EXPONENTS OF CLASS GROUPS OF CERTAIN IMAGINARY QUADRATIC FIELDS

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ABSTRACT. Let \( n > 1 \) be an odd integer. We prove that there are infinitely many imaginary quadratic fields of the form \( \mathbb{Q}(\sqrt{x^2 - 2y^n}) \) whose ideal class group has an element of order \( n \). This family gives a counter example to a conjecture by H. Wada [17] on the structure of ideal class groups.

1. Introduction

Let \( x, y \) and \( n \) be positive integers. We consider the family of imaginary quadratic fields

\[ K_{x,y,n,\mu} = \mathbb{Q}(\sqrt{x^2 - \mu y^n}) \]

with the conditions: \( \gcd(x, y) = 1 \), \( y > 1 \), \( \mu \in \{1, 2, 4\} \) and \( x^2 \leq \mu y^n \).

Let \( \mathcal{H}(K_{x,y,n,\mu}) \) and \( \mathcal{C}(K_{x,y,n,\mu}) \) be respectively denote the class number and (ideal) class group of \( K_{x,y,n,\mu} \). For \( \mu \in \{1, 4\} \), there are many results concerning the divisibility of \( \mathcal{H}(K_{x,y,n,\mu}) \).

In 1922, T. Nagell [13] proved that \( \mathcal{H}(K_{x,y,n,1}) \) is divisible by \( n \) if both \( n \) and \( y \) are odd, and \( \ell \mid x \), but \( \ell^2 \nmid x \) for all prime divisors \( \ell \) of \( n \). Let \( s \) be the square factor of \( x^2 - y^n \), that is

\[ x^2 - y^n = -s^2 D, \]

where \( D > 0 \) is the square-free part of \( x^2 - y^n \). For \( s = 1 \), N. C. Ankeny and S. Chowla [1] proved that \( \mathcal{H}(K_{x,3,n,1}) \) is divisible by \( n \) if both \( n \) and \( x \) are even, and \( x < (2 \times 3^{n-1})^{1/2} \). In 1998, M. R. Murty [11] considered the divisibility of \( \mathcal{H}(K_{1,y,n,1}) \) by \( n \), when \( s = 1 \) and \( n \geq 5 \) is odd. In the same paper, he further discussed this result when \( s < \frac{y^{n/2}}{2^{n/2}} \). He also discussed a more general case, that is the divisibility of \( \mathcal{H}(K_{x,y,n,1}) \) in [12]. K. Soundararajan [15] (resp. A. Ito [6]) studied the divisibility of \( \mathcal{H}(K_{x,y,n,1}) \) by \( n \) under the condition \( s < \sqrt{(y^n - x^2) / (y^{n/2} - 1)} \) (resp. each prime divisor of \( s \) divides \( D \) also). Furthermore, Y. Kishi [9] (resp.
A. Ito [6], and M. Zhu and T. Wang [18] studied the divisibility by \( n \) of \( H(K_{2^k,3,n,1}) \) (resp. \( H(K_{2^k,p,n,1}) \) with \( p \) odd prime and \( H(K_{2^k,y,n,1}) \) with \( y \) odd integer). Recently, K. Chakraborty et al. [2] discussed the divisibility by \( n \) of \( H(K_{p,q,n,1}) \) when both \( p \) and \( q \) are odd primes, and \( n \) is an odd integer.

On the other hand, B. H. Gross and D. E. Rohrlich [14] (res. J. H. E. Cohn [3], and K. Ishii [8]) proved the divisibility by \( n \) of \( H(K_{1,2,n,2}) \) except for \( n = 4 \), and \( H(K_{1,y,n,4}) \) for even \( n \). Further, S. R. Louboutin [10] proved that \( C(K_{1,y,n,4}) \) has an element of order \( n \) if at least one odd prime divisor of \( y \) is equal to 3 \((\text{mod} \ 4)\). Recently, A. Ito [7] discussed the divisibility of \( H(K_{3^e,y,n,4}) \) by \( n \) under certain conditions.

