Quantum random walks and their convergence
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Abstract
Using coordinate-free basic operators on toy Fock spaces [1], quantum random walks are defined following the ideas in [3, 1]. Strong convergence of quantum random walks associated with bounded structure maps is proved under suitable assumptions, extending the result obtained in [5] in case of one dimensional noise. To handle infinite dimensional noise we have used the coordinate-free language of quantum stochastic calculus developed in [2].

1 Toy Fock spaces and quantum random walks
1.1 Toy Fock spaces

Let \( K = L^2(\mathbb{R}_+, k_0) \) where \( k_0 \) is a Hilbert space with an orthonormal basis \( \{ e_i : i \geq 1 \} \). We note that, for any \( n \geq 0 \), the \( n \)-fold symmetric tensor product of \( K = L^2(\mathbb{R}_+, k_0) \) and their direct sum can canonically be embedded in the symmetric Fock space \( \Gamma := \Gamma(K) \). For any partition \( S \equiv (0 = t_0 < t_1 < t_2 \cdots) \) of \( \mathbb{R}_+ \), the Fock space \( \Gamma(S) \) can be viewed as the infinite tensor product \( \bigotimes_{n \geq 1} \Gamma_n \) of symmetric Fock spaces \( \{ \Gamma_n = \Gamma(K_{(t_{n-1}, t_n]}) \}_{n \geq 1} \) with respect to the stabilizing sequence \( \Omega = \{ \Omega_n : n \geq 1 \} \), where \( \Omega_n = \Omega_{(t_{n-1}, t_n]} \) is the vacuum vector in \( \Gamma_n \).

For any \( 0 \leq s \leq t \) and \( i \geq 1 \) we define a vector \( \chi^i_{(s,t]} := \frac{1_{(s,t]} \otimes e_i}{\sqrt{t-s}} \in \mathcal{K}(s,t] \). It is clear that \( \{ \chi^i_{(s,t]} \}_{i \geq 1} \) is an orthonormal family in \( \mathcal{K}(s,t] \) and hence in \( \Gamma(s,t] \). Here we note that the Hilbert subspace \( k_{(s,t]} \) of \( \Gamma(s,t] \) spanned by these orthonormal vectors is canonically isomorphic to \( k_0 \). Let us consider the subspace \( \tilde{k}_{(s,t]} = \mathbb{C} \Omega_{(s,t]} \oplus k_{(s,t]} \) of \( \Gamma \) and denote the space \( \tilde{k}_{(t_{n-1}, t_n]} \) by \( k_n \), which is isomorphic to \( k_0 := \mathbb{C} \oplus k_0 \).

Now we are in a position to define the toy Fock spaces.

Definition 1.1. The toy Fock space associated with the partition \( S \) of \( \mathbb{R}_+ \) is defined to be the subspace \( \Gamma(S) := \bigotimes_{n \geq 1} \tilde{k}_n \) with respect to the stabilizing vector

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\( \Omega = \otimes_{n \geq 1} \Omega_n \).

For notational simplicity we write \( \chi^i_n \) for the vector \( \chi^i_{(t_{n-1},t_n)} \). Let \( \square \) be the set of all finite subsets of \( \mathbb{N} \times \mathbb{N} \). Thus an element \( A \in \square \) is given by \( A = \{m_1, i_1; m_2, i_2; \cdots; m_n, i_n\} \) for some \( n \) with \( 1 \leq m_1 < m_2 \cdots m_n < \infty \). For \( A \in \square \), we associate a vector

\[
\chi_A = \Omega_1 \otimes \Omega_2 \otimes \cdots \otimes \chi^i_{m_1} \otimes \cdots \otimes \chi^i_{m_n} \otimes \Omega_{m_n+1} \cdots
\]

in the toy Fock space \( \Gamma(S) \). Clearly this family \( \{\chi_A : A \in \square\} \) forms an orthonormal basis for \( \Gamma(S) \). Let \( P(S) \) be the orthogonal projection of \( \Gamma \) onto the toy Fock space \( \Gamma(S) \). Without loss of generality now onwards let us consider toy Fock spaces \( \Gamma(S_h) \) associated with regular partition \( S_h = (0, h, \cdots) \) for some \( h > 0 \) and denote the orthogonal projection by \( P_h \). The projection \( P_h \) is given by

\[
P_h = P_0 \oplus \bigoplus_{n \geq 1} \sum_{1 \leq m_1 < m_2 \cdots m_n} \sum_{i_1, i_2, \cdots, i_n \geq 1} \otimes^\oplus_{i=1} \chi^i_{m_i} = \Lambda_i \chi^i_{m_i},
\]

where \( P_0 \) is the orthogonal projection of the symmetric Fock space \( \Gamma \) onto the one dimensional Hilbert space \( \mathbb{C} \Omega \). A simple computation shows that, for \( f \in \mathcal{K} \), given by \( f = \sum_{i \geq 1} f_i \otimes e_i \) with \( f_i \in L^2(\mathbb{R}_+) \),

\[
P_h(f) = \Lambda_i \chi^i_{m_i},
\]

and furthermore,

\[
P_h(e) = P_h(e(f_{(k-1)h}))(\Lambda_i \chi^i_{m_i}) \text{ and } P_h(e(f_{kh})) = \Lambda_k \bigoplus \sum_{i \geq 1} \frac{1}{\sqrt{h}} \int_{(k-1)h}^{kh} \frac{f_i(s)ds}{\sqrt{h}} \otimes e_i.
\]

Let us consider the subspace \( \mathcal{M} \) of \( L^2(\mathbb{R}_+, \mathbf{k}_0) \), given by

\[
\mathcal{M} = \{f \in L^2(\mathbb{R}_+, \mathbf{k}_0) : f_i \in C^1(\mathbb{R}_+) \text{ and } f_i = 0 \text{ for all but finitely many } i\}.
\]

Clearly \( \mathcal{M} \) is a dense subspace, so the algebraic tensor product \( \mathbf{h}_0 \otimes \mathcal{E}(\mathcal{M}) \) is dense in \( \mathbf{h}_0 \otimes \Gamma \). For \( f \in \mathcal{M} \) we define a constant \( c_f := \sum_{i \geq 1} \sup_{\tau} |f_i'(\tau)| \) where \( f_i' \) denotes the first derivative of the function \( f_i \). We have the following estimates which will be needed later.
Lemma 1.2. For any $f \in \mathcal{M}, k \geq 1$,
$$
\|(1 - P_h[k])e(f[k])\| \leq h(c_f + \|f\|_\infty)\|e(f[k])\|.
$$

Proof. (a). We have
$$
\|(1 - P_h[k])e(f[k])\| = \|(P_0 + P_1 - P_h)\|e(f[k]) + [1 - P_0 - P_1]e(f[k])\|
\leq \|f[k] - P_h f[k]\| + \|[1 - P_0 - P_1]e(f[k])\|.
$$
It is clear that $\|[1 - P_0 - P_1]e(f[k])\| \leq h\|f\|^2_\infty\|e(f[k])\|$. Let us consider the first term,
$$
\|f[k] - P_h f[k]\|^2 = \sum_{i \geq 1} \|1[f_i(1 - \frac{1}{h} \int_{[0]} f_i(s)ds)]^2
= \sum_{i \geq 1} \int_{[0]} dr|f_i(r) - \frac{1}{h} \int_{[0]} f_i(s)ds|^2
\leq \sum_{i \geq 1} \frac{1}{h^2} \int_{[0]} dr\int_{[0]} (f_i(r) - f_i(s)ds)^2
\leq \sum_{i \geq 1} \frac{1}{h^2} \int_{[0]} dr\int_{[0]} h\sup_{\tau} |f_i'(\tau)|ds|^2.
$$

Since $c_f = \sum_{i \geq 1} \sup_{\tau} |f_i'(\tau)|$, the required estimate follows.

By this lemma it can be proved that the family of orthogonal projections $P_h$ converges strongly to identity operator in $\Gamma$ as $h$ tends to 0.

1.2 Coordinate-free basic operators

Here we define basic operators associated with toy Fock space $\Gamma(S_h)$ using the fundamental processes in coordinate-free language of quantum stochastic calculus, developed in [2] and obtained some useful estimates. For $S \in \mathcal{B}(\mathcal{H}_h), R \in \mathcal{B}(\mathcal{H}_h, \mathcal{H}_h \otimes \mathcal{K}_0)$ and $T \in \mathcal{B}(\mathcal{H}_0 \otimes \mathcal{K}_0)$ let us define four basic operators as follows, for $k \geq 1$,

$$
N_1^1[k] = SP_0[k] = P_0[k] \Lambda_1^1[k],
N_1^2[k] = \frac{\Lambda_2^1[k]}{\sqrt{h}} P_1[k],
N_1^3[k] = \frac{\Lambda_3^1[k]}{\sqrt{h}} P_1[k],
N_1^4[k] = P_1[k] \Lambda_4^1[k] P_1[k] P_h[k].
$$

(1. 1)
where

\[
\begin{align*}
\Lambda_S^1[k] &= \mathcal{I}_S((k-1)h, kh), \\
\Lambda_R^2[k] &= a_R((k-1)h, kh), \\
\Lambda_R^3[k] &= a_R^1((k-1)h, kh), \\
\Lambda_T^4[k] &= \Lambda_T((k-1)h, kh).
\end{align*}
\]

(1. 2)

For definition of coordinate-free fundamental processes \( \mathcal{I}_S, a_R, a_R^1 \) and \( \Lambda_T \) we refer to \cite{2}. All these maps \( \mathcal{B}(h_0) \ni S \mapsto \Lambda_S^1[k], \mathcal{B}(h_0, h_0 \otimes k_0) \ni R \mapsto \Lambda_R^2[k], \Lambda_R^3[k] \) and \( \mathcal{B}(h_0 \otimes k_0) \ni T \mapsto \Lambda_T^4[k] \) are linear, and hence the maps \( \mathcal{B}(h_0) \ni S \mapsto N_S^1[k], \mathcal{B}(h_0, h_0 \otimes k_0) \ni R \mapsto N_R^2[k], N_R^3[k] \) and \( \mathcal{B}(h_0 \otimes k_0) \ni T \mapsto N_T^4[k] \) are so. It is clear that the subspace \( \Gamma(S_h) \) is invariant under all these operators \( N^l \) and their action on \( h_0 \otimes \Gamma \); for \( u \in h_0, f \in L^2(\mathbb{R}_+, k_0) \) are given by

\[
\begin{align*}
N_S^1[k]ue(f[k]) &= Su \otimes \Omega[k], \\
N_R^2[k]ue(f[k]) &= \frac{\Lambda_R^2[k]}{\sqrt{h}} u \otimes f[k] \\
&= \frac{1}{\sqrt{h}} \int_{[k]} R^*(uf(s))ds\Omega[k], \\
N_R^3[k]ue(f[k]) &= \frac{\Lambda_R^3[k]}{\sqrt{h}} u \otimes \Omega[k] \\
&= \frac{1}{\sqrt{h}} \int_{[k]} Ru, \\
N_T^4[k]ue(f[k]) &= \Lambda_T^4[k] P_h[k] f[k] \\
&= (1_{h_0} \otimes 1_{[k]})Tu \otimes P_h f(\cdot).
\end{align*}
\]

