,\varepsilon}\|_{L^2(D_{i,\varepsilon})} \leq C d_{i,\varepsilon} \|\nabla g\|_{L^2(D_{i,\varepsilon})}^2,$$

(4.22)

where $g_{i,\varepsilon}$ and $\bar{g}_{i,\varepsilon}$ are the mean values of $g$ over $\square_{i,\varepsilon}$ and $\partial D_{i,\varepsilon}$, respectively, i.e.

$$g_{i,\varepsilon} := \frac{1}{\varepsilon^d} \int_{\square_{i,\varepsilon}} g \, dx, \quad \bar{g}_{i,\varepsilon} := \frac{1}{\varepsilon^d} \int_{\partial D_{i,\varepsilon}} g \, ds.$$

The constant $C > 0$ depends only on $d$.

**Proof.** The estimate (4.21) is proven in [36, Lemma 2.1]. Let us prove (4.22). We denote by $D$ the unit ball in $\mathbb{R}^n$. One has the following trace and Poincaré inequalities on $D$:

$$\forall u \in H^1(D) : \|u\|_{L^2(\partial D)}^2 \leq C \|u\|^2_{H^1(D)},$$

(4.23)

$$\forall u \in H^1(D) \text{ with } \int_D u(x) \, dx = 0 : \|u\|^2_{L^2(D)} \leq C \|\nabla u\|^2_{L^2(D)}.$$

(4.24)

Making the change of variables $D \ni y = \frac{x - x_{i,\varepsilon}}{d_{i,\varepsilon}}$ with $x \in D_{i,\varepsilon}$ we reduce (4.23)–(4.24) to

$$\forall u \in H^1(D_{i,\varepsilon}) : \|u\|_{L^2(\partial D_{i,\varepsilon})}^2 \leq C \left( d_{i,\varepsilon}^{-1} \|u\|^2_{L^2(D_{i,\varepsilon})} + d_{i,\varepsilon} \|\nabla u\|^2_{L^2(D_{i,\varepsilon})} \right),$$

(4.25)

$$\forall u \in H^1(D_{i,\varepsilon}) \text{ with } \int_{D_{i,\varepsilon}} u(x) \, dx = 0 : \|u\|^2_{L^2(D_{i,\varepsilon})} \leq C d_{i,\varepsilon}^2 \|\nabla u\|^2_{L^2(D_{i,\varepsilon})},$$

(4.26)
One has:
\[ \| g - g_{i,\ell} \|_{L^2(\partial D_{i,\ell})} \leq \| g - g_{i,\ell} \|_{L^2(\partial D_{i,\ell})} + |g_{i,\ell} - \bar{g}_{i,\ell}| \cdot |\partial D_{i,\ell}|^{1/2} \]
\[ = \| g - g_{i,\ell} \|_{L^2(\partial D_{i,\ell})} + \int_{\partial D_{i,\ell}} (g_{i,\ell} - g) \, |\partial D_{i,\ell}|^{-1/2} \leq 2\| g - g_{i,\ell} \|_{L^2(\partial D_{i,\ell})}, \]
whence, using (4.25)-(4.26) with \( u := g - g_{i,\ell} \), we deduce
\[ \| g - g_{i,\ell} \|_{L^2(\partial D_{i,\ell})} \leq C \left( d_{\epsilon}^{-1} \| g - g_{i,\ell} \|_{L^2(D_{i,\ell})} + d_{\epsilon} \| \nabla g \|_{L^2(D_{i,\ell})}^2 \right) \leq C d_{\epsilon} \| \nabla g \|_{L^2(D_{i,\ell})}^2. \]
The lemma is proven. \( \square \)

The next estimate plays a crucial role for the proof of Theorems 2.8. We denote
\[ Y_{i,\ell} := \left\{ x \in \mathbb{R}^n : d_{\epsilon} < |x - x_{i,\ell}| < \frac{\epsilon}{2} \right\}. \tag{4.27} \]

**Lemma 4.7.** One has
\[ \forall u \in H^1(Y_{i,\ell}) : \quad \| u \|_{L^2(Y_{i,\ell})} \leq C \left( P_{\epsilon}^{-1} \gamma_{\ell} \| u \|_{L^2(\partial D_{i,\ell})} + Q_{\epsilon}^{-1} \| \nabla u \|_{L^2(Y_{i,\ell})}^2 \right), \tag{4.28} \]
where the constant \( C > 0 \) depends only on \( n \).

**Proof.** Evidently, it is enough to prove (4.28) only for \( C^\infty(\overline{Y_{i,\ell}}) \)-functions. We introduce spherical coordinates \( (r, \phi) \) in \( Y_{i,\ell} \), where \( r \in [d_{\epsilon}, \epsilon/2] \) is the radial coordinate, which stands for the distance to \( x_{i,\ell} \), while \( \phi = (\phi_1, \ldots, \phi_{n-1}) \) are the angular coordinates (here \( \phi_j \in [0, \pi] \) as \( j = 1, \ldots, n-2, \phi_{n-1} \in [0, 2\pi] \)). Let \( u \in C^\infty(\overline{Y_{i,\ell}}) \). One has:
\[ u(r, \phi) = u(d_{\epsilon}, \phi) + \int_{d_{\epsilon}}^r \frac{\partial u(\tau, \phi)}{\partial \tau} \, d\tau, \tag{4.29} \]
whence
\[ |u(r, \phi)|^2 \leq 2|u(d_{\epsilon}, \phi)|^2 + 2 \int_{d_{\epsilon}}^r \left| \frac{\partial u(\tau, \phi)}{\partial \tau} \right|^2 \, d\tau \leq 2|u(d_{\epsilon}, \phi)|^2 + 2I_{\epsilon} \int_{d_{\epsilon}}^{\epsilon/2} \left| \frac{\partial u(\tau, \phi)}{\partial \tau} \right|^2 \tau^{n-1} \, d\tau, \]
where \( I_{\epsilon} := \int_{d_{\epsilon}}^{\epsilon/2} \tau^{n-1} \, d\tau. \)

Multiplying this inequality by \( r^{n-1} \, dr \, d\phi \), where \( d\phi = \left( \prod_{j=1}^{n-2} ( \sin \phi_j )^{n-1-j} \right) d\phi_1 \ldots d\phi_{n-1} \), and integrating over \( r \in (d_{\epsilon}, \epsilon/2), \phi_j \in (0, \pi), j = 1, \ldots, n-2, \phi_{n-1} \in (0, 2\pi) \), we get
\[ \| u \|_{L^2(Y_{i,\ell})}^2 \leq 2 \left( \int_S |u(d_{\epsilon}, \phi)|^2 \, d\phi + I_{\epsilon} \int_{d_{\epsilon}}^{\epsilon/2} \left| \frac{\partial u(\tau, \phi)}{\partial \tau} \right|^2 \tau^{n-1} \, d\tau \, d\phi \right) \]
\[ \times \left( \int_{d_{\epsilon}}^{\epsilon/2} r^{n-1} \, dr \right) \leq \frac{\epsilon^n}{2n-1} \left( d_{\epsilon}^{-n+1} \| u \|_{L^2(\partial D_{i,\ell})}^2 + I_{\epsilon} \| \nabla u \|_{L^2(Y_{i,\ell})}^2 \right) \tag{4.30} \]
(\text{here } S := \{ \phi = (\phi_1, \ldots, \phi_{n-1}) : \phi_j \in (0, \pi), j = 1, \ldots, n-2, \phi_{n-1} \in (0, 2\pi) \}). One has
\[ d_{\epsilon}^{-n+1} = \kappa_n \gamma_{\ell}^{-1} \epsilon^{-n}, \tag{4.31} \]
\[ \begin{align*}
&n \geq 3 : \quad I_{\epsilon} \leq \int_{d_{\epsilon}}^{\infty} \tau^{n-1} \, d\tau = (n-2)^{-1} d_{\epsilon}^{2-n} = \kappa_n Q_{\epsilon}^{-1} \epsilon^{-n}, \tag{4.32} \\
&n = 2 : \quad I_{\epsilon} \leq \int_{d_{\epsilon}}^{1} \tau^{n-1} \, d\tau = |\ln d_{\epsilon}| = 2\pi Q_{\epsilon}^{-1} \epsilon^{-2}. \tag{4.33} 
\end{align*} \]
The desired estimate \((4.28)\) follows immediately from \((4.30)-(4.33)\). \hfill \square

Swapping \(r\) and \(d_ε\) in \((4.29)\) and then repeating verbatim the proof of \((4.28)\) we get the following estimate below; it will be used in the proof of Theorem 7.1.

Lemma 4.8. One has

\[ \forall u ∈ H^1(Ω_ε) : \quad γ_e ∥u∥^2_{L^2(∂Ω_ε)} ≤ C \left( P_e ∥u∥^2_{L^2(Ω_ε)} + P_e Q_e^{-1} ∥∇u∥^2_{L^2(Ω_ε)} \right) , \]  

where the constant \(C > 0\) depends only on \(n\).

5. Realisation of the abstract scheme

In this section we apply the abstract Theorems 3.1, 3.3 and 3.6 to the homogenization of the problem \((1.7)\). Through this section we always assume that

\[ P < ∞ \text{ or } Q < ∞. \]  

(5.1)

We denote

\[ H_ε := L^2(Ω_ε), \quad H := L^2(Ω). \]

Recall that the forms \(a_ε\) and \(a\) in \(H_ε\) and \(H\) are defined by \((2.1)\) and \((2.2)\), respectively; \(A_ε\) and \(A\) are non-negative, self-adjoint unbounded operators associated with these forms. We introduce the spaces \(H^1_ε\), \(H^1\) as in \((3.1)\) (cf. \((3.2)\)):

\[ H^1_ε = \{ u ∈ H^1(Ω_ε) : u |_{∂Ω} = 0 \}, \quad ∥u∥^2_{H^1_ε} = ∥u∥^2_{H^1(Ω_ε)} + γ_e \sum_{i ∈ I_ε} ∥u∥^2_{L^2(∂Ω_ε)}, \]

\[ H^1 = H^1_0(Ω), \quad ∥f∥^2_{H^1} = ∥f∥^2_{H^1(Ω)} + V∥f∥^2_{L^2(Ω)}. \]  

(5.2)

Finally, we introduce the space \(H^2\) as in \((3.1)\):

\[ H^2 = H^2(Ω) ∩ H^1_0(Ω), \quad ∥f∥_{H^2} = ∥(−Δ + 1)f∥_{L^2(Ω)}. \]  

(5.3)

Note that, due to uniform regularity of \(Ω\), the norm \(∥·∥_{H^2}\) is equivalent to the standard norm in \(H^2(Ω)\); see the estimate \((5.28)\) below.

