The spectrum of the Heisenberg ferromagnet from geometric considerations

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The Heisenberg model of quantum magnetism accurately models many physical systems. The Heisenberg model also has a Hamiltonian that involves only two-body interactions, and allows quantum information to be stored in its ground space with the possibility of quantum error correction. In this context, the spectral gap of a Heisenberg model quantifies the robustness of the quantum data stored therein. Most prior work on Heisenberg models focus on systems with low dimensions. Here, we study the spectrum of Heisenberg models with varying geometries. To this end, we use the connection between Heisenberg models and graph Laplacians. We exactly solve Heisenberg models where every pair of spins interact equally. We also bound the spectral gaps of Heisenberg ferromagnets either with long-range interactions or with infinite dimensions. Moreover, we bound the largest and some of the smallest energy eigenvalues for Heisenberg ferromagnets with varying geometries. Our findings suggest that a wide range of magnetic phenomena can be understood using graph theory.

The Heisenberg model (HM) is a quantum theory of magnetism [1], and is prevalent in many naturally occurring physical systems such as in various cuprates [2,3], in solid Helium-3 [4], and more generally in systems with interacting electrons [5]. The HM can also be engineered in ultracold atomic gases [6] and quantum dots [7]. Apart from describing actual physical systems, the HM is also especially suited for storing quantum information. This is because by storing quantum information in the symmetric subspace of the underlying spins, the quantum data necessarily resides in a decoherence-free-subspace [8] of any spin-half HM, and is immune from error. In fact, by using spin-half Heisenberg ferromagnets, quantum information stored in the symmetric subspace of the underlying spins necessarily resides in the system’s ground space [9]. In practice, there are often small unpredictable perturbations to the system Hamiltonian, and necessitates using quantum error correction [10]. Fortunately, there exists the possibility of quantum error correction within the symmetric subspace by using permutation-invariant codes [9,11–13]. Moreover, if the spectral gap of a Heisenberg ferromagnet can be made to grow with the system size, then excitations from the ground space ought to be suppressed and thereby enable robust storage of quantum information. In this sense, the Heisenberg model could allow us to sidestep the need to realize complex four-way interactions to store quantum information in Kitaev’s physical model [14,15]. Given the widespread applicability of magnetic material in classical information processing [16,17], quantum magnets based on the HM could similarly enable quantum technologies. In addition, the HM also can be used for quantum computation [18], quantum simulation, and has connections with other fields of physics such as string theory [19–23]. All this points to the importance of understanding the Heisenberg model.

The HM has been extensively studied since its introduction in 1929. Most studies focus on one-dimensional [2,24–30] and two-dimensional HMs [3,31,33], usually relying on the Bethe ansatz [34]. Recently, the validity of the spin-wave approximation has been studied with respect to Heisenberg lattices of arbitrary dimensions [35], with lower bounds on the free energy per site of HMs evaluated. However, while lower bounds on the free energy of the Heisenberg ferromagnet can arise from lower bounds on the spectrum of the Heisenberg ferromagnet, the converse does not hold. As such, bounds on the spectrum of HMs with arbitrary dimensions is as yet unavailable. This gap impedes a complete understanding of spontaneous magnetization [36] and the physics of spin liquids [37,38].

In this paper, we bound the energy eigenvalues of Heisenberg models with a wide range of geometries of the model’s underlying interactions when the Heisenberg Hamiltonian (HH) is ferromagnetic. In a nutshell, we exactly solve the mean-field HH where every exchange constant is identical, and we provide new bounds on the energy eigenvalues of other HHs from the geometric considerations of the HM’s underlying interactions. To the best of our knowledge, this is the first time graph-theoretic methods are used to obtain bounds on the eigenvalues of the HH. Our analysis relies mainly on the fact that a ferromagnetic HH can be written as a sum of graph Laplacians. The graphs of each of these Laplacians are in turn symmetric products of the underlying graph of interactions.

