ON STABILITY, FLUCTUATIONS, AND QUANTUM MECHANICS

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ABSTRACT. We review an important stability approach to quantization by Rusov and Vlasenko and indicate possible comparison of fluctuations to standard situations involving a quantum potential.

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1. INTRODUCTION

In [59, 60] (which are the same modulo typos and conclusions) one sketches how the work of Chetaev [14, 15, 16] (based in particular on classical results of Poincaré [55] and Lyapunov [44]) might allow one to relate stability of classical Hamiltonian systems to quantum mechanics. We review here some of the arguments (cf. also [1, 2, 5, 25, 26, 58] for generalities on the Poincaré-Chetaev equations).

One recalls that holonomic systems involve an agreement of the degrees of freedom with the number of independent variables (cf. [64]). Then following [14] consider a holonomic system with Hamiltonian coordinates

\[ \frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}; \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j} \]
and think of perturbations (1A) \( q_j = q_j(t) + \xi_j \) and \( p_j = p_j(t) + \eta_j \). Denoting then \( q_j \sim q_j(t) \) and \( p_j \sim p_j(t) \) one has

\[
\frac{d(q_j + \xi_j)}{dt} = \frac{\partial H(t, q_i + \xi_i, p_i + \eta_i)}{\partial p_j}; \quad \frac{d(p_j + \eta_j)}{dt} = -\frac{\partial H(t, q_i + \xi_i, p_i + \eta_i)}{\partial q_j}
\]

(note that no connection is made a priori to a Schrödinger equation - SE). Expanding and using (1.1) gives

\[
\frac{d\xi_j}{dt} = \sum \left( \frac{\partial^2 H}{\partial p_j \partial q_i} \xi_i + \frac{\partial^2 H}{\partial p_j \partial p_i} \eta_i \right) + X_j;
\]

\[
\frac{d\eta_j}{dt} = -\sum \left( \frac{\partial^2 H}{\partial q_j \partial q_i} \xi_i + \frac{\partial^2 H}{\partial q_j \partial p_i} \eta_i \right) \eta_i + Y_j
\]

where the \( X_j, Y_j \) are higher order terms in \( \xi, \eta \). The first approximations (with \( X_j = Y_j = 0 \)) are referred to as Poincaré variational equations. Now given stability questions relative to functions \( Q_s \) of \((t, q, p)\) one writes

\[
x_s = Q_s(t, q_i + \xi_i, p_i + \eta_i) - Q_s(t, q_i, p_i) = \sum \left( \frac{\partial Q_s}{\partial q_i} \xi_i + \frac{\partial Q_s}{\partial p_i} \eta_i \right) + \cdots
\]

which implies

\[
\frac{dx_s}{dt} = \sum \left( \frac{\partial Q'_s}{\partial q_i} \xi_i + \frac{\partial Q'_s}{\partial p_i} \eta_i \right) + \cdots
\]

where

\[
Q'_s = \frac{\partial Q_s}{\partial t} + \sum \left( \frac{\partial Q_s}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial Q_s}{\partial p_i} \frac{\partial H}{\partial q_i} \right)
\]

