ASKEY-WILSON FUNCTIONS AND QUANTUM GROUPS.

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Abstract. Eigenfunctions of the Askey-Wilson second order \( q \)-difference operator for \( 0 < q < 1 \) and \( |q| = 1 \) are constructed as formal matrix coefficients of the principal series representation of the quantized universal enveloping algebra \( \mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C})) \). The eigenfunctions are in integral form and may be viewed as analogues of Euler’s integral representation for Gauss’ hypergeometric series. We show that for \( 0 < q < 1 \) the resulting eigenfunction can be rewritten as a very-well-poised \( {}_8\phi_7 \)-series, and reduces for special parameter values to a natural elliptic analogue of the cosine kernel.

Dedicated to Mizan Rahman

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1. Introduction

The aim of this paper is to simplify the quantum group construction of explicit eigenfunctions of the second order Askey-Wilson \( q \)-difference operator, and to extend the results to the interesting and less well studied \( |q| = 1 \) case.

The approach is based on the known fact from [11], [15] and [6] that the second order Askey-Wilson difference operator arises as radial part of the quantum Casimir element of \( \mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C})) \) when the radial part is computed with respect to Koornwinder’s twisted primitive elements. Using this result, we construct nonpolynomial eigenfunctions of the Askey-Wilson second order difference operator as matrix coefficients of the principal series representation of \( \mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C})) \). The two cases \( 0 < q < 1 \) and \( |q| = 1 \) will be treated separately. The theory for \( 0 < q < 1 \) is related to the noncompact quantum group \( \mathcal{U}_q(\mathfrak{su}(1, 1)) \), while for \( |q| = 1 \) it is related to the noncompact quantum group \( \mathcal{U}_q(\mathfrak{su}(2, \mathbb{R})) \).

This approach was considered in [8] for \( 0 < q < 1 \) using an explicit realization of the principal series representation on \( L^2(\mathbb{Z}) \). The resulting eigenfunction then appears as a non-symmetric Poisson type kernel involving nonterminating \( {}_2\phi_1 \) series. With the help of a highly nontrivial summation formula, proved by Rahman in the appendix of [8] (see [7] for extensions), this eigenfunction was expressed as one of the explicit \( {}_8\phi_7 \)-solutions
of the Askey-Wilson second order difference operator from \[5\]. This eigenfunction was called the Askey-Wilson function in \[9\], since it is a meromorphic continuation of the Askey-Wilson polynomial in its degree. In this paper we start by reproving this result, now using an explicit realization of the principal series representation of \(U_q(\mathfrak{sl}(2, \mathbb{C}))\) as difference operators acting on analytic functions on the complex plane. Koornwinder's twisted primitive element then acts as a first order difference operator, hence eigenvectors are easily constructed (for the positive discrete series, this was observed by Rosengren in \[17\]). The corresponding matrix coefficients lead to explicit integral representations for eigenfunctions of the Askey-Wilson second order difference operator. These matrix coefficients can be rewritten as the explicit \(s\phi_7\)-series representation of the Askey-Wilson function by a residue computation.

We also show that for a special choice of parameter values, the Askey-Wilson function reduces to an elliptic analogue of the cosine kernel. This is the analogue of the classical fact that the Jacobi function reduces to the cosine kernel for special parameter values, see e.g. \[10\]. We give two proofs, one proof uses an explicit expansion formula of the Askey-Wilson function in Askey-Wilson polynomials from \[21\], the other proof uses Cherednik’s Hecke algebra techniques from \[2\] and \[22\].

In the second part of the paper we consider the quantum group techniques for \(|q| = 1\). In this case, the approach is similar to the construction of quantum analogues of Whittaker vectors and Whittaker functions from \[12\]. The role of \(q\)-shifted factorials, or equivalently \(q\)-gamma functions, is now taken over by Ruijsenaars' hyperbolic gamma function. The hyperbolic gamma function is directly related to Barnes’ double gamma function, as well as to Kurokawa’s double sine function, see \[19\] and references therein. The quantum group technique applied to this particular set-up leads to an eigenfunction of the Askey-Wilson second order difference operator for \(|q| = 1\), given explicitly as an Euler type integral involving hyperbolic gamma functions.

The emphasis in this paper lies on exhibiting the similarities between the \(0 < q < 1\) case and the \(|q| = 1\) case as much as possible. Other approaches might very well lead to eigenfunctions for the Askey-Wilson second order difference operator for \(|q| = 1\) which are “more optimal”, in the sense that they satisfy two Askey-Wilson type difference equations in the geometric parameter, one with respect to base \(q = \exp(2\pi i \tau)\), the other with respect to the “modular inverted” base \(\bar{q} = \exp(2\pi i/\tau)\), cf. \[12\] for \(q\)-Whittaker functions. Such eigenfunctions are expected to be realized as matrix coefficients of the modular double of the quantum group \(U_q(\mathfrak{sl}(2, \mathbb{C}))\) (a concept introduced by Faddeev in \[3\]), and are expected to be closely related to Ruijsenaars’ \[19\], \[20\] R-function. The R-function is an eigenfunction of two Askey-Wilson type second order difference operators in the geometric parameter, which is explicitly given as a Barnes’ type integral involving hyperbolic gamma functions. I hope to return to these considerations in a future paper.

Acknowledgments: It is a pleasure to dedicate this paper to Mizan Rahman. His important contributions to the theory of basic hypergeometric series and, more concretely, his kind help in the earlier stages of the research on the Askey-Wilson functions in \[8\], have played, and still play, an important role in my research on Askey-Wilson functions.

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2. Generalized gamma functions

In this section we discuss $q$-analogues of the gamma function for deformation parameter $q$ in the regions $0 < |q| < 1$ and $|q| = 1.$

2.1. The $q$-gamma function for $0 < |q| < 1$. Let $\tau$ be a fixed complex number in the upper half plane $\mathbb{H}$. The corresponding deformation parameter $q = q_\tau = \exp(2\pi i \tau)$ has modulus less than one. We write $q^u = \exp(2\pi i u)$ for $u \in \mathbb{C}$.

Let $b, b_j \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. The $q$-shifted factorial is defined by

$$ (b; q)_n = \prod_{j=0}^{n-1} (1 - bq^j), \quad (b_1, \ldots, b_m; q)_n = \prod_{j=1}^{m} (b_j; q)_n. $$

In $q$-analysis the function

$$ x \mapsto \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x} $$

is known as the $q$-gamma function, see e.g. [4]. For our purposes, it is more convenient to work with the function

$$ (2.1) \Gamma_\tau(x) := \frac{q^{-\frac{x^2}{16}}}{(-q^{\frac{1}{2}(x+1)}; q)_\infty}. $$

Observe that $\Gamma_\tau(x)$ is a zero-free meromorphic function with simple poles located at $-1 + \tau^{-1} + 2\mathbb{Z}_{\leq 0} + 2\tau^{-1}\mathbb{Z}$. It furthermore satisfies the difference equation

$$ (2.2) \Gamma_\tau(x + 2) = 2 \cos(\pi(x + 1)\tau/2) \Gamma_\tau(x). $$

Note furthermore that for $\tau \in i\mathbb{R}_{>0}$, i.e. $0 < q < 1$, the function $\bar{\Gamma}_\tau(z)$ satisfies $\Gamma_\tau(x) = \Gamma_\tau(\bar{x})$, where the bar stands for the complex conjugate.

Observe that the above defined $q$-gamma type functions are not $\tau^{-1}$-periodic. It is probably for this reason that formulas in $q$-analysis are usually expressed in terms of $q$-shifted factorials instead of $q$-gamma functions. For our present purposes the expressions in terms of $q$-gamma type functions are convenient because it clarifies the similarities with the $|q| = 1$ case.

2.2. The gamma function for $|q| = 1$. In this subsection we take $\tau \in \mathbb{R}_{<0}$, whence $q = q_\tau = \exp(2\pi i \tau)$ satisfies $|q| = 1$. As in the previous subsection, we write $q^u = \exp(2\pi i u)$ for $u \in \mathbb{C}$.

It is easy to verify that the integral

$$ (2.3) \gamma_\tau(z) = \frac{1}{2i} \int_0^\infty dy \left( \frac{z}{y} - \frac{\sinh(\tau y z)}{\sinh(y) \sinh(\tau y)} \right) $$

converges absolutely in the strip $|\text{Re}(z)| < 1 - \tau^{-1}$. For $z \in \mathbb{C}$ in this strip we set

$$ (2.4) \Gamma_\tau(z) = \exp(i\gamma_\tau(z)). $$
Ruijsenaars’ \[18\] Sect. 3] hyperbolic gamma function \( G(z) = G(a_+, a_-; z) \) with \( a_+, a_- > 0 \) is related to \( G_\tau \) by
\[
G_\tau(z) = G(a_+, a_-; ia_- z/2), \quad \tau = -a_- / a_+.
\]
In the following proposition we recall some of Ruijsenaars’ results \[18\] Sect. 3] on the hyperbolic gamma function.

**Proposition 2.1.** (i) The function \( G_\tau(z) \) satisfies the difference equation
\[
G_\tau(z + 2) = 2 \cos(\pi (z + 1) \tau/2) G_\tau(z).
\]
In particular, \( G_\tau(z) \) admits a meromorphic continuation to the complex plane \( \mathbb{C} \), which we again denote by \( G_\tau(z) \).

(ii) The zeros of \( G_\tau(z) \) are located at \( 1 - \tau^{-1} + 2 \mathbb{Z}_{\geq 0} + 2 \tau^{-1} \mathbb{Z}_{\leq 0} \), and the poles of \( G_\tau(z) \) are located at \( -1 + \tau^{-1} + 2 \mathbb{Z}_{\leq 0} + 2 \tau^{-1} \mathbb{Z}_{\geq 0} \).

(iii) \( G_\tau(-z) G_\tau(z) = 1 \) and \( G_\tau(z) = G_{\tau-1}(-\tau z) \).

(iv) The function \( \gamma_\tau(z) \) has an analytic continuation to the cut plane \( \mathbb{C} \setminus \{[-\infty, -1 + \tau^{-1}] \cup [1 - \tau^{-1}, \infty)\} \), which we again denote by \( \gamma_\tau(z) \). Set \( r = \max(1, -\tau^{-1}) \) and choose \( \epsilon > 0 \), then
\[
| \mp \gamma_\tau(z) + \frac{\pi \tau z^2}{8} - \frac{\pi}{24} (\tau + \tau^{-1}) | = \mathcal{O}(\exp((\epsilon - \pi/r)|\text{Im}(z)|)), \quad \text{Im}(z) \to \pm \infty
\]
uniformly for \( \text{Re}(z) \) in compacts of \( \mathbb{R} \).

From the explicit expression for \( G_\tau(z) \) with \( |\text{Re}(z)| < 1 - \tau^{-1} \) and the first order difference equation for \( G_\tau \), we have \( \overline{G_\tau(z)} = G_\tau(\bar{z}) \).

As Ruijsenaars verifies in \[19\] Appendix A], the hyperbolic gamma function is a quotient of Barnes’ double gamma function, and it essentially coincides with Kurokawa’s double sine function.

The hyperbolic gamma function is the important building block for \( q \)-analysis with \( |q| = 1 \). It was used in \[13\] and \[14\] to construct for \( |q| = 1 \) explicit integral solutions of the \( q \)-Bessel difference equation and of the \( q \)-hypergeometric difference equation. Ruijsenaars’ \[19\] used hyperbolic gamma functions to construct an eigenfunction for the Askey-Wilson second order difference operator for \( |q| = 1 \) as an explicit Barnes’ type integral. In this paper we construct an eigenfunction of the Askey-Wilson second order difference operator for \( |q| = 1 \) as an Euler type integral, using representation theory of quantum groups.

