$SL(\infty,R)$ Kac-Moody symmetry of $W_\infty$ gravity

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ABSTRACT

Two-dimensional gravity in the light-cone gauge was shown by Polyakov to exhibit an underlying $SL(2,R)$ Kac-Moody symmetry, which may be used to express the energy-momentum tensor for the metric component $h_{++}$ in terms of the $SL(2,R)$ currents via the Sugawara construction. We review some recent results which show that in a similar manner, $W_\infty$ and $W_{1+\infty}$ gravities have underlying $SL(\infty,R)$ and $GL(\infty,R)$ Kac-Moody symmetries respectively.

* Supported in part by the U.S. Department of Energy, under grant DE-FG05-91ER40633.
Contribution to the proceedings of the Trieste Summer Workshop on High-Energy Physics, Trieste, August 1991.
1. Introduction

Einstein gravity, with action $\int \sqrt{g} R \, d^n x$, becomes trivial in $n = 2$ dimensions because the integrand is then a total derivative. In fact the action is then proportional to the Gauss-Bonnet expression for the topological invariant $\chi$ – the Euler number. To circumvent this problem, Polyakov proposed that one could construct a theory of two-dimensional gravity with dynamics by taking the action to be given instead as the induced quantum effective action for a non-critical matter system in a curved two-dimensional background [1]. In the light-cone gauge, where the dynamics of the metric is described just by its $h_{++}$ component, this formulation of two-dimensional gravity has a “hidden” $SL(2, R)$ Kac-Moody symmetry, which enables one to write the $h_{++}$ gauge field in terms of $SL(2, R)$ Kac-Moody currents [1]. The energy-momentum tensor for $h_{++}$ may then be written in terms of the Sugawara energy-momentum tensor for these currents [2].

In this paper, we review some recent results in which the above construction is generalised to induced $W_\infty$ and $W_{1+\infty}$ gravities. One can show that in light-cone gauge, these theories exhibit underlying $SL(\infty, R)$ and $GL(\infty, R)$ symmetries respectively [3]. For technical reasons, which will be explained later, it is easier to describe the $W_{1+\infty}$ case first. We shall then indicate briefly how the construction goes for $W_\infty$ gravity.

The details of the derivation of the Kac-Moody symmetries for $W_{1+\infty}$ and $W_\infty$ gravity are necessarily rather involved, because of the complexity of the algebras. Since the full details may be found in [3], we shall concentrate here on the basic ideas, and the main results. The method used in [3] for deriving these results closely parallels the approach used in [1] to derive the $SL(2, R)$ symmetry of two-dimensional gravity. In this paper, we therefore review the two-dimensional gravity calculation in some detail, and then indicate the key steps in the generalisation to $W_{1+\infty}$ and $W_\infty$ gravity. A complete description may be found in [3].

2. The $SL(2, R)$ symmetry of two-dimensional gravity

The induced Polyakov action for two-dimensional gravity is obtained as the quantum effective action for a matter system in a general curved two-dimensional background. Since we shall be interested in gravity in the light-cone gauge, we may impose this gauge choice at the outset and start by considering a gauged classical matter Lagrangian of the form

$$L = L_{\text{mat}} + h T_{\text{mat}}.$$  \hspace{1cm} (1)

Here $L_{\text{mat}}$ denotes the Lagrangian for the matter fields; $T_{\text{mat}}$ is their energy-momentum tensor; and $h$ denotes the $h_{++}$ component of the metric in the light-cone gauge. The Lagrangian (1) can be thought of as describing the gauging of the Virasoro algebra, realised
on the matter fields in $L_{\text{mat}}$, that is generated by the current $T_{\text{mat}}$. Under the residual diffeomorphisms preserving the light-cone gauge, the gauge field $h$ transforms as

$$
\delta h = \bar{\partial} k + k \partial h - h \partial k,
$$

and the matter fields transform in the standard way dictated by their tensorial structure. The action given by (1) is then classically invariant under these local gauge transformations.