Given \( 1 \leq s \leq 2k \) and \( 1 \leq i, j \leq k \) one can express the \( \xi, \eta \) in terms of \( x_s \) and write (1B) \( (dx_s/dt) = X_s \) (normal form) with \( X_s(0) = 0 \). For equations (1B) with \( 1 \leq s \leq n \), for sufficiently small perturbations \( \epsilon_j, \epsilon'_j \) one assumes there exists some system of initial values \( x_{s0} \) with \( \sum x_{s0}^2 < A \) for an arbitrarily small \( A \) (with perturbations \( \epsilon_j, \epsilon'_j \leq E_j, E'_j \)). Further for arbitrarily small \( E_j, E'_j \) one assumes it is possible to find \( A \) as above such that there exists one or more values \( \epsilon_j, \epsilon'_j \) with absolute values \( \leq E_j, E'_j \). Under these conditions the initial values of \( x_s \) play the same role for stability as the \( \epsilon_j, \epsilon'_j \) and one assumes this to hold. One assumes also convergent power series for the \( X_s \) etc. Then Lyapunov stability means that for arbitrary small \( A \) there exists \( \lambda \) such that for all perturbations \( x_{s0} \) satisfying \( \sum x_{s0}^2 \leq \lambda \) and for all \( t \geq t_0 \) one has \( \sum x_s^2 < A \) (i.e. the unperturbed motion is stable). Next one considers \( t \geq t_0 \) and \( \sum x_s^2 \leq H \) and looks for a sign definite (Lyapunov) function \( V \) (with \( V' = \partial_i V + \sum_{1}^{n} X_j (\partial V/\partial x_j) \)) then sign definite of opposite sign or zero). If such a function exists the unperturbed motion is stable (see [14] for proof).
We pick up the story now in [15] where relations between optics and mechanics are illuminated. Take a holonomic mechanical system with coordinates \( q_i \) and conjugate momenta \( p_i \) with \( n \) degrees of freedom. Assume the holonomic constraints are independent of time and the forces acting on the system are represented by a potential function \( U(q_i) \). Let (1C) \( T = (1/2) \sum_{i,j} g_{ij} p_i p_j \) denote the kinetic energy where the \( g_{ij} = g_{ji} \) are not dependent explicitly on time. Hamilton’s equations have the form

\[
2T = \sum g_{ij} \frac{\partial S}{\partial q_i} \frac{\partial S}{\partial q_j} = 2(U + E)
\]

where \( E \) represents a kinetic energy constant (the sign of \( U \) is changed in Section 2). Here the integral of (1.7) is (1D) \( S(q_i, \alpha_i) + c \) with the \( \alpha_i \) constants and (1E) \( ||\partial^2 S/\partial q_i \partial \alpha_j|| \neq 0 \) while (1F) \( E = E(\alpha_i) \). According to the Hamilton-Jacobi theory the general solution of the motion equations is given via (1G) \( p_i = \partial S/\partial q_i \) and \( \beta_i = -t(\partial E/\partial \alpha_i) + \partial S/\partial \alpha_i \) where the \( \beta_i \) are constants. In order to determine a stable solution one looks at the Poincaré variations

\[
\frac{d\xi_i}{dt} = \sum_j \left( \frac{\partial^2 H}{\partial q_i \partial p_j} \xi_j + \frac{\partial^2 H}{\partial p_j \partial p_i} \eta_j \right);
\]

\[
\frac{d\eta_i}{dt} = -\sum \left( \frac{\partial^2 H}{\partial q_i \partial q_j} \xi_j + \frac{\partial^2 H}{\partial p_j \partial q_i} \eta_j \right)
\]

where \( H \) should be defined here via (1H) \( H = T - U \). For a stable unperturbed motion the differential equations for Poincaré variations (1.8) must be reducible by nonsingular transformation to a system of linear differential equations with constant coefficients all of whose characteristic values must be zero (recall that the Lyapunov characteristic value \( X[f] \) of \( f \) is \( X[f] = -\lim_{t \to \infty} [\log(|f(t)|)]/t \) - cf. [44, 45]). In such perturbed motion, because of (1G) one has (recall \( p_i \sim \partial S/\partial q_i \))

\[
\eta_i = \sum_j \frac{\partial^2 S}{\partial q_i \partial q_j} \xi_j \quad (i = 1, \cdots, n)
\]

Hence

\[
\frac{d\xi_i}{dt} = \sum_{j,s} \xi_s \frac{\partial}{\partial q_s} \left( g_{ij} \frac{\partial S}{\partial q_j} \right) \quad (i = 1, \cdots, n)
\]

Note here that (1.8) involves \( \sum g_{ij} p_i p_j - U \) so

\[
\frac{\partial H}{\partial p_i} = \sum g_{ij} p_j; \quad \frac{\partial H}{\partial q_j} = \sum \frac{\partial g_{ij}}{\partial q_j} p_i p_j - \frac{\partial U}{\partial q_j}
\]
and (1.10) says

\[
\tag{1.10} \frac{d\xi_i}{dt} = \sum_s \xi_s \left( \frac{\partial g_{ij}}{\partial q_s} \frac{\partial S}{\partial q_j} + g_{ij} \frac{\partial^2 S}{\partial q_s \partial q_j} \right) = \sum_s \xi_s \frac{\partial g_{ij}}{\partial q_s} \frac{\partial S}{\partial q_j} + \sum g_{ij} \eta_j
\]

