A characterization of round spheres in space forms

Francisco Fontenele* and Roberto Alonso Núñez

Abstract. Let $Q^{n+1}_c$ be the complete simply-connected $(n+1)$-dimensional space form of curvature $c$. In this paper we obtain a new characterization of geodesic spheres in $Q^{n+1}_c$ in terms of the higher order mean curvatures. In particular, we prove that the geodesic sphere is the only complete bounded immersed hypersurface in $Q^{n+1}_c$, $c \leq 0$, with constant mean curvature and constant scalar curvature. The proof relies on the well known Omori-Yau maximum principle, a formula of Walter for the Laplacian of the $r$-th mean curvature of a hypersurface in a space form, and a classical inequality of Gårding for hyperbolic polynomials.

1 Introduction

A question of interest in differential geometry is whether the geodesic sphere is the only compact oriented hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with constant $r$-th mean curvature $H_r$, for some $r = 1, \ldots, n$ ($H_1$, $H_2$, and $H_n$ are the mean curvature, the scalar curvature, and the Gauss-Kronecker curvature, respectively – see the definitions in Section 2). When $r = 1$ this question is the well known Hopf conjecture, and when $r = 2$ it is a problem proposed by Yau [29, Problema 31, p. 677].

As proved by Alexandrov [1] for $r = 1$, and by Ros [22, 23] (see also [16, 18]) for any $r$, the above question has an affirmative answer for embedded

*Partially supported by CNPq (Brazil)

12010 Mathematics Subject Classification. Primary 53C42, 14J70; Secondary 53C40, 53A10.

2Key words and phrases. Hypersurfaces in space forms, scalar curvature, Laplacian of the $r$-th mean curvature, hyperbolic polynomials.
hypersurfaces. In the immersed case, the question has a negative answer when \( r = 1 \) (by the examples of non-spherical compact hypersurfaces with constant mean curvature in the Euclidean space constructed by Wente [27] and by Hsiang, Teng and Yu [14]), and an affirmative answer when \( r = n \) (by a theorem of Hadamard). The problem is still unsolved for \( 1 < r < n \). For partial answers when \( r = 2 \) (Yau’s problem), see [5, 17, 20].

Because of the difficulty of the above question, it is natural to attempt to obtain the rigidity of the sphere in \( \mathbb{R}^{n+1} \) under geometric conditions stronger than \( H_r \) be constant for some \( r \). In this regard, Gardner [13] proved that if a compact oriented hypersurface \( M^n \) in \( \mathbb{R}^{n+1} \) has two consecutive mean curvatures \( H_r \) and \( H_{r+1} \) constant, for some \( r = 1, \ldots, n-1 \), then it is a geodesic sphere. For generalizations of this result see [3, 15, 26].

In [7], Cheng and Wan proved that a complete hypersurface \( M^3 \) with constant scalar curvature \( R \) and constant mean curvature \( H \neq 0 \) in \( \mathbb{R}^4 \) is a generalized cylinder \( S^k(a) \times \mathbb{R}^{3-k} \), for some \( k = 1, 2, 3 \) and some \( a > 0 \) (see [19] for results of this nature in higher dimensions). From this result one obtains the following improvement, when \( n = 3 \) and \( r = 1 \), in the theorem of Gardner referred to above: The geodesic spheres are the only complete bounded immersed hypersurfaces in \( \mathbb{R}^4 \) with constant scalar curvature and constant mean curvature (compare with Corollary 1.2).

Our main result (Theorem 1.1) provides a new characterization of geodesic spheres in space forms. There are many results of this nature in the literature, most of which assuring that a compact hypersurface that satisfies certain geometric conditions is a geodesic sphere. What makes special the characterization provided by Theorem 1.1 is that in it the geometric conditions are imposed on a complete hypersurface (that is bounded when \( c \leq 0 \), and contained in a spherical cap when \( c > 0 \)), and not on a compact one.

In the theorem below, as well as in the remaining of this work, \( Q_{c}^{n+1} \) stands for the \((n + 1)\)-dimensional complete simply-connected space of constant sectional curvature \( c \).

**Theorem 1.1.** Let \( M^n \) be a complete Riemannian manifold with scalar curvature \( R \) bounded from below, and let \( f : M^n \rightarrow Q_{c}^{n+1} \) be an isometric immersion. In the case \( c \leq 0 \), assume that \( f(M^n) \) is bounded, and in the case \( c > 0 \), that \( f(M^n) \) lies inside a geodesic ball of radius \( \rho < \pi/2\sqrt{c} \). If the mean curvature \( H \) is constant and, for some \( r = 2, \ldots, n \), the \( r \)-th mean curvature \( H_r \) is constant, then \( f(M^n) \) is a geodesic sphere of \( Q_{c}^{n+1} \).

The following results follow immediately from the above theorem. Notice
that the hypothesis in Theorem 1.1 that the scalar curvature of \( M^n \) is bounded from below is superfluous when \( r = 2 \).

**Corollary 1.2.** Let \( f : M^n \to \mathbb{Q}^{n+1}_c \) be an isometric immersion of a complete Riemannian manifold \( M^n \) in \( \mathbb{Q}^{n+1}_c \). In the case \( c \leq 0 \), assume that \( f(M^n) \) is bounded, and in the case \( c > 0 \), that \( f(M^n) \) lies inside a geodesic ball of radius \( \rho < \pi/2\sqrt{c} \). If the mean curvature \( H \) and the scalar curvature \( R \) are constant, then \( f(M^n) \) is a geodesic sphere of \( \mathbb{Q}^{n+1}_c \).

