On the Calculation of Invariant Tensors in Gauge Theories

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We present an efficient method for finding the independent invariant tensors of a gauge theory. Our method uses a theorem relating invariant tensors and D-flat directions in field space. We apply our method to several examples—SO(3) with symmetric tensors, SU(2) with a dimension 4 representation, and SU(3) with matter in the sextet—and find the set of independent invariant tensors in these theories.

A gauge theory is specified by the gauge group and the representation of the matter fields under the gauge group, but all observables, including the physical spectrum, are gauge invariant combinations of fields. The structure of these objects is found by contracting the gauge-covariant gauge invariant combinations of fields. The structure of but all observables, including the physical spectrum, are representation of the matter fields under the gauge group, expected to be a basis set of invariants such that all other invariants can be generated from these basis invariants.

These motivate us to understand the basis set of invariant tensors for a general gauge theory.

Invariant tensors are known for the fundamental representations of the classical groups\(^1\). However, the tensors for many other representations are not classified. For many groups such as \(E_6\), even the full set of tensors for the fundamental representation have not been found \(^1\).

A brief example will suffice to show the kinds of difficulties that may occur. It is known that in the group \(SO(3)\), the invariant tensors are \(\delta^{ij}\), \(\epsilon^{ijk}\). These tell us that in a theory where all fields \(V_I^j\) (we are using lower case letters for the gauge representation, and upper case to label the fields—a flavor index) are in the fundamental, the complete set of invariant polynomials are generated by \(V_I^j V_J^k \delta^{ij}\) and \(V_I^j V_J^k V_L^\ell \epsilon^{ijk}\). But in a theory with \(SO(3)\) gauge symmetry and with fields \(V_I^j\) in the symmetric tensor representation, one can produce an infinite set of invariants by contracting an arbitrarily long sequence \(V_I^{j_1} V_{j_2}^{j_k} \ldots V_L^{j_\ell}\) (and there exist further invariants involving epsilon tensors). Only a small set of these are independent, but finding these is nontrivial.

In this paper, we present a new approach to finding a minimal set of invariants for more complicated gauge theories. Our approach will be to use the connection between invariant tensors and D-flat directions in field space, which was originally described for supersymmetric field theories (specifically in the context of dualities in these theories \(^3\)). This theorem asserts that the independent gauge invariant polynomial invariants of the theory are in 1-1 correspondence with the orbits of constant field configurations where configurations differing by a complex gauge transformation are identified \(^3\)\(^4\). It is clear that at any point on configuration space, we can calculate the value of any gauge invariant combination of the fields. The theorem states that this can be reversed; a knowledge of the values of all the independent gauge invariant polynomials is sufficient to reconstruct the orbits of constant field configurations quotiented by complex gauge transformations.

This theorem is often used in simple theories to characterize the field space in terms of the known operators. Here we will reverse the implication, and use the field space to find a complete set of gauge invariant objects in various theories.

The procedure for finding the tensors is then as follows. We take a set of fields in the relevant representation, and set each component to an arbitrary constant value. We then use a complex gauge transformation to set some field components to zero. If the gauge transformations are completely fixed by this procedure, then the nonzero components parametrize the orbits, and we must find a set of invariants such that each of these parameters can be written as a linear combination of invariants. Such a set of invariants would then be a basis set of invariants for the theory.

In practice, we find that the full gauge symmetry is not easy to fix with a single field. In each case, we find a remnant discrete symmetry, and occasionally, a larger continuous symmetry. One possibility is to use further fields to completely gauge fix the symmetry, but in each case we analyze below, we find that the residual symmetry is simple enough that we can find the complete set of combinations which are invariant under the residual symmetry. These combinations parametrize the gauge fixed space, and we must find a set of invariants such that each of these combinations can be written as a linear combination of invariants. Such a set of invariants would then be a basis set of invariants for the theory.

We now show the practicality of this approach by explicitly finding the basis set of invariants for three gauge theories—\(SO(3)\) with symmetric tensor matter, \(SU(2)\) with matter in the dimension 4 representation, and \(SU(3)\) with matter in the sextet. To our knowledge, the last two are completely new analyses (the first case has been analyzed previously in \(^1\)).
I. SU(2) WITH FIELDS IN THE DIMENSION 4 REPRESENTATION

We will take as our first example a theory with a gauge group $SU(2)$ and a field content where there are $N$ fields in a representation of dimension 4; this is the simplest case for which the independent set of invariants has (to our knowledge) not been worked out.

The fields can be represented as three-index tensors $V_{abc}$ where $a, b, c = 1, 2$ are acted on by the gauge symmetry, and $I = 1..N$ labels the different fields (we shall consistently use lower case indices for gauge indices and upper case indices to label the different fields, similar to a flavor index). The fields can also be represented as a column vector with four elements:

$$V^I = \begin{pmatrix} V^I_{111} \\ V^I_{112} \\ V^I_{222} \\ V^I_{222} \end{pmatrix}$$

(1)

The invariant tensor is $\epsilon^{abc}$, but one can write an infinite set of invariants, and it is hard to find relations between them. We therefore find the gauge-fixed configuration space, and attempt to characterize this space by invariants.

