THE GROTHENDIECK RING OF VARIETIES IS NOT A DOMAIN

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1. The Grothendieck ring of varieties

Let $k$ be a field. By a $k$-variety we mean a geometrically reduced, separated scheme of finite type over $k$. Let $\mathcal{V}_k$ denote the category of $k$-varieties. Let $K_0(\mathcal{V}_k)$ denote the free abelian group generated by the isomorphism classes of $k$-varieties, modulo all relations of the form $[X - Y] = [X] - [Y]$ where $Y$ is a closed $k$-subvariety of a $k$-variety $X$. Here, and from now on, $[X]$ denotes the class of $X$ in $K_0(\mathcal{V}_k)$. The operation $[X] \cdot [Y] := [X \times_k Y]$ is well-defined, and makes $K_0(\mathcal{V}_k)$ a commutative ring with 1. It is known as the Grothendieck ring of $k$-varieties. A completed localization of $K_0(\mathcal{V}_k)$ is needed for the theory of motivic integration, which has many applications: see [Loo00] for a survey.

Our main result is the following.

Theorem 1. Suppose that $k$ is a field of characteristic zero. Then $K_0(\mathcal{V}_k)$ is not a domain.

Remark. We conjecture that the result holds also for fields $k$ of characteristic $p$. But we use a result whose proof relies on resolution of singularities and weak factorization of birational maps, which are known only in characteristic zero.

2. Abelian varieties of $GL_2$-type

If $A$ is an abelian variety over a field $k_0$, and $k$ is a field extension of $k_0$, then $\text{End}_k(A)$ denotes the endomorphism ring of the base extension $A_k := A \times_{k_0} k$, that is, the ring of endomorphisms defined over $k$.

Lemma 2. Let $k$ be a field of characteristic zero, and let $\overline{k}$ denote an algebraic closure. There exists an abelian variety $A$ over $k$ such that $\text{End}_k(A) = \text{End}_{\overline{k}}(A) \cong \mathcal{O}$, where $\mathcal{O}$ is the ring of integers of a number field of class number 2.

Let us precede the proof of Lemma 2 with a few paragraphs of motivation. Our strategy will be to find a single abelian variety $A$ over $\mathbb{Q}$ such that the base extension $A_k$ works over $k$.

Suppose that $A$ is a nonzero abelian variety over $\mathbb{Q}$. Let $\text{Lie} A$ be its Lie algebra, which is a $\mathbb{Q}$-vector space of dimension $\dim A$. If $\text{End}_\mathbb{Q}(A)$ is an order in a number field $F$, then the $\mathcal{O}$-action makes $\text{Lie} A$ a vector space over $\mathcal{O} \otimes \mathbb{Q} = F$; hence $[F : \mathbb{Q}] \leq \dim A \text{Lie} A = \dim A$. If moreover equality holds, then $A$ is said to be of $GL_2$-type. (The terminology is due to the following: If $A$ is of $GL_2$-type, then the action of the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on a Tate module $T_\ell A$ can be viewed as a representation $\rho_\ell : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell)$.)

Date: April 15, 2002.

1991 Mathematics Subject Classification. Primary 14A10; Secondary 14G35.

Key words and phrases. Grothendieck ring of varieties, modular abelian variety, stable birational equivalence, Albanese variety.

This research was supported by NSF grant DMS-9801104, and a Packard Fellowship.
Because \( \mathbb{Q} \) has class number 1, we must take \([F : \mathbb{Q}] \geq 2\) to find an \( A \) over \( \mathbb{Q} \) as in Lemma \( \ref{lemma:extension} \). The inequality \([F : \mathbb{Q}] \leq \dim A\) then forces \( \dim A \geq 2\). Moreover, if we want \( \dim A = 2\), then \( A \) must be of \( \text{GL}_2\)-type.

Abelian varieties of \( \text{GL}_2\)-type are closely connected to modular forms. For each \( N \geq 1 \), let \( \Gamma_1(N) \) denote the classical modular group, let \( X_1(N) \) denote the corresponding modular curve over \( \mathbb{Q} \), and let \( J_1(N) \) be the Jacobian of \( X_1(N) \). G. Shimura \cite{Shimura1971}, Theorem 7.14, attached to each weight-2 newform \( f \) on \( \Gamma_1(N) \) an abelian variety quotient \( A_f \) of \( J_1(N) \). It is known that \( \dim A_f = [F : \mathbb{Q}] \), where \( F \) is the number field generated over \( \mathbb{Q} \) by the Fourier coefficients of \( f \). These coefficients can also be identified with the endomorphisms of \( A_f \) induced by the Hecke correspondences on \( X_1(N) \); hence \( A_f \) is \( \text{GL}_2\)-type. Conversely, it is conjectured that each abelian variety of \( \text{GL}_2\)-type is \( \mathbb{Q}\)-isogenous to some \( A_f \). See \cite{Ribet1980} for more details. The \( \dim A = 1 \) case of this conjecture is the statement that elliptic curves over \( \mathbb{Q} \) are modular, which is known \cite{B成功}.  

