The Decay of Unstable Noncommutative Solitons

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Abstract

We study the classical decay of unstable scalar solitons in noncommutative field theory in $2 + 1$ dimensions. This can, but does not have to, be viewed as a toy model for the decay of D-branes in string theory. In the limit that the noncommutativity parameter $\theta$ is infinite, the gradient term is absent, there are no propagating modes and the soliton does not decay at all. If $\theta$ is large, but finite, the rotationally symmetric decay channel can be described as a highly excited nonlinear oscillator weakly coupled to a continuum of linear modes. This system is closely akin to those studied in the context of discrete breathers. We here diagonalize the linear problem and compute the decay rate to first order using a version of Fermi’s Golden Rule, leaving a more rigorous treatment for future work.

* Dedicated to Rudolf Haag, on the occasion of his eightieth birthday, with admiration and affection

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1 Introduction

Solitons are interesting objects. Originally discovered in experiments on shallow water waves one and a half centuries ago, they have since then become a rich and fascinating area of theoretical and mathematical study. Today, solitons are playing an increasingly important role in modern field and string theories.

A general issue in the analysis of solitons is stability, in both the dynamical (small perturbations of the initial conditions within a theory), and the structural (small deformations of the theory) sense. Stability can often be established on the basis of topological properties of the solution, or by the integrability of the underlying theory. On the other hand, a problem that has attracted much less attention concerns the dynamical long-time evolution of decaying, unstable solitons. The reason is apparently not because this question fails to be interesting, but rather because it is hard.

In this paper, we study the decay dynamics of a particularly simple class of solitons that were discovered only very recently, coined noncommutative solitons [20]. The corresponding model can be derived from the action functional

\[ S = \int dx \, dy \, dt \left[ \frac{1}{2} (\partial_t \Phi)^2 - \frac{1}{2} (\nabla \Phi)^2 - V(\Phi) \right], \]  

which describes a real scalar field \( \Phi(x, y, t) \) in 2 + 1 dimensions. Noncommutativity is introduced in the multiplication rule for \( \Phi \), as we review below. We will only consider space-space noncommutativity in this paper, governed by a single noncommutativity parameter \( \theta \in [0, \infty) \).

“Noncommutative field theories” of this kind appear for instance as world-volume theories of D-branes in string theory by turning on a large Neveu-Schwarz-Neveu-Schwarz B-field. In general, the resulting theory is more complicated than (1), and will include gauge fields and so on, but here, we will restrict ourselves to (1). We will not indulge here in a general discussion of noncommutative geometry [1], and its role in modern physics. As starting points for references and citations we refer to the short list [2, 3, 4, 5, 6], as well as to the reviews [7, 8, 9].

1.1 Motivation from string theory

Our interest in this problem arose from recent studies of tachyon condensation in string theory, hence let us briefly review this background motivation here. Tachyon condensation in string theory is formalized in Sen’s conjectures [10, 11]. Roughly, given a string
theory background with an unstable D-brane or D-brane system, i.e., given a solution of string theory with an open string tachyon, one should look for another solution of string theory that is related to the first solution by “tachyon condensation”. This solution corresponds to a different background with other branes, and can either be stable or contain other tachyons. In bosonic string theory, all $D^p$-branes are unstable, and open string theory is conjectured to have a solution corresponding to the condensation of all these tachyons, characterized by:

(S1) There are no perturbative open string excitations around this solution, which is hence also known as the “closed string vacuum of open string field theory”.

(S2) The energy difference between the solution with open strings and the closed string vacuum is exactly equal to the D-brane tension computed in perturbative string theory.

(To avoid confusion, we note that the bosonic string in 26 dimensions also has a tachyon in the closed string sector, whose role is, however, much less understood.)

A useful analogy in this context is to think of D-branes as “tachyonic solitons”. For example, a codimension-one brane in bosonic string theory is analogous to the solitonic lump solution in a scalar $\phi^3$ theory. This soliton and the D-brane are of course unstable. In superstring theory, a codimension-one brane is analogous to the stable solitonic kink in a scalar field theory with a $\phi^4$ double-well potential. The topological charge of such a soliton that guarantees stability is analogous to the Ramond-Ramond charge of a brane. These analogies break down at some level because string field theory contains many more fields than just the tachyon, and they are all excited in the vacua representing the endpoint of tachyon condensation.

Tachyon condensation can be formulated on the worldsheet (as worldsheet RG flow) or in spacetime in open string field theory [12]. This has been quite successful and can be regarded as one of the major advances in the field over the last few years. Until recently, however, the research has focused on finding the endpoint of the condensation, and not on the decay of the unstable brane (the condensation of the tachyon) as a dynamical process in time.

Very recently, Sen [13] has made precise proposals for the construction of the time-dependent solutions corresponding to the condensation process. This is based mainly on a worldsheet analysis, and is motivated in part by the S-brane proposal of
Gutperle and Strominger [16]. Again, there are many different possible processes one can study. The generic features characterizing the solutions are

(S3) If the initial configuration is an unstable brane configuration of non-zero codimension (say a Dp-brane in bosonic string theory with $p < 25$) then the energy stays localized in the plane of the original brane, in other words, energy is not carried away from this plane by classical radiation.

(S4) When computing the energy-momentum tensor of the solution one finds that, asymptotically for late times, the energy density of the solution is constant (in the plane of the original brane) and the pressure vanishes exponentially in time.

It is important to emphasize that these are supposed to be classical solutions of string theory. In particular, the radiation in (S3) are open strings. It is very likely that in fact closed strings (which appear at one loop level in open string theory) do carry away energy [16,17,18]. Furthermore, one may note that (S3) is in some sense a consequence of (S1). Namely, if one thinks of a lower-dimensional brane as a tachyonic soliton, then far away this solution looks like the closed string vacuum which does not allow open string excitations. Hence, there are no modes that could carry away energy. Also, (S2) and (S4) are of course partly trivial consequences of energy conservation.

All these questions are rather difficult to study in open string field theory. The underlying idea behind the present work was to study the dynamics of such decays in the so-called noncommutative (NC) field theory limit of string theory, in the hope that one could reproduce some of the above properties in a simple manner.

The reason for this hope is the fact that solitons in noncommutative field theories (NC solitons) have many properties in common with D-branes as tachyonic solitons. NC solitons and instantons in gauge theories have been studied independently of strings for quite some time, most notably by Nekrasov and Schwarz [19], and more recently by Harvey et al. [21], Gross and Nekrasov [22,24], Polychronakos [23], and others. In NC scalar field theories, solitons were discovered by Gopakumar, Minwalla and Strominger [20]. For a summary of the relation to D-branes, in particular as far as their charges and descent relations are concerned, see Harvey’s lectures [25].

An explanation of why NC solitons (in the infinite $\theta$ limit) are useful for describing tachyon condensation in string theory was given by Witten [26]. The point of [26] is that in the Seiberg-Witten limit, the open string field algebra factorizes into the Moyal algebra built on tachyon vertex operators and another algebra that “contains
all the stringy mystifications”. Therefore, in describing tachyon condensation, one can concentrate on the tachyon, the remaining pieces being added “trivially” at the end.

Given all this, it becomes a natural problem, from the point of view of string theory, to study the decay of unstable NC solitons, using (S1)-(S4) as guiding questions, as a toy model for the dynamics of open string tachyons on unstable D-branes.

1.2 Summary of Results

We will see that it is in fact not possible to reproduce the qualitative behavior described above in the simple model (1). A priori, one would not expect (S3) simply because of the failure of (S1) in the context of NC tachyon condensation. Indeed, while NC solitons have localized energy, far away the field is in the usual vacuum that has local propagating excitations. However, we note that intuitively (S1) is sufficient but not necessary for (S3), and in this sense, finding (S3) would have been a nice substitute for (S1). And recall that the very existence of NC solitons at finite \( \theta \) relies on the fact that the kinetic term is a small perturbation that does not destroy the qualitative properties at infinite \( \theta \). One could expect something similar in the time-dependent case.

What we will find is that, except for infinite \( \theta \), the energy that initially is stored in the soliton does not stay localized, but is carried away to infinity in the form of classical radiation. Nevertheless, the qualitative properties of the decay do depend on the noncommutativity parameter \( \theta \). In particular, the decay rate is exponentially suppressed in \( \theta \), for large \( \theta \).

We will proceed as follows. We start with a review of NC scalar solitons and their basic properties in section 2. It turns out that restricted to rotationally symmetric configurations, the system is similar to certain nonlinear lattices which exhibit the phenomenon of “discrete breathers”, and we review those in section 3 for completeness. We will then present some numerical results in section 4 in order to illustrate the qualitative properties of the decay. In section 5 we turn to reproducing this behavior through an approximate calculation of the decay rate. The technical core is here the explicit diagonalization of the linear problem in section 6 that puts us in a position to give a closed expression for the estimated decay rate in section 7. The approximation method relies on a certain version of Fermi’s Golden Rule that we will explain.

To conclude this introduction, we note that a different toy model for tachyon condensation in string theory, based on conventional field theory, has been considered by Zwiebach [28]. It would be interesting to compare the dynamics of the decay with our
present results. Also, Moeller and Zwiebach [30] have studied the time evolution of decaying solitons in $p$-adic string theory. The relation to our work is not clear. Finally, we note that some of the problems that we study here have also been broached in [31] and [32]. The stability of non-commutative scalar solitons is also discussed in [33, 34].