More or less any matter system that is classically conformally invariant may be chosen in (1) in order to obtain the induced two-dimensional gravity theory. A suitable choice would be to take the matter to be a complex fermion $\psi$, with $L_{\text{mat}} = \bar{\psi} \partial \psi$, and $T_{\text{mat}} = \frac{1}{2} \bar{\psi} \psi - \frac{1}{2} \psi \bar{\psi} \bar{\psi}$. If the matter is now quantised then the quantum effective action $\Gamma[h]$ may be defined as

$$
e^{-\Gamma[h]} \equiv \int D\psi e^{\frac{-1}{\pi} \int L}.
$$

This effective action transforms anomalously under the diffeomorphisms (2), as may be seen by deriving the appropriate Ward identity. We do this by observing that

$$
\frac{\partial \Gamma}{\partial h(z)} = \frac{1}{\pi} e^{\Gamma} \int D\psi T(z) e^{\frac{-1}{\pi} \int L},
$$

and hence if we take the $\bar{\partial}_z$ derivative the only contributions will come from terms where it acts on the singularities in the operator-product expansion of $T(z)$ with the $T(w)$ terms in the exponential. Thus we have

$$
\bar{\partial}_z \frac{\partial \Gamma}{\partial h(z)} = -\frac{1}{\pi^2} e^{\Gamma} \int D\psi \left( \partial T(w) + \frac{2T(w)}{(z-w)^2} + \frac{c/2}{(z-w)^4} \right) e^{\frac{-1}{\pi} \int L}.
$$

Using $\bar{\partial}_z (z-w)^{-1} = \pi \delta^{(2)}(z-w)$, it follows that

$$
\frac{\partial \delta \Gamma}{\partial h} - h \partial \frac{\delta \Gamma}{\partial h} - 2 \partial h \frac{\delta \Gamma}{\partial h} = \frac{c}{12\pi} \partial^3 h.
$$

Multiplying by a diffeomorphism parameter $k$ and integrating, we see that this implies that under (2), $\Gamma[h]$ transforms anomalously, as

$$
\delta \Gamma[h] = -\frac{c}{12\pi} \int k \partial^3 h.
$$

Viewing $\Gamma[h]$ as a (non-local) action for the light-cone metric component $h$, we may now quantise $h$ itself, and calculate its correlation functions. Consider the $(N+1)$-point function

$$
\langle h(z) h(x_1) \cdots h(x_N) \rangle \equiv \int D'h e^{-\Gamma[h]} h(z) h(x_1) \cdots h(x_N).
$$
two-point and three-point functions are given by
\[
\frac{c}{12\pi} \partial_z^3 \langle h(z)h_1 \cdots h_N \rangle = \sum_{r=1}^N \left\{ \delta\delta^{(2)}(z - x_r) \langle h_1 \cdots h_{r-1} \rangle \right\}.
\]

Acting on this with \(-\frac{c}{12} \partial_z^2\), and using the anomalous Ward identity (6), one obtains [1]
\[
\frac{c}{12\pi} \partial_z^3 \langle h(z)h_1 \cdots h_N \rangle = \sum_{r=1}^N \left\{ \delta\delta^{(2)}(z - x_r) \langle h_1 \cdots h_{r-1} \rangle \right\} - \partial\delta^{(2)}(z - x_r) \langle h(z)h_1 \cdots h_{r-1} \rangle - 2\delta^{(2)}(z - x_r) \langle \partial hh_1 \cdots h_{r-1} \rangle \right\}.
\]

(9)

where \(h_r\) denotes \(h(x_r)\), and we have used the result, proven by functional integration by parts, that for any operator \(O\) depending on \(h\), \(\langle (\delta\Gamma/\delta h) O \rangle = \langle \delta\Gamma/\delta h \rangle\). The symbol \(\partial\delta^{(2)}(z - x_r)\) indicates that this particular gauge field is omitted in the product. From the identity \(\partial_z^3 \langle \frac{z-x_r}{z-x_r} \rangle = 2\pi\delta^{(2)}(z - x_r)\), one may integrate (9) to obtain, after a rescaling \(h \rightarrow (6/c)h\) [1],
\[
\langle h(z)h(x_1) \cdots h(x_r) \rangle = \sum_{r=1}^N \left\{ -\frac{c}{6} \frac{(z-x_r)^2}{(z-x_r)^2} \langle h(x_1) \cdots h(x_{r-1}) \rangle \right\} - \left[ \frac{2(z-x_r)}{z-x_r} + \frac{(z-x_r)^2}{z-x_r} \partial_{x_r} \right] \langle h(x_1) \cdots h(x_N) \rangle \right\}.
\]

