On a generalization of the Gauss’s formula

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Abstract

In this paper we study a group theoretical generalization of the well-known Gauss’s formula that uses the generalized Euler’s totient function introduced in [11].

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1 Introduction

The Euler’s totient function (or, simply, the totient function) \( \varphi \) is one of the most famous functions in number theory. Notice that the totient \( \varphi(n) \) of a positive integer \( n \) is defined to be the number of positive integers less than or equal to \( n \) that are coprime to \( n \). The totient function is important mainly because it gives the order of the group of all units in the ring \((\mathbb{Z}_n, +, \cdot)\). Alternatively, \( \varphi(n) \) can be seen as the number of generators or as the number of elements of order \( n \) of the finite cyclic group \((\mathbb{Z}_n, +)\).

Recall also a well-known arithmetical identity involving the totient function, namely the Gauss’s formula

\[
\sum_{d|n} \varphi(d) = n, \; \forall \; n \in \mathbb{N}^*.
\]

Many generalizations of the totient function are known (for example, see [2, 3, 5, 8] and the special chapter on this topic in [6]). From these, the most significant is probably the Jordan’s totient function (see [7]).
The starting point for our discussion is given by the paper [11], where a new group theoretical generalization of the totient function has been studied. This is founded on the remark that $\varphi(n)$ counts in fact the number of elements of order $\exp(Z_n)$ in $(Z_n, +)$. Consequently, it makes sense to define

$$\varphi(G) = |\{a \in G \mid o(a) = \exp(G)\}|$$

for any finite group $G$. It is obvious that $\varphi(Z_n) = \varphi(n)$, for all $n \in \mathbb{N}^*$, and so a generalization of the classical totient function $\varphi$ is obtained. We observe that for $G \cong Z_n$ the Gauss’s formula can be rewritten as

$$(2) \quad \sum_{H \leq G} \varphi(H) = |G|.$$ 

This leads to the natural problem

*which are the finite groups $G$ satisfying the equality (2)?*

Its study is the main goal of the current paper. We show that the cyclic groups are the unique abelian groups with this property. Inspired by some particular cases, we conjecture that this is also true for nilpotent groups. Moreover, we give examples of non-nilpotent groups $G$ satisfying (2). Several open problems on this topic are also formulated.

Most of our notation is standard and will not be repeated here. Basic definitions and results on groups can be found in [4, 9]. For subgroup lattice concepts we refer the reader to [7, 10].

## 2 Main results

For a finite group $G$ let us denote

$$S(G) = \sum_{H \leq G} \varphi(H).$$

In this way, we are interested to describe the class $\mathcal{C}$ consisting of all finite groups $G$ for which $S(G) = |G|$. 

Obviously, the finite cyclic groups are contained in $\mathcal{C}$, by the Gauss’s formula. On the other hand, we easily obtain $S(Z_2 \times Z_2) = 7 \neq 4 = |Z_2 \times Z_2|$, proving that $\mathcal{C}$ is not closed under direct products or extensions.
For a detailed study of the class $\mathcal{C}$, we must look first at some basic properties of the map $S$. We remark that it satisfies the inequality

$$S(G) \geq \sum_{H \in C(G)} \varphi(H) = \sum_{H \in C(G)} \varphi(|H|),$$

where $C(G)$ denotes the poset of cyclic subgroups of $G$. Another easy but very important property of $S$ is the following.

**Proposition 1.** $S$ is multiplicative, that is if $(G_i)_{i=1}^k$ is a family of finite groups of coprime orders, then we have:

$$S(\prod_{i=1}^k G_i) = \prod_{i=1}^k S(G_i).$$

**Proof.** Since the groups $(G_i)_{i=1}^k$ are of coprime orders, we infer that every subgroup $H$ of $G = \prod_{i=1}^k G_i$ can be written as $H = \prod_{i=1}^k H_i$ with $H_i \leq G_i$, $\forall i = 1, k$. By Lemma 2.1 of [11], we know that $\varphi$ is multiplicative and therefore

$$\varphi(H) = \prod_{i=1}^k \varphi(H_i).$$

Then one obtains

$$S(\prod_{i=1}^k G_i) = \sum_{H \leq G} \varphi(H) = \sum_{i=1}^k \sum_{H_i \leq G_i} \varphi(H_1)\varphi(H_2)\cdots\varphi(H_k) =$$

$$= \prod_{i=1}^k \left( \sum_{H_i \leq G_i} \varphi(H_i) \right) = \prod_{i=1}^k S(G_i),$$

as desired. \qed

In particular, Proposition 1 shows that the computation of $S(G)$ for a finite nilpotent group $G$ is reduced to $p$-groups.

