Secret-Key Generation in Many-to-One Networks: An Integrated Game-Theoretic and Information-Theoretic Approach

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Abstract

This paper considers secret-key generation between several agents and a base station that observe independent and identically distributed realizations of correlated random variables. Each agent wishes to generate the longest possible individual key with the base station by means of public communication. All keys must be jointly kept secret from all external entities. Also each agent has a level of security clearance; this setup requires that keys generated by agents at a given level must be kept secret from the agents at a strictly superior level. In this many-to-one secret-key generation setting, it can be shown that agents with the same level of security clearance can take advantage of a collective protocol to increase the sum-rate of their generated keys. However, when each agent is only interested in maximizing its own secret-key rate, agents may be unwilling to participate in a collective protocol. Furthermore, when such a collective protocol is employed, how to fairly allocate individual key rates arises as a valid issue. This paper studies this tension between cooperation and self-interest with a game-theoretic treatment. The work establishes that, for each level of security clearance, cooperation is in the best interest of all individualistic agents and that there exists individual secret-key rate allocations that incentivize the agents to follow the protocol. Additionally, an explicit low-complexity coding scheme based on polar codes and hash functions that achieves such allocations is proposed.

Index Terms
Multiterminal secret-key generation, strong secrecy, coalitional game theory, hash functions, polar codes

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I. Introduction

Multiterminal settings subject to limited total resources bring about issues pertaining to competition, and fairness among users. Such issues are typically studied by means of game theory; see, for instance, references [2]–[4] which deal with the Gaussian multiple access channel, and [5]–[10] which deal with interference channels.

In this paper, we study a multiterminal secret-key generation problem that involves selfish users, and propose a solution based on cooperative game theory, more specifically, based on forming coalitions. We refer to [11]–[13] for an introduction to coalitional game theory, and to [14] for a review of some of its applications to telecommunications. Our setting can be explained as follows. Each agent wishes to generate an individual key of maximal length with the base station to securely and individually report information, using a one-time pad for instance. There are many such agents and a single base station. The generated keys must be jointly kept secret from all external entities. Additionally, we consider multiple levels of security clearance exist in the system; each agent has a pre-defined security clearance level, and it is required that keys generated by agents at a given level must be kept secret from the agents at a strictly superior level. We consider a source model for secret-key generation [15], [16], i.e., the agents and the base station observe independent and identically distributed (i.i.d.) realizations of correlated random variables, and can communicate over a public noiseless channel. It can be shown that when agents are altruistic, the agents that share the same security clearance level increase the sum of their key lengths by agreeing to participate in a joint protocol, in contrast to operating separately on their own. However, in the case when each agent is interested in maximizing its own key length only, as we consider, there exists a tension between cooperation and the sole interest of a given agent. Moreover, assuming that the agents collaborate to maximize the sum of their key lengths, another issue is to determine a fair allocation of individual key lengths, so that no agent has any incentive to deviate from the protocol.

Note that when there is a single level of security clearance, when the agents are not assumed selfish, and when fairness issues are ignored, the secret-key generation model we consider reduces to the one studied in [17] and is related to multiple-key generation in a network with trusted helpers [18]–[20]. Note also that once the secret-key generation protocol is done, the subsequent
transmission to the base station of messages protected by means of a one-time pad with the generated secret keys can be viewed as a noiseless multiple access wiretap channel [21].

Our contributions are three-fold. (i) We formally introduce an integrated game-theoretic and information-theoretic formulation of the problem in Section II. We then derive properties of the defined game and propose rate allocations as candidates for fair solutions in Section III. (ii) By adding the constraint that the agents are selfish, compared to the model in [17], we are able to derive a secret-key capacity region for an arbitrary number of agents, whereas without this consideration, the secret-key capacity region of the model is unknown, even for a single level of security clearance [17], [19]. (iii) We provide in Section IV an explicit and low-complexity coding scheme based on polar codes for source coding and hash functions to implement the solutions proposed in Section II. Note that few explicit coding schemes have been proposed for multiterminal secret-key generation problems [22]–[24], however, the coding schemes in these references do not seem to easily apply to our setting.

For clarity of exposition we first treat the case of a single level of security clearance in Sections III and IV and then extend our result to an arbitrary number of levels in Section V. We provide concluding remarks in Section VI.

II. PROBLEM STATEMENT

We define an auxiliary secret-key generation model with no selfishness constraint in Section II-A. In Section II-B, we explain our objective using the model of Section II-A to which selfishness constraints are added, and describe the integrated game-theoretic and information-theoretic problem formulation. Additional definitions that will be useful in our analysis are presented in Section II-C.

Notation: For any \( a \in \mathbb{N}^* \), define \([1, a]\) \(= [1, a] \cap \mathbb{N}\). For a given set \( S \), we let \( 2^S \) denote the power set of \( S \). Finally, \( \times \) denotes the Cartesian product.

A. An auxiliary secret-key generation model (without selfishness constraints)

Let \( Q \in \mathbb{N}^* \) and define \( Q \triangleq [1, Q] \). For \( q \in Q \), let \( L_q \in \mathbb{N}^* \) and let \( L_q \) be a set of \( L_q \) agents. In the following, we consider \( Q \) sets \((L_q)_{q \in Q}\) of agents and one base station. We also use the notation \( L_Q \triangleq \bigcup_{q \in Q} L_q \) to denote all the agents in the \( Q \) sets.
Fig. 1. Many-to-one secret-key generation setting. (b) provides details of the setting described in (a) at a given level $q \in \mathcal{Q}$. 
Definition of the source model. Define \( X_{\mathcal{L}_Q} \) as the Cartesian product of \( \sum_{q=1}^{Q} L_q \) finite alphabets \( X_l, l \in \mathcal{L}_Q \). Consider a discrete memoryless source (DMS) \( (X_{\mathcal{L}_Q} \times X_0, p_{X_{\mathcal{L}_Q}X_0}) \), where \( X_0 \) is a finite alphabet and \( X_{\mathcal{L}_Q} \triangleq (X_l)_{l \in \mathcal{L}_Q} \). For \( l \in \mathcal{L}_Q \), Agent \( l \) observes the component \( X_l \) of the DMS, and the base station observes the component \( X_0 \). The source is assumed to follow the following Markov chain: for any \( S, T \subset \mathcal{L}_Q \) such that \( S \cap T = \emptyset \),

\[
X_S - X_0 - X_T.
\]

(1)

Note that such a source model has already been considered in [23]–[25]. Assuming all random variables are binary, an instance of this model is \( X_l \triangleq X_0 \oplus B_l, \forall l \in \mathcal{L}_Q \), where the \( B_l \)'s are independent Bernoulli random variables and \( \oplus \) is the modulo-two addition. The source’s statistics are assumed known to all parties, and communication is allowed over an authenticated noiseless public channel.

Description of the objectives for the agents. The goal of Agent \( l \in \mathcal{L}_Q \) is to generate an individual secret-key with the base station. The index \( q \in \mathcal{Q} \) is meant to describe different sets of agents that do not have the same security constraints. In particular, we require that for any \( q \in \mathcal{Q} \), the secret keys generated by the agents in \( \mathcal{L}_q \) are secret from the agents in \( \bigcup_{i \in [q+1, Q]} \mathcal{L}_i \) but need not to be secret from the agents in \( \bigcup_{i \in [1, q]} \mathcal{L}_i \). One can interpret it in terms of levels of security clearance, where \( \mathcal{L}_q \), \( q \in \mathcal{Q} \), represents a set of agents that share the same level of security clearance, and for \( q' < q \), \( \mathcal{L}_{q'} \), represents another set of agents with a higher security clearance than \( \mathcal{L}_q \). We formalize the definition of a secret-key generation protocol for this setting, which is depicted in Figure 1.

Definition 1. Let \( q \in \mathcal{Q} \). For \( l \in \mathcal{L}_q \), let \( K_l \) be a key alphabet of size \( 2^{NR_l} \) and define \( K_{\mathcal{L}_q} \) as the Cartesian product of \( K_l, l \in \mathcal{L}_q \). A \( (2^{NR_l})_{l \in \mathcal{L}_q}, N \) secret-key generation strategy for the agents in \( \mathcal{L}_q \) is as follows.

1) The base station observes \( X_0^N \) and Agent \( l, l \in \mathcal{L}_q \), observes \( X_l^N \).

2) The agents in \( \mathcal{L}_q \) and the base station communicate, possibly interactively, over the public channel. The global public communication between the agents in \( \mathcal{L}_q \) and the base station is denoted by \( A_q \in A_q \), for some discrete alphabet \( A_q \).
3) Agent \( l, \ l \in \mathcal{L}_q \), computes \( K_l(X_l^N, A_q) \in \mathcal{K}_l \).

4) The base station computes \( \hat{K}_l(X_0^N, A_q) \in \mathcal{K}_l, \ l \in \mathcal{L}_q \).

In the following, we use the notation \( \mathcal{K}_{\mathcal{L}_q} \equiv (K_l)_{l \in \mathcal{L}_q} \).

**Definition 2.** Let \( q \in \mathcal{Q} \). A secret-key rate tuple \( (R_l)_{l \in \mathcal{L}_q} \) is achievable if there exists a sequence of \( ((2^{NR_l})_{l \in \mathcal{L}_q}, N) \) secret-key generation strategies for the agents in \( \mathcal{L}_q \) such that

\[
\lim_{N \to \infty} P(\hat{K}_{\mathcal{L}_q} \neq K_{\mathcal{L}_q}) = 0 \text{ (Reliability)},
\]

\[
\lim_{N \to \infty} I(K_{\mathcal{L}_q}; A_q, (X_{\mathcal{L}_i}^N)_{i \in [q+1,Q]}) = 0 \text{ (Collective Strong Secrecy)},
\]

\[
\lim_{N \to \infty} \log \left| \mathcal{K}_{\mathcal{L}_q} \right| - H(K_{\mathcal{L}_q}) = 0 \text{ (Key Uniformity)}.
\]

The secrecy constraint (3) ensures that the keys generated by the agents in \( \mathcal{L}_q \) are independent from the public communication and from the observations of the agents in \( (\mathcal{L}_i)_{i \in [q+1,Q]} \), i.e., the agents that have a strictly lower security clearance level. Note, however, that (3) does not mean that the key of a particular agent in \( \mathcal{L}_q \) is secret from the agents at a similar or superior security clearance level, i.e., the agents in \( (\mathcal{L}_i)_{i \in [1,q]} \). Moreover, the keys generated are almost jointly independent, so that the simultaneous use of the keys by the agents in \( \mathcal{L}_q \) is secure. Finally, observe that when \( Q = 1 \), we recover the setting in [17].

Observe that in the presented setting we implicitly assumed that for any \( q \in \mathcal{Q} \), the agents in \( \mathcal{L}_q \) are willing to agree on a common secret key generation protocol. In the next subsection, we study a similar setting but with the additional constraint that the agents are selfish.

