PERIODIC ORBITS OF HAMILTONIAN SYSTEMS LINEAR AND HYPERBOLIC AT INFINITY

BAŞAK Z. GÜREL

Abstract. We consider Hamiltonian diffeomorphisms of symplectic Euclidean spaces, generated by compactly supported time-dependent perturbations of hyperbolic quadratic forms. We prove that, under some natural assumptions, such a diffeomorphism must have simple periodic orbits of arbitrarily large period when it has fixed points which are not necessary from a homological perspective.

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1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. In this paper we consider time-dependent Hamiltonians $H$ on $\mathbb{R}^{2n}$ which, outside a compact set, are autonomous and coincide with a hyperbolic quadratic form (i.e., a non-degenerate quadratic form whose Hamiltonian vector field has no purely imaginary eigenvalues). We prove that, under some additional conditions, the Hamiltonian diffeomorphism $\varphi_H$ must have simple (i.e., uniterated) periodic orbits of arbitrarily large (prime) period when it has certain “homologically unnecessary” fixed points. In particular, $\varphi_H$ then has infinitely many periodic orbits. To be more precise, this result holds provided that $\varphi_H$ has at least one non-degenerate (or even homologically non-trivial) fixed point with non-zero

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mean index, and the quadratic form (i.e., the corresponding linear Hamiltonian vector field) has only real eigenvalues. (See Remark 3.3 for the case of complex eigenvalues.)

Our main motivation for studying this question comes from a variant of the Conley conjecture, applicable to manifolds for which the standard Conley conjecture fails. Recall in this connection that the latter asserts the existence of infinitely many periodic orbits for every Hamiltonian diffeomorphism of a closed symplectic manifold. This is the case for manifolds with spherically-vanishing first Chern class (of the tangent bundle) and also for negative monotone manifolds; see [CGG, GG1, He] and also [FH, Gi2, GG4, Hi, LeC, SZ]. However, the Conley conjecture, as stated, fails for some simple manifolds such as $S^2$: an irrational rotation of $S^2$ about the $z$-axis has only two periodic orbits, which are also the fixed points; these are the poles. In fact, any manifold that admits a Hamiltonian torus action with isolated fixed points also admits a Hamiltonian diffeomorphism with finitely many periodic orbits. In particular, $\mathbb{C}P^n$, the Grassmannians, and, more generally, most of the coadjoint orbits of compact Lie groups as well as symplectic toric manifolds all admit Hamiltonian diffeomorphisms with finitely many periodic orbits.

A viable alternative to the Conley conjecture for such manifolds is the conjecture that a Hamiltonian diffeomorphism with more fixed points than necessarily required by the (weak) Arnold conjecture has infinitely many periodic orbits. (It is possible that in this conjecture one might need to impose some kind of non-degeneracy condition (e.g., homological non-triviality) on the fixed points, as is the case for the version considered in this paper.) For $\mathbb{C}P^n$, the expected threshold is $n + 1$. This conjecture is inspired by a celebrated theorem of Franks stating that a Hamiltonian diffeomorphism (or, even, an area preserving homeomorphism) of $S^2$ with at least three fixed points must have infinitely many periodic orbits, [Fr1, Fr2]; see also [FH, LeC] for further refinements and [BH, CKRTZ, Ke] for symplectic topological proofs. We will refer to this analogue of the Conley conjecture as the HZ-conjecture, for, to the best of our knowledge, the first written account of the assertion is in [HZ, p. 263].

We find it useful to view the HZ-conjecture in a broader context. Namely, it appears that the presence of a fixed point that is unnecessary from a homological or geometrical perspective is already sufficient to force the existence of infinitely many periodic orbits. For instance, a theorem from [GG5] asserts that, for a certain class of closed monotone symplectic manifolds including $\mathbb{C}P^n$, any Hamiltonian diffeomorphism with a hyperbolic fixed point must necessarily have infinitely many periodic orbits. (Note that the original HZ-conjecture, at least for non-degenerate Hamiltonian diffeomorphisms of $\mathbb{C}P^n$, would follow if one could replace a hyperbolic fixed point with a non-elliptic one in this theorem.) Furthermore, there are obvious analogues of the HZ-conjecture for symplectomorphisms or non-contractible periodic orbits of Hamiltonian diffeomorphisms. These analogues are also of interest and in some instances more accessible than the original HZ-conjecture; see, e.g., [Ba, GG2, GG5, Gii].

The generalized HZ-conjecture is also the central theme of this paper, although here we focus on a different aspect of the problem. Our main result, Theorem 1.1, can be viewed as a “local version” of this conjecture, and it holds in all dimensions. Namely, we prove a variant of the HZ-conjecture for Hamiltonians on $\mathbb{R}^{2n}$ which are compactly supported perturbations of certain quadratic forms. Working with $\mathbb{R}^{2n}$
allows us to circumvent a number of symplectic topological obstacles to proving the HZ-conjecture and concentrate on what we interpret as the dynamical part of the problem, which is still quite non-trivial. This is a key difference, technical and conceptual, between the present work and the approach taken in [GG5] where the symplectic topology of the ambient manifold plays a central role. We use Floer theoretical techniques in the proofs. Deferring a more detailed discussion of our method to Section 1.2, we merely mention at this point that, for technical reasons, the quadratic form needs to be hyperbolic. Finally, it should also be noted that Hamiltonian systems on \( \mathbb{R}^{2n} \) with a controlled (e.g., asymptotically linear) behavior at infinity have been extensively studied in the context of Hamiltonian mechanics by classical variational methods; see, e.g., [Ab, AZ, An, Co, MW, Ra, ZL, Zo] and references therein. However, to the best of our knowledge, there is no overlap between that approach and the present work, including the results.

1.2. Main results. To state the main results of the paper, recall that the mean index \( \Delta_H(x) \in \mathbb{R} \) of a periodic orbit \( x \) of the Hamiltonian flow of \( H \) measures, roughly speaking, the total angle swept out by certain eigenvalues with absolute value one of the linearized flow \( d\varphi^t_H \) along \( x \); see [Lo, SZ] and also [EP, Section 3.3] and references therein for a more detailed discussion. For instance, the mean index is zero when \( d\varphi^t_H \) has no eigenvalues on the unit circle for any \( t \neq 0 \), and hence the orbit is hyperbolic. Finally, denote by \( \text{Fix}(\varphi_H) \) the collection of fixed points of \( \varphi_H \).

**Theorem 1.1.** Let \( H: S^1 \times \mathbb{R}^{2n} \to \mathbb{R} \) be a Hamiltonian which is equal to a hyperbolic quadratic form \( Q \) at infinity (i.e., outside a compact set) such that \( Q \) has only real eigenvalues. Assume that \( \varphi_H \) has a non-degenerate fixed point with non-zero mean index and \( \text{Fix}(\varphi_H) \) is finite. Then \( \varphi_H \) has simple, i.e., uniterated, periodic orbits of arbitrarily large period.

As a consequence, \( \varphi_H \) has infinitely many simple periodic orbits regardless of whether \( \text{Fix}(\varphi_H) \) is finite or not. In fact, the non-degeneracy condition in Theorem 1.1 can be relaxed and replaced by a much weaker, albeit more technical, condition that the point is isolated and homologically non-trivial, i.e., its local Floer homology is non-zero. This is Theorem 4.1.

**Remark 1.2.** This theorem and Theorem 1.4 below as well as their generalizations discussed in Section 4, also hold when the quadratic form \( Q \) has complex eigenvalues \( \sigma \), provided that \( |\text{Re } \sigma| > |\text{Im } \sigma| \); see Remark 3.3.

**Remark 1.3.** Viewing Theorem 1.1 from the perspective of the generalized HZ-conjecture, observe that the non-degenerate (or homologically non-trivial) fixed point with non-zero mean index is the “unnecessary” point. Furthermore, the presence of one such point \( x \) implies the existence of at least two other (homologically non-trivial) orbits. Indeed, the Floer homology for all iterations of \( H \) is concentrated in degree zero (see Section 3.1), and once \( k \) is so large that the index of the iterated orbit \( x^k \) is outside the range \([-n, n]\), another orbit must take over generating the homology. Furthermore, there should be at least one more periodic orbit to cancel out the contribution of \( x^k \) to the homology in higher degrees.

Hypothetically, results similar to Theorem 1.1 and other theorems discussed in this section hold when a hyperbolic quadratic form is replaced by any (autonomous) quadratic form \( Q \) without non-trivial periodic orbits. For instance, in this case,
one can expect to have infinitely many periodic orbits whenever $\varphi_H$ has a non-degenerate fixed point with mean index different from $\Delta_Q(0)$ or has at least two non-degenerate fixed points; cf. Remark 4.6. (The latter conjecture, which was the starting point of this work, is due to Alberto Abbondandolo.)

