Theory of Turbulent Accretion Disks

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Abstract

In low–mass disks, turbulent torques are probably the most important way of redistributing angular momentum. Here we present the theory of turbulent accretion disks. We show the molecular viscosity is far too small to account for the evolutionary timescale of disks, and we describe how turbulence may result in enhanced transport of (angular) momentum. We then turn to the magnetorotational instability, which thus far is the only mechanism that has been shown to initiate and sustain turbulence in disks. Finally, we present both the basis and the structure of $\alpha$ disk models.

1 Introduction

It is an observational fact that at least part of the mass in protostellar disks is accreted onto the central objet. In a Keplerian accretion disk, the specific angular momentum increases with radius. Therefore, a particle can be accreted only if its angular momentum is removed or transferred to particles located at larger radii. Whether angular momentum is removed from or redistributed within the disk depends on whether the disk is subject to external or internal torques. Possible external torques can either be magnetic (when an outflow is present) or tidal (in binary systems), whereas possible internal torques can either be gravitational (massive disks) or turbulent. These mechanisms have been discussed by Papaloizou & Lin (1995).

During the early stages of disk evolution, when the disk is still embedded (class 0/1 object) and has a significant mass compared to the central star, there may exist strong disk winds and bipolar outflows (e.g. Reipurth et al. 1997) with associated magnetic fields. During this stage, a hydromagnetic disk wind may be an important means of angular momentum removal for the system (Blandford & Payne 1982). Because of the action of magnetic torques, material ejected from the disk is able to carry away significantly more angular momentum than it originated with in the disk. Therefore, even a modest ejection rate can lead to a significant accretion rate through the disk. However, observations indicate that outflows may exist only in the early stages of disk evolution, so that this mechanism cannot account for angular momentum transport during the whole life of accretion disks. In addition, it may affect only the inner parts.
When the mass of the disk is significant compared to that of the star, gravitational instabilities may develop, leading to outward angular momentum transport (Papaloizou & Savonije 1991; Heemskerk et al. 1992; Laughlin & Bodenheimer 1994; Pickett et al. 1998) that results in additional mass growth of the central star. This redistribution of mass may occur on the dynamical timescale (a few orbits) of the outer part of the disk and thus may be quite rapid: on the order of $10^5$ yr for a disk radius of 500 AU. The parameter governing the importance of disk self–gravity is the Toomre parameter, $Q \sim M_\ast H/(M_d r)$, with $M_\ast$ being the central mass, $M_d$ being the disk mass contained within radius $r$ and $H$ being the disk semi–thickness. Typically $H/r \sim 0.1$ (Stapelfeldt et al. 1998) such that the condition for the importance of self–gravity, $Q \sim 1$, gives $M_d \sim 0.1 M_\ast$. During the period of global gravitational instability, it is reasonable to suppose that the disk mass is quickly redistributed and reduced by accretion onto the central object such that the effects of self–gravity become negligible.

If the disk surrounds a star which is in a pre–main sequence binary system, tidal torques transport angular momentum outward, from the disk rotation to the orbital motion of the binary. However, although tidal effects are important for truncating protostellar disks and for determining their size, it is unlikely that tidally–induced angular momentum transport plays a dominant role in the evolution of protostellar disks (see Terquem 2001 and references therein). In a non self–gravitating disk, the amount of transport provided by tidal waves is probably too small to account for the lifetime of protostellar disks. In addition, tidal effects tend to be localized in the disk outer regions.

When the disk mass is such that self–gravity can be ignored and the jet activity has significantly decreased, turbulent torques may become the most important way of redistributing angular momentum in the disk. Historically, the first angular momentum transport mechanism to be considered was through the action of viscosity (von Weizsäcker 1948). However, in order to result in evolution on astronomically interesting timescales, it is necessary to suppose that an anomalously large viscosity is produced through the action of some sort of turbulence.

In this paper we present the theory of turbulent accretion disks. We first calculate the molecular viscosity in a disk, and show this is far too small to account for the evolutionary timescale of disks. We then describe how turbulence may result in enhanced transport of (angular) momentum. About ten years ago, a linear magnetic instability was identified (Balbus & Hawley 1991) that results in MHD turbulence. We present this instability. We then describe both the basis and the structure of $\alpha$ disk models, which were introduced by Shakura & Sunyaev (1973).

2 Fundamental of hydrodynamics

2.1 Fluid mechanics vs. kinetic theory

The fundamental equations of fluids can be derived by considering them as either a collection of particles (kinetic theory) or as a smooth continuum. This latter approach is justified when the mean free path $\lambda$ of the particles is very small compared to the macroscopic length scale $L$ of interest in the fluid. This condition is not met in stellar systems (galaxies or star clusters), where $\lambda \gg L$, nor in planetary rings, where $\lambda \sim L$, but it is fulfilled in disks. The conservation equations are obtained more straightforwardly when considering the fluid as a continuum, which is the approach we will use here.

This does not lead to a precise expression for the transport coefficients, however, in contrast to kinetic theory. Transport of energy, mass and momentum occurs in a gas which is out of
equilibrium (i.e. in which the distribution function is not a Maxwell–Boltzmann distribution) through molecular collisions. Most of the time, the departure from equilibrium is tiny, so that the distribution function is nearly maxwellian. Within the context of the kinetic theory, in which molecular collisions are explicitly calculated, the Chapman–Enskog procedure gives the transport coefficients by considering small variations of the distribution function around the Maxwell–Boltzmann distribution. Such a calculation is not possible when fluids are viewed as continua, as in this case molecular collisions are not explicitly calculated. It is possible however to get a phenomenological expression for the transport coefficients in this context, as we shall see below.

For more details on the kinetic theory of gases, the reader is referred to, e.g., Huang (1987) and Shu (1992). For the description of fluids as continua, which is developed below, a more complete presentation can be found, e.g., in Acheson (1990), Batchelor (1967), Landau & Lifchitz (1987) and Shu (1992).

2.2 Conservation laws

2.2.1 Mass conservation

If there is no sink or source of matter, the time rate of mass variation within a fixed volume $V$ is equal to minus the flux of mass (mass advected by the flow) through the surface $S$ which encloses the volume:

$$\frac{\partial}{\partial t} \int_V \rho dV = - \int_S \rho \mathbf{v} \cdot \mathbf{n} dS,$$

(1)

where $\rho(r,t)$ and $\mathbf{v}(r,t)$ denote the density and velocity vector of the fluid at location $r$ and time $t$, and $\mathbf{n}$ is the unit normal to the surface oriented outward.

Since this has to be satisfied for any volume $V$, and using the divergence theorem, we get:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0.$$

(2)

The mass conservation equation derived above is also know as the continuity equation. It states that mass variation in a fixed volume is due to a continuous flow of mass through the surfaces which enclose the volume, and not to ‘jumps’ from one location to another remote one.

Equation (2) has been derived by considering the variation of mass of a volume fixed in space. There are two contributions to the time rate of mass change in this volume: $\rho \nabla \cdot \mathbf{v}$, which is due to the change of the volume itself, and $\mathbf{v} \cdot \nabla \rho$, which is due to the mass flowing through the volume.

We can also consider the time rate of mass change of a volume of fluid that we follow along its trajectory. For that we define the Lagrangian derivative $d/dt$, which is the time rate of change in a frame moving with a particular fluid element along its trajectory. The density $\rho$ depends both on $r$ and $t$ and, if $r$ is the location of the moving fluid element, it depends also on $t$. Then

$$\frac{d \rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_i} \frac{dx_i}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho,$$

(3)
were the \( x_i \) \( (i = 1, 2, 3) \) denote the coordinates and we adopt the standard Einstein notation of summation over repeated indices.

Equation (2) can therefore be rewritten as:

\[
\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0. \tag{4}
\]

This expresses the fact that the mass change in a volume moving with the fluid is due only to the change in the volume.

