CAUSAL APPROACH TO SUPERSYMMETRY: CHIRAL SUPERFIELDS

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Abstract

We construct quantized free superfields and represent them as operator-valued distributions in Fock space starting with Majorana fields. The perturbative construction of the S-matrix for interacting theories is carried through by extending the causal method of Epstein and Glaser to superspace. We propose a scaling and singular order of distributions in superspace by a procedure which scales both commutative and non-commutative variables. Using this singular order the chiral (Wess-Zumino) model appears to be super-renormalizable.
1 Introduction

In the presentation of supersymmetry one mostly starts by representing the supersymmetric algebra on classical fields [1-3]. Quantum fields only appear at a later stage when propagators and Feynman rules are derived. One exception is the third volume of Weinberg’s book [4] where some quantum fields are directly constructed without introducing classical fields. In any case one is surprisingly brief about quantum superfields. Some work concerning the axiomatic supersymmetric quantum field theory is contained in [5] and [6]. We have looked in vain for commutation relations or a concrete Fock space representation of the free fields. If one does not want to rely on formal path-integral methods, this is the essential basis of perturbation theory. Furthermore, the complete understanding of the free quantum fields is indispensable for the construction of gauge theories in the spirit of a recent monograph [7]. For this reason we discuss the quantized Majorana field in the next section which gives the fermionic component of the superfield. The quantized chiral superfield is introduced in sect.3 by applying finite susy-transformations to the scalar component. The (anti-) commutation relations of the component fields yield the commutation relations for the chiral superfield. Since the component fields have their standard representations in the bosonic or fermionic sectors of Fock space, the whole superfield is represented there as well. A direct representation of the superfield without using component fields is also possible, we will consider this subject elsewhere.

The construction of the free asymptotic superfield $\Phi$ enables us to set up perturbation theory for the S-matrix, if a first order coupling $T_1[\Phi]$ is given. For this purpose we use the causal method of Epstein and Glaser [8] which we have to extend to superspace. We observe that the causal structure in $x$-space can be established as in the case of ordinary field theory [7, 9]. The subtle step in the construction of time-ordered products is the splitting of causal distributions $d(x, \theta, \bar{\theta})$ into advanced and retarded parts. The latter is obtained (in the so-called regular case) as the following distributional limit

$$r(x, \theta, \bar{\theta}) = \lim_{\lambda \to 0} \chi_0 \left( \frac{v \cdot x}{\lambda} \right) d(x, \theta, \bar{\theta}), \quad (1.1)$$

where $\chi_0(t)$ is a $C^\infty$-function on $\mathbb{R}^1$ which switches from 0 to 1 in the intervall $0 < t < 1$ and $v$ is a fixed time-like vector. The existence of the limit (1.1) depends on the behaviour of $d$ under scaling transformation $x' = \lambda x$ for $\lambda \to 0$. However, the scaling limit of $d$ only exists if the Grassmann variables $\theta, \bar{\theta}$ are also scaled according to

$$\theta' = \sqrt{\lambda} \theta, \quad \bar{\theta}' = \sqrt{\lambda} \bar{\theta}. \quad (1.2)$$

This leads us to modified definitions of quasi-asymptotics and singular order $\omega$ on superspace in sect.4. The resulting formula for $\omega$ shows that for the Wess-Zumino model only the second order vacuum polarisation graphs allows renormalisation, apart from the vacuum graph. This is the non-renormalisation theorem in the framework of the causal theory.

2 The quantized Majorana field

We start from the quantized Dirac field

$$\psi(x) = (2\pi)^{-3/2} \int d^3p \left[ b_s(\vec{p}) u_s(\vec{p}) e^{-ipx} + d^+_s(\vec{p}) v_s(\vec{p}) e^{ipx} \right] \quad (2.1)$$
\[ i \gamma^\mu \partial_\mu \psi(x) = m \psi(x) \]  

in the chiral representation

\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix},
\]

where \( \sigma_k \) are the Pauli matrices. The \( u \)- and \( v \)-spinors are given by

\[
u_s(\vec{p}) = \frac{1}{2\sqrt{E(E + m)}} \begin{pmatrix} (m + \sigma_\mu p^\mu) \chi_s \\ (m + \hat{\sigma}_\mu p^\mu) \chi_s \end{pmatrix}, \quad \psi_s(\vec{p}) = \frac{1}{2\sqrt{E(E + m)}} \begin{pmatrix} (m + \sigma_\mu p^\mu) \chi_s \\ -(m + \hat{\sigma}_\mu p^\mu) \chi_s \end{pmatrix}.
\]

Here

\[
\sigma_\mu p^\mu = E + \vec{\sigma} \cdot \vec{p} = \sigma p, \quad \hat{\sigma}_\mu p^\mu = E - \vec{\sigma} \cdot \vec{p} = \hat{\sigma} p,
\]

where \( E = \sqrt{\vec{p}^2 + m^2} \) and \( \chi_s \) is the spin basis

\[
\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

The charge-conjugate spinor field is defined by

\[
\psi_C = C \gamma^0 \psi^{+T} = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi^{+T} = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \psi^{+T},
\]

where \( C \) is the charge-conjugation matrix which can be expressed by \( \sigma_2 \) in the chiral representation and the transposition \( T \) refers always to the spinor components, only. The Majorana field \( \psi^M \) is defined by the property that it agrees with its charge-conjugate

\[
\psi_C^M(x) = \psi^M(x).
\]

This implies the identification

\[
d_s(\vec{p}) = -s b_{-s}(\vec{p})
\]

in (2.1). Hence, the Majorana field is given by

\[
\psi^M(x) = (2\pi)^{-3/2} \int d^3p \left[ u_s(\vec{p}) e^{-i\vec{p} \cdot \vec{x}} b_s(\vec{p}) - s v_s(\vec{p}) e^{i\vec{p} \cdot \vec{x}} b^+_s(\vec{p}) \right],
\]

where the emission and absorption operators obey the usual anti-commutation relations

\[
\{b_s(\vec{p}), b^+_s(\vec{q})\} = \delta_{ss'} \delta^3(\vec{p} - \vec{q}),
\]

and all other anti-commutators vanish. The fermionic Fock space is then constructed in the usual manner by applying \( b^{+\prime}s \) to the vacuum \( \Omega \).