We proceed to sketch our findings with a little more detail. First, we exactly solve the spectral problem for the mean-field Heisenberg ferromagnet. In this case, the symmetric product of the graphs are precisely the Johnson graphs. These Johnson graphs are well-studied in the field of association schemes, and hence their graph spectra is well understood. Translating the language of association schemes into physics, we obtain the spectrum of the mean-field HM. Second, we bound the spectral gaps of (i) $D$-dimensional Heisenberg ferromagnets with long range interactions and (ii) certain infinite dimensional HMs. Here, the spectral gap of a Hamiltonian is the energy difference between its ground state and first excited state. The bounds in (i) arise mainly because the HM with long-range interactions dominates a mean-field Heisenberg model with sufficiently small exchange constant. To show (ii), we bound the expansion constants of the symmetric product of infinite dimension graphs based on the graph’s expansion constant [39]. Third, we bound (iii) some of the smallest eigenvalues and (iv) the largest eigenvalue of the HH when the Heisenberg ferromagnet has an arbitrary geometry. These...
bounds depend on the geometric properties of the graph of underlying interactions. The lower bounds in (iii) rely on lower bounds on the eigenvalues of graph Laplacians from the geometric properties of the graph. The upper bounds in (iii) are based on the generalized diameters of the symmetric products of the underlying graph of interactions. These generalized diameters can be thought of as the widths of multidimensional bodies. Bounds in (iv) arise from bounds on the largest eigenvalue of Laplacians and the geometry of the underlying graph of interactions.

Since we investigate the spectrum of HHs with graphs of varying dimensions, we need to explain what these graphs and their dimensions are. Here, a graph comprises of vertices from 1 to n which label the particles, and edges \{i,j\} which label the interaction between particles i and j. A graph’s dimension generalizes from the dimension of continuous manifolds. The edge-boundary of any set of vertices X denoted by \partial X is the set of edges in G with exactly one vertex in X. Suppose that every set X with k vertices in G satisfies the bound |\partial X| \geq ck^{1-1/\delta} for some positive constant c for every k \leq n/2. Then we say that G has a dimension of \delta with isoperimetric number c. This is analogous to the situation where a manifold with fixed volume k and a surface area of at least ck^{1-1/\delta} for some positive constant c has a dimension of \delta. The dimension of a physical system is then the dimension of the corresponding graph of interactions.

The central object in this paper is the HH. It is the mathematical embodiment of the HM’s energy level structure, and contains all information necessary to derive every property of the HM. More precisely, the HH for spin-half particles is a matrix given by

$$\hat{H} = -\sum_{\{i,j\}} J_{i,j} \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z - \frac{1}{2},$$

(1)

where \sigma_i^x, \sigma_i^y and \sigma_i^z as the usual Pauli matrices acting on the i-th particle, the sets \{i,j\} are included in the sum whenever particles i and j interact, and J_{i,j} is an exchange constant which quantifies the strength and nature of the coupling between the particles. We write the Hamiltonian in this way because it is convenient to set the ground state energy to be zero. We restrict our attention to ferromagnetic HHs, where every exchange constant is non-negative. We also assume the absence of an external magnetic field.

Finding the spectrum of the HH is akin to finding the resonance frequencies for its associated graph and its symmetric products. This is because of a one-to-one correspondence between ferromagnetic HHs and the symmetric product of their underlying graphs of interactions; every ferromagnetic HH with graph \(G\) can be written as sum of pairwise orthogonal matrices, with each matrix associated with the symmetric products of \(G\) \cite{2}. The k-th symmetric product of a graph \(G\) with vertices V and edges E denoted by \(G^{(k)}\) is a graph with the following properties. First, \(G^{(k)}\) has as its vertices all possible subsets of V of size k. Second, the edges of \(G^{(k)}\) are the sets \{X,Y\} where (i) X and Y are subsets of V with k vertices, (ii) X and Y have k−1 common elements, and (iii) their symmetric difference, the union of the sets without their intersection, is an edge in E. In short, \{X,Y\} is an edge in \(G^{(k)}\) only if the symmetric difference of X and Y is an edge in E, i.e. \(X \Delta Y \subseteq E\). Then, the HH on n spins with every non-zero exchange constant equal to 1, which we call the normalized Hamiltonian, can also be expressed as

$$\hat{H}_1 = L_0 + \cdots + L_n.$$  

(2)

This normalized Hamiltonian is just a sum of pairwise orthogonal matrices \(L_k\) \cite{40}. Here, each \(L_k\) is the Laplacian of the graph \(G^{(k)}\) and has rank \(\binom{n}{k}\). If we interpret \(G^{(k)}\) as a discrete manifold, the eigenvectors and eigenvalues of \(L_k\) are its normal modes and associated resonance frequencies.