We begin by considering a single field $V^1$. By a complex gauge transformation, one can set the second and third components to zero, and set the first component to 1. The field then has the form

$$V^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ d \end{pmatrix}$$

(2)

This breaks the continuous gauge symmetry but preserves a discrete symmetry, which can be understood as follows: if we interchange every gauge 1 index with a 2 index, this is equivalent to taking $\epsilon^{ab} \rightarrow -\epsilon^{ab}$. Then any invariant with $4n+2$ fields will pick up a minus sign, while any invariant with $4n$ fields is unchanged. This then indicates that the combined transformation of interchanging every 1 index with a 2 index, and multiplying every field by an overall factor of $i$ should be a symmetry (this is in fact the gauge symmetry corresponding to a rotation by $\pi$ around the x-axis).

Under this symmetry, we have

$$V^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ d \end{pmatrix} \rightarrow \begin{pmatrix} id \\ 0 \\ 0 \\ i \end{pmatrix}$$

(3)

We can further use a gauge transformation by $e^{i\alpha}$ (i.e. a gauge transform by the $L_3$ subgroup of $SU(2)$) to transform

$$V^I = \begin{pmatrix} V^I_{111} \\ V^I_{112} \\ V^I_{222} \\ V^I_{222} \end{pmatrix} \rightarrow \begin{pmatrix} e^{3i\alpha}V^I_{111} \\ e^{i\alpha}V^I_{112} \\ e^{-i\alpha}V^I_{222} \\ e^{-3i\alpha}V^I_{222} \end{pmatrix}$$

(4)

A suitable choice of complex $\alpha$ allows us to bring

$$V^1 = \begin{pmatrix} 1 \\ 0 \\ d \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} id \\ 0 \\ 0 \\ i \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(5)

We have therefore produced a configuration of the form (2) but with a change in sign of $d$. This implies that the sign of $d$ can be changed by a gauge transformation. The gauge invariant combination is $(d)^2$.

We now look for a $SU(2)$ invariant tensor such that gauge fixing the field to be of the form (2) allows us to deduce the value of $(d)^2$.

There is no nonzero bilinear invariant involving $V^1$ alone. We can however find an invariant of degree 4 in $V^1$ as follows: we first construct a symmetric combination of two fields

$$W_{ab}^{IJ} = (V^I)_{acd}(V^J)_{b\cd} + (V^I)_{bcd}(V^J)_{a\cd}$$

(6)

(as always in $SU(2)$, indices are raised and lowered by the epsilon tensor). We can then construct the invariant

$$I_{4}^{IJKL} = W_{ab}^{IJ}W_{KL}^{cd}$$

(7)

We now evaluate

$$I_{4}^{1111} = -8(d)^2$$

(8)

The knowledge of the invariant $I_{4}^{1111}$ is therefore sufficient to deduce the value of $(d)^2$, and therefore is hence sufficient to completely parametrize the gauge-inequivalent configurations of a single field. By the theorem cited in the introduction, $I_{4}^{IJKL}$ is a basis set of invariants for a single field in the dimension 4 representation of $SU(2)$.

We now consider multiple fields. These can be brought by a complex gauge transformation to the form

$$V^I = \begin{pmatrix} 1 \\ 0 \\ 0 \\ d \end{pmatrix}$$

$$V^I = \begin{pmatrix} a^I \\ b^I \\ c^I \\ d^I \end{pmatrix}$$

for $I > 1$

(9)

Exactly as above, this parametrization breaks the gauge symmetry but preserves a discrete $Z_2$ gauge sym-
the group of which are invariant both under this \( Z \) symmetry. Under this symmetry, we have
\[
V^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ d^1 \\ -d^1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ (d^1)^{-1}d^1 \\ -(d^1)^{-1}d^1 \end{pmatrix}
\]
\[
V^f = \begin{pmatrix} a^f \\ b^f \\ c^f \\ d^f \end{pmatrix} \rightarrow \begin{pmatrix} (d^f)^{-1}a^f \\ -(d^f)^{-1}b^f \\ (d^f)^{-1}c^f \\ -(d^f)^{-1}d^f \end{pmatrix}
\]

We now find combinations of the nonzero parameters which are gauge invariant under the residual \( Z_2 \). It is convenient, and no more difficult, to find combinations which are invariant both under this \( Z_2 \) and the \( L_3 \) subgroup of \( SU(2) \).

