The extended moduli space of special Lagrangian submanifolds

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Abstract

It is well known that the moduli space of all deformations of a compact special Lagrangian submanifold $X$ in a Calabi-Yau manifold $Y$ within the class of special Lagrangian submanifolds is isomorphic to the first de Rham cohomology group of $X$. Reinterpreting the embedding data $X \subset Y$ within the mathematical framework of the Batalin-Vilkovisky quantization, we find a natural deformation problem which extends the above moduli space to the full de Rham cohomology group of $X$.

§1. Introduction

Let $Y$ be a Calabi-Yau manifold of complex dimension $m$ with Kähler form $\omega$ and a nowhere vanishing holomorphic $m$-form $\Omega$. A compact real $m$-dimensional submanifold $X \hookrightarrow Y$ is called special Lagrangian if $\omega|_X = 0$ and $\text{Im } \Omega|_X = 0$. According to McLean [5], the moduli space of all deformations of $X$ inside $Y$ within the class of special Lagrangian submanifolds is a smooth manifold whose tangent space at $X$ is isomorphic to the first de Rham cohomology group $H^1(X, \mathbb{R})$.

Moduli spaces of special Lagrangian submanifolds are playing an increasingly important role in quantum cohomology and related topics. On physical grounds, Strominger, Yau and Zaslow argued [10] that whenever a Calabi-Yau manifold $Y$ has a mirror partner $\hat{Y}$, then $Y$ admits a foliation $\rho : Y^{2m} \rightarrow B^m$ by special Lagrangian tori $T^m$ and $\hat{Y}$ is the compactification of the family of dual tori $\tilde{T}^m$ along the fibres of the projection $\rho$ (for the mathematical account of this construction see [7]).

Recently, the mirror conjecture has been extended by Vafa [13], also on physical grounds, to Calabi-Yau manifolds $\hat{Y}$ equipped with stable vector bundles $W$. According to Vafa, the mirror partner of such a pair $(\hat{Y}, W)$ must be a triple $(Y, X, L)$ consisting of a Calabi-Yau manifold $Y$, a compact special Lagrangian submanifold $X \hookrightarrow Y$ and a flat unitary line bundle $L$ on $X$ together with an isomorphism between the moduli space (with typical tangent space $H^1(\hat{Y}, \text{End } W)$) of all deformations of the holomorphic vector bundle $W \rightarrow \hat{Y}$, and the moduli space (with typical tangent space $H^1(X, \mathbb{R}) \otimes \mathbb{C}$) associated with McLean’s deformations of the embedding $X \hookrightarrow Y$ and deformations of the flat unitary line bundle $L$ on $X$. Actually, Vafa conjectures that much more must be true:

$$H^*(\hat{Y}, \text{End } W) = H^*(X, \mathbb{R}) \otimes \mathbb{C}.$$
This raises a problem of finding a geometric interpretation of the full de Rham co-
homology group of a special Lagrangian submanifold $X \hookrightarrow Y$. Its solution is the main
theme of the present paper. By moving into the mathematical realm of Batalin-Vilkovisky
quantization, we devise, out of the same data $X \hookrightarrow Y$, a deformation problem whose
moduli space has the typical tangent space isomorphic to $H^*(X, \mathbb{R})$ thereby extending
McLean’s moduli space to the full de Rham group. The idea is very simple.

First, out of $Y$ we construct a real $(2m|2m)$-dimensional supermanifold $\mathcal{Y} := \Pi \Omega^1 Y$,
where $\Omega^1 Y$ is the real cotangent bundle and $\Pi$ denotes the parity change functor. The
supermanifold $\mathcal{Y}$ comes equipped with a complex structure and a nowhere vanishing ho-
morphic section, $\hat{\Omega} \in \Gamma(\mathcal{Y}, \text{Ber}_{\text{holo}}(\mathcal{Y}))$, of the holomorphic Berezinian bundle induced by
the holomorphic $m$-form form $\Omega$ on $Y$. We note that if $\mathcal{X} \subset \mathcal{Y}$ is an $(m|m)$-dimensional
real slice, then $\hat{\Omega}$ restricts to $\mathcal{X}$ as a global no-where vanishing section of the bundle
$\mathbb{C} \otimes \text{Ber}(\mathcal{X})$.

Second, we construct a real $(2m|2m + 1)$-dimensional supermanifold $\hat{\mathcal{Y}} := \mathcal{Y} \times \mathbb{R}^{0|1}$
and note that it comes canonically equipped with an odd exact contact structure repre-
sented by a 1-form $\hat{\theta}$. We also note that the Kähler form $\omega$ on $Y$ gives rise to an even
smooth function $\hat{\omega}$ on $\hat{\mathcal{Y}}$ and call an $(m|m)$-dimensional sub-supermanifold $\mathcal{X} \hookrightarrow \hat{\mathcal{Y}}$ special
Legendrian if the following conditions hold

$$
\hat{\theta}|_{\mathcal{X}} = 0, \quad \hat{\omega}|_{\mathcal{X}} = 0, \quad \text{Im}(\hat{\Omega}|_{p(\mathcal{X})}) = 0,
$$

where $p$ denotes the natural projection $\hat{\mathcal{Y}} \rightarrow \mathcal{Y}$.

If $\mathcal{X} \hookrightarrow \hat{\mathcal{Y}}$ is special Legendrian, then $\mathcal{X}_{\text{red}} \hookrightarrow Y$ is special Lagrangian. If $X \hookrightarrow Y$ is
special Lagrangian (with normal bundle denoted by $N$), then the associated superman-
ifold $\mathcal{X} := \Pi N^*$ is a special Legendrian sub-supermanifold of $\hat{\mathcal{Y}}$. However, the corre-
spondence between the special Lagrangian sub-supermanifolds $X \hookrightarrow Y$ and special Legendrian
sub-supermanifolds $\mathcal{X} \hookrightarrow \hat{\mathcal{Y}}$ is not one-to-one — passing from the Calabi-Yau manifold
$Y$ to the associated contact supermanifold $\hat{\mathcal{Y}}$ brings precisely the right amount of new
degrees of freedom to extend McLean’s moduli space $H^1(X, \mathbb{R})$ to the full de Rham group
$H^*(X, \mathbb{R})$.

**Main theorem.** Let $X \hookrightarrow Y$ be a compact special Lagrangian submanifold of a
Calabi-Yau manifold and let $\mathcal{X} = \Pi N^* \hookrightarrow \hat{\mathcal{Y}}$ be the associated special Legendrian sub-
supermanifold of the contact supermanifold. The maximal moduli space $\mathcal{M}$ of deforma-
tions of $\mathcal{X}$ inside $\mathcal{Y}$ within the class of special Legendrian sub-supermanifolds is a smooth
supermanifold whose tangent superspace at $\mathcal{X}$ is canonically isomorphic to $\Pi H^*(X, \mathbb{R})$.

In view of Vafa’s conjectures, it is important to study geometric structures induced
on $\mathcal{M}$ from the original data $X \hookrightarrow Y$ (cf. [3, 11]). It should be also noted that

The paper is organised as follows. In §2 and 3 we study extended moduli spaces of
general compact submanifolds $X$ in a manifold $Y$. In §4 we specialize to the case
when the ambient manifold $Y$ has a symplectic structure and the submanifolds $X \hookrightarrow Y$

\footnote{This also raises much easier problems of extending the moduli space of holomorphic vector bundles $W$ on $Y$ with typical tangent space $H^1(\hat{Y}, \text{End} W)$ to a supermanifold with typical tangent superspace $H^1(\hat{Y}, \text{End} W)$, and extending the moduli space of flat unitary line bundles on $X$ with typical tangent space $H^1(X, \mathbb{R})$ to a supermanifold with typical tangent superspace $H^*(X, \mathbb{R})$; the latter two extensions are very straightforward within the Batalin-Vilkovisky formalism and will be discussed elsewhere.}
are Lagrangian. In §5 we consider another special case when $Y$ is a complex manifold equipped with a nowhere-vanishing holomorphic volume form $\Omega$ and $X \hookrightarrow Y$ is a real slice of $Y$ satisfying $\text{Im} \Omega|_X = 0$. Finally, in §6 we combine all the previous results to prove the Main Theorem.

