ON HERMITE–HADAMARD TYPE INEQUALITIES FOR THE PRODUCT OF TWO CONVEX MAPPINGS DEFINED ON TOPOLOGICAL GROUPS

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Abstract. We study Hermite–Hadamard type inequalities for the product of two midconvex and quasi-midconvex functions and give some applications of our results.

1. Introduction

Let \( f : I \to \mathbb{R} \) be a convex mapping defined on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). The following double inequality:

\[
\frac{f(a) + f(b)}{2} \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

is known as Hermite–Hadamard inequality. The history of this inequality goes back to the papers of Hermite [9] and Hadamard [8] in 1883 and 1893, respectively. This inequality produces some classical inequalities of means for particular choice of the mapping \( f \). Inequality (1.1) has attracted a number of mathematicians and it has been generalized, extended, and refined in a number of ways (see e.g. [3, 4, 7]). Also some mappings naturally connected with (1.1) are defined and the properties of these mappings are discussed by many mathematicians (see e.g. [5, 6]). We only discuss recent studies in this paper.

A generalization of the left-hand side of (1.1) for convex functions defined on a convex subset of \( \mathbb{R}^n \) is the following inequality from [13]

\[
f(0) \leq \frac{1}{\mu(X)} \int_X f(x) \, dx,
\]

where \( X \subset \mathbb{R}^n \) is a convex bounded symmetric set that is, if \( x \in X \) then \( -x \in X \), \( f \) is a lower semicontinuous convex function \( f : X \to \mathbb{R} \) and \( \mu(X) \) is the volume of the set \( X \).

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In [12], Morassaei established the Hermite–Hadamard type inequality for mid-convex and quasi-midconvex functions in topological groups and discussed some of the properties of the mapping naturally connected with the Hermite–Hadamard inequality for globally midconvex functions defined in a topological group. Some of the main results of [12] are stated in the following theorems.

**Theorem 1.1.** [12] Let $G$ be a locally compact group and $\Omega \subset G$ an open symmetric set relative to $a \in G$ with $0 < \mu(\Omega) < \infty$. Let $f: \Omega \to \mathbb{R}$ be measurable and locally midconvex in $a$ and $f \in L_1(\Omega)$. If $\omega: \Omega \to \mathbb{R}$ is non-negative, symmetric with respect to $a$ and $\omega \in L_1(\Omega)$ such that $f \omega \in L_1(\Omega)$; then

$$ f(a) \int_{\Omega} \omega(az) \, d\mu(z) \leq \int_{\Omega} f(az) \omega(az) \, d\mu(z), $$

where $\mu$ is the Haar measure.

**Theorem 1.2.** [12] Suppose that $G$ is a locally compact group and $\Omega \subset G$ an open symmetric set relative to $a \in G$ with $0 < \mu(\Omega) < \infty$ and $e \in \Omega$. Let $f$ be measurable and quasi-midconvex real-valued function on $\Omega$ such that $f \in L_2(\Omega)$. If $\omega: \Omega \to \mathbb{R}$ is non-negative and symmetric with respect to $a$ and $\omega \in L_2(\Omega)$, then

$$ f(a) \int_{\Omega} \omega(az) \, d\mu(z) \leq \int_{\Omega} f(az) \omega(az) \, d\mu(z) + I(a), $$

where

$$ I(a) = \frac{1}{2} \int_{\Omega} |f(az) - f(az^{-1})| \omega(az) \, d\mu(z). $$

Furthermore, $I(a)$ satisfies the following inequality

$$ 0 \leq I(a) \leq \min \left\{ \frac{1}{\sqrt{2}} \left( \int_{\Omega} f^2(az) \, d\mu(z) - \int_{\Omega} f^2(az) f(az^{-1}) \, d\mu(z) \right)^{\frac{1}{2}}, \left( \int_{\Omega} \omega^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \right\}. $$

**Theorem 1.3.** Let $G$ be a locally compact group and $\Omega \subset G$ an open symmetric set relative to $a \in G$ with $0 < \mu(\Omega) < \infty$. Let $f$ be measurable real valued $P$-function on $\Omega$ such that $f \in L_1(\Omega)$. If $\omega: \Omega \to \mathbb{R}$ is non-negative symmetric with respect to $a$, $\omega \in L_1(\Omega)$ and $f \omega \in L_1(\Omega)$, then

$$ f(a) \int_{\Omega} \omega(az) \, d\mu(z) \leq 2 \int_{\Omega} f(az) \omega(az) \, d\mu(z). $$