As has been pointed out above, the proof of Theorem 1.1 is based on Floer theory. However, for a general quadratic form $Q$, even when the Floer homology exists, continuation maps fail to have the desired properties and the homology is not invariant under iterations. This is the case, for instance, for positive or negative definite $Q$ (see Remark 3.6) and the main reason why we restrict our attention to hyperbolic quadratic forms. Even for such forms some foundational aspects of Floer theory have to be reexamined. We do this in Section 3, using, as one could expect, a version of the maximum principle.

The condition that the fixed point is non-degenerate (and that it has non-zero mean index) is essential in Theorem 1.1. For instance, starting with the flow of $Q(x,y) = xy$ on $\mathbb{R}^2$, it is easy to introduce degenerate (homologically trivial) fixed points by slightly perturbing the flow away from the saddle. This way one can create an arbitrarily large number of fixed points without generating infinitely many periodic orbits. In fact, we expect some form of non-degeneracy (e.g., homological non-triviality) to be essential in the HZ-conjecture beyond the case of $S^2$.

In low dimensions, Theorem 1.1 combined with simple index analysis implies the HZ-conjecture in its original form for Hamiltonians in question. To state the result, recall first that $\varphi_H$ is said to be strongly non-degenerate if all iterations of $\varphi_H$ are non-degenerate.

**Theorem 1.4.** Let $H: S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$, with $2n = 2$ or 4, be a Hamiltonian which is equal to a hyperbolic quadratic form $Q$ at infinity such that $Q$ has only real eigenvalues. Assume that $\varphi_H$ is strongly non-degenerate and has at least two fixed points, and $\text{Fix}(\varphi_H)$ is finite. Then $\varphi_H$ has simple periodic orbits of arbitrarily large period.

Note that strong non-degeneracy is a $C^\infty$-generic condition in the class of Hamiltonians under consideration. Let us also point out that, in contrast with many closed manifolds (see, e.g., [GG2]), the existence of infinitely many periodic orbits is obviously not a $C^\infty$- or even $C^2$-generic property of Hamiltonians in Theorem 1.4: one has to have an extra periodic orbit which serves as a seed eventually “spawning an infinitude of off-springs”.

In dimension two, the strong non-degeneracy requirement can be relaxed. It suffices to just assume that $\varphi_H$ has at least two isolated homologically non-trivial fixed points; see Theorem 4.5. (Also, note that in this case the eigenvalues of a hyperbolic quadratic form are automatically real.) However, in dimension four, non-degeneracy enters the proof in a crucial way. Finally, note that the two-dimensional case of Theorem 1.4 is intimately related to the Franks’ theorem; see Remarks 4.6 and 4.7.

**Remark 1.5.** A more general version of Theorem 1.4 for $2n = 2$ was proved in [Ab, Theorem 5.1.9].

1.3. **Organization of the paper.** In Section 2, we set conventions and notation, and briefly recall some of the tools used in the paper and provide relevant references. We establish a version of the maximum principle and show that the Floer homology
as well as the relevant continuation maps are defined for the class of Hamiltonians
in question in Section 3. Finally, in Section 4, we prove Theorems 1.1 and 1.4.

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2. Conventions and notation

Throughout the paper, we will be working with the symplectic manifold \((\mathbb{R}^{2n}, \omega)\),
where \(\omega\) is the standard symplectic form. All Hamiltonians \(H\) considered here are
assumed to be one-periodic in time, i.e., \(H : S^1 \times \mathbb{R}^{2n} \to \mathbb{R}\), and we set \(H_t = H(t, \cdot)\)
for \(t \in S^1 = \mathbb{R}/\mathbb{Z}\). The Hamiltonian vector field \(X_H\) of \(H\) is defined by \(i_{X_H} \omega = -dH\).
The (time-dependent) flow of \(X_H\) is denoted by \(\varphi_H^t\) and its time-one map
by \(\varphi_H\). Such time-one maps are referred to as Hamiltonian diffeomorphisms.
The action of a one-periodic Hamiltonian \(H\) on a loop \(\gamma : S^1 \to \mathbb{R}^{2n}\) is defined by
\[
A_H(\gamma) = -\int z \omega + \int_{S^1} H_t(\gamma(t)) \, dt,
\]
where \(z : D^2 \to M\) is such that \(z|_{S^1} = \gamma\). The least action principle asserts that
the critical points of \(A_H\) on the space of all smooth maps \(\gamma : S^1 \to \mathbb{R}^{2n}\) are exactly
the one-periodic orbits of \(\varphi_H\).

Let \(K\) and \(H\) be two one-periodic Hamiltonians. The “composition” \(K \sharp H\) is
defined by the formula
\[
(K \sharp H)_t = K_t + H_t \circ (\varphi_K^t)^{-1}
\]
and the flow of \(K \sharp H\) is \(\varphi_K^t \circ \varphi_H^t\). We set \(H^{\sharp k} = H^k\sharp \cdots \sharp H\) (\(k\) times).
Abusing terminology, we will refer to \(H^{\sharp k}\) as the \(k\)th iteration of \(H\). Clearly, \(H^{\sharp k} = kH\)
when \(H\) is autonomous. (Note that the flow \(\varphi_{H^{\sharp k}}^t = (\varphi_H^t)^k; t \in [0, 1]\), is homotopic
with fixed end-points to the flow \(\varphi_H^t, t \in [0, k]\). Also, in general, \(H^{\sharp k}\) is not
one-periodic, even when \(H\) is.) Furthermore, setting
\[
\|F\|_B = \int_{S^1} \sup_B |F| \, dt
\]
for a bounded set \(B \subset \mathbb{R}^{2n}\), we have \(\|H^{\sharp k}\|_B = k\|H\|_B\) when \(H\) is autonomous.
Note that \(\|F\|_B\) is a variant of the Hofer norm. (When \(F\) is compactly supported
on \(\mathbb{R}^{2n}\), we will also use the notation \(\|F\|_{L^2}\) with the obvious meaning.)

The \(k\)th iteration of a one-periodic orbit \(\gamma\) of \(H\) will be denoted by \(\gamma^k\). More
specifically, \(\gamma^k(t) = \varphi_{H^{\sharp k}}^t(\gamma(0))\), where \(t \in [0, 1]\). We can think of \(\gamma^k\) as the \(k\)-
periodic orbit \(\gamma(t), t \in [0, k]\) of \(H\). Hence, there is an action-preserving one-to-one
correspondence between one-periodic orbits of \(H^{\sharp k}\) and \(k\)-periodic orbits of \(H\).

The action spectrum \(S(H)\) of \(H\) is the set of critical values of \(A_H\). This is a
zero measure, closed (hence nowhere dense) set; see, e.g., [HZ]. Clearly, the action
functional is homogeneous with respect to iteration:
\[
A_{H^{\sharp k}}(\gamma^k) = kA_H(\gamma).
\]

A periodic orbit \(\gamma\) of \(H\) is said to be non-degenerate if the linearized return
map \(d\varphi_H : T_{\gamma(0)}M \to T_{\gamma(t)}M\) has no eigenvalues equal to one. A Hamiltonian is
called non-degenerate if all its one-periodic orbits are non-degenerate and strongly
non-degenerate if all \(k\)-periodic orbits (for all \(k\)) are non-degenerate.
Let $\gamma$ be a non-degenerate periodic orbit. The Conley–Zehnder index $\mu_{cz}(H, \gamma) \in \mathbb{Z}$ is defined, up to a sign, as in [Sa, SZ]. (When $H$ is clear from the context we use the notation $\mu_{cz}(\gamma).$) More specifically, in this paper, the Conley–Zehnder index is the negative of that in [Sa]. In other words, we normalize $\mu_{cz}$ so that $\mu_{cz}(\gamma) = n$ when $\gamma$ is a non-degenerate maximum of an autonomous Hamiltonian with small Hessian. Furthermore, recall that the mean index $\Delta_H(\gamma)$ is defined regardless of whether $\gamma$ is degenerate or not, and $\Delta_H(\gamma)$ depends continuously on $H$ and $\gamma$ in the obvious sense. When $\gamma$ is non-degenerate, we have

$$0 \leq |\Delta_H(\gamma) - \mu_{cz}(H, \gamma)| < n.$$

Furthermore, the mean index is also homogeneous with respect to iteration:

$$\Delta_{H^k}(\gamma^k) = k\Delta_H(\gamma).$$

3. Maximum principle and Floer homology

Our goal in this section is to show that the Floer homology is defined and has the standard properties for the class of Hamiltonians in question. In our setting, essentially the only issue to deal with is the compactness of moduli spaces of Floer trajectories, which we establish by proving a version of the maximum principle.

