A Komar-like integral for mass and angular momentum of asymptotically AdS black holes in Einstein gravity

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\textbf{Abstract}

The purpose of this paper is to enhance the conventional Komar integral to asymptotically anti-de Sitter (AdS) black holes. In order to do so, we first obtain a potential that is the linear combination of the usual Komar potential with two third-order derivative terms generated by the action of the d’Alembertian operator and the exterior derivative upon a Killing vector. Then this higher-order corrected potential is extended to the Einstein gravity with a negative cosmological constant, yielding the potential that is the linear combination of the usual Komar one with it acted on by the d’Alembertian. The surface integral of the improved Komar potential can serve as a formula for conserved charges of asymptotically AdS spacetimes. Finally, we make use of such a formula to compute the mass and the angular momentum of Schwarzschild-AdS black holes, regular AdS black holes, asymptotically AdS Kerr-Sen black holes, Kerr-NUT-AdS black holes, and Kerr-AdS black holes in arbitrary dimensions. The results coincide with the ones in the literature.

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1 Introduction

There has been a great deal of interest in the asymptotically anti-de Sitter (AdS) black holes over the past several decades, particularly since the discovery of the well-known anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1]. Up to now, rotating black hole solutions with a cosmological constant have been found in different dimensions within the framework of the Einstein gravity theory. All of them satisfy exactly the vacuum Einstein field equation in the presence of the cosmological constant. In 1968, Carter first presented a generalization of the four-dimensional rotating Kerr black hole by including the cosmological constant [2]. Since this black hole possesses asymptotically de Sitter (dS) or AdS boundary conditions, it is usually referred to as Kerr-dS or Kerr-AdS black hole in the literature. After this, by introducing a NUT charge parameter and acceleration in the Kerr-AdS metric, the four-dimensional accelerating Kerr-NUT-AdS black hole solution was found in [3] [4]. Many years later, Hawking, Hunter and Taylor-Robinson found the five-dimensional generalization of the four-dimensional Kerr-(A)dS black hole, as well as the solutions with just one nonzero angular momentum parameter in all dimensions [5]. Subsequently, Gibbons, Lü, Page and Pope further constructed the general Kerr-(A)dS black holes with arbitrary angular momenta in all higher dimensions [6] [7], which can be regarded as the extensions of the Ricci-flat rotating Myers-Perry black holes [8] in the appearance of the cosmological constant. Soon afterwards the higher-dimensional NUT charge generalizations for the Kerr-(A)dS black holes were obtained in [9].

For the sake of understanding the asymptotically AdS black holes or the spacetimes with other asymptotical structures, in particular, their first law of thermodynamics together with other thermodynamic properties of relevance to energy and angular momentum, a question desired to address is to seek proper definitions for the conserved charges. Till now, some research has been devoted to this question from different perspectives, and a lot of progress has already been made, enriching our understanding on the conserved charges of gravity theories. For the asymptotically AdS spacetimes, there have existed a number of methods that can be adopted to define their conserved charges in the literature, for example, the well-known covariant phase space approach [10] [11] [12], the KBL superpotential method [13] [14], the conformal definition of conserved charges proposed by Ashtekar, Magnon and Das (AMD) [15] [16], the Brown-York approach [17], the counterterm method [18] [19], the Abbott-Deser-Tekin (ADT) formalism [20] [21] [22] [23] together with its off-shell generaliza-
tion \cite{24}, the topological regularization method \cite{25, 26, 27}, the Barnich-Brandt-Compere (BBC) formalism \cite{28, 29, 30, 31}, the field-theoretic approach \cite{32, 33, 34}, and the method proposed in \cite{35, 36}.

By contrast with the aforementioned methods, the so-called Komar integral, which is a surface integral with respect to a 2-form potential made up of the first-order derivative of a Killing vector field \cite{37}, may be thought of as the simplest formalism for conserved charges. Thanks to such a merit, it is of great convenience to use the Komar integral to evaluate the mass and the angular momentum of various solutions of gravity theories, particularly for the ones with asymptotically flat structure. Naturally, one expects that the simple formulation is also applicable to asymptotically AdS spacetimes. Nevertheless, when the Komar integral is applied to compute the mass of the asymptotically AdS black holes (for instance, see the calculations of the mass for the static and spherically symmetric Schwarzschild-AdS black holes in Subsection 3.1), unfortunately, it breaks down, arising from that the result turns out to be infinite. This implies that there must exist divergent terms at infinity. In this sense, the regularization of such terms requires at least that the usual Komar integral has to be modified to keep on its success in the asymptotically AdS spacetimes. What is more, although the original Komar integral can produce the physically meaningful mass together with the angular momentum of the asymptotically flat rotating black holes, such as the Myers-Perry black holes in all dimensions, the expression with respect to the mass differs from that for the angular momentum by a factor two in form \cite{38}. Therefore, to obtain a unified formulation for both the energy and the angular momentum, the original Komar integral should be improved.

In the present paper, we pursue the goal of providing a natural generalization of the conventional Komar integral so that it can work for the conserved charges of various spacetimes with an AdS asymptotic in the framework of the Einstein gravity theory. In order to do so, a proper modification to the ordinary Komar potential \cite{37} has to be made. Inspired by the expression of the Komar potential in differential forms, being the exterior derivative of a 1-form Killing vector field, a good candidate for the generalization of the Komar potential may be the one encompassing the higher-order derivatives of the Killing vector. Fortunately, we find that the approach to generate conserved currents associated with an arbitrary vector field put forward in the works \cite{39, 40} can assist us to accomplish the higher-order corrections to the Komar potential. The relevant idea behind these works goes as follows. If both the current and the vector field are treated as 1-forms, by letting the
degree-preserving d’Alembertian operation together with the codifferential and the exterior derivative (both of them have to be in pair) act on the 1-form vector field, one is able to obtain a series of 1-forms being the even-order derivatives of the vector field. By means of the linear combination of all the 1-forms, then conserved currents associated with the vector can be constructed.

Within what follows, on basis of the strategy of generating conserved currents in [39, 40], we obtain an identically conserved current consisting of the second- and fourth-order derivatives of an arbitrary Killing vector field. In terms of this current, a potential is immediately read off, which is the linear combination of the usual Komar potential with two third-order derivative terms generated by the action of the d’Alembertian and the exterior derivative on the 1-form Killing vector. Due to the structure of the potential, it can be regarded as the generalization of the usual Komar potential with higher-order corrections. Accordingly, we refer to the surface integral of this potential as a Komar-like integral. Furthermore, the generalized Komar potential is extended to the Einstein gravity theory endowed with the Einstein-Hilbert-Λ Lagrangian, yielding a simplified and specific potential. The surface integral with respect to the Komar-like potential gives rise to a unified definition for the energy and the angular momentum of asymptotically AdS black holes. As applications, we compute the mass and the angular momentum of the Schwarzschild-AdS black holes in arbitrary dimensions, the four-dimensional regular AdS black holes [57], the four-dimensional asymptotically AdS Kerr-Sen black holes [58], the Kerr-NUT-AdS black holes [3, 4, 9], and the Kerr-AdS black holes in arbitrary dimensions [6, 7].