The second term here is \([\partial^2 H/\partial p_i \partial p_j] \eta_j\) and we want to identify the term \(\xi_s(\partial g_{ij}/\partial q_s)(\partial S/\partial q_j)\) with \(\partial^2 H/\partial q_i \partial q_j\). However we can see that \(\partial U/\partial q_i = 0\) so \(\xi_s(\partial^2 H/\partial q_s \partial p_i) = \xi_s(2\partial^2 T/\partial q_s \partial p_i) = \xi_s(\partial g_{ij}/\partial q_s) p_j\) confirming (1.10). Here the \(q_i, \alpha_i\) are represented by their values in an unperturbed motion.

Now for a stable unperturbed motion let (1.10) be reducible by a nonsingular linear transformation (1I)

\[
x_i = \sum \gamma_{ij} \xi_j
\]

with a constant determinant \(\Gamma = |\gamma_{ij}|\). If \(\xi_{ir}\) \((r = 1, \ldots, n)\) are a normal system of independent solutions of (1.10) then (1J)

\[
x_{ir} = \sum \gamma_{ij} \xi_{jr}
\]

will be the solution for the reduced system. For a stable unperturbed motion all the characteristic values of the solutions \(x_{ir}\) \((i = 1, \ldots, n)\) are zero and consequently

\[
\tag{1.11} ||x_{sr}|| = C^* = ||\gamma_{sj}|| ||\xi_{jr}|| = \Gamma C \exp \left[ \int \sum \frac{\partial}{\partial q_i} \left( g_{ij} \frac{\partial S}{\partial q_j} \right) dt \right]
\]

Consequently for a stable perturbed motion (cf. [14, 44, 45])

\[
\tag{1.12} \sum \frac{\partial}{\partial q_i} \left( g_{ij} \frac{\partial S}{\partial q_j} \right) = 0
\]

Finally one considers a solution (1K) \(\Phi(-Et + S)\) of the HJ equation and for a stable unperturbed solution, because of (1.12), (1.7), and (1G), one has

\[
\tag{1.13} \sum \frac{\partial}{\partial q_i} \left( g_{ij} \frac{\partial \Phi}{\partial q_j} \right) = \Phi' \sum \frac{\partial}{\partial q_i} \left( g_{ij} \frac{\partial S}{\partial q_j} \right) + \Phi'' \sum g_{ij} \frac{\partial S}{\partial q_i} \frac{\partial S}{\partial q_j} = \frac{2(U + E)}{E^2} \frac{\partial^2 \Phi}{\partial t^2}
\]

which is a wave equation

\[
\tag{1.14} \frac{2(U + E)}{2E^2} \frac{\partial^2 \Phi}{\partial t^2} = \sum \frac{\partial}{\partial q_i} \left( g_{ij} \frac{\partial \Phi}{\partial q_j} \right)
\]

This indicates the analogy between Cauchy’s theory of light and stable motions of holonomic conservative systems (cf. [14, 15, 16]).

2. STABILITY APPROACH

Following Rusov and Vlasenko one writes an integral of the Hamilton-Jacobi (HJ) equation in the form (2A) \(S = f(t, q_i, \alpha_i) + A (i = 1, \ldots, n)\) with the \(\alpha_i\) arbitrary constants. The general solution is then (2B) \(p_i = \partial S/\partial q_i\) with \(\beta_i = \partial S/\partial \alpha_i\) where the \(\beta_i\) are new constants of integration. The canonical equations of motion are \(dq_i/dt = \partial H/\partial p_i\) and \(dp_i/dt = \partial H/\partial q_i\).
\(-\partial H/\partial q_i\), where \(H\) is the Hamiltonian and under perturbations of the \(\alpha_i, \beta_i\) one writes \(\xi_i = \delta q_i = q_i - q_i(t)\) and \(\eta_i = \delta p_i = p_i - p_i(t)\) and derives equations of first approximation

