Determinants in QCD at finite temperature

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Abstract

We compute the functional determinant for the fluctuations around the most general self-dual configuration with unit topological charge for 4D SU(2) Yang-Mills with one compactified direction. This configuration is called “instanton with non-trivial holonomy” or “Kraan-van-Baal-Lee-Lu caloron”. It is a generalization of the usual instantons for the case of non-zero temperature. We extend the earlier results of Diakonov, Gromov, Petrov and Slizovsky to arbitrary values of parameters.

1 Introduction

Since the pioneering work of Callan, Dashen and Gross [1], where it was proposed to approximate the QCD path-integral by a superposition of instantons, many authors succeeded in developing and applying the instanton liquid model [2]. This model has many lattice and phenomenological confirmations. An instanton-like lumpy structure has been observed in lattice studies using various cooling techniques [3]. The instanton liquid model successfully explains the chiral symmetry breaking [4], describes hadronic correlators and details of hadronic structure [5] and solves the $U(1)_A$ problem [6].

However, the standard instanton liquid model could not describe confinement [7]. In [8] it was shown analytically that consideration of the more general solutions with non-trivial holonomy (KvBLL calorons) [9, 10] leads to the existence of two phases with phase transition temperature $T_c \simeq 1.1A$. In [11] a dilute gas of KvBLL calorons was studied in detail. This approach is also motivated by lattice observations [12]. See also recent review [13] of this activity.

The KvBLL caloron [9] is a generalization of the BPST instanton [14] and Harrington-Shepard instanton [15] to nontrivial holonomy. It is a self-dual gauge field configuration, periodical in one Euclidean time direction with period $1/T$, where $T$ is a temperature. It is characterized by an additional gauge invariant – holonomy or eigenvalues of the Wilson line that goes along the time direction. The fascinating feature of KvBLL caloron is that it consists of two BPS dyons for $SU(2)$ gauge group (see fig.1). Recently the higher charge calorons were obtained [16, 17].

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Figure 1: The action density of KvBLL caloron as a function of \( z, x \) at fixed \( t = y = 0 \). At large separations \( r_{12} \) the caloron is a superposition of two BPS dyon solutions (left, \( r_{12} = 1.5/T \)). At small separations they merge (right, \( r_{12} = 0.6/T \)).

In general, to take into account the effect of quantum fluctuations around a classical solution one expands the Euclidean action as follows [18]

\[
Z_{\text{cl}} = e^{-S_{\text{cl}}} \int d(\text{collective coordinates}) \cdot \text{Jacobian} \cdot \text{Det}^{1/2}(W_{\mu\nu}) \cdot \text{Det}(-D_{\mu}^{2}),
\]

which is the single-pseudoparticle contribution to the partition function. Here \( D_{\mu} \) is a covariant derivative in the adjoint representation, \( \text{Det}(W_{\mu\nu}) \) denotes the non-zero mode determinant of the quadratic form of the Euclidean action, parametrized by collective coordinates of the classical solution. For self-dual background fields one can show [19] that \( \text{Det}(W_{\mu\nu}) = \text{Det}^{4}(-D_{\mu}^{2}) \). Thus the determinant \( \text{Det}(-D_{\mu}^{2}) \) determines the weight of the quasi-particle or the probability with which it occurs in the partition function of the theory. The quantum determinant for the case of zero temperature was computed by ‘t Hooft [6] in 70’s and it still plays an important role in phenomenological and theoretical studies of strong interaction physics. The finite temperature generalization was performed by Gross, Pisarski and Yaffe [20]. They found the weight of the instanton with a trivial holonomy i.e. with unit value of the Wilson loop going along the periodic Euclidean time direction. More recently exact analytical expressions were derived for the determinants in the fundamental and adjoint representations of the \( SU(2) \) gauge group and arbitrary holonomy in [8, 21]. Unfortunately, these expressions were extremely cumbersome and occupied a significant amount of a hard disk space. Nevertheless, in [21] the results of numerical evaluation were presented. Several approximate results have recently been established for the \( SU(N) \) case [22].

