On the Schur indices of cuspidal unipotent characters

Meinolf Geck

To John Thompson on his 70th birthday

Abstract. In previous work of Gow, Ohmori, Lusztig and the author, the Schur indices of all unipotent characters of finite groups of Lie type have been explicitly determined except for six cases in groups of type $F_4$, $E_7$ and $E_8$. In this paper, we show that the Schur indices of all cuspidal unipotent characters for type $F_4$ and $E_8$ are 1, assuming that the group is defined over a field of “good” characteristic. This settles four out of the six open cases. For type $E_7$, we show that the Schur indices are at most 2.

2000 Mathematics Subject Classification: Primary 20C15; Secondary 20G40

1. Introduction

By the work of Feit, Gow, Ohmori, Turull, ..., one expects that the Schur index of an irreducible character of a finite simple group is 1 or 2. This paper is a contribution to a solution of this problem, as far as the unipotent characters of a finite group of Lie type are concerned. By Lusztig [17], the Schur indices of all rational-valued unipotent characters are generally 1. However, there are exceptions in the finite unitary groups (Ohmori [21]) and the Ree groups $2F_4(q)$ ([6]) where the Schur indices are 2. In [6], we were also able to compute the Schur indices of some non-rational valued unipotent characters, but not all. This paper deals with some of the remaining cases, which arise in groups of type $F_4$, $E_7$ and $E_8$.

Our principal tool are Kawanaka’s generalized Gelfand–Graev characters [12]. These are characters induced from certain unipotent subgroups. It is known that every unipotent character occurs with “small” multiplicity in such an induced character. We shall show here that certain generalized Gelfand–Graev representations can actually be realized over $\mathbb{Q}$ (in type $F_4$ or $E_8$) or over a quadratic extension of $\mathbb{Q}$ (in type $E_7$). The method for doing this is inspired by Ohmori [20], who considered the case of ordinary Gelfand–Graev representations. Furthermore, using the explicit multiplicity formulas obtained by Kawanaka [12] and Lusztig [15],
we check that the unipotent characters that we wish to consider all occur with multiplicity 1 in those generalized Gelfand–Graev representations.

Combining the previous work by Ohmori, Lusztig and the author with the results obtained in this paper, the known results on Schur indices of unipotent characters can be summarized as follows. Let $G$ be a simple algebraic group and $F: G \to G$ be an endomorphism such that $G^F$ is a finite (twisted or untwisted) group of Lie type. Let $\rho$ be a unipotent irreducible character of $G^F$. Then the Schur index of $\rho$ is given as follows.

1. Assume that $\rho$ is cuspidal and rational-valued.
   - (a) If $G^F$ is of type $^2A_{n-1}$ where $n = s(s + 1)/2$ for some $s \geq 1$, then the Schur index of the unique cuspidal unipotent character of $G^F$ is 1 if $n$ is congruent to 0 or 1 modulo 4, and 2 otherwise; see Ohmori [21].
   - (b) If $G^F$ is a Ree group of type $^2F_4$, then the Schur index of the unique cuspidal unipotent character which occurs with even multiplicity in all Deligne–Lusztig generalized characters $RT_1$ is 2; see [6], Theorem 1.6.
   - (c) In all other cases, the Schur index of $\rho$ is 1; see Lusztig [17], Theorem 0.2.

2. Assume that $\rho$ is cuspidal and not rational-valued. Then the Schur index of $\rho$ is 1, except possibly for the two cuspidal unipotent characters in type $E_7$ (where the Schur index is at most 2) or the two cuspidal unipotent characters with character field $\mathbb{Q}(\sqrt{-1})$ in type $E_8$ over a field of characteristic 5; see Table 1 and Corollary 1.5 in [6], and Sections 2, 3 in the present paper. See Gow [9], p. 119, for groups of type $^2B_2$ and $^2G_2$.

3. Assume that $\rho$ is unipotent but not necessarily cuspidal. Then $\rho$ occurs with non-zero multiplicity in the Harish-Chandra induction $R^G_L(\psi)$ where $L$ is an $F$-stable Levi complement in some $F$-stable proper parabolic subgroup of $G$ and $\psi$ is a cuspidal unipotent character of $L^F$. In this situation, the Schur index of $\rho$ equals that of $\psi$; see [6], Proposition 5.6.

Thus, there are two remaining cases to be dealt with:

- the characters $E_7[\pm \xi]$ in type $E_7$;
- the characters $E_8[\pm \sqrt{-1}]$ for a group of type $E_8$ in characteristic 5.

For each of these cases, we show that the determination of the Schur index can be reduced to the problem of computing explicitly the values of certain induced characters. The latter problems are discussed in [7] and [10].

2. On the rationality of generalized Gelfand–Graev representations

The aim of this section is to show that certain generalized Gelfand–Graev representations of a finite group of Lie type can be realized over $\mathbb{Q}$. For this purpose,
we have to recall in some detail the construction of these representations. We shall freely use standard results and notations concerning (connected reductive) linear algebraic groups and their Lie algebras (see Carter [2] and the references there).

Let \( G \) be a connected reductive group defined over the finite field \( \mathbb{F}_q \), with corresponding Frobenius map \( F: G \to G \). We fix an \( F \)-stable Borel subgroup \( B \subseteq G \) and write \( B = U.T \) where \( U \) is the unipotent radical of \( B \) and \( T \) is an \( F \)-stable maximal torus. Let \( X = \text{Hom}(T, k^\times) \) be the character group of \( T \) and \( \Phi \subseteq X \) be the root system of \( G \) with respect to \( T \). Then \( B \) determines a positive system \( \Phi^+ \subseteq \Phi \) and a corresponding set of simple roots \( \Pi \subseteq \Phi^+ \). We have

\[
G = \langle T, X_\alpha \mid \alpha \in \Phi \rangle \quad \text{and} \quad U = \prod_{\alpha \in \Phi^+} X_\alpha
\]

where \( X_\alpha \) is the root subgroup corresponding to \( \alpha \in \Phi \). (Here, it is understood that the product is taken over some fixed order of the roots.) For each \( \alpha \in \Phi \), let \( \alpha^\vee \in Y = \text{Hom}(k^\times, T) \) be the corresponding coroot. Given \( \lambda \in X \) and \( \mu \in Y \), let \( \langle \lambda, \mu \rangle \) be the unique integer such that \( \gamma^{\langle \lambda, \mu \rangle} = (\lambda \circ \mu)(\gamma) \) for all \( \gamma \in k^\times \). This defines a non-degenerate pairing \( \langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z} \); the matrix

\[
C := \langle (\alpha, \beta^\vee) \rangle_{\alpha, \beta \in \Pi}
\]

is the Cartan matrix of \( G \). Now, for any \( \alpha \in \Phi \), there is an isomorphism of algebraic groups \( x_\alpha : k^+ \to X_\alpha \) such that

\[
t x_\alpha(\xi)t^{-1} = x_\alpha(\alpha(t)\xi) \quad \text{for all } t \in T \text{ and } \xi \in k.
\]

Since we will only need to consider this case, let us assume that \( F \) is of split type, such that

\[
F(x_\alpha(\xi)) = x_\alpha(\xi^q) \quad \text{for all } \alpha \in \Phi, \xi \in k,
\]

\[
F(t) = t^q \quad \text{for all } t \in T.
\]

