Local Lagrangian Formalism and Discretization of the Heisenberg Magnet Model

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Abstract

In this paper we develop the Lagrangian and multisymplectic structures of the Heisenberg magnet (HM) model which are then used as the basis for geometric discretizations of HM. Despite a topological obstruction to the existence of a global Lagrangian density, a local variational formulation allows one to derive local conservation laws using a version of Nöther’s theorem from the formal variational calculus of Gelfand-Dikii. Using the local Lagrangian form we extend the method of Marsden, Patrick and Schkoller to derive local multisymplectic discretizations directly from the variational principle. We employ a version of the finite element method to discretize the space of sections of the trivial magnetic spin bundle \( N = M \times S^2 \) over an appropriate space-time \( M \). Since sections do not form a vector space, the usual FEM bases can be used only locally with coordinate transformations intervening on element boundaries, and conservation properties are guaranteed only within an element. We discuss possible ways of circumventing this problem, including the use of a local version of the method of characteristics, non-polynomial FEM bases and Lie-group discretization methods.

1 Introduction

The treatment of PDEs in invariant form finds its most natural language in the setting of jet bundles over an appropriate space-time. In this setting, governing equations of “motion” are prescribed by invariantly-defined differential operators, and although calculations are ultimately done in terms of partial derivatives, the invariant meaning of all the operations is clear. Moreover, the invariant formulation and structure of the underlying fiber bundle frequently translate to a nontrivial geometry of the space of solutions, conservation properties and other “geometric” features. Although these geometric properties can be discovered in the coordinate formulation, conceptual clarity of invariant geometric derivations is frequently a great asset in itself.

All of this is particularly so for Langrangian PDEs whose equations of motion are derived as extermality conditions on the action functional – an apparently coordinate-free form. Symplectic and multisymplectic structures immediately follow by well-known procedures, as do conserved quantities corresponding to continuous symmetries (Nöther’s theorem). The situation is less fortunate for Hamiltonian systems as here a number of “symmetries” have been broken relative to the Lagrangian setting. A splitting of space-time relies upon the choosing a preferred time direction and a nonunique complementary space – a literal symmetry breaking as the full group of space-time transformations no longer preserves the form of the equations. Furthermore, the system is no longer specified by prescribing a single quantity – apart from the energy functional a Poisson structure has to be specified independently\(^1\).

Although it may appear from familiar examples of classical mechanics that the Lagrangian and Hamiltonian pictures are equivalent, this is not always so. In fact there is a local nondegeneracy condition on the Poisson bracket that must be satisfied before a local Lagrangian density can be reconstructed. In general, an additional

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\(^1\)In fact, there is a certain nonuniqueness associated with the choice of the canonical 1-form in Lagrangian theories living on jet bundles \( N^{(r)} \) with \( r > 1 \) \([4,7]\), although this is not as bad as the number of choices that need to be made in the Hamiltonian setting.
vanishing condition on a topological obstacle (i.e. exactness of the symplectic form) has to be satisfied before the variational picture is restored globally. Even so, a local Lagrangian formulation carries enough information to reconstruct the local multisymplectic geometry and Nöther-type local conservation laws. A central result is the derivation of the local multisymplectic form for the Heisenberg Magnet model from its local Lagrangian formulation.

Even when Lagrangian and Hamiltonian theories are equivalent, their discretizations may not be so. While the discrete variational principle yields analogs of symplectic and multisymplectic geometries under minimal conditions on the discretization, the Hamiltonian case calls for a sophisticated theory of symplectic or Poisson integrators to preserve the corresponding geometries. Therefore, the search for an equivalent Lagrangian formulation naturally arises when considering conservative discretizations. In this paper we discuss potential ways of constructing geometric discretizations from the local variational picture.

In Section 2 we briefly recall the picture of geometric PDEs as conditions on sections of an appropriate fiber bundle and their jets. The formal calculus of variations of Gelfand-Dikii is a very convenient computational tool for Lagrangian systems in this setting, and a source of many explicit formulas, including local conserved densities and a formal analog of Nöther’s theorem. Section 3 contains the derivation of the local Lagrangian form naturally arising when considering conservative discretizations. In this paper we discuss potential ways of integrating to preserve the corresponding geometries. Therefore, the search for an equivalent Lagrangian form.

2 Setting

The geometric theory of differential operators and variational calculus is most naturally set within the framework of smooth fiber bundles $N \rightarrow M$ over some $n$-dimensional base space-time $M$ and the corresponding jet bundles $J^r N \rightarrow M$. Differential operators and action functionals of the Lagrangian formalism act on the space of sections of the bundle, each section $\sigma : M \rightarrow N$, $\pi \circ \sigma = \text{id}_M$ identifiable with its image $\Sigma = \sigma(M)$, a copy of $M$ horizontally embedded in $N$. Fixing a local trivialization $N \cong \text{loc}M \times F$ expresses $N$ as a product of $M$ with the $k$-dimensional typical fiber $F$. A choice of local coordinates, $x = (x_i), i = 1, \ldots, n$ on the base and $u = (u_j), j = 1, \ldots, r$ on the fiber, defines a local coordinate system $x \times u : N \rightarrow \mathbb{R}^{n+r}$, making $N$ locally diffeomorphic to the canonical projection $\pi \cong \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n$, while inducing a similar diffeomorphism $\pi(r) \equiv \mathbb{R}^{n+k} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ for the $N(r)$ with the adapted coordinates $((x_i), (u_j^{(\alpha)}))$, $|\alpha| \leq r^2$ on $N(r)$. The usual bundles $\wedge^kTN$ and $\Lambda^kN$ have as their spaces of sections the $k$-vector fields $\mathbf{V}^k(N) = \mathbf{S}(TN)$ and the differential $k$-forms $\Omega^kN = \mathbf{S}(\Lambda^kN)$. In addition, there is a distinguished subspace $\mathbf{V}^\alpha(N,M) \subset \mathbf{V}^\alpha(N) \equiv \mathbf{V}^\alpha(N)$ of fiber-preserving fields, which generate local fiber bundle automorphisms, and the subspace $\mathbf{V}^{\alpha}(N/M) \equiv \mathbf{S}(\mathbf{T}(N/M)) \subset \mathbf{V}(N,M)$ of sections of the vertical subbundle, which fix fibers. When restricted to $\Sigma$ the local coordinates $u$ define functions of $x$ by

$$ u_\Sigma : \mathbb{R}^n \xrightarrow{x_{-1}} M \xrightarrow{\sigma} \Sigma \xrightarrow{u} \mathbb{R}^k. $$

Similarly, restriction of the fiber coordinates $u^{(\alpha)}$ to $\Sigma(r)$ defines functions $u^{(\alpha)} \Sigma(r)$ that encode the infinitesimal behavior of the section: $\left( \frac{\partial}{\partial x} \right)^{\alpha} u_\Sigma(x) = u_\Sigma^{(\alpha)}(x)$, $|\alpha| \leq r$.

To the variational principle we admit local functionals $\mathbf{A}$ that to each $\sigma \in \mathbf{S}(N)$ assign the integral $\int \mathbf{A}(\sigma)$ of an $n$-form $\mathbf{A}(\sigma) \in \Omega^n M$, such that the integrand $\mathbf{A}_x(\sigma) \in \Lambda^n_x M$ at any $x \in M$ is determined by the $r$-jet $\sigma^{(r)}$ of the section at that point$^3$. In other words $\mathbf{A}$ is a differential operator $\mathbf{S}(N) \rightarrow \Omega^n M$ of order $\leq r$ and by definition can be factored through a map $\mathbf{A}$, induced on sections by a smooth bundle map $\mathbf{A} : N(r) \rightarrow \Lambda^n M$ over $M$.

$$
\begin{align*}
S(N(r)) & \xrightarrow{\mathbf{A}} \Omega^n M \\
J^r & \uparrow \\
S(N) & \xrightarrow{\mathbf{A}} \mathbb{R}
\end{align*}
$$

$^2$Definitions for multi-indices are as usual: $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \sum_i \alpha_i$, $\left( \frac{\partial}{\partial x} \right)^{\alpha} \equiv \frac{\partial}{\partial x_1}^{\alpha_1} \cdots \frac{\partial}{\partial x_n}^{\alpha_n}$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $\epsilon_i 1$ in the $i$-th place and zeros everywhere else.