### 3. The Askey-Wilson second order difference operator and quantum groups

Throughout this section we require that \( \tau \in \mathbb{C} \setminus \frac{1}{2} \mathbb{Z} \) and we write \( q = q_\tau = \exp(2\pi i \tau) \) for the corresponding deformation parameter. The condition on \( \tau \) implies \( q \neq \pm 1 \). As usual, we write \( q^u = \exp(2\pi i u) \) for \( u \in \mathbb{C} \).
Definition 3.1. The quantum group $U_q$ is the unital associative algebra over $\mathbb{C}$ generated by $K^{\pm 1}$, $X^+$ and $X^-$, subject to the relations

\begin{align*}
KK^{-1} &= K^{-1}K = 1, \\
KX^+ &= qX^+K, \quad KX^- = q^{-1}X^-K, \\
X^+X^- - X^-X^+ &= \frac{K^2 - K^{-2}}{q - q^{-1}}.
\end{align*}

It is well known that $U_q$ has the structure of a Hopf-algebra, and as such it is a quantum deformation of the universal enveloping algebra of the simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. The Hopf-algebra structure does not play a significant role in the present paper, so the definition of the Hopf-algebra structure is omitted here. I only want to stipulate that the upcoming definition of Koornwinder’s twisted primitive element is motivated by its transformation behaviour under the action of the comultiplication of $U_q$, see [11] for details.

It is convenient to work with an extended version of $U_q$, which we define as follows. Write $\mathcal{A} = \bigoplus_{x \in \mathbb{C}} \mathbb{C} \hat{x}$ for the group algebra of the additive group $(\mathbb{C}, +)$. Denote $\text{End}_{\text{alg}}(U_q)$ for the unital algebra homomorphisms $\phi : U_q \to U_q$. There exists an algebra homomorphism $\kappa : \mathcal{A} \to \text{End}_{\text{alg}}(U_q)$, with $\kappa(\hat{x}) = \kappa_x \in \text{End}_{\text{alg}}(U_q)$ for $x \in \mathbb{C}$ defined by

\begin{align*}
\kappa_x(K^{\pm 1}) &= K^{\pm 1}, \quad \kappa_x(X^\pm) = q^{\pm x}X^\pm.
\end{align*}

Note that for $m \in \mathbb{Z}$,

\begin{align*}
\kappa_m(X) &= K^mXK^{-m} \quad \forall X \in U_q,
\end{align*}

so the automorphisms $\kappa_x$ generalize the inner automorphisms $K^m(\cdot)K^{-m}$ of $U_q$ ($m \in \mathbb{Z}$).

The extended algebra $\tilde{U}_q$ is now defined as follows.

Definition 3.2. The unital, associative algebra $\tilde{U}_q$ is the vector space $\mathcal{A} \otimes U_q$ with multiplication defined by

\begin{align*}
(\hat{x} \otimes X)(\hat{y} \otimes Y) &= (\hat{x+y}) \otimes \kappa_{-y}(X)Y, \quad \forall x, y \in \mathbb{C}, \quad \forall X, Y \in U_q.
\end{align*}

The unit element is $\hat{0} \otimes 1$.

Observe that $\mathcal{A}$ and $U_q$ embed as algebras in $\tilde{U}_q$ by the formulas

\begin{align*}
a \mapsto a \otimes 1, \quad X \mapsto \hat{0} \otimes X
\end{align*}

for $a \in \mathcal{A}$ and $X \in U_q$. We will use these canonical embeddings to identify the algebras $U_q$ and $\mathcal{A}$ with their images in $\tilde{U}_q$. The commutation relations between $\mathcal{A}$ and $U_q$ within $\tilde{U}_q$ then become

\begin{align*}
\hat{x}X = \hat{x} \otimes X = \kappa_x(X)\hat{x}, \quad x \in \mathbb{C}, \ X \in U_q.
\end{align*}

The quantum Casimir element, defined by

\begin{align}
\Omega := X^+X^- + \frac{q^{-1}K^2 + qK^{-2} - 2}{(q - q^{-1})^2} \in U_q,
\end{align}

is a central element in $\tilde{U}_q$. It plays a crucial role in the study of the representations of $U_q$. The Casimir element is related to the number of independent operators in the representation, which is a fundamental property of quantum groups.
is an algebraic generator of the center $\mathcal{Z}(U_q)$ of $U_q$. Note that $\Omega$ is also in the center $\mathcal{Z}(\tilde{U}_q)$ of the extended algebra $\tilde{U}_q$.

We now consider the following explicit realization of $\tilde{U}_q$. Let $\mathcal{M}$ be the space of meromorphic functions on the complex plane $\mathbb{C}$. For any $\lambda \in \mathbb{C}$, the assignment
\[
(\pi_\lambda(X^\pm f))(z) = q^{\pm i\lambda} \left( q^{\frac{1}{2} - i\lambda} f(z \mp 1) - q^{\frac{1}{2} + i\lambda} f(z \mp 1) \right),
\]
\[
(\pi_\lambda(K^\pm 1)f)(z) = f(z \pm 1),
\]
\[
(\pi_\lambda(\hat{x})f)(z) = f(z + x), \quad x \in \mathbb{C},
\]
uniquely extends to a representation of $\tilde{U}_q$ on $\mathcal{M}$. The quantum Casimir element $\Omega$ acts as
\[
(\pi_\lambda(\Omega)) = \left( \frac{q^{i\lambda} - q^{-i\lambda}}{q - q^{-1}} \right)^2 \text{Id}.
\]

Observe furthermore that $\pi_\lambda(\hat{m}) = \pi_\lambda(K^m)$ for all $m \in \mathbb{Z}$.

In the present paper the quantum group input to the theory of $q$-special functions is based on the explicit connection between the radial part of the quantum Casimir element $\Omega$ and the second order Askey-Wilson difference operator. Here the \textit{Askey-Wilson second order difference operator} $D = D^{a,b,c,d}$, depending on four parameters $(a,b,c,d)$ called the Askey-Wilson parameters, is defined by
\[
(Df)(x) = A(x)(f(x + 2) - f(x)) + A(-x)(f(x - 2) - f(x)),
\]
with $A(x) = A(x; a, b, c, d)$ the explicit function
\[
A(x) = \frac{(1 - q^{a+x})(1 - q^{b+x})(1 - q^{c+x})(1 - q^{d+x})}{(1 - q^{2x})(1 - q^{2+2x})},
\]
cf. [1]. The radial part of $\Omega$ is computed with respect to elements $Y_\rho - \mu_\alpha(\rho) 1 \in U_q$ for $\alpha, \rho \in \mathbb{C}$, where $Y_\rho$ is Koornwinder’s \textit{twisted primitive element},
\[
Y_\rho = q^{\frac{1}{2}} X^+ K - q^{-\frac{1}{2}} X^- K + \left( \frac{q^{-\rho} + q^\rho}{q^{-1} - q} \right) (K^2 - 1)
\]
and $\mu_\alpha(\rho)$ is the constant
\[
\mu_\alpha(\rho) = \left( \frac{q^\rho(1 - q^\alpha) + q^{-\rho}(1 - q^{-\alpha})}{q - q^{-1}} \right).
\]
Consider the five dimensional space
\[
U_q^1 = \text{span}_\mathbb{C}\{X^+, X^-, K, K^{-1}, 1\} \subset U_q.
\]
The radial part computation of $\Omega$ leads to the following result.

\textbf{Proposition 3.3.} \textit{Let $\rho, \sigma, \alpha, \beta \in \mathbb{C}$. For all $x \in \mathbb{C}$,}
\[
\hat{x} \Omega K = \hat{x} \Omega(x) K \mod (Y_\rho - \mu_\alpha(\rho)) \hat{x} U_q^1 + \hat{x} U_q^1 (Y_\sigma - \mu_\beta(\sigma))
\]
with $\Omega(x) = \Omega(x; \alpha, \rho, \beta, \sigma)$ given explicitly by
\[
\Omega(x) = \frac{q^{\beta-1}}{(q-q^{-1})^2} \left\{ B(x)K^2 + (C(x) + (1-q^{1-\beta})^2)1 + D(x)K^{-2} \right\}
\]
where
\[
B(x) = B(x; \alpha, \rho, \beta, \sigma) = q^{-\beta} \frac{(1-q^{a+x})(1-q^{2-a+x})(1-q^{b+x})(1-q^{2-b+x})}{(1-q^{2x})(1-q^{2+2x})},
\]
\[
C(x) = C(x; \alpha, \rho, \beta, \sigma) = -A(x; a, b, c, d) - A(-x; a, b, c, d),
\]
\[
D(x) = D(x; \alpha, \rho, \beta, \sigma) = q^{-\beta} \frac{(1-q^{c-x})(1-q^{2-c-x})(1-q^{d-x})(1-q^{2-d-x})}{(1-q^{2-2x})(1-q^{2-2x})}.
\]
Here the Askey-Wilson parameters $(a, b, c, d)$ are related to the parameters $\alpha, \beta, \rho, \sigma$ by

(3.7) \hspace{1cm} (a, b, c, d) = (1 + \rho + \sigma, 1 - \rho + \sigma, 1 + \alpha + \rho - \beta - \sigma, 1 - \alpha - \rho - \beta - \sigma).

Proof. The proof generalizes the radial part computation by Koornwinder \[11\], where the case $\alpha = \beta = 0$ and $x \in \mathbb{Z}$ is considered (see also Noumi & Mimachi \[15\] and Koelink \[16\] for extensions to discrete values of $\alpha$ and $\beta$). In the present set-up the computation is a bit more complex, and we gather more precise information on the remainder. For the convenience of the reader, I have included the main steps of the proof as appendix. \[\square\]

Unless specified otherwise, we assume that the Askey-Wilson parameters $(a, b, c, d)$ are related to the four parameters $(\alpha, \rho, \beta, \sigma)$ by (3.7).

Proposition 3.3 allows us to identify specific eigenfunctions of $\pi_\lambda(\Omega)$ with eigenfunctions of the second order difference operator

(3.8) \hspace{1cm} (\mathcal{L}f)(x) = (\mathcal{L}^{\alpha, \rho, \beta, \sigma} f)(x) = B(x)f(x+2) + C(x)f(x) + D(x)f(x-2).

The second order difference operator $\mathcal{L}$ is gauge equivalent to the Askey-Wilson second order difference equation $\mathcal{D} = \mathcal{D}^{a, b, c, d}$, since $\mathcal{L} = \Delta \circ \mathcal{D} \circ \Delta^{-1}$ with $\Delta(x) = \Delta(x; a, b, c, d)$ any meromorphic function satisfying the difference equation

(3.9) \hspace{1cm} \Delta(x+2) = \frac{\sin(\pi(c+x)\tau)\sin(\pi(d+x)\tau)}{\sin(\pi(2-a+x)\tau)\sin(\pi(2-b+x)\tau)} \Delta(x)
\]
\[
= \frac{(1-q^{c+x})(1-q^{d+x})}{(1-q^{2-a+x})(1-q^{2-b+x})} q^2 \Delta(x).
\]

To make use of the above radial part computation, we first need to construct explicit eigenfunctions of $\pi_\lambda(Y_\rho)$ with eigenvalue $\mu_\alpha(\rho)$. As we will see in the proof of the following proposition, the operator $\pi_\lambda(Y_\rho)$ is a first order difference operator, hence eigenfunctions of $\pi_\lambda(Y_\rho)$ admit the following simple characterization.

