Prime number decomposition of the Fourier transform of a function of the greatest common divisor.

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Abstract

The discrete Fourier transform of the greatest common divisor is a multiplicative function, if taken with respect to the same order of the primitive root of unity, which is a well known fact. As such, the transform can be expressed in the prime factors of the argument, the explicit form of which is proven in this paper. Subsequently it is shown how the procedure can be generalized to the discrete Fourier transform of a function of the greatest common divisor. From this representation some interesting relations concerning the Euler totient function and generalizations thereof are established.

Keywords: Discrete Fourier Transform, gcd, prime number decomposition, multiplicative function

1. Introduction

The discrete Fourier transform of the greatest common divisor, gcd, is commonly defined as

\[ h_m(n) = \sum_{k=1}^{n} \gcd(k, n) e^{-2\pi i k \frac{m}{n}}. \]  

The function \( h_m(n) \) is multiplicative in the argument \( n \) for fixed \( m \) [1, 2]. Also, \( h_m(n) \) is integer valued. This follows from the fact that \( h_m(n) \) can be written as the Dirichlet convolution of the identity function and the Ramanujan sum, as proven in [1], and that the Ramanujan sum itself is integer, as proven in [3]. We come back to this later, see equation 26 below.

Although the multiplicativity of \( h_m(n) \) already is established, as described above, we outline a direct proof below because this property is central in the current work. Hence the following proposition is formulated.
Proposition 1. The discrete Fourier transform \( h_m(n) \) defined in equation 1 is a multiplicative function in the argument \( n \) for a fixed value of the index \( m \).

Sketch of proof of proposition 1. Assume two integer numbers \( u \), and \( v \), that are coprime, i.e., \( \gcd(u, v) = 1 \), as required by the property of multiplicativity, then

\[
\begin{align*}
h_m(u)h_m(v) &= \sum_{k=1}^{u} \gcd(k, u)e^{-m\frac{2\pi i k}{u}} \sum_{l=1}^{v} \gcd(l, v)e^{-m\frac{2\pi i l}{v}} \\ &= \sum_{k=1}^{u} \sum_{l=1}^{v} \gcd(kv, u)e^{-m\frac{2\pi i (kv+lu)}{uv}} \\ &= \sum_{a=1}^{uv} \gcd(a, uv)e^{-m\frac{2\pi ia}{uv}} \\ &= h_m(uv)
\end{align*}
\]

Here equation 3 follows from equation 2 because of elementary properties of the greatest common divisor: \( \gcd(k, u) = \gcd(kv, u) = \gcd(kv + lu, u) \), and similarly \( \gcd(l, v) = \gcd(kv + lu, v) \), where the summation indices \( k \), and \( l \) are limited to the ranges in their respective summation. Subsequently, the multiplicativity of the greatest common divisor itself, \( \gcd(kv + lu, u)\gcd(kv + lu, v) = \gcd(kv + lu, uv) \) was used. Furthermore, equation 4 follows from equation 3 because of the Chinese remainder theorem which, in this case, says there is a unique mapping between the numbers \( k \) modulo \( u \) and \( l \) modulo \( v \), and the number \( a \) modulo \( uv \).

\( \square \)

It is clear that the reasoning in the proof of proposition 1 also holds if one replaces the greatest common divisor by a multiplicative function of the greatest common divisor everywhere in equations 2 to 4, i.e., replaces \( \gcd(k, u) \) by \( f(\gcd(k, u)) \).

2. Prime number representation of the discrete Fourier transform of the greatest common divisor

A fundamental property of multiplicative functions[4] is that the function value is entirely expressible in function values of powers of the prime-factors of the argument. The explicit form for the case where this function is the discrete Fourier transform of the greatest common divisor, i.e., \( h_m(n) \), is given in theorem 1 below. We define the symbol \( \theta_{t,s} = H[t-s] \), where \( H[w] \) is the Heaviside step function for integral arguments \( w \). Hence, \( \theta_{t,s} = 0 \) if \( t < s \), and \( \theta_{t,s} = 1 \) if \( t \geq s \). The comma-symbol between the indices is added to avoid confusion in case where an index is made up of an expression. We also use the function \( \min(r, s) \) indicating the minimum of the integral arguments \( r \) and \( s \).
Theorem 1. If the factorization of the number \( n \), the argument, in its prime factors \( p_j, j = 1, \ldots, r \), is given as \( n = \prod_{j=1}^{r} p_j^{s_j} \), where \( s_j \), with \( s_j \geq 1 \), is the multiplicity of prime factor \( p_j \), and the order \( m \) of the primitive root of unity, where \( 1 \leq m \leq n \), is written as \( m = u \prod_{i=1}^{t} p_i^{s_i} \), with \( t_i \geq 0 \), \( \gcd(u, p_i) = 1 \), and \( 1 \leq i \leq r \), then the discrete Fourier transform of the greatest common divisor can be expressed in prime factors as

\[
\sum_{k=1}^{n} \gcd(k, n)e^{-k \frac{2\pi i m}{n}} = \prod_{i=1}^{r} \left( \left( \min(t_i, s_i) + 1 \right) \varphi(p_i^{s_i}) + \theta_{t_i, s_i} p_i^{s_i-1} \right)
\]

where \( \varphi(n) \) is the Euler totient function.