Equation (10) is a recursion relation that gives the \((N+1)\)-point correlation function in terms of the \(N\)-point and \((N-1)\)-point correlation functions. Starting from \(\langle 1 \rangle = 1\) and \(\langle h(z) \rangle = 0\), one may solve iteratively for the higher-point functions. For example, the two-point and three-point functions are given by
\[
\langle h(x)h(y) \rangle = -\frac{c}{6} \frac{(x-y)^2}{(x-y)^2},
\]
\[
\langle h(x)h(y)h(z) \rangle = \frac{c}{3} \frac{(x-y)(y-z)(x-z)}{(x-y)(y-z)(z-x)}.
\]

(11)

(12)

The condition on \(h\) such that there is no anomaly in the variation of \(\Gamma[h]\) given by (7) is
\[
\partial^3 h = 0.
\]

(13)

This equation may equivalently be obtained as the equation of motion following from the induced action \(\Gamma[h]\). Its general solution may be written as
\[
h(z, \bar{z}) = J^{(1)}(\bar{z}) - 2J^{(0)}(\bar{z})z + J^{(-1)}(\bar{z})z^2.
\]

(14)

Using this expansion, one may re-express (10) as a recursion relation for the correlation functions of the coefficient operators \(J^i\). The result is [1]
\[
\langle J^i(\bar{z}) J^{j_1}(\bar{x}_1) \cdots J^{j_N}(\bar{x}_N) \rangle = \sum_{r=1}^N \left\{ -\frac{c}{12} \frac{\delta_i^{(2)}}{(z-x_r)^2} \langle J^{j_1}(\bar{x}_1) \cdots J^{j_r}(\bar{x}_r) \cdots J^{j_N}(\bar{x}_N) \rangle \right\} - \frac{\delta^{j_r}_{k_r}}{z-x_r} \langle J^{k_r}(\bar{x}_r) J^{j_1}(\bar{x}_1) \cdots J^{j_r}(\bar{x}_r) \cdots J^{j_N}(\bar{x}_N) \rangle \right\},
\]

(15)
where $\eta^{ij}$ is the inverse Cartan-Killing metric and $f^{ijk}$ are the structure constants of $SL(2, R)$. The two-point and three-point correlation functions (11) and (12) become

$$\langle J^i(\bar{x})J^j(\bar{y}) \rangle = -\frac{\eta^{ij}}{(\bar{x} - \bar{y})^2}, \quad (16)$$

$$\langle J^i(\bar{x})J^j(\bar{y})J^k(\bar{z}) \rangle = \frac{f^{ijk}}{(\bar{x} - \bar{y})(\bar{y} - \bar{z})(\bar{z} - \bar{x})}. \quad (17)$$

These correlation functions, and the recursion relation, are precisely those for the currents of an $SL(2, R)$ Kac-Moody algebra, arising from an $SL(2, R)$ WZW model.

3. The $GL(\infty, R)$ symmetry of $W_{1+\infty}$ gravity

The generalisation from two-dimensional light-cone gravity to $W_{1+\infty}$ gravity is made by introducing additional gauge fields $A_i$ for all the higher-spin currents of $W_{1+\infty}$, and replacing the Lagrangian (1) by

$$L = L_{\text{mat}} + \sum_{i=-1}^{\infty} A_i \tilde{V}^i. \quad (18)$$

Here $\tilde{V}^i(z)$ is the spin-$(i+2)$ current of $W_{1+\infty}$, and $A_i$ is its corresponding gauge field. Details of the $W_\infty$ and $W_{1+\infty}$ algebras may be found in [4], and also in the lectures on $W$ algebras and $W$ gravity in this volume [5]. We are adopting the notation of [5] here, and using $\tilde{V}^i$ to denote the currents of $W_{1+\infty}$, and $V^i$ to denote those for $W_\infty$.