We are now in the position to calculate the anti-commutator between the positive and negative frequency parts in (2.11)

\[
\{\psi_M^-(x)_a, \psi_M^+(y)_b\} = (2\pi)^{-3} \int d^3p \left[ u_{sa}(\vec{p}) s v_{-sb}(\vec{p}) T e^{-i\vec{p}(x-y)} \right].
\]
To evaluate this we regard the r.h.s. as a $4 \times 4$ matrix, $a, b = 1, \ldots, 4$. The matrix multiplication of the spin basis (2.7) gives the following $4 \times 4$ matrix

$$
\begin{pmatrix}
X_s \\
-\chi_s
\end{pmatrix}
\begin{pmatrix}
\chi_{-s} \\
-\chi_{-s}
\end{pmatrix}
= 
\begin{pmatrix}
0 & 1 & 0 & 1 \\
-1 & 0 & -1 & 0 \\
0 & 1 & 0 & 1 \\
-1 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
-\varepsilon & -\varepsilon \\
-\varepsilon & -\varepsilon
\end{pmatrix}
$$

(2.14)

where the antisymmetric matrix

$$
\varepsilon_{ab} = 
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
$$

(2.15)

was introduced. Using

$$
\hat{\sigma}_\mu = \varepsilon^{\sigma T} \varepsilon^{-1}
$$

(2.16)

and multiplying the various $2 \times 2$ matrices we finally get for the anti-commutator (2.13)

$$
= (2\pi)^{-3} \int \frac{d^3p}{2E} e^{-i{p}(x-y)} \begin{pmatrix}
-m\varepsilon & \sigma p\varepsilon \\
-\hat{\sigma} p\varepsilon & m\varepsilon
\end{pmatrix}.
$$

(2.17)

Without the matrix this would be equal to the positive frequency part of the Jordan-Pauli distribution $-iD^{(+)}(x-y)$. The negative frequency part is treated in the same way, yielding the following total anti-commutator

$$
\{\psi_M(x)_a, \psi_M(y)_b\} = 
\begin{pmatrix}
im\varepsilon & \sigma \partial_x \varepsilon \\
-\hat{\sigma} \partial_x \varepsilon & -im\varepsilon
\end{pmatrix}
D(x-y).
$$

(2.18)

Since $\hat{\sigma}\varepsilon = \varepsilon^{\sigma T} = -(\sigma\varepsilon)^T$, the r.h.s. shows the correct invariance under transposition and $x \leftrightarrow y$.

For supersymmetry one usually goes over to 2-component Weyl spinors in the notation of van der Waerden. Concerning notation we mostly follow the criterion of laziness: we prefer that notation which requires less keys in TEX on the computer. Instead of dotted greek indices we therefore use latin ones with a bar. The Majorana field is then written as

$$
\psi_M = \begin{pmatrix}
X_a \\
\bar{\chi}_a
\end{pmatrix}.
$$

(2.19)

The lower spinor is obtained from the upper one according to (2.8-9)

$$
\bar{\chi}^b = -i\sigma_2 \bar{\chi}_a \chi_a^+ T.
$$

(2.20)

Here we have written $-\hat{\sigma}_2$ instead of $\sigma_2$ in order to have the indices at the correct places. It is essential to write the antisymmetric matrix in (2.20) with $\hat{\sigma}_2$ because the Pauli matrices have mixed indices in contrast to the $\varepsilon$-tensor (2.15). Lowering the index $b$ by multiplying with $\varepsilon_{ab}$ and multiplying the $2 \times 2$ matrices on the r.h.s., we find the simple result

$$
\bar{\chi}_a = (\chi^+ T)_a = \chi_a^+.
$$

(2.21)

We have omitted the transposition because it has no consequence if we consider one spinor component only. It is not a definition but a derived fact that charge conjugation of the quantized Majorana field is represented in Fock space simply by hermitian conjugation.
From (2.18) we can now read off the various anti-commutators
\[
\{ \chi_a(x), \chi_b(y) \} = i m \epsilon_{ab} D(x - y) 
\]
(2.22)
\[
\{ \chi_a(x), \chi^b(y) \} = -i m \delta^b_a D(x - y) 
\]
(2.23)
\[
\{ \chi_a(x), \tilde{\chi}^b(y) \} = -\sigma^\alpha_{abc} \tilde{\chi}^c D(x - y) 
\]
(2.24)
\[
\{ \chi_a(x), \tilde{\chi}_b(y) \} = \sigma_{ab} D(x - y), 
\]
(2.25)
\[
\{ \tilde{\chi}^a(x), \tilde{\chi}^b(y) \} = i m \tilde{\epsilon}^{ab} D(x - y) 
\]
(2.26)
because the indices are raised and lowered with the anti-symmetric \( \epsilon \)-tensor (2.15). The change of sign in (2.23) and (2.25) is due to the relation
\[
\epsilon_{ab} = -\tilde{\epsilon}^{ab}, 
\]
since in (2.18) all \( \epsilon \)-matrices are equal to the original one (2.15). The relation (2.26) can also be derived from (2.22) by means of charge conjugation (2.21). The Lorentz index in \( \sigma^\mu \) is written upstairs from now on for notational reasons.

