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Tome 32, n° 2 (2020), p. 525-543.

<http://jtnb.centre-mersenne.org/item?id=JTNB_2020__32_2_525_0>
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Résumé. Dans cet article, nous donnons une preuve adélique de la formule de Chevalley–Gras pour les corps de nombres qui, elle-même, est une généralisation de la formule du nombre de classes ambiges. L'idée est de réduire cette formule au théorème de la norme de Hasse et à des lois de réciprocité globaux. Nous donnons également une preuve adélique de la formule de Chevalley–Gras pour les groupes des classes des diviseurs de degré 0 dans le cas des corps de fonctions, qui étend un résultat de Rosen.

Abstract. In this article we give an adelic proof of the Chevalley–Gras formula for global fields, which itself is a generalization of the ambiguous class number formula. The idea is to reduce the formula to the Hasse norm theorem and to the local and global reciprocity laws. We also give an adelic proof of the Chevalley–Gras formula for the class group of divisors of degree 0 in the function field case, which extends a result of Rosen.

1. Introduction

Let $K/k$ be a cyclic extension of number fields with Galois group $G$. Let $m$ be a modulus of $k$, which gives rise to a modulus $m_K$ of $K$. The ray class group $\text{Cl}^{m_K}$ modulo $m_K$ admits a $G$-module structure. The Chevalley–Gras formula describes an explicit relationship between the generalized ambiguous ray class number $|\text{Cl}^{m_K}_K / C|^G$ and $|\text{Cl}^m_k / N(C)|$, where $C \subset \text{Cl}^{m_K}_K$ is any $G$-submodule and $N$ is the norm map from $K$ to $k$. In the case when the submodule $C$ and the modulus $m$ are trivial, the formula then relates the class numbers $|\text{Cl}^G_K|$ and $|\text{Cl}_K|$, which is the classical ambiguous class number formula due to Chevalley [3, p. 406]. A proof of Chevalley’s formula can be found in Gras’ book [7, Lemma 6.1.2 and Remark 6.2.3] or in Lang’s book [10, Chapter 13, Section 4, Lemma 4.1]. Lemmermeyer [11] gives an elementary proof which follows closely the approach taken by Lang, but avoiding the machinery of cohomologies. The existing proofs of Chevalley’s formula are reduced to a result of the Herbrand quotient of global units.

2020 Mathematics Subject Classification. 11R29, 11R37, 11R34.

Mots-clés. class groups, ambiguous class number formulas, class field theory.

Li is supported by Anhui Initiative in Quantum Information Technologies (Grant No. AHY150200), NSFC (Grant No. 11571328) and the Fundamental Research Funds for the Central Universities (No. WK0010000058). Yu was partially supported by the grants MoST 103-2918-I-001-009 and 104-2115-M-001-001-MY3.
In [5, Théorème 4.3], Gras gave a formula for narrow class groups with arbitrary C. In [6, Théorème 2.7] (also see the English translation [8, Section 2]), he proved this formula for ray class groups. His proof is based on Chevalley’s formula. Recently, a generalization of Chevalley’s class number formula to dihedral extensions has been investigated by Caputo and Nuccio [2].

In this article we give an adelic proof of the Chevalley–Gras formula over global fields. More precisely, using the adelic language, we reduce the formula to the Hasse norm theorem and to the local and global index theorems, which is shorter and more conceptual.

In the function field case, the class group of divisors of degree 0 deserves a special attention. The ambiguous class number formula (the case C = 0) for functions fields was obtained by Rosen [15]. We also give an adelic proof of the formula with an arbitrary G-submodule C.

In the last section we add an elementary exposition of a cohomological variant for S-ray class groups, for the sake of completeness. This formulation is valid for an arbitrary Galois extension K/k, and is essentially equivalent to Chevalley’s original formula in the cyclic case, thanks to the theorem on the Herbrand quotient of global units.

2. The Chevalley–Gras formula

In this section, we recall the definition of the S-ray class group and prove the Chevalley–Gras formula. We then give some special cases of this formula for future convenience. In Example 2.6, we use this formula to reprove a classical result of Rédei on the 4-rank of the narrow class group of quadratic fields as this approach does not seem to appear in the literature.

2.1. Notation and S-ray class groups. Let F be a global field, that is, F is either a number field (a finite extension of \( \mathbb{Q} \)) or a global function field (a finite extension of \( \mathbb{F}_p(t) \) for some prime p). Let \( V_F \) denote the set of all places of F, and let \( V_{F,\infty} \) (resp. \( V_{F,f} \)) denote the subset of archimedean (resp. finite) places. (So in the function field case, \( V_{F,\infty} = \emptyset \).) For each place \( w \in V_F \), the completion of F at w is denoted by \( F_w \). Let \( \mathcal{O}_w \) denote the ring of integers of \( F_w \) if w is finite. The canonical embedding from F to the completion \( F_w \) is also denoted by w.

The letter S always denotes a non-empty finite set of places of F containing \( V_{F,\infty} \). Denote by \( \mathcal{O}_{F,S} \) the ring of S-integers of F, which consists of all elements \( a \in F \) such that \( a \in \mathcal{O}_w \) for all \( w \notin S \). An S-modulus \( \mathfrak{m} \) is a formal product \( \mathfrak{m}_\infty \cdot \mathfrak{m}_f \), where \( \mathfrak{m}_f \) is a nonzero integral ideal of \( \mathcal{O}_{F,S} \) and \( \mathfrak{m}_\infty \) is a formal product of some real places if F is a number field and \( \mathfrak{m}_\infty \) is always 1 otherwise. Let \( S(\mathfrak{m}) := \{ w \in V_F \mid w | \mathfrak{m} \} \) be the support of \( \mathfrak{m} \). Let \( I_F \) be the free abelian group generated by \( V_{F,f} \), and \( I_F^{S(\mathfrak{m})} \) the subgroup
generated by $V_{F,f} \setminus S(m)$. The ideal $m_f$ corresponds to an effective divisor in $I_F$ whose support is disjoint from $S$ (in $V_{F,f}$). Let

$$F^m := \{ x \in F^\times | x \equiv 1 \mod m_f \text{ and } w(x) > 0 \text{ for each real place } w|_{m_\infty} \}. $$

Let $i : F^\times \to I_F$ be the natural map defined by $a \mapsto \sum_{w \in V_{F,f}} \operatorname{ord}_w(a)w$. Note that $i(F^m) \subseteq I_F^{S(m)}$. The ray class group of $F$ modulo $m$ is defined as

$$\operatorname{Cl}_F^m := I_F^{S(m)}/i(F^m).$$

The $S$-ray class group of $F$ modulo $m$ is defined as

$$(2.1) \quad \operatorname{Cl}_{F,S}^m := \operatorname{Cl}_F^m/\langle \text{image of } S \cap V_{F,f} \rangle. $$

Since $S$ is non-empty, $\operatorname{Cl}_{F,S}^m$ is finite.

Alternatively, let $I_F^{S(m) \cup S} \subseteq I_F^{S(m)}$ be the subgroup generated by $V_{F,f} \setminus (S \cup S(m))$. Then we have a projection $\text{pr} : I_F^{S(m)} \to I_F^{S(m)}/(S \cap V_{F,f}) \cong I_F^{S(m) \cup S}$. Composing with $i$, we obtain a map $i_S : F^m \to I_F^{S(m) \cup S}$, which maps $a$ to $\operatorname{div}^S(a) := \sum_{w \not\in S} \operatorname{ord}_w(a)w$. Then one can define $\operatorname{Cl}_{F,S}^m$ by

$$I_F^{S(m) \cup S}/i_S(F^m),$$

and this agrees with the definition (2.1). The group $I_F^{S}$ can also be naturally identified with the ideal group of $\mathcal{O}_{F,S}$. Under this identification, the map $i_S : F^m \to I_F^{S(m) \cup S}$ sends $a$ to the principal ideal $a\mathcal{O}_{F,S}$. Put $I_{F,S}^m := i_S(F^m)$, the subgroup of principal $S$-ideals modulo $m$, and then we have $\operatorname{Cl}_{F,S}^m = I_F^{S(m) \cup S}/I_{F,S}^m$. In the case where $m = 1$, this is the $S$-ideal class group of $F$ and is denoted by $\operatorname{Cl}_{F,S}$.