**Proposition 3.4.** Let $\alpha, \rho \in \mathbb{C}$. A meromorphic function $f \in \mathcal{M}$ is an eigenfunction of $\pi_\lambda(Y_\rho)$ with eigenvalue $\mu_\alpha(\rho)$ if and only if

$$
\begin{align*}
f(z+2) = & \frac{\sin(\pi(-i\lambda - \alpha - \rho + z)\tau)\sin(\pi(-i\lambda + \alpha + \rho + z)\tau)}{\sin(\pi(i\lambda - \rho + 1 + z)\tau)\sin(\pi(i\lambda + \rho + 1 + z)\tau)} \frac{\sin(\pi(c+x)\tau)\sin(\pi(d+x)\tau)}{\sin(\pi(2-a+x)\tau)\sin(\pi(2-b+x)\tau)} \Delta(x), \\
& \text{for } z \in \mathbb{Z}, \lambda \notin \mathbb{Z},
\end{align*}
$$

where $\Delta(x) = \Delta(x; a, b, c, d)$. \[\square\]
Proof. A direct computation shows that $\pi_\lambda(Y_\rho) \in \text{End}(\mathcal{M})$ is the explicit first order difference operator

$$\left(\pi_\lambda(Y_\rho)f\right)(z) = \frac{q^{-\rho}}{q - q^{-1}} \left\{ (q^{\rho-1-i\lambda-z} - 1)(1 - q^{\rho+1+i\lambda+z})f(z + 2) - (q^{\rho+i\lambda-z} - 1)(1 - q^{\rho-i\lambda+z})f(z) \right\},$$

and, more generally,

$$\left(\left(\pi_\lambda(Y_\rho) - \mu_\alpha(\rho)\right)f\right)(z) = \frac{q^{-\rho}}{q - q^{-1}} \left\{ (q^{\rho-1-i\lambda-z} - 1)(1 - q^{\rho+1+i\lambda+z})f(z + 2) - (q^{\alpha+\rho+i\lambda-z} - 1)(1 - q^{\alpha+\rho-i\lambda+z})q^{-\alpha}f(z) \right\}.$$

The eigenvalue equation $\pi_\lambda(Y_\rho)f = \mu_\alpha(\rho)f$ is thus equivalent to the first order difference equation

$$f(z + 2) = \frac{(1 - q^{\alpha+\rho+i\lambda-z})(1 - q^{\alpha+\rho-i\lambda+z})}{(1 - q^{\rho-1-i\lambda-z})(1 - q^{\rho+1+i\lambda+z})}q^{-\alpha}f(z).$$

Rewriting this formula yields the desired result. \hfill \Box

4. The Askey-Wilson function for $0 < q < 1$.

In this section we take $\tau \in i\mathbb{R}_{>0}$, so that $0 < q = q_\tau = \exp(2\pi i \tau) < 1$. The assignment

$$(4.1) \quad (K^{\pm 1})^* = K^{\pm 1}, \quad (X^\pm)^* = -X^\mp$$

uniquely extends to a unital, anti-linear, anti-algebra involution on $\mathcal{U}_q$. This particular choice of *-structure corresponds classically to choosing the real form $\mathfrak{su}(1, 1)$ of $\mathfrak{sl}(2, \mathbb{C})$.

The elements $K^m$ ($m \in \mathbb{Z}$), the quantum Casimir element $\Omega$ (see (3.1)), and the special family $Y_\rho$ ($\rho \in \mathbb{R}$) of Koornwinder’s twisted primitive elements (3.4) are *-selfadjoint elements in $\mathcal{U}_q$. The eigenvalue $\mu_\alpha(\rho)$ (see (3.4)) is real for $\alpha, \rho \in \mathbb{R}$. We consider now an explicit *-unitary pairing for the representation $\pi_\lambda$.

Lemma 4.1. Let $\lambda \in \mathbb{R}$. Suppose that $f, g \in \mathcal{M}$ are $\tau^{-1}$-periodic and analytic on the strip \( \{z \in \mathbb{C} \mid |\text{Re}(z)| \leq 1\} \). Then

$$\langle \pi_\lambda(X)f, g \rangle = \langle f, \pi_\lambda(X^*)g \rangle, \quad \forall X \in \mathcal{U}_q^1,$$

with the pairing $\langle \cdot, \cdot \rangle$ defined by

$$\langle f, g \rangle := \int_0^1 f(y/\tau)\overline{g(y/\tau)}dy.$$

Proof. This is an easy verification for the basis elements $1, K^{\pm 1}, X^\pm$ of $\mathcal{U}_q^1$, using Cauchy’s Theorem to shift contours. \hfill \Box

Remark 4.2. This lemma can be applied recursively. Let $f, g \in \mathcal{M}$ be $\tau^{-1}$-periodic and analytic on the strip \( \{z \in \mathbb{C} \mid |\text{Re}(z)| \leq k\} \) with $k \in \mathbb{Z}_{>0}$. For any $X = X_1X_2\cdots X_m \in \mathcal{U}_q$ with $m \leq k$ and $X_i \in \mathcal{U}_q^1$,

$$\langle \pi_\lambda(X)f, g \rangle = \langle f, \pi_\lambda(X^*)g \rangle.$$
In particular, the subspace $O_{\tau^{-1}}$ of entire, $\tau^{-1}$-periodic functions is an $*$-unitary $\pi_{\lambda}$-invariant subspace of $M$ with respect to the pairing $\langle \cdot, \cdot \rangle$. This subspace is the algebraic version of the principal series representation of the $*$-algebra $(U_q, *)$.

To combine this lemma with the radial part computation of the quantum Casimir element (see Proposition 3.3.1), we need to construct meromorphic $\tau^{-1}$-periodic eigenfunctions of $\pi_{\lambda}(Y_\rho)$ ($\rho \in \mathbb{R}$) which are analytic in a large enough strip around the imaginary axis. We claim that the meromorphic function

$$ f_\lambda(z) = f_\lambda(z;\alpha,\rho) := \frac{\Gamma_{2\tau}(-1 - \frac{1}{2\tau} + \alpha + \rho - i\lambda + z)\Gamma_{2\tau}(-\frac{1}{2\tau} + \rho - i\lambda - z)}{\Gamma_{2\tau}(1 - \frac{1}{2\tau} + \alpha + \rho + i\lambda - z)\Gamma_{2\tau}(\frac{1}{2\tau} + \rho + i\lambda + z)} $$

meets these criteria for special values of the parameters. By Proposition 3.4 and the $\mu$ version of the principal series representation of the $C$ for some nonzero constant $\alpha,\rho,\beta,\sigma$ $(\rho \neq 0)$. Furthermore, observe that the poles of $f_\lambda(z;\alpha,\rho)$ are located at

$$ i\lambda - \alpha - \rho + 2\mathbb{Z}_{\leq 0} + \mathbb{Z}\tau^{-1}, \quad -i\lambda + \rho + 1 + 2\mathbb{Z}_{\geq 0} + \mathbb{Z}\tau^{-1}, $$

so $f_\lambda(z)$ is analytic on the strip $\{z \in \mathbb{C} \mid \text{Re}(z) \leq \rho\}$ when $\alpha > 0$, $\rho \geq 0$ and $\lambda \in \mathbb{R}$.

**Definition 4.3.** Let $\lambda \in \mathbb{R}$, $\alpha,\beta \in 2\mathbb{Z}_{\geq 0}$ and $\rho,\sigma \in \mathbb{R}_{\geq 3}$. For $x \in \mathbb{C}$ with $|\text{Re}(x)| \leq 2$ we define $\phi_\lambda(x) = \phi_\lambda(x;\alpha,\rho,\beta,\sigma)$ by

$$ \phi_\lambda(x) := \langle \pi_{\lambda}(\hat{K})f_\lambda(\cdot;\beta,\sigma), f_\lambda(\cdot;\alpha,\rho) \rangle. $$

Note that the matrix coefficient $\phi_\lambda(x)$ is given explicitly by

$$ \phi_\lambda(x) = \int_{-1}^{1} \frac{\Gamma_{2\tau}(-\frac{1}{2\tau} + \beta + \sigma - i\lambda + x + \frac{\theta}{\tau})\Gamma_{2\tau}(-1 - \frac{1}{2\tau} + \sigma - i\lambda - x - \frac{\theta}{\tau})}{\Gamma_{2\tau}(-\frac{1}{2\tau} + \beta + \sigma + i\lambda - x - \frac{\theta}{\tau})\Gamma_{2\tau}(1 - \frac{1}{2\tau} + \sigma + i\lambda + x + \frac{\theta}{\tau})} \times \frac{\Gamma_{2\tau}(-1 + \frac{1}{2\tau} + \alpha + \rho + i\lambda - \frac{\theta}{\tau})\Gamma_{2\tau}(\frac{1}{2\tau} + \rho + i\lambda + \frac{\theta}{\tau})}{\Gamma_{2\tau}(1 + \frac{1}{2\tau} + \alpha + \rho - i\lambda + \frac{\theta}{\tau})\Gamma_{2\tau}(\frac{1}{2\tau} + \rho - i\lambda - \frac{\theta}{\tau})} \, dy, $$

and that $\phi_\lambda(x)$ is analytic on the strip $\{x \in \mathbb{C} \mid |\text{Re}(x)| \leq 2\}$. The quantum group interpretation of this explicit integral leads to the following result.

**Theorem 4.4.** Let $\lambda \in \mathbb{R}$, $\alpha,\beta \in 2\mathbb{Z}_{\geq 0}$ and $\rho,\sigma \in \mathbb{R}_{\geq 3}$. The matrix coefficient $\phi_\lambda(x) = \phi_\lambda(x;\alpha,\rho,\beta,\sigma)$ satisfies the second order difference equation

$$ (L\phi_\lambda)(x) = E(\lambda)\phi_\lambda(x) $$

for generic $x \in i\mathbb{R}$, with the eigenvalue $E(\lambda) = E(\lambda;\beta)$ given by

$$ E(\lambda) = -1 - q^{2-2\beta} + q^{1-\beta}(q^{2\lambda} + q^{-2\lambda}). $$
Proof. For the duration of the proof we use the shorthand notations \( f(z) = f_{\lambda}(z; \beta, \sigma) \) and \( g(z) = f_{\lambda}(z; \alpha, \rho) \). By the conditions on the parameters, \( Y_{\rho} \) is \(*\)-selfadjoint, \( \mu_{\alpha}(\rho) \) is real and the meromorphic functions \( f(z) \) and \( g(z) \) are analytic on the strip \( \{ z \in \mathbb{C} : |\text{Re}(z)| \leq 3 \} \). By Lemma 4.1 we thus obtain for any \( X \in \mathcal{U}_{q}^{\lambda} \) and \( x \in i\mathbb{R} \),

\[
(\pi_{\lambda}(Y_{\rho} - \mu_{\alpha}(\rho))\hat{\pi}X)f, g) = (\pi_{\lambda}\hat{\pi}Xf, \pi_{\lambda}(Y_{\rho} - \mu_{\alpha}(\rho))g) = 0,
\]

and obviously also \((\pi_{\lambda}(\hat{\pi}X(Y_{\sigma} - \mu_{\beta}(\sigma)))f, g) = 0\). We conclude from Proposition 3.3 that

\[
(\pi_{\lambda}(\hat{\pi}\Omega K)f, g) = (\pi_{\lambda}(\hat{\pi}\Omega(x)KF)f, g)
\]

\[
= \frac{q^{\beta-1}}{(q - q^{-1})^{2}}(L\phi_{\lambda})(x) + \frac{q^{\beta-1}(1 - q^{1 - \beta})^{2}}{(q - q^{-1})^{2}}\phi_{\lambda}(x).
\]

On the other hand, (3.2) implies that

\[
(\pi_{\lambda}(\hat{\pi}\Omega K)f, g) = \left(\frac{q^{\lambda} - q^{-\lambda}}{q - q^{-1}}\right)^{2}\phi_{\lambda}(x),
\]

hence \((L\phi_{\lambda})(x) = E(\lambda)\phi_{\lambda}(x)\).