We consider the conjugacy classes of unipotent elements in \( G \). For this purpose, we assume throughout that the characteristic \( p \) of \( \mathbb{F}_q \) is a good prime for \( G \). Recall that this means that \( p \) is good for each simple factor involved in \( G \), and that the conditions for the various simple types are as follows.

\[
\begin{align*}
A_n & : \text{no condition}, \\
B_n, C_n, D_n & : \ p \neq 2, \\
G_2, F_4, E_6, E_7 & : \ p \neq 2, 3, \\
E_8 & : \ p \neq 2, 3, 5.
\end{align*}
\]

Then it is known (see Kawanaka [12] and the references there; see also Premet [22]) that one can naturally attach to each unipotent class of \( G \) a so-called weighted Dynkin diagram, that is, an additive map

\[
d : \Phi \to \mathbb{Z} \quad \text{such that } \ d(\alpha) \in \{0, 1, 2\} \text{ for all } \alpha \in \Pi.
\]

The assignment from unipotent classes to additive maps as above is injective, but not surjective in general. Complete lists of weighted Dynkin diagrams for the various types of simple algebraic groups can be found in Carter [2], §13.1.
Given such a weighted Dynkin diagram $d$, the corresponding unipotent class is determined as follows. We set
\[
L_d := \langle T, X_\alpha \mid \alpha \in \Phi, d(\alpha) = 0 \rangle \quad \text{and} \quad U_{d,i} := \prod_{\alpha \in \Phi^+, \ d(\alpha) \geq i} X_\alpha
\]
for $i = 1, 2, 3, \ldots$. Then $P_d := U_{d,1}L_d$ is a parabolic subgroup of $G$, with unipotent radical $U_{d,1}$ and Levi complement $L_d$. By Kawanaka [12], Theorem 2.1.1, there is a unique unipotent class $C$ in $G$ such that $C \cap U_{d,2}$ is dense in $U_{d,2}$; furthermore, $C \cap U_{d,2}$ is a single $P_d$-conjugacy class and we have
\[
C_G(u) \subseteq P_d \quad \text{for all } u \in C \cap U_{d,2}.
\]
Then $C$ is the unipotent class attached to the given weighted Dynkin diagram $d$.

In order to define the generalized Gelfand–Graev representation associated with an element $u \in C \cap U_{d,2}$, we need to introduce some further notation. Let $g$ be the Lie algebra of $G$ over $k = \mathbb{F}_q$. Then $g$ is also defined over $\mathbb{F}_q$ and we have a corresponding Frobenius map $F: g \to g$. Let $t \subseteq g$ be the Lie algebra of $T$. We have a Cartan decomposition
\[
g = t \oplus \bigoplus_{\alpha \in \Phi} k e_\alpha \quad \text{where } F(t) = t \text{ and } F(e_\alpha) = e_\alpha \text{ for all } \alpha \in \Phi.
\]

We shall set $c_\alpha := \kappa(e_\alpha^*, e_\alpha)$ for any $\alpha \in \Phi^+$, where $\kappa: g \times g \to k$ is a non-degenerate $G$-invariant, associative bilinear form and $x \mapsto x^*$ is an opposition $\mathbb{F}_q$-automorphism of $g$, that is, an automorphism such that $t^* = t$ and $e_\alpha^* \in \mathbb{F}_q e_{-\alpha}$ for all $\alpha \in \Phi$ (see Kawanaka [12], §3.1).

Finally, we fix a non-trivial homomorphism $\chi: \mathbb{F}_q^+ \to \mathbb{C}^\times$.

**Definition 2.1** (Kawanaka). Consider a unipotent element $u \in C \cap U_{d,2}$: write
\[
(*) \quad u \in \left( \prod_{\alpha \in \Phi^+} x_\alpha(\eta_\alpha) \right) \cdot U_{d,3}^{F \cdot} \quad \text{where } \eta_\alpha \in \mathbb{F}_q.
\]

With this notation, we define a map $\varphi_u: U_{d,2}^F \to \mathbb{C}^\times$ by
\[
\varphi_u\left( \prod_{\alpha \in \Phi^+, d(\alpha) \geq 2} x_\alpha(\xi_\alpha) \right) = \chi\left( \sum_{\alpha \in \Phi^+} c_\alpha \eta_\alpha \xi_\alpha \right) \quad \text{where } \xi_\alpha \in \mathbb{F}_q.
\]

Since $\kappa(e_\alpha^*, e_\beta) = 0$ unless $\alpha = -\beta$, this definition coincides with Kawanaka’s original definition in [12], (3.1.5); see also [5], Chapter 1. The map $\varphi_u$ actually is a group homomorphism, that is, a linear character of $U_{d,2}^F$. Inducing that character from $U_{d,2}^F$ to $G^F$, we obtain
\[
\text{Ind}_{U_{d,2}^F}^{G^F}(\varphi_u) = [U_{d,1}^F : U_{d,2}^F]^{1/2} \cdot \Gamma_u
\]
where $\Gamma_u$ is the character of the generalized Gelfand–Graev representation associated with $u$; see Kawanaka [12], (3.1.12). Note that the index $[U_{d,1}^F : U_{d,2}^F]$ is an even power of $q$, so the square root exists.

**Remark 2.2.** In the above setting, let us choose another element $u' \in C \cap U_{d,2}^F$,

$$u' \in \left( \prod_{\alpha \in \Phi^+ \text{ s.t. } d(\alpha) = 2} x_\alpha(\eta'_\alpha) \right) \cdot U^F_{d,3} \quad \text{where } \eta'_\alpha \in \mathbb{F}_q.$$  

Then we can also apply the above constructions to $u'$. Thus, we obtain a corresponding generalized Gelfand–Graev character $\Gamma_{u'}$ such that

$$\text{Ind}_{U_{d,2}^F}^{G^F}(\varphi_{u'}) = [U_{d,1}^F : U_{d,2}^F]^{1/2} \cdot \Gamma_u.$$ 

Using the defining formula for $\Gamma_u$, it is straightforward to check that $\Gamma_u = \Gamma_{u'}$ if $u'$ and $u$ lie in the same $P_{d}^F$-orbit. (Indeed, if $u' = g u g^{-1}$ where $g \in P_{d}^F$, then $\varphi_{u'}(x) = \varphi_u(g^{-1} x g)$ for all $x \in U_{d,2}^F$ and so the induced characters are equal.)

Now let $C$ be the $G$-conjugacy class of $u$. Then $C^F$ splits into orbits under the finite group $G^F$, and it is well-known that these $G^F$-orbits are parametrized by the $F$-conjugacy classes of $C_G(u)/C_G(u)^\circ$. Since $C_G(u) \subseteq P_d$, a complete set of representatives for the $G^F$-orbits in $C^F$ can be found inside $C \cap U_{d,2}^F$.

Assume now that $G$ is simple of adjoint type and write $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ where $l$ is the rank of $G$. Then we have $X = \mathbb{Z}^l$. Let $\{\omega_1, \ldots, \omega_l\}$ be the dual basis of $Y$, that is, we have $\langle \alpha_i, \omega_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq l$. Now we can state the following result which generalizes the arguments in Ohmori [20], Lemma 2 and Proposition 1(i). (See also Example 2.4 below.)

