On Energy of the Friedman Universes in Conformally Flat Coordinates

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Recently many authors have calculated energy of the Friedman universes by using coordinate-dependent double index energy-momentum complexes in Cartesian comoving coordinates \((t, x, y, z)\) and concluded that the flat and closed Friedman universes are energy-free. We show in this paper by using Einstein canonical energy-momentum complex and by doing calculations in conformally flat coordinates that such conclusion is incorrect. The results obtained in this paper are compatible with the results of the our previous paper \([8]\) where we have used coordinate-independent averaged energy-momentum tensors to analyze Friedman universes.

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I. INTRODUCTION

A spacetime is conformally flat if there exist coordinates \((\tau, x, y, z)\) in which the line element \(ds^2\) reads

\[
ds^2 = \Omega^2(\tau, x, y, z)(d\tau^2 - dx^2 - dy^2 - dz^2)
\]

where \(\eta_{ik}\) means the Minkowskian metric, i.e., \(\eta_{ik} = \text{diag}(1, -1, -1, -1)\) \([13]\). \(\Omega(\tau, x, y, z)\) is a sufficiently smooth and positive-definite function called conformal factor.

We will call the coordinates \((\tau, x, y, z)\) the conformally flat or conformally inertial coordinates.

The conformally flat coordinates are determined up to 15-parameters group of the conformal transformations. This 15-parameters Lie group contains, as a subgroup, the 10-parameters Poincare’ group \([1, 2]\).

It is obvious that the conformally flat coordinates are geometrically and physically distinguished like inertial coordinates \((t, x, y, z)\) in a Minkowskian spacetime \([14]\).

The necessary and sufficient condition iff a four (or more) dimensional spacetime could be conformally flat is vanishing of its Weyl conformal curvature tensor \([3]\). Physically, the Weyl tensor describes source-free, i.e., independent of matter, gravitational field.

If a spacetime is neither flat nor asymptotically flat (at spatial or at null infinity) but it is only conformally flat, then one should prefer conformally flat coordinates to analyse physical properties of the spacetime. Especially, one should prefer the conformally flat coordinates in order to analyse energy and momentum of such spacetime by using coordinate-dependent, double index energy-momentum complexes, matter and gravitation.

In this context we would like to remark that already in the case of a Minkowskian spacetime the energy-momentum complexes can be reasonably used only in an “affine” coordinates in which the metric components are constant, e.g., in an inertial (= Lorentzian) coordinates \((t, x, y, z)\) in which the line element \(ds^2\) reads

\[
ds^2 = dt^2 - dx^2 - dy^2 - dz^2.
\]

On the other hand, in an asymptotically flat spacetime one can reasonably use these complexes only in an asymptotically flat (= asymptotically inertial or asymptotically Lorentzian) coordinates. So, in the case of a conformally flat spacetime one should use the energy-momentum complexes in the conformally flat coordinates, i.e., in the conformally inertial coordinates.

It is commonly known that the Friedman universes are conformally flat \([4, 5, 6]\). So, it is natural to analyse of their energetic content in conformally flat coordinates \((\tau, x, y, z)\).

Recently many authors have calculated energy of the Friedman (and also more general, only spatially homogeneous) universes \([7]\) mainly by using coordinate-dependent double index energy-momentum complexes. These authors have performed their calculations not in the conformally flat coordinates \((\tau, x, y, z)\) but in the so-called Cartesian comoving coordinates.
coordinates \((t, x, y, z)\) in which the line element \(ds^2\) of the Friedman universes has the form

\[
ds^2 = dt^2 - \frac{a^2(t)(dx^2 + dy^2 + dz^2)}{[1 + k(x^2 + y^2 + z^2)]^2},
\]

where \(a = a(t)\) is the scale factor, and \(k = 0, 1\) means the normalized curvature of the slices \(t = \text{const.}\). \(t\) denotes the universal time parameter called cosmic time.

In the Cartesian comoving coordinates \((t, x, y, z)\) only spatial part of the full metric is conformally flat.

The above mentioned authors have concluded that the closed Friedman universes have zero net global energy, and that the flat Friedman universes are energy free, locally and globally \([15]\). For an open Friedman universes one gets divergent global results in the Cartesian comoving coordinates \((t, x, y, z)\).

In other comoving coordinates the results are dramatically different (see, e.g., \([8]\)).

Of course, the problem of the global quantities of the Friedman and more general, only spatially homogeneous, universes is not well-posed physical problem because one cannot measure the global energy and momentum in the case. The global energy and momentum, and global angular momentum also, have physical meaning only in the case of an asymptotically flat spacetime (at spatial or at null infinities) where these global quantities can be measured. So, the calculations of the global energy and momentum, and global angular momentum also, of an universe can have only some mathematical sense.

