A double coset ansatz for integrability in AdS/CFT

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Abstract: We give a proof that the expected counting of strings attached to giant graviton branes in $AdS_5 \times S^5$, as constrained by the Gauss Law, matches the dimension spanned by the expected dual operators in the gauge theory. The counting of string-brane configurations is formulated as a graph counting problem, which can be expressed as the number of points on a double coset involving permutation groups. Fourier transformation on the double coset suggests an ansatz for the diagonalization of the one-loop dilatation operator in this sector of strings attached to giant graviton branes. The ansatz agrees with and extends recent results which have found the dynamics of open string excitations of giants to be given by harmonic oscillators. We prove that it provides the conjectured diagonalization leading to harmonic oscillators.

Keywords: D-branes, AdS-CFT Correspondence, 1/N Expansion

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1 Introduction

The AdS/CFT correspondence [1] gives an equivalence between $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory in four dimensions and ten dimensional string theory in $AdS_5 \times S^5$. This allows the construction of quantum states in the $\mathcal{N} = 4$ super-Yang-Mills [2, 3], which are dual to half-BPS rotating branes (giant gravitons [4–6]) in the string theory. The construction of states uses the representation theory of symmetric and unitary groups, and in particular inter-relations between them encoded in Schur-Weyl duality. States $\chi_R(Z)$ are associated with Young diagrams $R$. When $R$ has order one rows of length order $N$, they are dual to multiple giants consisting of large branes in the $AdS_5$ space. States associated with long columns are dual to branes large in the $S^5$ space. The construction of states corresponding to strings attached to giants was undertaken in [7–16]. A particularly simple limit arises when the lengths of the columns are separated by order $N$ and the action of
the one-loop dilatation operator simplifies significantly [17–22]. The diagonalization of the one-loop dilatation operator reveals a new integrable sector, with the appearance of harmonic oscillator spectra describing excitations of strings attached to the giants. In this new sector both planar and non-planar diagrams contribute at large $N$. Integrability in the planar limit was discovered in [23, 24] and is discussed in the recent review [25].

The half-BPS giants are constructed from products of traces of powers of one matrix $Z$. General multi-trace operators made from $n$ copies of $Z$ can be parameterized by permutations, since upper indices are some permutation of the lower indices. The Young diagram operators are obtained by summing over permutations, with weights given by characters in irreducible representations (irreps) $R$ of the permutations. If there are $n$ copies of $Z$ involved, then the permutations are in the symmetric group $S_n$ of order $n!$. Permutations related by conjugation give the same trace, so conjugacy classes of $S_n$ give a natural parameterization of traces. Going to a representation basis of Young diagrams gives a simple way to implement finite $N$ relations, by restricting the Young diagrams to have no more than $N$ rows [3].

The states for attached strings can be constructed by replacing some of the $Z$ matrices with another “impurity” matrix $Y$. Each matrix $Y$ generates a 1-bit string [7, 26–28] with angular momentum in the $y$-direction. If there are $n$ copies of $Z$ involved and $m$ of $Y$, then the traces are parameterized by permutations $S_{m+n}$, but there are equivalences under conjugation by elements in $S_n \times S_m$. Representation theory again gives a natural basis $\chi_{R,(r,s)\mu\nu}(Z,Y)$ for these conjugacy classes, which naturally incorporates finite $N$ effects. The labels include $R$, a Young diagram corresponding to an irrep of $S_{m+n}$ and a pair $(r,s)$ of Young diagrams for $S_n \times S_m$. There are additional multiplicity labels $\mu, \nu$, which each run over the multiplicity with which $(r,s)$ appears when the irrep $R$ of $S_{m+n}$ is decomposed under the action of the subgroup $S_n \times S_m$.

Recent progress in the study of perturbations $\chi_{R,(r,s)\mu\nu}(Z,Y)$ of giants $\chi_R(Z)$ has found that the calculation of the spectrum of the one-loop dilatation operator reduces to systems of harmonic oscillators. The harmonic oscillator dynamics consists of motion of $p$ particles along the real line, their coordinates being given by the lengths of the Young diagram $R$ (which has $p$ long rows or long columns) interacting via quadratic pair-wise interaction potentials. Arriving at this harmonic oscillator dynamics requires a diagonalization in the space of labels $(s, \mu, \nu)$. There are $U(1)^p$ conserved charges in the system which forces the Young diagram $r$ to be completely determined by $R$.

This diagonalization in the $(s, \mu, \nu)$ sector has been achieved in various special cases in earlier papers. The numerical studies of [18, 20] considered $m = 2, 3, 4$ $Y$s and demonstrated a linear spectrum. An analytic approach which solved the problem when $R$ has 2 rows or columns and $m$ is general was given in [21] for operators built from 2 scalars $Z, Y$ and in [17] for operators built using 3 scalars $Z, Y, X$. The general problem for $p$ rows or columns was studied in [22] using a numerical approach. A key idea was Schur-Weyl duality (also developed further in [29]) which enabled a simple evaluation of the action of the dilatation operator. For specific examples involving 3, 4 and 5 rows the diagonalization was performed numerically, demonstrating a concrete connection to the Gauss Law constraints discussed in section 2. Based on these numerical results, [22] conjectured the expression (4.31), where
integers \( n_{ij} \) giving the number of strings stretched between branes \( i \) and \( j \), appears in a factored form of the action of the one-loop dilatation operator. In this paper, we prove this expression (4.31).

In parallel developments, the problem of diagonalizing the free field inner product for multi-matrix operators, in a way that preserves global symmetries was done in [30, 31]. The group-theoretic construction of these diagonal bases relied on the notion of Fourier transform on a finite group. It also showed the intimate relation between the counting of operators, refined according to global symmetries, and the actual construction of these operators. Often the counting, when expressed in the right group-theoretic language, provides natural hints for the actual construction of these operators. This theme was developed further in [32] to study eighth-BPS operators at weak coupling. At leading order in large \( N \), these are just symmetrized traces made from three matrices \( X, Y, Z \). A systematic procedure to construct \( 1/N \)-corrected BPS operators was given using the permutation group algebra. This procedure found another use for the concept of “counting to construction,” whereby tools which give an elegant counting provide the necessary hints for the construction of the operators. Another relevant development appeared in [33], where permutation group methods for graph counting were reviewed and extended for various applications in counting Feynman graphs. Double cosets involving permutation groups played a significant role.

This paper starts with a general proof that the counting of states that can be constructed from restricted Schur operators matches the expectation from the Gauss Law. We focus on the case where there are \( p \) (order one) giants, large either in the \( AdS_5 \) or the \( S^5 \), which are distinct. They have attached strings made of one type of building block, namely one impurity \( Y \). We express the general counting of these brane-string configurations in terms of graphs, which we call “Gauss graphs.” In this formulation, it becomes apparent that the number of these Gauss graphs is equal to the number of points in a double coset. The counting of these points is shown to be equal to the expected counting of operators in the restricted Schur construction. This is the first step of the counting to construction philosophy applied to these brane-string configurations.

Fourier transformation applied to the double coset, gives a basis of functions, constructed from representation theory. This naturally leads to an explicit formula for the wavefunction in the \((s, \mu, \nu)\) sector. This wavefunction is labeled by elements of the double coset. The full wavefunction is labeled by \( R \) and an element \( \sigma \) of the double coset. The action of the one-loop dilatation operator takes the simple form (4.12).

Section 2 describes how the Gauss Law constraints, as applied to the string-brane configurations, lead to a graph counting problem. The result of this graph counting shows that the number of Gauss graphs is equal to the number of points on a double coset of permutation groups. The counting of the relevant restricted Schur polynomials is shown to match the size of this double coset, demonstrating that the physics of the Gauss Law for the compact branes correctly matches the construction of operators by associating impurity insertions to the attached strings as conjectured by [13]. The mathematical equivalence leading to the identity is related to Schur-Weyl duality, a theorem that has proved, in many instances, to be a central instrument of gauge-string duality [22, 29, 34–36].

Section 3 considers Fourier transformation on the double coset which appears in sec-
tion 2, and proposes Gauss graph operators in $\mathcal{N} = 4$ SYM that utilize the Fourier coefficients that arise in the expansion of the delta function on the double coset. These Gauss graph operators are labeled by elements of the double coset.

Section 4 proves that the one-loop dilatation operator acts diagonally in these double coset elements, to produce a differential operator acting on the $R$ label (4.31). The structure of this differential operator as an element of $U(p)$ has been previously recognized in [19] and is related to a system of $p$ particles in a line with 2-body harmonic oscillator interactions.

Finally, a comment on notation is in order. In what follows, we will explicitly indicate all sums over multiplicity labels and over representation labels. For state labels we use the usual summation convention, that is, repeated indices are summed.

2 Gauss Law: graphs and counting

Our goal in this section is to argue that the number of states of an excited system of separated giant gravitons\footnote{None of the giant worldvolumes are coincident.} is equal to the number of restricted Schur polynomials, labeled by Young diagrams with widely separated corners.

2.1 Restricted Schur polynomials

The restricted Schur polynomial is given by [13, 44]

\[
\chi_{R,(r,s)\mu\nu}(Z,Y) = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \text{Tr}_R \left( P_{R,(r,s)\mu\nu} \Gamma_R(\sigma) \right) \text{Tr}_{V^\otimes n \otimes Y^\otimes m}(\sigma Z^\otimes n Y^\otimes m)
\]

(2.1)

where $R$ is a Young diagram with $m + n$ boxes, equivalently a partition of $(m + n)$, which we express as $R \vdash (m + n)$; $r, s$ are Young diagrams with $m$ and $n$ boxes respectively, equivalently $r \vdash m, s \vdash n$. The operator $P_{R,(r,s)\mu\nu}$ is defined using an $S_n \times S_m$ irrep $(r, s)$ subduced by $R$, i.e. when the irrep $R$ of $S_{m+n}$ is decomposed into irreps of the $S_n \times S_m$ subgroup, $(r, s)$ is one of the irreps that appears in the decomposition. The labels $\mu, \nu$ run over the multiplicity with which $(r, s)$ appears in this restriction, a multiplicity which is equal to the Littlewood-Richardson number $g(r, s; R)$. A basis of states in the irrep of $S_{m+n}$ corresponding to Young diagram $R$ can be given in terms of standard tableaux which are labelings of the Young diagram with integers 1 to $m+n$ [22, 29, 37]. These integers in the standard tableaux are strictly decreasing down the columns and along the rows.