We give a result similar to [12] for the product of two convex functions defined on a convex bounded symmetric subset $X$ of $\mathbb{R}^n$. We will also give our results for the product of two midconvex and quasi-midconvex mappings defined on topological groups in Section 3. Applications of the obtained results are given in Section 3 as well.
2. A Secondary Result

**Theorem 2.1.** Let \( f, g \) be two convex functions defined on a convex bounded symmetrical subset \( X \) of \( \mathbb{R}^n \). Then

\[
(2.1) \quad f(0)g(0) \leq \frac{1}{2\mu(X)} \int_X [f(x)g(x) + f(x)g(-x)]dx
\]

\[
= \frac{1}{2\mu(X)} \int_X [f(x)g(x) + f(-x)g(x)]dx.
\]

**Proof.** Consider the transformation \( h: \mathbb{R}^n \to \mathbb{R}^n, h = (h_1, \ldots, h_n) \), given by

\[
h_i(x_1, \ldots, x_n) = -x_i \quad i = 1, 2, \ldots, n.
\]

Then \( h(X) = X \) and

\[
\frac{D(h_1, \ldots, h_n)}{D(x_1, \ldots, x_n)} = \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix} = (-1)^n.
\]

Thus we have, by the change of variables that

\[
\int_X f(x_1, \ldots, x_n)g(x_1, \ldots, x_n)dx_1 \cdots dx_n
\]

\[
= \int_X f(h_1(x_1, \ldots, x_n), \ldots, h_n(x_1, \ldots, x_n))g(h_1(x_1, \ldots, x_n), \ldots, h_n(x_1, \ldots, x_n))
\]

\[
\times \left| \frac{D(h_1, \ldots, h_n)}{D(x_1, \ldots, x_n)} \right| dx_1 \cdots dx_n = \int_X f(-x_1, \ldots, -x_n)g(-x_1, \ldots, -x_n)dx_1 \cdots dx_n,
\]

\[
\int_X f(x_1, \ldots, x_n)g(-x_1, \ldots, -x_n)dx_1 \cdots dx_n
\]

\[
= \int_X f(h_1(x_1, \ldots, x_n), \ldots, h_n(x_1, \ldots, x_n))g(-h_1(x_1, \ldots, x_n), \ldots, -h_n(x_1, \ldots, x_n))
\]

\[
\times \left| \frac{D(h_1, \ldots, h_n)}{D(x_1, \ldots, x_n)} \right| dx_1 \cdots dx_n = \int_X f(-x_1, \ldots, -x_n)g(x_1, \ldots, x_n)dx_1 \cdots dx_n.
\]

Now by the convexity of \( f \) and \( g \) on \( X \), we get

\[
f(0, \ldots, 0)g(0, \ldots, 0) = f\left(\frac{1}{2}(x_1 - x_1), \ldots, \frac{1}{2}(x_n - x_n)\right)g\left(\frac{1}{2}(x_1 - x_1), \ldots, \frac{1}{2}(x_n - x_n)\right)
\]

\[
= f\left(\frac{1}{2}(x_1 - x_1, \ldots, x_n) + (-x_1, \ldots, -x_1)\right)g\left(\frac{1}{2}(x_1, \ldots, x_n) + (-x_1, \ldots, -x_n)\right)
\]

\[
\leq \frac{1}{2}f(x_1, \ldots, x_n) + f(-x_1, \ldots, -x_n)\left[f(x_1, \ldots, x_n) + g(x_1, \ldots, -x_n)\right]
\]

\[
= \frac{1}{2}f(x_1, \ldots, x_n)g(x_1, \ldots, x_n) + f(-x_1, \ldots, -x_n)g(-x_1, \ldots, -x_n)
\]

\[
+ f(-x_1, \ldots, -x_n)g(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)g(-x_1, \ldots, -x_n)
\]

which gives by integration on \( X \) that

\[
(2.2) \quad \int_X f(0, \ldots, 0)g(0, \ldots, 0)dx_1 \cdots dx_n
\]
\[ f(\bar{a}, \bar{z}, \bar{a}^{-1}) \leq \frac{1}{4} \left[ 2 \int_X f(x_1, \ldots, x_n) g(x_1, \ldots, x_n) \, dx_1 \ldots dx_n + \int_X f(x_1, \ldots, x_n) g(-x_1, \ldots, -x_n) \, dx_1 \ldots dx_n + \int_X f(-x_1, \ldots, -x_n) g(x_1, \ldots, x_n) \, dx_1 \ldots dx_n \right]. \]

Hence (2.1) follows from (2.2).

\[ \square \]

3. Main Results

Now we prove Hermite–Hadamard type inequalities for products of midconvex and quasi-convex functions defined in a topological groups. Before we proceed to prove our results we give some definitions from \[1\] \[11\] \[12\].