In the case of an universe a physical sense can have only local quantities, eg., energy density and its flux and global quantities of an isolated part of the universe, e.g., global energy of the Solar System. If we use a coordinate-dependent double index energy-momentum complex, then all these quantities should be calculated in a privileged coordinates, e.g., in the case of a Friedman universe one should use with this aim the geometrically and physically favorized conformally flat coordinates \((\tau, x, y, z)\).

We would like to emphasize that the global result \(E = 0\) obtained in the Cartesian comoving coordinates \((t, x, y, z)\) for a closed Friedman universe is obtained iff we admit the limiting process \(r \to \infty\) during integration over slice \(t = \text{const}\), where \(r = \sqrt{x^2 + y^2 + z^2}\) is the radial coordinate. But if \(r \to \infty\), then the spatial conformal factor \(\frac{a^2(t)}{(1 + rz)^2}\) goes to zero in the case giving a singularity.

Resuming, one can doubt in physical validity of the conclusion that the closed and flat Friedman universes (and also more general, only spatially homogeneous Kasner and Bianchi universes) are energy free.

In this context, we would like to remark that by using our coordinate independent averaged relative energy-momentum tensors \([8]\) or superenergy tensors \([9]\) one can do mathematically correct and coordinate independent local analysis of the Friedman and more general universes. One can also formally calculate, correctly from the mathematical point of view, the global, integral quantities for such universes.

It is interesting that following this way one gets positive-definite energy values for the all Friedman universes and also for Kasner and Bianchi type I universes \([10]\). So, in our opinion, all these universes needn’t be energetic nonentity.

In this paper we present the results of the analysis of the energetic content of the Friedman universes in the distinguished conformally flat coordinates \((\tau, x, y, z)\). These coordinates are the most suitable to this goal if one uses an energy-momentum complex. Our analysis will be done with the help of the most important in general relativity Einstein’s canonical double index energy-momentum complex

\[
\mathcal{E} \mathbf{K}_i{}^k := \sqrt{|g|} \left( T_i{}^k + \mathbf{E} t_i{}^k \right) = \mathbf{F} \mathbf{U}_i{}^{[kl]},
\]

where \(\mathbf{F} U_i{}^{[kl]} = (-1)^F U_i{}^{[kl]}\) mean Fried’s superpotentials, and \(\mathbf{E} t_i{}^k\) are the components of the canonical Einstein’s energy-momentum pseudotensor of the gravitational field \([10, 11]\). \(T_i{}^k\) denote the components of the symmetric energy-momentum tensor of matter.

As we will see, by using this energy-momentum complex in the conformally flat coordinates \((\tau, x, y, z)\), one cannot assert that the Friedman universes have zero net energy, locally or globally.

The analogous result one can obtain by using any other reasonable double index energy-momentum complex.

We hope that this paper and the our previous paper \([8]\) convincingly show that the Friedman universes are not energetic nonentity, neither locally nor globally.

Finishing this Section we would like to emphasize an important superiority of the conformally flat coordinates \((\tau, x, y, z)\) over the Cartesian comoving coordinates \((t, x, y, z)\). Namely, solving the energy-momentum problem of the Friedman universes in Cartesian comoving coordinates \((t, x, y, z)\) one uses only the line element (3) independently of the Einstein equations and their solutions. On the other hand, the results obtained in conformally flat coordinates \((\tau, x, y, z)\) explicitly depend not only on the Friedman-Lemaître-Robertson-Walker line element \(ds^2\) but also on the solutions of the Einstein equations.

In order to establish our attention we will consider in this paper only dust Friedman universes.
The paper is organized as follows. In Section 2 we give dust Friedman universes in conformally flat coordinates $(\tau, x, y, z)$, and in Section 3 we will analyse the energy and its flux for dust Friedman universes in these coordinates. Our analysis will be performed with the help of the Einstein canonical energy-momentum complex. Finally, in Section 4 we give our conclusion.

II. DUST FRIEDMAN UNIVERSES IN THE CONFORMALLY FLAT COORDINATES $(\tau, x, y, z)$

A. Closed dust Friedman universes ($k = 1$)

Let us consider the Friedman-Lemaitre-Robertson-Walker (FLRW) like line element

$$ds^2 = a^2(\eta)\left\{d\eta^2 - d\chi^2 - \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)\right\},$$

(5)

with the following ranges of the coordinates ($\eta, \chi, \theta, \varphi$):

$$0 < \chi < \pi, \quad 0 < \theta < \pi, \quad 0 < \varphi < 2\pi, \quad \chi - \pi < \eta < \pi - \chi.$$  

(6)

Physically, the coordinate $\eta$ is the conformal time, $\chi$ is a radial coordinate, and $\theta, \varphi$ are ordinary spherical angular coordinates (see, e.g., [12]).

