Predefined-Time Consensus of High-Order Multi-Agent Systems

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ABSTRACT
This paper addresses the distributed predefined-time leader-following consensus problem for a class of high-order multi-agent systems (MAS) with perturbed nonlinear agents’ dynamics and where the topology of the network contains a directed spanning tree, with the leader as the root. The proposed control method exhibits three main advantages: first, to our best knowledge, it is the first time that predefined-time convergence in a consensus problem is achieved for agents with high-order nonlinear dynamics, using a robust leader-following protocol, which allows an effective rejection of matched disturbances in the agents model. Second, the proposed controller provides continuous and smooth control signals of lower magnitude than existing approaches. Third, the proposed consensus protocol does not have parameters to be adjusted depending on the connectivity of the considered communication graph.

KEYWORDS
Predefined-time consensus; time base generators (TBGs); multi-agent systems; distributed protocol; high-order systems.

1. Introduction

Cooperative control of Multi-Agent Systems (MAS) is a broad topic involving many different related research problems, such as consensus, formation control, flocking, coverage control, among others; attracting considerable attention over the last decades due to their broad applications in different research areas (Chen, Lu, Yu, & Hill, 2013; Lewis, Zhang, Hengster-Movric, & Das, 2014; Ren & Cao, 2010). A fundamental problem in cooperative control of MAS is to design distributed consensus protocols in order to make the autonomous agents to agree on some variable of interest (for instance, achieving a common internal state). In this problem, each agent applies a local controller that only uses information obtained from local interactions between neighboring agents. There are two main consensus problems, the leader-following case, in which all agents converge to the state of the leader, and the leaderless case, in which the agents agree to a consensus state resulting from the local interactions and control protocols.

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With the aim of addressing the distributed consensus problem satisfying real-time constraints, a great deal of work has been recently published, proposing distributed consensus algorithms with finite-time and fixed-time convergence, see e.g., the works of Aldana-López, Gómez-Gutiérrez, Defoort, Sánchez-Torres, and Muñoz-Vázquez (2019); Mondal, Su, and Xie (2017); Ning, Jin, Zheng, and Man (2018). In both kind of convergence, the settling time is a finite value, but in the fixed-time convergence the settling time is uniformly bounded, meaning that the system converges to its equilibrium before an estimated bound that is independent of the initial state. Nevertheless, in the existing approaches of finite and fixed-time consensus, the convergence bound estimates are too conservative, leading to over-engineering the system to satisfy the real-time constraints (Zuo, Han, Ning, Ge, & Zhang, 2018), resulting in large control efforts as a drawback. In addition, to satisfy time constraints, such methods require a lower estimate of the algebraic connectivity of the network (Aldana-López et al., 2019). To overcome these problems, distributed algorithms with predefined-time convergence have been proposed. Unlike finite and fixed-time consensus, predefined-time consensus means that an agreement state of the MAS is achieved in a prespecified time, introduced as a parameter of the protocol, and that settling time is achieved independently of the agents’ initial state. Then, the settling time is constant for all the initial states in predefined-time convergence and there is no slack between the real and the desired settling time, which is not possible to obtain with autonomous finite and fixed-time algorithms. The hard time constraint of predefined-time consensus facilitates a user to set the MAS convergence time and it increases the potential of the engineering applications of consensus.

In this work, we address the distributed predefined-time consensus problem for high-order MAS with nonlinear dynamics and affected by disturbances. We consider a leader-following configuration where the topology among the followers and the leader contains a directed spanning tree, with the leader as the root. We introduce a novel methodology to solve the predefined-time consensus problem for a MAS of high-order dynamics. Our methodology consists in tracking reference signals by using feedback controllers. The references are defined by time base generators (TBGs), which are continuous time-dependent polynomial functions that converge to zero in a specified time (Becerra, Vázquez, Arechavaleta, & Delfin, 2018; Morasso, Sanguineti, & Spada, 1997). We propose two consensus protocols in which only a leader agent gives the reference to the high-order MAS. The first one is a linear control protocol where the tracking error of the high-order TBG trajectories between neighbor agents is feedbacked. The second one is a robust consensus protocol, based on the super-twisting controller to provide closed-loop stability and robustness against disturbances. The proposed protocols can be applied to nonlinear high-order MAS that can be transformed to the normal form (e.g. the Brunovsky’s canonical form by state feedback linearization (Khalil & Grizzle, 2002), in which agents dynamics are represented as a chain of integrators. Convergence at the predefined-time and global closed-loop stability are demonstrated theoretically and illustrated through simulations. Furthermore, comparisons between our proposal and existing predefined-time protocols are provided through simulations, showing that the proposed approach provides smoother and lower control efforts than other approaches. Besides, the proposed control protocols do not require a priori information of the network connectivity, which is also a robustness property that allows the method to work properly for small or large number of agents without readjusting controller parameters.
1.1. Related Work

To date, few contributions addressing the predefined-time consensus problem have been reported in the literature, mainly focusing on the MAS with first-order and second-order agents, as all the references described in this paragraph. A class of distributed linear protocols were developed for linear MAS over both undirected and directed communication networks (C. Liu, Zhou, & Liu, 2014) [103x723]Wang, Song, Hill, & Krstic, 2019] [103x703]Yong, Guangming, & Huiyang, 2012a, 2012b]. By using time-varying control gains, the agents in the network are forced to reach consensus at any pre-set time from any initial condition. Following another approach, the predefined-time consensus problem has been transformed into a motion planning problem in which the developed consensus protocols are based on a time-varying sampling sequence convergent to an off-line desired settling time (Y. Liu & Zhao, 2017) [103x535]Y. Liu, Zhao, Ren, & Chen, 2018] [103x522]Y. Liu, Zhao, Shi, & Wei, 2016] [103x521]Zhao & Liu, 2017b]. With a pre-specified settling time, these protocols solve the consensus problem of linear MAS over undirected and directed topologies, and directed switching topologies.

Time-varying gains derived from TBGs have been used to solve different consensus problems with prescribed convergence for agents with first order dynamics (Kan, Yucelen, Doucette, & Pasiliao, 2017) [103x496]Ning, Han, & Zuo, 2019] [103x495]Yucelen, Kan, & Pasiliao, 2018]. In particular, the rendezvous problem by Kan et al. (2017) and the leader-following consensus problem presented by Ning et al. (2019) [103x509]Yucelen et al. (2018), have been addressed. The consensus algorithms with time-varying gains present important drawbacks, for instance, the time-varying gain becomes singular as the prescribed-time is reached (Kan et al., 2017) [103x483]Yong et al., 2012b] [103x481]Yucelen et al., 2018] or the time-varying gain is piecewise constant (Y. Liu et al., 2018) with Zeno behavior (Zhang, Johansson, Lygeros, & Sastry, 2001). Commonly, a parameter is added to the controller to avoid the singularity of the time-varying gain. Unfortunately, with such modification, consensus cannot be reached in a constant time, but all agents arrive to a value around the consensus state in fixed-time. Following with MAS of first-order dynamics, TBGs have also been used to impose a reference profile of a consensus error and consequently achieve predefined-time convergence (Colunga, Vázquez, Becerra, & Gómez-Gutiérrez, 2018a, 2018b). In those works, a predefined-time distributed consensus protocol that requires to give the consensus value to each agent as a parameter of the controller was introduced.

