We consider one-loop light-by-light-scattering contributions to the Lamb shift of the $1s, 2s, 2p$ states in light muonic hydrogen like atoms at $Z \leq 10$. The contributions are of the order $\alpha^5m_\mu$ (with diverse dependence on the nuclear charge $Z$). Those include the contributions of the so-called Wichmann-Kroll potential ($\alpha(Z\alpha)^4m_\mu$), the virtual Delbrück scattering ($\alpha^2(Z\alpha)^3m_\mu$), etc. The results are obtained in a nonrelativistic approximation. For the calculation of the virtual-Delbrück-scattering contribution, we have constructed an effective potential in the coordinate space which may be applied to other calculations in muonic atoms.

I. INTRODUCTION

Muonic atoms give an opportunity to develop and test a bound-state QED theory and probe a nuclear structure with a specific range of parameters not available with ordinary [electronic] atoms. Recently the accuracy of the measurement of the $2s - 2p$ Lamb shift in some light hydrogen like muonic atoms has been dramatically improved [1, 2]. The QED theory of the energy levels in muonic atoms is somewhat different from that in ordinary atoms. The Bohr radius in muonic atoms is comparable with the Compton wave length of an electron. Because of that, an important role is played by the diagrams with the closed electron loops. Those contributions are specific for muonic atoms. The most important are those due to vacuum polarization. Their contribution to the energy is of the order $\alpha(Z\alpha)^2m$.

Effects of the virtual light-by-light scattering contribute to higher orders. There are three types of such contributions, characteristic diagrams which are presented in Fig. 1. They are all of the order $\alpha^2m_\mu$, but their dependence on the value of the nuclear $Z$ charge is different.

The $\alpha(Z\alpha)^4m$ contribution (see the graph 1:3 in Fig. 1) is the so-called Wichmann-Kroll (WK) contribution, which has been studied for a while (see, e.g., [3, 4]). A number of the results have been achieved for muonic atoms using certain numerical approximations of the exact WK potential. In particular, the approximations, introduced in [5] and [6] on the basis of the results of numerical integration in [6] were numerously applied (e.g., in [3, 7, 8]). The result for the $2p - 2s$ Lamb shift with the accuracy sufficient for applications in kHz was found in [8] and confirmed in [9–11]. In [10, 11] the result was also confirmed by direct calculations. The WK contributions to the $n = 2$ Lamb shift for some other light muonic atoms are obtained in, e.g., [12, 14].

The $\alpha^2(Z\alpha)^3m$ term is due to the virtual Delbrück scattering (see the 2:2 diagram in Fig. 1). It has also been studied for quite a long period (see, e.g., [3, 4]). Still, some questions have been resolved only recently [10].

The initial calculations were based on a so-called scattering approximation [12] (where the Coulomb muon propagator is substituted for a free one). The substitution by itself is incorrect (see, e.g., discussion in [14, 10]):
however, the formulas which were eventually used in the numerical calculations were nevertheless correct (see below). Results on the contribution to the Lamb shift in some light atoms were published, e.g., in \[10, 11\], but they were not very accurate.

The third type of contributions (see the 3:1 plot in Fig. 1) have not been calculated until recently. It was studied in \[10, 11\], where also the virtual-Delbrück-scattering contribution was found with a sufficient accuracy for several light muonic atoms.

A kind of theorem on the 2:2 and 3:1 contributions was announced in \[11\] and proven in \[10\]. The papers considered an approximation of a static muon, where its nonrelativistic propagator is presented with a \(\delta\) function over the energy. It was proven that the approximation is a valid one. We discuss the accuracy of the approximation in this paper (see Sec. III). Using that approximation \[10, 11\], the results on the 2:2 and 3:1 contributions to the Lamb shift in muonic hydrogen, deuterium and helium ions have been found (see \[14\] for \(\mu\)T). It was also demonstrated that the related limit can be achieved both from the diagrams with the bound-muon Green’s function (as shown in Fig. 1) and from those with the free Green’s function (as were used in the scattering approximation in \[3, 9\]). As far as the static-muon approximation is applicable, one may use both types of diagrams with the same result, which validates the working formulas used in \[3, 9\].

