SPECIAL CASES AND A DUAL VIEW ON THE LOCAL FORMULAS FOR EHRHART COEFFICIENTS FROM LATTICE TILES

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Abstract. McMullen’s formulas or local formulas for Ehrhart coefficients are functions on rational cones that determine the \(i\)-th coefficient of the Ehrhart polynomial as a weighted sum of the volumes of the \(i\)-dimensional faces of a polytope. This work focuses on the RS-\(\mu\)-construction as given in [13]. We give an explicit description of the construction from the dual point of view, i.e. given the cone of feasible directions instead of the normal cone as input value. We further show some properties of the construction in special cases, namely in case of symmetry and for the codimension one case.

1. Introduction

For a lattice polytope \(P\) in a Euclidean space \(V\) with lattice \(\Lambda\), the Ehrhart polynomial \(E_P\) counts the number of lattice points in the \(t\)-th dilate of \(P\),

\[ E_P(t) = |\Lambda \cap tP| = e_dt^d + e_{d-1}t^{d-1} + \ldots + e_1t + e_0, \]

for all \(t \in \mathbb{Z}_{\geq 0}\) and with \(d = \dim(P)\) (cf. Ehrhart 1962 [7]).

In 1983, McMullen [10] showed the existence of so-called McMullen’s formulas or local formulas for Ehrhart polynomials.

Definition 1. A real valued function \(\mu\) on rational cones in \(V\) is called a McMullen’s formula or local formula for Ehrhart coefficients, if for any lattice polytope \(P\) with Ehrhart polynomial \(E_P(t) = e_dt^d + e_{d-1}t^{d-1} + \ldots + e_1t + e_0\), we have

\[ e_i = \sum_{f \in \mathcal{F}_i} \mu(N_f) \text{vol}(f), \]

for all \(i \in \{0, \ldots, d\}\) with \(\mathcal{F}_i\) the set of all \(i\)-dimensional faces of \(P\).

Here, \(N_f\) is the (outer) normal vector of the face \(f\), i.e. the cone over the outer normal vectors of all facets meeting in \(f\). It is a convex rational cone, which means that it can be written as \(\{a_1v_1 + \ldots + a_kv_k \mid a_i \in \mathbb{R}_{\geq 0}\}\) for some vectors \(a_1, \ldots, a_k \in \Lambda\). The volume \(\text{vol}(f)\) denotes the relative volume of \(f\) with respect to the induced lattice in the affine span of \(f\), as defined in Section 2.

There have been several nice constructions of McMullen’s formulas, for example the ones by Pommersheim and Thomas, see [12], and Berline and Vergne, see [4]. This work will focus on the construction given in [13] by the author together with Achill Schürmann. Following the notation of Federico Castillo in his talk at Osaka University in 2018, we will refer to our construction as the RS-\(\mu\)-construction, in analogy to the BV-\(\alpha\)-construction by Berline and Vergne (cf. [5]). Each construction has different advantages. This construction, for instance, does not require a

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prior decomposition into unimodular cones and can be described using elementary geometric means. In contrast to [13], it might sometimes be more convenient to consider \( \mu \) not as a function on the normal cones of a polytope but from the dual point of view, given cones of feasible directions as input values. Note that both cones, the normal and the cone of feasible directions of a face, hold the same local information about the face and can be recovered from one another. In Section 2, we give a detailed construction of the RS-\( \mu \)-values given a full dimensional cone. To give more insight on how to compute the values, we then follow the construction given a polytope with cones of feasible directions and alongside compute the values in an example. In Section 3, we focus on the connection between the RS-\( \mu \)-values and the number of lattice points, thus giving a motivation for the idea behind the construction and an outline of the proof. In Section 4, we focus on symmetry and show a new result about the codimension one case under central symmetry.