**Corollary 1.3.** Let \( f : M^n \to \mathbb{Q}^{n+1}_c \) be an isometric immersion of a compact Riemannian manifold \( M^n \) in \( \mathbb{Q}^{n+1}_c \). In the case \( c > 0 \), assume that \( f(M) \) is contained in an open hemisphere of \( S^{n+1}_c \). If the mean curvature \( H \) is constant and, for some \( r = 2, ..., n \), the \( r \)-th mean curvature \( H_r \) is constant, then \( f(M^n) \) is a geodesic sphere of \( \mathbb{Q}^{n+1}_c \).

**Remark 1.4.** The examples of Wente [27] and Hsiang, Teng and Yu [14], referred to in the second paragraph of this section, show that the hypothesis that \( H_r \) is constant for some \( r, 2 \leq r \leq n \), can not be removed from Theorem 1.1. It is surely a difficult question to know whether the theorem holds without the assumption that \( H \) is constant (cf. Yau’s problem mentioned in the beginning of this section). We do not know whether Theorem 1.1 (for \( r \geq 3 \)) holds without the hypothesis that the scalar curvature of \( M \) is bounded below.

The proof of Theorem 1.1 relies on the well known Omori-Yau maximum principle [8, 21, 28], a formula of Walter [25] for the Laplacian of the \( r \)-th mean curvature of a hypersurface in a space form, and a classical inequality of Gårding [12] for hyperbolic polynomials.

## 2 Preliminaries

Given an isometric immersion \( f : M^n \to N^{n+k} \) of a \( n \)-dimensional Riemannian manifold \( M^n \) into a \( (n+k) \)-dimensional Riemannian manifold \( N^{n+k} \), denote by \( \sigma : TM \times TM \to TM^\perp \) the (vector valued) second fundamental form of \( f \), and by \( A_\xi \) the shape operator of the immersion with respect to a (locally defined) unit normal vector field \( \xi \). From the Gauss formula one obtains, for all smooth vector fields \( X \) and \( Y \),

\[
\langle A_\xi X, Y \rangle = \langle \sigma(X, Y), \xi \rangle. \tag{2.1}
\]
In the particular case that $M$ and $N$ are orientable and $k = 1$, one may choose a global unit normal vector field $\xi$ and so define a (symmetric) 2-tensor field $h$ on $M$ by $h(X, Y) = \langle \sigma(X, Y), \xi \rangle$. Then, by (2.1),

$$h(X, Y) = \langle AX, Y \rangle, \quad X, Y \in \mathfrak{X}(M),$$

where $A = A_\xi$ is the shape operator of the immersion with respect to $\xi$. If we assume further that $N^{n+1}$ has constant sectional curvature, it follows from the symmetry of $h$ and the Codazzi equation that the covariant derivative $\nabla h$ of $h$ is symmetric. Hence, $\nabla^2 h := \nabla(\nabla h)$ is symmetric in the first three entries. The following lemma shows what happens when we interchange vectors in its third and fourth entries. In its statement, as well as in the remaining of the work, we denote by $h_{ij}$, $h_{ijk}$ and $h_{ijkl}$ the components of $h$, $\nabla h$ and $\nabla^2 h$, respectively, in a local orthonormal frame field $\{e_1, \ldots, e_n\}$, i.e.,

$$h_{ij} = h(e_i, e_j), \quad h_{ijk} = \nabla h(e_i, e_j, e_k), \quad h_{ijkl} = \nabla^2 h(e_i, e_j, e_k, e_l).$$

**Lemma 2.1.** For any local orthonormal frame field $\{e_1, \ldots, e_n\}$ on $M^n$, we have

$$h_{ijkl} - h_{ijlk} = \sum_m R_{klim} h_{mj} + \sum_m R_{kljm} h_{im},$$

for all $i, j, k, l \in \{1, \ldots, n\}$, where $R$ is the Riemannian curvature tensor of $M^n$ and, for example, $R_{klim} = \langle R(e_k, e_l)e_i, e_m \rangle$.

Formula (2.3) above is well known. For a proof see, for instance, [6, p. 1167].

Given an isometric immersion $f : M^n \to N^{n+1}$, denote by $\lambda_1, \ldots, \lambda_n$ the principal curvatures of $M^n$ with respect to a global unit normal vector field $\xi$ (i.e., the eigenvalues of the shape operator $A = A_\xi$). It is well known that if we label the principal curvatures at each point by the condition $\lambda_1 \leq \cdots \leq \lambda_n$, then the resulting functions $\lambda_i : M \to \mathbb{R}, i = 1, \ldots, n$, are continuous.

The $r$-th mean curvature $H_r, 1 \leq r \leq n$, of $M^n$ is defined by

$$\binom{n}{r} H_r = \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r}. \quad (2.4)$$

Notice that $H_1$ is the mean curvature $H$ ($= \frac{1}{n} \text{tr} A$, where $\text{tr} A$ is the trace of $A$) and $H_n = \lambda_1 \lambda_2 \ldots \lambda_n$ is the Gauss-Kronecker curvature of the immersion. In
the particular case that $N^{n+1}$ has constant sectional curvature, the function $H_2^2$ is up to a constant the (normalized) scalar curvature $R$ of $M^n$. In fact, if $N^{n+1}$ has constant sectional curvature $c$ and if $\{e_1, ..., e_n\}$ is an orthonormal basis for the tangent space at a given point of $M^n$ such that $Ae_i = \lambda_i e_i$, $i = 1, ..., n$, then the sectional curvature $K(e_i, e_j)$ of the plane spanned by $e_i$ and $e_j$ is, by the Gauss equation, given by

$$K(e_i, e_j) = c + \lambda_i \lambda_j,$$

and so

$$R = \frac{1}{(n/2)} \sum_{i<j} K(e_i, e_j) = \frac{1}{(n/2)} \sum_{i<j} (c + \lambda_i \lambda_j) = c + H_2. \quad (2.5)$$

