Quantization of Friedmann-Robertson-Walker spacetimes in the presence of a negative cosmological constant and radiation

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In the present work, we quantize three Friedmann-Robertson-Walker models in the presence of a negative cosmological constant and radiation. The models differ from each other by the constant curvature of the spatial sections, which may be positive, negative or zero. They give rise to Wheeler-DeWitt equations for the scale factor which have the form of the Schrödinger equation for the quartic anharmonic oscillator. We find their eigenvalues and eigenfunctions by using a method first developed by Chhajlany and Malnev. After that, we use the eigenfunctions in order to construct wave packets for each case and evaluate the time-dependent expected value of the scale factors. We find for all of them that the expected values of the scale factors oscillate between maximum and minimum values. Since the expectation values of the scale factors never vanish, we conclude that these models do not have singularities.

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One of the motivations for the quantization of cosmological models was that of avoiding the initial *Big Bang* singularity. Since the pioneering work in quantum cosmology due to DeWitt [1], workers in this field have been attempting to prove that quantum cosmological models entail only regular space-times. An important contribution to this issue was given by Hartle and Hawking [2], who proposed the *no-boundary* boundary condition, which selects only regular space-times to contribute to the wave-function of the Universe, derived in the path integral formalism. Therefore, by construction, the *no-boundary* wave-functions are everywhere regular and predict a non-singular initial state for the Universe. Using that boundary condition, in certain particular cases the *no-boundary* wave-function can be explicitly computed [2, 3, 4]. Another way by which one may compute the wave-function of the Universe is by directly solving the Wheeler-DeWitt equation [1]. The wave-function of the Universe for some important models have been computed using this approach [2, 3, 4]. We should mention another boundary condition, the *tunelling* boundary condition, proposed by Vilenkin [7], to be imposed on solutions to the Wheeler-DeWitt equations. The *tunelling* wave-function was also shown to give rise to models free from the initial *Big Bang* singularity [7]. More recently, the absence of singularities in quantum cosmological models has been investigated by using the DeBroglie-Bohm interpretation of quantum mechanics [8]. In this interpretation, one may compute the dynamical trajectories for the quantum variables of the system. In particular, since for most of the quantum cosmological models one uses minisuperspaces [9], one computes, through the DeBroglie-Bohm interpretation, the dynamical trajectories for the scale factor. Then, for the great majority of cases studied so far, the scale factor never vanishes, which implies that the model has no initial singularities [10, 11]. Although we shall restrict our attention to a model in quantum general relativity, it is important to mention that several works in loop quantum cosmology have also shown that the wave-function of the Universe is free from initial singularities [12, 13].

Several important theoretical results and predictions in quantum cosmology have been obtained with a negative cosmological constant. Considering a subset of all four-dimensional spacetimes with constant negative curvature and compact space-like hypersurfaces, Carlip and coworkers showed how to compute the sum over topologies leading to the *no-boundary* wave-function [14, 15]. These spacetimes are curved only due to the presence of a negative cosmological constant. In Ref. [14] it was shown how to obtain a vanishing cosmological constant as a prediction from the *no-boundary* wave-function and in Ref. [15] it was shown how to obtain predictions about the topology of the
Universe from the no-boundary wave-function. We may also mention the result in Ref. [3], where the WKB no-boundary wave-function of a homogeneous and isotropic Universe with a negative cosmological constant was computed. Due to the regularity condition imposed upon the space-times contributing to the no-boundary wave-function, it was shown that only a well defined, discrete spectrum for the cosmological constant is possible. It was also found that among the space-times contributing to wave function, there were two complex conjugate ones that showed a new type of signature change.

It is important to mention that although recent observations point toward a positive cosmological constant, it is still possible that at the very early Universe the cosmological constant be negative. Besides that, we think it is important to understand more about such models which represent bound Universes (analogous to uni-dimensional atoms, in the present situation).

In the present paper, we use the formalism of quantum cosmology in order to quantize three Friedmann-Robertson-Walker models in the presence of a negative cosmological constant and radiation. The radiation is treated by means of the variational formalism developed by Schutz [16]. The models differ from each other by the constant curvature of their spatial sections, which may be positive, negative or zero. They give rise to Wheeler-DeWitt equations for the scale factor, which have the form of the Schrödinger equation for the quartic anharmonic oscillator. We find the eigenvalues and eigenfunctions of those equations by using a method first developed by Chhajlany and Malnev [17]. Then we use the eigenfunctions in order to construct wave packets for each case and evaluate the expectation value of the scale factors as a function of time. In Sec. II we introduce the classical models and solve them analytically, briefly commenting on the general behavior of the classical solutions. In Sec. III we quantize the model by solving the corresponding Wheeler-DeWitt equation. The wave-functional depends on the scale factor $a$ and on the canonical variable associated to the fluid, which in the Schutz variational formalism plays the role of time $T$. We separate the wave-functional in two parts, one depending solely on the scale factor and the other depending only on the time. The solution in the time sector of the Wheeler-DeWitt equation is trivial, leading to imaginary exponentials of the type $e^{-iE\tau}$, where $E$ is the system energy and $\tau = -T$. The scale factor sector of the Wheeler-DeWitt equation gives rise to the eigenvalue equation for the quartic anharmonic oscillator. We find semi-analytic solutions formed by the product of a decaying exponential term with a polynomial of fixed degree [17]. In Sec. IV we construct wave packets from the eigenfunctions, for each case, and compute the time-dependent, expectation values of the scale factors. We find in all cases that the expectation values of the scale factors show bounded oscillations. Since the expectation values of the scale factors never vanish, we conclude
that these models do not have singularities. We also observe that the energy levels depend on
the value of the curvature constant $k$ of the spatial sections. The model with $k < 0$ has the most
bounded energy levels, followed by the one with $k = 0$; the model with $k > 0$ have less bounded
energy levels. Finally, in Sec. V we summarize the main points and results of our paper.

II. THE CLASSICAL MODELS

Friedmann-Robertson-Walker cosmological models are characterized by the scale factor $a(t)$ and
have the following line element,

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right),$$

(1)

where $d\Omega^2$ is the line element of the two-dimensional sphere with unitary radius, $N(t)$ is the lapse
function and $k$ gives the type of constant curvature of the spatial sections. The curvature is positive
for $k = 1$, negative for $k = -1$ and zero for $k = 0$. Here, we are using the natural unit system,
where $\hbar = c = G = 1$. The matter content of the model is represented by a perfect fluid with four-
velocity $U^\mu = \delta^\mu_0$ in the comoving coordinate system used, plus a negative cosmological constant.
The total energy-momentum tensor is given by,

$$T_{\mu, \nu} = (\rho + p)U_\mu U_\nu - pg_{\mu, \nu} - \Lambda g_{\mu, \nu},$$

(2)

where $\rho$ and $p$ are the energy density and pressure of the fluid, respectively. Here, we assume that
$p = \rho/3$, which is the equation of the state for radiation. This is justified because the Universe is
initially dominated by radiation. Einstein’s equations for the metric (11) and the energy momentum
tensor (2) are equivalent to the Hamilton equations generated by the super-hamiltonian constraint

$$\mathcal{H} = -\frac{p_a^2}{12} - 3ka^2 + \Lambda a^4 + p_T,$$

(3)

where $p_a$ and $p_T$ are the momenta canonically conjugated to $a$ and $T$ the latter being the canonical
variable associated to the fluid [11].