2. Construction

Fix an ambient Euclidean space \( V \) and a lattice \( \Lambda \) of full rank. Let \( Q \) be a polyhedron and \( f \) a face of \( Q \). We define the cone of feasible directions of \( Q \) in the face \( f \), \( \text{fcone}(Q,f) \) for short, as

\[
\text{fcone}(Q,f) = \{ x \in V \mid \exists \varepsilon > 0: s + \varepsilon x \in Q \}
\]

for any point \( s \) in the relative interior \( \text{int}(f) \) of \( f \). Let further \( \text{lineal}(Q) \) be the lineality space of \( Q \), i.e. the biggest linear subspace contained in \( Q \), and denote \( \text{lat}(Q) := \Lambda \cap \text{lineal}(Q) \).

The construction we will give relies on a choice of certain lattice tiles, namely fundamental domains as defined below. Different choices lead to different values and thus give an infinite family of constructions.

**Definition 2.** For a polyhedron \( Q \subseteq V \) with induced sublattice \( L = \text{lineal}(Q) \cap \Lambda \) in the lineality space of \( Q \), a fundamental domain \( T(Q) \) is a subset of \( \text{lineal}(Q) \) with the following properties:

- \( \bigcup_{x \in L} x + T(Q) = \text{lineal}(Q) \),
- \( (x + T(Q)) \cap (y + T(Q)) = \emptyset \), for \( x, y \in L \) with \( x \neq y \) and
- every intersection of \( T(Q) \) with a linear subspace is measurable.

Examples of fundamental domains are given in Section 4.

The relative volume of a subset \( A \subseteq V \) denotes the volume of \( A \) in the affine span \( \text{aff}(A) \) normalized such that any fundamental domain in \( \text{aff}(A) \) has volume 1. Note that it is a lower dimensional volume if the affine span of \( A \) is.

We now give an explicit description of the construction of regions that determine the RS-\( \mu \)-values defined on full dimensional cones. In the subsequent section, we will give the whole construction in the way it occurs when starting with a polytope and determining the RS-\( \mu \)-values on its cones.

2.1. Construction on Cones. We will inductively define a map \( R \) from rational cones to subsets of \( V \), associating a region to each cone. From these regions, the RS-\( \mu \)-values can be computed via volume computations. The author is well aware that this is a formal and very compact description of the RS-\( \mu \)-construction. For a step-by-step construction with examples and pictures, see Section 2.2. The aim of this subsection is to give a clear, short and formally precise definition of the construction.
Let $C$ be a full dimensional rational cone in $V$. For non-full dimensional cones we intersect $V$ with the linear span of $C$ and consider that as our ambient space. For each subspace $A \subseteq V$ we assume to have chosen a fundamental domain $T(A)$ and denote $T(B) := T(\text{lineal}(B))$ for arbitrary subsets $B \subseteq V$.

If $C = V$ is the whole space, we set
\[ R(V) := T(V). \]
Otherwise, if $C \subsetneq V$, we assume we have constructed the regions $R(\text{fcone}(C, F))$ for all faces $F < C$. Let $X_C^F$ be the set of all points $x$ in $\text{lat}(\text{fcone}(C, F))$ that fulfill the conditions:
\begin{itemize}
  \item $[x + (R(\text{fcone}(C, F)) \cap \text{int}(\text{fcone}(C, F)))] \subseteq \text{int}(C)$ and
  \item $(x + R(\text{fcone}(C, F))) \cap (x' + R(\text{fcone}(C, F'))) = \emptyset$ for all $F' < C$, with $F'$ incomparable to $F$ and $x' \in \text{lat}(\text{fcone}(C, F'))$.
\end{itemize}

Then we define
\[ R(C) := (T(C) + \text{lineal}(C)^\perp) \setminus \bigcup_{F < C} (X_C^F + R(\text{fcone}(C, F))). \]

From this we can compute the values for the relative domain volume in each region $R(C)$ for the cone $C$:
\[ v_C := \text{vol}(R(C) \cap ((C \cap \Lambda) + T)). \]
And further the correction volumes for each $F < C$:
\[ w_C^F := \text{vol}(R(C) \cap (\text{lineal}(\text{fcone}(C, F)) \cap C)) \]
for all faces $F < C$.

