NEW SIMPLE LIE SUPERALGEBRAS IN CHARACTERISTIC 3

ALBERTO ELDUQUE

Abstract. Symplectic (respectively orthogonal) triple systems provide constructions of Lie algebras (resp. superalgebras). However, in characteristic 3, it is shown that this role can be interchanged and that Lie superalgebras (resp. algebras) can be built out of symplectic triple systems (resp. orthogonal triple systems) with a different construction. As a consequence, new simple finite dimensional Lie superalgebras, as well as new models of some nonclassical simple Lie algebras, over fields of characteristic 3, will be obtained.

1. Introduction

Symplectic triple systems [YA75] are basic ingredients in the construction of some 5-graded Lie algebras. They consist of a vector space $T$ endowed with a trilinear product $[., ., .]$ and a nonzero alternating bilinear form $(., .)$ satisfying some conditions (Definition 2.1). Given such a system over a field $k$ of characteristic $\neq 2$, a Lie algebra can be defined on

$$g = (\text{sl}_2(k) \oplus \text{inder}(T)) \oplus (k^2 \otimes T),$$

(1.1)

where $\text{inder}(T) = \text{span}\{[xy] : x, y \in T\}$ and $k^2$ is the natural module for the three dimensional simple split Lie algebra $\text{sl}_2(k)$ (Theorem 2.3). The 5-grading is obtained by looking at the eigenspaces of the adjoint action of a Cartan subalgebra in $\text{sl}_2(k)$. Equivalent formulations of this construction have been given by means of Freudenthal triple systems [Fre54, Fre59] or some other ternary algebras [Fau71]. In this way, different models of the exceptional simple Lie algebras have been obtained. (For constructions based on different systems see, for instance, [Kan73] or [All79].)

Here, besides the classical exceptional simple Lie algebras, it will be shown that, over fields of characteristic 3, a one-parameter family of ten dimensional simple Lie algebras discovered by Kostrikin [Kos70], as well as a simple Lie algebra of dimension 29 considered by Brown [Bro82], can be obtained, respectively, from a family of two dimensional simple symplectic triple systems (Proposition 2.12) and from an eight dimensional simple symplectic triple system (Remark 2.33), thus providing very simple descriptions of these Lie algebras.

Date: July 9, 2018.

2000 Mathematics Subject Classification. Primary 17B50; Secondary 17B60, 17B25.

Key words and phrases. Simple, Lie, algebra, superalgebra, symplectic triple system, orthogonal triple system, Freudenthal.

Supported by the Spanish Ministerio de Educación y Ciencia and FEDER (MTM 2004-081150-C04-02 and ) and by the Diputación General de Aragón (Grupo de Investigación de Álgebra).
But the main point of this paper is the description of another feature that occurs in characteristic 3. It will be noticed that over fields of this characteristic, one can forget about the \( \mathfrak{sl}_2(k) \) and the \( k^2 \) in (1.1) and consider instead the direct sum

\[
\tilde{g} = \text{inder}(T) \oplus T
\]

with a natural multiplication. Then \( \tilde{g} \) becomes a Lie superalgebra (instead of a Lie algebra). In other words, any symplectic triple system becomes an anti-Lie triple system in characteristic 3. For specific symplectic triple systems, new simple Lie superalgebras will be obtained over fields of characteristic 3.

Dually, orthogonal triple systems (defined by Okubo \cite{Oku93}) consist of a vector space endowed with a triple product and a bilinear symmetric form. The defining relations of an orthogonal triple system are obtained by changing symmetry and skew-symmetry in the defining relations of a symplectic triple system (Definition 4.1). It turns out that one can consider for orthogonal triple systems the same construction as in (1.1) and this gives now 5-graded Lie superalgebras (instead of algebras). The basic classical exceptional Lie superalgebras in characteristic 0 are obtained in this way \cite{KO03, EKO03}.

Again, in characteristic 3, some new simple Lie superalgebras appear with this construction, which are associated to a family of orthogonal triple systems related to central simple Jordan algebras of degree 3 (Theorem 1.23).

And, as for symplectic triple systems, in characteristic 3 one can forget about \( \mathfrak{sl}_2(k) \) and \( k^2 \) in (1.1) and get \( \tilde{g} \) as in (1.2) which is, now, a Lie algebra (instead of a superalgebra). New constructions of some classical and nonclassical simple Lie algebras are obtained in this way (Examples 5.2).

Throughout the paper, all the vector spaces and algebras will be assumed to be finite dimensional over a ground field \( k \) of characteristic \( \neq 2 \).

In Section 2, the definition of symplectic triple systems will be recalled, as well as the construction of the Lie algebras \( g \) in (1.1). The relationship of symplectic triple systems with Freudenthal triple systems and with a class of ternary algebras defined by Faulkner will be given, and this will lead to the classification of the simple symplectic triple systems over algebraically closed fields. In characteristic 3, there appears a new family of two dimensional symplectic triple systems and a new eight dimensional simple symplectic triple system, which give rise, through the construction in (1.1), to the one-parameter family of Kostrikin simple Lie algebras and to the 29 dimensional simple Lie algebra of Brown.

Section 3 is devoted to the construction (1.2) of Lie superalgebras over fields of characteristic 3. New simple Lie superalgebras of dimension 18, 35, 54, 98 and 189 appear with this construction. Besides, one can relax the definition of a symplectic triple system by allowing the alternating bilinear form to be trivial. It will be shown that no new simple symplectic triple systems appear in characteristic \( \neq 3 \), and it is conjectured that only two dimensional simple systems are possible in characteristic 3.

Section 4 is devoted to orthogonal triple systems and the construction (1.1) of Lie superalgebras from them. The classification of the simple \((−1, −1)\)
balanced Freudenthal Kantor triple systems in \[\text{[EKO03]}\] immediately provides the classification of the simple orthogonal triple systems over fields of characteristic 0. All the simple systems that appear in this classification have their counterparts in prime characteristic, and in characteristic 3 a new family of simple orthogonal triple systems will be defined, in terms of central simple Jordan algebras of degree 3. However, there is no known classification of the simple orthogonal triple systems over fields of prime characteristic. The construction \([1.1]\) provides here, in particular, new simple Lie superalgebras of dimension 24, 37, 50 and 105 over fields of characteristic 3.

Finally, Section 5 will deal with the construction \([1.2]\) of Lie algebras in terms of orthogonal triple systems. Apart from some classical Lie algebras, new models of Kostrikin’s ten dimensional simple Lie algebras and of Brown’s 29 dimensional simple Lie algebra will be given.

In a future work, some of the new simple Lie superalgebras in characteristic 3 that will be constructed here, will be shown to appear in a natural extension of Freudenthal’s Magic Square, obtained by means of composition superalgebras \([\text{EO02]}\).

### 2. Symplectic triple systems

The main objects of study in this section are defined as follows:

**Definition 2.1.** Let \(T\) be a vector space endowed with a nonzero alternating bilinear form \((.,.) : T \times T \to k\), and a triple product \((.,.,.) : T \times T \times T \to T\): \((x, y, z) \mapsto [xyz]\). Then \((T, [.,.,.], (.,.))\) is said to be a symplectic triple system (see \([\text{YA75]}\)) if it satisfies the following identities:

\[
\begin{align*}
[xyz] &= [yxz] \quad (2.2a) \\
[xyz] - [xzy] &= (x|z)y - (y|z)x + 2(y|z)x \quad (2.2b) \\
[xy][uw] &= [[xy][vw] + [u[xy]w] + [uv[xyw]] \quad (2.2c) \\
([xy][v]) + (u|[xyv]) &= 0 \quad (2.2d)
\end{align*}
\]

for any elements \(x, y, z, u, v, w \in T\).

Note that \((2.2d)\) can be written as

\[
[xyz] - [xzy] = \psi_{x,y}(z) - \psi_{x,z}(y)
\]

with \(\psi_{x,y}(z) = (x|z)y + (y|z)x\). (If \((.,.)\) is nondegenerate, the maps \(\psi_{x,y}\) span the symplectic Lie algebra \(\text{sp}(T)\).)

Also, \((2.2c)\) is equivalent to \(d_{x,y} = [xy,]\) being a derivation of the triple system. Let \(\text{ider}(T)\) be the linear span of \(\{d_{x,y} : x, y \in T\}\), which is a Lie subalgebra of \(\text{End}(T)\). Equation \((2.2d)\) is equivalent to the condition \(d_{x,y} \in \text{sp}(T)\) for any \(x, y \in T\).

As remarked in \([\text{El04}, \text{Section 4}]\), equation \((2.2d)\) is a consequence of \((2.2b)\) and \((2.2c)\) unless the characteristic of the ground field \(k\) is 3 and the dimension of \(T\) is 2.

As usual, a homomorphism of symplectic triple systems is a linear map \(\varphi : (T, [.,.,.], (.,.)) \to (T', [.,.,.'], (.,.'))\) satisfying both \(((\varphi(x)|\varphi(y))') = (x|y)\) and \(\varphi([xyz]) = [\varphi(x)\varphi(y)\varphi(z)]'\) for any \(x, y, z \in T\). An ideal of a symplectic triple system is a subspace \(I\) of \(T\) such that \([TTI] + [ITI] \subseteq I\), and the system is said to be simple if \([TTT] \neq 0\) and it contains no proper ideal.
Proposition 2.4. Let \((T, \ldots, (.|.|))\) be a symplectic triple system and assume that the dimension of \(T\) is \(> 2\) if the characteristic of the ground field \(k\) is \(3\). Then \((T, \ldots, (.|.|))\) is simple if and only if the bilinear form \((.|.|)\) is nondegenerate.

Proof. Let \(T^{\perp} = \{ x \in T : (x|T) = 0 \}\) be the kernel of the bilinear map. Then by (2.2a) \([TTT^{\perp}] \subseteq T^{\perp}\) and, by (2.2b), \([TT^{\perp}T] \subseteq [TTT^{\perp}] + T^{\perp} \subseteq T^{\perp}\). Hence \(T^{\perp}\) is an ideal of \(T\). Therefore, if \(T\) is simple, then \(T^{\perp} = 0\) and \((.|.|)\) is nondegenerate.

Conversely, assume that \((.|.|)\) is nondegenerate and let \(I\) be an ideal of \(T\), then by (2.2a)

\[(x|z)y - (x|y)z + 2(y|z)x \in I\]

for any \(x, y, z \in T\) such that one of them is in \(I\). With \(y, z \in T\) and \(x \in I\), it follows that

\[(x|z)y - (x|y)z \in I, \tag{2.5}\]

while for \(y \in I\) and \(x, z \in T\) we get, after interchanging \(x\) and \(y\),

\[(x|y)z + 2(x|z)y \in I \tag{2.6}\]

for any \(y, z \in T\) and \(x \in I\). If the characteristic of \(k\) is \(\neq 3\), it follows from (2.5) and (2.6) that \((I|T)T \subseteq I\), so that either \(I \subseteq T^{\perp} = 0\) or \(I = T\). On the other hand, if char \(k = 3\) and \(I \neq 0\), for any \(0 \neq x \in I\), let \(z \in T\) such that \((x|z) \neq 0\), then (2.5) shows that \((kx)^{\perp} \subseteq I\). Therefore either \(I = T\) or \(I\) has codimension one. But in the latter case, if \(\dim T > 2\), then there are linearly independent elements \(x, y \in I\), and hence \(T = (kx)^{\perp} + (ky)^{\perp} \subseteq I\). Thus, either \(I = T\) or \(\dim T = 2\), as required. \(\square\)

Proposition 2.4 fails for two dimensional symplectic triple systems over fields of characteristic 3. Note that in this case, the expression \((x|z)y - (x|y)z + 2(y|z)x\) in the right hand side of (2.2a) becomes \((x|z)y + (y|x)z + (z|y)z\), which is identically zero (since it is skew-symmetric in three variables on a two dimensional vector space). Therefore, in this case, (2.2a) and (2.2b) simply say that the triple product is symmetric. The two dimensional symplectic triple systems in characteristic 3 are easily classified:

Proposition 2.7. Let \((T, \ldots, (.|.|))\) be a two dimensional symplectic triple system over a field \(k\) of characteristic 3. Then either:

(i) There is a symplectic basis \(\{a, b\}\) of \(T\) (that is, \((a|b) = 1\)) and a scalar \(\alpha \in k\) such that \([aaa] = \alpha b\) and all the other triple products of basis elements is 0; or

(ii) there is a symplectic basis \(\{a, b\}\) of \(T\) and a scalar \(0 \neq \epsilon \in k\) such that \([aaa] = 0 = [bbb]\) and

\[[aab] = [aba] = [baa] = \epsilon a, \quad [abb] = [bab] = [bba] = -\epsilon b. \tag{2.8}\]

Moreover, two symplectic triple systems in case (i) with scalars \(\alpha\) and \(\alpha'\) are isomorphic if and only if there is a scalar \(0 \neq \mu \in k\) with \(\alpha = \mu^3 \alpha'\), while the scalar \(\epsilon\) is an invariant of the isomorphism class of the symplectic triple system in case (ii).
Proof. The case of trivial product \([TTT] = 0\) is covered in item (i). Hence we assume that \([TTT] \neq 0\).

If there is an element \(a \in T\) such that \([aaa] \neq 0\), take \(c = [aaa]\). Then by (2.2)–c, for any \(x, y \in T\), \([xx][yy] = 3[[xx][yy]] = 0\), so that \([xc] = 0\) for any \(x \in T\) and hence \([TT]c = 0\). In particular, this shows that \(c \in T \setminus ka\). Then there is a scalar \(0 \neq \alpha \in k\) such that \((a|c) = \alpha\), and hence we get (i) with \(b = \alpha^{-1}c\).

Therefore we may assume now that \([xxx] = 0\) for any \(x \in T\) but \([TTT] \neq 0\). Then there is an element \(a \in T\) such that \([aaT] \neq 0\). If \(d_{a,a} = [aa]\) is not nilpotent, since \([aaa] = 0\) there is an element \(b \in T \setminus ka\) such that \([aab] = \lambda b\) for some \(0 \neq \lambda \in k\). Thus, using (2.2),

\[
\lambda[aab] = [ab[aab]] = 2([aab][ab] + [aa[abb]]
= 2\lambda[abb] + [aa[abb]],
\]
so that \([aa[abb]] = -\lambda[aab]\) and \([abb] = 0\) as \(0\) and \(\lambda\) the only eigenvalues of \(d_{a,a}\) (dim \(T = 2\)). In this case, \(0 \neq \lambda(a|b) = (a|[aab]) = -(\[aaa]\)|b) = 0\), a contradiction. Hence we conclude that \(d_{a,a}\) is nilpotent, so there is an element \(c \in T \setminus ka\) such that \([aac] = a\). From (2.2) it follows that

\[
[aac] = 2[a[aac]c] = -[aac],
\]
so \([acc] + c \in \ker d_{a,a} = ka\), and there is a scalar \(\mu \in k\) such that \([acc] = \mu a - c\). Now

\[
\mu a - c = [cca] = [cc[aac]] = 2([cca]ac)
= -[(\mu a - c)ac] = -\mu a + (\mu a - c) = -c.
\]

Hence \(\mu = 0\) and \([cca] = -c\). If \((a|c) = \alpha\), with \(b = \alpha^{-1}c\) we obtain case (ii) with \(\epsilon = \alpha^{-1}\).

Finally, unless \([TTT] = 0\), the ideal \(kb = [TTT]\) in case (i) is fixed under any automorphism, so any other symplectic basis with a similar multiplication is of the form \(\{a' = \mu a + \eta b, b' = \mu^{-1}b\}\) for some \(\mu, \eta \in k\) with \(\mu \neq 0\). In this basis, \([a'a'a'] = \mu^4\eta b'\), whence the condition for isomorphism for these algebras. Also, in case (ii), \(d_{x,y} = -\epsilon\psi_{x,y} = -\epsilon((x|.|)y + (y|.|)x)\) for any \(x, y \in T\). Hence \(\epsilon\) is an invariant of the isomorphism class. \(\square\)

Note that the symplectic triple system in case (i) of Proposition 2.7 is not simple, even though \((.|.\cdot\cdot)\) is nondegenerate.

The simple symplectic triple system in case (ii) will be denoted by \(T_{2,\epsilon}\).

Symplectic triple systems are strongly related to \(\Z_2\)-graded Lie algebras (or to Lie triple systems \([YA75]\)). The precise statement that will be most useful here is the following:

**Theorem 2.9.** Let \((T, [...], (.|.|))\) be a symplectic triple system and let \((V, \langle ., . \rangle)\) be a two dimensional vector space endowed with a nonzero alternating bilinear form. Define the \(\Z_2\)-graded algebra \(g = g(T) = g_0 \oplus g_1\) with

\[
\begin{cases}
g_0 = \mathfrak{sp}(V) \oplus \text{inder}(T) & \text{(direct sum of ideals)},
g_1 = V \otimes T,
\end{cases}
\]
and anticommutative multiplication given by:
• $g_0$ is a Lie subalgebra of $g$.
• $g_0$ acts naturally on $g_1$; that is
  \[ [s, v \otimes x] = s(v) \otimes x, \quad [d, v \otimes x] = v \otimes d(x), \]
  for any $s \in \text{sp}(V)$, $d \in \text{inder}(T)$, $v \in V$, and $x \in T$.
• For any $u, v \in V$ and $x, y \in T$:
  \[ [u \otimes x, v \otimes y] = (x|y)\gamma_{u,v} + \langle u|v\rangle d_{x,y} \quad (2.10) \]
  where $\gamma_{u,v} = \langle u|v\rangle + \langle v|u\rangle$ and $d_{x,y} = [xy]$.

