How to prestress compliant mechanisms for a targeted stiffness adjustment

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Received 16 January 2020, revised 6 April 2020 Accepted for publication 12 May 2020 Published 3 July 2020

Abstract

The paper deals with compliant mechanisms with a variable stiffness behavior. Devices with such a behavior are getting more and more attention in recent years, especially in contact applications such as human-robotic-interactions or haptic interfaces. In the work presented here the variable stiffness behavior is achieved by the targeted application of prestressing forces. An optimization based method is presented to identify suitable force application points. Furthermore, the amount and direction of force is determined in order to cause a specific change in stiffness. The method consists of a convex optimization formulation and is based on modal parameters. The method is intended to work for any given design of a compliant mechanism. In this publication the method is tested and evaluated using two compliant mechanisms—the cartwheel hinge and a shape-adaptive airfoil profile.

Keywords: compliant mechanism, variable stiffness device, prestressing forces

(Some figures may appear in colour only in the online journal)

1. Introduction

Compliant mechanisms (CMs) take advantage of the elastic deformation of their own structure in order to realize a motion. This kind of mechanism provides several advantages, such as the absence of friction and wear, high lightweight potential and cost-effective manufacturing processes [1]. However, an important side effect is that the inherent structural stiffness needs to be overcome to enable a motion [2]. Once a CM is designed, its stiffness is not intended to be changed in order to adapt to special operation conditions, although such an ability would be beneficial for several tasks like robotic grippers, haptic interfaces or for human-robotic interactions in general [3].

The stiffness of a CM is essentially ruled by its material, geometry and boundary conditions [2]. At least one of these properties must be adaptable to achieve a CM with variable stiffness behavior.

In terms of CMs, the most widely used approach to achieve variable stiffness by changing material properties is the use of shape memory materials, such as shape memory alloys (SMA) or shape memory polymers (SMP) [4]. These materials make use of a temperature-dependent Young’s modulus to realize a change in stiffness. Examples can be found, among others, in [5] and [6]. However, the thermally triggered shape memory effect is not well suited for rapid stiffness changes due to a slow system response [4].

CMs mainly use bending deformation to generate a motion [7]. Consequently, the geometric parameters that affect the structural stiffness are the length of the compliant parts and the area moment of inertia. First approaches to realize a variable moment of inertia are shown mostly on cantilever beam examples. In [8] a shape-adaptive bending plate is presented, which changes the cross-sectional area by bending the plate around its longitudinal axis. By actively changing the stiffness, a vibration reduction could be achieved. Further shape-adaptive systems working on the same principle can be found in [9]. Other approaches make use of so-called multi-layer beams to influence the moment of inertia. These beams are...
made up of several layers that can be connected together by different types of actuation [10, 11]. If the individual layers are connected, they behave like an entire bending beam, otherwise they behave like several individual beams. Various concepts of CMs have been presented which take advantage of changing the active spring length. Examples can be found in [3, 12–16]. In these mechanisms, additional components, typically sliders, modify the active length of the compliant parts to influence the stiffness of the mechanism. Although large changes in stiffness can be achieved, the number of components increases significantly in most of the above-mentioned examples.

Furthermore, the stiffness of CMs can be altered by changing the boundary conditions. In particular, we focus on the application of prestressing forces, but limit our considerations on the pre-buckling behavior only. The effect of prestressing forces on the bending behavior of structures, in general, is described by the so-called stress stiffening effect [2]. In order to describe this effect, the example of a cantilever beam under prestress is used. In figure 1(a), a vertical force $F$ is applied to the free end of the cantilever beam, resulting in a deflection $u$. Keeping $F$ constant and applying an additional tensile prestressing force $P$ results in a smaller deflection $u$ (see figure 1(b)). In other words, the transversal stiffness of the beam increases. Vice versa, the application of a compressive force (cf figure 1(c)) leads to a decreasing transversal stiffness.

In terms of CMs, first investigations of the effect of prestressing forces have been carried out in [17, 18] and [19]. The authors analyzed leaf springs and cross-spring hinges under prestressing forces and observed the above mentioned change in stiffness. Later, Awtar [20] characterized the behavior of various types of parallelogram flexures under axial forces mathematically. Prototypical implementations using prestressing forces can be found in [21–23]. They show promising stiffness changes. In existing examples, the force application points required to apply the prestressing forces are determined on the basis of experience. Therefore, the existing prototypes are currently limited to simple mechanism designs. When the complexity of the mechanisms increases, e.g. for Multiple Input/Multiple Output mechanisms (MIMO), the identification of proper force application points becomes more challenging.