Among these Young tableaux, if we consider all those that have the integers 1 to $m$ entered in fixed locations, and the integers $\{m+1 \cdots m+n\}$ in arbitrary locations, we get complete irreps of $S_n$. A useful way to think about this approach to the reduction from $S_{m+n}$ to $S_n$ is to use partially labeled Young tableaux [22, 29] where the remaining $n$ boxes are left unlabelled. The unlabelled boxes determine a Young diagram $r$ of $S_n$. The different partial labelings of the remaining boxes with $\{1, \cdots, m\}$ form the basis of a vector space which span the states in irreps $s$ of $S_m$. Given the way $(n, m)$ appear in (2.1), we may think of the $Y$'s as “impurities” which are replacing $Z$'s and correspondingly we may think of the labelings $1, \cdots, m$ as specifying an order of removing “$Y$-boxes” from the Young diagram $R$ (with $m + n$ boxes) to leave a $Z$-Young diagram $r$.\footnote{None of the giant worldvolumes are coincident.}
When the $m$ $Y$ boxes are thus assembled into an irreps of $S_m$, an irrep $s$ can occur with some multiplicity. The labels $\mu_1, \mu_2$ run over this multiplicity. Concretely we can write

$$P_{R,(r,s)\mu_1 \mu_2} = 1_r \otimes |s \mu_1 ; i\rangle \langle s \mu_2 ; i|$$

(2.2)

where the $s$ state label $i$ is summed.

For restricted Schur polynomials corresponding to a system of $p$ giant gravitons we need $R$ to have $p$ rows which each have $O(N)$ boxes. Further, for a system of separated giant gravitons none of the $p$ rows have the same length. We will focus on operators for which the row lengths in $R$ differ by $O(N)$ boxes in the large $N$ limit. In this situation a concrete construction of the projectors (2.2) has been given in [22]. Each removed box is represented by a vector in a $p$-dimensional vector space $V_p$. Since we are removing $m$ boxes, the different ways of removing these span a vector space $V_p^\otimes m$. If the box $k$ is removed from row $i$, then the vector in the $k$'th tensor factor has all zero entries except for the $i$th entry which is a 1. Introduce the vector $\vec{m}$ whose components $m_i$ record the number of boxes removed from row $i$ of $R$ to produce $r$. The $m_i$ also correspond to the number of open strings emanating from the $i$th giant. Working with a basis of $V_p^\otimes m$, where the states have fixed $\vec{m}$, leads to the consideration of projectors

$$P_{\vec{m};R,(r,s)\mu_1 \mu_2} = 1_r \otimes |\vec{m} \ s \mu_1 ; i\rangle \langle \vec{m} \ s \mu_2 ; i|$$

(2.3)

which represents a refinement of (2.2). In [22] it was argued that when the corners of $R$ are well separated the vector $\vec{m}$ is conserved by the dilatation operator so we can consider the action of dilatation operator on projectors of fixed $\vec{m}$. These conserved $U(1)^p$ charges will be explained more in section 4.

To count the number of restricted Schur polynomials it is useful to recall some facts about $V_p^\otimes m$ using Schur-Weyl duality. We will do this in section 2.3 below.

In what follows, it proves convenient to work with rescaled restricted Schur polynomials that have unit two point function. We denote the normalized operators by $O_{R,(r,s)\mu_1 \mu_2}$.

### 2.2 States consistent with the Gauss Law

A giant graviton has a compact world volume so that the Gauss Law implies the total charge on the giant’s world volume must vanish. Since the string end points are charged, this gives a constraint on the possible open string configurations that are allowed: the number of strings emanating from the giant must equal the number of strings terminating on the giant.

Each open string configuration corresponds to a graph, where the vertices represent the brane and the directed links represent oriented strings. Group theoretic graph counting techniques will be useful in counting these graphs (for a review and application to Feynman graphs in a variety of field theory problems see [33] while some key earlier literature is [38]). To provide a systematic description of these open string configurations, we describe the graphs using some numbers. Consider a case where there are a total of $m$ strings and $p$ branes. A convenient way to obtain a combinatoric description of the graphs we consider is to divide each string into two halves and label each half. Since
Figure 1. Any open string configuration can be mapped to a labeled graph as shown. The two bold horizontal lines are identified. The graph itself determines a permutation, so each open string configuration is mapped to a permutation. For the graph shown the permutation in cycle notation is \( \sigma = (24)(536) \). As another example, the configuration in which all open strings loop back to the brane they start from is described by the identity permutation. The figure shows a configuration for a three giant system with seven open strings attached.

the strings are oriented we can label the outgoing ends with numbers \( \{1, \cdots, m\} \) and the ingoing ends with these same numbers. How the halves are joined is specified by a permutation \( \sigma \in S_m \). Let \( (m_1, m_2, \cdots, m_p) \) be the number of strings emanating from the distinct branes labeled from 1 to \( p \), so that \( m_1 + m_2 + \cdots + m_p = m \). By the Gauss law, the numbers of strings ending at these branes is also given by the same ordered sequence of integers \( (m_1, m_2, \cdots, m_p) \). We can choose the labels of the half-strings such that the ones emanating from the first brane are labeled \( \{1, 2, \cdots, m_1\} \), those emanating from the next set are labeled \( \{m_1 + 1, \cdots, m_1 + m_2\} \) etc. Likewise the half-strings incident on the first brane are labeled \( \{1, 2, \cdots, m_1\} \), those incident on the second brane are labeled \( \{m_1 + 1, \cdots, m_1 + m_2\} \) etc. The structure of the graph is encoded in the permutation \( \sigma \in S_m \) which describes how the \( m \) outgoing half-strings are tied to the \( m \) ingoing half-strings. There is some redundancy in this coding, because the \( m_i \) strings emanating from the \( i \)'th brane are indistinguishable, and likewise the \( m_i \) strings incident on the \( i \)'th brane are indistinguishable. For an example of this labeling, see the graph shown in figure 1. From the figure 1 it is immediately clear that permutations which differ only by swapping end points that connect to the same vertex do not describe distinct configurations. Relabeling of the outgoing half-strings, by permutations in their symmetry group \( \prod_i S_{m_i} \), acts on the permutation \( \sigma \) describing the graph by a left multiplication, while relabeling the ingoing half-strings, by permutations in their symmetry group \( \prod_i S_{m_i} \), multiplies \( \sigma \) on the right. The open string configurations are thus in one-to-one correspondence with elements of the double coset

\[
H \backslash S_m / H
\]

where the group \( H \) is \( S_{m_1} \times S_{m_2} \times \cdots \times S_{m_p} \). This subgroup of \( S_m \) will appear extensively in what follows. Each element of the double coset gives a distinct graph of the type shown in figure 1. We call these “Gauss graphs”.
Using the Burnside Lemma, the number of open string configurations $N_C$ (equivalently number of Gauss graphs) is

$$N_C = \frac{1}{|H|^2} \sum_{\alpha_1 \in H} \sum_{\alpha_2 \in H} \sum_{\sigma_1 \in S_m} \delta(\alpha_2 \sigma_1^{-1} \alpha_1^{-1} \sigma_1)$$  \hspace{1cm} (2.5)

The delta function $\delta(\alpha)$ on the group is defined as 1 if $\alpha$ is the identity and 0 otherwise. We can rewrite this as

$$N_C = \frac{1}{|H|^2} \sum_{s \vdash m} \sum_{\alpha_1 \in H} \sum_{\alpha_2 \in H} \chi_s(\alpha_2) \chi_s(\alpha_1)$$  \hspace{1cm} (2.6)

The expression $s \vdash m$ indicates that $s$ is being summed over partitions of $m$, which describe Young diagrams corresponding to irreps of $S_m$. The sums over $\alpha_1$ and $\alpha_2$ produce projection operators which project onto the trivial representation of $H$. Let $\mathcal{M}_s^H$ is the multiplicity of the one-dimensional representation of $H$ when the irreducible representation $s$ of $S_m$ is decomposed into representations of the subgroup $H$. The above formula is equivalent to

$$N_C = \sum_{s \vdash m} (\mathcal{M}_s^H)^2$$  \hspace{1cm} (2.7)

We can also count using cycle indices

$$N_C = N(Z(H) \ast Z(H)) = \sum_{q \vdash m} Z^2_q \text{Sym}(q)$$  \hspace{1cm} (2.8)

We know the cycle index of a product is the product of cycle indices so that

$$Z(H) = \prod_i Z(S_{m_i})$$  \hspace{1cm} (2.9)

### 2.3 Two ways to decompose $V_p \otimes^m$ and refine by $U(1)^p$ charges

We can write

$$V_p = \bigoplus_{i=1}^p V_i$$  \hspace{1cm} (2.10)

The vector space $V_i$ is a one-dimensional space, spanned by the eigenstate of $E_{ii}$ with eigenvalue one. If $v_i \in V_i$ then

$$E_{ii} v_j = \delta_{ij} v_i$$
$$E_{ij} v_k = \delta_{jk} v_i$$  \hspace{1cm} (2.11)

In the restricted Schur polynomial construction of [22] for long rows, a state in $V_i$ corresponds to a $Y$-box in the $i$’th row.

We have

$$V_p \otimes^m = \bigoplus_{s \vdash m; \ c_1(s) \leq p} V_s^{U(p)} \otimes V_s^{S_m}$$

$$= \bigoplus_{s \vdash m; \ c_1(s) \leq p} \bigoplus_{\vec{m} \vdash \overrightarrow{m}} \bigoplus_{i=1}^m V_{m_i}^{U(1)} \otimes V_{s \to \vec{m}}^{U(p) \to U(1)^p} \otimes V_s^{S_m}$$  \hspace{1cm} (2.12)
Here \( \vec{m} \) is giving the U(1) charges, with \( \sum_{i=1}^{p} m_i = m \). In the first line, we used Schur-Weyl duality. In the second, we decompose the U(\( p \)) irrep \( R \) into U(1)\(^p \) irreps, summing over all the irreps of this subgroup, labeled by \( \vec{m} \). \( V^{U(1)}_{m_i} \) is the one-dimensional irrep which transforms with charge \( i \) under the \( i \)’th U(1). In the restricted Schur construction for long rows, these are the numbers of boxes in the \( i \)’th row. Each set of U(1) charges \( \vec{m} \) will come with a multiplicity label. These multiplicity labels span a vector space \( V_{\vec{m}} \to U(1)^p \). The dimension of that vector space is the number of times the irrep \( \vec{m} \) of U(1)\(^p \) appears when the irrep \( s \) of U(\( p \)) is decomposed under the subgroup U(1)\(^p \). These are the Kotska numbers \([37]\) denoted by \( K_{s \vec{m}} \). Since the restricted Schur polynomials are labeled by a pair of multiplicity labels, the total number of restricted Schurs is the sum of the squares of the Kostka numbers

\[
\text{Number of restricted Schur polynomials} = \sum_{\vec{m}, s} (K_{s \vec{m}})^2
\]

(2.13)

The goal of this section is to prove the equality of the number of configurations consistent with the Gauss Law, given by (2.7), and the number of restricted Schurs, given by (2.13). This equality is a consequence of Schur-Weyl duality, which we now develop more fully.