2 Noncommutative scalar field theory in 2+1 dimensions and its solitons

The theory of our interest (1) is defined over a model of 2+1-dimensional spacetime in which time is an ordinary variable and space is the two-dimensional quantum mechanical phase space of a nonrelativistic particle in one dimension. In other words, we base space on the Heisenberg-Born-Jordan commutation relation,

$$[x, y] = i\theta. \quad (2)$$

It is well-known since the times of Weyl, Wigner, and Moyal that the algebra of functions on this space is a deformation of the ordinary algebra $C(\mathbb{R}^2)$ along the parameter $\theta$. In one particular ordering prescription, the product can be written as

$$\Phi_1 \star \Phi_2(x, y) = e^{\frac{i}{2}(\partial_\xi \cdot \partial_{\eta_2} - \partial_{\eta_2} \cdot \partial_\xi)} \Phi_1(\xi_1, \eta_1) \Phi_2(\xi_2, \eta_2)|_{(\xi_1, \eta_1) = (\xi_2, \eta_2) = (x, y)}. \quad (3)$$

Upon representing the commutation relations (2) in the usual manner on Hilbert space $\mathcal{H}$, eq. (3) turns into a prescription for translating a function over phase space, such as our field $\Phi$, into a linear operator on $\mathcal{H}$. Moreover, derivatives become commutators,

$$\partial_x \longleftrightarrow \frac{i}{\theta}[y, \cdot], \quad \partial_y \longleftrightarrow -\frac{i}{\theta}[x, \cdot], \quad (4)$$

and the integral a trace,

$$\int dx \, dy \longleftrightarrow 2\pi \theta \text{tr}. \quad (5)$$

This yields, after setting

$$a = \frac{1}{\sqrt{2\theta}}(x + iy), \quad a^\dagger = \frac{1}{\sqrt{2\theta}}(x - iy), \quad (6)$$

the following expression for $S$.

$$S = 2\pi \theta \int dt \text{tr} \left[ \frac{1}{2}(\partial_t \Phi)^2 - \frac{1}{\theta}[a, \Phi][\Phi, a^\dagger] - V(\Phi) \right] \quad (7)$$

We also record the equations of motion

$$\partial_t^2 \Phi + \frac{2}{\theta}[a, [a^\dagger, \Phi]] + V'(\Phi) = 0. \quad (8)$$
2.1 Rotational symmetry

These expressions simplify further on rotationally symmetric field configurations,

\[ \Phi(t) = \sum_{n=0}^{\infty} \lambda_n(t) P_n. \]  

(9)

Here, \( P_n = |n\rangle \langle n| \) is the projector onto the \( n \)-th excited state in the harmonic oscillator basis for \( \mathcal{H} \). One finds

\[ [a, [a^\dagger, P_n]] = (2n + 1)P_n - (n + 1)P_{n+1} - nP_{n-1}. \]  

(10)

The equations of motion (8) become

\[ \ddot{\lambda}_n = \frac{2}{\theta} \left( (n+1)(\lambda_{n+1} - \lambda_n) - n(\lambda_n - \lambda_{n-1}) \right) - V'(\lambda_n) \]

\[ \ddot{\lambda}_0 = \frac{2}{\theta} (\lambda_1 - \lambda_0) - V'(\lambda_0). \]  

(11)

This is a hamiltonian system with Hamiltonian

\[ H = 2\pi \theta \sum_{n=0}^{\infty} \left[ \frac{1}{2} \kappa_n^2 + \frac{n}{\theta} (\lambda_n - \lambda_{n-1})^2 + V(\lambda_n) \right], \]  

(12)

where \( \kappa_n \) are the momenta canonically conjugate to the \( \lambda_n \).

To connect (11) and (12) to something more familiar, we note that using (3) to write the projector \( P_n \) as a function on \( \mathbb{R}^2 \), it has a peak around

\[ r^2 = \langle n | x^2 + y^2 | n \rangle \approx 2\theta n, \]  

(13)

and that substituting (13) into the radial part of the Laplace operator written in two-dimensional polar coordinates, one obtains

\[ \frac{1}{r} \partial_r r \partial_r = \frac{2}{\theta} \partial_n n \partial_n. \]  

(14)

Upon discretizing \( n \) and writing \( \partial_n \) as a difference operator, one reproduces (11) or (12).

In this paper, we study properties of the dynamical system (11).

2.2 Non-commutative solitons

We start with static solutions of (8) and (11), the noncommutative solitons. It was pointed out in [20] that in the limit \( \theta \to \infty \), eq. (8) simply reduces to \( V'(\Phi) = 0 \) and
can be solved by a projector $\Phi = \lambda P$ with $\lambda \in \mathbb{R}$ such that $V'(\lambda) = 0$. One then treats the kinetic term as a perturbation and shows that the solutions continue to exist for $\theta < \infty$. Since, according to Derrick’s theorem, the solutions do not exist in the classical or “continuum” limit $\theta \to 0$, there must be a critical $\theta_c$ beyond which the NC solitons cease to exist. Of course, existence is not controlled by $\theta$ alone, but rather by certain dimensionless quantities built out of $\theta$ and the parameters of $V$.

The NC scalar solitons which are related to projectors in the $\theta \to \infty$ limit are referred to as the GMS (Gopakumar-Minwalla-Strominger) solitons. Instead of reviewing the large number of works devoted to GMS solitons, let us quote an existence theorem proved in [37], see also [36].

**Theorem 1.** For every sequence $\lambda^{(0)} = \{\lambda_0^{(0)}, \lambda_1^{(0)}, \ldots\}$ with $V'(\lambda_n^{(0)}) = 0$ and $V''(\lambda_n^{(0)}) > 0$ for all $n$ (i.e., a stable static solution of (11) at $1/\theta = 0$), there exists a unique continuation $\lambda(\theta) = \{\lambda_0(\theta), \lambda_1(\theta), \ldots\}$ to static solutions of (11) for $0 \leq 1/\theta < 1/\theta_c$, with some $\theta_c$ which depends on the chosen initial sequence $\lambda^{(0)}$.

We sketch the proof given in [37]. We work on the Hilbert space $l_2(\mathbb{Z}_{\geq 0}) = \mathcal{H} = \{\lambda; \sum_{n=0}^{\infty} |\lambda_n|^2 < \infty\}$. Let $(A\lambda)_n = -(n+1)(\lambda_{n+1} + \lambda_n) - n(\lambda_n - \lambda_{n-1})$ be the discretized radial part of the Laplacian according to (14), with domain $\mathcal{D}$.

The equation we have to solve is then
\[
\epsilon A\lambda + V'(\lambda) = 0,
\]
where $\epsilon = 2/\theta$.

The first step of the proof is to show that the operator $\epsilon A + V''(\lambda^{(0)})$ is invertible with bounded inverse as a map from $\mathcal{D}$ to $\mathcal{H}$. This essentially follows from $V''(\lambda^{(0)}) > 0$.

One then rewrites (15) as
\[
\lambda = (\epsilon A + V''(\lambda^{(0)}))^{-1} \left[ V''(\lambda^{(0)})\lambda^{(0)} + V'(\lambda^{(0)}) - V'(\lambda) - V''(\lambda^{(0)})(\lambda^{(0)} - \lambda) \right] =: T_\epsilon(\lambda),
\]
and shows that $T_\epsilon : \mathcal{H} \to \mathcal{H}$ is a contraction in a neighborhood of $\lambda^{(0)}$ (for reasonable regularity assumptions on $V$). The Banach fixed point theorem then implies the existence of a solution, with smooth dependence on $\epsilon$ in the appropriate norms.

### 2.3 Stable and unstable solitons

In order to simplify the following discussion, we assume that the potential $V$ is such that it has one global minimum $\lambda_{\text{MIN}}$, one local minimum $\lambda_{\text{min}} \neq \lambda_{\text{MIN}}$, and one local maximum $\lambda_{\text{max}}$. Corresponding statements should hold for a general potential.
In ref. [37], certain statements are proved concerning the stability of the solutions whose existence for $1/\theta > 0$ is established by the above Theorem 1. Stability is here defined as positivity of the Hessian of the Hamiltonian $H(\kappa_n = 0, \lambda)$, eq. (12), viewed as a quadratic form. For example, if $V(\lambda)$ is as described above, then the solutions obtained as continuations of $\lambda^{(0)}$ to $1/\theta > 0$ are stable if and only if $\lambda^{(0)}$ is of the form $\lambda_0^{(0)} = \lambda_1^{(0)} = \cdots = \lambda_N^{(0)} = \lambda_{\min}$ for some $N$ and $\lambda_n^{(0)} = \lambda_{\MIN}$ for $n > N$.

It is interesting to note that the unstable direction of the unstable solitons of Theorem 1 (e.g., the one connected to $\lambda_0^{(0)} = \lambda_2^{(0)} = \cdots = \lambda_{\MIN}$, $\lambda_1^{(0)} = \lambda_{\min}$) is not rotationally symmetric. More precisely, there exists a non-rotationally symmetric instability whenever the sequence $\lambda_0(\theta), \lambda_1(\theta), \ldots$ at finite $\theta$ is not monotonic. The nature of the instability implies that the decay of these solitons cannot be studied with an ansatz of the form (9).

Here, we shall be interested in a different kind of unstable soliton, namely the one obtained from continuation to $1/\theta > 0$ of the static unstable solution $\lambda_0^{(0)} = \lambda_{\max}$, $\lambda_n^{(0)} = \lambda_{\MIN}$ for $n > 0$. It is in fact rather simple to adapt the above proof to show existence of these solitons as static solutions. The essential point is that $\varepsilon A + V''(\lambda^{(0)})$ is still invertible even if $V''(\lambda_n^{(0)})$ is negative for a finite number of $n$’s, but positive elsewhere. This is because it differs from a positive operator only by a finite rank perturbation, so that $\varepsilon A + V''(\lambda^{(0)})$ can still be inverted for $\varepsilon$ in a finite neighborhood of 0. This soliton is unstable, but the instability is rotationally symmetric. Moreover, the sequence of $\lambda_n$’s is monotonic, so that at least initially, the soliton does not suffer from the non-rotationally symmetric instability discussed above. This justifies an ansatz of the form (9).

Summarizing, we can imagine studying the decay and long time evolution of the following unstable initial conditions, slightly perturbing them away from the stationary point.

(I1) A GMS soliton at finite $\theta$ obtained from an unstable static solution at infinite $\theta$, say with $\lambda_0^{(0)} = \lambda_{\max}$, $\lambda_n^{(0)} = \lambda_{\MIN}$ for $n > 0$.