Along with \(J_ε : H → H_ε\) (see \((2.3)\)) we introduce the operator \(\overline{J}_ε : H_ε → H\) via

\[ (\overline{J}_ε u)(x) := \begin{cases} u(x), & x ∈ Ω_ε, \\ 0, & x ∈ Ω \setminus Ω_ε = \bigcup_{i ∈ I_ε} D_ε. \end{cases} \]  

(5.4)

It is known that there exists a linear operator \(\overline{J}_ε^1 : H^1(Ω_ε) → H^1(Ω)\) satisfying

\[ (∁1_ε u) |_{Ω_ε} = u, \quad ∥\overline{J}_ε^1 u∥_{H^1(Ω)} ≤ C∥u∥_{H^1(Ω_ε)}, \quad ∀u ∈ H^1(Ω_ε). \]  

(5.5)

As before, by \(C\) we denote a generic positive constant independent of \(ε\) and of functions occurring at the estimates where these constants occur, it may vary from line to line. For the construction of such an operator we refer, e.g., to \([42] \text{ Example 4.10}\)\(^2\). In the following, we shall use the notation \(\overline{J}_ε^1\) for the restriction of the operator \(\overline{J}_ε^1\) to \(\text{dom}(a_ε) = \{ u ∈ H^1(Ω_ε) : u |_{∂Ω} = 0 \}\). Evidently, \(\text{ran}(\overline{J}_ε^1) \subset H^1_0(Ω)\), thus \(\overline{J}_ε^1\) is a well-defined linear operator from \(H^1_ε\) to \(H^1\). It follows from \((5.5)\) that

\[ ∥\overline{J}_ε^1 u∥_{H^1(Ω)} ≤ C∥u∥_{H^1_ε}, \quad ∀u ∈ \text{dom}(a_ε). \]  

(5.6)

\(^2\)For the construction of the operator satisfying \((5.5)\), the holes have to be sufficiently far away from each other; namely, \(\text{dist}(D_ε \cup \bigcup_{j \neq i} D_ε) ≥ \overline{C}d_ε\) should hold. In our case (see \((2.4)\)) this inequality holds with \(\overline{C} = 2\).
To employ Theorems 3.1 and 3.3 we also need a suitable operator $J^1_\varepsilon : \mathcal{H}^1 \to \mathcal{H}^1_\varepsilon$. For $f \in \mathcal{H}^1_0(\Omega)$ we define $J^1_\varepsilon f$ as follows:

$$(J^1_\varepsilon f)(x) := f(x) + \sum_{i \in I_{\varepsilon}} \left[ (f_{i,\varepsilon} - f(x))\tilde{\phi}_{i,\varepsilon}(x) + f_{i,\varepsilon}G_{i,\varepsilon}(x)\phi_{i,\varepsilon}(x) \right], \ x \in \Omega_{\varepsilon}. \quad (5.7)$$

Here

- $f_{i,\varepsilon}$ is the mean value of the function $f(x)$ in the domain $\square_{i,\varepsilon}$, i.e.,
  $$f_{i,\varepsilon} := \varepsilon^{-n} \int_{\square_{i,\varepsilon}} f(x) \, dx,$$
- for $n \geq 3$ the function $\tilde{\phi}_{i,\varepsilon}$ is given by
  $$\tilde{\phi}_{i,\varepsilon} := \phi \left( \frac{|x - x_{i,\varepsilon}|}{\delta_{i,\varepsilon}} \right),$$
  where $\phi : [0, \infty) \to \mathbb{R}$ is a smooth function satisfying (2.15),
- for $n = 2$ the function $\tilde{\phi}_{i,\varepsilon}$ is given by
  $$\tilde{\phi}_{i,\varepsilon}(x) := \begin{cases} 
  \ln \frac{|x - x_{i,\varepsilon}|}{\delta_{i,\varepsilon}} - \ln \sqrt{\varepsilon d_{i,\varepsilon}} & \text{as } |x - x_{i,\varepsilon}| \in (d_{i,\varepsilon}, \sqrt{\varepsilon d_{i,\varepsilon}}), \\
  0 & \text{as } |x - x_{i,\varepsilon}| \geq \sqrt{\varepsilon d_{i,\varepsilon}},
  \end{cases}$$
- $G_{i,\varepsilon}, \phi_{i,\varepsilon} \in C^\infty(\overline{\Omega}_{\varepsilon})$ are given in (2.13), (2.14), respectively.

Note that

$$\text{supp}(\tilde{\phi}_{i,\varepsilon}(x)) \subset Y_{i,\varepsilon}, \quad \text{supp}(\phi_{i,\varepsilon}(x)) \subset Y_{i,\varepsilon}, \quad (5.8)$$

where $Y_{i,\varepsilon}$ is given in (4.27) (for $n = 2$ the first inclusion follows from (2.4)). It is easy to see that if $f \in \mathcal{H}^1_0(\Omega)$, then $J^1_\varepsilon f \in \mathcal{H}^1(\Omega_{\varepsilon})$ and $J^1_\varepsilon f = 0$ on $\partial \Omega$. Thus $J^1_\varepsilon$ is indeed a well-defined linear operator from $\mathcal{H}^1$ to $\mathcal{H}^1_{\varepsilon}$.

It is straightforward to check that

$$(u, J^1_\varepsilon f)_{\mathcal{H}^1_{\varepsilon}} - (J^1_\varepsilon u, f)_{\mathcal{H}^1} = 0, \quad \forall f \in \mathcal{H}, \ u \in \mathcal{H}_{\varepsilon}, \quad (5.9)$$

i.e., the condition (3.4) holds true with any $\delta_{i,\varepsilon} \geq 0$. Our goal is to show that the operators $J^1_\varepsilon, J^1_\varepsilon, J^1_{\varepsilon}, J^1_{\varepsilon}$ satisfy the conditions (3.5)–(3.7) with $\delta_{i,\varepsilon} \leq C\eta_{i,\varepsilon}$, here $\eta_{i,\varepsilon}$ is given in (2.8). We prove this in Subsections 5.1–5.4.

**Remark 5.1.** The function $J^1_\varepsilon f$ defined by (5.7) resembles special test-functions one uses in the so-called Tartar’s energy method (see, e.g., [15, Chapter 8]). For homogenization problems in domains with tiny holes this method was used by Cioranescu and Murat in [16]. Our present construction (5.7) is inspired by test-functions elaborated in [9, 10] for homogenization problems on manifolds with complicated microstructure. We adjusted $J^1_\varepsilon$ in such a way that for $f \in \mathcal{H}^1(\Omega)$ the function $u := J^1_\varepsilon f$ satisfies the Robin boundary conditions $\partial_n u + \gamma_{i,\varepsilon} u = 0$ on $\partial D_{i,\varepsilon}$ (this follows easily from the properties of the cut-off functions $\tilde{\phi}_{i,\varepsilon}$ and $\phi_{i,\varepsilon}$ and the behavior of $G_{i,\varepsilon}$ on $\partial D_{i,\varepsilon}$, see (5.5) below).

**Remark 5.2.** One can also define the operator $J^1_\varepsilon$ similarly to the operator $J^1_\varepsilon$, i.e.

$$J^1_\varepsilon f = f \mid_{\partial \Omega_{\varepsilon}}. \quad (5.10)$$
In Section 7 we show that, if \( I^0_\epsilon \) is defined by (5.10) and \( P, Q \) satisfy (1.13), then the conditions of Theorem 3.1 are fulfilled with \( \delta_\epsilon = C \eta''_\epsilon \to 0 \), where \( \eta''_\epsilon \) is defined by (7.1). In some cases \( \eta''_\epsilon \) gives better convergence rates comparing with the one in (2.9) and (2.11).

5.1. Check of condition (5.5). Let \( f \in \text{dom}(a) = H^1_0(\Omega) \). Recall that the set \( Y_{i,\epsilon} \) is defined by (4.27). Using \( |\phi_{i,\epsilon}| \leq 1, \phi_{i,\epsilon} \leq 1 \) and (5.8), we get

\[
\| f_{i,\epsilon} - f \|_{L^2(\Omega)}^2 + 2 \sum_{i \in I_\epsilon} |f_{i,\epsilon}|^2 \leq 2 \sum_{i \in I_\epsilon} \| f_{i,\epsilon} - f \|_{L^2(Y_{i,\epsilon})}^2 + 2 \sum_{i \in I_\epsilon} |G_{i,\epsilon}|^2 .
\]

The first term in the right-hand-side of (5.11) is estimated via the Poincaré inequality (4.1). Applying it for \( g := f_{i,\epsilon} - f \), we obtain

\[
\sum_{i \in I_\epsilon} \| f - f_{i,\epsilon} \|_{L^2(\Omega)}^2 \leq \pi^{-2} \epsilon^2 \sum_{i \in I_\epsilon} \| \nabla f \|_{L^2(\Omega)}^2 \leq \pi^{-2} \epsilon^2 \| \nabla f \|_{L^2(\Omega)}^2 .
\]

Now, we estimate the second term. The Cauchy-Schwarz inequality yields

\[
|f_{i,\epsilon}|^2 = \epsilon^{-2n} \left( \int_{\Omega} f(x) \, dx \right)^2 \leq \epsilon^{-n} \| f \|_{L^2(\Omega)}^2 .
\]

Let us prove that

\[
\| G_{i,\epsilon} \|_{L^2(Y_{i,\epsilon})}^2 \leq C \epsilon^n .
\]

Indeed, straightforward calculation gives

\[
\| G_{i,\epsilon} \|_{L^2(Y_{i,\epsilon})}^2 \leq CV^2 \epsilon^{2n} .
\]

\[
\| G_{i,\epsilon} \|_{L^2(Y_{i,\epsilon})}^2 = CV^2 \epsilon^{2n} .
\]

where \( C > 0 \) depends only on \( n \). From (1.11), (2.5) we conclude that \( \sup_{\epsilon \in (0, \varepsilon_0]} V_\epsilon Q^{-1/2} < \infty \) provided (5.1) holds. Hence (see the definition of \( Q_\epsilon \))

\[
V_\epsilon^2 \leq C d_\epsilon^{-n - 2} \epsilon^{-n} \text{ if } n \neq 2 .
\]

Using (5.16) and taking into account that \( 2d_\epsilon < \epsilon \leq \varepsilon_0 < 1 \), we deduce from (5.15):

\[
\| G_{i,\epsilon} \|_{L^2(Y_{i,\epsilon})}^2 \leq C \epsilon^{n+2} .
\]

(recall that \( \Lambda_\epsilon \) is given in (1.1)). Moreover, since \( \sup_{\epsilon \in (0, \varepsilon_0]} V_\epsilon < \infty \), we also get

\[
\| G_{i,\epsilon} \|_{L^2(Y_{i,\epsilon})}^2 \leq C \epsilon^{n+2} .
\]

(5.17)
Combining (5.17)–(5.18) and taking into account that \( \varepsilon < 1, \Lambda \varepsilon < 1 \) (and, consequently, \( \varepsilon \ln \varepsilon < 1, \Lambda \varepsilon \ln \Lambda \varepsilon < 1 \)), we arrive at the required estimate (5.14). Finally, by (5.13) and (5.14), we obtain the estimate for the second term on the right-hand-side of (5.11):
\[
\sum_{i \in I_e} |f_{i,e}|^2 \|G_{i,e}\|_{L^2(Y_{ie})}^2 \leq C \varepsilon^2 \sum_{i \in I_e} \|f\|_{L^2(D_{i,e})}^2 \leq C \varepsilon^2 \|f\|_{L^2(\Omega)}^2.
\]
(5.19)

It follows from (5.11), (5.12), (5.19) that
\[
\|I_1^1 f - J_1 f\|_{H^1} \leq C \varepsilon \|f\|_{H^1}.
\]
(5.20)

5.2. Check of condition (3.6). Let \( u \in \text{dom}(a) \). One has
\[
\|I_1^1 u - J_1 u\|_{H^1} = \sum_{i \in I_e} \|I_1^1 u\|_{L^2(D_{i,e})}^2.
\]
(5.21)

Applying Lemma 4.1 for \( D := \square_{i,e}, D_1 := D_{i,e}, D_2 := \square_{i,e}, g := J_1 u \), we get
\[
\|I_1^1 u\|_{L^2(\Omega)}^2 \leq C \left( \Lambda_e^{n/2} \|I_1^1 u\|_{L^2(D_{i,e})}^2 + \varepsilon \|
\right.
\]}
(5.22)

Using (5.6), (5.22), we can conclude (5.21) as follows:
\[
\|I_1^1 u - J_1 u\|_{H^1} \leq C \delta_{e,1} \|u\|_{H^1},
\]
(5.23)

where
\[
\delta_{e,1} := \max \left\{ \Lambda_e^{n/2}, (\varepsilon d_e)^{1/2} \right\}.
\]
(5.24)

5.3. Check of condition (3.7). Let \( u \in \text{dom}(a), f \in \text{dom}(A) = H^2(\Omega) \cap H^1_0(\Omega) \). One has:
\[
a_{i,e}[u, J_1^1 f] - a[J_1^1 u, f] = - \sum_{i \in I_e} \left( \nabla (I_1^1 u) \nabla f \right)_{L^2(D_{i,e})} + \sum_{i \in I_e} \left( \nabla u \nabla \left( (f_{i,e} - f) \phi_{i,e} \right) \right)_{L^2(\Omega)}
\]
\[
+ \sum_{i \in I_e} \left( \nabla u \nabla (f_{i,e} G_{i,e} \phi_{i,e}) \right)_{L^2(\Omega)} + \sum_{i \in I_e} \gamma_e \int_{\partial D_{i,e}} u \nabla f \cdot \nu \, ds - V(J_1^1 u, f)_{L^2(\Omega)}.
\]
(5.25)

Below we estimate each term on the right-hand-side of (5.25).