The bijective transformation

$$\tau + r = \tan\left(\frac{\eta + \chi}{2}\right), \quad \tau - r = \tan\left(\frac{\eta - \chi}{2}\right),$$

$$\theta' = \theta, \quad \varphi' = \varphi,$$

$$0 < \chi < \pi, \quad \chi - \pi < \eta < \pi - \chi,$$

(7)

with inverse

$$\eta = \arctan(\tau + r) + \arctan(\tau - r), \quad (-)\infty < \tau + r < \infty,$$

$$\chi = \arctan(\tau + r) - \arctan(\tau - r), \quad (-)\infty < \tau - r < \infty, \quad 0 < r < \infty,$$

$$\theta = \theta', \quad \varphi = \varphi', \quad 0 < \theta' < \pi, \quad 0 < \varphi' < 2\pi,$$

(8)

map this spacetime onto conformally flat spacetime with the following line element

$$ds^2 = \frac{4a^2(\tau, x, y, z)}{[1 + (\tau + r)^2][1 + (\tau - r)^2]}\eta_{ik}dx^i dx^k =: \Omega^2(\tau, x, y, z)\eta_{ik}dx^i dx^k,$$

(9)

where

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r \cos\theta,$$

$$r = \sqrt{x^2 + y^2 + z^2}.$$  

(10)

This means that the transformation (7) covers the region of the spacetime (5)-(6) with the conformally flat coordinates $(\tau, x, y, z)$. One can call these coordinates the conformally inertial coordinates.

If we omit the angular coordinates ($\theta, \varphi$) then this region will be the triangle

$$0 < \chi < \pi, \quad \chi - \pi < \eta < \pi - \chi$$

on the plane $(\eta, \chi)$.

Now let us consider a closed dust Friedman universe with the following line element in the same coordinates $(\eta, \chi, \theta, \varphi)$

$$ds^2 = a^2(\eta)\left\{d\eta^2 - d\chi^2 - \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)\right\},$$

(12)

with

$$a = a_0(1 + \cos\eta), \quad t = a_0(\eta + \pi + \sin\eta),$$

(13)
and with the following ranges of the coordinates \((\eta, \chi, \theta, \varphi)\)
\[
(-)\pi < \eta < \pi, \quad 0 < \chi < \pi, \quad 0 < \theta < \pi, \quad 0 < \varphi < 2\pi.
\] (14)

The coordinates \((\eta, \chi, \theta, \varphi)\) are comoving, i.e., the dust particles and so-called fundamental observers are at rest in these coordinates.

\(a(\eta)\) is the scale factor and \(t\) means the cosmic time; \(a_0 = \frac{4}{3}\pi \rho a^3 = \text{const}\) is the first integral of the Friedman equations in the case.

If we omit the angular coordinates \((\theta, \varphi)\), then this universe is a rectangle \((-)\pi < \eta < \pi, \quad 0 < \chi < \pi\) on the plane of the variables \(\eta, \chi\).

Comparing this rectangle with the previous triangle one can easily see that the conformally flat coordinates \((\tau, x, y, z)\) cover only this half of the closed dust Friedman universe which is determined by the following ranges of the coordinates \((\chi, \eta, \theta, \varphi)\)
\[
0 < \chi < \pi, \quad \chi - \pi < \eta < \pi - \chi, \quad 0 < \theta < \pi, \quad 0 < \varphi < 2\pi.
\] (15)

It is worth to emphasize that only one slice, \(\eta = 0\), of the closed dust Friedman universe is entirely covered by the conformally flat coordinates \((\tau, x, y, z)\). Any other slice \(\eta = \eta_0 \neq 0\) is only partially covered by these coordinates.

Applying an active point of view one can say that this distinguished slice \(\eta = 0\) is mapped onto subspace
\[
\tau = 0, \quad (-)\infty < x < \infty, \quad (-)\infty < y < \infty, \quad (-)\infty < z < \infty
\] (16)
of the conformally flat spacetime \((\tau, x, y, z)\) which has the line element (9) with
\[
a(\tau, x, y, z) = a_0 \{1 + \cos[\arctan(\tau + r) + \arctan(\tau - r)]\}.
\] (17)

The limiting values
\[
x = \mp \infty, \quad y = \mp \infty, \quad z = \mp \infty
\] (18)
are not admissible by the condition \(\Omega(0, x, y, z) > 0\).