We recall that for a group \( (G, \cdot, e) \), a topology on \( G \) is compatible with the group structure when the maps \( G \times G \rightarrow G \) : \( (x, y) \mapsto xy \) (multiplication) and \( G \rightarrow G \) : \( x \mapsto x^{-1} \) (inverse) are continuous. A group together with a topology compatible with its group structure is a topological group. A compact group is a topological group that is a compact topological space.

A Haar measure on compact group \( G \) is a measure \( \mu : \Sigma \rightarrow [0, \infty) \), with a \( \sigma \)-algebra \( \Sigma \) containing all Borel subsets of \( G \), such that \( \mu(G) = 1 \) and \( \mu(\gamma S) = \mu(S) \) for all \( \gamma \in G \), \( S \in \Sigma \), where \( \gamma S = \{ \gamma a : a \in S \} \).

**Definition 3.1.** Let \( G \) be a topological group, \( \Omega \) a non-empty open subset of \( G \) and \( f \) a real-valued function on \( \Omega \). We say that \( f \) is globally (right) midconvex if \( 2f(a) \leq f(az) + f(az^{-1}) \) for all \( a, z \in G \) such that \( a, az, az^{-1} \in \Omega \). We say that \( f \) is locally (right) midconvex for every \( a \in \Omega \) if there exists an open symmetric neighborhood \( V = V^{-1} \) of \( e \) such that \( 2f(a) \leq f(az) + f(az^{-1}) \) for all \( z \in V \) such that \( az, az^{-1} \in \Omega \).

**Definition 3.2.** Let \( G \) be a topological group, \( \Omega \) a non-empty open subset of \( G \) and \( f \) a real-valued function on \( \Omega \). The mapping \( f \) is called quasi-(right) midconvex, if \( f(az) \leq \max\{ f(a), f(az^2) \} \) for every \( a, z \in G \) so that \( a, az, az^2 \in \Omega \). Note that \( a \) is the midpoint of \( az^{-1} \) and \( az \), and \( az \) is the midpoint of \( a \) and \( az^2 \).

**Definition 3.3.** Let \( \Omega \) be an open subset of a topological group \( G \), and \( a \in G \). \( \Omega \) is said to be symmetric relative to \( a \), if \( a^{-1} \Omega \) is symmetric and \( e \in a^{-1} \Omega \).

**Definition 3.4.** Let \( G \) be a topological group and \( \Omega \subset G \) an open set. A function \( \omega : \Omega \rightarrow \mathbb{R} \) is called symmetric relative to \( a \in G \), if for every \( z \in G \), \( az, az^{-1} \in \Omega \) and \( \omega(az) = \omega(az^{-1}) \).

We now give our main result.

**Theorem 3.1.** Suppose that \( G \) is a locally compact group and \( \Omega \subset G \) an open symmetric set relative to \( a \in G \) with \( 0 < \mu(\Omega) < \infty \). Let \( f, g : \Omega \rightarrow \mathbb{R}_+ \) be measurable and locally midconvex in \( a \) and \( f, g \in L_1(\Omega) \). If \( \omega : \Omega \rightarrow \mathbb{R} \) is non-negative symmetric with respect to \( a \) and \( \omega \in L_1(\Omega) \) such that \( f g \omega \in L_1(\Omega) \), then
we have

\[(3.1) \quad f(a)g(a) \int_{\Omega} \omega(az) \, d\mu(z) \]

\[\leq \frac{1}{2} \left[ \int_{\Omega} f(az)g(az)\omega(az) \, d\mu(z) + \int_{\Omega} f(az)g(az^{-1})\omega(az) \, d\mu(z) \right] \]

\[= \frac{1}{2} \left[ \int_{\Omega} f(az)g(az)\omega(az) \, d\mu(z) + \int_{\Omega} f(az^{-1})g(az)\omega(az) \, d\mu(z) \right].\]

where \(\mu\) is the Haar measure.