(1. 3)

For any \( S_1, S_2 \in \mathcal{B}(h_0), R_1, R_2 \in \mathcal{B}(h_0, h_0 \otimes k_0) \) and \( T_1, T_2 \in \mathcal{B}(h_0 \otimes k_0) \) we observe the following simple but useful identities, which are easy to derive

\begin{itemize}
  \item \((N_S^2[k])^2 = (N_S^3[k])^2 = 0, \quad N_{S_1}^1[k] N_{S_2}^1[k] = N_{S_1S_2}^1[k], \)
  \item \(N_{R_1}^2[k] N_{R_2}^2[k] = N_{R_1R_2}^1[k], \quad N_{S_1}^3[k] N_{S_2}^3[k] = N_{S_1S_2}^3[k], \)
  \item \(N_{R_1}^2[k] N_{R_2}^2[k] = N_{T_1R_2}^2[k], \quad N_{S_1}^3[k] N_{S_2}^3[k] = N_{RS}^3[k], \)
  \item \(N_{T_1}^4[k] N_{T_2}^4[k] = N_{T_1T_2}^4[k], \quad N_{S_1}^1[k] N_{S_2}^1[k] = N_{S_1S_2}^1[k], \)
  \item \(N_{S_1}^1[k] N_{S_2}^1[k] = N_{S_1S_2}^1[k], \quad N_{S_1}^1[k] + N_{S_2}^1[k] = S \otimes P_h[k]. \)
\end{itemize}
From (1.3) we have
\[
\|N^1_S[k]u e(f[k])\| = \|Su\|,
\|N^2_R[k]u e(f[k])\| \leq \sqrt{h}\|R\| \|u\| \|f\|_\infty,
\|N^3_R[k]u e(f[k])\| \leq \|Ru\|
\|N^4_T[k]u e(f[k])\| \leq \sqrt{h}\|T\| \|u\| \|f\|.
\tag{1.4}
\]

Here we also note the following which can be verified easily using Lemma 2.12 and Lemma 2.14 in \[2\],
\[
\|A^1_R[k]u e(f[k])\|^2
= \|(1_{\mathbf{h}_0} \otimes 1_{[k]})Ru\|^2 \|e(f[k])\|^2 + \int_{[k]} R^*(u f(s)) ds \|e(f[k])\|^2;
\]
\[
\|A^2_T[k]u e(f[k])\|^2
= \int_{[k]} |T u f(s)|^2 ds \|e(f[k])\|^2 + \int_{[k]} \langle f(s), T f(s) \rangle ds u e(f[k]) \|^2.
\tag{1.5}
\]

In the above expression \(T(f(s)) \in \mathcal{B}(\mathbf{h}_0, \mathbf{h}_0 \otimes \mathbf{k}_0)\) and \(\langle f(s), T(f(s)) \rangle \in \mathcal{B}(\mathbf{h}_0)\) define by
\[
T f(s)u = T u \otimes f(s), \forall u \in \mathbf{h}_0 \text{ and } \langle (f(s), T f(s))u, v \rangle = \langle T f(s)u, v \otimes f(s) \rangle, \forall u, v \in \mathbf{h}_0.
\]

For the basic operators \(N^i\)'s we have the estimates:

**Lemma 1.3.** (a). For any \(k \geq 1\) and \(u \in \mathbf{h}_0, f \in \mathcal{M}\),

1. \(\|\{h^N_S[k] - \Lambda^1_S[k]\}u e(f[k])\| \leq h^{\frac{3}{2}} \|Su\| \|f\|_\infty \|e(f[k])\|
\]
2. \(\|\{\sqrt{h} N^2_R[k] - \Lambda^2_R[k]\}u e(f[k])\| \leq h^{\frac{5}{2}} \|R\| \|u\| \|f\|^2_\infty \|e(f[k])\|
\]
3. \(\|\{\sqrt{h} N^3_R[k] - \Lambda^3_R[k]\}u e(f[k])\| \leq 2h \|Ru\| \|f\|_\infty \|e(f[k])\|
\]
4. \(\|\{N^4_T[k] - \Lambda^4_T[k]\}u e(f[k])\| \leq 2h \|T\|(c_f + \|f\|^2_\infty) \|e(f[k])\|
\]

(b). For any \(k \geq 1\) and \(u, v \in \mathbf{h}_0, f, g \in \mathcal{M}\), we have

1. \(\langle ve(g[k]), \{h^N_S[k] - \Lambda^1_S[k]\}u e(f[k])\rangle \|
\leq h^{\frac{3}{2}} \|Su\| \|f\|_\infty \|e(f[k])\| \|e(g[k])\|,
2. \( \langle v e(g[k]), \{ \sqrt{h} N^2_R[k] - \Lambda^2_R[k] \} u e(f[k]) \rangle \) 
\[ \leq h^{\frac{3}{2}} R \| u \| \| f \|_{\infty}^2 \| g \|_{\infty} \| e(f[k]) \| \| v e(g[k]) \|, \]

3. \( \langle v e(g[k]), \{ \sqrt{h} N^2_R[k] - \Lambda^2_R[k] \} u e(f[k]) \rangle \) 
\[ \leq 2h^2 \| R u \| \| v \| \| f \|_{\infty} \| g \|_{\infty} \| e(f[k]) \|^2 \| e(g[k]) \|^2, \]

4. \( \langle v e(g[k]), \{ N^4_R[k] - \Lambda^4_R[k] \} u e(f[k]) \rangle \) 
\[ \leq h^2 \left( \| f \|_{\infty} + c_f \| g \|_{\infty} \right)^2 \| T \| \| u \| \| v \| \| e(f[k]) \|^2 \| e(g[k]) \|^2. \]

\textit{Proof. a.(1)} It is clear from the definition that 
\[ \| \{ h N^2_R[k] - \Lambda^1_R[k] \} u e(f[k]) \| = h \| S u(\Omega[k] - e(f[k])) \| \]
\[ = h \| S u \|| \Omega[k] - e(f[k]) \| \]
\[ \leq h^{\frac{3}{2}} \| S u \| \| f \|_{\infty} \| e(f[k]) \|. \]

(2) From the definitions, we have 
\[ \| \{ \sqrt{h} N^2_R[k] - \Lambda^2_R[k] \} u e(f[k]) \| = \| \int_{[k]} R^*(u f(s)) ds \ (\Omega[k] - e(f[k])) \| \]
\[ \leq \int_{[k]} \| R^*(u f(s)) ds \| (\Omega[k] - e(f[k])) \| \]
\[ \leq h^{\frac{3}{2}} \| R \| \| u \| \| f \|_{\infty} \| e(f[k]) \|. \]

(3) We have 
\[ \| \{ \sqrt{h} N^3_R[k] - \Lambda^3_R[k] \} u e(f[k]) \|^2 \]
\[ = \| (1_{h^0} \otimes 1_{[k]}) R u - \Lambda^3_R[k] u e(f[k]) \|^2 \]
\[ = \| (1_{h^0} \otimes 1_{[k]}) R u \|^2 + \| \Lambda^3_R[k] u e(f[k]) \|^2 \]
\[ - 2R e \langle (1_{h^0} \otimes 1_{[k]}) R u, \Lambda^3_R[k] u e(f[k]) \rangle \]

Now using (1.5) and the definition of \( \Lambda^3_R \) the above quantity is equal to 
\[ \| (1_{h^0} \otimes 1_{[k]}) R u \|^2 + \| (1_{h^0} \otimes 1_{[k]}) R u \|^2 \| e(f[k]) \|^2 \]
\[ + \| \int_{[k]} R^*(u f(s)) ds \|^2 \| e(f[k]) \|^2 - 2\| (1_{h^0} \otimes 1_{[k]}) R u \|^2 \]
\[ = \| (1_{h^0} \otimes 1_{[k]}) R u \|^2 \| e(f[k]) \|^2 - 1 + \| \int_{[k]} R^*(u f(s)) ds \|^2 \| e(f[k]) \|^2 \]
\[ \leq 2h^2 \| R \|^2 \| u \|^2 \| f \|_{\infty} \| e(f[k]) \|^2. \]
(4) We have
\[
\|\{ N^4_T[k] - \Lambda^4_T[k]\} u e(f_1[k]) \|^2
= \| (1_{h_0} \otimes 1_{[k]} ) T(u \otimes P_h f(\cdot)) \|^2 + \| \Lambda^4_T[k] u e(f_1[k]) \|^2
- 2 \Re \langle (1_{h_0} \otimes 1_{[k]} ) T(u \otimes P_h f(\cdot)), \Lambda^4_T[k] u e(f_1[k]) \rangle.
\]
By the definition of \( \Lambda^4_T \) (see [2])
\[
\langle (1_{h_0} \otimes 1_{[k]} ) T(u \otimes P_h f(\cdot)), \Lambda^4_T[k] u e(f_1[k]) \rangle.
\]
\[
= \langle (1_{h_0} \otimes 1_{[k]} ) T(u \otimes P_h f(\cdot)), a^1(T_{f_{[k]}}^4) u e(f_1[k]) \rangle
= \langle (1_{h_0} \otimes 1_{[k]} ) T(u \otimes P_h f(\cdot)), T_{f_{[k]}}^4 (u \Omega_{[k]}) \rangle
= \int_{[k]} \langle T(u P_h (f)(s)), T(u f(s)) \rangle \ ds.
\]
Thus using [1, 3] we obtained
\[
\|\{ N^4_T[k] - \Lambda^4_T[k]\} u e(f_1[k]) \|^2
= \int_{[k]} \| T(u P_h (f)(s)) \|^2 \ ds
+ \int_{[k]} \| T(u f(s)) \|^2 \| e(f_1[k]) \|^2 + \| \int_{[k]} \langle f(s), T(f(s)) \rangle \ ds \| e(f_1[k]) \|^2
- 2 \Re \int_{[k]} \langle T(u P_h (f)(s)), T(u f(s)) \rangle \ ds
\]
\[
= \int_{[k]} \| T(u f(s)) \|^2 \| e(f_1[k]) \|^2 - 1 + \| \int_{[k]} \langle f(s), T(f(s)) \rangle \ ds \| e(f_1[k]) \|^2
+ \int_{[k]} \| T(u \otimes (1 - P_h)(f)(s)) \|^2 \ ds
\]
\[
\leq 2 h^2 \| T \|^2 \| f \|^4_{\infty} \| u e(f_1[k]) \|^2 + \| T \|^2 \| u \|^2 \int_{[k]} \| (1 - P_h) (f)(s) \|^2 \ ds
\]
\[
\leq 2 h^2 \| T \|^2 \| f \|^4_{\infty} \| u e(f_1[k]) \|^2 + \| T \|^2 \| u \|^2 \| (1 - P_h)(f_1[k]) \|^2.
\]
Since \( \|(1 - P_h) e(f_1[k]) \|^2 \leq h^2 c_f \), the required estimate follows.