More recently A. Hoque and K. Chakraborty [4] proved that \( H(K_{1,y,3,2}) \) is divisible by 3 for any odd integer \( y \). In this paper, we show that \( C(K_{p,q,n,2}) \) has an element of order \( n \) when both \( p \) and \( q \) are odd primes, and \( n \) is an odd integer. Namely, we prove:

**Theorem 1.** Let \( p \) and \( q \) be distinct odd primes, and let \( n \geq 3 \) be an odd integer with \( p^2 < 2q^n \) and \( 2q^n - p^2 \neq \Box \). Assume that \( 3q^{n/3} \neq p + 2 \) whenever \( 3 \mid n \). Then \( C(K_{p,q,n,2}) \) has an element of order \( n \).

An immediate consequence of the above result is:

**Corollary 1.** Let \( p, q \) and \( n \) as in Theorem 1. Then there are infinitely many imaginary quadratic fields with discriminants of the form \( p^2 - 2q^n \) whose class number is divisible by \( n \).

The present family of imaginary quadratic fields provides a counter example of a conjecture (namely, Conjecture 2) given by H. Wada [17] in 1970.

2. Proof of Theorem 1

We begin the proof with the following crucial proposition.

**Proposition 1.** Let \( p, q \) and \( n \) as in Theorem 1, and let \( s \) be the positive integer such that

\[
p^2 - 2q^n = -s^2D,
\]

where \( D \) is a square-free positive integer. Then for \( \alpha = p + s\sqrt{-D} \), and for any prime divisor \( \ell \) of \( n \), \( 2\alpha \) is not an \( \ell \)-th power of an element in the ring of integers of \( K_{p,q,n,\mu} \).

**Proof.** Let \( \ell \) be a prime such that \( \ell \mid n \). Then \( \ell \) is odd since \( n \) is odd. From (2.1), we see that \( -D \equiv 3 \) \((\text{mod} \ 4)\), and thus if \( 2\alpha \) is an \( \ell \)-th
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power of an element in the ring of integers \( \mathcal{O}_{K_{p,q,n,2}} \) in \( K_{p,q,n,2} \), then we can write

\[
2\alpha = (u + v\sqrt{-D})^\ell
\]  

(2.2)

for some \( u, v \in \mathcal{O}_{K_{p,q,n,2}} \). Taking norm, we obtain

\[
8q^n = (u^2 + Dv^2)^\ell.
\]

(2.3)

This shows that \( \ell = 3 \) with \( 3 \mid n \). Thus (2.3) reduces to

\[
2q^m = u^2 + Dv^2,
\]

(2.4)

where \( n = 3m \) for some integer \( m > 0 \). Now (2.4) implies that \( u \) and \( v \) either both odd or both even since \( D \) is odd. If both \( u \) and \( v \) are even, then \( 2 \mid q \) which is a contradiction. It remains to treat the case when both \( u \) and \( v \) are odd. We compare the real and imaginary parts on both sides of (2.2), and get

\[
2p = u^3 - 3uv^2D
\]

(2.5)

and

\[
2s = 3u^2v - v^3D.
\]

(2.6)

We see that (2.5) implies \( u \mid 2p \). As \( u \) is an odd integer and \( p \) is odd prime, we must have \( u = \pm 1 \) or \( u = \pm p \).

If \( u = 1 \), then (2.5) would imply \( 2p = 1 - 3v^2D \), which is a contradiction. Similarly, if \( u = p \) then (2.5) gives

\[
2 = p^2 - 3v^2D.
\]

(2.7)

Reading (2.7) modulo 3, we see that \( p^2 \equiv 2 \pmod{3} \) which again leads to a contradiction.