In this paper we consider the effective potential for the virtual-Delbrück-scattering contribution to the Lamb shift in light muonic two-body atoms. We use the representation of the potential in momentum space in terms of an integral over Feynman parameters \[10\] and study the effective potential in the coordinate space by means of an analytic Fourier transform and subsequent numerical integrations over the Feynman parameters. For the effective potential in the coordinate space, we find both asymptotics (at \(r \ll 1/m_e\) and \(r \gg 1/m_e\)). (Here and throughout the paper we apply the relativistic units in which \(\hbar = c = 1\).) Eventually, we fit the numerical results and asymptotics, obtained here. The approximation is accurate at the level of \(10^{-3}\) in the area where the muon wave function of low states is localized.

Our main results are related to the virtual-Delbrück-scattering contribution to the Lamb shift; however, we present numerical results for all three light-by-light (LbL) contributions (see Fig. 1), because their comparison can be useful.

The 2\(p - 1s\) Lamb-shift interval cannot be successfully measured in all the two-body muonic atoms (because of the range of the interval); however, the theory of the Lyman-\(\alpha\) transition is very similar. The data on such gross-structure transitions play an important role in determination of the rms charge radius of a large variety of elements (see, e.g., \[14\]). In this paper we tabulate the virtual light-by-light-scattering contribution to the Lamb shift of the 1s, 2s, 2\(p\) states which is sufficient for the calculation of both the 2\(p - 1s\) interval and the energy of the 2\(p - 1s\) transition. The considered range of the nuclear charge is \(Z = 1, \ldots, 10\).

II. THE EFFECTIVE POTENTIAL AND THE STATIC-MUON APPROXIMATION

As demonstrated in \[10\], once we can neglect various contributions to the muon propagator, such as the binding energy and those related to momentum transfer between the muon and the electron loop in comparison with its energy transfer \(q_0\), we arrive at the nonrelativistic propagator reduced to \(\delta(q_0)\). For \(Zm_\mu/n \leq m_e\) (\(n\) is the principle quantum number), the energy transfer is determined by the \(m_e\) scale. In the opposite case, when \(Zm_\mu/n \geq m_e\), the characteristic value of \(q_0\) is determined by the value of the momentum (in the LbL loop), which in its turn is determined by the characteristic atomic momentum \(Zm_\mu\). That means that once \(Z\alpha \ll 1\), we can apply the static-muon approximation. (In \[10\] we considered a stronger condition \((Z\alpha)^2m_\mu \ll m_e\).) All that is related, indeed, to only 2:2 and 3:1 contributions. The standard WK contribution does not require any conditions on the muon but only on the static regime of the nucleus. Those conditions are weaker and the validity of the WK potential is due to relativistic-recoil effects, i.e., due to corrections which are of higher order in both small parameters of the two-body Coulomb problem, \(Z\alpha\) and \(m_\mu/M\), where \(M\) is the nuclear mass.

\[
\Delta E_{3:1}(ns) = \frac{1}{Z^2} \Delta E_{1:3}(ns),
\]

since the related integrands differ by their normalization only. Note that Eq. (1) is correct only under the static-muon approximation. The corrections beyond the approximation are of different orders for \(\Delta E_{3:1}\) and \(\Delta E_{1:3}\).

Once the static-muon approximation is applicable, we arrive at a "double-external-field" limit, the diagrams for which are presented in Fig. 2. In particular, that allows us to immediately set a relation between the 3:1 contribution and the 1:3 one (WK):

\[
\Delta E_{3:1}(ns) = \frac{1}{Z^2} \Delta E_{1:3}(ns),
\]

FIG. 2: "Double-external-field" approximation with a static nucleus and a static muon.
An effective potential for the 2:2 contribution, an evaluation of which is the main purpose of this paper, is considered in detail in the next section.