The Dirac equation (2.2) for the Majorana field (2.19) reads
\[
i \sigma^\mu \partial_\mu \chi = m \chi, 
\]
(2.27)
\[
\partial^\mu \chi + \hat{\sigma}^\mu \chi = \chi \sigma^\mu \partial_\mu, 
\]
(2.28)
Note that \( \sigma^\mu \partial_\mu = \sigma_0 \partial_0 - \hat{\sigma} \cdot \vec{d}, \hat{\sigma}^\mu \partial_\mu = \sigma_0 \partial_0 + \hat{\sigma} \cdot \vec{d} \). Using
\[
\hat{\sigma}^\mu = \epsilon^{\alpha \beta} \sigma^\alpha \sigma^\beta \sigma^\mu 
\]
the second equation can also be written as
\[
-i \sigma^\mu \partial_\mu \chi = m \tilde{\chi}. 
\]
(2.29)

To get an explicit representation of the Majorana field \( \chi(f) \) in Fock space we must smear the first two components in (2.11) with 2-component test functions \( f(x) \in \mathcal{S}(\mathbb{R}^4) \otimes \mathbb{C}^2 = \mathcal{E} \). The Majorana field is an operator in the antisymmetric Fock space \( \mathcal{F} \) over \( \mathcal{F}^1 = \mathcal{E} \). Using the well-known representation
\[
\left( b^*_s(\vec{p}) \phi \right)_n = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n} (-1)^{j+1} (\vec{p} - \vec{p}_j) \chi_s \otimes \varphi_{n-1}(p_1, s_1 \ldots \hat{p}_j, s_j \ldots p_n, s_n) 
\]
(2.30)
\[
\left( b_s(\vec{p}) \phi \right)_n = \sqrt{n+1} \varphi_{n+1}(p, s, p_1, s_1 \ldots p_n, s_n), \quad s_j = \pm 1, 
\]
(2.31)
of the emission and absorption operators we find by means of (2.4-5)
\[
\left( \chi(f) \phi \right)_n = \sqrt{2\pi} \left[ \sqrt{n+1} \int \frac{d^3 \vec{p}}{2\sqrt{E(E + m)}} \hat{f}(-\vec{p} \mp \sigma \vec{p} \varphi_{n+1}(p, p_1, \ldots p_n) 
\right. 
\]
\[
+ \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-1)^{j} \frac{(m + \hat{\sigma} p_j) \epsilon}{2\sqrt{E_j(E_j + m)}} \hat{f}(p_j) \otimes \varphi_{n-1}(p_1, \ldots \hat{p}_j, \ldots p_n), 
\]
(2.32)
\[
\left( \tilde{\chi}(f) \phi \right)_n = \sqrt{2\pi} \left[ \sqrt{n+1} \int \frac{d^3 \vec{p}}{2\sqrt{E(E + m)}} \hat{f}(\vec{p} \mp \hat{\sigma} p \varphi_{n+1}(p, p_1, \ldots p_n) 
\right. 
\]
\[
- \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-1)^{j} \frac{(m + \sigma p_j) \epsilon}{2\sqrt{E_j(E_j + m)}} \hat{f}(p_j) \otimes \varphi_{n-1}(p_1, \ldots \hat{p}_j, \ldots p_n). 
\]
(2.33)
The argument \( \hat{p}_j \) has to be left out.
3 Quantized chiral superfield

We want to represent the supersymmetric algebra

\[ \{ Q_a, \bar{Q}_b \} = 2 \sigma^\mu_{ab} P_\mu, \]
\[ \{ Q_a, Q_b \} = 0 = \{ \bar{Q}_a, \bar{Q}_b \} = [ Q_a, P_\mu ] \]

by operators in Fock space. \( Q_a \) and \( \bar{Q}_b \) transform according to the \(( \frac{1}{2}, 0)\) and \((0, \frac{1}{2})\) representations of the proper Lorentz group, respectively. As usual one rewrites (3.1) as a Lie-algebra commutator

\[ [ \xi^a Q_a, \bar{\xi}^b \bar{Q}_b ] = 2 \xi \sigma^\mu \bar{\xi} P \]

by introducing anti-commuting C-numbers \( \xi^a, \bar{\xi}^b \). Most authors discuss classical superfields first, we exclusively work with free quantized fields in Fock space. So it is our aim to construct a field \( \Phi \) with the transformation law

\[ \delta \xi \Phi = [ \xi Q + \bar{\xi} \bar{Q}, \Phi ] \]

under infinitesimal supersymmetric transformations. Taking as initial value a scalar field \( A(x) \), the solution is given by the finite transformation

\[ \Phi(x, \theta, \bar{\theta}) = e^{\theta Q + \bar{\theta} \bar{Q}} A(x) e^{-\theta \sigma \bar{\theta} P} \]

We assume \( A(x) \) to be a charged scalar field of mass \( m \), quantized in the usual way

\[ [ A(x), A^+(x') ] = -i D(x - x') \]

where \( D(x) \) is the causal Jordan-Pauli distribution and all other commutators vanish.

