Efficient evaluation of risk allocations

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Abstract

Expectations of marginals conditional on the total risk of a portfolio are crucial in risk-sharing and allocation. However, computing these conditional expectations may be challenging, especially in critical cases where the marginal risks have compound distributions or when the risks are dependent. We introduce a generating function method to compute these conditional expectations. We provide efficient algorithms to compute the conditional expectations of marginals given the total risk for a portfolio of risks with lattice-type support. We show that the ordinary generating function of unconditional expected allocations is a function of the multivariate probability generating function of the portfolio. The generating function method allows us to develop recursive and transform-based techniques to compute the unconditional expected allocations. We illustrate our method to large-scale risk-sharing and risk allocation problems, including cases where the marginal risks have compound distributions, where the portfolio is composed of dependent risks, and where the risks have heavy tails, leading in some cases to computational gains of several orders of magnitude. Our approach is useful for risk-sharing in peer-to-peer insurance and risk allocation based on Euler’s rule.

Keywords: Risk allocation, generating functions, conditional mean risk-sharing, fast Fourier transform, Euler risk allocation

1 Introduction

Risk allocations are essential in actuarial science and quantitative risk management. Roughly speaking, risk allocation refers to redistributing a total risk to its granular risks. Risk allocation is used in various contexts, such as insurance pricing, reinsurance, and regulatory capital requirements. For instance, one may allocate the total risk to the policyholders for peer-to-peer insurance pricing. For an insurance company with many lines of business, one may allocate the total risk to each line of business to determine the required capital for each line.

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Numerical evaluation of aggregate and compound distributions is a challenging problem in non-life actuarial science and quantitative risk management. Typical approaches for the evaluation of aggregate and compound distributions are using the direct convolution approach, the Panjer recursion, or the transform-based techniques like the fast Fourier transform (FFT), see [Wang, 1996, Embrechts and Frei, 2009, McNeil et al., 2015] for details. The direct convolution approach is computationally expensive, while the Panjer recursion is limited to compound distributions. The transform-based techniques are more efficient than the previous methods and can be used for a wide range of distributions. Thanks to efficient algorithms like the FFT, one can compute the aggregate distribution of a large portfolio of risks in a reasonable amount of time, essential for large-scale risk aggregation problems in actuarial science and quantitative risk management.

Numerical evaluation of risk allocations is a more challenging problem than computing the aggregate distribution since it requires computing (i) the conditional expectations of the marginals given the total risk and (ii) the probability mass function of the aggregate random variable. While there exists a growing literature on risk-sharing and risk allocation, along with the properties of risk-sharing rules, the literature on the numerical evaluation of risk allocations is scarce, and the methods are often limited to small pools or a few lines of business. Solving this problem for a large portfolio of risks is especially tedious, especially when the marginal risks have compound distributions or when the risks are dependent. Providing efficient algorithms to compute the conditional expectations of marginals given the total risk is essential for risk-sharing and risk allocation problems in practical situations, and we aim in this paper to address this limitation.

In this paper, we introduce a transform-based method to compute the conditional expectations of the marginals given the total risk, that is, \( E[X_i | S = s] \), where \( s \) lies on a lattice-type support. We show that the ordinary generating function of unconditional expected allocations is a function of the multivariate probability generating function of the portfolio. The generating function method allows us to develop transform-based techniques to compute unconditional expected allocations. We provide a recursive method to compute unconditional expected allocations when marginals are compound distributions. Our findings are helpful for risk-sharing and risk allocation problems, allowing computational gains that make analysis feasible for large portfolios of risks.

The main result of this paper is a representation of the ordinary generating function (OGF) for unconditional expected allocations in terms of the multivariate probability generating function (pgf) of the random vector \( X \). This representation enables new techniques to compute the unconditional expected allocation \( E[X_i \times 1_{\{S = s\}}] \), for \( i \in \{1, \ldots, n\} \) and \( s \in h\mathbb{N} \) efficiently. In some cases, we can invert the OGF of unconditional expected allocations analytically to obtain formulas expressed as partial sums depending on the pmf of \( S \). In other cases, we can use an efficient algorithm, such as the FFT, to compute the unconditional expected allocations directly from the OGF; this algorithm leads to significant computational gains that allow us to compute the unconditional expected allocations for large portfolios of risks with an accuracy unachievable with existing methods.

Our approach provides a generating function method instead of a direct computation method. The method of generating functions is standard in discrete mathematics and number theory. One can use generating functions to compute a term in the Fibonacci sequence, find the sequence average, study recurrence relations or prove combinatorial identities; see [Wilf, 2006] for details. Generating functions often provide convenient and elegant methods to compute or extract terms from a sequence where closed-form formulas would be tedious. A problem encountered within this paper is related to finding the number of partitions (restricted partitions in some cases, see Section 5.5 for an example.
with Bernoulli rvs) to determine if two agents can share a risk for a given outcome of the total risk. It isn’t surprising to observe that generating functions can solve these problems within the context of discrete rvs: these methods are also used to compute the pmf of aggregate rvs or compound distributions; see, for example, [Embrechts et al., 1993], [Grubel and Hermesmeier, 1999], and [Embrechts and Frei, 2009] for applications of generating functions for risk management. The latter authors use the expression transform approaches to refer to generating function methods, and we will follow their nomenclature in what follows.

The paper is organized as follows. Section 2 provides background on risk allocations and the problem of computing conditional expectations of marginals given the total risk, along with notation and a review of existing approaches. Section 3 introduces the ordinary generating function for unconditional expected allocations and reveals a relationship with the multivariate probability generating function. We also propose an efficient method to extract the unconditional expected allocations from the ordinary generating function. Section 4 presents expressions for the unconditional expected allocation in the case of (compound) Katz distributions. The results in Section 4 provide recursive formulas for unconditional expected allocations based on the pmf of the aggregate random variable, but we prove these results by using a transform-based approach and returning to the original space to compute conditional expectations. It is not always possible to analytically return to the original space; in this case, we explore applications of the FFT algorithm to compute the unconditional expected allocations in Section 5. In Section 6, we illustrate our method to dependent risks, sometimes obtaining recursive relations, other times using the FFT algorithm. Section 7 discusses further generalizations of our results for continuous random variables.

2 Numerical evaluation of risk allocations

2.1 Risk-sharing and risk allocation

Pooling or aggregating risks is a core principle in insurance and risk management, whether in traditional insurance settings or peer-to-peer insurance models. In both cases, the aim is to reduce the variability of the total risk by pooling the risks of different policyholders or lines of business. This aggregation is beneficial as long as the risks are not perfectly correlated or comonotonic due to the diversification effect. Understanding the total aggregated risk distribution is crucial for computing essential risk measures like the Value-at-Risk (VaR) or the Tail-Value-at-Risk (TVaR, also known as expected shortfall or conditional tail expectation in the finance and banking literature). These measures help risk managers understand the portfolio’s potential losses and set aside capital to cover these losses in compliance with regulatory requirements.

In peer-to-peer insurance, the concept of risk aggregation is applied by having policyholders pool their risks. Each member shares in the total risk and pays a premium that reflects their contribution to the pooled risk. These premiums are determined post-event, based on the actual losses experienced by the group and according to a predetermined risk-sharing rule. The conditional mean risk-sharing rule, introduced by [Denuit and Dhaene, 2012], is a popular choice in peer-to-peer insurance pricing. This rule states that the price for the $i$th participant is the expected contribution of risk $X_i$, given that the actual loss $S$ is $s$. The authors of [Denuit et al., 2022] list twelve desirable properties for risk-sharing rules and prove that the conditional mean risk-sharing rule satisfies eleven of them. In [Jiao et al., 2022], the authors provide an axiomatic approach to risk-sharing rules and
show that the conditional mean risk-sharing rule satisfies the axioms of actuarial fairness, risk
fairness, risk anonymity, and operational anonymity.

2.2 Notation

We now set the notation used throughout the paper. Let $\mathbb{N}$ be the set of non-negative integers
$\{0, 1, 2, \ldots \}$, while $\mathbb{N}_1$ be $\mathbb{N} \setminus \{0\}$. Consider a portfolio of $n$ risks $X = (X_1, \ldots, X_n)$ where each
random variable (rv) has a distribution supported on a lattice-type set $h\mathbb{N} = \{hk | k \in \mathbb{N}\}$, for
some fixed $h \in \mathbb{R}^+$. The multivariate cumulative distribution function (cdf) and probability mass
function (pmf) are respectively $F_X(x)$ and $f_X(x)$, for $x \in \{h\mathbb{N}\} \times \cdots \times \{h\mathbb{N}\} = \{h\mathbb{N}\}^n$ and
marginal cdfs and pmfs are noted $F_{X_i}(x) = \Pr(X_i \leq x)$ and $f_{X_i}(x) = \Pr(X_i = x)$, for $x \in h\mathbb{N}$
and $i \in \{1, \ldots, n\}$. Throughout the paper, we assume that $E[X_i] < \infty$, for $i \in \{1, \ldots, n\}$. The rv
representing the portfolio aggregate loss is $S = X_1 + \cdots + X_n$, with cdf $F_S$, pmf $f_S$ and expectation
$E[S] = \sum_{i=1}^n E[X_i] < \infty$.

2.3 Unconditional expected allocations

In this paper, we propose methods based on ordinary generating functions to compute the uncon-
ditional expected allocation, which we define as follows.

Definition 2.1 (Unconditional expected allocation). Let $X$ be a vector of rvs, each with distri-
butions supported on a lattice-type set $h\mathbb{N}$, and $S = X_1 + \cdots + X_n$. The unconditional expected
allocation of $X_i$ to a total outcome $S = s$ is defined as $E[X_i \times 1_{\{S=s\}}]$, for $i \in \{1, \ldots, n\}$ and
$s \in h\mathbb{N}$.

While most loss distributions in actuarial science and quantitative risk management have contin-
uous support, the assumption of lattice-type support is not too prohibitive. It suffices to discretize
the continuous support into intervals of size $h$. The value of $h$ can be set to the smallest denomina-
tion of a currency system, e.g. $h = 0.01$ for dollars. In that case, the risk allocations are accurate
to the nearest penny.

In the spirit of Definition 2.1, we introduce the unconditional expected cumulative allocation
defined by

$$E[X_i \times 1_{\{S \leq s\}}] = \sum_{x \in \{0, h, \ldots, s\}} E[X_i \times 1_{\{S=x\}}],$$

for $s \in h\mathbb{N}$ and $i \in \{1, \ldots, n\}$.

Although rarely considered directly, unconditional expected allocations are essential in peer-to-
peer insurance pricing and risk allocation based on Euler’s rule.

One is interested in computing a participant’s contribution according to a risk-sharing rule in
peer-to-peer insurance pricing schemes. The conditional mean risk-sharing rule, studied in, for
instance, [Denuit and Dhaene, 2012], is a popular choice, where the price for the $i$th participant is
the expected contribution of risk $X_i$, for $i \in \{1, \ldots, n\}$, given that the actual loss $S$ is $s$, that is,

$$E[X_i | S = s] = \frac{E[X_i \times 1_{\{S=s\}}]}{\Pr(S = s)},$$

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assuming $\Pr(S = s) \neq 0$, for $s \in h\mathbb{N}$. We note that $\sum_{i=1}^{n} E[X_i|S = s] = s$, meaning that, under the conditional mean sharing rule, the total contributions of all participants equal the total losses $s$, for $s \in h\mathbb{N}$.

Unconditional expected allocations also appear in risk allocation based on Euler’s rule. Regulatory capital requirements are risk measures based on the aggregate rv of an insurance company’s portfolio. One risk measure of theoretical and practical interest is the TVaR. Risk allocation is an important research area in actuarial science, quantitative risk management and operations research, which aims to compute the contribution of each risk based on the total required (or available) capital. When using the TVaR as a regulatory capital requirement risk measure, one may compute the contributions of each risk to the capital based on the Euler risk-sharing paradigm; see [McNeil et al., 2015] for details.

Following [Embrechts and Hofert, 2013], define the generalized inverse of $S$ at level $\kappa$ by

$$F_S^{-1}(\kappa) = \inf_{x \in \mathbb{R}} \{F_S(x) \geq \kappa\},$$

for $0 < \kappa < 1$. We will consider the Range-Value-at-Risk as a risk measure for capital requirements, which can be seen as a generalization of the VaR and the TVaR.

**Definition 2.2.** The Range-Value-at-Risk is defined as

$$R\text{VaR}_{\alpha_1, \alpha_2}(X) = \begin{cases} \frac{1}{\alpha_2 - \alpha_1} \int_{\alpha_1}^{\alpha_2} \text{VaR}_u(X) \, du, & \alpha_1 < \alpha_2, \\ \text{VaR}_{\alpha_1}(X), & \alpha_1 = \alpha_2, \end{cases}$$

for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$.

