Gauge linked time-dependent non-Hermitian Hamiltonians

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Abstract

In this work we address systems described by time-dependent non-Hermitian Hamiltonians under time-dependent Dyson maps. We shown that when starting from a given time-dependent non-Hermitian Hamiltonian which is not itself an observable, an infinite chain of gauge linked time-dependent non-observable non-Hermitian Hamiltonians can be derived from it. The matrix elements of the observables associated with all these non observable Hamiltonians are, however, all linked to each other, and in the particular case where global gauges exist, these matrix elements becomes all identical to each other. In this case, therefore, by approaching whatever the Hamiltonian in the chain we can get information about any other Hamiltonian. We then show that the whole chain of time-dependent non-Hermitian Hamiltonians collapses to a single time-dependent non-Hermitian Hamiltonian when, under particular choices for the time-dependent Dyson maps, the observability of the Hamiltonians is assured. This collapse thus shows that the observability character of a non-Hermitian Hamiltonian prevents the construction of the gauge-linked Hamiltonian chain and, consequently, the possibility of approaching one Hamiltonian from another.
I. INTRODUCTION

Non-Hermitian quantum mechanics has receiving increasing attention in the literature, and since the decisive contributions of [1] and [2], it has permeated virtually every field of physics [3]. Experimental observation of $\mathcal{PT}$-symmetry and $\mathcal{PT}$-symmetry breaking has been reported in a variety of systems [4], and a different physical phenomena have been investigated within $\mathcal{PT}$-symmetric system [5–8].

Beyond the widely accepted grounds for treating time-independent [or even time-dependent (TD) [9]] non-Hermitian Hamiltonians through time-independent Dyson maps, recent contributions [10–12] have advanced the grounds for treating time-independent and specially TD non-Hermitian Hamiltonians through TD Dyson maps. Although it has been demonstrated that a TD metric operator can not ensure the unitarity of the time-evolution simultaneously with the observability of a non-Hermitian Hamiltonian [13], in Ref. [10] it has been demonstrated that, in spite of the non-observability of the Hamiltonian under a TD metric operator, any other observable associated with this Hamiltonian is derived in complete analogy with the case where a time-independent Dyson map is considered. And beyond Ref. [10], in a more recent contribution [12], a method has been presented which enable us to account for the unitarity of the time-evolution simultaneously with the observability of a non-Hermitian Hamiltonian even for a TD Dyson map. The method relies on the construction of a Schrödinger-like equation from which the TD Dyson map is derived from the TD quasi-Hermitian Hamiltonian itself. Moreover, in spite of the time-dependence of the Dyson map the method ensures a time-independent metric operator, a necessary condition for the observability of a quasi-Hermitian Hamiltonian. Therefore, although in agreement with the main premise in Refs. [10, 13], that a time-independent metric operator is needed for assuring the unitarity of the time evolution simultaneously with the observability of a quasi-Hermitian Hamiltonian, in Ref. [12] a TD Dyson map is considered, and this is an important point since for a TD non-Hermitian Hamiltonian, a time-independent Dyson map is a rather restrictive choice.

In the present contribution we follow the path explored by Refs. [10–12] to advance some interesting properties derived from non-Hermitian Hamiltonians under TD metric operators. Working within the main premise of Ref. [10, 13], that a non-Hermitian Hamiltonian is not itself an observable when the unitarity of the time evolution is ensured, we first verify
that one can build from this Hamiltonian a whole chain of gauge linked time-dependent non-observable non-Hermitian Hamiltonians. We demonstrate that the matrix elements of the observables associated with the gauge linked non-observable Hamiltonians are all linked to each other, and in the particular case where global gauges arise, these matrix elements became all identical to each other. Then, by approaching a given Hamiltonian, we can obtain information about any other Hamiltonian in the chain and, it is worth noting that it is immediate to identify that one of the Hamiltonians in the chain is easier to approach than the other. However, working under the premises of Ref. [12], where the Schrödinger-like equation is considered for the derivation of a TD Dyson map, thus enabling us to ensure the unitarity of the time evolution simultaneously with the observability of a non-Hermitian Hamiltonian, we automatically prevent the possibility of the Hamiltonian chain: In other words, the whole chain reduces to a single TD observable non-Hermitian Hamiltonian, showing that the observability character prevents the construction of gauge-linked Hamiltonians and observables. In short, for a TD non-observable non-Hermitian Hamiltonian we can construct a whole chain of connected non-observables non-Hermitian Hamiltonians whose associated observables are all connected to each other; however, when the observability of these Hamiltonians are assured, through the construction of particular Dyson maps from the Schrödinger-like equations, the whole chain collapses.

In what follows we first revisit, in Section II, the developments advanced in Ref. [10] to treat TD non-Hermitian Hamiltonians under TD Dyson maps and metric operators. We then show how to construct from a TD non-Hermitian Hamiltonian an infinite chain of gauge linked TD non-observable non-Hermitian Hamiltonians. The observables associated with these non-observables Hamiltonians are then discussed, specially within the particular case where global gauge transformations exist. In Section III all the developments in Section II is revisited now within the construction in Ref. [12] where a TD non-Hermitian Hamiltonian is itself an observable under a TD Dyson map (but a time-independent metric operator) derived from the Schrödinger-like equation. Two illustrative examples are given in Section IV, the TD harmonic oscillator under TD non-Hermitian linear and parametric amplification processes, and finally, in Section V we present our conclusions.
II. TIME-DEPENDENT NON-HERMITIAN SYSTEMS

Our starting point is a non-Hermitian TD Hamiltonian $H_t \neq H_t^\dagger$ that satisfies the TD Schrödinger equation and the TD quasi-Hermiticity relation \cite{10}

$$H_t \psi_t = i\hbar \partial_t \psi_t, \quad H_t^\dagger \rho_t - \rho_t H_t = i\hbar \partial_t \rho_t,$$

(1)

respectively. Defining a time-dependent Dyson map $\eta_t$ via the relation $\rho_t := \eta_t^\dagger \eta_t$, it follows from (1) that the wave function $\phi_t = \eta_t \psi_t$ satisfies the TD Schrödinger equation for the Hermitian Hamiltonian $h_t = h_t^\dagger$ related to $H_t$ in a TD quasi Hermiticity manner

$$h_t \phi_t = i\hbar \partial_t \phi_t, \quad h_t = \eta_t H_t \eta_t^{-1} + i\hbar (\partial_t \eta_t) \eta_t^{-1}.$$  

(2)

The standard quasi-Hermiticity relations are obtained when $\eta$ and $\rho$ are time-independent.