We first form combinations that are invariant under the \( L_3 \) subgroup of \( SU(2) \); these are
\[
a^f d^f, a^f c^f K^L, b^f c^f, b^f b^f K^L
\]
as well as expressions where one or more of the indices is replaced by 1:
\[
d^f, c^f K^L, a^f d^f, b^f b^f K^L d^f
\]
Under the \( Z_2 \) action, these combinations are acted on as
\[
d^f \rightarrow -d^f \quad a^f (d^f) \leftrightarrow -d^f
\]
\[
a^f c^f \rightarrow -a^f c^f \quad b^f c^f \rightarrow -b^f c^f
\]
\[
a^f c^f K^L \leftrightarrow b^f b^f K^L d^f \quad b^f b^f K^L d^f \leftrightarrow c^f c^f K^L
\]

We make combinations which are even/odd under the \( Z_2 \) gauge symmetry; the gauge invariant even combinations are
\[
i_1^f = a^f (d^f) - d^f
\]
\[
i_2^f = a^f c^f - a^f d^f
\]
\[
i_3^f = b^f c^f - b^f d^f
\]
\[
i_4^{JK} = b^f b^f K^L d^f + c^f c^f K^L
\]
\[
i_5^{JKL} = a^f c^f K^L c^f + b^f b^f K^L d^f
\]

A product of two combinations odd under the \( Z_2 \) is even under the \( Z_2 \). We can therefore generate a new set of gauge invariant even combinations:
\[
i_6^f = d^f (a^f d^f + d^f)
\]
\[
i_7^f = d^f (a^f d^f + a^f d^f)
\]
\[
i_8^f = d^f (b^f c^f + b^f c^f)
\]
\[
i_9^{JK} = d^f (b^f b^f K^L d^f - c^f c^f K^L)
\]
\[
i_{10}^{JKL} = d^f (a^f c^f K^L c^f - b^f b^f K^L d^f)
\]

The combinations \( i_1^{10} \) parametrize the space of gauge inequivalent configurations.

We look for \( SU(2) \) invariants that can reproduce these combinations; that is, we look for \( SU(2) \) invariant polynomials in the fields, such that when these are evaluated on the gauge fixed configuration \( \Psi \), their values are sufficient to reconstruct the combinations \( i_2^f \) to \( i_{12}^f \). It is immediate that a set of such operators would completely parametrize the configuration space.

The flavor symmetry is a guide. For instance, the combinations with one free index, \( i_1^f, i_2^f, i_3^f \) should be reproduced from operators with one free index. One such operator is provided by the operator \( \Psi^{LKL} \) where three indices are replaced by 1. Another operator that we can consider is the antisymmetric bilinear
\[
\Psi_2^L = \Psi_{abc}^L V_{Jabc}^L
\]

Indeed, we find that on the configuration \( \Psi \)
\[
I_2^L = i_1^L, \quad I_4^{11L} = 4i_6^L
\]

Hence a knowledge of the invariants \( i_1^L, i_3^L \) indeed allows us to reproduce the combinations with one free flavor index \( i_1^f, i_3^f \).

The combinations with two free flavor indices are \( i_2^{IJ}, i_7^{IJ}, i_8^{IJ} \). We find that on the configuration \( \Psi \)
\[
I_2^{IJ} = i_2^{IJ} - 3i_7^{IJ}, \quad I_4^{11IJ} = 4i_7^{IJ} - 4i_8^{IJ}
\]

but these are insufficient to reproduce \( i_2^{IJ}, i_7^{IJ}, i_8^{IJ} \). We therefore need further invariants of degree 6 and 8.

A suitable choice are the invariants
\[
I_6^{LJKLMN} = W_{ab} W_{cd} V_{Mab}^{L} V_{Ncd}^{L} \tag{17}
\]
\[
I_8^{LJKLMNPQ} = W_{ab} W_{cd} V_{Mab}^{L} W_{ef} V_{Pab}^{L} V_{Qdef}^{L} \tag{18}
\]

We find on the configuration \( \Psi \)
\[
I_6^{1111IJ} = 16(d^f)^2 i_1^{IJ}, \quad I_8^{111111IJ} = -32(d^f)^4 i_1^{IJ} \tag{19}
\]
\[
I_6^{1111JK} = -8i_4^{JK} + .. \tag{20}
\]
\[
I_6^{1111JK} = -8i_9^{JK} + .. \tag{21}
\]

Hence a knowledge of the invariants \( i_1^L, i_3^L \) allows us to reproduce the combinations with two free flavor indices \( i_2^{IJ}, i_3^{IJ}, i_7^{IJ}, i_8^{IJ} \).

We have two combinations with three flavor indices i.e. \( i_4^{JK}, i_5^{JK} \). We find on the configuration \( \Psi \)
\[
I_4^{11JK} = -8i_4^{JK} + .. \tag{21}
\]
\[
I_6^{11JK} = -8i_9^{JK} + .. \tag{22}
\]

where we have omitted terms which are composed of products of invariants of lower degree. So we do not need further invariants to reproduce the combinations with three flavor indices.

