A NOTE ON SPECTRAL ANALYSIS IN AUTOMORPHIC REPRESENTATION
THEORY FOR GL₂: II

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Abstract. We generalize Zagier’s work on regularized integral to the singular case in the adelic setting. We develop necessary tools of treating various singular cases of regularized triple product formulas, which appear naturally in the work of Michel and Venkatesh on the subconvexity problem for GL₂.

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1. Introduction

Trying to enlarge the applicability of the Rankin-Selberg method, Zagier [7] invented the regularized integral in automorphic representation theory for PGL₂, which deals with certain non convergent integrals naturally appearing in the theory. This idea was generalized and successfully applied to the problem of subconvexity for GL₂ in analytic number theory by Michel and Venkatesh [5, §4.3]. Roughly speaking, three definitions of regularized integral are available:

1. Subtract from \( \varphi \) a non integral part \( \mathcal{E}(\varphi) \), considered to have integral 0 by abuse of orthogonality, and define

\[
\int \varphi_{\text{reg}} = \int \varphi - \mathcal{E}(\varphi).
\]

2. We introduced some suitable meromorphic function \( E(s) \) with constant residue 1 at 0, such that

\[
\int \varphi \cdot E(s)
\]
is convergent for certain range of $s$ and has a meromorphic continuation. We then define
\[ \int_{\text{reg}} \varphi = \text{Res}_{s=0} \int \varphi \cdot E(s). \]

(3) We find a/any measure-operator $\sigma_0$ such that the convolution $\sigma_0 \ast \varphi$ becomes integrable, and that the dual measure $\sigma_0^\vee$ with respect to change of variables satisfies $\sigma_0^\vee \ast 1 = R_0 \neq 0$. We then define
\[ \int_{\text{reg}} \varphi = R_0^{-1} \int \sigma_0 \ast \varphi. \]

The idea of (1) seems to come from physicists. (2) has a variation in a simpler case \[ \& \] §4.3.2, and finds its incarnation in the automorphic representation theory as the Rankin-Selberg methods, which provides the concrete computability of the regularized integral. Zagier’s original work provides the equivalence of (1) and (2). (3) is due to Michel and Venkatesh. It is an extension of the theory of regularized integral, not just another equivalent definition. (In fact, $\sigma_0 \ast \varphi$ is really a different way of making $\varphi$ integrable from the subtraction $\varphi - E(\varphi)$, which in the application to the subconvexity problem \[ \& \] is at least “more suited” \[ \& \] §4.3.6 from the strategical viewpoint.)

However, neither \[ \& \] 7 nor \[ \& \] §4.3] was capable of explicitly treating an exceptional case (we shall call it the singular case), in which $\varphi$ has the quasi-character $| \cdot | \alpha$ in its set of exponents \[ \& \] §4.3.3]. An example is given by the regularizing Eisenstein series Definition 2.14. It is easy to see that any measure $\sigma_0$ made from Hecke operators which makes the regularizing Eisenstein series integrable also makes $\sigma_0^\vee = \sigma_0$ annihilate 1 (c.f. Remark 2.15). Hence the definition (3) can not be extended to the singular case. In order to avoid the singular case, the authors of \[ \& \] applied the technic of deformation \[ \& \] §5.2.6], which not only introduced complications in many auxiliary computations \[ \& \] §3.1.11, 3.2.4, 3.2.8, 4.1.9 & 4.4.3], but also reduced the precision of various estimations. Hence it is reasonable to look for a treatment of the singular case. (3) failing, we turn back to the original idea of Zagier, i.e., (1) and (2). For example, in the line after \[ \& \] 18], the author excluded the discussion of the possibility of “$\alpha_i = 1$”, which corresponds to the singular case, as well as the possibility of “$\alpha_i = 0$”. This is unnatural, since the case “$\alpha_i = 0$” is in the range of integrability, being easily handled by subtracting a constant function to reduce to the case of non-existence of “$\alpha_i = 0$” in (2) (c.f. Theorem 2.11 (4) for a rigorous treatment): while in the case of “$\alpha_i = 1$” the formula in Theorem 2.11 (4) still makes sense, except that the pole of $R(s, \varphi)$ at $s = 1/2$ may not be simple. It will be consistent with (1) if we allow the subtraction of the regularizing Eisenstein series (c.f. Definition 2.15), as well as its derivatives. Moreover, in Remark 2.10 although $(T(p) - 1)^2$ annihilates both $E^\text{reg}(1/2, \varepsilon_0)$ and 1, it “kills twice” 1 while $(T(p) - 1)$ alone does not “kill” $E^\text{reg}(1/2, \varepsilon_0)$, which intuitively suggests defining
\[ \int_{[\text{PGL}_2]} E^\text{reg}(1/2, \varepsilon_0) = 0 \]
from the viewpoint of definition (3). However, such an extension of regularized integral to the singular case looses PGL$_2$-invariance (c.f. Remark 2.23). All these suggest an extension to the singular case of the regularized integral, at least for the beauty of the theory itself\[1\]. The main purpose of this paper is to further develop this idea, in order to handle various regularized triple products of Eisenstein series in the singular case, which appear naturally in the work of \[ \& \].

In Section 2, we give a treatment of the basic theory in the singular case. For completeness, we also include a full treatment in the “regular case” in the adelic setting, which was first developed in \[ \& \] §4.3]. But even in the regular case, we stick strictly to the original idea of Zagier, avoiding the definition (3). In particular, we establish the PGL$_2(\mathbb{A})$-invariance of the regularized integral directly in Proposition 2.20 (to be compared with \[ \& \] §4.3.6]). In Section 3, we treat various singular cases of products of Eisenstein series. The idea of “deformation”, utilized in \[ \& \], turns out to be essential, since the computation directly from the definition fails to give explicit results. In Section 4, we develop more tools, which, together with

\[1\] The analysis of this paper, together with some technics treating the finite places with “more representation theoretic than analytic” arguments, improves the $c$-power dependence at the finite places into logarithmic power dependence in \[ \& \]. This will be explained in a future paper.
2. Zagier’s Regularized Integral

2.1. Regularized Integral on \( \mathbb{R}_+ \).

**Definition 2.1.** Let \( a : \mathbb{R}_+ \to \mathbb{C} \) be a continuous function. It is regularizable if

1. for any \( N \gg 1 \) as \( t \to \infty \)
   \[
   a(t) = f(t) + O(t^{-N}), \quad f(t) = \sum_{i=1}^{l} \frac{c_i}{n_i!} t^{s_i + \alpha_i} \log^{n_i} t,
   \]
   where \( c_i, \alpha_i \in \mathbb{C}, n_i \in \mathbb{N} \);

2. for some \( \alpha \in \mathbb{R} \) as \( t \to 0^+ \)
   \[
   a(t) = O(t^\alpha).
   \]

In this case, we write for \( T > 0 \)
   \[
   h_T(s) = \sum_{i=1}^{l} \frac{c_i}{n_i!} \frac{\partial^{n_i}}{\partial s^{n_i}} \left( \frac{T^{s+\alpha_i}}{s+\alpha_i} \right) = \sum_{i=1}^{l} \frac{c_i}{n_i!} \sum_{m=0}^{n_i} \frac{(-1)^{n_i-m}}{m!} \frac{T^{s+\alpha_i} \log^{m} T}{(s+\alpha_i)^{n_i-m+1}}.
   \]

**Lemma 2.2.** For \( f(t) \) as above, if \( \lim_{T \to \infty} \int_{1}^{T} \frac{f(t) dt}{t^2} \) exists, then \( \Re \alpha_i < 1/2 \) for all \( 1 \leq i \leq l \).

**Proof.** If not, the condition implies that
   \[
   \int_{1}^{T} \frac{f(t) dt}{t^2} = \sum_{\alpha_j \neq 1/2} \sum_{m=0}^{n_j} \frac{(-1)^{n_j-m}}{(\alpha_j - \frac{1}{2})^{n_j-m+1}} T^{\alpha_j - \frac{1}{2}} (\log T)^m - \sum_{\alpha_j \neq 1/2} \frac{c_j}{(\alpha_j - \frac{1}{2})^{n_j+1}} \frac{(\log T)^{n_j+1}}{n_j+1}
   \]
   is bounded as \( T \to +\infty \). Let \( \sigma = \max_{j} \Re \alpha_j \). We distinguish two cases.

1. \( \sigma = 1/2 \). Let \( l = \max \{ n_j : \Re \alpha_j = 1/2, \alpha_j \neq 1/2 \} \cup \{ n_j + 1 : \alpha_j = 1/2 \} \). We divide both sides of the equation by \( (\log T)^l \) and let \( T \to +\infty \) to get
   \[
   \lim_{T \to \infty} \sum_{\alpha_j \neq 1/2} c_j T^{i \tau_j} = 0
   \]
   where \( \Im \alpha_j = \tau_j \) for \( j \) such that either \( \Re \alpha_j = 1/2, \alpha_j \neq 1/2 \) and \( n_j = \alpha_j = 1/2 \). In particular \( \tau_j \) are mutually distinct.

2. \( \sigma > 1/2 \). Let \( l = \max \{ n_j : \Re \alpha_j = \sigma \} \). We divide both sides of the equation by \( T^{\sigma - 1/2} (\log T)^l \) and let \( T \to +\infty \) to get an equation of the same form as \( (2.1) \).

We conclude because \( (2.1) \) contradicts the following Corollary 2.3. \( \square \)
Write $T = \mathbb{R}/\mathbb{Z}$. Let $\vec{\theta} = (\theta_1, \cdots, \theta_n) \in \mathbb{R}^n$. For any $\vec{x} \in \mathbb{R}^n$, we write by $[\vec{x}]$ its image in $T^n$. Define

$$T_{\vec{\theta}} = \{[t\vec{\theta}] : t \in \mathbb{R}\}.$$ 

It is a closed subgroup of $\mathbb{T}^n$. Furthermore, the one parameter subgroup $U_{\vec{\theta}} = \{t\vec{\theta} : t \in \mathbb{R}\}$ of $\mathbb{R}^n$ acts uniquely ergodically on $T_{\vec{\theta}}$ w.r.t. the Haar measure of $T_{\vec{\theta}}$. More precisely,

**Lemma 2.3.** For any $f \in C(T_{\vec{\theta}})$, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f([\vec{x}] + [t\vec{\theta}])dt = \int_{T_{\vec{\theta}}} f dm_{\vec{\theta}}$$

for all $[\vec{x}] \in T_{\vec{\theta}}$. Here $dm_{\vec{\theta}}$ is the normalized Haar measure on $T_{\vec{\theta}}$.

**Proof.** Consider the group of characters $\text{Ch}(\mathbb{T}^n)$ of $\mathbb{T}^n$ given by

$$e_{\vec{\theta}}(\vec{x}) = e(\vec{n} \cdot \vec{x}) = e(\sum_{i=1}^n n_i x_i), \vec{n} \in \mathbb{N}^n, \vec{x} \in \mathbb{R}^n$$

where $e(x) = e^{2\pi ix}$. By the duality theorem for locally compact abelian groups, the group of characters $\text{Ch}(T_{\vec{\theta}})$ is the quotient of $\text{Ch}(\mathbb{T}^n)$ by the subgroup of $e_{\vec{\theta}}$'s which vanish on $T_{\vec{\theta}}$. Obviously, we have

$$e_{\vec{\theta}}(T_{\vec{\theta}}) = 1 \Leftrightarrow e(t\vec{n} \cdot \vec{\theta}) = 1 \text{ for } \forall t \in \mathbb{R} \Leftrightarrow \vec{n} \cdot \vec{\theta} = 0.$$

So $\text{Ch}(T_{\vec{\theta}})$ are $e_{\vec{\theta}}$'s modulo the subgroup of $e_{\vec{\theta}}$'s with $\vec{n} \cdot \vec{\theta} = 0$. Let $|e_{\vec{n}}| \neq 0$ denote a non trivial equivalence class of $e_{\vec{n}}$ in the quotient group, we calculate

$$\left| \frac{1}{T} \int_0^T [e_{\vec{n}}]([\vec{x}] + [t\vec{\theta}])dt \right| = \left| \frac{e(\vec{n} \cdot \vec{x})}{T} \cdot \frac{e(T\vec{n} \cdot \vec{\theta}) - 1}{\vec{n} \cdot \vec{\theta}} \right| \leq \frac{2}{|T| \cdot |\vec{n} \cdot \vec{\theta}|} \to 0, T \to \infty.$$

The lemma is thus proved for $f = [e_{\vec{n}}]$, hence the $\mathbb{C}$-vector space generated by $\text{Ch}(T_{\vec{\theta}})$, which is also a $*$-subalgebra of $C(T_{\vec{\theta}})$. The lemma then follows by a standard application of the complex version of the Stone-Weierstrass theorem.

**Corollary 2.4.** Consider the function $f(x) = \sum_{k=1}^n a_k x^{i \theta_k}, x \in \mathbb{R}_+$, where $a_k \in \mathbb{C}, \theta_k \in \mathbb{R}, 1 \leq k \leq n$. If

$$\lim_{x \to +\infty} f(x)$$

exists, then $f(x)$ is a constant function.

**Proof.** Note that $f(e^{2\pi i t}) = \sum_{k=1}^n a_k e(t \theta_k), t \in \mathbb{R}$. If $f$ is not constant, we can find $t_1, t_2 \in \mathbb{R}$ s.t. $f(e^{2\pi i t_1}) \neq f(e^{2\pi i t_2})$. By continuity, we then find some neighborhood $U_1$ of $[t_1 \vec{\theta}]$, and $U_2$ of $[t_2 \vec{\theta}]$ such that

$$\sum_{k=1}^n a_k e(x_k) \neq \sum_{k=1}^n a_k e(y_k), \forall \vec{x} \in U_1, \vec{y} \in U_2.$$

But the flow $\gamma(t) = [t\vec{\theta}]$ meets both $U_1$ and $U_2$ infinitely often by the lemma, hence

$$\lim_{t \to \infty} f(e^{2\pi i t})$$

can not exist, contradicting the hypothesis.

**2.2. Regularized Integral for PGL$_2$.**

**Definition 2.5.** Let $\varphi : \text{GL}_2(\mathbb{F})\backslash \text{GL}_2(\mathbb{A}) \to \mathbb{C}$ be a continuous function. It is slowly increasing if for some $c \in \mathbb{R}$ and $g$ lying in some Siegel domain we have

$$|\varphi(g)| \ll \text{Ht}(g)^c, \text{Ht}(g) \to \infty.$$
Definition 2.6. Let \( \varphi : \text{GL}_2(\mathbb{F}) \backslash \mathbb{Z}(\mathbb{A}) \backslash \text{GL}_2(\mathbb{A}) \to \mathbb{C} \) be a slowly increasing function. Its regularizing kernel \( a(t, \varphi) \) is
\[
a(t, \varphi) = \int_{F(\mathbb{A}) \times F(\mathbb{A})} \varphi(n(x)a(t+y)k)dxdy,
\]
where \( t^+ = s_F(t) \) is the image of \( t \) under the section of the adelic norm map \( \mathbb{A}^\times \to \mathbb{R}_+ \) recalled in the beginning of this paper.

Definition 2.7. We call a slowly increasing function \( \varphi : \text{GL}_2(\mathbb{F}) \backslash \mathbb{Z}(\mathbb{A}) \backslash \text{GL}_2(\mathbb{A}) \to \mathbb{C} \) regularizable if its regularizing kernel \( a(t, \varphi) \) satisfies the condition (1) of Definition 2.1. In this case, we define for \( s \in \mathbb{C}, \Re s \gg 1 \)
\[
R(s, \varphi) = \int_0^\infty (a(t, \varphi) - f(t))t^{s-\frac{1}{2}} \frac{dt}{t}, \quad R^*(s, \varphi) = \Lambda_F(1 + 2s)R(s, \varphi).
\]
The space of regularizable functions is denoted by \( \mathcal{A}^{\text{reg}}(\text{GL}_2) \).

Remark 2.8. This is equivalent to saying \( a(t, \varphi) \) regularizable, i.e., the condition (2) of Definition 2.1
is automatically satisfied due to the following Corollary 2.10.

Lemma 2.9. If \( \gamma \in \text{GL}_2(\mathbb{F}) - \mathcal{B}(\mathbb{F}) \), then we have \( Ht(\gamma g) \leq Ht(g)^{-1} \).

Proof. This is [6, Lemma 3.19]. \( \square \)

Corollary 2.10. If \( \varphi \) is slowly increasing as in Definition 2.2, then we have
\[
|\varphi(g)| \ll Ht(g)^{\min(0, -c)}, Ht(g) \to 0.
\]

Proof. If \( c \leq 0 \), then it is easy to see that \( \varphi \) is bounded, since elements of bounded height in a Siegel domain form a compact subset and \( \varphi \) is continuous. The same argument shows that if \( c < 0 \) we can assume \( |\varphi(g)| \ll Ht(g)^c \) to hold in a whole Siegel domain \( S \) containing a fundamental domain. If \( Ht(g) \) is small, we take \( \gamma \in \text{GL}_2(\mathbb{F}) \) such that \( \gamma g \in S \), thus by the lemma we get
\[
|\varphi(g)| = |\varphi(\gamma g)| \ll Ht(\gamma g)^c \leq Ht(g)^{-c}.
\]

The following function together with its Taylor expansion plays an important role:
\[
(2.2) \quad \lambda_F(s) := \frac{\Lambda_F(-2s)}{\Lambda_F(2s)} = \frac{\Lambda_F^{-1}(0)}{s} + \sum_{n=0}^{\infty} \frac{s^n}{n!} \lambda_F^{(n)}(0).
\]

Recall the truncation operator \( \Lambda^c \) defined in [2] (5.5). Let \( f_0 \in \text{Ind}^{K}_{B(\mathbb{A}) \cap K}(1, 1) \) be constant equal to 1 on \( K \).

Theorem 2.11. (Adelic version of regularization due to Zagier [2])

1. Let \( \varphi : \text{GL}_2(\mathbb{F}) \backslash \mathbb{Z}(\mathbb{A}) \backslash \text{GL}_2(\mathbb{A}) \to \mathbb{C} \) be a slowly increasing function. For \( s \in \mathbb{C}, \Re s \gg 1 \) and any \( T \gg 1 \) we have
\[
\int_{[\text{PGL}_2]} \varphi(g) \Lambda^T E(s, f_0)(g) dg = \int_0^T a(t, \varphi) t^{s-\frac{1}{2}} \frac{dt}{t} - \int_T^\infty a(t, \varphi) t^{-s-\frac{1}{2}} \frac{dt}{t} \cdot \lambda_F(s - 1/2).
\]

2. If \( \varphi \) is, in addition, regularizable, then we have for \( T \gg 1 \)
\[
R^*(s, \varphi) + \Lambda_F(1 + 2s) h_T(s) + \Lambda_F(1 - 2s) h_T(-s) = \int_{[\text{PGL}_2]} \varphi(g) \Lambda^T E^*(s, f_0)(g) dg + \Lambda_F(1 + 2s) \int_T^\infty (a(t, \varphi) - f(t)) t^{s-\frac{1}{2}} \frac{dt}{t} + \Lambda_F(1 - 2s) \int_T^\infty (a(t, \varphi) - f(t)) t^{-s-\frac{1}{2}} \frac{dt}{t}.
\]
In particular, $R(s, \varphi)$ has a meromorphic continuation to $s \in \mathbb{C}$ with possible poles at $s = \pm 1/2, \pm \alpha_i, (\rho - 1)/2$ for $\rho$ running over the non-trivial zeros of $\zeta_F$, and satisfies the functional equation

$$R^*(s, \varphi) = R^*(-s, \varphi).$$

(3) Under the condition of (2), if $\Re\alpha_i < 0$ for all $1 \leq i \leq l$, then we have

$$R(s, \varphi) = \int_{[\text{PGL}_2]} \varphi(g) E(s, f_0)(g) dg, \quad \max_{1 \leq i \leq l} \alpha_i < \Re s < - \max_{1 \leq i \leq l} \alpha_i.$$

(4) Under the condition of (2), if $\varphi$ is integrable on $[\text{PGL}_2] := \text{GL}_2(F)Z(\mathbb{A})\backslash \text{GL}_2(\mathbb{A})$, then we have $\Re \alpha_i < 1/2$ for all $1 \leq i \leq l$, and

$$\int_{[\text{PGL}_2]} \varphi(g) dg = \frac{1}{\lambda_F^{(-1)}(0)} \left( \text{Res}_{s=\frac{1}{2}} R(s, \varphi) + c_i \delta_{\alpha_i = -\frac{1}{2}} \right), \quad \text{Vol}([\text{PGL}_2]) = \lambda_F^{(-1)}(0).$$

**Proof.** Since the proofs of (1) to (3) are quite similar to that in [7], we only mention some essential points of them. Only (4) needs more explanation.