We are now going to calculate (3.5) by means of Hausdorff’s formula

\[ \Phi(x, \theta, \bar{\theta}) = e^{\theta Q + \bar{\theta} \bar{Q}} A(x) e^{-\theta \sigma \bar{\theta} P} \]

where (3.3) has been used. By the Lie series we first compute

\[ A' \overset{\text{def}}{=} e^{-\theta \sigma \bar{\theta} P} A(x) e^{\theta \sigma \bar{\theta} P} = \]
\[ = A + [ i \theta \sigma \bar{\theta} \partial, A ] + \frac{1}{2} [ i \theta \sigma \bar{\theta} \partial, [ i \theta \sigma \bar{\theta} \partial, A ] ] \]

using \( P_\mu = -i \partial_\mu \). The first commutator gives \( i \theta \sigma^\mu \bar{\theta} \partial \mu A \) and then the double commutator becomes equal to

\[ -\frac{1}{2} \theta \sigma^\mu \bar{\theta} \theta \sigma^\nu \bar{\theta} [ \partial_\mu, ( \partial_\nu A ) ] = -\frac{1}{4} \theta \theta \bar{\theta} \bar{\theta} A, \]

here we have used the identity

\[ \theta \sigma^\mu \bar{\theta} \theta \sigma^\nu \bar{\theta} = \frac{1}{2} \eta^{\mu \nu} \theta \theta \bar{\theta}. \]

The result can also be written as follows

\[ A' = \left( e^{i \theta \sigma \bar{\theta} A(x)} \right) \overset{\text{def}}{=} A(x + i \theta \sigma \bar{\theta}) \overset{\text{def}}{=} A(y), \]
\[ A(y) = A(x) + i \theta \sigma \bar{\theta} \partial A(x) + \frac{1}{4} \theta \theta \bar{\theta} \bar{\theta} A(x), \]

with the natural variable

\[ y = x + i \theta \sigma \bar{\theta} \]
in super-space.

To continue the calculation of (3.7) we need the commutators of $\bar{Q}_\alpha, Q_\alpha$ with $A(x)$. Let us assume
\[ [\bar{Q}_\alpha, A(x)] = 0, \quad (3.13) \]
this defines the chiral superfield in contrast to the anti-chiral one, and
\[ [Q_\alpha, A(x)] = \sqrt{2}\psi_\alpha(x). \quad (3.14) \]
Like $Q_\alpha$ $\psi$ must be a 2-component spinor field transforming by the $(\frac{1}{2}, 0)$-representation, the factor $\sqrt{2}$ is conventional. Comparing (3.1) with (2.22-26) we see that the anti-commutation rules of the $Q$'s are the same as for (massless) Majorana fields. Therefore, we quantize the spinor field $\psi$ as a Majorana field with mass $m$:
\[ \{\psi_\alpha(x), \psi_\beta(x')\} = i m \varepsilon_{ab} D(x - x') \quad (3.15) \]
\[ \{\bar{\psi}^\alpha(x), \bar{\psi}^\beta(x')\} = i m \varepsilon^{ab} D(x - x') \quad (3.16) \]
\[ \{\psi_\alpha(x), \bar{\psi}^\beta(x')\} = \sigma^\mu_{ab} \partial^\mu D(x - x'). \quad (3.17) \]
The rule (3.14) can also be used for $A(y)$ (3.11), hence
\[ e^{\theta Q} A(y)e^{-\theta Q} = A(y) + \sqrt{2} \theta \psi(y) + \frac{1}{2} [\theta Q, [\theta Q, A]]. \quad (3.18) \]
The double commutator is equal to
\[ \frac{1}{2} \sqrt{2} [\theta^a Q_a, \theta^b \psi_b(y)] = -\frac{1}{\sqrt{2}} \theta^a \theta^b \{Q_a, \psi_b(y)\}. \quad (3.19) \]
The anti-commutator on the r.h.s. must be anti-symmetric in $a, b$, it can thus be written as
\[ \{Q_a, \psi_b(y)\} = \sqrt{2} \varepsilon_{ab} F(y), \quad (3.20) \]
where $F(y)$ is a scalar field. Then we arrive at the usual expression
\[ \Phi = A(y) + \sqrt{2} \theta \psi(y) + \theta \theta F(y). \quad (3.21) \]
If we use the expansion (3.11) for $A(y)$ and $\psi(y)$ we obtain
\[ \Phi(x, \theta, \bar{\theta}) = A(x) + i\theta \sigma \partial A - \frac{i}{4} \theta \theta \bar{\theta} \partial A(x) + \]
\[ + \sqrt{2} \theta^a (\psi_a(x) + i \theta \sigma \partial \psi_a) + \theta \theta F(x). \quad (3.22) \]
Using the identity
\[ \theta^a \theta^b = -\frac{1}{2} \varepsilon^{ab} \theta \theta, \quad (3.23) \]
the second term in the bracket is equal to
\[ -i \sqrt{2} \theta \partial \psi \sigma \bar{\theta}. \quad (3.23) \]
The scalar field $F(x)$ is not independent. Indeed, from (3.15) and (3.14) we find by means of the Jacobi identity
\[ \{\psi_\alpha(x), \psi_\beta(x')\} = \frac{1}{\sqrt{2}} \{\psi_\alpha(x), [Q_\beta, A(x')]\} = \]
\[
\frac{1}{\sqrt{2}}\left(\{Q_b, [A(x'), \psi_a(x)]\} - [A(x'), \{\psi_a(x), Q_b]\}\right) = \\
= -\varepsilon_{ab}[F(x), A(x')] = im\varepsilon_{ab}D(x - x').
\] 

Since the adjoint of (3.13) implies
\[
[F(x), A^+(x')] = 0
\]
in a similar way, we may conclude
\[
F(x) = -mA^+(x),
\]
up to a C-number field which we set equal to 0. All component fields, i.e. $A(x)$ and $\psi(x)$, have their natural representations as operator-valued distributions in a big Fock space, hence, the chiral superfield is represented as well. The $\theta, \bar{\theta}$ are merely spectators in this construction. Since we know all commutators of the supercharges $Q, \bar{Q}$ with the fundamental fields $A, \psi$, we are also able to find the representation of the supercharges, but this is not needed for the following.