\[
\frac{d\xi_i}{dt} = \sum \frac{\partial^2 H}{\partial q_j \partial p_i} \xi_j + \sum \frac{\partial^2 H}{\partial p_j \partial p_i} \eta_j
\]

\[
\frac{d\eta_i}{dt} = -\sum \frac{\partial^2 H}{\partial q_j \partial q_i} \xi_j - \sum \frac{\partial^2 H}{\partial p_j \partial q_i} \eta_j
\]

as in (1.8). By differentiating in \(t\) one obtains then (2C) \(C = \sum (\xi_s \eta'_s - \eta_s \xi'_s)\) where \(C\) is a constant. Also for given \(\xi_s, \eta_s\) there is always at least one solution \(\xi'_s, \eta'_s\) for which \(C \neq 0\). Stability considerations (as in (1.1)) then lead via (\(\star\)) \(\eta_i = \sum (\partial^2 S/\partial q_i \partial q_j) \xi_j\) and (2D) \(H = (1/2) \sum g_{ij} p_i p_j + U = T + U\) to

\[
\frac{d\xi_i}{dt} = \sum \xi_s \frac{\partial}{\partial q_s} \left( g_{ij} \frac{\partial S}{\partial q_j} \right)
\]

(note in Section 1 \(H \sim T - U\) following [15] but we take now \(U \to -U\) to agree with [59,60] - the sign of \(U\) is not important here). According to [59,60], based on results of Chetaev [15] (as portrayed in Section 1), it results that \(L = \sum (\partial/\partial q_i) [g_{ij}(\partial S/\partial q_j)] = 0\) (as in (1.12)) for stability (we mention e.g. [14,15,16,44,45,50] for stability theory, Lyapunov exponents, and all that). One also notes in [59,60] that a similar result occurs for \(U \to U^* = U + Q\) for natural \(Q\) and it is assumed that it is \(Q\) which generates perturbations \(\delta q, \delta p\).

Now one introduces (in an ad hoc manner) a function (2E) \(\psi = A \exp(ikS)\) in (1.12) where \(k\) is constant and \(A\) is a real function of the coordinates \(q_i\) only. There results

\[
\frac{\partial S}{\partial q_j} = \frac{1}{ik} \left( \frac{1}{\psi} \frac{\partial \psi}{\partial q_j} - \frac{1}{A} \frac{\partial A}{\partial q_j} \right)
\]

so that (1.12) becomes

\[
\sum_{i,j} \frac{\partial}{\partial q_i} \left[ g_{ij} \left( \frac{1}{\psi} \frac{\partial \psi}{\partial q_j} - \frac{1}{A} \frac{\partial A}{\partial q_j} \right) \right] = 0
\]

On the other hand for the perturbed motion (with \(U \to U^* = U + Q\) the HJ equation can be written in the form