2.2.2 Incompressibility

A fluid is said to be incompressible if a volume element is neither compressed nor dilated when moving with the flow. Let us consider a fixed volume \( V \) of the fluid bounded by the surface \( S \). The volume of fluid which leaves \( V \) through an infinitesimal surface \( dS \) per unit time is \( \mathbf{v} \cdot \mathbf{n} dS \), where \( \mathbf{n} \) is the unit normal to the surface oriented outward. Therefore

\[
\frac{dV}{dt} = \int_S \mathbf{v} \cdot \mathbf{n} dS = \int_V (\nabla \cdot \mathbf{v}) dV, \tag{5}
\]

where we have used the divergence theorem to get the final expression. The condition for the fluid to be incompressible is therefore:

\[
\nabla \cdot \mathbf{v} = 0, \tag{6}
\]

or, equivalently, using the mass conservation equation (4):

\[
\frac{d\rho}{dt} = 0. \tag{7}
\]

This expresses the fact that, in an incompressible fluid, the density of a fluid element is constant along its trajectory.

2.2.3 Equation of motion and momentum conservation

The product of the acceleration of a fluid element moving with the flow, \( d\mathbf{v}/dt \), by the mass density \( \rho \) is equal to the net force per unit volume exerted onto this element. This has contribution from external forces (e.g., gravitational, magnetic...) and/or from momentum transport within the flow due to molecular collisions (pressure and viscous forces). The local form of the equation of motion is then:

\[
\rho \frac{d\mathbf{v}}{dt} = \rho \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla P + \mathbf{F}_{\text{visc}} + \mathbf{F}, \tag{8}
\]

where \( P \) is the pressure, \( \mathbf{F}_{\text{visc}} \) is the viscous force per unit volume and \( \mathbf{F} \) is any other force per unit volume which does not arise from pressure and viscosity. When \( \mathbf{F}_{\text{visc}} = 0 \), this equation is known as Euler equation.

Using equation (2), we can rewrite the equation of motion under the form:
\[
\frac{\partial}{\partial t} (\rho \mathbf{v}) = -\nabla \cdot (\rho \mathbf{v} \mathbf{v}) - \nabla P + \mathbf{F}_{\text{visc}} + \mathbf{F},
\]

(9)

where, according to our notation, the \( i \) component of the vector \( \nabla \cdot (\rho \mathbf{v} \mathbf{v}) \) is \( \nabla \cdot (\rho \mathbf{v} \mathbf{v}) \).

This is the momentum conservation equation, which states that the time rate of momentum variation within a fixed volume is equal to minus the momentum which is advected by the flow through the surface which encloses the volume plus the net force exerted onto the volume.

### 2.3 Viscous forces: Molecular transport of momentum

The viscous force in a fluid is due to an irreversible transport of momentum from regions where the velocity is higher to regions where it is lower. A variational calculation shows that uniform rotation is an extremum for the energy. Sometimes it is a minimum, in which case dissipation of energy at fixed (angular) momentum results in uniform rotation (e.g. synchronous binaries). This tends to be the case when pressure is dominant against gravity. Sometimes the extremum is a maximum, in which case all the matter tends to accumulate at the center, with all the angular momentum at infinity. This tends to be the case when rotation is dominant against gravity, and this is what happens in disks (for further details, see Lynden–Bell & Kalnajs 1972; Lynden–Bell & Pringle 1974). The dissipation of energy results from molecular collisions, or friction.

We denote by \( -\sigma_{ij} \) the flux of the \( i \) component of the momentum in the \( j \) direction. The tensor \([\sigma]\) is called the *viscous stress tensor*. The \( i \) component of the viscous force per unit volume is then:

\[
F_{\text{visc},i} \equiv \frac{\partial \sigma_{ij}}{\partial x_j}.
\]

(10)

Friction occurs only when different parts of the fluid have different velocities. Therefore \( \sigma_{ij} \) should depend on the velocity gradients. If the velocity varies on a scale large compared to the mean free path, i.e. to the scale over which molecular transport arises, one can suppose that \( \sigma_{ij} \) depends only on the first derivatives of the velocity with respect to the coordinates. Furthermore, we suppose that the dependence is linear, i.e. we limit ourselves to *Newtonian fluids*. The most general form of \( \sigma_{ij} \) is then:

\[
\sigma_{ij} \propto \frac{\partial v_i}{\partial x_j} + A \frac{\partial v_j}{\partial x_i} + B \delta_{ij} \frac{\partial v_k}{\partial x_k},
\]

(11)

where \( A \) and \( B \) are constants to be determined, and \( \delta_{ij} \) is the Kronecker symbol. If the flow is uniformly rotating in the \((xy)\)-plane for instance, we must have \( \sigma_{xy} = 0 \). Since \( v_x \propto y \) and \( v_y \propto -x \) in that case, that implies \( A = 1 \). Therefore the tensor \([\sigma]\) is symmetrical. We note that:

\[
Tr [\sigma] = (2 + 3B) \nabla \cdot \mathbf{v},
\]

which shows that \( Tr [\sigma] \) is a measure of the volume change of a fluid element. It is an experimental fact that the stresses which change the volume of a fluid element give different viscous forces than the stresses that preserve the volume. Therefore we rewrite \( \sigma_{ij} \) under the form:
\[\sigma_{ij} \propto \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v} \right) + b\delta_{ij} \nabla \cdot \mathbf{v},\]

where the term in bracket on the right hand side is trace free, i.e. does not modify the volume of a fluid element.

The shear and bulk viscosities, that we denote \(\eta\) and \(\zeta\) respectively, are then experimentally defined as:

\[\sigma_{ij} = \eta \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v} \right) + \zeta \delta_{ij} \nabla \cdot \mathbf{v}.\]  

(12)

The bulk viscosity is associated with internal degrees of freedom of the molecules in the fluid. It becomes negligible if the equipartition between these different degrees of freedom is reached over a timescale shorter than the timescale between two collisions. Furthermore, for a perfect monoatomic gas it can be shown that \(\zeta = 0\) (Huang 1987). Therefore, from now on we shall ignore it.

With this expression for \(\sigma_{ij}\), we can rewrite the \(i\) component of the momentum equation (9) under the form:

\[\frac{\partial (\rho v_i)}{\partial t} + \frac{\partial}{\partial x_j} (\rho v_i v_j + T_{ij}) = F_i,\]

(13)

where

\[T_{ij} = P\delta_{ij} - \sigma_{ij}.\]

Equation (13) makes is clear that the flux of the \(i\) component of the momentum in the \(j\) direction, \(\rho v_i v_j + T_{ij}\), has contributions from both advection and molecular transport (pressure and viscous forces).

For an incompressible fluid in which \(\eta\) is constant, the corresponding equation of motion, derived from equation (8) with the above expression of the stress tensor, is called the Navier-Stokes equation.

We note that since friction converts mechanical energy into heat, the rate of change of the kinetic energy, \(dE_k/dt\), must be negative if only viscous forces are acting. That can be shown from equation (13) to be equivalent to the condition \(\eta > 0\) (e.g., Landau & Lifchitz 1987, § II.16).

### 2.4 Expression of the shear viscosity

The expression (12) for the stress tensor need not be exact. It has been derived phenomenologically assuming that \(\sigma_{ij}\) depends only on a linear combination of the first derivatives of the velocity with respect to the coordinates. However, the kinetic theory applied to dilute gases leads to the same expression for \(\sigma_{ij}\). Here we are going to derive an approximate expression for the shear viscosity \(\eta\) by using the kinetic theory of gases, and we will point out that it arises from correlations between the particle velocities.
2.4.1 Kinetic theory of gases

For simplicity, let us assume that the velocity in cartesian coordinates is \( \mathbf{v} = v_x(y) \mathbf{e}_x \), where \( \mathbf{e}_x \) is a unit vector along the \( x \)-axis. The expression for the stress tensor then gives:

\[
\sigma_{xy} = \sigma_{yx} = \eta \frac{dv_x}{dy},
\]

(15)

This relation had already been proposed by Newton and Hooke, but it is Maxwell who gave the derivation of the viscosity \( \eta \) that we develop now.