In this paper, we present an optimization-based method that realizes the search of force application points for the targeted application of prestressing forces on predefined CMs. Further, the force directions and amplitudes are identified in order to achieve a targeted change in stiffness. The method itself is defined to be valid for any design of CMs. However, we limit our considerations on CMs with selective compliance. This kind of mechanism is described in detail in the following chapter.

The outline of the paper is as follows: First we summarize the mathematical definition of selective CMs. Subsequently, we present a design criterion that is necessary for the optimization method described later on. Further, the numerical implementation is developed and the method is tested for a Single Input/Single Output mechanism (SISO) and a MIMO mechanism.
All parameters together

\[ k_i \in \mathbb{R}, i = 1, \ldots, m \] (8)

are called primary stiffness and are assigned to the desired displacement modes. Similarly, the quantities

\[ k_j \in \mathbb{R}, j = m + 1, \ldots, n \] (9)

refer to the undesired displacement modes and are denoted as secondary stiffness [24].

If a CM is to be deformable exclusively within its kinematics, so that \( \mathbf{u} \in \mathbb{R} \), an infinitely high secondary stiffness would be required. Of course, this cannot be realized with real structures. However, CMs with a sufficiently high secondary stiffness allow to neglect the unwanted deformations. The displacements are then limited to the mechanism’s kinematics. According to Hasse and Campanile, a CM which exhibits this behavior is denoted as a CM with selective compliance [25]. Design methods for such mechanisms can be found in [25]. From these design methods it results that the higher the secondary stiffness is compared to the primary stiffness, the more the mechanism tends to exhibit selective compliant behavior. Consequently, as an indicator of selective compliance, we can write:

\[ k_i \ll k_j. \] (10)

After the synthesis of a selective CM, a suitable set of orthogonal displacement modes can be found by solving the following eigenvalue problem:

\[ \mathbf{k} \varphi = \lambda \mathbf{k} \] (11)

Under the normalization condition \( \varphi^T \varphi = 1 \), the eigenvectors \( \varphi \) refer to the desired and undesired displacement modes collected in the basis \( \mathbf{X} \) of equation (5), so that:

\[ \mathbf{kX} = \lambda \mathbf{k} \] (12)

The eigenvalues \( \lambda_i \) (compare equation (11)) refer to the primary stiffness values \( k_i \) and the eigenvalues \( \lambda_j \) to the secondary stiffness values \( k_j \) [24].

Prestressing forces lead to a change of the stiffness matrix \( \mathbf{k} \) and therefore influence both the primary and the secondary stiffness. Our goal is to reduce the primary stiffness of a given CM through the targeted application of prestressing forces, while maintaining or improving its selective compliant behavior. The suitable force application points are found using an optimization-based method, which is explained in detail below.

3. Method

The optimization procedure presented in this work presupposes a defined CM with selective compliance. The procedure is based on a finite-element model with linear elastic material and linear geometric assumptions. We define a set of \( k \) different load cases \( \mathbf{f}_k \in \mathbb{R}^n \), \( k = 1, 2, \ldots, k \) to induce an initial stress state in the mechanism. The individual load cases differ from each other by different force directions and force application points, but not by different amounts of the force vector. The initial stress state changes the stiffness of the mechanism and thus change the stiffness matrix \( \mathbf{k} \in \mathbb{R}^{n \times n} \). The original stiffness matrix—which is described in section 2 and considers the material and geometry of the structure—is hereafter named material stiffness matrix \( \mathbf{k}_e \in \mathbb{R}^{n \times n} \). The difference between \( \mathbf{k} \) and \( \mathbf{k}_e \) is denoted as initial stress stiffness matrix \( \mathbf{k}_g \in \mathbb{R}^{n \times n} \) [28], so we can write:

\[ \mathbf{k} = \mathbf{k}_e + \mathbf{k}_g \] (13)

\( \mathbf{k}_g \) depends on the structure’s geometry and is directly proportional to the stress state in the structure. The stress, in turn, is linearly dependent on the external load under the validity of Hooke’s Law and geometric linear assumptions [28]. Therefore an initial stress stiffness matrix \( \mathbf{k}_g(\mathbf{f}_k) \) can be defined for each initial load case \( \mathbf{f}_k \). In addition, a scaling of the prestressing force using a scaling factor \( x_k \in \mathbb{R}, k = 1, 2, \ldots, k \) has the same scaling effect on the initial stress stiffness matrix:

\[ \mathbf{k}_g(\mathbf{f}_k) = x_k \cdot \mathbf{k}_g(\mathbf{f}_k). \] (14)

