Tropical limit and a micro-macro correspondence in statistical physics

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Abstract

Tropical mathematics is used to establish a correspondence between certain microscopic and macroscopic objects in statistical models. Tropical algebra gives a common framework for macrosystems (subsets) and their elementary constituent (elements) that is well-behaved with respect to composition. This kind of connection is studied with maps that preserve a monoid structure. The approach highlights an underlying order relation that is explored through the concepts of filter and ideal. Main attention is paid to asymmetry and duality between max- and min-criteria. Physical implementations are presented through simple examples in thermodynamics and non-equilibrium physics. The phenomenon of ultrametricity, the notion of tropical equilibrium and the role of ground energy in non-equilibrium models are discussed. Tropical symmetry, i.e. idempotence, is investigated.

Keywords: Tropical limit, filter, monoid, ultrametric, non-equilibrium.

1 Introduction

The distinction between macroscopic and microscopic representations of phenomena is one of the fundamental problems in physics. This subject has generated many profound questions and techniques whose relevance goes beyond statistical physics. For example, the transition between the molecular dynamics and the human (thermodynamic) length scales is still investigated [1, 2, 3] and gave rise to the notion of statistical entropy, which has proved to be a fundamental tool in many areas of modern sciences [4, 5, 6].

More broadly, the “micro/macro” paradigm involves many situations where different descriptions of a system and their relative complexity have concrete effects. This is also a practical issue, since complex systems are now pervasive in many branches of science [7] and a deeper understanding of (dis-)similarities between elementary and emergent phenomena is a key point. In the cases when a collective behaviour is not reducible to its individual constituents, one can recognize complexity in the composition of the elementary entities. Hence, it is possible that a
change in the composition rules affects the relative complexity. This approach can be used to highlight analogies between the micro- and the macro-sectors, rather than their differences.

In the present work, we follow this path and concentrate on a correspondence between micro- and macro-physics starting from associative rules. In particular, we argue that tropical algebra [8] provides one with a common framework to deal with both these descriptions. Tropical limit is usually derived from the real or complex setting by means of the change of variable

\[ |x| \mapsto \exp \left( \frac{X}{\varepsilon} \right) \]

(1.1)

that induces a tropical algebra in the limit \( \varepsilon \to 0 \)

\[ X \oplus Y := \lim_{\varepsilon \to 0^+} \varepsilon \cdot \ln \left( e^{\frac{X}{\varepsilon}} + e^{\frac{Y}{\varepsilon}} \right). \]

(1.2)

Elementary entities \( X \) are combined through \( \oplus \), which is a “shadow” of the usual composition for variables \( \exp \left( \frac{X}{\varepsilon} \right) \) with exponential complexity in \( X \). This is a hint on advantages of tropical limit: it forgets part of the information in a system in order to highlight an underlying structure, which is often more practical to manage and still non-trivial.

Tropical limit in statistical physics was discussed in [9]. There, it has been argued that the Boltzmann constant \( k_B \) is a proper parameter to highlight a combinatorial skeleton of some statistical models. In several ordinary cases, the tropical and low-temperature limits coincide [10, 11, 12]. More generally, phenomena such as exponential degenerations [13, 14, 15] and negative and limiting temperatures [16, 17] can be easily included in the limit \( k_B \to 0 \) and make it non-trivial. The analysis suggests that the tropical limit of such statistical models preserves an associated relational structure, namely an order relation. Algebraic methods for ordered sets and logic have recently been developed, see e.g. [18], and also tropical mathematics has benefited from these tools [19].

Our aim is to deepen these concepts in order to establish a correspondence between certain micro- and macro-systems in accordance with tropical composition, hence the order structure. More specifically, we look at monoids, that are defined by a set \( \Lambda \) and a composition \( \oplus \) (associative operation) on \( \Lambda \) with a distinguished element \( \infty \) that is neutral for \( \oplus \). The assumption of this simple algebraic structure emphasises the role of \( \infty \), which is an extremum for the associated order relation. Cases when it is the only extremum are relevant in terms of symmetry breaking between dual orders. As we will see, this last point draws attention to the issue of the ground energy in general statistical models.

For every tropical structure (idempotent monoid) \( (\Lambda, \oplus, \infty) \), one also gets another tropical structure, that is the set \( \mathcal{P}(\Lambda) \) of the subsets of \( \Lambda \). The tropical operation on \( \mathcal{P}(\Lambda) \) is the set-theoretic join \( \cup \) (respectively, intersection \( \cap \)) and the neutral element is \( \emptyset \) (respectively, \( \Lambda \)). If the elements of \( \Lambda \) represent physical microscopic systems, subsets of \( \Lambda \) are “macrosystems”. In this perspective, we address the question of structure-preserving connections (i.e., monoid homomorphisms) between a tropical system \( \Lambda \) and the associated power set \( \mathcal{P}(\Lambda) \).

For this purpose, the concept of filter will be pivotal. Filters, especially ultrafilters, play a fundamental part in mathematical logic [20]. Since basic rules of classical logic are strictly related to probability axioms [21], it is not surprising that a different way to deal with proba-
The occurrence in a tropical addition max instead of the standard real one + and

\[(\max) \quad \{\langle g',\ell \rangle p, \langle \cdot,\psi \rangle p\} \max \geq \langle g',\psi \rangle p\]

We will show that filters give a rigorous but flexible background to investigate tropical statistical systems and their physical aspects. First, ultrametricity comes into play, since filters and their properties are defined in terms of tropical sums, and their algebraic properties are translated into order-reversing dualities (e.g., set complements) that preserve algebraic properties. Second, ultrametricity is a consequence of tropicalization processes of real quantities, and the tropical approach leads to get information on conjunction or disjunction, depending on its presentation via the tropical sum $\max$ or $\min$, but not both of them simultaneously. This broken symmetry is algebraically translated in the absence of a subtraction and affects basic counting principles such as (1.3). On the order-theoretical side, this process affects order-reversing dualities (e.g., set complements) that preserve algebraic properties. In particular, the dual of the neutral element is not an element of the algebra itself, for instance $-\infty$ is an element of the $\max$-plus algebra $[R, \max, +]$ and $+\infty$ is not.

This language can be effective in applications, with special attention to non-equilibrium systems and their physical aspects. A prominent role is played by spin glasses, whose study has prompted many theoretical and computational techniques, with applications in quantum field theories, message passing, and artificial intelligence. A major breakthrough in the study of spin glasses was done by Parisi with the Replica Symmetry Breaking ansatz. This technique opened the way to the description of interesting phenomena for a class of spin glasses. One of these is derived from the overlap distribution among different replicas and is called ultrametricity. An ultrametric is a distance which satisfies a stronger version of the triangle inequality

\[d(\alpha,\beta) \leq \max\{d(\alpha,\gamma), d(\gamma,\beta)\}\]  

(1.4)

The occurrence in (1.4) of a tropical addition $\max$ instead of the standard real one $+$ and exponential degeneration in spin glasses are hints of a link between this kind of models and tropical algebraic structures. We will show that filters give a rigorous but flexible background to investigate tropical statistical systems and their physical aspects.
the tropical limit. We will refer to this procedure as a dequantification for probability weights. Other puzzling concepts, such as the \( n \to 0 \) limit for the dimension of the replica overlap matrix in spin glass theory, could take a concrete shape in this setting.

It should be remarked that we do not wish to deal with measure-theoretic properties of specific models. Many interesting results in this direction have been achieved during last decades, with particular regard to \( p \)-adic models [34] and Ruelle probability cascade [35]. Here we focus on the unifying role that tropical limit plays in explaining apparent contrasts between macro- and micro-physics.

The paper is organized as follows. In Section 2 we remind few notions on monoids and order theory in order to make this paper self-contained. In Section 3 we deal with some consequences of its physical identification with the limit \( k_B \to 0 \). The discussion suggests what types of algebraic connections can be drawn in the tropical setting. This is formalized in Section 4, where filters and ideals are introduced and a simple relation with ultrametric is observed. In Section 5 we investigate links between tropical structures via monoid homomorphisms and provide a characterization of linearly ordered tropical structure in this perspective. In Section 6 we encode the information on partition functions in a suitable form for a “perturbative” study of tropical limit, since usual analytic expansions may fail in the tropical (non-analytic) limit. Subsequent sections apply these tools to concrete physical issues. In Section 7.1 we use the filter language to introduce tropical equilibrium and discuss a basic non-equilibrium model with particular focus on the choice of the ground energy. This issue is deepened in Section 7.2 with the introduction of global or local tropicalizations and the study of global or local tropical actions. In Section 8 the issue of tropicalizations of more variables, as a whole or one at a time, is addressed. Idempotence is achieved as a global or a local tropical symmetry. Its implications in counting processes lead to tropical probability, that is explored in terms of global (Subsection 8.1) or local (Subsection 8.2) tropical symmetry. Finally, we draw conclusions and discuss future perspectives in Section 9.

2 Notation and definitions

Tropical geometry (see e.g. [8] for an introduction) is a recent branch of algebraic geometry originating from earlier studies in computer science, optimization and mathematical physics [36, 38, 19]. It is based on usual algebraic concepts, like polynomials, ideals and varieties, translated in the setting of an idempotent semiring. This means that all algebraic expressions involve operations on a semiring instead of a field \( K \) (usually \( \mathbb{R} \) or \( \mathbb{C} \), or finite fields \( \mathbb{F}_q \)). The basic example of a tropical structure is the max-plus semiring \( \mathbb{R}_{\text{max}} \): it is defined as a 5-uple \((\mathbb{R} \cup \{-\infty\}, \oplus, \odot, -\infty, 0)\) where \( \mathbb{R}_{\text{max}} := \mathbb{R} \cup \{-\infty\} \), \( a \oplus b := \max\{a, b\} \) is the tropical addition, \( a \odot b := a + b \) is the tropical multiplication, \(-\infty\) and \( 0 \) are the neutral elements of \( \oplus \) and \( \odot \) respectively. A common process to derive the max-plus semiring from \( \mathbb{R} \) or \( \mathbb{C} \) comes through the limit \( \varepsilon \to 0^+ \) for the following family of ring operations

\[
\begin{align*}
x \oplus_\varepsilon y & := \varepsilon \cdot \ln \left( e^{\frac{x}{\varepsilon}} + e^{\frac{y}{\varepsilon}} \right), \\
x \odot_\varepsilon y & := \varepsilon \cdot \ln \left( e^{\frac{x}{\varepsilon}} \cdot e^{\frac{y}{\varepsilon}} \right) = x + y.
\end{align*}
\]
The peculiarity of the max-plus semiring is that $\oplus$ is idempotent, i.e. $x \oplus x = x$ for all $x$ in $\mathbb{R}_{\max}$. So there exists a multiplicative inverse for each element in $\mathbb{R} = \mathbb{R}_{\max} \setminus \{-\infty\}$, but there is no additive inverse for elements in $\mathbb{R}$. In fact, $x \oplus y = -\infty$ implies $-\infty = x \oplus y = (x \oplus x) \oplus y = x \oplus (x \oplus y) = x \oplus (-\infty) = x$. This means that there is no subtraction. Consequently, the extension of several useful mathematical tools, e.g. differentiation, to the tropical setting requires particular attention. This purpose has generated many additional techniques, such as ultradiscrete methods (see e.g. [37, 38]).

An action of $\mathbb{R}_{\max}$ on a statistical model has relevant effects. In particular, the inclusion-exclusion formula (1.3) should be reconsidered since it involves subtraction. Even the equivalent expression $p(A \cup B) + p(A \cap B) = p(A) \cup p(B)$ in not informative in the tropical language, since $p(A \cap B) \leq p(A \cup B)$, so $p(A \cup B) \cap p(A \cap B) = p(A \cup B)$.

For our purposes, it is worth looking at a more general tropical semiring, that is a 5-uple $(\Lambda, \oplus, \odot, \infty, 0)$ where $\Lambda$ is a set, the addition $\oplus$ and multiplication $\odot$ are associative and commutative binary operations on $\Lambda$ with identities and the distributivity property holds. $\infty$ is the neutral element for $\oplus$ and 0 is the neutral element for $\odot$. All the elements of $\Lambda \setminus \{\infty\}$ are multiplicatively invertible and $\odot$ is idempotent, i.e. $a \odot a = a$ for all $a \in \Lambda$.

The tropical monoid $(\Lambda, \oplus, \infty)$ is a sub-structure associated to a tropical semiring that forgets $\odot$ and 0. A monoid homomorphism is a map $\psi : (\Lambda_1, \oplus_1, \infty_1) \rightarrow (\Lambda_2, \oplus_2, \infty_2)$ such that $\psi(\infty_1) = \infty_2$ and $\psi(x \oplus_1 y) = \psi(x) \oplus_2 \psi(y)$ for all $x, y \in \Lambda_1$. An erasing element of a tropical monoid $\Lambda$ is an element $\top \in \Lambda$ such that $a \oplus \top = \top$ for all $a \in \Lambda$. If $\Lambda$ has no erasing element, then we call it a grounded monoid. If one also looks at the multiplication, an erasing element $\top$ satisfies $a \odot \top \odot b = a \odot (\top \odot a^{-1} \odot b) = a \odot \top$ for all $a, b \in \Lambda$. By uniqueness of the erasing element, one has $a \cdot \top = \top$.

When no ambiguity arises, $\Lambda$ will denote both a tropical algebra and the underlying set with a slight abuse of notation. Well-known examples of tropical monoids with $\oplus = \max$ are $\mathbb{N}$, $\mathbb{Q}^\vee := \mathbb{Q} \cup \{-\infty\}$ and similarly $\mathbb{R}^\vee$, while $\mathbb{R}^\wedge := \mathbb{R} \cup \{+\infty\}$ involves $\oplus = \min$.