**B. Secret-key generation with selfish users**

We consider the secret-key generation problem described by Definitions 1 and 2 when the agents are selfish, i.e., they are solely interested in maximizing their own secret-key rate. At each level \( q \in \mathcal{Q} \), the agents can potentially form coalitions to achieve this goal, in the sense that subsets of agents can agree on a collective protocol to follow before the actual secret-key generation protocol occurs. Note that the model allows the agents to communicate with each other over the public channel and determine whether or not they want to be part of a coalition. However, we do not assume any privilege for coalitions, in particular, if the members of a given
coalition need to communicate with each other, they only have access to the aforementioned public communication channel. The questions we are interested in are the following. (i) Can selfish agents find a consensus about which coalitions to form? (ii) If such consensus exists, how should the value, i.e., the secret-key sum-rate, of each coalition be allocated among its agents?

**Remark 1.** Observe that cooperation across different levels is not relevant. Indeed, the agents in $L_1$ need to keep their keys secret from any agents in $(L_i)_{i \in [2,Q]}$ and for any communication strategies of those agents, so that cooperation for the agents in $L_1$ can only happen at Level $L_1$. We then iterate the argument for $q = 2$ to $q = Q$ to see that cooperation can only happen among agents at the same level.

We define a game corresponding to this problem as follows. Let $q \in Q$. Consider a sequence of individual strategy sets $(A_l)_{l \in L_q}$ and a sequence of payoff functions $(\pi_l)_{l \in L_q}$, where for $l \in L_q$, $A_l$ corresponds to the set of strategies that agent $l$ can adopt and $\pi_l(a_{L_q})$ represents the payoff of agent $l$, i.e., the rate of its secret key, when the strategies $a_{L_q} \in \times_{l \in L_q} A_l$ are played by the agents. We assume a decentralized setting in the sense that the base station does not influence the strategies of the agents, i.e., is not a player but a passive entity. We next wish to formulate a coalitional game by associating with each coalition of cooperating agents $S \subseteq L_q$ a certain worth $v_q(S)$. As detailed in Section III, such mapping $v_q$ provides with a tool to study the stability of coalitions formed by the agents, where stability of a coalition means that there is no incentive to merge with another coalition or to split into smaller coalitions. Two potential choices for the worth $v_q(S)$ of coalition $S \subseteq L_q$ are the following, [26], [27]

\[
\max_{a_S} \min_{a_{L_q}\setminus S} \sum_{i \in S} \pi_i(a_S, a_{L_q}\setminus S), \tag{5}
\]

\[
\min_{a_{L_q}\setminus S} \max_{a_S} \sum_{i \in S} \pi_i(a_S, a_{L_q}\setminus S), \tag{6}
\]

where the quantity in (5) corresponds to the payoff that coalition $S$ can ensure to its members regardless of the strategies adopted by the member of $L_q \setminus S$, and the one in (6) to the payoff that coalition $L_q \setminus S$ cannot prevent coalition $S$ to receive. See, for instance, [28] for a detailed
explanation of the subtle difference between these two notions in general. Observe also that for our problem both quantities are equal since for any \( S \subseteq \mathcal{L}_q \), there exists \( a^*_{\mathcal{L}_q \setminus S} \in \bigtimes_{i \in \mathcal{L}_q \setminus S} A_i \) such that for any strategies \( a_S \in \bigtimes_{i \in S} A_i \), we have

\[
\sum_{i \in S} \pi_i(a_S, a_{\mathcal{L}_q \setminus S}) \geq \sum_{i \in S} \pi_i(a_S, a^*_{\mathcal{L}_q \setminus S}).
\]

Indeed, consider \( a^*_{\mathcal{L}_q \setminus S} \) as the strategies consisting in publicly disclosing \( X_i^N \) for all agents \( i \in \mathcal{L}_q \setminus S \). According to the terminology of [27] the game is clear.

To summarize, for a DMS \( \left( \mathcal{X}_{\mathcal{L}_q} \times \mathcal{X}_0, p_{X_{\mathcal{L}_q}X_0} \right) \), the secret-key generation problem described in Definitions 1, 2, when the agents are selfish is cast as \( Q \) coalitional games \( (\mathcal{L}, v_q)_{q \in \mathcal{Q}} \) where the value functions are defined for \( q \in \mathcal{Q} \) as

\[
v_q: 2^{\mathcal{L}_q} \to \mathbb{R}^+, \mathcal{S} \mapsto \max_{a_S} \min_{a_{\mathcal{L}_q \setminus S}} \sum_{i \in S} \pi_i(a_S, a_{\mathcal{L}_q \setminus S}) \tag{7}
\]

such that for any \( S \subseteq \mathcal{L}_q \), \( v_q(S) \) corresponds to the maximal secret-key sum-rate achievable by coalition \( S \) when no specific strategy is assumed for the agents in \( \mathcal{L}_q \setminus S \).

C. Additional definitions

Before we study the model described in Section II-B, we provide additional definitions that will be useful in the sequel. These definitions apply for the setting described in Section II-A when no selfishness constraint holds on the users but when security constraints with respect to a subset of agents hold, i.e., the keys generated by a given subset of agents are required to be secret from the rest of the agents.

For a given level \( q \in \mathcal{Q} \), we formalize the definition of a secret-key generation protocol for a group of agents \( S \subseteq \mathcal{L}_q \) in the following definitions.

**Definition 3.** Let \( q \in \mathcal{Q} \) and \( S \subseteq \mathcal{L}_q \). For \( i \in S \), let \( K_i \) be a key alphabet of size \( 2^{N_{R_i}} \) and define \( K_S \) as the Cartesian product of \( K_i, i \in S \). A \( \left( (2^{N_{R_i}})_{i \in S}, N \right) \) secret-key generation strategy for the coalition of agents \( S \) is as follows.

1) The base station observes \( X_0^N \) and Agent \( i, i \in S \), observes \( X_i^N \).
2) The agents in $S$ and the base station communicate, possibly interactively, over the public channel. The global public communication between the agents in $S$ and the base station is denoted by $A_S \in A_S$, for some discrete alphabet $A_S$.

3) Agent $i$, $i \in S$, computes $K_i(X_i^N, A_S) \in K_i$.

4) The base station computes $\widehat{K}_i(X_0^N, A_S) \in K_i, i \in S$.

In the following, we use the notation $K_S \triangleq (K_i)_{i \in S}$.

**Definition 4.** Let $q \in Q$ and $S \subseteq L_q$. A secret-key rate tuple $(R_i)_{i \in S}$ is achievable if there exists a sequence of $((2^{NR_i})_{i \in S}, N)$ secret-key generation strategies for the coalition of agents $S$ such that

$$\lim_{N \to \infty} P[\widehat{K}_S \neq K_S] = 0 \text{ (Reliability)},$$

$$\lim_{N \to \infty} I\left(K_S; A_S, X_{L_q \setminus S}^N, (X_{L_j}^N)_{j \in [q+1, Q]}\right) = 0 \text{ (Collective Strong Secrecy)},$$

$$\lim_{N \to \infty} \log |K_S| - H(K_S) = 0 \text{ (Key Uniformity)}.$$  

The secrecy constraint (9) with respect to the agents outside of $S$ means that the agents in $S$ follow a protocol for secret-key generation under the information-theoretic constraint that the agent in $L_q \setminus S$ are not assumed to follow any specific communication strategy.

**Remark 2.** (9) and (10) can be combined in only one condition. If

$$\lim_{N \to \infty} N\mathbb{V}\left(p_{K_S A_S X_{L_q \setminus S}^N} | (X_{L_j}^N)_{j \in [q+1, Q]}, p_{U_S} P_{A_S, X_{L_q \setminus S}^N} (X_{L_j}^N)_{j \in [q+1, Q]}\right) = 0,$$

then (9) and (10) hold by [29, Lemma 2.7], [25, Lemma 1], where $p_{U_S}$ denotes the uniform distribution over $[1, 2^{\sum_{i \in S} |K_i|}]$ and $\mathbb{V}(\cdot, \cdot)$ denotes the variational distance between two distributions.

**III. Game analysis for $Q = 1$**

We set $Q = 1$, i.e., we consider only one security level, and drop all the indices on $q$. For any $S \subseteq \mathcal{L}$, we define the complement of $S$ as $S^c \triangleq \mathcal{L} \setminus S$. We will discuss the extension to
Q > 1 in Section V. In Section III-A, we study the properties of the game in Section II-B and, in Section III-B, we propose candidates for the secret-key rate allocation.

A. Properties of the game and characterization of its core

We first provide the following characterization of the value function \( v \) defined in (7).

**Theorem 1.** We have for any coalition \( S \subseteq \mathcal{L} \)

\[
\max_{a_S} \min_{a_{S^c}} \sum_{i \in S} \pi_i(a_S, a_{S^c}) = I(X_S; X_0|X_{S^c}).
\]  

(11)

Hence, for any \( S \subseteq \mathcal{L} \)

\[ v(S) = I(X_S; X_0|X_{S^c}). \]

**Proof.** Consider the secret-key generation problem described in Definitions 3 and 4 with \( Q = 1 \). The secret-key sum-rate capacity for coalition \( S \subseteq \mathcal{L} \), i.e., the maximal secret-key sum-rate \( \sum_{i \in S} R_i \) achievable by coalition \( S \), is

\[
C_S \triangleq I(X_S; X_0|X_{S^c}).
\]  

(12)

The converse follows from [15], [16] by considering two legitimate users, each observing \( X_S^N \) and \( X_0^N \), one Eavesdropper, observing \( X_{S^c}^N \), and by the Markov chain (1). The achievability part is more involved and will later follow from Corollary 2 derived in Section IV. We intentionally postpone its proof to streamline presentation.

Next, we observe that the game defined in (7) is superadditive in the sense that any two disjoint coalitions \( S, T \subseteq \mathcal{L} \), \( S \cap T = \emptyset \), obtain secret-key sum-rate capacities that cannot add up to a quantity strictly larger than the secret-key sum-rate capacity of the coalition \( S \cup T \). Note indeed that the secrecy constraints for coalitions \( S \) and \( T \), with \( S \cap T = \emptyset \), implies a secrecy
constraint for the coalition $S \cup T$:

\[
I \left( K_{S\cup T}; A_{S\cup T}X_{(S\cup T)^c}^N \right) = I \left( K_S; A_SA_TX_{(S\cup T)^c}^N \right) + I \left( K_T; A_TA_TX_{(S\cup T)^c}^N \mid K_S \right) \tag{13a}
\]

\[
\leq I \left( K_S; A_SA_TX_{(S\cup T)^c}^N \right) + I \left( K_T; A_TA_TX_{(S\cup T)^c}^N \mid K_S \right) \tag{13b}
\]

\[
\leq I \left( K_S; A_SX_{S^c}^N \right) + I \left( K_T; A_TX_{T^c}^N \right) , \tag{13c}
\]

where in (13a) we decompose $A_{S\cup T}$ in two parts $A_S$ and $A_T$, the public communication emitted by the agents in $S$ and $T$, respectively, (13b) holds by positivity of the mutual information and (13c) holds because $(A_T, X_{(S\cup T)^c}^N)$ is a function of $(A_S, X_{S^c}^N)$ and $(A_S, X_{(S\cup T)^c}^N, K_S)$ is a function of $(A_T, X_{T^c}^N)$.