§2. Extended Kodaira moduli spaces

2.1. Families of compact submanifolds. Let $Y$ and $M$ be smooth manifolds and let $\pi_1 : Y \times M \longrightarrow Y$ and $\pi_2 : Y \times M \longrightarrow M$ be the natural projections. A family of compact submanifolds of the manifold $Y$ with the moduli space $M$ is a submanifold $F \hookrightarrow Y \times M$ such that the restriction, $\nu$, of the projection $\pi_2$ on $F$ is a proper regular map. Thus the family $F$ has the structure of a double fibration

$$Y \overset{\mu}{\hookleftarrow} F \overset{\nu}{\longrightarrow} M,$$

where $\mu \equiv \pi_1 \mid_F$. For each $t \in M$, there is an associated compact submanifold $X_t$ in $Y$ which is said to belong to the family $F$. Sometimes we use a more explicit notation $\{X_t \hookrightarrow Y \mid t \in M\}$ to denote the family $F$ of compact submanifolds. The family $F$ is called maximal if for any other family $\tilde{F} \hookrightarrow Y \times \tilde{M}$ such that $\mu \circ \nu^{-1}(t) = \tilde{\mu} \circ \tilde{\nu}^{-1}(\tilde{t})$ for some points $t \in M$ and $\tilde{t} \in \tilde{M}$, there is a neighbourhood $U \subset \tilde{M}$ of the point $\tilde{t}$ and a smooth map $f : \tilde{U} \rightarrow M$ such that $\tilde{\mu} \circ \tilde{\nu}^{-1}(\tilde{t}') = \mu \circ \nu^{-1}((f(\tilde{t}'))) \text{ for every } \tilde{t}' \in \tilde{U}$.

Similar definitions can be made in the category of complex manifolds, category of (complex) (super)manifolds and category of analytic (super)spaces.

2.2. The Kodaira map. Consider a 1-parameter family, $F \hookrightarrow Y \times M$, of compact sub(super)manifolds in a (super)manifold $Y$, where $M = \mathbb{R}^{1|0}$ or $\mathbb{R}^{0|1}$ with the natural coordinate denoted by $t$ (such a family is often called a 1-parameter deformation of the sub(super)manifold $X = \mu \circ \nu^{-1}(0)$). There is a finite covering $\{U_i\}$ of $F$ such that the restriction to each $U_i$ of the ideal sheaf $J_F$ of $F \hookrightarrow Y \times M$ is finitely generated, say $J_{F|U_i} = \langle f^\alpha_i \rangle$, $\alpha = 1, \ldots, \text{codim } F$. It is easy to see that the family $\{\frac{\partial f^\alpha_i}{\partial t} \mod J_F\}$ defines a global section of the normal bundle, $N_F$, of the embedding $F \hookrightarrow Y \times M$ and hence gives rise to a morphism of sheaves,

$$k : \quad TM \longrightarrow \quad \nu_\ast N_F$$

$$\frac{\partial}{\partial t} \quad \longrightarrow \quad \{\frac{\partial f^\alpha_i}{\partial t} \mod J_F\}.$$ 

This morphism, or rather its restriction

$$k_t : T_t M \longrightarrow H^0(X_t, N_t),$$

where $N_t \simeq N_{F|\nu^{-1}(t)}$ is the normal bundle of $X_t \hookrightarrow Y$, is called the Kodaira map.

If $M$ is the 1-tuple point $\mathbb{R}[t]/t^2$, then the family $F$ is called an infinitesimal deformation of $X = \mu \circ \nu^{-1}(0)$ in $Y$. The Kodaira map establishes a one-to-one correspondence between all possible infinitesimal deformations of $X$ inside $Y$ and the vector superspace $H^0(X, N)$. Often in this paper we shall be interested in deformations of $X$ inside $Y$ within
a class of special (say, complex, Lagrangian, Legendrian, etc.) submanifolds. The associated set of all possible infinitesimal deformations of $X$ is a vector subspace of $H^0(X,N)$ called the Zariski tangent space at $X$ to the moduli space of (special) compact submanifolds. Note that we do not require that any element of the Zariski tangent space at $X$ necessarily exponentiates to a genuine 1-parameter deformation of $X$. Put another way, the Zariski tangent space to the moduli space $M$ makes sense even when $M$ does not exist as a smooth manifold!

2.3. From manifolds to supermanifolds. Given a compact submanifold, $X \hookrightarrow Y$, of a smooth manifold $Y$. The associated exact sequence

$$0 \rightarrow TX \rightarrow TY|_X \rightarrow N \rightarrow 0$$

implies the canonical map

$$\Lambda^*TY|_X \rightarrow \Lambda^*N \rightarrow 0$$

which in turn implies the canonical embedding, $\mathcal{X} \hookrightarrow \mathcal{Y}$, of the associated supermanifolds $\mathcal{X} := (X, \Lambda^*N)$ and $\mathcal{Y} := (Y, \Lambda^*TY)$ (cf. [8, 9]). In more geometrical terms, $\mathcal{X} \simeq \Pi N^*$, $\mathcal{Y} \simeq \Pi\Omega^1 Y$ and $\mathcal{X} \hookrightarrow \mathcal{Y}$ corresponds just to the natural inclusion, $\Pi N^* \subset \Pi\Omega^1 M|_X$. The supermanifold $\mathcal{Y}$ comes equipped canonically with an even Liouville 1-form $\theta$ defined, in a natural local coordinate system $(x^a, \psi_a \simeq \Pi \partial/\partial x^a)$ on $\mathcal{Y}$, as follows

$$\theta = \sum_{a=1}^n dx^a \psi_a, \quad n = \dim Y.$$ 

The odd two-form

$$\eta := d\theta = -\sum_{a=1}^n dx^a \wedge d\psi_a$$

is non-degenerate and hence equips $\mathcal{Y}$ with an odd symplectic structure.

A $(p,n-p)$-dimensional sub-supermanifold $\mathcal{X} \hookrightarrow \mathcal{Y}$ is called Lagrangian if $\eta|_X = 0$ (this implies, in particular, that $\theta|_X$ is closed). It is called exact Lagrangian if $\theta|_X$ is an exact 1-form on $\mathcal{X}$.

2.3.1. Lemma. For any submanifold $X \hookrightarrow Y$, the associated sub-supermanifold $\mathcal{X} \hookrightarrow \mathcal{Y}$ is exact Lagrangian.

Proof. Assume $\dim X = p$. We can always choose a local coordinate system $(U, x^a)$ in a tubular neighbourhood $U$ of (a part of) $X$ inside $Y$ in such a way that $X \cap U = \{x^a = 0, \ a = p+1, \ldots, n\}$. Then the normal bundle $N$ of $X \hookrightarrow Y$ is locally generated by $\partial/\partial x^a$ with $a = p+1, \ldots, n$. Hence $\mathcal{X} \hookrightarrow \mathcal{Y}$ is locally given by the equations $x^a = 0, \psi_b = 0$ where $a = p+1, \ldots, n$ and $b = 1, \ldots, p$. It is now obvious that $\theta|_X = 0$. Finally, $\dim \mathcal{X} = (p, n-p)$. $\square$

2.3.2. Remark. It also follows from the above proof that, for any submanifold $X \hookrightarrow Y$ with the normal bundle $N_X$, $\theta|_{\Pi N_X} = 0$. It is not hard to check that the reverse is also true: if $\mathcal{X} \hookrightarrow \mathcal{Y}$ is an $(p|n-p)$-dimensional sub-supermanifold such that $\theta|_X = 0$, then $\mathcal{X} = \Pi N^* X$ for some submanifold $X \hookrightarrow Y$.

