Chern-Simons Vortices

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Abstract. In (2 + 1) dimensions, the Maxwell term $-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$ can be replaced by the Chern-Simons three-form $(\kappa/4)\epsilon^{\alpha\beta\gamma}A_{\alpha}F_{\beta\gamma}$, yielding a novel type of ‘electromagnetism’. This has been proposed for studying the Quantum Hall Effect as well as High-Temperature Superconductivity. The gauge field can be coupled to a scalar field either relativistically or non-relativistically. In both cases, one admits finite-energy, vortex solutions.

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1. Introduction

The phenomenological description of ‘ordinary’ superconductivity is provided by Landau-Ginzburg theory [1]: the Cooper pairs formed by the electrons are represented by a scalar field, whose charge is twice that of the electron. The scalar fields interact through their electromagnetic fields, governed by the Maxwell equations. The theory admits static, vortex-like solutions [2].

Landau-Ginzburg theory is non-relativistic; a relativistic version is provided by the Abelian Higgs model [3], which admits again static and purely magnetic vortex-type solutions.

In the eighties, however, new phenomena were discovered, namely the Quantum Hall Effect [4] and High-Temperature Superconductivity [5]. None of these is described by Landau-Ginzburg theory. Hence the need for a new theory.

Both phenomena mentioned above are essentially planar. The mathematical room for a modification is provided by the three-form

$$\frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}A_{\alpha}F_{\beta\gamma},$$

introduced by Chern and Simons [6] in their study of secondary characteristic classes. This form, which only exists in (2 + 1) dimensions, can be added to the conventional Maxwell term $-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$ [7]. The use of the Chern-Simons term has in fact been proposed in the above contexts [8], [9].

Since the Chern-Simons term is of lower-order in derivatives, for large distances it dominates the Maxwell term. Thus, for describing long-range phenomena, the Maxwell term can even be dropped: the dynamics of the ‘electromagnetic’ field will be described by the Chern-Simons term (1.1) alone.

Below, we review various aspects of Chern-Simons gauge theory. First, we explain the construction of relativistic topological vortices. The non-relativistic limit is physically relevant owing to the intrinsically non-relativistic character of condensed matter physics. It also provides an explicitly solvable model. Finally, we consider spinorial models. Again, explicit solutions are found.

2. Relativistic Chern-Simons vortices

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Let us consider a complex scalar field, \( \psi \), coupled to an abelian gauge field, described by the vector potential \([1\text{-form}], A_\mu dx^\mu\), defined on \((2+1)\)-dimensional Minkowski space with the metric \((g_{\mu\nu}) = \text{diag}(1,-1,-1)\), the coordinates being \(x^0 = t\) and \((x_i) = \vec{x}\). The action \( S = \int d^3x \mathcal{L} \) is given by \([10]\)

\[
\mathcal{L} = \frac{1}{2} D_\mu \psi (D^\mu \psi)^* + \frac{1}{4\kappa} \epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} - U(\psi),
\]

where \( D_\mu \psi = \partial_\mu \psi - ieA_\mu \psi \) denotes the covariant derivative, the constant \( e \) is the electric charge of the field \( \psi \), and \( \kappa \) is a constant. The potential \( U(\psi) \) is chosen, for later convenience, to be

\[
U(\psi) = \frac{\lambda}{4} |\psi|^2 (|\psi|^2 - 1)^2.
\]

The Euler-Lagrange equations of the action (2.1) read

\[
\begin{aligned}
\frac{1}{2} D_\mu D^\mu \psi &= - \frac{\delta U}{\delta \psi^*} = - \frac{\lambda}{4} (|\psi|^2 - 1)(3|\psi|^2 - 1) \psi, \\
\frac{1}{2\kappa} \epsilon^{\mu\alpha\beta} F_{\alpha\beta} &= e j^\mu,
\end{aligned}
\]

where \( F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \) is the ‘electromagnetic’ field, and \( j^\mu \equiv (g, \vec{j}) \) is the current

\[
j^\mu = \frac{1}{2i} [\psi^* D^\mu \psi - \psi (D^\mu \psi)^*].
\]

The first of the equations in (2.3) is the Non-Linear Klein-Gordon equation (NLKG) familiar from the Abelian Higgs model [3]; the second, called Field-Current Identity (FCI), replaces the Maxwell equations.

It follows from the Bianchi identity, \( \epsilon^{\alpha\beta\gamma} \partial_\alpha F_{\beta\gamma} = 0 \), that the current (2.4) is conserved, \( \partial_\mu j^\mu = 0 \).

Due to the presence of the vector potential in the Chern-Simons term, the Lagrangian (2.1) is manifestly not invariant with respect to a local gauge transformation

\[
(2.5) \quad \psi(x) \rightarrow e^{ie\lambda(x)} \psi(x), \quad A_\mu \rightarrow A_\mu + \partial_\mu \lambda.
\]

Using the Bianchi identity, the change can be written, however, as \( \partial_\alpha ((\kappa/4) \epsilon^{\alpha\beta\gamma} F_{\beta\gamma} \lambda) \). This is a surface term, which does not change the equations of motion. The two other terms in (2.1) are gauge-invariant.