Estimate of \( I_1^1 \). One has using (5.6):
\[
|I_1^1|^2 \leq \sum_{i \in I_e} \|\nabla (I_1^1 u)\|_{L^2(D_{i,e})}^2 \sum_{i \in I_e} \|\nabla f\|_{L^2(D_{i,e})}^2 \leq \|I_1^1 u\|_{H^1(\Omega)}^2 \sum_{i \in I_e} \|\nabla f\|_{L^2(D_{i,e})}^2 \leq C \|u\|_{H^1}^2 \sum_{i \in I_e} \|\nabla f\|_{L^2(D_{i,e})}^2.
\]
(5.26)

Recall that \( \delta_{e,1} \) is defined in (5.24). Applying Lemma 4.1 for \( D := \square_{i,e}, D_1 := D_{i,e}, D_2 := \square_{i,e} \) and \( g := |\nabla f| \), one has (cf. (5.22))
\[
\sum_{i \in I_e} \|\nabla f\|_{L^2(D_{i,e})}^2 \leq C \delta_{e,1} \sum_{i \in I_e} \|f\|_{H^1(\Omega)}^2 \leq C \delta_{e,1} \|f\|_{H^2(\Omega)}^2.
\]
(5.27)

\[\text{In fact, since } \Lambda_e \to 0 \text{ as } \varepsilon \to 0, \text{ one has even better asymptotics } \|G_{i,e}\|_{L^2(Y_{ie})}^2 = \mathcal{O}(\varepsilon^{n+2}). \text{ However, it does not lead to an improvement of the resulting estimate (5.20) – this is hindered by the inequality (5.12).}\]
Furthermore, uniform regularity of $\Omega$ yields the estimate [12, Theorem 2]
\[
\forall f \in \text{dom}(A) : \|f\|_{H^2(\Omega)}^2 \leq C (\|Af\|_{H^2}^2 + \|f\|_{H^2}^2) \leq C \|f\|_{H^2}^2
\]  
(5.28)
(the last inequality in (5.28) follows from $(Af, f)_{H^2} \geq 0$). Combining (5.26)–(5.28), we get
\[
|I_\delta^1| \leq C\delta_{\epsilon,1} \|f\|_{H^2} \|u\|_{H^2}.
\]  
(5.29)

**Estimate of $I_\epsilon^2$.** We denote
\[
F_{i,\epsilon} := \left\{ x \in \mathbb{R}^n : d_{\epsilon} < |x - x_{i,\epsilon}| < \tilde{d}_{\epsilon} \right\}, \text{ where } \tilde{d}_{\epsilon} := \begin{cases} 2d_{\epsilon}, & n \geq 3, \\ \sqrt{\epsilon d_{\epsilon}}, & n = 2. \end{cases}
\]  
(5.30)
It is easy to see that $\text{supp}(\tilde{\phi}_{i,\epsilon}) \subset F_{i,\epsilon}$. Using this fact, $\|\nabla u\|_{L^2(\Omega)} \leq \|u\|_{H^2}$ and $|\tilde{\phi}_{i,\epsilon}(x)| \leq 1$, we get
\[
|I_\epsilon^2| = \left| \sum_{i \in I_\epsilon} \left[ (\nabla u, \tilde{\phi}_{i,\epsilon} \nabla f)_{L^2(F_{i,\epsilon})} + (\nabla u, (f - f_{i,\epsilon}) \nabla \tilde{\phi}_{i,\epsilon})_{L^2(F_{i,\epsilon})} \right] \right|
\leq \|u\|_{H^2} \left[ \left( \sum_{i \in I_\epsilon} \|\nabla f\|_{L^2(F_{i,\epsilon})}^2 \right)^{1/2} + \sum_{i \in I_\epsilon} \|(f - f_{i,\epsilon}) \nabla \tilde{\phi}_{i,\epsilon}\|_{L^2(F_{i,\epsilon})}^2 \right]^{1/2}.
\]  
(5.31)
Using Lemma 4.1 with $D := \Box_{i,\epsilon}$, $D_1 := F_{i,\epsilon}$, $D_2 := \Box_{i,\epsilon}$ and $g := |\nabla f|$, we obtain
\[
\|\nabla f\|_{L^2(F_{i,\epsilon})} \leq C\delta_{\epsilon,2} \|f\|_{H^2(\Box_{i,\epsilon})}, \text{ where } \delta_{\epsilon,2} := \begin{cases} \delta_{\epsilon,1}, & n \geq 3, \\ \max \left\{ \Lambda_{\epsilon}^{1/2}; \epsilon^{3/4}d_{\epsilon}^{1/4} \right\}, & n = 2, \end{cases}
\]  
(5.32)
where $\delta_{\epsilon,1}$ is given in (5.24). Taking into account (5.28), we get finally
\[
\left( \sum_{i \in I_\epsilon} \|\nabla f\|_{L^2(F_{i,\epsilon})}^2 \right)^{1/2} \leq C\delta_{\epsilon,2} \|f\|_{H^2}.
\]  
(5.33)
To proceed further we need the classical Hölder inequality
\[
\|FG\|_{L^1(Y_{i,\epsilon})} \leq \|F\|_{L^p(Y_{i,\epsilon})} \|G\|_{L^q(Y_{i,\epsilon})}, \quad \forall p, q \in [1, \infty], \ p^{-1} + q^{-1} = 1
\]
(for $p = \infty$ one has the convention $p^{-1} = 0$). Setting $p := p/2$, $q := q/2$, $F := |f - f_{i,\epsilon}|^2$ and $G := |
abla \tilde{\phi}_{i,\epsilon}|^2$, we obtain
\[
\|(f - f_{i,\epsilon}) \nabla \tilde{\phi}_{i,\epsilon}\|_{L^2(Y_{i,\epsilon})}^2 \leq \|f - f_{i,\epsilon}\|_{L^2(Y_{i,\epsilon})}^2 \|\nabla \tilde{\phi}_{i,\epsilon}\|_{L^2(Y_{i,\epsilon})}^2, \quad \forall p, q \in [2, \infty], \ \frac{1}{p} + \frac{1}{q} = \frac{1}{2}.
\]  
(5.34)
Recall that $\Lambda_{\epsilon}$ is given in [11]; due to (2.4), $|\ln \Lambda_{\epsilon}| \geq \ln 4 > 1$. We choose $p$, $q$ as follows:
\[
p = \frac{2n}{n - 4} \text{ if } n \geq 5, \quad p = 2|\ln \Lambda_{\epsilon}| \text{ if } n = 4, \quad p = \infty \text{ if } n = 2, 3, \quad p = \infty \text{ if } n = 2, 3,
\]  
\[
q = \frac{n}{2} \text{ if } n \geq 5, \quad q = \frac{2}{1 - |\ln \Lambda_{\epsilon}|^{-1}} \text{ if } n = 4, \quad q = 2 \text{ if } n = 2, 3
\]  
(5.35)
(thus \( p^{-1} + q^{-1} = 1/2 \), moreover, \( p \) satisfies (4.2)). Using (4.4) for \( g := f - f_{i,e} \) and \( p \) as in (5.35) and taking into account (4.9), we obtain the estimate

\[
\|f - f_{i,e}\|_{L^p(\Delta_{i,e})} \leq C\|f\|_{H^2(\Delta_{i,e})} \begin{cases} 
\epsilon^{-1} & n \geq 5, \\
|\ln \Lambda_\epsilon| \cdot \epsilon^2 |\ln |\Lambda_\epsilon|^{-1} - 1| & n = 4, \\
\epsilon^{-1/2} & n = 3, \\
1 & n = 2.
\end{cases}
\]  

(5.37)

Moreover, straightforward calculation gives

\[
\|\nabla \tilde{\phi}_{i,e}\|_{L^2(Y_{i,e})} \leq C \begin{cases} 
\delta_{i,e} & n \geq 5, \\
\delta_{i,e}^{1 - 2|\ln \Lambda_\epsilon|^{-1}} & n = 4, \\
\delta_{i,e}^{1/2} & n = 3, \\
|\ln \Lambda_\epsilon|^{-1/2} & n = 2,
\end{cases}
\]  

(5.38)

where \( q \) is defined by (5.36). Using (5.34), (5.37), (5.38) and \( \Lambda_\epsilon^{-2|\ln \Lambda_\epsilon|^{-1}} = \exp(2) \), we get

\[
\|(f - f_{i,e})\nabla \tilde{\phi}_{i,e}\|_{L^2(Y_{i,e})} \leq C \delta_{i,3} \|f\|_{H^2(\Delta_{i,e})}, \quad \text{where } \delta_{i,3} := \begin{cases} 
\Lambda_\epsilon & n \geq 5, \\
|\ln \Lambda_\epsilon| & n = 4, \\
\Lambda_\epsilon^{1/2} & n = 3, \\
|\ln \Lambda_\epsilon|^{-1/2} & n = 2.
\end{cases}
\]  

(5.39)

Taking into account (5.28), we get from (5.39):

\[
\left( \sum_{i \in I_\epsilon} \|f - f_{i,e}\nabla \tilde{\phi}_{i,e}\|_{L^2(Y_{i,e})}^2 \right)^{1/2} \leq C \delta_{i,3} \|f\|_{H^2}. \]  

(5.40)

Combining (5.31), (5.33), (5.40), we arrive at the estimate

\[
|I_{\epsilon}^2| \leq C \max \{\delta_{i,2}; \delta_{i,3}\} \|f\|_{H^2}\|u\|_{H^1}. \]  

(5.41)

Estimate of \( I_3^3 \). It is easy to see that

\[
(f_{i,e}(f) (x) = f_{i,e}(G_{i,e}(x) + 1), \quad x \in \partial D_{i,e}.
\]  

(5.42)

Moreover, since

\[
(\phi_{i,e} - 1) |_{\partial D_{i,e}} = \phi_{i,e} |_{\partial Y_{i,e} \setminus \partial D_{i,e}} = 0, \quad \frac{\partial \phi_{i,e}}{\partial n} |_{\partial D_{i,e}} = \frac{\partial \phi_{i,e}}{\partial n} |_{\partial Y_{i,e} \setminus \partial D_{i,e}} = 0,
\]  

one has

\[
\frac{\partial (f_{i,e}G_{i,e}\phi_{i,e})}{\partial n} = f_{i,e} \frac{\partial G_{i,e}}{\partial n} \quad \text{on } \partial D_{i,e},
\]  

(5.43)

\[
\frac{\partial (f_{i,e}G_{i,e}\phi_{i,e})}{\partial n} = 0 \quad \text{on } \partial Y_{i,e} \setminus \partial D_{i,e}.
\]  

(5.44)
Using (5.42)–(5.44) and integrating by parts we get
\[ I_3^\varepsilon = - \sum_{i \in I_\varepsilon} (u, \Delta (f_{i,\varepsilon} G_{i,\varepsilon} \Phi_{i,\varepsilon}))_{L^2(Y_{i,\varepsilon})} - \sum_{i \in I_\varepsilon} (u - u_{i,\varepsilon}, \Delta (f_{i,\varepsilon} G_{i,\varepsilon} \Phi_{i,\varepsilon}))_{L^2(Y_{i,\varepsilon})} \]

where
\[ I_3^\varepsilon = - \sum_{i \in I_\varepsilon} (u, \Delta (f_{i,\varepsilon} G_{i,\varepsilon} \Phi_{i,\varepsilon}))_{L^2(Y_{i,\varepsilon})} - \sum_{i \in I_\varepsilon} (u - u_{i,\varepsilon}, \Delta (f_{i,\varepsilon} G_{i,\varepsilon} \Phi_{i,\varepsilon}))_{L^2(Y_{i,\varepsilon})} \]

Integrating by parts yields
\[ \frac{\partial G_{i,\varepsilon}}{\partial n} + \gamma_{i,\varepsilon} (G_{i,\varepsilon} + 1) = 0 \quad \text{on } \partial D_{i,\varepsilon}, \quad (5.45) \]

whence
\[ \frac{\partial G_{i,\varepsilon}}{\partial n} = - \gamma_{i,\varepsilon} \quad \text{on } \partial D_{i,\varepsilon}, \quad (5.46) \]