\[
\frac{1}{2k^2} \sum_{i,j} g_{ij} \left[ \frac{1}{\psi} \frac{\partial \psi}{\partial q_i} - \frac{1}{A} \frac{\partial A}{\partial q_i} \right] \left[ \frac{1}{\psi} \frac{\partial \psi}{\partial q_j} - \frac{1}{A} \frac{\partial A}{\partial q_j} \right] = \partial_t S + U + Q
\]
with \( \partial_t S \) obtained via (2E). Adding (2.4) and (2.5) yields
\[
(2.6) \quad \frac{1}{2k^2} \sum_{i,j} \frac{\partial}{\partial q_i} \left( g_{ij} \frac{\partial \psi}{\partial q_j} \right) - \frac{i}{2k^2} A \sum_{i,j} \frac{\partial}{\partial q_i} \left( \frac{\partial A}{\partial q_j} \right) - \frac{1}{k^2} A \sum_{i,j} \frac{\partial A}{\partial q_i} \frac{\partial S}{\partial q_j} \frac{\partial A}{\partial q_i} - \frac{1}{ikA} \partial_t A - U - Q = 0
\]
as a necessary stability condition (in the first approximation). Note (2.6) will not contain \( Q \) if \( A \) is defined via
\[
(2.7) \quad \frac{1}{2k^2} \sum_{i,j} \frac{\partial}{\partial q_i} \left( g_{ij} \frac{\partial A}{\partial q_j} \right) + \frac{i}{kA} \sum_{i,j} g_{ij} \frac{\partial A}{\partial q_i} \frac{\partial S}{\partial q_j} \frac{\partial A}{\partial q_i} = 0
\]
which means
\[
(2.8) \quad Q = -\frac{1}{2k^2} \sum_{i,j} \frac{\partial}{\partial q_i} \left( g_{ij} \frac{\partial A}{\partial q_j} \right) ; \quad \partial_t A = -\sum_{i,j} g_{ij} \frac{\partial A}{\partial q_i} \frac{\partial S}{\partial q_j}
\]
A discussion of the physical content of (2.8) appears in [59, 60] and given (2.8) the stability condition (2.6) leads to
\[
(2.9) \quad \frac{i}{k} \partial_t \psi = -\frac{1}{2k^2} \sum_{i,j} \frac{\partial}{\partial q_i} \left( g_{ij} \frac{\partial \psi}{\partial q_j} \right) + U \psi
\]
which is of course a SE for \( k = 1/\hbar \) (this is the place where quantum mechanics somewhat abruptly enters the picture - see Remark 2.1). In fact for kinetic energy \( (2F) \quad T = (1/2m) [p_1^2 + p_2^2 + p_3^2] \) (2.9) leads to
\[
(2.10) \quad Q = -\frac{\hbar^2}{2m} \frac{\Delta A}{A}; \quad \partial_t A = -\frac{1}{m} \sum_{x_j} \frac{\partial A}{\partial x_j} \frac{p_j}{m}; \quad k = \frac{1}{\hbar}
\]
and (2.9) becomes
\[
(2.11) \quad i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \Delta \psi + U \psi
\]
Going backwards now put the wave function \( \psi = A \exp(iS/\hbar) \) in (2.11) to obtain via (1.12) and (2.8) the Bohmian equations
\[
(2.12) \quad \partial_t A = -\frac{1}{2m} [A \Delta S + 2 \nabla A \cdot \nabla S] = -\nabla A \frac{\nabla S}{m}; \quad \partial_t S = -\left[ \frac{(\nabla S)^2}{2m} + U - \frac{\hbar^2}{2m} \frac{\Delta A}{A} \right]
\]
where the quantum potential \( Q_P \) is naturally identified.

If one writes now \( P = \psi \psi^* = A^2 \) then (2.12) can be rewritten in a familiar form
\[
(2.13) \quad \partial_t P = -\nabla P \cdot \frac{\nabla S}{m}; \quad \partial_t S + \frac{(\nabla S)^2}{2m} + U - \frac{\hbar^2}{4m} \left[ \frac{\Delta P}{P} - \frac{1}{2} \frac{(\nabla P)^2}{P^2} \right] = 0
\]
That \( P \) is indeed a probability density is substantiated via a (somewhat vague) least action of perturbation principle of Chetaev [16] which takes the form (2G) \( \int Q |\psi|^2 dV = min \) where \( dV \) is a volume element for the phase space (\( \int |\psi|^2 dV = 1 \)) and this condition involves absolute stability (one assumes that the influence of perturbative forces generated by \( Q \) is proportional to the density \( |\psi|^2 = A^2 \)). Write now, using (2D)

\[
Q = -\partial_t S - U - T = -\partial_t S - U - \frac{1}{2} \sum g_{ij} \frac{\partial S}{\partial q_i} \frac{\partial S}{\partial q_j}
\]

Then if (2E) holds one can show that

\[
\frac{1}{2} \sum g_{ij} \frac{\partial S}{\partial q_i} \frac{\partial S}{\partial q_j} = - \frac{1}{2k^2\psi^2} \sum g_{ij} \frac{\partial \psi}{\partial q_i} \frac{\partial \psi}{\partial q_j} + \frac{1}{2k^2A^2} \sum g_{ij} \frac{\partial A}{\partial q_i} \frac{\partial A}{\partial q_j} + \frac{ik}{2k^2A^2} \sum g_{ij} \frac{\partial A}{\partial q_i} \frac{\partial S}{\partial q_j}
\]

Then for the first term on the right side substitute its value from the first stability condition (2.4), then insert this relation into (2.15) and put the result into the equation (2.14) corresponding to the variational principle; the result is then (2.6) and consequently the resulting structure expression and the necessary condition for stability coincide with (2.8) and (2.9). This leads one to conclude classical mechanics and the quantization (stability) conditions represent two complementary procedures for description of stable motions of a physical system in a potential field. The authors cite an impressive list of references related to experimental work supporting these kinds of conclusion.