2.4. The extended Kodaira map. Let $\mathcal{X}$ be a Lagrangian sub-supermanifold of a supermanifold $\mathcal{Y}$ equipped with an odd symplectic structure $\eta$. Then, as usually, one
gets an odd isomorphism \( j : \Omega^1 \mathcal{X} \xrightarrow{\eta^{-1}} \mathcal{N} \), where \( \mathcal{N} \) is the normal bundle of \( \mathcal{X} \hookrightarrow \mathcal{Y} \). In particular, there is a monomorphism of sheaves,

\[
i : \mathcal{O}_{\mathcal{X}}/\mathbb{R} \xrightarrow{j_{od}} \mathcal{N},
\]

where \( d \) is the exterior derivative.

Consider now a one (even or odd) parameter family of compact exact Lagrangian sub-supermanifolds of the supermanifold \( \mathcal{Y} = \Pi \Omega^1 \mathcal{Y} \), i.e., a double fibration

\[
\mathcal{Y} \leftarrow^\mu \mathcal{F} \twoheadrightarrow^\nu M,
\]

with \( \nu \) being a proper submersion and \( \mathcal{X}_t := \mu \circ \nu^{-1}(t) \) being a compact exact Lagrangian sub-supermanifold of \( (\mathcal{Y}, \eta) \) for every \( t \in M \subset \mathbb{R}^{1|0} \) or \( \mathbb{R}^{0|1} \).

**2.4.1. Lemma.** For the family \( \{ \mathcal{X}_t \hookrightarrow \mathcal{Y} | t \in M \} \) as above the Kodaira map \( k_t : T_t M \rightarrow H^0(\mathcal{X}_t, \mathcal{N}_t) \) factors as follows

\[
k_t : T_t M \xrightarrow{k'} H^0(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t})/\mathbb{R} \xrightarrow{i} H^0(\mathcal{X}_t, \mathcal{N}_t).
\]

**Proof.** Since \( \mu^*(\eta)|_{\mathcal{X}_t} = 0 \), we have

\[
\mu^*(\eta) = A \wedge dt
\]

for some 1-form \( A \) on \( \mathcal{F} \) whose restriction to \( \nu^{-1}(t) \) represents, under the isomorphism \( j : \Omega^1 \mathcal{X}_t \xrightarrow{\eta^{-1}} \mathcal{N}_t \), the normal vector field \( k_t(\partial/\partial t) \). On the other hand,

\[
\mu^*(\theta) = \Psi dt + dB,
\]

for some smooth functions \( \Psi \) and \( B \) on \( \mathcal{F} \) with parities \( \tilde{\Psi} = \tilde{t} + 1 \) and \( \tilde{B} = 1 \). Thus \( A = d\Psi \) completing the proof. \( \square \)

**2.4.2. Corollary.** Let \( \mathcal{X} \) be a compact exact Lagrangian sub-supermanifold of an odd symplectic supermanifold \( \mathcal{Y} \). Then the Zariski tangent space at \( \mathcal{X} \) to the moduli space of all deformations of \( \mathcal{X} \) within the class of exact Lagrangian sub-supermanifolds is isomorphic to \( H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})/\mathbb{R} \).

2.4.3. **Definition.** If \( \mathcal{Y} = \Pi \Omega^1 \mathcal{Y} \) and \( \mathcal{X}_t \simeq \Pi N^*_t \), where \( N_t \) is the normal bundle of some submanifold \( \mathcal{X}_t \hookrightarrow \mathcal{Y} \), then \( H^0(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t})/\mathbb{R} \simeq H^0(\mathcal{X}_t, \Lambda^* N_t)/\mathbb{R} \). The associated map \( k' : T_t M \rightarrow H^0(\mathcal{X}_t, \Lambda^* N_t)/\mathbb{R} \) is called the extended Kodaira map.

2.5. **Extended Kodaira moduli space.** Kodaira \([\text{I}]\) proved that if \( \mathcal{X} \hookrightarrow \mathcal{Y} \) is a compact complex submanifold of a complex manifold with \( H^1(\mathcal{X}, \mathcal{N}) = 0 \), then there exists a maximal moduli space \( M \) parametrizing all possible deformations of \( \mathcal{X} \) inside \( \mathcal{Y} \) whose tangent space at the point \( \mathcal{X} \) is isomorphic to \( H^0(\mathcal{X}, \mathcal{N}) \).

\(^{2}\)Here and elsewhere \( \tilde{\text{stands for the parity of the kernel symbol.}} \)
With the same data $X \hookrightarrow Y$ one associates a pair $\mathcal{X} = \Pi N^* \hookrightarrow \mathcal{Y} = \Pi \Omega^1 Y$ and asks for all possible holomorphic deformations of $\mathcal{X}$ inside $(\mathcal{Y}, \eta)$ within the class of complex exact Lagrangian sub-supermanifolds.

2.5.1. Theorem. Let $X \hookrightarrow Y$ be a compact complex submanifold of a complex manifold and $\mathcal{X} \hookrightarrow \mathcal{Y}$ the associated compact complex exact Lagrangian sub-supermanifold. If $H^1(X, \Lambda^k N) = 0$ for all $k \geq 1$, then there exists a maximal moduli space $\mathcal{M}$, called the extended Kodaira moduli space, which parametrizes all possible deformations of $\mathcal{X}$ inside $(\mathcal{Y}, \eta)$ within the class of complex exact Lagrangian sub-supermanifolds. Its tangent space at the point $X$ is canonically isomorphic to $\sum_{k \geq 1} H^0(X, \Lambda^k N)$, with the following $\mathbb{Z}_2$-grading: $[T_X \mathcal{M}]_0 = \sum_{k \in 2\mathbb{Z}+1} H^0(X, \Lambda^k N)$ and $[T_X \mathcal{M}]_1 = \sum_{k \in 2\mathbb{Z}} H^0(X, \Lambda^k N)$.

Proof is routine, cf. [4, 6].

2.5.2. Example. Let $X$ be a projective line $\mathbb{CP}^1$ embedded into a complex 3-fold $Y$ with normal bundle $N = \mathcal{O}(1) \oplus \mathcal{O}(1)$. In this case the Kodaira moduli space is a complex 4-fold $\mathcal{M}$ canonically equipped, according to Penrose, with a self-dual conformal structure, while the extended Kodaira moduli space $\mathcal{M}$ is a $(4|3)$-dimensional supermanifold isomorphic to $\Pi \Omega^2_+ M$, where $\Omega^2_+ M$ is the bundle of self-dual 2-forms on $M$.

§3. Restoring the lost constants

3.1. Odd contact structure. Let $X$ be a compact submanifold of a manifold $Y$. It is shown in §2 that the Zariski tangent space to the extended moduli space of Lagrangian deformations of $X = \Pi N^*$ inside $\mathcal{Y} = \Pi \Omega^1 Y$ is $\Pi H^0(X, \Lambda^* N)/\mathbb{R}$. One can easily restore the lost constants $\mathbb{R}$ by extending $\mathcal{Y}$ to an odd contact supermanifold $\hat{\mathcal{Y}}$ and studying Legendrian families of compact sub-supermanifolds in $\hat{\mathcal{Y}}$.

Consider $\hat{\mathcal{Y}} := \mathcal{Y} \times \mathbb{R}^{01}$ and define a 1-form on $\hat{\mathcal{Y}}$

$$\hat{\theta} = d\varepsilon + p^*(\theta),$$

where $p : \hat{\mathcal{Y}} \rightarrow \mathcal{Y}$ is the natural projection and $\varepsilon$ is the standard coordinate on $\mathbb{R}^{01}$. The form $\hat{\theta}$ defines an odd contact structure on $\hat{\mathcal{Y}}$.