Again when \( u = -1 \), the relations (2.5) and (2.4) give,

\[
3Dv^2 - 1 = 2p
\]

(2.8)

and

\[
Dv^2 + 1 = 2q^m.
\]

(2.9)

We now add (2.8) and (2.9), and that gives

\[
2Dv^2 = p + q^m.
\]

(2.10)

Now subtracting (2.9) from (2.8) which leads us to

\[
Dv^2 = 1 + p - q^m.
\]

(2.11)

In this case finally (2.10) and (2.11) together give \( 3q^m = p + 2 \). This contradicts the assumption.

We are now left with the case when \( u = -p \). In this case (2.4) and (2.5) become,

\[
Dv^2 + p^2 = 2q^m
\]

(2.12)
and

\[ 3Dv^2 - p^2 = 2. \]  \hspace{1cm} (2.13)

Reading (2.13) modulo \( D \), we have

\[ p^2 \equiv -2 \pmod{D}. \]  \hspace{1cm} (2.14)

Multiplying (2.12) by \( q^3 \) and then using (2.1), we see that

\[ Dv^2q^3 + p^2q^3 = Ds^2 + p^2. \]

Reading this modulo \( D \), we have (since \( p \nmid D \))

\[ q^3 \equiv 1 \pmod{D}. \]  \hspace{1cm} (2.15)

Applying (2.14) in (2.12), we see that

\[ q^m \equiv -1 \pmod{D}. \]  \hspace{1cm} (2.16)

(2.15) and (2.16) together imply \( D \mid 2 \) if \( m \) is a multiple of 3. This is a contradiction. Again if \( m = 1 \pmod{3} \) then applying (2.15) in (2.16), we arrive at

\[ q \equiv -1 \pmod{D}. \]

This contradicts to (2.15). Finally in case of \( m = 2 \pmod{3} \), (2.15) and (2.16) together imply

\[ q^2 \equiv -1 \pmod{D}, \]

which further implies (by (2.15))

\[ q \equiv -1 \pmod{D}. \]

Thus once again we arrive at a contradiction. This completes the proof. \( \Box \)

**Proof of Theorem 1**

Let \( s \) be the positive integer such that

\[ p^2 - 2q^n = -s^2D, \]

where \( D \) is a square-free positive integer. Let \( \alpha := p + s\sqrt{-D} \). Then

\[ N(\alpha) = \alpha\bar{\alpha} = 2q^n, \]

where \( \bar{\alpha} \) is the conjugate of \( \alpha \), and they are coprime.

We see that \( q \) splits completely in \( K_{p,q,n,2} = \mathbb{Q} (\sqrt{-D}) \), and thus we have,

\[ (q) = qq', \]

where \( q \) and its conjugate \( q' \) are prime ideals in \( O_{K_{p,q,n,2}} \) different from each other. Moreover, \( (2) = p^2 \) with \( p = (2, 1 + \sqrt{-D}) \).
We see that \((\alpha)\) is not divisible by any rational integer other than ±1, and thus we can consider the following decomposition of \((\alpha)\):

\[(\alpha) = pq^m.\]

Then \(N(\alpha) = 2q^m\) and hence \(n = m\).

We now put \(\mathfrak{A} = pq\). Then (since \(n\) is odd)

\[\mathfrak{A}^n = p^{n-1}(pq^n) = (2\alpha),\]

which is principal in \(\mathcal{O}_{K_{p,q,n,2}}\). Thus if \([\mathfrak{A}]\) denotes the ideal class containing \(\mathfrak{A}\), then by Proposition 1 we see that the order of \([\mathfrak{A}]\) is \(n\). This completes the proof.