We can develop the steps above at the level of a basis for \( V^{\otimes m} \). The reduction coefficients that will arise in the final step are the branching coefficients for irreps of U(\( p \)) into the irrep \( \vec{m} \) of \( H = U(1)^p \). Indeed we can write \(|I\rangle\) as a shorthand for the tensor basis \(|i_1, i_2, \ldots, i_p\rangle\). From Schur-Weyl duality, we know there is a change of basis to

\[
|I\rangle = \sum_{s, m_s, M_s} |s, M_s, m_s\rangle \langle s, M_s, m_s|I\rangle
\]

(2.14)

The label \( M_s \) is a state label for the U(\( p \)) irreps. It corresponds to semi-standard Young tableaux, as reviewed in appendix A of [22]. The label \( m_s \), a state label for the \( S_m \) irreps, can be described by standard Young tableaux. We can now decompose into U(1)\(^p \)

\[
|I\rangle = \sum_{\vec{m}, \nu} \sum_{s, m_s, M_s} C^{\vec{m}, \nu}_{M_s} |s, \vec{m}, m_s\rangle \langle s, M_s, m_s|I\rangle
\]

(2.15)

The coefficient \( C^{\vec{m}, \nu}_{M_s} \) gives the decomposition of a U(\( p \)) irrep into U(1)\(^p \) irreps, and contains a multiplicity label \( \nu \). This multiplicity label is labeling states in \( V^{U(1)^p} \to U(1)^p \).

There is an alternative way to decompose \( V^{\otimes m} \) into irreps of \( H = U(1)^p \) by using permutations in \( S_m \). Observe that when we choose charges \( \vec{m} \), then there are \( m_1 \) copies of \( v_1 \), \( m_2 \) copies of \( v_2 \) etc. One such state is

\[
|\vec{v}, \vec{m}\rangle \equiv |v_1^{\otimes m_1} \otimes v_2^{\otimes m_2} \otimes \cdots \otimes v_p^{\otimes m_p}\rangle
\]

(2.16)

A general state with these charges can be obtained by a permutation of the above.

\[
|v_{\sigma}\rangle \equiv \sigma |v_1^{\otimes m_1} \otimes v_2^{\otimes m_2} \otimes \cdots \otimes v_p^{\otimes m_p}\rangle
\]

(2.17)
where

\[ \sigma|v_{i_1} \otimes \cdots \otimes v_{i_p}\rangle = |v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)}\rangle \] (2.18)

Clearly not all \( \sigma \) give independent vectors

\[ |v_\sigma\rangle = |v_{\sigma\gamma}\rangle \] (2.19)

if \( \gamma \in H \).

We can write

\[ |v_\sigma\rangle = \frac{1}{|H|} \sum_{\gamma \in H} |v_{\sigma\gamma}\rangle \] (2.20)

In other words the states are in correspondence with \( S_m/H \). A convenient description of these states can be developed using representation theory, exploiting methods of [30, 31].

Look at the representation basis

\[ |v_{s,i,j}\rangle = \sum_{\sigma \in S_m} \Gamma_{ij}^{(s)}(\sigma)|v_\sigma\rangle \]

\[ = \frac{1}{|H|} \sum_{\sigma \in S_m} \sum_{\gamma \in H} \Gamma_{ij}^{(s)}(\sigma)|v_{\sigma\gamma}\rangle \]

\[ = \frac{1}{|H|} \sum_{\sigma \in S_m} \sum_{\gamma \in H} \Gamma_{ij}^{(s)}(\sigma)|v_\sigma\rangle \]

\[ = \frac{1}{|H|} \sum_{\sigma \in S_m} \sum_{\gamma \in H} \Gamma_{ik}^{(s)}(\sigma)\Gamma_{kj}^{(s)}(\gamma)|v_\sigma\rangle \]

\[ = \sum_{\sigma \in S_m} \sum_{\mu} \Gamma_{ik}^{(s)}(\sigma)B_{k\mu}^{s\rightarrow 1H}B_{j\mu}^{s\rightarrow 1H}|v_\sigma\rangle \] (2.21)

In the last line above we have decomposed the matrix elements of the \( H \) projector into products of branching coefficients using

\[ \frac{1}{|H|} \sum_{\gamma \in H} \Gamma_{ik}^{(s)}(\gamma) = \sum_{\mu} B_{k\mu}^{s\rightarrow 1H}B_{j\mu}^{s\rightarrow 1H} \] (2.22)

It is now natural to introduce

\[ |\vec{m}, s, \mu; i\rangle \equiv \sum_{j} B_{j\mu}^{s\rightarrow 1H}|v_{s,i,j}\rangle = \sum_{j} B_{j\mu}^{s\rightarrow 1H} \sum_{\sigma \in S_m} \Gamma_{ij}^{(s)}(\sigma)|v_\sigma\rangle \] (2.23)

In this construction, the \( \mu \) index is a multiplicity for reduction of \( S_m \) into \( H \) and the group-theoretic transformations \( B_{j\mu}^{s\rightarrow 1H} \) involved have to do with \( S_m \rightarrow H \). In the construction earlier we had \( C_{M,\nu}^{\vec{m}} \) associated to \( U(p) \rightarrow U(1)^p \), which are closer to Gelfand-Tsetlin bases used in [22].

We can now prove the equality of Kotska numbers (defined in terms of reduction multiplicities of \( U(p) \) to \( U(1)^p \)) and the branching multiplicity of \( S_m \rightarrow H \). The decomposition of \( V_{p \otimes m} \otimes m \) refined according to \( U(1)^p \) in the second way we have done it is

\[ V_{p \otimes m} = \bigoplus_{\vec{m}} \bigoplus_{s} V_{s}^{S_m} \otimes V_{s\rightarrow 1}^{S_m \rightarrow H(\vec{m})} \otimes_{i=1}^{p} V_{m_i}^{U(1)^p} \] (2.24)
Compare (2.12) to (2.24) to deduce

\[ \mathcal{M}^s_{1_H} \equiv |V^S_{s \to 1}^{H^{(\hat{m})}}| = |V^U_{s \to 1}^{U(1)^p}| \equiv K^s \hat{m} \]

which is the desired equality between Kotska numbers for \( U(p) \to U(1)^p \) and branching multiplicities for \( S_m \to H \). This completes the proof of the equality between (2.7) and (2.13).

### 3 Gauss operators

In the previous section we argued that Gauss graphs are described by elements of the double coset (2.4). In a number of problems related to the construction of gauge-invariant operators in the context of gauge-string duality, it is found that counting results for gauge invariant operators, once expressed in appropriate group theoretic language, lead naturally to methods for the explicit construction of these operators. This occurs notably in the study of eighth-BPS operators at zero Yang-Mills coupling, which involves diagonalizing the free field inner product for holomorphic gauge-invariant multi-matrix operators [30, 31, 39, 40].

The link between counting to construction often involves Fourier transforms on groups. This counting to construction philosophy was developed further in [32] in the context of eighth BPS operators at weak coupling. We may expect therefore that the double coset we have used to count the Gauss graphs should also play an important role in constructing the operators dual to the Gauss graph configurations. In this section we will construct a complete set of functions on the double coset, which give, as in usual Fourier analysis, an expansion for the delta function, in this case, on the double coset. This gives a natural guess for the operators dual to a given Gauss graph configuration. In the next section we will see that one loop dilatation operator acts diagonally on the operators labeled by these double coset elements which thus provide the diagonalization of the one-loop dilatation operator action on \( s, \mu, \nu \) labels of the restricted Schur operators \( O^{R,r,s,\mu,\nu} \). This gives an analytic confirmation of the numerical results obtained in [22] as well as a significant extension of these results to the general case.

The methods of representation theory used in this section have been used in the context of AdS/CFT for diagonalizing the free field inner product for multi-matrix operators [30, 31]. Recall that the matrix elements of irreducible representations \( s \vdash m \) give a basis of functions on \( S_m \). Given an object \( O^\tau \) determined by a permutation \( \tau \), we can form linear combinations \( O^s_{ij} \) labeled by an irrep \( s \) and state labels \( i, j \).

\[
O^s_{ij} = \sum_{\sigma \in S_m} \Gamma_{ij}^{(s)}(\sigma) O^\sigma
\]

This is an isomorphic description, which is not surprising given the familiar group theory identity \( m! = \sum_s d_s^2 \). Indeed these matrix elements provide a resolution of the delta-function on the group since

\[
\sum_s \frac{d_s}{m!} \Gamma_{ij}^{(s)}(\sigma) \Gamma_{ij}^{(s)}(\tau) = \delta(\sigma \tau^{-1})
\]

and indeed behave like Fourier coefficients.
Suppose we have some object determined by a permutation \( \tau \in S_m \), call it \( O_\tau \), but which is invariant under left and right multiplication of \( \tau \) by \( \gamma_1, \gamma_2 \) in the subgroup \( H \).

Here \( H = H(\bar{m}) = \prod_i S_m \).