The classical dynamics is governed by the Hamilton equations, derived from eq. (3), namely
\[
\begin{aligned}
\dot{a} &= \frac{\partial (N\mathcal{H})}{\partial p_a} = -\frac{N p_a}{6}, \\
p_a &= -\frac{\partial (N\mathcal{H})}{\partial a} = 6k a N - 4\Lambda a^3 N, \\
\dot{T} &= \frac{\partial (N\mathcal{H})}{\partial p_T} = N, \\
p_T &= -\frac{\partial (N\mathcal{H})}{\partial T} = 0.
\end{aligned}
\] (4)

We also have the constraint equation \( \mathcal{H} = 0 \).

Choosing the gauge \( N = 1 \), we have the following solutions for the system (4):

\[
\begin{aligned}
T(\tau) &= \tau + c_1, \\
a(\tau) &= \sqrt{\frac{6}{3k + \sqrt{9k^2 - 12\Lambda \beta}}} \text{sn} \left( \frac{\sqrt{18k + 6\sqrt{9k^2 - 12\Lambda \beta}} (\tau - \tau_0)}{6}, \sigma \right),
\end{aligned}
\] (5)

where \( c_1, \beta \) and \( \tau_0 \) are integration constants, \( \text{sn} \) is the Jacobi’s elliptic sine of modulus \( \sigma \) given by

\[
\sigma = \frac{\sqrt{2}}{2} \sqrt{\frac{-2\beta k + 3k^2 - k\sqrt{9k^2 - 12\Lambda \beta}}{\Lambda \beta}}.
\] (6)

In the case of the models studied here, for which \( \Lambda < 0 \), Eqs. (5) and (6) imply that the scale factor performs bounded oscillations, for all values of \( k \). When the scale factor vanishes we have the formation of a singularity which may be either a Big Bang or a Big Crunch. For the sake of completeness, we mention that for \( \Lambda = 0 \), the case studied in Ref. [6] is recovered.

III. THE QUANTIZATION OF THE MODELS

We wish to quantize the models following the Dirac formalism for quantizing constrained systems [19]. First we introduce a wave-function which is a function of the canonical variables \( \dot{a} \) and \( \dot{T} \),

\[
\Psi = \Psi(\dot{a}, \dot{T}).
\] (7)
Then, we impose the appropriate commutators between the operators $\hat{a}$ and $\hat{T}$ and their conjugate momenta $\hat{P}_a$ and $\hat{P}_T$. Working in the Schrödinger picture, the operators $\hat{a}$ and $\hat{T}$ are simply multiplication operators, while their conjugate momenta are represented by the differential operators

$$ p_a \rightarrow -i \frac{\partial}{\partial a}, \quad p_T \rightarrow -i \frac{\partial}{\partial T}. \quad (8) $$

Finally, we demand that $\mathcal{H}$, the super-hamiltonian operator corresponding to (3), annihilate the wave-function $\Psi$, which leads to Wheeler-DeWitt equation

$$ \left( \frac{1}{12} \frac{\partial^2}{\partial a^2} - 3ka^2 + \Lambda a^4 \right) \Psi(a, \tau) = -i \frac{\partial}{\partial \tau} \Psi(a, \tau), \quad (9) $$

where the new variable $\tau = -T$ has been introduced. In order to avoid possible contributions from boundary terms at spatial infinity, we shall consider compact tri-dimensional spatial sections in the cases $k = 0$ and $k = -1$.

The operator $\mathcal{H}$ is self-adjoint [6] with respect to the internal product,

$$ (\Psi, \Phi) = \int_0^\infty da \, \Psi(a, \tau)^* \Phi(a, \tau), \quad (10) $$

if the wave functions are restricted to the set of those satisfying either $\Psi(0, \tau) = 0$ or $\Psi'(0, \tau) = 0$, where the prime $'$ means the partial derivative with respect to $a$.

The Wheeler-DeWitt equation (9) is the Schrödinger equation for the quartic anharmonic oscillator and may be solved by writing $\Psi(a, \tau)$ as

$$ \Psi(a, \tau) = e^{-iE \tau} \eta(a) \quad (11) $$

where $\eta(a)$ depends solely on $a$. Then $\eta(a)$ satisfies the eigenvalue equation

$$ -\frac{d^2 \eta(a)}{da^2} + V_e(a) \eta(a) = 12E \eta(a), \quad (12) $$

where the effective potential $V_e(a)$ is given by

$$ V_e(a) = 36ka^2 - 12\Lambda a^4. \quad (13) $$

A. The Method of Chhajlany and Malnev

The method of Chhajlany and Malnev [17] starts with the addition of an extra term to the original anharmonic oscillator potential, so that the modified Hamiltonian admits a subset of manifestly normalizable solutions. In the case we are considering, the extra term to be added to
the effective potential \( V_\text{eff} \) is proportional to \( a^6 \). In terms of that new enlarged potential, the eigenvalue equation \( (12) \) may be re-written as

\[
\eta''(a) + (\varepsilon - \alpha a^2 - ba^4 - ca^6)\eta(a) = 0 \tag{14}
\]

where \( \varepsilon = 12E, \alpha = 36k, b = -12\Lambda, c \) is a parameter to be determined by the method. The Ansatz for the solution of Eq. \( (14) \) takes the form

\[
\eta(a) = N \exp\left(-\frac{c}{4}a^4 - \frac{\gamma}{2}a^2\right)v(a), \tag{15}
\]

and has finite norm for \( c > 0 \). Here, \( v(a) \) is a polynomial of a certain degree, yet to be chosen; the parameter \( \gamma \) is to be chosen according to our convenience, as we shall see; \( N \) is a normalization factor. The method is based on the fact, shown in Ref. \[17\], that the larger the degree of the polynomial \( v(a) \), the smaller \( c \) is. Therefore, if one increases the order of \( v(a) \), the energy eigenvalues predicted by the present method tend monotonically, from above, to the energy eigenvalues of the original problem. One important property of the method is that the convergence is very fast. This means that one does not need to use a polynomial of very large order to obtain a good agreement with the energies of anharmonic oscillators already computed in the literature by other methods.

The next step is the substitution of the Ansatz \( (15) \) into the differential equation \( (14) \), which gives rise to the following equation for the polynomial \( v(a) \):

\[
v''(a) + 2(ca^2 + \gamma a)v'(a) + (\varepsilon - \gamma + (\gamma^2 + \alpha - 3c)a^2)v(a) = 0. \tag{16}
\]

Next, writing \( v(a) = \sum_n \beta_n a^n \) and inserting it in Eq. \( (16) \) along with the condition

\[
2c\gamma = b \tag{17}
\]

on \( c \) and \( \gamma \), \[17\] we manage to find:

\[
(\varepsilon - \gamma)\beta_0 + 2\beta_2 = 0, \quad (\varepsilon - 3\gamma)\beta_1 + 6\beta_3 = 0, \tag{18}
\]

and the general recurrence relation for the polynomial coefficients \( \beta_n \),

\[
(n + 4)(n + 3)\beta_{n+4} + [\varepsilon - \gamma(2n + 5)]\beta_{n+2} + [\gamma^2 - \alpha - c(2n + 3)]\beta_n = 0, \tag{19}
\]
for $n \geq 0$. The degree of the polynomial $v(a)$ is fixed to be, say $K$, by imposing the following conditions in (19),

$$\beta_K \neq 0, \quad \beta_{K+2} = \beta_{K+4} = 0.$$  \hspace{1cm} (20)

Due to the nature of the recurrence relation (19), it is clear that by fixing $K$ to be even (odd) the resulting polynomial $v(a)$ will be even (odd). Then, the coefficients $\beta_n$, $n = 2, 4, ..., K$ ($n = 3, 5, ..., K$), will be determined in terms of $\beta_0$ ($\beta_1$) by the normalization condition. In the present situation, we restrict our attention to the case of an odd polynomial. It means that, $K = 2m + 1$ for $m = 0, 1, 2, ...$. This condition is imposed in order that our wave-function vanishes at $a = 0$.

