Acceleration of Relativistic Electron Beam
Trapped in Extraordinary Beat Wave

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Abstract. Relativistic electron beam acceleration due to the extraordinary beat wave induced by nonlinear electron Landau and cyclotron damping of electromagnetic waves is investigated theoretically on the basis of the relativistic equations of motion for beam electrons trapped in the beat wave. The equations of motion in the moving frame of reference with the velocity of the electron beam were analyzed analytically, where the relation between the moving and laboratory frames is given by the Lorentz transformation.

1. Introduction
Relativistic electron beam acceleration due to the extraordinary beat wave induced by nonlinear Landau and cyclotron damping of the electromagnetic waves in a magnetized plasma is investigated theoretically on the basis of the relativistic equations of motion for beam electrons trapped in the beat wave [1-3]. In order to investigate the highly relativistic electron beam, the equations of motion in the moving frame with the velocity of the electron beam \( v_b = (0,0,v_b) \) were analyzed analytically, where the relation between the moving and laboratory frames is given by the Lorentz transformation [4]. The magnitude of the perpendicular components of the electric and magnetic fields in the moving frame increases about \( \beta \) times that in the laboratory frame, where \( \beta = (1 - v_b^2/c^2)^{-1/2} \) and \( c \) is the light speed. The beat waves are excited by nonlinear electron Landau and cyclotron damping of the two electromagnetic waves and trap the beam electrons, satisfying the resonance condition in the moving frame:

\[
\omega_b - \omega_b' = m \omega_{ce}^l
\]

where \( \omega_{ce}^l = eB_0/\gamma m_c c \) is the relativistic electron cyclotron frequency for beam electrons in the moving frame, and \( \omega_{b}' = \omega_{b} - \omega_{b} \) and \( \omega_{b}' = \omega_{b} - \omega_{b}' \) are the wave frequency and wave vector of the beat wave, respectively. When \( m = 0 \), the trapped beam electrons are accelerated or decelerated by the parallel electric field of the beat wave. When \( m = 1 \), the trapped beam electrons are accelerated or decelerated by the Lorentz force arising from the perpendicular magnetic field of the beat wave. It is proved that the acceleration rate in the laboratory frame increases approximately in proportion to \( \beta \). The detailed acceleration mechanism was clarified and the qualitative agreement with the numerical results of the perturbation theory was obtained.
2. Lorentz transformation
The Lorentz transformation of the laboratory frame of reference \((x, y, z, t)\) to the moving frame of reference \((x', y', z', t')\) is represented as
\[
x' = x, \quad y' = y, \quad z' = \beta(x - v_y t), \quad t' = \beta(t - v_y z/c^2) \tag{2}
\]
\[
\tilde{\omega}_k = \beta(\omega_k - k_y v_y), \quad \tilde{k}_\perp = k_\perp \quad \tilde{k}_y = \beta(k_y - \omega_k v_y/c^2) \tag{3}
\]
where \(\omega_k\) and \(k = (k_x, k_y, k_z)\) are the wave frequency and wave vector in the laboratory frame, respectively, and \(\tilde{\omega}_k\) and \(\tilde{k} = (\tilde{k}_x, \tilde{k}_y, \tilde{k}_z)\) are the wave frequency and wave vector in the moving frame, respectively. The electric and magnetic fields of the beat wave in the moving frame are also given by the Lorentz transformation [4] and are expressed as follows:
\[
\tilde{E}^{(2)}_{k} = E^{(2)}_{ik} + \beta \left(1 - \frac{k_y^2 v_y}{\omega_k}\right) E^{(2)}_{ik} + \frac{\beta v_y}{\omega_k} k_\perp^2 E^{(2)}_{ik} \tag{4}
\]
\[
\tilde{B}^{(2)}_{k} = B^{(2)}_{ik} + \beta B^{(2)}_{ik} - \frac{\beta}{c} v_y E^{(2)}_{ik} \tag{5}
\]
Here, \(E^{(2)}_{k}\) and \(B^{(2)}_{k}\) refer to the laboratory frame, \(\tilde{E}^{(2)}_{k}\) and \(\tilde{B}^{(2)}_{k}\) refer to the moving frame. It is found from the above equations that the magnitude of the perpendicular components of the electric and magnetic fields in the moving frame increases about \(\beta\) times that in the laboratory frame.