In average, a molecule has a collision with another molecule after it travels through a distance \( \lambda \), which is the mean free path of the particles. We suppose that after the collision, the momentum of the molecule is the same as that of its new environment.

Let us consider the momentum which is transported during the time \( \delta t \) across a surface element \( \delta S \) perpendicular to the \( y \)-axis and with ordinate \( y \). There are \( nu \delta t \delta S/6 \) particles crossing \( \delta S \) from above during \( \delta t \), where \( n \) is the number density of particles, \( u \) is their random (thermal) velocity relative to the mean flow and the factor 6 comes about because there are three possible directions for the particles, each with two orientations. Each of these particles travel through \( \lambda \) before it suffers a collision below \( \delta S \), which results in its momentum varying by:

\[
m [v_x(y) - v_x(y + \lambda)] \simeq -m\lambda \frac{dv_x}{dy},
\]

to first order in \( \lambda/L \), where \( L \) is the scale of variation of the velocity. In other words, each particle carries below \( \delta S \) the excess of momentum \( m\lambda dv_x/\delta y \). Here \( m \) is the mass of a particle.

On the other hand, each particle traveling upward carries above \( \delta S \) the deficit of momentum \(-m\lambda dv_x/\delta y \). Therefore, the net \( x \)-component of the momentum which is carried downward during \( \delta t \) by the particles crossing \( \delta S \) is:

\[
\delta^2 p_x = 2 \left( \frac{1}{6} nu \delta t \delta S \right) \left( m\lambda \frac{dv_x}{dt} \right) = \frac{1}{3} nm\lambda \frac{dv_x}{dy} \delta S \delta t.
\]

(16)

Let us now consider a box with horizontal faces at \( y \) and \( y + \delta y \) and surface area \( \delta S \). The exchange of particles across the upper face during the time \( \delta t \) results in the momentum \( \delta^2 p_x(y + \delta y) \) being added to the volume, whereas the exchange across the lower surface results in the momentum \( \delta^2 p_x(y) \) being removed from the volume. Therefore, the time rate of change of the momentum content of the box is:

\[
\frac{1}{\delta t} \left[ \delta^2 p_x(y + \delta y) - \delta^2 p_x(y) \right] \simeq \frac{d}{dy} \left( \frac{1}{3} nm\lambda \frac{dv_x}{dy} \right) \delta S \delta y
\]

(17)

to first order in \( \lambda/L \). This is also \( F_{visc,x} \delta S \delta y \), where \( F_{visc,x} \) is the viscous force per unit volume. Therefore:

\[
F_{visc,x} \simeq \frac{d}{dy} \left( \frac{1}{3} nm\lambda \frac{dv_x}{dy} \right).
\]

(18)

Since \( F_{visc,x} = d\sigma_{xy}/dy \) and using equation (15), this leads to:
\[ \eta \simeq \frac{1}{3} \rho u \lambda, \]  
(19)

where \( \rho = nm \) is the mass density of the particles. We also define the kinematic viscosity \( \nu \equiv \eta/\rho \), i.e.

\[ \nu \simeq \frac{1}{3} u \lambda. \]  
(20)

The thermal velocity \( u \) is on the order of the sound speed.

We denote by \( R \) the Reynolds number, which measures the relative strength of the inertial and viscous forces. The inertial force is \( \sim \rho v^2/L \), where \( v \) is the flow velocity, and the viscous force is \( \sim \rho \nu v/L^2 \) (from eqs. [10] and [12]). Therefore,

\[ R = L v/\nu. \]  
(21)

### 2.4.2 Velocity correlation

Here again, for simplicity, we consider the case where the flow velocity is along the \( x \)–axis and the motion is in the \((xy)\)–plane. The components of the instantaneous velocity of a particle in the fluid are \((v_x + u_x, u_y)\), where \( u_x \) and \( u_y \) are the components of the random (thermal) velocity relative to the mean flow. We have \( < u_x >= < u_y >= 0 \), where the brackets denote a time average. The flux of the \( x \)–component of the momentum along the \( y \)–direction is:

\[ nm (v_x + u_x) u_y. \]

Averaged over a large number of particles, or, equivalently, over time, this gives:

\[ \rho \langle u_x u_y \rangle, \]

since \( < v_x u_y > = v_x < u_y >= 0 \). By construction, this quantity is a component of the stress tensor. Therefore,

\[ \sigma_{xy} = -\rho \langle u_x u_y \rangle, \]  
(22)

where the minus sign comes about because of the definition of the viscous stress tensor (see § 2.3). For a Maxwell–Boltzmann distribution function (or any \( xy \) symmetric function), \( < u_x u_y >= 0 \) and there is no transport. Equations (13) and (22) lead to:

\[ \langle u_x u_y \rangle = -\nu \frac{dv_x}{dy}. \]  
(23)

Note that in general the time average of the fluctuation velocity may not be zero. In an accretion disk for instance, there is a net radial drift of mass, i.e. the mean value of the radial component of the fluctuation velocity is finite. Here, if \( < u_y > \) were non zero for instance, the flux of the \( x \)–component of the momentum along the \( y \)–direction would have the extra contribution \( \rho v_x < u_y > \). This represents the transport of mean momentum by the fluctuations. Formally
however, what we call the stress tensor would still be given by equation (22), as this is the only contribution from friction between particles.

We denote by $C$ the correlation coefficient between the velocities $u_x$ and $u_y$:

$$C = \frac{\langle u_x u_y \rangle}{u^2},$$

where $u$ is the characteristic velocity of the particles in the fluid relative to the mean flow. We see from (22) that $C$ gives a measure of the stress tensor. If $c_s$ is the sound speed, then $u \sim c_s$. Therefore

$$C \sim \frac{\nu}{c_s^2} \frac{dv_x}{dy} \sim \frac{\lambda v_x}{L c_s} = \frac{\lambda}{L} M,$$

where we have used equation (20) and $dv_x/dy \sim v_x/L$. Here $M = v_x/c_s$ is the Mach number. Using equations (20) and (21), we can write $\mathcal{R} = ML/\lambda$, and therefore $C = M^2/\mathcal{R}$.

In an accretion disk, $M \sim 10^{-20}$ and the Reynolds number corresponding to the molecular viscosity is larger than $10^{14}$, so that $C \sim 10^{-12} \ll 1$!!

This is a consequence of the small value of the ratio of the mean free path to the scale of the mean flow ($\sim 10^{-12}$) and it means the state of the gas is not affected at all by the molecular transport of angular momentum. In other words, the flow and the molecular transport are completely decoupled.

### 3 Angular momentum transport by turbulence

The realization that molecular transport of angular momentum is so inefficient led the theorists to look for another mechanism of transport in accretion disks. Because Reynolds numbers are so high, it was thought that probably accretion disks would be subject to the same hydrodynamical nonlinear instabilities that shear flows experience in laboratory. The resulting turbulence would then transport angular momentum efficiently. Although today much doubt has been cast on hydrodynamical instabilities in disks, turbulence is still a strong candidate for transport since it has been shown relatively recently that a linear magnetohydrodynamical instability can develop in disks (see below). Therefore, we turn now to turbulent transport, and contrast it with molecular transport. Much of this section is based on Tennekes & Lumley (1972), which the reader is referred to for more details (see also Balbus & Hawley 1998).