In this article we argue the existence of a simple relation between determinants in the adjoint and fundamental representations. Namely, if the determinant in the fundamental representation is written in the form [21] (we take \( T = 1 \) and restore exact \( T \)-dependence in the last section only)

\[
\log \text{Det}(-\nabla^{2})|_{T=1} = \log \text{Det}(-\nabla^{2})|_{T=0} + A(v, r_{12})
\]

\[
+ V \left[ P \left( \frac{2\pi - v}{2} \right) - \frac{\pi^{2}}{12} \right] + \frac{\pi r_{12}}{2} P'' \left( \frac{2\pi - v}{2} \right),
\]

where \( P(v) = \frac{v^{2}(2\pi - v)^{2}}{12\pi^{2}} \) - a perturbative potential, \( v \) - a quantity connected with holonomy (when \( v = 0 \) and \( v = 2\pi \) the holonomy is trivial, see [8, 21] for more notations), and \( r_{12} \) is a distance between constituent dyons or \( r_{12} = \pi \rho^{2}T \), where \( \rho \) is an instanton size. The determinant in the adjoint representation is simply

\[
\log \text{Det}(-D_{\mu}^{2})|_{T=1} = \log \text{Det}(-D_{\mu}^{2})|_{T=0} + 16A(v, r_{12}) + \log \left( 1 + \frac{r_{12}v\nabla}{2\pi} \right)
\]

\[
+ VP(v) + 2\pi r_{12}P''(v),
\]

where we denote \( \nabla = 2\pi - v \). This connections is in spirit of the one found by Gross, Pisarski and Yaffe. This relation provides an independent check of the results of [8]. In particular the large \( r_{12} \)
asymptotic, found there, can be easily rederived on the base of this relation. All analytical and numerical results of [21] extend automatically to the isospin-1 case. The function $A(v, r_{12})$ is known with a good accuracy from [21].

We prove the relation (3) in the following way: using an exact expressions for the determinants [8, 21] we calculate analytically the expansion in powers of $1/r_{12}$ (see Appendix A), and check the relation up to the $1/r_{12}^{10}$. Then we check the relation numerically for several values of $r_{12}$ and $v$ with a precision $10^{-5}$. This calculation involves 3-fold integration of an expression of several Mb size which by itself is rather nontrivial and is possible due to the numerical and analytical power of Mathematica.

In section II we review old results related to KvBLL caloron, which are important for the derivation. In section III we derive the result basing on the $1/r_{12}$ expansion and in section IV we calculate the quantum weight of KvBLL caloron and make more accurate the estimate for the phase transition temperature made in [8].

2 Old results

Before proceeding to argue the relation (3) between determinants in different representations let us first remind results concerning determinants in the background of KvBLL caloron.

2.1 Zero temperature

When the size of KvBLL caloron $\rho$ or distance between constituent BPS dyons $r_{12} = \pi \rho^2 T$ is small compared to $1/T$ the caloron reduces to the usual BPST instanton. We recall the results by ’t Hooft for isospin-$1/2$ and isospin-$1$ zero-temperature instanton determinants [6]:

$$
\log \text{Det}(-\nabla^2)|_{T=0} = \frac{1}{6} \log \mu \rho + \alpha(1/2), \quad \alpha(1/2) = \frac{\gamma_E}{6} - \frac{17}{72} + \frac{\log \pi}{6} - \frac{\zeta'(2)}{\pi^2}
$$

$$
\log \text{Det}(-D^2)|_{T=0} = \frac{2}{3} \log \mu \rho + \alpha(1), \quad \alpha(1) = \frac{2\gamma_E}{3} - \frac{16}{9} + \frac{2\log 2}{3} + \frac{2 \log (2\pi)}{3} - \frac{4\zeta'(2)}{\pi^2},
$$

(4)

(5)

where $\mu$ is a Pauli-Villars regulator.

2.2 Nonzero temperature, trivial holonomy

The determinant in case of the trivial holonomy and non-zero temperature was calculated by Gross, Pisarski and Yaffe [20]. At the trivial holonomy the caloron becomes spherically symmetric. Consequently, the resulting expressions are much simpler. Nevertheless, even for this simpler case it has not been shown analytically that the isospin-$1$ and isospin-$1/2$ determinants are related.