(1) This is standard Rankin-Selberg unfolding together with

$$\Lambda_T E(s, f_0)(g) = \sum_{\gamma \in B(F) \setminus \text{GL}_2(F)} f_0, s(\gamma) g 1_{H(t, \gamma) \leq T} - \sum_{\gamma \in B(F) \setminus \text{GL}_2(F)} \mathcal{M} f_0, s(\gamma) g 1_{H(t, \gamma) > T}.$$

(2) It follows from rewriting the two terms at the right hand side of (1). For the first term we have

$$\int_0^T a(t, \varphi) t^{s-\frac{1}{2}} dt = R(s, \varphi) - \int_T^\infty (a(t, \varphi) - f(t)) t^{s-\frac{1}{2}} dt + \sum_{i=1}^l c_i \frac{\log^{n_i} t}{n_i} + h_T(s) t^{\frac{s}{2}}.$$

For the second term we have a similar equality.

(3) In the case $\Re \alpha_i = 0$ for all $i$, we let $T \to \infty$, taking into account Proposition [30] and $h_T(s) \to 0$, to get the asserted equation.

(4) The integrability of $\varphi$ implies $\Re \alpha_i < 1/2$ for all $i$ by Lemma [22]. We take residue at $s = 1/2$ on both sides of the equation obtained in (2) and divide by $\Lambda_F(1 + 2s)$ to see for $T \gg 1$

$$\text{Res}_{s=\frac{1}{2}} R(s, \varphi) + \text{Res}_{s=\frac{1}{2}} h_T(s) + \lambda_F^{(-1)}(0) h_T(-\frac{1}{2})$$

$$= \lambda_F^{(-1)}(0) \int_{D_T} \varphi(g) dg + \text{Res}_{s=\frac{1}{2}} \int_{\text{D} - D_T} \varphi(g) (E(s, f_0)(g) - E(s, f_0)_N(g)) dg$$

$$+ \lambda_F^{(-1)}(0) \int_T^\infty (a(t, \varphi) - f(t)) \frac{dt}{t^2},$$

where $D$ is the standard fundamental domain for $[\text{PGL}_2]$ and $D_T$ is the set of $g \in D$ such that $H(t, g) \leq T$.

It is easy to see that as $T \to \infty$

$$h_T(-\frac{1}{2}) \to 0; \quad \int_T^\infty (a(t, \varphi) - f(t)) \frac{dt}{t^2} \to 0.$$

By Proposition [5, 30] $E(s, f_0)(g) - E(s, f_0)_N(g)$ is of uniformly rapid decay with respect to $H(t, g), g \in D$ as $s$ remains in a compact neighborhood of $1/2$. Hence

$$\int_{\text{D} - D_T} \varphi(g) (E(s, f_0)(g) - E(s, f_0)_N(g)) dg$$

is holomorphic at $s = 1/2$. We compute $\text{Res}_{s=\frac{1}{2}} h_T(s)$ by noting that the second summand of

$$\frac{d^n}{ds^n} \left( \frac{T^s}{s} \right) = \frac{(-1)^n n!}{s^{n+1}} + \int_0^{\log T} t^n e^{st} dt$$

is
is holomorphic at $s = 0$, and get 
\[
\text{Res}_{s=0} h_T(s) = c_i \delta_{\alpha_i = -\frac{1}{2}}.
\]
\[\square\]

**Definition 2.12.** We define the regularized integral of a regularizable function $\varphi : [\text{PGL}_2] \to \mathbb{C}$ as
\[
\int_{[\text{PGL}_2]} \varphi(g) dg = \frac{1}{\lambda^{-1}_F(0)} \left( \text{Res}_{s=\frac{1}{2}} R(s, \varphi) + c_i \delta_{\alpha_i = -\frac{1}{2}} \right),
\]
where $c_i, \alpha_i, n_i$ are associated with $a(t, \varphi)$ as in Definition 2.7. We call the first term the principal part of the regularized integral, the second the degenerate part of the regularized integral. The regularized integral is linear and extends the integral on $A^{\text{reg}}(\text{GL}_2) \cap L^1(\text{GL}_2, 1)$.

### 2.3. Basic Properties.

**Definition 2.13.** Let $\omega$ be a unitary character of $\mathbf{F}^\times \setminus \mathbb{A}^\times$. Let $\varphi$ be a smooth function on $\text{GL}_2(\mathbf{F}) \setminus \text{GL}_2(\mathbb{A})$ with central character $\omega$. We call $\varphi$ finitely regularizable if there exist characters $\chi_i : \mathbf{F}^\times \setminus \mathbb{A}^\times \to \mathbb{C}^\times$, $\alpha_i \in \mathbb{C}, n_i \in \mathbb{N}$ and smooth functions $f_i \in \text{Ind}_{K(\mathbf{B})}^{K(\mathbf{A})}(\chi_i, \omega \chi_i^{-1})$ for $1 \leq i \leq l$, such that for any $M \geq 1$
\[
\varphi(n(x)a(y)k) = \varphi_N(n(x)a(y)k) + O(|y|^{-M}), \quad \text{as } |y| \to \infty,
\]
where we have written the essential constant term
\[
\varphi_N(n(x)a(y)k) = \varphi_N(a(y)k) = \sum_{i=1}^{l} \chi_i(y)|y|^{\frac{1}{2} + \alpha_i} \log^n|y|_{\mathbb{A}} f_i(k).
\]
In this case, we call $\mathcal{E}_\omega(\varphi) = \{ \chi_i : |\chi_i|^{\frac{1}{2} + \alpha_i} \in \mathcal{E}_\omega(\varphi) : \mathbb{R} \alpha_i \geq 0 \}$, $\mathcal{E}_\omega^+(\varphi) = \{ \chi_i : |\chi_i|^{\frac{1}{2} + \alpha_i} \in \mathcal{E}_\omega^+(\varphi) : \mathbb{R} \alpha_i < 0 \}$.

The space of finitely regularizable functions with central character $\omega$ is denoted by $A^{\text{reg}}(\text{GL}_2, \omega)$.

**Remark 2.14.** In the case $\omega = 1$, a finitely regularizable is smooth and regularizable in the sense of Definition 2.7. But a smooth regularizable function doesn’t need to be finitely regularizable.

**Definition 2.15.** In the case $\omega^{-1} T(t) = |t|^{\mu}$ for some $\mu \in \mathbb{R}$, we introduce the regularizing Eisenstein series for $f \in V_{\xi, \omega, \xi^{-1}}$ and $s$ in a neighborhood of $(1 + \mu)/2$
\[
E^{\text{reg}}(s, f)(g) = E(s, f)(g) - \frac{\lambda_F(1-2s-\mu)}{\lambda_F(1-2s+\mu)} \int f(k) \chi^{-1}(\det g) |\mu|^s_{\mathbb{A}} dk.
\]
It is holomorphic at $s = (1 + \mu)/2$.

**Remark 2.16.** Let $e_0 \in \text{Res}_{K}^{\text{GL}_2(\mathbb{A})}(1, 1)$ be the spherical function taking value 1 on $K$. Let $T(p)$ denote the order 1 Hecke operator at a finite place $p$ with cardinality of the residue field $q$. For $s \neq 0$, we have
\[
T(p)E^{\frac{1}{2}}_{\frac{1}{2} + s, e_0} = \lambda_F(p) E^{\frac{1}{2} + s, e_0}, \quad \lambda_F(s) = \frac{q^{1/2 + s + q^{-1/2 + s}}}{q^{1/2 + q^{-1/2}}};
\]
or
\[
T(p)E^{\frac{1}{2}}_{\frac{1}{2} + s, e_0} = \lambda_F(s) E^{\frac{1}{2} + s, e_0} + (\lambda_F(s) - 1) \cdot \lambda_F(s).
\]
The pole of $\lambda_F(s)$, defined in (2.27), at $s = 0$ is compensated by the zero of $\lambda_F(s) - 1$, hence we get
\[
T(p)E^{\frac{1}{2}}_{\frac{1}{2}, e_0} = E^{\text{reg}}_{\frac{1}{2}}(1, e_0) + \lambda_F^{(1)}(0) \cdot \lambda_F^{(1)}(0).
\]
In particular, $(T(p) - 1)^2$ annihilates $E^{\text{reg}}_{\frac{1}{2}, e_0}$.

**Remark 2.17.** (Violation of Covariance) Unlike the usual Eisenstein series, the map
\[
\pi^\infty(\xi, |1-i\mu|/2, \omega \xi^{-1}) \to C^\infty(\text{GL}_2, \omega), \quad f_{(1-i\mu)/2} \mapsto E^{\text{reg}}((1-i\mu)/2, f)
\]
is NOT $GL_2(\mathbb{A})$-covariant. But it’s still $K$-covariant.
Remark 2.18. By Proposition 2.10 and the definition of cusp forms, \( \mathcal{A}^\pi(\text{GL}_2, \omega) \) contains:

- \( \chi(\det g) \) for quasi-characters \( \chi \) such that \( \chi^2 = \omega \);
- smooth cusp forms, i.e., \( \mathcal{A}_{\text{cusp}}(\text{GL}_2, \omega) \);
- \( \frac{\partial^n}{\partial s^n} E(s, f) \) for some \( n \in \mathbb{N} \) and smooth \( f \in \text{Ind}_{\mathbb{B}(\mathbb{A})}^{\text{GL}_2(\mathbb{A})}(\xi, \omega \xi^{-1}) \) with \( Re(s) \geq 0 \) and \( s \neq -i\mu/2, 1/2 - i\mu/2 \) for \( \mu = \omega^{-1}(\xi) \) if \( \omega \xi^{-2} \) is trivial on \( \mathbb{A}^{(1)} \);
- \( \frac{\partial^n}{\partial s^n} |_{s = -i\frac{\mu}{2} + \frac{B}{2}} (s + i\frac{B}{2})E^*(s, f) \) for some \( n \in \mathbb{N} \) and smooth \( f, \mu \) the same as above;
- \( \frac{\partial^n}{\partial s^n} \mathcal{E}^\text{reg}(1 - i\frac{\mu}{2}, f) \) for some \( n \in \mathbb{N} \) and \( f, \mu \) the same as above;
- for some unitary character \( \omega \) of \( \mathbb{A} \), where \( \varphi_j \in \mathcal{A}^\pi(\text{GL}_2, \omega_j) \) with \( \omega = \Pi_j \omega_j \).

In the last case, we have \( \varphi_j^* = \Pi_j \varphi_j^* \). Note that we have excluded \( E(s, f) \) for \( Re(s) < 0 \). But they are actually present since they are related to the case \( Re(s) > 0 \) by functional equation.

Definition 2.19. Let \( \omega \) be a unitary character of \( \mathcal{F}^\times \setminus \mathbb{A}^\times \). The \( L^2 \)-residual space of central character \( \omega \), denoted by \( \mathcal{E}(\text{GL}_2, \omega) \), is the direct sum of the vector spaces \( \mathcal{E}^+ \) (GL_2, \omega) resp. \( \mathcal{E}^\text{reg} \) (GL_2, \omega) spanned by functions

\[
\frac{\partial^n}{\partial s^n} E^*(s, f), \text{ if } s \neq \frac{1 - i\mu}{2} \quad \text{resp.} \quad \frac{\partial^n}{\partial s^n} \mathcal{E}^\text{reg}(1 - i\frac{\mu}{2}, f)
\]

where \( s \in \mathbb{C}, Re(s) > 0 \) and for some unitary character \( \xi \) of \( \mathcal{F}^\times \setminus \mathbb{A}^\times \), \( f \in V^\infty_{\xi, \omega \xi^{-1}} \), and \( \mu \) as above.

We have the following simple fact of which we omit the proof.

Proposition 2.20. \( \mathcal{A}^\pi(\text{GL}_2, \omega) \) is stable under the right translation by \( \text{GL}_2(\mathbb{A}) \). Moreover, for any \( \varphi \in \mathcal{A}^\pi(\text{GL}_2, \omega) \) and any \( g \in \text{GL}_2(\mathbb{A}) \), their sets of exponents are the same:

\[
\mathcal{E}x(g.\varphi) = \mathcal{E}x(\varphi).
\]

Proof. Take \( f \in \text{Res}_K^{\text{GL}_2(\mathbb{A})}(\pi(\xi, \omega \xi^{-1})) \) and the flat section \( f_s \in \pi(\xi;|s|, \omega \xi^{-1};|s|) \) associated to it. It suffices to show that for any \( s_0 \in \mathbb{C}, n \in \mathbb{N} \) and fixed \( g \in \text{GL}_2(\mathbb{A}) \), the right translate by \( g \) of the partial derivative of this flat section, as a function on \( \text{GL}_2(\mathbb{A}) \)

\[
x \mapsto g. f^{(n)}(x) := \frac{\partial^n}{\partial s^n} f_s(xg)
\]

is a linear combination of such functions. The above function is clearly left invariant by \( N(\mathbb{A}) \), of central character \( \omega \). Taking \( x = a(y) \kappa \) for \( y \in \mathbb{A}^\times, \kappa \in \mathbb{K} \) and writing

\[
\kappa g = z' n' a(y') \kappa', \quad \text{for} \quad z' \in \mathbb{Z}(\mathbb{A}), n' \in N(\mathbb{A}), y' \in \mathbb{A}^\times, \kappa' \in \mathbb{K},
\]

where \( z', n', y', \kappa' \) are viewed as functions in \( \kappa \), we obtain

\[
\frac{\partial^n}{\partial s^n} f_s(a(y) \kappa g) = \sum_{k=0}^{n} \binom{n}{k} \xi(y) |y|^{\frac{1}{2} + s_0} (\log |y|_\mathbb{A})^k \cdot |y'|^{\frac{1}{2} + s_0} (\log |y'|_\mathbb{A})^{-k} \omega(z') \xi(y') f(\kappa').
\]

Although \( z', n', y', \kappa' \) are not uniquely determined by \( \kappa \), both \( |y'|_\mathbb{A} \) and \( \omega(z') \xi(y') f(\kappa') \) are, and define smooth functions on \( \mathbb{K} \). Moreover, the function

\[
f_\kappa(\kappa) := |y|^{\frac{1}{2} + s_0} (\log |y|_\mathbb{A})^{-k} \omega(z') \xi(y') f(\kappa') \in \text{Res}_K^{\text{GL}_2(\mathbb{A})}(\pi(\xi, \omega \xi^{-1})).
\]

Hence we get the relation

\[
\frac{\partial^n}{\partial s^n} f_s = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k}{\partial s^k} f_{k,s}
\]

and conclude. \( \square \)

Proposition 2.21. The vector space \( \mathcal{E}^+(\text{GL}_2, \omega) \) is stable under the right translation by \( \text{GL}_2(\mathbb{A}) \).
for some functions \( a_{\vec{k}}(s,g) \) holomorphic in \( s \) and smooth in \( g \), since

\[
a_{\vec{k}}(s,g) = \int f_s(\kappa g) e_{\vec{k},s} d\kappa.
\]

Moreover, we have \( a_{\vec{k}}(s,g) \ll_{s,f,N} \lambda_{\vec{k}}^{-N} \) for any \( N \in \mathbb{N} \), uniformly for \( s \) lying in any compact neighborhood of \( s_0 \). It follows that for any \( l \in \mathbb{N} \)

\[
a^{(l)}_{\vec{k}}(s,g) := \left. \frac{\partial^l}{\partial s^l} \right|_{s=s_0} a_{\vec{k}}(s,g) \ll_{g,f,N} \lambda_{\vec{k}}^{-N}.
\]

We thus get

\[
g \cdot f^{(n)}_{s_0} = \sum_{k=0}^{n} \sum_{\vec{k}} \binom{n}{k} a^{(n-k)}_{\vec{k}}(s_0,g) e_{\vec{k},s_0}^{(k)}.
\]

The inner sum coincides with the value at \( s = s_0 \) of a flat section \( f^{(k)}_{k,s} \) defined by

\[
f_k = \binom{n}{k} \sum_{\vec{k}} a^{(n-k)}_{\vec{k}}(s_0,g) e_{\vec{k}},
\]

which is smooth. We can also verify that

\[
\sum_{k=0}^{n} \frac{\partial^k}{\partial s^k} \big|_{s=s_0} M_s f_{k,s} = \sum_{k=0}^{n} \sum_{\vec{k}} \binom{n}{k} a^{(n-k)}_{\vec{k}}(s_0,g) \frac{\partial^k}{\partial s^k} \big|_{s=s_0} (\mu_{\vec{k}}(s) e_{\vec{k},s_0})
\]

\[
= \sum_{\vec{k}} \frac{\partial^n}{\partial s^n} \big|_{s=s_0} (a_{\vec{k}}(s,g)\mu_{\vec{k}}(s) e_{\vec{k},s_0}) = \frac{\partial^n}{\partial s^n} \big|_{s=s_0} M_s f_s,
\]

hence we deduce that

\[
g E^{(n)}_{N}(s_0,f) = \sum_{k=0}^{n} E^{(k)}_{N}(s_0,f_k).
\]

We conclude by

\[
g E^{(n)}(s_0,f) = \sum_{k=0}^{n} E^{(k)}(s_0,f_k)
\]

since both sides are orthogonal to the cusp forms and have the same constant term. \( \square \)

**Remark 2.22.** \( f_k \) constructed in both proofs of Proposition 2.20 and 2.21 coincide with each other.

**Remark 2.23.** The subspace \( E^{reg}(\text{GL}_2,\omega) \) is stable under right translation by \( K \) but not by \( \text{GL}_2(A) \). Take \( \omega = 1 \) for example. Choose a finite place \( p_0 \) with uniformizer \( \varpi_0 \). Let \( e_0 \in \text{Res}_{\text{GL}_2(A)}^{\text{GL}_2(K)} \pi(1,1) \) be the spherical function taking value 1 on \( K \). Let \( e_1 \) be defined by \( e_{1,v} = e_{0,v} \) for \( v \neq p_0 \); while \( e_{1,p_0} \) is unitary, \( K_0[p_0] \)-invariant and orthogonal to \( e_{0,p_0} \). For example, if \( q = \text{N}(p_0) \), we can take

\[
e_{1,p_0} = \sqrt{1+q^{-1}} \cdot \left\{ \sqrt{q+1} \cdot 1_{K_0[p]} - \frac{1}{\sqrt{q+1}} \cdot 1_{K_p} \right\}.
\]

It can be computed, writing \( \tilde{\lambda}_F(s) = \lambda_F(s - 1/2) \), that

\[
M_s e_{0,s} = \tilde{\lambda}_F(s) e_{0,-s}, \quad M_s e_{1,s} = \tilde{\lambda}_F(s) \mu_1(s) e_{0,-s}, \quad E^{\text{reg}}_{N}(1/2,e_0) = e_{0,1/2} - \lambda_{F}^{(-1)}(0) e_{0,-1/2}, \quad a(\varpi_0^{-1}).e_{0,s} = c_1(s)e_{1,s} + c_0(s)e_{0,s} \quad \text{where}
\]
\[ \mu_1(s) = q^{-2s} \frac{1 - q^{-(1-2s)}}{1 - q^{-(1+2s)}}, \quad c_1(s) = \frac{q^{s+1/2} - q^{-(s+1/2)}}{q^{1/2} + q^{-1/2}}, \quad c_0(s) = \frac{q^s + q^{-s}}{q^{1/2} + q^{-1/2}}. \]

We deduce that
\[ \varphi := a(\omega_0^{-1})_E \text{reg}(1/2, \varepsilon_0) - c_0(1/2) \text{reg}(1/2, \varepsilon_0) - c_1(1/2) \text{reg}(1/2, \varepsilon_1) \]
has constant term
\[ \varphi_N = \lambda_1^{(-1)}(0)(e_1^{(1)}(-1/2) - c_1(1/2)\mu_1^{(-1)}(1/2)) \cdot e_{1,-1/2} + \lambda_1^{(-1)}(0)(e_0^{(1)}(-1/2)e_{0,-1/2} \neq 0, \]
hence \( 0 \neq \varphi \notin \mathcal{E}(\text{GL}_2, 1). \) Consequently, \( a(\omega_0^{-1})_E \text{reg}(1/2, \varepsilon_0) \notin \mathcal{E}(\text{GL}_2, 1). \) We also deduce that
\[ \int_{[\text{PGL}_2]} \text{reg} \ a(\omega_0^{-1})_E \text{reg}(1/2, \varepsilon_0) = \int_{[\text{PGL}_2]} \varphi = \lambda_1^{(-1)}(0)(e_0^{(1)}(-1/2) \neq 0, \]
hence the regularized integral is in general not \( \text{PGL}_2(\mathbb{A}) \)-invariant (c.f. Proposition \[4.20\](1) below).

