Experimental application of decoherence-free subspaces in a quantum-computing algorithm

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Abstract

For a practical quantum computer to operate, it will be essential to properly manage decoherence. One important technique for doing this is the use of “decoherence-free subspaces” (DFSs), which have recently been demonstrated. Here we present the first use of DFSs to improve the performance of a quantum algorithm. An optical implementation of the Deutsch-Jozsa algorithm can be made insensitive to a particular class of phase noise by encoding information in the appropriate subspaces; we observe a reduction of the error rate from 35% to essentially its pre-noise value of 8%.

One of the great stumbling blocks to building quantum computers, with their oft-touted ability to resolve certain problems more efficiently than any classical algorithm [1,2] is the ubiquity of decoherence. Coupling of any element of a quantum computer to an environment destroys its unitary evolution, and introduces uncontrollable noise; at first, it was thought by many [3] that these errors would make quantum computation impossible in practice. Since then, a variety of techniques for correcting errors and/or building in immunity to certain classes of decoherence have been developed [4,5,6] and it has been proved that if errors are kept below a certain constant threshold, arbitrarily large quantum computers are possible [7]. One important technique involves computing within certain subspaces of the full system’s Hilbert space known as decoherence-free subspaces (DFSs) [5,6] which remain unaffected by the interaction with the environment. Such DFSs exist when the interaction Hamiltonian has an appropriate symmetry property. DFSs have been demonstrated in a linear-optical experiment [8] and in NMR [9] and recently to help circumvent the technical noise which had previously limited the operation of ion-trap quantum computers [11]. To date, no demonstration has been made of the usefulness of DFSs in the context of the implementation of an actual quantum-computing algorithm [12]. In this paper, we present a linear-optical

1 Just prior to submission of this manuscript, we learned that a similar demonstration has now
implementation of the two-qubit Deutsch-Jozsa algorithm \cite{10,2}, and demonstrate that when a certain class of noise is introduced into the system, greatly increasing the error rate of the algorithm, it is possible to ‘encode’ one logical qubit into two physical qubits and take advantage of DFSs, reducing the error rate to close to zero.

Linear optics is well known to be an extremely powerful arena for the transportation and manipulation of quantum information, \cite{12,13}. Although it is also well known that due to the linearity of optics, this arena does not allow for scalable construction of quantum gates \cite{14}, recent work has shown that the incorporation of detection and post-selection may in fact render all-optical quantum computers an attractive possibility \cite{15}. Work also proceeds on development of nonlinearities which would allow for the development of natural two-qubit gates in optics \cite{17}. While we do not yet have access to a truly scalable optical quantum-computer architecture, many of the elements of any such system would be identical to those used in simple linear-optical geometries \cite{14}. For this reason, linear optics remains an important domain for the study of quantum coherence and error correction, even while the ultimate fate of optical quantum computing is uncertain. Recently, striking demonstrations of quantum search algorithms \cite{19} have been carried out in linear-optical systems, as has the first verification of DFSs \cite{8}. Additionally, it is already clear that even if quantum computation never becomes truly practical, quantum information processing may have a great effect on the practice of communications and cryptography \cite{20,21}. Although some information-processing will be necessary in this area as well, the question of scalability is not crucial, and linear-optical quantum computation could well prove applicable for elements such as quantum repeaters \cite{22}. In this context, we have chosen to study the applicability of DFSs to a linear-optical implementation of the quantum Deutsch-Jozsa algorithm, despite the non-scalable nature of the present architecture.