We shall restrict ourselves here to the study of incompressible flows, as this simplifies the discussion. For our argument, it is sufficient to take into account only pressure and viscous forces, but in principle any other (external or inertial) force could be added. The equations describing the fluid are:

$$\frac{\partial \tilde{v}_i}{\partial t} + \tilde{v}_j \frac{\partial \tilde{v}_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \tilde{\sigma}_{ij},$$

and

$$\frac{\partial \tilde{v}_i}{\partial x_i} = 0,$$
where the tilde symbol above a variable means we consider the instantaneous value of the variable at the location \( x_i \) and time \( t \). Here

\[
\tilde{\sigma}_{ij} = -\tilde{p}\delta_{ij} + \eta \tilde{s}_{ij},
\]

with

\[
\tilde{s}_{ij} = \frac{\partial \tilde{v}_i}{\partial x_j} + \frac{\partial \tilde{v}_j}{\partial x_i}
\]

and \( \tilde{p} \) is the pressure. We suppose \( \eta \) is a constant.

We now use the so-called Reynolds decomposition, in which an instantaneous value is written as the sum of a mean value (denoted by a capital letter) plus a fluctuation (denoted by a small letter):

\[
\tilde{v}_i = V_i + v_i.
\]

This fluctuation is characteristic of the turbulence. This decomposition is meaningful only if the timescale on which the fluctuations vary and the evolution timescale of the flow are well separated. The mean values are then taken over a timescale large compared to the turbulence timescale but short compared to that of the flow evolution. As here we are not interested in the long term evolution of the flow, we neglect the derivative with respect to time of the mean values (i.e. we consider a quasi–steady state). To simplify the discussion, we suppose that the average of \( v_i \) over time is zero: \( < v_i > = 0 \).

Equation (25) averaged over time then leads to \( \partial V_i / \partial x_i = 0 \), i.e. the mean flow is incompressible. Equation (25) thus implies \( \partial v_i / \partial x_i = 0 \), i.e. the fluctuations are also incompressible.

Using

\[
\tilde{\sigma}_{ij} = \Sigma_{ij} + \sigma_{ij}
\]

with \( \langle \sigma_{ij} \rangle = 0 \), the equation of motion averaged over time gives:

\[
\frac{\partial}{\partial x_j} V_j V_i + \frac{\partial}{\partial x_j} \langle v_j v_i \rangle = \frac{1}{\rho} \frac{\partial}{\partial x_j} \Sigma_{ij},
\]

(26)

where we have used \( \partial V_i / \partial t = 0 \) and the incompressibility of the mean flow and the fluctuations. The term \( \langle v_j v_i \rangle \) represents the averaged transport of the fluctuations of the momentum by the fluctuations of the velocities. This is the turbulent transport. It is non zero only if the turbulent velocities in the different directions are correlated. It is an experimental fact that this is in general the case for shear turbulence (see below). Equation (26) shows that momentum is transferred between the fluctuations and the mean flow through the term \( \partial < v_j v_i > / \partial x_j \).

We can rewrite equation (26) under the form:

\[
\frac{\partial}{\partial x_j} V_j V_i = \frac{1}{\rho} \frac{\partial}{\partial x_j} (\Sigma_{ij} + \tau_{ij}),
\]

(27)

where \( \tau_{ij} = -\rho \langle v_j v_i \rangle \) is called the Reynolds stress tensor, or turbulent stress. This is the contribution from the fluctuations to the averaged total stress tensor \( \Sigma_{ij} + \tau_{ij} \). Note that if we had not supposed \( < v_i > = 0 \), we would also have had terms like \( V_j < v_i > \), representing transport (or advection) of mean momentum by the fluctuations. This is what Lynden–Bell & Kalnajs (1972) call lorry–transport, because it is a direct “shipment” by the equilibrium flow. Formally, this term is not part of what we call the turbulent stress however.
As we have no expression for the components of $\tau_{ij}$ (six of which are independent), the problem has more unknowns than equations. This is the well–known closure problem of turbulence. Since $\tau_{ij}$ appears in the same way as $\Sigma_{ij}$ in equation (27), it is tempting to express $\tau_{ij}$ by analogy with $\Sigma_{ij}$, which depends on the molecular motion as we have seen in the previous section. This is the basis for the mixing length theory, in which $\tau_{ij}$ is written exactly like the tensor deriving from molecular motions using a so–called turbulent viscosity $\nu_T$. By analogy with expression (20) for the molecular viscosity, it is supposed that $\nu_T \sim v_T \Lambda$, where $v_T$ is a characteristic velocity of the turbulent eddies and $\Lambda$ is the so–called mixing length, which is the “mean free path” of the eddies, i.e. the distance they travel through before they mix with their environment.

The same analogy is used in accretion disk theory through the $\alpha$ model that we shall describe in details below.

It is important to note that the basis for this analogy is very weak. For a thorough discussion of the differences between molecular and turbulent transport, we refer the reader to Tennekes & Lumley (1972). Here we recall the main points.

There is a fundamental difference between molecular and turbulent transport, in that turbulence is part of the flow whereas molecules are part of the fluid. We saw above that molecular transport is decoupled from the state of the flow. This is not true for turbulent transport, which depends completely on the flow for its existence. Turbulence can be sustained only if the eddies are able to tap the energy of the mean flow. Observations of laboratory shear flows for instance indicate that the eddies which are most efficient in tapping the energy of the mean flow are also most efficient in preserving a good correlation between the different components of the turbulent velocities, thus giving a large stress tensor. This is very different than for the molecules, whose velocities do not depend on non local properties of the mean flow.

And as a matter of fact, transport of momentum is observed to be a characteristic of turbulent shear flows. In the air for instance, the correlation coefficient between the velocities is on the order of $10^{-6}$ for the molecules and 0.4 for turbulent motions. This is a characteristic of most shear flows.

In addition, we saw when we derived the expression of the viscosity (19) that this requires $\lambda \ll L$. For a turbulent flow, that would mean that the scale of the turbulent eddies is very small compared to the scale over which the characteristics of the flow, like the mean velocity, vary. This is not necessarily satisfied in an accretion disk, where both may be on the order of the disk semi–thickness.

The conclusion is that shear turbulence is indeed a very efficient way of transporting momentum, but great care should be taken when expressing the resulting stress tensor by analogy with molecular transport.

In particular, while representing gross transport by an effective viscosity can often be useful, doing a detailed stability analysis on a viscous fluid model for turbulent flow is generally not self–consistent, and can be very misleading (e.g. Hawley, Balbus, & Stone 2001).

Note that the above discussion applies to shear turbulence only, i.e. flows where the fluctuations get their energy from the mean velocity gradients. Transport of momentum may not be nearly as efficient when the energy source for the fluctuations is a gradient of temperature or magnetic field for instance. As a matter of fact, there are strong indications that the transport of momentum associated with thermal convection is orders of magnitude weaker than that associated with shear turbulence (Balbus & Hawley 1998 and references therein).
Although turbulence has been considered as a way of transporting angular momentum in accretion disks for more than fifty years, it is only relatively recently that an instability able to extract the energy of the shear and put it in the fluctuations has been elucidated. This instability, which requires a magnetic field, is described in the next section.

4 Magnetohydrodynamic instabilities

We consider now the stability of a disk containing a magnetic field. Balbus & Hawley (1991, 1998 and references therein) have shown that in under very general conditions a magnetized disk is subject to a linear instability which nonlinear development was subsequently found to result in turbulence (Hawley, Gammie & Balbus 1995; Brandenburg et al. 1995). Following the presentation by Balbus & Hawley (1998), we first briefly describe the waves which propagate in a fluid containing a magnetic field (more details can be found in, e.g., the textbooks by Jackson 1975, Shu 1992 or Sturrock 1994), and we then go on to describe the linear instability.