$^3$For a globally determined $r$. 

By a certain abuse of notation defined in \([10]\) we shall call \(\Lambda\) the symbol of the operator \(\tilde{\Lambda}\) and write

\[
\tilde{\Lambda}(\sigma) = \int \Lambda_x(\sigma^{(r)}),
\]

where in order to make sense of the integral we restrict integration to be over nice (see \([3]\)) compact domains \(\Sigma_M \subset M\).

The symbol \(\Lambda\) is identified with a section of the vector subbundle \(\Lambda^{0,n}N^{(r)} = \pi^{(r)}*\Lambda^n M \subset \Lambda^n N^{(r)}\) of horizontal forms vanishing on the fibers of the vertical subbundle \(T(N^{(r)}/M)\). Then the action of \(\tilde{\Lambda}\) can be defined simply by integration of its symbol over holonomic horizontal submanifolds \(\Sigma^{(r)} = \sigma^{(r)}(\Sigma_M) \subset N^{(r)}\):

\[
\tilde{\Lambda}_\Sigma \equiv \tilde{\Lambda}(\sigma) = \int_{\Sigma^{(r)}} \Lambda.
\]

In terms of this geometric description, variational calculus investigates the response of \(\tilde{\Lambda}\) to infinitesimal variations of the holonomic submanifold \(\Sigma^{(r)}\).

A generic variation \(Z_\Sigma\) of \(\Sigma\) is represented by a (local) 1-parameter bundle automorphism group \(\phi^r_M\), with the induced (local) base automorphism group \(\phi^r_N\), by restriction \(Z_\Sigma = Z|_\Sigma\) of the infinitesimal generator \(Z \in \varepsilon N(M,M)\). In coordinates we have

\[
Z = \frac{d}{dt}|_{t=0} \phi^r_N = X(x) + V(x,u) = \sum_i X_i(x) \frac{\partial}{\partial x_i} + \sum_j V_j(x,u) \frac{\partial}{\partial u_j}.
\]

The action of \(Z\) on \(\varepsilon N\) naturally splits into the horizontal \(X\) and vertical \(Y\) parts. The horizontal part acts on functions \(A_0 = C^\infty(N)\) by differentiation along \(\Sigma\), i.e. via pushing \(\frac{\partial}{\partial x_i}\) forward to \(\Sigma\), while the vertical part acts by differentiating along the vertical curves \(\sigma^i(x)\) via the natural action of \(\phi^r_N\) sections \(\sigma^i = \phi^r_N \circ \sigma \circ \phi_M^r\):

\[
X(x,u_\Sigma(x)) = \sigma^r_x \frac{\partial}{\partial x} = \sum_i X_i \left(\frac{\partial}{\partial x_i} + \sum_j u_{\Sigma,j} \frac{\partial}{\partial u_j}\right),
\]

\[
Y(x,u_\Sigma(x)) = \sum_j \frac{\partial}{\partial u_j} u_{\Sigma,j}(x) \frac{\partial}{\partial u_j} = \sum_j \left(V_j(x,u_\Sigma(x)) - \sum_i u_{\Sigma,j,i}(x) X_i(x)\right) \frac{\partial}{\partial u_j}.
\]

Geometrically, \(Y\) is the projection of \(Z\) onto the vertical subspace at \(T_\sigma(x)(N/M)\ parallel\ to \(T_x\Sigma\), while \(X\) is the projection onto \(T_x\Sigma\).

The action of \(Z\) on the higher jets \(\sigma^{(r)}(x)\) is easily computed by interchanging the order of differentiation, for instance,

\[
Y^{(1)} = Y + \sum_{j,i} \frac{\partial}{\partial u_j} u_{\Sigma,j,i}(x) \frac{\partial}{\partial u_j} = Y + \sum_{j,i} \frac{\partial}{\partial u_j} \frac{\partial}{\partial u_j} u_{\Sigma,j}(x) \frac{\partial}{\partial u_j} = Y + \sum_{j,i} \frac{\partial}{\partial u_j} Y_j(x,u_\Sigma(x)) \frac{\partial}{\partial u_j}.
\]

Thus, the first jet components of \(Y^{(1)}\) are obtained from components of \(Y\) by horizontal differentiation along \(T_x\Sigma\), using the second jet information \(u_{\Sigma,j,i}(x)\) to determine the horizontal subspace \(T_\Sigma^{(1)}\). The general rule follows recursively.

Jet bundles allow one to abstract from particular sections by encoding their infinitesimal information pointwise, similar to the way tangent bundles allow one to manipulate vectors without referring to local integral curves. In general, \(\frac{\partial}{\partial x_i}\) will act on algebras of function \(f \in A_r = C^\infty(N^{(r)})\) by the horizontal holonomic lift \(\partial_i = \text{Hor}(\frac{\partial}{\partial x_i})\) and has the property that it reduces to the total derivative \(\frac{\partial}{\partial x_i}\) upon restriction to a holonomic section: \(\partial_i = \sigma^{(r)} \frac{\partial}{\partial x_i}\). Since, as seen in the example above, \(\partial_i\) uses the higher jet data, it acts on \(A_r\) as a map to \(A_{r+1}\) on coordinates we have \(\partial_i u_j^{(r)} = u_j^{(r+1)}\). Therefore the operators \(\partial_i\) must be regarded as derivations of the algebra \(A = \text{inj lim} A_r \cong \bigcup A_r\) of functions on the infinite jet space \(\varepsilon N = \text{proj lim} N^{(r)}\). All derivatives \(\text{Der}(A)\), formally viewed as vector fields \(V(\varepsilon N)\) on \(\varepsilon N\) are given by the infinite series \(\varepsilon W = \sum_i W_i \frac{\partial}{\partial x_i} + \sum_{j,\alpha} W_{j,\alpha} \frac{\partial}{\partial u_j^{(\alpha)}}\), and derivations \(\partial_i\) lie within the subspace \(V(\varepsilon N,M)\) of formally fiber preserving fields, thus generating the submodule \(\text{Hor}(\varepsilon N)\) over \(K\). Likewise the vertical holonomic lifts \(\text{Ver}(Y)\) of fields on \(Y \in V(N/M)\) generate \(\text{Ver}(\varepsilon N) \in V(N/M) \subset V(\varepsilon N,M)\) as a module over \(\varepsilon N\):

\[
\partial_i = \frac{\partial}{\partial x_i} + \sum_{j,\alpha} u_j^{(r+1)} \frac{\partial}{\partial u_j^{(\alpha)}}; \quad \text{Ver}(Y) = \sum_{j,\alpha} Y_j^{(\alpha)} \frac{\partial}{\partial u_j^{(\alpha)}} \in V(\varepsilon N,M), \quad Y_j \in A_0.
\]
It is easily checked that $\text{Ver}(\mathcal{N})$ commutes with all $\partial_i$.

Geometrically, fields in $\text{Hor}(\mathcal{N})$ reduce to $V(M)$ and fields in $\text{Ver}(\mathcal{N})$ reduce to $V(N/M)$ upon restriction to holonomic submanifolds $\Sigma^{(r)} \subset N^{(r)} \subset \mathcal{N}$. In particular, the above $Z$, when regarded in $V(N/M)$ uniquely splits into a “horizontal” and “vertical” part,

$$X = \sum_i X_i \left( \frac{\partial}{\partial x_i} + u_{j,i} \frac{\partial}{\partial u_{j,i}} \right), \quad Y = \sum_j Y_j \frac{\partial}{\partial u_j} = \sum_j \left( V_j - \sum_i u_{j,i} \right) \frac{\partial}{\partial u_j}.$$ 

Each part uniquely lifts to $\mathcal{N} = \text{Hor}(\mathcal{Z}) \oplus \text{Ver}(\mathcal{Z}) \in \text{Hor}(\mathcal{N}) \oplus \text{Ver}(\mathcal{N})$, and together define a unique lift $\mathcal{Z} = \text{Hor}(\mathcal{Z}) \oplus \text{Ver}(\mathcal{Z}) \in \text{Hor}(\mathcal{N}) \oplus \text{Ver}(\mathcal{N})$.