It follows that in conformally flat coordinates it is possible to calculate integrals only over the distinguished spatial slice \(\eta = 0\) [17]. This fact is very important, e.g., for formal calculating global energy and momentum of a dust closed Friedman universe.

It is very interesting that in the conformally flat coordinates \((\tau, x, y, z)\) the initial singularity at \(\eta = (-)\pi\) and the final singularity at \(\eta = \pi\) are removed to \(\tau = (-)\infty\) and to \(\tau = \infty\) respectively, i.e., we have no cosmological singularity in the case at a finite moment of the conformal time coordinate \(\tau\).

Matter and comoving (= fundamental) observers are not at rest in the conformally flat coordinates \((\tau, x, y, z)\). They both move with the same 4-velocity
\[
u^0 = \frac{1 + x^2 + y^2}{2a(\tau, x, y, z)}, \quad u^1 = \frac{\sin \theta \cos \varphi \cdot \tau \cdot r}{a(\tau, x, y, z)},
\]
\[
u^2 = \frac{\sin \theta \cos \varphi \cdot \tau \cdot r}{a(\tau, x, y, z)}, \quad u^3 = \frac{\cos \theta \cdot \tau \cdot r}{a(\tau, x, y, z)},
\] (19)
where
\[
a = a_0 \{1 + \cos[\arctan(\tau + r) + \arctan(\tau - r)]\},
\]
\[
\sin \theta = \frac{\sqrt{x^2 + y^2}}{r}, \quad \cos \theta = \frac{z}{r}, \quad \cos \varphi = \frac{x}{\sqrt{x^2 + y^2}},
\]
\[
\sin \varphi = \frac{y}{\sqrt{x^2 + y^2}}, \quad r = \sqrt{x^2 + y^2 + z^2}.
\] (20)

Only fundamental observers which lie in the distinguished slice \(\eta = 0\) remain also at rest in the conformally flat coordinates \((\tau, x, y, z)\) in the slice \(\tau = 0\).
B. Open dust Friedman universe \((k = -1)\)

Now, let us consider an open dust Friedman universe endowed with the same comoving coordinates \((\eta, \chi, \theta, \varphi)\) as in the closed case.

We have (see, e.g., [12])

\[
ds^2 = a^2(\eta) \left\{ d\eta^2 - d\chi^2 - \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2) \right\},
\]

where \(a_0 = \frac{4}{3} \pi \rho a^3 = \text{const}\), and

\[
0 < \eta < \infty, \quad 0 < \chi < \infty, \quad 0 < \theta < \pi, \quad 0 < \varphi < 2\pi.
\]  

Then, the transformation

\[
r = a_0 e^\eta \sinh \chi, \quad \tau = a_0 e^\eta \cosh \chi, \quad \tau > a_0^2, \quad r > 0,
\]

with inverse

\[
\eta = \ln \left( \frac{2\sqrt{\tau^2 - r^2}}{a_0} \right), \quad \tau^2 - r^2 > \frac{a_0^2}{4}, \quad \theta = \theta', \quad \varphi = \varphi',
\]

brings the line element (21)-(22) to the conformally flat form

\[
ds^2 = \left( 1 - \frac{a_0}{2\sqrt{\tau^2 - r^2}} \right)^4 \eta_{ik} dx^i dx^k =: \Omega^2(\tau, x, y, z) \eta_{ik} dx^i dx^k.
\]  

Here the conformal factor \(\Omega = \left( 1 - \frac{a_0}{2\sqrt{\tau^2 - r^2}} \right)^2\), and \(\tau^2 - r^2 > \frac{a_0^2}{4}\). \(\tau = \sqrt{x^2 + y^2 + z^2}\), \(x = r \sin \theta \cos \varphi\), \(y = r \sin \theta \sin \varphi\), \(z = r \cos \theta\).

From an active point of view the transformation (23) maps the open dust Friedman universe (21)-(22) onto interior of the future light cone \(\tau^2 - x^2 - y^2 - z^2 = 0\) of a Minkowskian spacetime which line element in an inertial coordinates reads

\[
ds^2 = \eta_{ik} dx^i dx^k.
\]  

Under this mapping a slice \(0 < \eta = \eta_0\) of the open dust Friedman universe is mapped onto hyperboloid \(\tau^2 - r^2 = B^2\), \(B^2 := \frac{a_0^2}{4}\) in the spacetime with the line element (26).