Therefore we are led to consider \( A_f \) of dimension 2, where \( f \) is a newform as above.

**Proof of Lemma \( \ref{lemma:inclusion} \)** Tables \cite{Stein} show that there exists a weight-2 newform \( f = \sum_{n=1}^{\infty} a_n q^n \) on \( \Gamma_0(276) \) (hence also on \( \Gamma_1(276) \)) such that \( \mathbb{Q}(\{a_n : n \geq 1\}) = \mathbb{Q}(\sqrt{10}), \ a_{17} = 4 - \sqrt{10} \), and \( a_{19} = 2 + \sqrt{10} \). Let \( A = A_f \) be the corresponding abelian variety over \( \mathbb{Q} \). Then \( \dim A = [\mathbb{Q}(\sqrt{10}) : \mathbb{Q}] = 2 \). Also, \( \text{End}_\mathbb{Q}(A) \) is an order of \( \mathbb{Q}(\sqrt{10}) \) containing \( 4 - \sqrt{10} \), so \( \text{End}_\mathbb{Q}(A) \) is the maximal order \( \mathbb{Z}[\sqrt{10}] \) of \( \mathbb{Q}(\sqrt{10}) \). The class number of \( \mathbb{Q}(\sqrt{10}) \) is 2.

It remains to show that \( \text{End}_k(A) = \mathbb{Z}[\sqrt{10}] \) for any field extension \( k \) of \( \mathbb{Q} \). For any place of \( k \) at which \( A \) has good reduction, \( \text{End}_k(A) \) injects into the endomorphism ring of the reduction. We will use this to bound \( \text{End}_k(A) \). The abelian variety \( A \) has good reduction at all primes not dividing 276, so in particular it has good reduction at 17 and 19. Let \( A_{17} \) and \( A_{19} \) denote the resulting abelian varieties over \( \mathbb{F}_{17} \) and \( \mathbb{F}_{19} \). The places 17 and 19 of \( \mathbb{Q} \) extend to places of \( k \) taking values in \( \mathbb{F}_{17} \) and \( \mathbb{F}_{19} \). Thus \( \text{End}_k(A) \) injects into \( \text{End}_{\mathbb{F}_{17}}(A_{17}) \) and \( \text{End}_{\mathbb{F}_{19}}(A_{19}) \).

By the work of Eichler and Shimura (see Theorem 4 in D. Rohrlich’s article in \cite{CSS97}), the characteristic polynomial \( P_{17}(x) \) of Frobenius on \( A_{17} \) equals

\[
N_{\mathbb{Q}(\sqrt{10})/\mathbb{Q}}(x^2 - a_{17}x + 17) = x^4 - 8x^3 + 40x^2 - 136x + 289.
\]

This is irreducible over \( \mathbb{Q} \), and its middle coefficient is prime to 17, so \( A_{17} \) is a simple ordinary abelian surface. Checking the criterion in \cite{Hida2002} (see especially Theorem 6 and the last paragraph of Section 2), we find that \( \text{End}_{\mathbb{F}_{17}}(A_{17}) \otimes \mathbb{Q} \simeq \mathbb{Q}[x]/(P_{17}(x)) \). Similarly, \( \text{End}_{\mathbb{F}_{19}}(A_{19}) \otimes \mathbb{Q} \simeq \mathbb{Q}[x]/(P_{19}(x)) \). The ratio of the discriminants of \( P_{17}(x) \) and \( P_{19}(x) \) is not a square in \( \mathbb{Q} \), so \( \mathbb{Q}[x]/(P_{17}(x)) \) and \( \mathbb{Q}[x]/(P_{19}(x)) \) are distinct number fields of degree 4. But \( \text{End}_k(A) \otimes \mathbb{Q} \) embeds into both, so \( \dim_{\mathbb{Q}}(\text{End}_k(A) \otimes \mathbb{Q}) \leq 2 \). On the other hand, \( \mathbb{Z}[\sqrt{10}] \subseteq \text{End}_k(A) \), so \( \text{End}_k(A) = \mathbb{Z}[\sqrt{10}] \).

**Remark.** The case \( k = \mathbb{C} \) of Lemma \( \ref{lemma:complex} \) has an easy proof: let \( A \) be an elliptic curve over \( \mathbb{C} \) with complex multiplication by \( \mathbb{Z}[\sqrt{-5}] \).

