Fixed energy universality of Dyson Brownian motion

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Abstract: We consider Dyson Brownian motion for classical values of $\beta$ with deterministic initial data $V$. We prove that the local eigenvalue statistics coincide with the GOE/GUE in the fixed energy sense after time $t \gtrsim 1/N$ if the density of states of $V$ is bounded above and below down to scales $\eta \ll t$ in a window of size $L \approx \sqrt{t}$. Our results imply that fixed energy universality holds for essentially any random matrix ensemble for which averaged energy universality was previously known. Our methodology builds on the homogenization theory developed in [17] which reduces the microscopic problem to a mesoscopic problem. As an auxiliary result we prove a mesoscopic central limit theorem for linear statistics of various classes of test functions for classical Dyson Brownian motion.

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1 Introduction

In the pioneering work \cite{71}, Wigner introduced what are known as the Wigner random matrix ensembles. These ensembles consist of $N \times N$ real symmetric ($\beta = 1$) or complex Hermitian ($\beta = 2$) random matrices $W = (w_{ij})$ whose entries are centered and independent (up to the symmetry constraint $W = W^*$) with variance

$$E[(w_{ij})^2] = \frac{1 + \delta_{ij}}{N}, \quad \beta = 1, \quad E[|w_{ij}|^2] = \frac{1}{N}, \quad \beta = 2. \quad (1.1)$$

If the $w_{ij}$’s are independent real (resp., complex) Gaussians then the ensemble is called the Gaussian Orthogonal Ensemble (resp., Gaussian Unitary Ensemble) (GOE/GUE). Wigner conjectured that in the limit $N \to \infty$ the local eigenvalue statistics are universal in that they depend only on the symmetry class of the matrix ensemble (real symmetric or complex Hermitian) and are otherwise independent of the underlying distribution of the matrix entries. After Wigner’s seminal work, Gaudin, Dyson and Mehta explicitly calculated the eigenvalue correlation functions in the Gaussian cases.

Mehta formalized the universality conjecture in the book \cite{57} and stated that the correlation functions of general Wigner matrices should coincide with the GOE/GUE in the limit $N \to \infty$. There are several possible topologies in which this convergence could hold. Perhaps the most natural topology to consider is pointwise convergence of the correlation functions. However, this cannot hold for random matrix ensembles with discrete entries. One suitable topology is that of vague convergence of the correlation functions around an energy $E$, which we will call fixed energy universality. A weaker topology can be constructed by averaging over energies in a small window near $E$ and asking for vague convergence of the energy-averaged quantities. We will call this averaged or unfixed energy universality. Finally, one can also ask for the vague convergence of the eigenvalue gaps with a fixed label (i.e., vague convergence of the random variable $\lambda_{N/2+1} - \lambda_{N/2}$) which we call gap universality.

There has recently been spectacular progress in proving the Wigner-Dyson-Mehta conjecture for a wide variety of random matrix ensembles. Bulk universality for Wigner matrices of all symmetry classes was proven in the works \cite{32–34,36,39,41}. Parallel results were established in certain cases in \cite{65,66}, with the key result being a “four moment comparison theorem.” In this paper we are interested in the robust three-step approach to universality formulated and developed in the works \cite{32–34,36,39,41}. This approach consists of:

1. A high probability estimate of the eigenvalue density down to the almost-optimal scale $\eta \sim N^{\varepsilon}/N$. This establishes eigenvalue rigidity; that is, the bulk eigenvalues are close to their expectations

$$|\lambda_i - E[\lambda_i]| \leq \frac{N^{\varepsilon}}{N} \quad (1.2)$$

with overwhelming probability. Moreover, the expectations are determined by the quantiles of the macroscopic eigenvalue density.

2. Proving bulk universality for random matrix ensembles with a small additive Gaussian component. This is usually established by studying the rate of convergence of Dyson Brownian motion to local equilibrium.

3. A comparison or stability argument comparing a given random matrix ensemble to one with a small Gaussian component.

For complex Hermitian ensembles, Step 2 can be established by using an explicit algebraic formula, the Brézin-Hikami formula, to analyze the correlation functions. This idea was used by Johansson \cite{47} and Ben Arous-Peche \cite{10} who established bulk universality for ensembles with an order 1 Gaussian...
component, i.e., establishing that the time to equilibrium is at most order 1 in this case. The optimal time to equilibrium in the second step, i.e., for \( t \geq 1/N \), was established in [34] where the Brézin-Hikami formula and estimates from the local semicircle law were the key tools. In this special algebraic case, the second step yields fixed energy universality and so the WDM conjecture was established for complex Hermitian matrices in this strong sense [34, 35, 65].

An analogue of the Brézin-Hikami formula is unknown in the real symmetric case and therefore the approach [34] could not be extended to real symmetric Wigner ensembles. A new approach based on the local relaxation flow of Dyson Brownian motion (DBM) was developed in the works [36, 37, 40]. DBM is defined by applying an independent (up to the symmetry constraint \( H = H^\dagger \)) Ornstein-Uhlenbeck process to every matrix element; Dyson computed the flow on the eigenvalues and found that they satisfy a closed system of stochastic differential equations. The approach of [36, 37, 40] is based on this representation and applies to all symmetry classes as well as sample covariance matrices and sparse ensembles. However, it yields only averaged energy universality, albeit with the averaging taken over a very small window.

The second step was finally completed for real symmetric Wigner ensembles in the sense of fixed energy in [17]. By developing a sophisticated homogenization theory for a discrete parabolic equation derived from DBM, the authors proved that after a time \( t = o(1) \), the local statistics of Dyson Brownian motion started from a Wigner ensemble coincide with that of the GOE in the fixed energy sense. As the third step in the three-step strategy described above is insensitive to the mode of convergence of the correlation functions, this proved the Wigner-Dyson-Mehta conjecture in the fixed energy sense for all symmetry classes.

The time to equilibrium proven in the work [17] is relatively long, \( t \sim N^{-\varepsilon} \) for a small \( \varepsilon > 0 \), which moreover depends on the choice of test function. This limits the applicability of the work [17] in proving fixed energy universality for other ensembles. For example, it does not imply fixed energy universality for sparse random graphs, for which averaged energy universality is known [1, 32, 33, 45]. Moreover, the approach relies on the fact that the global eigenvalue density of the initial data is given by the semicircle law.

The analysis of DBM developed in the works [36, 37, 40] is in some sense global as it relies on the fact that DBM with initial data a Wigner matrix will follow the semicircle law. In the work [39] the correlation functions were expressed as time averages of random walks in a random environment. This allows for a local analysis of the dynamics and various tools from PDE (such as Hölder regularity via the di-Giorgi-Nash-Moser method) and stochastic analysis can be applied.

In the work [52] the time to equilibrium of DBM for a wide class of initial data (going beyond the Wigner class) was studied (see also [38] for related results). For random matrix ensembles that have a local density down to scales \( \eta \gtrsim 1/N \), it was proven that the time to local equilibrium is \( t \gtrsim 1/N \), in the sense of both averaged energy and gap universality.

There have been several recent works extending the Wigner-Dyson-Mehta conjecture beyond the class of Wigner matrices, such as to sparse random graphs [1, 7, 8, 32, 33, 44, 45], matrices with correlated entries [2, 6, 22], deformed Wigner ensembles [53, 54], certain classes of band matrices [13] and the general Wigner-type matrices of [3–5]. These works generally follow the three-step strategy outlined above. In many of these cases, the works [38, 52] essentially complete the second step of this approach. As the results [38, 52] imply averaged energy universality, any work relying on [38, 52] for the second step establishes the Wigner-Dyson-Mehta conjecture in only the averaged energy sense.

In the current work we establish that the time to local equilibrium for DBM is \( t \gtrsim N^{-1} \) for a wide class of initial data, in the fixed energy sense. The main assumption on the initial data is that the density of states is bounded above and below down to scales \( \eta \ll t \) in a window of size \( L \gg \sqrt{t} \). As a consequence, fixed energy universality is established for essentially all random matrix ensembles for which previously only averaged energy universality could be proven.

One of the key insights of [17] is that the difference of two coupled DBM flows obeys a discrete nonlocal parabolic equation. One of the main results of [17] is a homogenization theory for this parabolic equation. This theory shows that the solution of the discrete parabolic equation is given by the discretization of the continuum limit, and this reduces the problem of microscopic statistics to an easier mesoscopic problem.
Our approach follows the same high-level strategy in that we couple two DBM flows and develop a homogenization theory for the resulting parabolic equation. The generator of the parabolic equation of [17] is hard to control. To deal with this we modify the coupling of [17] and introduce a continuous interpolation. This gives us a family of parabolic equations whose generators have better properties.

The homogenization theory of [17] was based around a Duhamel expansion and estimating the coefficients of the generator. The short range part of the generator is quite singular and was controlled using an energy estimate and the discrete Di-Giorgi-Nash-Moser theorem of [39]. This caused some restriction on the time to equilibrium that could be proven.

Our method is based around the standard $\ell^2$-energy method and a discrete Sobolev inequality. The energy method gives us an estimate on the time average of the discrete $H^{1/2}$ norm of the difference between the fundamental solution of the discrete equation and its continuum limit. This allows us to get a time-averaged $\ell^\infty$ estimate on the fundamental solution of the discrete parabolic equation via a discrete Sobolev inequality. We then use the semigroup property to remove the time average.

In order to carry this out one needs a good ansatz for comparison with the discrete fundamental solution. We substitute the particle location coming from the DBM into the fundamental solution of the continuum limit. With this approach a martingale term, as well as other errors of lower order, arises in the energy method, but we are able to control them using heat kernel bounds for our process. This ansatz first appeared in [18] and has been used independently in [12] to analyze extremal gap statistics of Wigner ensembles.

We find that the limiting hydrodynamic equation is a fairly simple nonlocal parabolic equation describing a symmetric jump process on $\mathbb{R}$. The heat kernels of such processes have been studied recently in, e.g., [23, 24] and we partially rely on their work in our analysis of the limiting equation.

Our homogenization theory has an advantage over [17] in that our estimates hold with overwhelming probability (i.e., $\geq 1 - N^{-D}$ for any large $D > 0$). The homogenization theory [17] relied on certain level repulsion estimates and as a consequence the main estimates were only known to hold with polynomially high probability (i.e., $\geq 1 - N^{-\epsilon}$ for some small $\epsilon > 0$). While this is not significant to the application of universality, we believe that this improvement is important for future applications. Moreover our method is robust in that it essentially relies only on rigidity; in many random matrix ensembles optimal level repulsion estimates (on which the previous methods [17, 38, 52] relied) are not known and can be hard to establish.

As mentioned above, the homogenization theory reduces the microscopic problem of fixed energy universality to a problem involving linear mesoscopic statistics. Central limit theorems for mesoscopic linear statistics of Wigner matrices were established first in certain scales in [20, 21, 56] and then down to the almost-optimal scale $\eta = N^\epsilon/N$ in [43]. Mesoscopic statistics of compactly supported test functions for the special case of $\beta = 2$ for DBM with deterministic initial data was established in [30]. The analysis in [30] relied on the Brézin-Hikami formula special to the $\beta = 2$ case and cannot be applied here. Moreover the test function coming from the homogenization theorem is not of compact support - only its derivative is - and has no spatial decay, which presents a serious complication. The mesoscopic results [20, 21, 30, 43, 56] all apply only to functions with either compact support or at least some spatial decay as $|x| \to \infty$.

In the present work we establish a mesoscopic central limit theorem for DBM for a certain class of non-compactly supported test functions which have no spatial decay. As an aside, we remark that if our methods are restricted to the compactly supported case, then we can prove that if the scale of the function is less than $t$, then the linear statistic coincides with the GOE. Here, in the compactly supported case one can remove the restrictions that we encounter in the non-compactly supported case. This is an extension of some of the results of [30] to $\beta = 1$. Our main interest in a mesoscopic central limit theorem is to analyze the statistic coming from the homogenization theory, and so we only settles for a few remarks concerning test functions of compact support - see Section 6.

The works [38, 52] establishing averaged energy universality for DBM relied heavily on the discrete Di-Giorgi-Nash-Moser theorem of [39]. As a consequence, the rate of convergence was somewhat non-explicit. While in this work we do not attempt to derive optimal error bounds, our result improves on [38, 52] in the sense that our bounds can be quantified explicitly in terms of the parameters of the model.
1.1 Applications

Our homogenization theory also allows us to establish universality of the space-time DBM process — that is, up to an explicit deterministic shift in space and a re-scaling in space and time, the multitime correlation functions of DBM coincide with the GOE/GUE.

Averaged energy universality for general one-cut \( \beta \)-ensembles was established first in [14–16, 39]. Further results for bulk universality for multi-cut potentials were established in [9, 62]. Fixed energy universality for one-cut \( C_4 \) potentials was announced in [31] and can be proven using the methods of [17]. Previously, Shcherbina had established fixed energy universality for analytic potentials in the multi-cut case in [62]. For completeness we sketch how our methods can be adapted to re-prove the results of [17, 31] and moreover establish a polynomial error estimate.

1.1.1 Fixed energy universality for general random matrix ensembles

Many of the recent works on universality of general random matrix ensembles have relied on the aforementioned three-step strategy to proving universality. In the second step these works relied on [38, 52] for universality for the Gaussian divisible ensembles. The works [38, 52] provided both gap universality and averaged energy universality for the Gaussian divisible ensembles; consequently this form of universality has been proven, for example, for the adjacency matrices of sparse random graphs [1, 7, 8, 32, 33, 44, 45], matrices with correlated entries [2, 6, 22] and the general Wigner-type matrices of [3–5]. By instead relying on the current work, fixed energy universality is established for all of these ensembles.

1.1.2 Eigenvalue interval probabilities

Fixed energy universality has several other consequences which we now outline. It establishes the existence of the local density of states on microscopic scales as well as the universality of the Jimbo-Miwa-Mori-Sato formula for the gap probability. In addition, it implies universality of the distribution of the smallest singular value of various random matrix ensembles, including the adjacency matrices of a wide variety of sparse random graphs which is of interest in computer science.

1.1.3 Invertibility of symmetric random matrices

The invertibility problem in random matrix theory is typically divided into two components [68]. The first is whether a random matrix is invertible with high probability, and the second is to determine the typical size of the norm of the inverse, or size of the smallest singular value. A motivating problem of the former type is the conjecture that an iid Bernoulli matrix is singular with probability less than \((2 + o(1))^{-N}\). Komlos [50] first proved that the singularity probability is vanishing. An exponential bound was first obtained in [48] and later improved in [19, 63].

The size of the inverse is related to the condition number which plays a crucial role in applied linear algebra. For example, the condition number controls the complexity or numerical accuracy in solving the linear equation \(Ax = b\). Von Neumann and his collaborators speculated [69] that the least singular value satisfies \(s_{\text{min}}(A) \sim N^{-1}\) for matrices with iid entries.

By now a large literature has emerged on the invertibility problem for both symmetric and iid ensembles. We refer to the surveys [58, 60, 70] and the references therein, and mention only a few specific results placing the present work in context. The invertibility of dense Erdős-Rényi graphs was established in [26] and was later extended to the sparse regime in [27]. The first estimate of the form \(s_{\text{min}}(H) \sim N^{-1}\) for symmetric matrices was first obtained in [68]. In the iid case, it is even known that the distribution of the (properly rescaled) smallest singular value at the hard edge is universal [64]. However, in the sparse regime little is known about the size of the smallest singular value in the symmetric case.

It is an open conjecture that the adjacency matrix of a random \( d \)-regular graph is invertible with high probability for \( d \geq 3 \) [27, 42, 70]. This problem is of interest in universal packet recovery [29]. In the case of random \( d \)-regular directed graphs, substantial progress has been made by [25, 55]. It is also
conjectured that the adjacency matrices of more general sparse graphs outside the Erdös-Renyi class should be invertible with high probability [27].

The invertibility of many classes of random matrices is in fact a corollary of previous works by two of the current authors [45,52] as well as others [1,3,6–8,22]. These classes include, for example, adjacency matrices of random regular graphs, matrices with correlated entries and general sparse random matrices. To fix ideas we consider the $d$-regular random graph with adjacency matrix $A$, where

$$d \geq N^\epsilon.$$  

(1.3)

The invertibility of $A + G$ where $G$ is a small GOE component follows from Section 5 of [52] as well as the local law of [8]. The comparison methods of [7,45] then allow the invertibility to be transferred back to $A$. This proves the conjecture of [27,42,70] in the regime (1.3). This methodology extends to the other random matrix ensembles considered in [45], as well as those considered in [1,3,6,22] as long as $0$ lies in the bulk of the spectrum.

This strategy yields an additional effective estimate on the size of the inverse in all cases, which was previously known only in the non-sparse regime. That is, there is a $c > 0$ so that for all sufficiently small $\epsilon > 0$,

$$\mathbb{P}(|A^{-1}| \geq N^\epsilon / N) \leq C(\epsilon) N^{-c\epsilon}.$$ 

(1.4)

For example, one can take $A$ to be the (properly rescaled so that the spectrum lies in a window of order 1) adjacency matrix of a sparse $d$-regular or Erdős-Rényi graph.

The current work goes beyond this and establishes universality of the smallest singular value of many random matrix ensembles. Our work implies, for example, that for each $t \in \mathbb{R}$,

$$\lim_{N \to \infty} \mathbb{P}(|A^{-1}| \geq tN) - \mathbb{P}(|H^{-1}| \geq tN) = 0,$$ 

(1.5)

where $A$ is the (again, properly rescaled) adjacency matrix of a random $d$-regular graph and $H$ is a GOE matrix (again for $d$ in the regime (1.3)).

1.2 Overview

The remainder of the paper is as follows. In Section 2 we introduce precisely our model and state our main results, applications and auxiliary results. Section 3 contains the main part of the homogenization results and Section 4 contains the proofs of certain a-priori bounds on the heat kernel. We study and prove regularity of the limiting continuum equation in Section 5. In Section 6 we prove our results on mesoscopic linear statistics for DBM. Section 7 contains the proof of fixed energy universality using the homogenization theory and the central limit theorem for mesoscopic linear statistics. In Section 8 we sketch the proof of fixed energy universality for $\beta$-ensembles.

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2 Model

Let $V$ be a deterministic diagonal matrix and let $W$ be a standard GOE matrix. We consider the following model

$$H_t = V + \sqrt{t}W.$$ 

(2.1)

We make the following assumptions on $V$.

Definition 2.1. Let $G = G_N$ and $g = g_N$ be $N$-dependent parameters. For definiteness we assume that there is a $\delta > 0$ s.t. $N^{-\delta} \geq g \geq N^{\delta}/N$ and $G \leq N^{-\delta}$. This $\delta$ will not be important in the method or the main results. We say that $V$ is $(g,G)$-regular if

$$c \leq \text{Im} \left[ m_V(E + i\eta) \right] \leq C$$ 

(2.2)

for $|E| \leq G$ and $g \leq \eta \leq 10$, and if there is a $C_V > 0$ s.t.

$$||V|| \leq N^{C_V}.$$ 

(2.3)
Remark. The assumption (2.3) is technical and can be removed with some minor work. We omit this from the current paper.

We will be considering times satisfying $gN^\sigma \leq t \leq N^{-\sigma}G^2$. We also introduce here the frequently used notation $z = E + i\eta$ for $E, \eta \in \mathbb{R}$.

2.1 Free convolution

In this section we introduce the free convolution. The semicircle law is given by

$$\rho_{sc}(E) := \frac{1}{2\pi}1_{|E| \leq 2}\sqrt{4 - E^2}. \quad (2.4)$$

It describes the limiting eigenvalue density of the GOE. The eigenvalue density of $H_t$ does not follow the semicircle law and is given by a free convolution. We define the free convolution of $V$ with the semicircle law at time $t$ via its Stieltjes transform which we denote by $m_{fc,t}$. The function $m_{fc,t}$ is defined as the unique solution to

$$m_{fc,t}(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{V_i - z - tm_{fc,t}(z)}; \quad \text{Im} \left[ m_{fc,t}(z) \right], \quad \eta \geq 0. \quad (2.5)$$

The free convolution law is defined by

$$\rho_{fc,t}(E) := \lim_{\eta \to 0} \frac{1}{\pi} \text{Im} \left[ m_{fc,t}(E + i\eta) \right]. \quad (2.6)$$

The free convolution is well-studied. For example, it is known that a unique solution to (2.5) exists and that $\rho_{fc,t}$ is analytic on the interior of its support. We refer to [11] for further details. We will also denote the free convolution law at time $t$ by $\rho_{fc,t}(E) := \rho_V \boxplus \rho_{sc,t}$.

2.2 Fixed energy universality

Let $p_{H_t}^{(N)}$ denote the symmetrized eigenvalue density of $H_t$. The $k$-point correlation functions are defined by

$$p_{H_t}^{(k)}(\lambda_1, \cdots, \lambda_k) := \int p_{H_t}^{(N)}(\lambda_1, \cdots, \lambda_N) d\lambda_{k+1} \cdots d\lambda_N. \quad (2.7)$$

The corresponding objects for the GOE are denoted $p_{GOE}^{(N)}$ and $p_{GOE}^{(k)}$. The following is our main result which states that the $k$-point correlation functions of $H_t$ converges to those of the GOE in the fixed energy sense.

**Theorem 2.2.** Let $V$ be a deterministic $(g,G)$-regular diagonal matrix. Let $\sigma > 0$ and let

$$gN^\sigma \leq t \leq N^{-\sigma}G^2. \quad (2.8)$$

Let $0 < q < 1$ and let $|E| \leq qG$. There is a constant $\kappa > 0$ so that the following holds. For every $k$ and smooth test function $O \in C^\infty_c(\mathbb{R})$ there is a constant $C > 0$ such that

$$\left| \int O(\alpha_1, \cdots, \alpha_k) p_{H_t}^{(k)}(E + \frac{\alpha_1}{N\rho_{fc,t}(E)}, \cdots, E + \frac{\alpha_k}{N\rho_{fc,t}(E)} \right) d\alpha_1 \cdots d\alpha_k$$

$$- \int O(\alpha_1, \cdots, \alpha_k) p_{GOE}^{(k)}\left(E + \frac{\alpha_1}{N\rho_{sc}(E)}, \cdots, E + \frac{\alpha_k}{N\rho_{sc}(E)} \right) d\alpha_1 \cdots d\alpha_k \right| \leq CN^{-\kappa} \quad (2.9)$$

2.2.1 Applications to other ensembles

Theorem 2.2 implies fixed energy universality for a wide variety of ensembles appearing in random matrix theory. Recall the three-step strategy to proving universality for random matrix ensembles outlined in the introduction. Many recent works in random matrix theory used the results of [38, 52] to complete the second step. The input of [38, 52] is to provide universality of DBM started from the chosen random matrix ensemble in either the fixed gap sense or averaged energy sense. The third step is relatively insensitive to the type of universality proven in the second step. Therefore, if one uses Theorem 2.2 instead of [38, 52] one can prove fixed energy universality for following ensembles.
1. Sparse random matrix ensembles such as the adjacency matrices of random graphs [1, 7, 8, 32, 33, 44, 45]

2. The general Wigner-type matrices of [3–5].

3. Matrices with correlated entries [2, 6, 22].

4. Deformed Wigner ensembles [53, 54].

Lastly, while fixed energy universality of generalized Wigner matrices was settled in [17], our methods yield a polynomial rate of convergence which was previously unknown.

2.3 Further results

2.3.1 Multitime correlation functions

The eigenvalues $\lambda_i$ of $H_t$ at each fixed time are equal in distribution to the unique strong solution of the system of the following system of SDEs, known as Dyson Brownian motion:

$$d\lambda_i = \sqrt{\frac{2}{N}} dB_i + \frac{1}{N} \sum_j \frac{1}{\lambda_i - \lambda_j} dt$$

(2.10)

with initial data $\lambda_i(0) = V_i$. Theorem 2.2 implies that at each fixed time, the correlation functions of $\{\lambda_i(t)\}_i$ coincide with the GOE. Our methods also allow us to consider multitime correlation functions. For simplicity we just state the result for two times $t_a < t_b$. One can also consider any finite set of times $t_a, \ldots, t_k$. Given two times $t_a < t_b$ let $p_{t_a,t_b}(\lambda_1(t_a), \cdots, \lambda_N(t_a), \lambda_1(t_b), \cdots, \lambda_N(t_b))$ denote the symmetrized density of $\{\lambda_i(t_a), \lambda_j(t_b)\}_{i,j}$. The multitime $k$-point correlation function is defined by

$$p^{(k)}_{t_a,t_b}(\lambda_1(t_a), \cdots, \lambda_k(t_a), \lambda_1(t_b), \cdots, \lambda_k(t_b))$$

$$:= \int p_{t_a,t_b}(\lambda_1(t_a), \cdots, \lambda_N(t_a), \lambda_1(t_b), \cdots, \lambda_N(t_b)) d\lambda_{k+1}(t_a) \cdots d\lambda_N(t_a) d\lambda_{k+1}(t_b) \cdots d\lambda_N(t_b).$$

(2.11)

Denote the analogous object for the GOE by $p^{(k)}_{t_a,t_b,GOE}$ (i.e., start the process $\lambda_i$ from the GOE ensemble). Fix an energy $|E(t_a)| \leq qG$ and define $E(t)$ for $t > t_a$ by

$$\check{c}_t E = \text{Re}[m_{fc,t}(E)]$$

(2.12)

Theorem 2.3. Let $V$ be as above and let $qN^\sigma \leq t_a \leq N^{-\sigma}G^2$. Let $O$ be a smooth compactly supported test function. There is a constant $\kappa > 0$ so that for any $t_a \leq t_b \leq N^\kappa/N$ we have

$$\left| \int O(\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k) p^{(k)}_{t_a,t_b}(E(t_a) + \frac{\check{c}_t}{N \rho_{fc,t_a}(E(t_a))} E(t_a)^\gamma, E(t_b) + \frac{\check{c}_t}{N \rho_{fc,t_b}(E(t_b))} E(t_b)^\gamma) d\alpha d\beta \right|$$

$$- \int O(\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k) p^{(k)}_{t_a,cit,b,GOE}(E' + \frac{\check{c}_t}{N \rho_{fc}(E')} E'^\gamma, E'' + \frac{\check{c}_t}{N \rho_{fc}(E'')} E''^\gamma) d\alpha d\beta \right| \leq N^{-\kappa},$$

(2.13)

for any fixed $E' \in (-2, 2)$. Above the constant $c_t$ is defined by $c_t := (\rho_{fc}(0)/\rho_{fc,t_b}(E(t_b)))^2$.

Remark. One can replace $\rho_{fc,t_b}(E(t_b))$ by $\rho_{fc,t_a}(E(t_a))$ as the difference is $o(1)$.

2.3.2 Jimbo-Miwa-Mori-Sato formula

Once one establishes fixed energy universality for a random matrix ensemble, it is a standard argument to determine the distribution of the number of eigenvalues in an interval of size $c/N$. More precisely, Theorem 2.2 implies that for intervals $I_1, \cdots, I_k$ and integers $n_1, \cdots, n_k$ the probability

$$\mathbb{P} \left[ \left\{ \lambda_i \in E + \frac{I_j}{N \rho(E)} \right\} \right] = n_j, 1 \leq j \leq k$$

(2.14)
converges to that of the GOE where \( \rho(E) \) is the eigenvalue density of the ensemble under consideration.

For example, for the adjacency matrices of a class of sparse random graphs, we have

\[
\lim_{N \to \infty} P \left[ \left\{ \lambda_i \in \frac{[0,1]}{N \pi \rho_{sc}(0)} \right\} = 0 \right] = E_1(0, t)
\]  

(2.15)

where \( E_1 \) is an explicit function of a solution to Painlevé equation.

### 2.3.3 Invertibility of symmetric random matrices

The result (2.15) provides explicit information on the distribution of the size of the inverse of various random matrix ensembles. For example, (2.15) implies that for the adjacency matrices \( A \) of sparse Erdős-Rényi and \( d \)-regular graphs we have for every \( t > 0 \),

\[
\lim_{N \to \infty} |P[\|A^{-1}\| \geq Nt] - P[\|H^{-1}\| \geq Nt]| = 0
\]  

(2.16)

where \( H \) is a GOE matrix. From previous results in the literature [7, 8, 45, 52] it is easily deduced that the adjacency matrix of a sparse Erdős-Rényi or \( d \)-regular graph is invertible with high probability. The result (2.16) is finer, in that it demonstrates that the limiting distribution of the size of the inverse, or equivalently, the size of the smallest singular value of \( A \), is universal.

### 2.3.4 Fixed energy universality for \( \beta \)-ensembles

Our methods also imply fixed energy universality for a class of \( \beta \)-ensembles. A \( \beta \)-ensemble is a measure on the simplex \( 0 \leq \cdots \leq \lambda_N \) with probability density proportional to

\[
e^{-\beta N \sum \lambda_i^2} \lambda^{\frac{1}{2} V(\lambda_1)+\beta \sum_{i<j} \log|\lambda_i-\lambda_j|}.
\]  

(2.17)

We assume that \( V \) is a \( C^4 \) real function with second derivative bounded below and growth condition

\[
V(x) > (2+\alpha) \log(1+|x|)
\]  

(2.18)

for all large \( x \) and an \( \alpha > 0 \). The averaged density of the empirical spectral measure converges weakly to a continuous function \( \rho_V \), the equilibrium density with compact support. We assume that \( \rho_V \) is supported on a single interval \([A,B]\) and that \( V \) is regular in the sense of [51]. We denote the \( k \)-point correlation functions by \( p^{(k)}_V \) and those for the Gaussian \( \beta \)-ensemble (for \( V(x) = x^2/2 \)) by \( p^{(k)}_\beta \).

Under these conditions fixed energy universality was announced in [31] and can be proven using the methods of [17]. Previously, M. Shcherbina established fixed energy universality for multi-cut analytic \( \beta \)-ensembles in [62]. The following result is an improved version of the result in [31] in that it provides an error estimate \( N^{-\kappa} \) to the fixed energy universality. Similarly to [31], our methodology is based on the homogenization idea initiated in [17,31].

**Theorem 2.4.** Let \( V \) be as above and assume \( \beta \geq 1 \). Let \( E \in (A,B) \) and \( E' \in (-2,2) \). Let \( O \) be a smooth test function. There is a \( \kappa > 0 \) such that

\[
\left| \int O(\alpha_1, \ldots, \alpha_k) p^{(k)}_V \left( E + \frac{\alpha_1}{N \rho_V(E)}, \ldots E + \frac{\alpha_k}{N \rho_V(E)} \right) d\alpha_1 \cdots d\alpha_k - \int O(\alpha_1, \ldots, \alpha_k) p^{(k)}_\beta \left( E' + \frac{\alpha_1}{N \rho_{sc}(E')}, \ldots E' + \frac{\alpha_k}{N \rho_{sc}(E')} \right) d\alpha_1 \cdots d\alpha_k \right| \leq CN^{-\kappa}
\]  

(2.19)

**Remark.** It is also possible to deduce analogous results for multitime correlation functions in the following sense. If one modifies (2.10) to

\[
d\lambda_i = \sqrt{\frac{2}{N}} dB_i + \frac{1}{N} \sum_j \frac{1}{\lambda_i - \lambda_j} dt - \frac{V'(\lambda_i)}{2} dt
\]  

(2.20)

then the \( \beta \)-ensemble with potential \( V \) is left invariant by this flow. One can prove that the multitime correlation functions coincide with the process (2.10) started from a Gaussian \( \beta \)-ensemble.
2.3.5 Mesoscopic statistics for DBM

Our methodology of proving fixed energy universality reduces the microscopic problem to a problem involving mesoscopic linear statistics. In order to complete the proof of fixed energy universality we are forced to calculate mesoscopic statistics for DBM. Mesoscopic statistics have received some attention in the literature recently and we therefore state our result as it may be of independent interest.

**Theorem 2.5.** Let \( \varphi \) be a smooth test function satisfying

\[
\varphi'(x) = 0, \quad |x| > Ct', \quad |\varphi^{(k)}| \leq C/(t')^k, \quad k = 0, 1, 2,
\]

where \( t' = N^\alpha/N \). Let \( V \) be \((g, G)\)-regular and let \( |E| \leq qG \). Let \( t = N^\omega/N \) satisfy \( gN^\sigma \leq t \leq G^2 N^{-\sigma} \). Assume that \( \alpha > 0 \) satisfies \( \omega/2 < \alpha < \omega \). Then the mesoscopic statistic

\[
\sum_i \varphi(\lambda_i)
\]

converges weakly to a Gaussian. If \( \varphi \) is not compactly supported, then the variance is bounded below by \( c|\log(t'/t)| \).

**Remark.** Our results are more general — see Section 6. We calculate the characteristic function with an explicit rate of convergence in a growing neighborhood of the origin.

If \( \varphi \) is compactly supported, then with some modifications of our methods one can remove the unnatural restriction \( \alpha > \omega/2 \). As the above theorem will suffice in our application to fixed energy universality we do not provide the details.

2.4 Local law and rigidity

In this section we recall the local law for \( H_t \). These a-priori estimates are the key technical input of our methods. For times of order 1 the local law was established in \([53, 54]\). The argument was adapted to short times in \([52]\). The empirical Stieltjes transform of \( H_t \) will be denoted by

\[
m_N(z) := \frac{1}{N} \sum \frac{1}{\lambda_i - z}.
\]

Under the above hypotheses we have the following rigidity and local law estimates. We need some notation. For any \( 0 < q < 1 \) let

\[
\mathcal{I}_q := [-qG, qG].
\]

Let \( \varepsilon > 0 \) and \( 0 < q < 1 \). We consider the spectral domain

\[
\mathcal{D}_{\varepsilon, q} = \left\{ z = E + i\eta : E \in \mathcal{I}_q, N^{10C_V} \geq \eta \geq N^\varepsilon/N \right\}
\]

\[
\cup \left\{ z : D + i\eta : |E| \leq N^{10C_V}, N^{10C_V} \geq \eta \geq c \right\}
\]

We have

**Theorem 2.6.** Fix \( \varepsilon > 0 \) and \( 0 < q < 1 \). Let \( \sigma > 0 \) be such that \( gN^\sigma \leq t \leq N^{-\sigma} G^2 \). For any \( D > 0 \) and \( \delta > 0 \) we have

\[
P \left[ \sup_{z \in \mathcal{D}_{\varepsilon}} \left| m_N(z) - m_{N_0,t}(z) \right| \geq \frac{N^{\delta}}{N^\eta} \right] \leq CN^{-D}.
\]

We fix now a certain index set. Let \( 0 < q < 1 \). Let

\[
\mathcal{C}_q := \{ i : V_i \in \mathcal{I}_q \}.
\]
2.4.1 Classical eigenvalue locations

Given a probability measure \( \rho(x)dx \) and matrix size \( N \), we define the classical eigenvalues \( \gamma_i = \gamma_{i,N} \) in the following manner. If \( N \) is even then
\[
\gamma_i = \inf \left\{ x : \int_{-\infty}^{x} \rho(E)dE \geq \frac{i + 1}{N} \right\}.
\]
(2.28)

and if \( N \) is odd then
\[
\gamma_i = \inf \left\{ x : \int_{-\infty}^{x} \rho(E)dE \geq \frac{i + 1/2}{N} \right\}.
\]
(2.29)

We denote the classical eigenvalue locations of the free convolution law at times \( t \) by \( \gamma_{i,F} \) and the classical eigenvalue locations of the semicircle law by \( \gamma_{i,sc} \). The above definition is slightly nonstandard, but we take it so that
\[
\gamma_{i,sc} = 0,
\]
(2.30)

which will turn out to be convenient later.

2.4.2 Rigidity estimates

We have the following rigidity result for the eigenvalues.

**Theorem 2.7.** Fix \( 0 < q < 1 \) and let \( t \) be as above. For any \( \varepsilon > 0 \) and \( D > 0 \) we have
\[
\mathbb{P} \left[ \sup_{i \in C_q} |\lambda_i(t) - \gamma_{i}(t)| \geq \frac{N\varepsilon}{N} \right] \leq N^{-D}.
\]
(2.31)

We also have
\[
\mathbb{P} \left[ \sup_{j} |\lambda_j - V_j| \geq 3\sqrt{t} \right] \leq N^{-D}.
\]
(2.32)

From the above theorem we see that for any \( q_1 \) with \( q < q_1 < 1 \) we have for \( N \) large enough,
\[
i \in C_q \implies \gamma_{i}(t) \in T_{q_1}.
\]
(2.33)

2.5 Proof sketch

In this section we sketch a proof of fixed energy universality. The DBM flow starting from \( V \) is given by the SDE
\[
dx_i = \sqrt{\frac{2}{N}} dB_i + \frac{1}{N} \sum_j \frac{1}{x_i - x_j} dt \quad x_i(0) = V_i,
\]
(2.34)

where the \( dB_i \) are standard Brownian motions.

1. **Regularization.** Before coupling, we first allow the DBM (2.34) to run freely for an initial time interval of length \( t_0 \), where \( t_0 \) satisfies the compatibility conditions \( g \ll t_0 \ll G^2 \). This is needed for several reasons. Firstly, if \( t_0 \), we can apply the results [52] which state that rigidity holds wrt the free convolution law. Secondly, this regularizes the DBM flow in the sense that the free convolution will be regular on this scale; for example \( \rho_c(E) \leq C/t_0 \).

2. **Matching and coupling.** For times \( t \geq t_0 \) we couple the DBM flow to another DBM flow started from an independent GOE ensemble. That is, we define the process
\[
dy_i(t) = \sqrt{\frac{2}{N}} dB_i + \frac{1}{N} \sum_j \frac{1}{y_i - y_j} dt
\]
(2.35)

where initially \( y_i(t_0) \) is distributed as a GOE ensemble independent from \{\( x_i \}\}. The point is that the Brownian motions in (2.34) and (2.35) are the same. This idea first appeared in [17].
Moreover, we re-scale and shift the DBM flow so that the classical eigenvalue locations match those of the semicircle law near a chosen energy $E$. Due to the regularity of the free convolution law, we can match up to $\sqrt{Nt_0}$ eigenvalues.

At this point we are now regarding $t_0$ as a fixed a-priori scale on which the DBM flow is regular. By running the coupling for times $t$ satisfying $t_0 \leq t \leq t_0 + t_1$ with $t_1 \ll t_0$, the DBM flow will not see the non-matching eigenvalues.

3. **Discrete parabolic equation.** We see that the difference $w_i(t) := x_i(t) - y_i(t)$ satisfies the parabolic equation

$$\partial_t w_i = (Lw)_i$$

where

$$(Lw)_i := \frac{1}{N} \sum_j \frac{w_j - w_i}{(x_i - x_j)(y_i - y_j)}.$$

As $N \to \infty$, a natural limit for this equation is

$$\partial_t f(x) = \int \frac{f(y) - f(x)}{(x-y)^2} \rho_{t_0,t}(y) dy \sim \int \frac{f(y) - f(x)}{(x-y)^2} dy$$

In order to justify the replacement of $\rho_{t_0,t}$ by a constant we will use its regularity on the scale $t_0$ and a short-range approximation of the DBM flow. We omit the details in this simple sketch.

4. **Homogenization theory** We may now write

$$x_i(t_0 + t) - y_i(t_0 + t) = \sum_j U_{t_0}^C(t_0,t_0 + t)(x_i(t_0) - y_i(t_0)),$$

where $U^C$ is the semigroup for the equation (2.37). We need to develop a homogenization theory in order to calculate $U_{t_0}^C$. We let $p_t(x,y)$ denote the fundamental solution of (2.38). Let

$$f_t(t) := p_t(x_i(t), \gamma_a)$$

and

$$u_i(t) := Nu_{t_0}^C(t_0 + t, t_0)$$

Our main calculation is

$$d\|f - u\|^2_2 = -c\langle(f - u), B(f - u)\rangle dt + dM + \frac{o(1)}{t^2} dt,$$

where $M$ is a martingale. Integrating this back in time and dropping the time average for simplicity we will obtain

$$\langle(f - u), B(f - u)\rangle \leq \frac{o(1)}{t^2} + \frac{1}{t} \|u(0) - f(0)\|^2_2$$

By suitably smearing out the initial data slightly we can take $\|u(0) - f(0)\|^2_2 = o(t^{-1})$. Hence we obtain

$$\langle(f - u), B(f - u)\rangle \leq \frac{o(1)}{t^2}.$$  (2.44)

Finally by a discrete $H^{1/2} - \ell^\infty$ Sobolev inequality we obtain

$$\|u_i(t) - f_i(t)\|_{\ell^\infty} \leq \frac{o(1)}{t},$$  (2.45)

which gives us an $\ell^\infty$ estimate for $U^C$ as desired.
5. **Cut-offs.** The natural size of $w_i = NU_{ij}^\nu(t_0, t_0 + t)$ is $1/t$ for indices $i$ near $a$. Hence the estimate (2.45) determines the object $U_{ij}^\nu$ beyond its natural scale and we can use it to control the terms in the sum (2.39) for $j$ near $i$. We need to implement a cut-off to deal with the remaining terms in the sum, i.e. when $j$ is far from $i$. This involves using an a-priori upper bound for $U_{ij}^\nu$ which says that

$$NU_{ij}^{(B)}(t_0, t_0 + t) \leq N^\varepsilon p(t, \gamma_j) \leq N^\varepsilon \frac{t}{t^2 + (\gamma_i - \gamma_j)^2} \tag{2.46}$$

for any small $\varepsilon > 0$. This allows us to remove terms in (2.39) for which the classical eigenvalues of $x_i$ and $y_j$ match. For other terms we actually need to replace the processes (2.34) and (2.35) by short-range approximations. We omit the details from this sketch.

6. **Mesoscopic linear statistics.** The homogenization theory roughly proves that

$$x_i(t_0 + t) - y_i(t_0 + t) = \sum_{|i-j| \leq Nt} \zeta_i(\gamma_i - \gamma_j)(x_j(t_0) - y_j(t_0)) =: \zeta_x - \zeta_y \tag{2.47}$$

This reduces the microscopic problem to a simpler mesoscopic one. We prove that

$$|\psi(\lambda)| := |\mathbb{E}[e^{i\lambda \zeta_x}]| \leq e^{-\lambda^2 c_x \log(N)} + N^{-\varepsilon} \tag{2.48}$$

for some $\varepsilon > 0$ and a constant $c_x > 0$.

7. **Fourier cut-off.** We now proceed similarly to [17]. It suffices to consider for smooth test functions $Q$ sums of terms of the form,

$$\mathbb{E}[Q(N(x_i - E), N(x_j - x_i))]. \tag{2.49}$$

The homogenization theory shows that

$$\mathbb{E}[Q(N(x_i - E), N(x_j - x_i))] = \mathbb{E}[Q(N(y_i - E + (\zeta_x - \zeta_y)), N(y_j - y_i))] + o(1). \tag{2.50}$$

By Fourier duality we have

$$\mathbb{E}[Q(N(y_i - E + (\zeta_x - \zeta_y)), N(y_j - y_i))] = \int \mathbb{E}[\hat{Q}(\lambda, N(y_j - y_i))e^{i\lambda N(y_i - E + -\zeta_y)}]\psi(\lambda)d\lambda. \tag{2.51}$$

Here $\hat{Q}$ denotes the Fourier transform of $Q$ in the first variable. By (2.48) we can cut off the Fourier support of $Q$ in the range $|\lambda| \geq \delta$ for any small fixed $\delta > 0$. Therefore it suffices to consider observables $Q$ with Fourier support contained in $|\lambda| \leq \delta$.

8. **Reverse heat-flow.** Running the same argument with a third ensemble $z$ distributed as the GOE shows that

$$\mathbb{E}[Q(N(z_i - E), N(z_j - z_i))] = \mathbb{E}[Q(N(y_i - E + (\zeta_z - \zeta_y)), N(y_j - y_i))] + o(1). \tag{2.52}$$

As in [17] we see that from (2.50) and (2.52) that fixed energy universality will follow if we can prove that the function

$$F(a) := \mathbb{E}[Q(N(y_i - E + (a - \zeta_y)), N(y_j - y_i))] \tag{2.53}$$

is approximately constant, for $Q$ a function of small Fourier support. The argument to prove this is the same as in [17]. What is new is that we have analyzed the mesoscopic statistic $\zeta_x$ and used it to complete the Fourier cut-off in the previous step. In [17] a Fourier cut-off was also used, but only for $\delta$ a large constant; here, $\delta$ is allowed to be any small constant. In [17] this caused some restriction in the following argument on which $t$ one can prove fixed energy universality for. Here this restriction is removed due to the Fourier cut-off $\delta$.

We would like to prove that $F(a)$ is constant. Define $F_h(a) := F(a + h) - F(a)$. By translation invariance of the local GOE statistics we know that $\mathbb{E}[F_h(\zeta_x)] = o(1)$. We will prove that $\zeta_x$ is
close to a Gaussian with variance \( c_2 \log(N) \). In order to conclude that \( F_h \) is small we run the reverse heat flow argument of [17]. We see that

\[
\left| \tilde{F}_h(\lambda)e^{-c_2 \log(N)\lambda^2} \right| \leq N^{-\varepsilon}
\]  

for some \( \varepsilon > 0 \), independent of the \( \delta \) chosen above. By the Fourier support restriction on \( Q \) we see that \( |\tilde{F}_h(\lambda)| \leq N^{\delta^2 + \varepsilon}. \) Hence for \( \delta \) small enough we get that \( \tilde{F}_h = o(1) \) and we conclude that \( F_h \) is small. This proves fixed energy universality.

### 2.6 Notation

We will use the following notion of overwhelming probability.

**Definition 2.8.** We say that an event \( \mathcal{F} \) holds with overwhelming probability if for any \( D > 0 \) we have \( \mathbb{P}[\mathcal{F}^c] \leq N^{-D} \) for large enough \( N \). If we have a family of events \( \{\mathcal{F}(u)\}_u \) then we will say that \( \{\mathcal{F}(u)\}_u \) holds with overwhelming probability if \( \sup_u \mathbb{P}[\mathcal{F}^c(u)] \leq N^{-D} \) for large enough \( N \).

For two positive \( N \)-dependent quantities \( a_N \) and \( b_N \) we say that \( a_N \approx b_N \) if there are constants \( c \) and \( C \) s.t. \( ca_N \leq b_N \leq Ca_N \).

In our work we use \( C \) to denote a positive constant that can change from line to line. The constant \( C \) will typically only depend on the constants appearing in the assumptions on \( V \).

For \( A, B \in \mathbb{R} \) we denote

\[
\left[ [A, B] \right] := [A, B] \cap \mathbb{Z}.
\]

### 3 Homogenization

In this section we prove a homogenization result for DBM. This reduces the problem of fixed energy universality of the model \( H_t \) to a problem involving mesoscopic statistics. Given a real symmetric matrix \( M \) with eigenvalues \( \lambda_1(M) \leq \cdots \leq \lambda_N(M) \), we define Dyson Brownian motion with \( \beta = 1 \) and initial data \( M \) to be the process satisfying

\[
d\lambda_i = \sqrt{\frac{2}{N}} dB_i + \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} dt, \quad \lambda_i(0) = \lambda_i(M)
\]  

(3.1)

At each fixed time \( t \geq 0 \), the particles \( \{\lambda_i(t)\}_i \) are equal in distribution to the eigenvalues of the matrix \( M + \sqrt{t}W \), where \( W \) is a GOE matrix independent of \( M \). It well-known that there is a unique strong solution to the above system of SDEs and the sample paths are continuous a.s. Recall that we want to study the eigenvalues of the matrix \( H_t := V + \sqrt{t}W \). We will do this by studying the DBM flow for \( t \) in the regime

\[
t_0 \leq t \leq t_0 + t_1.
\]

(3.2)

Here \( t_0 \) and \( t_1 \) are times defined by \( t_i = N^{\omega_i}/N \) and \( \omega_1 < \omega_0 \). The time \( t_0 \) satisfies \( gN^\sigma \leq t_0 \leq N^{-\sigma}G^2 \).

Our study requires the choice of an index \( i_0 \in C_q \), with \( C_q \) defined in Section 2.4. We will compare eigenvalues near \( t_0 \) to the GOE. At time \( t_0 \) the eigenvalue density of \( H_{t_0} \) is given by the free convolution law \( \rho_{\text{fc}, t_0} \) as defined in Section 2.1. In this section we are going to assume that

\[
\gamma_{i_0}(t_0) = 0, \quad \rho_{\text{fc}, t_0}(0) = \rho_{\text{we}}(0).
\]

(3.3)

In applications of the homogenization theorem this will be implemented by a re-scaling and shift of \( V \) and a re-scaling of time. For times \( t \geq t_0 \) we define \( x_i(t) \) to be the solution of

\[
dx_i(t) := \sqrt{\frac{2}{N}} dB_i + \frac{1}{N} \sum_{j \neq i} \frac{1}{x_i(t) - x_j(t)} dt
\]

(3.4)

with initial data \( x_i(t_0) = \lambda_i(H_{t_0}) \).
We now introduce the coupled GOE process. For times \( t \geq t_0 \) define \( y_i(t) \) as the solution to
\[
dy_i(t) = \sqrt{\frac{2}{N}} dB_i + \frac{1}{N} \sum_{j \neq i} \frac{1}{y_i(t) - y_j(t)} dt \tag{3.5}
\]
where the initial data \( y_i(t_0) \) are the eigenvalues of a GOE matrix independent of \( \{x_i(t)\}_i \). Above, the Brownian motions are the same as those appearing in (3.4). At times \( t \geq t_0 \) the particles \( \{y_i(t)\}_i \) are distributed as
\[
\{y_i(t)\}_i \overset{d}{=} \left\{ \lambda_i \left( \sqrt{1 + (t - t_0)} W' \right) \right\}_i. \tag{3.6}
\]
At times \( t \geq t_0 \) the \( y_i(t) \) satisfy a rigidity estimate with respect to the classical eigenvalue locations
\[
\sqrt{1 + (t - t_0)} \gamma_i^{(sc)} \quad \text{where} \quad \gamma_i^{(sc)} \text{ denote the classical eigenvalue locations of the semicircle law } \rho_{sc}.
\]
For our purposes we adopt the convention \( N/2 := \lceil N/2 \rceil \) as well as \( \gamma^{(sc)}_{N/2} = 0 \).

We now state the following theorem. It is the main result of this section.

**Theorem 3.1.** Fix \( t_0 = N^{\omega_0}/N \) satisfying \( gN^\sigma \leq t_0 \leq N^{-\sigma}G^2 \) for \( \sigma > 0 \). Let \( t_1 = N^{\omega_1}/N \) with \( 0 < \omega_1 < \omega_0/2 \). Let
\[
t_2 = \max\{N^{-\omega_1/15} t_1, N^{-4(\omega_0/2 - \omega_1)/3} t_1\}. \tag{3.7}
\]
Let \( \epsilon_b < \min\{N^{\omega_0/2 - \omega_1}, 3, \omega_1/60\} \). Let \( i_0 \in \mathcal{O}_q \). Assume that
\[
\gamma_{i_0}(t_0) = 0, \quad \rho_{c,i_0}(0) = \rho_{sc}(0). \tag{3.8}
\]
With overwhelming probability we have the following estimates. For every \( |u| \leq t_2 \) and \( |i| \leq Nt_1N^{\epsilon_b} \) we have
\[
(x_{i_0+i}(t_0 + t_1 + u) - \gamma_{i_0}(t_0 + t_1 + u)) - y_{N/2+i}(t_0 + t_1 + u)
= \sum_{|j| \leq N^{-1 + (\omega_0/2 - \omega_1)/3}} \zeta \left( \frac{i-j}{N}, t_1 \right) \left[ x_{i_0+j}(t_0) - y_{N/2+j}(t_0 + t_1) \right] + \frac{N^{\epsilon}}{N} O \left( \frac{N^{\omega_1/3}}{N^{\omega_0/6}} + \frac{1}{N^{\omega_1/60}} \right) \tag{3.9}
\]

Theorem 3.1 will be a consequence of Theorem 3.6 below. For the mesoscopic statistic \( \zeta(x, t_1) \) we have the following properties.

**Proposition 3.2.** The function \( \zeta(x, t) \) satisfies for \( 0 \leq t \leq 1 \) and \( x \in \mathbb{R} \) the following. We have,
\[
\int \zeta(x, t) dx = 1, \quad 0 \leq \zeta(x, t) \leq C \frac{t}{x^2 + t^2} \tag{3.10}
\]
and
\[
|\partial_x^k \zeta(x, t)| \leq C \frac{t}{x^2 + t^2}, \quad k = 1, 2, 3. \tag{3.11}
\]

The function \( \zeta \) will be made more explicit below.

### 3.1 Re-indexing

In this subsection we are going to make some assumptions which will greatly simplify notation. In Appendix C we present an argument which reduces the general case to these assumptions.

Let \( i_0 \) be as in Theorem 3.1. We assume that \( N \) is odd and that \( i_0 = (N + 1)/2 \). Note that with our convention, \( \gamma^{(sc)}_{i_0} = 0 \).

Presently the eigenvalues are labelled by the integers \([1, N]\). We re-label the eigenvalues so that they are indexed by \([-((N-1)/2), (N-1)/2]\]. The eigenvalues are then \( x_{-((N-1)/2)} \leq x_{-(N-3)/2} \leq \cdots \leq x_{(N-1)/2} \) and \( i_0 = 0 \). We furthermore have that \( \gamma_{i_0}(t_0) = 0 \).

We adopt these assumptions for the remainder of Section 3, apart from Section 3.8 which is where we prove Theorem 3.1 and must therefore unravel the re-labelling. We also adopt this convention for Section 4. It will not be used in the other sections.
3.2 Interpolation

In the work [17], the parabolic equation satisfied by the differences \( u_i := x_i - y_i \) was directly considered. The jump rates of the generator of this equation are hard to control as they involve both of the differences \( (x_i - x_j) \) and \( (y_i - y_j) \). In this paper we define a continuous interpolation which allows us to consider a family of parabolic equations whose generators are easier to control.

We now introduce this interpolation. For \( 0 \leq \alpha \leq 1 \), we define \( z_i(t, \alpha) \) as the solution

\[
dz_i(t, \alpha) = \sqrt{\frac{2}{N}} dB_i + \left( \frac{1}{N} \sum_j \frac{1}{z_i(t, \alpha) - z_j(t, \alpha)} \right) dt, \quad z_i(0, \alpha) = \alpha x_i(t_0) + (1 - \alpha) y_i(t_0). \tag{3.12}
\]

Note that \( z_i(t, 0) = y_i(t_0 + t) \) and \( z_i(t, 1) = x_i(t_0 + t) \). Note that we have effectively introduced a time shift which sets \( t_0 = 0 \).

We will soon see that like the \( x_i \) and \( y_i \), the \( z_i(t, \alpha) \) satisfy a rigidity estimate. However, we have some freedom in choosing the measure with which to construct the classical eigenvalue locations. One choice is the free convolution of the empirical measure of the initial data \( z_i(0, \alpha) \) with the semicircle law. However, this law is somewhat singular for short times \( t \), and does not reflect the fact that at time \( t_0 \), the particles \( x_i(t_0) \) satisfy a rigidity estimate wrt \( \rho_c(t_0) \) which has some regularity properties (e.g., \( |\rho_c(t_0)| \leq C/t_0 \)).

To compensate for this, we construct a measure \( \nu(dx, \alpha) \) that has a density near 0 that is at least as smooth as \( \rho_c(t_0) \). The construction is described in detail in Appendix A, and for now we just sketch its construction. This measure constructed is random but has good properties with overwhelming probability.

We need \( \nu(dx, \alpha) \) to satisfy two properties which motivate its construction. Firstly, we would like it to have a smooth density near 0 which is at least as regular as \( \rho_c(t_0) \). Secondly, we need the initial data \( z_i(0, \alpha) \) to be approximated by \( \nu(dx, \alpha) \) down to the optimal scale \( \eta \geq 1/N \), so that at later times \( t \), the particles \( z_i(t, \alpha) \) follow the free convolution of \( \nu(dx, \alpha) \) with the semicircle distribution.

We now sketch the construction of the measure \( \nu(dx, \alpha) \): complete details are given in Appendix A. The construction requires the choice of a parameter \( 0 < q^* < 1 \) which we now fix. This \( q^* \) is the same as that which appears in Appendix A. In the interval \([-q^*G, q^*G]\), one can construct, using the inverse function theorem, a density whose quantiles (in this case defined by starting the integration of the density from 0) equal \( \alpha \gamma_i(t_0) + (1 - \alpha) \gamma_i^{(sc)} \). This density has as good regularity properties as \( \rho_c(t_0) \). Since \( z_i(0, \alpha) \) satisfy a rigidity estimate in the interval \([-q^*G, q^*G]\), this density gives the required approximation for \( z_i(t_0) \) in this interval. To approximate the \( z_i(t_0) \) outside the interval \([-q^*G, q^*G]\) we can take \( \nu(dx, \alpha) \) to consist of a Dirac delta mass at each \( z_i(t, \alpha) \) such that \( |z_i(t, \alpha)| > q^*G \). Since the delta functions are outside of the interval \([-q^*G, q^*G]\) we do not not affect the regularity inside this interval. Clearly \( \nu(dx, \alpha) \) gives a good approximation to \( z_i(t, \alpha) \).

We now record formally some of the needed properties of the measure \( \nu(dx, \alpha) \). Again, we mention that the explicit construction appears in Appendix A. One of the key properties will be that, although the measure \( \nu(dx, \alpha) \) and its free convolution are random, for \( \alpha = 0, 1 \) the quantitative properties inside \( |x| \leq q^*G \) coincide with \( \rho_{\text{fc},t} \) and \( \rho_{\text{sc}} \) and are deterministic. In particular, certain quantiles of the measure at later times are deterministic up to an error term. This summarized in Lemma 3.4 below.

Let \( k_0 \) be the largest index so that

\[
|\gamma_{k_0}(t_0)| \leq q^*G, \quad |\gamma_{-k_0}(t_0)| \leq q^*G, \quad |\gamma_{-k_0}^{(sc)}| = |\gamma_{k_0}^{(sc)}| \leq q^*G. \tag{3.13}
\]

Note that \( k_0 = NG \). The measure \( \nu(dx, \alpha) \) has a nonvanishing density on the interval

\[
G_\alpha := [\alpha \gamma_{-k_0}(t_0) + (1 - \alpha) \gamma_{-k_0}^{(sc)}(t_0), \alpha \gamma_{k_0}(t_0) + (1 - \alpha) \gamma_{k_0}^{(sc)}]
\]

but has a singular part which may overlap with an \( o(G) \) portion of \( G_\alpha \) at its boundary; for any \( 0 < q < 1 \), \( \nu(dx, \alpha) \) is purely a.c. on \( qG_\alpha \).

---

3. The solution map of initial data to sample paths is continuous and therefore we can assume that a.s. we have a solution for every \( \alpha \in [0, 1] \).
We now define $\rho(E, t, \alpha)$ to be the free convolution of $\rho(E, 0, \alpha)\,\mathrm{d}E := \nu(\mathrm{d}E, \alpha)$ and the semicircle distribution with Stieltjes transform $m(z, t, \alpha)$. We have abused notation here slightly as at time $t = 0$, the measure $\nu(\mathrm{d}x, \alpha)$ is the sum of an absolutely continuous part and a sum of delta functions and it is only for times $t > 0$ that it has a true density. However, this will not affect anything as whenever we write $\rho(E, 0, \alpha)$ we only be referring to $E$ near 0 where the measure $\nu(\mathrm{d}x, \alpha)$ is purely a.c.

The following holds for the free convolutions. We defer the proof to Appendix A.

**Lemma 3.3.** Let $\delta > 0$. All of the following holds for $|E| \leq N^{-\delta}t_0$ and $t \leq N^{-\delta}t_0$, and $N^\delta/N \leq \eta \leq 10$, and with overwhelming probability. For the Stieltjes transform we have

$$|m(t, E, \alpha) - m(t, 0, \alpha) - (m(t, E, 0) - m(t, 0, 0))| \leq C\log(N) \left( \frac{|E|}{t_0} + \frac{t}{t_0} \right)$$

(3.15)

We have

$$|\partial_z m(z, t, \alpha)| \leq \frac{C}{t_0}.$$  

(3.16)

For the free convolution laws we have

$$\left| \frac{\mathrm{d}}{\mathrm{d}E} \rho(E, t, \alpha) \right| \leq \frac{C}{t_0}, \quad \rho(0, 0, \alpha) = \rho(0, 0, 0) = \rho_{sc}(0),$$

(3.17)

and

$$|\rho(E, t, \alpha) - \rho_{sc}(0)| \leq C\log(N) \frac{|E| + t}{t_0}.$$  

(3.18)

Moreover, for $0 < q < 1$ and $E \in qG_\alpha$, $N^\delta/N \leq \eta \leq 10$,

$$|m(t, z, \alpha)| \leq C\log(N), \quad \varepsilon \leq \text{Im} \left[ m(t, z, \alpha) \right] \leq C.$$  

(3.19)

The classical eigenvalue locations of the measure $\rho(E, t, \alpha)$ are denoted by $\gamma_i(t, \alpha)$ and are defined by

$$\frac{1}{2} + \frac{i}{N} = \int_{-\infty}^{\gamma_i(t, \alpha)} \rho(E, t, \alpha)\mathrm{d}E.$$  

(3.20)

We have that $\gamma_0(0, \alpha) = 0$ by the definition of $\rho(E, 0, \alpha)$. We will also need to relate $\gamma_i(t, 0)$ and $\gamma_i(t, 1)$ back to $\gamma_i^{[sc]}$ and $\gamma_i(t)$, respectively.

**Lemma 3.4.** We have for any $\varepsilon > 0$ and $\omega_1 < \omega_0/2$ with overwhelming probability,

$$\sup_{0 \leq t \leq 10t_1} |\gamma_0(t, 1) - \gamma_0(t_0 + t)| \leq \frac{N^\varepsilon N^{\omega_1}}{N^{N^{\omega_0}/2}}$$

(3.21)

and

$$\sup_{0 \leq t \leq 10t_1} |\gamma_0(t, 0)| \leq \frac{N^\varepsilon N^{\omega_1}}{N^{N^{\omega_0}/2}}.$$  

(3.22)

For $|j|, |k| \leq N^{\omega_0/2}$ we have with overwhelming probability,

$$\gamma_k(t, \alpha) - \gamma_j(t, \alpha) = \frac{k - j}{\rho_{sc}(0)} + \mathcal{O}\left( \frac{1}{N} \right),$$

(3.23)

for any $t \leq 10t_1$, with $\omega_1 \leq \omega_0/2$ and $0 \leq \alpha \leq 1$.

We again defer the proof to Appendix A.

We define the empirical Stieltjes transforms by

$$m_N(z, t, \alpha) = \frac{1}{N} \sum_{i=1}^N \frac{1}{z_i(t, \alpha) - z}.$$  

(3.24)

The measures $\rho(E, t, \alpha)$ are $\alpha$ dependent. We are eventually going to introduce some short-range and long-range cut-offs, and differentiate certain objects in $\alpha$. As the cut-offs are inherently discrete
(they involve eigenvalue indices), we want to choose them independent of \( \alpha \) to interact nicely with the differentiation. This requires the introduction of the following index sets. Let \( k_1 \) be the largest index so that

\[
\bigcup_{0 \leq \alpha \leq 1} [\alpha \gamma_{-k_1}(t_0) + (1 - \alpha) \gamma_{-k_1}(t_0), \alpha \gamma_{k_1}(t_0) + (1 - \alpha) \gamma_{k_1}^{(sc)}(t_0)] \subseteq \bigcap_{0 \leq \alpha \leq 1} \mathcal{G}_\alpha \cap \{-\mathcal{G}_\alpha\}
\]

where \( \mathcal{F} := \{x : -x \in \mathcal{F}\} \).

