Additive arithmetic functions meet the inclusion–exclusion principle: Asymptotic formulas concerning the GCD and LCM of several integers

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Abstract. We obtain asymptotic formulas for the sums $\sum_{n_1,\ldots,n_k \leq x} f((n_1,\ldots,n_k))$ and $\sum_{n_1,\ldots,n_k \leq x} f([n_1,\ldots,n_k])$, involving the GCD and LCM of the integers $n_1,\ldots,n_k$, where $f$ belongs to certain classes of additive arithmetic functions. In particular, we consider the generalized omega function $\Omega_p(n) = \sum_{p^\nu \mid n} \nu^2$ investigated by Duncan (1962) and Hassani (2018), and the functions $A(n) = \sum_{p^\nu \mid n} \nu p$, $A^*(n) = \sum_{p^\nu \mid n} p$, $B(n) = A(n) - A^*(n)$ studied by Alladi and Erdős (1977). As a key auxiliary result, we use an inclusion–exclusion-type identity.

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1 Motivation

In the paper, we use the following notation: $\mathbb{N} = \{1,2,\ldots\}$, $\mathbb{N}_0 = \{0,1,2,\ldots\}$, $(n_1,\ldots,n_k)$ and $[n_1,\ldots,n_k]$ denote the greatest common divisor (GCD) and least common multiple (LCM) of $n_1,\ldots,n_k \in \mathbb{N}$, $1(n) = 1$ ($n \in \mathbb{N}$), $\mu$ is the Möbius function, $\tau(n) = \sum_{d \mid n} 1$, $\omega(n) = \sum_{p \mid n} 1$, $\Omega(n) = \sum_{p^\nu \mid n} 1$, $A(n) = \sum_{p^\nu \mid n} \nu p$, $A^*(n) = \sum_{p \mid n} p$, $B(n) = A(n) - A^*(n)$, $f \ast g$ denotes the convolution of the arithmetic functions $f$ and $g$, $\gamma$ is Euler’s constant, and $\left\langle \nu \right\rangle$ are the (classical) Eulerian numbers.

Let $f: \mathbb{N} \to \mathbb{C}$ be an arithmetic function, and let $k \in \mathbb{N}$. We are interested in asymptotic formulas for the sums

$$G_{f,k}(x) := \sum_{n_1,\ldots,n_k \leq x} f((n_1,\ldots,n_k)) \quad \text{and} \quad L_{f,k}(x) := \sum_{n_1,\ldots,n_k \leq x} f([n_1,\ldots,n_k]).$$

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By using the general identity
\[ G_{f,k}(x) = \sum_{d \leq x} (\mu * f)(d) \left\lfloor \frac{x}{d} \right\rfloor ^k, \]
valid for every function \( f \) (see Lemma 1), it is possible to deduce asymptotic formulas for \( G_{f,k}(x) \) for various functions \( f \). For example, if \( k \geq 3 \), then
\[ \sum_{n_1, \ldots, n_k \leq x} \frac{1}{(n_1, \ldots, n_k)} = \frac{\zeta(k+1)}{\zeta(k)} x^k + O(x^{k-1}), \quad (1.1) \]
\[ \sum_{n_1, \ldots, n_k \leq x} \tau((n_1, \ldots, n_k)) = \zeta(k) x^k + O(x^{k-1}), \quad (1.2) \]
where \( \tau(n) \) is the divisor function. See [11, Sect. 1] and [19, Thm. 3.6].

It is more difficult to obtain asymptotic formulas for the sums \( L_{f,k}(x) \) concerning the LCM of integers. If the function \( f \) is multiplicative, then \( f([n_1, \ldots, n_k]) \) and \( f((n_1, \ldots, n_k)) \) are multiplicative functions of \( k \) variables. Therefore the multiple Dirichlet series
\[ \sum_{n_1, \ldots, n_k = 1}^{\infty} \frac{f([n_1, \ldots, n_k])}{n_1^{s_1} \cdots n_k^{s_k}} \]
can be expanded into an Euler product, and the multiple convolution method can be used to deduce asymptotic formulas. For example, the counterpart of (1.2) is
\[ \sum_{n_1, \ldots, n_k \leq x} \tau([n_1, \ldots, n_k]) = x^k Q_k(\log x) + O(x^{k-1+\theta+\epsilon}), \]
where \( k \geq 2 \), \( Q_k(t) \) is a polynomial in \( t \) of degree \( k \), and \( \theta \) is the exponent in the Dirichlet divisor problem. See [19, Thm. 3.4]. This approach does not furnish a formula with remainder term as a counterpart of (1.1).

In [11, Thm. 2.3], it was only proved that for \( k \geq 3 \),
\[ S_k(x) := \sum_{n_1, \ldots, n_k \leq x} \frac{1}{[n_1, \ldots, n_k]} \asymp (\log x)^{2^k - 1} \]
and conjectured that
\[ S_k(x) = P_{2^k-1}(\log x) + O(x^{-r}), \]
where \( P_{2^k-1}(t) \) is a polynomial in \( t \) of degree \( 2^k - 1 \), and \( r \) is a positive real number. This conjecture was proved in [6] by a different method using analytic techniques. See [6, 11, 18, 19] for more detail.

In this paper, we obtain asymptotic formulas for the sums \( G_{f,k}(x) \) and \( L_{f,k}(x) \) for certain classes of additive functions. In particular, we consider the generalized omega function \( \Omega_k(n) = \sum_{p^r \parallel n} \nu^f \) investigated by Duncan [5] and Hassani [9] and the functions \( A(n) = \sum_{p^r \parallel n} \nu p, A^*(n) = \sum_{p \parallel n} p, B(n) = A(n) - A^*(n) \) studied by Alladi and Erdős [1]. A key identity of our approach is based on the application of the inclusion–exclusion principle to additive functions. See Proposition 1, which may be known in the literature, but we could not find any reference. Other key results used in the proofs are Saffari’s estimate obtained for the sum \( \sum_{n \leq x} \omega(n) \) and an estimate for \( \sum_{n \leq x} A(n) \), where \( A(n) \) is the Alladi–Erdős function. See [1, 16, 17]. The main results on the asymptotic formulas are stated in Section 3. Some preliminary lemmas used to the proofs are included in Section 4, and the proofs of the main results are given in Section 5.
2 Additive functions and the inclusion–exclusion principle

We recall that an arithmetic function \( f : \mathbb{N} \to \mathbb{C} \) is additive if \( f(mn) = f(m) + f(n) \) for all \( m, n \in \mathbb{N} \) with \( (m, n) = 1 \). If \( f \) is additive, then \( f(n) = \sum_{p^r \mid n} f(p^r) \). Some examples of additive functions are \( \log n \), \( \omega(n) \) and \( \Omega(n) \).

If \( f \) is additive, then \( f((m, n)) = f(m) + f(n) \) for all \( m, n \in \mathbb{N} \). To see this, it suffices to consider the case where \( m \) and \( n \) are powers of the same prime \( p \), that is, \( m = p^a \) and \( n = p^b \), where \( a, b \in \mathbb{N} \). Now we trivially have \( f(p^{\max(a, b)}) + f(p^{\min(a, b)}) = f(p^a) + f(p^b) \). In a similar way, if \( f \) is additive, then for all \( n_1, n_2, n_3 \in \mathbb{N} \),

\[
\begin{align*}
f([n_1, n_2, n_3]) &= f(n_1) + f(n_2) + f(n_3) - f((n_1, n_2)) - f((n_1, n_3)) - f((n_2, n_3)) \\
&\quad + f((n_1, n_2, n_3)).
\end{align*}
\]

We generalize these identities to several integers.