(b) The estimates (1) and (2) follow directly from (a).

(3) From the definitions
\[
\langle v e(g_{[k]}), \{ \sqrt{h} \ N^3_R[k] - \Lambda^3_R[k]\} u e(f_1[k]) \rangle
= \langle v e(g_{[k]}), \sqrt{h} \ N^3_R[k] u e(f_1[k]) \rangle - \langle v e(g_{[k]}), \Lambda^3_R[k] u e(f_1[k]) \rangle
= \langle v e(g_{[k]}), (1_{h_0} \otimes 1_{[k]}) R u \rangle - \langle \Lambda^3_R[k] v e(g_{[k]}), u e(f_1[k]) \rangle
= \int_{[k]} \langle R u, v g(s) \rangle \ ds (1 - \langle e(g_{[k]}), e(f_1[k]) \rangle).
\]
Thus we have obtained the required estimate,

$$
|\langle \text{ve}(g[k]), \{\sqrt{h} N_{[k]}^4[k] - \Lambda^4_T[k]\} \text{ve}(f[k]) \rangle |
\leq h^2 \|Ru\| \|v\| \|f\|_\infty \|g\|_\infty \|\text{ve}(f[k])\|^2 \|\text{ve}(g[k])\|^2.
$$

4. By definition of $N_{[k]}^4$ and $\Lambda^4_T$

$$
\langle \text{ve}(g[k]), \{N_{[k]}^4[k] - \Lambda^4_T[k]\} \text{ve}(f[k]) \rangle
= \langle \text{ve}(g[k]), \Lambda^4_T[k] \text{ve}(f[k]) \rangle - \langle \text{ve}(g[k]), \Lambda^4_T[k] \text{ve}(f[k]) \rangle
$$

$$
= \int_{[k]} \langle v g(s), T(u(P_h f)(s))ds - \int_{[k]} \langle v g(s), T(u f(s))ds \langle \text{ve}(g[k]), \text{ve}(f[k]) \rangle.
$$

$$
= \int_{[k]} \langle v g(s), T(u ((P_h - 1)f)(s))ds
$$

$$
+ \int_{[k]} \langle v g(s), T(u f(s))ds [1 - \langle \text{ve}(g[k]), \text{ve}(f[k]) \rangle].
$$

So we get

$$
|\langle \text{ve}(g[k]), \{N_{[k]}^4[k] - \Lambda^4_T[k]\} \text{ve}(f[k]) \rangle |
\leq \left( \int_{[k]} \|vg(s)\|^2 ds \right)^{\frac{1}{2}} \left( \int_{[k]} \|T[u((P_h - 1)f[k])s]\|^2 ds \right)^{\frac{1}{2}}
$$

$$
+ \int_{[k]} \|vg(s)\|\|T(u f(s))\|ds \|1 - \langle \text{ve}(g[k]), \text{ve}(f[k]) \rangle\|
$$

$$
\leq h \|v\|\|g\|_\infty \|T\|\|u\|\|((P_h[k] - 1)f[k]\|\|1 - \langle \text{ve}(g[k]), \text{ve}(f[k]) \rangle\|.
$$

Using the estimates of $\|(P_h[k] - 1)f[k]\|$ and $\|1 - \langle \text{ve}(g[k]), \text{ve}(f[k]) \rangle\|$ the required estimate follows. \hfill \Box

**Remark 1.4.** The estimates in the above Lemma will also hold if we replace the initial Hilbert space $h_0$ by $\{h_0 \otimes \Gamma(k-1)h]\}$ and take $S \in B(h_0 \otimes \Gamma(k-1)h]$),

$$
R \in B(h_0 \otimes \Gamma(k-1)h), h_0 \otimes \Gamma(k-1)h_0 \otimes k_0) \text{ and } T \in B(h_0 \otimes \Gamma(k-1)h \otimes k_0).
$$

### 1.3 Quantum random walk

Let $\mathcal{A} \subseteq B(h_0)$ be a von Neumann algebra. Let us consider the Hilbert von Neumann module $\mathcal{A} \otimes k_0$. Suppose we are given with a family of $*$-homomorphisms $\{\beta(h)\}_{h>0}$
from $A$ to $A \otimes B(k_0)$. For $h > 0$, $\beta(h)$ can be written as
\[
\beta(h, x) = \begin{pmatrix}
\beta_1(h, x) & (\beta_2(h, x))^* \\
\beta_3(h, x) & \beta_4(h, x)
\end{pmatrix}, \forall x \in A,
\]
where the components $\beta_i(h)$'s are contractive maps and $\beta_1(h) \in B(A), \beta_4(h) \in B(A, A \otimes B(k_0))$ and $\beta_2(h), \beta_3(h) \in B(A, A \otimes k_0)$. The *-homomorphic properties of $\beta(h)$ can be translated into the following properties of $\beta_i(h)$'s.

- $\beta_1(h, x^*) = (\beta_1(h, x))^*$, $\beta_4(h, x^*) = (\beta_4(h, x))^*$, $\beta_3(h, x^*) = \beta_2(h, x)$,
- $\beta_1(h, xy) = \beta_1(h, x)\beta_1(h, y) + (\beta_2(h, x))^*\beta_3(h, y)$,
- $\beta_2(h, xy) = \beta_1(h, x)(\beta_2(h, y))^* + (\beta_2(h, x))^*\beta_4(h, y)$,
- $\beta_3(h, xy) = \beta_3(h, x)\beta_1(h, y) + \beta_4(h, x)\beta_3(h, y)$,
- $\beta_4(h, xy) = \beta_3(h, x)(\beta_2(h, y))^* + \beta_4(h, x)\beta_4(h, y)$.

We define a family of maps $P_t^{(h)}: A \otimes \mathcal{E}(\mathcal{K}) \to A \otimes \Gamma$ as follows. We subdivide the interval $[0, t]$ into $[k] \equiv [(k - 1)h, kh]$, $1 \leq k \leq n$ so that $t \in ((n - 1)h, nh]$ as earlier and set for $x \in A$, $f \in \mathcal{K}$
\[
\begin{align*}
P_0^{(h)}(xe(f)) &= xe(f) \\
P_{kh}^{(h)}(xe(f)) &= \sum_{l=1}^{4} P_{(k-1)h}^{(h)} N_l^{(h)}(\beta_i(h, x), k)[e(f)]
\end{align*}
\]
and $P_t^{(h)} = P_{nh}^{(h)}$.

Now setting a family of linear maps $p_t^{(h)}: A \to A \otimes B(\Gamma)$, by
\[
p_t^{(h)}(x) e(f) := P_t^{(h)}(xe(f))u, \forall u \in k_0 \text{ we have }
\]
\[
\begin{align*}
p_0^{(h)}(x)e(f) &= xe(f) \\
p_t^{(h)}(x)e(f) &= p_{nh}^{(h)}(x)e(f) = \sum_{l=1}^{4} N_l^{(h)} p_{(n-1)h}^{(h)}(\beta_i(h, x))[n]e(f).
\end{align*}
\]

As per our convention $p_{(n-1)h}^{(h)}$ appear above are identified with their ampliations $p_{(n-1)h}^{(h)} \otimes 1_{k_0}$ as well as $p_{(n-1)h}^{(h)} \otimes 1_{B(k_0)}$. For $k \geq 1$, $l = 1, 2, 3$ and 4, $N_l^{(h)} p_{(n-1)h}^{(h)}(\beta_i(h, x))$ are defined in terms of
\[
\begin{align*}
\Lambda_1^{1}(p_{(k-1)h}^{(h)}(\beta_i(h, x)))[k], & \quad \Lambda_2^{2}(p_{(k-1)h}^{(h)}(\beta_2(h, x)))[k], \quad \Lambda_3^{3}(p_{(k-1)h}^{(h)}(\beta_3(h, x)))[k], \\
\Lambda_4^{4}(p_{(k-1)h}^{(h)}(\beta_4(h, x)))[k] \text{ where, for example } \Lambda_2^{2}(p_{(k-1)h}^{(h)}(\beta_2(h, x)))[k] \text{ carries the meaning of } a_2^{2}(p_{(k-1)h}^{(h)}(\beta_2(h, x)))[k] \text{ with initial Hilbert space } k_0 \otimes \Gamma_{(k-1)h}.
\end{align*}
\]
For notational simplicity, for any bounded $*$-preserving map

$$\alpha : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(k_0), \quad \alpha(x) = \begin{pmatrix} \alpha_1(h, x) & (\alpha_2(h, x))^* \\ \alpha_3(h, x) & \alpha_4(h, x) \end{pmatrix},$$

we write $N_{\alpha(h,x)}[k]$ for $\sum_{l=1}^4 N_l^\alpha(h,x)[k]$ and $\Lambda_{\alpha(h,x)}[k]$ for $\sum_{l=1}^4 A_l^\alpha(h,x)[k]$. Now for each $k \geq 1$ defining a linear map $\rho_k(h, x) = N_{\beta(h,x)}[k], p_{nh}^{(h)}$ can be written as $p_{nh}^{(h)} = \rho_1(h) \cdots \rho_n(h)$. By the properties of the family $\{\beta(h)\}$ and $\{N_l^\alpha[k]\}$, each $\rho_k(h)$ is a $*$-homomorphism and hence $p_t^{(h)}$ is so. We call this family of homomorphisms $\{p_t^{(h)} : t \geq 0\}$ a quantum random walk. In the next section we shall construct quantum random walks associated with uniformly continuous QDS on a von Neumann algebra and show the strong convergence. Let us conclude this section with the following observation which will be needed latter.