We can write

\[
O_\tau = \frac{1}{|H|^2} \sum_{\gamma_1, \gamma_2 \in H} O_{\gamma_1 \gamma_2} \tau
= \frac{1}{|H|^2} \sum_s \frac{d_s}{m!} \sum_{\gamma_1, \gamma_2} \Gamma_{ij}^{(s)}(\gamma_1 \gamma_2) O_{ij}^s
= \frac{1}{|H|^2} \sum_s \frac{d_s}{m!} \sum_{\gamma_1, \gamma_2} \Gamma_{ik}^{(s)}(\gamma_1) \Gamma_{kl}^{(s)}(\tau) \Gamma_{ij}^{(s)}(\gamma_2) O_{ij}^s
= \sum_s \frac{d_s}{m!} \Gamma_{kl}^{(s)}(\tau) B_{k\mu_1}^{s-1} H B_{l\mu_2}^{s-1} H O_{ij}^s
= \sum_s \left( \frac{d_s}{m!} \Gamma_{kl}^{(s)}(\tau) B_{k\mu_1}^{s-1} H B_{l\mu_2}^{s-1} H \right) \left( \frac{d_s}{m!} B_{\bar{m}1}^{s-1} H B_{\bar{m}2}^{s-1} H O_{ij}^s \right)
= \sum_s \frac{d_s}{m!} \Gamma_{kl}^{(s)}(\tau) B_{k\mu_1}^{s-1} H B_{l\mu_2}^{s-1} H O_{\mu_1 \mu_2}^s
\]

(3.3)

We have introduced branching coefficients for the trivial irrep of \( H \) inside the representation \( s \) of \( S_m \). These \( B_{ij}^{s-1} H \) give the expansion of the \( \mu \)’th occurrence of the identity irrep of \( H \) when irrep \( s \) of \( S_m \) is decomposed into irreps of the subgroup \( H \), in terms of the states labeled \( i \) in \( s \). We also defined the linear combinations

\[
O_{\mu_1 \mu_2}^s = \sqrt{\frac{d_s}{m!} B_{\bar{m}1}^{s-1} H B_{\bar{m}2}^{s-1} H} O_{ij}^s
\]

(3.4)

labeled by the irrep label \( s \) and a multiplicity label for the decomposition to the identity irrep of \( H \). These provide the representation theoretic basis for the double coset in accordance with (2.7).

We now show that the group-theoretic coefficients

\[
C_{\mu_1 \mu_2}^s(\tau) = |H| \sqrt{\frac{d_s}{m!} \Gamma_{kl}^{(s)}(\tau) B_{k\mu_1}^{s-1} H B_{l\mu_2}^{s-1} H}
\]

(3.5)

provide an orthogonal transformation between double coset elements \( \sigma \) and \( O_{\mu_1 \mu_2}^s \). The introduction of the normalization \(|H|\) is for convenience. We can show that

\[
C_{\mu_1 \mu_2}^s(\tau) C_{\mu_1 \mu_2}^s(\sigma) = |H|^2 \sum_s \frac{d_s}{m!} B_{k\mu_1}^{s-1} H B_{l\mu_2}^{s-1} H \Gamma_{kl}^{(s)}(\tau) B_{k\mu_1}^{s-1} H B_{l\mu_2}^{s-1} H \Gamma_{pq}^{(s)}(\sigma)
= \sum_s \sum_{\gamma_1, \gamma_2} \frac{d_s}{m!} \Gamma_{kp}^{(s)}(\gamma_1) \Gamma_{lq}^{(s)}(\gamma_2) \Gamma_{kl}^{(s)}(\tau) \Gamma_{pq}^{(s)}(\sigma)
= \sum_s \sum_{\gamma_1, \gamma_2} \frac{d_s}{m!} \chi_s(\gamma_1 \sigma \gamma_2^{-1} \tau^{-1})
= \sum_{\gamma_1, \gamma_2} \delta(\gamma_1 \sigma \gamma_2^{-1} \tau^{-1})
\]

(3.6)
This expresses orthogonality since the right hand side is a delta function on the double coset, and shows that a representation theoretic way of counting the number of elements in the double coset is

$$\sum_s (M_{1H}^s)^2$$  \hspace{1cm} (3.7)

in agreement with (2.7), which we previously obtained by applying the Burnside Lemma.

In view of this discussion, a very natural form for the operators dual to Gauss configuration $\sigma$, up to normalization, is

$$O_{R,r}(\sigma) = \frac{|H|}{\sqrt{m!}} \sum_{j,k} \sum_{s+m, \mu_1, \mu_2} \sqrt{d_s} \Gamma^{(s)}_{jk}(\sigma) B_{j\mu_1}^{s\rightarrow 1H} B_{k\mu_2}^{s\rightarrow 1H} O_{R,(r,s)\mu_1\mu_2}$$  \hspace{1cm} (3.8)

The overall factor has been chosen to ensure a convenient normalization. Indeed, the two point function of Gauss graph operators is

$$\langle O_{R,r}(\sigma_1) O_{T,t}^\dagger(\sigma_2) \rangle = \frac{|H|^2}{m!} \sum_{s+u, \mu_1\mu_2, \nu_1\nu_2} \sqrt{d_s d_u} \Gamma^{(s)}_{jk}(\sigma_1) B_{j\mu_1}^{s\rightarrow 1H} B_{k\mu_2}^{s\rightarrow 1H} \times \Gamma^{(u)}_{lm}(\sigma_2) B_{l\nu_1}^{u\rightarrow 1H} B_{m\nu_2}^{u\rightarrow 1H} \langle O_{R,(r,s)\mu_1\mu_2} O_{T,(t,u)\nu_1\nu_2}^\dagger \rangle$$  \hspace{1cm} (3.9)

Now, use (see appendix A)

$$\langle O_{R,(r,s)\mu_1\mu_2} O_{T,(t,u)\nu_1\nu_2}^\dagger \rangle = \delta_{RT} \delta_{rt} \delta_{su} \delta_{\mu_1\nu_1} \delta_{\mu_2\nu_2}$$  \hspace{1cm} (3.10)

to obtain

$$\langle O_{R,(r,s)\mu_1\mu_2} O_{T,(t,u)\nu_1\nu_2}^\dagger \rangle = \delta_{RT} \delta_{rt} \delta_{su} \delta_{\mu_1\nu_1} \delta_{\mu_2\nu_2}$$  \hspace{1cm} (3.10)

The right hand side is the delta function on the double coset, setting $\sigma_1 = \sigma_2$. Thus if $\sigma_1$ and $\sigma_2$ represent the same double coset element, the two point function is one and if they represent distinct coset elements, it vanishes.

4 Dilatation operator

In this section we will review the exact action of the one loop dilatation operator on restricted Schur polynomials [20]. We then review how this action simplifies when acting on restricted Schurs with long rows and well separated corners [21, 22]. Using this simplified action we prove that the Gauss graph operators diagonalize the dilatation operator’s $Y$ labels.
4.1 Action of the dilatation operator

When acting on restricted Schurs the one loop dilatation operator takes the form [20]

\[
DO_{R,(r,s)\mu_1\mu_2}(Z,Y) = \sum_{T,(t,u)\nu_1\nu_2} N_{R,(r,s)\mu_1\mu_2;T,(t,u)\nu_1\nu_2} O_{T,(t,u)\nu_1\nu_2}(Z,Y)
\]

where

\[
N_{R,(r,s)\mu_1\mu_2;T,(t,u)\nu_1\nu_2} = -\frac{2g_{YM}}{d_R d_T d_u(n+m)\sqrt{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_u}} \times \text{Tr}\left(\left[\Gamma^{(R)}((1,m+1)), P_{R;\mu_1\mu_2} \right] I_{R^T T'} \left[\Gamma^{(T)}((1,m+1)), P_{R;\nu_1\nu_2} \right] I_{T'R} \right)
\]

The trace above is over the direct sum representation \( R \oplus T \) where \( R, T \) are Young diagrams with \( m + n \) boxes. \( R' \) is one of the irreps subduced from \( R \) upon restricting to the \( S_{n+m-1} \) subgroup of \( S_{n+m} \) obtained by keeping only permutations that obey \( \sigma(1) = 1 \). \( T' \) is subduced by \( T \) in the same way. \( I_{R^T T'} \) is an intertwining map (see appendix D of [22] for details on its properties) from irrep \( R' \) to irrep \( T' \). It is only non-zero if \( R' \) and \( T' \) have the same shape. Thus, to get a non-zero result we need \( R = T = R \) and \( T \) must differ at most by the placement of a single box. \( d_a \) denotes the dimension of symmetric group irrep \( a \). \( f_S \) is the product of the factors in Young diagram \( S \) and \( \text{hooks}_S \) is the product of the hook lengths of Young diagram \( S \). Finally, \( c_{RR'} \) is the factor of the corner box that must be removed from \( R \) to obtain \( R' \).

When acting on Schurs labeled by Young diagrams \( R \) with long rows and well separated corners, it is possible to compute \( N_{R,(r,s)\mu_1\mu_2;T,(t,u)\nu_1\nu_2} \) explicitly [21, 22]. We will now review the relevant steps in this evaluation. We consider \( n \gg m \) and assume that \( R \) has \( p \) long rows. We hold \( p \) fixed and order 1 as we take \( N \to \infty \). In this limit the difference in the lengths of the corresponding rows of \( R \) and \( r \) can be neglected.

In the construction of the projectors we removed \( m \) boxes from \( R \) to produce \( r \) with each box represented by a vector in \( V_p \). To evaluate the action of the dilatation operator, it is convenient to remove \( m + 1 \) boxes again associating each with a vector in \( V_p \). This allows a straight forward evaluation of the action of \( \Gamma^{(R)}((1,m+1)) \) and \( \Gamma^{(T)}((1,m+1)) \).