The HH’s representation using Eq. (2) is crucial precisely because of its connections with Laplacians in graph theory \cite{41}. By denoting \(X\) as a state with the spins labeled by \(X\) in the up state and the remaining spins in the down state, where \(X\) is a subset of vertices in \(G\), the Laplacians of \(G^{(k)}\) are

$$L_k = \sum_{|X|=k} |\partial X\rangle \langle X| = \sum_{X \Delta Y \subseteq E} (|X\rangle \langle Y| + |Y\rangle \langle X|).$$  

(3)

When the graph \(G\) is connected, each \(L_k\) has exactly one eigenvalue equal to zero with eigenvector equal to the all ones vector on its support \(\mathbb{1}\). Hence the ground state energy of \(\hat{H}_1\) is zero with degeneracy \(n+1\), and the ground space is spanned by the Dicke states \(|D_n^0\rangle\) \cite{9}, where \(|D_n^0\rangle\) is a normalized superposition of all \(|X\rangle\) for which \(X\) is a subset of \(\{1,\ldots,n\}\) of size \(k\). We emphasize that the Laplacians \(L_k\) and \(L_{n-k}\) are unitarily equivalent. To see this, denote \(\overline{X}\) as the set complement of \(X \subseteq V\), and note that \(L_k = U_k L_{n-k} U_j^\dagger\) where \(U_k = \sum_{|X|: |X|=k} |X\rangle \langle X|\). Hence it suffices to only study Laplacians \(L_k\) for which \(k \leq \frac{n}{2}\).

We begin with an exact solution for a mean-field HM. Such a HM has \(n\) spins, and every pair of spin interacts with exactly the same exchange constant \(J\). In this case, the normalized Hamiltonian is \(\hat{H}_1 = -\sum_{i<j} J_{i,j} \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z - \frac{1}{2}.\) Moreover, the graph of interactions is precisely the complete graph on \(n\) vertices. The complete graph has infinite dimensions, and its symmetric products are the Johnson graphs for which

![Graph](image-url)
TABLE I: Table of eigenvalues of the mean-field HH when $J = 1$.

| $n$ | Eigenvalues | Multiplicities |
|-----|-------------|----------------|
| 2   | 0, 2        | 3, 1           |
| 3   | 0, 3        | 4, 4           |
| 4   | 0, 4, 6     | 5, 9, 2        |
| 5   | 0, 5, 8     | 6, 16, 10      |
| 6   | 0, 6, 10, 12 | 7, 25, 27, 5   |
| 7   | 0, 7, 12, 15 | 8, 36, 56, 28  |
| 8   | 0, 8, 14, 18, 20 | 9, 49, 100, 84, 14 |
| 9   | 0, 9, 16, 21, 24 | 10, 64, 162, 192, 84 |
| 10  | 0, 10, 18, 24, 28, 30 | 11, 81, 245, 375, 270, 42 |

FIG. 2: Here, we depict the spectrum of a mean-field Heisenberg model with 40 spins when $J = 1$.

The spectral problem has been exactly solved using association schemes \cite{42,43}. In particular, $L_k$ has exactly one eigenvalue equal to zero, and its other eigenvalues are $j(n + 1 - j)$ with multiplicities $m_j = \binom{n}{j} - \binom{n}{j-1}$ for $j = 1, \ldots, k$ \cite{44} Section 12.3.2]. Hence the positive eigenvalues of $\hat{H}_1$ are

$$j(n + 1 - j)$$

with multiplicities

$$(n + 1 - 2j)m_j,$$

where $j = 1, \ldots, \lfloor n/2 \rfloor$. Thus, the mean-field Heisenberg ferromagnet’s spectral gap grows linearly with the number of spins in the system. Properties of the Johnson scheme given in Ref. \cite{45} imply that the Laplacians $L_k$ have the spectral decomposition