**Proof.** Since \(f\) and \(g\) are midconvex in \(a\), we have \(2f(a) \leq f(az) + f(az^{-1})\) and \(2g(a) \leq g(az) + g(az^{-1})\) for any \(z \in \Omega\). From these inequalities we get

\[4f(a)g(a) \leq f(az)g(az) + f(az^{-1})g(az) + f(az)g(az^{-1}) + f(az^{-1})(g(az^{-1}).\]

Since \(\omega\) is non-negative and symmetric relative to \(a\), we have

\[4f(a)g(a)\omega(az) \leq f(az)g(az)\omega(az) + f(az^{-1})g(az)\omega(az^{-1})\]

\[+ f(az)g(az^{-1})\omega(az) + f(az^{-1})g(az^{-1})\omega(az^{-1}).\]

Integrating this inequality over \(\Omega\), we get

\[4f(a)g(a) \int_{\Omega} \omega(az) \, d\mu(z) \]

\[\leq \int_{\Omega} f(az)g(az)\omega(az) \, d\mu(z) + \int_{\Omega} f(az^{-1})g(az)\omega(az^{-1}) \, d\mu(z) \]

\[+ \int_{\Omega} f(az)g(az^{-1})\omega(az) \, d\mu(z) + \int_{\Omega} f(az^{-1})g(az^{-1})\omega(az^{-1}) \, d\mu(z) \]

\[= \int_{\Omega} f(z)g(z)\omega(z) \, d\mu(z) \]

\[+ \int_{\Omega} f(z)g(z^{-1})\omega(z) \, d\mu(z) \]

\[= \int_{G} f(z)g(z)\omega(z) \chi_{\alpha^{-1}\Omega}(z) \, d\mu(z) \]

\[+ \int_{G} f(z)g(z^{-1})\omega(z) \chi_{\alpha^{-1}\Omega}(z) \, d\mu(z) \]

that is

\[(3.2) \quad f(a)g(a) \int_{\Omega} \omega(az) \, d\mu(z) \]

\[\leq 2 \int_{G} f(z)g(z)\omega(z) \chi_{\alpha^{-1}\Omega}(z) \, d\mu(z) + 2 \int_{G} f(z)g(z^{-1})\omega(z) \chi_{\alpha^{-1}\Omega}(z^{-1}) \, d\mu(z)\]
we also have
\[ 2 \int_{\omega^{-1}\Omega} f(z)g(z)\omega(z) d\mu(z) + 2 \int_{\omega^{-1}\Omega} f(z)g(z^{-1})\omega(z) d\mu(z) \]
\[ = \frac{1}{2} \left[ \int_{\Omega} f(az)g(az)\omega(az) d\mu(z) + \int_{\Omega} f(az)g(az^{-1})\omega(az) d\mu(z) \right]. \]

Since
\[ \int_{\Omega} f(az)g(az^{-1})\omega(az) d\mu(z) = \int_{\Omega} f(az)g(az^{-1})\omega(az^{-1}) d\mu(z) \]
\[ = \int_{\omega^{-1}\Omega} f(z)g(z^{-1})\omega(z^{-1}) d\mu(z) = \int_{\omega^{-1}\Omega} f(z)g(z^{-1})\omega(z^{-1})\chi_{\omega^{-1}\Omega}(z) d\mu(z) \]
\[ = \int_{\omega^{-1}\Omega} f(z)g(z^{-1})\omega(z^{-1})\chi_{\omega^{-1}\Omega}(z) d\mu(z) = \int_{\omega^{-1}\Omega} f(az^{-1})g(az)\omega(az) d\mu(z), \]
we also have
\[ f(a)g(a) \int_{\Omega} \omega(az) d\mu(z) \]
\[ \leq \frac{1}{2} \left[ \int_{\Omega} f(az)g(az)\omega(az) d\mu(z) + \int_{\Omega} f(az^{-1})g(az)\omega(az) d\mu(z) \right]. \]

Consequently inequality (3.1) follows from (3.2) and (3.3). \qed

**Remark 3.1.** If we take \( a = e \) and \( \omega \equiv 1 \) on \( \Omega \) in Theorem 3.1, then we have
\[ f(e)g(e) \leq \frac{1}{2\mu(\Omega)} \left[ \int_{\Omega} f(z)g(z)\omega(z) d\mu(z) + \int_{\Omega} f(z)g(z^{-1})\omega(z) d\mu(z) \right] \]
\[ = \frac{1}{2\mu(\Omega)} \left[ \int_{\Omega} f(z)g(z)\omega(z) d\mu(z) + \int_{\Omega} f(z^{-1})g(z)\omega(z) d\mu(z) \right] \]
which is similar to (2.1).

**Remark 3.2.** If \( g \equiv 1 \) on \( \Omega \) in Theorem 3.1, then we have
\[ f(a) \int_{\Omega} \omega(az) d\mu(z) \leq \int_{\Omega} f(az)g(az)\omega(az) d\mu(z) \]
which is a similar result to that in [12, Theorem 1].

**Remark 3.3.** If we take \( a = e, \omega \equiv 1 \) and \( g \equiv 1 \) on \( \Omega \) in Theorem 3.1, then we have \( f(e) \leq \frac{1}{\mu(\Omega)} \int_{\Omega} f(z) d\mu(z) \) which is the same result to that in [12, Remark 1].