The plan of the present paper is as follows. In section 2 we make use of the d’Alembertian operator, the exterior derivative and the co-exterior derivative to act on an arbitrary Killing vector field to produce a conserved current with the linear combination of no more than fourth-order derivative terms of this vector. Subsequently, by specifying the current to Einstein gravity in the presence of a negative cosmological constant, we obtain a generalized Komar potential by means of the equation of motion and the properties of Killing vectors. The surface integral of the potential yields a formula of conserved charges for asymptotically AdS spacetimes. Section 3 is devoted to the applications of the formula for conserved charges into the calculations for the mass and the angular momentum of black holes with an AdS asymptotic. The last section is our conclusions. In Appendix A to see more clearly the contributions from the higher-order corrections to potentials, the inclusions of more than third-order derivatives of the Killing vector in the potentials are investigated.
2 General formalism

In the present section, by following the work [40], which provides a way of constructing conserved currents starting from the action of differential operators on an arbitrary vector field, we manage to find out a generalized Komar potential for solutions with asymptotically AdS structure in the context of D-dimensional Einstein gravity. Then the surface integral with respect to the generalized potential will be adopted to define the conserved charges such as the mass and the angular momentum. According to the work [40], the differential operators \( \text{d} \) and \( \hat{\delta} \), together with \( \Box = \nabla^\mu \nabla_\mu \), participate in the construction of the conserved current corresponding to the 1-form vector field. For convenience, here we present their definitions through the action upon an arbitrary \( p \)-form \( F = (p!)^{-1} F_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \), under the metric signature \((- , + , \cdots , +)\). The exterior derivative \( \text{d} \) and the degree-preserving d’Alembertian \( \Box \) are defined in components through \( (\text{d} F)_{\mu_0 \cdots \mu_p} = (p + 1) \nabla_{[\mu_0} F_{\mu_1 \cdots \mu_p]} \) and \( (\Box F)_{\mu_1 \cdots \mu_p} = \nabla^\rho \nabla_\rho F_{\mu_1 \cdots \mu_p} \), respectively, while the co-exterior derivative \( \hat{\delta} \) is defined in terms of the combination of the Hodge dual \( \ast \) and the exterior derivative, taking the form \( \hat{\delta} = (-1)^{pD+D+1} \ast \text{d} \ast \). Here and below the Hodge dual acts on the \( p \)-form \( F \) as \( (\ast F)_{\nu_1 \cdots \nu_D} = (p!)^{-1} F^{\nu_1 \cdots \nu_p} \epsilon_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{D-p}} \) with the completely skew-symmetric Levi-Civita tensor given by \( \epsilon_{\mu_0 \cdots \mu_{D-1}} = \sqrt{-g} D! \delta^0_{[\mu_0} \cdots \delta^{D-1}_{\mu_{D-1}]} \). According to the definitions, one directly obtains the identities \( \text{d}^2 = 0 = \hat{\delta}^2 \). More details on \( \text{d}, \hat{\delta} \) and \( \Box \) can be found in [39] and references therein. Furthermore, on basis of the relationship between the generalized Komar potential and the potential given in [44, 45], we shall propose a suitable substitute for the Killing potential.

We start with the vacuum Einstein gravity theory in \( D \) dimensions, equipped with the Einstein-Hilbert-\( \Lambda \) Lagrangian of the form

\[
\mathcal{L}_{EH} = \sqrt{-g} (R - 2\Lambda) .
\]  

(2.1)

The variation of the Lagrangian (2.1) with respect to the metric tensor \( g^{\mu\nu} \) gives rise to the gravitational field equation

\[
R_{\mu\nu} = - (D - 1) \ell^2 g_{\mu\nu} ,
\]

(2.2)

where \( \ell^{-1} \) stands for the radius of curvature for the maximally symmetric AdS spaces (for the metric of the \( D \)-dimensional AdS space see Eq. (3.6) without the \( m \) parameter in Subsection 3.1), related to the cosmological constant \( \Lambda \) in the manner

\[
\ell^2 = - \frac{2\Lambda}{(D - 1)(D - 2)} .
\]

(2.3)
In what follows, along the lines of the works \cite{39, 40}, we focus on giving a generalized Komar integral for the asymptotically AdS solutions of the field equation (2.2), with the Riemann curvature tensor at spatial infinity

\[ R_{\rho\sigma}^{\mu\nu} \big|_\infty = -2\ell^2 \delta_{[\rho}^{\mu} \delta_{\sigma]}^{\nu} \]. \tag{2.4} 

Within the work \cite{40}, it has been demonstrated that a conserved current with respect to a vector field can be constructed through the operation of the \((\Box, \hat{\delta}, d)\) operators, as well as their combinations, on such a vector if both the current and vector field are treated as 1-forms. This is attributed to the fact that the second-order derivative operations \(\Box, \hat{\delta}d\) and \(d\hat{\delta}\) are the three basic differential operators leaving the form degree of any differential form unchanged so that their action on the 1-form vector field yields a 1-form as well. Inspired with this strategy of constructing currents, for concreteness, we consider the conserved current \(J_{\text{gen}}^{(4th)}(\xi)\) associated with an arbitrary Killing vector \(\xi\), generated only by acting the \((\Box, \hat{\delta}, d)\) operators together with their combinations upon \(\xi\). Since \(J_{\text{gen}}^{(4th)}(\xi)\) is expected to be a natural and simple generalization of the usual Komar current \(J_{\text{Komar}} = -\hat{\delta}d\xi\) \cite{37}, it is reasonable to assume that \(J_{\text{gen}}^{(4th)}(\xi)\) is still linear in \(\xi\). Apart from this, when the Komar integral based on the current \(J_{\text{Komar}}\) is applied to compute the mass of asymptotically AdS black holes, it has been shown before that such an integral fails to yield a finite physical result because of the appearance of the divergent terms at spatial infinity, while the higher-order derivatives of the Killing vector have great potential to cancel out such divergent terms. Thus it is supposed that higher-order derivative terms of \(\xi\) are contained within \(J_{\text{gen}}^{(4th)}(\xi)\). However, the simplest and most natural higher-order derivative generalization to \(J_{\text{Komar}}\) is to merely bring fourth-order derivatives of \(\xi\) into \(J_{\text{gen}}^{(4th)}(\xi)\) (for discussions on this see Appendix A). Consequently, under the aforementioned assumptions, the current \(J_{\text{gen}}^{(4th)}(\xi)\), consisting of terms proportional to no more than fourth-order derivatives of \(\xi\), has the most general structure

\[ J_{\text{gen}}^{(4th)}(\xi) = J_{\text{gen}}^{(2th)}(\xi) + J_{\text{gen}}^{(4th)}(\xi), \tag{2.5} \]

where the second-order derivative 1-form \(J_{\text{gen}}^{(2th)}(\xi)\), which was utilized to reformulate the conventional Komar current in \cite{40}, is given by

\[ J_{\text{gen}}^{(2th)}(\xi) = \lambda_{11}\Box\xi + \lambda_{12}\hat{\delta}d\xi + \lambda_{13}d\hat{\delta}\xi, \tag{2.6} \]
while the 1-form $J_{\text{gen}}^{(4th)}(\xi)$ consisting of the fourth-order derivatives of the Killing vector reads as follows:

$$J_{\text{gen}}^{(4th)}(\xi) = \lambda_{21} \Box d\xi + \lambda_{22} d\Box d\xi + \lambda_{23} d\Box d\xi$$

$$+ \lambda_{24} \Box d\xi + \lambda_{25} d\Box d\xi + \lambda_{26} d\Box d\xi$$

$$+ \lambda_{27} d\Box d\xi + \lambda_{28} d\Box d\xi + \lambda_{29} d\Box d\xi.$$  \hfill (2.7)

In Eqs. (2.6) and (2.7), $\lambda_{1i}$’s and $\lambda_{2j}$’s are constant parameters, and the coderivative $\hat{\delta}$ has to be paired with the exterior derivative $d$ to ensure that the action of the operators preserves the form degree of the 1-form Killing vector field. Although $J_{\text{gen}}^{(4th)}(\xi)$ appears in a complicated form, by substituting the identities $\hat{\delta}\xi = 0$ and $\hat{\delta}d\xi = 2\Box \xi$ for Killing vectors into the above equation, we simplify it as

$$J_{\text{gen}}^{(4th)}(\xi) = \left( \frac{\lambda_{11}}{2} + \lambda_{12} \right) \hat{\delta}d\xi + (2\lambda_{22} + \lambda_{28}) \hat{\delta}d\Box d\xi$$

$$+ \lambda_{29} \hat{\delta}d\Box d\xi + (2\lambda_{21} + \lambda_{24}) d\Box d\xi.$$  \hfill (2.8)