(I2) A GMS soliton at finite $\theta$ obtained from a stable configuration at infinite $\theta$, say $\lambda_0^{(0)} = \lambda_2^{(0)} = \cdots = \lambda_{\MIN}$, $\lambda_1^{(0)} = \lambda_{\min}$.

(I3) The unstable maximum, $\lambda_n = \lambda_{\max}$ for all $n$.

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1We thank M. Salmhofer for this suggestion.
The questions to ask about the decay are quite different from case to case. In view of the relation between the noncommutative tachyon and string field theory explained in [26], one can draw an analogy between (I1) and the decay of an unstable D23 brane in bosonic string theory, in which one looks at the directions transverse to the brane. In view of (S3), then, one can ask if and how fast the energy that initially is localized is transported away to infinity. We will see that while energy is indeed dissipating whenever $1/\theta > 0$, the decay is very slow.

We emphasize, again, that the action (1) is not a good model for the tachyon of string field theory, even in the large B-field limit. Rather, as explained in [26], not only does the algebra factorize, but also the equations of motion do, and the tachyon has no dynamics in this picture. Thus, in the large B-field limit, one always obtains the noncommutative “field” theory at infinite $\theta$. It is a nice, but rather trivial observation of consistency that in this limit the energy does stay localized [31].

We find it harder to give an interpretation of the decays that start from (I2). We will not study these decays here, but we expect that some generalization of the technology that we develop could be applied in that case.

We finally comment on the initial conditions (I3). Again by analogy with string field theory, one might regard (I3) simply as the decay of the open string vacuum (or of a lower dimensional D-brane in which one looks in the directions along the brane). Then, one could ask for comparison with property (S4) of open string tachyons, in other words, can one see the formation of “tachyon matter” [13, 14, 15]. We will not study this here, by let us observe that if the perturbation away from the stationary point is uniform, i.e., independent of $n$, then the equations of motion can be trivially solved to yield uniform oscillations. So, starting from (I3), one may want to ask what happens under non-uniform perturbations. It also makes little sense to restrict to rotationally symmetric decays in this case.

In this paper, we will only deal with initial conditions of the form (I1).

3 Discrete breathers

In order to explain some intuition about the behavior of the system (11), we here digress on the theory of discrete breathers.

Discrete breathers are localized periodic solutions of lattices of oscillators, in which localization is not due to disorder (as in the usual case of Anderson localization),
but to nonlinearities. A useful review is [43]. The original discovery can be traced back to [38, 39], in which numerical simulations on these lattices were performed. A rigorous existence proof was given in [40], further and related earlier results appear in [41, 42, 43, 44, 45].

As an example, let us consider a Hamiltonian similar to (12),

\[
H = \sum_{n=-\infty}^{\infty} \left[ \frac{1}{2} \kappa_n^2 + V(\lambda_n) + \epsilon_n(\lambda_n - \lambda_{n-1})^2 \right].
\]  

For \( \epsilon_n = 0 \), this is just an infinite collection of independent classical oscillators. For oscillations around a minimum \( \lambda_{MIN} \) of \( V \) (we have in mind a potential similar to the one described in the previous section), we introduce canonical action-angle variables \((J_n, \phi_n)\), with

\[
J_n = \frac{1}{2\pi} \oint \sqrt{2(E(J_n) - V(\lambda_n))} \, d\lambda_n,
\]

and we let \( \phi_n = 0 \) correspond to the turning point of the oscillation, \( \kappa_n = 0 \). These coordinates are valid until \( \omega(J_n) = \frac{dE(J_n)}{dJ_n} = 0 \). Such oscillations are localized at whichever site of the lattice is excited.

On the other hand, if we couple the oscillators with \( \epsilon_n = \epsilon \neq 0 \) but small and constant, the lattice has propagating small oscillations (the phonons). The spectrum of the linearized problem is

\[
\omega_{Ph}^2(k) = \omega_0^2 + 4\epsilon(1 - \cos k),
\]

where \( \omega_0^2 = V''(\lambda_{MIN}) \), and \( k \in [0, 2\pi] \).

The linear spectrum is therefore bounded and given a nonlinear oscillation of the problem at \( \epsilon = 0 \) that is not in resonance with \( \omega_0 \), i.e., \( m\omega(J_n) \neq \omega_0 \) for all \( m \in \mathbb{Z} \), one can avoid resonances with the phonons, \( m\omega(J_n) \neq \omega_{Ph}(k) \) for all \( m \in \mathbb{Z} \) and \( k \in [0, 2\pi] \), if \( \epsilon \) is sufficiently small. It is therefore plausible that such an oscillation can continue to exist and be localized for \( \epsilon \neq 0 \). This intuitive argument can be turned into a rigorous existence theorem for discrete breathers [40].

**Theorem 2.** Every periodic solution of the decoupled nonlinear problem with frequency \( \omega \) satisfying the nonresonance condition \( m\omega \neq \omega_0 \) for all \( m \in \mathbb{Z} \) and the anharmonicity condition \( \frac{d\omega(J_n)}{dJ_n} \neq 0 \) for all \( n \in \mathbb{Z} \) and with initial conditions coded by a sequence
**$\eta_n \in \{0, 1\}$,**

\begin{align*}
\phi_n(0) &= 0 & \text{if } \eta_n = 1 \\
\phi_n(0) &= \pi & \text{if } \eta_n = -1 \\
\lambda_n(0) &= \lambda_{MIN} & \text{if } \eta_n = 0,
\end{align*}

*can be uniquely continued to a periodic solution of the full nonlinear problem $\epsilon_n = \epsilon$, for $\epsilon \geq 0$ sufficiently small. These oscillations are localized in the sense that $|\lambda_n(t) - \lambda_{MIN}|$ decays exponentially in $n$ whenever $\eta_n \to 0$.***

The proof, which can be found in [40], uses an implicit function theorem and is rather elementary. A more geometric formulation is given in [41]. The periodic solutions given by the theorem are dynamically stable, in the sense that small perturbations of the initial conditions remain small for all times. This is discussed in an abstract setting in [42].

We note that the solution that one wishes to continue has to be strictly periodic at $\epsilon = 0$, i.e., all excited oscillators have to be in resonance and they also have to be in phase, as coded by (20). This resonance condition appears to be opposite to the one found in the formulation of the KAM theorem. However, it is completely in agreement with the intuitive argument. If two frequencies are not rationally related, there will always be a resonance of some higher harmonics with the phonon band, leading to decay.

**What can discrete breathers teach us about NC soliton decay?**

The analogy with discrete breathers leads to the question whether breather-like solutions could exist in NC field theories. The answer seems to be no.

The obvious difference is that in (12) the coupling $\epsilon_n = n/\theta$ is not bounded as soon as $\theta < \infty$. The linear spectrum, which will be discussed in detail in the next section, is therefore unbounded. This leads to a failure of the intuitive argument given above for existence of breathers. Any nonlinear oscillation will be in resonance with the phonons, no matter how small $1/\theta$. There could be some hope if the linear spectrum was pure point, as in Anderson localization. However, as we will see below, the spectrum is purely absolutely continuous.

It is a little more difficult to understand what goes wrong with the proof of [40]. But recall that one of the assumptions for the implicit function theorem is continuity of
the derivative, and this seems to fail here, because the derivative with respect to $1/\theta$ is unbounded. While there might still be some hope that one could use slightly stronger versions of the implicit function theorem to prove existence, this will generically be restricted to very special conditions, and one cannot expect these breathers to be stable. In any case, our numerical results and the approximation computation of the decay rate clearly disfavor the existence of “noncommutative breathers”.

4 Numerical Results

We now turn to presenting some results obtained by studying the system (11) using numerics on the computer. We use the potential

$$V(\Phi) = 4\Phi^2(\Phi - 2)^2$$

and initial conditions of the form (11). Specifically, we take

$$\lambda_0(0) = \lambda_{max} = 1$$
$$\lambda_n(0) = \lambda_{MIN} = 0 \quad (n > 0)$$
$$\dot{\lambda}_n(0) = 0 \quad (\text{for all } n).$$

For nonzero $1/\theta$, this is not a stationary point, and will lead to some decay.

We have to work with a finite system $0 \leq n \leq N$. Therefore, in order to see a decay, if it occurs, we have to allow for a leak of energy. We do this by imposing “absorbing boundary conditions”, i.e., the discrete version of

$$\frac{\partial}{\partial t} + \frac{1}{\sqrt{r}} \frac{\partial}{\partial r} \sqrt{r} = 0,$$

(recall that outgoing spherical waves in two dimensions behave asymptotically like $e^{iE(t-r)/\sqrt{r}}$), which in view of (13) reads

$$\dot{\lambda}_{N+1} + \sqrt{\frac{2(N+1)}{\theta}}(\lambda_{N+1} - \lambda_N) + \frac{1}{\sqrt{8\theta(N+1)}}\lambda_{N+1} = 0.$$

We start by plotting, as a function of time, the energy $E_{loc}$ that is stored in the first four (say) oscillators, i.e.,

$$E_{loc}(t) = \sum_{n=0}^{3} \left[\frac{1}{2}\dot{\lambda}_n^2 + V(\lambda_n) + \frac{n}{\theta}(\lambda_n - \lambda_{n-1})^2\right],$$

divided by the initial energy, $E_{loc}(0)$. For different values of $\theta$, our results are shown in figs. [14].
Figure 1: Energy \((25)\) stored in the first four oscillators of the system \((11)\), truncated to 21 oscillators with boundary conditions \((24)\) \((N = 20)\). The initial conditions are \((22)\) for \(n \leq N\) and \(\lambda_{N+1}(0) = 0\), and \(\theta = 0.1\).

Figure 2: Same as fig. 1 for \(\theta = 0.2\).
Figure 3: Same as fig. for $\theta = 0.35$.

Figure 4: Same as fig. for $\theta = 0.7$. 
While the energy seems to always dissipate, there are apparently significant differences between large and small $\theta$. For small $\theta$, the system dissipates most of the energy at early times. For large $\theta$, the system loses a small fraction of its initial energy rather rapidly at early times, and after a certain “transient” time the energy dissipation rate approaches a constant but rather small value. To see this more clearly, we estimate the energy that is left after the transient time by averaging $E_{loc}(t)$ over a certain late time interval, and plot the result as a function of $\theta$. The result is shown in fig. 5.