The multiplicity \( s_j \) of prime \( p_j \) in \( n \) is naturally larger than 0, but \( m \) is not necessarily divisible by \( p_j \). Therefore, a multiplicity \( t_i \) can be equal to zero, indicating that prime factor \( p_i \) not is present in the prime factorization of \( m \). Also, since \( m \) ranges from 1 to \( n \), the multiplicity \( t_i \) of prime \( p_i \) in \( m \) can be higher then the multiplicity \( s_i \) of the same prime in \( n \).

Two different proofs will be given, an inductive proof and a constructive proof. The inductive proof provides more insight in the nature of the problem. The constructive proof will be generalized to the case of the discrete Fourier transform of a function of the greatest common divisor, further down in this work. In both cases it will be used that \( h_m(n) \) is a multiplicative function, meaning that \( h_m(n) = \prod_{n=1}^{m} h_m(p_i^{s_i}) \), implying that theorem 1 only needs to be proven for the case where \( n \) equals the power of a single prime factor, i.e., \( n = p^s \). The index \( i \) can then be omitted because only one factor is considered.

Proof 1, inductive proof. The two cases \( \gcd(m, n) > 1 \) and \( \gcd(m, n) = 1 \) are considered separately, starting with the case \( \gcd(m, n) > 1 \). Induction with respect to \( s \) will be applied.

Case 1 (\( \gcd(m, n) > 1 \)).

Step 1 (Base step). When \( s = 1 \) it follows that \( n = p \) and then also \( m = p \), because that is the only possible value for \( m \) where \( \gcd(m, p) > 1 \). Then also \( t = 1 \), and the right hand side of equation 6 becomes \( 2 \varphi(p) + p^0 = 2p - 1 \). For the left hand side one obtains

\[
\sum_{k=1}^{p} \gcd(k, p)e^{-k \frac{2\pi i p}{p}} = \sum_{k=1}^{p} \gcd(k, p) = (p - 1)1 + p = 2p - 1
\]

showing that equation 6 holds for \( s = 1 \).

Step 2 (Induction step). Assume that equation 6 holds for \( s = s' \), then with \( n = p^{s'} \) and \( m = up^t \) the induction hypothesis reads

\[
\sum_{k=1}^{p^{s'}} \gcd(k, p^{s'})e^{-k \frac{2\pi i p^{s'}}{p^{s'}}} = (\min(t, s') + 1) \varphi(p^{s'}) + \theta_{t, s'} p^{s'-1}
\]

Now consider the left hand side of equation 8 with \( s' \) replaced by \( s' + 1 \), and partition the summation over \( k = 1 \cdots p^{s'+1} \) into two subsets, I and II, where subset I consists of all multiples of \( p \) and subset II of the remaining values. Hence subset I = \( \{p, 2p, \cdots p^{s}p\} \)
and subset II \{1, 2, \cdots p - 1, p + 1, \cdots 2p - 1, 2p + 1, \cdots p' + p - 1\}. Consider the partial sum over \( k \) in subset I using \( k = jp \), this gives

\[
\sum_{k \in I} \gcd(k, p') e^{-k \frac{\varphi(p')}{p' + 1} up'} = \sum_{j=1}^{p'} \gcd(jp, p') e^{-jp \frac{\varphi(p')}{p' + 1} up'}
\]

(9)

\[
= p \sum_{j=1}^{p'} \gcd(j, p') e^{-j \frac{\varphi(p')}{p' + 1} up'}
\]

(10)

\[
= p \left[ (\min(t, s') + 1) \varphi(p') + \theta_{t,s'}p' - 1 \right]
\]

(11)

\[
= \left( \min(t, s') + 1 \right) \varphi(p' + 1) + \theta_{t,s'}p'
\]

(12)

where in the step to equation 10 it has been used that \( \gcd(jp, p') = p \gcd(j, p'), j = 1 \cdots p' \), and equation 11 is obtained with the induction hypothesis. Equation 12 follows by using that the totient function obeys \( p \varphi(p') = \varphi(p' + 1) \) for \( s' \geq 1 \).