**Proposition 2.3.** Let $G$ be simple of adjoint type and assume that there exist integers $n_j \in \mathbb{Z}$ ($1 \leq j \leq l$) such that

$$\sum_{j=1}^{l} n_j \langle \alpha, \omega_j \rangle = 1 \quad \text{for all } \alpha \in \Phi^+ \text{ such that } d(\alpha) = 2 \text{ and } \eta_\alpha \neq 0,$$

where the coefficients $\eta_\alpha$ are defined as in Definition 2.1 (*). Then the character $[U_{d,1}^F : U_{d,2}^F]^{1/2} \cdot \Gamma_u$ can be realized over $\mathbb{Q}$. If, moreover, we also have

$$\sum_{j=1}^{l} n_j \langle \alpha, \omega_j \rangle = 0 \quad \text{for all } \alpha \in \Phi^+ \text{ such that } d(\alpha) = 0,$$

then $[U_{d,1}^F : U_{d,2}^F]^{1/2} \cdot \Gamma_u$ can be realized over $\mathbb{Q}$, for every $u' \in C \cap U_{d,2}^F$.

**Proof.** Let $\nu$ be a generator for the multiplicative group of $\mathbb{F}_q$. We claim that there exists an element $t \in T^F$ such that

$$\alpha(t) = \nu \quad \text{for all } \alpha \in \Phi^+ \text{ with } d(\alpha) = 2 \text{ and } \eta_\alpha \neq 0.$$
This is seen as follows. By [2], Proposition 3.1.2, the map
\[ h: \prod_{i=1}^t k^i \rightarrow T, \quad (x_1, \ldots, x_t) \mapsto \prod_{i=1}^t \omega_i(x_i), \]
is an $\mathbb{F}_q$-isomorphism. In particular, we have $T^F = \{ h(x_1, \ldots, x_t) \mid x_i \in \mathbb{F}_q^* \}$. Now set $t = h(\nu^1, \ldots, \nu^q) \in T^F$. Let $\alpha \in \Phi^+$ be such that $d(\alpha) = 2$ and $\eta_\alpha \neq 0$. Then we have
\[ \alpha(t) = \prod_{j=1}^q x_j^{(\alpha, \omega_j)} = \prod_{j=1}^q \nu^{n_j(\alpha, \omega_j)} = \nu^{\sum_{j=1}^q n_j(\alpha, \omega_j)} = \nu, \]
as required. Thus, (†) is proved.

Now let $H := \langle t \rangle$. Then $H$ is a group of order $q - 1$ which normalizes $U_{d,2}^F$. Let us induce $\varphi_u$ from $U_{d,2}^F$ to the semidirect product $U_{d,2}^F \rtimes H$ and denote the induced character by $\psi_u$. We will show that $\psi_u$ is a rational-valued irreducible character. First note that $\psi_u$ has non-zero values only on elements in $U_{d,2}^F$. Furthermore, by Mackey’s formula, the restriction of $\psi_u$ to $U_{d,2}^F$ is given by $\sum_{h \in H} \varphi_u^h$ where $\varphi_u^h(x) = \varphi_u(hxh^{-1})$ for all $x \in U_{d,2}^F$. Now let
\[ x := \prod_{\alpha \in \Phi^+, d(\alpha) \geq 2} x_\alpha(\xi_\alpha) \in U_{d,2}^F \quad \text{for any } \xi_\alpha \in \mathbb{F}_q. \]
Then we have
\[ txt^{-1} = \prod_{\alpha \in \Phi^+, d(\alpha) \geq 2} tx_\alpha(\xi_\alpha)t^{-1} = \prod_{\alpha \in \Phi^+, d(\alpha) \geq 2} x_\alpha(\alpha(t)\xi_\alpha) \]
and so, using (†),
\[ \varphi_u^t(x) = \chi \left( \sum_{\alpha \in \Phi^+, d(\alpha) \geq 2} c_\alpha \eta_\alpha \alpha(t)\xi_\alpha \right) = \chi \left( \nu \sum_{\alpha \in \Phi^+, d(\alpha) \geq 2} c_\alpha \eta_\alpha \xi_\alpha \right). \]
Similarly, for any $1 \leq i \leq q - 1$, we obtain
\[ \varphi_u^t(x) = \chi(\nu^i\gamma_x) \quad \text{where} \quad \gamma_x := \sum_{\alpha \in \Phi^+, d(\alpha) \geq 2} c_\alpha \eta_\alpha \xi_\alpha \in \mathbb{F}_q. \]
This shows, first of all, that the characters $\varphi_u^t (1 \leq i \leq q - 1)$ are pairwise different and, hence, the character $\psi_u$ is seen to be irreducible. Furthermore, we obtain that
\[ \psi_u(x) = \sum_{h \in H} \varphi_u^h(x) = \sum_{i=1}^{q-1} \varphi_u^t(x) = \sum_{i=1}^{q-1} \chi(\nu^i\gamma_x) = \begin{cases} q - 1 & \text{if } \gamma_x = 0, \\ -1 & \text{if } \gamma_x \neq 0. \end{cases} \]
In particular, this shows that $\psi_u(x) \in \mathbb{Z}$ for all $x \in U_{d,2}^F$. Thus, $\psi_u$ is a rational-valued irreducible character of $U_{d,2}^F \rtimes H$. In order to show that $\psi_u$ can be realized over $\mathbb{Q}$, we note that (again by Mackey’s formula), the restriction of $\psi_u$ to $H$ is the
character of the regular representation of $H$. Hence the trivial character $1_H$ occurs with multiplicity 1 in that restriction. By Frobenius reciprocity, $\psi_u$ itself occurs with multiplicity 1 in the character of $U_{d,2}^F H$ obtained by inducing $1_H$. Since the latter character clearly is realized over $\mathbb{Q}$, a standard argument on Schur indices shows that $\psi_u$ can be realized over $\mathbb{Q}$ (see Isaacs [11], Corollary 10.2). Finally, by the transitivity of induction, the character

$$\text{Ind}_{U_{d,2}}^{G_F}(\varphi_u) = [U_{d,1}^F : U_{d,2}^F]^{1/2} \cdot \Gamma_u$$

can be realized over $\mathbb{Q}$, as claimed.

Now assume that the integers $n_j$ satisfy the additional equations described above and consider an arbitrary element $u' \in C \cap U_{d,2}^F$. Thus, we have $u' = gug^{-1}$ where $g \in P_d$. Let

$$u' \in \left( \prod_{\alpha \in \Phi^+} x_{\alpha}(\eta'_\alpha) \right) \cdot U_{d,3}^F \text{ where } \eta'_\alpha \in \mathbb{F}_q.$$ 

We claim that if $\eta'_\alpha \neq 0$, then $\alpha = \alpha_1 + \beta$ where $d(\beta) = 0$, $d(\alpha_1) = 2$ and $\eta_{\alpha_1} \neq 0$. To see this, it is enough to consider the special case where $g \in T$ or where $g \in X_{\beta}$ with $d(\beta) \geq 0$. The assertion is clear if $g \in T$. On the other hand, if $g \in X_{\beta}$ where $d(\beta) \geq 0$, then Chevalley’s commutator relations show that $gug^{-1}$ will lie in a product of root subgroups $X_{i;\beta+\alpha_1}$ where $d(\alpha_1) \geq 2$ and $i = 0, 1, 2, \ldots$, and the assertion also follows. We can now conclude that the integers $n_j$ automatically have the property that

$$\sum_{j=1}^{l} n_j (\alpha, \omega_j) = 1 \text{ for all } \alpha \in \Phi^+ \text{ such that } d(\alpha) = 2 \text{ and } \eta'_\alpha \neq 0.$$ 