For the comparison with the results for \(|q| = 1\), see Section 5, it is convenient to note that the matrix coefficient \(\phi_{\lambda}(x)\) can be rewritten as

\[
\phi_{\lambda}(x) = C \int_{0}^{1} \frac{\Gamma_{2\tau}(\frac{1}{2\tau^2} + \beta + \sigma - i\lambda + x + \frac{\beta}{\tau})\Gamma_{2\tau}(\frac{1}{2\tau} + \sigma - i\lambda + x + \frac{\beta}{\tau})}{\Gamma_{2\tau}(\frac{1}{2\tau} + \beta + \sigma + i\lambda - x - \frac{\beta}{\tau})\Gamma_{2\tau}(\frac{1}{2\tau} + \sigma + i\lambda + x + \frac{\beta}{\tau})} \times \frac{\Gamma_{2\tau}(-1 + \frac{1}{2\tau} + \alpha + \rho + i\lambda - \frac{\beta}{\tau})\Gamma_{2\tau}(-1 + \frac{1}{2\tau} + \rho + i\lambda + \frac{\beta}{\tau})}{\Gamma_{2\tau}(1 - \frac{1}{2\tau} + \alpha + \rho - i\lambda - \frac{\beta}{\tau})\Gamma_{2\tau}(1 - \frac{1}{2\tau} + \rho - i\lambda + \frac{\beta}{\tau})} dy
\]

for some \(x\)-independent nonzero constant \(C\). Formula (4.7) follows from (4.5) and the difference equation

\[
\Gamma_{2\tau}(x + \frac{1}{2\tau}) = \exp(-\frac{\pi i x}{2}) \Gamma_{2\tau}(x - \frac{1}{2\tau}).
\]

Corollary 4.5. Let \( \lambda \in \mathbb{R}, \alpha, \beta \in 2\mathbb{Z}_{>0} \) and \( \rho, \sigma \in \mathbb{R}_{\geq 3} \). Let the Askey-Wilson parameters \((a, b, c, d)\) be given by (3.7). The function \( F_{\lambda}(x) = \Lambda(x; a, b, c, d) \) defined by \( F_{\lambda}(x) = \Delta(x)^{-1}\phi_{\lambda}(x) \), with \( \Delta(x) = \Delta(x; a, b, c, d) \) the \( \tau^{-1} \)-periodic, meromorphic function

\[
\Delta(x) = \frac{\Gamma_{2\tau}(\frac{1}{2\tau} + \frac{1}{2\tau} + a - x)\Gamma_{2\tau}(\frac{1}{2\tau} + \frac{1}{2\tau} + b - x)}{\Gamma_{2\tau}(\frac{1}{2\tau} + c - x)\Gamma_{2\tau}(\frac{1}{2\tau} - d - x)},
\]

satisfies the Askey-Wilson second order difference equation

\[
(DF_{\lambda})(x) = E(\lambda)F_{\lambda}(x)
\]

for generic \( x \in i\mathbb{R} \) and generic \( \rho \) and \( \sigma \).

Proof. Note that \( \Delta(x) \) can be rewritten as

\[
\Delta(x) = \frac{(q^{2-c-x}, q^{2-d-x}, q^{2})_{\infty}}{(q^{a-x}, q^{b-x}, q^{2})_{\infty}} q^{-\frac{a+c+x}{2}}
\]
for some nonzero \((x\text{-independent})\) constant \(C\). Since \(\beta \in 2\mathbb{Z}_{>0}\), the gauge factor \(\Delta(x)\) is \(\tau^{-1}\)-invariant. Furthermore, \(\Delta(x)^{-1}\) is regular at \(x \in \pm 2 + i\mathbb{R}\) and \(x \in i\mathbb{R}\) under generic conditions on the parameters \(\rho\) and \(\sigma\). The proof is completed by observing that \(\Delta(x)\) satisfies the difference equation \(\text{(3.9)}\).

We end this section by extending these results to continuous parameters \(\alpha\) and \(\beta\). Changing integration variable and substituting the expression for \(\Gamma_{2\tau}\) in terms of \(q\)-shifted factorials, we can rewrite the matrix coefficient \(\phi(x)\) (see \(\text{(4.5)}\)) as

\[
\phi(x) = C \frac{q^{\frac{\beta x}{2\pi i}}}{2\pi i} \int_{\mathbb{T}} \frac{(q^{1+\beta+\sigma+i\lambda-x}/z, q^{2+\sigma+i\lambda+x}; q^2)_{\infty}}{(q^{1+\rho+i\lambda}/z, q^{2+\sigma+i\lambda+x}; q^2)_{\infty}} z^{\frac{\alpha-x}{2}} dz
\]

for some \(x\text{-independent nonzero constant}\) \(C\), where \(\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}\) is the positively oriented unit circle in the complex plane. To allow \(\alpha, \beta\) to be continuous parameters, we need to get rid of the term \(z^{\frac{\alpha-x}{2}}\) in the integrand. This can be achieved by rewriting \(\phi(x)\) as

\[
\phi(x) = C q^{\frac{\beta x}{2\pi i}} \int_{\mathbb{T}} \frac{(q^{1+\beta+\sigma+i\lambda-x}/z, q^{1+\alpha+\rho-\beta-i\lambda}/z; q^2)_{\infty}}{(q^{1-\rho+i\lambda}/z, q^{1+\rho+i\lambda}; q^2)_{\infty}} \times \frac{(q^{1-\sigma+i\lambda}/z, q^{2+\sigma+i\lambda+x}; q^2)_{\infty}}{(q^{1+\beta+\sigma-i\lambda+x}/z, q^{2+\alpha+\rho-\beta-i\lambda}; q^2)_{\infty}} dz
\]

with \(C\) again some (different) irrelevant \(x\text{-independent nonzero constant}\). The integral formula \(\text{(4.11)}\) follows from \(\text{(4.10)}\) by substitution of the identity

\[
z^{\frac{\alpha-x}{2}}(q^{1+\rho-\beta}/z; q^2)_{\infty} = (-q^{\lambda})^{\frac{\beta-\alpha}{2}} q^{\frac{(\beta-\alpha)^2}{4}} \frac{(q^{1-\sigma+i\lambda}/z, q^{2+\alpha+\rho-\beta-i\lambda}/z; q^2)_{\infty}}{(q^{1-\rho+i\lambda}/z; q^2)_{\infty}}
\]

which in turn is a direct consequence of the functional equation

\[
\theta(q^{2k}z) = (-z)^{-k}q^{-k^2}\theta(z), \quad k \in \mathbb{Z}
\]

for the modified Jacobi theta function \(\theta(z) = (qz, q/z; q^2)_{\infty}\). By \(\text{(4.9)}\), the eigenfunction \(F_\lambda(x) = F_\lambda(x; a, b, c, d)\) of the Askey-Wilson second order difference equation \(D\) (see Corollary \(\text{4.5}\)) is equal to

\[
F_\lambda(x) = F_\lambda(x; a, b, c, d) := \frac{(q^{1+\rho+\sigma-x}/z, q^{1-\rho+\sigma-x}; q^2)_{\infty}}{(q^{1-\alpha-\beta+i\lambda-x}/z, q^{1+\alpha+\rho + \beta + \sigma-x}; q^2)_{\infty}} \times \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{(q^{1+\beta+\sigma+i\lambda-x}/z, q^{1+\alpha+\rho-\beta-i\lambda}/z; q^2)_{\infty}}{(q^{1-\rho+i\lambda}/z, q^{1+\rho+i\lambda}; q^2)_{\infty}} \times \frac{(q^{1-\sigma+i\lambda}/z, q^{2+\sigma+i\lambda+x}; q^2)_{\infty}}{(q^{1+\beta+\sigma-i\lambda+x}/z, q^{2+\alpha+\rho-i\lambda}; q^2)_{\infty}} dz
\]

up to some nonzero \(x\text{-independent multiplicative constant}\).
Define for the Askey-Wilson parameters \((a, b, c, d)\) given by (3.7), dual Askey-Wilson parameters \((\bar{a}, \bar{b}, \bar{c}, \bar{d})\) by
\[
(\bar{a}, \bar{b}, \bar{c}, \bar{d}) = (1 - \beta, 1 + \beta + 2\sigma, 1 + \alpha + 2\rho, 1 - \alpha).
\]
This notion of dual Askey-Wilson parameters coincides with the notion of dual parameters as used in e.g. [13].

In the following theorem we express \(F_\lambda(x)\) in terms of basic hypergeometric series by shrinking the radius of the integration circle \(\mathbb{T}\) to zero while picking up residues. Recall that the very-well-poised \(s\phi_7\) series is defined by
\[
sW_7(u; b_1, b_2, b_3, b_4, b_5; q, z) = \sum_{k=0}^{\infty} \frac{(1 - uq^{2k})(u, b_1, b_2, b_3, b_4, b_5; q)kz^k}{(q, qu/b_1, qu/b_2, qu/b_3, qu/b_4, qu/b_5; q)_k^n}
\]
for \(|z| < 1\), see [14].

**Theorem 4.6.** Let \(\lambda \in \mathbb{R}, \alpha, \beta \in \mathbb{R}_{>0}\) and \(\rho, \sigma \in \mathbb{R}_{\geq 3}\). Under the parameter correspondence (3.7), the function \(F_\lambda(x) = F_\lambda(x; a, b, c, d)\) given by (4.12) can be expressed in terms of basic hypergeometric series as
\[
F_\lambda(x) = C \frac{(q^{2-a-d+2\lambda+x}, q^{2-a-d+2\lambda-x}; q^2)_\infty}{(q^{2-d+x}, q^{2-d-x}; q^2)_\infty} \times sW_7(q^{-2+a+b+c+2\lambda}; q^{a+x}, q^{a-x}, q^{a+2i\lambda}, q^{b+2i\lambda}, q^{c+2i\lambda}, q^{2}, q^{2-d-2i\lambda})
\]
with the (irrelevant) generically nonzero, \(x\)-independent constant \(C\) given by
\[
C = \frac{(q^{2+2\sigma}, q^{2+\alpha+2\rho-\beta}, q^{2+\alpha+2\rho+\beta+2\sigma}, q^{1+\beta+2\lambda}, q^{1+\alpha-2\lambda}, q^{2})_\infty}{(q^2, q^{1+\alpha+2i\lambda}, q^{1+\beta+2\sigma-2i\lambda}, q^{3+\alpha+2\rho+2\sigma+2i\lambda}, q^{2})_\infty}.
\]
Furthermore, \(F_\lambda(x)\) is \(\tau^{-1}\)-periodic and satisfies the Askey-Wilson difference equation \((DF_\lambda)(x) = E(\lambda)F_\lambda(x)\) for generic \(x \in i\mathbb{R}\).

