Changing times to optimise reachability in temporal graphs

Jessica Enright and Kitty Meeks

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Abstract

Temporal graphs (in which edges are active only at specified time steps) are an increasingly important and popular model for a wide variety of natural and social phenomena. We propose a new extension of classical graph modification problems into the temporal setting, and describe several variations on a modification problem in which we assign times to edges so as to maximise or minimise reachability sets within a temporal graph.

We give an assortment of complexity results on these problems, showing that they are hard under a variety of restrictions. In particular, if edges can be grouped into classes that must be assigned the same time, then our problem is hard even on directed acyclic graphs when both the reachability target and the classes of edges are of constant size, as well as on an extremely restrictive class of trees. We further show that one version of the problem is \text{W[1]}-hard when parameterised by the vertex cover number of the instance graph.

In the case that each edge is active at a unique timestep, we identify some very restricted cases in which the problem is solvable in polynomial time; however, list versions of both problems (each edge may only be assigned times from a specified lists) remain \text{NP}-complete in this setting even if the graph is of bounded degree and the reachability target is a constant.

1 Introduction

Graph modification problems are a classical problem in the study of graph algorithms: we are typically allowed to modify the input graph by inserting or deleting edges or vertices, and the goal is to obtain a graph that has some specified property with a set of modification operations of minimum cost [6, 18]. Problems of this kind have a wide range of applications (from dealing with noisy graph data, through analysing the robustness of a network to failures, to determining optimal intervention strategies in a network), and there are numerous results on their computational complexity in the literature [2, 3, 4, 5, 7, 15, 17, 18].

Temporal (or dynamic) graphs have emerged recently as a useful structure for representing real-world situations, and as a rich source of new algorithmic problems [1, 8, 11, 9, 12, 13, 14]. A temporal graph changes over time: each edge in the graph is only active at certain timesteps. Some notions from the study of static graphs transfer immediately to the temporal setting, but others – including the very basic notions of connectivity and reachability – become much more complex in the temporal setting.

In this paper, we introduce the idea of modifying times: in a temporal graph, in addition to inserting and deleting edges and vertices, we can also modify the sets of timesteps at which each edge is active in order to obtain a graph with the desired temporal properties.

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For this first case study on the feasibility of altering a graph’s temporal properties by altering the timesteps at which edges are active, we address the problem of changing the reachability sets of vertices in the graph. We chose this problem for two reasons: first of all, reachability is a concept that behaves very differently in the temporal and static settings, and secondly there are many applications where could be desirable to alter reachability in a (temporal) graph, whether to maximise the dissemination of information or to minimize the spread of a contagion. For example, we might seek to minimise the maximum reachability set of any vertex by changing the timing of international flights during a global pandemic, or the timing of livestock markets in anticipation of potential disease outbreaks; on the other hand, we might want to schedule networking events so that each participant’s information can reach as many others as possible. In many of the potential applications, it is natural to imagine that certain edges subsets of edges must be active at the same collection of timesteps, for example if they are contacts that will occur at the same event (whenever this might be scheduled), and our model allows for this possibility.

We demonstrate that the problems of minimising and maximising the worst-case reachability are extremely challenging: we identify a few simple cases in which the problems are tractable, but under most natural restrictions on the inputs both remain \( \text{NP} \)-complete.

The rest of the paper is organised as follows. We describe our model in more detail in Section 1.1, give formal definitions of the problems we consider in Section 1.2, and summarise our results in Section 1.3. In Section 2 we consider the case in which each edge can be scheduled independently, and in Sections 3 and 4 we give further hardness results in the case that specified sets of edges must be active at the same times. Due to space constraints, full details of most of the proofs are postponed to the appendix.

1.1 Our model, notation, and some simple observations on reachability

A vertex \( v \) is reachable from vertex \( u \) in a graph (digraph) if there is path (directed path) from \( u \) to \( v \) in that graph (digraph). The reachability set of a vertex in a digraph is the set of all vertices reachable from that vertex. A directed acyclic graph (DAG) is a directed graph that does not contain any directed cycle.

In its most general form, a temporal graph is a (directed) graph \( G = (V, E) \) with a function \( T \) that maps each edge to a list of time steps at which that edge is active. Note that these time steps need not give a continuous interval for any particular edge. A strict temporal path from \( u \) to \( v \) in a temporal graph \( (G = (V, E), T) \) is a (directed) path from \( u \) to \( v \) composed of edges \( e_0, e_1...e_k \) such that each edge \( e_i \) is assigned a time \( t(e_i) \) from its image in \( T \) where \( t(e_i) < t(e_{i+1}) \) for \( 0 \leq i < k \). We say a vertex \( v \) is temporally reachable from \( u \) in a temporal graph \( (G = (V, E), T) \), if there is a strict temporal path from \( u \) to \( v \); we adopt the convention that every vertex \( v \) is temporally reachable from itself. A weak temporal path, and the concept of weak temporal reachability are defined analogously, where we replace the requirement \( t(e_i) < t(e_{i+1}) \) with a weak inequality. Both notions have been considered in the literature on temporal graphs (see, for example [11, 13, 14, 19]); in this paper we consider the strict notion of temporal reachability, although many of our results can also be adapted to the weak case.

As we are only interested in reachability in the sense defined above, we do not care about the absolute times at which edges are active, simply the order in which they are active. Moreover, we will assume that it is determined in advance which subsets of edges will be active simultaneously, and the number of timesteps at which each such subset is active: this can be represented by a multiset \( \mathcal{E} = \{E_1, \ldots, E_h\} \) of subsets of the edge-set of \( G \), where each element of \( \mathcal{E} \) is a subset of edges which is active simultaneously, and the multiplicity of each element in \( \mathcal{E} \) is equal to the number of timesteps at which the subset is active. This model is appropriate when each
element of $E$ corresponds to the connections that will be active when a particular event (e.g. a livestock market, a networking event, or a flight) takes place: we can potentially change the relative timing of the various events, but each event will cause a fixed subset of edges to be active.

We further restrict the way in which we may alter timing by requiring that each element of $E$ is assigned a distinct timestep. Some restriction of this kind is required to avoid our reachability minimisation problem becoming trivial in the setting where we require strict temporal reachability: we could always minimise reachability by assigning all edge subsets the same timestep. On the other hand, for reachability maximisation, we will always be able to find a solution that does not re-use timesteps and which is at least as good as any solution that does. (These properties are reversed in the case of weak reachability, so in this setting it is also desirable to prevent multiple subsets being assigned to the same timestep.) With these restrictions, the timesteps at which each edge is active can be specified uniquely by the multiset $t$ of the timestep assignments to each edge.

We call the set of vertices that are temporally reachable from vertex $u$ in a temporal graph $(G, E, t)$ the *temporal reachability set of $u$*, written $\text{reach}_{G, E, t}(u)$. Note that the temporal reachability of any vertex $u$ in the temporal graph $(G, E, t)$ is a subset of its reachability set in the static underlying graph $G$. The *maximum temporal reachability* of a temporal graph is the maximum cardinality of the temporal reachability set of any vertex in the graph, and the *minimum temporal reachability* of a temporal graph is the minimum cardinality of the temporal reachability set of any vertex in the graph. Note that the temporal reachability set of any vertex can be computed in linear time using a breadth-first search, and so the maximum/minimum reachability can be computed in polynomial time by considering each vertex in turn.

We now make some simple observations about reachability sets.

**Observation 1.** Let $G$ be an undirected graph, $v$ a vertex of degree one in $G$, and $u$ the unique neighbour of $v$. Then $\text{reach}_{G, E, t}(v) \subseteq \text{reach}_{G, E, t}(u)$.

**Observation 2.** Let $uv$ be an edge in the undirected graph $G$, and assume that $uv$ is first active at time $t_1$, and no other edge incident with $v$ is active at any time before or equal to $t_1$. Then $\text{reach}_{G, E, t}(v) \subseteq \text{reach}_{G, E, t}(u)$.

For the next observation, we need one more piece of notation: we write $N_G(v)$ for the set of vertices in $G$ that are adjacent to $v$.

**Observation 3.** For any undirected temporal graph $(G, E, t)$ and $v \in V(G)$, we have $\text{reach}_{G, E, t}(v) \supseteq N_G(v) \cup \{v\}$.

In a directed graph $G$, we write $N_G^{\text{out}}(v)$ for the set of vertices $\{u : \overrightarrow{vu} \in E(G)\}$. We can now make an analogous observation for the directed case.