The time-dependent quasi-Hermiticity relation in Eq. (1) ensures that the time-dependent probabilities in the Hermitian and non-Hermitian systems are related as

$$\langle \phi_t | \tilde{\phi}_t \rangle = \langle \psi_t | \rho_t \tilde{\psi}_t \rangle := \langle \psi_t | \tilde{\psi}_t \rangle_{\rho_t},$$  

(3)

and consequently, that any observable $o_t$ in the Hermitian system has an observable counterpart

$$O_t = \eta_t^{-1} o_t \eta_t,$$  

(4)

in the non-Hermitian system in complete analogy with the scenario where the Hamiltonian $H$ and the Dyson map $\eta$ are time-independent operators.

Defining now the two Hilbert spaces $\mathcal{H}(\phi)$ with inner product $\langle \phi_t | \tilde{\phi}_t \rangle$ and $\mathcal{H}(\psi)$ with inner product $\langle \psi_t | \rho_t \tilde{\psi}_t \rangle = \langle \psi_t | \tilde{\psi}_t \rangle_{\rho_t}$, it is easily seen that the operators $u_{t,t'}$ and $U_{t,t'} = \eta_t^{-1} u_{t,t'} \eta_t'$ taking a wave function from time $t'$ to $t$ generate unitarity time-evolution operators, i.e. preserve probabilities, in those two spaces. We simple verify

$$\langle \phi_t | \tilde{\phi}_t \rangle = \langle u_{t,t'} \phi_{t'} | u_{t,t'}^\dagger \tilde{\phi}_{t'} \rangle = \langle \phi_{t'} | u_{t,t'}^\dagger u_{t',t} \tilde{\phi}_{t'} \rangle$$

$$= \langle \phi_{t'} | \tilde{\phi}_{t'} \rangle,$$  

(5)

and

$$\langle \psi_t | \rho_t \tilde{\psi}_t \rangle = \langle \eta_t^{-1} u_{t,t'} \eta_{t'} \psi_{t'} | \eta_t^\dagger \eta_t \eta_t^{-1} u_{t,t'} \eta_{t'} \tilde{\psi}_{t'} \rangle$$

$$= \langle \psi_{t'} | \eta_t^\dagger \eta_t \eta_t^{-1} u_{t,t'} \eta_{t'} \tilde{\psi}_{t'} \rangle$$

$$= \langle \psi_{t'} | \rho_{t'} \tilde{\psi}_{t'} \rangle.$$  

(6)
The evolution operator \( u_{t',t} \) satisfies the TD Schrödinger equation (2) associated to \( h_t \) and the standard relations in the usual manner

\[
h_t u_{t,t'} = i\hbar \partial_t u_{t,t'}, \quad u_{t,t'} u_{t',t''} = u_{t,t''}, \quad \text{and} \quad u_{t,t} = I,
\]

and also \( U_{t,t'} \) is easily shown to satisfy the TD Schrödinger equation (1) associated to the Hamiltonian \( H_t \)

\[
i\hbar \partial_t U_{t,t'} = i\hbar \eta^{-1} (\partial_t U_{t,t'}) \eta - i\hbar \eta^{-1} (\partial_t \eta) \eta^{-1} u_{t',t} \eta
\]
\[
= \eta^{-1} h_t u_{t,t'} \eta - i\hbar \eta^{-1} (\partial_t \eta) U_{t,t'}
\]
\[
= \eta^{-1} h_t \eta U_{t,t'} - i\hbar \eta^{-1} (\partial_t \eta) U_{t,t'}
\]
\[
= H_t U_{t,t'}.
\]

Since observables need to be self-adjoint operators, as in Eq. (4), the Hamiltonian \( H_t \) satisfying the TD Schrödinger equation (1) and generating the time-evolution (8) is not an observable quantity as pointed out in [2]. Instead the operator

\[
H'_t = \eta^{-1} h_t \eta = H_t + i\hbar \eta^{-1} \partial_t \eta,
\]

is an observable quantity in the Hilbert space \( \mathcal{H}(\psi) \). The operator \( H'_t \) is of course not a Hamiltonian in that space, but it can be used to set up a system of new TD quasi-Hermitian operators

\[
H'_t \psi'_t = i\hbar \partial_t \psi'_t, \quad (H'_t)^\dagger \rho'_t - \rho'_t H'_t = i\hbar \partial_t \rho'_t, \quad (10a)
\]
\[
h'_t \phi'_t = i\hbar \partial_t \phi'_t, \quad h'_t = \eta'_t H'_t (\eta'_t)^{-1} + i\hbar (\partial_t \eta'_t) (\eta'_t)^{-1}, \quad (10b)
\]

with \( \phi'_t = \eta'_t \psi'_t, \rho'_t := (\eta'_t)^\dagger \eta'_t \) and new Hilbert spaces \( \mathcal{H}(\phi') \) and \( \mathcal{H}(\psi') \).

**A. Gauge symmetrically linked Hamiltonian chain**

Apart from being linked by \( H'_t \), at this point the two systems \( \mathcal{H}(\phi), \mathcal{H}(\psi) \) and \( \mathcal{H}(\phi'), \mathcal{H}(\psi') \) are unrelated. In order to achieve that we assume that \( \mathcal{H}(\phi) \) and \( \mathcal{H}(\phi') \) are related to each other by a gauge transformation. In other words we assume that \( \phi'_t \) and \( \phi_t \) are related to each other by a unitary operator \( A_t \) as \( \phi'_t = A_t \phi_t \). The substitution of this relation into (10b) leads to the standard expression

\[
h'_t = A_t h_t A_t^{-1} + i\hbar (\partial_t A_t) A_t^{-1},
\]
and consequently to
\[ i\hbar \partial_t A_t = h'_t A_t - A_t h'_t, \] (12)
where \( h'_t \) follows by substituting Eq. (9) into Eq. (10b), thus giving
\[ h'_t = \eta'_t \eta_t^{-1} h_t \eta_t (\eta'_t)^{-1} + i\hbar (\partial_t \eta'_t) (\eta'_t)^{-1}. \] (13)

Thus we have now related the wavefunction of all four Hilbert spaces to each other
\[ \psi'_t = (\eta'_t)^{-1} \phi'_t = (\eta'_t)^{-1} A_t \phi_t = (\eta'_t)^{-1} A_t \eta_t \psi_t. \] (14)

