Approximate solutions of functional equations

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Received 21 May 2011, in final form 22 August 2011
Published 16 September 2011
Online at stacks.iop.org/JPhysA/44/405205

Abstract

Approximate solutions to functional evolution equations are constructed through a combination of series and conjugation methods, and relative errors are estimated. The methods are illustrated, both analytically and numerically, by construction of approximate continuous functional iterates for \(x^{1-s} \sin x\) and \(\lambda x (1-x)\). Simple functional conjugation by these functions, and their inverses, substantially improves the numerical accuracy of formal series approximations for their continuous iterates.

PACS numbers: 02.30.Ks, 02.30.Mv, 02.30.Lt, 05.45.-a, 11.10.Gh, 11.15.Ha, 11.25.Hf

(Some figures may appear in colour only in the online journal)

1. Overview

There are several circumstances where one encounters the functional evolution equation \([1, 13]\),

\[ x_{t+t} = x_t \circ x_t, \]

(1)

corresponding to an Abelian flow in the subscript parameter, with initial condition \(x_{t=0} = \text{id}\), the identity map. Here, ‘\(\circ\)’ is functional composition. For example, such an equation governs the changes in scale that underlie the renormalization group \([10, 8]\), or the rectification of local trajectories in dynamical systems \([3]\). While (1) does give rise to a local differential equation, in principle, the actual problem of interest may instead present the result of a fixed ‘large’ step in \(t\), say \(x_1\), from which all continuous iterates \(x_t\) are to be determined.

In this situation, a useful direct approach is to construct an \(N\)th-order formal series approximation for \(x_t\) around a fixed point of \(x_1\), say at \(x = 0\), by series analysis of the \(s = 1\) case of (1), written as

\[ x_1 (x_t (x)) = x_t (x_1 (x)). \]

(2)
With initial conditions corresponding to $x_t=0 = \text{id}$, the result has the form

$$x_t^{(N \text{approx})} (x) = \sum_{k=1}^{N} c_k (t) x^k,$$

(3)

where $c_1 (0) = 1$, $c_{k-1} (0) = 0$, so $x_t^{(N \text{approx})} (x) = x$. Note that it is only necessary to construct accurate approximations to $x_t$ for any unit interval in $t$, for then the composition rules $x_{t+t} = x_t \circ x_t$ and $x_{t-1+t} = x_{t-1} \circ x_t$ can be used to reach higher or lower values of $t$.

Now, for a given $N$ the series (3) may not produce accurate results for values of $t$, or for an interval of initial $x$, of interest. Quite possibly, this may be overcome if the series is convergent, by taking larger $N$, but if the series is only a formal one, perhaps asymptotic, with zero radius of convergence, larger $N$ may not be a viable option. So what can be done then?

Formally, for any fixed $N$, with $x_n = x_1 \circ \cdots \circ x_1$, it follows that

$$x_N = \lim_{n \to \pm \infty} x_n \circ x_t^{(N \text{approx})} \circ x_{-n}$$

(4)

is a solution to (2). This is a consequence of

$$x_1 \circ \left(x_n \circ x_t^{(N \text{approx})} \circ x_{-n}\right) = \left(x_{n+1} \circ x_t^{(N \text{approx})} \circ x_{-n-1}\right) \circ x_1.$$

(5)

In the limit $n \to \pm \infty$, equation (2) is obtained, if either of those limits exist. For a specified class of problems it might be possible to give an elegant proof that either limit exists by using various fixed point theorems from functional analysis [2, 11], but that is not our objective here.

The objective here is to estimate the numerical accuracy obtained by conjugating (3) a finite number of times, $n$, with the given, exactly known, ‘finite step’ functions $x_{n+1}$. That is, our concern here is the relative error obtained as a function of $n$, prior to taking the limit $n \to \infty$. In many cases, $n$-fold conjugation with the given $x_1$, and its inverse $x_1$, dramatically improves the numerical accuracy of the series approximation, with the error vanishing exponentially in either $N \ln n$ or $Nn$. This behavior came to light in follow-up studies of earlier work [6, 7, 5, 8, 4].