For the isospin-$1/2$ the result reads

$$
\log \det(-\nabla^2)|_{T=1} = \log \det(-\nabla^2)|_{T=0} + A(r_{12}) - \frac{\pi r_{12}}{6}
$$

(6)

where $r_{12} = \pi \rho^2$ can be interpreted as a distance between dyons when the holonomy becomes nontrivial (we take $T = 1$). As it was verified numerically, the isospin-$1$ determinant can be written in the form

$$
\log \det(-D^2) = \log \det(-D^2)|_{T=0} + 16A(r_{12}) + \frac{4\pi r_{12}}{3},
$$

(7)

where $A(r_{12})$ has the following asymptotics

$$
A(r_{12}) = -\frac{\pi r_{12}}{36} + \mathcal{O}\left(\frac{r_{12}^{3/2}}{T}\right) = \frac{1}{18} - \frac{\gamma_E}{6} - \frac{\pi^2}{216} - \frac{\log(r_{12}/\pi)}{12} + \mathcal{O}\left(\frac{1}{r_{12}}\right)
$$

(8)
2.3 Non-trivial holonomy, isospin-1/2

The task of calculating the determinant in the background of the caloron with nontrivial holonomy is more complicated because the field configuration has no spherical symmetry and has an additional parameter \( v \), that is connected with the value of the holonomy (when \( v = 0, 2\pi \) the holonomy becomes trivial). In [21] an expression for the isospin-1/2 was found for all distances \( r_{12} \) and holonomies \( 0 \leq v \leq 2\pi \)

\[
\log \text{Det}(-\nabla^2) = \log \text{Det}(-\nabla^2)_{|_{T=0}} + A(v, r_{12}) + \left[ P \left( \frac{\nabla^2}{2} \right) - \frac{\pi^2}{12} \right] V + P'' \left( \frac{\nabla}{2} \right) \frac{\pi r_{12}}{2} \quad (9)
\]

where the function \( A(v, r_{12}) \) is fitted in eq.(26) and has the following large \( r_{12} \) asymptotics (we specify more terms in eq.(27) of Appendix A)

\[
A(v, r_{12}) = \log \left( \frac{2\pi}{6} - \frac{v \log v}{12\pi} - \frac{\nabla \log v}{12\pi} + \frac{1}{18} - \frac{\gamma}{6} - \frac{\pi^2}{216} \right) - \frac{\log(r_{12}/\pi)}{12} + O \left( \frac{1}{r_{12}^2} \right) \quad (10)
\]

and for small \( r_{12} \) it is

\[
A(v, r_{12}) = \frac{(3v\nabla - 2\pi^2)r_{12}}{72\pi} + O \left( \frac{1}{r_{12}^2} \right) \quad (11)
\]

Note that eq.(9) is a generalization of eq.(6) to arbitrary values of the holonomy.

2.4 Non-trivial holonomy, isospin-1

Isospin-1 or ghost determinant plays an important role since it determines the statistical weight of the configuration. In [8] an analytical expression for its large-\( r_{12} \) asymptotics was found:

\[
\log \text{Det}(-D^2) = V P(v) + \frac{2}{3} \log \mu + \frac{3\pi - 4v}{3\pi} \log v + \frac{3\pi - 4v}{3\pi} \log v + \frac{5}{3} \log(2\pi) + 2\pi P''(v) r_{12} + O \left( \frac{1}{r_{12}} \right)
\]

\[
+ \frac{1}{r_{12}} \left[ \frac{1}{v} + \frac{1}{v^2} + \frac{23\pi}{54} - \frac{8\gamma_E}{3\pi} - \frac{74}{9\pi} - \frac{4}{3\pi} \log \left( \frac{\nabla \nabla r_{12}^2}{\pi^2} \right) \right] + c_1 + O \left( \frac{1}{r_{12}^2} \right)
\]

where

\[
c_1 = \log 2 + \frac{5}{3} \log \pi - \frac{8}{9} - 2\gamma_E - \frac{2\pi^2}{27} - \frac{4\zeta'(2)}{\pi^2} \quad . \quad (13)
\]

The most nontrivial part is a constant \( c_1 \). It can be easily rederived independently, using the result for isospin-1/2 (10) and the relation (3).