Similarly we may now also relate all four unitary time-evolution operators to each other as
\[
U'_{t,t'} = (\eta'_t)^{-1} u'_{t,t'} \eta'_t
= (\eta'_t)^{-1} A_t u_{t,t'} A^{-1}_t \eta'_t
= (\eta'_t)^{-1} A_t \eta_t U_{t,t'} \eta_t^{-1} A^{-1}_t \eta'_t.
\] (15)

Back to Eq. (10b) we note that \( H'_t \) is not an observable in the space \( \mathcal{H}(\psi') \), contrarily to the operator
\[ H''_t = (\eta'_t)^{-1} h'_t \eta'_t = H'_t + i\hbar (\eta'_t)^{-1} \partial_t \eta'_t, \] (16)
which can then be used to set up another pair of Schrödinger equations
\[ H''_t \psi''_t = i\hbar \partial_t \psi''_t, \quad (H''_t)^{-1} \rho''_t - \rho''_t H''_t = i\hbar \partial_t \rho''_t, \] (17a)
\[ h''_t \phi''_t = i\hbar \partial_t \phi''_t, \quad h''_t = \eta''_t H''_t (\eta''_t)^{-1} + i(\partial_t \eta''_t) (\eta''_t)^{-1}, \] (17b)
thus defining another pair of Hilbert spaces \( \mathcal{H}(\phi''), \mathcal{H}(\psi'') \). To link together the two systems \( \mathcal{H}(\phi'), \mathcal{H}(\psi') \) and \( \mathcal{H}(\phi''), \mathcal{H}(\psi'') \), we assume that \( \phi''_t = A'_t \phi'_t \), with \( A'_t \) being another unitary operator, such that, similarly to Eqs. (11), (12), and (13), we now have
\[ h''_t = A'_t h'_t (A'_t)^{-1} + i\hbar (\partial_t A'_t) (A'_t)^{-1} \] (18a)
\[ = \eta''_t (\eta'_t)^{-1} h'_t \eta'_t (\eta''_t)^{-1} + i\hbar (\partial_t \eta''_t) (\eta''_t)^{-1}, \] (18b)
and consequently
\[ i\hbar \partial_t A'_t = h''_t A'_t - A'_t h''_t. \] (19)

We can then build a whole chain of Hamiltonians starting from \( H_t \) and going through \( H'_t, H''_t, \ldots \) with their associated Hermitian counterparts \( h_t, h'_t, h''_t, \ldots \) derived through the
time-dependent Dyson maps $\eta_t, \eta'_t, \eta''_t, \ldots$ and the wave functions $\phi_t, \phi'_t, \phi''_t, \ldots$ related to each other by the unitary operators $A_t, A'_t, \ldots$. This construction, leading to the Hamiltonians derived from $H_t$: $H_t \rightarrow H'_t = H_t + i\hbar \eta_{t}^{-1} \partial_t \eta_t \rightarrow H''_t = H'_t + i\hbar (\eta'_t)^{-1} \partial_t \eta'_t \rightarrow \ldots$, can also be carried out in reverse by looking for the derivation of $H_t$ itself from

$$\tilde{H}_t = H_t - i\hbar (\tilde{\eta}_t)^{-1} \partial_t \tilde{\eta}_t, \quad (20)$$

and then the derivation of $\tilde{H}_t$ from

$$\tilde{\tilde{H}}_t = \tilde{H}_t - i\hbar \left(\frac{\tilde{\eta}_t}{\eta_t}\right)^{-1} \partial_t \tilde{\eta}_t, \quad (21)$$

and so on, thus leading to the infinite chain of non-Hermitian Hamiltonians

$$\ldots \rightarrow \tilde{\tilde{H}}_t \rightarrow \tilde{H}_t \rightarrow H_t \rightarrow H'_t \rightarrow H''_t \rightarrow \ldots, \quad (22)$$

which are related with their Hermitian counterparts

$$\ldots \rightarrow \tilde{h}_t = \tilde{\eta}_t H_t (\tilde{\eta}_t)^{-1} \rightarrow h_t = \eta_t H'_t (\eta_t)^{-1} \rightarrow h'_t = \eta'_t H''_t (\eta'_t)^{-1} \rightarrow \ldots, \quad (23)$$

where, as to be discussed bellow, the Hamiltonian $\tilde{h}_t$, on the border between the prime and the bar Hamiltonians, differs from all others because it does not involve a time derivative of the Dyson map operator.

Thus, following the same procedure leading from $H_t$ to $H'_t$ through $\eta_t$, and so on, by defining $\tilde{H}_t$ in the way written above we immediately obtain $H_t$ through $\tilde{\eta}_t$ and thus all the Hamiltonians preceding $\tilde{H}_t$ as given by the chain in Eq. (22). The Hamiltonian $\tilde{H}_t$ is then constructed from $H_t$ and the Dyson map $(\tilde{\eta}_t)^{-1}$ previously to the transformation that this map performs on the Schrödinger equation for $\tilde{H}_t$ leading to that for $H_t$, the same applying for all the Hamiltonians preceding $H_t$. Differently, the Hamiltonian $H'_t$ is constructed from $H_t$ and the Dyson map $\eta_t$ afterwards the transformation this map performs on the Schrödinger equation for $H_t$, the same applying for all the Hamiltonians following from $H_t$. However, for both cases, the Hamiltonians preceding $H_t$ or following from $H_t$, they are fully determined only after we have computed the time-dependent parameters defining their respective Dyson maps through the Hermiticity of the required Hermitian counterparts.

### B. Observables

Let us now turn to the observables $\ldots, \tilde{O}_t, O'_t, \ldots$ associated with the non-Hermitian Hamiltonians $\ldots, \tilde{H}_t, H_t, H'_t, \ldots$ composing the chain in Eq. (22). Considering, for example,
the observables associated with $H_t$, given by Eq. (4), their matrix elements in the space $\mathcal{H}(\psi)$ are related to the matrix elements of their Hermitian counterparts $a_t$ in the space $\mathcal{H}(\phi)$, as well as in that preceding it, $\mathcal{H}(\tilde{\phi})$, or following it, $\mathcal{H}(\phi')$, through the gauge operators $\tilde{A}_t$ and $A_t$ in the form

$$
\langle \psi_t | O_t \tilde{\psi}_t \rangle = \langle \phi_t | a_t \tilde{\phi}_t \rangle = \langle \phi_t | (\tilde{A}_t)^\dagger a_t A_t \tilde{\phi}_t \rangle
$$

