Cosmological solutions for a model with a $1/H^2$ term

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Abstract. We derive the cosmological solutions of a five-dimensional model with a $1/H^2$ term ($H^2 \equiv H_{MNPQ}H^{MNPQ}$), where $H_{MNPQ}$ is the 4-form field strength. The behaviours of the scale factors and the scalar potential in the effective theory are examined. We show that, as a consequence, the universe changes from showing decelerated expansion to showing accelerated expansion in the Einstein frame of the four-dimensional theory.

Keywords: extra dimensions, cosmology with extra dimensions, cosmological applications of theories with extra dimensions

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Observational evidence indicated that dark energy which may generate the current cosmic acceleration dominates about 70% of the critical energy density [1, 2]. WMAP supports strongly the notion that the dark energy exists at the present time [3, 4]. In cosmology and particle physics, it is an important problem to establish what the origin for driving cosmic acceleration is. Higher dimensional theories, such as string/M-theory and braneworlds, include scalar fields or antisymmetric tensor fields which play an important role in cosmological models. In this paper, we focus on a particular model with an inverse square field strength and investigate the cosmological evolution of the model.

We consider the five-dimensional model with a $1/H^2$ term, where $H^2 \equiv H_{MNPQ}H^{MNPQ}$ and $H_{MNPQ}$ denotes the 4-form field strength for the 3-form antisymmetric tensor field. The action is given by

$$S = \int d^4x \, dy \, \sqrt{-g} \left( \frac{1}{2} \mathcal{R} + 2 \cdot 4! \cdot \frac{1}{H^2} \right), \quad (1)$$

where $y$ means the fifth dimension, the fundamental scale is set to unity and $g$ is the determinant of the five-dimensional metric. The introduction of the $1/H^2$ term is motivated by the particular model of [5]. The model indicated that the $1/H^2$ term is indispensable when solving the cosmological constant problem without fine-tuning of parameters in the Lagrangian. As mentioned in [5, 6], the unusual $1/H^2$ term makes sense only if $H^2$ must develop a vacuum expectation value of the order of the fundamental scale. We expect that the term may be generated by dynamics of quantum gravity. If the VEV of $1/H^2$ is a significant effect in the history of universe, it is natural to think that the term affects cosmic evolution. Thus, we have an interest in the evolution of the universe in the present model.

We assume that the form of the metric is given by

$$ds^2 = -dt^2 + a^2(t) \, dx^2 + b^2(t) \, dy^2, \quad (2)$$

where $a(t)$ is the scale factor of three-dimensional space and $b(t)$ is the scale factor of an extra dimension. The compact space of the extra dimension is assumed to be a maximally symmetric space such as $S^1$. This metric corresponds to the higher dimensional Kasner metric with asymmetric scale factors [7].

Variation with respect to the metric leads to the Einstein equation

$$\mathcal{R}_{MN} - \frac{1}{2} g_{MN} \mathcal{R} = 2 \cdot 4! \left( \frac{8}{H^4} H_{MPQR} H^{PQR}_N + g_{MN} \frac{1}{H^2} \right), \quad (3)$$

where $M, N = 0, 1, 2, 3, 5$ denote the five-dimensional indices. The field equation for the 3-form antisymmetric tensor $A_{PQR}$ is

$$\partial_M \left( \frac{\sqrt{-g} H^{MPQR}}{H^4} \right) = 0. \quad (4)$$

Referring to the Freund–Rubin ansatz [5, 8], it is assumed that the 4-form field strength is

$$H_{\mu
u\rho\sigma} \propto \sqrt{-g} \, b^{-2/3} \, \epsilon_{\mu
u\rho\sigma}, \quad (5)$$

where $\mu, \nu, \ldots = 0, 1, 2, 3$ denote the indices of the four dimensions. Since equation (5) leads to $\sqrt{-g} H^{\mu
u\rho\sigma}/H^4 = \text{const}$, equation (4) is satisfied.
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The $(00)$, $(ij)$, $(55)$ components of the Einstein equation are

$$3 \left( \frac{\dot{a}}{a} \right)^2 + 3 \frac{\dot{a} \dot{b}}{a b} = \frac{6}{\alpha} b^{-2/3},$$

$$2 \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \left( \frac{\dot{a}}{a} \right)^2 + 2 \frac{\dot{a} \dot{b}}{b} = \frac{6}{\alpha} b^{-2/3},$$

$$3 \frac{\ddot{a}}{a} + 3 \left( \frac{\dot{a}}{a} \right)^2 = \frac{2}{\alpha} b^{-2/3},$$

where the dot means the derivative with respect to $t$ and $\alpha$ is a positive constant originating from the proportionality constant in equation (5). Since it is difficult to solve equations (6)–(8), we define a new variable $\tau$ as

$$d\tau = a^{-3} b^{-1} \, dt.$$  \hspace{1cm} (9)