![Figure 5: The energy left in the first four oscillators after the initial transient time apparent in figs. 1-4, obtained by averaging $E_{loc}(t)/E_{loc}(0)$ from $t = 30$ to $t = 50$, for varying $\theta$.](image)

We see that the turnover happens roughly at $\theta \approx 0.23$. This agrees rather well with the estimated critical $\theta_c$ below which the unstable NC soliton is expected to disappear. From [36], the most naive estimate is

$$\theta_c \sim \frac{\lambda}{|\chi'(\lambda)|_{\chi''(\lambda) = 0}} \approx 0.26$$

for the potential [24]. We also note for comparison that the period of linear oscillations is $T_0 = 2\pi/\omega_0 \approx 1.1$, where $\omega_0^2 = V''(0)$.

In the remainder of the paper, we will compute the rate of energy dissipation to a first approximation in the coupling $1/\theta$. 

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5 The Setup

We start from the classical Hamilton function

\[ H = \sum_{n=0}^{\infty} \left[ \frac{1}{2} \kappa_n^2 + V(\lambda_n) + \frac{n}{\theta} (\lambda_n - \lambda_{n-1})^2 \right], \] (27)

describing an infinite collection of oscillators, labeled by \( n = 0, 1, \ldots \), with \( n \)-dependent coupling. We want to understand the late time behavior at large \( \theta \), if at \( t = 0 \) we excite the zeroth oscillator highly into the non-linear regime. The numerics indicate that all other oscillators remain weakly excited during the decay process, so that we may linearize around the minimum of \( V \). Furthermore, for large \( \theta \), the zeroth oscillator is weakly coupled to all the other ones (the other oscillators are strongly coupled as \( n \to \infty \)). Accordingly, we split the Hamilton \( (27) \) into three pieces

\[ H = H_0 + H_{\text{cont}} + H_{\text{coup}}, \] (28)

where \( H_0 \) describes the zeroth oscillator that we want to excite,

\[ H_0 = \frac{1}{2} \kappa_0^2 + V(\lambda_0), \] (29)

\( H_{\text{cont}} \) describes the remaining oscillators linearized around the minimum, which for convenience we assume to be at \( \lambda_{MIN} = 0 \),

\[ H_{\text{cont}} = \sum_{n=1}^{\infty} \left[ \frac{1}{2} \kappa_n^2 + \frac{1}{2} \omega_0^2 \lambda_n^2 + \frac{n+1}{\theta} (\lambda_{n+1} - \lambda_n)^2 \right], \] (30)

and \( \omega_0^2 = V''|_{MIN} \). Finally, \( H_{\text{coup}} \) describes the coupling between \( H_0 \) and \( H_{\text{cont}} \)

\[ H_{\text{coup}} = \frac{1}{\theta} (\lambda_1 - \lambda_0)^2. \] (31)

At this level, the equations of motion are

\[ \ddot{\lambda}_0 + V'(\lambda_0) = \frac{2}{\theta} (\lambda_1 - \lambda_0) \]

\[ \ddot{\lambda}_1 + \omega_0^2 \lambda_1 - \frac{2}{\theta} [2(\lambda_2 - \lambda_1)] = -\frac{2}{\theta} (\lambda_1 - \lambda_0) \] (32)

\[ \ddot{\lambda}_n + \omega_0^2 \lambda_n - \frac{2}{\theta} [(n+1)(\lambda_{n+1} - \lambda_n) - n(\lambda_n - \lambda_{n-1})] = 0 \quad \text{for } n \geq 2 \]

Our strategy is as follows. We intend to treat the RHS of the first two equations in \( (32) \) as a perturbation. Without this term, the first line can be trivially integrated. By
our assumptions on the potential $V$, the solution, which we call $\tilde{\lambda}_0$, will be periodic, and we call the corresponding frequency $\alpha = \frac{dH_0(J_0)}{dJ_0}$, where $J_0$ is the action variable. We then feed this oscillation into the last two equations of (32), which we write as

$$\ddot{\lambda}_n + \omega^2_0 \lambda_n + \frac{2}{\theta} (A\lambda)_n = \frac{2}{\theta} \delta_{n,1} \tilde{\lambda}_0,$$  

(33)

where $A$ is the linear operator

$$(A\lambda)_n = -(n+1)(\lambda_{n+1} - \lambda_n) + n(\lambda_n - \lambda_{n-1}),$$  

(34)

supplemented with appropriate boundary conditions (see below). We can view (33) as a radiation problem with $\tilde{\lambda}_0$ as the source. It is then natural to decompose $\tilde{\lambda}_0$ into Fourier modes,

$$\tilde{\lambda}_0(t) = \sum_{m=-\infty}^{\infty} a_m e^{im\alpha t},$$  

(35)

and to first compute the energy radiated into the continuum modes of $A$ by a harmonic source. Summing over all Fourier modes, this will give an estimate for the rate of energy loss of the zeroth oscillator due to the coupling to the continuum modes. The expectation is that for large $\theta$, the energy loss per period is small, so that the coupling to the continuum essentially amounts to adding a friction term for $H_0$, with dissipation rate depending on the remaining energy. One can in principle study the resulting one-dimensional dissipative system, but we have not done this here. On the other hand, for small $\theta$, in particular $\theta < \theta_c$, the energy loss per period will become comparable to $H_0$. In fact, this will give an estimate for the critical $\theta_c$.

Most of what follows is concerned with the operator $A$. In the following section 6, we will diagonalize $A$, which means finding the generalized eigenfunctions and in particular their asymptotics at $n \to \infty$. This will allow in section 7 to compute the power radiated into the continuum by a harmonic source. Finally, we estimate the Fourier coefficients of $\tilde{\lambda}_0$ to obtain a decay rate of the soliton.

6 The operator $A$

The operator $A$ defined by (34) has the form of a discrete version of the familiar class of Sturm-Liouville operators, and we will draw on some of the theory pertaining to the latter (see, for instance, [46]). First, however, we will obtain some elementary results on the nature of the spectrum of $A$. 

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6.1 The spectrum of $A$ is unbounded and absolutely continuous

We have

**Proposition 1.** *The spectrum of $A$ is equal to $[0, \infty)$ and purely absolutely continuous.*

To prove the first part of the statement, we recall that the spectrum of $A_N$, 

$$(A_N \lambda)_n = N(2\lambda_n - \lambda_{n+1} - \lambda_{n-1})$$

on $l_2(\mathbb{Z})$ is equal to $[0, 4N]$. The spectrum is absolutely continuous and the generalized eigenvectors are $\lambda_n = e^{ikn}$ with spectral parameter $\kappa = 2N(1 - \cos k)$.

Now, for fixed $\kappa \in [0, \infty)$, we consider $N \in \mathbb{Z}_{\geq 0}$ large enough such that $\kappa = 2N(1 - \cos k)$ has a real solution $k(N) > 0$. For given $N, M \in \mathbb{Z}$, we can find a normalized 'almost-eigenvector' $\lambda = (\lambda_n) \in l_2(\mathbb{Z}_{\geq 0})$ with support in $[N, N + M]$ and

$$||(A_N - \kappa)\lambda|| \leq \text{const. } N\left(\frac{k(N)}{M} + \frac{1}{M^2}\right),$$

where the constant is independent of $N$ and $M$. (For instance, we can take $\lambda_n = f(n)e^{ik(N)n}$, where $f$ is normalized, has support in $[N, N + M]$, and is sufficiently smooth.) We also have an estimate on the first derivative $\lambda'$, $(\lambda')_n = \lambda_n - \lambda_{n+1}$, e.g.,

$$||\lambda'|| \leq \text{const. } \left(k(N) + \frac{1}{M}\right).$$

Then, using

$$((A - A_N)\lambda)_n = \frac{n - N}{N}(A_N \lambda)_n + \lambda_n - \lambda_{n+1},$$

we find

$$||(A - \kappa)\lambda|| \leq \text{const. } \left[\frac{M\kappa}{N} + k(N) + \frac{1}{M} + \left(\frac{M}{N} + 1\right)\left(\frac{Nk(N)}{M} + \frac{N}{M^2}\right)\right].$$

For large $N$, we have $k(N) \sim (\kappa/N)^{1/2}$, and we let $M \sim N^{3/4}$. Thus,

$$||(A - \kappa)\lambda|| \leq \text{const. } \left[\kappa N^{-1/4} + \kappa^{1/2}N^{-1/2} + N^{-3/4} + \kappa^{1/2}N^{-1/4} + N^{-1/2}\right].$$

Making $N$ large, the Weyl criterion implies that $\kappa$ is in the spectrum of $A$. Positivity of $A$ is trivial.

To prove absolute continuity, we use the so-called positive commutator method or Mourre theory [47]. The point of Mourre theory is that one can prove absolute continuity of parts of the spectrum of a self-adjoint operator such as $A$ by simply exhibiting another self-adjoint operator $K$ such that the commutator $i[A, K]$ is positive on the part of the spectrum in question.

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A positive commutator. We write

\[ A = 2n + 1 - nT_+ - T_- n, \]

where \((n\lambda)_n = n\lambda_n, (T_+\lambda)_n = \lambda_{n-1},\) and \((T_-\lambda)_n = \lambda_{n+1}.\) We have

\[ [n, T_\pm] = \pm T_\pm, \quad [T_+, T_-] = -\delta_{n,0}. \]

Define \(K = \frac{i}{2}(nT_+ - T_- n)\) and find

\[ i[A, K] = -[2n + 1 - nT_+ - T_- n, \frac{i}{2}(nT_+ - T_- n)] \]
\[ = -n[n, T_+] + [n, T_-]n \]
\[ - \frac{1}{2}(nT_-[T_+, n] + [n, T_-]nT_+ - [T_-, n]T_+ n - T_- n[n, T_+]) \]
\[ = -nT_+ - T_- n - \frac{1}{2}(-nT_-T_+ - T_- nT_+ - T_- T_+ n - T_- nT_+) \]
\[ = 2n + 1 - nT_+ - T_- n = A \geq 0. \]

This shows absolute continuity away from 0. But 0 is not an eigenvalue, because any solution of \(n(\lambda_n - \lambda_{n+1}) - (n + 1)(\lambda_{n+1} - \lambda_n) = 0\) is constant and either trivial or nonnormalizable.