When we move to the combinations with four indices, we find that \( i_1^{JKL} \) cannot be reproduced by the invariants we have. We need a different invariant with six fields, which is
\[
I_6^{LJKLMN} = W_{ab} W_{cd} V_{Mab}^{L} V_{Ncd}^{L} \tag{23}
\]

We find that on the configuration \( \Psi \)
\[
I_6^{11IJLKL} = 48i_1^{IJLKL} - 16(i_5^{IJLKL} + i_7^{IJLKL} + i_8^{IJLKL})
\]
\[
I_6^{11IJLKL} = -24i_1^{IJLKL} + 16(i_5^{IJLKL} + i_7^{IJLKL} + i_8^{IJLKL})
\]
which allows us to solve for $I_{10}^{JKL}$. Finally, it is straightforward to show that the combination $I_{10}^{JKL}$ with four flavor indices can be reproduced from $I_{4}^{IKL}$, $I_{8}^{J111 KL}$.

We find then that the invariants

$$I_{2}^{J}, I_{4}^{11J}, I_{6}^{1JKLMN}, I_{6}^{JKLMN}, I_{8}^{JKLMPQ}$$

are sufficient to reconstruct the gauge invariant parameter space of this theory. The theorem from the introduction then tells us that these are a complete set of independent polynomial invariants for the dimension 4 representation of $SU(2)$.

II. $SO(3)$ WITH FIELDS IN THE DIMENSION-5 REPRESENTATION

We will take as our next example a theory with a gauge group $SO(3)$ and a field content where there are $N$ fields in a representation of dimension 5; such a field is a symmetric tensor $V_{ij}$ of $SO(3)$, where we take $i, j = 1..3$, and $I = 1..N$ is a flavor index labeling the different fields. The field is a traceless symmetric tensor of $SO(3)$, and can therefore be written as

$$V^{I} = \begin{pmatrix} V_{11}^{I} & V_{12}^{I} & V_{13}^{I} \\ V_{21}^{I} & V_{22}^{I} & V_{23}^{I} \\ V_{31}^{I} & V_{32}^{I} & V_{33}^{I} \end{pmatrix}$$

with $V_{11}^{I} + V_{22}^{I} + V_{33}^{I} = 0$.

It will prove convenient to define a product

$$(A \cdot B)_{ij} = A_{ik} B_{kj}$$

as well as a trace

$$Tr(A) = A_{ij} \delta^{ij}$$

We may then write a sequence of invariants

$$I_{2}^{IJ} = Tr(V^{I} \cdot V^{J})$$
$$I_{3}^{JK} = Tr(V^{I} \cdot V^{J} \cdot V^{K})$$
$$I_{4}^{JKLM} = Tr(V^{I} \cdot V^{J} \cdot V^{K} \cdot V^{L})$$
$$I_{5}^{JKLMN} = Tr(V^{I} \cdot V^{J} \cdot V^{K} \cdot V^{L} \cdot V^{M})$$

and so on.

To find the independent set of invariants, we now find the gauge-fixed configuration space, and attempt to characterize this space by invariants.

We first consider the case where the matter content is a single field $V^{I}$. We can use complex gauge transformations to bring this to the form of a diagonal matrix

$$V^{I} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

This generically fixes the continuous gauge symmetry, but the ordering of the three diagonal elements can be changed by a gauge transformation. There is therefore a residual discrete $Z_{2} \times Z_{2}$ gauge symmetry. In the special case when two eigenvalues coincide, the continuous symmetry is partially unbroken and there is a residual $U(1)$ symmetry.

The invariants are the symmetric combinations

$$i_{1} = a + b + c = 0$$
$$i_{2} = a^{2} + b^{2} + c^{2}$$
$$i_{3} = a^{3} + b^{3} + c^{3}$$

We look for $SO(3)$ invariants which can reproduce these combinations. We find

$$I_{2}^{11} = i_{2}, \quad I_{3}^{111} = i_{3}$$

The invariants $I_{2}^{IJ}, I_{3}^{IJK}$ are hence sufficient to completely parametrize the gauge-invariant configurations of a single field, and are hence form a complete set of invariants for one field.

We now consider a generic configuration of multiple fields $V^{I}$. We gauge fix $V^{1}$ as before. The configuration is now

$$V^{1} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a - b \end{pmatrix}$$

$$V^{I} = \begin{pmatrix} V_{11}^{I} & V_{12}^{I} & V_{13}^{I} \\ V_{21}^{I} & V_{22}^{I} & V_{23}^{I} \\ V_{31}^{I} & V_{32}^{I} & V_{33}^{I} \end{pmatrix}$$

for $I > 1$

We first consider the special case where two eigenvalues of $V^{1}$ coincide. Here we have that

$$V^{1} = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$V^{I} = \begin{pmatrix} V_{11}^{I} & V_{12}^{I} & V_{13}^{I} \\ V_{21}^{I} & V_{22}^{I} & V_{23}^{I} \\ V_{31}^{I} & V_{32}^{I} & V_{33}^{I} \end{pmatrix}$$

for $I > 1$

This particular configuration preserves a $U(1)$ subgroup of the $SU(2)$, so we could proceed by further fixing the gauge to find the completely gauge-fixed hypersurface in field space. But the gauge group is simple enough at this point that we can straightforwardly write down a set of independent polynomial combinations (invariant under the $U(1)$ symmetry) which parametrize the configuration space.