Next consider the partial summation over subset II, which is such that \( \gcd(k, p' + 1) = 1 \). Upon rewriting the summation over subset II as the difference of the complete summation and the summation over subset I, one obtains

\[
\sum_{k=1}^{p'+1} \gcd(k, p' + 1) e^{-k \frac{\varphi(p')}{p' + 1} up'} = \sum_{k=1}^{p'+1} e^{-k \frac{\varphi(p')}{p' + 1} up'}
\]

(13)

\[
= \sum_{k=1}^{p'+1} e^{-k \frac{\varphi(p')}{p' + 1} up'} - \sum_{k \in I} e^{-k \frac{\varphi(p')}{p' + 1} up'}
\]

(14)

\[
= p' \varphi(p' + 1) + \theta_{t,s'}p'
\]

(15)

where it has been used that the first summation on the right hand side of equation 14 equals zero when \( t < s' + 1 \), by summing the geometric series. On the other hand, the summand equals 1 if \( t \geq s' + 1 \), resulting in the summation being equal to the number of terms, in this case. The second sum also equals zero, except when \( t \geq s' \). Here the summand equals 1 if \( t \geq s' \), because the summation index contains a factor \( p \), and the summation results in the number of terms, \( p' \), in this case. Collecting terms from equations 12 and 15, and cancelling the term \( \theta_{t,s'}p' \), gives

\[
\sum_{k=1}^{p'+1} \gcd(k, p' + 1) e^{-k \frac{\varphi(p')}{p' + 1} up'}
\]

(16)

\[
= (\min(t, s') + 1) \varphi(p' + 1) + p' \varphi(p' + 1) \theta_{t,s'+1}
\]

(17)

\[
= \left( \min(t, s' + 1) + 1 \right) \varphi(p') - \theta_{t,s'+1}(p' + 1 - p') + p' \varphi(p' + 1) \theta_{t,s'+1}
\]

(18)
In the step from equation 16 to 17 the property \(\min(t, s) = \min(t, s + 1) - \theta_{t,s+1}\) of the minimum-function has been used, as well as the specific form \(\varphi(p^s) = p^s - p^{s-1}\) of the totient function. Equation 18 completes the induction step, which started with equation 8.

**Case 2** \((\gcd(m, n) = 1)\). Since \(n = p^s\), with \(p\) a prime number, and \(s > 0\), the requirement \(\gcd(m, p^s) = 1\) implies that \(t = 0\) in equation 6, which implies that the term with \(\theta_{t,s}\) always equals 0. Also \(\min(0, s) = 0\) for all \(s > 0\). Hence it has to be proven that

\[
\sum_{k=1}^{p^s} \gcd(k, p^s)e^{-k\frac{2\pi i}{p^s}m} = \varphi(p^s)
\]  

which is done in the following two steps.

**Step 3** (Base step, \(s = 1\)). Direct evaluation of the left hand side of equation 19, with \(s = 1\), gives

\[
\sum_{k=1}^{p} \gcd(k, p)e^{-k\frac{2\pi i}{p}m} = \sum_{k=1}^{p} e^{-k\frac{2\pi i}{p}m} - \sum_{k=p}^{p} e^{-k\frac{2\pi i}{p}m} = 0 - 1 + p = \varphi(p)
\]

which proves the base step.

**Step 4** (Induction step). The equation to be proven, equation 19, is identical to equation 8 with \(t = 0\) and \(\theta_{t,s'} = 0\), thus we can reuse the strategy used there. Assume that equation 19 holds for \(s = s'\), and consider the left hand side of equation 19 for \(s = s' + 1\). Upon again partitioning the summation into the same two subsets I and II, as above, and repeating steps 9 to 15, where now all terms containing \(\theta\)-symbols or min-functions vanish, one obtains

\[
\sum_{k=1}^{p^{s'+1}} \gcd(k, p^{s'+1})e^{-k\frac{2\pi i}{p^{s'+1}}m} = \varphi(p^{s'+1})
\]

which completes the induction step.

All cases being considered this finalizes the proof.

**Proof 2, constructive proof.** The proof starts from a result [1], that expresses the discrete Fourier transform of a function of the greatest common divisor in the Dirichlet convolution of that function and the Ramanujan sum [5].