Finally, for any \( 0 < q < 1 \), we define

\[
\hat{\mathcal{C}}_q = \{ j : |j| \leq qk_1 \}.
\]

The cardinality \( \hat{\mathcal{C}}_q \) satisfies \( |\hat{\mathcal{C}}_q| \approx qGN \). This definition is just so that eigenvalues \( z_k(t, \alpha) \), for \( k \in \hat{\mathcal{C}}_q \) have nice qualitative properties uniformly in \( \alpha \) (the constants degenerate as \( q \to 1 \)). For example, the optimal rigidity estimate holds for any \( k \in \hat{\mathcal{C}}_q \). Moreover, the classical eigenvalue locations \( \gamma_k(t, \alpha) \) for \( k \in \hat{\mathcal{C}}_q \) will all be contained in a symmetric interval \([-q', q']\) for some \( q' \) on which all the densities \( \rho(E, t, \alpha) \) have good properties.

We will also use tacitly that for any \( q \) we have for \( j, k \in \hat{\mathcal{C}}_q \) that

\[
c \frac{|j - k|}{N} \leq |\gamma_j(t, \alpha) - \gamma_k(t, \alpha)| \leq C \frac{|j - k|}{N}.
\]

We have the following rigidity and local law estimates.

**Lemma 3.5.** Let \( \varepsilon > 0 \), \( \delta > 0 \), \( \delta_1 > 0 \), \( D > 0 \) and \( 0 < q < 1 \). We have

\[
P \left[ \sup_{0 \leq t \leq N^{-\delta}t_0} \sup_{i \in \hat{\mathcal{C}}_q} \sup_{0 \leq \alpha \leq 1} |z_i(t, \alpha) - \gamma_i(t, \alpha)| \geq \frac{N^\varepsilon}{N} \right] \leq N^{-D}.
\]

We have also,

\[
P \left[ \sup_{N^\delta/N \leq t \leq 10} \sup_{0 \leq t \leq N^{-\delta}t_0} \sup_{0 \leq \alpha \leq 1} \sup_{E \in \mathcal{E}_\alpha} |m_N(z, t, \alpha) - m(z, t, \alpha)| \geq \frac{N^\varepsilon}{N^\eta} \right] \leq N^{-D}.
\]

We defer the proof to Appendix A.

### 3.2.1 Reformulation of homogenization

In order to clarify what is precisely the input of the homogenization theorem we are going to recast the above results about rigidity, the free convolution measures, etc., as hypotheses and use this to prove a result from which Theorem 3.1 follows.

We take the processes \( z_i(t, 0) \) and \( z_i(t, 1) \) defined by (3.12) as given. We assume that they satisfy the following.

(i) \( z_i(0, 0) \) is a GOE ensemble.

(ii) There is a law \( \rho(E) \) with Stieltjes transform \( m(z) \) and parameters \( t_0 = N^{-\omega_0}/N \) and \( G \) with \( t_0 \leq N^{-\sigma}G^2 \) so that the following hold.

\[
c \leq \rho(E) \leq C, \quad |\partial_z^km(z)| \leq C/(t_0)^k, \quad |E| \leq G, \quad k = 1, 2.
\]

The classical eigenvalue locations of \( \rho(E) \) satisfy \( \gamma_0 = 0 \) and \( \rho(0) = \rho_{sc}(0) \).

(iii) Since \( z_1(t, 0) \) and \( z_1(t, 1) \) are defined, we can define the interpolations \( z_1(t, \alpha) \) as above, as well as the interpolating measures \( \rho(E, t, \alpha) \) (in the definition of the \( \rho(E, t, \alpha) \), the law \( \rho(E) \) takes the place of \( \rho_{t_1}(t_0) \)). Suppose that for all of these processes we have the rigidity and local law estimates

\[
P \left[ \sup_{0 \leq t \leq N^{-\delta}t_0} \sup_{i \in \hat{\mathcal{C}}_q} \sup_{0 \leq \alpha \leq 1} |z_i(t, \alpha) - \gamma_i(t, \alpha)| \geq \frac{N^\varepsilon}{N} \right] \leq CN^{-D}
\]
and
\[ P \left[ \sup_{0 \leq t \leq N^{-\delta_1} t_0} \sup_{N^{\delta_1-1} \leq \alpha \leq 10} \sup_{0 \leq \alpha \leq 1} E \left| \hat{\gamma}_m (z, t, \alpha) - m (z, t, \alpha) \right| \geq \frac{N^2}{Nt} \right] \leq C N^{-D} \] (3.32)
for any \( \delta, \varepsilon, \delta_1, D > 0 \) and \( 0 < q < 1 \).

(iv) There is a \( C > 0 \) so that for any \( D > 0 \),
\[ P \left[ \sup_{0 \leq t \leq t_0} \sup_i |z_i(t, 1)| \geq N^C \right] \leq N^{-D}. \] (3.33)
for large enough \( N \).

We note that under (ii), the constructed \( \rho(E, t, \alpha) \) obey the estimates of Lemma 3.3. Moreover the estimate (3.23) also holds. Under the above assumptions we will prove the following, from which Theorem 3.1 will be deduced.

**Theorem 3.6.** Let \( z_i(t, 0) \) and \( z_i(t, 1) \) be defined as at the start of Section 3.2.1. Suppose that assumptions (i)-(iv) hold. Let \( 0 < \varepsilon_b < \varepsilon_a \). Let \( t_2 := t_1 N^{-\varepsilon_2} \) with \( \omega_1 - \varepsilon_1 > 0 \). Let \( \varepsilon > 0 \). There is an event with overwhelming probability on which the following holds. For any \( |i| \leq N t_1 \varepsilon_b \) and \( |u| \leq t_2 \) we have
\[
\begin{align*}
&z_i(t_1 + u, 1) - z_i(t_1 + u, 0) = (\gamma_0(t_1 + u, 1) - \gamma_0(t_1 + u, 0)) \\
&\quad + \sum_{|j| \leq N t_1 \varepsilon_a} \zeta \left( \frac{i - j}{N}, t_1 \right) (z_j(0, 1) - z_j(0, 0)) \\
&\quad + \frac{N^\varepsilon}{N} \left( \frac{N\omega_A}{N^2} + \frac{1}{\sqrt{N\varepsilon}} + \frac{1}{\sqrt{N\varepsilon}} \right) + \frac{1}{N^{\varepsilon_a}} + N^{\varepsilon_2 + \varepsilon_a} \left( \frac{(N t_1)^2}{N^2} + \frac{1}{(N t_1)^{1/10}} + N^{\varepsilon_a - \varepsilon_2/2} \right)
\end{align*}
\] (3.34)

**Remark.** If the \( z_i(t, 1) \) obey the optimal local law wrt \( \rho(E) \) and its Stieltjes transform, then the estimate (3.21) holds if we define \( \gamma_0(t) \) by \( \gamma_0(0) = 0 \) and \( \hat{\gamma}_0(t) = -\text{Re} [m(\gamma_0(t))] \) with \( m \) the Stieltjes transform of the free convolution of \( \rho(E) \) at time \( t \). The estimate (3.22) already holds. Hence if the local law holds, then the classical eigenvalue locations appearing above can be replaced by the deterministic counterparts which do not depend on the realization of the initial data \( z_i(0, 1) \).

### 3.3 Short-range DBM

The centered \( \tilde{z}_i \)'s are given by
\[ \tilde{z}_i(t, \alpha) := z_i(t, \alpha) - \gamma_0(t, \alpha). \] (3.35)

We also define the classical locations of the centered \( \tilde{z}_i(t, \alpha) \) by
\[ \gamma_i(t, \alpha) = \gamma_i(t, \alpha) - \gamma_0(t, \alpha). \] (3.36)

Note that \( \gamma_i(t, \alpha) \) satisfies (see [54])
\[ \hat{\gamma}_i(t, \alpha) = -\text{Re} [m(\gamma_i(t, \alpha), t, \alpha)] \] (3.37)
and so for \( i \in \mathcal{C}_q \) we have
\[ |\hat{\gamma}_i(t, \alpha)| \leq C \log(N) \] (3.38)
by Lemma 3.3. The \( \tilde{z}_i(t, \alpha) \) satisfy the equations
\[ d\tilde{z}_i(t, \alpha) = \frac{dB_i}{\sqrt{N}} + \left( \frac{1}{N} \sum_j \frac{1}{\tilde{z}_i(t, \alpha) - \tilde{z}_j(t, \alpha)} + \text{Re} [m(\gamma_0(t, \alpha), t, \alpha)] \right) dt \] (3.39)

We introduce the following cut-off dynamics for the \( \tilde{z}_i \)'s. Its definition will use the parameters
\[ \omega_\varepsilon > 0, \quad \omega_A > 0, \quad 0 < q_* < 1. \] (3.40)
Before defining the short-range approximation we outline the role of each of these parameters in the definition. The parameter $\ell = N^{\omega_\ell}$ is the most fundamental. It is the “range” of the short-range approximation. In our short-range dynamics we allow particles $i$ and $j$ to interact iff $|i - j| \leq \ell$. In order for this approximation to be effective we need $\ell \gg N t_1$: that is $\ell$ must exceed the “range” of DBM which is $t_1$.

For particles $i$ near $0$ we can use the rigidity estimates to replace the long-range part of the dynamics (i.e., the force coming from particles $j$ s.t. $|i - j| > \ell$) by a deterministic quantity. Since the free convolution law is regular the dependence of this deterministic quantity on the particle index $i$ is smooth; we can therefore replace it by something independent of the particle index $i$ if $|i| \leq N^{\omega_A}$, as long as we choose $\omega_A$ to be smaller than the regularity scale of the free convolution law which is governed by $t_0$.

Finally, since the rigidity estimates only hold for particles near $0$ we need to make a different cut-off for particles away from $0$; this is the role of $q_\ast$. We will not make a short range cut-off for particles $i \notin \mathcal{C}_{q_\ast}$.

We now turn to the definition of the short-range approximation. We first introduce some notation. Let $\omega_\ell > 0$ be as above. For each $0 < q < 1$, define the short range index set $\mathcal{A}_q$ by

$$\mathcal{A}_q := \{(i,j) : |i - j| \leq N^{\omega_\ell}\} \cup \{(i,j) : ij > 0, i \notin \mathcal{C}_{q}, j \notin \mathcal{C}_{q}\}.$$  

We introduce the following notation. Let

$$\sum_{j}^{\mathcal{A}_q(i)} = \sum_{j: (i,j) \in \mathcal{A}_q} \quad \text{and} \quad \sum_{j}^{\mathcal{A}_q(i)} = \sum_{j: (i,j) \notin \mathcal{A}_q}.$$  

For a fixed $i$ let $j_{\leq, i}$ be the smallest index s.t. $(i, j_{\leq, i}) \in \mathcal{A}_{q_\ast}$ and $j_{\geq, i}$ be the largest index s.t. $(i, j_{\geq, i}) \in \mathcal{A}_{q_\ast}$. Then define the interval

$$\mathcal{I}_i(\alpha,t) = [\gamma_{j_{\leq, i}}(\alpha,t), \gamma_{j_{\geq, i}}(\alpha,t)].$$  

The interval $\mathcal{I}_i(\alpha,t)$ corresponds to the classical spatial locations of the particles $j$ that are allowed to interact with particle $i$.

Let $\omega_A > 0$ be as above. We define the short-range approximation $\hat{z}_i(t,\alpha)$ as the solution to the following system of SDEs. For $|i| \leq N^{\omega_A}$ let

$$d\hat{z}_i(t,\alpha) = \sqrt{\frac{2}{N}} dB_i + \frac{1}{N} \sum_{j}^{\mathcal{A}_{q_\ast}(i)} \frac{1}{\hat{z}_i(t,\alpha) - \hat{z}_j(t,\alpha)} dt  \tag{3.44}$$

and for $|i| > N^{\omega_A}$ let

$$d\hat{z}_i(t,\alpha) = \sqrt{\frac{2}{N}} dB_i + \frac{1}{N} \sum_{j}^{\mathcal{A}_{q_\ast}(i)} \frac{1}{\hat{z}_i(t,\alpha) - \hat{z}_j(t,\alpha)} dt + \frac{1}{N} \sum_{j}^{\mathcal{A}_{q_\ast}(i)} \frac{1}{\hat{z}_i(t,\alpha) - \hat{z}_j(t,\alpha)} dt + \text{Re}[m(t, \gamma_0(\alpha,t), \alpha)] dt dt.  \tag{3.45}$$

The initial condition is $\hat{z}_i(0,\alpha) = \hat{z}_i(0,\alpha)$. Like the $\hat{z}_i(t,\alpha)$, the $\hat{z}_i(t,\alpha)$ retain the ordering $\hat{z}_i(t,\alpha) < \hat{z}_{i+1}(t,\alpha)$ for all positive times. The above parameters are chosen so that

$$0 < \omega_1 < \omega_\ell < \omega_A < \omega_0/2. \tag{3.46}$$

The following lemma shows that the $\hat{z}_i$’s are a good approximation for the $z_i$’s - that is $\hat{z}_i = z_i + o(1/N)$ with overwhelming probability.

**Lemma 3.7.** Let $\hat{z}_i(t,\alpha)$ be defined as in (3.44)-(3.45) and $\hat{z}_i(t,\alpha)$ be defined as in (3.39). Let $\varepsilon > 0$ and $D > 0$. Then we have

$$\mathbb{P} \left[ \sup_{0 \leq t \leq t_1} \sup_{0 \leq \alpha \leq 1} |\hat{z}_i(t,\alpha) - \hat{z}_i(t,\alpha)| \geq N^\varepsilon t_1 \left( \frac{N^{\omega_A}}{N^{\omega_0}} + \frac{1}{\sqrt{N^G}} \right) \right] \leq N^{-D}, \tag{3.47}$$

for large enough $N$. 

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Proof. Define \( w_i(t, \alpha) := \hat{z}_i(t, \alpha) - \tilde{z}_i(t, \alpha) \). The \( w_i \) satisfy the equations
\[
\hat{\partial}_t w_i = \sum_j A_{a_j}^{(i)} B_{ij}(w_j - w_i) + A_i
\]
where
\[
B_{ij} = \frac{1}{N} \frac{1}{(\hat{z}_i(t, \alpha) - \tilde{z}_j(t, \alpha))(\hat{z}_i(t, \alpha) - \tilde{z}_j(t, \alpha))}
\]
The error term \( A_i \) satisfies \( A_i = 0 \) for \( |i| > N^{\omega_A} \), and for \( |i| \leq N^{\omega_A} \) it is given by
\[
A_i = \frac{1}{N} \sum_j A_{a_j}^{(i)} \frac{1}{z_i(t, \alpha) - z_j(t, \alpha)} + \text{Re}[m(\gamma_0(t, \alpha), t, \alpha)]
\]
\[
\times \left( \frac{1}{N} \sum_j A_{a_j}^{(i)} \frac{1}{z_i(t, \alpha) - z_j(t, \alpha)} - \int_{I_i^c(t, \alpha)} \frac{\rho(x, t, \alpha)dx}{z_i(t, \alpha) - x} \right)
\]
\[
+ \left( \int_{I_i(t, \alpha)} \frac{\rho(x, t, \alpha)dx}{z_i(t, \alpha) - x} - \int_{I_i^c(t, \alpha)} \frac{\rho(x, t, \alpha)dx}{\gamma_i(t, \alpha) - x} \right)
\]
\[
+ (\text{Re}[m(\gamma_0(t, \alpha), t, \alpha)]) - (\text{Re}[m(\gamma(t, \alpha), t, \alpha)]) + \left( \int_{I_i} \frac{\rho(x, t, \alpha)dx}{\gamma_i(t, \alpha) - x} \right)
\]
\[
=: E_1 + E_2 + E_3 + E_4
\]
The proof of the lemma is as follows. Since both \( \hat{z} \) and \( \tilde{z} \) are ordered the kernel \( \hat{B}_{ij} \) are the coefficients of a jump process on \([-(N - 1)/2, (N - 1)/2] \). Hence the semigroup \( \mathcal{U}(\hat{B}) \) is a contraction on every \( \ell^p \) space. Since at time \( t = 0 \) we have \( \hat{z}_i(0, \alpha) = \tilde{z}_i(0, \alpha) \), we have \( w(t) = \int_0^t \mathcal{U}(\hat{B})(s, t) A(s) ds \) by the Duhamel formula. Therefore,
\[
||w(t)||_{\infty} \leq t \sup_{0 \leq s \leq t} ||A(s)||_{\infty}
\]
The remainder of the proof consists of estimating \( A_i \) using the rigidity estimates.

Let \( \varepsilon > 0 \). For the remainder of the proof we work on the event that the estimates (3.31) and (3.32) of Section 3.2.1 (iii) hold with this \( \varepsilon \) and a \( q \) satisfying \( q_* < q < 1 \), and a small \( \delta > 0 \) to be determined and large \( D > 0 \). We take \( \delta_1 \) to satisfy \( \omega_0 - \delta_1 > \omega_1 \).

We fix \( \eta > N^{2\delta}/N \) satisfying \( \eta \ll G \). We write the term \( E_1 \) as
\[
E_1 = \frac{1}{N} \sum_j A_{a_j}^{(i)} \frac{1}{z_i(t, \alpha) - z_j(t, \alpha)} \int_{I_i^c(t, \alpha)} \frac{\rho(x, t, \alpha)dx}{z_i(t, \alpha) - x}
\]
\[
\times \left( \frac{1}{N} \sum_j A_{a_j}^{(i)} \frac{1}{z_i(t, \alpha) - z_j(t, \alpha)} - \int_{I_i^c(t, \alpha)} \frac{\rho(x, t, \alpha)dx}{z_i(t, \alpha) - x} \right)
\]
\[
+ \left( \int_{I_i} \frac{\rho(x, t, \alpha)dx}{z_i(t, \alpha) - x} - \int_{I_i^c(t, \alpha)} \frac{\rho(x, t, \alpha)dx}{\gamma_i(t, \alpha) - x} \right)
\]
\[
+ (\text{Re}[m(\gamma_0(t, \alpha), t, \alpha)]) - (\text{Re}[m(\gamma(t, \alpha), t, \alpha)]) + \left( \int_{I_i} \frac{\rho(x, t, \alpha)dx}{\gamma_i(t, \alpha) - x} \right)
\]
\[
=: F_1 + F_2 + F_3 + F_4 + F_5
\]
Above, the intervals \( \hat{I}_1, \hat{I}_2 \) and \( \hat{I}_3 \) are defined as follows. They depend on \( i, \alpha \) and \( t \) but we suppress this for notational simplicity. Let \( \alpha > 0 \) be the first index not in \( \hat{C}_q \). Define \( \hat{I}_1 := [\gamma_{-\alpha}(t, \alpha), \gamma_{\alpha}(t, \alpha)] \setminus \hat{I}_1(t, \alpha) \). We define \( \hat{I}_2 := [\gamma_{-\alpha}(t, \alpha), \gamma_{\alpha}(t, \alpha)] \). Lastly, \( \hat{I}_3 := \hat{I}_2 \). Before estimating each of the \( F_k \) let us explain the motivation for the above decomposition. First we remark that since \( |i| \leq N^{\omega_A} \) the interval \( I_i(t, \alpha) \) has length \( |I_i(t, \alpha)| \approx \ell/N \) and is contained in \( [\gamma_{-\alpha}(t, \alpha), \gamma_{\alpha}(t, \alpha)] \). Moreover, \( |\gamma_{\pm\alpha}(t, \alpha) - \gamma_{\pm\alpha}(t, \alpha)| \approx G \).
We want to use rigidity to estimate (3.52). However, we only know that rigidity holds for the particles in \( \hat{\mathcal{C}}_q \). Hence we break up the terms in (3.52) into two parts. The first is \( F_1 \) which is estimated using rigidity. The remaining particles are distance at least \( G \) from \( \hat{z}_i(t, \alpha) \) and we can use the local law (3.32) on a scale \( \eta \ll G \) to estimate this contribution. The terms \( F_2 - F_5 \) just correspond to some gymnastics to rewrite these particles in a form that can be estimated by the local law (3.32).

The term \( F_1 \) is estimated using rigidity; using (3.31) we easily see that \( |F_1| \leq CN^\varepsilon/N^{\omega_t} \). For \( F_2 \) we use the fact that the restriction \( |i| \leq N^{\omega_t} \) and \( j \not\in \hat{\mathcal{C}}_q \) enforces that \( |z_i - z_j| \geq cG \). Since \( \eta \ll G \) we may bound \( F_2 \) by

\[
|F_2| \leq \eta C \sum_j \frac{1}{(z_i - z_j)^2 + cG^2} \leq C \frac{\eta}{G} \text{Im} \left[ m_N(z_i + icG) \right] \leq C \frac{\eta}{G}.
\]

(3.54)

By similar reasoning we get \( |F_3| \leq C\eta/G \). For \( F_4 \) we use the local law estimate (3.32) and get \( |F_4| \leq N^\varepsilon/(N\eta) \). Lastly for \( F_5 \) we use the optimal rigidity estimate (3.31) and get

\[
|F_5| \leq \frac{N^\varepsilon}{N\eta} C (\text{Im} \left[ m(z_i + in \eta/t, \alpha) \right] + \text{Im} \left[ m_N(z_i + in\eta) \right]) \leq C \frac{N^\varepsilon}{N\eta}.
\]

(3.55)

Hence,

\[
|E_1| \leq CN^\varepsilon \left( \frac{1}{N^{\omega_t}} + \frac{\eta}{G} + \frac{1}{N\eta} \right) \leq CN^\varepsilon \left( \frac{1}{N^{\omega_t}} + \frac{1}{\sqrt{NG}} \right)
\]

(3.56)

where we optimized and chose \( \eta = (G/N)^{1/2} \) (and chose \( \delta > 0 \) small enough to allow this choice).

We can bound

\[
|E_2| = \left| \int_{I_i(t, \alpha)} \frac{\rho(x, t, \alpha)dx}{z_i - x} - \int_{I_i(t, \alpha)} \frac{\rho(x, t, \alpha)dx}{\gamma_i - x} \right| \leq C \frac{N^\varepsilon}{N^{\omega_t}} \text{Im} \left[ m(\gamma_i + icN^{\omega_t}/N, t, \alpha) \right] \leq C \frac{N^\varepsilon}{N^{\omega_t}}.
\]

(3.57)

For \( E_3 \) we use (3.16) and get \( |E_3| \leq CN^{\omega_t}/N^{\omega_0} \). We now have to bound \( E_4 \). Note that \( I_i \) is almost symmetric about \( \gamma_i(t, \alpha) \). We have

\[
|\gamma_i + k - \gamma_i| = |\gamma_i - k - \gamma_i| \left( 1 + O \left( \frac{1}{N t_0} \right) \right),
\]

(3.58)

and so

\[
|E_4| \leq \int_{I_i} \frac{\rho(x, t, \alpha) - \rho(\gamma_i, t, \alpha)}{\gamma_i - x} dx | + C \frac{N^{\omega_t}}{N t_0} \leq C \frac{N^{\omega_t}}{N t_0}.
\]

(3.59)

This proves that with overwhelming probability we have

\[
||A||_x \leq CN^\varepsilon \left( \frac{N^{\omega_t}}{N^{\omega_0}} + \frac{1}{N^{\omega_t}} + \frac{1}{\sqrt{NG}} \right).
\]

(3.60)

This yields the claim via Duhamel’s formula (3.51).

\[\square\]

### 3.4 Derivation of parabolic equation

Define now

\[
u_i := \partial_\alpha \hat{z}_i(t, \alpha).
\]

(3.61)

The \( u_i \) satisfy the equation

\[
\partial_t u_i = \sum_j A_{\eta^{\omega_t}}(i) B_{ij}(u_j - u_i) + \xi_i =: -(Bu)_i + \xi_i
\]

(3.62)

where

\[
B_{ij} = \frac{1}{N} \frac{1}{(\hat{z}_i - \hat{z}_j)^2}
\]

(3.63)
and \( \xi_i = 0 \) for \(|i| \leq N^{\omega_A} \) and for \(|i| > N^{\omega_A} \),

\[
\xi_i = \frac{1}{N} \sum_j A_{q,i}(j) (\hat{\xi}_i \tilde{z}_j - \hat{\xi}_j \tilde{z}_i) + \hat{\partial}_\alpha (\text{Re}[m(t, \gamma_0(\alpha, t), \alpha)])
\]  
(3.64)

Note that for \( \alpha_1 \) and \( \alpha_2 \), the differences \( \tilde{u}_i := \tilde{z}_i(\alpha_1) - \tilde{z}_i(\alpha_2) \) satisfy

\[
\hat{c}_t \tilde{u}_i = \frac{1}{N} \sum_j \frac{\tilde{u}_j - \tilde{u}_i}{(\tilde{z}_i(\alpha_1) - \tilde{z}_j(\alpha_1))(\tilde{z}_i(\alpha_2) - \tilde{z}_j(\alpha_2))}
\]

with \( \tilde{u}_i(0) = (z_i(0,1) - z_i(0,0))(\alpha_1 - \alpha_2) \). Since \( |z_i(0,0)| + |z_i(0,1)| \leq N^C \) with overwhelming probability for some \( C > 0 \) by (3.33) we see that

\[
\|\hat{c}_\alpha \tilde{z}(\alpha, t)\|_\infty \leq N^C
\]

for \( 0 \leq t \leq 1 \) with overwhelming probability. It is not hard to see that

\[
|\xi_i| \leq 1_{\{ |i| > N^{\omega_A} \}} N^C
\]

for some \( C > 0 \) with overwhelming probability.

The parabolic equation (3.62) is the key starting point. We will treat \( \xi_i \) as an error term. Since it vanishes for indices \(|i| \leq N^{\omega_A} \) and the operator \( \mathcal{B} \) involves jumps only for particles distance \( N^{\omega_{\ell}} \ll N^{\omega_A} \) apart we expect that \( \xi_i \) will have a negligible contribution near 0. This is in fact true as we will see below.

### 3.5 The kernel \( U^{(B)} \)

At this point essentially the entire remainder of Section 3 and all of Section 4 are concerned only with properties of the semigroup \( U^{(B)} \) of the kernel \( \mathcal{B} \). The semigroup \( U^{(B)} \) depends on the \( \tilde{z}_i(t, \alpha) \). The method that we are going to present for analyzing the semigroup \( U^{(B)} \) is more general and works for any semigroup whose kernel consists of random coefficients satisfying a system of SDEs with certain properties and certain a-priori bounds.

In this section we will pass to a more general set-up involving the hypotheses (I)-(III) below. The set-up consists of a semigroup \( U^{(B)} \) and kernel \( \mathcal{B} \) with coefficients \( B_{ij} = N^{-1}(\tilde{z}_i(t, \alpha) - \tilde{z}_j(t, \alpha))^{-2} \). Here we are abusing notation slightly and re-using \( \tilde{z}_i, \mathcal{B}, U^{(B)} \), etc. In the next few subsections and in Section 4 we will then use these hypotheses to derive various facts about \( U^{(B)} \). The main result about \( U^{(B)} \) is Theorem 3.10 which will be proven at the end of Section 3.6. After proving Theorem 3.10, we will return to the previous set-up and use Theorem 3.10 to prove Theorem 3.6 in Section 3.7.

We let \( \tilde{z}_i(t, \alpha) \) (we will leave in the \( \alpha \) notation even though it is unnecessary - for the next few sections \( \alpha \) should be regarded as fixed) be the solution to

\[
d\tilde{z}_i(t, \alpha) = \sqrt{\frac{2}{N}} dB_i + \frac{1}{N} \sum_j A_{q,i}(j) \frac{1}{\tilde{z}_i - \tilde{z}_j} dt + 1_{\{|i| \leq N^{\omega_A} \}} F_i dt + 1_{\{|i| > N^{\omega_A} \}} J_i dt
\]

(3.68)

where \( F_i \) and \( J_i \) are adapted bounded processes. The parameters \( \omega_A, \omega_{\ell} \) and \( q_\alpha \) are the same as before, and \( A_{q,i} \) is defined as above. Previously we also introduced the index \( k_0 \) in the definition of \( C_q := \{ i : |i| \leq q k_0 \} \). Here, we take \( k_0 \) as a given parameter in the set-up and assume that \( k_0 \approx N G \). Let \( \rho(E, t, \alpha)\) be measures with densities on \(|E| \leq q G \) for any \( 0 < q < 1 \) (here, we just sent \( G \to \infty \) in order to simplify notation). Suppose that the following hold.

(I) We have \( \rho(0, 0, \alpha) = \rho_{\infty}(0) \) and \( \gamma_0(0, \alpha) = 0 \) and

\[
c \leq \rho(E, t, \alpha) \leq C, \quad |\tilde{c}_E \rho(E, t, \alpha)| \leq \frac{C}{t_0}, \quad |\tilde{c}_t \rho(E, t, \alpha)| \leq \frac{C}{t_0} |E| \leq q G, \quad 0 \leq t \leq 10t_1.
\]

(3.69)
Moreover, $G^2 \geq N^\sigma t_0 = N^{\omega_0}/N$ for some $\sigma > 0$ and $\omega_0 > 0$. We assume that the classical eigenvalue locations $\gamma_i(\alpha, t)$ satisfy
\[ i \in \hat{\mathcal{C}}_q \implies \gamma_i(\alpha, 0) \in [-Gq', Gq'] \] (3.70)
for some $0 < q' < 1$ depending on $q$. We also assume
\[ |\tilde{c}_t \gamma_i(t, \alpha)| \leq C \log(N) \] (3.71)
for $i \in \hat{\mathcal{C}}_q$.

(II) We have the rigidity estimate
\[ \mathbb{P} \left[ \sup_{i \in \hat{\mathcal{C}}_q} \sup_{0 \leq t \leq 10t_1} |\tilde{z}_i(t, \alpha) - \gamma_i(t, \alpha)| \geq \frac{N^\varepsilon}{N} \right] \leq N^{-D} \] (3.72)
for any $\varepsilon, D > 0$ and $0 < q < 1$.

(III) For the terms $F_i$ and $J_i$ we have for some fixed $C_J > 0$ and $\omega_F > 0$ and every $0 < q < 1$,
\[ \mathbb{P} \left[ \sup_{i \in \hat{\mathcal{C}}_q} \sup_{0 \leq t \leq 10t_1} |J_i| \geq C_J \log(N) \right] \leq N^{-D} \] (3.73)
and
\[ \mathbb{P} \left[ \sup_{i} \sup_{0 \leq t \leq 10t_1} |J_i| \geq N^{C_J} \right] \leq N^{-D} \] (3.74)
and
\[ \mathbb{P} \left[ \sup_{i} \sup_{0 \leq t \leq 10t_1} |F_i| \geq \frac{N^\varepsilon}{N^{\omega_F}} \right] \leq N^{-D} \] (3.75)
for any $\varepsilon, D > 0$.

Remark. In our case $F_i = 0$ but we have added it for the following reason. In our set-up we have $F_i = 0$ because we differentiated the short-range approximation to arrive at the parabolic equation (3.62). In other applications it is conceivable that one would like a homogenization result for the full process $d\tilde{z}_i$. This is covered by the above set-up by rigidity - in this case $\omega_F = \omega_\ell$ (i.e., the long-range $z_i - z_j$ terms cancel with $\tilde{c}_\gamma = -\text{Re}(m(\gamma))$). Finally, while the $\omega_\ell$ appearing in the definition of the $\tilde{z}_i$ and $\mathcal{B}$ are the same, this is not crucial as extra terms can just be absorbed into the $F_i$ term using rigidity and the smoothness of the density $p(E, t, \alpha)$.

With $\tilde{z}_i$ satisfying (I)-(III) we will consider the operator
\[ (Bu)_i := \sum_j A_{q_\alpha}^{(i)} B_{ij}(u_i - u_j) \] (3.76)
with semigroup $\mathcal{U}^{(B)}$. Before we write down the main result about the semigroup $\mathcal{U}^{(B)}$ we record some estimates on it. First, we have the following finite speed of propagation estimate.

Lemma 3.8. Let $0 \leq s \leq t \leq t_1$. Let $q_\alpha$ and $\omega_\ell$ be in the definition of the short-range set $A_{q_\alpha}$ for $\mathcal{B}$. Suppose that (I)-(III) hold. Let $0 < q_1 < q_2 < q_\alpha$ be given. Let $D > 0$ and $\varepsilon > 0$. For each $\alpha$ there is an event $\mathcal{F}_\alpha$ with probability $\mathbb{P}[\mathcal{F}_\alpha] \geq 1 - N^{-D}$ on which the following estimates hold. If $i \in \hat{\mathcal{C}}_{q_2}$ and $0 \leq s \leq t \leq 10t_1$, then
\[ |\mathcal{U}^{(B)}_{ji}(s, t, \alpha)| \leq \frac{1}{N^D}, \quad |i - j| > N^{\omega_\ell + \varepsilon}. \] (3.77)
If $i \notin \hat{\mathcal{C}}_{q_2}$ and $j \in \hat{\mathcal{C}}_{q_1}$ and $0 \leq s \leq t \leq 10t_1$ then
\[ |\mathcal{U}^{(B)}_{ji}(s, t, \alpha)| \leq \frac{1}{N^D}. \] (3.78)
Lemma 3.8 is an immediate consequence of Theorem 4.1. Similar estimates appeared earlier in [18] and our proof follows closely the one appearing there.

Lemma 3.8 contains two estimates. The first (3.77) is almost-optimal in that the kernel decays quickly when \(|i - j| \gtrsim \ell\), where \(\ell\) is the range of the jump kernel. Its proof requires the optimal rigidity estimate. We also need the second estimate (3.78) which is weaker but holds for particles \(i\) for which rigidity does not hold.

We also have the following estimate for the kernel \(U^{(B)}\) which says that for the purposes of upper bounds we can think of \(U^{(B)} \sim t/(x^2 + t^2)\).

**Lemma 3.9.** Let \(q_\ast\) be from the definition of \(A_{q_\ast}\) in the definition of \(B\). Suppose that (I)-(III) hold. Let \(0 < q_1 < q_\ast\), \(D > 0\) and \(\varepsilon > 0\). For each \(\alpha\) there is an event with \(\mathbb{P}[\mathcal{F}_\alpha] \geq 1 - N^{-D}\) on which the following estimates hold. For \(i, j \in \mathcal{C}_{q_1}\) and \(0 \leq s \leq t \leq 10t_1\) we have

\[
U^{(B)}_{ij}(s, t) \leq \frac{N^\varepsilon}{N} \frac{|t - s| \vee N^{-1}}{((i - j)/N)^2 + ((t - s) \vee N^{-1})^2}
\]  

(3.79)

**Remark.** Note that the above estimate will not hold for \(U^{(B)}\) if \((s - t) \gg N^{\omega_\ell}/N\).

The proof of the above estimate is deferred to the next section and is stated there as Theorem 4.7. Roughly, the proof consists of the following steps. First we derive the general estimate

\[
|U^{(B)}_{ij}(0, t)| \leq \frac{C}{Nt}
\]  

(3.80)

using the Nash method. This argument is similar to that in [39] - it is slightly different as we only have a short range operator \(B\) living on the scale \(N^{\omega_\ell}\), but this does not affect things as long as \(t \ll N^{\omega_\ell}/N\).

Then we decompose

\[
B = S + R
\]  

(3.81)

where \(S\) is a short-range operator on the scale \(\ell_2 \sim N\ell\). (Although \(B\) is already a short-range operator, we are interested in time scales \(Nt \ll \ell\), where \(\ell\) is the scale that \(B\) lives on - hence we must make a further long range/short range decomposition of \(B\).) We then prove finite speed estimates for \(S\) and use this together with a Duhamel expansion to derive the estimate.

### 3.6 Homogenization of \(U^{(B)}\)

In this section we will prove that \(U^{(B)}\) is given by a deterministic quantity, plus random corrections of lower order. This is the main calculation of Section 3. Fix \(\varepsilon_B > 0\) s.t.

\[
\omega_A - \varepsilon_B > \omega_\ell,
\]  

(3.82)

and let

\[
|a| \leq N^{\omega_A - \varepsilon_B}.
\]  

(3.83)

We consider a solution \(w\) of the equation

\[
\partial_t w_i = -(Bw)_i, \quad w_i(0) = N\delta_a(i).
\]  

(3.84)

Let \(\mu\) be the counting measure on \([[-(N - 1)/2, (N - 1)/2]]\) normalized to have mass 1. We introduce the \(\ell^p\)-norms

\[
||u||_p = \int |u_i|^pd\mu(i), \quad ||u||_{\infty} = \sup_i |u_i|.
\]  

(3.85)

The eigenvalue density is smooth on the scale \(t_0\). Our operator \(B\) instead lives on the scale \(N^{\omega_\ell}/N \ll t_0\), and we are working with times \(t \ll \ell/N \ll t_0\), and so our solutions \(w_i\) will never see the density fluctuations. Hence it makes sense to compare \(w\) with the solution (on \(\mathbb{R}\)) of

\[
\partial_t f(x) = \int_{|x-y| \leq \ell_1} \frac{f(y) - f(x)}{(x-y)^2} \rho_{sc}(0)dy
\]  

(3.86)
where
\[ \eta_\ell := \frac{N \omega_\ell}{N \rho_{sc}(0)}. \] (3.87)

Let \( p_t(x, y) \) be the kernel of the above equation. We define the “flat” classical eigenvalue locations by
\[ \gamma_j^{(f)} := \frac{j}{N \rho_{sc}(0)}. \] (3.88)

Note that for \( |j| \leq N^{\omega_0/2} \) we have
\[ |\gamma_j^{(f)} - \gamma_j(t, \alpha)| \leq C \frac{N}{\ell}. \] (3.89)

The main result of this section is the following.

**Theorem 3.10.** Suppose that (I)-(III) of Section 3.5 hold. Fix an index \(|a| \leq N^{\omega/2}\). Let \( i \) satisfy \(|i - a| \leq \ell/10\). Let \( t_1 \) be as above and let
\[ t_2 := N^{-\epsilon_2} t_1 \] (3.90)
for \( \omega_1 - \epsilon_2 > 0 \). Let \( \epsilon > 0 \) and \( D > 0 \). There is an event \( \mathcal{F}_\alpha \) with \( \mathbb{P}[\mathcal{F}_\alpha] \geq 1 - N^{-D} \) on which the following estimate holds. For every \( u \) with \(|u| \leq t_2\) we have
\[ \left| \mathcal{U}_a^{(B)}(0, t_1 + u) - \frac{1}{N} p_{t_1}(\gamma_i^{(f)}, \gamma_a^{(f)}) \right| \leq N \frac{N^{\epsilon_2}}{N t_1} \left( \frac{(N t_1)^2}{\ell^2} + \frac{1}{(N t_1)^{1/10}} + \frac{1}{N^2 \omega_0/3} \right) + N \frac{N^{\epsilon_2/2}}{N t_1}. \] (3.91)

In the remainder of Section 3.6 we will work under the assumption that (I)-(III) hold. The proof of the following lemma is deferred to Section 5.

**Lemma 3.11.** Let \( \epsilon_1 > 0 \) and \( D_1 > 0 \). We have for \( N^{-D_1} \leq t \leq N^{-\epsilon_1} \eta_\ell \),
\[ p_t(x, y) \leq C \frac{t}{(x - y)^2 + t^2}. \] (3.92)

For any \( \epsilon_2 > 0 \) if \(|x - y| > N^{\epsilon_2} \eta_\ell \) and \( N^{-D_1} \leq t \leq N^{-\epsilon_1} \eta_\ell \),
\[ p_t(x, y) \leq \frac{1}{N D_2} \] (3.93)
for any \( D_2 > 0 \).

For spatial derivatives we have, for \( N^{-D_1} \leq t \leq N^{-\epsilon_1} \eta_\ell \),
\[ p_t^{(k)}(x, y) \leq C \frac{1}{t \ell} p_t(x, y) + \frac{1}{N D_2}, \] (3.94)
and
\[ p_t^{(k)}(x, y) \leq \frac{1}{N D} \] (3.95)
for any \( D_2 \) if \(|x - y| > N^{\epsilon_2} \eta_\ell \).

For the time derivative we have for \( N^{-D_1} \leq t \leq N^{-\epsilon_1} \eta_\ell \),
\[ |	ilde{\gamma}_t p_t(x, y)| \leq \frac{C}{x^2 + y^2} + N^{-D_2}. \] (3.96)

**Remark.** The short time cut-off \( t \geq N^{-D_1} \) is technical. In our application we will only take \( p_t \) with \( t \geq N^{-1} \).

We need to introduce two auxiliary scales \( s_0 \) and \( s_1 \). They will satisfy
\[ N^{-1} \ll s_0 \ll s_1 \ll t_1 \ll t_0. \] (3.97)

Define now
\[ f(x, t) = \sum_j \frac{1}{N} p_{s_0 + t - s_1}(x, \gamma_j^{(f)}) w_j(s_1), \] (3.98)
\[ f_i(t) := f(\hat{z}_i(t, \alpha), t). \] (3.99)

We are going to compare \( w_i(t) \) to \( f_i(t) \). A more natural choice would perhaps be \( f_i(t) = p_t(\hat{z}_i, \gamma_j^{(f)}) \). We explain here the motivation for the above choice of \( f_i \) and the introduction of the scales \( s_1 \) and \( s_0 \). Our method relies on differentiating the \( \ell^2 \) norm of the difference \( w_i - f_i \), and then integrating it back. We therefore require an estimate on the \( \ell^2 \) norm of the difference at the beginning endpoint of the time interval over which the integration occurs. One choice could be \( t = 0 \) for the start point of this interval. However, \( w_i \) is quite singular at this point so we allow it to evolve for a short time \( s_1 \) before comparing \( f_i \) to \( w_i \). At this point one might want to take \( f(t) = p_{t-s_1} \ast w(s_1) \). However at \( t = s_1 \), the kernel \( p_{t-s_1} \) is a \( \delta \)-function and so this convolution operation does not make sense. We therefore introduce the regularizing scale \( s_0 \). By the standard energy estimate \( w(s_1) \) has some smoothness on the scale \( s_1 \). This allows us to control the \( \ell^2 \) distance between \( w(s_1) \) and its convolution with the approximate \( \delta \)-function \( p_{s_0} \).

An additional technical complication is that the standard energy estimate involves a time average and so we will have to average the startpoint \( s_1 \) over the interval \([s_1, 2s_1]\).

We have the normalization condition
\[ \sum_j \frac{1}{N} p_t(\hat{z}_i, \gamma_j^{(f)}) = 1 + O((Nt)^{-1}) \] (3.100)
and also for \( \ell_1 \gg Nt \),
\[ \sum_{|i-j| \leq \ell_1} \frac{1}{N} p_t(\hat{z}_i, \gamma_j^{(f)}) = 1 + O((Nt)^{-1}) + O((Nt)/\ell_1) \] (3.101)
and
\[ \sum_{|i-j| > \ell_1} \frac{1}{N} p_t(\hat{z}_i, \gamma_j^{(f)}) \leq C \frac{Nt}{\ell_1} \] (3.102)
The following lemma provides an estimate on the \( \ell^2 \) norm of the difference \( w - f \). The error is in terms of the scales \( s_0, \ell \) and \( s_1 \) as well as a quantity which can be controlled via the standard energy estimate for \( w \).

**Lemma 3.12.** Let \( w \) be as in (3.84) and \( f \) as in (3.98). For any \( \varepsilon_1 > 0 \) and \( \varepsilon_2 > 0 \) and \( D > 0 \) there is for each \( \alpha \) an event \( \mathcal{F}_\alpha \) with \( \mathbb{P}[\mathcal{F}_\alpha] \geq 1 - N^{-D} \) on which the following holds,
\[ ||w(s_1) - f(s_1)||_2^2 \leq s_0 C \sum_{|i| \leq N^\omega_{A-B} + N^{\omega_2}} \sum_{|i-j| \leq \ell} \frac{(w_i(s_1) - w_j(s_1))^2}{(i-j)^2} + N^{\varepsilon_1} \left( \frac{1}{(Ns_0)^2} + \frac{(Ns_0)^2}{\ell^2} \right) \frac{1}{s_1}. \] (3.103)

**Proof.** For notational simplicity let
\[ \sum' := \sum_{|i| \leq N^\omega_{A-B} + N^{\omega_2}} \sum. \] (3.104)
We also drop the argument \( s_1 \) and write \( w_i = w_i(s_1) \), \( f_i = f_i(s_1) \). With overwhelming probability we have,
\[ \frac{1}{N} \sum_i (w_i - f_i)^2 = \frac{1}{N} \sum' \left( w_i - \sum_j \frac{p_{s_0}(\hat{z}_i, \gamma_j^{(f)})}{N} w_j \right)^2 + O(N^{-D}) \]
\[ \leq \frac{C}{N} \sum' \left( \sum_{|i-j| \leq \ell} \frac{p_{s_0}(\hat{z}_i, \gamma_j^{(f)})}{N} (w_i - w_j) \right)^2 + C \left( \frac{1}{(Ns_0)^2} + \frac{(Ns_0)^2}{\ell^2} \right) ||w||_2^2 \] (3.105)
In the first line we used the decay estimates from Lemmas 3.8 and 3.11 to change the sum from $\sum$ to $\sum'$. In the inequality we used the normalization condition (3.101), as well as (3.102) which together with Young’s inequality shows that

$$\frac{1}{N} \sum_i \left( \sum_{|j-i| \geq \ell} \frac{1}{N} p_i(\tilde{z}_i, \gamma_j^{(f)}) w_j \right)^2 \leq C \frac{(N \ell)^2}{\ell^2} \|w\|_{L^2}^2. \quad (3.106)$$

We then bound the first term in (3.105) by

$$\frac{1}{N} \sum_i \left( \sum_{|j-i| \leq \ell} \frac{p_{s_0}(\tilde{z}_i, \gamma_j^{(f)})}{N} (w_i - w_j) \right)^2 \leq \frac{1}{N} \sum_i \left( \sum_{|j-i| \leq \ell} \frac{p_{s_0}^2(\tilde{z}_i, \gamma_j^{(f)}) |i-j|^2}{N^2} \right) \left( \sum_{|j-i| \leq \ell} \frac{(w_i - w_j)^2}{|i-j|^2} \right). \quad (3.107)$$

We have

$$\sum_{j:|j-i| \leq \ell} \frac{p_{s_0}^2(x_i, \gamma_j^{(f)}) |i-j|^2}{N^2} \leq CN \int \frac{(s_0)^2 x^2}{(x^2 + (s_0)^2)^2} dx \leq CN s_0. \quad (3.108)$$

Lastly we can estimate the $\ell^2$ norm of $w$ using Lemma 3.9 by

$$\|w\|_{L^2}^2 \leq \frac{N^{\varepsilon_1}}{N} \sum_i \frac{(s_1)^2}{((t-a)/N)^2 + (s_1)^2} \leq \frac{N^{\varepsilon_1}}{s_1}. \quad (3.109)$$

These inequalities yield the claim.

The main calculation of the homogenization theorem is the following. We use the Ito lemma to differentiate $\|w - f\|_{L^2}^2$. Roughly what we find is that

$$d \|w - f\|_{L^2}^2 = -\|w - f\|_{H^{1/2}}^2 dt + \text{lower order} \quad (3.110)$$

where the lower order terms contains a martingale term as well as other errors\(^1\). Integrating this back gives us control over the homogeneous $H^{1/2}$ norm of $w - f$.

**Lemma 3.13.** Let $w$ be as in (3.84) and $f$ as in (3.98) with parameters $s_1$ and $s_0$. For $t \geq s_1$ we can write the Ito differential of $\|w(t) - f(t)\|_{L^2}$ in the form

$$d \frac{1}{N} \sum_i (w_i - f_i)^2 = -\langle w(t) - f(t), B(w(t) - f(t)) \rangle dt + X_t dt + dM_t \quad (3.111)$$

where $M_t$ is a martingale and $X_t$ is a process implicitly defined by the above equality. We have the following estimates for $X_t$ and $M_t$. Let $\varepsilon > 0$ and $D > 0$ be given. For each $\alpha$ there is an event $\mathcal{F}_\alpha$ with $\mathbb{P}[\mathcal{F}_\alpha] \geq 1 - N^{-D}$ on which the following estimates hold. For $s_1 \leq t \leq 9t_1$ we have for $X_t$

$$|X_t| \leq \frac{1}{5} \langle w - f, B(w - f) \rangle + \frac{C}{t + s_1} \left( \frac{1}{\sqrt{N(t-s_1+s_0)}} + \frac{1}{N^{\alpha}} \right). \quad (3.112)$$

For any $u_1$ and $u_2$ with $9t_1 > u_2 > u_1 \geq s_1$ we have

$$\left| \int_{u_1}^{u_2} dM_t \right| \leq \frac{N^{\varepsilon}}{N} \frac{1}{(u_1 + s_1)^{3/2}} \frac{1}{(u_1 - s_1 + s_0)^{1/2}}. \quad (3.113)$$

\(^1\) A similar idea was independently discovered in a forthcoming work by Jun Yin and Antti Knowles [49].
Proof. The estimates
\[
f'(t, \hat{z}_i) \leq \frac{C}{(t - s_1 + s_0)} f(t, \hat{z}_i) + N^{-D} \quad f''(t, \hat{z}_i) \leq \frac{C}{(t - s_1 + s_0)^2} f(t, \hat{z}_i) + N^{-D}
\] (3.114)
and
\[
f(t, \hat{z}_i) \leq \frac{C}{t - s_1 + s_0}
\] (3.115)
are immediate corollaries of Lemma 3.11. Since \(|w(s_1)|_\infty \leq C(s_1)^{-1}\) we obtain
\[
f(t, \hat{z}_i) \leq \frac{C}{t + s_1}
\] (3.116)
In the calculations below we implicitly use that for any \(\varepsilon_1 > 0\),
\[
|f_i| \leq \frac{1}{ND}, \quad w_i \leq \frac{1}{ND}, \quad \text{if } |i| \geq |a| + N^{\omega_\ell + \varepsilon_1}
\] (3.117)
with overwhelming probability by the finite speed estimates of Lemma 3.8 above. For example by our assumption on \(a\) this holds for
\[
|i| > N^{\omega_\ell - \varepsilon} + N^{\omega_\ell + \varepsilon_1}, \quad \text{or } |i| > N^{\omega_\ell}.
\] (3.118)
It will be convenient to use the notation \(f_i^{(k)} := f^{(k)}(\hat{z}_i)\). It is clear that similar estimates to (3.117) hold for the derivatives \(f'_i\) and \(f''_i\) and \((\partial_t f)_i\).

We calculate by the Ito formula,
\[
d \frac{1}{N} \sum_i (w_i - f_i)^2 \\
= \frac{2}{N} \sum_i (w_i - f_i) \left[ \partial_t w_i dt - (\partial_t f)(t, \hat{z}_i) dt - f'_i(t, \hat{z}_i) d\hat{z}_i - f''_i(t, \hat{z}_i) \frac{dt}{2N} \right] + (f'(t, \hat{z}_i))^2 \frac{dt}{2N}.
\] (3.119)
Let us start with the Ito terms. Using (3.114) and (3.116) we obtain
\[
\left| \frac{1}{N^2} \sum_i (f'_i)^2 \right| \leq \frac{C}{N(t + s_0 - s_1)^2} \frac{1}{t + s_1} \frac{1}{N^2} \sum_i f_i \leq \frac{C}{N(t + s_0 - s_1)^2} \frac{1}{t + s_1}
\] (3.120)
and similarly,
\[
\left| \frac{1}{N^2} \sum_i (w_i - f_i) f''_i \right| \leq \frac{C}{N(t + s_0 - s_1)^2} \frac{1}{t + s_1} \frac{1}{N^2} \sum_i w_i + f_i \leq \frac{C}{N(t + s_0 - s_1)^2} \frac{1}{t + s_1}.
\] (3.121)
We write
\[
\sum_i (w_i - f_i)(\partial_t w_i - (\partial_t f)_i) = \frac{1}{N} \sum_i (w_i - f_i) \left( \frac{A_{\partial w}(i)}{N} \sum_j \frac{w_j - w_i}{(\hat{z}_j - \hat{z}_i)^2} - \frac{A_{\partial w}(i)}{N} \sum_j \frac{f_j - f_i}{(\hat{z}_j - \hat{z}_i)^2} \right)
\] (3.122)
\[
+ \frac{1}{N} \sum_i (w_i - f_i) \left( \frac{A_{\partial w}(i)}{N} \sum_j \frac{f_j - f_i}{(\hat{z}_j - \hat{z}_i)^2} - \int_{|\hat{z}_j - y| \leq \eta_\ell} \frac{f(y) - f(\hat{z}_j)}{(\hat{z}_j - y)^2} \rho_{\omega_\ell}(0) dy \right)
\] (3.123)
The term (3.122) equals
\[
\sum_i (w_i - f_i) \left( \frac{A_{\partial w}(i)}{N} \sum_j \frac{w_j - w_i}{(\hat{z}_j - \hat{z}_i)^2} - \frac{A_{\partial w}(i)}{N} \sum_j \frac{f_j - f_i}{(\hat{z}_j - \hat{z}_i)^2} \right) = -\frac{1}{2} \langle w - f, B(w - f) \rangle.
\] (3.124)
Note that this term is negative and is the first term appearing on the RHS of (3.111). It will be used to account for terms on which we cannot use rigidity. For \(0 < \omega_{\ell,2} < \omega_{\ell}\) define
\[
A_2 := \{(i,j) : |i - j| \leq N^{\omega_{\ell,2}} \} \cup \{(i,j) : ij > 0, i, j \notin \mathcal{C}_{q_n}\}.
\] (3.125)
We write the term (3.123) as

\[
\frac{1}{N} \sum_i (w_i - f_i) \left( \frac{1}{N} \sum_j A_{q^*,(i)} f_j - f_i \right) \int_{|z_i - y| \leq \eta} \frac{f(y) - f(\hat{z}_i)}{(\hat{z}_i - y)^2} \rho_{\text{sc}}(0) dy
\]

\[
= \frac{1}{N} \sum_i (w_i - f_i) \left( \frac{1}{N} \sum_j A_{q^*,(i)} f_j - f_i \right) \left( \frac{1}{N} \sum_j \frac{f_j - f_i}{(\hat{z}_j - \hat{z}_i)^2} \right)
\]

\[
+ \frac{1}{N} \sum_i (w_i - f_i) \left( \frac{1}{N} \sum_j A_{q^*,(i)} \frac{f_j - f_i}{(\hat{z}_j - \hat{z}_i)^2} - \int_{|z_i - y| \leq \eta} \frac{f(y) - f(\hat{z}_i)}{(\hat{z}_i - y)^2} \rho_{\text{sc}}(0) dy \right)
\]

(3.126)

We first deal with (3.126). We will later use rigidity to deal with (3.127). Write

\[
v_i := w_i - f_i.
\]

Using a second order Taylor expansion for \( f_i \) we have for \( |i| \leq N^{\omega \alpha} \),

\[
\frac{1}{N} \sum_j A_{q^*,(i)} f_j - f_i \left( \frac{1}{N} \sum_j \frac{f_j - f_i}{(\hat{z}_j - \hat{z}_i)^2} \right) = \frac{1}{N} \sum_j A_{q^*,(i)} \frac{f_j'}{\hat{z}_j - \hat{z}_i} + O(\eta).
\]

(3.129)

To estimate the remainder term we used the fact that \( ||f''||_{\infty} \leq C(t - s_1 + s_0)^{-2}(t + s_1)^{-1} \) as well as the fact that since \( |i| \leq N^{\omega \alpha} \), the cardinality of \( \{ j : (j, i) \in A_2 \} \) is less than \( CN^{\omega \alpha \epsilon} \). For \( |i| > N^{\omega \alpha} \) we just use

\[
\frac{1}{N} \sum_j A_{q^*,(i)} f_j - f_i \left( \frac{1}{N} \sum_j \frac{f_j - f_i}{(\hat{z}_j - \hat{z}_i)^2} \right) = \frac{1}{N} \sum_j A_{q^*,(i)} \frac{f_j'}{\hat{z}_j - \hat{z}_i} + O(N^\epsilon).
\]

(3.130)

Using (3.129) and (3.130) and the estimate (3.117) we can write the term (3.126) as

\[
\frac{1}{N} \sum_i (w_i - f_i) \left( \frac{1}{N} \sum_j A_{q^*,(i)} f_j - f_i \right) = \frac{1}{N} \sum_i v_i \left( \frac{1}{N} \sum_j A_{q^*,(i)} f_j' \right) \left( \frac{1}{N} \sum_j \frac{f_j'}{\hat{z}_j - \hat{z}_i} \right) + O\left( \frac{N^{\omega \epsilon}}{N(t - s_1 + s_0)^2 t + s_1} \right).
\]

(3.131)

We then write the first term on the RHS of (3.131) as

\[
\frac{1}{N^2} \sum_i v_i \left( \frac{1}{N} \sum_j A_{q^*,(i)} f_j' \right) \left( \frac{1}{N} \sum_j \frac{f_j'}{\hat{z}_j - \hat{z}_i} \right) = \frac{1}{2 N^2} \sum_{(i,j) \in A_2} \frac{(v_i - v_j)f_j' + v_j(f_j' - f_i')}{\hat{z}_j - \hat{z}_i}.
\]

(3.132)

Using again the estimates (3.117) we bound the second term by

\[
\left| \frac{1}{2 N^2} \sum_{(i,j) \in A_2} \frac{(v_i - v_j)f_j' - f_i'}{\hat{z}_j - \hat{z}_i} \right| \leq \left| \frac{1}{2 N^2} \sum_{|i-j| \leq N^{\omega \epsilon}, |j| \leq 2N^{\omega \alpha}} \frac{v_j(f_j' - f_j')}{\hat{z}_j - \hat{z}_i} \right| + N^{-D}
\]

\[
\leq C \frac{N^{\omega \epsilon}}{N(t - s_1 + s_0)^2 t + s_1}.
\]

(3.133)

The second inequality used (3.114). We bound the first term on the RHS of (3.132) using Schwarz by

\[
\left| \frac{1}{2 N^2} \sum_{(i,j) \in A_2} \frac{(v_i - v_j)f_j'}{\hat{z}_j - \hat{z}_i} \right| \leq \frac{1}{10 N^2} \sum_{(i,j) \in A_2} \frac{(v_i - v_j)^2}{(\hat{z}_j - \hat{z}_i)^2} + C \frac{N^{\omega \epsilon}}{N(t - s_1 + s_0)^2 t + s_1} \sum_j 1
\]

\[
\leq \frac{1}{10 N^2} \sum_{(i,j) \in A_2} \frac{(v_i - v_j)^2}{(\hat{z}_j - \hat{z}_i)^2} + C \frac{N^{\omega \epsilon}}{N(t - s_1 + s_0)^2 t + s_1}
\]

(3.134)
where we used again the decay estimate (3.117), the estimate for \( f_j^t \) and the fact that the cardinality of the set \( \{ j : (i,j) \in A_2 \} \) is bounded by \( CN^{\omega^t} \) for \( |i| \leq N^{\omega^t} \). The first term will be absorbed into the \( \langle (w-f), B(w-f) \rangle \) term.