In the conformally flat coordinates \((\tau, x, y, z)\) the dust matter filling the open Friedman universe and comoving fundamental observers also are not at rest. Namely, they have the following 4-velocity in these coordinates

\[
u^0 = \frac{\tau}{a}, \quad \nu^1 = \frac{x}{a}, \quad \nu^2 = \frac{y}{a}, \quad \nu^3 = \frac{z}{a},
\]

where

\[
a = a_0 \left( \frac{\tau^2 - r^2 + a_0^2/4}{a_0\sqrt{\tau^2 - r^2}} - 1 \right), \quad r^2 = x^2 + y^2 + z^2.
\]

C. Flat dust Friedman universes \((k = 0)\)

Finally, let us consider a flat Friedman universe filled with dust matter in the Cartesian comoving coordinates \((t, x, y, z)\).
We have (see, e.g., [12])

\[ ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \]  

where

\[
 a(t) = At^{2/3}, \quad A = \frac{4}{3}\pi \rho a^3 = \text{const} > 0, \quad 0 < t < \infty. \tag{30}
\]

The parameter \( t \) is the \textit{cosmic time} and \( a(t) \) denotes as usual the \textit{scale factor}.

In order to pass to the conformally flat coordinates \((\tau, x, y, z)\) it is sufficient in the case only to change the time coordinate \( t \) onto \textit{conformal time} \( \tau \) following the scheme

\[ d\tau = \frac{dt}{a(t)}. \tag{31} \]

From (30)-(31) it follows that

\[ \tau = \frac{3}{A} t^{1/3} \equiv t = \frac{A^3}{27^3}, \tag{32} \]

and

\[ a(\tau) := a[t(\tau)] = \frac{A^3}{9} \tau^2, \quad 0 < \tau < \infty. \tag{33} \]

Substituting into line element (29) \( dt^2 = a^2(\tau)d\tau^2 \) we get

\[ ds^2 = a^2(\tau)(d\tau^2 - dx^2 - dy^2 - dz^2), \tag{34} \]

i.e., we get the line element (29)-(30) in the \textit{conformally flat form} with the \textit{conformal factor}

\[ \Omega = \Omega(\tau) = a(\tau) = \frac{A^3}{9} \tau^2, \quad 0 < \tau < \infty. \tag{35} \]

From geometrical point of view the flat dust Friedman universe in conformally flat coordinates \((\tau, x, y, z)\) is identical with the upper half \((\tau > 0)\) of the conformally flat spacetime which has the following line element

\[ ds^2 = a^2(\tau)(d\tau^2 - dx^2 - dy^2 - dz^2). \tag{36} \]

It is interesting that in this case the conformally flat coordinates \((\tau, x, y, z)\) are also \textit{comoving coordinates}, like initial Cartesian coordinates \((t, x, y, z)\).

The 4-velocity of a particle of the dust which fills the flat Friedman universe (identical with the 4-velocity of a fundamental observer) reads in the conformally flat coordinates \((\tau, x, y, z)\)

\[ u^i = \frac{\delta^i_0}{a(\tau)} \equiv u_i = a(\tau)\eta_{0i}. \tag{37} \]

It results that the dust and the fundamental observers both \textit{are at rest} in these coordinates, like as in the Cartesian comoving coordinates \((t, x, y, z)\).

\section*{III. ENERGY OF THE FRIEDMAN UNIVERSES IN THE CONFORMALLY FLAT COORDINATES \((\tau, x, y, z)\)}

In this Section we will consider the energetic content of the Friedman universes in the physically and geometrically distinguished \textit{conformally flat coordinates} \((\tau, x, y, z)\). We will use in our analysis the double index Einstein’s canonical energy-momentum complex, matter and gravitation,

\[ E K_i^k := \sqrt{|g|} (T_i^k + E t_i^k). \tag{38} \]
Here $T_i^k$ are the components of the symmetric energy-momentum tensor of matter and $E_t^k$ mean the components of the so-called Einstein gravitational energy-momentum pseudotensor (see, e.g., [10, 11, 12]).

It is known that

$$\sqrt{|g|}(T_i^k + E_t^k) = E U_i^{[kl]},$$

(39)

where $E U_i^{[kl]} = (-)E U_i^{[lk]}$ are Freud’s superpotentials which in a coordinate basis read

$$E U_i^{[kl]} = \alpha \left\{ \frac{g_{ia}}{\sqrt{|g|}} \left[ (-g) \left( g^{kb} g^{00} - g^{ia} g^{kb} \right) \right] \right\}, \quad \alpha = \frac{1}{16\pi},$$

(40)

and that the equations (39) represents special form of the Einstein equations (in mixed form and multiplied by $\sqrt{|g|}$).