3 Derivation of the relation

In this section we derive the relation (3) by comparing the large-\( r_{12} \) asymptotics. We shall check this relation up to the 10th order in \( 1/r_{12} \). The method of the calculation is taken from [8].

A derivative of the determinant with respect to a parameter of the background field is

\[
\frac{\partial \log \det(-D^2)}{\partial r_{12}} \equiv - \int \text{Tr} (\partial_{r_{12}} A_\mu J_\mu) \quad (14)
\]

Here \( J_\mu \) is a vacuum current related to Green function of the covariant Laplas operator in the background of KvBLL caloron. One of the results of [8] and [21] is an expression for the vacuum current \( J_\mu \) that is a rational function of \( r, s, R = e^{i\nabla}, S = e^{i\nabla}, E_0 = e^{2i\pi z_0} \) and \( v \), where \( r, s \) are distances from the BPS dyons (see. fig.2), \( 1/v \) and \( 1/\nabla \) are their core sizes.
The main point in the expansion is to divide space into tree domains: two balls of radius $R$, such that $1/v, 1/\sqrt{\nu} \ll R \ll r_{12}$, surrounding the centers of the constituent BPS dyons, and all the rest space. Then we expand the expression in powers of $1/r_{12}$ near each core and integrate it over the core domains. The expression outside the cores has an exponential precision and the only source of the $1/r_{12}$ terms here is the nontrivial domain of integration.

The vacuum current of the isospin-1 can be naturally divided into tree pieces $J_\mu = J^{r,s,m}_\mu$, i.e.

$$\frac{\partial}{\partial r_{12}} \log \det^{r,s,m}_\text{core} \equiv - \int_\text{core} \text{Tr} (\partial_\nu A_\mu J^{r,s,m}_{\mu})$$

(15)

where integration is over two ball of the radius $R$. The total $\frac{\partial}{\partial r_{12}} \log \det(-D^2)$ is a sum of these three contributions and a contribution that comes from the integration over the rest space $\frac{\partial}{\partial r_{12}} \log \det(-D^2)_\text{far}$. In Appendix A the expansion of these contributions in powers of $1/r_{12}$ is given. One can make sure that at all orders the following equalities hold:

$$\frac{\partial}{\partial r_{12}} \log \det^r_\text{cores} = 4 \frac{\partial A}{\partial r_{12}} - \frac{2 \log \frac{r_{12}}{\sqrt{\nu}}}{3r_{12}^2 \pi} + \frac{1}{3r_{12}} + \frac{\pi}{8r_{12}^2} - \frac{23}{18r_{12}^2 \pi} + (R^n \text{ terms})$$

(16)

$$\frac{\partial}{\partial r_{12}} \log \det^{s,m}_\text{cores} = 12 \frac{\partial A}{\partial r_{12}} - \frac{2 \log \frac{r_{12}}{\sqrt{\nu}}}{r_{12}^2 \pi} + \frac{1}{r_{12}} + \frac{3\pi}{8r_{12}^2} + \frac{5}{6r_{12}^2 \pi} - \frac{2\pi}{r_{12}(r_{12}\sqrt{\nu} + 2\pi)} + (R^n \text{ terms})$$

(17)

here we do not write $R^n$ terms as they all get cancelled with the similar terms in the contribution of the 'far' region.

$$\frac{\partial}{\partial r_{12}} \log \det^{r,s,m}_\text{far} \equiv \frac{8 \log \frac{r_{12}}{\sqrt{\nu}}}{3r_{12}^2 \pi} + \frac{8 - 9\pi^2}{18\pi r_{12}^2} + 2\pi P''(\nu) + (R^n \text{ terms})$$