The same conclusion can also be reached by looking at the field equations: the FCIs are invariant, since both the field strength and the current are invariant; both sides of the NLKG get in turn be multiplied by \( e^{ie\lambda} \).

**Finite-energy configurations**

Let us consider a static field configuration \((A_\mu, \psi)\). The energy, defined as the space integral of the time-time component of the energy-momentum tensor associated with the Lagrangian (2.1), is

\[
(2.6) \quad E \equiv \int d^2 \vec{x} T^{00} = \int d^2 \vec{x} \left[ \frac{1}{2} D_t \psi (D^t \psi)^* - \frac{1}{2} e^2 A_0^2 |\psi|^2 + \kappa A_0 B + U(\psi) \right],
\]

where \( B = -F_{12} \) is the magnetic field. Note that this expression is not positive definite. Observe, however, that the static solutions of the equations of movement (2.3) are stationary points of the energy. Variation of (2.6) w. r. t. \( A_0 \) yields one of the equations of motion, namely

\[
(2.7) \quad -e^2 A_0 |\psi|^2 + \kappa B = 0.
\]
Eliminating $A_0$ from (2.6) using this equation, we obtain the positive definite energy functional

$$E = \int d^2\vec{r} \left[ \frac{1}{2} D_\mu \psi (D^\mu \psi)^* + \frac{\kappa^2}{2\epsilon^2} \frac{B^2}{|\psi|^2} + U(\psi) \right].$$

We are interested in static, finite-energy configurations. Finite energy at infinity is guaranteed by the conditions \(^{(1)}\)

\begin{align*}
  i.) & \quad |\psi|^2 - 1 = o(1/r), \\
  ii.) & \quad B = o(1/r), \quad r \to \infty. \\
  iii.) & \quad \bar{D}\psi = o(1/r).
\end{align*}

Therefore, the $U(1)$ gauge symmetry is broken for large $r$. In particular, the scalar field $\psi$ is covariantly constant, $\bar{D}\psi = 0$. This equation is solved by parallel transport,

$$\psi(\vec{x}) = \exp \left[ i \int_{\vec{x}_0}^{\vec{x}} eA_i d\vec{x}^i \right] \psi_0,$$

which is well-defined whenever

$$\oint eA_i d\vec{x} = \int_{\mathbb{R}^2} d^2\vec{x} eB \equiv e\Phi = 2\pi n, \quad n = 0, \pm 1, \ldots.$$

Thus, the magnetic flux is quantized.

By i.), the asymptotic values of the Higgs field provide us with a mapping from the circle at infinity into the vacuum manifold, which is again a circle, $|\psi|^2 = 1$. Since the vector potential behaves asymptotically

$$A_j \simeq \frac{i}{e} \partial_j \log \psi,$$

the integer $n$ in Eq. (2.10) is precisely the winding number of this mapping; $n$ is called the topological charge (or vortex number).

Spontaneous symmetry breaking generates mass \([12]\). This occurs in a somewhat unusual guise, though. Expanding $j^\mu$ around the vacuum expectation value of $\psi$ we find $j^\mu = -eA^\mu$ so that the FCI in (2.3) is approximately

$$\frac{1}{2\kappa} e^{\mu\alpha\beta} F_{\alpha\beta} \simeq -e^2 A^\mu.$$

Hence $F_{\alpha\mu} = -\left( \frac{e^2}{\kappa} \right) \epsilon_{\alpha\mu\beta} A^\beta$. Inserting here $F_{\alpha\beta}$ and deriving by $\partial^\alpha$ we find that the gauge field $A^\mu$ satisfies the Klein-Gordon equation

$$\Box A^\mu = - \left( \frac{e^2}{\kappa} \right)^2 A^\mu,$$

showing that the mass of the gauge field is

$$m_A = \frac{e^2}{\kappa}.$$

The Higgs mass is found in turn in the usual way: the expectation value of the Higgs field can be chosen as $\psi_0 = (1, 0)$. Expanding $\psi$ as $(\psi_r, \psi_\theta) = (1 + \varphi, \theta)$ we get

$$U = U(1) + \frac{\delta U}{\delta |\psi|} \bigg|_{|\psi|=1} \varphi + \frac{1}{2} \frac{\delta^2 U}{\delta |\psi|^2} \bigg|_{|\psi|=1} \varphi^2,$$

since $|\psi| = 1$ is a critical point of $U$. We conclude that the mass of the Higgs particle is

$$m_\psi^2 = \frac{\delta^2 U}{\delta |\psi|^2} \bigg|_{|\psi|=1} = 2\lambda.$$

(This result can also be seen by considering the radial equation (2.17) below).