The Dirichlet convolution, denoted \(f * g\), of two arithmetic functions, \(f\) and \(g\), is defined as

\[
f * g(n) = \sum_{d|n} f\left(\frac{n}{d}\right)g(d)
\]
and the Ramanujan sum as

\[ c_n(m) = \sum_{k=1}^{n} e^{ \frac{k \pi i}{\phi(n)n} m } . \] (25)

For notational reasons we use \( r_m(n) = c_n(m) \) for the Ramanujan sum if it is used in combination with Dirichlet convolution, but also use the more common notation \( c_n(m) \) in other cases. The result referred to above \([1]\) can now be written as

\[ \sum_{k=1}^{n} f(\gcd(k,n))e^{-\frac{k \pi i}{\phi(n)n} m} = f * r_m(n) \] (26)

The case \( f(n) = \text{id}(n) \), reading

\[ \sum_{k=1}^{n} \gcd(k,n)e^{-\frac{k \pi i}{\phi(n)n} m} = \text{id} * r_m(n), \] (27)

is what is needed here. Using von Sterneck’s arithmetic function \([6]\), the Ramanujan sum can be written as\([7]\)

\[ r_m(n) = \mu\left( \frac{n}{\gcd(m,n)} \right) \frac{\varphi(n)}{\varphi(\frac{n}{\gcd(m,n)})} \] (28)

where \( \mu(n) \) is the Möbius function. With this representation of the Ramanujan sum equation 27 becomes

\[ \sum_{k=1}^{n} \gcd(k,n)e^{-\frac{k \pi i}{\phi(n)n} m} = \sum_{d|n} \frac{n}{d} \mu\left( \frac{d}{\gcd(m,d)} \right) \frac{\varphi(d)}{\varphi(\frac{d}{\gcd(m,d)})} . \] (29)

As before, only the case where \( n \) is a power of a prime number \( p \) needs to be considered. In that case the divisors of \( n \) also are powers of \( p \). Hence, take \( n = p^s \), and \( d = p^b \), \( b = 0, \cdots, s \). Again two cases are considered, namely \( \gcd(m,p^s) > 1 \) and \( \gcd(m,p^s) = 1 \)

**Case 3 (\( \gcd(m,p^s) = 1 \)).** Then for all divisors \( d \) of \( n \) it holds that \( \gcd(m,d) = 1 \). In the summation on the right hand side of equation 29 the only non-zero contributions arise from the terms where \( d = 1 \), i.e., \( b = 0 \), and \( d = p, \) i.e., \( b = 1 \). For all other values of \( b \) the Möbius function gives zero. Thus one obtains from equation 29, with \( n = p^s \)

\[ \sum_{k=1}^{n} \gcd(k,n)e^{-\frac{k \pi i}{\phi(n)n} m} = p^s \frac{\varphi(1)}{\varphi(1)} + \frac{p^s}{p} \frac{(-1)^{\varphi(p)}}{\varphi(p)} \] (30)

\[ = p^s - p^{s-1} \] (31)

\[ = \varphi(p^s) \] (32)

which proves equation 6 of theorem 1 for this case.
Case 4 \((\gcd(m, p^s) > 1)\). This means that \(m\) contains a power of \(p\), i.e., \(m = u p^t, t \geq 1\), and \(\gcd(u, p) = 1\). Equation 29 can subsequently be written as

\[
\sum_{k=1}^{n} \gcd(k, n) e^{-\frac{2\pi i}{n} k} = \sum_{b=0}^{s} p^b \mu\left(\frac{p^b}{\gcd(\gcd(u p^t, p^s))}\right) \frac{\varphi(p^b)}{\varphi\left(\frac{p^b}{\gcd(u p^t, p^s)}\right)}.
\]

(33)

The totient function has the property \(\varphi(p^b) = p^b - p^{b-1}, b \geq 1\), whereas \(\varphi(p^0) = 1\).

Therefore the term \(b = 0\) will be taken out of the summation in equation 33, the value of this term equals \(p^s\), irrespective of \(t\).

Now partition the sum over \(b\) into two subranges, \(b = 0, \cdots, t\) and \(b = t + 1, \cdots, s\), where the second subrange is absent if \(t \geq s\). Furthermore, use that \(\gcd(\gcd(u p^t, p^b)) = p^{\min(t, b)}\)

\[
\mu\left(\frac{p^b}{\gcd(\gcd(u p^t, p^s))}\right) = \begin{cases} 
\mu(1) & b \leq t \\
\mu(p^{b-t}) & b = t + 1 \\
0 & b > t + 1
\end{cases}
\]

(34)

then equation 33 becomes

\[
\sum_{k=1}^{n} \gcd(k, n) e^{-\frac{2\pi i}{n} k} = p^s + \sum_{b=1}^{t} \frac{p^s}{p^b} \varphi(p^b) + \sum_{b=t+1}^{s} \frac{p^s}{p^b} \mu(p^{b-t}) \frac{\varphi(p^b)}{\varphi\left(\frac{p^b}{\gcd(u p^t, p^s)}\right)}
\]