**REMARK 2.1.** The arguments in [59, 60] have seemed to be independent of the nature of the perturbations beyond the important relation (1.9). However the emergence of \( Q \) as a quantum potential provides \( 2\nabla A/A = \nabla P/P \sim \delta p \) as a “standard” momentum fluctuation. This seems to suggest some equivalence to standard perturbative models with a quantum potential (cf. [9, 10]) and perhaps forecasts the uncertainty principle in some sense (see below). The technique could perhaps provide an alternative approach to some results of [27, 28] for example involving the generation of the SE from Hamiltonian theory via metaplectic coverings and short time propagators, etc. The results reviewed here seem however to be perhaps too general although very attractive.

### 3. THE QUANTUM POTENTIAL

From Sections 1-2 we have the suggestion that given a stable Hamiltonian system with perturbations \( \delta q \) and \( \delta p \) generated by a “potential” \( Q \sim \delta U \) it follows that there is a Schrödinger equation (SE) with \( Q \) as the quantum potential (QP) which describes the motion. It seems therefore appropriate
to examine this in the light of other manifestations of the QP as in e.g. \[9, 10, 11, 12, 17, 21, 22, 23, 24, 30, 31, 32, 33, 35, 42, 43, 56\]. We note that following \[10\] one can reverse some arguments involving the exact uncertainty principle (cf. \[9, 31, 32, 33, 56\]) to show that any SE described by a QP based on \(|\psi|^2 = P\) can be modeled on a quantum model of a classical Hamiltonian \(H\) perturbed by a term \(H_Q\) based on Fisher information, namely

\[
H_Q = \frac{c}{2m} \int \frac{(\nabla P)^2}{P} dx = \frac{c}{2m} \int P(\delta p)^2
\]

where \(\delta p = \nabla P/P\). This does not of course deny the presence of “related” \(x \sim q\) oscillations \(\delta x \sim \delta q\) and in fact in Olavo \[54\] (cf. also \[9\]) Gaussian fluctuations in \(\delta q\) are indicated and related to \(\delta p\) via an exact uncertainty relation \((3A)\) \((\delta p)^2 \cdot (\delta q)^2 = \hbar^2/4\). We note that the arguments establishing exact uncertainty stipulate that the position uncertainty must be entirely characterized by \(P = |\psi|^2\) (cf. \[9, 31, 32, 33, 56\]).

**REMARK 3.1.** We recall here \[34\] (cf. also \[57\]) were it is shown that quantum mechanics can be considered as a classical theory in which a Riemannian geometry is provided with the distance between states defined with natural units determined via Planck’s constant (which is the inverse of the scalar curvature).

In \[6\] one shows that non-relativistic quantum mechanics for a free particle emerges from classical mechanics via an invariance principle under transformations that preserve the Heisenberg inequality. The invariance imposes a change in the laws of classical mechanics corresponding to the classical to quantum transition. Some similarities to the Nottale theory of scale relativity in a fractal space-time are also indicated (cf. \[9, 13, 52, 51\]). There are relations here to the Hall-Reginatto treatment which postulates that the non-classical momentum fluctuations are entirely determined by the position probability (as mentioned above). In Brenig’s work one derives this from an invariance principle under scale transformations affecting the position and momentum uncertainties and preserving the Heisenberg inequality. One modifies the classical definition of momentum uncertainty in order to satisfy the imposed transformation rules and this modification is also constrained by conditions of causality and additivity of kinetic energy used by Hall-Reginatto. This leads to a complete specification of the functional dependance of the supplementary term corresponding to the modification which turns out to be proportional to the quantum potential. We give a brief sketch of this as follows and refer to \[6, 9\] for more details. Thus one wants to preserve \((\Delta x)(\Delta p) \geq \hbar^2/4\) for \(x \sim x_k, p \sim p_k (k = 1, 2, 3)\).
and is led to the following transformation (\(\alpha \in \mathbb{R}\))

\[
(\Delta x')^2 = e^{-\alpha}(\Delta x)^2; \quad (\Delta p')^2 = e^{-\alpha}(\Delta p)^2 + \frac{h^2}{4}\frac{(e^\alpha - e^{-\alpha})}{(\Delta x)^2}
\]