From the commutation rules of the component fields it is now easy to determine the commutator of the chiral superfield. This is best done with the expression (3.21). Taking (3.25) into account we find
\[
[\Phi(y, \theta), \Phi(y', \theta')] = \theta'\theta'[A(y), -mA^+(y')] + \theta\theta[-mA^+(y), A(y')] + \\
+2\theta^a\theta^b\{\psi_a(y), \psi_b(y')\} = im(\theta - \theta')^2D(y - y').
\] 

The square can also be regarded as the 2-dimensional $\delta$-distribution
\[
(\theta - \theta')^2 = \delta^2(\theta - \theta')
\]
because its integral $\int d^2(\theta - \theta')$ is one. By the translation formula
\[
D(y - y') = D(x + i\theta\sigma\bar{\theta} - x' - i\theta'\sigma\bar{\theta}') = \\
= \left[\exp i(\theta\sigma\bar{\theta} - \theta'\sigma\bar{\theta}')\partial_x\right]D(x - x')
\]
we can go over to ordinary variables in superspace:
\[
[\Phi(x, \theta, \bar{\theta}), \Phi(x', \theta', \bar{\theta}')] = im\delta^2(\theta - \theta')\left[\exp i(\theta\sigma\bar{\theta} - \theta'\sigma\bar{\theta}')\partial_x\right]D(x - x').
\]

The adjoint of the chiral superfield is the anti-chiral one. To see this one has to use adjoining relations like
\[
(\theta\psi)^+ = \bar{\theta}\bar{\psi}
\]
which follow from (2.20). In the same way as above one obtains the following further commutation relations
\[
[\Phi(x, \theta, \bar{\theta}), \Phi^+(x', \theta', \bar{\theta}')] = -i\left[\exp i(\theta\sigma\bar{\theta} + \theta'\sigma\bar{\theta}' - 2\theta\sigma\bar{\theta}')\partial_x\right]D(x - x')
\]
\[
[\Phi^+(x, \theta, \bar{\theta}), \Phi^+(x', \theta', \bar{\theta}')] = -im\delta^2(\theta - \theta')\left[\exp -i(\theta\sigma\bar{\theta} - \theta'\sigma\bar{\theta}')\partial_x\right]D(x - x').
\]

The commutators of the various emission and absorption parts are of the same form, only $D$ must be substituted by $D^+$ or $D^-$. If one expands the exponentials one gets up to two derivatives on the Jordan-Pauli distribution $D(x - x')$. Considered as distributions in $x$-space alone, these are rather singular distributions and it is not at all clear why supersymmetric theories have a tame ultraviolet behaviour. The point is that we have to consider all distributions in superspace. Then, as we will see in the next section, the distributions in the commutation relations are as good as in ordinary field theory.
4 Causal perturbation theory

Our goal is to construct the S-functional $S(g)$ as a formal power series

$$S(g) = \sum_{n=0}^{\infty} \frac{1}{n!} \int dX_1 \ldots dX_n T_n(X_1, \ldots, X_n) g(X_1) \ldots g(X_n). \quad (4.1)$$

Here $X_j = (x^\mu_j, \theta_j, \bar{\theta}_j)$ denotes the superspace coordinates, $dX_j = d^4x^\mu_j d^2\theta_j d^2\bar{\theta}_j$, and $g(X_j)$ is a test-function on superspace, the physical S-matrix is obtained in the adiabatic limit $g \to 1$. The time-ordered products $T_n$ are constructed inductively, $T_1(X)$ must be given. For the Wess-Zumino model it reads

$$T_1(X) = i\left(\frac{\lambda^3}{3!} : \Phi(X)^3 : \delta^2(\bar{\theta}) + \text{h.c.}\right). \quad (4.2)$$

The double dots denote normal ordering of the free superfields.

The essential property of $T_1(X)$ is the causal commutator in x-space

$$[T_1(X_1), T_1(X_2)] = 0, \quad (x_1 - x_2)^2 < 0. \quad (4.3)$$

This follows from the causal support of $D(x)$ in the commutation relations. In n-th order we require the following causal factorization

$$T_n(X_1, \ldots, X_n) = T_m(X_1, \ldots, X_m) T_{n-m}(X_{m+1}, \ldots, X_n) \quad (4.4)$$

for $\{x_1, \ldots, x_m\} > \{x_{m+1}, \ldots, x_n\}$ and arbitrary $\theta$'s. The last inequality means

$$\{x_1, \ldots, x_m\} \subset \{\{x_{m+1}, \ldots, x_n\} + \bar{V}_-\} = 0, \quad (4.5)$$

where $\bar{V}_-$ is the closed backward light-cone. In addition the $T_n$'s are assumed to be symmetric in all arguments and translation invariant.