**Corollary 2.** Let $G$ be a finite nilpotent group and $G_i$, $i = 1, 2, ..., k$, be the Sylow subgroups of $G$. Then

$$S(G) = \prod_{i=1}^k S(G_i).$$
Proof. The equality follows immediately from Proposition 1, since a finite nilpotent group is the direct product of its Sylow subgroups.

Notice that for a finite abelian $p$-group $G$ the value $\varphi(G)$ has been precisely computed in Theorem 2.3 of [11]. This is essential to show the following result.

**Theorem 3.** Let $G$ be a finite abelian group. Then $S(G) \geq |G|$, and we have equality if and only if $G$ is cyclic.

Proof. Remark first that we can assume $G$ to be a $p$-group, by Corollary 2. Let $(p^{\alpha_1}, p^{\alpha_2}, \ldots, p^{\alpha_r})$ be the type of $G$ and assume that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{s-1} < \alpha_s = \alpha_{s+1} = \cdots = \alpha_r$. Then we have

$$\varphi(G) = |G| \left( 1 - \frac{1}{p^{r-s+1}} \right) \geq |G| \left( 1 - \frac{1}{p} \right).$$

On the other hand, it is well-known that $G$ has $\frac{p^r - 1}{p - 1}$ maximal subgroups, namely $p^{r-1}$ subgroups isomorphic to $M_1 = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_r}}$, $p^{r-2}$ subgroups isomorphic to $M_2 = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_r}}$, $\ldots$, and one subgroup isomorphic to $M_r = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p^{\alpha_r}}$. One obtains

$$S(G) \geq \varphi(G) + \sum_{i=1}^{r} p^{r-i} \varphi(M_i) + 1 \geq |G| \left( 1 - \frac{1}{p} \right) + \sum_{i=1}^{r} p^{r-i} \frac{|G|}{p} \left( 1 - \frac{1}{p} \right) + 1 =$$

$$= |G| \frac{p^r + p^2 - p - 1}{p^2} + 1.$$

If $r \geq 2$, then

$$\frac{p^r + p^2 - p - 1}{p^2} \geq \frac{2p^2 - p - 1}{p^2} > 1,$$

implying that

$$S(G) > |G| + 1 > |G|.$$

Consequently, $G$ belongs to $C$ if and only if $r = 1$, i.e. if and only if it is cyclic. \qed
Next we will focus on extending the above result from abelian \( p \)-groups to arbitrary \( p \)-groups, and consequently to arbitrary nilpotent groups. By a direct calculation, we infer that for all non-abelian \( p \)-groups \( G \) of order \( p^3 \) (whose classification is well-known – see e.g. [9], II) we have

\[
S(G) > |G|.
\]

This inequality also holds for other classes of non-abelian \( p \)-groups \( G \), determined by the existence of abelian subgroups of a given structure.

**Theorem 4.** Let \( G \) be a non-abelian \( p \)-group of order \( p^n \), \( n \geq 4 \). If \( G \) has an abelian subgroup of order \( p^m \) and rank \( r \) with \( m + r \geq n + 2 \), then \( S(G) > |G| \), i.e. \( G \) is not contained in \( \mathcal{C} \). In particular, if \( G \) has an elementary abelian maximal subgroup, then it does not belong to \( \mathcal{C} \).

**Proof.** Let \( A \) be an abelian subgroup of order \( p^m \) and rank \( r \) of \( G \), and assume that \( m + r \geq n + 2 \). By the proof of Theorem 3, we infer that

\[
S(G) > S(A) \geq \frac{p^m p^r + p^2 - p - 1}{p^2} + 1 = \\
= p^{m+r-2} + p^{m-2}(p^2 - p - 1) + 1 \geq \\
\geq p^{m+r-2} + p^{m-2} + 1 > p^{m+r-2} \geq \\
\geq p^n,
\]

as claimed. \( \square \)

**Theorem 5.** Let \( G \) be a non-abelian \( p \)-group of order \( p^n \), \( n \geq 4 \). If \( G \) has a cyclic maximal subgroup, then \( S(G) > |G| \), i.e. \( G \) is not contained in \( \mathcal{C} \).

**Proof.** By Theorem 4.1 of [9], II, we know that \( G \) is isomorphic to

- \( M(p^n) = \langle x, y \mid x^{p^{n-1}} = y^p = 1, y^{-1}xy = x^{p^{n-2}+1} \rangle \)

when \( p \) is odd, or to one of the following groups

- \( M(2^n) \)
- the dihedral group \( D_{2^n} \),
- the generalized quaternion group

\[
Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^4 = 1, xy^{-1} = x^{2^{n-1}-1} \rangle,
\]
the quasi-dihedral group \( S_{2n} = \langle x, y \mid x^{2^{2n-1}} = y^2 = 1, \ y^{-1}xy = x^{2^{2n-2}-1} \rangle \)

when \( p = 2 \).