To guarantee that $\hat{\delta}J_{\text{gen}}^{(4th)}(\xi) = (2\lambda_{21} + \lambda_{24}) \hat{\delta}d\Box d\xi = 0$ holds for an arbitrary Killing vector $\xi$, it is demanded that $\lambda_{24} = -2\lambda_{21}$. In such a setting, for convenience, the identically conserved current $J_{\text{gen}}^{(4th)}(\xi)$ is recast into the following form

$$J^{(4th)}(\xi) = k_{1} \hat{\delta}d\xi + 2k_{2} \hat{\delta}d\Box d\xi + k_{3} \hat{\delta}d\Box d\xi,$$  \hfill (2.9)

with arbitrary constant parameters $k_{i}$’s. As usual, in terms of the relation between the current and potential $J^{(4th)}(\xi) = -\hat{\delta}K^{(3th)}(\xi)$, the 2-form potential $K^{(3th)}(\xi)$ corresponding to the $J^{(4th)}(\xi)$ current is straightforwardly read off as

$$K^{(3th)}(\xi) = -k_{1} d\xi - 2k_{2} d\Box d\xi - k_{3} d\Box d\xi.$$  \hfill (2.10)

In contrast with the ordinary Komar potential $K_{\text{Komar}} = d\xi$ being the first-order derivative of the Killing vector [37], here the 2-form $K^{(3th)}(\xi)$ differs from $K_{\text{Komar}}$ by encompassing two additional third-order derivative terms $d\Box d\xi$ and $d\Box d\xi$. Such terms are generated through the action of the differential operators on the 1-form Killing vector field, which is in accordance with the spirit of constructing the original Komar potential. Naturally, $K^{(3th)}(\xi)$ can be regarded as the higher-order derivative generalization of the ordinary Komar potential.
Furthermore, by making use of the Weitzenböck identity [39], we arrive at

\[ K^{(3th)}(\xi) = -k_1 d\xi - 2(k_2 + k_3)d\Box\xi \]

\[ -2k_3 R_{\mu}^{\rho} \nabla_{\rho} \xi_{\nu} dx^{\mu} \wedge dx^{\nu} \]

\[ + k_3 R_{\mu\nu}^{\rho\sigma} \nabla_{\rho} \xi_{\sigma} dx^{\mu} \wedge dx^{\nu}. \]  

(2.11)

Here it should be emphasized that \( K^{(3th)}(\xi) \) in Eq. (2.11) is general and it is irrelevant to equations of motion for fields. The constant parameters \( k_1, k_2 \) and \( k_3 \) can be determined by specifying \( K^{(3th)}(\xi) \) to a gravity theory under consideration. All of them at least guarantee that the conserved charges defined in terms of the 2-form \( K^{(3th)}(\xi) \), such as the mass and the angular momentum, are convergent at infinity. This will be demonstrated in the following part of this section, as well as within Appendix A.

With the generic Komar-like potential \( K^{(3th)}(\xi) \) in hand, we concentrate on figuring out the potential with respect to the Einstein gravity characterized by Lagrangian (2.1). In the framework of this gravity theory, under the equation of motion (2.2), as well as the identity \( \Box\xi = (D - 1)\ell^2 \xi \) arising from the relationship \( \nabla_{\rho} \nabla_{\mu} \xi_{\nu} = -R_{\mu\rho\nu}^{\sigma} \xi^{\sigma} \) and the field equation (2.2), the potential \( K^{(3th)}(\xi) \) turns into

\[ K_{AdS}^{(3th)} = c_2 (d\xi + c_1 R_{\mu\nu}^{\rho\sigma} \nabla_{\rho} \xi_{\sigma} dx^{\mu} \wedge dx^{\nu}), \]

(2.12)

with \( c_2 = -[k_1 + 2(D - 1)k_2\ell^2] \) and \( c_1 = k_3/c_2 \). At this stage there are actually only two undetermined constant parameters. As what will be demonstrated in Appendix A, the \( c_1 \) parameter can be completely fixed by analyzing the asymptotical behaviour of the spacetimes. For asymptotically AdS spacetime metrics obeying the falloff condition (2.4), due to the fact that the determinant of the metric suffers from divergence at spatial infinity, the necessary but not the sufficient condition for the potential \( K_{AdS}^{(3th)} \) to satisfy is that it has to vanish at spatial infinity to guarantee that the conserved charges defined in terms of the surface integrals of \( K^{(3th)}_{AdS} \) are finite. As a consequence, with the help of the equation \( K_{AdS}^{(3th)} |_{\infty} = c_2 (1 - 2c_4\ell^2) d\xi \), such a condition gives rise to

\[ c_1 = \frac{1}{2\ell^2}. \]

(2.13)

Under the \( c_1 \) parameter given by Eq. (2.13), it will be shown in the next section that the divergent Komar integral for the asymptotically AdS black holes can be indeed regularized by the higher-order derivative term \( \Box d\xi \). Meanwhile, on the other hand, it seems not to be
crucial for the determination of the global factor $c_2$ attributed to that it can be incorporated into the formula for the conserved charges. Actually, when the surface integral in terms of $K_{AdS}^{(3th)}$ is adopted to calculate the conserved charges of black holes, for instance, the mass of Schwarzschild-AdS black holes, in order to cover smoothly the results in the absence of the cosmological constant or to reproduce the same mass as that via other standard methods for conserved quantities, such as the covariant phase space approach [10, 11, 12], the AMD formalism [15, 16], the ADT method [20, 21, 22, 23] and the BBC formalism [28, 29, 30, 31], for example, it will be confirmed in the following subsection (3.1) that the $c_2$ parameter is presented by

$$c_2 = \frac{1}{2(D - 3)}.$$  

Consequently, by substituting Eqs. (2.13) and (2.14) into Eq. (2.10), we obtain the generalized Komar potential $\mathcal{K}$, given by

$$\mathcal{K} = \frac{1}{2(D - 3)} \left( d\xi - \frac{\Box d\xi}{2\ell^2} \right),$$

which is the linear combination of the ordinary Komar potential $K_{Komar}$ with its second-order derivative yielded under the action of the degree-preserving d’Alembertian operation. Here the Komar-like potential $\mathcal{K}$ is just the natural generalization of $K_{Komar}$ that we desire to find in the present paper.

When the generalized Komar potential $\mathcal{K}$ is established, a formula for the conserved charges of the Einstein gravity theory admitting asymptotically AdS spacetime metrics can be proposed as the surface integral with respect to $\mathcal{K}$, being of the form

$$Q = \frac{1}{8\pi} \int_{\partial \Sigma} \star \mathcal{K}.$$  

As usual, in Eq. (2.16), $\partial \Sigma$ denotes the $(D - 2)$-dimensional $t = const$ surface at spatial infinity ($r = \infty$) if the conserved charge $Q$ represents the mass or the angular momentum, under the coordinate system $\{ t, r, x^i \}$ ($i = 1, 2, \cdots, D - 2$), where $r$ stands for the radial coordinate. However, when the spacetime metrics are asymptotically flat, the higher-order derivative term $\Box d\xi$ involved in the potential $\mathcal{K}$ can be excluded. Thus the surface integral (2.16) turns into the conventional Komar one with a vanishing cosmological constant. This is attributed to the fact that the $\Box d\xi$ term decreases fast enough so that it vanishes at spatial infinity.

1Throughout the present paper, we take into account the units in which $G = c = 1$.  