**Observation 4.** For any undirected temporal graph $(G, E, t)$ and $v \in V(G)$, we have $\text{reach}_{G, E, t}(v) \supseteq N_G^{\text{out}}(v) \cup \{v\}$.

Finally, we define the *edge-class interaction graph of $(G, E)$* to be the graph with vertex-set $[h]$ in which vertices $i$ and $j$ are adjacent if and only if there exist $e_i \in E_i$ and $e_j \in E_j$ such that $e_i$ and $e_j$ are incident.

**Observation 5.** Suppose that $i$ and $j$ are non-adjacent vertices in the edge-class interaction graph of $(G, E)$, and that the sets $E_i$ and $E_j$ are active at consecutive timesteps. Then swapping the timesteps assigned to $E_i$ and $E_j$ does not change the reachability set of any vertex in $G$.
1.2 Problems considered

Our main focus in this paper is the following problems.

**Min-Max Reachability Temporal Ordering**

*Input:* A graph $G = (V, E)$, a multiset $E = \{E_1, \ldots, E_h\}$ of subsets of $E$, and a positive integer $k$

*Question:* Is there a bijective function $t : E \rightarrow [h]$ such that maximum temporal reachability of $(G, E, t)$ is at most $k$?

**Max-Min Reachability Temporal Ordering**

*Input:* A graph $G = (V, E)$, a multiset $E = \{E_1, \ldots, E_h\}$ of subsets of $E$, and a positive integer $k$

*Question:* Is there a bijective function $t : E \rightarrow [h]$ such that minimum temporal reachability of $(G, E, t)$ is at least $k$?

We also consider a natural “list” generalisation of both problems, where restrictions are placed on the allowable timings for each set of edges. This models the situation in which we do not have complete freedom to schedule events: for example, we might not be allowed to change the timing of a specific event more than some specified amount from some default schedule.

**List Min-Max Reachability Temporal Ordering**

*Input:* A graph $G = (V, E)$, a multiset $E = \{E_1, \ldots, E_h\}$ of subsets of $E$, subsets $L_1, \ldots, L_h$ of $[h]$, and a positive integer $k$

*Question:* Is there a bijective function $t : E \rightarrow [h]$ such that $t(E_i) \in L_i$ for each $i$ and the maximum temporal reachability of $(G, E, t)$ is at most $k$?

**List Max-Min Reachability Temporal Ordering** is defined analogously. We observe that all four of the problems are clearly contained in NP: a function $t$ acts as a certificate.

We make three simple observations about situations in which all of these problems admit efficient algorithms; $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree respectively of the graph $G$.

**Observation 6.** Min-Max Reachability Temporal Ordering and List Min-Max Reachability Temporal Ordering are solvable in polynomial-time when $k \leq \Delta(G) + 1$.

*Proof.* By Observation 3 we know that the maximum reachability set of $(G, E, t)$ is at least $\Delta(G) + 1 > k$, so we must have a no-instance.

**Observation 7.** Max-Min Reachability Temporal Ordering and List Max-Min Reachability Temporal Ordering are solvable in polynomial-time when $k \leq \delta(G) + 1$.

*Proof.* By Observation 3 we know that the minimum reachability set of $(G, E, t)$ is at least $\delta(G) + 1 \geq k$, so we must have a yes-instance.

**Observation 8.** Min-Max Reachability Temporal Ordering, List Min-Max Reachability Temporal Ordering, Max-Min Reachability Temporal Ordering and List Max-Min Reachability Temporal Ordering all belong to FPT when parameterised by the number $h$ of timesteps.
Proof. Given any instance of List Min-Max Reachability Temporal Ordering, there are at most $h!$ bijective functions $t$ from $E_1, \ldots, E_h$ such that $t(E_i) \in L_i$ for each $i$; we can consider each of these in turn and compute the maximum temporal reachability of the corresponding temporal graph in polynomial time. As Min-Max Reachability Temporal Ordering is a special case of List Min-Max Reachability Temporal Ordering, the corresponding result for Min-Max Reachability Temporal Ordering follows immediately. An identical argument applies in the case of Max-Min Reachability Temporal Ordering and List Max-Min Reachability Temporal Ordering.

1.3 Summary of results

We begin by considering the case in which each edge is active at a unique timestep and no edges are active simultaneously. In this setting, we prove the following results.

- The list versions of both problems are NP-complete, even if $G$ has bounded degree and $k$ is a constant.

- Min-Max Reachability Temporal Ordering and Max-Min Reachability Temporal Ordering are solvable in polynomial time in this setting if either $G$ is a DAG, or $G$ comes from a special class of trees.

When larger sets of simultaneously active edges may be specified, we show the following.

- Min-Max Reachability Temporal Ordering and Max-Min Reachability Temporal Ordering are both NP-hard even if $G$ comes from the same class of trees considered above; as a corollary to this result we see that Min-Max Reachability Temporal Ordering is W[1]-hard when parameterised by the vertex cover number of $G$.

- Min-Max Reachability Temporal Ordering is NP-hard even if $G$ is a DAG, the simultaneously active edge-sets have constant size, and $k$ is a constant.

Finally, we consider placing restrictions on the edge-class interaction graph, and show that List Min-Max Reachability Temporal Ordering remains NP-hard even if this graph is a star and the underlying graph $G$ is a linear forest.

2 Reachability when each edge is active at a unique timestep

In this section, we consider the restricted setting in which each edge in the graph is active at a unique timestep, and no edges are active simultaneously: this corresponds to the setting in which the collection of edge-sets $E$ consists of pairwise disjoint singleton sets. We begin by demonstrating that both list problems remain NP-complete in this setting, even if $k$ is a constant and $G$ has bounded degree.

Theorem 2.1. List Min-Max Reachability Temporal Ordering is NP-complete, even if all elements of $E$ are pairwise disjoint and $|E_i| = 1$ for all $i$, $G$ has maximum degree at most five, $|L_i| \leq 2$ for all $i$, and $k$ is at most 21.

Proof. We give a reduction from the following problem, shown to be NP-complete in [16].

\[ (3,4)\text{-SAT} \]

\textbf{Input}: A CNF formula $\Phi$ in which every clause contains exactly three distinct variables, and every variable appears in at most four clauses.

\textbf{Question}: Is $\Phi$ satisfiable?
Suppose that our instance \( \Phi \) has variables \( x_1, \ldots, x_n \) and clauses \( C_1, \ldots, C_m \). We construct an instance \( (G, \{E_1, \ldots, E_h\}, \{L_1, \ldots, L_h\}, k) \) of LIST MIN-MAX REACHABILITY TEMPORAL ORDERING, which is a yes-instance if and only if \( \Phi \) is a yes-instance for \( (4,3)\)-SAT.

We construct \( G \) as follows. For each variable \( x_i \) (with \( 1 \leq i \leq n \)), we have a path \( P[i] \) on three vertices; we refer to the middle vertex of \( P[i] \) as \( a_i \) and, abusing notation, we will refer to the endpoints of this path as \( x \) and \( \neg x \) respectively. For each clause \( C_j \) (with \( 1 \leq j \leq m \)) we have a vertex \( c_j \) in \( G \); for each literal \( \ell \) in \( C_j \), we have a two-edge path from \( c_j \) to \( \ell \) in \( G \). We refer to the internal vertex of the path from \( c_j \) to \( \ell \) as \( u_{j,\ell} \). To complete the construction of \( G \), for each clause \( C_j \) and literal \( \ell \) in \( C_j \), we add a set \( X(c_j, \ell) \) of pairwise non-adjacent vertices, each of which is adjacent only to \( u_{j,\ell} \). The number of vertices in \( X(c_j, \ell) \) is equal to

\[
3 - |\{C_r : r > j \text{ and } \ell \text{ appears in } C_r\}|.
\]

Note that, as each variable appears in at most four clauses, \( 0 \leq |X(c_j, \ell)| \leq 3 \) for each \( j, \ell \). It is then easy to verify that the maximum degree of \( G \) is at most 5, as each clause contains exactly 3 literals and no literal appears in more than 4 clauses.