(18)

adding up contributions from the 'far' and the 'core' regions we have

$$\frac{\partial}{\partial r_{12}} \log \det(-D^2) = \partial_{r_{12}}\left(16A + \frac{1}{3} \log r_{12} + \log (2\pi + r_{12}\sqrt{\nu}) + 2\pi P''(\nu)\right).$$

(19)

Finally we integrate it up to the small values of $r_{12}$, where KvBLL caloron reduces to the ordinary BPST instanton and the determinant is known. Matching with eq.(5) we conclude

$$\log \det(-D^2) = \log \det(-D^2)|_{T=0} + 16A(\nu, r_{12}) + \log \left(1 + \frac{r_{12}\sqrt{\nu}}{2\pi}\right) + 2\pi P''(\nu)r_{12} + VP(\nu)$$

(20)

The claim is that this answer is exact. It gives right large-$r_{12}$ asymptotics (13) and is consistent with the trivial holonomy results (6), (7). Moreover, we have tested it numerically for several values of $r_{12}$ and $\nu$ with a precision of order $10^{-5}$. We consider this as a serious proof of the relation (20).
Figure 3: Free energy of the caloron gas in units of $T^3V$ at $T = 1.5\Lambda$ (dotted), $T = 1.325\Lambda$ (solid) and $T = 1.25\Lambda$ (dashed) as function of the asymptotic value of $A_4$ in units of $T$.

4 Quantum weight

In this section we renew the main result of [8] – the quantum weight of KvBLL caloron. The concept of the quantum weight is discussed in detail, for example, in [8]. For a self-dual configuration it reads

$$Z = \int \prod_{i=1}^{p} d\xi_{i} e^{-S_{cl}} \left( \frac{\mu}{g\sqrt{2\pi}} \right)^{p} J \det^{-1}(-D^2)$$

where $\xi_{i}$ are coordinates on the moduli space of the configuration, $g$ is a gauge coupling, and $J$ is a measure on the moduli space. It can be expressed it terms of a metric on the moduli space

$$J = \sqrt{\det g_{ij}},$$

in [9] it was found that

$$J = 8(2\pi)^3 \rho^3 \left(1 + \frac{r_{12}}{2\pi} \sqrt{\nu} \right).$$

The total number of zero modes is 8. The associated collective coordinates are $z_{\mu}$ - position of KvBLL caloron center, one gauge orientation and two angles that determine the orientation is space, combined into $\mathcal{O}$, and the instanton size $\rho$. One can parameterize the moduli space by two 3D coordinates of dyons and two color orientations of the dyons. It turns our that the determinant does not depend on the color orientations.

$$\int \prod_{i=1}^{8} d\xi_{i} J = \int d^4z d^d\mathcal{O} \, dp \, \rho^3 \left(1 + \frac{r_{12}}{2\pi} \sqrt{\nu} \right) 16 (2\pi)^{10} = \int d^3z_1 \, d^3z_2 \left(1 + \frac{r_{12}}{2\pi} \sqrt{\nu} \right) \frac{1}{r_{12}} 16 (2\pi)^7$$

combining this with (21) and (3) we arrive at

$$Z_{KvBLL} = \int d^3z_1 \, d^3z_2 \, T^6 \, C_{A} \left(\frac{8\pi^2}{g^2}\right)^4 \left(\frac{\Lambda_{E} \gamma_{E}}{4\pi T}\right) \left(\frac{1}{Tr_{12}}\right)^4 \times \exp \left[-V \, P(v) - 16A(v, r_{12}) - 2\pi r_{12} P''(v)\right],$$

$$C_{A} = 2^8 \pi^2 \exp \left(\frac{16}{9} - 8\gamma_{E} + \frac{4\zeta'(2)}{\pi^2} + \frac{2}{3} \log 2 \right).$$

Surprisingly, the moduli space measure and the third term in the expression (20) for $\det(-D^2)$ exactly cancel each other.