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In order to see the universality of the Komar-like potential (2.15), we move on to comparing it with the ones obtained by other approaches. First, in comparison with the superpotential $K_{\mu\nu}$ given by Eq. (35) in [41], here represented by the 2-form $K_{\mu\nu}$ in the framework of the Einstein gravity armed with the Lagrangian (2.1), we follow the work [41] (see [42, 43] as well) to compute $K_{\mu\nu}$ and find that

$$K_{\mu\nu} = \frac{c_2}{\ell^2} \left( R_{\rho\sigma}^{\mu\nu} - 4 R_{[\rho}^{\mu} \delta_{\sigma]}^{\nu} \right) \nabla^\rho \xi^\sigma + \frac{(D - 2) \ell^2 + 2 c_2 R}{2 \ell^2} \nabla^\mu \xi^\nu$$

$$= \frac{1}{D - 3} \left( \nabla [\mu \xi^\nu] + \frac{1}{2 \ell^2} R_{\rho\sigma}^{\mu\nu} \nabla^\rho \xi^\sigma \right) = K^{\mu\nu}.$$  (2.17)

It is worth noting that the equation of motion (2.2) for the field has been used in order to obtain the second equality in the above equation. Second, the potential $K^{\mu\nu}$ is equivalent with the KBL superpotential in [13, 14] and the potential via the topological regularization method in even dimensions [25, 26, 27, 47, 48, 49]. Third, as what has been shown in [41], the perturbation of $K^{\mu\nu}$ about the AdS spacetimes coincides with the Iyer-Wald potential [11], the ADT potential [20, 21, 22, 23, 24], the potential via the field-theoretical approach [33], and the potential given by [42, 43]. According to these, we further conclude that it is reasonable to modify the usual Komar potential as the one $K^{(3th)}(\xi)$.

What is more, we perform a comparison between the generalized potential $K$ and the Komar-like potential put forward within the works [44, 45], here denoted by $\tilde{K}^{\mu\nu}$. The potential $\tilde{K}^{\mu\nu}$, which can be regarded as the generalization of the results for the four-dimensional Einstein gravity in [46], was constructed out of the equation of motion and the property $\Box \xi^\mu = - R^\mu_{\nu\rho\sigma} \xi^\nu$ for the divergence-free Killing vector $\xi^\mu$. It is expressed as the linear combination of the ordinary Komar potential $K_{\mu\nu}^\text{Komar}$ with the 2-form Killing potential $\omega^{\mu\nu}$ defined through the relation $\xi^\mu = - \nabla_\nu \omega^{\mu\nu}$, that is,

$$\tilde{K}^{\mu\nu} = \nabla [\mu \xi^\nu] - (D - 1) \ell^2 \omega^{\mu\nu}.$$  (2.18)

Obviously, the potential $\tilde{K}^{\mu\nu}$ takes a similar structure to the one $K^{\mu\nu}$. In fact, we shall see below that both of them are equivalent at spatial infinity within the scope of the Einstein gravity. However, an advantage of $K^{\mu\nu}$ over $\tilde{K}^{\mu\nu}$ is that one does not have to solve the equation $\xi^\mu = - \nabla_\nu \omega^{\mu\nu}$.

With the help of the Bianchi identity $\nabla_\nu R^{\mu\nu}_{\rho\sigma} = 0$ and the field equation (2.2), we
compute the divergence of the skew-symmetric tensor $R^{\mu\nu}_{\rho\sigma} \nabla^\rho \xi^\sigma$ in $K^{\mu\nu}$, leading to

$$
\nabla_\nu (R^{\mu\nu}_{\rho\sigma} \nabla^\rho \xi^\sigma) = R^{\mu\nu}_{\rho\sigma} \nabla_\nu \nabla^\rho \xi^\sigma - 2 (\nabla_\rho R^{\mu}_{\nu}) \nabla^\rho \xi^\sigma
$$

$$
= R^{\mu\nu}_{\rho\sigma} R^{\rho\sigma}_{\lambda\nu} \xi^\lambda.
$$

(2.19)

By utilizing Eqs. (2.4) and (2.19), we further obtain

$$
\frac{1}{2\ell^2} \nabla_\nu (R^{\mu\nu}_{\rho\sigma} \nabla^\rho \xi^\sigma) \bigg|_\infty = (D - 1) \ell^2 \xi^\mu
$$

$$
= -(D - 1) \ell^2 \nabla_\nu \omega^{\mu\nu}.
$$

(2.20)

As a consequence of Eq. (2.20), we conclude that the potentials $K^{\mu\nu}$ without the $c_2$ factor and $\tilde{K}^{\mu\nu}$ coincide with each other at spatial infinity. Furthermore, as what has been mentioned above, to completely fix the potential $\tilde{K}^{\mu\nu}$, a necessary procedure is to solve the Killing potential $\omega^{\mu\nu}$ from the equation $\xi^\mu = -\nabla_\nu \omega^{\mu\nu}$, which is of great difficulty to handle, particularly for rotating spacetimes in higher dimensions. As a solution, we propose that the 2-form $R^{\mu\nu}_{\rho\sigma} \nabla^\rho \xi^\sigma$ can be a proper substitute for the Killing potential $\omega^{\mu\nu}$ involved in the potential $\tilde{K}^{\mu\nu}$, that is,

$$
\omega^{\mu\nu} \rightarrow - \frac{1}{2(D - 1)\ell^4} R^{\mu\nu}_{\rho\sigma} \nabla^\rho \xi^\sigma.
$$

(2.21)

3 Applications in black holes with an AdS asymptotic

In this section, we are going to focus on applications of the Komar-like integral (2.16) in the computations for conserved charges of various asymptotically AdS black holes, including the $D$-dimensional Schwarzschild-AdS black hole, the four-dimensional regular AdS black hole, the four-dimensional asymptotically AdS Kerr-Sen black hole together with its ultra-spinning generalization, the four-dimensional Kerr-NUT-AdS black hole, and the Kerr-AdS black hole in arbitrary dimensions. By contrast with other existing approaches, the calculations are much simpler. All the results support that the surface integral in terms of the improved Komar potential with higher-order corrections indeed yield the physical mass and angular momentum in a unified way within the scope of the Einstein gravity in the presence of the negative cosmological constant.
3.1 Mass of Schwarzschild-AdS black holes in arbitrary dimensions and four-dimensional regular AdS black holes

In this subsection, on basis of the formula (2.16) for conserved charges, we will calculate the mass of $D$-dimensional Schwarzschild-AdS black holes, as well as four-dimensional spherically symmetric regular AdS black holes.

Without loss of generality, let us take into consideration the static Schwarzschild-like metric ansatz being of the form

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + Y^2(r)h_{ij}(x^k)dx^idx^j,$$

(3.1)

where $h_{ij}$ is the metric for $(D - 2)$-dimensional space. Our goal is to apply the formula (2.16) to compute the mass of the solutions equipped with the metric ansatz (3.1). The associated Killing vector is chosen as $\xi^\mu = -\delta_t^\mu$. Under this to evaluate the $(t, r)$ component for the covariant derivative with respect to the vector $\xi^\mu$, we have

$$\nabla^t\xi^r = -\nabla^r\xi^t = \frac{1}{2}XF'.$$

(3.2)

Here and in the remainder of this subsection, the function with prime denotes its derivative with respect to the radial coordinate $r$. The $(t, r, \rho, \sigma)$ component of the Riemann curvature tensor $R_{\rho\sigma}^{tr}$ is read off as

$$R_{\rho\sigma}^{tr} = -R_{\rho\sigma}^{rt} = -\frac{1}{2} \left( F' X' + 2XF'' \right) \delta_t^\rho \delta_r^\sigma.$$  

(3.3)

Substituting Eqs. (3.2) and (3.3) into the Komar-like potential $K_{\mu\nu}$ in Eq. (2.15), we obtain its $(t, r)$ component of the form

$$K_{tr} = c_2XF' \left( 1 - \frac{XF''}{2\ell^2} - \frac{F'X'}{4\ell^2} \right).$$

(3.4)

As a consequence, under the formula (2.16) for the conserved charges, the mass is defined through the integral of $K_{tr}$ over the $(D - 2)$-dimensional space with the metric $h_{ij}$, presented by

$$M = \frac{c_2V_{D-2}}{8\pi} \lim_{r \to \infty} F' |Y|^{D-2} \sqrt{|X|} \left( 1 - \frac{XF''}{2\ell^2} - \frac{F'X'}{4\ell^2} \right),$$