Consequently

\[
(\Delta x')^2(\Delta p')^2 = e^{-2\alpha}(\Delta x)^2(\Delta p)^2 + \frac{h^2}{4}(1 - e^{-2\alpha})
\]

Thus if \((\Delta x)^2(\Delta p)^2 = h^2/4\) it remains so for any \(\alpha\) and for \(\alpha \to \infty\) one has \((\Delta x')^2(\Delta p')^2 \to h^2/4\) for any value of \((\Delta x)^2(\Delta p)^2 \geq (h^2/4)\). Now one considers a probability density \(P\) and an action variable \(S\) with functionals of the form (3.3) \(\mathcal{A} = \int d^dxF(x, P, \nabla P, ..., S, \nabla S, ...)\) where classically (3C) \(\partial_t \mathcal{A} = \{\mathcal{A}, H_C\}\) with (3D) \(H_C = \int d^d x[P|\nabla S|^2/2m]\) a Hamiltonian functional and

\[
\{\mathcal{A}, \mathcal{B}\} = \int d^d x \left[ \frac{\delta \mathcal{A}}{\delta P} \frac{\delta \mathcal{B}}{\delta \nabla S} - \frac{\delta \mathcal{A}}{\delta \nabla S} \frac{\delta \mathcal{B}}{\delta P} \right]
\]

This provides an infinite Lie algebra structure for functionals (3B). The time transformations are generated by \(H_C\) applied to \(P(x)\) and \(S(x)\) and yields the continuity equation and the HJ equation

\[
\partial_t S = -\nabla \cdot \left( \frac{P \nabla S}{m} \right); \quad \partial_t P = -\frac{[\nabla S]^2}{2m}
\]

where \(\nabla S = p\) is the classical momentum. Now consider space dilatations \(x \to \exp(-\alpha/2)x\) with

\[
P'(x) = e^{3\alpha/2}P(e^{\alpha/2}x); \quad S'(x) = e^{-\alpha}S(e^{\alpha/2}x)
\]

noting that they keep the dynamical equations (3.5) invariant. For simplicity assume that the average momentum of the particle is zero; general results can then be retrieved by a Galilean transformation. Then the classical uncertainty for a momentum component is (3E) \(\Delta p_{cl,k}^2 = \int d^d x P(\partial_k S)^2\)

and, dropping the index \(k\), via (3.6) \(\Delta p_{cl}^2\) transforms as (3F) \(\Delta'(p_{cl})^2 = e^{-\alpha}\Delta p_{cl}^2\) while (3G) \(\Delta'(x')^2 = e^{-\alpha}\Delta x^2\) (with \(\Delta x^2\) still unspecified). Evidently (3F) shows that (3.2) does not hold but rather corresponds to the first term on the right in (3.2). Hence one must modify (3E) in order to get a quantity \(\Delta p^2\) satisfying (3.2). This leads to

\[
\Delta_{pq,k}^2 = \int d^d x P(x)(\partial_k S(x))^2 + h^2\Omega_k \quad (k = 1, 2, 3)
\]

Now impose the condition that the rules (3H) should transform \(\Delta_{pq}^2\) as prescribed by (3.2) and this will reduce the set of possible functional forms of \(\Omega\). There results (cf. [4] for details)

\[
\Delta(p_{q})^2 = e^{-\alpha}\Delta_{pq}^2 + h^2\Omega' \Rightarrow \Delta(p_{q}')^2 = e^{-\alpha}\Delta_{pq}^2 + h^2(\Omega' - e^{-\alpha}\Omega)
\]
Identifying this with (3.2) yields then

\[
(3.9) \quad Q' - e^{-\alpha} Q = \frac{1}{4\Delta x^2} (e^\alpha - e^{-\alpha}) \Rightarrow Q' - \frac{1}{4\Delta (x')^2} = e^{-\alpha} \left( Q - \frac{1}{4\Delta x^2} \right)
\]