$$
= \langle \phi_t | A_t a_t A_t^\dagger \tilde{\phi}_t \rangle. \quad (24)
$$

Similar relations hold for the matrix elements of all other observables ..., $\tilde{O}_t, O'_t, ...$, related to the non-Hermitian Hamiltonians ..., $\tilde{H}_t, H'_t, ...$. In the same way that the matrix elements of the observables ..., $\tilde{O}_t, O_t, O'_t, ...$ can be computed in whatever the Hilbert space ..., $\mathcal{H}(\tilde{\phi}), \mathcal{H}(\phi), \mathcal{H}(\phi')$, ..., these matrix elements are all connected to each other, since the space states as well as the observables are also all connected to each other [the former as given, for example, by Eq. (4)], and the latter as given by $O'_t = (\eta_t)^{-1} \eta_t O_t \eta_t^{-1} \eta'_t]$. This shows that by approaching a given Hamiltonian in the chain, we thus obtain information about any other Hamiltonian, and considering again the observable $O_t$, it is immediate to relate its matrix elements in its own space $\mathcal{H}(\psi)$ with those computed for example in $\mathcal{H}(\psi')$ as

$$
\langle \psi_t | O_t \tilde{\psi}_t \rangle_{\psi_t} = \langle \psi'_t | (\eta'_t)^\dagger A_t \eta_t O_t \eta_t^{-1} A_t^\dagger \eta'_t \tilde{\psi}_t \rangle
$$

$$
= \langle \psi'_t | (\eta'_t)^\dagger A_t \eta_t O'_t (\eta_t)^{-1} A_t^\dagger \eta'_t \tilde{\psi}_t \rangle. \quad (25)
$$

Regarding the matrix elements of the Hamiltonian $H'_t$ in the space $\mathcal{H}(\psi)$, where it is an observable, they are related to the matrix elements of $h_t$ in the space $\mathcal{H}(\phi)$, as well as in that preceding it, $\mathcal{H}(\tilde{\phi})$, or following it, $\mathcal{H}(\phi')$, through the gauge operators $\tilde{A}_t$ and $A'_t$:

$$
\langle \psi_t | H_t \tilde{\psi}_t \rangle = \langle \phi_t | h_t \tilde{\phi}_t \rangle = \langle \phi_t | (\tilde{A}_t)^\dagger h_t A_t \tilde{\phi}_t \rangle
$$

$$
= \langle \phi_t | (A_t^\dagger)^{-1} h_t (A_t)^{-1} \tilde{\phi}_t \rangle. \quad (26)
$$

The matrix elements of the Hamiltonian $H'_t$ in the space $\mathcal{H}(\psi')$ are related to the matrix elements of $h'_t$ and $h_t$ in the space $\mathcal{H}(\phi')$ in the form

$$
\langle \psi'_t | H'_t \tilde{\psi}'_t \rangle = \langle \phi'_t | h'_t \tilde{\phi}'_t \rangle = \langle \phi'_t | (\eta'_t)^{-1} h'_t \eta'_t - i h (\eta_t)^{-1} \partial_t \eta'_t] (\eta_t)^{-1} \tilde{\phi}'_t \rangle
$$

$$
= \langle \phi'_t | \eta'_t \eta_t^{-1} h_t \eta_t (\eta'_t)^{-1} \tilde{\phi}'_t \rangle, \quad (27)
$$

clearly showing that $H'_t$ is not an observable in its own space $\mathcal{H}(\psi')$. 

8
C. Global gauge transformations

From Eqs. (12) and (19) we conclude that under global gauge transformations, where all the operators ..., $\tilde{A}_t$, $A_t$, $A'_t$, ... are time-dependent or constant phase factors, proportional to the identity, such that $i\hbar \partial_\tau A_t = [h'_t - h_t] A_t$, it follows that

$$A_t = A'_t \exp \left( -\frac{i}{\hbar} \int^t_\tau (h'_\tau - h_\tau) d\tau \right), \tag{28}$$

with similar expressions for $A'_t, A''_t, ...$. Therefore, global gauge operators $A'_t, A''_t, ...$, demand the neighboring Hermitian Hamiltonians to differ from each other only by a $C$-number, i.e., $h'_t - h_t = C_t$, $h''_t - h'_t = C'_t$, ..., such that $A_t = A'_t \exp \left[ -\frac{i}{\hbar} \int^t_\tau C_\tau d\tau \right]$, and so on. Moreover, global gauges also demand the Dyson maps to satisfy equations of the form

$$\partial_\tau \eta'_t = -\frac{i}{\hbar} \eta'_t \left[ (\eta'_t)^{-1} h'_t \eta'_t - \eta^{-1}_t h_t \eta_t + C_t \right], \tag{29}$$

with similar expressions for all other Dyson maps.

We thus verify that, under global gauge transformations the matrix elements of the observable $O_t$ in the space $\mathcal{H}(\psi)$, as given by Eq. (24), are related to the matrix elements of their Hermitian counterparts $o_t$ in the spaces $\mathcal{H}(\phi), \mathcal{H}(\tilde{\phi})$, and $\mathcal{H}(\phi')$, in the simplified form

$$\langle \psi_t | O_t \tilde{\psi}_t \rangle_{\rho_t} = \langle \phi_t | o_t \tilde{\phi}_t \rangle = \langle \tilde{\phi}_t | o_t \tilde{\phi}_t \rangle = \langle \phi'_t | o_t \tilde{\phi}'_t \rangle. \tag{30}$$

In addition, the matrix elements of all the observables ..., $O_t, O_t, O'_t, ...$ (associated with the non-Hermitian Hamiltonians ..., $\tilde{H}_t, H_t, H'_t, ...$), in their respective spaces ..., $\tilde{H}(\tilde{\psi}), \mathcal{H}(\psi), \mathcal{H}(\psi'), ...$, all equal each other under global gauge transformations:

$$\langle \tilde{\psi}_t | \tilde{O}_t \tilde{\psi}_t \rangle_{\tilde{\rho}_t} = \langle \psi'_t | O'_t \tilde{\psi}'_t \rangle_{\tilde{\rho}'_t} = \langle \psi'_t | O'_t \tilde{\psi}'_t \rangle_{\tilde{\rho}'_t} = \langle \tilde{\phi}_t | o_t \tilde{\phi}_t \rangle. \tag{31}$$

Regarding the matrix elements of the Hamiltonian $H'_t$ in the the space $\mathcal{H}(\psi)$, under global gauges they are related to the matrix elements of $h_t$ in the space $\mathcal{H}(\phi), \mathcal{H}(\tilde{\phi})$, and $\mathcal{H}(\phi')$, in the form

$$\langle \psi_t | H'_t \tilde{\psi}_t \rangle_{\rho_t} = \langle \phi_t | h_t \tilde{\phi}_t \rangle = \langle \tilde{\phi}_t | h_t \tilde{\phi}_t \rangle = \langle \phi'_t | h_t \tilde{\phi}'_t \rangle. \tag{32}$$
whereas the matrix elements of the Hamiltonian $H'_t$ in the space $\mathcal{H}(\psi')$ are still given by Eq. (27).