The fields in $V^{I}$ can be organized into combinations with specific charges $2, 1, 0, -1, -2$ under the $U(1)$; these
are
\[
T_{2}^I = 2V_{12}^I + i(V_{11}^I - V_{22}^I),
T_{1}^I = V_{13}^I - iV_{23}^I, \\
T_{0}^I = V_{11}^I + V_{22}^I,
\]
(36)
where the subscripts denote the respective charges. The negatively charged fields are the complex conjugates of the positive charges.

In addition to the \(U(1)\), there is also a discrete \(Z_2\) symmetry corresponding to charge conjugation
\[
T_{2}^I \leftrightarrow -T_{-2}^I, \quad T_{1}^I \leftrightarrow -T_{-1}^I, \quad T_{0}^I \leftrightarrow T_{0}^I
\]
(37)

The combinations of fields invariant under the \(U(1)\) symmetry are
\[
T_{2}^I T_{-2}^J, \quad T_{1}^I T_{-1}^J, \quad T_{0}^I, \quad T_{2}^I T_{-1}^J T_{-1}^K T_{1}^L, \quad T_{-2}^I T_{1}^J T_{1}^K.
\]
(38)
Under the \(Z_2\) action, these combinations are acted on as
\[
T_{0}^I \leftrightarrow T_{0}^I, \\
T_{2}^I T_{-2}^J \leftrightarrow T_{2}^I T_{-2}^J, \\
T_{1}^I T_{-1}^J \leftrightarrow T_{1}^I T_{-1}^J, \\
T_{2}^I T_{-1}^J T_{1}^K \leftrightarrow -T_{-2}^I T_{1}^J T_{1}^K.
\]

We make combinations which are even/odd under the \(Z_2\); the gauge invariant even combinations are
\[
i_1^I = T_{0}^I, \\
i_2^I = T_{2}^I T_{-2}^J + T_{2}^I T_{-2}^J, \\
i_3^I = T_{1}^I T_{-1}^J + T_{1}^I T_{-1}^J, \\
i_4^{IK} = T_{2}^I T_{-1}^J T_{-1}^K - T_{-2}^I T_{1}^J T_{1}^K.
\]
(39)
A product of two combinations odd under the \(Z_2\) is even under the \(Z_2\). The only such combination which cannot be written in terms of the already obtained even combinations is
\[
i_5^{IKL} = (T_{2}^I T_{-2}^J - T_{2}^I T_{-2}^J)(T_{1}^K T_{-1}^L - T_{1}^K T_{-1}^L).
\]
(40)
The combinations \(i_1^I, i_2^I, i_3^I, i_4^{IK}, i_5^{IKL}\) completely parametrize the gauge-invariant orbits of the configuration space.

We now promote these to \(SO(3)\) invariants; that is, we look for \(SO(3)\) invariants which reduce to the combinations \((39, 40)\) on the gauge-fixed configuration space of equation \((44)\). Once again, we use the flavor symmetry as a guide.

The combination with one free index i.e. \(i_1^I\) should be reproduced from operators with one free flavor index. One such operator is provided by the operator \(I_{2}^{IJ}\) where one of the fields is taken to be \(V^I\). Indeed, we find
\[
I_{2}^{IJ} = 3a_i i_1^I.
\]
(41)
Hence a knowledge of the invariant \(I_{2}^{IJ}\) allows us to reproduce the combinations with one free flavor index \(i_1^I\).

The combinations with two free flavor indices i.e. \(i_2^I, i_3^I\) should be reproduced from \(SO(3)\) invariants with two free flavor indices \(IJ\). Indeed, we find
\[
I_{2}^{IJ} = \frac{1}{4}(i_2^I + i_3^I + \ldots)
\]
(42)
\[
I_{3}^{IJ} = \frac{a}{4}(i_2^I - 2i_3^I) + \ldots
\]
(43)
where the ellipses indicate terms with (already determined) lower degree combinations like \(i_1^I i_1^I\). These two invariants therefore determine \(i_2^I, i_3^I\).

For \(i_4^{IK}\), we have \(JK\) symmetric. This will be reproduced by an \(SO(3)\) invariant with three flavor indices \(IJK\), where the \(JK\) indices are symmetrized. Such a \(SO(3)\) invariant, in general, when expanded in terms of the fields, would produce linear combinations of \(i_4^{IJK} + i_4^{IKJ} + i_4^{KJI}\) and \(-i_4^{IJK} + i_4^{IKJ} + i_4^{KJI}\), both of which satisfy these symmetries. We therefore need at least two different \(SO(3)\) invariants with this symmetry structure.

There is one such invariant of degree 3 which is completely symmetric
\[
I_{3}^{IJK} = -\frac{1}{4}(i_2^I + i_4^{IK} + i_4^{KIJ})
\]
(44)
where the ellipses indicate terms with (already determined) lower degree combinations. We need at least one more invariant, and so we now consider invariants of degree 4.