For convenience, we also let $\mathcal{O}_w := F_w$ if $w \in V_{F,\infty}$, and for each $w|m$ we also write

$$1 + m\mathcal{O}_w := \begin{cases} 1 + m_f\mathcal{O}_w & \text{if } w|m_f; \\ (F_w^\times)^2 & \text{if } w|m_\infty. \end{cases}$$

Let $K/k$ be a finite Galois extension of global fields with Galois group $G$. Let $(S_K, m_K)$ be a pair consisting of a finite set $S_K$ of places and a modulus $m_K$ of $K$ as above. Suppose that both $S_K$ and $m_K$ are $G$-invariant. Then the $S_K$-ray class group $\operatorname{Cl}_{K,S_K}^{m_K}$ admits an action of $G$. So for any $G$-submodule $C$ of $\operatorname{Cl}_{K,S_K}^{m_K}$, one may look for a relationship between $|\{(\operatorname{Cl}_{K,S_K}^{m_K} / C)^G\}|$ and $|\operatorname{Cl}_{K,S}^{m}/N(C)|$, for a suitable pair $(S, m)$ for $k$ related to $(S_K, m_K)$, where $N$ is the norm map from $K$ to $k$. This question is answered mostly when $K/k$ is cyclic and remains open in general, even for the abelian case.

Suppose $(S, m)$ is a pair for $k$. Let $S_K$ be the set of places of $K$ over $S$ and let $m_K := m_{K,\infty} \cdot m_{K,f}$, where $m_{K,\infty}$ is the set of real places of $K$ over the support of $m_\infty$ and $m_{K,f} := m_f\mathcal{O}_{K,S}$. Then $(S_K, m_K)$ is a $G$-invariant pair, and we say $(S_K, m_K)$ is induced by $(S, m)$. In this case, we also write
We denote the canonical surjection $\text{Cl}_{K,S}$ for $\text{Cl}_{K,S}^{m}$ and call it the $S$-ray class group of $K$ modulo $m$. When $m = 1$, we write also $\text{Cl}_{K,S}$ for $\text{Cl}_{K,S}^{m}$ and call it the $S$-ideal class group of $K$.

2.2. The main formula. Let $K/k$ be a cyclic extension of global fields with group $G = \langle \sigma \rangle$, where $\sigma$ is a generator. Let $N = N_{K/k}$ be the norm map from $K$ to $k$. Let $S \supset V_{k,\infty}$ be a finite non-empty set of places of $k$ and $m$ an $S$-modulus. Let $\text{Cl}_{K,S}^{m} := \text{Cl}_{K,S}^{m}$ be the $S$-ray class group modulo $m$, where $(S_K, m_K)$ is the pair induced by $(S, m)$.

For $v \in V_{k,f}$, denote by $e_v$ and $f_v$ the ramification index and inertia degree of $v$ in $K/k$ respectively. In the number field case, if $v$ is real and every place $w|v$ of $K$ is complex, we say that $v$ is ramified in $K$, and put $e_v = 2$ and $f_v = 1$, otherwise, we put $e_v = f_v = 1$. The following theorem which we call the Chevalley–Gras formula over global fields is proved by Gras in the number field case; see [8, Theorem 3.6].

**Theorem 2.1.** Let $K/k$ be a cyclic extension of global fields with Galois group $G$. Let $m$ be a modulus of $k$, and let $S \supset V_{k,\infty}$ be a finite non-empty set of places of $k$ such that $S \cap S(m_f) = \emptyset$. Let $C$ be a $G$-submodule of the $S$-ray class group $\text{Cl}_{K,S}^{m}$. Let $D$ be any subgroup of $I_K^{S(m)}$ such that the image of $D$ in $\text{Cl}_{K,S}^{m}$ is equal to $C$. Then

$$\frac{|(\text{Cl}_{K,S}^{m} / C)^G|}{|\text{Cl}_{K,S}^{m} / N(C)|} = \prod_{v \in S \setminus S(m)} e_v f_v \prod_{v \in S(m)} [1 + mO_v : N(\prod_{w|v} (1 + mO_w))] \prod_{v \notin S \cup S(m)} e_v \frac{[K : k][\Lambda : \Lambda \cap N(K^m)]}{[K : k][\Lambda : \Lambda \cap N(K^m)]}.
$$

Here $\Lambda = \{ x \in k^m \mid (x)O_{k,S} = N(d)O_{k,S} \text{ in } I_K^{S(m)} \text{ for some } d \in D \}$.

**Remark 2.2.**

1. The group $\Lambda$ depends on the choice of $D$, however, we will see in (2.5) that the index $[\Lambda : \Lambda \cap N(K^m)]$ depends only on $C$.

2. The group $N(\prod_{w|v} (1 + mO_w))$ equals $N_{K_w/k_v}(1 + mO_w)$ for any $w|v$ as $K/k$ is Galois.

**Proof.** We first express $\text{Cl}_{K,S}^{m} / C$ in terms of ideles. Let $A_K^{\times}$ denote the idele group of $K$. Let $A_K^{m} = \{ (a_w)_w \in A_K^{\times} | a_w \in 1 + mO_w \text{ for each } w|m \}$. We denote the canonical surjection $A_K^{m} \to I_{K}^{S(m)}$ by $\pi$. The kernel of $\pi$ is $U_{K}^{m} := \prod_{w|m} O_w^{\times} \prod_{w|m} 1 + mO_w$. Put

$$U_{K,S}^{m} = U_{K}^{m} \prod_{w \in S_K, w|m_{\infty}} K_w^{\times} = \prod_{w|m, w \not\in S_K} O_w^{\times} \prod_{w|m} 1 + mO_w \prod_{w \in S_K, w|m_{\infty}} K_w^{\times}.
$$
Then $\pi$ induces an isomorphism
\[ A^m_K/K^m U^m_{K, S} \cong \text{Cl}^m_{K, S}. \]
By the approximation theorem, $A^m_K K^\times = A^\times_K$. Note that $A^m_K \cap K^\times U^m_{K, S} = K^m U^m_{K, S}$. So the inclusion $A^m_K \subset A^\times_K$ induces an isomorphism
\[ \text{Cl}^m_{K, S} \cong A^\times_K / K^\times U^m_{K, S}. \]
Put $\tilde{D} = \pi^{-1}(D)K^\times U^m_{K, S}$. It follows that
\[ \text{Cl}^m_{K, S} / \mathcal{C} \cong A^\times_K / \tilde{D}. \]
Since $H^1(G, A^\times_K) = 0$, the fact $G = \langle \sigma \rangle$ is cyclic implies that $(A^\times_K)^{1-\sigma}$ is the kernel of the norm from $A^\times_K$ to $A^\times_k$. So there is an exact commutative diagram
\[
\begin{array}{cccccc}
1 & \longrightarrow & \tilde{D} \cap (A^\times_K)^{1-\sigma} & \longrightarrow & \tilde{D} & \overset{N}{\longrightarrow} & N(\tilde{D}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & (A^\times_K)^{1-\sigma} & \longrightarrow & A^\times_K & \overset{N}{\longrightarrow} & N(A^\times_K) & \longrightarrow & 1.
\end{array}
\]
The snake lemma gives the short exact sequence
\[ 1 \longrightarrow (\text{Cl}^m_{K, S} / \mathcal{C})^{1-\sigma} \longrightarrow \text{Cl}^m_{K, S} / \mathcal{C} \longrightarrow N(A^\times_K) / N(\tilde{D}) \longrightarrow 1, \]
as one has $(A^\times_K)^{1-\sigma} / (A^\times_K)^{1-\sigma} \cap \tilde{D}) \cong (\text{Cl}^m_{K, S} / \mathcal{C})^{1-\sigma}$. For any finite $G$-module $M$, one has $|M^G| = |M/M^{1-\sigma}|$ by the exact sequence
\[ 0 \longrightarrow M^G \longrightarrow M \overset{1-\sigma}{\longrightarrow} M \longrightarrow M/M^{1-\sigma} \longrightarrow 0. \]
Thus we obtain the equality
\[ |(\text{Cl}^m_{K, S} / \mathcal{C})^G| = |N(A^\times_K) / N(\tilde{D})|. \]
Recall that Hasse's norm theorem says that $k^\times \cap N(A^\times_K) = N(K^\times)$. So given an element $N(x) = aN(d) \in N(A^\times_K) \cap k^\times N(\tilde{D})$ with $x \in A^\times_K$, $a \in k^\times$ and $d \in \tilde{D}$, we have $a = N(y)$ for some $y \in K^\times$. Hence $N(A^\times_K) \cap k^\times N(\tilde{D}) = N(K^\times)N(\tilde{D}) = N(\tilde{D})$. Therefore the natural map
\[ N(A^\times_K) / N(\tilde{D}) \longrightarrow k^\times N(A^\times_K) / k^\times N(\tilde{D}) \]
is an isomorphism. The global index theorem [9, Chapter IX, Section 5] says that $|A^\times_k / k^\times N(A^\times_K)| = |G| = |K : k|$. This implies that $k^\times N(A^\times_K) / k^\times N(\tilde{D})$ is a subgroup of $A^\times_k / k^\times N(\tilde{D})$ with index $[K : k]$. Therefore
\[ |(\text{Cl}^m_{K, S} / \mathcal{C})^G| = [K : k]^{-1} |A^\times_k / k^\times N(\tilde{D})|. \]
To compute $A^\times_k / k^\times N(\tilde{D})$, we consider the exact sequence
\[
(2.2) \quad 1 \rightarrow k^\times N(\tilde{D}) U^m_{k, S} / k^\times N(\tilde{D}) \rightarrow A^\times_k / k^\times N(\tilde{D}) \rightarrow A^\times_k / k^\times U^m_{k, S} N(\tilde{D}) \rightarrow 1.
\]
We claim that the last term is isomorphic to $\text{Cl}_{k,S}^m / \text{N}(\mathcal{C})$. By the identification $\mathbb{A}_k^m / k^m U_{k,S}^m \cong \text{Cl}_{k,S}^m$, we have that $\text{N}(\mathcal{C}) \subset \text{Cl}_{k,S}^m$ is the image of $\text{N}(\pi^{-1}(\mathcal{D}))$ in $\text{Cl}_{k,S}^m$. Hence