The squared norm $|A|^2$ of the shape operator $A$ is defined as the trace of $A^2$. It is easy to see that

$$|A|^2 = \sum_i \lambda_i^2. \quad (2.6)$$

From (2.4), (2.5) and (2.6) we obtain the following useful relation involving the mean curvature $H$, the norm $|A|$ of the shape operator $A$ and the normalized scalar curvature $R$:

$$n^2 H^2 = \left( \sum_{i=1}^n \lambda_i \right)^2 = \sum_{i=1}^n \lambda_i^2 + \sum_{i \neq j} \lambda_i \lambda_j = |A|^2 + n(n-1)(R-c). \quad (2.7)$$

In terms of the $r$-th symmetric function $\sigma_r : \mathbb{R}^n \to \mathbb{R}$,

$$\sigma_r(x_1, \ldots, x_n) = \sum_{i_1 < \ldots < i_r} x_{i_1} \ldots x_{i_r}, \quad (2.8)$$

equality (2.4) can be rewritten as

$$\binom{n}{r} H_r = \sigma_r \circ \overrightarrow{\lambda}, \quad (2.9)$$

where $\overrightarrow{\lambda} = (\lambda_1, ..., \lambda_n)$ is the principal curvature vector of the immersion. In order to unify the notation, we define $H_0 = 1 = \sigma_0$ and $H_r = 0 = \sigma_r$, for all $r \geq n + 1$. 

5
As one might expect, the knowledge of the properties of the symmetric functions is very important to the study of the higher order mean curvatures of a hypersurface. In order to state a property of the symmetric functions that will be relevant to us, we will summarize below some of the results of the classical article by Gårding [12] on hyperbolic polynomials (see also [4, p. 268] and [10, p. 217]).

Let \( P : \mathbb{R}^n \to \mathbb{R} \) be a homogenous polynomial of degree \( m \) and let \( a = (a_1, \ldots, a_n) \) be a fixed vector of \( \mathbb{R}^n \). We say that \( P \) is hyperbolic with respect to the vector \( a \), or in short, that \( P \) is \( a \)-hyperbolic, if for every \( x \in \mathbb{R}^n \) the polynomial in \( s \), \( P(sa + x) \), has \( m \) real roots. Denote by \( \Gamma_P \) the connected component of the set \( \{ P \neq 0 \} \) that contains \( a \). In [12], Gårding proved that \( \Gamma_P \) is an open convex cone, with vertex at the origin, and that the homogenous polynomial of degree \( m - 1 \) defined by

\[
Q(x) = \left. \frac{d}{ds} \right|_{s=0} P(sa + x) = \sum_{j=1}^{n} a_j \frac{\partial P}{\partial x_j}(x)
\]

is also \( a \)-hyperbolic. Moreover, \( \Gamma_P \subset \Gamma_Q \).

As can be easily seen, the \( n \)-th symmetric function \( \sigma_n \) is hyperbolic with respect to the vector \( a = (1, \ldots, 1) \). Applying the results of the previous paragraph to \( \sigma_n \), and observing that

\[
\sigma_r(x) = \left. \frac{1}{(n-r)!} \frac{d^{n-r}}{ds^{n-r}} \right|_{s=0} \sigma_n(sa + x), \quad r = 1, \ldots, n - 1,
\]

one concludes that \( \sigma_r \), \( 1 \leq r \leq n \), is hyperbolic with respect to \( a = (1, \ldots, 1) \) and that

\[
\Gamma_1 \supset \Gamma_2 \supset \ldots \supset \Gamma_n,
\]

where \( \Gamma_r := \Gamma_{\sigma_r} \).

In [12], Gårding established an inequality for hyperbolic polynomials involving their completely polarized forms. A particular case of this inequality, from which the general case is derived, says that

\[
\frac{1}{m} \sum_{k=1}^{n} y_k \frac{\partial P}{\partial x_k}(x) \geq P(y)^{\frac{1}{m}} P(x)^{1-\frac{1}{m}}, \quad \forall x, y \in \Gamma_P.
\]

As observed in [4, p. 269], the above inequality is equivalent to the assertion that \( P^{1/m} \) is a concave function on \( \Gamma_P \). In particular, we have the following result, which will play an important role in the proof of Theorem 1.1.

**Proposition 2.2.** For each \( r = 1, 2, \ldots, n \), the function \( \sigma_r^{1/r} \) is concave on \( \Gamma_r \).
3 The Laplacian of the $r$-th mean curvature

The symmetric functions $\sigma_r$, $1 \leq r \leq n$, defined by (2.8), arise naturally from the identity

$$\prod_{s=1}^{n}(x_s + t) = \sum_{r=0}^{n} \sigma_r(x)t^{n-r}, \quad (3.1)$$

which is valid for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Differentiating this identity with respect to $x_j$, one obtains

$$\prod_{s \neq j}(x_s + t) = \sum_{r=0}^{n} \frac{\partial \sigma_r}{\partial x_j}(x)t^{n-r}, \quad j = 1, ..., n. \quad (3.2)$$

Differentiation of (3.2) with respect to $x_i$, for $i \neq j$, yields

$$\prod_{s \neq i, j}(x_s + t) = \sum_{r=0}^{n} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(x)t^{n-r}, \quad i \neq j. \quad (3.3)$$

From the identities

$$\sigma_r(x) = x_i \sigma_{r-1}(\hat{x}_i) + \sigma_r(\hat{x}_i), \quad x \in \mathbb{R}^n, \quad 1 \leq i, r \leq n, \quad (3.4)$$

where, for instance, $\sigma_{r-1}(\hat{x}_i) = \sigma_{r-1}(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$, one obtains, for all $r = 2, ..., n$,

$$\frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(x) = \begin{cases} \sigma_{r-2}(\hat{x}_i, \hat{x}_j), & i \neq j, \\ 0, & i = j. \end{cases} \quad (3.5)$$

In [25] Walter established a formula for the Laplacian of the $r$-th mean curvature of a hypersurface in a space of constant sectional curvature. For convenience of the reader, we state and prove that formula below. Recall that the Laplacian $\Delta u$ of a $C^2$-function $u$ defined on a Riemannian manifold $(M, \langle , \rangle)$ is the trace of the 2-tensor field Hess $u$, called the Hessian of $u$, defined by Hess $u(X, Y) = \langle \nabla_X \nabla u, Y \rangle$, for all $X, Y \in \mathfrak{X}(M)$.