**Proposition 2.24.** Let \( \varphi \in \mathcal{A}^\text{r}(\text{GL}_2, \omega). \)

1. We can always find (not unique) \( \mathcal{E}(\varphi) \in \mathcal{E}(\text{GL}_2, \omega) \) such that \( \varphi - \mathcal{E}(\varphi) \in L^1(\text{GL}_2, \omega). \)
2. If for any \( \chi \in \mathcal{E}(\varphi) \), we have \( \Re \chi \neq 1/2 \), then there is a unique function \( \mathcal{E}(\varphi) \in \mathcal{E}(\text{GL}_2, \omega) \) such that \( \varphi - \mathcal{E}(\varphi) \in L^2(\text{GL}_2, \omega). \) Moreover, for any \( X \) in the universal enveloping algebra of \( \text{GL}_2(\mathbb{A}_\infty) \), we have \( X\varphi - X\mathcal{E}(\varphi) \in L^2(\text{GL}_2, \omega) \), i.e., \( \mathcal{E}(X, \varphi) = X\mathcal{E}(\varphi). \)

**Proof.** By Lemma \[2.2\] and Proposition \[5.3\] it is not difficult to see that \( \mathcal{E}(\text{GL}_2, \omega) \cap L^2(\text{GL}_2(\mathbb{F}) \backslash \text{GL}_2(\mathbb{A}), \omega) = \{0\} \), which implies the uniqueness of \( \mathcal{E}(\varphi) \) in (2). For the existence, we find \( \chi_i, \alpha_i, \nu_i, f_i \) as in Definition \[2.13\]. Then we take, writing \( \mu_i = \mu(\omega^{-1} \chi_i^2), \)

\[ \mathcal{E}(\varphi) = \sum_{\Re \alpha_j > 0} \frac{\partial^{n_j}}{\partial s^{n_j}} \text{E}(\alpha_j, \omega, \omega^{-1} \chi_j; f_j) + \sum_{\Re \alpha_j > 0} \frac{\partial^{n_j}}{\partial s^{n_j}} \text{E}^\text{reg}(\alpha_j, \omega, \omega^{-1} \chi_j; f_j). \]

Thus for any \( \chi \in \mathcal{E}(\varphi - \mathcal{E}(\varphi)) \), we have \( \Re \chi \leq 1/2 \) resp. \( \Re \chi < 1/2 \) under the condition in (2). Hence \( \varphi - \mathcal{E}(\varphi) \in L^1(\text{GL}_2, \omega) \) resp. \( L^2(\text{GL}_2, \omega) \). For the “moreover” part, it suffices to see that the differential operator \( X \) does not increase the real part of elements in \( \mathcal{E}(\varphi) \), which is essentially due to the following calculation:

\[ y \frac{d}{dy}(y^{\sigma} \log^k y) = y^{\sigma} \log^k y + k y^{\sigma} \log^{k-1} y, \forall y > 0, \sigma \in \mathbb{C}, k \in \mathbb{N}. \]

\[ \square \]

**Definition 2.25.** In the case (2), we call \( \mathcal{E}(\varphi) \) the \( L^2 \)-residue of \( \varphi \). For definiteness, we shall write \( \mathcal{E}(\varphi) \) to be the one given by \[2.20\].

**Proposition 2.26.**

1. For any \( \mathcal{E} \in \mathcal{E}(\text{GL}_2, 1) \), we have
\[ \int_{[\text{PGL}_2]} \mathcal{E}(g) dg = 0. \]
If moreover \( \mathcal{E} \in \mathcal{E}^+(\text{GL}_2, 1) \), then for any \( g_0 \in \text{GL}_2(\mathbb{A}) \), we have
\[ \int_{[\text{PGL}_2]} g_0 \mathcal{E}(g) dg = 0. \]

2. Let \( \varphi \in \mathcal{A}^\text{r}(\text{GL}_2, 1) \). For any \( \mathcal{E} \in \mathcal{E}(\text{GL}_2, 1) \) such that \( \varphi - \mathcal{E} \in L^1([\text{PGL}_2]) \) we have
\[ \int_{[\text{PGL}_2]} \varphi(g) dg = \int_{[\text{PGL}_2]} (\varphi - \mathcal{E})(g) dg. \]
In particular, \( \int_{[\text{PGL}_2]} \) is always \( K \)-invariant. It is \( \text{GL}_2(\mathbb{A}) \)-invariant on the subspace of \( \varphi \in \mathcal{A}^\text{r}(\text{GL}_2, 1) \) such that \( \mathcal{E}(\varphi) \) does not contain \( |.|^\mathbb{A} \).
We conclude by Definition 2.12. □

which is regular at \( \Lambda(1_s/n) \).

Remark 2.27. For (1), the second assertion follows from the first by Proposition 2.21. We calculate \( a(t, E) \) for \( E(g) = \frac{\partial^n}{\partial s^n} E(s, f)(g_0) \) with \( s \neq 1/2 - i\mu(\chi) \) resp. \( \frac{\partial^n}{\partial s^n} E^{\text{reg}}(\frac{1}{2} - i\mu(\chi), f)(g) \) for some \( f \in V^\infty_{\chi,\chi^{-1}} \) with \( \Re s > 0 \), a unitary character \( \chi \) of \( F^x \backslash A^x \) if \( \chi|_{F^x \backslash A^{(1)}} = 1 \). Due to the integral \( \int_{F^x \backslash A^{(1)}} d^x y \), it is easy to see that \( a(t, \varphi) \) is non-vanishing only if \( \chi \) is trivial on \( F^x \backslash A^{(1)} \), in which case \( \mu(\omega \chi^2) = 2\mu(\chi) \). We also notice that we can interchange the order of \( M \) and \( \int_K dk \) since they commute with each other.

- \( E(g) = \frac{\partial^n}{\partial s^n} E(s, f)(g) \) with \( s \neq 1/2 - i\mu(\chi) \): We get

\[
a(t, E) = \zeta^s_F(1) \int_K f(k) dk \left\{ t\log^n t + \frac{\partial^n}{\partial s^n} \right|_{s=\frac{1}{2}-i\mu(\chi)} \left( \frac{t^{\frac{1}{2}-s-i\mu(\chi)} - 1}{\Lambda_F(1 - 2s - 2i\mu(\chi))} \right) \right\},
\]

and conclude by the fact that \( a(t, E) \) has no constant term as a function of \( t \).

- \( E(g) = \frac{\partial^n}{\partial s^n} E^{\text{reg}}(\frac{1}{2} - i\mu(\chi), f)(g) \): We get

\[
a(t, E) = \zeta^s_F(1) \int_K f(k) dk \left\{ (-1)^l \frac{n^l}{l + 1} \log^{l+1} t \cdot \frac{\partial^n}{\partial s^n} \right|_{s=\frac{1}{2} - i\mu(\chi)} \left( \frac{s\Lambda_F(2s)}{\Lambda_F(2 - 2s)} \right) \right\},
\]

and conclude the same way as in the previous case.

For (2), the first part is trivial. For the second part, we note that \( \varphi - E(\varphi) \in L^1 \) implies \( g_0 \varphi - g_0 E(\varphi) \in L^1 \) for any \( g_0 \in \text{GL}_2(\mathbb{A}) \), and \( g_0 \varphi - E(\varphi) \) has regularized integral 0 by (1) if either \( E \in E^{\pm}(\text{GL}_2, 1) \) or \( g_0 \in K \). □

Remark 2.27. The above proof of (2) is to be compared with [5, §4.3.6], where another simpler but indirect proof was given for the “regular case”.

3. Product of Two Eisenstein Series: Singular Cases

3.1. Deformation Technics. Above all, we have the following result in the regular case (c.f. [5, §3]).

**Lemma 3.1.** Let \( \xi_j, \xi_j' \) be (unitary) Hecke characters with \( \xi_1 \xi_1' \xi_2 \xi_2' = 1 \) and write \( \pi_j = \pi(\xi_j, \xi_j') \), \( j = 1, 2 \). For \( f_j \in \pi_j \), we shall write \( E^s \) for \( E \) or \( E^{\text{reg}} \), whichever is regular at \( s = 1/2 \). If \( \pi_1 \neq \pi_2 \), then for any \( n_1, n_2 \in \mathbb{N} \), we have

\[
\int_{[p\text{GL}_2]}^{\text{reg}} E^{(n_1)}(0, f_1) E^{(n_2)}(0, f_2) = 0, \quad \int_{[p\text{GL}_2]}^{\text{reg}} E^{(n_1)}(1/2, f_1) E^{(n_2)}(1/2, f_2) = 0.
\]

**Proof.** \( R(s, E^{(n_1)}(0, f_1) E^{(n_2)}(0, f_2)) \) resp. \( R(s, E^{(n_1)}(1/2, f_1) E^{(n_2)}(1/2, f_2)) \) represents the value at \( s_1 = s_2 = 0 \) of \( \partial^{n_1} \partial^{n_2} \) of

\[
\Lambda(1/2 + s + s_1 + s_2, \xi_1 \xi_2) \Lambda(1/2 + s + s_1 - s_2, \xi_1 \xi_2') \Lambda(1/2 + s - s_1 + s_2, \xi_1' \xi_2) \Lambda(1/2 + s - s_1 - s_2, \xi_1' \xi_2') \quad \text{resp.}
\]

\[
\Lambda(3/2 + s + s_1 + s_2, \xi_1 \xi_2) \Lambda(1/2 + s + s_1 - s_2, \xi_1 \xi_2') \Lambda(1/2 + s - s_1 + s_2, \xi_1' \xi_2) \Lambda(-1/2 + s - s_1 - s_2, \xi_1' \xi_2'),
\]

which is regular at \( s = 1/2 \) by assumption. The degenerate part is also easily seen to be 0 by assumption. We conclude by Definition 2.12. □
In other cases beyond the above one, it seems to be difficult to obtain simple formulas by definition. However, the idea of deformation does provide simple and useful formulas. In general, if \( \varphi \in A^r(PGL_2), E \in \mathcal{E}(PGL_2) \) are given, so that \( \varphi - E \in L^1([PGL_2]) \), and if we can find continuous families \( \varphi_s \in A^r(PGL_2), E_s \in \mathcal{E}(PGL_2) \) which coincide with \( \varphi, E \) at \( s = 0 \), then we have

\[
(3.1) \quad \int_{[PGL_2]}^{\text{reg}} \varphi - \int_{[PGL_2]}^{\text{reg}} E = \lim_{s \to 0} \left( \int_{[PGL_2]}^{\text{reg}} \varphi_s - \int_{[PGL_2]}^{\text{reg}} E_s \right).
\]

All the formulas we are going to obtain will follow this principle together with (suitable simple variants of Lemma 3.1). Since the computation is long, we shall only give detail in the most complicated cases. The notations in Lemma 3.1 will be used unless otherwise explicitly reset.

3.2. Unitary Series.

**Definition 3.2.** If \( f \in \text{Res}^{GL_2(A)}_K(\pi(1, 1)) \), we define for any \( s \in \mathbb{C} \) an operator \( \mathcal{M}_s : \text{Res}^{GL_2(A)}_K(\pi(1, 1)) \to \text{Res}^{GL_2(A)}_K(\pi(1, 1)) \) (abuse of notations) by requiring

\[
\mathcal{M}_sf(a(\omega)\kappa) = \xi_2(\omega)|g|^{-1/2} M_s f(\kappa), \quad \text{i.e.,} \quad \mathcal{M}_sf = (\mathcal{M}_s f)_{-s};
\]

resp. \( \mathcal{M}_s = \mathcal{M}_s \circ (I - P_K e_\xi) \), if \( \xi_1 = \xi_2 = \xi \) (since \( \mathcal{M}_s \) is “diagonalizable”)

\[
\mathcal{M}_sf = \sum_{n=0}^{\infty} \frac{s^n}{n!} M_0^{(n)} f, \quad \text{resp.} \quad \mathcal{M}_{1/2+s} f = \sum_{n=0}^{\infty} \frac{s^n}{n!} M_1^{(n)} f.
\]

Here \( P_K \), with \( dk \) the probability Haar measure on \( K \), is defined to be the map

\[
\pi(\xi, \xi) \to \mathbb{C}, f \mapsto \int_K f(\kappa) \xi^{-1}(\kappa) \, dk,
\]

and \( e_\xi = \xi \circ \det \in \text{Res}^{GL_2(A)}_K(\pi(1, 1)) \).

**Lemma 3.3.** Let \( f_1, f_2 \in \text{Res}^{GL_2(A)}_K(\pi(1, 1)) \). For \( 0 \neq s \in \mathbb{C} \) small, we have

\[
\int_{[PGL_2]}^{\text{reg}} E(s, f_1)^{(1)}(0, f_2) = 0.
\]

**Proof.** This is a variant of Lemma 3.1. □

We continue to use the notations in the previous lemma. We can write

\[
E_N(s, f_1)^{(1)}(0, f_2) = 2(f_1 f_2)_1^{(1)} + 2(M_{s} f_1 f_2)_1^{(1)} + (f_1 M_0^{(1)} f_2)_1^{(1)}.
\]

We tentatively define

\[
\mathcal{E}^{\text{reg}}(s) := s^{-1} \left\{ 2E^{\text{reg}, (1)}(1/2 + s, f_1 f_2) + 2E^{\text{reg}, (1)}(1/2 - s, M_{s} f_1 f_2) + E^{\text{reg}, (1)}(1/2 + s, f_1 M_0^{(1)} f_2) + E^{\text{reg}, (1)}(1/2 - s, M_{s} f_1 M_0^{(1)} f_2) \right\}
\]

Applying Lemma 3.3 with \( n = 0, 1 \) together with (3.1), we get

\[
\int_{[PGL_2]}^{\text{reg}} E^{(1)}(0, f_1)^{(1)}(0, f_2) = \lim_{s \to 0} \int_{[PGL_2]}^{\text{reg}} s^{-1} E(s, f_1)^{(1)}(0, f_2) - \mathcal{E}^{\text{reg}}(s)
\]

\[
= \frac{1}{s} \left\{ 2 \frac{\lambda_{F}^{(1)}(s)}{\lambda_{F}^{(1)}}(0) P_K(f_1 f_2) + 2 \frac{\lambda_{F}^{(1)}(-s)}{\lambda_{F}^{(1)}}(0) P_K(M_{s} f_1 f_2)
\right.
\]

\[
+ \frac{\lambda_{F}(s)}{\lambda_{F}^{(1)}}(0) P_K(f_1 M_0^{(1)} f_2) + \frac{\lambda_{F}(-s)}{\lambda_{F}^{(1)}}(0) P_K(M_{s} f_1 M_0^{(1)} f_2)
\right\}.
\]
Taking Laurent expansions, we verify that the function in $s$ in the range of the above limit is regular at $s = 0$, unlike its appearance. The properties

$$\text{P}_K(f_1 M_0^{(k)} f_2) = \text{P}_K(f_2 M_0^{(k)} f_1), \forall k \in \mathbb{N}; \quad M_0^{(2)} = M_0^{(1)} \circ M_0^{(1)}$$

coming from $\mathcal{M}_s \circ \mathcal{M}_{-s} = 1$

must be used. Taking limit as $s \to 0$, we obtain (2) of the following:

**Theorem 3.4.** If $\pi(\xi_1, \xi_2)$ is spherical, we also write the regularized integral of the product of two unitary Eisenstein series is computed as:

1. If $\pi_1 = \pi(\xi_1, \xi_2), \pi_2 = \pi(\xi_1^{-1}, \xi_2^{-1})$ resp. $\pi_2 = \pi(\xi_2^{-1}, \xi_1^{-1})$ and $\xi_1 \neq \xi_2$, then

$$\int_{[\text{PGL}_2]} \text{E}^{\text{reg}}(0, f_1) \text{E}(0, f_2) = \frac{2\lambda_\mathcal{F}^{(0)}(0)}{\lambda_\mathcal{F}^{(-1)}(0)} \text{P}_K(f_1 f_2) - \text{P}_K(M_0^{(1)} f_1 \cdot M_0 f_2), \quad \text{resp.}$$

$$\frac{\lambda_\mathcal{F}^{(0)}(0)}{\lambda_\mathcal{F}^{(-1)}(0)} (\text{P}_K(f_1 M_0 f_2) + \text{P}_K(f_2 M_0 f_1)) - \text{P}_K(M_0^{(1)} f_1 \cdot f_2).$$

2. If $\pi_1 = \pi(\xi, \xi), \pi_2 = \pi(\xi^{-1}, \xi^{-1})$, then

$$\int_{[\text{PGL}_2]} \text{E}^{(1)}(0, f_1) \text{E}^{(1)}(0, f_2) = \frac{4\lambda_\mathcal{F}^{(2)}(0)}{\lambda_\mathcal{F}^{(-1)}(0)} \text{P}_K(f_1 f_2) + \frac{4\lambda_\mathcal{F}^{(2)}(0)}{\lambda_\mathcal{F}^{(-1)}(0)} \text{P}_K(f_1 \cdot M_0^{(1)} f_2) + \frac{\lambda_\mathcal{F}^{(0)}(0)}{\lambda_\mathcal{F}^{(-1)}(0)} \text{P}_K(M_0^{(1)} f_1 \cdot M_0^{(1)} f_2) - \frac{1}{3} \text{P}_K(M_0^{(3)} f_1 \cdot f_2) - \text{P}_K(M_0^{(2)} f_1 \cdot M_0^{(1)} f_2).$$

**Remark 3.5.** It is possible to get formulas for all derivatives, exploiting more the relation $\mathcal{M}_s \circ \mathcal{M}_{-s} = 1$. Since we don’t have applications of these formulas, we do not include them here.