(3.5)

where the volume for the $(D - 2)$-dimensional space $V_{D-2} = \int \sqrt{h}dx^{D-2}$ with the determinant of the metric tensor $h = det(h_{ij})$. 
For concreteness, we take into account the mass of the \( D \)-dimensional Schwarzschild-AdS black holes. In such a case, all the undetermined quantities in the line element (3.1) are given by

\[
F(r) = 1 - \frac{2m}{r^{D-3}} + \ell^2 r^2, \quad X(r) = 1, \\
Y(r) = r, \quad d\Omega_{D-2}^2 = h_{ij}dx^idx^j.
\]  

(3.6)

In the above equation, the integration constant \( m \) is related to the mass, and \( d\Omega_{D-2}^2 \) is the line element on a \((D-2)\)-dimensional unit sphere. By making use of Eq. (3.5), we obtain the mass of the Schwarzschild-AdS black holes \( M_{SA_{D\text{s}}} \), being of the form

\[
M_{SA_{D\text{s}}} = \frac{mc^2(D-2)(D-3)V_{D-2}}{4\pi}.
\]

(3.7)

Here the volume of the \((D-2)\)-dimensional unit sphere \( V_{D-2} \) is given by

\[
V_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)}.
\]

(3.8)

When the \( c_2 \) parameter is given by Eq. (2.14), the mass for the Schwarzschild-AdS black holes \( M_{SA_{D\text{s}}} \) takes the same value as that in the existing literature \[16, 30, 50, 51, 52, 53, 54, 55\]. Besides, it returns to the standard result for the mass of the Schwarzschild black holes when \( \ell = 0 \). In other words, these confirm the value of \( c_2 \). On the other hand, if the third-order derivatives in \( M \) are neglected, it is observed from Eq. (3.5) that \( M_{SA_{D\text{s}}} \) diverges at infinity since \( \left(r^{D-2}F'\right) \to \infty \). This demonstrates that the higher-order derivative terms just cancel out the divergent ones existing in the original Komar potential.

As another application to the static spacetime, one is able to adopt straightforwardly the formula (3.5) to calculate the mass of the so-called Bardeen-type regular AdS black holes constructed in the works \[56, 57\]. For generality, we consider the metric in \[57\], given by

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.9}
\]

with the function

\[
f(r) = 1 - \frac{2m}{r} - \frac{2q^3r^k}{\alpha r^{(r+q^l)^{k/l}} + \ell^2 r^2}.
\]

(3.10)

Here, \( m, q, \alpha, k \) and \( l \) are constant parameters. The direct calculations on base of Eq. (3.5) give rise to the mass \( M_{Reg_{AdS}} \) for the asymptotically AdS regular black holes, presented by means of

\[
M_{Reg_{AdS}} = m + \frac{q^3}{\alpha}.
\]

(3.11)
$M_{\text{RegAdS}}$ coincides with the AMD mass in [57]. Obviously, the inclusion of the $q^3/\alpha$ term modifies the mass of the four-dimensional Schwarzschild-AdS black holes.

### 3.2 Mass and angular momentum for Kerr-Sen-AdS$_4$ black holes

In this subsection, we are going to compute the mass and angular momentum for the four-dimensional asymptotically AdS Kerr-Sen (Kerr-Sen-AdS$_4$) black holes given by [58]. We adopt the line element in a non-rotating frame at infinity, taking the following form in Boyer-Lindquist coordinates

$$ds^2 = -\frac{\Delta_r}{\Sigma} \left[ dt - \Upsilon_\theta \left( \frac{d\phi}{\Xi} - a \ell^2 \frac{dt}{\Xi} \right) \right]^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2$$

$$+ \frac{\Delta_\theta \sin^2 \theta}{\Sigma} \left[ adt - \Upsilon_r \left( \frac{d\phi}{\Xi} - a \ell^2 \frac{dt}{\Xi} \right) \right]^2. \quad (3.12)$$

In Eq. (3.12), the functions $\Upsilon_\theta$, $\Upsilon_r$, $\Delta_\theta$, $\Sigma$, and $\Delta_r$ are given by

$$\Upsilon_\theta = a \sin^2 \theta,$$

$$\Upsilon_r = r^2 + 2br + a^2,$$

$$\Delta_\theta = 1 - a^2 \ell^2 \cos^2 \theta,$$

$$\Sigma = r^2 + 2br + a^2 \cos^2 \theta,$$

$$\Delta_r = \Upsilon_r [1 + \ell^2 (r^2 + 2br)] - 2mr,$$  

respectively. The constant parameter $\Xi = 1 - \ell^2 a^2$, and $(m, a, b)$ are integration constants. When $b = 0$, the metric ansatz [3.12] turns into that of the four-dimensional Kerr-AdS black hole solution [2].

The mass $M_{KSA_{\text{AdS}}}$ and the angular momentum $J_{KSA_{\text{AdS}}}$ for the Kerr-Sen-AdS$_4$ black hole are associated with the Killing vectors $\xi^\mu_{(t)} = (-1, 0, 0, 0)$ and $\xi^\mu_{(\phi)} = (0, 0, 0, 1)$, respectively. In this setting, on basis of Eq. (2.15) to evaluate the $(t, r)$ component of the Komar-like potential, we arrive at

$$\sqrt{-g} K^{tr} (\xi^\mu_{(t)}) = \frac{m (3 \Delta_\theta - \Xi) \sin \theta}{\Xi^2} + O \left( \frac{1}{r} \right),$$

$$\sqrt{-g} K^{tr} (\xi^\mu_{(\phi)}) = \frac{3m \sin \theta \Upsilon_\theta}{\Xi^2} + O \left( \frac{1}{r} \right),$$

where $\sqrt{-g} = (\Upsilon_r - a \Upsilon_\theta) \sin \theta / \Xi$. The integration of Eq. (3.14) with respect to the coordinate $\theta$ further leads to

$$M_{KSA_{\text{AdS}}} = \frac{m}{\Xi^2}, \quad J_{KSA_{\text{AdS}}} = \frac{ma}{\Xi^2}. \quad (3.15)$$
On the other hand, like in [58], if one computes the mass and angular momentum of the Kerr-Sen-AdS\textsubscript{4} black hole in the rotating frame, that is, $\phi \to \phi + a\ell^2 t$ in the metric (3.12), the mass $M_{KSAdS}$ becomes $m/\Xi$, while $J_{KSAdS}$ remains the same value.

It is worth noting that the scalar fields and the U(1) gauge field are not involved in the computations for the conserved charges of the Kerr-Sen-AdS\textsubscript{4} black hole. As a matter of fact, one can follow the work [59] to confirm that those fields decrease fast enough at asymptotic infinity so that they make no contribution to the charges. This holds true to the four-dimensional Kerr-Newman-AdS black holes [2,3]. Therefore, their mass and angular momentum are consistent with the ones in Eq. (3.15).

What is more, in terms of the metric given by Eq. (13) in [58], one makes use of the generalized Komar potential (2.15) with $\ell$ substituted by $l^{-1}$ to compute the mass and the angular momentum of the ultra-spinning Kerr-Sen-AdS\textsubscript{4} black hole. One can obtain the mass $M = \mu m/(2\pi)$ and the angular momentum $J = \mu ml/(2\pi)$ in [58].