$$L_k = \sum_{j=1}^{k} j(n + 1 - j)P_{k,j}$$

where $P_{k,j} = \frac{1}{\binom{n}{j}} \sum_{z=0}^{n-1} h_{k,j}(z) A_{k,z}$ are pairwise orthogonal projectors, $A_{k,z} = \sum_{x,y \subseteq \{1, \ldots, n\}} |X\rangle\langle Y|$ is the $z$-th generalized adjacency matrix of the Johnson graph associated with $L_k$ relating $k$-sets a distance of $z$ apart, and $h_{k,j}(z)$ is the Hahn polynomial given by $h_{k,j}(z) = m_j \sum_{a=0}^{z} (-1)^{a} \binom{\binom{n}{j} - 1}{a} \binom{\binom{n}{j}}{a} \binom{z}{a}$.

We now study HMs with long-range interactions, where every pair of spins separated by distance $r$ interacts with exchange constant at least $J_r^{-\delta}$. To unravel this HH’s spectrum, we utilize the spectrum of the mean-field HH with an inequality from matrix analysis. For any Hermitian matrix $A$ of size $n$, let $\lambda_j(A)$ be its eigenvalues where $\lambda_0(A)$ is its smallest eigenvalue of $A$ and $\lambda_{n-1}(A) \geq \cdots \geq \lambda_1(A) \geq \lambda_0(A)$. By Weyl’s monotonicity theorem, $\lambda_j(A + B) \geq \lambda_j(A)$ whenever $A$ and $B$ are positive semidefinite matrices with the smallest eigenvalue of $B$ equal to zero. Now let $\hat{H}_I$ denote the normalized Hamiltonian of the mean-field HM, and maximize $J_0$ for which the matrix $B$ in the decomposition $\hat{H} = J_0 \hat{H}_I + B$ is positive-semidefinite. Then $B$ will have smallest eigenvalue equal to zero and $\lambda_j(\hat{H}) \geq J_0 \lambda_j(\hat{H}_I)$. In particular, the $J_0$ that optimizes this bound is the smallest exchange constant in the long-range interaction model. We can apply this inequality to the HM has a geometry of a $D$-dimensional lattice with $n$ spins arranged in a grid with lattice spacing $a$. In this case, the largest separation of the spins is $n^{1/D}a\sqrt{D}$ and hence every exchange constant is at least $J_0 = J(n^{1/D}a\sqrt{D})^{-\alpha}$. The spectral gap of such a Heisenberg ferromagnet thus satisfies the bound

$$g \geq J_0 n = J(a\sqrt{D})^{-\alpha} n^{1-\alpha/D}.$$  

Similarly for general $D$-dimensional systems where pairs of spins are at most a distance of $cn^{1/D}$ apart for some positive constant $c$, the spectral gap similarly satisfies the bound $g \geq Jc^{-\alpha} n^{1-\alpha/D}$. Hence if the dimension $D$ of the system is larger than the exponent $\alpha$, the spectral gap of such HMs grows with the number of the system’s spins.

We now turn our attention to ferromagnetic HMs with $n$ spins and infinite dimensions. In particular, we obtain lower bounds on the spectral gap of the normalized Hamiltonian $\hat{H}_I$ when $n \geq 4$ and the graph’s isoperimetric number $c$ is at least $n(\frac{1}{2} + \varepsilon)$ for some positive $\varepsilon$. Note that because some pairs of spins need not interact, such a HH is no longer strongly interacting, and the spectral gap of $\hat{H}_I$ cannot be obtained from that of the mean-field HH. Instead, we appeal to the bound $\lambda_1(L) \geq \beta - \sqrt{\beta^2 - \theta^2} \geq \frac{\theta}{2\beta}$ \cite{66,67}. This bound applies to infinite dimensional graphs with at least four vertices and with isoperimetric number, maximum vertex degree and given by $\theta$, $\beta$ and Laplacian $L$ respectively. Now, the isoperimetric number of $G(k)$ is at least $n(\frac{k-1}{n-k+1})$ for $1 \leq k \leq n/2$ \cite{59}, and the maximum vertex degree of $G(k)$ is trivially at most $k(n-k)$. Combining these inequalities, we find that $\lambda_1(L_k) \geq \frac{n^2}{2(n-k)}(\frac{k-1}{n-k+1})^2$. Thus the spectral gap of $\hat{H}_I$ is at least $3\varepsilon^2$ which does not depend on $n$.