3.1.1. Lemma. For any submanifold $X \hookrightarrow Y$, the associated sub-supermanifold $\mathcal{X} = \Pi N^* \hookrightarrow \hat{\mathcal{Y}}$ is Legendrian with respect to the odd contact structure $\theta$.

Proof. $\hat{\theta}|_{\mathcal{X}} = d\varepsilon|_{\mathcal{X}} + p^*(\theta)|_{\mathcal{X}} = 0 + 0 = 0$. $\square$

Thus one can associate with data $X \hookrightarrow Y$ the moduli space $\mathcal{M}$ of all possible deformations of $\mathcal{X}$ inside $\hat{\mathcal{Y}}$ within the class of Legendrian sub-supermanifolds.

3.1.2. Proposition. The Zariski tangent space to $\mathcal{M}$ at $\mathcal{X}$ is $\Pi H^0(X, \Lambda^* N)$.

Proof. If $\hat{\mathcal{Y}} \xleftarrow{\hat{\mu}} \mathcal{F} \xrightarrow{\nu} M \subset \mathbb{R}^{10}$ or $\mathbb{R}^{01}$ is a 1-parameter family of compact Legendrian sub-supermanifolds, then

$$\hat{\mu}^*(\hat{\theta}) = \Psi dt$$
for some $\Psi \in \Gamma(\mathcal{F}, \mathcal{O}_F)$ with $\bar{\Psi} = \tilde{t} + 1$ (compare this with $\mu^\ast(\eta) = d\Psi \wedge dt$ in 2.4.1). The restriction of $\Psi$ to $\nu^{-1}(t)$ represents the image of $\partial / \partial t$ under the extended Kodaira map. □

3.2. Remark. If $\{\mathcal{X}_t \hookrightarrow \hat{Y} | t \in \mathcal{M}\}$ is a family of compact Legendrian sub-supermanifolds, then $\{p(\mathcal{X}_t) \hookrightarrow Y | t \in \mathcal{M}\}$ is a family of exact Lagrangian sub-supermanifolds.

3.3. An important observation. If $(Y, \omega)$ is a symplectic manifold and $X \hookrightarrow Y$ a Lagrangian submanifold with respect to $\omega$, then the normal bundle $N$ is canonically isomorphic to $\Omega^1 X$. Thus the associated extended Zariski tangent space is isomorphic to $\Omega^* X$.

§ 4. Even + odd symplectic structures

4.1. Isotropic Lagrangian sub-supermanifolds. In this section we assume that $Y$ is an even $2m$-dimensional symplectic manifold. The symplectic 2-form $\omega$ on $Y$ gives rise to a global even function $\hat{\omega}$ on the associated odd symplectic supermanifold $\hat{Y} = \Pi \Omega^1 Y$ (and hence on $\hat{\mathcal{Y}} = \mathcal{Y} \times \mathbb{R}^{0|1}$) defined, in a natural local coordinate system $(x^a, \psi_a = \Pi \partial / \partial x^a)$, as follows

$$\hat{\omega} = \sum_{a,b=1}^{2m} \omega^{ab}(x) \psi_a \psi_b,$$

where $\omega^{ab}(x)$ is the matrix inverse to the matrix $\omega_{ab}(x)$ of components of $\omega$ in the basis $dx^a$. The latter function gives rise to an odd Hamiltonian vector field $Q$ on $\hat{Y}$ (or a contact vector field $Q$ on $\hat{\mathcal{Y}}$) defined by

$$Q \downarrow \eta = d\hat{\omega},$$

and is given, in a natural local coordinate system, by

$$Q = \sum_{a,b} \omega^{ab} \psi_b \frac{\partial}{\partial x^a} + \sum_{a,b,c,d,e} w^{ad} \frac{\partial \omega_{bc}}{\partial x^a} \omega^{ce} \psi_c \psi_e \frac{\partial}{\partial \psi_b}.$$

Differentials forms on $Y$ can be identified with smooth functions on the supermanifold $\Pi TY$, $\Gamma(Y, \Omega^* Y) = \Gamma(\Pi TY, \mathcal{O}_{\Pi TY})$. Under this identification the de Rham differential $d : \Omega^* Y \rightarrow \Omega^* Y$ corresponds to an odd vector field $d$ on $\Pi TY$ satisfying $d^2 = 0$. The even symplectic form $\omega$ establishes an isomorphism $\phi : \Pi TY \rightarrow \mathcal{Y}$ and hence maps $d$ into an odd vector field $\phi_* d$ on $Y$ which, as it is not hard to check, coincides precisely with $Q$. This observation implies, in particular, that $Q^2 = 0$ and $Q \bar{\omega} = 0$.

4.1.1 Definition. A Lagrangian (resp. Legendrian) sub-supermanifold of $(\mathcal{Y}, \eta)$ (resp. $(\hat{\mathcal{Y}}, \hat{\theta})$) is called $\omega$-isotropic if $\hat{\omega} |_{\mathcal{X}} = 0$ (resp. $\hat{\omega} |_{p(\mathcal{X})} = 0$).

If $\mathcal{X} \hookrightarrow \mathcal{Y}$ is $\omega$-isotropic, then $Q |_{\mathcal{X}} \in \Gamma(\mathcal{X}, T\mathcal{X})$.

4.1.2 Lemma. Let $(Y, \omega)$ be a symplectic manifold and let $\mathcal{Y} := \Pi \Omega^1 Y$.

(i) If $\mathcal{X}$ is a compact $(m|m)$-dimensional $\omega$-isotropic Lagrangian submanifold $(\mathcal{Y}, \eta)$, then $\mathcal{X}_{\text{red}}$ is a compact Lagrangian submanifold of $(Y, \omega)$.
(ii) Let \( X \) be a compact Lagrangian submanifold of \((Y, \omega)\). Then the associated compact \((m|m)\)-dimensional sub-supermanifold \( \mathcal{X} := \Pi N^*_X \hookrightarrow Y \) is \( \omega \)-isotropic. Moreover, under the isomorphism \( \Gamma(\mathcal{X}, \mathcal{O}_X) = \Gamma(X, \Omega^*X) \) the vector field \( Q|_X \) goes into the usual de Rham differential \( d \) on \( X \).

Proof is very straightforward when one uses Darboux coordinates.

4.2 Normal exponential map. Let \( X \) be an \( r \)-dimensional compact manifold of an \( n \)-dimensional manifold \( Y \) and let \( \mathcal{X} = \Pi N^*_X \) be the associated Lagrangian sub-supermanifold of the odd symplectic supermanifold \( (Y = \Pi \Omega^1Y, \eta) \).

4.2.1. Lemma. There exist

- a tubular neighbourhood \( U \) of \( \mathcal{X} \) in \( Y \),
- a tubular neighbourhood \( \mathcal{V} \) of \( 0_{\mathcal{X}} \) in \( \Pi \Omega^1\mathcal{X} \), where \( 0_{\mathcal{X}} \simeq \mathcal{X} \) is the zero section of the bundle \( \Pi \Omega^1\mathcal{X} \to \mathcal{X} \),
- a diffeomorphism \( \exp : \mathcal{V} \to U \),

such that

(i) \( \exp|_{0_{\mathcal{X}}} : \mathcal{X} \to \mathcal{X} \) is the identity map, and

(ii) \( \exp^*(\theta) - \theta_0 = dF \) for some \( F \in \Gamma(\mathcal{X}, \mathcal{O}_X) \), where \( \theta \) is the Liouville form on \( \Pi \Omega^1Y \) and \( \theta_0 \) is the Liouville form on \( \Pi \Omega^1\mathcal{X} \). In particular, \( \exp^*(\eta) = \eta_0 \), where \( \eta_0 \) is the natural odd symplectic structure on \( \Pi \Omega^1\mathcal{X} \).