## 3. Abelian varieties and projective modules

Let \( A \) be an abelian variety over a field \( k \), and let \( \mathcal{O} = \text{End}_k(A) \). Given a finite-rank projective right \( \mathcal{O} \)-module \( M \), we define an abelian variety \( M \otimes_\mathcal{O} A \) as follows: choose a finite presentation \( \mathcal{O}^m \to \mathcal{O}^n \to M \to 0 \), and let \( M \otimes_\mathcal{O} A \) be the cokernel of the homomorphism
\( A^m \rightarrow A^n \) defined by the matrix that gives \( \mathcal{O}^m \rightarrow \mathcal{O}^n \). It is straightforward to check that this is independent of the presentation, and that \( M \mapsto (M \otimes_{\mathcal{O}} A) \) defines a fully faithful functor \( T \) from the category of finite-rank projective right \( \mathcal{O} \)-modules to the category of abelian varieties over \( k \). (Essentially the same construction is discussed in the appendix by J.-P. Serre in [Lau01].)

**Lemma 3.** Let \( k \) be a field of characteristic zero. There exist abelian varieties \( A \) and \( B \) over \( k \) such that \( A \times A \simeq B \times B \) but \( A_k \ncong B_k \).

**Proof.** Let \( A \) and \( \mathcal{O} \) be as in Lemma 2. Let \( I \) be a nonprincipal ideal of \( \mathcal{O} \). Since \( \mathcal{O} \) is a Dedekind domain, the isomorphism type of a direct sum of fractional ideals \( I_1 \oplus \ldots \oplus I_n \) is determined exactly by the nonnegative integer \( n \) and the product of the classes of the \( I_i \) in the class group \( \text{Pic}(\mathcal{O}) \). Since \( \text{Pic}(\mathcal{O}) \simeq \mathbb{Z}/2 \), we have \( \mathcal{O} \oplus \mathcal{O} \simeq I \oplus I \) as \( \mathcal{O} \)-modules. Applying the functor \( T \) yields \( A \times A \simeq B \times B \), where \( B := I \otimes_{\mathcal{O}} A \). Since \( \text{End}_k(A) \) also equals \( \mathcal{O} \), we have \( B_k \ncong I_k \otimes_{\mathcal{O}} A_k \). Since \( T \) for \( k \) is fully faithful, \( A_k \ncong B_k \). \( \square \)

4. Rings related to the Grothendieck ring of varieties

For any extension of fields \( k \subseteq k' \), there is a ring homomorphism \( K_0(V_k) \rightarrow K_0(V_{k'}) \) mapping \([X]\) to \([X_{k'}]\).

Let \( k \) be a field of characteristic zero. Smooth, projective, geometrically integral \( k \)-varieties \( X \) and \( Y \) are called *stably birational* if \( X \times \mathbb{P}^m \) is birational to \( Y \times \mathbb{P}^n \) for some integers \( m, n \geq 0 \). The set \( \text{SB}_k \) of equivalence classes of this relation is a monoid under product of varieties over \( k \). Let \( \mathbb{Z}[\text{SB}_k] \) denote the corresponding monoid ring.

When \( k = \mathbb{C} \), there is a unique ring homomorphism \( K_0(V_k) \rightarrow \mathbb{Z}[\text{SB}_k] \) mapping the class of any smooth projective integral variety to its stable birational class [LL01]. (In fact, this homomorphism is surjective, and its kernel is the ideal generated by \( L := [A^1] \).) The proof in [LL01] requires resolution of singularities and weak factorization of birational maps [AKMW00, Theorem 0.1.1], [W01, Conjecture 0.0.1]. The same proof works over any algebraically closed field of characteristic zero.

The set \( \text{AV}_k \) of isomorphism classes of abelian varieties over \( k \) is a monoid. The Albanese functor mapping a smooth, projective, geometrically integral variety to its Albanese variety induces a homomorphism of monoids \( \text{SB}_k \rightarrow \text{AV}_k \), since the Albanese variety is a birational invariant, since formation of the Albanese variety commutes with products, and since the Albanese variety of \( \mathbb{P}^n \) is trivial. Therefore we obtain a ring homomorphism \( \mathbb{Z}[\text{SB}_k] \rightarrow \mathbb{Z}[\text{AV}_k] \).

5. Zerodivisors

**Proof of Theorem 2.** Let \( A \) and \( B \) be as in Lemma 3. Then \( ([A] + [B])([A] - [B]) = 0 \) in \( K_0(V_k) \). On the other hand, \([A] + [B]\) and \([A] - [B]\) are nonzero, because their images under the composition

\[
K_0(V_k) \rightarrow K_0(V_{k'}) \rightarrow \mathbb{Z}[\text{SB}_{k'}] \rightarrow \mathbb{Z}[\text{AV}_{k'}]
\]

are nonzero. (The Albanese variety of an abelian variety is itself.) \( \square \)

**Acknowledgements**

I thank Eduard Looijenga and Arthur Ogus for discussions. The package GP-PARI was used to perform the calculations in the last paragraph of the proof of Lemma 3.
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