3. Numerical Examples

Here, we provide some numerical values to corroborate Theorem 1. All the computations in this paper were done using PARI/GP (version 2.7.6) [16]. Table 1 gives the list of imaginary quadratic fields \(K_{p,q,n,2}\) corresponding to the distinct primes \(p\) and \(q\) not larger than 17, and odd integer \(3 \leq n \leq 19\). We see that absolute discriminants are not exceeding \(5 \times 10^{23}\), and the corresponding class numbers are very large which can go up to (about) \(5 \times 10^{11}\). It is noted that this list does not exhaust all the imaginary quadratic fields \(K_{p,q,n,2}\) of discriminants not exceeding \(5 \times 10^{23}\). In the Table 1 we use * mark in the column for class number to indicate the failure of the assumption “3q^n/3 \(\neq p + 2\)” of Theorem 1.

Table 1: Numerical examples of Theorem 1.

| \(n\) | \(p\) | \(q\) | \(p^2 - 2q^n\) | \(h(-D)\) | \(n\) | \(p\) | \(q\) | \(p^2 - 2q^n\) | \(h(-D)\) |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 3 | 5 | -241 | 12 | 3 | 3 | 7 | -677 | 30 |
| 3 | 3 | 11 | -2653 | 24 | 3 | 3 | 13 | -4385 | 96 |
| 3 | 3 | 17 | -9817 | 48 | 3 | 5 | 3 | -29 | 6 |
| 3 | 5 | 7 | -661 | 18 | 3 | 5 | 11 | -2637 | 36 |
| 3 | 5 | 13 | -4369 | 48 | 3 | 5 | 17 | -9801 | 72 |
| 3 | 7 | 3 | -5 | 2* | 3 | 7 | 5 | -201 | 12 |
| 3 | 7 | 11 | -2613 | 24 | 3 | 7 | 13 | -4345 | 48 |
| 3 | 7 | 17 | -9777 | 60 | 3 | 11 | 5 | -129 | 12 |
| 3 | 11 | 7 | -565 | 12 | 3 | 11 | 13 | -4273 | 24 |
| 3 | 11 | 17 | -9705 | 72 | 3 | 13 | 5 | -81 | 1* |
| 3 | 13 | 7 | -517 | 12 | 3 | 13 | 11 | -2493 | 24 |
| 3 | 13 | 17 | -9657 | 48 | 3 | 17 | 7 | -397 | 6 |
| 3 | 17 | 11 | -2373 | 24 | 3 | 17 | 13 | -4105 | 48 |
| 5 | 3 | 5 | -6241 | 40 | 5 | 3 | 7 | -3905 | 240 |
| 5 | 3 | 11 | -322093 | 150 | 5 | 3 | 13 | -742577 | 800 |
| 5 | 3 | 17 | -2839705 | 800 | 5 | 5 | 3 | -461 | 30 |
| 5 | 5 | 7 | -33589 | 150 | 5 | 5 | 11 | -322077 | 280 |
| 5 | 5 | 13 | -742561 | 500 | 5 | 5 | 17 | -2839069 | 1760 |
| 5 | 7 | 3 | -437 | 20 | 5 | 7 | 5 | -6201 | 80 |