The direct sum $\text{Hor}(\mathcal{N}) \oplus \text{Ver}(\mathcal{N}) \subset V(N/M)$, generates a split of the space of $1$-forms $\Omega^0,\mathcal{N} = \text{inj lim} \Omega^1,\Sigma^{(r)}$ and of the usual “de Rham” differential $\delta$ into the vertical and horizontal parts: $\Omega^1,\mathcal{N} = \Omega^1,\mathcal{N} \oplus \Omega^0,\mathcal{N}$ and $\delta = \delta + \delta$ respectively, where $\Omega^1,\mathcal{N} = \delta \text{A}$ consists of forms vanishing upon contraction with horizontal fields, and $\Omega^0,\mathcal{N} = \delta \text{A}$ vanishing upon contraction with vertical vector fields. The geometric meaning of the horizontal and vector differentials is that $d$ represents differentiations along the horizontal subspace at each level $\Sigma^{(r)}$ and reduces to the differential in $\Omega^1,M$ upon restriction to a holonomic section, while $\delta$ vanishes, measuring the “virtual variations” transversal to holonomic $\Sigma^{(r)}$.

Operating formally we can deduce the properties of the calculus defined on $\Omega^0,\mathcal{N}$ by the pair of differentials $\delta, d$. In particular, $\delta$ and $d$ anti-commute with one another, and on the generators we have

$$dx_i = dx_i, \quad du^{(\alpha)} = \sum_i u^{(\alpha,\epsilon_1)} dx_i, \quad df = \sum_i \partial_i f dx_i,$$

$$\delta u^{(\alpha)} = du^{(\alpha)} - \sum_i u^{(\alpha,\epsilon_1)} dx_i, \quad \delta x_i = 0, \quad \delta f = \sum_{j,\alpha} \frac{\partial f}{\partial u^{(\alpha)}_j} \delta u^{(\alpha)}_j,$$

and the module $\Omega^0,\mathcal{N}$ decomposes with respect to $\delta$ and $d$

$$\Omega^0,\mathcal{N} = \bigoplus_{k+l=m} \Omega^{k,l} = \bigoplus_{k+l=m} \Omega^{0,k} \wedge \Omega^{l,0}, \quad \delta : \Omega^{k,l} \rightarrow \Omega^{k+1,l}, \quad d : \Omega^{k,l} \rightarrow \Omega^{k,l+1}$$

into exterior subalgebras of vertical $k$-forms $\Omega^{k,0} = \text{A}[\delta u^{(\alpha_1)}_l, \ldots, \delta u^{(\alpha_k)}_l]$, and horizontal $l$-forms $\Omega^{0,l} = \text{A}[dx_1, \ldots, dx_l]$ (in particular, $\Omega^0,\mathcal{N} = 0$ for $l > n$), with each subalgebra $\Omega^{k,l} \subset \mathcal{N}$ filtered $\Omega^0,\mathcal{N} \subset \ldots \subset \Omega^{k,l} \subset \ldots$ by the subalgebras of $k,l$-forms $\Omega^{k,l} = \text{A}[\delta u^{(\alpha_1)}_l, \ldots, \delta u^{(\alpha_k)}_l, dx_1, \ldots, dx_l]$ of differential degree $\text{deg} \sum_{i,\alpha} |\alpha_i| \leq r$.

The fields $V(N)$ act on $\Omega^0,\mathcal{N}$ by Lie derivatives $\mathcal{L}_{\mathcal{V}}(\Omega) = \mathcal{W}(\Omega) = (d\mathcal{W}) + i(\mathcal{W}) \delta \Omega$, with the only nonvanishing contractions being $i(\partial_i) dx_i = 1$ and $i(\partial_i) \delta u^{(\alpha)}_j = 1$. The easily observed commutation relations simplify calculaions of $\mathcal{L}_{\text{Ver}(\mathcal{N})}$ and $\mathcal{L}_{\text{Hor}(\mathcal{N})}$:

$$\delta i(\text{Hor}(\mathcal{N})) + i(\text{Hor}(\mathcal{N})) \delta = di(\text{Ver}(\mathcal{N})) + i(\text{Ver}(\mathcal{N})) d = 0,$$

$$\mathcal{Z}(\Omega) = (\delta i(\text{Ver}(\mathcal{Z})) + i(\text{Ver}(\mathcal{Z})) \delta) \Omega + (di(\text{Hor}(\mathcal{Z})) + i(\text{Hor}(\mathcal{Z})) d) \Omega.$$ 

This forms the basis of formal variational calculus, developed in detail in [3, 7], here we sketch the necessary results.

The main result is the formula for the variational derivative [3], which computes the action of $Z$ in $\Omega^0,n$ on the symbol of the action functional $\Lambda \in \Omega^{0,n} \Sigma^{(r)}$ via the holonomic lift:

$$Z(\Lambda) \equiv \mathcal{Z}(\Lambda) = i(\text{Ver}(\mathcal{Z})) \delta \Lambda + di(\text{Hor}(\mathcal{Z})) \Lambda,$$

since $i(\text{Ver}(\mathcal{N})) \Omega^0,l = d\Omega^{k,n} = 0$ for any $k,l$. This shows in what sense the vertical part of $\mathcal{Z}$ acts on $\Lambda$ as on a function, and the horizontal part $\mathcal{N}$ acts on $\Lambda$ as on a horizontal $n$-form (see [3, eq. (4.37)]). In coordinates we have

$$\Lambda = \Lambda dx_1 \wedge \cdots \wedge dx_n, \quad \Lambda \in \mathcal{A}, \quad \delta \Lambda = \sum_{j,\alpha} \frac{\partial \Lambda}{\partial u^{(\alpha)}_j} \delta u^{(\alpha)}_j \wedge dx_1 \wedge \cdots \wedge dx_n =$$

$$\sum_{j,\alpha} \frac{\partial \Lambda}{\partial u^{(\alpha)}_j} \delta(x^{\alpha}) \wedge dx_1 \wedge \cdots \wedge dx_n.$$
By a formal differential analog of integration by parts we can split δΛ as follows:

\[
\delta \Lambda = \frac{\delta \Lambda}{\delta u_j} \wedge dx_1 \wedge \cdots \wedge dx_n - \\
\frac{\delta \Lambda}{\delta u_j} \in \Omega^0_{0,n}
\]

\[
d \left( \sum_i \Theta_i (-1)^{i-1} dx_1 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge dx_n \right) \in \Omega^1_{0,n} \oplus d\Omega^{1,n-1} \mathcal{N}.
\] (3)

The geometric meaning of the variational derivative operator is that it provides a formal adjoint to the operator of vertical lift on vectors \(Y \in V(N/M)\), modulo the horizontal differential \(d\):

\[
i(\text{Ver}(Y)) \delta \Lambda = i(Y) \delta \Lambda_0 - i(Y) d\Theta = i(Y) \delta \Lambda_0 + d \left( i(Y) \Omega^{(1)} \right),
\] (4)

where \(\delta \Lambda_0 \in \Omega^1_{0,n} \mathcal{N}\) is the variational derivative. Integrating \(\Omega\) over some \(\Sigma^{(r)}\), since \(Y\) vanishes on the boundary of \(\partial \Sigma\), the adjoint meaning becomes exact and the equations of motion defining the extrema of \(\Lambda\) are given locally by

\[
\frac{\delta \Lambda}{\delta u_j} = 0.
\] (5)