III. THE EFFECTIVE POTENTIAL FOR THE VIRTUAL-DELBRÜCK-SCATTERING CONTRIBUTION

Following [10], the contribution of virtual Delbrück scattering to the Lamb shift in light muonic atoms can be presented in terms of a certain potential. In the momentum space the result reads [11]

$$\Delta E_{2:2} = \int \frac{d^3q}{(2\pi)^3} V_{2:2}(q^2) F(q^2)$$  \hspace{1cm} (2)

The potential $V_{2:2}(q^2)$ is presented in momentum space as an integral over the Feynman parameters [11]

$$V_{2:2}(q^2) = \frac{3}{4\pi} \alpha^2(Z\alpha)^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 du \int_0^1 dv \int_0^1 dw \int_0^1 dt \sum_{k=1,2} \left\{ \frac{E_{2:2}^{(k)}(s_{2:2}^{(k)} q^2 + m_r^2)^2}{s_{2:2}^{(k)} q^2 + m_r^2} + \frac{C_{2:2}^{(k)} q^2}{s_{2:2}^{(k)} q^2 + m_r^2} + \frac{D_{2:2}^{(k)} q^2}{s_{2:2}^{(k)} q^2 + m_r^2} \right\},$$  \hspace{1cm} (4)

where $E_{2:2}^{(k)}$, $C_{2:2}^{(k)}$, $D_{2:2}^{(k)}$, and $s_{2:2}^{(k)}$ are bulky dimensionless functions of those parameters considered in [10]. The parameter $k$ is to distinguish two diagrams contributing to $V_{2:2}$: $k = 1$ stands for the left 2:2 graph (see Fig. 2) and $k = 2$ is for the right one.

The dependence on $q^2$ is simple, which allows us to immediately perform the Fourier transformation

$$V_{2:2}(r) = \frac{4\pi}{r} \int_0^\infty \frac{dq}{(2\pi)^3} q \sin(qr)V_{2:2}(q^2)$$  \hspace{1cm} (5)

and to obtain a result in the coordinate space, which reads

$$V_{2:2}(r) = \frac{3}{4\pi} \alpha^2(Z\alpha)^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 du \int_0^1 dv \int_0^1 dw \int_0^1 dt \sum_{k=1,2} \exp \left( -\frac{m_r r}{s_{2:2}^{(k)}} \right) \left\{ \frac{E_{2:2}^{(k)}}{4\pi s_{2:2}^{(k)} r} + \frac{C_{2:2}^{(k)}}{s_{2:2}^{(k)} m_r r} - \frac{D_{2:2}^{(k)}}{s_{2:2}^{(k)} r} \right\}.$$  \hspace{1cm} (6)

The explicit representation of the potential $V_{2:2}(r)$ is cumbersome and for practical applications we further look for an efficient approximate formula. To derive it we first find the value of the potential in certain points in the coordinate space (see Fig. 3) and then fit them with a Padé approximation.

To improve the accuracy of the fit, prior to fitting, we look for the asymptotics. The potential behaves as $\propto r^{-1}$ at short distances, as one should expect from [8], while at long distances it is $\propto r^{-4}$. The general situation is illustrated in the plot in Fig. 3. The range of characteristic values of $x$, which are of interest for light muonic atoms, is summarized in Table 1.

The short-distance asymptotic coefficient can be directly established from [6] in a rather straightforward way. The result of the numerical integration reads

$$V_{2:2}(r \ll 1/m_r) \simeq -0.027565(13) \frac{\alpha^2(Z\alpha)^2}{r}.$$  \hspace{1cm} (7)