**Proposition 1.** Let \( f : \mathbb{N} \to \mathbb{C} \) be an additive function, and let \( k \in \mathbb{N} \) and \( n_1, \ldots, n_k \in \mathbb{N} \). Then

\[
f([n_1, \ldots, n_k]) = \sum_{1 \leq j \leq k} (-1)^{j-1} \sum_{1 \leq i_1 < \cdots < i_j \leq k} f((n_{i_1}, \ldots, n_{i_j})).
\]

**Proof.** It suffices to prove identity (2.1) if \( n_1 = p^{\nu_1}, \ldots, n_k = p^{\nu_k} \) are powers of the same prime \( p \) with \( \nu_1, \ldots, \nu_k \in \mathbb{N}_0 \), that is,

\[
f(p^{\max(\nu_1, \ldots, \nu_k)}) = \sum_{1 \leq j \leq k} (-1)^{j-1} \sum_{1 \leq i_1 < \cdots < i_j \leq k} f(p^{\min(\nu_{i_1}, \ldots, \nu_{i_j})}).
\]

By symmetry we can assume that \( \nu_1 \leq \nu_2 \leq \cdots \leq \nu_k \). Then the left-hand side of (2.2) is \( f(p^{\nu_\ell}) \). Let \( 1 \leq \ell \leq k \). On the right-hand side of (2.2), for a fixed \( j \), the term \( f(p^{\nu_{\ell}}) \) appears if \( i_1 = \ell < i_2 < \cdots < i_j \leq k \). This happens \( \binom{k-\ell}{j-1} \) times. Hence on the right-hand side the coefficient of \( f(p^{\nu_{\ell}}) \) is

\[
\sum_{1 \leq j \leq k} (-1)^{j-1} \binom{k-\ell}{j-1} = \sum_{1 \leq j \leq k-\ell+1} (-1)^{j-1} \binom{k-\ell}{j-1} = \begin{cases} 1 & \text{if } \ell = k, \\ 0 & \text{if } \ell \neq k, \end{cases}
\]

which completes the proof, similarly as in the proof of the inclusion–exclusion principle. \( \square \)

**Remark 1.** Consider the additive function \( \omega(n) \). For given \( n_1, \ldots, n_k \in \mathbb{N} \), let \( A_j \) be the set of prime factors of \( n_j \) \((1 \leq j \leq k)\). Then \( \omega([n_1, \ldots, n_k]) = \#(A_{i_1} \cup \cdots \cup A_{i_j}) \), and (2.1) reduces to the classical inclusion–exclusion principle for the sets \( A_j \).

Recall that a function \( g : \mathbb{N} \to \mathbb{C} \) is multiplicative if \( g(mn) = g(m)g(n) \) for all \( m, n \in \mathbb{N} \) with \( (m, n) = 1 \). If \( g \) is multiplicative, then \( g(n) = \prod_{p^r \mid n} g(p^r) \). If \( f \) is additive, then the function \( g(n) = 2f(n) \) is multiplicative. Conversely, if \( g \) is multiplicative (and positive), then the function \( f(n) = \log g(n) \) is additive. If \( g \) is multiplicative, then similarly as before,

\[
g((m, n))g([m, n]) = g(m)g(n)
\]

for all \( m, n \in \mathbb{N} \). This is well known and is included in many textbooks. See, for example, [12, Ex. 1.9]. Also see [10] for the related notion of semimultiplicative (Selberg multiplicative) functions and [2] for some other similar two-variable identities.

We have the following result, more general than (2.3).
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Corollary 1. Let \( g : \mathbb{N} \to \mathbb{C} \) be a nonvanishing multiplicative function, and let \( k \in \mathbb{N} \) and \( n_1, \ldots, n_k \in \mathbb{N} \). Then
\[
g([n_1, \ldots, n_k]) = \prod_{1 \leq j \leq k} \left( \prod_{1 \leq i_1 < \ldots < i_j \leq k} g(n_{i_1}, \ldots, n_{i_j}) \right) ^{(-1)^{j-1}}.
\]

Proof. Apply (formally) Proposition 1 to the additive function \( f(n) = \log g(n) \).

If \( g(n) = n \), then the following formula (see, e.g., [15, Ex. 3.4.56]) gives the LCM of several integers in terms of GCDs:
\[
[n_1, \ldots, n_k] = \frac{n_1 \cdot \ldots \cdot n_k}{\left( \frac{n_1, n_2}{n_1, n_3} \right) \cdot \left( \frac{n_1, n_2, n_3}{n_1, n_2, n_3, n_4} \right) \cdot \ldots \cdot \left( \frac{n_{k-1}, n_k}{n_{k-1}, n_k} \right)}.
\]

3 Asymptotic formulas for multivariable sums

3.1 The class \( \mathcal{F}_0 \) of omega-type functions

Denote by \( \mathcal{F}_0 \) the class of additive functions \( f : \mathbb{N} \to \mathbb{C} \) such that \( f(p) = 1 \) for every prime \( p \) and \( f(p') \ll \nu^\ell \) uniformly for the primes \( p \) and \( \nu \geq 2 \), where \( \ell \in \mathbb{N}_0 \) is some integer. For example, the functions \( \omega(n) \) and \( \Omega(n) \) are in \( \mathcal{F}_0 \). More generally, the function \( \Omega_{\ell}(n) = \sum_{p^\ell \mid n} \nu^\ell \) with \( \ell \in \mathbb{N}_0 \) is in the class \( \mathcal{F}_0 \). Note that \( \Omega_0(n) = \omega(n) \), \( \Omega_1(n) = \Omega(n) \). The function \( \Omega_{\ell}(n) \) was defined by Duncan [5], who also obtained an asymptotic formula for \( \sum_{n \leq x} \Omega_{\ell}(n) \).

Another example of a function in the class \( \mathcal{F}_0 \) is \( T_{\nu}(n) = \sum_{p^\nu \mid n} (\nu^{-1} \nu^\ell) \), where \( (\nu^{-1} \nu^\ell) = \frac{n+k-1}{n+1} \times \cdots \times \frac{n+k-1}{k!} \) is the number of \( k \) combinations with repetitions of \( n \) elements. Observe that \( T_0(n) = \omega(n) \) and \( T_1(n) = \Omega(n) \).

Saffari [16] proved the estimate
\[
\sum_{n \leq x} \omega(n) = x \log \log x + M x + x \sum_{j=1}^N \frac{a_j}{(\log x)^j} + O \left( \frac{x}{(\log x)^{N+1}} \right)
\]
for every fixed integer \( N \geq 1 \), where \( M \) is the Mertens constant defined by
\[
M = \gamma + \sum_p \left( \log \left( 1 - \frac{1}{p} \right) + \frac{1}{p} \right) \approx 0.2614,
\]
and the constants \( a_j \) (\( 1 \leq j \leq N \)) are given by
\[
a_j = - \int_1^\infty \frac{t - |t|}{t^2 \log t} \, dt;
\]
in particular, \( a_1 = \gamma - 1 \). Also see [3, Sect. 4.3.11].

By using the proximity of the functions \( f \in \mathcal{F}_0 \) and the function \( \omega \) we first prove the following result.

Theorem 1. Let \( f \) be a function in class \( \mathcal{F}_0 \). Then
\[
\sum_{n \leq x} f(n) = x \log \log x + C_f x + x \sum_{j=1}^N \frac{a_j}{(\log x)^j} + O \left( \frac{x}{(\log x)^{N+1}} \right)
\]

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for every fixed $N \geq 1$, where the constant $C_f$ is given by

$$C_f = \gamma + \sum_p \left( \log \left( 1 - \frac{1}{p} \right) + \left( 1 - \frac{1}{p} \right) \sum_{\nu=1}^{\infty} \frac{f(p^\nu)}{p^\nu} \right),$$

and the constants $a_j$ ($1 \leq j \leq N$) are as in (3.2).

Note that a weaker asymptotic formula for the sum $\sum_{n \leq x} f(n)$ is given in [4, Thm. 6.19] under the more restrictive conditions: $f$ is an additive function such that $f(p) = 1$ for all primes $p$ and $f(p^\nu) - f(p^{\nu-1}) = O(1)$ uniformly for the primes $p$ and $\nu \geq 2$; they are satisfied by the functions $\omega(n)$ and $\Omega(n)$, but not by $\Omega_{\ell}(n)$ and $T_{\ell}(n)$ with $\ell \geq 2$.

**Corollary 2.** The estimate of Theorem 1 holds for the function $f(n) = \Omega_{\ell}(n)$ ($\ell \in \mathbb{N}_0$) with the constant

$$C_{\Omega_{\ell}} = \gamma + \sum_p \left( \log \left( 1 - \frac{1}{p} \right) + \frac{1}{p} \left( 1 - \frac{1}{p} \right)^{-\ell} \sum_{\nu=0}^{\ell-1} \frac{\binom{\ell}{\nu}}{p^{\nu \ell}} \right),$$

where $\binom{\ell}{\nu}$ are the Eulerian numbers to be defined in Section 4.2, and the inner sum is considered to be 1 if $\ell = 0$.