Then $g(T)$ is a Lie algebra. Moreover, $g(T)$ is simple if and only if so is $(T,[\cdot],[\cdot])$.

Conversely, given a $\mathbb{Z}_2$-graded Lie algebra $g = g_0 \oplus g_1$ with
\[
\begin{align*}
  g_0 &= \text{sp}(V) \oplus \mathfrak{s} \quad \text{(direct sum of ideals)}, \\
  g_1 &= V \otimes T \quad \text{(as a module for } g_0),
\end{align*}
\]
where $T$ is a module for $\mathfrak{s}$, the $g_0$-invariance of the Lie bracket implies that equation (2.10) is satisfied for an alternating bilinear form $(.,.) : T \times T \to k$ and a symmetric bilinear map $d_\cdot : T \times T \to \mathfrak{s}$. Then, if $(.,.)$ is not 0 and a triple product on $T$ is defined by means of $[xyz] = d_{x,y}(z)$, $(T,[\cdot],[\cdot])$ is a symplectic triple system.

Proof. This is proved in [Eld04], following the ideas in [YA75], with the exception of the assertion about the simplicity of $g$. For this, first note that if $T$ is not simple, then $T^\perp$ is a proper ideal of $T$ (see 2.3 and its proof), and then $d_{T,T^\perp}$ is an ideal of $\text{inder}(T)$ and $a = d_{T,T^\perp} \oplus (V \otimes T^\perp)$ is a proper ideal of $g$. Conversely, assume that $T$ is simple. Since $g_1$ is not the adjoint module for $g_0$, it is enough to prove that $g$ has no proper homogeneous ideal. Let $a = a_0 \oplus a_1$ be an homogeneous ideal of $g$. If $a_1 = 0$, then $a$ is an ideal of $g_0$ and $[a,g_1] = 0$. But $g_0$ acts faithfully on $g_1$, hence $a = 0$. Otherwise, $a_1 \neq 0$, by $\text{sp}(V)$-invariance, $a_1 = V \otimes I$ for some subspace $I$ of $T$, which is closed under the action of $\text{inder}(T)$. Thus, $[III] \subseteq I$. By $\text{sp}(V)$-invariance too, $a_0 = (a_0 \cap \text{sp}(V)) \oplus (a_0 \cap \text{inder}(T))$. Let $\{v, w\}$ be a symplectic basis of $V$, then for any $x \in T$ and $y \in I$, $x \otimes v, y \otimes w = d_{x,y} + (x|y)\gamma v,w \in a_0$. Hence $d_{I,T} \subseteq a$ and, hence, $[d_{I,T},a] = [d_{I,T}, V \otimes I] = V \otimes [III] \subseteq a$. Thus $[III] \subseteq I$ and $I$ is an ideal of $T$. But then $I = T$, $a_1 = g_1$ and $g_0 = [g_1,g_1] \subseteq a$, so that $a = g$. \qed

The argument in the proof above is similar to the one in [EKO03 Theorem 2.2].

As a noteworthy example, consider the simple symplectic triple system $T_{2,ε}$ that appears in Proposition 2.7 where the characteristic of the ground field $k$ is 3. Here, as remarked in the proof of 2.7, $d_{x,y} = -εψ_{x,y}$ for any $x, y \in T_{2,ε}$, so that $\text{inder}(T) = \text{sp}(T)$. Therefore, the simple Lie algebra $g(T_{2,ε})$ is given by:
\[
g(T_{2,ε}) = \left( \text{sp}(V_1) \oplus \text{sp}(V_2) \right) \oplus \left( V_1 \otimes V_2 \right) \quad (2.11)\]
where $V_1$ and $V_2$ are two dimensional vector spaces endowed with nonzero alternating bilinear forms $\langle .,. \rangle_i$ ($i = 1, 2$) and the Lie bracket is determined
by the Lie multiplication in $\mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)$, the natural action of $\mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)$ on $V_1 \otimes V_2$ and by

$$[a \otimes x, b \otimes y] = \langle x|y\rangle 2\gamma_{a,b} - \epsilon\langle a|b\rangle \gamma_{x,y},$$

for any $a,b \in V_1$ and $x,y \in V_2$, where $\gamma_{a,b} = \langle a|b\rangle_1 + \langle b|a\rangle_1 a$ and $\gamma_{x,y} = \langle x|y\rangle_2 + \langle y|x\rangle_2 x$.

There is a great similarity of the above defined Lie algebra and the exceptional simple classical Lie superalgebras $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ in [Sch79 pp. 17-18], where three two dimensional space are used. These Lie superalgebras constitute the family $D(2,1;a)$ in Kac’s classification [Kac77].

Kostrikin defined in [Kos70] a parametric family $L(\epsilon)$ ($\epsilon \neq 0$) of ten dimensional simple Lie algebras over fields of characteristic 3, which appear as subalgebras of the contact algebra $K_3^\epsilon$. Here these simple Lie algebras have a natural interpretation:

**Proposition 2.12.** Let $k$ be a field of characteristic 3. Then, for any $0 \neq \epsilon \in k$, the Lie algebra $\mathfrak{g}(T_{2,\epsilon})$ and $L(\epsilon)$ are isomorphic.

**Proof.** This has been checked in [Bro90], although in terms of Freudenthal triple systems instead of symplectic triple systems (see Theorem 2.16 below). Actually, the Freudenthal triple system that appears in the second paragraph of [Bro90] p. 29 with parameter $c$ corresponds through 2.16 to the symplectic triple system $T_{2,\epsilon}$ with $\epsilon = -(c + 1)$, Brown shows that this system corresponds to the Kostrikin algebra $L\left(-\frac{1}{c+1}\right)$, which in turn is isomorphic to $L(-(c + 1)) = L(\epsilon)$ ([Kos70]).

**Remark 2.13.** Therefore, the definition of $\mathfrak{g}(T_{2,\epsilon})$ in (2.11) gives a nice description of the simple Lie algebra $L(\epsilon)$. Also, take symplectic bases $\{v_i, w_i\}$ of $V_i$ ($i = 1, 2$) and the elements

$$e_1 = \frac{1}{2} \gamma_{v_2,v_2}, \quad f_1 = -\gamma_{w_2,w_2}, \quad h_1 = [e_1, f_1] = -\gamma_{v_2,w_2},$$

$$e_2 = v_1 \otimes w_2, \quad f_2 = -w_1 \otimes v_2, \quad h_2 = [e_2, f_2] = \gamma_{v_1,w_1} + \epsilon \gamma_{v_2,w_2},$$

which satisfy that

$$[h_1, e_1] = 2e_1, \quad [h_1, e_2] = -e_2, \quad [h_2, e_1] = -2\epsilon e_1, \quad [h_2, e_2] = (\epsilon - 1)e_2,$$

and this shows that $\mathfrak{g}(T_{2,\epsilon})$ is the contragredient Lie algebra (see [VK71 §3]) associated to the Cartan matrix $A = \begin{pmatrix} -2 & -1 \\ \epsilon & \epsilon - 1 \end{pmatrix}$.

For $\epsilon = 1$ ($L(1)$ is shown in [Kos70] to present some specific features), after scaling the last row, this is the matrix $C_{2,\infty} = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$, while for $\epsilon \neq 1$, after scaling the last row, we get the Cartan matrix $C_{2,\frac{\epsilon}{\epsilon - 1}} = \begin{pmatrix} 2 & -1 \\ \frac{2}{\epsilon} & -2 \end{pmatrix}$. In particular, for $\epsilon = -1$, we get the Cartan matrix associated to the simple classical Lie algebra of type $C_2$ and, indeed, $\mathfrak{g}(T_{2,-1})$ is naturally isomorphic to $\mathfrak{sp}(V_1 \perp V_2)$. \hfill \Box

For later use, let us define an eight dimensional module for the simple Lie algebra $L(1) = \mathfrak{g}(T_{2,1})$ (char $k = 3$), which will provide an important
example of a simple symplectic triple system. With the previous notation, consider the \( \mathbb{Z}_2 \)-graded vector space \( W = W_0 \oplus W_1 \), with

\[
\begin{cases}
W_0 = V_1 \otimes \mathfrak{sp}(V_2), \\
W_1 = V_2,
\end{cases}
\]

and give \( W \) the structure of a graded module for the \( \mathbb{Z}_2 \)-graded Lie algebra \( \mathfrak{gl}(T_{2,1}) \) with the natural action of \( \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2) \) (that is, \( \rho_a(\alpha_1 \otimes \alpha_2) = s_1(\alpha_1) \otimes s_2, \rho_{s_2}(\alpha_1 \otimes t_2) = a_1 \otimes [s_2, t_2], \rho_{s_1}(a_2) = 0 \) and \( \rho_{s_1}(a_2) = s_2(a_2) \), for any \( a_1 \in V_1, a_2 \in V_2, s_1 \in \mathfrak{sp}(V_1) \) and \( s_2, t_2 \in \mathfrak{sp}(V_2) \)). Finally, for any \( a_1, b_1 \in V_1, a_2, b_2 \in V_2 \) and \( s_2 \in \mathfrak{sp}(V_2) \), define the action of \( \alpha_1 \otimes \alpha_2 \in \mathfrak{gl}(T_{2,1}) \) on \( W \) by means of:

\[
\begin{align*}
\rho_{a_1 \otimes a_2}(b_1 \otimes s_2) &= -\langle a_1 | b_1 \rangle_1 s_2(a_2) \in W_1, \\
\rho_{a_1 \otimes a_2}(b_2) &= -a_1 \otimes \gamma_{a_2, b_2} \in W_0.
\end{align*}
\]

Using that \( [s_2, \gamma_{a_2, b_2}] = \gamma_{s_2(a_2), b_2} + \gamma_{a_2, s_2(b_2)} \) and that, since the characteristic is 3, \( (a_2 | b_2)_2 = \gamma_{s_2(a_2), b_2} - \gamma_{a_2, s_2(b_2)} \) for any \( a, b \in V_2 \) and \( s_2 \in \mathfrak{sp}(V_2) \), it follows easily that \( \rho : \mathfrak{gl}(T_{2,1}) \to \mathfrak{gl}(W) \) is indeed a representation.

**Proposition 2.14.** Let \( k \) be a field of characteristic 3 and let \( \rho : \mathfrak{gl}(T_{2,1}) \to \mathfrak{gl}(W) \) be the representation defined above. Then \( T = W \), endowed with the alternating bilinear form given by

\[
\begin{cases}
(W_0 | W_1) = 0, \\
(a_1 \otimes s_2 | b_1 \otimes t_2) = \langle a | b \rangle_1 \text{trace}(s_2 t_2), \\
(a_2 | b_2) = \langle a_2 | b_2 \rangle_2,
\end{cases}
\]

and with the triple product \( [ABC] = d_{A,B}(C) \), where \( d : W \times W \to \mathfrak{gl}(W) \), \( (A, B) \mapsto d_{A,B} \) is the symmetric bilinear map determined by

\[
\begin{align*}
d_{a_1 \otimes s_2, b_1 \otimes t_2} &= \rho_{\text{trace}(s_2 t_2)} \gamma_{a_1, b_1} - \langle a_1 | b_1 \rangle_1 [s_2, t_2], \\
d_{a_1 \otimes s_2, a_2} &= \rho_{a_1 \otimes s_2(a_2)}, \\
d_{a_2, b_2} &= -\rho_{\gamma_{a_2, b_2}},
\end{align*}
\]

for any \( a_1, b_1 \in V_1, a_2, b_2 \in V_2 \) and \( s_2, t_2 \in \mathfrak{sp}(V_2) \), is a symplectic triple system.

**Proof.** This is a straightforward computation, using that for any \( s_2, t_2, r \in \mathfrak{sp}(V_2) \),

\[
s_2 t_2 = \frac{1}{2} \text{trace}(s_2 t_2) + \frac{1}{2} [s_2, t_2] = -\text{trace}(s_2 t_2) - [s_2, t_2] \in \mathfrak{gl}(V_2),
\]

\[
[[s_2, t_2], r_2] = s_2(t_2 r_2 + r_2 t_2) + (t_2 r_2 + r_2 t_2)s_2
\]

\[
- (s_2 r_2 + r_2 s_2)t_2 - t_2(r_2 s_2 + s_2 r_2)
\]

\[
= 2 \text{trace}(t_2 r_2)s_2 - 2 \text{trace}(s_2 r_2)t_2 = \text{trace}(s_2 r_2)t_2 - \text{trace}(t_2 r_2)s_2.
\]

The symplectic triple systems are strongly related to other triple systems too, in particular to Freudenthal triple systems and to Faulkner ternary algebras \([\text{Fau71]}\) (or balanced symplectic algebras \([\text{FF72}]\)).

Let \( T \) be a vector space endowed with a nondegenerate alternating bilinear form \( (\cdot, \cdot) : T \times T \to k \), and a triple product \( T \times T \times T \to T, (x, y, z) \mapsto xyz \).
Then \((T, xyz, (\cdot, \cdot))\) is said to be a Freudenthal triple system (see \cite{Mey68, Fer72, Bro84}) if it satisfies:

\begin{align}
xyz & \text{ is symmetric in its arguments,} \\
(x|yst) & \text{ is symmetric in its arguments,} \\
(xyy)z + (yxx)yz + (xyy|z)x + (yxx|z)y & + (x|z)xyy + (y|z)yxx = 0,
\end{align}

for any \(x, y, z, t \in T\).

The next result relates the symplectic triple systems and the Freudenthal triple systems, its proof may be found in \cite[Theorem 4.7]{Eld04}.

**Theorem 2.16.** Let \((\cdot, \cdot)\) be a nondegenerate alternating bilinear form on the vector space \(T\) and let \(xyz\) and \([xyz]\) be two triple products on \(T\) related by \(xyz = [xyz] - \psi_{xy}(z)\) for any \(x, y, z \in T\). Then \((T, [...], (\cdot, \cdot))\) is a symplectic triple system if and only if either \(xyz = 0\) for any \(x, y, z \in T\), or \((T, xyz, (\cdot, \cdot))\) is a Freudenthal triple system.

Also, let \(T\) be a vector space endowed with an alternating bilinear form \((\cdot, \cdot)\) and a triple product \((xyz)\) satisfying (\cite{Fau71} or \cite{FF72}, Section 3)):

\begin{align}
\langle xyz \rangle &= \langle yxz \rangle + (x|y)z \\
\langle xyz \rangle &= \langle xzy \rangle + (y|z)x \\
\langle\langle xyz\rangle vw \rangle &= \langle\langle xvw \rangle yz \rangle + \langle\langle yvw \rangle z \rangle + \langle\langle zwv \rangle x \rangle
\end{align}

for any \(x, y, z, v, w \in T\). Then \((T, (\cdot, \cdot), (\cdot, \cdot))\) is called a Faulkner ternary algebra. These triple systems have been called balanced symplectic algebras in \cite{FF72}.

**Theorem 2.18.** Let \((\cdot, \cdot)\) be an alternating bilinear form on the vector space \(T\) and let \([xyz]\) and \((xyz)\) be two triple products related by \([xyz] = -2(xyz) + |x|y)z\) for any \(x, y, z \in T\). Then \((T, [...], (\cdot, \cdot))\) satisfies (2.2a-d) if and only if \((T, xyz, (\cdot, \cdot))\) is a Faulkner ternary algebra. In particular, the symplectic triple systems are in bijection with the Faulkner ternary algebras with nonzero alternating bilinear form.

**Proof.** For any \(x, y, z \in T\),

\begin{align}
\langle xyz \rangle - \langle yxz \rangle - (x|y)z &= -\frac{1}{2}(\langle yzx \rangle - \langle xzy \rangle - (z|x)y - (z|y)x - (x|y)z) \\
&= \frac{1}{2}\left(\langle yzx \rangle - \langle xzy \rangle - \psi_{xy}(z) + \psi_{xz}(y)\right), \\
\langle xyz \rangle - \langle xzy \rangle - (y|z)x &= -\frac{1}{2}(\langle yzx \rangle - \langle zyx \rangle - (z|x)y + (z|y)x - (y|z)x) \\
&= -\frac{1}{2}\left(\langle yzx \rangle - \langle zyx \rangle\right),
\end{align}

so that \((2.2a)\) and \((2.2b)\) are equivalent to \((2.17a)\) and \((2.17b)\).

On the other hand, if \((2.17b)\) and \((2.17c)\) are satisfied, \([FF72] (3.1)\) shows that

\begin{align}
\langle xyzw | y \rangle + (x|ywz) &= 0
\end{align}

(2.19)
for any \( x, y, z, t \in T \), but
\[
-2\left( (xyz) + (x(ywz)) \right) = (zw) + (zwx) - (y[wz]) + (x[ywz]) - (wx(z)) + (w[xz]) - (v[wxy]) + (v[wz]) - (y([vwx]) - (y[vwz])) + (y[vwz]) x.
\]

Hence (2.2a) and (2.2d) are equivalent to (2.17a) and (2.19). Finally, assuming (2.2a) and (2.2d) are equivalent to (2.17a) and (2.19).

\[
4\left( (xyz)vw - (xvw)yz - (x(yvw)z) - (xyzw) \right) = -2\left( [vw]xyz - (v[w]xyz) - (y[ wz])z - (y[z]xv) + (y[ zw])x \right)
- \left( (yvw)zx + [(yvw)z]x - (y[zvw])x \right)
= \left( (vw)xyz - (y[vwz])x - (y[vwx])x \right) + \left( (y[vwz])x \right) x
\]

which shows that, if (2.2b) is satisfied, then (2.2c) and (2.19) are equivalent. \( \square \)

**Remark 2.20.** See [Bro84] for a proof, in characteristic 3, of the relationship between Freudenthal triple systems and Faulkner ternary algebras. Also, [FF72, Lemma 3.2 and Corollary 1] deals with this relationship in characteristic \( \neq 2, 3 \).