Continued on next page
| \( n \) | \( p \) | \( q \) | \( p^2 - 2q^2 \) | \( h(-D) \) | \( n \) | \( p \) | \( q \) | \( p^2 - 2q^2 \) | \( h(-D) \) |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 7 | 11 | -322053 | 320 | 5 | 7 | 13 | -742537 | 380 |
| 5 | 7 | 17 | -2839065 | 1120 | 5 | 11 | 3 | -365 | 20 |
| 5 | 11 | 5 | -6129 | 60 | 5 | 11 | 7 | -33493 | 70 |
| 5 | 11 | 13 | -742465 | 480 | 5 | 11 | 17 | -283993 | 800 |
| 5 | 13 | 7 | -34445 | 80 | 5 | 13 | 11 | -321933 | 440 |
| 5 | 13 | 17 | -28394545 | 960 | 5 | 17 | 3 | -197 | 10 |
| 5 | 17 | 5 | -5961 | 60 | 5 | 17 | 7 | -3325 | 120 |
| 5 | 17 | 11 | -321913 | 240 | 5 | 17 | 13 | -742297 | 480 |
| 7 | 3 | 5 | -156241 | 168 | 7 | 3 | 7 | -1647677 | 1260 |
| 7 | 3 | 11 | -3897433 | 2926 | 7 | 3 | 13 | -125497025 | 11648 |
| 7 | 3 | 17 | -820677237 | 10724 | 7 | 5 | 3 | -365 | 20 |
| 7 | 5 | 7 | -6129 | 896 | 7 | 5 | 11 | -38974317 | 3268 |
| 7 | 5 | 13 | -125497009 | 5824 | 7 | 5 | 17 | -820677321 | 26432 |
| 7 | 7 | 3 | -38971293 | 3696 | 7 | 7 | 5 | -156201 | 308 |
| 7 | 7 | 11 | -321913 | 3606 | 7 | 7 | 13 | -742297 | 5768 |
| 7 | 13 | 3 | -156241 | 168 | 7 | 13 | 5 | -1647677 | 18144 |
| 7 | 13 | 7 | -6129 | 728 | 7 | 13 | 11 | -38974317 | 3360 |
| 7 | 13 | 17 | -820677177 | 1680 | 7 | 17 | 3 | -4085 | 56 |
| 7 | 17 | 5 | -155961 | 224 | 7 | 17 | 7 | -1646797 | 658 |
| 7 | 17 | 11 | -38974053 | 3780 | 7 | 17 | 13 | -125496745 | 7392 |
| 7 | 17 | 13 | -389740533 | 3780 | 7 | 17 | 17 | -1646797 | 7392 |
| 9 | 3 | 5 | -3906241 | 3606 | 9 | 3 | 7 | -80707205 | 11448 |
| 9 | 3 | 11 | -4715895373 | 29556 | 9 | 3 | 13 | -21208998737 | 162432 |
| 9 | 3 | 17 | -237175752985 | 337176 | 9 | 5 | 3 | -39341 | 198 |
| 9 | 5 | 7 | -80707189 | 7272 | 9 | 5 | 11 | -4715895357 | 67716 |
| 9 | 5 | 13 | -21208998721 | 62136 | 9 | 5 | 17 | -237175752969 | 349164 |
| 9 | 7 | 3 | -39317 | 162 | 9 | 7 | 5 | -3906201 | 2448 |
| 9 | 7 | 11 | -4715895333 | 28512 | 9 | 7 | 13 | -21208998607 | 57744 |
| 9 | 7 | 13 | -237175752945 | 463824 | 9 | 11 | 3 | -39245 | 288 |
| 9 | 11 | 5 | -3906129 | 1692 | 9 | 11 | 7 | -80707093 | 3852 |
| 9 | 11 | 13 | -21208998625 | 79200 | 9 | 11 | 17 | -237175752873 | 284256 |
| 9 | 13 | 3 | -39197 | 108 | 9 | 13 | 5 | -3906081 | 1512 |
| 9 | 13 | 7 | -39197 | 408 | 9 | 13 | 11 | -47158953213 | 33300 |
| 9 | 13 | 11 | -237175752825 | 228096 | 9 | 17 | 3 | -39077 | 140 |
| 9 | 17 | 5 | -3905961 | 1308 | 9 | 17 | 7 | -80707093 | 5184 |
| 9 | 17 | 11 | -4715895093 | 35712 | 9 | 17 | 13 | -21208998457 | 74376 |
| 11 | 3 | 5 | -97656241 | 3608 | 11 | 3 | 7 | -3954653477 | 46332 |
| 11 | 3 | 11 | -57062341213 | 286770 | 11 | 3 | 13 | -3584320788065 | 2956800 |
| 11 | 3 | 17 | -68543792615257 | 2056120 | 11 | 5 | 3 | -35429 | 704 |
| 11 | 5 | 7 | -3954653461 | 36432 | 11 | 5 | 11 | -57062341197 | 519200 |
| 11 | 5 | 13 | -3584320788049 | 875072 | 11 | 5 | 17 | -68543792615241 | 6392760 |
| 11 | 7 | 3 | -35425 | 528 | 11 | 7 | 5 | -97656201 | 6864 |
| 11 | 7 | 11 | -57062341173 | 340032 | 11 | 7 | 13 | -3584320788025 | 1146464 |
| 11 | 7 | 17 | -68543792615217 | 5876112 | 11 | 11 | 3 | -354173 | 528 |
| 11 | 11 | 5 | -97656129 | 8712 | 11 | 11 | 7 | -3954653365 | 39776 |