We obtain bounds on the largest eigenvalue of ferromagnetic HMs with graphs having dimension $\delta$, isoperimetric number $c$, and maximum vertex degrees $\beta$. Note that obtaining bounds on $\hat{A} = h_{\beta-1}(\hat{H}_I)$, the largest eigenvalue of the normalized HH $\hat{H}_I$, amounts to obtaining bounds on the largest eigenvalues of $L_k$. Now the largest eigenvalue of the Laplacian of any graph is at least its maximum vertex degree \cite{49} and at most twice its maximum vertex degree \cite{49,50}. Thus the largest eigenvalue of $L_k$ is at least $ck^{1-1/\delta}$ and at
most \(2k\beta\) for \(1 \leq k \leq n/2\). Hence
\[
c[n/2]^{1-1/\delta} \leq \lambda \leq n\beta. 
\] (8)

We now obtain lower bounds on the smaller eigenvalues of the HH. For this, we use lower bounds on the smaller eigenvalues of graph Laplacians. These bounds depend only on the geometric properties of graphs. Namely, if a graph with \(m\) edges, dimension \(\delta > 2\) and isoperimetric number \(c\) has as its minimum and maximum vertex degrees \(b\) and \(\beta\) respectively, then its Laplacian \(L\) has eigenvalues that satisfy the bound
\[
\lambda_j(L) \geq \frac{bc^2}{16c^2(b^2 - 2)(b - 1)} \left(1 + \frac{i}{6m}\right)^{2/\delta}. 
\] (9)

The proof relies mainly on lower bounds on the eigenvalues of normalized Laplacians through Sobolev inequalities on graphs. We subsequently use matrix inequalities relating the eigenvalues of a graph’s Laplacian and normalized Laplacian.

To obtain lower bounds on the eigenvalues of HH, it remains to derive the geometric properties of \(G^{[k]}\) from the geometric properties of \(G\) to obtain bounds on the eigenvalues of \(L_k\). The minimum and maximum vertex degrees of \(G^{[k]}\) which we denote by \(b_k\) and \(\beta_k\) respectively clearly depend on the dimensionality of \(G\). This is because \(b_k \geq c_k^{1-1/\delta}\) and obviously \(\beta_k \leq k\beta\). Moreover it has been shown that if deleting any \(k-1\) vertices from \(G\) yields a vertex-induced subgraph with isoperimetric dimension and number \(\delta_k\) and \(c_k/k\) respectively, then \(G^{[k]}\) has dimension and isoperimetric number \(\delta_k\) and \(c_k\) respectively \([39]\). Hence a computer with memory polynomial in the number of spins \(n\) can compute lower bounds on the eigenvalues of \(H\). Only bounds on the isoperimetric properties of every vertex-induced subgraph of \(G\) with up to half of its vertices deleted needs to be used. The potentially exponential number of ways to pick these subgraphs, and the exponential number of ways to pick subsets of vertices from these subgraphs leads to a run-time that is exponential in \(n\). In contrast, computing the eigenvalues of \(L_k\) directly requires a computer with memory and run-time both exponential in \(n\). Alternatively, one can directly compute approximations of the few smallest eigenvalues of \(L_k\) in nearly-linear time in \(n^k\) \([51]\), which also requires exponential time and memory for \(k\) linear in \(n\).