The large-distance asymptotic behavior is not that simple to establish from [6]. Considering the LbL contributions (see Fig. 1) in the $t$ channel, we note that some pure photonic intermediate states are possible there, which sets the branch point for $t = -q^2$ to zero and eventually leads to a certain $r^{-4}$ behavior at large distances.
TABLE I: Characteristic differences of the wave functions of the low states (1s, 2s, 2p) in light two-body muonic atoms. Here, \( \kappa = Z \alpha m_e/m_e \) is the characteristic momentum of the muonic states in the units of \( m_e \), while \( x_n = n^2/\kappa \) is the characteristic radius of the \( nl \) state in units of \( \alpha e = \hbar/m_e c \).

```
| Ion | Z | \kappa | x_1  | x_2  |
|-----|---|--------|------|------|
| 1H  | 1 | 1.356  | 0.737| 2.950|
| 2H  | 1 | 1.428  | 0.700| 2.800|
| 3H  | 1 | 1.454  | 0.688| 2.751|
| 4He | 2 | 2.908  | 0.344| 1.375|
| 5He | 2 | 2.935  | 0.341| 1.363|
| 6Li | 3 | 4.443  | 0.225| 0.900|
| 7Li | 3 | 4.555  | 0.224| 0.898|
| 9Be | 4 | 5.960  | 0.167| 0.671|
| 10B | 5 | 7.460  | 0.134| 0.536|
| 11B | 5 | 7.467  | 0.133| 0.536|
| 12C | 6 | 8.968  | 0.111| 0.446|
| 13C | 6 | 8.975  | 0.111| 0.446|
| 14N | 7 | 10.48  | 0.095| 0.382|
| 15N | 7 | 10.48  | 0.095| 0.382|
| 16O | 8 | 11.99  | 0.083| 0.334|
| 17O | 8 | 11.99  | 0.083| 0.334|
| 18O | 8 | 12.00  | 0.083| 0.333|
| 19F | 9 | 13.50  | 0.074| 0.296|
| 20Ne| 10| 15.00  | 0.067| 0.267|
| 21Ne| 10| 15.01  | 0.066| 0.267|
| 22Ne| 10| 15.01  | 0.066| 0.266|
```

With the asymptotic coefficients in hand, we fit the numerical results. The fit reads

\[
V_{2:2}^{\text{approx}}(r) = - \alpha^2 (Z \alpha)^2 m_e \frac{7.236 + 0.3099 x + 2.561 x^2}{262.5 + 902.0 x + 751.7 x^2 + 458.6 x^3 + 2.62 x^4 + 100 x^5},
\]

(9)

where \( x = m_c r \). The fit has \( \chi^2 = 9.5 \) for 22 degrees of freedom. We estimate the accuracy of the fit as \( 1 \times 10^{-3} \) for \( x \leq 1 \). In the interval of \( 1 < x < 10 \) the uncertainty gradually increases to a few percent level. For higher \( x \), thanks to the correct asymptotic behavior, the error does not exceed that level.

As an independent test of our fit, we compare the results obtained by using the fit for the \( n = 2 \) Lamb shift in the lightest two-body muonic atoms with the direct ones \( \text{[10, 11]} \) (see Table II). The results are in perfect agreement within our estimation of the uncertainty of the fit as \( 10^{-3} \).

The virtual-Delbrück-scattering situation is very different from the WK one. As mentioned, the WK potential \( V_{1:3}(r) \) \( \text{[17]} \) is valid when one can neglect the recoil effects, i.e., it is a result of an expansion not only in \( Z \alpha \), but also in \( m_s/M \). Because of the recoil nature of the corrections, the WK potential is applicable in both ordinary and muonic atoms. In the former we are interested in a large range of distances at \( x \gg 1 \), while the latter deals only with \( x \sim 1 \) or \( x \ll 1 \). The 2:2 potential is applicable only for muonic atoms \( \text{[10, 11]} \) and therefore the area with \( x \gg 1 \) and even with \( x \geq 1 \) is of low interest. It still may appear in evaluation of the energy for the highly excited states with \( n^2/Z \gg 1 \), but most of the applications rely on a study of the lower states with \( n = 1, 2 \). For such states the accuracy of the Padé approximation \( \text{[13]} \) is at the level of \( 10^{-3} \). Note, that this is the accuracy of the approximation of \( V_{2:2}(r) \) potential. Meanwhile, the very applicability of that potential due to the static muon approximation has lower accuracy (see above).

As an example of applicability of the \( x \gg 1 \) area to practical cases, we mention neutral antiprotonic helium, where the characteristic size of the antiproton orbit is comparable with the 1s orbit of an electron in a hydrogen atom (see, e.g., \( \text{[22]} \)).

IV. NUMERICAL RESULTS

The purpose of the paper is a derivation of an effective potential for the 2:2 contribution to the muonic-atom Lamb shift at medium \( Z \), which has been done in the previous section. It is interesting to compare the numerical
results with those from other LbL terms, and in particular, with the WK ones.