In summary, the estimates (3.131)-(3.134) prove that for the term (3.126) we have

\[
\left| \frac{1}{N} \sum_i (w_i - f_i) \left( \frac{1}{N} \sum_j A_{2, (i)} \frac{f_j - f_i}{(z_i - \hat{z}_i)^2} \right) \right| \leq \frac{1}{10} N^2 \sum_{(i,j) \in A_2} \frac{(v_i - v_j)^2}{(z_i - \hat{z}_j)^2} + C \frac{N^{\omega^t} \cdot 1}{N(t - s_1 + s_0)^2 t + s_1}. \tag{3.135} \]

In order to complete the bound of (3.123) we need to estimate (3.127). This term will be estimated by rigidity. Due to the decay estimates (3.117) we can safely ignore the terms with \( |i| > N^{\omega^t} \); i.e., for \( |i| > N^{\omega^t} \) the term inside the brackets is estimated by

\[
\left| \frac{1}{N} \sum_j A_{2, (i)} \frac{f_j - f_i}{(z_i - \hat{z}_j)^2} - \int_{|\hat{z}_i - y| \leq \eta} \frac{f(y) - f(\hat{z}_i)}{(z_i - y)^2} \rho_{sc}(0) dy \right| \leq N^{\omega^t}. \tag{3.136} \]

For the terms with \( |i| \leq N^{\omega^t} \) we use the rigidity estimates (3.72) of Section 3.5 (II). We write

\[
\left| \frac{1}{N} \sum_j A_{2, (i)} \frac{f_j - f_i}{(z_i - \hat{z}_j)^2} - \int_{|\hat{z}_i - y| \leq \eta} \frac{f(y) - f(\hat{z}_i)}{(z_i - y)^2} \rho_{sc}(0) dy \right| \leq C N^\varepsilon \frac{1}{N^{\omega^t} \cdot 1} \frac{1}{t - s_1 + s_0 \cdot 1}. \tag{3.139} \]

The term (3.138) is estimated using a second order Taylor expansion. We write it as

\[
\int_{|y - \hat{z}_i| \leq \eta} \frac{f(y) - f(\hat{z}_i)}{(z_i - y)^2} \rho_{sc}(0) dy = \int_{|\hat{z}_i - y| \leq \eta} \frac{f'(\hat{z}_i)}{(z_i - \hat{z}_i)} \rho_{sc}(0) dy + O \left( \frac{N^{\omega^t} \cdot 1}{N(t - s_1 + s_0)^2 t + s_1} \right). \tag{3.140} \]

We then have that

\[
\int_{|y - \hat{z}_i| \leq \eta} \frac{f'(\hat{z}_i)}{(z_i - \hat{z}_i)} \rho_{sc}(0) dy = 0. \tag{3.141} \]

Combining (3.117) with (3.127) for the terms with \( |i| > N^{\omega^t} \) and then (3.139) and (3.140) for the remaining terms yields the following estimate for (3.127).

\[
\left| \frac{1}{N} \sum_i (w_i - f_i) \left( \frac{1}{N} \sum_j A_{2, (i)} \frac{f_j - f_i}{(z_i - \hat{z}_i)^2} - \int_{|\hat{z}_i - y| \leq \eta} \frac{f(y) - f(\hat{z}_i)}{(z_i - y)^2} \rho_{sc}(0) dy \right) \right| \leq C N^\varepsilon \frac{1}{N^{\omega^t} \cdot 1} \frac{1}{t - s_1 + s_0 \cdot 1} + C \frac{N^{\omega^t} \cdot 1}{N(t - s_1 + s_0)^2 t + s_1}. \tag{3.142} \]

In summary, we see that (3.124) and the estimates (3.135) and (3.142) imply

\[
\frac{1}{N} \sum_i (w_i - f_i)(\hat{c}_t w_i - (\hat{c}_t f)_i) = -\frac{1}{2} \langle w-f, B(w-f) \rangle + Y_i \tag{3.143} \]
where
\[ |Y_t| \leq \frac{1}{10} \langle w - f, B(w - f) \rangle + C N^{\varepsilon} N^{\omega_{t,2}} \frac{1}{t - s_1 + s_0} \frac{1}{t + s_1} + C \frac{N^{\omega_{t,2}}}{N(t - s_1 + s_0)^2} \frac{1}{t + s_1}. \]  

(3.144)

The remaining term to deal with is
\[
\frac{1}{N} \sum_i (w_i - f_i) f_i' d\hat{z}_i = d M_t + \frac{1}{N} \sum_i (w_i - f_i) f_i' \left\{ \frac{1}{N} \sum_j A_{2(i)} \frac{1}{\hat{z}_i - \hat{z}_j} + \frac{1}{N} \sum_j A_{4 \setminus A_2(i)} \frac{1}{\hat{z}_i - \hat{z}_j} + 1 \right\} dt.
\]

(3.145)

The martingale term is
\[
d M_t = \frac{1}{N} \sum_i (w_i - f_i) f_i' \sqrt{\frac{2}{N}} dB_i
\]

(3.146)

which we estimate later. The first non-martingale term appearing on the RHS of (3.145) is identical to (3.132) (we comment here that they actually appear with the same sign and so do not cancel as one might hope) and so we have, proceeding as above,
\[
\left| \frac{1}{N} \sum_i (w_i - f_i) f_i' \frac{1}{N} \sum_j A_{2(i)} \frac{1}{\hat{z}_i - \hat{z}_j} \right| \leq \frac{1}{10} \langle (w - f), B(w - f) \rangle + C \frac{N^{\omega_{t,2}}}{N(t - s_1 + s_0)^2} \frac{1}{t + s_1}.
\]

(3.147)

Using (3.117) and (3.74) we can drop all terms in (3.145) with \(|i| > N^{\omega_A} \); i.e., we have with overwhelming probability
\[
\left| \sum_{|i| > N^{\omega_A}} (w_i - f_i) f_i' \left\{ \frac{1}{N} \sum_j A_{4 \setminus A_2(i)} \frac{1}{\hat{z}_i - \hat{z}_j} + J_i \right\} \right| \leq \frac{1}{N^D}.
\]

(3.148)

For the \( F_i \) terms we have by (3.75) with overwhelming probability,
\[
\left| \sum_{|i| \leq N^{\omega_A}} (w_i - f_i) f_i' F_i \right| \leq C \frac{N^{\varepsilon}}{N^{\omega_F} (t - s_1 + s_0)(t + s_1)}
\]

(3.149)

For \(|i| \leq N^{\omega_A} \), since
\[
0 = \int_{\eta_{r,2}} \rho_{\varepsilon}(0) dy,
\]

(3.150)

we have by the rigidity estimate (3.72)
\[
\left| \frac{1}{N} \sum_j A_{4 \setminus A_2(i)} \frac{1}{\hat{z}_i - \hat{z}_j} \right| = \left| \frac{1}{N} \sum_j A_{4 \setminus A_2(i)} \frac{1}{\hat{z}_i - \hat{z}_j} - \int_{\eta_{r,2} \leq |y - \hat{z}_i| \leq \eta_{r}} \frac{1}{\hat{z}_i - \hat{z}_j} - \rho_{\varepsilon}(0) dy \right| \leq C \frac{N^{\varepsilon}}{N^{\omega_{r,2}}}.
\]

(3.151)

Hence, ignoring the martingale term (and slightly abusing notation) we have obtained the following bound for (3.145).
\[
\left| \frac{1}{N} \sum_i (w_i - f_i) f_i' d\hat{z}_i \right| \leq \frac{\langle (w - f), B(w - f) \rangle}{10} + C \frac{N^{\varepsilon}}{t + s_1 \left(t - s_1 + s_0\right)} \left( \frac{N^{\omega_{t,2}}}{N(t - s_1 + s_0)} + \frac{1}{N^{\omega_{r,2}}} + \frac{1}{N^{\omega_F}} \right)
\]

(3.152)

for any \( \varepsilon > 0 \). The equality (3.111) and estimate (3.112) follow from (3.120), (3.121), (3.143), (3.144) and (3.152), after optimizing and choosing \( N^{\omega_{t,2}} \approx \sqrt{N(t - s_1 + s_0)} \).

The quadratic variation of the martingale term satisfies
\[
d \langle M \rangle t = \frac{1}{N^3} \sum_i (w_i - f_i)^2 (f_i')^2 dt \leq C \frac{1}{N^2} \frac{1}{(t + s_1)^3} \frac{1}{(t - s_1 + s_0)^2} dt
\]

(3.153)
with overwhelming probability. Hence by the BDG inequality,
\[
E \left[ \sup_{u_2:9t_1 \geq u_2 \geq u_1} \left| \int_{u_1}^{u_2} dM_t \right|^p \right] \leq C_p \frac{1}{N^p} \frac{1}{(u_1 + s_1)^{3p/2}} \frac{1}{(u_1 - s_1 + s_0)^{p/2}} \tag{3.154}
\]
and so
\[
\sup_{u_2:9t_1 \geq u_2 \geq u_1} \left| \int_{u_1}^{u_2} dM_t \right| \leq \frac{N^\varepsilon}{N} \frac{1}{(u_1 + s_1)^{3/2}} \frac{1}{(u_1 - s_1 + s_0)^{1/2}} \tag{3.155}
\]
with overwhelming probability. A simple argument using a union bound over \(u_1\) in a set of cardinality at most \(N^2\) extends this estimate to all \(s_1 < u_1 < u_2 < 9t_1\). This yields (3.113).

Lemma 3.13 yields the following corollary.

**Corollary 3.14.** Let \(w\) be as in (3.84) and \(f\) as in (3.98) with parameters \(s_1\) and \(s_0\). Let \(\varepsilon > 0\) and \(D > 0\). For each \(\alpha\) there is an event \(\mathcal{F}_{\alpha}\) with \(\mathbb{P}[\mathcal{F}_{\alpha}] \geq 1 - N^{-D}\) on which the following holds.
\[
\int_{s_1}^{2t_1} \langle (w - f), \mathcal{B}(w - f) \rangle ds \leq \| (w - f)(s_1) \|_2^2 + \frac{N^\varepsilon}{s_1} \left\{ \frac{1}{(N s_0)^{1/2}} + \frac{1}{N^\omega_F} \right\} \tag{3.160}
\]

Putting together the last two lemmas yields the following homogenization theorem. It is essentially Theorem 3.10 but with a time average. In the next subsection we will remove the time average.

**Theorem 3.15.** Let \(a\) and \(i\) satisfy
\[
|a| \leq N^{\omega \Lambda - \varepsilon_B}, \quad |i - a| \leq \ell/10. \tag{3.157}
\]

For any \(\varepsilon > 0\) and \(D > 0\) there is an event \(\mathcal{F}_{\alpha}\) with \(\mathbb{P}[\mathcal{F}_{\alpha}] \geq 1 - N^{-D}\) on which
\[
\frac{1}{t_1} \int_0^{t_1} \left( U_{t_1+u}^{(B)}(i,a) - \frac{1}{N} p_{t_1+u}(\gamma_i^f, \gamma_a^f) \right)^2 du \leq \frac{N^\varepsilon}{(N t_1)^2} \left\{ \frac{(N t_1)^2}{\ell^2} + \frac{s_1^2}{s_1} + \frac{t_1}{s_1} \left( \frac{1}{(N s_0)^{1/2}} + \frac{1}{N^\omega_F} + \frac{s_0}{s_1} \right) \right\} \tag{3.158}
\]

**Proof.** Define \(w\) and \(f\) as in (3.84) and (3.98), except replace \(s_1\) by an auxiliary \(s_1' \in [s_1, 2s_1]\). The reason for doing this is that will eventually have to average \(s_1'\) over \([s_1, 2s_1]\).

We estimate for \(0 \leq u \leq t_1\),
\[
\left( U_{t_1+u}^{(B)}(i,a) - \frac{1}{N} p_{t_1+u}(\gamma_i^f, \gamma_a^f) \right)^2 \leq C \left( \frac{1}{N} w_{t_1+u}(i) - \frac{1}{N} f_{t_1+u}(i) \right)^2 \tag{3.159}
\]
\[
+ C \left( \frac{1}{N} p_{t_1+u}(\gamma_i^f, \gamma_a^f) - \frac{1}{N} p_{t_1+u-s_1'+s_0}(\hat{z}_i, \gamma_a^f) \right)^2 \tag{3.160}
\]
\[
+ C \left( \frac{1}{N} p_{t_1+u-s_1'+s_0}(\hat{z}_i, \gamma_a^f) - \frac{1}{N} f_{t_1+u}(i) \right)^2 \tag{3.161}
\]
The terms (3.160) and (3.161) are estimated using essentially the regularity of \(p_t(x, y)\). The remaining term is estimated using the last corollary.

We can estimate the term (3.160) using the results from Lemma 3.11 and the optimal rigidity estimate (3.72) from Section 3.5 (II). We obtain,
\[
\left( \frac{1}{N} p_{t_1+u}(\gamma_i^f, \gamma_a^f) - \frac{1}{N} p_{t_1+u-s_1'+s_0}(\hat{z}_i, \gamma_a^f) \right)^2 \leq C \left( \frac{1}{N} p_{t_1+u}(\gamma_i^f, \gamma_a^f) - \frac{1}{N} p_{t_1+u}(\hat{z}_i, \gamma_a^f) \right)^2 \tag{3.162}
\]
\[
+ C \left( \frac{1}{N} p_{t_1+u}(\gamma_i^f, \gamma_a^f) - \frac{1}{N} p_{t_1+u-s_1'+s_0}(\gamma_i^f, \gamma_a^f) \right)^2 \leq \frac{N^\varepsilon}{(N t_1)^2} + \frac{C}{(N t_1)^2} \frac{s_1^2}{t_1}. \tag{3.162}
\]
For (3.161) we have, using the normalization $N^{-1} \sum w(j) = 1$, the profile from Lemma 3.9, and the estimates from Lemma 3.11,

$$
\left( \frac{1}{N} p_{t_1+u-s_1+so}(\tilde{z}_i, \gamma_a(j)) - \frac{1}{N} f_{t_1+u}(i) \right)^2
\leq N^\varepsilon \left( \frac{1}{N^2} \sum_j w_s(j) \left( p_{t_1+u-s_1+so}(\tilde{z}_i, \gamma_a(j)) - p_{t_1+u-s_1+so}(\tilde{z}_i, \gamma_j(j)) \right) \right)^2
\leq CN^\varepsilon \left( \frac{1}{Nt_1} \int_{|x| \leq t_1} \frac{s_1|x|}{x^2 + s_1^2} dx \right)^2 + CN^\varepsilon \left( \frac{1}{Nt_1} \int_{|x| > t_1} \frac{s_1}{s_1^2 + x^2} dx \right)^2 \leq \frac{CN^2\varepsilon}{(Nt_1)^2} \frac{s_1^2}{t_1^2}. \tag{3.163}
$$

Above, we used the fact that

$$
|p_{t_1+u-s_1+so}(\tilde{z}_i, \gamma_a(j)) - p_{t_1+u-s_1+so}(\tilde{z}_i, \gamma_j(j))| \leq \frac{C}{t_1} \min \left\{ \left| \frac{j-a}{Nt_1} \right|, 1 \right\}. \tag{3.164}
$$

By a Sobolev inequality whose proof we defer to Appendix D we have

$$
\left( \frac{1}{N} w_{t_1+u(i)} - \frac{1}{N} f_{t_1+u}(i) \right)^2 \leq \frac{N^\varepsilon}{N^2} \langle (w - f)(t_1 + u), B(w - f)(t_1 + u) \rangle + C \left( \frac{1}{N^2} \sum_{|j-i| \leq \ell} w_{t_1+u(j)} - \frac{1}{N} \sum_{|j-i| \leq \ell} f_{t_1+u}(j) \right)^2. \tag{3.165}
$$

We have

$$
\frac{1}{N} \sum_{|j-i| \leq \ell} w_{t_1+u(j)} = \frac{1}{\ell} + O \left( \frac{Nt_1}{\ell^2} \right). \tag{3.166}
$$

Similarly, (using (3.100)) we have

$$
\frac{1}{N} \sum_{|j-i| \leq \ell} f_{t_1+u}(j) = \frac{1}{\ell} + O \left( \frac{1}{\ell Nt_1} + \frac{Nt_1}{\ell^2} \right). \tag{3.167}
$$

Hence,

$$
\left( \frac{1}{N} w_{t_1+u(i)} - \frac{1}{N} f_{t_1+u}(i) \right)^2 \leq \frac{N^\varepsilon}{N^2} \langle (w - f)(t_1 + u), B(w - f)(t_1 + u) \rangle + C \frac{1}{\ell^2(Nt_1)^2} + C \frac{(Nt_1)^2}{\ell^4}. \tag{3.168}
$$

We now apply Corollary 3.14 to obtain that there is an event with overwhelming probability (which depends on the choice of $s_1'$), such that

$$
\int_{s_1'}^{2t_1} \langle (w - f), B(w - f) \rangle ds \leq \frac{N^\varepsilon}{s_1} \left( \frac{1}{(Ns_0)^{1/2}} + \frac{1}{N^{\omega_F}} \right) + \| (w - f)(s_1') \|_2^2. \tag{3.169}
$$

We can average over $s_1' \in [s_1, 2s_1]$ (even though the event described above is $s_1'$-dependent, since each holds with overwhelming probability and $U^{(B)}$ and $p_t$ are bounded, we can apply Lemma E.1) and obtain that with overwhelming probability,

$$
\frac{1}{t_1} \int_0^{t_1} \left( U^{(B)}_{t_1+u}(i, a) - \frac{1}{N} p_{t_1+u}(\gamma_a(j), \gamma_a(j)) \right)^2 du \leq \frac{1}{N^2t_1s_1} \int_0^{s_1} \| (w - f)(s_1 + u) \|_2^2 du
+ \frac{CN^\varepsilon}{(Nt_1)^2} \left( \frac{1}{(Nt_1)^2} + \frac{s_1^2}{t_1^2} + \frac{1}{\ell^2} + \frac{(Nt_1)^4}{\ell^4} + \frac{t_1}{s_1(Ns_0)^{1/2}} + \frac{t_1}{s_1 N^{\omega_F}} \right). \tag{3.169}
$$
Note that on the RHS the choice of $f$ itself has an $s_1 + u$ dependence. By Lemma 3.12 we have

$$
\int_0^{s_1} ||(w-f)(s_1+u)||_2^2 du \leq N^\varepsilon \left( \frac{1}{(Ns_0)^2} + \frac{(Ns_0)^2}{\ell^2} \right)
$$

$$
+ C s_0 \int_0^{s_1} \sum_{|i| \leq N^\varepsilon A} \sum_{|i-j| \leq \ell} \frac{(w_i(s_1+u) - w_j(s_1+u))^2}{(i-j)^2} du.
$$

(3.170)

With overwhelming probability we have

$$
\int_0^{s_1} \sum_{|i| \leq N^\varepsilon A} \sum_{|i-j| \leq \ell} \frac{(w_i(s_1+u) - w_j(s_1+u))^2}{(i-j)^2} du \leq N^\varepsilon \int_0^{s_1} \langle w, Bw(s_1+u) \rangle du \leq N^\varepsilon ||w(s_1)||_2^2 \leq \frac{CN^{2\varepsilon}}{s_1}
$$

(3.171)

where in the second inequality we used the standard energy estimate $\partial_t ||w||_2^2 = -\langle w, Bw \rangle$. The claim now follows after simplifying the errors. In particular we simplify using

$$
\frac{1}{\ell^2} \leq \frac{(Ns_0)^2}{\ell^2} \leq s_0, \quad \frac{1}{(Ns_0)^2} \leq \frac{1}{s_1 (Ns_0)^{1/2}}
$$

(3.172)

3.6.1 Removal of time average

Let $\varepsilon_2 > 0$ and let

$$
t_2 := t_1 N^{-\varepsilon_2}.
$$

(3.173)

In this section we show how to remove the time average in Theorem 3.15. More precisely, we prove the following theorem. It is deduced from Theorem 3.15 using only the fact that $U^{(B)}$ is a semigroup, the decay properties of $U^{(B)}$ given by Lemma 3.9 and the regularity of $p_t(x, y)$.

**Theorem 3.16.** Let $a$ satisfy

$$
|a| \leq N^{\omega_A - \varepsilon_B} / 2
$$

(3.174)

and $i$ satisfy

$$
|i - a| \leq \frac{\ell}{20}.
$$

(3.175)

For any $\varepsilon > 0$ and $D > 0$ there is an event $\mathcal{F}_a$ with $P[\mathcal{F}_a] \geq 1 - N^{-D}$ on which

$$
\left| U^{(B)}_{t_1+2t_2} (i, a) - \frac{1}{N} P_{t_1} (\gamma_i^{(f)}, \gamma_a) \right|
$$

$$
\leq CN^\varepsilon \frac{N^{2\varepsilon}}{Nt_1} \left( \frac{s_1^2}{t_1^4} + \frac{(N t_1)^4}{\ell^4} + \frac{t_1}{s_1} \left( \frac{1}{(Ns_0)^{1/2}} + \frac{1}{N^{\omega_B} + \frac{s_0}{s_1}} \right) \right)^{1/2}
$$

$$
+ \frac{N^\varepsilon}{Nt_1} N^{-2\varepsilon/2}.
$$

(3.176)

**Proof.** Theorem 3.15 implies that we have with overwhelming probability

$$
\frac{1}{t_2} \int_0^{t_2} \left| U^{(B)}_{t_1+u} (j, k) - \frac{1}{N} P_{t_1+u} (\gamma_j^{(f)}, \gamma_k) \right| du 
$$

$$
\leq N^\varepsilon \frac{N^{2\varepsilon}}{Nt_1} \left( \frac{s_1^2}{t_1^4} + \frac{(N t_1)^4}{\ell^4} + \frac{t_1}{s_1} \left( \frac{1}{(Ns_0)^{1/2}} + \frac{1}{N^{\omega_B} + \frac{s_0}{s_1}} \right) \right)^{1/2},
$$

(3.177)

for $k \leq N^{\omega_A - \varepsilon_B}$ and $|j - k| \leq \ell/10$. For notational simplicity let us denote

$$
\Phi := N^{2\varepsilon} \left( \frac{s_1^2}{t_1^4} + \frac{(N t_1)^4}{\ell^4} + \frac{t_1}{s_1} \left( \frac{1}{(Ns_0)^{1/2}} + \frac{1}{N^{\omega_B} + \frac{s_0}{s_1}} \right) \right)^{1/2}
$$

(3.178)
By the semigroup property we can write for any $0 \leq u \leq t_2$,
\[
U_{ai}(B)(0, t_1 + 2t_2) = \sum_j U_{aj}(B)(t_1 + u, t_1 + 2t_2) U_{ji}(B)(0, t_1 + u)
\tag{3.179}
\]
and so we can take an average over $u$ and obtain
\[
U_{ai}(B)(0, t_1 + 2t_2) = \frac{1}{t_2} \int_0^{t_2} \sum_j U_{aj}(B)(t_1 + u, t_1 + 2t_2) U_{ji}(B)(0, t_1 + u) du.
\tag{3.180}
\]
We now rewrite the RHS as
\[
U_{ai}(B)(0, t_1 + 2t_2) = \sum_j \frac{1}{t_2} \int_0^{t_2} U_{aj}(B)(t_1 + u, t_1 + 2t_2) \left( U_{ji}(B)(0, t_1 + u) - \frac{1}{N} p_{t_1+u}(\gamma_j, \gamma_i) \right) du
\tag{3.181}
\]
\[
+ \frac{1}{t_2} \int_0^{t_2} U_{aj}(B)(t_1 + u, t_1 + 2t_2) \frac{1}{N} \left( p_{t_1+u}(\gamma_j, \gamma_i) - p_{t_1+u}(\gamma_j, \gamma_i) \right) du
\tag{3.182}
\]
\[
+ \sum_j \frac{1}{t_2} \int_0^{t_2} U_{aj}(B)(t_1 + u, t_1 + 2t_2) \left( \frac{1}{N} p_{t_1+u}(\gamma_a, \gamma_i) \right) du
\tag{3.183}
\]
We have the estimate
\[
|U_{aj}(B)(t_1 + u, t_1 + 2t_2)| \leq \frac{1}{N} \frac{N^2 t_2}{((a - j)^2 + t_2^2}
\tag{3.184}
\]
from which we see that for any $\delta > 0$,
\[
\sum_{j: |j - a| > N t_2 \delta} |U_{aj}(B)(t_1 + u, t_1 + 2t_2)| \leq \frac{N^\varepsilon}{N^\delta}
\tag{3.185}
\]
and also
\[
\sum_{j: |j - a| \leq N t_2 \delta} U_{aj}(B)(t_1 + u, t_1 + 2t_2) = 1 + N^\varepsilon O \left( \frac{1}{N^\delta} \right).
\tag{3.186}
\]
Fix a $\delta > 0$ s.t. $\delta < \varepsilon_2$. We also have the estimate
\[
|U_{ji}(B)(0, t_1 + u)| + \left| \frac{1}{N} p_{t_1+u}(\gamma_j, \gamma_i) \right| \leq \frac{N^\varepsilon}{N t_1}.
\tag{3.187}
\]
We use these to estimate the term (3.181) by
\[
\left| \sum_j \frac{1}{t_2} \int_0^{t_2} U_{aj}(B)(t_1 + u, t_1 + 2t_2) \left( U_{ji}(B)(0, t_1 + u) - \frac{1}{N} p_{t_1+u}(\gamma_j, \gamma_i) \right) du \right|
\leq \sum_{j: |j - a| > N t_2 \delta} \frac{1}{t_2} \int_0^{t_2} U_{aj}(B)(t_1 + u, t_1 + 2t_2) \frac{N^\varepsilon}{N t_1} du
\]
\[
+ \sum_{j: |j - a| \leq N t_2 \delta} \frac{N^2 t_2}{N ((j - a)^2/2 + t_2^2} \frac{1}{t_2} \int_0^{t_2} U_{ji}(B)(0, t_1 + u) - \frac{1}{N} p_{t_1+u}(\gamma_j, \gamma_i) du
\]
\[
\leq \frac{N^2 \Phi}{N^2 t_1^2 \delta} + \frac{N^2 \varepsilon}{N t_1} 
\tag{3.188}
\]
Note that we are allowed to apply the estimate (3.177) because $|j - a| \leq N t_2 \delta < \ell$ which implies $|j - i| < \ell/10$. For $|j - a| \leq N^\delta (N t_2)$ we have the estimate
\[
\frac{1}{N} \left| p_{t_1+u}(\gamma_j, \gamma_i) - p_{t_1+u}(\gamma_a, \gamma_i) \right| \leq \frac{N^\varepsilon}{N t_1} \frac{N^2 \delta}{N t_1}.
\tag{3.189}
\]
Therefore we can estimate (3.182) by

\[ \left| \sum_{j} \frac{1}{t_2} \int_{t_0}^{t_2} U_{a_j}^{(B)}(t_1 + u, t_1 + 2t_2) \frac{1}{N} \left( p_{t_1 + u}(\gamma_{i_1}^{(f)}, \gamma_{i_1}^{(i)}) - p_{t_1 + u}(\gamma_{a}^{(f)}, \gamma_{a}^{(i)}) \right) du \right| \]

\[ \leq \sum_{j:|j-a| > Nt_2N^\delta} \frac{1}{t_2} \int_{t_0}^{t_2} U_{a_j}^{(B)}(t_1 + u, t_1 + 2t_2) \frac{N^\varepsilon}{Nt_1} du \]

\[ + \sum_{j:|j-a| \leq Nt_2N^\delta} \frac{1}{t_2} \int_{t_0}^{t_2} U_{a_j}^{(B)}(t_1 + u, t_1 + 2t_2) N^\varepsilon \frac{1}{Nt_1} \frac{1}{Nt_1} du \]

\[ \leq N^{2\varepsilon} \frac{1}{Nt_1} \left( \frac{1}{N^\varepsilon} + \frac{N^\delta}{N^\varepsilon} \right). \tag{3.190} \]

Lastly, since \( \sum_j U_{a_j}^{(B)}(t_1 + u, t_1 + 2t_2) = 1 \) we get

\[ \sum_{j} \frac{1}{t_2} \int_{t_0}^{t_2} U_{a_j}^{(B)}(t_1 + u, t_1 + 2t_2) \left( \frac{1}{N} p_{t_1 + u}(\gamma_{i_1}^{(f)}, \gamma_{i_1}^{(i)}) \right) du = \frac{1}{t_2} \int_{t_0}^{t_2} \left( \frac{1}{N} p_{t_1 + u}(\gamma_{i_1}^{(f)}, \gamma_{i_1}^{(i)}) \right) \]

\[ = \frac{1}{N} p_{t_1 + 2t_2}(\gamma_{i_1}^{(f)}, \gamma_{i_1}^{(i)}) + \frac{1}{Nt_1} O(N^{-\varepsilon}z). \tag{3.191} \]

Here we used

\[ \frac{1}{N} \left| p_{t_1 + u}(\gamma_{i_1}^{(f)}, \gamma_{i_1}^{(i)}) - p_{t_1}(\gamma_{i_1}^{(f)}, \gamma_{i_1}^{(i)}) \right| \leq C \frac{N^{-\varepsilon}z}{Nt_1}. \tag{3.192} \]

This yields the claim after taking \( \delta = \varepsilon z/2 \).

At this point we just have to choose the parameters \( s_0 \) and \( s_1 \) to conclude the homogenization result for \( U^{(B)} \).

**Proof of Theorem 3.10.** We use the result of Theorem 3.16 and just make a choice of \( s_0 \) and \( s_1 \). First we optimize over \( s_0 \) and take

\[ (N_{s_0}) = (N_{s_1})^{2/3}. \tag{3.193} \]

The error then simplifies to

\[ N^\varepsilon \frac{N^{\varepsilon z}}{Nt_1} \left\{ \frac{s_0^2}{t_1^2} + \frac{(N_{s_1})^4}{\ell^4} + \frac{t_1}{s_1} \left( \frac{1}{N^\omega_F} + \frac{1}{(N_{s_1})^{1/3}} \right) \right\}^{1/2} + N^\varepsilon \frac{N^{-\varepsilon z/2}}{Nt_1}. \tag{3.194} \]

To optimize over \( s_1 \) we have two cases. If \( \omega_F \geq \omega_1 3/10 \) then we take \( N_{s_1} = (N_{s_1})^{9/10} \). If \( \omega_F \leq \omega_1 3/10 \) then we take \( N_{s_1} = Nt_1 N^{-\omega_F/3} \). The error simplifies to

\[ N^\varepsilon \frac{N^{\varepsilon z}}{Nt_1} \left\{ \frac{(N_{s_1})^4}{\ell^4} + \frac{1}{(N_{s_1})^{1/5}} + \frac{1}{N^\omega_F 2/3} \right\}^{1/2} + N^\varepsilon \frac{N^{-\varepsilon z/2}}{Nt_1}. \tag{3.195} \]

This is the claim.

**3.7 Completion of proof of Theorem 3.6**

First of all we see by the definition of \( z_i \) and by Lemma 3.7 that with overwhelming probability,

\[ z_i(t_1 + u, 1) - z_i(t_1 + u, 0) = (\tilde{z}_i(t_1 + u, 1) - \tilde{z}_i(t_1 + u, 0)) + (\gamma_0(t_1 + u, 1) - \gamma_0(t_1 + u, 0)) \]

\[ = (\tilde{z}_i(t_1 + u, 1) - \tilde{z}_i(t_1 + u, 0)) + (\gamma_0(t_1 + u, 1) - \gamma_0(t_1 + u, 0)) \]

\[ + O \left( N^\varepsilon t_1 \left( \frac{N^\omega_F}{N^\omega_0} + \frac{1}{N^\omega_F} + \frac{1}{N^{1/3}} \right) \right). \tag{3.196} \]

With \( u_i = \tilde{z}_i \) we have

\[ (\tilde{z}_i(t_1 + u, 1) - \tilde{z}_i(t_1 + u, 0)) = \int_0^1 u_i(t_1 + u, \alpha) d\alpha. \tag{3.197} \]
Recall \( u \) satisfies \( \partial_t u = Bu + \xi \) with \( \xi \) defined as in Section 3.4. By the bound (3.67), the assumption (3.33) and Lemma 3.8 we see that for any small \( \delta_B > 0 \)

\[
\sup_{|i| \leq N^{\omega_A-3\delta_B}, |u| \leq t_1} |u_i(t_1 + u) - v_i(t_1 + u)| \leq \frac{1}{N^{10}}
\]

(3.198)

with overwhelming probability where \( v_i \) is defined by

\[
\partial_t v = Bv, \quad v_i(0) = 1_{|i| \leq N^{\omega_A}} u_i(0).
\]

(3.199)

Fix an \( \varepsilon_a > 0 \) and consider the solution

\[
\partial_t w = Bw, \quad w_i(0) = 1_{|i| \leq (Nt_1)^{N^{\varepsilon_b}}} u_i(0).
\]

(3.200)

Since \( |u_i(0)| \leq N^{\varepsilon}/N \) for any \( |i| \leq N^{\omega_b}/2 \) with overwhelming probability, we see by Lemma 3.9 that for \( |i| \leq Nt_1N^{\varepsilon_b} \) with \( \varepsilon_b < \varepsilon_a \),

\[
|v_i(t_1 + u) - w_i(t_1 + u)| \leq \sum_{N^{\omega_1+\varepsilon_a}<|j| \leq N^{\omega_A}} |U_{ij}^{(B)}(0, t_1 + u)u_j(0)| \leq \frac{N^{\varepsilon}}{Nt_1} \sum_{|j| > N^{\omega_1+\varepsilon_a}} \frac{1}{(i-j)^2} \leq C \frac{N^{\varepsilon}}{NN^{\varepsilon_a}}.
\]

(3.201)

Therefore,

\[
z_i(t_1 + u, 1) - z_i(t_1 + u, 0) = (\gamma_0(t_1 + u, 1) - \gamma_0(t_1 + u, 0)) + \int_0^1 \sum_{|j| \leq Nt_1N^{\varepsilon_a}} U_{ij}^{(B)}(0, t_1 + u, \alpha)(z_j(0, 1) - z_j(0, 0)) d\alpha + \frac{N^{\varepsilon}}{N^2} \mathcal{O} \left( N^{\omega_1} \left( N^{\omega_A} + \frac{1}{N^{\omega_b}} + \frac{1}{\sqrt{NG}} \right) + \frac{1}{N^{\varepsilon_a}} \right)
\]

(3.202)

with overwhelming probability for \( |i| \leq Nt_1N^{\varepsilon_b} \). Then Theorem 3.6 now follows from an application of Theorem 3.10 with \( \omega_F = \infty \). \( \square \)

### 3.8 Proof of Theorem 3.1

Theorem 3.6 implies, after re-writing \( z_i(t, 1) \) in terms of \( x_i \), that with overwhelming probability we have,

\[
(x_{i_0+i_1}(t_0 + t_1 + u) - \gamma_{i_0}(t_0 + t_1 + u)) - y_{N/2+i_2}(t_0 + t_1 + u)
\]

\[
= \sum_{|j| \leq Nt_1N^{\varepsilon_a}} \zeta(i - j, t_1) \left[ (x_{i_0+j}(t_0 + t_1) - \gamma_{i_0}(t_0 + t_1)) - y_{N/2+j}(t_0 + t_1) \right]
\]

\[
+ \frac{N^{\varepsilon}}{N^2} \mathcal{O} \left( N^{\omega_1} \left( \frac{1}{N^{\omega_b}} + \frac{1}{N^{\varepsilon_a}} \right) + N^{\varepsilon_2+\varepsilon_a} \left( \frac{(Nt_1)^2}{\ell^2} + \frac{1}{(Nt_1)^{1/10}} \right) + N^{\varepsilon_a-\varepsilon_2/2} + N^{\omega_1} \frac{N^{\varepsilon_b}}{N^{\omega_0}} \right),
\]

(3.203)

for any \( |i| \leq Nt_1N^{\varepsilon_b} \) and \( |u| \leq t_2 \). We also used Lemma 3.4 to replace the classical eigenvalue locations from the interpolating measures with those coming from the original free convolution \( \rho_{\ell, t} \) and the semicircle law. We choose \( \omega_A = (\omega_F + \omega_0)/2 \). The error simplifies to

\[
\frac{N^{2\varepsilon}}{N} \mathcal{O} \left( N^{\omega_1} \left( \frac{1}{N^{\omega_b}} \right) + \frac{1}{N^{\varepsilon_a}} + N^{\varepsilon_2+\varepsilon_a} \left( \frac{(Nt_1)^2}{\ell^2} + \frac{1}{(Nt_1)^{1/10}} \right) + N^{\varepsilon_a-\varepsilon_2/2} \right).
\]

(3.204)

There are two cases. First if \( \omega_1 \geq 10\omega_0/21 \) then the error simplifies to

\[
\frac{N^{2\varepsilon}}{N} \mathcal{O} \left( N^{\omega_1} \left( \frac{1}{N^{\omega_b}} \right) + \frac{1}{N^{\varepsilon_a}} + N^{\varepsilon_2+\varepsilon_a} \left( \frac{(Nt_1)^2}{\ell^2} \right) + N^{\varepsilon_a-\varepsilon_2/2} \right).
\]

(3.205)
In this case we then take $\varepsilon_2 = 4(\omega_\ell - \omega_1)/3$ and then $\varepsilon_a = (\omega_\ell - \omega_1)/3$. The error simplifies to

$$\frac{N^{3\varepsilon}}{N^\calO\left(\frac{N^{\omega_1/3}}{N^{\omega_1/6}}\right)}.$$  

(3.206)

We then take $\omega_\ell = \omega_0/2 - \varepsilon$ and so the error is

$$\frac{N^{4\varepsilon}}{N^\calO\left(\frac{N^{\omega_1/3}}{N^{\omega_0/6}}\right)} \leq \frac{N^{4\varepsilon}}{N^\calO\left(\frac{N^{\omega_1/3}}{N^{\omega_0/6}} + \frac{1}{N^{\omega_1/60}}\right)}.$$  

(3.207)

In the case $\omega_1 < 10\omega_0/21$ we take $\omega_\ell = 21\omega_1/20 < \omega_0/2$. The error simplifies to

$$\frac{N^{2\varepsilon}}{N^\calO\left(\frac{1}{N^{\omega_1/60}} + \frac{1}{N^{\varepsilon_a}} + \frac{N^{|\varepsilon_a|}}{N^{\omega_1/10}} + \frac{N^{|\varepsilon_a|}}{N^{\omega_1/60}}\right)}.$$  

(3.208)

Choose $\varepsilon_2 = \omega_1/15$ and $\varepsilon_a = \omega_1/60$. The error then simplifies to

$$\frac{N^{3\varepsilon}}{N^\calO\left(\frac{1}{N^{\omega_1/60}}\right)} \leq \frac{N^{4\varepsilon}}{N^\calO\left(\frac{1}{N^{\omega_1/60}} + \frac{N^{\omega_1/3}}{N^{\omega_0/6}}\right)}.$$  

(3.209)

$\square$

4 Finite speed estimates

4.1 Estimate for short-range operator

In this section we work in the set-up of Section 3.5 and assume that (I)-(III) hold. Let $\hat{z}_i$ be defined as in that section. Fix a parameter $\ell_3 = N^{\omega_\ell,3}$ satisfying

$$0 < \omega_{\ell,3} \leq \omega_\ell.$$  

(4.1)

For the current Section 4.1 we fix a $0 < q < 1$ and let $\calA_3$ be the set

$$\calA_3 := \{(i, j) : |i - j| \leq N^{\omega_\ell,3}\} \cup \{(i, j) : ij > 0, i \notin \hat{\calC}_q, j \notin \hat{\calC}_q\}.$$  

(4.2)

Define the operator

$$(E_3 u)_i := \sum_j \frac{1}{N (\hat{z}_j - \hat{z}_i)^2}.$$  

(4.3)

We want to prove the following theorem. Lemma 3.8 is an immediate consequence. The method is based on that appearing in [18].

**Theorem 4.1.** Let $\ell_3$ as above. Let $D_1, D_2 > 0$, and let $\varepsilon > 0$. Let $0 < q_3 < q$. We assume that Section 3.5 (I)-(III) hold. Fix a time $u \leq 10\ell_1$. There is an event $F(\alpha, u)$ s.t. $\mathbb{P}[F(\alpha, u)] \geq 1 - N^{-D_1}$ for $N$ large enough (independent of $\alpha$) such that all of the following estimates hold. For every $u \leq s \leq t \leq (u + 2\ell_3/N) \wedge 10\ell_1$ we have the estimate

$$U_{(B^3)}^\alpha(s, t) \leq \frac{1}{ND_2}.$$  

(4.4)

provided one of the following three criteria holds.

(i) $a \in \hat{\calC}_{q_3}$ and $|a - b| > N^{\omega_3,3 + \varepsilon}$.

(ii) $b \in \hat{\calC}_{q_3}$ and $|a - b| > N^{\omega_3,3 + \varepsilon}$.

(iii) $a \notin \hat{\calC}_{q_3}$, $b \notin \hat{\calC}_{q_3}$ and $ab < 0$.  

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Hence, the same estimate holds for any $0 \leq s \leq t \leq 10t_1$ that satisfy $t - s \leq \ell_3/N$, with overwhelming probability.

**Remark.** In the original set-up with the interpolating $\hat{z}_i(t, \alpha)$, for technical reasons we cannot prove the theorem for all $\alpha$ on the same event. This did not affect the main results of the previous section as we always integrated over $\alpha$.

In the proof of Theorem 4.1 we take $u = 0$ for notational simplicity. The first step in proving Theorem 4.1 is to establish the estimate for $s = 0$, which is the content of the following lemma. We then use the semigroup property to extend the estimate to all $s$.

**Lemma 4.2.** Fix $0 < q_3 < q$. Let $\varepsilon > 0$ and $D_1, D_2 > 0$. Assume that Section 3.5 (I)-(III) hold. There is an event $\mathcal{F}_a$ with $\mathbb{P}[\mathcal{F}_a] \geq 1 - N^{-D_1}$ on which the following estimates hold. For every $0 \leq t \leq \ell_3/N \wedge 10t_1$ we have the estimate

$$U_{\alpha a}^{(B)}(0, t) \leq \frac{1}{N^{D_2}}.$$  \hfill (4.5)

provided one of the following three criteria holds.

(i) $a \in \mathcal{C}_{q_4}$ and $|a - b| > N^{\omega_{\ell, 4} + \varepsilon}$.

(ii) $b \in \mathcal{C}_{q_4}$ and $|a - b| > N^{\omega_{\ell, 3} + \varepsilon}$.

(iii) $a \notin \mathcal{C}_{q_4}$, $b \notin \mathcal{C}_{q_4}$ and $ab < 0$.

**Proof.** Define for $t \geq 0$, $f_i(t) := U_{\alpha a}^{(B)}(0, t)$; i.e., $f_i$ satisfies the equation

$$\partial_t f_i = (B_3 f)_i, \quad f_i(0) = \delta_a.$$  \hfill (4.6)

WLOG, take $\varepsilon > 0$ s.t. $\omega_{\ell, 3} + \varepsilon < \omega_0/2$. We can assume $a \geq 0$. Fix $q_4$ satisfying $q_3 < q_4 < q$. It then suffices to prove the statement for the following two cases.

1. $a \in \mathcal{C}_{q_4}$, $|a - b| > N^{\omega_{\ell, 4} + \varepsilon}$.

2. $a \notin \mathcal{C}_{q_4}$, $b \in \mathcal{C}_{q_4}$ or $b < 0$.

Let us first consider the case $a \in \mathcal{C}_{q_4}$. Let $\nu > 0$ and define

$$\phi_k := e^{\nu\psi(\hat{z}_k(t, \alpha) - \gamma_a(t, \alpha))/2}$$  \hfill (4.7)

where $\psi$ is the following smooth function. Fix a scale $\ell_4 = N^{\omega_{\ell, 4}} > 0$ and a $\delta_1 > 0$ s.t.

$$\delta_1 + \omega_{\ell} < \omega_0, \quad 0 < \omega_{\ell, 4} \leq \omega_{\ell, 3}.$$  \hfill (4.8)

Assume $0 < \varepsilon < \delta_1$. We choose $\psi$ s.t.

$$\psi(x) = -x, \quad |x| \leq \frac{N^{\delta_1 + \omega_{\ell}}}{N},$$  \hfill (4.9)

and

$$\psi(x) = \mp \left( \frac{N^{\delta_1 + \omega_{\ell}}}{N} + \frac{\ell_4}{2N} \right), \quad \pm x > \frac{N^{\delta_1 + \omega_{\ell}}}{N} + \frac{\ell_4}{N}.$$  \hfill (4.10)

We can choose $\psi$ so that $|\psi'| \leq 1$ and $|\psi''| \leq CN/\ell_4$.

Our proof is based around a Gronwall argument and we will need to take an expectation of a martingale. For this we need to introduce the following stopping time $\tau_r$. Let $q < q_r < 1$ and $\varepsilon_r > 0$ with $\varepsilon_r < \varepsilon/100$. Let $\tau_i$, $i = 1, 2, 3$ be the stopping time

$$\tau_1 := \inf\{t > 0 : \exists i \in \mathcal{C}_{q_4} : |\hat{z}_i(t) - \gamma_i(t)| > N^{\varepsilon_r}/N\}$$

$$\tau_2 := \inf\{t > 0 : \exists i \in \mathcal{C}_{q_4} : |J_i| > C_f \log(N)\}$$

$$\tau_3 := \inf\{t > 0 : \exists i : |F_i| \geq \log(N)\}.$$  \hfill (4.11)
We set \( \tau_r := \tau_1 \land \tau_2 \land \tau_3 \land (10t_1) \). We know that \( \tau_r = 10t_1 \) with overwhelming probability by the assumptions (II) and (III) of Section 3.5.

Define now \( v_k(t) = \phi_k f_k \) and \( F = \sum_k v_k^2(t)1_{(\tau_r > 0)} \). By the same calculation as in [18] we obtain

\[
dF(t) = -\frac{1}{2} \sum_{(i,j) \in A_3} \frac{1}{N} \frac{(v_i - v_j)^2}{(\dot{z}_i - \dot{z}_j)^2} dt
- \sum_{(j,k) \in A_3} \frac{1}{N} \frac{1}{(\dot{z}_j - \dot{z}_k)^2} \left[ \frac{\phi_i}{\phi_j} + \frac{\phi_j}{\phi_k} - 2 \right] v_kv_j dt
+ \sum_{k} \nu v_k^2 \psi_k^2 d(\dot{z}_k - \dot{\gamma}_a)
+ \sum_{k} v_k^2 (\nu^2 (\psi_k')^2 + \nu \psi_k''') \frac{dt}{N}.
\] (4.12)

We now deal with each term individually, applying Gronwall at the end of the proof. In the remainder of the argument we work on times \( t < \tau_r \). We start with (4.13). Fix \( q_5 \) satisfying \( q_4 < q_5 < q \). By rigidity and choice of \( \psi \) we have that the term

\[
\frac{\phi_i}{\phi_j} + \frac{\phi_j}{\phi_k} - 2, \quad (j, k) \in A_3
\] (4.16)

vanishes unless \( j, k \in \hat{C}_{q_5} \). In this case by rigidity and the fact that \( |\psi'| \leq 1 \) we have that

\[
\left| \frac{\phi_i}{\phi_j} + \frac{\phi_j}{\phi_k} - 2 \right| \leq C \nu^2 |\dot{z}_j - \dot{z}_k|^2
\] (4.17)
as long as we choose \( \nu \) so that \( \nu \ell_3 \leq CN \). Hence,

\[
\left| \sum_{(j,k) \in A_3} \frac{1}{N} \frac{1}{(\dot{z}_j - \dot{z}_k)^2} \left[ \frac{\phi_i}{\phi_j} + \frac{\phi_j}{\phi_k} - 2 \right] v_kv_j \right| \leq C \nu^2 \ell_3 \sum_k v_k^2.
\] (4.18)

Above we used the fact that the cardinality of the set \( \{ j : (j, k) \in A_3 \} \) is bounded by \( C\ell_3 \) if \( k \in \hat{C}_{q_5} \). The Ito terms (4.15) are bounded by

\[
\left| \sum_k v_k^2 (\nu^2 (\psi_k')^2 + \nu \psi_k''') \frac{dt}{N} \right| \leq C \left( \frac{\nu^2}{N} + \frac{\nu}{\ell_4} \right) \sum_k v_k^2.
\] (4.19)

We now deal with the terms (4.14). By rigidity we have \( \psi_k' = 0 \) if \( k \notin \hat{C}_{q_5} \). We can therefore assume \( k \in \hat{C}_{q_5} \). We fix a \( \delta_2 > 0 \) s.t.

\[
\delta_2 < \omega_{\ell,3}.
\] (4.20)

From the definition of \( \dot{z}_k \) process and the definition of \( \tau_r \) we see that we can write for \( k \in \hat{C}_{q_5} \),

\[
d(\dot{z}_k - \dot{\gamma}_a) = \sum_{|j-k| \leq N^{s_2}} \frac{1}{N} \frac{1}{2z_k - \dot{z}_j} dt + X_idt + \sqrt{\frac{2}{N}}dB_k
\] (4.21)

where we have the bound \( |X_i| \leq C \log N \). The first term on the RHS of (4.21) corresponds to

\[
\nu \sum_k v_k^2 \psi_k' \sum_{|j-k| \leq N^{s_2}} \frac{1}{N} \frac{1}{2z_k - \dot{z}_j} = \nu \left[ \sum_{|j-k| \leq N^{s_2}} \psi_k' \frac{\psi_j' - \psi_k'}{2z_k - \dot{z}_j} + \nu \left[ \sum_{|j-k| \leq N^{s_2}} \frac{v_k^2 \psi_j' - \psi_k'}{N} \frac{1}{2z_k - \dot{z}_j} \right] \right]
\] (4.22)

The second term of (4.22) is bounded by

\[
\left| \nu \frac{1}{2} \sum_{|j-k| \leq N^{s_2}} \frac{v_k^2 \psi_j' - \psi_k'}{N} \frac{1}{2z_k - \dot{z}_j} \right| \leq C \nu N^{\delta_2} \ell_4 \sum_k v_k^2.
\] (4.23)
We use the Schwarz inequality to bound the first term of (4.22) by
\[
\left| \frac{\nu}{2} \sum_{|j-k| \leq N^{\delta_2}} \frac{\psi'_k v_k^2 - v_j^2}{\zeta_k - \zeta_j} \right| \leq \frac{1}{100} \sum_{|j-k| \leq N^{\delta_2}} \frac{(v_k - v_j)^2}{N(\zeta_k - \zeta_j)^2} + \frac{C\nu^2 N^{\delta_2} \sum_k v_k^2}{N}. \tag{4.24}
\]

The first term on the RHS is absorbed into the term (4.12). Collecting everything we have proven that under the assumption \(\nu \ell_3 \leq CN\), we have
\[
\hat{\delta}_i \mathbb{E}[F(t)] \leq C \left( \frac{\nu^2 \ell_3}{N} + \frac{\nu N^{\delta_2}}{\ell_4} + \nu \log(N) \right) \mathbb{E}[F(t)]. \tag{4.25}
\]

We can take \(\ell_4 = \ell_3\) and \(\nu = N/(\ell_3 N^{\varepsilon/2})\). Then by Gronwall we see that for \(t \leq \ell_3/N\) we get
\[
\mathbb{E}[F(t)] \leq C \mathbb{E}[F(0)] \leq C \tag{4.26}
\]
where the second inequality follows from rigidity, the definition of \(\tau\) and the initial condition \(f_k(0) = \delta_{ak}\). By construction, for any \(\varepsilon > 0\) we have that if \(j \leq a - N^{\omega_{\ell,3} + \varepsilon}\),
\[
\psi(\hat{\zeta}_j - \hat{\gamma}_a) \geq \frac{c}{N} \min\{N^{\omega_{\ell,3} + \varepsilon}, N^{\omega_{\ell,3} + \delta_1}\} = \frac{c}{N} N^{\omega_{\ell,3} + \varepsilon} \tag{4.27}
\]
and hence
\[
\nu \psi(\hat{\zeta}_j - \hat{\gamma}_a) > c N^{\varepsilon/2}. \tag{4.28}
\]

We conclude the claim for \(b \leq a - N^{\omega_{\ell,3} - \varepsilon}\) and \(a \in \mathcal{C}_{q_3}\) from Markov’s inequality. For \(b > a + N^{\omega_{\ell,3} + \varepsilon}\) the argument is similar; one just replaces \(\psi\) by \(-\psi\).

We now consider the case \(a \notin \mathcal{C}_{q_3}\). Recall that we assumed \(a \geq 0\). The argument is identical except one considers, instead of \(\psi\) above,
\[
\varphi(x) := \psi(x - \hat{\gamma}_d(t, \alpha)) \tag{4.29}
\]
where \(d\) is an index chosen in the following way. If \(d_1 > 0\) is the largest index in \(\mathcal{C}_{q_3}\) and \(d_2 > 0\) is the largest index in \(\mathcal{C}_{q_3}\) (recall \(q_3 < q_4\)) then \(d = (d_1 + d_2)/2\). One then defines
\[
\varphi_k := \varphi(\hat{\zeta}_k), \quad \phi_k := e^{\varphi_k/2}, \quad v_k := e^{\varphi_k/2} f_k, \quad F(t) = \sum_k v_k^2. \tag{4.30}
\]

With this choice rigidity implies that \(\varphi'_k = 0\) for \(k \notin \mathcal{C}_{q_4}\). Rigidity also implies that the term
\[
\frac{\phi_k}{\phi_j} + \frac{\phi_j}{\phi_k} - 2, \quad (j, k) \in \mathcal{A}_3 \tag{4.31}
\]
vanishes unless both \(j, k \in \mathcal{C}_{q_4}\). With these considerations the argument can proceed exactly as above. We again arrive at (4.25) and choose \(\ell_4 = \ell_3\) and \(\nu = N/(\ell_3 N^{\varepsilon/2})\). We see that for \(k \in \mathcal{C}_{q_3}\) we have
\[
\psi(\hat{\zeta}_k - \hat{\gamma}_d) \geq c N^{\delta_1 + \omega_{\ell,3}} / N \tag{4.32}
\]
and so \(\nu \psi(\hat{\zeta}_k - \hat{\gamma}_d) \geq c N^{\varepsilon/2}\). We conclude as before. Note that now we only need that \(\psi(\hat{\zeta}_a - \hat{\gamma}_d) < 0\) which follows by the ordering of eigenvalues and rigidity to satisfy
\[
\mathbb{E}[F(0)] \leq C. \tag{4.33}
\]

**Proof of Theorem 4.1.** For notational simplicity we set \(u = 0\). Let \(\varepsilon\) and \(q_3\) be as in the statement of Theorem 4.1. Wlog, we can assume that \(\omega_{i,3} + \varepsilon < \omega_0/2\). We can assume that the estimates of Lemma 4.2 hold for \(q_3\) satisfying \(q_3 < q_4\) and \(\varepsilon = \varepsilon/2\). For any \(i\) we can write
\[
U_{bi}^{(B)_i}(0,t) = \sum_j U_{bj}^{(B)_i}(s,t)U_{ji}^{(B)_i}(0,s) \geq U_{bi}^{(B)_i}(s,t)U_{ai}^{(B)_i}(0,s). \tag{4.34}
\]

We just need to find an \(i\) s.t. the LHS is bounded above and \(U_{ai}^{(B)_i}(0,s)\) is bounded below. Fix \(q_5\) satisfying \(q_3 < q_5 < q_4\). As before, it suffices to assume \(a \geq 0\) and to consider the following two cases.
1. $a \in \hat{\mathcal{C}}_{q_5}$, and $|b - a| > N^{\omega_{\ell,3} + \varepsilon}$.

2. $a \notin \hat{\mathcal{C}}_{q_5}$ and $b \in \hat{\mathcal{C}}_{q_1}$ or $b < 0$.

Let us first consider the case $a \in \hat{\mathcal{C}}_{q_5}$. Since the estimates of Lemma 4.2 hold, and $a \in \hat{\mathcal{C}}_{q_5}$, we have that

$$U^{(B)}_{ai}(0,s) \leq \frac{1}{N^{D_2}}, \quad |i - a| > N^{\omega_{\ell,3} + \varepsilon/2}.$$  \hspace{1cm} (4.35)

Since

$$\sum_i U^{(B)}_{ai}(0,s) = 1,$$

this implies that there is an $i_0$ s.t. $|i_0 - a| \leq N^{\omega_{\ell,3} + \varepsilon/2}$ and $U^{(B)}_{ai_0}(0,s) \geq N^{-1}$. Moreover, $i_0 \in \hat{\mathcal{C}}_{q_4}$.

Then since $|a - b| > N^{\omega_{\ell,3} + \varepsilon}$ we see that $|i_0 - b| > N^{\omega_{\ell,3} + \varepsilon/2}$. Hence,

$$U^{(B)}_{a_i}(0,t) \leq \frac{1}{N^{D_2}}.$$  \hspace{1cm} (4.37)

Therefore,

$$U^{(B)}_{ba}(s,t) \leq \frac{1}{N^{D_2-1}}.$$  \hspace{1cm} (4.38)

Let us now consider the case $a \notin \hat{\mathcal{C}}_{q_5}$ and $b \in \hat{\mathcal{C}}_{q_1}$ or $b < 0$. Wlog we can take $a > 0$. Fix $q_5$ s.t. $q_3 < q_5 < q_5$. If $i$ is such that either $i \in \hat{\mathcal{C}}_{q_5}$ or $i \leq 0$, then $U^{(B)}_{ai}(0,s) \leq N^{-D_2}$. Hence, there is an $i_0$ s.t. $U^{(B)}_{ai_0} \geq N^{-1}$ and $i_0 > 0$ and $i_0 \notin \hat{\mathcal{C}}_{q_5}$. But then since $b \in \hat{\mathcal{C}}_{q_3}$ or $b < 0$ we get that

$$U^{(B)}_{a_i}(0,t) \leq \frac{1}{N^{D_2}}$$  \hspace{1cm} (4.39)

and this yields the claim as before.

\[ \square \]

### 4.2 Kernel estimate

In this section we prove Lemma 3.9. It is split into two parts, an energy estimate and a Duhamel expansion.

#### 4.2.1 Energy estimate

Let $B$ is as in Section 3.3. In this subsection our goal is to prove the following energy estimate for $U^{(B)}$. The argument is very similar to that in [39]. The major difference is that in the duality part of Nash’s argument we have to be a little careful with the support of the functions, as we do not know that rigidity holds for all particles $i$. To compensate for this we use the finite speed estimates from the previous section.

We recall the semigroup $U^{(B)}$ for the short-range operator $B$ associated with in short-range set $A_{q_5}$ with parameters $q_s$ and $\omega_\ell$ from Section 3.5.

**Lemma 4.3.** Fix $0 < q_3 < q_s$. Let $a \in \hat{\mathcal{C}}_{q_3}$. Let $\varepsilon > 0, D > 0$. Assume that Section 3.5 (I)-(III) hold. There is an event $F_\alpha$ which holds with probability $P[F_\alpha] \geq 1 - N^{-D}$ on which the following estimates hold. For every $0 \leq s \leq t \leq 10t_1$ and every $i$,

$$U^{(B)}_{ai}(s,t) \leq \frac{N^\varepsilon}{N(t-s)}.$$  \hspace{1cm} (4.40)

**Proof.** Recall from [39] the inequality

$$||u||_i^2 ||u||_2^2 \leq C \sum_{i,j \in \mathbb{Z}} \frac{(u_i - u_j)^2}{(i-j)^2}$$  \hspace{1cm} (4.41)

which holds for sequences $u : \mathbb{Z} \to \mathbb{R}$. Fix $q_3 < q_5 < q_s$. Let $g(u) = U^{(B)}(s,u)g(s)$ where $g(s)$ has support only in the indices $i \in \hat{\mathcal{C}}_{q_5}$ and

$$||g(s)||_1 = 1.$$  \hspace{1cm} (4.42)
Extending \( g(u) \) by 0 to all of \( \mathbb{Z} \) we apply the above inequality to \( g(u) \) and obtain, with overwhelming probability,
\[
\|g(u)\|_2^2 \leq C \sum_{i,j \in \mathbb{Z}} \frac{(g_i(u) - g_j(u))^2}{(i-j)^2} \leq C \sum_{|i-j| \leq \kappa, i,j \notin \hat{C}_{q_s}} \frac{(g_i(u) - g_j(u))^2}{(i-j)^2} + C \frac{N}{\ell} \|g(u)\|_2^2 + \frac{1}{ND}.
\]
\[
\leq N^\varepsilon \langle g(u), B g(u) \rangle + C \|g(u)\|_2^2 \frac{N}{\ell} + \frac{1}{ND}. \tag{4.43}
\]
We used the fact that for \( 0 \leq u_1 \leq u_2 \leq t_1 \),
\[
\|U_{ij}^{(B)}(u_1,u_2)\| \leq \frac{1}{ND}, \quad \text{for } (i,j) \in \{i,j : i \in \hat{C}_{q_5}, j \notin \hat{C}_{q_5} \} \cup \{i,j : j \in \hat{C}_{q_5}, i \notin \hat{C}_{q_5} \}, \tag{4.44}
\]
which holds due to Theorem 4.1, with overwhelming probability. Therefore
\[
\hat{c}_i \|g(u)\|_2^2 = -\langle g(u), B g(u) \rangle \leq -N^{-\varepsilon} \|g(u)\|_2^2 + C \|g(u)\|_2^2 \frac{N}{\ell} + \frac{1}{ND}. \tag{4.45}
\]
Above, we used the Holder inequality \( \|g(u)\|_2 \leq \|g(u)\|_1^{1/3} \|g(u)\|_4^{2/3} \leq \|g(s)\|_1^{1/3} \|g(u)\|_4^{2/3} = \|g(u)\|_4^{2/3} \) for \( u \geq s \). Since \( t_1 N/\ell \ll 1 \) we see that this implies
\[
\|g(t)\|_2 \leq C N^\varepsilon (t-s)^{-1/2}. \tag{4.46}
\]
We have therefore proven that
\[
\|U^{(B)}(s,t)g\|_2 \leq (t-s)^{-1/2} N^\varepsilon \|g\|_1 \tag{4.47}
\]
for every \( g \) supported in \( i \in \hat{C}_{q_5} \).

The above argument clearly also applies to \( (U^{(B)}(s,t))^T \) (in particular note that the bound (4.44) is symmetric in \( i \) and \( j \)).

Fix now \( q_4 \) satisfying \( q_3 < q_4 < q_5 \). Let now \( f \) be supported in \( i \in \hat{C}_{q_4} \) and have \( \|f\|_2 = 1 \). Then we have with overwhelming probability,
\[
\|U^{(B)}(s,t)f\|_\infty = \sup_{\|g\|_1 = 1} \langle U^{(B)}(s,t)f, g \rangle \leq \sup_{\|g\|_1 = 1, q_5 = 1, i \notin \hat{C}_{q_5}} \langle U^{(B)}(s,t)f, g \rangle + \frac{1}{ND} \tag{4.48}
\]
where we used the fact that \( |\langle U^{(B)}(s,t)f, g \rangle| \leq N^{-D} \) for any \( D > 0 \) for \( i \notin \hat{C}_{q_5} \) which holds due to Theorem 4.1. For \( g \) as in the RHS of (4.48) we have,
\[
\langle U^{(B)}(s,t)f, g \rangle = \langle f, (U^{(B)}(s,t))^T g \rangle \leq \|f\|_2 \|(U^{(B)}(s,t))^T g\|_2 \leq N^\varepsilon (t-s)^{-1/2}. \tag{4.49}
\]
This proves that for \( f \) supported in \( i \in \hat{C}_{q_4} \) we have
\[
\|U^{(B)}(s,t)f\|_\infty \leq N^\varepsilon (t-s)^{-1/2} \|f\|_2. \tag{4.50}
\]
Lastly, let \( f \) have support in \( \hat{C}_{q_3} \) and \( \|f\|_1 = 1 \). Applying what we have proved above we have with overwhelming probability, with \( u = s + (t-s)/2 \),
\[
\|U^{(B)}(s,t)f\|_\infty = \|U^{(B)}(u,t)U^{(B)}(s,u)f\|_\infty \leq \|U^{(B)}(u,t)1_{\hat{C}_{q_4}}U^{(B)}(s,u)f\|_\infty + \frac{1}{ND} \leq C \frac{N^\varepsilon}{(t-s)^{1/2}} \|U^{(B)}(s,u)f\|_2 + \frac{1}{ND} \leq C \frac{N^\varepsilon}{(t-s)^{1/2}} \|f\|_1. \tag{4.51}
\]
This yields the claim.
4.2.2 Duhamel expansion

We want to prove the following.

**Lemma 4.4.** Let $D > 0$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and $0 < q_3 < q_*$. Assume that Section 3.5 (I) - (III) hold. Let $\ell_3$ be a scale satisfying

$$N^{\varepsilon_3} \leq \ell_3 \leq N^{1}. \quad (4.52)$$

There exists an event $\mathcal{F}_\alpha = \mathcal{F}_\alpha(\ell_3)$ with probability $\mathbb{P}[\mathcal{F}_\alpha] \geq 1 - N^{-D}$ on which the following estimates hold. For every $0 \leq s \leq t \leq 10t_1$ which satisfy $N(t - s) \leq N^{-\varepsilon_3} \ell_3$ and indices $a$ and $p$ satisfying $a, p \in \hat{\mathcal{C}}_{q_3}$ and

$$|a - p| \geq N^{\varepsilon_1} \ell_3 \quad (4.53)$$

we have

$$U_{ap}(B)(s, t) \leq N^{\varepsilon_3} \frac{N(t - s) + 1}{(a - p)^2}. \quad (4.54)$$

By taking a sequence of at most $N$ scales $\ell_3 = N^\varepsilon, N^{\varepsilon + 1}, \ldots$ we easily see that Lemma 4.4 implies the following estimate.