Clearly, the Range-Value-at-Risk becomes the Value-at-Risk $\text{VaR}_\kappa(X)$ if $\alpha_1 = \alpha_2 = \kappa$. Further, if $\alpha_1 = \kappa$ and $\alpha_2 = 1$, then the corresponding RVaR is also $\text{TVaR}_\kappa(X)$, also called the expected shortfall. We have

$$R\text{VaR}_{\alpha_1, \alpha_2}(S) = \frac{1}{\alpha_2 - \alpha_1} \left\{ F_S^{-1}(\alpha_1)[F_S(F_S^{-1}(\alpha_1)) - \alpha_1] + E \left[ S \times 1\{F_S^{-1}(\alpha_1) < S \leq F_S^{-1}(\alpha_2)\} \right] \\
+ F_S^{-1}(\alpha_2)[\alpha_2 - F_S(F_S^{-1}(\alpha_2))] \right\}. $$

Applying Euler’s rule for risk allocation [Tasche, 1999], the contribution of $X_i$ to the RVaR of the aggregate random variable is $R\text{VaR}_{\alpha_1, \alpha_2}(X_i; S) = E[X_i|S = \text{VaR}_{\alpha_1}(S)]$ for $\alpha_1 = \alpha_2$ and

$$R\text{VaR}_{\alpha_1, \alpha_2}(X_i; S) = \frac{1}{\alpha_2 - \alpha_1} \left( E \left[ X_i \times 1\{S = F_S^{-1}(\alpha_1)\} \right] \frac{F_S(F_S^{-1}(\alpha_1)) - \alpha_1}{\Pr(S = F_S^{-1}(\alpha_1))} \\
+ E \left[ X_i \times 1\{F_S^{-1}(\alpha_1) < S \leq F_S^{-1}(\alpha_2)\} \right] \frac{\alpha_2 - F_S(F_S^{-1}(\alpha_2))}{\Pr(S = F_S^{-1}(\alpha_2))} \right),$$

for $\alpha_1 < \alpha_2$. Notice that two of the three expected values in (3) are unconditional expected allocations, and we can compute the third one using unconditional expected allocations.
The Euler-based RVaR decomposition is a top-down risk allocation method of risk allocation. By the additive property of the expected value, the full allocation property [McNeil et al., 2015] holds:

\[ \text{RVaR}_{\alpha_1, \alpha_2}(S) = \sum_{i=1}^{n} \text{RVaR}_{\alpha_1, \alpha_2}(X_i; S) \]  

for any pair \((\alpha_1, \alpha_2)\) such that \(0 \leq \alpha_1 \leq \alpha_2 \leq 1\).

The relations in (2) and (3) require the computation of the unconditional expected allocation \(E[X_i \times 1_{\{S=s\}}]\) for \(s \in h\mathbb{N}\). One, therefore, seeks an efficient method to compute these values.

### 2.4 Existing approaches to compute unconditional expected allocations

One finds two common approaches to computing unconditional expected allocations in the actuarial science and quantitative risk management literature. The first approach is a direct method for computing unconditional expected allocations through summation or integration. Letting \(S_{-i} = \sum_{j=1, j \neq i}^{n} X_j\), we have

\[ E[X_1 \times 1_{\{S=s\}}] = \sum_{x \in \{0, h, 2h, \ldots, s\}} xf_{X_1, S_{-1}}(x, s-x), \quad s \in h\mathbb{N}. \]

The direct summation method is used in [Cossette et al., 2018] for discrete rvs when the dependence structure is an Archimedean copula. In Section 5 of [Bargès et al., 2009], the authors use this approach in a continuous setting to compute TVaR-based allocations for a mixture of exponential distributions linked through an FGM copula. The second approach is based on size-biased transforms, used notably in [Furman and Landsman, 2005, Furman and Landsman, 2008]; see also [Arratia et al., 2019] for a review of the size-biased transform and its applications. Under that approach,

\[ E[X_1 \times 1_{\{S=s\}}] = E[X_1] \Pr(\tilde{X}_1 + S_{-1} = s), \quad s \in h\mathbb{N}, \]

where \(\tilde{X}_1\) is the size-biased transform of \(X_1\) with pmf

\[ f_{\tilde{X}_1}(x) = xf_{X_1}(x)/E[X_1], \quad x \in h\mathbb{N}. \]

The authors of, for instance, [Denuit and Dhaene, 2012, Denuit, 2020, Denuit and Robert, 2020], use the size-biased transform method to derive properties and results about the conditional mean risk-sharing rule.

To simplify the notation in the theory developed in the remainder of this paper, we set \(h = 1\); that is, we consider only rvs which have integer support. Therefore, for the remainder of this paper, the \(n\)-variate random vector \(X\) takes values in \(\mathbb{N}^n\). One may transform a rv with lattice-type support \(h\mathbb{N}\) into one of integer support \(\mathbb{N}\) by multiplying the rv by the constant \(h^{-1}\). By linearity of the unconditional expected allocation, one may easily recover unconditional expected allocations for the original rv.

Typically, conditional mean risk-sharing and risk allocation are used for small pools or a few lines of businesses. However, the efficient methods based on OGFs proposed in this paper enable these techniques to be used even for a large portfolio of risks. Once equipped with generating functions for unconditional expected allocations, risk managers can perform risk allocation at the customer level.
3 Ordinary generating functions for unconditional expected allocations

3.1 Ordinary generating functions

Ordinary generating functions are a useful mathematical tool since they capture every sequence value into one formula. See Chapter 7 of [Graham et al., 1994] or the monograph [Wilf, 2006] for details on generating functions, and Chapter 3 of [Sedgewick and Flajolet, 2013] for efficient algorithms to extract the values of the sequence. Following Section 3.1 of [Sedgewick and Flajolet, 2013], we define ordinary generating functions.

**Definition 3.1** (Ordinary generating function). For a sequence \( \{a_k\}_{k \in \mathbb{N}} \), the function

\[
A(z) = \sum_{k=0}^{\infty} a_k z^k, \quad |z| \leq 1,
\]

is its ordinary generating function (OGF). We use the notation \([z^k]A(z)\) to refer to the coefficient \(a_k, k \in \mathbb{N}\).

The following lemma summarizes the relevant operations one can perform on generating functions (see Theorem 3.1 and Table 3.2 of [Sedgewick and Flajolet, 2013] for details).

**Lemma 3.2.** If \( A(z) = \sum_{k=0}^{\infty} a_k z^k \) and \( B(z) = \sum_{k=0}^{\infty} b_k z^k \) are two OGFs, then the following operations produce OFGs with the corresponding sequences:

1. **Addition** \( A(z) + B(z) = \sum_{k=1}^{\infty} (a_k + b_k) z^k \).
2. **Right shift** \( zA(z) = \sum_{k=1}^{\infty} a_{k-1} z^k \).
3. **Index multiply** \( A'(z) = \sum_{k=0}^{\infty} (k+1) a_{k+1} z^k \).
4. **Convolution** \( A(z)B(z) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} a_j b_{k-j} \right) z^k \).
5. **Partial sum** \( A(z)/(1 - z) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} a_j \right) z^k \).

When \( a_k \geq 0 \) for \( k \in \mathbb{N} \) and \( \sum_{k=0}^{\infty} a_k = 1 \), then the OGF is the pgf of a discrete rv \( X \), denoted as \( P_X \), where the values of the pmf of the rv \( X \) is \( f_X(k) = \Pr(X = k) = a_k, k \in \mathbb{N} \). The expression in (7) becomes

\[
P_X(z) = \sum_{k=0}^{\infty} f_X(k) z^k, \quad |z| \leq 1.
\]

The pgf is an essential tool in all areas of probability, statistics, and actuarial science, as explained, for example, in Section 5.1 in [Grimmett and Stirzaker, 2020], and Sections 1.2 and 2.4 in [Panjer and Willmot, 1992].
In this paper, we rely on a multivariate ordinary generating function capturing the values of the pmf of a discrete random vector. As described in Section 34.2.1 of [Johnson et al., 1997], the multivariate pgf of a vector of discrete rvs $X = (X_1, \ldots, X_n)$, with multivariate pmf $f_X$, is

$$P_X(z_1, \ldots, z_n) = E \left[ z_1^{X_1} \times \cdots \times z_n^{X_n} \right] = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} z_1^{k_1} \times \cdots \times z_n^{k_n} f_X(k_1, \ldots, k_n),$$

for $|z_j| \leq 1$, $j \in \{1, \ldots, n\}$. For more details on multivariate pgfs or ordinary functions and their properties, see Appendix A in [Axelrod and Kimmel, 2015], and Chapter 3 of [Flajolet and Sedgewick, 2009].

The following theorem, a generalization of Theorem 1 of [Wang, 1998], shows the usefulness of the multivariate pgf to capture at once the dependence relations between the components of $X$ and aggregation of subsets of them.

**Theorem 3.3.** Let $A = \{A_1, \ldots, A_m\}$ be a partition of the set $\{1, \ldots, n\}$, for $m \leq n$. Define the random vector $Y_A = (Y_{A_1}, \ldots, Y_{A_m})$ with $Y_{A_j} = \sum_{i \in A_j} X_i$, for $j \in \{1, \ldots, m\}$. Then the multivariate pgf of $Y_A$ is

$$P_{Y_A}(t_1, \ldots, t_m) = P_X \left( \prod_{j=1}^{m} t_j^{1_{\{i\in A_j\}}}, \ldots, \prod_{j=1}^{m} t_j^{1_{\{n \in A_j\}}} \right), \quad |t_j| \leq 1, \ j \in \{1, \ldots, m\}. \quad (10)$$

If the margins of $X$ correspond to risks related to individual business units, then one may apply Theorem 3.3 to obtain the pgf of random vectors at the department or division level within the organizational hierarchy of the entire business. If the margins of $X$ correspond to individual risks in an insurance portfolio, then Theorem 3.3 provides an expression for the pgf of total risks aggregated by coverage type or geographic regions.

Here are special cases of partitions useful in the context of Theorem 3.3. When $m = 1$, such that $A_1 = \{1, \ldots, n\}$ and $Y_{A_1} = S$, the result in (10) leads to Theorem 1 of [Wang, 1998]. For $m = 2$, we have $A_1 \subset \{1, \ldots, n\}$ and $A_2 = A_1^C = \{1, \ldots, n\} \setminus A_1$, that is, a subset of $\{1, \ldots, n\}$ and its complement. Finally, if $m = n$, then $A_j = \{j\}$ and $Y_{A_j} = X_j$, for $j \in \{1, \ldots, n\}$. Remark that for each product in the arguments of (10), only one value of $t_j$, for $j \in \{1, \ldots, m\}$, remains since $A$ is a partition of a set.

As noted in Section 4.2 of [Wang, 1998] and Section 5.1 of [Grimmett and Stirzaker, 2020], one may use pgfs to extract factorial moments, mixed moments and pmfs. In the remainder of this section, we add unconditional expected allocations to this list.

### 3.2 Ordinary generating functions for unconditional expected allocations

In this paper, our interest is that of computing unconditional expected allocations; hence, we define the function $P_S^{[i]}(t)$ as the OGF of the sequence of unconditional expected allocations for the rv $X_i$, that is,

$$P_S^{[i]}(t) := \sum_{k=0}^{\infty} t^k E \left[ X_i \times 1_{\{S=k\}} \right], \quad (11)$$
for \( i \in \{1, \ldots, n\} \).

Aiming to simplify the presentation, unless otherwise specified, we develop formulas for \( i = 1 \) for the remainder of this paper. One may obtain the other unconditional expected allocations by appropriate reindexing.

The following theorem is at the basis of the results in this paper and provides a link between the OGF of the unconditional expected allocations of the rv \( X_1 \) and the multivariate pgf of \( X \).

**Theorem 3.4.** If \( X \) is a vector of rvs with multivariate pgf \( \mathcal{P}_X \) and \( S \) is the aggregate loss rv, then the expression of \( \mathcal{P}_S^{[1]} \) is given by

\[
\mathcal{P}_S^{[1]}(t) = \left[ t_1 \times \frac{\partial}{\partial t_1} \mathcal{P}_X(t_1, \ldots, t_n) \right]_{t_1=\ldots=t_n=t} .
\]  

**Proof.** Applying Theorem 3.3 with \( m = 2, A_1 = 1 \) and \( A_2 = \{2, \ldots, n\} \), the pgf of \((X_1, S-1)\) is

\[
\mathcal{P}_{X_1, S-1}(t_1, t_{-1}) = E \left[ t_1 X_1 + t_{-1} X_2 + \cdots + X_n \right] = \mathcal{P}_X(t_1, t_{-1}, \ldots, t_{-1})
\]

for \(|t_1| \leq 1\) and \(|t_{-1}| \leq 1\). We define

\[
\mathcal{P}_{X_1, S-1}^{[1]}(t_1, t_{-1}) = \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty k_1 t_1^{k_1} t_{-1}^{k_2} f_{X_1, S-1}(k_1, k_2),
\]

which becomes

\[
\mathcal{P}_{X_1, S-1}^{[1]}(t_1, t_{-1}) = t_1 \frac{\partial}{\partial t_1} \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty t_1^{k_1} t_{-1}^{k_2} f_{X_1, S-1}(k_1, k_2) = t_1 \times \frac{\partial}{\partial t_1} \mathcal{P}_{X_1, S-1}(t_1, t_{-1}).
\]

Finally, it follows from the same arguments as in Theorem 3.3 that \( \mathcal{P}_S^{[1]}(t) = \mathcal{P}_{X_1, S-1}^{[1]}(t, t) \), which becomes

\[
\mathcal{P}_S^{[1]}(t) = \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty k_1 t_1^{k_1} t_2^{k_2} f_{X_1, S-1}(k_1, k_2) = \sum_{k=0}^\infty t^k \sum_{k_1=0}^k k_1 f_{X_1, S-1}(k_1, k - k_1) = \sum_{k=0}^\infty t^k E \left[ X_1 \times 1_{\{S=k\}} \right],
\]  

where (13) is the power series representation in (11) of unconditional expected allocations, as desired. \( \square \)

From the uniqueness theorem of pgfs (see, for instance, Section 5.1 of [Grimmett and Stirzaker, 2020]), one can recover the values of \( E \left[ X_1 \times 1_{\{S=k\}} \right], k \in \mathbb{N} \), by differentiating

\[
[t^k] \mathcal{P}_S^{[1]}(t) = E \left[ X_1 \times 1_{\{S=k\}} \right] = \frac{1}{k!} \frac{d^k}{dt^k} \mathcal{P}_S^{[1]}(t) \bigg|_{t=0},
\]

or using an algorithm to extract the coefficients of a polynomial. The entire Section 3.3 provides a method using FFT to extract the coefficients from the OGF for unconditional expected allocations.
Consequently, (13) is a powerful tool to capture every unconditional expected allocation for \(X_1\) within a single function.