### 3.3 Conserved quantities of four-dimensional Kerr-NUT-AdS black holes

The metric ansatz of the four-dimensional Kerr-NUT-AdS black holes, which has been known for some time [3,4,9], can be written as the same form as Eq. (3.12). However, the functions $(\Upsilon_r, \Delta_r, \Delta_\theta, \Upsilon_\theta, \Sigma)$ are replaced with the ones $(\hat{\Upsilon}_r, \hat{\Delta}_r, \hat{\Delta}_\theta, \hat{\Upsilon}_\theta, \hat{\Sigma})$, respectively. Here the functions $\hat{\Upsilon}_r$ and $\hat{\Delta}_r$ with respect to the coordinate $r$ are presented by

\[
\hat{\Upsilon}_r = r^2 + (a + n)^2, \\
\hat{\Delta}_r = r^2 + \ell^2 r^2 r^2(r^2 + 6n^2 + a^2) - 2mr \\
+ (3\ell^2 n^2 + 1)(a^2 - n^2),
\]

while the functions $(\hat{\Delta}_\theta, \hat{\Upsilon}_\theta, \hat{\Sigma})$ are given by

\[
\hat{\Delta}_\theta = 1 - a\ell^2 \cos \theta (4n + a \cos \theta), \\
\hat{\Upsilon}_\theta = a \sin^2 \theta + 2n(1 - \cos \theta), \\
\hat{\Sigma} = r^2 + (n + a \cos \theta)^2.
\]
In Eqs. (3.16) and (3.17), \( n \) is the NUT charge parameter. Accordingly, the \((t, r)\) components of the potential \( K^{\mu\nu} \) related to the Killing vectors \( \xi^\mu_{(t)} \) and \( \xi^\mu_{(\phi)} \) are read off as

\[
\sqrt{-g} \hat{K}^{tr} \left( \xi^\mu_{(t)} \right) = \frac{m \left( 3a \ell^2 \hat{\Upsilon}_\theta + 2\Xi \right) \sin \theta}{\Xi^2} + O \left( \frac{1}{r} \right),
\]

\[
\sqrt{-g} \hat{K}^{tr} \left( \xi^\mu_{(\phi)} \right) = \frac{3m \sin \theta \hat{\Upsilon}_\theta}{\Xi^2} + O \left( \frac{1}{r} \right). \tag{3.18}
\]

Furthermore, in terms of the formulation (2.16) for conserved quantities to compute the mass \( M_{K\text{NUT}} \) and the angular momentum \( J_{K\text{NUT}} \) for the four-dimensional Kerr-NUT-AdS black hole, we have

\[
M_{K\text{NUT}} = \frac{m(1 + 3an\ell^2)}{\Xi^2}, \quad J_{K\text{NUT}} = \frac{m(a + 3n)}{\Xi^2}. \tag{3.19}
\]

Letting the NUT parameter \( n \) in Eq. (3.19) vanish, we see that the mass \( M_{K\text{NUT}} \) and the angular momentum \( J_{K\text{NUT}} \) becomes the ones of the four-dimensional Kerr-AdS black hole, respectively. Although here we only consider the computations of the mass and the angular momentum for the four-dimensional Kerr-NUT-AdS black holes, we expect that the formula (2.16) are applicable to the higher-dimensional ones. This will be verified in the future work.

### 3.4 Conserved charges for \( D \)-dimensional Kerr-AdS black holes

In the present subsection, the mass and angular momenta for the generic \( D \)-dimensional Kerr-AdS black holes constructed in \[6, 7\] will be computed in terms of the formula (2.16). As is known, these black holes can be seen as the higher-dimensional generalizations of the four- and five-dimensional Kerr-AdS black holes given by \[2, 5\], as well as the extensions including a cosmological constant to the asymptotically flat Myers-Perry black holes in \( D \) dimensions \[8\].

The metric tensors for the \( D \)-dimensional Kerr-AdS black holes satisfy the field equation (2.2). They possess \( n = (D - \varepsilon - 1)/2 \) (\( \varepsilon = 1 \) for \( D \) even and \( \varepsilon = 0 \) for \( D \) odd) independent rotations in \( n \) orthogonal 2-planes, characterized by \( n \) parameters \( a_i \) (\( 1 \leq i \leq n \)) and \( n \) azimuthal angles \( \phi_i \), which are \( 2\pi \)-periodic. In the coordinate system \( \{ t, r, \mu_1, \cdots, \mu_{n+\varepsilon-1}, \phi_1, \cdots, \phi_n \} \), the line element of the spacetime for the \( D \)-dimensional
Kerr-AdS black hole is of the form

\[ ds^2_{(D)} = ds^2_{(D)} + \frac{2m^2 U}{H(V - 2m)} d\tau^2 + \frac{2m H}{r^2 UV} \left( W dt - \sum_{i=1}^{n} a_i \mu_i^2 d\phi_i \right)^2, \]  

(3.20)
in which the line element \( ds^2_{(D)} \) is read off as

\[ ds^2_{(D)} = -HW dt^2 + \sum_{i=1}^{n} \mu_i^2 (r^2 + a_i^2) \frac{d\phi_i^2}{\Xi_i} + \sum_{i=1}^{n+\varepsilon} (r^2 + a_i^2) \mu_i^2 \frac{d\phi_i^2}{\Xi_i} - \frac{\ell^2}{HW} \left( \sum_{i=1}^{n+\varepsilon} \frac{r^2 + a_i^2}{\Xi_i} \mu_i \mu_i \right)^2 + \frac{r^2 U}{H} dr^2. \]  

(3.21)

In Eqs. (3.20) and (3.21), the constant parameters \( \Xi_i = 1 - a_i^2 \ell^2 \) \((1 \leq i \leq n)\), while \( \Xi_{n+1} = 1 \), arising from that \( a_{n+1} = 0 \), when \( D \) is even. The four functions \((H, U, V, W)\) are presented respectively by

\[ H = 1 + \ell^2 r^2, \quad U = \sum_{i=1}^{n+\varepsilon} \frac{\mu_i^2}{r^2 + a_i^2}, \]
\[ V = r^{-\varepsilon} H \prod_{i=1}^{n} (r^2 + a_i^2), \]
\[ W = \sum_{i=1}^{n+\varepsilon} \frac{\mu_i^2}{\Xi_i}. \]  

(3.22)

The \( \mu_i \) variables are constrained by \( \sum_{i=1}^{n+\varepsilon} \mu_i^2 = 1 \), from which \( \mu_{n+\varepsilon} \) can be solved as \( \mu_{n+\varepsilon} = \sqrt{1 - \sum_{k=1}^{n+\varepsilon-1} \mu_k^2} \). Under such a choice of \( \mu_{n+\varepsilon} \), each \( \mu_i \) is assumed to range over \( 0 \leq \mu_i \leq 1 \) \((i = 1, 2, \cdots, n)\), while the \( \mu_{n+1} \) variable runs from -1 to 1 in even \( D \) case. The value of \( \sqrt{-g} \) is written as

\[ \sqrt{-g} = \frac{r^3 UV \prod_{j=1}^{n} \mu_j}{H \mu_{n+\varepsilon} \prod_{k=1}^{n+\varepsilon} \Xi_k}. \]  

(3.23)

For the Kerr-AdS black hole (3.20), the Killing vectors associated with the mass \( M_{KAdS} \) and the angular momentum \( J_{KAdS}^{(i)} \) along the \( \phi_i \) direction are chosen as \( \xi_{(t)}^\mu = -\delta_t^\mu \) and \( \xi_{(\phi_i)}^\mu = \delta_{\phi_i}^\mu \), respectively. By utilizing Eq. (2.15) for the superpotential \( K^{\mu\nu} \), we have

\[ \sqrt{-g} K_{KAdS}^{tr} (\xi_{(t)}^\mu) = \frac{m_i (D - 1) W - 1}{\mu_{n+\varepsilon} \prod_{k=1}^{n} \Xi_k} \prod_{j=1}^{n} \mu_j + O \left( \frac{1}{r} \right), \]
\[ \sqrt{-g} K_{KAdS}^{tr} (\xi_{(\phi_i)}^\mu) = \frac{ma_i (D - 1) \mu_i^2 \prod_{j=1}^{n} \mu_j}{\mu_{n+\varepsilon} \Xi_i \prod_{k=1}^{n} \Xi_k} + O \left( \frac{1}{r} \right). \]  