**Lemma 4.5.** Let $D > 0$ and $\varepsilon_1, \varepsilon_2$ and $0 < q_3 < q_*$. There is an event $\mathcal{F}_\alpha$ with probability $\mathbb{P}[\mathcal{F}_\alpha] \geq 1 - N^{-D}$ on which the following holds. For every $0 \leq s \leq t \leq 10t_1$ and pair of indices $a, p$ satisfying $a, p \in \hat{\mathcal{C}}_{q_3}$ and

$$|a - p| \geq N^{\varepsilon_1} [1 \vee (N(t - s))] \quad (4.55)$$

we have

$$U_{ap}(B)(s, t) \leq N^{\varepsilon_3} \frac{N(t - s) + 1}{(a - p)^2}. \quad (4.56)$$

**Proof of Lemma 4.4.** We can work under the assumption that the estimates of Lemma 3.5 hold. We assume $a < p$. The proof for $a > p$ is identical. Write

$$B = S + \mathcal{R} \quad (4.57)$$

where $S = B_3$ is defined as at the start of Section 4.1 with $\ell_3$ as in the statement of Lemma 4.4, and $\mathcal{R}$ is defined implicitly by the above equality. For notational simplicity we set $s = 0$, but this has no effect on the proof. For each $M$ we have

$$U^{(B)}(0, t) = U^{S}(0, t) + \sum_{i=1}^{M} \int_{0 \leq s_{i-1} \leq \cdots \leq s_{i} \leq t} U^{S}(s_{i}, t)R U^{S}(s_{i-1}, s_{i}) \cdots R U^{S}(0, s_{1})ds_{1} \cdots ds_{i}$$

$$+ \int_{0 \leq s_{1} \ldots \leq s_{M+1} \leq t} U^{(B)}(s_{M+1}, t)R U^{S}(s_{M}, s_{M+1}) \cdots R U^{S}(0, s_{1})ds_{1} \cdots ds_{M+1}$$

$$= U^{S}(0, t) + \sum_{i=1}^{M} \int_{0 \leq s_{1} \ldots \leq t} A_{i}ds_{1} \cdots ds_{i} + B_{M+1}. \quad (4.58)$$

By Theorem 4.1 $U_{ap}^{S}(0, t) \leq N^{-100}$. We next deal with the term $B_{M+1}$. Using the estimates of Lemma 3.5, it is easy to check that for every $i$ we have

$$\sum_{j} |R(i, j)| \leq C \frac{N}{\ell_3} \quad (4.59)$$

and so

$$||\mathcal{R}||_{p \to \ell^p} \leq C \frac{N}{\ell_3} \quad (4.60)$$

for every $p$. Hence,

$$|B_{M+1}(i, j)| \leq C_M N^2 \left( \frac{Nt}{\ell_3} \right)^M \leq \frac{1}{N^{100}} \quad (4.61)$$
for $M$ a large constant depending only on $\varepsilon_1$. By Lemma 4.6 below we see that
\[
\left| \int_{0 \leq s_1 \leq \ldots \leq t} A_i(a, p) ds_1 \cdots ds_i \right| \leq C \frac{N^t}{(a - p)^{2}} \left( \frac{Nt}{\ell_3} \right)^{i-1} \tag{4.62}
\]
This concludes the proof. \hfill \Box

**Lemma 4.6.** Let $A_i$, $R$, etc. be as above. Let $a < p$, $M$ be as above. Define $L_i$ as
\[
L_i := p - \frac{p - a}{100M}. \tag{4.63}
\]
Then for $j \leq L_i$ we have
\[
A_i(j, p) \leq C_M \frac{N^i}{(p - a)^2 \ell_3^{i-1}}. \tag{4.64}
\]

**Proof.** The proof is by induction on $i$. Define
\[
R := N^{\varepsilon_1/4} \ell_3. \tag{4.65}
\]
Fix $q_3 < q_5 < q_s$. Theorem 4.1 implies that for $(a, b)$ s.t. $|a - b| > R$ and either $a$ or $b \in \mathcal{C}_{q_5}$, we have with overwhelming probability
\[
|U^S_{ab}(s, t)| \leq \frac{1}{Nb} \tag{4.66}
\]
for any $D > 0$, and $0 \leq s \leq t \leq \ell_3$. We have that
\[
A_1(j, p) = \sum_{k,l} U^S_{jk} R_{kl} U^S_{lp}. \tag{4.67}
\]
Since the matrix elements of $R$ are bounded by (say) $N^2$, it suffices by (4.66) to consider only terms in (4.67) that satisfy $|l - p| \leq R$. Since $j \leq L_1$ we can apply Theorem 4.1 and ignore terms satisfying
\[
k \geq L_1 + R. \tag{4.68}
\]
For such $l$ and $k$, using the fact that $j \leq L_1$ and $R \ll |p - a|$ we see that $|l - k| \geq c(p - a)$ and so
\[
|A_1(j, p)| \leq \sum_{k,l} U^S_{jk} \frac{N}{(p - a)^2} U^S_{lp} \leq \frac{N}{(p - a)^2}, \tag{4.69}
\]
where in the second inequality we used the $\ell^p \to \ell^p$ boundedness of $U^S$. Now for $A_{i+1}$ and $j \leq L_{i+1}$ we write
\[
A_{i+1}(j, p) = \sum_{k} \sum_{l \geq L_i} U^S(j, k) R(k, l) A_i(l, p) + \sum_{k} \sum_{l \leq L_i} U^S(j, k) R(k, l) A_i(l, p) \tag{4.70}
\]
We start with estimating the first sum. By Theorem 4.1 we can restrict the $k$ summation to terms satisfying
\[
k \leq L_{i+1} + R. \tag{4.71}
\]
Then since $l \geq L_i$ we get that $|k - l| \geq c|p - a|$ and so
\[
\left| \sum_{k} \sum_{l \geq L_i} U^S(j, k) R(k, l) A_i(l, p) \right| \leq C \frac{N}{(p - a)^2} \sum_{k,l} U^S(j, k) A_i(l, p) \leq C \frac{N}{(p - a)^2} ||U^S||_{\infty \to \infty} ||A_{i-1}||_{1 \to 1}
\leq C \frac{N}{(p - a)^2} \frac{N^i}{\ell_3^{i}}. \tag{4.72}
\]
For the second sum in (4.70) we apply the induction assumption and obtain
\[
\left| \sum_{k} \sum_{l \leq L_i} U^S(j, k) R(k, l) A_i(l, p) \right| \leq \sum_{k} \sum_{l \leq L_i} U^S(j, k) R(k, l) C \frac{N^i}{\ell_3^{i-1}(p - a)^2} \leq C ||U^S R||_{\infty \to \infty} \frac{N^i}{\ell_3^{i-1}(p - a)^2}
\leq C \frac{N}{\ell_3^{i-1}(p - a)^2}. \tag{4.73}
\]
This completes the proof. \hfill \Box
4.3 Profile of random kernel

Combining Lemma 4.3 and 4.5 we easily obtain the following.

**Theorem 4.7.** Fix $0 < q_3 < q_*$. Let $D$ and $\varepsilon > 0$. Assume that Section 3.5 (I)-(III) hold. There is an event $F_\alpha$ with probability $P[F_\alpha] \geq 1 - N^{-D}$ on which the following estimate holds. We have for every $0 \leq s \leq t \leq t_1$ and $j, k \in \mathbb{C}_{q_3},$

$$U_{i,j}^{(B)}(s, t) \leq N^2 \frac{|t - s| \vee N^{-1}}{(i - j)/N)^2 + ((t - s) \wedge N^{-1})^2} \quad (4.74)$$

5 Regularity of hydrodynamic equation

In this section we analyze the limiting equation (3.86), deriving in particular the estimates in Lemma 3.11. We introduce the kernel

$$K_{\eta_j} f(x) := \int_{|x - y| \leq \eta_j} \frac{f(x) - f(y)}{(x - y)^2} dy. \quad (5.1)$$

The integral is understood in a principal value sense.

By [24, Theorem 1.4], for the heat kernel corresponding to $K_1$, we have the estimates:

$$p_1(t, x, y) \approx \frac{1}{t} \wedge \frac{t}{|x - y|^2}, \quad 0 < t \leq 1, \quad (5.2)$$

for $|x - y| < 1$ and

$$p_1(t, x, y) \approx e^{-c|x-y| \log \frac{|x-y|}{t}}, \quad 0 < t \leq 1, \quad (5.3)$$

for $|x - y| \geq 1$.

If $f(t, x)$ satisfies $\partial_t f(t, x) = K_1 f(t, x)$, then $g(t, x) = f(t/\eta_t, x/\eta_t)$ satisfies $\partial_t g(t, x) = K_{\eta_t} g(t, x)$, so from (5.2), (5.3) we obtain:

$$p_{\eta_t}(t, x, y) = (1/\eta_t)p_1(t/\eta_t, x/\eta_t, y/\eta_t) \approx \frac{1}{t} \wedge \frac{t}{|x - y|^2}, \quad 0 \leq t \leq \eta_t, \quad (5.4)$$

when $|x - y| < \eta_t$ and

$$p_{\eta_t}(t, x, y) \approx \frac{1}{\eta_t} e^{-c|x-y|/\eta_t \log \frac{|x-y|}{t}}, \quad (5.5)$$

for $t \leq \eta_t$ and $|x - y| \geq \eta_t$, (3.92) and (3.93) follow directly from this. In the rest of the proof, we will also use that the upper bound in (5.4) remains true for all $x, y, t \leq \eta_t$ (See [23, Proposition 2.2, i]):

$$p_{\eta_t}(t, x, y) \leq \frac{C}{t} \wedge \frac{t}{|x - y|^2}, \quad 0 < t \leq \eta_t. \quad (5.6)$$

We will first estimate the spatial derivatives of $p_{\eta_t}(t, x, y)$. Letting $\Lambda = (-\Delta)^{1/2}$ be the half-Laplacian with kernel

$$\int \frac{f(x) - f(y)}{(x - y)^2} dy.$$

The corresponding heat kernel is

$$p_{\Lambda}(t, x, y) := (e^{t\Lambda} \delta)(x) = \frac{1}{\pi t^2 + (x - y)^2}. \quad (5.7)$$

We have the Duhamel formula

$$p_{\eta_t}(t, x, y) = (e^{tK_{\eta_t} \delta})(x) = p_{\Lambda}(t, x, y) + \int_0^t e^{(t-s)\Lambda} (K_{\eta_t} - \Lambda) f_{\eta_t}(s) ds. \quad (5.8)$$
Here we have denoted for simplicity

\[ f_{\eta_t}(x, s) := p_{\eta_t}(s, x, y). \]

Since \( e^{t\Lambda} \) is smoothing, the equality (5.8) shows in particular that \( p_{\eta_t}(t, \cdot, y) \) is in \( C^\infty(\mathbb{R}) \).

We will estimate the first spatial derivative by differentiating (5.8). The operators \( \Lambda, K_{\eta_t} \) are translation invariant, so for general \( k \) we have

\[
\partial_x^k \int_0^t e^{(t-s)\Lambda}(K_{\eta_t} - \Lambda)f_{\eta_t}(s) \, ds = \int_0^t e^{(t-s)\Lambda}\partial_x^k(K_{\eta_t} - \Lambda)f_{\eta_t}(s) \, ds.
\]

By direct computation we have

\[
(K_{\eta_t} - \Lambda)f(x) = \frac{2}{\eta_t} f'(x) - \int_{|x-z| > \eta_t} \frac{f(z)}{(x-z)^2} \, dz.
\]

Differentiating, we find,

\[
\partial_x^k(K_{\eta_t} - \Lambda)f(x) = \frac{2}{\eta_t} f^{(k)}(x) + (-1)^{k+1} k! \int_{|x-z| > \eta_t} \frac{f(z)(x-z)^{-k-2}}{s^2 + (x-y)^2} \, dz.
\]

We will first derive an estimate on the second term. In order to estimate the first we later derive a Hölder estimate for \( p_{\eta_t}(t, x, y) \).

We first derive the following estimate.

**Lemma 5.1.** We have,

\[
\int_{|x-z| \geq \eta_t} \frac{|f_{\eta_t}(z)|}{|x-z|^{k+2}} \, dz \leq C \int_{|z-x| \leq \eta_t} \frac{1}{|z|^{k+2}} \frac{s}{|z|^{k+2} + (z-x)^2} \, dz \leq \frac{C}{\eta_t^{k+1}} \left( \frac{s}{s^2 + (x-y)^2} + \frac{\eta_t}{\eta_t^2 + (x-y)^2} \right).
\]

for \( 0 \leq s \leq \eta_t \).

**Proof.** By (5.6) we can estimate

\[
\int_{|x-z| \geq \eta_t} \frac{|f_{\eta_t}(z)|}{|x-z|^{k+2}} \, dz \leq C \int_{|z-x| \leq \eta_t} \frac{1}{|z|^{k+2}} \frac{s}{|z|^{k+2} + (z-x)^2} \, dz + C \int_{|z-x| \geq \eta_t} \frac{1}{|z|^{k+2}} \frac{s}{|z|^{k+2} + (z-x)^2} \, dz.
\]

For the first term we have

\[
\int_{|z-x| \leq \eta_t} \frac{1}{|z|^{k+2}} \frac{s}{|z|^{k+2} + (z-x)^2} \, dz \leq C \int_{|z-x| \leq \eta_t} \frac{s}{|z|^{k+2} + (z-x)^2} \, dz \leq \frac{C}{\eta_t^{k+1}} \left( \frac{s}{s^2 + (x-y)^2} + \frac{\eta_t}{\eta_t^2 + (x-y)^2} \right).
\]

For the second term we first consider the case \( |x-y| \geq \eta_t/2 \). We have,

\[
\int_{|z-x| \geq \eta_t/2} \frac{1}{|z|^{k+2}} \frac{s}{|z|^{k+2} + (z-x)^2} \, dz \leq \frac{C}{|x-y|^{k+2}} \int_{|z-x| \geq \eta_t/2} \frac{s}{|z|^{k+2} + (z-x)^2} \, dz \leq \frac{C}{\eta_t^{k+1}} \left( \frac{s}{s^2 + (x-y)^2} \right).
\]

In the case \( |x-y| \leq \eta_t/2 \) we have \( |z-(x-y)| \geq c|z| \) for \( |z| \geq \eta_t \) and so

\[
\int_{|z-x| \geq \eta_t/2} \frac{1}{|z|^{k+2}} \frac{s}{|z|^{k+2} + (z-x)^2} \, dz \leq \frac{Cs}{\eta_t^{k+1}} \int_{|z-x| \geq \eta_t/2} \frac{s}{|z|^3} \, dz \leq \frac{Cs}{\eta_t^{k+1}} \left( \frac{s}{s^2 + (x-y)^2} \right).
\]

because \( \eta_t \geq s \) and \( \eta_t \geq c|x-y| \).

We now derive the following Hölder estimate for \( p_{\eta_t} \).

**Lemma 5.2.** For any \( 0 \leq \alpha < 1 \) and \( 0 \leq t \leq \eta_t \) we have

\[
|p_{\eta_t}(t, x, y) - p_{\eta_t}(t, z, y)| \leq \frac{C|x-z|^\alpha}{t^{1+\alpha}}.
\]
Proof. We have
\[ p_{\eta_t}(t, x, y) = p_{\Lambda}(t, x, y) + \int_0^t (e^{(t-s)\Lambda} g(s))(x)ds \]  
(5.16)
where we denoted \( g(s, w) = [(K - \Lambda) f_{\eta_t}(s)](w) \). The first term satisfies the desired bound so we need only estimate the second term. From (5.6) and (5.10) we have the estimate
\[ |g(s, w)| \leq C \left( \frac{s}{s^2 + w^2} + \frac{\eta_t}{\eta_t^2 + w^2} \right). \]  
(5.17)
We estimate
\[ \left| (e^{(t-s)\Lambda} g(s))(x) - (e^{(t-s)\Lambda} g(s))(z) \right| \]
\[ \leq \frac{1}{\pi} \int \frac{t - s}{(t-s)^2 + (x-w)^2} - \frac{t - s}{(t-s)^2 + (z-w)^2} \left| g(s, w) \right| dw \]
\[ \leq C(t - s) \int \frac{|x - z|^\alpha (|x - w|^{2\alpha} + |z - w|^{2\alpha})}{((t-s)^2 + (x-w)^2)((t-s)^2 + (z-w)^2)} |g(s, w)| dw \]
\[ \leq \frac{C|x - z|^\alpha}{(t-s)^\alpha} \int \frac{t - s}{(t-s)^2 + (x-w)^2} |g(s, w)| dw \]
\[ + \frac{C|x - z|^\alpha}{(t-s)^\alpha} \int \frac{t - s}{(t-s)^2 + (z-w)^2} |g(s, w)| dw \]  
(5.18)
Using (5.17) and the bound
\[ \int \frac{a}{a^2 + w^2} \frac{b}{b^2 + (w-c)^2} dw \leq C \frac{b \vee a}{(b \vee a)^2 + c^2} \]  
(5.19)
we get
\[ \left| (e^{(t-s)\Lambda} g(s))(x) - (e^{(t-s)\Lambda} g(s))(z) \right| \leq \frac{C|x - z|^\alpha}{(t-s)^\alpha} \frac{1}{t}. \]  
(5.20)
The claim follows after integrating in s. \( \square \)

Lemma 5.3. Let \( 0 < \alpha < 1 \). We have,
\[ |\partial_x p_{\eta_t}(t, x, y)| \leq C \left( \frac{1 + |\log \delta|/t^{\eta_t}}{t^2 + (x-y)^2} + \frac{\delta^\alpha}{t} \right), \]  
(5.21)
for any \( t \leq \eta_t \) and \( 0 < \delta < 1 \).

Proof. We have
\[ \partial_x p_{\eta_t}(t, x, y) = \partial_x p_{\Lambda}(t, x, y) - \int_0^t e^{(t-s)\Lambda} \partial_x [(K - \lambda) f_{\eta_t}](x)ds. \]  
(5.22)
We write
\[ e^{(t-s)\Lambda} \partial_x [(K - \lambda) f_{\eta_t}](x) = - \int (\partial_x \frac{t - s}{(t-s)^2 + (w-x)^2}) f_{\eta_t}(w) dw \]
\[ + \int \frac{t - s}{(t-s)^2 + (w-x)^2} g_1(s, w) dw \]  
(5.23)
(5.24)
where
\[ g_1(s, w) := \int_{|w| \geq |w-z| \geq \eta_t} \frac{1}{(w-z)^2} f_{\eta_t}(z) dz. \]  
(5.25)
From (5.10) we have
\[ |g_1(s, w)| \leq \frac{C}{\eta_t^2} \left( \frac{s}{s^2 + (w-y)^2} + \frac{\eta_t}{\eta_t^2 + (w-y)^2} \right). \]  
(5.26)
For (5.24) we have
\[
\left| \int \frac{t-s}{(t-s)^2 + (w-x)^2} g_1(s,w) \, dw \right| \leq \frac{C}{\eta^2} \int \frac{t-s}{(t-s)^2 + (w-(x-y))^2} \left( \frac{s}{s^2 + w^2} + \frac{\eta}{\eta^2 + w^2} \right) \, dw.
\] (5.27)

Using (5.19) and \( t \leq \eta \) one easily concludes
\[
\frac{C}{\eta^2} \int \frac{t-s}{(t-s)^2 + (w-(x-y))^2} \left( \frac{s}{s^2 + w^2} + \frac{\eta}{\eta^2 + w^2} \right) \, dw \leq \frac{C}{t^2 + (x-y)^2}.
\] (5.28)

For the term (5.23) we first note that
\[
\int \partial_w \frac{t-s}{(t-s)^2 + (w-x)^2} f_{\eta t}(s,w) \, dw = \int \partial_w \frac{t-s}{(t-s)^2 + (w-x)^2} (f_{\eta t}(s,w) - f_{\eta t}(s,x)) \, dw.
\]

Hence, using (5.6) and Lemma 5.2 we can estimate
\[
\left| \int_0^t \partial_w \frac{t-s}{(t-s)^2 + (w-x)^2} f_{\eta t}(s,w) \, dwds \right| \leq C \int_0^{t(1-\delta)} \int \frac{(t-s)|x-w|}{((t-s)^2 + (w-x)^2)^2 s^2 + (w-y)^2} \, dwds
\]
\[+ C \int_t^{-t(1-\delta)} \frac{(t-s)|x-w|}{((t-s)^2 + (w-x)^2)^2} s^{-1-\alpha} \, dwds \] (5.29)

For (5.29), we use (5.19) to obtain
\[
\int_0^{t(1-\delta)} \frac{(t-s)|x-w|}{((t-s)^2 + (w-x)^2)^2 s^2 + (w-y)^2} \, dwds \leq \int_0^{t(1-\delta)} \frac{1}{t-s (t-s)^2 + (w-x)^2} \frac{t-s}{s^2 + (w-y)^2} \, dwds
\]
\[\leq \frac{C}{t^2 + (x-y)^2} \int_0^{t(1-\delta)} s^{-1-\alpha} \, dwds \leq C \log \delta \cdot \frac{t}{t^2 + (x-y)^2}.
\]

For (5.30), we use
\[
\frac{|w-x|^{1+\alpha}}{(t-s)^2 + (w-x)^2} \leq \frac{1}{(t-s)^{1-\alpha}}
\]
to obtain
\[
\int_t^{-t(1-\delta)} \frac{(t-s)|x-w|^{1+\alpha}}{((t-s)^2 + (w-x)^2)^2 s^{-1-\alpha}} \, dwds \leq \int_t^{-t(1-\delta)} (t-s)^{-1+\alpha} \int_{(t-s)^2 + (w-x)^2} \frac{t-s}{s^{1-\alpha}} \, dwds
\]
\[\leq \frac{C \delta^\alpha}{1-\alpha} t.
\]

This yields the claim. \( \square \)

We now can conclude with estimates of the spatial derivatives of \( p_{\eta t}(t,x,y) \).

**Theorem 5.4.** Fix \( D_1 > 0 \) and \( D_2 > 0 \) and \( \delta > 0 \). Let \( \eta_t = N^{x_{\eta t}}/N \) for each \( k \) we have
\[
\left| \partial_x^k p_{\eta t}(t,x,y) \right| \leq \frac{C}{t^k t^2 + (x-y)^2} + N^{-D_2},
\] (5.31)
for \( N^{-D_1} \leq t \leq N^{-\delta} \eta_t \). For \( |x-y| > N^{\epsilon} \eta_t \) we have
\[
\left| \partial_x^k p_{\eta t}(t,x,y) \right| \leq N^{-D_2}.
\] (5.32)

**Proof.** The bound (5.31) for \( k = 1 \) follows by taking \( \delta = N^{-D} \) in (5.21). For \( k \geq 2 \) we have by the Chapman-Kolmogorov equation and translation invariance:
\[
\partial_x^{(k-1)} p_{\eta t}(t,x,y) = \int p(t/2,x,w)p^{(k-1)}(t/2,w,y) \, dy.
\]
Differentiating once more, we have
\[
\partial_x^{(k)} p_{\eta}(t, x, y) = \int \partial_x p(t/2, x, w) p^{(k-1)}(t/2, w, y) \, dy. \tag{5.33}
\]

The bounds (5.31) for higher \( k \) then follow by strong induction and (5.19).

For (5.32) we use translation invariance and the Chapman-Kolmogorov equation to write for \( x > 0 \),
\[
\partial_x p_{\eta}(t, x, 0) = \int_{w \leq x/2} \partial_w p_{\eta}(t/2, x, w) p_{\eta}(t/2, w, 0) \, dw \\
- \int_{w > x/2} \partial_w p_{\eta}(t/2, x, w) p_{\eta}(t/2, w, 0) \, dw + \int_{w \leq x/2} \partial_w p_{\eta}(t/2, x, w) p_{\eta}(t/2, w, 0) \, dw \\
+ p_{\eta}(t/2, x, 0) \partial_x p_{\eta}(t/2, x, 2, 0). \tag{5.34}
\]

If \( |x| > N^\varepsilon \eta \), we see that each term is \( O(N^{-D}) \) using (5.32). The higher order derivatives can be handled similarly.

Finally we handle the time derivative.

**Theorem 5.5.** Fix \( D_1, D_2 > 0 \) and \( \delta > 0 \). For \( N^{-D_1} \leq t \leq N^{-\delta} \eta \) we have
\[
|\partial_t p_{\eta}(t, x, y)| \leq \frac{C}{t^2 + (x - y)^2} + N^{-D_2}. \tag{5.35}
\]

**Proof.** We write
\[
\partial_t p_{\eta}(t, x, 0) = \int_{|x - z| \leq \eta} \frac{p_{\eta}(t, x, 0) - p_{\eta}(t, z, 0)}{(x - z)^2} \, dz = \int_{|x - z| \leq t} \frac{p_{\eta}(t, x, 0) - p_{\eta}(t, z, 0)}{(x - z)^2} \, dz \tag{5.36}
\]
\[
+ \int_{|x - z| > t} \frac{p_{\eta}(t, x, 0) - p_{\eta}(t, z, 0)}{(x - z)^2} \, dz \tag{5.37}
\]

An argument similar to the proof of Lemma 5.1 yields
\[
\left| \int_{|x - z| > t} \frac{p_{\eta}(t, x, 0) - p_{\eta}(t, z, 0)}{(x - z)^2} \, dz \right| \leq C \int_{|x - z| > t} \frac{1}{(x - z)^2} \left( \frac{t}{t^2 + x^2} + \frac{t}{t^2 + z^2} \right) \, dz \leq \frac{C}{t^2 + x^2}. \tag{5.38}
\]

We now turn to (5.36). Note that
\[
\sup_{|u - x| \leq t} \frac{1}{t^2 + u^2} \leq \frac{C}{t^2 + x^2}. \tag{5.39}
\]

Hence by a second order Taylor expansion,
\[
\int_{|x - z| \leq t} \frac{p_{\eta}(t, x, 0) - p_{\eta}(t, z, 0)}{(x - z)^2} \, dz \\
= \int_{|x - z| \leq t} \frac{\partial_x p_{\eta}(t, x, 0)}{(x - z)} + O \left( \frac{1}{t} \frac{1}{t^2 + x^2} + N^{-D} \right) \, dz = O \left( \frac{1}{t^2 + x^2} + N^{-D} \right). \tag{5.40}
\]

This yields the claim.

\[
\square
\]

\section{6 Mesoscopic linear statistics}

Let \( \varphi_N \) be a sequence of twice differentiable functions such that:
\[
\| \varphi_N \|_{L^\infty}, \| \varphi_N' \|_{L^1} \leq C, \tag{6.1}
\]
\[
\text{supp} \varphi_N'(x) \cap [-t^{1/4}, t^{1/4}] \subset (-t_1 N^\alpha, t_1 N^\alpha), \tag{6.2}
\]
for some $0 < r < \omega_1/100$.

We will assume throughout that $t_1 = N^{\omega_1}/N$. As for the parameter $t$, we assume $t_1 N^r < t < 1$, and we will denote $t = N^{\omega_0}/N$.

Finally, since the spectrum of $H_t$ is contained in $-N^{C_V} - 1, N^{C_V} + 1$ with overwhelming probability, there is no loss of generality in making the following assumption on the support of $\varphi_N$:

$$\text{supp} \varphi_N \subset [-N^{C_V} - 2, N^{C_V} + 2]. \quad (6.3)$$

We can now state the main result of this section.

**Theorem 6.1.** Let $\varphi_N$ be a sequence of real-valued $C^2(\mathbb{R})$ functions satisfying (6.1), (6.2), (6.3) in addition to the following growth conditions on the derivatives:

$$\|\varphi_N^{(k)}\|_{L^\infty} \leq Ct_1^{-k+1}, \quad k = 1, 2, \quad (6.4)$$

and

$$\int \int \left( \frac{\varphi_N(x) - \varphi_N(y)}{x - y} \right)^2 \, dx \, dy \geq c. \quad (6.5)$$

Let the parameters $t = N^{\omega_0}/N$ and $t_1 = N^{\omega_1}/N$ satisfy

$$\omega_0 > \omega_1 > \omega_0/2 \quad (6.6)$$

Then, uniformly in $|x| \leq N^{\omega_1/8 - \omega_0/16}$,

$$\mathbb{E}[e^{ix(\text{tr} \varphi_N)(H_t) - \text{tr} \varphi_N(H_t))}] = \exp \left( -\frac{x^2}{2} V(\varphi_N) \right) + \mathcal{O}(N^{\omega_0/4 - \omega_1/2}).$$

Here $V(\varphi_N)$ is a quadratic functional in $\varphi_N$ such that

$$V(\varphi_N) = \frac{-1}{\pi^2} \int_{-Ct}^{Ct} \varphi_N(\tau)(H\varphi_N'(\tau)) \, d\tau + \mathcal{O}(1)$$

and

$$= \frac{1}{2\pi^2} \int_{-Ct}^{Ct} \int_{-Ct}^{Ct} \left( \frac{\varphi_N(\tau) - \varphi_N(s)}{\tau - s} \right)^2 \, ds \, d\tau + \mathcal{O}(1). \quad (6.7)$$

Here $C > 0$ is some (small) constant, and $H$ denotes the Hilbert transform (see 6.120). In particular,

$$V(\varphi_N) \geq c \log t/t_1 + \mathcal{O}(1)$$

if $\varphi_N = \sum x(y/(t_1N^\omega))p_{t_1}(0, y) \, dy$.

If $\text{supp} \varphi_N \subset (-N^\gamma t_1, N^\gamma t_1)$, then we have the more precise evaluation:

$$V(\varphi_N) = \frac{-1}{\pi^2} \int \varphi_N(\tau)(H\varphi_N'(\tau)) \, d\tau + \mathcal{O}(N^{\omega_0/2 - \omega_1} N^{2r})$$

and

$$= \frac{1}{2\pi^2} \int \left( \frac{\varphi_N(\tau) - \varphi_N(s)}{\tau - s} \right)^2 \, ds \, d\tau + \mathcal{O}(N^{\omega_0/2 - \omega_1} N^{2r}). \quad (6.8)$$

**Remark.** We make several comments concerning Theorem 6.1 and the many conditions in the statement.

1. The first inequality of (6.6) ensures that the scale of the function is smaller than the time scale of DBM. See the remark after the statement of Theorem 2.5.

2. The second inequality of (6.6) is technical, but removing it requires substantial modification of some of the estimates below. It is used in particular to simplify the handling of some error terms in the proof of Proposition 6.5, which is key in deriving the theorem.

3. The condition (6.5) ensures the limiting random variable is non-degenerate, that is, its variance is bounded below. It is used only in the final integration at (6.46). Our method can be extended to cover the case of vanishing variance, but we will have no need for such an extension.
4. A typical situation in which Theorem 6.1 holds with the approximation (6.8) is when

$$\varphi_N(x) = \varphi(x/t_1),$$

where $\varphi$ is some smooth function either compactly supported or vanishing as $|x| \to \infty$. This setting has been studied extensively in the random matrix theory literature (see for example [30], [56], [43]), and is typically what is being referred to when one speaks of “linear statistics of mesoscopic observables”. The more general theorem above is essential for the main result of this paper.

5. If the functions $\varphi_N$ do not have spatial decay, the variance of the linear statistics grows logarithmically. This should be compared to the well-known fact that the variance of the number of eigenvalues in an interval grows like $\log N$. See [28, 59]. A function $\varphi_N$ with “large” (compared to $t$) support, but whose derivative is supported in a region of size $t_1$ is, up to a linear transformation, an approximation on scale $t_1$ of an indicator function.

A central role will be played by the resolvent matrix

$$G(z) = (H_t - z)^{-1},$$

where $z = \tau + i\eta \in \mathbb{C}$. The normalized trace of $G$ is denoted by $m_N(z)$:

$$m_N(z) = \frac{1}{N} \text{tr}(G(z)).$$

The latter quantity closely approximates the Stieltjes transform $m_{t_1}(z)$ of the deformed semicircle law.

Let $H^{(j)}$ be the $(j,j)$-submatrix of $H_t$, that is, the $(N - 1) \times (N - 1)$ matrix obtained from the Wigner matrix $H$ by removing the $j$th row and column. We introduce the following notation for the resolvent of $H^{(j)}$ and its normalized trace:

$$G^{(j)}(z) := (H^{(j)}_t - z)^{-1}, \quad m^{(j)}_N(z) := \frac{1}{N} \text{tr}(H^{(j)}_t - z)^{-1}.$$  

Following [61], we reserve special symbols for two quantities involving $G$ and $G^{(j)}$ which will play a role in the computations to come. First, we denote, for $j = 1, \ldots, N$,

$$A_j = A_j(z) := -\frac{1}{G_{jj}(z)}.$$  

Next, we let $h^{(1)} := (h_{1i})_{1 \leq i \leq N}$, and then define

$$B_j = B_j(z) := \langle (G^{(j)}(z))^2 h^{(j)}, h^{(j)} \rangle,$$  

where $\langle u, v \rangle$ denotes the inner product of the vectors $u, v \in \mathbb{C}^N$. The importance of these quantities for us comes mainly through the identity (6.25).

Following [52], we also define

$$g_i(z) = \frac{1}{V_i - z - t m_{t_1}(z)},$$

so that

$$m_{t_1}(z) = \frac{1}{N} \sum_{i=1}^{N} g_i(z).$$

Finally, we define

$$R_2(z) = \frac{1}{N} \sum_{i=1}^{N} g_i(z)^2, \quad \tilde{R}_2(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\text{E}[A_j(z)]^2}.$$  

We will often deal with centered random variables. For a random variable $X$ with $\text{E}[X] < \infty$, we denote by

$$X^\circ := X - \text{E}[X],$$

the corresponding centered random variable.
6.1 Estimates for \( A_j \) and \( B_j \)

The following definition will be useful.

**Definition 6.2 (Stochastic Domination).** Let

\[
X = (X^{(N)}(u) : N \in \mathbb{N}, u \in U^{(N)}), \quad Y = (Y^{(N)} : N \in \mathbb{N}, u \in U^{(N)})
\]

be two families of nonnegative random variables, where \( U^{(N)} \) is a possibly \( N \)-dependent parameter set. We say that \( X \) is stochastically dominated by \( Y \), uniformly in \( u \), if for all small \( \epsilon > 0 \) and all (large) \( D \) we have

\[
\sup_{u \in U^{(N)}} \mathbb{P}(X^{(N)} > N^\epsilon Y^{(N)}(u)) \leq N^{-D}
\]

for all \( N \geq N_0(\epsilon, D) \). If \( X \) is stochastically dominated by \( Y \), uniformly in \( u \), we write

\[
X \prec Y.
\]

(6.16)

For complex valued \( Y \), we write \( Y = \mathcal{O}_<(X) \) if \( |Y| < X \).

Recall the definition of \( \mathcal{D}_{\epsilon,q} \). We define \( \Omega_N \) as the union of this region with its reflection about the real axis, with a choice \( \epsilon = \xi \) to be determined.

\[
\Omega_N = (\mathcal{D}_{\xi,q} \cup \overline{\mathcal{D}_{\xi,q}})
\]

\[
= \left\{ z = E + i\eta : E \in \mathcal{I}_q, N^{10CV} \geq |\eta| \geq N^\xi/N \right\} \cup \{ z : E + i\eta : |E| \leq N^{2CV}, N^{CV} \geq |\eta| \geq c \}.
\]

(6.17)

Since \( m_N(z) = \overline{m_N(z)} \) and \( m_{fc,t}(z) = \overline{m_{fc,t}(z)} \), the local law extends to \( z \in \Omega_N \).

The following estimates for the quantities \( A \) and \( B \) defined in (6.12) and (6.13):

**Theorem 6.3.** We have, uniformly in \( z \in \Omega_N \),

\[
\mathbb{E}_j A_j(z) = z + tm_{fc,t}(z) - V_j + \mathcal{O}_<(t(N|\eta|)^{-1}),
\]

(6.18)

\[
A_j = \mathcal{O}_<(\sqrt{t}N^{-1/2} + t(N|\eta|)^{-1/2})
\]

(6.19)

\[
B_j(z) = t\partial_z m_{fc,t}(z) + \mathcal{O}_<(t|\eta|^{-1}(N|\eta|)^{-1/2}).
\]

(6.20)

**Proof.** Recall the definition of \( A_j(z + i\eta) \) above. By the Schur complement formula [52, Lemma 7.7],

\[
A_j(z) = z - h_{jj} + \langle G^{(j)} h^{(j)}, h^{(j)} \rangle.
\]

(6.21)

Taking the partial expectation over \( h^{(j)} = (h_{ji})_{i \neq j} \), we have

\[
\mathbb{E}_j A_j(z) = z - V_j + \frac{t}{N} \text{tr} G^{(j)}(z).
\]

(6.22)

Using the local law, we obtain

\[
\mathbb{E}_j A_j(z) = z - V_j + tm_{fc,t}(z) + \mathcal{O}_<(t(N|\eta|)^{-1}),
\]

which is (6.18).

The estimate (6.19) is proved in [52, Lemma 7.9] using the local law.

For (6.20), note that

\[
B_j = \langle (G^{(j)})^2 h^{(j)}, h^{(j)} \rangle = \partial_z \langle G^{(j)} h^{(j)}, h^{(j)} \rangle.
\]

Taking the expectation with respect to \( h^{(j)} \) first, and then using the local law, we find

\[
\mathbb{E}[B_j] = t\mathbb{E}[\partial_z m_N^{(j)}] = t\partial_z \int_{|z-\zeta|=\eta/2} \frac{\mathbb{E}[m_N^{(j)}(\zeta)]}{z-\zeta} d\zeta
\]

(6.23)

\[
= t\partial_z m_{fc,t}(z) + \mathcal{O}_<(t|\eta|^{-1}(N|\eta|)^{-1/2}).
\]

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For the second moment, we have

\[
\mathbb{E}[|B_j|^2] = \mathbb{E}\left[ \sum_{i,r,k}^{(j)} G_{ir}^{(j)} G_{rk}^{(j)} (h_{ij} h_{kj} - \delta_{ik} t N^{-1}) \right]^2 + O(t^2 |\eta|^{-2} (N |\eta|)^{-2})
\]

\[
= \mathbb{E} \sum_{i,l,k,r,m,n}^{(j)} G_{il}^{(j)} G_{lk}^{(j)} G_{rm}^{(j)} G_{mn}^{(j)} \mathbb{E}_j [(h_{ij} h_{kj} - N^{-1} t \delta_{ik})(h_{jr} h_{nj} - N^{-1} t \delta_{rn})] + O(t (N |\eta|)^{-2})
\]

\[
= \frac{2t^2}{N^2} \text{Etr}(G^{(j)} |\eta|^4 + O(t (N |\eta|)^{-2}) = O(t^2 |\eta|^{-2} (N |\eta|)^{-1}). (6.24)
\]

(6.20) now follows from the large deviation type estimates in [52, Lemma 7.7], and the local law.

We will repeatedly use the identity:

\[
N(m_N - m_N^{(j)}) = G_{jj} \left( 1 + \sum_{i,l,k}^{(j)} h_{ij} G_{il}^{(j)} G_{lk}^{(j)} h_{kj} \right) = -A_j^{-1}(1 + B_j). \tag{6.25}
\]

The following lemma collects the main estimates we need for this quantity.

**Lemma 6.4.** Uniformly for in $\tau + i \eta \in \Omega_N$,

\[
A_j^{-1}(1 + B_j) = \frac{1 + t \hat{c}_{\text{mfc},t}^r}{\mathbb{E}[A_j]} + O_{\prec}(|\eta|^{-1} (N |\eta|)^{-1/2}), \tag{6.26}
\]

\[
\left( (A_j^{-1}) (1 + B_j) \right)^{\circ} = \frac{B_j^\circ}{\mathbb{E}[A_j]} - \frac{A_j^\circ (1 + \mathbb{E}[B_j])}{\mathbb{E}[A_j]^2} + \frac{1}{\mathbb{E}[A_j]^2} (A_j^\circ B_j^\circ)^{\circ} + \frac{1}{\mathbb{E}[A_j]} \left( \frac{(A_j^\circ)^2}{A_j} (1 + B_j) \right)^{\circ}. \tag{6.27}
\]

**Proof.** The first estimate follows directly from (6.19), (6.20) and the stability estimate [52, Eqn (7.8)]

\[
|V_i - z - t m_{\text{fc},t}(z)| \geq c \max(t, |\eta|). \tag{6.28}
\]

We begin by using the expansion

\[
\frac{1}{A} = \frac{1}{\mathbb{E} A} \cdot \frac{1}{1 + \frac{A}{\mathbb{E} A}} = \frac{1}{\mathbb{E} A} \left( 1 - \frac{A^\circ}{\mathbb{E} A} + \frac{(A^\circ)^2}{(\mathbb{E} A)^2} - \ldots + (-1)^k \frac{1}{\mathbb{E} A^k} \frac{(A^\circ)^k}{1 + \frac{A}{\mathbb{E} A}} \right). \tag{6.29}
\]

For (6.27), we expand using (6.29) with $k = 2$,

\[
\left( (A_j^{-1}) (1 + B_j) \right)^{\circ} = \frac{B_j^\circ}{\mathbb{E}[A_j]} - \frac{1}{\mathbb{E}[A_j]^2} (A_j^\circ (1 + B_j))^\circ + \left( \frac{(A_j^\circ)^2 (1 + B_j)}{\mathbb{E} A_j + A_j^\circ} \right)^{\circ} \tag{6.30}
\]

\[
= \frac{B_j^\circ}{\mathbb{E}[A_j]} - \frac{A_j^\circ (1 + \mathbb{E}[B_j])}{\mathbb{E}[A_j]^2} + \frac{1}{\mathbb{E}[A_j]^2} (A_j^\circ B_j^\circ)^{\circ} + \frac{1}{\mathbb{E}[A_j]} \left( \frac{(A_j^\circ)^2}{A_j} (1 + B_j) \right)^{\circ}. \tag{6.31}
\]

\]

6.2 Computation of the characteristic function

We derive an equation for the derivative of the characteristic function of the linear statistic. Let $z = \tau + i \eta$. Recall the definition of $C_V$ in (2.3). Without loss of generality, we can assume $C_V \geq 5$. We let $\chi$ be a smooth cut-off function such that $\chi(x) = 1$, for $|x| \leq N^{10C_V} - 1$ and $\chi(x) = 0$, for $|x| \geq N^{10C_V}$. Next, define the almost analytic extension of $\phi_N$ to $\mathbb{C}$.

\[
\hat{\phi}_N(z) = \chi(\eta) (\phi(z) + i \eta \phi_N'(z)).
\]

The Helffer-Sjöstrand formula is the following representation of $\phi_N$:

\[
\phi_N(\lambda) = \frac{1}{\pi} \int \hat{z} \hat{\phi}_N(\tau + i \eta) \frac{e^{\lambda \tau - i \eta}}{\lambda - \tau - i \eta} d\tau d\eta = \frac{1}{\pi} \int \frac{i \eta \phi_N''(\tau) \chi(\eta) + i (\phi_N(\tau) + i \eta \phi_N'(\tau)) \chi'(\eta)}{\lambda - \tau - i \eta} d\tau d\eta. \tag{6.31}
\]

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We write
\[ e(x) := \exp \left( ix \left( \text{tr}[\varphi_N] - \mathbb{E}[\varphi_N] \right) \right), \quad \psi(x) := \mathbb{E}[e(x)]. \] (6.32)

By (6.31), the derivative \( \psi'(x) \) equals
\[ \frac{i}{\pi} \int_{\mathbb{R}^2} (i \eta \varphi_N^\prime(\tau) \chi(\eta) + i (\varphi_N(\tau) + i \eta \varphi_N^\prime(\tau)) \chi'(\eta)) E(z) \, d\tau d\eta, \] (6.33)
\[ E(z) := N \mathbb{E}[e(x)(m_N(\tau + i \eta) - \mathbb{E}m_N(\tau + i \eta))]. \] (6.34)

The rest of this section is concerned with computing \( E(z) \). Let
\[ e_j(x) := \exp \left( ix \int_{\mathbb{R}^2} \partial_{\bar{z}} \tilde{\varphi}_N(\tau) \text{tr}G^{(j)}(\tau + i \eta)^{o} \, d\eta d\tau \right). \]

We write
\[ \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\varphi}_N(z) \mathbb{E}[\text{tr}G(\tau + i \eta) e^\circ] \, dz = \int \partial_{\bar{z}} \tilde{\varphi}_N \sum_{j=1}^{N} \mathbb{E} \left[ G_{jj}(\tau + i \eta) e^\circ \right] \, dz \]
\[ = \int \partial_{\bar{z}} \tilde{\varphi}_N \sum_{j=1}^{N} \mathbb{E} \left[ G_{jj}(\tau + i \eta) e_j^\circ \right] \, dz + \int \partial_{\bar{z}} \tilde{\varphi}_N \sum_{j=1}^{N} \mathbb{E} \left[ G_{jj}^\circ(\tau + i \eta - e_j) \right] \, dz. \] (6.35)

In view of (6.35), we define
\[ T_1(\tau, \eta) := \sum_{j=1}^{N} \mathbb{E} \left[ G_{jj}^\circ(\tau + i \eta) e_j^\circ \right], \quad T_2(\tau, \eta) := \sum_{j=1}^{N} \mathbb{E} \left[ G_{jj}^\circ(\tau + i \eta - e_j) \right]. \] (6.36)

We compute these two terms in Propositions 6.6 and 6.5. The result of Proposition 6.6 is
\[ T_1 = t \tilde{R}_2(z) \cdot \mathbb{E}[e(x) \text{tr}G(\tau + i \eta)^{o}] + O_{<} \left( N^{-1/2} |\eta|^{-3/2} \right) + |x|O_{<} \left( |\eta|^{-1} N^{-1/2} \|\varphi_N^{\prime o}\|_{L^1}^{1/2} \|\varphi_N^\prime\|_{L^1}^{1/2} \right). \]

By the definition of \( T_1 \) and \( T_2 \) (6.36), we have, for \( z \in \Omega_N \),
\[ (1 - t \tilde{R}_2(z)) \mathbb{E}[e^\circ(x) \text{tr}G(z)] = T_2(z) + O_{<} \left( N^{-1/2} |\eta|^{-3/2} \right) + |x|O_{<} \left( |\eta|^{-1} N^{-1/2} \|\varphi_N^{\prime o}\|_{L^1}^{1/2} \|\varphi_N^\prime\|_{L^1}^{1/2} \right), \]
so, since \( |1 - t \tilde{R}_2| \geq c \) by [52, Eqn (7.10)] and Proposition 6.3,
\[ \mathbb{E}[e^\circ(x) \text{tr}G(z)] = \frac{T_2(z)}{1 - t \tilde{R}_2(z)} + O_{<} \left( N^{-1/2} |\eta|^{-3/2} \right) + |x|O_{<} \left( |\eta|^{-1} N^{-1/2} \|\varphi_N^{\prime o}\|_{L^1}^{1/2} \|\varphi_N^\prime\|_{L^1}^{1/2} \right). \] (6.37)

Write:
\[ \int \partial_{\bar{z}} \tilde{\varphi}_N(z) \mathbb{E}[e^\circ(x) \text{tr}G(z)] \, dz = \int_{\Omega_N} \partial_{\bar{z}} \tilde{\varphi}_N(z) \mathbb{E}[e^\circ(x) \text{tr}G(z)] \, dz + \int_{\partial \Omega_N} \partial_{\bar{z}} \tilde{\varphi}_N(z) \mathbb{E}[e^\circ(x) \text{tr}G(z)] \, dz \]
\[ =: I_1 + I_2. \]

For \( I_2 \), note that \( \Omega_N^c \cap \text{supp} \chi(\eta) \subset \{ z : |\text{Im} z| < N^{-1+\xi} \} \), so we have
\[ I_2 = 2 \int_{0 < \eta < N^{-1+\xi}} i \eta \varphi_N^\prime(\tau + i \eta) \chi(\eta) \mathbb{E}[e(x)(\text{Im} \text{tr}G(z))^o] \, dz \] (6.38)
\[ = O_{<} \left( N^{-1+\xi} \|\varphi_N^\prime\|_{L^1} \right). \]

For \( I_1 \), we use (6.37):
where
\[
\Delta_1 = \mathbb{E}[e^z(x)\text{tr}G(z)] - \frac{T_2(z)}{1-tR_2(z)}
\]
is a holomorphic function in \(\Omega_N\) satisfying the bounds:
\[
\Delta_1 = O_\prec((N^{-1/2}\eta)^{-3/2}) + |x|O_\prec\left(|\eta|^{-1}N^{-1/2}\|\varphi''_N\|_{L^1}^{1/2}\|\varphi'_N\|_{L^1}^{1/2}\right).
\]
Using integration by parts in \(\tau = \text{Re}z\) when \(|\eta| \geq \|\varphi''_N\|_{L^1}\) as in the proof of Lemma 6.7 (see (6.63)), it is easily shown that
\[
\int_{\Omega_N} \partial_z \tilde{\varphi}_N(z) \Delta_1 \, dz = (1 + |x|)O_\prec((N^{-1/2}\log N)\|\varphi''_N\|_{L^1}^{1/2}\|\varphi'_N\|_{L^1}^{3/2}). \tag{6.41}
\]

We compute the main term in \(I'_1\). We need an expression for \(T_2\). The next proposition will be proved in the following sections.

**Proposition 6.5.** Let
\[
S_{2,1}(z, z') = \frac{t^2}{N} \sum_{j=1}^{N} g_j(z)^2 g_j(z') \partial_{z'} \frac{m_{t,\ell}(z) - m_{t,\ell}(z')}{z - z'}, \tag{6.42}
\]
\[
S_{2,2}(z, z') := \frac{t^2}{N} \sum_{j=1}^{N} g_j(z)^2 g_j(z')^2 (1 + t\partial_z m_{t,\ell}(z')) \frac{m_{t,\ell}(z) - m_{t,\ell}(z')}{z - z'}, \tag{6.43}
\]
and
\[
S_{2,3}(z, z') := \frac{t}{N} \sum_{j=1}^{N} g_j(z)^2 g_j(z')^2 (1 + t\partial_z m_{t,\ell}(z')). \tag{6.44}
\]

The quantity \(I'_1\) (6.39) is given by
\[
I'_1 = -\frac{2ix}{\pi} \mathbb{E}[e(x)] \int_{\Omega_N} \int_{\Omega_N} \partial_z \tilde{\varphi}_N(z) \partial_z \tilde{\varphi}_N(z') \frac{1}{1-tR_2(z)} S_{2,1}(z, z') \, dz \, dz' + \frac{2ix}{\pi} \mathbb{E}[e(x)] \int_{\Omega_N} \int_{\Omega_N} \partial_z \tilde{\varphi}_N(z) \partial_z \tilde{\varphi}_N(z') \frac{1}{1-tR_2(z)} (S_{2,2}(z, z') + S_{2,3}(z, z')) \, dz \, dz'
\]
\[
+ |x|O(t^{1/2}N^{-1/2+2\epsilon})\|\varphi''_N\|_{L^1}\|\varphi'_N\|_{L^1}
\]
\[
+ (1 + |x|)^2 O((N^{-1/2}(\log N)^2)\|\varphi''_N\|_{L^1}^{1/2}\|\varphi'_N\|_{L^1}^{1/2}
\]
\[
+ |x|O(N^{-1/2}\log N)(t^{1/2}\|\varphi''_N\|_{L^1}\|\varphi'_N\|_{L^1} + t^{-1/2}\|\varphi'_N\|_{L^1}^{2}).
\]

Recall the definition of \(\psi\) in (6.32). By Proposition 6.5, (6.38), (6.41), we have
\[
\psi'(x) = \frac{i}{\pi} \int \partial_z \tilde{\varphi}_N(z) \mathbb{E}[\text{tr}G(z)e^z] \, dz
\]
\[
= -xV(\varphi_N)\psi(x)
\]
\[
+ |x|O(t^{1/2}N^{-1/2+2\epsilon})\|\varphi''_N\|_{L^1}\|\varphi'_N\|_{L^1}
\]
\[
+ |x|(1 + |x|)O((N^{-1/2}(\log N)^2)\|\varphi''_N\|_{L^1}^{1/2}\|\varphi'_N\|_{L^1}^{1/2}
\]
\[
+ |x|O(N^{-1/2}\log N)(t^{1/2}\|\varphi''_N\|_{L^1}\|\varphi'_N\|_{L^1} + t^{-1/2}\|\varphi'_N\|_{L^1}^{2}).
\]

where
\[
V(\varphi_N) := -\frac{2}{\pi^2} \int_{\Omega_N} \int_{\Omega_N} \partial_z \tilde{\varphi}_N(z) \partial_z \tilde{\varphi}_N(z') \frac{1}{1-tR_2(z)} S_{2,1}(z, z') \, dz \, dz' + \frac{2}{\pi^2} \int_{\Omega_N} \int_{\Omega_N} \partial_z \tilde{\varphi}_N(z) \partial_z \tilde{\varphi}_N(z') \frac{1}{1-tR_2(z)} S_{2,2}(z, z') \, dz \, dz'
\]
\[
+ \frac{2}{\pi^2} \int_{\Omega_N} \int_{\Omega_N} \partial_z \tilde{\varphi}_N(z) \partial_z \tilde{\varphi}_N(z') \frac{1}{1-tR_2(z)} S_{2,3}(z, z') \, dz \, dz.
\]
By our assumptions, the error term in (6.45) is bounded by
\[ N^{2\xi}(1+|x|)O\left(\frac{t^{1/2}}{N^{1/2}t_1}\right) + N^{2\xi}|x|(1+|x|)O\left(\frac{1}{(Nt_1)^{1/2}}\right) = (1+|x|)O(N^{\omega_0/2-\omega_1+3\xi}) + \frac{1}{|x|^2}O(N^{-\omega_1/2+3\xi}). \]
Integrating (6.45) from \( x = 0 \) to \( |x| \leq N^{\omega_1/4-\omega_0/8-3\xi} \) using (6.5), we find:
\[ \psi(x) = \exp\left(-\frac{x^2}{2}V(\varphi_N)\right) + O(N^{\omega_0/4-\omega_1/2}), \quad \text{(6.46)} \]
which is the assertion of Theorem 6.1.

### 6.3 Computation of \( T_1 \)

**Proposition 6.6.** We have the estimate:
\[ T_1 = t\tilde{R}_2(z) \cdot \mathbb{E}[\epsilon(x)trG(\tau + i\eta)^2] + O_\infty(N^{-1/2}\eta^{-3/2}) + |x|O_\infty\left(|\eta|^{-1}N^{-1/2}\|\varphi_N^r\|^2\|\varphi_N^r\|_{L^1}^{1/2}\right), \quad \text{(6.47)} \]
uniformly for \( z = D_{\xi,q} \cup 1_{\xi,q} \).

We choose \( k = 3 \) in (6.29) and write:
\[ T_1 = \sum_{j=1}^{N} \frac{\mathbb{E}[e_j^0\mathbb{E}_j[A_j^0(\tau + i\eta)^3]]}{\mathbb{E}[A_j(\tau + i\eta)]^2} - \sum_{j=1}^{N} \frac{\mathbb{E}[e_j^0\mathbb{E}_j[(A_j^0(\tau + i\eta))^2]]}{\mathbb{E}[A_j(\tau + i\eta)]^3} + \sum_{j=1}^{N} \frac{1}{\mathbb{E}[A_j(\tau + i\eta)]^3} \mathbb{E}\left[ e_j^0(A_j^0)^3 \right], \quad \text{(6.48)} \]
where we have denoted by \( \mathbb{E}_j \) integration over the first row of \( H \) and have used that \( e_j \) is independent of this row. The first term on the right of (6.48) will be seen to be the main term in (6.47). To deal with the second term, we compute
\[ \mathbb{E}_j[(A_j^0)^2] = \mathbb{E}_j[(-\sqrt{t}w_{jj} + \sum_{kl} h_{jk}G_{kl}^{(j)}h_{lj} - \frac{t}{N}\mathbb{E}[trG^{(j)}]^2] = \frac{t}{N} + \sum_{kl} h_{jk}h_{lj} - \frac{t}{N}\mathbb{E}[trG^{(j)}]^2 + N^{-1}t\mathbb{E}_j[(trG^{(j)} - \mathbb{E}trG^{(j)})^2]. \quad \text{(6.49)} \]

We further compute, using the local law:
\[ \mathbb{E}_j[\sum_{kl} G_{kl}^{(j)}(h_{jk}h_{lj} - N^{-1}t\delta_{kl})^2] = \frac{t^2}{N^2} \sum_{kl} G_{kl}^{(j)}G_{kl}^{(j)} = \frac{t^2}{N}\bar{c}_z m_N + O_\infty(t^2N^{-2}|\eta|^{-2}). \quad \text{(6.50)} \]
Inserting (6.49), (6.50) into (6.48) and using \( |e_j^0| \leq 2, \mathbb{E}e_j^0 = 0 \), we find:
\[ T_1(\tau, \eta) = \sum_{j=1}^{N} \frac{\mathbb{E}[e_j^0\mathbb{E}_j[A_j^0(\tau + i\eta)]]}{\mathbb{E}[A_j(\tau + i\eta)]} + \sum_{j=1}^{N} \frac{1}{\mathbb{E}[A_j]} \cdot O(t^2N^{-2}|\eta|^{-2}) + \sum_{j=1}^{N} \frac{1}{\mathbb{E}[A_j]} \cdot O(t^{3/2}N^{-3/2} + t^3N^{-2}|\eta|^{-2}). \quad \text{(6.51)} \]
For the last term we have also used (6.19).

Note that
\[ \mathbb{E}_j(A_j)^0 = \frac{t}{N}trG^{(j)} - \frac{t}{N}\mathbb{E}trG^{(j)} = t(m_N^{(j)})^0, \]

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and so (6.26) implies
\[ N |E[E_j((A_j)^\circ)e_j] - E[(tm_N)^\circ e_j]| \leq 2tE[|(A_j^{-1}(1 + B_j))^\circ|] < tN^{-1/2}|\eta|^{-3/2}. \]  

(6.53)

It now follows from (6.28) that
\[ T_1 = \sum_{j=1}^{N} \frac{tE[e_j^m m_N]}{E[A_j(\tau + i\eta)]^2} + N \sum_{j=1}^{N} \frac{1}{|E[A_j(\tau + i\eta)]|^2} \cdot O(tN^{-3/2}|\eta|^{-3/2}). \]

We now replace \( e_j \) in (6.51) by \( e(x) \). Using that \(|e^{i\alpha} - e^{i\beta}| \leq |\alpha - \beta|\), we find
\[ |E[(m_N)^\circ e_j] - E[(m_N)^\circ e]| \leq C(1 + |x|)E \left[ \left| \int_{C} \tilde{\varphi}_N(z')N[(m_N^{(j)})^\circ(z') - (m_N)^\circ(z')]\,dz' \right| |(m_N)^\circ(z)\right]. \]

(6.54)

Here \( z' = s + i\eta' \).

To evaluate (6.54), we use \((m_N)^\circ < (N|\eta|)^{-1}\), together with the following lemma, for which we will also have use in the next section.

Lemma 6.7. With overwhelming probability
\[ \int_{C} \tilde{\varphi}_N(z')N[(m_N^{(j)})^\circ(z') - (m_N)^\circ(z')]\,dz' < |x|O(\|\varphi_N^\circ\|_{L^1}\|\varphi_N'\|_{L^1}/N)^{1/2}. \]

Proof. Let \( \epsilon > 0 \) be a parameter to be determined later. Split the integral into two regions, using the real-valuedness of \( \varphi_N \):
\[ \int_{C} \tilde{\varphi}_N(z')N[(m_N^{(j)})^\circ(z') - (m_N)^\circ(z')]\,dz' = \text{Re} \int_{D_{\epsilon,\eta}} \tilde{\varphi}_N(z')N[(m_N^{(j)})^\circ(z') - (m_N)^\circ(z')]\,dz' + \text{Re} \int_{D_{\epsilon,\eta}^c} \tilde{\varphi}_N(z')N[(m_N^{(j)})^\circ(z') - (m_N)^\circ(z')]\,dz'. \]

(6.55)

For the first integral, we simply estimate the real part by the full modulus. Our task is thus to estimate the sum
\[ N \left| \int_{D_{\epsilon,\eta}} \tilde{\varphi}_N(z')((m_N^{(j)})^\circ(z') - (m_N)^\circ(z'))\,dz' \right| \geq N |\chi(\eta')| N^{-1} \left| \int_{D_{\epsilon,\eta}^c}(m_N^{(j)})^\circ(z')\,dz' \right|. \]

(6.56)

(6.57)

(6.58)

(6.59)

Since \( \{\chi'(\eta') \neq 0\} \subset \{|\eta'| \geq N^{10C'} \}, \) using (6.26), (6.56) is
\[ \int_{D_{\epsilon,\eta}} i\eta' \varphi_N''(s)N((m_N^{(j)})^\circ(z') - (m_N)^\circ(z'))\,dz' + N^{-4C'}(\varphi_N^\circ\|_{L^1} + \|\varphi_N'\|_{L^1}). \]  

(6.60)

The error term here is \( O(N^{-2}) \). Introducing a new parameter \( \epsilon_2 \), we split the \( \eta' \) integral in the first term in (6.60) into the regions
\[ \{N^{\epsilon}/N < |\eta'| \leq N^{\epsilon^2}/N\}, \quad \{N^{\epsilon^2}/N \leq |\eta'| \leq N^{10C'}\}. \]

(6.61)

(6.62)
In the region (6.61), we use (6.26) to find a bound of

\[ \int_{\{N^s/N < |\eta'| < N^{s+2}/N\}} |\eta'||\phi_N''(s)|O(N^{-1/2}|\eta'|^{-3/2}) \, d\eta' \leq CN^{s+1/2-1}\|\phi_N''\|_{L^1}. \]

In (6.62), integration by parts in s gives that the term (6.56) is bounded by

\[ \frac{N}{\int |\varphi_N''(s)| |\varphi_N''(s)| \, d\eta' \leq N|\varphi_N''(s)| |\varphi_N''(s)| - \frac{1}{2} |\varphi_N''(s)| |\varphi_N''(s)| - \frac{1}{2} |\varphi_N''(s)| |\varphi_N''(s)|. \]

(6.63)

Optimizing \( \epsilon_2 \), we find that (6.56) is bounded by \( O(N^{-1/2}\|\varphi_N''\|_{L^1}) \). For (6.57), we use the assumption (6.2) on the support of \( \varphi_N' \). The integration is over

\[ \{0 < |\eta'| < N^s/N\} \cup \{10 < |\eta'| < N^{10C_V}\}. \]

In the first region, we have by monotonicity – see [52, Lemma 7.19] for details – \( |\text{Im} \, m_N^0(z')|, |\text{Im} \, (m_N^1)^0(z')| < (N\eta')^{-1} \), so this term is

\[ O_\epsilon(\int_{0 < |\eta'| < N^s/N} |\varphi_N''(s)| \, d\eta') = O(N^{-1+\epsilon}\|\varphi_N''\|_{L^1}). \]

For the integral over \( |\eta'| > 10 \), we integrate by parts and use \( \partial_s \text{Im} \, m_N = -\partial_s \text{Re} \, m_N \) to find the estimate

\[ N \int_{|\eta'| > 10} |\varphi_N'(s)| \partial_s \text{Im} \, m_N^0(z') = O(N^{-1/2}\|\varphi_N''\|_{L^1}). \]

Recalling (6.3), the term (6.58) is estimated by

\[ N \int_{|\eta'| > 10} |\varphi_N'(s)| \partial_s \text{Re} \, m_N^0(z') = O(N^{-1/2}\|\varphi_N''\|_{L^1}). \]

(6.64)

Assuming (without loss of generality) that \( C_V \geq 5 \), this is \( O(N^{-2}) \). Using \( |m_N^0(\eta')| < (N\eta')^{-1} \), the term (6.59) is \( O(N^{-1}\|\varphi_N''\|_{L^1}) \). Combining all the above, we find that, for any \( \epsilon > 0 \):

\[ \left| \int_{C} \partial_s \tilde{\varphi}_N(z') N[(m_N^1)^0(z') - (m_N^0)^0(z')] \, d\eta' \right| = O_\epsilon(N^{-1/2}\|\varphi_N''\|_{L^1}) \|\varphi_N''\|_{L^1}^{1/2} + N^{-1+\epsilon}\|\varphi_N''\|_{L^1}. \]

(6.64)

The result now follows by optimizing in \( \epsilon \).