**Proposition 3.1.** Let $M^n$ be an orientable hypersurface of an orientable Riemannian manifold $N^{n+1}_c$ of constant sectional curvature $c$. Then, for every
\[ r = 1, \ldots, n \text{ and every } p \in M^n, \]
\[
\binom{n}{r} \Delta H_r = n \sum_j \frac{\partial \sigma_r}{\partial x_j} (\lambda) \text{Hess}(e_j, e_j) - \sum_{i<j} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j} (\lambda_i - \lambda_j)^2 K_{ij} \\
+ \sum \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j} (\lambda)(h_{iik} h_{jjk} - h_{iijk}),
\] (3.6)

where \( \lambda_1, \ldots, \lambda_n \) are the principal curvatures of \( M^n \) at \( p \), \( \lambda = (\lambda_1, \ldots, \lambda_n) \), \{\( e_1, \ldots, e_n \)\} is an orthonormal basis of \( T_p M \) that diagonalizes the shape operator \( A \), and \( K_{ij} \) is the sectional curvature of \( M^n \) in the plane spanned by \( \{e_i, e_j\} \).

**Proof.** Extend the orthonormal basis \( \{e_1, \ldots, e_n\} \) of \( T_p M \) to a local orthonormal frame field, still denoted by \( \{e_1, \ldots, e_n\} \), through parallel transport of the \( e_i \)'s along the geodesics emanating from \( p \). From (2.2), one obtains
\[
(A + tI)e_j = \sum_l (h_{lj} + t \delta_{lj})e_l, \quad 1 \leq j \leq n, \quad t \in \mathbb{R}.
\] (3.7)

Denoting by \( V_1, \ldots, V_n \) the columns of the matrix \( (h_{lj} + t \delta_{lj}) \), one has
\[
e_k(V_j) = \sum_l h_{ljk} E_l, \quad j, k = 1, \ldots, n,
\] (3.8)

where \( \{E_1, \ldots, E_n\} \) is the canonical basis of \( \mathbb{R}^n \). Then, by (3.8) and multilinearity of the determinant,
\[
e_k(\det (A + tI)) = \sum_{j,l} h_{ljk} \det (V_1, \ldots, V_{j-1}, E_l, V_{j+1}, \ldots, V_n).
\] (3.9)

Differentiating the above equality with respect to \( e_k \) and using (3.8), we obtain at \( p \)
\[
e_k e_k(\det (A + tI)) = \sum_{i\neq j} (h_{iik} h_{jjk} - h_{iijk}^2) \prod_{s \neq i,j} (\lambda_s + t) \\
+ \sum_j h_{jjk} \prod_{s \neq j} (\lambda_s + t).
\] (3.10)

By Lemma 2.1 and Codazzi equation, we have at \( p \)
\[
h_{jjk} = h_{jkjk} = h_{jjk} + \sum_m \mathcal{R}_{jkjm} h_{mk} + \sum_m \mathcal{R}_{jkm} h_{jm} \\
= h_{kkjj} + (\lambda_j - \lambda_k) \mathcal{R}_{jkkj}.
\] (3.11)
Covariant differentiation of the equality \( nH = \sum_k h_{kk} \) gives
\[
nHess H(e_j, e_j) = \sum_k h_{kkjj}, \quad j = 1, \ldots, n. \tag{3.12}
\]
Since the Laplacian of a function is the trace of its Hessian, and \( \nabla_e e_j(p) = 0 \), \( 1 \leq i, j \leq n \), summing over \( k \) in (3.10), and using (3.11) and (3.12), we arrive at
\[
\Delta \det (A + tI) = \sum_j nH_{jj} \prod_{s \neq j}(\lambda_s + t) + \sum_{j \neq k} (\lambda_j - \lambda_k)R_{jkkj} \prod_{s \neq j}(\lambda_s + t)
+ \sum_k \sum_{i < j} (h_{iik}h_{jjk} - h_{ijk}^2) \prod_{s \neq i, j}(\lambda_s + t), \tag{3.13}
\]
where \( H_{jj} = Hess H(e_j, e_j) \). Since
\[
\prod_{s \neq j}(\lambda_s + t) = [(\lambda_k - \lambda_j) + (\lambda_j + t)] \prod_{s \neq j,k}(\lambda_s + t),
\]
the second term on the right hand side of (3.13) can be written as
\[
\sum_{j \neq k} (\lambda_j - \lambda_k)R_{jkkj} \prod_{s \neq j}(\lambda_s + t) = \sum_{j \neq k} (\lambda_j - \lambda_k)^2 R_{jkkj} \prod_{s \neq j,k}(\lambda_s + t)
- \sum_{j \neq k} (\lambda_k - \lambda_j)R_{jkkj} \prod_{s \neq k}(\lambda_s + t).
\]
Since \( R_{jkkj} = R_{kjjk} \), the second term on the right hand side of the above equality is minus the term on the left. Hence
\[
\sum_{j \neq k} (\lambda_j - \lambda_k)R_{jkkj} \prod_{s \neq j}(\lambda_s + t) = - \sum_{j < k} (\lambda_j - \lambda_k)^2 R_{jkkj} \prod_{s \neq j,k}(\lambda_s + t). \tag{3.14}
\]
It now follows from (3.2), (3.3), (3.13) and (3.14) that
\[
\Delta \det (A + tI) = \sum_{r=0}^n \left\{ \sum_j nH_{jj} \frac{\partial \sigma_r}{\partial x_j}(\vec{\lambda}) - \sum_{i < j} (\lambda_i - \lambda_j)^2 R_{ijji} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda})
+ \sum_{i,j,k} (h_{iik}h_{jjk} - h_{ijk}^2) \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda}) \right\} t^{n-r}. \tag{3.15}
\]
On the other hand, taking $x_s = \lambda$ in (3.1) and using (2.9), one obtains

