Examples of Pairs of Ordered Congruent-like \( n \)-gons with Different Areas

Michele Gaeta and Giovanni Vincenzi

Michele Gaeta (m.gaeta23@studenti.unisa.it) was born on 20 January 1999 in Salerno (Italy). He graduated with honors in math at University of Salerno in July 2020. Currently he is attending a master's degree in math at University of Salerno.

Giovanni Vincenzi (vincenzi@unisa.it) obtained his Ph.D. in Algebra in 1994 at the University of Naples “Federico II”, (Italy). He is currently Professor at the University of Salerno (Italy). He has various interests in Mathematics with a particular predilection for its foundations, historical, and didactic aspects.

Let’s consider a cycle of \( n \) rods of possible varying lengths \((n \text{ positive integer})\), attached with hinges. Clearly, if we have more than three rods we can move them, and we can obtain different shapes (\( SSS \) congruence doesn’t hold past \( n = 3 \)). For example, when \( n = 4 \) we have a model, that an engineer would call articulated quadrilaterals, that is a “four-bar linkages.” These have numerous practical applications. This is because if we fix the side \( AB \), then the paths which can be followed by \( C \) and \( D \) are clearly defined.

![Figure 1. Pair of articulated quadrilaterals. Their sides are in the same order and have the same lengths; but their inner angles are not pairwise congruent.](image)

What happens to a pair of articulated \( n \)-gons if we restrict the angles to be the same? Well, obviously if they are the same in the corresponding locations (so \( SASASAS \ldots \) generalizes), then they will be congruent. What if we permute the angles? Here the answer is not so easy, because it depends on the number \( n \) of the sides (angles).

This kind of problems have been recently studied in [3], where the authors studied convex \( n \)-gons and introduced the formal definition of ordered congruent-like \( n \)-gons.

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Here we highlight that the term “ordered” just refers to the sequence of the sides of the \( n \)-gons and not to their angles. Intuitively, two ordered congruent-like \( n \)-gons can be considered as an articulated \( n \)-gon such that by moving its sides, it is possible to find a new configuration in which every angle may change position but “preserves” its magnitude (see [3, Figure 3]; Figure 2 or Figure 3).

**Figure 2.** Pairs of ordered congruent-like hexagons \( \mathcal{E} \) and \( \mathcal{E}' \). They have been constructed considering two quadrilaterals, with a common side and suitable angles (green angles are congruent; orange angles are not congruent to blue angles). In the transition from left to right, the quadrilateral \( Q_1 \) has been reflected, so \( \mathcal{E} \) and \( \mathcal{E}' \) are not congruent.

In [3] the authors showed, by an articulated proof which splits into many different sub cases corresponding to the possible permutations of the angles, that two ordered congruent-like quadrilaterals are congruent. The case \( n = 5 \) is still open; while even if it seems strange at first glance, when \( n > 5 \) it is not difficult to find pairs of non-congruent \( n \)-gons that are ordered congruent-like (see Figure 3).

**Figure 3.** Pairs of ordered congruent-like \( n \)-gons \( \mathcal{P}_n \) and \( \mathcal{P}'_n \).

We note that the concept of “pairs of ordered congruent-like \( n \)-gons” can be considered as a particular case of a more general notion of “pairs of congruent-like \( n \)-gons,” in which both the order of the side and the angles may be not preserved. Examples of “pairs of (non-ordered) congruent like \( n \)-gons” that are not congruent can be constructed for every \( n > 3 \) (see [1, Figure 1] for an easy construction when \( n > 4 \)). We refer the reader to [1] and [2] for a deeper discussion of (non-ordered) congruent-like polygons, and the case \( n = 4 \).

We highlight that at the moment all constructions of pairs of ordered congruent-like \( n \)-gons’ that are known produce pairs of polygons with equal areas. It is natural to ask: is this forced? We can summarize what we have just resumed about ordered congruent-like \( n \)-gons issues in the following scheme:
Now, we will see that for every $n > 5$ there are pairs of ordered congruent-like $n$-gons with different areas, thus, the last cell of the above table may be fill with “Yes”.

**The construction**

We begin the construction with the following trapezoid (see Figure 5).

Figure 5. Quadrilateral $P_2$ has been obtained from the trapezoid $P_1$ by a rotation of $30^\circ$ of the sides $AD$ and $CB$.

Starting from the trapezoid $P_1 = (ABCD)$ that has been obtained as a union of a rectangle with the sides of length 3 and 6 and half equilateral triangle for which the height is 3, we can turn simultaneity (clockwise) both the sides $DA$ and $CB$ around $D$ and $C$ (respectively) of $30^\circ$. Thus, a new quadrilateral $P_2 = (A'B'CD)$ is obtained.

First, we show that $Area(ABCD) < Area(A'B'CD)$. By construction $CD = 6$, $CB' = CB = 2\sqrt{3}$ and $D\hat{C}B'$ is right, so that

$$CB' = CD \tan(B'DC) \text{ and } DB' = \sqrt{(36 + 12)} = \sqrt{48}.$$

It follows that $\tan(B'DC) = \frac{2\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$, and hence $B'DC = 30^\circ$. Therefore $A'DB'$ is a right triangle. Now we can easily compute the areas of the two quadrilaterals:
• \( \text{Area}(ABCD) = APDC + \frac{1}{2}3PB = 18 + \frac{1}{2}3\sqrt{3} \)

• \( \text{Area}(A'B'CD) = CDB' + A'DB' = \frac{1}{2}6 \cdot 2\sqrt{3} + \frac{1}{2}3\sqrt{48} = 6\sqrt{3} + 6\sqrt{3} = 12\sqrt{3} \).

It follows that \( \text{Area}(ABCD) < \text{Area}(A'B'CD) \).

We also note that \( \alpha' + \beta' = 150^\circ \).

Now, to determine a pair of hexagons as required, it is enough to consider the union of two copies of \( P_1 \) and two copies of \( P_2 \) as shown in Figure 6.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{Pairs of ordered congruent-like hexagons with different area.}
\end{figure}

The general case \( (n > 6) \) can be easily derived from the hexagons adjoining a suitable \( (n - 4) \)-gon (see \( P_3 \) in Figure 7).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Pairs of ordered congruent-like \( n \)-gons with different areas.}
\end{figure}

As an exercise, the reader can try to find two congruent-like heptagons with different areas.

\textbf{Remark.} It should be noted that the polygons constructed as above, say \( \mathcal{V}_n \) and \( \mathcal{V'}_n \), are very special pairs of congruent-like polygons: they are articulated \( n \)-gons with different areas with the property that they can be obtained from each other by a deformation that preserves the order and the length of their sides, and such that each ordered (clockwise) sequence of the consecutive inner angles of \( \mathcal{V}_n \) is equal to some ordered (clockwise or
counterclockwise) sequence of the consecutive inner angles of $V'_n$. Pairs of polygons of this kind can provide some useful hints for a lesson on polygons. For example, it could be observed that to determine the area of a $n$-gon, it is not enough to know all its sides and angles; but we also need to know which are their mutual positions. The use of dynamic geometry software could be useful for this aim.

We conclude by observing that it does not seem easy to find similar constructions for pentagons.

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**References**

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