We set \( h = \left| E(G) \right| \), and for \( 1 \leq i \leq h \) we set \( E_i = \{e_i\} \) (where \( e_1, \ldots, e_h \) is an enumeration of the edges of \( G \)); abusing notation, we will refer to \( E_i \) and \( e_i \) interchangeably. We now define the lists \( L_1, \ldots, L_h \). The list for each edge in the path \( P[i] \) (for \( 1 \leq i \leq n \)) is \( \{h - 2n + 2i - 1, h - 2n + 2i\} \). Each other edge has a singleton list containing some element of \( \{h - 2n\} \). Thus we can specify the remaining lists by fixing an order on the remaining \( h - 2n \) edges. We begin with all edges incident with some clause vertex \( c_j \); we order them first by increasing order of the index \( j \), and for edges incident with the same vertex \( c_j \) we order them by the index of the unique variable \( x_i \) such that the other endpoint is \( u_{j,\ell} \) with \( \ell \in \{x_i, \neg x_i\} \). (Note that this does indeed uniquely specify an ordering on this subset of edges, as we may assume without loss of generality that no variable occurs more than once in the same clause.) Next, we list the remaining edges incident with some \( u_{j,\ell} \). The remaining edges incident with a specific vertex \( u_{j,\ell} \) will occur consecutively, with all edges incident with some vertex in \( X(c_j, \ell) \) occurring (in an arbitrary order) before \( u_{j,\ell} \); we order these groups of edges first by increasing index of \( c_j \), and then within those corresponding to the same clause \( c_j \) by increasing index of the unique variable \( x_i \) such that \( \ell \in \{x_i, \neg x_i\} \).

Finally, we set \( k = 21 \). This completes the construction of our instance of LIST MIN-MAX REACHABILITY TEMPORAL ORDERING. For convenience, we write \( V_P \) and \( E_P \) for the sets of vertices and edges respectively belonging to some path \( P[i] \).

**Claim 2.1.1.** For any \( t : E \to [h] \) with \( t(E_i) \in L_i \) for all \( i \):

1. if the maximum reachability of \( (G, E, t) \) is at least 22, then there is a clause vertex \( c_j \) that has reachability set of size at least 22, and
2. for any clause vertex \( c_j \):
   
   (a) \( \text{reach}_{G \setminus E_P, E, t}(c_j) \) has size exactly 16, and
   
   (b) if \( C_j = (\ell_1 \lor \ell_2 \lor \ell_3) \), then \( \text{reach}_{G \setminus E_P, E, t}(c_j) \cap V_P = \{\ell_1, \ell_2, \ell_3\} \).

**Proof of claim.** To see that point (1) is true, suppose for a contradiction that there is some vertex \( w \in V(G) \) with reachability set of size at least 22, but that no vertex \( c_j \) has a reachability set of this size. We first observe that \( w \notin V_P \) for any \( i \): for any \( w \in V(P[i]) \), \( \text{reach}_{G, E, t}(w) \subseteq V(P[i]) \), so the reachability set of \( w \) has size at most 3. Secondly, we may assume without loss of generality that \( w \notin X_{j,\ell} \) for any \( j, \ell \); by Observation [1] such a vertex cannot have the unique largest reachability set in the graph. So it remains to argue that we cannot have
Thus we see that specifically, we have\[\{\}\]

By (1), this means that

Moreover, for any arbitrary clause $\pi_{\ell}$

Recall that, by construction, $|X_{j,\ell_i} \cup \{u_{r,\ell_i} : r > j\}| = 3$. Thus we see that

Point (b) follows immediately from the same observation about $\text{reach}_{G \setminus E_P}(c_j)$. □ (Claim)

We now consider how $\text{reach}_{G \setminus E_P}(c_j)$ differs from $\text{reach}_{G,\ell}(c_j)$. As all edges in $E_P$ are active strictly after all other edges, any additional vertices in $\text{reach}_{G,\ell}(c_j)$ must belong to $V_P$. Moreover, for any $C_j = (\ell_1 \lor \ell_2 \lor \ell_3)$ where $\ell_r \in \{x_i, \neg x_i\}$, it is clear that

$$\{\ell_1, \ell_2, \ell_3, a_{i_1}, a_{i_2}, a_{i_3}\} \subseteq \text{reach}_{G,\ell}(c_j) \cap V_P \subseteq \{\ell_1, \ell_2, \ell_3, a_{i_1}, a_{i_2}, a_{i_3}, \neg \ell_1, \neg \ell_2, \neg \ell_3\}.$$  

Specifically, we have $\neg \ell_i \in \text{reach}_{G,\ell}(c_j)$ if and only if $\ell_r a_{i_r}$ is active strictly before $a_{i_r} (\neg \ell_r)$. Thus we see that

$$|\text{reach}_{G,\ell}(c_j)| = 19 + |\{r : r \in \{1, 2, 3\} \text{ and } \ell_r a_{i_r} \text{ is active strictly before } a_{i_r} (\neg \ell_r)\}|.$$  

Now, suppose that $\Phi$ is a yes-instance, so there exists a satisfying assignment $B : \{x_1, \ldots, x_n\} \to \{\text{TRUE}, \text{FALSE}\}$. We define a mapping $\pi : E_P \to [2n]$ as follows:

$$\pi(a_i, x_i) = \begin{cases} 2i & \text{if } \ell_i \in \{x_i, \neg x_i\} \text{ and } \ell_i \text{ evaluates to TRUE under } B \\ 2i - 1 & \text{otherwise.} \end{cases}$$

The mapping $\pi$ determines the order in which the edges in $E_P$ are active. Now, consider an arbitrary clause $C_j = (\ell_1 \lor \ell_2 \lor \ell_3)$. Since $B$ is a satisfying assignment, we know that at least one of the literals in $C_j$ is true; without loss of generality assume that $\ell_1$ is true, where $\ell_1 \in \{x_i, \neg x_i\}$. By definition of $\pi$, this means that $\ell_1 a_{i_1}$ is active strictly after $(\neg \ell_1)a_{i_1}$, so

$$|\{r : r \in \{1, 2, 3\} \text{ and } \ell_r a_{i_r} \text{ is active strictly before } a_{i_r} (\neg \ell_r)\}| \leq 2.$$  

By (1), this means that

$$|\text{reach}_{G,\ell}(c_j)| \leq 19 + 2 = 21.$$  

As this holds for all clauses, we can conclude that we have a yes-instance for List MIN-MAX REACHABILITY TEMPORAL ORDERING.

Conversely, suppose that we have an ordering of the edges in $E_P$ such that no vertex has reachability set larger than 21. We infer a truth assignment $B : \{x_1, \ldots, x_n\} \to \{\text{TRUE}, \text{FALSE}\}$ from this ordering as follows:

$$B(x_i) = \begin{cases} \text{TRUE} & \text{if } x_i a_{i_1} \text{ is active after } (\neg x_i) a_{i_1} \\ \text{FALSE} & \text{otherwise.} \end{cases}$$

We can rewrite (1) in terms of this truth assignment to see that, for any clause $C_j = (\ell_1 \lor \ell_2 \lor \ell_3)$ where $\ell_r \in \{x_i, \neg x_i\}$, we have

$$|\text{reach}_{G,\ell}(c_j)| = 19 + |\{r : r \in \{1, 2, 3\} \text{ and } B(\ell_r) = \text{FALSE}\}|.$$  

Since no vertex has a reachability set larger than 21, this implies that at most two of $\ell_1, \ell_2, \ell_3$ can be false, or equivalently that our arbitrarily chosen clause must contain at least one literal which is true; in other words, $B$ is a satisfying assignment for $\Phi$, as required. □
An analogous result for **List Max-Min Reachability Temporal Ordering** can be obtained with a minor adaptation of the reduction above. We reverse the roles of true and false in the reduction, adding sufficiently many pairwise adjacent neighbours to every vertex other than the clause vertices $c_1, \ldots, c_m$ (with the incident edges for these new vertices assigned fixed timesteps before any of the other edges) that clause vertices are the only vertices whose reachability set might be smaller than 20.

**Theorem 2.2.** **List Max-Min Reachability Temporal Ordering** is NP-complete, even if $|E_i| = 1$ for all $i$, $G$ has maximum degree at most 25, and $k$ is at most 20.

**Proof.** We again give a reduction from (4,3)-SAT. Suppose that our instance $\Phi$ has variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$. We construct an instance $(G', \tilde{E}, \tilde{L}, k')$ of **List Min-Max Reachability Temporal Ordering** which is a yes-instance if and only if $\Phi$ is a yes-instance for (4,3)-SAT.

The construction of $G'$ is based closely on the construction given in the proof of Theorem 2.1. Let $G$ denote the graph we construct from $\Phi$ by following the construction in the proof of Theorem 2.1. We obtain $G'$ from $G$ by, for each $v \in V(G)$ that is not a clause vertex $c_j$ for some $1 \leq j \leq m$, adding a set $Y_v$ of 19 new neighbours of $v$, all of which are pairwise adjacent. As before, each edge of $G'$ belongs to one unique element of $E$, and the only edges whose list of permitted timesteps has cardinality greater than one are those belonging to some path $P[i]$. Indeed, we assign each edge in $E(G') \setminus E(G)$ a unique timestep in the range $\{1, \ldots, |E(G) \setminus E(G')| \}$, and adjust the lists previously assigned to each edge in $G$ by adding $|E(G) \setminus E(G')|$ to each element in the list. We complete the construction of our instance by setting $k = 20$.