Eqs. (19) and (20) require that the coefficient $\beta_K$ must vanish; then

$$\gamma^2 = \alpha + c(2K + 3).$$  \hspace{1cm} (21)

Combining this with (17), we obtain a cubic algebraic equation in the parameter $c$,

$$4c^3(2K + 3) + 4\alpha c^2 - b^2 = 0.$$  \hspace{1cm} (22)

The solutions of this equation depend on the known parameters $b$ and $K$. We must find the real, positive root to this equation so that the Ansatz Eq. (15) be normalizable. That real positive root, as proved in Ref. [17], is a monotonically decreasing function of $K$. Therefore, the greater the polynomial degree, the better the agreement between the energy eigenvalues obtained by this method and the actual energy eigenvalues.

Now, by setting the condition $\beta_{K+2} = 0$ in Eq. (19) we may determine the corresponding energy eigenvalues $\varepsilon$ and polynomial coefficients $\beta_n$. The $(m+1)$ allowed energy levels of the anharmonic oscillator are obtained as the roots of the equation $D = 0$, where $D$ is the following $(m+1) \times (m+1)$ determinant:

$$
\begin{vmatrix}
(\varepsilon - 3\gamma) & 6 & 0 & 0 & \cdots & \cdots & 0 \\
\gamma^2 - \alpha - 5c & (\varepsilon - 7\gamma) & 20 & 0 & \cdots & \cdots & 0 \\
0 & \gamma^2 - \alpha - 9c & (\varepsilon - 11\gamma) & 42 & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & (\gamma^2 - \alpha - (2K + 3)c) & (\varepsilon - (2K + 5)\gamma) & (K + 4)(K + 3) \\
\end{vmatrix}
$$

(23)

The lowest real, positive root will correspond to the ground state energy level and the excited levels will be given by the sequence of higher real, positive roots. Next, we must substitute these values in the set of Eqs. (19) in order to evaluate the coefficients $\beta_n$ and obtain the appropriate
polynomial $v_l(a)$; the index $l = 0, 1, 2, ..., m$ represents the energy level, for each of which we shall have an eigenfunction $\eta_l(a)$ and a wave-function $\Psi_l(a, \tau) = \exp(-i E_l \tau) \eta_l(a)$, according to Eqs. (11) and (15).

We construct a general solution to the Wheeler-DeWitt equation (9) by taking linear combinations of the $\Psi_l(a, \tau)$'s,

$$\Theta(a, \tau) = \sum_{l=0}^{m} A_l(E) \eta_l(a) e^{-iE_l \tau},$$

where the coefficients $A_l(E)$ will be fixed later. With those combinations we compute the expected value for the scale factor $a$, following the many worlds interpretation of quantum mechanics [20]. In the present situation, we may write the expected value for the scale factor $a$ is

$$\langle a \rangle (\tau) = \frac{\int_0^{\infty} a \left| \Theta(a, \tau) \right|^2 da}{\int_0^{\infty} \left| \Theta(a, \tau) \right|^2 da}.$$  

IV. RESULTS

We shall treat, now, each model separately depending on the constant curvature of the spatial sections. The difference from one model to the other will appear in the value of the parameter $\alpha$ in Eq. (14). For all models we shall use the value of $\Lambda = -0.1$, therefore one has $b = 1.2$ in Eq. (14). Also, we shall fix the polynomial degree to be $K = 45$, for all models. It means that, we shall have 23 energy levels and 23 eigenfunctions $\eta_l(a)$. A precision of at least 15 significant digits had to be used, in order to guarantee the orthogonality of the set of functions $\eta_l(a)$'s Eq. (15). The symbolic system Maple has been used, and the precision of calculations was chosen so that the largest number of energy levels be achieved and the corresponding (approximate) eigenfunctions be sufficiently orthogonal.

A. The model with $k = 1$.

In this model $\alpha = 36$ and the spatial sections are $S^3$'s. Using the values of $K$ and $b$, we solved Eq. (22) to find $c = 0.090068960666669615962974$. Computations have been performed with 20 significant digits. Now, introducing all these quantities in the determinant $D$, Eq. (23), we obtain the first 23 energy levels; they are listed in Table I. The first lowest energy levels are in agreement with the ones computed, pertubatively, by Landau for the quartic anharmonic potencial, equivalent to the
present case \[21\]. After that, we substitute these values in the set of Eqs. (19) and compute the coefficients \(\beta_n\). With these \(\beta_n\), we write the following \(\eta(a, \tau)\), according to Eq. (15),

\[
\eta_l(a) = N_l e^{-0.02251724016740390744 a^4 - 3.330781189986629985 a^2 v_l(a)},
\]

(26)

where

\[
v_l(a) = \sum_{i=0}^{22} A_{l,2i+1} a^{2i+1}.
\]

(27)

The coefficients \(N_l\) are normalization coefficients and \(i = 0, 1, ..., m\). The \(N_l\)’s and the \(A_{l,2i+1}\)’s for the present model are listed in the appendix A, Tables VIII to XIII.

Next, we construct the wave-packet \(\Theta(a, \tau)\) with the aid of the \(\eta_l(a)\), according to Eqs. (24), Eq. (26), and the energy levels in Table I. Finally, using the wave-packet \(\Theta(a, \tau)\) we compute the expected value for the scale factor \(a\), Eq. (25). The result is shown in Fig. 1; it can be seen that \(\langle a \rangle\) does not vanish, therefore we may say that the quantization of this model removed the singularities it had at the classical level. It is clear from Fig. 1 also, that \(\langle a \rangle\) performs bounded oscillations. That means that the spatial sections \(S^3\)’s oscillate between finite maximum and minimum radius.