Then we get the value for $C$ as
\[ \mu(C) := v_C - \sum_{F < C} w_C^F \cdot \mu(F). \]

2.2. Construction on fcones of Polytopes. To give a feeling for how the construction works given an actual polytope, we go through the construction from the perspective of a general, full dimensional lattice polytope $P$. Simultaneously, we compute the values for one particular example. Again, for each subspace $A \subseteq V$ we choose and fix a fundamental domain $T(A)$ and set $T := T(V)$.

To avoid unreadable expressions, we expand $\mu, R, v$ and $w$ to functions on the faces of $P$ by setting $\mu(f) := \mu(\text{fcone}(P, f)), R(f) := R(\text{fcone}(P, f)), v^f := v_{\text{fcone}(P, f)}$ and $w_g := w_{\text{fcone}(P, g)}$. In the example we use the same notation and keep in mind that all fcones are taken with respect to the example simplex $S$.

Example. Let $S = \text{conv}(v_1, v_2, v_3) \subseteq \mathbb{R}^2$ be the simplex with vertices $v_1 = (1, 0), v_2 = (2, 1)$ and $v_3 = (0, 2)$. We consider it as a lattice polytope in $\mathbb{R}^2$ with respect to the lattice $\mathbb{Z}^2$.

The first value we compute is $\mu(P)$ for $P$ as a face of itself. Since $\text{fcone}(P, P) = V$, we get
\[ R(P) = R(V) = T. \]
and
\[ v_P = \text{vol}(R(P) \cap ((V \cap \Lambda) + T)) = \text{vol}(T) = 1, \]
which directly determines the RS-$\mu$-value as
\[ \mu(P) = v_P = 1. \]

Remark. Using this to compute the $d$-th Ehrhart coefficient, we get
\[ e_d = \sum_{f \in F_d} \mu(f)\text{vol}(f) = \mu(P)\text{vol}(P) = 1 \cdot \text{vol}(P) = \text{vol}(P) \]
as desired, since the highest Ehrhart coefficient is known to be the relative volume of the polytope.

Example. The region $R(S) = R(\mathbb{R}^2)$ is given as the fundamental domain $T$ of $\mathbb{Z}^2$, which we choose to be the square with edgelength 1 and the origin as barycentre. Then we have $\mu(S) = v_S = \text{vol}(T) = 1$.

Now, let $F < P$ be a facet of $P$. Then $H^+ := \text{fcone}(P, F)$ is a halfspace containing a hyperplane $H$ and $R(H^+)$ is defined as
\[ R(H^+) = (T(F) + H^+) \setminus (X_{H^+} + T), \]
where
\[ X_{H^+} = \{ x \in \Lambda \mid (x + T_V) \subseteq \text{int}(H^+) \}. \]
That means that $R(H^+)$ equals the strip $(T(F) + H^+)$ minus all fundamental domains that lie entirely inside of $H^+$.

Then we have
\[ v_{H^+} = \text{vol}(R(H^+) \cap ((H^+ \cap \Lambda) + T)) \]
and
\[ w_{H^+}^V = \text{vol}(R(H^+) \cap H^+). \]
That yields the RS-$\mu$-value as
\[ \mu(H^+) = v_{H^+} - w_{H^+}^V \cdot \mu(V) = v_{H^+} - w_{H^+}^V. \]

Example. $S$ has three facets, namely the edges $f_1, f_2$ and $f_3$. Then for $i \in \{1, 2, 3\}$, $R(f_i) = \text{Strip}(f_i) \setminus \left( X_{S,f_i}^V + T \right)$, where $\text{Strip}(f_i) := T(f_i) + \text{lineal}(\text{fcone}(S, f_i))^\perp$.

An illustration of the construction of the region $R(f_1)$ is given in Figure 1. The area of $v_{f_1}$ is depicted in Figure 2 and the area of $w_{S,f_1}^V$ in Figure 3. Altogether we get
\[ \mu(f_1) = v_{f_1} - w_{S,f_1}^V = 2 - 3/2 = 1/2. \]

Using the results from Section 1.2 we know that the values for all facets equal 1/2.