2. A rational illustration

As a tractable example [14, 12, 10, 16, 9], consider

$$x_1 (x) = \frac{x}{1-x}, \quad x_{-1} (x) = \frac{x}{1+x}.$$  

(6)

In this particular simple case, an exact solution to (1) is

$$x_t (x) = \frac{x}{1-xt}.$$  

(7)

But to illustrate the series method, consider (2) given only $x_1$ in (6). Recursive analysis in powers of $x$ immediately gives $c_k (t) = t^{k-1}$, where we have defined the form and scale of the $t$ parameterization by the choice $c_2 (t) = t$, to be in agreement with (7). For instance, with $x_t (x) = x + tx^2 + c_3 (t) x^3 + O(x^4)$ we find

$$\frac{x_t (x)}{1-x} - x_t \left(\frac{x}{1-x}\right) = (t^2 - c_3 (t))x^4 + O(x^5).$$

(8)

Thus, (2) is satisfied if and only if $c_3 (t) = t^2$. So it goes to higher orders in $x$, with each $c_{k+1}$ determined by $c_{j<k}$ to satisfy (2)—not just by direct expansion of (7).
The result for the approximation is therefore

\[ x_t^{(N \text{approx})} (x) = \sum_{k=1}^{N} t^{k-1} x^k = \frac{(1 - t^n x^N) x}{1 - tx} . \]  

(9)

Then, by \( n \)-fold conjugation of this approximation with \( x_1 \) and its inverse, obtain

\[ x_n \circ x_t^{(N \text{approx})} \circ x_{-n} (x) \equiv x \left( 1 - \left( \frac{tx}{1 + nx} \right)^N \right) / \left( 1 - tx + nx \left( \frac{tx}{1 + nx} \right)^N \right) . \]  

(10)

This indeed converges to the exact result, (7), as \( n \to \infty \) for any \( \text{fixed } N > 1 \). But what is the relative error for finite \( n \)?

Since we know the exact answer in this elementary case, this error is not difficult to compute. For any fixed \( N \) and finite \( n \) the relative error is

\[ R_n (x, N, n) \overset{\text{defn}}{=} \frac{x_t (x) - x_n \circ x_t^{(N \text{approx})} \circ x_{-n} (x)}{x_t (x)} = \frac{1 - tx + nx \left( \frac{tx}{1 + nx} \right)^N}{1 - tx + nx} . \]  

(11)

For \( N > 1 \) this indeed vanishes as \( n \to \infty \), for any \( \text{fixed } x \), so long as \( tx \neq 1 \). However, more importantly, for \( n \) finite but large compared to both \( t \) and \( 1/x \), this error goes to zero as the \( (N - 1) \)st power of \( n \),

\[ R_n (x, N, n) \approx \frac{t^n x}{1 - tx n^{N-1}} . \]  

(12)

Therefore, in principle, one can obtain numerical results to any desired accuracy by repeated conjugation of the approximate series with the given step-function \( x_1 \). It is remarkable that it is only required to take any fixed \( N \geq 2 \) to produce such results, although in practice, as is manifested in (12), computational efficiencies can be improved by taking larger \( N \) since the desired accuracy is then reached for smaller \( n \).

The result for the relative error can be understood in terms of Schröder theory [14] as it applies to this simple example. The continuous iterate \( x_t \) given in (7) may be constructed from a Schröder function, \( \Psi \), and its inverse. An exact result for this particular example is given by

\[ \Psi (x) = \exp \left( -\frac{1}{x} \right) , \quad \Psi^{-1} (x) = -\frac{1}{\ln x} . \]  

(13)

In Schröder’s conjugacy framework, the general iterate is \( x_t (x) = \Psi^{-1} (e^t \Psi (x)) \), or,

\[ \Psi \left( x_t (x) \right) = e^t \Psi (x) . \]  

(14)

Thus, \( w \equiv \Psi (x) \) is just the change of variable that reduces the effect of evolution in \( t \) to nothing but a multiplicative rescaling, \( w_t = e^t w \).