The exact function $A(r_{12}, v)$ was fitted in [21] by

$$A(v, r_{12}) \simeq -\frac{1}{12} \log \left(1 + \frac{r_{12} T}{3}\right) - \frac{r_{12} \alpha}{216\pi(1 + r_{12} T)} + \frac{0.00302 \, r_{12} (\alpha + 9\nu \sqrt{\nu} / T)}{2.0488 + r_{12}^2 T^2}.$$
where \( \alpha = 18v \log \frac{5}{7} + 18v \log \frac{9}{7} - 216.6117T \). This expression has a maximum absolute error \( 5 \times 10^{-3} \).

### 4.1 Estimation of the \( T_c \)

We made slightly more accurate the crude estimation of the free energy of ensemble of K\(v\)BLL calorons without taking into account an interaction made in [8]. We do not repeat the details here and just give the result.

To obtain a phase transition one has to consider a gas of the calorons. The density of the calorons increases when the temperature becomes smaller. At some critical temperature \( T_c \) the density becomes sufficient to override the perturbative potential \( P(v) \) and nontrivial values of holonomy become preferable (see fig.3). Our new estimate for \( T_c \) is 1.3\( \Lambda \) which is slightly bigger then in [8].

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### A Series expansion with respect to \( 1/r_{12} \)

In this appendix we give results of the expansion in powers of \( 1/r_{12} \). This expressions are used to obtain eq.(16) and eq.(17).

\[
\frac{\partial A(v,r_{12})}{\partial r_{12}} = -\frac{1}{24r_{12}} + \left[ \frac{25}{144} + \frac{\gamma_e}{12\pi} + \frac{23\pi}{1728} + \frac{\log(vr_{12}/\pi)}{12\pi} \right] \frac{1}{r_{12}} + \frac{1}{12\pi r_{12}^2} - \frac{1}{24\pi r_{12}^2 v^2} (27)
\]

We divide \( \frac{\partial \log \text{det}^1}{\partial r_{12}} \) into two parts \( \frac{\partial \log \text{det}^1}{\partial r_{12}} \) and \( \frac{\partial \log \text{det}^2}{\partial r_{12}} \).

\[
\frac{\partial \log \text{det}^1}{\partial r_{12}} = \left[ \frac{3}{8} \frac{\pi^2}{v} \right] \frac{1}{r_{12}^2} + \left[ \frac{3}{8} \frac{\pi^2}{9} - 11\pi^4 \right] \frac{1}{r_{12}^4} + \left[ \frac{2873}{6720} - \frac{7\pi^2}{144} + \frac{437\pi^4}{75600} + \frac{2\pi^6}{2205} \right] \frac{1}{r_{12}^6} \tag{28}
\]

\[
\frac{\partial \log \text{det}^2}{\partial r_{12}} = \left[ \frac{33}{70} \frac{19\pi^2}{144} + \frac{10693\pi^4}{302400} - \frac{547\pi^6}{52920} \right] \frac{1}{r_{12}^4 v^2} + \left[ \frac{215}{149\pi^2} + \frac{149\pi^4}{288} - \frac{1651\pi^6}{21600} + \frac{1321\pi^8}{26460} \right] \frac{1}{r_{12}^6 v^3} + \left[ \frac{353\pi^2}{70} \frac{38039\pi^4}{138240} + \frac{8023081\pi^6}{121927680} + \frac{143\pi^8}{32256} \right] \frac{1}{r_{12}^8} + \left[ \frac{288}{71680} \frac{377\pi^2}{32} + \frac{412879\pi^4}{460800} + \frac{38671\pi^6}{564480} + \frac{7049\pi^8}{108000} \right] \frac{1}{r_{12}^8 v^2} + \left[ \frac{153973}{337920} \frac{407\pi^2}{96} + \frac{471609\pi^4}{1382400} - \frac{368419\pi^6}{145152} + \frac{2376127\pi^8}{4536000} - \frac{8461\pi^{10}}{1372140} \right] \frac{1}{r_{12}^{10} v^3} + \left[ \frac{77537}{168960} \frac{111\pi^2}{16} + \frac{50991947\pi^4}{4838400} + \frac{630755\pi^6}{72576} + \frac{14417821\pi^8}{4536000} + \frac{27623\pi^{10}}{177870} \right] \frac{1}{r_{12}^{10} v^5} + \left[ \frac{5}{v} + \frac{1}{v^{2}} \right] \frac{1}{r_{12}^{10} v^8} \tag{27}
\]
\[\frac{\partial \log \det_{\text{core}}^2}{\partial r_{12}} = \left[ \frac{5}{2\pi} + \frac{\gamma_E}{\pi} + \frac{3\pi}{36} + \log(v R/\pi) \right] \frac{1}{r_{12}} + \frac{1}{\pi} r_{12}^2 v - \frac{2}{2\pi} r_{12} v^2 + \left[ \frac{1}{3\pi} - \frac{\pi^2}{180} \right] \frac{1}{r_{12} v^3} \] (29) 