Superadditivity implies that there is an interest in forming a large coalition to obtain a larger secret-key sum rate, however, large coalition might not be in the individual interest of the agents, in the sense that increasing the secret-key key sum-rate of a given coalition might not lead to an increased individual secret-key rate for all the player of the coalitions. A useful concept to overcome this complication is the core of the game.

**Definition 5** (e.g. [30]). *The core of a superadditive game $(L, v)$ is defined as follows.*

\[
\mathcal{C}(v) \triangleq \left\{ (R_i)_{i \in L} : \sum_{i \in L} R_i = v(L) \text{ and } \sum_{i \in S} R_i \geq v(S), \forall S \subset L \right\}. \tag{14}
\]

Observe that for any point in the core, the grand coalition, i.e., the coalition $L$, is in the best interest to all agents, since the set of inequalities in (14) ensures that no coalition of agents can increase its secret-key sum-rate by leaving the grand coalition. Observe also that for any point in the core the maximal secret-key sum rate $v(L)$ for the grand coalition is achieved. In general, the core of a game can be empty. However, we will show that the game we have defined has a non-empty core.

Definition 5 further clarifies the choice of the value function $v$. A coalition $S$ wishes to be associated with a value $v(S)$ as large as possible, while the agents outside $S$ wish $v(S)$ to be as small as possible to demand a higher share of $v(L)$. The latter achieve their goal by waiving a threat argument, which consists in arguing that they could adopt the strategy that minimizes
by publicly disclosing their source observations, whereas coalition $\mathcal{S}$ achieves its goal by arguing that it can always achieve the secret-key sum-rate capacity of Theorem 1, irrespective of the strategy of agents in $\mathcal{S}^c$. This formulation is analogous to the one for the Gaussian multiple access channel problem studied in [2], and the Gaussian multiple access wiretap channel problem studied in [31], where users can also form coalitions to request a larger communication sum-rate by means of jamming threats, and is generically termed as alpha effectiveness or alpha theory [26]–[28].

We now introduce the notion of convexity for a game to better understand the structure of the core of our game.

**Definition 6 ([32])**. A game $(\mathcal{L}, v)$ is convex if $v : 2^\mathcal{L} \to \mathbb{R}^+$ is supermodular, i.e.,

$$\forall \mathcal{U}, \mathcal{V} \subseteq \mathcal{L}, v(\mathcal{U}) + v(\mathcal{V}) \leq v(\mathcal{U} \cup \mathcal{V}) + v(\mathcal{U} \cap \mathcal{V}).$$

(15)

The intuition behind this definition is that supermodularity provides a stronger incentive to form coalition than superadditivity. Indeed, supermodularity of a function $v : 2^\mathcal{L} \to \mathbb{R}^+$ can equivalently be defined as follows [32]

$$\forall l \in \mathcal{L}, \forall \mathcal{T} \subseteq \mathcal{L} \setminus \{l\}, \forall \mathcal{S} \subseteq \mathcal{T}, v(\mathcal{S} \cup \{l\}) - v(\mathcal{S}) \leq v(\mathcal{T} \cup \{l\}) - v(\mathcal{T}),$$

(16)

which means that, in addition to superadditivity, the contribution of a single agent to a given coalition increases with the size of the coalition it joins. We also refer to [32] for other interpretations of supermodularity.

**Proposition 1.** The game $(\mathcal{L}, v)$ defined in (7) is convex.

**Proof.** For any $\mathcal{S} \subseteq \mathcal{L}$, we have

$$I(X_S; X_0|X_{\mathcal{S}^c}) = H(X_S|X_{\mathcal{S}^c}) - H(X_S|X_{\mathcal{S}^c}X_0)$$

(17a)

$$= H(X_S|X_{\mathcal{S}^c}) - H(X_S|X_0)$$

(17b)

$$= H(X_\mathcal{L}) - H(X_{\mathcal{S}^c}) - H(X_S|X_0),$$

(17c)
where we have used the Markov chain (1) in the second equality. Then, \( S \mapsto -H(X_S | X_0) \) is supermodular because for any \( \mathcal{U}, \mathcal{V} \subseteq \mathcal{L} \),

\[
H(X_{\mathcal{U} \cup \mathcal{V}} | X_0) + H(X_{\mathcal{U} \cap \mathcal{V}} | X_0) = H(X_{\mathcal{U}} | X_0) + H(X_{\mathcal{V} \setminus \mathcal{U}} | X_{\mathcal{U}} X_0) + H(X_{\mathcal{U} \cap \mathcal{V}} | X_0) \leq H(X_{\mathcal{U}} | X_0) + H(X_{\mathcal{V} \setminus \mathcal{U}} | X_{\mathcal{U} \cap \mathcal{V}} X_0) + H(X_{\mathcal{U} \cap \mathcal{V}} | X_0) = H(X_{\mathcal{U}} | X_0) + H(X_{\mathcal{V}} | X_0),
\]

where (18c) holds because conditioning reduces entropy. Consequently, \( S \mapsto -H(X_{S^c}) \) is also supermodular since for any supermodular function \( w, S \mapsto w(S^c) \) is supermodular. Hence, by (17c) we conclude that \( v \) is supermodular.

A consequence of Proposition 1 is that the core of our game is non-empty.

**Corollary 1.** By [32], any convex game has non-empty core. Hence, by Proposition 1, our game defined in (7) has a non-empty core \( \mathcal{C}(v) \).

**Remark 3.** From a geometric point of view, \( \{(R_l)_{l \in \mathcal{L}} : \sum_{i \in S} R_i \geq v(S), \forall S \subset \mathcal{L}\} \) is a contrapolymatroid [33] when \( v \) is convex, and its intersection with the hyperplane \( \{(R_l)_{l \in \mathcal{L}} : \sum_{l \in \mathcal{L}} R_l = v(\mathcal{L})\} \) forms the core of \( v \) [32]. See Example 2 and Figure 2 for an illustration.

**Remark 4.** In the case of a convex game, the core coincides with the bargaining set for the grand coalition [34] and thus admits an alternative interpretation, in terms of stable allocations resulting from a sequence of “threats” and “counter-threats”, see [34], [35] for further details.

We provide an alternative characterization of the core that will turn out to be useful in the following. It can also be viewed as a converse for our problem since the secret-key rate-tuples in the core are upper-bounded.
Theorem 2. The core of the game \((\mathcal{L}, v)\) defined in (7) is given by

\[
\mathcal{C}(v) = \left\{ (R_l)_{l \in \mathcal{L}} : I(X_S; X_0) - I(X_S; X_{S^c}) \leq \sum_{i \in S} R_i \leq I(X_S; X_0), \forall S \subseteq \mathcal{L} \right\}. \quad (19)
\]

Proof. We have the following equivalences

\[
\left( \sum_{l \in \mathcal{L}} R_l = v(\mathcal{L}) \text{ and } \sum_{i \in S} R_i \geq v(S), \forall S \subset \mathcal{L} \right) \iff \left( \sum_{i \in S} R_i = v(\mathcal{L}) - \sum_{i \in S^c} R_i \text{ and } \sum_{i \in S} R_i \geq v(S), \forall S \subset \mathcal{L} \right) \iff \left( v(\mathcal{L}) - v(S^c) \geq \sum_{i \in S} R_i \geq v(S), \forall S \subset \mathcal{L} \right). \quad (20a, 20b, 20c)
\]

Finally, for any \(S \subseteq \mathcal{L}\), we have

\[
v(\mathcal{L}) - v(S^c) = I(X_\mathcal{L}; X_0) - I(X_{S^c}; X_0 | X_S)
\]

\[
= I(X_S; X_0), \quad (21a)
\]

and by the Markov chain (1)

\[
v(S) = I(X_S; X_0) - I(X_S; X_{S^c}). \quad \blacksquare
\]

B. Candidates for the secret-key rate allocation

Although \(\mathcal{C}(v)\) has been shown to be non-empty in Section III-A, a remaining issue is now to choose a specific rate-tuple allocation in the core. Shapley introduced a solution concept to ensure fairness according to the following axioms.

(i) Efficiency axiom, i.e., the secret-key sum-rate capacity for the grand coalition is achieved;

(ii) Symmetry axiom, i.e., any two agents that equally contribute to any coalition in the sense that for any \(i, j \in \mathcal{L}\), for any \(S \subseteq L\) such that \(i \neq j\) and \(i, j \notin S\), \(v(S \cup \{i\}) = v(S \cup \{j\})\), obtain the same individual secret-key rate;

(iii) Dummy axiom, i.e., any agent that does not bring value to any coalition he can join, in the sense, for any \(i \in \mathcal{L}\), for any \(S \subseteq L\) such that \(i \notin S\), \(v(S \cup \{i\}) = v(S)\), receives a null secret-key rate;
(iv) Additivity axiom, i.e., for any two games \( v \) and \( u \) played by the agents, the individual secret-key length obtained by an agent for the game \( u + v \), is the sum of secret-key lengths when \( u \) and \( v \) are played separately. In our setting, the later axiom could correspond to several key generation protocols performed by the same agents with the source statistics varying for each protocol. Moreover, it would mean that even if the agents do not know in advance the number \( P \) of secret-key generation protocols they are going to be involved in and which particular source statistics will be associated with each protocol, they are going to obtain the same individual key lengths as if they had to perform the \( P \) protocols simultaneously, in the sense of performing one protocol whose value function is the sum of \( P \) value functions.

**Example 1.** We have the following intuitive property. If there exist \( i, j \in L \) such that \( i \neq j \) and \( p_{X_i|X_0} = p_{X_j|X_0} \), then Agent \( i \) and Agent \( j \) satisfy the symmetry axiom described above.

**Proof.** Let \( i, j \in L \), and \( S \subseteq L \) such that \( i \neq j \) and \( i, j \notin S \). Define \( S^c \triangleq S \setminus \{i, j\} \). We have

\[
v(S \cup \{i\}) = H(X_L) - H(X_{(S \cup \{i\})^c}) - H(X_{S \cup \{i\}|X_0})
\]

\[
= H(X_L) - H(X_jX_{S^c}) - H(X_i|X_0) - H(X_S|X_0)
\]

\[
= H(X_L) - H(X_iX_{S^c}) - H(X_j|X_0) - H(X_S|X_0)
\]

\[
= v(S \cup \{j\}),
\]

where (22a) holds by (17c), (22b) holds by definition of \( S^c \) and by the Markov chain (1), (22c) holds because by the Markov chain (1) \( p_{X_iX_{S^c}X_0} = p_{X_i|X_0}p_{X_{S^c}|X_0} = p_{X_j|X_0}p_{X_{S^c}|X_0} = p_{X_jX_{S^c}X_0} \), which implies by marginalization over \( X_0 \), \( p_{X_iX_{S^c}} = p_{X_jX_{S^c}} \), which in turn implies \( H(X_iX_{S^c}) = H(X_jX_{S^c}) \), (22d) holds similar to (a) and (b).