As mentioned above, \( R \) and \( T \) agree after removing a single box. The \( R' \) and \( T' \) subspaces are obtained by removing this single box from \( R \) and \( T \) respectively. To produce a map from \( R' \) to \( T' \) we simply need a map from the vector corresponding to the box removed from \( R \) to the vector corresponding to the box removed from \( T \). This map is \( E^{(1)}_{ij} \) if we remove the box from row \( i \) of \( R \) and row \( j \) of \( T \). Using the identification

\[
(1,m+1) = \text{Tr}(E^{(1)} E^{(m+1)})
\]

We easily find, for example (repeated indices are summed)

\[
E^{(1)}_{ji} \Gamma^{(R)}((1,m+1)) = E^{(1)}_{ji} E^{(1)}_{kl} E^{(m+1)}_{lk} = E^{(1)}_{ji} E^{(m+1)}_{li}
\]

An easy way to understand this result is to recognize that \( E^{(1)}_{ji} = E^{(1)}_{ji} E^{(m+1)}_{li} \) so that \( \Gamma^{(R)}((1,m+1)) \) simply swapped the column labels. This simple action is a direct consequence of the simplified action of the symmetric group when the corners of \( R \) are well
separated. After performing these manipulations we are left with a trace over products of $E_{ij}$s acting in slots 1 and $m + 1$ and the operators $P_{R; (r,s); u \nu_1}^{(r,s); u \nu_1}$ and $P_{R; (t,u); v \nu_2}^{(t,u); v \nu_2}$. The trace thus factorizes into a trace over irrep $r$ and a trace over $V_p^{\otimes m}$. After performing these traces we have

$$DO_{R,(r,s)\mu_1\mu_2} = -g_{YM}^2 \sum_{u \nu_1 \nu_2} \sum_{i < j} \delta_{\vec{m}, \vec{n}} M^{(ij)}_{s \mu_1 \mu_2; u \nu_1 \nu_2} \Delta_{ij} O_{R,(r,u)\nu_1 \nu_2}$$  \hspace{1cm} (4.4)$$

where $\Delta_{ij}$ acts only on the Young diagrams $R, r$ and

$$M^{(ij)}_{s \mu_1 \mu_2; u \nu_1 \nu_2} = \frac{m}{\sqrt{d_s d_u}} \left( \langle \vec{m}, s, \mu_2 ; a | E^{(1)}_{ij} | \vec{m}, u, \nu_2 ; b \rangle + \langle \vec{m}, s, \mu_2 ; a | E^{(1)}_{jj} | \vec{m}, u, \nu_2 ; b \rangle \right)$$  \hspace{1cm} (4.5)$$

where $a$ and $b$ are summed. $a$ labels states in irrep $s$ and $b$ labels states in irrep $t$. The action of operator $\Delta_{ij}$ is most easily split up into three terms

$$\Delta_{ij} = \Delta_{ij}^+ + \Delta_{ij}^0 + \Delta_{ij}^-$$  \hspace{1cm} (4.6)$$

Denote the row lengths of $r$ by $r_i$. The Young diagram $r_{ij}^+$ is obtained by removing a box from row $j$ and adding it to row $i$. The Young diagram $r_{ij}^-$ is obtained by removing a box from row $i$ and adding it to row $j$. In terms of these Young diagrams we have

$$\Delta_{ij}^0 O_{R,(r,s)\mu_1\mu_2} = -(N + r_i + r_j) O_{R,(r,s)\mu_1\mu_2}$$  \hspace{1cm} (4.7)$$

$$\Delta_{ij}^+ O_{R,(r,s)\mu_1\mu_2} = \sqrt{(N + r_i)(N + r_j)} O_{R,(r,s)\mu_1\mu_2}$$  \hspace{1cm} (4.8)$$

$$\Delta_{ij}^- O_{R,(r,s)\mu_1\mu_2} = \sqrt{(N + r_i)(N + r_j)} O_{R,(r,s)\mu_1\mu_2}$$  \hspace{1cm} (4.9)$$

Notice that $\Delta_{ij}$ acts on $r$ i.e. on $Z_s$ and $M^{(ij)}_{s \mu_1 \mu_2; u \nu_1 \nu_2}$ on $Y_s$. Note that it is a consequence of the fact that $R$ and $r$ change in exactly the same way that $\vec{m}$ is preserved by the dilatation operator. As a matrix $\Delta_{ij}$ has matrix elements

$$\Delta_{ij}^{R; T,t} = \sqrt{(N + r_i)(N + r_j)} (\delta_{T,R_{ij}} \delta_{t,r_{ij}^+} + \delta_{T,R_{ij}} \delta_{t,r_{ij}^-}) - (N + r_i + r_j) \delta_{T,R} \delta_{t,r}$$  \hspace{1cm} (4.10)$$

In terms of these matrix elements we can write (4.4) as

$$DO_{R,(r,s)\mu_1\mu_2} = -g_{YM}^2 \sum_{T, (r,u) \nu_1 \nu_2} \sum_{i < j} \delta_{\vec{m}, \vec{n}} M^{(ij)}_{s \mu_1 \mu_2; u \nu_1 \nu_2} \Delta_{ij}^{R; T,t} O_{T,(r,u)\nu_1 \nu_2}$$  \hspace{1cm} (4.11)$$

### 4.2 Diagonalization

Given the factorized dilatation operator (4.4), we can diagonalize on the $s \mu_1 \mu_2; u \nu_1 \nu_2$ and the $R, r; T, t$ labels separately. In this section we are mainly concerned with describing the result of diagonalizing on the $s \mu_1 \mu_2; u \nu_1 \nu_2$ labels. This result was obtained analytically for two rows. For more than two rows the results are numerical, motivating a conjecture we describe in this section. In the next section we will provide an analytic treatment valid for any number of rows, thereby proving the conjecture.
After diagonalization on the $s_{µ1µ2}; w\land ν2$ labels one obtains a collection of decoupled eigenproblems in the $R, r; T, t$ labels. There is one eigenproblem for each Gauss graph that can be drawn and the structure of each problem is naturally read from the Gauss graph. To obtain the problem associated to a particular Gauss graph, count the number $n_{ij}$ of strings (of either orientation) stretching between branes $i$ and $j$. For example, the Gauss graph of figure 1 has $n_{12} = 1$, $n_{13} = 3$ and $n_{23} = 1$. The action of the dilatation operator on the Gauss graph operator is

$$DO_{R,r}(σ) = -g^2_{YM} \sum_{i<j} n_{ij}(σ) Δ_{ij} O_{R,r}(σ)$$ (4.12)

To obtain anomalous dimensions one needs to solve an eigenproblem on the $R, r$ labels. We have anticipated the fact that it is the Gauss graph operators defined above that accomplish this diagonalization. This is one of the key results of this article and will be proved in the next section. Towards this end, it is useful to develop a formula for $n_{ij}$ in terms of $σ$.

For $i < j$, let $n^+_{ij}$ be the number of strings going from $i$ to $j$ and $n^-_{ij}$ the number from $j$ to $i$. Since $n_{ij}$ is orientation blind we have $n_{ij} = n^+_{ij} + n^-_{ij}$. If $k$ is in the range $\{m_1 + \cdots + m_i - 1 + 1, \cdots, m_1 + \cdots + m_j\}$, then $n^+_{ij}$ is the number of $σ(i)$ lying in the range $\{m_1 + \cdots + m_j - 1 + 1, \cdots, m_1 + \cdots + m_j\}$.

$$n^+_{ij}(σ) = \sum_{k=m_1+\cdots+m_i-1+1}^{m_1+\cdots+m_j} \sum_{l=m_1+\cdots+m_j-1+1}^{m_1+\cdots+m_j} \delta(σ(k), l)$$ (4.13)

Equivalently if we say that $S_1, S_2, \cdots, S_p$ are, respectively, the first $m_1$ positive integers, the next $m_2$, and so on, then

$$n^+_{ij}(σ) = \sum_{k \in S_i} \sum_{l \in S_j} \delta(σ(k), l)$$ (4.14)

Similarly the number of strings going the other way is

$$n^-_{ij}(σ) = \sum_{k \in S_i} \sum_{l \in S_j} \delta(σ(l), k)$$ (4.15)

### 4.3 Action on Gauss graph operators

Having defined the kets $|\vec{m}, s, µ; i\rangle$, we will now think about the bras $⟨\vec{m}, u, ν; j|$.

The following definition

$$⟨\vec{m}, u, ν; j| = \frac{d_u}{m!H} \sum_{τ ∈ S_m} ⟨\vec{m}| n_{ij} τ^{-1}(n) B_{\nu v}^{i-1} H$$ (4.16)

will give the correctly normalized relation

$$⟨\vec{m}, u, ν; j|\vec{m}, s, µ; i\rangle = δ_{\vec{m}\vec{m}} δ_{uu} δ_{jj} δ_{νν}$$ (4.17)
Figure 2. The number of open strings emanating on each brane is described by \( \vec{m} \). The permutation \( \sigma \) specifies how these strings are to be terminated on the branes.

To see this calculate

\[
\langle \vec{m}, u, \nu; j | \vec{n}, s, \mu; i \rangle = \frac{d_u}{m!|H|} \sum_{\tau, \sigma} \langle \vec{v}, \vec{m} | \tau^{-1} \Gamma_{jk}^{(u)}(\tau) B_{k\nu}^{u^{-1}H} \Gamma_{il}^{(s)}(\sigma) B_{l\mu}^{s^{-1}H} | \vec{v}, \vec{n} \rangle
\]

\[
= \frac{d_u}{m!|H|} \sum_{\tau, \sigma} \Gamma^{(u)}_{jk}(\tau) B_{k\nu}^{u^{-1}H} \Gamma^{(s)}_{il}(\sigma) B_{l\mu}^{s^{-1}H} \langle \vec{v}, \vec{m} | \tau^{-1} \sigma | \vec{v}, \vec{n} \rangle
\]

\[
= \frac{d_u}{m!|H|} \sum_{\tau, \sigma} \sum_{\gamma \in H} \delta(\tau^{-1} \sigma \gamma) \delta_{\vec{m}, \vec{n}} \Gamma^{(u)}_{jk}(\tau) B_{k\nu}^{u^{-1}H} \Gamma^{(s)}_{il}(\sigma) B_{l\mu}^{s^{-1}H} \delta_{\vec{m}, \vec{n}}
\] (4.18)

The \( |v_i > \) in \( V_p \) are normalized as \( \langle v_i | v_j \rangle = \delta_{ij} \), so that the \( |\vec{v}, \vec{m} > \) defined in (2.16) obey

\[
\langle \vec{v}, \vec{m} | \sigma | \vec{v}, \vec{n} \rangle = \delta_{\vec{m}, \vec{n}} \sum_{\gamma \in H} \delta(\sigma \gamma)
\] (4.19)