Now, Theorem 2.18, together with the classification of the simple Faulkner ternary algebras with nonzero alternating bilinear form in [FF72, Theorem 4.1] over fields of characteristic \( \neq 2, 3 \) (which is based on the classification in [Mey68] of the Freudenthal triple systems) immediately yields the classification of the simple symplectic triple systems over algebraically closed fields. Here we follow the notations in [McC04] concerning Jordan algebras.

**Theorem 2.21.** Let \( k \) be an algebraically closed field of characteristic \( \neq 2, 3 \) and let \((T, [\cdot, \cdot, \cdot]), (\cdot, \cdot)\) be a simple symplectic triple system. Then either:

(i) \([xyz] = \psi_{x,y}(z)\) for any \( x, y, z \in T \); or

(ii) there exists a Jordan algebra \( J \) such that either \( J = 0 \), or \( J = J^{ord}(q,c) \) is the Jordan algebra of a nondegenerate quadratic form \( q \) with basepoint \( e \) (McC04 II.3.3), with trace form \( t(a,b) \), where we define \( a \times b = 0 \) for any \( a, b \in J \), or a Jordan algebra \( J = J^{ord}(n,c) \) of a nondegenerate cubic form \( n \) with basepoint \( c \) (McC04 II.4.3), trace form \( t(a,b) \) and cross product \( a \times b \), such that, up to isomorphism,

\[
T = \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} : \alpha, \beta \in k, \ a, b \in J \right\}, \tag{2.22}
\]
and for \( x_i = \left( \begin{smallmatrix} a_i \cr b_i \cr \beta_i \end{smallmatrix} \right) \), \( i = 1, 2, 3 \):

\[
\begin{align*}
(x_1|x_2) &= \alpha_1\beta_2 - \alpha_2\beta_1 - t(a_1, b_2) + t(b_1, a_2), \\
[x_1|x_2|x_3] &= \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} \quad \text{with} \\
\gamma &= -3(\alpha_1\beta_2 + \alpha_2\beta_1) + t(a_1, b_2) + t(a_2, b_1)\alpha_3 \\
&\quad + 2\left(\alpha_1 t(b_2, a_3) + \alpha_2 t(b_1, a_3) - t(a_1 \times a_2, a_3)\right) \\
c &= -3(\alpha_1\beta_2 + \alpha_2\beta_1) + t(a_1, b_2) + t(a_2, b_1)\alpha_3 \\
&\quad + 2\left(\alpha_1 t(b_2, a_3) - \beta_2\alpha_3 a_1 + (t(b_1, a_3) - \beta_1\alpha_3) a_2\right) \\
&\quad + 2\left(\alpha_1 (b_2 \times b_3) + \alpha_2 (b_1 \times b_3) + \alpha_3 (b_1 \times b_2)\right) \\
&\quad - 2\left((a_1 \times a_2) \times b_3 + (a_1 \times a_3) \times b_2 + (a_2 \times a_3) \times b_1\right) \\
\delta &= -\gamma^\sigma, \quad d = -c^\sigma, \quad \text{where} \quad \sigma = (\alpha\beta)(ab) \quad \text{(that is, \( \gamma^\sigma \) and \( c^\sigma \) are obtained from \( \gamma \) and \( c \) by interchanging \( \alpha \) and \( \beta \) and also \( a \) and \( b \) throughout).}
\end{align*}
\tag{2.23}
\]

Remark 2.24. The case \( J = 0 \) above is missing in [FF72, Theorem 4.1], it corresponds to the case in which \( M^+ = 0 = M^- \) in the arguments of [FF72, Section 4].

Remark 2.25. Over an algebraically closed ground field \( k \) of characteristic \( \neq 2, 3 \), the Jordan algebras \( \text{Jord}(n, c) \) of a nondegenerate cubic form with basepoint are (see [Jac66, McC04]), up to isomorphism, either the ground field \( k \) with \( n(\alpha) = \alpha^3 \), so that \( t(\alpha, \beta) = 3\alpha\beta \) for any \( \alpha, \beta \in k \), or the cartesian product \( k \times \text{Jord}(q, e) \), for a nondegenerate quadratic form \( q \), or the Jordan algebra \( J = H_3(C) \) of \( 3 \times 3 \)-hermitian matrices over \( C \), where \( C \) is either \( k, k \times k, \text{Mat}_2(k) \) or the algebra of split octonions. In all cases, \( c \) is the identity element 1.

A natural question now is, given the symplectic triple systems in Theorem 2.21 what do the simple Lie algebras \( g(T) \) look like? In order to answer this question, and for future use, we will first consider different constructions of symplectic triple systems.

Examples 2.26. Let \( (V, \langle .|\cdot \rangle) \) be a two dimensional vector space endowed with a nonzero alternating bilinear form.

**Symplectic case:** Let \( (T, [\cdot|\cdot], \langle .|\cdot \rangle) \) be a symplectic triple system in item (i) of Theorem 2.21 then the Lie algebra \( g(T) = g_0 \oplus g_1 \) defined in 2.9 satisfies that \( g_0 = \text{sp}(V) \oplus \text{sp}(T) \) and \( g_1 = V \otimes T \). Moreover, \( g_0 \) embeds naturally in the symplectic Lie algebra \( \text{sp}(V \perp T) \) of the orthogonal direct sum \( V \perp T \), and so does \( g_1 = V \otimes T \) by means of \( u \otimes x \rightarrow \Psi_{u,x} : v \mapsto \langle u|v\rangle x, \ y \mapsto (x|y) u, \) for any \( u, v \in V \) and \( x, y \in T \). This gives an isomorphism \( g(T) \simeq \text{sp}(V \perp T) \). \( \Box \)
Special case: Let $W$ be a nonzero vector space over a ground field $k$ and let $W^*$ be its dual vector space. Consider the direct sum $T = W \oplus W^*$, endowed with the triple product determined by $[WWT] = 0 = [W^*W^*T]$ and

$$[xfy] = f(x)y + 2f(y)x, \quad (2.27a)$$
$$[xfg] = -f(x)g - 2g(x)f, \quad (2.27b)$$

for any $x, y \in W$ and $f, g \in W^*$, and of the alternating bilinear form such that $(W|W) = 0 = (W^*|W^*)$ and $(f|x) = -(x|f) = f(x)$ for any $x \in W$ and $f \in W^*$. Then $(T, [\cdot, \cdot, \cdot])$ is easily checked to be a symplectic triple system, where $W$ and $W^*$ are invariant under $\text{inder}(T)$. Also note that for any $x \in W$ and $f \in W^*$, the element $d_{x,f}W \in \mathfrak{gl}(W)$ (determined by (2.27a)) has trace equal to $(2 + \dim W)f(x)$, which is 0 if the characteristic of $k$ divides $2 + \dim W$.

Moreover, let $g(T) = (\mathfrak{sp}(V) \oplus \text{inder}(T)) \oplus (V \otimes T)$ be the associated $\mathbb{Z}_2$-graded Lie algebra (Theorem 2.4), and identify the Lie algebra $\mathfrak{gl}(V \oplus W)$ of endomorphisms of $V \oplus W$ with the set of $2$ by $2$ matrices $(\begin{array}{cc} A & B \\ C & D \end{array})$, where $A \in \mathfrak{gl}(V)$, $D \in \mathfrak{gl}(W)$, $B \in \text{Hom}_k(W, V)$ and $C \in \text{Hom}_k(V, W)$. Then if $\mathfrak{z}$ denotes the center of the special linear Lie algebra $\mathfrak{sl}(V \oplus W)$ ($\mathfrak{z} = 0$ if char $k$ does not divide $\dim(V \oplus W) = 2 + \dim W$, and $\mathfrak{z} = k \text{id}$ otherwise) and $\mathfrak{psl}(V \oplus W) = \mathfrak{sl}(V \oplus W)/\mathfrak{z}$ denotes the corresponding projective special linear Lie algebra, a straightforward computation shows that the linear map $\Phi: \mathfrak{g}(T) \rightarrow \mathfrak{psl}(V \oplus W)$ such that

$$\Phi(s) = \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} + \mathfrak{z},$$
$$\Phi(d) = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} + \mathfrak{z} & \text{if char } k \text{ divides } 2 + \dim W, \\ \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} - \frac{\text{trace}(d)}{2+\dim W} \begin{pmatrix} \text{id} & 0 \\ 0 & \text{id} \end{pmatrix} & \text{otherwise}, \end{cases}$$
$$\Phi(a \otimes x) = \begin{pmatrix} 0 & (a,x) \\ (a,x) & 0 \end{pmatrix} + \mathfrak{z},$$
$$\Phi(a \otimes f) = \begin{pmatrix} 0 & 2f(x) \\ 0 & 0 \end{pmatrix} + \mathfrak{z},$$

for any $s \in \mathfrak{sp}(V)$, $d \in \text{inder}(T)$, $a \in V$, $x \in W$ and $f \in W^*$, is a Lie algebra isomorphism. $\square$

Orthogonal case: Consider now a vector space $W$ of dimension $\geq 3$ over our ground field $k$, endowed with a nondegenerate symmetric bilinear form $q_W(\cdot, \cdot)$. On the tensor product $V \otimes V$ there is another nondegenerate symmetric bilinear form determined by $q_{V \otimes V}(a \otimes u, b \otimes v) = \langle a|b \rangle \langle u|v \rangle$.

Recall that if $(U, q)$ is a vector space endowed with a nondegenerate bilinear form, then the orthogonal Lie algebra

$$\mathfrak{so}(U, q) = \{ f \in \mathfrak{gl}(U) : q(f(x), y) + q(x, f(y)) = 0, \forall x, y \in U \}$$

is spanned by the linear operators $\sigma_{x,y} : z \mapsto q(x, z)y - q(y, z)x$, for $x, y \in U$, which satisfy that $[\sigma_{x,y}, \sigma_{z,t}] = \sigma_{\sigma_{x,y}(z),t} + \sigma_{z,\sigma_{x,y}(t)}$ for any $x, y, z, t \in U$. 

\[\text{End of page} 12\]
$U$. Moreover, if $U$ is the orthogonal direct sum $U = U_1 \perp U_2$ of two subspaces, then $\mathfrak{so}(U_i, q_i)$ ($q_i = q|_{U_i}$) is naturally embedded in $\mathfrak{so}(U, q)$, $i = 1, 2$, and $\mathfrak{so}(U, q) = \mathfrak{so}(U_1, q_1) \oplus \mathfrak{so}(U_2, q_2) \oplus \sigma_{U_1, U_2}$. Besides, $\sigma_{U_1, U_2}$ is linearly isomorphic to $U_1 \otimes U_2$ and

$$[\sigma_{x_1, x_2}, \sigma_{y_1, y_2}] = \sigma_{x_1, x_2}(y_1, y_2) + \sigma_{y_1, x_1}(x_2, y_2)$$

for any $x_1, y_1 \in U_1$ and $x_2, y_2 \in U_2$.

The vector space $(V \otimes V) \oplus W$ is endowed with the nondegenerate symmetric bilinear form which is the orthogonal sum of $q_{V \otimes V}$ and $q_{V}$, and hence its orthogonal Lie algebra is

$$\mathfrak{so}(V \otimes V \oplus W) = \mathfrak{so}(V \otimes V, q_{V \otimes V}) \oplus \mathfrak{so}(W, q_{V}) \oplus \sigma_{V \otimes V, W}.$$ But $\mathfrak{so}(V \otimes V, q_{V \otimes V}) = \mathfrak{sp}(V) \oplus \mathfrak{sp}(V)$, where the first (respectively, second) copy of $\mathfrak{sp}(V)$ acts on the first (resp., second) slot of $V \otimes V$. Therefore, with the natural identifications, one has

$$\mathfrak{so}(V \otimes V \oplus W) = \left( \mathfrak{sp}(V) \oplus \mathfrak{sp}(V) \oplus \mathfrak{so}(W, q_{V}) \right) \oplus (V \otimes V \otimes W),$$

and

$$[a \otimes u \otimes x, b \otimes v \otimes y] = q_{V}(x, y)\sigma_{a \otimes u, b \otimes v} + q_{V \otimes V}(a \otimes u, b \otimes v)\sigma_{x, y}, \quad (2.28)$$

for any $a, b, u, v \in V$ and $x, y \in W$. But $q_{V \otimes V}(a \otimes u, b \otimes v) = \langle a \mid b \rangle \langle u \mid v \rangle$ and

$$\sigma_{a \otimes u, b \otimes v}(c \otimes w) = \langle a \mid c \rangle \langle u \mid w \rangle b \otimes v - \langle b \mid c \rangle \langle v \mid w \rangle a \otimes u$$

$$= \frac{1}{2}(\langle u \mid v \rangle \gamma_{a, b}(c) \otimes w + \frac{1}{2}(a \mid b)c \otimes \gamma_{u, v}(w),$$

for any $a, b, c, u, v, w \in V$. (To check the last equality, it is enough by Zariski density to assume that $\{a, b\}$ and $\{u, v\}$ are bases of $V$, and then even that $\langle a \mid b \rangle = 1 = \langle u \mid v \rangle$. Now one can just express $c$ (respectively $w$) as a linear combination of $a$ and $b$ (resp., of $u$ and $v$) and expand.)

Hence equation $[2.28]$ becomes

$$[a \otimes u \otimes x, b \otimes v \otimes y] = \frac{1}{2}(\langle u \mid v \rangle q_{W}(x, y)\gamma_{a, b} + \frac{1}{2}(a \mid b)q_{W}(x, y)\gamma_{u, v} + \langle a \mid b \rangle \langle u \mid v \rangle\sigma_{x, y}),$$

for any $a, b, u, v \in V$ and $x, y \in W$. According to Theorem $2.9$, $T = V \otimes W$ is a symplectic triple system, endowed with the nondegenerate alternating form given by $(u \otimes x \mid v \otimes y) = \frac{1}{2}(\langle u \mid v \rangle q_{W}(x, y)$, and with the triple product given by

$$[(u \otimes x) \otimes (v \otimes y) \otimes (w \otimes z)] = \frac{1}{2}q_{W}(x, y)\gamma_{u, v}(w) \otimes z + \langle u \mid v \rangle \sigma_{x, y}(z), \quad (2.29)$$

for any $u, v, w \in V$ and $x, y, z \in T$. Besides, by Proposition $2.4$ this symplectic triple system is simple. Moreover, the Lie algebra $\mathfrak{g}(T)$ is isomorphic to the orthogonal Lie algebra $\mathfrak{so}(V \otimes V \oplus W, q).$ \hfill $\square$

**G$_2$-case**: Assume here that the characteristic of the ground field $k$ is $\neq 3$. Let $k[x, y]$ be the polynomial algebra in two variables $x$ and $y$, and identify $\mathfrak{sl}_2(k)$ ($\simeq \mathfrak{sp}(V)$) with the subalgebra of the Lie algebra of derivations of $k[x, y]$ spanned by $\{x_0 \frac{\partial}{\partial x}, y_0 \frac{\partial}{\partial x}, x_0 \frac{\partial}{\partial y}, y_0 \frac{\partial}{\partial y}\}$. Let $V_n = k_n[x, y]$ be the linear space of the degree $n$ homogeneous polynomials. Then $V_n$ is invariant under the action of $\mathfrak{sl}_2(k)$ and if the characteristic of $k$ is either $0$ or $> n$, then it
is irreducible. In particular, since $\text{char } k \neq 3$, $V_3$ is an irreducible module of dimension 4 for $\mathfrak{sl}_2(k)$.

For any $f \in V_n$ and $g \in V_m$, the transvection $(f, g)_q$ is defined, assuming no conflict with the characteristic of $k$, by [Dix84]:

$$(f, g)_q = \begin{cases} 0 & \text{if } q > \min(n, m), \\
\frac{(n-q)!}{n!} \frac{(m-q)!}{m!} \sum_{i=0}^{q} \left( (-1)^i \frac{\partial^q f}{\partial x^i \partial y^{q-i}} \frac{\partial^q g}{\partial x^i \partial y^{q-i}} \right) & \text{otherwise,}
\end{cases}$$

so that $(f, g)_q \in V_{n+m-2q}$. We may identify $(V, \langle \cdot, \cdot \rangle)$ with $(V_1, \langle \cdot, \cdot \rangle)$ (BDE03 Lemma 2.2)). Then (BDE03 Theorem 3.2), the simple Lie algebra of type $G$ dimension 4 for $\mathfrak{sl}_2$,.

Theorem 2.30. Let $k$ be an algebraically closed field of characteristic $\neq 2, 3$ and let $(T, \ldots, \langle \cdot, \cdot \rangle)$ be a simple symplectic triple system. Then:

(a) If $[xy]_3 = q_{x,y}(z)$ for any $x, y, z \in T$ (item (i) in 2.21), then $\mathfrak{g}(T)$ is isomorphic to the symplectic Lie algebra $\mathfrak{sp}(V \perp T)$ as in Examples 2.26 (symplectic case).