$$\text{Cl}_{k,S}^m / \text{N}(\mathcal{C}) \cong \mathbb{A}_k^m / \text{N}(\pi^{-1}(\mathcal{D})) k^m U_{k,S}^m.$$  

Then the inclusion $\mathbb{A}_k^m \hookrightarrow \mathbb{A}_k^\times$ induces an isomorphism

$$\mathbb{A}_k^m / \text{N}(\pi^{-1}(\mathcal{D})) k^m U_{k,S}^m \cong \mathbb{A}_k^\times / \text{N}(\pi^{-1}(\mathcal{D})) k^m U_{k,S} = \mathbb{A}_k^\times / k^m U_{k,S} \text{N}(\mathcal{D}).$$

The first term of (2.2) can be computed by the exact sequence

$$(2.3) \quad 1 \longrightarrow U_{k,S}^m \cap k^\times \mathcal{N}(\mathcal{D}) / \text{N}(U_{k,S}^m) \longrightarrow U_{k,S}^m / \text{N}(U_{k,S}^m) \longrightarrow k^\times \mathcal{N}(\mathcal{D}) U_{k,S} / k^\times \mathcal{N}(\mathcal{D}) \longrightarrow 1.$$  

Let $G_v$ and $I_v$ be the decomposition group and inertia group of $v$ respectively. For each place $v$ of $k$, we choose a place $w$ of $K$ above $v$. By local class field theory, $H^2(G_v, K_w^\times) = k^\times / \text{N}_{K_w/k_v}(K_w^\times) \cong G_v$ and $H^2(G_v, O_w^\times) / \text{N}_{K_w/k_v}(O_w^\times) \cong I_v$. It follows from the cyclicity of $G$ and Shapiro’s Lemma that

$$U_{k,S}^m / \text{N}(U_{k,S}^m) \cong \prod_{v \notin S \setminus S(m)} H^2(G_v, K_w^\times) \times \prod_{v \in S(m) \setminus S} (1 + mO_v) \times \prod_{v \in S(m)} H^2(G_v, O_w^\times) \cong \prod_{v \notin S \setminus S(m)} G_v \times \prod_{v \in S(m)} \text{N}_{K_w/k_v}(1 + mO_v) \times \prod_{v \notin S \setminus S(m)} I_v.$$  

Here we use our condition that $S$ is disjoint with $S(m)$ and the fact that $1 + mO_v = \mathbb{R}_{>0} = \text{N}_{K_w/k_v}(1 + mO_v)$ if $v | m_\infty$. This contributes to the numerator of the right hand side term in the theorem. In order to prove the theorem, it suffices to show that the first term of (2.3) is isomorphic to $\Lambda / \Lambda \cap N(\mathbb{K})^m$.

Recall that $\pi$ is the natural projection $\mathbb{A}_K^m \to I_K^S(m)$. Write $\mathcal{D} = \pi^{-1}(\mathcal{D})$ for simplicity. As $U_{k,S}^m \text{N}(\mathcal{D}) \subset \mathbb{A}_K^m$, it is direct to check that

$$(2.4) \quad \Lambda = k^m \cap U_{k,S}^m \text{N}(\mathcal{D}) = k^\times \cap U_{k,S}^m \text{N}(\mathcal{D}).$$

Given $x = u \text{N}(\mathfrak{d}) \in \Lambda$ with $u \in U_{k,S}^m$ and $\mathfrak{d} \in \mathcal{D}$, we define a function $f$ as follows:

$$f : \Lambda \longrightarrow U_{k,S}^m \cap k^\times \mathcal{N}(\mathcal{D}) U_{k,S}^m / \text{N}(U_{k,S}^m), \quad x \mapsto u \bmod \text{N}(U_{k,S}^m).$$

We need to show that $f$ is well-defined. Suppose $x = u \text{N}(\mathfrak{d}) = u' \text{N}(\mathfrak{d}') \in \Lambda$ with $u, u' \in U_{k,S}^m$ and $\mathfrak{d}, \mathfrak{d}' \in \mathcal{D}$. Then $u'/u = \text{N}(\mathfrak{d}/\mathfrak{d}') \in \text{N}(\mathcal{D}) \cap U_{k,S}^m \subset \text{N}(\mathbb{A}_K^m) \cap U_{k,S}^m$. By Lemma 2.3(1), the last group coincides with $\text{N}(U_{k,S}^m)$. So $f$ is a well-defined map.
It is clear that \( f \) is a group homomorphism. We show that \( f \) is surjective. Let \( u = tN(d\bar{a}) \) be an element of \( U_{K,S}^m \cap k^\times N(DU_{K,S}^m) \) with \( t \in k^\times, \bar{a} \in \tilde{D} \) and \( a \in U_{K,S}^m \). Then \( t = uN(a)^{-1}N(\bar{a})^{-1} \) with \( uN(a)^{-1} \in U_{K,S}^m \) and \( N(\bar{a})^{-1} \in N(\tilde{D}) \). Note that \( t \) is in fact in \( k^m \). This shows \( t \in \Lambda \) by (2.4).

We have \( f(t) = uN(a)^{-1} \mod N(U_{K,S}^m) \equiv u \mod N(U_{K,S}^m) \). This proves the surjectivity.

The kernel of \( f \) by definition coincides with \( \Lambda \cap N(U_{K,S}^m \tilde{D}) \). Lemma 2.3(3) shows that it also equals \( \Lambda \cap N(K^m) \). Thus, as desired, \( f \) induces an isomorphism

\[
\Lambda/\Lambda \cap N(K^m) \cong U_{K,S}^m \cap k^\times N(DU_{K,S}^m)/N(U_{K,S}^m).
\]

Observe that the term \( k^\times N(DU_{K,S}^m) \) is independent of the choice of \( D \) as \( k^\times N(K^\times \tilde{D}U_{K,S}^m) = k^\times N(\tilde{D}) \). This finishes the proof of the theorem. \( \square \)

**Lemma 2.3.** We have the following equalities:

1. \( N(A_n^m) \cap U_{K,S}^m = N(U_{K,S}^m) \);
2. \( N(K^\times) \cap N(A_n^m) = N(K^m) \);
3. \( \Lambda \cap N(U_{K,S}^m \tilde{D}) = \Lambda \cap N(K^m) = \Lambda \cap N(A_n^m) \).