The above properties allow the inductive construction of the $T_n$'s by the method of Epstein and Glaser. This construction goes as follows: Assuming $T_m(X_1, \ldots, X_m)$ for $m \leq n - 1$ to be known, then we can compute

$$A'_n(X_1, \ldots, X_n) = \sum_{P_2} \check{T}_{n_1}(X) T_{n-n_1}(Y, X_n) \quad (4.6)$$

$$R'_n(X_1, \ldots, X_n) = \sum_{P_2} \check{T}_{n-n_1}(Y, X_n) \check{T}_{n_1}(X) \quad (4.7)$$

where the sums run over all partitions

$$P_2 : \quad \{X_1, \ldots, X_n\} = X \cup Y, \quad X \neq \emptyset \quad (4.8)$$

into disjoint subsets with $|X| = n_1 \geq 1$, $|Y| \leq n - 2$. The $\check{T}_n$ are obtained by inversion of the perturbation series $S(g)^{-1}$, they can be expressed by the $T_n$'s as follows

$$\check{T}_n(X) = \sum_{r=1}^{n} (-)^r \sum_{P_r} T_{n_1}(X_1) \ldots T_{n_r}(X_r),$$

where the second sum runs over all partitions $P_r$ of $X$ into $r$ disjoint subsets.
If the sums in (4.6-7) are extended over all partitions $P_2 \subseteq P_0$, including the empty set $X = \emptyset$, then we get the distributions

$$A_n(X_1, \ldots X_n) = \sum_{P_0} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, X_n)$$

$$= A_n' + T_n(X_1, \ldots X_n), \quad (4.8)$$

$$R_n(X_1, \ldots X_n) = \sum_{P_0} \tilde{T}_{n-n_1}(Y, X_n) \tilde{T}_{n_1}(X)$$

$$= R_n' + T_n(X_1, \ldots X_n). \quad (4.9)$$

These two distributions are not known by the induction assumption because they contain the unknown $T_n$. Only the difference

$$D_n = R_n' - A_n' = R_n - A_n \quad (4.10)$$

is known. Here the causal structure comes in: $D_n$ has causal support, $R_n$ is the retarded part and $-A_n$ the advanced part

$$\text{supp } R_n(X_1, \ldots X_n) \subseteq \Gamma_{n-1}^+(x_n)$$

$$\text{supp } A_n(X_1, \ldots X_n) \subseteq \Gamma_{n-1}^-(x_n), \quad (4.11)$$

with

$$\Gamma_{n-1}^\pm(x_n) = \{(X_1, \ldots X_n) | x_j \in \bar{V}_\pm(x_n), \forall j = 1, \ldots n-1\}, \quad (4.12)$$

where

$$\bar{V}_-(x) = \{y | (y-x)^2 \geq 0, y^0 \leq x^0\}$$

is the closed backward cone in Minkowski space and $\bar{V}_+$ the closed forward cone. These support properties follow from causality (4.4) in the same way as for ordinary quantum field theory. What remains to be done is the splitting of $D_n$ (4.4) into retarded and advanced parts, then $T_n$ follows from (4.8-9)

$$T_n = R_n - R_n' = A_n - A_n'. \quad (4.13)$$

The distribution splitting is achieved by expanding $D_n$ in normally ordered form using Wick’s theorem for superfields and splitting the numerical distributions. To split a numerical distribution $d(X)$ on superspace, it is necessary to analyse the behaviour in the neighbourhood of $X = 0$ by means of scaling transformations

$$x'_j = \lambda x_j, \quad \theta' = \sqrt{\lambda} \theta, \quad \bar{\theta}' = \sqrt{\lambda} \bar{\theta}. \quad (4.14)$$

The $\theta$’s must be scaled this way to get non-trivial limits for $\lambda \to 0$ (cf. (3.28)). As in ordinary field theory we introduce the following definitions:

**Definition 1.** The distribution $d(X)$, tempered in $m$-dimensional $x$-space, has a quasi-asymptotics $d_0(X)$ at $X = 0$ with respect to a positive continuous function $g(\lambda), \lambda > 0$, if the limit

$$\lim_{\lambda \to 0} g(\lambda) \lambda^m d(\lambda X) = d_0(X) \neq 0 \quad (4.15)$$

exists.

There is an equivalent definition for the Fourier transform with respect to $x$:
**Definition 2.** The distribution \( \hat{d}(p, \theta, \tilde{\theta}) \) has quasi-asymptotics \( \hat{d}_0(p, \theta, \tilde{\theta}) \) if the limit

\[
\lim_{\lambda \to 0} \varrho(\lambda)\hat{d}\left(\frac{p}{\lambda}, \sqrt{\lambda}\theta, \sqrt{\lambda}\tilde{\theta}\right) = \hat{d}_0(p, \theta, \tilde{\theta}) \neq 0
\]

exists. If the function \( \varrho(\lambda) \) satisfies

\[
\lim_{\lambda \to 0} \frac{\varrho(a\lambda)}{\varrho(\lambda)} = a^\omega
\]

for each \( a > 0 \), then \( d \) is called singular of order \( \omega \) at \( X = 0 \). \( \varrho(\lambda) \) is called power counting function.

It is easy to check that the distribution on the r.h.s. of the commutation relation (3.31) has \( \omega = -2 \), as in ordinary field theory. The distributions in (3.29) and (3.32) have even \( \omega = -3 \) due to the additional \( \delta \)-distributions. If a causal distribution \( d(X) \) has negative \( \omega \) it can be split trivially by multiplying with a step-function \( \Theta(x^0) \). This can again be proven as in ordinary field theory. For \( \omega \geq 0 \) the splitting must be carefully done, in the usual terminology the graph needs renormalization. The non-renormalization property of the Wess-Zumino model with coupling (4.2) in the causal approach is now contained in the following theorem:

**Theorem:** The singular order of a super-graph of order \( n \) for the Wess-Zumino model satisfies

\[
\omega \leq 4 - \frac{1}{2}(e + e') + l_+ - \frac{5}{2}n,
\]

where \( e \) and \( e' \) is the number of external \( \Phi \)- and \( \Phi^+ \)-legs and \( l_+ \) the number of \( \Phi \Phi^+ \)-contractions.