Solving the differential equations as shown in the appendix, the scale factors can be expressed in terms of $\tau$:

$$a(\tau) = \left[ \sinh \left( \frac{2}{3} |C| \tau \right) \right]^{-3/28} \exp \left( -\frac{1}{7} C \tau \right),$$

$$b(\tau) = \left[ \sinh \left( \frac{2}{3} |C| \tau \right) \right]^{-15/56} \exp \left( \frac{9}{14} C \tau \right),$$

where $C$ is an integration constant.

We would like to consider the universe of this model in the Einstein frame. In order to transform the action with metric (2) to the Einstein frame, the conformal transformation is applied as follows:

$$\bar{g}^{(4)}_{\mu\nu} = b g^{(4)}_{\mu\nu}.$$  \hspace{1cm} (12)

That is, the effective Lagrangian

$$L_{\text{eff}} = \sqrt{-\bar{g}^{(4)}} \left( \frac{1}{2} R^{(4)} - 3 b^{-1} \bar{g}^{(4)\mu\nu} \partial_\mu b^{1/2} \partial_\nu b^{1/2} + 2 \cdot 4! b^{-1} \frac{1}{H^2} \right),$$

where the barred quantities represent ones in the Einstein frame. Thus the scale factor $b$ behaves as a scalar field in the four-dimensional theory. Moreover, according to the assumption of a compact extra dimension with maximally symmetric space, the contributions of the curvature of the extra dimension are vanishing in $L_{\text{eff}}$.

By introducing the scalar field $\Phi$ as

$$b = e^{\sqrt{2/3} \Phi},$$

we can obtain the Lagrangian with the canonically kinetic term

$$L_{\text{eff}} = \sqrt{-\bar{g}^{(4)}} \left( \frac{1}{2} R^{(4)} - \frac{1}{2} \bar{g}^{(4)\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{2}{\alpha} e^{-\sqrt{50/27} \Phi} \right),$$

where we used (5). The effective scalar potential for a scalar field $\Phi$ is

$$V_{\text{eff}} = \frac{2}{\alpha} e^{-\sqrt{50/27} \Phi}.$$  \hspace{1cm} (16)

1 Since the action is $S = \int d^4x \sqrt{-g^{(5)}} L = \int d^4x \sqrt{-\bar{g}^{(4)}} L_{\text{eff}}$, we can obtain (13) by substituting (12) into $L_{\text{eff}}$. 

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The exponent of the scalar potential plays an important role in the evolution of universe. We discuss it later.

In order to investigate the cosmological evolution of the present cosmological model, we examine the sign of the derivative with respect to the time of the scale factors for a four-dimensional observer. From equation (12), the time and the scale factor in the Einstein frame are given by

$$d\bar{t} = b^{1/2} dt, \quad \bar{a} = b^{1/2} a.$$  \hspace{1cm} (17)

Before examining the behaviour of $\bar{a}$, it is necessary to see the relation between $\bar{t}$ and $\tau$. From equations (9) and (17), one gets

$$d\bar{t} = a^3 b^{3/2} d\tau = \left[ \sinh^2 \left( \frac{2}{3} |C| \tau \right) \right]^{-81/112} \exp \left( \frac{15}{28} C \tau \right) d\tau.$$  \hspace{1cm} (18)

We can explore the asymptotic relation for the limit $|\tau| \to 0$ and $|\tau| \to \infty$.

For $|\tau| \to 0$, according to equation (18), $\tau \to -0$ maps to $\bar{t} \to -\infty$ via $\bar{t} \sim -\tau^{-25/56}$ and $\tau \to -0$ maps to $\bar{t} \to +\infty$ via $\bar{t} \sim (-\tau)^{-25/56}$.