If we now consider a codimension two face $f$ of $P$, then we have exactly two facets $F_1$ and $F_2$ of $P$ that meet in $f$. That means that $W := \text{fcone}(P, f)$ is a wedge defined by the intersection of the halfspaces $H_1^+ := \text{fcone}(P, F_1)$ and $H_2^+ := \text{fcone}(P, F_2)$, whose lineality spaces are the hyperplanes $H_1 := \text{lineal}(\text{fcone}(P, F_1))$ and $H_2 := \text{lineal}(\text{fcone}(P, F_2))$, respectively. Hence, the lineality space of $\text{fcone}(P, f)$ is the line $\text{lineal}(\text{fcone}(P, f)) = H_1 \cap H_2$.

As above, for $\text{fcone}(P, P) = V$ we have
\[ X_{V}^W = \{ x \in \Lambda \mid (x + T_V) \subseteq \text{int}(W) \}, \]
Figure 1. Construction of the regions $R(f_1)$ and $R(v_2)$ for the edge $f_1$ (above) and the vertex $v_2$ (below).

for $H_1^+$ we have

$$X_{H_1^+}^W = \{ x \in \text{lat}(H_1^+) \mid (x + R(H_1^+)) \cap \text{int}(H_1^+) \subseteq \text{int}(W) \text{ and } (x + R(H_1^+)) \cap (x' + R(H_1^+)) = \emptyset \text{ for all } x' \in \text{lat}(H_1^+) \}$$

and analogously for $H_2^+$

$$X_{H_2^+}^W = \{ x \in \text{lat}(H_2^+) \mid (x + R(H_2^+)) \cap \text{int}(H_2^+) \subseteq \text{int}(W) \text{ and } (x + R(H_2^+)) \cap (x' + R(H_1^+)) = \emptyset \text{ for all } x' \in \text{lat}(H_1^+) \}.$$ 

Then the region $R(W)$ is given by

$$R(W) = (T(W) + \text{lineal}(W)^+)^- \setminus \left( (X_V^W + T) \cup (X_{H_1^+}^W + R(H_1^+)) \cup (X_{H_2^+}^W + R(H_2^+)) \right).$$

The relative domain volume is

$$v_W = \text{vol}(R(W) \cap ((W \cap \Lambda) + T))$$

and we have to consider three correction volumes:

$$w_{H_1^+}^W = \text{vol}(R(W) \cap (H_1 \cap C)),$$

$$w_{H_2^+}^W = \text{vol}(R(W) \cap (H_2 \cap C))$$

and

$$w_V^W = \text{vol}(R(W) \cap W).$$

The RS-$\mu$-value for $W$ then is

$$\mu(W) = v_W - w_{H_1^+}^W \cdot \mu(H_1^+) - w_{H_2^+}^W \cdot \mu(H_2^+) - w_V^W \cdot \mu(V).$$
Example. To compute the value corresponding to the vertex $v_2$ of $S$, we construct the region
\[
R(v_2) = \mathbb{R}^2 \setminus \left( (X_{S}^{v_2} + T) \cup (X_{f_1}^{v_2} + T(f_1)) \cup (X_{f_3}^{v_2} + T(f_3)) \right)
\]
as shown in Figure 1 bottom. We then get $v_{v_2} = 7/4$ (cf. Figure 2) and as correction volumes we have the 1-dim. relative volumes $w_{v_2}^{f_1} = 1/2$ and $w_{v_2}^{f_2} = 1/2$ and the 2-dim. relative volume $w_{v_2}^{S} = 7/8$ (cf. Figure 3). Altogether we get the RS-$\mu$-value for the fcone of $S$ at the vertex $v_2$ as
\[
\mu(v_2) = v_{v_2} - w_{v_2}^{f_1} \cdot \mu(f_1) - w_{v_2}^{f_2} \cdot \mu(f_3) - w_{v_2}^{S} \cdot \mu(S)
= \frac{7}{4} - \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} - \frac{7}{8} \cdot 1 = \frac{3}{8}.
\]
Analogously, we can compute $\mu(v_3) = 1/4$ and $\mu(v_1) = 3/8$. For reasons of symmetry, the latter has to equal the value $\mu(v_2)$, see Section 4.1 for an explanation.