Furthermore, due to the linear assumptions, the different load cases \( \mathbf{f}_k \) can be superimposed, so that we can write:

\[ \mathbf{k} = \mathbf{k}_e + \sum_{\xi=1}^{k} x_{\xi} \cdot \mathbf{k}_g(\mathbf{f}_k). \] (15)

The external force vector \( \mathbf{f} \) results from the sum of the scaled initial load cases \( \mathbf{f}_k \):

\[ \mathbf{f} = \sum_{\xi=1}^{k} x_{\xi} \mathbf{f}_k \] (16)

It should be noted that the external force vector used here only contains the total prestressing forces. Other external forces that lead to a deflection of the mechanism are not considered here.

Under consideration of equation (7) and (15), the primary stiffness values \( \lambda_i \) can be determined by solving the following linear equation system:
\[ \varphi_j^T \left( k_e + \sum_{\xi=1}^{k} x_\xi \cdot k_{g,\xi} (f_\xi) \right) \varphi_j = \lambda_j \text{ with } i = 1, \ldots, m. \]  

(17)

Analogously, the secondary stiffness values \( \lambda_j \) can be calculated by

\[ \varphi_j^T \left( k_e + \sum_{\xi=1}^{k} x_\xi \cdot k_{g,\xi} (f_\xi) \right) \varphi_j = \hat{\lambda}_j \text{ with } j = m + 1, \ldots, n. \]  

(18)

Further, we can rewrite equation (17)

\[ \hat{\lambda}_{e,j} + \sum_{\xi=1}^{k} x_\xi \cdot \hat{\lambda}_{g,j,\xi} = \hat{\lambda}_j \]  

(19)

with

\[ \hat{\lambda}_{e,j} = \varphi_j^T k_e \varphi_j \text{ and } \hat{\lambda}_{g,j,\xi} = \varphi_j^T k_{g,\xi} (f_\xi) \varphi_j. \]  

(20)

Similarly, we can rewrite equation (18)

\[ \hat{\lambda}_{e,j} + \sum_{\xi=1}^{k} x_\xi \cdot \hat{\lambda}_{g,j,\xi} = \hat{\lambda}_j \]  

(21)

with

\[ \hat{\lambda}_{e,j} = \varphi_j^T k_e \varphi_j \text{ and } \hat{\lambda}_{g,j,\xi} = \varphi_j^T k_{g,\xi} (f_\xi) \varphi_j. \]  

(22)

Assuming that \( \varphi_i \) and \( \varphi_j \) are determined using equation (11) for the unloaded mechanism, all variables of equations (17) and (18), besides \( x_\xi \), can be calculated prior to the optimization formulation. The scaling factors \( x_\xi \) are determined by solving the optimization problem. These values represent the design variables and are collected in the design vector \( x \in \mathbb{R}^k \). The formal description of the optimization problem is as follows:

\[
\begin{align*}
\text{minimize} & \quad \sum_{\xi=1}^{k} |x_\xi| \\
\text{s.t.} & \quad \lambda_{d,i} - \hat{\lambda}_i (x) = 0 \\
\varphi_i^T f (x) = 0 \\
\varphi_j^T f (x) = 0 \\
\beta \cdot \hat{\lambda}_i (x) \cdot \hat{\lambda}_{e,j} - \hat{\lambda}_j (x) \cdot \hat{\lambda}_{e,j} & \leq 0 \\
\forall j^* & \in \{j \} : j^* \leq m + \eta
\end{align*}
\]

(23)

The objective function minimizes the sum of the absolute value of the scaling factors in order to reduce the total amount of applied prestressing forces. This minimization is performed under the constraint that the values of the primary stiffness are equal to respective target values \( \lambda_{d,i} \). A suitable target value, for example, can be obtained from a predefined percentage of the primary stiffness values \( \lambda_{e,\xi} \) of the unloaded mechanism. Further restrictions are that the applied forces do not deflect the mechanism in the primary displacement modes nor in the considered secondary displacement modes. Therefore, the modal component of the force for the corresponding displacement modes is set to zero. Obviously, not all displacement modes can be considered, otherwise all modal forces would be equal to zero, which would result in no prestressing forces being applied. Therefore, we consider all primary displacement modes as well as the secondary displacement modes \( \varphi_j \) with the \( \eta \) lowest eigenvalues. According to experience, \( \eta = 1 \) to 3 should be adequate for most cases. The last constraint concerns the issue that the ratio between secondary and primary stiffness values for the prestressed mechanism should be at least the same as in the unloaded case. This constraint ensures that the selective compliant behavior in the unstressed and prestressed mechanism is at least equal. A further increase of the ratio can be achieved by selecting a factor \( \beta \geq 1 \). For \( \beta = 1 \), the two ratios are at least equal.