Thus, we can apply the above arguments to $u'$ as well and conclude that the character $[U_{d,1}^F : U_{d,2}^F]^{1/2} \cdot \Gamma_{u'}$ can be realized over $\mathbb{Q}$. 

\[\square\]

Example 2.4. (Cf. Ohmori [20], p. 151.) Assume that $G$ is simple of adjoint type. The weighted Dynkin diagram corresponding to the conjugacy class of regular unipotent elements in $G$ is the map $d_0: \Phi \to \mathbb{Z}$ such that $d_0(\alpha_i) = 2$ for all $1 \leq i \leq l$. In this case, we have

$$P_{d_0} = B, \quad L_{d_0} = T \quad \text{and} \quad U_{d_0,1} = U_{d_0,2} = U;$$

furthermore, $u = \prod_{i=1}^{l} x_{\alpha_i}(\eta_i)$, where $\eta_i \in \mathbb{F}_q^\times$, is a representative in that class. Now the system of equations in Proposition 2.3 is simply given by

$$\sum_{j=1}^{l} n_j (\alpha_i, \omega_j) = 1 \quad \text{for all } 1 \leq i \leq l.$$ 

Thus, we have $n_j = 1$ for $1 \leq j \leq l$. Consequently, $\Gamma_u$ (which is just an ordinary Gelfand–Graev character) can be realized over $\mathbb{Q}$. 

Example 2.5. Assume that $G$ is simple of adjoint type and that $d$ is a weighted Dynkin diagram of the following special type: There exists some $i_0 \in \{1, \ldots, l\}$ such that

$$d(\alpha_{i_0}) = 2 \quad \text{and} \quad d(\alpha_i) = 0 \quad \text{for} \; i \in \{1, \ldots, l\} \setminus \{i_0\}.$$ 

We claim that, in this case, the generalized Gelfand–Graev character $\Gamma_u$ can be realized over $\mathbb{Q}$, for any $u \in C \cap U_d$. This is seen as follows. First note that $U_{d,1} = U_{d,2}$. Now let $\eta_\alpha$ be defined as in Definition 2.1 and consider the two sets of equations in Proposition 2.3. The second set of equations reads

$$\sum_{j=1}^{l} n_j \langle \alpha_i, \omega_j \rangle = 0 \quad \text{for all} \; i \in \{1, \ldots, l\} \setminus \{i_0\}.$$ 

This yields $n_i = 0$ for all $i \neq i_0$. Then the first set of equations reads

$$n_{i_0} \langle \alpha, \omega_{i_0} \rangle = 1 \quad \text{for all} \; \alpha \in \Phi^+ \; \text{such that} \; d(\alpha) = 2 \; \text{and} \; \eta_\alpha \neq 0.$$ 

However, writing any such $\alpha$ as a linear combination of simple roots, we see that $\alpha_{i_0}$ always occurs with coefficient 1 and so the above equations hold with $n_{i_0} = 1$. Thus, the systems of equations in Proposition 2.3 have a unique solution.

We close this section with a general remark concerning the Schur indices of unipotent characters. This remark shows that it will be enough to consider the unipotent characters of simple groups of adjoint type.

Remark 2.6. Assume that $G$ is simple of adjoint type. Let $G_1$ be an algebraic group over $\mathbb{F}_q$ such that $G_1/Z(G_1)$ is simple of the same type as $G$. We denote the Frobenius map on $G_1$ again by $F$. Then we have a surjective homomorphism of algebraic groups $\varphi: G_1 \rightarrow G$ which is defined over $\mathbb{F}_q$ and such that $\ker(\varphi) = Z(G_1)$. Let $\chi$ be a unipotent character of $G^F$. Denote by $\chi_1$ the character of $G_1^F$ which is obtained by first restricting $\chi$ to $\varphi(G_1^F)$ and then pulling back to $G_1^F$ via $\varphi$. By Proposition 7.10, the map $\chi \mapsto \chi_1$ defines a bijection between the unipotent characters of $G^F$ and $G_1^F$, respectively. One can express this by saying that the unipotent characters are “insensitive” to the center of $G$.

For a unipotent character $\chi \in \text{Irr}(G^F)$, denote by $Q(\chi) = Q(\chi(g) \mid g \in G^F)$ the character field of $\chi$ and by $m_Q(\chi)$ the Schur index of $\chi$; we use similar notations for a unipotent character of $G_1^F$. We claim that

$$Q(\chi) = Q(\chi_1), \quad \text{and} \quad m_Q(\chi) = m_Q(\chi_1).$$

Proof. First note that we clearly have $Q(\chi_1) \subseteq Q(\chi)$. To show that we have equality, let $E \supseteq \mathbb{Q}$ be a finite Galois extension such that any unipotent character of $G^F$ and of $G_1^F$ can be realized over $E$. Given any $\tau \in \text{Gal}(E/\mathbb{Q})$ and any unipotent character $\chi \in \text{Irr}(G^F)$, we denote by $\chi^\tau$ the irreducible character obtained by algebraic conjugation; it is still a unipotent character of $G^F$ (see the remarks in \cite{6}, (5.1)). Now the map $\chi \mapsto \chi_1$ certainly is compatible with field automorphisms. Thus, we have $\chi^\tau_1 = (\chi^\tau)_1$. Since the map $\chi \mapsto \chi_1$ is a bijection, we conclude that $\chi^\tau_1 = \chi_1$ if and only if $\chi^\tau = \chi$. This implies that $Q(\chi) = Q(\chi_1)$, as required.
Now consider the Schur indices. First note that we certainly have \( m_\mathbb{Q}(\chi_1) \leq m_\mathbb{Q}(\chi) \). (Indeed, if \( \chi \) can be realized over some extension field of \( \mathbb{Q} \), then \( \chi_1 \) is automatically realized over the same field.) To show the reverse inequality, we argue as follows. The character \( \chi_1 \) can be realized over an extension field \( E_1 \supseteq \mathbb{Q} \) such that \( [E_1 : \mathbb{Q}] = m_\mathbb{Q}(\chi_1) \). We now regard \( \chi_1 \) as a character of \( \varphi(G^F) \). By Frobenius reciprocity, \( \chi \) occurs with multiplicity 1 in the character obtained by inducing \( \chi_1 \) from \( \varphi(G^F) \) to \( G^F \). Since that induced character can also be realized over \( E_1 \), a standard argument on Schur indices ([11], Corollary 10.2) shows that \( m_\mathbb{Q}(\chi) \leq m_\mathbb{Q}(\chi_1) \), as required.

\[ \square \]

### 3. Simple groups of type \( G_2, F_4 \) and \( E_8 \)

Throughout this section, let \( G \) be simple of (adjoint) type \( G_2, F_4 \) or \( E_8 \), defined over the finite field \( \mathbb{F}_q \) with corresponding Frobenius map \( F: G \to G \). Then \( F \) is of split type, in the sense of the previous section. The cuspidal unipotent characters of \( G^F \) and the previously known results on their Schur indices are listed in Table 1.