Using (5.43), (5.44), (5.47) and Gauss’s divergence theorem, we can transform (5.46) as follows:

Evidently, one has
\[ I_3^3 = \sum_{i \in I_\varepsilon} (V_{i,\varepsilon} - V) u_{i,\varepsilon} f_{i,\varepsilon} \varepsilon^n - \sum_{i \in I_\varepsilon} V (\bar{f}_{i,\varepsilon} - f_{i,\varepsilon})_{L^2(\square_{i,\varepsilon})} + \sum_{i \in I_\varepsilon} (u - u_{i,\varepsilon}, \Delta (f_{i,\varepsilon} G_{i,\varepsilon} \Phi_{i,\varepsilon}))_{L^2(Y_{i,\varepsilon})} \]

Then, using (4.1), (5.13) and similarly the Cauchy-Schwarz inequality for \( u_{i,\varepsilon} \), we get
\[ |I_3^3| \leq \sum_{i \in I_\varepsilon} |V_{i,\varepsilon} - V|^2 \left( \sum_{i \in I_\varepsilon} |f_{i,\varepsilon}|^2 \right) \frac{1}{2} \left( \sum_{i \in I_\varepsilon} |u_{i,\varepsilon}|^2 \right) \frac{1}{2} \]

\[ + V \left( \sum_{i \in I_\varepsilon} \|f - f_{i,\varepsilon}\|^2_{L^2(\square_{i,\varepsilon})} \right) \frac{1}{2} \left( \sum_{i \in I_\varepsilon} \|\bar{f}_{i,\varepsilon}\|^2_{L^2(\square_{i,\varepsilon})} \right) \frac{1}{2} \]

\[ \leq |V_{i,\varepsilon} - V|^2 \|f\|^2_{L^2(\Omega)} \|\bar{f}_{i,\varepsilon}\|^2_{L^2(\square_{i,\varepsilon})} + V \varepsilon \|\nabla f\|^2_{L^2(\Omega)} \|\bar{f}_{i,\varepsilon}\|^2_{L^2(\Omega)}, \]
whence, taking into account (5.6), we conclude
\[ |I^3_{\varepsilon}| \leq C \max\{ |V_{\varepsilon} - V|; \varepsilon \} \|f\|_{H^1} \|u\|_{H^1}. \] (5.48)

We denote
\[ \overline{\mathcal{Y}}_{i,\varepsilon} := \left\{ x \in \mathbb{R}^n : \frac{\varepsilon}{4} < |x| < \frac{\varepsilon}{2} \right\}. \] (5.49)

One has supp\((D^a\phi_{i,\varepsilon}) \subset \overline{\mathcal{Y}}_{i,\varepsilon},|a|=1,2.\) Using this and the fact that \(\Delta G_{i,\varepsilon} = 0\) for \(x \neq x_{i,\varepsilon},\) we estimate the term \(I^3_{\varepsilon}\) as follows:
\[ I^3_{\varepsilon} := \left| \sum_{i \in I_\varepsilon} (u - u_{i,\varepsilon}) 2f_{i,\varepsilon} \nabla G_{i,\varepsilon} \cdot \nabla \phi_{i,\varepsilon} + f_{i,\varepsilon} G_{i,\varepsilon} \Delta \phi_{i,\varepsilon} \right|_{L^2(\mathcal{Y}_{i,\varepsilon})}^2 \leq \left( \sum_{i \in I_\varepsilon} \|u - u_{i,\varepsilon}\|_{L^2(\mathcal{Y}_{i,\varepsilon})}^2 \right)^{1/2} \times \left( \sum_{i \in I_\varepsilon} 2|f_{i,\varepsilon}|^2 \left( \|2\nabla G_{i,\varepsilon} \cdot \nabla \phi_{i,\varepsilon}\|_{L^2(\mathcal{Y}_{i,\varepsilon})}^2 + \|G_{i,\varepsilon} \Delta \phi_{i,\varepsilon}\|_{L^2(\mathcal{Y}_{i,\varepsilon})}^2 \right) \right)^{1/2}. \]

Using (4.1) and (5.6), we get
\[ \sum_{i \in I_\varepsilon} \|u - u_{i,\varepsilon}\|_{L^2(\mathcal{Y}_{i,\varepsilon})}^2 \leq \sum_{i \in I_\varepsilon} \|\mathcal{J}_\varepsilon u - u_{i,\varepsilon}\|_{L^2(\mathcal{Y}_{i,\varepsilon})}^2 \leq C\varepsilon^2 \sum_{i \in I_\varepsilon} \|\nabla(\mathcal{J}_\varepsilon u)\|_{L^2(\mathcal{Y}_{i,\varepsilon})}^2 \leq C\varepsilon^2 \|u\|_{H^1}. \] (5.50)
Furthermore, taking into account (2.7), one can easily get the pointwise estimate
\[ |G_{i,\varepsilon}(x)| \leq C\varepsilon \delta_{\varepsilon A}, \quad |\nabla G_{i,\varepsilon}(x)| \leq C\varepsilon, \quad |\Delta \phi_{i,\varepsilon}(x)| \leq C\varepsilon^{-2}, \quad |\nabla \phi_{i,\varepsilon}(x)| \leq C\varepsilon^{-1}, \] (5.51)
where
\[ \delta_{\varepsilon A} := \begin{cases} \varepsilon, & n \geq 3, \\ |\varepsilon| \ln |\varepsilon|, & n = 2 \end{cases} \] (5.52)

Using (5.13), (5.51) and \(|\overline{\mathcal{Y}}_{i,\varepsilon}| \leq C\varepsilon^n,\) one gets
\[ |f_{i,\varepsilon}|^2 \left( \|2\nabla G_{i,\varepsilon} \cdot \nabla \phi_{i,\varepsilon}\|_{L^2(\overline{\mathcal{Y}}_{i,\varepsilon})}^2 + \|G_{i,\varepsilon} \Delta \phi_{i,\varepsilon}\|_{L^2(\overline{\mathcal{Y}}_{i,\varepsilon})}^2 \right) \leq C(1 + \varepsilon^{-2}(\delta_{\varepsilon A})^2) \|f\|_{L^2(\mathcal{Y}_{i,\varepsilon})}. \] (5.53)

Taking into account
\[ \varepsilon \leq |\ln \varepsilon_0|^{-1}\delta_{\varepsilon A} \quad \text{as } n = 2 \] (5.54)
(recall that \(\varepsilon \leq \varepsilon_0 < 1\)) we infer from (5.50), (5.53) that
\[ |I^3_{\varepsilon}| \leq C\varepsilon \|f\|_{H^1} \|u\|_{H^1}. \] (5.55)

Finally, using Lemma 4.5 for \(g := f\) and \(g := \mathcal{J}_\varepsilon u\) and (5.6), one gets the estimate for \(I^3_{\varepsilon}:\)
\[ |I^3_{\varepsilon}| \leq C\varepsilon^2 \|\nabla f\|_{L^2(\mathcal{Y}_{i,\varepsilon})} \|\nabla(\mathcal{J}_\varepsilon u)\|_{L^2(\mathcal{Y}_{i,\varepsilon})} \leq C\varepsilon^2 \|f\|_{H^1} \|u\|_{H^1}, \] (5.56)
and, as a result (see (5.48), (5.55), (5.56)), we arrive at the estimate for \(I^3:\)
\[ |I^3| \leq C \max\{ |V_{\varepsilon} - V|; \varepsilon; \delta_{\varepsilon A}; \delta_{\varepsilon A}^2 \} \|f\|_{H^1} \|u\|_{H^1} \leq C \max\{ |V_{\varepsilon} - V|; \delta_{\varepsilon A} \} \|f\|_{H^1} \|u\|_{H^1}. \] (5.57)
(in the last inequality in (5.57) we use \(\varepsilon < 1\) and (5.54)). Combining (3.29), (4.1), (5.57) and taking into account (3.3), we get
\[ |a_{\varepsilon}[u, J^3_{\varepsilon} f] - a[J^3_{\varepsilon} u, f]| \leq C \max\{ \delta_{\varepsilon 1}; \delta_{\varepsilon 2}; \delta_{\varepsilon 3}; \delta_{\varepsilon 4}; |V_{\varepsilon} - V| \} \|f\|_{H^2} \|u\|_{H^1}. \] (5.58)
5.4. **Conclusion.** Combining (5.20), (5.23), (5.58) and taking into account (5.54), we conclude that the conditions (3.5)–(3.7) of Theorem 3.1 hold with
\[ \delta_\varepsilon := C \max \{ \delta_{\varepsilon,1}; \delta_{\varepsilon,2}; \delta_{\varepsilon,3}; \delta_{\varepsilon,4}; |V_\varepsilon - V| \} , \]
where \( \delta_{\varepsilon,k}, k = 1,2,3,4 \) are defined in (5.24), (5.32), (5.39), (5.52), respectively. It is straightforward to check that
\[ \delta_\varepsilon \leq C \eta_\varepsilon, \] (5.59)
where \( \eta_\varepsilon \) is defined by (2.8), Q.E.D.

**Remark 5.3.** Recall that \( \partial \Omega \) is assumed to be uniformly regular [12]. We need it to guarantee the fulfillment of (5.28). However, it is well-known that (5.28) remains valid under less restrictive assumptions on \( \Omega \), for example, if \( \partial \Omega \) is compact and belongs to the \( C^{1,1} \) class or if \( \Omega \) is a convex domain with Lipschitz boundary [28 Theorems 2.2.2.3 and 3.2.1.2].

### 6. Proof of the main results

6.1. **Proof of Theorem 2.1** In Section 5 we established the fulfillment of the properties (3.4)–(3.7) with \( \delta_\varepsilon \leq C \eta_\varepsilon \). Then, by Theorem 3.1, we immediately arrive at the required estimate (2.9). Theorem 2.1 is proven.

6.2. **Proof of Theorem 2.2** Let \( g \in L^2(\Omega) \), and \( f := (A + I)^{-1}g \). One has
\[ \|(A_\varepsilon + I)^{-1}J_\varepsilon g - J_\varepsilon (A + I)^{-1}g\|_{H^1(\Omega)} \]
\[ \leq \|(A_\varepsilon + I)^{-1}J_\varepsilon g - J_\varepsilon^1 (A + I)^{-1}g\|_{H^1(\Omega)} + \|(J_\varepsilon^1 - J_\varepsilon)(A + I)^{-1}g\|_{H^1(\Omega)} \]
\[ \leq \|(A_\varepsilon + I)^{-1}J_\varepsilon g - J_\varepsilon^1 (A + I)^{-1}g\|_{H_1^1} + \|(J_\varepsilon^1 - J_\varepsilon)f\|_{H^1(\Omega)}. \] (6.1)

In Section 5 we proved that (3.4)–(3.7) hold with \( \delta_\varepsilon \leq C \eta_\varepsilon \). Hence by Theorem 3.3 we have
\[ \|(A_\varepsilon + I)^{-1}J_\varepsilon g - J_\varepsilon^1 (A + I)^{-1}g\|_{H_1^1} \leq C \eta_\varepsilon \|g\|_{L^2(\Omega)}. \] (6.2)

Now, we estimate the second term in the right-hand-side of (6.2). Recall that \( Y_{i,\varepsilon} \) is given in (4.27) and the inclusions (5.8) hold. One has:
\[ \|(J_\varepsilon^1 - J_\varepsilon)f\|_{H^1(\Omega)} \leq 2 \sum_{i \in I_\varepsilon} \|f - f_{i,\varepsilon}\|_{H^1(\Omega)} + 2 \sum_{i \in I_\varepsilon} \|G_{i,\varepsilon}^1 f_{i,\varepsilon}\|_{H^1(\Omega)}. \] (6.3)