The operator-product expansions for the $W_{1+\infty}$ currents take the form

$$\tilde{V}^i(z)\tilde{V}^j(w) \sim -\sum_{\ell \geq 0} \tilde{f}^{ij}_{2\ell}(\partial_z, \partial_w) \frac{\tilde{V}^{i+j-2\ell}(w)}{z-w} \tilde{c}_i \delta^{ij}(\partial_z)^{2i+3} \frac{1}{z-w}, \quad (19)$$

with

$$\tilde{f}^{ij}_{2\ell}(m, n) = \frac{1}{2(2\ell + 1)!} \tilde{\phi}^{ij}_{2\ell} M^{ij}_{2\ell}(m, n), \quad (20)$$

where

$$\tilde{\phi}^{ij}_{2\ell} = 4F_3 \left[ \begin{array}{c} -i - \frac{1}{2}, \frac{1}{2}, -\ell - \frac{1}{2}, -\ell \\ -i - \frac{1}{2}, -j - \frac{1}{2}, i + j - 2\ell + \frac{5}{2} \end{array} ; 1 \right] \quad (21)$$

and

$$M^{ij}_{2\ell}(m, n) = \sum_{k=0}^{2\ell+1} (-)^k \binom{2\ell + 1}{k} (2i - 2\ell + 2)_k (2j + 2 - k)_{2\ell+1-k} m^{2\ell+1-k} n^k. \quad (22)$$
The central charges \( \tilde{c}_i \) are given by

\[
\tilde{c}_i = \frac{2^{2i-2}((i+1)!)^2}{(2i+1)!!(2i+3)!!}c.
\]  

(23)

(Fuller details may be found in [4,5].)

It is clear that the same steps as those described in section 2 for pure gravity may be followed here for \( W_{1+\infty} \) gravity. The analogue of the anomalous Ward identity (6) turns out to be

\[
\bar{\partial} \frac{\delta \Gamma}{\delta A_i} + \sum_{\ell \geq 0} \tilde{f}^{ij}_{2\ell}(\partial_i - \partial_A) \left( \frac{\delta \Gamma}{\delta A_{i+j-2\ell}} A_j \right) = \frac{\tilde{c}_i}{\pi} \partial^{2i+3} A_i,
\]

where the derivative \( \partial \) in \( \tilde{f}^{ij}_{2\ell}(\partial_i - \partial_A) \) acts on everything to its right, whilst \( \partial_A \) acts only on the explicit \( A_j \) term in the following parentheses. The analogue of the diffeomorphism transformation (2) is

\[
\delta A_i = \bar{\partial} k_i + \sum_{\ell \geq 0} \sum_{j=-1}^{i+2\ell+1} \tilde{f}^{ij}_{2\ell} (\partial_i A_j, \partial_k) A_j k_{i-j+2\ell},
\]

(25)

where \( \partial_A \) acts only on \( A_j \), and \( \partial_k \) acts only on the transformation parameters \( k_{i-j+2\ell} \). From (24) and (25), it follows that under a spin-\( i+2 \) transformation with parameter \( k_i \), the effective action \( \Gamma[A] \) has an anomalous variation

\[
\delta k_i \Gamma[A] = -\frac{\tilde{c}_i}{\pi} \partial^{2i+3} A_i.
\]

(26)

The anomalous Ward identity (24) may be used to obtain a recursion relation for the correlation functions for the gauge fields of quantum \( W_{1+\infty} \) gravity. Repeating the steps described in section 2 for pure gravity, we find [3]

\[
\langle A_i(z) A_{j_1}(x_1) \cdots A_{j_N}(x_N) \rangle =
\]

\[
- \sum_{r=1}^{N} \tilde{c}_{jr}(2jr + 2)! \delta^{i,jr} \frac{(z - x_r)^{2i+2}}{(\bar{z} - \bar{x}_r)^2} \langle A_{j_1}(x_1) \cdots A_{j_r}(x_r) \cdots A_{j_N}(x_N) \rangle
\]

\[
- \sum_{k \geq 1} \frac{[(i+k)/2]}{\tilde{c}_{k}(2k+2)!} \sum_{\ell=0}^{[i+k]/2} \sum_{r=1}^{N} \delta_{i+k-2\ell,jr} \tilde{f}^{ik}_{2\ell}(\partial_z, -\partial_A) \delta_{i,j+2\ell} \frac{(z - x_r)^{2i+2}}{(\bar{z} - \bar{x}_r)^2} \langle A_k(x_r) A_{j_1}(x_1) \cdots A_{j_r}(x_r) \cdots A_{j_N}(x_N) \rangle,
\]