\(^{(1)}\) These conditions are in no way necessary; they yield the so-called topological solitons. Non-topological solutions are constructed in Ref. \([11]\).
Radially symmetric solutions\(^{10}\)

For the radially symmetric Ansatz
\[
(2.16) \quad A_0 = A_0(r), \quad A_r = 0, \quad A_\theta = A(r), \quad \psi(r) = f(r)e^{-in\theta},
\]
the equations of motion (2.3) read
\[
(2.17) \quad \frac{1}{r} \frac{dA}{dr} + \frac{e^2}{\kappa} f^2 A_0 = 0, \quad \frac{dA_0}{dr} + \frac{e^2}{\kappa} f^2 \left( \frac{n}{e} + A \right) = 0, \quad \frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \frac{e^2}{r^2} \left( \frac{n}{e} + A \right)^2 f + e^2 A_0^2 f = -\frac{\lambda}{4} f (1 - f^2)(1 - 3f^2),
\]
with asymptotic conditions
\[
(2.18) \quad \lim_{r \to \infty} A(r) = -\frac{n}{e}, \quad \lim_{r \to \infty} f(r) = 1, \quad \lim_{r \to 0} A(r) = 0, \quad \lim_{r \to 0} f(r) = 0.
\]
This is either seen by a direct substitution into the equations, or by re-writing the energy as
\[
(2.19) \quad E = 2\pi \int_0^\infty dr \left\{ \frac{r}{2} (f')^2 + \frac{a^2}{2r} f^2 + \frac{\kappa^2}{2e^4 f^2} \frac{(a')^2}{r} + U(f) \right\},
\]
where \(a = eA + n\). The upper equation in (2.17) is plainly the radial form of (2.7). Then variation of (2.19) with respect to \(a\) and \(f\) yields the two other equations in (2.17).

Approximate solutions can be obtained by inserting the asymptotic value, \(f \simeq 1\), into the first two equations:
\[
\frac{a'}{r} + \frac{e^2}{\kappa} A_0 = 0, \quad A_0' + \frac{e^2 a}{\kappa} r = 0,
\]
from which we infer that
\[
\frac{d^2 A_0}{d\rho^2} + \frac{1}{\rho} \frac{dA_0}{d\rho} - A_0 = 0, \quad \rho \equiv (e^2 / \kappa) r.
\]
This is the modified Bessel equation [Bessel equation of imaginary argument] of order zero. Hence
\[
(2.20) \quad A_0 = CK_0\left(\frac{e^2}{\kappa} r\right).
\]
Similarly, for \(a = n/e + A\) we find, putting \(\alpha = a/r\),
\[
\alpha'' + \frac{\alpha'}{\rho} - \left(1 + \frac{1}{\rho^2}\right)\alpha = 0,
\]
which is Bessel’s equation of order 1 with imaginary argument. Thus \(\alpha = CK_1(\rho)\) so that
\[
(2.21) \quad A = -\frac{n}{e} + C\frac{e^2}{\kappa} r K_1\left(\frac{e^2}{\kappa} r\right).
\]
Another way of deriving this result is to express \(A\) from from the middle equation in (2.17),
\[
A = -\frac{n}{e} - \frac{\kappa}{e^2} r \frac{d}{dr} A_0.
\]
The consistency with (2.21) follows from the recursion relation \( K_0' = -K_1 \) of the Bessel functions.

An even coarser approximation is obtained by eliminating the \( a' \) term by setting \( a = ur^{-1/2} \) and dropping the terms with inverse powers of \( r \). Then both equations reduce to

\[
(2.22) \quad u'' = \left( \frac{e^2}{\kappa} \right)^2 u \quad \implies \quad A_0 = a = \frac{C}{\sqrt{r}} e^{-m_A r},
\]

which shows that the fields approach their asymptotic values exponentially, with characteristic length determined by the gauge field mass.

The deviation of \( f \) from its asymptotic value, \( \varphi = 1 - f \), is found by inserting \( \varphi \) into the last eqn. of (2.17); developing to first order in \( \varphi \) we get

\[
(2.23) \quad \varphi'' + \frac{1}{r} \varphi' - 2\lambda \varphi \simeq 0 \quad \implies \quad \varphi = C K_0(\sqrt{2\lambda} r),
\]

whose asymptotic behaviour is again exponential with characteristic length \((m_\psi)^{-1}\),

\[
(2.24) \quad \varphi = \frac{C}{\sqrt{r}} e^{-m_\psi r}.
\]