An especially convenient corollary holds for allocating a rv independent from the remaining risks.

**Corollary 3.5.** If \(X_1\) and \(S_{−1}\) are independent, then the expression of \(\mathcal{P}^{[1]}_{S}\) in (12) becomes

\[
\mathcal{P}^{[1]}_{S}(t) = t \mathcal{P}_{X_1}(t) \mathcal{P}_{S_{−1}}(t). \tag{14}
\]

Aiming for a more efficient method to compute the expectations, we find an OGF for the unconditional expected cumulative allocation defined in (1).

**Corollary 3.6.** If \(X\) is a vector of rvs with multivariate pgf \(\mathcal{P}_{X}\) and \(S\) is the aggregate rv, then the function

\[
\mathcal{P}^{[1]}_{S}(t)/(1 − t) = \frac{1}{1 − t} \left[ t_1 \times \frac{\partial}{\partial t_1} \mathcal{P}_{X}(t_1, \ldots, t_n) \right]_{t_1=\cdots=t_n=t}
\]

is the OGF of the sequence of cumulative unconditional expected allocations \(\{E[X_1 \times 1_{\{S\leq k\}}]\}_{k\in\mathbb{N}}\).

**Proof.** Applying operation 5 of Lemma 3.2, we have

\[
\frac{\mathcal{P}^{[1]}_{S}(t)}{1 − t} = \sum_{k=0}^{\infty} t^k \left( \sum_{j=0}^{k} E \left[ X_1 \times 1_{\{S=j\}} \right] \right) = \sum_{k=0}^{\infty} t^k E \left[ X_1 \times 1_{\{S\leq k\}} \right]. \tag{15}
\]

\[\square\]

### 3.3 Outline of the FFT approach to compute unconditional expected allocations

Equipped with an OGF for unconditional expected allocations, one may seek to solve for the unconditional expected allocations analytically. In some cases, analytical inversion of the OGF will be possible, but otherwise, we may resort to numerical algorithms to compute the unconditional expected allocations. This section provides an algorithm to recover the unconditional expected allocations based on their OGF.

A significant advantage of working with pgfs (and more generally, with OGFs) is that the FFT algorithm of [Cooley and Tukey, 1965] provides an efficient method to extract the values of OGFs, as explained in Chapter 30 of [Cormen et al., 2009]. See also [Embrechts et al., 1993] for applications of the FFT algorithm in actuarial science and quantitative risk management.

Define the characteristic function of \(S\) as

\[
\phi_S(t) := E \left[ e^{itS} \right] = \mathcal{P}_{S} \left( e^{it} \right), \quad |t| \leq 1
\]

and analogously, the characteristic version of the OGF for unconditional expected allocations,

\[
\phi^{[1]}_{S}(t) := \sum_{k=0}^{\infty} e^{itk} E \left[ X_1 \times 1_{\{S=k\}} \right] = \mathcal{P}^{[1]}_{S} \left( e^{it} \right), \quad |t| \leq 1.
\]
In this section, we aim to recover the values of $E[X_1 \times 1_{\{S=k\}}]$ using the discrete Fourier transform (DFT). Set $f_X = (f_X(0), f_X(1), \ldots, f_X(k_{max} - 1))$ for a truncation point $k_{max} \in \mathbb{N}$. Here we assume that $f_X(k) = 0$ for $k \geq k_{max}$ such that there is no truncation error. The DFT of $f_X$, noted $\hat{f}_X = (\hat{f}_X(0), \hat{f}_X(1), \ldots, \hat{f}_X(k_{max} - 1))$, is

$$\hat{f}_X(k) = \sum_{j=0}^{k_{max}-1} f_X(j)e^{i2\pi j k_{max}}, \quad k = 0, \ldots, k_{max} - 1. \quad (16)$$

The inverse DFT can recover the original sequence with

$$f_X(k) = \frac{1}{k_{max}} \sum_{j=0}^{k_{max}-1} \hat{f}_X(j)e^{-i2\pi j k_{max}}, \quad k = 0, \ldots, k_{max} - 1. \quad (17)$$

The authors of [Embrechts and Frei, 2009] explain how computing the pmf of a compound sum is more efficient with the FFT than using Panjer recursion or direct convolution. We now show how to apply the FFT algorithm to compute unconditional expected allocations. Let $\mu_{1:k} = E[X_1 \times 1_{\{S=k\}}]$ for $k = 0, \ldots, k_{max} - 1$ and $\mu_1 = (\mu_{1:0}, \ldots, \mu_{1:(k_{max} - 1)})$ with the obvious case $\mu_{1:0} = 0$. Then, the discrete Fourier transform of $\mu_1$, noted $\tilde{\mu}_1 = (\tilde{\mu}_{1:0}, \ldots, \tilde{\mu}_{1:(k_{max} - 1)})$, is

$$\tilde{\mu}_{1:j} = P_S^{[1]}(e^{i2\pi j/k_{max}}), \quad j = 0, \ldots, k_{max} - 1. \quad (18)$$

For notational convenience, we write the vector $\{e^{i2\pi j/k_{max}}\}_{0 \leq j \leq k_{max} - 1}$ as $\tilde{\mathbf{e}}_1$. Then, we have that $\tilde{\mu}_1 = P_S^{[1]}(\tilde{\mathbf{e}}_1)$. Computing the inverse DFT of (18) yields the values of $E[X_1 \times 1_{\{S=k\}}]$ for $k = 0, \ldots, k_{max} - 1$. If $k_{max}$ is a power of 2, algorithms like the FFT of [Cooley and Tukey, 1965] are especially efficient.

Note that computing the cumulative unconditional expected allocations is trickier since division by $(1 - t)$ is undefined for $|t| = 1$. One, therefore, requires simplifications before applying the FFT algorithm to the OGF of cumulative unconditional expected allocations. In practice, one only obtains a slight numerical advantage from using the FFT algorithm for cumulative unconditional expected allocations, which has algorithmic complexity $O(n \log n)$. Suppose one computes unconditional expected allocations with the FFT algorithm and takes the cumulative sum of the result. In that case, the algorithmic complexity remains $O(n \log n)$.

One consideration when using the FFT algorithm is that one must select a truncation point large enough such that $f_S(k_{max}) = 0$. One could have a large value of $k_{max}$ if $S$ is a large portfolio or if individual risks have heavy tails. In the context of peer-to-peer insurance with a stop loss reinsurance contract with trigger $\omega$, we have $f_S(x) = 0$ for $x > \omega$, and $E \left[ X_1 \times 1_{\{S=k\}} \right] = E \left[ X_1 \times 1_{\{S=\omega\}} \right]$ for all $k \geq \omega$; thus stop loss contracts sets an upper bound to the truncation point required.

If $X_1$ is a discrete rv, independent of $S_{-1}$, then the OGF for unconditional expected allocations is given by (14). If we have no closed-form solution for $P_{X_1}(t)$, then one can compute the DFT of $tP_{X_1}(t)$ by using the pmf of $X_1$ and the properties of OGFs. One can compute the pgf of $X_1$ as $P_{X_1}(t) = \sum_{k=0}^{\infty} t^k f_X(k)$ and $tP_{X_1}(t) = \sum_{k=0}^{\infty} t^k k f_X(k)$. It follows that one can compute the DFT of $tP_{X_1}(t)$ as the DFT of the vector $\{k f_X(k)\}_{k \in \mathbb{N}}$. We can compute the DFT of $tP_{X_1}(t)/(1 - t)$ as the DFT of the partial sum of the vector $\{k f_X(k)\}_{k \in \mathbb{N}}$.

To apply the generating function approach with the FFT algorithm (or other efficient convolution algorithms) when the rvs are continuous, one must discretize their continuous cdfs for a step size
$h \in \mathbb{R}^+$. For a brief presentation of the upper, lower, and mean preserving discretization methods and their applications with the FFT algorithm, see, for instance, Section 5 of [Bargès et al., 2009] and Section 2 of [Embrechts and Frei, 2009]. Stochastic order properties for each of these three methods are examined in Chapter 1 of [Müller and Stoyan, 2002].

4 Implications for Katz distributions

Let $M$ be a positive discrete rv following a distribution belonging to the Katz family of distributions (see [Katz, 1965], Section 2.5.4 of [Winkelmann, 2008] and Section 2.3.1 of [Johnson et al., 2005]), also referred to as the $(a, b, 0)$ family of distributions in [Klugman et al., 2018]. The pmf of $M$ satisfies the recursive relation

$$f_M(k) = \frac{a + b}{k} f_M(k - 1), \quad k \in \mathbb{N}_1,$$

where $a < 1$ and $b > 0$. The expectation is $E[M] = b/(1 - a)$, while the variance is $Var(M) = b/((1 - a)^2)$. One derives the following result for the pgf of $M$ from (19).

**Lemma 4.1.** If $M$ follows a Katz distribution, then its pgf satisfies the differential equation $P'_M(t) = \frac{a + b}{1 - at} P_M(t)$.

**Proof.** See Section 4.5.1 of [Dickson, 2017].

Note that the solution to the differential equation in Lemma 4.1, as provided in equation (2.41) of [Johnson et al., 2005], is $P_M(t) = [(1 - a)/(1 - at)]^{\frac{b}{a + 1}}$, for $|t| \leq 1$ and $a \neq 0$.

Members of the Katz family are the Poisson distribution (with $a = 0$ and $b = \lambda$), the binomial distribution (with $a = -q/(1 - q)$ and $b = (n + 1)q/(1 - q)$) and the negative binomial distribution with pmf given by

$$f_M(k) = \binom{r + k - 1}{k} q^r (1 - q)^k, \quad k \in \mathbb{N},$$

with $a = 1 - q$ and $b = (r - 1)(1 - q)$. Note that each distribution has different starting values within the recursive relation (provided in the references above), but the starting values aren’t required in the current paper.

4.1 Allocations and family of Katz distributions

The following theorem presents an efficient formula to compute unconditional expected allocations.

**Theorem 4.2.** Let $X_1$ follow a Katz distribution independent of $S_{-1}$. For $|a| < 1$ and $k \in \mathbb{N}_1$, we have

$$[t^k] P_S^{[1]}(t) = E\left[X_1 1_{\{S=k\}}\right] = (a + b) \sum_{j=0}^{k-1} a^j f_S(k - 1 - j),$$

where $f_S(k) = f_M(k)$.
and

\[
[t^k] \left\{ \frac{\mathcal{P}_S^{[1]}(t)}{1-t} \right\} = E \left[ X_1 \times 1_{\{S \leq k\}} \right] = (a + b) \sum_{j=0}^{k-1} a^j f_S(k-1-j) \tag{21a}
\]

\[
= (a + b) \sum_{j=0}^{k-1} \frac{1 - a^{j+1}}{1-a} f_S(k-1-j). \tag{21b}
\]

**Proof.** Applying Corollary 3.5 and Lemma 4.1, the OGF for unconditional expected allocations is

\[
\mathcal{P}_S^{[1]}(t) = t \mathcal{P}'_{X_1}(t) \mathcal{P}_{S-1}(t) = t \frac{a + b}{1-at} \mathcal{P}_{X_1}(t) \mathcal{P}_{S-1}(t) = t \frac{a + b}{1-at} \mathcal{P}_S(t). \tag{22}
\]

Then, (20) follows from Property 4 of OGFs in Lemma 3.2. The relation in (21a) follows from another application of Property 4 of Lemma 3.2 to (20). Alternatively, the OGF for cumulative unconditional expected allocations is

\[
\mathcal{P}_S^{[1]}(t) = \frac{t}{1-t} \mathcal{P}_S(t) = \frac{a + b}{a - 1} \mathcal{P}_S(t) \left( \frac{1}{1-at} - \frac{1}{1-t} \right) = \frac{a + b}{a - 1} \mathcal{P}_S(t) \sum_{k=0}^{\infty} t^k \left( a^k - 1 \right) \tag{23}
\]

for \( |t| < 1 \). Then, (21b) also follows from the convolution property of OGFs in Lemma 3.2. \( \square \)

Notice that (20) and (21b) require the same number of computations, so it isn’t more complex to compute cumulative unconditional expected allocations than individual valued allocations. We also have the relationship

\[
(a - 1) E \left[ X_1 \times 1_{\{S \leq k\}} \right] = a E \left[ X_1 \times 1_{\{S = k\}} \right] - (a + b) f_S(k-1), \quad |a| < 1, \quad k \in \mathbb{N}. \tag{24}
\]

We list the implications of Theorem 4.2 in Table 1, which hold whenever \( X_1 \) and \( (X_2, \ldots, X_n) \) are independent, even if the random vector \((X_2, \ldots, X_n)\) has a complicated dependence structure. The following two examples are special cases of Theorem 4.2 with practical interest. Both examples show that by first writing the problem in the transformed space, we can then invert the OGF back to the original probability space to obtain closed-form or recursive-type expressions for unconditional expected allocations. In these cases, we do not require the FFT algorithm to compute the unconditional expected allocations directly (although one may need to use the FFT algorithm to compute the pmf of \( S \)).

