A three-dimensional non-hydrostatic mathematical model with mixed triangle and quadrilateral grids

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Abstract. The three-dimensional numerical models based on the hydrostatic pressure hypothesis can’t accurately and effectively simulate the cases of estuaries or natural rivers with the presence of short wave flow, stratified gravity flow, sudden change of local topography or flow near underwater buildings. Therefore it is useful to resort to a more accurate model in which the hydrostatic assumption is removed. Based on the mixed grids of triangle and quadrilateral, a three-dimensional non hydrostatic mathematical model is presented. The control equations are discretized by the semi-implicit fractional step method. The pressure is divided into the hydrostatic pressure and non-hydrostatic pressure terms. At each step, the water level is calculated by solving the sparse equations, and then the non-hydrostatic pressure is obtained by the pressure Poisson equation. Three typical examples were adopted to simulate strong three-dimensional flow. As a result, the results show that the model can accurately and effectively simulate strong three-dimensional flow with a few layers.

Keywords: 3-D Numerical Model; Non-Hydrostatic; Free Surface Flows

1. Introduction
The three-dimensional numerical simulation is of great significance in practical engineering. At present, the three-dimensional numerical models based on the hydrostatic pressure hypothesis were widely used to solve practical engineering problems, namely the three-dimensional shallow water model. In the case of the short-wave flow, stratified gravity flow, sudden change of local topography or flow near underwater buildings, the effect of vertical acceleration is more. The hydrostatic pressure assumption model will introduce large errors. It is urgently needing to establish a more precise three-dimensional non-hydrostatic mathematical model.

The non-hydrostatic mathematical model decomposes the pressure term of Reynolds time averaged N-S equation into the non-hydrostatic pressure and hydrostatic pressure term, and uses the step-by-step method to solve the N-S equation[1-3]. The non-hydrostatic model obtains the free surface by solving the continuity equation of water depth average, while the model based on the Reynolds time averaged N-S equation needs to adopt the specific free surface capture method. Therefore, compared with the model based on Reynolds time averaged N-S equation, the non-hydrostatic model only needs a few layers of grid in the vertical direction, and the calculation stability and efficiency are greatly improved. These characteristics enable it to be effectively used to simulate strong three-dimensional flow. Recent years, their non-hydrostatic mathematical model based on the semi-implicit fractional step method[4], the explicit projection method[5], and the implicit method[6-7] have been developed. Using a few of vertical layers, these models can accurately predict strong three-dimensional flow.
The accuracy and calculation efficiency of the three-dimensional non-hydrostatic mathematical models are determined by the grid system, construction ideas and processing modes in the control equation. Due to the maximum flexibility of the triangle grid, and high efficiency of the quadrilateral grid, a 3D non-hydrostatic mathematical model is developed with the triangular and quadrilateral mixed grids in this paper. The friction step method is used to solve the N-S equations, which is divided into two main steps. Firstly, the momentum equations without pressure gradient term are solved. Secondly, the momentum equations and continuity equation with only pressure term are solved. To minimize the number of vertical layers and subsequently the computational cost, the higher order approximation method is used to calculate the vertical velocity in the top layer. The preconditioned conjugate gradient method is adopted to solve the pressure Poisson equation. In order to obtain the maximum flexibility and high efficiency in the complex computational region, the control equations are discretized on the mixed triangle and quadrilateral grids.

2. Governing equation

The conservative governing equations can be expressed as[9]

\[
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho u v) + \frac{\partial}{\partial z} (\rho u w) = -g \frac{\partial P}{\partial x} \rho + \frac{\partial}{\partial x} (\rho \nu) \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} (\rho \nu) \frac{\partial u}{\partial y} + \frac{\partial}{\partial z} (\rho \nu) \frac{\partial u}{\partial z} \tag{1}
\]

\[
\frac{\partial v}{\partial t} + \frac{\partial}{\partial x} (\rho u v) + \frac{\partial}{\partial y} (\rho u v^2) + \frac{\partial}{\partial z} (\rho u v w) = -g \frac{\partial P}{\partial y} \rho + \frac{\partial}{\partial x} (\rho \nu) \frac{\partial v}{\partial x} + \frac{\partial}{\partial y} (\rho \nu) \frac{\partial v}{\partial y} + \frac{\partial}{\partial z} (\rho \nu) \frac{\partial v}{\partial z} \tag{2}
\]