But suppose that the exact \( x_t \) were supplanted by an approximation of the form \( x_t^{(N \text{approx})} = x_t + \alpha x_t^{N+1} + O(x_t^{N+2}) \), for some coefficient \( \alpha \). Then,

\[ \Psi \left( x_t^{(N \text{approx})} \right) = (1 + \alpha x_t^{N-1} + O(x_t^N)) \exp(-1/x_t) . \]  

(15)

Alternatively, with \( x_t = -1/\ln \Psi (x_t) = -1/\ln w_t \),

\[ w_t^{(\text{approx})} \equiv \Psi \left( x_t^{(N \text{approx})} \right) \approx \left( 1 + \frac{\alpha}{(-\ln w_t)^{N-1}} + O \left( \frac{1}{(-\ln w_t)^N} \right) \right) w_t . \]  

(16)

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3 This result is for a particular choice (namely, \( \lambda = e \)) of the ‘multiplier’ in Schröder’s equation, \( \lambda \Psi (x) = \Psi \left( x_t (x) \right) \). The multiplier is undetermined when \( d^n \Psi (x)/dx^n \big|_{x=0} \) is zero, or undefined, for all \( n \). In such cases, it is usually possible to choose \( \lambda \) just as a matter of taste.
Then, it follows from the multiplicative rescaling behavior of \( w \) that under the variable change \( x_t \to u_1 \) the conjugated approximation \( x_n \circ x_1^{\text{approx}} \circ x_{n-1} \) presents itself as

\[
u_n \circ x_1^{\text{approx}} \circ w = \left( 1 + \frac{\alpha}{(n - \ln w_1)^{N-1}} + O\left( \frac{1}{(n - \ln w_1)^N} \right) \right) u_1.
\]

This gives a relative error with the same power-law asymptotics\(^4\) as (12), namely,

\[ R \sim \frac{\alpha}{n^{N-1}}. \]

3. The roots of \( \sin \)

A more interesting example is provided by the sine function, where our notation for the continuous iterate is now \( \sin_n(x) \) with \( \sin_1(x) = \sin(x) \) and \( \sin_{-1}(x) = \arcsin(x) \). In this notation, the Abelian functional composition equation is

\[ \sin_t \circ \sin_r = \sin_{r+t}. \]

Again specializing to \( t = 1 \), written as

\[ \sin(\sin_t(x)) = \sin_t(\sin(x)), \]

we find a formal series solution,

\[
\sin_t(x^{\text{approx}}) = x - \frac{1}{3!}t x^3 + \frac{(5t - 4)t}{5!} x^5 - \frac{(175t^2 - 536t + 164)t}{3 \times 7!} x^7 \\
+ \frac{(25t - 24)(8 - 7t)^2 t}{9!} x^9 + O(x^{11}).
\]

This is sufficient for the numerical work to follow (other results for this example are available online: http://server.physics.miami.edu/curtright/Schroeder.html & http://server.physics.miami.edu/curtright/TheRootsOfSin.pdf).

In this case, the series (20) is probably not convergent for almost all \( t \) [15, 16, 9], although it obviously is convergent for \( t \in \mathbb{Z} \). Rather, for generic \( t \) the series appears to be asymptotic. For example, for \( t = 1/2 \), using Mathematica to extend the series (20) to \( O(x^{31}) \) or so, the smooth behavior of the series coefficients \( c_k \) for \( k \leq 61 \) suggests a finite radius of convergence \( \approx 4/3 \), as estimated by \( 1/|c_{61}|^{1/6} \). But then less smooth behavior sets in for the \( c_{k>61} \) and numerical estimates of the radius begin to fall toward zero, as would be expected for an asymptotic series.

Also note in passing that (20) implies the corresponding Schrödner function has an essential singularity at \( x = 0 \), as given explicitly by (see footnote 3)

\[ \Psi(x) \to 0.665 \exp\left( \frac{3}{x^2} + \frac{79}{1050} x^2 + \frac{29}{2625} x^4 + O(x^5) \right). \]