(3.24)
The results in Eq. (3.24) were also obtained by using the KBL superpotential method in [52], as well as by means of the BBC formalism in [30]. Substituting Eq. (3.24) into the Komar-like integral (2.16) to calculate the mass $M_{KAdS}$ and the angular momentum $J_{KAdS}^{(i)}$, one obtains the following results:

$$M_{KAdS} = \frac{V_{D-2}}{4\pi} \frac{m}{\prod_{j=1}^{n} \Xi_j} \left( \sum_{i=1}^{n} \frac{1}{\Xi_i} - \frac{1 - \epsilon}{2} \right),$$

$$J_{KAdS}^{(i)} = \frac{V_{D-2}}{4\pi} \frac{m a_i}{\Xi_i \prod_{j=1}^{n} \Xi_j}. \quad (3.25)$$

Here the volume of the $(D-2)$-dimensional sphere $V_{D-2}$ is presented in (3.8). It should be pointed out that the following identity has been used in the calculations

$$\int \frac{\mu_i^2 \prod_{j=1}^{n} \mu_j}{\mu_{n+\epsilon}} \left( \prod_{k=1}^{n+\epsilon-1} d\mu_k \right) \prod_{l=1}^{n} d\phi_l = \frac{2V_{D-2}}{D-1}, \quad (3.26)$$

when the integer $i$ ranges over $1 \leq i \leq n$, together with the following integral derived from Eq. (3.26)

$$\int \frac{\mu_i^2 \prod_{j=1}^{n} \mu_j}{\mu_{n+1}} \left( \prod_{k=1}^{n} d\mu_k \right) \prod_{l=1}^{n} d\phi_l = \frac{V_{D-2}}{D-1} \quad (3.27)$$

in even $D$ case. What is more, due to that $(\Box d\xi(\phi_i))^{tr}$ multiplied by the factor $\sqrt{-g}$ vanishes at infinity, the third-order derivative term in the Komar-like potential (2.15) makes no contributions to the angular momentum. In this sense, the usual Komar integral is enough to yield the angular momentum of $D$-dimensional Kerr-AdS black holes [51].

When all the rotation parameters $a_i$’s vanish, $M_{KAdS}$ coincides with the mass $M_{SAdS}$ for the $D$-dimensional Schwarzschild-AdS black holes given by Eq. (3.7). The mass $M_{KAdS}$ and the angular momentum $J_{KAdS}^{(i)}$ in Eq. (3.25) are congruent with the results via other methods in the literature [27, 30, 50, 51, 52, 53, 54, 55].

4 Summary

Within this paper, for the sake of generalizing the usual Komar integral to the Einstein gravity theory equipped with the Lagrangian (2.11), we modify the usual Komar current as the identically conserved one $\mathcal{J}^{(4th)}$ given by Eq. (2.9). This extended current being comprised of second- and fourth-order derivative terms of the Killing vector field $\xi$ is generated
via the action of the three differential operators $(\Box, d, \delta)$ on $\xi$. Through the well-known relationship between the conserved current and potential, we then derive the potential $K^{(3\text{th})}$ in Eq. (2.10). Such a potential can be interpreted as the higher-order derivative generalization of the usual Komar potential, due to the fact that it contains third-order derivative terms $d\Box \xi$ and $\Box d\xi$ apart from the Komar potential $d\xi$. Particularly, by applying the potential with higher-order corrections to the usual Komar one into the Einstein gravity with a negative cosmological constant, we obtain the Komar-like potential $\mathcal{K}$ in Eq. (2.15), which is equivalent with some potentials in the literature. The surface integral with respect to the potential $\mathcal{K}^{\mu\nu}$ further gives rise to a unified definition (2.16) for the mass and the angular momentum of spacetimes with an AdS asymptotic. As applications of the formula (2.16), we compute the mass and the angular momentum of some asymptotically AdS black holes, such as the $D$-dimensional Schwarzschild-AdS black hole, the Bardeen-type regular AdS black hole, the four-dimensional asymptotically AdS Kerr-Sen black hole, the Kerr-NUT-AdS black hole, and the Kerr-AdS black holes in arbitrary dimensions. All the results are consistent with the ones via other methods.

There are several issues associated with the present paper which should deserve further investigation. Firstly, as is known, the usual Komar current and potential can be derived from the variation of the Einstein-Hilbert-Λ Lagrangian under the guidance of Noether theorem. Thus, in parallel with this, it is better to find an effective Lagrangian, from which the current $J^{(4\text{th})}$ and the potential $K^{(3\text{th})}$ are able to be derived by following the standard Noether method. Secondly, it is of great interest to extend the higher-order corrected potential $K^{(3\text{th})}$ to investigate the definitions of the conserved charges for spacetimes with asymptotical structures different from the asymptotically AdS one. Thirdly, to see further the universality of the Komar-like potential $\mathcal{K}$, it can be utilized to understand the thermodynamical properties of various spacetimes with an AdS asymptotic, such as the derivation of the Smarr formula for the first law and enthalpy.

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A Inclusions of more than fourth-order derivatives within conserved currents

According to the rules in the work [40], all the currents generated under the action of the \((\Box, d, \delta)\) operators on an arbitrary Killing vector contain only even-order derivatives of this vector. As a consequence, except for the derivative of order two, the lowest differential order derivative of the Killing vector is the fourth-order one. To this point, for simplicity, one could give priority to the fourth-order derivatives of the Killing vector when the higher-order corrections to the ordinary Komar current are under consideration. In principle, if these are inadequate to meet the requirement for the convergence of conserved charges, the sixth- or higher-order ones can be continuously considered to be incorporated into the conserved currents. However, in this appendix, it will be demonstrated that the fourth-order derivatives are sufficient within the context of the Einstein gravity theory in the presence of a negative cosmological constant.

We first take into consideration of the inclusion of sixth-order derivative terms of the Killing vector \(\xi\). In this general case, the most generic 1-form \(J_{\text{gen}}^{(6\text{th})}(\xi)\), generated by the action of the \((\Box, d, \delta)\) operators upon \(\xi\), is decomposed as

\[
J_{\text{gen}}^{(6\text{th})}(\xi) = J_{\Box}^{(6\text{th})}(\xi) + J_{d}^{(6\text{th})}(\xi) + J_{\delta}^{(6\text{th})}(\xi).
\]  

(A.1)

Here the 1-form \(J_{\Box}^{(6\text{th})}(\xi)\) is presented by

\[
J_{\Box}^{(6\text{th})}(\xi) = \alpha_{31}\Box\delta\Box d\xi + \alpha_{32}\Box^{2}\delta\Box d\xi + \alpha_{33}\Box\delta\Box d\xi \\
+ \alpha_{34}\Box\delta\Box\Box d\xi + \alpha_{35}\Box\delta\Box d\xi + \alpha_{36}\Box^{3}\xi,
\]

(A.2)

the closed 1-form \(J_{d}^{(6\text{th})}(\xi)\) is of the form

\[
J_{d}^{(6\text{th})}(\xi) = \beta_{31}\delta\Box\delta\Box d\xi + \beta_{32}\delta\Box\Box\Box d\xi + \beta_{33}\delta\Box\Box\Box d\xi,
\]

(A.3)

and the identically conserved \(J_{\delta}^{(6\text{th})}(\xi)\) is given by

\[
J_{\delta}^{(6\text{th})}(\xi) = \lambda_{31}(\delta d)^{3}\xi + \lambda_{32}\delta\Box d\delta\Box d\xi + \lambda_{33}\delta\Box d\Box\Box d\xi \\
+ \lambda_{34}\delta\Box d\Box\Box d\xi + \lambda_{35}\delta\Box d^{2}\Box d\xi + \lambda_{36}\delta\Box^{2}d\Box d\xi \\
+ \lambda_{37}\delta\Box\Box d\Box d\xi + \lambda_{38}\delta\Box d\Box d\Box d\xi.
\]

(A.4)