**Proposition 2** (e.g. [36]). Given a coalitional game \((L, v)\), there exists a unique \( L \)-tuple \( \left( R^{\text{Shap}}_l \right)_{l \in L} \) that satisfies the efficiency, symmetry, dummy, and additivity axiom described above. \( \left( R^{\text{Shap}}_l \right)_{l \in L} \) is called the Shapley value.

For convex games, the Shapley value is in the core, and is explicited in the following proposition.
Proposition 3. The Shapley value of \((\mathcal{L}, v)\) defined in (7) is in \(C(v)\) and is given by

\[
\forall l \in \mathcal{L}, R_l^{\text{Shap}} = \sum_{S \subseteq \mathcal{L} \setminus \{l\}} \frac{|S|! (L - |S| - 1)!}{L!} (v(S \cup \{l\}) - v(S))
\]

\[= I (X_l; X_0) - \frac{1}{L} \sum_{S \subseteq \mathcal{L} \setminus \{l\}} \left( \frac{L - 1}{|S|} \right) I (X_l; X_S). \tag{23a}\]

Proof. The fact that the Shapley value belongs to the core follows by [32] from the convexity of \((\mathcal{L}, v)\) proved in Proposition 1. (23a) is also from [32]. (23b) is obtained by remarking that for any \(l \in \mathcal{L}\), for any \(S \subseteq \mathcal{L} \setminus \{l\}\)

\[
v(S \cup \{l\}) - v(S) = H(X_{Sc}) + H(X_S|X_0) - H(X_{(S\cup\{l\})c}) - H(X_{S\cup\{l\}}|X_0)
\]

\[= H(X_{Sc\cap\{l\}}|X_{Sc\setminus\{l\}}) - H(X_l|X_0X_S)
\]

\[= H(X_l|X_{Sc\setminus\{l\}}) - H(X_l|X_0)
\]

\[= I (X_l; X_0) - I (X_l; X_{Sc\setminus\{l\}}), \tag{24a}\]

where (24a) holds by (17c), (24c) holds because \(l \notin S\) and by the Markov chain (1). Finally, we conclude by observing that

\[
\sum_{S \subseteq \mathcal{L} \setminus \{l\}} \frac{|S|! (L - |S| - 1)!}{L!} = \sum_{k=0}^{L-1} \frac{L! (L - 1)!}{L!} = 1, \tag{25}\]

and that a change of variables yields

\[
\sum_{S \subseteq \mathcal{L} \setminus \{l\}} \frac{|S|! (L - |S| - 1)!}{L!} I (X_l; X_{Sc\setminus\{l\}}) = \sum_{S \subseteq \mathcal{L} \setminus \{l\}} \frac{|S|! (L - |S| - 1)!}{L!} I (X_l; X_S). \tag{26}\]

\[
\text{Remark 5. Geometrically, the Shapley value corresponds to the center of gravity of the vertices of } C(v) \text{ [32]. See Example 2 and Figure 2 for an illustration.}
\]

Observe that (23b) quantifies the difference of key length obtained for Agent \(l, l \in \mathcal{L}\), between the case \(L = 1\) and the case \(L > 1\). Note also that the term \(\frac{1}{L} \sum_{S \subseteq \mathcal{L} \setminus \{l\}} \left( \frac{L - 1}{|S|} \right) I (X_l; X_S)\) is upper-bounded by \(I(X_l; X_{\mathcal{L} \setminus \{l\}})\) according to Theorem 2 since the Shapley value belongs to
the core.

Other solution concepts than the Shapley value can be considered to choose a “fair” point in the core. In particular, the additivity axiom might not always be relevant in our problem, for instance, if the agents only perform a single secret-key generation protocol. We do not intend to provide an exhaustive list of such concepts, we will, however, describe a solution concept that has attracted a certain interest in many studies, the nucleolus.

**Definition 7 ([37]).** Define the set 
\[ \mathcal{Y} \triangleq \{ y = (y_i)_{i \in \mathcal{L}} \in \mathbb{R}^L_+ : \sum_{i \in \mathcal{L}} y_i = v(\mathcal{L}) \}. \]
For \( y \in \mathcal{Y} \), for \( S \in 2^\mathcal{L} \), define the excess \( e(y, S) \triangleq v(S) - \sum_{i \in S} y_i \), and define the vector \( \theta(y) = (\theta_i(y))_{i \in [1, 2^L]} \in \mathbb{R}^{2^L} \) as \( (e(y, S))_{S \in 2^\mathcal{L}} \) sorted in nonincreasing order, i.e., for \( i, j \in [1, 2^L], i < j \Rightarrow \theta_i(y) \geq \theta_j(y) \). The nucleolus is defined as

\[ \{ y_0 \in \mathcal{Y} : \theta(y_0) \preceq \theta(y), \forall y \in \mathcal{Y} \}, \]  

where “\( \preceq \)” denote the lexicographic order, i.e., for \( y^{(1)}, y^{(2)} \in \mathcal{Y}, \ (y^{(1)} \preceq y^{(2)}) \iff (y^{(1)} = y^{(2)} \text{ or } \exists i_0, (\forall j < i_0, y^{(1)}_j = y^{(2)}_j \text{ and } y^{(1)}_{i_0} < y^{(2)}_{i_0})) \).

A possible interpretation of the nucleolus is to see the excess \( e(y, S) \triangleq v(S) - \sum_{i \in S} y_i \) for some \( y \in \mathcal{Y}, S \in 2^\mathcal{L} \), as an indicator of dissatisfaction of coalition \( S \) associated with \( y \) (the higher the excess, the higher the dissatisfaction). One thus might want to choose the \( y \) that minimizes the maximal excess, i.e., the first component of \( \theta \). If several choices for \( y \) are possible, one can decide to select \( y \) such that the second largest excess, i.e., the second component of \( \theta \), is minimized. One can then continue until a unique choice for \( y \) is obtained as stated in Proposition 4. This interpretation appears, for instance, in [30].

**Proposition 4 ([37]).** For a convex game, the nucleolus is a singleton and belongs to the core.

The nucleolus has, however, no closed-form formula and involves the resolution of successive minimization problems. We illustrate this concept in the following example. For completeness and to compute the nucleolus in Example 2, we summarize in Algorithm 1 a concise description of the method described in [38], [39].
Fig. 2. Core, Shapley value, and nucleolus of the game described in Example 2.

Remark 6. In the case of a convex game, the nucleolus coincides with the kernel [30] and thus admits another interpretation, see [30, Section 5] for further details.

Example 2. Let $X_0$ be a Bernoulli random variable with parameter $q \in [0, 1/2]$. Define $X_l \triangleq X_0 \oplus B_l, \forall l \in \mathcal{L}$, where the $B_l$'s are independent Bernoulli random variables with parameter $p_l \in [0, 1/2]$. Let $H_b(\cdot)$ denote the binary entropy and define for any $x \in [0, 1], \bar{x} = 1 - x$. For any $S \subseteq \mathcal{L}$, we have the following formula

\begin{align*}
v(S) &= H(X_{\mathcal{L}}) - H(X_{\mathcal{S}^c}) - H(X_S|X_0) \\ &= H(X_{\mathcal{L}}) - H(X_{\mathcal{S}^c}) - \sum_{i \in S} H_b(p_i) \\ &= -\sum_{\mathcal{T} \subseteq \mathcal{L}} f_{\mathcal{L}}(\mathcal{T}) \log f_{\mathcal{L}}(\mathcal{T}) + \sum_{\mathcal{T} \subseteq \mathcal{S}^c} f_{\mathcal{S}^c}(\mathcal{T}) \log f_{\mathcal{S}^c}(\mathcal{T}) - \sum_{i \in S} H_b(p_i),
\end{align*}
Algorithm 1 Nucleolus Computation

1: \( k \leftarrow 0 \)
2: \( \mathcal{E}_0 \leftarrow \emptyset \)
3: while the system \((S_k)\) has rank \(< L\) do
4: \( k \leftarrow k + 1 \)
5: Solve the following linear program and let \( z_k^* \) denote the value of the objective function obtained

\[
\begin{align*}
\text{Minimize} & \quad z \quad \text{subject to} \\
& \quad z + \sum_{i \in S} x_i \geq v(S), \forall S \subset \mathcal{L} \text{ s.t. } S \not\in \bigcup_{j=0}^{k-1} \mathcal{E}_j \quad (E_k) \\
& \quad \left( z_j^* + \sum_{i \in S} x_i = v(S), \forall S \in \mathcal{E}_j, j \in [1, k-1] \right) \quad (S_k) \\
& \quad \sum_{i \in \mathcal{L}} x_i = v(\mathcal{L}) 
\end{align*}
\]

6: Define \( \mathcal{E}_k \triangleq \{ S \subset \mathcal{L} : (E_k) \text{ holds with equality} \} \)
7: end while
8: return the nucleolus \( (R_{\text{Nucl}}^i)_{i \in \mathcal{L}} = (x_i)_{i \in \mathcal{L}} \)

where (28a) holds by (17c), (28b) holds by independence of the \( B_i \)’s, and where in (28c) we have defined for any \( S \subset \mathcal{L} \)

\[
f_S : 2^S \to \mathbb{R}_+, T \mapsto q \prod_{i \in T} p_i \prod_{j \in S \setminus T} \bar{p}_j + \bar{q} \prod_{i \in \mathcal{T}} \bar{p}_i \prod_{j \in S \setminus \mathcal{T}} p_j. \tag{29}
\]

Assume now that \( L = 3 \), and \( (q_1 p_1 p_2 p_3) = (0.40 0.20 0.27 0.25) \). We obtain \( v(\{1\}) \approx 0.17134 \), \( v(\{2\}) \approx 0.08205 \), \( v(\{3\}) \approx 0.10142 \), \( v(\{1, 2\}) \approx 0.28771 \), \( v(\{1, 3\}) \approx 0.31679 \), \( v(\{2, 3\}) \approx 0.20155 \), \( v(\{1, 2, 3\}) \approx 0.46921 \). Using Algorithm 1 and Proposition 3, we obtain the following secret-key rates

\[
R_{\text{Nucl}}^1 \in [0.2109, 0.2110], \quad R_{\text{Shap}}^1 \in [0.2165, 0.2166], \\
R_{\text{Nucl}}^2 \in [0.1172, 0.1173], \quad R_{\text{Shap}}^2 \in [0.1142, 0.1143], \\
R_{\text{Nucl}}^3 \in [0.1410, 0.1411], \quad R_{\text{Shap}}^3 \in [0.1384, 0.1385].
\]

The core of the game, as well as the Shapley value and the nucleolus are depicted in Figure 2.
IV. HOW TO ACHIEVE ANY POINT OF THE CORE FOR THE CASE $Q = 1$

We still assume $Q = 1$ in this section. We will discuss the extension of this section to $Q > 1$ in Section V. We have seen in Section III that the grand coalition, i.e., the coalition $\mathcal{L}$, is in the best interest of all agents, and we have characterized the acceptable operating points as the core of the game. Assuming that the grand coalition agrees on an operating point in the core, we now would like to answer whether there exists a secret-key generation protocol for this specific operating point. We show in this section the following three results. In Theorem 3, we claim that the coding scheme presented in Section IV-A achieves for the grand coalition a region that contains the core $\mathcal{C}(v)$. The proof is presented in Section IV-B. In Theorem 4, we provide an achievable region for any coalition $S \subset \mathcal{L}$ of agents. The coding scheme and its analysis partly rely on Theorem 3 and are discussed in Appendix C. Finally, we complete the proof of Theorem 1 with Corollary 2 obtained from Theorem 4.