Define the multisymplectic form \(\Omega\) and the vertical part of \(\partial_i\) by

\[
\Omega = \delta \Theta \in \Omega^{2,n-1}, \quad \Omega^{(i)} = \delta \Theta^{(i)} \in \Omega^{2,0}, \quad X_i = \partial_i - \frac{\partial}{\partial x_i} = \sum_{j, \alpha} u_j^{(\alpha)} \in V(\mathcal{N}/M).
\] (6)

On any holonomic \(\Sigma^{(r)}\) satisfying (3) we have the equivalent covariant Hamiltonian system, which, when \(\Lambda\) does not depend on the base coordinates \(x\) explicitly, has the form\(^5\)

\[
\sum_i i(X_i) \Omega^{(i)} = -\delta \mathcal{H}, \quad \mathcal{H} = \sum_j i(X_j) \Theta^{(j)} - \Lambda \in \mathcal{L},
\] (7)

obtained by a version of the Legendre transform. Among other things, it implies the preservation of the multisymplectic form

\[
\sum_i X_i (\Omega^{(i)}) = -\delta^2 \mathcal{H} = 0,
\] (8)

as well as the local conservation laws (formal Nöther’s theorem):

\[
\sum_k \partial_k T_{jk} = 0, \quad T_{jj} = \mathcal{H} - \sum_{k \neq j} i(X_k) \Theta^{(k)}, \quad T_{jk} = i(X_j) \Theta^{(k)}, j \neq k.
\] (9)

Later we will have a chance to use the momenta defined as \(\mathcal{L}\)

\[
P_{j,(\alpha)} = \frac{\delta \Lambda}{\delta u_j^{(\alpha)}} = \sum_{\beta} (-1)^{|\beta|} \frac{|\beta|}{|\alpha|} (\alpha_1 + \beta_1) \cdots (\alpha_n + \beta_n) \frac{\partial \beta^\Lambda}{\partial u_j^{(\alpha+\beta)}},
\] (10)

in particular, \(P_{j,(0)} = \frac{\delta \Lambda}{\delta u_j}\).

The connection with the usual time formalism is done as follows \(\mathcal{L}\). Fix a space-time split of \(M = M^{n-1} \times M^1\) with a compatible coordinate system \((x_1, \ldots, x_n-1, t = x_n)\). For any section \(\Sigma\) fix the time slice \(\Sigma(t)\) over the hyperplane \(M^{n-1} \times t\) defined by \(x_n = t\). For the holonomic lift \(\Sigma^{(r)}(t)\) of any such slice, we define the symplectic form \(\omega\), the corresponding primitive 1-form \(\theta\) and the Hamiltonian functional \(\mathcal{H}\) with the corresponding densities

\[
\tilde{\omega}(\sigma) = \int_{\Sigma(t)} \omega, \quad \tilde{\theta}(\sigma) = \int_{\Sigma(t)} \theta, \quad \tilde{\mathcal{H}}(\sigma) = \int_{\Sigma(t)} T_{nn} dx_1 \wedge \cdots \wedge dx_{n-1},
\] (11)

provided the Lagrangian \(\Lambda\) is regular \(\mathcal{L}\). Then the Hamiltonian evolutionary equations are given as usual by

\[
\tilde{\mathcal{H}} = i(X_i) \tilde{\omega},
\]

and the Lagrangian density is recovered by a version of the inverse Legendre transform:

\[
\Lambda = (-\mathcal{H} + i(X_i) \theta) \wedge dx_n.
\] (12)

\(^5\)Which depends on the choice of trivialization.
3 Local Lagrangian of HM System

In this section we derive a local Lagrangian form of the action functional of the Heisenberg magnet (HM) model since the symplectic form corresponding to the Poisson bracket of HM is not exact and a global Lagrangian does not exist. Throughout this section we explicitly work in coordinates.

The HM model is a Hamiltonian system defined on the product fiber bundle \( N = M \times \mathbb{R}^3 \), where the space-time base is simply \( M = \mathbb{R}^2 \) with coordinates \((x_1, x_2)\) on the base and \( \vec{S} = (S_1, S_2, S_3) \) on the fiber (e.g. [2]).

A solution of the system is a section \( \Sigma \in N \) represented by the restriction of the fiber coordinates \( \vec{S} : M \to \mathbb{R}^3 \) and satisfying the following equation

\[
\partial_2 S_a - \sum_{b,c} \epsilon_{abc} S_b \partial_1^2 S_c = 0.
\]

We now reconstruct the Hamiltonian structure of HM in two different coordinate systems on the fiber.

3.1 Hamiltonian form of HM

After the space-time split \( M \cong M^{n-1} \times M^1 = \mathbb{R} \times \mathbb{R} \) with coordinates \( x = x_1 \) and \( t = x_2 \), we have the instantaneous state of the system \( \Sigma(t) = \sigma(t)(M^{n-1}) \) at time \( t \in M^1 \) given by a section \( \sigma(t) \) of the instantaneous bundle \( N(t) = M^{n-1} \times \mathbb{R}^3 \to M^{n-1} \) and represented by a function \( \vec{S}(t) : M^{n-1} \to \mathbb{R}^3 \). The notation from Section 2 is modified accordingly: \( A_r(t), A(t), \Omega^{(r)}(t) \) etc. The Hamiltonian functional of the system is

\[
\bar{H}(\sigma(t)) = \int_{\Sigma(t)} \frac{1}{2} (S_{1.x}^2 + S_{2.x}^2 + S_{3.x}^2) \, dx,
\]

prescribed by the horizontal 1-form \( \mathbf{H} = H \, dx = \frac{1}{2} (S_{1.x}^2 + S_{2.x}^2 + S_{3.x}^2) \, dx \in \Omega^{0,1} N(t) \in \Omega^{0,1} \mathbb{N}(t) \) with the density \( H \in A_1(t) \). There is a local Poisson bracket on each fiber defined in coordinates by

\[
\{S_a, S_b\} = \Psi_{a,b}(\vec{S}) = -\sum_c \epsilon_{abc} S_c,
\]

(13)

which can be viewed as a section over \( N(t) \) of the exterior power of the vertical bundle \( \Psi : N(t) \to \wedge^2 T(N(t)/M) \). Since we are interested in a local description, we shall avoid the development of the usual infinite-dimensional formalism via the introduction of a Poisson bracket as a local bilinear operator \( \Psi(\vec{S}(x)) \delta(x - y) \) on the loop algebra of state variations (see [5]), and instead immediately write the equations of motion in the local form for any point \( x \in M(t) \):

\[
\vec{S}_t(x) = \{H, \vec{S}\}(x) \equiv \sum_{a,b} \Psi_{a,b}(S(x)) \frac{\delta \vec{S}}{\delta S_a}(x) \frac{\delta H}{\delta S_b}(x) = \vec{S}(x) \times \vec{S}_{xx}(x),
\]

(14)

where as functions in \( A_0(t) \) the coordinate functions \( \vec{S} \) have the natural variational derivatives as defined in Section 2 of \( \frac{\delta S}{\delta S_t} = \sum_b \delta_{a,b} S_b \).

To restore independence of the fiber coordinates in (14) we can state that the vector product in (14) is defined with respect to the standard metric on \( \mathbb{R}^3 \). There is a more natural way, however: the bracket (14) is the Lie-Poisson bracket on the space \( so(3)^* \cong \mathbb{R}^3 \) dual to the Lie algebra \( so(3) \) [9], and therefore variational derivatives of functions in \( A(t) = \mathcal{O}^{0,n}\mathbf{N}(t) \) lie in \( \mathcal{O}^{1,0}\mathbf{N}(t) \cong A(t) \otimes T^* so(3)^* \cong A(t) \otimes so(3)^* \), on which the bivector field \( \Psi \in A(t) \otimes \wedge^2 T so(3)^* \) acts naturally and independently of any coordinate system.