The Deutsch-Jozsa algorithm is designed to distinguish between two classes of functions

been performed in liquid-state NMR: J. Ollerenshaw, D. Lidar, and L. Kay, in preparation.
(“oracles”) on N-bit binary inputs. “Constant” functions return the same value (0 or 1) for all $2^n$ possible inputs, while “balanced” functions return 0 for half the possible inputs and 1 for the other half. Clearly, a classical algorithm would on some occasions require as many as $2^n - 1 + 1$ queries to unambiguously determine to which class a given oracle belongs. By contrast, Deutsch and Jozsa showed \([10]\) that a quantum algorithm requires only one such query. In the 2-qubit Deutsch-Jozsa algorithm \([2]\), the oracle is a function on a single bit. It takes as input a query bit $x$ and a signal bit $y$; its action is to perform the unitary mapping $|x, y\rangle \rightarrow |x, y \oplus f(x)\rangle$. To perform the algorithm, the input is prepared in $H|0\rangle \otimes H|1\rangle = \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)$ which the oracle maps to $\frac{1}{2}(|0\rangle \otimes (|f(0)\rangle - |\overline{f(0)}\rangle) + |1\rangle \otimes (|f(1)\rangle - |\overline{f(1)}\rangle) = \frac{1}{2}(|0\rangle e^{i\pi f(0)} + |1\rangle e^{i\pi f(1)}) \otimes H|1\rangle$. A Hadamard on the query qubit then transforms it into $|f(0) \oplus f(1)\rangle$, which is equal to 0 for constant and 1 for balanced functions. Thus measurement in the computational basis allows one to determine a global property of $f(x)$, namely $f(0) \oplus f(1)$, in a single evaluation of the function. Furthermore, the signal qubit is in fact superfluous after the oracle \([23]\). Thus only one logical qubit is needed after the operation of the oracle. If some source of decoherence is present during the propagation from the oracle to the final Hadamard, one may consider encoding this logical qubit in some decoherence-free subspace of the two physical qubits.

In this experiment we represent the four basis states of two logical qubits ($|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$, where the first bit corresponds to the query and the second to the signal) by a photon traveling down one of four optical rails numbered 1, 2, 3 and 4 respectively. It is possible to implement a universal set of one- and two-qubit operations in a four-rail representation \([14]\). For example a NOT gate on the query qubit can be realized by simultaneously swapping rails 1 and 3 and rails 2 and 4. A CNOT gate on the signal qubit is implemented by swapping rails 3 and 4. To perform a Hadamard gate on the query qubit, we combine rails 1 and 3 and rails 2 and 4 at two 50/50 beam-splitters; a $\pi$ phase shift is also needed on two of the arms. Analogous gates can be constructed for the other qubit. The transformations introduced by the four possible functions can also be implemented in this representation by four different settings of an oracle operating as follows: if $f(0)$ is 1, rails 1
and 2 are swapped; if \( f(1) \) is 1, rails 3 and 4 are swapped. Thus the task of distinguishing balanced from constant oracles reduces to that of determining whether the number of swaps was odd or even.

The schematic diagram of the interferometer is shown in Fig. 1. Each photon is sent along rail 2 corresponding to the logical state \( |01 \rangle \). The two pairs of 50/50 beam splitters \( A1, A2 \) and \( B1, B2 \) implement the two Hadamard gates on the query and signal qubits respectively, preparing the qubits for the oracle’s action. The last two 50/50 beam splitters \( C1 \) and \( C2 \) realize the Hadamard gate on the query qubit after the oracle. Rails 1-4 illuminate photodiodes \( PD1 - PD4 \). A photon reaching PD1 or PD2 indicates that the value of the query qubit after the algorithm, \( f(0) \oplus f(1) \), is 0. This constitutes a determination that the oracle is constant, while the other two detectors indicate balanced oracles.