(b) If $(T, \ldots, \langle \cdot, \cdot \rangle)$ is the symplectic triple system associated to a Jordan algebra $J$ which is either 0 or the Jordan algebra of a nondegenerate quadratic form with basepoint $J = \mathfrak{Jord}(q,e)$, then $T$ is isomorphic to the symplectic triple system in Examples 2.26 (special case), for a suitable vector space $W$ with $\dim W = 1 + \dim J$. In particular, $\mathfrak{g}(T)$ is isomorphic to $\mathfrak{psl}(V \oplus W) \simeq \mathfrak{psl}_{3+\dim J}(k)$.

(c) If $(T, \ldots, \langle \cdot, \cdot \rangle)$ is the symplectic triple system associated to the Jordan algebra of a nondegenerate cubic form of type $J = k \times \mathfrak{Jord}(q,e)$, then $(T, \ldots, \langle \cdot, \cdot \rangle)$ is isomorphic to the symplectic triple system in Examples 2.26 (orthogonal case), for a vector space $W$ endowed with a nondegenerate symmetric bilinear form, with $\dim W = 1 + \dim J$. In particular, $\mathfrak{g}(T)$ is isomorphic to the orthogonal Lie algebra $\mathfrak{so}_5((V \otimes V) \oplus W) \simeq \mathfrak{so}_{5+\dim J}(k)$.
(d) If \( (T, [...], (., .)) \) is the symplectic triple system associated to the Jordan algebra \( J = k \) with cubic form \( n(\alpha) = \alpha^3 \), then \( T \) is isomorphic to the symplectic triple system defined on \( V_3 \) in Examples 2.26 (orthogonal case). In particular, \( \mathfrak{g}(T) \) is the simple Lie algebra of type \( G_2 \).

(e) If \( (T, [...], (., .)) \) is the symplectic triple system associated to the Jordan algebra \( H_3(C) \), where \( C \) is either \( k, k \times k, \text{Mat}_2(k) \) or the algebra of split octonions, then \( \mathfrak{g}(T) \) is isomorphic, respectively, to the simple Lie algebras of type \( F_4, E_6, E_7 \) and \( E_8 \).

**Proof.** Part (a) is clear from the arguments in Example 2.26 (symplectic case).

Also, if either \( J = 0 \) or \( J = J_{\text{ord}}(q, e) \) for a nondegenerate quadratic form \( q \) and basepoint \( e \), then one has \( T = W \oplus \tilde{W} \) with \( W = \{ \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} : \alpha \in k, \ a \in J \} \) and \( \tilde{W} = \{ \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} : \beta \in k, \ b \in J \} \). The subspaces \( W \) and \( \tilde{W} \) are maximal isotropic subspaces of \( T \) and we will identify \( \tilde{W} \) with the dual \( W^* \) by means of \( x_2 \in \tilde{W} \mapsto (x_2,) : W \rightarrow k \). Moreover, the expression of the triple product in \( \mathfrak{g}(T) \) gives \( [WWT] = 0 = [W^*W^*T] \). Besides, for any \( x = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} \in W, \ y = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} \in \tilde{W} \), and \( f = \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix} \in W^* \) we have

\[
[xf y] = \left( \begin{array}{cc}
-3\alpha\beta + t(a, b) & 2t(b, c) - \beta(\alpha + t(a, b)) + 2t(b, c) - \beta\gamma
\\
0 & 0
\end{array} \right)
\]

that is, equation \( \mathfrak{g}(T) \) is satisfied for any \( x, y, f \in W \) and \( f \in W^* \). Also, since \( ([xf y] | g) + (y|[xf y]) = 0 \) for any \( x, y \in W \) and \( f, g \in W^* \), or by direct computation as above, it follows that equation \( \mathfrak{g}(T) \) is satisfied too. Hence part (b) follows.

Now, in item (ii) of Theorem 2.21, the dimension of \( T \) is always even. If \( \dim T = 4 \), then either \( T \) is associated to the unique Jordan algebra \( J = J_{\text{ord}}(q, e) \) of a nondegenerate quadratic form with basepoint of dimension 1, which is already known to be isomorphic to the symplectic triple system in Examples 2.26 (orthogonal case) with \( \dim W = 2 \); or \( J = k \) with cubic form \( n(\alpha) = \alpha^3 \) and basepoint 1. Then, necessarily, this case corresponds to the \( G_2 \)-case in Examples 2.26.

On the other hand, if \( \dim T > 4 \), then either \( T \) is associated to the Jordan algebra of hermitian matrices \( H_3(C) \), and hence the associated Lie algebra \( \mathfrak{g}(T) \) is \( F_4, E_6, E_7 \) or \( E_8 \) (this goes back to Freudenthal, see also [YA75]); or to the unique Jordan algebra \( J = k \oplus J_{\text{ord}}(q, e) \) with \( \dim J = \frac{1}{2}(\dim T - 1) \). The only possibility left here is case (c). \( \square \)

**Remark 2.31.** For a different construction of the exceptional symplectic triple systems in item (e) of Theorem 2.30, see [Eld04].

Note that the symplectic triple systems that appear in Theorem 2.31 make sense too over fields of characteristic 3, with the exceptions in case (ii) of \( J = 0 \), because then \( [., .] \) is the trivial product by equations \( 2.27a \) and \( 2.27b \), or \( J = k \) with \( n(\alpha) = \alpha^3 \) for any \( \alpha \in k \), because then \( t(\alpha) = 3\alpha = 0 \) for any \( \alpha \in k \), so that the trace form is trivial (this has to do with the fact that the exceptional Lie algebra of type \( G_2 \) is no longer simple in characteristic 3 –see, for instance, [BEMN02]). Theorem 2.16 and Brown’s
classification of the Freudenthal triple systems in characteristic 3 \[\text{Bro84}\] yield:

**Theorem 2.32.** Let \( k \) be an algebraically closed field of characteristic 3 and let \( (T, \{\ldots, \cdot, \ldots\}) \) be a simple symplectic triple system, then either:

(i) \([xyz] = \psi_{x,y}(z)\) for any \( x, y, z \in T \); or

(ii) there exists a Jordan algebra \( J \) of either a nondegenerate quadratic form with basepoint, or of a nondegenerate cubic form with basepoint, such that \( T \) is given by the formulas in \((2.22)\) and \((2.23)\); or

(iii) \( \dim T = 2 \), and hence \( T \) is described in item (ii) of Proposition 2.7; or

(iv) up to isomorphism, \( T \) is the symplectic triple system described in Proposition 2.14.

**Proof.** According to \[\text{Bro84}\], the only differences in characteristic 3 are given by the two dimensional symplectic triple systems and a unique exceptional symplectic triple system of dimension 8 which, therefore, must be the one in Proposition 2.14. Note that if \( T \) is this symplectic triple system, \( \text{ind}(T) \) is the Lie algebra \( L(1) \) of Kostrikin, so this symplectic triple system is indeed different from the previous ones. \( \square \)

**Remark 2.33.** The associated Lie algebras \( g(T) \) of the algebras in item (iii) of Theorem 2.32 are given in equation \((2.11)\), Proposition 2.12 and Remark 2.13. On the other hand, the simple Lie algebra \( g(T) \) associated to the eight-dimensional symplectic triple system in item (iv) of Theorem 2.32 is a simple Lie algebra of dimension 29, which is specific of characteristic 3 (see \[\text{Bro90}, \text{Bro82}\]). This Lie algebra is the contragredient Lie algebra with Cartan matrix \(
\begin{pmatrix}
 2 & -1 & 0 \\
 -1 & 2 & -1 \\
 0 & -1 & 0
\end{pmatrix}
\).

As for the Lie algebras associated to the symplectic triple systems in items (i) and (ii) of Theorem 2.32, everything works as for characteristic \( \neq 2, 3 \), with the only exception of the symplectic triple system associated to the Jordan algebra \( J = H_3(k \times k) = \text{Mat}_3(K)^+ \). What happens here is that the split Lie algebra of type \( E_6 \) in characteristic 3 (obtained by taking a Chevalley basis of the corresponding complex Lie algebra, then the \( \mathbb{Z} \)-algebra spanned by this basis, and finally tensoring with the field) is no longer simple, but has a one-dimensional center (see, for instance, \[\text{VK71}, \S3\]). Modulo this center, one gets a simple Lie algebra of dimension 77, which is precisely the Lie algebra \( g(T) \) in this case. This can be deduced from \[\text{Eld04}\], we will not go into details.

### 3. Lie superalgebras and symplectic triple systems

An interesting feature in characteristic 3 is that any symplectic triple system is an anti-Lie triple system, that is, the odd part of a Lie superalgebra. More precisely, we can forget about \( \mathfrak{sp}(V) \) and \( V \) in Theorem 2.9 and get, not a \( \mathbb{Z}_2 \)-graded Lie algebra, but a Lie superalgebra. The precise statement is the following:
Theorem 3.1. Let \((T, [\cdot, \cdot], (\cdot, \cdot))\) be a symplectic triple system over a field of characteristic 3. Define the superalgebra \(\tilde{g} = \tilde{g}(T) = \tilde{g}_0 \oplus \tilde{g}_1\), with:

\[
\tilde{g}_0 = \text{inder}(T), \quad \tilde{g}_1 = T,
\]

and superanticommutative multiplication given by:

- \(\tilde{g}_0\) is a Lie subalgebra of \(\tilde{g}\);
- \(\tilde{g}_0\) acts naturally on \(\tilde{g}_1\), that is, \([d, x] = d(x)\) for any \(d \in \text{inder}(T)\) and \(x \in T\);
- \([x, y] = d_{x,y} = [xy],\) for any \(x, y \in T\).

Then \(\tilde{g}(T)\) is a Lie superalgebra. Moreover, \(T\) is simple if and only if so is \(\tilde{g}(T)\).

Proof. For any \(x, y, z \in T\), using the symmetry in the first two variables of \([\cdot, \cdot, \cdot]\) \((2.2a)\), the skew-symmetry of \((\cdot, \cdot)\) and \((2.2b)\), one gets:

\[
[x, y, z] + [y, z, x] + [z, x, y] = [xyz] + [yzx] + [zxy]
= [xyz] + [yzx] - 2zxy
= ([xyz] - [zxy]) + ([zyx] - [zxy])
= (x|z)y - (x|y)z + 2(y|z)x + (z|x)y - (z|y)x + 2(y|x)z
= 3((y|z)x - (x|y)z) = 0.
\]

This shows that \(\tilde{g}(T)\) is a Lie superalgebra.

Moreover, any ideal \(I\) of \(T\) induces the ideal \(d_{I,T} \oplus I\) of \(\tilde{g}(T)\). Conversely, if \(a = a_0 \oplus a_1\) is a nonzero ideal of \(\tilde{g}(T)\), then \(I = a_1 \neq 0\), as \(\tilde{g}_0\) acts faithfully on \(\tilde{g}_1\). Then \([TTI] = [\tilde{g}_0, a_1] \subseteq a_1 = I\), while \([ITI] = [a_1, \tilde{g}_1], \tilde{g}_1] \subseteq a_1 = I\). Hence \(I\) is a nonzero ideal of \(T\). This proves the last part of the Theorem. □

Let \(k\) be a field of characteristic 3 and consider the symplectic triple systems in Examples 2.26.

In the symplectic case, \([xyz] = \psi_{x,y}(z)\) for any \(x, y, z \in T\), so that \(\text{inder}(T) = \mathfrak{sp}(T)\) and \(\tilde{g}(T) = \mathfrak{sp}(T) \oplus T\), which is isomorphic to the orthosymplectic Lie superalgebra \(\mathfrak{osp}(W)\) of the superspace \(W = W_0 \oplus W_1\), with \(W_0 = ke\) (a vector space of dimension 1), \(W_1 = T\) and supersymmetric bilinear form given by extending \((\cdot, \cdot)\) on \(T\) by means of \((e|e) = 1\) and \((e|T) = 0\).

In the special case: \(T = W \oplus W^*\) with triple product determined in (2.27). The restriction map \(\text{inder}(T) \rightarrow \mathfrak{gl}(W), d \mapsto d|W\), gives an isomorphism between \(\text{inder}(T)\) and either \(\mathfrak{sl}(W)\) or \(\mathfrak{gl}(W)\) (depending on the dimension of \(W\) being or not congruent to 1 modulo 3). Let \(\tilde{W}\) be the superspace with \(\tilde{W}_0 = ke\) and \(\tilde{W}_1 = W\), and identify the special linear superalgebra \(\mathfrak{sl}(\tilde{W})\) with the vector space of \(2 \times 2\) matrices \(\{(\frac{\text{trace} A f}{x, A f}) : x \in W, f \in W^*, A \in \mathfrak{gl}(W)\}\). Let \(\tilde{g}\) be the center of \(\mathfrak{sl}(\tilde{W})\) (\(\tilde{g} = 0\) if \(\dim W \neq 1\) modulo 3, and \(\dim \tilde{g} = 1\) otherwise), and let \(\mathfrak{psl}(W) = \mathfrak{sl}(\tilde{W})/\tilde{g}\) be the corresponding projective special linear Lie superalgebra. Then the linear
map \( \tilde{\Phi} : \tilde{\mathfrak{g}}(T) \to \mathfrak{psl}(\tilde{W}) \), such that:

\[
\tilde{\Phi}(x, f) = \begin{pmatrix} 0 & -f \\ x & 0 \end{pmatrix} + \tilde{3},
\]

\[
\tilde{\Phi}(d) = \begin{cases} 
\begin{pmatrix} 0 & 0 \\ 0 & d_{|W} \end{pmatrix} + \tilde{3} & \text{if } \dim W \equiv 1 \pmod{3}, \\
\frac{\text{trace } d_{|W}}{1 - \dim W} d_{|W} + \frac{\text{trace } d_{|W}}{1 - \dim W} id & \text{otherwise,}
\end{cases}
\]

for any \( x \in W, f \in W^* \) and \( d \in \text{ind}(T) \), is an isomorphism of Lie superalgebras.

In the orthogonal case, \( T = V \otimes W \), where \( V \) is a two dimensional vector space endowed with a nonzero alternating bilinear form \( \langle \cdot, \cdot \rangle \), and \( W \) is a vector space of dimension \( \geq 3 \) endowed with a nondegenerate symmetric bilinear form \( q_W \). In this case, from Examples 2.26 (orthogonal case):

\[
\tilde{\mathfrak{g}}(T) = \left( \mathfrak{sp}(V) \oplus \mathfrak{so}(W, q_W) \right) \oplus (V \otimes W),
\]

with 2.26 yielding

\[ [u \otimes x, v \otimes y] = \frac{1}{2}q_W(x, y)\gamma_{u,v} + \langle u|v\rangle\sigma_{x,y}, \]

for any \( u, v \in V \) and \( x, y \in W \). Then \( \tilde{\mathfrak{g}}(T) \) is easily checked to be isomorphic to the orthosymplectic Lie superalgebra of the vector superspace \( \tilde{W} \), with \( \tilde{W}_0 = W \) and \( \tilde{W}_1 = V \), endowed with the nondegenerate supersymmetric bilinear form given by \( q_W \) on \( \tilde{W}_0 \) and \( \langle \cdot, \cdot \rangle \) on \( \tilde{W}_1 \).

Also, if \( T \) is a simple symplectic triple system of dimension 2, then by Theorem 2.11 there is a scalar \( 0 \neq \epsilon \in k \) such that \( T \simeq T_{2,\epsilon} \). From equation (2.11), we conclude that \( \tilde{\mathfrak{g}}(T)_0 \simeq \mathfrak{sl}_2(k) \) and \( \tilde{\mathfrak{g}}(T)_1 \) is the two dimensional irreducible module for \( \tilde{\mathfrak{g}}(T)_0 \). Hence \( \tilde{\mathfrak{g}}(T) \) is isomorphic to the orthosymplectic Lie superalgebra \( \mathfrak{osp}_{1,2}(k) \).