**Theorem 3.** Consider a DMS $(\mathcal{X}_L \times \mathcal{X}_0, p_{X_L X_0})$ such that $\forall l \in \mathcal{L}, |\mathcal{X}_l| = 2$. Any rate tuple in

$$R_L \triangleq \left\{ (R_i)_{i \in \mathcal{L}} : 0 \leq \sum_{i \in S} R_i \leq I(X_S; X_0), \forall S \subseteq \mathcal{L} \right\}$$

is achievable by the grand coalition, in the sense of Definition 2, with the coding scheme of Section IV-A. Moreover, by Theorem 2 we have

$$R_L \supseteq \mathcal{C}(v).$$

**Theorem 4.** Consider a DMS $(\mathcal{X}_L \times \mathcal{X}_0, p_{X_L X_0})$ such that $\forall l \in \mathcal{L}, |\mathcal{X}_l| = 2$ and the Markov chain (1) holds. Any rate tuple in

$$R_S \triangleq \left\{ (R_i)_{i \in \mathcal{S}} : 0 \leq \sum_{i \in T} R_i \leq I(X_T; X_0 | X_{S^c}), \forall T \subseteq S \right\}.$$ \hspace{1cm} (32)

is achievable in the sense of Definition 2 by the coalition of agents $S \subset \mathcal{L}$.

**Proof.** See Appendix C. \hfill ■

**Corollary 2.** Theorem 4 implies the achievability part of Theorem 1 when $Q = 1$, i.e., the secret-key sum rate $I(X_S; X_0 | X_{S^c})$ is achievable by the coalition $S \subseteq \mathcal{L}$. 
Proof. See Appendix D.

Remark 7. Note that Theorem 3 does not require the Markov chain (1). Note also that Theorem 3 and Theorem 4 extend to prime size alphabets by using [40, Lemma 7] in place of Lemma 2.

Remark 8. For $L = 2$, [17, Theorem 3] provides a coding scheme to achieve the region $\mathcal{R}_L$ in Theorem 3, and for arbitrary $L$, [17, Theorem 1] provides a coding scheme that achieves the sum-rate $v(\mathcal{L})$. However, in contrast to our solutions, these coding schemes, which rely on certain existence results from [25], are neither explicit nor low-complexity, require time-sharing (for [17, Theorem 3]), and only provide weak secrecy.

A. Coding Scheme

The principle of the coding scheme is to separately deal with reliability and secrecy, as it can be done for secret-key generation between two users [41], albeit with additional complications. More specifically, a reconciliation step is first performed to allow the base station to reconstruct the observations $X^N_L$ of the agents. Then, during a privacy amplification step, each agent extracts from its observations a key that can be reconstructed at the base station. The reconciliation step itself does not present any difficulty, the main complications, compared to a two-user scenario, are (i) to deal with a distributed setting in the privacy amplification step and (ii) to analyze the combination of the reconciliation and privacy amplification steps, as detailed in the next section.

Our coding scheme operates over $B$ blocks of length $N$, where $N$ is a power of 2. We define $B \triangleq [1, B]$. We omit indexation of the variables over blocks because encoding is identical for all blocks. The reconciliation step, described in Algorithm 2, makes use of polar codes. In particular we introduce the following notation. For $n \in \mathbb{N}$ and $N \triangleq 2^n$, let $G_n \triangleq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \otimes n$ be the source polarization transform defined in [42]. For any $l \in \mathcal{L}$, we define the polar transform of $X^N_l$ by

$$U^N_l \triangleq X^N_l G_n,$$

moreover, for any set $\mathcal{I} \subseteq [1, N]$, we define $U^N_{\mathcal{I}} \triangleq ((U_l)_i)_{i \in \mathcal{I}}$. For any $l \in \mathcal{L}$, for any $\mathcal{S} \subseteq \mathcal{L}$,
we also define the following “high entropy” and “very high entropy” sets.

$$\mathcal{H}_{X_i|X_0X_{1:i-1}X_S} \triangleq \left\{ i \in [1, N] : H((U_i)_i|(U_i)^{i-1}X_0^i X_{1:i-1} X_S^N) \geq \delta_N \right\},$$  \hspace{1cm} (34)

$$\mathcal{V}_{X_i|X_0X_{1:i-1}X_S} \triangleq \left\{ i \in [1, N] : H((U_i)_i|(U_i)^{i-1}X_0^i X_{1:i-1} X_S^N) \geq 1 - \delta_N \right\},$$  \hspace{1cm} (35)

where we have defined $X_{1:i-1} \triangleq (X_j)_{j \in [1:i-1]}$ and $X_S \triangleq (X_j)_{j \in S}$. An interpretation of these sets that will be used in our analysis can be summarized in the following two lemmas.

**Lemma 1** (Source coding with side information [42]). Consider a discrete memoryless source with joint probability distribution $p_{XY}$ over $\mathcal{X} \times \mathcal{Y}$ with $|\mathcal{X}| = 2$ and $\mathcal{Y}$ finite. Define $A^N \triangleq X^NG_n$, and for $\delta_N \triangleq 2^{-N^\beta}$ with $\beta \in [0, 1/2]$, the set $\mathcal{H}_{X|Y} \triangleq \left\{ i \in [1, N] : H(A_i|A^{i-1}Y^N) > \delta_N \right\}$. Given $A^N[\mathcal{H}_{X|Y}]$ and $Y^N$ it is possible to form $\hat{A}^N$ by the successive cancellation decoder of [42] such that $\lim_{N \to \infty} \mathbb{P}[\hat{A}^N \neq A^N] = 0$. Moreover, $\lim_{N \to \infty} |\mathcal{H}_{X|Y}|/N = H(X|Y)$.

**Lemma 2** ([24]). Consider a discrete memoryless source with joint probability distribution $p_{XZ}$ over $\mathcal{X} \times Z$ with $|\mathcal{X}| = 2$ and $\mathcal{Z}$ finite. Define $A^N \triangleq X^NG_n$, and for $\delta_N \triangleq 2^{-N^\beta}$ with $\beta \in [0, 1/2]$, the set $\mathcal{V}_{X|Z} \triangleq \left\{ i \in [1, N] : H(A_i|A^{i-1}Z^N) > 1 - \delta_N \right\}$. $A^N[\mathcal{V}_{X|Z}]$ is almost uniform and independent from $Z^N$ in the sense $\lim_{N \to \infty} \mathbb{V}(p_{A^N[\mathcal{V}_{X|Z}]Z^N}, p_U p_{Z^N}) = 0$, where $p_U$ is the uniform distribution over $\{0, 1\}^{|\mathcal{V}_{X|Z}|}$. Moreover, $\lim_{N \to \infty} |\mathcal{V}_{X|Z}|/N = H(X|Z)$.

Hence, by Lemma 1, the vector $U^N_l[\mathcal{H}_{X_i|X_0X_{1:i-1}X_S}]$, $l \in \mathcal{L}$, ensures near lossless reconstruction of $X^N_i$ given $(X_0^i, X_{1:i-1}^i, X_S^N)$. By Lemma 2, the vector $U^N_l[\mathcal{V}_{X_i|X_0X_{1:i-1}X_S}]$, $l \in \mathcal{L}$, is almost uniform and independent from $(X_0^i, X_{1:i-1}^i, X_S^N)$. Note also that by definition $\mathcal{V}_{X_i|X_0X_{1:i-1}X_S} \subset \mathcal{H}_{X_i|X_0X_{1:i-1}X_S}$. We refer to [24], [40] for further discussion of theses sets.

The privacy amplification step, described in Algorithm 3, relies on two-universal hash functions [43], [44].

**Definition 8.** A family $\mathcal{F}$ of two-universal hash functions $\mathcal{F} = \{ f : \{0,1\}^N \to \{0,1\}^r \}$ is such that

$$\forall x, x' \in \{0,1\}^N, x \neq x' \implies \mathbb{P}[F(x) = F(x')] \leq 2^{-r},$$  \hspace{1cm} (36)
Algorithm 2 Reconciliation protocol

1: for Agent \( l \in \mathcal{L} \) do
2:  for Block \( b \in \mathcal{B} \) do
3:   Compute \( U^N_l \triangleq X^N_l G_n \)
4:  Transmit \( A_l \triangleq U^N_l [H_{X_l|X_0X_1\ldots l-1}] \) to the base station over the public channel
5: end for
6: end for
7: Let \( A_L \triangleq (A_l)_{l \in \mathcal{L}} \) denote the public communication in a Block \( b \in \mathcal{B} \).
8: for Block \( b \in \mathcal{B} \) do
9:   Given \( A^b_L \) and \( X^N_0 \) observed in Block \( b \), the base station reconstructs \( X^N_L \) for Block \( b \) using the successive cancellation algorithm for source coding with side information of [42].
10: end for

Algorithm 3 Privacy amplification protocol

1: for Block \( b \in \mathcal{B} \) do
2:  for Agent \( l \in \mathcal{L} \) do
3:   Compute \( K_l \triangleq F_l(X^N_l) \)
4:  Publicly transmit the choice of \( F_l \) to the base station
5: end for
6: for \( l \in \mathcal{L} \) do
7:  The base station computes \( K_l \triangleq F_l(X^N_l) \)
8: end for
9: end for

where \( F \) is a function uniformly chosen in \( \mathcal{F} \).

For \( l \in \mathcal{L} \), we let \( F_l : \{0,1\}^N \rightarrow \{0,1\}^{r_l} \), be uniformly chosen in a family \( \mathcal{F}_l \) of two-universal hash functions. Note that \( r_l \) represents the key length obtained by Agent \( l \). The main difficulty in the analysis of the privacy amplification step is to find the admissible values, in the sense of Definition 2, for \( r_l \). We leave these quantities unspecified in this section, and will specify them in Section IV-B.