Owing antisymmetry of the Freud superpotentials one can easily obtain from (39) the following local energy-momentum conservation laws, for matter and gravitation

$$E \mathcal{K}_i^{k, l} = 0.$$  

(41)

By using integral Stokes theorem one can obtain from (41) reasonable integral conservation laws for a closed system in an asymptotically flat coordinates.

Of course, one can consider in GR many other energy-momentum complexes. But the Einstein expressions is the best one of the all variety of the energy-momentum complexes (see, e.g., [11]). In consequence, we confine in this paper, like in our previous papers, only to this double index energy-momentum complex [18].

For a conformally flat spacetime with

$$g_{ik} = \Omega^2 \eta_{ik} \equiv g^{ik} = \Omega^{-2} \eta^{ik}, \quad \Omega = \Omega(\tau, x, y, z),$$

$$\sqrt{|g|} = \Omega^2,$$

(42)

one obtains from (39)-(40)

$$E \mathcal{K}_i^{k} = 4\alpha \left( \delta_i^k \eta^{lb} - \delta_i^l \eta^{kb} \right) \left( \Omega_l \Omega_b + \Omega \Omega_{lb} \right).$$

(43)

As a trivial conclusion we get from (43)

$$E \mathcal{K}_0^0 = 0$$

(44)

iff $\Omega = \Omega(\tau) \equiv \Omega(\tau)$. We have the situation of such a kind in the case of a flat Friedman universe.

Note that in this case the component $E \mathcal{K}_0^0$ has physical meaning of the total “energy density”, matter and gravitation, for comoving observers which have 4-velocities $u^i = \Omega_{\tau}$. In consequence, there exist observers with 4-velocities with $u^i$. In consequence, there exist observers with 4-velocities

$$u^i = \left( \frac{1}{a\sqrt{1 - v^2}} , \frac{v_x}{a\sqrt{1 - v^2}} , \frac{v_y}{a\sqrt{1 - v^2}} , \frac{v_z}{a\sqrt{1 - v^2}} \right),$$

(45)

$$v_x = \frac{dx}{d\tau}, \quad v_y = \frac{dy}{d\tau}, \quad v_z = \frac{dz}{d\tau}, \quad \frac{dv_x}{d\tau} = \frac{v_x^2}{v_x^2 + v_y^2 + v_z^2},$$

(46)

for which the “energy density” $\epsilon := E \mathcal{K}_i^k u^i u_k$ and its flux (Poynting’s vector)

$$P^i = (\delta^i_k - u^i u_k) E \mathcal{K}_i^{k} u^l$$

(47)
are different from zero.

Namely, we have for such observers

\[ \epsilon = \frac{(-)^8}{27} \alpha A^6 r^2 \frac{v^2}{(1 - v^2)} < 0, \]

\[ P^0 = \frac{4\alpha(\dot{a}^2 + a\ddot{a})v^2}{a(\tau)(1 - v^2)^{3/2}}, \quad P^3 = \frac{4\alpha(\dot{a}^2 + a\ddot{a})v^3}{a(\tau)(1 - v^2)^{3/2}}, \]  

(48)

where

\[ a(\tau) = \frac{A^3}{9} \tau^2 > 0, \quad \dot{a} = \frac{2A^3}{9} \tau > 0, \quad \ddot{a} = \frac{2A^3}{9} > 0, \quad \beta = 1, 2, 3. \]  

(49)

The formal integral

\[ E = \int_{r=\text{const}} \epsilon dxdydz \]  

(50)

is divergent to minus infinity.

We would like to remark that the spatial velocity \( v^2 = v_x^2 + v_y^2 + v_z^2 \) of these observers can be infinitesimally small, i.e., these observers can infinitesimally differ from comoving observers.

Only for comoving observers which have their 4-velocity of the form \( u^i = \frac{\delta i}{\Omega} \) we have

\[ \epsilon = E K^0_0 = 0 \quad \rightarrow \quad E = 0. \]  

(51)

So, the physical situation in this case is qualitatively and quantitatively different than in the case of a Minkowskian spacetime endowed with an inertial coordinates \( (t, x, y, z) \). Namely, in Minkowskian spacetime covered by an inertial coordinates \( (t, x, y, z) \) the canonical energy-momentum complex \( E \dot{K}^{ik} \) (and other energy-momentum complexes also) identically vanishes and for any observers we have \( \epsilon = 0, \ P^i = 0 \).