3.1. Floer homology. Let $Q$ be a hyperbolic quadratic form on $\mathbb{R}^{2n}$, i.e., $Q$ is non-degenerate and has no eigenvalues on $i\mathbb{R}$. (Recall that throughout the paper by eigenvalues of $Q$ we mean the eigenvalues of the linear Hamiltonian vector field $X_Q$.) Assume further that all eigenvalues of $Q$ are real. (See Remark 3.3 for a variant of the maximum principle when $Q$ has complex eigenvalues.) Denote by $\mathcal{H}_Q$ the set of one-periodic Hamiltonians $H : S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$ which are compactly supported time-dependent perturbations of $Q$. Let $J = J_\epsilon$ be a time-dependent almost complex structure compatible with $\omega$. We are interested in solutions $u : \mathbb{R} \times S^1 \to \mathbb{R}^{2n}$ of the Floer equation

$$\partial_s u + J(u)\partial_t u = -\nabla_H(u), \quad \text{(3.1)}$$

where $u = u(s, t)$ with coordinates $(s, t)$ on $\mathbb{R} \times S^1$ and the gradient is taken with respect to the one-periodic in time metric $\langle \cdot, \cdot \rangle = \omega(\cdot, J_\epsilon \cdot)$ on $\mathbb{R}^{2n}$.

In this setting we have:

**Theorem 3.1.** Let $Q$ be a hyperbolic quadratic form on $(\mathbb{R}^{2n}, \omega)$ with only real eigenvalues. Then there exists a linear complex structure $J_Q$ compatible with $\omega$ such that whenever $J \equiv J_Q$ and $H \equiv Q$ outside an open ball $B$ with respect to the metric $\langle \cdot, \cdot \rangle_Q := \omega(\cdot, J_Q \cdot)$, any solution of (3.1) for the pair $(H, J)$ that is asymptotic to periodic orbits of $H$ in $B$ is necessarily contained in $B$.

More generally, consider now solutions $u : \Omega \to \mathbb{R}^{2n}$ of (3.1), where $\Omega \subset \mathbb{R} \times S^1$ is an open connected subset. Theorem 3.1 is an immediate consequence of the following proposition which we will refer to as the maximum principle.

**Proposition 3.2** (Maximum Principle). Let $Q$ be a hyperbolic quadratic form on $(\mathbb{R}^{2n}, \omega)$ with only real eigenvalues. Then there exists a linear complex structure $J_Q$ compatible with $\omega$ such that for any solution $u$ (with domain $\Omega \subset \mathbb{R} \times S^1$) of the Floer equation (3.1) for $(Q, J_Q)$, the function $\rho = \|u\|^2/2$, where the norm is induced by the metric $\langle \cdot, \cdot \rangle = \omega(\cdot, J_Q \cdot)$, cannot attain a maximum at an interior point of $\Omega$ unless $\rho$ is constant.
**Proof of Proposition 3.2.** Below we first introduce $J_Q$ and then prove that $\rho$ is subharmonic on $\Omega$, i.e., $\Delta \rho \geq 0$, where the Laplacian is taken with respect to metric $\omega(\cdot, J_Q)$.

Since we will be changing the basis and the inner product on the ambient space throughout the proof, it is more convenient to work with a hyperbolic quadratic form $Q$ on a finite dimensional symplectic vector space $(V^{2n}, \omega)$. Equip $V$ with a symplectic basis $(\partial_{p_1}, \partial_{q_1}, \cdots, \partial_{p_n}, \partial_{q_n})$ such that in the corresponding coordinates $(p, q)$ on $\mathbb{R}^{2n}$, the quadratic form $Q$ is expressed as

$$Q(p, q) = \langle Ap, q \rangle.$$  \hfill (3.2)

Here $A$ is a non-degenerate lower triangular $n \times n$ matrix, and $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$. Indeed, since $Q$ is non-degenerate with only real eigenvalues, it can be expressed in some symplectic basis $(\partial_p, \partial_q)$, as the direct sum of the following normal forms

$$\sigma \sum_{i=1}^m p_i q_i - \sum_{i=1}^{m-1} p_i q_{i+1},$$

where $\sigma$ ranges over the positive eigenvalues of $Q$ and $m$ is the multiplicity of $\sigma$; see [Ar, Wi]. We emphasize that $p$‘s and $q$‘s are treated here as vectors in $\mathbb{R}^n$ using the bases $(\partial_p)$ and $(\partial_q)$, respectively. (It is clear from this formula that $A$ is indeed lower triangular.) Note that with this choice all diagonal entries of $A$, i.e., the eigenvalues of $A$, are positive.

Let $A = D + E$ where $D$ is the diagonal part of $A$ and $E$ is the strictly lower triangular part. By rescaling the basis vectors $(\partial_p, \partial_q)$, while still keeping the basis symplectic and keeping (3.2), we can make $E$ arbitrarily small. (We will specify shortly how small $E$ has to be. Here we merely note that the rescaling does not affect $D$ and that, in fact, $E$ is required to be small compared to $D$.) We keep the notation $(\partial_p, \partial_q)$ for the new basis and $(p, q)$ for the resulting linear coordinates.

The complex structure $J_Q$ is defined by the requirement $J_Q \partial_p = -\partial_q$. This structure is compatible with $\omega$, and we denote by $\langle \cdot, \cdot \rangle_Q$ the resulting inner product $\omega(\cdot, J_Q \cdot)$ on $V$, i.e., $\langle \cdot, \cdot \rangle_Q := \omega(\cdot, J_Q \cdot)$. From now on we identify $(V, \omega)$ with the standard symplectic $\mathbb{R}^{2n}$ using the basis $(\partial_p, \partial_q)$. Under this identification, $J_Q$ becomes the standard complex structure on $\mathbb{C}^n$, and $\langle \cdot, \cdot \rangle_Q$ turns into the standard inner product. Note also that the restriction of $\langle \cdot, \cdot \rangle_Q$ to the subspaces generated by $\partial_p$ and $\partial_q$, respectively, is the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$. Finally, we emphasize that all these structures, except for $\omega$, depend on the choice of the basis $(\partial_p, \partial_q)$ which is to be finalized below (after we state how small $E$ needs to be).

In what follows, we calculate the Laplacian with respect to the metric $\langle \cdot, \cdot \rangle_Q$, where we set $u(s, t) = (p, q)$ and use the Floer equation (3.1):

$$\Delta \rho = \rho_{ss} + \rho_{tt}$$

$$= \|u_s\|^2 + \|u_t\|^2 - \langle u_s, \partial_s \nabla Q(u) \rangle_Q + \langle u_t, J_Q \partial_t \nabla Q(u) \rangle_Q$$

$$= \|u_s\|^2 + \|u_t\|^2 + \langle A^2 p, p \rangle + \langle A^2 q, q \rangle$$

$$+ \langle p, (A - A^T) q_s \rangle - \langle q, (A - A^T) p_s \rangle$$

$$= \|u_s\|^2 + \|u_t\|^2 + \|Dp\|^2 + \|Dq\|^2 + \langle E^2 p, p \rangle + \langle E^2 q, q \rangle$$

$$+ \langle (DE + ED) p, p \rangle + \langle (DE + ED) q, q \rangle$$

$$+ \langle p, (E - E^T) q_s \rangle - \langle q, (E - E^T) p_s \rangle.$$  \hfill (3.3)
Next, we specify the requirements on $E$. To this end, let $\lambda := \min \lambda_i > 0$, where $\lambda_i$’s are the eigenvalues of $A$ (or $D$). Then we have

$$\|Dx\| \geq \lambda^2 \|x\|^2 \text{ for any } x \in \mathbb{R}^n. \quad (3.4)$$

Now, $E$ is required to be so small that (i), (ii) and (iii) below hold:

(i) $|\langle E^2 x, x \rangle| \leq \frac{\lambda^2}{10} \|x\|^2$ for any $x \in \mathbb{R}^n$,

(ii) $|\langle DE x, x \rangle| \leq \frac{\lambda^2}{20} \|x\|^2$ and $|\langle ED x, x \rangle| \leq \frac{\lambda^2}{20} \|x\|^2$ for any $x \in \mathbb{R}^n$,

(iii) $|\langle x, (E - E^T) y \rangle| \leq \frac{\lambda}{8} \|x\| \|y\|$ for any $x$ and $y \in \mathbb{R}^n$.

Using (3.4), and (i) and (ii) for $x = p$ and $x = q$, and (iii) for $(x, y) = (p, q)$ and $(x, y) = (q, p)$ in (3.3), it is straightforward to show that

$$\Delta \rho \geq \frac{3 \lambda^2}{10} \|u\|^2 + \|u_s\|^2 + \frac{\lambda^2}{2} \|u\|^2 - \frac{\lambda}{4} \|u\| \|u_s\|$$

$$\geq \frac{3 \lambda^2}{10} \|u\|^2 \left( \|u_s\| - \frac{\lambda}{\sqrt{2} \|u\|} \right)^2$$

$$\geq \frac{3 \lambda^2}{10} \|u\|^2 \geq 0. \quad (3.5)$$

Remark 3.3. It is not hard to see that Proposition 3.2 still holds when the quadratic form $Q$ has complex eigenvalues $\sigma$, provided that $|\text{Re } \sigma| > |\text{Im } \sigma|$ or, equivalently, $\text{Re } \sigma^2 > 0$ for all eigenvalues. However, in general, without this assumption (or when $Q$ is elliptic, but not positive definite), there seems to be no reason to expect the maximum principle to hold. There are also several other variants of the maximum principle which hold for solutions of the Floer equation for $Q$. For instance, it holds for the functions $\|p\|^2$ and $\|q\|^2$ separately.