A permutation outside \( H \) would lead to overlaps \( \langle v_i | v_j \rangle \) for \( i \neq j \), which is zero. Thus, we have

\[
\langle \vec{m}, u, \nu; j | \vec{n}, s, \mu; i \rangle = \frac{1}{m!|H|} \frac{d_u}{m!|H|} \sum_{\tau, \sigma} \sum_{\gamma \in H} \Gamma^{(u)}_{jk}(\sigma) B_{k\nu}^{u^{-1}H} \Gamma^{(s)}_{il}(\sigma) B_{l\mu}^{s^{-1}H} \delta_{\vec{m}, \vec{n}}
\]

\[
= \frac{1}{|H|} \delta_{su} \delta_{ji} \delta_{jl} \sum_{\gamma \in H} \Gamma^{(u)}_{jk}(\gamma) B_{k\nu}^{u^{-1}H} B_{l\mu}^{s^{-1}H} \delta_{\vec{m}, \vec{n}}
\]

\[
= \frac{1}{|H|} \delta_{ij} \delta_{su} \sum_{\gamma \in H} \Gamma^{(u)}_{lk}(\gamma) B_{k\nu}^{u^{-1}H} B_{l\mu}^{s^{-1}H} \delta_{\vec{m}, \vec{n}}
\]

\[
= \delta_{ij} \delta_{su} B_{la}^{u^{-1}H} B_{k\nu}^{u^{-1}H} B_{k\nu}^{u^{-1}H} B_{l\mu}^{s^{-1}H} \delta_{\vec{m}, \vec{n}}
\]

\[
= \delta_{ij} \delta_{su} \delta_{\mu\nu} \delta_{\vec{m}, \vec{n}}
\] (4.20)

which completes the demonstration.
We will now calculate the matrix elements of $D$ in the Gauss graph basis, showing how the matrix $M_{ij|\mu_1\mu_2;\nu_1\nu_2}$ appears.

\[
\langle T^i_z(\sigma_2)DO_{R_T,T}(\theta) \rangle = \frac{|H|^2}{m!} \sum_{s,u-m \mu_1 \mu_2 \nu_1 \nu_2} \sum_{s,u-m \mu_1 \mu_2 \nu_1 \nu_2} \sqrt{d_s d_u} \Gamma_{jk}^{(s)}(\sigma_2) B_{j\mu_1}^{s \rightarrow 1H} B_{k\mu_2}^{s \rightarrow 1H} \Gamma_{l\mu_1}^{(u)}(\sigma_1) B_{l\nu_1}^{u \rightarrow 1H} B_{m\nu_2}^{u \rightarrow 1H} \\
\times \langle T^i_z(\sigma_2)DO_{R_T,T}(\theta) \rangle
\]

\[
= -\frac{|H|^2}{m!} \sum_{s,u-m \mu_1 \mu_2 \nu_1 \nu_2} \sum_{s,u-m \mu_1 \mu_2 \nu_1 \nu_2} \sqrt{d_s d_u} \Gamma_{jk}^{(s)}(\sigma_2) B_{j\mu_1}^{s \rightarrow 1H} B_{k\mu_2}^{s \rightarrow 1H} \Gamma_{l\mu_1}^{(u)}(\sigma_1) B_{l\nu_1}^{u \rightarrow 1H} B_{m\nu_2}^{u \rightarrow 1H} \\
\times \sum_{i<j} g^2_{YM} M_{ij|\mu_1\mu_2;\nu_1\nu_2} \Delta_{ij}^{R_T,T} m \langle \tilde{m}, s, \mu_2; a| E^{(1)}_{ii} | \tilde{m}, u, \nu_2; b \rangle \langle \tilde{m}, u, \nu_1; b| E^{(1)}_{jj} | \tilde{m}, s, \mu_1; a \rangle \\
+ \langle \tilde{m}, s, \mu_2; a| E^{(1)}_{jj} | \tilde{m}, u, \nu_2; b \rangle \langle \tilde{m}, u, \nu_1; b| E^{(1)}_{ii} | \tilde{m}, s, \mu_1; a \rangle \quad (4.21)
\]

Focus on the evaluation of

\[
\sum_{s,u-m \mu_1 \mu_2 \nu_1 \nu_2} \sum_{s,u-m \mu_1 \mu_2 \nu_1 \nu_2} \Gamma_{jk}^{(s)}(\sigma_2) B_{j\mu_1}^{s \rightarrow 1H} B_{k\mu_2}^{s \rightarrow 1H} \Gamma_{l\mu_1}^{(u)}(\sigma_1) B_{l\nu_1}^{u \rightarrow 1H} B_{m\nu_2}^{u \rightarrow 1H} \\
\times m \langle\tilde{m}, s, \mu_2; a| E^{(1)}_{ii} | \tilde{m}, u, \nu_2; b \rangle \langle \tilde{m}, u, \nu_1; b| E^{(1)}_{jj} | \tilde{m}, s, \mu_1; a \rangle \\
+ \langle \tilde{m}, s, \mu_2; a| E^{(1)}_{jj} | \tilde{m}, u, \nu_2; b \rangle \langle \tilde{m}, u, \nu_1; b| E^{(1)}_{ii} | \tilde{m}, s, \mu_1; a \rangle \quad (4.22)
\]

To evaluate (4.22) start by considering

\[
\sum_u \langle \tilde{m}, u, \nu_2; b \rangle \langle \tilde{m}, u, \nu_1; b| E^{u \rightarrow 1H}_{l\nu_1} B^{u \rightarrow 1H}_{m\nu_2} \Gamma_{l\mu_1}^{(u)}(\sigma_2) \\
= \frac{1}{|H| m!} \sum_u \sum_{\sigma, \tau \in S_m} d_u \langle \tilde{m}, u, \nu_2; b \rangle \langle \tilde{m}, u, \nu_1; b| E^{u \rightarrow 1H}_{l\nu_1} B^{u \rightarrow 1H}_{m\nu_2} \Gamma_{l\mu_1}^{(u)}(\sigma_2) \\
= \frac{1}{|H| m} \sum_u \langle \tilde{m}, u, \nu_2; b \rangle \langle \tilde{m}, u, \nu_1; b| E^{u \rightarrow 1H}_{l\nu_1} B^{u \rightarrow 1H}_{m\nu_2} \Gamma_{l\mu_1}^{(u)}(\sigma_2) \\
\]

\[
= \frac{|H|^2}{m!} \sum_u \sum_{\gamma_1, \gamma_2 \in H} \delta(\gamma_1 \gamma_2 - 1) |\tilde{m}, u, \nu_2; b \rangle \langle \tilde{m}, u, \nu_1; b| E^{u \rightarrow 1H}_{l\nu_1} B^{u \rightarrow 1H}_{m\nu_2} \Gamma_{l\mu_1}^{(u)}(\sigma_2) \\
= \frac{1}{|H|^3} \sum_{\sigma, \tau \in S_m} \sum_{\gamma_1, \gamma_2 \in H} \delta(\gamma_1 \gamma_2 - 1) |\tilde{m}, u, \nu_2; b \rangle \langle \tilde{m}, u, \nu_1; b| E^{u \rightarrow 1H}_{l\nu_1} B^{u \rightarrow 1H}_{m\nu_2} \Gamma_{l\mu_1}^{(u)}(\sigma_2) \\
\]
Using this twice we get for the first term in (4.22)

\[ T_1 = \frac{m}{|H|^6} \sum_{\gamma_1 \cdots \gamma_4} \sum_{\sigma, \beta, \tau} \langle v_\beta | E_i^{(1)} | v_\alpha \rangle \langle v_\tau | E_j^{(1)} | v_\alpha \rangle \delta(\gamma_1 \sigma_2^{-1} \gamma_2^{-1} \tau^{-1} \sigma) \delta(\gamma_3 \sigma_1 \gamma_4^{-1} \beta^{-1} \alpha) \]

\[ = \frac{m}{|H|^6} \sum_{\beta, \tau} \langle \bar{v} \beta^{-1} E_{ij} \tau \gamma_2 \sigma_2 \bar{\gamma}^{-1} | \bar{v} \rangle \langle \bar{v} \tau^{-1} E_{jj} \beta \gamma_4 \sigma_4 \gamma_3^{-1} | \bar{v} \rangle \]

\[ = \frac{m}{|H|^4} \sum_{\beta, \tau} \langle \bar{v} | E_i^{\beta^{-1}(1)} \beta^{-1} \gamma_2 \sigma_2 \bar{\gamma}^{-1} | \bar{v} \rangle \langle \bar{v} | E_j^{\tau^{-1}(1)} \tau^{-1} \beta \gamma_4 \sigma_4 \gamma_3^{-1} | \bar{v} \rangle \quad (4.24) \]

We dropped the $\gamma_1, \gamma_3$ and and picked up $|H|^2$ using invariance of $|\bar{v}\rangle$ under $H$.

Now consider $E^{\sigma(1)} | \bar{v}\rangle$ (or equivalently $\langle \bar{v} | E^{\sigma^{-1}(1)}$). This gives $|\bar{v}\rangle$ if $\sigma(1)$ belongs to the set $S_i$ of integers between $m_1 + m_2 + \cdots + m_{i-1} + 1$ and $m_1 + m_2 + \cdots + m_i$ both inclusive.

The above expression will be zero unless $\beta^{-1}(1) \in S_i$ and $\tau^{-1}(i) \in S_j$. We also note that

\[ \langle \bar{v} \sigma | \bar{v} \rangle = \sum_{\gamma \in H} \delta(\sigma \gamma) \quad (4.25) \]

So we can write

\[ T_1 = \frac{m}{|H|^4} \sum_{\beta, \tau} \sum_{\gamma} \delta(\beta^{-1} \tau \gamma_2 \sigma_2 \gamma_3) \delta(\tau^{-1} \beta \gamma_4 \sigma_1^{-1} \gamma_1) \]

\[ \sum_{k \in S_i} \delta(\beta^{-1}(1), k) \sum_{l \in S_j} \delta(\tau^{-1}(1), l) \quad (4.26) \]

The delta functions in the second line imply

\[ \beta^{-1} \tau(l) = k, \]

\[ \tau^{-1} \beta(k) = l; \quad \text{for } l \in S_j, k \in S_i \quad (4.27) \]