There are two fits for the WK potential for the muonic atoms, which are available in literature. (The potential is valid by itself for ordinary and muonic atoms; however, the purpose of the fit determines the range of the distances of interest (see above).) One of them is [6]

\[
V_{1;3}(r) = \frac{0.3617 \alpha(Z\alpha)^2 Z\alpha}{\pi} \exp\left(0.3728 x \right) \\
- \sqrt{2.906 + 11.4 \pi + 4.417 x^2}.
\] (10)

Another fit applied in numerical calculations in muonic atoms is [3]

\[
V_{1;3} = \frac{\alpha(Z\alpha)^3}{\pi^3 r} \left\{ \begin{array}{ll}
-0.1755 + 0.1559 x + 0.0880 x^2 & \text{for } x \geq 1 \\
+0.649 - 0.208 x & \text{for } x \leq 1
\end{array} \right.
\] (11)

Both fits are based on numerical calculations by Vogel for the interval of $0.1 < x \leq 1$ and in that area the fits well agree with the numerical results (at the level of $10^{-3}$). They both utilize the known leading asymptotic term at low $x$. They are different in area $x > 1$. The advantage of (11) is more smooth behavior around $x = 1$ and therefore a better extrapolation to the low end of the $x > 1$ interval, while the fit in (11) accommodates the asymptotic term at $x \gg 1$ and is better at high end of the interval.

We use our own fit of Vogel’s data [6]

\[
V_{\text{app, WK}}(x) = \frac{\alpha(Z\alpha)^3}{r} \left\{ \begin{array}{c}
5.026 + 0.02676x + 0.2829x^2 \\
240.0 + 725.4x + 542.2x^2 + 649.8x^3 + 150.2x^4 + 9.457x^5 + 100x^6
\end{array} \right. \]
(12)

(which fits the data for $0.1 < x \leq 1$ with a fractional uncertainty better than $10^{-3}$ and correctly reproduces the asymptotics at low $r$ [17] (see also [22, 24]) and at high $r$ [17] (see also [22, 24]). In contrast to the fit [11] from [3], our fit in (12) has smooth behavior at 6 around $x = 1$.

The application of the fits to the $n = 2$ Lamb shift in muonic hydrogen is rather questionable (see Table I), since we essentially need to integrate over an interval outside of the data area of [6], which was used to derive the fit. The smooth behavior at around $x = 1$ and a correct $x \gg 1$ asymptotics (mentioned above) should deliver a reasonable result, but its accuracy is unclear.

Previously, while calculating the results for muonic hydrogen, deuterium, and helium [10, 11, 14] we have used a direct calculation instead of the fits. To verify the accuracy of the previous fits and our fit, we compare our results of a direct calculation and the results from the fits for $2s, 2p$ for a few light atoms where the characteristic values of $x$ are the largest (see Table III). The error of our fit is about 1%, while for the others it is at a few percent level. Eventually we estimate the accuracy of our
fit as follows; at $0.1 < x < 1$ it is below $1 \times 10^{-3}$, and it gradually reduces for $x < 0.1$ and $x > 1$ down to a 1% level.

The results for $n = 1, 2$ states in a two-body muonic atom are summarized in Tables IV and V for all three LbL contributions (the 1:3, 2:2, 3:1 ones). The uncertainty of the fits is discussed above, as well as the uncertainty of the static-muon approximation.

V. CONCLUSIONS

In conclusion, we have derived a representation for an effective potential induced by the virtual Delbrück scattering in the leading nonrelativistic approximation. We have obtained its numerical values in a number of points in the coordinate space and found an efficient Padé approximation. The accuracy of the Padé approximation is the highest for $m_e r < 1$, which allowed us to find the contributions to the Lamb shift of the low states in light two-body muonic atoms. We estimate the accuracy of the numerical evaluation as at the level of one part in a thousand, which is higher than the accuracy of the leading nonrelativistic approximation by itself.