The bracket (14) is nondegenerate on the leaves of a 2-dimensional foliation of \( \mathbb{R}^3 \) consisting of spheres about the origin, orbits of the coadjoint action of \( SO(3) \) [9]. On each such leaf a symplectic form dual to the bracket, the Kirillov form, is defined. Since, as is easily verified, the dynamics of (14) preserves the leaves \( \{|\vec{S}|^2 = 0\} \), the system can be considered on the bundle \( N(t) = M^{n-1} \times S^2 \) with the unit sphere as the fiber, provided the initial conditions are chosen as a section of this new bundle. It is well-known that the symplectic form dual to \( \Psi \) is the area form on the sphere, which can be written down as soon as local coordinates have been chosen on \( S^2 \). For instance, the unit sphere with a deleted north pole can be parametrized by the inverse stereographic projection from the pole \((0, 0, 1)\). Following [9] we introduce the following transformation to complex coordinates:

\[
w = w_1 + iw_2, \quad S = S_1 + iS_2, \quad Q = S_3, \quad R = \frac{1 + |w|^2}{2} > 0, \quad \mathbf{S} = (S, \overline{S}, Q)^T, \quad \mathbf{W} = (w, \overline{w})^T,
\]

the correspondence is then established as follows:

\[
S = \frac{2w}{1 + |w|^2}, \quad \overline{S} = \frac{2\overline{w}}{1 + |w|^2}, \quad Q = \frac{|w|^2 - 1}{|w|^2 + 1} = \frac{R - 1}{R},
\]

\[
|w|^2 = \frac{1 + Q}{1 - Q}, \quad R = \frac{1}{1 + Q}, \quad w = \frac{S}{R}, \quad \overline{w} = \overline{S}R.
\]

(15)
The Poisson brackets of the modified coordinate functions \( S = (S, \overline{S}, Q) \) and the Hamiltonian density

\[
\{S, \overline{S}\} = 2iQ, \quad \{S, Q\} = -iS, \quad \{\overline{S}, Q\} = i\overline{S}, \quad H = \frac{1}{2}(|S_x|^2 + Q_z^2),
\]

(16)
generate the equations of motion equivalent to (13). The coordinate transformations (15) generate the bracket of the \( W \)-coordinates:

\[
\{w, \overline{w}\} = R_2 \{S, \overline{S}\} + \overline{S} R_3 \{S, Q\} + S R_3 \{Q, \overline{S}\} = -2iR^2.
\]

This bracket is clearly non-degenerate and defines a symplectic (Kirillov) form

\[
\omega = \frac{\delta w \wedge \delta \overline{w}}{2iR^2} = \frac{2\delta w \wedge \delta \overline{w}}{i(1 + |w|^2)^2},
\]

(17)
which after scaling \( z = \frac{w}{\sqrt{iR}} \) reduces to the canonical form \( \frac{\delta z \wedge \delta \overline{z}}{i} \). Since exterior differentiation commutes with pullback, we obtain a primitive for \( \omega \) by pulling back a primitive \( \theta \) of the canonical form:

\[
\theta = \frac{z \delta \overline{z} - \overline{z} \delta z}{2i} = \frac{1}{2i} \left( \frac{w}{R} - \frac{|w|^2}{4R^2} \right) \delta \overline{w} - \frac{w |w|^2}{4R^2} \delta w - \left( \frac{\overline{w}}{R} - \frac{|\overline{w}|^2}{4R^2} \right) \delta w - \frac{|w|^2}{4R^2} \delta \overline{w} = \frac{w \delta \overline{w} - \overline{w} \delta w}{2iR}. \]

The density in (16) transforms to

\[
H = \frac{w_x \overline{w}_x}{2R^2},
\]

(18)
which generates equations of motion in the new bracket

\[
w_t = i - \frac{R w_{xx} + \overline{w}_x^2}{R} = i \frac{-w_{xx} - |w|^2 w_{xx} + 2 \overline{w}_x^2}{2R} = -iw_{xx} + i \frac{\overline{w}_x^2}{R}.
\]

(19)

### 3.2 Lagrangian and Multisymplectic formulation of HM

We now have all the elements necessary to reconstruct the Lagrangian and multisymplectic form of the system in the complex plane. With respect to the symplectic structure (17) the equations of motion (19) are

\[
i(\mathbf{W}) \omega = \frac{\delta H}{\delta \mathbf{W}}.
\]

From (12), letting \( \theta \) be defined as in (3.1), \( X_t = \partial_t = w_t \frac{\partial}{\partial w} + \overline{w}_t \frac{\partial}{\partial \overline{w}} \), and \( H = H dx \) using (15), one obtains

\[
\mathbf{A} = -(H + i(X_t) \theta) dx \wedge dt = \left( \frac{w_x \overline{w}_x}{2R^2} + \frac{w \overline{w}_x - \overline{w} w_t}{2iR} \right) dx \wedge dt.
\]

(20)

The multisymplectic form now follows from (7). Computing the variation of the Lagrangian one obtains

\[
\delta \mathbf{A} = \left\{ \frac{w_x \overline{w}_x}{2R^3} + \partial_x \left( \frac{w_x}{2R^2} \right) - \frac{w_x}{2iR} + \frac{w \overline{w}_x - \overline{w} w_t}{4iR^2} - \partial_t \left( \frac{w}{2iR} \right) \right\} \delta w \wedge dx \wedge dt
\]

\[
+ \left\{ \frac{w_x \overline{w}_x}{2R^3} + \partial_x \left( \frac{w_x}{2R^2} \right) + \frac{w_x}{2R} + \frac{w \overline{w}_x - \overline{w} w_t}{4iR^2} + \partial_t \left( \frac{w}{2iR} \right) \right\} \delta \overline{w} \wedge dx \wedge dt
\]

\[
+ \partial \left\{ \frac{w_x}{2R^2} \delta w \wedge dt + \frac{w_x}{2R} \delta \overline{w} \wedge dt + \frac{w}{2iR} \delta w \wedge dx - \frac{w}{2iR} \delta \overline{w} \wedge dx \right\},
\]

(21)
yielding the fundamental differential forms

\[
\Theta = -\frac{w_x}{2R^2} \delta w \wedge dt - \frac{w_x}{2R^2} \delta \overline{w} \wedge dt - \frac{w}{2iR} \delta w \wedge dx + \frac{w}{2iR} \delta \overline{w} \wedge dx,
\]

(22)

\[
\Omega = \delta \Theta \frac{w_x - \overline{w}_x}{2R^3} \delta w \wedge \delta \overline{w} \wedge dx + \frac{1}{2R^2} \delta w \wedge \delta w \wedge dx + \frac{1}{2R^2} \delta w \wedge \delta \overline{w} \wedge dx + \frac{1}{2iR^2} \delta w \wedge \delta \overline{w} \wedge dx,
\]

(23)
and, by using (7), the covariant Hamiltonian

$$\mathcal{H} = \left(\frac{w_x w_x}{2R^2} + \frac{w_l - w_t}{2iR}\right) dx \wedge dt + \left(\frac{w_l w_t - w_x}{2iR} - \frac{w_x w_x}{R^2}\right) dx \wedge dt = -\frac{w_x w_x}{2R^2} dx \wedge dt = -\mathbf{H}. \quad (24)$$

Comparing the coefficients of $\delta w \wedge dx \wedge dt$ and $\delta \mathbf{w} \wedge dx \wedge dt$ in

$$\delta \mathcal{H} = \frac{w_l w_x}{2R^3} \delta w \wedge dx \wedge dt + \frac{w_x w_x}{2R^3} \delta \mathbf{w} \wedge dx \wedge dt - \frac{w_x}{2R^2} \delta w_x \wedge dx \wedge dt - \frac{w_x}{2R^2} \delta \mathbf{w}_x \wedge dx \wedge dt,$$

to those in

$$dx \wedge i(\tilde{\partial}_x) \Omega = \frac{R w_x}{2R^3} w_x \delta w \wedge dx \wedge dt + \frac{w_x}{2R^2} \delta w_x \wedge dx \wedge dt - \frac{w_x}{2R^2} \delta \mathbf{w}_x \wedge dx \wedge dt,$$

we obtain the equations of motion (19), while the coefficients of $\delta w_x \wedge dx \wedge dt$ and $\delta \mathbf{w}_x \wedge dx \wedge dt$ yield the tautological identities

$$-\frac{w_x}{2R^2} = -\frac{w_l}{2R^2}, \quad -\frac{w_x}{2R^2} = -\frac{w_t}{2R^2}. \quad (25)$$