We now obtain upper bounds on the smaller eigenvalues of the HH. For this, we use diameter of graphs and their generalizations. The diameter of a graph is the length of its shortest path. The \(j\)-diameter of a graph generalizes the length of the shortest path between \((j + 1)\) vertices, maximized over all choices of \((j + 1)\) vertices. Intuitively, the \(j\)-diameter of a body, is its width when it is interpreted to have \(j\) dimensions. Upper bounds on the eigenvalues of \(L_k\) can be obtained \(d_{j,k}\), the \(j\)-diameter of \(G^{[k]}\). Namely, whenever \(d_{j,k} \geq 2\), then
\[
\lambda_j(L_k) \leq \frac{1}{k} \left(1 - 2/\left(1 + \frac{n}{k}\right)^{1/(d_{j,k}-1)}\right), 
\] (10)

where \(\lambda_k\) is the largest eigenvalue of \(L_k\). Since the \(j\)-diameter of \(G^{[k]}\) may be unwieldy to calculate, we outline a polynomial time algorithm to obtain lower bounds on it. By definition, the \(j\)-diameter of \(G^{[k]}\) is at least the length of the shortest path in the graph \(G^{[k]}\) between any distinct subsets \(X_1, \ldots, X_{j+1}\) each with \(k\) vertices from \(V\). Evaluating the length of this shortest path requires evaluating all possible pairwise distances amongst the sets \(X_1, \ldots, X_{j+1}\), of which there are \(\binom{j+1}{2}\) choices. The distance between \(X = \{x_1, \ldots, x_k\}\) and \(Y = \{y_1, \ldots, y_k\}\) with respect to \(G^{[k]}\) is in turn the sum of the distances with respect to \(G\) between \(x_j\) and \(y_{\pi(j)}\) minimized over all permutations \(\pi\) that permute \(k\) symbols. Note that there are \(k!\) possible permutations and \(k\) distances to sum for each instance. Hence, evaluating \(k(k!\)\) distances of paths in the graph \(G\) suffices to compute the length between \(X\) and \(Y\) in the graph \(G^{[k]}\). Now Dijkstra’s algorithm computes the distance between any pair of vertices in \(G\) in \(O(n^2)\) time \([52]\). Hence a time complexity of \(O(n^2k(k!)^2)\) is required given every random selection of \(X_j\)’s to yield a lower bound for \(d_{j,k}\). Thus for constant \(k\), upper bounds on the eigenvalues of the ferromagnetic HH can be computed in quadratic time in \(jn\). Such an algorithm would outperform a direct solver for Laplacians \([51]\) whenever \(k \geq 3\).

In this paper, we obtain many bounds on the spectrum of the ferromagnetic HHs. For this, we rely on tools from graph theory and matrix analysis. The relevance of these bounds go beyond that of the HM. In particular, the Mott insulating phase of the Hubbard model is a Heisenberg antiferromagnet \([53]\), and describes transitions between conducting and insulating systems. This leads to widespread interest in Heisenberg antiferromagnets. Since the Hamiltonians of these antiferromagnets differ from the ferromagnets only by a sign, our results for the eigenvalues and eigenprojectors of the Heisenberg ferromagnet can also be translated to Heisenberg antiferromagnets. Moreover, with these bounds on the eigenvalues of the Heisenberg ferromagnet and antiferromagnet, bounds on thermodynamic quantities of the corresponding Heisenberg models such as free energy can be evaluated. Looking ahead, we expect that advances in theory of the symmetric product of graphs will give better bounds for the spectrum of the Heisenberg ferromagnet. Other than the spin-\(\frac{1}{2}\) Heisenberg ferromagnet, we anticipate that graph-theoretic techniques will also apply in other physical systems.

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Appendix A: Decomposition of the HH as a sum of Laplacians

Here we prove the decomposition. The first step is to notice that the swap operator two qubits can be written as

\[
(\ket{0}\bra{0})(\ket{0}\bra{0}) + (\ket{0}\bra{1})(\ket{1}\bra{0}) + (\ket{1}\bra{0})(\ket{0}\bra{1}) + (\ket{1}\bra{1})(\ket{1}\bra{1}),
\]

and is identical to the sum

\[
\frac{\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z + 1}{2},
\]

Then denoting the operator that swaps qubits \(i\) and \(j\) as \(\pi_{i,j}\), we have the identity

\[
\pi_{i,j} - 1 = \frac{\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z - 1}{2}.
\]

This allows us to rewrite the normalized HH with a graph \(G = (V, E)\) in terms of swap operators, so that

\[
\hat{H}_1 = \sum_{\{i,j\} \in E} (1 - \pi_{i,j}).
\]