We can replace the two delta functions in the last line with a delta function constraining $\beta^{-1} \tau$, i.e. $\sum_{l \in S_j} \sum_{k \in S_i} \delta(\beta^{-1} \tau(k), l)$. This can be done in the current context, because the rest of the expression only depends on $\beta^{-1} \tau$. If we replace $\beta \rightarrow \beta^{-1} \alpha; \tau^{-1} \rightarrow \tau^{-1} \alpha$ with $\alpha \in \mathbb{Z}_m$, this amounts to replacing the $1$ by $\alpha(1)$. By summing over $\alpha$ in $\mathbb{Z}_m$ we can replace the $1$ by a sum over $i$ from $1$ to $m$ (normalized by $1/m$). So we are lead to write

\[ T_1 = \frac{1}{|H|^4} \sum_{\beta, \tau} \sum_{\gamma \in H} \delta(\beta^{-1} \tau \gamma_2 \sigma_2 \gamma_3) \delta(\tau^{-1} \beta \gamma_4 \sigma_1^{-1} \gamma_1) \sum_{l \in S_j} \sum_{k \in S_i} \delta(\beta^{-1} \tau(k), l) \]

\[ = \frac{m!}{|H|^4} \sum_{\beta} \sum_{\gamma \in H} \delta(\beta^{-1} \gamma_2 \sigma_2 \gamma_3) \delta(\beta \gamma_4 \sigma_1^{-1} \gamma_1) \sum_{l \in S_j} \sum_{k \in S_i} \delta(\beta^{-1}(1), l) \]

\[ = \frac{m!}{|H|^4} \sum_{\beta} \sum_{\gamma \in H} \delta(\beta^{-1} \gamma_2 \sigma_2 \gamma_3) \delta(\beta \gamma_4 \sigma_1^{-1} \gamma_1) \ n_{ij}^+(\beta^{-1}) \]

\[ = \frac{m!}{|H|^4} \sum_{\gamma_1, \gamma_2} \delta(\gamma_1 \sigma_2 \gamma_2^{-1}) \ n_{ij}^+(\sigma_1) \]

\[ = \frac{m!}{|H|^4} \sum_{\gamma_1, \gamma_2} \delta(\gamma_1 \sigma_2 \gamma_2^{-1}) \ n_{ij}^+(\sigma_1) \quad (4.28) \]
We have recognized the definition of $n_{ij}^+(\sigma)$ and the fact that it is invariant under left and right multiplication by $H$. In the second term of (4.22) we have $i, j$ exchanged and $n_{ij}^+ = n_{ji}^-$. Combining the two terms we would get

$$m! |H|^2 \sum_{\gamma_1, \gamma_2} \delta(\gamma_1 \sigma_2 \gamma_2^{-1}) \ n_{ij}(\sigma_1)$$

Plugging this into (4.21) we find

$$\langle O_{T,s}(\sigma_2)O_{R,r}(\sigma_1) \rangle = -g_{YM}^2 \sum_{i<j} n_{ij}(\sigma_1) \Delta_{ij}^{R,r;T,s}$$

which proves that the Gauss graph operators indeed diagonalize the impurity labels. We can also write (4.30) as

$$DO_{R,r}(\sigma_1) = -g_{YM}^2 \sum_{i<j} n_{ij}(\sigma_1) \Delta_{ij} O_{R,r}(\sigma_1)$$

This last eigenproblem has been considered in detail in [19]. Taking a large $N$ continuum limit, the above discrete problem becomes a differential equation, equivalent to a set of decoupled oscillators. The same spectrum is obtained by solving the discrete problem or the large $N$ continuum differential equation.

The discussion above has focused on the case that $R$ has $p$ long rows. These operators are dual to giant gravitons wrapping an $S^3 \subset AdS_5$. The case that $R$ has $p$ long columns, which is dual to giant gravitons wrapping an $S^3 \subset S_5$, is easily obtained from the above results. The $\Delta_{ij}$ for this case is obtained by replacing the $r_i \rightarrow -r_i$ and $r_j \rightarrow -r_j$ in (4.7), (4.8) and (4.9). The final result (4.31) is unchanged when written in terms of the new $\Delta_{ij}$.

5 Outlook

There are a number of natural ways in which this work can be extended. We have limited ourselves to restricted Schur polynomials labeled by Young diagrams $R$ that have well separated corners, corresponding to giant gravitons that are well separated in spacetime. We conjecture that the permutation $\sigma$ specifying the brane-string configuration obeying the Gauss Law, and appearing in the operators $O_{R,r}(\sigma)$ will continue to provide a diagonalization of the dilatation operator to all orders in the loop expansion in this distant corners limit. The action will be diagonal in $\sigma$ but there will be a mixing of the $R$ label which involves the movement of more boxes at higher orders. Proving (or disproving) this conjecture would give important information on the structure of higher loop corrections to the dilatation operator.

Another fascinating generalization is to consider the case where some of the branes are coincident, in which case some of the row lengths of $R$ will be equal. This case is particularly interesting as it corresponds to non-abelian brane worldvolume theories. A first step would be to give a general account of the counting of restricted Schurs in terms of the Gauss Law for these non-abelian brane worldvolumes. For initial studies in this
direction see [13]. In line with the counting to construction philosophy we have followed in this article, a general proof of this counting should contain the hints of the corresponding operator construction. Implementing this will require some work in making the action of the one-loop dilatation operator more explicit.

The counting of BPS states in [41] was expressed in terms of bit strings $Y^k$, built using $k$-bits at a time. In this article we have focused on a description of strings by assembling single bit strings. The precise relation between these two descriptions will be interesting to clarify.

We have considered operators built from $Zs$ and dilutely doped with a single type of impurity $Y$. The one loop dilatation operator has same form in the $sl(2)$ sector [42], where we dope with covariant derivatives, so our double coset ansatz works in that case too. In general we could build operators with impurities that include more types of scalars together with covariant derivatives and fermions. This would allow a complete description of the one loop, large $N$ but non-planar dilatation operator. The one loop planar dilatation operator is integrable [43]. Is this complete one loop non-planar dilatation operator integrable in this sector of perturbations around well-separated half-BPS giants? Is there a double coset ansatz that can be used to diagonalize the problem?

An interesting concept we have found very useful in this paper is what we may call the counting to construction philosophy. This is the expectation that once we have proved that some framework based on groups (e.g permutation groups) or algebras (e.g Brauer algebras) correctly counts the quantum states, expected from gauge-string duality for example, then the same framework will contain the information for constructing the states, often via tools related to Fourier transformation on the groups or algebras along, frequently, with Schur-Weyl duality. The link between enumeration and construction is also an active theme of research in areas such as the mathematical classification of molecular structures using double cosets, see for example [45]. This theme also appears in the categorification of numeric to homological invariants in the context of knot theory [46] and branes [47], with interesting links to Schur-Weyl duality and representation theory [48]. It is clear that there is much to be understood about the interplay of this theme with gauge-string duality.

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A Conventions

In this appendix we will spell out our conventions for $(\chi_{R,(r,s)\mu\nu}(Z,Y))^\dagger$. It is straightforward to check that

$$\text{Tr} \left( \sigma Z^\otimes n Y^\otimes m \right)^\dagger = \text{Tr} \left( \sigma^{-1} Z^\dagger^\otimes n Y^\dagger^\otimes m \right)$$
Using this we find
\[
\chi_{R,(r,s)\mu\nu}(Z,Y)^{\dagger} = \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \text{Tr}_R \left( P_{R \rightarrow (r,s)\mu\nu} \Gamma_R(\sigma) \right) \text{Tr}_{V_N^{\otimes n+m}} \left( \sigma^{-1} Z^{\dagger} \otimes nY^{\dagger} \otimes m \right)
\]
\[
= \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \text{Tr}_R \left( P_{R \rightarrow (r,s)\mu\nu} \Gamma_R(\sigma^{-1}) \right) \text{Tr}_{V_N^{\otimes n+m}} \left( \sigma Z^{\dagger} \otimes nY^{\dagger} \otimes m \right)
\]
\[
= \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \text{Tr}_R \left( P_{R \rightarrow (r,s)\mu\nu} \Gamma_R(\sigma)^T \right) \text{Tr}_{V_N^{\otimes n+m}} \left( \sigma Z^{\dagger} \otimes nY^{\dagger} \otimes m \right)
\]
\[
= \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \text{Tr}_R \left( P_{R \rightarrow (r,s)\nu\mu} \Gamma_R(\sigma) \right) \text{Tr}_{V_N^{\otimes n+m}} \left( \sigma Z^{\dagger} \otimes nY^{\dagger} \otimes m \right)
\]
\[
= \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \text{Tr}_R \left( P_{R \rightarrow (r,s)\mu\nu} \Gamma_R(\sigma) \right) \text{Tr}_{V_N^{\otimes n+m}} \left( \sigma Z^{\dagger} \otimes nY^{\dagger} \otimes m \right) \quad (A.1)
\]

Thus, following the original derivation of the two point function \([44]\) we find
\[
\langle \chi_{T,(t,u)\alpha\beta} \chi_{R,(r,s)\mu\nu}(Z,Y)^{\dagger} \rangle = n!m! \text{Tr}(P_{T \rightarrow (t,u)\alpha\beta} P_{R \rightarrow (r,s)\nu\mu})
\]
\[
= \frac{\text{hooks}_R}{\text{hooks}_t \text{hooks}_u} \delta_{TR} \delta_{rt} \delta_{us} \delta_{\alpha\beta} \delta_{\nu\mu} \quad (A.2)
\]

This is the convention followed in this article, and it matches \([44]\). Our motivation for adopting this convention, is that we get the natural orthogonality (3.11) between Gauss graph operators. In \([22]\) a different definition
\[
\chi_{R,(r,s)\mu\nu}(Z,Y)^{\dagger} \equiv \frac{1}{n!m!} \sum_{\sigma \in S_{n+m}} \text{Tr}_R \left( P_{R \rightarrow (r,s)\mu\nu} \Gamma_R(\sigma) \right) \text{Tr}_{V_N^{\otimes n+m}} \left( \sigma Z^{\dagger} \otimes nY^{\dagger} \otimes m \right) \quad (A.3)
\]
was used. This implies
\[
\langle \chi_{T,(t,u)\alpha\beta} \chi_{R,(r,s)\mu\nu}(Z,Y)^{\dagger} \rangle = \frac{\text{hooks}_R}{\text{hooks}_t \text{hooks}_u} \delta_{TR} \delta_{rt} \delta_{us} \delta_{\alpha\beta} \delta_{\nu\mu} \quad (A.4)
\]

This convention looks natural if one interprets multiplicity labels as Chan-Paton factors and is the motivation for adopting this convention in \([22]\).

B Counting operators

In this appendix we will give some examples of the counting arguments constructed in section 2.