(27)

after a rescaling of the gauge fields \( A_i \) according to

\[
A_i \rightarrow \frac{1}{\tilde{c}_i(2i+2)!} A_i.
\]

(28)
This recursion relation may be solved iteratively for the correlation functions for $W_{1+\infty}$ gravity. For example, we find [3]

$$\langle A_i(x, \bar{x})A_j(y, \bar{y}) \rangle = \tilde{c}_i(2i + 2)! \frac{(x - y)^{2i+2}}{(\bar{x} - \bar{y})^2}$$  \hspace{1cm} (29)

for the two-point function. For the three-point function, we find [3]

$$\langle A_i(x, \bar{x})A_j(y, \bar{y})A_k(z, \bar{z}) \rangle = \tilde{N}_{ijk} \frac{(x - y)^{i+j-k+1}(y - z)^{j+k-i+1}(z - x)^{k+i-j+1}}{(\bar{x} - \bar{y})(\bar{y} - \bar{z})(\bar{z} - \bar{x})},$$  \hspace{1cm} (30)

when $i + j + k$ is even, and zero when $i + j + k$ is odd. $\tilde{N}_{ijk}$ is defined by

$$\tilde{N}_{ijk} \equiv \frac{(2i + 2)!(2j + 2)!(2k + 2)!(i + j - k + 1)!(j + k - i + 1)!(k + i - j + 1)!}{(i + j - k + 1)!(j + k - i + 1)!(k + i - j + 1)!} \tilde{P}_{ijk},$$  \hspace{1cm} (31)

with $\tilde{P}_{ijk}$ given by

$$\tilde{P}_{ijk} = \frac{1}{2^i} \tilde{P}^{ij}_{i+j-k}.$$  \hspace{1cm} (32)

$\tilde{P}_{ijk}$ is manifestly symmetric in $i$ and $j$. Although it is not manifest, it is in fact totally symmetric in $i$, $j$ and $k$ [3].

From (26), we see that the conditions on the gauge fields for the transformation rules to be anomaly free are

$$\partial^{2i+3}A_i = 0,$$  \hspace{1cm} (33)

generalising (13) for pure gravity. Thus we may expand the spin-$(i + 2)$ gauge field $A_i$ as a polynomial of degree $(2i + 3)$ in $z$:

$$A_i(z, \bar{z}) = \sum_{m=-i-1}^{i+1} \binom{2i + 2}{i + 1 + m} J_m^i(\bar{z})(-z)^{i+1+m}.$$  \hspace{1cm} (34)

Substituting this into the two-point function (29) and three-point function (30) for the gauge fields $A$, we obtain the two-point and three-point functions for the “expansion coefficients” $J_m^i(\bar{z})$. For the two-point function, we find [3]

$$\langle J_m^i(\bar{x})J_n^j(\bar{y}) \rangle = \frac{\tilde{K}_{mn}^{ij}}{(\bar{x} - \bar{y})^2},$$  \hspace{1cm} (35)

where the $\tilde{K}_{mn}^{ij}$ are given by

$$\tilde{K}_{mn}^{ij} = (-1)^{i+1+m} \tilde{c}_i(i + 1 + m)!(i + 1 - m)! \delta^{ij} \delta_{m+n,0}.$$  \hspace{1cm} (36)
After some algebra, we find that the three-point function for $J^i_m$ can be written as [3]

$$
\langle J^i_m(\bar{x})J^j_n(\bar{y})J^k_p(\bar{z}) \rangle = \frac{\bar{Q}^{ijk}_{mnp}}{(\bar{x} - \bar{y})(\bar{y} - \bar{z})(\bar{z} - \bar{x})},
$$