As a consequence of Theorem 3.1, the total and filtered Floer homology groups of $H \in \mathcal{H}_\omega$, denoted by $\text{HF}(H)$ and $\text{HF}^{(a, b)}(H)$, respectively, are defined and have properties similar to those for closed symplectically aspherical manifolds; see, e.g., [HZ, MS]. (For the sake of simplicity all homology groups are taken over $\mathbb{Z}_2$.) Likewise, the local Floer homology $\text{HF}(H, \gamma)$ of $H$ at an isolated periodic orbit $\gamma$ is also defined and has the usual properties; see, e.g., [FL1, FL2, Gi2, GG3]. (Here $J$ is an $\omega$-compatible almost complex structure which is generic within the class of almost complex structures equal outside a compact set to some $J_0$ as in Theorem 3.1. We will discuss the dependence of the Floer homology on $J$ shortly.)

Our next goal is to define the continuation maps induced by homotopies of Hamiltonians in $\mathcal{H}_\omega$. To this end, we say that a homotopy $F_s = Q + f_s$ in $\mathcal{H}_\omega$ from $H_0$ to $H_1$ is compactly supported if $\bigcup_s \text{supp } f_s$ is bounded. (Observe that this is not automatically the case.) Then we have a continuation map

$$\Psi: \text{HF}^{(a, b)}(H_0) \to \text{HF}^{(a, b) + C}(H_1) \quad (3.6)$$
for any $H_0$ and $H_1$ in $\mathcal{H}_Q$, induced by a homotopy $F_s$ in $\mathcal{H}_Q$. Here $(a, b) + C$ stands for $(a + C, b + C)$ and

$$C \geq \int_{-\infty}^{\infty} \int_{S^1 \times \mathbb{R}^n} \sup_{\mathbb{R}^n} \partial_s F_s \, dt \, ds = \int_{-\infty}^{\infty} \int_{S^1 \times \mathbb{R}^n} \sup_{\mathbb{R}^n} \partial_s f_s \, dt \, ds$$

with $\partial_s F_s \equiv 0$ when $|s|$ is large; see [Gi1, Section 3.2.2]. Note that the supremum in (3.7) exists since the homotopy is compactly supported.

We now have $HF(H) = HF(Q) \cong \mathbb{Z}_2$, concentrated in degree $\mu_{cz}(Q, 0) = 0$. It is clear that the filtered Floer homology $HF^{(a, b)}(H)$ is independent of the almost complex structure $J$ as long as $J_Q$ is fixed. However, it is not obvious at all whether this homology is independent of the choice of $J_Q$. In what follows, we will always have $J_Q$ fixed and suppress this hypothetical dependence in the notation.

### 3.2. Continuation maps beyond $\mathcal{H}_Q$.

The class $\mathcal{H}_Q$ is not closed under iterations. For instance, $H^{22} \in \mathcal{H}_{2Q}$ when $H \in \mathcal{H}_Q$. To incorporate iterations into the picture, we consider a broader class $\hat{\mathcal{H}}_Q$ which is the union of the classes $\mathcal{H}_{kQ}$ for all real $k > 0$. Clearly, this class is now closed under iterations. Moreover, one can see from the proof of Proposition 3.2 that there exists a common almost complex structure, $J_Q$, for which Theorem 3.1 holds for all Hamiltonians in $\mathcal{H}_Q$ or, to be more precise, any $J_Q$ can also be taken as $J_Q$. (The reason is that $J_Q$ is determined by the requirement that the off-diagonal part $E$ of $A$ is small compared to $D$, rather than just small. Thus, if conditions (i), (ii) and (iii) are satisfied for $Q$, they are also automatically satisfied in the same basis for $kQ$ for any $k > 0$.) From now on we fix $J_Q$.

As above, we say that a homotopy $F_s = k(s)Q + f_s$ in $\hat{\mathcal{H}}_Q$ is compactly supported if $\bigcup_s \text{supp} f_s$ is bounded and call the closure of this union the support of the homotopy. A homotopy is called slow if it is compactly supported and, say,

$$\frac{|k'(s)|}{(k(s))^2} \leq \frac{3\lambda^2}{20} \inf_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|x\|^2}{|Q(x)|},$$

where $\lambda = \min \lambda_i$ as in Section 3.1; cf. [CFH]. Clearly, the right hand side in (3.8) is positive. (In fact, the infimum in (3.8) is equal to $1/\lambda_{\text{max}}$ where $\lambda_{\text{max}}$ is the largest of the absolute values of the eigenvalues of $Q$ with respect to $\|x\|^2$. This follows from the Courant–Fischer minimax theorem; see, e.g., [De, Chapter 5].) Recall also that a homotopy $F_s$ from $H_0$ to $H_1$ is called linear if $F_s = (1 - g(s))H_0 + g(s)H_1$, where $g: \mathbb{R} \to \mathbb{R}$ is an increasing smooth function equal to zero for $s \ll 0$ and one for $s \gg 1$.

**Theorem 3.4.** Let $F_s$ be a slow homotopy in $\hat{\mathcal{H}}_Q$ from $H_0$ to $H_1$, supported in a ball $B$ with respect to the metric $\langle \cdot, \cdot \rangle_Q = \omega(\cdot, J_Q \cdot)$. The continuation map $\Psi$ as in (3.6) is defined, where $C$ satisfies

$$C \geq \int_{-\infty}^{\infty} \int_{S^1 \times B} \sup_{B} \partial_s F_s \, dt \, ds$$

This map is independent of the slow homotopy. Furthermore, for a linear slow homotopy, we can take $C = \|H_1 - H_0\|_B := \int_{S^1} \sup_B |H_1 - H_0| \, dt$.

It is not hard to see that any compactly supported homotopy in $\hat{\mathcal{H}}_Q$ can be reparametrized to make it slow without changing the right hand side in (3.9). Note also that, although the notion of a slow homotopy is independent of the size of the
support, the lower bound in (3.9) does depend in general on the ball $B$ containing the support and increases with the size of $B$. However, one can show that the continuation map $\Psi$ is independent of the ball $B$ in the following sense: whenever for a fixed homotopy and two different balls $C$ satisfies (3.9) for both of the balls, the resulting continuation map is independent of the ball. (In what follows we will not use this fact.)

The continuation maps $\Psi$ have properties similar to their counterparts in the ordinary Floer homology. For instance, the continuation map induced by a concatenation of homotopies is equal to the composition of the continuation maps, and continuation maps commute with the maps in the long exact sequence in filtered Floer homology. (See [Gi1] for a detailed account on the so-called $C$-bounded homotopies in filtered Floer homology.) Note, however, that here, as in (3.6), the almost complex structure $J_s$ is independent of $s$ outside $B$.

**Remark 3.5.** We emphasize that the continuation map $\Psi$ is not necessarily defined when the homotopy is not slow.

**Proof of Theorem 3.4.** To prove that $\Psi$ is well-defined, it suffices to show that the maximum principle, Proposition 3.2, extends to solutions of the Floer equation for slow homotopies $Q_s = k(s)Q$ connecting $Q_0 = k_0Q$ and $Q_1 = k_1Q$, where $k_0 = k(0)$ and $k_1 = k(1)$. To this end, note that $Q_s = k(s)(Ap, q)$ and recall that, as was noted above, we can take $J_{kQ}$ to be $J_Q$. Calculating the Laplacian with respect to the metric $\langle \cdot, \cdot \rangle_Q = \omega(\cdot, J_Q \cdot)$ in this setting, we obtain

$$
\Delta \rho = \|u_s\|^2 + \|u_t\|^2 + k(s) \left( \langle p, (A - A^T) q_s \rangle - \langle q, (A - A^T) p_s \rangle \right) \\
+ (k(s))^2 \left( \langle A^2 p, p \rangle + \langle A^2 q, q \rangle \right) - 2k'(s) \langle Ap, q \rangle \\
= \|u_s\|^2 + \|u_t\|^2 + (k(s))^2 \left( \|Dp\|^2 + \|Dq\|^2 + \langle E^2 p, p \rangle + \langle E^2 q, q \rangle \right) \\
+ (k(s))^2 \left( \langle (DE + ED) p, p \rangle + \langle (DE + ED) q, q \rangle \right) \\
+ k(s) \left( \langle p, (E - E^T) q_s \rangle - \langle q, (E - E^T) p_s \rangle \right) \\
- 2k'(s) \langle Ap, q \rangle \\
\geq \frac{3\lambda^2}{10} (k(s))^2 \|u_t\|^2 + \left( \|u_s\| - \frac{\lambda |k(s)|}{\sqrt{2}} \|u_t\| \right)^2 - 2|k'(s)| \cdot |Q(u)|.
$$

Hence $\Delta \rho \geq 0$ by (3.8).