From idempotence and associativity, it follows that the tropical addition $\oplus$ induces an order on $\Lambda$, that is

$$x \preceq y \iff x \oplus y = y$$  \hspace{1cm} (2.2)

and $\infty$ is the minimum element of $\Lambda$ with respect to this order. We adopt the following notation for ordered sets: a partially ordered set (or poset) is a pair $(\Lambda, \preceq)$ where $\preceq$ is a binary reflexive, antisymmetric and transitive relation on $\Lambda$. A poset is totally ordered if any two elements $x, y \in \Lambda$ are comparable, i.e. $x \preceq y$ or $y \preceq x$ for all $x, y \in \Lambda$. The minimum (respectively, maximum) of a poset, if it exists, is the unique element $\bot$ (respectively, $\top$), such that $\bot \preceq x$ (respectively, $x \preceq \top$) for all $x \in \Lambda$. The infimum of a subset $\lambda \subseteq \Lambda$, if it exists, is the unique element $\inf(\lambda) \in \Lambda$ such that i.) $\inf(\lambda) \preceq x$ for all $x \in \lambda$, and ii.) if $y \in \Lambda$ is such that $y \preceq x$ holds for all $x \in \lambda$, then $y \preceq \inf(\lambda)$. The supremum of a subset $\lambda \subseteq \Lambda$ is the infimum of $\lambda$ in the poset $(\Lambda, \succeq)$ given by the inverse relation of $\preceq$. A join-semilattice (respectively, meet-semilattice) is a poset $(\Lambda, \preceq)$ such that each pair of elements of $\Lambda$, or equivalently each finite subset of $\Lambda$, has a supremum (respectively, an infimum). A lattice is simultaneously a join- and a meet-semilattice. A complete lattice is a lattice $(\Lambda, \preceq)$ where arbitrary (even infinite) suprema and infima exist.

With previous definitions, we will call grounded poset a poset $(\Lambda, \preceq, \bot)$ with a minimum $\bot$. 

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and without maximum. Note that a grounded tropical monoid can not be a complete lattice.

Indeed, \( \sup \Lambda \) does not exists, since it would be an erasing element.

Now we can introduce filters as follows. Given a poset \((\Lambda, \preceq)\), a **filter** on \(\Lambda\) is a collection \(\mathcal{F} \subseteq \Lambda\) of elements of \(\Lambda\), such that the following properties hold:

1. \( \mathcal{F} \neq \emptyset \);
2. for all \( x, y \in \mathcal{F} \), there exists \( z \in \mathcal{F} \) such that \( z \preceq x \) and \( z \preceq y \) (\(\mathcal{F}\) is downward directed);
3. for all \( x \in \mathcal{F} \), if \( x \preceq y \) then \( y \in \mathcal{F} \) (\(\mathcal{F}\) is upward closed).

If the following additional property holds

\[
A \in \mathcal{F} \iff A^c \notin \mathcal{F}, \quad A \in \mathcal{P}(\Omega),
\]

one talks of \(\mathcal{F}\) as an **ultrafilter** on \(\Omega\). It is equivalent to the property that, for every filter \(\mathcal{V}\) such that \(\mathcal{F} \subseteq \mathcal{V}\), one has \(\mathcal{F} = \mathcal{V}\), e.g. \(\mathcal{F}\) is a maximal filter. A filter such that \(\bot \notin \mathcal{F}\), i.e. \(\mathcal{F} \neq \Lambda\), is said **proper**. Note that if \((\Lambda, \preceq)\) is a lattice, then property 2. in the definition of filters can be restated using property 3. as \(\inf\{x, y\} \in \mathcal{F}\) for all \(x, y \in \mathcal{F}\). A filter is called a **principal filter** if it is of the form

\[
\mathcal{F}_x := \{y \in \Lambda : x \preceq y\}
\]

for some \(x \in \Lambda\). If \(\mathcal{B} \subseteq \Lambda\) verifies properties 1. and 2. and \(\bot \notin \mathcal{B}\), then \(\mathcal{B}\) is a **proper filter base**. One can extend a filter base \(\mathcal{B}\) to \(\mathcal{F} := \{C \in \Lambda : \exists B \in \mathcal{B}, B \preceq C\}\), which is a filter.

The dual notion of a filter is an **ideal** on \(\Lambda\), that is a collection \(\mathcal{I} \subseteq \Lambda\) of elements of \(\Lambda\) such that:

1. \(\mathcal{I} \neq \emptyset\);
2. for all \( x, y \in \mathcal{I} \), there exists \( z \in \mathcal{I} \) such that \( x \preceq z \) and \( y \preceq z \) (\(\mathcal{I}\) is upward directed);
3. for all \( x \in \mathcal{I} \), if \( y \preceq x \) then \( y \in \mathcal{I} \) (\(\mathcal{I}\) is downward closed).

An ideal of the form

\[
\mathcal{I}_y := \{x \in \Lambda : x \preceq y\},
\]

is called a **principal ideal**. If \(\mathcal{B} \subseteq \Lambda\) verifies the properties 1. and 2. and \(\top \notin \mathcal{B}\), then \(\mathcal{B}\) is a **proper ideal base**. One can extend an ideal base \(\mathcal{B}\) to \(\mathcal{I} := \{C \in \Lambda : \exists B \in \mathcal{B}, C \preceq B\}\), which is an ideal.

In particular, for any tropical semiring one has \(a \oplus (a \oplus b) = (a \oplus a) \oplus b = a \oplus b\), then \(a \preceq a \oplus b\). In the same way, \(b \preceq a \oplus b\). If \(z \in \Lambda\) is such that \(a \preceq z\) and \(b \preceq z\) then \(a \oplus z = z\) and \(b \oplus z = z\), thus \((a \oplus b) \oplus z = a \oplus (b \oplus z) = a \oplus z = z\). So \(a \oplus b \preceq z\). This corresponds to the fact that

\[
a \oplus b = \sup\{a, b\}
\]

and \(\Lambda\) is a join-semilattice.
Tropical limit and the role of Boltzmann constant

The physical parameter which controls the occurrence of real or tropical features is the Boltzmann constant \([9]\). More to the point, given a physical statistical system specified by a partition function and an associated free energy, its \textit{tropical limit} is defined as the simultaneous limit for \(k_B\) and Avogadro number \(N_A\) such that their product is kept constant, i.e.

\[
k_B \rightarrow 0^+, \quad N_A \rightarrow \infty: \quad k_B \cdot N_A =: R \text{ is constant.} \quad (3.1)
\]

A deeper understanding of effects of the limit \(k_B \rightarrow 0\) is necessary since \(k_B\) is the fundamental unit that connects the microscopic and macroscopic worlds. As a first check, it should be noted that this is physically reasonable since \(k_B \ll 1\) and \(N_A \gg 1\), while the universal gas constant \(R \sim \mathcal{O}(1)\) is the order of unity, so it is a macroscopically distinguishable quantity.

One can also draw an analogy with the \(\hbar \rightarrow 0\) limit when one has a quantum theory and a consistent procedure, driven by \(\hbar\), that shut off quantum effects. This goes beyond a formal analogy since the limit \(k_B \rightarrow 0\) has been discussed in stochastic processes \([39]\), viewed as the limit of vanishing white noise, and in the analysis of thermodynamic complementarity and fluctuation theory \([40, 41]\). These approaches establish a stronger correspondence between the semiclassical limit \(\hbar \rightarrow 0\) in quantum mechanics and the \(k_B \rightarrow 0\) limit. However, the corresponding “classical” theory has been interpreted as a thermodynamic limit, where a large number of particles \(N\) are involved. It should be stressed that our approach is conceptually different from thermodynamic limit(s), which involves the approximation of random variables with their averages. By contrast, tropical limit is intended to provide a setting where statistical properties can be studied exactly through certain algebraic rules.

Different statistical models can have the same tropical limit. In this sense, the limit \(k_B \rightarrow 0\) provides a tropical classification that leaves control parameters (e.g., temperature \(T\)) as free variables. At the same time, the action on the reference cardinality \(N_A\) in (3.1) results in a different counting process and this affects statistics of a generic system through arithmetic. The scaling of \(N_A\), as well as the algebraic structure that arises in the process, is not explicit in the deterministic theory obtained in \([40, 41]\), even if some similarities can be drawn. The implications of the tropical limit in arithmetic and its connections with the thermodynamic formalism have been discussed in a different framework, namely number theory, see e.g. \([12]\) and references therein. However, these works introduce the tropical limit as a low temperature limit, i.e. \(T \rightarrow 0\). There are both conceptual and practical differences with our approach. First, the limit \(T \rightarrow 0\) reduces the model to a part of the boundary of the space of thermodynamic variables given by the evaluation at \(T = 0\). As already remarked, we preserve the set of free parameters and, hence, thermodynamic relations, through the limit \(k_B \rightarrow 0\). This corresponds to a change in the algebraic structure associated to the variable space.

Secondly, the scaling \(N_A \rightarrow \infty\) and its consequences on the arithmetic process of counting affect the degenerations of energy levels, so it relates to the phenomenon of exponential degenerations. For instance, one can consider the Boltzmann formula for the microcanonical ensemble \(S_B = k_B \cdot \ln \Omega\). The number of microstates \(\Omega\) is the cardinality of the set of microstates compatible with some constraints, i.e. fixed energy and particle number. Non-trivial situations
arise if one assumes that $k_B \to 0$ implies a different way of counting. The cardinality $\Omega$ thereby depends on $k_B$ and exponential degenerations

$$\Omega(k_B) \sim \exp \left( \frac{S}{k_B} \right)$$

(3.2)

make the tropical method non-trivial.

A possible connection can also be drawn to the definition of temperatures in small systems. At least two proposals have received attention, that are Boltzmann and Gibbs (or Hertz) temperatures [42], and there is still much debate on which definition should be adopted [43, 42]. In both cases, they are deduced from entropy via $T = \left( \frac{\partial S}{\partial E} \right)^{-1}$. One has the Gibbs temperature $T_G$ if $S = S_G(E) := k_B \cdot \ln \Omega(E)$ is Gibbs’ entropy, where $\Omega(E)$ is the number (or in general a measure) of microstates with energy $\tilde{E} \leq E$. The Boltzmann temperature comes from Boltzmann’s entropy $S_B(E) = k_B \cdot \ln \omega(E)$, where $\omega(E) = \frac{\partial \Omega(E)}{\partial E}$ is the number of states with energy $\tilde{E} = E$. $T_G$ is positive while $T_B$ can assume negative values, for example in cases of bounded spectrum. In the limit $k_B \to 0$, assuming the scaling (3.2) and additional hypotheses, e.g. non-vanishing heat capacity, the two definitions can coincide. When $T_B$ equals $T_G$ at $k_B = 0$, only boundary terms in $\Omega(E)$ are relevant for temperature, since they represent microstates defining $\omega(E)$. Boltzmann’s entropy is assumed real, then $\omega(E) \geq 0$ for all $E$ and $\Omega(E)$ increases with $E$. So these boundary terms are the dominant ones.

The correspondence between expressions “$\leq E$” and “$= E$” in previous statements is a focal point in our discussion. Indeed, we will consider microsystems defined by fixed energy and associated statistical data (e.g., degeneration of the energy level). On the other hand, we will denote collections of more microsystems as macrosystems. Hence, the previous observation relates microsystems to certain macrosystems and will be used to realize a tropical correspondence.

Before we proceed to this issue, it is worth making a few remarks on the meaning of the tropical limit at the fundamental (set-theoretic) level. The change in the counting process also prompts the use of a different notion of cardinality or different notion of sets. In this regard, it is worth noting that the limit $k_B \to 0$ can be interpreted as a fuzzyfication, see [46, 47] for a detailed introduction about fuzzy mathematics and [6] for its latest developments. Fuzzy variables for statistical purposes has been used in theoretical computer science [48]. For example, the fuzzy $c$-means in the context of clustering (see e.g. [49]) have a direct analogue in our statistical setting, as can be noted identifying $m - 1$ with $k_B$ in [49].

Boltzmann constant has also a key role in representing information. For example, if $x$ in (2.1) is interpreted as the number of available digits in a certain representation of a number, then the scaling $x \to \frac{x}{k_B}$ describes a generalized change of base. In this aspect, Boltzmann constant is recognized as a bridge between Gibbs and Shannon entropy and, more generally, between thermodynamics and information theory [50].

In all these perspectives, what is left in the limit $k_B \to 0$ is the underlying relational order. We explore some aspects of this relation in the following section.
4 Filters and ultrametricity

One of the most important geometric aspects of the tropical algebra is ultrametricity. An ultrametric on a set $\Omega$ is a metric where the triangle inequality is tropicalized, i.e.

$$d(x, y) \leq d(x, z) \oplus d(z, y) = \max\{d(x, z), d(z, y)\}, \quad x, y, z \in \Omega. \quad (4.1)$$

Ultrametricity is a phenomenon that characterizes many hierarchical models. Accordingly, it has important applications in statistical and complex systems. For example, ultrametricity was recognized in spin glasses [51] and some models have been proposed for a better understanding of this pattern, for example by means of the $p$-adic metric [52, 53, 34]. We remind that, if $\Omega \equiv \mathbb{Q}$ and $p$ is a prime number, then the $p$-adic norm

$$||x||_p = p^{-a} \iff x = p^a \cdot \frac{q}{r}, \quad \gcd(q, p) = \gcd(r, p) = 1, \quad (4.2)$$

with the assumption $||0||_p = 0$, induces the $p$-adic ultrametric $||x - y||_p$ on $\mathbb{Q}$.

For our purposes, the poset $(\mathcal{P}(\Omega), \subseteq)$ of the subsets of a set $\Omega$ ordered by inclusion has a cardinal role. In such a case, the duality between filters and ideals is also restated as follows.

**Lemma 1.** Given a set $\Omega$, $\mathcal{I} \subseteq \mathcal{P}(\Omega)$ is an ideal if and only if $\mathcal{F} := \{\Omega \setminus A : A \in \mathcal{I}\}$ is a filter.

**Proof:** The proof is a straightforward consequence of the definitions of filter and ideal, see also [54]. \[\square\]

There is a connection between metric and topology, which is expressed by the fact that the set $\mathcal{B} = \{S(x_0, r) : x_0 \in \Omega, r \in \mathbb{R}_+\}$ of open balls $S(x_0, r) := \{x \in \Omega : d(x, x_0) < r\}$ for the metric $d$ is a base for a topology on $\Omega$. In the particular case of ultrametrics, a similar relation can be found with filters (or ideals). For notational convenience, let us denote $\mathbb{R}_+^\backslash := \{x \in \mathbb{R} : x > 0\} \cup \{+\infty\}$ and $d(x_0, \cdot) := \{d(x_0, x) : x \in \Omega\}$.

**Proposition 1.** Let $d$ be an ultrametric on a set $\Omega$. Then, the set $\mathcal{B} := \{S(x_0, r) : x_0 \in \Omega, r \in d(x_0, \cdot)\}$ of ultrametric balls $S(x_0, r) := \{x \in \Omega : d(x, x_0) \leq r\}$ is an ideal base. Moreover, the resulting ideal is proper if and only if $d(\Omega^2)$ is grounded as a sublattice of $\mathbb{R}_{>0}$, i.e. $\max\{d(x, y) : x, y \in \Omega\}$ does not exists. Vice versa, if $\mathcal{I}$ is an ideal on $\Omega = \bigcup_{A \in \mathcal{I}} A$, then the function

$$(x, y) \mapsto d(x, y) := \begin{cases} 0, & x = y \\ \inf \{d(A) : A \in \mathcal{I}, \{x, y\} \subseteq A\}, & x \neq y \end{cases} \quad (4.3)$$

is an ultrametric on $\Omega$, for each decreasing function $d : (\mathcal{I}, \subseteq) \longrightarrow (\mathbb{R}_+^\backslash, \leq)$ such that $\inf d(\mathcal{I}) > 0$.