D. Practical Application of a Gauge Linked Hamiltonian Chain

Before addressing the method of constructing observables TD non-Hermitian Hamiltonians which simultaneously imply the unitarity of the Schrödinger time-evolution [12], it is worth stressing that the Hamiltonian chain has a very clear practical application: As long as the matrix elements of their associated observables are all linked together, it becomes simpler to pick up the Hamiltonian $\bar{H}_t$ among all those in the chain. In fact, the computation of its Hermitian counterpart $\bar{h}_t = \bar{\eta}_t H_t (\bar{\eta}_t)^{-1}$, unlike all other Hermitian counterparts, does not involve a time derivative of the corresponding Dyson map, what is generally a difficult task demanding the Gauss decomposition of this TD operator. In the two illustrative examples given below—the TD harmonic oscillator under TD linear and nonlinear amplification processes—we explore this practical feature.

III. OBSERVABILITY OF THE NON-HERMITIAN HAMILTONIANS IN THEIR OWN SPACES SIMULTANEOUSLY TO THE UNITARITY OF THE TIME EVOLUTION

It has been demonstrated in Ref. [2] that a TD Dyson map can not ensure the unitarity of the time-evolution simultaneously with the observability of the Hamiltonian. The developments in Ref. [2] has been extended to demonstrate that despite the nonobservability of the non-Hermitian Hamiltonian under a TD Dyson map, a TD Dyson equation and a TD quasi-Hermiticity relation can be solved consistently, showing that any other observable in the non-Hermitian system is derived in complete analogy with the time-independent scenario [10]. Solutions to the proposed TD Dyson equation and quasi-Hermiticity relation have been presented in the literature [10, 11].

More recently, in Ref. [12] an strategy has been presented for the derivation of a time-dependent Dyson map which ensures simultaneously the unitarity of the time evolution and the observability of a quasi-Hermitian Hamiltonian. This time-dependent Dyson map is derived deterministically, except for its initial condition, through a Schrödinger-like equation
governed by the non-Hermitian Hamiltonian itself. The Schrödinger-like equation follows by imposing, in the Eq. (2) for $h_t$, the gauge-like term $i (\partial_t \eta_t) \eta_t^{-1}$ to be equal to $\eta_t H_t \eta_t^{-1}$, thus leading to

$$i\hbar \partial_t \eta_t = \eta_t H_t. \tag{33}$$

which is indeed similar to the Schrödinger equation written in the dual Hilbert space. Evidently, the above constructed equation ensures the similarity transformation

$$h_t = 2\eta_t H_t \eta_t^{-1}, \tag{34}$$

and by demanding $h_t$ to be Hermitian, we derive the quasi-Hermiticity relation

$$H_t^\dagger \rho_t = \rho_t H_t, \tag{35}$$

which consistently implies the time-independency of the metric operator $\rho_t = \rho(t_0)$ in spite of the time-dependency of the Dyson map. In fact, the time-independency of the metric is a necessary condition for the observability of the non-Hermitian $H_t$.

Here, considering our gauge linked quasi-Hermitian Hamiltonian chain, when imposing the Schrödinger-like equation of the form (33) for all the TD Dyson maps (i.e., $i\hbar \partial_t \eta'_t = \eta'_t H'_t$, $i\hbar \partial_t \eta''_t = \eta''_t H''_t$, ...), we verify that all the non-Hermitian Hamiltonian in the chain (22) automatically reduces to a single Hamiltonian $H_t$ apart from constant factors; to be precise, we obtain ..., $\tilde{H}_t = H_t/4$, $\tilde{H}_t = H_t/2$, $H'_t = 2H_t$, $H''_t = 4H_t$, ... . We have thus found here the interesting property that an observable quasi-Hermitian Hamiltonian is unique, whereas the lack of its observability enable us to construct the whole chain of gauge symmetrically linked quasi-Hermitian Hamiltonians ..., $\tilde{H}_t, H_t, H'_t, ...$, with the associated observables..., $\tilde{O}_t, O_t, O'_t, ...$. In the case where global gauge transformations are required, the matrix elements of all the observables in their respective spaces equal each other, as in Eq. (??), despite the fact that the Hamiltonians themselves are not observables.

IV. THE LINEAR TIME-DEPENDENT NON-HERMITIAN HAMILTONIAN FOR BOSONIC OPERATORS

As an illustrative example of the theory presented above we consider a TD harmonic oscillator under a non-Hermitian linear amplification process, described by the Hamiltonian ($\hbar = 1$)
\[ H_t = \omega_t a^\dagger a + \alpha_t a + \beta_t a^\dagger, \]  

(36)

where \( a \) and \( a^\dagger \) are bosonic annihilation and creation operators and the TD parameters \( \omega_t, \alpha_t, \beta_t \) are complex functions. It is evident that \( H(t) \) is not an Hermitian operator when \( \omega_t \notin \mathbb{R} \) and/or \( \alpha_t \neq \beta_t^* \). It becomes \( \mathcal{PT} \)-symmetric when demanding \( \omega_t \) to be an even function in \( t \) or a generic function of \( it \), simultaneously with demanding \( \alpha_t, \beta_t \) to be odd functions in \( t \) or pure-imaginary generic functions of \( it \).

