On the odderon intercept in the perturbative QCD

M.A.Braun
Department of High Energy Physics, University of St. Petersburg, 198904 St. Petersburg, Russia.

Abstract.

The odderon intercept is calculated directly using the wave function recently constructed by R.A.Janik and J.Wosiek. The results confirm their reported value. It is also found that their solution for \( q_3 = 0 \) does not satisfy the Bose-symmetry requirements. Introduction of terms into the trial wave function with an asymptotical behavior similar to the Janik-Wosiek wave functions does not seem to improve variational estimates significantly. The diffusion parameter is found to be of the order 0.6.

SPbU-IP-1998/8
1 Introduction. Basic equations.

1. Recently R.A.Janik and J. Wosiek (JW) published a report on the solution of the odderon problem in the perturbative QCD [1]. They derived the odderon intercept from the spectrum of the integral of motion $q_3$ introduced by L.N.Lipatov for the three-Reggeon system [2], combined with their earlier solution of the appropriate Baxter equation [3]. If one relates the odderon intercept to the odderon “energy” per pair of Reggeons $\epsilon$ as

$$\alpha_O(0) = 1 - (3\alpha_s/2\pi)\epsilon$$

the result of JW for the ground state is

$$\epsilon = 0.16478...$$

Thus they confirm our old conclusion that the odderon intercept lies below unity [3]. Their exact value for it is somewhat higher than obtained in variational calculations, which gave larger value for $\epsilon$: 0.29 [4] and 0.223 [5,6].

In view of a somewhat indirect way of finding the odderon energy, JW suggested verifying their result by means of direct calculations of the energy with their wave function, using the technical machinery developed in the variational approach. Also, to prove that their result indeed gives the minimal value for the energy, it is desirable to repeat variational calculation using more general trial functions of the form suggested by the JW solution. This short note presents results of these calculations.

2. The odderon energy can be sought as a ratio

$$\epsilon = E/D$$

where $E$ and $D$ are energy and normalization functionals, quadratic in the odderon wave function $Z(r, \phi)$ [5]. Explicitly

$$E = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu \epsilon_n(\nu)|\alpha_n(\nu)|^2$$

Here

$$\epsilon_n(\nu) = 2 \text{Re} \left( \psi\left(1 + \frac{|n|}{2} + i\nu\right) - \psi(1) \right)$$

$\alpha_n(\nu)$ is essentially a double Fourier transform of $Z(r, \phi)$:

$$\alpha_n(\nu) = \int_0^{2\pi} d\phi e^{-i\nu\phi}$$
\[
\left( i\nu + \frac{n+1}{2} + re^{i\phi} (h - i\nu - \frac{n-1}{2}) \right) \left( i\nu - \frac{n-1}{2} \right) (-\bar{h} + i\nu - \frac{n-1}{2}) Z(r, \phi)
\]  

(6)

where \( h \) and \( \bar{h} \) are the two conformal weights, which are taken to be equal to 1/2 for the ground state. \( D \) is obtained substituting \( \epsilon_n(\nu) \) by unity in (1).

The odderon wave function \( Z \) can be considered as a function of \( z = r \exp i\phi \) and its conjugate. It has a form

\[
Z(z, z^*) = |z(1-z)|^{1/3} \Phi(z, z^*)
\]

(7)

Bose symmetry requires invariance of \( \Phi \) under substitutions

\[
z \rightarrow 1-z, \quad z \rightarrow 1/z
\]

(8)

Function \( \Phi \) was explicitly constructed by JW (in the form of a power series).

Once \( \Phi \) is known, calculation of \( \epsilon \) reduces to, first, performing the double Fourier transform (6) and, second, doing the integration over \( \nu \) and summation over \( n \) in (4).

All the difficulty in the calculation resides at the first stage. Actually one has to go to quite high values of \( n \) and \( \nu \) to get reasonable accuracy. The Fourier transform to such high values of \( n \) and \( \nu \) requires much effort. It also requires a very precise knowledge of the wave function \( \Phi \)

## 2 Odderon ground state.

1. Because of the latter requirement our first step was to obtain the odderon wave function with a higher precision than reported in [1]. To this end we set up a program which essentially repeats the procedure employed in [1] and allows to reduce discontinuities of \( \Phi \) calculated in different variables to values of the order \( 10^{-9} \). The only difference with [1] is that we use the basic functions \( u_i, i = 1, 2, 3 \), multiplied by certain factors to make them real at points where the transfer matrices are calculated. This substantially facilitates achieving the desired accuracy.