\[
\frac{\partial w}{\partial t} + \frac{\partial}{\partial x} (\rho u w) + \frac{\partial}{\partial y} (\rho u v w) + \frac{\partial}{\partial z} (\rho u w^2) = -g \frac{\partial P}{\partial z} \rho + \frac{\partial}{\partial x} (\rho \nu) \frac{\partial w}{\partial x} + \frac{\partial}{\partial y} (\rho \nu) \frac{\partial w}{\partial y} + \frac{\partial}{\partial z} (\rho \nu) \frac{\partial w}{\partial z} \tag{3}
\]

where \( u \) is the velocity component in \( x \)-coordinates; \( v \) is the velocity component in \( y \)-coordinates; \( w \) is the velocity component in \( z \)-coordinates. \( P \) is the pressure term, \( \nu \) and \( \nu' \) are vertical and horizontal turbulent eddy viscosity, respectively. \( f \) is the Coriolis parameter, \( \rho \) is the density.

Integrate the continuity equation from the surface \( z = \eta \) to the bottom \( z = -h \) to obtain the free surface control equation

\[
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \int_{-h}^{\eta} udz + \frac{\partial}{\partial y} \int_{-h}^{\eta} vdz = 0 \tag{5}
\]

where \( \eta \) is the water surface above horizontal datum.

3. Numerical methods

3.1. The definition of variables

Due to the maximum flexibility of the unstructured grid, the triangle and quadrilateral mixed grids were used to discretize the complex boundary in computed domain. The control volume and definition of variables are shown in Figure 1. The detailed definition of multi-layer was adopted from ref [8] in the vertical direction.

3.2. Discretization and algorithm

In order to avoid the influence of free surface wave velocity, bottom friction, vertical viscosity and wind stress, on the stability of the model, a semi-implicit fractional step method[4] is used to solve the continuity control equation and Navier-Stokes equation on the triangle and quadrilateral mixed grids. In the first step of hydrostatic pressure calculation, the water level gradient term and non-hydrostatic pressure term in the momentum governing equation and the horizontal velocity term in the surface control equation are discretized by the \( \theta \)-method[9]. In addition, the bottom friction, vertical viscosity and wind stress terms are discretized by the full implicit method. In order to obtain the velocity and
water level in the predicted step, the implicit non-hydrostatic pressure term is ignored in these equations. In the second step, the predicted water level and velocity obtained in the first step are modified to meet the continuity equation by considering the influence of implicit non-hydrostatic term, and then the final result is obtained. The higher order approximation method is used to calculate the vertical velocity in the top layer. In fact, the discretization and algorithm is adapted from the early 3-d N-S solver [9].

\[ \frac{\partial u}{\partial x} = \tau_{xz} \], \[ \frac{\partial v}{\partial y} = \tau_{yz} \]

At the surface, the wind stress is determined by

\[ \rho \gamma \frac{\partial u}{\partial z} = \tau_{wx} \], \[ \rho \gamma \frac{\partial v}{\partial z} = \tau_{wy} \]

At the bed, the bed shear stress is obtained

\[ \gamma \frac{\partial u}{\partial z} = \tau_{bx} \], \[ \gamma \frac{\partial v}{\partial z} = \tau_{by} \]

where \( \tau_{wx}, \tau_{wy}, \tau_{bx}, \) and \( \tau_{by} \) are the wind stress and bed shear stress term in \( x \)-, \( y \)-coordinates, respectively, \( k_i \) is the roughness length.

3.4. Stability Analysis

The vertical diffusion term and free surface gradient term are calculated by the \( \theta \)– method in the model, so the stability of the model is not affected by these terms. However, the stability of the model is limited because the convection term and the horizontal diffusion term of the momentum equation are discretized by the conservative Euler method. The stability limit of convection diffusion term
based on unstructured grid can be approximately obtained from the two-dimensional (horizontal vertical) stability of Cartesian grid. Fletcher (1997) gives the limitation of time step for calculating convection term and horizontal diffusion term of momentum equation by using explicit central difference scheme

$$\left( \frac{|U| \Delta t}{\delta_j} + \frac{|w| \Delta t}{\Delta z} \right)^2 \leq \frac{2\nu_h \Delta t}{\delta_j^2} \leq 1 \quad (9)$$

3.5. Model characteristics

The projection of 3D computing domain on the plane is discretized by orthogonal unstructured triangle and quadrilateral mixed grids. The model can overcome the shortcomings of the poor adaptability of structured grid to the complex computing domain. The model has strong adaptability to the complex computing domain.