4 When \( x_1(x) \) is analytic about \( x = 0 \), with \( x_1(0) = 0 \) and real \( x'_1(0) = \lambda \neq \pm 1 \), there is a solution of Schröder’s equation which is also analytic, with \( \Psi(x) = x + O(x^2) \). Under the exact change of variables given by \( w = \Psi(x) \) the series approximation \( x_1^{\text{approx}}(x) \) therefore becomes \( w_3 + \gamma w_2 + O(w) \) for some coefficient \( \gamma \), where \( w_1 = x'_1 w \), while the exact \( x_{3n} \) become \( w_{3n} = \lambda^{3n} w \). So under this change of variables the conjugation \( x_n \circ x_1^{\text{approx}} \circ x_{n-1} \) presents itself as

\[ \lambda^n((\lambda^{-n} w + \gamma \lambda^{2-2n} w^2 + O(\lambda^{3-3n} w^3))) = (1 + \gamma \lambda^{-n} w + O(\lambda^{2-2n} w^2)) w). \]

This gives a relative error \( R \sim \frac{\alpha}{n^{N-1}} \) which vanishes exponentially in \( n \), either as \( n \to +\infty \) for \( |\lambda| > 1 \), or as \( n \to -\infty \) for \( |\lambda| < 1 \). Thus, for such analytic \( \Psi \) cases, the conjugations would converge more rapidly to the exact result than for the rational and sine examples discussed in the text. For a thorough discussion of an analytic case, with \( x'_1(0) \neq \pm 1 \), see section IV in the text.
Nonetheless, the conjugation method produces approximations $\sin_{n,t} \approx A_{n,t}$ which provide compelling numerical evidence for the existence of a limit, hence an exact result for $\sin_{n}$, as $n \to \infty$. But note it is important in this case to take the conjugations to be of the form

$$A_{n,t} = \sin_{-n} \circ \sin_{n}^{(\text{approx})} \circ \sin_{n},$$

the general rule being to act first with functions that are smaller in magnitude than the identity map, thereby leading to evaluation of the truncated series at points closer to $x = 0$.

The improvement wrought by conjugation of the truncated ninth-order series in (20) is easily seen in the accompanying figures, for the case $t = 1/2$. The series itself is not credible beyond $x = \pi/2$, but a single conjugation forces the expected periodicity in $x$ and gives the more plausible green curve seen in figure 1. Repeated conjugation does not produce any discernible differences with the green curve to the accuracy of figure 1, but when the graph is magnified, as it is in figure 2, numerical evidence for convergence of the sequence of conjugations is quite compelling.

Proceeding in this way leads to the set of curves for various values of $t$ shown in figure 3. Each of these curves results from the 5-fold conjugation $A_{5,t}$ of the truncated ninth-order series (20). Note once again, as previously remarked in a general context, it is only necessary to construct accurate approximations to $\sin_{n}$ for any unit interval in $t$, for then the composition rules $\sin_{1+t} = \sin \circ \sin_{t}$ and $\sin_{-1+t} = \arcsin \circ \sin_{t}$ can be used to construct the curves for higher or lower values of $t$. Also note from the numerics the obvious inference that $\sin_{n}(x)$ becomes the periodic triangular ‘sawtooth’ function as $t \to 0$, with $\lim_{t \to 0} \sin_{n}((2k + 1)\pi/2) = (-1)^k \pi/2$.

As constructed, $\sin_{n}(x)$ for $t \geq 0$ is guaranteed to be real for all real values of $x$, but certainly it is not obvious for generic $t$ what numerical values are actually attained at the extrema for $x = 1/2 \pi \mod \pi$. It suffices here for us to point out that the maxima are approximated by the simple expression

$$\sin_{n}(\pi/2) \approx \left(\frac{\pi}{2}\right)^{1-\sqrt{t}}.$$
Figure 2. The $t = 1/2, n = 1, 2, 3, 4$ and 5-fold conjugations (22), in green, orange, blue, red, and purple, respectively.

Figure 3. Various $\sin_t (x)$ as given by 5-fold conjugations $A_{5,t} (x)$ of the ninth-order series (20), for $t \leq 1$.