Concerning high-order MAS, there exist results based on autonomous protocols (Ning & Han, 2019) [103x276]Tian, Zuo, & Wang, 2017] [103x275]Zhang & Duan, 2019], which are not able to achieve a constant convergence time and only an overestimation of it is possible. To our best knowledge only three works address consensus in predefined-time for high-order MAS (Wang & Song, 2018) [103x237]Zhao & Liu, 2017a] [103x236]Zhao, Liu, Wen, Ren, & Chen, 2019]. A multi-leaders approach has been presented by Zhao and Liu (2017a), in which the followers are driven to the convex hull spanned by the leaders at a specified settling time if the undirected fixed topology is connected. A distributed time-varying control approach in a leader-following consensus scheme based on a finite-time observer has been investigated by Wang, Song, 2018), demonstrating consensus in a pre-specified finite-time under fixed directed topologies having a spanning tree. Recently, a specified-time consensus protocol in a leaderless scheme has been developed by Zhao et al. (2019) for MAS with general linear dynamics over directed graphs containing a directed spanning tree and based on a motion planning strategy.

To the authors’ opinion, only the works of (Wang & Song, 2018) [103x289]Zhao et al., 2019) represent the closest approaches to the problem addressed in this paper. Nevertheless,
in contrast to them, in our approach neither Zeno behavior is produced nor singularities occur in the control signal, avoiding thus the main drawbacks of consensus approaches based on time-varying gains. Moreover, unlike the aforementioned works [Wang & Song 2018, Zhao & Liu 2017a, Zhao et al. 2019], we address the predefined consensus problem for the case where agents are affected by disturbances. In addition, it is shown that our consensus protocols produce continuous and smoother signals of lower magnitude than those approaches, which is more suitable for certain applications, e.g. in formation control of MAS, where the consensus signal provides velocity references to be tracked by the agents (Oh, Park, & Ahn 2015). Finally, those approaches require to readjust some controller parameters depending on the algebraic connectivity of the network (Aldana-López et al. 2019), which is not the case of our approach.

The rest of this paper is organized as follows. Section 2 recalls basic concepts. Section 3 introduces the model of the MAS and defines the predefined-time consensus problem. In Section 4 a linear-feedback controller and a robust controller are proposed for the predefined-time consensus problem in a leader-following scheme. Section 5 reports simulations of the proposed approach and comparisons with other protocols in the literature. Finally, Section 6 presents some conclusions.

2. Theoretical Preliminaries

Let us first introduce some notation. \( I_n \) represents the identity matrix of dimension \( n \times n \), \( 0_{n \times n} \) denotes the zero matrix of dimension \( n \times n \). Let \( 1_n \) and \( 0_n \) be the \( n \)-dimensional column vectors with all entries equal to one and zero, respectively. \( A \otimes B \) denotes the Kronecker product of matrices \( A \) and \( B \).

2.1. Time Base Generators

Time base generators (TBGs) are parametric functions of time, particularly designed to stabilize a system in such a way that its state describes a convenient transient profile. TBGs have been previously used by Becerra et al. (2018) to achieve predefined-time convergence of first-order and higher-order systems. For the case of first-order systems, TBGs have been used for the control of robots at kinematic level by Jarquín, Arechavaleta, and Parra-Vega (2011).

Definition 2.1. (Becerra et al., 2018) A TBG of order \( r \) and settling time \( t_f \) is defined as a continuous and differentiable time-dependent function \( h(t) \), described as

\[
h(t) = \begin{cases} 
\tau(t) \cdot c & \text{if } t \in [0, t_f] \\
0 & \text{otherwise},
\end{cases}
\]

where \( \tau(t) = [t^r, t^{r-1}, \ldots, t, 1] \) is the time basis vector and \( c \) is a vector of coefficients of proper dimensions.

Definition 2.2. (Becerra et al. 2018) Consider a control system of order \( n \). For the design of a predefined-time controller, a collection of \( n \) TBGs of order \( r \geq 2n + 1 \) is
designed, fulfilling the following conditions at initial time and settling time $t_f$

$$\forall k \in \{1, \ldots, n\}, \ \forall j \in \{0, \ldots, n\},$$

$$h_k^{(j)}(t)_{t=0} = \begin{cases} 1 & \text{if } j = k - 1 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

$$h_k^{(j)}(t)_{t \geq t_f} = 0,$$

where $h_k(t) = \tau(t) \cdot c_k$ denotes the $k$-th TBG in the collection for $t \in [0, t_f]$, and $h_k^{(j)}(t)$ denotes its $j$-th time derivative.

The TBGs and their derivatives are grouped in a time-varying matrix as

$$H(t) = \begin{bmatrix}
h_1(t) & h_2(t) & \cdots & h_n(t) \\
h_1'(t) & h_2'(t) & \cdots & h_n'(t) \\
\vdots & \vdots & \ddots & \vdots \\
h_1^{(n-1)}(t) & h_2^{(n-1)}(t) & \cdots & h_n^{(n-1)}(t)
\end{bmatrix}, \quad (3)$$

then $H(0) = I_n$, and $H(t \geq t_f) = 0_{n \times n}$ according to the constraints (2).

Predefined-time controllers for a single high-order system have been introduced by Becerra et al. (2018), using time-varying controllers based on the TBGs. There, the TBGs were used as time-varying gains and reference trajectories to be tracked by means of feedback controllers. Moreover, a couple of procedures for calculating the coefficients $c_k$, fulfilling the required constraints, were introduced. In particular, when a high degree polynomial is used, optimal coefficients can be computed by minimizing a cost function.

### 2.2. Algebraic Graph Theory

In a MAS, the communication between agents is represented by a graph. Let us recall some basic definitions on graph theory (Horn & Johnson, 2012; Li & Duan, 2014; Yu, Wen, Chen, & Cao, 2017).

**Definition 2.3.** A communication graph is a tuple $G = (\mathcal{V}, \mathcal{E}, A)$ that consists of a set of vertices representing agents $\mathcal{V} = \{v_1, \ldots, v_N\}$, a set of edges representing communication channels $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ with non-negative entries $a_{ij}$; in particular, $a_{ij} > 0$ if $(v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0$ if $(v_i, v_j) \notin \mathcal{E}$. The set of neighbors of agent $i$ is denoted by $N_i = \{j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$.