**Example 4.3 (Poisson distributions).** Assume that \( X_1, \ldots, X_n \) are independent with \( X_i \sim \text{Pois}(\lambda_i) \), \( i \in \{1, \ldots, n\} \). Then, \( S \sim \text{Pois}(\lambda_S) \), with \( \lambda_S = \lambda_1 + \cdots + \lambda_n \). From (20) of Theorem 4.2, we recover the result presented in Section 10.3 of [Marceau, 2013, page 413],

\[
[t^k] \mathcal{P}_S^{[1]}(t) = E \left[ X_1 \times 1_{\{S = k\}} \right] = \lambda S^{k-1} e^{-\lambda_S} = \frac{\lambda_1^k}{\lambda_S} k \Pr(S = k), \quad k \in \mathbb{N}.
\]

Thus, we have \( E[X_1 | S = k] = \lambda_1 / \lambda_S k \), which is a linear function of \( k \), hence the contribution under the conditional mean risk-sharing rule coincides with the contribution under the proportional (or linear) allocation rule.
Example 4.4 (Negative binomial distributions). Assume that $X_1, \ldots, X_n$ are independent with $X_i \sim NB(r_i, q_i)$, $i \in \{1, \ldots, n\}$. Inserting $a = (1 - q_1)$ and $b = (r_1 - 1)(1 - q_1)$ into (22), we have

$$P_S^{[1]}(t) = t \cdot \frac{r_1(1 - q_1)}{1 - (1 - q_1)t} P_S(t) = \frac{r_1(1 - q_1)}{q_1} t \left( \frac{q_1}{1 - (1 - q_1)t} \right)^{r_1+1} P_{S-1}(t), \quad |t| \leq 1.$$  

We can define

$$P_{S^*}(t) := t \left( \frac{q_1}{1 - (1 - q_1)t} \right)^{r_1+1} P_{S-1}(t),$$

which corresponds to the pgf of a rv $S^*$ whose pmf is the convolution of the pmfs of $n$ negative binomial distributed rvs, shifted to the right by one. It follows that $P_S^{[1]}(t) = r_1(1 - q_1)/q_1 P_{S^*}(t)$, which we can invert to the original space and apply Theorem 1 of [Furman, 2007] to obtain

$$[t^k]P_S^{[1]}(t) = E[X_1 \times 1_{\{S=k\}}] = r_1 \frac{1 - q_1}{q_1} R \sum_{\ell=0}^{\infty} \delta_{t}(r + \ell + k - 1) P_{\ell+1}(1 - q^*)^{k-1},$$

where $q^* = \min(q_1, \ldots, q_n)$, $r = r_1 + \cdots + r_n$,

$$R = \left( \frac{q^*}{1 - q^*} \right)^n \prod_{j=1}^{n} \frac{(1 - q_j)}{q_j}$$

$$\delta_{\ell+1} = 1 \frac{1}{\ell + 1} \sum_{i=1}^{\ell+1} i \xi_i \delta_{\ell+1-i}; \quad \delta_0 = 1; \quad \ell \in \mathbb{N}_0$$

and

$$\xi_i = \frac{r_1 + 1}{i} \left( 1 - \frac{q^*}{q^*(1 - q_1)} \right) + \sum_{j=2}^{n} \frac{r_j}{i} \left( 1 - \frac{q^*}{q^*(1 - q_j)} \right).$$

For binomial distributions, one requires the success probability to satisfy $q < 1/2$ such that $|a| < 1$. Alternately, if $q > 1/2$, one could express the problem in terms of failure probability $1 - q$ and then apply Theorem 4.2.

|   | Poisson | Negative binomial | Binomial |
|---|--------|------------------|---------|
| $a$ | 0      | $1 - q$          | $-q/(1-q)$, for $0 < q < 1/2$ |
| $b$ | $\lambda$ | $(r-1)(1-q)$ | $(n+1)q/(1-q)$ |
| $E[X_1 \times 1_{\{S=k\}}] (\lambda)$ | $\mu S(k-1)$ | $r \sum_{j=1}^{k} (1 - q)^j f_S(k - j)$ | $n \sum_{j=1}^{k} (1 - q)^j f_S(k - j)$ |
| $E[X_1 \times 1_{\{S \leq k\}} (\mu)] (\mu)$ | $\lambda f_S(k-1)$ | $r \sum_{j=1}^{k} (1 - q)^j f_S(k - j)$ | $n \sum_{j=1}^{k} (1 - q)^j f_S(k - j)$ |
| $E[X_1 \times 1_{\{S \leq k\}} (\mu)] (\nu)$ | $\lambda f_S(k-1)$ | $r \sum_{j=1}^{k} (1 - q)^j f_S(k - j)$ | $n \sum_{j=1}^{k} (1 - q)^j f_S(k - j)$ |

Table 1: Implications of Theorem 4.2 for all distributions.

4.2 Allocations and family of compound Katz distributions

Let $M$ be a frequency rv with support on $\mathbb{N}$. Let $\{B_1, B_2, \ldots\}$ form a sequence of independent, identically distributed and non-negative severity rvs, independent of $M$. Within the context of the
current paper, we assume that the severity rvs take values in \( \mathbb{N} \). In this section, we consider cases where the rv \( X \) is defined as a random sum, that is,

\[
X = \begin{cases} 
0, & M = 0 \\
\sum_{j=1}^{M} B_j, & M > 0.
\end{cases}
\] (25)

It follows from (25) that the pmf of \( X \) is

\[
f_X(k) = \begin{cases} 
\Pr(M = 0), & k = 0 \\
\sum_{j=1}^{\infty} \Pr(M = j) \Pr(B_1 + \cdots + B_j = k), & k \in \mathbb{N}.
\end{cases}
\]

Evaluation of \( \Pr(B_1 + \cdots + B_j = k) \) is analytically and computationally expensive since direct computation results from \( j - 1 \) convolutions. Fortunately, [Panjer, 1981] and others have developed efficient recursive relationships to compute the pmf of \( X \) when \( M \) is a Katz distribution; we often refer to these relations as Panjer recursions. We are now interested in the OGF for unconditional expected allocations for compound Katz distributions such that we may have an efficient algorithm for unconditional expected allocations.

Theorem 4.5. Let \( X_1 \) be a rv having a compound Katz distribution with frequency rv \( M_1 \) having cdf in the Katz family of distributions with parameter \(|a| < 1 \) and discrete severity rv \( B_1 \), with \( X_1 \) independent of \( S_{-1} \). The OGF of unconditional expected allocations is

\[
P_S^{[1]}(t) = tP_{B_1}^l(t)P_{M_1}(P_{B_1}(t))P_{S_{-1}}(t).
\] (26)

Further, if \(|aP_{B_1}(t)| < 1 \) for all \(|t| < 1 \), then

\[
P_S^{[1]}(t) = tP_{B_1}^l(t)\frac{a+b}{1-aP_{B_1}(t)}P_{S}(t).
\] (27)

Proof. The pgf of the compound rv \( X_1 \) is \( P_{X_1}(t) = P_{M_1}(P_{B_1}(t)) \), then (26) follows directly from (14). The relation in (27) follows from the chain rule and Lemma 4.1.

Example 4.6 (Independent compound Poisson distributions). Let \( X_1 \) be a rv whose distribution belongs to the class of compound Poisson distributions, whose severity distribution is discrete with support \( \mathbb{N} \). We have

\[
P_S^{[1]}(t) = \lambda_1 tP_{B_1}^l(t)P_{M_1}(P_{B_1}(t))P_{S_{-1}}(t) = \lambda_1 tP_{B_1}^l(t)P_{S}(t).
\] (28)

It follows that

\[
[t^k]P_S^{[1]}(t) = E[X_1 \times 1_{\{S=k\}}] = \lambda_1 \sum_{l=1}^{k} f_{B_1}(l)f_{S}(k-l), \quad k \in \mathbb{N}_1
\]

and

\[
[t^k] \left\{ \frac{P_S^{[1]}(t)}{1-t} \right\} = E[X_1 \times 1_{\{S\leq k\}}] = \lambda_1 \sum_{l=1}^{k} E \left[ B_1 \times 1_{\{B_1\leq l\}} \right] f_{S}(k-l), \quad k \in \mathbb{N}_1
\]

\[
= \lambda_1 \sum_{l=1}^{k} f_{B_1}(l)F_{S}(k-l), \quad k \in \mathbb{N}_1.
\]

Remark 4.7. The results of Theorem 4.5 are analogous to Section 4 of [Denuit and Robert, 2020] when the severity follows a discrete distribution. One can recover continuous versions of the results from [Denuit and Robert, 2020] using the continuous version of the OGF for cumulative unconditional expected allocations; see Section 7 for details.
4.3 Algorithm for a sum of independent compound Poisson distributed rvs

Consider a portfolio of $n$ independent participants, where $X_i$ is a compound Poisson distributed rv with frequency parameter $\lambda_i$ and discrete severity rv $B_i$ for $i = 1, \ldots, n$. We have $P_S(t) = \prod_{i=1}^{n} P_{M_i}(P_{B_i}(t))$. For Poisson distributions, the OGF of unconditional expected allocations for the $i$th risk, $i \in \{1, \ldots, n\}$, is $\lambda_i t P'_{B_i}(t) P_S(t)$.

We use the FFT algorithm to compute the unconditional expected allocations. The most computationally intensive step is using the FFT algorithm to compute the values of $f_S$. Fortunately, since the term $P_S(t)$ is present for the OGF of the probability masses of $S$, of the unconditional expected allocations and of the cumulative unconditional expected allocations, one must only compute the DFT of $P_S(t)$ once. In Algorithm 1, we present a method to compute the unconditional expected allocations efficiently using the FFT algorithm. One can change line 6 by the cumulative sum of the vector to compute cumulative unconditional expected allocations.