The advantage of the formulation, presented in equation (23), is that it is a convex optimization formulation. This means that the global optimal value is found by a proper search algorithm. In addition, convex optimization formulations can be solved very efficiently. An optimization problem is convex if both the objective function and the constraints are convex. The objective function—the sum of absolute values—in equation (23) is according to Boyd [29] convex. Furthermore, the constraints are linear functions which are convex by definition [29].

4. Implementation

An overview of the entire numerical procedure is given in figure 2. First, the description of a design of a CM must be provided. After the discretization of this design, the material stiffness matrix \( k_e \) can be calculated. Afterwards, the eigenvectors of \( k_e \) are determined and categorized into primary \( \varphi_i \) and secondary \( \varphi_j \) displacement modes. Subsequently, initial load cases are defined and the corresponding initial stress stiffness matrices are calculated. Using these quantities, \( \hat{\lambda}_{e,i} \), \( \hat{\lambda}_{e,j} \), \( \hat{\lambda}_{g,i,\xi} \) and \( \hat{\lambda}_{g,j,\xi} \) can be computed.

After defining an arbitrary starting point for the design variables collected in \( x \), the quantities \( \lambda_i \), \( \lambda_j \) and the force vector \( f \) can be calculated. This enables the evaluation of the objective function and the constraint functions. In order to find an improved scaling of the load cases, the aforementioned quantities are transferred to an optimization algorithm. In our case, we use the optimization algorithm SDPT3 [30] and SeDuMi [31] for convex optimization problems, both implemented in the freely available software package CVX [32, 33] for MATLAB. Both optimization algorithms show—as expected for convex optimization—identical results. These search algorithms show a very high time efficiency in solving convex optimization problems, especially when dealing with a large number of design variables. However, the presented optimization problem can also be solved by search algorithms included in the Matlab Optimization Toolbox, such as Sequential-Quadratic-Programming (SQP) or Interior-Point-Method. The mentioned calculation steps are performed within an iterative loop as shown in figure 2. The design variables are updated

1 A MATLAB-code for the exemplary implementation can be found at www.tu-chemnitz.de/mh/mp/sns
between these loops. The process continues until convergence is achieved.

5. Examples and FEA verification

This chapter demonstrates the ability of the procedure described above in detail, using two planar CMs with selective compliance from literature: the cartwheel hinge [26] and a belt rib structure of a shape-adaptive airfoil [34]. For the cartwheel hinge, a distinction is made between two cases: in the first case we only analyze the structure itself without the influence of any clamping; in the second case we investigate the constrained hinge. In addition, the linear assumptions made are verified by a 3-d nonlinear finite element analysis (FEA) of the cartwheel hinge.

5.1. Unconstrained cartwheel hinge

The cartwheel hinge is a SISO mechanism with a good selective compliant behavior. The mechanism is intended to deform in one rotational DoF only. The cartwheel hinge in its initial and deformed state is shown in figure 3(a) in a perspective view. It should be noted that the deflection \( \theta_0 \) of the mechanism as well as the clamping in the figure are for illustration purposes only and have no further significance for the procedure.

In figure 3(b) the discretized structure is illustrated in top-view. For discretization, a two-dimensional beam element according to the linear Bernoulli-Euler beam theory of 2nd order [35] is used. One beam element connects two element nodes, each consisting of two translational and one rotational DoFs. The \( w \) total element nodes are marked by circles in figure 3(b), while the line thickness illustrates the thickness \( t \) of the beam elements. The geometric parameters of the cartwheel hinge as well as the number of nodes and elements \( v \) can be found in table 1.