The water level function method is used to capture the free surface directly. When the vertical layer is divided into one layer, the model will be calculated in two dimensions. Therefore, different accuracy calculation models can be used for different problems and areas.

The explicit Euler method is used to discretize the internal gravity wave term of the horizontal momentum equation. The stability of the model is limited by the internal gravity wave term.

4. Model applications

In this chapter, the numerical model established in the previous chapter will be verified by an example. Examples include: (1) small amplitude uni-nodal wave; (2) Propagation and deformation of regular waves on submerged breakwaters. Compared with the analytical solutions and calculated data, the model established in this paper can be used to simulate the complicated three-dimensional water wave flow.

**Figure 2.** The comparisons of surface elevations between analytical solutions and calculated data, at \((x, y) = (0.5) \text{ m}\).
4.1. Small amplitude uni-nodal wave

The classical small amplitude wave motion is widely used in the detection model to simulate the non-hydrostatic pressure flow and the characteristics of mass and energy conservation. Whether the small amplitude wave motion can be simulated successfully is one of the important marks to distinguish the hydrostatic model from the non-hydrostatic model.

A uni nodal wave is used to verify non-hydrostatic pressure models in the closed pool with length 10m, width 10m and depth 10m [10]. The initial water level

$$\eta(x) = A \cos(kx), \quad 0 < x < L$$

where $A$ is the wave amplitude, $k = 2\pi / nL$, and $n = 2$.

The closed pool is discretized with 1928 triangular grids and 5 layers vertically. In order to get higher accuracy, the time step is 0.001s. Figure 2 is the Comparisons of the surface elevations between analytical solutions and numerical results, at $(x, y) = (0.5)$. It can be seen that the analytical solution is in good agreement with the calculation results of the non-hydrostatic pressure model. Figure 3 is the

![Figure 3](image-url)
comparison of flow field between analytical solution and non-hydrostatic pressure mode at $5T/8$ and $T/8$. It can be seen that the velocity vector diagram of non-hydrostatic pressure is also in good agreement, which verifies the correctness of the non-hydrostatic pressure model established.

![Figure 4. The schematic diagram of regular wave propagation](image)

4.2. Propagation and deformation of regular waves on submerged breakwaters

It is a complex process for regular waves to propagate across submerged breakwaters. In the area of constant water depth in front of submerged breakwaters, waves propagate at a constant speed and keep the wave shape unchanged. In the front slope of submerged breakwaters, due to the shallower water depth, the wave nonlinearity increases, and the higher harmonics are generated. In the back slope of submerged breakwater, with the increase of water depth, wave nonlinearity weakens, and higher harmonic is released. The simulation of this complex process is often used to investigate the performance of the numerical model of non-hydrostatic free surface flow[10].

The schematic diagram of calculation area see Figure 4. The whole calculation area is 35m long, and the last 10m is sponge absorption layer. The front slope of submerged dike is 1:20, and the back slope is 1:10. Incident wave height is 2cm, wave period is 2.02s. At the inlet boundary, the incident wave is generated by giving the inlet and outlet flow velocity:

$$u|_{\text{inflow}} = \frac{\alpha H}{2kh} f_r \sin \omega t$$

The horizontal grid scale of the calculation domain is 0.01m, and the vertical grid is divided into 4 layers, each layer is 0.1m thick. Time step is 0.01s.
Figure 5 shows the comparisons of the surface elevations between experimental data and numerical results. It can be seen from figure 5 that the measured values are in good agreement with the calculation results, which shows the ability of the non-hydrostatic model established in this paper to simulate the wave propagation deformation on the non-flat bottom slope.

5. Conclusions
A three-dimensional non-hydrostatic mathematical model with mixed triangle and quadrilateral grids is developed. Firstly, the hydrostatics or predicted velocity field satisfying the free surface equation is calculated. Then the non-hydrostatic pressure field is used to modify it to enhance the local continuity.
For the maximum flexibility and high computing efficiency, the friction step method is used to solve the N-S equations on the triangular and quadrilateral mixed meshes. The model has been applied to test cases where non-hydrostatic pressure has been shown to have an important effect on the velocity circulation patterns. The model can effectively and accurately simulate strong three-dimensional flow. Numerical examples of surface wave propagation are provided to illustrate this. The development of efficient and high-precision three-dimensional free surface flow model based on parallel computing technology is the direction to be tried.

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