The form of this equation indicates the existence of a relation between \( Q \) and \( \Delta x^2 \) that is scale independent, namely (3I) \( Q_k = 1/4\Delta x_k^2 \); this is the only possibility for which the relation between \( \Delta p_k^2 \) and \( \Delta x_k^2 \) is independent of \( \alpha \). In conclusion the supplementary term necessary to obtain a definition of \( \Delta p^2_q \) compatible with (3.2) is inversely proportional to \( \Delta x^2 \) as in (3I). Compatibility with the Hall-Reginatto methods and techniques is then explained (cf. [6]) and one is led to the form

\[
(3.10) \quad Q_k = \beta \int d^3 x \left[ \partial_k P(x)^{1/2} \right]^2
\]

leading to (for \( \beta = 1 \))

\[
(3.11) \quad H_q = \int d^3 x \, \left[ \frac{P(x) |\nabla S(x)|^2}{2m} + \frac{\hbar^2}{2m} |\nabla P^{1/2}(x)|^2 \right]
\]

and one has for \( P = R^2 \) (from \( \psi = R\exp(iS/\hbar) \)) the formula \( |\nabla P^{1/2}| = (1/2)|\nabla P|/P \Rightarrow |\nabla P^{1/2}| = (1/4)|\nabla P|^2 \). Hence the last term in (3.11) coincides with a quantum potential times \( P \) via

\[
(3.12) \quad \frac{\hbar^2}{2m} |\nabla P^{1/2}|^2 = \frac{\hbar^2}{8m} \left( \frac{\nabla P}{P} \right)^2 ; \quad Q = -\frac{\hbar^2}{2m} \Delta P^{1/2};
\]

\[
PQ = -\frac{\hbar^2}{8m} \left[ 2\Delta P - \frac{(\nabla P)^2}{P} \right] ; \quad \int PQ d^3 x = \frac{\hbar^2}{8m} \int d^4 x \left( \frac{\nabla P}{P} \right)^2
\]

and this is the desired quantum addition to the classical Hamiltonian.

**REMARK 3.2.** We note that in work of Grössing (cf. [12, 30]) one deals with subquantum thermal oscillations leading to momentum fluctuations (3J) \( \delta p = -(\hbar/2)(\nabla P/P) \) where \( P \) is a position probability density with \( -\nabla \log(P) = \beta \nabla Q \) for \( Q \) a thermal term (thus \( P = e^{\exp(-\beta Q)} \) where \( \beta = 1/kT \) with \( k \) the Boltzman constant). This leads also to consideration of a diffusion process with osmotic velocity \( u \propto -\nabla Q \) and produces a quantum potential

\[
(3.13) \quad Q = \frac{\hbar^2}{4m} \left[ \nabla^2 \tilde{Q} - \frac{1}{D} \partial_t \tilde{Q} \right]
\]

where \( \tilde{Q} = Q/kT \) and \( D = \hbar/2m \) is a diffusion coefficient. Consequently (cf. [12] one has a Fisher information (3K) \( F \propto \beta^2 \int \exp(-\beta Q(\nabla Q)^2) d^3 x \). As in the preceding discussions the fluctuations are generated by the position probability density and one expects a connection to (Bohmian) quantum mechanics (cf. [9, 17, 23, 24]).
REMARK 3.3. There is considerable literature devoted to the emergence of quantum mechanics from classical mechanics. There have been many studies of stochastic and hydrodynamic models, or fractal situations, involving such situations and we mention in particular [9, 10, 11, 12, 13, 17, 18, 23, 24, 29, 30, 31, 32, 33, 12, 43, 47, 17, 18, 39, 51, 52, 54, 56, 61, 62, 63]; a survey of some of this appears in [9]. For various geometrical considerations related to the emergence question see also [3, 7, 8, 19, 35, 36, 37, 38, 39, 40, 41, 46].
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