We now turn to a more formal analysis of the discrete Sturm-Liouville operator \(A.\)

### 6.2 Boundary Conditions

\(A\) is naively a second order difference operator, and it might seem that we need to specify boundary conditions at \(n = 0.\) However, from the definition (34), it is obvious that \(n = 0\) is a singular point of the difference operator (the order of the operator jumps there). Therefore, \(we do not need to specify boundary conditions,\) if we consider the operator on \(l_2(\{0, 1, \ldots\})\). We will denote this operator (called \(A\) in the previous subsection) acting on \(l_2(\mathbb{Z}_{\geq 0})\) as in (34) without boundary conditions, by \(A_0.\) An alternative definition of \(A_0\) would be by a quadratic form.

For our radiation problem (32), (33), we want to treat \(\tilde{\lambda}_0\) as an inhomogeneous term added to a homogeneous linear equation (the LHS of (33)). In order for (33) to coincide with (32), we have to define \(A\) acting in \(l_2(\{1, 2, \ldots\})\) as in (34), with in addition the boundary condition

\[ \lambda_1 - \lambda_0 = 0. \]
This boundary condition makes $A$ self-adjoint. But note that, as we review below, the solution of the radiation problem (computation of the resolvent of $A$) requires the knowledge of the solutions of the finite difference equation with arbitrary boundary conditions. Also note that the particular form of the boundary condition is a result of the way we have separated the perturbation in (32). We could, for instance, also have left $\frac{2}{b}\lambda_1$ on the LHS of the equations. Our results should not depend on this choice.

6.3 The generalized eigenfunctions

We know already that the spectrum of $A_0$ is $[0, \infty)$ and absolutely continuous. The same statement holds obviously for $A$. Here, we want to complement this result by explicitly diagonalizing $A$. From the spectral theory of Sturm-Liouville operators, it is well-known that we will need to know the two linearly independent solutions of

$$ (A\lambda)_n = \kappa \lambda_n $$

for arbitrary $\kappa \in \mathbb{C}$. These two solutions will not necessarily be $l_2$ normalizable and satisfy linearly independent boundary conditions at $n = 1$. Specifically, we call the two solutions of (46), $\lambda_n = \Phi_\kappa(n)$ and $\lambda_n = \Psi_\kappa(n)$, satisfying the boundary conditions

$$ \Phi_\kappa(0) = 1 \quad \Phi_\kappa(1) = 1 - \kappa $$
$$ \Psi_\kappa(0) = 0 \quad \Psi_\kappa(1) = 1. $$

The reason for choosing (47) was that $\Phi_\kappa$ then also is a solution of $A_0\Phi_\kappa = \kappa \Phi_\kappa$, but this turns out to be unimportant. The reason for choosing (48) as boundary conditions for $\Psi_\kappa$ is that in this way, the Wronskian of the finite difference equation, which is given in the standard Sturm-Liouville fashion, is equal to one,

$$ W(\Phi_\kappa, \Psi_\kappa) = (n + 1)(\Phi_\kappa(n)\Psi_\kappa(n + 1) - \Psi_\kappa(n)\Phi_\kappa(n + 1)) $$
$$ = \text{const.} = 1. $$

We note that the equation (46) has been studied in [32]. It seems, however, that one of the functions given in [32] is incorrect, and moreover, the asymptotics of the solutions in the regime where we will need them have not been worked out. Quite a
bit of work on \cite{16} shows that the two solutions are as follows.

\[ \Phi_\kappa(n) = \sum_{k=0}^{n} \frac{(-\kappa)^k}{k!} \binom{n}{k} \]  

(50)

\[ \Psi_\kappa(n) = \sum_{k=0}^{n} \frac{(-\kappa)^k}{k!} \left[ \sum_{s=0}^{k} \binom{k}{s} \binom{n}{s} (H_{n-s} + H_{k-s} - 2H_s) \right], \]  

(51)

where \( H_k \) are the “harmonic numbers”\(^2\)

\[ H_k = \sum_{m=1}^{k} \frac{1}{m}. \]  

(52)

It is trivial that (50) and (51) are entire functions of \( \kappa \in \mathbb{C} \).

We see from (50), (51) that \( \Phi_\kappa(n) \) is a polynomial in \( \kappa \) of order \( n \), while \( \Psi_\kappa(n) \) is a polynomial of order \( n - 1 \) (it is easy to see that the coefficient of \( (-\kappa)^n \) vanishes). In fact, \( \Phi_\kappa(n) \) is nothing but a \textit{Laguerre polynomial}, which is also a particular \textit{confluent hypergeometric function}. Namely,

\[ \Phi_\kappa(n) = L_n(\kappa) = _1F_1(-n, 1; \kappa). \]  

(53)

A similar, but more complicated statement can be made for \( \Psi_\kappa(n) \), see below.

\section*{6.4 Asymptotics}

We wish to determine the asymptotic behavior of \( \Phi_\kappa(n) \) and \( \Psi_\kappa(n) \) for \( \kappa \) fixed, and \( n \gg |\kappa| \) large, for \( \kappa \) in the positive half plane \( \mathbb{C}_+ := \{ \kappa \in \mathbb{C} | \text{Re}(\kappa) > 0 \} \). It was noticed before that for large \( n \), the operator \( A \), eq. (34), can be approximated by a differential operator, \( A \approx -\frac{1}{2r} \partial_r r \partial_r \), where \( n = r^2/2 \). Let us also recall that the \textit{Bessel functions} of order zero,

\[ J_0(x) = \sum_{k=0}^{\infty} \frac{(-x^2/4)^k}{(k!)^2} \]  

(54)

\[ Y_0(x) = \frac{2}{\pi} \left[ J_0(x)(\ln \frac{x}{2} + \gamma) - \sum_{k=1}^{\infty} \frac{(-x^2/4)^k}{(k!)^2} H_k \right], \]  

(55)

where \( \gamma = \lim_{n \to \infty} (H_n - \ln n) \) is the \textit{Euler-Mascheroni constant}, are two linearly independent solutions of \textit{Bessel’s differential equation}

\[ g'' + \frac{1}{x} g' + g = 0. \]  

(56)

\(^2H_0 = 0\)
We choose the standard branch for the logarithm in (55). Furthermore, it is easy to see that both (54) and the series appearing in (55) are entire functions of $x \in \mathbb{C}$. Hence, it is clear that $J_0(x)$ and $Y_0(x)$ are analytic in $\mathbb{C}_+$. Comparing (46) with (56), we expect an asymptotic expansion of $\Phi_\kappa(n)$ and $\Psi_\kappa(n)$ in terms of the Bessel functions $J_0(x)$ and $Y_0(x)$, with $x = \sqrt{4n\kappa}$. Here, and in all similar cases considered below, we choose the branch of the square root for which the signs of $\text{Im}(x)$ and $\text{Im}(\kappa)$ are equal.

6.4.1 Asymptotics of $\Phi_\kappa(n)$

For $\Phi_\kappa$, such an asymptotic expansion is indeed known (see, e.g., [48]). It can be derived as follows. Because of (53), $\Phi_\kappa(n) = L_n(\kappa)$ satisfies the Laguerre differential equation

$$\kappa f'' + (1 - \kappa)f' + nf = 0,$$

(57)

where differentiation is with respect to $\kappa$. If we make the usual substitution $f(\kappa) = e^{\kappa/2}g(\kappa)$ with $\kappa = x^2/4n$ in eq. (57), we obtain

$$g'' + \frac{1}{x}g' + \left(1 - \frac{x^2}{4n^2} + \frac{1}{2n}\right)g = 0,$$

(58)

where differentiation is with respect to $x$.

For $|x| \ll n$, which is equivalent to $n \gg |\kappa|$, this reduces to (50), up to a perturbation of order $O(\kappa n)$. Thus, in this limit, any solution $f(\kappa)$ of (57) is asymptotic to a linear combination of Bessel functions $J_0(x)$ and $Y_0(x)$, where the deviation from the exact solution is an asymptotic series in powers of $|\kappa| n$. The particular linear combination at hand can be determined from the behavior at $x, \kappa \to 0$.

From the expressions for $\Phi_\kappa(n)$ in eq. (50) and $J_0(x)$ in eq. (54) it is easy to see that

$$\lim_{n \to \infty} \Phi_{\kappa n}(n) = J_0(x) ,$$

(59)

where $\kappa_n := \frac{x^2}{4n}$, for any fixed value of $x \in \mathbb{C}_+$. Thus, by what has been stated above, it follows that for any fixed $\kappa \in \mathbb{C}_+$ and $n \gg |\kappa|$ large,

$$\Phi_\kappa(n) \sim e^{\kappa/2}J_0(x(n, \kappa)) ,$$

(60)

with $x(n, \kappa) = \sqrt{4n\kappa}$. 23
6.4.2 Asymptotics of $\Psi_\kappa(n)$

The asymptotics of $\Psi_\kappa(n)$ for large $n \gg |\kappa|$ requires more work. It is natural to expect a connection to the second solution of Laguerre’s differential equation (57). Surprisingly, it turns out that this solution is nowhere to be found in the usual references, including [48, 49, 50]. The point is that eq. (57) is a very special case of the confluent hypergeometric equation in which both parameters are integer, and the “logarithmic solution” given in the references reduces to the ordinary one in this case. One has to go back to [51] to find that a second solution is given by

$$K_n(\kappa) = L_n(\kappa) \ln \kappa + \sum_{k=1}^{n} \frac{(-\kappa)^k}{k!} \binom{n}{k} (H_{n-k} - 2H_k - H_n) +$$

$$+ (-1)^n \sum_{k=n+1}^{\infty} \frac{n! (k-n-1)!}{k!} \kappa^k,$$  \hspace{1cm} (61)

which one can easily verify by direct substitution, and where we choose the standard branch of the logarithm. It is easy to see that both series on the RHS of (61) define entire functions of $\kappa \in \mathbb{C}$.