**Definition 2.4.** A graph $G$ is called directed if the edges are ordered pairs, i.e., $(v_i, v_j)$ and $(v_j, v_i)$ denote different edges. A graph $G$ is called undirected if the edges are unordered pairs, i.e., $(v_i, v_j)$ and $(v_j, v_i)$ denote the same edge.

In the MAS framework, $(v_i, v_j)$ denotes that agent $v_j$ can obtain information from agent $v_i$.

**Definition 2.5.** A directed path from vertex $v_i$ to $v_j$ is a sequence of distinct vertices $v_i, v_{i_1}, \ldots, v_{i_r}, v_j \in \mathcal{V}$ and edges $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_r}, v_j) \in \mathcal{E}$.

A graph $G$ is said to be strongly connected if there exists a directed path between
any two distinct vertices \( v_i \) and \( v_j \) in \( \mathcal{V} \).

A directed graph \( \mathcal{G} \) is said to have a directed spanning tree if \( \mathcal{G} \) has at least one vertex \( v_i \), named root, such that for any other vertex \( v_j \in \mathcal{V} \setminus \{v_i\} \) there is a directed path from \( v_i \) to \( v_j \).

It is known that a strongly connected directed graph contains at least one directed spanning tree (Li & Duan, 2014; Yu et al., 2017).

**Definition 2.6.** Let \( \mathcal{G} \) be a graph with \( N \) vertices. The Laplacian matrix of \( \mathcal{G} \) is defined as the \( N \times N \) matrix \( \mathbf{L} = [l_{ij}] \), where

\[
    l_{ij} = \begin{cases} 
    -a_{ij}, & \text{if } i \neq j, \\
    \sum_{k=1,k \neq i}^{N} a_{ik}, & \text{if } i = j.
    \end{cases}
\]  

(4)

### 3. MAS Definition and Problem Statement

**Definition 3.1.** A multi-agent system (MAS) consists of a collection of \( N \) agents named followers, whose dynamics are described by nonlinear systems with relative degree \( n \), an agent named leader, and a communication graph \( \mathcal{G} \) with \( N + 1 \) vertices, each one associated to a different agent. It is assumed that the model of each follower agent is in the normal form (Khalil & Grizzle, 2002), e.g. for the \( i \)-th agent

\[
\begin{align*}
\dot{x}_{i1} &= x_{i2}, \\
& \quad \vdots \\
\dot{x}_{i(n-1)} &= x_{in}, \\
\dot{x}_{in} &= f_i(x_i, \varphi_i) + g_i(x_i, \varphi_i)u_i(t) + \rho_i(t), \\
\dot{\varphi}_{i1} &= q_{i1}(x_i, \varphi_i), \\
& \quad \vdots \\
\dot{\varphi}_{ir} &= q_{ir}(x_i, \varphi_i), \\
y_i &= x_{i1},
\end{align*}
\]  

(5)

where \( x_i = [x_{i1}, \ldots, x_{in}]^T \in \mathbb{R}^n \) is the agent’s state, \( u_i(t) \in \mathbb{R} \) is the agent’s control input, \( f_i(x_i, \varphi_i) \) and \( g_i(x_i, \varphi_i) \) are smooth nonlinear functions, \( \rho_i(t) \in \mathbb{R} \) is a time-varying bounded disturbance, \( \varphi_i = [\varphi_{i1}, \ldots, \varphi_{ir}]^T \in \mathbb{R}^n \) is the agent’s zero dynamics, and \( y_i \in \mathbb{R} \) is the agent’s output. It is assumed that the relative degree is well defined \( (g_i(x_i, \varphi_i) \neq 0) \), and the zero dynamics is stable.

Furthermore, the leader is an agent whose dynamics are given as an integrator chain, i.e.

\[
\begin{align*}
\dot{x}_{lk} &= x_{l(k+1)}, & k &= 1, \ldots, n - 1 \\
\dot{x}_{ln} &= u_l(t),
\end{align*}
\]  

(6)

where \( x_l(t) \in \mathbb{R}^n \) is the leader’s state and \( u_l(t) \in \mathbb{R} \) is the leader’s control input.

The input-output dynamics (5) of each \( i \)-th follower agent can be conveniently
represented as an $n$-integrators chain as

$$\dot{x}_{ik} = x_{i(k+1)}, \quad k = 1, \ldots, n - 1 \quad (7)$$

$$\dot{x}_{in} = v_i(t) + \rho_i(t),$$

by applying the control input

$$u_i = (-f_i(x_i, \varphi_i) + v_i)/g_i(x_i, \varphi_i), \quad (8)$$

where $v_i(t) \in \mathbb{R}$ is an auxiliary control input. Then the $i$-th agent’s dynamic (7) can be expressed in a vectorial form as

$$\dot{x}_i = Ax_i + B(v_i(t) + \rho_i(t)), \quad i \in \{1, \ldots, N\} \quad (9)$$

with adequate constant matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^n$, where $(A, B)$ is controllable.

Similarly, the leader’s dynamics (6) can be written as

$$\dot{x}_l = Ax_l + Bu_l(t), \quad (10)$$

The class of agents that can be represented in the form of (7) is broad. In particular, any SISO linear time-invariant system $\dot{x}_i = \bar{A}x_i + \bar{B}(u_i(t) + \rho_i(t))$ can also be transformed into an $n$-order integrator system (7) provided it is controllable, observable and has no transmission zeros, by transforming the system into the so-called observability canonical form (Kailath, 1980) and applying an input that cancels the open-loop dynamics of the $n$-th state equation.

Now, let us introduce the concept of consensus error in the leader-following scheme.

**Definition 3.2.** Consider a MAS, where each follower is already in its integrator chain form (7), i.e., the control law (8) is applied to each follower agent. The **consensus error** for each $i$-th follower is defined as

$$e^f_i(t) = [e_{i1}, \ldots, e_{in}]^T = \sum_{j \in N_i \setminus \{l\}} a_{ij} (x_j(t) - x_i(t)) - b_i (x_i(t) - x_l(t)), \quad (11)$$

where $a_{ij}$ are the entries of the graph adjacency matrix, $N_i \setminus \{l\}$ denotes the set of neighbors of agent $i$-th excepting the leader, and $b_i = a_{il}$ represents the adjacency to the leader ($b_i = a_{il} > 0$ if agent $i$ is a neighbor of the leader, $b_i = a_{il} = 0$ otherwise).