B. The model with \(k = 0\)

In this model \(\alpha = 0\) and the spatial sections are some closed three-dimensional solid with zero curvature, locally isometric to \(R^3\) \[22\]. Here, like in the previous case, we have used 20 significant digits. Introducing the values of \(K\) and \(b\) in Eq. (22) we obtain \(c = 0.15701453260387612225\). Now, using all these quantities in the determinant \(D\) Eq. (23), we obtain the first 23 energy levels. They are shown in Table I. The first lowest energy levels are in agreement with the ones computed, pertubatively, by Landau for the quartic anharmonic potential, equivalent to the present case \[21\]. After that, we substitute these values in the set of Eqs. (19) and compute the coefficients \(\beta_n\). With these \(\beta_n\), we write the following \(\eta(a, \tau)\) Eq. (15),

\[
\eta_l(a) = N_l e^{-0.039253633150969030562 a^4 - 1.91065116728306000300 a^2 v_l(a)},
\]

(28)

where the \(v_l(a)\) have the general expression given in Eq. (27) and the coefficients \(N_l\) are normalization coefficients. The \(N_l\)’s and the \(A_{l,2i+1}\)’s for the present model are listed in appendix A, Tables VIII to XIII.
Next, we construct the wave-packet $\Theta(a, \tau)$, with the aid of the $\eta_l(a)$ and the energy levels, according to Eqs. (24) and (28), as well as Table I. Using the wave-packet $\Theta(a, \tau)$ we compute the expected value for the scale factor $a$, as in Eq. (25). The result is shown in Fig. 2. It can be seen that $\langle a \rangle$ does not assume the value zero; therefore we may say that the quantization of this model removed the singularities it had at the classical level. It is clear, also, from Fig. 2 that $\langle a \rangle$ has bounded oscillations, that is, oscillates between finite maximum and minimum values.

C. The model with $k = -1$

In this model $\alpha = -36$ and the spatial sections are some closed three-dimensional solid with negative constant curvature, locally isometric to $H^3$. Here, we have used 15 significant digits. Introducing the values of $K$ and $b$, we solved Eq. (22) to find $c = 0.41011969406177$. Now, using all these quantities in the determinant $D$ Eq. (23), we obtain the first 23 energy levels; they are listed in Table I. After that, we substitute these values in the set of Eqs. (19) and compute the coefficients $\beta_n$, used in

$$\eta_l(a) = N_l e^{-0.102527992350000 a^4 - 0.731507545000000 a^2} v_l(a), \quad (29)$$

where the $v_l(a)$ have the general expression given in Eq. (27) and the $N_l$'s are normalization coefficients. The $N_l$'s and the $A_l,2l+1$'s for the present model are listed in the appendix A, Tables XIV to XVIII. Due to numerical inconsistencies we have considered, in the present case, 18 $v_l(a)$'s corresponding to the first 18 energy levels.

Next, we construct the wave-packet $\Theta(a, \tau)$ with the aid of the $\eta_l(a)$, according to Eqs. (24) and (29) and the energy levels in Table I. Finally, using the wave-packet $\Theta(a, \tau)$ we compute the expected value for the scale factor $a$, according to (25). The result is shown in Fig. 3. It can be seen that $\langle a \rangle$ does not assume the value zero. Therefore, we may say that the quantization of this model removed the singularities it had at the classical level. It is also clear that $\langle a \rangle$ performs bounded oscillations.

From Table I we observe that the energy levels depend on the value of the curvature constant of the spatial sections. The model with negative constant curvature has the most bounded energy levels, then one has the model with zero curvature and finally the model with positive constant curvature has the less bounded energy levels.

Observing Eqs. (26), (28) and (29), we see that the wave-functions $\Psi(a, \tau)$ for all three cases
are exponentially damped as \(a \to \infty\) and behave as powers of \(a\) in the limit \(a \to 0\). Following Hawking and Page [24], we may say that this behavior of the \(\Psi(a, \tau)\)'s makes them \textit{excited states} of wormholes.

| Level | \(k=1\) | \(k=0\) | \(k=-1\) |
|-------|---------|---------|---------|
| \(E_1\) | 1.5103016760578712464 | 0.34040279742876372340 | -10.5531383749465 |
| \(E_2\) | 3.5509871014722954423 | 1.04960188616133576012 | -8.85096087898748 |
| \(E_3\) | 5.6323931685080087038 | 1.924878225962522139 | -7.22478356024350 |
| \(E_4\) | 7.725653179671366439 | 2.9223145294773461677 | -5.80294992656624 |
| \(E_5\) | 9.85778177762723638293 | 4.02273829686363472 | -4.22519889374100 |
| \(E_6\) | 12.018664367453930157 | 5.26079073554842275 | -2.87075159719689 |
| \(E_7\) | 14.207548216681935132 | 6.478734067709572007 | -1.63482529734928 |
| \(E_8\) | 16.42373543030010131 | 7.8108774877366064367 | -0.536430728059501 |
| \(E_9\) | 18.666574187471793427 | 9.2122719218977134548 | 0.481759230982801 |
| \(E_{10}\) | 20.935469528987590984 | 10.67458823118187238 | 1.5831085800971 |
| \(E_{11}\) | 23.2290089586946986517 | 12.19412672497696030 | 2.8331076555370 |
| \(E_{12}\) | 25.54220854858484858 | 13.7676252568273494 | 4.21398303307806 |
| \(E_{13}\) | 27.82969749251834013 | 15.392811056829087358 | 5.70783697327336 |
| \(E_{14}\) | 30.260978882469751228 | 17.06694219332377073 | 7.302861347368 |
| \(E_{15}\) | 32.651310592907906010 | 18.78809172182546958 | 8.99091730871327 |
| \(E_{16}\) | 35.06640708164574204 | 20.554451348143454694 | 10.765748496350 |
| \(E_{17}\) | 37.502643477205654295 | 22.364363322485552407 | 12.622505157144 |
| \(E_{18}\) | 39.963027226267456788 | 24.21637679099429384 | 14.5570952852202 |
| \(E_{19}\) | 42.4436440402959303694 | 26.19932260627772146 | 16.5661310993634 |
| \(E_{20}\) | 44.946892054208137000 | 28.041419503470901850 | 18.6464752057244 |
| \(E_{21}\) | 47.4792993831837116 | 30.0121104895025902151 | 20.7956557497686 |
| \(E_{22}\) | 50.0168754545613080 | 32.020172678051097447 | 23.0112498043436 |
| \(E_{23}\) | 52.581852999800707709 | 34.0644852960309900 | 25.2911792258758 |

TABLE I: The lowest calculated energy levels for the cases \(k=0\), \(k=1\), and \(k=-1\) (in all cases, \(\Lambda = -0.1\)).

V. CONCLUSIONS.

In the present paper, the formalism of quantum cosmology was employed to quantize three Friedmann-Robertson-Walker models in the presence of a negative cosmological constant and radiation. The variational formalism of Schutz [16] allowed us to ascribe dynamical degrees of freedom to the radiation fluid. The models differ from each other by the constant curvature of the spatial sections, which may be positive, negative or zero. The quantization of the models gave rise to Wheeler-DeWitt equations, for the scale factor, which had the form of the Schrödinger equation for the quartic anharmonic-oscillator. We found the approximate eigenvalues and eigenfunctions of those equations by using a method first developed by Chhajlany and Malnev [17]. After that,
we used the eigenfunctions in order to construct wave-packets for each case and evaluate the time dependent, expected value of the scale factors. We found for all of them that the expected values of the scale factors evolve with bounded oscillations. Since the expectation value of the scale factors never vanish, we concluded that these models do not have singularities. We also observed that the energy levels depend on the value of the curvature constant of the spatial sections. The model with negative curvature constant has the most bounded energy levels, whereas the model with positive constant curvature has the less bounded energy levels.