Using the previous lemma in (6.54), we have

\[ |E[(m_N)^0 e_j] - E[(m_N)^0 e]| = O((N|\eta|)^{-1})\|\varphi_N''\|_{L^1}^{1/2} \|\varphi_N''\|_{L^1}^{1/2} N^{-1/2}. \]

By (6.28) and (6.18), we can estimate

\[ \frac{t}{N} \sum_{j=1}^{N} \frac{1}{|E A_j|^2} < 1. \]

(6.65)
From this we get that, for all $\tau + i\eta \in \Omega_N$,

$$T_1(\tau, \eta) = \mathbb{E}[e(x)\text{tr}G(\tau + i\eta)^{\circ}] \cdot \frac{t}{N} \sum_{j=1}^{N} \frac{1}{|E[A_j]|^2} \left( \frac{1}{O_\prec(N^{-1/2}|\eta|^{-3/2})} + \frac{1}{O_\prec(1/\|\varphi_\prec\|_{L_1}^2)} \right)$$

$$= t \tilde{R}_2(z) \cdot \mathbb{E}[e(x)\text{tr}G(\tau + i\eta)^{\circ}] + O_\prec(N^{-1/2}(|\eta|^{-3/2} + t^{-1/2})) + \mathbb{E}[|x|O_\prec(1/\|\varphi_\prec\|_{L_1}^2) - \|\varphi_N\|_{L_1}^2/\|\varphi_\prec\|_{L_1}^{1/2})].$$

This is the claim of Proposition 6.6.

### 6.4 Computation of $T_2$

We now compute $T_2$. Recall from the definition (6.36) that

$$T_2(\tau, \eta) = \sum_{j=1}^{N} \mathbb{E}\left[ G^{\circ}_{j \tau j}(\tau + i\eta)(e - e_j) \right].$$

By (6.29), we have:

$$\mathbb{E}\left[ G^{\circ}_{j \tau j}(\tau + i\eta)(e - e_j) \right] = \frac{1}{|E[A_j]|^2} \mathbb{E}\left[ A^{\circ}_j(\tau + i\eta)(e - e_j) \right] - \frac{1}{|E[A_j]|^2} \mathbb{E}\left[ \frac{1}{E[A_j] + A_j}(A^{\circ}_j(\tau + i\eta))^2(e - e_j) \right].$$

Using the expansion

$$\exp(iX^{(j)}) - \exp(iX) = \exp(iX^{(j)}) \cdot (1 - \exp(i(X - X^{(j)}))) = \exp(iX^{(j)})(i(X^{(j)} - X) + O(|X - X^{(j)}|^2),$$

we have, by Lemma (6.7),

$$e_j(x) - e(x) - \frac{i\pi}{\pi} e_j(x) \int \partial_t \varphi(N) \left( N \cdot \left( m^{(j)}_N - m_N \right) (s + i\eta') \right)^{\circ} dz'$$

$$= |x|^2 O_\prec(1/\|\varphi_N\|_{L_1}^{1/2}) \cdot (N^{-1/2}|\eta|^{-3/2}),$$

with overwhelming probability.

Using (6.68) and (6.19) in (6.67), we get the following expression for $T_2$, which holds for $\tau + i\eta \in \Omega_N$:

$$T_2 = - \sum_{j=1}^{N} \frac{1}{|E[A_j]|^2} \left( \frac{i\pi}{\pi} \right) \int \partial_t \varphi(N) \mathbb{E}\left[ e_j(x) \left( N \cdot \left( m^{(j)}_N - m_N \right) (s + i\eta') \right)^{\circ} A_j(\tau + i\eta)^{\circ} \right] dz'$$

$$+ \frac{1}{N} \sum_{j=1}^{N} \frac{t|x|^2}{|E[A_j]|^2} \cdot O((N|\eta|^{-1/2} + t^{-1/2}N^{-1/2})\|\varphi_N\|_{L_1}^{1/2} \|\varphi_\prec\|_{L_1}^{1/2}).$$

We now compute the main term in (6.69). We begin by splitting:

$$\int_{\Omega_N} \partial_t \varphi(N) \mathbb{E}\left[ e_j(x) \left( N \cdot \left( m^{(j)}_N - m_N \right) (s + i\eta') \right)^{\circ} A_j(\tau + i\eta)^{\circ} \right] dz'$$

$$+ \int_{\Omega_N} \partial_t \varphi(N) \mathbb{E}\left[ e_j(x) \left( N \cdot \left( m^{(j)}_N - m_N \right) (s + i\eta') \right)^{\circ} A_j(\tau + i\eta)^{\circ} \right] dz'.$$
The term (6.73) is estimated in the same way as second term in (6.55). Together with (6.19), This gives a bound of \(O_{\prec}(t(N|\eta|)^{-1/2} + t^{1/2}N^{-1/2})N^{-1+\xi})\|\varphi''_N\|_{L^1}.\) We see that the total contribution to \(T_2\) of the sum over \(j\) of (6.73) is bounded by

\[\frac{1}{N} \sum_{j=1}^{N} \frac{t|x|}{|E[A_j(\tau + i\eta)]|^2}(O((N|\eta|)^{-1/2}) + O(t^{-1/2}N^{-1/2}))N^\xi \|\varphi''_N\|_{L^1}.\] (6.74)

For the first term (6.72), we use the expansion (6.30). The main terms are

\[T_{2,1} = \sum_{j=1}^{N} \frac{i x}{\pi} \int_{\Omega_N} \partial_\xi \varphi_N(z') \frac{1}{E[A_j(z')]^2} \mathbb{E} \left[ e_j(x) \frac{B_j(z')}{E[A_j(z')]^2} A_j(z)^0 \right] dz',\]

\[T_{2,2} = \sum_{j=1}^{N} \frac{i x}{\pi} \int_{\Omega_N} \partial_\xi \varphi_N(z') \frac{1}{E[A_j(z')]^2} \mathbb{E} \left[ e_j(x) A_j^0(z') (1 + EB_j(z')) A_j(z)^0 \right] dz'.\] (6.75)

The remaining terms will be shown to be error terms:

\[T_{2,3} = \sum_{j=1}^{N} \frac{x}{E[A_j(z')]^2} \int_{\Omega_N} \partial_\xi \varphi_N(z') \frac{1}{E[A_j(z')]^2} \mathbb{E}[A_j^0(z')B_j^0(z')A_j(z)] dz',\]

\[T_{2,4} = \sum_{j=1}^{N} \frac{x}{E[A_j(z')]^2} \int_{\Omega_N} \partial_\xi \varphi_N(z') \frac{1}{E[A_j(z')]^2} \mathbb{E} \left[ (A_j^0(z'))^2 A_j(z') (1 + B_j(z')) A_j^0(z) \right] dz'.\] (6.77)

Collecting the error terms obtained so far and using (6.28), we find

\[I_1 = \int_{\Omega_N} \partial_\xi \varphi_N(z) \frac{T_2(z)}{1 - tR(z)} dz = \int_{\Omega_N} \partial_\xi \varphi_N(z) \left( T_{2,1}(z) + T_{2,2}(z) + T_{2,3}(z) + T_{2,4}(z) \right) dz + \int_{\Omega_N} \partial_\xi \varphi_N(z) \Delta_{1,1}(z) dz,\] (6.78)

where \(\Delta_{1,1}\) is \(1/(1 - tR(z))\) times the difference between \(T_2\) and the main term (6.69), restricted to the region \(\Omega_N.\) \(|\Delta_{1,1}|\) is bounded by the sum of the errors (6.70), (6.71) and (6.74).

We have:

\[\int_{\Omega_N} \partial_\xi \varphi_N(z) \Delta_{1,1}(z) dz = \int_{\Omega_N} i\varphi''_N(\tau) \eta \chi(\eta) \Delta_{1,1}(z) dz \]

\[+ \int_{\Omega_N} i\varphi''_N(\tau) \chi(\eta) \Delta_{1,1}(z) dz \]

\[- \int_{\Omega_N} \varphi''_N(\tau) \eta \chi(\eta) \Delta_{1,1}(z) dz.\] (6.81)

We first estimate (6.79). After integration by parts in \(\tau,\) and using \(|\partial_\xi \Delta_{1,1}(z)| \leq 2|\eta|^{-1} \max_{|w-z|=|\eta|/2} |\Delta_{1,1}(w)|,\) this is bounded by

\[\int_{\{z:N^{-1+\xi}|\eta|<|\eta|^{10CV}\}} |\varphi''_N(\tau) \eta \chi(\eta)||\partial_\xi \Delta_{1,1}(z)| dz \]

\[\leq \|\varphi''_N\|_{L^1} \int_{N^{-1+\xi}|\eta|<|\eta|^{10CV}} \left( \frac{1}{N} \sum_{j=1}^{N} \frac{tN^\xi|x|}{|E[A_j(\tau + i\eta)]|^2} O((N|\eta|)^{-1/2} + t^{-1/2}N^{-1/2})\|\varphi''_N\|_{L^1} \right) d\eta\] (6.82)

\[+ \|\varphi''_N\|_{L^1} \int_{N^{-1+\xi}|\eta|<|\eta|^{10CV}} \left( \frac{1}{N} \sum_{j=1}^{N} \frac{|x|}{|E[A_j(\tau + i\eta)]|^3} O(tN^{-1/2}|\eta|^{-1} + tN^{-1/2})\|\varphi''_N\|_{L^1}^{1/2}\|\varphi''_N\|_{L^2}^{1/2} \right) d\eta\] (6.83)
Split the $\eta$ integral (6.82) into $\{\eta \leq t\}$, $\{\eta > t\}$, and

$$\frac{1}{N} \sum_{j=1}^{N} \frac{1}{\mathbb{E}[A_j]^{1/2}} \leq C \log N/(\max(t, |\eta|)).$$

This gives the estimate

$$|x|\|\varphi'_{N}\|_{L^1} \cdot \mathcal{O}(t^{1/2}N^{-1/2+2\varepsilon}).$$

(6.84)

With $t = N^{\omega_0}/N$ and $\|\varphi'_{N}\|_{L^1} \leq N^{1/\omega_1}$, this is $\mathcal{O}(N^{\omega_0}/N^{\omega_1})$, which is $\mathcal{O}(N^{-\varepsilon})$ if $\varepsilon$ is small enough.

By direct computation and (6.65), the term (6.83) is $|x|\mathcal{O}(tN^{-1/2} \log N)\|\varphi'_{N}\|_{L^1}^{3/2}$. For the terms (6.80), (6.81), the integrands are supported in the region $\{z : N^{10C_{\varepsilon}} - 1 < |\text{Im } z| < N^{10C_{\varepsilon}}\}$. In this region, we use the bound $|\mathbb{E}A_j(\tau + i\eta)| \geq c|\eta|$, to obtain a bound of the form $C\|\varphi'_{N}\|_{L^1}N^{-2}$. The remaining terms $T_{2,1}, T_{2,2}, T_{2,3}, T_{2,4}$ are computed in the following sections.

### 6.5 Computation of $T_{2,1}$.

We now compute the term $T_{2,1}$ (6.75). Since $c_j(x)$ is independent of $(h_{ij})_{i=1}^{N}$, we first compute

$$\mathbb{E}_j[A_j^{\tau}(\tau + i\eta) \cdot \mathbb{E}[A_j(s + i\eta')]].$$

(6.85)

For simplicity of notation, we will write $G(s)$ for $G(s + i\eta')$ and $G(\tau)$ for $G(\tau + i\eta)$. Similar notational simplifications apply to $G^{(j)}(s + i\eta'), m_{N}^{(j)}(s + i\eta'), m_{N}^{(j)}(s + i\eta')$, etc.

The result of the following computation is:

**Proposition 6.8.** Uniformly for $\tau + i\eta, s + i\eta' \in \Omega_N$,

$$NE_j[A_j^{\tau}(\tau + i\eta) \cdot \mathbb{E}[A_j(s + i\eta')]] = \frac{2t^2}{\mathbb{E}[A_j(s + i\eta')]} \cdot \partial_{\tau} m_{N}(\tau - s + i(\eta - \eta')) + g_j(s) \cdot \mathcal{O}_\varepsilon(t^2N^{-1}|\eta'|^{-2}|\eta|^{-1}).$$

(6.86)

**Proof.** We first recenter around the conditional expectations $\mathbb{E}_jA_j, \mathbb{E}_jB_j$ instead of the full expectations, using the identity

$$\mathbb{E}_j[(A - \mathbb{E}[A])(B - \mathbb{E}[B]) = (\mathbb{E}_j[A - \mathbb{E}[A]])(\mathbb{E}_j[B] - \mathbb{E}[B]) + (\mathbb{E}_j[A] - \mathbb{E}[A])(\mathbb{E}_j[B] - \mathbb{E}[B]).$$

This produces an error $\mathcal{O}(t^2N^{-1}|\eta'|^{-2}|\eta|^{-1})$. We then write

$$NE_j[(A_j(z) - \mathbb{E}_jA_j(z))(B_j(z') - \mathbb{E}_jB_j(z'))]$$

$$= NE_j \left( \sum_{i,k} G_{ik}^{(j)}(\tau)(h_{ij}h_{kj} - N^{-1}t\delta_{ik}) \right) \times \left( \sum_{k=1}^{N} \frac{1}{\mathbb{E}[A_j(z')]} G_{kk}^{(j)}(s)G_{km}^{(j)}(s)(h_{ij}h_{mj} - N^{-1}t\delta_{im}) \right)$$

$$= \frac{2t^2}{NE_j[A_j(s + i\eta')]} \text{tr}(G^{(j)}(\tau)(G^{(j)}(s))^2).$$

(6.87)

Now we use

$$\frac{1}{N} \text{tr}G^{(j)}(\tau)(G^{(j)}(s))^2 = \partial_{z'} \text{tr}(G^{(j)}(\tau)G^{(j)}(z))|_{z=s+i\eta'}.$$

We can write this as

$$\frac{1}{N} \text{tr}G^{(j)}(\tau) - \text{tr}G^{(j)}(s) = \partial_{s'} \frac{f(z) - f(z')}{z - z'} = \int_{0}^{1} (1 - \alpha) f''(z' + \alpha(z - z')) d\alpha.$$

(6.88)
If $\eta\eta' > 0$ and $|\eta - \eta'| < \max(|\eta|, |\eta'|)/2$, we use (6.88) with

$$f(z) = m_N^{(j)}(z) - m_{fc,t}(z)$$

to find

$$\partial_s m_{fc,t}(\tau) - m_{fc,t}(s) \over \tau - s + i(\eta - \eta') + O(\max_{\alpha \in [z',z]} |\alpha\eta + (1 - \alpha)\eta'|^{-3}) = \partial_s m_{fc,t}(\tau) - m_{fc,t}(s) \over \tau - s + i(\eta - \eta') + O(\max_{\alpha \in [z',z]} |\alpha\eta + (1 - \alpha)\eta'|^{-3}) = \partial_s m_{fc,t}(\tau) - m_{fc,t}(s) \over \tau - s + i(\eta - \eta') + O(\max_{\alpha \in [z',z]} |\alpha\eta + (1 - \alpha)\eta'|^{-3}).$$

If $|\eta - \eta'| > \max(|\eta|, |\eta'|)/2$, we perform the differentiation

$$\frac{-\partial_s m_N^{(j)}(s)}{(\tau - s) + i(\eta - \eta')} + \frac{m_N^{(j)}(z) - m_N^{(j)}(z')}{((\tau - s) + i(\eta - \eta'))^2}.$$

Using the local law, we replace $m_N^{(j)}(s), m_N^{(j)}(\tau)$ by $m_{fc,t}(s), m_{fc,t}(\tau)$ with an error $O(N^{-1}|\eta'|^{-2}|\eta|^{-1})$.

If $\eta\eta' < 0$, applying the local law again we find

$$\partial_s m_{fc,t}(\tau) - m_{fc,t}(s) \over \tau - s + i(\eta - \eta') + O(\max_{\alpha \in [z',z]} |\alpha\eta + (1 - \alpha)\eta'|^{-3}) = \partial_s m_{fc,t}(\tau) - m_{fc,t}(s) \over \tau - s + i(\eta - \eta') + O(\max_{\alpha \in [z',z]} |\alpha\eta + (1 - \alpha)\eta'|^{-3}).$$

Using Proposition 6.8 in the main term of (6.75), and using (6.18) to replace $1/E[A_j], 1/E[A_j]^2$ by $g_j, g_j^2$ we find, for $\tau + i\eta \in \Omega_N$:

$$T_{2,1}(z) = -\frac{2ix}{\pi} \sum_{j=1}^N \int_{\Omega_N} \left( g_j(z)^2 g_j(z') \partial_z \varphi_N(z') \partial_s m_{fc,t}(\tau) - m_{fc,t}(s) \over \tau - s + i(\eta - \eta') \right) |6.89|$$

$$- \frac{2ix}{\pi} \sum_{j=1}^N \int_{\Omega_N} \left( g_j(z)^2 g_j(z') i\eta' \varphi_N'(s) \partial_s m_{fc,t}(\tau) - m_{fc,t}(s) \over \tau - s + i(\eta - \eta') \right) |6.90|$$

$$+ \frac{1}{\pi} \sum_{j=1}^N \left( \frac{|x|}{|E[A_j(\tau + i\eta)|^2]} \int_{\Omega_N} \frac{1}{E[A_j(s + i\eta')]} \cdot O(\max_{\alpha \in [z',z]} |\alpha\eta + (1 - \alpha)\eta'|^{-3}) \right) |6.91|$$

$$+ \frac{1}{\pi} \sum_{j=1}^N \left( \frac{|x|}{|E[A_j(\tau + i\eta)|^2]} \int_{\Omega_N} \frac{1}{E[A_j(s + i\eta')]} \cdot O(\max_{\alpha \in [z',z]} |\alpha\eta + (1 - \alpha)\eta'|^{-3}) \right) |6.92|$$

Note that above, we have omitted the terms with support in the region $\{\chi(\ell a') \neq 0\}$, as they are smaller than the terms displayed.

For the term (6.91), we use (6.65) and (6.28) to find an estimate

$$|x|O(\log N(N|\eta|)^{-1})||\varphi_N^\prime||_1.$$

To deal with the remaining terms, we use the following estimates:

**Proposition 6.9.** If $\eta, \eta' \in \Omega_N$ and $\eta\eta' > 0$, then

$$\partial_s m_{fc,t}(\tau) - m_{fc,t}(s) \over \tau - s + i(\eta - \eta') = O(|\eta|^{-1}|\eta'|^{-1}).$$

If $\eta\eta' < 0$, then

$$\partial_s m_{fc,t}(\tau) - m_{fc,t}(s) \over \tau - s + i(\eta - \eta') = O(|\eta - \eta'|^{-1}|\eta'|^{-1}).$$

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Proof. By the representation (6.88) the left side of (6.95) is

\[ \int_0^1 \alpha m_{k,t}^{(2)}(z' + \alpha(z - z')) \, d\alpha. \]  

(6.97)

This is bounded by

\[ \max_{\zeta \in [z,z']} |m_{k,t}^{(\eta)}(\zeta)| \leq C|\eta|^{-1}|\eta'|^{-1}. \]

For (6.96), we simply perform the differentiation:

\[
\frac{\partial_z m_{k,t}(\tau) - m_{k,t}(s)}{\tau - s + i(\eta - \eta')} = - \frac{\partial_z m_{k,t}(s)}{\tau - s + i(\eta - \eta')} + \frac{m_{k,t}(\tau) - m_{k,t}(s)}{(\tau - s + i(\eta - \eta'))^2} = \mathcal{O}(|\eta'|^{-1}|\eta - \eta'|^{-1}) + \mathcal{O}(|\eta - \eta'|^{-2}).
\]

By (6.95), (6.96), and (6.65) the term (6.92) is bounded by \(|x| \mathcal{O}(\log N |\eta|^{-1})\|\varphi_{N}''\|_{L^1}. For the term (6.90), we use (6.64), (6.95), (6.96), and integrate by parts in \(s'\) when \(|\eta'| \leq \|\varphi_{N}''\|_{L^1}\) to find an error

\[ |x|(1 + |x|)\mathcal{O}(\log N |\eta|^{-1/2}|\varphi_{N}''|_{L^1}^{1/2}|\varphi_{N}'''|_{L^1}^{3/2} N^{-1/2}) \]

We have shown the main term in (6.75) is

\[- \frac{2ix}{\pi} \mathbb{E}[e(x)] \int_{\Omega_N} S_{2,1}(z, z') \partial_z \hat{\varphi}_N(z') dz' + |x|(1 + |x|)\mathcal{O}(\log N N^{-1/2} |\eta|^{-1})\|\varphi_{N}''\|_{L^1}\|\varphi_{N}'''\|_{L^1}^{1/2},\]

with

\[ S_{2,1}(z, z') = \frac{t^2}{N} \sum_{j=1}^{N} g_j(z)^2 g_j(z') \partial_z m_{k,t}(z) - m_{k,t}(z') \]

(6.98)

Multiplying \(T_{2,1}(z)/(1 - t \hat{R}(z))\) by \(\partial_z \hat{\varphi}_N(z)\), and integrating we have:

\[
\int_{\Omega_N} \partial_z \hat{\varphi}_N(z) \frac{T_{2,1}(z)}{1 - t \hat{R}(z)} \, dz = - \frac{2ix}{\pi} \mathbb{E}[e(x)] \int_{\Omega_N} \frac{1}{1 - t \hat{R}(z)} \partial_z \hat{\varphi}_N(z) \int_{\Omega_N} \partial_z \hat{\varphi}_N(z') S_{2,1}(z, z') \, dz' \, dz 
+ \int_{\Omega_N} \frac{1}{1 - t \hat{R}(z)} i\eta \chi(\eta) \varphi_{N}''(\tau) \Delta_{2,1}(z) \, dz,
\]

where

\[ \Delta_{2,1}(z) := T_{2,1} + \frac{\mathbb{E}[e(x)] \xi}{\pi} \int_{\Omega_N} \partial_z \hat{\varphi}_N(z') S_{2,1}(z, z') \, dz' \]

is analytic in \( \text{Im } z > 0 \) and \( \text{Im } z < 0 \) and

\[ \frac{\Delta_{2,1}(z)}{1 - \hat{R}(z)} = \mathcal{O}(|\eta|^{-1} \log N N^{1/2} |x|(1 + |x|)\|\varphi_{N}''\|_{L^1}^{1/2}\|\varphi_{N}'''\|_{L^1}^{3/2}). \]

Integrating by parts in \( \tau \) in the integral (6.99) and using

\[ \partial_\tau \frac{\Delta_{2,1}(z)}{1 - \hat{R}(z)} = \mathcal{O}(|\eta|^{-2} \log N N^{1/2} |x|(1 + |x|)\|\varphi_{N}''\|_{L^1}^{1/2}\|\varphi_{N}'''\|_{L^1}^{3/2}), \]

we find

\[
\int_{\Omega_N} \partial_\tau \hat{\varphi}_N(z) \frac{T_{2,1}(z)}{1 - t \hat{R}(z)} \, dz = - \frac{2ix}{\pi} \mathbb{E}[e(x)] \int_{\Omega_N} \frac{1}{1 - t \hat{R}(z)} \partial_\tau \hat{\varphi}_N(z) \int_{\Omega_N} \partial_\tau \hat{\varphi}_N(z') S_{2,1}(z, z') \, dz' \, dz 
+ \mathcal{O}((\log N)^2/N^{1/2}) |x|(1 + |x|)\|\varphi_{N}''\|_{L^1}^{1/2}\|\varphi_{N}'''\|_{L^1}^{3/2}. \]

(6.100)
6.6 Computation $T_{2,2}$, $T_{2,3}$, $T_{2,4}$

The computation of $T_{2,2}$ is almost identical (but simpler) to that in Proposition 6.8.

**Proposition 6.10.** There are constants for $t + i\eta, s + i\eta' \in \Omega_N$,

$$\begin{align*}
N &\mathbb{E}_j[A_j(\tau + i\eta)(1 + \mathbb{E}B_j(z'))A_j^\dagger(s + i\eta')] \\
&= (1 + t\partial_z m_{fc,t}(z')) \cdot \left(2t^2m_{fc,t} - \frac{m_{fc,t}(s)}{\tau - s + i(\eta - \eta')}\right) \cdot (1 + \mathcal{O}(tN^{-1}|\eta'|^{-2}) + t) \\
&+ \mathcal{O}(t^2N^{-1}|\eta|^{-1}|\eta'|^{-1}).
\end{align*}$$

(6.101)

We have shown that

$$T_{2,2}(z) = \frac{2ix}{\pi} \mathbb{E}[e(x)] \int_{\Omega_N} \partial_z \bar{\varphi}_N(z') \int (S_{2,2}(z, z') + S_{2,3}(z, z')) dz' \, dz + \Delta_{2,2}(z)$$

where

$$S_{2,2}(z, z') = \frac{t^2}{N} \sum_{j=1}^{N} g_j(z)^2 g_j(z')^2 \left(1 + t\partial_z m_{fc,t}(z')\right) \frac{m_{fc,t}(z) - m_{fc,t}(z')}{z - z'},$$

$$S_{2,3}(z, z') = \frac{t}{N} \sum_{j=1}^{N} g_j(z)^2 g_j(z')^2 \left(1 + t\partial_z m_{fc,t}(z')\right).$$

$\Delta_{2,2}(z)$ is analytic in $\text{Im} \, z \neq 0$ and

$$\begin{align*}
|\Delta_{2,2}(z)| &= \frac{|x|}{N} \sum_{j=1}^{N} \frac{1}{|\mathbb{E}[A_j(z)]|^2} \int_{\Omega_N} \frac{1}{|\mathbb{E}[A_j(z')]|^2} \mathcal{O}(t^2N^{-1}|\eta|^{-1}|\eta'|^{-1})|\varphi''(s)| \, dz' \\
&+ \frac{|x|}{N} \sum_{j=1}^{N} \frac{1}{|\mathbb{E}[A_j(z)]|^2} \int_{\Omega_N} \frac{1}{|\mathbb{E}[A_j(z')]|^2} \left(t^2 \min(|\eta|^{-1}, |\eta'|^{-1}) + t\right) \mathcal{O}(tN^{-1}(|\eta|^{-1} + |\eta'|^{-1}))|\varphi''(s)| \, dz' \\
&+ \frac{|x|}{N} \sum_{j=1}^{N} \mathbb{E}[e(x) - e_j(x)] \int_{\Omega_N} g_j(z')^2 |\varphi''(s)| \left(2t^2m_{fc,t}(z) - m_{fc,t}(z') + t\right) \, dz' \\
&= |x|(1 + |x|)\mathcal{O}(\eta|^{-1}\log NN^{-1/2}\|\varphi_N''\|_{L^1}^{1/2}\|\varphi_N''\|_{L^1}^{3/2})
\end{align*}$$

We have used (6.65) and (6.28).

Using the derivative bound

$$\partial_\tau \frac{\Delta_{2,2}(z)}{1 - tR_2(z)} = |x|(1 + |x|)\mathcal{O}(\eta|^{-2}\log NN^{1/2}\|\varphi_N''\|_{L^1}^{1/2}\|\varphi_N''\|_{L^1}^{3/2}),$$

we have

$$\int_{\Omega_N} \partial_z \bar{\varphi}_N(z) \frac{\Delta_{2,2}(z)}{1 - tR_2(z)} \, dz = \int_{\Omega_N} i\varphi''(\tau) \eta N(\eta) \partial_\tau \frac{\Delta_{2,2}(z)}{1 - tR_2(z)} \, dz$$

$$= |x|(1 + |x|)\mathcal{O}(\log N)^2N^{1/2}\|\varphi_N''\|_{L^1}^{5/2}\|\varphi_N''\|_{L^1}^{1/2}.$$ (6.102)

For $T_{2,3}$, we use (6.19), (6.20), to estimate the integrand by

$$\begin{align*}
\frac{1}{N} \sum_{j=1}^{N} \frac{|x|}{|\mathbb{E}[A_j(z)]|^2 \mathbb{E}[A_j(z')]|^2} \mathcal{O}\left(t^2|\eta'|^{-1/2}|\eta|^{-1/2} + t^3|\eta'|^{-1/2} + t^3|\eta|^{-1/2} + t\right) N^{-1/2}|\eta'|^{-3/2}.
\end{align*}$$

(6.103)
The terms to estimate are

\[ T_{2,3}(z) = \sum_{j=1}^{N} \frac{x}{E[A_j(z)z]^2} \int_{\Omega_N} i\eta' \chi(\eta') \phi_N^*(s) \frac{1}{E[(A_j(z'))^2]} \frac{1}{E[A_j(z')B_j^*(z')A_j(z)]} dz' \]  \hspace{1cm} (6.104)

\[ + \sum_{j=1}^{N} \frac{x}{E[A_j(z)z]^2} \int_{\Omega_N} i\phi_N(s) \chi(\eta') \frac{1}{E[(A_j(z'))^2]} \frac{1}{E[A_j(z')B_j^*(z')A_j(z)]} dz' \]  \hspace{1cm} (6.105)

\[ - \sum_{j=1}^{N} \frac{x}{E[A_j(z)z]^2} \int_{\Omega_N} \eta' \phi_N(s) \chi(\eta') \frac{1}{E[(A_j(z'))^2]} \frac{1}{E[A_j(z')B_j^*(z')A_j(z)]} dz'. \]  \hspace{1cm} (6.106)

Use \{\zeta' : |\text{Im} \ z'| \geq N^{10C_V} - 1\} on the support of the integrands, and (6.103) to estimate the terms (6.105), (6.106) by

\[ N^{-10C_V} \sum_{j=1}^{N} \frac{|x|}{E[A_j(z)]^2} |\text{Im} \ z'|^{-1/2} (\|\phi_N\|_{L^1} + \|\varphi_N\|_{L^1}). \]  \hspace{1cm} (6.107)

Inserting the bound (6.103) into (6.104), using \|E[A_j(z)]\| \geq c \eta, for |\eta| \leq t and \|E[A_j(z)]\| \geq |\eta| for |\eta| \geq t, we find

\[ \frac{1}{N} \sum_{j=1}^{N} \frac{t^{-1} |x|}{E[A_j(z)]^2} N^{-1/2} O(t^2 |\eta|^{-1/2} \log N + t^{3/2} \log N + t^2 |\eta|^{-1/2} + t^{3/2}) \|\varphi_N\|_{L^1}. \]  \hspace{1cm} (6.108)

Integrating \((\partial_z \varphi_N(z)T_{2,3}(z))/(1 - t R_2(z))\) over \(\Omega_N\), and integrating by parts:

\[ \int_{\Omega_N} i\eta \chi(\eta') \phi_N^*(\tau) \frac{T_{2,3}(z)}{1 - t R_2(z)} \, dz = \int_{\Omega_N} i\phi_N(\eta \chi(\eta)) \phi_N'(\tau) \frac{T_{2,3}(z)}{1 - t R_2(z)} \, dz \]

\[ + N^{-1+\xi} \int i\chi(\eta) \phi_N'(\tau) \frac{T_{2,3}(\tau + iN^{-1+\xi})}{1 - t R_2(\tau + iN^{-1+\xi})} \, d\tau. \]

Using (6.107), (6.108), these terms are bounded by

\[ |x| O(t^{1/2} N^{-1/2} \log N \|\varphi_N\|_{L^1} \|\varphi_N\|_{L^1}) = |x| O(N^{-1/2+1/2} \omega_0 \omega_1 \log N) \|\varphi_N\|_{L^1}. \]  \hspace{1cm} (6.109)

The contribution to

\[ \int_{\Omega_N} \partial_z \phi_N(z) \frac{T_{2,3}(z)}{1 - t R_2(z)} \, dz \]

of the terms involving \(\chi'(\eta)\varphi_N(\tau), \chi'(\eta)\varphi_N(\tau)\) is easily estimated using the support property of \(\chi',\)

and found to be

\[ |x| O(N^{-2C_V}) \|\varphi_N\|_{L^1}. \]

For \(T_{2,4}\), we again have three terms

\[ T_{2,4} = \sum_{j=1}^{N} \frac{x}{E[A_j(z)]^2} \int_{\Omega_N} i\phi_N^* (s) \chi(\eta') \frac{(A_j(z'))^2}{A_j(z')} (1 + B_j(z')) A_j^*(z) \, dz' \]  \hspace{1cm} (6.110)

\[ + \sum_{j=1}^{N} \frac{x}{E[A_j(z)]^2} \int_{\Omega_N} i\phi_N (s) \chi(\eta') \frac{(A_j(z'))^2}{A_j(z')} (1 + B_j(z')) A_j^*(z) \, dz' \]  \hspace{1cm} (6.111)

\[ - \sum_{j=1}^{N} \frac{x}{E[A_j(z)]^2} \int_{\Omega_N} i\phi_N^* (s) \chi(\eta') \frac{(A_j(z'))^2}{A_j(z')} (1 + B_j(z')) A_j^*(z) \, dz'. \]  \hspace{1cm} (6.112)

We use (6.19), (6.65) to find that the integrand in (6.110)-(6.112) is bounded by

\[ \frac{|x|}{N} \sum_{j=1}^{N} \frac{t}{E[A_j(z)]^2} \frac{1}{E[A_j(z')]^2} \frac{1}{|A_j(z')|} O \left( t^2 |\eta'|^{-1} N |\eta|^{-1/2} + t^{3/2} |\eta'|^{-1} N^{-1/2} + t (N |\eta|)^{-1/2} + t^{1/2} N^{-1/2} \right). \]  \hspace{1cm} (6.113)
For (6.110), we first integrate by parts to write this term as

\[
\sum_{j=1}^{N} \frac{x}{|E[A_j(z)]|^2} \int_{\Omega_N} \frac{i \varphi_N'(s) \partial_{\eta'}(\eta' \chi(\eta'))}{E[A_j(z')]} \left( \frac{(A_j(z'))^2}{A_j(z')} (1 + B_j(z')) A_j(z) \right) \, dz' - \sum_{j=1}^{N} \frac{x N^{-1+\xi}}{|E[A_j(z)]|^2} \int_{\Omega_N} \frac{i \varphi_N'(s) \chi(\eta')}{E[A_j(s+iN^{-1+\xi})]} \left( \frac{(A_j(s+iN^{-1+\xi}))^2}{A_j(s+N^{-1+\xi})} (1 + B_j(s+iN^{-1+\xi})) A_j(z) \right) \, ds.
\]

(6.114)

Using (6.113) in (6.114), together with the estimate (6.28) when \(|\eta'| \leq t\) and \(|A_j(z')| \geq c|\eta'|\) when \(|\eta'| \geq t\), (6.110) is bounded by

\[
\frac{1}{N} \sum_{j=1}^{N} \frac{|x|}{|E[A_j(z)]|^2} O(\log N(|\eta|)^{-1/2} + t^{-1/2} N^{-1/2} \log N + t^{-1/2} N^{-1/2}) \|\varphi_N'\|_{L^1}.
\]

(6.115)

Using (6.113) again and \(\eta : \chi'(\eta) \neq 0\) \(\subset \{|\eta| \geq N^{C_\nu} - 1\}\), the terms (6.111) and (6.112) are estimated by

\[
|x|O(N^{-2C_\nu} \|\varphi_N'\|_{L^1}).
\]

Using the bound (6.115), we now conclude as in the case of \(T_{2,3}\), by integrating by parts in \(\tau\):

\[
\int_{\Omega_N} \partial_\tau \varphi_N(z) \frac{T_{2,4}(z)}{1 - t R_2(z)} \, dz = \frac{|x|}{N} \sum_{j=1}^{N} \int_{\Omega_N} \partial_\eta (\eta \chi(\eta)) \varphi_N'(\tau) \frac{|T_{2,4}(z)|}{1 - t R_2(z)} \, dz + O(N^{-2})
\]

(6.116)

Collecting the error terms (6.84), (6.100), (6.102), (6.109), (6.116), we obtain

\[
I_1' = -\frac{2ix}{\pi} E[e(x)] \int_{\Omega_N} \int_{\Omega_N} \partial_\partial \varphi_N(z) \partial_\partial \varphi_N(z') \frac{1}{1 - t R_2(z)} S_{2,1}(z, z') \, d z d z' + \frac{2ix}{\pi} E[e(x)] \int_{\Omega_N} \int_{\Omega_N} \partial_\partial \varphi_N(z) \partial_\partial \varphi_N(z') \frac{1}{1 - t R_2(z)} (S_{2,2}(z, z') + S_{2,3}(z, z')) \, d z d z'
\]

(6.117)

(6.118)

This ends the proof of Proposition 6.5.

6.7 Variance term

In this section, we give an asymptotic approximation of the expression \(V(\varphi_N)\) defined in (6.45). This quantity represents the variance of the limiting random variable for the linear statistics of \(\varphi_N\). The result is as follows

**Proposition 6.11.** Recall the definition of \(V(\varphi_N)\) in (6.45). Then

\[
V(\varphi_N) = -\frac{1}{\pi^2} \int_{-Ct}^{Ct} \varphi_N(\tau)(H \varphi_N)(\tau) \, d \tau + O(1).
\]

(6.119)

Here, \(H f\) denotes the Hilbert transform:

\[
(H f)(x) = \lim_{\epsilon \to 0} \int f(y) \text{Re} \frac{1}{(x - y) + i\epsilon} \, dy.
\]

(6.120)
In particular, for
\[ \varphi_N(x) = \int_0^x \chi(y/(t_1 N^\alpha)) p_t(0, y) dy, \]
we have
\[ V(\varphi_N) \geq c \log(t/t_1) \cdot (1 + o(1)). \]

Moreover, if
\[ \text{supp} \varphi_N \subset (-N^r t_1, N^r t_1), \]
then
\[ V(\varphi_N) = -\frac{1}{\pi^2} \int \varphi_N(\tau) (H \varphi'_N)(\tau) d\tau + O(N^{\omega_0/2 - \omega_1 + \xi}) \]
for any \( \xi > 0 \).

We begin by reducing the domain of integration. Define
\[ \Omega_N^* = \{ z = E + i \eta : E \in \mathcal{I}_q, N^{-1+\xi} < |\eta| \leq N^{10C_N} \}. \]

Note that
\[ \{ \partial_z \tilde{\varphi}_N \neq 0 \} \cap (\Omega_N \backslash \Omega_N^*) \subset \{ N^{10C_N} - 1 < |\eta| < N^{10C_N} \}. \] (6.121)
If either \( z \) or \( z' \) lies in the latter region, then
\[ \frac{1}{1 - t R_2(z)} S_{2,1}(z, z') = \frac{t^2}{N(1 - t R_2(z))} \sum_{j=1}^N g_j(z)^2 g_j(z') \partial_z \frac{m_{t, c}(z) - m_{t, c}(z')}{z - z'} = O(t^2 N^{-4C_N}). \] (6.122)

Similarly,
\[ \frac{1}{1 - t R_2(z)} S_{2,2}(z, z') = O(t^2 N^{-4C_N}). \] (6.123)

With (6.122), (6.123), it is easy to show that the domain of integration \( \Omega_N \times \Omega_N \) in (6.117), (6.118) can be replaced by \( \Omega_N^* \times \Omega_N^* \) with an error \( O(N^{-2}) \).

Next, we have the following:

**Proposition 6.12.**
\[ \frac{1}{N} \sum_{j=1}^N \frac{g_j(z)^2}{1 - t R_2(z)} g_j(z') = \partial_z \frac{m_{t, c}(z) - m_{t, c}(z')}{z - z' + t(m_{t, c}(z) - m_{t, c}(z'))}. \] (6.124)

**Proof.** By [52, Eqn. (7.24)]
\[ \partial_z m_{t, c}(z) = \frac{R_2(z)}{1 - t R_2(z)}, \quad 1 + t \partial_z m_{t, c}(z) = \frac{1}{1 - t R_2(z)}. \] (6.125)

By partial fractions, we have
\[ \frac{1}{N} \sum_{j=1}^N \frac{g_j(z)^2}{1 - t R_2(z)} g_j(z') = \partial_z \frac{1}{N} \sum_{j=1}^N g_j(z) g_j(z') = \frac{1}{N} \partial_z \frac{m_{t, c}(z) - m_{t, c}(z')}{z - z' + t(m_{t, c}(z) - m_{t, c}(z'))}. \]

The two integrals appearing in the definition of \( V(\varphi_N) \) are
\[ I_{1,1} := \int_{\Omega_N^*} \int_{\Omega_N^*} \partial_z \tilde{\varphi}_N(z) \partial_z \tilde{\varphi}_N(z') \frac{1}{1 - t R_2(z)} S_{2,1}(z, z') dz dz', \] (6.126)
\[ I_{1,2} := \int_{\Omega_N^*} \int_{\Omega_N^*} \partial_z \tilde{\varphi}_N(z) \partial_z \tilde{\varphi}_N(z') \frac{1}{1 - t R_2(z)} S_{2,2}(z, z') dz dz', \] (6.127)
\[ I_{1,3} := \int_{\Omega_N^*} \int_{\Omega_N^*} \partial_z \tilde{\varphi}_N(z) \partial_z \tilde{\varphi}_N(z') \frac{1}{1 - t R_2(z)} S_{2,3}(z, z') dz dz. \] (6.128)
By Proposition 6.12,
\[
I_{1,1} = \frac{t^2}{N} \int_{\Omega_N^*} \int_{\Omega_N^*} \partial_z \tilde{\varphi}_N(z) \partial_z \tilde{\varphi}_N(z') \partial_z \sum_{j=1}^{N} g_j(z) g_j(z') \partial_s \left( \frac{m_{fc,t}(z) - m_{fc,t}(z')}{z - z'} \right) \, dz \, dz'.
\] (6.129)

Similarly, we have:
\[
\frac{S_{2,2}(z, z')}{{\Gamma} - tR_2(z)} = \frac{t^2}{N} \partial_z \tilde{\varphi}_N(z) \partial_z \tilde{\varphi}_N(z') \partial_z \sum_{j=1}^{N} g_j(z) g_j(z') \frac{m_{fc,t}(z) - m_{fc,t}(z')}{z - z'} \, dz \, dz'.
\] (6.130)

\[
\frac{S_{2,2}(z, z')}{{\Gamma} - tR_2(z)} = \frac{t}{N} \partial_z \tilde{\varphi}_N(z) \partial_z \tilde{\varphi}_N(z') \partial_z \sum_{j=1}^{N} g_j(z) g_j(z') \, dz \, dz'.
\] (6.131)

Integrating by parts in \( s = \text{Re} z' \). The boundary term is only non-zero in the region (6.121), where we can use (6.122).
\[
I_{1,2} = -\frac{t^2}{N} \int_{\Omega_N^*} \int_{\Omega_N^*} \partial_z \tilde{\varphi}_N(z) \partial_z \tilde{\varphi}_N(z') \partial_z \sum_{j=1}^{N} g_j(z) g_j(z') \frac{m_{fc,t}(z) - m_{fc,t}(z')}{z - z'} \, dz \, dz'.
\] (6.132)

Note that the second integral (6.131) is equal to \( I_{1,1} \).

We begin by computing the \( z \) integral in (6.130). The integrand is \( \partial_z \tilde{\varphi}_N(z) \) multiplying a function analytic in each of \( \{ \text{Im} z > 0 \} \) and \( \{ \text{Im} z < 0 \} \). Let \( \Omega \subset \mathbb{C} \) be a domain. For \( F \) a \( C^1(\Omega) \) function, Green’s theorem in complex notation is
\[
\int_{\Omega} \partial_z \tilde{F}(z) \, dz = \frac{-i}{2} \int_{\partial\Omega} F(z) \, dz.
\] (6.133)

We split the integral (6.130) into the two regions \( \Omega_N^* \cap \{ \text{Im} z > 0 \} \), \( \Omega_N^* \cap \{ \text{Im} z < 0 \} \) and apply Green’s theorem to each. The first region is a rectangle in the upper half-plane. The integrand in the resulting line integral, \( \tilde{\varphi}_N \), is zero on the “top” segment \( [-qG + iN^{10C\Gamma}, qG + iN^{10C\Gamma}] \).

We label the terms corresponding to three other boundary line integrals by (+) to denote \( \text{Im} z > 0 \) and number them according to the corresponding boundary segments as (1) for \( [-qG + iN^{-1+\xi}, qG + iN^{-1+\xi}] \); (2) for \( [qG + iN^{-1+\xi}, qG + iN^{10C\Gamma}] \); and (3) for \( [-qG + iN^{10C\Gamma}, qG + iN^{10C\Gamma}] \):
\[
2i \int_{\Omega_N^* \cap \{ \text{Im} z > 0 \}} \partial_z \tilde{\varphi}_N(z) \frac{t^2}{N} \partial_z \sum_{j=1}^{N} g_j(z) g_j(z') \frac{m_{fc,t}(z) - m_{fc,t}(z')}{z - z'} \, dz.
\] (6.134)

\[
\int_{-qG}^{qG} \left( \varphi_N(\tau) + iN^{-1+\xi} \varphi_N'(\tau) \right) \partial_\tau \frac{m_{fc,t}(\tau + iN^{-1+\xi}) - m_{fc,t}(z')}{\tau + iN^{-1+\xi} - z' + t(m_{fc,t}(\tau + iN^{-1+\xi}) - m_{fc,t}(z'))} \, d\tau.
\] (6.135)

\[
\int_{qG}^{qG} \varphi_N(qG) \int_{N^{-1+\xi}}^{N^{10C\Gamma} + 1} \chi(\eta) \partial_\eta \frac{m_{fc,t}(qG + i\eta) - m_{fc,t}(z')}{qG + i\eta - z' + t(m_{fc,t}(qG + i\eta) - m_{fc,t}(z'))} \, d\eta.
\] (6.136)

Similarly, the second region \( \Omega_N^* \cap \{ \text{Im} z < 0 \} \) is labelled by (−) in indices. The sides are labelled in counter-clockwise orientation as (1), \( [qG-iN^{-1+\xi}, -qG-iN^{-1+\xi}] \); (2), \( [-qG-iN^{-1+\xi}, -qG-iN^{10C\Gamma}] \);
To summarize, we have shown so far

\[ 2t \int_{\Omega_N^q \cap \{\text{Im } z < 0\}} \partial_z \bar{\varphi}_N(z) \frac{t^2}{N} \partial_z \sum_{j=1}^N g_j(z)g_j(z') \frac{m_{lc,t}(z) - m_{lc,t}(z')}{z - z'} \, dz = t^2 \int_{qG} (\varphi_N(\tau) - iN^{-1+\xi} \varphi''_N(\tau)) \partial_z \frac{m_{lc,t}(\tau) - m_{lc,t}(z')}{\tau - z'} + t(m_{lc,t}(\tau) - iN^{-1+\xi} - m_{lc,t}(z')) \, dx \]

(6.137)

\[ + m_{lc,t}(\tau - iN^{-1+\xi} - m_{lc,t}(z')) \, dx \]

(6.138)

\[ + t^2 \varphi_N(-qG) \int_{-N^{10Cv} - 1}^{-N^{-1+\xi}} \chi(\eta) \partial_z \frac{m_{lc,t}(-qG + i\eta) - m_{lc,t}(z')}{qE + i\eta - z'} + t(m_{lc,t}(-qG + i\eta) - m_{lc,t}(z')) \, d\eta \]

(6.139)

\[ + t^2 \varphi_N(qG) \int_{-N^{10Cv} - 1}^{-N^{-1+\xi}} \chi(\eta) \partial_z \frac{m_{lc,t}(qG + i\eta) - m_{lc,t}(z')}{qE + i\eta - z'} + t(m_{lc,t}(qG + i\eta) - m_{lc,t}(z')) \, d\eta \]

(6.140)

We now insert \((-i/2)I_{1,\pm,k}(\zeta')\), \(k = 1, 2, 3\), into the integral (6.130), and apply Green’s theorem in each of the regions \(\Omega_N^q \cap \{\text{Im } z' > 0\}\) and \(\Omega_N^q \cap \{\text{Im } z' < 0\}\). We label the oriented sides of that region as previously:

\[ -(i/2) \int_{\Omega_N^q \cap \{\text{Im } z' < 0\}} \partial_z \bar{\varphi}_N(z') I_{1,\pm,k}(\zeta') \, dz' \]

(6.141)

\[ = \frac{1}{4} \int_{-qG}^{qG} (\varphi''_N(s) + iN^{-1+\xi} \varphi''_N(s)) I_{1,\pm,k}(s + iN^{-1+\xi}) \, ds \]

(6.142)

\[ + \frac{1}{4} \int_{N^{-1+\xi}}^{N^{10Cv}} \chi(\eta')(\varphi'_N(qG) + i\eta' \varphi''_N(qG)) I_{1,\pm,k}(qG + i\eta') \, d\eta' \]

(6.143)

By the support condition (6.2), the terms (6.142) and (6.143) are 0 for any \(k\) and choice of \(\pm\). We denote the remaining term (6.141) by \(I_{1,\pm,k,\pm}\). Similarly, applying Green’s theorem to \(\Omega_N^q \cap \{\text{Im } z' < 0\}\):

\[ -(i/2) \int_{\Omega_N^q \cap \{\text{Im } z' > 0\}} \partial_z \bar{\varphi}_N(z') I_{1,\pm,j}(\zeta') \, dz' \]

(6.144)

\[ = \frac{1}{4} \int_{-qG}^{qG} (\varphi'_N(s) - iN^{-1+\xi} \varphi''_N(s)) I_{2,\pm,k}(x' - iN^{-1+\xi}) \, dx' \]

To summarize, we have shown so far

\[ \frac{t^2}{N} \int_{\Omega_N^q} \int_{\Omega_N^q} \partial_z \bar{\varphi}_N(z) \partial_{z'} \partial_z \bar{\varphi}_N(z') \partial_{z'} \sum_{j=1}^N g_j(z)g_j(z') \frac{m_{lc,t}(z) - m_{lc,t}(z')}{z - z'} \, dz \, dz' = \sum_{k=1}^3 \sum_{\alpha,\beta \in \{\pm\}} I_{1,\alpha,k,\beta} \]

Only the terms \(I_{1,\pm,1,\pm}\) contribute to the variance. This is the content of the following.

**Proposition 6.13.** Recall the parameter \(\sigma > 0\) in our statement of the local law. For any choice of \(I_{2,\pm,k,\pm}\) with \(k \neq 1\), we have

\[ |I_{1,\pm,k,\pm}| \leq O(t^\sigma \log N(\|\varphi'_N\|_{L^1} + N^{-1+\xi}\|\varphi''_N\|_{L^1})). \]
Proof. By [52, Eqn (7.25)],
\[ |\partial_z m_{fc,t}(z)| \leq Ct^{-1}. \] (6.145)

Compute the derivative:
\[
\partial_z \frac{m_{fc,t}(z) - m_{fc,t}(z')}{z - z' + t(m_{fc,t}(z) - m_{fc,t}(z'))} = -\frac{m_{fc,t}(z) - m_{fc,t}(z')}{(z - z' + t(m_{fc,t}(z) - m_{fc,t}(z')))^2} (1 + t\partial_z m_{fc,t}(z))
\]
\[
+ \frac{1}{z - z' + t(m_{fc,t}(z) - m_{fc,t}(z'))}.
\]

Note that \( t(m_{fc,t}(z) - m_{fc,t}(z')) = \mathcal{O}(t) \) for \( z, z' \in D_{\epsilon,q} \). So if \(|\text{Re}z| \geq q \geq t^{1/2}N^{\sigma/2} \) and \(|\text{Re}z'| = \mathcal{O}(t_1N^\gamma)\)

Then
\[
\partial_z \frac{m_{fc,t}(z) - m_{fc,t}(z')}{z - z' + t(m_{fc,t}(z) - m_{fc,t}(z'))} < t^{-1}t^{-1/2}N^{-\sigma/2} \leq t^{-3/2+\sigma/2}.
\] (6.146)

Similarly:
\[
\left| \frac{m_{fc,t}(z) - m_{fc,t}(z')}{z - z'} \right| < \min(t^{-1/2+\sigma/2}, |\eta|^{-1}).
\]

By (6.28), and \( \frac{1}{N} \sum |g_j(z)| \leq C \log N [52, \text{Eqn. (7.36)}] \), so
\[
\left| \frac{1}{N} \sum_{j=1}^N g_j(z)^2 (1 + t\partial_z m_{fc,t}(z))g_j(z') \right| \leq C|\eta|^{-2} \log N.
\]

Combining this with (6.146), we have
\[
\left| \frac{1}{N} \sum_{j=1}^N g_j(z)^2 (1 + t\partial_z m_{fc,t}(z))g_j(z') \right| < \min(|\eta|^{-2} \log N, t^{-3/2+\sigma/2}).
\]

Inserting this into (6.134), (6.135), (6.139), (6.140), for \( z' \) such that \( s = \text{Re}z' \in \text{supp} \varphi_N' \) and \( k = 2, 3 \):
\[
|I_{1,\pm,k,\pm}| \leq t^2 \log N(\|\varphi_N'\|_{L^1} + N^{-1+\xi}\|\varphi_N''\|_{L^1}) \int_{N^{-1+\xi}}^{N^{10C_Y}} \min(t^{-2+\sigma}, |\eta|^{-3})d\eta
\]
\[
\leq t^2 \log N(\|\varphi_N'\|_{L^1} + N^{-1+\xi}\|\varphi_N''\|_{L^1}).
\]

For brevity of notation, we let \( s^\pm = s \pm iN^{-1+\xi} \), \( \tau^\pm = \tau \pm iN^{-1+\xi} \). We have so far shown that
\[
= \frac{t^2}{4} \int_{-qG}^{qG} \int_{-qG}^{qG} \hat{\varphi}_N(\tau)\hat{\varphi}_N'(s)\partial_\tau \frac{m_{fc,t}(\tau^+) - m_{fc,t}(s^+)}{\tau - s + t(m_{fc,t}(\tau^+) - m_{fc,t}(s^+))} \frac{m_{fc,t}(\tau^+) - m_{fc,t}(s^+)}{\tau - s} d\tau ds
\] (6.147)

\[
= \frac{t^2}{4} \int_{-qG}^{qG} \int_{-qG}^{qG} \hat{\varphi}_N(\tau)\hat{\varphi}_N'(s)\partial_\tau \frac{m_{fc,t}(\tau^-) - m_{fc,t}(s^-)}{\tau - s + t(m_{fc,t}(\tau^-) - m_{fc,t}(s^-))} \frac{m_{fc,t}(\tau^-) - m_{fc,t}(s^-)}{\tau - s} d\tau ds
\] (6.148)

\[
= \frac{t^2}{4} \int_{-qG}^{qG} \int_{-qG}^{qG} \hat{\varphi}_N(\tau)\hat{\varphi}_N'(s)\partial_\tau \frac{m_{fc,t}(\tau^+) - m_{fc,t}(s^+)}{\tau - s + t(m_{fc,t}(\tau^+) - m_{fc,t}(s^+))} \frac{m_{fc,t}(\tau^+) - m_{fc,t}(s^+)}{\tau - s} d\tau ds
\] (6.149)

\[
= \frac{t^2}{4} \int_{-qG}^{qG} \int_{-qG}^{qG} \hat{\varphi}_N(\tau)\hat{\varphi}_N'(s)\partial_\tau \frac{m_{fc,t}(\tau^-) - m_{fc,t}(s^-)}{\tau - s + t(m_{fc,t}(\tau^-) - m_{fc,t}(s^-))} \frac{m_{fc,t}(\tau^-) - m_{fc,t}(s^-)}{\tau - s} d\tau ds
\] (6.150)

\[
+ \mathcal{O}(t^2 \log N(\|\varphi_N'\|_{L^1} + N^{-1+\xi}\|\varphi_N''\|_{L^1})).
\] (6.152)

The main terms are (6.150), (6.151). These are of order \( \log N \) for the functions we are interested in. The other two terms are bounded by a constant.
Proposition 6.14. Let $s^\pm = s \pm iN^{-1+\xi}$, $\tau^\pm = \tau \pm iN^{-1+\xi}$. There is a constant $C$ such that
\begin{align}
|t^2 \int_{-qG}^{qG} \int_{-qG}^{qG} \tilde{\varphi}_N(\tau) \varphi'_N(s) \frac{m_{fc,t}(\tau^+) - m_{fc,t}(s^+)}{\tau - s + t(m_{fc,t}(\tau^+) - m_{fc,t}(s^+))} \frac{m_{fc,t}(\tau^+) - m_{fc,t}(s^+)}{\tau - s} \, d\tau d\tau| &\leq C (6.153) \\
t^2 \int_{-qG}^{qG} \int_{-qG}^{qG} \tilde{\varphi}_N(\tau) \varphi'_N(s) \frac{m_{fc,t}(\tau^+) - m_{fc,t}(s^+)}{\tau - s + t(m_{fc,t}(\tau^+) - m_{fc,t}(s^+))} \frac{m_{fc,t}(\tau^+) - m_{fc,t}(s^+)}{\tau - s} \, d\tau d\tau| &\leq C. (6.154)
\end{align}

Proof. First note the estimate
\begin{equation}
\left| \frac{m_{fc,t}(\tau^+) - m_{fc,t}(s^+)}{\tau - s + t(m_{fc,t}(\tau^+) - m_{fc,t}(s^+))} \right| \leq Ct^{-2}, (6.155)
\end{equation}
for $|\tau - s| \leq Ct$, which follows from the alternate representation
\begin{equation}
\frac{1}{N} \sum_{j=1}^{N} g_j(s)^2 (1 + t \partial_s m_{fc,t}(s)) g_j(s), (6.156)
\end{equation}
(6.154), (6.28), and the bound [52, Eqn (7.24)]
\begin{equation}
\frac{1}{N} \sum_{j=1}^{N} \left| V_j - z - t m_{fc,t}(z) \right| \leq Ct^{-1}. (6.157)
\end{equation}

Define $\zeta(z) := z + t m_{fc,t}(z)$. We begin by noting that (6.125) implies, for $z = \tau \pm iN^{-1+\xi}$, $z' = s + iN^{-1+\xi} \in \Omega_N$
\begin{align}
|\text{Re}(\zeta(z) - \zeta(z'))| &= |\text{Re} \int_{\tau}^{s} \partial_x \zeta(x) + iN^{-1+\xi} \, dx| = |\int_{\tau}^{s} \text{Re} \frac{1}{1 - tR_2(x + iN^{-1+\xi})} \, dx| \\
&= \left| \int_{\tau}^{s} \text{Re} \frac{1 - tR_2(x + iN^{-1+\xi})}{|1 - tR_2(x + iN^{-1+\xi})|^2} \, dx \right| \geq C|\tau - s|. (6.158)
\end{align}

In the second to last step we have used (6.157) as well as the lower bound $\text{Re}(1 - tR_2(z)) \geq c$, (see [52, Lemma 7.2]).

We then estimate the integral in (6.153) as
\begin{align}
\int_{-qG}^{qG} |\varphi'_N(s)| \int_{|\tau - s| < Mt} \left| \frac{m_{fc,t}(\tau^+) - m_{fc,t}(s^+)}{|\tau - s|} \right| d\tau d\tau &
\end{align}
\begin{align}
&\quad + t^2 \int_{-Q}^{Q} |\varphi'_N(s)| \int_{|\tau - s| \geq Mt} \left| \frac{m_{fc,t}(\tau^+) - m_{fc,t}(s^+)}{|\tau - s|} \right| d\tau d\tau, (6.159)
\end{align}
where $M$ is some constant. In the range $\{\tau : |\tau - s| < Mt\}$, we use (6.145) in the inner integral, to obtain a bound of constant order. Using that $|m_{fc,t}| \leq C \in Q_t$, and (6.158), we have for $|\tau - s| \geq Mt$:
\begin{equation}
\left| \frac{m_{fc,t}(\tau^+) - m_{fc,t}(s^+)}{|\tau - s|} \right| \leq C \frac{t^{-1}}{|s - \tau|}, (6.160)
\end{equation}
while
\begin{equation}
\frac{|m_{fc,t}(\tau^+) - m_{fc,t}(\tau^-)|}{|\tau - s|} \leq C \frac{t^{-1}}{|s - \tau|},
\end{equation}
Integrating over $\{\tau : |\tau - s| \geq Mt\}$ then again gives a constant bound for the $\tau$ integral. Since $\|\varphi'_N\|_{L^1} \leq C$ by assumption, we are done.
Summing the two terms (6.150), (6.151), we find a kernel multiplying $\tilde{\varphi}_N(s)\tilde{\varphi}_N(\tau)$, equal to
\[
2t^2 \text{Re} \frac{m_{c,t}(\tau^+)}{\tau - s - 2iN^{-1+\xi}} - \frac{m_{c,t}(s^-)}{\tau - s - 2iN^{-1+\xi}} \frac{m_{c,t}(\tau^+)}{\tau - s - 2iN^{-1+\xi}} - \frac{m_{c,t}(s^-)}{\tau - s - 2iN^{-1+\xi}}.
\]
Recall:
\[
\lim_{\varepsilon \to 0} m_{c,t}(x + i\varepsilon) = (H\rho_{c,t})(x) + i\pi \rho(x),
\]
so
\[
m_{c,t}(\tau^+) - m_{c,t}(s^-) = 2i\pi \rho_{c,t}(\tau) + \max_z |m'_{c,t}(z)| \cdot O(|\tau - s|),
\]
so the kernel is
\[
-2\text{Re} \frac{1}{\tau - s + 2iN^{-1+\xi}} + \max_z |m'_{c,t}(z)| \frac{t^2}{t^2 + |\tau - s|^2} O\left(\frac{|\tau - s|}{|\tau - s + 2iN^{-1+\xi}|}\right),
\]
when $|\tau - s| \leq Mt$, provided $M$ is sufficiently small. Integrating $\tilde{\varphi}_N(\tau)\tilde{\varphi}_N(s)$ against the error term, using (6.145), and splitting the $\tau$ integral according to $|\tau - s| \leq Mt$ and $|\tau - s| > Mt$, we find an error term of
\[
C(\|\varphi_N\|_{L^1} + N^{-1+\xi}\|\varphi'_N\|_{L^1})\|\varphi_N\|_{L^1} \leq C.
\]
The main term of (6.147) is then
\[
\frac{1}{2} \int_{-tM < \tau < tM} \tilde{\varphi}_N(\tau) \tilde{\varphi}_N(s) \Re \frac{1}{\tau - s + 2iN^{-1+\xi}} \, ds \, d\tau &= \frac{1}{2} \int_{-tM < \tau < tM} \varphi_N(\tau) \varphi_N(s) \Re \frac{1}{\tau - s + 2iN^{-1+\xi}} \, ds \, d\tau + O(1),
\]
for the second step, we have used
\[
\text{Re} \int_{|\tau - s| \leq N} \frac{1}{\tau - s + 2iN^{-1+\xi}} \, d\tau = \int_{|\tau - s| \leq N} \frac{\tau - s}{(\tau - s)^2 + 4N^{-2+2\xi}} \, d\tau = 0,
\]
to write:
\[
\text{Re} \int_{-tM < \tau < tM} \varphi_N(\tau) \varphi'_N(s) \Re \frac{1}{\tau - s + 2iN^{-1+\xi}} \, ds \, d\tau = \text{Re} \int_{-tM < \tau < tM} \varphi_N(\tau) (\varphi'_N(s) - \varphi'_N(\tau)) \Re \frac{1}{\tau - s + 2iN^{-1+\xi}} \, ds \, d\tau.
\]
The difference between the last expression and $\int_{-tM < \tau < tM} \varphi_N(\tau) (H\varphi'_N)(\tau) \, d\tau$ is
\[
\text{Re} \int_{-tM < \tau < tM} \varphi_N(\tau) \int (\varphi'_N(s) - \varphi'_N(\tau)) \Re \frac{2iN^{-1+\xi}}{(\tau - s + 2iN^{-1+\xi})(\tau - s)} \, ds \, d\tau
\]
We split the inner integral into $|\tau - s| \leq \delta$ and $|\tau - s| > \delta$ to find the estimate
\[
t |\varphi'_N|_{C^\alpha} \delta^\alpha + \frac{N^{-1+\xi}}{\delta} |\varphi'_N|_{L^1}.
\]
Optimizing in $\delta$, and using $\|\varphi'_N\|_{L^1} \leq C$ we get a bound of
\[
(t |\varphi'_N|_{C^\alpha})^{\frac{1}{1+\alpha}} N^{(-1+\xi)(\frac{\alpha}{1+\alpha})}.
\]

The right side is bounded by
\[ CN^{\frac{1}{2m}} \alpha^{-1} \frac{1}{1 + \frac{1}{2m} \alpha^{1/2}}. \]

Using condition (6.6), for \( \alpha > 0 \) small enough, this is \( O(N^{-c}) \).