(37)

when $i + j + k$ is even, and zero when $i + j + k$ is odd. The coefficients $\bar{Q}^{ijk}_{mnp}$ are given by

$$
\bar{Q}^{ijk}_{mnp} = \delta_{m+n+p,0} \times \sum_{q \geq 0} \frac{(i+1+m)!(j+1-n)!(i+1-n)!(k+1+p)!(k+1-p)!\bar{P}_{ijk}(-)^j+1-m+p+q}{(j-k-m+q)!(i+1+m-q)!(j+i+p+q)!(k+1-p-q)!(k+i-j+1-q)!q!}
$$

(38)

By analogy with the pure-gravity case described in section 2, we should expect that $\tilde{K}^{ij}_{mn}$ in the two-point function (35) and $\tilde{Q}^{ijk}_{mnp}$ in the three-point function (37) should be related to the Cartan-Killing metric and structure constants of the Lie algebra from some underlying Kac-Moody symmetry of $W_{1+\infty}$ gravity. This indeed turns out to be the case; the Lie algebra in question here is $GL(\infty, R)$ [3]. It can in fact be described as the algebra generated by the “wedge” of $W_{1+\infty}$ generators $\tilde{V}^i_m$ [4](where $m$ is the Laurent-mode index in the expansion of the spin-$(i + 2)$ $W_{1+\infty}$ current $V^i(z)$), with $m$ is restricted to lie in the range

$$
-(i + 1) \leq m \leq (i + 1), \quad i \geq -1.
$$

(39)

Thus we may think of the “expansion coefficients” $J^i_m(\bar{z})$ in (34) as being $GL(\infty, R)$ Kac-Moody currents, with the underlying Lie algebra being described by the subset of $W_{1+\infty}$ generators $X^i_m = \tilde{V}^i_m$ specified by (39).

Because $GL(\infty, R)$ is infinite dimensional some care has to be taken when defining the Cartan-Killing metric, since a naive definition such as $\text{tr}(X^i_mX^j_n)$ would diverge when the trace over infinite-dimensional matrices is taken. The way around this is to observe that the trace operation is simply a procedure for projecting onto the singlet term in the product of the two adjoint representations generated by $X^i_m$ and $X^j_n$. Thus if we can find a way of recognising the singlet term without taking the trace then we can read off the Cartan-Killing metric as the coefficient of the singlet, thereby avoiding the divergent coefficient that it would acquire if we were to take the trace. There is indeed a simple way to do this for the $GL(\infty, R)$ algebra described by the subset of $W_{1+\infty}$ generators $\tilde{V}^i_m$ specified by (39) [3]. The method exploits the fact that one can define a product operation for the generators of $W_{1+\infty}$. (This “lone-star” product $\tilde{V}^i_m \star \tilde{V}^j_n$ is associative, which implies that it defines a Lie bracket $[\tilde{V}^i_m, \tilde{V}^j_n] = \tilde{V}^i_m \star \tilde{V}^j_n - \tilde{V}^j_n \star \tilde{V}^i_m$ that automatically satisfies the Jacobi identity.)

The zero-mode of the spin-1 current, $V^0_1$, turns out to commute with every generator in $W_{1+\infty}$. Thus it behaves like the identity operator, and is a singlet under $GL(\infty, R)$. The Cartan-Killing metric for $GL(\infty, R)$ may therefore be defined as the coefficient of $X^i_0$ in the lone-star product $X^i_m \star X^j_n$ of $GL(\infty, R)$ generators.
As shown in [3], the Cartan-Killing metric \( \tilde{\eta}_{mn} \) for \( GL(\infty, R) \), defined by the procedure described above, is related to the coefficients \( \tilde{K}_{mn} \) appearing in (35) by \( \tilde{K}_{mn} = \frac{1}{\tilde{\eta}} \eta_{mn} \). After some algebra, one can show by substituting the expansion (34) into the recursion relation (27) that the coefficients \( J^i_{m}(z) \) satisfy the recursion relation [3]

\[
\langle J^i_n(z) J^j_{m_1} (\bar{x}_1) \cdots J^j_{m_N} (\bar{x}_N) \rangle = \frac{1}{8^r} \sum \frac{\tilde{\eta}_{ijr}}{(z - x_r)^2} \langle J^j_{m_1} (\bar{x}_1) \cdots J^j_{m_r} (\bar{x}_r) \cdots J^j_{m_N} (\bar{x}_N) \rangle \\
- \sum_k \sum_{m_{-k}} \sum_{k=0}^{k+1} \frac{f_{ijk} m_k}{(z - x_r)} \langle J^j_{m_k} (\bar{x}_r) J^j_{m_1} (\bar{x}_1) \cdots J^j_{m_r} (\bar{x}_r) \cdots J^j_{m_N} (\bar{x}_N) \rangle,
\]