| 11 | 13 | 5 | -3584320787953 | 797720 | 11 | 11 | 17 | -68543792615145 | 3800544 |
| 11 | 13 | 11 | -35425 | 660 | 11 | 13 | 5 | -97656081 | 9944 |
| 11 | 13 | 17 | -3954653317 | 37268 | 11 | 13 | 11 | -57062341053 | 570240 |
| 11 | 13 | 17 | -68543792615097 | 4511232 | 11 | 17 | 3 | -354005 | 352 |
| 11 | 17 | 5 | -97655961 | 7216 | 11 | 17 | 7 | -3954653197 | 25872 |
| n  | p  | q  | \(p^* - 2q^*\) | \(h(-D)\) | n  | p  | q  | \(p^* - 2q^*\) | \(h(-D)\) |
|----|----|----|----------------|-----------|----|----|----|----------------|-----------|
| 11 | 17 | 11 | -570623409933 | 353760    | 11 | 17 | 13 | -3584320787885 | 1076416   |
| 13 | 3  | 5  | -2441406241    | 29432     | 13 | 3  | 7  | -1937780208055 | 435908    |
| 13 | 5  | 7  | -193778020789  | 35742     | 13 | 5  | 11 | -6004542428737 | 4188392   |
| 13 | 5  | 13 | -60575021318481 | 14981136  | 13 | 5  | 17 | -198991560658118497 | 89333920 |
| 7  | 3  | 5  | -3188477       | 1012      | 7  | 3  | 7  | -2441406201     | 35864     |
| 13 | 11 | 11 | -60045424287813 | 4855608  | 13 | 11 | 13 | -605750213184457 | 10482888  |
| 13 | 11 | 17 | -19899156065811825 | 104113152 | 13 | 11 | 17 | -198991560658116753 | 70607680  |
| 13 | 3  | 11 | -605750213184385 | 14582404  | 13 | 3  | 17 | -3188477        | 1716      |
| 13 | 5  | 13 | -605750213184481 | 353760    | 13 | 5  | 17 | -1937780208055 | 435908    |
| 13 | 7  | 3  | -9495123019381293 | 44413440  | 13 | 7  | 11 | -6904542287693  | 6    |
| 13 | 7  | 13 | -605750213184457 | 4435704   | 13 | 7  | 13 | -3188477        | 1716      |
| 13 | 7  | 17 | -19899156065811825 | 104113152 | 13 | 7  | 17 | -198991560658116753 | 70607680  |
| 13 | 11 | 3  | -9495123019381293 | 44413440  | 13 | 11 | 3  | -3188477        | 1716      |
| 13 | 11 | 5  | -605750213184481 | 353760    | 13 | 11 | 5  | -3188477        | 1716      |
| 13 | 11 | 7  | -6904542287693  | 670400    | 13 | 11 | 7  | -3188477        | 1716      |
| 13 | 13 | 3  | -9495123019381293 | 44413440  | 13 | 13 | 3  | -3188477        | 1716      |
| 17 | 3  | 5  | -1525878906241  | 133620    | 17 | 3  | 7  | -46526102797405 | 212000    |
| 17 | 3  | 7  | -2279779037042621 | 17489568  | 17 | 3  | 11 | -12231818086829902573 | 5849478624 |
| 17 | 5  | 3  | -23245252885    | 50388     | 17 | 5  | 7  | -46526102797405 | 212000    |
| 17 | 7  | 3  | -23245252885    | 50388     | 17 | 7  | 7  | -46526102797405 | 212000    |
| 17 | 11 | 3  | -23245252885    | 50388     | 17 | 11 | 3  | -23245252885    | 50388     |
| 17 | 11 | 5  | -1525878906096129 | 890792    | 17 | 11 | 5  | -1525878906096129 | 890792    |
| 17 | 13 | 3  | -23245252885    | 50388     | 17 | 13 | 3  | -23245252885    | 50388     |
| 17 | 13 | 5  | -1525878906096129 | 890792    | 17 | 13 | 5  | -1525878906096129 | 890792    |
| 17 | 17 | 3  | -23245252885    | 50388     | 17 | 17 | 3  | -23245252885    | 50388     |
| 17 | 17 | 5  | -1525878906096129 | 890792    | 17 | 17 | 5  | -1525878906096129 | 890792    |
| 17 | 17 | 7  | -23245252885    | 50388     | 17 | 17 | 7  | -23245252885    | 50388     |
| 17 | 17 | 11 | -12231818086829902573 | 4729953024 | 17 | 17 | 11 | -12231818086829902573 | 4729953024 |