We now proceed to computing \( I_{1,1} (6.129) \). Recall the definition of the constant \( K \) introduced in Proposition 6.14. Note first that by the support assumption on \( \varphi_N \), \( \partial_\zeta \varphi_N = 0 \) in \( \{|E| > Mt\} \) we can replace the integration domain \( \Omega_N^{**} \) by:
\[ \Omega_N^{**} = \Omega_N \cap \{ E + i\eta : |E| \leq Mt \}. \]

We use Green’s theorem (6.132) on \( \Omega_N^{**} \cap \{ \text{Im} z > 0 \} \). The boundary of this region consists of four segments. The function \( \tilde{\varphi}_N(z) \) is zero on the top segment is \( [-tM + iN^{10\text{CV}}, tM + iN^{10\text{CV}}] \). We number the remaining parts of the boundary as: (1) \([-tM + iN^{-1+\xi}, tM + iN^{-1+\xi}]\); (2) \([-tM + iN^{10\text{CV}}, -tM + iN^{-1+\xi}]\); and (3) \([tM + iN^{-1+\xi}, tM + iN^{10\text{CV}}]\). The resulting of our application of Green’s theorem to
\[ \int_{\partial \Omega_N^{**} \cap \{ \text{Im} z > 0 \}} \frac{\partial_\zeta \tilde{\varphi}_N(z) \partial_\zeta N \sum_{j=1}^{N} g_j(z)g_j(z') \partial_\zeta' \frac{m_{fc,t}(z) - m_{fc,t}(z')}{z - z'} \, dz \, dz' \quad (6.163) \]
is a sum of line integrals which we label according to the corresponding parts of the boundary. For (1), we have:
\[ I_{2,+,1} = t^2 \int_{-Mt}^{Mt} (\varphi_N(\tau) + iN^{-1+\xi} \varphi'_N(\tau)) \partial_\zeta \frac{m_{fc,t}(\tau + iN^{-1+\xi}) - m_{fc,t}(\tau')}{\tau + iN^{-1+\xi} - z'} \, d\tau \times \partial_\zeta' \frac{m_{fc,t}(\tau + iN^{-1+\xi}) - m_{fc,t}(\tau')}{\tau + iN^{-1+\xi} - z'} \, d\tau; \quad (6.164) \]
for (2), we have the sum:
\[ I_{2,+,2} = -t^2 \varphi_N(-tM) \int_{N^{-1+\xi}}^{N^{10\text{CV}}} \partial_\zeta \frac{m_{fc,t}(-tM + i\eta) - m_{fc,t}(z')}{-2tM + i\eta - z'} \, d\eta \times \partial_\zeta' \frac{m_{fc,t}(-tM + i\eta) - m_{fc,t}(z')}{-2tM + i\eta - z'} \, d\eta; \quad (6.165) \]
for (3):
\[ I_{2,+,3} = t^2 \varphi_N(tM) \int_{N^{-1+\xi}}^{N^{10\text{CV}}} \partial_\zeta \frac{m_{fc,t}(tM + i\eta) - m_{fc,t}(z')}{tM + i\eta - z'} \, d\eta \times \partial_\zeta' \frac{m_{fc,t}(tM + i\eta) - m_{fc,t}(z')}{tM + i\eta - z'} \, d\eta; \quad (6.166) \]

We similarly define \( I_{2,-,k}, k = 1, 2, 3 \) as the line integrals along the boundary of \( \Omega_N^{**} \cap \{ \text{Im} z < 0 \} \). We now insert \( I_{2,\pm,k}(z') \) into the integral (6.131) and apply Green’s theorem to obtain
\[ \begin{align*}
\frac{1}{2i} \int_{\Omega_N^{**} \cap \{ \text{Im} z > 0 \}} \partial_\zeta \tilde{\varphi}_N(z') I_{2,\pm,k}(z') \, dz' \\
\frac{1}{4} \int_{-qG} (\varphi_N(s) + iN^{-1+\xi} \varphi'_N(s)) I_{2,\pm,k}(s + iN^{-1+\xi}) \, ds \\
+ \frac{1}{4} \int_{N^{10\text{CV}}} \varphi_N(-qG) I_{2,\pm,k}(-qG + i\eta') \, d\eta' \\
+ \frac{1}{4} \int_{N^{-1+\xi}}^{N^{10\text{CV}}} \varphi_N(qG) I_{2,\pm,k}(qG + i\eta') \, d\eta' \\
:= I_{2,\pm,k,+,1} + I_{2,\pm,k,+,2} + I_{2,\pm,k,+,3},
\end{align*} \quad (6.167) \]
Applying Green’s theorem to the \( z' \) integral in the region \( \Omega_N^{**} \cap \{ \text{Im} \ z' < 0 \} \), we obtain:

\[
\frac{1}{2i} \int_{\Omega_N^{**} \cap \{ \text{Im} \ z' > 0 \}} \partial_{z'} \varphi_N(z') I_{2,\pm,k}(z') \, dz' = \frac{1}{4} \int_{qG}^{qG} (\varphi_N(s) - iN^{-1+\xi}\varphi'_N(s)) I_{2,\pm,k}(s - iN^{-1+\xi}) \, ds \\
+ \frac{1}{4} \int_{-N^{1+\xi}}^{N^{10CV}} \varphi_N(-qG) I_{2,\pm,k}(-qG - i\eta') \, d\eta' \\
+ \frac{1}{4} \int_{-N^{1+\xi}}^{N^{10CV}} \varphi_N(qG) I_{2,\pm,k}(qG - i\eta') \, d\eta'
\]

(6.168)

So far, we have

\[
\frac{t^2}{N} \int_{\Omega_N^{**}} \partial_{z'} \varphi_N(z) \partial_{z'} \varphi_N(z') \partial_z \sum_{j=1}^{N} g_j(z) g_j(z') \partial_z \frac{m_{f,t}(z) - m_{f,t}'(z')}{z - z'} \, dz \, dz' = \sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{\alpha,\beta \in \{\pm\}} I_{2,\alpha,k,\beta,j}.
\]

The main contribution comes from the terms \( I_{2,\pm,1,\pm,1} \). The remaining terms are polynomially smaller.

**Proposition 6.15.** For any choice of \( k, j \) with \( (k, j) \neq (1, 1) \), \( \alpha, \beta \in \{\pm\} \),

\[
|I_{2,\alpha,k,\beta,j}| = O(1).
\]

**Proof.** We start with \( k = 1, j = 2, 3 \). By symmetry, it suffices to deal with \( j = 2 \). That is, we estimate

\[
I_{2,\pm,1,\pm,2} = \pm \frac{\varphi_N(qG)}{4} \int_{N^{-1+\xi}}^{M_t} \int_{-t R_2(\tau \pm iN^{-1+\xi})}^{M_t} \frac{\varphi_N(\tau) \pm iN^{-1+\xi}\varphi'_N(\tau)}{1 - t R_2(\tau \pm iN^{-1+\xi})} S_{2,1}(\tau \pm iN^{-1+\xi}, qG + i\eta') \, d\tau \, d\eta'.
\]

(6.169)

Compute the kernel

\[
\frac{1}{1 - t R_2(\tau \pm iN^{-1+\xi})} S_{2,1}(\tau \pm iN^{-1+\xi}, qG + i\eta')
\]

\[
= -t^2 \frac{m_{f,t}(z) - m_{f,t}(z') - m_{f,t}(z) - m_{f,t}(z') \pm iN^{-1+\xi}}{(z - z' + t(m_{f,t}(z) - m_{f,t}(z')))^2 (z - z')^2}
\]

where \( z = \tau \pm iN^{-1+\xi}, z' = qG + i\eta' \). Since \( |\tau| \leq 2M_t \), we have\n
\[
|\tau - qG| \geq (1/2)qG \geq t^{1/2}N^{\sigma/2},
\]

and so, as in Proposition 6.13,

\[
|t^2 \frac{m_{f,t}(z) - m_{f,t}(z') \pm iN^{-1+\xi}}{(z - z' + t(m_{f,t}(z) - m_{f,t}(z')))^2 (z - z')^2} | \leq Ct^{1/2+\sigma/2}.
\]

Inserting this into (6.169), we find

\[
|I_{2,\pm,1,\pm,2}|
\]

\[
\leq C(1 + \|\varphi'_N\|_{L^2} N^{-1+\xi}) t^{1/2+\sigma/2} \int_{N^{-1+\xi}}^{M_t} \int_{-t R_2(\tau \pm iN^{-1+\xi})}^{M_t} \frac{1}{|\tau + iN^{-1+\xi} - qG - i\eta'|^2 + |\tau + iN^{-1+\xi} - qG - i\eta'|} \, d\tau \, d\eta'.
\]

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Recalling that $|m'_{\kappa,t}(z)| \leq Ct^{-1}$, this quantity is bounded by

$$C \log N(1 + \|\varphi_N\|_{L^\infty N^{-1+\xi}}) t^{1/2+\sigma} N^\sigma.$$ 

We have shown

$$|I_{2,\pm,1,\pm,2}|, |I_{2,\pm,1,\pm,3}| = O(\log N(1 + \|\varphi_N\|_{L^\infty N^{-1+\xi}}) t^{1/2+\sigma} N^\sigma). \quad (6.170)$$

We not estimate $I_{2,\pm,2,\pm,1}$:

$$t^2 \varphi_N(-Mt) \int_{N^{-1+\xi}}^{N^{10CV}} \int_{qG} (\varphi_N(s) \pm iN^{-1+\xi} \varphi'_N(s)) \frac{1}{1 - tR_2(-Mt \pm i\eta)} S_{2,1}(-Mt + i\eta, s \pm iN^{-1+\xi}) \, ds \, d\eta.$$ 

The kernel is

$$\frac{1}{1 - tR_2(-Mt \pm i\eta)} S_{2,1}(-Mt + i\eta, s \pm iN^{-1+\xi})$$

$$= -t^2 \frac{(m_{\kappa,t}(z) - m_{\kappa,t}'(z') - m_{\kappa,t}'(z)'(z - z')) m_{\kappa,t}(z) - m_{\kappa,t}'(z') - m_{\kappa,t}'(z)'(z - z'))}{(z - z' + t(m_{\kappa,t}(z) - m_{\kappa,t}'(z'))^2)} \quad (6.172),$$

with $z = -qG \pm i\eta$ and $z' = s \pm iN^{-1+\xi}$. The $I_{2,\pm,2,\pm,1}$ can be performed in the same way whether $\Im z \Im z' > 0$ or $\Im z \Im z' < 0$, except in the region

$$\{(z, z') : |s + tM| \leq Mt/2, N^{-1+\xi} < \eta < tM/10\}. \quad (6.173)$$

If $|s + tM| \geq Mt/2$ and $\eta \geq Mt/10$, we use the estimate (6.158)

$$\frac{1}{|z - z' + t(m_{\kappa,t}(z) - m_{\kappa,t}'(z'))|} \leq \frac{C}{|tM + s|},$$

and

$$|z - z'| = |tM - s + iN^{-1+\xi} + i\eta + \tau| \geq \eta \quad (6.174)$$

to find the bound

$$t^2 \int_{\eta \geq Mt/10} \int_{|s + tM + s| \geq Mt/2} \frac{1 + |m_{\kappa,t}(z)||z - z'|}{|z - z' + t(m_{\kappa,t}(z) - m_{\kappa,t}'(z'))^2|} \, ds \, d\eta$$

$$\leq Ct^2 \int_{N^{-1+\xi} < \eta \leq Mt/10} t^{-1}(\eta^{-2} + |m_{\kappa,t}(z)|\eta^{-1} + |m_{\kappa,t}'(z')|\eta^{-3/2} t^{1/2}) \, ds$$

$$= O(1). \quad (6.175)$$

To pass to the last line, we have used $|m_{\kappa,t}'(z)| \leq C\eta^{-1}$. For the case $|s + tM| \leq Mt/2$ and $\eta \geq Mt/10$, we have the bound

$$t^2 \int_{\eta \geq Mt/10} \int_{|s + tM + s| \leq Mt/2} \frac{1 + |m_{\kappa,t}(z)||z - z'|}{\eta^2|z - z'|^2} \, ds \, d\eta$$

$$\leq Ct^2 \int_{|s + tM| \leq Mt/2} t^{-3} \, d\eta$$

$$= O(1). \quad (6.176)$$

If $|s + tM| \geq Mt/2$ and $\eta \leq Mt/10$, use $|z - z'| \geq |s + tM|$ to find the estimate

$$t^2 \int_{\eta \leq N^{-1+\xi} < \eta \leq Mt/10} \int_{|s + tM + s| \geq Mt/2} \frac{1 + |m_{\kappa,t}(z)||z - z'|}{|tM + s|^2|z - z'|^2} \, ds \, d\eta$$

$$\leq Ct^{-1} \int_{\eta \leq N^{-1+\xi} < \eta \leq Mt/10} (1 + t|m_{\kappa,t}'(z)| + t|m_{\kappa,t}'(z')|) \, d\eta$$

$$= O(1). \quad (6.177)$$
At this point, we have obtained estimates for $I_{2,±2,±1}$ in the complement of (6.173).

We now estimate the contribution to $I_{2,0,2,±1}$ from the region (6.173), when $\alpha$ and $\beta$ are of opposite signs. This term is somewhat delicate. It will suffice to deal with $I_{2,+,2,−1}$. We split the $s$ integral into the regions $\{s : |s + Mt| \leq Mt/2\}$ and its complement. In the first region $\varphi_N(s) \equiv \varphi_N(−Mt)$.

When $\eta < Mt/10$ as well, we expand the kernel to second order. For this, $\text{Im} z \text{Im} z’ < 0$, so we have the expansion

$$m_{fc,t}(z') = m_{fc,t}(\bar{z}) + m_{fc,t}(\bar{z})(z' - \bar{z}) + \mathcal{O}(\max_{z} |m''_{fc,t}(z)||z - \bar{z}|^2)$$

(6.178)

Using (6.178) and the lower bound

$$|m_{fc,t}(z) - m_{fc,t}(\bar{z})| = 2\pi \rho_{fc,t}(z) \geq c > 0,$$

the kernel (6.172) is given by:

$$-\frac{1}{(z - z')^2} \left(1 - \frac{m'_{fc,t}(\bar{z})}{2\pi i \rho_{fc,t}(\bar{z})} (z - \bar{z}) - \frac{m'_{fc,t}(\bar{z})}{2\pi i \rho_{fc,t}(\bar{z})} (z' - \bar{z}) - \frac{m'_{fc,t}(\bar{z})}{2\pi i \rho_{fc,t}(\bar{z})} (z' - \bar{z}) \right)$$

$$+ \left(\frac{m'_{fc,t}(\bar{z})}{4\pi \rho_{fc,t}(\bar{z})} (z' - \bar{z}) (z - \bar{z}) + \frac{m'_{fc,t}(\bar{z}) m''_{fc,t}(\bar{z})}{4\pi \rho_{fc,t}(\bar{z})} (z' - \bar{z})(z - \bar{z})\right)$$

$$\times \left(1 - \frac{m'_{fc,t}(\bar{z})}{\pi i \rho_{fc,t}(\bar{z})} (z' - \bar{z}) + \frac{z - \bar{z}}{\pi i \rho_{fc,t}(\bar{z})} \right)$$

$$+ \mathcal{O} \left(\max_{z} |m''_{fc,t}(z)||(|z - z'|^2 + |z - \bar{z}'|^2) \right),$$

(6.179)

(6.180)

(6.181)

(6.182)

(6.183)

(6.184)

for $|tM - s| \leq Mt/2$ and $\eta \leq Mt/10$.

The cancellation that arises from performing the $s$ integral first in (6.179), (6.180) is crucial. For example, the contribution to $I_{2,+,2,−1}$ from $\{|s - Mt| \leq Mt/2\}$, $\eta < Mt/10$ of the term $1/(z - z')^2$ is

$$\varphi_N(−Mt)^2 \int_{N−1+\xi}^{Mt/10} \int_{|s|+Mt|\leq Mt/2} \frac{1}{(−Mt + i\eta - s + iN−1+\xi)^2} d\eta d\eta$$

$$\leq C \varphi_N(−Mt)^2 \int_{N−1+\xi}^{Mt/10} \frac{tM}{(tM/2)^2 + (\eta + N−1+\xi)^2} d\eta.$$ 

$$= \mathcal{O}(1).$$

To estimate the remaining terms, letting $z = −tM + i\eta$ and $z' = s - iN−1+\xi$, we compute:

$$\int_{|s|+Mt|\leq Mt/2} \frac{z' - \bar{z}}{(z - z')^2} d\eta = - \int_{|s|+Mt|\leq Mt/2} \frac{1}{z - \bar{z}} d\eta + \int_{|s|+Mt|\leq Mt/2} \frac{z - \bar{z}}{(z - z')^2} d\eta,$$

$$\int_{|s|+Mt|\leq Mt/2} \frac{1}{z - z'} d\eta = \log \left(\frac{tM}{tM} + i\eta + iN−1+\xi \right) = \mathcal{O}(\eta/t), \quad |\eta| \leq Mt/10,$$

$$\int_{|s|+Mt|\leq Mt/2} \frac{z - \bar{z}}{(z - z')^2} d\eta = \frac{2i\eta tM}{(tM/2)^2 + (\eta + N−1+\xi)^2}. $$

We have used the principal determination of the logarithm in (6.183).
Using (6.183), (6.184), and (6.145),
\[
\int_{N^{-1+\xi} \int s:|s+tM/2| \leq Mt/2} \left(1 + \frac{1}{2\pi i t \rho_{c,t}(z)} \right) m'_{c,t}(\tilde{z}) \frac{(z'-\tilde{z})}{(z-\tilde{z})^2} \, ds \, d\eta \\
= \int_{N^{-1+\xi} \int s:|s+tM/2| \leq Mt/2} O(t^{-1}|\eta| tM) \frac{m'_{c,t}(\tilde{z})}{(z-\tilde{z})^2} \, ds \, d\eta \\
= \int_{N^{-1+\xi} \int s:|s+tM/2| \leq Mt/2} \frac{tM}{(tM/2)^2 + (\eta + N^{-1+\xi})^2} + O(t^{-2}|\eta|) \, d\eta = O(1),
\]
Moreover, for \( z = -tM + i\eta, z' = s - iN^{-1+\xi}, \) we have
\[
|\tilde{z} - z', |z - \tilde{z}| \leq 2|z - z'|,
\]
so
\[
\left| \int_{N^{-1+\xi} \int s:|s+tM/2| \leq Mt/2} \frac{(m'_{c,t}(\tilde{z}))^2}{4\pi^2 \rho_{c,t}(z)} \frac{(z'-\tilde{z})(z-\tilde{z})}{(z-\tilde{z})^2} \left(1 + m'_{c,t}(\tilde{z})(z'-\tilde{z}) - \frac{z'-\tilde{z}}{t} \right) \, ds \, d\eta \right| \\
\leq \int_{N^{-1+\xi} \int s:|s+tM/2| \leq Mt/2} t^{-2}(1 + t^{-1}(|z - z'| + |\tilde{z} - z'|)) \, d\eta = O(1),
\]
Similar estimates hold for the other terms containing a quadratic expression in \( z' - z, \tilde{z} - z' \) or \( z - \tilde{z} \) in (6.179), (6.180).

For the error term (6.181), we use (6.185) and the estimate
\[
|\partial_{\bar{z}}^2 m_{c,t}(z)| \leq \frac{1}{N} \sum_{j=1}^{N} \left| g_j(z) \right| \left(1 + t\partial_{z} m_{c,t}(z) \right) \left(1 - tR_2(z) \right)^2 \\
\leq Ct^{-2}.
\]
The last step in (6.187) follows from (6.157) and (6.28). (See also [52, Lemma 7.2].) The result is
\[
\int_{N^{-1+\xi} \int s:|s+tM/2| \leq Mt/2} \max_{z} |m''_{c,t}(z)| (|z - z'|^2 + |\tilde{z} - z'|^2) \, d\eta \leq C \int_{N^{-1+\xi} \int s:|s+tM/2| \leq Mt/2} \, d\eta = O(1).
\]
At this point all terms in the expansion (6.179), (6.180) are accounted for.

To estimate the contribution from the region (6.173) to \( I_{2,\alpha,2,\beta,1} \) when \( \alpha \) and \( \beta \) have the same sign, we use (6.155)
\[
\left| \partial_{z} m_{c,t}(z) - m_{c,t}(z') \right| \leq Ct^{-2},
\]
and the estimate
\[
\left| \partial_{z'} m_{c,t}(z) - m_{c,t}(z') \right| \leq Ct^{-2},
\]
which follows from (6.88) and the estimate (6.187). We have:
\[
|I_{2,+,2,+1}| \leq Ct^{-2} \varphi(-Mt) \int_{N^{-1+\xi} \int s:|s+tM| \leq Mt/2} |\tilde{\varphi}(s)| \, d\eta \leq C.
\]
The same bound holds for $I_{2, -2, -1}$.

Replacing $z = -Mt + i\eta$ by $z = Mt + i\eta$, we obtain the bounds $|I_{2, \pm 3, \pm 1}| = \mathcal{O}(1)$. Turning to $I_{2, \pm 3, \pm 2}$, we have to estimate:

$$
\pm \varphi(-Mt)\varphi_N(-qG) \int_{-N^{1+\epsilon}}^{1} \int_{-N^{1+\epsilon}}^{1} \frac{1}{1 - tR_2(-Mt + i\eta)} S_{2, 1}(-tN^\sigma \pm i\eta, -qG \pm i\eta') d\eta d\eta'.
$$

Note that

$$
|z - z' - t(m_{fc, l}(z) - m_{fc, l}(z'))| \geq cqG \quad (6.189)
$$

for $z = -Mt + i\eta$ and $z' = -qG + i\eta'$, so by (6.146), (6.9) the integrand is bounded by $Ct^{1/2+\sigma/2}|\eta|^{-1}|\eta'|^{-1}$.

Performing the double integration, we obtain a bound of

$$
Ct^{1/2+\sigma/2}(\log N)^2.
$$

This last estimate depended only on the lower bound (6.189), so we have the same estimate for $I_{2, \pm 2, \pm 3}$, $I_{2, \pm 3, \pm 2}$, $I_{2, \pm 3, \pm 3}$.

Denote $\tau = t \pm iN^{-1+\epsilon}$ and $s = s \pm iN^{-1+\epsilon}$. We have shown

$$
\int \int \phi(\tau)\varphi_N(s) \frac{m_{fc, l}(\tau) - m_{fc, l}(s)}{\tau - s} d\tau ds < \mathcal{O}(1).
$$

The main terms here are (6.193) and (6.194). For the remaining terms we have

**Proposition 6.16.** We have the estimate: There is a constant $C$ such that

$$
|\int_{-qG}^{qG} \int_{-Mt}^{Mt} \varphi_N(\tau) \varphi_N(s) \frac{m_{fc, l}(\tau) - m_{fc, l}(s)}{\tau - s} d\tau ds| \leq C,
$$

(6.196)

$$
|\int_{-qG}^{qG} \int_{-Mt}^{Mt} \varphi_N(\tau) \varphi_N(s) \frac{m_{fc, l}(\tau) - m_{fc, l}(s)}{\tau - s} d\tau ds| \leq C.
$$

(6.197)

**Proof.** We deal with the first quantity. The second quantity is estimated similarly. The kernel part of the integrand is

$$
-2m_{fc, l}(\tau) + \frac{m_{fc, l}(\tau) - \int_{\tau} m_{fc, l}(\tau) (\tau - s) m_{fc, l}(s) - \int_{\tau} m_{fc, l}(\tau) (\tau - s)^2}{(\tau - s)^2}.
$$

In the region $\{\tau : |\tau - s| \leq Mt\}$, we use (6.155), and (6.188). So (6.196) is bounded by

$$
t^2 \int_{-Mt}^{Mt} \left( \int_{|\tau - s| \leq Mt} \frac{1 + |m_{fc, l}(\tau)||\tau - s|}{|\tau - s|^4} d\tau \right) ds \leq C.
$$

(6.198)
The sum of the remaining terms (6.193), (6.194) is

$$\frac{\iota^2}{2} \int_{-tM}^{tM} \int_{-\eta G}^{\eta G} \varphi_{N}(\tau) \varphi_{N}(s) \Re \partial_{\tau} \frac{m_{fc,t}(\tau^{-}) - m_{fc,t}(s^{+})}{\tau - s + t(m_{fc,t}(\tau^{-}) - m_{fc,t}(s^{+}))} \partial_{s} \frac{m_{fc,t}(\tau^{-}) - m_{fc,t}(s^{+})}{\tau - s - 2iN^{-1+\xi}} \, d\tau \, ds.$$  

Using the expansion (6.179), (6.180) in the region $\{ \tau : |\tau| \leq 2Mt \}$:

$$\iota^2 \Re \partial_{\tau} \frac{m_{fc,t}(\tau^{-}) - m_{fc,t}(s^{+})}{\tau - s + t(m_{fc,t}(\tau^{-}) - m_{fc,t}(s^{+}))} \partial_{s} \frac{m_{fc,t}(\tau^{-}) - m_{fc,t}(s^{+})}{\tau - s - 2iN^{-1+\xi}}$$

$$= - \frac{1}{(\tau - s - 2iN^{-1+\xi})^{2}} + \Delta_{3},$$

where $\Delta_{3}(z, z')$ is an error term. The most serious terms in $\Delta_{3}$ are handled using the computation

$$\int_{-2tM}^{2tM} \int_{-tM}^{tM} \frac{m'_{fc,t}(z)}{\rho_{fc,t}(z)} \varphi_{N}(s) \varphi_{N}(\tau) \frac{1}{s - \tau - 2iN^{-1+\xi}} \, ds \, d\tau = e \int \int f(\xi)g(\lambda) K(\xi - \lambda) \, d\xi \, d\lambda,$$

where $f$ and $g$ are the inverse Fourier transforms of $1_{[-2tM,2tM]}(\tau) \frac{m'_{fc,t}(z)}{\rho_{fc,t}(z)} \varphi(\tau)$ and $1_{[-tM,tM]}(s) \varphi_{N}(s)$, respectively, and

$$K(\xi, \lambda) = K(\xi - \lambda)$$

$$:= i1_{(-\infty,0)}(\xi - \lambda)e^{-2N^{-1+\xi}|\xi - \lambda|},$$

so that

$$\hat{K}(x) = \frac{1}{x - 2iN^{-1+\xi}}.$$

From the Fourier representation, the Plancherel theorem and the simple estimates

$$\|f\|_{L^{2}} = O(t^{-1/2}), \|g\|_{L^{2}} = O(t^{1/2}),$$

the term (6.199) is $O(1)$. All other error terms are then easily estimated, using $z - \bar{z} = 2N^{-1+\xi}$ and the trivial bound

$$\int_{-2tM}^{2tM} \int_{-tM}^{tM} |\varphi_{N}(s)||\varphi_{N}(\tau)| \frac{1}{|s - \tau - 2iN^{-1+\xi}|^{2}} \, ds \, d\tau = O(\log N).$$

As in (6.198) contribution from the region $\{|\tau| \geq 2Mt\} \subset \{ \tau : |\tau - s| \geq Mt \}$, as well as the error terms, are $O(1)$. Adding the contributions from the two main terms, we find:

$$\int \int_{\Omega_{N} \times \Omega_{N}} \partial_{\tau} \varphi_{N}(z) \partial_{\tau} \varphi_{N}(z') \frac{1}{1 - tR(z)} S_{2,2}(z, z') \, dz \, dz'$$

$$= \frac{1}{2} \int_{-tM}^{tM} \int_{-2tM}^{2tM} \varphi_{N}(\tau) \varphi_{N}(s) \frac{2}{(\tau - s)^{2} - N^{-2+\xi}} \, d\tau \, ds = O(1)$$

$$= \frac{1}{2} \int_{-tM}^{tM} \int_{-2tM}^{2tM} \varphi_{N}(\tau) \varphi_{N}(s) \Re \partial_{s} \frac{1}{\tau - s - iN^{-1+\xi}} \, d\tau \, ds = O(1).$$

This is the same quantity as in (6.162), and so this ends the computation of the term (6.131).

It remains to estimate $I_{1,3}$. Integrating by parts in $z$ and $z'$, we have

$$I_{1,3} = \frac{i}{N} \sum_{j=1}^{N} \int_{\Omega_{N}^{*}} \int_{\Omega_{N}^{*}} g_{j}(z)g_{j}(z') \partial_{\tau} \varphi(z) \partial_{s} \varphi(z') \, dz \, dz' + O(N^{-2}).$$
As for $I_{1,1}$ and $I_{1,2}$, we use Green’s theorem to the domains $\Omega_N \cap \{ \text{Im } z > 0 \}$, $\Omega_N \cap \{ \text{Im } z < 0 \}$, $\Omega_N \cap \{ \text{Im } z' > 0 \}$, $\Omega_N \cap \{ \text{Im } z' < 0 \}$. By the support properties of $\partial_s \tilde{\varphi}(\tau)$, we only find contributions from the segments $[-qG \pm iN^{-1+\xi}, qG \pm iN^{-1+\xi}]$. Denoting $\tau^\pm = \tau \pm iN^{-1+\xi}$, $s^\pm = s \pm iN^{-1+\xi}$, the result is

$$
I_{1,3} = \frac{t}{4N} \sum_{j=1}^{N} \int_{-qG}^{qG} \int_{-qG}^{qG} g_j(\tau^+) g_j(s^+) \partial_s \tilde{\varphi}_N(\tau^+) \partial_\tau \tilde{\varphi}_N(s^+) \, ds \, d\tau
+ \frac{t}{4N} \sum_{j=1}^{N} \int_{-qG}^{qG} \int_{-qG}^{qG} g_j(\tau^-) g_j(s^-) \partial_s \tilde{\varphi}_N(\tau^-) \partial_\tau \tilde{\varphi}_N(s^-) \, ds \, d\tau
- \frac{t}{4N} \sum_{j=1}^{N} \int_{-qG}^{qG} \int_{-qG}^{qG} g_j(\tau^+) g_j(s^-) \partial_s \tilde{\varphi}_N(\tau^+) \partial_\tau \tilde{\varphi}_N(s^-) \, ds \, d\tau
- \frac{t}{4N} \sum_{j=1}^{N} \int_{-qG}^{qG} \int_{-qG}^{qG} g_j(\tau^-) g_j(s^+) \partial_s \tilde{\varphi}_N(\tau^-) \partial_\tau \tilde{\varphi}_N(s^+) \, ds \, d\tau + O(N^{-2}).
$$

(6.202)

By (6.157), we have

$$
\left| \frac{t}{N} \sum_{j=1}^{N} g_j(z) g_j(z') \right| \leq C,
$$

so from (6.202), we obtain

$$
|I_{1,3}| \leq C \| \varphi'_N \|_{L^1} \leq C.
$$

(6.203)

Combining the results (6.162), (6.200), (6.203) we find

$$
V(\varphi_N) = \frac{2}{\pi^2} (-I_{1,1} + I_{1,2} + I_{1,3})
= \frac{2}{\pi^2} (-2 \cdot (6.129) + (6.131))
= \frac{1}{\pi^2} \int_{-Mt}^{Mt} \varphi_N(\tau) H \varphi_N'(\tau) \, d\tau + O(1).
$$

If the function $\varphi_N$ is compactly supported:

$$
\text{supp} \varphi_N \subset (-t_1 N^1, t_1 N^1),
$$

the terms $I_{1,2}, I_{1,3}$ are small for large $N$. Indeed, by (6.157), (6.28):

$$
|I_{1,2}| \leq C t^{-1} \int_{\Omega_N} \int_{\Omega_N} |\partial_z \varphi_N(z)| |\partial_z \varphi_N(z')| \left| \left| \frac{m_{k,t}(z) - m_{k,t}(z')}{|z - z'|} \right| \right| \, dz \, dz'
\leq C t^{-1} \| \varphi_N \|_{L^1} \log N
\leq CN^\omega N^{-\omega' \log N},
$$

with a similar bound holding for $I_{1,3}$. For $I_{1,1}$, the support of $\tilde{\varphi}_N$ means that we can apply Green’s theorem to find

$$
I_{1,1} = \frac{t^2}{2} \text{Re} \int_{-N^t t_1}^{N^t t_1} \int_{-N^t t_1}^{N^t t_1} \tilde{\varphi}_N(s) \tilde{\varphi}_N(\tau) \partial_\tau \left( \frac{m_{k,t}(\tau^+) - m_{k,t}(s^+)}{\tau - s + l(m_{k,t}(\tau^+) - m_{k,t}(s^+))} \right) \partial_s \left( \frac{m_{k,t}(\tau^+) - m_{k,t}(s^+)}{\tau - s + l(m_{k,t}(\tau^+) - m_{k,t}(s^+))} \right) \, ds \, d\tau
- \frac{t^2}{2} \text{Re} \int_{-N^t t_1}^{N^t t_1} \int_{-N^t t_1}^{N^t t_1} \tilde{\varphi}_N(s) \tilde{\varphi}_N(\tau) \partial_\tau \left( \frac{m_{k,t}(\tau^-) - m_{k,t}(s^-)}{\tau - s + l(m_{k,t}(\tau^-) - m_{k,t}(s^-))} \right) \partial_s \left( \frac{m_{k,t}(\tau^-) - m_{k,t}(s^-)}{\tau - s + l(m_{k,t}(\tau^-) - m_{k,t}(s^-))} \right) \, ds \, d\tau.
$$

(6.204)

By (6.155), (6.145), the first term in (6.204) is bounded by

$$
C t^{-1} \| \varphi_N \|_{L^1}.
$$
By (6.178), the kernel in the second term in (6.204) is
\[ -\frac{1}{(\tau - s + 2iN^{-1+\xi})^2} + \mathcal{O}(\max_z |m'_{\ell,t}(z)| + 1/t) \frac{1}{|\tau - s + 2iN^{-1+\xi}|}, \]
so that
\[ I_{1,1} = \frac{1}{2} \text{Re} \int_{-N^{t_1}}^{N^{t_1}} \int_{-N^{t_1}}^{N^{t_1}} \varphi_N'(\tau)\varphi_N(s) \frac{1}{(\tau - s + 2iN^{-1+\xi})^2} \, ds \, d\tau + \mathcal{O}(t^{-1} \log N\|\varphi_N\|_{L^1}). \]
(6.205)

After integration by parts in \( s \), the main term in (6.205) is
\[ -\frac{1}{2} \text{Re} \int_{-N^{t_1}}^{N^{t_1}} \varphi_N'(s) \frac{\varphi_N(\tau) - \varphi_N(s)}{\tau - s + 2iN^{-1+\xi}} \, ds \, d\tau. \]
(6.206)

We have added in the term
\[ -\lim_{R \to \infty} \text{Re} \int \varphi_N(s) \varphi_N(s) \int_{\{\tau : 1/R < |\tau - s| \leq R\}} \frac{1}{\tau - s + 2iN^{-1+\xi}} \, d\tau \, ds = 0. \]

By the same computation as for (6.162), this quantity is
\[ -\frac{1}{2} \int \varphi_N(\tau) H(\varphi_N)(\tau) \, d\tau + (t\|\varphi_N'\|_{C^0}) \frac{1}{\tau^{1+\xi}} N^{(-1+\xi)\frac{n}{2\pi}}. \]

Reversing the integration by parts in (6.206), we obtain the second expression on the right side in (6.8).

6.8 Mean

In this section, we compute the next order correction to the deformed (average) semicircle law:

**Theorem 6.17.** Let \( \varphi_N \) be a sequence of functions as in Theorem 6.1. Let \( \lambda_i \) denote the eigenvalues of the deformed model \( H_t = V + \sqrt{t}W \). Then
\[ \mathbb{E} \sum_{j=1}^{N} \varphi_N(\lambda_i) - N \int \varphi_N(x) \rho_{\ell,t}(x) \, dx = -t^2 \int \varphi_N(x)(R_2 m'_{\ell,t})(x+i0) - (R_2 m'_{\ell,t})(x-i0) \, dx \]
\[ + \mathcal{O}(t N^{-\xi} + N^{-1/2} t^{-1/2}) \|\varphi'_N\|_{L^1} + \mathcal{O}(\|\varphi''_N\|_{L^1}/N)^{1/2}. \]
(6.207)

**Proof.** Using the Helffer-Sjöstrand representation (6.31), the difference (6.207) can be rewritten as
\[ \frac{N}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\varphi}_N(z) \mathbb{E}[m_N(z) - m_{\ell,t}(z)] \, dz. \]

Proceeding as in Section 6.2, we replace the domain of integration by \( \Omega_N \):
\[ N \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{\varphi}_N(z) \mathbb{E}[m_N(z) - m_{\ell,t}(z)] \, dz = N \int_{\Omega_N} \partial_{\bar{z}} \tilde{\varphi}_N(z) \mathbb{E}[m_N(z) - m_{\ell,t}(z)] \, dz + \mathcal{O}(\|\varphi''_N\|_{L^1}/N)^{1/2}. \]
(6.208)

We now compute
\[ N(m_N(z) - m_{\ell,t}(z)) = \sum_{j=1}^{N} \left( G_{jj}(z) - \frac{1}{V_j - z - tm_{\ell,t}(z)} \right) \]
(6.209)
for \( z \in \Omega_N \). By (6.21), we have
\[ G_{jj}(z) = \frac{1}{V_j - z - m_{\ell,t}(z)} - \frac{t(m_{\ell,t}(z) - m_N^{(j)}(z) + \sqrt{tw_{jj}}}{V_j - z - tm_{\ell,t}(z)} \]
\[ + \frac{(t(m_{\ell,t}(z) - m_N^{(j)}(z) + \sqrt{tw_{jj}})^2}{(V_j - z - tm_{\ell,t}(z))^2} - \frac{t(m_{\ell,t}(z) - m_N^{(j)}(z) + \sqrt{tw_{jj}})^3}{(V_j - z - tm_{\ell,t}(z))^2 A_j}. \]
Putting this in (6.209) and taking expectations, we obtain:

\[ N\mathbb{E}[m_N(z) - m_{fc,t}(z)] = -t \sum_{j=1}^{N} g_j(z)\mathbb{E}[m_N^{(j)}(z) - m_{fc,t}(z)] \tag{6.210} \]
\[ + \frac{t}{N} \sum_{j=1}^{N} g_j(z)^2 + \Delta_{\text{mean}}, \tag{6.211} \]

where there error

\[ \Delta_{\text{mean}}(z) = \sum_{j=1}^{N} \mathbb{E} \left( \frac{(t(m_{fc,t}(z) - m_N^{(j)}(z)))^2}{(V_j - z - tm_{fc,t}(z))^2} \right) \]
\[ - \sum_{j=1}^{N} \mathbb{E} \left( \frac{(t(m_{fc,t}(z) - m_N^{(j)}(z)) + \sqrt{t}w_{jj})^3}{(V_j - z - tm_{fc,t}(z))^2 A_j} \right) \]

is analytic in \( \Omega_N \) with

\[ |\Delta_{\text{mean}}(z)| \leq C \sum_{j=1}^{N} \frac{t^2|m_{fc,t}(z) - m_N^{(j)}(z)|^2}{|V_j - z - tm_{fc,t}(z)|^2} \]
\[ + C \sum_{j=1}^{N} \mathbb{E} \left( \frac{t^3|m_{fc,t}(z) - m_N^{(j)}(z)|^3}{|V_j - z - tm_{fc,t}(z)|^2 |A_j|} \right) \]
\[ + C \sum_{j=1}^{N} \mathbb{E} \left( \frac{t^{3/2}|w_{jj}|^3}{|V_j - z - tm_{fc,t}(z)|^2 |A_j|} \right) \tag{6.212} \]

Next, we write

\[ t \sum_{j=1}^{N} g_j(z)\mathbb{E}[m_N^{(j)}(z) - m_{fc,t}(z)] = t \sum_{j=1}^{N} g_j(z)\mathbb{E}[m_N^{(j)}(z) - m_N(z)] + Ntm_{fc,t}(z)\mathbb{E}[m_N(z) - m_{fc,t}(z)]. \]

Using this in (6.210), we find

\[ N(1 + tm_{fc,t}(z))\mathbb{E}[m_N(z) - m_{fc,t}(z)] = -t \sum_{j=1}^{N} g_j(z)\mathbb{E}[m_N^{(j)}(z) - m_N(z)] + \frac{t}{N} \sum_{j=1}^{N} g_j(z)^2 + \Delta_{\text{mean}}(z). \]

By (6.25), (6.26):

\[ N\mathbb{E}[m_N^{(j)}(z) - m_N(z)] = \frac{1 + t\partial_z m_{fc,t}(z)}{\mathbb{E}[A_j(z)]} + \mathcal{O}(|\eta|^{-1} (N|\eta|)^{-1}). \tag{6.213} \]

Together with (6.18), this shows

\[ N(1 + tm_{fc,t}(z))\mathbb{E}[m_N(z) - m_{fc,t}(z)] = -\frac{t^2}{N} \sum_{j=1}^{N} g_j(z)^2 \partial_z m_{fc,t}(z) + \Delta_{\text{mean}}. \tag{6.214} \]

Since \( |m_{fc,t}(z)| \leq Ct^{-1/2} \), we have \( 1 + tm_{fc,t}(z) = 1 + \mathcal{O}(t^{1/2}) \), and so we may divide both sides of (6.214) by \( 1 - tm_{fc,t}(z) \):

\[ N\mathbb{E}[m_N(z) - m_{fc,t}(z)] = -t^2 \partial_z m_{fc,t}(z)R_2(z) + \frac{\Delta_{\text{mean}}(z)}{1 + m_{fc,t}(z)} \tag{6.215} \]

for \( z \in \Omega_N \). We have so far shown that

\[ N \int_{\Omega_N} \partial_z \bar{\phi}(z)\mathbb{E}[m_N(z) - m_{fc,t}(z)] \, dz = -t^2 \int_{\Omega_N} \partial_z \bar{\phi}(z)R_2(z)\partial_z m_{fc,t}(z) \, dz \]
\[ + \int_{\Omega_N} \partial_z \bar{\phi}(z) \frac{\Delta_{\text{mean}}(z)}{1 + m_{fc,t}(z)} \, dz. \tag{6.217} \]
By the local law, on $\Omega_N$, we have
\[
\sum_{j=1}^N \frac{t^2|m_{t,c,t}(z) - m^{(j)}_N(z)|^2}{|V_j - z - m_{t,c,l}(z)|^2} = \mathcal{O}(t(N|\eta|)^{-1}|\eta|^{-1}),
\]
\[
\sum_{j=1}^N \frac{t^3|m_{t,c,t}(z) - m^{(j)}_N(z)|^3}{|V_j - z - m_{t,c,l}(z)|^2|A_j|} = \mathcal{O}(t(N|\eta|)^{-2}|\eta|^{-1}),
\]
\[
\sum_{j=1}^N \frac{t^{3/2}|w_{jj}|^{3/2}}{|V_j - z - m_{t,c,l}(z)|^2|A_j|} = \frac{t^{3/2}}{N}\sum_{j=1}^N |g_j(z)|^3\mathcal{O}(N^{-1/2}).
\] (6.218)

It follows:
\[
\int_{\Omega_N} i\eta \chi(\eta)\varphi''(\tau) \frac{\Delta_{\text{mean}}(z)}{1 + tm_{t,c,l}(z)} \, dz = -\int_{\Omega_N} i\eta \chi(\eta)\varphi'(\tau) \partial_z \frac{\Delta_{\text{mean}}(z)}{1 + tm_{t,c,l}(z)} \, dz
\]

The last quantity is bounded by
\[
\|\varphi''_N\|_L^1 \int_{N^{-1} + \xi}^{10} (|\eta|^{-1}(N|\eta|)^{-1} + \frac{t^{3/2}}{N}\sum_{j=1}^N |g_j(z)|^3\mathcal{O}(N^{-1/2})) \, d\eta \leq C(tN^{-\xi} + N^{-1/2}t^{-1/2})\|\varphi''_N\|_L^1.
\] (6.219)

The remaining part of the error term (6.217) is
\[
\int_{\Omega_N} i\chi'(\eta)(\varphi_N(\tau) + i\eta\varphi_N(\tau)) \frac{\Delta_{\text{mean}}(z)}{1 + tm_{t,c,l}(z)} \, dz
\] (6.220)

Since $\{\eta : \chi' \neq 0\} \subset [N^{10C_V} - 1, N^{10C_V}]$, (6.220) gives an error of
\[
C\|\varphi_N\|_{L^1}N^{-20C_V} + \|\varphi''_N\|_{L^1}N^{-10C_V} = CN^{-2}\|\varphi''_N\|_{L^1}.
\] (6.221)

Adding the errors (6.208), (6.219) and (6.221), we find
\[
N \int_{\Omega_N} \partial_z \bar{\varphi}_N(z) E[m_N(z) - m_{t,c,l}(z)] \, dz = -t^2 \int_{\Omega_N} \partial_z \bar{\varphi}_N(z) R_2(z) \partial_z m_{t,c,l}(z) \, dz
\] (6.222)
\[
+ \mathcal{O}(tN^{-\xi}N^{-1/2}t^{-1/2})\|\varphi''_N\|_{L^1} + \mathcal{O}(\|\varphi''_N\|_{L^1}/N)^{1/2}.
\] (6.223)

A simple computation using $|tm_{t,c,l}(z)| \leq C$ shows that
\[
t^2 \int_{\Omega_N} \partial_z \bar{\varphi}_N(z) R_2(z) \partial_z m_{t,c,l}(z) \, dz = \mathcal{O}(N^{-1+\xi})\|\varphi''_N\|_{L^1} + \mathcal{O}(tN^{-C_V})(1 + \|\varphi''_N\|_{L^1}).
\]

The desired result is now obtained by applying Green’s theorem to
\[
t^2 \int_{\Omega_N} \partial_z \bar{\varphi}_N(z) R_2(z) \partial_z m_{t,c,l}(z) \, dz = t^2 \int \varphi(x)((R_2m_{t,c,l}'')(x + i0) - (R_2m_{t,c,l}'')(x - i0)) \, dx.
\]

\[
\Box
\]

6.9 $\beta$-ensembles

In this section, we obtain a result similar to Theorem 6.1 for linear statistics of $\beta$ ensembles. A result covering the case of quadratic potentials and general test functions was previously proved in [17, Theorem 5.4]. Our statement is somewhat simpler because we are considering functions with small support.
Theorem 6.18. Let supp $\rho_V = [A,B]$. Let $\varphi_N$ be a sequence of real-valued $C^2$ functions on $\mathbb{R}$ with support in $[2A-B, 2B-A]$, satisfying (6.1), (6.2), in addition to the following growth conditions on the derivatives:

$$\|\varphi_N^{(k+1)}\|_{L^\infty} \leq C_k t_1^{-k}, \quad k = 0, 1.$$ 

Let the parameters $t, t_1$ be chosen as in Theorem 6.1. There is an $\epsilon > 0$ such that uniformly in $|x| \leq N^\epsilon$:

$$\mathbb{E}[e^{ix(\tau_1 \varphi_N(H_1) - N(\tau_1 \varphi_N)\rho_V(x)dx)}] = \exp \left(-\frac{x^2}{2} V(\varphi_N) + i\delta(\varphi_N)\right) + O_N(N^{-1+20\epsilon})\|\varphi_N^\prime\|_{L^1} + O_N(N^{-2\epsilon}),$$

where the variance $V(\varphi_N)$ is given by

$$V(\varphi_N) = \frac{1}{2\beta \pi^2} \int_A^B \left( \varphi_N(x) - \varphi_N(y) \right)^2 \frac{AB + xy - \frac{1}{2}(A+B)(x+y)}{(x-A)(B-x)(y-A)(B-y)} \, dx dy,$$

and

$$\delta(\varphi_N) = \frac{ix}{2\pi^2} \left( \frac{2}{\beta} - 1 \right) \int_A^B \varphi_N(x) \left( \frac{(H\rho_V)'(\tau + i0)}{\rho_V(\tau + i0)} - \frac{(H\rho_V)'(\tau - i0)}{\rho_V(\tau - i0)} \right) dx.$$ 

Proof. The proof uses the loop-equation computation in [17]. It was carried out there for the special case $V(x) = \frac{x^2}{2}.$

For a sequence of functions $\varphi_N$, consider the complex weighted measures

$$\mu_{V, \varphi_N}(dx) = \int \frac{\mu_{V, \varphi_N}(dx)}{Z(x)} e^{ixS_N(\varphi_N)} \mu_V(dx),$$

$$Z(x) = \mathbb{E}^{\mu_V} \left[ e^{ixS_N(\varphi_N)} \right],$$

$$S_N(\varphi_N) = \sum_{j=1}^N \varphi_N(x_j) - N \int \varphi_N(x) \rho_V(x) \, dx.$$ 

We denote by $\rho_1^{(N,h)}$ the 1-point function of $\mu_{V,h}$. Define the Stieltjes transforms

$$m_{N,h}(z) = \int \frac{\rho_1^{(N,h)}(x)}{x - z} \, dx, \quad m_V(z) = \int \frac{\rho_V(x)}{x - z} \, dx.$$ 

We study the asymptotic behavior of $m_{N,\varphi_N}(z) - m_V(z)$. Define the quantities:

$$b(z) = 2m_V(z) - \partial_E \tilde{V}(z) := 2m_V(z) - V(E) + i\eta V'(E) - \frac{\eta^2}{2} V''(E)$$ 

(6.225)

$$c_N(z) = -\frac{2ix}{\beta N} \int \frac{\varphi_N(s)}{z - s} \rho_V(s) \, ds + \frac{1}{N} \left( \frac{2}{\beta} - 1 \right) m_{N,\varphi_N}(z)$$

$$+ \int \frac{\partial_E \tilde{V}(z) - V'(s)}{z - s} \left( \rho_1^{(N,\varphi_N)}(s) - \rho_V(s) \right) \, ds.$$ 

(6.226)

Lemma 6.19. Let supp $\rho_V = [A,B]$, and let $\kappa > 0$ small, $\xi > 0$ be arbitrary. Uniformly in

$$\Omega_N := \{ z = E + i\eta : N^{-1+\xi} \leq |\eta| \leq N^{-\xi}, E \in (A + \kappa, B - \kappa) \},$$

we have

$$m_{N,\varphi_N}(z) - m_V(z) = -\frac{c_N(z)}{b(z)} + O_N \left( \frac{N^\xi \omega_N(z)}{|b_N(z)|} \right),$$

(6.228)

where the error $\omega_N(z)$ is given by

$$\omega_N(z) = \frac{N^{-2+\xi/20}}{|Z(x)|^2} \left( \frac{1}{|\eta|} \|\varphi_N^\prime\|_{L^1} + |\eta|^{-2} \|\varphi_N^\prime\|_{L^1} \right).$$ 

(6.229)
Proof. The proof in the case of quadratic $V$ is given in [17, Theorem 5.4]. The same proof can be applied in our case, with minor modifications. Here, we merely point out these differences.

The main difference with the argument in [17] is that we are dealing with the case of general $V$, not just the Gaussian case. In particular, for non-analytic $V$, we use the analytic extension $\tilde{V}$. From the equilibrium relation, we have

$$V'(E) = -2\text{Re}m_V(E + i0),$$

so

$$|\text{Im } b(z)| = |2\text{Im } m(z)| + O(N^{-\xi}) > c$$

in $\Omega_N$. The lower bound (6.230) is essential to the rest of the argument, and explains our choice of upper bound for $|\eta|$ in the definition of the region $\Omega_N$.

First, we have the rigidity estimate

$$|\mu_{V,\varphi_N}| \left( |x_k - \gamma^{(V)}_k| > N^{-\frac{2}{3}} + \xi/k^{1/3} \right) \leq \frac{e^{-c_N}}{|Z(x)|},$$

(6.231)

where $\gamma_k$ is the $k$-th classical location for $\rho_V$. This follows from the result for general potential in [15].

From this, we have the following estimates as in [17, Lemma 5.3]: for each $\epsilon > 0$, and $0 < |\eta| < 1$,

$$\int \frac{\varphi_{1}^{N}(s)}{z-s} \rho_{1}^{N,\varphi_N}(s) ds - \int \frac{\varphi_{1}^{N}(s)}{z-s} \rho_{V}(s) ds = \frac{N^{-1+\epsilon}}{|Z(x)|} \mathcal{O} \left( \int \frac{|\varphi_{1}^{N}(s)|}{|z-s|^2} ds + \int \frac{|\varphi_{1}^{N}(s)|}{|z-s|^3} ds \right),$$

(6.232)

$$m_{N,\varphi}(z) - m_{V}(z) = \mathcal{O} \left( \frac{N^{-1+\epsilon}}{|\eta|Z(x)|^2} \right),$$

(6.233)

$$m_{N,\varphi}(z) - m_{V}(z) = \mathcal{O} \left( \frac{N^{-1+\epsilon}}{|\eta|^2|Z(x)|^2} \right),$$

(6.234)

$$\frac{1}{N^2} \text{Var}_{\mu_{V,\varphi_N}} \left( \sum_{k=1}^{N} \frac{1}{z-x_k} \right) = \mathcal{O} \left( \frac{N^{-2+2\epsilon}}{|\eta|^2|Z(x)|^2} \right).$$

(6.235)

The proof given there depends on (6.231).

Next, we have the loop equation [15, Eqn. (6.18)]:

$$(m_{N,\varphi}(z) - m_{V}(z))^2 + b_N(z) \cdot (m_{N,\varphi}(z) - m_{V}(z))$$

$$+ \frac{2\pi}{\beta N} \int \frac{\varphi_{1}^{N}(s)}{z-s} \rho_{1}^{N,\varphi_N}(s) ds - \frac{1}{N} \left( \frac{2}{\beta} - 1 \right) m_{N,\varphi}(z)$$

$$+ \int \frac{\partial E \tilde{V}(z) - V'(s)}{z-s} \rho_{1}^{N,\varphi_N}(s) \rho_{V}(s) ds$$

$$= \frac{1}{N^2} \text{Var}_{\mu_{V,\varphi_N}} \left( \sum_{k=1}^{N} \frac{1}{z-x_k} \right) + \mathcal{O}(e^{-c_N}).$$

(6.236)

Using the estimates (6.232), (6.235) in (6.236) leads to following equation for $X_N := m_{N,\varphi_N}(z) - m_{V}(z)$:

$$X_N^2(z) + b_N(z)X_N(z) + c_N(z) = \mathcal{O}(\omega_N(z)).$$

(6.237)

We can now argue as in [17] that $X_N(z)$ is the root of (6.237) corresponding to $X_N(z) \to 0$. We choose $\epsilon < \xi/100$ to obtain the error bound (6.229).

We use the Helffer-Sjöstrand formula:

$$\varphi_{N}(\lambda) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\tilde{\varphi}_N(z)}{\lambda - \tau - i\eta} \, dz, \quad z = \tau + i\eta$$

(6.238)
where $\hat{\tau}_z = \frac{1}{2}(\partial_\tau + i\hat{\eta})$ and $\hat{\varphi}_N$ is the almost-analytic extension of $\varphi_N$:

$$\hat{\varphi}_N(z) = (\varphi_N(\tau) + i\eta \varphi'_N(\tau))\chi(\eta),$$

where $0 \leq \chi(\eta) \leq 1$ is a cutoff function with $\chi(\eta) = 1$, $|\eta| \leq N^{-\xi}/2$, and $\chi(\eta) = 0$, $|\eta| > N^{-\xi}$.

The representation (6.238) allows us to derive the following

**Proposition 6.20.**

$$i\mathbb{E}[e^{izS(\varphi_N)}S(\varphi_N)] = -x(V(\varphi_N) + i\delta(\varphi_N))\mathbb{E}[e^{izS(\varphi_N)}]$$

$$+ |x|O\left(\frac{N^{-1+2\Omega}}{|Z(x)|^2}\right) (1 + \|\varphi'_N\|_{L^1}) \|\varphi'_N\|_{L^1} + |x|O\left(\frac{N^{-2\Omega}}{|Z(x)|^2}\right) \|\varphi'_N\|_{L^1}^2.$$  \hspace{1cm} (6.239)

**Proof.**

We compute the quantity

$$S(\varphi_N) = \frac{1}{\pi} \int_{\mathbb{R}^2} (i\eta\varphi''_N(\tau)\chi(\eta) + i(\varphi_N(\tau) + i\eta\varphi'_N(\tau))\chi'(\eta))N(m_N,\varphi_N(z) - m_V(z)) \, d\eta d\tau.$$  \hspace{1cm} (6.240)

We let $\epsilon > 10\xi$, and split the integral (6.240) into two regions, $\Omega_N \cap \{|\eta| > N^{-1+\epsilon}\}$ and its complement. For the integral over $(\Omega_N \cap \{|\eta| > N^{-1+\epsilon}\})^c$, note that $\chi' = 0$ in this region, and use (6.233):

$$\int_{\Omega_N \cap \{|\eta| > N^{-1+\epsilon}\}} i\eta\varphi''_N(\tau)\chi(\eta)N(m_N,\varphi_N(z) - m_V(z)) \, d\tau d\eta$$

$$= \int_{\{|\eta| < N^{-1+\epsilon}\}} |\eta|\varphi''_N(\tau)\chi(\eta)O_\varphi\left(\frac{N^\xi}{|\eta||Z(x)|^2}\right) \, d\tau d\eta.$$  \hspace{1cm} (6.241)

We let

$$\Delta_V(z) = m_N(z) - m_V(\varphi_N(z)) + \frac{c(z)}{b(z)}.$$  

That is, $\Delta_V(z)$ is the error term in (6.228). It is analytic in $\Omega_N$, and $|\Delta_V(z)| = O_\varphi(N^\xi \omega_N(z)/|b_N(z)|)$.