(35)

\[
= p^s + \sum_{b=1}^{t} \frac{p^s}{p^b} (p^b - p^{b-1}) - \sum_{b=t+1}^{s} \frac{p^s}{p^{t+1}} \varphi\left(\frac{p^{t+1}}{\gcd(u p^t, p^s)}\right) (1 - \theta_{t,s})
\]

(36)

\[
= (\min(t, s) + 1) \varphi(p^s) + \theta_{t,s} p^{s-1}
\]

(37)

Because \(\gcd(m, p^s) > 1\) the first summation in equation 35 contains at least one term, corresponding to the case \(t = 1\). In the second summation only the term with \(b = t + 1\) remains, again because of the properties of the Möbius function, except in the case where \(t \geq s\), when this term is not present at all, resulting in the factor \(1 - \theta_{t,s}\) in equation 36. After cancelling a factor \(p^{b-1}\) in the remaining summation on the right hand side of equation 36 the summand no longer depends on \(b\) resulting in a factor \(\min(t, s)\). The \(\min\) function results from the fact that the summation over \(b\) runs up to \(t\) except when \(s < t\), in which case it runs up to \(b = s\). Equation 37 is the desired result.

\[ \square \]

3. Discussion

In this section a number of interesting consequences of theorem 1 are discussed, some of which are generalizations of known results.
Corollary 1. If \( \gcd(m, n) = 1 \) then the discrete Fourier transform of the greatest common divisor, as given in equation 1, is equal to the Euler totient function, i.e., if \( \gcd(m, n) = 1 \) then \( h_m(n) = \varphi(n) \).

Proof. From theorem 1 it follows that if \( \gcd(m, n) = 1 \), then \( t_i = 0 \), implying that \( \theta_s, t_v = 0 \), and \( \min(t_i, s_i) = 0 \), \( i = 1, \cdots, r \). Because of the multiplicativity of the totient function one obtains

\[
\sum_{k=1}^{n} \gcd(k, n)e^{-\frac{k \pi i}{n} m} = \prod_{i=1}^{r} \varphi(p_i s_i) \tag{38}
\]

\[
= \varphi(n). \tag{39}
\]

The result of corollary 1 was known for the special case \( m = 1 \), see [1].

Corollary 2. If \( \gcd(m, n) = 1 \) then the Ramanujan sum \( r_m(n) = \mu(n) \)

Proof. From corollary 1 and equation 27 it follows that if \( \gcd(m, n) = 1 \) then

\[
id * r_m(n) \quad = \quad \varphi(n) \tag{40}
\]

which combined with the well known result that \( id * \mu(n) = \varphi(n) \) proves corollary 2.

Corollaries 1 and 2 above both apply to the case where \( \gcd(m, n) = 1 \), in which case the function \( h_m(n) \) becomes entirely independent of \( m \). As a matter of fact, also for general \( m \) does \( h_m(n) \) only depend on prime factors \( p_i \) of \( n \). The only specific property of \( m \) that affects the result is the multiplicity \( t_i \) of \( p_i \) in \( m \), as can be seen from theorem 1.

4. Generalizations

It is possible to generalize theorem 1 to the case of the discrete Fourier transform of a function of the greatest common divisor. Hence we define the discrete Fourier transform of a function of the greatest common divisor as follows. Given an arithmetic function \( f(n) : \mathbb{N} \rightarrow \mathbb{C} \), the discrete Fourier transform of this function of the greatest common divisor with respect to the \( m \)-th order of the \( n \)-th root of unity, \( 1 \leq m \leq n \), is given by

\[
\hat{f}_m(n) = \sum_{k=1}^{n} f(\gcd(k, n))e^{-\frac{k \pi i}{n} m} \tag{41}
\]

As already made clear in the introduction, \( \hat{f}_m(n) \) is a multiplicative function for fixed \( m \) if \( f \) is a multiplicative function. Moreover, if \( f \) is integer valued then \( \hat{f}_m(n) \) is integer valued, which also follows from [1]. Taking the the identity function \( id(n) := n \) for \( f(n) \), equation 41 gives the discrete Fourier transform of the gcd, \( h_m(n) \).