Using (5.12), (5.19) and \( |\phi_{i,\varepsilon}| \leq 1, |\phi_{i,\varepsilon}| \leq 1 \), we get
\[ \sum_{i \in I_\varepsilon} \|f - f_{i,\varepsilon}\|_{L^2(\Omega)} \leq C \varepsilon^2 \sum_{i \in I_\varepsilon} \|\nabla f\|_{L^2(\Omega)} \leq C \varepsilon^2 \|f\|_{H^1}, \] (6.4)
\[ \sum_{i \in I_\varepsilon} \|G_{i,\varepsilon}^1 f_{i,\varepsilon}\|_{L^2(\Omega)} \leq C \varepsilon^2 \sum_{i \in I_\varepsilon} \|f_{i,\varepsilon}\|_{L^2(\Omega)} \leq C \varepsilon^2 \|f\|_{H^1}. \] (6.5)

Moreover, one has (see (5.33), (5.40) and note supp(\( \phi_{i,\varepsilon} \)) \( \subset F_{i,\varepsilon} \), where the set \( F_{i,\varepsilon} \) is defined in (5.30):\n\[ \sum_{i \in I_\varepsilon} \|\nabla((f - f_{i,\varepsilon}) \phi_{i,\varepsilon})\|_{L^2(\Omega)} \leq 2 \sum_{i \in I_\varepsilon} \|\nabla f\|_{L^2(\Omega)} + 2 \sum_{i \in I_\varepsilon} \|f - f_{i,\varepsilon}\|_{L^2(\Omega)} \]
\[ \leq C \max \{ (\delta_{\varepsilon,2})^2; (\delta_{\varepsilon,3})^2 \} \|f\|_{H^1}, \] (6.6)
where $\delta_{1,2}$, $\delta_{1,3}$ are given in (5.32), (5.39), respectively. Finally, using (5.13) and taking into account that $|\phi_{i,\varepsilon}| \leq 1$, $|\nabla \phi_{i,\varepsilon}| \leq C\varepsilon^{-1}$, we obtain

$$
\sum_{i \in \mathcal{I}_e} \| \nabla \left( f_{i,\varepsilon} G_{i,\varepsilon} \phi_{i,\varepsilon} \right) \|_{L^2(\Omega_{i,\varepsilon})}^2 \leq C\varepsilon^{-n} \sum_{i \in \mathcal{I}_e} \| f \|_{L^2(\mathcal{D}_{i,\varepsilon})}^2 \left( \| \nabla G_{i,\varepsilon} \|_{L^2(\Omega_{i,\varepsilon})}^2 + \varepsilon^{-2} \| G_{i,\varepsilon} \|_{L^2(\Omega_{i,\varepsilon})}^2 \right),
$$
(6.7)

where $\bar{Y}_{i,\varepsilon}$ is given in (5.49); here we have used that $\text{supp}(\nabla \phi_{i,\varepsilon}) \subset \bar{Y}_{i,\varepsilon}$. Straightforward calculation yields

$$
\| \nabla G_{i,\varepsilon} \|_{L^2(\Omega_{i,\varepsilon})}^2 \leq \| \nabla G_{i,\varepsilon} \|_{L^2(\mathbb{R}^n \setminus \mathcal{D}_{i,\varepsilon})}^2 = V_\varepsilon^2 Q_\varepsilon^{-1} \delta_{1,4}^2, \quad n \geq 3,
$$
(6.8)

$$
\| \nabla G_{i,\varepsilon} \|_{L^2(\Omega_{i,\varepsilon})}^2 \leq \| \nabla G_{i,\varepsilon} \|_{L^2(B_1(x_{i,\varepsilon}) \setminus \mathcal{D}_{i,\varepsilon})}^2 = V_\varepsilon^2 Q_\varepsilon^{-1} \varepsilon^2, \quad n = 2,
$$
where $B_1(x_{i,\varepsilon})$ is the unit ball with the center at $x_{i,\varepsilon}$. Moreover, using the pointwise estimates (5.51) and $|\bar{Y}_{i,\varepsilon}| \leq C\varepsilon^n$, we get

$$
\| G_{i,\varepsilon} \|_{L^2(\mathcal{D}_{i,\varepsilon})}^2 \leq C\varepsilon^{n+2} \delta_{1,4}^2,
$$
(6.9)

where $\delta_{1,4}$ is given in (5.52). It follows from (6.7)–(6.9) that

$$
\sum_{i \in \mathcal{I}_e} \| \nabla \left( f_{i,\varepsilon} G_{i,\varepsilon} \phi_{i,\varepsilon} \right) \|_{L^2(\Omega_{i,\varepsilon})}^2 \leq C \max \left\{ V_\varepsilon^2 Q_\varepsilon^{-1} ; \delta_{1,4}^2 \right\} \| f \|_{L^2(\Omega)}^2.
$$
(6.10)

Combining (6.3)–(6.6), (6.10) and taking into account (3.3), (5.59) and $\varepsilon \leq C\delta_{1,4}$, we arrive at

$$
\| (J_1 - f_\varepsilon) g \|_{H^1(\Omega)} \leq C \max \left\{ V_\varepsilon Q_\varepsilon^{-1/2} ; \delta_{1,2} ; \delta_{1,3} ; \delta_{1,4} \right\} \| f \|_{H^2} \leq C \max \left\{ V_\varepsilon Q_\varepsilon^{-1/2} ; \eta_\varepsilon \right\} \| g \|_{L^2(\Omega)}.
$$
(6.11)

The required estimate (2.11) follows from (6.1), (6.2), (6.11) and the definition (2.10) of $\eta_\varepsilon$.

### 6.3. Proof of Theorem 2.4

Let $g \in L^2(\Omega)$, and $f := (A + I)^{-1} g$. One has

$$
\| (A_\varepsilon + I)^{-1} J_\varepsilon g - (I + G_\varepsilon) J_\varepsilon (A + I)^{-1} g \|_{H^1(\Omega)} \leq \| (A_\varepsilon + I)^{-1} J_\varepsilon g - J_\varepsilon^1 (A + I)^{-1} g \|_{H^1(\Omega)} + \| (J_\varepsilon^1 - (I + G_\varepsilon) J_\varepsilon) f \|_{H^1(\Omega)}.
$$
(6.12)

Again, using Theorem 3.3 and taking into account (2.19), we have

$$
\| (A_\varepsilon + I)^{-1} J_\varepsilon g - J_\varepsilon^1 (A + I)^{-1} g \|_{H^1(\Omega)} \leq C\bar{\eta}_\varepsilon \| g \|_{L^2(\Omega)},
$$
(6.13)

where $\bar{\eta}_\varepsilon$ is defined by (2.16).

Now, we estimate the second term on the right-hand-side of (6.12). One has

$$
(J_\varepsilon^1 - (I + G_\varepsilon) J_\varepsilon) f = \sum_{i \in \mathcal{I}_e} (f_{i,\varepsilon} - f) G_{i,\varepsilon} \phi_{i,\varepsilon} + \sum_{i \in \mathcal{I}_e} (f_{i,\varepsilon} - f) G_{i,\varepsilon} \bar{\phi}_{i,\varepsilon}.
$$
(6.14)

Using (2.19), (5.59), (6.4), (6.6), we get the estimate

$$
\left\| \sum_{i \in \mathcal{I}_e} (f - f_{i,\varepsilon}) \bar{\phi}_{i,\varepsilon} \right\|_{H^1(\Omega)} = \left( \sum_{i \in \mathcal{I}_e} \| (f - f_{i,\varepsilon}) \bar{\phi}_{i,\varepsilon} \|_{H^1(\Omega_{i,\varepsilon})}^2 \right)^{1/2} \leq C \max \{ \varepsilon ; \delta_{1,2} ; \delta_{1,3} \} \| f \|_{H^2} \leq C\bar{\eta}_\varepsilon \| f \|_{H^2} = C\bar{\eta}_\varepsilon \| g \|_{H^2}.
$$
(6.15)

To proceed further, we observe that

$$
\begin{align*}
\text{n} \geq 3 : & \quad \| G_{i,\varepsilon} \|_{L^\infty(\mathbb{R}^n \setminus \mathcal{D}_{i,\varepsilon})} \\
\text{n} = 2 : & \quad \| G_{i,\varepsilon} \|_{L^\infty(B_1(x_{i,\varepsilon}) \setminus \mathcal{D}_{i,\varepsilon})} \\
& \leq \| G_{i,\varepsilon} \|_{L^\infty(\partial \mathcal{D}_{i,\varepsilon})} = V_\varepsilon Q_\varepsilon^{-1} = \frac{P_\varepsilon}{P_\varepsilon + Q_\varepsilon} \leq 1.
\end{align*}
$$
(6.16)
Using (4.1), (6.16), and \( |\phi_{i,\varepsilon}| \leq 1 \), we get
\[
\left\| \sum_{i \in I_{\varepsilon}} (f_{i,\varepsilon} - f) G_{i,\varepsilon} \phi_{i,\varepsilon} \right\|_{L^2(\Omega_{\varepsilon})} \leq \left( \sum_{i \in I_{\varepsilon}} \left\| f_{i,\varepsilon} - f \right\|^2_{L^2(\Omega_{\varepsilon})} \right)^{1/2} \leq C \varepsilon \| \nabla f \|_{L^2(\Omega)} \leq C \eta_{\varepsilon} \| f \|_{H^1}. \tag{6.17}
\]
Furthermore, we have
\[
\left\| \nabla \left( \sum_{i \in I_{\varepsilon}} (f_{i,\varepsilon} - f) G_{i,\varepsilon} \phi_{i,\varepsilon} \right) \right\|_{L^2(\Omega_{\varepsilon})} \leq \left( \sum_{i \in I_{\varepsilon}} \left\| G_{i,\varepsilon} \phi_{i,\varepsilon} \nabla f \right\|^2_{L^2(\Omega_{\varepsilon})} \right)^{1/2} + \left( \sum_{i \in I_{\varepsilon}} \left\| (f - f_{i,\varepsilon}) \nabla (G_{i,\varepsilon} \phi_{i,\varepsilon}) \right\|^2_{L^2(\Omega_{\varepsilon})} \right)^{1/2}.
\]

Estimate of \( I^1_\varepsilon \). One has (cf. (5.34)):
\[
I^1_\varepsilon \leq \left( \sum_{i \in I_{\varepsilon}} \left\| \nabla f \right\|^2_{L^p(Y_{\varepsilon}^i)} \left\| G_{i,\varepsilon} \phi_{i,\varepsilon} \right\|^2_{L^q(Y_{\varepsilon}^i)} \right)^{1/2} , \quad p, q \in [2, \infty], \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}. \tag{6.18}
\]
We choose \( p \) and \( q \) as follows:
\[
p = \frac{2n}{n - 2}, \quad q = n \quad \text{if} \quad n \geq 3 \quad \text{and} \quad p = 6, \quad q = 3 \quad \text{if} \quad n = 2,
\]
that is \( p^{-1} + q^{-1} = 1/2 \), moreover, \( p \) satisfies (4.3). Then, using the estimate (4.3), we get
\[
\left\| \nabla f \right\|_{L^p(\Omega_{\varepsilon})} \leq C \| f \|_{H^2(\Omega_{\varepsilon})} \begin{cases} \varepsilon^{-1} & n \geq 3, \\ \varepsilon^{-2/3} & n = 2. \end{cases} \tag{6.19}
\]
Moreover, using (2.18) and \( |\phi_{i,\varepsilon}| \leq 1 \), we obtain via direct calculation (see the similar calculations in (5.15), where \( q = 2 \)):
\[
\left\| G_{i,\varepsilon} \phi_{i,\varepsilon} \right\|_{L^q(Y_{\varepsilon}^i)} \leq \left\| G_{i,\varepsilon} \right\|_{L^q(Y_{\varepsilon}^i)} \leq C \begin{cases} \varepsilon^{n/(n-2)}, & n \geq 4, \\ \varepsilon^3 |\ln \varepsilon|^{1/3}, & n = 3, \\ \varepsilon^{8/3} |\ln \varepsilon|, & n = 2. \end{cases} \tag{6.20}
\]
Combining (6.18), (6.19), (6.20), we arrive at the estimate
\[
I^1_\varepsilon \leq C \| f \|_{H^2(\Omega)} \begin{cases} \varepsilon^{2/(n-2)}, & n \geq 4, \\ \varepsilon^2 |\ln \varepsilon|^{1/3}, & n = 3, \\ \varepsilon^2 |\ln \varepsilon|, & n = 2. \end{cases} \tag{6.21}
\]