**Problem statement.** Consider a MAS as in Definition 3.1 and assume that the control law (8) is applied to each agent. The predefined-time consensus problem consists in designing a protocol in the form $v_i = \eta_i(e^f_i, t)$ for each follower agent, such that the state of all the agents reach a consensus state, given by the leader’s state $x_l$, in a predefined time $t_f$ from any initial state $x_i(0)$, i.e., $\forall i \in \{1, \ldots, N\}, x_i(t) \to x_l(t)$ as $t \to t_f$.

**Assumption 1.** If the topology $\mathcal{G}$ is an undirected graph, then it is connected. If the topology $\mathcal{G}$ is a directed graph, then it has an spanning tree in which the leader acts as the root. The leader vertex will be denoted as $l$. 

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4. Predefined-Time Consensus Protocols

In this section, two protocols for the predefined-time consensus problem are introduced. First, a linear feedback-based consensus protocol is presented. Later, a consensus protocol based on the super-twisting controller is proposed, providing robustness against disturbances while maintaining the predefined-time convergence property.

Before introducing the protocols, let us first demonstrate that Assumption 1 implies that the consensus is reached when the consensus errors are null for all the agents.

**Lemma 4.1.** Consider a high-order MAS modeled as in Definition 3.1, fulfilling Assumption 1, and the control law (8) for each follower agent. If for each i-th follower $e^f_i(t) = 0$, then $x_i(t) = x_i(t), \ i.e., \ consensus \ is \ reached.$

**Proof.** Let us define $M = \text{diag}(b_1, \ldots, b_N)$ (a diagonal matrix with entries $b_1, \ldots, b_N$) and $m = M \cdot 1_N = [b_1, \ldots, b_N]^T$. Moreover, let us denote $x(t) = [x^T_1(t), \ldots, x^T_N(t)]^T$ and $e^f(t) = [e^f_1(t), \ldots, e^f_N(t)]^T$. By using the Laplacian matrix $L$ and (11), $e^f(t)$ can be expressed as

\[
e^f(t) = -(L \otimes I_n)x(t) - (M \otimes I_n)(x(t) - (1_N \otimes x_i(t)))
\]

\[
= -(L \otimes I_n + M \otimes I_n)x(t) + (M \cdot 1_N \otimes x_i(t))
\]

\[
= -(L \otimes I_n + M \otimes I_n)x(t) + (m \otimes x_i(t))
\]

\[
= -(L \otimes I_n + M \otimes I_n)x(t) + (m \otimes I_n)(1 \otimes x_i(t)). \quad (12)
\]

Lemma 1 from Shao, Shi, and Cao (2018) for high order systems ensures $(L \otimes I_n + M \otimes I_n)^{-1}(m \otimes I_n) = 1_N \otimes I_n$, which holds by Assumption 1. By using this result and the hypothesis $e^f_i(t) = 0_n$, the equation (12) can be solved for $x(t)$ as

\[
x(t) = (L \otimes I_n + M \otimes I_n)^{-1}(m \otimes I_n)(1 \otimes x_i(t))
\]

\[
= (1_N \otimes I_n)(1 \otimes x_i(t))
\]

\[
= 1_N \otimes x_i(t). \quad (13)
\]

Therefore, consensus of the high-order MAS in the leader-following scheme is achieved when, for each i-th follower, $e^f_i(t) = 0$.

Now, in order to solve the predefined-time consensus problem, our approach is to design a protocol $v_i$ for each i-th agent, that enforces the consensus error’s transient behavior $e^f_i(t) = H(t)e^f_i(0)$, named the TBG reference trajectory for the i-th follower agent, where $H(t)$ is defined as in (3) and $e^f_i(0)$ denotes the initial consensus error. In this context, the tracking error for the i-th agent is defined as

\[
\xi_i(t) = [\xi_{i1}, \ldots, \xi_{in}]^T = e^f_i(t) - H(t)e^f_i(0). \quad (14)
\]

Thus, the consensus protocols to be defined for each agent, $v_i$, will enforce $\xi_i(t) = 0_n \ \forall t \geq 0$. In fact, since $H(t) = 0_{nxn} \ \forall t \geq t_f$, then $\xi_i(t) = 0_n \ \forall t \geq 0$ implies $e^f_i(t) = H(t)e^f_i(0) = 0_n \ \forall t \geq t_f$, i.e., if the tracking error is null then the consensus error is null at $t_f$, consequently, if this occurs for all the agents then the consensus is reached at $t_f$. 

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It is possible to perfectly track the TBG reference trajectory from the initial time 
(i.e., \(\xi_i(t) = 0\), \(\forall t \geq 0\)) due to its coherence with the initial consensus error and the 
error dynamics. In detail, notice that \(H(t) e_i^f(0) = e_i^f(0)\) at \(t = 0\), since \(H(0) = I_n\) by 
Definition 2.2 and (3). Moreover, the definition of \(H(t)\) implies that \(\dot{e}_{ij}^f(t) = e_{i(j+1)}^f(t)\) 
\(\forall t > 0, \forall j \in \{1,...,(n-1)\}\), i.e., the reference trajectory represents an \(n\)-integrator 
chain, similar to the dynamics of the followers and the leader.

4.1. Predefined-Time Consensus for a Linear Leader-Following Scheme

The following theorem proposes a feedback-based protocol able to drive a high-order 
MAS to consensus in predefined-time, providing closed-loop stability of the tracking 
error.

**Theorem 4.2.** Consider a high-order MAS modeled as in Definition 3.1 with \(\rho_i = 0\), 
fulfilling Assumption 1, and the control law (8) for each follower agent. Consider TBG 
functions for a system of order \(n\) as in (2), gathered in the matrix \(H(t)\) as in (3), 
and define the time-varying gain vector \(K_i(t) = [h_1^n(t),h_2^n(t),...h_n^n(t)]\). For each agent 
i, define \(\beta_i = (b_i + \sum_{j \in N_i \backslash \{i\}} a_{ij})\) and consider its tracking error (14) and the vector 
\(\xi_{i2:in} = [\xi_2,...,\xi_{in}]^T\).

Considering the linear controller defined for each follower agent \(i\) as

\[
v_i = \beta_i^{-1} \left( b_i u_l + \sum_{j \in N_i \backslash \{i\}} a_{ij} v_j - K_i(t) e_i^f(0) + K_{f,i} \xi_{i2:in} \right),
\]

there exist gains \(K_{fr} \in \mathbb{R}^{n-1}\) such that the agents’ tracking errors \(\xi_i(t)\) are globally 
asymptotically stable. Furthermore, predefined-time convergence of the followers’ state 
\(x_i(t)\) to the leader’s state \(x_l(t)\) is achieved at time \(t_f\), independently of their initial 
states.