**Table 1:** Schur indices of cuspidal unipotent characters in type \( G_2, F_4 \) and \( E_8 \) (notation of Carter [2], §13.9; see also [3], Table 1)

| Type   | Character   | Schur index |
|--------|-------------|-------------|
| \( G_2 \) | \( G_2[1], G_2[-1] \) | 1 (see Lusztig [17]) |
|        | \( G_2[\theta], G_2[\theta^2] \) | 1 (see [4]) |
| \( F_4 \) | \( F_4^I[1], F_4^I[-1], F_4[I] \) | 1 (see Lusztig [17]) |
|        | \( F_4[i], F_4[-i] \) | ? (1 in char. 2, 3; see [3]) |
|        | \( F_4[\theta], F_4[\theta^2] \) | 1 (see [4]) |
| \( E_8 \) | \( E_8^I[1], E_8^{II}[1], E_8[-1] \) | 1 (see Lusztig [17]) |
|        | \( E_8[-\theta], E_8[-\theta^2] \) | ? (1 in char. 2, 3, 5; see [3]) |
|        | \( E_8[\theta], E_8[\theta^2] \) | ? (1 in char. 2, 3, 5; see [3]) |
|        | \( E_8[i], E_8[-i] \) | ? (1 in char. 2, 3; see [3]) |
|        | \( E_8[\zeta], E_8[\zeta^2], E_8[\zeta^3], E_8[\zeta^4] \) | 1 (see [4]) |

\[ i := \sqrt{-1}, \quad \theta := \exp(2\pi i/3), \quad \zeta := \exp(2\pi i/5). \]

Now let us assume that \( G \) is defined over a field of good characteristic. All the cuspidal unipotent characters of \( G^F \) have the same “unipotent support” in the sense of Lusztig [13]; see also [8], Proposition 4.2. Hence there is a unique unipotent class \( C_0 \) in \( G \) such that all the cuspidal unipotent characters of \( G^F \) have unipotent support \( C_0 \). Let \( e_0 = \dim \mathfrak{B}_u \) (the dimension of the variety of Borel subgroups containing an element \( u \in C_0 \)). Then, by [3], Theorem 3.7, \( q^{e_0} \) is the
exact power of $q$ dividing the degrees of the unipotent characters of $G^F$ having unipotent support $C_0$. By inspection of the tables in [2], Chapter 9, we find that $e_0 = 1, 4$ or $16$ for $G$ of type $G_2$, $F_4$ or $E_8$, respectively. Furthermore, for $G$ of type $G_2$ or $F_4$, the class $C_0$ is uniquely determined by this condition. In type $E_8$, we use the additional information that $C_G(u)/C_G(u)^0$ must be non-trivial (see once more the formula in [3], Theorem 3.7). Then $C_0$ is uniquely determined. The weighted Dynkin diagram $d_0$ associated to $C_0$ is specified in Table 2. Furthermore, we have $C_G(u)/C_G(u)^0 \cong S_3$, $S_4$ or $S_5$ for $G$ of type $G_2$, $F_4$ or $E_8$, respectively.

Table 2: Weighted Dynkin diagrams for the unipotent supports of cuspidal unipotent characters in type $G_2$, $F_4$ and $E_8$

![Diagram of Dynkin diagrams]

Now let us fix a cuspidal unipotent character $\rho$ of $G^F$. By Lusztig, [15], Theorem 11.2, there exists some $u \in C_0 \cap U_{d_0,2}$ such that $\rho$ occurs with “small” multiplicity in $\Gamma_u$. In the present situation, Kawanaka actually obtained an explicit multiplicity formula. To state that formula, we need to recall some facts about the parametrization of unipotent characters; see Lusztig [13], Main Theorem 4.23. Let $F_0$ be the unique family of unipotent characters which contains the cuspidal unipotent characters. Let $G_0$ be the finite group attached to that family; we have $G_0 \cong C_G(u)/C_G(u)^0$ in the present situation. Let $M_0$ be the set of all pairs $(x, \sigma)$ where $x \in G_0$ (up to conjugacy) and $\sigma \in \text{Irr}(C_G(u)(x))$. Then we have a bijection

$$M_0 \rightarrow F_0, \quad (x, \sigma) \mapsto \rho_{(x, \sigma)}.$$  

(which satisfies some further conditions which we do not need to recall here). On the other hand, there is a well-defined “split” element $u_0 \in C_0 \cap U_{d_0,2}$; see Shoji [28], Remark 5.1. Having fixed that element, there is a canonical parametrization of the $G^F$-conjugacy classes contained in $C_0^F$ by the conjugacy classes of $G_0$. We shall denote by $\Gamma_{u_y}$ the generalized Gelfand–Graev character with respect to a unipotent element $u_y \in C_0 \cap U_{d_0,2}^F$ in the $G^F$-class parametrized by $y \in G_0$. Having fixed this notation, we can now state the following result.

**Theorem 3.1** (Kawanaka). Recall that $G$ is a group of type $G_2$, $F_4$ or $E_8$ in good characteristic. Let $\langle \rho_{(x, \sigma)}, \Gamma_{u_y} \rangle$ be the multiplicity of $\rho_{(x, \sigma)}$ in the generalized
Gelfand–Graev character $\Gamma_{u_y}$, where $x, y \in G_0$ and $\sigma \in \text{Irr}(C_{\hat{G}_0}(x))$. Then we have
\[
\langle \rho(x, \sigma), \Gamma_{u_y} \rangle = \begin{cases} 
\sigma(1) & \text{if } x = y \pmod{G_0}, \\
0 & \text{otherwise}.
\end{cases}
\]

Proof. Kawanaka obtained an explicit formula for the values of all the generalized Gelfand–Graev characters of $G^F$; see [12], Corollary 3.2.7 and Lemma 3.3.10. Furthermore, in [12], §4.2, he obtained explicit formulas for the values of the unipotent characters on unipotent elements. (In the latter reference, it is actually assumed that the characteristic is large. But this hypothesis is only used in reference to results on the Green functions of $G^F$, which are now known to hold in general; see Shoji [24], Theorem 5.5.) The above multiplicity formula is obtained by a simple formal rewriting of Kawanaka’s formulas. In large characteristic, that formula could also be deduced from Lusztig’s more general results in [15].

Corollary 3.2. Recall that $G$ is a group of type $G_2$, $F_4$ or $E_8$ in good characteristic. Then every cuspidal unipotent character of $G^F$ has Schur index 1.

Proof. By inspection of the tables in [2], §13.9, all cuspidal unipotent characters of $G^F$ are labelled by pairs $(x, \sigma) \in M_0$ where $\sigma(1) = 1$. Thus, by Theorem 3.1 every such character occurs with multiplicity 1 in some generalized Gelfand–Graev character attached to an element in $C_0 \cap U_{d_0,2}$. Now we notice that the weighted Dynkin diagrams in Table 2 are of the type considered in Example 2.5. Thus, all the generalized Gelfand–Graev characters associated with unipotent elements in $C_0 \cap U_{d_0,2}$ can be realized over $\mathbb{Q}$. Hence, the assertion follows by a standard argument on Schur indices (see Isaacs [11], Corollary 10.2).

The above result also covers some of the cases (at least in good characteristic) already dealt with by Lusztig [17] or the author [6]. In combination with the previously known results listed in Table 1 we see that there is only one case left that remains to be considered:

the characters $E_8[\pm i]$ for a group of type $E_8$ in characteristic 5.