4.1 Waves propagating in a magnetized fluid

Here we take into account only pressure and magnetic forces, as these are the only important forces for our argument. The governing equations are the induction equation:

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}),
\]  

(28)

which ensures that \( \nabla \cdot \mathbf{B} = 0 \), the equation of continuity:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,
\]  

(29)

and the equation of motion:

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P + \frac{1}{\mu_0 \rho} (\nabla \times \mathbf{B}) \times \mathbf{B},
\]  

(30)

where the last term on the right hand side is the Lorentz force per unit volume in SI units (\( \mu_0 \) is the permeability of vacuum). Note that it can also be written:

\[
\frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla \left( \frac{B^2}{2\mu_0} \right) + \left( \frac{\mathbf{B}}{\mu_0} \cdot \nabla \right) \mathbf{B}
\]  

(31)

which shows that the magnetic force if the sum of a magnetic pressure and a magnetic tension (first term and second term on the right hand side, respectively).

We first consider a nonrotating fluid and we look for small perturbations to an equilibrium state characterized by the quantities \( \rho_0, \mathbf{v}_0 = 0, P_0 \) and \( \mathbf{B}_0 \). We denote the Eulerian perturbations by a prime, so that \( \rho = \rho_0 + \rho' \) etc, and we suppose they vary on a scale much smaller than the scale of variation of the equilibrium quantities. The linearized equations (28), (29) and (30) are then:
\[ \frac{\partial B'}{\partial t} = \nabla \times (v' \times B_0), \quad (32) \]
\[ \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot v' = 0, \quad (33) \]
\[ \frac{\partial v'}{\partial t} = -\frac{1}{\rho_0} \nabla P' + \frac{1}{\mu_0 \rho_0} (\nabla \times B') \times B_0. \quad (34) \]

Since the coefficients of the above equations are constant, we perform a Fourier transform for the variables \( r \) and \( t \) and look for solutions under the form \( \exp[i(\omega t - k \cdot r)] \) with \( \omega \) and \( |k| \) real constants. The above equations can then be rewritten by replacing \( \partial / \partial t \) by \( i\omega \) and \( \nabla \) by \( -i k \).

To close the system of equations, we need an equation of state for the fluid. We consider adiabatic perturbations (i.e. we suppose there is no dissipation of heat), so that:

\[ \frac{P'}{P} = \gamma \frac{\rho'}{\rho}, \quad (35) \]

where \( \gamma \) is the ratio of specific heats (\( \gamma = 5/3 \) for a perfect gas).

The system of equations \((32)–(33)\) gives a dispersion relation of order six. It can be simplified by noting that the solutions associated with motions perpendicular to the \((k, B_0)\) plane and those associated with motions in this plane can be decoupled.

The first type of solutions is an Alfvén wave, and its dispersion relation is:

\[ \omega^2 = k^2 v_A^2 \cos^2 \psi, \quad (36) \]

where \( \psi \) is the angle between \( k \) and \( B_0 \) and

\[ v_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}} \]

is the Alfvén speed. These are transverse waves (with no motion in the \((k, B_0)\) plane) which propagate along the field lines. They do not involve any compression across the field lines and their restoring force is the magnetic tension. These waves are analogous to the waves which propagate along a stretched string. Within the factor \( \cos \psi \), we see that the Alfvén speed is the square root of the magnetic tension divided by a mass density, just like the phase speed in the case of a string.

The second type of solutions is associated with motions only in the \((k, B_0)\) plane. Its dispersion relation is:

\[ \omega^2 = \frac{k^2}{2} \left\{ (v_A^2 + c_s^2) \pm \left[ (v_A^2 + c_s^2)^2 - 4v_A^2 c_s^2 \cos^2 \psi \right]^{1/2} \right\}, \quad (37) \]

where \( c_s^2 = \gamma P_0 / \rho_0 \) is the adiabatic sound speed. The upper sign, which gives a larger phase speed \( \omega / k \), corresponds to the fast MHD wave, whereas the lower sign corresponds to the slow MHD wave. The fast mode is also referred to as magneto–acoustic wave. If either \( c_s \ll v_A \), \( v_A \ll c_s \) or \( \cos \psi \ll 1 \), then

\[ \omega_+^2 = k^2 \left( v_A^2 + c_s^2 \right) \]
and
\[ \omega_2 = \frac{k^2 v_A^2 c_s^2 \cos^2 \psi}{v_A^2 + c_s^2}, \]
where the subscripts + and - denote the fast and slow waves, respectively. For the fast mode, the magnetic and thermal pressures act in phase. If \( v_A \gg c_s \), the slow mode is an acoustic wave propagating along the field lines, whereas if \( v_A \ll c_s \) it degenerates into an Alfven mode in its dispersion properties (the eigenvector is distinct from that of the Alfven mode however, as the motions are not in the same plane). For the fast mode it is the opposite.

In the absence of rotation, these modes are stable. However, Balbus & Hawley have shown that when rotation is introduced, the slow mode can become unstable, and this what we describe now.

### 4.2 Linear magnetorotational instability

We consider the simplest system in which the instability can develop. This is an axisymmetric disk with a vertical uniform magnetic field. For the original presentation, which includes the case of a radial field, see Balbus & Hawley (1991), and for the stability of a toroidal field see Balbus & Hawley (1992b), Fogglioz & Tagger (1995), Terquem & Papaloizou (1996) and Ogilvie & Pringle (1996).

Since it is the slow mode which is destabilized, one can consider an incompressible fluid. That amounts to taking the limit \( c_s \to \infty \) in the above equations, so that the fast mode disappears (its frequency becomes infinite).

The system of equations describing the fluid is then
\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}),
\]
(38)
\[
\nabla \cdot \mathbf{v} = 0,
\]
(39)
\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P + \frac{1}{\mu_0 \rho} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla \Psi,
\]
(40)
where \( \Psi \) is the gravitational potential due to the central star.

We use the cylindrical coordinates \((r, \phi, z)\) and we denote the unit vectors in the three directions by \( \mathbf{e}_r \), \( \mathbf{e}_\phi \) and \( \mathbf{e}_z \). The components of the vector \((\mathbf{v} \cdot \nabla) \mathbf{v}\) which appears in the equation of motion are:

\[
v_r \frac{\partial v_r}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_r}{\partial \phi} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\phi^2}{r},
\]
\[
v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_\phi}{\partial \phi} + v_z \frac{\partial v_\phi}{\partial z} + \frac{v_r v_\phi}{r},
\]
\[
v_r \frac{\partial v_z}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_z}{\partial \phi} + v_z \frac{\partial v_z}{\partial z}.
\]

The equilibrium quantities, that we suppose uniform, are \( \rho_0, P_0 \), and \( \mathbf{B}_0 = B_0 \mathbf{e}_z \). The velocity is \( \mathbf{v}_0 = r \Omega(r) \mathbf{e}_\phi \). We now suppose that this equilibrium is slightly perturbed, so that \( \rho = \rho_0 + \rho' \)
etc, where the prime denotes the Eulerian perturbations, and we look for solutions proportional to \(i(\omega t - k_z z - k_r r)\). Here we consider axisymmetric perturbations, but more general solutions can be obtained (see the above mentioned papers).