**Proof.** Part I. First, let us prove closed-loop stability of the tracking error of each 
agent \(\xi_i\).

By taking the time derivative of the consensus error of the \(i\)-th follower (11), using the 
dynamics of the followers (9) and the leader (10), and assuming \(\rho_i = 0\), the 
dynamics of the consensus error of the \(i\)-th follower is expressed as

\[
\dot{e}_i^f(t) = A e_i^f(t) + B \left( -\beta_i v_i + \sum_{j \in N_i} a_{ij} v_j + b_i u_l \right).
\]

The time derivative of the tracking error (14) requires to compute \(\dot{H}(t)\). By employing the matrices \(A\) and \(B\) in (9), we obtain

\[
BK_i(t) = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\begin{bmatrix}
h_1^n(t),h_2^n(t),...,h_n^n(t)
\end{bmatrix} = 
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
h_1^n(t) & h_2^n(t) & \cdots & h_n^n(t)
\end{bmatrix}.
\]
and

\[ AH(t) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0 
\end{bmatrix} \begin{bmatrix}
h_1(t) & h_2(t) & \cdots & h_n(t) \\
h_1(t) & h_2(t) & \cdots & h_n(t) \\
\vdots & \vdots & \ddots & \vdots \\
h_1^{(n-1)}(t) & h_2^{(n-1)}(t) & \cdots & h_n^{(n-1)}(t) \\
0 & 0 & \cdots & 0 
\end{bmatrix} \]

Taking the time derivative of (3), it results

\[ \dot{\mathbf{H}}(t) = \begin{bmatrix}
\dot{h}_1(t) & \dot{h}_2(t) & \cdots & \dot{h}_n(t) \\
\dot{h}_1(t) & \dot{h}_2(t) & \cdots & \dot{h}_n(t) \\
\vdots & \vdots & \ddots & \vdots \\
\dot{h}_1^{(n-1)}(t) & \dot{h}_2^{(n-1)}(t) & \cdots & \dot{h}_n^{(n-1)}(t) \\
0 & 0 & \cdots & 0 
\end{bmatrix}. \]

Then, we readily derive that

\[ \dot{\mathbf{H}}(t) = AH(t) + BK_i(t). \]  

Then, using (16) and (17), the time derivative of the tracking error (14) can be expressed as

\[ \dot{\xi}_i(t) = \dot{e}_i^f(t) - \dot{\mathbf{H}}(t)\tilde{e}_i^f(0), \]

\[ = A\xi_i(t) + B\left(-\beta_i v_i + \sum_{j \in N_i \setminus \{i\}} a_{ij} v_j + b_i u_i - K_i(t)e_i^f(0)\right). \]

Given the canonical form of \( A \) and \( B \), the tracking error dynamics are represented as

\[ \dot{\xi}_{ik}(t) = \xi_{i(k+1)}, \quad k = 1, \ldots, n - 1, \]

\[ \dot{\xi}_{in}(t) = -\beta_i v_i + \sum_{j \in N_i \setminus \{i\}} a_{ij} v_j + b_i u_i - K_i(t)e_i^f(0). \]  

Substituting the control protocol (15) into the above expression yields

\[ \dot{\xi}_{ik} = \xi_{i(k+1)}, \quad \text{for } k = 1, \ldots, n - 1, \]

\[ \dot{\xi}_{in} = -K_{fr}\xi_{i2:in}, \]

which can be enforced to exhibit global asymptotic stability through an appropriate choice of the gain vector \( K_{fr} \), since this dynamics can be seen as a controllable chain of \( n - 1 \) integrators with a feedback input \(-K_{fr}\xi_{i2:in}\). Then, for each \( i \)-th agent, \( \xi_i(t) \)
is globally asymptotically stable.

Part II. Now let us prove predefined-time consensus.

Recall that at initial time $t = 0$, $H(0) = I_n$ and thus $\xi_i(0) = 0_n$ according to (14). This implies that $\xi_i(t) = 0_n \forall t \geq 0$ by the stability of (19). Thus, it follows that $e_i^f(t) = H(t)e_i^f(0) \forall t \geq 0$ in accordance to the definition of the tracking error (14). Finally, since $H(t_f) = 0_{n \times n}$, then $e_i^f(t_f) = 0_n$. Since this occurs for all the agents, Lemma 4.1 implies that consensus is reached at $t_f$.

4.2. Robust Predefined-Time Consensus for a Leader-Following Scheme

In order to effectively deal with disturbances while achieving predefined-time convergence, a robust protocol is proposed in the sequel based on the super-twisting controller (STC), which is known to be able to effectively reject matched uncertainties/disturbances (Moreno & Osorio, 2012).

As explained before, the predefined-time consensus problem can be solved by designing a protocol $v_i$ for each follower agent such that $\xi_i(t) = 0_n \forall t \geq 0$. Since the tracking error exhibits high-order dynamics, in order to apply the STC technique, a sliding surface (an algebraic variety in the state space containing the origin) is firstly designed in such a way that $\xi_i(t)$ is asymptotically stable when confined to the sliding surface, later, the STC is designed in order to maintain the tracking error on the surface (Chalanga, Kamal, Fridman, Bandyopadhyay, & Moreno, 2016). The sliding surface designed for the $i$-th follower agent is characterized by a variable $s_i(\xi_i(t))$, in such a way that $s_i(\xi_i(t)) = 0$ when the tracking error is evolving on the surface. In particular, we define

$$s_i(t) = [K_{fr}, 1] \xi_i(t) = K_{fr}\xi_{i1:i(n-1)} + \xi_{in}, \quad \text{(20)}$$

where $K_{fr} \in \mathbb{R}^{n-1}$ is a gain vector and $\xi_{i1:i(n-1)} = [\xi_{i1}, \ldots, \xi_{i(n-1)}]^T$.

**Theorem 4.3.** Consider a high-order perturbed MAS modeled as in Definition 3.1, fulfilling Assumption 1, and the control law (8) for each follower agent. Consider TBG functions for a system of order $n$ as in (2), gathered in the matrix $H(t)$ as in (3), and define the time-varying gain vector $K_i(t) = [h_i^{(1)}(t), \ldots, h_i^{(n)}(t)]$. For each agent $i$, define $\beta_i = (b_i + \sum_{j \in N \setminus \{i\}} a_{ij})$ and consider its tracking error (14), the vector $\xi_{i2:in} = [\xi_{i2}, \ldots, \xi_{in}]^T$ and the sliding surface (20).

Considering the nonlinear controller defined for each follower agent $i$ as

$$v_i = \beta_i^{-1} (\mu_i + \nu_i), \quad \text{(21)}$$

$$\mu_i = b_i u_i + \sum_{j \in N \setminus \{i\}} a_{ij} v_j - K_i(t)e_i^f(0) + K_{fr}\xi_{i2:in},$$

$$\nu_i = k_1 |s_i|^{1/2} \text{sign}(s_i) - w_i,$$

$$\dot{w}_i = -k_2 \text{sign}(s_i),$$

there exist gains $K_{fr} \in \mathbb{R}^{n-1}$, $k_1 > 0$ and $k_2 > 0$ such that the agents’ tracking errors $\xi_i(t)$ are globally asymptotically stable. Furthermore, predefined-time convergence of
the followers’ state $x_i(t)$ to the leader’s state $x_l(t)$ is achieved at time $t_f$, independently of their initial states.