Estimate of \( I^2_\varepsilon \). One has (cf. (5.34)):
\[
I^2_\varepsilon \leq \left( \sum_{i \in I_{\varepsilon}} \left\| f - f_{i,\varepsilon} \right\|^2_{L^p(Y_{\varepsilon}^i)} \left\| \nabla (G_{i,\varepsilon} \phi_{i,\varepsilon}) \right\|^2_{L^q(Y_{\varepsilon}^i)} \right)^{1/2} , \quad p, q \in [2, \infty], \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}. \tag{6.22}
\]
We choose \( p \) and \( q \) by (5.35) and (5.36), respectively. The first factor on the right-hand-side of (6.22) is already estimated in (5.37). The second factor can be estimated via simple
calculation (taking into account (2.19)):

\[ \| \nabla (G_{i,\varepsilon}\phi_{i,\varepsilon}) \|_{L^2(\Omega_{\varepsilon})} \leq C \left( \| \nabla G_{i,\varepsilon} \|_{L^2(\Omega_{\varepsilon})} + \varepsilon^{-1} \| G_{i,\varepsilon} \|_{L^4(\Omega_{\varepsilon})} \right) \leq C \begin{cases} \varepsilon^{n/(n-2)}, & n \geq 5, \\ \varepsilon^{2-4|\ln \Lambda_{\varepsilon}|^{-1}}, & n = 4, \\ \varepsilon^{3/2}, & n = 3, \\ \varepsilon, & n = 2 \end{cases} \]

(recall that \( \overline{\Omega_{i,\varepsilon}} \) is defined by (5.49), \( \text{supp}(\nabla \phi_{i,\varepsilon}) \subset \overline{\Omega_{i,\varepsilon}} \), and \( |\nabla \phi_{i,\varepsilon}| \leq C \varepsilon^{-1} \). Combining (5.28), (5.37), (6.22), (6.23), and taking into account that \( \varepsilon^{-2|\ln \Lambda_{\varepsilon}|^{-1}} \leq C \) as \( n = 4 \) (this estimate follows from (2.19)), we get

\[ \mathcal{I}_{\varepsilon}^2 \leq C \| f \|_{H^2} \begin{cases} \varepsilon^{2/(n-2)}, & n \geq 5, \\ \varepsilon |\ln \varepsilon|, & n = 4, \\ \varepsilon, & n = 2,3. \end{cases} \]

It follows from (6.21), (6.24) that

\[ \| \nabla \left( \sum_{i \in I_{\varepsilon}} (f_{i,\varepsilon} - f) G_{i,\varepsilon}\phi_{i,\varepsilon} \right) \|_{L^2(\Omega_{\varepsilon})} \leq C \eta_{\varepsilon} \| f \|_{H^2} = C \eta_{\varepsilon} \| g \|_{L^2(\Omega)} \]

(in the case \( n = 2 \) we use \( \varepsilon \leq C |\ln \varepsilon| \) with \( C = |\ln \varepsilon_0|^{-1} \).

Combining (6.12)–(6.15), (6.17), (6.25), we arrive at the required estimate (2.17).

### 6.4. Proof of Theorem 2.7

Let us apply Theorem 3.6. We have been already proven that the estimate (3.13) holds with \( \rho_{\varepsilon} = \tilde{\rho}_{\varepsilon} = C \eta_{\varepsilon} \). Moreover, due to (5.9), one has \( I_{\varepsilon}^+ = \tilde{I}_{\varepsilon} \) and thus

\[ \left( (\Lambda_{\varepsilon} + 1)^{-1} I_{\varepsilon} - I_{\varepsilon} (\Lambda + 1)^{-1} \right)^* = \tilde{I}_{\varepsilon} (\Lambda_{\varepsilon} + 1)^{-1} - (\Lambda + 1)^{-1} \tilde{I}_{\varepsilon}, \]

whence the estimate (3.14) also holds with \( \tilde{\rho}_{\varepsilon} = C \eta_{\varepsilon} \).

Since \( \| u \|_{L^2(\Omega_{\varepsilon})} = \| I_{\varepsilon} u \|_{L^2(\Omega)} \), we conclude that the estimate (3.16) holds with \( \tilde{\mu}_{\varepsilon} = 1, \tilde{\nu}_{\varepsilon} = 0 \). Finally, one has

\[ \| f \|_{H^2}^2 = \| I_{\varepsilon} f \|_{H^2}^2 + \sum_{i \in I_{\varepsilon}} \| f \|_{L^2(D_i,\varepsilon)}^2, \]

Using Lemma 4.1 (for \( D := D_{i,\varepsilon}, D_1 := D_{i,\varepsilon}, D_2 := \square_{i,\varepsilon} \setminus \overline{\square_{i,\varepsilon}} \)), we get

\[ \sum_{i \in I_{\varepsilon}} \| f \|_{L^2(D_i,\varepsilon)}^2 \leq C \sum_{i \in I_{\varepsilon}} \left( \Lambda_{\varepsilon}^{n/2} \| f \|_{L^2(\Omega_{\varepsilon} \setminus \overline{\square_{i,\varepsilon}})}^2 + \varepsilon d_{\varepsilon} \| \nabla f \|_{L^2(\Omega_{\varepsilon} \setminus \overline{\square_{i,\varepsilon}})}^2 \right) \leq C \left( \Lambda_{\varepsilon}^{n/2} \| I_{\varepsilon} f \|_{H^2}^2 + \varepsilon d_{\varepsilon} \| f \|_{H^2}^2 \right) \]

(here we use \( |\square_{i,\varepsilon} \setminus D_{i,\varepsilon}| \geq C \varepsilon^n \)). It follows from (6.26)–(6.27) that

\[ \| f \|_{H^2}^2 \leq \left( 1 + C \Lambda_{\varepsilon}^{n/2} \right) \| I_{\varepsilon} f \|_{H^2}^2 + C \varepsilon d_{\varepsilon} \| f \|_{H^2}^2, \]

i.e., the estimate (3.15) holds with \( \mu_{\varepsilon} := 1 + C \Lambda_{\varepsilon}^{n/2} \) and \( \nu_{\varepsilon} := C \varepsilon d_{\varepsilon} \). Then, applying Theorem 3.6, we arrive at

\[ \tilde{d}_{H} (\sigma(\Lambda_{\varepsilon}), \sigma(\Lambda)) \leq C \left( \varepsilon d_{\varepsilon} + \sqrt{(\varepsilon d_{\varepsilon})^2 + \eta_{\varepsilon}^2 (1 + C \Lambda_{\varepsilon}^{n/2})} \right). \]

Evidently, \( \varepsilon d_{\varepsilon} < \eta_{\varepsilon} \); also, we have \( \Lambda_{\varepsilon} < 1 \). Hence (6.28) implies the required estimate (2.22).
6.5. Proof of Theorem 2.8

Let \( v \in L^2(\Omega_\varepsilon) \). We set \( u := (A_\varepsilon + I)^{-1}v \). Lemma 4.7 yields

\[
\sum_{i \in I_\varepsilon} \|u\|_{L^2(Y_{i,\varepsilon})}^2 \leq C \max \left\{ P^{-1}_\varepsilon, Q^{-1}_\varepsilon \right\} a_\varepsilon[u, u] \tag{6.29}
\]

(recall that \( Y_{i,\varepsilon} \) is given in (4.27)). We denote \( R_{i,\varepsilon} := \square_{i,\varepsilon} \setminus D_{i,\varepsilon} \cup Y_{i,\varepsilon} \). Using Lemma 4.1 for \( D := \square_{i,\varepsilon}, D_1 := R_{i,\varepsilon}, D_2 := Y_{i,\varepsilon}, \ g := \bar{j}_\varepsilon^2 u \) (note that \( |R_{i,\varepsilon}| \leq Ce^\eta, |Y_{i,\varepsilon}| \geq Ce^\eta \)) and (6.29), we obtain

\[
\sum_{i \in I_\varepsilon} \|u\|_{L^2(R_{i,\varepsilon})}^2 \leq C \sum_{i \in I_\varepsilon} \left( \|u\|_{L^2(Y_{i,\varepsilon})}^2 + \varepsilon^2 \|\nabla (\bar{j}_\varepsilon^2 u)\|_{L^2(D_{i,\varepsilon})}^2 \right)
\]

\[
\leq C \max \left\{ P^{-1}_\varepsilon, Q^{-1}_\varepsilon \right\} a_\varepsilon[u, u] + C\varepsilon^2 \|\nabla (\bar{j}_\varepsilon^2 u)\|_{L^2(\Omega)}^2. \tag{6.30}
\]

Finally, by Lemma 4.5 applied for \( g := \bar{j}_\varepsilon^2 u \) we have

\[
\|u\|_{L^2(\Omega) \setminus \bigcup_{i \in I_\varepsilon} D_{i,\varepsilon}} \leq C\varepsilon \|\nabla (\bar{j}_\varepsilon^2 u)\|_{L^2(\Omega)}. \tag{6.31}
\]

Combining (5.6), (6.29)–(6.31), we get

\[
\| (A_\varepsilon + I)^{-1}v \|_{L^2(\Omega_\varepsilon)}^2 = \sum_{i \in I_\varepsilon} \|u\|_{L^2(Y_{i,\varepsilon})}^2 + \sum_{i \in I_\varepsilon} \|u\|_{L^2(R_{i,\varepsilon})}^2 + \|u\|_{L^2(\Omega_\varepsilon) \setminus \bigcup_{i \in I_\varepsilon} D_{i,\varepsilon}}^2
\]

\[
\leq C \max \left\{ P^{-1}_\varepsilon, Q^{-1}_\varepsilon \right\} a_\varepsilon[u, u] + C\varepsilon \|\nabla (\bar{j}_\varepsilon^2 u)\|_{L^2(\Omega)}^2
\]

\[
\leq C \max \left\{ P^{-1}_\varepsilon, Q^{-1}_\varepsilon \right\} \|u\|_{H^\varepsilon}^2
\]

\[
= C \max \left\{ P^{-1}_\varepsilon, Q^{-1}_\varepsilon \right\} \|A_\varepsilon u + u\|_{L^2(\Omega_\varepsilon)}^2
\]

\[
\leq C \max \left\{ P^{-1}_\varepsilon, Q^{-1}_\varepsilon \right\} \|A_\varepsilon u + u\|_{L^2(\Omega_\varepsilon)} \|u\|_{L^2(\Omega_\varepsilon)}
\]

\[
= C \max \left\{ P^{-1}_\varepsilon, Q^{-1}_\varepsilon \right\} \|v\|_{L^2(\Omega_\varepsilon)} \| (A_\varepsilon + I)^{-1}v \|_{L^2(\Omega_\varepsilon)}. \tag{6.32}
\]

(note that the constant \( C \) in (6.32) changes from line to line). The required estimate (2.23) follows immediately from (6.32). Theorem 2.8 is proven.

7. Case (1.13) revisited

In this section we obtain another \((H^1 \rightarrow L^2)\) operator estimate for \( P, Q \) satisfying (1.13). In some cases (see the discussion after Theorem 7.1) the new estimate gives better convergence rate than the estimates (2.9) and (2.11).

We define for \( P < \infty \):

\[
\eta''_\varepsilon := \max \left\{ P_\varepsilon Q^{-1/2}_\varepsilon; |P_\varepsilon - P|_\varepsilon ; \varepsilon; A'^{-1/2}_\varepsilon \right\}. \tag{7.1}
\]

It is easy to see that \( \eta''_\varepsilon \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \) provided (1.13) holds.