3. Equations of motion for beam electrons trapped in the beat wave
3.1. Equations of motion
The trapped beam electrons in the moving frame are governed by the following equation of motion:
\[
\frac{dp'}{dt'} = -e\tilde{E}^{(2)}_{k} - \frac{e}{\gamma' m c} p' \times \left(B_0 + \tilde{B}^{(2)}_{k}\right) \tag{6}
\]
where \(p' = p'_0 + p'^{(2)}\), \(\gamma' = \left(1 + p'^2/m_c^2 c^2\right)^{1/2} = \left(1 - v'^2/c^2\right)^{-1/2}\), and \(\frac{dp'_0}{dt'} = -\frac{e}{\gamma' m c} p'_0 \times B_0\). Then, equation (6) becomes
\[
\frac{dp'^{(2)}}{dt'} = -e\tilde{E}^{(2)}_{k} - \frac{e}{\gamma' m c} p'_0 \times \tilde{B}^{(2)}_{k} - \frac{e}{\gamma' m c} p'^{(2)} \times B_0 - \frac{e}{\gamma' m c} p'^{(2)} \times \tilde{B}^{(2)}_{k} \tag{7}
\]
It is considered that \(\tilde{E}^{(2)}_{k}, \tilde{B}^{(2)}_{k}\) \(\propto \exp\left[i\left(\tilde{k}_n z' - \tilde{k}_y t'\right)\right], p'_0 \propto \exp\left(i\omega'_0 t'\right), p'_0 = \gamma' m v'_0\), and
\[
p'^{(2)} = R_0 \exp\left[i\left(\tilde{k}_n z' - \tilde{k}_y t'\right)\right] + R_0 \exp\left[i\left(\tilde{k}_n z' - \tilde{k}_y t' + \omega'_v t'\right)\right]. \tag{8}
\]
Here, the dependence of \(\tilde{E}^{(2)}_{k}, \tilde{B}^{(2)}_{k}\) on \(x'\) and \(y'\) is not taken into account for the simplified analysis.

3.2. Nonlinear electron Landau damping
In the case of \(m = 0\), the resonance condition of nonlinear electron Landau damping of \(\tilde{\omega}_k, -\tilde{k}_y v'_0 = 0\) is satisfied. Considering that \(z' = v'_y t' + z'_1\), the parallel component of equation (7) that is proportional to \(\sin\left(\tilde{k}_n z'_1 + \phi\right) = \sin\left(\tilde{k}_n z'_1 + \phi\right)\) becomes
\[
\frac{dp'^{(2)}}{dt'} = -eQ_0 \sin\left(\tilde{k}_n z'_1 + \phi\right) \tag{8}
\]
where $Q_0 = |E_{kz}^{(2)}|$. This equation shows that the trapped beam electrons are accelerated or decelerated by the parallel electric field of the beat wave.

### 3.3. Nonlinear electron cyclotron damping

In the case of $m = 1$, $\tilde{k}_{nz} - \omega_k t = \tilde{k}_{nz} - \omega_k t'$ holds owing to the resonance condition of nonlinear electron cyclotron damping of $\omega_k - \tilde{k}_{nz} = \omega_k$. Then, $p^{(n)}$ becomes $p^{(n)} = R \exp[i(\tilde{k}_{nz} - \omega_k t')] + R \exp(\tilde{k}_{nz} t')$. Consequently, the parallel component that is proportional to $\sin(\tilde{k}_{nz} t' + \phi')$ is expressed as

$$\frac{dp^{(n)}_{\parallel}}{dt'} = -eQ_1 \sin(\tilde{k}_{nz} t' + \phi')$$

where $e' = \frac{e}{\gamma' m' c}$ and $Q_1 = \sin(\tilde{k}_{nz} t' + \phi')$, and $Q_1$ is expressed approximately as $Q_1 = -\beta |E_{kz}^{(2)}|$. This equation shows that the trapped beam electrons are accelerated or decelerated by the Lorentz force arising from the perpendicular magnetic fields of the beat wave.