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APPENDIX A: COEFFICIENTS OF THE POLYNOMIALS $v_i(a)$

The following Tables contain the coefficients $A_{i,2i+1}$ introduced in eq. (27), for each case considered. Tables II to VII refer to the case $\Lambda = -0.1, k = 1$; Tables VIII to XIII refer to the case $\Lambda = -0.1, k = 0$; Tables XIV to XVIII refer to the case $\Lambda = -0.1, k = -1$. The normalization coefficients $N_l$ can be found in the respective captions.
TABLE III: The case $\Lambda = -0.1$, $k = 1$: coefficients $A_{2,2+i}$, $A_{3,2+i}$, $A_{3,2+i}$ and $A_{4,2+i}$ of the polynomials $v_1$, $v_2$, $v_3$ and $v_4$. The coefficients of the normalization are $N_1 = 5.03405139329824531$, $N_2 = 7.177893398758523789$, $N_3 = 8.100976169443530531$ and $N_4 = 8.829373452955623879$.

| $i$ | Coefficients $A_{2,2+i}$ | Coefficients $A_{3,2+i}$ | Coefficients $A_{4,2+i}$ | Coefficients $A_{5,2+i}$ |
|----|-----------------|-----------------|-----------------|-----------------|
| 0  | 1               | 1               | 1               | 1               |
| 1  | 3.0101778378780650567 | 3.711919912957988682 | 7.91550514702414092 | -12.12052513947670289 |
| 2  | 0.0458146435579258289 | -1.5414190599141135 | 7.85390535600127347 | 27.5270869798758433 |
| 3  | 0.004285639735431414758 | -0.163534878078269419 | 2.510476704539982642 | -10.55181090705699599 |
| 4  | 0.000265117720931812184 | -0.0145119948729966878 | 0.345238199577501333 | -3.812327087882482085 |
| 5  | 0.000143448997309090358 | -0.00691166587792663458 | 0.029286470133405913 | -0.5163403766138148 |
| 6  | 0.00000566644141038910 | -0.000401229566177813 | 0.000174607918283983 | -0.002402694824121028 |
| 7  | 0.000000800935087394 | -0.000015951604288011 | 0.00007740861677853 | -0.00263473533609904 |
| 8  | 0.000000046865857116 | 0.00000000472918331013 | 0.000000267318849027 | -0.00009874718661166 |
| 9  | 0.0069636666 x 10^{-11} | -0.000000011224053333 | 0.00000003741958296 | -0.000000318530880166 |
| 10 | 0.31473593 x 10^{-11} | -2.276166337 x 10^{-11} | 0.000000016444845275 | -0.000000008143729283 |
| 11 | 1.7264391 x 10^{-15} | -1.745255 x 10^{-17} | 3.01217556 x 10^{-11} | -0.00000000672249001 |
| 12 | 3.3556 x 10^{-17} | -5.1566 x 10^{-15} | 4.572534 x 10^{-13} | -2.79071173 x 10^{-11} |
| 13 | -4.521 x 10^{-20} | -6.6200 x 10^{-16} | 5.5476 x 10^{-15} | -3.080602 x 10^{-13} |
| 14 | -1.770 x 10^{-19} | -3.510 x 10^{-18} | 3.324 x 10^{-17} | -4.272 x 10^{-15} |
| 15 | -7.90 x 10^{-20} | -7.71 x 10^{-18} | -5.55 x 10^{-17} | -5.055 x 10^{-17} |
| 16 | -2.516632615944480508 x 10^{-20} | -3.81038361305366486 x 10^{-19} | -2.663403598954915162 x 10^{-18} | -5.495815805952724946 x 10^{-18} |
| 17 | -8.262464107607218199 x 10^{-21} | -1.27992858350292166 x 10^{-19} | -8.32099352590616564 x 10^{-19} | -1.53571922610499537 x 10^{-18} |
| 18 | -2.85367698695939083 x 10^{-17} | -4.13650685318377254 x 10^{-18} | -2.49510190728201979 x 10^{-17} | -4.2585768282465659 x 10^{-17} |
| 19 | -9.7900912539081576539 x 10^{-20} | -1.238554129519897984 x 10^{-19} | -7.169067141131138842 x 10^{-19} | -1.145428666427046249 x 10^{-18} |
| 20 | -2.807252483462929951 x 10^{-22} | -5.37902797674888446 x 10^{-21} | -1.169492030115953062 x 10^{-20} | -2.9760389894189501 x 10^{-20} |
| 21 | -8.876902812953990115 x 10^{-23} | -1.05109408483794670566 x 10^{-22} | -5.28866640263849473 x 10^{-21} | -7.6874152728580365 x 10^{-21} |
| 22 | -2.390616197064350868 x 10^{-23} | -2.869415147657205262 x 10^{-22} | -1.384743125110228663 x 10^{-21} | -1.89760890496289124 x 10^{-21} |

TABLE II: The case $\Lambda = -0.1$, $k = 1$: coefficients $A_{2,2+i}$, $A_{3,2+i}$, $A_{3,2+i}$ and $A_{4,2+i}$ of the polynoms $v_1$, $v_2$, $v_3$ and $v_4$. The coefficients of the normalization are $N_1 = 5.03405139329824531$, $N_2 = 7.177893398758523789$, $N_3 = 8.100976169443530531$ and $N_4 = 8.829373452955623879$. Able to
1. Coefficients $A_{0,2i+1}$

| i | Coefficients $A_{0,2i+1}$ |
|---|--------------------------|
| 0 | 1                        |
| 1 | -34.920367184849838455   |
| 2 | 301.1502108113170573     |
| 3 | -1074.561450716837436    |
| 4 | 1821.664178261444414     |
| 5 | -5456.135676676769056    |
| 6 | 626.289529144679131      |
| 7 | -86.46315919380308       |
| 8 | -7370.7089394744189      |
| 9 | 1.1600434769019113       |
| 10| 0.19134771783165755      |
| 11| 0.00005345650555767      |
| 12| 0.000013563752576736     |
| 13| 0.000003272617412782     |
| 14| 0.000000896059659689     |
| 15| -0.000000027611096515    |
| 16| -0.237793458450459745    |
| 17| 8.057519851399088103     |
| 18| 6.059794167995191810     |
| 19| 1.02159940980083814      |
| 20| 1.82940531826615080      |
| 21| 3.729203265719669990     |
| 22| 5.806765162285714334     |

2. Coefficients $A_{1,2i+1}$

| i | Coefficients $A_{1,2i+1}$ |
|---|--------------------------|
| 0 | 1                        |
| 1 | -57.19116754952899458    |
| 2 | 904.650193166384671      |
| 3 | -6231.178646784948620    |
| 4 | 22665.28968957207490     |
| 5 | -28511.53588496471157    |
| 6 | 32533.6397401910935      |
| 7 | -21615.83588433699227    |
| 8 | 10272.4802043182441      |
| 9 | -9.00318288026029698     |
| 10| 4.08708184714602549     |
| 11| -0.039480925883413581    |
| 12| 1.230836510872739247    |
| 13| -0.012565434538789      |
| 14| 0.005820703446505691    |
| 15| -0.004070326072981824   |
| 16| 0.000012625433567895    |
| 17| 0.000003063058048013    |
| 18| 0.000001027611096515    |
| 19| -0.0000001027611096515  |
| 20| 0.0000001027611096515   |
| 21| 0.0000001027611096515   |
| 22| 0.0000001027611096515   |