Using the discretized structure, the elastic stiffness matrix \( k_e \) can be calculated. By solving the eigenvalue problem of equation (11), the primary displacement mode \( \phi_1 \) and the secondary displacement modes \( \phi_j \) can be determined. We take only the first secondary displacement mode (with the lowest eigenvalue) into account, so that \( \eta = 1 \). Each of the load cases \( f_\xi \) are expressed by a vector containing the node forces \( F_{x} \) and \( F_{y} \) (in \( x-, \) and \( y- \) direction, cf figures 3(b)) and (a) torque \( M \) for all element nodes:

\[
\mathbf{f}_\xi = \begin{bmatrix}
F_{x1} \\
F_{y1} \\
M_1 \\
F_{x2} \\
\vdots \\
M_w
\end{bmatrix},
\]  

(24)

In the considered example, the load cases consist of translational node forces only, i.e. \( M_{1,...,w} = 0 \). The forces of each load case are defined as a collinear force pair, which is hence self-equilibrated. This allows for a calculation without any clamping. However, for the method itself the occurring three rigid-body displacement modes must be neglected. These rigid-body modes describe the displacement of the rigid-body in space, which is why the associated modal stiffness value is almost equal to zero [36]. When calculating a constrained structure that is in a static equilibrium, no rigid-body modes occur.

In figure 3(c) a choice of force pairs including the top left node are illustrated by grey lines. Each line represents one load case in which only the force values of the nodes connected by a line are nonzero. Each pair introduces a unit force into the structure, in our case of one Newton. Any possible

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**Table 1. Parameters of the cart wheel hinge.**

| Parameter       | Value  |
|-----------------|--------|
| Beam thickness  | \( t = 0.8 \text{ mm} \) |
| Height          | \( h = 90 \text{ mm} \) |
| Length          | \( l = 80 \text{ mm} \) |
| Width           | \( b = 20 \text{ mm} \) |
| Young’s modulus | \( E = 185000 \text{ Nmm}^{-1} \) |
| Poisson’s ratio  | \( \nu = 0.36 \) |
| Number of element nodes | \( w = 51 \) |
| Number of elements | \( v = 52 \) |
connection between two nodes represents one load case. As a consequence, the number $k$ of load cases is dependent on the discretization of the structure and can be calculated using the following equation:

$$k = \frac{w(w-1)}{2}$$  \hspace{1cm} (25)

In our case $k$ results in 1275. However, an additional user-defined selection may also be suitable in order to avoid the application of prestressing forces on particularly sensitive regions of the structure or to reduce the computational effort.

An initial stress stiffness matrix can now be calculated for each load case. Further we can determine $\lambda_{e,i}$, $\lambda_{e,j}$, $\lambda_{g,i,\xi}$ and $\lambda_{g,j,\xi}$, where $i = 1$ and $j = 2$. The target stiffness value $\lambda_d$ results in $0.5\lambda_{e,1}$ in order to halve the initial primary stiffness value. Furthermore, $\beta$ is set to one, so that the selective compliant behavior of the prestressed state is at least equal to the unloaded case. The result of the optimization can be found in figure 3(d). The combination of two scaled load cases is the optimum solution. The design variables that scale all other load cases are negligibly small and thus eliminate these load cases.

First of all, it can be observed that the resulting force pairs introduce a compressive prestress into the mechanism. Since the stiffness of the mechanism is to be reduced, this corresponds to the effect of stress stiffening. The quality of the optimization results can be evaluated by analyzing the modal parameters and the mode shapes of the unstressed (indices un) and prestressed (indices pre) cartwheel hinge. For this purpose, an eigenvalue analysis is performed for both stress states. It is important to note that for modal analysis—although unusual in terms of CMs with selective compliance (cf chapter 2)—we take the initial stress stiffness matrix into account. This allows us to investigate the influence of the prestressing forces on both eigenvalues and eigenvectors. The resulting mode shapes are illustrated in figure 4. A grey dashed line shows the initial shape of the cartwheel hinge. The black solid line with circle markers represents the primary displacement mode of the unstressed cartwheel hinge, while the grey dotted line with circle markers illustrate the primary displacement mode of the prestressed cartwheel hinge. Both mode shapes are almost identical. However, slight deviations occur. This is
mainly due to the fact that the stiffness matrix is reduced to $k_e$ in the unloaded case, since the initial stress stiffness matrix results in zero without any prestressing. Consequently—by solving the eigenvalue problem taking into account the initial stress stiffness matrix—the deflection modes change in different prestressing states. However, figure 4 shows that this effect is negligibly small for this example and that the resulting deflection mode is almost identical to the desired displacement mode. The primary stiffness values and the secondary values are listed in table 2.