Proof. There is a tubular neighbourhood \( U \) of \( \mathcal{X}_{red} \) in \( Y \) which can be identified via the normal exponential map with a tubular neighbourhood \( V \subset N \) of the zero section of the normal bundle \( N \) of \( X \) in \( Y \). These neighbourhoods and the exponential map have a canonical extension to the map \( \exp : U \to \mathcal{V} \) which has the property (i). We only have to check the validity of (ii). Let \((x^\alpha, x^{\dot{\alpha}})\), \( \alpha = 1, \ldots, r, \dot{\alpha} = r + 1, \ldots, n \) be a local trivialisation of \( N \), where \( x^\alpha \) are local coordinates on the base of \( N \) and \( x^{\dot{\alpha}} \) are the fibre coordinates. In the associated local coordinate system \((x^\alpha, x^{\dot{\alpha}}, \psi_\alpha := \Pi \partial/\partial x^\alpha, \psi_{\dot{\alpha}} := \Pi \partial/\partial x^{\dot{\alpha}})\) on \( V \subset \Pi \Omega^1\mathcal{X} \) the zero section \( 0_{\mathcal{X}} \) is given by the equations \( x^{\dot{\alpha}} = \psi_\alpha = 0 \). We have

\[
\exp^*(\theta) = dx^\alpha \Pi \frac{\partial}{\partial x^\alpha} + dx^{\dot{\alpha}} \Pi \frac{\partial}{\partial x^{\dot{\alpha}}} = dx^\alpha \psi_\alpha + dx^{\dot{\alpha}} \psi_{\dot{\alpha}},
\]

and

\[
\theta_0 = dx^\alpha \Pi \frac{\partial}{\partial x^\alpha} + d\psi_\alpha \Pi \frac{\partial}{\partial \psi_\alpha} = dx^\alpha \psi_\alpha - d\psi_{\dot{\alpha}} x^{\dot{\alpha}}.
\]

Hence \( \exp^*(\theta) - \theta_0 = d(\psi_{\dot{\alpha}} x^{\dot{\alpha}}) \). Since \( \psi_{\dot{\alpha}} x^{\dot{\alpha}} \) is an invariant, the statement follows. \( \Box \)

4.2.2. Remark. The above Lemma establishes a one-to-one correspondence between nearby to \( \mathcal{X} \) exact Lagrangian sub-supermanifolds and global exact differential forms on \( \mathcal{X} \). Note, however, that this correspondence is not canonical but depends on the choice of the normal exponential map \( \exp : \mathcal{V} \to U \). If \( f \) is a global odd section of \( \mathcal{O}_\mathcal{X} \) (such that
\( df \in V \subset \Pi \Omega^1 \mathcal{X} \) and \( \mathcal{X}_{df} \to \mathcal{Y} \) is the associated Lagrangian sub-supermanifold, then we have a diffeomorphism

\[
\exp_{df} : \mathcal{X} \xrightarrow{df} V \xrightarrow{\exp} \mathcal{X}_{df}.
\]

Consider now a particular case when \( \mathcal{Y} \) is a \( 2m \)-dimensional symplectic manifold \((\mathcal{Y}, \omega)\) and \( X \hookrightarrow \mathcal{Y} \) is a compact Lagrangian submanifold with respect to \( \omega \). In this case the normal bundle \( N \) of \( X \) in \( Y \) is isomorphic to \( \Omega^1 X \) and hence the total space of \( N \) is naturally a symplectic manifold implying that \( \Pi \Omega^1 \mathcal{X} \) (with \( \mathcal{X} = \Pi N^* \)) comes canonically equipped with an odd vector field \( Q_0 \) such that \( Q_0^2 = 0 \) (cf. subsection 4.1). Since the normal exponential map \( N \supset V \xrightarrow{\exp} U \subset \mathcal{Y} \) can be chosen to be a symplectomorphism, the associated extended exponential map \( \Pi \Omega^1 \mathcal{X} \supset V \xrightarrow{\exp} U \subset \Pi \Omega^1 \mathcal{Y} \) can be chosen to satisfy the additional property

\[
\exp_*(Q_0) = Q.
\]

Note also that the isomorphism \( N = \Omega^1 X \) implies \( \mathcal{X} = \Pi T X \) which in turn implies \( \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) = \Omega^* X \). Then we have

4.2.3. Lemma. For any \((\mathcal{U}, \mathcal{V}, \exp)\) as above and any exact Lagrangian submanifold \( \mathcal{X}_{df} \hookrightarrow \mathcal{U} \), the function \( \exp_{df}^* (\tilde{\omega} |_{\mathcal{X}_{df}}) \in \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) = \Omega^* X \) defines a closed (non-homogeneous, in general) differential form on \( \mathcal{X} \).

Proof. Let \( \phi_{df} : \Pi \Omega^1 \mathcal{X} \to \Pi \Omega^1 \mathcal{X} \) be a translation by \( df \) along the fibres of the projection \( \Pi \Omega^1 \mathcal{X} \to \mathcal{X} \). In the natural coordinates on \( \Pi \Omega^1 \mathcal{X} \) we have

\[
[Q_0 - (\phi_{df})_* Q_0] \exp^*(\tilde{\omega}) = \left[ \sum_{\alpha} \left( \frac{\partial f}{\partial x^\alpha} + \sum_{\beta} \psi_{\beta} \frac{\partial^2 f}{\partial x^\beta \partial \psi_\alpha} \right) \frac{\partial}{\partial x^\alpha} - \sum_{\alpha, \beta} \psi_{\beta} \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} \frac{\partial}{\partial \psi_\alpha} \right] \sum_{\gamma} \psi_\gamma \psi_\sigma = 0,
\]

and hence

\[
Q_0 |_X \left( \exp_{df}^* (\tilde{\omega} |_{\mathcal{X}_{df}}) \right) = Q_0 |_X \left( \phi_{df}^* \circ \exp^* (\tilde{\omega}) \right) |_X = [\phi_{df}^* \circ (Q_0 \exp^* (\tilde{\omega}))] |_X = [\phi_{df}^* \circ \exp^* (\tilde{\omega})] |_X = 0.
\]

Then the statement follows from the fact that under the isomorphism \( \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X}) = \Omega^* X \) the vector field \( Q_0 |_X \in \Gamma(\mathcal{X}, T \mathcal{X}) \) goes into the de Rham differential on \( \Omega^* X \) (cf. Lemma 4.1.2(ii)). \( \square \)

4.3. Moduli space of isotropic sub-supermanifolds. Given a compact Lagrangian submanifold \( \mathcal{X} \) of a symplectic manifold \( (\mathcal{Y}, \omega) \). With these data one may associate the extended moduli space \( \mathcal{M} \) of all possible deformations of \( \mathcal{X} = \Pi N^* \) inside \( \mathcal{Y} \) within the class of Legendrian, \( \omega \)-isotropic sub-supermanifolds.
4.3.1. Theorem. The Zariski tangent space to $M$ is $\Pi H^0(X, \Omega^\ast X_{\text{closed}})$, where $\Omega^\ast X_{\text{closed}}$ is the sheaf of closed differential forms on $X$.