At least, this is true for $0 \leq t \leq 1$, where the relative error between the exact (numerical) value of $\sin_t (\pi/2)$ and this approximation is less than about 3 parts per mille. (More accurate numerics are available online http://server.physics.miami.edu/curtright/SineMapIterationExtrema.pdf.) The square root branch point exhibited in this approximate expression is perhaps the most direct numerical evidence that the iterates are not analytic at $t = 0$ for all $x.$
The graphs in figure 2 give a sense of the overall relative error, but lacking closed-form expressions for either the iterates or the conjugations of the series approximations, closed-form results for the error are not available for generic $t$. For $t \in \mathbb{Z}$, however, precise calculation is indeed possible since both exact results and convergent series are known. It suffices here to consider just one exact case, $t = 1$. Defining the relative error as before,

$$R_1(x, n) = \frac{\sin x - \sin_{-n} \circ \sin_1^{\text{(approx)}} \circ \sin_n(x)}{\sin x},$$

we have computed numerically the error involved in various conjugations of the ninth-order series (see figure 4). As previously remarked, conjugation by the sine guarantees that the approximations are periodic. The results are shown here.

It would be interesting to compute relative errors for other, generic $t$, but at this stage it is only possible for us to compute the relative successive differences,

$$S_t(x, n) = \frac{\sin_{-n} \circ \sin_t^{\text{(approx)}} \circ \sin_{n+1}(x) \circ \sin_{-n+1}(x) - \sin_{-n} \circ \sin_t^{\text{(approx)}} \circ \sin_{n-1}(x)}{\sin_{-n} \circ \sin_t^{\text{(approx)}} \circ \sin_n(x)}.$$

For example, consider $t = 1/2$ using the ninth-order series (see figure 5).

This numerical data support the proposition that these continuous iterates of the sine function are well defined and straightforward to compute using the methods in this paper.

4. The logistic map

The sequence of conjugations converges more rapidly in situations where the underlying Schröder function is analytic about the fixed point, in contrast to (13) and (21). Instead of
power-law behavior, for such situations the relative error vanishes exponentially in \( n \), the number of conjugations. The general theory is well illustrated by the logistic map,

\[
x_1(x) = \lambda x (1 - x), \quad \text{for } 0 \leq x \leq \lambda/4 \quad \text{and} \quad 0 \leq \lambda \leq 4.
\]

The result of the theory is as follows.

**Theorem 1.** Relative error after \( n \)-fold conjugation of the truncated series \( (3) \) is given by

\[
R_t(x, \lambda, N, n) = \frac{x_t(x) - x_n(x_t^{(N, \text{approx})}(x_{-n}(x)))}{x_t(x)} = \left(\frac{1}{\lambda}\right)^n r(x, t, \lambda, N, n)
\]

\[
= \left(\frac{1}{\lambda}\right)^n \lambda^{-t} e_{N+1}(t, \lambda) x^N + O(x^{N+1}).
\]

**Proof.** To sketch a proof, and to see more clearly the assumptions involved, as well as to obtain expressions for \( r(x, t, \lambda, N, n) \), write the truncated series as

\[
x_{t}^{(N, \text{approx})}(x) = x_t(x) - \lambda^{N+1} \delta_N(x, t),
\]

where \( \delta_N(x, t) \) represents the exact difference, whose expansion in \( x \) begins \( O(x^0) \). Thus, the conjugation gives exactly

\[
x_{n}(x_t^{(N, \text{approx})}(x_{-n}(x))) = x_n(x_t(x_{-n}(x))) - (x_{-n}(x))^{N+1} \delta_N(x_{-n}(x), t)).
\]

Now expand the RHS in powers of \( (x_{-n})^{N+1} \delta_N \),

\[
x_n(x_t(x_{-n}(x))) - (x_{-n}(x))^{N+1} \delta_N(x_{-n}(x), t)) = x_n(x_t(x_{-n}(x)))
\]

\[
= (x_{-n}(x))^{N+1} \delta_N(x_{-n}(x), t) x'_n(x_t(x_{-n}(x))) + O((x_{-n}^2)^{N+1} \delta_N^2).
\]