**Proof.** The proof is inductively. In the inductive step we must form tensor products of two distributions

\[
T^1_r(x_1, \theta_1, \tilde{\theta}_1, \ldots x_r, \theta_r, \tilde{\theta}_r)T^2_s(y_1, \theta'_1, \tilde{\theta}'_1, \ldots y_s, \theta'_s, \tilde{\theta}'_s).
\]

Let us assume that in the normal ordering \( l_ \Phi \Phi \)- and \( \Phi^+ \Phi^+ \)-contractions are performed. Using translation invariance, the resulting numerical distributions are of the following form

\[
t_1(x_1 - x_r, \ldots x_r - x_{r-1} - x_r; \theta, \tilde{\theta}) \prod_{j=1}^l D^{(+)}(x_{r_j} - y_{s_j}; \theta_{r_j}, \tilde{\theta}_{r_j}, \theta'_{s_j}, \tilde{\theta}'_{s_j}) \times
\]

\[
\times t_2(y_1 - y_{s_1}, \ldots y_{s_1} - y_{s_1}; \theta', \tilde{\theta}').
\]

We introduce relative coordinates

\[
\xi_j = x_j - x_r, \eta_j = y_j - y_s, \eta = x_r - y_s,
\]

and compute the Fourier transform in \( x \)-space. Since the products go over into convolutions, we get

\[
\hat{t}(p_1, \ldots p_{r-1}, q_1, \ldots q_{s-1}, q; \theta, \tilde{\theta}, \theta', \tilde{\theta}') = \int \hat{t}((\xi, \eta; \theta, \tilde{\theta}, \theta', \tilde{\theta}')e^{ip \xi + i\eta \eta}d^{4r-4} \xi d^{4s} \eta
\]

\[
= \int \prod_{j=1}^l d\kappa_j \delta \left( q - \sum_{j=1}^l \kappa_j \right) \hat{t}_1(\ldots p_i - \kappa_{r(i)} \ldots; \theta, \tilde{\theta})
\]

\[
\times \prod_{j=1}^l \hat{D}^{(+)}(\kappa_j; \theta_{r_j}, \tilde{\theta}_{r_j}, \theta'_{s_j}, \tilde{\theta}'_{s_j}) \hat{t}_2(\ldots q_i + \kappa_{s(i)} \ldots; \theta', \tilde{\theta}').
\]
Applying this to a test function \(\varphi\), we have
\[
\langle \hat{t}, \varphi \rangle = \int d^{4r-2}p' d^{4s-4}q' d^{4r} \theta d^{4s} \theta' \hat{t}_1(p', \theta, \bar{\theta}) \times \\
\times \hat{t}_2(q', \theta', \bar{\theta}') \psi(p', q', \theta, \bar{\theta}, \theta', \bar{\theta}')
\]
with
\[
\psi(\ldots) = \int d^4q \prod_j d\kappa_j \delta(q - \sum_j \kappa_j) \varphi(\ldots p'_j + \kappa_{r(i)}, \ldots q'_j - \kappa_{s(i)}, \ldots q; \theta, \bar{\theta}, \theta', \bar{\theta}') \\
\times \prod_{j=1}^l \hat{D}^{(+)}(\kappa_j; \theta_{r_j}, \bar{\theta}_{r_j}, \theta'_{s_j}, \bar{\theta}'_{s_j}).
\]  
(4.22)

In order to determine the singular order of \(\hat{t}\) we have to consider the scaled distribution
\[
\langle \hat{t} \left( p' \sqrt{\lambda} \theta, \ldots \right), \varphi \rangle = \lambda^{4(r+s-1)} \lambda^{-2(r+s)} \langle \hat{t}(p), \varphi \left( \lambda p, \frac{\theta}{\sqrt{\lambda}}, \ldots \right) \rangle = \lambda^{4(r+s-2)} \int d^{4r-2}p' d^{4s-4}q' d^{4r} \theta d^{4s} \theta' \hat{t}_1(p', \theta, \bar{\theta}) \times \\
\times \hat{t}_2(q', \theta', \bar{\theta}') \psi(p', q', \theta, \bar{\theta}, \theta', \bar{\theta}')
\]
(4.24)

where
\[
\psi_\lambda = \int d^4q \prod_{j=1}^l d\kappa_j \delta^4(q - \sum_j \kappa_j) \varphi(\ldots, \lambda(p'_j + \kappa_{r(i)}), \ldots, \lambda(q'_j - \kappa_{s(i)}), \ldots; \lambda q; \frac{\theta}{\sqrt{\lambda}}, \frac{\bar{\theta}}{\sqrt{\lambda}}, \frac{\theta'}{\sqrt{\lambda}}, \frac{\bar{\theta}'}{\sqrt{\lambda}}) \\
\times \prod_{j=1}^l \hat{D}^{(+)}(\kappa_j; \theta_{r_j}, \bar{\theta}_{r_j}, \theta'_{s_j}, \bar{\theta}'_{s_j}).
\]  
(4.25)

We introduce scaled variables \(\tilde{\kappa}_j = \lambda \kappa_j, \tilde{q} = \lambda q, \tilde{\theta} = \theta / \sqrt{\lambda}\) etc. and note that
\[
\lim_{\lambda \to 0} \lambda^{-3} \hat{D}^{(+)}(\kappa_j, \tilde{\theta} \sqrt{\lambda}, \ldots) = \hat{D}_0^{(+)}(\tilde{\kappa}, \tilde{\theta}, \ldots).
\]