FIGURE 2. Values for the relative domain volume $v_f$ for certain faces $f < g$ of $S$.

FIGURE 3. Values for the correction volume $w_g^f$ for certain faces $f < g$ of $S$.

For the general faces $f$ of $P$ the region of $f$ is computed as
\[
R(f) := (T(f) + \text{lineal}(\text{fcone}(P, f))^\perp) \setminus \bigcup_{g > f} (X_{g}^{f} + R(g))
\]
the relative domain volume is
\[
v_f := \text{vol}(R(f) \cap ((f \cap \Lambda) + T))
\]
and the correction volume for each face $g$ of $P$ with $g > f$ is
\[
w_g^f := \text{vol}(R(f) \cap (\text{lineal}(\text{fcone}(P, g)) \cap f)).
\]
Then we get the value for $fcone(P, f)$ as

$$
\mu(f) := v_f - \sum_{g > f} w_g \cdot \mu(g).
$$

3. The geometrical connection between the RS-$\mu$-values and the number of lattice points

This section gives a motivation behind the definition of the RS-$\mu$-values and, along the way, an outline of the proof. Complete proofs can be found in [13].

Let $P = \text{conv}(v_1, \ldots, v_m)$ be a full dimensional lattice polytope in $d$-dimensional Euclidean space $V$. We use the same notation as in Section 2.2, where we write $\mu(f)$ for $\mu(fcone(P, f))$ and the same for $R$, $v$ and $w$.

For a face $f$ of $P$ and $t \in \mathbb{Z}_{\geq 0}$ define the set $X(tf)$ of all feasible lattice points in $t \cdot f$ as

$$
X(tf) := \bigcap_{i=1}^{m} \left( X_{f_i} + tv_i \right).
$$

After the construction of all regions for $P$, we get the following tiling of $V$

**Theorem 1 ([13 Theorem 1]).** Let $P \subseteq V$ be a full dimensional lattice polytope. There exists a $t_0 \in \mathbb{Z}_{>0}$ such that for each $t \geq t_0$ we have a tiling of $V$ into translated regions of the form

$$
\{ x + R(f) : f \leq P, x \in X(tf) \}.
$$

Example. The tiling of $8 \cdot S$ for the example polygon $S$ from Section 2.2 can be seen in Figure 4 on the left; in the picture, each region is translated by a unique lattice point marked $\bullet$ and the union of all these lattice points within a certain face $f$ form the set $X(8f)$.

![Figure 4](image-url)
The crucial step between counting the number of lattice points in $P$ and the RS-$\mu$-value given by volumes is to interpret the number of lattice points in $tP$ as the volume of all translates of $T$ by the lattice points in $P$:

$$|\Lambda \cap tP| = \sum_{x \in \Lambda \cap tP} \text{vol}(x + T) = \text{vol}((\Lambda \cap tP) + T).$$

The first part of the equation holds, since by definition $\text{vol}(T) = 1$ for any fundamental domain $T$ of $\Lambda$, and the second part follows from $(x + T) \cap (y + T) = \emptyset$ for all $x, y \in \Lambda$ with $x \neq y$. We call the set $DC := (\Lambda \cap tP) + T$ a (fundamental) domain complex of $tP$ (cf. Figure 4, middle). By taking the volume of the respective part of the domain complex in each region of the tiling in Theorem 1, we get

$$|\Lambda \cap tP| = \text{vol}(DC) = \sum_{f \leq P} \sum_{x \in X(f)} \text{vol}((x + R(f)) \cap DC).$$

It turns out that (*) is exactly the value $v_f$ as defined in Section 2, which thus yields

$$|\Lambda \cap tP| = \sum_{f \leq P} \sum_{x \in X(f)} v_f = \sum_{f \leq P} v_f \cdot |X(f)|, \quad (1)$$

see Figure 4, right.