At this step for the integral of motion \( q_3 \) and wave function parameters \( \alpha, \beta \) and \( \gamma \) we obtained the following values (precision \( 10^{-9} \))

\[
iq_3 = 0.205257506, \quad \alpha = 0.709605410, \quad \beta = -0.689380668, \quad \gamma = 0.145651837
\]

(9)

However even with these high precision values direct calculation of \( \Phi \) in one of the three sets of variables \( z, 1-1/z \) or \( 1/(1-z) \) fails in the vicinity of the point
$z_0 = \exp i\pi/3$ where none of the series converges absolutely. In spite of the fact that $z_0$ is only a single point in the $r, \phi$ plane, this makes it practically impossible to calculate the double Fourier transform for $|n| > 10$ and/or $|\nu| > 5$. To overcome this difficulty we had to redevelop the function $\Phi$ around the intermediate point $(1/2)z_0$. With this redevelopment 100 terms in the series proved to be sufficient to obtain reliable results.

The Fourier transform itself was performed by interpolating $Z$ quadratically on the $r, \phi$ grid and doing the integrals in $r$ and $\phi$ analytically. Reasonable results are obtained already with a 160 x 160 grid. We however also used 320 x 320 and 640 x 640 grids to analyse the precision achieved at this step.

Even with a 640 x 640 grid the numerical Fourier transform becomes unreliable for $|n| > 30$ and/or $|\nu| > 15$. For such high values of $|n|$ and $|\nu|$ we used asymptotic formulas for the Fourier transform, which can easily be obtained from the expansion of $\Phi$ around $z = 1$. Our final cutoffs were chosen to be $|n| < 300$ and $|\nu| < 150$, which proved to be quite sufficient for the determination of $\epsilon$ with a precision 0.001.

2. Results of our numerical calculation of $D$ and $E$ in the region $|n| < 30$ and $|\nu| < 15$ using an $N \times N$ grid in the $r, \phi$ plane are presented in the Table 1. for $N = 160, 320$ and 640

| N   | D       | E       |
|-----|---------|---------|
| 160 | 1.642162| 0.255480|
| 320 | 1.642085| 0.254852|
| 640 | 1.642056| 0.254632|

Table 1. $D$ and $E$ for the ground state

To these values one has to add the contributions from the asymptotic region described in the preceding section. They are

$$\Delta D = 0.002398, \quad \Delta E = 0.018451$$

(10)

Taking the results at $N = 640$ as the most accurate ones we finally have

$$D = 1.644454, \quad E = 0.273083, \quad \epsilon = 0.1660$$

(11)

Thus our result for $\epsilon$ coincides with the value found in [1] up to 0.001. With all the difficulties involved in the numerical calculations we consider this agreement quite
satisfactory. So direct calculation of the odderon energy confirms the result found by JW.

3. Our previous variational calculations gave very small change in energy as more analytic terms were added. To analyse the reason of the about 30% improvement given by the JW wave function, we tried to extend our variational calculations to more general trial functions as compared to [5,6], whose form is suggested by the latter function. At $q_3 \neq 0$ it possesses an asymptotical behavior at $z \to 0$ of the same sort as the trial wave function introduced by P.Gauron, L.N.Lipatov and B.Nicolescu in [5], except of a term which behaves as $r^{5/3} \cos 2\phi$. To investigate the importance of this behaviour we introduced a term into the trial wave function

$$ca(z, z^*)^{-1/6}b(z, z^*)^2$$

(12)

Here $a$ is the argument in the trial wave function of [5]:

$$a = x/y, \quad x = |z|^2|z_1|^2, \quad y = (1 + |z|^2)(1 + |z_1|^2)(|z|^2 + |z_1|^2)$$

(13)

where $z_1 = 1 - z$. The argument $b$ is

$$b = w/y, \quad w = (1 - |z|^2)(1 - |z_1|^2)(|z|^2 - |z_1|^2)$$

(14)

It is invariant under $z \to 1 - z$ and changes sign under $z \to 1/z$, so that $b$ is invariant.