The left side of table 2 shows the modal parameters of the unloaded case, while the right side represents those of the prestressed state. In the middle column, the two cases are compared and the main aspects of the optimization problem are evaluated. It becomes clear that the primary stiffness value is halved as a result of the prestressing forces. This corresponds to the expectations, as it is specified by the definition of $\lambda_d$ in the constraints of the optimization formulation. However, small deviations may occur due to rounding errors, numerical errors and the fact that the deflection modes of the prestressed and unstressed case differ slightly (as mentioned above). Furthermore, table 2 examines the relationship between secondary stiffness and primary stiffness for both stress states. It can be clearly observed that this ratio even increases in the prestressed case and thus improves the selective compliant behavior.

In the next section, an optimization of a constrained cartwheel hinge is presented. Further, the results of the optimization are evaluated using a nonlinear 3-d FEA.

### 5.2. Constrained cartwheel hinge

Other than for the unconstrained cartwheel hinge described above, the initial load cases are no longer defined in force pairs, but single forces are applied. The directions of the individual forces are defined in such a way that each force vectors points to the positions of one of the remaining element nodes. The total number of load cases results in $k = 616$. The other assumptions made above, such as the structural model used, the optimization formulation or predefined parameters, do not change. The cartwheel hinge including its clamping is shown in figure 5(a).

The results are shown in figure 5(b). Compared to the unconstrained case, the force directions of the solution change slightly. Due to the clamping, the shape of the primary displacement mode in the unstressed state changes and therefore the result of the optimization. The primary displacement mode is compared in figure 6 in the unloaded (black colored line) and prestressed case (grey dotted line with circle markers).

The results of the eigenvalue analysis are shown in table 3. It can be observed that the primary stiffness is approximately halved. The ratios between secondary and primary stiffness also meet the requirements of the optimization constraints.

The amount of the determined forces results in $F_1 \approx F_2 \approx 41N$. In order to check this value, the linear buckling force of the cartwheel hinge can be determined by solving the following generalized eigenvalue problem for the optimized load case [37]:

$$ k_e \Psi = -\mu \cdot k_g \Psi. $$

(Figures and diagrams are not reproduced here but should be referenced as necessary for understanding the context and content of the text.)
with expectations, since the stiffness is to be halved by the
displacement mode, so that

The critical buckling load

\[ \lambda_{1,\text{crit}} = 81.4 \text{ N}, \]

which is about twice the optimized load. This is in line
with expectations, since the stiffness is to be halved by the
optimized prestressing forces and the stiffness becomes zero
at the critical buckling load.

In order to verify the linear assumptions, a structural non-
linear 3-d FEA is performed with the commercial simulation
software Ansys 19.2 to calculate the stiffness of the cartwheel
hinge with different amounts of prestressing forces.

The geometric and material parameters used for the calcu-
lation of the cartwheel hinge are listed in table 1. The material
used is a spring steel, which is described by an elastic isotropic
constitutive law. The structure is discretized by hexaeder solid
elements with quadratic interpolation. Geometrical nonlinear-
ities are taken into account, whereas thermal effects and non-
linearities resulting from the material are neglected. The struc-
ture is fixed at the lower end, as shown in figure 7(a). Two load
steps are used for calculation. In the first one, an imperfec-
tion is applied by means of a small load \( F_{\text{imp}} = 1 \text{ N} \) (cf figure
7(a)). In the second load step, \( F_{\text{imp}} \) is kept constant while the
prestressing forces \( F_1 \) and \( F_2 \) are applied. The imperfection
force is defined as a force with constant direction and amount.
In our FEA, this also applies to the prestressing forces. As
there are no large deformations in this example, the directions
of the forces do not change much. However, for prototypes
the prestressing forces are intended to point during the entire
deforation range to the corresponding nodes.

The rotational stiffness—which is related to the primary
displacement mode—is defined by a lever arm, \( F_{\text{imp}} \) and the
increasing angle resulting from a deflection. In figure 7(b) the
rotational stiffness in dependence of the prestressing force is
shown. Until shortly before the critical buckling load a lin-
ear course can be observed. Analyzing the stiffness in the
diagram without prestressing forces, \( F_1 = F_2 = 0 \text{ N} \), and with
the optimized force values, \( F_1 \approx F_2 \approx 41 \text{ N} \), it becomes clear
that the stiffness is about halved under acceptable tolerances.
Nonlinear effects, slight differences in geometry due to the
conversion of beam elements into solid elements, and the
aforementioned deviations, can explain minor deviations.