The penetration depths of the gauge and scalar fields are therefore

\[
(2.25) \quad \eta = \frac{1}{m_A} = \frac{e^2}{\kappa} \quad \text{and} \quad \xi = \frac{1}{m_\psi} = \frac{1}{\sqrt{2\lambda}},
\]

respectively. For small \( r \) instead, inserting the developments in powers of \( r \), we find

\[
(2.26) \quad f(r) \sim f_0 r^{|n|} + \ldots,
\]

\[
A_0 \sim \alpha_0 - \frac{e f_0^2 \kappa}{2\kappa |n|} r^{2|n|} + \ldots,
\]

\[
A \sim -\frac{e^2 f_0^2 \alpha_0}{2\kappa (|n| + 1)} r^{2|n|+2} + \ldots,
\]

where \( \alpha_0 \) and \( f_0 \) are constants. In summary,

\[
(2.27) \quad |\psi(r)| \equiv f(r) \propto \begin{cases} r^{|n|} & r \sim 0 \\ 1 - C r^{-1/2} e^{-m_\psi r} & r \to \infty \end{cases}
\]

\[
|E(r)| = |A_0'(r)| \propto \begin{cases} r^{2|n|-1} & r \sim 0 \\ C r^{-1/2} e^{-m_A r} + \text{lower order terms} & r \to \infty \end{cases}
\]

\[
|B(r)| = \frac{|A'|}{r} \propto \begin{cases} r^{2|n|} & r \sim 0 \\ C r^{-3/2} e^{-m_A r} + \text{lower order terms} & r \to \infty \end{cases}
\]

**Self-duality**

Let us suppose that the fields have equal masses, \( m_\psi^2 = m_A^2 \equiv m^2 \) and hence equal penetration depths.

Then the Bogomolny trick applies, i.e., the energy can be rewritten in the form

\[
(2.28) \quad E = \int d^2 \vec{r} \left[ \frac{1}{2} (D_1 + iD_2) \psi \right]^2 + \frac{1}{2} \left| \frac{\kappa}{e} \frac{B}{\psi} \right| \pm \frac{e^2}{2\kappa} \psi^* (1 - |\psi|^2)^2 \right] \equiv \int d^2 \vec{r} \frac{eB}{2} (1 - |\psi|^2).
\]
The last term can also be presented as

$$\pm \frac{eB}{2} \mp \frac{1}{2} \nabla \times \vec{J}.$$  

The integrand of the $B$-term yields the magnetic flux; the second is transformed, by Stokes’ theorem, into the circulation of the current at infinity which vanishes since all fields drop off at infinity by assumption. Its integral is therefore proportional to the magnetic flux, $\pm e\Phi/2$. Since the first integral is non-negative, we have, in conclusion,

$$E \geq \frac{e|\Phi|}{2} = \pi |n|,$$

equality being only attained if the ‘self-duality’ equations

$$D_1 \psi = \mp iD_2 \psi \quad \text{and} \quad eB = \pm \frac{m^2}{2} |\psi|^2 (1 - |\psi|^2)$$

hold. It is readily verified that the solutions of equations (2.7) and (2.31) solve automatically the non-linear Klein-Gordon equation.

For the radial Ansatz (2.16) the self-duality equations become

$$f' = \pm \frac{a}{r} f, \quad \frac{a'}{r} = \pm \frac{1}{2} m^2 f^2 (f^2 - 1),$$

where we introduced again $a = eA + n$. Deriving the first of these equations and using the second one, we for $f$ we get the ‘Liouville - type’ equation

$$\Delta \log f = \frac{m^2}{2} f^2 (f^2 - 1).$$

Another way of obtaining the first-order eqns. (2.32) is to rewrite, for

$$U(f) = (m^2/8) f^2 (f^2 - 1)^2,$$

the energy as

$$\pi \int_0^\infty r dr \left\{ \left[ f' + \frac{a}{r} f \right]^2 + \frac{1}{m^2 f^2} \left[ \frac{a'}{r} + \frac{m^2}{2} f^2 (f^2 - 1) \right]^2 \right\} \pm \pi (af^2)|_0^\infty \mp \pi a|_0^\infty.$$

The boundary conditions read

$$a(\infty) = 0, \quad f(\infty) = 1,$$

$$a(0) = n, \quad f(0) = 0,$$

and thus $E \geq \pi |n|$ as before, with equality attained iff the equations (2.32) hold.

For $n = 0$ the only solution is the vacuum,

$$f \equiv 1, \quad A \equiv 1.$$

To see this, note that the boundary conditions at infinity are $f(\infty) = 1$ and $A(\infty) = 0$. Let now $f(r)$, $A(r)$ denote an arbitrary finite-energy configuration and consider

$$f_r(r) = f(r), \quad A_r(r) = \tau A(r)$$
where $\tau > 0$ is a real parameter. This provides us with a 1-parameter family configurations with finite energy

\begin{equation}
E_\tau = 2\pi \int_0^\infty dr \left\{ \frac{r}{2} \left( f' \right)^2 + \tau^2 \left[ \frac{a^2}{2r} f'^2 + \frac{r}{2m^2} \left( \frac{a'}{r} \right)^2 \right] + U(f) \right\},
\end{equation}

which is a monotonic function of $\tau$, whose minimum is at $\tau = 0$ i.e. for $a \equiv 0$. Then Eq. (2.32) implies that $f' \equiv 0$ so that $f \equiv 1$ is the only possibility.