**Proof:** For the sake of clarity, the proof is presented in Appendix A. \[\square\]

It should be noted that previous proposition does not hold for a general metric. For example, let

$$\Omega := \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \left\{ 3 - \frac{1}{m} : m \in \mathbb{N} \right\} \quad (4.4)$$
with the usual euclidean metric $d_E$. Note that $d_E(\Omega^2)$ is grounded since $\sup d_E(\Omega^2) = 3 \notin d_E(\Omega^2)$. So $1 = d_E(1, 2)$ belongs to both $d_E(1, \cdot)$ and $d_E(2, \cdot)$. Hence, one has $S(1, 1) \cup S(2, 1) = \Omega$ that is not contained in any ball of the type $S(x_0, r)$ with $x_0 \in A$ and $r \in d_E(x_0, \cdot)$.

We remark that the function (4.3), $d$ is allowed to assume the value $+\infty$. In such a case, if $d(A) = +\infty$ for all $A \in I$ such that $\{x, y\} \subseteq A$, then $x$ and $y$ are at infinite distance. This can be an interesting eventuality, but if one wants to avoid it one can use the function $g := g \circ d$ instead of $d$, where

$$g(x) := \left(1 + (x^{-1})^{-1}\right)$$

for all $x \in \mathbb{R}^+_\infty$. So $g$ is decreasing, bounded by 1 from above and strictly positive since $0 < \inf d(I) \leq d(A)$ for all $A \in I$.

If one starts with an ultrametric to get an associated ideal, the corresponding filter (Lemma 1) can be used to recover the original ultrametric. Indeed, one has the following

**Corollary 1.** If $\mathcal{F}$ is a filter on $\Omega$ and $\emptyset = \bigcap_{G \in \mathcal{F}} G$, then the function

$$(x, y) \mapsto D(x, y) := \begin{cases} 0, & x = y \\ \inf \{d(G) : G \in \mathcal{F}, \{x, y\} \cap G = \emptyset\}, & x \neq y \end{cases}$$

is an ultrametric on $\Omega$, for each increasing function $d : (\mathcal{F}, \subseteq) \rightarrow (\mathbb{R}^+_\infty, \leq)$ such that $\inf d(\mathcal{I}) > 0$. Moreover, if $d$ is an ultrametric on $\Omega$, $\mathcal{I}_d$ is the ideal generated by ultrametric balls, $\mathcal{F}_d$ is the corresponding filter, then $D_d = d$ if

$$d(F) = d_d(F) := \sup \{d(x, y) : x, y \notin F\}.$$  

**Proof:** Let $\mathcal{F}$ and $d$ be as in the claim and say $D(x, y) := \{d(G) : G \in \mathcal{F}, \{x, y\} \cap G = \emptyset\}$, $x \neq y$. Then, consider the corresponding ideal $I$ as in Lemma 1 and the function $d_I : I \rightarrow \mathbb{R}^+_\infty$ defined as $d_I(A) := d(\Omega \setminus A)$. In particular, $d_I$ is decreasing, and $d(\mathcal{F}) = d_I(I)$ implies $\inf d_I(I) = \inf d(\mathcal{F}) > 0$. From these data one can produce the mapping $D_I$ and the function $d$ as in Proposition 1. By definitions, one has a correspondence between $G \in \mathcal{F}$, $\{x, y\} \cap G = \emptyset$ and $\Omega \setminus G \in I$, $\{x, y\} \subseteq \Omega \setminus G$, hence $D(x, y) = D_I(x, y)$ and $D(x, y) = \inf D(x, y) = d(x, y)$. So $D$ is an ultrametric.

Now let $d$ be an ultrametric on $\Omega$, $\mathcal{I}_d$ the ideal generated by the base of ultrametric balls and $\mathcal{F}_d$ the corresponding filter. Let us consider the function $d_d$ as in (4.7). The corresponding ultrametric $D(x, y)$ obtained from (4.6) coincides with $d$. In fact, if $x \neq y$ and $\{x, y\} \cap G = \emptyset$, then one has $d(x, y) \leq d_d(G)$ thus $d(x, y) \leq D(x, y)$. Moreover, $\{x, y\} \cap (\Omega \setminus S(x, d(x, y))) = \emptyset$, so $D(x, y) \leq \sup \{d(u, v) : u, v \notin (\Omega \setminus S(x, d(x, y)))\} = \sup \{d(u, v) : u, v \in S(x, d(x, y))\} = d(x, y)$. Hence, $D(x, y) = d(x, y)$.

**4.1 Discussion on the conditions in Proposition 1**

An example of the relation between filters, ideals and ultrametric is the one of $p$-adic norm (4.2). One has $d(\mathbb{Q}^2) \subseteq \{0\} \cup \{p^n : n \in \mathbb{Z}\}$ by definition. Vice versa, for all $n \in \mathbb{Z}$, one has $d(p^{-n}, 0) = ||p^{-n}||_p = p^n$ and $d(0, 0) = 0$. Thus $d(\mathbb{Q}^2) = \{0\} \cup \{p^n : n \in \mathbb{Z}\}$ is grounded
and the set of $p$-adic balls generates a proper ideal. On the other hand, the filter generated by $p$-adic balls and the function $d_p(G) := \sup \{||x-y||_p : x, y \notin G\}$ returns the $p$-adic ultrametric $D_p(x, y) := ||x-y||_p$.

The condition $\inf \mathfrak{d}(\mathcal{I}) > 0$ (respectively, $\inf \mathfrak{d}(\mathcal{F}) > 0$) in Proposition 1 (respectively, Corollary 1) is due to the choice $\max$ for the tropical sum in (4.1) and can be relaxed with a different presentation of $\ominus$, as it will be shown in next section.

The condition is sufficient to avoid the degeneration of $d$. If this request is not satisfied, degeneration could occur. For example, let us take $\Omega = \mathbb{N}$ and $\mathcal{I}_{\text{fin}} = \{A \subseteq \mathbb{N} : |A| < \infty\}$, that is the ideal of finite subsets of $\mathbb{N}$. Let $\mathfrak{d}(A) := 1 - \sum_{\alpha \in A} \frac{1}{2^n}$, $A \in \mathcal{I}$. Note that $\mathfrak{d}(\mathcal{I}) \subseteq \mathbb{R}_{>0}$ and $\mathfrak{d}$ is decreasing, but sets $A_n := [n] = \{1, \ldots, n\}$, $n \geq 3$, satisfy $\{1, 2\} \in A_n, A_n \in \mathcal{I}_{\text{fin}}$ and $\mathfrak{d}(A_n) = 1 - \sum_{i=1}^n \frac{1}{2^n} \xrightarrow{n \to \infty} 0$. So $d(1, 2) = 0$. Anyway, this condition is not necessary. For example, in the $p$-adic case (4.2) one has $\mathfrak{d}(\mathbb{Q} \setminus S(x, p^{-n})) = \sup \{||x-y||_p : ||x-y||_p \leq p^{-n}\} = p^{-n} \xrightarrow{n \to +\infty} 0$. Thus $\inf \mathfrak{d}(\mathcal{F}) = 0$, but the resulting $D_p(x, y) = ||x-y||_p$ is not degenerate.

The monotony condition on $\mathfrak{d}$ is sufficient but not necessary too. For example, $\mathfrak{d}(F) = \sup\{d(x,y) : x, y \notin F\}$ in Corollary 1 is not increasing, in fact it is decreasing. Nevertheless, it generates an ultrametric since it is derived from an ultrametric. In general, non-monotone functions do not return ultrametrics. An example is the filter $\mathcal{F}_{\text{fin}}$ associated to $\mathcal{I}_{\text{fin}}$ with any function $\mathfrak{d} : \mathcal{F}_{\text{fin}} \to \mathbb{R}$ such that $\mathfrak{d}(F) > 1 + \mathfrak{d}(G)$ for all $F, G \in \mathcal{F}$ with $\{1, 2\} \cap F = \emptyset$ and $\{1, 2\} \cap G \neq \emptyset$. Such functions are not increasing. One also finds $d(1, 3) \leq \mathfrak{d}(\mathbb{N}\setminus\{1, 3\})$ and $d(2, 3) \leq \mathfrak{d}(\mathbb{N}\setminus\{2, 3\})$, thus $\max\{d(1, 3), d(2, 3)\} \leq \max\{\mathfrak{d}(\mathbb{N}\setminus\{1, 3\}), \mathfrak{d}(\mathbb{N}\setminus\{2, 3\})\} \leq d(1, 2) - 1$. Hence $\max\{d(1, 3), d(2, 3)\} < d(1, 2)$ and ultrametric triangle inequality does not hold.

### 4.2 The case of finite sets

The image $d(\Omega^2)$ is not grounded in case of finite sets $\Omega$, so the resulting condition returns the trivial filter (and ideal) $\mathcal{P}(\Omega)$. Anyway, filters on a finite set $\Omega$ are easily described by the following well-known result (see e.g. [20]).

**Proposition 2.** If $\#\Omega < \infty$ and $\mathcal{F}$ is a filter on $\Omega$, then there exists a unique non-empty subset $\zeta \subseteq \Omega$ such that $\mathcal{F} = \{\Theta \subseteq \Omega : \zeta \subseteq \Omega\}$. Moreover, $\mathcal{F}$ is an ultrafilter if and only if $\#\zeta = 1$.

Previous proposition means that filters on finite sets are principal. Non-principal (ultra-)filters appear for infinite sets and play an important role in mathematical logic through the concept of $\mathcal{F}$-limit [20]. On the other hand, principal filters satisfies $\zeta(\mathcal{F}) = \bigcap_{H \in \mathcal{F}} H \neq \emptyset$. In that case, if $z \in \zeta(\mathcal{F})$, then $\mathfrak{D}(z, y) = \emptyset$ in Corollary 1, so it is consistent to assign $d(z, y) = \inf \emptyset := \sup \mathbb{R}_{>0} = +\infty$. If one acts via $g$ in (4.5), the resulting ultrametric is $g(d(x,y)) \in [0,1]$, that vanishes if and only if $x=y$ and equals 1 if and only if $\{x, y\} \cap \zeta(\mathcal{F}) \neq \emptyset$.

The importance of filters on finite sets also lies in its connections with the statistical amoeba formalism developed in [55]. The instability domain for a statistical amoeba is the locus of points $x$ in the parameter space $\mathbb{R}^n$ where the family

$$\mathcal{N}_k(x) := \{\mathcal{I} \subseteq [N] : \#\mathcal{I} = k, Z_k(\mathcal{I}; x) < 0\} \quad (4.8)$$
has maximal cardinality, where $k < N + \frac{1}{2}$ and
\[
Z_k(\mathcal{I}; x) := -\sum_{\alpha \in \mathcal{I}} e^{f_\alpha(x)} + \sum_{\beta \notin \mathcal{I}} e^{f_\beta(x)}.
\] (4.9)

These quantities were introduced in order to study real points where the partition function becomes singular and explore the associated metastability. In particular, the results in [55] mean that $N_k(x)$ is induced by an ultrafilter if and only if $x$ belongs to the instability domain $D_{k-}$.

5 Monoid homomorphisms and a set/element correspondence

The combinatorial data that remain in the tropical limit of a statistical system are based on an order relation, which is captured by the concepts of filters and ideals. In the tropical perspective, these order relations separate the notions of a function and of its expressions. Indeed, let us consider tropical polynomials, that are algebraic expressions of the form
\[
\bigoplus_{I \in \mathbb{I}} a_I \odot X^{\odot I}.
\] (5.1)

where $\mathbb{I}$ is a finite set of multi-indices $I := (i_1, \ldots, i_n)$, $\{a_I\}_{I \in \mathbb{I}}$ are coefficients and $X^{\odot I} := X_1^{\odot i_1} \odot \cdots \odot X_n^{\odot i_n}$. The expression (5.1) is equivalent to
\[
\left(\bigoplus_{I \in \mathbb{I}} a_I \odot X^{\odot I}\right) \oplus a_H \odot X^{\odot H}
\] (5.2)

for any $j \in \mathbb{N} \cup \{0\}$ and $a_H \in \{a_I\}_{I \in \mathbb{I}}$. Such a redundancy in the description of algebraic functions in many different but equivalent forms is distinctive of tropical algebra and can be referred as tropical symmetry.

So, if $F(x) : \Omega \rightarrow \Lambda$ is a tropical expression from a set $\Omega$ in a tropical monoid $\Lambda$, then all the expressions in the set $\mathcal{I}_{F(x)} := \{G(x) : G(x) \preceq F(x)\}$ satisfy $F(x) \oplus \mathcal{I}_{F(x)} = \{F(x)\}$. The set $\mathcal{I}_{F(x)}$ is a principal ideal for the poset $\Lambda$. Then, the equivalence of (5.1) and (5.2) can be stated using the language of filters and ideals.

For tropical monoids, an erasing element $\top$ is the maximum for the associated order, so grounded monoids are grounded posets. As already remarked in the Introduction, monoids give a special role to the neutral element $\infty$ of $\oplus$, which is an extremal for the corresponding order. The fact that $\infty$ is the only extremal element in grounded posets is fundamental in the symmetry breaking between max and min. Lemma 1 basically identifies ideals and filters on the power set $\mathcal{P}(\Omega)$, which is not a grounded monoid since it has both a minimum $\emptyset$ and a maximum $\Omega$.

In case of grounded monoids, the relation between filters and ideal is more involved and has relevant physical interpretations connected with the max/min duality.

A first implication is that the absence of an erasing element is an obstruction for the definition of a dual operation $\loplus$ on $\Lambda$ such that $(\Lambda, \loplus)$ is a tropical monoid and the order $\succeq$ induced by $\loplus$ is the opposite of $\preceq$. For instance, that gives a split between the max-plus and the min-
plus algebras. In applications, it distinguishes different constraints on physical systems, since the stability of a configuration is often determined by extremality conditions. This is the case of a system defined by a Lagrangian $L := \frac{1}{2}m\|\dot{x}\|^2 - \phi(x)$, local minima for the potential $\phi(x)$ are stable equilibrium points, while local maxima are unstable equilibrium points. This symmetry breaking leads to an orientation expressed by the order relation $\leq$, which is the “tropical skeleton” discussed in the Introduction.