Proof. Let $\{X_t \hookrightarrow \hat{Y}| t \in M\}$ be a 1-parameter family of $\omega$-isotropic Legendrian submanifolds, and let

$$\hat{Y} \leftarrow^\mu F \to^\nu M,$$

be an associated 1-parameter family of Lagrangian, $\omega$-isotropic sub-supermanifolds. The vector field $V_f$ on $\hat{Y}$ gives rise to a vector field on $\hat{Y} \times M$ (denoted by the same letter) which is tangent to $F \hookrightarrow \hat{Y} \times M$. We have

$$V_f \upharpoonright F \mu^\ast(\eta) = (V_f \upharpoonright F \Psi \upharpoonright d) \wedge dt,$$

implying

$$\mu^\ast(df) = (V_f \upharpoonright F \Psi)dt.$$

Since $\mu^\ast(df) = d(f \upharpoonright F) = 0$, we get $V_f \upharpoonright F \Psi = 0$. Finally, the required statement follows 4.1.1(ii) which says that $V_f \upharpoonright F$ is essentially the de Rham differential. $\square$

§5. Moduli spaces of special real slices

5.1. Batalin-Vilkovisky structures. Let $\mathcal{Y}$ be an $(n|n)$-dimensional compact supermanifold equipped with an odd symplectic form $\eta$ and an even nowhere-vanishing section $\mu$ of the Berezinian bundle $\text{Ber}(\mathcal{Y})$. Such data have been extensively studied by A.S. Schwarz in [8, 9] in the context of Batalin-Vilkovisky quantization.

The volume form $\mu$ induces the Berezin integral, $\int_\mu f$, on smooth functions $f$ on $\mathcal{Y}$. In particular, $\mu$ gives rise to a divergence operator $\text{div} \, V$ on smooth vector fields $V$ on $\mathcal{Y}$ which can be characterized by the formula [2]

$$\int (\text{div} \, V) f \mu = - \int V(f) \mu.$$

If $x^a$ is a local coordinate system on $\mathcal{Y}$ and $D^a(dx^a)$ the associated local basis of $\text{Ber}(\mathcal{Y})$, then $\mu = \rho(x) D^a(dx^a)$ for some even nowhere-vanishing even function $\rho(x)$ and

$$\text{div} \, V = \frac{1}{\rho} (-1)^{\tilde{a}(1+\tilde{V})} \frac{\partial (V^a \rho)}{\partial x^a},$$

where $\tilde{a}$ is the parity of $x^a$ and $V^a$ are the components of $V$ in the basis $\partial/\partial x^a$. Another possible definition, which also works in the holomorphic category, is

$$\text{div} \, V = \frac{L_V \mu}{\mu},$$

where $L_V$ stands for the Lie derivative along the vector field $V$.

In particular, if $V_f$ is the hamiltonian vector field on $\mathcal{Y}$ associated to a smooth function $f \in \Gamma(\mathcal{Y}, \mathcal{O}_Y)$, then one defines a second order operator,

$$\Delta f := \frac{1}{2} \text{div} \, V_f.$$
Note that this operator depends solely on \( \mu \) and \( \eta \). The situations when \( \Delta^2 = 0 \) are of special interest in the context of Batalin-Vilkovisky quantization. The data \((\mathcal{Y}, \eta, \mu)\) with property \( \Delta^2 = 0 \) are sometimes called \( SP \)-manifolds [3, 4] or Batalin-Vilkovisky supermanifolds [5]. The structure \((\mathcal{Y}, \eta, \mu)\) which arises in the context of Calabi-Yau manifolds does actually satisfy the requirement \( \Delta^2 = 0 \), see §6.

5.2. Integration on Lagrangian sub-supermanifolds. Let \( \mathcal{Y} \) again be an \((n|n)\)-dimensional compact oriented supermanifold equipped with an odd symplectic form \( \eta \) and an even nowhere-vanishing section \( \Omega \) of the Berezinian bundle \( \text{Ber}(\mathcal{Y}) \), and let \( \mathcal{X} \hookrightarrow \mathcal{Y} \) be a compact \((r|n-r)\)-dimensional Lagrangian sub-supermanifold. Then the extension

\[
0 \longrightarrow \mathcal{T}\mathcal{X} \longrightarrow \mathcal{T}\mathcal{Y}|_{\mathcal{X}} \longrightarrow \mathcal{N} \longrightarrow 0
\]

and the isomorphism \( \mathcal{N} \cong \Pi \Omega^1 \mathcal{X} \) imply

\[
\text{Ber}(\mathcal{Y})|_{\mathcal{X}} = \text{Ber}(\mathcal{X})^{\otimes 2}.
\]

Thus the volume form \( \widehat{\Omega} \) on \( \mathcal{Y} \) induces a volume form on \( \mathcal{X} \) which we denote by \( \widehat{\Omega}^{1/2} \). A possible problem with taking the square root is overcome with the assumption that \( \mathcal{Y} \) is oriented; a clear and explicit construction of \( \widehat{\Omega}^{1/2} \) is given by A.S. Schwarz in [8].

As an example, let us consider the case when \( \mathcal{Y} = \Pi \Omega^1 \mathcal{Y} \), where \( \mathcal{Y} \) is an \( n \)-dimensional compact manifold equipped with a nowhere-vanishing \( n \)-form \( \Omega \). The latter gives rise, via the isomorphism \( \text{Ber}(\mathcal{Y}) \cong \text{Det}(\mathcal{Y})^{\otimes 2} \), to a volume form \( \widehat{\Omega} \) on \( \mathcal{Y} \). If \( x^a \) is a local coordinate system in which \( \Omega = \alpha(x) dx^1 \wedge \ldots \wedge dx^n \), then, in the associated local coordinate system \((x^a, \psi_a := \Pi \partial/\partial x^a)\) on \( \mathcal{Y} \),

\[
\widehat{\Omega} = \alpha^2(x) D^*(dx^a, d\psi_a).
\]

In particular, if \( \mathcal{X} \hookrightarrow \mathcal{Y} \) is a Lagrangian sub-supermanifold given locally by the equations \( x^b = 0, \psi_e = 0, b = r + 1, \ldots, n, e = 1, \ldots, r \), then \( \widehat{\Omega}^{1/2} = \alpha(x)|_{\mathcal{X}} \Pi \partial/\partial x^a \).

There is a natural morphism of sheaves,

\[
F : \Omega^* \mathcal{Y} \longrightarrow \mathcal{O}_\mathcal{Y}
\]

\[
\sum w_{a_1 \ldots a_k} dx^{a_1} \wedge \ldots \wedge dx^{a_k} \longrightarrow \sum \alpha(x)^{-1} w_{a_1 \ldots a_k} \varepsilon^{a_1 \ldots a_k a_{k+1} \ldots a_n} \psi_{a_{k+1}} \ldots \psi_n,
\]

where \( \varepsilon^{a_1 \ldots a_k a_n} \) is the antisymmetric tensor with \( \varepsilon^{1 \ldots n} = 1 \). One has [12, 3, 4],

\[
F(dw) = \Delta F(w),
\]

where \( \Delta \) is the Batalin-Vilkovisky operator on \((\mathcal{Y}, \widehat{\Omega}, \eta)\).