The result of Corollary 2 was proved by Hassani [9], also by using Saffari’s estimate for the sum $\sum_{n \leq x} \omega(n)$, but invoking some different arguments and without referring to the Eulerian numbers.

**Corollary 3.** The estimate of Theorem 1 holds for the function $f(n) = T_{\ell}(n)$ ($\ell \in \mathbb{N}_0$) with the constant

$$C_{T_{\ell}} = \gamma + \sum_p \left( \log \left( 1 - \frac{1}{p} \right) + \frac{1}{p} \left( 1 - \frac{1}{p} \right)^{-\ell} \right).$$

Next, we deduce the following estimates for the sums $G_{f,k}(x)$ with $k \geq 2$.

**Theorem 2.** Let $f$ be a function in class $F_0$, and let $k \in \mathbb{N}$, $k \geq 2$. Then

$$\sum_{n_1, \ldots, n_k \leq x} f((n_1, \ldots, n_k)) = D_{f,k} x^k + \begin{cases} O(x^{k-1}) & \text{if } k \geq 3, \\ O(x \log \log x) & \text{if } k = 2, \end{cases}$$

where the constant $D_{f,k}$ is given by

$$D_{f,k} = \sum_p \left( 1 - \frac{1}{p^k} \right) \sum_{\nu=1}^{\infty} \frac{f(p^\nu)}{p^{\nu k}}. \tag{3.3}$$

**Corollary 4.** The estimate of Theorem 2 holds for the functions $f(n) = \Omega_{\ell}(n)$ and $f(n) = T_{\ell}(n)$ ($\ell \in \mathbb{N}_0$) with the constants

$$D_{\Omega_{\ell},k} = \sum_p \frac{1}{p^k} \left( 1 - \frac{1}{p^k} \right)^{-\ell} \sum_{\nu=0}^{\ell-1} \frac{\binom{\ell}{\nu}}{p^{\nu \ell}},$$

where the inner sum is considered to be 1 if $\ell = 0$, and

$$D_{T_{\ell},k} = \sum_p \frac{1}{p^k} \left( 1 - \frac{1}{p^k} \right)^{-\ell}.$$
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In particular, Theorem 2 holds for the functions \( \omega(n) \) and \( \Omega(n) \) with the constants

\[
D_{\omega,k} = \sum_p \frac{1}{p^k}, \quad D_{\Omega,k} = \sum_p \frac{1}{p^k - 1}.
\]

In what follows, we obtain our estimates for the sums \( L_{f,k}(x) \) involving the LCM of integers.

**Theorem 3.** Let \( f \) be a function in class \( \mathcal{F}_0 \). Then for every \( k \in \mathbb{N}, \, k \geq 2 \),

\[
\sum_{n_1, \ldots, n_k \leq x} f([n_1, \ldots, n_k]) = k x^k \log \log x + E_{f,k} x^k + k x^k \sum_{j=1}^{N} \frac{a_j}{(\log x)^j} + O\left(\frac{x^k}{(\log x)^{N+1}}\right),
\]

for every fixed \( N \geq 1 \), where the constant \( E_{f,k} \) is given by

\[
E_{f,k} = k C_f - \sum_{j=2}^{k} (-1)^j \binom{k}{j} D_{f,j},
\]

and the constants \( a_j \) (\( 1 \leq j \leq N \)) are as in (3.2).

**Corollary 5.** The estimate of Theorem 3 holds for the functions \( \Omega_{\ell}(n) \) and \( T_{\ell}(n) \) with \( \ell \in \mathbb{N}_0 \). In particular, it holds for the functions \( \omega(n) \) and \( \Omega(n) \) with the constants

\[
E_{\omega,k} = k \gamma + \sum_p \left( k \log \left( 1 - \frac{1}{p} \right) + 1 - \left( 1 - \frac{1}{p} \right)^k \right)
\]

and

\[
E_{\Omega,k} = k \gamma + \sum_p \left( k \log \left( 1 - \frac{1}{p} \right) + \frac{k}{p - 1} - \sum_{j=2}^{k} (-1)^j \binom{k}{j} \frac{1}{p^j - 1} \right).
\]

Finally, we remark that it is also possible to apply our results to the functions \( f(n) = c \log g(n) \), where \( g(n) \) are certain multiplicative functions, and \( c \) is a constant. For example, let \( g(n) = \tau(n) \). Then the function \( f(n) = \log \tau(n)/\log 2 \) is in the class \( \mathcal{F}_0 \), and our results can be applied. We obtain from Theorem 3 the following asymptotic formula.

**Corollary 6.** If \( k \in \mathbb{N} \), then

\[
\sum_{n_1, \ldots, n_k \leq x} \log \tau([n_1, \ldots, n_k]) = k(\log 2)x^k \log \log x + A_k x^k + k(\log 2) \sum_{j=1}^{N} \frac{a_j x^k}{(\log x)^j}
\]

\[
+ O\left(\frac{x^k}{(\log x)^{N+1}}\right)
\]

for every fixed \( N \geq 1 \), where the constant \( A_k \) is given by

\[
A_k = k C - \sum_{j=2}^{k} (-1)^j \binom{k}{j} D_{j},
\]

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the sum being 0 if $k = 1$, with

$$C = (\log 2) \gamma + \sum_{p} \left( (\log 2) \log \left( 1 - \frac{1}{p} \right) + \left( 1 - \frac{1}{p} \right) \sum_{\nu=1}^{\infty} \frac{\log(\nu + 1)}{p^\nu} \right)$$

and

$$D_j = \sum_{p} \left( 1 - \frac{1}{p^j} \right) \sum_{\nu=1}^{\infty} \frac{\log(\nu + 1)}{p^{\nu j}} \quad (2 \leq j \leq k).$$

In the particular case $k = 1$, the result of Corollary 6 was obtained by Hassani [8, Thm. 1.1], and a weaker asymptotic formula is given in [4, Probl. 6.12].

3.2 The class $\mathcal{F}_1$ of Alladi–Erdős-type functions

Let $\mathcal{F}_1$ denote the class of additive functions $f : \mathbb{N} \rightarrow \mathbb{C}$ such that $f(p) = p$ for every prime $p$ and $f(p^\nu) \ll \nu^\ell p^\nu$ uniformly for the primes $p$ and $\nu \geq 2$, where $\ell \in \mathbb{N}_0$ is an integer. For example, the functions $A_{\ell}(n) = \sum_{p^\nu \mid n} \nu^\ell p^\nu$ and $A_{\ell}(n) = \sum_{p^\nu \mid n} \nu^\ell p^\nu$ with $\ell \in \mathbb{N}_0$ are in $\mathcal{F}_1$. In particular, $A_1(n) = A(n) := \sum_{p^\nu \mid n} \nu p$ and $A_0(n) = A^*(n) := \sum_{p \mid n} p$ are the Alladi–Erdős functions.

It is known that

$$\sum_{n \leq x} A(n) = \sum_{p \leq x} \frac{x}{p^\nu} = \frac{\pi^2 x^2}{12 \log x} + O\left( \frac{x^2}{(\log x)^2} \right),$$

which can be proved by using a strong form of the prime number theorem. See [1] and [17, p. 62, 467]. Also see [20, Thm. 2] for a simple approach leading to a slightly weaker error term. The same formula (3.4) holds for $\sum_{n \leq x} A^*(n) = \sum_{p \leq x} p[x/p]$.

We point out the following result.

Theorem 4. Let $f$ be a function in class $\mathcal{F}_1$. Then

$$\sum_{n \leq x} f(n) = \frac{\pi^2 x^2}{12 \log x} + O\left( \frac{x^2}{(\log x)^2} \right).$$

Next, we deduce the corresponding estimates for the sums $G_{f,k}(x)$ with $k \geq 2$.