These can be used to define momenta as in (19):

$$\mathcal{H} = -2R^2 \mathbf{p} \mathbf{d}x \wedge dt = -(1 + |\mathbf{w}|^2)|\mathbf{p}|^2 dx \wedge dt, \quad \mathbf{H} = -\delta w \wedge \delta \mathbf{p} \wedge dx \wedge dt + \frac{1}{2R^2} \delta \mathbf{w} \wedge \delta \mathbf{w} \wedge dx \wedge dt, \quad (26)$$

and the components of (7) take the form

$$dx \wedge i(\tilde{\partial}_x) \Omega = w_x \delta w \wedge dx \wedge dt - p_x \delta \mathbf{w} \wedge dx \wedge dt + \mathbf{w}_x \delta \mathbf{p} \wedge dx \wedge dt - \mathbf{w}_x \delta \mathbf{w} \wedge dx \wedge dt,$$

$$dt \wedge i(\tilde{\partial}_t) \Omega = \frac{w_t}{2R^2} \delta w \wedge dx \wedge dt - \frac{w_t}{2R^2} \delta \mathbf{w} \wedge dx \wedge dt,$$

yielding the equations

$$-2R^2 \mathbf{p} \mathbf{d}x \wedge dt = -2R^2 |\mathbf{w}|^2 \delta w \wedge dx \wedge dt - 2Rw |\mathbf{p}|^2 \delta \mathbf{w} \wedge dx \wedge dt - 2R^2 \mathbf{p} \delta \mathbf{w} \wedge dx \wedge dt - 2R^2 \mathbf{p} \delta \mathbf{w} \wedge dx \wedge dt,$$

yielding the equations

$$-2R^2 \mathbf{p} = w_l, \quad -2R^2 \mathbf{p} = w_x, \quad -2R^2 |\mathbf{p}|^2 = -\frac{w_t}{2R^2} - p_x, \quad -2Rw |\mathbf{p}|^2 = \frac{w_t}{2R^2} - \mathbf{p} \mathbf{d}x \wedge dt,$$

which are equivalent to (19). This is the local multisymplectic formulation of HM.

To relate our formalism to (1) we introduce two matrices $\mathbf{K}$ and $\mathbf{M}$ and a coordinate vector $\mathbf{z} = (w, \mathbf{w}, \mathbf{p}, \mathbf{p})$ (since $w_l, \mathbf{w}_l$ never show up explicitly, we can get away with fewer fiber coordinates):

$$\mathbf{K} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{M} = \frac{1}{2R^2} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (28)$$

If $\boldsymbol{\Omega} = \Omega_t dt + \Omega_x dx, \boldsymbol{\Omega}$, where $\Omega_t, \Omega_x \in \Omega^{(2,0)}$ are fiber 2-forms, then $\mathbf{K}$ and $\mathbf{M}$ are simply matrices for $\Omega_t$ and $-\Omega_x$ in $\mathbf{z}$-coordinates. Indeed,

$$dx \wedge i(\tilde{\partial}_x) \Omega = -i(\tilde{\partial}_x) \Omega_t dx \wedge dt = -\mathbf{z}^T \mathbf{K} \cdot \delta \mathbf{z} \wedge dx \wedge dt, \quad dt \wedge i(\tilde{\partial}_t) \Omega = i(\tilde{\partial}_t) \Omega_t dx \wedge dt = -\mathbf{z}^T \mathbf{M} \cdot \delta \mathbf{z} \wedge dx \wedge dt,$$

and the canonical equations (7) reduce to

$$\mathbf{K} \cdot \mathbf{z}_x + \mathbf{M} \cdot \mathbf{z}_t = \nabla_x \mathcal{H}, \quad (29)$$

where $\nabla_x \mathcal{H}$ is a column vector of partial derivatives.

Finally, the conservation laws (20) yield the usual energy ($H$) and momentum ($P$) conservation laws:

$$T_{tt} = \frac{w_x w_x}{2R^2} = -H, \quad T_{tx} = -\frac{w_l w_x + w_t w_x}{2R^2}, \quad (30)$$

$$T_{xx} = \frac{w_x w_x - w_l w_l}{2iR} = P, \quad T_{xx} = -\frac{w_l w_x + w_t w_x}{2R^2} + \frac{w_l w_l}{2iR}. \quad (31)$$
It is important that the definition of these quantities ultimately relies on the choice of the primitive form \( \theta \), which is defined only locally. In general, we have no global definition for the conserved densities \( T_{jk} \) nor for the Lagrangian density itself. The functionals defined by these densities are known as multivalued functionals since changes in local choices of \( \theta \) result in differences by a multiple of a topological term, much like the multivalued function in \( z \). This phenomenon was investigated, among others, in [9, 13], where [9] sets a general task of relating the properties of critical points of such functionals to the topology of the underlying infinite-dimensional manifold of sections, extending the classical Morse theory, and, similarly, in the finite-dimensional case \( F \), replacing functions of the Morse theory by nontrivial elements \( H^1(F, \mathbb{R}) \), i.e. close but non-exact forms. In our situation the finite-dimensional case \( (\theta) \) fully determines the infinite-dimensional counterpart. The essential idea of [9] is to find a covering manifold on which the 1-form becomes exact, defining a global function, to which the classical Morse theory applies. At the moment it is not clear to us what covering of the sphere \( S^2 \) will trivialize \( \theta \). Such a covering is unlikely to be obtained by reduction of the symmetry groups \( SO(3) \) and \( SU(2) \) since, being 3-dimensional they do not carry a natural symplectic structure themselves, and being compact they do not have factor-spaces \( F \) with a nontrivial \( H^2(F, \mathbb{R}) \). The necessary covering must be similar to an infinitely-sheeted covering of the sphere completely (no branch points excluded). Further, the classical theory due to Lyusternik-Shnirelmann extends the Morse theory in the infinite-dimensional case to allow degenerate functionals, and relates the numbers of its critical points of different index to a topological invariant called the Lyusternik-Shnirelmann category (see the recent work by Walter Craig on water waves). This theory does require a globally-defined Lagrangian function, which the Novikov-Morse theory promises to provide. Given this, we should be able to estimate the number of solutions of a given kind (say, periodic ones) of HM on a given domain (e.g., with periodic boundary conditions) including the higher-dimensional spatial cases. Similar considerations apply to linearizations of HM near fixed points or periodic solutions, promising to be a tool of stability analysis. These investigations will be continued elsewhere.

4 HM discretizations

Given a PDE in variational form there are approaches to its discretizations that more or less automatically produce discrete analogs of the preservation of the multisymplectic form [8], and frequently result in superior discrete conservation laws (Cf. [9] which suggests a discrete analog of Nöther’s theorem). While [8] developed a finite-difference approach to variational PDEs, we believe a more natural approach is to rely on the finite element method (FEM) involving space-time meshes. One of the reasons is that the equation already possesses a discrete conservation laws (Cf. (9) which suggests a discrete analog of Nöther’s theorem). While [8] developed produce discrete analogs of the preservation of the multisymplectic form (8), and frequently result in superior

4.1 Finite element method for HM

Using (20) the continuous action functional is (with \( \mathcal{L} \equiv \Lambda \) as defined in (11))

\[
\mathcal{L} = \int \Lambda = \int \left( \frac{w_\theta w_t - \mathcal{M} w_t}{2R} - \frac{w_x w_\theta}{2R^2} \right) dx \wedge dt.
\]

Introducing a rectangular space-time mesh \( \tilde{M} \) on \( M \) with elements \( \{e\} \), we take a basis of piecewise linear on \( e \) functions \( P^1 \) (more generally they can be piecewise polynomial) [12] which, on the canonical square element with coordinates \((x_i, \eta) \in [0,1] \times [0,1] \), have the form

\[
\phi_1 = \frac{1}{4}(1 - \xi)(1 - \eta), \quad \phi_2 = \frac{1}{4}(1 + \xi)(1 - \eta), \quad \phi_3 = \frac{1}{4}(1 + \xi)(1 + \eta), \quad \phi_4 = \frac{1}{4}(1 - \xi)(1 + \eta).
\]