3. Coefficients $A_{2,2i+1}$

| i | Coefficients $A_{2,2i+1}$ |
|---|--------------------------|
| 0 | 1                        |
| 1 | -61.9718580954994902     |
| 2 | 1609.191274988781390     |
| 3 | -8907.85467124312174     |
| 4 | 45909.9494932741828     |
| 5 | -7849.032812948048188    |
| 6 | 18585.5240611622577     |
| 7 | -2172.52704719975453    |
| 8 | 7447.62801493655355     |
| 9 | -10573.923814700024037  |
| 10| 95064.18500322058688     |
| 11| 29885.4836569894002     |
| 12| 155.362845455656845     |
| 13| -1.2081193574639017     |
| 14| 0.91640484473517166    |
| 15| 0.5732655069009697      |
| 16| 1.9360858040813        |
| 17| 6.628014036355538       |
| 18| 3.08411426642694425     |
| 19| 5.48356569840002       |
| 20| -0.329879278090832502   |
| 21| 1.212352904234874       |
| 22| 1.008072758398060134    |

Table IV: The case $\Lambda = -0.1$, $k = 1$: coefficients $A_{0,2i+1}, A_{1,2i+1}, A_{2,2i+1}$ of the polynomials $v_9, v_{10}, v_{11}$ and $v_{12}$. The coefficients of the normalization are $N_9 = 12.5774391607750885, N_{10} = 11.75422030737378725, N_{11} = 12.13088884440861937 and N_{12} = 12.489270658573768104.

Table V: The case $\Lambda = -0.1$, $k = 1$: coefficients $A_{13,2i+1}, A_{14,2i+1}, A_{15,2i+1}$ and $A_{16,2i+1}$ of the polynomials $v_{13}, v_{14}, v_{15}$ and $v_{16}$. The coefficients of the normalization are $N_{13} = 12.831657345615050646, N_{14} = 13.160199699505029079, N_{15} = 13.481183098459831301 and N_{16} = 13.783901938149622931.
The case \( \Lambda = -0.1, k = 1 \): coefficients \( A_{17,2}+1 \), \( A_{18,2}+1 \), \( A_{19,2}+1 \) and \( A_{20,2}+1 \) of the polynomials \( v_1 \), \( v_2 \), \( v_1 \) and \( v_2 \). The coefficients of the normalization are \( \Lambda N_1 = 14.088835285318067966, N_{18} = 14.3719296753422158, N_{19} = 14.659291915176228163 \) and \( N_{20} = 14.930156447596541712 \).

\[
\begin{array}{cccc}
1 & Coefficients \( A_{17,2}+1 \) & Coefficients \( A_{18,2}+1 \) & Coefficients \( A_{19,2}+1 \) & Coefficients \( A_{20,2}+1 \) \\
1 & 1 & 1 & 1 & 1 \\
1 & -71.674505764424687952 & -76.59527362548310578 & -81.55650761532004390 & -86.560039218429771602 \\
2 & 1445.281293047251820 & 1657.605624254726463 & 1886.381424330086194 & 2132.22070120935152 \\
3 & -1295.762078598028761 & -1602.740104975024638 & -1956.862930953026290 & -2295.352.087190666129 \\
4 & 6283.794759697998382 & 8430.333013974434112 & 111087.7291195923833 & 1329875.057268366759 \\
5 & -18396.59722144101012 & -26953.846547304347888 & -385321.01208259817624 & -593010930907933 \\
6 & 14749.3874624855152 & 506351.9196649401080 & 874970.6232112823525 & 1329875.057268366759 \\
7 & 441540.3408409095578 & 791825.4272176973838 & -1361742.4045763478949 & -2259352.807190666129 \\
8 & 837990.6636160926593 & 738000.6402916308159 & 1497172.72406876630 & 2766070.843988238668 \\
9 & -290696.9867814435541 & -551618.427857578888 & -1187417.298035770539 & -2414563.481435768021 \\
10 & 160169.9016629087112 & 279218.5188632714734 & 687499.121336089958 & 1574324.417942725784 \\
11 & -31286.437651828955577 & -101454.73705241262099 & -291829.66149555115473 & -764549.93448862478493 \\
12 & 6933.883828175577839 & 26135.50163444661231 & 90489.25667174200654 & 276970.7755248607752 \\
13 & -827.92773276506180597 & -4632.644705966838681 & -20212.25469193666141 & -74446.7271855648475 \\
14 & 56.9715363260658843 & 529.70990976132116387 & 3156.5078170969347902 & 14630.127421681508902 \\
15 & -0.60818646671781787 & -13.32977267104135576 & -524.2366718306025380 & -2042.410751456805246 \\
16 & -0.13804878749944466 & 0.4327963029543647888 & 18.9093917267498872 & 191.55565880401964646 \\
17 & 0.00170555532620603351 & 0.5081247743296758777 & -0.031938001414568406 & -10.64041982516368418 \\
18 & 0.00253407041024959341 & -0.000089078034172207 & -0.0221195795277456563 & -0.21985595919567848 \\
19 & 0.00006045936716730944 & -0.000000404028802230226 & 0.00552696995868830 & 0.00705595832765000 \\
20 & 0.000000757016026240530 & -0.0000000052983757787 & 0.0000248478140916501 & -0.0000249470801656134 \\
21 & 0.000000000442918495059 & -0.0000001559026864238 & 0.0000002521547855223 & -0.0000006806919974417 \\
22 & 1.65463760926912201 & 3.1204878790252230 & 3.000000000010666093145985327 & 0.0000000003279910124338324555 \\
\end{array}
\]

TABLE VII: The case \( \Lambda = -0.1, k = 1 \): coefficients \( A_{21,2}+1 \), \( A_{22,2}+1 \) and \( A_{23,2}+1 \) of the polynomials \( v_1 \), \( v_2 \) and \( v_2 \). The coefficients of the normalization are \( N_{21} = 15.196350124389910569, N_{22} = 14.43515484930478896 \) and \( N_{23} = 15.645567647056785661 \).
The case $\Lambda = 3$.

TABLE VIII: The case $\Lambda = -0.1, k = 0$: coefficients $A_{2,1+i}$, $A_{2,2+i}$, $A_{3,3+i}$ and $A_{4,4+i}$ of the polynomials $v_1$, $v_2$ and $v_3$. The coefficients of the normalization are $N_1 = 2.14071273145309126$, $N_2 = 3.2975672607529412541$, $N_3 = 4.1572580515879983879$ and $N_4 = 4.87799685243088671$.