A consideration of the combination \(i_5^{IKL}\) suggests that we should look at combinations where the four indices form two antisymmetrized pairs. We choose such a combination
\[
I_{4}^{IJKL} = Tr(V^I \cdot V^J \cdot V^K \cdot V^L)
\]
(45)
and we find
\[
I_{4}^{IJK1} + I_{4}^{IKJ1} = -\frac{3}{4}(2i_{4}^{IJK} - i_{4}^{IKJ} - i_{4}^{KIJ})
\]
(46)
Hence the invariants \(I_{3}^{IJK}, I_{4}^{IJKL}\) allow us to solve for \(i_{4}^{IJK}\)

Finally we consider \(i_5^{IKL}\). Here \(IJ\) and \(KL\) are antisymmetrized. We should look for \(SO(3)\) invariants with four flavor indices \(IJKL\), where the \(IJ, KL\) indices are antisymmetrized. However, such an \(SO(3)\) invariant, in general, when expanded in terms of the fields, will produce linear combinations of

(a) the combination completely antisymmetric in \(IJKL\):
\[
i_{5}^{IKLJ} + i_{5}^{KJIL} - i_{5}^{IKJL} + i_{5}^{JIKL} + i_{5}^{ILJK} - i_{5}^{IJKL}
\]
(b) another combination, still symmetric in \((IJ) \leftrightarrow (KL)\):
\[
i_{5}^{IKLJ} + i_{5}^{KJIL} + (1/2)(i_{5}^{IKJL} - i_{5}^{IKJL} - i_{5}^{IJKL} + i_{5}^{IJKL})
\]
and (c) one combination antisymmetric in \((IJ) \leftrightarrow (KL)\)

\[ i_{5}^{IJKL} - i_{5}^{KLIJ} \]

We should therefore have at least three \(SO(3)\) invariants with four flavor indices \(IJKL\), where the \(IJ, KL\) indices are antisymmetrized. One such invariant is provided by \(i_{5}^{IJKL}\). We therefore need two invariants of degree 5.

We therefore define

\[ I_{5}^{ijkl} = i_{5}^{ijkl} \quad (47) \]
\[ I_{6}^{ijkl} = i_{5}^{ijkl} \quad (48) \]

We find that \(i_{5}^{ijkl}\) is proportional to the completely antisymmetric combination (a), \(I_{5}^{ijkl}\) is proportional to the combination (b), and \(I_{5}^{ijkl} - I_{5}^{klij}\) is proportional to combination (c). Hence \(i_{5}^{ijkl}\) can indeed be written as a combination of these invariants.

The configuration space with the enhanced \(U(1)\) symmetry is therefore completely characterized by the invariants

\[ I_{2}^{I}, I_{3}^{IJK}, I_{5}^{ijkl}, I_{6}^{ijkl} \]

More generally, the eigenvalues of \(V^I\) are all different, and we have

\[ V^I = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & -a & -b \end{pmatrix} \quad (49) \]

The parameters \(a, b\) are reproduced from \(I_{2}^{I}, I_{3}^{IJK}\), following the analysis for a single field.

This parametrization of the configuration space preserves two \(Z_2\) symmetries:

\[ (Z_2)_A : V_{12}^I \rightarrow -V_{12}^I, V_{13}^I \rightarrow -V_{13}^I \]
\[ (Z_2)_B : V_{12}^I \rightarrow -V_{12}^I, V_{23}^I \rightarrow -V_{23}^I \quad (50) \]

To find the moduli space, we should find the combinations of \(V^I\) which are gauge invariant under these discrete symmetries. Once again, the remaining symmetry is simple enough that we can just do this by inspection. We find that the polynomials invariant under the two discrete symmetries are generated by

\[ V_{11}^I, V_{22}^I, V_{12}^I V_{12}^J, V_{13}^I V_{13}^J, V_{23}^I V_{23}^J, V_{12}^I V_{13}^J V_{23}^K \]

We now promote these to \(SO(3)\) invariants; that is, we look for \(SO(3)\) invariants which reduce to these combinations on the configuration space of equation (49). We first check whether the invariants we have found are sufficient to do this.

We start with invariants with one flavor index \(I\). We find

\[ I_{3}^{I} = aV_{11}^I + bV_{22}^I + (a + b)(V_{11}^I + V_{22}^I) \quad (51) \]
\[ I_{3}^{11I} = a^2V_{11}^I + b^2V_{22}^I + (a + b)^2(-V_{11}^I - V_{22}^I) \quad (52) \]

which can be used to solve for \(V_{11}^I, V_{22}^I\) in terms of \(I_{3}^{I}, I_{3}^{11I}\). This inversion fails only if \(b^2 = 0\) and \(a = b\). Hence \(I_{3}^{I}, I_{3}^{11I}\) are sufficient to solve for \(V_{11}^I, V_{22}^I\).

Our final result is then that every point on the gauge invariant configuration space can be reproduced by a knowledge of the invariants

\[ I_{2}^{I}, I_{3}^{IJK}, I_{5}^{ijkl}, I_{6}^{ijkl} \quad (53) \]

Hence the theorem described in the introduction ensures that any gauge invariant polynomial in this theory can be generated by these invariants.