Observe that, in $G'$, every vertex that does not belong to $\{c_1, \ldots, c_m\}$ has degree at least 19 so, by Observation 4, has reachability set of cardinality at least 20. Moreover, as all edges in $E(G') \setminus E(G)$ are active strictly before any edge in $E(G)$, and no edge in $E(G') \setminus E(G)$ is incident with any clause vertex $c_j$, any vertex that is temporally reachable from $c_j$ in $(G', \tilde{E}, t)$ is temporally reachable from $c_j$ in $(G, \tilde{E}, \tilde{t})$, where $\tilde{E}$ and $\tilde{t}$ are the restrictions of $E$ and $t$ to $G$. Thus we see that $(G', \tilde{E}, \tilde{L}, k')$ is a yes-instance if and only if there is some bijection $\tilde{\ell} : \tilde{E} \to [\tilde{L}]$ such that $|\text{reach}_{G,\tilde{E},\tilde{\ell}}(c_j)| \geq 20$ for every clause vertex $c_j$.

We know from the proof of Theorem 2.1 that, if $C_j = (\ell_1 \lor \ell_2 \lor \ell_3)$, then

$$|\text{reach}_{G,\tilde{E},\tilde{\ell}}(c_j)| = 19 + \{|r : r \in \{1, 2, 3\} \text{ and } \ell_r a_{i_r} \text{ is active strictly before } a_{i_r}(\neg \ell_r)\}|.$$  

Now suppose that $\Phi$ is a yes-instance, and let $B$ be a satisfying assignment. We fix an ordering of the edges so that $\ell_r a_{i_r}$ is active strictly before $a_{i_r}(\neg \ell_r)$ if and only if $B(\ell)$ evaluates to true; since every clause must contain at least one literal that evaluates to true under $B$, we see that $|\text{reach}_{G,\tilde{E},\tilde{\ell}}(c_j)| \geq 20$ for each $c_j$, as required. Conversely, suppose that $|\text{reach}_{G,\tilde{E},\tilde{\ell}}(c_j)| \geq 20$ for each clause $c_j$. We now define a truth assignment by setting

$$B(x_i) = \begin{cases} \text{TRUE} & \text{if } x_i a_i \text{ is active strictly before } (\neg x_i) a_i \\ \text{FALSE} & \text{otherwise}. \end{cases}$$

Since $|\text{reach}_{G,\tilde{E},\tilde{\ell}}(c_j)| \geq 20$, it follows that $C_j$ contains at least one literal that evaluates to true under this assignment, so $\Phi$ is a yes-instance. \hfill \square

Now we proceed to show that, with disjoint singleton edge classes, **Min-Max Reachability Temporal Ordering** and **Max-Min Reachability Temporal Ordering** are tractable in some special cases. We begin with the setting in which the underlying graph $G$ is a DAG. A **topological ordering** of the vertices in a DAG is an ordering in which every vertex $v$ precedes all vertices that are reachable from $v$ in the DAG; such an ordering can be found for any DAG in linear time.
**Theorem 2.3.** Let \((G, \mathcal{E}, k)\) be an instance of Min-Max Reachability Temporal Ordering, where \(\mathcal{E}\) is a set of pairwise disjoint singleton sets, and \(G = (V, \overrightarrow{E})\) is a DAG. Then \((G, \mathcal{E}, k)\) is a yes-instance if and only if \(k\) is strictly greater than \(\Delta\), the maximum out-degree of \(G\).

**Proof.** Let \(\Delta\) be the maximum out-degree of \(G\). We know by Observation [4] that, for any bijection \(t : \mathcal{E} \rightarrow \left|\mathcal{E}\right|\), the maximum reachability of \((G, \mathcal{E}, t)\) will be at least \(\Delta + 1\), so if \(k \leq \Delta\) we will certainly have a no-instance.

Conversely, we argue that there is an ordering \(t : \mathcal{E} \rightarrow \left|\mathcal{E}\right|\) such that the maximum reachability of \((G, \mathcal{E}, t)\) is at most \(\Delta + 1\). Fix a topological ordering \(\{v_1, \ldots, v_m\}\) of the vertices of \(G\). We now define a related ordering of the edges: fix any ordering \(\{e_1, \ldots, e_m\}\) of \(E(G)\) such that \(\overrightarrow{v_iu}\) precedes \(\overrightarrow{w_jv}\) whenever \(v_i\) precedes \(v_j\) in the topological ordering. If \(E_i = \{e_i\}\) for each \(i\), we now define \(t(e_i) = m + 1 - i \in [m]\).

We claim that there is no strict temporal path on more than one edge in \((G, \mathcal{E}, t)\). Suppose, for a contradiction that \(u, v, w\) is a strict temporal path in \((G, \mathcal{E}, t)\). In this case we must have \(t(\overrightarrow{vu}) > t(\overrightarrow{uw})\), which by definition of \(t\) means that \(\overrightarrow{vu}\) precedes \(\overrightarrow{uw}\) in our edge ordering; this implies that \(v\) precedes \(u\) in the topological ordering of the vertices, but as \(v\) is reachable from \(u\) this is not possible. Thus we can conclude that the longest temporal path in \((G, \mathcal{E}, t)\) consists of a single edge, so the reachability set of any vertex \(v\) is contained in \(\{v\} \cup N^\text{root}(v)\), as required. \(\square\)

By reversing the edge ordering used in the proof of Theorem 2.3 we obtain the following result on Max-Min Reachability Temporal Ordering.

**Theorem 2.4.** Let \((G, \mathcal{E}, k)\) be the input to an instance of Max-Min Reachability Temporal Ordering, where \(\mathcal{E}\) is a set of pairwise disjoint singleton sets, and \(G = (V, \overrightarrow{E})\) is a DAG. Then \((G, \mathcal{E}, k)\) is a yes-instance if and only if the maximum reachability of the static graph \(G\) is at least \(k\).

**Proof.** It is clear that the maximum reachability of the temporal graph can never exceed that of the static graph, so it remains to show that there is some function \(t\) such that these two quantities are equal. As in the proof of Theorem 2.3 we fix an ordering \(e_1, \ldots, e_m\) of the edges of \(G\) which corresponds to a topological ordering of the vertices. If \(E_i = \{e_i\}\) for each \(i\), we now define \(t(e_i) = i\) (so we have reversed the ordering from Theorem 2.3). Now consider an arbitrary pair of vertices \(u\) and \(v\), such that \(v\) is in the static reachability set of \(u\); it remains to show that \(v\) is in \(\text{reach}_{G,E,t}(u)\). Since \(v\) is in the static reachability set of \(u\), there is a directed path \(u = w_0, w_1, \ldots, w_r = v\) in \(G\) such that \(\overrightarrow{w_{i-1}w_i} \in E(G)\) for \(1 \leq i \leq r\). It follows from the definition of a topological ordering that \(u = w_0 < w_1 < \cdots < w_r = v\) in such an ordering; the construction of our function \(t\) therefore implies that \(t(\overrightarrow{w_{i-1}w_i}) < t(\overrightarrow{w_iw_{i+1}})\) for \(1 \leq i \leq k - 1\), so we have a strict temporal path from \(u\) to \(v\) in \((G, \mathcal{E}, t)\), as required. \(\square\)

The tractability of both problems in this setting now follows immediately.

**Corollary 2.5.** Min-Max Reachability Temporal Ordering and Max-Min Reachability Temporal Ordering are solvable in polynomial time if \(\mathcal{E}\) is a set of pairwise disjoint singleton sets, and \(G\) is a DAG.

We conclude this section by showing that Min-Max Reachability Temporal Ordering and Max-Min Reachability Temporal Ordering are polynomial-time solvable on a special class of trees.

**Theorem 2.6.** Min-Max Reachability Temporal Ordering is solvable in polynomial time if \(\mathcal{E}\) consists of pairwise disjoint singleton sets, and \(G\) is a tree obtained from a path by adding at least two additional leaves adjacent to each endpoint.
Proof. We show that we have a yes-instance if and only if $k \geq \Delta + 1$, where $\Delta$ denotes the maximum degree of $G$. One direction is trivial by Observation 3: we will define function $t : E(G) \rightarrow |E(t)|$ so that the maximum reachability of $(G, E, t)$ is at most $\Delta + 1$.