**Theorem 3.2.** Suppose that \( G \) is a locally compact group and \( \Omega \subset G \) an open symmetric set relative to \( a \in G \) with \( 0 < \mu(\Omega) < \infty \) and \( e \in \Omega \). Let \( f \) and \( g \) be measurable and quasi-midconvex non-negative real-valued functions on \( \Omega \) such that \( fg \in L_2(\Omega) \). If \( \omega: \Omega \to \mathbb{R} \) is non-negative and symmetric with respect to \( a \) and \( \omega \in L_2(\Omega) \), we have
\[ f(a)g(a) \int_{\Omega} \omega(az) d\mu(z) \]
\[
\leq \frac{1}{2} \int_{\Omega} f(az)g(az)\omega(az)\,d\mu(z) + \frac{1}{2} \int_{\Omega} f(az)g(az^{-1})\omega(az)\,d\mu(z) + I(a)
\]
\[
= \frac{1}{2} \int_{\Omega} f(az)g(az)\omega(az)\,d\mu(z) + \frac{1}{2} \int_{\Omega} f(az^{-1})g(az)\omega(az)\,d\mu(z) + I(a),
\]
where
\[
I(a) = \frac{1}{2} \int_{\Omega} f(az)|g(az) - g(az^{-1})|\omega(az)\,d\mu(z)
\]
\[
+ \frac{1}{2} \int_{\Omega} |f(az) - f(az^{-1})|g(az)\omega(az)\,d\mu(z)
\]
\[
+ \frac{1}{4} \int_{\Omega} |f(az) - f(az^{-1})||g(az) - g(az^{-1})|\omega(az)\,d\mu(z).
\]
Furthermore,
\[
0 \leq I(a) \leq \min\{A_1, A_2, A_3, A_4\},
\]
where
\[
A_1 = \frac{1}{2} \left( \int_{\Omega} (f(az^{-1})(g(az) - g(az^{-1})))^2\,d\mu(z) \right)^{1/2} \left( \int_{\Omega} \omega^2(az)\,d\mu(z) \right)^{1/2}
\]
\[
+ \frac{1}{2} \left( \int_{\Omega} (g(az^{-1})(f(az) - f(az^{-1})))^2\,d\mu(z) \right)^{1/2} \left( \int_{\Omega} \omega^2(az)\,d\mu(z) \right)^{1/2}
\]
\[
+ \frac{1}{4} \left( \int_{\Omega} ((f(az^{-1}) - f(az))(g(az) - g(az^{-1})))^2\,d\mu(z) \right)^{1/2} \left( \int_{\Omega} \omega^2(az)\,d\mu(z) \right)^{1/2},
\]
\[
A_2 = \frac{1}{\sqrt{2}} \left( \int_{\Omega} f^2(az)\,d\mu(z) \right)^{1/2} \left( \int_{\Omega} g^2(az)\omega^2(az)\,d\mu(z) \right)^{1/2}
\]
\[
- \int_{\Omega} g(az)g(az^{-1})\omega^2(az)\,d\mu(z) + \frac{1}{\sqrt{2}} \left( \int_{\Omega} g^2(az)\,d\mu(z) \right)^{1/2}
\]
\[
\times \left( \int_{\Omega} f^2(az)\omega^2(az)\,d\mu(z) - \int_{\Omega} f(az)f(az^{-1})\omega^2(az)\,d\mu(z) \right)^{1/2}
\]
\[
+ \frac{1}{2} \left( \int_{\Omega} f^2(az)\,d\mu(z) - \int_{\Omega} f(az)f^{-1}(az)\,d\mu(z) \right)^{1/2}
\]
\[
\times \left( \int_{\Omega} g^2(az)\omega^2(az)\,d\mu(z) - \int_{\Omega} g(az)g(az^{-1})\omega^2(az)\,d\mu(z) \right)^{1/2},
\]
\[
A_3 = \frac{1}{\sqrt{2}} \left( \int_{\Omega} g^2(az)\,d\mu(z) \right)^{1/2} \left( \int_{\Omega} g^2(az)g(az^{-1})\,d\mu(z) \right)^{1/2}
\]
\[
+ \frac{1}{\sqrt{2}} \left( \int_{\Omega} f^2(az)\,d\mu(z) \right)^{1/2} \left( \int_{\Omega} f^2(az)g(az^{-1})\,d\mu(z) \right)^{1/2}
\]
\[
+ \frac{1}{2} \left( \int_{\Omega} g^2(az)\,d\mu(z) \right)^{1/2} \left( \int_{\Omega} g^2(az)g(az^{-1})\,d\mu(z) \right)^{1/2}
\]
Since $\omega \leq (3.8)$ 0