### 3.4. Trapping frequency

For the nonrelativistic motion of the trapped beam electrons in the moving frame, $(\gamma' = 1, \ p_i^{(n)} = \gamma' m_i v_i^{(n)} = m_i dz_i / dt', \ |\tilde{k}_{nz}| \ll 1)$, the trapping frequency of the trapped beam electrons in the moving frame can be given by $\omega_b = \left(\frac{eQ_0 \tilde{k}_z}{m_e}\right)^{1/2}$. On the basis of equation (2), $t' = t / \beta$ can be found to hold approximately. Then the trapping frequency of the trapped beam electrons in the laboratory frame becomes

$$\omega_b = \frac{\omega_b}{\beta} = \left(\frac{eQ_0 m_{\parallel}^2}{m_e \beta^2}\right)^{1/2}.$$ (10)

### 4. Solution for relativistic trapped beam electrons

Next we derive the analytical and approximate solution for the relativistic motion of the trapped beam electrons in the moving frame. In the case of $\gamma' > 1$ and $p_i^{(n)} = p_i^{(n)}$, equations (8) and (9) can be rewritten as $m_i c^2 \frac{d\gamma'}{dt'} = eQ_0 \frac{d \cos(\tilde{k}_{nz} t')}{dt'}$, where $\gamma' = \left[1 - (v_i^{(n)}/c)^2\right]^{-1/2}$, $\phi' = 0$ and $m = 0, 1$.

Thus the solution of the above equation can be obtained as follows:

$$\gamma' = -\frac{2eQ_0}{\tilde{k}_z m'^2 c^2} \left(\sin^2 \frac{1}{2} \tilde{k}_{nz} t' - \sin \frac{1}{2} \delta_m\right) + 1$$ (11)

where it is assumed that $\gamma' m_i c^2 = m_i c^2, \ |\tilde{k}_z| \ll \delta_m < 0$ at $t' = 0$, thereby the trapped beam electrons are accelerated and attain the maximum energy in the moving frame given by

$$\gamma' m_i c^2 = \frac{2eQ_0}{\tilde{k}_z} \sin \frac{1}{2} \delta_m + m_i c^2$$ (12)

at $z' = 0$. In the case of $\gamma' = 1$, it can be proved that $z' = 0$ at $t' = \delta_m / \omega_k$. From equations (2), (3), (10) and (12), the maximum relativistic energy and the trapping frequency in the laboratory frame can be obtained approximately as
\[ \beta y' m_c^2 = A_\beta \beta^2 m_c^2 \left( U_\kappa / n_m c^2 \right)^{1/2} \sin^2 \frac{1}{2} \delta_m + \beta m_c^2, \quad \omega_b = \omega_{\beta'/\beta} = B_{\beta'} \omega_{\beta'/\beta} \left( U_\kappa / n_m c^2 \right)^{1/4} \] (13)

where \( A_\beta = 2 \omega_{\beta'/\beta}, \quad A_\beta = 2 \omega_{\beta'/\beta}, \quad B_{\beta'} = \left( \omega_{\beta'/\beta} \beta' \right)^{1/2}, \quad B_\beta = \left( \omega_{\beta'/\beta} \beta \right)^{1/2}, \quad U_\kappa = \left| E_\kappa^{(2)} \right|^2 / 4\pi \)

is the energy density of the beat wave, \( \omega_{\beta'/\beta} = e B_{\beta'}/m_c \) is the nonrelativistic electron cyclotron frequency, and \( \omega_{\beta'/\beta} = \left( 4 \pi n_e c^2 / m_e \right)^{1/2} \) is the electron plasma frequency of the beam electrons. Further, since it can be proved that \( Q_m / k^* \propto \beta \) holds approximately, \( d \left( \beta y' m_c^2 \right) / dt = \left( e Q_m / k^* \right) d \cos \left( k z^* \right) / dt' \propto \beta \) is proved. Then it is shown that the acceleration rate in the laboratory frame increases approximately in proportion to \( \beta \).