The conclusion is that only well-known simple Lie superalgebras appear as \( \tilde{\mathfrak{g}}(T) \) for these ‘classical’ symplectic triple systems. However, the Lie superalgebra \( \tilde{\mathfrak{g}}(T) \) for the remaining simple symplectic triple systems in Theorem 2.32 have no counterpart in characteristic 0 (see [Kac77, Scho79]), thus providing new simple Lie superalgebras in characteristic 3:

**Theorem 3.2.** Let \( k \) be an algebraically closed field of characteristic 3. Then there are simple finite dimensional Lie superalgebras \( \tilde{\mathfrak{g}} \) over \( k \) satisfying:

1. \( \dim \tilde{\mathfrak{g}} = 18 (= 10 + 8) \), \( \tilde{\mathfrak{g}}_0 \) is the Kaplansky Lie superalgebra \( L(1) \) and \( \tilde{\mathfrak{g}}_1 \) is its 8-dimensional irreducible module in Proposition 2.14.
2. \( \dim \tilde{\mathfrak{g}} = 35 (= 21 + 14) \), \( \tilde{\mathfrak{g}}_0 \) is the symplectic Lie algebra \( \mathfrak{sp}_6(k) \) and \( \tilde{\mathfrak{g}}_1 \) is a 14-dimensional irreducible module for \( \tilde{\mathfrak{g}}_0 \).
3. \( \dim \tilde{\mathfrak{g}} = 54 (= 34 + 20) \), \( \tilde{\mathfrak{g}}_0 \) is the projective special Lie algebra \( \mathfrak{psl}_6(k) \) and \( \tilde{\mathfrak{g}}_1 \) is a 20-dimensional irreducible module for \( \tilde{\mathfrak{g}}_0 \).
4. \( \dim \tilde{\mathfrak{g}} = 98 (= 66 + 32) \), \( \tilde{\mathfrak{g}}_0 \) is the orthogonal Lie algebra \( \mathfrak{so}_{12}(k) \) and \( \tilde{\mathfrak{g}}_1 \) is a 32-dimensional irreducible module for \( \tilde{\mathfrak{g}}_0 \) (spin module).
5. \( \dim \tilde{\mathfrak{g}} = 189 (= 133 + 56) \), \( \tilde{\mathfrak{g}}_0 \) is the simple Lie algebra of type \( E_7 \) and \( \tilde{\mathfrak{g}}_1 \) is a 56-dimensional irreducible module for \( \tilde{\mathfrak{g}}_0 \).
Proof. It is enough to consider the simple symplectic triple systems in Theorem 2.32 not considered in the previous remarks. These are the 8-dimensional simple symplectic triple system in Proposition 2.14 and the simple symplectic triple systems corresponding to the Jordan algebra $J = H_3(C)$ for $C = k, k \times k, \text{Mat}_2(k)$ or the split octonions. The irreducibility in (i) follows from the description before Proposition 2.14, and for the remaining four cases it can be checked from the description of these symplectic triple systems in [Eld04]. □

Remark 3.3. For the Lie superalgebras in items (ii)–(v), the odd part $\tilde{g}_1$ is given by a simple symplectic triple system $T$, and the action of $\tilde{g}_0 = \text{inder}(T)$ on $\tilde{g}_1 = T$ coincides with the action inside the Lie algebra $g(T)$ in Theorem 2.9. Since in all these cases, $g(T)$ is a restricted Lie algebra, it follows that $\tilde{g}_1$ is a restricted irreducible $\tilde{g}_0$-module. Actually, with some care (using for instance [Eld04, §4]), it is checked that, with the ordering of the simple roots as in [Bou02], $\tilde{g}_1$ is the irreducible restricted $\tilde{g}_0$-module of highest weight $\omega_3$ in cases (ii) and (iii), any of $\omega_5$ or $\omega_6$ in case (iv) and $\omega_7$ in case (v). (See also [Kru04].)

It must be remarked here that, in the proof of Theorem 3.1, the fact that the alternating bilinear form $(.,.)$ on $T$ is nonzero has played no role. This suggests the next definition.

Definition 3.4. Let $T$ be a vector space over a field $k$ endowed with a triple product $T \times T \times T \to T, (x, y, z) \mapsto [xyz]$. Then $(T, [..])$ is said to be a null symplectic triple system if it satisfies the following identities:

$$
[xyz] = [yxz] = [xzy], \quad (3.5a)
$$

$$
[xy[uvw]] = [[xyu]vw] + [u[xyv]w] + [uv[xyw]]. \quad (3.5b)
$$

That is, the triple product is symmetric on its arguments, and for each $x, y \in T$, the linear $d_{x,y} = [xy]$ is a derivation. Again, the linear span of these maps $d_{x,y}$’s will be denoted by $\text{inder}(T)$.

For Freudenthal triple systems, Kantor also considered the possibility of the alternating form to be zero in [Kan90].

Note that the construction in Theorem 2.4 may be modified for null symplectic triple systems to give, with the same kind of arguments:

Proposition 3.6. Let $(T, [..])$ be a null symplectic triple system and let $(V, \langle.,.\rangle)$ be a two dimensional vector space endowed with a nonzero alternating bilinear form. Define the $\mathbb{Z}_2$-graded algebra $g = g(T) = g_0 \oplus g_1$ with

$$
\begin{cases}
  g_0 = \text{inder}(T), \\
  g_1 = V \otimes T,
\end{cases}
$$

and anticommutative multiplication given by:

- $g_0$ is a Lie subalgebra of $g$;
- $g_0$ acts naturally on $g_1$, that is, $[d, v \otimes x] = v \otimes d(x)$ for any $d \in \text{inder}(T), v \in V$, and $x \in T$;
Theorem 3.9. Given a null symplectic triple system \( (T, [\cdot, \cdot], \langle\cdot, \cdot\rangle) \), consider the Lie algebra \( \mathfrak{g}(T) \) defined in Proposition 3.6. If \( \{v, w\} \) is a symplectic basis of \( (V, \langle\cdot, \cdot\rangle) \), then the decomposition \( \mathfrak{g}(T) = (v \otimes T) \oplus \text{inder}(T) \oplus (w \otimes T) \) is a 3-grading of \( \mathfrak{g}(T) \). This shows that the Lie algebras \( \mathfrak{g}(T) \), for null symplectic triple systems, are precisely the 3-graded Lie algebras \( \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) such that \( \mathfrak{g}_0 \) and \( \mathfrak{g}_1 \) are isomorphic as modules over \( \mathfrak{g}_0 \).

Over fields of characteristic \( \neq 3 \), there are no simple null symplectic triple systems:

**Theorem 3.9.** Let \( k \) be a field of characteristic \( \neq 2, 3 \). Then there are no simple null symplectic triple systems over \( k \).

**Proof.** Assume, on the contrary, that \( (T, [\cdot, \cdot]) \) is a simple null symplectic triple system over \( k \). We will get a contradiction following several steps:

1. Let us prove first that for any \( x \in T \), \( d^2_{x,x} = 0 \):

   In fact, from (3.5) we conclude that for any \( x, y \in T \), \( [d_{x,y}, d_{x,x}] = 2d_{[xy],x} \). But also, \( [d_{x,y}, d_{x,x}] = -[d_{x,x}, d_{x,y}] = -d_{[xxx],y} - d_{x,[xy]} \). Therefore

   \[
   3d_{x,[xy]} = -d_{[xxx],y},
   \]

   for any \( x, y \in T \). If this is applied to \( x \), one gets by (3.5) that \( 3[x[x, y]] = -[x[y, x]] = -3[x[x, y]] \) (since \( d_{x,y} \) is a derivation). Therefore, \( 6[x[x, y]] = 0 \) and hence \( d^2_{x,x} = 0 \).

   By linearization of \( d^2_{x,x} = 0 \) we get

   \[
   d_{x,y}d_{y,x} + d_{y,y}d_{x,x} + 4d^2_{x,y} = 0
   \]

   for any \( x, y \in T \).
(ii) For any \( x, y \in T \), \( d_{x,x}d_{y,y}d_{x,x} = -d_{[xy],[xy]} \): 

Since \( d_{x,x}^2 = 0 \), \([d_{x,x}, [d_{x,x}, d_{y,y}]] = -2d_{x,x}d_{y,y}d_{x,x}\). But, by the derivation property and the symmetry of the triple product, also \([d_{x,x}, [d_{x,x}, d_{y,y}]] = 2[d_{x,x}, [d_{xy}, y]] = 2d_{[xy],[xy]}\), whence the result. As a consequence, we get that for any \( x \in T \): 
\[
[[xxT][xxT]T] \subseteq [xxT].
\] (3.12)

(iii) There are elements \( x \in T \) such that \( d_{x,x} = 0 \):

Take \( 0 \neq S \) a minimal nonzero subspace of \( T \) with the property that \([SST] \subseteq S\). By equation (3.12), for any \( 0 \neq x \in S \), \( S_x = [xxT] \subseteq S \) and also \([S_xS,T] \subseteq S_x\). Therefore, either \( d_{x,x} = 0 \) and we are done, or \( S_x = S \) and hence there is an element \( y \in S \) such that \([xy]=x\). But then \([xxx]=d_{x,x}(x)=d_{x,x}^2(y)=0\), so by (3.10) \( 3d_{x,x}=3d_{x,x}[xy]=0 \), and thus \( x=d_{x,x}(y)=0 \), a contradiction.

(iv) \( T \) is the linear span of the elements \( x \) with \( d_{x,x} = 0 \):

By (ii), if \( d_{x,x} = 0 \), then also \( d_{[yxy],[yxy]} = 0 \). Thus the span of \( \{ x \in T : d_{x,x} = 0 \} \) is \( (T) \)-invariant, and hence an ideal of \( T \).

(v) A subtriple \( S \) of \( T \) is said to be solvable if \( 0 \) belongs to the chain of subtriples defined by \( S^{(0)} = S \) and \( S^{(i+1)} = [S^{(i)}S^{(i)}] \) for any \( i \geq 0 \). Then, if \( S \) is a solvable subtriple of \( T \), the subalgebra of the associative algebra \( \text{End}_k(T) \) generated by \( d_{S,S} \) is nilpotent:

This can be proved by induction on the minimal natural number \( n \) such that \( S^{(n)} = 0 \). This is trivial for \( n = 0 \). Otherwise, let \( R \) be the subalgebra of \( \text{End}_k(T) \) generated by \( d_{S,S} \) and let \( A \) be the subalgebra generated by \( d_{S,S(1)} \). By induction hypothesis, \( A \) is nilpotent. Consider also the subalgebra \( B \) of \( \text{End}_k(T) \) generated by \( d_{S,S(1)} \). Let us first check that \( B \) is nilpotent.

Since \([d_{S,S},d_{S,S(1)}] \subseteq d_{S,(1),S(1)} \), it follows that \( RA \subseteq AR + A \), and hence \( I = AR + A \) is a nilpotent ideal of \( R \). Now, for any \( s,t \in S \) and \( s_1,t_1 \in S(1) \), \([d_{s,s_1},d_{t,t_1}] = d_{[s_1t],t_1} + d_{t_1[ss_1t]} \) and \([d_{ss_1t},t_1] \) belongs to \( d_{S,(1),S(1)} \). Therefore, the generating subspace \( (d_{S,S(1)} + B \cap I)/B \cap I \) of \( B/B \cap I \) is closed under commutation. Also, for any \( x \in S \) and \( y \in S(1) \), \( d_{x,y}^2 = -\frac{1}{2}(d_{x,x}d_{y,y} + d_{y,y}d_{x,x}) \in RA + AR \subseteq I \) (by (3.11)). Hence, by Engel’s Theorem [Jac66, Theorem II.1], the subalgebra \( B/B \cap I \) is nilpotent and, since \( I \) is already known to be nilpotent, we conclude that the subalgebra \( B \) is nilpotent.

On the other hand, \([d_{S,S},d_{S,S(1)}] \subseteq d_{S,S(1)} \), so that \( RB \subseteq BR + B \). Hence \( J = BR + B \) is a nilpotent ideal of \( R \). Also, \([d_{S,S},d_{S,S}] \subseteq d_{S,S(1)} \subseteq B \subseteq J \). Thus, \( R/J \) is a commutative algebra generated by nilpotent elements, and hence \( R/J \) is nilpotent, and so is \( R \), as required.

(vi) To get to a contradiction, let \( S \) be a maximal solvable subtriple of \( T \). Since \( T \) is simple, \( S \neq T \). By (v), \( d_{S,S} \) acts nilpotently on \( T \), so that, because of (iv), there is an element \( w \in T \setminus S \) such that \( d_{w,w} = 0 \) and a natural number \( n \) such that \( d_{w,w}^n = 0 \), but \( d_{w,w}^{n+1} \notin S \), and \( d_{w,w} \notin S \). Hence, there are elements \( x_1, \ldots, x_n \in S \) such that \( x = d_{x_1,x_1} \cdots d_{x_n,x_n} \notin S \), \( d_{S,S}(x) \subseteq S \) and, because \( d_{w,w} = 0 \) and (ii), also \( d_{x,x} = 0 \). Take \( S' = S + kx \). Then
$[S'S'S'] \subseteq [SSS] + [SSx] \subseteq S$. Hence $S'$ is solvable and larger than $S$, a contradiction. □

**Remark 3.13.** This proof is inspired in similar proofs for Jordan pairs [Loo75, Ch. 14]. Actually, observe that if \{v, w\} is any basis of $V$, then the decomposition $g(T) = (v \otimes T) \oplus \text{inder} T \oplus (w \otimes T)$ is a 3-grading of $g(T)$, and hence the pair $(v \otimes T, w \otimes T)$ is a Jordan pair (characteristic $\neq 3$), which is simple if so is $T$. One can use then known results on Jordan pairs to get Theorem 3.9, but we have preferred to give a self contained proof.

By Theorem 2.18 the null symplectic triple systems correspond bijectively to the Faulkner ternary algebras with trivial alternating bilinear form. Therefore, we get the following improvement of [FF72, Lemma 3.1], which in turn, shows that [FF72] contains a complete classification of the simple Faulkner ternary algebras over algebraically closed fields of characteristic $\neq 2, 3$.

**Corollary 3.14.** Let $k$ be a field of characteristic $\neq 2, 3$. Then a Faulkner ternary algebra $(T, \langle...,\rangle, (.,|.)$ over $k$ is simple if and only if $(.,.|.)$ is nondegenerate.

**Proof.** By Theorem 3.9 if $(T, \langle...,\rangle, (.,|.)$ is simple, then $(.,.|.)$ is not 0 and now Proposition 2.4 (or [FF72] Lemma 3.1) applies. □

The comments in the paragraph previous to Proposition 2.7 show that any two dimensional simple symplectic triple system over a field of characteristic 3 is actually a simple null symplectic triple system. The following conjecture sounds plausible.

**Conjecture 3.15.** The dimension of any simple null symplectic triple system over a field of characteristic 3 is two.

### 4. Orthogonal Triple Systems

If the symmetry and skew-symmetry in the definition of a symplectic triple system (Definition 2.1) are interchanged, one obtains the definition of orthogonal triple systems, which first appeared in [Oku93 Section V]:

**Definition 4.1.** Let $T$ be a vector space endowed with a nonzero symmetric bilinear form $(.|.) : T \times T \to k$, and a triple product $T \times T \times T \to T$: $(x, y, z) \mapsto [xyz]$. Then $(T, [...],[.|.])$ is said to be an orthogonal triple system if it satisfies the following identities:

\[
\begin{align*}
[xy] &= 0 \quad &\text{(4.2a)} \\
[xy] &= (x|y)y - (y|y)x \quad &\text{(4.2b)} \\
[xy][uv] &= [[xy][uv]] + [u[xy][uv]] + [uv[xy][uv]] \quad &\text{(4.2c)} \\
([xy][uv]) + (u|[xy][uv]) &= 0 \quad &\text{(4.2d)}
\end{align*}
\]

for any elements $x, y, u, v, w \in T$. 
Note that (4.2b) can be written as
\[ [xyy] = \sigma_{x,y}(y) \] (4.3)
with \( \sigma_{x,y}(z) = (x|z)y - (y|z)x. \) (If \( .|. \) is nondegenerate, the maps \( \sigma_{x,y} \) span the orthogonal Lie algebra \( \mathfrak{so}(T) \).)

Also, as for the symplectic case, (4.2c) is equivalent to \( d_{x,y} = [xy.] \) being a derivation of the triple system. Let \( \text{inder}(T) \) be the linear span of \( \{d_{x,y} : x, y \in T\} \), which is a Lie subalgebra of \( \text{End}(T) \). Equation (4.2d) is equivalent to \( d_{x,y} \in \mathfrak{so}(T) \) for any \( x, y \in T \).

Here condition (4.2d) is always a consequence of the previous ones. This is trivial if \( \dim T = 1 \); otherwise take linearly independent elements \( u, v \in T \).

Then (4.2c) with \( w = v \) gives, using (4.2b), that
\[
\left( ([xyu]|v) + (u|[xyv]) \right)v = 2([xyv]|v)u,
\]
whence the claim. Anyway, we have preferred to keep (4.2d) in the definition.

The definition of homomorphism between two orthogonal triple systems and of ideal and simplicity are the natural ones.

Orthogonal triple systems are strongly related to another class of triple systems: the \((-1, -1)\) balanced Freudenthal Kantor triple systems (see [EKO03] and the references there in).

With the same kind of arguments, as in the proof of Proposition 2.4 (see also [EKO03, Theorem 2.2]) one gets:

**Proposition 4.4.** Let \((T, [\ldots], (.|.))\) be an orthogonal triple system. Then \((T, [\ldots], (.|.))\) is simple if and only if the bilinear form \((.|.)\) is nondegenerate.

And by mimicking the proof of Theorem 2.9 or using [EKO03, Theorem 2.1], it is shown how orthogonal triple systems are related to a specific class of Lie superalgebras:

**Theorem 4.5.** Let \((T, [\ldots], (.|.))\) be an orthogonal triple system and let \((V, (.|.)\)) be a two dimensional vector space endowed with a nonzero alternating bilinear form. Define the superalgebra \( \mathfrak{g} = \mathfrak{g}(T) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) with
\[
\begin{align*}
\mathfrak{g}_0 &= \text{sp}(V) \oplus \text{inder}(T) \quad \text{(direct sum of ideals)}, \\
\mathfrak{g}_1 &= V \otimes T,
\end{align*}
\]
and superanticommutative multiplication given by:

- \( \mathfrak{g}_0 \) is a Lie subalgebra of \( \mathfrak{g} \);
- \( \mathfrak{g}_0 \) acts naturally on \( \mathfrak{g}_1 \), that is,
  \[ [s, v \otimes x] = s(v) \otimes x, \quad [d, v \otimes x] = v \otimes d(x), \]
  for any \( s \in \text{sp}(V), d \in \text{inder}(T), v \in V, \) and \( x \in T \);
- for any \( u, v \in V \) and \( x, y \in T \):
  \[ [u \otimes x, v \otimes y] = (x|y)\gamma_{u,v} - (u|v)d_{x,y} \] (4.6)
  where \( \gamma_{u,v} = (u.|.)v + (v.|.)u \) and \( d_{x,y} = [xy.] \).