### 3.3. Singular Series.

**Lemma 3.6.** Let $f, f_1, f_2 \in \text{Res}^{\text{GL}_2(\mathbb{A})}_{\mathcal{K}} \pi(1, 1)$. For $0 \neq s \in \mathbb{C}$ small, we have for any $n, n_1, n_2 \in \mathbb{N}$

$$\int_{[\text{PGL}_2]} \text{E}^{\text{reg}, (n)} \left( \frac{1}{2} + s, f \right) = - \frac{\lambda_\mathcal{F}^{(n)}(s)}{\lambda_\mathcal{F}^{(-1)}(0)} \text{P}_K(f);$$

$$\int_{[\text{PGL}_2]} \text{E}^{\text{reg}, (n_1)} \left( \frac{1}{2} + s, f_1 \right) \text{E}^{\text{reg}, (n_2)} \left( \frac{1}{2}, f_2 \right) = 0.$$

**Proof.** The first formula follows immediately from Proposition 2.20 and definition. The second one is a variant of Lemma 3.1.

We continue to use the notations in the previous section and lemma. Denote $e = e_1$. We can write

$$\text{E}^{\text{reg}, (n)} \left( \frac{1}{2}, f \right) = f_1^{(n)} + \sum_{k=0}^{n} \frac{n}{k} (-1)^k (\tilde{\mathcal{M}}_{1/2}^{(n-k)} f)^{(k)}_{1/2}$$

$$+ \text{P}_K(f) \cdot \frac{(-1)^{n+1} \lambda_\mathcal{F}^{(-1)}(0)}{n+1} e_{-1/2} + \sum_{k=1}^{n} \frac{n}{k} (-1)^k \lambda_\mathcal{F}^{(n-k)}(0) e_{k-1/2}.$$
from which one easily deduce $E_N^{\text{reg}}(1/2 + s, f_1)E_N^{\text{reg} (n_2)}(1/2, f_2)$. We tentatively define

$$E^{\text{reg}}(s) := E^{(n_2)}\left(\frac{3}{2} + s, f_1 f_2\right) + \sum_{k=0}^{n_2} \binom{n_2}{k} (-1)^k E^{\text{reg} (k)}\left(\frac{1}{2} + s, f_1 M^{(n_2-k)}_{1/2} f_2\right)$$

$$+ P_K (f_2) \cdot \left\{ \frac{(-1)^{n_2+1} \lambda^{-1}_{F}(0)}{n_2 + 1} E^{\text{reg} (n_2+1)}\left(\frac{1}{2} + s, f_1\right) \right\}$$

$$+ \sum_{k=1}^{n_2} \binom{n_2}{k} (-1)^k \lambda^{-1}_{F}(0) E^{\text{reg} (k)}\left(\frac{1}{2} + s, f_1\right) + E^{\text{reg} (n_2)}\left(\frac{1}{2} - s, f_2 M^{1/2}_1 f_1\right)$$

$$+ \lambda_F(s) P_K (f_1) \cdot \left\{ E^{\text{reg} (n_2)}\left(\frac{1}{2} - s, f_2\right) - E^{\text{reg} (n_2)}\left(\frac{1}{2}, f_2\right) \right\}.$$ 

Applying Lemma 66 with $n_1 = 0$ together with (3.1), we get

$$\int_{[\text{PGL}_2]} E^{\text{reg}}\left(\frac{1}{2}, f_1\right) E^{\text{reg} (n_2)}\left(\frac{1}{2}, f_2\right) = \lim_{s \to 0} \int_{[\text{PGL}_2]} E^{\text{reg} (n_2)}\left(\frac{1}{2} + s, f_1\right) E^{\text{reg} (n_2)}\left(\frac{1}{2}, f_2\right) - E^{\text{reg} (s)}$$

$$= \lim_{s \to 0} \sum_{k=0}^{n_2} \binom{n_2}{k} \left(\frac{(-1)^k}{\lambda^{-1}_{F}(0)} \lambda^{-1}_{F}(s) P_K (f_1 M^{(n_2-k)}_{1/2} f_2) + \frac{\lambda^{-1}_{F}(s)}{\lambda^{-1}_{F}(0)} P_K (f_2 M^{1/2}_1 f_1) \right)$$

$$+ P_K (f_1) P_K (f_2) \cdot \left\{ \frac{(-1)^{n_2+1} \lambda^{-1}_{F}(s)}{n_2 + 1} + \sum_{k=1}^{n_2} \binom{n_2}{k} (-1)^k \frac{\lambda^{-1}_{F}(0) \lambda^{-1}_{F}(s)}{\lambda^{-1}_{F}(0)} + \lambda_F(s) \lambda^{-1}_{F}(s) \right\}.$$ 

Taking Laurent expansions, we verify that the function in $s$ in the range of the above limit is regular at $s = 0$, unlike its appearance. The symmetry

$$P_K (f_1 M^{(k)}_{1/2} f_2) = P_K (f_2 M^{(k)}_{1/2} f_1), \forall k \in \mathbb{N}$$

must be used. Moreover, it can be differentiated $n_1$ times to deduce (3) of the following:

**Theorem 3.7.**

1. If $\pi_1 \neq \pi_2$ and $\xi_1 = \xi_1', \xi_2 \neq \xi_2'$ resp. $\xi_1 \neq \xi_1', \xi_2 = \xi_2'$ resp. $\xi_1 = \xi_1', \xi_2 = \xi_2'$, $\xi_1 \xi_2 \neq 1$ and $\xi_1^{\xi_2} \xi_2 = 1$, then for any $n_1, n_2 \in \mathbb{N}$

$$\int_{[\text{PGL}_2]} E^{\text{reg} (n_1)}\left(\frac{1}{2}, f_1\right) E^{\text{reg} (n_2)}\left(\frac{1}{2}, f_2\right) = 0 \quad \text{resp.} \quad \int_{[\text{PGL}_2]} E^{\text{reg} (n_1)}\left(\frac{1}{2}, f_1\right) E^{\text{reg} (n_2)}\left(\frac{1}{2}, f_2\right) = 0$$

$$\int_{[\text{PGL}_2]} E^{\text{reg} (n_1)}\left(\frac{1}{2}, f_1\right) E^{\text{reg} (n_2)}\left(\frac{1}{2}, f_2\right) = 0.$$ 

2. If $\pi_1 = \pi(\xi_1, \xi_2)$, $\pi_2 = \pi(\xi_1^{-1}, \xi_2^{-1})$ resp. $\pi_2 = \pi(\xi_1^{-1}, \xi_2^{-1})$ with $\xi_1 \neq \xi_2$, then for any $n_1, n_2 \in \mathbb{N}$

$$\int_{[\text{PGL}_2]} E^{\text{reg} (n_1)}\left(\frac{1}{2}, f_1\right) E^{\text{reg} (n_2)}\left(\frac{1}{2}, f_2\right) = 0,$$ 

is a linear combination with coefficients depending only on $n_1, n_2$ and $\lambda_F(s)$ of

$$P_K (M^{(n_1+n_2+1)}_{1/2} f_1 \cdot f_2); \quad P_K (M^{(l)}_{1/2} f_1 \cdot f_2) = P_K (f_1 \cdot M^{(l)}_{1/2} f_2), 0 \leq l \leq \max(n_1, n_2).$$

3. If $\pi_1 = \pi(\xi, \xi)$, $\pi_2 = \pi(\xi^{-1}, \xi^{-1})$, then for any $n_1, n_2 \in \mathbb{N}$

$$\int_{[\text{PGL}_2]} E^{\text{reg} (n_1)}\left(\frac{1}{2}, f_1\right) E^{\text{reg} (n_2)}\left(\frac{1}{2}, f_2\right)$$

is a linear combination with coefficients depending only on $n_1, n_2$ and $\lambda_F(s)$ of

$$P_K (\bar{M}^{(l)}_{1/2} f_1 \cdot f_2) = P_K (f_1 \cdot \bar{M}^{(l)}_{1/2} f_2), 0 \leq l \leq \max(n_1, n_2); \quad P_K (\bar{M}^{(n_1+n_2+1)}_{1/2} f_1 \cdot f_2); \quad P_K (f_1) P_K (f_2).$$
4. Towards Singular Triple Product of Eisenstein Series

4.1. Some Complement of Regularized Integral. Let $\xi_1, \xi_2, \omega$ be Hecke characters with $\xi_1 \xi_2 \omega = 1$. Let $f \in \pi(\xi_1, \xi_2)$ and $\varphi \in C^\infty(\text{GL}_2, \omega)$, i.e., a smooth function on $\text{GL}_2(\mathbb{F}) \backslash \text{GL}_2(\mathbb{A})$ with central character $\omega$. Suppose $\varphi$ is finitely regularizable defined in Definition 2.13.

**Proposition 4.1.** For $\Re s \gg 1$ sufficiently large,

$$R(s, \varphi; f) := \int_{\mathbb{F}^* \backslash \mathbb{A}} \int_{\mathbb{K}} (\varphi_N - \varphi_N^*) (a(y) \kappa) f(\kappa) |y|^s \frac{d\kappa d^\times y}{\mathbb{A}}$$

is absolutely convergent. It has a meromorphic continuation to $s \in \mathbb{C}$. If in addition

$$\Theta := \max \{\Re \alpha_j\} < 0,$$

then we have, with the right hand side absolutely converging

$$R(s, \varphi; f) = \int_{[\text{PGL}_2]} \varphi \cdot E(s, f), \quad \Theta < \Re s < -\Theta.$$

In the above region, the possible poles of $R(s, \varphi; f)$ are

1. $1/2 + i\mu_j (\xi_1 \xi_2^{-1})$ if $\xi_1 \xi_2^{-1}$ is trivial on $\mathbb{A}^{(1)}$;
2. $(\rho - 1)/2$ where $\rho$ runs over the non-trivial zeros of $L(s, \xi_1 \xi_2^{-1})$.

In particular $R(s, \varphi; f)$ is holomorphic for $0 \leq \Re s < \min(-\Theta, 1/2)$.

**Proof.** The proof is quite similar to that of Theorem 2.11 (3), except that $\mathcal{M}_f$ is no longer explicitly computable. In fact, we have for $T > 1, \Re s \gg 1$, using the standard Rankin-Selberg unfolding

$$\int_{[\text{PGL}_2]} \varphi \cdot \Lambda^T E(s, f) = R(s, \varphi; f)$$

$$= \int_{\mathbb{F}^* \backslash \mathbb{A}^*} \left( \int_{\mathbb{K}} (\varphi_N - \varphi_N^*) (a(y) \kappa) f(\kappa) d\kappa \right) \xi_1(y) |y|^{s-1/2} 1_{|y|_\mathbb{A} < T} d^\times y$$

$$- \int_{\mathbb{F}^* \backslash \mathbb{A}^*} \left( \int_{\mathbb{K}} (\varphi_N - \varphi_N^*) (a(y) \kappa) M_f(\kappa) d\kappa \right) \xi_2(y) |y|^{-s-1/2} 1_{|y|_\mathbb{A} < T} d^\times y$$

$$+ \text{Vol}(\mathbb{F}^* \backslash \mathbb{A}^{(1)}) \left( \sum_{j=1}^{l} \int_{\mathbb{K}} f_j(\kappa) f(\kappa) d\kappa \cdot 1_{\chi_j \xi_1(\mathbb{A}^{(1)}) = 1} \cdot \frac{\partial^{n_j}}{\partial s^{n_j}} \left( \frac{T^{s+\alpha_j + i\mu_j}}{s + \alpha_j + i\mu_j} \right) \right)$$

$$- \sum_{j=1}^{l} \int_{\mathbb{K}} f_j(\kappa) M_f(\kappa) d\kappa \cdot 1_{\chi_j \xi_2(\mathbb{A}^{(1)}) = 1} \cdot (-1)^{n_j} \frac{\partial^{n_j}}{\partial s^{n_j}} \left( \frac{T^{-s + \alpha_j + i\mu_j}}{s + \alpha_j + i\mu_j} \right),$$

where $\mu_j$ resp. $\mu_j'$ is such that

$$\chi_j \xi_1(y) = |y|^{i\mu_j}, \quad \text{resp.} \quad \chi_j \xi_2(y) = |y|^{i\mu_j'}.$$

We conclude by first shifting $s$ to the desired region, then letting $T \to \infty$. The possible poles are encoded in the possible poles of $\mathcal{M}_f$, which are included in those of $L(1 + 2s, \xi_1 \xi_2^{-1})^{-1}$ in the above region (c.f. for example [3], Corollary 3.7, 3.10 & Lemma 3.18]).

**Proposition 4.2.** Let notations be as in the previous proposition with $\Theta \leq -1/2$. Recall

$$\varphi_N^*(n(x)a(y)k) = \varphi_N^*(a(y)k) = \sum_{j=1}^{l} \chi_j(y) |y|^{s+\alpha_j} \log^n |y|^{\Lambda} f_j(k).$$

(1) If $\xi_1 \neq \xi_2$, then

$$\left( \frac{\partial^n R}{\partial s^n} \right)^{\text{hol}} \left( \frac{1}{2}, \varphi; f \right) = \int_{[\text{PGL}_2]} \varphi \cdot E^{(n)} \left( \frac{1}{2}, f \right) - \sum_{j} \left( \frac{\lambda_{\varphi}^{(n+\alpha_j)}}{\lambda_{\varphi}^{(-1)}} \right)(0) \mathcal{P}_K(f_j, f),$$

where $\lambda_{\varphi}^{(n+\alpha_j)}$ and $\lambda_{\varphi}^{(-1)}$ are the regularized zeta functions of $\varphi$ for $n+\alpha_j$ and $-1$ respectively.
where the summation is over $j$ such that $\xi_1\chi_j(\mathcal{H}^{(1)}) = 1, \alpha_j + i\mu(\xi_1\chi_j) = 1/2$.

(2) If $\xi_1 = \xi_2 = \xi$, then
\[
\left( \frac{\partial^n R}{\partial s^n} \right)_{\text{hol}}^{0.5}(\varphi; f) = \int_{[\text{PGL}_2]} \varphi \cdot E^{\text{reg},(n)}(\frac{1}{2} + s, f) - \sum_j \frac{\lambda_F^{(n+n_j)}(0)}{\lambda_F^{(-1)}(0)} P_K(f_j f) + \lambda_F^{(n)}(0) \cdot P_K(f \cdot (\xi^{-1} \circ \det)) \cdot \int_{[\text{PGL}_2]} \varphi \cdot (\xi \circ \det),
\]
where the summation over $j$ is as in the previous case.

Proof. The case (1) being simpler, we only give details for (2). By twisting, we may assume $\xi = 1$. Let $s$ be small with $\Re s < 0$. The $L^2$-residue of $\varphi \cdot E(1/2 + s, f)$ is given by
\[
\mathcal{E}(s) := \sum_j E^{(n_j)}(s + 1 + \alpha_j, f_j f),
\]
where the summation is over $j$ such that $\Re \alpha_j > -1$. Define
\[
\mathcal{E}^{\text{reg}}(s) := \sum_j E^{\text{reg},(n_j)}(s + 1 + \alpha_j, f_j f) + \sum_j^{\ast} E^{(n_j)}(s + 1 + \alpha_j, f_j f)
\]
where $\sum_j'$ is the summation as in the statement and $\sum_j^{\ast}$ is the rest. By the previous proposition, we have
\[
R(\frac{1}{2} + s, \varphi; f) = \int_{[\text{PGL}_2]} \varphi \cdot E(\frac{1}{2} + s, f) = \int_{[\text{PGL}_2]} \varphi \cdot E^{\text{reg}}(\frac{1}{2} + s, f) + \lambda_F(s) P_K(f) \cdot \int_{[\text{PGL}_2]} \varphi
\]
\[
= \int_{[\text{PGL}_2]} (\varphi \cdot E^{\text{reg}}(\frac{1}{2} + s, f) - E^{\text{reg}}(s)) - \int_{[\text{PGL}_2]} (\mathcal{E}(s) - \mathcal{E}^{\text{reg}}(s))
\]
\[
+ \lambda_F(s) P_K(f) \cdot \int_{[\text{PGL}_2]} \varphi.
\]
Since $E^{\text{reg},(n)}(s)$ is the $L^2$-residue of $\varphi \cdot E^{\text{reg},(n)}(\frac{1}{2} + s, f)$, we can compare the holomorphic part of both sides and conclude by
\[
\mathcal{E}(s) - \mathcal{E}^{\text{reg}}(s) = \sum_j E^{(n_j)}(s + 1 + \alpha_j, f_j f).
\]

\[
\square
\]

4.2. A Triple Product Formula.

Theorem 4.3. Let $f_j \in \pi(1,1), j = 1, 2, 3$. Then for any $n \in \mathbb{N}$
\[
\int_{[\text{PGL}_2]} E^*(0, f_1) \cdot E^*(0, f_2) \cdot E^{\text{reg},(n)}(\frac{1}{2}, f_3)
\]
is the sum of
\[
\left( \frac{\partial^n R}{\partial s^n} \right)_{\text{hol}}^{0.5}(\frac{1}{2}, E^*(0, f_1) \cdot E^*(0, f_2); f_3)
\]
and a weighted sum with coefficients depending only on $\lambda_F(s)$ of
\[
P_K (\mathcal{M}^{(l)}_0 f_1 \cdot f_2) P_K(f_3), \quad 0 \leq l \leq 3;
\]
\[
P_K (f_1 \cdot f_2 \cdot \widetilde{\mathcal{M}}^{(l)}_{1/2} f_3), \quad 0 \leq l \leq \max(2, n) \quad \& \quad l = n + 3;
\]
\[
P_K ((f_1 \mathcal{M} f_2 + f_2 \mathcal{M} f_1) \cdot \widetilde{\mathcal{M}}^{(l)}_{1/2} f_3), \quad 0 \leq l \leq \max(1, n) \quad \& \quad l = n + 2;
\]
\[
P_K (\mathcal{M}_0 f_1 \cdot \mathcal{M}_0 f_2 \cdot \widetilde{\mathcal{M}}^{(l)}_{1/2} f_3), \quad 0 \leq l \leq n \quad \& \quad l = n + 1.
\]
Proof. We shall only point out how the computation is effectuated, since the formulas are quite long.

\[
E(f_1, f_2) := (\Lambda^*F)^2 \cdot \left\{ \text{E}^{\text{reg}}(2) \left( \frac{1}{2}, f_1 f_2 \right) + \frac{1}{2} \text{E}^{\text{reg}}(1) \left( \frac{1}{2}, f_1 \cdot M_0(1) f_2 + M_0(1) f_1 \cdot f_2 \right) \right. \\
+ \frac{1}{4} \text{E}^{\text{reg}} \left( \frac{1}{2}, M_0(1) f_1 \cdot M_0(1) f_2 \right) \right\}
\]

is the L²-residue of E*(0, f_1) · E*(0, f_2). Let \( \varphi := E^\ast(0, f_1) \cdot E^\ast(0, f_2) - E(f_1, f_2) \), then we need to compute

\[
\int_{[\text{PGL}_2]} \varphi \cdot \text{E}^{\text{reg}}(n) \left( \frac{1}{2}, f_3 \right) + \int_{[\text{PGL}_2]} \text{E}(f_1, f_2) \cdot \text{E}^{\text{reg}}(n) \left( \frac{1}{2}, f_3 \right).
\]

The first term is computed by Proposition 4.2 (2), involving

\[
\int_{[\text{PGL}_2]} \varphi = \int_{[\text{PGL}_2]} E^\ast(0, f_1) \cdot E^\ast(0, f_2) = \frac{(\Lambda^*F)^2}{4} \int_{[\text{PGL}_2]} E^{(2)}(0, f_1) \cdot E^{(1)}(0, f_2),
\]

which is treated in Theorem 3.4 (2). The second term is treated in Theorem 3.7 (3). \( \square \)

5. APPENDIX: Bounds of Smooth Eisenstein Series

5.1. General Remarks. We take the notations and assumptions in \( \mathcal{O} \). Namely we fix a section \( s_F : \mathbb{R}_+ \to F^{1} \backslash \mathbb{A}^s \) and assume the Hecke characters \( \omega, \xi \) to be trivial on the image of \( s_F \). We then have the definition of the Eisenstein series \( E(s, \xi, \omega \xi^{-1}; f) \) for \( f \in \mathcal{V}_{s,\xi,\omega \xi^{-1}}^{\infty} \).