The uncertainty of the Padé approximation for the potential is the best for $m_e r < 1$ (at the level of $10^{-3}$), and it gradually increases to the few-percent level for $m_e r \approx 10$. The data of the numerical evaluation of the potential itself at higher $m_e r$ are not accurate enough; however, the Padé approximation is constrained by the long-distance asymptotic behavior, which we have established by an independent evaluation.

In particular, we have tabulated the related contributions to the Lamb shift of the 1s, 2s, 2p states in muonic atoms with $Z \leq 10$. Those states are sufficient for two important problems, namely, for a theory of the $n=2$ Lamb shift and of the Lyman-$\alpha$ interval.

We have also compared the results for the virtual-Delbrück-scattering contribution and the Wichmann-Kroll one. At $Z = 1$ they are comparable (being of opposite signs). They increase with the value of $Z$, but the Wichmann-Kroll one increases faster. At $Z = 10$ the virtual-Delbrück-scattering contribution is between 10 and 20% of the Wichmann-Kroll contribution depending on the state.

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### Table III: The WK contributions to the 2s and 2p Lamb shift in light muonic atoms. The results of direct calculations are taken from [10, 11]. The uncertainty of the fits for $x > 1$ is a priori unclear and not shown.

| Atom, state | $x$ | contribution [meV] |
|-------------|-----|-------------------|
| $\mu$H (2s) | 2.95 | 0.001 240 | 0.001 238 | 0.001 243 | 0.001 2472(7) |
| $\mu$H (2p) |      | 0.000 2196 | 0.000 2196 | 0.000 2270 | 0.000 228 87(4) |
| $\mu$D (2s) | 2.80 | 0.001 362 | 0.001 364 | 0.001 3693(7) |
| $\mu$D (2p) |      | 0.000 2609 | 0.000 2691 | 0.000 271 23(4) |
| $\mu$He$^+$ (2s) | 1.36 | 0.037 67 | 0.037 30 | 0.037 69 | 0.037 833(22) |
| $\mu$He$^+$ (2p) | | 0.017 68 | 0.017 82 | 0.017 8676(15) |

### Table IV: The LbL contributions to the Lamb shift of the 1s state in a light two-body muonic atom. The contributions are given in units of $\alpha^3 (Z\alpha)^2 m_e$ and meV. The results are given for the total LbL contribution and for its components (see Fig. 1). We present in the table the central values, while the accuracy of the calculation is discussed in the text.