**Theorem 7.1.** Let \( P, Q \) satisfy (1.13). Then one has

\[
\left\| (A_\varepsilon + I)^{-1}J_\varepsilon - J_\varepsilon (A + I)^{-1} \right\|_{L^2(\Omega) \rightarrow H^1(\Omega_\varepsilon)} \leq C\eta''_\varepsilon. \tag{7.2}
\]
Before to present the proof of Theorem 7.1, we compare it with the estimates (2.9) and (2.11). Let \( P, Q \) satisfy (1.13). In this case \( P_e = P = V \), whence
\[
V_e - V = P_e - P - \kappa_e, \quad \text{where} \quad \kappa_e := \frac{P_e^2}{Q_e} \frac{1}{1 + P_e Q_e^{-1}}. \tag{7.3}
\]
Using (7.3), we get
\[
\eta_e = \max \left\{ |P_e - P| + O(P_e^2 Q_e^{-1}); \epsilon; R_e \Lambda_e^{n/2} \right\}, \quad R_e := \left\{ \begin{array}{ll} \Lambda_e^{1-n/2}, & n \geq 5, \\
\Lambda_e^{-1} \ln \Lambda_e |, & n = 4, \\
\Lambda_e^{-1}, & n = 3, \\
\Lambda_e^{-1} \ln \Lambda_e |^{-1/2}, & n = 2, \end{array} \right. \tag{7.4}
\]
\[
\eta'_e = \max \left\{ r_e P_e Q_e^{-1/2}, |P_e - P| + O(P_e^2 Q_e^{-1}); \epsilon; R_e \Lambda_e^{n/2} \right\}, \quad r_e := (1 + P_e Q_e^{-1})^{-1}. \tag{7.5}
\]
Taking into account that \( R_e \to \infty \) and \( r_e \to 1 \) as \( \epsilon \to 0 \), we conclude from (7.4)–(7.5) that the estimate (7.2) gives a better convergence rate than the estimates (2.9) and (2.11) if the term \( \Lambda_e^{n/2} \) prevails in \( \eta'_e \), i.e., if \( P_e Q_e^{-1/2} = o(\Lambda_e^{n/2}) \), \( |P_e - P| = o(\Lambda_e^{n/2}) \), and \( \epsilon = o(\Lambda_e^{n/2}) \).

We illustrate the above conclusion with an example. Let \( n \geq 5 \), \( \gamma_e = e^{n-(n-1)s} \) and \( d_e = e^s \) with \( s \in (1, \frac{n}{n-2}) \). We have \( P_e = \gamma_e > 0 \) and \( Q_e \sim \infty \) which corresponds to the dashed bold open interval on Figure 2. One has:
\[
\eta_e = \max \left\{ \mu \epsilon e^{-n-2}; \epsilon; \epsilon^{s-1} \right\}, \quad \eta'_e = \max \left\{ v \epsilon e^{-n+2}; \mu \epsilon e^{-n-2}; \epsilon; \epsilon^{s-1} \right\},
\]
\[
\eta''_e = \max \left\{ v \epsilon e^{-n+2}; \epsilon; \epsilon^{(s-1)n} \right\},
\]
where \( v := \sqrt{\frac{\mu e}{n-2}}, v_e \sim v, \mu_e \sim v^2 \). Evidently, \( \eta''_e \sim o(\eta_e) \) and \( \eta''_e \sim o(\eta'_e) \) for \( s \in (1, \frac{n+2}{n}) \).

**Proof of Theorem 7.1** As before, let \( H_e := L^2(\Omega_e) \) and \( H := L^2(\Omega) \), and the forms \( a_0 \) and \( a \) in \( H_e \) and \( H \) be defined by (2.1) and (2.2), respectively. Again we introduce the spaces \( H^1_0, H^1 \) as in (5.2), and the space \( H^2 \) as in (5.3). Finally, let the operators \( J_e, J \) be defined by (2.3), (5.4) and (5.5), respectively, while this time the operator \( J_1 \) be defined by (5.10). Below we demonstrate that the conditions of Theorem 5.1 hold with \( \delta_e = C \gamma''_e \).

Recall that (3.4) is fulfilled for any \( \delta_e \geq 0 \). Moreover (see (5.23)), the condition (3.6) is fulfilled with \( \delta_e = C \delta_{e,1} \), where \( \delta_{e,1} \) is given in (5.24). Also, since now \( J_1 f = J f \) for \( f \in \text{dom}(a) \), we conclude that the condition (3.5) holds with any \( \delta_e \geq 0 \).

We examine the fulfilment of the condition (5.7). Let \( f \in H^2 \) and \( u \in H^1_0 \). Recall that \( f_{i,e} \) and \( u_{i,e} \) stand for the mean values of \( f \) and \( J_{i,e} u \) over \( \square_{i,e} \). We also denote by \( \tilde{f}_{i,e} \) and \( \tilde{u}_{i,e} \) the mean values of \( f \) and \( \tilde{J}_{i,e} u \) over \( \partial D_{i,e} \) i.e.
\[
\tilde{f}_{i,e} := \frac{1}{\zeta d_e d_{e,i}} \int_{\partial D_{i,e}} f ds, \quad \tilde{u}_{i,e} := \frac{1}{\zeta d_e d_{e,i}} \int_{\partial D_{i,e}} J_{i,e} u ds.
\]
Taking into account the definition of \( P_e \) and that \( V = P \) (this follows from (1.13)), we have
\[
a_e[u, f] - a[f, f] = - \sum_{i \in I_e} (\nabla J_{i,e} u, \nabla f)_{L^2(D_{i,e})} + \gamma_e \sum_{i \in I_e} (u, f)_{L^2(\partial D_{i,e})} - P(J_{i,e} u, f)_{L^2(\Omega)} = \sum_{k=1}^7 L_{e,k},
\]
where $L_{e,1} = - \sum_{i \in I_e} (\nabla f^1_i u_i, \nabla f)_{L^2(D_{i\varepsilon})}$, and

\[ L_{e,2} = (P_\varepsilon - P) \sum_{i \in I_e} u_{i\varepsilon} f_{i\varepsilon} \epsilon^n, \quad L_{e,3} = P_\varepsilon \sum_{i \in I_e} (\tilde{u}_{i\varepsilon} - u_{i\varepsilon}) f_{i\varepsilon} \epsilon^n, \]

\[ L_{e,4} = P_\varepsilon \sum_{i \in I_e} u_{i\varepsilon} (f_{i\varepsilon} - f) \epsilon^n, \quad L_{e,5} = \gamma_\varepsilon \sum_{i \in I_e} (u, f - \tilde{f}_{i\varepsilon})_{L^2(\partial D_{i\varepsilon})}, \]

\[ L_{e,6} = \sum_{i \in I_e} \left( |f_{i\varepsilon} u_{i\varepsilon} f_{i\varepsilon} - f|_{L^2(D_{i\varepsilon})} \right), \quad L_{e,7} = -P \sum_{i \in I_e} (u, f)_{L^2(\Omega \setminus \cup_{i \in I_e} D_{i\varepsilon})}. \]

We have already examined the terms $L_{e,1}$ and $L_{e,7}$ in (5.29) and (5.56), respectively:

\[ |L_{e,1}| \leq C d_\varepsilon,1 \|f\|_{H^1} \|u\|_{H^1}, \quad (7.6) \]

\[ |L_{e,7}| \leq C \varepsilon^2 \|f\|_{H^1} \|u\|_{H^1}, \quad (7.7) \]

Using the Cauchy-Schwarz inequality (5.13) and a similar inequality for $u_{i\varepsilon}$, one has

\[ |L_{e,2}| \leq |P_\varepsilon - P| \left( \sum_{i \in I_e} \epsilon^n |f_{i\varepsilon}|^2 \right)^{1/2} \left( \sum_{i \in I_e} \epsilon^n |u_{i\varepsilon}|^2 \right)^{1/2} \]

\[ \leq |P_\varepsilon - P| \|f\|_{L^2(\Omega)} \|f^1_i u\|_{L^2(\Omega)} \leq C |P_\varepsilon - P| \|f\|_{H^1} \|u\|_{H^1}, \quad (7.8) \]

(the last step in (7.8) relies on (5.6)). Similarly, using the Cauchy-Schwarz inequality for $\tilde{f}_{i\varepsilon}$ and taking into account the definition of $P_\varepsilon$, we obtain

\[ |L_{e,3}| \leq P_\varepsilon \left( P_\varepsilon^{-1} \gamma_\varepsilon \sum_{i \in I_e} \|f\|_{L^2(\partial D_{i\varepsilon})}^2 \right)^{1/2} \left( \sum_{i \in I_e} \epsilon^n |u_{i\varepsilon} - u_{i\varepsilon}|^2 \right)^{1/2} \cdot \]

Note that, due to (2.5), $\sup_{\varepsilon \in (0, \varepsilon_0]} Q^{-1}_\varepsilon < \infty$ if $Q \neq 0$ (cf. (1.13)). Then, using the estimate (4.34) (for $f$) and the estimate (4.21) (for $f^1_i u$), we obtain

\[ |L_{e,3}| \leq C P_\varepsilon \left( \sum_{i \in I_e} \|f\|_{L^2(\partial D_{i\varepsilon})}^2 + \|f\|_{L^2(\partial D_{i\varepsilon})}^2 \right)^{1/2} \left( \epsilon^2 + Q^{-1}_\varepsilon \sum_{i \in I_e} \|\nabla f\|_{L^2(\partial D_{i\varepsilon})}^2 \right)^{1/2} \]

\[ \leq C P_\varepsilon \left( \epsilon^2 + Q^{-1}_\varepsilon \right)^{1/2} \|f\|_{H^1(\Omega)} \|f^1_i u\|_{L^2(\Omega)} \leq C \max\{P_\varepsilon; P_\varepsilon Q^{-1}_\varepsilon/2\} \|f\|_{H^1} \|u\|_{H^1}, \quad (7.9) \]

(in the last step we again use (5.6)). Further, using the estimate (4.21) for $f$, the Cauchy-Schwarz inequality for $u_{i\varepsilon}$ and (5.6), we derive

\[ |L_{e,4}| \leq P_\varepsilon \left( \epsilon^n \sum_{i \in I_e} |f_{i\varepsilon} - f|_{L^2(\partial D_{i\varepsilon})}^2 \right)^{1/2} \left( \sum_{i \in I_e} \epsilon^n |u_{i\varepsilon}|^2 \right)^{1/2} \]

\[ \leq C P_\varepsilon \left( \epsilon^2 + Q^{-1}_\varepsilon \right)^{1/2} \|f\|_{L^2(\Omega)} \|f^1_i u\|_{L^2(\Omega)} \leq C \max\{P_\varepsilon; P_\varepsilon Q^{-1}_\varepsilon/2\} \|f\|_{H^1} \|u\|_{H^1}. \quad (7.10) \]

To estimate $L_{e,5}$ we utilize (5.6), (4.22) and (4.34):

\[ |L_{e,5}| \leq \left( \gamma_\varepsilon \sum_{i \in I_e} \|f - \tilde{f}_{i\varepsilon}\|_{L^2(\partial D_{i\varepsilon})}^2 \right)^{1/2} \left( \gamma_\varepsilon \sum_{i \in I_e} \|u\|_{L^2(\partial D_{i\varepsilon})}^2 \right)^{1/2} \]

\[ \leq C \left( \gamma_\varepsilon d_\varepsilon \sum_{i \in I_e} \|\nabla f\|_{L^2(D_{i\varepsilon})}^2 \right)^{1/2} \left( P_\varepsilon \sum_{i \in I_e} \left( \|u\|_{L^2(\partial D_{i\varepsilon})}^2 + Q^{-1}_\varepsilon \|\nabla u\|_{L^2(\partial D_{i\varepsilon})}^2 \right) \right)^{1/2} \]