$$\det(A + tI) = \sum_{r=0}^{n-1} \left( \frac{n}{r} \right) H_r t^{n-r},$$

(3.16)
and so

$$\Delta \det(A + tI) = \sum_{r=0}^{n-1} \left( \frac{n}{r} \right) \Delta H_r t^{n-r}.$$  

(3.17)

Comparing (3.15) and (3.17), one obtains (3.6).

4 Complete and bounded hypersurfaces

In the proof of Theorem 1.1 we will use, besides Propositions 2.2 and 3.1, the following result.

Proposition 4.1. Let $M^n$ be a complete Riemannian manifold with sectional curvature $K$ bounded from below and $f : M^n \to Q^{n+k}_c$ an isometric immersion of $M^n$ into the $(n + k)$-dimensional complete simply-connected space $Q^{n+k}_c$ of constant sectional curvature $c$. In the case $c \leq 0$, assume that $f(M^n)$ is bounded, and in the case $c > 0$, that $f(M^n)$ lies inside a geodesic ball of radius $\rho < \pi/\sqrt{c}$. Then, there exist $p \in M$ and a unit vector $\xi_0 \in (f_* T_p M)^\perp$ such that, for any unit vector $v \in T_p M$,

$$\langle A \xi_0 v, v \rangle > \begin{cases} 0, & c \geq 0 \\ \sqrt{-c}, & c < 0 \end{cases}$$

(4.1)

We believe that the above proposition is known, but since we were unable to find a reference for it in the literature, we will prove it below. The main ingredient in this proof is the following well known maximum principle due to Omori and Yau [8, 21, 28] (see [11, Theorem 3.4] for a conceptual refinement of this principle):

Omori-Yau Maximum Principle. Let $M^n$ be a complete Riemannian manifold with sectional curvature (resp. Ricci curvature) bounded from below, and let $f : M \to \mathbb{R}$ be a $C^2$-function bounded from above. Then, for every $\varepsilon > 0$, there exists $x_\varepsilon \in M$ such that

$$f(x_\varepsilon) > \sup f - \varepsilon, \quad \|\nabla f(x_\varepsilon)\| < \varepsilon, \quad \text{Hess} f(x_\varepsilon)(v, v) < \varepsilon \|v\|^2 \quad (\text{resp. } \Delta f(x_\varepsilon) < \varepsilon).$$
The following lemma, which will also be used in the proof of Proposition 4.1, expresses the gradient and Hessian of the restriction of a function to a submanifold in terms of the space gradient and Hessian (see [9, p. 46] for a proof). In its statement, we will use the symbol $\nabla$ for the gradient of any function involved.

**Lemma 4.2.** Let $f : M^n \rightarrow N^{n+k}$ be an isometric immersion of a Riemannian manifold $M^n$ into a Riemannian manifold $N^{n+k}$, and let $g : N \rightarrow \mathbb{R}$ be a function of class $C^2$. Then, for all $p \in M$ and $v, w \in T_pM$, one has

$$f_*(\nabla(g \circ f)(p)) = \left[\nabla g(f(p))\right]^\top,$$

(4.2)

$$\text{Hess} (g \circ f)_p(v, w) = \text{Hess}_{f(p)}(f_*v, f_*w) + \left\langle \nabla g(f(p)), \sigma_p(v, w) \right\rangle,$$

(4.3)

where $\sigma$ is the second fundamental form of the immersion, $f_*$ is the differential of $f$ and $"^\top"$ means orthogonal projection onto $f_*(T_pM)$.

**Proof of Proposition 4.1.** By hypothesis, $f(M)$ is contained in some closed ball $\overline{B}_\rho(q_0)$ of center $q_0$ and radius $\rho$, with $\rho < \pi/2\sqrt{c}$ if $c > 0$. Let $r(\cdot) = d(\cdot, q_0)$ be the distance function from the point $q_0$ in $\overline{Q}^{n+k}$ and let $g = r \circ f$. Since $g$ is bounded from above (for $f(M) \subset \overline{B}_\rho(q_0)$) and the sectional curvatures of $M$ are bounded from below, the Omori-Yau maximum principle assures us that, for every $\varepsilon > 0$, there exist $x_\varepsilon \in M$ such that

$$g(x_\varepsilon) > \sup g - \varepsilon, \quad \|\nabla g(x_\varepsilon)\| < \varepsilon, \quad \text{Hess} g_{x_\varepsilon}(v, v) < \varepsilon\|v\|^2, \quad \forall v \in T_{x_\varepsilon}M.$$

From the last two inequalities and Lemma 4.2 we obtain

$$\varepsilon > \|\nabla g(x_\varepsilon)\| = \|\nabla r(f(x_\varepsilon))\|^\top$$

(4.4)

and, for every $v \in T_{x_\varepsilon}M$,

$$\varepsilon\|v\|^2 > \text{Hess} g_{x_\varepsilon}(v, v) = \text{Hess} r_{f(x_\varepsilon)}(f_*v, f_*v) + \left\langle \sigma_{x_\varepsilon}(v, v), \nabla r(f(x_\varepsilon)) \right\rangle,$$

(4.5)

where the superscript $"^\top"$ indicates orthogonal projection on $f_*(T_{x_\varepsilon}M)$.