One source of decoherence in such systems is the phase noise introduced by fluctuating optical path lengths between the different sections of the apparatus, created either by variations in distance or by temperature variations and turbulent air flow. In real optical systems, the stability of certain path-length differences may be larger than that of others, either because of the physical proximity of certain paths to one another or because of the particular sources of mechanical or thermal noise. This may lead to a situation where the dominant source of decoherence exhibits a particular symmetry which can be exploited for computing within DFSs. To simulate the effects such processes could have in larger-scale, distributed quantum-information systems, we introduced a high degree of turbulence by placing the tip of a hot soldering iron below two of the optical rails. These two optical paths (rails 2 and 3) were spatially superposed in this region, distinguished only by their polarisation; for this reason, they experienced essentially the same random phase shifts under the influence of the turbulent air flow, relative to the other optical rails. Since the outputs of the optical Deutsch-Jozsa setup are the outputs of two parallel interferometers, which measure the phase of rail 2 with respect to that of rail 4 and rail 3 with respect to rail 1, this phase noise destroys the interference on which the success of the algorithm relies. On the other hand, inspection of the optical schematic makes the physical process behind the algorithm evident:
rails 1 and 3 are prepared in phase with one another, while rails 2 and 4 are also prepared in phase, but 180° out of phase with the former pair. Thus, constructive interference is observed either between 1 and 3 or between 2 and 4. If a single pair (1 and 2 or 3 and 4) is swapped by the oracle, destructive interference is instead observed at both interferometers, while if an even number of swaps occurs, constructive interference is restored. In other words, so long as each interferometer compares an output of each of the potential swap regions in the oracle with one from the other, it is possible to distinguish a balanced oracle (one swap) from a constant oracle (zero or two swaps). The strategy to deal with phase noise impressed symmetrically on paths 2 and 3 now becomes clear: instead of interfering 2 with 4 and 1 with 3, one can accomplish the same task by interfering 2 with 3 and 1 with 4. In this way, the random phase appears at both inputs to the same interferometer, and has no effect on the measured results.

This modification can be expressed as an encoding of the data into a pair of DFSs. Since our engineered phase noise has identical effects on the two states of odd parity (|01⟩ and |10⟩, stored on rails 2 and 3 respectively) and on the two states of even parity (|00⟩ and |11⟩, stored on rails 1 and 4), each fixed-parity subspace can store a single logical qubit in a decoherence-free fashion. The action of the soldering iron tip may be modelled by the evolution operator \( \exp(i\sigma_z^1\sigma_z^2\delta\phi) \), where \( \delta\phi \) is a random, fluctuating phase. In a subspace with a definite eigenvalue of \( \sigma_z^1\sigma_z^2 \), the random phase, \( \delta\phi \), does nothing but impress an overall global phase on the quantum state, leaving the information within the subspace unaffected. Since the 2-qubit Deutsch-Jozsa algorithm relies on a single qubit (query qubit) after the oracle has completed its action, this single qubit may be encoded in either of these DFSs, providing immunity to parity-dependent phase noise which occurs between the oracle and the final Hadamard gates. As shown in Fig. 1c, a CNOT after the oracle encodes the query qubit into these DFSs, and a second CNOT after the final Hadamard can be used for decoding. The decoding CNOT is unnecessary since measurements are only performed on the query qubit. Swapping rails 3 and 4 performs the encoding; or equivalently, beamsplitters \( C_1 \) and \( C_2 \) may be replaced by \( D_1 \) and \( D_2 \).
The actual experimental setup is shown in Fig. 2. The light source was a diode laser operating at 780 nm. To implement the four different oracle settings a specific kind of variable beam splitter (VBS) was designed. This variable beam splitter consists of a half-waveplate between two polarizing beam splitters (PBS); any desired reflectivity can be obtained with this optical arrangement. For realizing our oracles, a pair of these VBSs was used, and each was adjusted either for maximum or minimum reflectivity, essentially acting as a swap or the identity. Hadamards were constructed using similar VBSs. After the oracle rails 2 and 3 were combined into the same spatial mode in a PBS to guarantee the collective phase shift for these beams in presence of decoherence, and then separated out by another PBS. The transformation between two different encodings was performed by applying another VBS to either swap rails 3 and 4 or not. The experimental setup was designed such that in all of these interferometers the spatial path lengths are always balanced. The average fringe visibility for all four output ports and all possible settings of oracle and encoding was measured to be about 95%. This setup consists of 16 different possible Mach-Zehnder interferometers, of which two are in operation for any given oracle/encoding combination.