For $|\tau| \to \infty$, we must carefully handle equation (18). In the case of $C > 0$, $\tau \to +\infty$ maps to $\bar{t} \to -0$ via $\bar{t} \sim -\exp(-3C\tau/7)$ and $\tau \to -\infty$ maps to $\bar{t} \to +0$ via $\bar{t} \sim \exp(3C\tau/2)$. On the other hand, in the case of $C < 0$, $\tau \to +\infty$ maps to $\bar{t} \to -0$ via $\bar{t} \sim -\exp(-3|C|\tau/2)$ and $\tau \to -\infty$ maps to $\bar{t} \to +0$ via $\bar{t} \sim \exp(3|C|\tau/7)$. Although the dependence of $\tau$ is different, regardless of the sign of the integration constant $C$, it turns out that the sign and direction of $\bar{t}$ and $\tau$ are reversed.

From equations (9) and (17), the derivatives of $\bar{a}$ are given by

$$\frac{d}{d\bar{t}} \bar{a} = a^{-3} b^{-3/2} \left( H_a + \frac{1}{2} H_b \right),$$  \hspace{1cm} (19)

$$\frac{d^2}{d\bar{t}^2} \bar{a} = a^{-6} b^{-3} \left\{ -2 \left( H_a + \frac{1}{2} H_b \right)^2 + H_a' + \frac{1}{2} H_b' \right\},$$  \hspace{1cm} (20)

where $H_a = a'/a$ and $H_b = b'/b$, the prime denotes the derivative of $\tau$.

Following equations (19) and (20), the relevant terms which determine the signs of $d\bar{a}/d\bar{t}$ and $d^2\bar{a}/d\bar{t}^2$ can be expressed as

$$\text{sgn} \left( \frac{d}{d\bar{t}} \bar{a} \right) = \text{sgn} \left( -9 \coth \frac{2}{3} |C| \tau + \frac{5}{|C|} \right),$$  \hspace{1cm} (21)

$$\text{sgn} \left( \frac{d^2}{d\bar{t}^2} \bar{a} \right) = \text{sgn} \left( 3 \coth^2 \frac{2}{3} |C| \tau + 90 \frac{C}{|C|} \coth \frac{2}{3} |C| \tau - 109 \right),$$  \hspace{1cm} (22)

where $\text{sgn}(x)$ denotes the sign function. The signs of $d\bar{a}/d\bar{t}$ and $d^2\bar{a}/d\bar{t}^2$ are explicitly shown below.

| $\tau$ | $-\infty$ | $\tau_1$ | 0 | $\tau_2$ | $+\infty$ |
|--------|-----------|--------|---|--------|--------|
| $d\bar{a}/d\bar{t}$ | + | 0 | - | - |
| $d^2\bar{a}/d\bar{t}^2$ | - | 0 | + | 0 |

According to equation (22), the values of $\tau_1$ and $\tau_2$ depend on the sign of $C$. For $C > 0$, we have $\tau_1 = -(3/4C) \log 2(2 + \sqrt{3})/7$ and $\tau_2 = (3/4C) \log 7(2 + \sqrt{3})/2$. For $C < 0$, we have $\tau_1 = -(3/4|C|) \log 7(2 + \sqrt{3})/2$ and $\tau_2 = (3/4|C|) \log 2(2 + \sqrt{3})/7$. 

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The survey of the behaviours of $\bar{a}$ for $\tau$ is illustrated in figure 1. Here $d\bar{a}/d\bar{t} > 0$ and $d\bar{a}/d\bar{t} < 0$ correspond to expansion and contraction, respectively. On the other hand, $d^2\bar{a}/d\bar{t}^2 > 0$ and $d^2\bar{a}/d\bar{t}^2 < 0$ correspond to acceleration and deceleration, respectively.

From the above table, the expansion occurs during $-\infty < \tau < -0$ and $+0 < \tau < +\infty$ and the contraction does during $+0 < \tau < +\infty$ ($-\infty < \bar{t} < -0$). In addition, we can see that the periods of $\tau_1 < \tau < -0$ and $+0 < \tau < \tau_2$ are accelerated phase, the periods of $-\infty < \tau < \tau_1$ and $\tau_2 < \tau < +\infty$ are the decelerated phase. As shown in figure 1, the era of accelerated expansion exists in the range $\bar{t}(\tau = \tau_1) < \bar{t} < +\infty$. Interestingly, it turns out that the universe changes from the decelerated expansion to accelerated expansion during $+0 < \bar{t} < +\infty$ ($-\infty < \tau < -0$).