Remark 5.1. We will sometimes omit \( \xi, \omega \xi^{-1} \) and write \( E(s, f) \) when it is clear from the context.

In \( \mathcal{O} \), we studied the size of \( E(s, \xi, \omega \xi^{-1}; f) \). For the purpose of the present paper, we need something finer. Precisely, we shall decompose it as

\[
E(s, \xi, \omega \xi^{-1}; f) = E_N(s, \xi, \omega \xi^{-1}; f) + \left( E(s, \xi, \omega \xi^{-1}; f) - E_N(s, \xi, \omega \xi^{-1}; f) \right)
\]

with

\[
E_N(s, \xi, \omega \xi^{-1}; f)(g) := \int_{F \backslash \mathbb{A}} E(s, \xi, \omega \xi^{-1}; f)(a(n)g) dx, n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}
\]

and study the growth in \( g \) of \( E_N(s, \xi, \omega \xi^{-1}; f) \) and \( E(s, \xi, \omega \xi^{-1}; f) - E_N(s, \xi, \omega \xi^{-1}; f) \) separately, as well as all their derivatives with respect to \( s \).

The study of the constant term is reduced to the study of the intertwining operator, which is already done in \( \mathcal{O} \). We focus on

\[
E(s, \xi, \omega \xi^{-1}; f)(g) - E_N(s, \xi, \omega \xi^{-1}; f)(g) = \sum_{\alpha \in \mathcal{F}^s} W(s, \xi, \omega \xi^{-1}; f)\alpha(x)g) dx, a(x) = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}
\]

and

\[
W(s, \xi, \omega \xi^{-1}; f)(g) = \int_{F \backslash \mathbb{A}} \psi(-x)E(s, \xi, \omega \xi^{-1}; f)(n(x)g) dx,
\]

where \( \psi \) is the standard additive character of \( F \backslash \mathbb{A} \). We are thus reduced to the study of the Whittaker functions \( W(s, \xi, \omega \xi^{-1}; f) \). If we were only interested in \( W(s, \xi, \omega \xi^{-1}; f) \) itself, then its behavior is already completely clear by \( \mathcal{O} \) or more generally with the “singular” cases by \( \mathcal{O} \) Proposition 2.2. However, we need a bit more for our purpose in this paper. Namely, we also need to estimate \( \partial^r \partial^s \mathcal{W}(s, \xi, \omega \xi^{-1}; f)(g) \).

Then not the results of \textit{loc.cit.} but the method serves, i.e., the method of integral representation of Whittaker functions.

If \( \Phi \in \mathcal{S}(\mathbb{A}^s) \) is a Schwartz function, we can define the following (non flat) section in \( \mathcal{V}_{s,\xi,\omega \xi^{-1}}^{\infty} \)

\[
f_\Phi(s, \xi, \omega \xi^{-1}; g) = \xi(\det g) \cdot \det g^{\frac{1}{2} + s} \mathcal{S} \Phi((0, t) g) \omega^{-1} \xi^2(t) |t|^{1+2s} dt
\]

first defined for \( \Re s > 0 \) then meromorphically continued to \( s \in \mathbb{C} \). Given \( f \in \mathcal{V}_{s,\xi,\omega \xi^{-1}}^{\infty} \), we want to give an explicit \( \Phi \) associated with \( f \). For simplicity of notations, we may assume \( f \) to be a pure tensor. We then construct \( \Phi = \otimes \Phi_p \) place by place:
(1) At $F_v = \mathbb{C}$ resp. $F_v = \mathbb{R}$ and for $f_v$ spherical resp. not spherical, we choose $\Phi_v$ using the construction in [6, Lemma 3.5 (1)] resp. [3, Lemma 3.8 (1)] for spherical resp. smooth functions.

(2) At $v < \infty$ and for $f_v$ not spherical, we choose $\Phi_v$ by

$$\Phi_v((0,1)\kappa) = C(\psi_v)\xi_v(d\det\kappa)^{-1}f_v(\kappa), \kappa \in K_v$$

i.e.

$$\Phi_v(u,x) = C(\psi_v)\xi_v^2(\kappa)f_v\left(\frac{x}{1}ight), \forall u \in o_v, x \in o_v;$$

$$\Phi_v(u,uy) = C(\psi_v)\xi_v^2(\kappa)f_v\left(\frac{-1}{y}\right), \forall u \in o_v, y \in o_v;$$

and $\Phi_v(x,y) = 0$ for $\max(|x|_v,|y|_v) \neq 1$.

(3) At $v = \infty$ and for $f_v$ spherical, we choose $\Phi_v$ by

$$\Phi_v = C(\psi_v) \cdot f_v(1) \cdot 1_{s^2},$$

Let $S = S(f)$ be the set of places $v$ such that $f_v$ is not spherical. Then we get

$$f_\Phi(s,\xi,\omega \xi^{-1};g) = D(F)^2\Lambda^N(1 + 2s, \omega^{-1}\xi^2) \prod_{v \in S} K_{v,a_v}(s,\omega_v^{-1}\xi^2_v) \cdot f(g).$$

We can thus deduce the bounds of $W(s,\xi,\omega \xi^{-1};f)$ from those of

$$W_\Phi(s,\xi,\omega \xi^{-1};g) = \xi(d\det g)\det g_\Lambda^{\frac{1}{2} + s} \int_{\Lambda^s} G_2(\varpi) \Phi(t, \frac{1}{t}) \omega^{-1}\xi^2(t)|t|^2s dt,$$

where the partial Fourier transforms are defined as in [3, (3.3)].

In Section 4.2, we will bound (5.2) locally place by place. We then use the obtained bound to get a bound for the sum $\Sigma_{a \in F^\times}|W_\Phi(s,\xi,\omega \xi^{-1};a(g))|$, using a convergence lemma treated in Section 4.4. We will treat all bounds with uniformity for $s$ with real part lying in any compact interval, so that the bounds for the derivatives in $s$ follow automatically by Cauchy’s integral formulae.

In Section 4.3, we will determine the behavior of the constant term based on [1].

5.2. Bounds of Non Constant Terms.

5.2.1. Archimedean Places. We omit the subscript $v$ since we work locally. The local integral representation has the form

$$W_\Phi(s,\xi,\omega \xi^{-1};a(y)\kappa) = \omega^{-1}(y)|y|_F^{\frac{1}{2} - \varepsilon} \int_{F^\times} G_2(\varpi) \Phi(t, \frac{y}{t}) \omega^{-1}\xi^2(t)|t|^2s dt,$$

for $y \in F^\times, \kappa \in K$.

We are thus reduced to studying the integral at the right hand side. By [3, Proposition 4.1] as well as its counterpart in the singular cases, it is easy to see the rapid decay at $\infty$ of

$$|W_\Phi(s,\xi,\omega \xi^{-1};a(y)\kappa)| \ll |y|^{-N}, \forall N \in \mathbb{N},$$

and the polynomial increase at $0$ of

$$|W_\Phi(s,\xi,\omega \xi^{-1};a(y)\kappa)| \ll |y|^{\frac{1}{2} - |R| - \varepsilon}, \forall \varepsilon > 0.$$

As for the implied constants in the above estimations, one naturally guess it is related to the Schwartz norms of $G_2(\varpi)\Phi$). Then we need to related these norms to the Schwartz norms of $\Phi$ itself. According to this strategy, we state the following two lemmas and the desired proposition.

**Lemma 5.2.** For any Schwartz norm $S^*$ there is a Schwartz norm $S^{**}$ such that

$$\sup_{\kappa \in K} S^*(G_2(\varpi)\Phi) \ll S^{**}(\Phi).$$
Lemma 5.3. For the real part of $s$ lying in a fixed compact interval, any Schwartz function $\Phi \in S(F^2)$ and any integer $N \in \mathbb{N}$, there is a Schwartz norm $S^*$ such that as $|y|_F \to \infty$

$$\left| \int_{F^*} \Phi(t, \frac{y}{t}) \omega^{-1} \xi^2(t) |t|_F^{2s} d^*t \right| \ll S^*(\Phi)|y|_F^{-N};$$

while for any $\epsilon > 0$ there is a Schwartz norm $S^{**}$ such that as $|y|_F \to 0$

$$\left| \int_{F^*} \Phi(t, \frac{y}{t}) \omega^{-1} \xi^2(t) |t|_F^{2s} d^*t \right| \ll \epsilon S^{**}(\Phi) \max(|y|_F^{-\epsilon}, |y|_F^{-2R_s-\epsilon}).$$

Proposition 5.4. Let the real part of $s$ vary in a fixed compact interval. For any integer $N \in \mathbb{N}$, as $|y| \to \infty$ and uniformly in $\kappa$, there is a Schwartz norm $S^*$ such that

$$|W\Phi(s, \xi, \omega\xi^{-1}; a(y)\kappa)| \ll S^*(\Phi)|y|_F^{-N};$$

while for any $\epsilon > 0$, as $|y| \to 0$ and uniformly in $\kappa$, there is a Schwartz norm $S^{**}$ such that

$$|W\Phi(s, \xi, \omega\xi^{-1}; a(y)\kappa)| \ll \epsilon S^{**}(\Phi)|y|_F^{-|R_s|-\epsilon}.$$

We recall the definition of Schwartz norms on $\mathbb{R}^d$ for positive integers $d$.

Definition 5.5. For $l \in [1, \infty], \vec{p}, \vec{m} \in \mathbb{N}^d$, we put the semi-norm $S_{l}^{\vec{p}, \vec{m}}$ on $S(\mathbb{R}^d)$ by

$$S_{l}^{\vec{p}, \vec{m}}(\Phi) = \|\partial^{\vec{m}} \Phi\|_{l}.$$

Here we have written:

- $\|\cdot\|_l$ is the $L^l$-norm on $\mathbb{R}^d$.
- For $\vec{x} = (x_i)_{1 \leq i \leq d} \in \mathbb{R}^d$, $\vec{p} = (p_i)_{1 \leq i \leq d} \in \mathbb{N}^d$, $\vec{x}^{\vec{p}} = \prod_{i=1}^{d} x_i^{p_i}$.
- For $\vec{m} = (m_i)_{1 \leq i \leq d} \in \mathbb{N}^d$, $\vec{x} = (x_i)_{1 \leq i \leq d} \in \mathbb{R}^d$,

$$\partial^{\vec{m}} = \prod_{i=1}^{d} \frac{\partial^{m_i}}{\partial x_i^{m_i}}.$$  

Remark 5.6. Since $\mathbb{C} \simeq \mathbb{R}^2$, we put the semi-norms for $S(\mathbb{R}^2)$ on $S(\mathbb{C})$.

Remark 5.7. If we do not specify the parameters of a Schwartz norm $S^*$, we mean the max of a finite collection of Schwartz norms. This applies to Lemma 5.3, 5.4 and Proposition 5.4.

We first treat Lemma 5.2.

Proposition 5.8. The topology on $S(\mathbb{R}^d)$ defined by the system of semi-norms $S_{l}^*$ does not depend on $l \in [1, \infty]$.

Proof. In the case $d = 1$, we have for $l \in [1, \infty)$ and any $\Phi \in S(\mathbb{R})$

$$\int_{\mathbb{R}} |\Phi(x)| |dx| \leq S_{\infty}^{0,0}(\Phi) \int_{-1}^{1} dx + S_{\infty}^{2,0}(\Phi) \int_{|x| > 1} |x|^{-2} dx,$$

from which we deduce by replacing $\Phi(x)$ with $x^p \Phi^{(m)}(x)$ that

$$S_{l}^{\vec{p}, \vec{m}}(\Phi) \ll_{l} S_{\infty}^{\vec{p}, \vec{m}}(\Phi) + S_{\infty}^{\vec{p}+2, m}(\Phi).$$

In the opposite direction, from Hölder inequality

$$|\Phi(y) - \Phi(x)| = \left| \int_{x}^{y} \Phi'(t) dt \right| \leq \|\Phi'\|_{l} \cdot \left( \int_{x}^{y} |dt| \right)^{\frac{1}{l-1}}$$

and $(a + b)^l \leq 2^{l-1}(a^l + b^l)$ we deduce

$$|\Phi(y)|^l \leq 2^{l-1} (|\Phi(x)|^l + \|\Phi'\|_{l}^l \cdot |y - x|^{l-1}).$$
Integrating both sides against \( \min(1, |x - y|^{-l-1}) dx \) gives
\[
(2 + \frac{2}{l})|\Phi(y)|^l \leq 2^{l-1} \left( \|\Phi\|_l + \|\Phi'\|_l \cdot \int_{\mathbb{R}} \min(|x|^{l-1}, |x|^{-2}) dx \right).
\]
Hence we get (a Sobolev inequality) and conclude the case \( d = 1 \) by
\[
\|\Phi\|_\infty \ll_l \|\Phi\|_l + \|\Phi'\|_l.
\]
For general \( d \), one deduces easily by induction
\[
S_{l,d}^\kappa,\overline{\kappa} (\Phi)^l \ll_{l,d} \sum_{\ell \in \{0,2\}^d} S_{\infty}^\ell \kappa,\overline{\kappa} (\Phi)^l,
\]
\[
\|\Phi\|_{\infty}^l \ll_{l,d} \sum_{\ell \in \{0,1\}^d} S_{l}^\kappa,\sigma (\Phi)^l.
\]
\[\square\]

**Proof.** (of Lemma 5.2) By the above proposition, the problem is reduced to the uniform continuity of
\[
\tilde{\mathcal{F}}_2(\cdot) \circ R(\kappa) : \mathcal{S}(\mathbb{F}^2) \rightarrow \mathcal{S}(\mathbb{F}^2)
\]
with respect to \( \kappa \in K \). The continuity of \( \tilde{\mathcal{F}}_2(\cdot) \) follows by considering the \( S_2^\kappa \) semi-norms. The uniform continuity of \( R(\kappa) \) follows by considering the \( S_\infty^\kappa \) semi-norms. \[\square\]

We then turn to Lemma 5.3. Actually, we are going to reduce to the situation of Mellin transform on \( \mathbb{R}_+ \), which we shall study at the first place. For any \( c \in \mathbb{R} \), define
\[
B_c(\mathbb{R}_+) = \left\{ f : C^\infty(\mathbb{R}_+) : \sup_{y > 0} |f(k)(y)y^{\sigma+k}| < \infty, \forall \sigma > c \right\}.
\]
\[
H_c(\mathbb{C}) = \left\{ M \text{ holomorphic in } \mathbb{R}s > c : \sup_{\Re s = \sigma} |s(s+1) \cdots (s+k-1) M(s)| < \infty, \forall \sigma > c \right\}.
\]

**Definition 5.9.** For any fixed \( l \in [0, \infty] \), we put a system of semi-norms \( B_{l}^{k,\sigma} \) with \( k \in \mathbb{N}, \sigma \in (c, \infty) \) on \( B_c(\mathbb{R}_+) \) by
\[
B_{l}^{k,\sigma}(f) = \left( \int_0^\infty |f(k)(y)y^{\sigma+k}|^l \frac{dy}{y} \right)^{\frac{1}{l}},\ l \neq \infty;\ B_{\infty}^{k,\sigma}(f) = \sup_{y > 0} |f(k)(y)y^{\sigma+k}|.
\]

**Definition 5.10.** For any fixed \( l \in [0, \infty] \), we put a system of semi-norms \( H_{l}^{k,\sigma} \) with \( k \in \mathbb{N}, \sigma \in (c, \infty) \) on \( H_c(\mathbb{C}) \) by
\[
H_{l}^{k,\sigma}(M) = \left( \int_{\Re s = \sigma} |s(s+1) \cdots (s+k-1) M(s)|^l \frac{ds}{2\pi i} \right)^{\frac{1}{l}},\ l \neq \infty;\ H_{\infty}^{k,\sigma}(M) = \sup_{\Re s = \sigma} |s(s+1) \cdots (s+k-1) M(s)|.
\]

**Proposition 5.11.** The topology on \( B_c(\mathbb{R}_+) \) defined by \( B_{l}^{k,\sigma} \) does not depend on \( l \). More precisely, for any \( f \in B_c(\mathbb{R}_+) \) we have for \( 1 \leq l < \infty \) and \( \epsilon > 0 \) small with \( \sigma - \epsilon > c \)
\[
B_{l}^{k,\sigma}(f) \ll_{\epsilon,l} B_{\infty}^{k,\sigma+\epsilon}(f) + B_{\infty}^{k,\sigma-\epsilon}(f);
\]
\[
B_{\infty}^{k,\sigma}(f) \ll_{\epsilon,l,\sigma+k} B_{l}^{k,\sigma}(f) + B_{l}^{k+1,\sigma}(f).
\]

**Proof.** The first inequality follows from
\[
\int_0^\infty |f(k)(y)y^{\sigma+k}|^l \frac{dy}{y} \leq B_{\infty}^{k,\sigma+\epsilon}(f)^l \int_1^\infty y^{-\epsilon} \frac{dy}{y} + B_{\infty}^{k,\sigma-\epsilon}(f)^l \int_0^1 y^\epsilon \frac{dy}{y}.
\]
For the second inequality, we first note that for any \( x, y > 0 \)
\[
f(k)(y)y^{\sigma+k} - f(k)(x)x^{\sigma+k} = \int_y^x f(k+1)(u)u^{\sigma+k+1} \frac{du}{u} + (\sigma + k) \int_y^x f(k)(u)u^{\sigma+k} \frac{du}{u}.
\]
We can bound the integrals using Hölder inequality as

\[ \left| \int_y^x f^{(k)}(u)u^{\sigma+k} \frac{du}{u} \right| \leq B_{l,k}^{k,\sigma}(f) \cdot |\log(y/x)|^{l-1}, \]

from which we deduce

\[ |f^{(k)}(y)^{\sigma+k}| \leq |f^{(k)}(x)x^{\sigma+k}| + \left[ B_{l,k}^{k+1,\sigma}(f) + |\sigma+k| \cdot B_{l,k}^{k,\sigma}(f) \right] \cdot |\log(y/x)|^{l-1}. \]

Raising to the power \( l \geq 1 \) and use \((a+b)^l \leq 2^{l-1}(a^l + b^l)\) gives

\[ |f^{(k)}(y)^{\sigma+k}| \leq 2^{l-1} \left\{ |f^{(k)}(x)x^{\sigma+k}| + \left[ B_{l,k}^{k+1,\sigma}(f) + |\sigma+k| \cdot B_{l,k}^{k,\sigma}(f) \right]^l \cdot |\log(y/x)|^{l-1} \right\}. \]