Algorithm 1: Conditional means for compound Poisson distributions.

| Input: Parameters $\lambda_i, f_{B_i}$ for $i = 1, \ldots, n$. |
| Output: Unconditional expected allocations $E[X_i|S = k]$ for $k = 0, \ldots, k_{\text{max}} - 1$ and $i = 1, \ldots, n$. |
| 1 for $i = 1, \ldots, n$ do |
| 2 Compute $\hat{f}_{X_i} = P_{X_i}(\hat{e}_1)$ or with (16); |
| 3 Compute the DFT of $S$ as the element-wise product $\hat{f}_S = \prod_{i=1}^{n} \hat{f}_{X_i}$; |
| 4 Compute $f_S$ by taking the inverse DFT of $\hat{f}_S$; |
| 5 for $i = 1, \ldots, n$ do |
| 6 Compute the DFT $\hat{\phi}_{B_i}$ of the vector $\{(k + 1)f_{B_i}(k + 1)\}_{0 \leq k \leq k_{\text{max}} - 1}$; |
| 7 Compute element-wise $\hat{\mu}_i = \lambda_i \hat{e}_1 \times \hat{\phi}_{B_i} \times \hat{f}_S$; |
| 8 Compute $\mu_i$ as the inverse DFT of $\hat{\mu}_i$; |
| 9 Compute $\{E[X_i|S = k]\}_{0 \leq k \leq k_{\text{max}} - 1}$ by the element-wise division $\mu_i/f_S$; |
| 10 Return $\{E[X_i|S = k]\}_{0 \leq k \leq k_{\text{max}} - 1}$ for $i = 1, \ldots, n$. |

5 Applications of the FFT algorithm

In this section, we present a few applications that use the FFT algorithm to compute unconditional expected allocations and observe their implications for risk-sharing. We start with a small portfolio of risks, where the FFT algorithm is not essential but will explain the method and point out numerical considerations. Then, we consider a larger portfolio to show that the method scales well to problems with many agents. We examine the numerical comparison of direct, recursive, and transform-based approaches and apply arithmetization techniques to a problem involving heavy-tailed risks. To the best of our knowledge, our method is the first to efficiently handle the large and heavy-tailed portfolios examined in this section unless each risk is identically distributed, testifying to the utility of our approach in practical situations.
Table 2: Values of $\lambda_i$ and $f_{C_i}$ for each participant $i \in \{1, \ldots, 4\}$ for a small pool of four participants.

|   | $\lambda_i$ | $f_{C_i}(1)$ | $f_{C_i}(2)$ | $f_{C_i}(3)$ | $f_{C_i}(4)$ |
|---|-------------|---------------|---------------|---------------|---------------|
| 1 | 0.08        | 0.1           | 0.2           | 0.4           | 0.3           |
| 2 | 0.08        | 0.15          | 0.25          | 0.3           | 0.3           |
| 3 | 0.1         | 0.1           | 0.2           | 0.3           | 0.4           |
| 4 | 0.1         | 0.15          | 0.25          | 0.3           | 0.3           |

5.1 Small portfolio of independent compound Poisson distributed rvs

We replicate Case 1 of the application in Section 6.1 of [Denuit, 2019]. Consider four participants in a pool, and each participant contributes risk $X_i$ that follows a compound Poisson distribution, with parameter $\lambda_i$ and a discrete severity whose pmf is $f_{C_i}$ with support $\{1, 2, 3, 4\}$, for $i \in \{1, \ldots, 4\}$. We present the values of $\lambda_i$ and $f_{C_i}$ for each participant $i \in \{1, \ldots, 4\}$ in Table 2. We provide the R code in Appendix A.1, the numerical values that follow come from R version 4.0.4. Besides the setup and validation code, the actual computation of conditional means takes fewer than 15 lines (even if the number of participants grows). We recover the values in [Denuit, 2019].

In Figure 1, we present three graphs: the pmf of $S$, the total unconditional expected allocations for a given outcome of $S$, and the total conditional means for a given outcome of $S$. One should have $\sum_{i=1}^n E[X_i \times 1_{\{S=k\}}] = k \Pr(S = k)$, which is what we observe in the middle plane of Figure 1.

To compute the unconditional expected allocations, we must divide the total unconditional expected allocations by the pmf of $S$. If the pmf of $S$ is very small for some values of $k$, the unconditional expected allocations may be inaccurate. This occurs since machines have finite precision, and exact zeroes are not always represented accurately due to underflow issues. The FFT algorithm introduces small numerical errors during the computation of the pmf of $S$ and the unconditional expected allocations. Still, these errors are negligible since they occur for events whose probability is close to the machine precision. Therefore, one should only consider the unconditional expected allocations for values of $k$ where $\Pr(S = k)$ is not too small, which are the values of $k$ where the unconditional expected allocations are accurate.

While the transform-based approach proposed in this paper provides accurate unconditional expected allocations for the important values of $k$, it is useful to identify the values of $k$ where the unconditional expected allocations are inaccurate. One way to validate the accuracy of unconditional expected allocations is to validate that the full allocation property in (4) holds. In this case, we expect $\sum_{i=1}^n E[X_i | S = k] = k$ for all $k \in \mathbb{N}$. We plot the curve $\sum_{i=1}^n E[X_i | S = k]$ in the right plane of Figure 1. That curve is linear between $k = 0$ and $k = 37$. However, one has $\sum_{i=1}^n E[X_i | S = 38] = 38.05$, which is slightly higher than 38. The FFT method of computing unconditional expected allocations provides inaccurate values when the mass function is under machine precision; for example, we have $\sum_{i=1}^n E[X_i | S = 43] = 116$ and $\sum_{i=1}^n E[X_i | S = 63] = -146$. However, we have $\Pr(S = 43) = 1.7 \times 10^{-17}$ and $\Pr(S = 63) = 3.3 \times 10^{-19}$, that is, they are numerically indiscernible from zero because of underflow. We will investigate the impact of underflow on the unconditional expected allocations in Section 5.3, where we show that the FFT algorithm is still more useful than the direct and recursive methods for large portfolios of risks.
Probability mass function of $S$

$Pr(S=k)$

0 20 40 60
0
0.2
0.4
0.6

Total expected allocations

$\sum_{i=1}^{n} E[X_i \times 1_{\{S=k\}}]$

0 20 40 60
0
0.1
0.2
0.3

Total conditional means

$\sum_{i=1}^{n} E[X_i | S=k]$

0 20 40 60
−100
0
100

Figure 1: Left: pmf of $S$. Middle: $\sum_{i=1}^{n} E[X_i \times 1_{\{S=k\}}]$. Right: $\sum_{i=1}^{n} E[X_i | S=k]$.

5.2 Large portfolio of independent compound Poisson distributed rvs

We consider a portfolio or pool of 10,000 risks in the second application. Each risk $X_i$ is independent and follows a compound Poisson distribution with parameter $\lambda_i$, with severity rv $B_i \sim NB(r_i, q_i)$, implying that $E[X_i] = \lambda_i r_i (1 - q_i)/q_i$, for $i = 1, \ldots, 10,000$. We set each risk to have different triplets of parameters. For illustration purposes we simulate the triplets of parameters $(\lambda_i, r_i, q_i)$ for $i = 1, \ldots, 10,000$ according to $\lambda_i \sim \text{Exp}(10)$, $r_i \sim \text{Unif}(\{1, 2, 3, 4, 5, 6\})$ and $q_i \sim \text{Unif}(\{0.4, 0.5\})$ such that on average, $\lambda_i = 0.1$, $r_i = 3.5$ and $q_i = 0.45$. We present the simulated parameters and expected values for the first eight contracts in Table 3.

| $i$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\lambda_i$ | 0.161152 | 0.031859 | 0.027368 | 0.238748 | 0.115137 | 0.470203 | 0.146247 | 0.011747 |
| $q_i$ | 0.489756 | 0.423367 | 0.455898 | 0.451500 | 0.486834 | 0.440405 | 0.440082 | 0.481335 |
| $r_i$ | 2   | 6   | 1   | 4   | 5   | 6   | 3   | 1   |
| $E[X_i]$ | 0.335788 | 0.260354 | 0.032668 | 1.160162 | 0.728190 | 2.987289 | 0.558214 | 0.012658 |

Table 3: First eight sets of parameters.

We present the code in Appendix A.2, using R version 4.0.4. That code computes conditional means for each of the 10,000 unique risks and takes approximately 16 seconds on a personal computer (with an Intel@Core™i5-7600K CPU @ 3.80GHz CPU).

In Figure 2 we present the pmf of the conditional means $E[X_i | S]$ for $i \in \{1, \ldots, 8\}$. Note that all eight pmfs share the same values on the y-axis but differ on the x-axis. This is because the relationship giving the probability masses for conditional means is

$$Pr(E[X_i | S] = E[X_i | S = k]) = \sum_{j \in \mathbb{N} : E[X_i | S = j] = E[X_i | S = k]} Pr(S = j),$$

for all $i \in \{1, \ldots, 10,000\}$ and $k \in \mathbb{N}$. In the case where each risk is heterogeneous, we have that $Pr(E[X_i | S] = E[X_i | S = k]) \approx Pr(S = k)$; the only difference is the domains of $E[X_i | S]$ for $i \in \{1, \ldots, 10,000\}$. Indeed, although the pmf for the conditional means of risk $j = 2$ (in red) appears to be a single point mass, we observe by magnifying that the pmf shares the probability values from the other pmfs. Also, as shown by the authors of [Denuit and Robert, 2021b] under
mild technical conditions, the conditional means converge to the expected value. For illustration purposes, we add vertical dashed lines at the expected values.

\[
\Pr(E[X_i|S] = x) \quad i = 1, \ldots, 8
\]

Figure 2: Probability mass function of the conditional means \(E[X_i|S]\), for the contracts \(i \in \{1, \ldots, 8\}\). The vertical lines go through the values \(E[X_i]\), for \(i \in \{1, \ldots, 8\}\).

### 5.3 Numerical comparison of approaches

In this section, we compare the time required and the accuracy of the numerical approximation from the FFT algorithm. We will compare the direct approach, the recursive approach, and the FFT algorithm. To do this, we consider a portfolio of two risks \(X_1\) and \(X_2\) with compound Poisson distributions and Pareto severities, where the Poisson rates are respectively \(\lambda_1 = 0.05\) and \(\lambda_2 = 0.02\), the Pareto shapes are respectively \(\alpha_1 = 4\) and \(\alpha_2 = 5\) and the Pareto scales are respectively \(\beta_1 = 2000\) and \(\beta_2 = 3000\). The expected values are \(E[X_1] = 33.3333\) and \(E[X_2] = 15\). Since our approach works with discrete random variables, we consider discretized Pareto severities with different discretization steps using the moment matching method; see, for instance, Appendix E.2 of [Klugman et al., 2018] for discretization methods. Selecting a small discretization step will yield more accurate approximations but require more computation time.

We discretize up to the truncation point of \(k_{max} = 2^{18} = 262144\), such that, numerically, \(F_{B_1}(k_{max}) = F_{B_2}(k_{max}) = 1\), so there is no aliasing error (up to machine precision). In Table 4, we present the computation time for the direct, recursive, and FFT approaches for different discretization steps \(h\). We observe that the FFT approach is significantly faster than the direct and recursive approaches, and the time increase is more significant for the direct and recursive approaches. The reason is that the direct approach has a complexity of \(O(1/h^3)\), the recursive approach has a complexity of \(O(1/h^2)\), and the FFT approach has a complexity of \(O(- \log(h)/h)\). In this case, the FFT approach enables one to compute the conditional means at the level of a penny in less than a minute, which is not possible with the direct and recursive approaches in a reasonable time.
Table 4: Computation time for direct, recursive and FFT approaches.

| h    | 1000  | 100   | 10    | 1     | 0.1   | 0.01  |
|------|-------|-------|-------|-------|-------|-------|
| Direct | 0.1446 | 15.4318 | 1987.458 | –    | –    | –    |
| Recursive | 0.0253 | 0.970  | 94.300 | 9766.796 | –    | –    |
| Transform | 0.0003 | 0.003  | 0.025  | 0.238 | 5.835 | 59.955 |

We will now study the accuracy of the numerical approximation for the direct, recursive, and FFT approaches. To allow for exact computation of the conditional means and to compare the approaches with the exact values, we consider a discretization step of $h = 100$. Denote by $g_{i;k}^D$, $g_{i;k}^R$, and $g_{i;k}^T$ the conditional means of $X_i$ given $S = k$ computed using the direct, recursive, and FFT approaches, respectively. The conditional mean risk-sharing rules satisfy the full allocation property (4). We compute the stopping time based on the transform-based conditional means $\gamma^*(k_{\text{max}}) = \inf\{k \in h\mathbb{N} : |g_{1;k}^T + g_{2;k}^T - k| \leq \varepsilon\}$, where the conditional means are computed with marginals truncated at $k_{\text{max}}$. We assume that the conditional means are accurate up to the tolerance $\varepsilon$ for $k \leq \gamma^*$, and calculate the errors based on this assumption. The direct approach is exact up to the truncation point, that is, for $k_{\text{max}} > \gamma^*(k_{\text{max}})$, or if the truncation point is such that the probabilities sum to one (at least up to machine precision). The recursive and FFT approaches are approximations due to wrap-around, aliasing, and FFT underflow errors.

To compute the errors of different computation methods, we consider two error measures: the sum of absolute errors and the supremum of absolute errors. We compute the errors for the direct, recursive and transform approaches with different truncation points. We will compute the exact conditional means using the direct approach with a truncation point of $2^{18}$, and we will denote the exact conditional means by $g_{i;k}^*$. The two error measures we consider are the sum of absolute errors and the supremum of absolute errors, defined as

$$E_{\text{sum},j}^m := \sum_{k \in \{0, h, 2h, \ldots, \gamma^*\}} |g_{j;h}^m - g_{j;k}^m|$$

and

$$E_{\text{sup},j}^m := \sup_{k \in \{0, h, 2h, \ldots, \gamma^*\}} |g_{j;h}^* - g_{j;k}^*|,$$

for $j \in \{1, 2\}$ and $m \in \{D, R, T\}$. We present, in Table 5, the errors for the direct, recursive, and transform approaches with different truncation points. When the truncation point is too small (e.g., $2^{14}$), the discretization probabilities for the severity distributions do not sum to one, which leads to errors in all approaches (mostly due to errors from computing the probability mass of $X_1$ and $X_2$). Note that, since the balance property is satisfied (up to $\varepsilon$), we will have $E_{\text{sum},1}^m \approx E_{\text{sum},2}^m$ and $E_{\text{sup},1}^m \approx E_{\text{sup},2}^m$, and we only report the errors for $X_1$ in Table 5.