Although we cannot obtain the explicit form of the scale factor $\bar{a}(\bar{t})$, the asymptotic behaviour of $\bar{a}$ can be evaluated. We consider the range of $+0 < \bar{t} < +\infty$ ($-\infty < \tau < -0$), since $\bar{t}$ is the proper time of the Einstein frame for the four-dimensional observer.

For $\bar{t} \to +0$ ($\tau \to -\infty$), equations (10) and (18) lead to

$$\bar{a} \sim \bar{t}^{1/3}. \quad (23)$$

At the initial time, $\bar{a}$ evolves through the equation of state corresponding to $\omega = 1$. In the general case, the scale factor is given by $a \sim t^{2/(\omega+1)}$ for the equation of state $p = \omega \rho$, where $p$, $\rho$ are the pressure and the energy density.

For $\bar{t} \to +\infty$ ($\tau \to -0$), we obtain

$$\bar{a} \sim \bar{t}^{27/25}. \quad (24)$$

For sufficiently late time, $\bar{a}$ evolves through the equation of state corresponding to $\omega = -31/81$. This era is an accelerating phase due to $-31/81 < -1/3$. Thus it turns out that the transition from $\omega = 1$ (stiff) to $\omega = -31/81$ (quintessence) corresponds to that from decelerated expansion to accelerated expansion.

By analysing the properties of the effective scalar potential in equation (16), we confirm the results of equations (23) and (24). We consider the evolution of the scalar field in the present model. The scalar field $\Phi$ and the velocity $d\Phi/d\bar{t}$ are given by

$$\Phi = \sqrt{3} \left( -\frac{15}{28} \log \left| \sinh \frac{2|C|}{3} \tau \right| + \frac{9}{14} C \tau \right), \quad (25)$$

$$\frac{d\Phi}{d\bar{t}} = \frac{|C|}{14} \left( \sinh^2 \frac{2|C|}{3} \tau \right)^{81/112} e^{-\left(15/28\right)C \tau} \left( -5 \coth \frac{2|C|}{3} \tau + 9 \frac{C}{|C|} \right). \quad (26)$$

The asymptotic behaviours of the scalar field and the velocity are examined.
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Figure 2. The illustrations represent the evolution of scalar field $\Phi$. Depending on the sign of $C$, the initial start points of $\Phi$ are different. For $C > 0$, starting from $\Phi \to -\infty$, the scalar field goes to $+\infty$ and rolls downhill. For $C < 0$, starting from $\Phi \to +\infty$, the scalar field climbs up the hill and turns around, and rolls downhill.

At initial time $\bar{t} \to +0 (\tau \to -\infty)$, the start point of $\Phi$ depends on the sign of the integration constant $C$:

$$
C > 0 \quad \Phi \sim C\tau \to -\infty
$$
$$
C < 0 \quad \Phi \sim -\frac{1}{17}|C|\tau \to +\infty.
$$

(27)

Furthermore, for the same limit $\tau \to -\infty$, the asymptotic velocities are

$$
C > 0 \quad \frac{d\Phi}{d\bar{t}} \sim +e^{-43C\tau/28} \to +\infty
$$
$$
C < 0 \quad \frac{d\Phi}{d\bar{t}} \sim -e^{-12|C|\tau/7} \to -\infty.
$$

(28)

Since the kinetic energy $|d\Phi/d\bar{t}|^2$ dominates over potential energy $V(\Phi)$, we obtain $\omega = 1$ because $p = \rho$. This is consistent with the result of equation (23).

At the sufficiently late time $\bar{t} \to +\infty (\tau \to -0)$, from equations (25) and (26), we get

$$
\Phi \to +\infty, \quad \frac{d\Phi}{d\bar{t}} \to +0,
$$

(29)

regardless of the sign of $C$. Since the velocity is very small, this implies that the kinetic energy and potential energy are compatible. In this case, the value of $\omega$ is determined by the exponent of the exponential potential. In the general case, the potential of $e^{-\lambda\psi}$ ($\psi$ is the canonically normalized field) has $\omega = \lambda^2/3 - 1$. According to equation (16), in the case of the present model with $\lambda = \sqrt{50}/27$, we obtain

$$
\omega = -\frac{31}{81}.
$$

(30)

Obviously, this result is consistent with the behaviour of the scale factor of equation (23).