Proof. Part I. First, let us prove closed-loop stability of the tracking error of each agent $\xi_i$.

By taking the time derivative of the consensus error of the $i$-th follower (11) and using the perturbed dynamics of the followers (9) and the leader (10), the dynamics of the consensus error of the $i$-th follower is expressed as

$$\dot{e}_i^f(t) = A e_i^f(t) + B \left( -\beta_i v_i + \sum_{j \in N_i} a_{ij} v_j - \beta_i \rho_i + \sum_{j \in N_i \setminus \{l\}} a_{ij} \rho_j + b_i u_l \right). \quad (22)$$

By employing the expression $\dot{H}(t) = AH(t) + BK_i(t)$, derived in the proof of Theorem 4.2, and introducing (22), the time derivative of the tracking error (14) can be expressed as

$$\dot{\xi}_i(t) = \dot{e}_i^f(t) - \dot{H}(t) e_i^f(0),$$

$$= A \xi_i(t) + B \left( -\beta_i v_i + \sum_{j \in N_i \setminus \{l\}} a_{ij} v_j - \beta_i \rho_i + \sum_{j \in N_i \setminus \{l\}} a_{ij} \rho_j + b_i u_l - K_i(t) e_i^f(0) \right).$$

Given the canonical form of $A$ and $B$, the tracking error dynamics are represented as

$$\dot{\xi}_{ik}(t) = \xi_{i(k+1)}, \quad k = 1, \ldots, n - 1,$$

$$\dot{\xi}_{in}(t) = -\beta_i v_i + \sum_{j \in N_i \setminus \{l\}} a_{ij} v_j - \beta_i \rho_i + \sum_{j \in N_i \setminus \{l\}} a_{ij} \rho_j + b_i u_l - K_i(t) e_i^f(0). \quad (23)$$

By using this expression, the dynamics of the variable $s_i(t)$ (20) is computed as

$$\dot{s}_i(t) = K_{fr} \xi_{2:n} + \left( -\beta_i v_i + \sum_{j \in N_i \setminus \{l\}} a_{ij} v_j - \beta_i \rho_i + \sum_{j \in N_i \setminus \{l\}} a_{ij} \rho_j + b_i u_l - K_i e_i^f(0) \right).$$

Substituting the control protocol (21) into the above expression yields

$$\dot{s}_i = -k_1 |s_i|^{1/2} \text{sign}(s_i) + w_i + \psi_i,$$

$$\dot{\psi}_i = -k_2 \text{sign}(s_i),$$

where $\psi_i = -\beta_i \rho_i + \sum_{j \in N_i \setminus \{l\}} a_{ij} \rho_j$. Let $z_i = w_i + \psi_i$, then the previous equation can be rewritten as

$$\dot{s}_i = -k_1 |s_i|^{1/2} \text{sign}(s_i) + z_i,$$

$$\dot{z}_i = -k_2 \text{sign}(s_i) + \dot{\psi}_i. \quad (24)$$

The works of Chalanga and Plestan (2017), Moreno and Osorio (2012) have been dedicated to analyze the stability of the dynamics (24), which is properly the super-twisting formula, considering a bounded continuously differentiable disturbance $\psi_i$, i.e., $|\psi_i|_{\max} < L$ and $|\dot{\psi}_i|_{\max} < M$ for some constants $L > 0, M > 0$. Some conditions must be accomplished in order to guarantee convergence to the origin ($s_i = 0, z_i = 0$).
in spite of the disturbance, and for that, the bound $M$ is particularly important. The uncertain system (24) converges globally to the origin in finite time if $k_2 > M$ and $k_1 > 1.41\sqrt{k_2} + M$, and a bound of the convergence time to the sliding surface is well characterized (Chalanga & Plestan, 2017).

We can use that result to set control gains $k_1$ and $k_2$, guaranteeing that the dynamics of the tracking error for each agent is eventually constrained to the sliding surface, i.e., $s_i = \dot{s}_i = 0$, regardless of the presence of disturbances. In such case, from (20) it follows that

$$\xi_{in} = -K_{fr}\xi_{i1;i(n-1)}.$$  

Then, considering this expression and the first $n - 1$ equations of tracking error dynamics (23), the behavior of the tracking error on the sliding surface results in

$$\dot{\xi}_{ik} = \xi_{i(k+1)}, \quad \text{for } k = 1, \ldots, n-2,$$

$$\dot{\xi}_{i(n-1)} = \xi_{in} = -K_{fr}\xi_{i1;i(n-1)},$$

which can be enforced to exhibit global asymptotic stability through an appropriate choice of the gain vector $K_{fr}$. Then, for each $i$-th agent, $\xi_i(t)$ is globally asymptotically stable.

**Part II.** Based on the previous expressions, now let us prove predefined-time consensus. At initial time $t = 0$, $H(0) = I_n$ and thus $\xi_i(0) = 0_n$ according to (14). This implies that $s_i(0) = 0$ according to (20). Since $s_i$ is initially null, the finite-time stability of (24) implies that $s_i(t) \simeq 0 \forall t > 0$, regardless of $\psi_i(t) \neq 0$ (disturbances are rapidly rejected by the super-twisting control, as discussed above). Then, $\xi_i(t) = 0_n \forall t \geq 0$ by the stability of (25). Thus, it follows that $e_i^f(t) = H(t)e_i^f(0) \forall t \geq 0$ in accordance to the definition (14). Finally, since $H(t_f) = 0_{n \times n}$, then $e_i^f(t_f) = 0_n$. Since this occurs for all the agents, Lemma 4.1 implies that consensus is reached at $t_f$. \qed

As reviewed in the introduction, to the best of our knowledge, only the works of Wang and Song (2018); Zhao and Liu (2017a); Zhao et al. (2019) have investigated high-order predefined-time protocols. The proposed consensus protocol (21) has the advantage, over those protocols, of providing robustness against large matched perturbations while the control effort is continuous. Moreover, the protocols proposed in this work do not require information about the network’s connectivity, as required by the protocols introduced in the literature.