We shall now outline a strategy for dealing with this case. That strategy is analogous to that in [6], (6.5). For the remainder of this section, let $G$ be a simple group of type $E_8$ over $\mathbb{F}_q$ where $q$ is odd and $p \equiv 1 \pmod{4}$. Let us label the simple roots of $G$ as in Table 2 and let $\{\omega_1, \ldots, \omega_8\}$ be the dual basis of $Y$. Then we have $T = \{h(x_1, \ldots, x_8) \mid x_i \in k^x\}$ as in the proof of Proposition 2.3. Now let $\nu$ be a generator of the multiplicative group of $\mathbb{F}_q$ and consider the semisimple element $s := h(1, 1, 1, 1, 1, \nu^{(q-1)/4}, 1, 1) \in T^F$.

Thus, $s$ is an element of order 4. Its centralizer is given by $G_1 = \langle T, X_\alpha \mid \alpha \in \Phi_1 \rangle$ where $\Phi_1 = \{\alpha \in \Phi \mid \alpha(s) = 1\}$. It is readily checked that a system of simple roots of $\Phi_1$ is given by $\Pi_1 = (\Pi \setminus \{\alpha_6\}) \cup \{-\alpha_6\}$, where
\[
\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8
\]
is the unique highest root in \( \Phi \). The root system \( \Phi_1 \) is of type \( D_5 \times A_3 \). Now, by Lusztig [14, Proposition 21.3], there is a well-defined unipotent class \( C_1 \subseteq G_1 \) which supports a “cuspidal local system”. That unipotent class is determined explicitly by [18, Corollary 4.9], as far as the \( D_5 \)-factor is concerned. (We necessarily have the class of regular unipotent elements in the \( A_3 \)-factor; see [14, §18].) Using [2, §13.9], we see that the weighted Dynkin diagram \( d_1 \) of \( C_1 \) is given as follows:

\[
\begin{array}{ccccccc}
D_5 \times A_3 & \alpha_1 & 2 & \alpha_3 & 2 & \alpha_4 & 0 & \alpha_5 & 2 & \alpha_7 & 2 & \alpha_8 & 2 & -\alpha_0 \\
\end{array}
\]

By [1, §11.3.2], we can explicitly identify an adjoint group of type \( D_5 \) with a 10-dimensional orthogonal group. Determining the Jordan normal form of the matrix corresponding to the element

\[
u_1 := x_{\alpha_1(1)}x_{\alpha_5(1)}x_{\alpha_2(1)}x_{\alpha_3+\alpha_4(1)}x_{\alpha_4+\alpha_5(1)} \cdot x_{\alpha_7(1)}x_{\alpha_8(1)}x_{-\alpha_0(1)},
\]

we see that \( u_1 \in C_1 \cap U_{d_1,2} \). Now, we have \( C_{G_1}(u_1)/C_{G_1}(u_1) \cong \mathbb{Z}/4\mathbb{Z} \); see [6, Table 2], and the references there. Let \( \psi \) be one of the two faithful irreducible characters of this group and define \( \Gamma(C_1, \psi) \) as in [15, (7.5)]. Thus, \( \Gamma(C_1, \psi) \) is a certain linear combination (where the coefficients involve the values of \( \psi \)) of the generalized Gelfand–Graev characters of \( G_{d_1,2} \) associated with the various unipotent elements in \( C_1 \cap U_{d_1,2} \). Now we can state the following result.

**Proposition 3.3.** In the above setting, assume that the following conditions hold:

(a) The values of \( \Gamma(C_1, \psi) \) are given by Lusztig’s formula [14, Corollary 7.6]. (This is known to be true if the characteristic is “large”.)

(b) The assertion in [16, Theorem 0.8], holds for cuspidal character sheaves in \( G \). (This is known to be true if the characteristic is good; see [24, Theorem 4.1].)

Then the Schur index of the cuspidal unipotent character \( E_8[\pm i] \) is 1.

**Proof.** We begin by showing that the generalized Gelfand–Graev characters of \( G_{d_1,2} \) which are associated with the unipotent elements in \( C_1 \cap U_{d_1,2} \) can be realized over \( \mathbb{Q} \). For this purpose, we use an argument analogous to that in Proposition 2.3.

We shall consider a system of equations as in Proposition 2.3 (but with respect to the unipotent element \( u_1 \in G_1 \) specified above):

\[
\sum_{j=1}^{8} n_j \langle \alpha, \omega_j \rangle = \begin{cases} 
1 & \text{for } \alpha \in \{\alpha_1, \alpha_5, \alpha_2, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_7, \alpha_8, -\alpha_0\}, \\
0 & \text{for all } \alpha \in \Phi^+ \text{ such that } d_1(\alpha) = 0.
\end{cases}
\]

One readily checks that this system has a unique solution:

\[(n_1, n_2, \ldots, n_8) = (1, 1, 1, 0, 1, -5, 1, 1).\]
Thus, \( t := h(\nu, \nu, \nu, 1, \nu, \nu^{-5}, \nu, \nu) \in T^F \) is a semisimple element of order \( q - 1 \).

Now let \( u \in C_1 \cap U^F_{d_1,2} \). Then, as in the proof of Proposition 2.3, we see that \( \alpha(t) = \nu \) (if \( \alpha \) is involved in the expression of \( u \) as a product of root subgroup elements) or \( \alpha(t) = 1 \) (if \( d_1(\alpha) = 0 \)). Let \( H = \langle t \rangle \) and denote by \( \psi_u \) the character obtained by inducing \( \varphi_u \) from \( U^F_{d_1,2} \) to \( U^F_{d_1,2}.H \). Following exactly the same kind of arguments as in the proof of Proposition 2.3, we see that \( \psi_u \) is an irreducible character which can actually be realized over \( \mathbb{Q} \). Consequently, the generalized Gelfand–Graev character \( \Gamma_u \) of \( G^F_1 \) can also be realized over \( \mathbb{Q} \), as claimed.

We now follow the line of argument in [6], (6.5). First note that the character values of \( E_8[\pm i] \) generate the field \( \mathbb{Q}(i) \); see [6], Table 1. Thus, we must show that \( E_8[\pm i] \) can be realized over \( \mathbb{Q}(i) \). Let \( \Sigma \) be the \( G \)-conjugacy class of \( su_1 \). Then \( \Sigma \) supports a “cuspidal local system” \( \mathcal{E} \); see [6], Table 2 and the references there. Using [13], Main Theorem 4.23, the explicit description of the Fourier matrix for type \( E_8 \), and our assumption (b), we see that

\[
\langle E_8[\pm i], \chi_{(\Sigma, \mathcal{E})} \rangle = \frac{1}{4} \varepsilon \quad \text{where } \varepsilon \in \mathbb{C}, |\varepsilon| = 1;
\]

here, \( \chi_{(\Sigma, \mathcal{E})} \) is a characteristic function of \( (\Sigma, \mathcal{E}) \) and \( \langle \ , \ \rangle \) denotes the usual hermitian product on the space of complex-valued class functions on \( G^F \). On the other hand, since \( C_{G_1}(u_1)/C_{G_1}^o(u_1) \cong \mathbb{Z}/4\mathbb{Z} \), the set \( C_1 \cap U^F_{d_1,2} \) splits into four classes in \( G^F_1 \), with representatives \( u_1, u_2, u_3, u_4 \in C_1 \cap U^F_{d_1,2} \) say. Then, our assumption (a) implies that we have