| $i$ | $A_{2,1+i}$ | $A_{2,2+i}$ | $A_{3,3+i}$ | $A_{4,4+i}$ |
|-----|-------------|-------------|-------------|-------------|
| 0   | 1           | 1           | 1           | 1           |
| 1   | -1.08965425492456695152 | -8.96949789386262545150 | 11.39095646258845371 | 13.17110380190125843 |
| 2   | 5.911356981069691861 | 14.52671429815456250 | 37.43081469087727242 | 45.23851745858884849 |
| 3   | 0.0484611386652386 | -4.415628113958991496 | 19.826241454993944663 | 51.36446972843141913 |
| 4   | -0.215771209614105729 | -2.23197712685588547 | 18.04829935024544045 | 46.03458678122724646 |
| 5   | -0.3127220167580372258 | 0.27549695190852529 | 2.1125585993972344 | 2.58982135729574534 |
| 6   | -0.0412722900677451245 | -0.205810920259844444 | 0.785122583951114951 | -1.38914857702272466 |
| 7   | 0.00127929585070893232 | 0.0255756358764110922 | 0.0979267596427372845 | 0.16916975977788552 |
| 8   | 0.000563080616555213 | -0.0022905100058957609 | 0.175672672335277223 | 0.3593942450920524321 |
| 9   | 0.000583139342052616344 | -0.0011865097096258307 | -0.00016062543266016517 | -0.0086526254284178584 |
| 10  | 0.000696206748764433456 | -0.00019645179227763463 | 0.00013699986960627451 | 0.0001419512567595559 |
| 11  | 0.0000497188710906381 | -0.00002037783497853086 | 0.00006066258836413128 | 0.000085157834071062795 |
| 12  | 0.00003507598484858766 | -0.00001560785248176592 | 0.000006055075353417073 | 0.00001470192457568935 |
| 13  | 0.00001469035529729898 | -0.0000098395147832486714 | 0.000000412578365955358 | 0.000001696317936031525 |
| 14  | 0.00000507317209762660 | -0.00000359585745284892617 | 0.0000002058721988055916 | 0.000000147024231242269 |
| 15  | 1.6590527773310810 | -0.000000019375154995018 | 0.000000007906547610537999 | 0.0000000495731508608849 |
| 16  | 3.9269913818710 | -3.101671168297710 | 2.283566945664 | 2.228356798048 | 2.383100028125 |
| 17  | 7.2437011819801515 | -6.19567120424224 | 5.10657179871839 | 5.13881025721858 | 5.13088102572185 |
| 18  | 1.0204303657410 | -1.3966489411910 | 2.2828058250801 | 2.2828058250801 | 2.33880000000000 |
| 19  | 7.66182381019 | -7.45425482021029 | 8.0399262210 | 8.0399262210 | 8.0399262210 |
| 20  | 3.4326710123 | -3.75392922 | 4.02607128 | 4.02607128 | 4.02607128 |
| 21  | 1.6918510126 | -7.75210126 | 5.97618056 | 5.97618056 | 5.97618056 |

TABLE IX: The case $\Lambda = -0.1, k = 0$: coefficients $A_{3,2+i}$, $A_{4,3+i}$, $A_{7,3+i}$ and $A_{8,4+i}$ of the polynomials $v_5$, $v_6$ and $v_5$. The coefficients of the normalization are $N_5 = 5.5127641866644550623$, $N_6 = 6.0885073250502348952$, $N_7 = 6.619917572251377252$ and $N_8 = 7.1165584094285250131$. 
The case Λ = N/11.

\[ \begin{align*}
1 & \quad \text{Coefficients A}_{9,2i+1} \\
0 & \quad 1 \\
1 & \quad -16.513892676521366889 \\
2 & \quad 68.5001612448738624 \\
3 & \quad -106.15659827876660383 \\
4 & \quad 66.82399151564388053 \\
5 & \quad -11.49015358076526177 \\
6 & \quad -3.173797462037087022 \\
7 & \quad 0.695068922441349115 \\
8 & \quad 0.1374915281783708335 \\
9 & \quad -0.008532843172313451 \\
10 & \quad -0.003790640422242626 \\
11 & \quad 0.0002429952828120116 \\
12 & \quad 0.000931759560788945 \\
13 & \quad 0.0004735617668937723 \\
14 & \quad 0.0000360070572626666 \\
15 & \quad 0.0000056363520928940 
\end{align*} \]

\[ \begin{align*}
1 & \quad \text{Coefficients A}_{10,2i+1} \\
0 & \quad 1 \\
1 & \quad -17.84437892635281758835 \\
2 & \quad 0.484596746394590593 \\
3 & \quad 0.0001445693495898953 \\
4 & \quad 0.000756513570199314 \\
5 & \quad 0.0001023884181599884 \\
6 & \quad -0.00001924785025817540602 \\
7 & \quad 0.000036195499028517150800 \\
8 & \quad 0.00000906556489975039291 \\
9 & \quad -0.0000000101518887756666 \\
10 & \quad -4.07278543068301981 \\
11 & \quad 0.6549057349229941 \\
12 & \quad 6.94619389320176157 \\
13 & \quad 3.9087806723168925 \\
14 & \quad 1.16031824410403455 \\
15 & \quad -1.24923699 
\end{align*} \]
TABLE XII: The case $\lambda = -0.1$, $k = 0$: coefficients $A_{17,2+1}$, $A_{18,2+1}$, $A_{19,2+1}$ and $A_{20,2+1}$ of the polynomials $v_{17}$, $v_{18}$ and $v_{19}$. The coefficients of the normalization are $N_{17} = 10.705458524372789228$, $N_{18} = 11.04230246761782828$, $N_{19} = 11.371947653621139153$ and $N_{20} = 11.69401933142212328$.

| Coefficients $A_{17,2+1}$ | Coefficients $A_{18,2+1}$ | Coefficients $A_{19,2+1}$ | Coefficients $A_{20,2+1}$ |
|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| $0$                         | $1$                         | $1$                         | $1$                         |
| $1$                         | $-58.11355691172124372$     | $-62.1296918889134863$      | $-66.2186458921318159$      |
| $2$                         | $0.5132093662252903$        | $1.099.85539665470655$      | $1.264.17551657305455$      |
| $3$                         | $7.313.840515021451148$     | $9.023.72844485212616$      | $-1.1017.89241056715348$    |
| $4$                         | $0.592.4954838429114$       | $4.077.82414467133701$      | $3.526.494870461818902$     |
| $5$                         | $-7.915.520455075840984$    | $-1.145.47115152991911$     | $-1.62494.0515996880322$    |
| $6$                         | $1.3096.2151831157913$      | $2.1469.8260538202037$      | $3.30602.32813557261476$    |
| $7$                         | $-16.347.0755991158433$     | $-2.8466.3898640108375$     | $-4.72495.1669318632112$    |
| $8$                         | $1.39290.5072888326392$     | $2.61701.5262705165357$     | $4.92106.9297617039263$     |
| $9$                         | $-9.769.47311186724690$     | $-1.18784.02174790301995$   | $-3.82727.2036221572606$    |
| $10$                        | $1.416.1342023307200$       | $9.0628.2505972566811$      | $2.26391.61037314154891$    |
| $11$                        | $-1.4516.300381609262037$   | $-4.0270.4020873652472836$  | $-1.03199.15685937234746$   |
| $12$                        | $1.3511.1925509582125$      | $1.2483.50321078413188$     | $3.6573.960225725437450$    |
| $13$                        | $-7.637.249518578900371$    | $-2.9709.518564472614691$   | $-1.1017.92405541099820$    |
| $14$                        | $1.117.25747223624190$      | $5.40.3340521327758122$     | $2.2194.268027452868640$    |
| $15$                        | $-1.47103658868635365$      | $-7.19725186739563823$      | $-3.97093963510504406$      |
| $16$                        | $0.7678441910736955467$     | $5.711996038388740561$      | $3.48.59056506191514270$    |
| $17$                        | $-0.24795719287515729020$   | $-0.5352277631450223233$    | $-4.84.6619450371115201$    |
| $18$                        | $-0.19.00985830247051695$   | $0.42142612954096613408$    | $0.364573570530886688$      |
| $19$                        | $0.00978280717534190649$    | $-0.00045572234106708401$   | $-0.01975901102718214400$   |
| $20$                        | $0.0009186464427472925383$  | $0.0001408421034337756998$  | $0.000279209021144666174$   |
| $21$                        | $0.0009034387787024970856$  | $0.000099929862799863221994$| $-0.00091366927542067051$   |
| $22$                        | $0.0009001741293063128279$  | $0.000000177756487795262963$| $0.000018649345249322889$   |