The max\min splitting also concerns the ultrametric triangle inequality (4.1). Its form with $\oplus = \min$ is

$$\tilde{d}(x, z) \oplus \tilde{d}(z, y) = \min \{ \tilde{d}(x, z), \tilde{d}(z, y) \} \leq \tilde{d}(x, y), \quad x, y, z \in \Omega. \quad (5.3)$$

Both (4.1) and (5.3) are summarized in the expression

$$\tilde{d}(x, z) \oplus \tilde{d}(z, y) \oplus \tilde{d}(x, y) = \tilde{d}(x, z) \oplus \tilde{d}(x, y), \quad x, y, z \in \Omega. \quad (5.4)$$

On the other hand, filters and ideals distinguish these forms. The next result is a straightforward consequence of previous observations and stresses the effect of dual presentations (orders) of grounded posets on the geometry of the system (non-degeneracy conditions for the ultrametric).

**Corollary 2.** Let $I$ be an ideal on $\Omega$, $\bigcup_{A \in I} A = \Omega$ and $\mathfrak{d} : I \rightarrow \mathbb{R}_+^\wedge$ be any increasing function. Then the function

$$\mathfrak{d}(x, y) := \begin{cases} 0, & x = y \\ \sup \{ \mathfrak{d}(A) : A \in I, \{x, y\} \subseteq A \}, & x \neq y \end{cases} \quad (5.5)$$

is an ultrametric in the form (5.3). In particular, if $d$, $I_d$, $F_d$ and $\mathfrak{d}_d$ are as in Corollary 1, then $\mathfrak{d}_d$ induces the presentations (4.1) and (5.3) for the same ultrametric structure, when applied to $I_d$ and $F_d$ respectively.

**Proof:** Let $I$ be an ideal and consider any increasing positive function $\mathfrak{d} : (I, \subseteq) \rightarrow (\mathbb{R}_+^\wedge, \leq)$.

One can see that $d$ is symmetric and $\mathfrak{D}(x_1, x_2) \neq \emptyset$, so $d(x_1, x_2)$ is strictly positive if $x_1 \neq x_2$. One also gets $\min \{ d(x, y), d(y, z) \} \leq d(x, z)$ by using the same arguments as in Proposition 1. From Corollary 1, $\mathfrak{d}_d$ returns the form (4.1) for the ultrametric structure of $d$ when (4.3) is applied to the ideal $I_d$. \hfill \square

### 5.1 Duality and the role of linear orders

We now focus on a class of tropical algebras that offers a simple but flexible framework for physical applications. So, we consider a tropical monoid $\Lambda$ and introduce

$$i(y) := \{ x \in \Lambda : x \prec y \}, \quad I_\Lambda := \{ i(y) : y \in \Lambda \}$$

$$\phi(y) := \{ x \in \Lambda : y \preceq x \}, \quad F_\Lambda := \{ \phi(y) : y \in \Lambda \}. \quad (5.6)$$

In principle the order $\preceq$ can be general. In fact, this freedom allows one to deal with a broader class of orders, including set-theoretic inclusion $\subseteq$. Anyway, the use of filters of the form (5.6) gives a special role to totally (or linearly) ordered sets. Indeed, one can characterize totally ordered sets through the following simple property.
Proposition 3. The mapping \( \iota : (\Lambda, \oplus, \infty) \rightarrow (\mathcal{I}_\Lambda, \cup, \emptyset) \) defined in (5.6) is a monoid homomorphism if and only if \((\Lambda, \oplus, \infty)\) is totally ordered. The mapping \( \phi : (\Lambda, \oplus, \infty) \rightarrow (\mathcal{F}_\Lambda, \cap, \Lambda) \) is always a monoid homomorphism.

Proof: \( \emptyset = \iota(\infty) \) belongs to \( \mathcal{I}_\Lambda \) and it is a neutral element for \( \cup \). So \((\mathcal{I}_\Lambda, \cup, \emptyset)\) is a tropical monoid if and only if, for all \( x, y \in \Lambda \), there exists \( z \in \Lambda \) such that \( \iota(x) \cup \iota(y) = \iota(z) \). Let us assume that \( \Lambda \) is not totally ordered, then there exist \( x, y \in \Lambda \) such that \( x \oplus y \notin \{x, y\} \).

In particular \( x \neq y \), \( x \notin \iota(y) \) and \( y \notin \iota(x) \). Let us take \( z \in \Lambda \) such that \( \iota(x) \cup \iota(y) = \iota(z) \). From previous observation one has \( \{x, y\} \cap \iota(z) = \emptyset \). So \( x \in \iota(x \oplus y) \) but \( x \notin \iota(z) = \iota(x) \cup \iota(y) \), hence \( \iota(x \oplus y) \neq \iota(x) \cup \iota(y) \). Thus \( \iota \) is not a monoid homomorphism. If instead \( \Lambda \) is a linearly ordered set, then it is easy to see that \( \iota(a) \cup \iota(b) = \iota(\sup\{a, b\}) = \iota(a \oplus b) \), so \((\mathcal{I}_\Lambda, \cup, \emptyset)\) is a tropical monoid and \( \iota \) is a monoid homomorphism.

Now let us consider the tropical monoid \((\mathcal{F}_\Lambda, \cap, \Lambda)\). If \( z \in \phi(x) \cap \phi(y) \) then \( x \preceq z \) and \( y \preceq z \), hence \( x \oplus y \preceq z \) and \( z \in \phi(x \oplus y) \). Thus \( \phi(x) \cap \phi(z) \subseteq \phi(x \oplus y) \). If \( z \in \phi(x \oplus y) \) then \( x \preceq x \oplus y \preceq z \) so \( z \in \phi(x) \). Similarly \( z \in \phi(y) \), hence \( z \in \phi(x) \cap \phi(y) \). This means that \( \phi(x) \cap \phi(y) = \phi(x \oplus y) \). Moreover \( \phi(\infty) = \Lambda \), hence \( \phi : (\Lambda, \oplus, \infty) \rightarrow (\mathcal{F}_\Lambda, \cap, \Lambda) \) is always a monoid homomorphism.

Set complementation \( \epsilon : (\mathcal{P}(\Lambda), \cup, \emptyset) \rightarrow (\mathcal{P}(\Lambda), \cap, \Lambda) \) is a monoid isomorphism. In the case of totally ordered lattices \( \Lambda \), the restriction \( \hat{\epsilon} := \epsilon|_{\mathcal{I}(\Lambda)} \) acts as \( \hat{\epsilon}(\iota(x)) = \phi(x) \) and is a monoid isomorphism that allows to identify \((\mathcal{I}_\Lambda, \cup, \emptyset)\) and \((\mathcal{F}_\Lambda, \cap, \Lambda)\) through \( \phi = \hat{\epsilon} \circ \iota \). Thus, linearly ordered sets imply the compatibility of set complementation and algebraic characteristics.

Proposition 3 also underlines a correspondence between elements and \( \) (a class of) subsets of a monoid \( \Lambda \) that is compatible with tropical composition. As already mentioned in the Introduction, such a set/element correspondence is manifest in the physical identification of the tropical limit with the scaling \( N_\Lambda \rightarrow \infty \). In this context, the set/element correspondence suggested in Section 3 involves the tropical totally ordered monoid \( \mathbb{N}^\vee \) and the bijection between \( N_\Lambda \in \mathbb{N}^\vee \) and \( \phi(N_\Lambda) = \{n \in \mathbb{N}^\vee : N_\Lambda \leq n\} \in \mathcal{F}_\mathbb{N} \) in (5.6).

One can also look at the set of principal ideals

\[ \phi_c(y) := \{x \in \Lambda : x \preceq y\}, \quad \mathcal{J}_\Lambda := \{\phi_c(y) : y \in \Lambda\}. \quad (5.7) \]

All posets \( \phi_c \) have both a minimum \( \bot \) and a maximum \( y \), hence none of them is grounded. Clearly, \( \phi_c \) is never a monoid homomorphism between \((\Lambda, \oplus, \bot)\) and \((\mathcal{P}(\Lambda), \cup, \emptyset)\) since \( \phi_c(\bot) = \{\bot\} \neq \emptyset \). Anyway, it induces a monoid homomorphism \( \hat{\phi}_c := (\Lambda, \oplus, \bot) \rightarrow (\mathcal{P}(\Lambda \setminus \{\bot\}), \cup, \emptyset) \) by “grounding”, i.e. considering \( \hat{\phi}_c(y) = \phi_c(y) \setminus \{\bot\} \). This gives the same result as \( \phi \) in (5.6) with the opposite order \( \succeq \). Proposition 3 shows that the slight difference between (grounded) \( \prec \) in (5.6) and (non-grounded) \( \preceq \) in (5.7) brings to different constraints in order to preserve algebraic properties.

5.2 Linear orders via homomorphisms

If the poset \( \Lambda \) is not totally ordered, then there could exist \( y \in \Lambda \) such that \( \phi_c(y) \) is not grounded even if \( \Lambda \) be a grounded monoid. In such a case, there exist \( y, \top, \top_1 \in \Lambda \) such that \( \max \phi_c(y) = \top \prec \top_1 \).
In the meantime, non-totally ordered sets give one the freedom to have several totally ordered substructures (chains) in the same framework. A way to extract totally ordered sets from posets is given by monoid homomorphisms from a totally ordered set $\Delta$. So, for a monotone increasing $\vartheta : \Delta \rightarrow \Lambda$ one can define the maps

$$
\iota_{\vartheta} (a) := \iota(\vartheta (a)), \quad \phi_{\vartheta} (a) := \phi_{\vartheta}(\vartheta (a)) .
$$

(5.8)

Let us consider the totally ordered tropical monoid $\mathbb{R}_{\text{max}}$ and an increasing map $\vartheta : \mathbb{R}^\uparrow \rightarrow \Lambda$. Note that one has $\iota_{\vartheta} (a) \cup (\iota_{\vartheta} (a) \cup \iota_{\vartheta} (b)) = \iota_{\vartheta} (a) \cup \iota_{\vartheta} (b)$ for all $a, b \in \mathbb{R}^\uparrow$, hence $\iota_{\vartheta} (a) \subseteq \iota_{\vartheta} (a) \cup \iota_{\vartheta} (b)$. Thus one has $\iota_{\vartheta} (a) \subseteq \iota_{\vartheta} (\max\{a, b\})$ and $\iota_{\vartheta} (b) \subseteq \iota_{\vartheta} (a \oplus b)$, which implies $\iota_{\vartheta} (a) \cup \iota_{\vartheta} (b) \subseteq \iota_{\vartheta} (\max\{a, b\})$. But $\iota_{\vartheta} (\max\{a, b\}) \in \{\iota_{\vartheta} (a), \iota_{\vartheta} (b)\}$, then $\iota_{\vartheta} (\max\{a, b\}) \subseteq \iota_{\vartheta} (a) \cup \iota_{\vartheta} (b)$.

This means that $\iota_{\vartheta} (a \oplus b) = \iota_{\vartheta} (a) \cup \iota_{\vartheta} (b)$ holds for all increasing functions $\vartheta$ from $\mathbb{R}^\uparrow$ to $\Lambda$. The only condition needed to get a monoid homomorphism is $\iota_{\vartheta} (-\infty) = \{-\infty\}$. Physically, one can say that $\iota_{\vartheta}$ is a monoid homomorphism if and only if it preserves the choice of the vacuum.

Having remarked the opportunity to extract totally ordered sets from general posets using monoid homomorphisms, a natural question in the light of algebraic correspondence between elements and subsets is the extension of homomorphism from sets to their power sets. At this purpose, it is worth referring to a tropical monoid $(\Lambda, \oplus, \infty)$ as an almost complete lattice if the extension

$$
\tilde{\Lambda} := \Lambda \cup \{\top\}
$$

with the relation $a \preceq \top$ for all $a \in \Lambda$ is a join-complete semilattice, i.e. it admits arbitrary sup (and sums).

For example, the tropical monoid $(\Lambda, \oplus, \bot) = (\mathcal{P}_\text{fin}(\mathbb{N}), \cup, \emptyset)$ of finite subsets of $\mathbb{N}$ is grounded since $\mathbb{N}$ is not finite. If $\{A_n\}$ is any collection of elements in $\mathcal{P}_\text{fin}$ and $\bigcup^n \downarrow A_n$ is not finite, then $\mathbb{N} = \sup A_n$ in $\mathcal{P}_\text{fin}(\mathbb{N}) \cup \{\mathbb{N}\}$. So it is also almost complete with the extension $\top = \mathbb{N}$.

On the other hand, let us take $(\mathbb{R}^\uparrow : = \mathbb{R}^\uparrow \setminus \{0\}, \sup, -\infty)$. One can consider an extension $\mathbb{R}^\uparrow \cup \{+\infty\}$ as in (5.9). If there exists $\omega := \sup\{a \in \mathbb{R}^\uparrow : a < 0\} \in \mathbb{R}^\uparrow \cup \{+\infty\}$, then $\omega$ cannot be neither negative, since it would be $\omega < -\frac{\omega}{2} < 0$, nor positive, since $0 < -\frac{\omega}{2} < \omega$ would be an upper bound smaller than the least upper bound. So $\omega$ does not exists.

Even if the distinction of complete semilattices and complete lattices is inessential from the perspective of order theory, it becomes relevant when one looks at the algebraic structure, including homomorphisms. These maps distinguish between an operation and the dual one. Also note that all join-complete semilattices are also almost complete. The following proposition clarifies some simple properties of the link between the tropical structure of $\Lambda$ and of its power set.

**Proposition 4.** The tropical monoid $(\Lambda, \oplus, \infty)$ is an almost complete lattice if and only if $\iota(y)$ is an almost complete lattice for all $y \in \Lambda$. Moreover, if $\psi : (\Lambda, \oplus, \infty) \rightarrow (\tilde{\Lambda}, \oplus, \infty)$ is a monoid homomorphism and $\phi(\Lambda)$ is grounded, then $\Lambda$ is grounded too.

**Proof:** Clearly $\iota(y)$ is an almost complete lattice if and only if $\phi_{\iota}(y)$ is a join-complete semilattice.