Integrating both sides against \( \min((x/y)^{\ell}, (x/y)^{-\ell})dx/x \leq dx/x \) gives

\[ \frac{2}{cl} \cdot |f^{(k)}(y)^{\sigma+k}| \leq 2^{l-1} \left\{ B_{l,k}^{k,\sigma}(f)^l + \left[ B_{l,k}^{k+1,\sigma}(f) + |\sigma+k| \cdot B_{l,k}^{k,\sigma}(f) \right]^l \cdot \int_0^\infty \min(x^\epsilon, x^{-\epsilon})|\log x|^{l-1} dx \right\}. \]

We conclude since \( \int_0^\infty \min(x^\epsilon, x^{-\epsilon})|\log x|^{l-1} dx \leq \infty \). \( \square \)

**Proposition 5.12.** The topology on \( H_c(\mathbb{C}) \) defined by \( H_{l,k}^{k,\sigma} \) does not depend on \( l \). More precisely, for any \( M \in H_c(\mathbb{C}) \) we have for \( 1 \leq l < \infty \) and \( \epsilon > 0 \) small with \( \sigma - \epsilon > c \)

\[ H_{l,k}^{k,\sigma}(M) \ll_{k,l} H_{l,k}^{k,\sigma}(M) + H_{\infty}^{k+2,\sigma}(M); \]

\[ H_{\infty}^{k,\sigma}(M) \ll_{\epsilon,l,k} H_{l,k}^{k,\sigma}(M) + H_{l,k}^{k+1,\sigma+\epsilon}(M) + H_{l,k}^{k,\sigma-\epsilon}(M) + H_{l,k}^{k+1,\sigma-\epsilon}(M). \]

**Proof.** The first inequality follows from

\[
\int_{\mathbb{R} = \sigma} |s(s+1) \cdots (s+k-1)M(s)|^l \frac{ds}{2\pi i} \leq H_{\infty}^{k,\sigma}(M)^l \int_{\mathbb{R} = \sigma} \frac{ds}{2\pi i} \left. \right|_{\Im s \leq 1} + H_{\infty}^{k+2,\sigma}(M)^l \int_{\mathbb{R} = \sigma} \frac{ds}{2\pi i} \left. \right|_{\Im s > 1} \frac{1}{(s+k)(s+k+1)^l} \]

For the second inequality, we first note that for any \( s_0 \) with \( \Re s_0 = \sigma \)

\[ s_0(s_0+1) \cdots (s_0+k-1)M(s_0) = \int_{\mathbb{R} = \sigma + \epsilon} \frac{s(s+1) \cdots (s+k-1)M(s)}{s-s_0} \frac{ds}{2\pi i} - \int_{\mathbb{R} = \sigma - \epsilon} \frac{s(s+1) \cdots (s+k-1)M(s)}{s-s_0} \frac{ds}{2\pi i} \]

To bound the integrals, we apply Hölder inequality to get

\[
\int_{\mathbb{R} = \sigma + \epsilon} \frac{|s(s+1) \cdots (s+k-1)M(s)|}{s-s_0} \frac{ds}{2\pi i} \leq H_{\infty}^{k+1,\sigma+\epsilon}(M)^l \frac{1}{\epsilon} \left. \right|_{\Im s \geq 1} + H_{\infty}^{k,\sigma+\epsilon}(M)^l \frac{1}{\epsilon} \left. \right|_{\Im s \leq 1} \frac{ds}{2\pi i} \]

and conclude by the similar bound on \( \Re s = \sigma - \epsilon \). \( \square \)

**Proposition 5.13.** The two maps

\[ B_c(\mathbb{R}^+) \to H_c(\mathbb{C}), f \mapsto \mathfrak{F}(f)(s) := \int_0^\infty f(y)y^{s-1} \frac{dy}{y}, \text{ for } \Re s > c; \]

\[ H_c(\mathbb{C}) \to B_c(\mathbb{R}^+), M \mapsto f_M(y) := \int_{\Re s = \sigma} M(s)y^{s-1} \frac{ds}{2\pi i}, \forall \sigma > c \]

are continuous with respect to the above topologies defined by semi-norms.
Proof. By integration by parts we get
\[
\Im(f(s)) = \frac{(-1)^k}{s(s + 1) \cdots (s + k - 1)} \int_0^\infty f^{(k)}(y) y^{s + k} dy,
\]
from which it follows readily that \(H^k(f) \leq B_1^k(f)\). Similarly we pass the derivatives under the integral to get
\[
f_M^{(k)}(y) = (-1)^k \int_{y = \sigma} s(s + 1) \cdots (s + k - 1) M(s) y^{s - k} ds.
\]
from which it follows readily that \(B^k_\infty(f) \leq H_1^k(M)\). □

**Definition 5.14.** We write the multiplicative group \(\mathbb{F}^1 = \{ x \in \mathbb{F} : |x|_\mathbb{F} = 1 \} \). For any function \( f \) on \( \mathbb{F}^1 \) and any character \( \xi \in \hat{\mathbb{F}}^1 \) we define a function on \( \mathbb{R}_+ \)
\[
f_\xi(t) = f(t; \xi) = \int_{\mathbb{F}^1} f(tu) \xi(u) du, t > 0
\]
where \( du \) is the probability Haar measure on \( \mathbb{F}^1 \). Concretely:

1. If \( \mathbb{F} = \mathbb{R} \) then \( \mathbb{F}^1 = \{ \pm 1 \} \), \( \hat{\mathbb{F}}^1 = \{ \xi_+, \xi_- \} \) with \( \xi_+ \equiv 1 \) and \( \xi_-(-1) = -1 \). We then define
\[
f(t; +) = f_+(t) = \frac{1}{2} (f(t) + f(-t)),
\]
\[
f(t; -) = f_-(t) = \frac{1}{2} (f(t) - f(-t)).
\]

2. If \( \mathbb{F} = \mathbb{C} \) then \( \mathbb{F}^1 = \{ e^{i\theta} : \theta \in \mathbb{R}/2\pi \mathbb{Z} \}, \hat{\mathbb{F}}^1 = \{ \xi_n : n \in \mathbb{Z} \} \) with \( \xi_n(e^{i\theta}) = e^{in\theta} \). We then define
\[
f(t; n) = f_n(t) = \int_{\mathbb{R}/2\pi \mathbb{Z}} f(te^{i\theta}) e^{in\theta} d\theta.
\]

**Proposition 5.15.** \( f_\xi \in B_0(\mathbb{R}_+) \) for any \( f \in \mathcal{S}(\mathbb{F}) \) and \( \xi \in \hat{\mathbb{F}}^1 \). The map
\[
\mathcal{S}(\mathbb{F}) \to B_0(\mathbb{R}_+), f \mapsto f_\xi
\]
is continuous. Moreover, in the case \( \mathbb{F} = \mathbb{C} \), for any \( k, l \in \mathbb{N}, \sigma > 0 \) there is a finite collection of norms
\( S^*_\infty \) independent of \( n (n \neq 0 \) if \( l \neq 0 \) \) such that
\[
B^k_\infty(f_n) \leq k_\sigma |n|^{-l} S^*_\infty(f).
\]
Proof. In the case \( \mathbb{F} = \mathbb{R} \), we have
\[
k^k \frac{dt}{dt} f(t; +) = \frac{1}{2} \left( t^k f^{(k)}(t) + (-1)^k t^k f^{(k)}(-t) \right),
\]
\[
k^k \frac{dt}{dt} f(t; -) = \frac{1}{2} \left( t^k f^{(k)}(t) + (-1)^k t^k f^{(k)}(-t) \right),
\]
from which it is easy to see
\[
B^k_\infty(f_{\pm}) \leq S^{[k+\sigma]k}_\infty(f) + S^{[k+\sigma]k}_\infty(f), \forall k \in \mathbb{N}, \sigma > 0.
\]
In the case \( \mathbb{F} = \mathbb{C} \), with the Cartesian & Polar coordinates
\[
(x, y) = (t \cos \theta, t \sin \theta), (z, \bar{z}) = (x + iy, x - iy)
\]
we have
\[
\frac{\partial}{\partial \theta} = i \left( z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right); \frac{\partial}{\partial t} = z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}}.
\]
By induction on \( k \in \mathbb{N} \), it is easy to see
\[
k^k \frac{dt}{dt} = P_k(z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}})
\]
for some polynomial $P_k \in \mathbb{Z}[X]$ and any $k \in \mathbb{N}$. It follows that
\[
t^k f^{(k)}_n(t) = \int_{\mathbb{R}/2\pi\mathbb{Z}} (P_k(z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}}) f)(t e^{i\theta}) e^{i n \theta} \frac{d\theta}{2\pi} = \frac{(-1)^l}{n!} \int_{\mathbb{R}/2\pi\mathbb{Z}} \left( z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right)^l P_k(z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}}) f)(t e^{i\theta}) e^{i n \theta} \frac{d\theta}{2\pi}.
\]
Hence, we deduce that
\[
B_{\infty}^{k,\sigma}(f_n) \leq |n|^{-l} \left\{ \left( z \bar{z} \right)^{\frac{1}{2}} \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right)^l P_k(z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}}) f \right\}_{\infty} + \left\{ (z \bar{z})^{\frac{1}{2}} \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right)^l P_k(z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}}) f \right\}_{\infty}.
\]
The right hand side is obviously bounded by some Schwartz norm of $f$.

Proof. (of Lemma 5.3) We only treat the case $F = \mathbb{C}$, the real case being similar and simpler. Writing
\[
f(y) = \int_{\mathbb{C}^\times} \Phi(t, y/t) e^{-i \xi^2(t)} |t|_{\mathbb{C}}^{2s} d^\times t,
\]
we can take its Fourier expansion on $\mathbb{C}^\times$
\[
f(te^{i\theta}) = \sum_{n \in \mathbb{Z}} f_n(t) e^{-i n \theta}, t \in \mathbb{R}_+.
\]
Extending each $\xi_n \in \hat{\mathbb{C}^\times}$ to $\mathbb{C}^\times$ by triviality on $\mathbb{R}_+$ we have the Mellin transform
\[
\mathfrak{M}(f_n)(s_1) = \int_{\mathbb{C}^\times} f(y) \xi_n(y) |y|_{\mathbb{C}}^{2s} d^\times y = \int_{\mathbb{C}^\times \times \mathbb{C}^\times} \Phi(t, y) e^{-i \xi^2(y)} |t|_{\mathbb{C}}^{2s} d^\times y = \int_{\mathbb{R}_+ \times \mathbb{R}_+} \Phi_d e^{-i \xi \xi_n(t_1, t_2)} |t_1|_{\mathbb{C}}^{2s} |t_2|_{\mathbb{C}}^{2s} d^\times t_1 d^\times t_2.
\]
Considering the $H_\infty$ semi-norms it is easy to see $\mathfrak{M}(f_n) \in H_{\text{max}(0,-4R\delta)}(\mathbb{C})$. We can also bound
\[
H_{\infty}^{k,\sigma}(\mathfrak{M}(f_n)) \ll_{k,\sigma} \min(1, |n|^{-2}) S_1^1(\Phi), \forall \sigma > \max(0,-4R\delta).
\]
As $R\delta$ lies in a compact interval, the orders of $S_1^1$ can be made uniform (but depends on $\sigma$). Hence $f_n \in B_{\text{max}(0,-4R\delta)}(\mathbb{R}_+)$ and for any $\sigma > \max(0,-4R\delta)$ we get
\[
|t^\sigma f(te^{i\theta})| \leq \sum_n B_{\infty}^{0,\sigma}(f_n) \ll S_1^1(\Phi) \sum_n \min(1, |n|^{-2}).
\]
We conclude by noting $t^\sigma = |te^{i\theta}|_{\mathbb{C}}^{\sigma/2}$.

Obviously, Proposition 5.4 is a direct consequence of Lemma 5.2 and 5.3.

5.2.2. Non Archimedean Places. We continue to omit the subscript $v$ for simplicity of notations.

**Definition 5.16.** Let $d \geq 1$ be an integer. For any $\Phi \in S(\mathbb{F}_v^d)$ we define its support index $D(\Phi) \in \mathbb{Z}$, additive invariance index $\delta(\Phi) \in \mathbb{Z}$ and multiplicative invariance index $m(\Phi) \in \mathbb{N}$ as follows.

1. $D(\Phi)$ is the largest integer $D$ such that
   \[
   \Phi(\vec{x}) \neq 0 \Rightarrow \vec{x} \in p^D \times \cdots \times p^D.
   \]
2. $\delta(\Phi)$ is the smallest integer $\delta$ such that
   \[
   \Phi(\vec{x} + i) = \Phi(\vec{x}), \forall \vec{x} \in \mathbb{F}_v^d, i \in p^\delta \times \cdots \times p^\delta.
   \]
For any \( \kappa \in \text{GL}_d(\mathfrak{o}) \) we have \( D(R(\kappa), \Phi) = D(\Phi, \delta(\kappa)) = \Phi(\kappa) - \Phi \), and \( m(R(\kappa), \Phi) = m(\Phi) \).

**Proposition 5.17.** The three indices satisfy the following relations.

1. \( m(\Phi) \leq \delta(\Phi) - D(\Phi) \).
2. For any \( \kappa \in \text{GL}_d(\mathfrak{o}) \), we have \( D(R(\kappa), \Phi) = D(\Phi) \), \( \delta(R(\kappa), \Phi) = \delta(\Phi) \) and \( m(R(\kappa), \Phi) = m(\Phi) \).

3. Let \( \mathfrak{F}(\cdot) \) denote the Fourier transform

\[
\mathfrak{F}(\Phi)(\vec{x}) = \int_{\mathbb{F}^d} \Phi(\vec{y}) \psi(-\vec{y} \cdot \vec{x}) d\vec{y}.
\]

Then we have

\[
D(\Phi) + \delta(\mathfrak{F}(\Phi)) = \delta(\Phi) + D(\mathfrak{F}(\Phi)) = -\epsilon(\psi).
\]

More generally, let \( I = \{i_1, \ldots, i_j\} \subset \{1, \ldots, d\} \). We define the partial Fourier transform \( \mathfrak{F}_I(\cdot) = \mathfrak{F}_{i_1}(\mathfrak{F}_{i_2}(\cdots \mathfrak{F}_{i_j}(\cdot))) \). Then we have

\[
\delta(\mathfrak{F}_I(\Phi))) \leq \max(\delta(\Phi), -\epsilon(\psi) - D(\Phi));
\]

\[
D(\mathfrak{F}_I(\Phi)) \geq \min(D(\Phi), -\epsilon(\psi) - \delta(\Phi)).
\]

**Proof.** (0) and (1) are obvious from definition. (3) follows easily from (2). We prove (2) as follows. From

\[
\mathfrak{F}(\Phi)(\vec{x} + \vec{t}) = \int_{\mathbb{F}^d} \Phi(\vec{y}) \psi(-\vec{y} \cdot \vec{x}) \psi(-\vec{y} \cdot \vec{t}) d\vec{y},
\]

we see that for \( \vec{t} \in \mathbb{F}^{-\epsilon(\psi)} \) hence \( \psi(-\vec{y} \cdot \vec{t}) = 1 \) and \( \mathfrak{F}(\Phi)(\vec{x} + \vec{t}) = \mathfrak{F}(\Phi)(\vec{x}) \). Thus

\[
\delta(\mathfrak{F}(\Phi)) \leq -\epsilon(\psi) - D(\Phi).
\]

On the other hand, if \( \vec{x} \notin \mathbb{F}^{-\epsilon(\psi)} \) then at least one component, say \( x_1 \notin \mathbb{F}^{-\epsilon(\psi)} \), i.e., \( v(x_1) < -\epsilon(\psi) - \delta(\Phi) \). As \( t_1 \) runs under the condition \( v(t_1) = -\epsilon(\psi) - 1 - v(x_1) + \delta(\Phi) \), \( x_1 t_1 \) runs under the condition \( v(x_1 t_1) = -\epsilon(\psi) - 1 \). Hence at least for one \( t_1 \neq 1 \). Writing \( \vec{t} = (t_1, 0, \ldots, 0) \), we get

\[
\mathfrak{F}(\Phi)(\vec{x}) = \int_{\mathbb{F}^d} \Phi(\vec{y} + \vec{t}) \psi(-\vec{y} \cdot \vec{x}) d\vec{y} = \int_{\mathbb{F}^d} \Phi(\vec{y}) \psi(-\vec{y} \cdot \vec{x}) d\vec{y} \cdot \psi(t_1 \vec{x}) = \psi(t_1 x_1) \mathfrak{F}(\Phi)(\vec{x}).
\]

Hence \( \mathfrak{F}(\Phi)(\vec{x}) = 0 \) and

\[
D(\mathfrak{F}(\Phi)) \geq -\epsilon(\psi) - \delta(\Phi).
\]

Replacing \( \Phi \) with \( \mathfrak{F}(\Phi) \) in the above argument gives the inequalities in the opposite direction.

**Definition 5.18.** For \( l \in [1, \infty] \), \( \vec{d} \in \mathbb{R}_{\geq 0}^d \), we put the semi-norm \( S_l^d \) on \( \mathcal{S}^d(\mathbb{F}^d) \) by

\[
S_l^d(\Phi) = \| \vec{x}^d | \mathbb{F} \cdot \Phi \|_l.
\]

Here we have written:

- \( \| \cdot \|_l \) is the \( L^l \)-norm on \( \mathbb{F}^d \).
- For \( \vec{x} = (x_i)_{1 \leq i \leq d} \in \mathbb{F}^d \), \( \vec{d} = (d_i)_{1 \leq i \leq d} \in \mathbb{R}_{\geq 0}^d \), \( |\vec{x}^d|_{\mathbb{F}} = \Pi_{i=1}^d |x_i|^{d_i} \).

We shall also write \( |\vec{d}| = \sum_i d_i \).

**Proposition 5.19.** We have the following relations of norms for any \( \Phi \in \mathcal{S}(\mathbb{F}^d) \).

1. \( \| \Phi \|_\infty \leq q \frac{d(\Phi)}{\vec{d}} \| \Phi \|_l \) and \( \| \Phi \|_l \leq q \frac{d(\Phi)}{\vec{d}} \| \Phi \|_\infty \).
2. \( S_l^d(\Phi) \leq q^{-|\vec{d}|} D(\Phi) \| \Phi \|_l \).
Proof. It suffices to prove the case \(d = 1\). Let \(x_0 \in F\) be such that \(|\Phi(x_0)| = \|\Phi\|_\infty\), then

\[
\|\Phi\|_\infty \approx \text{Vol}(p^\delta(\Phi))^{-1} \int_{x_0 + p^\delta(\Phi)} |\Phi(x)|^i dx \leq q^{\delta(\Phi)} \|\Phi\|_i
\]

and we get the first inequality. The second follows from

\[
\|\Phi\|_i \leq \|\Phi\|_\infty \int_{\supp(\Phi)} dx \leq \|\Phi\|_\infty \cdot \text{Vol}(p^D(\Phi)) = q^{-D(\Phi)} \|\Phi\|_i.
\]

For the last, we deduce it from

\[
S_i^\delta(\Phi) = \int_{p^D(\Phi)} |x\|^i |\Phi(x)|^i dx \leq \sup_{x \in p^D(\Phi)} |x\|^i \cdot \|\Phi\|_i = q^{-iD(\Phi)} \|\Phi\|_i.
\]

\[\blacksquare\]

**Definition 5.20.** For any \(c \in \mathbb{R}\), we define \(B_c(\mathbb{Z}; \varpi)\) to be the space of functions \(f : \varpi^\mathbb{Z} \to \mathbb{C}\) satisfying

1. \(\lim_{n \to -\infty} f(\varpi^n)q^{-n\sigma} = 0\) for any \(\sigma > 0\).
2. \(\lim_{n \to +\infty} f(\varpi^n)q^{-n\sigma} = 0\) for any \(\sigma > c\).