### Table 2. Eigenvalue analysis of the unconstrained cartwheel hinge (neglecting rigid-body modes).

| Unstressed Cartwheel Hinge | Comparison | Prestressed Cartwheel Hinge |
|----------------------------|------------|-----------------------------|
| \( \lambda_{1,\text{un}} = \lambda_{e,1} = 1.7197 \) | \( \lambda_{1,\text{un}} \approx 2 \cdot \lambda_{1,\text{pre}} \) | \( \lambda_{1,\text{pre}} = 0.8575 \) |
| \( \lambda_{2,\text{un}} = \lambda_{e,2} = 41.84 \) | \( \beta \cdot \lambda_{e,2} \leq \lambda_{2,\text{pre}} \) | \( \lambda_{2,\text{pre}} = 38.1537 \) |
| \( F_1 = F_2 = 0 \text{ N} \) |                               | \( F_1 \approx F_2 \approx 77 \text{ N} \) |

### Table 3. Eigenvalue analysis of the constrained cartwheel hinge (neglecting rigid-body modes).

| Unstressed Cartwheel Hinge | Comparison | Prestressed Cartwheel Hinge |
|----------------------------|------------|-----------------------------|
| \( \lambda_{1,\text{un}} = \lambda_{e,1} = 0.3906 \) | \( \lambda_{1,\text{un}} \approx 2 \cdot \lambda_{1,\text{pre}} \) | \( \lambda_{1,\text{pre}} = 0.1948 \) |
| \( \lambda_{2,\text{un}} = \lambda_{e,2} = 37.48 \) | \( \beta \cdot \lambda_{e,2} \leq \lambda_{2,\text{pre}} \) | \( \lambda_{2,\text{pre}} = 35.64 \) |

### Figure 6. Constrained cartwheel hinge: initial shape (grey dashed line); primary displacement mode in unstressed state (black solid line with cross markers); primary displacement mode in prestressed state (grey dotted line with circle markers).

\[ \Psi = \varphi_1 \]  \hspace{1cm} (27)
The two analyzed cases of the cartwheel hinge—constrained and unconstrained—show that the optimization procedure works as expected. In the following, the performance of the optimization method is tested on a MIMO mechanism.

### 5.3. Shape-adaptive airfoil profile

In the following, the optimization method is applied to a MIMO mechanism with selective compliance. The structure used describes a planar mechanism in the shape of an airfoil profile. In initial state, the outer shape corresponds to the so-called NACA0012 (National Advisory Committee for Aeronautics) profile and can be deformed into the NACA2412 profile, which is referred to the single primary displacement mode. The mechanism itself is designed to exhibit only this primary stiffness mode, i.e., \( m = 1 \). Both profiles are standardized shapes for the cross section of airfoils. Further information can be found in [38]. However, the mechanism itself is intended to be a support in the belt rib concept. In this concept, several supports are interconnected by an enclosing shell. The idea is that the mechanism does not extend along the entire wing but only at individual points in order to save as much weight as possible. Further information about the belt rib concept and the initial design of the mechanism used can be found in [34] and [39].

In our consideration, we concentrate only on the stiffness adaptation of the given structure, which is illustrated in figure 9.

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**Table 4.** Parameters of the airfoil structure.

| Parameter                      | Value     |
|--------------------------------|-----------|
| Beam thickness \( t_{\text{out}} \) | 0.5 mm    |
| Width \( b \)                   | 20 mm     |
| Number of elements \( m \)      | 76        |
| Young’s modulus \( E_{\text{out}} \) | 120000 N mm\(^{-1}\) |
| Young’s modulus \( E_{\text{inner}} \) | 3100 N mm\(^{-1}\) |

**Table 5.** Eigenvalue analysis of the unconstrained airfoil structure (neglecting rigid-body modes).

| Unstressed Cartwheel Hinge | Comparison | Prestressed Cartwheel Hinge |
|----------------------------|------------|-----------------------------|
| \( \lambda_{1,\text{un}} = \tilde{\lambda}_{c,1} = 0.2843 \) | \( \lambda_{1,\text{un}} \approx 2 \cdot \lambda_{1,\text{pre}} \) | \( \lambda_{1,\text{pre}} = 0.1414 \) |
| \( \lambda_{2,\text{un}} = \tilde{\lambda}_{c,2} = 1.9628 \) | \( \beta \cdot \frac{\lambda_{1,\text{un}}}{\lambda_{c,2}} \leq \frac{\lambda_{2,\text{un}}}{\lambda_{c,1}} \) | \( \lambda_{2,\text{pre}} = 1.6504 \) |

**Table 6.** Eigenvalue analysis of the unconstrained airfoil structure with \( \beta = 2.3 \) (neglecting rigid-body modes).