Let $\Lambda$ be an almost complete lattice and take its extension $\tilde{\Lambda}$ as (5.9). If $\mathcal{S} \subseteq \phi_{\iota}(y)$ then all elements $z \in \mathcal{S}$ satisfy $z \preceq y$, so $\sup \mathcal{S}$ exists in $\tilde{\Lambda}$ and $\sup \mathcal{S} \preceq y$. In particular, $\sup \mathcal{S} \in \tilde{\Lambda}$.
\( \phi_c(y) \) and \( \phi_c(y) \) is join-complete. Vice versa, let \( \phi_c(x) \) be a join-complete semilattice for all \( x \in \Lambda \) and take any \( S \subseteq \Lambda \). If there exists \( y \in \Lambda \) such that \( y_1 \preceq y \) for all \( y_1 \in S \), then \( S \subseteq \phi_c(y) \), which is join-complete. Hence \( \sup S \) exists and it is an element of \( \phi_c(y) \). Otherwise, for all \( y \in \Lambda \) there exists \( y_1 \in S \) such that \( y_1 \not\preceq y \). So \( T \in \hat{\Lambda} \) is the only upper bound for \( S \), hence it is \( \sup S = T \). This is independent of the choice of \( S \) among all subsets of \( \Lambda \) such that \( \sup S \notin \Lambda \). Finally, if \( T \in T \subseteq \hat{\Lambda} \) then \( \sup T = T \). Thus \( \hat{\Lambda} \) is a join-complete semilattice.

Now, let us assume that \( \psi : (\Omega, \oplus, \top) \rightarrow (\hat{\Lambda}, \oplus, \top) \) is a monoid homomorphism. If \( \psi(\Lambda) \) is grounded, then for all \( a \in \Lambda \) there exists \( b \in \hat{\Lambda} \) such that \( \psi(a) \neq \psi(a) \oplus \psi(b) = \psi(a \oplus b) \), so \( a \neq a \oplus b \). Hence \( \hat{\Lambda} \) is grounded.

Finally, next result extends the link in Proposition 4 to the case of monoid homomorphisms.

**Proposition 5.** Let \( (\Delta, \max, -\infty) \) be a tropical monoid whose induced order is total. So,

1. if \( \psi : (\Delta, \max, -\infty) \rightarrow (\Lambda, \oplus, \top) \) is a monoid homomorphism, then \( \psi(\Delta) \) is totally ordered.
2. If \( \phi_\theta (\Delta, \max, -\infty) \rightarrow (\{ \Lambda \subseteq \Lambda : \top \in A \}, \cup, \{ \top \}) \) in (5.8) is a monoid homomorphism, then \( \Theta : \Delta \rightarrow \Lambda \) is a monoid homomorphism too.
3. If \( \Lambda \) is an almost complete lattice, then for every monoid homomorphism \( \psi : \Delta \rightarrow \mathcal{P}(\Lambda) \) there exist a sublattice \( \Delta_0 \subseteq \Delta \) and monoid homomorphisms \( \vartheta : \Delta_0 \rightarrow \Lambda \) and \( \iota : \Delta_0 \rightarrow \mathcal{P}(\Lambda) \) such that \( \iota = \iota_\vartheta = \iota \circ \vartheta \) and \( \vartheta \) covers \( \psi \), namely \( \psi(a) \subseteq \iota(a) \cup \{ \vartheta(a) \} = \vartheta_\vartheta(a) \) for all \( \psi(a) \) having an upper bound.

**Proof:** For the sake of clarity, the proof is presented in Appendix B. \( \square \)

The last proposition identifies a simple connection between homomorphisms to a set and to the associated power set. Statement 2. in Proposition 5 describes the extension of a monoid homomorphism to subsets. Vice versa, statement 3. allows one to extract data from a monoid homomorphism to a power set and reduce to the underlying set.

It should be remarked that, if \( \Lambda \) is not an almost complete lattice, statement 3. does not necessarily hold. Let us take \( \Lambda \equiv \mathbb{R}_+^\ast \), which is not almost complete, and a mapping \( \psi : \mathbb{R}_+^\ast \rightarrow \mathcal{P}(\mathbb{R}_+^\ast) \) defined by \( \psi(a) = \{ b \in \mathbb{R}_+^\ast : b \leq a \} \). If such a function \( \vartheta \) would exist, then \( \vartheta(0) \in \iota_\vartheta(\varepsilon) \) for all \( \varepsilon > 0 \). In particular \( \vartheta(0) \leq \sup \psi(\varepsilon) = \varepsilon \) for all \( \varepsilon > 0 \). In the same way one can see that \( \varepsilon = \sup \{ \psi(\varepsilon) \} \leq \vartheta(0) \) for all \( \varepsilon < 0 \). Thus the only possibility for \( \vartheta(0) \) is 0, which is not in \( \mathbb{R}_+^\ast \). The role of almost complete lattices when one looks at chains of a poset \( \Lambda \), in light of Propositions 4 and 5, supports the choice of macrosystems (5.6) for the tropical correspondence.

## 6 Nested tropical limit

Filters on finite sets, whose explicit form is remarked in Proposition 2, also appear in relation to a sort of “perturbative” construction for the tropical limit in statistical physics. Indeed, let us take a partition function

\[
\mathcal{Z}(x) = \sum_{\alpha=1}^{N} e^{f_{\alpha}(x)} (6.1)
\]
relative to (free) energies \( f_\alpha(x), \alpha \in [N] \). Let us introduce the nesting form of type A of the partition function \( Z(x) \) is the set of data generated by the base case

\[
\mathcal{M}_0(x) := \{ \alpha \in [N] : f_\alpha(x) = \max \{ f_\beta(x) : \beta \in [N] \} \},
\]

\[
\mathcal{S}_0(x) := [N] \setminus \mathcal{M}_0,
\]

\[
\mu_0(x) := f_{\alpha_0}(x), \quad \alpha_0 \in \mathcal{M}_0,
\]

\[
\nu_0(x) := \# \mathcal{M}_0(x)
\]  

(6.2)

and recursively extended as follows

\[
\mathcal{M}_\ell(x) := \{ \alpha \in \mathcal{S}_{\ell-1}(x) : f_\alpha(x) = \max \{ f_\beta(x) : \beta \in \mathcal{S}_{\ell-1}(x) \} \},
\]

\[
\mathcal{S}_\ell(x) := [N] \setminus \left( \bigcup_{h=0}^{\ell} \mathcal{M}_h \right),
\]

\[
\mu_\ell(x) := f_{\alpha_\ell}(x), \quad \alpha_\ell \in \mathcal{M}_\ell,
\]

\[
\nu_\ell(x) := \# \mathcal{M}_\ell(x)
\]

(6.3)

up to \( \ell = L \), which is the minimum integer such that \( \mathcal{S}_L(x) = \emptyset \). The nesting form of type B of the partition function \( Z(x) \) is the nesting form of the partition function relative to energies \(-f_\alpha(x), \alpha \in [N]\).

The nesting form of type B can be also obtained by substitution of max with min in (6.2), (6.3). In the finite \( N \) case that we are dealing with, it is defined as

\[
m_\ell(x) := \mathcal{M}_{L-\ell}(x), \quad s_\ell(x) := [N] \setminus \left( \bigcup_{h=0}^{\ell} m_h \right),
\]

\[
\kappa_\ell(x) := \mu_{L-\ell}(x), \quad \lambda_\ell(x) := \nu_{L-\ell}(x)
\]  

(6.4)

for \( 0 \leq \ell \leq L \). The filter \( \mathcal{F} := \{ B \subseteq [N] : m_0(x) \subseteq B \} \) is the set of sure events with respect to the usual probability \( W_\alpha = \# (\{ \alpha \cap m_0(x) \}) \), also introduced in [9].

The data \( \{ \mu_\ell(x), \nu_\ell(x) : \ell \in \{ 0, \ldots, L \} \} \) can be visualized from the partition function in the following equivalent form:

\[
Z(x) = e^{\mu_0(x)} \cdot \left( \nu_0 + e^{-\mu_0(x)} \cdot \sum_{\alpha \in \mathcal{S}_0(x)} e^{f_\alpha(x)} \right)
\]

\[
= e^{\mu_0(x)} \cdot \left( \nu_0 + e^{\mu_1(x) - \mu_0(x)} \cdot \left( \nu_1 + e^{-\mu_1(x)} \cdot \sum_{\alpha \in \mathcal{S}_1(x)} e^{f_\alpha(x)} \right) \right)
\]

\[
= e^{\mu_0(x)} \cdot \left( \nu_0 + e^{\mu_1(x) - \mu_0(x)} \cdot \left( \nu_1 + e^{\mu_2(x) - \mu_1(x)} \cdot \nu_2 + \ldots \right) \right)
\]

\[
\ldots \left( \nu_{L-2} + e^{\mu_{L-1}(x) - \mu_{L-2}(x)} \cdot \left( \nu_{L-1} + e^{\mu_L(x) - \mu_{L-1}(x)} \cdot \nu_L \right) \right) \right)\).
\]  

(6.5)

This method can be considered as an alternative perturbation expansion of the tropical limit. Indeed, the function \( e^{-\frac{1}{x}} \) is not analytic around \( k = 0 \), thus standard perturbative tools (e.g., series expansion) fail in this case.
Proposition 6. Zeroth and first order corrections of \( F(k_B) \) correspond to data \( \kappa_0 \) and \( \lambda_0 \) respectively. The \( m \)-th order coefficient in the Taylor expansion of \( F \) near \( k = 0 \) vanishes, for all \( m \geq 2 \).

\[ \kappa_0 \text{ and } \lambda_0 \] respectively. The \( m \)-th order coefficient in the Taylor expansion of \( F \) near \( k = 0 \) vanishes, for all \( m \geq 2 \).

Proof: This is a simple calculation that is presented in Appendix C.

So zeroth and first order corrections to tropical free energy give data \( (\kappa_0, \lambda_0) \) relative to the first level in the nesting form of \( B \)-type. They correspond to tropical free energy and statistical prefactors \( -\ln \lambda_0 \) for entropy in [9]. The standard perturbative approach allows one to get contributions up to first order, i.e., relatively to first level of the nesting form. Nevertheless, this process breaks out at higher levels. Contributions beyond the first order are included in subsequent levels. One can apply series expansion along with this nested structure to recover data (6.4). One gets \( \kappa_\ell \) (from the 0-th order of the \((\ell + 1)\)-th level) and \( \lambda_\ell \) (from the first order of the \((\ell + 1)\)-th level).

If this construction is done with finitely many levels, it returns a finite set of data. In such a case, the recursion as a reversal symmetry, that is, the nesting forms of type \( A \) and \( B \) relative to \( Z(x) \) coincide. The situation for a countable infinite number of levels, \( \alpha \in \mathbb{N} \), is more subtle. An explanatory example involves the \( p \)-adic numbers. Let us consider “positional” weight \( \mu_\ell(x) := -\ell \cdot \ln p \) that depend on \( \ell \in \mathbb{N}_0 \) only, and cardinalities \( \nu_\ell(x) \in \{0, 1, \ldots, p - 1\} \). The corresponding \( A \)-type partition function

\[
Z(x) = 1 \cdot (\nu_0(x) + p^{-1} \cdot (\nu_1(x) + p^{-1} \cdot (\nu_2(x) + p^{-1} \cdot (\ldots
\]

is the base \( p \) expansion of a real number. On the other hand, \( B \)-type nesting results in \( \mu_\ell(x) \equiv \ell \cdot \ln p, \ell \in \mathbb{N}_0 \), and

\[
Z(x) = 1 \cdot (\nu_0(x) + p \cdot (\nu_1(x) + p \cdot (\nu_2(x) + p \cdot (\ldots
\]

which is a \( p \)-adic number.

Cases with an infinite number of levels can be put into bijection with other continued nested expansions. Furthermore, the nested structure (6.5) relies on the linear order \( \leq \) on \( \mathbb{R} \), so it could be generalized using more general partial orders. These kinds of extensions, and the reduction to the cases of linear order (Proposition 5), deserve more attention in order to better understand tropicalization methods, their expansions and potential applications. This would go beyond the scope of this work and will be studied in a separate paper.

7 Tropicalization(s) and the role of ground energy

We can now use the filter language to discuss simple models in non-equilibrium physics. Two elementary remarks on \( \mathbb{R}_{\max} = (\mathbb{R}^{\lor}, \max, -\infty) \) lead to major implications. Firstly, \( \mathbb{R}^{\lor} \) is a
grounded poset, and this is linked to the asymmetry between max and min. Secondly, a presentation of the tropical algebra has a particular symmetry, i.e. idempotence, which will be relevant in the study of tropicalization processes.

7.1 A simple example: duality and non-equilibrium

Here, we look at the connections between max/min duality, non-equilibrium systems and ground energy in more detail. To this end, we start from a simple thermodynamic example. Let us take $N$ microsystems, where each system is defined by a triple $(E_\alpha, S_\alpha, T_\alpha)$, $\alpha \in [N]$. Here $E_\alpha$ is the energy of the microsystem, $S_\alpha = k_B \cdot \ln g_\alpha(k_B)$ is its entropy and $T_\alpha$ is a “temperature”. Concrete instances of such a model could describe microscopic ensembles at the moment when they are put in contact, so they are not in equilibrium. However, we do not force the temperature to be Boltzmann or Gibbs temperature and we do not put constraints on $T_\alpha$ in order to work in full generality. These quantities define a micro-free energy $F_\alpha = F(E_\alpha, S_\alpha, T_\alpha) := E_\alpha - T_\alpha \cdot S_\alpha$. One can give the same tropical framework to all the dependent variables, i.e. micro-free energies, via

$$\beta : \left\{ \frac{F_\alpha}{T_\alpha}, \alpha \in [N] \right\} \subseteq \mathbb{R} \mapsto \mathbb{R}^\wedge \quad (7.1)$$

where the immersion $\mathbb{R} \mapsto \mathbb{R}^\wedge$ acts as the identity on $\mathbb{R}$. So we can think at dependent variables as real or tropical ones. The function $\beta$ will be called tropical temperature (even if it plays the role of an inverse temperature) and the $N$ microsystems are tropically thermalized. Generalizing the results in [9], the tropical free energy is defined as

$$F_{\text{trop}} := \bigoplus_{\alpha \in [N]} \beta \left( \frac{E_\alpha - S_\alpha \cdot T_\alpha}{T_\alpha} \right). \quad (7.2)$$

The tropical addition in (7.2) is identified with min, so it corresponds to a $B$-type model. The duality between $A$-type and $B$-type nesting forms can be now restated in more geometric terms.

Indeed, the corresponding $A$-type model is obtained by means of a projective-like transformation, that are

$$\tilde{E}_\alpha := S_\alpha, \quad \tilde{S}_\alpha := E_\alpha, \quad \tilde{T}_\alpha := \frac{1}{T_\alpha}. \quad (7.3)$$

Under previous transformation, (7.2) becomes

$$\min_{\alpha \in [N]} \beta \left( \frac{-T_\alpha \cdot S_\alpha + E_\alpha}{T_\alpha} \right) = \min_{\alpha \in [N]} \beta \left( -\tilde{E}_\alpha + \tilde{S}_\alpha \cdot \tilde{T}_\alpha \right)$$

$$= - \max_{\alpha \in [N]} \beta \left( \tilde{E}_\alpha - \tilde{S}_\alpha \cdot \tilde{T}_\alpha \right). \quad (7.4)$$

So one can rewrite the tropical free energy (7.2) of the $B$-type as an $A$-type expression (7.4).