III. SU(3) WITH SEXTETS

We now consider a theory with a \(SU(3)\) symmetry and fields in the sextet representation \(V_{ij}^I\).

We can form an infinite set of invariants by contracting these fields with the epsilon tensor \(\epsilon_{ijk}\). We now find an independent set of tensors by finding a set that can parametrize the gauge fixed configuration space.

We begin by considering a single field. By a gauge transformation, we can bring it to the form

\[ V^I = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \quad (54) \]

This form of the field preserves two \(U(1)\) symmetries; the first is

\[ V_{11} \rightarrow e^{2\alpha}V_{11}, \quad V_{22} \rightarrow e^{-2\alpha}V_{22}, \quad V_{33} \rightarrow V_{33} \]
\[ V_{13} \rightarrow e^{i\alpha}V_{13}, \quad V_{23} \rightarrow e^{-i\alpha}V_{23}, \quad V_{12} \rightarrow V_{12} \quad (55) \]

and the second is

\[ V_{11} \rightarrow e^{2\alpha}V_{11}, \quad V_{33} \rightarrow e^{-2\alpha}V_{33}, \quad V_{22} \rightarrow V_{22} \]
\[ V_{12} \rightarrow e^{i\alpha}V_{12}, \quad V_{23} \rightarrow e^{-i\alpha}V_{23}, \quad V_{13} \rightarrow V_{13} \quad (56) \]

These symmetries alter the eigenvalues without changing the form: the first one takes \(a \rightarrow e^{2\alpha}a, b \rightarrow e^{-2\alpha}b, c \rightarrow c\), and the second takes \(a \rightarrow e^{2\alpha}a, c \rightarrow e^{-2\alpha}c, b \rightarrow b\). The only gauge invariant combination is the product \(abc\).
We define the invariant (we use dotted indices for the complex conjugate representation)
\[ I^{iJK}_3 \equiv \epsilon^{ikm} \epsilon^{jm} v_{ij}^{I} V^{I}_{kl} V^{K}_{mn} \]  
(57)
and we find
\[ I^{111}_3 = 6abc \]  
(58)
This invariant therefore reproduces the gauge-fixed configuration space for a single field, and is therefore a complete set of invariants for a single sextet of SU(3).

We now consider multiple fields. We can bring these to the form
\[
V^I = \begin{pmatrix}
    a & 0 & 0 \\
    0 & a & 0 \\
    0 & 0 & a
\end{pmatrix}
\]
(59)

where we have used the U(1) symmetries to further simplify the form of the first field.

The form of the configuration space still preserves an SO(3) symmetry, and each sextet of SU(3) decomposes to a 1+5 of SO(3). Fortunately, we have analyzed this system already in the previous section, and so we can write down the invariants. The only new invariant is the singlet, which is the trace of the matrix. Combining with the previously derived SO(3) invariant combinations for this theory are

\[
i^{I}_1 \equiv \text{Tr}(V^I) \]
\[i^{JJ}_2 \equiv \text{Tr}((V^I \cdot V^J) \]
\[i^{JK}_3 \equiv \text{Tr}(V^I \cdot V^J \cdot V^K) \]
\[i^{JKLM}_4 \equiv \text{Tr}(V^I \cdot V^J \cdot V^K \cdot V^L) \]
\[i^{JKLMN}_5 \equiv \text{Tr}(V^I \cdot V^J \cdot V^K \cdot V^L \cdot V^M) \]
(60)

We now find SU(3) invariants which reproduce these combinations on the configuration space (59).

From the invariant that we have already defined, we obtain
\[ I^{11}_3 = 2a^2 i^J_2 \]
\[ I^{JJ}_3 = -a i^J_2 + .. \]
(61)

which reproduces all combinations with one, two or three free flavor indices.

For the remaining combinations, we need to consider invariants containing six fields contracted with 4 epsilon tensors. The structure of the combinations above suggests that we should look at combinations where there are two pairs of three fields, where the three fields are antisymmetrized. This suggests the invariant
\[ I^{JKLMN}_6 = \epsilon^{abc} V_{ad}^{I} V_{be}^{J} V_{cf}^{K} \epsilon^{im} \epsilon^{jlm} V_{kl}^{I} V_{mn}^{K} \]  
(62)

We find
\[ J^{[1]}_{6} = \frac{\epsilon^{abc} V_{ad}^{I} V_{be}^{J} V_{cf}^{K}}{6} \]  
(63)
\[ J^{[JKLM]}_{6} = 6 a i^{JK}_3 \]  
(64)
\[ J^{[[LM]]}_{6} = 4 a i^{JK}_3 \]  
(65)

We thus find that every point on the gauge-fixed configuration space can be reproduced by a knowledge of the invariants
\[ I^{JKLMN}_3 \cdot I^{JKLMN}_6 \]  
(66)

Hence the theorem described in the introduction ensures that any gauge invariant polynomial in this theory can be generated by these invariants.