When the truncation point $k_{\text{max}}$ is too small, $F_{B_1}(k_{\text{max}})$ and $F_{B_2}(k_{\text{max}})$ are smaller than one and we have $\gamma^*(k_{\text{max}}) > k_{\text{max}}$, leading to errors in all approaches (the largest errors coming from the direct approach). In this application, we have for $k_{\text{max}} = 2^{15}$ that $\gamma^*(k_{\text{max}})$ larger than $k_{\text{max}}$ for a single step, leading to large errors. For $k_{\text{max}} = 2^{16}$ and $k_{\text{max}} = 2^{17}$, we have $\gamma^*(k_{\text{max}}) < k_{\text{max}}$, so there are no errors for the direct approach. The wrap-around errors from the recursive and FFT approaches are similar. While the errors are larger for $k_{\text{max}} = 2^{17}$ than for $k_{\text{max}} = 2^{16}$, more values...
Table 5: Errors for the recursive and FFT approaches with different truncation points.

| $k_{max}$ | $2^{14} = 16384$ | $2^{15} = 32768$ | $2^{16} = 65536$ | $2^{17} = 131072$ |
|-----------|----------------|----------------|----------------|----------------|
| $F_{B_1}(k_{max})$ | 0.9998599 | 0.9999891 | 0.9999992 | 0.9999999 |
| $F_{B_2}(k_{max})$ | 0.9999112 | 0.9999958 | 0.9999998 | 1.0000000 |
| $E_{sum,1}^D$ | 14120.947 | 783.3588 | 0.000000 | 0.000000 |
| $E_{sum,1}^D$ | 2465.381 | 783.3588 | 0.000000 | 0.000000 |
| $E_{sum,1}^R$ | 35620.298 | 239.3113 | 0.541361 | 1.245401 |
| $E_{sum,1}^R$ | 1265.880 | 239.2936 | 0.017958 | 0.027168 |
| $E_{sum,1}^T$ | 35620.312 | 239.3079 | 0.536580 | 1.243309 |
| $E_{sum,1}^T$ | 1265.880 | 239.2937 | 0.018011 | 0.027106 |
| $\gamma^*(k_{max})$ | 20800 | 32800 | 56800 | 64000 |
| $F_S(\gamma^*(k_{max}))$ | 0.9999963 | 0.9999994 | 0.9999999 | 1.0000000 |

are computed, and the extra values are computed for a higher $k$ where the errors are larger. Still, the errors are small, and the worst-case error corresponds to three pennies for the recursive and FFT approaches, which is acceptable for most applications. Since the FFT approach is significantly faster than the recursive approach, we recommend using the FFT approach for computing the conditional means.

5.4 Portfolio of heavy tailed risks

Next, we consider the computation of unconditional expected allocations for a portfolio of heavy-tailed risks. In particular, we consider risks whose variance does not exist; hence, the central limit theorem results of [Denuit and Robert, 2021b] do not hold because the variance of the sum of each rv does not exist. We consider a portfolio of size $n \in \{3, 100, 1000\}$ and compare the behaviour of the first three contracts. Our goal is to illustrate empirically that the conditional mean for each contract converges to their marginal mean. We set $X_i, i \in \{1, \ldots, n\}$, to follow an arithmetized Pareto distribution defined using the moment matching method. Further, we select parameters $\alpha_i \in [1.3, 1.9], \lambda_i \sim Unif([5, 15])$, such that $E[X_i] \approx 10$ for $i \in \{1, 2, 3\}$. We write the approximate symbol since the mean may not be preserved exactly due to truncation since Pareto rvs are heavy-tailed. For the remaining risks $X_i, i \in \{4, \ldots, 1000\}$, we simulate the parameters according to $\alpha_i \sim Unif([1.3, 1.9])$ and $\lambda_i \sim Unif([5, 15])$, implying $50/9 \leq E[X_i] \leq 50$ for $i \in \{4, \ldots, 1000\}$, and the variance does not exist for any risk in the portfolio. We provide the R code in Appendix A.3.

In Figure 3, we present the cdf of the conditional means for risks $X_1, X_2$ and $X_3$. The dashed, dotted, and dash-dotted lines present the cdf of conditional means for $n = 3, 100$ and $1000$ respectively. Due to the heavy-tailed risks, one must select a large truncation point $k_{max}$ to avoid aliasing (see, for instance, [Grubel and Hermesmeier, 1999] and [Embrechts and Frei, 2009] for discussions on aliasing with FFT methods for aggregation). Hence, we compute $1000 \times 2^{20}$ values, which takes approximately 9 minutes on a personal laptop. To facilitate comparisons, we present the cdf of $X_i (n = 1)$ in black and the expected value of $X_i$ in green (vertical line), $i \in \{1, 2, 3\}$. Each cdf crosses...
once. Therefore, according to the Karlin-Novikoff criteria, given that they share the same mean, the conditional means are ordered under the convex order, as expected; see, for instance, [Denuit and Dhaene, 2012]. One may observe that the cdfs of the conditional means approach the cdf of a degenerate rv at the mean. The conditional mean of $X_3$ approaches the degenerate rv at its mean faster since its tail is lighter than $X_1$ or $X_2$. Indeed, one observes that the cdf of $E[X_3|S=x]$ is almost vertical, while the cdf of $E[X_1|S=x]$ is not.

![Figure 3: Cumulative distribution function of conditional means for $n = 1, 3, 100, 1000$.](image)

According to this application, one observes that the conditional mean $E[X_1|S]$ converges in distribution to the expected value $E[X_1]$ as the portfolio size increases. However, future research remains to show that this conjecture is true in general, that is, providing a law of large numbers result for the conditional mean, generalizing the results of [Denuit and Robert, 2020] and [Denuit and Robert, 2021b].

### 5.5 Small portfolio of heterogeneous losses

Let $I = (I_1, \ldots, I_n)$ be a vector of independent Bernoulli rvs with marginal probabilities $q_i \in (0, 1)$, for $i \in \{1, \ldots, n\}$. Further define the rv $X_i = b_i \times I_i$, with $b_i \in \mathbb{N}_1$, for $i \in \{1, \ldots, n\}$. This model is sometimes called the individual risk model (with a fixed payment amount) and has applications, for instance, in life insurance, where death benefits are usually known in advance, or for insurance-linked securities in situations where investors recover their initial investment unless a trigger event occurs before the maturity date. The interested reader may refer to [Klugman et al., 2018] for detailed examples of the individual risk model. The multivariate pgf of $X = (X_1, \ldots, X_n)$ is

$$\mathcal{P}_X(t_1, \ldots, t_n) = \prod_{i=1}^{n} (1 - q_i + q_i t_i^{b_i}),$$  \hspace{1cm} (29)

while the OGF of the sequence of unconditional expected allocations for risk $X_1$ is

$$\mathcal{P}_S^{[1]}(t) = q_1 b_1 t^{b_1} \prod_{i=2}^{n} (1 - q_i + q_i t^{b_i}).$$

To compute exact values of the pmf and unconditional expected allocations using the FFT approach, one must select $k_{\text{max}} \geq 1 + \sum_{i=1}^{n} b_i$ (or select the smallest $m$ such that $2^m \geq 1 + \sum_{i=1}^{n} b_i$).

Let us discuss some of the theoretical difficulties with computing the conditional means in the context of this application. To do so, we will need some notation. The cardinality of a set $\mathcal{A}$
is denoted by $|A|$. Define the set $B = \{(x_1, \ldots, x_n) : x_i \in \{0, b_i\}, 1 \leq i \leq n\}$ as all distinct possible outcomes of $X$. Note that $|B| = 2^n$. Define $B_k = \{(x_1, \ldots, x_n) \in B : \sum_{i=1}^n x_i = k\}$, for $k = 0, 1, \ldots, s_{\text{max}}$, where $s_{\text{max}} = \sum_{i=1}^n b_i$. Note that $|B_0| = 1$ and $|B_{s_{\text{max}}}| = 1$.

The sets $B_k$ and $B_{k'}$ are mutually exclusive, i.e. $B_k \cap B_{k'} = \emptyset$, for $k \neq k' \in \{1, \ldots, n\}$. Also, $\bigcup_{k=0}^{s_{\text{max}}} B_k = B$. When $B_k$ is empty ($B_k = \emptyset$), we have $|B_k| = 0$, meaning the event $\{S = k\}$ is impossible. Such situations may occur when the number of contracts is small and the coverage amounts are heterogeneous. We say that $k$ is a possible outcome of the total losses $S$ if $|B_k| > 0$.

Fix $k \in \{0, 1, \ldots, s_{\text{max}}\}$ such that $|B_k| = 1$, and let $(x_1, \ldots, x_n)$ be the element of that singleton. This implies that the conditional expectation is given by $E[X_i|S = k] = x_i$, for $x_i \in \{0, b_i\}$, which means that the support of $E[X_i|S]$ is 0 or its full coverage $b_i$, for $i \in \{1, \ldots, n\}$. In other words, the support of $E[X_i|S]$ is the same as the support of $X_i$; and a participant in a pool has not benefited from a diversification of its risk, no matter the size $n$ of the portfolio. As $|B_k|$ increases, the support of $E[X_i|S]$ increases more elements, and these are the situations where insurance provides more value to customers. Counting the number of partitions of a set is a difficult problem in number theory. Fortunately, the OGF method provides a numerical solution to compute the unconditional expected allocations without further notions of number theory. See also Example 4.1 of [Denuit et al., 2021] for a situation where some participants do not diversify due to partitions of odd numbers.

We consider a portfolio of $n = 6$ risks. We present the parameters for this example in Table 6, and the code to replicate this study is in Appendix A.4.

| $i$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ |
|-----|-----|-----|-----|-----|-----|-----|
| $b_i$ | $1$ | $3$ | $10$ | $4$ | $5$ | $10$ |
| $q_i$ | $0.8$ | $0.2$ | $0.3$ | $0.05$ | $0.15$ | $0.25$ |

Table 6: Marginal parameters for a small portfolio of heterogeneous losses.

In Figure 4, we present the conditional means along with the pmf and cdf of conditional means for risks $X_1$, $X_2$ and $X_3$. We describe each panel in the following:

- The left panel presents the values of $E[X_i|S]$, for $i \in \{1, 2, 3\}$. Note that for the claim severity values in Table 6, we have $|B_2| = |B_{31}| = 0$, hence the events $S = 2$ and $S = 31$ are impossible and we have $\Pr(S = 2) = \Pr(S = 31) = 0$. When computed using the pgf in (34) and the FFT algorithm, we have $\Pr(S' = 2) = \Pr(S' = 31) \approx 10^{-16}$ since this is the underflow error using double precision with IEEE 754. Hence, the conditional means should be 0 for $k = 2$ and $k = 31$; dividing two underflowed values generates erratic results. These values should be rejected from the analysis, but we show them in red as a warning of numerical problems with the FFT method if one is not wary of underflow versus true zeroes when using the FFT method. As in other applications, one should observe the total conditional means (row 4 of Figure 4) and retain the values that form a step function with steps of 1. Conditional means that deviate from their expected total should be discarded due to underflow or division by zero. However, the events which cause numerical issues have zero or negligible probability (under $10^{-16}$); hence, the expectations of interest do not suffer from underflow.

Also of interest is the shape of conditional means as a function of $k$. For $i = 1$, we have unpredictable unconditional expected allocations since the outcome 1 is often a part of $B_k$, $k \in \mathbb{N}_1$, and $q_1$ is greater than $q_i, i \in \{2, \ldots, 6\}$. For $i = 2$, we have predictable unconditional
expected allocations since 3 is a part of \( B_k \) for cyclical values of \( k \), and 3 does not divide the other values of \( b_i, i \in \{1, 3, 4, 5, 6\} \). Finally, we have \( b_3 = 10 \) and \( b_6 = 10 \). Hence, the conditional allocations are often shared between risks \( X_3 \) and \( X_6 \), though not perfectly since \( q_3 \neq q_6 \).

In row 3 of Figure 4, we have a mass around 5.6 since the outcomes \((X_1 = 1, X_3 = 0, X_4 = 4, X_5 = 5, X_6 = 0)\) and \((X_1 = 0, X_3 = 0, X_4 = 0, X_5 = 0, X_6 = 10)\) also yields \( S = 10 \), so these events diversify the event \((X_1 = 0, X_3 = 10, X_4 = 0, X_5 = 0, X_6 = 0)\).

- The middle panel presents the pmf of \( E[X_i|S] \), for \( i \in \{1, 2, 3\} \). The support of this rv is the set of values \( \{E[X_i|S = k], k \in \mathbb{N}\} \), for \( i \in \{1, 2, 3\} \). Notice that the support of this rv is sparse for small portfolios with heterogeneous values of \( b_i, i \in \{1, \ldots, n\} \).
- The right panel presents the cdf of \( E[X_i|S] \), for \( i \in \{1, 2, 3\} \), simplifying the interpretation of the middle panel since the probability masses may appear close together.