This implies for \(\lambda \to 0\)
\[
\psi_\lambda \to \frac{1}{\lambda^4} \psi \left( \lambda p', \lambda q', \frac{\theta}{\sqrt{\lambda}}, \frac{\bar{\theta}}{\sqrt{\lambda}}, \frac{\theta'}{\sqrt{\lambda}}, \frac{\bar{\theta}'}{\sqrt{\lambda}} \right).
\]  
(4.26)

Using scaled variables \(\lambda p' = \tilde{p}, \lambda q' = \tilde{q}, \theta / \sqrt{\lambda} = \tilde{\theta}, \ldots\) again, we find
\[
\langle \hat{t} \left( \frac{p}{\lambda}, \sqrt{\lambda} \theta, \ldots \right), \varphi \rangle \to \frac{\lambda^4}{\lambda^4} \int d^{4r-2}p' d^{4s-4}q' d^{4r} \theta d^{4s} \theta' \hat{t}_1 \left( \frac{\tilde{p}}{\lambda}, \sqrt{\lambda} \theta, \sqrt{\lambda} \bar{\theta} \right) \times \\
\times \hat{t}_2 \left( \frac{\tilde{q}}{\lambda}, \sqrt{\lambda} \bar{\theta}', \sqrt{\lambda} \bar{\theta}' \right) \psi(\tilde{p}, \tilde{q}, \theta, \bar{\theta}, \theta', \bar{\theta}') .
\]  
(4.27)

By the induction hypothesis, \(\hat{t}_1\) and \(\hat{t}_2\) have singular order \(\omega_1, \omega_2\) with power counting functions \(\varrho_1(\lambda), \varrho_2(\lambda)\), respectively. Then the following limit exists:
\[
\lim_{\lambda \to 0} \lambda^{l-4} \varrho_1(\lambda) \varrho_2(\lambda) \langle \hat{t} \left( \frac{p}{\lambda}, \sqrt{\lambda} \theta, \ldots \right), \varphi \rangle = \langle \hat{t}_0(p, \theta, \ldots), \varphi \rangle.
\]  
(4.28)
Hence, the singular order of $\hat{t}(p, \theta, \bar{\theta})$ satisfies
\[ \omega \leq \omega_1 + \omega_2 + l - 4. \] (4.29)
If instead of the $l$ $\Phi\Phi$-contractions we make $l_+$ $\Phi\Phi^+$-contractions, then in the same way we obtain
\[ \omega \leq \omega_1 + \omega_2 + 2l_+ - 4. \] (4.30)

The results (4.29-30) suggest the following ansatz
\[ \omega \leq 4 + l + l' + 2l_+ + \alpha n. \] (4.31)

$l$ is the number of $\Phi\Phi$- $l'$ the number of $\Phi^+\Phi^+$- and $l_+$ the $\Phi\Phi^+$-contractions, 4 is a consequence of translation invariance. The term $\alpha n$ (with $\alpha$ still unknown) is possible because the order of perturbation theory is additive. Consider now a supergraph with $n_1$ $\Phi^3$-vertices and $n_2$ $\Phi^{+3}$-vertices, $n_1 + n_2 = n$. Then for the number $e$ of external $\Phi$-legs and $e'$ external $\Phi^+$-legs we have the relations
\[ 3n_1 = e + 2l + l_+, \quad 3n_2 = e' + 2l' + l_+. \] (4.32)
Solving for $l$ and $l'$ and substituting into (4.31), we obtain
\[ \omega \leq 4 + \frac{3}{2}(n_1 + n_2) - \frac{1}{2}(e + e') - l_+ + 2l_+ + \alpha n \]
\[ = 4 - \frac{1}{2}(e + e') + l_+ + \left(\frac{3}{2} + \alpha\right)n. \] (4.33)
The value of $\alpha$ follows from the beginning of the induction: for $n = 1$ $\omega$ must be zero. Taking one $\Phi^3$-vertex we have
\[ \omega = 4 - \frac{3}{2} + \frac{3}{2} + \alpha = 0, \]
which gives $\alpha = -4$. Hence
\[ \omega \leq 4 - \frac{1}{2}(e + e') + l_+ - \frac{5}{2}n, \] (4.34)
this completes the proof.

5 Conclusions

The minus sign $-5n/2$ in (4.34) shows that the theory is super-renormalizable, because for $n$ big enough $\omega$ becomes negative. This is obviously a consequence of the purely polynomial $\Phi^3$-structure of the vertex. For an ordinary scalar $\varphi^3$-theory which is super-normalizable as well one has
\[ \omega \leq 4 - e - n. \] (5.1)
The bigger factor $-5/2$ in (4.34) shows that the Wess-Zumino model has still a better ultraviolet behaviour. In fact, apart from the vacuum graph, there is only one graph with non-negative $\omega$: this is the second-order loop with one $\Phi^3$, and one $\Phi^{+3}$-vertex which has
\[ \omega = 4 - 1 + 2 - 5 = 0. \] (5.2)
If one adds an additional inner line, $l_+$ increases by 1 but the order $n$ by 2, consequently $\omega$ becomes more negative. Therefore, the second-order loop graph (5.2) is the only one which
requires normalisation. This is the non-renormalisation property for time-ordered products $T_n(X)$ on superspace. The formula (4.34) gives an upper bound for the singular order. Indeed, for certain graphs $\omega$ is actually smaller as a consequence of supersymmetry. But this is not important because $\omega$ is anyway negative so that no renormalisation is needed. The singular order considered here, which is the essential parameter for the distribution splitting, is not the same as the degree of divergence $D$ considered by other authors [2, 3]. The latter refers in our language to the $T_n$ after integration over $\theta$'s. We will return to this point elsewhere.

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