Next, we consider the case where the TD Dyson map \( \eta_t \) is not derived from the Schrödinger-like equation (33), so that the Hamiltonian \( H_t \) is not an observable and a whole chain of gauge linked Hamiltonians can be derived from it. Thus, considering the following ansatz for an Hermitian time-dependent Dyson map

\[ \bar{\eta}_t = \exp \left( \bar{\gamma}_t a + \bar{\gamma}_t^* a^\dagger \right), \]  

(37)

we may construct the operator \( \bar{H}_t \) using Eq. (20), which will be fully defined only after the calculation of the time-dependent parameters \( \bar{\gamma}_t \) and \( \bar{\lambda}_t \). Then, by transforming the Schrödinger equation for \( \bar{H}_t \) through the Dyson map \( \bar{\eta}_t \), we obtain the Schrödinger equation for

\[ \bar{h}_t = \bar{\eta}_t H_t (\bar{\eta}_t)^{-1} \]
\[ = \omega_t a^\dagger a + \bar{u}_t a + \bar{v}_t a^\dagger + \bar{f}_t, \]  

(38)

with

\[ \bar{u}_t = \omega_t \bar{\gamma}_t + \alpha_t, \]  

(39a)

\[ \bar{v}_t = -\omega_t (\bar{\gamma}_t)^* + \beta_t, \]  

(39b)

\[ \bar{f}_t = -\omega_t |\bar{\gamma}_t|^2 - \alpha_t (\bar{\gamma}_t)^* + \beta_t \bar{\gamma}_t. \]  

(39c)

By imposing \( \bar{h}_t \) to be Hermitian, i.e., \( \omega_t, \bar{f}_t \in \mathbb{R} \) and \( \bar{v}_t = \bar{u}_t^* \), we obtain \( \bar{\gamma}_t = \left[ |\beta_t|^2 - \omega_t \right] / 2\omega_t \) and \( \alpha_t \beta_t \in \mathbb{R} \); consequently, it follows that \( \bar{f}_t = \left[ |\alpha_t|^2 + |\beta_t|^2 - 2\alpha_t \beta_t \right] / 4\omega_t \).

Next, we transform the Schrödinger equation for \( H_t \) using the invariant form for the Hermitian time-dependent Dyson map

\[ \eta_t = \exp \left( \gamma_t a + \gamma_t^* a^\dagger \right), \]  

(40)
to obtain the Schrödinger equation for
\[
\dot{h}_t = \eta_t H'_t \eta_t^{-1} = \omega_t a_+^t a + u_t a + v_t a_+^t + f_t, \tag{41}
\]
with \(H'_t\) given by Eq. (9) and
\[
u_t = \omega_t \gamma_t + \alpha_t + i \partial_t \gamma_t, \tag{42a}
\]
\[
v_t = -\omega_t \gamma_t^* + \beta_t + i \partial_t \gamma_t^*, \tag{42b}
\]
\[
f_t = -\omega_t |\gamma_t|^2 - \alpha_t \gamma_t^* + \beta_t \gamma_t + \frac{i}{2} (\gamma_t \partial_t \gamma_t^* - \gamma_t^* \partial_t \gamma_t). \tag{42c}
\]
For \(h_t\) to be Hermitian we require \(\bar{f}_t \in \mathbb{R}\) and \(v_t = u_t^*\), what lead to the equation \(\partial_t \gamma_t = i \omega_t \gamma + i [\alpha_t - \beta_t^*]/2\), and consequently to \(u_t = \bar{u}_t = v_t^* = \bar{v}_t^* = [\alpha_t + \beta_t^*]/2\) and \(f_t = -\alpha_t \gamma_t^* + \beta_t \gamma_t + \text{Re}[\alpha_t \gamma_t^* - \beta_t \gamma_t]/2\), such that
\[
h_t = \bar{h}_t + \bar{C}_t, \tag{43}
\]
with \(\bar{C}_t = f_t - \bar{f}_t\).

We can go further by computing \(h'_t = \eta'_t H''_t (\eta'_t)^{-1}\) from the invariant form \(\eta'_t = \exp[\gamma'_t a + (\gamma'_t)^* a^\dagger]\) and \(H''_t\) given by Eq. (16); we obtain
\[
h'_t = \omega_t a_+^t a + u_t a + v_t a_+^t + f'_t, \tag{44}
\]
with
\[
f'_t = \omega_t \left(|\gamma'_t|^2 + 2 |\gamma'_t|^2 - \text{Re} \left[\gamma_t (\gamma'_t)^*\right]\right) + \frac{1}{2} \text{Re}(\alpha_t \gamma_t^* - \beta_t \gamma_t) - i \text{Im}[\alpha_t (\gamma'_t)^* - \beta_t \gamma'_t], \tag{45}
\]
such that, similarly to Eq. (43), it follows that
\[
h'_t = h_t + C_t, \tag{46}
\]
with \(C_t = f'_t - f_t\).

Therefore, the gauge operators \(\ldots, \bar{A}_t, A_t, \ldots\) connecting the adjacent spaces \(\ldots, \mathcal{H}(\bar{\phi}), \mathcal{H}(\phi), \mathcal{H}(\phi'), \ldots\) are global operators of the form \(\ldots, \bar{A}_t = \exp\left(i \int_0^t \bar{C}_t d\tau\right), A_t = \exp\left(i \int_0^t C_t d\tau\right), \ldots\) which makes the matrix elements of the observables \(\ldots, \bar{O}_t = \bar{\eta}_t^{-1} O_t \bar{\eta}_t, O_t = \ldots\)
\[ \eta_t^{-1} \eta_t, \ldots \text{ exactly the same in whatever the Hilbert space } \mathcal{H}(\bar{\phi}), \mathcal{H}(\phi), \mathcal{H}(\phi'), \ldots \]

Considering, for example, the field quadratures

\[ x_k = \frac{a - (-1)^k a^\dagger}{2i k^{k-1}}, \quad k = 1, 2, \quad (47) \]

we obtain the observables

\[ \bar{X}_{k,t} = (\bar{\eta}_t)^{-1} x_k \bar{\eta}_t = x_k + \frac{\bar{\gamma}_t + (-1)^k (\gamma_t)^*}{2i k^{k-1}}, \quad (48a) \]
\[ X_{k,t} = \eta_t^{-1} x_k \eta_t = x_k + \frac{\gamma_t + (-1)^k \gamma_t^*}{2i k^{k-1}}, \quad (48b) \]

such that the quasi-Hermitian operators \( \bar{X}_{1,t}, X_{1,t}, (\bar{X}_{2,t}, X_{2,t}) \) equal their \( L^2 \)-counterparts except for time-dependent functions which equal zero when \( \bar{\gamma}_t, \gamma_t \) go through a pure real (pure imaginary). To compute the matrix elements of these observables we follow the reasonings in Ref. [14], where the solution of the Schrödinger equation for the Hermitian TD Hamiltonian is a displaced Fock state \( |m\rangle \) apart from a TD global phase factor

\[ |\phi_{m,t}\rangle = e^{i \varphi_{m,t}} D (\theta_t) |m\rangle, \quad (49) \]

where \( D (\theta_t) = \exp \left[ \theta_t a^\dagger - \theta_t^* a \right] \) is the displacement operator, with \( \theta_t = \theta_0 \exp (-i \chi_t), \quad \chi_t = \int_0^t \omega_{\tau} d\tau \), and