| Ion | $Z$ | $\Delta E_{1,1} \alpha^3 (Z\alpha)^2 m_e$ | $\Delta E_{2,1} \alpha^3 (Z\alpha)^2 m_e$ | $\Delta E_{3,1} \alpha^3 (Z\alpha)^2 m_e$ | $\Delta E_{LbL}(1s) \alpha^3 (Z\alpha)^2 m_e$ | $\Delta E_{LbL}(1s)$ [meV] |
|-----|----|-------------------------------------|-------------------------------------|-------------------------------------|-------------------------------------|-------------------|
| 1H  | 1  | 0.005 804 | 0.008 095 | 0.005 804 | 0.003 513 | 0.006 903 |
| 2H  | 1  | 0.006 073 | 0.008 410 | 0.006 073 | 0.003 736 | 0.007 734 |
| 3H  | 1  | 0.006 167 | 0.008 520 | 0.006 167 | 0.003 814 | 0.008 038 |
| 3He | 2  | 0.040 28  | 0.010 67  | 0.010 07  | 0.024 13  | 0.203 40  |
| 4He | 2  | 0.044 49  | 0.012 43  | 0.012 44  | 0.076 30  | 1.474 61  |
| 6Li | 3  | 0.1118   | 0.047 97  | 0.076 30  | 1.474 61  |
| 7Li | 3  | 0.1120   | 0.048 02  | 0.076 30  | 1.474 61  |
| 9Be | 4  | 0.2227   | 0.071 50  | 0.165 11  | 5.704 51  |
| 10B | 5  | 0.3737   | 0.096 03  | 0.292 60  | 15.81 52  |
| 11B | 5  | 0.3738   | 0.096 06  | 0.292 60  | 15.83 52  |
| 12C | 6  | 0.5656   | 0.1213    | 0.460 00  | 35.87 52  |
| 13C | 6  | 0.5657   | 0.1213    | 0.460 00  | 35.90 52  |
| 14N | 7  | 0.7988   | 0.1471    | 0.668 11  | 71.00 52  |
| 15N | 7  | 0.7989   | 0.1471    | 0.668 11  | 71.04 52  |
| 16O | 8  | 1.073    | 0.1731    | 0.917 00  | 127.4 52  |
| 17O | 8  | 1.073    | 0.1732    | 0.917 00  | 127.5 52  |
| 18O | 8  | 1.074    | 0.1732    | 0.917 00  | 127.5 52  |
| 19F | 9  | 1.390    | 0.1995    | 1.207 21  | 212.5 52  |
| 20Ne| 10 | 1.747    | 0.2260    | 1.539 33  | 334.5 52  |
| 21Ne| 10 | 1.747    | 0.2260    | 1.539 33  | 334.6 52  |
| 22Ne| 10 | 1.747    | 0.2260    | 1.539 33  | 334.7 52  |
| Ion | $Z$ | $\Delta E_{1:3}$ | $\Delta E_{2:2}$ | $\Delta E_{3:1}$ | $\Delta E_{\text{LbL}(1s)}$ | $\Delta E_{\text{LbL}(1s)}$ |
|-----|-----|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1H  | 1   | 0.00006323      | -0.00009114     | 0.00006323      | 0.00003532      | 0.00006941      |
| 2H  | 1   | 0.00006592      | -0.00009498     | 0.00006592      | 0.00003687      | 0.00007631      |
| 3H  | 1   | 0.00006686      | -0.00009632     | 0.00006686      | 0.00003740      | 0.00007880      |
| 3He | 2   | 0.0004404       | -0.0000188      | 0.0001010       | 0.00002317      | 0.001953        |
| 4He | 2   | 0.0004431       | -0.0000207      | 0.0001108       | 0.00002332      | 0.001983        |
| 6Li | 3   | 0.01319         | -0.000236       | 0.001465        | 0.008416        | 0.1625          |
| 7Li | 3   | 0.01321         | -0.000246       | 0.001468        | 0.008432        | 0.1633          |
| 9Be | 4   | 0.02839         | -0.0009825      | 0.001774        | 0.02034         | 0.7027          |
| 10B | 5   | 0.05096         | -0.001383       | 0.002039        | 0.03917         | 2.117           |
| 11B | 5   | 0.05099         | -0.001384       | 0.002040        | 0.03919         | 2.121           |
| 12C | 6   | 0.08168         | -0.001821       | 0.002269        | 0.06575         | 5.127           |
| 13C | 6   | 0.08172         | -0.001821       | 0.002270        | 0.06578         | 5.132           |
| 14N | 7   | 0.1210          | -0.002287       | 0.002470        | 0.1006          | 10.69           |
| 15N | 7   | 0.1210          | -0.002288       | 0.002470        | 0.1006          | 10.70           |
| 16O | 8   | 0.1693          | -0.002778       | 0.002645        | 0.1442          | 20.03           |
| 17O | 8   | 0.1693          | -0.002778       | 0.002646        | 0.1442          | 20.04           |
| 18O | 8   | 0.1694          | -0.002779       | 0.002646        | 0.1442          | 20.06           |
| 19F | 9   | 0.2269          | -0.003290       | 0.002801        | 0.1968          | 34.64           |
| 20Ne| 10  | 0.2938          | -0.003819       | 0.002938        | 0.2585          | 56.20           |
| 21Ne| 10  | 0.2938          | -0.003819       | 0.002937        | 0.2586          | 56.23           |
| 22Ne| 10  | 0.2938          | -0.003820       | 0.002938        | 0.2586          | 56.24           |

**TABLE V**: The LbL contributions to the Lamb shift of the 2s state in a light two-body muonic atom. The contributions are given in units of $\alpha^3(Z\alpha)^2m_r$ and meV. The results are given for the total LbL contribution and for its components (see Fig. [I]). We present in the table the central values, while the accuracy of the calculation is discussed in the text.
TABLE VI: The LbL contributions to the Lamb shift of the $2p$ state in a light two-body muonic atom. The contributions are given in units of $\alpha^3(Z\alpha)^2 m_r$ and meV. The results are given for the total LbL contribution and for its components (see Fig. 1). We present in the table the central values, while the accuracy of the calculation is discussed in the text. That is a nonrelativistic calculation and therefore the results for $2p_{1/2}$ and $2p_{3/2}$ are the same.