**Lemma 1.5.** For any $t \geq 0, t \in ((n - 1)h, nh]$ for some $n \geq 1$ and $x \in \mathcal{A}, u \in h_0$ and $f \in \mathcal{K}$

$$\mathcal{P}_t^{(h)}(x e(f))u = xue(f) + \sum_{k=1}^n \mathcal{P}_t^{(h)}(\mathcal{P}_{(k-1)h}^{(h)}N_{\beta(h,x)-b(x)}[k]e(f))u + F(h, x, u, f), \quad (1.8)$$

where $b(x) = \begin{pmatrix} b_1(x) & (b_2(x))^* \\ b_3(x) & b_4(x) \end{pmatrix} = x \otimes 1_k$ and

$F(h, x, u, f) = -\sum_{k=1}^n \mathcal{P}_t^{(h)}(\mathcal{P}_{(k-1)h}(x(1 - P_h[k])e(f))u.$ Moreover, for any $f \in \mathcal{M}$

$$\|F(h, x, u, f)\|^2 \leq h \langle c(f, t)\|x\|^2 \|u\|^2, \quad (1.9)$$

where $c(f, t) = 2t(c_f + \|f\|_\infty)\|e(f)\|.$

**Proof.** Since for any $k \geq 1$,

$$N_{b(x)}[k] = \sum_{l=1}^4 N_{b(l)}^l[k] = N_x^1[k] + N_x^4[k] = x \otimes P_h[k],$$

We get

$$\mathcal{P}_t^{(h)}(x e(f))u = \mathcal{P}_t^{(h)(x e(f))u}$$

$$= xue(f) + \sum_{k=1}^n (\mathcal{P}_{kh}^{(h)} - \mathcal{P}_{(k-1)h}^{(h)})(x e(f))u$$

$$= xue(f) + \sum_{k=1}^n \mathcal{P}_{(k-1)h}^{(h)}N_{\beta(h,x)-b(x)}[k]e(f)u$$

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Using the uniform bound for estimate follows.

We have

\[
\| \sum_{k=1}^{n} p_{(k-1)h}^{(h)}(x \otimes 1 - N_{b(x)}[k])e(f)u
\]

\[
= xu e(f) + \sum_{k=1}^{n} p_{(k-1)h}^{(h)} N_{\beta(h,x)-b(x)}[k] e(f)u + F(h, x, u, f).
\]

In order to obtained (1.9) let us consider the following. For any \(1 \leq m \leq n\) setting

\[Z_m = \sum_{k=1}^{m} p_{(k-1)h}^{(h)}(x)(1 - P_{h}[k]),\]

we have

\[\|Z_mue(f_{mh})\| \leq \sum_{k=1}^{m} \|p_{(k-1)h}^{(h)}(x)ue(f_{(k-1)h})\| \|1 - P_{h}[k]\|e(f_{[k]})\|e(f_{[kh,mh]})\|\]

Now using Lemma 1.2(a) and the fact that \(p_{kh}^{(h)}\)'s are homomorphisms,

\[\|Z_mue(f_{mh})\| \leq \sum_{k=1}^{m} h(c_f + \|f\|_\infty)\|x\| \|ue(f_{mh})\| \leq t(c_f + \|f\|_\infty)\|x\| \|ue(f_{mh})\|.
\]

We have

\[\|F(h, x, u, f)\|^2 \leq \sum_{k=1}^{n} \|p_{(k-1)h}^{(h)}(x)ue(f_{(k-1)h})\|^2 \|1 - P_{h}[k]\|e(f_{[k]})\|^2 \|e(f_{[kh]})\|^2 + 2Re \sum_{k=1}^{n} \langle Z_{k-1}ue(f_{(k-1)h}), p_{(k-1)h}^{(h)}(x)ue(f_{(k-1)h}) \rangle\]

\[\langle e(f_{[k]}), (1 - P_{h}[k])e(f_{[k]}) \rangle \|e(f_{[kh]})\|^2 \leq \sum_{k=1}^{n} \|x\|^2 \|ue(f_{(k-1)h})\|^2 \|1 - P_{h}[k]\|e(f_{[k]})\|^2 \|e(f_{[kh]})\|^2 + 2 \sum_{k=1}^{n} \|Z_{k-1}ue(f_{(k-1)h})\| \|x\| \|ue(f_{(k-1)h})\| \|1 - P_{h}[k]\|e(f_{[k]})\|^2 \|e(f_{[kh]})\|^2.
\]

Using the uniform bound for \(\|Z_{k-1}ue(f_{(k-1)h})\|\) and Lemma 1.2(a) the required estimate follows.

By above Lemma and the definition \(p_{t}^{(h)}\) we have

\[p_{t}^{(h)}(xe(f))u = p_{t}^{(h)}(x)e(f) = xe(f)\]

\[+ \sum_{k=1}^{n} N_{\beta(h,x)-b(x)}[k]ue(f) + F(h, x, u, f) \quad (1.10)\]
2 EH flow as a strong limit of Quantum random walks

Here, we shall construct quantum random walk and prove the strong convergence extending the ideas in [3].

Let $T_t$ be a uniformly continuous conservative QDS on von Neumann algebra $\mathcal{A}$ with the generator $L$. Then (for detail see [2]):

(i) There exists a Hilbert space $k_0$ and structure maps $(\mathcal{L}, \delta, \sigma)$, where $\delta \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes k_0)$ and $\sigma \in \mathcal{B}(\mathcal{A}, \mathcal{A} \otimes \mathcal{B}(k_0))$.

(ii) The map $\Theta = \left( \begin{array}{ccc} \theta_1 & (\theta_2(\cdot))^* \\ \theta_3 & \delta^t \\ \theta_4 & \sigma \end{array} \right) : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(k_0)$ is a bounded CCP map (where $\delta^t(x) = (\delta(x^*))^*$ for all $x \in \mathcal{A}$) with the structure

\[
\theta(x) = V^*(x \otimes 1_{k_0})V + W(x \otimes 1_{k_0}) + (x \otimes 1_{k_0})W^*, \forall x \in \mathcal{A}, \quad (2.1)
\]

where $V, W \in \mathcal{B}(h_0 \otimes \hat{k}_0)$, and the estimate

\[
\|\Theta(x)\xi\| \leq \|(x \otimes 1_{\mathcal{H}})D\xi\|, \forall x \in \mathcal{A}, \xi \in h_0 \otimes \hat{k}_0, \forall x \in \mathcal{A}, \xi \in h_0 \otimes \hat{k}_0, \quad (2.2)
\]

where $D \in \mathcal{B}(h_0 \otimes k_0, h_0 \otimes \mathcal{H})$, $\mathcal{H} = \hat{k}_0 \oplus k_0 \oplus \hat{k}_0$.

(iii) Let $\tau \geq 0$ be fixed. There exists a unique solution $J_t$ of the equation,

\[
J_t = \text{id}_{\mathcal{A} \otimes \Gamma} + \int_0^t J_s \Lambda_{\Theta}(ds), \quad 0 \leq t \leq \tau \quad (2.3)
\]

(here we have written $\Lambda_{\Theta}(ds)$ for $\Lambda_{\theta_1}(ds) + 2\Lambda_{\theta_3}(ds) + 3\Lambda_{\theta_4}(ds)$)

as a regular adapted process mapping $\mathcal{A} \otimes \mathcal{E}(\mathcal{C})$ into $\mathcal{A} \otimes \Gamma$ and satisfies

\[
\sup_{0 \leq t \leq \tau} \|J_t(x \otimes e(f))u\| \leq C'(f)\|x \otimes 1_{\Gamma_{\tau}(L^2([0, \tau], \mathcal{H}))}\|E_{\tau}u\|,
\]

where $f \in \mathcal{C}, E_{\tau} \in \mathcal{B}(h_0, h_0 \otimes \Gamma_{\tau}(L^2([0, \tau], \mathcal{H})))$, $C'(f)$ is some constant and $\Gamma_{\tau}(L^2([0, \tau], \mathcal{H}))$ is the free Fock space over $L^2([0, \tau], \mathcal{H})$.

For $m \geq 0$, let us consider the ampliation

\[
\Theta_{(m)} : \mathcal{A} \otimes \mathcal{B}(k_0^{\tilde{m}}) \to \mathcal{A} \otimes \mathcal{B}(k_0^{\tilde{m}}) \otimes \mathcal{B}(\hat{k}_0)
\]

of the map $\Theta$ given by

\[
\Theta_{(m)}(X) = Q_m^* \left( \Theta \otimes \text{id}_{\mathcal{B}(k_0^{\tilde{m}})}(X) \right) Q_m \quad (2.4)
\]
where $Q_m : h_0 \otimes k_0^{\otimes 3} \otimes \hat{k}_0 \rightarrow h_0 \otimes k_0 \otimes \hat{k}_0^{\otimes 3}$ is the unitary operator which interchanges the second and third tensor components. From the structure (2.1) of the map $\Theta$,

$$\Theta_{(m)}(X) = Q_m^*(V^* \otimes 1_{\hat{k}_0^{\otimes 3}})Q_m(X \otimes 1_{k_0})Q_m^*(V \otimes 1_{k_0})Q_m + Q_m(W \otimes 1_{\hat{k}_0^{\otimes 3}})Q_m(X \otimes 1_{k_0}) + (X \otimes 1_{k_0})Q_m(W^* \otimes 1_{\hat{k}_0^{\otimes 3}})Q_m.$$

For $\xi \in h_0 \otimes k_0^{\otimes 3} \otimes \hat{k}_0$,

$$\|\Theta_{(m)}(X)\xi\|^2 \leq 3\left(\|V\|^2\|(X \otimes 1_{k_0})Q_m^*(V \otimes 1_{k_0})Q_m\xi\|^2 + \|W\|^2\|(X \otimes 1_{k_0})\xi\|^2 + \|(X \otimes 1_{k_0})Q_m(W^* \otimes 1_{\hat{k}_0^{\otimes 3}})Q_m\xi\|^2\right).$$

Setting

$$D_m \xi = \sqrt{3} \left(\|V\|Q_m^*(V \otimes 1_{k_0})Q_m \xi \otimes \|W\|\xi \otimes Q_m^*(W^* \otimes 1_{\hat{k}_0^{\otimes 3}})Q_m \xi\right),$$

$D_m \in \mathcal{B}(h_0 \otimes k_0^{\otimes 3} \otimes \hat{k}_0, h_0 \otimes k_0^{\otimes 3} \otimes \mathcal{H})$ (where $\mathcal{H} = \hat{k}_0 \oplus \hat{k}_0 \oplus \hat{k}_0$ as earlier) and

$$\|\Theta_{(m)}(X)\xi\| \leq \|(X \otimes 1_{\mathcal{H}})D_m \xi\|, \forall X \in \mathcal{A} \otimes \mathcal{B}(k_0^{\otimes 3}). \quad (2.5)$$

Thus $\|\Theta_{(m)}\| \leq \|D_m\|$, by definition $\|D_m\|^2 \leq 3(\|V\|^2 + \|W\|^2), \forall m \geq 0$ and hence $\Theta$ can be extend as a map $\bigoplus_{m \geq 0} \Theta_{(m)}$ from $\mathcal{A} \otimes \mathcal{B}(\Gamma \mathcal{H}(k_0))$ into itself with $\|\bigoplus_{m \geq 0} \Theta_{(m)}\| \leq 3(\|V\|^2 + \|W\|^2)$, we denote this map by same symbol $\Theta$.