**IV. SUMMARY AND CONCLUSION**

We have discussed a new method to efficiently find a set of independent invariant tensors in gauge theories. We have done this by using a theorem, familiar from supersymmetric field theories, that relates D-flat directions to the invariants in a gauge theory. Specifically, this theorem asserts that the constant configurations, identified by complex gauge transformations, are in 1-1 correspondence with the gauge invariant operators in the theory.

We have shown that this provides a straightforward method to find the independent invariant tensors. We have explicitly applied these methods to three gauge theories – SO(3) with fields in the symmetric tensor representation, SU(2) with a dimension 4 representation, and SU(3) with matter in the sextet – and in each case, we have found the set of independent polynomial invariants. This shows the practicality of the approach.

Our methods are general, and as far as we can see, can be applied to any group with any matter content. The immediate ones which would be interesting to analyze are exceptional groups with matter in the (anti)-fundamental representation. Knowing the invariant tensors would also help in looking for dual pairs in supersymmetric gauge theories with exceptional gauge groups.

We hope to return to this topic in future work.

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[1] P. Pouliot, J. Phys. A 34, 8631 (2001) doi:10.1088/0305-4470/34/41/317 [hep-th/0107151].
[2] N. Seiberg, “Electric - magnetic duality in supersymmetric nonAbelian gauge theories,” Nucl. Phys. B 435, 129 (1995) [hep-th/9411149].
[3] D. Kutasov, A. Schwimmer and N. Seiberg, “Chiral Rings, Singularity Theory and Electric-Magnetic Duality,” Nucl. Phys. B 459, 455 (1996) [hep-th/9510222].
[4] P. Brax, C. Grojean and C. A. Savoy, “Anomaly matching and syzygies in N = 1 gauge theories,” Nucl. Phys. B 561, 77 (1999) [hep-ph/9808345].
[5] P. Pouliot, “Molien function for duality,” JHEP 9901, 021 (1999) [hep-th/9812015].
[6] H. Weyl, “The Classical Groups, Their Invariants and Representations,” Princeton University Press, 1946.
[7] W. Fulton and J. Harris, “Representation Theory, A First Course,” Springer-Verlag, 1991.
[8] P. Olver, “Classical Invariant Theory,” London Mathematical Society Student Texts #44, 1999.
[9] J. Dieudonne and J. Carrell, “Invariant Theory, Old and New,” Academic Press, 1971.
[10] R. Howe, “Perspectives on Invariant Theory,” The Schur Lectures (1992), Israel Mathematical Conference Proceedings, 1995.
[11] G. Gurevich, “Foundations of the Theory of Algebraic Invariants,” P. Noordhoff - Groningen, The Netherlands, 1964.
[12] G. Schwarz, “Invariant theory of $G_2$,” Bull. Am. Math. Soc. 9, 335 (1983).
[13] G. Schwarz, “Invariant theory of $G_2$ and $Spin_7$,” Comment. Math. Helvetici 63, 624 (1988).
[14] G. Schwarz, “Representations of Simple Lie Groups with a Free Module of Covariants,” Inventiones Math. 50, 1 (1978).
[15] G. Schwarz, “Representations of Simple Lie Groups with Regular Rings of Invariants,” Inventiones Math. 49, 167 (1978).
[16] V. Popov, “Syzygies in the Theory of Invariants,” Math. USSR Izvestiya 22, 507 (1984).
[17] Y. Gufan, A. V. Popov, G. Sartori, V. Talamini, G. Valent and E. Vinberg, “Geometric Invariant Theory Approach to the Determination of Ground States D-wave Condensates in Isotropic Space,” Jour. Math. Phys. 42, 1533 (2001).
[18] D. Berger, J. N. Howard and A. Rajaraman, LHEP 1, no. 2, 14-19 (2018) doi:10.31526/LHEP.2.2018.04 [arXiv:1806.04332 [hep-th]].
[19] F. Buccella, J. P. Derendinger, S. Ferrara and C. A. Savoy, “Patterns Of Symmetry Breaking In Supersymmetric Gauge Theories,” Phys. Lett. B 115, 375 (1982).
[20] C. Procesi and G. W. Schwarz, “The Geometry Of Orbit Spaces And Gauge Symmetry Breaking In Supersymmetric Gauge Theories,” Phys. Lett. B 161 (1985) 117.
[21] R. Gatto and G. Sartori, “Consequences Of The Complex Character Of The Internal Symmetry In Supersymmetric Theories,” Commun. Math. Phys. 109, 327 (1987).
[22] T. Gherghetta, C. Kolda and S. P. Martin, “Flat directions in the scalar potential of the supersymmetric standard model,” Nucl. Phys. B 468, 37 (1996) [hep-ph/9510370].
[23] P. Brax and C. A. Savoy, “Supersymmetric flat directions and analytic gauge invariants,” hep-th/0104077.
[24] M. A. Luty and W. I. Taylor, “Varieties of vacua in classical supersymmetric gauge theories,” Phys. Rev. D 53, 3399 (1996) [hep-th/9506098].