**Proof.** Recall \( A_n^m = \{(a_w) \in A_K^\times \mid a_w \in 1 + mO_w \text{ for each } w \mid \mathfrak{m}\} \). As mentioned in Remark 2.2, \( N(\prod_w K_w^\times) = N_{K_w/k_w}(K_w^\times) \) for each \( w \) as \( K/k \) is Galois. For a place \( w \mid \mathfrak{m} \), it is easy to see that \( N_{K_w/k_w}(K_w^\times) \cap O_w^\times = N_{K_w/k_w}(O_w^\times) \). We obtain \( N(\prod_w K_w^\times) \cap O_w^\times = N(\prod_w O_w^\times) \). For a place \( w \mid \mathfrak{m} \), we have the trivial equality \( N_{K_w/k_w}(1 + mO_w) \cap 1 + mO_w = N_{K_w/k_w}(1 + mO_w) \). It follows that \( N(A_n^m) \cap U_{K,S}^m = N(U_{K,S}^m) \). This proves (1).

To prove (2), let \( K^\times \times A_n^m \) denote the direct product of \( K^\times \) and \( A_n^m \). Consider the exact commutative diagram

\[
\begin{array}{ccccccccc}
1 & & 1
\downarrow & & \downarrow
\hline
K^\times & & N(K^\times) & & \Lambda \cap N(K^m)
\downarrow_{x \mapsto (x,x)} & & \downarrow_{x \mapsto (x,x)}
\hline
K^\times \times A_n^m & & N(K^\times) \times N(A_n^m)
\downarrow_{(a,b) \mapsto ab^{-1}} & & \downarrow_{(a,b) \mapsto ab^{-1}}
\hline
K^\times A_n^m & & N(K^\times)N(A_n^m)
\downarrow & & \downarrow
\hline
1 & & 1
\end{array}
\]

Note that \( K^\times \cap A_n^m \) by definition is \( K^m \). By the approximation theorem, \( K^\times A_n^m = A_K^\times \). Since \( H^1(G, A_K^\times) = H^1(G, K^\times) = 0 \) and \( G \) is cyclic, the
The snake lemma gives an exact sequence

\[(K^\times)^{1-\sigma} \times (\mathbb{A}_K^m \cap (\mathbb{A}_K^\times)^{1-\sigma}) \rightarrow (\mathbb{A}_K^\times)^{1-\sigma} \rightarrow N(K^\times) \cap N(\mathbb{A}_K^m)/N(K^m) \rightarrow 0.\]

The first arrow is surjective by the weak approximation theorem. Thus the last term is 0. This proves (2).

Now let’s prove (3). Hasse’s norm theorem says that \(k^\times \cap N(\mathbb{A}_K^\times) = N(K^\times)\). Recall that \(\Lambda = k^\times \cap U_{K,S} N(\overline{D})\) by (2.4). By (2), we have

\[
\Lambda \cap N(U_{K,S}^m \overline{D}) = k^\times \cap N(U_{K,S}^m \overline{D}) \subset N(K^\times) \cap N(U_{K,S}^m \overline{D}) \subset N(K^\times) \cap N(\mathbb{A}_K)
\]

\[= N(K^m).\]

This proves the inclusion \(\Lambda \cap N(U_{K,S}^m \overline{D}) \subset \Lambda \cap N(K^m)\). To show the other inclusion, note that \(U_{K,S}^m N(\overline{D}) \cap N(\mathbb{A}_K^m) = N(U_{K,S}^m \overline{D})\) by (1). Then

\[
\Lambda \cap N(K^m) \subset \Lambda \cap N(\mathbb{A}_K^m) = k^\times \cap N(U_{K,S}^m \overline{D}) = \Lambda \cap N(U_{K,S}^m \overline{D}).
\]

The last equality follows from (2.4).

The second equality in (3) follows from

\[
\Lambda \cap N(\mathbb{A}_K^m) \subset \Lambda \cap N(K^\times) \cap N(\mathbb{A}_K^m) = \Lambda \cap N(K^m).
\]

This completes the proof of the lemma. \(\Box\)

Remark 2.4. The idea of our adelic proof of Theorem 2.1 comes from [17], which shows that Chevalley’s ambiguous class number formula follows immediately from the Hasse norm theorem, and the local and global norm index theorems. When the extension \(K/k\) is abelian, we know that the analogous statements for the local and global norm index theorems hold true; see [9, Chapter IX, Sections 3 and 5]. However, to extend Chevalley’s formula to abelian extensions, the assumption that \(K/k\) is cyclic is crucial in the argument used in [17].

2.3. Examples. We list some special cases of the Chevalley–Gras formula in the number field case.

Example 2.5.

(1) If \(m = 1\) and \(S\) is the set of infinite places, then \(\text{Cl}_K^m\) is equal to \(\text{Cl}_K\), the class group of \(K\). The theorem says

\[
\frac{|(\text{Cl}_K/\mathcal{C})^G|}{|\text{Cl}_K|} = \frac{\prod_{v \leq \infty} e_v}{[K : k][\Lambda : \Lambda \cap N(K^\times)]}.
\]

If we let \(\mathcal{C}\) and \(D\) be trivial, then \(\Lambda = \mathcal{O}_k^\times\), the unit group of \(\mathcal{O}_k\). The formula becomes the ambiguous class number formula for the class group

\[
\frac{|(\text{Cl}_K)^G|}{|\text{Cl}_K|} = \frac{\prod_{v \leq \infty} e_v}{[K : k][\mathcal{O}_k^\times : \mathcal{O}_k^\times \cap N(K^\times)]}.
\]
(2) If \( m \) is the product of all the real places of \( k \) and \( S \) is the set of infinite places, then \( \text{Cl}^m_{K,S} \) is the narrow class group \( \text{Cl}^+_{K} \) of \( K \). Similarly, \( \text{Cl}^m_{K,S} = \text{Cl}^+_{K} \). Note that \( K^m \) is equal to \( K^+ \), the group of totally positive elements of \( K^\times \). The theorem says

\[
\frac{|(\text{Cl}^+_{K}/\mathcal{C})^G|}{|\text{Cl}^+_{K}/N(\mathcal{C})|} = \prod_{v \mid \infty} e_v |K:k|[\Lambda : \Lambda \cap N(K^+)].
\]

(2.6)

If we further let \( \mathcal{C} \) and \( D \) be trivial, then \( \Lambda = (\mathcal{O}_K^\times)^+ \). The formula becomes the ambiguous class number formula for narrow class groups which was first proved by Chevalley in [3, p. 406].

We now use the formula (2.6) to reprove a classical result of Rédei.

**Example 2.6** (4-rank of narrow class groups of quadratic fields). Let \( K \) be a quadratic number field with discriminant \( d \). Let \( T = \{p_1, \ldots, p_t\} \) be the set of prime numbers ramified in \( K \). Let \( G = \text{Gal}(K/\mathbb{Q}) = \langle \sigma \rangle \). For \( a \in \text{Cl}^+_K \), \( N_{K/\mathbb{Q}}(a) = aa^\sigma = 1 \) as \( \mathbb{Q} \) has class number 1. This implies that \( |\text{Cl}^+_K[2]| := |\{a \in \text{Cl}^+_K \mid a^2 = 1\}| = |(\text{Cl}^+_K)^G| \). The latter term has cardinality \( 2^{t-1} \) by Chevalley’s formula (2.6). In other words, the 2-rank of \( \text{Cl}^+_K \) is \( t - 1 \). The following \( \mathbb{F}_2 \)-matrix is the Rédei matrix:

\[
R := \left( \log(p_i,d)_{p_j} \right)_{1 \leq i,j \leq t}.
\]

Here \( \log : \{\pm 1\} \to \mathbb{F}_2 \) is the logarithm map and \( (p_i,d)_{p_j} \) is the quadratic Hilbert symbol of \( p_i \) and \( d \) at the prime \( p_j \). Note that the sum of each row of this matrix is zero by the product formula of Hilbert symbols. A theorem of Rédei [16, Theorem 3.1] says that the 4-rank of \( \text{Cl}^+_K \) is \( t - 1 - \text{rank}(R) \).