A closed form expression in the case where \( f \) is completely multiplicative will be proven below. The following lemma covers the general case of a multiplicative function of the greatest common divisor, in which case the result still contains a summation. When specializing to a completely multiplicative function this summation can be done.
Lemma 1. If the factorization of the number \( n \) in its prime factors \( p_j, j = 1, \ldots, r \), is given as \( n = \prod_{j=1}^{r} p_j^{s_j} \), where \( s_j \), with \( s_j \geq 1 \), is the multiplicity of prime factor \( p_j \), and the number \( m \), where \( 1 \leq m \leq n \), is written as \( m = \prod_{i=1}^{r} p_i^{t_i} \), with \( t_i \geq 0 \), \( \gcd(u, p_i) = 1 \), and \( 1 \leq i \leq r \), then the discrete Fourier transform of a multiplicative function \( f \) of the greatest common divisor is given by

\[
\sum_{k=1}^{n} f(\gcd(k, n)) e^{-k \frac{2\pi i}{n} m} = \\
\prod_{i=1}^{r} \left[ f(p_i^{s_i}) + (p_i - 1) \sum_{b=1}^{\min(t_i, s_i)} p_i^{b-1} f(p_i^{s_i-b}) - f(p_i^{s_i-t_i-1})p_i^{t_i}(1 - \theta_{t_i, s_i}) \right] \tag{42}
\]

Proof. The proof is similar to the proof of theorem 1, we repeat the constructive proof based on equation 26 and von Sterneck’s formula, equation 28, where now the identity function \( \text{id}(n) \) is replaced by \( f(n) \),

\[
\sum_{k=1}^{n} \gcd(k, n) e^{-k \frac{2\pi i}{n} m} = \\
\sum_{d|n} \frac{\varphi(d)}{\varphi(\gcd(m, d))} = \\
\frac{1}{\varphi(\gcd(m, d))} \left( 1 - \frac{\varphi(d)}{\varphi(\gcd(m, d))} \right). \tag{43}
\]

Steps similar to those from equation 33 to 37 for the case \( \gcd(m, n) > 1 \) and from equation 30 to equation 31 for the case \( \gcd(m, n) = 1 \) produce the result of lemma 1.

The result of theorem 1 is recovered from lemma 1 by taking \( f = \text{id} \). In that case the summand of the summation in equation 42 becomes independent of the summation index which means that the summation results in a factor \( \min(t, s) \). After regrouping terms equation 6 of theorem 1 is obtained.

In order to make further progress it is needed to make assumptions about the function \( f \). We will consider the natural case of specializing to a completely multiplicative function. In this case the summation in lemma 1 can be explicitly performed.

Theorem 2. The discrete Fourier transform of a completely multiplicative function \( f \), not equal to the identity function \( \text{id} \), of the greatest common divisor is, under the conditions of lemma 1, given by

\[
\sum_{k=1}^{n} f(\gcd(k, n)) e^{-k \frac{2\pi i}{n} m} = \\
\prod_{i=1}^{r} \left[ f(p_i^{t_i}) - f(p_i^{s_i-t_i-1})p_i^{t_i}(1 - \theta_{t_i, s_i}) \right. \\
+ \left. (p_i - 1) f(p_i^{s_i-1}) - f(p_i^{\min(t_i, s_i)})p_i^{\min(t_i, s_i)} - f(p_i^{\min(t_i, s_i)}) \right] \tag{44}
\]

Proof. If the function \( f \) is completely multiplicative then \( f(n^*) = f^*(n) \), and the series in equation 42, lemma 1, becomes geometric. As a consequence the summation can be performed leading to the given closed form expression. The absence of the summed term for the case \( t_i = 0 \) has been made explicit with factor \( 1 - \delta_{0t_i} \).
If the function \( f \) equals the identity function the closed form formula for a geometric series diverges. However the series can still be summed as proven in theorem 1.

The sum function of an arithmetic function \( t(n) \) is defined as the Dirichlet product of \( t(n) \) with the constant one function \( 1(n) := 1 \) as

\[
S^t(n) = (1 * t)(n)
\]  
(45)

If the function \( f \) is written as the sum function of some function \( t \), \( f = 1 * t \), then the following corollary can be formulated.

**Corollary 3.** If the function \( f(n) \) is the sum function of the function \( t(n) \), i.e., \( f(n) = 1 * t(n) \), then, for those values of \( m \) that are such that \( \gcd(m, n) = 1 \), the discrete Fourier transform of the sum function of the greatest common divisor gives back the function \( t \), i.e., \( \hat{f}_m(n) = t(n) \).

**Proof.** From equations 41 and 26 it follows that \( \hat{f}_m(n) = r_m * f(n) \), which is what is proven in [1]. From corollary 2 it then follows that \( \hat{f}_m(n) = \mu(n) * 1 * t(n) = t(n) \) because the Möbius function is the Dirichlet inverse of the constant one function.