Now, the facts that we can take $C$ satisfying (3.9) for a general slow homotopy and that $C = \|H_1 - H_0\|_Y$ satisfies (3.9) for a linear slow homotopy are established by a standard calculation (see, e.g., [Gi1, Sc]) combined with the observation that homotopy trajectories (i.e., solutions of (3.1) for the pair $(F_s, J_s)$) are confined to $B$ due to the maximum principle. □

**Remark 3.6.** The maximum principle is also known to hold for positive definite quadratic Hamiltonians; see [McD, Vi] and also [Sc]. This fact underlies the definition of symplectic homology and, in fact, was the motivation of our approach in this paper. However, it is worth pointing out that in this case the continuation map between a Hamiltonian equal to $kQ$ at infinity and the one equal to $(k + 1)Q$ at infinity is defined only in one direction and this map, depending on $Q$, may be zero. This is the main reason why our approach to the proof of Theorem 1.1 does not carry over to positive definite quadratic Hamiltonians.
4. Proofs and generalizations

4.1. Proof of Theorem 1.1. As has been mentioned in the introduction, we establish a more general result. To state it, recall again that an isolated periodic orbit is said to be homologically non-trivial if the local Floer homology of $H$ at $x$ is non-zero. For instance, a non-degenerate fixed point is homologically non-trivial. More generally, an isolated fixed point with non-vanishing topological index is homologically non-trivial; for this index is equal, up to a sign, to the Euler characteristic of the local Floer homology. The notion of homological non-triviality seems to be particularly well-suited for use in the context of HZ- and Conley conjectures; see, e.g., Remark 1.3. Theorem 1.1 is an immediate consequence of the following result.

**Theorem 4.1.** Let $H: S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian which is equal to a hyperbolic quadratic form $Q$ at infinity (i.e., outside a compact set) such that $Q$ has only real eigenvalues. Assume that $\varphi_H$ has an isolated homologically non-trivial fixed point $x$ with non-zero mean index and $\text{Fix}(\varphi_H)$ is finite. Then $\varphi_H$ has simple periodic orbits of arbitrarily large period.

**Proof of Theorem 4.1.** In what follows, for the sake of brevity, we suppress the $t$-dependence when taking a supremum or specifying the support of a function. For instance, when we say that a function is supported in $Y \subset \mathbb{R}^{2n}$, we mean that the support is in $S^1 \times Y$. Likewise, two functions are equal on $Y$ means that they are equal on $S^1 \times Y$, etc. Finally, the supremum, without a set specified, will stand for the supremum over $\mathbb{R}^{2n}$.

Let $H = Q + f$ as in the statement of the theorem. Pick a polyball $P = B^n \times B^n$ containing $\text{supp} f$ and a ball $V \supset P$. Throughout the proof, as in Section 3, we assume that the off-diagonal part $E$ of $A$ is small enough when compared to the diagonal part $D$. In particular, every integral curve of the flow of $Q$ intersect $P$ along a connected set. Before we actually turn to the proof of the theorem, we need to first modify $H$, without essentially changing its dynamics, to control the energy shift resulting from the homotopy between different iterations of $H$.

**Lemma 4.2.** There exist constants $C_1 > 0$ and $C_2 > 0$, depending only on the quadratic form $Q$ and the ball $V$, such that for every $\epsilon \in (0, 1]$ there exists an autonomous Hamiltonian $\tilde{Q}$ with the following properties:

(i) $\tilde{Q} = Q$ on $V$,
(ii) $\tilde{Q} = \epsilon Q$ outside a ball $V_\epsilon \supset V$ of radius $R = C_1/\sqrt{\epsilon}$,
(iii) $\sup |\tilde{Q}| = C_2$,
(iv) The Hamiltonian flow of $\tilde{Q}$ has no periodic orbits other than the origin, and every integral curve of its flow intersects $P$ along a connected set.

The essential point here is that the constants $C_1$ and $C_2$ are independent of $\epsilon$ while $R = C_1/\sqrt{\epsilon}$ (but not, say, of order $1/\epsilon$). We will prove this lemma by giving an explicit construction of $\tilde{Q}$ after the proof of Theorem 4.1. One can think of $\tilde{Q}$ as a family of Hamiltonians smoothly parametrized by $\epsilon$ with $\tilde{Q} = Q$ for $\epsilon = 1$.

Consider now the Hamiltonian

$$\tilde{H} = \tilde{Q} + f = \epsilon Q + (\tilde{Q} - \epsilon Q) + f = \epsilon Q + h,$$

where $h = (\tilde{Q} - \epsilon Q) + f$ is supported in $V_\epsilon$. Observe that $\tilde{H} \in \mathcal{H}_Q$. Furthermore, $\tilde{H} = H$ in $V$, the ball where the Hamiltonians have non-trivial dynamics. Moreover,
for every period, $\tilde{H}$ and $H$ have exactly the same periodic orbits by Lemma 4.2 (iv), and the orbits have the same actions and indices. In fact, one might expect these Hamiltonians to have exactly the same filtered Floer homology with isomorphism induced by a slow linear homotopy. However, we have not been able to prove this fact.

The next lemma concerns the iterations $\tilde{H}^{2k}$ and an estimate, independent of $k$, of the difference $\tilde{H}^{2(k+\ell)} - \tilde{H}^{2k}$, which will be essential for the proof of Theorem 4.1.

**Lemma 4.3.** The Hamiltonian $\tilde{H}^{2k}$ satisfies the following conditions:

1. $\tilde{H}^{2k} \in \tilde{H}_Q$ and is equal to $k\epsilon Q$ outside the ball $B_k$ of radius $\|\varphi_{\epsilon Q}^{(k-1)}\|R$ centered at the origin, where $\varphi_{\epsilon Q}^{(k-1)}$ is viewed as a linear operator.
2. Assume that $k$, $\ell$ and $\epsilon$ are such that $\|\varphi_{\epsilon Q}^{(k+\ell-1)}\| \leq 2$. Then
   \[
   \|\tilde{H}^{2(k+\ell)} - \tilde{H}^{2k}\|_{B_{k+\ell}} \leq C_3 \ell, \tag{4.1}
   \]
   where $C_3$ is independent of $k$, $\ell$ and $\epsilon$, and the norm is as defined in (2.2).

**Proof of Lemma 4.3.** Denote by $B_1$ the ball $V_\epsilon$ from Lemma 4.2, i.e., $B_1$ is the ball of radius $R = C_1/\sqrt{\epsilon}$ centered at the origin. Consider the nested sets

\[ Y_k = \bigcup_{t \in [0, 1]} \varphi_{\epsilon Q}^{(k-1)t}(B_1) \text{ for } k \in \mathbb{N}. \]

Let $B_k = B(R_k)$ be the ball of radius $R_k = \|\varphi_{\epsilon Q}^{(k-1)}\|R$. Clearly, $B_k \supset Y_k$.

Recall that $\tilde{H} = \epsilon Q + h$, where $h = (\tilde{Q} - \epsilon Q) + f$ is supported in $B_1$. Observe that $\tilde{H}^{2k}$ can be expressed as

\[
\tilde{H}^{2k} = k\epsilon Q + \sum_{j=0}^{k-1} h \circ (\varphi_{\tilde{H}}^t)^{-j} + \epsilon \sum_{j=0}^{k-1} (Q \circ (\varphi_{\tilde{H}}^t)^{-j} - Q) = k\epsilon Q + h_k,
\]

We now show that $\text{supp } h_k \subset Y_k$, which settles (i). Since $\text{supp } h \subset B_1$, a point $x$ can be in $\text{supp}(h \circ (\varphi_{\tilde{H}}^t)^{-j})$ only if $(\varphi_{\epsilon Q}^t)^{-j}(x) \in B_1$ for some $\tau \in [0, t]$. This implies that

\[ x \in ((\varphi_{\epsilon Q}^t)^{-j})^{-1}\vphantom{((\varphi_{\epsilon Q}^t)^{-j})^{k-1}}(B_1) = \varphi_{\epsilon Q}^{jt}(B_1) \subset \bigcup_{t \in [0, 1]} \varphi_{\epsilon Q}^{jt}(B_1) = Y_j. \]