Example. Equation (1) for $4S$ gives

$$|\mathbb{Z}^2 \cap 4S| = 109 = 1 \cdot 70 + 2 \cdot 6 + 2 \cdot 6 + 1 \cdot 7 + 7 \cdot 4 \cdot 1 + 7 \cdot 4 \cdot 1 + 9 \cdot 2 \cdot 1$$

$$= \sum_{f \leq S} v_f \cdot |X(8f)|,$$

where the sum runs over the faces $S, f_1, f_2, f_3, v_1, v_2, v_3$ in that order.

To extract the value for the local formula from Equation (1), we need to determine the difference between $|X(tf)|$ and $\text{vol}(f)$. It can be shown (see Lemma 8) that by the construction of the correction volumes $w$ we have that

$$\text{vol}(tf) = \sum_{g \leq f} w_g^f \cdot |X(tf)|,$$

where we formally define $w_f^f = 1$.

We want the RS-$\mu$-value to be defined such that

$$|\Lambda \cap tP| = \sum_{f \leq P} \mu(f) \cdot \text{vol}(f)$$

holds. Using Equation (2), we get

$$\sum_{f \leq P} \mu(f) \cdot \text{vol}(f) = \sum_{f \leq P} \mu(f) \cdot \left(\sum_{g \leq f} w_g^f \cdot |X(tf)|\right).$$
By combinatorially reordering the right hand side, we get
\[
\sum_{f \leq P} \mu(f) \cdot \text{vol}(f) = \sum_{f \leq P} \left[ \mu(f) \cdot |X(tf)| + \mu(f) \cdot \sum_{h > f} w_h^f \cdot |X(tf)| \right]
\]
\[
= \sum_{f \leq P} \left[ \left( \mu(f) + \sum_{h > f} w_h^f \right) \cdot |X(tf)| \right]
\]

Hence, we want to define the RS-\(\mu\)-value such that the following equation holds
\[
\sum_{f \leq P} v_f \cdot |X(tf)| = |\Lambda \cap tP| = \sum_{f \leq P} \left[ \left( \mu(f) + \sum_{h > f} w_h^f \right) \cdot |X(tf)| \right].
\]

Comparing coefficients we get
\[
v_f = \mu(f) + \sum_{h > f} w_h^f,
\]
which directly leads to the definition of the RS-\(\mu\)-values as
\[
\mu(f) = v_f - \sum_{h > f} w_h^f.
\]

4. Symmetry and special cases

Symmetry of the RS-\(\mu\)-values can be achieved by choosing symmetric fundamental domains, for example by taking Dirichlet–Voronoi cells, as we will show in Section 4.1.1. We will finish this article by proving that given central symmetry, the values on halfspaces are determined to be 1/2. That implies that the value is always 1/2 for fcones of a facet of a polytope.

4.1. Dirichlet–Voronoi cells and symmetry of RS-\(\mu\). Possibly the most natural choice of fundamental domains are Dirichlet–Voronoi cells. Given a space \(V\) and an inner product \(\langle \cdot, \cdot \rangle\) with induced norm \(\| \cdot \|\), the Dirichlet–Voronoi cell of a sublattice \(L \subseteq \Lambda\) is defined as
\[
\text{DV}(L, \langle \cdot, \cdot \rangle) := \{ x \in \text{lineal}(L) : \| x \| \leq \| x - a \| \text{ for all } a \in L \}.
\]

In this definition, it is not yet a fundamental domain of the lattice \(L\), but by considering the Dirichlet–Voronoi cell half open, it can be seen as a fundamental domain of the lattice. Dirichlet–Voronoi cells are naturally centrally symmetric and can be forced to have certain symmetries by choosing a suitable inner product.