In Eqs. (A.2), (A.3) and (A.4), \(\alpha_{3i}\)'s, \(\beta_{3i}\)'s and \(\lambda_{3i}\)'s represent constant parameters. For a particular Killing vector satisfying \(\delta J_{\Box}^{(6\text{th})}(\xi) = -\delta J_{d}^{(6\text{th})}(\xi)\), the 1-form \(J_{\text{gen}}^{(6\text{th})}(\xi)\) is conserved. However, in order to guarantee that \(\delta J_{\text{gen}}^{(6\text{th})}(\xi) = 0\) holds identically for any Killing
vector, it is required that both the 1-forms $J_{\square}^{(6th)}(\xi)$ and $J_d^{(6th)}(\xi)$ vanish. As a consequence, we obtain the identically conserved current comprised of the sixth-order derivatives of the Killing vector $J^{(6th)}(\xi) = J_{\delta}^{(6th)}(\xi)$. For convenience, making use of the identity $\hat{\delta}d\xi = 2\square\xi$ to reformulate the current $J^{(6th)}(\xi)$ results in

$$J^{(6th)}(\xi) = -k_{61}\hat{\delta}\square d\xi - k_{62}\hat{\delta}\square d\xi - k_{63}\hat{\delta}d\square d\xi - k_{64}\hat{\delta}d\square d\xi - k_{65}\hat{\delta}\square d\xi,$$

where $k_{6i}$’s denote arbitrary constant parameters. The relationship $J^{(6th)}(\xi) = -\hat{\delta}K^{(5th)}(\xi)$ gives rise to the potential

$$K^{(5th)}(\xi) = k_{61}\square d\xi + k_{62}\square d\xi + k_{63}\hat{\delta}d\square d\xi + k_{64}\hat{\delta}d\square d\xi + k_{65}\square d\xi.$$

Next, as a particular case of Eq. (A.6), we switch to dealing with the potentials within the framework of the Einstein gravity theory endowed with the Lagrangian (2.1). Under the identity $\square\xi = (D - 1)\ell^2\xi$, the potential $K^{(5th)}(\xi)$ turns into

$$K^{(5th)}_{gr}(\xi) = k_{61}(D - 1)\ell^2\square d\xi + (k_{62} + 2k_{63})(D - 1)^2\ell^4d\xi + k_{64}\hat{\delta}\square d\xi + k_{65}\square d\xi.$$  

Applying the terms proportional to $\square d\xi$ and $d\xi$ in $K^{(5th)}_{gr}(\xi)$ can be covered by $K^{(3th)}(\xi)$ in (2.10). What is more, due to that

$$\hat{\delta}\square d\xi = -2 \left( R_{\alpha\beta}^{\lambda\varnothing} R_{\lambda\mu}^{\varnothing\rhoans} \nabla_{\varnothing} + \xi_{\alpha} R_{\lambda\mu}^{\varnothing\rhoans} \nabla_{\varnothing} R_{\rho\varnothing}^{\lambda\varnothing} \right) dx^\mu \wedge dx^\nu,$$

by making use of the falloff condition (2.4) for the Riemann curvature tensor and the equation $(R_{\rho\sigma}^{\nu\rho} \nabla_{\nu}) R_{\lambda\alpha}^{\varnothing\rhoans} \bigg|_\infty = 0$, we observe that $(\hat{\delta}\square d\xi) \bigg|_\infty = 4(D - 1)\ell^4d\xi$, proportional to $d\xi$. This implies that $\hat{\delta}\square d\xi$ overlaps $d\xi$ in $K^{(3th)}(\xi)$ at infinity. Therefore, the linear combination of $K^{(3th)}(\xi)$ with $K^{(5th)}_{gr}(\xi)$ can be equivalently described by the potential $K^{(5th)}_{gr}(\xi)$ with no more than fifth-order derivatives of the Killing vector, being of the form

$$K^{(5th)}_{gr}(\xi) = c_2(\xi - c_1\square d\xi + c_3\square d\xi).$$

Alternatively, it is expressed in components as

$$K^{(5th)}_{gr}^{\mu\nu} = 2c_2 \left( \nabla_\mu \xi_\nu + c_1 R_{\rho\varnothing}^{\mu\rhoans} \nabla_\nu + c_3 R_{\alpha\beta}^{\mu\rhoans} R_{\rho\varnothing}^{\alpha\beta} \nabla_\nu \xi_{\varnothing} \right) - 2c_2 c_3 (\square R_{\rho\varnothing}^{\mu\rhoans}) \nabla_\nu \xi_{\varnothing}.$$  

(A.10)
In order to guarantee that the integral of $K_{gr}^{(5th)tr}$ over the surface at infinity is convergent, Eq. (A.10) shows that the constant parameters $c_1$ and $c_3$ have to be constrained through 
\[
(\sqrt{-g}K_{gr}^{(5th)tr})|_\infty = C,
\]
where $C$ is a certain finite constant parameter. Because of the divergence of $\sqrt{-g}$ at infinity for the asymptotically AdS spacetimes (this can be seen from the metric for the AdS space, given by Eq. (3.6) in the absence of the $m$ parameter), we have the necessary but not the sufficient condition $K_{gr}^{(5th)tr}|_\infty = 0$ for the fulfillment of the convergence of the conserved charges, which is written as
\[
\left[ (1 - 2c_1\ell^2 + 4c_3\ell^4)\nabla^t \xi^r - c_3(\Box R_{\rho\sigma})\nabla^\rho \xi^\sigma \right]|_\infty = 0,
\] (A.11)
with the help of the falloff condition (2.4). It is easy to check that the above equation holds identically under the following setting
\[
c_1 = \frac{1}{2\ell^2}, \quad c_3 = 0.
\] (A.12)
As a consequence, under the condition that the choice of the parameters in Eq. (A.12) is able to ensure that the relation $(\sqrt{-g}K_{gr}^{(5th)tr})|_\infty = C$ holds, one can conclude that it is unnecessary to introduce the fifth-order derivatives of the Killing vector to cancel out the divergent terms from the usual Komar potential in the context of the Einstein gravity with a negative cosmological constant. Obviously, such a conclusion holds for the cases in which more than fifth-order derivatives are included in the potentials, arising from the fact that the 2-form $\Box d\xi$ involved in the potential $K^{(3th)}_{\text{AdS}}(\xi)$, fulfilling $\Box d\xi|_\infty = 2\ell^2 d\xi$, is proportional to the conventional first-order derivative Komar potential at infinity so that the linear combination $d\xi - c_1 \Box d\xi$ disappears inevitably at infinity under $c_1 = 1/(2\ell^2)$.

However, generally speaking, for a potential $K^{\mu\nu}_{2N+1}$ that is the linear combination of all the $(2i+1)$-th-order (the integer $i$ is restricted to $0 \leq i \leq N$ for some given positive integer $N$) derivatives of a Killing vector, the vanishing of $K_{2N+1}^{tr}$ at infinity does not of itself guarantee that it multiplied by the factor $\sqrt{-g}$ is certainly finite at infinity. In such a case, the more than $(2N+1)$-th-order derivatives of the Killing vector should be taken into consideration. Anyway, our fundamental requirement is to find out the potential $K^{\mu\nu}_{2N+1}$ as simple as possible that renders $(\sqrt{-g}K^{tr}_{2N+1})|_\infty$ finite. For instance, if the third-order derivative potential $K_{\text{AdS}}^{(3th)}$ in Eq. (2.12) still fulfills $(\sqrt{-g}K_{\text{AdS}}^{(3th)tr})|_\infty \to \infty$, we can go on to consider the fifth-order derivative one $K_{gr}^{(5th)}$ in Eq. (A.9) or the one of higher order.

As a matter of fact, in Section 3 it has been shown that the surface integrals of the $K_{\text{AdS}}^{(3th)}$ potential with $c_1 = 1/(2\ell^2)$ are able to yield the physical charges of the asymptotically AdS
black holes in the context of the Einstein gravity. To this point, it is advisable to consider only the fourth-order corrections to the ordinary Komar current in Section 1.

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