| Ion | Z  | $\Delta E_{1:3}$ | $\Delta E_{2:2}$ | $\Delta E_{3:1}$ | $\Delta E_{\text{LbL}(1s)}$ | $\Delta E_{\text{LbL}(1s)}$ |
|-----|----|------------------|------------------|------------------|-----------------------------|-----------------------------|
|     |    | $[\alpha^3(Z\alpha)^2 m_r]$ | $[\alpha^3(Z\alpha)^2 m_r]$ | $[\alpha^3(Z\alpha)^2 m_r]$ | $[\alpha^3(Z\alpha)^2 m_r]$, meV | $[\alpha^3(Z\alpha)^2 m_r]$, meV |
| $^1\text{H}$ | 1 | 0.000 1116 | -0.000 3265 | 0.000 1555 | -0.000 095 43 | -0.000 1875 |
| $^2\text{H}$ | 1 | 0.000 1300 | -0.000 3543 | 0.000 1300 | -0.000 094 24 | -0.000 1951 |
| $^3\text{H}$ | 1 | 0.000 1353 | -0.000 3642 | 0.000 1353 | -0.000 093 55 | -0.000 1971 |
| $^3\text{He}$ | 2 | 0.002 065 | -0.001 848 | 0.000 5161 | 0.000 7332 | 0.006 180 |
| $^4\text{He}$ | 2 | 0.002 095 | -0.001 867 | 0.000 5237 | 0.000 7518 | 0.006 394 |
| $^6\text{Li}$ | 3 | 0.008 568 | -0.004 338 | 0.000 9520 | 0.005 182 | 0.100 01 |
| $^7\text{Li}$ | 3 | 0.008 597 | -0.004 349 | 0.000 9552 | 0.005 203 | 0.100 07 |
| $^9\text{Be}$ | 4 | 0.021 43 | -0.007 548 | 0.001 339 | 0.015 22 | 0.525 8 |
| $^{10}\text{B}$ | 5 | 0.041 72 | -0.011 30 | 0.001 669 | 0.032 09 | 1.734 |
| $^{11}\text{B}$ | 5 | 0.041 76 | -0.011 31 | 0.001 670 | 0.032 12 | 1.738 |
| $^{12}\text{C}$ | 6 | 0.070 29 | -0.015 51 | 0.001 952 | 0.056 73 | 4.423 |
| $^{13}\text{C}$ | 6 | 0.070 33 | -0.015 52 | 0.001 954 | 0.056 76 | 4.429 |
| $^{14}\text{N}$ | 7 | 0.107 6 | -0.020 08 | 0.002 196 | 0.089 72 | 9.535 |
| $^{15}\text{N}$ | 7 | 0.107 6 | -0.020 08 | 0.002 197 | 0.089 75 | 9.543 |
| $^{16}\text{O}$ | 8 | 0.154 0 | -0.024 93 | 0.002 406 | 0.131 5 | 18.27 |
| $^{17}\text{O}$ | 8 | 0.154 0 | -0.024 94 | 0.002 407 | 0.131 5 | 18.28 |
| $^{18}\text{O}$ | 8 | 0.154 1 | -0.024 94 | 0.002 408 | 0.131 5 | 18.29 |
| $^{19}\text{F}$ | 9 | 0.209 8 | -0.030 03 | 0.002 590 | 0.184 2 | 32.10 |
| $^{20}\text{Ne}$ | 10 | 0.275 1 | -0.035 31 | 0.002 751 | 0.242 5 | 52.72 |
| $^{21}\text{Ne}$ | 10 | 0.275 1 | -0.035 32 | 0.002 751 | 0.242 5 | 52.74 |
| $^{22}\text{Ne}$ | 10 | 0.275 1 | -0.035 32 | 0.002 751 | 0.242 6 | 52.76 |