5.2.1. Lemma Let \( \mathcal{Y} \) be a manifold \( \mathcal{Y} \) equipped with a nowhere vanishing volume form \( \Omega \) and let \( \mathcal{Y} = \Pi \Omega^1 \mathcal{Y} \). If, for any compact submanifold \( X \hookrightarrow \mathcal{Y} \), the function \( \Phi \in \Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \) is such that

\[
\int_{\Pi \mathcal{X}} \Phi \widehat{\Omega}^{1/2} = 0,
\]

then \( \Phi = \Delta \Psi \) for some \( \Phi \in \Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \).
Proof. We may assume for simplicity that $\Phi$ is homogeneous in odd coordinates $\psi_a$, i.e. that $\Phi = F(w)$ for some $k$-form on $Y$. According to A.S. Schwarz $[8]$, 

$$\int_{\Pi N^*} \Phi \hat{\Omega}^{1/2} = \int_X w.$$ 

Since this vanishes for any compact submanifold $X \hookrightarrow Y$, $w = ds$ for some $(k-1)$-form $s$ on $Y$. Then $\Phi = F(w) = F(ds) = \Delta F(s)$. $\square$

### 5.3. Holomorphic volume forms

Let $Y$ be an $m$-dimensional complex manifold equipped with a nowhere-vanishing holomorphic $m$-form $\Omega$. Then the associated $(m|m)$-dimensional complex supermanifold $\mathcal{Y} = \Pi \Omega^1_c Y$ comes equipped with two natural odd symplectic structures. The first one is holomorphic and is represented, in a natural local coordinate system $(z^\alpha, \zeta^\alpha := i\Pi \partial/\partial x^\alpha)$, $\alpha = 1, \ldots, m$, by the odd holomorphic 2-form 

$$\eta_c = d \left( \sum_\alpha d\bar{z}^\alpha \zeta_\alpha \right) = - \sum_\alpha d(x^\alpha + ix^\bar{\alpha}) \wedge d(\psi_{\bar{\alpha}} + i\psi_\alpha),$$

where $z^\alpha = x^\alpha + ix^\bar{\alpha}$, $\psi_\alpha = \Pi \partial/\partial x^\alpha$ and $\psi_{\bar{\alpha}} = \Pi \partial/\partial x^{\bar{\alpha}}$. The second one is real and comes from the identification of the real $(2m|2m)$-dimensional supermanifold underlying $\mathcal{Y}$ (which we denote by the same letter $\mathcal{Y}$) with the real cotangent bundle $\Pi \Omega^1 Y$. It is given by 

$$\eta = d \sum_\alpha (dx^\alpha d\psi_\alpha + dx^{\bar{\alpha}} \psi_{\bar{\alpha}}).$$

Clearly, $\eta = \text{Im} \eta_c$.

The holomorphic $m$-form $\Omega$ induces, via the isomorphism 

$$\text{Ber}_c(\mathcal{Y}) = [\Omega^m_c Y]^{\otimes 2},$$

a holomorphic volume form $\hat{\Omega}$ on $\mathcal{Y}$.

A compact real $(m|m)$-dimensional sub-supermanifold $\mathcal{X} \hookrightarrow \mathcal{Y}$ is called a real slice if the sheaf $\mathbb{C} \otimes \mathcal{T} \mathcal{X}$ is isomorphic to the sheaf of smooth sections of $\mathcal{T}_c \mathcal{Y}|_X$. In this case $\hat{\Omega}$ induces $[1]$ a smooth section, $\hat{\Omega}|_X$, of the complexified Berezinian bundle $\mathbb{C} \otimes \text{Ber}(\mathcal{X})$. A real slice $\mathcal{X} \hookrightarrow \mathcal{Y}$ is called special if $\text{Im}(\hat{\Omega}|_X) = 0$. In this case $\text{Re}(\hat{\Omega}|_X)$ is a nowhere-vanishing real volume form on $\mathcal{X}$. If $\mathcal{X} \hookrightarrow \mathcal{Y}$ is also Lagrangian with respect to the real odd symplectic structure $\eta$, then $\eta_c|_X$ is non-degenerate and hence makes $\mathcal{X}$ into an odd symplectic manifold. According to 5.1, the data $(\text{Re}(\hat{\Omega}|_X), \eta_c|_X)$ induces on the structure sheaf of $\mathcal{X}$ a second-order differential operator $\Delta$. Note that if $X \hookrightarrow Y$ is real slice of the manifold $Y$ such that $\text{Im} \Omega|_X = 0$, then the associated sub-supermanifold $\mathcal{X} := \Pi N^* \hookrightarrow \mathcal{Y}$ is a special Lagrangian real slice.

### 5.3.1. Theorem

Let $Y$ be a complex manifold $Y$ equipped with a nowhere-vanishing holomorphic $m$-form $\Omega$, $X$ a compact real slice of $Y$ such that $\text{Im} \Omega|_X = 0$, and $\mathcal{X} = \Pi N^*$ the associated special Lagrangian real slice in $\mathcal{Y} = \Pi \Omega^1 Y$. Then the Zariski tangent space

---

In this and the next sections the subscript $c$ is used to distinguish holomorphic objects from the real ones. In particular, $\Omega^1_c Y$ denotes the bundle of holomorphic 1-forms on $Y$ as opposite to $\Omega^1 Y$ which denotes the bundle of real smooth 1-forms on the real manifold underlying $Y$. 

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to the moduli space of all possible deformations of $X$ inside $Y$ within the class of special Lagrangian real slices is isomorphic to the kernel of the operator $\Delta : \Gamma(X, \mathcal{O}_X)/\mathbb{R} \to \Gamma(X, \mathcal{O}_X)$.

**Proof.** Consider a 1-parameter family, $\{X_t \hookrightarrow Y \mid t \in \mathbb{R}\}$, of special Lagrangian real slices in $Y$ such that $X_{t=0} = X$. Let $z^\alpha = x^\alpha + i\dot{x}^\alpha$ be a local coordinate system on $Y$ in which $X$ is given by $\dot{x}^\alpha = 0$. Then in the associated local coordinate system $(z^\alpha, \zeta_\alpha := i\partial/\partial z^\alpha - \psi_\dot{\alpha} + i\psi_\alpha)$ on $Y$, the equations of $X_t \hookrightarrow Y$ are

$$x^\dot{\alpha} = \frac{\partial \Phi}{\partial \psi_\dot{\alpha}}, \quad \psi_\alpha = -\frac{\partial \Phi}{\partial x^\alpha}$$

for some 1-parameter family of smooth functions $\Phi = \Phi(x^\alpha, \psi_\dot{\alpha}, t)$ satisfying the boundary condition $\Phi(x^\alpha, \psi_\dot{\alpha}, 0) = 0$. The image of $\partial/\partial t$ under the extended Kodaira map is represented by the function (see Lemma 2.4.1)

$$k_{t=0} \left( \frac{\partial}{\partial t} \right) = \frac{\partial \Phi}{\partial t} \bigg|_{t=0} = \Psi.$$

If $\hat{\Omega} = \rho(z^\alpha, \zeta_\alpha)D^*(dz^\alpha, d\zeta_\alpha)$ for some holomorphic function $\rho(z^\alpha, \zeta_\alpha)$, then

$$\hat{\Omega}|_{X_t} = \rho \left( x^\alpha + i\frac{\partial \Phi}{\partial \psi_\dot{\alpha}}, \psi_\dot{\alpha} - i\frac{\partial \Phi}{\partial x^\alpha} \right) D^* \left( d(x^\alpha + i\frac{\partial \Phi}{\partial \psi_\dot{\alpha}}), d(\psi_\dot{\alpha} - i\frac{\partial \Phi}{\partial x^\alpha}) \right) = \rho \left( x^\alpha + i\frac{\partial \Phi}{\partial \psi_\dot{\alpha}}, \psi_\dot{\alpha} - i\frac{\partial \Phi}{\partial x^\alpha} \right) \text{Ber} \left( \delta_\beta^\alpha + i\frac{\partial^2 \Phi}{\partial x^\alpha \partial \psi_\dot{\beta}} - \delta_\beta^\alpha \frac{\partial^2 \Phi}{\partial x^\alpha \partial \psi_\dot{\beta}} \right) D^*(dx^\alpha, d\psi_\dot{\alpha}).$$

Hence,

$$\frac{d\text{Im}(\hat{\Omega}|_{X_t})}{dt} \bigg|_{t=0} = \frac{1}{\rho_0} \sum_\alpha \left( \frac{\partial \Psi}{\partial x^\alpha} \frac{\partial \rho_0}{\partial x^\alpha} - \frac{\partial \Psi}{\partial x^\alpha} \frac{\partial \rho_0}{\partial \psi_\dot{\alpha}} + \rho_0 \frac{\partial^2 \Psi}{\partial x^\alpha \partial \psi_\dot{\alpha}} \right) \rho_0 D^*(dx^\alpha, d\psi_\dot{\alpha}) = (\text{div } H_\Psi) \text{ Re } \hat{\Omega}|_{X_t} = (\Delta \Psi) \text{ Re } \hat{\Omega}|_{X_t},$$

where $\rho_0 = \text{Re } \rho(x_\alpha, \psi_\dot{\alpha})$. Hence $\Delta \Psi = 0$. $\square$