At $q_3 = 0$ a term appears in the solution $\Phi$ which is unique in the $r, \phi$ plane and blows up as $r^{-1/3}$ at $r \to 0$. Inspection of the functionals $D$ and $E$ shows however that it is admissible in spite of the apparent singularity at small $r$ in (6). As mentioned, we could not satisfy the Bose symmetry requirements with this solution. Nevertheless we tried to estimate its possible significance and so included a term proportional to $a^{-1/6}$ into the trial function.

Our final trial function thus included 5 terms, three old ones of the same form as in [5,6] and two new ones described above.

Calculations with this generalized trial function showed first of all that the term with $A^{-1/6}$ presents difficulties for numerical integration, of the same sort that we encountered in studying the excited odderon, only much worse. In fact, with this term added, we could only calculate the double Fourier transform reliably for $|n| < 8, |\nu| < 5$. On the other hand, the term with $b$ lead to no difficulties whatsoever. However in both cases one finds that the new added terms bear no influence on the value of energy. At
the minimum value for the functional $E$ the coefficients before them turn out to be quite small and the value itself is only a few percent lower than without the new terms. So our conclusion is that simple addition of new terms into the trial wave functions even with two independent arguments $a$ and $b$ described above does not improve variational estimates. It is a factorized form of the Janik-Wosiek wave function which allows to make energy substantially lower. These calculations also make us believe that the JW function indeed belongs to the odderon ground state.

3 Other eigenvalues of $q_3$

1. We have also tried to check the result of [1] for the excited state with the next higher value of $i q_3$. Unfortunately in this case calculations proved to be still more difficult and we could not arrive at a result of a convincing accuracy.

   Our precise calculations of the wave function gave for this state:

   $i q_3 = 2.343921063, \quad \alpha = 0.391855163, \quad \beta = -0.0533712012, \quad \gamma = 0.918477570$ (15)

   Numerical calculation of $D$ and $E$ in the region $|n| < 15, |\nu| < 15$ gave results presented in Table 2.

   \[
   \begin{array}{|c|c|c|}
   \hline
   N & D & E \\
   \hline
   160 & 2.92863 & 6.08989 \\
   320 & 2.80693 & 5.27381 \\
   640 & 3.05939 & 6.96764 \\
   \hline
   \end{array}
   \]

   As one observes the achieved accuracy does not exceed 15%. Analysing these numbers one can see that all the error comes from the region of maximal $|n|$ and $|\nu|$ where the double Fourier transform is apparently performed inaccurately. From these numbers we can only conclude that for this excited state

   $\epsilon \simeq 2. \pm 0.3$ (16)

   In [1] the found value is $1.71231...$. Our result does not contradict this number.

2. We have also studied a possible solution for $q_3 = 0$, reported in [1]. However in this case we were not able to construct a wave function unique in the $r, \phi$ plane and
satisfying the necessary symmetry requirements. If, following [1] we seek $\Phi$ in the form

$$\Phi(z, z^*) = \bar{u}Au$$  \hspace{1cm} (17)

where $u_i(z)$ are the three basic solutions of the eigenvalue equation for $q_3$ and require the matrix $A$ to have certain matrix elements equal to zero to make $\Phi$ finite, then at $q_3 = 0$ we find it impossible for $A$ to be invariant under the basic symmetry transformations of $z$. This seems to be related to the fact that at $q_3 = 0$ one may also take $A_{33} \neq 0$. In any case there does not seem to exist solutions for $A$ which preserve the required Bose symmetry.

Thus our conclusion is that the $q_3 = 0$ state reported in [1] does not correspond to any physical odderon state.

### 4 "Moving” odderon

For conformal weights $h = \frac{1}{2} + i\sigma$ the odderon energy is supposed to behave at small $\sigma$ as

$$\epsilon(\sigma) = \epsilon_0 + a\sigma^2$$  \hspace{1cm} (18)

where $\epsilon_0$ is the value (2) and $a$ is a parameter which determines diffusion of the odderon wave function in the momentum space. This parameter has been long known for the pomeron to be $14\zeta(3)$ (in units $3\alpha_s/\pi$). It is of certain interest to find $a$ for the odderon.