For every $v \in T_{x_\varepsilon}M$, write

$$f_*v = v_1 + v_2,$$

(4.6)
where \( v_1 \) and \( v_2 \) are the components of \( f_*v \) that are parallel and orthogonal, respectively, to \( \nabla r(f(x_\varepsilon)) \). Recalling that \( \nabla_{\nabla r} \nabla r = 0 \), where \( \nabla \) is the Riemannian connection of \( \mathbb{Q}^{n+k}_c \), one has

\[
\text{Hess } r_{f(x_\varepsilon)}(f_*v, f_*v) = \text{Hess } r_{f(x_\varepsilon)}(v_1 + v_2, v_1 + v_2) = \text{Hess } r_{f(x_\varepsilon)}(v_2, v_2). \tag{4.7}
\]

Note that \( v_2 \) is tangent to the geodesic sphere \( S \) of \( \mathbb{Q}^{n+k}_c \) centered at \( q_0 \) that contains \( f(x_\varepsilon) \). Applying \( (4.3) \) for the inclusion \( \iota : S \to \mathbb{Q}^{n+k}_c \) and \( g = r \), one obtains

\[
\text{Hess } r_{f(x_\varepsilon)}(v_2, v_2) = (Bv_2, v_2), \tag{4.8}
\]

where \( B \) is the shape operator of \( S \) with respect to \(-\nabla r\). Since the principal curvatures of a geodesic sphere of radius \( t \) in \( \mathbb{Q}^{n+k}_c \) are constant and given by

\[
\mu_c(t) = \begin{cases} 
\sqrt{c} \cot(\sqrt{c} t), & c > 0, \ 0 < t < \pi/\sqrt{c}, \\
1/t, & c = 0, \ t > 0, \\
\sqrt{-c} \coth(\sqrt{-c} t), & c < 0, \ t > 0,
\end{cases} \tag{4.9}
\]

it follows from \( (4.7) \) and \( (4.8) \) that

\[
\text{Hess } r_{f(x_\varepsilon)}(f_*v, f_*v) = \mu_c(r(f(x_\varepsilon)))\|v_2\|^2. \tag{4.10}
\]

As \( \|\nabla r\| \equiv 1 \), by \( (4.6) \) one has \( v_1 = (f_*v, \nabla r(f(x_\varepsilon)))\nabla r(f(x_\varepsilon)) \). Then, by \( (4.4) \),

\[
\|v_1\| = |(f_*v, \nabla r(f(x_\varepsilon))^\top)| \leq \|f_*v\|\|\nabla r(f(x_\varepsilon))^\top\| < \varepsilon\|v\|. \tag{4.11}
\]

From \( (4.6) \) and \( (4.11) \), we obtain

\[
\|v_2\|^2 = \|f_*v\|^2 - \|v_1\|^2 = \|v\|^2 - \|v_1\|^2 > (1 - \varepsilon^2)\|v\|^2. \tag{4.12}
\]

Hence, by \( (4.5) \), \( (4.10) \) and \( (4.12) \),

\[
\varepsilon\|v\|^2 > \mu_c(r(f(x_\varepsilon)))(1 - \varepsilon^2)\|v\|^2 + \langle \sigma_{x_\varepsilon}(v, v), \nabla r(f(x_\varepsilon)) \rangle.
\]

Since \( \mu_c \) is decreasing and \( r(f(x_\varepsilon)) \leq \rho \), it follows that

\[
\varepsilon\|v\|^2 > \mu_c(\rho)(1 - \varepsilon^2)\|v\|^2 + \langle \sigma_{x_\varepsilon}(v, v), \nabla r(f(x_\varepsilon)) \rangle
\]

\[
= \mu_c(\rho)(1 - \varepsilon^2)\|v\|^2 + \langle \sigma_{x_\varepsilon}(v, v), \nabla r(f(x_\varepsilon))^\perp \rangle,
\]

12
where $\nabla r(f(x_\varepsilon))^\perp$ is the component of $\nabla r(f(x_\varepsilon))$ that is orthogonal to $f_*(T_xM)$. Setting $\xi_\varepsilon = -\nabla r(f(x_\varepsilon))^\perp/||\nabla r(f(x_\varepsilon))^\perp||$, it follows from (2.1) and the above inequality that

$$\langle A_{\xi_\varepsilon}v, v \rangle = \langle \sigma_{x_\varepsilon}(v, v), \xi_\varepsilon \rangle > \frac{\mu_c(\rho)(1 - \varepsilon^2) - \varepsilon}{||\nabla r(f(x_\varepsilon))^\perp||}, \quad (4.13)$$

for all $v \in T_{x_\varepsilon}M$, $||v|| = 1$. Since, by (4.4), the term on the right hand side of (4.13) tends to $\mu_c(\rho)$ when $\varepsilon \to 0$, and, by (4.9), $\mu_c(\rho) > 0$ for $c \geq 0$ and $\mu_c(\rho) > \sqrt{-c}$ for $c < 0$, (4.1) is fulfilled choosing $p = x_\varepsilon$ and $\xi_0 = \xi_\varepsilon$, where $\varepsilon$ is any positive number sufficiently small.