**Proof.** The expression for \(F_\lambda\) follows by shrinking the radius of the integration contour \(\mathbb{T}\) to zero while picking up residues. It is actually a special case of [14] Exerc. 4.4, p.122], in which one should replace the base \(q\) by \(q^2\) and the parameters \((a, b, c, d, f, g, h, k)\) by
\[
(q^{1-\beta-\sigma-i\lambda+x}, q^{1-\rho+i\lambda}, q^{1+\rho+i\lambda}, q^{\sigma-i\lambda-x}, q^{2+\sigma+i\lambda+x}, q^{1+\beta-i\lambda+x}, q^{1+\beta-\sigma-i\lambda+x}, q^{1-\beta-2i\lambda}).
\]
We have seen that the difference equation \((DF_\lambda)(x) = E(\lambda)F_\lambda(x)\) for generic \(x \in i\mathbb{R}\) is valid under the extra assumption \(\alpha, \beta \in 2\mathbb{Z}_{>0}\). For \(\alpha, \beta \in \mathbb{R}_{>0}\) this difference equation has been proved by Ismail and Rahman [5] using the explicit expression of \(F_\lambda(x)\) as very-well-poised \(s\phi_7\) series.

Using the explicit expressions of \(F_\lambda(x)\), the conditions on \(x, \lambda\) and the four parameters \(\alpha, \rho, \beta, \sigma\) can be relaxed by meromorphic continuation. The resulting function \(F_\lambda\) is a meromorphic, \(\tau^{-1}\)-periodic eigenfunction of the Askey-Wilson second order difference operator \(D = D^{a, b, c, d}\) with eigenvalue \(E(\lambda) = E(\lambda; \beta)\).
Remark 4.7. Some special cases of the matrix coefficients $\phi_\lambda$ were explicitly expressed in terms of very-well-poised $8\phi_7$ series in [8] using the realization of the representation $\pi_\lambda$ on the representation space $l^2(\mathbb{Z})$. In this approach the basic hypergeometric series manipulations are much harder, since one needs a highly nontrivial evaluation of a non-symmetric Poisson type kernel involving nonterminating $2\phi_1$-series which is due to Rahman, see the appendix of [8] and [7].

Remark 4.8. The explicit $8\phi_7$ expression (4.13) of $F_\lambda(x)$ and its analytic continuation was named the Askey-Wilson function in [9]. Suslov [23], [24] established Fourier-Bessel type orthogonality relations for the Askey-Wilson function. Koelink and the author [8], [9] defined a generalized Fourier transform involving the Askey-Wilson function as the integral kernel, and established its Plancherel and inversion formula. This transform, called the Askey-Wilson function transform, arises as Fourier transform on the noncompact quantum group $SU_q(1,1)$ (see [8]), and may thus be seen as a natural analogue of the Jacobi function transform.

5. The expansion formula and the elliptic cosine kernel

In this section we still assume that $\tau \in i\mathbb{R}_{>0}$, so $0 < q = q_\tau = \exp(2\pi i \tau) < 1$. To keep contact with the conventions of the previous section, we keep working in base $q^2$.

First we recall the (normalized) Askey-Wilson polynomials [1]. The Askey-Wilson polynomials are defined by

$$E_m(x) = E_m(x; a, b, c, d) := 4\phi_3 \left( q^{-2m}, q^{2m-2+a+b+c+d}, q^{a+x}, q^{a-x}; q^2, q^2 \right), \quad m \in \mathbb{Z}_{\geq 0},$$

with

$$4\phi_3 \left( \frac{a_1, a_2, a_3, a_4}{b_1, b_2, b_3}; q, z \right) = \sum_{k=0}^\infty \frac{(a_1, a_2, a_3, a_4; q)_k}{(q, b_1, b_2, b_3; q)_k} z^k, \quad |z| < 1.$$

The Askey-Wilson polynomial $E_m(x)$ is a polynomial in $q^x + q^{-x}$ of degree $m$, normalized by $E_m(a) = 1$. They satisfy the orthogonality relations

$$\int_0^1 E_m(x/\tau) E_n(x/\tau) \frac{(q^{2x}, q^{-2x}; q_\tau^2)_\infty}{(q^{a+x}, q^{a-x}, q^{b+x}, q^{b-x}, q^{c+x}, q^{c-x}, q^{d+x}, q^{d-x}, q^2)_\infty} dx = 0, \quad m \neq n,$$

provided that $\text{Re}(a), \text{Re}(b), \text{Re}(c), \text{Re}(d) > 0$. Observe that the Askey-Wilson polynomial $E_m(x)$ is regular at the special choice

$$(a, b, c, d) = (0, 1, 1, 1 - \frac{1}{2\tau})$$

of Askey-Wilson parameters. The above orthogonality relations also extend (by continuity) to the Askey-Wilson parameters (5.1), leading to

$$\int_0^1 E_m(x/\tau; 0, \frac{1}{2\tau}, 1, 1 - \frac{1}{2\tau}) E_n(x/\tau; 0, \frac{1}{2\tau}, 1, 1 - \frac{1}{2\tau}) \frac{1}{2\tau} dx = 0, \quad m \neq n,$$
hence we conclude that
\begin{equation}
E_m(x; 0, \frac{1}{2\tau}, 1, 1 - \frac{1}{2\tau}) = \cos(2\pi m\tau x), \quad m \in \mathbb{Z}_{\geq 0},
\end{equation}
which is the usual cosine kernel from the Fourier theory on the unit circle. In this section we derive an analogous result for the Askey-Wilson function.

In the previous section we have introduced the dual Askey-Wilson parameters associated to \(a, b, c\) and \(d\). They can be alternatively expressed as
\[(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = (\frac{1}{2}(a+b+c+d), \frac{1}{2}(a+b-c-d), \frac{1}{2}(a-b+c+d)).\]

Note that the special choice (5.1) of Askey-Wilson parameters is self-dual, \((a, b, c, d) = (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})\). For our present purposes it is convenient to use yet another normalization of the Askey-Wilson function, namely
\[
\mathcal{E}^+(\mu, x) = \mathcal{E}^+(\mu; a, b, c, d)
\]
\[
:= \left( \frac{q^{2a-a-d+\mu+x}; q^{2a-a-d}; q^{2a-a-d}; q^2}{(q^{2a+b+c-d}; q^{2a-a-d}; q^{2a-a-d}; q^2)} \right)_{\infty}
\times \frac{W_7(q^{2a-a-d}; q^{2a-a-d}; q^{2a-a-d}; q^{2a-a-d}; q^{2a-a-d}; q^{2a-a-d}; q^{2a-a-d})}{\prod_{m=0}^{\infty} E_m(x; a, b, c, 2 - d) E_m(\mu; \tilde{a}, \tilde{b}, \tilde{c}, 2 - \tilde{d})}
\]
\[
\times \frac{(1 - q^{2a-a-d}; q^{2a-a-d}; q^{2a-a-d}; q^2)_{\infty}}{(1 - q^{a+b+c-d}; q^{a+b}; q^{a+b}; q^2)} (1 - q^{2a-a-d}; q^{2a-a-d}; q^{2a-a-d}; q^2)_{m} (-1)^m q^{(1-a-d)m} q^{m^2},
\]

For fixed \(\lambda\), the eigenfunction \(F_{\lambda}(\cdot)\) of the Askey-Wilson second order difference operator \(\mathcal{D}\) is a constant multiple of \(\mathcal{E}^+(2i\lambda, \cdot)\). The present normalization of the Askey-Wilson function is convenient due to the properties
\begin{equation}
\mathcal{E}^+(\mu, x; a, b, c, d) = \mathcal{E}^+(x, \mu; \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})
\end{equation}
and \(\mathcal{E}^+(-\tilde{a}, -a) = 1\). The property (5.3) is called \textit{duality} and can be proved using a transformation formula for very-well-poised \(\phi_7\) series, see [9] for details. Furthermore,
\begin{equation}
\mathcal{E}^+(\tilde{a} + 2m, x) = E_m(x), \quad m \in \mathbb{Z}_{\geq 0},
\end{equation}
see e.g. [9 (3.5)], thus the Askey-Wilson function \(\mathcal{E}^+(\mu, x)\) provides a natural meromorphic continuation of the Askey-Wilson polynomial in its degree.

The meromorphic continuation of \(\mathcal{E}^+(\mu, x)\) in \(\mu\) and \(x\) can be established by the integral representation of the Askey-Wilson function (see the previous section), or by the expression of the Askey-Wilson function as a sum of two balanced \(\phi_3\)'s (see e.g. [9 (3.3)]). For our present purposes, it is most convenient to consider the meromorphic continuation via the expansion formula of the Askey-Wilson function in Askey-Wilson polynomials, given by
\begin{equation}
\mathcal{E}^+(\mu, x) = \left( \frac{q^{2a-a-d}; q^{2a-a-d}; q^{2a-a-d}; q^2}{(q^{2a+b+c-d}; q^{2a-a-d}; q^{2a-a-d}; q^2)} \right)_{\infty}
\times \sum_{m=0}^{\infty} E_m(x; a, b, c, 2 - d) E_m(\mu; \tilde{a}, \tilde{b}, \tilde{c}, 2 - \tilde{d})
\times \frac{(1 - q^{2m+a+b+c-d})(q^{a+b+c-d}; q^{a+b}; q^{a+b}; q^2)_{m}}{(1 - q^{a+b+c-d})(q^{2a+b-d}; q^{2a+c-d}; q^2)_{m}} (-1)^m q^{(1-a-d)m} q^{m^2},
\end{equation}
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see [21 Thm. 4.2]. The sum converges absolutely and uniformly on compacta of \((\mu, x) \in \mathbb{C} \times \mathbb{C}\) due to the Gaussian \(q^{m^2}\). The expansion formula (5.5) shows that the Askey-Wilson function \(E^+(\mu, x)\) is well defined and regular at the special choice (5.1) of Askey-Wilson parameters. In fact, for this special choice of parameters, the Askey-Wilson function can be expressed in terms of the (renormalized) Jacobi theta function

\[ \vartheta(x) = (-q^{1+x}, -q^{1-x}; q^2)_{\infty} \]

as follows.

**Proposition 5.1.** We have the identity

\[ E^+_0(\mu, x) = (-q, -q, -q, -q^2, q^2)_{\infty}^{2} \left( \frac{\vartheta(\mu + x) + \vartheta(\mu - x)}{2 \vartheta(\mu) \vartheta(x)} \right) \cdot \sum_{m=1}^{\infty} \cos(2\pi m \tau \mu) \cos(2\pi m \tau x) \frac{q^{m^2}}{2} \cdot \vartheta(\mu + x) + \vartheta(\mu - x) \].

**Proof.** To simplify notations, we write

\[ E^+_0(\mu, x) = E^+(\mu, x; 0, \frac{1}{2\tau}, 1, 1 - \frac{1}{2\tau}) \]

for the duration of the proof.

We substitute the special choice (5.1) of Askey-Wilson parameters in the expansion formula for \(E^+(\mu, x)\). By simple \(q\)-series manipulations and by (5.2), we obtain the explicit formula

\[ E^+_0(\mu, x) = \frac{(-q, -q, -q, -q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \left( 1 + 2 \sum_{m=1}^{\infty} \cos(2\pi m \tau \mu) \cos(2\pi m \tau x) q^{m^2} \right) \cdot \vartheta(\mu + x) + \vartheta(\mu - x) \cdot \vartheta(\mu) \vartheta(x) \].