The matrix \( R \) is the transpose of the matrix in [16, Theorem 3.1]. One can check that the logarithm Hilbert symbol \( \log(p_i,d)_{p_j} \) coincides with the logarithm Kronecker symbol \( \left( \frac{p_j^*}{p_i} \right) \in \mathbb{F}_2 \) when \( i \neq j \). Here \( p^* = (-1)^{p-1/2}p \) for \( p \) odd. If \( 2 \mid d, 2^* \) is the number such that \( d = \prod_{p \mid d} p^{*} \).

The proof in [16] uses the explicit construction of the 2-Hilbert class field. We give a proof by applying (2.6) to \( K/\mathbb{Q} \). By definition, the 4-rank of \( \text{Cl}^+_{K} \) is \( \text{rk}_4 \text{Cl}^+_K = \dim_{\mathbb{F}_2} \text{Cl}^+_K[4]/\text{Cl}^+_K[2] \). As we mentioned, \( a^\sigma = a^{-1} \) for \( a \in \text{Cl}^+_K \). It follows that

\[
a \bmod \text{Cl}^+_K[2] \in (\text{Cl}^+_K/\text{Cl}^+_K[2])^G \iff a^\sigma a^{-1} = a^{-2} \in \text{Cl}^+_K[2] \iff a \in \text{Cl}^+_K[4].
\]

This shows \( \text{Cl}^+_K[4]/\text{Cl}^+_K[2] = (\text{Cl}^+_K/\text{Cl}^+_K[2])^G \). We now use (2.6) to compute the order of this group.

Take \( \mathcal{C} = \text{Cl}^+_K[2] \) in (2.6). It is well known that \( \mathcal{C} \) is generated by the ramified prime ideals. We add a proof for this fact here for the sake of completeness. Suppose \( I \in \mathcal{I}_K \) such that its image \( \text{cl}(I) \) is in \( \text{Cl}^+_K[2] = (\text{Cl}^+_K)^G \). Then \( I^\sigma I^{-1} \) is generated by some totally positive element \( \alpha \in K^\times \). Since
Let $D$ be the subgroup of $I_K$ generated by the ramified prime ideals. We have shown that $D$ generates $C$. The group $\Lambda$ of (2.6) is then the subgroup of $\mathbb{Q}^\times$ generated by the ramified prime ideals. We let $\mathbb{Q}$ be a global function field with Galois group $G$. Let $\mathbb{A}^0_K$ be the kernel of the degree map $\deg_K : \mathbb{A}^\times_K \rightarrow \mathbb{Z}$, 

$$\deg_K : \mathbb{A}^\times_K \rightarrow \mathbb{Z}, \quad (x_w)_w \mapsto \sum_w \text{ord}_w(x_w)[k_w : \mathbb{F}_{q'}],$$

where $k_w$ is the residue field of $w$. Let $U_K = \prod_w \mathcal{O}^\times_w$. The class group of divisors and the class group of divisors of degree 0 of $K$ are defined respectively by 

$$\text{Cl}_K = \mathbb{A}^\times_K / U_K K^\times \quad \text{and} \quad \text{Cl}_K^0 = \mathbb{A}^0_K / U_K K^\times.$$

It is well known that $\text{Cl}_K^0$ is finite. The degree map induces the exact sequence

$$0 \rightarrow \text{Cl}_K^0 \rightarrow \text{Cl}_K \xrightarrow{\deg_K} \mathbb{Z} \rightarrow 0.$$ 

See [1, Chapter V, Theorem 5] for the surjectivity of $\deg_K$. We define $\text{Cl}_k$, $\text{Cl}_k^0$, $\deg_k$, $U_k$ for $k$ in the same way. Let $N$ denote the norm map from $K$ to $k$. For a prime divisor $w \in \mathbb{A}^\times_K / U_K$ of $K$, by definition $N(w) = v[k_w : k_v]$ where $v$ is the prime divisor of $k$ below $w$. This implies $\deg_k(N(\mathbb{A}^\times_K)) = [\mathbb{F}_{q'} : \mathbb{F}_{q}]\mathbb{Z}$.

Let $\mathcal{C}$ be a $G$-submodule of $\text{Cl}_K^0$. Choose any subgroup $D$ of $\mathbb{A}^0_K$ such that the image of $D$ in $\text{Cl}_K^0$ is equal to $\mathcal{C}$, and put $\Lambda := k^\times \cap N(D)U_k$ in $\mathbb{A}^\times_k$. Note that $\Lambda$ depends on the choice of $D$, however, its image in
The Chevalley–Gras formula over global fields

\[ k^\times \cap N(D)N(K^\times)U_k/N(K^\times) \]
depends only on \( \mathcal{C} \). In particular, the index
\[ [\Lambda : \Lambda \cap N(K^\times)] \]
depends only on \( \mathcal{C} \). Let \( d(K/k) \in \mathbb{Z} \) denote the positive
generator of the ideal \( \text{deg}_K(\text{Cl}_K^0) \) of \( \mathbb{Z} \).

**Theorem 3.1.** With notation as above, one has

\[ |(\text{Cl}_K^0/\mathcal{C})^G| = |\text{Cl}_k^0/N(\mathcal{C})| \frac{[\mathbb{F}_{q'} : \mathbb{F}_q] \prod_v e_v}{[K : k][\Lambda : \Lambda \cap N(K^\times)]} d(K/k). \]

**Remark 3.2.** Putting \( \mathcal{C} = 0 \) and \( D = 0 \), we obtain the following formula

\[ |(\text{Cl}_K^0)^G| = |\text{Cl}_k^0| \frac{[\mathbb{F}_{q'} : \mathbb{F}_q] \prod_v e_v}{[K : k][\mathbb{F}_{q'}^\times : \mathbb{F}_q^\times \cap N(K^\times)]} d(K/k). \]

When \( q' = q \), this recovers the ambiguous class number formula obtained
by Rosen (see [15, Theorem 8 and Proposition 2]). It is shown in [15, p. 164]
that the invariant \( d(K/k) \) divides another invariant \( \delta(K/k) \) which is easier
to compute. Rosen also computed \( d(K/k) \) in some special cases; see [15,
Theorem 4]. For example, if the cyclic extension \( K/k \) is everywhere, then \( d(K/k) = [K : k] \); see [15, Corollary to Theorem 4].

**Lemma 3.3.** Let \( \sigma \) be a generator of \( G \). For any \( G \)-submodule \( \mathcal{C} \subset \text{Cl}_K^0 \),
we have

\[ d(K/k) = |(\text{Cl}_K/\mathcal{C})^{1-\sigma} / (\text{Cl}_K^0/\mathcal{C})^{1-\sigma}|. \]

**Proof.** This follows from the exact sequences

\[
\begin{array}{ccccccccccc}
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & (\text{Cl}_K^0/\mathcal{C})^G & \rightarrow & (\text{Cl}_K/\mathcal{C})^G & \rightarrow & d(K/k)\mathbb{Z} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{Cl}_K^0/\mathcal{C} & \rightarrow & \text{Cl}_K/\mathcal{C} & \rightarrow & \text{deg}_K \mathbb{Z} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & (\text{Cl}_K^0)^{1-\sigma} & \rightarrow & (\text{Cl}_K/\mathcal{C})^{1-\sigma} & \rightarrow & \mathbb{Z}/d(K/k)\mathbb{Z} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