Suppose that the vertices of the path are $v_0, v_1, \ldots, v_r$. Our function $t$ corresponds to the following order on the edges: first the pendant edges incident with the endpoints, then the edges $(v_1, v_2)$ and $(v_{r-2}, v_{r-1})$, then the edges $(v_0, v_1)$ and $(v_{r-1}, v_r)$. Now we order every edge $(v_i, v_{i+1}), 2 \leq i < r - 3$ where $i$ is even, followed by all of the remaining edges on the path in arbitrary order. We claim that the reachability set of any vertex in $(G, E, t)$ has size at most $\max\{4, \Delta + 1\}$; the result will follow immediately, as $\Delta \geq 3$.

To see that this claim holds, first consider the leaf $v_0$ incident at the temporally first edge between $v_0$ and one of its leaves: it will have the largest reachability set of the leaves of $v_0$. It reaches all the leaves adjacent to $v_0$, as well as $v_0$ and $v_1$, but it does not reach $v_2$, and therefore reaches no other vertices. It therefore reaches $|N(v_0)| + 1$ vertices. Similarly, the leaf incident with the temporally first edge incident with $v_k$ reaches all of the leaves of $v_k$, as well as $v_k$ itself and $v_{k-1}$, a total of $|N(v_k)| + 1$ vertices. Vertices $v_0$ and $v_k$ reach only their neighbours. Consider now vertices $v_1, v_2, \ldots, v_{k-2}$. Because of our parity alternation approach, the longest temporal path originating at any of these vertices will be of two edges in one direction on the path, and on edge in the other, giving us a possible maximum reachability set of 4 vertices.

We obtain an analogous result for Max-Min Reachability Temporal Ordering on the same subclass of trees; in this case we show that the minimum reachability set always has cardinality exactly 3.

**Theorem 2.7.** Max-Min Reachability Temporal Ordering is solvable in polynomial time if $E$ consists of pairwise disjoint singleton sets, and $G$ is a tree obtained from a path by adding additional leaves adjacent to each endpoint.

Proof. We show that we have a yes instance if and only if $k \leq 3$. We argue that for all functions $t : E(G) \rightarrow |E(t)|$, the size of the smallest reachability in $(G, E, t)$ is at most 3. Say that the vertices of the path are $v_0, v_1, \ldots, v_r$, the leaves adjacent to $v_0$ are $u_0, u_1, \ldots, u_{d(v_0)-1}$, and the leaves adjacent to $v_r$ are $v_0, w_1, \ldots, w_{d(v_r)-1}$.

We now argue by contradiction. Suppose we have a function $t : E(G) \rightarrow |E(t)|$ such that the size of the smallest reachability in $(G, E, t)$ is at least 4. Let be $(u, v_0)$ the temporally last edge between $v_0$ and a leaf. If $t((u, v_0)) > t((v_0, v_1))$, then $u$ would reach only itself and $v_0$, a contradiction, so $t((u, v_0)) < t((v_0, v_1))$, and similarly $t((w, v_r)) < t((v_r, v_{r-1})), \forall w \in w_0, w_1, \ldots, w_{d(v_r)-1}$. Now consider the sequence of edges from $v_0$ to $v_r$. If $t((v_i, v_{i+1})) < t((v_i, v_{i+1}))$, for $0 < i < v_{r-1}$, then $v_{i+1}$ reaches only the three vertices in its closed neighbourhood (a contradiction). Otherwise, there must be some smallest $i$, with $0 < i < r - 1$, such that $t((v_i, v_{i+1})) < t((v_{i+1}, v_i))$: then vertex $v_{i+1}$ reaches only its closed neighbourhood, again a contradiction. Thus for all functions $t : E(G) \rightarrow |E(G)|$, the size of the smallest reachability set in $(G, E, t)$ is at most 3. This is achieved for any such tree by any $t$ that orders the edges to pendant leaves first, followed by the edges along the path in order from one end of the path to the other.

## 3 Reachability with sets of simultaneously active edges

In this section we demonstrate that our problems become even harder when the input can contain non-singleton edge-sets. We begin by showing that Min-Max Reachability Temporal Ordering remains NP-complete on trees if larger sets of simultaneously active edges are allowed; in fact, our proof demonstrates hardness for the same class of trees on which the problem was shown to be polynomially solvable in when each edge is active at a unique timestep.
Theorem 3.1. Min-Max Reachability Temporal Ordering remains NP-complete even if $G$ is a tree obtained from a path by adding additional leaf vertices adjacent to its endpoints.

Proof. We prove this result by means of a reduction from the following problem, shown to be NP-complete in [10].

**Clique**

*Input:* A graph $G = (V, E)$, and a positive integer $k$

*Question:* Does $G$ contain a clique on $k$ vertices as a subgraph?

Let $(G,k)$ be the input to an instance of Clique, and suppose that $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$. We will construct an instance $(G', \{E_1, \ldots, E_h\}, k')$ of MIN-MAX REACHABILITY TEMPORAL ORDERING, such that $(G', \{E_1, \ldots, E_h\}, k')$ is a yes-instance for MIN-MAX REACHABILITY TEMPORAL ORDERING if and only if $(G,k)$ is a yes-instance for Clique.

We construct $G'$ as follows. Let $P$ be a path on $k+1$ vertices, whose endpoints are denoted $s$ and $r$ respectively. We obtain $G'$ from $P$ by adding $n \left(\frac{k}{2} + 1\right)$ new leaves $\{u_i^s: 1 \leq i \leq n, 1 \leq j \leq \left(\frac{k}{2}\right) + 1\}$ adjacent to $s$ and $m$ new leaves $\{w_1, \ldots, w_m\}$ adjacent to $r$. Note that $G'$ has $k + 1 + m + n \left(\frac{k}{2} + 1\right) = O(m + k^2n)$ vertices.

We now define the edge subsets $E = \{E_1, \ldots, E_n\}$: we have one subset corresponding to each vertex of $G$. We set

$$E_i = \{e \in E(P)\} \cup \{su_i^j: 1 \leq j \leq \left(\frac{k}{2}\right) + 1\} \cup \{rw_j: e_j \text{ incident with } v_i\}.$$  

To complete the construction of our instance of MIN-MAX REACHABILITY TEMPORAL ORDERING, we set $k' = |G'| - \left(\frac{k}{2}\right)$. It is clear that we can construct $(G', E, k')$ from $(G,k)$ in polynomial time.

We begin by arguing that $s$ is the only vertex in $G'$ whose reachability set can contain more than $k'$ vertices, regardless of the choice of ordering.

**Claim 3.1.1.** Fix an arbitrary bijective function $t: E \to [n]$. Then, for any vertex $x \in V(G') \setminus \{s\}$, $|\text{reach}_{G', E, t}(x)| \leq k'$.

Proof of claim. Fix $x \in V(G') \setminus \{s\}$. It suffices to demonstrate that we can find $\left(\frac{k}{2}\right)$ vertices in $V(G')$ that are not in $\text{reach}_{G', E, t}(x)$.

First suppose that $x = u_i^j$ for some $i$ and $j$, and set $U = \{u_\ell^j: \ell \neq j\}$. Note that $|U| = \left(\frac{k}{2}\right)$. Since both edges on the unique path from $x$ to any $u \in U$ belong only to the edge subset $E_i$, there cannot be a strict temporal path from $x$ to $u$. Thus $U$ is a set of $\left(\frac{k}{2}\right)$ vertices which are not contained in the temporal reachability set of $x$.

Now suppose that $x \neq u_i^j$ for any $i, j$. Fix $i$ such that $E_i = t^{-1}(1)$. We claim that $U = \{u_i^j: 1 \leq j \leq \left(\frac{k}{2}\right) + 1\}$ does not lie in the temporal reachability set of $x$. To see that this is the case, note that edges incident with vertices in $U$ are only active at timestep 1, and so if any such edge belongs to a strict temporal path it must be the first edge on such a path; hence there can only be a strict temporal path from some vertex $y$ to a vertex $u \in U$ if $y$ is adjacent to $u$. However, by choice of $x$ (which is neither $u_i^j$ for any $i, j$, or $s$) we know that $x$ is not adjacent to any vertex in $U$, and hence no vertex in $U$ is in the reachability set of $x$. Since $|U| = \left(\frac{k}{2}\right) + 1$, this completes the proof of the claim.

\[\square\] (Claim)
We will say that the bijective function \( t : \mathcal{E} \rightarrow [n] \) is good for \( s \) if \( |\text{reach}_{G',\mathcal{E},t}(s)| \leq k' \). It follows from Claim 3.1.1 that \((G',\mathcal{E},k')\) is a yes-instance if and only if some function \( t \) is good for \( s \). It therefore remains to show that there is a function \( t \) which is good for \( s \) if and only if \( G \) contains a clique of on \( k \) vertices.