**Figure 4:** Left: conditional means. Middle: pmf of conditional means. Right: cdf of conditional means.
Note that as more participants enter the pool, more risks may diversify; that is, $B_k$ has a higher cardinality for all $k \in \mathbb{N}_1$. The risks diversify, and the pmf of unconditional expected allocations is less sparse. In Figure 5, we replicate the above study but add 69 participants where we sample the parameters according to $q_i \sim \text{Unif}([0,1])$ and $b_i \sim \text{Unif}([1,2,\ldots,10])$. We present the results for risk 3 (with $b_3 = 10$) and the total pool in Figure 5. Once again, we observe numerical issues for large values of $k$ in the left panel. However, the middle panel is much less sparse than in Figure 4.

Figure 5: Pool of 75 participants. Left: conditional means. Middle: pmf of conditional means. Right: cdf of conditional means.

6 Sum of dependent rvs

One may also use the methods described in this paper to compute unconditional expected allocations for dependent rvs. One obtains convenient results when the multivariate pgf is simple to differentiate, which is sometimes the case for mixture models (which include common shock models). The results from this section supplement the literature on risk allocation or risk sharing for mixture models as studied in Section 3 of [Cossette et al., 2018] or Section 4 of [Denuit and Robert, 2021a].

6.1 Multivariate Poisson distribution constructed with common shocks

As a first example, we present a common shock model. Multivariate Poisson distributions based on common shocks are studied notably in [Teicher, 1954] and [Mahamumulu, 1967]. The interested reader may also consult [Lindskog and McNeil, 2003] for actuarial applications of common shock Poisson models.
Example 6.1 (Hierarchical common Poisson shocks). Let \( Y_A \sim \text{Pois}(\lambda_A) \) for \( A \in \{1, 2\}^3 \cup \{0, 1, 2\} \) be independent rvs. We construct dependent rvs through the common shock framework \( X_{ijk} = Y_{ijk} + Y_i + Y_j + Y_k \) for \( (i, j, k) \in \{1, 2\}^3 \). This is a special case of the multivariate Poisson distribution from [Mahamunulu, 1967], and we illustrate the dependence structure in Figure 6. Let

\[
S = \sum_{(i,j,k) \in \{1,2\}^3} X_{ijk}.
\]

Then, one may verify that \( S \) follows a compound Poisson distribution, so one may use Panjer recursion or FFT to compute the values of the pmf of \( S \). Further, the OGF for the unconditional expected allocations of risk \( X_{ijk} \), for \( (i, j, k) \in \{1, 2\}^3 \), is

\[
\mathcal{P}_S^{(ijk)}(t) = \left( \lambda_{ijk} t + \lambda_i t^2 + \lambda_j t^4 + \lambda_k t^8 \right) \mathcal{P}_S(t).
\]

For \( (i, j, k) \in \{1, 2\}^3 \), we deduce that

\[
E \left[ X_{ijk} \times 1_{\{S=k\}} \right] = \begin{cases} 0, & k = 0 \\ \lambda_{ijk} f_S(k - 1), & k = 1 \\ \lambda_{ijk} f_S(k - 1) + \lambda_i f_S(k - 2), & k = 2, 3 \\ \lambda_{ijk} f_S(k - 1) + \lambda_j f_S(k - 2) + \lambda_i f_S(k - 4), & k = 4, \ldots, 7 \\ \lambda_{ijk} f_S(k - 1) + \lambda_j f_S(k - 2) + \lambda_i f_S(k - 4) + \lambda_0 f_S(k - 8), & k = 8, 9, \ldots 
\end{cases}
\]

More general Poisson common shock models, as proposed in [Mahamunulu, 1967], yield similar expressions for expected and cumulative unconditional expected allocations.

6.2 Multivariate mixed Poisson distribution

Next, we consider a multivariate mixed Poisson distribution. We induce dependence using a mixture random vector \( \Theta = (\Theta_1, \ldots, \Theta_n) \) with \( E[\Theta_i] = 1 \) for \( i = 1, \ldots, n \). Consider a vector of conditionally independent rvs \( (X_i | \Theta_i = \theta_i) \sim \text{Poisson}(\lambda_i \theta_i) \) for \( i = 1, \ldots, n \). The multivariate pgf of \( (X_1, \ldots, X_n) \) is

\[
\mathcal{P}_X(t_1, \ldots, t_n) = E_{\Theta} \left[ e^{\Theta_1 \lambda_1 (t_1 - 1)} \ldots e^{\Theta_n \lambda_n (t_n - 1)} \right] = \mathcal{M}_\Theta(\lambda_1 (t_1 - 1), \ldots, \lambda_n (t_n - 1)),
\]

(30)

where \( \mathcal{M}_\Theta \) is the multivariate moment generating function (mgf) of \( \Theta \). Then, combining Theorem 3.4 and (30), we find that

\[
\mathcal{P}_S^{[1]}(t) = \lambda_1 t \left. \frac{\partial}{\partial x} \mathcal{M}_\Theta(x, \lambda_2 (t - 1), \ldots, \lambda_n (t - 1)) \right|_{x = \lambda_1 (t - 1)}.
\]

(31)
Example 6.2 (Poisson-gamma common mixture). We consider a mixture distribution from a bivariate gamma common shock model described in [Mathai and Moschopoulos, 1991]. Let us define three independent rvs $Y_i, i \in \{0, 1, 2\}$ where $Y_0 \sim \text{Gamma}(\gamma_0, \beta_0)$, and $Y_i \sim \text{Gamma}(r_i - \gamma_0, r_i)$ for $i \in \{1, 2\}$ with $0 \leq \gamma_0 \leq \min(r_1, r_2)$. Let $\Theta = \beta_0/r_1Y_0 + Y_1$ for $i = 1, 2$. Then the pair of rvs $(\Theta_1, \Theta_2)$ follows a bivariate gamma distribution with marginals $\Theta_1 \sim \text{Ga}(r_i, r_i)$, $i = 1, 2$ and $\gamma_0$ is a dependence parameter. The bivariate mgf of the pair of rvs $(\Theta_1, \Theta_2)$ is

$$M_{\Theta_1, \Theta_2}(x_1, x_2) = \left(1 - \frac{x_1}{r_1}\right)^{(r_1 - \gamma_0)} \left(1 - \frac{x_2}{r_2}\right)^{(r_2 - \gamma_0)} \left(1 - \frac{x_1}{r_1} \frac{x_2}{r_2}\right)^{-\gamma_0}$$ (32)

and its derivative with respect to $x_1$ is

$$\frac{\partial}{\partial x_1}M_{\Theta_1, \Theta_2}(x_1, x_2) = \left(\frac{r_1 - \gamma_0}{r_1} \frac{1}{1 - x_1/r_1} + \frac{\gamma_0}{r_1} \frac{1}{1 - x_1/r_1 - x_2/r_2}\right)M_{\Theta_1, \Theta_2}(x_1, x_2).$$ (33)

Consequently, the mixed Poisson distributed random vector $(X_1, X_2)$ follows a bivariate negative binomial distribution. It follows from (30) and (32) that

$$P_S(t) = (1 - \zeta_1(t - 1))^{-(r_1 - \gamma_0)} (1 - \zeta_2(t - 1))^{-(r_2 - \gamma_0)} (1 - \zeta_{12}(t - 1))^{-\gamma_0},$$

where $\zeta_1 = \lambda_1/r_1$, $\zeta_2 = \lambda_2/r_2$ and $\zeta_{12} = \lambda_1/r_1 + \lambda_2/r_2$. We recognize that $S$ is the sum of three independent negative binomial rvs with parameters $(r_1 - \gamma_0, 1/(1 - \zeta_1))$, $(r_2 - \gamma_0, 1/(1 - \zeta_2))$ and $(\gamma_0, 1/(1 - \zeta_{12}))$. The expression of the pmf $f_S$ of $S$ is given in Theorem 1 of [Furman, 2007]. From (31) and (33), we get the following expression for the OGF for unconditional expected allocations:

$$P_{S}^{[1]}(t) = \lambda_1 t \left(\frac{1 - \gamma_0/r_1}{1 - \zeta_1(t - 1)} + \frac{\gamma_0/r_1}{1 - \zeta_{12}(t - 1)}\right)P_S(t).$$

Finally, we can recover the unconditional expected allocations using the FFT algorithm or with the recursive-type formula

$$[t^k]P_{S}^{[1]}(t) = E \left[X_1 \times 1_{\{S=k\}}\right] = \lambda_1 \sum_{j=0}^{k-1} \left[\left(1 - \frac{\gamma_0}{r_1}\right) \frac{1}{1 + \zeta_1} \left(\frac{\zeta_1}{1 + \zeta_1}\right)^j + \frac{\gamma_0}{r_1} \frac{1}{1 + \zeta_{12}} \left(\frac{\zeta_{12}}{1 + \zeta_{12}}\right)^j\right] f_S(k-1-j).$$

One may develop similar expressions for cumulative unconditional expected allocations, applying the cumulative operator to the geometric series or to the pmf of $S$.

6.3 Multivariate Bernoulli distributions defined with Archimedean copulas

Finally, we consider a multivariate Bernoulli distribution whose dependence structure is defined with an Archimedean copula. Let $(I_1, \ldots, I_n)$ form a random vector, where the marginal distributions are Bernoulli with success probability $q_i \in (0, 1)$, for $i \in \{1, \ldots, n\}$. Following [Marshall and Olkin, 1988], we define the random vector according to $P(I_i = 1|\Theta = \theta) = r_i^\theta$, where $\Theta$ is a mixing rv with a distribution defined on a strictly positive support. The relationship between the parameters $r_i$ and $q_i$ is

$$P(I_i = 1) = E_{\Theta}[r_i^\theta] = L_{\Theta}(-\ln r_i),$$

from which it follows that $r_i = \exp\{-L_{\Theta}^{-1}(q_i)\}$, where $L_{\Theta}(t)$ and $L_{\Theta}^{-1}(t)$ are respectively the Laplace-Stieltjes transform and the inverse Laplace-Stieltjes transform of the mixing rv. Further define the
rv $X_i = b_i \times I_i$, with $b_i \in \mathbb{N}_1$ for $i \in \{1, \ldots, n\}$. Note that the rvs $(X_i|\Theta = \theta)$ are conditionally independent, for $i \in \{1, \ldots, n\}$ and $\theta > 0$. It follows that the multivariate pgf of $X = (X_1, \ldots, X_n)$ is

$$P_X(t_1, \ldots, t_n) = E \left[ \prod_{i=1}^{n} (1 - r_i^\theta + r_i^\theta t_i^{b_i}) \right] = \int_0^{\infty} \prod_{i=1}^{n} (1 - r_i^\theta + r_i^\theta t_i^{b_i}) dF_\Theta(\theta).$$

We note that the underlying dependence structure in this model is an Archimedean copula; see, for instance, [Marshall and Olkin, 1988], Section 4.7.5.2 of [Denuit et al., 2006] or Section 7.4 of [McNeil et al., 2015] for the frailty construction of Archimedean copulas using common mixtures.

We consider the case where $\Theta$ is a discrete rv with support $\mathbb{N}_1$. Following the computational strategy from [Cossette et al., 2018], we select a threshold value $\theta^* = F^{-1}_\Theta(1 - \varepsilon)$ for a small $\varepsilon > 0$ and we have

$$P_S(t) = \sum_{\theta=1}^{\theta^*} Pr(\Theta = \theta) \prod_{i=1}^{n} (1 - r_i^\theta + r_i^\theta t_i^{b_i}). \quad (34)$$

Note that when the components of the random vector are independent, the rv $S$ follows a generalized Poisson-binomial distribution [Zhang et al., 2018]. In the case of (34), we notice the pgf of a mixture of generalized Poisson-binomial distributions, where the mixture rv comes from the frailty construction of Archimedean copulas.

The OGF of the sequence of unconditional expected allocations for risk $X_1$ is

$$P_S^{[1]}(t) = \sum_{\theta=1}^{\theta^*} Pr(\Theta = \theta) r_1^\theta b_1 t^{b_1} \prod_{i=2}^{n} (1 - r_i^\theta + r_i^\theta t_i^{b_i}).$$

**Example 6.3.** We consider a portfolio of $n = 6$ risks, with $\Theta$ following a shifted geometric rv with pmf $f_\Theta(k) = (1 - \alpha)\alpha^{k-1}$, for $k \in \mathbb{N}_1$, with $\alpha = 0.5$. It follows that the underlying dependence structure is an Ali-Mikhail-Haq copula. Following [Cossette et al., 2018], we select a threshold $\varepsilon = 10^{-10}$, such that $\theta^* = 34$. The indemnity payments are the same as in Table 6. We present the validation curve, the pmf for the conditional means of risk $X_3$, and the pmf of $S$ in Figure 7 for $\alpha \in \{0, 0.1, 0.5, 0.8, 0.95\}$. Increasing the dependence parameter increases the probability of zero contributions and full ($X_3 = b_3$) contributions. For other allocation values, the support of $E[X_3|S]$ tends to cluster around the same value of 6 since increasing the dependence also increases the probability of mutual occurrence. Indeed, the probabilities for the outcomes $X_1 = 1, X_4 = 4$ and $X_5 = 5$ become more likely (resp. 0.006, 0.007, 0.011, 0.02 and 0.032 for $\alpha = 0, 0.1, 0.5, 0.8$ and 0.95), so more diversification occurs when the total costs are divisible by 10, as $\alpha$ increases.