Theorem 5. Let $f$ be a function in class $\mathcal{F}_1$. Then

$$\sum_{n_1, n_2 \leq x} f((n_1, n_2)) = x^2 \log \log x + D_{f,2} x^2 + O\left( \frac{x^2}{\log x} \right),$$

where

$$D_{f,2} = \gamma + \sum_{p} \left( \log \left( 1 - \frac{1}{p} \right) + \left( 1 - \frac{1}{p^2} \right) \sum_{\nu=1}^{\infty} \frac{f(p^\nu)}{p^{2\nu}} \right),$$

and if $k \geq 3$, then

$$\sum_{n_1, \ldots, n_k \leq x} f((n_1, \ldots, n_k)) = D_{f,k} x^k + \begin{cases} O(x^{k-1}) & \text{if } k \geq 4, \\ O(x^2 \log \log x) & \text{if } k = 3, \end{cases}$$

where $D_{f,k} = D_{f,k}$ is defined by (3.3).
Corollary 7. The estimate of Theorem 5 holds for the function \( f(n) = A_\ell(n) \) \((\ell \in \mathbb{N})\) with the constants
\[
\overline{D}_{A_\ell,2} = \gamma + \sum_p \left( \log \left( 1 - \frac{1}{p} \right) \right) + \frac{1}{p} \left( 1 - \frac{1}{p^2} \right)^{-\ell} \sum_{\nu=0}^{\ell-1} \frac{\binom{\ell}{\nu}}{p^{2\nu}},
\]
\[
\overline{D}_{A_\ell,k} = \sum_p \frac{1}{p^{k-1}} \left( 1 - \frac{1}{p^k} \right)^{-\ell} \sum_{\nu=0}^{\ell-1} \frac{\binom{\ell}{\nu}}{p^{\nu}} \quad (k \geq 3),
\]
the inner sums being considered 1 if \( \ell = 0 \).

In particular, it holds for the function \( A(n) = A_1(n) \) with
\[
\overline{D}_{A,2} = \gamma + \sum_p \left( \log \left( 1 - \frac{1}{p} \right) \right) + \frac{p}{p^2 - 1} \approx 0.4971, \quad \overline{D}_{A,k} = \sum_p \frac{p}{p^{k-1}} \quad (k \geq 3),
\]
and it holds for the function \( A^*(n) = A_0(n) \) with the constants \( \overline{D}_{A^*,2} = M \) (the Mertens constant) and \( \overline{D}_{A^*,k} = \sum p^{1/p^{k-1}} \) \((k \geq 3)\).

Now we obtain the estimate for the sums \( L_{f,k}(x) \) involving the LCM of integers.

Theorem 6. Let \( f \) be a function in class \( \mathcal{F}_1 \). Then for every \( k \in \mathbb{N} \),
\[
\sum_{n_1, \ldots, n_k \leq x} f([n_1, \ldots, n_k]) = \frac{\pi^2 k x^{k+1}}{12 \log x} + O \left( \frac{x^{k+1}}{(\log x)^2} \right).
\]

3.3 The function \( B(n) \)

In this section, we consider the function \( B(n) = A(n) - A^*(n) \), where \( B(p) = 0 \) for every prime \( p \). It would be possible to define and study here another class of additive functions \( f \) with \( f(p) = 0 \) for the primes \( p \) and with adequate order conditions on \( f(p^\nu) \) with \( \nu \geq 2 \), but we confine ourselves to the function \( B(n) \).

Alladi and Erdős [1, Thm. 1.5] proved that
\[
\sum_{n \leq x} B(n) = x \log \log x + O(x).
\]

We improve this estimate as follows.

Theorem 7. We have
\[
\sum_{n \leq x} B(n) = x \log \sqrt{x} + Fx + O \left( \frac{x}{\log x} \right),
\]
where
\[
F := \gamma + \sum_p \left( \log \left( 1 - \frac{1}{p} \right) + \frac{1}{p-1} \right) \approx 1.0346.
\]

For the sums involving the GCD and LCM, we have the following results.

Theorem 8. Let \( k \in \mathbb{N} \), \( k \geq 2 \), be fixed. Then
\[
\sum_{n_1, \ldots, n_k \leq x} B([n_1, \ldots, n_k]) = x^k \sum_p \frac{1}{p^{k-1}(p^k - 1)} + \begin{cases} O(x^{k-1}) & \text{if } k \geq 3, \\ O(x \log \log x) & \text{if } k = 2. \end{cases}
\]
Theorem 9. Let \( k \in \mathbb{N}, k \geq 2 \), be fixed. Then
\[
\sum_{n_1, \ldots, n_k \leq x} B([n_1, \ldots, n_k]) = kx^k \log \log x + H_k x^k + O\left(\frac{x^k}{\log x}\right),
\]
where
\[
H_k := kF + \sum_{j=2}^{k} \binom{k}{j} \sum_p \frac{1}{p^{j-1}(p^j - 1)},
\]
and \( F \) is the constant defined in Theorem 7. For instance,

| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|----|
| 1.816 | 2.435 | 2.907 | 3.255 | 3.5004 | 3.658 | 3.74 | 3.758 | 3.719 |

4 Preliminaries to the proofs

First, we prove the next identity, already mentioned in Section 1.

Lemma 1. Let \( f \) be an arithmetic function, and let \( k \in \mathbb{N} \). Then
\[
G_{f,k}(x) := \sum_{n_1, \ldots, n_k \leq x} f((n_1, \ldots, n_k)) = \sum_{d \leq x} (\mu * f)(d) \left[ \frac{x}{d} \right]^k.
\]

Proof. Since \( f(n) = \sum_{d|n} (\mu * f)(d) \), we have
\[
G_{f,k}(x) = \sum_{n_1, \ldots, n_k \leq x} d((n_1, \ldots, n_k)) \sum_{d|n} (\mu * f)(d) = \sum_{n_1=d_1, \ldots, n_k=d_k \leq x} (\mu * f)(d)
\]
\[
= \sum_{d \leq x} (\mu * f)(d) \sum_{j_1, \ldots, j_k \leq x/d} 1 = \sum_{d \leq x} (\mu * f)(d) \left[ \frac{x}{d} \right]^k. \quad \Box
\]

4.1 Properties of additive functions

The next result is well known.

Lemma 2. If \( f \) is an additive function, then
\[
(\mu * f)(n) = \begin{cases} f(p^\nu) - f(p^\nu-1) & \text{if } n = p^\nu \ (\nu \geq 1), \\ 0 & \text{otherwise}. \end{cases}
\]

Proof. We have
\[
f(n) = \sum_{p^\nu|n} f(p^\nu) = \sum_{p^\nu|n} (f(p^\nu) - f(p^\nu-1)) = \sum_{d|n} g(d),
\]
where
\[
g(d) = \begin{cases} f(p^\nu) - f(p^\nu-1) & \text{if } d = p^\nu \ (\nu \geq 1), \\ 0 & \text{otherwise}. \end{cases}
\]
Hence \( f = g * 1 \), and by Möbius inversion we have \( \mu * f = g \). \quad \Box

We will use the following identity.
Additive arithmetic functions meet the inclusion–exclusion principle

Lemma 3. Let \( f \) be an additive function, and let \( k \in \mathbb{N} \). Then

\[
\sum_{n_1, \ldots, n_k \leq x} f ((n_1, \ldots, n_k)) = \sum_{p \leq x, \nu \geq 1} \left( f (p^\nu) - f (p^{\nu-1}) \right) \left\lfloor \frac{x}{p^\nu} \right\rfloor^k.
\]

Proof. It immediately follows from Lemmas 1 and 2. \( \square \)

4.2 An identity involving the Eulerian numbers

Let \( \binom{n}{k} \) denote the (classical) Eulerian numbers defined as the numbers of permutations \( h \in S_n \) with \( k \) descents. Here a number \( i \) is called a descent of \( h \) if \( h(i) > h(i+1) \). In the paper, we use the identity

\[
\sum_{k=1}^{\infty} k^n x^k = \frac{x}{(1-x)^{n+1}} \sum_{k=0}^{n-1} \binom{n}{k} x^k \quad (n \in \mathbb{N}_0, \ |x| < 1),
\]

where in the right-hand side the sum is considered to be 1 if \( n = 0 \).