| Unstressed Cartwheel Hinge | Comparison | Prestressed Cartwheel Hinge |
|----------------------------|------------|-----------------------------|
| \( \lambda_{1,\text{un}} = \tilde{\lambda}_{c,1} = 0.2843 \) | \( \lambda_{1,\text{un}} \approx 2 \cdot \lambda_{1,\text{pre}} \) | \( \lambda_{1,\text{pre}} = 0.1459 \) |
| \( \lambda_{2,\text{un}} = \tilde{\lambda}_{c,2} = 1.9628 \) | \( \beta \cdot \frac{\lambda_{1,\text{un}}}{\lambda_{c,2}} \leq \frac{\lambda_{2,\text{un}}}{\lambda_{c,1}} \) | \( \lambda_{2,\text{pre}} = 2.1964 \) |

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Figure 7. Constrained cartwheel hinge: a) FEA boundary conditions b) FEA results.
In addition, the discretization of the structure is shown. As already mentioned above, the mechanism consists of an inner structure (cf figure 8(a) black color) and an outer enclosing shell (compare figure 8(a) grey color). These substructures have different material parameters as listed in table 4. Further essential parameters are also listed in table 4. While the outer shell exhibits a constant beam thickness $t_{\text{out}}$, the thicknesses of the inner structure beams are variable.
The optimization is similar to the optimization of the unconstrained cartwheel hinge described above. The same calculation model is used, the primary stiffness is to be halved and $\beta$ is equal to one. The structure is considered without any clamping and the individual load cases are defined as force pairs, which are shown exemplarily for one element node in figure 8(b). The total of $k = 2016$ initial load cases are reduced to only three load cases as a result of the optimization. These loads are illustrated in figure 8(c). If several load cases are applied to one element node as a result of the optimization, the corresponding forces are summed and only the resulting force is shown (cf figure 8(c) right side).

An eigenvalue analysis—similar to the cartwheel hinge examples—is performed on the basis of the optimization results and summarized in table 5. The eigenvectors are illustrated in figure 9. The primary stiffness is almost halved in the prestressed state compared to the unstressed state. Further, the ratio between the lowest eigenvalue of the secondary stiffness to the eigenvalue of the primary stiffness is, in the prestressed state, at least at the same level as in the unstressed state. This example shows in particular the performance of the optimization method on MIMO mechanisms.

In the preceding considerations, it is shown that the condition describing the selective compliant behavior in the unstressed and prestressed state is at least equivalent, i.e. $\beta = 1$. In the following, it is briefly shown that the method presented here is also able to increase the selective compliant behavior. For this purpose, we consider the airfoil structure again and set the value of $\beta$ to 2.3. Figure 10 shows the resulting load cases. The results of the eigenvalue analysis are given in table 6. This example shows that an increase of the selective compliant behavior is possible with the presented method. However, it should be noted that a further increase of the value $\beta$ may lead to more and more complex load cases.

6. Discussion and conclusion

The paper deals with the integration of a variable stiffness behavior into existing selective CMs, through the targeted application of prestressing forces. A convex optimization method, based on modal parameters, is presented to find the most suitable force application points. Within the optimization, scaling factors that determine the applied amount of prestressing forces are minimized. Additionally, a target stiffness value is achieved and the selective compliant behavior of the mechanism in the prestressed state is kept on the same level or even is increased in comparison to the unstressed state.

The optimization formulation is demonstrated using two example mechanisms: the cartwheel hinge and an airfoil mechanism with selective compliance. The results underline the feasibility and potential of the presented method as well as the validity of the assumptions made.

In future investigations, the procedure should be extended by a penalty function to reduce the number of force application points to a minimum. Further steps involve the implementation of physical prototypes. Both passive and active elements can be used to apply the prestressing forces. Passive elements could realize a constant stiffness change of a predesign CM. For example, a prestressed wire or an appropriate spring can be used to apply the calculated collinear force pairs. It should be noted, that large deformations can cause a change in the distance between the force application points of a force pair. To compensate for this, the stress-strain plateau of superelastic shape memory alloys can be used as a material for the wire. Alternatively, a spring can be specifically designed for this propose. With active elements, such as pneumatic muscles, the change in length can be equalized by control. Using active elements, the performance of the selective CMs with variable stiffness can be investigated in the context of the research topic variable stiffness actuation (VSA). Using suitable force application points and a controlled actuator, variable stiffness devices can be designed which exhibit a tailored spring characteristic or which are able to adapt smoothly to any unpredictable environmental impact.

Acknowledgments

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)—project number 418362853.

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