\[\frac{\partial \log \det_{\text{core}}^m}{\partial r_{12}} = \left[ \frac{1}{18} + \frac{\gamma_E}{3\pi} + \frac{108}{3\pi} + \log(v R/\pi) \right] \frac{1}{r_{12}} + \frac{1}{3\pi r_{12} v} - \frac{1}{3\pi r_{12} v} + \left[ \frac{1}{9\pi} - \frac{\pi^3}{540} \right] \frac{1}{r_{12} v^3} \] (30) 

\[+ (v \leftrightarrow -v) + O\left(\frac{1}{r_{12}}\right)\]

\[+ \frac{\partial \log \det_{\text{core}}^2}{\partial r_{12}} = \left[ \frac{1}{18} + \frac{\gamma_E}{3\pi} + \frac{108}{3\pi} + \log(v R/\pi) \right] \frac{1}{r_{12}} + \frac{1}{3\pi r_{12} v} - \frac{1}{3\pi r_{12} v} + \left[ \frac{1}{9\pi} - \frac{\pi^3}{540} \right] \frac{1}{r_{12} v^3} \] (31) 

\[+ \frac{\partial \log \det_{\text{core}}^m}{\partial r_{12}} = \left[ \frac{1}{18} + \frac{\gamma_E}{3\pi} + \frac{108}{3\pi} + \log(v R/\pi) \right] \frac{1}{r_{12}} + \frac{1}{3\pi r_{12} v} - \frac{1}{3\pi r_{12} v} + \left[ \frac{1}{9\pi} - \frac{\pi^3}{540} \right] \frac{1}{r_{12} v^3} \] (32) 

\[+ (v \leftrightarrow -v) + O\left(\frac{1}{r_{12}}\right)\]

Analogously, we divide \(\frac{\partial \log \det^2_{\text{core}}}{\partial r_{12}}\) into two parts \(\frac{\partial \log \det^m_{\text{core}}}{\partial r_{12}}\) and \(\frac{\partial \log \det^m_{\text{core}}}{\partial r_{12}}\). It is very convenient to extract the factor \((1 + \frac{\pi v}{2\pi})\) from the denominator of this contribution before making an expansion.

References

[1] C. G. Callan (Jr), R. F. Dashen, and D. J. Gross. Phys. Rev., D17:2717, 1978.

[2] D. Diakonov and V. Y. Petrov. Nucl. Phys., B245:259, 1984. & E. V. Shuryak and J. J. M. Verbaarschot. Phys. Rev., D52:295–306, 1995. & T. Schäfer and E. V. Shuryak. Phys. Rev., D53:6522–6542, 1996.
[3] E.-M. Ilgenfritz, M. L. Laursen, G. Schierholz, M. Müller-Preussker, and H. Schiller. Nucl. Phys., B268:693, 1986. & M. Teper. Phys. Lett., B171:86, 1986. & J. Hoek, M. Teper, and J. Waterhouse. Nucl. Phys., B288:589, 1987. & M. I. Polikarpov and A. I. Veselov. Nucl. Phys., B297:34, 1988. & M. C. Chu, J. M. Grandy, S. Huang, and J. W. Negele. Phys. Rev., D49:6039–6050, 1994. & T. A. DeGrand, A. Hasenfratz, and T. G. Kovacs. Nucl. Phys., B520:301–322, 1998.

[4] T. Banks and A. Casher. Nucl. Phys., B169:103, 1980.

