Quantum Turbulence in Coflow of Superfluid $^4$He

S. Ikawa$^1$ · M. Tsubota$^{1,2}$

Abstract We study numerically nonuniform quantum turbulence of coflow in a square channel by the vortex filament model. Coflow means that superfluid velocity $v_s$ and normal fluid velocity $v_n$ flow in the same direction. Quantum turbulence for thermal counterflow has been long studied theoretically and experimentally. In recent years, experiments of coflow have been performed to observe different features from thermal counterflow. By supposing that $v_s$ is uniform and $v_n$ takes the Hagen–Poiseuille profile, simulations find that quantized vortices are distributed inhomogeneously. Vortices like to accumulate on the surface of a cylinder with $v_s \simeq v_n$. Consequently, the vortex configuration becomes degenerate from three-dimensional to two-dimensional.

Keywords Superfluid $^4$He · Quantized vortex · Quantum turbulence · Coflow

1 Introduction

Quantum turbulence is one of the most important issues in low-temperature physics and has been studied theoretically and experimentally for thermal counterflow where the superfluid and normal fluid flow oppositely. In recent years, experiments of coflow where the superfluid and normal fluid flow in the same direction are performed to observe features different from thermal counterflow [1]. For example, the vortex line density is proportional to the $3/2$ power of the velocity and independent of temperature.
The motivation of the present paper is to find numerically some behavior of vortices characteristic of coflow.

According to the two-fluid model, superfluid $^4$He at a finite temperature consists of an intimate mixture of a viscous normal fluid component (density $ρ_n$ and velocity $v_n$) and inviscid superfluid components (density $ρ_s$ and velocity $v_s$) [2]. In superfluid $^4$He, the circulation of a superfluid vortex, called a quantized vortex, is quantized by the quantum circulation $κ = h/m_4$, where $h$ is Planck’s constant and $m_4$ is the mass of a $^4$He atom. Quantum turbulence generally occurs by tangling of quantized vortices [3].

We perform numerical simulation for coflow in a square channel; the applied superfluid velocity is prescribed to be uniform flow and normal fluid velocity to be Hagen–Poiseuille flow. Vortices are distributed inhomogeneously and tend to accumulate through the mutual friction on the surface of a cylinder where superfluid velocity equals the normal fluid velocity. As a result, the vortex configuration becomes degenerate from three-dimensional to two-dimensional. How strongly the vortices accumulate depends on temperature and the averaged velocity.

The contents of this paper are as follows. In Sect. 2, we shall clarify the formulation of the model and introduce the equation of motion. In Sect. 3, we show the results for coflow. Section 4 is devoted to the conclusion and the future work.

### 2 Formulation

We perform numerical simulation using the vortex filament model with the full Biot–Savart law [4,5]. A point $s$ on the vortex filament is represented in a parametric form $s = s(ξ, t)$, where $t$ is time and $ξ$ is the one-dimensional coordinate along the filament. The equation of motion of $s$ is given by

$$\dot{s} = v_s + αs' × (v_n - v_s) - α's' × [s' × (v_n - v_s)],$$

where $α$ and $α'$ are the temperature-dependent mutual friction coefficients and the prime denotes derivatives of $s$ with respect to $ξ$.

The first term of the right-hand side of Eq. (1) is superfluid velocity, given by

$$v_s = v_{s,ω} + v_{s,b} + v_{s,a}.$$  \hspace{1cm} (2)

Here $v_{s,ω}$ is the velocity field caused by vortex filaments, $v_{s,b}$ the boundary induced field, and $v_{s,a}$ the applied uniform velocity field. The velocity field $v_{s,ω}$ is represented by the Biot–Savart law

$$v_{s,ω}(r) = \frac{κ}{4π} \int _{L} \frac{(s_1 - r) × ds_1}{|s_1 - r|^3},$$

where $s_1$ refers to a point on the vortex filament and the integration is performed along the vortex filaments. The velocity field $v_{s,b}$ is obtained by a simple procedure; it is just the field produced by an image vortex that is constructed by reflecting the filament into the surface and reversing its direction.
The second and third terms of the right-hand side of Eq. (1) are caused by mutual friction. The second term makes a curved vortex balloon out or collapse inward. As discussed in [6], when the relative velocity $v_{ns} = v_n - v_s$, flows against $v_{s,ω}$, the mutual friction always shrinks the curved vortex locally. On the other hand, $v_{ns}$ flowing along $v_{s,ω}$ yields a critical radius of curvature $R_c$. When the local radius $R$ at a point on the vortex is smaller than $R_c$, the curved vortex shrinks locally, while the curved vortex balloons out when $R > R_c$.