TABLE XIII: The case $\lambda = -0.1$, $k = 0$: coefficients $A_{21,2+1}$, $A_{22,2+1}$ and $A_{23,2+1}$ of the polynomials $v_{21}$, $v_{22}$ and $v_{23}$. The coefficients of the normalization are $N_{21} = 12.099134872819683019$, $N_{22} = 12.317752618954874001$ and $N_{23} = 12.61990574174860855$. 
The coefficients of the normalization are

\[ A_0, A_1, \ldots, A_{n-1} \]

for the case \( \Lambda = -0.1, k = -1 \): coefficients \( A_{2i+1}, A_{2i+2}, A_{2i+3} \) and \( A_{2i+4} \) of the polynomials \( v_1, v_2, v_3 \) and \( v_4 \). The coefficients of the normalization are

\[ N_1 = 0.10799425740054 \times 10^{-6}, \quad N_2 = 0.29609588075209 \times 10^{-5}, \quad N_3 = 0.51825990768902 \times 10^{-4} \quad \text{and} \quad N_4 = 0.68364982251735 \times 10^{-3}. \]
| \( j \) | \( A_{0,2i+1} \) | \( A_{0,2i+1} \) | \( A_{1,2i+1} \) | \( A_{1,2i+1} \) |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | 0.232010916759383 | 10.6841664013405 | 0.232010916759383 | 0.232010916759383 |
| 2 | 1.82534502790314 | 0.6732685171793 | 1.82534502790314 | 1.82534502790314 |
| 3 | 0.26450448980053 | 0.26450448980053 | 0.26450448980053 | 0.26450448980053 |
| 4 | 0.78625345700393 | 0.78625345700393 | 0.78625345700393 | 0.78625345700393 |
| 5 | 0.222581678877701 | 0.222581678877701 | 0.222581678877701 | 0.222581678877701 |
| 6 | 1.07872355607127 | 1.07872355607127 | 1.07872355607127 | 1.07872355607127 |
| 7 | 0.0835207257699909 | 0.0835207257699909 | 0.0835207257699909 | 0.0835207257699909 |
| 8 | 0.0039984508814047 | 0.0039984508814047 | 0.0039984508814047 | 0.0039984508814047 |
| 9 | 0.0002036371912035 | 0.0002036371912035 | 0.0002036371912035 | 0.0002036371912035 |
| 10 | 0.00001358650299393 | 0.00001358650299393 | 0.00001358650299393 | 0.00001358650299393 |
| 11 | 0.000009602874509464 | 0.000009602874509464 | 0.000009602874509464 | 0.000009602874509464 |
| 12 | 0.00000112062211165345 | 0.00000112062211165345 | 0.00000112062211165345 | 0.00000112062211165345 |
| 13 | 0.00000023438137405986 | 0.00000023438137405986 | 0.00000023438137405986 | 0.00000023438137405986 |
| 14 | 0.00000001244844412243 | 0.00000001244844412243 | 0.00000001244844412243 | 0.00000001244844412243 |
| 15 | 0.0000000041671109562926 | 0.0000000041671109562926 | 0.0000000041671109562926 | 0.0000000041671109562926 |
| 16 | 0.00000000076317977012007 | 0.00000000076317977012007 | 0.00000000076317977012007 | 0.00000000076317977012007 |
| 17 | 0.0000000000191622815810 | 0.0000000000191622815810 | 0.0000000000191622815810 | 0.0000000000191622815810 |
| 18 | 0.00000000015194457568067 | 0.00000000015194457568067 | 0.00000000015194457568067 | 0.00000000015194457568067 |
| 19 | 0.00000000035609137563205 | 0.00000000035609137563205 | 0.00000000035609137563205 | 0.00000000035609137563205 |
| 20 | 0.00000000053433833531015 | 0.00000000053433833531015 | 0.00000000053433833531015 | 0.00000000053433833531015 |
| 21 | 0.00000000071291018812278 | 0.00000000071291018812278 | 0.00000000071291018812278 | 0.00000000071291018812278 |
| 22 | 0.00000000095688493503208 | 0.00000000095688493503208 | 0.00000000095688493503208 | 0.00000000095688493503208 |

**TABLE XVI:** The case \( \lambda = -0.1, k = -1 \): coefficients \( A_{0,2i+1}, A_{10,2i+1}, A_{11,2i+1} \) and \( A_{12,2i+1} \) of the polynomials \( v_9, v_{10}, v_{11} \) and \( v_{12} \). The coefficients of the normalization are \( N_9 = 2.9703125957288, N_{10} = 4.35635774907061, N_{11} = 5.3968132616858 \) and \( N_{12} = 6.24220211798654 \).
| \( t \) | Coefficients \( A_{17,2i+1} \) | Coefficients \( A_{18,2i+1} \) |
|---|---|---|
| 0 | 1 | 1 |
| 1 | −24.5135934862227 | −28.3826830252342 |
| 2 | 171.297655578830 | 231.563657333425 |
| 3 | −532.032302271353 | −851.104989577650 |
| 4 | 879.049849725663 | 1700.00285906748 |
| 5 | −837.566899410330 | −2028.92845736494 |
| 6 | 466.195823019623 | 1512.52406344436 |
| 7 | −137.342049715716 | −704.22275510342 |
| 8 | 8.5966426834457 | 188.892562917903 |
| 9 | 7.3616712174631 | −17.5315261277321 |
| 10 | −2.1248762290393 | −5.41899629313905 |
| 11 | 0.061635310901618 | 1.9359317266694 |
| 12 | 0.061218659707561 | −0.16397783115011 |
| 13 | −0.00928671964286124 | −0.0263879713884429 |
| 14 | 0.0001214729454678 | 0.00637464986776314 |
| 15 | 0.00016743351198480 | −0.00018679286772932 |
| 16 | −0.00000107860479624729 | −0.000064625942914273 |
| 17 | −0.00000014792486677497 | 0.000000596587830124041 |
| 18 | 0.000000075304283408867 | 0.000000160392422977485 |
| 19 | 0.000000006058647233395 | −0.0000000400489171784274 |
| 20 | −0.000000000414054631266702 | 0.0000000080165107135159 |
| 21 | −9.3995623753296 × 10^{-12} | 8.5494253394270 × 10^{-11} |
| 22 | 8.3296478661547 × 10^{-13} | −3.37515341995877 × 10^{-12} |

**TABLE XVIII:** The case \( \Lambda = -0.1, k = -1 \): coefficients \( A_{17,2i+1} \) and \( A_{18,2i+1} \) of the polynomials \( v_{17} \) and \( v_{18} \). The coefficients of the normalization are \( N_{17} = 9.43144441465169 \) and \( N_{18} = 9.7506466119335 \).

**FIG. 1:** Behavior of the expectation value of the scalar factor for \( \Lambda = -0.1, k = 1 \).
FIG. 2: Behavior of the expectation value of the scalar factor for $\Lambda = -0.1$, $k = 0$.

FIG. 3: Behavior of the expectation value of the scalar factor for $\Lambda = -0.1$, $k = -1$. 