Then the approximations \( w \approx \sum_v w^v\phi_v \) are defined by collocation at element nodes (1 per element vertex, 4 per element), and for polynomial terms of \( \mathcal{L} \) they reduce to polynomials in the nodal values \( w^v \), while the rational terms have no canonical representation. First, we consider the product approximation [2] \( \frac{1}{R^v} \approx r^v\phi_v = \frac{1}{R^v}\phi_v \), and
obtain the following discrete action functional:

\[
L = \frac{1}{2} \int \left[ \sum_{u,v,\pi} i r^u \phi_a w^u w^v (\phi_e, \partial_t \phi_e - \partial_t \phi_e, \phi_{e,\pi}) - \sum_{u,v,\pi,\tau} (r^u r^v w^u w^v \phi_a \partial_e \phi_e, \partial_e \phi_{e,\pi}) \right] dx \wedge dt =
\]

\[
\sum_c \left[ \sum_{a,\pi,b=1}^4 \frac{1}{2} \int (\phi_{e,a} \partial_t \phi_{e,\pi} - \partial_t \phi_{e,\pi}, \partial_t \phi_{e,a}) \phi_{e,b} dx \wedge dt - \sum_{a,\pi,b=1}^4 \frac{1}{2} \int \partial_x \phi_{e,a} \partial_x \phi_{e,\pi} \phi_{e,b} dx \wedge dt \right],
\]

composed of element functionals \( L_e \):

\[
L = \sum_e L_e(w_e, \overline{w_e}), \quad L_e(w_e, \overline{w_e}) = i \sum_{a,\pi,b=1}^4 w_e^a w_e^b \partial_x \phi_{e,a} A_e^{a,\pi,b} - \sum_{a,\pi,b=1}^4 w_e^a w_e^b \partial_x \phi_{e,a} \phi_{e,b}, \quad H_e^{a,\pi,b,\tau} = \frac{1}{2} h_x \int (\partial_t \phi_{e,a} \partial_t \phi_{e,\pi} \phi_{e,b} dx \wedge dt,
\]

called element Lagrangians. The coefficients \( A_e^{a,\pi,b} \) and \( H_e^{a,\pi,b,\tau} \) are independent of the element \( e \) taken to be a rectangle with sides \( h_x, h_t \), and which can be computed by reduction to the canonical integrals:

\[
A_e^{a,\pi,b} = \hat{A}_{e,a,b,\pi} = \frac{1}{2} h_x \int (\phi_{e,a} \partial_t \phi_{e,\pi} - \partial_t \phi_{e,\pi}, \partial_t \phi_{e,a}) \phi_{e,b} dx \wedge dt,
\]

employing the obvious symmetries:

\[
\hat{A}_{e,a,b,\pi} = -\hat{A}_{e,a,b,\pi} \in \mathbb{R}, \quad \hat{H}_{e,a,b,\pi} = \hat{H}_{e,a,b,\pi} = \hat{H}_{e,a,b,\pi} \in \mathbb{R}.
\]

Being polynomials of degree \( \leq 8 \) in \( \xi, \eta \) these can be computed exactly (to round-off) using 5-point Gaussian quadrature, and as in any FEM code are never derived explicitly but assembled at run-time. While \( \hat{H} \) is a full 4-th order tensor, the 3-rd order tensor \( A \) has the following sparsity structure:

\[
\hat{A}_{e,a,b,\pi} \rightarrow \begin{pmatrix}
0 & 0 & \times & \times \\
0 & 0 & \times & \times \\
\times & \times & 0 & 0 \\
\times & \times & 0 & 0
\end{pmatrix}.
\]

As an alternative to the product approximation, we can discretize the rational terms by using their element averages. The element average of a function is defined to be the average of its approximation in the space spanned by \( \{ \phi_e \} \):

\[
\bar{f}_e = \frac{1}{|e|} \int e f^a \phi_e = \frac{1}{|e|} \int e \sum_{a=1}^4 f^a \phi_e, \quad f^a = \sum_{a=1}^4 f^a \phi_e, \quad |e| = \int e = h_x h_t.
\]

\( f^\pi_a \) denotes the nodal value of \( f \) at the \( a \)-th vertex of element \( e \). Computing the element averages easily yields:

\[
\hat{\phi}_a = \frac{1}{4}, \quad a = 1, \ldots, 4; \quad \text{then}
\]

\[
\bar{f}_e = \frac{1}{4} (f^1 + f^2 + f^3 + f^4).
\]

Defining element moment matrices \( \hat{K}^{(0)}, \hat{K}^{(1)}, \hat{K}^{(2)} \):

\[
\hat{K}^{(0)} = \int \phi_e \phi_e \quad \hat{K}^{(1)} = \int \phi_e \partial_t \phi_e \quad \hat{K}^{(2)} = \int \partial_x \phi_e \partial_x \phi_e.
\]

we have

\[
\hat{K}^{(0)} = \frac{h_x h_t}{36} \begin{pmatrix}
4 & 2 & 2 & 1 \\
2 & 4 & 1 & 2 \\
2 & 1 & 4 & 2 \\
1 & 2 & 2 & 4
\end{pmatrix}, \quad \hat{K}^{(1)} = \frac{h_x}{12} \begin{pmatrix}
-2 & -1 & 2 & 1 \\
-1 & -2 & 1 & 2 \\
-2 & -1 & 2 & 1 \\
-1 & -2 & 1 & 2
\end{pmatrix}, \quad \hat{K}^{(2)} = \frac{h_t}{6h_x} \begin{pmatrix}
2 & -2 & 1 & -1 \\
-2 & 2 & -1 & 1 \\
1 & -1 & 2 & -2 \\
-1 & 2 & -2 & 1
\end{pmatrix}.
\]
Now substituting approximations \( w \doteq \sum w^a \phi_a \) and \( \bar{R} \doteq R(\bar{w}) \) into \( \mathcal{L} \) \((\bar{R} \) is defined in the element interior only, which is enough for the integral to make sense), we obtain another discrete action functional:

\[
L = -\int \sum_{\alpha, \pi} \left( \frac{w^a \overline{w}^\pi \phi_a}{2R^2} + \frac{w^a \overline{w}^\pi (\phi_a x^\alpha - \phi_a x^\alpha \phi_a)}{2i\bar{R}} \right) \, dx \wedge dt = \sum_{\alpha} \sum_{\pi} \left( \frac{w^a \overline{w}^\pi}{2R^2} (\bar{K}(2) - \bar{K}(1) \frac{R}{\bar{R}} - \bar{K}(1)) \right).
\]

Noting that

\[
\bar{K}^{(2)} - (\bar{K}^{(1)})^T = \frac{h_x}{4} \begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix} = \bar{J},
\]

and letting \( \bar{K} = \bar{K}^{(2)} \) we can write

\[
L = \sum_{\alpha} L_{\alpha}(w, \bar{w}), \quad L_{\alpha}(w, \bar{w}) = \sum_{\alpha, \pi} \frac{w^a \overline{w}^\pi}{2R} \left( i \partial_{\alpha} \bar{\pi} - \bar{K}_a \bar{\pi} \right).
\]