We prescribe the Hagen–Poiseuille profile $u_p$ for $v_n$ fixed for the simulation, though the normal fluid flow in the experiment [1] may be actually turbulent. When the normal fluid flows along the $x$ direction, the $x$ component of $v_n$ is given by

$$u_p(y, z) = u_0 \sum_{m=1,3,5,...}^{∞} (-1)^{(m-1)/2} \left[ 1 - \frac{\cosh(mπz/2a)}{\cosh(mπb/2a)} \right] \frac{\cos(mπy/2a)}{m^3},$$

where $u_0$ is a normalization factor and $a$ and $b$ are halves of the channel width along the y and z axes, respectively [7].

To characterize the development of vortices, we introduce the vortex line density (VLD) as

$$L = \frac{1}{Ω} \int_L dξ,$$

where the integral is performed along all vortices in the sample volume $Ω$.

In this study, our calculation is performed under the following conditions. A vortex filament is represented by a string of discrete points. The numerical space resolution, namely the minimum distance between neighboring points, is $Δξ = 8.0 \times 10^{-4}$ cm, the time resolution is $Δ = 1.0 \times 10^{-4}$ s, the computational box is $0.1 \times 0.1 \times 0.1$ cm$^3$, and both the values of $a$ and $b$ in Eq. (4) are 0.05 cm. We consider the case of $\bar{v}_n = \bar{v}_s$, where $\bar{v}_n$ and $\bar{v}_s$ are the spatially averaged normal fluid velocity and the spatial average of the applied superfluid velocity $v_{s,ω}$, respectively. The periodic boundary conditions are used along the flow direction $x$, whereas the solid boundary conditions are applied to channel walls. The effects of reconnection are artificially performed, whenever two vortices approach more closely than $Δξ$. The initial state consists of eight randomly oriented vortex rings of radius 0.023 cm (Fig. 3a).

### 3 Results and Discussion

#### 3.1 Distribution of Localized Vortices and Development of Vortex

We perform numerical simulation for coflow in a square channel and find two different states of vortices, namely the diffusive state and the localized state. The features of two states are shown in the figures; Fig. 1 shows the time development of the VLD and Fig. 2 shows the snapshots of the vortices. In the diffusive state, the VLD fluctuates irregularly and the vortices are diffusive (Figs. 1a, 2a). In the localized state, the VLD just increases (Fig. 1b) and the vortices localize in a region shown in Fig. 2b,c.
Fig. 1 Temporal development of the VLD shows two different kinds of behavior depending on the vortex states. In (a) with $T=1.35\,\text{K}$ and $\bar{v}_n(=\bar{v}_s)=1.0\,\text{cm/s}$ the VLD fluctuates irregularly, when the vortices are diffusive as shown in Fig. 2a. In (b) with $T=1.95\,\text{K}$ and $\bar{v}_n = 1.0\,\text{cm/s}$ the VLD just increases, when the vortices are localized as shown in Fig. 2b and c (Color figure online).

In the localized state, the development of the vortices consists of two stages. As shown in Fig. 1b, the first stage is $0\,\text{s} \leq t < 0.5\,\text{s}$ and the second is $0.5\,\text{s} \leq t$. In the first stage (Fig. 3a, b, c), the vortices experience many reconnections, while they
Fig. 2 Snapshots of vortices viewed along the flow direction: (a) a snapshot at $t = 15$ s of the dynamics of Fig. 1a; (b) a snapshot at $t = 20$ s of the dynamics of Fig. 1b. The distribution of vortices has two states, which are diffusive (a) and localized (b). (c) is the snapshot of (b) viewed from the side.

Fig. 3 Simulations of the time development of vortex tangle in coflow viewed along the flow direction ($T = 1.95$ K, $v_n = 1.0$ cm/s): (a) $t = 0$ s, (b) $t = 0.1$ s, (c) $t = 0.2$ s, and (d) $t = 0.75$ s.

are attracted to a localized region by the mutual friction. Some small vortex loops are made by reconnections and balloon out to a localized region. Then, the VLD increases rapidly. In the second (Fig. 3d), the vortices protruding from the localized region towards the walls continue to extend and wrap the localized region. Since the
reconnections seldom occur in this stage, the VLD increases much slower than that of the first stage.