For any fixed $m \geq 0$ let us look at the following qsde on $\mathcal{A} \otimes \mathcal{B}(k_0^{\otimes 3}) \otimes \Gamma$

$$\eta_{m,t} = id_{\mathcal{A} \otimes \mathcal{B}(k_0^{\otimes 3})} \otimes \mathcal{I} + \int_0^t \eta_{m,s} \Lambda_{\Theta}(ds), 0 \leq t \leq \tau. \quad (2.6)$$

Since we have the estimate, for any $X \in \mathcal{A} \otimes \mathcal{B}(k_0^{\otimes 3}), \xi \in h_0 \otimes k_0^{\otimes 3} \otimes \hat{k}_0$

$$\|\Theta(X)\xi\| = \|\Theta_{(m)}(X)\xi\| \leq \|(X \otimes 1_{\mathcal{H}})D_m \xi\|,$$

by a simple adoption of the proof of the existence of solution $J_t$ of the qsde (2.3) (Theorem 3.3.6 (i) in [2]), it can be shown that

(i) the qsde (2.6) admit a unique solution $\eta_{m,t}$ as an adapted regular process mapping $\mathcal{A} \otimes \mathcal{B}(k_0^{\otimes 3}) \otimes \mathcal{E}(\mathcal{C})$ into $\mathcal{A} \otimes \mathcal{B}(k_0^{\otimes 3}) \otimes \Gamma$.

(ii) $\eta_{m,t}$ satisfies the estimate

$$\sup_{0 \leq t \leq \tau} \|\eta_{m,t}(X \otimes e(f))\xi\| \leq C'(f)\|(X \otimes 1_{\Gamma \mathcal{H}(L^2([0,\tau],\mathcal{H}))})E_{\tau}\xi\|, \quad (2.7)$$

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where $f \in \mathcal{C}$ and $\mathcal{C}'(f)$ is some constant. The operator $E_\tau$ appears above is an element of 
$\mathcal{B}(h_0 \otimes k_0^{\otimes n}, h_0 \otimes k_0^{\otimes n} \otimes \Gamma_f(L^2([0, \tau], \mathcal{H})))$, define as follows:

$$E_\tau \xi = \bigoplus_{n \geq 0} (n!)^{\frac{1}{2}} E_n^{(n)} \xi,$$

where $E_n^{(n)} \in \mathcal{B}(h_0 \otimes k_0^{\otimes n}, h_0 \otimes k_0^{\otimes n} \otimes (L^2([0, \tau], \mathcal{H}))^{\otimes n})$ given by, for $\xi \in h_0 \otimes k_0^{\otimes n}$

$$E_n^{(0)} \xi = \xi,$$

$$(E_n^{(1)}(\xi))(s) = D(\xi \otimes \hat{f}(s)||\hat{f}_0(\xi)||) \quad \text{and iteratively}$$

$$(E_n^{(n)}(\xi))(s_1, s_2, \ldots, s_n) = (D_m \otimes 1_{L^2([0, \tau], \mathcal{H})^{\otimes n-1}}) Q_n$$

$$(\xi \otimes \hat{f}(s_1)\hat{f}_0(s_1)||)$$

( $Q_n : h_0 \otimes k_0^{\otimes n} \otimes L^2([0, \tau], \mathcal{H})^{\otimes (n-1)} \otimes k_0 \rightarrow h_0 \otimes k_0^{\otimes n} \otimes k_0 \otimes L^2([0, \tau], \mathcal{H})^{\otimes (n-1)}$ is the unitary operator which interchanges the third and fourth tensor components).

It is clear that $J_t \otimes id_{\mathcal{B}(k_0^{\otimes n})} \equiv \Upsilon_m(J_t \otimes id_{\mathcal{B}(k_0^{\otimes n})}) \Upsilon_m : \mathcal{A} \otimes \mathcal{B}(k_0^{\otimes n}) \otimes \mathcal{E}(\mathcal{C}) \rightarrow \mathcal{A} \otimes \mathcal{B}(k_0^{\otimes n}) \otimes \mathcal{E}(\mathcal{C})$, where $\Upsilon_m : h_0 \otimes k_0^{\otimes n} \otimes \mathcal{E}(\mathcal{C}) \rightarrow h_0 \otimes \mathcal{E}(\mathcal{C}) \otimes k_0^{\otimes n}$

satisfies the qSDE (2.7) and hence $\eta_{m,t} = J_t \otimes id_{\mathcal{B}(k_0^{\otimes n})}$. By definition of $E_\tau$, it can be easily seen that $\|E_\tau\|$ uniformly bounded for $m \geq 0$ and hence the estimate (2.7) allow us to extend $\{J_t\}$ as a regular adapted process $\{\bigoplus_{m \geq 0} J_t \otimes id_{\mathcal{B}(k_0^{\otimes n})}\}$ mapping $\mathcal{A} \otimes \mathcal{B}(\Gamma_f(k_0)) \otimes \mathcal{E}(\mathcal{C})$ into $\mathcal{A} \otimes \mathcal{B}(\Gamma_f(k_0)) \otimes \mathcal{E}(\mathcal{C})$, we denote this family by same symbol $J_t$. For a given $f \in \mathcal{C}$ this $J_t$ satisfies

$$\|J_t(Xe(f)\xi)\| \leq D'||X'||\|\xi\|, \forall X \in \mathcal{A} \otimes \mathcal{B}(\Gamma_f(k_0)) \text{ and } \xi \in h_0 \otimes \Gamma_f(k_0)), \quad (2.8)$$

for some constant $D'$ independent of $X$ and $\xi$.

**Construction of quantum random walks**

In particular suppose we are given with a conservative uniformly continuous QDS $T_t$ on a von Neumann algebra $\mathcal{A}$ with generator $\mathcal{L}$ given by

$$\mathcal{L}(x) = R^*(x \otimes 1_{k_0})R - \frac{1}{2} R^*Rx - \frac{1}{2} xR^*R, \forall x \in \mathcal{A} \quad (2.9)$$

for some Hilbert space $k_0$ and $R \in \mathcal{A} \otimes k_0$.

**Theorem 2.1.** Let $\mathcal{L}$ be given by (2.9). Then there exists a $*$-homomorphic family $\{\beta(h)\}_{h>0}$ such that the family of linear maps $E(h) : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}(k_0)$ given by, for
\( x \in \mathcal{A} \)

\[
E(h, x) = \begin{pmatrix}
  h^{-2} [\beta_1(h, x) - x - h\theta_1(x)] & h^{-2} (\beta_2(h, x) - \sqrt{h}\theta_2(x))^* \\
  h^{-2} [\beta_3(h, x) - \sqrt{h}\theta_3(x)] & h^{-1} [\beta_4(h, x) - x \otimes 1_{k_0} - \theta_4(x)]
\end{pmatrix},
\]

is uniformly norm bounded, also as maps from \( \mathcal{A} \otimes \mathcal{B}(\mathcal{K}_0) \) into itself, have uniform norm bound i.e. \( \|E(h)\| \leq M \), for some constant \( M \) independent of \( h \).

In particular, it follows that for any \( l \)

\[
\|b_l(X) - h^\varepsilon \theta_l(X)\| \leq M \|X\| h^{1+\varepsilon}, \forall X \in \mathcal{A} \otimes \mathcal{B}(\mathcal{K}_0) \quad (2.10)
\]

where \( \varepsilon_1 = 1, \varepsilon_2 = \varepsilon_3 = \frac{1}{2} \) and \( \varepsilon_4 = 0 \).

**Proof.** Here the map \( \Theta \) is given by \( \Theta(x) = \begin{pmatrix} \theta_1(x) & (\theta_2(x))^* \\ \theta_3(x) & \theta_4(x) \end{pmatrix} = \begin{pmatrix} \mathcal{L}(x) & \delta_1(x) \\ \delta(x) & \sigma(x) \end{pmatrix}, \)

\( \forall x \in \mathcal{A}, \) where \( \delta(x) = (x \otimes 1_{k_0})R - Rx, \delta_1(x) = (\delta(x)^*)^* = R^*(x \otimes 1_{k_0}) - xR^* \) and \( \sigma = 0 \). Setting \( \tilde{R} = \begin{pmatrix} 0 & -R^* \\ R & 0 \end{pmatrix} \) from \( \mathcal{K}_0 \) to itself. It is clear that \( \tilde{R} \) is a bounded skew symmetric operator thus it generate a one parameter unitary group \( \{e^{it\tilde{R}}\} \).

For \( h > 0 \), we consider the unitary operator \( U(h) = e^{\sqrt{h}\tilde{R}} \) which can be written as \( \begin{pmatrix} \cos(\sqrt{h}|R|) - \sqrt{h}D(h)R^* \\ \sqrt{h}RD(h) & \cos(\sqrt{h}|R^*|) \end{pmatrix} \) where \( D(h) = \sin(\sqrt{h}|R|)(\sqrt{h}|R|)^{-1} \) and \( |R|, |R^*| \) denote the positive square root of \( R^*R \) and \( RR^* \) respectively. It can easily be observed that

\[
\begin{align*}
\| \cos(\sqrt{h}|R|) - 1_{\mathcal{K}_0} + \frac{h}{2}|R|^2 \| & \leq h^2\|R\|^4, \\
\| \cos(\sqrt{h}|R|) - 1_{\mathcal{K}_0} \| & \leq h\|R\|^2, \\
\| \cos(\sqrt{h}|R^*|) - 1_{\mathcal{K}_0 \otimes \mathcal{K}_0} \| & \leq h\|R\|^2, \\
\|D(h) - 1_{\mathcal{K}_0} \| & \leq h\|R\|^2, \\
\|\cos(\sqrt{h}|R|)\| & \leq 1, \\
\|D(h)\| & \leq 1.
\end{align*}
\]