Let us assume henceforth that $n \neq 0$. No analytic solution has been found so far. To study the large-$r$ behaviour, put $\varphi \equiv 1 - f$. Inserting $f \simeq 1$, Eq. (2.32) reduces to

\begin{equation}
\varphi' = \mp \frac{a}{r}, \quad \frac{a'}{r} = \mp m^2 \varphi.
\end{equation}

From here we get

\[ \varphi'' + \frac{1}{r} \varphi' - m^2 \varphi = 0 \quad \Rightarrow \quad \varphi = C K_0(mr). \]
\[ a'' - \frac{1}{r} a' - m^2 a = 0 \quad \Rightarrow \quad a = C mr K_1(mr). \]

Thus, for large $r$,

\begin{equation}
f \simeq 1 - C K_0(mr) \quad \text{and} \quad A \simeq -\frac{n}{e} + C mr K_1(mr)
\end{equation}

with some constant $C$. For small $r$ instead, Eq. (2.32), yields, to $O(r^{5|n|+1})$,

\begin{equation}
f(r) = f_0 r^{|n|} - \frac{f_0^2 m^2}{2(2|n| + 2)} r^{3|n|+2} + O(r^{5|n|+2}).
\end{equation}
\begin{equation}
A = - \frac{f_0^2 m^2}{2(2|n| + 2)e} r^{2|n|+2} + \frac{f_0^2 m^2}{2(4|n| + 2)e} r^{4n+2} + O(r^{4|n|+4}).
\end{equation}

The result is consistent with (2.26) since the constant $\alpha_0$ is now $\alpha_0 = m/2e = e/2\kappa$.

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**Fig. 1.** The scalar field of the radially symmetric charge-2 relativistic vortex.
Fig. 2. The magnetic field of the radially symmetric charge-2 relativistic vortex. Note that $B = 0$ where the scalar field vanishes: the vortex has a doughnut-like shape.

Let us mention that Eq. (2.33) is actually valid in full generality, without the assumption of radial symmetry. Expressing in fact the vector-potential from $(D_1 \pm iD_2)\psi = 0$ as

$$e\vec{A} = \vec{\nabla} \left( \text{Arg} \psi \right) \pm \vec{\nabla} \times \log |\psi|$$

and inserting into the second equation in (2.31), we get again (2.33), with $|\psi|$ replacing $f$. Index-theoretical calculations show that, for topological charge $n$, Eqn. (2.33) admits a $2|n|$ parameter family of solutions [10].

3. Non-relativistic vortices

The non-relativistic limit of the above system is found [13,14] by setting

$$\psi = e^{-imc^2t}\Psi + e^{+imc^2t}\bar{\Psi},$$

where $\Psi$ and $\bar{\Psi}$ denote the particles and antiparticles, respectively. Inserting this into the action, dropping the oscillating terms and only keeping those of order $1/c$, we see that both the particles and antiparticles are conserved. We can therefore consistently set $\bar{\Psi} = 0$. The remaining matter Lagrangian reads

$$\mathcal{L}_{\text{matter}} = i\Psi^* D_t \Psi - \frac{1}{2m} |\vec{D}\Psi|^2 + \frac{\Lambda}{2}(\Psi^*\Psi)^2,$$

where $\Lambda = e^2/mc|\kappa|$. One can show that the theory is indeed non-relativistic [15], [16]. In what follows, we put $c = 1$.

Let the constant $\Lambda$ be arbitrary. Variation of $\int \mathcal{L}_{\text{matter}}$ w. r. t. $\Psi^*$ yields the gauged non-linear Schrödinger equation

$$i\partial_t \Psi = \left[ -\frac{\vec{D}^2}{2m} - eA_0 - \Lambda \Psi^*\Psi \right] \Psi.$$  

A fully self-consistent system is obtained by adding the matter action to the Chern-Simons action (1.1); the variational equations of this latter are clearly the FCIs in Eq. (2.3), written in non-relativistic notations as

$$B \equiv e^{ij}\partial_i A_j = -\frac{e}{\kappa} \varrho, \quad E^i \equiv -\partial_i A^0 - \partial_0 A^i = \frac{e}{\kappa} e^{ij} J^j,$$
where the current is now

\[ j^\mu \equiv (\varrho, \vec{\jmath}) = (\Psi^* \Psi, \frac{1}{2m \iota} [\Psi^* \tilde{D} \Psi - \Psi (\tilde{D} \Psi)^*]). \]

This is legitimate, since the Chern-Simons term is also non-relativistic: it is in fact invariant with respect to any coordinate transformation.