Hence, the first term in $h_k$ is supported in $Y_k$. Dealing with the second term in $h_k$, we first note that

\[
\epsilon(Q \circ (\varphi_{\tilde{H}}^t)^{-j} - Q) = \epsilon(Q \circ (\varphi_{\tilde{H}}^t)^{-j} - Q \circ (\varphi_{\epsilon Q}^t)^{-j})
\]

since $Q$ is autonomous. Now, it is clear that $(\varphi_{\tilde{H}}^t)^{-j}(x) \neq (\varphi_{\epsilon Q}^t)^{-j}(x)$ only when the integral curve of $\epsilon Q$ through $x$ for $[-jt, 0]$ enters $B_1$, i.e., $(\varphi_{\epsilon Q}^t)^{-j}(x) \in B_1$ for some $\tau \in [0, t]$. Hence, similarly to the first term, the second term in $h_k$ is also supported in $Y_k$, and we have $\text{supp } h_k \subset Y_k$. 
To establish (ii), denote by $B(1)$ the unit ball and observe that for any $j = k, \ldots, k + \ell - 1$, we have
\[
\sup_{B_{k+\ell}} |h \circ (\varphi^k_H)^{-j}| = \sup_{B_1} |\tilde{Q} - \epsilon Q + f| \\
\leq \sup_{B_1} |\tilde{Q}| + \epsilon \sup_{B_1} |Q| + \sup_{B_1} |f| \\
\leq C_2 + \epsilon C_1^2 \sup_{B(1)} |Q| + \sup_{B(1)} |f| \\
= C_2 + C_1^2 \sup_{B(1)} |Q| + \sup_{B(1)} |f|.
\]
Furthermore, by the energy conservation law, we have
\[
\sup_{B_{k+\ell}} |Q \circ (\varphi^k_H)^{-j} - Q| = \sup_{B_{k+\ell}} |Q \circ (\varphi^k_H)^{-j} - Q \circ (\varphi^j_Q)^{-j}| \\
= \sup_{B_1} |Q \circ (\varphi^k_H)^{-j} - Q \circ (\varphi^j_Q)^{-j}| \\
\leq 2 \sup_{B_1} |Q| \\
\leq 2 C_1^2 \epsilon \sup_{B(1)} |Q|.
\]

Now, recall that $k$, $\ell$ and $\epsilon$ are such that $\|\varphi^{(k+\ell-1)}_Q\| \leq 2$. Thus $B_{k+\ell} \subset 2B_1$. Setting $M = \sup_{B(1)} |Q|$ and using the above estimates, we then have:
\[
\sup_{B_{k+\ell}} |\hat{H}^{(k+\ell)} - \hat{H}^{2k}| = \sup_{B_{k+\ell}} \left|k\ell Q + \sum_{j=k}^{k+\ell-1} h \circ (\varphi^k_H)^{-j} + \epsilon \sum_{j=k}^{k+\ell-1} (Q \circ (\varphi^k_H)^{-j} - Q)\right| \\
\leq k\ell \sup_{2B_1} |Q| + \sum_{j=k}^{k+\ell-1} \sup_{B_{k+\ell}} \left|h \circ (\varphi^k_H)^{-j}\right| \\
+ \epsilon \sum_{j=k}^{k+\ell-1} \sup_{B_{k+\ell}} \left(Q \circ (\varphi^k_H)^{-j} - Q\right) \\
\leq 4k\ell \epsilon C_1^2 M + \ell (C_2 + C_1^2 M + \sup_{B_1} |f|) + 2k\ell C_1^2 \epsilon M \\
\leq (7C_1^2 M + C_2 + \sup_{B_1} |f|) \ell =: C_3 \ell
\]
where $C_3$ is independent of $k$, $\ell$ and $\epsilon$. Finally, we have
\[
\|\hat{H}^{(k+\ell)} - \hat{H}^{2k}\|_{B_{k+\ell}} = \int_{S^1} \sup_{B_{k+\ell}} |\hat{H}^{(k+\ell)} - \hat{H}^{2k}| dt \leq C_3 \ell,
\]
which concludes the proof of Lemma 4.3. \qed

From now on we will work with the Hamiltonians $\hat{H}$, and at this stage we prefer not to specify the parameter $\epsilon$ yet. These Hamiltonians have the same periodic orbits with the same actions and indices, and up to the point when the homotopy between the iterated Hamiltonians is considered the argument applies to any of the Hamiltonians $\tilde{H}$.

It is worth mentioning again that the Hamiltonians $\tilde{H}^{2k}$ are not one-periodic in time even though $\tilde{H}$ is. This issue, however, is quite standard and can be dealt
with in a straightforward way. Namely, consider a Hamiltonian $G = K + g$, where
$g = g_t$ is time-dependent for $t \in [0, 1]$ and $K$ is any autonomous Hamiltonian. The Hamiltonian
diffeomorphism $\varphi_G$ can be generated by a one-periodic Hamiltonian

$$\tilde{G} = K + \lambda(t)g_{\lambda(t)} \circ \varphi^\lambda_{K(t)-1},$$

where $\lambda: [0, 1] \to [0, 1]$ is an increasing function equal to zero for $t \approx 0$ and one for $t \approx 1$. We apply this procedure to $\tilde{H}^{2k}$ with $K = k \epsilon Q$ and $g = h_k$. The actions, the
Conley-Zehnder indices, and the mean indices of the periodic orbits do not change.
The change of the set $B_k$ can be made arbitrarily small, of the order $\|1 - \lambda(t)\|_{L^1}$.
As a consequence, the upper bound (4.1) can also be adjusted by an arbitrarily small
amount independent of $k$. In what follows, we will treat the Hamiltonians
$\tilde{H}^{2k}$ as one-periodic in time, allowing for these straightforward modifications.

Now we are in a position to proceed with the proof. It suffices to show that
there exist arbitrarily large primes which occur as periods of simple periodic orbits.
Arguing by contradiction, assume that only finitely many prime numbers are at-
tained as the periods. From now on, we always denote by $p$ or $p_i$ a prime number
greater than the largest period. Let $\tilde{H} = \bar{Q} + f$ where $\bar{Q}$ is any Hamiltonian from
Lemma 4.2. Then for any such prime $p$ all $p$-periodic orbits of $\varphi_{\tilde{H}}$ are iterations
of fixed points of $\varphi_{\bar{H}}$, and hence $S(\tilde{H}^{2p}) = pS(H)$. Recall in this connection that
$\varphi_{\bar{H}}$ is assumed to have finitely many fixed points. Next, let us note that all suffi-
ciently large prime numbers are admissible in the sense of [GG3]. Thus, under such
iterations of $\tilde{H}$, the orbit $x$ stays isolated, and

$$HF(\tilde{H}^{2p}, x^p) = HF(H^{2p}, x) = HF(H, x)$$

up to, in the second equality, a shift of degree determined by the order of iteration
$p$; see [GG3, Theorem 1.1]. In particular, in our case, $HF(H^{2p}, x^p) \neq 0$ since
$HF(H, x) \neq 0$.

As has been mentioned above, $\Delta_{\tilde{H}}(x) = \Delta_H(x)$, and let us assume that $\Delta_H(x)$ is
negative for the argument is similar if $\Delta_H(x) > 0$. Moreover, let us assume for the sake
of simplicity that $A_H(x) = 0$ and hence $A_{\tilde{H}}(x) = 0$. (The general case can be dealt
with in a similar fashion and requires only notational modifications.) Consequently,
$A_{\tilde{H}^{2p}}(x^p) = 0$ for all iterations $p$. Let $a > 0$ be outside $S(\tilde{H}) = S(H)$ such that $0$
is the only point in $(-a, a) \cap S(H)$ and therefore in $(-ap, ap) \cap S(\tilde{H}^{2p})$. Then we have

$$HF_*^{(-ap, ap)}(\tilde{H}^{2p}) = HF_*(H^{2p}, x^p) \oplus \ldots,$$  \hspace{1cm} (4.3)

where the dots represent the local Floer homology contributions from the fixed
points with zero action other than $x$. Furthermore, we henceforth focus on degrees
* such that $|*| > n$. This guarantees that the fixed points with zero mean index
do not contribute to $HF_*(\tilde{H}^{2p})$, for their local Floer homology groups are
supported in $[-n, n]$, where the support is, by definition, the set of degrees for
which the local Floer homology groups are non-zero. Thus all terms on the right
hand side of (4.3) come from fixed points with non-zero mean index. Moreover, we
can further restrict * so that only the fixed points $\gamma$ having the same mean index
as $x$ contribute to the right hand side of (4.3). This is possible since the supports
of local Floer homology groups coming from fixed points with other non-zero mean
indices are separated from $supp HF_*(H^{2p}, x^p) \subset [p\Delta_H(x) - n, p\Delta_H(x) + n]$ whenever
$p$ is sufficiently large.
From now on, we work with primes $p > 2$ which are as large as is needed above. Let us order these prime numbers as $p_1 < p_2 < \ldots$. In what follows, $p_i$ always denotes a prime from this sequence.