Now we define a \( \ast \)-homomorphism \( \beta(h) \) from \( \mathcal{A} \) to \( \mathcal{A} \otimes \mathcal{B}(\mathcal{K}_0) \) implemented by the unitary \( U(h) \), i.e. for \( x \in \mathcal{A}, \beta(h, x) := \beta(h)(x) = (U(h))^* (x \otimes 1_{k_0}) U(h) \). So for any \( x \in \mathcal{A}, \beta(h, x) = \begin{pmatrix} \beta_1(h, x) & (\beta_2(h, x))^* \\ \beta_3(h, x) & \beta_4(h, x) \end{pmatrix} \)

\[
= \begin{pmatrix}
  \{\cos(\sqrt{h}|R|) x \cos(\sqrt{h}|R|) \}
  & \{ -\sqrt{h} \cos(\sqrt{h}|R|) x D(h) R^* \\
  + hD(h) R^* (x \otimes 1_{k_0}) RD(h) \} \\
  \{-\sqrt{h} RD(h) x \cos(\sqrt{h}|R|) \}
  & \{ h RD(h) x D(h) R^* \\
  + \sqrt{h} \cos(\sqrt{h}|R^*|) (x \otimes 1_{k_0}) RD(h) \} \\
  + \sqrt{h} \cos(\sqrt{h}|R^*|) (x \otimes 1_{k_0}) RD(h) \}
\end{pmatrix}.
\]
We have
\[ \beta_1(h, x) - x - h\theta_1(x) \]
\[ = \cos(\sqrt{h}|R|)x\cos(\sqrt{h}|R|) + hD(h)R^*(x \otimes 1_{k_0})RD(h) \]
\[ - x - h \left( R^*(x \otimes 1_{k_0})R - \frac{1}{2}|R|^2x - \frac{1}{2}|x|^2 \right) \]
\[ = \left[ \cos(\sqrt{h}|R|) - 1_{h_0} + \frac{1}{2}|R|^2 \right] x \cos(\sqrt{h}|R|) \]
\[ + x \left[ \cos(\sqrt{h}|R|) - 1_{h_0} + \frac{1}{2}|R|^2 \right] + \frac{1}{2}|R|^2x \left[ 1_{h_0} - \cos(\sqrt{h}|R|) \right] \]
\[ + h[D(h) - 1_{h_0}]R^*(x \otimes 1_{k_0})RD(h) + hR^*(x \otimes 1_{k_0})RD(h). \]

By (2.11) we get
\[ \|\beta_1(h, x) - x - h\theta_1(x)\| \leq 5h^2\|R\|\|x\|. \] (2.12)

By definition we have
\[ \beta_2(x^*) - \sqrt{h}\theta_2(x^*) = \beta_3(x) - \sqrt{h}\theta_3(x) \]
\[ = \sqrt{h} \left[ -RD(h)x\cos(\sqrt{h}|R|) + \cos(\sqrt{h}|R^*|)(x \otimes 1_{k_0})RD(h) - (x \otimes 1_{k_0})R + Rx \right] \]
\[ = \sqrt{h} \left[ -RD(h)x[\cos(\sqrt{h}|R|) - 1_{h_0}] - R[D(h) - 1_{h_0}] + \cos(\sqrt{h}|R^*|)(x \otimes 1_{k_0})RD(h) - 1_{h_0} + \cos(\sqrt{h}|R^*|) - 1_{h_0}] \right]. \]

Using (2.11) we get
\[ \|\beta_2(x^*) - \sqrt{h}\theta_2(x^*)\| = \|\beta_3(x) - \sqrt{h}\theta_3(x)\| \]
\[ \leq \sqrt{h} \left[ \|RD(h)x[\cos(\sqrt{h}|R|) - 1_{h_0}]\| + \|R[D(h) - 1_{h_0}]\| \right] \]
\[ + \|\cos(\sqrt{h}|R^*|)(x \otimes 1_{k_0})RD(h) - 1_{h_0}\| + \|\cos(\sqrt{h}|R^*|) - 1_{h_0} \otimes 1_{k_0} \| \right] \]
\[ \leq 4h^{\frac{3}{2}}\|R\|^3\|x\|. \]

Now let us consider \( \beta_4(x) - \theta_4(x) \), we have
\[ \|\beta_4(x) - \theta_4(x)\| \]
\[ = \|hRD(h)xRD(h)R^* + \cos(\sqrt{h}|R^*|)(x \otimes 1_{k_0}) \cos(\sqrt{h}|R^*|) - (x \otimes 1_{k_0})\| \]
\[ \leq h\|RD(h)xRD(h)R^*\| + \|\cos(\sqrt{h}|R^*|) - 1_{h_0} \otimes 1_{k_0} \| \]
\[ + \|\cos(\sqrt{h}|R^*|) - 1_{h_0} \otimes 1_{k_0} \| \]
\[ \leq 3h\|R\|^3\|x\|. \]
Thus for $l = 1, 2, 3$ and 4,

$$
\|\beta_l(h, x) - b_l(x) - h^{e_l} \theta_l(x)\| \leq M \|x\| h^{1+e_l}, \forall x \in \mathcal{A},
$$

(2.13)

where constant $M = 5(\|R\|^2 + \|R\|^3 + \|R\|^4)$.

For $m \geq 0$, let us consider the amplification of the maps $\Theta, b$ and $\beta$ as maps from $\mathcal{A} \otimes \mathcal{B}(\Gamma_f(\hat{k}_0))$ into itself. For $X \in \mathcal{A} \otimes \mathcal{B}(\hat{k}_0)$

$$
\Theta(X) = \Theta_m(X) = Q_m \left( \Theta \otimes id_{\mathcal{B}(\hat{k}_0)} \right)(X) Q_m,
$$

where $Q_m : h_0 \otimes \hat{k}_0 \otimes \hat{k}_0 \rightarrow h_0 \otimes \hat{k}_0 \otimes \hat{k}_0$ is the unitary operator which interchanges the second and third tensor components. This operator $Q_m = 1_{h_0} \otimes \hat{k}_0 \otimes \hat{k}_0$ where $q_m : h_0 \otimes \hat{k}_0 \otimes \hat{k}_0 \rightarrow h_0 \otimes \hat{k}_0 \otimes \hat{k}_0$ is defined as $Q_m$. By definition we have

$$
\theta_1(X) = (R^* \otimes 1_{\hat{k}_0})q_m(X \otimes 1_{\hat{k}_0})q_m^*(R \otimes 1_{\hat{k}_0}) - \frac{1}{2} \|R\| \otimes 1_{\hat{k}_0}X - \frac{1}{2} X(\|R\| \otimes 1_{\hat{k}_0})
$$

$$
\theta_2(X)^* = \left[ X(R^* \otimes 1_{\hat{k}_0})q_m - (R^* \otimes 1_{\hat{k}_0})q_m(X \otimes 1_{\hat{k}_0}) \right] q_m^\ast
$$

$$
\theta_3(X) = q_m^\ast \left[ (R \otimes 1_{\hat{k}_0})X - q_m(X \otimes 1_{\hat{k}_0})q_m^\ast(R \otimes 1_{\hat{k}_0}) \right]
$$

$$
\theta_4(X) = 0
$$

and components of $\beta(h, X)$ are

$$
\beta_1(h, X) = (\cos(\sqrt{h}|R|) \otimes 1_{\hat{k}_0})X(\cos(\sqrt{h}|R|) \otimes 1_{\hat{k}_0})
$$

$$
+ h(D(h)R^* \otimes 1_{\hat{k}_0})q_m(X \otimes 1_{\hat{k}_0})q_m^\ast(RD(h) \otimes 1_{\hat{k}_0})
$$

$$
\beta_2(h, X)^* = \left[ -\sqrt{h}(\cos(\sqrt{h}|R|) \otimes 1_{\hat{k}_0})X(D(h)R^* \otimes 1_{\hat{k}_0}) \right] q_m
$$

$$
+ \sqrt{h}(D(h)R^* \otimes 1_{\hat{k}_0})q_m(X \otimes 1_{\hat{k}_0})q_m^\ast(\cos(\sqrt{h}|R|) \otimes 1_{\hat{k}_0})
$$

$$
\beta_3(h, X) = q_m^\ast \left[ -\sqrt{h}(RD(h) \otimes 1_{\hat{k}_0})X(\cos(\sqrt{h}|R|) \otimes 1_{\hat{k}_0}) \right]
$$

$$
+ \sqrt{h}(\cos(\sqrt{h}|R|)^\ast \otimes 1_{\hat{k}_0})q_m(X \otimes 1_{\hat{k}_0})q_m^\ast(RD(h) \otimes 1_{\hat{k}_0})
$$

$$
\beta_4(h, X) = q_m^\ast \left[ h(RD(h) \otimes 1_{\hat{k}_0})X(D(h)R^* \otimes 1_{\hat{k}_0}) \right]
$$

$$
+ \left(\cos(\sqrt{h}|R|)^\ast \otimes 1_{\hat{k}_0})q_m(X \otimes 1_{\hat{k}_0})q_m^\ast(\cos(\sqrt{h}|R|) \otimes 1_{\hat{k}_0}) \right] q_m.
$$
By same argument as for (2.13) one has
\[
\|\beta_1(h, X) - b_t(X) - h\theta_1(X)\| \leq C'h^{1+\varepsilon_i}\|X\|, \forall X \in \mathcal{A} \otimes \mathcal{B}(\mathcal{K}_0^{\infty}),
\]
for some constant $C'$ independent of $h > 0$, $m \geq 0$. Thus $\|E(h)(X)\| \leq M\|X\|, \forall X \in \mathcal{A} \otimes \mathcal{B}(\Gamma_f(\mathcal{K}_0))$, for some constant $M$ independent of $h$.

Now consider the quantum random walks \( \{p_t^{(h)} : h > 0\} \) associated with the *-homomorphic family \( \{\beta(h)\} \) in the above theorem. In the next section we shall prove that this quantum random walks \( \{p_t^{(h)} : h > 0\} \) converges strongly.

**Convergence of quantum random walk**

Let \( \mathcal{B} = \mathcal{A} \otimes \mathcal{B}(\Gamma_f(\mathcal{K}_0)) \otimes \mathcal{B}(\Gamma) \), which can be decomposed as
\[
\mathcal{B} = \left( \mathcal{A} \otimes \bigoplus_{m \geq 0} \mathcal{B}(\mathcal{K}_0^{\infty}) \right) \otimes \mathcal{B}_c \text{ for some subspace } \mathcal{B}_c.
\]
Now let us consider the extensions of all these maps \( \Theta, \beta(h), b, p_t^{(h)} \) and \( \mathcal{P}_t^{(h)} \) as bounded linear maps from \( \mathcal{B} \) into itself, given by, for example extension of \( p_t^{(h)} \) is \( \bigoplus_{m \geq 0} p_t^{(h)} \otimes \text{id}_{\mathcal{B}(\mathcal{K}_0^{\infty})} \oplus \mathcal{B}_c \). We denote these extensions by same symbols as the original maps. From the Theorem 2.1 it follows that
\[
\|\beta_t(h, X) - b_t(X) - h\varepsilon_t\theta_t(X)\| \leq C'\|X\|h^{1+\varepsilon_i}, \forall X \in \mathcal{A} \otimes \mathcal{B}(\mathcal{K}_0^{\infty}). \tag{2.14}
\]

For $h \geq 0$ we define a map \( \Theta(h) := \begin{pmatrix} h\theta_1 & h^{1/2}(\theta_2(\cdot))' \\ h^{1/2}\theta_3 & \theta_4 \end{pmatrix} \) from \( \mathcal{A} \) to \( \mathcal{A} \otimes \mathcal{B}(\mathcal{K}_0) \), as the map \( \Theta, \Theta(h) \) also extend as a bounded map from \( \mathcal{B} \) into itself. Here we have the following observations which will be needed later for proving the convergence of quantum random walk \( p_t^{(h)} \).