B. Coding Scheme Analysis

Success of the reconciliation step is ensured by the result on source coding with side information in Lemma 1. We thus focus on the privacy amplification step.

In the following we use the following notation. The indicator function is denoted by \( \mathbb{1}\{\omega\} \), which is equal to 1 if the predicate \( \omega \) is true and 0 otherwise. For a discrete random variable \( X \)
distributed according to \( p_X \) over the alphabet \( \mathcal{X} \), we let
\[
\mathcal{T}_\varepsilon^N(X) \triangleq \left\{ x^N \in \mathcal{X}^N : \left| \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}\{x_i = a\} - p_X(a) \right| \leq \varepsilon p_X(a), \forall a \in \mathcal{X} \right\} \tag{37}
\]
denote the \( \varepsilon \)-letter-typical set associated with \( p_X \) for sequences of length \( N \), see, for instance, [45], and define \( \mu_X \triangleq \min_{x \in S_X} p(x) \), where \( S_X \triangleq \{ x \in \mathcal{X} : p(x) > 0 \} \). Additionally, the min-entropy of \( X \) is defined as
\[
H_\infty(X) \triangleq - \log \left( \max_x p_X(x) \right). \tag{38}
\]

We will need the following two lemmas. Lemma 3 is a refined version of [46] meant to relate a min-entropy to a Shannon entropy, which is easier to study, and Lemma 4 can be interpreted as quantifying how much information is revealed about \( X_N^S \) knowing the overall public communication \( A_L \).

**Lemma 3** ([47, Lemma 1.1] [46]). Let \( \varepsilon > 0 \). Consider a DMS \( (\mathcal{X} \times \mathcal{Z}, p_{XZ}) \) and define the random variable \( \Theta \) as
\[
\Theta \triangleq \mathbf{1}\{(X^B, Z^B) \in \mathcal{T}_\varepsilon^B(XZ)\} \mathbf{1}\{Z^B \in \mathcal{T}_\varepsilon^B(Z)\}, \tag{39}
\]
Then, \( \mathbb{P}[\Theta = 1] \geq 1 - \delta_\varepsilon^0(B) \), with \( \delta_\varepsilon^0(B) \triangleq 2|S_X|e^{-\varepsilon^2 B \mu_X / 3} + 2|S_{XZ}| e^{-\varepsilon^2 B \mu_{XZ} / 3} \). Moreover, if \( z^B \in \mathcal{T}_\varepsilon^B(Z) \), then
\[
H_\infty(X^B | Z^B = z^B, \Theta = 1) \geq B(1 - \varepsilon) H(X | Z) + \log(1 - \delta_\varepsilon^1(B)), \tag{40}
\]
where \( \delta_\varepsilon^1(B) \triangleq 2|S_{X,Z}| e^{-\varepsilon^2 B \mu_{X,Z} / 6} \).

**Lemma 4.** For any \( S \subseteq \mathcal{L} \), we have
\[
H(X_N^S | A_L) \geq NI(X_S; X_0) + o(N). \tag{41}
\]

**Proof.** See Appendix A. \( \blacksquare \)
We are now equipped to show (43d). Let $S \subseteq \mathcal{L}$ and let
\[ \Theta \triangleq 1 \{ (X_S^{NB}, A^B_L) \in \mathcal{T}_{2\epsilon}^B(X_S^N A_L) \} 1 \{ A^B_L \in \mathcal{T}_\epsilon^B(A_L) \}. \tag{42} \]
For $a^B_L \in \mathcal{T}_\epsilon^B(A_L)$, we have
\[ H_\infty(X_S^{NB}|A^B_L = a^B_L, \Theta = 1) \geq B(1 - \epsilon)H(X_S^N|A_L) + \log(1 - \delta_1^1(N, B)) \tag{43a} \]
\[ \geq (1 - \epsilon)(NB \cdot I(X_S; X_0) + o(N) \cdot B) + \log(1 - \delta_1^1(N, B)) \tag{43b} \]
\[ = (1 - \epsilon) \cdot NB \cdot I(X_S; X_0) + \delta_2^2(N, B), \tag{43c} \]
where (43b) holds by Lemma 3 applied to the DMS $(X^N_S \times A_L, p_{X^N_S A_L})$, (43c) holds by Lemma 4, and in (43d) we have defined
\[ \delta_2^2(N, B) \triangleq o(N) \cdot B(1 - \epsilon) + \log(1 - \delta_1^1(N, B)). \tag{44} \]

**Remark 9.** Unfortunately, unlike the two-user secret-key generation setting considered in [46], it is not possible to use [46, Lemma 10] to obtain a tight lower bound on $H_\infty(X_S^{NB}|A^B_L = a^B_L)$.
We use Lemmas 3, 4 to circumvent this issue.

We will then need the following version of the leftover hash lemma [48], [49].

**Lemma 5** (Leftover hash lemma for concatenated hash functions). Let $X_L \triangleq (X_l)_{l \in \mathcal{L}}$ and $Z$ be random variables distributed according to $p_{X_L Z}$ over $X_L \times Z$. For $l \in \mathcal{L}$, let $F_l : \{0, 1\}^{n_l} \rightarrow \{0, 1\}^{r_l}$ be uniformly chosen in a family $\mathcal{F}_l$ of two-universal hash functions. Define $s_L \triangleq \prod_{l \in \mathcal{L}} s_l$, where $s_l \triangleq |\mathcal{F}_l|$, $l \in \mathcal{L}$, and for any $S \subseteq \mathcal{L}$, define $r_S \triangleq \sum_{i \in S} r_i$. Define also $F_L \triangleq (F_l)_{l \in \mathcal{L}}$ and
\[ F_L(X_L) \triangleq (F_1(X_1)||F_2(X_2)|| \ldots ||F_L(X_L)), \tag{45} \]
where $||$ denotes concatenation. Then, for any $z \in Z$, we have
\[ \forall (p_{F_L(X_L), F_L|Z=z}, p_{U_K P_{U_F}}) \leq \sum_{S \subseteq \mathcal{L}} 2^{r_S - H_\infty(X_S|Z=z)}, \tag{46} \]
where $p_{U_k}$ and $p_{U_F}$ are the uniform distribution over $[1, 2^{r_c}]$, and $[1, s_L]$, respectively.

A consequence of (46) is

$$
\mathbb{V}(p_{F_L}(X_L), F_L; Z) \leq \sum_{S \subset L} \frac{2^{r_S} - H_\infty(X_S|Z)}{4^{r_S} - H_\infty(X_S|Z)}
$$

where we have used the average conditional min-entropy of $X$ given $Z$ defined as in [49] by

$$
H_\infty(X|Z) \triangleq -\log(\mathbb{E}_{p_Z} \max_x p_{X|Z}(x)).
$$

Proof. See Appendix B.

Combining (43d) and Lemma 5, we are able to determine the admissible values for $r_l$, $l \in L$, as follows.

$$
\mathbb{V}(p_{F_L}(X_L^{N,L}), F_L; A_L^B, p_{U_k} p_{U_F} p_{A_L^B}) \leq \mathbb{V}(p_{F_L}(X_L^{N,L}), F_L; A_L^B \Theta, p_{U_k} p_{U_F} p_{A_L^B \Theta})
$$

$$
= \mathbb{E}_{p_{\Theta}} \left[ \mathbb{V}(p_{F_L}(X_L^{N,L}), F_L; A_L^B \Theta, p_{U_k} p_{U_F} p_{A_L^B \Theta}) \right]
$$

$$
\leq 2\mathbb{P}[\Theta = 0] + \mathbb{V}(p_{F_L}(X_L^{N,L}), F_L; A_L^B \Theta = 1, p_{U_k} p_{U_F} p_{A_L^B \Theta = 1})
$$

$$
\leq 2\mathbb{P}[\Theta = 0] + \mathbb{E}_{p_{A_L^B \Theta = 1}} \left[ \mathbb{V}(p_{F_L}(X_L^{N,L}), F_L; A_L^B \Theta = 1, p_{U_k} p_{U_F}) \right]
$$

$$
\leq 2\mathbb{P}[\Theta = 0] + \mathbb{E}_{p_{A_L^B \Theta = 1}} \left[ \sum_{S \subset L} \frac{2^{r_S} - H_\infty(X_S|Z)}{2^{r_S} - H_\infty(X_S|Z)} \right]
$$

$$
\leq 2\mathbb{P}[\Theta = 0] + \mathbb{E}_{p_{A_L^B \Theta = 1}} \left[ \sum_{S \subset L, S \neq \emptyset} \frac{2^{r_S} - (1-\epsilon) \cdot NB \cdot I(X_S; X_0) - \delta_2^2(N, B)}{2^{r_S} - (1-\epsilon) \cdot NB \cdot I(X_S; X_0) - \delta_2^2(N, B)} \right]
$$

$$
= 2\mathbb{P}[\Theta = 0] + \sum_{S \subset L, S \neq \emptyset} \frac{2^{r_S} - (1-\epsilon) \cdot NB \cdot I(X_S; X_0) - \delta_2^2(N, B)}{2^{r_S} - (1-\epsilon) \cdot NB \cdot I(X_S; X_0) - \delta_2^2(N, B)}
$$

$$
\leq 2\delta_2^0(N, B) + \sum_{S \subset L, S \neq \emptyset} \frac{2^{r_S} - (1-\epsilon) \cdot NB \cdot I(X_S; X_0) - \delta_2^2(N, B)}{2^{r_S} - (1-\epsilon) \cdot NB \cdot I(X_S; X_0) - \delta_2^2(N, B)}
$$

where (49b) holds by marginalization over $\Theta$ and the triangle inequality, (49d) holds because
\( V(\cdot, \cdot) \) is upper bounded by 2, (49f) holds by Lemma 5 with the substitutions \( z \leftarrow (a^B_L, \Theta = 1) \) and \( X_L \leftarrow X_L^{NB} \), (49g) holds by (43d), (49i) holds by Lemma 3.

Finally, we conclude that Theorem 3 holds by Remark 2 and (49i).

V. Extension to \( Q > 1 \)

To extend Sections III and IV to \( Q > 1 \), we consider the setting described in Definitions 1 and 2 when \( Q = 1 \) and when, additionally, an eavesdropper that observes the public communication is also in possession of correlated source observations. We describe the model and present the results in Section V-A and Section V-B, respectively.

Note that to obtain the extension to \( Q > 1 \), it will then be sufficient to apply the results of Section V-B to each security clearance level \( q \in Q \) by considering for the agents in \( L_q \), the DMS \((\mathcal{X}_{L_q} \times \mathcal{X}_0 \times \mathcal{X}_{L_{q+1}Q})\), where \( \mathcal{L}_{q+1}Q \triangleq \bigcup_{i \in [q+1, Q]} \mathcal{L}_i \), with the assumption that an eavesdropper observes the components \( X_{L_{q+1}Q} \) of the DMS.