The subspace \(B_c(\mathbb{Z}; \varpi) \subset B_c(\mathbb{Z}; \varpi)\) is defined by replacing (1) with

(1') \(f(\varpi^n) = 0\) for \(n < -1\).

**Definition 5.21.** For any \(c \in \mathbb{R}\) we define \(H_c(\mathbb{C}; q)\) to be the space of meromorphic functions \(M : \mathbb{C} \to \mathbb{C}\) satisfying

1. \(\text{M}(s + i\frac{2\pi}{\log q}) = \text{M}(s)\) for all \(s \in \mathbb{C}\).
2. \(\text{M}(s)\) is holomorphic for \(\Re s > c\).

**Definition 5.22.** For any \(l \in [1, \infty]\) we put a system of semi-norms \(B_c^l\) for \(\sigma > c\) on \(B_c(\mathbb{Z}; \varpi)\) by

\[
B_c^l(f) = \left( \sum_{n \in \mathbb{Z}} q^{-n\sigma} |f(\varpi^n)|^l \right)^{\frac{1}{l}}, l \neq \infty; B_c^\infty(f) = \sup_{n \in \mathbb{Z}} q^{-n\sigma} |f(\varpi^n)|.
\]

**Definition 5.23.** For any \(l \in [1, \infty]\) we put a system of semi-norms \(H_c^l\) for \(\sigma > c\) on \(H_c(\mathbb{C}; q)\) by

\[
H_c^l(M) = \left( \int_0^{2\pi} |M(\sigma + i\tau)|^l \log q d\tau \right)^{\frac{1}{l}}, l \neq \infty; H_c^\infty(M) = \max_{\Re s = \sigma} |M(s)|.
\]

**Proposition 5.24.** The topology on \(B_c(\mathbb{Z}; \varpi)\) defined by \(B_c^l\) does not depend on \(l\). More precisely, for any \(f \in B_c(\mathbb{Z}; \varpi)\) we have for \(1 \leq l < \infty\) and \(\epsilon > 0\) small with \(\sigma - \epsilon > c\)

\[
B_c^l(f) \ll_{\epsilon,l} B_c^{\sigma-\epsilon}(f) + B_c^{\sigma+\epsilon}(f);
B_c^\infty(f) \leq B_c^\infty(f).
\]

**Proof.** The first follows from

\[
B_c^l(f)^l \leq B_c^{\sigma-\epsilon}(f) \sum_{n \geq 0} q^{-n\epsilon} + B_c^{\sigma+\epsilon}(f) \sum_{n < 0} q^{n\epsilon}.
\]

The second is obvious by positivity. \[\blacksquare\]

**Proposition 5.25.** The topology on \(H_c(\mathbb{C}; q)\) defined by \(H_c^l\) does not depend on \(l\). More precisely, for any \(M \in H_c(\mathbb{C}; q)\) we have for \(1 \leq l < \infty\) and \(\epsilon > 0\) small with \(\sigma - \epsilon > c\)

\[
H_c^l(M) \leq H_c^\infty(M);
H_c^\infty(M) \ll_{\epsilon} H_c^{\sigma-\epsilon}(M) + H_c^{\sigma+\epsilon}(M).
\]
Proof. The first is obvious. For the second, let \( s_0 \) be any complex number with \( \Re s_0 = \sigma \) and \( 0 < \Im s_0 < 2\pi/\log q \). Selecting the contour joining \( \sigma + \epsilon, \sigma + \epsilon + i2\pi/\log q, \sigma - \epsilon + i2\pi/\log q \) and \( \sigma - \epsilon \), we see

\[
\log q \cdot M(s_0) = \int_{0}^{2\pi} \frac{M(\sigma + \epsilon + i\tau) \log q d\tau}{\sigma + \epsilon + i\tau - s_0} - \int_{0}^{2\pi} \frac{M(\sigma - \epsilon - i\tau) \log q d\tau}{\sigma - \epsilon - i\tau - s_0}.
\]

Using Hölder inequality we deduce

\[
\log q \cdot |M(s_0)| \leq \frac{1}{\epsilon} H^\sigma(\epsilon)(M) \cdot \left( \int_{0}^{2\pi} \frac{\log q d\tau}{\sigma + \epsilon + i\tau} \right)^{\frac{1}{\epsilon}} + \frac{1}{\epsilon} H^{\sigma - \epsilon}(\epsilon)(M) \cdot \left( \int_{0}^{2\pi} \frac{\log q d\tau}{\sigma - \epsilon - i\tau} \right)^{\frac{1}{\epsilon}}.
\]

We conclude by taking sup with respect to \( \Re s_0 = \sigma \). \( \square \)

**Proposition 5.26.** The two maps

\[
B_{c}(\mathbb{Z}; \omega) \to H_{c}(\mathbb{C}; q), f \mapsto \mathcal{M}(f)(s) = \sum_{n \in \mathbb{Z}} f(\omega^{n})q^{-ns}, \text{ for } \Re s > c;
\]

\[
H_{c}(\mathbb{C}; q) \to B_{c}(\mathbb{Z}; \omega), M \mapsto f_{M}(\omega^{n}) = \int_{0}^{2\pi} M(\sigma + i\tau)q^{n(\sigma + i\tau)} \frac{\log q d\tau}{2\pi}, \forall \sigma > c
\]

are continuous with respect to the above topologies defined by semi-norms.

Proof. The continuity follows from

\[
H^\omega(\mathcal{M}(f)) \leq \sum_{n \in \mathbb{Z}} |f(\omega^{n})|q^{-ns} = B^\sigma(f);
\]

\[
B_{\infty}(f_{M}) \leq \sup_{n \in \mathbb{Z}} \int_{0}^{2\pi} |M(\sigma + i\tau)q^{ns}| \frac{\log q d\tau}{2\pi} = H^\epsilon(\sigma,M).
\]

\( \square \)

Note that the abstract part of Definition 5.14 still makes sense in the current case, i.e., for any function \( f : F \to \mathbb{C} \) and \( \xi \in \widehat{F} \) we can define

\[
f_{\xi}(\omega^{n}) = \int_{F} f(\omega^{n}u)\xi(\omega)d\omega, n \in \mathbb{Z}.
\]

**Proposition 5.27.** For any \( f \in S(F) \), \( f_{\xi} \neq 0 \) only for \( \xi \) satisfying \( \xi(\xi) \leq m(f) \), hence for only finitely many \( \xi \). We have \( f_{\xi} \in B^{\sigma}_{0}(\mathbb{Z}; \omega) \) and

\[
B_{\infty}(f_{\xi}) \leq S^\sigma(f), \forall \sigma > 0.
\]

Proof. Obvious. \( \square \)

**Lemma 5.28.** For any \( \Phi \in S(F^{2}) \), \( s \in \mathbb{C} \), \( \sigma > \max(0, -2\Re s) \) and \( \epsilon > 0 \) with \( \sigma - \epsilon, \sigma + 2\Re s - \epsilon > 0 \), there is \( N = N(\epsilon, \sigma, 2\Re s) > 0 \) such that with implied constant depending only on \( \epsilon \)

\[
\left| \int_{F^{x}} \Phi(t, \frac{y}{t^{s}})t^{-\epsilon} \xi_{\omega}(t)|t|^{2s} d^{s} t \right| \ll_{\epsilon} q^{-N(\Phi) + m(\Phi)}||\Phi||_{\infty} |y|^{-\sigma} 1_{y \in F^{2}(\Phi)}.
\]

Proof. Writing

\[
f(y) = \int_{F^{x}} \Phi(t, \frac{y}{t^{s}})t^{-\epsilon} \xi_{\omega}(t)|t|^{2s} d^{s} t,
\]

we have for any \( \xi_{1} \in \widehat{F} \) and \( s_{1} \in \mathbb{C} \) with \( \Re s_{1} > \max(0, -2\Re s) \)

\[
\mathcal{M}(f_{\xi_{1}})(s_{1}) = \int_{F^{x} \times F^{x}} \Phi(t, y) t^{-\epsilon} \xi_{\omega}(t)|t|^{2s} s_{1}(y)|y|^{2s} d^{s} t d^{s} y
\]

\[
= \mathcal{M}(\Phi_{\omega}^{-1} \xi_{\omega}(\xi_{1}))(s_{1} + 2s + i\mu(\omega^{-1} \xi_{2}^{2}s_{1})),
\]

where...
where the second Mellin transform is the natural two dimensional one. By Proposition 5.24 \( \Phi_{\omega^{-1}\xi^2\xi_1, \xi_1} \neq 0 \) only if \( c(\omega^{-1}\xi^2\xi_1), c(\xi_1) \leq m(\Phi) \). In particular, the number of such \( \xi_1 \) is bounded by \( q^{m(\Phi)} \). From the Mellin inversion for \( \sigma > \max(0,-2\Re s) \)

\[
f(\varpi^n u) = \sum_{\xi_1} f_{\xi_1}(\varpi^n)\xi_1(u)^{-1} = \sum_{\xi_1} \xi_1(u)^{-1} \int_0^{\infty} \mathcal{M}(f_{\xi_1})(\sigma + i\tau) q^{n(\sigma + i\tau)} \frac{\log q dx}{2\pi}
\]

we can successively apply Propositions 5.23 \& 5.24 \& 5.27 and 5.19 to get

\[
|\eta|^F |f(y)| \leq \sum_{\xi_1} H_1^\sigma(\mathcal{M}(f_{\xi_1})) \leq \sum_{\xi_1} H_\infty(\mathcal{M}(f_{\xi_1})) \\
\leq \sum_{\xi_1} H_{\sigma+2\Re s, \sigma}(\mathcal{M}(\Phi_{\omega^{-1}\xi^2\xi_1, \xi_1})) \leq \sum_{\xi_1} B_{\sigma+2\Re s, \sigma}(\Phi_{\omega^{-1}\xi^2\xi_1, \xi_1}) \\
\leq (S_{\infty}^{\sigma+2\Re s, \sigma} + S_{\infty}^{\sigma+2\Re s, \sigma} + S_{\infty}^{\sigma+2\Re s, \sigma} + S_{\infty}^{\sigma+2\Re s, \sigma}) (\Phi) \sum_{\xi_1} 1 \\
\leq q^{-N(D(\Phi) + m(\Phi))}\|\Phi\|_{\infty}.
\]

Finally, it is obvious that \( f(y) \neq 0 \) implies the existence of some \( t \in F^x \) such that \( (t, y/t) \) lies in the support of \( \Phi \) hence in \( p^{D(\Phi)} \times p^{D(\Phi)} \), thus \( y \in p^{2D(\Phi)} \).

**Proposition 5.29.** For any \( \Phi \in S(F^2) \), let

\[
D = \min(D(\Phi), -c(\psi) - \delta(\Phi)); \delta = \max(\delta(\Phi), c(\psi) - D(\Phi)).
\]

Then for any \( \sigma > |\Re s| \) and \( \epsilon > 0 \) with \( \sigma - \epsilon > |\Re s| \), there is \( N = N(\epsilon, \sigma + \Re s, \sigma - \Re s) > 0 \) continuous in \( \epsilon, \sigma \pm \Re s \) such that

\[
|W_\Phi(s, \xi, \omega \xi^{-1}; a(y)\kappa)| \ll_q q^{-(N+1)D+2\delta}\|\Phi\|_2 \cdot |y|^{-\sigma}\delta_{1 y \in p^{2D}}.
\]

Moreover, at an unramified place with \( c(\psi) = c(\xi) = c(\omega \xi^{-1}) = 0 \) and \( \Phi = 1_{F^\times} \) we have for any \( \epsilon > 0 \)

\[
|W_\Phi(s, \xi, \omega \xi^{-1}; a(y)\kappa)| \leq 2 \epsilon^{-\frac{1}{\log q}} \left( \sup_{x > 0} \frac{x}{e^x} \right) |y|^{-\epsilon} \delta_{1 y \in F^\times + 1 y \in \mathbb{F}}.
\]

**Proof.** The first part is a direct consequence of the previous lemma and the following inequalities deduced from Propositions 5.17 \& 5.19

\[
D(\mathfrak{g}_2(R(\kappa), \Phi)) \geq \min(D(\Phi), -c(\psi) - \delta(\Phi)) = D;
\]

\[
m(\mathfrak{g}_2(R(\kappa), \Phi)) \leq \delta(\mathfrak{g}_2(R(\kappa), \Phi)) - D(\mathfrak{g}_2(R(\kappa), \Phi)) \leq \delta - D;
\]

\[
\|\mathfrak{g}_2(R(\kappa), \Phi)\|_{\infty} \leq q^{\delta(\mathfrak{g}_2(R(\kappa), \Phi))}\|\mathfrak{g}_2(R(\kappa), \Phi)\|_2 \leq q^{\delta}\|\Phi\|_2.
\]

The “moreover” part follows from a direct computation (or [1, Theorem 4.6.5])

\[
W_\Phi(s, \xi, \omega \xi^{-1}; a(\varpi^n)) = q^{-\frac{\alpha n + 1 - \beta n + 1}{\alpha - \beta}} 1_{n \geq 0}, \text{ with } \alpha = \xi(\varpi)q^{-s}, \beta = \omega \xi^{-1}(\varpi)q^s,
\]

which implies for \( n \geq 1 \)

\[
|W_\Phi(s, \xi, \omega \xi^{-1}; a(\varpi^n))| \leq (n + 1)|\varpi^n|^2 |\Re s| = \frac{n + 1}{\epsilon n \log q} \cdot \frac{\epsilon n \log q}{q^n} \cdot |\varpi^n|^2 |\Re s| - \epsilon
\]

is bounded as in the statement. \qed
5.2.3. Global Bound. Writing $D_v = \min(D(\Phi_v), -c(\psi_v) - \delta(\Phi_v))$ for $v < \infty$, it follows from Proposition 5.31 and 5.28 that for any $\epsilon > 0$ and $N \gg 1$

$$\left| W_\Phi(s, \xi, \omega\xi^{-1}; na(y)\kappa) \right| \ll c_{s,N,\Phi} \prod_{v|\infty} \min(|y_v|^{-|\Re s|-\epsilon},|y_v|^{-N}) \prod_{v < \infty} |y_v|^{-|\Re s|-\epsilon} \prod_{y_v \in \mathbb{P}_v^2S_v}.$$ 

The bound is uniform when $\Re s$ lies in a fixed compact interval. Together with Lemma 5.32, we obtain

**Proposition 5.30.** For any $\Phi \in \mathcal{S}(\mathbb{A}^2)$, $m \in \mathbb{N}$ and any $\sigma > |\Re s|$, we have

$$\left| \frac{\partial^m}{\partial s^m} (E(s, \xi, \omega\xi^{-1}; \Phi)(g) - EN(s, \xi, \omega\xi^{-1}; \Phi)(g)) \right| \leq \sum_{\alpha \in \mathbb{F}^m} \left| \frac{\partial^m}{\partial s^m} W_\Phi(s, \xi, \omega\xi^{-1}; a(\alpha)g) \right| \ll_{s,\Phi} \operatorname{Ht}(g)^{-\frac{s}{2} - \epsilon},$$

which is of rapid decay with respect to $\operatorname{Ht}(g)$.

5.3. Behavior of Constant Term. Consider $f \in V_\infty^\infty$ and take its Fourier expansion into $K$-isotypic types

$$f = \sum_{\vec{n}} \hat{f}_{\vec{n}} \cdot e_{\vec{n}}(\xi, \omega\xi^{-1})$$

for some $\hat{f}_{\vec{n}} \in \mathbb{C}$, where $e_{\vec{n}}(\xi, \omega\xi^{-1})$ is unitary with $K$-type parametrized by $\vec{n}$ as in [6, Section 3.5]. For $\Re s \gg 1$, we have

$$EN(s, \xi, \omega\xi^{-1}; f) = f(s, \xi, \omega\xi^{-1}) + \sum_{\vec{n}} \hat{f}_{\vec{n}} \cdot \mu(s, \xi, \omega\xi^{-1}; \vec{n})e_{\vec{n}}(-s, \omega\xi^{-1}, \xi),$$

where $\mu(s, \xi, \omega\xi^{-1}; \vec{n})$ are the explicit coefficients of the intertwining operator on the $K$-type $\vec{n}$ part as in loc.cit.

**Proposition 5.31.** The possible poles of the collection of meromorphic functions $\{\mu(s, \xi, \omega\xi^{-1}; \vec{n}) : \vec{n}\}$ are

- The pole of $\Lambda(1 - 2s, \omega\xi^{-2})$ at $s = (1 + i\mu(\omega\xi^{-2}))/2$ with order at most 1, when $\omega\xi^{-2}$ is trivial on $\Lambda^{(1)}$ and $\vec{n} = 0$. We call this pole the spherical pole.
- The (both trivial and non-trivial) zeros of $L(1 + 2s, \omega^{-1}\xi^2)$ with order at most that of the zero.

**Proof.** This can be seen either from our explicit computation of $\mu(s, \xi, \omega\xi^{-1}; \vec{n})$ in [6], or from (5.1) and (5.4) and

$$\mathcal{M}_{\Phi}(s, \xi, \omega\xi^{-1}) = f_{\Phi}(-s, \omega\xi^{-1}, \xi),$$

which is meromorphic with a simple pole at $s = (1 + i\mu(\omega\xi^{-2}))/2$ (the other pole is cancelled by that of $f_a(s, \xi, \omega\xi^{-1})$) by Tate’s theory. \hfill \Box

**Lemma 5.32.** Assume $\mu(s, \xi, \omega\xi^{-1}; \vec{n}_0)$ has a pole at $s_0$ with order $n$ (n = 0 if it is holomorphic at $s_0$) for some $\vec{n}_0$. Define

$$||\vec{n}|| = \prod_v (|n_v| + 1) \text{ for } \vec{n} = (n_v)_v.$$ 

Then as $s$ lying in any small compact neighborhood $K$ of $s_0$ where no other pole occur, we have

$$|(s - s_0)^n \mu(s, \xi, \omega\xi^{-1}; \vec{n})| \ll ||\vec{n}||^N$$

for some $N$ and the implied constant depending only on $K$, $\xi$ and $\omega\xi^{-1}$ (i.e., independent of $\vec{n}$).

**Proof.** This follows from the explicit computation of $\mu(s, \xi, \omega\xi^{-1}; \vec{n})$ in [6] together with the following obvious bound

$$\prod_{k=1}^n \left| \frac{k - s}{k + s} \right| \leq \prod_k \left( 1 + \frac{|s|}{|k + s|} \right) \leq \exp \left\{ \sum_k \frac{|s|}{|k + s|} \right\} \ll (n + 1)^N.$$ 

\hfill \Box
Proposition 5.33. Under the condition of the lemma we have

\[(s - s_0)^n E_N(s, \xi, \omega \xi^{-1}; f) = (s - s_0)^n f(s, \xi, \omega \xi^{-1}) + \sum_{\tilde{n}} \tilde{f}_\tilde{n} (s - s_0)^n \mu(s, \xi, \omega \xi^{-1}; \tilde{n}) e_{\tilde{n}}(-s, \omega \xi^{-1}, \xi).\]

Consequently, for any \(m \in \mathbb{N}, y \in \mathbb{A}^\times\) and \(\kappa \in K\) we can write

\[\frac{d^m}{ds^m} \big|_{s=s_0} (s - s_0)^n E_N(s, \xi, \omega \xi^{-1}; f)(a(y)\kappa) = \frac{m!}{(m-n)!} \left|y\right|_A^{\frac{1}{2}+s_0} \xi(y) \left(\log |y|_A\right)^{m-n} f(\kappa) + \sum_{k=0}^{m} \left|y\right|_A^{\frac{1}{2}-s_0} \omega \xi^{-1}(y) \left(\log |y|_A\right)^k f_k(\kappa),\]

where \(\left(\log |y|_A\right)^{m-n}\) is understood as 0 if \(m < n\) and where the maps

\[V_{\xi, \omega \xi^{-1}} \rightarrow V_{\omega \xi^{-1}, \xi}, f \rightarrow f_k\]

are \(K\)-maps and continuous with respect to the Sobolev norms on \(K_\infty\).