**Lemma 2.2.** For any $l, X \in \mathcal{A} \otimes \mathcal{B}(\mathcal{K}_0^{\infty}), \xi \in \mathcal{h}_0 \otimes \mathcal{K}_0^{\infty}$ and $f \in \mathcal{M}$ we have

1. \( \|\sum_{k=1}^{n} N_{P_{(k-1)h}[\beta(h, X) - b(h, X) - \Theta(h, X)][k] \xi e(f)}\| \leq \sqrt{h}C_1(f, t)\|X\|\|\xi\|\)

2. \( \|\sum_{k=1}^{n} \left[ N_{P_{(k-1)h}[\Theta(h, X)][k]} - A\right] p_k^{(h)}(\Theta(h, X))[k] \xi e(f)\|^2 \)

\( \leq hC_2(f, t)\|X\|^2\|\xi\|^2, \)

3. \( \|\sum_{k=1}^{n} p_k^{(h)}(X)(1 - P_h[k]) \xi e(f))\|^2 \leq h c(f, t)\|X\|^2\|\xi\|^2. \)

where constants $c(f, t)$ is as in Lemma 2.3, $C_1(f, t) = t(1 + \|f\|_\infty)\|e(f)\|$ and $C_2(f, t) = (1 + t)^2(\|f\|_\infty + \|f\|_\infty^2)(1 + \|\Theta\|^2)\|e(f)\|^2$. 

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Proof. (1). For any \( l \) we have

\[
\| \sum_{k=1}^{n} N^l_{p_{(k-1)h}}[\beta_l(h,X) - b_l(X) - h^{\varepsilon_l}\theta_l(X)] [k] \xi_l(f) \| \\
\leq \sum_{k=1}^{n} \| N^l_{p_{(k-1)h}}[\beta_l(h,X) - b_l(X) - h^{\varepsilon_l}\theta_l(X)] [k] \xi_{k-1}(f[k]) \| \| e(f[h]) \|
\]

where \( \xi_{k-1} = \xi_l(f[(k-1)h]) \) is a vector in the initial Hilbert space \( h_0 \otimes \Gamma_n(k_0) \otimes \Gamma_{(k-1)h} \). For any \( l \), from (2.14) and contractivity of \( p_{l}^{(h)} \), we get

\[
\| p_{(k-1)h}[\beta_l(h,X) - b_l(X) - h^{\varepsilon_l}\theta_l(X)] \| \leq Ch^{1+\varepsilon_l} \| X \|,
\]

hence by (1.4) the above quantity is dominated by

\[
\sum_{k=1}^{n} h^{2} C(1 + \| f \|_{\infty}) \| X \| \| \xi_l(f) \| \text{ and required estimate follows.}
\]

(2). By Lemma 1.3 the terms correspond to \( l = 1,2 \) can be estimated as,

\[
\| \sum_{k=1}^{n} \left[ h^{\varepsilon_l} N^l_{p_{(k-1)h}}[\theta_l(X)] [k] - \Lambda^l_{p_{(k-1)h}}[\theta_l(X)] [k] \right] \xi_l(f) \|
\leq \sum_{k=1}^{n} \| \left[ h^{\varepsilon_l} N^l_{p_{(k-1)h}}[\theta_l(X)] [k] - \Lambda^l_{p_{(k-1)h}}[\theta_l(X)] [k] \right] \xi_{k-1}(f[k]) \| \| e(f[h]) \|
\leq \sum_{k=1}^{n} h^{2} \| p_{(k-1)h}[\theta_l(X)] \| \| \xi_{k-1} \| (\| f \|_{\infty} + \| f \|_{\infty}^{2}) \| e(f[(k-1)h]) \|
\leq (\| f \|_{\infty} + \| f \|_{\infty}^{2}) \| \Theta \| \sum_{k=1}^{n} h^{2} \| X \| \| \xi_l(f) \|.
\]

Thus the required estimate follows. Now consider other two terms correspond to \( l = 3 \) and 4. Setting for \( 1 \leq m \leq n \)

\[
Z_m = \sum_{k=1}^{n} \left[ \sqrt{h} N^l_{p_{(k-1)h}}[\theta_l(X)] [k] - \Lambda^l_{p_{(k-1)h}}[\theta_l(X)] [k] \right],
\]

by Lemma 1.3 (a), we have

\[
\| Z_m u e(f_{mh}) \|
\leq \sum_{k=1}^{m} \| \left[ \sqrt{h} N^l_{p_{(k-1)h}}[\theta_l(X)] [k] - \Lambda^l_{p_{(k-1)h}}[\theta_l(X)] [k] \right] \xi_{(k-1)}(f[k]) \| \| e(f_{(kh, mh)}) \|
\leq \sum_{k=1}^{m} h \| p_{(k-1)h}[\theta_l(X)] \| \| f \|_{\infty} \| e(f[k]) \| \| e(f_{(kh, mh)}) \|.
\]

Thus

\[
\| Z_m u e(f_{mh}) \| \leq t \| \Theta \| \| f \|_{\infty} \| X \| \| \xi_l(f_{mh}) \|.
\] (2.15)
We have the following equality,

\[
\| Z_n \xi e(f) \|^2
= \sum_{k=1}^{n} \left\| \sqrt{h} N_{(k-1)h}(\theta(X)) [k] - \Lambda_{P_{(k-1)h}(\theta(X))}^{(h)} [k] \right\| \xi_{(k-1)} e(f_{[k]}) \|^2 \| e(f_{[k]}) \|^2 \\
+ 2 \operatorname{Re} \sum_{k=1}^{n} \langle Z_{k-1} \xi_{k-1} e(f_{[k]}), \left[ \sqrt{h} N_{(k-1)h}(\theta(X)) [k] - \Lambda_{P_{(k-1)h}(\theta(X))}^{(h)} [k] \right] \xi_{k-1} e(f_{[k]}), \| e(f_{[k]}) \|^2 \rangle.
\]

By the estimate in Lemma 1.3,

\[
\| \sum_{k=1}^{n} \left[ \sqrt{h} N_{(k-1)h}(\theta(X)) [k] - \Lambda_{P_{(k-1)h}(\theta(X))}^{(h)} [k] \right] \xi e(f) \|^2 \\
\leq \sum_{k=1}^{n} h^2 \| P_{(k-1)h}(\theta(X)) \xi_{k-1} \| \| f \|_\infty^2 \| e(f_{(k-1)h}) \|^2 \\
+ 2 \sum_{k=1}^{n} h^2 \| Z_{k-1} \xi_{k-1} \| \| P_{(k-1)h}(\theta(X)) \| \| \xi_{k-1} \| \| f \|_\infty^2 \| e(f_{(k-1)h}) \|^2.
\]

Now using (2.15), above quantity is less than or equal to

\[
\sum_{k=1}^{n} h^2 \| f \|_\infty^2 \| X \|^2 \| \xi e(f) \|^2 \\
+ 2 \sum_{k=1}^{n} h^2 t \| \Theta \|^2 \| f \|_\infty^3 \| X \|^2 \| \xi e(f) \|^2
\]

and required estimate follows.

(3.) The proof is same as for estimate (1.9) in Lemma 1.5.

Now we shall prove the strong convergence of the quantum random walks \( p_{(h,t)} \). Note that \( J_t : \mathcal{A} \otimes \mathcal{E}(\mathcal{K}) \to \mathcal{A} \otimes \Gamma \) is the unique solution of the qsde

\[
J_t = id_{\mathcal{A} \otimes \Gamma} + \int_0^t J_s \Lambda \Theta (ds).
\]

(2.16)

We define a family of maps \( J_{(h,t)}^{(h)} \) by

\[
J_{0}^{(h)}(xe(f))u = xue(f) \\
J_{t}^{(h)}(xe(f))u = J_{nh}(xe(f))u = xue(f) + \sum_{k=1}^{n} J_{(k-1)h}(\Lambda_{\Theta}(x)[k]e(f))u
\]
for \( t \in ((n - 1)h, nh] \). Thus by definition
\[
J_t^{(h)}(xe(f))u = J_{nh}^{(h)}(xe(f))u = xue(f) + \sum_{k=1}^{n} \Lambda_{j(k-1)h}(\Theta(x))[k]ue(f). \tag{2. 17}
\]

For \( u \in h_0, f \in \mathcal{M} \) the adapted process \( J_t \) satisfies
\[
J_t(xe(f))u = xue(f) + \int_0^t J_s \Lambda_{\Theta}(ds)(xe(f))u
\]
and the map \( t \mapsto J_t(xe(f))u \) is continuous. Thus by definition of this integral
\[
\lim_{h \to 0} \|J_t(xe(f))u - J_t^{(h)}(xe(f))u\| = 0
\]
and hence
\[
\lim_{h \to 0} \|j_t(x)e(f) - j_t(x)^{(h)}e(f)\| = 0. \tag{2. 18}
\]

Now we are in position to prove the following result

**Theorem 2.3.** Let \( p_t^{(h)} \) be the quantum random walk associated with \( \beta(h) \). Then for each \( x \in \mathcal{A} \) and \( t \geq 0, p_t^{(h)}(x) \) converges strongly to \( j_t(x) \). Thus \( j_t : \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}(\Gamma) \) is a \(*\)-homomorphic flow.

**Proof.** In order to prove
\[
\lim_{h \to 0} \|p_t^{(h)}(xe(f)) - j_t(x)e(f)\| = 0, \quad \forall u \in h_0, f \in \mathcal{M}, \tag{2. 19}
\]
by (2. 18) it is sufficient to show that
\[
\lim_{h \to 0} \|p_t^{(h)}(xe(f)) - j_t^{(h)}(xe(f))\| = 0 \forall u \in h_0, f \in \mathcal{M}. \tag{2. 20}
\]

For any fixed \( h > 0, f \in \mathcal{M} \) let us define a family of bounded linear maps
\[
W_t^{(h)} : \mathcal{A} \to \mathcal{A} \otimes \Gamma
\]
given by, for \( x \in \mathcal{A} \) and \( u \in h_0, \)
\[
W_t^{(h)}(x)u = p_t^{(h)}(x)u - j_t^{(h)}(x)u = [p_t^{(h)}(xe(f)) - j_t^{(h)}(xe(f))]u =: Y_t^{(h)}(xe(f))u.
\]

Here, recall that \( \{J_t\} \) extend as a regular adapted process mapping \( \mathcal{A} \otimes \mathcal{B}(\Gamma_0(\hat{k}_0)) \otimes \mathcal{E}(\mathcal{C}) \) into \( \mathcal{A} \otimes \mathcal{B}(\Gamma_0(\hat{k}_0)) \otimes \Gamma \) and hence for each \( X \in \mathcal{A} \otimes \mathcal{B}(\Gamma_0(\hat{k}_0)) \) the family \( \{j_t(X)\} \) define by \( j_t(X)\xi e(f) = J_t(X \otimes e(f))\xi, \forall \xi \in h_0 \otimes \Gamma_0(\hat{k}_0), f \in \mathcal{C}, \) is a regular
(h₀ ⊗ Γᵣ, K)-adapted process. For a given f ∈ M ⊆ C by estimate \(2.8\), \(Wₜ^{(h)}\) extend as a bounded linear map from \(A ⊗ B(Γᵣ(k₀))\) into \(A ⊗ B(Γᵣ(k₀)) ⊗ Γ\).