The vortex distribution of coflow is different from that of nonuniform thermal counterflow [8]. This is because the coflow has the region where the mutual friction does not work. To see where the vortices accumulate, we consider the localized induction approximation (LIA). Then, the second term of the right-hand side of Eq. (1) is \( \alpha s' \times (v_n - v_{s,a} - \beta s' \times s'') \), where \( \beta \) is the quantity proportional to the quantum circulation [6]. The mutual friction vanishes in the region where \( v_n - v_{s,a} - \beta s' \times s'' \) vanishes. For the sake of simplicity, if the term \( \beta s' \times s'' \) is negligible, the region is like a square pipe as shown by the light symbols in Fig. 4. However, this region is modified to the dark symbols by the term \( \beta s' \times s'' \); the position around a corner is shifted inward because the radius of curvature is small and the position around a side is shifted outward because that is large.

### 3.2 Parameter Dependence of Localization

Whether the vortices are localized or diffusive depends on the parameters. We introduce a dimensionless variable \( L_{\text{in}}/L_{\text{out}} \) to characterize the behavior of the vortices:

\[
\frac{L_{\text{in}}}{L_{\text{out}}} = \frac{\int L_{\text{in}} d\xi}{\int L_{\text{out}} d\xi}.
\]

Here \( L_{\text{in}} \) is obtained by the integration along all vortices \( L_{\text{in}} \) in the cylindrical region which is between central axis and the radius 0.045 cm, and \( L_{\text{out}} \) is obtained by the integration along all other vortices \( L_{\text{out}} \) in the region between the radius 0.045 cm and the wall. The time development of \( L_{\text{in}}/L_{\text{out}} \) is shown in Fig. 5a. We define that the vortices are localized when \( L_{\text{in}}/L_{\text{out}} \) exceeds 5 after the time passes sufficiently, and that the vortices are diffusive when \( L_{\text{in}}/L_{\text{out}} \) is less than 5. The value of \( L_{\text{in}}/L_{\text{out}} \)
depends strongly on temperature and the averaged velocity $\bar{v}(=\bar{v}_n = \bar{v}_s)$, and the resulting phase diagram is shown in Fig. 5b. The vortices tend to be localized for higher temperature and faster velocity, and diffusive for lower temperature and slower velocity.

Mutual friction depends on temperature and velocity, because the coefficient $\alpha$ is dependent on temperature and increasing the velocity makes the relative velocity faster everywhere. If the temperature is lower and velocity is slower, the velocity caused by mutual friction is dominated by the self-induced velocity $\beta s' \times s''$ and the vortices almost move freely. Consequently, whether the vortices are localized or diffusive depends on temperature and the velocity.

4 Conclusions

We found that the vortex development of coflow has two states: one is diffusive and the other is localized. At the localized state, the vortices accumulate on the surface of a cylinder where the mutual friction vanishes. Whether the vortices are localized or diffusive depends on temperature and the averaged velocity.

The future work is to perform a numerical simulation of coflow under turbulent normal fluid flow. In the experiment [1], the normal fluid flow seems turbulent. We will reproduce this situation by calculating the motion of the vortex filament with the turbulent normal fluid flow.

Acknowledgments M. T. was supported by JSPS KAKENHI Grant No. 26400366 and MEXT KAKENHI ‘Fluctuation & Structure’ Grant No. 26103526.

References

1. E. Varga, S. Babuin, L. Skrbek, Phys. Fluids 27, 065101 (2015)
2. R.J. Donnelly, Quantized Vortices in Helium II (Cambridge University Press, Cambridge, 1995), pp. 42–45
3. R.P. Feynman, Progress in Low Temperature Physics, vol. 1 (Elsevier, Amsterdam, 1955)
4. K.W. Schwarz, Phys. Rev. B 38, 2398 (1988)
5. H. Adachi, S. Fujiyama, M. Tsubota, Phys. Rev. B 81, 104511 (2010)
6. K.W. Schwarz, Phys. Rev. B 31, 5782 (1985)
7. R.W. Johnson, *The Handbook of Fluid Dynamics* (CRC, Boca Raton, 1998)
8. S. Yui, M. Tsubota, Phys. Rev. B 91, 184504 (2015)