A. Model

We first define as in Section II-A an auxiliary secret-key generation problem without selfishness constraints. As alluded to earlier, we only consider one level, i.e., \( Q = 1 \). Consider a DMS \((\mathcal{X}_L \times \mathcal{X}_0 \times \mathcal{X}_{L_{q+1}Q})\), where \( Z \) is an additional, compared to the source model considered in Section II-A when \( Q = 1 \), component of the source observed by an eavesdropper. The source is assumed to follow the following Markov chain: for any \( S, T \subseteq L \) such that \( S \cap T = \emptyset \),

\[
X_S - X_0 - (X_T Z). \tag{50}
\]

We then consider the same Definitions 3, 4 as in Section II-A for \( Q = 1 \), i.e., the secrecy constraint (9) of Definition 4, becomes for a coalition \( S \subset L \)

\[
\lim_{N \to \infty} I(K_S; A_S X_{Sc}^N Z^N) = 0. \tag{51}
\]

Next, we define our object of study as the secret-key generation problem we have just defined when \( S = L \) and when the users are selfish. Similar to Section II-B, we wish to understand whether the agents can find a consensus about the coalitions to form, and how the secret-sum
rate of each coalition should be allocated among its agents. Following Section II-B, we cast the
problem as a coalitional game \((\mathcal{L}, v^Z)\) where the value of coalition \(S \subseteq \mathcal{L}\) is defined as the
\textit{maximal secret-key sum-rate} that coalition \(S\) can obtain \textit{regardless of the strategies adopted by}
the member of \(S^c\).

B. Results

Similar to the proof of Theorem 1 by using Corollary 4, stated below, in place of Corollary 2,
one can show the following characterization of the value function \(v^Z\).

\[
v^Z : 2^\mathcal{L} \to \mathbb{R}^+; S \mapsto I(X_S; X_0|X_{S^c}Z).
\]  

(52)

Similar to Proposition 1, one can show that the game \((\mathcal{L}, v^Z)\) is convex, and similar to
Theorem 2 that its core is given by

\[
\mathcal{C}(v^Z) = \left\{ (R_l)_{l \in \mathcal{L}} : I(X_S; X_0) - I(X_S; X_{S^c}Z) \leq \sum_{i \in S} R_i \leq I(X_S; X_0|Z), \forall S \subseteq \mathcal{L} \right\}.
\]  

(53)

Moreover, similar to Proposition 3, the Shapley value is in \(\mathcal{C}(v^Z)\) and given by

\[
\forall l \in \mathcal{L}, R^\text{Shap}_l = I(X_l; X_0|Z) - \frac{1}{L} \sum_{S \subseteq \mathcal{L} \setminus \{l\}} \binom{L - 1}{|S|}^{-1} I(X_l; X_S|Z).
\]  

(54)

Finally, it is possible to achieve any point of the core \(\mathcal{C}(v^Z)\) with an explicit and low-complexity
coding scheme, by deducing from Theorem 4 the following corollary.

**Corollary 3.** Consider a DMS \((\mathcal{X}_L \times X_0 \times Z, p_{X_LX_0Z})\) such that \(\forall l \in \mathcal{L}, |X_l| = 2\) and the Markov
chain (50) holds. Any rate tuple in

\[
\mathcal{R}^Z_S \triangleq \left\{ (R_l)_{l \in S} : 0 \leq \sum_{i \in T} R_i \leq I(X_T; X_0|X_{S^c}Z), \forall T \subseteq S \right\}
\]  

(55)

is achievable by the coalition of agents \(S \subseteq \mathcal{L}\). Moreover,

\[
\mathcal{C}(v^Z) \subseteq \mathcal{R}^Z_S,
\]  

(56)
Finally, from Corollary 3, we deduce, similar to the proof of Corollary 2 the following corollary, which allows us to establish (52).

**Corollary 4.** Corollary 3 implies that for any $S \subseteq \mathcal{L}$, the secret-key sum rate $I(X_S; X_0|X_\mathcal{L}, Z)$ is achievable by Coalition $S$.

**VI. Concluding Remarks**

We have studied a pairwise secret-key generation source model between multiple agents who are subject to different levels of security clearance, and a base station. Although cooperation among agents sharing the same level of security clearance can increase their individual key length, it can, at the same time, lead to conflict of interests between agents. We have proposed an integrated information-theoretic and game-theoretic formulation of the problem. Specifically, we have cast the problem as a coalitional game in which the value function is determined under information-theoretic guarantees, i.e., the value associated with a coalition is computed with no restrictions on the strategies that the users outside the coalition could adopt. We have shown that the game associated with our problem is convex, and characterized its core, which is interpreted as a converse for our setting. We have shown that for a given level of security clearance the grand coalition is in the best interest of all agents and stable, in the sense that any coalition of agents has a disincentive to leave the grand coalition. We have also characterized the Shapley value, and used it as a possible solution concept to ensure fairness among agents. Finally, we have proposed an explicit and low-complexity coding scheme relying on polar codes for source coding and hash functions to achieve any point of the core, including the Shapley value and the nucleolus. Under the proposed coalitional game theory framework, we thus obtain the secret-key capacity region for our problem. It contrasts with the fact that no tight outer bound is known for the model we consider when the selfishness constraints are removed, even when only a single level of security clearance is considered.

The alpha theory framework is general and could be applied to other security problems involving a tension between cooperation and self-interest. The challenge is in characterizing a value function for this framework. For instance, in our setting, being able to determine the
value function in the non-degraded setting remains open, and is, unfortunately, at least as difficult as determining the secret-key capacity for the two-user secret generation model of [16].

APPENDIX A

PROOF OF LEMMA 4

Let $S \subseteq \mathcal{L}$. We have

\begin{equation}
H(X_S^N | A_L) = H(X_S^N A_S A_{\mathcal{L} \setminus S}) - H(A_L) \tag{57a}
\end{equation}

\begin{equation}
= H(X_S^N A_{\mathcal{L} \setminus S}) - H(A_L) \tag{57b}
\end{equation}

\begin{equation}
= H(X_S^N) + H(A_{\mathcal{L} \setminus S} | X_S^N) - H(A_L) \tag{57c}
\end{equation}

\begin{equation}
\geq H(X_S^N) + H(A_{\mathcal{L} \setminus S} | X_S^N) - \log |A_L| \tag{57d}
\end{equation}

\begin{equation}
= H(X_S^N) + H(A_{\mathcal{L} \setminus S} | X_S^N) - N H(X_{\mathcal{L}} | X_0) + o(N) \tag{57e}
\end{equation}

\begin{equation}
= NI(X_S; X_0) + H(A_{\mathcal{L} \setminus S} | X_S^N) - N H(X_{\mathcal{L} \setminus S} | X_0 X_S) + o(N), \tag{57f}
\end{equation}

where (57b) holds because $A_S$ is a function of $X_S^N$, (57e) holds because

\begin{equation}
\log |A_L| = \sum_{l \in \mathcal{L}} \log |A_l| \tag{58a}
\end{equation}

\begin{equation}
= \sum_{l \in \mathcal{L}} |\mathcal{H}_{X_l | X_0 X_{1:l-1}}| \tag{58b}
\end{equation}

\begin{equation}
= \sum_{l \in \mathcal{L}} NH(X_l | X_0 X_{1:l-1}) + o(N) \tag{58c}
\end{equation}

\begin{equation}
= NH(X_{\mathcal{L}} | X_0) + o(N), \tag{58d}
\end{equation}

where (58c) holds by [42], [50], [51, Theorem 3.5].

We lower-bound the second term in the right-hand side of (57f) as follows

\begin{equation}
H(A_{\mathcal{L} \setminus S} | X_S^N) \geq \sum_{j \in \mathcal{L} \setminus S} H(A_j | A_{1:j-1} X_S^N) \tag{59a}
\end{equation}

\begin{equation}
\geq \sum_{j \in \mathcal{L} \setminus S} H(A_j | X_0^N X_{1:j-1} X_S^N) \tag{59b}
\end{equation}
\[ \sum_{j \in \mathcal{L}\setminus \mathcal{S}} H(U_j^N [\mathcal{H}_{X_j | X_0^{1:j-1}} | X_0^N X_{1:j-1}^N X_S^N]) \tag{59c} \]

\[ \geq \sum_{j \in \mathcal{L}\setminus \mathcal{S}} H(U_j^N [\mathcal{V}_{X_j | X_0^{1:j-1}X_S}] | X_0^N X_{1:j-1}^N X_S^N) \tag{59d} \]

\[ \geq \sum_{j \in \mathcal{L}\setminus \mathcal{S}} \sum_{i \in \mathcal{V}_{X_j | X_0^{1:j-1}X_S}} H((U_j)_i | (U_j)^i_{1:j-1} X_0^N X_{1:j-1}^N X_S^N) \tag{59e} \]

\[ \geq \sum_{j \in \mathcal{L}\setminus \mathcal{S}} \sum_{i \in \mathcal{V}_{X_j | X_0^{1:j-1}X_S}} (1 - \delta_N) \tag{59f} \]

\[ = \sum_{j \in \mathcal{L}\setminus \mathcal{S}} |\mathcal{V}_{X_j | X_0^{1:j-1}X_S}| (1 - \delta_N) \tag{59g} \]

\[ = \sum_{j \in \mathcal{L}\setminus \mathcal{S}} NH(X_j | X_0^{1:j-1}X_S) + o(N) \tag{59h} \]

\[ = \sum_{j \in \mathcal{L}\setminus \mathcal{S}} NH(X_j | X_0 X_{[1:j-1]\setminus \mathcal{L}\setminus \mathcal{S}} X_S) + o(N) \tag{59i} \]

\[ = NH(X_{\mathcal{L}\setminus \mathcal{S}} | X_0 X_S) + o(N), \tag{59j} \]

where (59a) and (59e) hold by the chain rule and because conditioning reduces entropy, (59d) holds because \( \mathcal{H}_{X_j | X_0^{1:j-1}} \supset \mathcal{H}_{X_j | X_0^{1:j-1}X_S} \supset \mathcal{V}_{X_j | X_0^{1:j-1}X_S} \), (59f) holds by definition of \( \mathcal{V}_{X_j | X_0^{1:j-1}X_S} \), (59h) holds by Lemma 2, (59j) holds by the chain rule.

Finally, combining (57f) and (59j) proves Lemma 4.