Using the well known Jacobi triple product identity

\[ \sum_{m=-\infty}^{\infty} q^{m^2} = 1 + 2 \sum_{m=1}^{\infty} \cos(2\pi m \tau x) q^{m^2} = (q^2; q^2)_{\infty} \vartheta(x) \]

and the elementary identity

\[ \cos(2\pi m \tau \mu) \cos(2\pi m \tau x) = \frac{1}{2} \left( \cos(2\pi m \tau (\mu + x)) + \cos(2\pi m \tau (\mu - x)) \right) \]

we deduce that

\[ E^+_0(\mu, x) = \frac{(-q, -q, -q, -q^2; q^2)_{\infty}}{2} \left( \frac{\vartheta(\mu + x) + \vartheta(\mu - x)}{\vartheta(\mu) \vartheta(x)} \right) \cdot \sum_{m=-\infty}^{\infty} q^{-(x+\kappa \tau^{-1})^2} \cdot \sqrt{-2i \tau} (q^2; q^2)_{\infty} \vartheta(\mu + x) + \vartheta(\mu - x) \cdot \vartheta(\mu) \vartheta(x) \].

Simplifying the multiplicative constant yields the desired result. \(\square\)

**Remark 5.2.** The Jacobi theta function \(\vartheta(x)\) is the natural \(\tau^{-1}\)-periodic analogue of the Gaussian \(q^{-x^2}\), since

\[ \sum_{k=-\infty}^{\infty} q^{-(x+\kappa \tau^{-1})^2} = \sqrt{-2i \tau} (q^2; q^2)_{\infty} \vartheta(x) \]

by the Jacobi triple product identity and the Jacobi inversion formula. In fact, in [21] and [22] it is shown that the function \((q^{2-d+x}, q^{2-d-x}, q^2)_{\infty}\), which reduces to \(\vartheta(x)\) for the Askey-Wilson parameters (5.1), plays the role of the Gaussian in the Askey-Wilson theory.
If one replaces the theta functions by Gaussians in the right hand side of (5.6), then we obtain up to a multiplicative constant
\[
\frac{1}{2} \left( \frac{q^{-(\mu+x)^2} + q^{-(\mu-x)^2}}{q^{-\mu^2-2x^2}} \right) = \frac{1}{2} \left( q^{-2\mu x} + q^{2\mu x} \right) = \cos(4\pi \tau \mu x),
\]
which is essentially the classical cosine kernel. Thus the right hand side of (5.6) is an elliptic analogue of the cosine kernel.

**Remark 5.3.** Using the quasi-periodicity
\[
\vartheta(x + 2) = q^{-1-x} \vartheta(x)
\]
of the Jacobi theta function, we obtain as a consequence of (5.6),
\[
\mathcal{E}^+(2m, x; 0, 1, 1 - \frac{1}{2\tau}) = \cos(2\pi m \tau x), \quad m \in \mathbb{Z}_{\geq 0},
\]
which is in accordance with (5.2) and (5.4).

**Remark 5.4.** The orthogonality relations for the Askey-Wilson polynomials with Askey-Wilson parameters (5.1) are equivalent to the $L^2$-theory of the classical Fourier transform on the unit circle. On the other hand, the $L^2$-theory of the Askey-Wilson function transform, see [9], does not reduce to the $L^2$-theory of the classical Fourier theory on the real line for the Askey-Wilson parameters (5.1). Instead one obtains a Fourier type transform with integral kernel given by the elliptic cosine function (5.6). For the corresponding $L^2$ theory, the transform is defined on a weighted $L^2$-space consisting of functions that are supported on a finite closed interval and an infinite, unbounded sequence of discrete mass points. This transform, as well as the general Askey-Wilson function transform, still has many properties in common with the classical Fourier transform on the real line, see e.g. [9], [21] and [22].

Cherednik’s [2] Hecke algebra approach to $q$-special functions leads to a direct proof that the right hand side of the expansion formula (5.5) is an eigenfunction of the Askey-Wilson second order difference operator $\mathcal{D}$, see [22]. The expansion formula (5.5) may thus be seen as the explicit link between Cherednik’s approach and Ismail’s and Rahman’s [5] construction of eigenfunctions of $\mathcal{D}$ in terms of very-well-poised $8\varphi_7$ series. We end this section by sketching a proof of Proposition 5.1 using Cherednik’s Hecke algebra approach.

The affine Hecke algebra techniques for Askey-Wilson polynomials are developed in full detail in [16], and for Askey-Wilson functions in [22]. We first recall one of the main results from [22], specialized to the present rank one situation.

Define two difference-reflection operators by
\[
(T_0^{c,d} f)(x) = -q^{-2+c+d} f(x) + \frac{(1-q^{c-x})(1-q^{d-x})}{(1-q^{2-2x})} (f(2-x) - f(x)),
\]
\[
(T_1^{a,b} f)(x) = -q^{a+b} f(x) + \frac{(1-q^{a+x})(1-q^{b+x})}{(1-q^{2x})} (f(-x) - f(x)).
\]

(5.9)
The connection with affine Hecke algebras follows from the fact that $T_0 = T_0^{a,b}$ and $T_1 = T_1^{a,b}$ satisfy Hecke type quadratic relations. These relations imply that the operators $T_0$ and $T_1$ are invertible. Consider the (invertible) operator

$$Y = Y^{a,b,c,d} := T_1^{a,b} \circ T_0^{c,d}.$$  

**Remark 5.5.** The operator $Y + Y^{-1}$, acting on even functions, is essentially the Askey-Wilson second order difference operator $D$, see e.g. [10, Prop. 5.8].

Theorem 5.17 in [22] states that for generic Askey-Wilson parameters $(a, b, c, d)$, there exists a unique meromorphic function $E(\cdot, \cdot) = E(\cdot, \cdot; a, b, c, d)$ on $\mathbb{C} \times \mathbb{C}$ satisfying the following six conditions:

1. $E(\mu, x)$ is $\tau^{-1}$-periodic in $\mu$ and $x$,
2. $(\mu, x) \mapsto (q^{2-d+\mu}, q^{2-d-\mu}, q^{2-d+x}, q^{2-d-x}, q^2) \infty E(\mu, x)$ is analytic,
3. For fixed generic $\mu \in \mathbb{C}$, $E(\mu, \cdot)$ is an eigenfunction of $Y^{a,b,c,d}$ with eigenvalue $q^{a-\mu}$,
4. For fixed generic $x \in \mathbb{C}$, $E(\cdot, x)$ is an eigenfunction of $Y^{a,b,c,d}$ with eigenvalue $q^{a-x}$,
5. $(T_1^{a,b} E(\mu, \cdot))(x) = -q^{a+b} (T_1^{a,b} E(\cdot, x))(\mu)$,
6. $E(-\tilde{a}, -a) = 1$.

The existence of a kernel $E$ satisfying the above six conditions is proved by explicitly constructing $E$ as series expansion in nonsymmetric analogues of the Askey-Wilson polynomials, see [22 (6.6)]. This expansion formula for $E$ is very similar to the expansion formula (5.3) of the Askey-Wilson function $E^+$ in Askey-Wilson polynomials. In fact, a comparison of the formulas leads to the explicit link

$$E^+(\mu, x) = (C_{a,b}^+ E(\mu, \cdot))(x), \quad C_{a,b}^+ := \frac{1}{1-q^{a+b} (1 + T_1^{a,b})},$$

see [22, Thm. 6.20]. These results allow us to study the Askey-Wilson function $E^+$ using the characterizing conditions 1–6 for the underlying kernel $E$, instead of focussing on the explicit expression for $E^+$.

The kernel $E(\mu, x)$ is regular at the Askey-Wilson parameters (5.11). The resulting kernel

$$E_0(\mu, x) = E(\mu, x; 0, 1, 1, 1 - \frac{1}{2\tau})$$

is the unique meromorphic kernel satisfying the six conditions 1–6 for the special Askey-Wilson parameters (5.11). Observe that the operators $T_0, T_1$ and $Y$ for the special Askey-Wilson parameters (5.11) reduce to

$$(T_0 f)(x) = f(2 - x), \quad (T_1 f)(x) = f(-x), \quad (Y f)(x) = f(2 + x),$$

hence $E_0$ is the unique meromorphic kernel satisfying the six conditions

1'. $E_0(\mu, x)$ is $\tau^{-1}$-periodic in $\mu$ and $x$,
2'. $(\mu, x) \mapsto \vartheta(\mu) \vartheta(x) E_0(\mu, x)$ is analytic,
3'. $E_0(\mu, x + 2) = q^{-\mu} E_0(\mu, x)$,
4'. $E_0(\mu + 2, x) = q^{-x} E_0(\mu, x)$,
5'. $E_0(\mu, -x) = E_0(-\mu, x)$,
6’. $\mathcal{E}_0(0, 0) = 1$.

We conclude that

$$\mathcal{E}_0(\mu, x) = (-q, -q; q^2)_\infty \frac{\vartheta(\mu + x)}{\vartheta(\mu) \vartheta(x)},$$

since the right hand side satisfies 1’–6’ due to the quasi-periodicity (5.8) of $\vartheta(x)$. Thus $\mathcal{E}_0(\mu, x)$ is an elliptic analogue of the exponential kernel $\exp(-4\pi i \tau \mu x)$, cf. Remark 5.2.

Using the notation (5.7), we conclude that

$$\mathcal{E}_0^+(\mu, x) = (C_0^+, \mathcal{E}_0(\mu, \cdot))(x) = \frac{1}{2} (\mathcal{E}_0(\mu, x) + \mathcal{E}_0(\mu, -x)),$$

which is the desired formula (5.6).

6. The Askey-Wilson function for $|q| = 1$.

In this section we take $-\frac{1}{2} < \tau < 0$, so that $q = q_\tau = \exp(2\pi i \tau)$ has modulus one and $q \neq \pm 1$. The assignment

$$(K^{\pm 1})^* = K^{\mp 1}, \quad (X^{\pm})^* = -X^{\pm}$$

uniquely extends to a unital, anti-linear, anti-algebra involution on $\mathcal{U}_q$. This particular choice of $*$-structure corresponds to the real form $\mathfrak{so}(2, \mathbb{R})$ of $\mathfrak{sl}(2, \mathbb{C})$. The $*$-unitary sesquilinear form for the representations $\pi_\lambda (\lambda \in \mathbb{R})$ of $\mathcal{U}_q$ (see Section 3) is

$$\langle f, g \rangle' = \int_{-i\infty}^{i\infty} f(z)\overline{g(z)} \, dz,$$

where $f$ and $g$ are meromorphic functions which are regular on a large enough strip around the imaginary axes and decay sufficiently fast at $\pm i\infty$. Koornwinder’s twisted primitive element $iY_\rho \in \mathcal{U}_q$ is $*$-selfadjoint for $\rho \in \mathbb{R}$. Thus in principle we are all set to extend the construction of eigenfunctions of the Askey-Wilson second order difference operator $\mathcal{D}$ to the $|q| = 1$ case by simply replacing the role of the $q$-gamma function $\Gamma_{2\tau}$ by $G_{2\tau}$. We need to be careful though due to the following differences with the $0 < q < 1$ case:

a. The analogue of the explicit eigenfunction of $iY_\rho$ (cf. (4.2)), given now as quotient of hyperbolic gamma functions $G_{2\tau}$, has more singularities.

b. No $\tau^{-1}$-periodicity conditions have to be imposed. Consequently, the parameters $\alpha$ and $\beta$ do not need to be discretized for the $|q| = 1$ case.

c. We have to take the decay rates at $\pm i\infty$ of integrands into account.