In figure 2, the evolutions of a scalar field $\Phi$ are shown. Equation (27) indicates that the initial start points of $\Phi$ are different for the two cases. By combining the result with equations (27)–(29), we can understand the evolutions of $\Phi$. For $C > 0$, starting from $\Phi \to -\infty$, the scalar field rolls down toward $\Phi \to +\infty$. For $C < 0$, starting from $\Phi \to +\infty$, the scalar field climbs up the hill and turns around; it rolls down toward
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$\Phi \to +\infty$. There exists a transition from decelerated expansion to accelerated expansion for each case.

In conclusion, we have derived the cosmological solutions of a five-dimensional model with a $1/H^2$ term. It was pointed out that in the Einstein frame the universe changes from decelerated expansion to accelerated expansion. From the analysis of asymptotic behaviours of the scale factors, it was shown that the transition is from the phase of $\omega = 1$ to the phase of $\omega = -31/81$. Furthermore, we analysed the properties of the effective scalar potential and a scalar field in the effective Lagrangian. As a consequence, we obtained the same results for the evolutions of the universe. Thus we constructed a toy model in accordance with the current cosmic acceleration by introducing an inverse square field strength term. We expect that the term can be generated from the dynamics of the underlying theories. The present model does not have the inflationary period in the early universe. The model should be improved; for example, extension to higher dimensions is possible. It may be necessary to consider drastic models in order to explain recent astronomical observations in the framework of cosmology and particle physics. When proposing the model of the cosmic acceleration, the present model with the $1/H^2$ term may have several new possibilities.

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Appendix

From equations (6)–(8), we derive equations (10) and (11) [7]. The following transformations are performed:

$$a = e^X, \quad b = e^Y.$$  \hspace{1cm} (A.1)

Substituting equation (A.1) into equations (6)–(8), suitable linear combinations yield the following equations:

$$\dddot{X} + \dddot{Y} \left( 3\dot{X} + \dot{Y} \right) = \frac{8}{3\alpha} e^{-(2/3)Y},$$ \hspace{1cm} (A.2)

$$\dddot{Y} + \dddot{X} \left( 3\dot{X} + \dot{Y} \right) = \frac{20}{3\alpha} e^{-(2/3)Y},$$ \hspace{1cm} (A.3)

$$\dot{X}^2 + \dot{X} \dot{Y} = \frac{2}{\alpha} e^{-(2/3)Y}. \hspace{1cm} (A.4)

Using equation (9), we can rewrite equations (A.2)–(A.4) as follows:

$$X'' = \frac{8}{3\alpha} e^Z,$$ \hspace{1cm} (A.5)

$$Y'' = \frac{20}{3\alpha} e^Z,$$ \hspace{1cm} (A.6)

$$X'' + X' Y' = \frac{2}{\alpha} e^Z,$$ \hspace{1cm} (A.7)

$$Z = 6X + \frac{4}{3}Y.$$ \hspace{1cm} (A.8)
where the prime means the derivative with respect to $\tau$. Using equations (A.5), (A.6) and (A.8), we have
\[ Z'' = \frac{224}{9\alpha}e^Z \rightarrow Z'^2 = \frac{448}{9\alpha}e^Z + C_1, \]  
(A.9)
where $C_1$ is a constant. Furthermore, equations (A.5) and (A.6) yield
\[ Y = \frac{4}{9}X + C\tau + C_2, \]  
(A.10)
where $C, C_2$ are constants. From equations (A.9) and (A.10), equation (A.7) leads to
\[ C_1 = \frac{16}{9}C^2. \]  
(A.11)
Substituting equation (A.11) into equation (A.9), we obtain
\[ Z'^2 = \frac{448}{9\alpha}e^Z + \frac{16}{9}C^2. \]  
(A.12)
Solving it, we have
\[ e^Z = \frac{\alpha C^2}{28} \frac{1}{\sinh^2 \left( \frac{2}{9}|C|\tau \right)}. \]  
(A.13)
Using equations (A.1), (A.8), (A.10) and equation (A.13), we can obtain equations (10) and (11). Note that the absolute in argument of equation (A.13) stems from $\sqrt{C^2} = |C|$ when soling equation (A.12); $|C|$ cannot be removed by rescaling, because, over the range $-\infty < \tau < +\infty$, the behaviours of the scale factors depends on the sign of $C$ through equation (A.10). Consequently, as shown in figure 2, the sign of $C$ controls the revolutions of the scalar field. The situations are similar to those of the model of [7].

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