First we deal with the Gauss graph counting problem. To approach this numerically we have found it easiest to implement (2.5) in GAP. We have counted the number $N_C$ of open string configurations for the stated $\vec{m}$ shown below.
| Total Number of Strings (m) | Valencies ($\vec{m} = \{m_i\}$) | Configurations $N_C$ |
|---------------------------|---------------------------------|---------------------|
| 4                         | $\{2,1,1\}$                    | 7                   |
| 5                         | $\{3,1,1\}$                    | 7                   |
| 5                         | $\{4,1\}$                      | 2                   |
| 5                         | $\{3,2\}$                      | 3                   |
| 5                         | $\{2,2,1\}$                    | 11                  |
| 3                         | $\{1,1,1\}$                    | 6                   |
| 8                         | $\{4,2,1,1\}$                  | 68                  |

To count the number of restricted Schur polynomials, according to (2.13) we should sum the squares of the Kostka numbers. The Kostka numbers are easily evaluated with the help, for example, of the Symmetrica program [49]. We will write the Kostka numbers as

$$\begin{align*}
\otimes \otimes \otimes \otimes &= \otimes \otimes + 2 \otimes \otimes + \otimes + \otimes \\
1^2 + 2^2 + 1^2 + 1^2 &= 7
\end{align*}$$

The left hand side of this equation determines $\vec{m} = (2,1,1)$. The right hand side shows the non zero irreps $s$ that we can obtain. The coefficient of each term is the Kostka number. Thus, for example $K_{(2,1,1)}(2,1,1) = 2$. For each line of the table above it is now a simple matter to check that we reproduce $N_C$:

$$\begin{align*}
\otimes \otimes \otimes &= \otimes \otimes + 2 \otimes \otimes + \otimes + \otimes \\
1^2 + 2^2 + 1^2 + 1^2 &= 7
\end{align*}$$

$$\begin{align*}
\otimes \otimes \otimes &= \otimes \otimes + 2 \otimes \otimes + \otimes + \otimes \\
1^2 + 2^2 + 1^2 + 1^2 &= 7
\end{align*}$$

$$\begin{align*}
\otimes \otimes \otimes &= \otimes \otimes + 2 \otimes \otimes + \otimes + \otimes \\
1^2 + 1^2 &= 2
\end{align*}$$

$$\begin{align*}
\otimes \otimes \otimes &= \otimes \otimes + 2 \otimes \otimes + \otimes + \otimes \\
1^2 + 1^2 + 1^2 &= 3
\end{align*}$$

$$\begin{align*}
\otimes \otimes \otimes &= \otimes \otimes + 2 \otimes \otimes + \otimes + \otimes \\
1^2 + 2^2 + 1^2 + 2^2 + 1^2 &= 11
\end{align*}$$

$$\begin{align*}
\otimes \otimes \otimes &= \otimes \otimes + 2 \otimes \otimes + \otimes + \otimes \\
1^2 + 2^2 + 1^2 &= 6
\end{align*}$$
C Examples of the Gauss graph operators

In this section we will use (3.8) to explicitly construct some examples of Gauss operators. We have two goals in mind: to demonstrate how the formula (3.8) is used and to make contact with operators already constructed in the literature.

C.1 BPS operators

The BPS operator is associated with the open string configuration that has all strings looping back to the brane they start from. This corresponds to taking the identity for $\sigma$. In this case

$$\Gamma_{jk}^{(s)}(1) = \delta_{jk}$$

so that

$$O(1) = \frac{|H|}{m!} \sum_{s \vdash m} \sum_{\mu_1, \mu_2} \sqrt{d_{s}} B_{j \mu_1}^{s \rightarrow 1 \mu} B_{j \mu_2}^{s \rightarrow 1 \mu} O_{R,(r,s)\mu_1 \mu_2} = \frac{|H|}{m!} \sum_{s \vdash m} \sum_{\mu} \sqrt{d_{s}} O_{R,(r,s)\mu \mu}$$

This is exactly what [20] has found based on numerical studies.

C.2 Two row operators

In this case we have no multiplicity label so that, up to a normalization factor of $\frac{|H|}{m!}$ which we drop, we have

$$O(\sigma) = \sum_{s \vdash m} \sqrt{d_{s}} \Gamma^{(s)}_{jk}(\sigma) B_{j \mu_1}^{s \rightarrow 1 \mu} B_{k \mu_2}^{s \rightarrow 1 \mu} O_{R,(r,s)}$$

$$= \prod_{i} \frac{1}{m_i!} \sum_{s \vdash m} \sum_{\alpha \in \Pi_i, S_{m_i}} \sqrt{d_{s}} \Gamma^{(s)}_{jk}(\sigma) \Gamma^{(s)}_{k \mu}(\alpha) O_{R,(r,s)}$$

$$= \prod_{i} \frac{1}{m_i!} \sum_{s \vdash m} \sum_{\alpha \in \Pi_i, S_{m_i}} \sqrt{d_{s}} \chi_{s}(\sigma \alpha) O_{R,(r,s)} \quad (C.1)$$

where $\chi_{s}(\sigma)$ is the character of $\sigma \in S_{m}$ in irrep $s$.

Consider $m = 3$, $m_1 = 1$ and $m_2 = 2$. There are two possible Gauss operators. For $\sigma = 1$ we obtain the BPS operator as discussed above. The other configuration, which is
non-BPS, is obtained for \( \sigma = (23) \). In this case
\[
\frac{1}{2!} \text{Tr} \left( \Gamma^{((12)(23))} \right) = 1
\]
\[
\frac{1}{2!} \text{Tr} \left( \Gamma^{((12)(23))} \right) = -\frac{1}{2}
\]
so that
\[
O((23)) = O_{R,\{r\},(23)} - \frac{1}{\sqrt{2}} O_{R,\{r\},(23)}
\]
which is in perfect agreement with section 5.1 of [20]. Now consider \( m = 4 \) and \( m_1 = 2 \), \( m_2 = 2 \). There are three possible Gauss operators. For \( \sigma = 1 \) we again obtain the BPS operator. For \( \sigma = (23) \), using
\[
\frac{1}{2!} \text{Tr} \left( \Gamma^{((12)(34))} \right) = 1
\]
we find
\[
O((23)) = O_{R,\{r\},(23)} - \frac{1}{\sqrt{2}} O_{R,\{r\},(23)}
\]
The last configuration is obtained for \( \sigma = (14)(23) \). In this case
\[
\frac{1}{2!} \text{Tr} \left( \Gamma^{((12)(34))} \right) = 1
\]
\[
\frac{1}{2!} \text{Tr} \left( \Gamma^{((12)(34))} \right) = -1
\]
so that
\[
O((23)) = O_{R,\{r\},(23)} + \sqrt{2} O_{R,\{r\},(23)} - \sqrt{3} O_{R,\{r\},(23)}
\]
These results are in complete agreement with section 5.2 of [20].

Next consider removing \( m = 8 \) boxes with \( m_1 = m_2 = 4 \). The relevant dilatation operator equation, given in equation (4.3) of [21], is
\[
DO_{j,j^3}(b_0, b_1) = g_{\text{YM}}^2 \left[ \frac{1}{2} \left( m - \frac{(m+2)(j^3)^2}{j(j+1)} \right) \Delta O_{j,j^3}(b_0, b_1)
\right.
\]
\[
+ \sqrt{(m+2j+4)(m-2j+2)} \frac{(j+j^3+1)(j-j^3+1)}{2(j+1)} \Delta O_{j+1,j^3}(b_0, b_1)
\]
\[
+ \sqrt{(m+2j+2)(m-2j+2)} \frac{(j+j^3)(j-j^3)}{2j} \Delta O_{j-1,j^3}(b_0, b_1)
\]
(C.2)
where
\[ \Delta O(b_0, b_1) = \sqrt{(N + b_0)(N + b_0 + b_1)}(O(b_0 + 1, b_1 - 2) + O(b_0 - 1, b_1 + 2)) \]
\[- (2N + 2b_0 + b_1)O(b_0, b_1). \] 
(C.3)

The case we study has operators with \( j^3 = 0 \) and \( j = 0, 1, 2, 3, 4 \) in the notation of [21]. The above operator is easily diagonalized numerically giving eigenvalues \( 0, 2, 4, 6, 8 \). It is simple to test that our Gauss graph operators
\[ O(\sigma) = \sum_{s \vdash 8} \sum_{\alpha \in S_4 \times S_4} \sqrt{d_s} \chi_s(\alpha \sigma) O_{R,(r,s)} \]
are eigenfunctions with the following eigenvalues
\[ \sigma = 1 \leftrightarrow 0 \]
\[ \sigma = (45) \leftrightarrow 2 \]
\[ \sigma = (45)(63) \leftrightarrow 4 \]
\[ \sigma = (45)(63)(72) \leftrightarrow 6 \]
\[ \sigma = (45)(63)(72)(81) \leftrightarrow 8 \] 
(C.4)

### C.3 Three row operators

Now consider a three row example for which we remove three boxes \( m = 3 \) and \( m_1 = m_2 = m_3 = 1 \). In this case our Gauss graph operators are
\[ O(\sigma) = \sum_{s \vdash 3} \sum_{\mu_1, \mu_2} \sqrt{d_s} \chi_s(\sigma) B_{j \mu_1}^{s-1} B_{k \mu_2}^{s-1} O_{R,(r,s)} \]

The subgroup \( H = S_1 \times S_1 \times S_1 \) has a single element, the identity. The branching coefficients are thus
\[ B = 1 \]
\[ B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]
\[ B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

Note that we have simplified the notation by dropping from the superscript the specification \( s \rightarrow 1_H \), taking it as understood that these are reduction coefficients of the trivial irrep of \( H \) appearing in the \( s \) specified by the Young diagram. Each \( \sigma \in S_3 \) gives a different Gauss graph operator. There are six possible open string configurations that are given in figure 3 and they correspond as:
\[ \sigma = 1 \leftrightarrow (a) \]
\[ \sigma = (12) \leftrightarrow (b) \]
\[ \sigma = (13) \leftrightarrow (c) \]
\[ \sigma = (23) \leftrightarrow (d) \]
\[ \sigma = (123), (321) \leftrightarrow (e), (f) \] 
(C.5)
Figure 3. Gauss graphs for three strings and three branes.

The last line corresponds to a degeneracy because there are two configurations differing only in orientation of the open strings. The above correspondence is from comparing to the numerical results of [22] and there is again a perfect agreement. In particular, one can easily test that the above Gauss operators are simultaneous eigenfunctions of the matrices $M^{12}$, $M^{23}$, $M^{13}$ of section 3.2 of [22].

C.4 Four row operators

Four rows with $m = 4$ and $m_1 = 1 = m_2 = m_3 = m_4$ is also easy to compare. The group $H = S_1 \times S_1 \times S_1 \times S_1$ again just has a single element (the identity) so that branching coefficients are again trivial to compute

\[
B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}
\]

There are 24 operators in this case and here too we have exact, complete, agreement with the numerical results of [22].

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