Next, we let \(X\) denote any subset of vertices \(V = \{1, \ldots, n\}\). Then for any distinct \(i\) and \(j\) from the set \(V\), we have

\[
\pi_{i,j}|X\rangle = \begin{cases} 
|X\rangle & , i, j \in X \\
|X\rangle & , i, j \notin X \\
|X\triangle \{i, j\}\rangle & , \{i, j\} \in \partial X 
\end{cases}.
\]

This allows us to obtain

\[
\sum_{\{i,j\} \in E} \pi_{i,j}|X\rangle = \sum_{\{i,j\} \in \partial X} \pi_{i,j}|X\rangle + \sum_{\{i,j\} \notin \partial X} \pi_{i,j}|X\rangle
\]

\[
= \sum_{\{i,j\} \in \partial X} |X\triangle \{i, j\}\rangle + \sum_{\{i,j\} \notin \partial X} |X\rangle
\]

\[
= \sum_{\{i,j\} \in \partial X} |X\triangle \{i, j\}\rangle + (m - |\partial X|)|X\rangle,
\]

where \(m\) denotes the number of edges in \(E\). Hence

\[
\hat{H}_1|X\rangle = \sum_{\{i,j\} \in E} |X\rangle - \sum_{\{i,j\} \in E} \pi_{i,j}|X\rangle
\]

\[
= |\partial X||X\rangle - \sum_{\{i,j\} \in \partial X} |X\triangle \{i, j\}\rangle.
\]

Clearly if \(Y\) is a subset of \(V\) that has a different size from \(X\), then \(\langle Y|\hat{H}_1|X\rangle = 0\). This immediately implies that \(\hat{H}_1\) can be written as a sum of orthogonal matrices, each of them supported on the space spanned by \(|X\rangle\) where \(X\) have constant size. Next, note that \(\langle X|\hat{H}_1|X\rangle = |\partial X|\), which implies that the diagonal entries of \(L_x\) are given by the sizes of the corresponding edge-boundaries of \(k\)-sets. Finally, note that if \(Y\) has the same size as \(X\), then \(\langle Y|\hat{H}_1|X\rangle = 0\) whenever \(X\triangle Y \in E\) and \(\langle Y|\hat{H}_1|X\rangle = 0\) whenever \(X\triangle Y \notin E\). This proves that decomposition given in Eq. \(A4\) and Eq. \(A5\).
Appendix B: Lower bounds on the eigenvalues of a Laplacian

In this section, we prove the inequality shown in Eq. (9). We begin by reviewing the definitions of the Laplacian and the normalized Laplacian of a graph. For our purposes, the graphs we consider are connected, i.e., there is a spanning tree that covers all the vertices in the graph.

Let $D_G = \sum_{v \in V} d_v |v|$ denote the degree matrix of a graph $G = (V, E)$. Let $A_G$ denote the adjacency matrix of a graph, which means that it is a matrix with matrix elements equal to either 0 or 1, and where $\langle u | A_G | v \rangle = 1$ iff the vertex $u$ is adjacent to $v$. Let $L_G$ denote the Laplacian of a graph, which can be written as $D_G - A_G$. When a graph is connected, its degree matrix is non-singular, and we can write its normalized Laplacian of $G$ as

$$\tilde{L}_G = D_G^{-1/2}L_GD_G^{-1/2} \quad (B1)$$

We can obtain bounds on the eigenvalues of the normalized Laplacian in terms of the eigenvalues of the Laplacian.

**Lemma 1.** Namely if the graph has minimum and maximum vertex degrees given by $b$ and $\beta$ respectively,

$$b \lambda_j(\tilde{L}_G) \leq \lambda_j(L_G) \leq \beta \lambda_j(\tilde{L}_G). \quad (B2)$$