The experiment was performed by measuring the signals at detectors PD1 through PD4 as the half waveplates were adjusted to cycle through all four oracles and both encodings. The intensities at detectors PD1 through PD4 were normalized to their sum, to yield the probabilities of a photon reaching each of the detectors. These normalized intensities are plotted in Fig. 3 for all 4 oracle settings, in both the standard algorithm and the DFS encoding. Ideally, all the photons should arrive at detectors PD1 and PD2 for constant functions and at detectors PD3 and PD4 for balanced functions. In figure 4, we plot the probability of a photon reaching either PD1 or PD2, and the probability of a photon reaching either PD3 or PD4. The average error rates were measured to be about 8% in the absence of added noise. The sources of errors in this experiment were mostly due to imperfect visibility, (due to alignment and waveplate setting), and uncertainty and drift in the optical phases setting, when a 12° phase error on one beam correspond to a 2% error rate. The drift of the interferometer during measurement was kept low by balancing all path
lengths as mentioned above and enclosing the interferometer. Introduction of the turbulent airflow increased the average error rates to 35% for the standard algorithm. When the DFS encoding was used in the presence of turbulence, however, the error rates dropped to 7%, essentially equal to the value in the absence of noise.

This work demonstrates that a simple modification of a quantum algorithm may be used to encode information into DFSs, and significantly reduce the error rate introduced by realistic, physical noise sources, provided that these sources have certain symmetry properties. Phase noise is an everpresent issue in coherent optical systems, and often exhibits certain correlations which should be exploitable in this manner. We also note that since the noise characteristics are intimately tied to the particular physical realization of a quantum circuit, it may often prove easier to design the decoherence-free process by direct consideration of the multiple interferometers which constitute the optical device, than by contemplation of the very general quantum circuits.

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I. FIGURE CAPTIONS

Fig 1. Schematic of an interferometer which implements the 2-qubit Deutsch-Jozsa algorithm (a). All beam splitters are 50/50. With beam splitters C1 and C2 in place, the
standard algorithm (b) is performed. In this work, we show that an alternate encoding (c) is preferable in the presence of random noise as indicated on rails 2 and 3; replacing beam splitters C1 and C2 with beam splitters D1 and D2 implements this modified algorithm.

Fig 2. Experimental setup. Variable-reflectivity beam splitters are implemented using a pair of polarizing beam splitters (PBS) and a half waveplate. The “preparation” portion of the interferometer produces the same superposition as the pair of Hadamards in Fig. 1. The oracle consists of two variable beam splitters which can each be set to swap two rails or leave them unchanged, plus two half-waveplate used to induce $\pi$ phase shifts on two of the outputs. The random noise is generated by inducing turbulent airflow under rails 2 and 3 while they are spatially superposed.

Fig 3. Experimental data: Normalized intensity is a measure of the fraction of photons reaching each detector, PD1 through PD4. Data are shown for both the DFS and standard encodings, for each of the four oracles (00, 01, 10, and 11); C indicates “constant” oracles while B indicates “balanced” oracles. The bottom plot shows the same data in the presence of noise. Note that the noise has a much more significant effect in the case of the standard encoding.

Fig 4. The probability of the algorithm returning a 0 or a 1 for each of the oracles, in each encoding, with and without the addition of phase noise. The data are extracted by summing the normalized intensities from Fig. 3 for PD1 and PD2 (dashed line, indicating constant oracles) and for PD3 and PD4 (solid line, indicating balanced oracles). Note that the success rate is close to 1 even in the presence of noise when the DFS encoding is used.
a) Laser

A1  B1

A2  B2

Oracle

Random Noise

1 2 3 4

1 2 3 4

A1

B1

A2

B2

Oracle

Random Noise

|00⟩ |01⟩ |10⟩ |11⟩

|0⟩ |1⟩

|0⟩ |1⟩

Photodiodes

1 2 3 4

H x y

|0⟩ |1⟩

H y y\oplus f(x)

H
Different oracle settings in time (s)