Thus, integrating by parts:

$$\int_{\Omega_N \cap \{|\eta| > N^{-1+\epsilon}\}} \eta\varphi''_N(\tau)\chi(\eta)\Delta_V(z) \, dz$$

$$= \int_{\Omega_N \cap \{|\eta| > N^{-1+\epsilon}\}} \partial_\eta(\eta\chi(\eta))\varphi'_N(\tau)\Delta_V(z) \, dz + O(N^{-1+\xi}) \int |\varphi'_N(\tau)| |\Delta_V(\tau + iN^{-1+\xi})| \, d\tau$$

$$= \frac{|\varphi'_N\|_{L^1}^2}{|Z(x)|^2} O_\varphi(N^{-1+\epsilon+\xi/20}) \int_{N^{-1+\epsilon}}^{N^{-\epsilon}} |\partial_\eta(\eta\chi(\eta))| (\|\varphi''_N\|_{L^1}/|\eta|^{-1} + \|\varphi'_N\|_{L^1}/|\eta|^{-2}) \, d\eta$$

$$= O_\varphi\left(\frac{N^{-1+2\Omega}}{|Z(x)|^2}\right) \|\varphi'_N\|_{L^1} \|\varphi''_N\|_{L^1} + O_\varphi\left(\frac{N^{-2\Omega}}{|Z(x)|^2}\right) \|\varphi'_N\|_{L^1}^2.$$  \hspace{1cm} (6.242)

For the remainder of the error term, we have

$$\int_{\Omega_N \cap \{|\eta| > N^{-1+\epsilon}\}} (\varphi_N(\tau) + i\eta\varphi'_N(\tau))\chi'(\eta)\Delta_V(z) \, dz$$

$$= \int_{\Omega_N \cap \{|\eta| > N^{-1+\epsilon}\}} (|\varphi_N(x)| + |\varphi'_N(x)|)\chi'(y)\frac{N^{1+\xi} \omega_N(z)}{|b(z)|} \, dz.$$  \hspace{1cm} (6.243)

$$= O_\varphi\left(\frac{N^{-1+3\xi}|z|^2}{B(\lambda)}\right) \|\varphi'_N\|_{L^1} (1 + \|\varphi'_N\|_{L^1}).$$

Finally,

$$\int_{\Omega_N \cap \{|\eta| > N^{-1+\epsilon}\}} \partial_\tau \hat{\varphi}_N(z) \frac{c(z)}{b(z)} \, dz = \int_{\Omega_N \cap \{|\eta| > N^{-1+\epsilon}\}} \eta\varphi''_N(\tau)\frac{c(z)}{b(z)} \, dz = O(N^{-1+\xi}|x|) \|\varphi_N\|_{L^1}.$$  \hspace{1cm} (6.244)
Combining the estimates (6.241), (6.242), (6.243), (6.244), we have:

\[ S(\varphi_N) = -\frac{1}{2\pi} \int_C \frac{c(z)}{b(z)} \, d\bar{z} \, \bar{\varphi}_N(z) \, dz + O \left( \frac{N^{-1+20\epsilon}}{|Z(x)|^2} \right) (1 + \|\varphi_N\|_{L^1}) \|\varphi_N\|_{L^1} + O \left( \frac{N^{-2\epsilon}}{|Z(x)|^2} \right) \|\varphi_N\|_{L^1}^2. \]

To compute the main term, we use:

\[ \lim_{\epsilon \to 0} b(\tau \pm i\epsilon) = \pm 2\pi i \rho_V(\tau), \tag{6.245} \]

\[ \lim_{\epsilon \to 0} \int \frac{\varphi_N'(s)}{\tau \pm i\epsilon - s} \rho_V(s) \, ds = \mp i\pi \varphi_N(\tau) + (H_V \varphi_N)(\tau), \tag{6.246} \]

where

\[ H_V f(\tau) := \text{p.v.} \int \frac{f(s)}{\tau - s} \rho_V(s) \, ds. \tag{6.248} \]

Applying Green's theorem to the region \{z : |\text{Im } z| > \epsilon\}, we obtain

\[ \lim_{\epsilon \to 0} -\frac{1}{\pi} \int_{|\text{Im } z| > \epsilon} \frac{\partial \bar{z} \varphi_N(z)}{b(z)} \, d\bar{z} \, \bar{\varphi}_N(z) \, dz = -\frac{x}{\beta \pi^2} \int \varphi_N(\tau)(H_V \varphi_N)(\tau) \frac{1}{\rho_V(\tau)} \, d\tau \]

\[ -\frac{1}{2\pi^2 i} \int \varphi_N(\tau) \left( \int \frac{V'(\tau) - V'(s)}{\tau - s} \left( \rho_1^{(N,A)}(s) - \rho_V(s) \right) \frac{1}{\rho_V(\tau)} \, dx \right) \]

\[ + \frac{i}{2\pi} \left( \frac{2}{\beta} - 1 \right) \int \varphi_N(\tau) \left( \frac{(H \rho_V)'(\tau + i0)}{\rho_V(\tau + i0)} - \frac{(H \rho_V)'(\tau - i0)}{\rho_V(\tau - i0)} \right) \, d\tau. \tag{6.249} \]

To simplify this expression, write

\[ \rho_V(\tau) = \sigma(\tau) \omega(\tau), \]

where \( \sigma(\tau) = \sqrt{(\tau - A)(B - \tau)} 1_{[A,B]}(\tau) \), and \( \omega(\tau) > 0 \) on \( \text{supp} \rho_V \). Using this notation, we write:

\[ (H_V \varphi_N)(\tau) = \text{p.v.} \int \frac{\varphi_N'(s)}{\tau - s} \rho_V(s) \, ds = \omega(\tau) \text{p.v.} \int_A^B \frac{\varphi_N'(s)}{\tau - s} \sigma(s) \, ds - \int_A^B \varphi_N'(s) \frac{\omega(\tau) - \omega(s)}{\tau - s} \sigma(s) \, ds. \tag{6.250} \]

For a \( C^1 \) function on \([A,B]\), we define the finite Hilbert transform by

\[ H_{AB} f(\tau) = \text{p.v.} \int \frac{f(s)}{\tau - s} \, ds. \]

The quantity \( \omega(\tau) \) can be expressed in terms of \( V \) using Tricomi's inversion formula for \( H_{AB} \) [67, p. 179] (see also [46, Eqn. (3.9)]):

\[ \omega(\tau) = \frac{1}{2\pi} \int_A^B \frac{V'(\tau) - V'(s)}{\tau - s} \frac{1}{\sigma(s)} \, ds. \tag{6.251} \]

An alternative formulation of the relation (6.251) is

\[ H_{AB}(\omega \sigma)(s) = -\frac{V'(s)}{2}. \tag{6.252} \]

By the equilibrium relation, we have

\[ H_{AB}(\rho_1^{(N,A)} - \rho_V)(\tau) \]

\[ = \text{p.v.} \int_A^B \frac{1}{\tau - s} (\rho_1^{(N,A)}(s) - \rho_V(s)) \, ds \]

\[ = -\frac{ix}{\beta N} \varphi_N'(\tau). \tag{6.253} \]
This relation can be inverted as [67, p. 178]:

\[ (\rho_1^{(N,\varphi_N)} - \rho_V)(\tau) = \frac{1}{\sigma(\tau)} \frac{ix}{N\beta} \int_A^B \frac{\varphi'_N(s)}{\tau - s} \sigma(s) \, ds. \]

With this notation, we write (6.250) as:

\[ \omega(\tau)\sigma(\tau) \cdot \frac{1}{\sigma(\tau)} \text{p.v.} \frac{ix}{N\beta} \int_A^B \frac{\varphi'_N(s)}{\tau - s} \sigma(s) \, ds - \frac{ix}{\beta N} \int_A^B \frac{\varphi'_N(s)\omega(s)}{\tau - s} \sigma(s) \, ds = \omega(\tau)\sigma(\tau) \cdot (\rho_1^{(N,\varphi_N)} - \rho_V)(\tau) + H_{AB}[\omega\sigma H_{AB}(\rho_1^{(N,\varphi_N)} - \rho_V)](\tau). \] (6.253)

Returning to (6.249), we compute, using (6.252)

\[ \frac{1}{2} \int \frac{V'(\tau) - V'(s)}{\tau - s} (\rho_1^{(N,\varphi_N)}(s) - \rho_V(s)) \, ds \]

\[ = \frac{V'(\tau)}{2} H_{AB}(\rho_1^{(N,\varphi_N)} - \rho_V)(\tau) - H_{AB}[ (V'/2)(\rho_1^{(N,\varphi_N)} - \rho_V) ](\tau) \]

\[ = - H_{AB}(\omega\sigma)(\rho_1^{(N,\varphi_N)} - \rho_V)(\tau) + H_{AB}[ H_{AB}(\omega\sigma)(\rho_1^{(N,\varphi_N)} - \rho_V) ](\tau). \] (6.254)

Using the general convolution relation [67, Eqn. (4), p. 174]:

\[ H[\phi_1 H \phi_2 + \phi_2 H \phi_1] = H\phi_1 H \phi_2 - \phi_1 \phi_2, \]

the two terms (6.253) and (6.254) sum to zero. Using this in (6.249), we find:

\[ \lim_{\epsilon \to 0} \frac{-1}{\pi} \left( \frac{x}{|\lim_{z \to |z| \epsilon} \hat{\varphi}_N(z) \frac{c(z)}{b(z)} \right) dz \]

\[ = - \frac{x}{\beta \pi^2} \int A^B \varphi_N(\tau) \text{p.v.} \int A^B \frac{\varphi'_N(s)}{\tau - s} \sigma(s) \, ds \frac{1}{\sigma(\tau)} \, d\tau \]

\[ + \frac{i}{2\pi^2} \left( \frac{2}{\beta} - 1 \right) \int A^B \varphi_N(\tau) \left( \frac{H\rho_V}{\tau + i0} \frac{1}{\rho_V(\tau + i0)} - (H\rho_V)'(\tau - i0) \frac{1}{\rho_V(\tau - i0)} \right) \, d\tau. \] (6.255)

For the term (6.255), a final simplification is possible. Using

\[ \text{p.v.} \int A^B \frac{1}{\tau - s} \frac{1}{\sqrt{(\tau - A)(B - \tau)}} \, d\tau = 0, \quad s \in [A, B], \]

the integral in this term is rewritten as

\[ \int A^B \varphi_N(\tau) \int A^B \frac{\varphi'_N(\tau) - \varphi'_N(s)}{\tau - s} \sigma(s) \, ds \frac{1}{\sigma(\tau)} \, d\tau. \]

Integrating by parts in \( s \), we obtain the expression \( V(\varphi_N) \).

To calculate the characteristic function, differentiate \( Z(x) \) and use

\[ \frac{d}{dx} Z(x) = E^{\mu\nu} [e^{ixS(\varphi_N)} i S(\varphi_N)] \]

\[ = -x(V(\varphi_N) + i\delta(\varphi_N)) Z(x) \]

\[ + \mathcal{O} \left( \frac{N^{-1+2\xi}}{|Z(x)|^2} \right) (1 + \|\varphi'_N\|_{L^1}) \|\varphi'_N\|_{L^1} + \mathcal{O} \left( \frac{N^{-2\xi}}{|Z(x)|^2} \right) \|\varphi'_N\|^2_{L^1}. \] (6.256)

To avoid the extra factors of \( |Z(x)| \) appearing in the error term, we consider

\[ g(x) = e^{x^2(V(\varphi_N) + i\delta(\varphi_N))} Z(x)^2. \]

Then from (6.256), and recalling the assumptions (6.2), (6.1) on \( \varphi_N \), we obtain

\[ g'(x) = e^{x^2(V(\varphi_N) + i\delta(\varphi_N))} \mathcal{O}(N^{-1+2\xi} \|\varphi'_N\|_{L^1} + N^{-2\xi}). \]

Integrating \( g'(x) \) from 0 to \( x \) with \( |x| \leq N^\xi \), we obtain the theorem.  

\[ \square \]
7 Proof of main results

7.1 Proof of Theorem 2.2

In this section we prove Theorem 2.2, fixed energy universality for Dyson Brownian motion. We follow closely Section 4 of [17], taking advantage of the new input of the mesoscopic CLT of Section 6. Let $V$ be $(g,G)$-regular and fix $t_0 = \frac{N^{\omega_0}}{N}$ with $N^\sigma g \leq t_0 \leq N^{-\sigma}G^2$. Let $t_1 = \frac{N^{\omega_1}}{N}$ with $\omega_1 < \omega_0/3$. Let $|E| < qG$ be given and let $t_0$ be the index s.t. the classical eigenvalue $\gamma_{t_0}(t_0 + t_1)$ is closest to $E$.

Let $x_i(t)$ denote Dyson Brownian motion with initial data $V$. Consider the auxiliary process

$$\hat{x}_i(t) := a(x_i(t/a^2) - b)$$

(7.1)

where

$$a = \frac{\rho_{sc}(0)}{\rho_{sc,t_0}(\gamma_{t_0}(t_0))}, \quad b = \gamma_{t_0}(t_0).$$

(7.2)

Then the process $\hat{x}_i(t)$ is DBM started from initial data $a(V - b)$. Note that since $V$ is $(g,G)$-regular, the initial data $a(V - b)$ is $(cg,cG)$-regular for some $c > 0$. Define $\tilde{t}_0 := t_0/a^2$ and $\tilde{t}_1 := t_1/a^2$. At time $\tilde{t}_0$ we have that the free convolution law for $\hat{x}_i$ satisfies $\hat{\rho}_{sc,:0}(0) = \rho_{sc}(0)$ and $\hat{\gamma}_{t_0}(t_0) = 0$. The process $\{\hat{x}_i(t)\}_i$ satisfies the hypotheses of Theorem 3.1. Therefore, we have a coupling to a process $\{y_i(t)\}_i$ that is a DBM started from initial data $y_i(0)$ a GOE matrix independent from $\{x_i(t)\}_i$. Moreover, we have that

$$\hat{x}_{i_0+i} (\tilde{t}_0 + \tilde{t}_1) - \gamma_{i_0}(\tilde{t}_0 + \tilde{t}_1) = \gamma_{N/2+i}(\tilde{t}_1) + \frac{1}{N} \sum_{|j| \leq N^{6\lambda_1/60}} p(j/N)(\hat{x}_{i_0+j}(\tilde{t}_0) - y_{N/2+j}(0)) + \frac{1}{N} O(1),$$

(7.3)

where $y_i$ is GOE ensemble at time 0 and $c_1 > 0$. Above, the function $p(x)$ satisfies

$$|p(x)| \leq C \frac{t_1}{t_1^2 + x^2}, \quad |p'(x)| \leq C \frac{1}{t_1^2 + x^2}.$$  

(7.4)

The constant $c_1 > 0$ is fixed for the purposes of this section and depends only on $\omega_0$ and $\omega_1$. Hence, we see that

$$a(x_{i_0+i}(t_0 + t_1) - \gamma_{i_0}(t_0 + t_1)) = \gamma_{N/2+k}(t_0) - \gamma_{N/2+k}(t_0) + \frac{1}{N} O(1) = \frac{k}{N} \rho_{sc}(0) + \frac{1}{N} O(1)$$

(7.5)

Note that

$$a(\gamma_{i_0+k}(t_0) - \gamma_{i_0}(t_0)) = \gamma_{N/2+k}^{(sc)} - \gamma_{N/2}^{(sc)} + \frac{1}{N} O(1) = \frac{k}{N} \rho_{sc}(0) + \frac{1}{N} O(1)$$

(7.6)

for $|k| \leq N^{6\omega_1/60}$. With overwhelming probability we have

$$\sum_{|j| \leq N^{6\lambda_1/60}} p(j/N)(a(x_{i_0+j}(t_0) - \gamma_{i_0}(t_0)) - y_{N/2+j}(0)) = \frac{1}{N} O(1)$$

(7.7)

where $\chi_1(x)$ is a smooth cut-off function identically 1 for $|x| \leq 1$ and 0 for $|x| > 2$.

Arguing as in [17] with overwhelming probability we can rewrite

$$\sum_j \chi_1 \left( \frac{j/N}{(t_1 N^{2\lambda_1/300})} \right) p(j/N)(a(x_{i_0+j}(t_0) - \gamma_{i_0}(t_0)) - y_{N/2+j}(0)) = (\zeta_x - \zeta_y + O(1))$$

(7.8)
for a constant $c_2 > 0$ and some mesoscopic linear statistics $\zeta_x$ and $\zeta_y$. The functions $\zeta_x$ and $\zeta_y$ are of the form

$$\zeta_x = \sum_j G(a(x_j(t_0) - \gamma_{i_0}(t_0))) - G(a(\gamma_j(t_0) - \gamma_{i_0}(t_0))), \quad \zeta_y = \sum_j G(y_j(0)) - G(\gamma^{(\infty)})$$

for a function $G$ which is defined by

$$G(x) := \int_0^x \chi_1(s/(t_1 N_2^{\omega_1/300}))p(s)ds. \quad (7.10)$$

Note that by rigidity we have

$$|\zeta_x| \leq C N^\varepsilon \sum_j \frac{t_1}{t_1^2 + (j/N)^2} \leq C N^\varepsilon \quad (7.11)$$

for any $\varepsilon > 0$, with overwhelming probability.

For simplicity we will only consider the 2-point function. It suffices to calculate

$$\sum_{i,j} \mathbb{E}[Q(Na(x_i(t_0 + t_1) - E), Na(x_j(t_0 + t_1) - x_i(t_0 + t_1))] \quad (7.12)$$

for compactly supported smooth $Q : \mathbb{R}^2 \to \mathbb{R}$.

### 7.1.1 Reduction to observables with small Fourier support

By the homogenization result and rigidity we can write for any sufficiently small $\delta_R > 0$,

$$\sum_{i,j} \mathbb{E}[Q(Na(x_i(t_0 + t_1) - E), Na(x_j(t_0 + t_1) - x_i(t_0 + t_1))] = \sum_{|i|,|j| \leq N^{\delta_R}} \mathbb{E}[Q(N(y_{N/2,i}(\hat{t}_1) - a(E - \gamma_{i_0}(t_0 + t_1))) + \zeta_x - \zeta_y, N(y_{N/2,j}(\hat{t}_1) - y_{N/2+1}(\hat{t}_1))] + \mathcal{O}(N^{2\delta_R - c_3}),$$

\hspace{1cm} (7.13)

for $c_3 = \min\{c_1, c_2, \omega_1/350\}$. For simplicity denote $d = a(E - \gamma_{i_0}(t_0 + t_1))$. Note $|d| \leq C$. We now make a Fourier cut-off of $Q$. We denote by $Q(\lambda, y)$ the Fourier transform of $Q$ in the first variable. We let $\psi(\lambda)$ be the Fourier transform of $\zeta_x$. Since $\zeta_x$ is independent of $\{y_i(t)\}_i$, we can write

$$\sum_{|i|,|j| \leq N^{\delta_R}} \mathbb{E}[Q(N(y_{N/2,i}(\hat{t}_1) - d) + \zeta_x - \zeta_y, N(y_{N/2,j}(\hat{t}_1) - y_{N/2+1}(\hat{t}_1)))] = \sum_{|i|,|j| \leq N^{\delta_R}} \int d\lambda \psi(\lambda) \mathbb{E} \left[ Q(\lambda, N(y_{j}(\hat{t}_1) - y_i(\hat{t}_1))) e^{i\lambda[N(y_{N/2+1} - d) + \zeta_y]} \right]. \quad (7.14)$$

The particles $x_i(t_0)$ are distributed as the eigenvalues of

$$V + \sqrt{t_0}W \overset{d}{=} V + \sqrt{t_0 - \omega_3} W + \sqrt{\omega_3 W'} =: \hat{V} + \sqrt{\omega_3} W',$$

\hspace{1cm} (7.15)

where $W$ and $W'$ are independent GOE matrices. We choose $t_3 = N^{\omega_3}/N$ with $\omega_1 < \omega_3 < 2\omega_1$. We take $\omega_3 = \omega_1(2 - 1/5)$. We calculate

$$|\psi(\lambda)| = \mathbb{E}[e^{i\lambda \zeta_x}] = \mathbb{E} \left[ \mathbb{E} \left[ e^{i\lambda \zeta_x} |\hat{V}| \right] \right]. \quad (16.16)$$

The matrix $\hat{V}$ is $(qq, qG)$-regular with overwhelming probability for any $0 < q < 1$. By our choice of $\omega_3$ we can apply Section 6 and conclude that with overwhelming probability,

$$\mathbb{E} \left[ e^{i\lambda \zeta_x} |\hat{V}| \right] \leq e^{-c_4 \lambda^2 \log(N) + N^{-c_4}}, \quad |\lambda| \leq N^{c_4} \quad (17.17)$$

The particles $x_i(t_0)$ are distributed as the eigenvalues of

$$V + \sqrt{t_0}W \overset{d}{=} V + \sqrt{t_0 - \omega_3} W + \sqrt{\omega_3 W'} =: \hat{V} + \sqrt{\omega_3} W',$$

\hspace{1cm} (7.15)

where $W$ and $W'$ are independent GOE matrices. We choose $t_3 = N^{\omega_3}/N$ with $\omega_1 < \omega_3 < 2\omega_1$. We take $\omega_3 = \omega_1(2 - 1/5)$. We calculate

$$|\psi(\lambda)| = \mathbb{E}[e^{i\lambda \zeta_x}] = \mathbb{E} \left[ \mathbb{E} \left[ e^{i\lambda \zeta_x} |\hat{V}| \right] \right]. \quad (16.16)$$

The matrix $\hat{V}$ is $(qq, qG)$-regular with overwhelming probability for any $0 < q < 1$. By our choice of $\omega_3$ we can apply Section 6 and conclude that with overwhelming probability,
for some constants \(c_4, c_5 > 0\). Fix \(\delta_F > 0\) and let \(\chi_2\) be a smooth compactly supported function s.t. \(\chi_2(\lambda) = 1\) for \(|\lambda| \leq \delta_F/2\) and \(\chi_2(\lambda) = 0\) for \(|\lambda| > \delta_F\). For any \(\varepsilon > 0\) we have,

\[
\sum_{|i|,|j| \leq N^{4R}} \int d\lambda \psi(\lambda) E \left[ \hat{Q}(\lambda, N(y_j(t_1) - y_i(t_1))) e^{i\lambda[N(y_{N/2+i}-d)+\zeta_s]} \right] = \sum_{|i|,|j| \leq N^{4R}} \int d\lambda \psi(\lambda) E \left[ \chi_2(\lambda) \hat{Q}(\lambda, N(y_j(t_1) - y_i(t_1))) e^{i\lambda[N(y_{N/2+i}-d)+\zeta_s]} \right] + O(N^{2\delta_F+\varepsilon}(N^{-c_3} + N^{-c_4} + N^{-c_5\delta_F^2/4})).
\]

(7.18)

Above we estimated the region \(\delta_F/2 < |\lambda| \leq N^\varepsilon\) using (7.17). The region \(|\lambda| > N^\varepsilon\) is estimated using the fact that \(\hat{Q}\) is Schwartz and so \(|\hat{Q}(\lambda, y)| \leq C_M/(1 + |\lambda|^M)\) for any \(M > 0\).

Let \(Q_1 : \mathbb{R}^2 \to \mathbb{R}\) be the function with Fourier transform in the first variable \(\hat{Q}_1(\lambda, y) = \hat{Q}(\lambda, y)\chi_1(\lambda)\). We see that we have proven

\[
\sum_{i,j} E\left[ Q(N(x_i(t_0 + t_1) - E), N(x_j(t_0 + t_1) - x_i(t_0 + t_1))) \right] = \sum_{|i|,|j| \leq N^{4R}} E\left[ Q_1(N(y_{N/2+i}-d) + \zeta_z - \zeta_y, N(y_{N/2+j}-d) + \zeta_x - \zeta_y) \right] + O\left( N^{2\delta_F+\varepsilon}(N^{-c_3} + N^{-c_4} + N^{-c_5\delta_F^2/4}) \right).
\]

(7.19)

Note that we have the following bound for \(Q_1\). For any \(M > 0\) there is a constant \(C(M, Q, \delta_F)\) s.t.

\[
|\partial_x^\alpha Q_1(x, y)| + |\partial_y^\alpha Q_1(x, y)| \leq C(Q, M, \delta_F) \frac{1}{1 + |x|^M} (\delta_F)^\alpha
\]

(7.20)

for every \(\alpha\).

### 7.1.2 Reduction to constancy of \(G\)

If we repeat the same argument with an auxilliary DBM started from a GOE ensemble, which we denote by \(z_i(t)\) then we see that

\[
\sum_{i,j} E\left[ Q(N(z_i(t_1)), N(z_j(t_1) - z_i(t_1))) \right] = \sum_{|i|,|j| \leq N^{4R}} E\left[ Q_1(N(y_{N/2+i}+s) + \zeta_x - \zeta_y, N(y_{N/2+j}+s) - y_{N/2+i}+s)) \right] + O\left( N^{2\delta_F+\varepsilon}(N^{-c_3} + N^{-c_4} + N^{-c_5\delta_F^2/4}) \right),
\]

(7.21)

for a constant \(c_z > 0\). Define the function

\[
F(s) := \sum_{|i|,|j| \leq N^{4R}} E\left[ Q_1(Ny_{N/2+i}(t_1)+s - \zeta_y, N(y_{N/2+j}(t_1)+s) - y_{N/2+i}(t_1)) \right].
\]

(7.22)

From (7.19) and (7.21) and the fact that \(|\zeta_z| + |\zeta_x| \leq N^\varepsilon\) with overwhelming probability we see that in order to prove fixed energy universality it suffices to show that \(|F(s) - F(0)| = o(1)\) for \(|s| \leq N^{\delta_F/2}\).

### 7.1.3 Preliminary estimates for reverse heat flow

Define

\[
F_h(s) := F(s+h) - F(s).
\]

(7.23)

We will eventually expand \(F_h(s)\) in a power series. In this section we establish estimates on the terms in that power series. We follow closely [17].
For $\alpha \in \mathbb{N}_{\geq 0}$ define
\[
F^{\alpha}(s) := \sum_{|i|,|j| \leq N^{\delta R}} \mathbb{E}[(c_{x}^{\alpha} Q_{1})(N y_{N/2+i}(\hat{t}_{1}) + s - \zeta_{y}, N(y_{j}(\hat{t}_{1}) - y_{i}(\hat{t}_{1})))],
\] (7.24)

and
\[
F^{\alpha}_{h}(s) := F^{\alpha}(s + h) - F^{\alpha}(s).
\] (7.25)

First, the argument of [17] using the translation invariance of the GOE statistics gives
\[
\left| \sum_{i,j} \mathbb{E}[(c_{x}^{\alpha} Q_{1})(N z_{i}(\hat{t}_{1}) + s + h, N(z_{j}(\hat{t}_{1}) - z_{i}(\hat{t}_{1})))]ight| - \sum_{i,j} \mathbb{E}[(c_{x}^{\alpha} Q_{1})(N z_{i}(\hat{t}_{1}) + s, N(z_{j}(\hat{t}_{1}) - z_{i}(\hat{t}_{1})))]| \leq C(Q, \delta_{F}(\delta_{F})^{\alpha} N^{-1/4} (7.26)
\]

for $|s|, |h| \leq N^{1/4}$.

Next, by rigidity, using (7.20) we have
\[
\left| \sum_{i,j} \mathbb{E}[(c_{x}^{\alpha} Q_{1})(N z_{i}(\hat{t}_{1}) + s, N(z_{j}(\hat{t}_{1}) - z_{i}(\hat{t}_{1})))]ight| - \sum_{|i|,|j| \leq N^{\delta R}} \mathbb{E}[(c_{x}^{\alpha} Q_{1})(N z_{i}(\hat{t}_{1}) + s, N(z_{j}(\hat{t}_{1}) - z_{i}(\hat{t}_{1})))]| \leq C(Q, \delta_{F}, M)(\delta_{F})^{\alpha} N^{-M \delta R} (7.27)
\]

for any $M > 0$ and $|s| \leq 10N^{\delta R}/2$. By homogenization we have
\[
\left| \sum_{|i|,|j| \leq N^{\delta R}} \mathbb{E}[(c_{x}^{\alpha} Q_{1})(N z_{i}(\hat{t}_{1}) + s, N(z_{j}(\hat{t}_{1}) - z_{i}(\hat{t}_{1})))]ight| - \sum_{|i|,|j| \leq N^{\delta R}} \mathbb{E}[(c_{x}^{\alpha} Q_{1})(N z_{i}(\hat{t}_{1}) + s + \zeta_{x} - \zeta_{y}, N(y_{j}(\hat{t}_{1}) - y_{i}(\hat{t}_{1})))]| \leq C(Q, \delta_{F})(\delta_{F})^{\alpha} N^{2\delta R - c_{3}}. (7.28)
\]

We have the estimate
\[
\hat{\zeta}_{x}(\lambda) = e^{-c_{x} \lambda^{2} \log(N) + i\lambda b_{N}} + \mathcal{O}(N^{-c_{5}}), \quad |\lambda| \leq N^{c_{5}} (7.29)
\]
for constants $c_{x}, c_{5} > 0$ and $b_{N}$ satisfying $|b_{N}| \leq N^{\delta R}/100$. Let $\zeta$ be a Gaussian with variance $c_{x} \log(N)$ and mean $b_{N}$. We see that
\[
|\mathbb{E}[F_{h}^{\alpha}(s + \zeta)] - \mathbb{E}[F_{h}^{\alpha}(s + \zeta_{x})]| \leq C(Q, \delta_{F})(\delta_{F})^{\alpha} N^{2\delta R - c_{5}}. (7.30)
\]

Collecting (7.26) (7.27) (7.28) and (7.30), we see that
\[
|\mathbb{E}[F_{h}^{\alpha}(s + \zeta)]| \leq C(Q, M, \delta_{F})(\delta_{F})^{\alpha} N^{2\delta R}(N^{-c_{5}} + N^{-c_{3}} + N^{-M \delta R}) (7.31)
\]

for any $M > 0$ and $|s|, |h| \leq 5N^{\delta R}/2$.

### 7.1.4 Reverse heat flow

Following [17] we can write
\[
F_{h}(0) = \sum_{\alpha = 1}^{\infty} \frac{(c_{x} \log(N))^{\alpha}}{\alpha!} \mathbb{E}[F_{h}^{\alpha}(\zeta + b_{N})].
\] (7.32)

Hence,
\[
|F_{h}(0)| \leq C N^{\delta_{2} c_{3}} N^{2\delta R}(N^{-c_{5}} + N^{-c_{3}} + N^{-M \delta R}),
\] (7.33)

for $|h| \leq N^{\delta R}/2$. 

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Take $\delta_F > 0$ and small enough so that $\delta_F^2 c_z \leq \min\{c_5, c_3, 1/10\}/10$. Now take $\delta_R > 0$ as

$$\delta_R = \min\{c_3, c_4, c_5 \delta_F^2/4, c_6 \delta_{F}^2/4, c_5\}/10.$$  \hspace{1cm} (7.34)

Now take $M$ large enough so that $M \delta_R > 10$. We see that we have proven that there is an $a > 0$ so that

$$|F_h(0)| \leq N^{-a}$$  \hspace{1cm} (7.35)

for $|h| \leq a$, and

$$\sum_{i,j} \mathbb{E}[Q(N(x_i(t_0 + t_1) - E), N(x_j(t_0 + t_1) - x_i(t_0 + t_1)))]$$

$$= \sum_{|i,j| \leq N^4 \delta_R} \mathbb{E}[Q_1(N(y_{N/2+i}(t_1/a^2) - d) + \zeta_x - \zeta_y, N(y_{N/2+j}(t_1/a^2) - y_{N/2+i}(t_1/a^2))] + \mathcal{O}(N^{-a}),$$

and

$$\sum_{i,j} \mathbb{E}[Q(N(z_i(t_1/a^2) - E), N(z_j(t_1/a^2) - z_i(t_1/a^2)))]$$

$$= \sum_{|i,j| \leq N^4 \delta_R} \mathbb{E}[Q_1(N(y_{N/2+i}(t_1/a^2)) + \zeta_x - \zeta_y, N(y_{N/2+j}(t_1/a^2) - y_{N/2+i}(t_1/a^2))] + \mathcal{O}(N^{-a}).$$

This yields fixed energy universality.

### 7.2 Multitime correlation functions

The proof of Theorem 2.3 is nearly identical to that of Theorem 2.2. It suffices to calculate observables of the form, e.g.,

$$\sum_{i,j} \mathbb{E}[Q(x_i(t_a) - E(t_a), (x_j(t_b) - E(t_b)) - (x_i(t_a) - E(t_a))].$$  \hspace{1cm} (7.36)

with the energies $E(t)$ defined as in the theorem statement. Since the mesoscopic part $\zeta_x, \zeta_y$ of the homogenization estimates

$$x_i(t) - E(t) = y_i(t) + \zeta_x - \zeta_y + o(1)$$  \hspace{1cm} (7.37)

are the same for $t_a$ and $t_b$, the proof given above applies to observables of the form (7.36).

### 8 General $\beta$-ensembles

In this section we prove fixed energy universality for $\beta$-ensembles, $\beta \geq 1$. The strategy is similar to the case of classical DBM; however the coupling must change as we lack a suitable matrix model representation for the DBM flow on $\beta$-ensembles.

#### 8.1 DBM flow for general $\beta$

We let $x_i$ be a general $\beta$-ensemble with potential $V$ satisfying the hypotheses in Section 2.3.4. We let $y_i$ be an independent Gaussian $\beta$-ensemble. We consider the coupled flows

$$dx_i = \frac{\sqrt{2d}B_i}{\sqrt{2N}} + \frac{1}{N} \sum_j \frac{1}{x_i - x_j} dt - \frac{V'(x_i)}{2} dt$$  \hspace{1cm} (8.1)

and

$$dy_i = \frac{\sqrt{2d}B_i}{\sqrt{2N}} + \frac{1}{N} \sum_j \frac{1}{y_i - y_j} dt - \frac{y_i}{2} dt.$$  \hspace{1cm} (8.2)

These flows leave the distribution of $\{x_i\}_i$ and $\{y_i\}_i$ invariant (however, they obviously do not leave the joint distribution of $\{x_i, y_i\}_i$ invariant). For notational simplicity we only consider eigenvalues...
near the index \( i_0 = N/2 \); the general case proceeds via the same proof. Note that we do not need to perform the re-indexing argument of Section 3.1 because we are in the one-cut case: we only ever consider \( i_0 \in [(\alpha N, (1 - \alpha)N)] \) for a fixed \( \alpha > 0 \), where both the \( \beta \)-ensemble \( V \) and the Gaussian \( \beta \)-ensemble both exhibit bulk statistics.

We can re-scale and translate the \( x_i \) so that the equilibrium density satisfies
\[
\int_{-\infty}^{0} \rho_{V}(x) dx = 1/2, \quad \rho_{V}(0) = \rho_{sc}(0).
\] (8.3)

If \( \gamma^{(V)} \) and \( \gamma^{(sc)} \) are the classical eigenvalue locations of \( \rho_V \) and \( \rho_{sc} \) respectively, then we have
\[
|\gamma_i^{(V)} - \gamma_i^{(sc)}| \leq \frac{C}{N}, \quad \text{for } |i| \leq \sqrt{N}
\] (8.4)
where we have once again shifted indices so that \( \gamma_0^{(V)} = \gamma_0^{(sc)} = 0 \) and the indices run over \([-N/2, N/2]\).

Recall the equilibrium equation
\[
\frac{V'(x)}{2} = -\int \rho_{V}(y) \frac{dy}{y - x}, \quad x = -\int \rho_{sc}(y) \frac{dy}{y - x}.
\] (8.5)

Fix now a parameter \( \ell = N^{-\omega} \) with \( \omega \ell < 1/2 \) and a \( t_1 = N^{-\omega_1}/N \) with \( \omega \leq \omega \ell / 4 \) We define the following short-range index set \( \mathcal{E} \) by
\[
\mathcal{E} := \{ |i - j| \leq \ell \} \cup \{ |i| > N/4, \, |j| > N/4 \}.
\] (8.6)

We introduce the notation
\[
\mathcal{E}^{(i)} := \sum_{j : (i, j) \in \mathcal{E}} \mathcal{E}^{(j)} := \sum_{j : (i, j) \notin \mathcal{E}}
\] (8.7)

We consider the process \( \hat{x}_i \) defined by
\[
d\hat{x}_i = \frac{\sqrt{2}dB_i}{\sqrt{3}N} + \frac{1}{N} \sum_{j} \frac{1}{x_i - x_j + \varepsilon_{ij}} dt + 1_{\{|i| > N/3\}} \left( \sum_{j} \frac{1}{x_i - x_j} - \frac{V'(x_i)}{2} \right) dt
\] (8.8)
where
\[
\varepsilon_{ij} = N^{-500}, \quad i > j, \quad \varepsilon_{ij} = -N^{-500}, \quad i < j.
\] (8.9)
We will need some level repulsion estimates and the following event \( \mathcal{F}_{\varepsilon} \). For \( \varepsilon > 0 \) we let \( \mathcal{F}_{\varepsilon}(t) \) be the event that for all \( i \) we have
\[
|x_i(t) - \gamma_i^{(V)}| + |y_i(t) - \gamma_i^{(sc)}| \leq \frac{N^\varepsilon}{N^{2/3}(N/2 - |i| + 1)^{1/3}}
\] (8.10)
and let
\[
\mathcal{F}_{\varepsilon} := \bigcap_{0 \leq t \leq 1} \mathcal{F}_{\varepsilon}(t).
\] (8.11)

By the rigidity estimates from [16] and the argument in Appendix B we see that \( \mathcal{F}_{\varepsilon} \) holds with overwhelming probability. The following level repulsion estimates follow from [39].

**Lemma 8.1.** There is an \( \varepsilon_{LR} > 0 \) and a small \( \delta > 0 \) so that the following holds. For \( s \geq e^{-N^{\varepsilon_{LR}}} \) we have for any \( \varepsilon > 0 \),
\[
\mathbb{P}[\mathcal{F}_{\delta}(t) \cap |x_i(t) - x_{i+1}(t)| \leq s/N] \leq CN^\varepsilon s^2, \quad \mathbb{P}[\mathcal{F}_{\delta}(t) \cap |y_i(t) - y_{i+1}(t)| \leq s/N] \leq CN^\varepsilon s^2.
\] (8.12)
For any \( s > 0 \) we have
\[
\mathbb{P}[\mathcal{F}_{\delta}(t) \cap |x_i(t) - x_{i+1}(t)| \leq s/N] \leq CN^3 s^2, \quad \mathbb{P}[\mathcal{F}_{\delta}(t) \cap |y_i(t) - y_{i+1}(t)| \leq s/N] \leq CN^3 s^2.
\] (8.13)

We have the following estimate. It is proven by working on the event \( \mathcal{F}_{\varepsilon} \) and combining the proof of (3.7) of [17] (see also the related Lemma 4.4 of [52]) with the proof of Lemma 3.7.

**Lemma 8.2.** Let \( \varepsilon > 0 \). There is an event with probability at least \( 1 - N^{-300} \) on which
\[
\sup_{0 \leq t \leq 10t_1} \sup_i |\hat{x}_i - x_i| \leq t_1 N^\varepsilon \left( \frac{\ell}{N} + \frac{1}{\ell} \right).
\] (8.14)
8.1.1 Finite speed estimates and profile of $\mathcal{U}^{(B)}$

We define now the operator $\mathcal{B}$ by

$$
(Bu)_i := \frac{1}{N} \sum_j \frac{\xi_{(i)}}{(x_i - x_j + \varepsilon_{ij})(y_i - y_j + \varepsilon_{ij})}u_j - u_i.
$$

(8.15)

Let

$$
B_{ij} := \frac{1}{N} \frac{1}{(x_i - x_j + \varepsilon_{ij})(y_i - y_j + \varepsilon_{ij})}.
$$

(8.16)

We need a finite speed estimate analogous to Theorem 4.1. The main difference is that we can only prove the estimate on an event of polynomially high probability, instead of overwhelming probability.

**Lemma 8.3.** Let $0 < q < 1$ and let $|a| \leq qN/4$. Let $\delta > 0$. There is an event $\mathcal{F}_a$ with $\mathbb{P}[\mathcal{F}_a] \geq 1 - N^{-c\delta}$ on which the following holds. For all $0 \leq s \leq t \leq \ell/N$, we have

$$
\mathcal{U}^{(B)}_{ij}(s, t) + \mathcal{U}^{(B)}_{ji}(s, t) \leq e^{-N^{-c\delta}}, \quad i \leq a - \ell N^\delta, \text{ and } j \geq a + \ell N^\delta.
$$

(8.17)

**Proof.** It is more convenient to adapt the proof of [39] instead of the proof of Theorem 4.1. We can assume that the event $\mathcal{F}_{\varepsilon_1}$ holds with a small $\varepsilon_1 > 0$. Fix $\delta > 0$. Let $b := a - \ell N^\delta$. Define

$$
\phi_j := e^{\nu \psi_j}, \quad \psi_j := \frac{1}{N} \min\{(j - b)_+, (a - b)\}.
$$

(8.18)

Note that

$$
|\psi_j - \psi_k| \leq \frac{|j - k|}{N}
$$

(8.19)

for any $j, k$. Let $r_j(u)$ satisfy

$$
\partial_u r = Br
$$

(8.20)

with initial condition $r_j(s) = 1_{s \leq b}$, i.e.,

$$
r_j(u) = \sum_{k \leq b} \mathcal{U}^{(B)}_{jk}(s, u).
$$

(8.21)

Define $f(u)$ by

$$
f(u) = \sum_j \phi_j r_j^2(u).
$$

(8.22)

Following [39], we differentiate $f$ and obtain

$$
f'(u) = 2 \sum_i \psi_i \sum_j r_j(u) B_{ij}(r_j - r_i) = \sum_{(i, j) \in \mathcal{E}} B_{ij}(r_j - r_i)[r_i \psi_i - r_j \psi_j]
$$

$$
= \sum_{(i, j) \in \mathcal{E}} B_{ij}(r_j - r_i) \phi_i(r_i - r_j) + \sum_{(i, j) \in \mathcal{E}} B_{ij}(r_j - r_i)[\phi_i - \phi_j]r_j.
$$

(8.23)

As in [39], we use Schwarz on the second sum, absorbing the part quadratic in $r$ into the first sum which is negative. We obtain

$$
f'(u) \leq C \sum_i \sum_j B_{ij} \phi_i^{-1} r_j^2 (\phi_i - \phi_j)^2.
$$

(8.24)

By definition of $\psi$ we see that $\psi_j = \psi_i$ if $|i - b| \geq 2\ell N^\delta$ and $(i, j) \in \mathcal{E}$. Hence,

$$
f'(u) \leq C \sum_{i: |i - b| \leq 2\ell N^\delta} \sum_{j: |j - b| \leq \ell} B_{ij} \phi_i^2 \left(\frac{\phi_j}{\phi_i} - 1\right)^2.
$$

(8.25)
On the condition that $\nu \ell \leq CN$ we get for any $\varepsilon > 0$
\[
f'(u) \leq C \sum_{i:|i-b|\leq 2\ell N^4} \sum_{j:|j-\ell|\leq \ell} B_{ij} \frac{|i-j|^2}{N^2} \phi_i r_i^2 \nu^2
\]
\[
= CV^2 \sum_{i:|i-b|\leq 2\ell N^4} \phi_i r_i^2 \sum_{j:|j-\ell|\leq \ell} B_{ij} \frac{|i-j|^2}{N^2} + CV^2 \sum_{i:|i-b|\leq 2\ell N^4} \phi_i r_i^2 \sum_{j:|j-\ell|>N^\varepsilon} B_{ij} \frac{|i-j|^2}{N^2}. \tag{8.26}
\]
The second term we can estimate by rigidity and obtain
\[
\nu^2 \sum_{i:|i-b|\leq 2\ell N^4} \phi_i r_i^2 \sum_{j:|j-\ell|\leq \ell} B_{ij} \frac{|i-j|^2}{N^2} \leq CV^2 \ell \nu f(u). \tag{8.27}
\]
The first term we estimate by
\[
\nu^2 \sum_{i:|i-b|\leq 2\ell N^4} \phi_i r_i^2 \sum_{j:|j-\ell|\leq \ell} B_{ij} \frac{|i-j|^2}{N^2} \leq f(u) \nu^2 \left( \sum_{i:|i-b|\leq 2\ell N^4} \sum_{|j-\ell|\leq \ell} B_{ij} \frac{|i-j|^2}{N^2} \right). \tag{8.28}
\]
We obtain by Gronwall that, using $t \leq \ell/N$,
\[
f(u) \leq \exp \left[ CV^2 \ell^2 \nu^2 + CV^2 \int_0^{\ell/N} \sum_{i:|i-b|\leq 2\ell N^4} \sum_{|j-\ell|\leq \ell} B_{ij} \frac{|i-j|^2}{N^2} dt \right] f(s). \tag{8.29}
\]
By the level repulsion estimates of Lemma 8.1, the fact that we are working on $\mathcal{F}_{x_1}$, and Markov’s inequality there is an event with probability at least $1 - N^{-\varepsilon/2}$ on which
\[
\int_0^{\ell/N} \sum_{i:|i-b|\leq 2\ell N^4} \sum_{|j-\ell|\leq \ell} B_{ij} \frac{|i-j|^2}{N^2} \leq CV_{N^4} \ell^2 N^6. \tag{8.30}
\]
Hence, if we take
\[
\nu = \frac{N}{\ell N^{2\varepsilon} N^6/2} \tag{8.31}
\]
we see that
\[
f(u) \leq CV \ell f(s) \leq CN \tag{8.32}
\]
with probability at least $1 - N^{-\varepsilon/2}$. By definition we have for $j \geq b$ and by our choice of $\nu$,
\[
\nu \psi_j = \nu \frac{\ell N^6}{N} = N^{\delta/2 - 2\varepsilon}. \tag{8.33}
\]
Taking $\varepsilon = \delta/8$ we get the claim. 

It is not too hard to adapt the arguments of Section 4.2.2 to prove the following.

Lemma 8.4. Let $\varepsilon > 0$ and $\delta > 0$. Fix $a$, $b$ with $a \leq b$ and $|a|, |b| \leq q_1 N/4$ with $0 < q_1 < 1$. Assume $N^{\varepsilon_1} < b - a < N^{1-\varepsilon_1}$. There is an event $\mathcal{F}_{ab}$ with $\mathbb{P}[\mathcal{F}_{ab}] \geq 1 - N^{-\varepsilon} + N^{-\varepsilon\delta}$ on which the following holds.
\[
\mathcal{U}_{ij}(B)(s, t) + \mathcal{U}_{ji}(B)(s, t) \leq \frac{N^\varepsilon}{\mathbb{N}} \frac{t - s + 1/N}{((b-a)/N)^2} \tag{8.34}
\]
for every $i \leq a$ and $j \geq b$ and $0 \leq s \leq t \leq \min\{|a-b|N^{-\delta}/N, 10\ell_1\}$.

Arguing in a dyadic fashion, this implies the following estimate.

Lemma 8.5. Let $\varepsilon > 0$ and $\delta > 0$. Let $|a| < q_1 N/4$ with $0 < q_1 < 1$. There is an event $\mathcal{F}_a$ with $\mathbb{P}[\mathcal{F}_a] \geq 1 - N^{-\varepsilon} + N^{-\varepsilon\delta}$ on which the following estimates hold.
\[
\mathcal{U}_{ij}(B)(s, t) + \mathcal{U}_{ji}(B)(s, t) \leq \frac{N^\varepsilon}{\mathbb{N}} \frac{t - s + 1/N}{((a-j)/N)^2} \tag{8.35}
\]
for any $(i, j, s, t)$ satisfying $i \leq a$ and $j \geq a + N^\varepsilon$ and $0 \leq s \leq t \leq \min\{N^{-\delta}|i-j|/N, 10\ell_1\}$. 

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The proof of the following is a straightforward modification of Lemma 4.3.

**Lemma 8.6.** Let $\varepsilon > 0$. Let $0 < q < 1$. There is an event $\mathcal{F}$ with $\mathbb{P}[\mathcal{F}] \geq 1 - N^{-c\varepsilon}$ on which we have the following estimates. For all $|a| \leq qN/4$ and $|b| \leq qN/4$, and $0 \leq s \leq t \leq \ell/N$,

$$U_{ab}^{(B)}(s,t) \leq \frac{N^\varepsilon}{N(t-s)}.$$  \hspace{1cm} (8.36)

Combining the previous two lemmas yields the following estimate.

**Lemma 8.7.** Let $\varepsilon > 0$ and $0 < q_1 < q_2 < 1$. Fix a with $|a| \leq q_1 N/4$. Fix $0 \leq s \leq t_1$. There is an event $\mathcal{F}_a(s)$ with probability $\mathbb{P}[\mathcal{F}_a(s)] \geq 1 - N^{-c\varepsilon}$ on which the following estimates hold.

$$U_{ja}^{(B)}(s,t) + U_{aj}^{(B)}(s,t) \leq \frac{N^\varepsilon}{N} \frac{t-s+1/N}{((a-j)/N)^2 + (t-s)^2 + 1/N^2}$$  \hspace{1cm} (8.37)

for every $|j| \leq q_2 N/4$ and $t$ satisfying $s \leq t \leq 10t_1$.

**Remark.** Alternatively, one may fix $t$ and let $s$ vary instead. We will also later need the following slight variant.

**Lemma 8.8.** Let $\varepsilon > 0$, and fix $0 < q_1 < q_2 < 1$. Fix a with $|a| \leq q_1 N/4$. Fix a scale $t_3 = N^{\omega_3}/N$, with $\omega_3 \leq \omega_1$. Fix $s$. There is an event with probability at least $1 - N^{-c\varepsilon}$ on which

$$U_{ja}^{(B)}(u,t) + U_{aj}^{(B)}(u,t) \leq \frac{N^\varepsilon}{N} \frac{t_3}{((a-j)/N)^2 + t_3^2}$$  \hspace{1cm} (8.38)

for every $j$ and $u, t$ satisfying $|j| \leq q_2 N/4$ and $s \leq u \leq t \leq s + 10t_3$, and $|t-u| \geq t_4/10$.

### 8.1.2 Parabolic equation

Let $u_i = \hat{x}_i - \hat{y}_i$. Then $u_i$ satisfies the equation

$$\partial_t u_i = \sum_j \mathcal{E}_{\langle i \rangle} B_{ij}(u_j - u_i) + \xi_i + A_i$$  \hspace{1cm} (8.39)

where

$$\xi_i = \sum_j \mathcal{E}_{\langle i \rangle} B_{ij}((x_i - x_j) - (\hat{x}_i - \hat{x}_j) - (y_i - y_j) + (\hat{y}_i - \hat{y}_j))$$  \hspace{1cm} (8.40)

and

$$A_i := 1_{\{|i| > N/5\}} \left( \sum_j \frac{1}{x_i - x_j} - \frac{V(x_i)}{2} \right) - 1_{\{|i| > N/5\}} \left( \sum_j \frac{1}{y_i - y_j} - \frac{y_i}{2} \right).$$  \hspace{1cm} (8.41)

Write $\xi_i$ as

$$\xi_i = \xi_i^{(1)} + \xi_i^{(2)}$$  \hspace{1cm} (8.42)

where

$$\xi_i^{(1)} := \sum_{j:|i-j| \leq N^\varepsilon} \mathcal{E}_{\langle i \rangle} B_{ij}((x_i - x_j) - (\hat{x}_i - \hat{x}_j) - (y_i - y_j) + (\hat{y}_i - \hat{y}_j)),$$  \hspace{1cm} (8.43)

and

$$\xi_i^{(2)} := \sum_{j:|i-j| > N^\varepsilon} \mathcal{E}_{\langle i \rangle} B_{ij}((x_i - x_j) - (\hat{x}_i - \hat{x}_j) - (y_i - y_j) + (\hat{y}_i - \hat{y}_j)).$$  \hspace{1cm} (8.44)

Due to rigidity and Lemma 8.2 we have with probability at least $1 - N^{-290}$ that

$$||\xi_i^{(2)}||_\infty \leq N^\varepsilon N t_1 \left( \frac{\ell}{N} + \frac{1}{\ell} \right).$$  \hspace{1cm} (8.45)
Using Lemmas 8.5 and 8.6 we see that with probability at least $1 - N^{-c\varepsilon}$,

$$
\sup_{0 \leq t \leq t_1} \left| \int_0^t \sum_j \mathcal{U}_{aj}^{(B)}(s,t) \xi_j(s) ds \right| \leq \sum_{|j-a| \leq Nt_1N^c} \int_0^{10t_1} |\xi_j(s)| ds + \sum_{|j-a| > Nt_1N^c} \int_0^{10t_1} Nt_1|\xi_j(s)| \frac{1}{(j-a)^2}. \\
(8.46)
$$

Hence, using Lemma 8.2, the inequality (8.45) and Markov inequality to deal with the $\xi^{(1)}$ part we see that there is, for each index $a$, an event with probability at least $\mathbb{P}[\mathcal{F}_a] \geq 1 - N^{-c\varepsilon}$ on which

$$
\sup_{0 \leq t \leq t_1} \left| \int_0^t \sum_j \mathcal{U}_{aj}^{(B)}(s,t) \xi_j(s) ds \right| \leq \frac{N^{4c}}{N}(Nt_1)^3 \left( \frac{\ell}{N} + \frac{1}{\ell} \right). \\
(8.47)
$$

Define $v_i$ by

$$
\partial_t v = \mathcal{B}v, \quad v(0) = u(0). \\
(8.48)
$$

For each $|a| \leq N/6$ we see that by (8.47) (using Lemma 8.3 to deal with the $A_i$ term) and the Duhamel formula that there is an event $\mathcal{F}_a$ with probability at least $1 - N^{-c\varepsilon}$ on which

$$
\sup_{0 \leq t \leq t_1} |v_a(t) - u_a(t)| \leq \frac{N^{c\varepsilon}}{N}(Nt_1)^3 \left( \frac{\ell}{N} + \frac{1}{\ell} \right). \\
(8.49)
$$

### 8.1.3 Initial data cut-offs

As in Section 3, we can perform initial data cut-offs. Let $\varepsilon_0 > 0$. We have the following.

**Lemma 8.9.** Let $\varepsilon > 0$. Let $|a| \leq \sqrt{N}$. Let $\varepsilon > 0$. There is an event $\mathcal{F}_a$ with probability $\mathbb{P}[\mathcal{F}_a] \geq 1 - N^{-c\varepsilon}$ on which the following estimates hold. For all $0 \leq t \leq t_1$,

$$
x_a(t) - y_a(t) = \sum_{|j-a| \leq Nt_1N^c} \mathcal{U}_{aj}^{(B)}(0,t)(x_j(0) - y_j(0)) + \frac{N^{c\varepsilon}}{N}(Nt_1)^3 \left( \frac{\ell}{N} + \frac{1}{\ell} \right). \\
(8.50)
$$

### 8.1.4 Homogenization

We now proceed as in Section 3.6. Fix an $\varepsilon_B > 0$. Fix an index $a$ s.t.

$$
|a| \leq N^{1/2-\varepsilon_B}. \\
(8.51)
$$

Define $w_i$ by

$$
\partial_t w = \mathcal{B}w, \quad w_0(0) = N\delta_a(i). \\
(8.52)
$$

Let $p_t(x,y)$ be as in Section 3.6. Fix

$$
N^{-1} \ll s_0 \ll s_1 \ll t_1. \\
(8.53)
$$

Recall the flat eigenvalue locations $\gamma_j^{(i)}$. We define $f(x,t)$ by

$$
f(x,t) = \sum_j \frac{1}{N} p_{s_0+t-s_1-s_2}(x, \gamma_j^{(i)}) w_j(s_1) \\
(8.54)
$$

and $f_i(t)$ by $f_i(t) := f(y_i(t),t)$. Note that here it ends up being more convenient to use $y_i$ and not $\hat{y}_i$ above, as opposed to in Section 3.

The proof of Lemma 3.12 goes through without change and we have the following.

**Lemma 8.10.** Let $\varepsilon > 0$ and $\varepsilon_B > 0$ and $\varepsilon_1 > 0$. There is an event $\mathcal{F}_a$ with $\mathbb{P}[\mathcal{F}_a] \geq 1 - N^{-c\varepsilon}$ on which the following estimate holds.

$$
||w(s_1) - f(s_1)||_2^2 \leq s_0 C \sum_{|i| \leq N^{1/2-\varepsilon_B + N^{c\varepsilon+1} |i-j|} \leq \ell} \left( \frac{w_i(s_1) - w_j(s_1)}{(i-j)^2} \right)^2 + N^{\varepsilon} \left( \frac{1}{(N s_0)^2} + \frac{(N s_0)^2}{\ell^2} \right) \frac{1}{s_1}. \\
(8.55)
$$
Due to the lack of overwhelming probability in our events, we need to argue somewhat differently in order to prove the analog of Lemma 3.13. In particular, we will take the expectation of a martingale, and so we need to introduce the following stopping time denoted by \( \tau \). It is constructed as the minimum of the following 6 stopping times. The definition is a little complicated as we need a variety of estimates to hold for the calculations of Lemma 3.13.

Let \( \varepsilon > 0 \) and \( \varepsilon_1 > 0 \). First we define the stopping time \( \tau_1 \) by
\[
\tau_1 := \inf \{ u \geq s_1 : \exists i, |i| \geq N^{1/2} : w_i > e^{-N^{1/2}} \text{ or } f_i > e^{-N^{1/2}} \} \tag{8.56}
\]
and then the stopping time \( \tau_2 \) by
\[
\tau_2 := \inf \left\{ u \geq s_1 : \exists i \in \mathbb{Z} : w_i(u) \geq \frac{N^{\varepsilon\tau}(u - s_1 + 1/N)}{(u - s_1)^2 + 1/N^2} \right\}. \tag{8.57}
\]
We define the stopping time
\[
\tau_3 := \inf \left\{ u \geq s_1 : \exists i \in \mathbb{Z} : |\dot{y}_i(u) - \gamma_i^{(sc)}| \text{ or } |y_i(u) - \gamma_i^{(sc)}| \text{ or } |\dot{x}_i(u) - \gamma_i^{(V)}| \text{ or } |x_i(u) - \gamma_i^{(V)}| > \frac{N^{\varepsilon_1/10}}{(N/2 - |i|)^{1/3}N^2/3} \right\} \tag{8.58}
\]
and the stopping time
\[
\tau_4 := \inf \left\{ u \geq s_1 : \int_{s_1}^{u} \sum_{i,j} \frac{1}{N(t - s_1 + s_0)^2(t + s_1)^2} \frac{1}{|y_i - y_j + \varepsilon_{ij}|} + \frac{1}{|x_i - x_j + \varepsilon_{ij}|} \right\} \tag{8.59}
\]
and the stopping time
\[
\tau_5 := \inf \left\{ u : u \geq s_1 : \int_{s_1}^{u} \sum_{i,j} \frac{1}{|x_i - x_j|} + \frac{1}{|y_i - y_j|} + \frac{1}{|y_i - y_j + \varepsilon_{ij}|} \frac{1}{|x_i - x_j + \varepsilon_{ij}|} \right\} \tag{8.60}
\]
and finally
\[
\tau_6 = \inf \left\{ u : u \geq s_1 : \int_{s_1}^{u} \sum_{i,j} \frac{1}{|y_i - y_j|} - \frac{1}{y_i - y_j + \varepsilon_{ij}} \right\} \tag{8.61}
\]
We set
\[
\tau := \tau_1 \wedge \tau_2 \wedge \tau_3 \wedge \tau_4 \wedge \tau_5 \wedge \tau_6 \wedge 10t_1. \tag{8.62}
\]
Lemmas 8.3 and 8.7 (for \( \tau_6 \) see the proof of (3.7) of [17]) imply that
\[
P[\tau < 10t_1] \leq N^{-\varepsilon_1} + N^{-\varepsilon_2}. \tag{8.63}
\]
For \( s_1 \leq t \leq \tau \) we have
\[
f(x,t) \leq C N^{\varepsilon\tau} \sum_j \frac{(t + s_0)}{(j/N - x)^2 + (t + s_0)^2 j^2/N^2 + (s_1)^2} \leq C N^{\varepsilon\tau} \int \frac{t + s_0}{(y - x)^2 + (t + s_0)^2 y^2 + (s_1)^2} dy \leq C N^{\varepsilon\tau} \frac{(t + s_0)}{x^2 + (t + s_0) (y - x)^2 + (s_1)^2} \tag{8.64}
\]
Here we used that for \( s \leq t \) and \( x > 0 \), that if \( x > t \),
\[
\int \frac{t}{(x - y)^2 + t^2 y^2 + s^2} dy = \int_{y \leq x/2} + \int_{y \geq 3x/2} + \int_{y < x/2 < 3x/2} \left( \frac{t}{x^2} \right) \frac{s}{y^2 + s^2} dy + \frac{s}{x^2} \int_{y < x/2 < 3x/2} \frac{t}{(y - x)^2 + t^2} dy \leq C \frac{t}{x^2} \left( \frac{s}{x^2} \right) \frac{t}{y^2 + s^2} dy + \frac{s}{x^2} \int_{y < x/2 < 3x/2} \frac{t}{(y - x)^2 + t^2} dy \tag{8.65}
\]
and if $|x| \leq t$,
\[
\int \frac{t}{(x-y)^2 + t^2} \frac{s}{y^2 + s^2} \, dy \leq \frac{C}{t} \int \frac{s}{y^2 + s^2} \, dy \leq \frac{Ct}{t^2 + x^2}.
\] (8.66)

**Lemma 8.11.** We have
\[
d \frac{1}{N} \sum_i (w_i - f_i)^2 = - \frac{1}{2} \langle w - f, B(w - f) \rangle + X_t \, dt + dM_t.
\] (8.67)

The term $M_t$ is a martingale. For $X_t$ we have
\[
\int_{s_1}^u X_t \, dt \leq \frac{1}{5} \int_{s_1}^u \langle (w - f), B(w - f) \rangle \, dt + \frac{N^{8\epsilon + \epsilon}}{s_1} \left( \frac{1}{(N s_0)^{1/2}} \right),
\] for any $u$ with $s_1 \leq u \leq \tau$.