Next, recall that $\Delta_H(x) > 0$ and let $m \in \mathbb{N}$ be such that $m > n/\Delta_H(x)$. Then, using the fact that $p_{i+m} - p_i \geq 2m$, we see that the supports of $HF(H^{2p_i}, \gamma_{p_i})$ and $HF(H^{2p_{i+m}}, \gamma_{p_{i+m}})$ are disjoint for all $i$ and for all fixed points $\gamma$ of $\varphi_H$ with $A_H(\gamma) = 0$ and $\Delta_H(\gamma) = \Delta_H(x)$. This is because

\[ [p_i, \Delta_H(x) - n, p_i \Delta_H(x) + n] \cap [p_{i+m}, \Delta_H(x) - n, p_{i+m} \Delta_H(x) + n] = \emptyset, \]

where the first interval contains $\text{supp } HF(H^{2p_i}, \gamma_{p_i})$ and the second one contains $\text{supp } HF(H^{2p_{i+m}}, \gamma_{p_{i+m}})$. Moreover, for any $p_i$, there exists an integer $s_i$ such that $HF_{s_i}(H^{2p_i}, x^{p_i}) \neq 0$ as is mentioned earlier and proved in [GG3]. Thus we see that

\[
HF_{s_i}(H^{2p_i}, x^{p_i}) \neq 0 \quad \text{and} \quad HF_{s_i}(H^{2p_{i+m}}, \gamma_{p_{i+m}}) = 0 \tag{4.4}
\]

for all fixed points $\gamma$ as above, since $s_i$ is outside $\text{supp } HF(H^{2p_{i+m}}, \gamma_{p_{i+m}})$ for all such $\gamma$.

Choose $p_i$ so large that $p_i \alpha > 6C_3(p_{i+m} - p_i)$ where $C_3$ is introduced in Lemma 4.3. The latter is guaranteed for large primes by the fact that $p_{i+1} - p_i = o(p_i)$; see [BHP]. (Obviously, one can write $p_{i+m} - p_i$ as a telescoping sum of the differences of two consecutive primes, and hence, by a simple inductive argument, $p_{i+m} - p_i = o(p_i)$.) Now, pick $\alpha > 0$, depending on $m$ and $i$, such that

\[-p_i \alpha < -\alpha < -\alpha + 2C_3(p_{i+m} - p_i) < 0 < \alpha < \alpha + 2C_3(p_{i+m} - p_i) < p_i \alpha.\]

For instance, $\alpha$ satisfying $p_i \alpha - 4C_3(p_{i+m} - p_i) < \alpha < p_i \alpha - 2C_3(p_{i+m} - p_i)$ would work. As a consequence, we also have

\[-p_{i+m} \alpha < -\alpha < \alpha + C_3(p_{i+m} - p_i) < 0 < \alpha + C_3(p_{i+m} - p_i) < p_{i+m} \alpha.\]

Finally, let us specify $Q$ and, in turn, $\tilde{H}$. To this end, we choose $\epsilon > 0$ so small that $\|\varphi_{\epsilon(p_{i+m} - 1)}\| \leq 2$ and hence (4.1) is satisfied with $k = p_i$ and $k + l = p_{i+m}$ in the second assertion of Lemma 4.3. Set $\delta := C_3(p_{i+m} - p_i)$. Then, for a linear homotopy, which we may assume to be slow in the sense of Section 3.2, from $\tilde{H}^{2p_i}$ to $\tilde{H}^{2p_{i+m}}$, we have the induced map

\[
HF^{(-\alpha, \alpha)}(\tilde{H}^{2p_i}) \to HF^{(-\alpha, \alpha) + \delta}(\tilde{H}^{2p_{i+m}}).
\]

Here the fact that $\delta$ is the correct action shift follows from Theorem 3.4 and Lemma 4.3. Likewise, the linear-homotopy map from $\tilde{H}^{2p_{i+m}}$ to $\tilde{H}^{2p_i}$ results in another action shift in $\delta$. Consider now the following commutative diagram:

\[
\begin{array}{ccc}
0 \neq HF_{s_i}^{(-\alpha, \alpha) + \delta}(\tilde{H}^{2p_i}) & \cong & HF_{s_i}^{(-\alpha, \alpha) + 2\delta}(\tilde{H}^{2p_i}) \\
\downarrow & & \downarrow \\
0 & \neq & HF_{s_i}^{(-\alpha, \alpha) + \delta}(\tilde{H}^{2p_{i+m}})
\end{array}
\]

Here the top group is zero due to our choice of the degree $s_i$. On the other hand,

\[
HF_{s_i}^{(-\alpha, \alpha)}(\tilde{H}^{2p_i}) = HF_{s_i}(H^{2p_i}, x^{p_i}) \oplus \ldots \neq 0,
\]

and the horizontal arrow is induced by the natural quotient-inclusion map; see, e.g., [G1]. This is, indeed, an isomorphism by the stability of filtered Floer homology (see, e.g., [GG3]) because 0 is the only action value in the intervals $(-\alpha, \alpha)$ and
\((-\alpha, \alpha) + 2\delta\). To summarize, a non-zero isomorphism factors through a zero group in the diagram. This contradiction completes the proof of Theorem 4.1, modulo a proof of Lemma 4.2 which is given below.

\[ \square \]

Remark 4.4. Notice that we have actually established the existence of a simple periodic orbit of either \(H^{p_l}\) or \(H^{p_l+m}\). In particular, starting with a sufficiently large prime number, among every \(m\) consecutive primes, there exists at least one prime which is the period of a simple periodic orbit of \(\varphi_H\).

Furthermore, for an infinite sequence of simple \(p_l\)-periodic orbits \(x_l\) of \(\varphi_H\) found this way, where \(p_l \to \infty\), we have \(\Delta_{H^{p_l}}(x_l)/p_l \to \Delta_H(x)\). Hence, in some sense, the mean index \(\Delta_H(x)\) is an accumulation point in the union of normalized index spectra for \(H\) and its all iterations. (Of course, \(\Delta_H(x)\) could possibly be isolated, but then \(\Delta_{H^{p_l}}(x_l)/p_l = \Delta_H(x)\).) A similar fact also holds for the action.

Finally note that the condition that the eigenvalues \(\sigma\) of \(Q\) are real can be relaxed and replaced by the requirement that \(|\text{Re } \sigma| > |\text{Im } \sigma|\); cf. Remark 3.3.

Proof of Lemma 4.2. We construct the function \(Q\) in three steps and then show that \(Q\) has the required properties.

**Step 1.** Set \(c = \sup_V |Q|\). Let \(\eta: \mathbb{R} \to \mathbb{R}\) be a smooth function such that

- \(\eta\) is odd,
- \(\eta(x) = x\) when \(|x| \leq c\),
- \(\eta(x) = cx\) when \(|x| \geq c' = 2c/\epsilon\),
- \(\eta' \geq \epsilon/2\).

It is easy to see that \(\eta\) with these properties exists. Note that to have a monotone function \(\eta\) such that \(\eta(x) = x\) when \(|x| \leq c\) and \(\eta(x) = cx\) when \(|x| \geq c'\), we must have \(c' > c/\epsilon\). This is the main reason why the radius \(R\) in the statement of the lemma must be of order \(1/\sqrt{\epsilon}\). As the first step in the construction of \(Q\), we replace \(Q\) by \(\eta \circ Q\).

**Step 2.** In the second step, we appropriately cut off \(\eta \circ Q\) and define a new Hamiltonian \(\tilde{Q}\) which is a linear transition from \(\eta \circ Q\) to \(cQ\) in the \(q\)-direction. Namely, let \(r > 0\) be the radius of the ball \(V\) and set \(a_0 = r/\sqrt{\epsilon}\) and \(a_1 = 2a_0\). (So, \(r < a_0 < a_1\).) The modification of \(\eta \circ Q\) takes place on the domain \(a_0 \leq ||q|| \leq a_1\).

To this end, choose a smooth monotone increasing function \(\phi: [0, \infty) \to \mathbb{R}\) such that \(\phi(x) = 0\) when \(x \leq a_0\) and \(\phi(x) = 1\) when \(x \geq a_1\), and \(|\phi'| \leq 2/|a_1 - a_0| = 2/a_0\).

We then set

\[ \tilde{Q} = \phi(||q||)cQ + (1 - \phi(||q||))\eta \circ Q. \]

**Step 3.** In the third step, we suitably cut off \(\tilde{Q}\) and finally define the desired Hamiltonian \(Q\). To this end, let \(b_0 = \max\{r, 32c/\lambda r \sqrt{\epsilon}\}\) and \(b_1 = 2b_0\). Here, as in Section 3.1, \(\lambda = \min \lambda_i > 0\), where \(\lambda_i\)'s are the eigenvalues of \(A\). (The reason for this choice of \(b_0\) will be clear at the end of the proof.) Choose a smooth monotone increasing function \(\psi: [0, \infty) \to \mathbb{R}\) such that \(\psi(x) = 0\) when \(x \leq b_0\) and \(\psi(x) = 1\) when \(x \geq b_1\).