Then \( \mathfrak{g}(T) \) is a Lie superalgebra. Moreover, \( \mathfrak{g}(T) \) is simple if and only if so is \((T, [\ldots], (.|.))\).
Conversely, given a Lie superalgebra $g = g_0 \oplus g_1$ with
\[
\begin{align*}
g_0 &= \mathfrak{sp}(V) \oplus \mathfrak{s} \quad \text{(direct sum of ideals),} \\
g_1 &= V \otimes T \quad \text{(as a module for } g_0),
\end{align*}
\]
where $T$ is a module for $\mathfrak{s}$, by $g_0$-invariance of the Lie bracket, equation (4.6) is satisfied for a symmetric bilinear form $(\cdot,\cdot) : T \times T \to k$ and a skew-symmetric bilinear map $d_{\cdot,\cdot} : T \times T \to \mathfrak{s}$. Then, if $(\cdot,\cdot)$ is not 0 and a triple product on $T$ is defined by means of $[xyz] = d_{x,y}(z)$, $(T,\ldots,\cdot,\ldots)$ is an orthogonal triple system.

Over fields of characteristic 0, the classification of the simple $(-1,-1)$ balanced Freudenthal Kantor triple systems in [EKO03, Theorem 4.3] immediately implies the following classification of the simple orthogonal triple systems.

**Theorem 4.7.** Let $k$ be a field of characteristic zero and let $(T,\ldots,\cdot,\ldots)$ be a simple orthogonal triple system over $k$. Then either:

(i) $[xyz] = (x|z)y - (y|z)x = \sigma_{x,y}(z)$ for any $x,y,z \in T$ (orthogonal type). In this case $\text{inder}(T) = \mathfrak{so}(T)$.

(ii) There is a quadratic étale algebra $K$ over $k$ such that $T$ is a free $K$-module of rank at least 3, endowed with a nondegenerate hermitian form $h : T \times T \to K$ ($h(x,y) = h(y,x)$, $h(rx,y) = rh(x,y)$, for any $x,y \in T$ and $r \in K$, where $r \mapsto \bar{r}$ is the standard involution on $K$) such that
\[
\begin{align*}
(x|x) &= h(x,x) \\
[xyz] &= h(z,x)y - h(z,y)x + \frac{1}{2}(h(x,y) - h(y,x))z
\end{align*}
\]
for any $x,y,z \in T$ (unitarian type). In this case $\text{inder}(T)$ is the unitarian Lie algebra $u(T,h) = \{f \in \text{End}_K(T) : h(f(x),y) + h(x,f(y)) = 0, \text{ for any } x,y \in T\}$.

(iii) There is a quaternion algebra $Q$ over $k$ such that $T$ is a free left $Q$-module of rank $\geq 2$, endowed with a nondegenerate hermitian form $h : T \times T \to Q$ satisfying (4.8) (symplectic type). In this case $\text{inder}(T)$ is the direct sum of the symplectic Lie algebra $\mathfrak{sp}(T,h) = \{f \in \text{End}_Q(T) : h(f(x),y) + h(x,f(y)) = 0, \text{ for any } x,y \in T\}$ and of the three dimensional simple Lie algebra $[Q,Q]$.

(iv) $\dim_k T = 4$ and there is a nonzero skew-symmetric multilinear map $\Phi : T \times T \times T \times T \to k$ such that, for any $x,y,z,t \in T$,
\[
[xyz] = \{xyz\} + \sigma_{x,y}(z),
\]
where $\{\ldots\}$ is defined by means of $\Phi(x,y,z,t) = (\{xyz\}|t)$. In this case there is a scalar $0 \neq \mu \in k$ such that $\{a_1a_2a_3\} \{b_1b_2b_3\} = \mu \det\{a_i|b_j\}$ ($T_\mu$-type). For $\mu = 1$, $\text{inder}(T)$ is a three dimensional simple Lie algebra, while for $\mu \neq 0,1$, $\text{inder}(T) = \mathfrak{so}(T)$.

(v) $\dim_k T = 7$ and there is an eight-dimensional Cayley algebra $C$ over $k$ with trace $t$ and a nonzero scalar $\alpha \in k$ such that $T = C_\alpha = \{x \in \mathfrak{sp}(V) : x|\cdots|x|x = \alpha\}$.\]
where $D_{x,y} = [L_x, L_y] + [L_x, R_y] + [R_x, R_y] \in \text{der}(C)$ ($L_x$ and $R_x$ denote the left and right multiplications by $x$ in $C$) (G-type). In this case $\text{inder}(T)$ is the Lie algebra of derivations of $C$: $\text{der}(C)$ (viewed as a Lie subalgebra of $\mathfrak{gl}(C_0)$), which is a simple Lie algebra of type $G_2$.

(vi) $\dim_k T = 8$ and $T$ is endowed with a 3-fold vector cross product $X$ of type $I$ (see [Eld96] and the references there in) relative to a nondegenerate symmetric bilinear form $b(.,.)$ (so that $b(X(x, y, z), X(x, y, z)) = \det(b(a_i, a_j))$ for any $a_1, a_2, a_3 \in T$) such that

\[ [xyz] = X(x, y, z) + 3\tau_{x,y}(z) \quad (4.11) \]

for any $x, y, z \in T$, where $\tau_{x,y} = b(x,.)y - b(y,.)x$ and $(.,.) = 3b(.,.)$ (F-type). In this case $\text{inder}(T)$ is isomorphic to the orthogonal Lie algebra $\mathfrak{so}(W, b)$, where $W$ is any regular seven dimensional subspace of $T$ relative to $b$. Besides, $T$ is the spin module for $\text{inder}(T)$.

Remark 4.12. Two orthogonal triple systems in different items in Theorem 4.7 are never isomorphic, and the conditions for isomorphism among orthogonal triple systems in the same item can be read off [EKO03, Theorem 4.3]. Only the F-type has been described in a slightly different way as it is in [EKO03], where our $X$ denotes what appears as $\frac{1}{3}X$ there.

As for symplectic triple systems, the definition of orthogonal triple systems makes sense even if the symmetric bilinear form involved is trivial.

Definition 4.13. Let $T$ be a vector space over a field $k$ endowed with a triple product $T \times T \times T \to T$, $(x, y, z) \mapsto [xyz]$. Then $(T, [.,.])$ is said to be a null orthogonal triple system if it satisfies the following identities:

\[ [xyy] = [yyx] = 0, \quad (4.14a) \]

\[ [xy[wv]] = [[xyu]vw] + [u[xyv]w] + [uv[xyw]]. \quad (4.14b) \]

The following counterpart to Proposition 3.6 follows with the same sort of arguments used there:

Proposition 4.15. Let $(T, [.,.])$ be a null orthogonal triple system and let $(V, (.,.))$ be a two dimensional vector space endowed with a nonzero alternating bilinear form. Define the superalgebra $\mathfrak{g} = \mathfrak{g}(T) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with

\[
\begin{cases}
\mathfrak{g}_0 = \text{inder}(T), \\
\mathfrak{g}_1 = V \otimes T,
\end{cases}
\]

and superanticommutative multiplication given by:

- $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}$,
\[ \mathfrak{g}_1 \text{ acts naturally on } \mathfrak{g}, \text{ that is, } \left[ d, v \otimes x \right] = v \otimes d(x) \text{ for any } d \in \text{inder}(T), v \in V, \text{ and } x \in T; \]

- for any \( u, v \in V \) and \( x, y \in T \):
  \[ [u \otimes x, v \otimes y] = (u|v)d_{x,y} \tag{4.16} \]

where \( d_{x,y} = [xy] \).

Then \( \mathfrak{g}(T) \) is a Lie superalgebra. Moreover, \( \mathfrak{g}(T) \) is simple if and only if so is \( \langle T, \[\ldots\], \[\:] \rangle \).

Conversely, given a Lie superalgebra \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \), with \( \mathfrak{g}_0 = s \) and \( \mathfrak{g}_1 = V \otimes T \), where \( T \) is a module for \( s \) and the multiplication of odd elements is given by \( \mathfrak{g}(T) \) for a bilinear symmetric map \( d_{\ldots} : T \times T \rightarrow s \), \( (x,y) \mapsto d_{x,y} \); then \( T \) is a null orthogonal triple system with the triple product defined by \( [xyz] = d_{x,y}(z) \), for any \( x, y, z \in T \).

**Remark 4.17.** As for symplectic triple systems, given a null orthogonal triple system \( \langle T, \[\ldots\], \[\:] \rangle \), consider the Lie superalgebra \( \mathfrak{g}(T) \) defined in Proposition 4.15. If \( \{v,w\} \) is a symplectic basis of \( \langle V, \[\ldots\] \rangle \), then the decomposition \( \mathfrak{g}(T) = (v \otimes T) \oplus \text{inder}(T) \oplus (w \otimes T) \) is a consistent 3-grading of \( \mathfrak{g}(T) \) \( (\mathfrak{g}_0 = \mathfrak{g}_0 \text{ and } \mathfrak{g}_1 = \mathfrak{g}_1 \oplus \mathfrak{g}_1) \). This shows that the Lie superalgebras \( \mathfrak{g}(T) \), for null orthogonal triple systems, are precisely the consistently 3-graded Lie superalgebras \( \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) such that \( \mathfrak{g}_1 \) and \( \mathfrak{g}_1 \) are isomorphic as modules over \( \mathfrak{g}_0 \).

The classification in Theorem 4.17 can be completed by means of the following:

**Theorem 4.18.** Let \( \langle T, \[\ldots\], \[\:] \rangle \) be a simple null orthogonal triple system over a field \( k \) of characteristic 0. Then \( \dim_k T = 4 \) and there is a non-degenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \) and a nonzero multilinear skew-symmetric map \( \Phi : T \times T \times T \times T \rightarrow k \) such that

\[ \Phi(x, y, z, t) = ([xyz]|t) \tag{4.19} \]

for any \( x, y, z, t \in T \). Conversely, any triple system defined by means of \( \Phi \) is a simple null orthogonal triple system.

Moreover, given two such null orthogonal triple systems \( \langle T_1, \[\ldots\], \[\:] \rangle \) and \( \langle T_2, \[\ldots\], \[\:] \rangle \) over \( k \) with associated multilinear maps \( \Phi_1 \) and \( \Phi_2 \) and non-degenerate symmetric bilinear forms \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_2 \), let \( \mu_1 \) \( (i = 1, 2) \) be the nonzero scalars such that

\[ ([x_1x_2x_3]|y_1y_2y_3)_i = \mu_i \det\left( (x_i|y_j)_i \right) \]

for any \( x_i, y_i \in T_i \). Then \( \langle T_1, \[\ldots\], \[\:] \rangle \) is isomorphic to \( \langle T_2, \[\ldots\], \[\:] \rangle \) if and only if there is a similarity \( \varphi : \langle T_1, \[\ldots\], \[\:] \rangle \rangle \rightarrow \langle T_2, \[\ldots\], \[\:] \rangle \rangle \) of norm \( \alpha \) (that is, \( \varphi(x)|\varphi(y)\rangle_2 = \alpha(x|y)_1 \) for any \( x, y \in T_1 \)) such that \( \mu_1 = \mu_2 \alpha^2 \).

**Proof.** Assume first that \( \dim_k T = 4 \) and that there are maps \( \Phi \) and \( \langle \cdot, \cdot \rangle \) such that \( \Phi \) is satisfied. Then \( \langle \cdot, \cdot \rangle \) is skew-symmetric (4.14a), since so is \( \Phi \). Then, if \( \{e_1, e_2, e_3, e_4\} \) is any basis of \( T \) with \( \Phi(e_1, e_2, e_3, e_4) = 1 \), and \( \{e^1, e^2, e^3, e^4\} \) is its dual basis relative to \( \langle \cdot, \cdot \rangle \), the map \( d_{e_1, e_2} = [e_1 e_2] \) annihilates \( e_1 \) and \( e_2 \) and takes \( e_3 \) to \( e_4 \) and \( e_4 \) to \( -e^3 \). This argument shows easily that the maps \( d_{e_i, e_j} \) \( (1 \leq i < j \leq 4) \) are linearly independent, and
hence \( \text{dim } d_{T,T} = 6 \). On the other hand, because of (4.19), \( d_{T,T} \subseteq \mathfrak{so}(T) \) so, by dimension count, \( d_{T,T} = \mathfrak{so}(T) \), which is contained in the special linear Lie algebra \( \mathfrak{sl}(T) \). But \( \Phi \) is invariant under \( \mathfrak{sl}(T) \), so both \( \Phi \) and \( (\cdot,\cdot) \) are invariant under \( d_{T,T} \). This shows that \([\cdot]\) is invariant too under \( d_{T,T} \), which is equivalent to condition (4.14) and, therefore, \((T,[\cdot])\) is a null orthogonal triple system, which is irreducible as a \( \text{ind}(T) = \mathfrak{so}(T) \)-module, and hence simple.

Now, if \( \varphi : (T_1,[\cdot,\cdot]_1) \to (T_2,[\cdot,\cdot]_2) \) is an isomorphism between two of these null orthogonal triple systems, then for any \( a_1,a_2 \in T_1 \), \([\varphi(a_1)\varphi(a_2)]_2 = \varphi[a_1,a_2]_1 \varphi^{-1} \), so that \( \varphi \) induces an isomorphism \( \varphi : \mathfrak{so}(T_1) \to \mathfrak{so}(T_2) \), \( d \mapsto \varphi d \varphi^{-1} \). But then \((\cdot,\cdot)_1\) and \((\varphi(\cdot),\varphi(\cdot))_2\) are both invariant under the action of \( \mathfrak{so}(T_1) \). Therefore, there is a nonzero scalar \( \alpha \in k \) such that \((\varphi(x)|\varphi(y))_2 = \alpha (x|y)_1\) for any \( x,y \in T_1 \), and \( \varphi \) is a similarity of norm \( \alpha \).

Moreover, for any \( x_1,x_2,x_3,y_1,y_2,y_3 \in T_1 \):

\[
\alpha \mu_1 \det((x_i|y_j)_1) = \alpha ([x_1x_2x_3]_1[y_1y_2y_3]_1)_1 = \varphi([x_1x_2x_3]_1)[\varphi(y_1y_2y_3)]_1_2 = ([\varphi(x_1)\varphi(x_2)\varphi(x_3)]_2[\varphi(y_1)\varphi(y_2)\varphi(y_3)]_2)_2 \]
\[
= \mu_2 \det((\varphi(x_i)|\varphi(y_j)_2) = \mu_2 \alpha^3 \det((x_i|y_j)_1)
\]

so that \( \mu_1 = \mu_2 \alpha^2 \). Conversely, assume that \( \psi : (T_1,(\cdot,\cdot)_1) \to (T_2,(\cdot,\cdot)_2) \) is a similarity of norm \( \alpha \) such that \( \mu_1 = \mu_2 \alpha^2 \). Then, since the skew-symmetric multilinear map on \( T_1 \) given by \( \Phi'_1(x,y,z,t) = \Phi_2(\psi(x),\psi(y),\psi(z),\psi(t)) \) is (as \( \text{dim}_k T_1 = 4 \)) a scalar multiple of \( \Phi_1 \), there is a nonzero scalar \( \beta \in k \) such that \( \Phi_2(\psi(x),\psi(y),\psi(z),\psi(t)) = \beta \Phi_1(x,y,z,t) \) for any \( x,y,z,t \in T_1 \), and hence we conclude that

\[
[\psi(x)\psi(y)\psi(z)]_2 = \frac{\beta}{\alpha} \psi([xyz]_1)
\]

for any \( x,y,z \in T_1 \). But now, for any \( x_1,x_2,x_3,y_1,y_2,y_3 \in T_1 \):

\[
\alpha \mu_1 \det((x_i|y_j)_1) = \alpha ([x_1x_2x_3]_1[y_1y_2y_3]_1)_1 = (\psi([x_1x_2x_3]_1)[\psi(y_1y_2y_3)]_1)_2 = \left( \frac{\alpha}{\beta} \right)^2 \mu_2 \det((\psi(x_i)|\psi(y_j)_2) = \left( \frac{\alpha}{\beta} \right)^2 \mu_2 \alpha^3 \det((x_i|y_j)_1)
\]

and, since \( \mu_1 = \mu_2 \alpha^2 \), we obtain \( \left( \frac{\alpha}{\beta} \right)^2 = 1 \) or \( \beta = \pm \alpha \). In case \( \beta = \alpha \), \( \psi \) is an isomorphism between \((T_1,[\cdot,\cdot]_1)\) and \((T_2,[\cdot,\cdot]_2)\), while if \( \beta = -\alpha \), \( \psi \) satisfies \( \psi([xyz]_1) = -[\psi(x)\psi(y)\psi(z)]_2 \) for any \( x,y,z \in T_1 \). But we can always take an orthogonal map \( \sigma \) of \((T_1,[\cdot,\cdot]_1)\) (that is \((\sigma(x)\sigma(y))_1 = (x|y)_1\) for any \( x,y \)) with \( \text{det } \sigma = -1 \). Then, for any \( x,y,z,t \in T_1 \), \( \Phi_1(\sigma(x),\sigma(y),\sigma(z),\sigma(t)) = -\Phi_1(x,y,z,t) \), so that \( \sigma([xyz]_1) = -[\sigma(x)\sigma(y)\sigma(z)]_1 \) and the composite map \( \psi \sigma \) is an isomorphism.
Finally, let us show that these four dimensional simple null orthogonal triple systems are the only ones. To do so, we may assume that \( k \) is algebraically closed. Note that, by Proposition \( 4.15 \), any simple null orthogonal triple system \((T, [...])\) gives rise to a simple Lie superalgebra \( g = g(T) = g_0 \oplus g_1 \), where \( g_0 = \text{inder}(T) \) and \( g_1 = V \otimes T \), with \((V, [,])\) is a two dimensional vector space endowed with a nonzero alternating bilinear form. The simplicity of \( T \) shows that \( g_1 \) is a sum of two copies of an irreducible module for \( g_0 \). A close look at the classification in [Kac77] reveals that the only possibility for \( g(T) \) is to be isomorphic to \( \text{psl}_2(k) \), so that \( \text{inder}(T) \sim \text{sl}_2(k) \oplus \text{sl}_2(k) \sim \text{sp}(V) \oplus \text{sp}(V) \) and \( T \) is the four dimensional irreducible module \( V \otimes V \) (the symmetric bilinear form is just the tensor product of the alternating forms on each copy of \( V \)). Hence the formula \( \Phi(x, y, z, t) = ([xyz] | t) \) defines a multilinear skew-symmetric map and the proof is complete.