5 Proof of Theorem 1.1.

Since $H$ is constant and $R$ is bounded from below, from (2.7) one obtains that $|A|^2$ is bounded, and so that the sectional curvatures of $M^n$ are bounded from below. Then, by Proposition 1.1 there exist a point $p \in M$ and a unit vector $\xi_0 \in (f_*(T_pM))^\perp$ such that

$$\langle A_{\xi_0}v, v \rangle > \alpha_c ||v||^2, \quad v \in T_pM, \quad (5.1)$$

where

$$\alpha_c = \begin{cases} 0, & c \geq 0, \\ \sqrt{-c}, & c < 0. \end{cases} \quad (5.2)$$

Choosing the unit normal vector field $\xi$ such that $\xi(p) = \xi_0$, by (5.1) the principal curvatures of $M$ at $p$ satisfy

$$\lambda_i(p) > \alpha_c \geq 0, \quad i = 1, \ldots, n. \quad (5.3)$$

By Proposition 3.1 one has, as $H$ and $R_\varepsilon$ are constant,

$$\sum_{i<j} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda})(\lambda_i - \lambda_j)^2 K_{ij} = \sum_{i,j,k} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda})(h_{iik}h_{jjk} - h_{ij}^2), \quad (5.4)$$

where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$. From (5.3) one obtains that $H_\varepsilon > 0$ and that $\vec{\lambda}(p)$ belongs to the Gårding’s cone $\Gamma_\varepsilon$ (see Section 2). Then, since $M$ is connected, $\vec{\lambda}(q) \in \Gamma_\varepsilon$, $\forall q \in M$. 

13
By Proposition 2.2, \( W_r = \sigma_r^{1/r} \) is a concave function on \( \Gamma_r \). Thus,

\[
\sum_{i,j} y_i y_j \frac{\partial^2 W_r}{\partial x_i \partial x_j}(x) \leq 0,
\]

(5.5)

for all \( x \in \Gamma_r \) and \((y_1, \ldots, y_n) \in \mathbb{R}^n\). A simple computation shows that

\[
\frac{\partial^2 W_r}{\partial x_i \partial x_j} = \frac{1}{r} \sigma_r^{1/r-2} \left( \frac{1}{r} \frac{\partial \sigma_r}{\partial x_i} \frac{\partial \sigma_r}{\partial x_j} + \sigma_r \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j} \right).
\]

(5.6)

Using (5.6) in (5.5), we conclude that

\[
\sigma_r(x) \sum_{i,j} y_i y_j \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(x) \leq \frac{r-1}{r} \left( \sum_{j} \frac{\partial \sigma_r}{\partial x_j}(x) \right)^2,
\]

(5.7)

for all \( x \in \Gamma_r \) and \((y_1, \cdots, y_n) \in \mathbb{R}^n\). Taking \( x = \vec{\lambda} \) and \( y_i = h_{iik}, i = 1, \ldots, n \), in (5.7), one obtains

\[
\binom{n}{r} H_r \sum_{i,j} h_{iik} h_{jjk} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda}) \leq \frac{r-1}{r} \left( \sum_{j} h_{jjk} \frac{\partial \sigma_r}{\partial x_j}(\vec{\lambda}) \right)^2, \quad \forall k.
\]

(5.8)

We claim that in a basis that diagonalizes \( A \),

\[
\sum_j h_{jjk} \frac{\partial \sigma_r}{\partial x_j}(\vec{\lambda}) = \binom{n}{r} e_k(H_r).
\]

(5.9)

The claim can be proved using the formula [24, p. 225]

\[
\binom{n}{r} e_k(H_r) = \text{trace}(P_{r-1} \nabla e_k A),
\]

where \( P_{r-1} \) is the \((r-1)\)-th Newton tensor associated with the shape operator \( A \) of \( M \). Alternatively, (5.9) can be obtained from the computations made in the proof of Proposition 3.1. In fact, by (3.2) and (3.9) we have

\[
e_k(\det(A + tI)) = \sum_j h_{jjk} \prod_{s \neq j}(\lambda_s + t).
\]

\[
= \sum_{r=0}^{n} \left( \sum_j h_{jjk} \frac{\partial \sigma_r}{\partial x_j}(\vec{\lambda}) \right) t^{n-r}.
\]

(5.10)
On the other hand, by (3.16) one has
\[ e_k \left( \det (A + tI) \right) = \sum_{r=0}^{n} \binom{n}{r} e_k(H_r)t^{n-r}. \quad (5.11) \]

Comparing (5.10) and (5.11), one obtains (5.9).

Since \( H_r \) is a positive constant, from (5.8) and (5.9) one obtains
\[ \sum_{i,j} h_{ijk} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda}) \leq 0, \quad k = 1, ..., n. \]

Using this information in (5.4), we conclude that the inequality
\[ \sum_{i<j} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda}(p))\left(\lambda_i - \lambda_j\right)^2K_{ij} \leq -\sum_{i,j,k} h_{ijk} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda}) \]
holds at every point of \( M \). Since, by (3.3) and (5.3),
\[ \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda}(p)) = \begin{cases} \sigma_{r-2}(\hat{\lambda}_i(p), \hat{\lambda}_j(p)) > 0, & i \neq j, \\ 0, & i = j, \end{cases} \quad (5.12) \]
it follows that
\[ \sum_{i<j} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda}(p))(\lambda_i(p) - \lambda_j(p))^2K_{ij}(p) \leq 0. \quad (5.13) \]

Since, by (5.3) and the Gauss equation,
\[ K_{ij}(p) = c + \lambda_i(p)\lambda_j(p) > c + \alpha_c^2 \geq 0, \quad i \neq j, \]
it follows from (5.12) and (5.13) that
\[ \lambda_1(p) = \cdots = \lambda_n(p) = H. \quad (5.14) \]

The above argument in fact shows that every point \( q \in M \) for which \( \lambda_i(q) > \alpha_c, \forall i \), is umbilical. Since \( H > \alpha_c \) by (5.3) and (5.14), one then has that the set \( B \) of all the umbilical points of \( M \) is open. Since \( B \) is also nonempty (for \( p \in M \)) and closed (by the continuity of the principal curvature functions), one concludes that \( B = M \) from the connectedness of \( M \). Hence,
\[ \lambda_1 = \cdots = \lambda_n = H > \alpha_c, \quad (5.15) \]
at any point of $M$. It now follows from (5.2), (3.15) and the Gauss equation that the sectional curvature of $M$ satisfies $K = c + H^2 > 0$. In particular, $M$ is compact. It now follows from the classification of the umbilical hypersurfaces in a simply connected space form (see, for instance, [2, p. 25]) that $f(M)$ is a hypersphere of $\mathbb{Q}^{n+1}$.

\[
\square
\]

References

[1] A.D. Aleksandrov, *Uniqueness theorems for surfaces in the large*, Vestnik Leningrad Univ. Math., 13 (1958), 5-8.