Thus, by using double index energy-momentum complexes, one cannot assert that the flat Friedman universes are energetic nonentity, like a Minkowskian spacetime. All depends in the case on family of the used observers.

2. An open dust Friedman universe.

In this case all the components of the canonical energy-momentum complex \( E \dot{K}^{ik} \) are different from zero in the conformally flat coordinates \( (\tau, x, y, z) \). So, an open dust Friedman universe surely is not an energetic nonentity.

If one calculates the “total energy density” \( \epsilon = E \dot{K}^{0k} u^i u_k \), matter and gravitation, for family of the observers which are at rest in the conformally flat coordinates \( (\tau, x, y, z) \), i.e., for observers which have their 4-velocities of the form \( u^i = \frac{\delta i}{\Omega} \) in these coordinates, then one gets

\[ \epsilon = E K^0_0 = \frac{3}{2} \alpha a_0^2 \frac{(2\sqrt{\tau^2 - r^2} - a_0)^2}{(\tau^2 - r^2)} \left[ \frac{r^2}{(\tau^2 - r^2)^3} - \frac{\tau^2(a_0 - 2\sqrt{\tau^2 - r^2})}{a_0(\tau^2 - r^2)^3} \right]. \]  

(52)

This expression is negative-definite and the formal integral

\[ E = \int_{\tau^2 - r^2 = B^2} E K^0_0 d^3 S \]  

(53)

over hypersurface \( \tau^2 - r^2 = B^2 \), \( B := \frac{4\alpha}{\Omega} v^0 > \frac{a_0}{\Omega} \) is divergent to minus infinity \([19]\).

The integral (53) has mathematical meaning of the global energy, matter and gravitation, contained in the hypersurface \( \tau^2 - r^2 = B^2 \), \( B > \frac{a_0}{\Omega} \) [for observers which are at rest in the conformally flat coordinates \( (\tau, x, y, z) \) in which the line element \( ds^2 \) is given by (25)].

3. A closed dust Friedman universe.

In this case also all the components of the canonical energy-momentum complex \( E \dot{K}^{0k} \) are different from zero in the conformally flat coordinates \( (\tau, x, y, z) \). Thus, this universe, like an open Friedman universe, has non-zero
“energy density” for an arbitrary set of observers, i.e., a closed dust Friedman universe is not an energetic nonentity.

Concerning global energy of a closed dust Friedman universe we must remember that this notion has only some mathematical meaning, and that the conformally flat coordinates \((\tau, x, y, z)\) cover entirely only one distinguished slice \(\eta = 0\) of a closed dust Friedman universe. In conformally flat coordinates \((\tau, x, y, z)\) this slice is given by

\[
\tau = 0, \quad (-)\infty < x < \infty, \quad (-)\infty < y < \infty, \quad (-)\infty < z < \infty.
\]

At the moment \(\tau = 0\) the fundamental observers which were at rest in the initial coordinates \((\eta, \chi, \theta, \varphi)\) are also at rest in the conformally flat coordinates \((\tau, x, y, z)\). It is easily seen from the formulas (19)-(20) of the Section 2A. So, for these observers the component \(E_{K_0^0}(\tau = 0, x, y, z)\) represents total “energy density”, matter and gravitation, at the moment \(\tau = 0\).

By a simple calculation one can easily get that this component \(E_{K_0^0}(\tau = 0, x, y, z)\) reads

\[
E_{K_0^0} = \frac{(-384\alpha a_0^2(r^2 - 1)}{(r^2 + 1)^4}. \tag{55}
\]

Formal calculation of the energy contained inside of the distinguished slice \(\tau = 0\) in the conformally flat coordinates \((\tau, x, y, z)\) gives

\[
E = \int_{\tau=0} E_{K_0^0} d\tau dy dz = (-)1536\pi \alpha a_0^2 \int_0^A \frac{(r^4 - r^2)}{(r^2 + 1)^2} dr = \frac{512\pi \alpha a_0^2 A^3}{(1 + A^2)^3} > 0. \tag{56}
\]

A can be arbitrary big but it always should be finite because \(A \to \infty\) leads us to \(\Omega \to 0\), i.e., the limiting process \(A \to \infty\) leads to a singularity.

Despite that, if we take the formal limit \(A \to \infty\), then we will get \(E = 0\).

But one cannot conclude from this result that the closed dust Friedman universe really has zero net global energy. The reasons are the following. At first, one cannot calculate analogous global integral over any other spatial slice \(\eta = \eta_0 = \text{const} \neq 0, \quad (-)\pi < \eta_0 < \pi\) of the closed dust Friedman universe because other slices are not entirely covered by the conformally flat coordinates \((\tau, x, y, z)\). We have already mentioned about this important fact in Section 2A. Secondly, we have no global conservation laws in the domain of the relativistic cosmology.