There appears the natural open question of the classification of the simple (null) orthogonal triple systems over fields of prime characteristic. All the examples in Theorems 4.7 and 4.18 have their counterparts in prime characteristic, with only a couple of changes needed for the simple orthogonal triple systems of \( G \) and \( F \) types in characteristic 3. The \( F \)-type orthogonal triple system becomes a simple null orthogonal triple system in characteristic 3. On the other hand, given a Cayley algebra over a field \( k \) of characteristic 3, the linear span of the derivations \( D_{x,y} \)'s form the unique simple ideal \( \text{inder}(C) \), of dimension 7, of the fourteen dimensional Lie algebra of derivations \( \text{der}(C) \), as shown in [BEMN02], where it is proved that \( \text{inder}(C) \) is a form of the simple Lie algebra \( \text{psl}_3(k) \) (the simple Lie algebra of type \( A_2 \) in characteristic 3). Moreover, any form of \( \text{psl}_3(k) \) is the inner derivation algebra of a Cayley algebra. Hence, over fields of characteristic 3, the simple orthogonal triple systems of \( G \)-type make sense, but their inner derivation algebras are seven dimensional, instead of fourteen dimensional, and thus \( \dim g(T) = (3 + 7) + (2 \times 7) = 24 \).

The proofs of Theorems 4.7 and 4.18 are based in the classification by Kac of the simple finite dimensional Lie superalgebras over algebraically closed field of characteristic 0, so different methods are needed in prime characteristic.

Here, we will content ourselves with showing that over fields of characteristic 3, there is a new family of simple orthogonal triple systems related to Jordan algebras of nondegenerate cubic forms with basepoint.

**Examples 4.20.** Let \( J = J_{\text{ord}}(n, 1) \) be the Jordan algebra of a nondegenerate cubic form \( n \) with basepoint 1, over a field \( k \) of characteristic 3, and assume that \( \dim_k J \geq 3 \). Then any \( x \in J \) satisfies a cubic equation \([\text{McC04}, \text{II.4.2}]
\begin{align*}
x^3 - t(x)x^2 + s(x)x - n(x)1 &= 0, \quad (4.21)
\end{align*}
where \( t \) is its trace linear form, \( s(x) = t(x^2) \) is the spur quadratic form and the multiplication in \( J \) is denoted by \( x \cdot y \).
Let \( J_0 = \{ x \in J : t(x) = 0 \} \) be the subspace of zero trace elements. Since \( \text{char } k = 3 \), \( t(1) = 0 \), so that \( k1 \in J_0 \). Consider the quotient space \( \hat{J} = J_0/k1 \). For any \( x \in J_0 \), \( s(x) = -\frac{1}{2}t(x^2) \) and, by linearization of (4.21), we get that for any \( x, y \in J_0 \):

\[
\begin{align*}
y^2 \cdot x - (x \cdot y) \cdot y & \equiv -2t(x)y - t(y)x \mod k1, \\
& \equiv t(x, y)y - t(y, x) \mod k1.
\end{align*}
\]

(4.22)

Let us denote by \( \hat{x} \) the class of \( x \) modulo \( k1 \). Since \( J_0 \) is the orthogonal of \( k1 \) relative to the trace bilinear form \( t(a, b) = t(a \cdot b) \), \( t \) induces a nondegenerate symmetric bilinear form on \( \hat{J} \) defined by \( t(\hat{x}, \hat{y}) = t(x, y) \) for any \( x, y \in J_0 \).

Now, consider for any \( x, y, z \in J_0 \) the inner derivation of \( J \) given by \( D_{x,y} : x \mapsto x \cdot (y \cdot z) - y \cdot (x \cdot z) \) (see \( \text{Jac6} \)). Since the trace form is invariant under the Lie algebra of derivations, \( D_{x,y} \) leaves \( J_0 \) invariant, and obviously satisfies \( D_{x,y}(1) = 0 \), so it induces a map \( d_{x,y} : \hat{J} \to \hat{J}, \hat{z} \mapsto D_{x,y}(\hat{z}) \) and a well defined bilinear map \( \hat{J} \times \hat{J} \to \mathfrak{gl}(\hat{J}), (\hat{x}, \hat{y}) \mapsto d_{x,y} \). Consider now the triple product \( […] \) on \( \hat{J} \) defined by

\[
[\hat{x} \hat{y} \hat{z}] = d_{x,y}(\hat{z})
\]

for any \( x, y, z \in J_0 \). This is well defined and satisfies (4.2a), because of the skew-symmetry of \( d_{\ldots} \). Also, (4.22) implies that

\[
[\hat{x} \hat{y} \hat{y}] = d_{x,y}(\hat{y}) = t(x, y)\hat{y} - t(y, x)\hat{x}
\]

\[
= t(\hat{x}, \hat{y})\hat{y} - t(\hat{y}, \hat{y})\hat{x},
\]

so that (4.2b) is satisfied too, relative to the trace bilinear form. Since \( D_{x,y} \) is a derivation of \( J \) for any \( x, y \in J \), (4.2a) follows immediately, while (4.2d) is a consequence of the previous ones.

Therefore, by the nondegeneracy of the trace form

\( (\hat{J}, […] , t(\ldots)) \) is a simple orthogonal triple system over \( k \).

There are two possibilities for such Jordan algebras, either \( J = k \times \text{ord}(q, e) \), where \( \text{ord}(q, e) \) is the Jordan algebra of a nondegenerate quadratic from \( q \) with basepoint \( e \), or \( J \) is a central simple Jordan algebra of degree 3.

In the first case, \( \text{ord}(q, e) = ke \oplus W \), where \( W = \{ x \in \text{ord}(q, e) : q(x, e) = 0 \} \), and \( \hat{J} = J_0/k1 \) can be identified with \( W \). Moreover, for any \( x, y, z \in W \) (see \( \text{McC01} \) II.3.3):

\[
x \cdot (y \cdot z) - y \cdot (x \cdot z) = x \cdot (-\frac{1}{2}q(y, z)e) - y \cdot (-\frac{1}{2}q(x, z)e)
\]

\[
= q(y, z)x - q(x, z)y,
\]

and

\[
t(x, y) = t(x \cdot y) = -\frac{1}{2}t(q(x, y)e) = -q(x, y).
\]

Therefore, \( [xyz] = t(x, z)y - t(y, z)x \) and \( (\hat{J}, […] , t(\ldots)) \) is a simple orthogonal triple system of orthogonal type. Hence nothing new appears in this case.

In the second case (Jordan type), \( J \) is a central simple Jordan algebra of degree 3 so, after a scalar extension, \( J \) is the algebra of hermitian matrices \( H_3(C) \), where \( C \) is either \( k, k \times k, \text{Mat}_2(k) \) or the algebra of split octonions.
Besides, according to \[Jac66\] Theorem VI.9 and Theorem IX.17, the Lie algebra of derivations of $J$, $\text{der}(J)$, is isomorphic, respectively, to $\mathfrak{so}_3(k)$, $\mathfrak{psl}_3(k)$, $\mathfrak{sp}_6(k)$, or the 52 dimensional simple Lie algebra of type $F_4$. Also, since $J_0$ is an invariant subspace for $\text{der}(J)$, $D_{J_0,J_0}$ is an ideal of $\text{der}(J)$. Hence, by simplicity, $\text{inder}(\hat{J})$ is isomorphic either to $\mathfrak{so}_3(k)$, $\mathfrak{sp}_6(k)$ or the Lie algebra of type $F_4$ if $\dim C = 1$, 4 or 8. Finally, if $\dim C = 2$, then $J \cong \text{Mat}_3(k)^+$, where $x \cdot y = \frac{1}{2}(xy + yx)$ for any $x, y \in \text{Mat}_3(k)$, so

$$D_{x,y} : z \mapsto \frac{1}{3} \left( x(yz + zy) + (yz + zy)x - (y(xz + zx) - (xz + zx)y \right) = \frac{1}{7}[[x, y], z],$$

for any $x, y$ and, hence, $D_{J_0,J_0}$ is isomorphic to the seven dimensional simple Lie algebra $\mathfrak{psl}_3(k)$, and so is $\text{inder}(\hat{J})$.

Therefore, for $J = H_3(k)$, dim $\hat{J} = 4$ and $t([\hat{x} \hat{y} \hat{z}]$, $\hat{u})$ gives a skew-symmetric nonzero four-linear map (because the trace is $\text{der}(J)$-invariant). Hence $(\hat{J}, \ldots, t(\ldots))$ is a simple orthogonal triple system of $D_4$-type, as the dimension of $\text{inder}(\hat{J}) \cong \mathfrak{so}_3(k)$ is 3 (see [1.7]). Again, nothing new appears in this case.

For $J = \text{Mat}_3(k)^+$, up to isomorphism $\hat{J} = \mathfrak{psl}_3(k)$ and $d_{\hat{x}, \hat{y}} = \frac{1}{3} \text{ad}([\hat{x}, \hat{y}]$ for any $\hat{x}, \hat{y} \in \hat{J}$ (brackets in the Lie algebra $\mathfrak{psl}_3(k)$). It follows that $(\hat{J}, \ldots, t(\ldots))$ is a simple orthogonal triple system of $G$-type.

Finally, for $J = H_3(C)$, with $\dim C = 4$ or 8, dim $\hat{J} = 13$ or 25 respectively and the situation has no counterpart in characteristic 0. \hfill \square

Summarizing some of the previous work, some other simple Lie superalgebras in characteristic 3, with no counterpart in characteristic 0, have been found:

**Theorem 4.23.** Let $k$ be an algebraically closed field of characteristic 3. Then there are simple finite dimensional Lie superalgebras $\mathfrak{g}$ over $k$ satisfying:

(i) $\dim \mathfrak{g} = 24 \ (= (3 + 7) + (2 \times 7))$, $\mathfrak{g}_0$ is the direct sum of $\mathfrak{sl}_2(k)$ and $\mathfrak{psl}_3(k)$ and, as a $\mathfrak{g}_0$-module, $\mathfrak{g}_1$ is the tensor product of the natural two dimensional module for $\mathfrak{sl}_2(k)$ and the adjoint module for $\mathfrak{psl}_3(k)$ (G-type).

(ii) $\dim \mathfrak{g} = 37 \ (= 21 + (2 \times 8))$, $\mathfrak{g}_0 = \mathfrak{so}_7(k)$ and, as a $\mathfrak{g}_0$-module, $\mathfrak{g}_1$ is the direct sum of two copies of its spin module (F-type).

(iii) $\dim \mathfrak{g} = 50 \ (= (3 + 21) + (2 \times 13))$, $\mathfrak{g}_0$ is the direct sum of $\mathfrak{sl}_2(k)$ and $\mathfrak{sp}_6(k)$ and, as a $\mathfrak{g}_0$-module, $\mathfrak{g}_1$ is the tensor product of the natural two dimensional module for $\mathfrak{sl}_2(k)$ and of a 13 dimensional irreducible module for $\mathfrak{sp}_6(k)$ (namely, the quotient of the space of zero trace symmetric matrices in $\text{Mat}_6(k)$ relative to the symplectic involution modulo the scalar matrices).

(iv) $\dim \mathfrak{g} = 105 \ (= (3 + 52) + (2 \times 25))$, $\mathfrak{g}_0$ is the direct sum of $\mathfrak{sl}_2(k)$ and of the central simple Lie algebra of type $F_4$ and, as a $\mathfrak{g}_0$-module, $\mathfrak{g}_1$ is the tensor product of the natural two dimensional module for $\mathfrak{sl}_2(k)$ and a 25 dimensional irreducible module for $F_4$. \hfill $\square$
5. Lie algebras and orthogonal triple systems

Characteristic 3 is special too for orthogonal triple systems, because it turns out that any such system is a Lie triple system, that is, the odd part of a \(\mathbb{Z}_2\)-graded Lie algebra. More precisely:

**Theorem 5.1.** Let \((T, [\ldots], (\ldots))\) be either an orthogonal triple system or a null orthogonal triple system over a field of characteristic 3. Define the \(\mathbb{Z}_2\)-graded algebra \(\tilde{g} = \tilde{g}(T) = \tilde{g}_0 \oplus \tilde{g}_1\), with:

\[\tilde{g}_0 = \text{inder}(T), \quad \tilde{g}_1 = T,\]

and anticommutative multiplication given by:

- \(\tilde{g}_0\) is a Lie subalgebra of \(\tilde{g}\);
- \(\tilde{g}_0\) acts naturally on \(\tilde{g}_1\), that is, \([d,x] = d(x)\) for any \(d \in \text{inder}(T)\) and \(x \in T\);
- \([x,y] = d_{x,y} = [xy,]\), for any \(x,y \in T\).

Then \(\tilde{g}(T)\) is a Lie algebra. Moreover, \(T\) is simple if and only if so is \(\tilde{g}(T)\).

**Proof.** As in the proof of Theorem 3.1 if \((T, [\ldots], (\ldots))\) is an orthogonal triple system and \(x, y, z \in T\), in \(\tilde{g}(T)\):

\[
[[x,y],z] + [[y,z],x] + [[z,x],y] = [xyz] + [y]zx + [zxy] = [xyz] + [y]zx - 2[zxy]
\]

\[
= \left([xyz] + [xzy]\right) - \left([y]zx + [zxy]\right) \quad \text{(by (4.2a))}
\]

\[
= (\sigma_{x,y}(z) + \sigma_{x,z}(y)) - (\sigma_{z,y}(x) + \sigma_{z,x}(y)) \quad \text{(by (4.2b))}
\]

\[
= 3(\sigma_{x,y}y + \sigma_{x,z}y) - (\sigma_{z,y}x + \sigma_{z,x}y) = 0
\]

and the same argument works for null orthogonal triple systems with trivial \((\ldots)\). The rest of the proof follows exactly as in Theorem 3.1. \(\square\)

Let \(k\) be a field of characteristic 3 and consider the analogues over \(k\) of the orthogonal triple systems that appear in Theorems 4.7 and 4.18. We finish the paper by computing the corresponding Lie algebras \(\tilde{g}\). Besides some classical Lie algebras, the simple Lie algebras \(L(\epsilon)\) of Kostrikin, as well as Brown’s 29-dimensional simple Lie algebra, appear again.

**Examples 5.2.** Orthogonal type: Here \([xyz] = \sigma_{x,y}(z)\) for any \(x, y, z \in T\), so that \(\text{inder}(T) = \mathfrak{so}(T)\) and \(\tilde{g}(T) = \mathfrak{so}(T) \oplus T\), which is isomorphic to the orthogonal Lie algebra of a space which is the orthogonal sum of \(T\) and a one dimensional subspace.