Now we give an adelic proof of Theorem 3.1. The reader will realize that
the proof is analogous to that of Theorem 2.1.
Proof. Put $\bar{D} = DK^\times U_K$. The facts $H^1(G, A_K^\times) = \hat{H}^{-1}(G, A_K^\times) = 0$ and $(A_K^\times)^{1-\sigma} \subset A_K^0$ give the exact commutative diagram

$$
\begin{array}{cccc}
1 & \longrightarrow & \bar{D} \cap (A_K^\times)^{1-\sigma} & \longrightarrow & \bar{D} & \longrightarrow & N(\bar{D}) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & (A_K^\times)^{1-\sigma} & \longrightarrow & A_K^0 & \longrightarrow & N(A_K^0) & \longrightarrow & 1.
\end{array}
$$

As $(A_K^\times)^{1-\sigma} / \bar{D} \cap (A_K^\times)^{1-\sigma} \cong (\text{Cl}_K / \mathcal{C})^{1-\sigma}$, the snake lemma gives the short exact sequence

$$0 \longrightarrow (\text{Cl}_K / \mathcal{C})^{1-\sigma} \longrightarrow \text{Cl}_K^0 / \mathcal{C} \longrightarrow N(A_K^0) / N(\bar{D}) \longrightarrow 0.$$ 

We remark that $(\text{Cl}_K / \mathcal{C})^{1-\sigma}$ is finite although $\text{Cl}_K / \mathcal{C}$ is infinite. By the above lemma,

$$|([\text{Cl}_K^0 / \mathcal{C}]^G| = |N(A_K^0) / N(\bar{D})| \cdot |(\text{Cl}_K / \mathcal{C})^{1-\sigma} / (\text{Cl}_K^0 / \mathcal{C})^{1-\sigma}| = d(K/k)|N(A_K^0) / N(\bar{D})| = d(K/k)|N(A_K^0) k^\times / N(\bar{D}) k^\times|.$$ 

We prove the last equality as follows. Let $N(x) = N(d)a \in N(A_K^0) \cap N(\bar{D})k^\times$ with $x \in A_K^0, d \in \bar{D}$ and $a \in k^\times$. Then $a = N(xd^{-1}) \in k^\times \cap N(A_K^0) \subset k^\times \cap N(A_K^0) = k^\times \cap N(K^\times)$ by Hasse’s norm theorem. Then the inclusion $N(A_K^0) \subset N(A_K^0) k^\times$ induces an isomorphism

$$N(A_K^0) / N(\bar{D}) \cong N(A_K^0) k^\times / N(\bar{D}) k^\times.$$ 

Consider the short exact sequence

$$0 \longrightarrow N(A_K^0) k^\times / N(\bar{D}) k^\times \longrightarrow A_K^0 / N(\bar{D}) k^\times \longrightarrow A_K^0 / N(A_K^0) k^\times \longrightarrow 0.$$ 

Suppose that $k = \mathbb{F}_q(E)$ is the function field of some curve $E$. We apply the degree map to the Artin reciprocity map $\text{Art} : A_K^\times / k^\times N(A_K^\times) \cong G$ and obtain the following exact commutative diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & A_K^0 / k^\times N(A_K^0) & \longrightarrow & A_K^\times / k^\times N(A_K^\times) & \longrightarrow & \mathbb{Z} / [\mathbb{F}_q', \mathbb{F}_q] \mathbb{Z} & \longrightarrow & 0 \\
0 \longrightarrow & \text{Gal}(K / \mathbb{F}_q(E)) & \longrightarrow & G & \longrightarrow & \text{Gal}(\mathbb{F}_q' / \mathbb{F}_q) & \longrightarrow & 0.
\end{array}
$$

Note that the isomorphism $\varphi_2$ is induced by the Frobenius map. The commutativity of the right diagram follows from [1, Chapter VIII, Theorem 10]. By the Corollary of [1, Chapter VIII, Theorem 10], the map $\varphi_1$ is surjective and hence is an isomorphism.
It follows from (3.1) that

$$\left| \mathbb{A}_k^0 / k^\times N(\mathbb{A}_K^0) \right| = \frac{[K : k]}{[\mathbb{F}_q' : \mathbb{F}_q]}.$$ 

To prove the theorem, it remains to show that

$$\left| \mathbb{A}_k^0 / N(D) k^\times \right| = \left| \text{Cl}_k^0 / N(C) \right| \prod_{v} e_v \left[ \Lambda : \Lambda \cap N(K^\times) \right].$$

The proof is the same as that following equation (2.2) in the proof of Theorem 2.1, and is omitted. \hfill \Box

4. A cohomological variant for $S$-ray class groups

Let $K/k$ be a finite Galois extension of global fields with Galois group $G$. As in Section 2.1, we let $\mathfrak{m}$ be a modulus of $k$, and $S$ be a non-empty finite set of places of $k$ containing all archimedean places which is disjoint from the support of $\mathfrak{m}_f$. In this section we shall discuss a cohomological variant of the ambiguous $S$-ray class number formula of $K/k$; see Theorem 4.1. This formulation has been generalized to an arbitrary algebraic torus $T$ over $k$ by Gonzalez-Aviles [4] when the modulus $\mathfrak{m}$ is trivial, where the present formula may be viewed as the special case $T = \mathbb{G}_{m,k}$. Furthermore, when $K/k$ is cyclic, we explain that Theorem 4.1 is essentially equivalent to Chevalley’s ambiguous class number formula (the case $C = 0$ in Theorem 2.1), thanks to the theorem on the Herbrand quotient of global $S$-units. The argument of the proof of Theorem 4.1 is slightly different from Lang’s exposition [10, Chapter XIII, Section 4].

We keep the notation of Section 2.1. Recall that we write $\text{Cl}_{K,S}^m$ for $\text{Cl}_{K,S}^{m,k}$. Let $E_{K,S}^m$ be the intersection of the group of $S$-units of $K$ with $K^m$. We have the exact sequence

$$1 \rightarrow E_{K,S}^m \rightarrow K^m \xrightarrow{i_S} P_K^{m,S} \rightarrow 1.$$ 

To state the main result, we need to separate the infinite part $\mathfrak{m}_\infty$ of the modulus $\mathfrak{m} = \mathfrak{m}_f \mathfrak{m}_\infty$. Write

$$\mathfrak{m}_\infty = \mathfrak{m}_\infty^r \mathfrak{m}_\infty^c \quad \text{and} \quad \mathfrak{m}_r := \mathfrak{m}_f \mathfrak{m}_\infty^r,$$

where $\mathfrak{m}_\infty^r$ is the product of the real places $v$ dividing $\mathfrak{m}_\infty$ such that $v$ is unramified in $K$ (i.e. $v$ stays real in $K$) and $\mathfrak{m}_\infty^c$ is the product of those $v$ such that $v$ becomes complex in $K$. Note that

$$\left( K^m \right)^G = K^m \cap k^\times = k^{m_r} \quad \text{whence} \quad \left( E_{K,S}^m \right)^G = E_{k,S}^{m_r}.$$
Lemma 4.4. Let \( K/k \) be a finite Galois extension of global fields with Galois group \( G \). Let \( m \) be a modulus of \( k \) and \( m_r \) be as in (4.2). Let \( S \supset V_{k,\infty} \) be a finite non-empty set of places of \( k \) such that \( S \cap S(m_f) = \emptyset \). Then

\[
\frac{|(\text{Cl}_{K,S}^m)^G|}{|\text{Cl}_{K,S}^m|} = \frac{|H^2(G, E_{K,S}^m)|}{|H^1(G, E_{K,S}^m)|} \prod_{v \not\in S \cup S(m)} e_v \cdot |H^1(G, K^m)| \left| \text{Im}(H^2(G, E_{K,S}^m) \to H^2(G, K^m)) \right|.
\]

When the support \( S(m) \) of \( m \) is empty, the term \( H^1(G, K^m) = H^1(G, K^\times) \) is trivial by Hilbert’s Theorem 90. For the general case, we have the following formula.

Proposition 4.2. Let the notation and the assumptions be the same as in Theorem 4.1. Then

\[
H^1(G, K^m) \cong \prod_{v \in S(m_f)} H^1(G, \prod_{w \mid v} (1 + mO_w)).
\]

Furthermore, if the extension \( K/k \) is cyclic, then

\[
|H^1(G, K^m)| = \prod_{v \in S(m_f)} [1 + mO_v : N(\prod_{w \mid v} (1 + mO_w))].
\]

The following proposition relates \( |\text{Cl}_{k,S}^m| \) with \( |\text{Cl}_{k,S}^{m_r}| \). One can find a proof using the language of ideals in [14, Chapter V, Theorem 1.7] for example; we shall present an adelic proof for the sake of completeness.