Note that \( \binom{n}{0} = \binom{n-1}{-1} = 1, \binom{n}{n} = 0 (n \geq 1) \), and the Eulerian numbers have the symmetry property \( \binom{n}{k} = \binom{n-1}{n-k} \) (\( n \geq 1, k \geq 0 \)) and satisfy the recurrence relation

\[
\binom{n}{k} = (k+1) \binom{n-1}{k} + (n-k) \binom{n-1}{k-1} \quad (n \geq 1, k \geq 0),
\]

where, by convention, \( \binom{n}{0} = 1 \) and \( \binom{n}{k} = 0 \) for \( k < 0 \) and \( n \geq 1 \). See, for example, [7, Chap. 6] and [14, Chap. 1].

We deduce the following estimates.

Lemma 4. For fixed \( k, \ell \in \mathbb{N} \),

\[
\sum_{n=1}^{\infty} \frac{n^\ell}{p^{nk}} \ll \frac{1}{p^k} \quad \text{as } p \to \infty,
\]

and

\[
\sum_{n=2}^{\infty} \frac{n^\ell}{p^{nk}} \ll \frac{1}{p^{2k}} \quad \text{as } p \to \infty.
\]

Proof. According to identity (4.1),

\[
\sum_{n=1}^{\infty} \frac{n^\ell}{p^{nk}} = \frac{1/p^k}{(1-1/p^k)^{\ell+1}} \sum_{n=0}^{\ell-1} \binom{\ell}{n} \frac{1}{p^{nk}} \ll \frac{1}{p^k} \sum_{n=0}^{\ell-1} \binom{\ell}{n} \frac{1}{p^{nk}} \ll \frac{1}{p^k},
\]

where \( \ell \) is fixed, we have finitely many terms, and the largest term with respect to \( p \) of the sum is that for \( n = 0 \). This gives (4.2). Similarly,

\[
\sum_{n=2}^{\infty} \frac{n^\ell}{p^{nk}} = -\frac{1}{p^k} + \frac{1/p^k}{(1-1/p^k)^{\ell+1}} \sum_{n=0}^{\ell-1} \binom{\ell}{n} \frac{1}{p^{nk}} \ll -\frac{1}{p^k} \left( 1 - \frac{1}{p^k} \right)^{\ell+1} + \frac{1}{p^k} \sum_{n=0}^{\ell-1} \binom{\ell}{n} \frac{1}{p^{nk}} \ll \frac{1}{p^{2k}},
\]

where since \( \binom{\ell}{0} = 1 \), the term \( 1/p^k \) cancels out, giving (4.3). \( \square \)

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4.3 Estimates of certain sums

The estimates of Lemma 4 are not sufficient for our proofs. We need good estimates of the sums $\sum_{n > z} n^\ell / p^{nk}$, where $z \geq 1$ is a real number.

**Lemma 5.**

(i) Let $k, \ell \in \mathbb{N}$, let $p$ be a prime, and let $z \geq 1$ be a real number satisfying $z > \ell \max(1, \log z)/(k \log p)$. Then

$$\sum_{n > z} n^\ell / p^{nk} \leq \frac{z^\ell}{p^{kz}} \left( \frac{1}{k \log p - \ell / z} + 1 \right).$$

(ii) In particular, if $k \log p > \max(\ell \max(1, \log z)/z, \ell / z + 1/2)$, then

$$\sum_{n > z} n^\ell / p^{nk} \leq \frac{3z^\ell}{p^{kz}}. \tag{4.4}$$

Note that (4.2) and (4.3) are particular cases of (4.4). The proof of Lemma 5 uses the following van der Corput-type bound.

**Lemma 6.** Let $a < b$ be real numbers, and let $g \in C^1[a, b]$ be such that $g \geq 0$, $g'$ is nondecreasing, and there exists $\lambda_1 > 0$ such that $g'(x) \geq \lambda_1$ for all $x \in [a, b]$. Then

$$\int_a^b e^{-g(t)} \, dt \leq \frac{e^{-g(a)}}{\lambda_1}.$$

**Proof of Lemma 6.** Use the following inequality due to Ostrowski (see, e.g., [13, (3.7.35)]). Let $a < b$ be real numbers, let $F$ be a real monotone function such that $F(a)F(b) \geq 0$ and $|F(a)| \geq |F(b)|$, and let $G$ be a complex function, both integrable on $[a, b]$. Then

$$\left| \int_a^b F(t)G(t) \, dt \right| \leq |F(a)| \max_{a \leq z \leq b} \left| \int_a^z G(t) \, dt \right|.$$

Choosing $F(t) = 1/g'(t)$ and $G(t) = g'(t)e^{-g(t)}$ and proceeding as in van der Corput’s first derivative test, we have

$$\int_a^b e^{-g(t)} \, dt \leq \frac{1}{g'(a)} \max_{a \leq z \leq b} \int_a^z g'(t)e^{-g(t)} \, dt \leq \frac{1}{\lambda_1} \max_{a \leq z \leq b} \left( e^{-g(a)} - e^{-g(z)} \right) \leq \frac{e^{-g(a)}}{\lambda_1}$$

as stated. \qed

**Proof of Lemma 5.** Apply Lemma 6 with $g(t) := tk \log p - \ell \log t$, for which $g'(t) = k \log p - \ell / t \geq k \log p - \ell / z$ for all $t \geq z$. Note that the condition $z > \ell \max(1, \log z)/(k \log p)$ ensures that $g(z) > 0$ and $g'(z) > 0$. Then Lemma 6 yields

$$\sum_{n > z} n^\ell / p^{nk} = \sum_{n > z} e^{-g(n)} \leq \int_z^\infty e^{-g(t)} \, dt + e^{-g(z)} \leq \frac{e^{-g(z)}}{k \log p - \ell / z} + e^{-g(z)} = \frac{z^\ell}{p^{kz}} \left( \frac{1}{k \log p - \ell / z} + 1 \right).$$

The second part of the lemma follows by noticing that if $k \log p > \ell / z + 1/2$, then $1/(k \log p - \ell / z) < 2$. \qed
Lemma 7. Fix $k, \ell \in \mathbb{N}$. We have, as $x \to \infty$,

\[
\sum_{p \leq x} \sum_{\nu > \log x / \log p} \frac{\nu^\ell}{p^{\nu k}} \ll \frac{1}{x^{k-1}(\log x)^2}, \tag{4.5}
\]

\[
\sum_{p \leq x^{1/2}} \sum_{\nu > \log x / \log p} \frac{\nu^\ell}{p^{\nu}} \ll \frac{1}{x^{1/2}(\log x)^2}. \tag{4.6}
\]

Proof. Assume that $x \geq e^{2\ell/k}$ and apply Lemma 5 with $z = \log x / \log p$. Note that the condition $z > \ell / (k \log p)$ is fulfilled as soon as $x > e^{\ell/k}$. Now Lemma 5 yields

\[
\sum_{\nu > \log x / \log p} \frac{\nu^\ell}{p^{\nu}} \leq \frac{1}{x^k} \left( \frac{\log x}{\log p} \right)^\ell \left( k \log p - \frac{\ell \log p}{\log x} \right)^{-1} = \frac{1}{k x^k} \frac{(\log x)^\ell}{(\log p)^{\ell+1}} \left( 1 - \frac{\ell}{k \log x} \right)^{-1}
\]

\[
\leq \frac{2(\log x)^\ell}{k x^k (\log p)^{\ell+1}},
\]

where we used the inequality $(1 - a)^{-1} \leq 2$ for $0 \leq a \leq 1/2$. Hence

\[
S := \sum_{p \leq x} \sum_{\nu > \log x / \log p} \frac{\nu^\ell}{p^{\nu k}} \ll \frac{(\log x)^\ell}{x^k} \sum_{p \leq x} \frac{1}{(\log p)^{\ell+1}}. \tag{4.7}
\]

We also have

\[
\sum_{p \leq x} \frac{1}{(\log p)^{\ell+1}} = \left( \sum_{p \leq \sqrt{x}} + \sum_{\sqrt{x} < p \leq x} \right) \frac{1}{(\log p)^{\ell+1}} \leq \frac{\pi(\sqrt{x})}{(\log 2)^{\ell+1}} + \frac{2^{\ell+1} \pi(x)}{(\log x)^{\ell+1}} \ll \frac{x}{(\log x)^{\ell+2}}, \tag{4.8}
\]

using the Chebyshev estimate $\pi(x) \ll x / \log x$.