To this aim we first found the parameters of the odderon wave function for various (small) $\sigma$ using the same method as employed for $h = 1/2$. Our results are presented in Table 3. The value of $iq_3$ turned out to be real for arbitrary $\sigma$, whereas, with $\alpha$ chosen to be real, both $\beta$ and $\gamma$ result complex. We chose $\alpha = 1$

| $\sigma$ | $iq_3$      | $\beta$            | $\gamma$            |
|---------|-------------|---------------------|---------------------|
| 0.01    | 0.205306079 | -0.971740164-i0.014404102 | 0.205305637-i0.00425478 |
| 0.1     | 0.210089247 | -0.995153863-i0.142974530 | 0.210052319-i0.003938872 |
| 0.2     | 0.224303315 | -1.060790013-i0.281079150 | 0.224222881-i0.006006327 |
| 0.3     | 0.247227544 | -1.156524786-i0.415163678 | 0.247186043-i0.004529717 |
| 0.5     | 0.316528176 | -1.395571390-i0.695891904 | 0.316214188+i0.014095104 |
| 1.0     | 0.619239545 | -2.044631201-i1.672240784 | 0.591391973+i0.183611401 |

Table 3. Odderon parameters for $h = \frac{1}{2} + i\sigma$
Inspecting these figures one immediately notes that $|\gamma| = iq_3$. This relation was predicted (for real $\gamma$) by L.N.Lipatov [7].

With the odderon parameters found we calculated the odderon energies directly, using the same technique as for $\sigma = 0$. With $\sigma$ different from zero calculation becomes still more cumbersome and time and memory consuming due to lack of certain symmetries and overall complex arithmetics. For these reasons we had to limit ourselves with a maximal $160 \times 160$ grid in the $r, \phi$ plane and neglected the contribution from the asymptotical region $n > 30 |\nu| > 15$. Our results are shown in Table 4 together with the ones just obtained via the solution of the Baxter equation [8]

**Table 4. Odderon energies for** $h = \frac{1}{2} + i\sigma$

| $\sigma$ | $\epsilon$ | $\epsilon$ [8] |
|---------|------------|----------------|
| 0.0     | 0.1534     | 0.16478        |
| 0.1     | 0.1597     |                |
| 0.3     | 0.2085     | 0.21777        |
| 0.5     | 0.2980     | 0.30523        |
| 1.0     | 0.6269     | 0.63228        |

Our energies lie a little below the ones obtained from the Baxter equation, which is natural since we have neglected the asymptotic part of the $n, \nu$ region in (4). Having this in mind we find a complete agreement between our direct calculation results and the ones based on the Baxter equation.

From our energies we find for the parameter $a$ in (18)

$$a = 0.61$$

More precise energies found in [8] lead to

$$a = 0.605$$

Note however that already at $\sigma = 1$ the approximation (18) breaks down and more powers of $\sigma^2$ are needed to describe the energy behaviour. It is interesting that the parameter $a$ for the odderon is much smaller than the one for the pomeron. In fact their ratio is of the same order as the ratio of corresponding energies.
5 Acknowledgements

The author expresses his deep gratitude to Prof. J.Wosiek whose comments initiated this investigation and whose advice accompanied its fulfillment. He is also grateful to Prof. L.N.Lipatov for very interesting discussions. He is grateful to both of them for communicating their yet unpublished results.

6 References.

1. R.A.Janik and J.Wosiek, Cracow preprint TPJU-2/98, hep-th/9802100.
2. L.N.Lipatov, Phys. Lett. B309 (1993) 304.
3. R.A.Janik and J.Wosiek, Phys. Rev. Lett. 76 (1977) 2935; R.A.Janik, Acta Phys. Polon. 27 (1996) 1819.
4. N.Armesto and M.A.Braun, Z.Phys., C63 (1997) 709.
5. P.Gauron, L.N.Lipatov and B.Nicolescu, Phys. Lett. B304 (1993) 334; Z.Phys. C63 (1994) 253.
6. M.A.Braun, St. Petersburg preprint SPbU-IP-1998/3, hep-ph/9801352
7. L.N.Lipatov, private communication.
8. J.Wosiek, private communication