It is worth noting that some aspects have to be considered during the application of the proposed consensus protocols. First, all the clocks of the agents in the network must be synchronized to achieve predefined-time convergence. Second, physical constraints of the systems must be taken into account to set $t_f$, considering that a small $t_f$ will result in large control efforts. Thus, the maximum allowable input of each agent must be taken into consideration to set $t_f$, however, this is not in the scope of this work. Further analysis is required to obtain a relation between the maximum control effort $\max(|v|)$ as a function of the predefined settling time $t_f$ and the initial consensus error, however, we already know that this relation is linear for unperturbed cases when the settling time $t_f$ is fixed. Third, it is assumed that the state of each agent is available and transmitted to its neighbors without any time-delay or packet dropouts.
Table 1.: Initial states of the 8 agents with third-order dynamics.

|   | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     |
|---|-------|-------|-------|-------|-------|-------|-------|-------|
| $x_{i1}(0)$ | -1.66 | 1.89  | 0.60  | -1.07 | -0.38 | -1.51 | -0.92 | -0.96 |
| $x_{i2}(0)$ | -0.67 | -1.39 | -0.60 | -1.51 | 1.53  | -1.62 | 1.72  | -0.40 |
| $x_{i3}(0)$ | -1.81 | -0.63 | 0.94  | 1.17  | 0.17  | 0.74  | 1.57  | -1.78 |

5. Simulations

In this section, simulations are performed to illustrate the effectiveness and advantages of the proposed TBG-based consensus protocols. In particular, we present results using the nonlinear controller (21), which enhances the performance of the linear controller (15) due to its robustness properties. In the following simulations, a MAS of 8 third-order agents is considered, where the agents’ dynamics are described by (5) with $f(x_i) = x_{i1}x_{i2}\sin(x_{i3}) + 0.1x_{i1}x_{i3}$ and $g(x_i) = -2$ for agents $\{1, 3, 5, 7\}$, and $f(x_i) = 0$ and $g(x_i) = 1$ for agents $\{2, 4, 6, 8\}$. None of the agents exhibit zero dynamics (i.e., the variables $\varphi_i$ in (5) do not exist). The convergence time is preset to $t_f = 5$ seconds. The implementation of the proposed TBG-tracking protocols require the computation of the TBGs references and the time-dependent gain, i.e., to design the functions $h_1(t), \ldots, h_n(t)$ fulfilling (2), and to evaluate $H(t)$ and $K_i(t)$ during the system evolution. The following TBG functions are used, $h_1(t) = 20(t/t_f)^7 - 70(t/t_f)^6 + 84(t/t_f)^5 - 35(t/t_f)^4 + 1$, $h_2(t) = 10t^7/t_f^7 - 36t^6/t_f^6 + 45t^5/t_f^5 - 20t^4/t_f^4 + t^3/t_f^3 + t^2/t_f^2 + t/t_f$, which fulfill with (2).

The communication topologies shown in fig. 1 will be used (Yong et al., 2012b). The first one $\mathcal{G}_1$ is a connected undirected graph, the second one $\mathcal{G}_2$ is a directed graph having a spanning tree. The initial states of the eight agents are randomly selected in the range $(-2, 2)$ and are shown in Table 1. The simulations were implemented in MATLAB using the Euler forward method to approximate the time-derivatives with a time step of 0.1ms.

Figure 1.: Communication graphs obtained from (Yong et al., 2012b). Left: undirected graph $\mathcal{G}_1$. Right: directed graph $\mathcal{G}_2$. 
5.1. Robust Predefine-Time Consensus

In this subsection, the application of the proposed robust predefined-time TBG controller (21) is illustrated considering disturbances \( p_i(t) = \alpha_i(1 + 1 \sin(5t)) \), with \( \alpha_i \) randomly selected in \((0, 1)\). We evaluated the consensus protocol for both communication topologies \( G_1 \) and \( G_2 \). A third-order leader (root node) is considered, having communication only with the first follower agent, i.e., \( b_1 = 1 \) and \( b_i = 0, \forall i = \{2, \ldots, 8\} \). The leader’s behavior is modeled as in (6), and its state is maintained constant and equal to \( x_l(t) = [-1, 0, 0]^T \) by setting its control input as \( u_l = 0 \). The gains of the robust controller are set as \( k_1 = 3, k_2 = 7 \) and \( K_{fr} = [1, 2] \).

![Figure 2: State evolution (left) and consensus error trajectories (right) of the third-order perturbed MAS under (21), for \( G_1 \) and \( t_f = 5 \) s. The consensus state at \( t_f \) is \( x_l(5) = x_f(5) = [-1, 0, 0]^T \). In the right figure, the continuous lines represent the evolution of the errors whereas the TBG references are drawn with dashed lines.](image)

![Figure 3: Sliding surfaces evolution (left) and auxiliary control inputs response (right) of the third-order perturbed MAS under (21), for \( G_1 \) and \( t_f = 5s \).](image)

Simulations results for the communication graph \( G_1 \) are shown in figs. 2-3. In order to show the robustness against errors in the initial value information, we introduced errors in the initial state used by the control law to evaluate \( e_i^f(0) \), in particular we introduced an error of 10% of the absolute values of the real initial state \( x_i(0) \) for each agent. The consensus performance under the robust TBG controller is shown in fig. 2. It can be observed in the figure to the left that, in spite of the disturbances \( \rho(t) \), the followers achieve consensus to the leader’s state \( x_l = [-1, 0, 0]^T \) at the predefined time 5 seconds. Fig. 2 (right) shows that the consensus error trajectories initiating out of the TBG references, due to the error on the initial state, are forced to follow the TBG references. This effect is even clearer in fig. 3 (left), where the sliding surface of each agent’s control do not initiate in zero as would be if the control law considers the real initial state, but they converge to zero rapidly. Finally, observe in fig. 3 (right) that the auxiliary control inputs \( v_i \) evolve smoothly over time and keep oscillating after \( t_f \) in order to reject the disturbance \( \rho(t) \).
Figs. 4-5 present the results for the communication graph $G_2$. In this case, the same time-varying disturbance $\rho(t)$ was introduced, but no error was introduced in the initial state control computation, i.e., the real initial state was used in the controller to compute $e^i(0)$. As shown in fig. 4 (left), the followers achieve consensus to the leader’s state $x_l = [-1, 0, 0]^T$ at the predefined time 5 seconds. The consensus error trajectories shown in fig. 4 (right) start over the TBG references and they converge to the origin in the predefined-time. Since the control law uses the real initial state of the agents, the sliding surfaces initiate on zero as can be seen in fig. 5 (left) and after a small transient (due to the disturbance), they return to zero. Fig. 5 (right) shows the auxiliary control inputs $v_i$ of each agent, which in this case initiate null, since there is not initial tracking error, and the evolution of these control inputs are also smooth over time.

Figure 4.: State evolution (left) and consensus error trajectories (right) of the third-order perturbed MAS under (21), for $G_2$ and $t_f = 5$ s. The consensus state at $t_f$ is $x_i(5) = x_l(5) = [-1, 0, 0]^T$. In the right figure, the continuous lines represent the evolution of the errors whereas the TBG references are drawn with dashed lines.