Since it consists of exact trajectories that obey \( (1) \), the first term gives \( x_n(x_t(x_{-n}(x))) = x_t(x) \), while the second term involves

\[
x'_n(x_t(x_{-n}(x))) = \frac{1}{dx} x_n(x_t(x_{-n}(x))) = \frac{1}{dx} x_t(x).
\]

again using \( (1) \). Thus,

\[
x_t(x) - x_t^{(N, \text{approx})}(x) = (x_{-n}(x))^{N+1} \delta_N(x_{-n}(x), t) \frac{dx_t(x)/dx}{dx_{-n}(x)/dx} + O((x_{-n}^2)^{N+1} \delta_N^2).
\]

To proceed, we require that the unit step function is such that \( x_{-n} \) flows toward the fixed point at the origin for the point \( x \) under consideration (so we suppose \( |\lambda| > 1 \), but if not, just interchange \( x_1 \) and \( x_{-1} \). We also suppose that \( n \) has been chosen large compared to \( t \) so that

\[
x_t(x) \equiv \lambda^{t-n} e_n(x, t)
\]

is small, where \( e_n(x, t) = x + O(x^2) \). If these requirements are met, then

\[
x_t(x) - x_t^{(N, \text{approx})}(x) = \lambda^{-n(N+1)} e_n^{(N+1)}(x, 0) \delta_N(\lambda^{-n} e_n(x, 0), t) \frac{dx_t(x)/dx}{\lambda^{t-n} e_n(x, t)/dx}
\]

\[
+ O(\lambda^{-2n(N+1)} e_n^{2N+2} \delta_N^2).
\]
where \( \delta \eta (x, t) / dx = 1 + O(x) \). The result for the relative error is therefore of the form in the statement of the theorem, with
\[
r(x, t, \lambda, N, n) = \frac{\delta \eta (\lambda^{-n}\delta \eta (x, 0, t))}{\lambda^2 \delta \eta (x, t) / dx} \frac{dx_t (x) / dx}{x_r (x)} + O[\lambda^{-n(N+2)} \delta \eta^2] .
\]
(35)

For small \( x \), we have
\[
\delta \eta (\lambda^{-n}\delta \eta (x, 0, t)) = c_{N+1} (t, \lambda) + O(x),
\]
\[
\delta \eta (\lambda^{-n}\delta \eta (x, 0, t)) = \frac{dx_t (x) / dx}{x_r (x)} = \frac{1}{x} (1 + O(x)),
\]
(36)
and therefore
\[
r(x, t, \lambda, N, n) = \lambda^{-c_{N+1} (t, \lambda)} x^N + O(x^{N+1}),
\]
(37)
again as previously stated.

The form given in (35) enables analysis of the size of the error as a function of \( x \). Moreover, the form in (37) suggests an approximate scaling law for the error
\[
R_t (x, \lambda, N, n) \approx \lambda^N R_t (x, \lambda, N, n + 1),
\]
(38)
at least for \( x \) near the origin. It is interesting to check whether this is true for larger \( x \). In fact, it is.

To be specific, consider the case \( \lambda = 2 \) which can be solved in closed form to obtain
\[
\Psi (x) = -\frac{1}{2} \ln (1 - 2x), \quad \Psi^{-1} (x) = \frac{1}{2} (1 - e^{-2x}),
\]
(39)
\[
x_r (x) = \frac{1}{2} (1 - (1 - 2x)^2), \quad \frac{dx_r (x)}{dx} = 2^2 (1 - 2x)^{-1+x},
\]
(40)
\[
\delta \eta (x, t) = 2^{n-t-1} (1 - (1 - 2x)^{2^t}), \quad \frac{dx_t (x, t) / dx}{x_r (x)} = (1 - 2x)^{-1+2^{-x}},
\]
(41)
as well as
\[
\delta \eta (x, t) = \left( 2^n \frac{1}{2} (1 - (1 - 2x)^{2^t}) \right)^{N+1},
\]
(42)
\[
\delta \eta (x, t) = \left( \frac{(-2)^N}{(N+1)!} \prod_{j=0}^{N} (2^t - j) \right) \times \text{hypergeom}([1, N+1 - 2^t], [2 + N], 2x).
\]
(43)
This last result may be obtained by solving (2) recursively for \( x_1 (x) = 2x (1 - x) \)—not just by direct expansion of (40)—to obtain coefficients
\[
c_k (t) = \frac{(-2)^{n-k-1} (2^t - j)}{k!} \prod_{j=0}^{N} (2^t - j).
\]
(44)
These \( \lambda = 2 \) results are all well behaved enough for the steps in the proof of the theorem to be valid for \( x < 1/2 \). However, at the upper end of the interval, \( 0 \leq x \leq 1/2 \), some additional consideration is needed. Expression (33) is 1/2 at \( x = 1/2 \), independent of \( r \) and \( n \), and therefore not small. That is to say, at \( x = 1/2 \) the \( 2^{-nN} \) prefactor in \( R_t (x, \lambda, N, n) \) is
not present to suppress the error. On the other hand, the ratio \( (dx_t(x)/dx) / (d\delta(x,t)/dx) \) in (34), and in (35), always vanishes at \( x = 1/2 \), for any \( t \), so the leading contribution to the relative error is actually zero at that point. Thus, the upper end of the \( x \) interval does not pose a problem after all. In fact, as is evident in the graphs to follow, or by direct calculation, the maximum magnitude of the leading contribution to \( R \) occurs for \( x < 1/2 \), for which the \( 2^{-nN} \) prefactor is present.