### Appendix B

**Proof of Lemma 5**

We first prove the result when \( Z = \emptyset \). For \( X_{\mathcal{L}}, X_{\mathcal{L}}', F_{\mathcal{L}}, F_{\mathcal{L}}' \) independent, we compute the following collision probability

\[ \mathbb{P}[(F_{\mathcal{L}}(X_{\mathcal{L}}), F_{\mathcal{L}}) = (F_{\mathcal{L}}'(X'_{\mathcal{L}}), F_{\mathcal{L}}')] \tag{60a} \]

\[ = \mathbb{P}[F_{\mathcal{L}} = F_{\mathcal{L}}'] \mathbb{P}[F_{\mathcal{L}}(X_{\mathcal{L}}) = F_{\mathcal{L}}(X_{\mathcal{L}}')] \tag{60b} \]

\[ = \prod_{l \in \mathcal{L}} \mathbb{P}[F_l = F_{l}'] \mathbb{P}[F_{\mathcal{L}}(X_{\mathcal{L}}) = F_{\mathcal{L}}(X_{\mathcal{L}}')] \tag{60c} \]

\[ = s_{\mathcal{L}}^{-1} \sum_{x_{\mathcal{L}}, x'_{\mathcal{L}}} \mathbb{P}[F_{\mathcal{L}}(x_{\mathcal{L}}) = F_{\mathcal{L}}(x'_{\mathcal{L}})] \mathbb{P}[X_{\mathcal{L}} = x_{\mathcal{L}}, X_{\mathcal{L}}' = x'_{\mathcal{L}}] \tag{60d} \]
\[ s_L^{-1} \sum_{x_L, x'_L} \mathbb{P}[F_L(x_L) = F_L(x'_L)] \mathbb{P}[X_L = x_L] \mathbb{P}[X'_L = x'_L] = s_L^{-1} \sum_{S \subseteq \mathcal{L}} \sum_{x_L} \sum_{x'_L} \mathbb{P}[F_L(x_L) = F_L(x'_L)] \mathbb{P}[X_L = x_L] \mathbb{P}[X'_L = x'_L] \]  

(60e) 

\[ = s_L^{-1} \sum_{S \subseteq \mathcal{L}} \sum_{x_L} \sum_{x'_L} \mathbb{P}[F_L(x_L) = F_L(x'_L)] \mathbb{P}[X_L = x_L] \mathbb{P}[X'_L = x'_L] \]  

(60f) 

\[ = s_L^{-1} \sum_{S \subseteq \mathcal{L}} \sum_{x_L} \sum_{x'_L} \prod_{l \in \mathcal{L}} \mathbb{P}[F_l(x_l) = F_l(x'_l)] \mathbb{P}[X_L = x_L] \mathbb{P}[X'_L = x'_L] \]  

(60g) 

\[ \leq s_L^{-1} \sum_{S \subseteq \mathcal{L}} \sum_{x_L} \sum_{x'_L} 2^{-r_S} \mathbb{P}[X_L = x_L] \mathbb{P}[X'_L = x'_L] \]  

(60h) 

\[ \leq s_L^{-1} \sum_{S \subseteq \mathcal{L}} \sum_{x_L} 2^{-r_S} \mathbb{P}[X_L = x_L] 2^{-H_{\infty}(p_{X_{S^c}})} \]  

(60i) 

\[ \leq s_L^{-1} \sum_{S \subseteq \mathcal{L}} 2^{-r_S-H_{\infty}(p_{X_{S^c}})} \]  

(60j) 

\[ = s_L^{-1} \sum_{S \subseteq \mathcal{L}} 2^{-r_S - H_{\infty}(p_{X_{S^c}})} , \]  

(60k) 

where (60h) holds by the two-universality of the \( F_l \)'s, \( l \in \mathcal{L} \), (60i) holds by marginalization over \( X'_L \), (60j) holds by definition of the min-entropy.

Then, viewing \( \mathbb{V}(p_{F_L(X_L), F_{L'}, p_{U_k} p_{U_{L'}}}) \) as a scalar product between \( (p_{F_L(X_L), F_L} - p_{U_k} p_{U_L}) \) and its sign, by Cauchy-Schwarz inequality, we have

\[ \mathbb{V}(p_{F_L(X_L), F_{L'}, p_{U_k} p_{U_{L'}}})^2 \leq s_L 2^{r_L} \sum_{m_L, f_L} \left[ p_{F_L(X_L), F_L}(m_L, f_L) - \frac{1}{s_L 2^{r_L}} \right]^2 \]  

(61a) 

\[ = s_L 2^{r_L} \left[ \sum_{m_L, f_L} p_{F_L(X_L), F_L}(m_L, f_L) \right]^2 - 1 \]  

(61b) 

\[ = s_L 2^{r_L} \mathbb{P}[(F_L(X_L), F_L) = (F'_L(X'_L), F'_L)] - 1 \]  

(61c) 

\[ \leq 2^{r_L} \sum_{S \subseteq \mathcal{L}} 2^{-r_S - H_{\infty}(p_{X_{S^c}})} - 1 \]  

(61d) 

\[ \leq \sum_{S \subseteq \mathcal{L}} 2^{r_{S^c} - H_{\infty}(p_{X_{S^c}})} \]  

(61e) 

\[ = \sum_{S \subseteq \mathcal{L}} 2^{r_{S^c} - H_{\infty}(p_{X_{S^c}})} , \]  

(61f)
where (61d) holds by (60k).

We now introduce the random variable $Z$ correlated to $X_L$ and proceed as in [49]. Let $z \in Z$ and $X_L^{(z)}$ be defined by $p_{X_L^{(z)}} = p_{X_L|Z=z}$. We have

$$\mathbb{V}(p_F(x_L), F_L, Z, p_U, p_F, p_Z) = \mathbb{E}_Z \left[ \mathbb{V}(p_{F_L}(X_L^{(z)}), F_L, Z, p_U, p_F) \right]$$

(62a)

$$\leq \mathbb{E}_Z \left[ \sum_{S \subseteq L} 2^{r_S - H_\infty(X_S|Z=z)} \right]$$

(62b)

$$\leq \sqrt{\sum_{S \subseteq L} 2^{r_S - H_\infty(X_S)}},$$

(62c)

where (62b) holds by (61f), (62c) holds by Jensen’s inequality and since, by definition, $\mathbb{E}_Z[2^{-H_\infty(X_S|Z=z)}] = 2^{-H_\infty(X_S|Z)}, S \subseteq L$.

**APPENDIX C**

**PROOF OF THEOREM 4**

We consider the coding scheme of Section IV-A. The only difference with the proof of Theorem 3 is that instead of Lemma 4, we now need to lower bound for any $T \subseteq S$, the quantity $H(X_N^T|X_S^N, A_S)$. We do it as follows.

$$H(X_N^T|X_S^N, A_S)$$

(63a)

$$= H(X_T^N X_S^N A_S) - H(X_S^N A_S)$$

(63b)

$$= H(X_T^N X_S^N A_S \setminus T) - H(X_S^N A_S)$$

(63c)

$$= H(X_T^N | X_S^N) + H(A_S \setminus T | X_T^N X_S^N) - H(A_S | X_S^N).$$

(63d)

We lower bound $H(A_S \setminus T | X_T^N X_S^N)$ in the right hand side of (63d) as follows.

$$H(A_S \setminus T | X_T^N X_S^N) \geq N H(X_S \setminus T | X_0 X_T X_S^c) + o(N)$$

(64a)

$$= N H(X_S \setminus T | X_0) + o(N),$$

(64b)

where (64a) holds similarly to (59j) proved in Appendix A by conditioning on $X_S^N$, (64b) holds by the Markov chain (1). We then upper bound $H(A_S | X_S^N)$ in the right hand side of (63d) as
follows.

\[
H(A_S | X^{N}_{S^c}) \leq \log |A_S| \tag{65a}
\]

\[
= \sum_{i \in S} \log |A_i| \tag{65b}
\]

\[
= \sum_{i \in S} |H_{X_i | X_0 X_{1:i-1}}| \tag{65c}
\]

\[
= \sum_{i \in S} NH(X_i | X_0 X_{1:i-1}) + o(N) \tag{65d}
\]

\[
= NH(X_S | X_0) + o(N) \tag{65e}
\]

\[
= NH(X_T | X_0) + H(X_{S \setminus T} | X_0) + o(N), \tag{65f}
\]

where (65d) holds by [50, Theorem 3.5], (65e) and (65f) hold by the Markov chain (1). Hence, we obtain

\[
H(X^{N}_T | X^{N}_{S^c} A_S) \geq NH(X_T | X_{S^c}) - NH(X_T | X_0) + o(N) \tag{66a}
\]

\[
= NI(X_T; X_0 | X_{S^c}) + o(N), \tag{66b}
\]

where (66a) holds by combining (63d), (64b), and (65f), (66b) holds by the Markov chain (1).

**APPENDIX D**

**PROOF OF COROLLARY 2**

We first show the following lemma and then use a similar argument than in [52].

**Lemma 6.** Let \( S \subseteq \mathcal{L} \). The set function \( w : 2^S \rightarrow \mathbb{R}^+ \), \( T \mapsto I(X_T; X_0 | X_{S^c}) \) is submodular, i.e., \( -w \) is supermodular.

**Proof.** Let \( U, V \subseteq S \). We have

\[
I(X_{U \cup V}; X_0 | X_{S^c}) + I(X_{U \cap V}; X_0 | X_{S^c}) \tag{67a}
\]

\[
= I(X_U; X_0 | X_{S^c}) + I(X_V \setminus U; X_0 | X_{S^c} X_U) + I(X_{U \cap V}; X_0 | X_{S^c}) \tag{67b}
\]

\[
= I(X_U; X_0 | X_{S^c}) + H(X_V \setminus U | X_{S^c} X_U) - H(X_V \setminus U | X_0 X_{S^c} X_U) + I(X_{U \cap V}; X_0 | X_{S^c}) \tag{67c}
\]
\[ \begin{align*}
&\leq I(X_U; X_0|X_S^c) + I(X_{V\setminus U}; X_0|X_S^c, X_{U\cap V}) + I(X_{U\setminus V}; X_0|X_S^c) \\
&= I(X_U; X_0|X_S^c) + I(X_V; X_0|X_S^c),
\end{align*} \tag{67d} \]

where (67d) holds because \( H(X_{V\setminus U}|X_S^c, X_U) \leq H(X_{V\setminus U}|X_S^c, X_{U\setminus V}) \) and because \( H(X_{V\setminus U}|X_0, X_S^c, X_U) = H(X_{V\setminus U}|X_0, X_S^c, X_{U\cap V}) \) by the Markov chain (1).

For \( S \subset \mathcal{L}, (S, w) \) defines a concave game by submodularity of \( w \) shown in Lemma 6. Consequently, its core \( \mathcal{C}(w) \triangleq \{(R_i)_{i \in S} : \sum_{i \in S} R_i = v(S) \text{ and } \sum_{i \in T} R_i \leq v(T), \forall T \subset S\} \) is non-empty by [32], i.e., there exists an achievable rate tuple in \( \mathcal{R}_S \) with sum-rate \( I(X_S; X_0|X_S^c) \).

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