Let \(P\) be a lattice polytope and \(\mathcal{G}\) a subgroup of all lattice symmetries of \(P\), i.e. \(\mathcal{G}\) is a finite matrix group with \(A \cdot P := \{ A \cdot x : x \in P \} = P\) and \(A \cdot \Lambda = \Lambda\) for all \(A \in \mathcal{G}\). Then we can define a \(\mathcal{G}\)-invariant inner product by taking
\[
\langle x, y \rangle_{\mathcal{G}} := x^t G y \quad \text{for all } x, y \in V,
\]
with the Gram matrix \(G\) given by
\[
G := \frac{1}{|\mathcal{G}|} \sum_{A \in \mathcal{G}} A^t A.
\]
Let $\| \cdot \|_G$ be the induced norm and let $D$ be the Dirichlet–Voronoi cell for $\Lambda$ given by the that particular inner product,

$$D := DV(\Lambda, \langle \cdot, \cdot \rangle_G) = \{ x \in V : \| x \|_G \leq \| x - p \|_G \text{ for all } p \in \Lambda \}.$$ 

Then $D$ is invariant under the action of $G$: Let $x \in D$, then for $A \in G$ we have

$$\| Ax \|_G = \| x \|_G \leq \| x - p \|_G = \| Ax - Ap \|_G \text{ for all } p \in \Lambda.$$ 

Since $AA = \Lambda$, we get $AD \subseteq D$ for all $A \in G$. Substituting $A$ by $A^{-1}$, we get $A^{-1}D \subseteq D$ which yields $D \subseteq AD$ and hence $AD = D$. Similarly, we see that for all faces $f$ in the same $G$-orbit the fcones and Dirichlet–Voronoi cells in $\Lambda \cap N_f^\perp$ are mapped onto each other. Hence, the used regions are invariant under the action of $G$ and $\mu$ is constant on $G$-orbits.

### 4.2. Codimension one faces under central symmetry.

It is known that the second highest Ehrhart coefficient always equals $1/2$ times the sum over the relative volumes of the facets of a polytope. A natural conjecture would be that all values of McMullen’s formulas corresponding to facets (in this case all values on halfspaces) have the value $1/2$. This is not true in general for the RS-$\mu$-values, but we show here that it does hold for the RS-$\mu$-values when all fundamental domains are centrally symmetric. Therefore, in the following let $T(A)$ be a centrally symmetric fundamental domain for each $A \subseteq V$. Again, we denote $T := T(V)$.

Now, let $F$ be a facet of $P$ with fcone $C := fcone(P, F)$. That means lineal$(C)$ is a hyperplane in $V$. Let $C^+$ be the open halfspace inside of $C$ and $C^-$ the open halfspace on the other side, i.e. the complement of $C$.

Then

$$R(C) = (T(C) + \text{lineal}(C)_{\perp}) \setminus (X_C^C + T)$$

$$= \text{Strip}(C) \cap \left( (\Lambda \setminus X_C^C) + T \right),$$

where

$$X_C^C = \{ x \in \Lambda \mid (x + T) \subseteq C^+ \}.$$ 

Taking a closer look at the construction in Section 2 we note that for computing $v_C$ and $w^*_K$ and hence $\mu(C)$ it is only necessary to consider $R(C)$ intersected with the
union of all fundamental domains that have a nonempty intersection with lineal(C).
We therefore consider the relevant part $\tilde{R}(C)$ of the region $R(C)$:

$$\tilde{R}(C) = \left\{ p \in \text{lat}(W) \mid (p + T) \cap \text{lineal}(C) \neq \emptyset \right\} + T \cap \text{Strip}(C)$$

This relevant part of $R(C)$ can be partitioned into three parts

$$\tilde{R}(C) = (X + T) \cap \text{Strip}(C)$$

$$:= X_0$$

$$\cup ((X \cap C^+) + T) \cap \text{Strip}(C)$$

$$:= X_+$$

$$\cup ((X \cap C^-) + T) \cap \text{Strip}(C),$$

$$:= X_-$$

where the unions are disjoint, since $X_0, X_+, X_-$ are. For an illustration see Figure 6.