§6. Existence of the extended moduli space of special Lagrangian submanifolds

**6.1. Initial data.** Let $X$ be a compact special Lagrangian submanifold of a Calabi-Yau manifold $Y$ equipped with the Kähler form $\omega$ and a holomorphic volume form $\Omega$, and let $X = \Pi N^* \hookrightarrow \hat{Y}$ be the associated special Legendrian sub-supermanifold of the contact supermanifold $\hat{Y}$ (see §1). With these data one naturally associates the moduli superspace $\mathcal{M}$ of all deformations of $X$ inside $\hat{Y}$ within the class of special Legendrian sub-supermanifolds.
6.2. Proposition. The Zariski tangent superspace to $\mathcal{M}$ at $\mathcal{X}$ is canonically isomorphic to $\Pi H^1(X, \mathbb{R})$.

Proof. It is not hard to check that under the isomorphism $\mathcal{O}_X = \Omega^* \mathcal{X}$ the Batalin-Vilkovisky operator $\Delta : \mathcal{O}_X \to \mathcal{O}_X$ goes into $2d^*d*$, where $d$ is the de Rham differential and $*$ is the Hodge duality operator. Then Theorems 4.3.1 and 5.3.1 imply that the Zariski tangent superspace is isomorphic to

$$
\Pi \Gamma(X, \Omega^* X_{closed}) \cap \Pi \Gamma(X, \Omega^* X_{c_closed}) = \Pi H^1(X, \mathbb{R}).
$$

6.3. Theorem. $\mathcal{M}$ is a smooth supermanifold.

Proof (after McLean [3]). Let $\mathcal{V}$ be a tubular neighbourhood of the zero section in $\Pi \Omega^1 \mathcal{X}$, $\mathcal{U}$ a tubular neighbourhood of $\mathcal{X}$ in $\hat{\mathcal{Y}}$ and $\exp : \mathcal{V} \to \mathcal{U}$ the normal exponential map constructed as in section 4.2. This map identifies nearby (to $\mathcal{X}$) special Legendrian sub-supermanifolds $\mathcal{X}_f$ of $\hat{\mathcal{Y}}$ with global odd sections $f$ of $\Gamma(\mathcal{X}, \mathcal{O}_X)$ and induces a diffeomorphism $\exp_f : \mathcal{X} \to \mathcal{X}_f$. Let $\mathcal{V}'$ be an open subset in $\Gamma(\mathcal{X}, \mathcal{O}_X)$ lying in the preimage of $\mathcal{V}$ under the map $d : \mathcal{O}_X \to \Omega^1 \mathcal{X}$. We define a non-linear map

$$
\phi : \mathcal{V}' \subset \Gamma(\mathcal{X}, \mathcal{O}_X) \to \Omega^* \mathcal{X} \bigoplus \Omega^* \mathcal{X}
$$

as follows

$$
\phi(f) = \left( \exp_f^*(\hat{\omega}), \quad \text{Im} \left( \frac{(p \circ \exp_f)^* (\hat{\Omega})_{|p(X_f)}}{\text{Re} (\hat{\Omega})_{|p(X_f)}} \right)^{1/2} \right),
$$

where $p : \hat{\mathcal{Y}} \to \mathcal{V} = \Pi \Omega^1 \mathcal{Y}$ is the natural projection. The square root in the above formula always exists (cf. section 5.2). Note that $\phi^{-1}(0, 0) = \mathcal{M}$.

It follows from Lemma 4.2.3 that $\exp_f^*(\hat{\omega}) \in \Omega^* \mathcal{X}$ is a closed differential form. Replacing $f$ with $tf$, we see that the map $\exp_f : \mathcal{X} \to \hat{\mathcal{Y}}$ is homotopic to the inclusion $\mathcal{X} \to \hat{\mathcal{Y}}$. Therefore, denoting by $[\ ]$ the cohomology class, we get $[\exp_f^*(\hat{\omega})] = [\hat{\omega}]_{|X} = 0$ and conclude that $\exp_f^*(\hat{\omega})$ is an exact differential form on $\mathcal{X}$.

Since $\hat{\Omega}$ is holomorphic, the integral $\int_{p(X_f)} \hat{\Omega}_{|p(X_f)}$ depends only on the homology class of $\mathcal{X}_{red}$ in $Y$ [4]. Analogously, for any compact $(r|m-r)$-dimensional Lagrangian sub-supermanifold $\mathcal{Z} \subset \mathcal{X}_f$, the integral $\int_{\mathcal{Z}} \hat{\Omega}^{1/2}$ (and hence its real and imaginary parts) depends only on the homology class of $\mathcal{Z}_{red}$ in $Y$. Since $\mathcal{Z}_{red}$ is homologous to an $r$-dimensional cycle in $\mathcal{X}$ and $\text{Im} (\hat{\Omega}^{1/2})$ vanishes, we conclude that $\int_{\mathcal{Z}} \text{Im} (\hat{\Omega}^{1/2}) = 0$ for any such $\mathcal{Z}$. Thus, for any smooth cycle $\mathcal{Z} \hookrightarrow X \subset Y$, we have

$$
0 = \int_{\mathcal{Z}} \text{Im} (\hat{\Omega}^{1/2})
= \int_{\Pi \mathcal{Z}} \left( \frac{(p \circ \exp_f)^* \text{Im} (\hat{\Omega}^{1/2})_{|\mathcal{Z}}}{\text{Re} (\hat{\Omega})_{|\Pi \mathcal{Z}}^{1/2}} \right) \text{Re} (\hat{\Omega})_{|\Pi \mathcal{Z}}^{1/2}
= \int_{\Pi \mathcal{Z}} \text{Im} \left( \frac{(p \circ \exp_f)^* (\hat{\Omega})_{|\mathcal{Z}}}{\text{Re} (\hat{\Omega})_{|\Pi \mathcal{Z}}} \right)^{1/2} \text{Re} (\hat{\Omega})_{|\Pi \mathcal{Z}}^{1/2},
$$

with respect to the odd symplectic structure induced on $p(X_f)$ from the holomorphic odd symplectic structure on $\mathcal{Y}$, see section 5.2.
where \( Z_f := p \circ \exp_f(\Pi N_f^*) \) and we used the fact that \( Z \) and \( (Z_f)_{red} \) are homologous in \( Y \).

By Lemma 5.2.1 and the fact that in our case \( \Delta = *d* \), the integrand of the last integral is a coexact differential form in \( \Omega^*X \).

Thus we proved that \( \phi \) maps \( \mathcal{V}' \subset \Omega^*X \) into the subset

\[
\Omega^*X_{exact} \bigoplus \Omega^*X_{coexact} \subset \Omega^*X \bigoplus \Omega^*X.
\]

Put another way, as a map from \( C^{1, \alpha} \) differential forms on \( X \) to exact and coexact \( C^{0, \alpha} \) differential forms, \( \phi \) is surjective. Then, by the Banach space implicit function theorem and elliptic regularity, the extended moduli space \( \mathcal{M} = \phi^{-1}(0, 0) \) is smooth with tangent space at 0 canonically isomorphic to the kernel of the following operator (see the proofs of Theorems 4.3.1 and 5.3.1),

\[
\frac{d}{dt}\phi(tf) \bigg|_{t=0} = (d, *d) : \Omega^*X \to \Omega^*X \bigoplus \Omega^*X
\]

which is precisely \( \Pi H^*(X, \mathbb{R}) \).

\( \square \)

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