From (4.7) and (4.8) we obtain that

\[
S \ll \frac{1}{x^{k-1}(\log x)^2}.
\]

This proves estimate (4.5). The proof of (4.6) is similar. \qed

5 Proofs of the asymptotic formulas

5.1 Proofs of the results in Section 3.1

Proof of Theorem 1. Using Lemma 3 for $k = 1$, and that $f(p) = 1$,

\[
\sum_{n \leq x} f(n) = \sum_{p^\nu \leq x} \left( f(p^\nu) - f(p^{\nu-1}) \right) \left[ \frac{x}{p^\nu} \right] = \sum_{p \leq x} \left[ \frac{x}{p} \right] + \sum_{p^\nu \leq x \atop \nu \geq 2} \left( f(p^\nu) - f(p^{\nu-1}) \right) \left[ \frac{x}{p^\nu} \right]
\]

\[
=: S_1 + S_2, \tag{5.1}
\]

where $S_1 := \sum_{p \leq x} [x/p] = \sum_{n \leq x} \omega(n)$ and Saffari’s estimate (3.1) can be applied.

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We estimate the sum
\[
S_2 := \sum_{\nu \geq 2} \sum_{\nu \leq x} \frac{f(p^\nu) - f(p^{\nu-1})}{p^\nu} \left\lfloor \frac{x}{p^\nu} \right\rfloor
\]
\[
= x \sum_{\nu \geq 2} \sum_{\nu \leq x} \frac{f(p^\nu) - f(p^{\nu-1})}{p^\nu} - \sum_{\nu \geq 2} \sum_{\nu \leq x} \left( f(p^\nu) - f(p^{\nu-1}) \right) \left( \frac{x}{p^\nu} - \left\lfloor \frac{x}{p^\nu} \right\rfloor \right). \tag{5.2}
\]

By the definition of the class \( \mathcal{F}_0 \) we have \( f(p^\nu) - f(p^{\nu-1}) \ll \nu^\ell \), uniformly for the primes \( p \) and \( \nu \geq 2 \), for some \( \ell \in \mathbb{N}_0 \), and the second sum in (5.2) is
\[
\ll \sum_{\nu \geq 2} \nu^\ell = \sum_{p \leq x^{1/2}} \sum_{2 \leq \nu \leq \log z / \log p} \nu^\ell \ll (\log x)^{\ell+1} \sum_{p \leq x^{1/2}} \frac{1}{(\log p)^{\ell+1}} \ll x^{1/2} \log x,
\]
by using (4.8).

We also have
\[
\sum_{p \leq x^{1/2}} \sum_{\nu \geq 2} \frac{|f(p^\nu) - f(p^{\nu-1})|}{p^\nu} \ll \sum_{p \leq x^{1/2}} \sum_{\nu \geq 2} \nu^\ell \ll \sum_{p} \frac{1}{p^2} < \infty
\]
by estimate (4.3) with \( k = 1 \) or by (4.4) applied for \( z = 2, k = 1 \). Hence the series
\[
\sum_{p \leq x^{1/2}} \sum_{\nu \geq 2} \frac{f(p^\nu) - f(p^{\nu-1})}{p^\nu} = \sum_{p} \left( \left( 1 - \frac{1}{p^2} \right) \sum_{\nu = 1}^{\infty} \frac{f(p^\nu)}{p^\nu} - \frac{1}{p^2} \right) =: B_f
\]
is absolutely convergent. Also,
\[
\sum_{\nu \geq 2} \sum_{\nu \leq x} \frac{f(p^\nu) - f(p^{\nu-1})}{p^\nu} \nu \leq 2 \ll \sum_{p \geq x^{1/2}} \sum_{\nu \geq 2} \frac{f(p^\nu) - f(p^{\nu-1})}{p^\nu}
\]
\[
= \sum_{p \leq x^{1/2}} \left( \sum_{\nu = 2}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^\nu} - \sum_{\nu > \log z / \log p} f(p^\nu) - f(p^{\nu-1}) \right)
\]
\[
= \sum_{p \leq x^{1/2}} \sum_{\nu = 2}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^\nu} - \sum_{p > x^{1/2}} \sum_{\nu = 2}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^\nu}
\]
\[
- \sum_{p \leq x^{1/2}} \sum_{\nu > \log z / \log p} \frac{f(p^\nu) - f(p^{\nu-1})}{p^\nu}, \tag{5.3}
\]
where
\[
\sum_{p > x^{1/2}} \sum_{\nu = 2}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^\nu} \ll \sum_{p > x^{1/2}} \sum_{\nu = 2}^{\infty} \frac{\nu^\ell}{p^2} \ll \sum_{p > x^{1/2}} \frac{1}{p^2} \ll \sum_{n > x^{1/2}} \frac{1}{n^2} \ll \frac{1}{x^{1/2}},
\]
again by (4.3) or (4.4).

Furthermore, the double sum in (5.3) is \( \ll 1/(x^{1/2}(\log x)^2) \) by estimate (4.6).
Putting these altogether gives

\[ S_2 = B_f + O(x^{1/2}). \quad (5.4) \]

Now the proof is complete by (5.1), (3.1), and (5.4). \(\square\)

**Proof of Corollary 2.** It follows from Theorem 1 and identity (4.1). \(\square\)

**Proof of Corollary 3.** It follows from Theorem 1 and the known identity

\[ \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k = \frac{1}{(1-x)^n} \quad (n \in \mathbb{N}_0, \ |x| < 1). \quad \square \quad (5.5) \]

**Proof of Theorem 2.** Using Lemma 3, we have

\[
\sum_{n_1, \ldots, n_k \leq x} f((n_1, \ldots, n_k)) = \sum_{p^\nu \leq x, \nu \geq 1} \left( f(p^\nu) - f(p^{\nu-1}) \right) \frac{x}{p^\nu}^k,
\]

where

\[
\left( \frac{x}{p^\nu} \right)^k = \left( \frac{x}{p^\nu} + O(1) \right)^k = \left( \frac{x}{p^\nu} \right)^k + O\left( \left( \frac{x}{p^\nu} \right)^{k-1} \right).
\]

We deduce

\[
\sum_{n_1, \ldots, n_k \leq x} f((n_1, \ldots, n_k)) = x^k \sum_{p^\nu \leq x, \nu \geq 1} \frac{f(p^\nu) - f(p^{\nu-1})}{p^\nu k} + x^{k-1} R_{f,k}(x),
\]

where

\[
R_{f,k}(x) \ll \sum_{p^\nu \leq x, \nu \geq 1} \frac{f(p^\nu) - f(p^{\nu-1})}{p^\nu (k-1)} \ll \sum_{p \leq x} \sum_{\nu = 1}^{\infty} \frac{\nu^\ell}{p^{\nu(k-1)}} \ll \sum_{p \leq x} \frac{1}{p^{k-1}},
\]

using (4.4) or (4.2), which is \(\ll \log \log x\) for \(k = 2\) and \(\ll 1\) for \(k \geq 3\).

Here

\[
\sum_{p, \nu = 1}^{\infty} \frac{|f(p^\nu) - f(p^{\nu-1})|}{p^\nu k} \ll \sum_{p} \sum_{\nu = 1}^{\infty} \frac{\nu^\ell}{p^\nu k} \ll \sum_{p} \frac{1}{p^k} < \infty,
\]

again by (4.2) or (4.4), where \(k \geq 2\), and hence the series

\[
\sum_{p} \sum_{\nu = 1}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^\nu k} = \sum_{p} \left( 1 - \frac{1}{p^k} \right) \sum_{\nu = 1}^{\infty} \frac{f(p^\nu)}{p^\nu k} =: D_{f,k}
\]

is absolutely convergent.