Previous transformation corresponds to the conjugation $c \circ \min \circ c = \max$, where $c : \mathbb{R}^\vee \mapsto \mathbb{R}^\wedge$ is the involution $c(x) = -x$. One can restore the presentation with addition $\oplus = \min$ via the simultaneous real transformation (7.3) and the tropical inversion

$$\tilde{\beta} = \left\{ \frac{1}{\beta} \right\} := -\beta : \mathbb{R} \mapsto \mathbb{R}^\vee. \quad (7.5)$$
That gives
\[ \min_{\alpha \in [N]} \beta \left( \frac{-T_\alpha \cdot S_\alpha + E_\alpha}{T_\alpha} \right) = \min_{\alpha \in [N]} \tilde{\beta} \left( \hat{E}_\alpha - \hat{S}_\alpha \cdot \hat{T}_\alpha \right). \] (7.6)

The inversion \( \hat{T}_\alpha := \frac{1}{T_\alpha} \) in (7.3) is the usual (real) one for real temperatures \( T_\alpha \), while \( \tilde{\beta} := \frac{1}{\beta} \) in (7.5) is the tropical inversion for tropical temperature \( \beta \). It should be stressed that (7.5) is not an involution on a set, but rather a homomorphism between two different grounded monoids, that are \( \mathbb{R}_{\text{max}} = (\mathbb{R}^\vee, \max, -\infty) \) and \( \mathbb{R}_{\text{min}} = (\mathbb{R}^\wedge, \min, +\infty) \).

So, the different behaviour for “positive” and “negative” \( \beta \) is resolved by a reparametrization. It is worth remarking that a different approach to the change of behaviour between positive and negative temperatures was proposed in [16], where the role of the real parameter \( \frac{1}{k_BT} \) instead of \( T \) was emphasized. On the other hand, the tropical approach and the \( \max \setminus \min \) duality in (7.5) let one have a common setting where these different physical regimes are preserved. In the particular case of equilibrium \( T_\alpha = T \) for a constant \( T \) one recovers the results in [9].

The map of a \( B \)-type non-equilibrium system to an \( A \)-model can be studied in terms of the choice of the ground energy. In fact, both models are invariant under a shift of free energies \( F_\alpha, \alpha \in [N] \). However, the \( B \)-model has not translation invariance for the energy spectrum \( \{E_\alpha : \alpha \in [N]\} \) in general. This means that, if temperatures \( T_\alpha \) do not coincide (non-equilibrium case) and one reparametrizes \( \tilde{E}_\alpha := E + E_\alpha \) for any constant \( E \in \mathbb{R} \), then \( \frac{E_\alpha - S_\alpha \cdot T_\alpha}{T_\alpha} < \frac{E_\gamma - S_\gamma \cdot T_\gamma}{T_\gamma} \) is in general not equivalent to \( \frac{E_\alpha - S_\alpha \cdot T_\alpha}{T_\alpha} < \frac{E_\gamma - S_\gamma \cdot T_\gamma}{T_\gamma} \). On the other hand, the \( A \)-type model has such a type of symmetry, since \( \tilde{E}_\alpha - \tilde{S}_\alpha \cdot \tilde{T}_\alpha < \tilde{E}_\gamma - \tilde{S}_\gamma \cdot \tilde{T}_\gamma \) implies \( (\tilde{E}_\alpha + \tilde{E}) - \tilde{S}_\alpha \cdot \tilde{T}_\alpha < (\tilde{E}_\gamma + \tilde{E}) - \tilde{S}_\gamma \cdot \tilde{T}_\gamma \) for all \( \tilde{E} \in \mathbb{R} \).

A more detailed discussion on this restricted translational invariance of the spectrum and the meaning of the duality \( E_\alpha \Leftrightarrow S_\alpha \) would go beyond the scope of this paper and will be discussed elsewhere.

### 7.2 Tropical action, normalization and the choice of ground energy

The previous example suggests a more general way to move from real to tropical entities. We will call a tropicalization of a set \( \mathcal{R} \) a map
\[ \tau : \mathcal{R} \rightarrow \Lambda \] (7.7)
where \( \Lambda \) is a tropical semiring. In particular, we are interested in tropicalizations of real variables \( \mathcal{R} \subseteq \mathbb{R}^n \). The results in Section 5 give the opportunity to consider simultaneously two objects associated to a monoid. The first is \( (\phi(y), \oplus, y) \in (5.6) \), \( y \in \Lambda \), that is a filter with respect to \( \preceq \).
The second is \( F_\Lambda \) in (5.6). Proposition 3 suggests to concentrate, firstly, on a totally ordered set in order to include the presentation \( \mathcal{I}_\Lambda \). In fact, this is the case of \( \beta \circ F \) in (7.1), where \( \Lambda = \mathbb{R}^\wedge \).
So one has at least two additional tropicalizations on the same poset \( \Lambda \), that are
\[ \iota_x : \mathcal{R} \rightarrow \phi(x) = \{y \in \Lambda : x \preceq y\}, \] (7.8)
\[ \phi : \mathcal{R} \rightarrow F_\Lambda = \phi(\Lambda). \] (7.9)
The map (7.8) has a physical relevance in terms of stability. Indeed, a physical system is considered stable if its energy spectrum is bounded from below. This condition is expressed by grounded posets \( \phi(x) \), where \( x \) plays the role of ground energy and bounds the elements of \( \iota_x(\mathcal{R}) \) from below. From the statistical point of view, the existence of such \( x \) means that the associated tropical probability distribution \( W_{n,\text{tr}} = \frac{F_n - F_{n,T}}{T} \) in [9] is normalizable.

The domains \( \Lambda \) and \( \mathcal{F}_\Lambda \) are homomorphic as tropical monoids by Proposition 3. Nevertheless, quite different conclusions can be drawn from these processes when one looks at the semiring action induced by \( \circ = + \). Indeed, the translation \( \varepsilon + \mathcal{R} := \{ \varepsilon + x : x \in \mathcal{R} \}, \varepsilon \in \mathbb{R} \), corresponds to actions on \( \phi(x) \) and \( \mathcal{F}_\Lambda \), namely

\[
\varepsilon \circ y := \varepsilon \circ \iota_x(y), \quad \varepsilon \circ \phi(x) := \{ \varepsilon \circ \tau(y) : \tau(y) \in \phi(x) \}.
\]

The action (7.10) is invertible with inverse \( \varepsilon^{-1} \circ y = (-\varepsilon) \circ \iota_x(y) \), for all \( \varepsilon \in \Lambda \setminus \{ \infty \} \). So it maps any filter \( \phi(x) \) to another one \( \phi(\varepsilon \circ x) \). If one tropicalizes \( \mathcal{R} \) via (7.8), then the associated tropical action will be called global. This means that (7.10) acts simultaneously on the whole set \( \mathcal{R} \) mapping it to another filter \( \varepsilon \circ \iota_x(\mathcal{R}) \). In particular, the tropicalization \( \beta \circ F \) given by the map (7.1) is global, with \( \Lambda = \mathbb{R}_{\min} \) and \( \circ = + \). If the systems are in equilibrium, i.e. \( T_\alpha = T \) is constant for all \( \alpha \), then the global tropical action describes a different choice of ground energy and a consequent shift of (free) energies by the same value. So one recovers the invariance of a physical description under different choices of the ground energy. For systems out of equilibrium (different \( T_\alpha \)), this tropicalization does not coincide with the shift of energy levels.

If instead one uses (7.9) to tropicalize \( \mathcal{R} \), then each element \( x \in \mathcal{R} \) is presented a filter \( \phi(x) \) on a certain poset. In the latter approach, the tropical action will be called local, that means that each real variable \( x \in \mathcal{R} \) actually represents the choice for the ground value of its image \( \phi(x) \). This freedom is important when one is interested in tropicalizations of probability distributions. Before moving on to this issue, it should be stressed that some effects of the invariance of micro-free energies under constant shifts have been studied in a geometric framework in [56]. More specifically, Gauss-Kronecker curvature for an ideal statistical mapping vanishes if and only if there exists a non-vanishing Killing vector field \( \sum_{i=1}^{n} c_i \frac{\partial}{\partial x^i} \) for the statistical hypersurface, whose coefficients \( c_i \) are constant. This corresponds to a translational symmetry, that is “global” (i.e., all \( c_i \) are equal) in the super-ideal case.

8 Global and local tropical symmetry

The tropicalization (7.1) of dependent variables \( \{ F_\alpha : \alpha \in [N] \} \) produces free energies with a tropical symmetry. This means that the value of the tropical macroscopic free energy (7.2) does not change if one creates a copy of a certain microsystem \( F_\alpha \). The creation of copies affects the counting, and this stresses the role of tropical limit in probability and statistics.

Both global (7.8) and local (7.9) tropicalizations induce tropical symmetry on former real variables. These procedures are connected (Proposition 3) as long as only one variable is involved. However, distinctive features can be extracted from each of these two processes in cases
of more variables, depending on tropicalizing them as a whole (the set $\mathcal{R}$) or individually (each element of $\mathcal{R}$ one at a time).

8.1 Global tropical symmetry and statistical amoebas

In [9] usual probabilities for events $X \subseteq [N]$

$$W(X) = \frac{\#(X \cap m_0(T))}{\#m_0(T)} \quad (8.1)$$

were identified, while “tropical” probabilities at $k_B \ll 1$ for states and energy levels are respectively

$$w_{\alpha,\text{tr}} = -S_\alpha + \frac{F_{\text{tr}}(T) - F_\alpha(T)}{T} - k_B \cdot \ln \left(\#m_0(T)\right), \quad W_{\alpha,\text{tr}} = w_{\alpha,\text{tr}} + S_\alpha. \quad (8.2)$$

At $k_B = 0$, the weights in (8.2) are tropically additive and normalized as $\bigoplus_{\alpha=1}^{N} W_{\alpha,\text{tr}} = 0$. In particular, one has $\#m_0(T) = 1$ at regular domains where only one phase $\alpha_0 \in [N]$ satisfies the minimum for the free energy. Here, one gets an ultrafilter probability

$$W(X) = \begin{cases} 1, & \text{if } \alpha_0 \in X \\ 0, & \text{if } \alpha_0 \notin X \end{cases} \quad (8.3)$$

This is a particular instance of a more general application of filters to probability. In fact, it is well known that a proper filter can be seen as a $\{0, 1\}$-finitely real additive measure on the set $\Omega$, i.e. a function $\tau : \Omega \to \{0, 1\}$ such that $\tau(\emptyset) = 0$ and

$$\tau \left( \bigcup_{h=1}^{\ell} \Omega_h \right) = \sum_{h=1}^{\ell} \tau(\Omega_h) \quad (8.4)$$

where $\ell \in \mathbb{N}$ and $\Omega_h$ are disjoint measurable subsets of $\Omega$. It is easily shown that the function

$$\tau(X) := \begin{cases} 1, & \text{if } X \in \mathcal{U} \\ 0, & \text{if } X^c \in \mathcal{U} \\ \text{undefined}, & \text{otherwise} \end{cases}, \quad X \subseteq \Omega \quad (8.5)$$

is a $\{0, 1\}$-valued measure where countably additivity is relaxed to finite additivity. Indeed, if $\tau(\Omega_1) = \tau(\Omega_2) = 1$ then both $\Omega_1$ and $\Omega_2$ belong to the filter, hence their intersection is in the filter too. The fact that the empty set $\emptyset$ does not belong to any proper filter implies that $\Omega_1 \cap \Omega_2 \neq \emptyset$. Thus, there is at most one nonvanishing term in the sum (8.4), where $\Omega_h$ are elements of the filter. If $\Omega$ is a finite set, then $\tau$ in (8.5) is a real probability measure, since in this case finite additivity is equivalent to the usual $\sigma$-additivity.

This interpretation of filters fits well with the tropicalization (7.8). If $\Omega \to \phi(y)$ and $\phi(y) \notin \mathbb{R}^\vee$, i.e. $y \neq -\infty$, then any element $\alpha$ of $\Omega$ can be interpreted as real, since $\Omega \subseteq \mathbb{R}$, or tropical, as $\iota_y(\alpha)$. At the same time, one finds both real (8.1) and tropical weights (8.2) and the latter are a result of the global tropicalization (7.1).

In this case, the limit process involves $k_B$: idempotence for the probability $W_\alpha = W(\{\alpha\})$
defined in [9] holds only at the lowest (zeroth) order in $k_B$ on the singular locus. This is evident in (8.2), where the statistical corrections depend on the cardinality $\lambda_0(T) = \# m_0(T)$ defined in (6.4). They correspond to first order corrections in $k_B \ll 1$ and are the only non-trivial purely perturbative corrections (Proposition 6). By the same token, the statistical weights (8.1) are equal to

$$\frac{1}{\lambda_0(T)}.$$ 

Thus, idempotence is lost from the point of view of usual probability weight (8.1).

Loss of idempotence is a remarkable phenomenon at the singular locus, where the free energy is non-differentiable and this can be seen as a phase transition. In the line of thought that associates phase transitions to a broken symmetry, at a critical temperature $T^*$ the broken tropical symmetry appears as loss of idempotence in statistical prefactors. The breaking of tropical symmetry on the singular locus is accidental, i.e. it occurs for certain values of parameters (e.g., temperature). Furthermore, it is physical in terms of observability by means of averages of observables with weights (8.1). In the context of tropical geometry [8], where one associates a simplicial complex to algebraic tropical functions, the accidental coincidence of phases is described by simplices with non-maximal dimension.

In general, the addition of a copy of a subsystem makes this broken symmetry systematic. In fact, one can consider the extension of $\{F_\alpha : \alpha \in [N]\}$ by a function $F_{N+1} = F_{\alpha_0}$ where $F_{\alpha_0}(T) \leq F_\beta(T)$ for all $\beta \in [N]$ and $T$ in a certain domain. The statistical factor for $W_{\alpha_0}(T)$ now involves both $\alpha_0 \in m_0(T)$ and $N + 1$, i.e.

$$W_{\alpha_0}(T) = \frac{1}{\lambda_0(T)} \rightarrow \frac{2}{\lambda_0(T) + 1}$$

(8.6)

for any $T$ in the domain. Contrary to the case of phase transitions, averages of observables are unchanged by the addition of a copy in regular domains, so they are not observable in this sense. However, tropical copies can still be identified on the singular locus.