The linearized system of equations is then:

\[
\begin{align*}
    i\omega B'_r &= -ik_z B'_r, \tag{41} \\
    i\omega B'_\phi &= -ik_z B'_\phi - ik_r r \Omega B'_z + \left(\frac{d(r \Omega)}{dr} - ik_r r \Omega\right) B'_r, \tag{42} \\
    i\omega B'_z &= \frac{B}{r} v'_r + ik_r B'_r, \tag{43} \\
    \left(\frac{1}{r} - ik_r\right) v'_r - ik v'_z &= 0, \tag{44} \\
    i\omega v'_r - 2\Omega v'_\phi &= \frac{ik_r P'}{\rho} - \frac{ik_z BB'_r}{\mu_0 \rho} + \frac{ik_z BB'_z}{\mu_0 \rho}, \tag{45} \\
    i\omega v'_\phi + \left(\frac{d(r \Omega)}{dr} + \Omega\right) v'_r &= -\frac{ik_z BB'_\phi}{\mu_0 \rho}, \tag{46} \\
    i\omega v'_z &= \frac{ik_z P'}{\rho}, \tag{47}
\end{align*}
\]

where, for simplicity, we have dropped the subscript '0' for the equilibrium quantities. We now consider large vertical wavenumbers, such that \(|k_z| \gg |k_r|\) and \(|k_z| \gg 1/r\). Then, the above system of equations leads to the following dispersion relation:

\[
\omega^4 - \omega^2 \left(2k^2 v_A^2 + \frac{d\Omega^2}{dr} + 4\Omega^2\right) + k^2 v_A^2 \left(k^2 v_A^2 + \frac{d\Omega^2}{dr}\right) = 0. \tag{48}
\]

There is instability if \(\omega\) has a negative imaginary part (the perturbation then grows exponentially with time). Equation (48) is a quartic for \(\omega^2\) which solutions are real. Therefore there is instability if \(\omega^2 < 0\), which requires:

\[
k^2 v_A^2 < -\frac{d\Omega^2}{dr}. \tag{49}
\]

This criterion has a very simple physical explanation. It states that there is an instability when the magnetic tension that acts on a segment of a field line is smaller than the net tidal force (i.e. centrifugal force minus gravitational force) acting on it.

For a given equilibrium field \(B\), and therefore a given Alfvén speed \(v_A\), there will always be a wavenumber \(k\) such that this inequality is satisfied provided the right hand side is positive. Therefore, all the disks with

\[
\frac{d\Omega^2}{dr} > 0 \tag{50}
\]

are unstable, and this is the criterion for instability.

The heart of the instability resides in the fact that a perturbed fluid element tends to conserve its angular velocity when a magnetic field is present. This is to be contrasted with a non magnetized
disk, in which a perturbed element tends to conserve its specific angular momentum. When displaced inward therefore, it has too much angular momentum for its new location (as the angular momentum increases outward in an accretion disk), and it moves back to its initial unperturbed position. When a magnetic field is present, the magnetic tension along the field line tends to enforce isorotation of the elements to which it is connected. A fluid element displaced inward has therefore a lower angular velocity than the elements at its new location, and thus not enough angular momentum for its new position. As a result it sinks further in. On the opposite, a fluid element connected to the same field line and displaced outward will tend to move further out. Angular momentum is transferred via magnetic torques from the inner fluid element to the outer fluid element. Note that the source of free energy for the instability is not in the magnetic field, but in the disk differential rotation. The magnetic field just provides a path to extract the energy.

From equation (48), we can write the negative values of $\omega^2$ as a function of $k^2 v_A^2$, and show that the maximum growth rate is:

$$|\omega_{\text{max}}| = \frac{1}{2} \left| \frac{d\Omega}{d\ln r} \right|. \quad (51)$$

For a Keplerian disk,

$$|\omega_{\text{max}}| = \frac{3}{4} \Omega \quad (52)$$

and is attained for

$$kv_A = \frac{\sqrt{15}}{4} \Omega. \quad (53)$$

This holds even if the field has radial and azimuthal components and if the perturbed quantities are allowed to vary with $r$ and $\phi$ provided we then replace $kv_A$ by $k \cdot v_A$. Note that the non-axisymmetric case is more subtle though, as in that case plane waves cannot be sustained, being sheared out by the differential rotation. If we write $k = k_r \mathbf{e}_r + (m/r) \mathbf{e}_\phi + k_z \mathbf{e}_z$, with $m$ being the azimuthal wavenumber, then $k_r$ is time dependent and

$$k_r(t) = k_0 - m \frac{d\Omega}{dr}$$

(Goldreich & Lynden–Bell 1965), which means that a disturbance always becomes trailing in a disk where the angular velocity decreases outward. If $k_r$ is initially large and positive (leading disturbance), then the mode is stable as indicated by (49). But as time goes on, $k_r$ decreases, so that $k$ enters a region of instability. As $k_r$ becomes negative and keeps decreasing though, the mode becomes stable again. Formally, the instability is therefore not purely exponential. The important question however is whether the mode can be amplified significantly before its wavelength becomes small enough to be affected by ohmic resistivity. This, of course, depends on the magnetic Reynolds number.

We have seen above that the so-called magnetorotational or Balbus–Hawley instability can develop in any disk in which the angular velocity decreases outward. In principle there is no other condition. However, it may be that the scale of the modes which are unstable according to (19) do not fit into the disk, i.e. they have a wavelength larger than the disk semithickness.
This is the case if $v_A/\Omega > H$. Since in a thin disk $\Omega H \sim c_s$ (see § 5.3.1), the disk will be stable if $v_A > c_s$.

Another condition which is implicit in the above presentation is, of course, that the magnetic field be coupled to the fluid. This may not be the case everywhere in protostellar disks, which are rather cold and dense (Gammie 1996; Fromang, Terquem & Balbus 2001).

Ohmic resistivity can prevent the development of the instability if the scale on which it acts is comparable to the wavelength of the unstable modes. The time it takes for the magnetic field to diffuse over a scale $\sim 1/k$ due to the effect of an Ohmic resistivity $\eta_B$ is $\sim 1/(\eta_B k^2)$. Since in the absence of resistivity it would grow on a timescale $\sim 1/\Omega$, we see that Ohmic dissipation prevents the instability if $\eta_B \sim \Omega/k^2$, i.e. if the magnetic Reynolds number is of order unity. For more details, the reader is referred to Balbus & Blaes (1994), Jin (1996) and Papaloizou & Terquem (1997). Note that Fleming, Stone & Hawley (2000) have shown that even though the linear instability is not affected at higher values of the magnetic Reynolds number, the nonlinear instability cannot be sustained for values as high as $10^4$ in some circumstances, which depend on the field topology for instance. However, the issue is not yet settled as other effects like Hall electromotive forces may resurrect the instability in regions where Ohmic resistivity acting alone would inhibit it (Wardle 1999, Balbus & Terquem 2001).

**4.3 Angular momentum transport in magnetized disks**

Numerical simulations have shown that this instability puts the energy it extracts from the disk differential rotation into fluctuations which transport angular momentum outward (see Balbus & Hawley 1998 and references therein). As noted above, transport of momentum is not a property of all types of turbulence. Furthermore, when it happens, it is not always against the velocity gradient, i.e. outward in an accretion disk. That has to be the case in an isolated dissipative system, but not if the energy driving the turbulence has an external source. We have already mentioned that the transport associated with thermal convection appears to be very weak compared to that associated with shear turbulence. In addition, there are strong indications that convection driven by external heating transports angular momentum inward in accretion disks (Ryu & Goodman 1992, Cabot 1996, Stone & Balbus 1996). In the case of the magnetorotational instability though, it can be shown that in the linear regime the transport is outward (Balbus & Hawley 1992a). This still holds in the nonlinear regime, as expected since the energy source for the turbulence is internal. Numerical simulations also show that most of the transport is done by the (magnetic) Maxwell stress, which dominates over the (hydrodynamic) Reynolds stress. Furthermore, magnetic fields are regenerated through a dynamo action so that the mechanism can sustain itself in an isolated system.

It is important to realize that this instability and the resulting turbulence do not tend to make the disk angular velocity uniform. This is because the angular velocity is imposed by the gravitational potential of the object around which the disk rotates. The turbulence can smear the shear out only on scales smaller than the disk semithickness $H$. To smear the shear out on a scale $l$, the turbulent velocity $v_t$ has indeed to be on the order of the shear over $l$, i.e. $l|\Omega|/dr| \sim l\Omega$, where $\Omega$ is the angular velocity in the disk. If $l \sim H$, then $v_t \sim c_s$, as in a thin disk $c_s \sim H\Omega$ (see § 5.3.1). Therefore, only supersonic turbulence could modify the shear on scales larger than $H$. However, such a turbulence could not be maintained as the supersonic fluctuations would dissipate into shocks. The state toward which the disk evolves as a result of the turbulence is not a state of uniform rotation but one where all the mass is at the center and all the angular momentum at infinity (Lynden–Bell & Pringle 1974).
Long before a mechanism for producing turbulence in accretion disks had been identified, Shakura & Sunyaev (1973) proposed a prescription for modeling turbulent disks. We now describe this prescription, and discuss its validity in the context of magnetic turbulence. We then develop disk models based on this so-called $\alpha$ prescription.