As an interesting example of the application of theorem 2 consider the power function

\[
id_k(n) = n^k.
\]  
(46)

This is a completely multiplicative function not equal to the identity function if \( k \neq 1 \), hence theorem 2 applies. On the other hand, if \( k = 1 \), the function \( id_k \) equals the identity function and theorem 1 applies. It can be verified that applying theorem 2 with \( f = id_k \) and then taking the limit \( k \to 1 \), using l’Hopital’s rule, reproduces the result of theorem 1. We will not reproduce this here.

The sum-function \( S^f \) of a multiplicative function \( f(n) \) can be expressed in the prime-factors of the argument as follows[7]

\[
S^f(n) = \prod_{i=1}^{r} \left[ 1 + f(p_i) + \cdots + f(p_i^{s_i}) \right]
\]  
(47)

With the help of lemma 1 it is possible to obtain a converse relation, as expressed by the following corollary.

**Corollary 4.** Given \( n = \prod_{i=1}^{r} p_i^{s_i} \), a multiplicative function \( f \) can be expressed in its sum-function \( S^f \) as

\[
f(n) = \prod_{i=1}^{r} \left[ S^f(p_i^{s_i}) - S^f(p_i^{s_i-1}) \right]
\]  
(48)
Proof. From lemma 1 and formula 26, taking $S^f$ for $f$, and choosing $m$ such that $\gcd(m, n) = 1$, it follows that

$$S^f * r_m(n) = \prod_{i=1}^{r} \left[ S^f(p_i^{s_i}) - S^f(p_i^{s_i-1}) \right] \tag{49}$$

because the condition $\gcd(m, n) = 1$ implies that $t_i = 0, i = 1, \ldots, r$, resulting in both the uncompleted sum and the symbol $\theta_{t_i, s_i}$ in equation 42 being zero. With corollary 2 equation 48 of corollary 4 follows directly from equation 49.

Note that corollary 4 applies generally, and has no dependence on the order of unity $m$ anymore. Corollary 4 reduces to the familiar expression for the Euler totient function, valid for $n \neq 1$,

$$\varphi(n) = \prod_{i=1}^{r} \left[ p_i^{s_i} - p_i^{s_i-1} \right] \tag{50}$$

if $\varphi$ is chosen for $f$ and one uses that the identity is the sum-function of the totient-function, as expressed by the well known formula,

$$\text{id} * \mu(n) = \varphi(n) \tag{51}$$

Conversely, equation 49 can be written as

$$f * \mu(n) = \prod_{i=1}^{r} \left[ f(p_i^{s_i}) - f(p_i^{s_i-1}) \right] \tag{52}$$

in which form it can be viewed as a generalization of formula 51. If the power function $\text{id}_k$, see expression 46 above, is chosen for the function $f$ in equation 52, the well known relation

$$\text{id}_k * \mu(n) = J_k(n) \tag{53}$$

for the Jordan function $J_k(n)$, defined as

$$J_k(n) = n^k \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i^k} \right) \tag{54}$$

is obtained. $J_k(n)$ is a generalization of the totient function with $\varphi(n) = J_1(n)$. 

11
5. The GCD sum function

The sum of greatest common divisors is called Pillai’s arithmetical function [8]. An expression for this sum exists [9], and reads, in the notation of the referred work,

\[
\sum_{j=1}^{p^\alpha} \gcd(j, p^\alpha) = (\alpha + 1)p^\alpha - \alpha p^{\alpha-1} = (\alpha + 1)\varphi(p^\alpha) + p^{\alpha-1} \tag{55}
\]

This sum is just a special case of the discrete Fourier transform of the greatest common divisor corresponding to the case \(\text{m} = \text{n}\). By virtue of lemma 1 and theorem 2 this sum can now be generalized to the sum of a function of the greatest common divisor. Hence we can formulate the following corollary.

**Corollary 5.** Given a multiplicative function \(f\), the sum over \(k, k = 1, \ldots, n\) of the function values \(f(\gcd(k, n))\) of the greatest common divisor can be expressed in the prime factors of the number \(n = \prod_{i=1}^{s} p_i^{s_i}\) in the following way

\[
\sum_{k=1}^{n} f(\gcd(k, n)) = \prod_{i=1}^{r} \left[ f(p_i^{s_i}) + (p_i - 1) \sum_{b=1}^{s_i} p_i^{b-1} f(p_i^{s_i-b}) \right] \tag{57}
\]

If the function \(f\) is completely multiplicative, but not equal to the identity function \(id\), the sum in expression 57 can be evaluated, yielding

\[
\sum_{k=1}^{n} f(\gcd(k, n)) = \prod_{i=1}^{r} \left[ f(p_i^{s_i}) + (p_i - 1)f(p_i^{s_i-1}) \frac{f(p_i^{s_i}) - p_i^{s_i}}{f(p_i^{s_i}) - p_i f(p_i^{s_i-1})} \right] \tag{58}
\]

**Proof of corollary 5.** By taking the discrete Fourier transform with \(m = n\) in expression 42, it follows that \(t_i = s_i, i = 1, \ldots, r\) implying that \(\min(t_i, s_i) = s_i\) and \(1 - \theta_{s_i, t_i} = 0\) and consequently the statements of corollary 5 follow immediately from lemma 1, and theorem 2. 