Figure 5.: Auxiliary control inputs response of the third-order perturbed MAS under (21), for $G_2$ and $t_f = 5$ s.

5.2. Comparison with existing approaches

For comparison purposes, simulations of the predefined-time consensus protocols for high order systems presented by Wang and Song (2018); Zhao et al. (2019) are given in this subsection. One of them is a leader-following scheme of Wang and Song (2018) based on a time-varying scaling function having a parameter to set, and a matrix defined by optimal control. The other scheme, by Zhao et al. (2019), is a leaderless discrete-time protocol based on the infinite frequency sampling of a time sequence. For a realistic implementation of the last scheme, we evaluate the sampling truncation based on a consensus error bound. The results are shown in figs. 6-7, where the plots
to the left show the state response and the plots to the right present the auxiliary control signals produced by the corresponding evaluated protocol. It can be observed in figs. 6 (left) and 7 (left) that the states of all agents converge, at the specified settling time $t_f = 5s$, to the consensus states $\mathbf{x}_i = [-1, 0, 0]^T$ and $\mathbf{x}_i^* = [3.01, 1.32, 0.23]^T$, respectively. Notice in figs. 6 (right) and 7 (right) that the magnitude of the auxiliary control efforts for both control schemes are initially large, and particularly, the control efforts with the second protocol (Zhao et al., 2019) become very large and the signals are not smooth due to its planned switching strategy. It is worth noting that the compared approaches were implemented without introducing the disturbance $\rho(t)$ because they are not able to work properly in such conditions. In comparison, fig. 5 shows that our proposed TBG controller (21) generates control signals that start in zero, provide a smooth continuous evolution and exhibit lower magnitudes compared with the existing predefined-time consensus approaches.

![Figure 6](image1.png)

**Figure 6.** State response (left) and auxiliary control inputs (right) of the third-order MAS under the control of Wang and Song (2018) (eq. (19)), for $\mathcal{G}_1$ and $t_f = 5s$. The consensus state at $t_f$ is $\mathbf{x}_i(5) = \mathbf{x}_l(5) = [-1, 0, 0]^T$.

![Figure 7](image2.png)

**Figure 7.** State response (left) and auxiliary control inputs (right) of the third-order MAS under control the of Zhao et al. (2019) (eq. (2)), for $\mathcal{G}_2$ and $t_f = 5s$. The consensus state at $t_f$ is $\mathbf{x}_i(5) = \mathbf{x}_i^* = [3.01, 1.32, 0.23]^T$.

In order to widely compare the performance of the proposed protocol and the existing ones (Wang & Song, 2018; Zhao et al., 2019), simulations were performed with ten different initial conditions that were randomly selected for the eight agents in such a way that the norm of $\mathbf{x}(0)$ is varied from 1 to 10. The leader state was kept constant $\mathbf{x}_l = [-1, 0, 0]^T$ for (21) and for the leader-following scheme of Wang and Song (2018). The same previous control gains and settling time $t_f = 5s$ were used for all the cases with the graph $\mathcal{G}_1$. For every experiment, the norm of the consensus error $\mathbf{e}(t_f)$ and the maximum absolute value of the auxiliary control input $\mathbf{v}$ were measured. During the simulations, all the controllers achieved consensus at the predefined-time, with errors lower than $(||\mathbf{e}(t_f)|| < 1 \times 10^{-4})$. The results for the magnitude of the auxiliary
control $v$ are shown in fig. 8 (left). Notice that the maximum auxiliary control efforts are significantly lower for the TBG-based proposed controller (21) with respect to the compared approaches (Wang & Song, 2018; Zhao et al., 2019). The three approaches were also compared by computing the maximum value of the control input $\max(|v|)$ for different values of the convergence time parameter $t_f$. Such evaluation was performed using the graph $G_1$ and the initial states given in Table 1. The results are presented in fig. 8 (right), where it can be seen that the maximum control effort is lower for our proposed controller than using the compared approaches. In a range, the maximum control effort is lower as the convergence time increases for our predefined-time approach and the pre-specified approach (Wang and Song (2018)), but the relation is far away from being proportional. Surprisingly, the control effort was higher as the convergence time increases for the specified-time approach (Zhao et al. (2019)) along all the evaluated range. Further analysis is required to formulate relations between the maximum control effort, the convergence time and the initial consensus error.

Figure 8.: Comparison of the maximum value of the absolute auxiliary control input $\max(|v|)$ as a function of the initial condition $x(0)$ in the plots to the left and as a function of the convergence time $t_f$ in plots to the right. Controllers: Robust predefined-time with TBG (21), Predefined-time with TBG (15), Pre-specified finite time (Wang and Song (2018)) and Specified-time (Zhao et al. (2019)).

Finally, another comparison was carried out with our approach and those of Wang and Song (2018) and Zhao et al. (2019) by increasing the number of agents, in particular for cases with 10, 20, 50, 100 and 200 second-order agents, considering a circular communication undirected graph (i.e., the $i$-th follower is connected to followers $i - 1$ and $i + 1$). The same initial state conditions were used for each approach, using values randomly selected between $-5$ to 5. The leader state was kept constant $x_l = [-1, 0, 0]^T$.
for our approach \cite{15} and the compared scheme of Wang and Song \cite{2018}. The same previous control gains and settling time \( t_f = 5s \) were used for all the cases. It can be seen in fig. 9 that the control effort for our proposed controller \cite{15} was almost constant for all the simulations and considerably lower than the comparison controllers as the number of agents increases. Moreover, as the number of agents increases, the approach of Zhao et al. \cite{2019} required readjustments of some control parameters to maintain the final consensus error lower than \(|e^T(t_f)| < 1 \times 10^{-2}\), which was not required for our proposed TBG controller.

6. Conclusions

In this work, a couple of distributed control laws to achieve predefined-time consensus have been proposed for a class of high-order MAS with nonlinear agents’ dynamics over undirected and directed communication graph topologies. The design of the proposed leader-following protocols combines the advantages of time base generators and feedback controllers to achieve closed-loop stability and robustness. The salient feature of this method is that for any connected undirected topology or directed topology having a spanning tree, consensus is achieved in a predefined time, independently of the initial conditions and detailed characteristics in the connectivity of the communication graph. Furthermore, the proposed leader-following protocol based on the super-twisting controller, provides robustness against matching disturbances, while maintaining the predefined-time convergence property. Another advantage of the proposed method is that the produced control efforts are continuous and smooth over time, exhibiting lower magnitudes than existing protocols for high-order predefined-time (preset-time) consensus. Contrary to existing predefined-time consensus protocols based on time-varying gains, in our approach no singularities appear in the control computation. As a future work, the proposed methodology will be extended for MAS with switching topologies and time-delays.

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