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Furthermore, 
\[ \sum_{p^\nu \leq x \atop \nu \geq 1} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{\nu k}} = \sum_{p \leq x} \sum_{1 \leq \nu \leq \log x / \log p} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{\nu k}} \]
\[ = \sum_{p \leq x} \left( \sum_{\nu = 1}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{\nu k}} \right) - \sum_{\nu > \log x / \log p} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{\nu k}} \]
\[ = \sum_{p \leq x} \sum_{\nu = 1}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{\nu k}} - \sum_{p > x} \sum_{\nu = 1}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{\nu k}} \]
\[- \sum_{p \leq x} \sum_{\nu > \log x / \log p} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{\nu k}}.\]

Here
\[ \sum_{p > x} \sum_{\nu = 1}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{\nu k}} \ll \sum_{p > x} \sum_{\nu = 1}^{\infty} \frac{\nu^\ell}{p^{\nu k}} \ll \sum_{p > x} \frac{1}{p^k} \ll \sum_{n > x} \frac{1}{n^k} \ll \frac{1}{x^{-1}}, \]
again by (4.2) or (4.4). Also,
\[ \sum_{p \leq x} \sum_{\nu > \log x / \log p} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{\nu k}} \ll \sum_{p \leq x} \sum_{\nu > \log x / \log p} \frac{\nu^\ell}{p^{\nu k}} \ll \frac{1}{x^{-1} \log^2 x}, \]
by (4.5). This completes the proof. \( \square \)

**Proof of Corollary 4.** It follows from Theorem 2 and identities (4.1) and (5.5). \( \square \)

**Proof of Theorem 3.** By Proposition 1 and by symmetry we have
\[ L_{f,k}(x) = \sum_{n_1, \ldots, n_k \leq x} f([n_1, \ldots, n_k]) \]
\[ = \sum_{n_1, \ldots, n_k \leq x} \sum_{1 \leq j \leq k} (-1)^{j-1} \sum_{1 \leq i_1 < \cdots < i_j \leq k} f((n_{i_1}, \ldots, n_{i_j})) \]
\[ = \sum_{1 \leq j \leq k} (-1)^{j-1} \binom{k}{j} \sum_{n_1, \ldots, n_k \leq x} f((n_1, \ldots, n_j)) \]
\[ = \sum_{1 \leq j \leq k} (-1)^{j-1} \binom{k}{j} \sum_{n_1, \ldots, n_j \leq x} f((n_1, \ldots, n_j)) \sum_{n_{j+1}, \ldots, n_k \leq x} 1, \]
where the last sum is \( \lfloor x \rfloor^{k-j} = x^{k-j} + O(x^{k-j-1}) \). Therefore by Theorems 1 and 2 we have
\[ L_{f,k}(x) = k \left( \sum_{n_1 \leq x} f(n_1) \right) (x^{k-1} + O(x^{k-2})) \]
\[ + \sum_{2 \leq j \leq k} (-1)^{j-1} \binom{k}{j} \left( \sum_{n_1, \ldots, n_j \leq x} f((n_1, \ldots, n_j)) \right) (x^{k-j} + O(x^{k-j-1})) \]
Additive arithmetic functions meet the inclusion–exclusion principle

\[
\begin{align*}
&= k \left( x \log \log x + C_k x + x \sum_{j=1}^{N} \frac{a_j}{(\log x)^j} + O\left( \frac{x}{(\log x)^{N+1}} \right) \right) \left( x^{k-1} + O(x^{k-2}) \right) \\
&\quad + \sum_{2 \leq j \leq k} (-1)^{j-1} \binom{k}{j} \left( D_{f,j} x^j + O(R_j(x)) \right) \left( x^{k-j} + O(x^{k-j-1}) \right),
\end{align*}
\]

where \( R_j(x) = x^{j-1} \) (\( j \geq 3 \)) and \( R_2(x) = x \log \log x \) (\( j = 2 \)). This gives the result. \( \square \)

5.2 Proofs of the results in Section 3.2

**Proof of Theorem 4.** Using Lemma 3 for \( k = 1 \), since \( f(p) = p \), we have

\[
\sum_{n \leq x} f(n) = \sum_{p \leq x, \nu \geq 1} \frac{x}{p^\nu} = \sum_{p \leq x} \left( \sum_{\nu \geq 1} \frac{x}{p^\nu} \right) = \sum_{p \leq x} \left( \frac{x}{p^\nu} \right) = T_1 + T_2,
\]

where \( T_1 := \sum_{p \leq x} \frac{x}{p} = \sum_{n \leq x} A(n) \), and we can apply estimate \((3.4)\).

If \( f \in \mathcal{F}_1 \), then \( f(p^\nu) - f(p^{\nu-1}) \ll \nu^\ell \nu^\nu \) for some \( \ell \in \mathbb{N}_0 \), and we show that the sum \( T_2 \) is negligible by comparison:

\[
T_2 := \sum_{p \leq x, \nu \geq 2} \left( f(p^\nu) - f(p^{\nu-1}) \right) \left( \frac{x}{p^\nu} \right) \ll \sum_{p \leq x, \nu \geq 2} \nu^\ell \sum_{\nu \geq 1} \frac{x}{\nu^\nu} \ll \sum_{\nu \geq 1} \nu^{\ell+1} \ll x^{3/2} \nu^{\ell+1}.
\]

**Proof of Theorem 5.** Similar to the proof of Theorem 2. By Lemma 3 we have

\[
L_{f,k}(x) := \sum_{n_1, \ldots, n_k \leq x} f(n_1, \ldots, n_k) = \sum_{p_1 \leq x, \nu_1 \geq 1} \frac{x}{p_1^{\nu_1}} = x^{k-1} V_{f,k}(x),
\]

where

\[
V_{f,k}(x) \ll \sum_{p_1 \leq x, \nu_1 \geq 1} \frac{f(p^\nu) - f(p^{\nu-1})}{p_1^{\nu(k-1)}} \ll \sum_{p \leq x} \sum_{\nu=1}^\infty \frac{\nu^\ell}{p^{\nu(k-2)}}.
\]

**Case 1.** If \( k \geq 3 \), then \( V_{f,k}(x) \ll \sum_{p \leq x} 1/p^{k-2} \) by \((4.4)\) or \((4.2)\), which is \( \ll \log \log x \) for \( k = 3 \) and \( \ll 1 \) for \( k \geq 4 \).

Also,

\[
\sum_{p} \sum_{\nu=1}^\infty \frac{|f(p^\nu) - f(p^{\nu-1})|}{p^{\nu k}} \ll \sum_{p} \sum_{\nu=1}^\infty \frac{\nu^\ell}{p^{\nu(k-1)}} \ll \sum_{p} \frac{1}{p^{k-1}} < \infty
\]
by (4.2), where \( k \geq 3 \), and hence the series
\[
\sum_{p} \sum_{\nu=1}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{\nu k}} = \sum_{p} \left( 1 - \frac{1}{p^k} \right) \sum_{\nu=1}^{\infty} \frac{f(p^\nu)}{p^{\nu k}} =: D_{f,k}
\]
is absolutely convergent.
Furthermore, as in the proof of Theorem 2,
\[
\sum_{p \leq x} \sum_{\nu \geq 1} f(p^\nu) - f(p^{\nu-1}) = \sum_{p} \sum_{\nu=1}^{\infty} f(p^\nu) - f(p^{\nu-1}) - \sum_{p > x} \sum_{\nu=1}^{\infty} f(p^\nu) - f(p^{\nu-1})
\]
\[
= - \sum_{p \leq x} \sum_{\nu > \log x / \log p} f(p^\nu) - f(p^{\nu-1}) / p^{\nu k},
\]
where
\[
\sum_{p > x} \sum_{\nu = 1}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{\nu k}} \ll \sum_{p > x} \sum_{\nu = 1}^{\infty} \frac{\nu^\ell}{p^{\nu(k-1)}} \ll \sum_{n > x} \sum_{\nu = 1}^{\infty} \frac{1}{n^{k-1}} \ll \frac{1}{x^{k-2}}.
\]

Also, by (4.5)
\[
\sum_{p \leq x} \sum_{\nu > \log x / \log p} f(p^\nu) - f(p^{\nu-1}) / p^{\nu k} \ll \sum_{p \leq x} \sum_{\nu > \log x / \log p} \frac{\nu^\ell}{p^{\nu(k-1)}} \ll \frac{1}{x^{k-2}(\log x)^2}.
\]