Recall that $\mu(C)$ is defined as

$$\mu(C) = v_C - \mu(W) \cdot w^{\perp}_W$$

$$= \text{vol}(\tilde{R}(C) \cap ((C \cap \text{lat}(W)) + T)) - 1 \cdot \text{vol}(\tilde{R}(C) \cap (W \cap C))$$

To show that $\mu(C) = 1/2$, we show that everything but half the lattice cell around the origin cancels out nicely. We use the fact that everything is centrally symmetric in the following sense:

Let $\sigma_0$ be the point reflection at the origin:

$$\sigma_0 : V \rightarrow V$$

$$v \mapsto -v$$

Then $\sigma_0(T) = T$ by assumption and since $C$ is a halfspace, we also have

$$\sigma_0(\text{lineal}(C)) = \text{lineal}(C),$$

$$\sigma_0(C^+) = C^-,$$

$$\sigma_0(C^-) = C^+,$$

$$\sigma_0(\text{Strip}(C)) = \text{Strip}(C)$$

$$\sigma_0(\tilde{R}(C)) = \tilde{R}(C)$$
Now observe that \( \text{vol}((X_0 + T) \cap \text{Strip}(C)) = \text{vol}(0 + T) = 1 \), since the intersection of a centrally symmetric fundamental domain with a linear subspace is always contained in the centrally symmetric fundamental domain of that subspace.

Since \( \sigma_0 \) does not change the volume, we can use it to gain information on the occurring volumes:

\[
\sigma_0((T \cap \text{Strip}(C)) \cap C^+) = T \cap \text{Strip}(C) \cap C^-
\]

and hence,

\[
\text{vol}(((X_0 + T) \cap \text{Strip}(C)) \cap C^-) = \frac{1}{2}.
\]

**Figure 7.** *Left:* \( v_C \), *middle:* \( w^C_W \) and *right:* \( w^C_W - v_C \)

Moreover, we have the disjoint union:

\[
\tilde{R}(C) \cap C^+ = ((X_0 + T) \cap \text{Strip}(C)) \cap C^+
\]

\[
\cup (X_+ + T) \cap \text{Strip}(C)) \cap C^+
\]

\[
\cup (X_- + T) \cap \text{Strip}(C)) \cap C^+
\]

and since \( \sigma_0(((X_- + T) \cap \text{Strip}(C)) \cap C^+) = ((X_+ + T) \cap \text{Strip}(C)) \cap C^+ \), the two have equal volume and we have

\[
w^C_V = \text{vol}(\tilde{R}(C) \cap (V \cap C^+))
\]

\[
= \text{vol}(((X_0 + T) \cap \text{Strip}(C)) \cap C^+)
\]

\[
+ \text{vol}((X_+ + T) \cap \text{Strip}(C)) \cap C^+)
\]

\[
+ \text{vol}((X_+ + T) \cap \text{Strip}(C)) \cap C^-
\]

\[
= \text{vol}(((X_0 + T) \cap \text{Strip}(C)) \cap C^+)
\]

\[
+ \text{vol}((X_+ + T) \cap \text{Strip}(C))
\]

see Figure 7 middle. Together with

\[
v_C = \text{vol}(\tilde{R}(C) \cap ((C \cap \Lambda) + T)) = \text{vol}(((X_0 + T) \cap \text{Strip}(C)) + \text{vol}((X_+ + T) \cap \text{Strip}(C)),
\]

(cf. Figure 7 left), we finally get

\[
\mu(C) = v_C - w^C_V
\]

\[
= \text{vol}(\tilde{R}(C) \cap ((C \cap \Lambda) + T) - \text{vol}(\tilde{R}(C) \cap (V \cap C))
\]

\[
= \text{vol}((X_0 + T) \cap \text{Strip}(C)) - \text{vol}(((X_0 + T) \cap \text{Strip}(C)) \cap C^+)
\]

\[
= \text{vol}((X_0 + T) \cap \text{Strip}(C)) \cap C^-
\]

\[
= \frac{1}{2}
\]

as we wanted to show (cf. Figure 7 right).
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