One needs to be careful with the decay rate (see c) in reproving the crucial Theorem 4.4 because acting by $\pi_\lambda(X^\pm)$ worsens the asymptotics at $\pm i\infty$ (the factor $q^z = \exp(2\pi i \tau z)$ is $O(\exp(-2\pi \tau \text{Im}(z)))$ as $\text{Im}(z) \to \infty$). The decay rate can be improved by considering different eigenfunctions of $iY_\rho$, but then the location of the singularities turns out to cause problems.

To get around these problems, we generalize the techniques of Section 4 to a nonunitary set-up. More concretely, we replace $\langle \cdot, \cdot \rangle'$ by a bilinear form, given as a contour integral over a certain deformation of $i\mathbb{R}$. With such a bilinear form, the singularity problems and the asymptotic problems can be resolved simultaneously.
The proper replacement of the involution $\star$ is the unique unital, linear, anti-algebra involution $\circ$ on $U_q$ satisfying
\[
(K^{\pm 1})^\circ = K^{\mp 1}, \quad (X^{\pm})^\circ = -X^{\pm}.
\]
For $f \in M$ we write $S(f) \subset \mathbb{C}$ for the singular set of $f$.

**Definition 6.1.** $f \in M$ is said to have (exponential) growth rate $\epsilon \in \mathbb{R}$ at $\pm i\infty$ when the following two conditions are satisfied:

1. For some compact subset $K_f \subset \mathbb{R}$,
$$S(f) \subset \{ z \in \mathbb{C} \mid \text{Im}(z) \in K_f \}.$$  

2. The function $f$ satisfies
$$|f(x + iy)| = \mathcal{O}(\exp(\epsilon |y|)), \quad y \to \pm \infty,$$ uniformly for $x$ in compacts of $\mathbb{R}$.

We call a contour $C$ a deformation of $i\mathbb{R}$ when $C$ intersects the line $l_c = \{ z \in \mathbb{C} \mid \text{Im}(z) = c \}$ in exactly one point $z_c(C)$ for all $c \in \mathbb{R}$, and $z_c(C) = ic$ for $|c| >> 0$. For $k \in \mathbb{Z}_{\geq 0}$ we define the strip of radius $k$ around $C$ by
$$\bigcup_{c \in \mathbb{R}} \{ z \in l_c \mid |z - z_c(C)| \leq k \}.$$ For $k = 0$, the strip of radius zero around $C$ is the contour $C$ itself.

**Lemma 6.2.** Suppose that $f, g \in M$ have growth rates $\epsilon_f$ and $\epsilon_g$ at $\pm i\infty$, respectively. Suppose furthermore that $\epsilon_f + \epsilon_g < 2\pi \tau$ and that $f$ and $g$ are analytic on the strip of radius one around a given (oriented) deformation $C$ of $i\mathbb{R}$. Then
\[
(\pi_\lambda(X)f, g)_C = (f, \pi_{-\lambda}(X^\circ)g)_C, \quad \forall X \in U^1_q,
\]
where
\[
(f, g)_C = \int_C f(z)g(z)dz.
\]

**Proof.** The proof follows by an elementary application of Cauchy’s Theorem. \qed

**Remark 6.3.** The condition on the growth rates in Lemma 6.2 may be weakened to $\epsilon_f + \epsilon_g < 0$ when $X$ is taken from the subspace span$_{\mathbb{C}}\{1, K^{-1}, K\}$ of $U^1_q$.

Koornwinder’s twisted primitive element $Y_\rho$ (see (3.4)) is not $\circ$-invariant,
\[
Y_\rho^\circ = -q^{\frac{1}{2}}K^{-1}X^+ + q^{-\frac{1}{2}}K^{-1}X^- + \left(\frac{q^{-\rho} + q^\rho}{q^{-1} - q}\right)(K^{-2} - 1).
\]

On the other hand, a direct computation shows that $\pi_{-\lambda}(Y_\rho^\circ)$ is still a first order difference operator. This leads to the following analogue of Proposition 3.4.
Proposition 6.4. Let $\alpha, \rho \in \mathbb{C}$. A meromorphic function $f \in \mathcal{M}$ is an eigenfunction of $\pi_{-\lambda}(Y^\rho_\rho)$ with eigenvalue $\mu_\alpha(\rho)$ if and only if

$$f(z - 2) = \frac{\sin(\pi(i\lambda + \alpha + \rho - z)\tau) \sin(\pi(-i\lambda + \alpha + \rho + z)\tau)}{\sin(\pi(i\lambda + \rho - 1 + z)\tau) \sin(\pi(-i\lambda + \rho + 1 - z)\tau)} f(z).$$

We define two meromorphic functions by

$$g_\lambda(z; \beta, \sigma) = \frac{G_{2\tau}(\frac{1}{2\tau} - 1 + \beta + \sigma - i\lambda + z)G_{2\tau}(\frac{1}{2\tau} + \sigma - i\lambda - z)}{G_{2\tau}(\frac{1}{2\tau} + 1 + \beta + \sigma + i\lambda - z)G_{2\tau}(\frac{1}{2\tau} + \sigma + i\lambda + z)},$$

$$h_\lambda(z; \alpha, \rho) = \frac{G_{2\tau}(\frac{1}{2\tau} - 1 + \alpha + \rho + i\lambda - z)G_{2\tau}(\frac{1}{2\tau} + \rho + i\lambda + z)}{G_{2\tau}(\frac{1}{2\tau} + 1 + \alpha + \rho - i\lambda + z)G_{2\tau}(\frac{1}{2\tau} + \rho - i\lambda - z)}.$$

The difference equation for $G_{2\tau}$, Proposition 5.4 and Proposition 6.4 imply

$$\pi_{-\lambda}(Y_\sigma)g_\lambda(\cdot; \beta, \sigma) = \mu_\beta(\sigma)g_\lambda(\cdot; \beta, \sigma), \quad \pi_{-\lambda}(Y^\rho_\rho)h_\lambda(\cdot; \alpha, \rho) = \mu_\alpha(\rho)h_\lambda(\cdot; \alpha, \rho).$$

We want to construct now eigenfunctions of the gauged Askey-Wilson second order difference operator $\mathcal{L}$ which are of the form

$$\psi_\lambda(x; \alpha, \rho, \beta, \sigma) = (\pi_{\lambda}(\hat{x}K)g_\lambda(\cdot; \beta, \sigma), h_\lambda(\cdot; \alpha, \rho))_C$$

for a suitable deformation $C$ of $i\mathbb{R}$. To make sense of this integral, we need to take the singularities and the asymptotic behaviour at $\pm i\infty$ of the integrand into account. The singularities can be located using the precise information on the zeros and poles of the hyperbolic gamma function $G_{2\tau}$, see Proposition 2.4(ii). It follows that the singularities of $z \mapsto g_\lambda(1 + x + z; \beta, \sigma)$ are contained in the union of the four half lines

$$-1 - \beta - \sigma + i\lambda - x + \mathbb{R}_{\leq 0}, \quad -1 + \beta + \sigma + i\lambda - x + \mathbb{R}_{\leq 0},$$

$$-\sigma - i\lambda - x + \mathbb{R}_{\geq 0}, \quad \sigma - i\lambda - x + \mathbb{R}_{\geq 0},$$

and the singularities of $z \mapsto h_\lambda(z; \alpha, \rho)$ are contained in the union of the four half lines

$$-1 + \frac{1}{\tau} - \rho - i\lambda + \mathbb{R}_{\leq 0}, \quad -1 + \rho - i\lambda + \mathbb{R}_{\leq 0},$$

$$-\alpha - \rho + i\lambda + \mathbb{R}_{\geq 0}, \quad -\frac{1}{\tau} + \alpha + \rho + i\lambda + \mathbb{R}_{\geq 0}.$$

We call the above eight half lines the singular half lines with respect to the given, fixed parameters $\tau, x, \lambda, \alpha, \rho, \beta, \sigma$. Each singular half line is contained in some horizontal line $l_c$ for some $c \in \mathbb{R}$. For generic parameters $\alpha, \rho, \beta, \sigma$, the eight singular half lines lie on different horizontal lines. Under these generic assumptions, there exists a deformation $C_x$ of $i\mathbb{R}$ which separates the four singular half lines with real part tending to $-\infty$ from the four singular half lines with real part tending to $\infty$. We take such contour $C = C_x$ in the definition (6.4) of $\psi_\lambda$. The resulting function $\psi_\lambda(x)$ is well defined and independent of the particular choice of the deformation $C_x$ of $i\mathbb{R}$, since $g_\lambda(\cdot; \beta, \sigma)$ (respectively $h_\lambda(\cdot; \alpha, \rho)$) has growth rate $\pi((1 - 2\text{Im}(\lambda))\tau - 1)$ (respectively $\pi((1 - 2\text{Im}(\lambda))\tau)$ at $\pm i\infty$. This follows from
the asymptotic behaviour of the hyperbolic gamma function $G_\tau$, see Proposition \[\text{(iv)}\].

The resulting function $\psi_\lambda(x)$ is analytic in $x$.

**Theorem 6.5.** For generic parameters $\alpha, \rho, \beta, \sigma$, the function $\psi_\lambda(\cdot) = \psi_\lambda(\cdot; \alpha, \rho, \beta, \sigma)$ defined by

$$
\psi_\lambda(x) = \int_{C_x} g_\lambda(1 + x + z; \beta, \sigma) h_\lambda(z; \alpha, \rho) \, dz
$$

\[\text{(6.5)}\]

is an eigenfunction of the gauged Askey-Wilson difference operator $L^{\alpha, \rho, \beta, \sigma}$ with eigenvalue $E(\lambda; \beta)$.

**Proof.** We adjust the proof of Theorem \[\text{4.4}\] to the present set-up. We simplify notations by writing $g(z) = g_\lambda(z; \beta, \sigma)$ and $h(z) = h_\lambda(z; \alpha, \rho)$. Choose the deformation $C_x$ of $i\mathbb{R}$ such that $g$ and $h$ are analytic on the strip of radius $\geq 4$ around $C_x$.

Observe that $\pi_\lambda(\hat{x}\Omega K)g$ has the same growth rate at $\pm i\infty$ as $g$. By \[\text{(3.2)}\] and the definition of $\psi_\lambda(x)$, we conclude that the integral $(\pi_\lambda(\hat{x}\Omega K)g, h)_{C_x}$ converges absolutely and equals

$$
\left( \frac{q^{i\lambda} - q^{-i\lambda}}{q - q^{-1}} \right)^2 \psi_\lambda(x).
$$

On the other hand, the radial part computation of $\Omega$ with respect to Koornwinder’s twisted primitive element yields

$$
\hat{x}\Omega K = \hat{x}\Omega(x)K + ((Y_\rho - \mu_\alpha(\rho))K^{-1})X\hat{x}K + \hat{x}Z(Y_\sigma - \mu_\beta(\sigma))
$$

for certain elements $X, Z \in U^1_q$, cf. Proposition \[\text{3.3}\]. Substituting this algebraic identity in $(\pi_\lambda(\hat{x}\Omega K)g, h)_{C_x}$ and using $\pi_\lambda(Y_\alpha)g = \mu_\beta(\sigma)g$, we have

$$
(\pi_\lambda(\hat{x}\Omega K)g, h)_{C_x} = (\pi_\lambda(\hat{x}\Omega(x)K)g, h)_{C_x} + (\pi_\lambda((Y_\rho - \mu_\alpha(\rho))K^{-1}X\hat{x}K)g, h)_{C_x},
$$

\[\text{(6.7)}\]

provided that both integrals on the right hand side of \[\text{(6.7)}\] converge absolutely.