**Case II.** If \( k = 2 \), then
\[
V_{f,2}(x) \ll \sum_{p \leq x} \sum_{\nu \leq \log x / \log p} \nu^\ell \ll \sum_{p \leq x} \left( \frac{\log x}{\log p} \right)^{\ell+1} \ll \frac{x}{\log x}
\]
by (4.8). Also,
\[
\sum_{p \leq x} \sum_{\nu \geq 1} f(p^\nu) - f(p^{\nu-1}) = \sum_{p \leq x} \frac{1}{p} + \sum_{p^2 \leq x} \sum_{\nu \geq 2} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{2\nu}},
\]
where
\[
\sum_{p \leq x} \frac{1}{p} = \log \log x + M + O \left( \frac{1}{\log x} \right),
\]
and
\[
\sum_{p} \sum_{\nu=2}^{\infty} \frac{|f(p^\nu) - f(p^{\nu-1})|}{p^{2\nu}} \ll \sum_{p} \sum_{\nu=2}^{\infty} \frac{\nu^\ell}{p^{\nu}} \ll \sum_{p} \frac{1}{p^2} < \infty
\]
by (4.2), and hence the series
\[
\sum_{p} \sum_{\nu=2}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{2\nu}} = \sum_{p} \left( -\frac{1}{p} + \left( 1 - \frac{1}{p^2} \right) \sum_{\nu=2}^{\infty} \frac{f(p^\nu)}{p^{2\nu}} \right)
\]
is absolutely convergent.
Furthermore, similarly as before,
\[
\sum_{p^\nu \leq x, \nu \geq 2} f(p^\nu) - f(p^{\nu-1}) = \sum_{p} \sum_{\nu=2}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{2\nu}} - \sum_{p > x^{1/2}} \sum_{\nu=2}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{2\nu}}
\]
\[
- \sum_{p \leq x^{1/2}} \sum_{\nu > \log x / \log \nu} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{2\nu}},
\]
where
\[
\sum_{p > x^{1/2}} \sum_{\nu=2}^{\infty} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{2\nu}} \ll \sum_{p > x^{1/2}} \sum_{\nu=2}^{\infty} \frac{1}{p^2} \ll \sum_{n > x^{1/2}} \frac{1}{n^2} \ll \frac{1}{x^{1/2}},
\]
and
\[
\sum_{p \leq x^{1/2}} \sum_{\nu > \log x / \log \nu} \frac{f(p^\nu) - f(p^{\nu-1})}{p^{2\nu}} \ll \sum_{p \leq x^{1/2}} \sum_{\nu > \log x / \log \nu} \frac{1}{p^{2\nu}} \ll \frac{1}{x^{1/2}(\log x)^2}
\]
by (4.6). □

**Proof of Theorem 6.** Similar to the proof of Theorem 3. By Proposition 1 and by symmetry we have

\[
L_{f,k}(x) := \sum_{n_1, \ldots, n_k \leq x} f([n_1, \ldots, n_k]) = \sum_{n_1, \ldots, n_k \leq x} \sum_{1 \leq j \leq k} (-1)^{j-1} \sum_{1 \leq i_1 < \cdots < i_j \leq k} f((n_{i_1}, \ldots, n_{i_j}))
\]
\[
= \sum_{1 \leq j \leq k} (-1)^{j-1} \binom{k}{j} \sum_{n_1, \ldots, n_k \leq x} f((n_1, \ldots, n_j))
\]
\[
= k \left( \sum_{n_1 \leq x} f(n_1) \right) (x^{k-1} + O(x^{k-2})) - \binom{k}{2} \left( \sum_{n_1, n_2 \leq x} f((n_1, n_2)) \right) (x^{k-2} + O(x^{k-3}))
\]
\[
+ \sum_{3 \leq j \leq k} (-1)^{j-1} \binom{k}{j} \left( \sum_{n_1, \ldots, n_j \leq x} f((n_1, \ldots, n_j)) \right) (x^{k-j} + O(x^{k-j-1})).
\]

Now Theorems 4 and 5 give the result. □

### 5.3 Proofs of the results in Section 3.3

**Proof of Theorem 7.** Since \(B(p) = 0\) and \(B(p^\nu) - B(p^{\nu-1}) = p\) for \(\nu \geq 2\), Lemma 3 with \(k = 1\) yields

\[
\sum_{n \leq x} B(n) = \sum_{p^\nu \leq x, \nu \geq 2} \frac{p x}{p^\nu} = \sum_{p \leq \sqrt{x}} \sum_{2 \leq \nu \leq \log x / \log p} \frac{x}{p^\nu}
\]
\[
= x \sum_{p \leq \sqrt{x}} \sum_{2 \leq \nu \leq \log x / \log p} \frac{1}{p^\nu} - \sum_{p \leq \sqrt{x}} \sum_{2 \leq \nu \leq \log x / \log p} \left( \frac{x}{p^\nu} - \frac{x}{p^\nu} \right)
\]
\[
:= x \Sigma_1 - \Sigma_2
\]
with

\[ \Sigma_1 = \sum_{p \leq \sqrt{x}} p \sum_{\nu=2}^{\infty} \frac{1}{p^{\nu}} - \sum_{p \leq \sqrt{x}} p \sum_{\nu \geq \log_2 x / \log p} \frac{1}{p^{\nu}} = \sum_{p \leq \sqrt{x}} \frac{1}{p-1} + O\left( \frac{1}{x} \sum_{p \leq \sqrt{x}} p \right) \]

\[ = \sum_{p \leq \sqrt{x}} \frac{1}{p} + \sum_{p} \frac{1}{p(p-1)} - \sum_{p > \sqrt{x}} \frac{1}{p(p-1)} + O\left( \frac{\pi(\sqrt{x})}{\sqrt{x}} \right) \]

\[ = \log \log \sqrt{x} + M + \sum_{p} \frac{1}{p(p-1)} + O\left( \frac{1}{\log x} \right) \]

\[ = \log \log \sqrt{x} + F + O\left( \frac{1}{\log x} \right) \]

and

\[ |\Sigma_2| \leq \log x \sum_{p \leq \sqrt{x}} \frac{p}{\log p} \ll \sqrt{x} \log x \sum_{p \leq \sqrt{x}} \frac{1}{\log p} \ll \frac{x}{\log x} \]

by (4.8), as required. \( \square \)

**Proof of Theorem 8.** By Lemma 3, similarly to the proofs above,

\[ \sum_{n_1, \ldots, n_k \leq x} B((n_1, \ldots, n_k)) \]

\[ = \sum_{p \leq \sqrt{x}} \sum_{\nu=2}^{\infty} \frac{1}{p^{\nu}} - x^k \sum_{p \leq \sqrt{x}} \sum_{\nu \geq \log_2 x / \log p} \frac{1}{p^{\nu}} \]

\[ + O\left( x^{k-1} \sum_{p \leq \sqrt{x}} \sum_{\nu \geq \log_2 x / \log p} \frac{1}{p^{\nu(k-1)-1}} \right) \]

\[ = x^k \sum_{p \leq \sqrt{x}} \frac{1}{p^{k-1}(p^{k-1})} - x^k \sum_{p > \sqrt{x}} \frac{1}{p^{k-1}(p^{k-1})} + O\left( \sum_{p \leq \sqrt{x}} \frac{p^{k-1}}{p^{k-1}} \right) \]

\[ + O\left( x^{k-1} \sum_{p \leq \sqrt{x}} \sum_{\nu=2}^{\infty} \frac{1}{p^{\nu(k-1)-1}} \right) \]

\[ = x^k \sum_{p} \frac{1}{p^{k-1}(p^{k-1})} + O\left( \frac{x}{\log x} \right) + O\left( \sqrt{x} \pi(\sqrt{x}) \right) + O\left( x^{k-1} \sum_{p \leq \sqrt{x}} \frac{p^{k-3}}{p^{k-1}-p} \right) \]

\[ = x^k \sum_{p} \frac{1}{p^{k-1}(p^{k-1})} + O\left( \frac{x}{\log x} \right) + O(x^{k-1}U_k(x)), \]

where \( U_k(x) = 1 \) (\( k \geq 3 \)) and \( U_2(x) = \log \log x \) (\( k = 2 \)), completing the proof. \( \square \)

**Proof of Theorem 9.** Similar to the proofs of Theorems 3 and 6 by using Proposition 1 and Theorems 7 and 8. We omit the details. \( \square \)
Additive arithmetic functions meet the inclusion–exclusion principle

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