Next, we add 69 participants to the pool to investigate the effect of reducing the sparsity of the possible unconditional expected allocations. We present the validation curve, the values of $Pr(E[X_3|S] = k)$ for $k \in \{0, \ldots, 441\}$ and $\alpha = \{0,0.1,0.5,0.8,0.95\}$ in Figure 8. Note that the pmf of $S$ does not always converge to a normal distribution; hence, central limit theorems do not apply. Indeed, the common mixture representation of the Ali-Mikhail-Haq copula generates multiple nodes for the pmf of $S$ in this example. However, the OGF method with the FFT algorithm lets us easily extract the exact values of the pmf of unconditional expected allocations. As we increase the dependence parameter, the probability masses of $S$ and $E[X_3|S]$ are less concentrated around their means; thus, the tail of the distributions have non-zero mass, so there are no numerical issues in the validation curve. The code for this example is provided in Appendix A.4.
Figure 7: Pool of six participants.

7 Discussion

We proposed a generating function method to compute the unconditional expected allocation, which has valuable applications in peer-to-peer insurance and risk allocation problems. The method simplifies solutions to risk allocation problems and enables FFT-based algorithms for fast computations.

The link between derivatives of pgfs and conditional distributions is not new. See, for instance, the use of derivatives to study conditional distributions with Poisson rvs [Subrahmaniam, 1966, Kocherlakota, 1988] or with phase-type distributions [Ren and Zitikis, 2017]. In a bivariate setting, [Kocherlakota, 1992] show that the conditional pgf of $X_1$ given the sum $S = X_1 + X_2 = s$ is

$$
P_{X_1|S}(t_1|k) = \frac{\partial^k}{\partial t^k} P_{X_1,X_2}(t_1 t_2, t_2) \bigg|_{t_1=t_2=0},$$

for $|t_1| \leq 1$ and $k \in \mathbb{N}$. However, computing conditional expected values would involve computing multiple partial derivatives of the bivariate pgf. The method proposed in this paper only requires one partial derivative, a more convenient and tractable task.

We remark that the generating function method provides a simpler proof of the size-biased transform method of computing unconditional expected allocations for discrete rvs. With $\tilde{X}$ representing the size-biased transform of the rv $X$, along with the definition of the size-biased transform in (6), the relationship between the pgfs of $X$ and $\tilde{X}$ is

$$
P_{\tilde{X}}(t) = E \left[ t^{\tilde{X}} \right] = \sum_{k=0}^{\infty} t^k f_{\tilde{X}}(k) = \frac{t}{E[X]} \sum_{k=0}^{\infty} k t^{k-1} f_X(k) = \frac{E[X]}{dt} \sum_{k=0}^{\infty} t^k f_X(k) = \frac{t}{E[X]} P'_X(t),$$

for $|t| \leq 1$. Alternatively, one can obtain the pgf of $\tilde{X}$ by applying operation 2 (right shift) and 3 (index multiply) of OGFs from Lemma 3.2. See Section 2.2.1 of [Arratia et al., 2019] for discussions.
on the characteristic function and pgfs of size-biased rv's. From (13), we have
\[
P^{[1]}_S(t) = E[X_1 t^S] = \sum_{k=0}^{\infty} t^k \sum_{k_1=0}^{k} k_1 f_{X_1,S-1}(k_1,k-k_1) = \sum_{k=0}^{\infty} t^k \sum_{k_1=0}^{k} \frac{f_{\bar{X}_1,S-1}(k_1,k-k_1)}{E[X_1]},
\]
then \(E[X_1]P^{[1]}_S(t)\) is the pgf of \(\bar{X}_1 + S_{-1}\), so (5) follows immediately.

Future research could involve developing methods to quantify or correct aliasing errors for heavy-tailed distributions. In Section 5.4, we use a very large truncation point \((k_{max} = 2^{20})\). As computer processors continue to perform faster computations, it is convenient to increase the truncation point; however, it may also be convenient to provide methods that reduce this error source for efficiency’s sake. The authors of [Grubel and Hermesmeier, 1999] quantify the aliasing error related to using the FFT algorithm to compute the pmf of compound distributions and propose a tilting procedure to reduce this error. Developing a similar theory for the OGFs of unconditional expected allocations and cumulative unconditional expected allocations will increase these methods’ efficiency.

Another research topic involves the allocation of tail variance. In [Furman and Landsman, 2006], the authors introduce the tail variance, defined by
\[
TV_{\kappa}(X) = Var(X|X > F_X^{-1}(\kappa)),
\]
with \(\kappa \in (0,1)\), and propose allocations via the tail covariance allocation rule,
\[
TCov_{\kappa}(X_1|S) = Cov(X_1, S|S > F_S^{-1}(\kappa)) = \sum_{j=1}^{n} Cov(X_1, X_j|S > F_S^{-1}(\kappa)).
\]
One can obtain efficient algorithms to compute the desired expectations once again. We have
\[
E \left[ X_1 X_j t^S \right] = \left\{ t_1 t_j \frac{\partial^2}{\partial t_1 \partial t_j} \mathcal{P}_{X_1,\ldots,X_n}(t_1,\ldots,t_n) \right\}_{t_1=\cdots=t_n=t}
\]
for \( j \in \{1, \ldots, n\} \setminus \{1\} \). The OGF for unconditional expected allocations for the second factorial moment is

\[
E [X_1(X_1 - 1)t^S] = \left\{ t_1^2 \frac{\partial^2}{\partial t_1^2} P_{X_1,S-1}(t_1, t_2) \right\}_{t_1=t_2=t},
\]

one can generalize the latter formula to \( k \)th factorial moments by taking subsequent derivatives. It follows that \( E[X_1X_j|S > k] \) and \( E[X_1^2|S > k] \) can be computed with

\[
E [X_1X_j \times 1_{\{S \leq k\}}] = [t^k] \left\{ \frac{E[X_1X_j t^S]}{1-t} \right\}
\]

and

\[
E [X_1^2 \times 1_{\{S \leq k\}}] = [t^k] \left\{ \frac{E[X_1(X_1 - 1)t^S] + P_{S,1}^{(1)}(t)}{1-t} \right\}.
\]

Finally, one can consider the implications of this method in the continuous case. Letting \( \mathcal{L}_{X_1, \ldots, X_n} \) denote the multivariate Laplace-Stieltjes transform of the vector \( (X_1, \ldots, X_n) \), one can show that

\[
-\frac{\partial}{\partial t_1} \mathcal{L}_{X_1, \ldots, X_n}(t_1, \ldots, t_n) \bigg|_{t_1=\cdots=t_n=t},
\]

for \( t \geq 0 \), is the Laplace transform of \( E[X_1 \times 1_{\{S = s\}}] \). One could use this formulation to obtain new closed-form expressions for unconditional expected allocations, compute unconditional expected allocations through numerical inversion of Laplace transforms, or develop asymptotic properties of unconditional expected allocations. We note that the Laplace transform of size-biased rvs in the context of continuous rvs is explored in, for instance, [Furman et al., 2020].

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A R code for the numerical applications

A.1 Small portfolio of independent compound Poisson rvs

```r
n_participants <- 4
kmax <- 2^6
cmax <- 4
lam <- list(0.08, 0.08, 0.1, 0.1)
fca <- list(c(0, 0.1, 0.2, 0.4, 0.3, rep(0, kmax - cmax - 1)),
           c(0, 0.15, 0.25, 0.3, 0.3, rep(0, kmax - cmax - 1)),
           c(0, 0.1, 0.2, 0.3, 0.4, rep(0, kmax - cmax - 1)),
           c(0, 0.15, 0.25, 0.3, 0.3, rep(0, kmax - cmax - 1)))
```
dft_fx <- list()
phic <- list()
mu <- list()
conditional_mean <- list()

for(i in 1:n_participants) {
  dft_fx[[i]] <- exp(lam[[i]] * (fft(fc[[i]]) - 1))
  phic[[i]] <- fft(c(1:(kmax-1) * fc[[i]][-1], 0))
}

dft_fs <- Reduce("*", dft_fx)
fs <- Re(fft(dft_fs, inverse = TRUE))/kmax
e1 <- exp(-2i*pi*(0:(kmax-1))/kmax)

for(i in 1:n_participants) {
  dft_mu <- e1 * phic[[i]] * lam[[i]] * dft_fs
  mu[[i]] <- Re(fft(dft_mu, inverse = TRUE))/kmax
  conditional_mean[[i]] <- mu[[i]]/fs
}
sapply(conditional_mean, ", [2] # Validation
conditional_mean_total <- Reduce("+", conditional_mean)
conditional_mean_total[1 + 1:10] # Validation

A.2 Large portfolio of independent compound Poisson rvs

set.seed(10112021)
n_participants <- 10000
kmax <- 2^13
lam <- list()
fc <- list()
mu <- list()

lambdas <- rexp(n_participants, 10)
rs <- sample(1:6, n_participants, replace = TRUE)
qs <- runif(n_participants, 0.4, 0.5)

# Assign parameters
for(i in 1:n_participants) {
  lam[[i]] <- lambdas[i]
  fci <- dnbinom(0:(kmax-2), rs[i], qs[i])
  fc[[i]] <- c(fci, 1 - sum(fci))
}

dft_fx <- list()
phic <- list()
cm <- list()

for(i in 1:n_participants) {
  dft_fx[[i]] <- exp(lam[[i]] * (fft(fc[[i]]) - 1))
  phic[[i]] <- fft(c(1:(kmax-1) * fc[[i]][-1], 0))
}

dft_fs <- Reduce("*", dft_fx)
fs <- Re(fft(dft_fs, inverse = TRUE))/kmax
e1 <- exp(-2i*pi*(0:(kmax-1))/kmax)

for(i in 1:n_participants) {

A.3 Portfolio of heavy-tailed risks

library(actuar)
n <- 3
xmax <- 2^15
kmax <- 2^20
alphas <- seq(1.3, 1.9, 0.3)
lambdas <- 10 * (alphas - 1)
phis <- rep(1, kmax)

for(i in 1:n) {
  fx <- discretize(ppareto(x, alphas[i], lambdas[i]), 0, xmax - 1,
                   method = "unbiased", lev = levpareto(x, alphas[i], lambdas[i]))
  phix <- fft(c(fx, rep(0, kmax - xmax)))
  phis <- phis * phix
}

fs3 <- Re(fft(phis, inverse = TRUE))/kmax
good_values <- (fs3 >= 0)

for(i in 1:n) {
  fx <- discretize(ppareto(x, alphas[i], lambdas[i]), 0, xmax - 1,
                   method = "unbiased", lev = levpareto(x, alphas[i], lambdas[i]))
  phix <- fft(c(fx, rep(0, kmax - xmax)))
  phi_deriv_x1 <- fft(c((1:(xmax - 1)) * fx[-1], rep(0, kmax - xmax + 1)))
  agf <- phis / phix * phi_deriv_x1 * exp(-2i*pi*(0:(kmax-1))/kmax)
  cm3[[i]] <- (Re(fft(agf, inverse = TRUE))/kmax / fs3)[1:xmax]
}

A.4 Archimedean copula example

set.seed(20220314)
n <- 6
# bi <- sample(1:10, n, replace = TRUE)
bis <- c(1, 3, 10, 4, 5, 10)
qis <- c(0.1, 0.15, 0.2, 0.25, 0.3)
kmax <- sum(bis) + 1
ffts <- matrix(exp(-2i * pi * (0:(kmax - 1))/kmax), n, length(bis))
alphas <- 0
eps_theta <- 1e-10
theta_max <- max(2, floor(log(eps_theta)/log(alphas)) + 1)
f_theta <- alphas**(1:theta_max - 1) * (1 - alphas)

LST_inv_geom <- function(u) log((1 - alphas)/u + alphas)
fft1 <- exp(-2i * pi * (0:(kmax - 1))/kmax)

qi <- runif(n)
ri <- exp(-LST_inv_geom(qi))

fgp_S <- function(s) {
marginals <- apply(sapply(1:n, function(k) 1 - ri[k]^(1:theta_max) + ri[k]^(1:theta_max) * s^bi[k]), 1, prod)
sum(f_theta * marginals)
}

fgp_S <- Vectorize(fgp_S)
phis <- fgp_S(fft1)

fs <- (Re(fft(phis, inverse = TRUE))/kmax)

pgf_alloc_i <- function(s, i) {
marginals <- bi[i] * ri[i]^(1:theta_max) * s^bi[i] * apply(sapply((1:n)[-i], function(k) 1 - ri[k]^(1:theta_max) + ri[k]^(1:theta_max) * s^bi[k]), 1, prod)
sum(f_theta * marginals)
}

pgf_alloc_i <- Vectorize(pgf_alloc_i)
phi_alloc_1 <- pgf_alloc_i(fft1, 1)
conditional_mean_1 <- (Re(fft(phi_alloc_1, inverse = TRUE))/kmax/fs)
round(conditional_mean_1, 3)
plot(conditional_mean_1, type = 's')

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