5 $\alpha$ disk models

5.1 Evolution of turbulent disks

Here we consider an axisymmetric disk rotating around a central object. We suppose the motion is in the plane of the disk, or, equivalently, we use the vertically averaged equations of mass conservation and motion. The velocity is $\mathbf{v} \equiv (u_r, r\Omega + u_\phi)$. The term $r\Omega$ is the circular velocity around the central mass, and $(u_r, u_\phi)$ are the components of the fluctuation velocity. Note that as the disk accretes, there is a net radial drift and the mean value of $u_r$ is non-zero. Here, the equation of mass conservation (2) and the azimuthal component of the equation of motion (8) are:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r\Sigma v_r) = 0,$$  

(54)

$$\Sigma \left( \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_r v_\phi}{r} \right) = 0,$$  

(55)

where $\Sigma$ is the surface mass density, i.e. the mass density integrated over the disk thickness. Here, according to the discussion in §2.4, we have neglected the viscous force arising from molecular viscosity. Multiplying equation (55) by $r$ and using (54), we obtain the angular momentum equation:

$$\frac{\partial}{\partial r} (r\Sigma (r\Omega + u_\phi)) + \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \Sigma (r\Omega + u_\phi) u_r \right) = 0.$$  

(56)

As pointed out by Balbus & Papaloizou (1999), to get a diffusion equation describing the disk evolution we need to smooth out the fluctuations over radius. To do so, we average the above equation over a scale large compared to that of the fluctuations, but small compared to the characteristic scale of the flow (i.e. $r$). Equation (54) then yields:

$$\frac{\partial}{\partial t} \left( \Sigma r^2 \Omega \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( \Sigma r^3 \Omega < u_r > + \Sigma r^2 < u_r u_\phi > \right) = 0,$$  

(57)

where the brackets denote the radial average and we have neglected $< u_\phi >$ compared to $r\Omega$ in the time derivative. This is justified because $| < u_\phi > | \ll r\Omega$ and the systematic, evolutionary time derivative of $< u_\phi >$ is limited. In the radial divergence term however, both $< u_r >$ and $< u_r u_\phi >$ are second order, and therefore we retain all the terms. This equation is the same as that describing a viscous flow with $\mathbf{v} \equiv (u_r, r\Omega)$ and a stress tensor $\sigma_{r\phi} \equiv -\Sigma < u_r u_\phi >$ as given by equation (22).

There are two contributions to the flux of angular momentum: the term $\Sigma r^2 \Omega < u_r >$ is the mean angular momentum advected through the disk by the velocity fluctuations (because of
the accretion of mass), whereas the term \( \Sigma r < u_r u_\phi > \) represents the angular momentum fluctuations transported by the velocity fluctuations.

Using equation (54) averaged over radius, we can rewrite equation (57) under the form:

\[
 r \Sigma < u_r > = - \left[ \frac{d}{dr} \left( r^2 \Omega \right) \right]^{-1} \frac{\partial}{\partial r} \left( \Sigma r^2 < u_r u_\phi > \right).
\]

Using equation (54) again to eliminate \( < u_r > \), this leads to the diffusion equation:

\[
 \frac{\partial \Sigma}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( \left[ \frac{d}{dr} \left( r^2 \Omega \right) \right]^{-1} \frac{\partial}{\partial r} \left( \Sigma r^2 < u_r u_\phi > \right) \right).
\]

In a steady state, equation (54) gives \( r \Sigma < u_r > \) constant. Then the accretion rate

\[
 \dot{M} \equiv -2\pi r \Sigma < u_r >
\]

is constant through the disk. Integration of the angular momentum equation (57) then yields:

\[
 \Sigma < u_r u_\phi > = \frac{\dot{M}}{2\pi \Omega} \left[ 1 - \left( \frac{R_i}{r} \right)^{1/2} \right],
\]

where \( R_i \) is the disk inner boundary. We have assumed here that the turbulent stress \( < u_r u_\phi > \) vanishes at the disk inner edge (i.e. there is no torque at the boundary) and that the disk is Keplerian, so that \( \Omega \propto r^{-3/2} \). The above equation shows that for the mass to be accreted inward (i.e. \( < u_r > \) negative), the flux of angular momentum due to the fluctuations has to be positive, i.e. the fluctuations have to transport angular momentum outward.

### 5.2 The α prescription

We pointed out that the angular momentum equation (57) is analogous to that describing a viscous flow with \( \mathbf{v} \equiv (u_r, r \Omega) \) and a stress tensor \( \sigma_{r\phi} \equiv -\Sigma < u_r u_\phi > \). Therefore, it is tempting to push the analogy further and express the turbulent stress \( -\Sigma < u_r u_\phi > \) as if it derived from an enhanced 'turbulent viscosity' \( \nu \), defined such that (equation [12]):

\[
 -\Sigma < u_r u_\phi > \equiv \Sigma \nu r \frac{d\Omega}{dr}.
\]

In a Keplerian disk, this gives:

\[
 < u_r u_\phi > \sim \nu \Omega.
\]

The equations presented in the previous section have been derived by Lynden–Bell & Pringle (1974; see also Pringle 1981) assuming a viscous flow with this expression of the stress tensor. The prescription proposed by Shakura & Sunyaev (1973) consists in writing \( \nu = v_t H \), where \( H \) is the disk thickness, assumed to be the maximum scale of the turbulent cells, and \( v_t \) is the turbulent velocity. They further define

\[
 \alpha \equiv \frac{v_t}{c_s},
\]
where $c_s$ is the sound velocity. Note that $\alpha < 1$, otherwise the fluctuations would dissipate into shocks in such a way as to restore $v_t < c_s$. Equation (62) can then be rewritten under the form:

$$< u_r u_\phi > \sim \alpha c_s^2.$$ 

(63)

Here we have used the fact that in a thin disk $H \sim c_s/\Omega$ (see § 5.3.1).

So far we have focussed on non magnetized disks. In these, there are strong indications that $< u_r u_\phi >$ is either zero or negative. Magnetism is needed to correlate the velocities. The above discussion does apply to magnetized disks provided we replace $< u_r u_\phi >$ by $< u_r u_\phi - u_{Ar} u_{A\phi} >$, where $(u_{Ar}, u_{A\phi})$ are the components of the fluctuations of the Alfven velocity (Shakura & Sunyaev 1973, Balbus & Hawley 1998). The extra term represents the Maxwell stress.

The validity of the $\alpha$ prescription in the context of magnetic turbulence was discussed by Balbus & Papaloizou (1999). They first pointed out that, as long as $< u_r u_\phi >$ (or $< u_r u_\phi - u_{Ar} u_{A\phi} >$) is positive, the disk dynamics is the same as if it were evolving under the action of a viscosity. In that case indeed, the diffusion coefficient in equation (58) is positive. We can then always define an $\alpha$ parameter according to (63), although $\alpha$ may not be constant through the disk.

Note however that this implicitly assumes it is possible to average the equations over radius in the way described in § 5.1. If the scale of the fluctuations and the characteristic disk scale are not well separated, such an average cannot be performed. Since the maximum scale of the fluctuations is of order the disk thickness $H$, this procedure requires that there is a scale large compared to $H$ and small compared to $r$. This condition may be only marginally fulfilled in protostellar disks, in which $H$ may be up to $0.1-0.2r$.