Proof. The lemma shows that \((s - s_0)^n \mu(s, \xi, \omega \xi^{-1}; \tilde{n})\) is polynomially increasing in \(\tilde{n}\). But \(\tilde{f}_\tilde{n}\) is rapidly decreasing in \(\tilde{n}\) by smoothness of \(f\). The sum over \(\tilde{n}\) is thus absolutely and uniformly convergent for \(s\) lying in \(K\), hence defines an analytic function in \(s\). By uniqueness of analytic continuation, we get the first equation. The rest follows from the polynomial increase of \((s - s_0)^n \mu(s, \xi, \omega \xi^{-1}; \tilde{n})\) and Cauchy’s integral formula for derivatives. \(\square\)

Remark 5.34. If \(s_0\) is the spherical pole, then \(n = 1\) and for \(m = 0\) we have \(f_k = 0\) unless \(k = 0\). Then

\[f_0(\kappa) = \lim_{s \rightarrow s_0} \frac{(s - s_0)A_F(2(s_0 - s))}{A_F(2 - 2(s_0 - s))} \int_K f(\kappa) d\kappa = -\frac{A_F(0)}{A_F(2)} \int_K f(\kappa) d\kappa.\]

5.4. A Convergence Lemma. Let \(r, N \in \mathbb{N}, r, N \geq 2\). It is well-known that there is an exact sequence

\[1 \rightarrow \Gamma(N) \rightarrow \text{SL}_r(\mathbb{Z}) \rightarrow \text{SL}_r(\mathbb{Z}/NZ) \rightarrow 1,\]

where \(\Gamma(N)\) is called the principal congruence subgroup of \(\text{SL}_r(\mathbb{Z})\) modulo \(N\). The pre-image of the upper resp. lower triangular subgroup of \(\text{SL}_r(\mathbb{Z}/NZ)\) in \(\text{SL}_r(\mathbb{Z})\) is denoted by \(\Gamma_0(N)\) resp. \(\Gamma_0(N)\), which includes \(\Gamma(N)\) as a normal subgroup. For \(\alpha \in \mathbb{Z}^{r-1}\) realized as a column vector, we define matrices, with \(I_k\) denoting the identity matrix of rank \(k\)

\[n_+^+(\alpha) = \begin{pmatrix} 1 & \alpha^T \\ 0 & I_{r-1} \end{pmatrix}, \quad n_-^-(\alpha) = \begin{pmatrix} 1 & \bar{\alpha}^T \\ 0 & I_{r-1} \end{pmatrix}.\]

Lemma 5.35. Any coset of \(\Gamma_0(N) \backslash \text{SL}_r(\mathbb{Z})\) resp. \(\Gamma_0(N) \backslash \text{SL}_r(\mathbb{Z})\) has a representative of the form \(N_-N_+\) resp. \(N_+N_-\), where \(N_-\) resp. \(N_+\) is lower resp. upper unipotent with off-diagonal entries lying in \([-N/2, N/2) \cap \mathbb{Z}\).

Proof. We treat the case for \(\Gamma_0(N)\). Take any \(A \in \text{SL}_r(\mathbb{Z})\). Let the first column of \(A\) be \((a_1, \cdots, a_r)^T \in \mathbb{Z}^r\). Then we have \(\text{lcd}(a_1, \cdots, a_r) = 1\), which implies the existence of \(u_i \in \mathbb{Z}\) such that \(\Sigma_{i=1}^r u_i a_i = 1\). In particular, we have \(\text{lcd}(u_1, u_2, \cdots, a_r) = 1\). For any \(k_j \in \mathbb{Z}, 2 \leq j \leq r\), the substitution

\[u'_i = u_i + \sum_{j=2}^r k_j a_j; \quad u'_j = u_j - k_j, 2 \leq j \leq r\]

still gives \(\Sigma_{i=1}^r u'_i a_i = 1\). By an iterative application of Dirichlet’s theorem on primes in arithmetic progression, we can choose \(k_j\)'s such that \(u'_i\) is a prime number as large as we want. In particular, we can make \(\text{lcd}(u'_1, N) = 1\). Since \(\text{lcd}(u'_1, \cdots, u'_r) = 1\), at least one of \(u'_j, 2 \leq j \leq r\), say \(u'_2\), is coprime with \(u'_1\). Hence \(\text{lcd}(u'_1, Nu'_2) = 1\) and we can find \(v_1, v_2 \in \mathbb{Z}\) such that \(u'_1 v_2 - Nu'_2 v_1 = 1\). The matrix

\[B = \begin{pmatrix} u'_1 & u'_2 & u'_3 & \cdots \\ Nu'_1 & v_2 & 0 & \cdots \\ 0 & 0 & I_{r-2} \end{pmatrix} \in \Gamma_0(N)\]
and gives, for some $A' = \tilde{A} - \tilde{\alpha} \tilde{\beta}_1^T \in \text{SL}_r(-1)(\mathbb{Z})$, that
\[ BA = \left( \frac{1}{\tilde{\alpha}} \tilde{\beta}_1^T \right) \Rightarrow A \in \Gamma_0(N) \left( \frac{1}{\tilde{\alpha}} A' \right) n_+^+(\tilde{\beta}_1). \]

Repeating the process on $A'$ or making an induction on $r$ we then find successively $\tilde{\beta}_k^r \in \mathbb{Z}^{r-k}$ such that for some lower unipotent $N_+$
\[ A \in \Gamma_0(N)N_1N_1^-, N_1^+ = \prod_{k=0}^{r-2} \left( \begin{array}{cc} I_k & 0 \\ 0 & n_{r-k}^+(\tilde{\beta}_k) \end{array} \right). \]

Taking $N_-$ resp. $N_+$ with off-diagonal entries lying in $[-N/2, N/2] \cap \mathbb{Z}$ such that $N_1^{\pm} \equiv N_- \pmod{N}$ resp. $N_+ \equiv N_+ \pmod{N}$, we then find $N_1^{\pm}N_1^{-1}, N_1^{+}N_1^{-1} \in \Gamma(N)$ and conclude by its normality. \hfill \Box

Let $[F : \mathbb{Q}] = r = r_1 + 2r_2$ where $r_1$ resp. $2r_2$ is the number of embeddings of $F$ into real resp. complex numbers. Recall that we have a canonical map by choosing one complex embedding $\sigma_{r_1+j}, 1 \leq j \leq r_2$ in a pair conjugate to each other by complex conjugation
\[ \sigma : F \to \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \simeq \mathbb{R}^r, \]
\[ \sigma(x) = (\sigma_1(x), \cdots, \sigma_{r_1+r_2}(x)) = (\sigma_1(x), \cdots, \sigma_{r_1}(x), \Re \sigma_{r_1+1}(x), \Im \sigma_{r_1+1}(x), \cdots, \Re \sigma_{r_1+r_2}(x), \Im \sigma_{r_1+r_2}(x)). \]
For every fractional ideal $\mathfrak{J}$, $\sigma(\mathfrak{J})$ is then a $\mathbb{Z}$-lattice of $\mathbb{R}^r$. For $c \gg 1$, we define a functions $f_c$ on $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ by
\[ f_c(\bar{x}) = \prod_{i=1}^{r_1} \min(1, |x_i|^e) \prod_{j=1}^{2r_2} \min(1, |x_{r_1+j}|^{2e}), \bar{x} = (x_i)_{1 \leq i \leq r_1 + r_2}. \]

**Lemma 5.36.** Let $\mathfrak{J} \subset \mathfrak{o}$ be an integral ideal. We have the following two estimations for $t > 0$.

1. $\sum_{\alpha \in \mathfrak{J}^{-1} - \{0\}} f_c(t \sigma(\alpha)) \ll_{\mathfrak{F}, e} |\mathfrak{J}/\mathfrak{J}|^{\frac{3e}{2}} t^{-c}$ if $c > r$.
2. $\sum_{\alpha \in \mathfrak{J}^{-1}} f_c(t \sigma(\alpha)) \ll_{\mathfrak{F}, e} t^{-r} |\mathfrak{J}/\mathfrak{J}|^{-1} \left( 1 + t \frac{|\mathfrak{J}|^2}{\sqrt{r}} \right)^{rc}$ if $c > 1$.

**Proof.** (1) On $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ we have a usual norm $\| \cdot \|_2$ given by
\[ \| \bar{x} \|_2 = \sqrt{\sum_{i=1}^{r_1 + r_2} |x_i|^2}, \bar{x} = (x_i)_{1 \leq i \leq r_1 + r_2}. \]
We then easily see, essentially by comparing $f_c$ with the infinity norm, that
\[ \sup_{\bar{x}} f_c(\bar{x}) |\bar{x}|_2 \leq (r_1 + r_2)^2. \]

If $\alpha_i \in \mathfrak{J}^{-1}$ is an integral basis such that $\mathfrak{J}^{-1} = \Sigma_{i=1}^r \mathbb{Z} \alpha_i$, then $\sigma(\alpha_i)$ is a basis of the lattice $\sigma(\mathfrak{J}^{-1})$. We can define another norm $\| \cdot \|_{\mathfrak{J}^{-1}}$ by
\[ \| \bar{x} \|_{\mathfrak{J}^{-1}} = \sqrt{\sum_{i=1}^{r} n_i^2}, \bar{x} = (n_i)_{1 \leq i \leq r} \in \mathbb{R}^r. \]
Or equivalently, if we write $A_{\mathfrak{J}^{-1}} = (\sigma(\alpha_1), \cdots, \sigma(\alpha_r)) \in \text{GL}_r(\mathbb{R})$, we have
\[ \| \bar{x} \|_{\mathfrak{J}^{-1}} = \| A_{\mathfrak{J}^{-1}} \bar{x} \|_2. \]
Fix a basis $e^*_i$ of $\sigma(\mathfrak{o})$ and write $A_\mathfrak{o} = (e^*_1, \cdots, e^*_r)$. By elementary divisor theorem, there are $\gamma, \gamma_3 \in \text{SL}_r(\mathbb{Z})$ and some $d_i \in \mathbb{Z} - \{0\}, d_i \mid d_{i+1}, |d_1 \cdots d_r| = |\mathfrak{o}/\mathfrak{J}| = |\mathfrak{J}^{-1}/\mathfrak{J}|$, such that
\[ A_{\mathfrak{J}^{-1}} = \gamma \text{diag}(d_1, \cdots, d_r) \gamma_3 A_\mathfrak{o}^{-1}. \]
Changing $\gamma$ is equivalent to changing the choice of basis $\alpha_i$ for $\mathfrak{J}^{-1}$. Applying Lemma \ref{lem:7.2} we can find $\gamma$ such that

$$\text{diag}(d_1, \ldots, d_r){^{-1}}\gamma\text{diag}(d_1, \ldots, d_r){\gamma}_3 = N_+ N_-$$

where $N_+$ resp. $N_-$ is upper resp. lower unipotent with entries lying in $[-\frac{d_r}{2d_1}, \frac{d_r}{2d_1}]$. We thus get a bound of the operator norm

$$\|A_{\mathfrak{J}^{-1}}\|_2 = \|\text{diag}(d_1, \ldots, d_r)N_+ N_- A_{\mathfrak{J}^{-1}}\|_2 \leq |d_r| \left( r + \frac{r(r-1)d_r^2}{8d_1^2} \right) \|A_{\mathfrak{J}^{-1}}\|_2.$$ 

We finally estimate and conclude by

$$\sum_{\alpha \in \mathfrak{J}^{-1} - \{0\}} f_c(t\sigma(\alpha)) = \sum_{\alpha \in \mathfrak{J}^{-1} - \{0\}} \|t\sigma(\alpha)\|_{\mathfrak{J}^{-1}}^{-c} \left( \frac{\|t\sigma(\alpha)\|_{\mathfrak{J}^{-1}}^{5 - 1} f_c(t\sigma(\alpha))}{\|t\sigma(\alpha)\|_2^c} \right)\|A_{\mathfrak{J}^{-1}}\|_2^c \left( r_1 + r_2 \sum_{\bar{n} \in \mathbb{Z}^r - \{0\}} \|\bar{n}\|_2^{-c} \right).$$

(2) If $\mathcal{L}$ is a lattice in $\mathbb{R}^r$ with a basis given by the column vectors in a matrix $A_{\mathcal{L}} \in \text{GL}_r(\mathbb{R})$, we define its diameter $d(\mathcal{L})$ associated to this basis as the diameter of the fundamental parallelogram spanned by this basis, i.e.,

$$d(\mathcal{L}) = \max_{-1 \leq \lambda_i \leq 1, 1 \leq i \leq r} \|A_{\mathcal{L}} \bar{\lambda}\|_2.$$

Thus we have obviously $d(\mathcal{L}) \leq \sqrt{r}\|A_{\mathcal{L}}\|_2$. The same argument in the last part of (1) gives, for some choice of basis of $\mathfrak{J}^{-1}$, that

$$d(\sigma(\mathfrak{J}^{-1})) \leq \sqrt{r}\|A_{\mathfrak{J}}\|_2^{-1} \left( r + \frac{r(r-1)d_r^2}{8d_1^2} \right).$$

It is also easy to see that for $\bar{x}, \bar{y}$ in the same translate of a parallelogram $x_0 + \mathcal{P}$ of $t\sigma(\mathfrak{J}^{-1})$, we have

$$\frac{f_c(\bar{x})}{f_c(\bar{y})} \leq 2^{2r_c} \left( \frac{r_1 + r_2 + \sqrt{r}d(\sigma(\mathfrak{J}^{-1}))}{r} \right)^{r_c} \int_{\mathbb{R}^r} f_c(\bar{x})d\bar{x}.$$

We thus conclude by

$$\text{Vol}(\mathbb{R}^r/t\sigma(\mathfrak{J}^{-1})) \sum_{\alpha \in \mathfrak{J}^{-1}} f_c(t\sigma(\alpha)) \leq 2^{2r_c} \left( \frac{r_1 + r_2 + \sqrt{r}d(\sigma(\mathfrak{J}^{-1}))}{r} \right)^{r_c} \int_{\mathbb{R}^r} f_c(\bar{x})d\bar{x} \ll_{\mathbf{F}, c} \left( 1 + \frac{|\alpha/\mathfrak{J}|^2}{\sqrt{r}} \right)^{r_c}.$$

We shall write, denoting by $p_v$ the prime ideal corresponding to $v < \infty$, $v(\mathfrak{J}) \in \mathbb{N}$ such that

$$\mathfrak{J} = \prod_{v < \infty} p_v^{v(\mathfrak{J})}.$$

A variant of the above estimations in the adelic language is the following lemma.

**Lemma 5.37.** Let $\mathfrak{J}$ be an integral ideal. Let $c_1, c_2 \in \mathbb{R}, c_2 - c_1 > r$, we have

$$\sum_{\alpha \in \mathbf{F}^r \setminus \{0\}} \prod_{v < \infty} \min(|\alpha y_v|_{c_1}^{-1}, |\alpha y_v|_{c_2}^{-1}) \prod_{v < \infty} |\alpha y_v|_{c_1}^{-1} \text{v}(\alpha y_v) \geq c_2 - c_1 - v(\mathfrak{J})$$

$$\ll_{\mathbf{F}, c_2 - c_1} \min \left( |y|_{\mathbf{A}}^{-c_1 - 1} |\alpha/\mathfrak{J}|^{-1} \left( 1 + |y|_{\mathbf{A}}^{\frac{1}{2} \frac{|\alpha/\mathfrak{J}|^2}{r}} \right)^{r(c_2 - c_1)} \right), |y|_{\mathbf{A}}^{-c_1 - c_2 + c_1} |\alpha/\mathfrak{J}|^{3(c_2 - c_1)}.$$

where $y = (y_v)_v \in \mathbf{A}^\times$, $r = [\mathbf{F} : \mathbb{Q}]$. 

Proof. The following exact sequence is split with a splitting section \( s : \mathbb{R}_+ \to A^\times \)
\[
1 \to A^{(1)} \to A^\times \xrightarrow{1/s} \mathbb{R}_+ \to 1
\]
such that the image of \( s \) is contained in \( A^\times_\infty = \prod_v F_v^\times \). For any \( t \in \mathbb{R}_+ \), we write \( t^+ = s(t) \) with \( t^+_v = t^{1/\sigma} \in \mathbb{R}_+ \subset F_v \), such that
\[
|t^+_v|_\alpha = \prod_{v|\infty} |t^+_v|_v = t.
\]
We may apply the compactness of \( F^\times \setminus A^{(1)} \), or proceed alternatively in the following more classical way. Let \( \text{Cl}(F) \) be the class group of \( F \) and choose an integral ideal \( \tau \) in each class \([\tau] \in \text{Cl}(F)\). Let \( \delta_r \in A^\times_{\fin} \) be a representative of \( \tau \) in the group of ideles. Since \( \text{Cl}(F) \simeq F^\times \setminus A^\times_{\fin} / \delta^\times \), there is a unique \( \tau \), and some \( \beta = \beta_0 u \in F^\times \) with freely chosen \( u \in \mathfrak{o}^\times \) such that \( y_{\fin} \in \beta \delta_r \mathfrak{o}^\times \). Let \( t = |y|_\Lambda \). We can write \( y = y_{\infty}y_{\fin} = t^+ y'_{\infty}y_{\fin} \), and find
\[
\prod_{v|\infty} |\beta^{-1} y_{v}(t^+_v)^{-1}|_v = \prod_{v|\infty} |u^{-1} \beta_0^{-1} y'_{v}|_v = t^{-1} |\beta^{-1} y|_\Lambda |\delta_r|_\Lambda^{-1} = |\mathfrak{o}/\tau|.
\]
By Dirichlet’s unit theorem, we can adjust \( u \in \mathfrak{o}^\times \) so that each \( |u^{-1} \beta_0^{-1} y'_{v}|_v \) remains in a compact set in \( \mathbb{R}_+ \) depending only on \( F \). We thus find
\[
\sum_{\alpha \in F^\times} \prod_{v|\infty} \min(|\alpha y_{v}|_{v}^{-c_1}, |\alpha y_{v}|_{v}^{-c_2}) \prod_{v<\infty} |\alpha y_{v}|_{v}^{-c_1} 1_{v(\alpha y_{v}) \geq -c(\psi_{v}) - v(\mathfrak{J})} = \sum_{\alpha \in F^\times} \prod_{v|\infty} \min(|\alpha \beta^{-1} y_{v}|_{v}^{-c_1}, |\alpha \beta^{-1} y_{v}|_{v}^{-c_2}) \prod_{v<\infty} |\alpha \delta_r|_{v}^{-c_1} 1_{v(\alpha ) \geq -c(\psi_{v}) - v(\mathfrak{J})}
\]
\[
= t^{-c_1} \sum_{\alpha \in (\mathfrak{o} F^{1/\sigma})^{-1}(\mathfrak{J})} \prod_{v|\infty} \min(1, |\alpha t_{v}|_{v}^{-c_2-c_1}, |u^{-1} \beta_0^{-1} y'_{v}|_{v}^{c_2-c_1}) \ll_{F,c_2-c_1} t^{-c_1} \sum_{\alpha \in (\mathfrak{o} F^{1/\sigma})^{-1}(\mathfrak{J})} f_{c_2-c_1} (t^+ \sigma(\alpha)).
\]
We then apply Lemma 3.36 to conclude. \( \square \)

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