To show that this is true, we first give a characterisation of the temporal reachability set of \( s \).

**Claim 3.1.2.** Fix an arbitrary bijective function \( t : \mathcal{E} \rightarrow [n] \). Then the only vertices of \( G' \) that do not belong to \( \text{reach}_{G',\mathcal{E},t}(s) \) are vertices \( w_i \) such that \( e_i = v_jv_\ell \) and \( t(E_j), t(E_\ell) \leq k \).

**Proof of claim.** First observe that, for any choice of \( t \), \( \text{reach}_{G',\mathcal{E},t}(s) \) contains

1. every vertex \( u_i^j \) (with \( 1 \leq i \leq n \) and \( 1 \leq j \leq \left( \frac{k}{2} \right) + 1 \)), and
2. every vertex of \( P \).

Now consider a vertex \( w_i \); by definition, \( w_i \) is in \( \text{reach}_{G',\mathcal{E},t}(s) \) if and only if there is a strict temporal path from \( s \) to \( w_i \). There is only one possible choice of path, and the first \( k \) edges on this path are active at every timestep. Thus we have a strict temporal path from \( s \) to \( w_i \) if and only if the edge \( rw_i \) is active at some timestep after the first \( k \). Since \( rw_i \) is active only at \( t(E_j) \) and \( t(E_\ell) \), this means that \( w_i \) is in the temporal reachability set of \( s \) if and only if at least one of \( t(E_j) \) and \( t(E_\ell) \) is strictly greater than \( k \). Conversely, the only vertices of \( G' \) that are not in \( \text{reach}_{G',\mathcal{E},t}(s) \) are the vertices \( w_i \) such that \( e_i = v_jv_\ell \) and \( t(E_j), t(E_\ell) \leq k \), as required. \( \square \) (Claim)

Now suppose that \( G \) contains a clique induced by the vertices \( \{v_{i_1}, \ldots, v_{i_k}\} \). We claim that any function \( t \) which maps \( \{E_{i_1}, \ldots, E_{i_k}\} \) to \([k]\) is good for \( s \). By Claim 3.1.2 we see that the vertices of \( G' \) that are not in \( \text{reach}_{G',\mathcal{E},t}(s) \) are the vertices \( w_i \) such that \( e_i = v_jv_\ell \) and \( E_j, E_\ell \in t^{-1}([k]) = \{E_{i_1}, \ldots, E_{i_k}\} \). In other words, the vertices not in the temporal reachability set correspond to edges in \( G \) which have both endpoints in the set \( \{v_{i_1}, \ldots, v_{i_k}\} \). Since, by assumption, this set of vertices induces a clique, we know that there are precisely \( \left( \frac{k}{2} \right) \) such vertices, so the reachability set misses \( \left( \frac{k}{2} \right) \) vertices and \( t \) is indeed good for \( s \).

Conversely, suppose that the function \( t \) is good for \( s \), and set \( C = \{i : t(E_i) \leq k\} \). We claim that \( \{v_i : i \in C\} \) induces a clique in \( G \). We know by Claim 3.1.2 that the only vertices that do not belong to \( \text{reach}_{G',\mathcal{E},t}(s) \) are vertices \( w_i \) such that \( e_i = v_jv_\ell \) and \( j, \ell \in C \). We therefore know, since \( t \) is good for \( s \), that there must be \( \left( \frac{k}{2} \right) \) unordered pairs \( \{j, \ell\} \subset C \) such that \( G \) contains an edge \( v_jv_\ell \). Since the total number of unordered pairs from \( C \) is equal to \( \left( \frac{k}{2} \right) \), it follows that there is an edge between every pair of vertices in the set \( \{v_i : i \in C\} \), implying that this set of \( k \) vertices does indeed induce a clique in \( G \), as required. \( \square \)

Notice that, in the proof of Theorem 3.1 the vertices \( s \) and \( t \) together with every second internal vertex on the path \( P \) form a vertex cover for \( G' \), meaning that the vertex cover number of \( G \) is at most \( k/2 + 1 \). Thus the reduction is in fact an fpt-reduction from \textsc{Clique} parameterised by \( k \) (the size of the desired solution) to \textsc{Min-Max Reachability Temporal Ordering}, restricted to trees, parameterised by the vertex cover number of the underlying graph. Since \textsc{Clique} is \( \text{W}[1] \)-complete with respect to this parameterisation, we obtain the following immediate corollary.

**Corollary 3.2.** \textsc{Min-Max Reachability Temporal Ordering} is \( \text{W}[1] \)-hard parameterised by the vertex cover number of \( G \), even if we require \( G \) to be a tree.
We can adapt the construction used to prove Theorem 5.1 to show that Max-Min Reachability Temporal Ordering is also NP-complete when the underlying graph comes from the same special class of trees. In this case, we reduce instead from Vertex Cover, and edges incident with $s$ are all active at all timesteps; we will achieve the target reachability for all vertices $\{w_1, \ldots, w_n\}$ if and only if there is a small enough vertex cover.

**Theorem 3.3.** Max-Min Reachability Temporal Ordering is NP-complete even if $G$ is a tree obtained from a path by adding additional leaf vertices adjacent to its endpoints.

**Proof.** We give a reduction from the following problem, shown to be NP-complete in [10].

**Vertex Cover**

**Input:** A graph $G = (V, E)$, and a positive integer $k$

**Question:** Is there a set $X \subseteq V(G)$, with $|X| = k$, such that every edge in $G$ has at least one endpoint in $X$?

Let $(G, k)$ be the input to our instance of Vertex Cover, and suppose that $V(G) = \{v_1, \ldots, v_m\}$ and $E(G) = \{e_1, \ldots, e_m\}$. We will construct an instance $(G', \mathcal{E}', k')$ of Max-Min Reachability Temporal Ordering which is a yes-instance if and only if $(G, k)$ is a yes-instance for Vertex Cover.

Let $P$ be a path on $n - k$ vertices; we will refer to the endpoints of $P$ as $s$ and $r$. We obtain $G'$ from $P$ by adding a set $W = \{w_1, \ldots, w_m\}$ of $m$ new leaves adjacent to $r$, and a set $U = \{u_1, \ldots, u_{m+1}\}$ of $m+1$ new leaves adjacent to $s$. We set $\mathcal{E} = \{E_1, \ldots, E_n\}$ where, for each $i$,

$$E_i = E(P) \cup \{su_j : 1 \leq j \leq m + 1\} \cup \{rw_j : v_i \text{ is incident with } e_j\}.$$

Finally, we set $k' = n + m - k + 1$.

Observe that, for every $v \in V(G') \setminus W$ and any bijection $t : \mathcal{E} \to [n]$, we have $|\text{reach}_{G', \mathcal{E}, t}(v)| \geq n + m - k + 1$: since all edges not incident with some element of $W$ are active at all timesteps, and the distance between any two vertices not in $W$ is at most $n - k$, we have $|\text{reach}_{G', \mathcal{E}, t}(v)| \geq V(P) \cup W$, so $|\text{reach}_{G', \mathcal{E}, t}(v)| \geq n - k + m + 1$. Thus we have a yes-instance to Max-Min Reachability Temporal Ordering if and only if there is a bijection $t : \mathcal{E} \to n$ such that $|\text{reach}_{G', \mathcal{E}, t}(w)| \geq n + m - k + 1$ for all $w \in W$.

Suppose first that $G$ contains a vertex-cover $X$ of size at $k$. We fix any function $t : \mathcal{E} \to [n]$ with the property that $t(E_i) \leq k$ whenever $v_i \in X$. Fix an arbitrary $w \in W$, and suppose that $w = v_iv_j$. Since $U$ is a vertex cover for $G$, we must have that at least one of $t(E_i)$ and $t(E_j)$ is at most $k$; thus the edge $wr$ is active during at least one of the first $k$ timesteps. Since the distance from $r$ to all vertices in $V(P) \cup U$ is at most $n - k$, and all edges along the paths from $r$ to such vertices are active at all timesteps, it follows that every vertex in $V(P) \cup U$ is in the reachability set of $w$. Hence $|\text{reach}_{G', \mathcal{E}, t}(w)| \geq 1 + n - k + m + 1 > n + m - k + 1$, as required.