Balbus & Papaloizou (1999) further noted that for the $\alpha$ prescription to apply, the disk had to behave viscously not only in its dynamics but also in its energetics. The key point here is that a viscous disk dissipates locally the energy it extracts from the shear, whether in a steady state or not. This may not be the case in a turbulent disk where, if the turbulent cascade is not efficient, part of the energy may be advected with the flow. As we have not addressed the energetics of viscous disks above, we will not go into the details of the discussion here. These can be found in Balbus & Papaloizou (1999), who showed that in disks subject to MHD turbulence the energy extracted from the shear is indeed dissipated locally (through the turbulent cascade) whether the disk is evolving or not. Note that this is in general not the case when the turbulence is due to self–gravitating instabilities. In that case indeed, part of the energy is transported by waves through the disk.

Most theoretical protostellar disk models have relied on the $\alpha$ parametrization. We have commented that MHD turbulence does lend itself to this prescription in thin disks. However, because MHD instabilities develop only in an adequately ionized fluid, they may not operate everywhere in protostellar disks (Gammie 1996). Therefore it is likely that the parameter $\alpha$ is not constant through these disks. It may even be that only parts of these disks can be described using this $\alpha$ prescription. However, we may still learn about disks from these models in the same way as we learned about stars from simple polytropic models. Therefore, we now describe these models.

5.3 Vertical structure

In a thin disk, the radial gradients of temperature and pressure are much smaller than the vertical gradients. The vertical and radial structures therefore decouple, and can be calculated
separately. Here we present the calculation of the vertical structure, i.e. we determine pressure, temperature and density as a function of height $z$. Once this is done, given an initial surface density distribution, the radial structure can be computed using the diffusion equation (58). The section below is based on Papaloizou & Terquem (1999), which the reader is referred to for more details.

### 5.3.1 Basic equations

The equations describing the disk vertical structure are the equation of vertical hydrostatic equilibrium:

$$\frac{1}{\rho} \frac{\partial P}{\partial z} = -\frac{GM_z z}{(r^2 + z^2)^{3/2}}, \quad (64)$$

and the energy equation, which states that the rate of energy removal by radiation is locally balanced by the rate of energy production by viscous dissipation:

$$\frac{\partial F}{\partial z} = \frac{9}{4} \rho \nu \Omega^2, \quad (65)$$

where $F$ is the radiative flux of energy through a surface of constant $z$ which is given by:

$$F = -\frac{16 \sigma T^3}{3 \kappa \rho} \frac{\partial T}{\partial z}. \quad (66)$$

Here $G$ is the gravitational constant, $M_*$ is the central mass, $P$ is the pressure, $T$ is the temperature, $\kappa$ is the opacity, which in general depends on both $\rho$ and $T$, and $\sigma$ is the Stefan–Boltzmann constant.

To close the system of equations, we relate $P$, $\rho$ and $T$ through the equation of state of an ideal gas:

$$P = \frac{\rho k T}{\mu m_H}, \quad (67)$$

where $k$ is the Boltzmann constant, $\mu$ is the mean molecular weight and $m_H$ is the mass of the hydrogen atom. If we limit our calculations to temperatures lower than 4,000 K (typical of protostellar disks), then, at the densities of interest, hydrogen is in molecular form. Since the main component of protostellar disks is hydrogen, we then have $\mu = 2$. We note that for the temperatures we consider transport of energy by convection can be neglected.

We denote the isothermal sound speed by $c_s$ ($c_s^2 = P/\rho$). We adopt the $\alpha$–parametrization of Shakura & Sunyaev (1973), so that the kinematic viscosity is written $\nu = \alpha c_s^2 / \Omega$ (equations [52] and [63]). In general, $\alpha$ is a function of both $r$ and $z$. With this formalism, equation (65) becomes:

$$\frac{\partial F}{\partial z} = \frac{9}{4} \alpha \Omega P. \quad (68)$$

Note that, in a thin disk, $z \ll r$. Then, if we set $|\partial P/\partial z| \sim P/H$ and $z \sim H$ in equation (54), we get $c_s \sim \Omega H$, where $\Omega$ is the Keplerian velocity ($\Omega^2 = GM_*/r^3$) and we have used $c_s^2 = P/\rho.$
The thin disk approximation then requires $c_s \ll r \Omega$, i.e. the angular velocity to be highly supersonic.

### 5.3.2 Boundary conditions

We have to solve three first order ordinary differential equations for the three variables $\mathcal{F}$, $P$ (or equivalently $\rho$), and $T$ as a function of $z$ at a given radius $r$. Accordingly, we need three boundary conditions at each $r$. We shall denote with a subscript $s$ values at the disk surface.

The first boundary condition is obtained by integrating equation (65) over $z$ between $-H$ and $H$. Since by symmetry $\mathcal{F}(z = 0) = 0$, this gives:

$$\mathcal{F}_s = \frac{3}{8\pi} \dot{M}_s \Omega^2,$$

where we have defined $\dot{M}_s \equiv 3\pi \langle \nu \rangle \Sigma$, with $\langle \nu \rangle$ being the vertically averaged viscosity. If the disk were in a steady state, $\dot{M}_s$ would not vary with $r$ and would be the constant accretion rate through the disk (eq. [60] and [62] with $R_i \ll r$). In general however, this quantity does depend on $r$.

Another boundary condition is obtained by integrating equation (64) over $z$ between $H$ and infinity. A detailed derivation of this condition is presented in the Appendix A of Papaloizou & Terquem (1999). Here we just give the result:

$$P_s = \frac{\Omega^2 H \tau_{ab}}{\kappa_s},$$

where $\tau_{ab}$ is the optical depth above the disk. This condition is familiar in stellar structure, where $\Omega^2 H$ would be replaced by the acceleration of gravity at the stellar surface. Since we have defined the disk surface such that the atmosphere above the disk is isothermal, we have to take $\tau_{ab} \ll 1$. Providing this is satisfied, the results do not depend on the value of $\tau_{ab}$ we choose.

A third and final boundary condition is given by the expression of the surface temperature (see Appendix A of Papaloizou & Terquem 1999 for a detailed derivation of this expression):

$$2\sigma \left(T_s^4 - T_b^4\right) - \frac{9\alpha k T_s \Omega}{8\mu m_H \kappa_s} - \frac{3}{8\pi} \dot{M}_s \Omega^2 = 0.$$  

Here the disk is assumed immersed in a medium with background temperature $T_b$. The surface opacity $\kappa_s$ in general depends on both $T_s$ and $\rho_s$ and we have used $c_s^2 = kT_s/(\mu m_H)$. The boundary condition (71) is the same as that used by Levermore & Pomraning (1981) in the Eddington approximation (their eq. [56] with $\gamma = 1/2$). In the simple case when $T_b = 0$ and the surface dissipation term involving $\alpha$ is set to zero, with $\dot{M}_s$ being retained, it simply relates the disk surface temperature to the emergent radiation flux.

### 5.3.3 Disk models

At a given radius $r$ and for a given value of the parameters $\dot{M}_s$ and $\alpha$ (which uniquely determine the disk model), the disk vertical structure is obtained by solving equations (64), (66) and (68) with the boundary conditions (69), (70) and (71).
Numerical results are published, e.g., by Papaloizou & Terquem (1999) (see also Lin & Papaloizou 1980, 1985; Bell & Lin 1994; Bell et al. 1997), in which the opacity is taken from Bell & Lin (1994). This has contributions from dust grains, molecules, atoms and ions. It is written in the form $\kappa = \kappa_i \rho^a T^b$ where $\kappa_i$, $a$ and $b$ vary with temperature. For a specified $\dot{M}_{st}$, they calculate the value of $H$, the vertical height of the disk surface, iteratively. Starting from an estimated value of $H$, after satisfying the surface boundary conditions, the equations are integrated down to the mid–plane $z = 0$. The condition that $\mathcal{F} = 0$ at $z = 0$ is not in general satisfied. An iteration procedure is then used to adjust value of $H$ until $\mathcal{F} = 0$ at $z = 0$ to a specified accuracy.

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