\[ \leq C P_\varepsilon Q^{-1}_\varepsilon \|f\|_{H^1} \|u\|_{H^1}. \quad (7.11) \]
The last step of the above estimate relies on $\sup_{\varepsilon \in (0,\delta_0)} Q_\varepsilon^{-1} < \infty$ (for $Q$ as in (1.13)), and the equalities
\[
\gamma_\varepsilon d_\varepsilon = P_\varepsilon Q_\varepsilon^{-1}(n - 2), \quad n \geq 3,
\gamma_\varepsilon d_\varepsilon = P_\varepsilon Q_\varepsilon^{-1} |\ln d_\varepsilon|^{-1}, \quad n = 2
\]
(note $|\ln d_\varepsilon|^{-1} < |\ln \varepsilon|^{-1}$). Finally, we estimate $L_{\varepsilon,0}$ via (4.1) (with $g := f - f_{i,\varepsilon}$) and (5.6):
\[
|L_{\varepsilon,0}| \leq P \left( \sum_{i \in I_\varepsilon} \|f - f_{i,\varepsilon}\|_{L^2(Q_\varepsilon)}^2 \right)^{1/2} \left( \sum_{i \in I_\varepsilon} \|\bar{I}_\varepsilon u\|_{L^2(Q_\varepsilon)}^2 \right)^{1/2} \\
\leq C \varepsilon \|\nabla f\|_{L^2(\Omega)} \|\bar{I}_\varepsilon u\|_{L^2(\Omega)} \leq C \varepsilon \|f\|_{H^1} \|u\|_{H^1}.
\] (7.12)
Combining (7.6)–(7.12) and taking into account that $\sup_{\varepsilon \in (0,\delta_0)} P_\varepsilon < \infty$, we arrive at
\[
|a_\varepsilon[u, J_\varepsilon] - a_\varepsilon[\bar{J}_\varepsilon u, f]| \leq C \max \left\{ P_\varepsilon Q_\varepsilon^{-1/2}; |P_\varepsilon - P|; \varepsilon; \delta_{\varepsilon,1} \right\} \|f\|_{H^1} \|u\|_{H^1}.
\]
Taking into account that $\delta_{\varepsilon,1} = \max\{\Lambda_{\varepsilon,1}^{-n/2}; (\varepsilon d_\varepsilon)^{1/2}\} \leq \max\{\Lambda_{\varepsilon,1}^{-n/2}; \varepsilon\}$, we conclude that the conditions (3.4)–(3.7) are fulfilled with $\delta_\varepsilon = C \eta_\varepsilon''$, where $\eta_\varepsilon''$ is defined by (7.1). Hence, by Theorem 3.6 we obtain the estimate
\[
\left\| (A_\varepsilon + 1)^{-1} J_\varepsilon - J_\varepsilon (A + 1)^{-1} \right\|_{H^1 \rightarrow H^1} \leq C \eta_\varepsilon''.
\] (7.13)
Since $\|\psi\|_{H^1(\Omega)} \leq \|\psi\|_{H^1}$ and $J_\varepsilon f = J_\varepsilon f$, we immediately get from (7.13) the required estimate (7.2). Theorem 3.3 is proven. \hfill \Box

Acknowledgements

The work of the first author is partly supported by the Czech Science Foundation (GAČR) through the project 21-07129S. The second author gratefully acknowledges financial support by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

References

[1] R. A. Adams, Sobolev spaces, Academic Press, New York-London, 1975.
[2] C. Anné, O. Post, Wildly perturbed manifolds: norm resolvent and spectral convergence, J. Spectr. Theory 11 (2021), 229–279.
[3] L.V. Berlyand, M.V. Goncharenko, Averaging the diffusion equation in a porous medium with weak absorption, J. Sov. Math. 52 (1990), 3428–3435.
[4] M.S. Birman, T.A. Suslina, Second order periodic differential operators. Threshold properties and homogenization, St. Petersburg Math. J. 15 (2004), 639–714.
[5] M.S. Birman, T.A. Suslina, Averaging of periodic differential operators taking a corrector into account. Approximation of solutions in the Sobolev class $H^1(\mathbb{R}^d)$, St. Petersburg Math. J. 18 (2007), 857–955.
[6] D. Borisov, R. Bunoiu, G. Cardone, On a waveguide with frequently alternating boundary conditions: homogenized Neumann condition, Ann. Henri Poincaré 11 (2010), 1591–1627.
[7] D. Borisov, G. Cardone, T. Durante, Homogenization and norm-resolvent convergence for elliptic operators in a strip perforated along a curve, Proc. Roy. Soc. Edinburgh Sect. A 146 (2016), 1115–1158.
[8] D. Borisov, G. Cardone, L. Faella, C. Perugia, Uniform resolvent convergence for strip with fast oscillating boundary, J. Differential Equations 255 (2013), 4378–4402.
[9] L. Boutet de Monvel, I.D. Chueshov, E.Ya. Khruslov, Homogenization of attractors for semilinear parabolic equations on manifolds with complicated microstructure, Ann. Mat. Pura Appl., IV. Ser. 172 (1997), 297–322.
[10] L. Boutet de Monvel, E.Ya. Khruslov, Averaging of the diffusion equation on Riemannian manifolds of complex microstructure, Trans. Mosc. Math. Soc. 1997, 137–161; translation from Tr. Mosk. Mat. O.-va. 58, 158–186 (1997).

[11] A. Brillard, Asymptotic analysis of two elliptic equations with oscillating terms, RAIRO, Modélisation Math. Anal. Numér. 22 (1988), 187–216.

[12] E.E. Browder, On the spectral theory of elliptic differential operators. I, Math. Annal. 142 (1961), 22–130.

[13] G. Cardone, A. Khrabustovskyi, $\delta'$-interaction as a limit of a thin Neumann waveguide with transversal window, J. Math. Anal. Appl. 473 (2019), 1320–1342.

[14] K. Cherednichenko, P. Dondl, F. Rössler, Norm-resolvent convergence in perforated domains, Asympt. Analysis 110 (2018), 163–184.

[15] D. Cioranescu, P. Donato, An introduction to homogenization, Oxford, Oxford University Press, 1999.

[16] D. Cioranescu, F. Murat, Un terme étrange venu d’ailleurs, Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. II (Paris, 1979/1980), Res. Notes in Math., vol. 60, Pitman, Boston, Mass.-London, 1982, pp. 98–138, 389–390.

[17] D. Cioranescu, J. Saint Jean Paulin, Homogenization in open sets with holes, J. Math. Anal. Appl. 71 (1979), 590–607.

[18] J.I. Díaz, D. Gómez-Castro, T.A. Shaposhnikova, Nonlinear reaction-diffusion processes for nanocomposites. Anomalous improved homogenization, De Gruyter, Berlin, 2021.

[19] J.I. Díaz, D. Gómez-Castro, T.A. Shaposhnikova, M.N. Zubova, Change of homogenized absorption term in diffusion processes with reaction on the boundary of periodically distributed asymptotic parameters of critical size, Electron. J. Differ. Equ. (2017), 178.

[20] J.I. Díaz, D. Gómez-Castro, T.A. Shaposhnikova, M.N. Zubova, Classification of homogenized limits of diffusion problems with spatially dependent reaction over critical-size particles, Appl. Anal. 98 (2019), 232–255.

[21] L.C. Evans, Partial Differential Equations, AMS, Providence, RI, 1998.

[22] D. Gómez, E. Pérez, T.A. Shaposhnikova, On homogenization of nonlinear Robin type boundary conditions for cavities along manifolds and associated spectral problems, Asymptotic Anal. 80 (2012), 289–322.

[23] D. Gómez, E. Pérez, T.A. Shaposhnikova, Spectral boundary homogenization problems in perforated domains with Robin boundary conditions and large parameters, Constanda, Christian (ed.) et al., Integral methods in science and engineering. Progress in numerical and analytic techniques, New York, Birkhäuser. 155–174 (2013).

[24] M. Goncharenko, The asymptotic behaviour of the third boundary-value problem solutions in domains with fine-grained boundaries, Homogenization and applications to material sciences. Proceedings of the international conference, Nice, France, June 6–10, 1995. Tokyo: Gakkotosho. GAKUTO Int. Ser., Math. Sci. Appl. 9, 203–213 (1995).

[25] M.V. Goncharenko, L.A. Khil’kova, Homogenized model of diffusion in porous media with nonlinear absorption on the boundary, Ukr. Math. J. 67 (2016), 1349–1366.

[26] G. Griso, Error estimate and unfolding for periodic homogenization, Asymptot. Anal. 40 (2004), 269–286.

[27] G. Griso, Interior error estimate for periodic homogenization, Anal. Appl. 4 (2006), 61–79.

[28] P. Grisvard, Elliptic problems in nonsmooth domains, Pitman, Boston, MA, 1985.

[29] I. Herbst, S. Nakamura, Schrödinger operators with strong magnetic fields: Quasi-periodicity of spectral orbits and topology, in: Differential Operators and Spectral Theory, in: Amer. Math. Soc. Transl. Ser. 2, vol. 189, Amer. Math. Soc., Providence, RI, 1999, pp. 105–123.

[30] S. Kaizu, The Robin problems on domains with many tiny holes, Proc. Japan Acad. Ser. A Math. Sci. 61 (1985), 39–42.

[31] S. Kaizu, A monotone boundary condition for a domain with many tiny spherical holds, Proc. Japan Acad., Ser. A 61 (1985), 140–143.

[32] S. Kaizu, An average effect of many tiny holes in nonlinear boundary value problems with monotone boundary conditions, Proc. Japan Acad., Ser. A 62 (1986), 133–136.

[33] S. Kaizu, The Poisson equation with semilinear boundary conditions in domains with many tiny holes, J. Fac. Sci., Univ. Tokyo, Sect. I A 36 (1989), 43–86.

[34] S. Kaizu, Behavior of solutions of the Poisson equation under fragmentation of the boundary of the domain, Japan J. Appl. Math. 7 (1990), 77–102.

[35] T. Kato, Perturbation Theory for Linear Operators, Berlin-Heidelberg-New York, Springer, 1966.
[36] A. Khrabustovskyi, Homogenization of spectral problem on Riemannian manifold consisting of two domains connected by many tubes, Proc. R. Soc. Edinb., Sect. A, Math. 143 (2013), 1255–1289.
[37] A. Khrabustovskyi, O. Post, Operator estimates for the crushed ice problem, Asymptotic Anal. 110 (2018), 137–161.
[38] A. Khrabustovskyi, O. Post, A geometric approximation of δ-interactions by Neumann Laplacians, arXiv:2104.10463 [math.SP].
[39] E.Ya. Khruslov, The asymptotic behavior of solutions of the second boundary value problem under fragmentation of the boundary of the domain, Math. USSR Sb. 35 (1979), 266–282.
[40] M. Lobo, O.A. Oleinik, M.E. Perez, T.A. Shaposhnikova, On homogenization of solutions of boundary value problems in domains, perforated along manifolds, Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 25 (1997), 611–629.
[41] V. A. Marchenko, E. Y. Khruslov, Boundary-value problems with fine-grained boundary, Mat. Sb. (N.S.) 65 (1964), 458–472.
[42] V. A. Marchenko, E. Y. Khruslov, Homogenization of partial differential equations, Birkhäuser, Boston, 2006.
[43] T.A. Mel’nyk, O.A. Sivak, Asymptotic analysis of a boundary-value problem with nonlinear multiphase boundary interactions in a perforated domain, Ukr. Math. J. 61 (2009), 592–612.
[44] D. Mugnolo, R. Nittka, O. Post, Norm convergence of sectorial operators on varying Hilbert spaces, Oper. Matrices 7 (2013), 955–995.
[45] O. Post, Spectral convergence of quasi-one-dimensional spaces, Ann. Henri Poincaré 7 (2006), 933–973.
[46] O. Post, Spectral analysis on graph-like spaces. Lecture Notes in Mathematics 2039, Springer, Berlin, 2012.
[47] O. Post, J. Simmer, Quasi-unitary equivalence and generalised norm resolvent convergence, Rev. Roum. Math. Pures Appl. 64 (2019), 373–391.
[48] J. Rauch, M. Taylor, Potential and scattering theory on wildly perturbed domains, J. Funct. Anal. 18 (1975), 27–59.
[49] M. Reed, B. Simon, Methods of Modern Mathematical Physics I: Functional Analysis, Academic Press, New York–London, 1972.
[50] T.A. Suslina, Spectral approach to homogenization of elliptic operators in a perforated space, Rev. Math. Phys. 30 (2018), 1840016.
[51] M. Taylor, Partial differential equations II. Qualitative studies of linear equations, Springer, New York, 2011.
[52] V.V. Zhikov, On operator estimates in homogenization theory, Dokl. Akad. Nauk 403 (2005), 305–308.
[53] V. V. Zhikov, Spectral method in homogenization theory, Proc. Steklov Inst. Math. 250 (2005), 85–94.
[54] V. V. Zhikov, S. E. Pastukhova, On operator estimates for some problems in homogenization theory, Russ. J. Math. Phys. 12 (2005), 515–524.
[55] V. V. Zhikov, S. E. Pastukhova, On operator estimates in homogenization theory, Russian Math. Surveys 71 (2016), 417–511.