The element Lagrangian \( L_{\alpha} \) corresponds to \( L_\Delta \) in [8] and represents one element’s contribution to the action. For a regular mesh the discrete Lagrangian is invariant under a mesh shift and as can be easily seen from [8], then the element Lagrangian is defined in terms of the canonical element Lagrangian \( \bar{L} \):

\[
\bar{L}(w, \bar{w}) = \bar{\hat{L}}(w^1, w^2, w^3, w^4), \quad \bar{L}(w^1, w^2, w^3, w^4) = \sum_{\alpha, \pi} \frac{w^a \overline{w}^\pi}{2R} \left( i \partial_{\alpha} \bar{\pi} - \bar{K}_a \bar{\pi} \right).
\]

where \((w^1, w^2, w^3, w^4)\) denote the four nodal points at a particular element, and \(\bar{R} \) is defined using the average of the four values \(w^1, w^2, w^3, w^4\). Defining \(J(w, \bar{w})\) and \(K(w, \bar{w})\) by their matrices

\[
\bar{J}(w, \bar{w}) = w \cdot \bar{J} \cdot \bar{w} = \sum_{\alpha, \pi} w_a \bar{J}_{a, \pi} \bar{w}_\pi, \quad \bar{K}(w, \bar{w}) = w \cdot \bar{K} \cdot \bar{w} = \sum_{\alpha, \pi} w_a \bar{K}_{a, \pi} \bar{w}_\pi,
\]

we obtain

\[
\bar{L}(w, \bar{w}) = \frac{i}{2R} \bar{J}(w, \bar{w}) - \frac{1}{2R^2} \bar{K}(w, \bar{w}), \quad \bar{J}(w, \bar{w}) = \frac{i h_x}{2R} \left[ \frac{(w^1 \overline{w}^1 - w^2 \overline{w}^2)}{4} + \frac{(w^3 \overline{w}^3 - w^4 \overline{w}^4)}{4} \right] -
\]

\[
\frac{1}{2R^2} \frac{h_x}{6h_x} \left[ 2|w^1|^2 + 2|w^2|^2 + 2|w^3|^2 + 2|w^4|^2 \right]
\]

\[
- 2(w^1 \overline{w}^2 + w^2 \overline{w}^1) + (w^3 \overline{w}^2 + w^2 \overline{w}^3) - (w^1 \overline{w}^4 + w^4 \overline{w}^1) - (w^2 \overline{w}^3 + w^3 \overline{w}^2) + (w^3 \overline{w}^4 + w^4 \overline{w}^3) - 2(w^3 \overline{w}^1 + w^1 \overline{w}^3).
\]

The discrete Euler-Lagrange field equations (DELF, [8]) are obtained by differentiating the discrete action, which depends on a particular nodal value \(w^a\) only through the element Lagrangians corresponding to four elements \(e_1^a, a = 1, \ldots, 4\) containing \(v\) as one of their vertices: \(L_{e_1^a}, L_{e_2^a}, L_{e_3^a}, L_{e_4^a}\). DELF equations are written simply as

\[
\partial L / \partial w^a = 0, \quad \partial L / \partial \bar{w}^a = 0, \quad \partial \bar{R} / \partial w^a = \frac{1}{8} \bar{w},
\]

determined using the canonical element derivative

\[
\partial \bar{L} / \partial w^a = \frac{i}{2R} \bar{J}_{a, \pi} \bar{w} - \frac{i \bar{w}}{16R^2} \bar{J}(w, \bar{w}) - \frac{1}{2R^2} \bar{K}_{a, \pi} \bar{w} + \frac{\bar{w}}{8R^2} \bar{K}(w, \bar{w}).
\]

### 4.2 Alternative approaches

Preliminary numerical experiments indicate that the FEM-based methods developed above can be unstable, apparently due to a relatively poor approximation of the rational terms in the polynomial bases. Here we sketch alternative approaches that may help avoid these difficulties.

First, consider a local quasi-linear form of the HM equations of motion. For this construction we rewrite the Lagrangian for HM and the equations of motion in the real coordinates \(w = a + ib\):

\[
\mathbf{A} = \int \left( \frac{w \overline{w}}{2R} - \frac{w \overline{w}}{2R^2} \right) \, dx \wedge dt = \int \left( \text{Im} \left( \frac{w \overline{w}}{R} \right) - \text{Re} \left( \frac{w \overline{w}}{2R^2} \right) \right) \, dx \wedge dt = 
\]

\[
\int \left( \frac{ab - \frac{a^2 + b^2}{R}}{2R^2} \right) \, dx \wedge dt.
\]
Applying the Euler-Lagrange operator explicitly we obtain
\[
\frac{\delta A}{\delta a} = \frac{b_t}{R} + \partial_t \left( \frac{b}{R^2} \right) - a \frac{ab_t - a_t b}{R^2} = \frac{2b_t}{R} - \frac{b}{R^2} (a a_t + b b_t) - \frac{a^2 b_t + a_t b}{R^2} = \frac{2b_t}{R} - \frac{(a^2 + b^2) b_t}{R^2} = \frac{b_t}{R^2},
\]
\[
\frac{\delta H}{\delta a} = \partial_x \left( \frac{a_x}{R^2} \right) + a \frac{a_x^2 + b^2}{R^3} = \frac{a_{xx}}{R^2} - 2 a x \left( a x + b b_x \right) + a \frac{a_x^2 + b^2}{R^3} = \frac{a_{xx}}{R^2} + a \frac{(b_x^2 - a_x^2) - 2a x b b_x}{R^3}.
\]
Since \( A \) is anti-symmetric in \( a \) and \( b \) while \( H \) is symmetric in the same variables, we easily obtain the corresponding expressions for \( \frac{\delta A}{\delta b} \) and \( \frac{\delta H}{\delta b} \):
\[
\frac{\delta A}{\delta b} = \frac{b_t}{R^2}, \quad \frac{\delta H}{\delta b} = \frac{b_{xx}}{R^2} + \frac{b (a_x^2 - b_x^2) - 2a a_x b_x}{R^3}.
\]
Thus, the Euler-Lagrange equations are
\[
\frac{\delta \Lambda}{\delta a} = \frac{b_t}{R^2} - \frac{a_{xx}}{R^2} + \frac{a (a_x^2 - b_x^2) + 2a x b b_x}{R^3} = 0, \quad \frac{\delta \Lambda}{\delta b} = -a_t - \frac{b_{xx}}{R^2} + \frac{b (b_x^2 - a_x^2) + 2a a_x b_x}{R^3} = 0, \quad (37)
\]
which are equivalent to \( [19] \) or
\[
\partial_t a + \partial_x b_x + \frac{b (b_x^2 - a_x^2) + 2a a_x b_x}{R} = 0, \quad \partial_t b - \partial_x a_x + \frac{a (a_x^2 - b_x^2) + 2a a_x b_x}{R} = 0. \quad (38)
\]
This equation has the quasi-linear and even the semi-linear form and is amenable to the method of characteristics.

Another approach, relying on the construction of a special class of section bases spaces relies on Lie-group methods. Limitations of space allow us only to sketch the approach. The group \( SO(3) \) acts transitively on \( S^2 \), therefore on each element \( e \) with canonical local coordinates \( e \) have \( \hat{S}(x, t) = g(x, t) \cdot \hat{S}_0 \), where \( g \in SO(3) \) and the action should be coadjoint (unlike the standard linear representation, found in the numerical literature), since it interacts naturally with the commutator. The group near the identity must be parameterized by its Lie algebra \( \cong \mathbb{R}^3 \), so that \( g(x, t) = e^{\text{exp}(x A(x, t) + t B(x, t))} \) where \( A \) and \( B \) are polynomial functions of \( x, t \) with values in \( so(3) \) and \( e^{\text{exp}} \) is an approximation to the exponential map (e.g. the Caley transform \( (I + \frac{1}{2} A)(I - \frac{1}{2} A)^{-1} \)). Using a convenient choice of \( A \) and \( B \) we should be able to reduce the spin length constraint (orbit thought \( \hat{S}_0 \) diffeomorphic to a the coset space by a close subgroup of rotations around \( \hat{S}_0 \)), but is linearization (ideal in \( so(3) \)). The spaces of sections (no longer linear) generated by \( A \) and \( B \) in a certain smoothness class replace the traditional FEM bases. The details are the subject of a forthcoming publication.

In this paper we have established the local Lagrangian and multisymplectic structure of the Heisenberg magnet model, and have shown how the powerful formalism of the variational calculus can naturally lead to the conservation properties and suggest natural discretizations.

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