Table 1 – continued from previous page
4. Concluding Remarks

We begin by observing that in Table 1 there are some values of $p$ and $q$ (see * mark) for which the class number of the corresponding imaginary quadratic fields are not divisible by a given odd integer $n \geq 3$. These are because of the failure of the assumption “$3q^{n/3} \neq p + 2$ when $3 \mid n$”. However the class number of $K_{19,7,3,2}$ is 12 that satisfies the divisibility property even though this assumption does not hold. Thus this assumption is neither necessary not sufficient. We have found only two pairs of values of $p$ and $q$ for which this assumption does not hold. Thus it may be possible to drop this assumption by adding some exceptions for the values of the pair $(p, q)$.

In the light of the numerical evidence we are tempted to state the following conjecture:

Conjecture 1. Let $p$ and $q$ be two distinct odd primes. For each odd integer $n$ and each positive integer $m$ such that $m$ is not a $n$-root of any rational integer, there are infinitely many imaginary quadratic fields of the form $\mathbb{Q}(\sqrt{p^2 - mq^n})$ whose class number is divisible by $n$.

For $m = 1, 4$, this conjecture is true (see [2] and references therein). Further more Corollary 1 concludes that the conjecture is true for any odd integer $n$ when $m = 2$.

Finally, we demonstrate the prime parts of the class groups of $K_{p,q,n,2}$ in Table 2. The class group of a number field can be expressed, by the structure theorem of abelian groups, as the direct product of cyclic groups of orders $h_1, h_2, \ldots, h_t$. We denote the direct product $C_{h_1} \times C_{h_2} \times \cdots \times C_{h_t}$ of cyclic groups by $[h_1, h_2, \ldots, h_t]$. By Gauss’s genus theory, if there are $r$ number of distinct rational primes that ramifies in $\mathbb{Q}(\sqrt{-D})$, for some square-free integer $D > 1$, then the 2-rank of its class group is $r - 1$. In other words, the 2-Sylow subgroup of its class group has rank $r - 1$. It is noted that the 2-Sylow subgroup of class group tends to $r - 2$ elementary 2-groups and one large cyclic factor.
collecting the other powers of 2 in the class number so that the 2-Sylow subgroup of the subgroup of squares is cyclic. On the other hand, \( r - 1 \) number of even integers are there among \( h_1, h_2, \ldots, h_t \). Sometimes, the structure of the class group of \( \mathbb{Q}(\sqrt{-D}) \) can be trivially determined by \( r \) and the class number, \( h = h_1 h_2 \cdots h_t \). In this case, the group is cyclic when \( r = 1 \) or \( r = 2 \) or it is of the type \((h_1, h_2, 2^{r_1}, \ldots, 2^{r_k})\) when \( r \geq 3 \). In this aspect, H. Wada [17] stated the following conjecture in 1970.