**Proof.** For $t \leq \tau$ the bounds (3.114) to (3.116) hold, and we also have
\[
|f_i| + w_i \leq e^{-N^{\epsilon/2}}, \quad |i| \geq N^{1/2}.
\] (8.69)

We will use these tacitly in the proof. Using the Ito formula we calculate
\[
d \frac{1}{N} \sum_i (w_i - f_i)^2 = \frac{1}{N} \sum_i \left[ (w_i - f_i) \left( \partial_t w_i \, dt - \partial_t f_i \, dt - f_i' \, dy_i' - f_i'' \, dt \right) - (f_i')^2 \, dt \right].
\] (8.70)

The Ito terms are handled as before and we get
\[
\int_{s_1 + s_2}^\tau \frac{1}{N} \sum_i (w_i - f_i)^2 \, dt + \frac{f_i'}{N} \, dt \leq \frac{1}{N^{8\epsilon + \epsilon}} \frac{1}{N s_0 s_1}.
\] (8.71)

We make the same calculation as in Lemma 3.13 and write
\[
\frac{1}{N} \sum_i (w_i - f_i) (\partial_t w_i - (\partial_t f_i)) = - \frac{1}{2} \langle w - f, B(w - f) \rangle
\]
\[
+ \frac{1}{N} \sum_i (w_i - f_i) \left( \partial x_i (f_j - f_i) - \int_{|y-y_i| \leq N^\epsilon} \frac{f(y) - f(y_i)}{(y_i - y)^2} \rho_{sc}(0) \, dy \right).
\] (8.72)

Fix $0 < \omega_{\ell,2} < \omega_{\ell}$ and define
\[
\mathcal{E}_2 := \{(i, j) : |i - j| \leq N^{\omega_{\ell,2}} \} \cup \{(i, j) : i j > 0, |i| \geq N/4, |j| \geq N/4\}.
\] (8.73)

We then write the second term in (8.72) as
\[
\frac{1}{N} \sum_i (w_i - f_i) \left( \partial x_i (f_j - f_i) - \int_{|y-y_i| \leq N^\epsilon} \frac{f(y) - f(y_i)}{(y_i - y)^2} \rho_{sc}(0) \, dy \right)
\]
\[
= \frac{1}{N} \sum_i (w_i - f_i) \left( \partial x_i (f_j - f_i) - \int_{|y-y_i| \leq N^\epsilon} \frac{f(y) - f(y_i)}{(y_i - y)^2} \rho_{sc}(0) \, dy \right).
\] (8.74)

Let $v_i := w_i - f_i$. We first turn to (8.74). For $|i| \leq N^{1/2}$ we have
\[
\sum_j B_{ij} (f_j - f_i) = \frac{1}{N} \sum_j \frac{f_i'}{x_i - x_i + \epsilon_{ij}} - \sum_j B_{ij} \epsilon_{ij} f_i'
\]
\[
= \frac{N^{2\epsilon + \epsilon}}{N(t - s_1 + s_0)^2(t + s_1)} O \left( N^{\omega_{\ell,2}} + \frac{1}{N} \sum_{|j-i| \leq N^{2\epsilon + \epsilon}} \frac{1}{|x_i - x_j + \epsilon_{ij}|} \right).
\] (8.76)
For $|i| > N^{1/2}$ we use

$$
\sum_j \frac{\mathcal{E}_{2, (i)} B_{ij} (f_j - f_i)}{N} = \frac{1}{N} \sum_j f'_j \left( \sum_j \frac{\mathcal{E}_{2, (i)}}{x_j - x_i} - \sum_j B_{ij} \varepsilon_{ij} + N^2 \mathcal{O} \left( \frac{1}{N} \sum_{|j-i| \leq N^{1/2}} \frac{1}{|x_j - x_i + \varepsilon_{ij}|} \right) \right). \tag{8.77}
$$

Using (8.76) and (8.77), and the definition of $\tau$ we see that for the term (8.74) we have

$$
\int_{s_1}^u \frac{1}{N} \sum_i (w_i - f_i) \left( \sum_j B_{ij} (f_j - f_i) \right) dt = \int_{s_1}^u \frac{1}{N^2} \sum_i \frac{\mathcal{E}_{2, (i)}}{x_i - x_j + \varepsilon_{ij}} dt + \mathcal{O} \left( \frac{N^{N^{1/2}} N^{4\epsilon_{\tau}}}{N s_1 s_0} \right), \tag{8.79}
$$

for any $u \leq \tau$. We now handle (8.78). We rewrite it as

$$
\frac{1}{N^2} \sum_i \mathcal{E}_{2, (i)} \frac{f'_i}{x_i - x_j + \varepsilon_{ij}} = \frac{1}{2} \frac{N^2}{N^{1/2}} \sum_{(i,j) \in \mathcal{E}_2} (v_i - v_j) \frac{f'_i}{x_j - x_i + \varepsilon_{ij}} + \frac{1}{2} \frac{N^2}{N^{1/2}} \sum_{(i,j) \in \mathcal{E}_2} v_j \frac{f'_j - f'_i}{x_j - x_i + \varepsilon_{ij}}. \tag{8.80}
$$

The second term on the RHS of (8.80) is bounded by

$$
\left| \frac{1}{2} \frac{N^2}{N^{1/2}} \sum_{(i,j) \in \mathcal{E}_2} v_j \frac{f'_j - f'_i}{x_j - x_i + \varepsilon_{ij}} \right| \leq \frac{N^{2\epsilon_{\tau}}}{N^{1/2}} \sum_{|i-j| \leq N^{1/2}} \frac{|y_i - y_j|}{x_i - x_j + \varepsilon_{ij}} + \mathcal{O} \left( N^{5\epsilon_{\tau}} \right) + \mathcal{O} \left( N^{5\epsilon_{\tau}} \right) \leq \mathcal{O} \left( N^{5\epsilon_{\tau}} \right).
$$

Hence, by the definition of $\tau$ we see that for any $u \leq \tau$

$$
\int_{s_1}^u \frac{1}{N^2} \sum_{(i,j) \in \mathcal{E}_2} \frac{v_j (f'_j - f'_i)}{x_j - x_i + \varepsilon_{ij}} dt \leq \mathcal{O} \left( N^{5\epsilon_{\tau}} \right). \tag{8.82}
$$

The first term on the RHS of (8.80) is bounded using the Schwarz inequality. We obtain

$$
\left| \frac{1}{2} \frac{N^2}{N^{1/2}} \sum_{(i,j) \in \mathcal{E}_2} \frac{(v_i - v_j) f'_j}{x_j - x_i + \varepsilon_{ij}} \right| \leq \frac{1}{N} \frac{10}{10} \sum_{(i,j) \in \mathcal{E}_2} B_{ij} (v_i - v_j)^2 + \frac{C}{N^2} \sum_i \frac{\mathcal{E}_{2, (i)}}{N^{2\epsilon_{\tau}}} \sum_j \frac{y_i - y_j + \varepsilon_{ij}}{x_i - x_j + \varepsilon_{ij}} \frac{2}{2} \frac{N^2}{N^{1/2}} \sum_{(i,j) \in \mathcal{E}_2} \frac{(v_i - v_j)^2}{x_j - x_i + \varepsilon_{ij}} + \mathcal{O} \left( N^{5\epsilon_{\tau}} \right) \leq \mathcal{O} \left( N^{5\epsilon_{\tau}} \right) \leq \mathcal{O} \left( N^{5\epsilon_{\tau}} \right) + \mathcal{O} \left( N^{5\epsilon_{\tau}} \right) + \mathcal{O} \left( N^{5\epsilon_{\tau}} \right).
$$

$$
\int_{s_1}^u \frac{1}{N^2} \sum_{(i,j) \in \mathcal{E}_2} \frac{v_j (f'_j - f'_i)}{x_j - x_i + \varepsilon_{ij}} dt \leq \mathcal{O} \left( N^{5\epsilon_{\tau}} \right). \tag{8.83}
$$
Hence by the definition of \( \tau \) we see that for any \( u \leq \tau \),

\[
\int_{s_1}^{u} \left[ \frac{1}{2} \sum_{(i,j) \in \mathcal{E}_2} \frac{(v_i - v_j)f_i'}{x_j - x_i + \epsilon_{ij}} \right] dt \leq \frac{1}{10} \int_{s_1}^{u} \langle w - f, \mathcal{B}(w - f) \rangle dt + \frac{N^{3\epsilon_\tau}N^{\omega_{\epsilon,2}}}{N s_1 s_0}.
\] (8.84)

This finishes the estimate for (8.74). The estimate for (8.75) is handled using rigidity in the same manner as the term (3.127) is handled in the proof of Lemma 3.13 (Note that for \( |i| \leq N^{1/2} \) we have that \( |\gamma_i^{(sc)} - \gamma_i^{(V)}| \leq C/N \) so the change from \((\hat{z}_i - \hat{z}_j)^{-2}\) to \((x_i - x_j + \epsilon_{ij})^{-1}(y_i - y_j + \epsilon_{ij})^{-1}\) does not affect anything; we use the exponential bound for the terms \( |i| > N^{1/2} \) so we can discard them). We obtain

\[
\int_{s_1 + s_2}^{T} \left[ \frac{1}{N} \sum_i (w_i - f_i) \left( \sum_j B_{ij}(f_j - f_i) - \int_{|y_i - y_j| \leq \eta} \frac{f(y) - f(y_i)}{(y_i - y)^2} \rho_{ac}(0) dy \right) \right] dt \leq \frac{N^{2\epsilon_\tau}}{s_1 N^{\omega_{\epsilon,2}}}. \] (8.85)

In summary we have so far proven that

\[
\frac{1}{N} \sum_i w_i(\hat{c}_iw_i - (\hat{c}_if)_i) = \frac{1}{2} \langle w - f, \mathcal{B}(w - f) \rangle + Y_t
\] (8.86)

where

\[
\int_{s_1}^{u} Y_t dt \leq \int_{s_1}^{u} \frac{1}{10} \langle w - f, \mathcal{B}(w - f) \rangle dt + N^{6\epsilon_\tau} \left( \frac{1}{N s_1 s_2 s_3} + \frac{N^{\omega_{\epsilon,2}}}{N s_0 s_1} \right). \] (8.87)

The remaining term to deal with is

\[
\frac{1}{N} \sum_i (w_i - f_i)f_i' dy_i = dM_t + \frac{1}{N} \sum_i (w_i - f_i)f_i' \left( \frac{1}{N} \sum_j \frac{1}{y_i - y_j - \frac{y_j}{2}} \right) dt
\]

\[
= dM_t + \frac{1}{N} \sum_i (w_i - f_i)f_i' \left( \frac{1}{N} \sum_j \frac{1}{y_i - y_j + \epsilon_{ij}} \right)
\]

\[
+ \frac{1}{N} \sum_j \frac{\mathcal{E}_2^{(i)}}{y_i - y_j - \frac{y_j}{2}}
\]

\[
+ \frac{1}{N} \sum_j \frac{\mathcal{E}_2^{(i)}}{y_i - y_j - \frac{y_j}{2}} dt. \] (8.88) (8.89) (8.90)

where the Martingale term is

\[
dM_t = \frac{1}{N} \sum_i (w_i - f_i)f_i' \frac{dB_i}{\sqrt{N}}. \] (8.91)

The term (8.89) can be discarded due to the definition of \( \tau_6 \). The term (8.88) is similar to (8.80), and a similar argument gives

\[
\int_{s_1}^{u} \left[ \frac{1}{N} \sum_i (w_i - f_i)f_i' \frac{\mathcal{E}_2^{(i)}}{y_i - y_j + \epsilon_{ij}} \right] dt \leq \int_{s_1}^{u} \langle w - f, \mathcal{B}(w - f) \rangle dt + C \frac{N^{\omega_{\epsilon,2}}N^{5\epsilon_\tau}}{N s_0 s_1}. \] (8.92)

The term (8.90) is easily handled with rigidity and we obtain

\[
\int_{s_1}^{u} \left[ \frac{1}{N} \sum_i (w_i - f_i)f_i' \left( \frac{1}{y_i - y_j} \right) \right] dt \leq \frac{N^{3\epsilon_\tau}}{s_1 N^{\omega_{\epsilon,2}}} + \frac{N^{3\epsilon_\tau}N^{\omega_{\epsilon,2}}}{s_1 N}. \] (8.93)

This completes the proof after taking \( N^{\omega_{\epsilon,2}} = (N s_0)^{1/2} \).

With this in hand the completion of the homogenization theorem is very similar to Section 3. Recall that we need to average over \([s_1, 2s_1]\). Therefore, as in Section 3 we introduce \( s' \in [s_1, 2s_1] \).
and define the object the function $f$ with $s_1'$ instead of $s_1$. Fix a small $\varepsilon_3 > 0$. First of all, let $\mathcal{F}$ be the event that the bounds of Lemma 8.8 hold at the scales $t_3 = s_1$ and also $t_3 = 8t_1$, with exponent $\varepsilon_3/2$. Let us also demand that on the event $\mathcal{F}$ we have $|\mathcal{U}^{(B)}_{a}| \leq e^{-N^{\varepsilon_3/10}}$ for $|i| > N^{1/2-\varepsilon_1/4}$. We have that $\mathcal{F}$ holds with probability $\mathbb{P}[\mathcal{F}] \geq 1 - N^{-\varepsilon_3}$.

By the proof of Theorem 3.15 we have for $|i-a| \leq \ell/10$,

$$
\frac{1}{t_1} \int_0^{t_1} 1_{\mathcal{F}} \left( \frac{1}{N} \left( w_{t_1+u}(i) - w_{t_1+u}(j) \right)^2 + \frac{1}{N} \left( f_{t_1+u}(i) - f_{t_1+u}(j) \right)^2 \right) du \leq \frac{N^{\varepsilon_3}}{(Nt_1)^2} + \frac{N^{\varepsilon_3}}{(Nt_1)^2} \frac{s_1^2}{t_1}.
$$

(8.94)

We used the bounds Lemma 8.8 at the scale $t_3 = s_1$ (which hold on $\mathcal{F}$) to bound the term analogous to (3.161). Let $\mathcal{F}_t$ be the event $\{\tau \geq 10t_1\}$, where $\tau$ is the same stopping time as above. Here the definition of $\tau$ is with $s_1'$ instead of $s_1$. Taking expectations we bound

$$
\mathbb{E} \left[ \frac{1}{t_1} \int_0^{t_1} dt \left( \frac{1}{N} w_{t_1+u}(i) - \frac{1}{N} f_{t_1+u}(i) \right)^2 \right] \leq \mathbb{E} \left[ \frac{1}{t_1} \int_{t_1}^{\tau} dt \left( \frac{1}{N} w_u(i) - \frac{1}{N} f_u(i) \right)^2 \right] + C \frac{N^{\varepsilon_3}}{(Nt_1)^2} \mathbb{P}[\{\tau < 10t_1\}] \leq \mathbb{E} \left[ \frac{1}{t_1} \int_{t_1}^{\tau} dt \left( \frac{1}{N} w_u(i) - \frac{1}{N} f_u(i) \right)^2 \right] + C \frac{N^{\varepsilon_3}}{(Nt_1)^2} N^{-\varepsilon r}. \quad (8.95)
$$

We then bound, as in the proof of Theorem 3.15

$$
\mathbb{E} \left[ \frac{1}{t_1} \int_{t_1}^{\tau} dt \left( \frac{1}{N} w_u(i) - \frac{1}{N} f_u(i) \right)^2 \right] \leq \mathbb{E} \left[ \frac{1}{t_1} \int_{s_1'}^{\tau} dt \left( \frac{N^\varepsilon}{N^2} w - f, B(w - f) \right) dt \right] + C \frac{N^{\varepsilon_3}}{t^2(Nt_1)^2} + C \frac{N^{\varepsilon_3}(Nt_1)^2}{t^4}. \quad (8.96)
$$

for any small $\varepsilon > 0$. Let $\mathcal{F}_2$ be the event that $||w(s_1)||_{\mathcal{H}^2}^2 \leq N^{\varepsilon_3}/s_1$. We have that $\mathcal{F} \subseteq \mathcal{F}_2$. Hence,

$$
\mathbb{E} \left[ \frac{1}{t_1} \int_{s_1'}^{\tau} dt \left( \frac{N^\varepsilon}{N^2} w - f, B(w - f) \right) dt \right] \leq \mathbb{E} \left[ \frac{1}{t_1} \int_{s_1'}^{\tau} dt \left( \frac{N^\varepsilon}{N^2} w - f, B(w - f) \right) dt \right] \quad (8.97)
$$

One can check that on the event that $\{\tau > s_1'\}$ the bound (8.55) holds with error $\varepsilon_r$. Applying first Lemma 8.11 and then the bound (8.55) we get,

$$
\mathbb{E} \left[ \frac{1}{t_1} \int_{s_1'}^{\tau} dt \left( \frac{N^\varepsilon}{N^2} w - f, B(w - f) \right) dt \right] \leq \mathbb{E} \left[ \frac{N^\varepsilon}{t_1} \int_{s_1'}^{\tau} dt m_{s_1'} \right] + C \frac{N^\varepsilon}{N^2t_1} \mathbb{E} \left[ \int_{s_1'}^{\tau} dt \left( w - f, s_1' \right) \right]^2 + C \frac{N^8}{N^2t_1} + C \frac{N^8}{N^2t_1} \quad (8.98)
$$

Above we used that

$$
\mathbb{E} \left[ \frac{N^\varepsilon}{t_1} \int_{s_1'}^{\tau} dt m_{s_1'} \right] = 0 \quad (8.99)
$$

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due to the fact that $\mathcal{F}_2$ is measurable wrt $\sigma(B_i(s), s \leq s_1)$. We now have due to the energy inequality and the definition of $\mathcal{F}_2$,

$$\frac{1}{s_1} \int_{s_1}^{2s_1} \mathbb{E} \left[ 1_{\mathcal{F}_2} \langle w(s'), Bw(s') \rangle \right] ds' = \mathbb{E} \left[ \frac{1}{s_1} \int_{s_1}^{2s_1} \langle w(s_1), Bw(s_1) \rangle ds_1 \right] \leq CN^{\varepsilon_3} \frac{1}{s_1^2} \quad (8.100)$$

Collecting everything we see that

$$\mathbb{E} \left[ \frac{1}{t_1} \int_{t_1}^{2t_1} \left( U_{t_1+a}^{(B)}(i, a) - \frac{1}{N} p_t(\gamma_i, \gamma_a) \right)^2 \right] \leq \frac{1}{(Nt_1)^2} \left\{ \frac{N^{2\varepsilon_3} N^{2\varepsilon_3}}{(Nt_1)^2} + \frac{N^{2\varepsilon_3} s_1^2}{t_1^2} + \frac{N^{2\varepsilon_3} N_{t_1}^4}{N_0^2} + \frac{N^{2\varepsilon_3} s_1}{s_1 t_1} + \frac{t_1 N^{2\varepsilon_3} N_0^2}{s_1 t_1^2} \right\} \quad (8.101)$$

This proves the following lemma (first fix $\varepsilon_\tau$ small, then $\varepsilon_3$ smaller depending on $\varepsilon_\tau$).

**Lemma 8.12.** Let $\varepsilon > 0$ be small enough. Let $i$ and $a$ satisfy $|a| \leq N^{1/2-\varepsilon_1}$ and $|i-a| \leq \ell/10$. There is an event $\mathcal{F}_{IA}$ with probability $\mathbb{P}[\mathcal{F}_{IA}] \geq 1 - N^{-\varepsilon}$ on which

$$\frac{1}{t_1} \int_{t_1}^{2t_1} \left| U_{t_1+a}^{(B)}(i, a) - \frac{1}{N} p_t(\gamma_i, \gamma_a) \right| du \leq \frac{N^{-\varepsilon}}{Nt_1} + \frac{N^{\varepsilon_3} N_0}{Nt_1} \left( \frac{1}{Nt_1} + \frac{t_1^{1/2} (N_0)}{s_1^{1/2} \ell} + \frac{t_1^{1/2}}{s_1^{1/2} (N_0)^{1/4}} + \frac{s_1}{t_1} + \frac{s_1^{1/2} t_1^{1/2}}{s_1^2} \right) \quad (8.102)$$

We can remove the time average in the same way as in Section 3.

**Lemma 8.13.** Let $\varepsilon > 0$ and $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$. Let $t_2 = t_1 N^{-\varepsilon_2}$. There is an event with probability $\mathbb{P}[\mathcal{F}_{IA}] \geq 1 - N^{-\varepsilon_2} - N^{-\varepsilon_3}$ on which

$$\left| U_{t_1+2t_2}^{(B)}(i, a) - \frac{1}{N} p_t(\gamma_i, \gamma_a) \right| \leq \frac{N^{\varepsilon_2} N^{-\varepsilon}}{Nt_1} + \frac{N^{\varepsilon_2} N^{\varepsilon_3}}{Nt_1} \left( \frac{1}{Nt_1} + \frac{t_1^{1/2} (N_0)}{s_1^{1/2} \ell} + \frac{t_1^{1/2}}{s_1^{1/2} (N_0)^{1/4}} + \frac{s_1}{t_1} + \frac{s_1^{1/2} t_1^{1/2}}{s_1^2} \right) \quad (8.103)$$

The proof is similar to that of Theorem 3.16. One needs to introduce an additional event $\mathcal{F}_3$ on which the bounds of Lemma 8.7 (see also the remark immediately subsequent to it) hold with $\varepsilon_3 > 0$. To get around the fact that the event of Lemma 8.12 depends on $\mathcal{F}_{IA}$, one applies Markov inequality on the event $\mathcal{F}_3$ to the term (3.181).

By choosing the scales $s_0, s_1$ appropriately, we obtain the following.

**Lemma 8.14.** Let $\sigma > 0$ and let $t_1$ satisfy

$$\frac{N^{\sigma}}{N} \leq t_1 \leq \frac{N^{1/4-\sigma}}{N}. \quad (8.104)$$

Let $\varepsilon > 0$. Let $\varepsilon > 0$ be sufficiently small. There is an event $\mathcal{F}_{IA}$ with $\mathbb{P}[\mathcal{F}_{IA}] \geq 1 - N^{-\varepsilon}$ on which we have, for every $|t_1 - t_1| \leq N^{\sigma_0}/N$,

$$\left| U_{t_1}^{(B)}(0, t) - \frac{1}{N} p_{t_1}(\gamma_i, \gamma_a) \right| \leq \frac{1}{Nt_1} \left( N^{-\varepsilon} + N^{-\sigma_0} \right). \quad (8.105)$$
Let now $\mathcal{F}_1$ be the event on which the estimates of Lemma 8.6 hold with $\varepsilon_1 > 0$, and $\mathcal{F}_2$ the event of Lemma 8.9 holds with $\varepsilon_2 > 0$. Denote by $\mathcal{F}_a$ the event of Lemma 8.14 with $\varepsilon_3 > 0$. We then write

$$x_a(t) - y_a(t) = \sum_{|j| - a| \leq N_t \epsilon^a} \mathbb{U}_{a_j}^{(B)}(0, t)(x_j(0) - y_j(0)) + \frac{N\varepsilon^2}{N^2} \mathcal{O}\left(\frac{1}{N^\varepsilon_a + N^{-c\sigma}}\right)$$

$$= \sum_{|j| - a| \leq N_t \epsilon^a} p_t(\gamma_a^{(f)}, \gamma_j^{(f)})(x_j(0) - y_j(0))$$

$$+ \sum_{|j| - a| \leq N_t \epsilon^a} \mathbf{1}_{\mathcal{F}_a}(\mathbb{U}_{a_j}^{(B)} - p_t(\gamma_a^{(f)}, \gamma_j^{(f)}))(x_j(0) - y_j(0)) \tag{8.106}$$

$$+ \sum_{|j| - a| \leq N_t \epsilon^a} \mathbf{1}_{\mathcal{F}_a}(\mathbb{U}_{a_j}^{(B)} - p_t(\gamma_a^{(f)}, \gamma_j^{(f)}))(x_j(0) - y_j(0)) \tag{8.107}$$

$$+ \frac{N\varepsilon^2}{N} \mathcal{O}\left(\frac{1}{N^\varepsilon_a + N^{-c\sigma}}\right) \tag{8.108}$$

For the term (8.106) we have

$$\left| \sum_{|j| - a| \leq N_t \epsilon^a} \mathbf{1}_{\mathcal{F}_a}(\mathbb{U}_{a_j}^{(B)} - p_t(\gamma_a^{(f)}, \gamma_j^{(f)}))(x_j(0) - y_j(0)) \right| \leq N_{\varepsilon_a + \varepsilon} \frac{1}{N} (N^{-c\varepsilon_3} + N^{-c\sigma}). \tag{8.109}$$

For the term (8.107) we have

$$\mathbb{E}\left[ \mathbf{1}_{\mathcal{F}_1} \left| \sum_{|j| - a| \leq N_t \epsilon^a} \mathbf{1}_{\mathcal{F}_a}(\mathbb{U}_{a_j}^{(B)} - p_t(\gamma_a^{(f)}, \gamma_j^{(f)}))(x_j(0) - y_j(0)) \right| \right] \leq \frac{N^\varepsilon}{N} N_{\varepsilon_a} N_{\varepsilon_1} N^{-c\varepsilon_3}. \tag{8.110}$$

Hence, first fixing $\varepsilon_3$ small, then taking $\varepsilon_a$ and $\varepsilon_1$ and $\varepsilon$ small depending on $\varepsilon_3$ and $\sigma$, and then taking $\varepsilon_2$ small, we obtain the following.

**Theorem 8.15.** Let $\varepsilon > 0$ sufficiently small. Let $t_1, t$ be as above. There is an event $\mathcal{F}_a$ with $\mathbb{P}[\mathcal{F}_a] \geq 1 - N^{-c\varepsilon}$ on which

$$x_a(t) - y_a(t) = \sum_{|j| - a| \leq N_t \epsilon^a} p_t(\gamma_a^{(f)}, \gamma_j^{(f)})(x_j(0) - y_j(0)) + \frac{1}{N} \mathcal{O}\left(\frac{1}{N_{\varepsilon_a} + N^{-c\sigma}}\right). \tag{8.111}$$

### 8.2 Proof of fixed energy universality

We now prove fixed energy universality for $\beta$-ensembles. For simplicity we just consider $E = 0$. Let $\varepsilon_h > 0$ be as in Theorem 8.15. Suppose that the events $\mathcal{F}_a$ defined in Theorem 8.15 hold with probability $\mathbb{P}[\mathcal{F}_a] \geq 1 - N_{\varepsilon_h}^{\varepsilon_h}$. Denote the error in the estimate by $\mathcal{O}(N^{-1})$. Let

$$\delta_1 = c_h \varepsilon_h / 10. \tag{8.112}$$

Let $\mathcal{F} = \bigcup_{|a| \leq N^\varepsilon_h} \mathcal{F}_a$. Then $\mathbb{P}[\mathcal{F}] \geq 1 - N_{\varepsilon_h}^{\varepsilon_h}/2$. As in the proof of Theorem 2.2 we consider

$$\sum_{i,j} \mathbb{E}[Q(Nx_i(t), Nx_j(t) - x_i(t))]. \tag{8.113}$$

We assume that the eigenvalues are labelled by $[-N/2, N/2]$ and that $\gamma^{(V)}_{N/2} = 0$. Let $\delta_R = \min\{\delta_1/10, \varepsilon_1/10\}$. Arguing as in the proof of Theorem 2.2 we can apply rigidity and the homogenization theorem and write

$$\sum_{i,j} \mathbb{E}[Q(Nx_i(t), Nx_j(t) - x_i(t))] = \sum_{|i|, |j| \leq N^{\delta_R}} \mathbb{E}[Q(Nx_i(t), Nx_j(t) - x_i(t))] + \mathcal{O}(N^{-D})$$

$$= \sum_{|i|, |j| \leq N^{\delta_R}} \mathbb{E}[Q(Ny_i(t) + \zeta_x - \zeta_y, N(y_j(t) - y_i(t)))] + \mathcal{O}(N^{-c}) \tag{8.114}$$

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for some $c > 0$. Similarly if $z_i(t)$ is an auxilliary GOE ensemble we can write

$$
\sum_{i,j} \mathbb{E}[Q(N(z_i(t) - E), z_j(t) - z_i(t))]
= \sum_{|i,j| \leq N^{\delta R}} \mathbb{E}[Q(N(y_i(t) - E) + \zeta_z - \zeta_y, N(y_j(t) - y_i(t)))] + O(N^{-c}),
$$

(8.115)

for any $|E| \leq N^{\delta R}/2/N$. Denote

$$
F_1(s) := \sum_{|i,j| \leq N^{\delta R}} \mathbb{E}[Q(N(y_i(t) + s - \zeta_y, N(y_j(t) - y_i(t))].
$$

(8.116)

We have

$$
\sum_{|i,j| \leq N^{\delta R}} \mathbb{E}[Q(N(y_i(t) + \zeta_x - \zeta_y, N(y_j(t) - y_i(t))]
- \sum_{|i,j| \leq N^{\delta R}} \mathbb{E}[Q(N(y_i(t) - E) + \zeta_z - \zeta_y, N(y_j(t) - y_i(t))]
= \int d\lambda \hat{F}_1(\lambda)(\psi_x(\lambda) - \hat{\psi}_x(\lambda)e^{i\lambda E})
$$

(8.117)

where $\psi_x$ and $\hat{\psi}_x$ are the Fourier transforms of $\zeta_x$ and $\zeta_y$. If we let $E = \mathbb{E}[\zeta_x] - \mathbb{E}[\zeta_z] = O(N^{\gamma})$ for any $\varepsilon > 0$ then by Section 6,

$$
|\psi_x(\lambda) - \hat{\psi}_x(\lambda)e^{i\lambda E}| \leq N^{-c_2}, \quad |\lambda| \leq N^{c_2}
$$

(8.118)

for some $c_2 > 0$. If we take $\delta_R < c_2/2$, then we see that

$$
\left| \int d\lambda \hat{F}_1(\lambda)(\psi_x(\lambda) - \hat{\psi}_x(\lambda)e^{i\lambda E}) \right| \leq N^{-c_2/3}.
$$

(8.119)

This proves fixed energy universality for $\beta$ ensembles.

### A Local laws and properties of free convolution

#### A.1 Free convolution properties

In this section we summarize the local laws for DBM as well as derive some properties of the free convolution. Recall that $H_t$ is defined as

$$
H_t : V + \sqrt{t} W.
$$

(A.1)

where $W$ is a GOE matrix. The Stieltjes transform of the free convolution is defined as the solution to the fixed point equation

$$
m_{V, t}(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{V_i - z - tm_{V, t}(z)}.
$$

(A.2)

We call $V (g, G)$-regular if

$$
c \leq \text{Im} \ [m_V(E + i\eta)] \leq C
$$

(A.3)

for $|E| \leq G$ and

$$
g \leq \eta \leq 10
$$

(A.4)

and $\|V\| \leq N^{C_V}$ for some $C_V > 0$. We collect some properties of the free convolution in the following lemma. It can be found in Section 7 of [52].
Lemma A.1. Let $V$ be $(g, G)$-regular. Let $0 < q < 1$. Let $\sigma > 0$ and let $t$ satisfy
\[ gN^\sigma \leq t \leq N^{-\sigma}G^2. \] (A.5)

For $|E| \leq qG$ we have
\[ c \leq \text{Im} [m_{fc,t}(z)] \leq C \] (A.6)
for $0 \leq \eta \leq 10$. We have
\[ |\partial_z m_{fc,t}(z)| \leq \frac{C}{t + \eta}. \] (A.7)

We also require an estimate for the second derivative of $m_{fc,t}(z)$. We easily calculate
\[
\partial_z m_{fc,t}(z) = \left(1 - t \int \frac{1}{(x - z - tm_{fc,t}(z))^2} d\mu_V(x)\right)^{-1}
\times \int \frac{1}{(x - z - tm_{fc,t}(z))^2} d\mu_V(x)
\] (A.8)
from which we see that
\[
\partial_z^2 m_{fc,t}(z) = (1 - tR_2)^{-2} tR_3 R_2 \left[1 + t\partial_z m_{fc,t}(z)\right] + (1 - tR_2)^{-1} R_3 \left[1 + t\partial_z m_{fc,t}(z)\right],
\] (A.9)
where
\[ R_k := \int \frac{1}{(x - z - tm_{fc,t}(z))^k} d\mu_V(x). \] (A.10)

Since $|R_2| \leq Ct^{-1}$ and $|R_3| \leq Ct^{-2}$ we see, using that $|1 - tR_2| \geq c$ (see [52]) and (A.7) that
\[ |\partial_z^2 m_{fc,t}(z)| \leq \frac{C}{t^2}. \] (A.11)

We have the following local law from Section 7 of [52].

Lemma A.2. Let $V$ be $(g, G)$-regular and let $\sigma > 0$, and $\varepsilon, \delta > 0$ and let $0 < q < 1$. Let $t$ satisfy $gN^\sigma \leq t \leq N^{-\sigma}G^2$. With overwhelming probability we have for $|E| \leq qG$ and $\eta \geq N^\delta/N$,
\[ |m_{fc,t}(z) - m_N(z)| \leq \frac{N^\varepsilon}{N\eta}. \] (A.12)

The above estimate also holds if $\eta \geq 10$ and for any $E$.

We have the following rigidity result.

Lemma A.3. Fix $0 < q < 1$. Let $\varepsilon > 0$. For every $i$ s.t. $|\gamma_i| \leq qG$ we have with overwhelming probability
\[ |x_i - \gamma_i| \leq \frac{N^\varepsilon}{N}. \] (A.13)

We have also the estimate
\[ |x_i - V_i| \leq C\sqrt{t} \] (A.14)
for every $i$.

The estimate (A.14) is just a consequence of the perturbation bound $\lambda_j(A - B) \leq \|A - B\|$ and the fact that for a GOE matrix, $\|W\| \leq 3$ with overwhelming probability. We will not use this estimate in this paper.

A.2 Rescaling and relabelling set-up

Recall that in Section 3 we fixed an index $i_0$ with $|\gamma_{i_0}(t_0)| \leq qG$, and that we have assumed that
\[ \rho_{fc,i_0}(0) = \rho_{sc}(0), \quad \gamma_{i_0}(t_0) = 0. \] (A.15)

Moreover, we assume $N$ is odd and. that $i_0 = (N + 1)/2$. Finally, we have relabelled the eigenvalues so that $i_0 = 0$ and they run over the index set $\left[\left[-(N - 1)/2, (N - 1)/2\right]\right]$. 

A.3 Construction of law of interpolating ensembles

We now wish to discuss rigidity for the interpolating ensemble. Define

\[ z_i(\alpha) = (1 - \alpha)y_i(t_0) + \alpha x_i(t_0) \quad (A.16) \]

where \( x_i(t_0) \) are the eigenvalues of \( H_{t_0} \) and \( y_i(t_0) \) are the eigenvalues of an independent GOE matrix. Fix now \( 0 < q^* < 1 \). Let \( k_0 \) be the largest possible natural number so that

\[ |\gamma_{k_0}(t_0)| \leq q^* G, \quad |\gamma_{-k_0}(t_0)| \leq q^* G, \quad |\gamma_{k_0}^{(sc)}| \leq q^* G. \quad (A.17) \]

First of all, extend \( \gamma_x(t_0) \) to all \( |x| \leq (N - 1)/2 \) by

\[ \frac{x}{N} = \int_0^{\gamma_x(t_0)} \rho_{ic,t_0}(E) dE, \quad (A.18) \]

and similarly for \( \gamma^{(sc)}_x \). We define a function \( f(x, \alpha) : [-k_0, k_0] \times [0, 1] \to \mathbb{R} \) as follows. For \( |x| \leq k_0 \),

\[ f(x, \alpha) = \alpha \gamma_x(t_0) + (1 - \alpha) \gamma^{(sc)}_x. \quad (A.19) \]

Then \( f(0, \alpha) = 0 \). For each \( \alpha \), \( f(\cdot, \alpha) \) is a bijection from

\[ f(\cdot, \alpha) : [-k_0, k_0] \to [\alpha \gamma_{-k_0}(t_0) + (1 - \alpha) \gamma^{(sc)}_{-k_0}, \alpha \gamma_{k_0}(t_0) + (1 - \alpha) \gamma^{(sc)}_{k_0}] =: \mathcal{G}_\alpha \quad (A.20) \]

Since

\[ \frac{d}{dx} \gamma_x(t_0) = \frac{1}{N} \frac{1}{\rho_{ic,t_0}(\gamma_x(t_0))} \quad (A.21) \]

we have

\[ \frac{c}{N} \leq f'(x, \alpha) \leq \frac{C}{N}. \quad (A.22) \]

Let \( g \) be inverse of \( f \),

\[ g(f(x, \alpha), \alpha) = x. \quad (A.23) \]

For each \( \alpha \) we now the function \( h(y, \alpha) \) on the interval \( \mathcal{G}_\alpha \) by

\[ h(y, \alpha) := \frac{1}{N} \frac{d}{dy} g(y, \alpha). \quad (A.24) \]

By elementary calculations we have the following explicit formula for \( h \):

\[ h(y, \alpha) = \frac{1}{N} \frac{1}{f'(g(y, \alpha), \alpha)} = \left( \frac{\alpha}{\rho_{ic,t_0}(\gamma(g(y, \alpha), t_0))} + \frac{1 - \alpha}{\rho_{sc}(\gamma^{(sc)}(g(y, \alpha)))} \right)^{-1} \quad (A.25) \]

In particular we see that

\[ c \leq h(y, \alpha) \leq C, \quad |h'(y, \alpha)| \leq \frac{C}{t_0}, \quad |h''(y, \alpha)| \leq \frac{C}{t_0^2} \quad h(0, \alpha) = \rho_{sc}(0), \quad (A.26) \]

and by definition,

\[ \int_{\mathcal{G}_\alpha} h(y, \alpha) dy = \frac{2k_0}{N}. \quad (A.27) \]

Note that by definition,

\[ \int_0^{f(x, \alpha)} h(x, \alpha) dx = \frac{x}{N}, \quad (A.28) \]

and so we can immediately see that

\[ \left| \frac{1}{N} \sum_{|i| \leq k_0} \frac{1}{z_i(\alpha) - z} - \int_{\mathcal{G}_\alpha} \frac{1}{x - z} h(x, \alpha) dx \right| \leq \frac{N^\varepsilon}{N \eta} \quad (A.29) \]
for any $\varepsilon > 0$ and $\eta \geq N^3/N$ for any $\delta > 0$ with overwhelming probability. We now define the following probability measure on $\mathbb{R}$. Let

$$\nu(dx, \alpha) = h(x, \alpha)dx + \frac{1}{N} \sum_{|i| > k_0} \delta_{z_i(\alpha)}.$$ (A.30)

Then $\nu(\alpha)$ is a probability measure and denoting,

$$m(z, 0, \alpha) = \int \frac{1}{x-z} \nu(dx)$$ (A.31)

we immediately see that

$$\left| \frac{1}{N} \sum_{i} \frac{1}{z_i(\alpha) - z} - m(z, 0, \alpha) \right| \leq \frac{N^\varepsilon}{N\eta}, \quad \eta \geq \frac{N^\delta}{N}$$ (A.32)

for any $\varepsilon, \delta > 0$ and with overwhelming probability. We denote the free convolution of $\nu(\alpha)$ with the GOE at time $t$ by $\rho(x,t,\alpha)d\alpha$. It is defined through its Stieltjes transform which satisfies the fixed point equation

$$m(z, t, \alpha) = \int \frac{1}{x-z-tm(z, t, \alpha)}d\nu(x, \alpha).$$ (A.33)

Let now $\varepsilon > 0$ be given and let $0 \leq t_1 \leq N^{-\varepsilon}t_0$. The proofs in [52] yield the following lemma.

**Lemma A.4.** Let $0 < q < 1$. Let $\varepsilon > 0$. Let $0 \leq t \leq t_1$. For $E \in q\mathcal{G}_\alpha$ and $0 \leq \eta \leq 10$ we have the following estimates with overwhelming probability. First we have,

$$c \leq \text{Im} [m(z, t, \alpha)] \leq C.$$ (A.34)

We have

$$c \leq \left| 1 - t \int \frac{1}{(x-z-tm(z, t, \alpha))^2}d\nu(x, \alpha) \right| \leq C$$ (A.35)

and

$$\left| t^2 \int \frac{1}{(x-z-tm(z, t, \alpha))^3}d\nu(x, \alpha) \right| \leq C.$$ (A.36)

Note that we always have following the a-priori bound for $m(z, t, \alpha)$. By Cauchy-Schwarz,

$$|m(z, t, \alpha)|^2 \leq \int \frac{1}{|x-z-tm(z, t, \alpha)|^2}d\nu(x, \alpha) = \frac{\text{Im} [m(z, t, \alpha)]}{\eta + t\text{Im} [m(z, t, \alpha)]} \leq \frac{1}{t}. $$ (A.37)

We have the following improved regularity of $m(z, t, \alpha)$.

**Lemma A.5.** Let $0 < q < 1$ and $0 \leq t \leq t_1$. For $E \in q\mathcal{G}_\alpha$ we have with overwhelming probability,

$$|\partial_z m(z, t, \alpha)| \leq \frac{C}{t_0 + \eta}.$$ (A.38)

**Proof.** We calculate

$$\partial_z m(z, t, \alpha) = \left( 1 - t \int \frac{1}{(x-z-tm(z, t, \alpha))^2}d\nu(x, \alpha) \right)^{-1} \times \int \frac{1}{(x-z-t)^2}d\nu(x, \alpha).$$ (A.39)

Since $|tm(z, t, \alpha)| \leq \sqrt{t} \ll G$, we see that

$$\text{Re}[z + tm(z, t, \alpha)] \in q_1\mathcal{G}_\alpha.$$ (A.40)
for any \( q \) s.t. \( q < q_1 < 1 \) and \( N \) large enough. It therefore suffices to prove
\[
|\partial_z m(z, 0, \alpha)| \leq \frac{C}{t_0} \tag{A.41}
\]
for any \( E \in q_1 \mathcal{G}_\alpha \). We write
\[
\partial_z m(z, 0, \alpha) = \int \frac{1}{(x-z)^2} h(x, \alpha) + \frac{1}{N} \sum_{|i| > k_0} \frac{1}{(z_i(\alpha) - z)^2} \tag{A.42}
\]
Optimal rigidity guarantees that for \( |i| > k_0, |z_i(\alpha) - z| > cG \) and so with overwhelming probability,
\[
\frac{1}{N} \sum_{|i| > k_0} \frac{1}{|z_i(\alpha) - z|^2} \leq \frac{C}{G} \text{Im} [m(0 + iG, 0, \alpha)] \leq \frac{C}{G} \leq \frac{C}{t_0} \tag{A.43}
\]
We write the other term as
\[
\int \frac{1}{(x-z)^2} h(x, \alpha) = \int_{|x-E| \leq t_0} \frac{1}{(x-z)^2} h(x, \alpha) dx + \int_{|x-E| > t_0} \frac{1}{(x-z)^2} h(x, \alpha) dx. \tag{A.44}
\]
We clearly have
\[
\left| \int_{|x-E| > t_0} \frac{1}{(x-z)^2} h(x, \alpha) dx \right| \leq \frac{C}{t_0}. \tag{A.45}
\]
We integrate the other term by parts and obtain
\[
\int_{|x-E| \leq t_0} \frac{1}{(x-z)^2} h(x, \alpha) dx = \frac{h(E + t_0, \alpha)}{-t_0 - i\eta} - \frac{h(E - t_0, \alpha)}{t_0 - i\eta} - \int_{|x-E| \leq t_0} \frac{1}{(x-z)^2} h'(x, \alpha) dx. \tag{A.46}
\]
Clearly
\[
\left| \frac{h(E + t_0, \alpha)}{-t_0 - i\eta} \right| \leq \frac{C}{t_0} \tag{A.47}
\]
We split the last term into its real and imaginary parts
\[
\int_{|x-E| \leq t_0} \frac{1}{(x-z)^2} h'(x, \alpha) dx = i \int_{|x-E| \leq t_0} \frac{\eta}{(x-E)^2 + \eta^2} h'(x, \alpha) dx + \int_{|x-E| \leq t_0} \frac{x-E}{(x-E)^2 + \eta^2} h'(x, \alpha) dx. \tag{A.48}
\]
The imaginary part is easily bounded by
\[
\left| \int_{|x-E| \leq t_0} \frac{\eta}{(x-E)^2 + \eta^2} h'(x, \alpha) dx \right| \leq \frac{C}{t_0} \int_{\mathbb{R}} \frac{\eta}{x^2 + \eta^2} dx \leq \frac{C}{t_0}. \tag{A.49}
\]
We use (A.11) to bound the real part by
\[
\left| \int_{|x-E| \leq t_0} \frac{x-E}{(x-E)^2 + \eta^2} h'(x, \alpha) dx \right| \leq \frac{C}{t_0} \int_{|x-E| \leq t_0} \frac{(x-E)^2}{(x-E)^2 + \eta^2} \left| h'(x, \alpha) - h'(E, \alpha) \right| dx \leq \frac{C}{t_0} \int_{|x-E| \leq t_0} \frac{(x-E)^2}{(x-E)^2 + \eta^2} dx \leq \frac{C}{t_0}. \tag{A.50}
\]
This yields the claim. \( \square \)

This allows us to conclude a few things about the densities \( \rho(t, \alpha) \).

**Lemma A.6.** Let \( 0 < q < 1 \). We have for \( 0 \leq t \leq t_1 \) and \( E \in q \mathcal{G}_\alpha \),

\[
c \leq \rho(E, t, \alpha) \leq C, \quad |\rho'(E, t, \alpha)| \leq \frac{C}{t_0}, \quad |\partial_t \rho(E, t, \alpha)| \leq \frac{C \log(N)}{t_0} \tag{A.51}
\]

Hence,
\[
\rho(E, t, \alpha) = \rho_{sc}(0) + O \left( \frac{t \log(N)}{t_0} + \frac{|E|}{t_0} \right). \tag{A.52}
\]
Proof. We only need to prove the statement about the time derivative. This follows immediately from the equation

\[ \partial_t m(z, t, \alpha) = \frac{1}{2} m(z, t, \alpha) \partial_z m(z, t, \alpha). \]  \hspace{1cm} (A.53)

The proof of the following result is a minor modification of Section 7 of [52]. Denote

\[ m_N(z, t, \alpha) = \sum_{i=1}^{N} \frac{1}{z_i(t, \alpha) - z}. \]  \hspace{1cm} (A.54)

**Lemma A.7.** Let \( 0 \leq \alpha \leq 1 \) and fix \( 0 < q < 1 \). Let \( \varepsilon, \delta > 0 \). Let \( 0 \leq t \leq t_1 \). The following estimates hold with overwhelming probability. For \( E \in qG_\alpha \) and \( \eta \geq N^3/N \) we have

\[ |m_N(z, t, \alpha) - m(z, t, \alpha)| \leq \frac{N\varepsilon}{N\eta}, \] \hspace{1cm} (A.55)

The above estimates also hold if \( \eta \geq 10 \).

From the above estimates we conclude the following rigidity result. We define the classical eigenvalue locations by

\[ \frac{i}{N} = \int_{0}^{\gamma_i(t, \alpha)} \rho(x, t, \alpha) dx. \] \hspace{1cm} (A.56)

Note that they satisfy

\[ \partial_t \gamma_i(t, \alpha) = \text{Re}[m(\gamma_i(t, \alpha), t, \alpha)] \] \hspace{1cm} (A.57)

and so

\[ |\partial_t \gamma_i(t, \alpha)| \leq C \log(N). \] \hspace{1cm} (A.58)

**Lemma A.8.** Let \( 0 < q < 1 \) and let \( \varepsilon > 0 \). Let \( i \) be such that \( \gamma_i(0, \alpha) \in qG_\alpha \). Then

\[ |z_i(t, \alpha) - \gamma_i(t, \alpha)| \leq \frac{N\varepsilon}{N} \] \hspace{1cm} (A.59)

with overwhelming probability. We have also with overwhelming probability,

\[ |z_i(t, \alpha) - (\alpha V_i + (1 - \alpha)\gamma_i^{(sc)})| \leq C \sqrt{t_0}. \] \hspace{1cm} (A.60)

We now make a slight digression on the classical eigenvalue locations of \( \rho(t, \alpha) \) which we denote by \( \gamma_i(t, \alpha) \). We want to elucidate the connection with the function \( f(x, \alpha) \). Fix \( 0 < q < 1 \). With overwhelming probability the eigenvalues \( \{z_i(0, \alpha) : i < -k_0\} \) are all to the left of the interval \( qG_\alpha \), and the eigenvalues \( \{z_i(0, \alpha) : i > k_0\} \) are all to the right of \( qG_\alpha \). Hence, with overwhelming probability \( \gamma_i(0, \alpha) = f(i, \alpha) \) for any \( i \) s.t. \( \gamma_i(\alpha) \in qG_\alpha \). We therefore also have \( \gamma_0(0, \alpha) = 0 \) with overwhelming probability.

We have the following lemma.

**Lemma A.9.** For \( t \leq 10t_1 \) and \( \varepsilon > 0 \) we have with overwhelming probability

\[ \sup_{0 \leq t \leq 10t_1} |\gamma_0(t_0 + t) - \gamma_0(t, 1)| \leq \frac{1}{N} \frac{N\varepsilon N^{\omega_1}}{N^{\omega_0/2}}. \] \hspace{1cm} (A.61)

and

\[ \sup_{0 \leq t \leq 10t_1} |0 - \gamma_0(t, 0)| \leq \frac{1}{N} \frac{N\varepsilon N^{\omega_1}}{N^{\omega_0/2}}. \] \hspace{1cm} (A.62)

With overwhelming probability,

\[ \gamma_k(t, \alpha) - \gamma_j(t, \alpha) = \frac{j - k}{N\rho_{sc}(0)} + O \left( \frac{1}{N} \right) \] \hspace{1cm} (A.63)

for \( |j| + |k| \leq N^{\omega_0/2} \) and \( t \leq 10t_1 \) with \( \omega_1 \leq \omega_0/2 \).
Proof. We start with (A.61) and (A.62). For $L > 0$ and small $c > 0$ we define

$$ I_1 := [\gamma - L(t,1), \gamma_L(t,1)], \quad I_3 := \mathbb{R}[\gamma - cG(t,1), \gamma_cG(t,1)] \quad I_2 := I_3 \setminus I_1. \quad (A.64) $$

Take $c$ small enough so that rigidity holds for $|i| \leq cGN$. We write,

$$ \hat{c}_t \gamma_0(t,1) = \text{Re}[m(\gamma_0(t,1), t, 1)] = \frac{1}{N} \sum_{|i| > L} \frac{1}{\gamma_0(t-1) - x_i(t+t_0)} $$

$$ + \left( \frac{1}{N} \sum_{|i| > L} \frac{1}{\gamma_0(t-1) - x_i(t+t_0)} - \frac{1}{N} \sum_{|i| > L} \frac{1}{\gamma_0(t-1) - x_i(t+t_0)} \right) $$

$$ + \left( \int_{I_3} \frac{\rho(x, t, 1)dx}{\gamma_0(t-1) - x} - \frac{1}{N} \sum_{L \ll |i| < cGN} \frac{1}{\gamma_0(t-1) - x_i(t+t_0)} \right) $$

$$ + \left( \int_{I_2} \frac{\rho(x, t, 1)dx}{\gamma_0(t-1) - x} - \frac{1}{N} \sum_{L \ll |i| < cGN} \frac{1}{\gamma_0(t-1) - x_i(t+t_0)} \right) + \left( \int_{I_1} \frac{\rho(x, t, 1)dx}{\gamma_0(t-1) - x} \right) $$

$$ = \frac{1}{N} \sum_{|i| > L} \frac{1}{\gamma_0(t-1) - x_i(t+t_0)} + A_1 + A_2 + A_3 + A_4 \quad (A.65) $$

By rigidity we have $|A_1| + |A_3| \leq N^\varepsilon/L$ for any $\varepsilon > 0$ with overwhelming probability. The same argument handling the error term $E_4$ in the proof of Lemma 3.7 yields $|A_4| \leq N^\varepsilon L/(Nt_0)$ with overwhelming probability. The error term $A_2$ is handled in the same way as $E_1$ in the proof of Lemma 3.7 and we see that $|A_2| \leq N^\varepsilon/\sqrt{NG}$ with overwhelming probability. Choosing $L = \sqrt{Nt_0}$ we see that

$$ \hat{c}_t \gamma_0(t,1) = \frac{1}{N} \sum_{|i| > \sqrt{Nt_0}} \frac{1}{\gamma_0(t-1) - x_i(t+t_0)} + N^\varepsilon \mathcal{O} \left( \frac{1}{\sqrt{Nt_0}} \right) \quad (A.66) $$

with overwhelming probability. Similarly we see that

$$ \hat{c}_t \gamma_0(t_0 + t) = \frac{1}{N} \sum_{|i| > \sqrt{Nt_0}} \frac{1}{\gamma_0(t_0 + t) - x_i(t+t_0)} + N^\varepsilon \mathcal{O} \left( \frac{1}{\sqrt{Nt_0}} \right) \quad (A.67) $$

with overwhelming probability. Hence, for $|t| \leq 10t_1$,

$$ |\gamma_0(t,1) - \gamma_0(t_0 + t)| \leq \frac{1}{N} N^\varepsilon N^{\omega_1} \quad (A.68) $$

The same argument applies to $\gamma_0^{(sc)}$ and $\gamma_0(t,0)$.

We now prove (A.63). We have,

$$ \frac{k - j}{N} = \int_{\gamma_j(t,\alpha)}^{\gamma_k(t,\alpha)} \rho(E, t, \alpha) dE $$

$$ = \rho_{sc}(0)(\gamma_k(t, \alpha) - \gamma_j(t, \alpha)) + \int_{\gamma_j(t,\alpha)}^{\gamma_k(t,\alpha)} \mathcal{O}(\gamma_k(t, \alpha) - \gamma_j(t, \alpha)|t/t_0). \quad (A.69) $$

Hence,

$$ \gamma_k(t, \alpha) - \gamma_j(t, \alpha) = \frac{j - k}{N\rho_{sc}(0)} + \mathcal{O} \left( \frac{1}{N} \right) \quad (A.70) $$

for $|j| + |k| \leq N^{\omega_0/2}$, and $t \leq 10t_1$. □
B Stochastic continuity

In Appendix A we proved rigidity for each fixed time \( t \geq t_0 \) and each fixed \( \alpha \) - i.e., Lemma A.8. In this appendix we go from the estimates of Lemma A.8 to estimates for all time \( t \) and \( \alpha \) simultaneously. We continue with the notation of Appendix A. Recall the definition of \( \hat{C}_q \) from Section 3.

**Lemma B.1.** Let \( D > 0 \) and \( \varepsilon > 0 \). Let \( 0 < q < q_* \). We have

\[
\mathbb{P} \left[ \sup_{i \in \mathbb{C}_q} \sup_{0 \leq \alpha \leq 1} \sup_{0 \leq t \leq 10t_1} |z_i(t, \alpha) - \gamma_i(t, \alpha)| \geq \frac{N^\varepsilon}{N} \right] \leq \frac{1}{ND}.
\]

(B.1)

Given \( \alpha_1 \) and \( \alpha_2 \), the difference

\[
u_i := z_i(t, \alpha_1) - z_i(t, \alpha_2)
\]

satisfies the parabolic equation

\[
\partial_t u = Lu
\]

(B.3)

where

\[
(Lu)_i := \frac{1}{N} \sum_j \frac{u_j - u_i}{(z_i(t, \alpha_1) - z_j(t, \alpha_1))(z_i(t, \alpha_2) - z_j(t, \alpha_2))}.
\]

(B.4)

Hence,

\[
\sup_t ||u(t)||_\infty \leq C|z(0, \alpha_1) - z(0, \alpha_2)||_\infty \leq C|\alpha_1 - \alpha_2| \sup_i |x_i(t_0)| + |y_i(t_0)|.
\]

(B.5)

With overwhelming probability we have that

\[
\sup_i |x_i(t_0)| + |y_i(t_0)| \leq CN^{C_V}
\]

(B.6)

for a fixed \( C_V > 0 \) by our assumptions on \( V \). Hence in order to prove Lemma B.1 we can just prove it for a set of \( \alpha \) of at most size \( N^{2C_V} \); i.e., we need only prove the following.

**Lemma B.2.** Fix \( 0 \leq \alpha \leq 1 \). We have,

\[
\mathbb{P} \left[ \sup_{i \in \mathbb{C}_q} \sup_{0 \leq t \leq 10t_1} |z_i(t, \alpha) - \gamma_i(t, \alpha)| \geq \frac{N^\varepsilon}{N} \right] \leq \frac{1}{ND}.
\]

(B.7)

**Proof.** Consider the equation for \( m_N \) for general \( \beta \geq 1 \),

\[
dm_N = \partial_z (m_N(m_N)) + \frac{\beta - 2}{N} \partial_z^2 m_N + \frac{1}{N^{3/2}(\lambda_i - z)^2} \sqrt{2\beta} dB_i
\]

(B.8)

For \( \eta \geq N^{-2} \) we see that for any \( t \) we have by the BDG inequality,

\[
\mathbb{P} \left[ \sup_{0 \leq s \leq N^{-100}} |m_N(z, t + s) - m_N(t)| \geq N^{-5} \right] \leq N^{-D}
\]

(B.9)

for any \( D > 0 \). From Appendix A we know that the local law holds on the domain \( D_{\varepsilon, q} \) defined in Section 2.4 on a set of times \( t = kN^{-100} \), for \( 0 \leq k \leq N^{100} 10t_1 \) with overwhelming probability. Hence we can extend the local law to all times \( t \) on the domain \( D_{\varepsilon, q} \). Rigidity is a consequence of this.

We also want to prove that

\[
\sup_{0 \leq \alpha \leq 1} \sup_{0 \leq t \leq 10t_1} |z_i(\alpha, t)| \leq N^C
\]

with overwhelming probability for some \( C > 0 \). This follows from the above argument again. First it suffices to prove it for fixed \( \alpha \). We can assume that it holds for a mesh of times \( t \) with overwhelming probability. Consider then \( m_N(z) \) with \( z = N^{2C} + iN^{-2} \). At each time in the mesh we have that \( |m_N(z)| \leq N^{-4} \). By the above argument it then holds for all times \( t \). Therefore a particle cannot cross \( E = N^{2C} \) or at some time \( t \), we would have \( \text{Im} \left[ m_N(N^{2C} + iN^{-2}) \right] \geq 1 \).
C Re-indexing argument

Recall our set-up in Section 3. We have the process \( x_i(t) \) that satisfies

\[
\mathrm{d}x_i = \sqrt{\frac{2}{N}} \mathrm{d}B_i + \frac{1}{N} \sum_j \frac{1}{x_i - x_j} \mathrm{d}t
\]

We fixed an index \( i_0 \). In this appendix we want to show that we can assume that \( N \) is odd and \( i_0 = (N + 1)/2 \). Our method is to construct an auxiliary DBM process \( x^* \) of \( 2N - 1 \) particles s.t. for every index \( j \) s.t. \( i_0 + j \in [1, N] \),

\[
\operatorname{sup}_{0 \leq t \leq 1} |x^*_{i+j}(t) - x_{i_0+j}(t)| \leq \frac{1}{N^{100}},
\]

with overwhelming probability. Similarly, we construct a process \( y^* \) of \( 2N \) particles s.t.

\[
\operatorname{sup}_{0 \leq t \leq 1} |y^*_{i+j}(t) - y_{i+j}(t)| \leq \frac{1}{N^{100}}.
\]

Then the argument of Section 3 goes through using the processes \( x^* \) and \( y^* \) instead of \( x \) and \( y \). We will also see that the estimate of Lemma B.1 also holds for the \( x^* \) and \( y^* \) (with an appropriate modification of \( \gamma^{(sc)} \) for the particles added to the GOE flow that has no effect on the rest of the paper).

We now construct \( x^* \). Let \([1, 2N - 1] = C_1 \cup C_2 \cup C_3 \) where \( C_1 = [1, N - i_0] \), \( C_2 = [N - i_0 + 1, 2N - i_0] \) and \( C_3 = [2N - i_0 + 1, 2N] \). Recall that by assumption there is a \( C_\nu \) s.t. \( |V_i| \leq N^{C_\nu} \). Let \( B^i \) be the Brownian motions for the \( x_i \) as above. Let \( \tilde{B}_i \) be independent standard Brownian motions except that

\[
\tilde{B}_i = B_{i-(N-i_0)}, \quad i \in C_2.
\]

We let \( x^* \) be the solution to

\[
\mathrm{d}x^*_i = \sqrt{\frac{2}{N}} \mathrm{d}B^i + \frac{1}{N} \sum_j \frac{1}{x^*_i - x^*_j} \mathrm{d}t
\]

with initial condition

\[
x^*_i(0) = \tilde{V}_i
\]

where

\[
\tilde{V}_i = \begin{cases} 
-2N^{C_\nu+200} + iN^5, & i \in C_1 \\
V_{i-(N-i_0)}, & i \in C_2 \\
N^{2C_\nu+200} + iN^5, & i \in C_3
\end{cases}
\]

Define now the process \( \hat{x} \) by

\[
\mathrm{d}\hat{x}_i = \frac{\mathrm{d}\tilde{B}_i}{\sqrt{N}} + 1_{i \in C_1 \cup C_3} \frac{1}{N} \sum_{j \in C_1 \cup C_3} \frac{1}{\hat{x}_j - \hat{x}_i} \mathrm{d}t + 1_{i \in C_2} \frac{1}{N} \sum_{j \in C_2} \frac{1}{\hat{x}_j - \hat{x}_i} \mathrm{d}t - \frac{\hat{x}_i}{2} \mathrm{d}t,
\]

with initial data

\[
\hat{x}_i(0) = \tilde{V}_i.
\]

Then the process \( \hat{x}_i \) decomposes into two independent processes \( \{\hat{x}_i\}_{i \in C_1 \cup C_3} \) and \( \{\hat{x}_i\}_{i \in C_2} \), and \( \hat{x}_i = x_{i-(N-i_0)}(t) \) a.s. for all times \( t \). For the process \( \{\hat{x}_i\}_{i \in C_1 \cup C_3} \) an easy argument using standard large deviations bounds on Brownian motion and the fact that initially the particles are far apart, so the interaction term is negligible gives that

\[
\mathbb{P}\left[ \sup_{0 \leq t \leq 1} |\hat{x}_i - \tilde{V}_i| \geq 10 \right] \leq N^{-D}
\]

for any \( D > 0 \) and \( i \in C_1 \cup C_3 \). In particular we see that with overwhelming probability, the \( \hat{x}_i \) retain their ordering \( \hat{x}_i < \hat{x}_{i+1} \).
Since with overwhelming probability the \( \hat{x}_i \) retain their ordering, the difference \( u_i = \hat{x}_i - x_i^* \) satisfies a parabolic equation with overwhelming probability,

\[
\partial_t u_i = \sum_{j=1}^{2N-1} \frac{u_j - u_i}{(\hat{x}_i - \hat{x}_j)(x_i^* - x_j^*)} + \xi_i \tag{C.11}
\]

where

\[
\xi_i := 1_{\{i \in C_1 \cup C_3\}} \frac{1}{N} \sum_{j \in C_2} \frac{1}{\hat{x}_j - \hat{x}_i} \text{d}t + 1_{\{i \in C_2\}} \frac{1}{N} \sum_{j \in C_1 \cup C_3} \frac{1}{\hat{x}_j - \hat{x}_i}. \tag{C.12}
\]

By the estimate (C.10) we see that if \( i \in C_1 \cup C_3 \) and \( j \in C_2 \) then

\[
\sup_{0 \leq t \leq 1} |\hat{x}_i(t) - \hat{x}_j(t)| \leq \frac{C}{N^{200}}. \tag{C.13}
\]

Therefore

\[
\sup_{0 \leq t \leq 1} ||\xi(t)||_{\infty} \leq \frac{1}{N^{190}}. \tag{C.14}
\]

By the Duhamel formula we conclude that

\[
\sup_{0 \leq t \leq t_1} ||x^*(t) - \hat{x}(t)||_{\infty} \leq \frac{1}{N^{180}} \tag{C.15}
\]

with overwhelming probability. This yields (C.2). We can make a similar construction for \( y \). The new process \( y^* \) satisfies the estimates of Lemma B.1, but replacing \( \gamma_j^{(sc)} \) by \( \hat{V}_j \) for the indices \( j \) for which \( \gamma_j^{(sc)} \) is undefined.

We remark that the \( N \) appearing in (C.5) and the corresponding definition of \( y^* \) no longer equals the number of particles which is \( 2N - 1 \). However, since \( N = 2N - 1 \) and the same factor appears in the definition of \( x^* \) and \( y^* \), this will not affect any of our methods.

## D Sobolev inequality

Let \( M_1 < M_2 \) be natural numbers. Let \( u_i : \mathbb{Z} \to \mathbb{R} \) be a sequence. By Cauchy-Schwarz,

\[
\left( \frac{1}{2M_1 + 1} \sum_{|i| \leq M_1} u_i - \frac{1}{2M_2 + 1} \sum_{|j| \leq M_2} u_j \right)^2 \leq \frac{1}{(2M_1 + 1)^2(2M_2 + 1)^2} \left( \sum_{|i| \leq M_1, |j| \leq M_2, i \neq j} \frac{(u_i - u_j)(i - j)}{(i - j)^2} \right)^2 \leq \frac{1}{(2M_1 + 1)^2(2M_2 + 1)^2} \left( \sum_{|i| \leq M_1, |j| \leq M_2, i \neq j} (u_i - u_j)^2 (i - j)^2 \right)
\]

Clearly,

\[
\left( \sum_{|i| \leq M_1, |j| \leq M_2, i \neq j} (i - j)^2 \right)^2 \leq C \left( \sum_{|i| \leq M_1} M_i^2 \right) \leq CM_1M_2^3. \tag{D.2}
\]

Therefore,

\[
\left( \frac{1}{2M_1 + 1} \sum_{|i| \leq M_1} u_i - \frac{1}{2M_2 + 1} \sum_{|j| \leq M_2} u_j \right)^2 \leq C_M \frac{M_2^3}{M_1} \left( \sum_{|i| \leq M_1, |j| \leq M_2} (u_i - u_j)^2 (i - j)^2 \right). \tag{D.3}
\]

We can iterate the above inequality to prove the following lemma.
Lemma D.1. Let \( u_i : \mathbb{Z} \to \mathbb{R} \) be a sequence and let \( N \) be a natural number. There is a universal constant \( C \) s.t.

\[
\left( u_0 - \frac{1}{2N+1} \sum_{|i| \leq N} u_i \right)^2 \leq C \log(N)^2 \left( \sum_{|i| \leq N, |j| \leq N} \frac{(u_i - u_j)^2}{(i-j)^2} \right).
\] (D.4)

Proof. Choose \( n \leq C \log(N) \) numbers \( M_i \) s.t. \( M_0 = 1 \), \( M_n = N \) and \( M_{i+1} \leq 3M_i \). Define \( A_i \) by

\[
A_i := \frac{1}{2M_i + 1} \sum_{|j| \leq M_i} u_j - \frac{1}{2M_{i+1} + 1} \sum_{|j| \leq M_{i+1}} u_j.
\] (D.5)

Then,

\[
\left( u_0 - \frac{1}{2N+1} \sum_{|i| \leq N} u_i \right)^2 = \left( \sum_{i=1}^{n-1} A_i \right)^2 \leq n^2 \sup_i (A_i)^2.
\] (D.6)

By (D.3),

\[
(A_i)^2 \leq C \sum_{|i| \leq N, |j| \leq N} \frac{(u_i - u_j)^2}{(i-j)^2}
\] (D.7)

which yields the claim. \( \square \)

E Fubini lemma

Lemma E.1. Let \( X(u) \) for \( 0 \leq u \leq 1 \) s.t. there is an event \( \mathcal{F} \) with \( \mathbb{P}[\mathcal{F}] \geq 1 - \varepsilon \) on which \( \sup_u |X(u)| \leq A \). Suppose that for each \( u \) there is an event \( \mathcal{F}_u \) with \( \mathbb{P}[\mathcal{F}_u \cap \mathcal{F}] \geq 1 - \varepsilon \) on which \( |X(u)| \leq Y(u) \). Then,

\[
\mathbb{P} \left[ \int_0^1 X(u) du > \int_0^1 Y(u) du + \delta \right] \leq \varepsilon \left( 1 + \frac{A}{\delta} \right)
\] (E.1)

Proof. We write

\[
1_{\mathcal{F}} \int_0^1 |X(u)| du = 1_{\mathcal{F}} \int_0^1 |X(u)| 1_{\mathcal{F}_u} du + \int_0^1 |X(u)| 1_{\mathcal{F}_u \cap \mathcal{F}_u} du
\]

\[
\leq \int_0^1 Y(u) du + A \int_0^1 1_{\mathcal{F}_u} du.
\] (E.2)

By Markov’s inequality,

\[
\mathbb{P} \left[ A \int_0^1 1_{\mathcal{F}_u} du > \delta \right] \leq \frac{A \varepsilon}{\delta}.
\] (E.3)

The claim follows. \( \square \)

Remark. We usually apply this with \( A = N^C \) for some fixed \( C \) and \( \varepsilon = N^{-D} \) for a large \( D \) and \( \delta = N^{-D/2} \).

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