Now by the Cauchy–Schwarz inequality, we observe from (3.7) that

\[ A_4 + \frac{3}{2} \int \Omega f(az)g(\omega(az)) \, d\mu(z) + \frac{3}{2} \int \Omega f(az)g(az^{-1}) \omega(az) \, d\mu(z). \]

**Proof.** Since $\Omega$ is a symmetric set relative to $a$, for any $z \in G$ and by the quasi-midconvexity of $f$ and $g$, we have

\[
\begin{align*}
f(a) &= \max \{ f(az), f(az^{-1}) \} = \frac{f(az) + f(az^{-1}) + |f(az) - f(az^{-1})|}{2}, \\
g(a) &= \max \{ g(az), g(az^{-1}) \} = \frac{g(az) + g(az^{-1}) + |g(az) - g(az^{-1})|}{2}.
\end{align*}
\]

Now, by the non-negativity of $f$ and $g$, we get

\[
\begin{align*}
(3.6) \quad f(a)g(a) &\leq \frac{1}{2}[f(az)g(az) + f(az)g(az^{-1}) + f(az)|g(az) - g(az^{-1})|] \\
&+ f(az^{-1})g(az) + f(az^{-1})g(az^{-1}) + f(az^{-1})|g(az) - g(az^{-1})| \\
&+ |f(az) - f(az^{-1})|g(az) + |f(az) - f(az^{-1})|g(az^{-1}) \\
&+ |f(az) - f(az^{-1})||g(az) - g(az^{-1})|.
\end{align*}
\]

Since $\omega$ is non-negative and symmetric relative to $a$, we have from \[3.3\]

\[
\begin{align*}
f(a)g(a) \int \Omega \omega(az) \, d\mu(z) \\
\leq \frac{1}{2} \int \Omega f(az)g(az)\omega(az) \, d\mu(z) + \frac{1}{2} \int \Omega f(az)g(az^{-1})\omega(az) \, d\mu(z) + I(a) \\
= \frac{1}{2} \int \Omega f(az)g(az)\omega(az) \, d\mu(z) + \frac{1}{2} \int \Omega f(az^{-1})g(az)\omega(az) \, d\mu(z) + I(a).
\end{align*}
\]

Hence, \[3.4\] is proved, where

\[
(3.7) \quad I(a) = \frac{1}{2} \int \Omega f(az)|g(az) - g(az^{-1})|\omega(az) \, d\mu(z) \\
+ \frac{1}{2} \int \Omega |f(az) - f(az^{-1})|g(az)\omega(az) \, d\mu(z) \\
+ \frac{1}{4} \int \Omega |f(az) - f(az^{-1})||g(az) - g(az^{-1})|\omega(az) \, d\mu(z).
\]

Now by the Cauchy–Schwarz inequality, we observe from \[3.7\] that

\[
\begin{align*}
(3.8) \quad 0 &\leq I(a) \leq \frac{1}{2} \left( \int \Omega (f(az)(g(az) - g(az^{-1})))^2 \, d\mu(z) \right)^{\frac{1}{2}} \left( \int \Omega \omega^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
&+ \frac{1}{2} \left( \int \Omega (g(az)(f(az) - f(az^{-1})))^2 \, d\mu(z) \right)^{\frac{1}{2}} \left( \int \Omega \omega^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
&+ \frac{1}{4} \left( \int \Omega ((f(az^{-1}) - f(az))(g(az) - g(az^{-1})))^2 \, d\mu(z) \right)^{\frac{1}{2}} \left( \int \Omega \omega^2(az) \, d\mu(z) \right)^{\frac{1}{2}}.
\end{align*}
\]
Again by the Cauchy–Schwarz inequality, we have from (3.7) the following inequality
\begin{align*}
0 \leq I(a) & \leq \frac{1}{2} \left( \int_{\Omega} f^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} \left( g(az) - g(az^{-1}) \right)^2 \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad + \frac{1}{2} \left( \int_{\Omega} g^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} (f(az) - f(az^{-1}))^2 \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad + \frac{1}{4} \left( \int_{\Omega} (f(az) - f(az^{-1}))^2 \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} g(az) - g(az^{-1}) \right)^{2} \left( \int_{\Omega} \omega^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad + \frac{1}{4} \left( \int_{\Omega} (f(az) - f(az^{-1}))^2 \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} g(az) - g(az^{-1}) \right)^{2} \left( \int_{\Omega} \omega^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad \leq \frac{1}{\sqrt{2}} \left( \int_{\Omega} f^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} g^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad + \frac{1}{\sqrt{2}} \left( \int_{\Omega} f^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} g^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad + \frac{1}{2} \left( \int_{\Omega} f^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} f^{-1}(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad \times \left( \int_{\Omega} g^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} g(az) - g(az^{-1}) \right)^{2} \left( \int_{\Omega} \omega^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad + \frac{1}{2} \left( \int_{\Omega} g^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} g(az) - g(az^{-1}) \right)^{2} \left( \int_{\Omega} \omega^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad \times \left( \int_{\Omega} f^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} f^{-1}(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad \leq \frac{1}{\sqrt{2}} \left( \int_{\Omega} f^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} g^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad + \frac{1}{\sqrt{2}} \left( \int_{\Omega} f^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} g^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad + \frac{1}{2} \left( \int_{\Omega} f^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} f^{-1}(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad \times \left( \int_{\Omega} g^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} g(az) - g(az^{-1}) \right)^{2} \left( \int_{\Omega} \omega^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad + \frac{1}{2} \left( \int_{\Omega} g^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} g(az) - g(az^{-1}) \right)^{2} \left( \int_{\Omega} \omega^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad \times \left( \int_{\Omega} f^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} f^{-1}(az) \, d\mu(z) \right)^{\frac{1}{2}}.
\end{align*}