Before we relate $K_n(\kappa)$ to $\Psi_\kappa(n)$, let us determine its asymptotics for large $n \gg |\kappa|$, for $\kappa \in \mathbb{C}_+$. We use the same logic as for $L_n(\kappa)$ and first consider the limit $n \to \infty$, $\kappa \to 0$, with $x^2 = 4\kappa n$ fixed. Defining $\kappa_n := \frac{x^2}{4n}$, we easily see that

$$K_n(\kappa_n) \sim L_n(\kappa_n) \ln \kappa_n - \frac{2}{(k!)^2} \sum_{k=1}^{\infty} \frac{(-\kappa_n n)^k}{k!} H_k$$

$$\sim \pi Y_0(x) - J_0(x)(\ln n + 2\gamma) \quad \text{for } n \to \infty.$$  \hspace{1cm} (62)

The same arguments that were used for $L_n(\kappa)$ then imply that for any $\kappa \in \mathbb{C}_+$, and $n \gg |\kappa|$ large,

$$K_n(\kappa) \sim \pi e^{\kappa/2} Y_0(x(n, \kappa)) - e^{\kappa/2} J_0(x(n, \kappa))(\ln n + 2\gamma),$$  \hspace{1cm} (63)

where $x(n, \kappa) = \sqrt{4\kappa n}$.

To relate $\Psi_\kappa(n)$ and $K_n(\kappa)$, we rewrite the sums in (61) in terms of $\tilde{s} = k - s$ and $\tilde{k} = k - \tilde{s} = s$. This yields

$$\Psi_\kappa(n) = \sum_{\tilde{s}=0}^{n} \frac{(-\kappa)^{\tilde{s}}}{\tilde{s}!} \sum_{\tilde{k}=0}^{\tilde{n}} \frac{(-\kappa)^{\tilde{k}}}{\tilde{k}!} \binom{n}{\tilde{k}} (H_{n-\tilde{k}} + H_{\tilde{s}} - 2H_{\tilde{k}}).$$  \hspace{1cm} (64)
Using this, we claim that for $\kappa \in \mathbb{C}_+$ and $n$ large,

$$
\left| \Psi_\kappa(n) - \sum_{s=0}^{n} \frac{(-\kappa)^s}{s!} \left( \sum_{k=0}^{n} \frac{(-\kappa)^k}{k!} \left( n_{n-k} - 2H_k \right) \right) \right| \leq C(8|\kappa|) \frac{n \log n}{n!}, \quad (65)
$$

for a constant $C$ that is uniform in $n$ and $|\kappa|$. Indeed, the LHS of (65) is bounded by

$$
\sum_{s=0}^{n} \frac{|\kappa|^s}{s!} \left[ \sum_{k=0}^{n} \frac{|\kappa|^k}{k!} \left( n_{n-k} + H_s - 2H_k \right) \right] \leq 2^{n+2} \log n \sum_{s=0}^{n} \sum_{r=n}^{n+s} \frac{|\kappa|^r}{r!} \left( \frac{\tilde{s}}{r} \right) \quad (66)
$$

$$
< Cn(2|\kappa|)^n \frac{n \log n}{n!} \sum_{r=n}^{2n} 2^r,
$$

since $H_l < 2 \log n$ for all $1 \leq l \leq n$, and $\binom{n}{k} \leq 2^n$. This straightforwardly implies (65).

By noticing that the last term in (66) is small in the limit $n \gg |\kappa|$, with absolute value bounded by

$$
\sum_{r=1}^{\infty} \frac{n!}{(n+r)!} \frac{(r-1)!}{(n+r)!} |\kappa|^{n+r} \leq \frac{|\kappa|^n}{n!} e|\kappa|, \quad (67)
$$

one then infers that for any $\kappa \in \mathbb{C}_+$, and large $n \gg |\kappa|$,

$$
\Psi_\kappa(n) \sim e^{-\kappa} \left[ \sum_{k=0}^{n} \frac{(-\kappa)^k}{k!} \left( n_{n-k} - 2H_k \right) \right] + \sum_{s=0}^{\infty} \frac{(-\kappa)^s}{s!} H_s \left[ \sum_{k=0}^{n} \frac{(-\kappa)^k}{k!} \left( \binom{n}{k} \right) \right] \quad (68)
$$

$$
\sim e^{-\kappa} \left[ K_n(\kappa) + L_n(\kappa)(H_n - \ln \kappa) \right] + L_n(\kappa) \sum_{s=0}^{\infty} \frac{(-\kappa)^s}{s!} H_s,
$$

with a small error bounded by the sum of (65) and (67). Using (68), we finally find

$$
\Psi_\kappa(n) \sim \pi e^{-\kappa/2} Y_0(x(n, \kappa)) + e^{-\kappa/2} F(\kappa) J_0(x(n, \kappa)), \quad (69)
$$

for $n \gg |\kappa|$, where we have defined

$$
F(\kappa) = e^\kappa \sum_{s=0}^{\infty} \frac{(-\kappa)^s}{s!} H_s - \gamma - \ln \kappa. \quad (70)
$$

It is possible to express $F(\kappa)$ in terms of the incomplete Gamma function, but we do not need this here. As a check, one may note that the Wronskian of the Bessel functions corresponding to (66) is $W(J_0(x(n, \kappa)), Y_0(x(n, \kappa))) = 2/\pi$, which after the appropriate coordinate transformations confirms $W(\Phi_\kappa(n), \Psi_\kappa(n)) = 1$. 

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6.5 Resolvent and density of states

We can now compute the resolvent of $A$ and the spectral measure (density of states) by following the usual Sturm-Liouville procedure. We introduce, for $\kappa \in \mathbb{C}_+$, $\text{Im}(\kappa) \neq 0$, the functions $f_\kappa(n)$ and $g_\kappa(n)$, satisfying $Af_\kappa = \kappa f_\kappa$ and $Ag_\kappa = \kappa g_\kappa$, and characterized by the property that $f_\kappa$ satisfies the boundary condition \((45)\), $f_\kappa(1) = f_\kappa(0)$, while $g_\kappa$ is normalizable at infinity. We can express $f_\kappa$ and $g_\kappa$ as linear combinations of $\Phi_\kappa$ and $\Psi_\kappa$ and we normalize them such that the Wronskian is equal to one,

$$W(g_\kappa, f_\kappa) = 1. \quad (71)$$

Using $f_\kappa$ and $g_\kappa$, the resolvent of $A$ can be written as

$$R_\kappa(n, m) = (A - \kappa)^{-1}(n, m) = \begin{cases} f_\kappa(n)g_\kappa(m) & \text{for } n \leq m \\ g_\kappa(n)f_\kappa(m) & \text{for } n \geq m \end{cases} \quad (72)$$

Using \((71)\) and \((72)\), it is easy to see that $f_\kappa$ is given by

$$f_\kappa = \Phi_\kappa + \kappa \Psi_\kappa. \quad (73)$$

To determine $g_\kappa$, we recall that the Bessel functions $J_0$ and $Y_0$ behave at infinity as

$$J_0(x) \sim \frac{2}{\pi x} \cos \left( x - \frac{\pi}{4} \right) \quad (74)$$

$$Y_0(x) \sim \frac{2}{\pi x} \sin \left( x - \frac{\pi}{4} \right). \quad (75)$$

Using \((60)\) and \((69)\), and $x = 2\sqrt{n\kappa}$, we see that the normalizable solution, behaving as $J_0(x) \pm iY_0(x)$ for $\text{Im}(\kappa) \gtrless 0$, is given by

$$g_\kappa = C_\pm \left[ \pi e^{-\kappa} \Phi_\kappa \pm i \Psi_\kappa \mp i e^{-\kappa} F(\kappa) \Phi_\kappa \right], \quad (76)$$

where $C_\pm$ is determined by the requirement on the Wronskian,

$$1 = W(g_\kappa, f_\kappa) = C_\pm \left[ \kappa \pi e^{-\kappa} \mp i \kappa e^{-\kappa} F(\kappa) \mp i \right]. \quad (77)$$

The density of states or spectral measure $\rho(\kappa)$ of $A$ is determined from the jump of the imaginary part of the resolvent across the real $\kappa$ axis. In the case at hand, one finds in the usual way,

$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} (g_{\kappa + i\epsilon} - g_{\kappa - i\epsilon}) = \rho(\kappa) f_\kappa, \quad (78)$$

which upon using \((73)\) and \((76)\) gives, for $\kappa > 0$,

$$\rho(\kappa) = \frac{e^{-\kappa}}{(\kappa \pi e^{-\kappa})^2 + (1 + \kappa e^{-\kappa} F(\kappa))^2}. \quad (79)$$

For illustrative purposes, we show a plot of $\rho(\kappa)$ in fig. 6.
7 Estimating the decay rate

Equipped with the analysis of the linear operator $A$, we now estimate the decay rate in the approximation explained in section 5.

7.1 A harmonic source

We first study the radiation problem (33) with a harmonic source.

$$\ddot{\lambda}_n + \omega_0^2 \lambda_n + \frac{2}{\vartheta}(A\lambda)_n = \frac{2}{\vartheta} \delta_{n,1} e^{i\omega t}. \tag{80}$$

There are in principle two ways to treat this problem. One is to make an ansatz of the form $\lambda_n(t) = e^{i\omega t} \mu_n$ and to solve for $\mu_n$, imposing outgoing boundary conditions at $n \to \infty$. The other is to use the spectral decomposition of $\lambda_n(t)$ in eigenmodes of $A$. As a check, we will compute with both methods.