\[
4 \chi_{(\Sigma, \mathcal{E})} = \sum_{r=1}^4 \sum_{\lambda \in \text{Irr}(Z_1)} \xi_{r,\lambda} \text{Ind}_{Z_1 \times U^F_{d_1,2}}^{G^F_1} (\lambda \boxtimes \varphi_{u_r})
\]

where \( Z_1 := Z(G_1) = \langle s \rangle \) and \( \xi_{r,\lambda} \in \mathbb{Z}[i] \); see the analogous equation in [6], (6.5), and note that \( Z(G_1)^o = \{1\} \). Now, since \( Z_1 \cap H = \{1\} \), it is easily shown that

\[
\text{Ind}_{Z_1 \times U^F_{d_1,2}}^{(Z_1 \times U^F_{d_1,2}).H} (\lambda \boxtimes \varphi_{u_r}) = \lambda \boxtimes \text{Ind}_{U^F_{d_1,2}}^{U^F_{d_1,2}} (\varphi_{u_r}) = \lambda \boxtimes \psi_{u_r}
\]

can be realized over \( \mathbb{Q}(i) \). Thus, we have

\[
4 \chi_{(\Sigma, \mathcal{E})} = \sum_\alpha r_\alpha \rho_\alpha \quad \text{where } r_\alpha \in \mathbb{Z}[i] \text{ and } \rho_\alpha \text{ are characters of } G^F \text{ which can be realized over } \mathbb{Q}(i). \quad \text{Since } \langle E_8[\pm i], 4\chi_{(\Sigma, \mathcal{E})} \rangle = \varepsilon \text{ has absolute value } 1, \text{ we conclude that the greatest common divisor of the multiplicities } \langle E_8[\pm i], \rho_\alpha \rangle \text{ must be } 1. \quad \text{Hence, a standard argument on Schur indices} \quad \text{(11), Corollary 10.2) implies that } E_8[\pm i] \text{ can be realized over } \mathbb{Q}(i), \text{ as desired.} \quad \square
\]

In principal, the hypothesis on \( \Gamma_{(C_1, \psi)} \) can be verified by an explicit computation for a group of type \( D_5 \times A_3 \) (in the same way as Kawanaka [12] verified that hypothesis for groups of exceptional type). The details will be discussed in [10].
4. Simple groups of type $E_7$

Let $G$ be a simple group of adjoint type $E_7$ in good characteristic. There are precisely two cuspidal unipotent characters of $G^F$ denoted by $E_7[±ξ]$ where $ξ = \sqrt{-q}$; see the table in [2], §13.9. By [3], Table 1, the character values of $E_7[±ξ]$ lie in the field $\mathbb{Q}(ξ)$ and the Schur index of $E_7[±ξ]$ is 1, 2 or 4. Furthermore, by [6], Example 6.4, we already know that the Schur index is 1 if $p \not≡ 1 \mod 4$ or if $q$ is not a square, where $p$ is the characteristic of $\mathbb{F}_q$. Thus, the remaining task is to determine the Schur index when $q$ is a square and $p ≡ 1 \mod 4$.

Let $C_0$ be the common unipotent support of the two cuspidal unipotent characters. Arguing as in the previous section (that is, checking the exact power of $q$ dividing the degree of $E_7[±ξ]$ and comparing with the tables in [2], Chapter 9), we find that the corresponding weighted Dynkin diagram $d_0$ is given as follows.

\[ E_7 \]
\[ \alpha_1 \quad 0 \quad \alpha_3 \quad \alpha_4 \quad 0 \quad \alpha_5 \quad \alpha_6 \quad \alpha_7 \]

\[ 0 \quad \alpha_2 \]

**Lemma 4.1.** Assume that $q$ is an even power of $p$. Let $u ∈ C_0 ∩ U_{d_0,2}^F$ and recall the definition of the linear character $ϕ_u$ from Section 4. Then there exists a field $K ⊆ \mathbb{Q}$ such that $[K : \mathbb{Q}] \leq 2$ and $\text{Ind}_{U_{d_0,2}^F}(ϕ_u)$ can be realized over $K$.

**Proof.** The following argument is inspired by Ohmori [20], Proposition 1.

By Mizuno [19], Lemma 28, a representative $u_0 ∈ C_0 ∩ U_{d_0,2}^F$ is given by

\[ u_0 = x_{20}(1)x_{21}(1)x_{24}(1)x_{28}(1)x_{30}(1), \]

where the subscripts correspond to the following roots in $Φ^+$:

- $20 : \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$,
- $21 : \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5$,
- $24 : \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$,
- $28 : \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$,
- $30 : \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7$.

We consider a set of equations similar to that in Proposition 2.8

\[ \sum_{j=1}^{7} n_j \langle \alpha, ω_j \rangle = \begin{cases} 2 & \text{for } \alpha ∈ Φ^+ \text{ labelled by } 20, 21, 24, 28 \text{ and } 30, \\ 0 & \text{for all } \alpha ∈ Φ^+ \text{ such that } d_0(\alpha) = 0. \end{cases} \]

One readily checks that this system has a unique solution:

\[ (n_1, n_2, \ldots, n_7) = (1, 0, 0, 1, 0, 1, 0). \]

Now let $ν$ be a generator of the multiplicative group of $\mathbb{F}_p ⊆ \mathbb{F}_q$. Since $q$ is an even power of $p$, we can find a square root of $ν$ in $\mathbb{F}_q$, which we denote by $ν^{1/2} ∈ \mathbb{F}_q$. Then the element

\[ t := h(ν^{a_1/2}, \ldots, ν^{a_7/2}) = h(ν^{1/2}, 1, ν^{1/2}, 1, ν^{1/2}, 1) ∈ T^F \]
has order $2(p-1)$ and, as in the proof of Proposition 2.3, we see that $\alpha(t) = \nu$ for all $\alpha \in \Phi^+$ such that $a_0(\alpha) = 2$ and $\eta_0 \neq 0$ where the coefficients $\eta_0$ are defined as in Definition 2.1. (with respect to the unipotent element $u \in C_0 \cap U_{d,2}^F$). Now let $H = \langle t \rangle$. Then $H$ is a group of order $2(p-1)$ which normalizes $U_{d,2}^F$. Let us induce $\varphi_u$ from $U_{d,2}^F$ to the semidirect product $U_{d,2}^F \rtimes H$ and denote the induced character by $\psi_u$. Arguing as in the proof of Ohmori [20], Proposition 1(ii), we see that $\psi_u$ is the sum of two rational-valued irreducible characters $\psi_1$ and $\psi_2$, where one of them, $\psi_1$, say, can be realized over $\mathbb{Q}$. By the Brauer–Speiser theorem (see [3], 74.27), the character $\psi_2$ has Schur index at most 2. Thus, there exists a field $K \supseteq \mathbb{Q}$ such that $[K : \mathbb{Q}] \leq 2$ and $\psi = \psi_1 + \psi_2$ can be realized over $K$. Consequently, by the transitivity of induction, the character

$$\text{Ind}_{U_{d,2}^F}^{G^F}(\varphi_u) = \text{Ind}_{U_{d,2}^F \rtimes H}^{G^F}(\psi_u)$$

can also be realized over $K$. \qed

**Remark 4.2.** We have remarked at the beginning of this section that it is enough to compute the Schur index of $E_7[\pm \xi]$ in the case where $q$ is a square and $p \equiv 1 \mod 4$. In this case, the character values of $E_7[\pm \xi]$ lie in the field $\mathbb{Q}(\sqrt{-1})$.