**Self-dual vortex solutions**

We would again like to find static soliton solutions. The construction of an energy-momentum tensor is now more subtle, because the theory is non-relativistic. This is nevertheless possible [15], [16]. One gets the energy functional

\[ E = \int d^2 x \left( \frac{|\tilde{D}_1 \Psi|^2}{2m} - \frac{\Lambda}{2} (\Psi^* \Psi)^2 \right). \]

Applying once more the Bogomolny trick, this is also written as

\[ E = \int d^2 x \left( \frac{|D_1 \pm i D_2 \Psi|^2}{2m} - \frac{1}{4} \left( \Lambda - \frac{e^2}{m|\kappa|} \right) (\Psi^* \Psi)^2. \]

Thus, for the specific value

\[ \Lambda = \frac{e^2}{m|\kappa|} \]

of the non-linearity, — the same as the one we obtained above by taking the non-relativistic limit of the self-dual relativistic theory — the second term vanishes. Then the energy is positive definite, whose absolute minimum — zero — is attained for "self-dual" fields i.e. for such that

\[ (D_1 \pm i D_2) \Psi = 0. \]

Separating the phase as \( \Psi = e^{i\omega} \sqrt{\varrho} \), we get

\[ \tilde{A} = \nabla \omega + \frac{1}{2e} \nabla \times \log \varrho, \]

where \( \varrho \) solves the Liouville equation,

\[ \Delta \log \varrho = \pm \frac{2e^2}{\kappa} \varrho, \]

cf. (2.33). Physically admissible solutions arise when the r.h.s. is negative. Hence the upper sign has to be chosen when \( \kappa < 0 \) and the lower sign when \( \kappa > 0 \). Then the general solution reads

\[ \varrho = \frac{4|\kappa|}{e^2} \frac{|f'|^2}{[1 + |f'|^2]^2}, \]

where \( f \) is a meromorphic function. In particular, for

\[ f(z) = \sum_{i=1}^{n} \frac{c_i}{z - z_i}, \]

where the \( c_i \) and the \( z_i \) are \( 2n \) complex numbers, we get a \( 4n \)-parameter family of physically admissible solutions, with vortex number \( n \).
Note that $\varrho$ vanishes at the poles of $f$; these points represent the ‘positions’ of the vortices. The singularity in $\varrho$ is precisely compensated by the singularity in the phase, leaving us with a regular magnetic field [15].

The number of independent parameters is in fact $4n - 1$, due to a global gauge freedom; index-theory calculations [17] show that this is indeed the maximal number of independent solutions.

For $f = c/z^n$ in particular, we obtain the radially symmetric solution

\[
\varrho(r) = \frac{4n^2|\kappa|}{e^2 r^2} \left[ \frac{|c|}{r^n + |c|} \right]^{-2},
\]

whose flux is *evenly* quantized,

\[
\Phi = -(\text{sgn} \kappa) \frac{2\hbar}{e} \times n.
\]

A few non-relativistic vortices are shown on FIGs. 3-6 below.

**Fig. 3** The non-relativistic radially symmetric $N = 1$ vortex has a maximum at $r = 0$.

**Fig. 4** The non-relativistic radially symmetric $N = 2$ vortex has a ‘doughnut-like’ shape. For $N \geq 2$, the particle density vanishes at $r = 0$. 
Fig. 5 The non-relativistic $N = 2$ vortex representing two separated 1-vortices with ‘positions’ at $x = \pm 1$ and $y = 0$.

Fig. 6 The non-relativistic $N = 4$ vortex representing two separated 2-vortices with ‘positions’ at $x = \pm 1$ and $y = 0$.

Let us mention that the only time-dependent solutions found so far are either obtained from the static solution by the action of the Schrödinger group (the fundamental symmetry group of the theory [13, 14, 15, 16]), or are stationary [18].

4. Spinor vortices

A. Relativistic spinor vortices

In Ref. [19] Cho et al. obtain, by dimensional reduction from Minkowski space, a (2 + 1)-dimensional
system. After some notational changes, their equations read

\[
\begin{align*}
\frac{1}{2} \kappa \epsilon^{\alpha\beta\gamma} F_{\beta\gamma} &= e \left( \bar{\psi}_+ \gamma^\alpha \psi_+ + \bar{\psi}_- \gamma^\alpha \psi_- \right), \\
(i c \gamma^a_{\pm} D_a - m) \psi_\pm &= 0,
\end{align*}
\]

where the two sets of Dirac matrices are

\[
(\gamma^a_{\pm}) = (\pm (1/c) \sigma^3, i\sigma^2, -i\sigma^1),
\]

and the $\psi_\pm$ denote the chiral components, defined as eigenvectors of the chirality operator

\[
\Gamma = \begin{pmatrix}
-i\sigma_3 & 0 \\
0 & i\sigma_3
\end{pmatrix}.
\]

Observe that, although the Dirac equations are decoupled, the chiral components are nevertheless coupled through the Chern-Simons equation. Stationary solutions, representing purely magnetic vortices, are readily found \[19\]. It is particularly interesting to construct static solutions. For $A_0 = 0$ and $\partial_t A_i = 0$, setting

\[
\psi_\pm = e^{-imt} \begin{pmatrix} F_\pm \\ G_\pm \end{pmatrix},
\]

the relativistic system (4.1) becomes, for $c = 1$,

\[
\kappa \epsilon^{ij} \partial_j A_i = -e(|F_+|^2 + |G_-|^2), \\
(D_1 + iD_2) F_\pm = 0, \quad (D_1 - iD_2) G_\pm = 0.
\]

Now, for $F_\pm = 0$ or $G_\pm = 0$, these equations are identical to those which describe the non-relativistic, self-dual vortices of Jackiw and Pi \[15\]!