7.1.1 Diagonalization of $A$

We introduce

$$\lambda_\kappa(t) = \sum_{n=1}^{\infty} f_\kappa(n) \lambda_n(t), \tag{81}$$
where \( f_\kappa \) is given by (73). The inverse transformation involves the density of states,\[
\lambda_n(t) = \int \rho(\kappa) d\kappa \; f_\kappa(n) \; \lambda_\kappa(t) .
\] (82)

Using eq. (81) in eq. (80) yields\[
\ddot{\lambda}_\kappa + \left( \frac{2}{\theta} \kappa + \omega_0^2 \right) \lambda_\kappa = be^{i\omega t} ,
\] (83)

where \( b = \frac{2}{\theta} a f_\kappa(1) \). This is a forced harmonic oscillator and can be solved in the usual way. Using (82) to plug the solution back into the expression for the energy,\[
E(t) = \sum_{n=1}^{\infty} \left[ \frac{1}{2} \dot{\lambda}_n^2 + \frac{1}{2} \omega_0^2 \lambda^2_n \right] + \frac{1}{2\theta} (\lambda, A\lambda) ,
\] (84)

one finds that at late times, \( E \) is proportional to \( t \) and the radiated power is given by\[
P = \lim_{t \to \infty} \frac{E(t)}{t} = \frac{\pi}{2\theta} \omega a^2 \rho(\kappa)
\] (85)

where \( \kappa = \frac{\theta}{2}(\omega^2 - \omega_0^2) \) has to be positive, and \( \rho(\kappa) \) is given by (79). The derivation of (85) is similar to that of Fermi’s Golden Rule.

### 7.1.2 Modification of boundary conditions

If we put \( \lambda_n(t) = e^{i\omega t} \mu_n \) into eq. (80), we obtain\[
-\omega^2 \mu_n + \frac{2}{\theta}(A\mu)_n + \omega_0^2 \mu_n = \frac{2}{\theta} \delta_{n,1} a .
\] (86)

Thus, for \( n \neq 1 \), \( \mu_n = (\mu_1) \) must be a linear combination of \( \Phi_\kappa \) and \( \Psi_\kappa \), with \( \kappa = \frac{\theta}{2}(\omega^2 - \omega_0^2) \). We require that \( \mu_n \) be outgoing at infinity, which from the behavior of \( J_0 \) and \( Y_0 \) means that \( \mu_n \) must behave as \( J_0(x) + iY_0(x) \). Hence,\[
\mu_n = D \left[ \pi e^{-\kappa} \Phi_\kappa(n) + i\Psi_\kappa(n) - i e^{-\kappa} F(\kappa) \Phi_\kappa(n) \right] ,
\] (87)

where \( D \) is determined by the equation at \( n = 1 \),\[
-\frac{2}{\theta} 2(\mu_2 - \mu_1) + (\omega_0^2 - \omega^2) \mu_1 = -\frac{2}{\theta} (\mu_1 - \mu_0) = \frac{2}{\theta} a ,
\] (88)

yielding\[
a = D \left[ \pi \kappa e^{-\kappa} - i\kappa e^{-\kappa} F(\kappa) - i \right] .
\] (89)
To finish the calculation of the energy output, one has to plug the solution into the expression for the “energy current”,

\[ T_{01} = -\frac{2}{\theta} (n + 1)(\lambda_{n+1} - \lambda_n)\dot{\lambda}_n. \]  

(90)

Using the asymptotics of \( \Phi_\kappa \) and \( \Psi_\kappa \) and of \( J_0 \) and \( Y_0 \), one obtains for the average radiated power

\[ P = \overline{T_{01}}(n \to \infty) = \frac{\pi}{2\theta} e^{-\kappa}|D|^2, \]  

(91)

which using (89), reproduces exactly (85),

\[ P = P(\omega, a, \theta) = \frac{\pi}{\theta} a^2 e^{-\kappa} \left( \frac{e^{-\kappa}}{\kappa} \right)^2, \]  

(92)

with \( \kappa = \frac{\theta}{2}(\omega^2 - \omega_0^2) > 0 \), and \( F(\kappa) \) is given by (70).

### 7.2 The decay rate of the NC soliton

As explained in section 5, the total decay rate of the NC soliton is obtained by decomposing the oscillation \( \tilde{\lambda}_0 \) into Fourier modes, (35), using (92) for each mode, and summing over Fourier modes, i.e.,

\[ P_{\text{tot}}(\alpha, \theta) = \sum_{m=-\infty}^{\infty} P(m\alpha, a_m, \theta), \]  

(93)

where

\[ a_m = \frac{\alpha}{2\pi} \int_0^{2\pi/\alpha} dt \, e^{-i\alpha t} \tilde{\lambda}_0(t). \]  

(94)

are the Fourier coefficients of \( \tilde{\lambda}_0 \). The precise values of the \( a_m \) of course depend on the details of the potential and the exact value of the basic frequency \( \alpha \) (which, if we enforce energy conservation, is not fixed), so that we need an estimate to make further progress on (93).

There are two regimes in which one can estimate the Fourier coefficients. The first is for almost harmonic oscillations, for which one can use perturbation theory to determine the \( a_m \). This will be the description when, for late times, the zeroth oscillator is in a small vicinity of the stationary point, and the soliton has almost completely decayed. The second regime is defined by \( \alpha \ll \omega_0 \). It describes the initial

\[^{3}\text{We define } T_{01} \text{ by the requirement that it satisfy } \dot{T}_{00}(n) = -(T_{01}(n) - T_{01}(n-1)) \text{ on a solution of the equations of motion, where } T_{00}(n) = \frac{1}{2} \dot{\lambda}_n^2 + \frac{\theta}{\pi} (\lambda_n - \lambda_{n-1})^2 + \frac{1}{2}\omega_0^2 \lambda_n^2 \text{ is the “energy density”}.\]
stages of the decay of the NC soliton, and it will be a consistent approximation if the energy loss per period is much smaller than the total energy in the zeroth oscillator, i.e., we require
\[ P_{\text{tot}}(\alpha, \theta) \ll \alpha H_0 \ll \omega_0 H_0 . \] (95)

In this regime, the coefficients (94) can be estimated as follows \[52\].

If the energy \( H_0 \) is exactly equal to the critical value \( V(\lambda_{\text{max}}) \), there is not only the static solution, but also the separatrix solution that is not periodic, but rather takes an infinite amount of time to fall off and come back to the top of the potential. We call this separatrix \( \bar{\lambda}(t) \).

For energies close to the critical point of \( V \), the oscillation, which has a frequency \( \alpha \ll \omega_0 \), is well approximated by the convolution of the separatrix with a sum of delta functions separated by \( 2\pi/\alpha \), which is a much longer time scale than the scale typical of \( \bar{\lambda} \), i.e.,
\[ \tilde{\lambda}_0(t) \approx \sum_{m=-\infty}^{\infty} \bar{\lambda}(t - 2\pi m/\alpha) . \] (96)
The Fourier transform of \( \tilde{\lambda}_0 \) is then given by the product of the Fourier transform of \( \bar{\lambda} \) with delta functions at \( m\alpha, m \in \mathbb{Z} \). In other words,
\[ a_m \approx \alpha \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \ e^{-im\alpha t} \bar{\lambda}(t) = \alpha \hat{\bar{\lambda}}(m\alpha) \] (97)
where \( \hat{\bar{\lambda}} \) is the Fourier transform of the separatrix.

Plugging (97) in (93), and using that \( \rho(\kappa) \) is finite everywhere to replace the sum over Fourier modes by an integral, we obtain the final expression for the decay rate of the NC soliton,
\[ P_{\text{tot}}(\alpha, \theta) \approx \frac{\pi}{2\theta} \alpha \int_{\omega_0}^{\infty} d\omega \omega|\hat{\bar{\lambda}}(\omega)|^2 \rho(\kappa) , \] (98)
in which \( \kappa = \frac{2}{\theta}(\omega^2 - \omega_0^2) \).

For some special potentials, one can explicitly compute the separatrix and its Fourier transform. For instance, for the quartic potential that we used in the numerics, \( V(\Phi) = 4\Phi^2(\Phi - 2)^2 \), one has\(^4\)
\[ \tilde{\lambda}(t) = \frac{\sqrt{2}}{\cosh 4t} + 1 , \] (99)

\(^4\)Another simple example that can be solved explicitly is the cubic potential.
and

\[
\hat{\lambda}(\alpha) = \frac{1}{4\sqrt{2}} \frac{1}{\cosh \frac{\pi \alpha}{8}}.
\]

(100)

For illustration, we show in fig. 7 the numerical evaluation of (98) for this separatrix. We see that the value of \(\theta\) at which the energy loss per period becomes appreciable is around \(\theta \approx 0.2\), in rough agreement with the estimate (26).

\[ P_{\text{tot}}(\theta, \alpha)/\alpha V(\lambda_{\text{max}}) \]

Figure 7: The energy loss per period of \(\tilde{\lambda}_0\) at initial stages of the NC soliton decay, as a function of \(\theta\). The potential is of the quartic form (21).

8 Conclusions

In this paper, we have studied some properties of the dynamical system (11), which is (the radial part of) the classical equations of motions for a scalar noncommutative field in 2 + 1 dimensions. We have in particular analyzed the long time evolution of initial conditions of the form (I1) corresponding to unstable noncommutative solitons. We have obtained numerical and analytical evidence that the energy is dissipated away to infinity in the form of classical radiation. For large, but finite values of the noncommutativity parameter \(\theta\), the decay rate of the soliton is exponentially small in \(\theta\). Our main technical result is the explicit diagonalization of the linear problem.
Our paper will end here, but we would like to mention a number of open problems that are worthwhile of further investigation, and to which we hope to return in the near future.