Thirdly, if we use an other set of observers, e.g., the set of observers which have their \(4\)-velocities at the moment \(\tau = 0\)

\[
u^0 = \frac{1}{\Omega \sqrt{1 - v^2}}, \quad \nu^1 = \frac{v}{\Omega \sqrt{1 - v^2}}, \quad \nu^2 = \nu^3 = 0, \tag{57}
\]

where \(v = \sqrt{\frac{dx}{d\tau}^2}\), then we will obtain for such observers (For simplicity we will put \(v = \text{const} > 0\))

\[
\epsilon = E_{K_i^k u^i u_k} = (-)\frac{384\alpha a_0^2}{(1 - v^2)} \left[ \frac{r^2(1 - v^2) + 2v^2 x^2 - 1}{(r^2 + 1)^4} \right]. \tag{58}
\]

It follows from the above expression that for these observers the “global energy” \(E\) contained in the subspace \(\tau = 0\) reads

\[
E = (-)\frac{384\alpha a_0^2}{(1 - v^2)} \int_0^\infty \int_0\pi \int_0^{2\pi} \frac{[(1 - v^2)r^2 + 2v^2 x^2 - 1]}{(1 + r^2)^4} r^2 \sin \theta dr d\theta d\varphi = 16\pi^2 a_0^2 v^2 \frac{v^2}{(1 - v^2)} > 0, \tag{59}
\]

i.e., it is positive-definite even for infinitesimally small \(v\).

Thus, the “global energetic content” in the subspace \(\tau = 0\) depends on the used set of the observers which are studying the closed dust Friedman universe.

Once more we met a situation which is qualitatively and quantitatively different than the situation in Minkowskian spacetime endowed with an inertial coordinates \((t, x, y, z)\).
IV. CONCLUSION

Our conclusion is that the Friedman universes are not energetic nonentity even if we analyse these universes only with the help of a double index energy-momentum complex. Because these universes are not asymptotically flat, such analysis should be performed in the geometrically and physically distinguished conformally flat coordinates \((\tau, x, y, z)\).

We hope that we have convincingly justified this conclusion in this paper.

The our conclusion is in full agreement with our previous analysis of the Friedman (and also more general) universes with the help of the averaged relative energy-momentum tensors \([8]\).

Of course, our conclusion contradicts the recently very popular opinion that the Friedman universes are energy-free. Such opinion originated from incomplete analysis of these universes performed in the Cartesian comoving coordinates \((t, x, y, z)\) in which only the spatial part of the FLRW line element is conformally flat.

By incomplete analysis we mean the fact of using only the comoving observers to analyse the energetic content of the Friedman (and also more general) universes. As we have seen, using an other set of the observers gives other, non-null local and global results for flat Friedman universes and non-null global results for a closed Friedman universe.

In fact, only by using the non-comoving observers one is able to show that the flat Friedman universes are not energetic nonentity neither locally nor globally and that the closed Friedman universes are not global energetic nonentity.

Limitation to the comoving observers only is not justified physically, e.g., an Earth’s observer is not a comoving observer in the real Universe.

We think that the conformally flat coordinates \((\tau, x, y, z)\) have much more profound geometrical and physical meaning than the Cartesian comoving coordinates. Thus, in order to correctly analyse the energy and momentum of the Friedman universes with the help of a coordinate-dependent energy-momentum complex one should work in these coordinates. We have done this in the present paper for the energy.

We hope that this paper and the our previous paper \([8]\) will finish discussion about energetic content of the Friedman universes.

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[13] We prefer signature+ − − − and we will use geometrized units in which $G = c = 1$.
[14] For example, they determine the same causal structure underlying spacetime as an inertial coordinates $(t, x, y, z)$ in a Minkowskian spacetime.
[15] I must say that in my old papers I also followed this conclusion. Now I think that it was incorrect.
[16] More general spatially homogeneous universes have not been considered yet.
[17] $\eta = 0$ corresponds to space $\tau = 0$ in the conformally flat coordinates $(\tau, x, y, z)$ as it was already mentioned before.
[18] But using of an other reasonable double index energy-momentum complex will lead us to analogous results.
[19] The hypersurface $\tau^2 - r^2 = B^2$ is an image in the conformally flat coordinates $(\tau, x, y, z)$ of the spatial slice $\eta = \eta_0$ of the Friedman universe in the initial coordinates $(\eta, \chi, \theta, \phi)$. 