**Proof.** Denoting the $i$-th largest singular value of a matrix $A$ of size $d_a$ as $s_i(A)$ with $s_1(A) \geq \cdots \geq s_{d_a}(A)$, we have from Ref [54, Problem III.6.5] the inequalities

$$s_i(AB) \leq s_i(A)s_1(B), \quad s_i(AB) \leq s_1(A)s_i(B). \quad (B3)$$

Applying the above inequalities iteratively, it follows that

$$s_i(\tilde{L}_G) = s_i(D_G^{-1/2}L_GD_G^{-1/2}) \leq s_i(D_G^{-1/2}L_G)s_1(D_G^{-1/2}) \leq s_1(D_G^{-1/2})s_1(L_G)s_1(D_G^{-1/2}) = s_i(L_G)s_1(D_G^{-1/2}). \quad (B4)$$

Similarly, we have

$$s_i(L_G) = s_i(D_G^{1/2}\tilde{L}_GD_G^{1/2}) \leq s_i(D_G^{1/2}L_G)s_1(D_G^{1/2}) \leq s_1(D_G^{1/2})s_1(\tilde{L}_G)s_1(D_G^{1/2}) = s_i(\tilde{L}_G)s_1(D_G). \quad (B5)$$

Since the matrices $D_G, D_G^{-1}, L_G$ and $\tilde{L}_G$ are positive semidefinite matrix, their singular values are equivalent to its eigenvalues. The largest eigenvalue of $D_G$ and $D_G^{-1}$ are $\beta$ and $b^{-1}$ respectively. Hence the inequalities $(B4)$ and $(B5)$ then give the result. \hfill \Box

We now review the connection between the isoperimetric properties of a graph and the eigenvalues of its normalized Laplacian. For a graph $G = (V, E)$, denote the volume of a subset of vertices $X$ as $vol(X) = \sum_{v \in X} d_v$. The isoperimetric inequality we focus on is

$$|\partial X| \geq c_\delta (vol(X))^{1-1/\delta} \quad (B6)$$

where $vol(X) \leq vol(\bar{X})$.

The Sobolev inequality on graphs has the form

$$\sum_{[u,v] \in E} |f(u) - f(v)|^2 \geq A \min_m \left( \sum_{v \in V} |f(v) - m|^2 d_v \right)^{2/\alpha}. \quad (B7)$$

Typically $A$ depends on $c_\delta$ and $\delta$. Chung and Yau proved when the above Sobolev inequality holds for a graph, the eigenvalues of the graph’s normalized Laplacians satisfy the lower bound

$$\lambda_j(\tilde{L}) \geq \frac{A}{e^{3/\delta}} (j/vol(G))^{2/\delta}. \quad (B8)$$

Now Ostrovskii’s Sobolev inequality has for $\delta > 2$, we get

$$A = \frac{c_\delta^2}{16} \left( \frac{\delta - 2}{\delta - 1} \right)^2. \quad (B9)$$

Hence

$$\lambda_j(\tilde{L}_G) \geq \frac{c_\delta^2}{16} \left( \frac{\delta - 2}{\delta - 1} \right)^2 \left( \frac{j}{vol(G)} \right)^{2/\delta}. \quad (B10)$$
Combining this with Lemma 1 we get

\[ \lambda_j(L_G) \geq b \frac{c_\delta^2}{e^{4/\delta}} \left( \frac{\delta - 2}{\delta - 1} \right)^2 \left( \frac{j}{\text{vol}(G)} \right)^{2/\delta}. \]  

(B11)

It remains to relate \( c_\delta \) to \( c \). Note that if a graph \( G \) has \( \delta \) and \( a \beta \) as its isoperimetric dimension and number respectively, then the vertex subsets \( X \) of \( V \) satisfy the bound

\[ |\partial X| \geq a \beta \min\{|X|,|V|-|X|\}^{1-1/\delta} \geq a \min\{\text{vol}_X, \text{vol}_{V\setminus X}\}^{1-1/\delta}. \]  

(B12)

Hence we can take \( c_\delta = c/\beta \), where \( \beta \) is the maximum vertex degree of the graph. This yields

\[ \lambda_j(L_G) \geq \frac{bc^2}{16e^2} \left( \frac{\delta - 2}{\delta - 1} \right)^2 \left( \frac{j}{6m} \right)^{2/\delta}, \]  

(B13)

where \( m \) is the number of edges in the graph.
FIG. 3: \( G^{(3)} \), the symmetric cube of the graph \( G \) depicted in Figure 1, is shown here.