[5] E. V. Shuryak and J. J. M. Verbaarschot. Nucl. Phys., B410:55–89, 1993. & T. Schäfer, E. V. Shuryak, and J. J. M. Verbaarschot. Nucl. Phys., B412:143–168, 1994. & T. Schäfer and E. V. Shuryak. Phys. Rev., D50:478–485, 1994.

[6] G. ’t Hooft. Phys. Rev. Lett., 37:8–11, 1976.

[7] D. Chen, R. C. Brower, J. W. Negele, and E. V. Shuryak. Nucl. Phys. Proc. Suppl., 73:512–514, 1999.

[8] D. Diakonov, N. Gromov, V. Petrov and S. Slizovskiy, Phys. Rev. D 70, 036003 (2004)

[9] T.C. Kraan and P. van Baal, Phys. Lett. B 428, 268 (1998) 268, hep-th/9802049; Nucl. Phys. B 533, 627 (1998), hep-th/9805168. & K. Lee and C. Lu, Phys. Rev. D 58, 025011 (1998), hep-th/9802108.

[10] T. C. Kraan and P. van Baal. Phys. Lett., B435:389–395, 1998.

[11] P. Gerhold, E. M. Ilgenfritz and M. Muller-Preussker, Nucl. Phys. B 760, 1 (2007) [arXiv:hep-ph/0607315];

[12] E.-M. Ilgenfritz, B. V. Martemyanov, M. Müller-Preussker, S. Shcheredin, and A. I. Veselov. Phys. Rev., D66:074503, 2002. & F. Bruckmann, E. M. Ilgenfritz, B. V. Martemyanov, and P. van Baal. Phys. Rev., D70:105013, 2004. & E.-M. Ilgenfritz, B. V. Martemyanov, M. Müller-Preussker, and A. I. Veselov. Phys. Rev., D73:094509, 2006. & E.-M. Ilgenfritz, B. V. Martemyanov, M. Müller-Preussker, and A. I. Veselov. Phys. Rev., D71:034505, 2005. & C. Gattringer. Phys. Rev., D67:034507, 2003. & C. Gattringer and S. Schaefer. Nucl. Phys., B654:30–60, 2003. & C. Gattringer, E.-M. Ilgenfritz, B. V. Martemyanov, M. Müller-Preussker, D. Peschka, R. Pullirsch, S. Schaefer, and A. Schäfer. Nucl. Phys. Proc. Suppl., 129:653–658, 2004. & F. Bruckmann, E.-M. Ilgenfritz, B. V. Martemyanov, M. Müller-Preussker, D. Nogradi, D. Peschka, and P. van Baal. Nucl. Phys. Proc. Suppl., 140:635–646, 2005. & E. M. Ilgenfritz, M. Müller-Preussker, and D. Peschka. Phys. Rev., D71:116003, 2005.

[13] P. van Baal, arXiv:hep-ph/0610409.

[14] A. Belavin, A. Polyakov, A. Schwartz and Yu. Tyupkin, Phys. Lett. 59, 85 (1975).

[15] B.J. Harrington and H.K. Shepard, Phys. Rev. D 17, 2122 (1978); ibid. 18, 2990 (1978).

[16] F. Bruckmann, D. Nogradi and P. van Baal, Nucl. Phys. B 698, 233 (2004) [arXiv:hep-th/0404210].

[17] F. Bruckmann and P. van Baal, Nucl. Phys. B 645, 105 (2002) [arXiv:hep-th/0209010].

[18] J.-L. Gervais and B. Sakita. Phys. Rev., D11:2943, 1975.

[19] L.S. Brown and D.B. Creamer, Phys. Rev. D 18, 3695 (1978).

[20] D.J. Gross, R.D. Pisarski and L.G. Yaffe, Rev. Mod. Phys. 53, 43 (1981).
[21] N. Gromov and S. Slizovskiy, Phys. Rev. D 71 (2005) 125019 [arXiv:hep-th/0504024].

[22] N. Gromov and S. Slizovskiy, Phys. Rev. D 73, 025022 (2006) [arXiv:hep-th/0507101].