**B. Non-relativistic spinor vortices**

Non-relativistic spinor vortices can also be constructed along the same lines \[20\]. Following Lévy-Leblond \[21\], a non-relativistic spin $\frac{1}{2}$ field $\psi = \begin{pmatrix} \Phi \\ \chi \end{pmatrix}$ where $\Phi$ and $\chi$ are two-component ‘Pauli’ spinors, is described by the 2 + 1 dimensional equations

\[
\begin{align*}
(\vec{\sigma} \cdot \vec{D}) \Phi + 2m \chi &= 0, \\
D_t \Phi + i(\vec{\sigma} \cdot \vec{D}) \chi &= 0.
\end{align*}
\]

These spinors are coupled to the Chern-Simons gauge field through the mass (or particle) density, $\varrho = |\Phi|^2$, as well as through the spatial components of the current,

\[
\vec{J} = i(\Phi^\dagger \vec{\sigma} \chi - \chi^\dagger \vec{\sigma} \Phi),
\]

according to the Chern-Simons FCI (2.3). The chirality operator is still given by Eq. (4.3) and is still conserved. Observe that $\Phi$ and $\chi$ in Eq. (4.6) are not the chiral components of $\psi$; these latter are defined by $\frac{1}{2}(1 \pm i\Gamma) \psi_\pm = \pm \psi_\pm$.

It is easy to see that Eq. (4.6) splits into two uncoupled systems for $\psi_+$ and $\psi_-$. Each of the chiral components separately describe (in general different) physical phenomena in 2 + 1 dimensions. For the ease of presentation, we keep, nevertheless, all four components of $\psi$. 
Now the current can be written in the form:

\[ \mathbf{J} = \frac{1}{2m} (\Phi^\dagger \mathbf{D} \Phi - (\mathbf{D} \Phi)^\dagger \Phi) + \nabla \times \left( \frac{1}{2m} \Phi^\dagger \sigma_3 \Phi \right). \]

Using \((\mathbf{D} \cdot \vec{\sigma})^2 = \mathbf{D}^2 + eB\sigma_3\), we find that the component-spinors satisfy

\[
\begin{aligned}
iD_t \Phi &= -\frac{1}{2m} \left[ \mathbf{D}^2 + eB\sigma_3 \right] \Phi, \\
iD_t \chi &= -\frac{1}{2m} \left[ \mathbf{D}^2 + eB\sigma_3 \right] \chi - \frac{e^2}{2m} (\vec{\sigma} \cdot \vec{E}) \Phi.
\end{aligned}
\]

Thus, \(\Phi\) solves a ‘Pauli equation’, while \(\chi\) couples through the Darwin term, \(\vec{\sigma} \cdot \vec{E}\). Expressing \(\vec{E}\) and \(B\) through the FCI, (2.3) and inserting into our equations, we get finally

\[
\begin{aligned}
iD_t \Phi &= \left[ -\frac{1}{2m} \mathbf{D}^2 + \frac{e^2}{2m} |\Phi|^2 \sigma_3 \right] \Phi, \\
iD_t \chi &= \left[ -\frac{1}{2m} \mathbf{D}^2 + \frac{e^2}{2m} |\Phi|^2 \sigma_3 \right] \chi - \frac{e^2}{2m} (\vec{\sigma} \times \vec{J}) \Phi.
\end{aligned}
\]

If the chirality of \(\psi\) is restricted to \(+1\) (or \(-1\)), this system describes non-relativistic spin \(+\frac{1}{2}\) fields interacting with a Chern-Simons gauge field. Leaving the chirality of \(\psi\) unspecified, it describes two spinor fields of spin \(\pm \frac{1}{2}\), interacting with each other and the Chern-Simons gauge field.

Since the lower component is simply \(\chi = -(1/2m)(\vec{\sigma} \cdot \mathbf{D})\Phi\), it is enough to solve the \(\Phi\)-equation. For

\[
\Phi_+ = \begin{pmatrix} \Psi_+ \\ 0 \end{pmatrix} \quad \text{and} \quad \Phi_- = \begin{pmatrix} 0 \\ \Psi_- \end{pmatrix}
\]

respectively — which amounts to working with the \(\pm\) chirality components — the ‘Pauli’ equation in (4.10) reduces to

\[
iD_t \Psi_\pm = \left[ -\frac{1}{2m} \mathbf{D}^2 + \frac{e^2}{2m} (\Psi_\pm |\Psi_\pm|^2 \sigma_3) \right] \Psi_\pm, \quad \lambda \equiv \frac{e^2}{2m}.
\]

which again (3.2), but with non-linearity \(\pm\lambda\), half of the special value \(\Lambda\) in (3.6) used by Jackiw and Pi. For this reason, our solutions (presented below) will be purely magnetic, \(A_t \equiv 0\), unlike in the case studied by Jackiw and Pi. In detail, let us consider the static system

\[
\begin{aligned}
\left[ -\frac{1}{2m} (\mathbf{D}^2 + eB\sigma_3) - eA_t \right] \Phi &= 0, \\
\mathbf{J} &= \frac{\kappa}{e} \nabla \times A_t, \\
\kappa B &= -e\theta,
\end{aligned}
\]

and try the first-order Ansatz

\[
(D_1 \pm iD_2) \Phi = 0.
\]