Now Ohmori [20], p. 154, points out that the character $\psi_2$ arising in the above proof has non-trivial local Schur indices at $\infty$ and at the prime $p$. This implies that $K$ is not contained in $\mathbb{Q}(\sqrt{-1})$ if $p \equiv 1 \mod 4$. Indeed, if we had $K \subseteq \mathbb{Q}(\sqrt{-1})$, then $\psi_2$ could be realized over $\mathbb{Q}(\sqrt{-1})$. Since $p \equiv 1 \mod 4$, the field $\mathbb{Q}(\sqrt{-1})$ is contained in the field of $p$-adic numbers. Hence, $\psi_2$ could be realized over the latter field, contradicting the fact that the local Schur index at $p$ is not one.

**Corollary 4.3.** Recall that $G$ is a group of type $E_7$ in good characteristic. Then the two cuspidal unipotent characters of $G^F$ have Schur index at most 2.

**Proof.** We have already remarked at the beginning of this section that the Schur index of $E_7[\pm \xi]$ is 1 if $q$ is not a square. So let us now assume that $q$ is an even power of $p$. Let $u_0 \in C_0 \cap U_{d,2}^F$. Then it is known that $C_G(u_0)/C_G(u_0) \cong \mathbb{Z}/2\mathbb{Z}$ (see the tables in [2], §13). Thus, $C_G^F$ splits into two classes in the finite group $G^F$. Let $u_1 \in C_0 \cap U_{d,2}^F$ be a representative of the second $G^F$-conjugacy class contained in $C_G^F$. Combining [8], Theorem 3.7 and Remark 3.8, we have that

$$\langle E_7[\pm \xi], \Gamma_{u_0} \rangle + \langle E_7[\pm \xi], \Gamma_{u_1} \rangle = 1.$$  

(This could also be deduced from Kawanaka’s explicit multiplicity formulas [12].) Thus, $E_7[\pm \xi]$ occurs with multiplicity 1 in $\Gamma_u$ for some $u \in C_0 \cap U_{d,2}^F$. Consequently, we have

$$\langle E_7[\pm \xi], \text{Ind}_{U_{d,2}^F}^{G^F}(\varphi_u) \rangle = \left[ U_{d,1}^F : U_{d,2}^F \right]^{1/2} =: m_0,$$

where the number $m_0$ on the right hand side is odd since we are in good characteristic. Let $K \supseteq \mathbb{Q}$ be as in Lemma 4.1. Then a standard result on Schur indices (see Isaacs [14], Lemma 10.4) shows that the Schur index of $E_7[\pm \xi]$ divides $[K : \mathbb{Q}] m_0$. 


However, as we have already remarked at the beginning of this section, the Schur index of $E_7[\pm\xi]$ must be 1, 2 or 4. Thus, since $m_0$ is odd, we can now conclude that the Schur index of $E_7[\pm\xi]$ divides $[K : \mathbb{Q}] \leq 2$. □

**Remark 4.4.** The Schur index of $E_7[\pm\xi]$ equals 1 or 2, according to whether

$$\left< E_7[\pm\xi], \text{Ind}_{E_{7}(q)}^{G^{\rho}}(\psi_1) \right>$$

is odd or even,

where $\psi_1$ is defined in the proof of Lemma 4.1. Thus, the problem is reduced to the computation of the above scalar product, and this will be discussed in [7].

**Acknowledgements.** I wish to thank the organizers of the “Finite Groups 2003” conference, C. Y. Ho, P. Sin, P. H. Tiep and A. Turull, for the invitation and the University of Gainesville for its hospitality.

**References**

[1] R. W. Carter, Simple groups of Lie type, Wiley, New York, 1972; reprinted 1989 as Wiley Classics Library Edition.

[2] R. W. Carter, Finite groups of Lie type: Conjugacy classes and complex characters, Wiley, New York, 1985; reprinted 1993 as Wiley Classics Library Edition.

[3] C. W. Curtis and I. Reiner, Methods of representation theory, vol. II, Wiley, New York, 1987; reprinted 1994 as Wiley Classics Library Edition.

[4] P. Deligne and G. Lusztig, Representations of reductive groups over finite fields, Ann. of Math. 103 (1976), 103–161.

[5] M. Geck, Verallgemeinerte Gelfand-Graev Charaktere und Zerlegungszahlen endlicher Gruppen vom Lie-Typ, Dissertation, RWTH Aachen, 1990.

[6] M. Geck, Character values, Schur indices and character sheaves, Represent. Theory 7 (2003), 19–55.

[7] M. Geck, The Schur indices of the cuspidal unipotent characters of the finite Chevalley groups $E_7(q)$, preprint (2003).

[8] M. Geck and G. Malle, On the existence of a unipotent support for the irreducible characters of finite groups of Lie type, Trans. Amer. Math. Soc. 352 (2000), 429–456.

[9] R. Gow, Schur indices of some groups of Lie type, J. Algebra 42 (1976), 102–120.

[10] D. HAZARD, Ph. D. thesis (in preparation), Université Lyon 1.

[11] I. M. Isaacs, Character Theory of Finite Groups, Academic Press, New York (1976); corrected reprint by Dover Publ., New York, 1994.

[12] N. Kawanaka, Generalized Gelfand-Graev representations of exceptional algebraic groups I, Invent. Math. 84 (1986), 575–616.

[13] G. Lusztig, Characters of reductive groups over a finite field, Annals Math. Studies, vol. 107, Princeton University Press, 1984.
[14] G. Lusztig, Character sheaves, IV, Advances in Math. 59 (1986), 1–63.
[15] G. Lusztig, A unipotent support for irreducible representations, Advances in Math. 94, 139–179 (1992).
[16] G. Lusztig, Remarks on computing irreducible characters, J. Amer. Math. Soc. 5, 971–986 (1992).
[17] G. Lusztig, Rationality properties of unipotent representations, J. Algebra 258 (2002), 1–22.
[18] G. Lusztig and N. Spaltenstein, On the generalized Springer correspondence for classical groups, Algebraic groups and related topics, Adv. Studies in Pure Math., vol. 6, Kinokuniya and North-Holland, Tokyo and Amsterdam, pp. 289–316.
[19] K. Mizuno, The conjugate classes of unipotent elements of the Chevalley groups $E_7$ and $E_8$, Tokyo J. Math. 3 (1980), 391–459.
[20] Z. Ohmori, On the Schur indices of certain irreducible characters of reductive groups over finite fields, Osaka J. Math 25 (1988), 149–159.
[21] Z. Ohmori, The Schur indices of the cuspidal unipotent characters of the finite unitary groups, Proc. Japan Acad. Ser. A Math. Sci. 72 (1996), 111–113.
[22] A. Premet, Nilpotent orbits in good characteristic and the Kempf–Rousseau theory, J. Algebra 260 (2003), 338–366.
[23] T. Shoji, Green functions of reductive groups over a finite field, Proc. Symp. Pure Math. 47 (1987), 289–302, Amer. Math. Soc.
[24] T. Shoji, Character sheaves and almost characters of reductive groups II, Advances in Math. 111 (1995), 314–354.

Meinolf Geck, Institut Girard Desargues, Université Lyon 1, 21 av Claude Bernard, 69622 Villeurbanne cedex, France
Email: geck@desargues.univ-lyon1.fr