Conversely, suppose that we have a bijection $t : \mathcal{E} \to [n]$ such that $|\text{reach}_{G', \mathcal{E}, t}(w)| \geq n + m - k + 1$ for every $w \in W$. For each $w \in W$, write $t_0(w)$ for the first timestep at which the edge $wr$ is active, and fix $w_j$ such that $t_0(w)$ is as large as possible. Note that $|W \cup V(P)| = m + n - k$, so as $|\text{reach}_{G', \mathcal{E}, t}(w_j)| \geq n + m - k + 1$ there must be some vertex $u \in \text{reach}_{G', \mathcal{E}, t}(w_j) \cap U$. As there are $n - k$ edges on the path from $r$ to $u$, and there must be some sequence $t_0(w_j) \leq t_1 < \cdots < t_{n-k}$ such that the $t_i$th edge on the path is active at time $t_i$, we must have $t_0(w_j) \leq n - (n - k) = k$.

Now set $X = \{v_i : t(E_i) \leq k\}$. It is clear that $X \subseteq V(G)$ and, by bijectivity of $t$, $|X| = k$. To show that $X$ is a vertex cover for $G$, we consider an arbitrary edge $e = v_pv_q$. By choice of $w_j$, we know that $t_0(w_i) \leq t_0(w_j) \leq k$, so there is at some timestep $t_{n-k}$ such that $u$ is active.
of $k$; it follows that at least one of $t(E_p)$ and $t(E_q)$ is at most $k$, so $\{v_p, v_q\} \cap X \neq \emptyset$. Thus $X$ is indeed a vertex cover for $G$. 

Finally, we show that, in contrast with Corollary 2.5 MIN-MAX REACHABILITY TEMPORAL ORDERING remains NP-complete when we allow edge subsets of size at most 3, even if the underlying graph $G$ is a DAG.

**Theorem 3.4.** MIN-MAX REACHABILITY TEMPORAL ORDERING is NP-complete, even $G$ is a DAG with bounded degree, $k$ is at most 9, and $|E_i| \leq 3$ for each $E_i \in \mathcal{E}$.

**Proof.** We provide a reduction from (4,3)-SAT. Let $\Phi = C_1 \land \cdots \land C_m$ be our instance of (4,3)-SAT, and suppose that the variables in $\Phi$ are $x_1, \ldots, x_n$. We construct an instance $(G, \mathcal{E}, k)$ (with the properties in the statement of the theorem) which is a yes-instance if and only if $\Phi$ is satisfiable.

The vertex-set of $G$ consists of two sets, $V_{\text{clause}} = \{c_j : 1 \leq j \leq m\}$, and $V_{\text{var}} = \{v_{x_1,1}, v_{x_1,2}, v_{x_1,3}, v_{x_2,1}, v_{x_2,2}, v_{x_2,3} : 1 \leq i \leq n\}$. $G$ contains directed edges $v_{x_1,1}v_{x_1,2}$, $v_{x_1,2}v_{x_1,3}$, $v_{x_2,1}v_{x_2,3}$ and $v_{x_2,2}v_{x_3,3}$ for each $1 \leq i \leq n$; for each $1 \leq j \leq m$, if $C_j = (\ell_1 \lor \ell_2 \lor \ell_3)$, we also have edges $c_jv_{\ell_1,1}$, $c_jv_{\ell_2,2}$, and $c_jv_{\ell_3,1}$.

We now define the set $\mathcal{E}$ of edge-classes. For each clause $C_j$ and literal $\ell$ appearing in $C_j$, we have four sets in $\mathcal{E}$:

- two copies of the set $\{c_j, v_{\ell,1}, v_{\ell,2}v_{\ell,3}, v_{\ell,1}v_{\ell,2}\}$, denoted $E^{(1)}_{C_j, \ell}$ and $E^{(2)}_{C_j, \ell}$, and
- two copies of the set $\{c_j, v_{\ell,1}, v_{\ell,1}v_{\ell,2}, v_{\ell,1}v_{\ell,3}\}$, denoted $E^{(1)}_{C_j, -\ell}$ and $E^{(2)}_{C_j, -\ell}$.

We complete the construction of our instance of MIN-MAX REACHABILITY TEMPORAL ORDERING by setting $k = 9$. It is straightforward to verify that $G$ is a DAG with bounded degree.

Note that the only vertices with reachability set of cardinality greater than 3 in the static graph $G$ are those corresponding to clauses, so it suffices to argue that there is a function $t : \mathcal{E} \rightarrow [4m]$ such that $|\text{reach}_{G, \mathcal{E}, t}(c_j)| \leq 9$ for all $1 \leq j \leq m$ if and only if $\Phi$ is satisfiable. If $C_j = (\ell_1 \lor \ell_2 \lor \ell_3)$, the reachability set of $c_j$ in $G$ is precisely

\[
\{c_j, v_{\ell_1,1}, v_{\ell_1,2}, v_{\ell_1,3}, v_{\ell_2,1}, v_{\ell_2,2}, v_{\ell_2,3}, v_{\ell_3,1}, v_{\ell_3,2}, v_{\ell_3,3}\},
\]

which has cardinality 10, so we have $|\text{reach}_{G, \mathcal{E}, t}(c_j)| \leq 9$ if and only if the temporal reachability set of $c_j$ in $(G, \mathcal{E}, t)$ is a strict subset of its reachability set in $G$.

Suppose now that $\Phi$ has a satisfying assignment $B : \{x_1, \ldots, x_n\} \rightarrow \{\text{TRUE, FALSE}\}$. Let $t$ be any bijection $\mathcal{E} \rightarrow [4m]$ such that $t(E^{(i)}_{C_j, \ell}) \leq 2m$ whenever $B(\ell)$ evaluates to TRUE, and $t(E^{(i)}_{C_j, \ell}) \geq 2m + 1$ whenever $B(\ell)$ evaluates to FALSE. Fix an arbitrary clause $C_j$. Since $B$ is a satisfying assignment for $\Phi$, we know that there is some literal $\ell$ appearing in $C_j$ which evaluates to TRUE under $B$. We claim that $v_{\ell,3} \notin \text{reach}_{G, \mathcal{E}, t}(c_j)$. To see that this is true, observe that the edge $v_{\ell,2}v_{\ell,3}$ appears only in the sets $E^{(1)}_{C_j, \ell}$ and $E^{(2)}_{C_j, \ell}$, where either $\ell$ or $-\ell$ appears in $C_r$; since we are assuming that $B(\ell)$ evaluates to TRUE, it follows from the definition that each such set is active only during the first $2m$ timesteps. On the other hand, $v_{\ell,1}v_{\ell,2}$ appears only in the sets $E^{(1)}_{C_j, -\ell}$ and $E^{(2)}_{C_j, -\ell}$ where either $\ell$ or $-\ell$ appears in $C_r$, and so is only at timesteps greater than or equal to $2m + 1$. Since the only directed path from $c_j$ to $v_{\ell,3}$ uses the edges $v_{\ell,1}v_{\ell,2}$ and $v_{\ell,2}v_{\ell,3}$ in this order, we see that there cannot be a strict temporal path from $c_j$ to $v_{\ell,3}$ in $(G, \mathcal{E}, t)$. Hence $|\text{reach}_{G, \mathcal{E}, t}(c_j)| \leq 9$, as required.