\[ \varphi_{m,t} = \int_0^t d\tau \langle m | D (\theta_{\tau}) (i \partial_{\tau} - h_{\tau}) D (\theta_{\tau}) |m\rangle = -m \chi_t - \int_0^t d\tau f_{\tau}. \quad (50) \]

is the well-known Lewis and Riesenfeld phase [15]. The state vector \( |\phi_{m,t}\rangle \) can be conveniently rewritten as

\[ |\phi_{m,t}\rangle = \Upsilon_t D (\theta_t) R (\omega_t) |m\rangle, \quad (51) \]

where \( \Upsilon_t = \exp \left( -i \int_0^t d\tau f_{\tau} \right) \) is a global phase factor and \( R (\omega_t) = \exp \left( -i \chi_t a^\dagger a \right) \) a rotation operator. Thus, for a generic superposition \( |\phi_t\rangle = \sum_m c_m |\phi_{m,t}\rangle \) it follows that \( |\phi_t\rangle = U_t |\phi_0\rangle = \bar{A}_t |\bar{\phi}_t\rangle \), with the evolution operator

\[ U_t = \Upsilon_t D (\theta_t) R (\omega_t) D^\dagger (\theta_0). \quad (52) \]

The base functions for all other Hermitian counterparts of the non-Hermitian Hamiltonians, differ from those for \( h_t \) —as given by Eq. (51)— only for the global phase factor
\( \Upsilon_t \), where the TD function \( f_t \) must be replaced by the corresponding function related to the Hermitian Hamiltonian. The TD frequency \( \omega_t \) remains unchanged for all Hermitian Hamiltonians as well as the TD parameter \( \theta_t \).

For the initial coherent state \( |\phi_0\rangle \), the expectation values of the observables \( \bar{O}_t \) and \( O_t \) are thus related with their Hermitian counterparts as

\[
\langle \bar{\psi}_t | \bar{O}_t | \bar{\psi}_t \rangle_{\bar{\rho}_t} = \langle \bar{\psi}_t | O_t | \psi_t \rangle_{\rho_t} = \langle \bar{e}^{-ixt} (\phi_0 - \theta_0) + \theta_t | e^{-ixt} (\phi_0 - \theta_0) + \theta_t \rangle,
\]

leading to the expectation values of the quadratures

\[
\langle \bar{\psi}_t | \bar{X}_{1,t} | \bar{\psi}_t \rangle_{\bar{\rho}_t} = \langle \psi_t | X_{1,t} | \psi_t \rangle_{\rho_t} = \text{Re} \left\{ e^{-ixt} (\phi_0 - \theta_0) + \theta_t \right\},
\]
(54a)

\[
\langle \bar{\psi}_t | \bar{X}_{2,t} | \bar{\psi}_t \rangle_{\bar{\rho}_t} = \langle \psi_t | X_{2,t} | \psi_t \rangle_{\rho_t} = \text{Im} \left\{ e^{-ixt} (\phi_0 - \theta_0) + \theta_t \right\}.
\]
(54b)

**A. Local gauges: The generalized time-dependent Swanson Hamiltonian**

We now address the case where a TD harmonic oscillator is under a TD non-Hermitian parametric amplification process, which correspond to the generalized time-dependent Swanson Hamiltonian \[11\]

\[
H_t = \omega_t (a^\dagger a + 1/2) + \alpha_t a^2 + \beta_t a^\dagger^2,
\]
(55)

where \( \omega_t, \alpha_t, \beta_t \in \mathbb{C} \). When \( \omega_t \notin \mathbb{R} \) or \( \alpha_t \neq \beta_t^* \) the above Hamiltonian is clearly not Hermitian, and it becomes \( \mathcal{PT} \)-symmetric when demanding \( \omega_t, \alpha_t, \beta_t \) to be even functions in \( t \) or generic functions of \( it \). Considering the Hermitian time-dependent Dyson map

\[
\bar{\eta}_t = \exp \left[ \bar{\epsilon}_t \left( a^\dagger a + 1/2 \right) + \bar{\mu}_t a^2 + (\bar{\mu}_t)^* a^\dagger^2 \right] = \exp \left( \bar{\lambda}_{+,t} K_+ \right) \exp \left( \ln \bar{\lambda}_{0,t} K_0 \right) \exp \left( \bar{\lambda}_{-,t} K_- \right).
\]
(56)

where \( K_+ = a^\dagger^2/2, K_- = a^2/2, K_0 = (a^\dagger a/2 + 1/4) \) form an \( SU(1,1) \)-algebra, with the TD coefficients

\[
\bar{\lambda}_{+,t} = -\Phi_t e^{-i\bar{\psi}_t},
\]
(57a)

\[
\bar{\lambda}_{-,t} = -\Phi_t e^{i\bar{\phi}_t},
\]
(57b)

\[
\bar{\lambda}_{0,t} = \Phi_t^2 - \bar{\chi}_t.
\]
(57c)
where $\bar{\Phi}_t = |z_t|/\Gamma_{-t}$ and $\tilde{\chi}_t = 2\bar{\Phi}_t/|z_t| - 1$, with $\Gamma_{\pm t} = 1 \pm (\Xi_t/\epsilon_t) \coth \Xi_t$, $\Xi_t = \sqrt{\epsilon_t^2 - 4|\vec{\mu}_t|^2}$, and $\bar{z}_t = 2\vec{\mu}_t/\epsilon_t = |z_t|e^{i\varphi_t}$. Now, using the relation

$$\tilde{\eta}_t \begin{pmatrix} a \\ a^\dagger \end{pmatrix} \tilde{\eta}_t^{-1} = \pm \frac{1}{\sqrt{\lambda_{0,t}}} \begin{pmatrix} -1 & \tilde{\lambda}_{+,t} \\ -\tilde{\lambda}_{-,t} & \tilde{\chi}_t \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix},$$

we obtain the transformed Hamiltonian

$$\tilde{h}_t = \tilde{\eta}_t H \tilde{\eta}_t^{-1} = \tilde{W}_t(a^\dagger a + 1/2) + \tilde{V}_t a^2 + \tilde{T}_t a^2.$$  