Using the Cauchy–Schwarz inequality again, from (3.7) we infer (3.10)
\begin{align*}
0 \leq I(a) & \leq \frac{1}{\sqrt{2}} \left( \int_{\Omega} f^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} g^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad + \frac{1}{\sqrt{2}} \left( \int_{\Omega} f^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} g^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad + \frac{1}{2} \left( \int_{\Omega} f^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} f^{-1}(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad \times \left( \int_{\Omega} g^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} g(az) - g(az^{-1}) \right)^{2} \left( \int_{\Omega} \omega^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad + \frac{1}{2} \left( \int_{\Omega} g^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} g(az) - g(az^{-1}) \right)^{2} \left( \int_{\Omega} \omega^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \\
& \quad \times \left( \int_{\Omega} f^2(az) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} f^{-1}(az) \, d\mu(z) \right)^{\frac{1}{2}}.
\end{align*}

Finally, by using the properties of absolute value, we have
\begin{align*}
0 \leq I(a) & \leq \frac{3}{\sqrt{2}} \int_{\Omega} f(az) \, d\mu(z) + \frac{3}{\sqrt{2}} \int_{\Omega} f(az) \omega(az) \, d\mu(z).
\end{align*}

Inequality (3.9) follows from (3.8)–(3.11).
Corollary 3.1. Suppose the assumptions of Theorem 3.2 are satisfied. If \( g \equiv 1 \) on \( \Omega \) in Theorem 3.2, we have

\[
(3.12) \quad f(a) \int_{\Omega} \omega(az) \, d\mu(z) \leq \int_{\Omega} f(az) \omega(az) \, d\mu(z) + I(a),
\]

where \( I(a) = \frac{1}{2} \int_{\Omega} |f(az) - f(az^{-1})| \omega(az) \, d\mu(z) \). Furthermore,

\[
0 \leq I(a) \leq \min\{B_1, B_2, B_3\},
\]

where

\[
B_1 = \frac{1}{\sqrt{2}} \sqrt{\mu(\Omega)} \left( \int_{\Omega} f^2(az) \omega^2(az) \, d\mu(z) - \int_{\Omega} f(az) f(az^{-1}) \omega^2(az) \, d\mu(z) \right)^{\frac{1}{2}},
\]

\[
B_2 = \frac{1}{\sqrt{2}} \left( \int_{\Omega} f^2(az) \, d\mu(z) - \int_{\Omega} f^2(az) f(az^{-1}) \, d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\Omega} \omega^2(az) \, d\mu(z) \right)^{\frac{1}{2}},
\]

\[
B_3 = \int_{\Omega} f(az) \omega(az) \, d\mu(z).
\]

Definition 3.5. The function \( f : \Omega \to \mathbb{R} \) is said to be a \( P \)-function in \( \Omega \), if \( f(a) \leq f(az) + f(az^{-1}) \) for all \( a \in \Omega \) and \( z \in G \) such that \( az, az^{-1} \in \Omega \).