[2] J. Berndt, S. Console and C. Olmos, *Submanifolds and holonomy*, Chapman and Hall/CRC Research Notes in Mathematics Series, 434, 2003.

[3] I. Bivens, *Integral formulas and hyperspheres in a simply connected space form*, Proc. Amer. Math. Soc., 88 (1983), 113-118.

[4] L. Caffarelli, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations III: Functions of the eigenvalues of the Hessian*, Acta Math., 155 (1985), 261-301.

[5] Q.-M. Cheng, *Complete hypersurfaces in a Euclidean space $\mathbb{R}^{n+1}$ with constant scalar curvature*, Indiana Univ. Math. J., 51 (2002), 53-68.

[6] Q.-M. Cheng, *Submanifolds with constant scalar curvature*, Proc. Roy. Soc. Edinburgh Sect. A, 132 (2002), 1163-1183.

[7] Q.-M. Cheng and Q.R. Wan, *Complete hypersurfaces of $\mathbb{R}^4$ with constant mean curvature*, Monatsh. Math., 118 (1994), 171-204.

[8] S.Y. Cheng and S.T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math., 28 (1975), 333-354.

[9] M. Dajczer, *Submanifolds and Isometric Immersions* Math. Lect. Ser. 13, Publish or Perish Inc. Houston, 1990.

[10] F. Fontenele and S. L. Silva, *A Tangency Principle and Applications*, Illinois J. Math., Vol. 45 (2001) 213-228.
[11] F. Fontenele and F. Xavier, *Good shadows, dynamics and convex hulls of complete submanifolds*, Asian J. Math., **15** (2011), 9-32.

[12] L. Gårding, *An inequality for hyperbolic polynomials*, J. Math. Mech., Vol. **8** (1959) 957-965.

[13] R. Gardner, *The Dirichlet integral in differential geometry*, Global Analysis, S.S. Chern and S. Smale, editors, Proc. Sympos. Pure Math. Vol. 15, Amer. Math. Soc., Providence, R.I., 1970, pp. 231-237.

[14] W.Y. Hsiang, Z.H. Teng and W.C. Yu, *New examples of constant mean curvature immersions of $(2k-1)$-spheres into Euclidean $2k$-space*, Ann. of Math. **117** (1983), 609-625.

[15] S.-E. Koh, *A characterization of round spheres*, Proc. Amer. Math. Soc., **126** (1998), 3657-3660.

[16] N.J. Korevaar, *Spheres theorem via Aleksandrov for constant Weingarten curvature hypersurfaces–Appendix to a note of A. Ros*, J. Differential Geom., **27** (1988), 221-223.

[17] H. Li, *Hypersurfaces with constant scalar curvature in space forms*, Math. Ann., **305** (1996), 665-672.

[18] S. Montiel and A. Ros, *Compact hypersurfaces: The Alexandrov theorem for higher order mean curvatures*. In: H.B. Lawson and K. Tenenblat (eds) Differential Geometry, a Symposium in honor of M. do Carmo, Pitman Monographs Vol. 52, pp. 279-296, 1991.

[19] R.A. Núñez, *On complete hypersurfaces with constant mean and scalar curvatures in Euclidean spaces*, Proc. Amer. Math. Soc., **145** (2017), 2677-2688.

[20] T. Okayasu, *On compact hypersurfaces with constant scalar curvature in the Euclidean space*, Kodai Math. J., **28** (2005), 577-585.

[21] H. Omori, *Isometric immersions of Riemannians manifolds*, J. Math. Soc. Japan, **19** (1967), 205-214.

[22] A. Ros, *Compact hypersurfaces with constant scalar curvature and a congruence theorem*, J. Differential Geom., **27** (1988), 215-220.
[23] A. Ros, *Compact hypersurfaces with constant higher order mean curvatures*, Rev. Mat. Iberoam., 3 (1987), 447-453.

[24] H. Rosenberg, *Hypersurfaces of constant curvature in space forms*, Bull. Sci. Math., 117 (1993), 211-239.

[25] R. Walter, *Compact hypersurfaces with a constant higher mean curvature function*, Math. Ann., 270 (1985) 125-145.

[26] Q. Wang, *Totally umbilical property and higher order curvature of hypersurfaces in a positive curvature space form*, Acta Math. Sinica (Chin. Ser.), 57 (2014) 47-50.

[27] H.C. Wente, *Counterexample to a conjecture of H. Hopf*, Pacific J. Math., 121 (1986) 193-243.

[28] S.T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math., 28 (1975), 201-228.

[29] S.T. Yau, *Problem section*, Seminar on Differential Geometry, Ann. of Math. Stud. 102, Princeton University Press, Princeton, NJ, 1982.

Francisco Fontenele
Departamento de Geometria
Universidade Federal Fluminense
Niterói, RJ, Brazil
fontenele@mat.uff.br

Roberto Alonso Núñez
Rua Mário Santos Braga s/n
24020-140 Niterói, RJ, Brazil
roberto78nunez@gmail.com