(59)

where, by defining $\omega_t = |\omega_t|e^{i\varphi_{\omega,t}}$, $\alpha_t = |\alpha_t|e^{i\varphi_{\alpha,t}}$, and $\beta_t = |\beta_t|e^{i\varphi_{\beta,t}}$, the coefficient functions which assure the Hermiticity of $\tilde{h}_t$, i.e., $\tilde{W}_t \in \mathbb{R}$ and $\tilde{T}_t = \tilde{T}_t^*$, are given by

$$\tilde{W}_t = \frac{1}{\chi_t - \tilde{\Phi}_t^2} \{ |\omega_t| (\tilde{\chi}_t + \tilde{\Phi}_t^2) \cos \varphi_{\omega,t} - 2\tilde{\Phi}_t |\alpha_t| \cos (\varphi_t - \varphi_{\alpha,t}) - |\beta_t| \tilde{\chi}_t \cos (\varphi_t + \varphi_{\beta,t}) \},$$

(60a)

$$\tilde{V}_t = \frac{1}{\chi_t - \tilde{\Phi}_t^2} \left( |\omega_t| \tilde{\Phi}_t e^{i(\varphi_t + \varphi_{\omega,t})} - |\alpha_t| e^{i\varphi_{\alpha,t}} - |\beta_t| \tilde{\Phi}_t^2 e^{-2i\varphi_t} \right),$$

(60b)

with $\tilde{\Phi}_t$ and $\varphi_t$ following from the system

$$[|\omega_t| \tilde{\Phi}_t \sin \varphi_{\omega,t} + |\alpha_t| \sin (\varphi_t - \varphi_{\alpha,t})] \left(1 - \tilde{\Phi}_t^2\right) + |\beta_t| \left((2\tilde{\chi}_t - 1) \tilde{\Phi}_t^2 - \tilde{\chi}_t^2\right) \sin (\varphi_t + \varphi_{\beta,t}) = 0,$$

(61a)

$$\left(\tilde{\chi}_t - 1\right) \tilde{\Phi}_t |\omega_t| \cos \varphi_{\omega,t} + |\alpha_t| \left(1 - \tilde{\Phi}_t^2\right) \cos (\varphi_t - \varphi_{\alpha,t}) + |\beta_t| \left(\tilde{\Phi}_t^2 - \tilde{\chi}_t^2\right) \cos (\varphi_t + \varphi_{\beta,t}) = 0.$$  

(61b)

Next, under the invariant Dyson map

$$\eta_t = \exp \left(\lambda_{+,t} K_+ \right) \exp \left(\ln \lambda_{0,t} K_0 \right) \exp \left(\lambda_{-,t} K_- \right),$$

(62)

we derive the Hamiltonian

$$h_t = \eta_t H \eta_t^{-1} + i\tilde{\eta}_t \eta_t^{-1} = W_t(a^\dagger a + 1/2) + V_t a^2 + T_t a^2.$$  

(63)

where the coefficient functions which assure its Hermiticity are given by

$$W_t = |\omega_t| \cos \varphi_{\omega,t} + \frac{2\tilde{\Phi}_t}{1 - \chi_t} [|\alpha_t| \cos (\varphi_t - \varphi_{\alpha,t}) - |\beta_t| \cos (\varphi_t + \varphi_{\beta,t})],$$

(64a)

$$V_t = \frac{1}{1 - \chi_t} \left(|\alpha_t| e^{i\varphi_{\alpha,t}} - |\beta_t| \chi_t e^{-i\varphi_{\beta,t}} - i |\omega_t| \tilde{\Phi}_t \sin \varphi_{\omega,t} e^{i\varphi_t} \right).$$

(64b)
with $\Phi_t$ and $\varphi_t$ following from the couple nonlinear equations

\[
\dot{\Phi}_t = \frac{2}{\chi_t - 1} \left\{ [\omega_t | \Phi_t \sin \varphi_{\omega, t} + \alpha_t | \sin (\varphi_t - \varphi_{\alpha, t})] (1 - \Phi_t^2) + \right. \\
\left. |\beta_t| \left[ (2\chi_t - 1) \Phi_t^2 - \chi_t^2 \right] \sin (\varphi_t + \varphi_{\beta, t}) \right\} \\
\dot{\varphi}_t = \frac{2}{(\chi_t - 1) \Phi_t} \left[ \alpha_t \left( 1 - \Phi_t^2 \right) \cos (\varphi_t - \varphi_{\alpha, t}) + |\beta_t| \left( \Phi_t^2 - \chi_t^2 \right) \cos (\varphi_t + \varphi_{\beta, t}) \right] \\
+ 2 |\omega_t| \cos \varphi_{\omega, t},
\]

which automatically reduces to those in Eq. (61) when considering $\dot{\Phi}_t = \dot{\varphi}_t = 0$. It is immediate to conclude that in this case the difference $\bar{h}_t - \bar{h}_t$ is not a c-number, and consequently, a local gauge transformation is required to link these two Hermitian Hamiltonians.

V. CONCLUSIONS

We have here considered the problem of TD non-Hermitian Hamiltonians under TD Dyson maps, a subject of significant importance that has received attention in the recent literature on non-Hermitian quantum mechanics [10–13]. We have first shown how to construct from a given TD non-observable non-Hermitian Hamiltonian an infinite chain of gauge linked Hamiltonians, whose associated observables and their matrix elements are all related to each other. In the particular case where the Hamiltonians are linked together by global gauges, the matrix elements of the observables associated with these TD non-observables non-Hermitian Hamiltonians became all identical to each other, making all these Hamiltonians equivalents. In such a case, by approaching whatever the Hamiltonian in the chain we can get information about any other Hamiltonian, and this property becomes all the more important when we find that among the infinite Hamiltonians in the chain one of them is definitely easier to treat: $\bar{H}_t$, whose Hermitian counterpart, $\bar{h}_t = \bar{\eta}_t \bar{H}_t (\bar{\eta}_t)^{-1}$, does not demand the time derivative of the corresponding Dyson map, $\bar{\eta}_t$, and consequently, the Gauss decomposition of this operator.

When, on the other hand, we ensure these TD non-Hermitian Hamiltonian to be observables, simultaneously to ensuring the unitarity of the time-evolution they govern, the whole chain collapse to a single TD observable non-Hermitian Hamiltonian. Therefore, the observability character of a TD non-Hermitian Hamiltonian prevents the possibility of gauge-linked associated Hamiltonians and observables. After going through these properties, we
then present two illustrative examples: the TD harmonic oscillators under TD linear and parametric non-Hermitian amplification processes, the linear case resulting in global gauge transformations.

The properties derived here help us to better understand systems described by time-dependent non-Hermitian Hamiltonians under time-dependent Dyson maps, which have only recently been studied and should play a central role in non-Hermitian quantum mechanics.

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