Theorem 3.3. Let \( G \) be a locally compact group and \( \Omega \subset G \) an open symmetric set relative to \( a \in G \) with \( 0 < \mu(\Omega) < \infty \). Let \( f, g \) be measurable non-negative real valued \( P \)-functions on \( \Omega \) such that \( fg \in L_1(\Omega) \). If \( \omega : \Omega \to \mathbb{R} \) is non-negative symmetric to \( a \), \( \omega \in L_1(\Omega) \) and \( fg \omega \in L_1(\Omega) \), we have

\[
f(a)g(a) \int_{\Omega} \omega(az) \, d\mu(z) \\
\leq 2 \int_{\Omega} f(az)g(az) \omega(az) \, d\mu(z) + 2 \int_{\Omega} f(az)g(az^{-1}) \omega(az) \, d\mu(z) \\
= 2 \int_{\Omega} f(az)g(az) \omega(az) \, d\mu(z) + 2 \int_{\Omega} f(az^{-1})g(az) \omega(az) \, d\mu(z).
\]

Proof. Since \( f \) and \( g \) are \( P \)-functions and \( \omega \) is non-negative and symmetric to \( a \), we have

\[
f(a)g(a) \omega(az) \leq (f(az) + f(az^{-1}))(g(az) + g(az^{-1})) \omega(az) \\
= f(az)g(az) \omega(az) + f(az^{-1})g(az^{-1}) \omega(az) \\
+ f(az)g(az^{-1}) \omega(az) + f(az^{-1})g(az) \omega(az).
\]

Integrating this inequality on \( \Omega \), we get

\[
f(a)g(a) \int_{\Omega} \omega(az) \, d\mu(z) \\
\leq \int_{\Omega} f(az)g(az) \omega(az) \, d\mu(z) + \int_{\Omega} f(az^{-1})g(az^{-1}) \omega(az) \, d\mu(z) \\
+ \int_{\Omega} f(az)g(az^{-1}) \omega(az) \, d\mu(z) + \int_{\Omega} f(az^{-1})g(az) \omega(az) \, d\mu(z).
\]
If have Hence, the proof of the theorem is completed.

\[ + a - a \ (3.13) \]

If \( \omega \) from Theorem 3.1 we have result of Theorem 2.1 holds.

A mapping in connection to Hadamard’s inequality

5. S. S. Dragomir, Quasi-convex functions and Hadamard’s inequalities

3. S. S. Dragomir, C. E. M. Pearce, Midconvex functions in locally compact groups

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\[ (0) \equiv 1 \]

\[ \int \omega(z) \, d\mu(z) \leq 4 \int f(z) \omega(z) \, d\mu(z). \]

COROLLARY 3.3. If we take \( a = e \) and \( \omega \equiv 1 \) on \( \Omega \) in Corollary 3.2 then we have

\[ f(e) \leq \frac{4}{\mu(\Omega)} \int \Omega f(z) \, d\mu(z). \]

Some of the applications of our results are given in the following remarks.

REMARK 3.4. Set \( G = \mathbb{R} \). Since \( \mathbb{R} \) is an abelian additive group, for all \( a, z \in \mathbb{R} \), \( a - z \) and \( a + z \) are points for which \( a \) is the midpoint. Now, if \( a - z = y \) and \( a + z = x \), then \( a = \frac{1}{2} (x + y) \). If we take \( \Omega = [a, b] \), we get \( a = 0 \) and \( y = -x \). Hence, from Theorem 3.1 we have

\[ (3.13) \quad f(0)g(0) \int_{-b}^{b} \omega(x) \, dx \leq \frac{1}{2} \left[ \int_{-b}^{b} f(x)g(x)\omega(x) \, dx + \int_{-b}^{b} f(x)g(-x)\omega(x) \, dx \right]\]

\[ = \frac{1}{2} \left[ \int_{-b}^{b} f(x)g(x)\omega(x) \, dx + \int_{-b}^{b} f(-x)g(x)\omega(x) \, dx \right]. \]

If \( \omega(x) \equiv 1 \) for all \( x \in [-b, b] \) in (3.13), we obtain

\[ f(0)g(0) \leq \frac{1}{4b} \left[ \int_{-b}^{b} f(x)g(x) \, dx + \int_{-b}^{b} f(x)g(-x) \, dx \right]\]

\[ = \frac{1}{4b} \left[ \int_{-b}^{b} f(x)g(x) \, dx + \int_{-b}^{b} f(-x)g(x) \, dx \right]. \]

REMARK 3.5. If in the Theorem 3.1 \( G = \mathbb{R}^n \) with the operation of additive and \( \Omega = X \) is an open bounded symmetric and convex subset of \( \mathbb{R}^n \), then the result of Theorem 2.1 holds.

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