We consider the actual system where the relativistic electron beam is injected axially into the magnetized plasma and the intense CO\(_2\) laser beam is launched. The acceleration quantities in the laboratory frame are deduced on the basis of equation (13) and shown in Table 1. They are estimated for the conditions of \( B_{\beta'} = 10 \text{T} \), \( m = 0 \), \( r_e = 0.17 \text{mm} \) (for the cases a, b, c), \( B_{\beta'} = 5 \text{T}, m = 1 \), \( r_e = 0.34 \text{mm} \) (for the cases d, e, f), \( \lambda_0 = 2 \pi c / \omega_0 = 9.92 \mu \text{m} \), \( \lambda_0 = 2 \pi c / \omega_0 = 10.7 \mu \text{m} \), \( \omega_{\beta'/\beta} / \omega_{\beta'/\beta} = 0.01 \), \( U_\kappa = 10^6 U_A \), and the wave energy density of the first electromagnetic wave \( U_k \) is set such that \( U_k = \beta n_m c^2 \), where \( r_e = v_t / \omega_{\beta'/\beta} \) is the Larmor radius for \( v_t = c \) and is less than the diameter of the laser beam \( d_1 = 1 \text{mm} \) [5]. The trapping frequency \( \omega_{\beta'/\beta} / 2\pi \), the acceleration time \( \tau_\alpha = \beta y' \Delta = \beta \delta_m / \omega_{\beta'/\beta} = \delta_m / \omega_{\beta'/\beta} \), the acceleration length \( z_\alpha = \tau_\alpha v_t = \tau_\alpha \delta_e / \delta_k = \tau_\alpha c k / k \), the energy gain \( \Delta W_e = \Delta (\beta m c^2) = \beta (y' - 1) m c^2 \), the acceleration gradient \( \Delta (\Delta W_e) / z_\alpha \) are estimated for the various values of \( \beta \) and the properly chosen values of the phase \( \delta_m \). The magnitude of the wave electric field \( |E_k| \) and the absolute intensity of the wave power \( U_k v_t \) are \( 3 \times 10^6 \text{V/m}, 96 \text{GW/cm}^2 \) (a), \( 9.5 \times 10^6 \text{V/m}, 1 \text{TW/cm}^2 \) (b), \( 9.5 \times 10^6 \text{V/m}, 96 \text{TW/cm}^2 \) (c), \( 3 \times 10^9 \text{V/m}, 10 \text{TW/cm}^2 \) (d), \( 6.7 \times 10^9 \text{V/m}, 48 \text{TW/cm}^2 \) (e), \( 2.1 \times 10^{11} \text{V/m}, 0.5 \text{PW/cm}^2 \) (f), respectively.

**Table 1.** The quantities for the acceleration by means of CO\(_2\) laser injection are estimated for the various values of \( \beta \).

| Cases | \( \beta \) | \( W_e \) | \( \omega_{\beta'/\beta} / 2\pi \) | \( \delta_m \) | \( \tau_\alpha \) | \( z_\alpha \) | \( \Delta W_e \) | \( \Delta (\Delta W_e) / z_\alpha \) |
|-------|-------------|----------|-----------------|-------------|-------------|-------------|-------------|-----------------|
| a     | 100         | 51MeV   | 7.9MHz          | 0.5\pi     | 32ns        | 0.38m       | 64KeV       | 170KeV/m       |
| B     | 1000        | 510MeV  | 450KHz          | 0.5\pi     | 0.56\mu s   | 6.7m        | 2.0MeV      | 300KeV/m       |
| C     | \( 10^5 \)  | 51GeV   | 1.4KHz          | 5.0 \times 10^{-2} | 5.6\mu s | 68m       | 2.5GeV      | 37MeV/m       |
| D     | \( 4 \times 10^4 \) | 20GeV | 31KHz | 4.0 \times 10^{-2} | 0.2ns | 2.5 \times 10^{-3} m | 1.3GeV | 530GeV/m      |
| E     | \( 2 \times 10^8 \) | 100GeV | 21MHz | 1.0 \times 10^{-2} | 76ps | 9.1 \times 10^{-4} m | 4.6GeV | 5.0TeV/m      |
| F     | \( 2 \times 10^6 \) | 1TeV   | 12MHz | 1.7 \times 10^{-3} | 23ps | 2.8 \times 10^{-4} m | 42GeV | 150TeV/m     |

**5. Conclusion**

It is verified theoretically from the relativistic equation of motion for the trapped beam electrons that the highly relativistic electron beam can be accelerated by the extraordinary beat wave induced by the intense laser beam. The authors wish to thank Professor Y. Kitagawa for useful discussions.

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