For the second term on the right hand side of \[\text{(6.7)}\], observe that the sum of the growth rates of $g$ and $h$ at $\pm i\infty$ equals $(2\tau - 1)\pi$. Furthermore,

$$(2\tau - 1)\pi < 4\pi \tau$$

since $-\frac{1}{2} < \tau < 0$, and

$$(Y_\rho - \mu_\alpha(\rho))K^{-1} \in U^1_q.
$$

Hence the integral

$$
(\pi_\lambda((Y_\rho - \mu_\alpha(\rho))K^{-1}X\hat{x}K)g, h)_{C_x}
$$

converges absolutely, and Lemma \[\text{6.2}\] shows that this integral equals

$$
(\pi_\lambda(X\hat{x}K)g, \pi_{-\lambda}(K)\pi_{-\lambda}(Y_\rho - \mu_\alpha(\rho))h)_{C_x} = 0,
$$

as desired.
where the last equality follows from the eigenvalue equation \( \pi_{-\lambda}(Y^\circ)h = \mu_\alpha(\rho)h \).

For the first term on the right hand side of (6.7), observe that \( \pi_{\lambda}(x_0 K)g \) has the same growth rate at \( \pm i\infty \) as \( g \). Hence the integral \( (\pi_{\lambda}(x_0 K)g,h)_{C_x} \) converges absolutely. By the definition of the gauged Askey-Wilson second order difference operator \( \mathcal{L} \), this integral equals

\[
\frac{q^{\beta-1}}{(q-q^{-1})^2}(\mathcal{L}\psi_{\lambda})(x) + \frac{q^{\beta-1}(1-q^{1-\beta})^2}{(q-q^{-1})^2} \psi_{\lambda}(x).
\]

Combining the results, we conclude that \( (\pi_{\lambda}(\hat{x}\Omega K)g,h)_{C_x} = (\pi_{\lambda}(\hat{x}\Omega(x)K)g,h)_{C_x} \), and for both sides we have obtained an explicit expression in terms of \( \psi_{\lambda} \). The resulting identity for \( \psi_{\lambda} \) yields the desired difference equation \( \mathcal{L}\psi_{\lambda} = E(\lambda)\psi_{\lambda} \).

\[\square\]

**Remark 6.6.** The eigenfunction \( \psi_{\lambda} \) of \( \mathcal{L} \) as given by the integral (6.3) looks very similar to the eigenfunction \( \phi_{\lambda} \) of \( \mathcal{L} \) (for the case \( \tau \in i\mathbb{R}_{>0} \)) as given by the integral (4.7), since the integrand of \( \phi_{\lambda} \) is essentially the integrand of \( \psi_{\lambda} \) with the hyperbolic gamma functions \( G_{2\tau} \) replaced by \( q \)-gamma functions \( \Gamma_{2\tau} \). Note though that the integration cycles are different.

We can reformulate the difference equation for \( \psi_{\lambda} \) in terms of the Askey-Wilson second order difference operator \( \mathcal{D} = D^{a,b,c,d} \) (see (3.3)) using an appropriate gauge factor, cf. Corollary 4.5. Using the parameter correspondence (3.7), we can for instance choose the gauge factor \( \delta(x) = \delta(x;\alpha,\rho,\beta,\sigma) \) by

\[\delta(x) = \frac{G_{2\tau}(-1 + \frac{1}{2\tau} + a - x)G_{2\tau}(-1 + \frac{1}{2\tau} + b - x)}{G_{2\tau}(1 + \frac{1}{2\tau} - c - x)G_{2\tau}(1 + \frac{1}{2\tau} - d - x)}.
\]

**Corollary 6.7.** For generic parameters \( \alpha,\rho,\beta,\sigma \), the function \( H_{\lambda} = H_{\lambda}(-;a,b,c,d) \in \mathcal{M} \) defined by \( H_{\lambda}(x) = \delta(x)^{-1}\psi_{\lambda}(x) \) satisfies the Askey-Wilson second order difference equation

\[(\mathcal{D}H_{\lambda})(x) = E(\lambda)H_{\lambda}(x),\]

where the Askey-Wilson parameters \( (a,b,c,d) \) are given by (3.7).

7. **Appendix**

In this appendix we sketch a proof of Proposition 3.3 following closely the arguments of Koornwinder [11] for the special case \( \alpha = \beta = 0 \).

Fix \( x \in \mathbb{C} \) and write \( x \equiv x' \) for \( x, x' \in \hat{U}_q^\infty \) if

\[x - x' \in (Y_\rho - \mu_{\alpha}(\rho)) \hat{x}U_q^1 + \hat{x}U_q^1(Y_\sigma - \mu_{\beta}(\sigma)).\]

In order to simplify notations, I use the notations \( \mu = \mu_{\alpha}(\rho) \) and \( \nu = \mu_{\beta}(\sigma) \) for the duration of the proof. To reduce \( \hat{x}\Omega K = \hat{x}K\Omega \) in the desired form, we need to concentrate on the part \( \hat{x}KX^+X^- \). Using

\[\hat{x}KX^+X^- = q^{\frac{1}{2}}x^+(q^{\frac{1}{2}}X^+K - q^{-\frac{1}{2}}X^-K)\hat{x}X^-
+ q^{-\frac{1}{2}}x^-\hat{x}(q^{-\frac{1}{2}}X^-K - q^{\frac{1}{2}}X^+K) + q^{2\tau}\hat{x}KX^-X^+\]
and the commutation relation between $X^-$ and $X^+$ in $\mathcal{U}_q$, we obtain

$$(1 - q^{2x}) \hat{x} KX^+ X^- \equiv q^{\frac{1}{2}+x} \left( \frac{q^{\rho} + q^{-\rho}}{q - q^{-1}} (K^2 - 1) + \mu \right) \hat{x} X^-$$

$$- q^{-\frac{1}{2}+x} X^- \hat{x} \left( \frac{q^{\sigma} + q^{-\sigma}}{q - q^{-1}} (K^2 - 1) + \nu \right) + q^{2x} \left( \frac{K^{-1} - K^3}{q - q^{-1}} \right) \hat{x},$$

hence we need to focus now only on the reduction of $\hat{x}X^-$ and $\hat{x}K^2X^-$. Using

$$\hat{x} X^- = q^{\frac{1}{2} - x} \left( q^{-\frac{1}{2}} X^- K - q^{\frac{1}{2}} X^+ K \right) K^{-1} \hat{x}$$

$$+ q^{\frac{3}{2} - 2x} K^{-1} \hat{x} \left( q^{\frac{1}{2}} X^+ K - q^{-\frac{1}{2}} X^- K \right) + q^{1-2x} K^{-1} \hat{x} X^-, $$

we obtain

$$(1 - q^{-2-2x}) \hat{x} X^- \equiv - q^{\frac{1}{2} - x} \left( \frac{q^{\rho} + q^{-\rho}}{q - q^{-1}} (K^2 - 1) + \mu \right) K^{-1} \hat{x}$$

$$+ q^{\frac{3}{2} - 2x} K^{-1} \hat{x} \left( \frac{q^{\sigma} + q^{-\sigma}}{q - q^{-1}} (K^2 - 1) + \nu \right),$$

and a similar computation yields

$$(1 - q^{-2-2x}) \hat{x} K^2 X^- \equiv - q^{\frac{3}{2} - x} \left( \frac{q^{\rho} + q^{-\rho}}{q - q^{-1}} (K^2 - 1) + \mu \right) K \hat{x}$$

$$+ q^{-\frac{3}{2} - 2x} K \hat{x} \left( \frac{q^{\sigma} + q^{-\sigma}}{q - q^{-1}} (K^2 - 1) + \nu \right).$$

Collecting all these results we arrive for generic $x \in \mathbb{C}$ at a formula of the form

$$\hat{x} \Omega K \equiv \hat{x} \left( \tilde{B}(x) K^3 + \tilde{C}(x) K + \tilde{D}(x) K^{-1} \right)$$

for explicit rational expressions $\tilde{B}(x), \tilde{C}(x)$ and $\tilde{D}(x)$. Using

$$q^{\rho} + q^{-\rho} - (q - q^{-1}) \mu = q^{\rho + \alpha} + q^{-\rho - \alpha},$$

$$q^{\sigma} + q^{-\sigma} - (q - q^{-1}) \nu = q^{\sigma + \beta} + q^{-\sigma - \beta},$$

the rational functions $\tilde{B}(x), \tilde{C}(x)$ and $\tilde{D}(x)$ are explicitly given by

$$\tilde{B}(x) = \left( q^{\frac{1}{2}+x} (q^{\rho} + q^{-\rho}) - q^{\frac{3}{2}+2x} (q^{\sigma} + q^{-\sigma}) \right) \left( q^{-\frac{3}{2} - 2x} (q^{\rho} + q^{-\rho}) - q^{-\frac{1}{2} - x} (q^{\rho} + q^{-\rho}) \right)$$

$$+ \frac{q^{2x}}{(q - q^{-1})(1 - q^{2x})} + \frac{q^{-1}}{(q - q^{-1})^2},$$
\[ \tilde{C}(x) = -\frac{2}{(q - q^{-1})^2} \]
\[ + \frac{(q^{\frac{1}{2}+x}(q^\alpha + q^{-\rho}) - q^{\frac{3}{2}+2x}(q^\sigma + q^{-\sigma})) (q^{-\frac{1}{2}-x}(q^{\rho+\alpha} + q^{-\rho-\alpha}) - q^{-\frac{1}{2}-2x}(q^{\sigma+\beta} + q^{-\sigma-\beta}))}{(q - q^{-1})^2(1 - q^{2x})(1 - q^{-2-2x})} \]
\[ + \frac{(q^{-\frac{1}{2}+2x}(q^{\sigma+\beta} + q^{-\sigma-\beta}) - q^{\frac{1}{2}+x}(q^{\rho+\alpha} + q^{-\rho-\alpha})) (q^{\frac{1}{2}-2x}(q^\alpha + q^{-\rho}) - q^{\frac{1}{2}-x}(q^\sigma + q^{-\sigma}))}{(q - q^{-1})^2(1 - q^{2x})(1 - q^{-2-2x})} , \]
and
\[ \tilde{D}(x) = \frac{q^{2x}}{(q - q^{-1})(1 - q^{2x})} + \frac{q}{(q - q^{-1})^2} \]
\[ + \frac{(q^{-\frac{1}{2}+2x}(q^{\rho+\alpha} + q^{-\rho-\alpha}) - q^{\frac{1}{2}+x}(q^{\rho+\alpha} + q^{-\rho-\alpha})) (q^{\frac{1}{2}-2x}(q^{\sigma+\beta} + q^{-\sigma-\beta}) - q^{\frac{1}{2}-x}(q^{\rho+\alpha} + q^{-\rho-\alpha}))}{(q - q^{-1})^2(1 - q^{2x})(1 - q^{-2-2x})} , \]
By a tedious computation, it can now be proven that
\[ \tilde{B}(x) = \frac{q^{\beta-1}}{(q - q^{-1})^2} B(x) , \]
\[ \tilde{C}(x) = \frac{q^{\beta-1}}{(q - q^{-1})^2} (-A(x) - A(x^{-1}) + (1 - q^{1-\beta})^2) , \]
\[ \tilde{D}(x) = \frac{q^{\beta-1}}{(q - q^{-1})^2} D(x) , \]
with the parameters \(a, b, c, d\) as in (3.7).

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