Putting all this together for the \( \lambda = 2 \) logistic map, the leading approximation to the relative error is

\[
R_t(x, \lambda = 2, N = 5, n) \approx \delta_N \left( \frac{1}{2} (1 - (1 - 2x)^{2^{-n}}), t \right) \times \frac{2^{n-N} (1 - (1 - 2x)^{2^{-n}})^{N+1} \times (1 - 2x)^{2(1-2^{-n})}}{(1 - (1 - 2x)^2)}. \tag{45}
\]

For comparison purposes, we plot (in figure 6) this last expression for various \( N \) and \( n \), and selected \( t \), especially to check (38), for \( \lambda = 2 \). That scaling law is seen to hold fairly well, even for \( x \approx 1/2 \).

Finally, we note there is no discernible difference, to the accuracy of this last plot, between the leading approximations and the exact results for the relative error as computed from (40) and the 5-, 6- and 7-fold conjugations of \( x^{(5 \text{ approx})} (x) \). The largest of these differences, between the exact and the leading approximation of the relative error for \( N = 5, n = 5 \), is shown in figure 7. So the exact relative errors are essentially indistinguishable from their leading approximations, at least for \( \lambda = 2 \) and \( t = 1/2 \) and \( t = 3/4 \), and the relative error after 7 conjugations of the fifth-order series is always less than 3 parts in \( 10^{12} \) for these two values of \( t \). Other values of \( t \) taken from the unit interval \([0, 1]\) are similarly well approximated by the combined series and conjugation methods.

More or less the same results can be obtained for other values of \( \lambda \) as well, but only for one other case, namely \( \lambda = 4 \), is it possible to compare to exact, closed-form results.
5. Summary

By combining series approximations with functional conjugations, accurate representations of continuous functional iterates were obtained for selected rational, sine and logistic maps, and relative errors were estimated. These examples illustrate both the simplicity and the power of the methods. Although the rational and sine examples have more singular underpinnings (their Schröder functions have essential singularities) than the selected logistic map (whose Schröder function is analytic), nevertheless, the same methods work well for all three cases. This should be true for many other examples. Indeed, it would appear to be a relatively straightforward task to construct accurate approximations for the continuous iterations of all the classical special functions. Many of these, like the sine iterates depicted above, are expected to exhibit smooth, intuitive iterations that are easy to grasp, conceptually. Others, such as those occur for the logistic map when the parameter leads to chaotic behavior, are expected to lead to much more exotic iterations. Despite the long history of and voluminous literature on functional equations (see [1, 13] for history and bibliographies), the full landscape of features to be encountered is still largely unexplored, in our opinion. Perhaps the methods elucidated in this paper will help to continue that exploration.

Acknowledgments

We thank David Fairlie and Andrzej Veitia for discussions related to this research. We also thank an anonymous referee for asking a question which led us to obtain (23). This work was supported in part by NSF Award 0855386, and in part by the US Department of Energy, Division of High Energy Physics, under contract DE-AC02-06CH11357.

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