Conversely, suppose that there is a bijection $t : \mathcal{E} \rightarrow [4m]$ such that the maximum reachability of $(G, \mathcal{E}, t)$ is at most 9. We define maxtime$_t(x_i)$ to be the latest timestep assigned by $t$ to any
edge-set of the form $E_{C_j, \ell}^{(r)}$ where $r \in \{1, 2\}$ and $\ell \in \{x_i, \neg x_i\}$. We now define a truth assignment as follows:

$$B(x_i) = \begin{cases} \text{TRUE} & \text{if } t^{-1}(\maxtime_l(x_i)) \text{ is of the form } E_{C_j, \neg}^{(r)}; \\ \text{FALSE} & \text{if } t^{-1}(\maxtime_l(x_i)) \text{ is of the form } E_{C_j, x_i}^{(r)}. \end{cases}$$

Now fix an arbitrary clause $C_j$ and suppose that the literal $\ell \in \{x_i, \neg x_i\}$ appears in $C_j$. We claim that, if $B(\ell)$ evaluates to FALSE, we have $v_{\ell,1}, v_{\ell,2}, v_{\ell,3} \in \text{reach}_{G, E, t}(c_j)$. By construction of $B$, we know that $t^{-1}(\maxtime_l(x_i))$ is of the form $E_{C_j, \ell}^{(r)}$ (for some clause $C_j$ which involves the variable $x_i$) and so includes the edge $\overline{v_{\ell,2}v_{\ell,3}}$. By definition of $\maxtime_l(x_i)$, this means there exist distinct timesteps $s_1, s_2 < \maxtime_l(x_i)$ such that $t(E_{C_j, \neg}^{(r)}) = s_r$ for $r \in \{1, 2\}$; without loss of generality we may assume that $s_1 < s_2$. Since $c_j, v_{\ell,1} \in E_{C_j, \neg}^{(1)} = t^{-1}(s_1)$ and $\overline{v_{\ell,1}v_{\ell,2}} \in E_{C_j, \ell}^{(2)} = t^{-1}(s_2)$, we have a strict temporal path $c_j, v_{\ell,1}, v_{\ell,2}, v_{\ell,3}$; so we do indeed have $v_{\ell,1}, v_{\ell,2}, v_{\ell,3} \in \text{reach}_{G, E, t}(c_j)$. Hence, if every literal in $C_j$ evaluates to FALSE under $B$, we would have $|\text{reach}_{G, E, t}(c_j)| = 10$, contradicting our assumption that the maximum reachability of $(G, E, t)$ is at most 9. Thus we can conclude that every clause contains at least one literal which evaluates to TRUE under $B$, and so $B$ is a satisfying assignment for $\Phi$. \qed

4 Restricting the edge-set interaction graph

In many of the reductions above, the expressive power of our problem even on highly restricted graph classes came from the fact that, with edge classes of size two or more, the decisions made at one location in the graph could have an effect on distant parts of the graph. Thus, the edge-class interaction graph in the instances we defined was typically much denser than the underlying graph $G$: for example, in the proof of Theorem 4.1 although the graph $G$ we construct is a tree, the edge-class interaction graph is a clique.

It is therefore natural to ask whether we can obtain efficient algorithms for the problem if we place structural restrictions on the edge-class interaction graph. In the next result, we show that even very strong restrictions of this kind are not enough to make List Min-Max Reachability Temporal Ordering tractable.

**Theorem 4.1.** List Min-Max Reachability Temporal Ordering is NP-complete, even if the edge-class interaction graph of the input is a star, and the underlying graph $G$ is a disjoint union of five-vertex paths.

**Proof.** We give a reduction from the NP-complete problem VERTEX COVER. Suppose that $(G, k)$ is the input to our instance of VERTEX COVER. We construct an instance $(H, E = \{E_1, \ldots, E_h\}, \{L_1, \ldots, L_h\}, \ell)$ of List Min-Max Reachability Temporal Ordering which is a yes-instance if and only if $(G, k)$ is a yes-instance for VERTEX COVER.

$H$ consists of $|E(G)|$ disjoint paths on 5 vertices, which are in one-to-one correspondence with the edges of $G$: we refer to the path corresponding to $e \in E(G)$ as $P_e$. If $e = uv$, we refer to the endpoints of $P_e$ as $u[e]$ and $v[e]$, and their unique neighbours on $P_e$ as $u'[e]$ and $v'[e]$ respectively. We call the edges of $P_e$ that are not incident with either $u[e]$ or $v[e]$ the middle edges of $P_e$, and we refer to the midpoint of $P_e$ (the vertex not adjacent to either $u[e]$ or $v[e]$) as $x_e$.

Suppose that $V(G) = \{v_1, \ldots, v_n\}$. We set $h = n + 1$, for each $1 \leq i \leq n$ we set $E_i = \{v_i[e]v_i'[e] : e \in E(G)\}$, and we set $E_{n+1}$ to be the set of all edges that are middle edges of some path. For $1 \leq i \leq n$, we set $L_i = [n+1] \setminus \{k+1\}$, and we set $L_{n+1} = \{k+1\}$. We complete the construction of our instance of List Min-Max Reachability Temporal Ordering by setting $\ell = 4$. 15
Suppose first that $G$ contains a vertex cover $X \subset V(G)$ with $|X| = k$. Let $t : E \to [n + 1]$ be any function such that $t(E_i) \leq k$ for all $v_i \in X$, and $t(E_{n+1}) = k$. We claim that the maximum temporal reachability of $(H, E, t)$ is at most 4. First observe that, for any $t$, the maximum temporal reachability of $(H, E, t)$ cannot be more than 5, since every connected component of $H$ contains exactly 5 vertices. Moreover, if any vertex has reachability set of size 5, it must be $x_e$ for some edge $e \in E(G)$: as both edges incident with $x_e$ occur simultaneously, there is no strict temporal path between the two connected components of $P_e \setminus \{x_e\}$, so no vertex other than $x_e$ can possibly reach all other vertices in its component. Suppose now that reach$_{H,E,t}(x_e) = 5$ for some $e = v_iv_j \in E(G)$. In this case, we must have $t(E_i), t(E_j) > k + 1$; by construction of $t$, this only happens if neither $v_i$ nor $v_j$ belongs to $X$. However, as $v_iv_j$ is an edge, this contradicts the assumption that $U$ is a vertex cover. Hence the maximum reachability of $(H, E, t)$ is at most 4, as required.

Conversely, suppose that we have a function $t : E \to [n + 1]$, with $t(E_i) \in L_i$ for all $i$, such that the maximum reachability of $(H, E, t)$ is at most 4. We claim that $G$ has a vertex cover of size at most $k$. Set $X = \{v_i \in V(G) : t(E_i) \leq k\}$. It is clear that $X \subset V(G)$ and $|X| = k$; it remains to demonstrate that $U$ is a vertex cover. To do this, fix an arbitrary edge $e = v_iv_j \in E(G)$. If neither $v_i$ nor $v_j$ belongs to $X$, we would have reach$_{H,E,t}(x_e) = V(P_e)$, since both middle edges of $P_e$ are active strictly before either of the remaining two edges. Thus the maximum reachability of $(H, E, t)$ would be at least 5, giving a contradiction. As our choice of $i$ was arbitrary, we may conclude that $X$ is indeed a vertex cover for $G$. \hfill \(\square\)

## 5 Conclusions and future work

We have shown that the problems **Min-Max Reachability Temporal Ordering**, **List Min-Max Reachability Temporal Ordering**, **Max-Min Reachability Temporal Ordering** and **List Max-Min Reachability Temporal Ordering** are extremely difficult: they remain $\text{NP}$-complete even under very strong restrictions on the input, and our only tractability results are on very restrictive classes of trees or directed acyclic graphs, together with the requirement that no edges are active simultaneously.

Our hardness results rule out many of the restrictions which one might imagine could give rise to efficient algorithms, but a few possibilities remain. One natural open question arising from our results is whether any of the four problems is solvable in polynomial time on arbitrary trees, in the case where no edges are active simultaneously. It also remains open whether **Min-Max Reachability Temporal Ordering** (respectively **Max-Min Reachability Temporal Ordering**) belongs to $\text{XP}$ (respectively $\text{FPT}$) parameterised by the vertex cover number of the underlying graph $G$.

Moreover, some of our results only apply to the list versions of the problem. It would be particularly interesting to know whether **Min-Max Reachability Temporal Ordering** and **Max-Min Reachability Temporal Ordering** remain $\text{NP}$-complete when no simultaneously active edges are allowed.

Another possible restriction of the problem would be to require that all sets of simultaneously active edges induce (disjoint unions of) cliques: this corresponds to an application in which all possible contacts between the individuals involved in a particular event are assumed. This a generalisation of the setting considered in Section 2 so our hardness results apply here, but it would be interesting to to investigate whether any of the tractability results for singleton edge classes can be extended to this more general setting.

In this work we have considered a notion of temporal reachability in which edges consecutive edges on a path from $u$ to $v$ must occur at non-simultaneous time-steps. However, many of our results can be generalised to the setting of weak reachability: in our model, the choice of
definition is only relevant if some edge-set \( C_i \) contains two or more incident edges. Thus, all the results from Section 2 also apply under weak reachability.

Given the plethora of hardness results we have seen here, a natural direction for future research would be to investigate the existence or otherwise of efficient approximation algorithms for these problems. For certain applications, it might also be relevant to consider the problem of minimising/maximising the average cardinality of the reachability set over all vertices of the graph, or indeed the expected size of the reachability set (perhaps given some distribution over starting vertices) in a probabilistic model where each edge has an associated transmission probability.

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