6. Example

Consider the case \(n = pq\), with \(p\) and \(q\) different prime numbers, and \(p < q\). The totient function then equals \(\varphi(pq) = (p-1)(q-1)\). View the discrete Fourier transform of the greatest common divisor as a mapping of the array of greatest common divisors, indexed by \(k\), into the array \(h_m(n)\), defined in equation 1, and indexed by the order of the root of unity \(m\). Thus, \(k\) and \(m\) run over the same set of values, namely \(1 \cdots n\). Applying theorem 1 results in the representation, given in table 1, of the function \(h_m(n)\) defined in equation 1.

It should be kept in mind that the explicit layout of such tables depends on the actual values of the prime factors of the number \(n\). In the layout of table 1, for example, it is assumed that \(p^2 > q\), but this could of course just as well be the other way round.
\[
gcd(k, n) = h_m(n)
\]

\[
\begin{array}{ccc}
 k, m & \gcd(k, n) & h_m(n) \\
 1 & 1 & \varphi(n) \\
 \vdots & \vdots & \vdots \\
 p - 1 & 1 & \varphi(n) \\
 p & p & (2p - 1)(q - 1) \\
 p + 1 & 1 & \varphi(n) \\
 \vdots & \vdots & \vdots \\
 2p - 1 & 1 & \varphi(n) \\
 2p & p & (2p - 1)(q - 1) \\
 2p + 1 & 1 & \varphi(n) \\
 \vdots & \vdots & \vdots \\
 q - 1 & 1 & \varphi(n) \\
 q & q & (p - 1)(2q - 1) \\
 q + 1 & 1 & \varphi(n) \\
 \vdots & \vdots & \vdots \\
 p^s - 1 & 1 & \varphi(n) \\
 p^s & p & (2p - 1)(q - 1) \\
 p^s + 1 & 1 & \varphi(n) \\
 \vdots & \vdots & \vdots \\
 n - 1 & 1 & \varphi(n) \\
 n & n & (2p - 1)(2q - 1)
\end{array}
\]

Table 1: Representation of \(h_m(n), n = pq\)

Note that there are, in this case, only four possible outcomes for \(h_m(n)\). It is a general property of \(h_m(n)\) that relatively few values are possible. Consider the case where \(n = p^s\), then, from theorem 1, the following cases for \(m\) can be distinguished.

\[
h_m(n) = \begin{cases} 
\varphi(p^s) & (\gcd(m, n) = 1) \\
(t + 1)\varphi(p^s) & (\gcd(m, n) = p^t, t < s) \\
(s + 1)\varphi(p^s) + p^{s-1} & (\gcd(m, n) = p^s)
\end{cases}
\]

(59)

Obviously, tables similar to table 1 can be constructed for all \(n\) given the prime factorization of \(n\). As another example, consider the case \(n = p^3q^2w\), with \(p, q,\) and \(w\) different prime numbers. This results in the representation given in table 2, this time including just one typical entry. The totient function now equals \(\varphi(n) = (p^3 - p^2)(q^2 - q)(w - 1)\).

7. Conclusion

The discrete Fourier transform of a function of the greatest common divisor has been expressed directly in function values of the prime factors of the argument. Thus, an alternative way of calculating the transform is established. Instead of evaluating a sum.
\[ k, m \quad \gcd(k, n) \quad h_m(n) \]

\begin{array}{ccc}
1 & 1 & \varphi(n) \\
p^2q^2 & p^2q^2 & 3\varphi(p^2) \left[3\varphi(q^2) + q\right] \varphi(w) \\
n & n & \left[4\varphi(p^3) + p^2\right] \left[3\varphi(q^2) + q\right] \left[2\varphi(w) + 1\right]
\end{array}

Table 2: \textbf{Representation of} \( h_m(n) \), \( n = p^3q^2w \)

of terms containing exponential factors, an expression based on function values of powers of prime factors has to be evaluated. Naturally this approach requires knowledge of the prime factors.

Specializing to the identity-function, relationships with the Möbius function and the Euler totient function have been obtained. Furthermore, by taking the order \( m \) equal to the root of unity \( n \), an expression in prime factors of the sum of function values of the greatest common divisor could be stated. Examples of explicit layouts, based on the order \( m \), have been presented.

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