Define

\[ \hat{Q} = \psi(||p||)cQ + (1 - \psi(||p||))\tilde{Q}. \]
Checking the conditions (i)-(iv). Since $\eta \circ Q = Q$ on $V$ by the definition of $c$, and $a_0 \geq r$ and $b_0 \geq r$, we clearly have $\tilde{Q} = Q$ on the ball $V$. Furthermore, $\tilde{Q} = \epsilon Q$ outside the ball of radius $R = \sqrt{a_1^2 + b_1^2}$. Let $V_{\epsilon}$ be this ball. It is clear from our choice of $a_1$ and $b_1$ that $R$ has the form $C_1/\sqrt{\epsilon}$, where $C_1$ is independent of $\epsilon$. This proves (i) and (ii).

To establish (iii), observe first that

$$\sup |\eta(x) - \epsilon x| = \sup |\eta(x) - \epsilon x| \leq \eta(c') + \epsilon c' = 2\epsilon c = 4\epsilon c / \epsilon = 4c.$$ (4.5)

Thus

$$\sup_{V_{\epsilon}} |\tilde{Q}| \leq \sup_{V_{\epsilon}} |\epsilon Q| + \sup_{V_{\epsilon}} |\tilde{Q}|
\leq 2 \sup_{V_{\epsilon}} |\epsilon Q| + \sup_{V_{\epsilon}} |\eta \circ Q|
\leq 3 \sup_{V_{\epsilon}} |\epsilon Q| + \sup_{V_{\epsilon}} |\eta \circ Q - \epsilon Q|
\leq 3\epsilon C_2 \sup_{B(1)} |Q| + 4c
= 3\epsilon C_2 \sup_{B(1)} |Q| + 4c =: C_2,$$

with $C_2$ independent of $\epsilon$. This is where replacing $Q$ by $\eta \circ Q$ in Step 1 is essential.

To verify the condition (iv), note that without loss of generality we may assume that the off-diagonal part of $A$ is so small that

$$L_{X_Q} \|p\|^2 \leq -\lambda \|p\|^2 \text{ and } L_{X_Q} \|q\|^2 \leq -\lambda \|q\|^2.$$

(Here we dropped the factor of 2 on the right hand side of the inequalities to account for the off-diagonal terms.) In particular, every integral curve of $Q$ enters the polyball $P$ through the "side" part, $\|p\| = const$, of the boundary $\partial P$ and leaves it through the "top", $\|q\| = const$, of $\partial P$.

We will show that

(a) the flow of $\tilde{Q}$ is equal to the flow of $Q$ on $P$ and on the disk $q = 0$, $\|p\| \leq b_0$,
(b) $L_{X_Q} \|q\|^2 \geq 0$ when $\|p\| \leq b_0$, with strict inequality when $q \neq 0$,
(c) $L_{X_Q} \|p\|^2 < 0$ when $\|p\| \geq b_0$.

It is not hard to see that (iv) readily follows from these assertions.

The assertion (a) is obvious since $\tilde{Q} = Q$ in the region where $|Q| \leq c$ and $\|q\| \leq a_0$ and $\|p\| \leq b_0$, containing $V \supset P$ and the disk $q = 0$, $\|p\| \leq b_0$. It remains to check (b) and (c), i.e., that the function $\|q\|^2$ increases along the flow of $\tilde{Q}$ when $\|p\| \leq b_0$, and the function $\|p\|^2$ decreases along the flow of $\tilde{Q}$ when $\|p\| \geq b_0$, unless $q = 0$.

To prove (b), first note that $\tilde{Q} = \hat{Q}$ in the region where $\|p\| \leq b_0$. Also, recall that $\eta' \geq \epsilon/2$ and, hence,

$$\epsilon \phi(|q|) + (1 - \phi(|q|)) (\eta' \circ Q) \geq \epsilon/2.$$

Therefore, since $\phi(|q|)$ is independent of $\|p\|$, we have

$$L_{X_Q} \|q\|^2 = L_{X_Q} \|q\|^2
= \epsilon \phi(|q|) L_{X_Q} \|q\|^2 + (1 - \phi(|p|)) (\eta' \circ Q) L_{X_Q} \|q\|^2
\geq \frac{\epsilon \lambda}{2} \|q\|^2 \geq 0.$$
Thus, in the region where \( a \) bounding from above the second term, our choices of fixed point must necessarily have infinitely many periodic orbits. However, the requirement that \( |\phi| \leq 2/(a_1 - a_0) = 2/a_0 \) enter the picture. Then, by (4.5), we have

\[
2|\epsilon Q - \eta \circ Q| \cdot |\phi'| \cdot ||p|| \leq 8c \frac{2}{|a_1 - a_0|} ||p|| = \frac{16c}{a_0} ||p|| \leq \frac{16c\sqrt{r}}{r} ||p||.
\]

Thus, in the region where \( ||p|| > b_0 = 32c/\lambda r \sqrt{c} \), we have

\[
\|L_{X_0} \|p\|^2 < -\epsilon \frac{\lambda}{2} ||p||^2 + \frac{16c\sqrt{c}}{r} ||p|| < 0,
\]

which completes the proof of (c). (Note that \( b_0 \) is chosen exactly to make (c) hold.)

4.2. Proof of Theorem 1.4. As has been pointed out in the introduction, we have a more general result in dimension two:

**Theorem 4.5.** Let \( H : S^1 \times \mathbb{R}^2 \to \mathbb{R} \) be a Hamiltonian which is equal to a hyperbolic quadratic form at infinity. Assume that \( \varphi_H \) has at least two isolated homologically non-trivial fixed points and \( \text{Fix}(\varphi_H) \) is finite. Then \( \varphi_H \) has simple periodic orbits of arbitrarily large period.

**Remark 4.6.** A similar two-dimensional result holds when \( H \) is elliptic quadratic at infinity. (In fact, the requirement that the fixed points be homologically non-trivial is not needed in this case.) Indeed, since the Hamiltonian is elliptic outside a compact set, a sufficiently large sub-level will be invariant under the flow. Then Franks’ theorem stating that an area-preserving map of the two-disk has either one or infinitely many periodic points, [Fr1], implies the result.

**Remark 4.7.** Using Theorem 4.5, we can also prove a weaker version of Franks’ theorem on \( S^2 \), asserting that a Hamiltonian diffeomorphism of \( S^2 \) with a hyperbolic fixed point must necessarily have infinitely many periodic orbits. However, the
argument is somewhat involved, and we omit it since this result also follows from
the main theorem of [GG5] and, as has been mentioned in the introduction, at least
two other symplectic proofs of Franks’ theorem are available, [BH, CKRTZ, Ke].

Proof of Theorem 4.5. Observe that if \( \varphi_H \) has a homologically non-trivial
fixed point with non-zero mean index, then the theorem follows from Theorem 4.1. So,
let us assume that there are at least two isolated homologically non-trivial fixed
points with zero mean index. Notice that all of these points cannot have non-zero
local Floer homology concentrated in degree zero: \( HF_*(H) = 0 \) when \( * \neq 0 \) and
\( HF_0(H) = \mathbb{Z}_2 \). Thus \( \varphi_H \) must have at least one fixed point with zero mean index
and non-zero local Floer homology in degree \( \pm 1 \). Such an orbit is a symplectically
degenerate maximum, and its presence implies that \( \varphi_H \) has simple periodic orbits
of arbitrarily large prime period; see [GG1, GG3] and also [Gi2, He, Hi]. (Strictly
speaking, the latter fact has been established only for (a broad class of) closed
symplectic manifolds. However, since the Hamiltonian in our case is a compactly
supported perturbation of a hyperbolic quadratic form on \( \mathbb{R}^{2n} \), having periodic
orbits only within the support of the perturbation, the proof in the case of closed
manifolds, for instance the one in [GG1], goes through word-for-word.) \( \square \)

Proof of Theorem 1.4 in dimension four. Recall that the homology is concentrated
in degree zero and \( HF_0(H) = \mathbb{Z}_2 \). Hence one of the fixed points of \( \varphi_H \) must
have non-zero Conley-Zehnder index. By a straightforward index analysis, it is
easy to see that, in dimension four, such an orbit must necessarily have non-zero
mean index. (Indeed, observe that, in dimension four, the mean index of a non-
degenerate fixed point is zero if and only if the linearization is hyperbolic or its
eigenvalues comprise two pairs “conjugate” to each other. It is clear that in both
cases the Conley-Zehnder index is zero.) Finally, applying Theorem 4.1, we obtain
the existence of simple periodic orbits with arbitrarily large prime period. \( \square \)

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Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA

E-mail address: basak.gurel@ucf.edu