A common property of all these \( p \)-groups \( G \) is that they have \( p+1 \) maximal subgroups, say \( M_1, M_2, ... , M_{p+1} \), and (at least) one of them is cyclic, say \( M_{p+1} \cong \mathbb{Z}_{p^{n-1}} \). Moreover, \( \Phi(G) \) is cyclic of order \( p^{n-2} \). Then, by applying the Inclusion-Exclusion Principle, one obtains

\[
(4) \quad S(G) = \varphi(G) + \sum_{i=1}^{p+1} S(M_i) - p \cdot S(\Phi(G)) = \varphi(G) + \sum_{i=1}^{p+1} S(M_i) - p^{n-1}.
\]

For \( M(p^n) \) it is easy to check that \( p \) maximal subgroups are cyclic, say \( M_i \cong \mathbb{Z}_{p^{n}} \), \( i = 2, 3, ..., p+1 \), and \( M_1 \cong \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}} \). Then \( \varphi(M(p^n)) = p \cdot \varphi(p^{n-1}) = p^n - p^{n-1} \) and (4) leads to

\[
S(M(p^n)) = p^n - p^{n-1} + S(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) + p \cdot p^{n-1} - p^{n-1} > p^n - p^{n-1} + p^n - p^{n-1} = 2 \cdot p^n - p^{n-1} > p^n,
\]

according to Theorem 3.

For every \( G \in \{ D_{2^n}, Q_{2^n}, S_{2^n} \} \) we have \( \varphi(G) = 2^{n-2} \). Then (4) can be rewritten as

\[
(5) \quad S(G) = 2^{n-2} + S(M_1) + S(M_2).
\]

The pair \( (M_1, M_2) \) of maximal subgroups of \( D_{2^n}, Q_{2^n} \) and \( S_{2^n} \) is \( (D_{2^n-1}, D_{2^n-1}), (Q_{2^n-1}, Q_{2^n-1}) \) and \( (D_{2^n-1}, Q_{2^n-1}) \), respectively. Clearly, in the first two cases (5) becomes a recurrence relation which easily leads to

\[
S(D_{2^n}) = 2^{n+1} + (n-3) \cdot 2^{n-2} > 2^n
\]

and

\[
S(Q_{2^n}) = (n+4) \cdot 2^{n-2} > 2^n,
\]

while for \( G = S_{2^n} \) one obtains

\[
S(S_{2^n}) = 2^{n-2} + S(D_{2^n-1}) + S(Q_{2^n-1}) = (2n+9) \cdot 2^{n-3} > 2^n.
\]

This completes the proof. \( \square \)
Inspired by the previous results, we came up with the following conjecture.

**Conjecture 6.** Let $G$ be a finite nilpotent group. Then $S(G) \geq |G|$, and we have equality if and only if $G$ is cyclic.

Obviously, Conjecture 6 can be reformulated in the next way: the cyclic groups are the unique finite nilpotent groups contained in $C$. It leads to the natural assumption that $C$ consists in fact only of the finite cyclic groups. This is not true, as shows the following elementary example.

**Example.** Let $G$ be the non-abelian group of order $pq$, where $p < q$ are primes and $p \mid q - 1$. The subgroup structure of $G$ is well-known: it possesses one subgroup of order 1, $q$ subgroups of order $p$, one subgroup of order $q$ and one subgroup of order $pq$. Then

$$S(G) = 1 + q \varphi(Z_p) + \varphi(Z_q) + \varphi(G) = 1 + q(p-1) + q - 1 = pq = |G|,$$

i.e. $G$ belongs to $C$.

In particular, the above example shows that the dihedral group $D_6$ is contained in $C$. In fact we are able to characterize the containment to $C$ for arbitrary dihedral groups $D_{2n} = \langle x, y \mid x^n = y^2 = 1, yxy = x^{-1} \rangle$, $n \geq 2$.

**Theorem 7.** The dihedral group $D_{2n}$ belongs to $C$ if and only if $n$ is odd.

*Proof.* Let $n = 2^k m$ with $k, m \in \mathbb{N}$ and $m$ odd. Then the lattice of divisors of $n$ can be written as the union of the sets $\mathcal{D}_i = \{ 2^i m' \mid m' \mid m \}$, $i = 0, 1, ..., k$.