**Conjecture 2** (Wada [17]). *All the class groups of imaginary quadratic fields are either cyclic or of the type \((h_1, h_2, 2^{r_1}, \ldots, 2^{r_k})\).*

In Table 2, we find a class group of the type \((h_1, h_2, 2^{r_1}, 2^{r_2}, \ldots, 2^{r_k})\) (see ** mark) which is not cyclic. This a counter example to Conjecture 2. We demonstrate the structures of class groups of \( K_{p,q,n,2} \) for some values of \( p, q \) and \( n \). In Table 2 by \((h_1, h_2, \cdots, h_t)\) we mean the group \( \mathbb{Z}_{h_1} \times \mathbb{Z}_{h_2} \times \cdots \times \mathbb{Z}_{h_t} \).

| \( p^2 - 2q^n \) | Structure of \( \mathcal{O}(K_{p,q,n,2}) \) | 2-parts | 3-parts | 5-parts | Remaining parts |
|-----------------|---------------------------------|--------|--------|--------|----------------|
| \( 11^2 - 2 \times 17^5 \) | 20, 10, 2, 2 \( \star \) | (2,2,4) | – | (5,5) | – |
| \( 7^2 - 2 \times 17^1 \) | 1084512, 6, 2, 2, 2, 2 \( \star \) | (2,2,2,2,32) | (3) | – | (11,13,79) |
| \( 13^2 - 2 \times 11^3 \) | 779550, 30, 3 \( \star \) | (2) | (3,3,3) | (5,25) | (23,139) |
| \( 13^2 - 2 \times 17^5 \) | 10105440, 12, 2, 2, 2 \( \star \) | (2,2,4,32) | (3,3) | (5) | (37,569) |
| \( 3^2 - 2 \times 5^4 \) | 565992, 6, 2, 2, 2 \( \star \) | (2,2,2,2) | (3,9) | – | (7,1123) |
| \( 3^2 - 2 \times 13^4 \) | 7991268142, 6, 2, 2, 2 \( \star \) | (2,2,2,2,16) | (3,3) | – | (7,2378357) |
| \( 17^2 - 2 \times 11^{21} \) | 286454952, 12, 2, 2, 2 \( \star \) | (2,2,2,2,4) | (3,9) | – | (7,568363) |
| \( 11^2 - 2 \times 7^{29} \) | 292374800, 10, 2, 2, 2 \( \star \) | (2,2,2,16) | – | (5,25) | (101,7237) |
| \( 13^2 - 2 \times 17^{25} \) | 606345093225000, 6, 2, 2, 2 \( \star \) | (2,2,2,2,8) | (3,3) | (3125) | (41,59,332617) |
| \( 5^2 - 2 \times 11^{27} \) | 381006021618, 6, 6, 2, 2, 2 \( \star \) | (2,2,2,2,2) | (3,3,81) | – | (11,211,92119) |
| \( 5^2 - 2 \times 13^{27} \) | 939278579820, 6, 2, 2, 2, 2 \( \star \) | (2,2,2,2,4) | (3,27) | (5) | (59,9011,32717) |
| \( 5^2 - 2 \times 17^{27} \) | 6275528336021332, 6, 2, 2, 2, 2 \( \star \) | (2,2,2,2,4) | (3,81) | – | (183167,10571179) |
| \( 17^2 - 2 \times 13^{24} \) | 875145176912, 6, 2, 2, 2, 2 \( \star \) | (2,2,2,2,8) | (3,27) | – | (12251,67493) |
| \( 17^2 - 2 \times 11^{29} \) | 12592335622520, 2, 2, 2, 2, 2 \( \star \) | (2,2,2,2,2,8) | (27) | (5) | (29,7411,54251) |

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