It is worthy of note that statistical amoebas [55] provide one with a geometric formulation for this limit procedure. Indeed, the instability domain $\mathcal{D}_{k-}$ for a statistical amoeba (4.9) is induced by the ultrafilter $\mathcal{U}(\alpha(x))$ through $\mathcal{N}_{k-}(x) = \mathcal{U}(\alpha(x)) \cap \{A \subseteq [N] : \# A = k\}$, where $\alpha(x)$ is the only index in $[N]$ such that $f_\alpha(x) > f_\beta(x)$ for all $\beta \neq \alpha(x)$. So the statistical amoeba can be used to study singularities of free energy (zeros of (4.9)), non-equilibrium domains (where (4.9) is negative) and emergence of a “macroscopic” behaviour in domains of maximal instability $\mathcal{D}_{k-}$ (where a filter measure (8.5) is defined). In this context, tropical limits are obtained via the scaling of independent variables, $x_i \rightarrow \frac{x_i}{k_B}$, or dependent ones $f_\alpha(x) \rightarrow \frac{f_\alpha(x)}{k_B}$.

### 8.2 Local tropical symmetry and the dequantification procedure

In Section 7 we have pointed out that the local tropicalization (7.9) describes the labeling of a set of systems $\phi(R)$ by their ground energy $x$, with $x \in R$. We can now explore the statistical effects of individual implementation of idempotence on real variables. Let us take a finite number, say $N$, of distinct microsystems $\Phi := \{f_\alpha(x) : \alpha \in [N]\}$ that defines the statistical model through the partition function (6.1).

The index space $[N]$ can be immersed in $\tilde{N} := [N] \times N$ identifying $\alpha \in [N]$ with $(\alpha, 1) \in \tilde{N}$. 

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The limit procedure can be implemented through the map
\[
T : \tilde{N} \rightarrow \tilde{N} \\
T(\alpha, n) = T_\alpha(n) := (\alpha, n + 1), \quad \alpha \in [N], \ n \in \mathbb{N}.
\]
(8.7)

So \( T_\alpha \) describes the addition of a tropical copy of the microsystem \( \alpha \) in the macrosystem, i.e. the disjoint union
\[
T_\alpha(\Phi) := \{ f_\beta(x) : \beta \in [N] \} \cup \{ f_\alpha(x) \}.
\]
(8.8)

One can consider the sets \([N], \Phi \) and \([N] \times \{1\} \subseteq \tilde{N}\) as minimal presentations of the macrosystem since microsystems are pairwise distinct. If \( X \subseteq \tilde{N} \), we will write \( T(X) := X \cup \{ T(x) : x \in X \} \).

Now let us identify a “tropical distribution” on \([N]\) starting from a standard (real, additive) one \( w_{k_B} : [N] \rightarrow [0; 1] \), where \( k_B \) is a parameter that controls the tropicalization process. From the point of view of standard probability, the creation of a copy of a dominant microsystem affects statistical weights as in (8.6). On the other hand, a tropical “probability” should not discern the addition of copies, since they define the same tropical system. Thus, we assume that the creation of copies does not affect the tropical system.

This request implies that we can consistently assign tropical weights to a set \( X \subseteq \tilde{N} \), starting from a real distribution \( w_{k_B} \), if \( X \) is closed under addition of copies. So, we will say that a set \( Y \subseteq \tilde{N} \) is T-closed if \( T(Y) = Y \). The T-closure \( \bar{X} \) of \( X \subseteq \tilde{N} \) is the smallest among all T-closed sets \( Y \subseteq \tilde{N} \) such that \( X \subseteq Y \). So a set \( Y \) is T-closed if and only if \( Y = \bar{Y} \). Since intersections of T-closed sets are T-closed, the T-closure of sets is well-defined and its explicit form is
\[
\bar{X} := \bar{T}(X) = \bigcap_{X \subseteq Y = \bar{Y}} Y.
\]
(8.9)

In this setting, T-closed sets represent tropically measurable set. One can get such a tropical measure from weights \( w_{k_B;\alpha} \) assigned to individual copies of the microsystem \( \alpha \). In the line of thoughts of [9], we consider \( N \in \mathbb{N} \) copies of the microsystem \( \alpha \) and the tropical limit of \( w_{k_B;\alpha} \) as the simultaneous limit \( k_B \rightarrow 0 \) and \( N \rightarrow \infty \). This clearly depends on the explicit dependence of weights \( w_{k_B;\alpha} \) from \( k_B \) and from the relation between \( k_B \) and \( N \). For the sake of concreteness, we look at Gibbs weights
\[
w_{k_B;\alpha} = w_{k_B;\alpha,1} := \exp \left( -\frac{f_\alpha(T)}{k_B} \right) \sum_{\beta \in [N]} \exp \left( -\frac{f_\beta(T)}{k_B} \right)
\]
(8.10)

and we adopt the prescription \( k_B := \frac{1}{N} \). Thus, the role of \( k_B \) in this process is to control the creation of tropical copies.

The addition of \( N-1 \) copies \( \alpha \cong (\alpha, 1) \rightarrow (\alpha, N) \) affects the weight \( w_{k_B;\alpha} \) for the microsystem \( \alpha \) as
\[
w_{k_B;\alpha,N} = \frac{N \cdot \exp (-N \cdot f_\alpha(T))}{(N - 1) \cdot \exp (-N \cdot f_\alpha(T)) + \sum_{\beta \in [N]} \exp (-N \cdot f_\beta(T))}.
\]
(8.11)
Now one can consider the limit $k_B \to 0^+$. If $\alpha \in m_0(T)$, then (8.11) becomes

$$w_{0,\alpha} := \lim_{N \to \infty} \frac{\mathcal{N} \cdot e^{-N \cdot f_\alpha(T)}}{(N - 1) \cdot e^{-N \cdot f_\alpha(T)} + \sum_{\beta \in m_0(T)} e^{-N \cdot f_\beta(T)} + \sum_{\gamma \notin m_0(T)} e^{-N \cdot f_\gamma(T)}}$$

$$= \lim_{N \to \infty} \frac{\mathcal{N}}{\lambda_0(T) - 1 + N} = 1. \quad (8.12)$$

If instead $\alpha \notin m_0(T)$, then

$$0 \leq w_{0,\alpha} = \lim_{N \to \infty} \frac{\mathcal{N} \cdot e^{-N \cdot f_\alpha(T)}}{(N - 1) \cdot e^{-N \cdot f_\alpha(T)} + \sum_{\beta \in m_0(T)} e^{-N \cdot f_\beta(T)} + \sum_{\gamma \notin m_0(T)} e^{-N \cdot f_\gamma(T)}}$$

$$\leq \lim_{N \to \infty} \frac{\mathcal{N}}{(N - 1) + \sum_{\beta \in m_0(T)} e^{-N \cdot (f_\alpha(T) - f_\beta(T))}} = 0 \quad (8.13)$$

so $w_{0,\alpha} = 0$. These limits rely on both the countable additivity of real probability and the exponential form (8.10) of Gibbs weights.

Similarly, one can consider $w_{N-1}(T^N(X))$ for $X \subseteq [N]$. Generally, $w_0$ is not a real additive distribution. Indeed, for any partition of $m_0(T)$ in two disjoint sets, say $X_1$ and $X_2$, $w_0(X_1) + w_0(X_2) \geq 1 \geq w_0(X_1 \cup X_2) = w_0(m_0(T))$. So $w_0$ is real additive if and only if, for each possible partition, exactly one set $X_1$ or $X_2$ is empty. This means that $\#m_0(T) = 1$, so one recovers the ultrafilter probability (8.3).

By contrast, $w_0$ is tropically additive ($\oplus = \max$) even at $\#m_0(T) \geq 1$. If $\{X_n\}$ is any family of pairwise disjoint subsets of $[N]$, then $w_0 \left( \bigcup_n X_n \right) = 1$ if and only if $X_n \cap m_0(T) \neq \emptyset$ for at least one $n$, that is $\max\{w_0(X_n)\} = 1$. So $w_0$ is a possibility distribution, in the sense that they concern the possibility for a certain set of events to happen.

It is worth remarking that this process involves weights $w_{k_B,\alpha}$ one at a time, thus one asks for idempotence for each $\alpha$ individually. This corresponds to making copies of the microsystems subsequent to the prior measurement $\alpha \mapsto w_\alpha$. If the process was “global”, then the same number of copies should be created for all the microsystems and usual probabilities (8.1) would be recovered at $k_B \to 0$.

Both the real weights (8.5) and the tropical $w_0$ take values in $\{0, 1\}$. In particular, $\tau$ is a weaker version of usual probability (8.1), since it only distinguishes between sure and not sure events. $\{w_{0,\alpha}\}$ is a weaker version of $W_{0,\tau}$ in (8.2), since it only provides information on the existence of an element of $m_0$ in $X$. In this regard, the procedure used to derive $w_0$ can be called dequantification. It should be stressed that the occurrence of a tropical possibility distribution is consistent with the choice of T-closed sets (8.9) as measurable sets.

Also the way in which the dequantification limit is approached is easily linked to a local tropicalization (7.9). In fact, one can first choose an enumeration for $\mathbb{Q}$, that is a bijection from $\mathbb{N}$ to $\mathbb{Q}$. Then the copying process (8.7) moves towards the choice $\Lambda = \mathbb{Q}$ in (5.6). Indeed, once the tropical limit is reached one has $F_\alpha(T) < F_\beta(T)$ if and only if $\phi(F_\beta(T)) \subset \phi(F_\alpha(T))$ and $\epsilon(F_\alpha(T)) \subset \epsilon(F_\beta(T))$, since the rationals are dense in $\mathbb{R}$.
9 Conclusions and future perspectives

This work was aimed at investigating the links between tropical limit, algebra and statistical physics. The above discussion suggests that some physical phenomena can take advantage from a tropical description. A simple algebraic assumption provides a framework where the concepts of dominance, hierarchical distance and composition can be discussed simultaneously. Connections with physical issues can be recognized when one deals with systems that exhibit ultrametricity, exponential degenerations of energy levels and metastability.

This opens the way to other questions and proposals. First, it is worth extending the correspondence between elements and subsets looking at other set-theoretic notions. In particular, given a family of sets $(\Omega_n)_{n \in \mathbb{I}}$ indexed by $\mathbb{I}$, one could consider, for any element $\alpha \in \bigcup_n \Omega_n$, a “dual” cardinality $\#\alpha$ related to the number of sets $\Omega_n$ containing $\alpha$. If one assumes that each total cardinality $\sum_{\Omega_n : \alpha \in \Omega_n} \#\alpha = 1$ is independent on the number of sets, e.g. $\#\alpha = \frac{1}{\# \{\Omega_n : \alpha \in \Omega_n\}}$, and the family $\{\Phi, \{\alpha\}, \{\alpha\}, \ldots, \{\alpha\}\}$ for $N$ copies of the $\alpha$-th microsystem (8.8) is considered, then $\#\beta = 1$ at $\beta \neq \alpha$ and $\#\alpha = \frac{1}{N+1}$. The limit $\frac{1}{N} \to 0$ for such a procedure could be formalized in order to understand better the physical meaning behind the $n \to 0$ limit for the dimension of the overlap matrix in replica trick and spin glasses [31], so it deserves a more detailed investigation.

On a broader level, these tools can be useful in the comprehensive study of different features of complexity. The main advantage pertains to the relation between structural complexity and algebraic rules. The former is the “hardware” of a system, e.g. the geometry of a complex networks, and ultrametricity often has a key part in this context. The latter define associative processes, that is the “software”, and give a basis for extended logics [18], including fuzzy logic. So, a tropical micro-macro correspondence and associated tools (e.g., perturbative tropical limit in Section 6) can help explain connections between the physical structure of complex systems and their underlying logic. This also comes with the dimensionality issue induced by the limit $k_B \to 0$ for Boltzmann constant, as already noticed in Sections 3 and 8.2. All of this could give new hints on the theoretical framework for the effectiveness of many methods of statistical physical in current learning models.

Acknowledgements

I am grateful to Prof. Boris Konopelchenko and Prof. Giulio Landolfi for their kind comments and continuous support. Part of this work was written during a research visit at the University of Loughborough. I would like to thank Prof. Eugene Ferapontov and Matteo Casati for kind hospitality.

A Proof of Proposition 1

Let us first assume that $d(\Omega^2)$ is grounded. Then, one can check that
1. \( \mathcal{B} \) is non-empty. Indeed \( \Omega \neq \emptyset \), \( 0 \in d(x_0, \cdot) \) and for all \( x_0 \in \Omega \) the singletons \( \{ x_0 \} = \{ x \in \Omega : d(x, x_0) \leq 0 \} = S(x_0, 0) \) belong to \( \mathcal{B} \).

2. \( \Omega \notin \mathcal{B} \). Indeed, let us assume the contrary and suppose that there exists \( x_0 \in \Omega \) and \( r \in d(x_0, \cdot) \) such that \( S(x_0, r) = \Omega \). Since there is no maximum for \( d(\Omega^2) \), there exists a pair \((x_1, x_2)\) such that \( d(x_1, x_2) > r \). But \( x_1, x_2 \in \Omega = S(x_0, r) \), then \( r < d(x_1, x_2) \leq \max\{d(x_1, x_0), d(x_2, x_0)\} \leq r \), contradiction.

3. The union of any two elements of \( \mathcal{B} \) is contained in an element of \( \mathcal{B} \). In fact, let us take \( S(x_0, r) \) and \( S(y_0, s) \) with \( x_0, y_0 \in \Omega \), \( r \in d(x_0, \cdot) \) and \( s \in d(y_0, \cdot) \). So define \( M := \max\{r, s, d(x_0, y_0)\} \). Clearly \( M \in \{r, s, d(x_0, y_0)\} \subseteq d(x_0, \cdot) \cup d(y_0, \cdot) \) and \( S(x_0, r) \subseteq S(x_0, M) \), as follows from the definition. Moreover, if \( y \in S(y_0, s) \), then \( d(y, x_0) \leq \max\{d(y, y_0), d(y_0, x_0)\} \leq \max\{s, d(x_0, y_0)\} \leq M \). Thus, \( S(x_0, r) \cup S(y_0, s) \subseteq S(x_0, M) = S(y_0, M) \), that is \( S(x_0, r) \cup S(y_0, s) \subseteq S(\bar{x}, M) \) with \( \bar{x} \in \{x_0, y_0\} \) and \( M \in d(\bar{x}, \cdot) \).