Unitarian type: Consider the split case: \(K = k \times k\). Hence, as in [EKO03, p.358], we may assume that \(T = W \oplus W^*\) for a vector space \(W\) with \(\dim_k W \geq 3\) and, after scaling, we get that the triple product is determined by \([WWWT] = 0 = [W^*W^*T]\) and by

\[
[xfy] = f(x)y - 2f(y)x, \quad (5.3a)
\]

\[
[xfg] = -f(x)g + 2g(x)f, \quad (5.3b)
\]
for any \( x, y \in W \) and \( f, g \in W^* \) (compare with the special case in Examples 2.20). The restriction map \( \text{ider}(T) \to \mathfrak{gl}(W) \), \( d \mapsto d|_W \) gives an isomorphism between \( \text{ider}(T) \) and either \( \mathfrak{sl}(W) \) or \( \mathfrak{gl}(W) \), depending on the dimension of \( W \), being or not congruent to 2 modulo 3. Let \( \bar{W} \) be the \( \mathbb{Z}_2 \)-graded vector space with \( \bar{W}_0 = k e \) and \( \bar{W}_1 = W \) and identify the special linear Lie algebra \( \mathfrak{sl}(\bar{W}) \) with the vector space of \( 2 \times 2 \) matrices \( \{ (\begin{array}{cc} -\text{trace} & A \\ A & \end{array}) : x \in W, f \in W^*, A \in \mathfrak{gl}(W) \} \). Let \( \tilde{\mathfrak{z}} \) be the center of \( \mathfrak{sl}(\bar{W}) \) (\( \tilde{\mathfrak{z}} = 0 \) if \( \dim W \neq 2 \) modulo 3, and \( \dim \tilde{\mathfrak{z}} = 1 \) otherwise), and let \( \mathfrak{psl}(\bar{W}) = \mathfrak{sl}(\bar{W})/\tilde{\mathfrak{z}} \) be the corresponding projective special linear Lie algebra. Then the linear map \( \tilde{\Phi} : \tilde{\mathfrak{g}}(T) \to \mathfrak{psl}(\bar{W}) \), such that:

\[
\tilde{\Phi}((x, f)) = \left( \begin{array}{cc} 0 & f \\ x & 0 \end{array} \right) + \tilde{\mathfrak{z}},
\]

\[
\tilde{\Phi}(d) = \left\{ \begin{array}{ll}
\left( \begin{array}{cc} 0 & 0 \\ 0 & d|_W \end{array} \right) + \tilde{\mathfrak{z}} & \text{if } \dim W \equiv 2 \pmod{3}, \\
\left( \begin{array}{cc} 0 & \text{trace} \cdot d|_W \\ \text{trace} \cdot d|_W & -d|_W \end{array} \right) + \tilde{\mathfrak{z}} & \text{otherwise},
\end{array} \right.
\]

for any \( x \in W \), \( f \in W^* \) and \( d \in \text{ider}(T) \), is an isomorphism of Lie algebras.

**Symplectic type:** Consider again the split case, where \( Q = \text{End}_k(V) \), for a two dimensional vector space \( V \) endowed with a nonzero alternating bilinear form \( (.,.) \). Then the arguments in [EKO03, p. 359] give that, up to isomorphism, \( T = V \otimes W \) for some even dimensional vector space \( W \) of dimension \( \geq 4 \), endowed with a nondegenerate alternating bilinear form \( \psi : W \times W \to k \) such that the triple product becomes (just take \( \varphi(u, v) = 2(u|v) \) in [EKO03]):

\[
[(a \otimes x)(b \otimes y)(c \otimes z)] = \psi(x, y)\gamma_{a,b}(c) \otimes z - 2(a|b)\gamma_{x,y}(z)
\]

for any \( a, b, c \in V \) and \( x, y, z \in W \), where \( \gamma_{a,b} = (a.|) + (|b.)a \) and \( \psi_{x,y} = \psi(x,.)y + \psi(y,.)x \). Besides, \( \text{ider}(T) \) is isomorphic naturally to the direct sum \( \mathfrak{sp}(V) \oplus \mathfrak{sp}(W) \) of the corresponding symplectic Lie algebra. Thus we may identify \( \tilde{\mathfrak{g}}(T) \) with the \( \mathbb{Z}_2 \)-graded Lie algebra

\[
\tilde{\mathfrak{g}} = (\mathfrak{sp}(V) \oplus \mathfrak{sp}(W)) \oplus (V \otimes W),
\]

where

\[
[a \otimes x, b \otimes y] = \psi(x, y)\gamma_{a,b} + (a|b)\psi_{x,y} \in \mathfrak{sp}(V) \oplus \mathfrak{sp}(W),
\]

for any \( a, b \in V \) and \( x, y \in W \). Thus \( \tilde{\mathfrak{g}} \) is isomorphic to the symplectic Lie algebra of the orthogonal sum of \( V \) and \( W \): \( \mathfrak{sp}(V \perp W) \).

**G-type:** In characteristic 3, for any \( x, y \) in a Cayley algebra \( C \):

\[
D_{x,y} = [L_x, L_y] + [L_x, R_y] + [R_x, R_y] = [L_x - R_x, L_y - R_y]
\]

\[
= [\text{ad} x, \text{ad} y] = \text{ad}[x, y],
\]

(because \( [L_x, R_x] = 0 \) for any \( x \) and \( [[x, y], z] + [[y, z], x] + [[z, x], y] = 6((xy)z - (y)z) = 0 \) [EKN02, proof of Theorem 1]). Therefore, the triple product on \( T = C_0 \) in (4.10) becomes

\[
[xyz] = \alpha[[x, y], z],
\]
and thus $\tilde{g}(T) = \text{iner}(T) \oplus T \cong C_0 \otimes_k k[e]$, where $k[e] = k1 \oplus ke$, with $e^2 = \alpha$, and $C_0$ is a Lie algebra with the usual bracket $[x, y] = xy - yx$, which is a form of $\mathfrak{psl}_3(k)$ (the split simple Lie algebra of type $A_2$ over $k$). Note that if $\alpha \in k^2$, $\tilde{g}(T) \cong C_0 \oplus C_0$.

**F-type:** Here $\tilde{g}(T)$ is a $\mathbb{Z}_2$-graded Lie algebra whose even part is a simple Lie algebra of type $B_3$ and the even part is its spin module. Therefore $\tilde{g}(T)$ is a form of the 29 dimensional simple Lie algebra discovered by Brown [Bro82].

**$D_{\mu}$-type:** Assume here that the ground field $k$ is algebraically closed and consider the simple Lie superalgebra $g = D(2, 1; \alpha) = \Gamma(1, \alpha, -(1 + \alpha))$, $\alpha \neq 0, -1$ (notation as in [Sch79, pp. 17-18]):

$$\begin{cases}
g_0 = \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2) \oplus \mathfrak{sp}(V_3), \\
g_1 = V_1 \otimes V_2 \otimes V_3,
\end{cases}$$

where the $V_i$'s are two dimensional vector spaces endowed with nonzero alternating bilinear forms $(.,.)$ (the same notation will be used for the three $V_i$'s), and the Lie bracket of even or even and odd elements is the natural one, while

$$\begin{align*}
[x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3] \\
= (x_2|y_2)(x_3|y_3)\gamma_{x_1,y_1} + \alpha(x_1|y_1)(x_3|y_3)\gamma_{x_2,y_2} - (1 + \alpha)(x_1|y_1)(x_2|y_2)\gamma_{x_3,y_3},
\end{align*}$$

for any $x_i, y_i \in V_i$, $i = 1, 2, 3$. According to Theorem 5.5 $T_\alpha = V_2 \otimes V_3$ is a simple orthogonal triple system with the triple product and symmetric bilinear form determined by

$$\begin{align*}
[(x_2 \otimes x_3)(y_2 \otimes y_3)(z_2 \otimes z_3)] \\
= \alpha(x_3|y_3)\gamma_{x_2,y_2}z_2 \otimes z_3 - (1 + \alpha)<x_2|y_2>z_2 \otimes \gamma_{x_3,y_3}(z_3),
\end{align*}$$

for arbitrary elements $x_i, y_i, z_i \in V_i$, $i = 2, 3$. These are precisely the orthogonal triple systems of $D_{\mu}$-type (see [EKO13]) with $\mu \neq 1$. Here the simple $\mathbb{Z}_2$-graded Lie algebra $\tilde{g}(T_\alpha)$ satisfies

$$\begin{cases}
\tilde{g}_0 = \mathfrak{sp}(V_2) \oplus \mathfrak{sp}(V_3), \\
\tilde{g}_1 = V_2 \otimes V_3,
\end{cases}$$

and the bracket of odd basis elements is given by:

$$[x_2 \otimes x_3, y_2 \otimes y_3] = \alpha(x_3|y_3)\gamma_{x_2,y_2} - (1 + \alpha)<x_2|y_2>\gamma_{x_3,y_3},$$

for $x_i, y_i \in V_i$, $i = 2, 3$. Proposition 2.12 shows that $\tilde{g}(T_\alpha)$ is isomorphic to the Kostrikin Lie algebra $L\left(\frac{\alpha}{\mu}\right)$. Thus we get all the Kostrikin algebras $L(\epsilon)$ for $\epsilon \neq 1$ ($L(1)$ will be obtained shortly). However, for $\alpha = -1$, we get a Lie superalgebra $g = g_0 \oplus g_1$ with

$$\begin{cases}
g_0 = \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2), \\
g_1 = V_1 \otimes V_2 \otimes V_3,
\end{cases}$$

(5.4)
(the $V_i$’s as before) and Lie bracket determined by

\[
[x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3] = \langle x_2 | y_2 \rangle \langle x_3 | y_3 \rangle \gamma_{x_1, y_1} - \langle x_1 | y_1 \rangle \langle x_3 | y_3 \rangle \gamma_{x_2, y_2},
\]  

(5.5)

for $x_i, y_i \in V_i$, $i = 1, 2, 3$. This Lie superalgebra is isomorphic to $\mathfrak{psl}_2(k)$ and shows that $T_{-1} = V_2 \otimes V_3$ is a simple orthogonal triple system with the triple product and symmetric bilinear form determined by

\[
[(x_2 \otimes x_3)(y_2 \otimes y_3)(z_2 \otimes z_3)] = -(x_3 | y_3) \gamma_{x_2, y_2} (z_2 \otimes z_3),
\]

and shows that $T$ is a simple orthogonal triple system with the triple product and symmetric bilinear form determined by

\[
(x_2 \otimes x_3 | y_2 \otimes y_3) = \langle x_2 | y_2 \rangle \langle x_3 | y_3 \rangle,
\]

for arbitrary elements $x_i, y_i, z_i \in V_i$, $i = 2, 3$. From [EKO03 Proposition 3.3] it follows that $T_{-1}$ corresponds to the $D_1$-type. The associated $\mathbb{Z}_2$-graded Lie algebra $\hat{g} = \hat{g}(T_{-1})$ satisfies that $\hat{g}_0 = \mathfrak{sp}(V_2)$ and $\hat{g}_1 = V_2 \otimes V_3$ (the direct sum of two copies of the natural module for $\hat{g}_0$), and this is isomorphic to $\mathfrak{psl}_2(k)$. Also, because of (5.21) and (5.3), it turns out that $T = V_1 \otimes V_2$, with triple product $[xyz] = d_{x,y,z}(z)$ determined by

\[
d_{x_1 \otimes x_2, y_1 \otimes y_2} = \langle x_1 | y_1 \rangle \gamma_{x_1, y_1} - \langle x_1 | y_1 \rangle \gamma_{x_2, y_2}
\]

for $x_i, y_i \in V_i$, $i = 1, 2$, is a simple null orthogonal triple system, and its associated Lie algebra $\hat{g}(T)$ is, because of Proposition 2.12, isomorphic to the Kostrikin algebra $L(1)$.

**Jordan type:** Let $J$ be a central simple Jordan algebra of degree 3, so that $\dim_k J = 6, 9, 15$ or 17, and let $(J, [\ldots], t(\ldots))$ be its associated orthogonal triple system of Jordan type as in Examples 4.20. Then the arguments given there show that $\text{der}(J) = [\text{der}(J), \text{der}(J)] \simeq \text{inder}(\hat{J})$ if $\dim_k J \neq 9$, and $\text{der}(J), \text{der}(J) = \text{inder}(\hat{J})$ if $\dim_k J = 9$. Associated to $J$ there is the Lie algebra $\mathfrak{L}_0(J) = L_x \oplus D_{J, J}$ (a subalgebra of $\mathfrak{gl}(J)$), where $L_x$ denotes the left multiplication by $x$ (see [Jac66 VI.9]). (Incidentally, this is the Lie algebra that appears in the second row in Tits construction of Freudenthal’s Magic Square if the characteristic were $\neq 3$ [Sch95 Theorem 4.13].) The derived algebra is $\mathfrak{L}_0'(J) = [\mathfrak{L}_0(J), \mathfrak{L}_0(J)] = L_{x_0} \oplus D_{J_0, J_0}$, because $D_{J_0, J_0}(J_0) = J_0$, and its center is $k L_1$. Then the natural map $\mathfrak{L}_0'(J) \to \hat{g}(\hat{J}) = \text{inder}(\hat{J}) \oplus \hat{J}$, which is the identity on $D_{J_0, J_0} = \text{inder}(\hat{J})$ and takes $L_x$, for $x \in J_0$, to $\hat{x}$, induces an isomorphism $\hat{g}(\hat{J}) \cong \mathfrak{L}_0'(J)/k L_1$.

If $\dim_k J = 6$, $\hat{J}$ is of $D_1$-type and $\hat{g}(\hat{J})$ is a form of $\mathfrak{psl}_3(k)$, while for $\dim_k J = 9$, $\hat{J}$ is of $G$-type and $\hat{g}(\hat{J})$ is a form of $\mathfrak{psl}_3(k) \oplus \mathfrak{psl}_3(k)$. If $\dim_k J = 15$ and $k$ is algebraically closed, $J$ is, up to isomorphism, the algebra of hermitian matrices in $\text{Mat}_6(k)$ relative to the standard symplectic involution, $D_{J, J} = D_{J_0, J_0} = \text{inder}(\hat{J})$ is given by the adjoint action on $J$ of the skew-hermitian matrices: $\mathfrak{sp}_6(k)$, and then $\mathfrak{L}_0'(J) \cong \mathfrak{sl}_6(k)$ and hence $\hat{g}(\hat{J})$ is isomorphic to $\mathfrak{psl}_6(k)$. Therefore, in general, if $\dim_k J = 15$, $\hat{g}(\hat{J})$ is a form of $\mathfrak{psl}_6(k)$. Finally, if $J$ is exceptional ($\dim_k J = 27$) and $k$ is algebraically closed, $\mathfrak{L}_0'(J)$ is the Lie algebra of type $E_6$ (which has a one dimensional center), and $\hat{g}(\hat{J})$ is then isomorphic to the simple finite dimensional contragredient Lie algebra $E_6'$ (notation as in [VK71 §3]).
References

[All79] B. N. Allison, *Models of isotropic simple Lie algebras*, Comm. Algebra 7 (1979), no. 17, 1835–1875.

[BDE03] Pilar Benito, Cristina Draper, and Alberto Elduque, *Models of the octonions and G2*, Linear Algebra Appl. 371 (2003), 333–359.

[BEMN02] Pablo Alberca Bjerregaard, Alberto Elduque, Cándido Martín González, and Francisco José Navarro Márquez, *On the Cartan-Jacobson theorem*, J. Algebra 250 (2002), no. 2, 397–407.

[Bou02] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), translated from the 1968 French original by Andrew Pressley, Springer-Verlag, Berlin, 2002.

[Bro82] Gordon Brown, *Properties of a 29-dimensional simple Lie algebra of characteristic three*, Math. Ann. 261 (1982), no. 4, 487–492.

[Bro84] ——, *Freudenthal triple systems of characteristic three*, Algebras Groups Geom. 1 (1984), no. 3, 399–441.

[Bro90] ——, *Structure of certain simple Lie algebras of characteristic three*, Lie algebra and related topics (Madison, WI, 1988), Contemp. Math., vol. 110, Amer. Math. Soc., Providence, RI, 1990, pp. 27–31.

[Dix84] J. Dixmier, *Certaines algèbres non associatives simples définies par la transvection des formes binaires*, J. Reine Angew. Math. 346 (1984), 110–128.

[Eld96] Alberto Elduque, *On a class of ternary composition algebras*, J. Korean Math. Soc. 33 (1996), no. 1, 183–203.

[Kan73] I. L. Kantor, *Models of the exceptional Lie algebras*, Dokl. Akad. Nauk SSSR 208 (1973), 1276–1279. MR MR0349779 (50 #2272)

[Fre54] N. Jacobson, *Structure theory for a class of Jordan algebras*, Proc. Nat. Acad. Sci. U.S.A. 55 (1956), 363–368.

[Fre59] ——, *Beziehungen der E_7 und E_8 zur Oktavenebene. II*, Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indag. Math. 16 (1954), 363–368.

[Jac66] N. Jacobson, *Structure theory for a class of Jordan algebras*, Proc. Nat. Acad. Sci. U.S.A. 55 (1966), 243–251.

[Kac77] V. G. Kac, *Lie superalgebras*, Advances in Math. 26 (1977), no. 1, 8–96.

[McC04] Kevin McCrimmon, *A taste of Jordan algebras*, Universitext, Springer-Verlag, New York, 2004.
[Mey68] Kurt Meyberg, *Eine Theorie der Freudenthalschen Tripelsysteme. I, II*, Nederl. Akad. Wetensch. Proc. Ser. A 71=Indag. Math. 30 (1968), 162–174, 175–190.

[Oku93] Susumu Okubo, *Triple products and Yang-Baxter equation. I. Octonionic and quaternionic triple systems*, J. Math. Phys. 34 (1993), no. 7, 3273–3291.

[Sch95] Richard D. Schafer, *An introduction to nonassociative algebras*, Dover Publications Inc., New York, 1995.

[Sch79] Manfred Scheunert, *The theory of Lie superalgebras*, Lecture Notes in Mathematics, vol. 716, Springer, Berlin, 1979.

[VK71] B. Ju. Veisfeiler and V. G. Kac, *Exponentials in Lie algebras of characteristic p*, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 762–788.

[YA75] Kiyosi Yamaguti and Hiroshi Asano, *On the Freudenthal’s construction of exceptional Lie algebras*, Proc. Japan Acad. 51 (1975), no. 4, 253–258.

Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain

E-mail address: elduque@unizar.es