Eq. (4.14) makes it possible to replace \(\mathbf{D}^2 = D_1^2 + D_2^2\) by \(\mp eB\), then the first equation in (4.13) can be written as

\[
\left[ -\frac{1}{2m} eB(\mp 1 + \sigma_3) - eA_t \right] \Phi = 0,
\]
while the current is
\begin{equation}
\vec{J} = \frac{1}{2m} \vec{\nabla} \times [\Phi^\dagger (\mp 1 + \sigma_3) \Phi].
\end{equation}

Now, due to the presence of \(\sigma_3\), both Eq. (4.16) and the second equation in (4.13) can be solved with a zero \(A_t\) and \(\vec{J}\); by choosing \(\Phi \equiv \Phi_+ \) (\(\Phi \equiv \Phi_-\)) for the upper (lower) cases respectively makes \((\mp 1 + \sigma_3)\Phi\) vanish. (It is readily seen from Eq. (4.15) that any solution has a definite chirality).

The remaining task is to solve the first-order conditions
\begin{equation}
(D_1 + iD_2)\Psi_+ = 0, \quad \text{or} \quad (D_1 - iD_2)\Psi_- = 0,
\end{equation}
which is done in the same way as in Section 3:
\begin{equation}
\vec{A} = \pm \frac{1}{2e} \vec{\nabla} \times \log g, \quad \Delta \log g = \pm \frac{2e^2}{\kappa} g.
\end{equation}
A normalizable solution is obtained for \(\Psi_+\) when \(\kappa < 0\), and for \(\Psi_-\) when \(\kappa > 0\). (These correspond precisely to having an attractive non-linearity in Eq. (4.12)). The lower components vanish in both cases, as seen from
\begin{equation}
\chi = -\frac{1}{2m} (\vec{\sigma} \cdot \vec{D}) \Phi.
\end{equation}
Both solutions only involve one of the 2+1 dimensional spinor fields \(\psi_{\pm}\), depending on the sign of \(\kappa\).

The physical properties such as symmetries and conserved quantities can be studied by noting that our equations are in fact obtained by variation of the 2+1-dimensional action given in [20], which can also be used to show that the coupled Lévy-Leblond — Chern-Simons system is, just like its scalar counterpart, Schrödinger symmetric [15].

A conserved energy-momentum tensor can be constructed and used to derive conserved quantities [20]. One finds that the ‘particle number’ \(N\) determines the actual values of all the conserved charges: for the radially symmetric solution, e.g., the magnetic flux, \(-eN/\kappa\), and the mass, \(\mathcal{M} = mN\), are the same as for the scalar soliton of [15]. The total angular momentum, however, can be shown to be \(I = \mp N/2\), half of the corresponding value for the scalar soliton. As a consequence of self-duality, our solutions have vanishing energy, just like the ones of Ref. [15].

C. The non-relativistic limit

Setting
\begin{equation}
\psi_+ = e^{-imc^2t} \left( \frac{\Psi_+}{\chi_+} \right) \quad \text{and} \quad \psi_- = e^{-imc^2t} \left( \frac{\bar{\chi}-}{\Psi_-} \right),
\end{equation}
Eq. (4.1) become
\begin{equation}
\begin{cases}
iD_t \Phi - c\vec{\sigma} \cdot \vec{D} \bar{\chi} = 0, \\
iD_t \bar{\chi} + c\vec{\sigma} \cdot \vec{D} \Phi + 2mce^2 \bar{\chi} = 0,
\end{cases}
\end{equation}
where \(\Phi = \left( \frac{\Psi_+}{\Psi_-} \right)\) and \(\bar{\chi} = \left( \frac{\bar{\chi}-}{\chi_+} \right)\). In the non-relativistic limit \(mc^2\bar{\chi} >> iD_t\bar{\chi}\), so that this latter can be dropped from the second equation. Redefining \(\bar{\chi} = \chi/c\bar{\chi}\) yields precisely our Eq. (4.6). This also explains, why one gets the same (namely the Liouville) equation both in the relativistic and the non-relativistic cases: for static and purely magnetic fields, the terms containing \(D_t\) are automatically zero.

In this paper, we only reviewed the Abelian Chern-Simons theories. Non-Abelian generalizations are studied in Refs. [14], [22] and [23].

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