On the other hand, for every divisor $d$ of $n$, $D_{2n}$ has one subgroup isomorphic to $\mathbb{Z}_d$, namely $\langle x^{\frac{n}{d}} \rangle$, and $\frac{n}{d}$ subgroups isomorphic to $D_{2d}$, namely $\langle x^{\frac{n}{d}}, x^{i-1}y \rangle$, $i = 1, 2, ..., \frac{n}{d}$. Recall that we have $\varphi(D_2) = 1$, $\varphi(D_4) = 4$, and

$$\varphi(D_{2n}) = \begin{cases} 0, & n \equiv 1 \pmod{2} \\ \varphi(n), & n \equiv 0 \pmod{2} \end{cases} \quad \forall n \geq 3$$

by Theorem 2.6 of [11]. It follows that

$$S(D_{2n}) = \sum_{H \leq D_{2n}} \varphi(H) = \sum_{d \mid n} \left( \varphi(\mathbb{Z}_d) + \frac{n}{d} \varphi(D_{2d}) \right) =$$
where
\[ \Sigma = \sum_{i=1}^{k} \sum_{m'|m} \frac{n}{2^{i}m'} \phi(D_{2i+1m'}). \]

Hence \( S(D_{2n}) = 2n \) if and only if \( \Sigma = 0 \). This happens if and only if \( k = 0 \), i.e. \( n \) is odd. \( \square \)

**Remark.** By Theorem 7, we have \( S(D_{2n}) = 2n \) for \( n \) odd. An explicit value of \( S(D_{2n}) \) for \( n \) even can be calculated, too. Let \( n \) as above and let \( m = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s} \) be the decomposition of \( m \) as a product of prime factors. We remark that \( \phi(D_{2i+1m'}) = \phi(2^i m') = 2^{i-1} \phi(m') \), excepting the case \( i = m' = 1 \) when \( \phi(D_{2i+1m'}) = 3 \). One obtains

\[ \Sigma = \frac{3n}{2} - \frac{n}{2} + \sum_{i=1}^{k} \sum_{m'|m} \frac{n}{2^{i}m'} \phi(m') = n + \frac{kn}{2} \sum_{m'|m} \phi(m') = \]

\[ = n + \frac{kn}{2} \prod_{i=1}^{s} \left( \alpha_i + 1 - \frac{\alpha_i}{p_i} \right) \]

and thus

\[ S(D_{2n}) = 3n + \frac{kn}{2} \prod_{i=1}^{s} \left( \alpha_i + 1 - \frac{\alpha_i}{p_i} \right). \]

For example, we can easily check that

\[ S(D_{12}) = 23. \]

Next we observe that both the non-abelian groups of order \( pq \) and the dihedral groups \( D_{2n} \) with \( n \) odd, which we verified to be contained in \( \mathcal{C} \), are semidirect products of a cyclic normal subgroup \( N \) by a cyclic subgroup \( H \) of prime order satisfying \( C_N(H) = 1 \). The containment of such a group to \( \mathcal{C} \) can be also characterized, extending the above results.
Theorem 8. Let $G$ be a finite non-abelian group and $N \cong \mathbb{Z}_n$ be a normal Hall subgroup of $G$ which has a complement $H$ of prime order $p$ such that $C_N(H) = 1$. Then $G$ belongs to $\mathcal{C}$ if and only if the number of complements of $N$ in $G$ is $n$.

Proof. Under our hypotheses, $L(G)$ consists of the subgroups of $N$, say $N_d$ with $d = |N_d|, d \mid n$, of the complements of $N$ in $G$, say $H_1 = H, H_2, \ldots, H_{n_p}$, and of the semidirect products $N_dH_i$, with $d \mid n, d \neq 1$ and $i = 1, \ldots, n_p$. Since $C_N(H) = 1$, every $N_dH_i$ with $d \neq 1$ is not cyclic and so it does not contain elements of order $dp = \exp(N_dH_i)$. Consequently, we infer that $\varphi(N_dH_i) = 0$ for all $d \mid n$ with $d \neq 1$ and all $i = 1, \ldots, n_p$. This leads to

$$S(G) = S(N) + \sum_{i=1}^{n_p} \varphi(H_i) = n + n_p(p - 1).$$

It is now obvious that

$$S(G) = np \iff n_p = n,$$

which ends the proof.

We conclude that at least two important classes of finite groups are contained in $\mathcal{C}$: cyclic groups and semidirect products of type indicated in Theorem 8. Remark that these groups $G$ are supersolvable and that $S(G)$ equals the sum of all values of $\varphi$ on the cyclic subgroups of $G$, that is (3) becomes an equality.

Finally, we remark that every subgroup and every quotient of such a group also belong to $\mathcal{C}$, that is $\mathcal{C}$ seems to be closed under subgroups and homomorphic images.

We end this paper by indicating several natural problems on the above class $\mathcal{C}$.

Problem 1. Prove or disprove Conjecture 6.

Problem 2. Give a complete description of $\mathcal{C}$ (in our opinion, it consists of the finite cyclic groups and of non-abelian semidirect products of a certain type, most probably metacyclic groups). It is true that $\mathcal{C}$ is contained in the class of finite supersolvable groups?

Problem 3. Study whether $\mathcal{C}$ is closed under subgroups and homomorphic images.
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