In the context of NC field theory, there are several other unstable solitons whose decays one might want to study. We have already mentioned the scalar soliton decays of type (I2) and (I3). Decays of the type (I2) are particularly interesting because they have to proceed through non rotationally invariant configurations \[37\]. Solution of this problem will involve solving the linear problem for nonzero angular momentum (i.e., finding the “discrete Bessel functions” for degree \(l \neq 0\), the function \(\Phi_\kappa\) and \(\Psi_\kappa\) corresponding to \(l = 0\)). Furthermore, it is not a priori clear into what decay products such a soliton will decay. For example, the soliton corresponding to \(\lambda_0^{(0)} = \lambda_2^{(0)} = \cdots = \lambda_{\text{MIN}}, \lambda_1^{(0)} = \lambda_{\text{min}}\) could decay into either the stable soliton corresponding to \(\lambda_0^{(0)} = \lambda_{\text{min}}, \lambda_n^{(0)} = \lambda_{\text{MIN}} (n > 0)\) or into the global vacuum. An additional question is whether there exist rotationally asymmetric stable solitons, and if so, whether they can be formed as decay products of the above process. It is natural to ask whether there is a simple criterion to decide this question.

NC gauge theories, with or without couplings to scalar fields, also admit solitons, which in fact have a more direct interpretation in terms of D-branes in string theory (see \([25]\) for a review). Some of these solitons are unstable and hence can decay. Our solution of the linear problem is directly relevant in this context, but the nonlinearities are rather different, leading to a markedly different qualitative behavior \([35]\).

Also from the point of view of string theory, it would be interesting to compare our results with the qualitative properties of the decay of the lump in \(\phi^3\) theory in 1 + 1 dimensions, which is another natural toy model for tachyon condensation. The fluctuations around the lump and the expression of the stable vacuum in terms of the fluctuations were studied in \([28]\). However, it seems to be unclear whether in the time evolution, the field eventually settles down in the stable vacuum, and if so, how the energy is dissipated \([29]\).

Probably the most immediate problem from a mathematical point of view is to substantiate our qualitative discussion with rigorous estimates of the decay. As we have mentioned, our system is similar to systems exhibiting the discrete breather phenomenon \([43]\). One can also expect a connection to the general theory of resonances in nonlinear field equations and their radiative decay, studied, for example, in \([53, 54]\). This is the general framework in which we expect a more rigorous treatment of our
problem.

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References

[1] A. Connes, “C* Algebras And Differential Geometry,” Compt. Rend. Acad. Sci. (Ser. I Math.) A 290, 599 (1980) [arXiv:hep-th/0101093].

[2] E. Witten, “Noncommutative Geometry And String Field Theory,” Nucl. Phys. B 268, 253 (1986).

[3] A. Connes, “Noncommutative Geometry,” Academic Press, San Diego (1994).

[4] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M theory as a matrix model: A conjecture,” Phys. Rev. D 55, 5112 (1997) [arXiv:hep-th/9610043].

[5] A. Connes, M. R. Douglas and A. Schwarz, “Noncommutative geometry and matrix theory: Compactification on tori,” JHEP 9802, 003 (1998) [arXiv:hep-th/9711162].

[6] N. Seiberg and E. Witten, “String theory and noncommutative geometry,” JHEP 9909, 032 (1999) [arXiv:hep-th/9908142].

[7] J. Fröhlich, O. Grandjean and A. Recknagel, “Supersymmetric quantum theory, non-commutative geometry, and gravitation,” [arXiv:hep-th/9706132]

[8] M. R. Douglas and N. A. Nekrasov, “Noncommutative field theory,” Rev. Mod. Phys. 73, 977 (2001) [arXiv:hep-th/0106048].

[9] R. J. Szabo, “Quantum field theory on noncommutative spaces,” [arXiv:hep-th/0109162]
[10] A. Sen, “Tachyon condensation on the brane antibrane system,” JHEP 9808, 012 (1998) [arXiv:hep-th/9805170].

[11] A. Sen, “Descent relations among bosonic D-branes,” Int. J. Mod. Phys. A 14, 4061 (1999) [arXiv:hep-th/9902105].

[12] A. Sen and B. Zwiebach, “Tachyon condensation in string field theory,” JHEP 0003, 002 (2000) [arXiv:hep-th/9912249].

[13] A. Sen, “Rolling tachyon,” JHEP 0204, 048 (2002) [arXiv:hep-th/0203211].

[14] A. Sen, “Tachyon matter,” [arXiv:hep-th/0203265]

[15] A. Sen, “Field theory of tachyon matter,” [arXiv:hep-th/0204143]

[16] M. Gutperle and A. Strominger, “Spacelike branes,” JHEP 0204, 018 (2002) [arXiv:hep-th/0202210].

[17] A. Buchel, P. Langfelder and J. Walcher, “Does the tachyon matter?,” Annals Phys. 302, 78 (2002) [arXiv:hep-th/0207235].

[18] A. Strominger, “Open string creation by S-branes,” [arXiv:hep-th/0209090]

[19] N. Nekrasov and A. Schwarz, “Instantons on noncommutative R**4 and (2,0) superconformal six dimensional theory,” Commun. Math. Phys. 198, 689 (1998) [arXiv:hep-th/9802068].

[20] R. Gopakumar, S. Minwalla, and A. Strominger, “Noncommutative solitons,” JHEP 05, 020 (2000) [hep-th/0003160].

[21] J. A. Harvey, P. Kraus, F. Larsen and E. J. Martinec, “D-branes and strings as non-commutative solitons,” JHEP 0007, 042 (2000) [arXiv:hep-th/0005031].

[22] D. J. Gross and N. A. Nekrasov, “Monopoles and strings in noncommutative gauge theory,” JHEP 0007, 034 (2000) [arXiv:hep-th/0005204].

[23] A. P. Polychronakos, “Flux tube solutions in noncommutative gauge theories,” Phys. Lett. B 495, 407 (2000) [arXiv:hep-th/0007043].

[24] D. J. Gross and N. A. Nekrasov, “Solitons in noncommutative gauge theory,” JHEP 0103, 044 (2001) [arXiv:hep-th/0010090].
[25] J. A. Harvey, “Komaba lectures on noncommutative solitons and D-branes,” hep-th/0102076.

[26] E. Witten, “Noncommutative tachyons and string field theory,” hep-th/0006071.

[27] A. Sen, “Some issues in non-commutative tachyon condensation,” JHEP 0011, 035 (2000) arXiv:hep-th/0009038.

[28] B. Zwiebach, “A solvable toy model for tachyon condensation in string field theory,” JHEP 09, 028 (2000) hep-th/0008227.

[29] B. Zwiebach, private communication

[30] N. Moeller and B. Zwiebach, “Dynamics with infinitely many time derivatives and rolling tachyons,” JHEP 0210, 034 (2002) arXiv:hep-th/0207107.

[31] C. Acatrinei and C. Sochichiu, “A note on the decay of noncommutative solitons,” arXiv:hep-th/0104263.

[32] C. Acatrinei, “Noncommutative radial waves,” arXiv:hep-th/0106006.

[33] M. G. Jackson, “The stability of noncommutative scalar solitons,” JHEP 0109, 004 (2001) arXiv:hep-th/0103217.

[34] L. Hadasz, U. Lindstrom, M. Rocek and R. von Unge, “Noncommutative multisolitons: Moduli spaces, quantization, finite Theta effects and stability,” JHEP 0106, 040 (2001) arXiv:hep-th/0104017.

[35] T. Erler, D. Gross, and J. Walcher, work in progress.

[36] B. Durhuus, T. Jonsson, and R. Nest, “Noncommutative scalar solitons: Existence and nonexistence,” Phys. Lett. B500, 320–325 (2001) hep-th/0011139.

[37] B. Durhuus, T. Jonsson and R. Nest, “The existence and stability of noncommutative scalar solitons,” arXiv:hep-th/0107121.

[38] S. Takeno, K. Kisoda, and A. J. Sievers, “Intrinsic localized vibrational modes in anharmonic crystals,” Prog. Theor. Phys. Suppl. 94, 242 (1988).

[39] D. K. Campbell and M. Peyrard, “Chaos and order in nonintegrable model field theories,” Chaos, ed. D. K. Campbell (New York, AIP), 305 (1990).
[40] R. S. MacKay and S. Aubry, “Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators,” Nonlinearity 7, 1623 (1994).

[41] J.-A. Sepulchre and R. S. MacKay, “Localized oscillations in conservative or dissipative networks of weakly coupled autonomous oscillators,” Nonlinearity 10, 679 (1997).

[42] R. S. MacKay and J.-A. Sepulchre, “Stability of discrete breathers,” Physica D 119, 148 (1998).

[43] S. Flach and C. R. Willis, “Discrete Breathers,” Phys. Rep. 295, 181 (1998).

[44] C. Albanese and J. Fröhlich, “Periodic solutions of some infinite-dimensional hamiltonians associated with nonlinear partial differential equations I,” Comm. Math. Phys. 116, 475 (1988).

[45] C. Albanese and J. Fröhlich, “Perturbation theory for periodic orbits in a class of infinite dimensional hamiltonian systems,” Comm. Math. Phys. 138, 193 (1991).

[46] R. K. Richtmyer, “Principles of Advanced Mathematical Physics”, Vol. 1, Springer, New York 1978.

[47] E. Mourre, “Absence of singular continuous spectrum for certain self-adjoint operators,” Comm. Math. Phys. 78, 391 (1981).

[48] A. Erdélyi (Ed.), W. Magnus, F. Oberhettinger, F. G. Tricomi, “Higher Transcendental Functions” (Bateman Manuscript Project), McGraw Hill, New York 1953.

[49] I. S. Gradshteyn and I. M. Ryzhik, “Tables of Integrals, Series, and Products,” 4th Edition, Academic Press, New York 1965.

[50] M. Abramowitz and I. A. Stegun, “Handbook of Mathematical Functions,” 9th Printing, Dover Publications, Inc., New York 1970.

[51] W. Magnus and F. Oberhettinger, “Formeln und Sätze für die speziellen Funktionen der mathematischen Physik,” 2. Auflage, Springer, Berlin 1948.

[52] D. Eardley, unpublished notes and private communication
[53] I. M. Sigal, “Non-linear wave and Schrödinger equations I. Instability of periodic and quasiperiodic solutions,” Comm. Math. Phys. 153, 297 (1993).

[54] A. Soffer and M. I. Weinstein, “Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations,” Invent. math. 136, 9 (1999).