So, let us consider the closure of \( \mathcal{B} \) under subsets, i.e. \( \mathcal{I} := \{ T : T \subseteq A, A \in \mathcal{B} \} \). It satisfies downward closedness by construction and \( \Omega \notin \mathcal{I} \) since \( \Omega \notin A \) for all \( A \in \mathcal{B} \). Moreover, if \( A, B \in \mathcal{I} \), then there exist \( x_A, x_B \in \Omega \), \( r_A \in d(x_A, \cdot) \) and \( r_B \in d(x_B, \cdot) \) such that \( A \subseteq S(x_A, r_A) \) and \( B \subseteq S(x_B, r_B) \). By previous observations, there exist \( x_0 \in \Omega \) and \( r \in d(x_0, \cdot) \) such that \( A \cup B \subseteq S(x_A, r_A) \cup S(x_B, r_B) \subseteq S(x_0, r) \), then \( A \cup B \in \mathcal{I} \). This means that \( \mathcal{I} \) is an ideal and \( \mathcal{F} := \{ \Omega \setminus A : A \in \mathcal{I} \} \) is a filter by Lemma 1. On the other hand, if \( d(\Omega^2) \) is not grounded, then there exists \((\bar{x}, \bar{y}) \in \Omega^2 \) such that \( d(x, y) \leq d(\bar{x}, \bar{y}) \) for all \( x, y \in \Omega \). This means that \( S(\bar{x}, d(\bar{x}, \bar{y})) = \{ x \in \Omega : d(x, \bar{x}) \leq d(\bar{x}, \bar{y}) \} = \Omega \). So \( \Omega \in \mathcal{B} \) and the closure of \( \mathcal{B} \) under subsets is the trivial ideal \( \mathcal{P}(\Omega) \).

Now, let \( \mathcal{I} \) be an ideal and consider any decreasing positive function \( \mathfrak{d} : (\mathcal{I}, \subseteq) \rightarrow (\mathbb{R}_R^+, \leq) \) with \( \inf \mathfrak{d}(\mathcal{I}) > 0 \). We will denote with \( \mathcal{D} : \Omega^2 \setminus \{(x, x) : x \in \Omega \} \rightarrow \mathcal{P}(\mathbb{R}) \) the map

\[
\mathcal{D}(x, y) := \{ \mathfrak{d}(A) : A \in \mathcal{I}, \{ x, y \} \subseteq A \} .
\] (A.1)

If \( x_1 \neq x_2 \) then there exist \( A_i \in \mathcal{I} \) such that \( x_i \in A_i, \ i \in \{1, 2\} \), since \( \Omega = \bigcup_{A \in \mathcal{I}} A \). Thus \( A_1 \cup A_2 \in \mathcal{I} \) from upward directedness of ideals and \( \{ x_1, x_2 \} \subseteq A_1 \cup A_2 \). This means that \( \mathcal{D}(x_1, x_2) \neq \emptyset \), so \( d(x_1, x_2) \geq \inf \mathfrak{d}(\mathcal{I}) > 0 \). Moreover, \( d \) is symmetric in its entries. Then, for all \( x \neq y \neq z \neq x \) in \( \Omega \) one finds

\[
\max\{d(x, y), d(y, z)\} = \max\{\inf \mathcal{D}(x, y), \inf \mathcal{D}(y, z)\} = \inf\{\max\{\mathfrak{d}(A), \mathfrak{d}(B)\} : A, B \in \mathcal{I}, \{ x, y \} \subseteq A, \{ y, z \} \subseteq B\}
\] (A.2)

as follows from the complete distributivity of \( \max \) over arbitrary inf in the lattice \( \{(x \in \mathbb{R}_R^+ : x \geq \inf \mathfrak{d}(\mathcal{I}) \}, \leq \)\). If \( A, B \in \mathcal{I} \), then \( A \cup B \in \mathcal{I} \), hence downward closedness of ideals implies
\( A \cup \{z\} \in \mathcal{I} \) for all \( z \in B \). Thus, decreasing monotony of \( \mathcal{D} \) gives

\[
\inf \{ \max \{ \mathcal{D}(A), \mathcal{D}(B) \} : A, B \in \mathcal{I}, \{x, y\} \subseteq A, \{y, z\} \subseteq B \} 
\geq \inf \{ \max \{ \mathcal{D}(A \cup \{z\}), \mathcal{D}(B \cup \{x\}) \} : A, B \in \mathcal{I}, \{x, y\} \subseteq A, \{y, z\} \subseteq B \}
= \inf \{ \mathcal{D}(A) : A \in \mathcal{I}, \{x, y, z\} \subseteq A \}
\geq \inf \{ \mathcal{D}(A) : A \in \mathcal{I}, \{x, z\} \subseteq A \} = d(x, z).
\tag{A.3}
\]

where last inequality comes from \( \{A \in \mathcal{I}, \{x, y, z\} \subseteq A\} \subseteq \{A \in \mathcal{I}, \{x, z\} \subseteq A\} \). Indeed, it is an equality because of decreasing monotony of \( \mathcal{D} \). Finally, with a slight abuse of notation, one can denote by the same symbol the extension of \( d \) to \( \Omega^2 \) such that \( d(x, x) = 0 \), \( x \in \Omega \). Thus \( d \) is symmetric in its arguments, vanishes if \( x = y \), is positive if \( x \neq y \), and verifies the ultrametric triangle inequality. So, \( d \) is an ultrametric.

## B Proof of Proposition 5

First, it is worth pointing out the following observation. Let \( (\Lambda, \preceq) \) be a join-complete semilattice. If \( z_1 := \sup \{ \sup X, \sup Y \} \), then \( x \preceq \sup X \preceq z_1 \) for all \( x \in X \) and \( y \preceq \sup Y \preceq z_1 \) for all \( y \in Y \). Thus \( u \preceq z_1 \) for all \( u \in X \cup Y \) and \( z_2 := \sup (X \cup Y \cup z_1) \). On the other hand, from \( \sup X \preceq \sup (X \cup Y) = z_2 \) and similarly \( \sup Y \preceq z_2 \) one has \( z_1 = \sup \{ \sup X, \sup Y \} \preceq z_2 \). This means that

\[
\sup \{ \sup X, \sup Y \} = \sup \{ X \cup Y \}.
\tag{B.1}
\]

So, let us move to the proof of the proposition. Let \( \Delta \) be a totally ordered set.

1. Let us assume that \( \psi : (\Delta, \max, -\infty) \to (\Lambda, \oplus, \bot) \) is a monoid homomorphism and take any \( \psi(a), \psi(b) \in \psi(\Delta) \). From \( \max \{a, b\} \in \{a, b\} \) one gets \( \psi(a) \oplus \psi(b) = \psi(\max \{a, b\}) \in \{\psi(a), \psi(b)\} \), thus \( \psi(\Delta) \) is totally ordered.

2. Now let \( \phi_\psi \) in (5.8) be a monoid homomorphism, so \( \vartheta(\max \{a, b\}) \in \phi_\psi(\max \{a, b\}) = \phi_\psi(a) \cup \phi_\psi(b) \). Hence \( \vartheta(\max \{a, b\}) \leq \sup \{\vartheta(a), \vartheta(b)\} \). Moreover, if \( u, v \in \Delta \) and \( \max \{u, v\} = v \), then \( \phi_\psi(u) \subseteq \phi_\psi(v) \). So \( \vartheta(u) \in \phi_\psi(u) \subseteq \phi_\psi(v) \) means that \( \vartheta(u) \leq \vartheta(v) \) and \( \vartheta \) is increasing. Thus, \( \vartheta(a) \leq \vartheta(\max \{a, b\}) \) and \( \vartheta(b) \leq \vartheta(\max \{a, b\}) \), i.e. \( \sup \{\vartheta(a), \vartheta(b)\} \leq \vartheta(\max \{a, b\}) \). One finally gets \( \sup \{\vartheta(a), \vartheta(b)\} = \vartheta(\max \{a, b\}) \), that means \( \vartheta(a) \oplus \vartheta(b) = \vartheta(\max \{a, b\}) \) by (2.6). Furthermore, from \( \vartheta(-\infty) \in \phi_\psi(-\infty) \) and the homomorphism condition \( \phi_\psi(-\infty) = \{\bot\} \) one deduces \( \vartheta(-\infty) = \bot \). So the mapping \( \vartheta \) is a monoid homomorphism and the poset \( \vartheta(\mathbb{R}^+) \) is a totally ordered set in \( \Lambda \).

3. Let us introduce \( \hat{\psi} := \text{id} \circ \psi : \Delta \to \mathcal{P}(\hat{\Lambda}) \), where \( \hat{\Lambda} \) is defined in (5.9) and \( \text{id} : \mathcal{P}(\Lambda) \to \mathcal{P}(\hat{\Lambda}) \) is the immersion \( S \subseteq \Lambda \to S \subseteq \hat{\Lambda} \). The mapping \( \hat{\vartheta}(a) := \sup \hat{\psi}(a), a \in \Delta \), satisfies \( \hat{\vartheta}(\bot) = \sup \hat{\psi}(\bot) = \sup \{\bot\} = \bot \) and

\[
\hat{\vartheta}(a) \oplus \hat{\vartheta}(b) = \sup \{\hat{\vartheta}(a), \hat{\vartheta}(b)\} = \sup (\hat{\psi}(a) \cup \hat{\psi}(b))
= \sup \hat{\psi}(\max \{a, b\}).
\tag{B.2}
\]

Previous equalities come from (2.6), (B.1), the definition of \( \hat{\psi} \) and the assumption that \( \psi \)
is a monoid homomorphism, which implies that \( \hat{\psi} \) is a monoid homomorphism too. Hence \( \hat{\psi} : \Delta \rightarrow \hat{\Lambda} \) is a monoid homomorphism. Let \( \Delta_0 := \hat{\psi}(\Lambda) \), that is a totally ordered subset of \( \Delta \) with \( -\infty \in \hat{\psi}(\Lambda) \subseteq \Delta_0 \). One can now define the restriction \( \hat{\psi} := \hat{\psi}|_{\Delta_0} \) and the map \( \hat{i} := \iota \circ \hat{\psi} : \Delta_0 \rightarrow \Lambda \). From part (1.), the set \( \hat{\psi}(\Delta_0) \) is totally ordered, so the restriction of \( \iota \) to \( \hat{\psi}(\Delta_0) \) is a monoid homomorphism as follows from Proposition 3. Thus \( \hat{i} \) is a monoid homomorphism too and \( \psi(a) \subseteq \hat{i}(a) \cup \{ \partial(a) \} = \phi_{\partial}(a) \).

C Proof of Proposition 6

Let us write \( k := k_B \) and \( \partial_k := \frac{d}{dk} \) for notational convenience and make explicit reference to the temperature \( T \) introducing \( F_\alpha := T \cdot f_\alpha \). One finds that tropical free energy corresponds to the 0-th order term in the expansion \( F = \min \{ F_\alpha : \alpha \in [N] \} = \kappa_0 \). First order correction corresponds to statistical prefactors in [9] and they coincide with results from standard perturbation theory. In fact, one has

\[
\left( \partial_k \frac{F}{T} \right) \bigg|_{k=0} = \left[ \partial_k (-k \ln Z) \right] \bigg|_{k=0} = - \ln \left( \lambda_0 + \sum_{\alpha \in \mathbb{S}_0} \exp \frac{T\kappa_0 - F_\alpha}{kT} \right) \bigg|_{k=0} - k \cdot \partial_k \ln Z \bigg|_{k=0} = - \ln \lambda_0 \quad (C.1)
\]

Now let us consider higher order contributions. One has \( \partial_k^m \frac{F(k)}{T} = \partial_k^m \left( \frac{F(k)}{T} - \kappa_0 \right) \), so we can take \( \kappa_0 = \frac{F_{\text{trop}}}{T} \equiv 0 \) without loss of generality. In particular, this means that \( \lim_{k \to 0^+} Z(k) = \lambda_0 \neq 0 \). Thus, one has

\[
\partial_k^m Z \bigg|_{k=0} = \sum_{\alpha \in \mathbb{S}_0} \partial_k^m \exp \left( -\frac{F_\alpha}{kT} \right) \bigg|_{k=0} = \sum_{\alpha \in \mathbb{S}_0} Q_m \left( k^{-1}; -\frac{F_\alpha}{T} \right) \exp \left. \frac{-F_\alpha}{kT} \right|_{k=0} = 0 \quad (C.2)
\]

where \( Q_m \left( k^{-1}; -\frac{F_\alpha}{T} \right) \) is a polynomial in \( k^{-1} \) with coefficients that depend on \( -\frac{F_\alpha}{T} \). Moreover \( \partial_k \left( \frac{1}{Z} \right) = - \frac{-Z \partial_k Z}{Z^2} \rightarrow 0 \). Now assume that \( \partial_k^l \frac{1}{Z} = 0 \) for all \( 1 \leq l \leq m - 1 \). Then

\[
0 = \partial_k^m \left( \frac{1}{Z} \cdot Z \right) = \sum_{l=0}^m \binom{m}{l} \partial_k^l \frac{1}{Z} \cdot \partial_k^{m-l} Z. \quad (C.3)
\]

From the inductive hypothesis one gets \( 0 = \frac{1}{Z} \cdot \partial_k^m Z + Z \cdot \partial_k^m \frac{1}{Z} \). But \( \partial_k^m Z = 0 \) at \( m \geq 1 \) from (C.2) and \( \lim_{k \to 0^+} Z(k) \neq 0 \). Hence \( \partial_k^m \frac{1}{Z} = 0 \) for all \( m \geq 1 \) by induction. So \( \partial_k^{m-l} Z \big|_{k=0} = 0 \) if \( m > l \), \( \partial_k^l \frac{1}{Z} \) is vanishing if \( l > 0 \) and finite at \( l = 0 \). Given that

\[
\partial_k^m \ln Z = \partial_k^{m-1} \frac{-\partial_k Z}{Z} = \sum_{l=0}^{m-1} \binom{m-1}{l} \partial_k^l \frac{1}{Z} \cdot \partial_k^{m-l} Z, \quad (C.4)
\]
this means that \( \partial_m^m \ln Z = 0 \) for all \( m \geq 1 \). Thus

\[
\partial_m^m \frac{F}{T} = \partial_m^m (-k \ln Z) = - \sum_{l=0}^{m} \binom{m}{l} \cdot \partial_k^k \cdot \partial_k^m \ln Z
\]

\[
= -k \cdot \partial_k^m \ln Z - m \cdot \partial_k^{m-1} \ln Z
\]

which vanishes at \( k_B \rightarrow 0^+ \) and \( m > 1 \).

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