(40)

where \( f_{ijk} \) are the structure constants of \( GL(\infty, R) \), read off from the wedge subalgebra of \( W_{1+\infty} \). Equation (40) is precisely the recursion relation for the Kac-Moody currents from a \( GL(\infty, R) \) WZW model. Thus, we have established that \( W_{1+\infty} \) gravity has an underlying \( GL(\infty, R) \) Kac-Moody symmetry.

4. Conclusions

We have seen in the previous section that Polyakov’s result for the underlying \( SL(2, R) \) Kac-Moody symmetry of two-dimensional gravity generalises to a \( GL(\infty, R) \) symmetry for \( W_{1+\infty} \) gravity. The case of \( W_\infty \) gravity can be handled in a very similar way. There is one further complication here, resulting from the fact that the corresponding “lone-star” product for \( W_\infty \) does not generate a spin-1 term on the right-hand side, and so one cannot identify the Cartan-Killing metric for the wedge subalgebra (which is \( SL(\infty, R) \) in this case) as the coefficient of \( V_0^{-1} \). The solution to this problem is to view \( W_\infty \) as a special case of a one-parameter family of parametrisations of the \( W_{1+\infty} \) algebra [3,4]. The \( W_\infty \) algebra, for which the spin-1 generator can be truncated from the \( W_{1+\infty} \) algebra, can then be approached via a limiting procedure, in which one rescales generators so as to retain a non-zero coefficient for \( V_0^{-1} \) in the \( W_\infty \) limit. The details are described in [3]. The conclusion is that \( W_\infty \) gravity has an underlying \( SL(\infty, R) \) Kac-Moody symmetry.

In the work of [2], the \( SL(2, R) \) Kac-Moody symmetry was exploited for studying aspects of two-dimensional quantum gravity. In particular, the energy-momentum tensor for the gauge field \( h \), which must be added to the Lagrangian to preserve general covariance, was expressed in terms of the \( SL(2, R) \) Kac-Moody currents by using the Sugawara construction. In principle, a similar procedure should be possible for \( W_\infty \) or \( W_{1+\infty} \) gravity. Indeed a generalisation of the Sugawara construction, known as Casimir algebras, has been given for \( W \)-extended conformal algebras [6]. Essentially, one builds the higher-spin currents by using symmetric invariant tensors of the Lie algebra underlying the \( W \) algebra. For example, \( W_N \) can be viewed as the Casimir algebra of \( SU(N) \), and the higher-spin currents can be built in terms of the \( SU(N) \) Kac-Moody currents \( J^A(z) \) by using the invariant \( d \) tensors of \( SU(N) \).
Thus we have $T = \eta_{AB} : J^A J^B :, W = d_{ABC} : J^A J^B J^C :, \text{ etc.}$ The procedure described in section 3 for extracting the Cartan-Killing metric for $GL(\infty, R)$ can be straightforwardly extended to obtain its arbitrary-rank symmetric $d$ tensors. The limiting procedure described above can be used to obtain the analogous tensors for $SL(\infty, R)$. Thus in principle it should be possible to repeat the steps of [2], and express the energy-momentum tensor for the gauge fields of $W_\infty$ or $W_{1+\infty}$ gravity in terms of the corresponding $SL(\infty, R)$ or $GL(\infty, R)$ Kac-Moody currents. The main outstanding problem seems to be that one obtains divergent results that would presumably have to be regularised in some way [3]; there are indications from other considerations that such a regularisation ought to be possible [7]. It may be that achieving a better understanding of how to do this will require finding a higher-dimensional interpretation for the infinite set of higher-spin currents of $W_\infty$.

Acknowledgments

I am very grateful to my collaborators in the work described in this review, namely Shawn Shen, Kaiwen Xu and Kajia Yuan, and to the organisers of the Trieste Summer School in High-Energy Physics for hospitality.

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