Viewing \(A ⊗ B(Γᵣ(k₀))\) and \(A ⊗ B(Γᵣ(k₀)) ⊗ Γ\) as subspaces of \(B\), let us denote by same symbol \(Wₜ^{(h)}\) to the canonical extentions of \(Wₜ^{(h)}\) as linear maps from \(B\) into itself preserving the norm.

In order to prove \(2.20\) we shall show that \(\|Wₜ^{(h)}\|\) (as maps from \(B\) into itself) converges to 0 as \(h\) tends to 0. For any \(X ∈ A ⊗ B(k₀ \{\bar{1}\})\) and \(ξ ∈ h₀ ⊗ k₀ \{\bar{1}\}\) by \(2.17\) and \(1.10\), we have

\[
Wₜ^{(h)}(X)ξ = \sum_{k=1}^{n} \left[ N_{p(k-1)h}(βₜ h X, bₜ X) \right] [k] - Λ_{j(k-1)h}(Θ(X)) [k] \right] ξᵣ \mathbf{e}(f)
- \sum_{k=1}^{n} pₜ h(X)(1 - Pₜ h[k])ξᵣ \mathbf{e}(f)

= \sum_{k=1}^{n} \left[ \left( N_{p(k-1)h}(βₜ h X, bₜ X) \right) [k] - N_{p(k-1)h}(Θ(X)) [k] \right] \mathbf{e}(f)
+ \left( N_{p(k-1)h}(Θ(X)) [k] - Λ_{j(k-1)h}(Θ(X)) [k] \right) \mathbf{e}(f)
- \sum_{k=1}^{n} pₜ h(X)(1 - Pₜ h[k])ξᵣ \mathbf{e}(f)

\]

Using linearity of \(N_{(\cdot)}[k]\) and \(Λ_{(\cdot)}[k]\),

\[
\|Wₜ^{(h)}(X)ξ\|^2 \leq 4 \left( \| \sum_{k=1}^{n} N_{p(k-1)h}(βₜ h X, bₜ X - Θ(X)) [k] ξ[k-1] \mathbf{e}(f) \|^2 \right)
+ \| \sum_{k=1}^{n} [N_{p(k-1)h}(Θ(X)) [k] - Λ_{p(k-1)h}(Θ(X)) [k]] \mathbf{e}(f) \|^2 \right)
+ \| \sum_{k=1}^{n} Pₜ h(X)(1 - Pₜ h[k])ξ[k-1] \mathbf{e}(f) \|^2 \right)
+ \| \sum_{k=1}^{n} Λ_{p(k-1)h}(Θ(X)) [k] ξ[k-1] \mathbf{e}(f) \|^2 \right)
= 4(I₁ + I₂ + I₃ + I₄).

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By Lemma 2.2 we have

\[ I_1 + I_2 + I_3 \leq \text{const}(f, t)\|X\|^2\|\xi\|^2h. \]

Now let us consider the terms in \( I_4 \). We have by estimate (3.13) in Proposition 3.3.5 in [2]

\[
\| \sum_{k=1}^{n} \Lambda_{\mu_{(k-1)h}-\nu(k-1)h}[\Theta(X)]^k \xi \mathbf{e}(f) \|^2 \\
= \| \sum_{k=1}^{n} Y_{(k-1)h}^r \Lambda_{\nu}[k] X \mathbf{e}(f) \xi \|^2 \\
= \| \int_{0}^{nh} Y_{s}^r \Lambda(\mathbf{e}(f)) \xi \|^2 \\
\leq 2c_f (1 + \|f\|^2_{\infty}) \int_{0}^{nh} \| (Y_{s}^r \otimes 1_{k_0})(\Theta(X) \mathbf{e}(f)) \| \xi \|^2 ds.
\]

It can be easily seen that

\[
(Y_{s}^r \otimes 1_{k_0})(\Theta(X) \mathbf{e}(f)) = [(Y_{s}^r \otimes id_{B_{k_0}})(\Theta(X) \mathbf{e}(f))] f(s) = [Y_{s}^r(\Theta(X) \mathbf{e}(f))] f(s),
\]

so the above quantity is equal to

\[
2c_f (1 + \|f\|^2_{\infty}) \int_{0}^{nh} \| (Y_{s}^r(\Theta(X) \mathbf{e}(f)) \| f(s) \xi \|^2 ds,
\]

\[
= 2c_f (1 + \|f\|^2_{\infty}) \int_{0}^{nh} \| W_s^r(\Theta(X)) \xi \otimes f(s) \|^2 ds
\]

\[
\leq 2c_f (1 + \|f\|^2_{\infty})^2 \sum_{k=1}^{n} h \| W_{(k-1)h}^r \|^2 \| \Theta(X) \|^2 \| \xi \|^2
\]

\[
\leq c_f \sum_{k=1}^{n} h \| W_{(k-1)h}^r \|^2 \| \Theta \|^2 \| X \|^2 \| \xi \|^2.
\]

Combining all the above estimates, we obtained

\[
\| W_t^r(X) \xi \|^2 \leq hC \| X \|^2 \| \xi \|^2 + D \sum_{k=1}^{n} h \| W_{(k-1)h}^r \|^2 \| X \|^2 \| \xi \|^2
\]

(2.21)

for some constant \( C \) and \( D \) independent of \( h \). For any \( X \in B \) and \( \xi \in h_0 \otimes \Gamma_{h}(k_0) \otimes \Gamma \)
we can write \( X = \bigoplus_{m \geq 0} X_m \oplus X' \) with \( X_m \in A \otimes B_{(k_0^m)} \), \( X' \in B_c \) and \( \xi = \bigoplus_{m \geq 0} \xi_m \oplus \xi' \) with \( \xi_m \in h_0 \otimes k_0^m \) and \( \xi' \) belong to orthogonal complement of
\( \mathfrak{h}_0 \otimes \mathfrak{k}_0^{\mathbb{D}} \) for all \( m \geq 0 \). Using the estimate (2.21) we have

\[
\| W_t^{(h)}(X) \xi \|^2 = \sum_{m \geq 0} \| W_t^{(h)}(X_m) \xi_m \|^2 \\
\leq hC \sum_{m \geq 0} \| X_m \|^2 \| \xi_m \|^2 + D \sum_{k=1}^n h \| W_{(k-1)h}^{(h)} \|^2 \sum_{m \geq 0} \| X_m \|^2 \| \xi_m \|^2 \\
\leq hC \| X \|^2 \| \xi \|^2 + D \sum_{k=1}^n h \| W_{(k-1)h}^{(h)} \|^2 \| X \|^2 \| \xi \|^2.
\]

Taking supremum over all \( \xi \in \mathfrak{h}_0 \otimes \Gamma \otimes \mathfrak{k}_0 \otimes \Gamma \), \( X \in B \) such that \( \| \xi \| \leq 1, \| X \| \leq 1 \) we get

\[
\| W_t^{(h)} \|^2 = \| W_{nh}^{(h)} \|^2 \leq hC + hD \sum_{k=1}^n \| W_{(k-1)h}^{(h)} \|^2. \tag{2.22}
\]

By definition \( \| W_0^{(h)} \|^2 = 0 \) so (2.22) gives \( \| W_t^{(h)} \|^2 \leq hC \) and

\[
\| W_{2h}^{(h)} \|^2 \leq hC + hD \| W_t^{(h)} \|^2 \leq ch(1 + hD).
\]

Then by induction it follows that

\[
\| W_t^{(h)} \|^2 = \| W_{nh}^{(h)} \|^2 \leq hC(1 + hD)^{n-1} \leq hCe^{Dt}
\]

and hence

\[
\lim_{h \to 0} \| W_t^{(h)} \|^2 = 0, \text{ in particular } \lim_{h \to 0} \| p_t^{(h)}(x)ue(f) - j_t^{(h)}(x)ue(f) \| = 0.
\]

Which says that for any \( u \in \mathfrak{h}_0 \) and \( f \in \mathcal{M} \), \( \{ p_t^{(h)}(x)ue(f) : h > 0 \} \) is Cauchy in \( \mathfrak{h}_0 \otimes \Gamma \). Since \( \| p_t^{(h)}(x) \| \leq \| x \| \) and algebraic tensor product \( \mathfrak{h}_0 \otimes \mathcal{E}(\mathcal{M}) \) is dense in \( \mathfrak{h}_0 \otimes \Gamma \) it follows that \( \{ p_t^{(h)}(x)\xi : h > 0 \} \) is Cauchy for all \( \xi \in \mathfrak{h}_0 \otimes \Gamma \) and hence for each \( x \in \mathcal{A} \), \( \{ p_t^{(h)}(x) \} \) converges strongly to \( j_t(x) \). Thus \( j_t : \mathcal{A} \to \mathcal{A} \otimes B(\Gamma) \) is a contractive \(*\)-homomorphic flow. \( \square \)

**Remark 2.4.** (i) It may be observed that in the above quantum stochastic dilation \( \{ j_t \} \) of the dynamical semigroup \( \{ T_t \} \) there is no “Poisson” term since \( \theta_4(x) = 0 \) for all \( x \in \mathcal{A} \). This is only to be expected since the choice of representation of \( \mathcal{A} \) is \( x \otimes 1_{\mathfrak{k}_0} \) for all \( x \in \mathcal{A} \). The more general case of dilation using the convergence of quantum random walks where the representation is non trivial (and therefore will have non zero “Poisson” component) is being investigated.

(ii) The method of proof employed above does not seem to be amenable to adaptation
for a dynamical semigroup with unbounded generator. On the other hand, one has example of the convergence of random walks to diffusion processes (which of course, has unbounded generators) in the classical case. For the handling of these cases, one may have to find different method to replace the proof of Theorem 2.3.

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