Proposition 4.3. Let the notation and the assumptions be the same as in Theorem 4.1. Then

1. \( |\text{Cl}_{k,S}^m| = |\text{Cl}_{k,S}| \cdot [E_{k,S} : E_{k,S}^{m_r}]^{-1} \cdot 2^{|S(\infty^0)} \cdot |(O_{k,S}/m_fO_{k,S})^\times| \);
2. \( |\text{Cl}_{k,S}^m| = |\text{Cl}_{k,S}^{m_r}| \cdot [E_{k,S}^{m_r} : E_{k,S}^{m}]^{-1} \cdot 2^{|S(\infty^0)}| \).

Lemma 4.4. We have

1. \( H^1(G, I_K^S) = 0 \) and \( (I_K^S)^G/I_k^S \cong \bigoplus_{v \not\in S} \mathbb{Z}/e_v\mathbb{Z}; \)
2. \( (P_K^m)^G/P_k^{m_r}\cong \text{Ker } \varphi, \text{ where } \varphi \text{ is the natural map } H^1(G, E_{K,S}^m) \to H^1(G, K^m); \)
3. \( |H^1(G, P_K^{m_r})| = |H^1(G, K^m)| \cdot |\text{Ker } \psi| \cdot |\text{Im } \varphi|^{-1}, \text{ where } \psi \text{ is the map } H^2(G, E_{K,S}^m) \to H^2(G, K^m). \)

Proof. (1). For each place \( v \) of \( k \), let \( G_v \) be a decomposition group of \( v \), which is uniquely determined up to conjugate. Since \( G \) acts transitively on the set of places of \( K \) above \( v \), \( I_K^S \cong \bigoplus_{v \not\in S} Zv = \bigoplus_{v \not\in S} \text{Ind}_{G_v}^G Z. \) By Shapiro’s Lemma, \( H^1(G, I_K^S) = \bigoplus_{v \not\in S} H^1(G_v, Z) = 0. \)

For each finite place \( v \) of \( k \), let \( p_v \) be the corresponding prime ideal of \( O_k \), and \( a_v := \prod_{\mathfrak{q}|p_v} \mathfrak{q} \) the prime ideal of \( O_K \) such that \( a_v^{e_v} = p_v O_K \). It is clear that \( (I_K^S)^G \) and \( I_K^S \) are free abelian groups generated by \( a_v \) and \( p_v \) for all \( v \not\in S \), respectively. Thus, \( (I_K^S)^G/I_k^S \cong \bigoplus_{v \not\in S} \mathbb{Z}/e_v\mathbb{Z}. \)
(2). Taking Galois cohomology of the exact sequence (4.1), we get the long exact sequence
\[ 1 \rightarrow (E^m_{K,S})^G \rightarrow (K^m)^G \rightarrow (P^m_{K,\mathcal{S}})^G \rightarrow H^1(G, E^m_{K,S}) \xrightarrow{\varphi} H^1(G, K^m). \]
We have \((K^m)^G = \mathbb{k}^m\). It follows that \((P^m_{K,\mathcal{S}})^G / P^m_{\mathcal{K},\mathcal{S}} \cong \text{Ker} \varphi\).

(3). Taking Galois cohomology of the exact sequence (4.1), we get the long exact sequence
\[ H^1(G, E^m_{K,S}) \xrightarrow{\varphi} H^1(G, K^m) \rightarrow H^1(G, K^m) \rightarrow H^1(G, P^m_{K,\mathcal{S}}) \rightarrow H^2(G, E^m_{K,S}) \xrightarrow{\psi} H^2(G, K^m). \]
and an exact sequence
\[ 0 \rightarrow \text{Im} \varphi \rightarrow H^1(G, K^m) \rightarrow H^1(G, P^m_{K,S}) \rightarrow \text{Ker} \psi \rightarrow 0. \]
From this the statement (3) follows. \(\square\)

**Proof of Theorem 4.1.** Consider the exact sequence of \(G\)-modules
\[ 0 \rightarrow P^m_{K,\mathcal{S}} \rightarrow I^S_{K,(m)\cup \mathcal{S}} \rightarrow \text{Cl}^m_{K,\mathcal{S}} \rightarrow 0. \]
Taking Galois cohomology, we obtain the following exact commutative diagram
\[
\begin{array}{ccccccccc}
0 & \rightarrow & P^m_{\mathcal{K},\mathcal{S}} & \rightarrow & I^S_{K,(m)\cup \mathcal{S}} & \rightarrow & \text{Cl}^m_{K,\mathcal{S}} & \rightarrow & 0 \\
& \downarrow & & \downarrow j & & \downarrow & & \downarrow & \\
0 & \rightarrow & (P^m_{K,\mathcal{S}})^G & \rightarrow & (I^S_{K,(m)\cup \mathcal{S}})^G & \rightarrow & (\text{Cl}^m_{K,\mathcal{S}})^G & \rightarrow & H^1(G, P^m_{K,S}) \rightarrow 0
\end{array}
\]
We remark that the map \(j\) is not injective in general. The snake lemma gives the exact sequence
\[ 0 \rightarrow \text{Ker} j \rightarrow (P^m_{K,\mathcal{S}})^G / P^m_{\mathcal{K},\mathcal{S}} \rightarrow (I^S_{K,(m)\cup \mathcal{S}})^G / I^S_{K,(m)\cup \mathcal{S}} \rightarrow (\text{Cl}^m_{K,\mathcal{S}})^G / \text{Im} j \rightarrow H^1(G, P^m_{K,S}) \rightarrow 0. \]
By Lemma 4.4, we have
\[
\begin{align*}
|\text{(Cl}^m_{K,\mathcal{S}})^G / \text{Cl}^m_{K,\mathcal{S}}| &= |(\text{Cl}^m_{K,\mathcal{S}})^G / \text{Im} j| \cdot |\text{Ker} j|^{-1} \\
&= \frac{|H^1(G, K^m)| \cdot |\text{Ker} \psi| \cdot |\text{Im} \varphi|^{-1}}{|\text{Ker} \varphi|} \cdot \prod_{v \notin (m)\cup \mathcal{S}} e_v \\
&= \frac{|H^1(G, K^m)| \cdot |\text{Ker} \psi|}{|H^1(G, E^m_{K,S})|} \cdot \prod_{v \notin (m)\cup \mathcal{S}} e_v \\
&= \frac{|H^1(G, K^m)| \cdot |H^2(G, E^m_{K,S})|}{|H^1(G, E^m_{K,S})||\text{Im} \psi|} \cdot \prod_{v \notin (m)\cup \mathcal{S}} e_v.
\end{align*}
\]
This completes the proof of Theorem 4.1. □

Proof of Proposition 4.2. The facts $H^1(G, \mathbb{A}_K^\times) = H^1(G, K^\times) = 0$ and $(\mathbb{A}_K^\times)^G = \mathbb{A}_k^\times$ will be used. Taking Galois cohomology of the short exact sequence

$$1 \rightarrow K^m \rightarrow K^\times \rightarrow K^\times/K^m \rightarrow 1,$$

we get the exact sequence

$$1 \rightarrow k^\times/k^m \rightarrow (K^\times/K^m)^G \rightarrow H^1(G, K^m) \rightarrow 1.$$

Recall that $\mathbb{A}_K^m = \{(a_w)_w \in \mathbb{A}_K^\times \mid a_w \in 1 + m\mathcal{O}_w \text{ for } w|\mathfrak{m}\}$. We have $K^m = K^\times \cap \mathbb{A}_K^m$ and $\mathbb{A}_K^\times = K^\times \mathbb{A}_K^m$ by the weak approximation theorem. This gives a natural isomorphism

$$K^\times/K^m \cong \mathbb{A}_K^\times/\mathbb{A}_K^m.$$

Taking Galois cohomology of $1 \rightarrow \mathbb{A}_K^m \rightarrow \mathbb{A}_K^\times \rightarrow \mathbb{A}_k^\times/\mathbb{A}_K^m \rightarrow 1$, we get the exact sequence

$$1 \rightarrow \mathbb{A}_k^\times/\mathbb{A}_K^m \rightarrow (\mathbb{A}_K^\times/\mathbb{A}_K^m)^G \rightarrow H^1(G, \mathbb{A}_K^m) \rightarrow 1.$$

This implies that

$$H^1(G, K^m) \cong H^1(G, \mathbb{A}_K^m).$$

(Note that this isomorphism can also be deduced from taking Galois cohomology of the short exact sequence $1 \rightarrow K^m \rightarrow \mathbb{A}_K^m \rightarrow \mathbb{A}_K^m/K^m = \mathbb{A}_K^\times/K^\times \rightarrow 1$ and using the facts that $(\mathbb{A}_K^\times/K^\times)^G = \mathbb{A}_k^\times/k^\times$ and $H^1(G, \mathbb{A}_K^\times/K^\times) = 0$.) Let $\mathbb{A}_K^{S(\mathfrak{m})}$ be the subgroup of $\mathbb{A}_K^\times$ such that

$$\mathbb{A}_K^\times = \mathbb{A}_K^{S(\mathfrak{m})} \times \prod_{w|\mathfrak{m}} K_w^\times$$

as a direct product.

We have

$$\mathbb{A}_K^m = \mathbb{A}_K^{S(\mathfrak{m})} \times \prod_{w|\mathfrak{m}} 1 + m\mathcal{O}_w \quad \text{and} \quad H^1(G, \mathbb{A}_K^{S(\mathfrak{m})}) = 0.$$

It follows that

$$H^1(G, \mathbb{A}_K^m) \cong \prod_{v|\mathfrak{m}} H^1(G, \prod_{w|v} 1 + m\mathcal{O}_w).$$

Observe that $H^1(G, \prod_{w|v} 1 + m\mathcal{O}_w) \cong H^1(G_v, 1 + m\mathcal{O}_{w_1})$ by Shapiro’s Lemma, where $w_1$ is a place of $K$ over $v$ and $G_v = \text{Gal}(K_{w_1}/k_v)$. Also note that $H^1(G_v, 1 + m\mathcal{O}_w) = 0$ when $v$ is real. This proves the first part of Proposition 4.2.

Assume that $K/k$ is cyclic. For a finite place $v$ of $k$, the Herbrand quotient of the $\text{Gal}(K_w/k_v)$-module $\mathcal{O}_w^\times$ is 1; see [9, Chapter IX, Section 3, Lemma 4].
Note that $1 + m\mathcal{O}_w$ has finite index in $\mathcal{O}_w^\times$. We then have
\[ |H^1(G_v, 1 + m\mathcal{O}_w)| = |H^2(G_v, 1 + m\mathcal{O}_w)| = [1 + m\mathcal{O}_v : N_{K_w/k_v}(1 + m\mathcal{O}_w)]. \]
This completes the proof of Proposition 4.2. \[ \square \]

**Proof of Proposition 4.3.** As in the proof of Theorem 2.1, we have
\[ \text{Cl}^m_{k,S} \cong \mathbb{A}_{k,S}^\times /k^\times U^m_{k,S}, \]
where
\[ U^m_{k,S} = \prod_{v \nmid m, v \notin S} \mathcal{O}_v^\times \prod_{v \mid m} 1 + m\mathcal{O}_v \prod_{v \in S, v \nmid m_\infty} k_v^\times. \]
It follows that
\[ \frac{|\text{Cl}^m_{k,S}|}{|\text{Cl}_{k,S}|} = |k^\times U_{k,S}/k^\times U^m_{k,S}|. \]
Consider the exact sequence
\[ (4.4) \quad 1 \longrightarrow U_{k,S} \cap k^\times U^m_{k,S}/U^m_{k,S} \longrightarrow U_{k,S}/U^m_{k,S} \longrightarrow k^\times U_{k,S}/k^\times U^m_{k,S} \longrightarrow 1. \]
Clearly the middle term has order
\[ 2^{|S(m_\infty)|} \cdot \prod_{v \mid m_f} |(\mathcal{O}_v^\times /1 + m\mathcal{O}_v)| = 2^{|S(m_\infty)|} \cdot \prod_{v \mid m_f} |(\mathcal{O}_v/m\mathcal{O}_v)^\times| \]
\[ = 2^{|S(m_\infty)|} \cdot |(\mathcal{O}_{k,S}/m_f\mathcal{O}_{k,S})^\times|. \]
The last equality follows from the Chinese remainder theorem.

Suppose $A, B$ and $C$ are subgroups of some abelian group (written multiplicatively) such that $C \subset A$. Then it is direct to check that the natural map $B \cap A \hookrightarrow A \cap BC$ induces an isomorphism
\[ B \cap A/B \cap C \cong A \cap BC/C. \]
Applying this to $A = U_{k,S}, B = k^\times$ and $C = U^m_{k,S}$ shows that the first term of (4.4) is isomorphic to $E_{k,S}/E^m_{k,S}$. This proves formula (1).

Formula (2) follows from applying (1) to the modulus $m$ and $m_r$ respectively. \[ \square \]

**Remark 4.5.** Suppose that $K/k$ is cyclic. We can identify $H^2(G, M)$ with the Tate cohomology $\tilde{H}^2(G, M) \cong \tilde{H}^0(G, M)$ by periodicity for any $G$-module $M$. The theorem on the Herbrand quotient of global units says (see [9, Chapter IX, Section 4, Corollary 2])
\[ \frac{|H^2(G, E_{K,S})|}{|H^1(G, E_{K,S})|} = \prod_{v \in S} |G_v| / [K : k]. \]
Here $G_v$ is the decomposition group of $v$ and $E_{K,S}$ is the group of $S$-units of $K$. Note that $E^m_{K,S}$ has finite index in $E_{K,S}$. So they have the same
Herbrand quotient. Since \( H^2(G, E_{K,S}^m) = E_{K,S}^{mr}/N(E_{K,S}^m) \) and \( H^2(G, K^m) = k^{mr}/N(K^m) \), we have

\[
|\text{Im} \psi| = |E_{K,S}^{mr}/N(K^m)| = |E_{K,S}^{mr} : E_{K,S}^{mr} \cap N(K^m)|.
\]

Thus, by Theorem 4.1 and Proposition 4.2, we obtain the ambiguous class number formula for \( S \)-ray class groups

\[
\frac{|(C_{K,S}^m)^G|}{|C_{K,S}^{mr}|} = \frac{\prod_{v \in S} e_v f_v \cdot \prod_{v \in S(m_f)} [1 + m\mathcal{O}_v : N(\prod_{w|v}(1 + m\mathcal{O}_w))] \cdot \prod_{v \notin S(m) \cup S} e_v}{[K : k][E_{K,S}^{mr} : E_{K,S}^{mr} \cap N(K^m)]}.
\]

To compare this formula to Theorem 2.1, we first note that \( E_{K,S}^{mr} \cap N(K^m) = E_{K,S}^m \cap N(K^m) \). Then by Proposition 4.3(2), the above formula gives

\[
\frac{|(C_{K,S}^m)^G|}{|C_{K,S}^{mr}|} = \frac{\prod_{v \in S \setminus S(m_\infty)} e_v f_v \cdot \prod_{v \in S(m_f)} [1 + m\mathcal{O}_v : N(\prod_{w|v}(1 + m\mathcal{O}_w))] \cdot \prod_{v \notin S(m) \cup S} e_v}{[K : k][E_{K,S}^m : E_{K,S}^m \cap N(K^m)]}.
\]

This is the formula in Theorem 2.1 when \( C = 0 \).

Acknowledgments. We thank Professor Georges Gras for his helpful comments and Professor Christian Maire for informing us about his thesis [13], which is related to our Section 4. We are also grateful to the anonymous referee for their helpful comments and encouragement to extend the previous results of Section 4 to \( S \)-ray class groups; these improve the paper significantly.

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