Cheeger-harmonic functions in metric measure spaces revisited

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Abstract. Let $(X, d, \mu)$ be a complete metric measure space, with $\mu$ a locally doubling measure, that supports a local weak $L^2$-Poincaré inequality. By assuming a heat semigroup type curvature condition, we prove that Cheeger-harmonic functions are Lipschitz continuous on $(X, d, \mu)$. Gradient estimates for Cheeger-harmonic functions and solutions to a class of non-linear Poisson type equations are presented.

1 Introduction

The study of Lipschitz continuity of Cheeger-harmonic functions was originated by Koskela et. al. [20], which can be viewed as a metric version of Yau’s gradient estimate ([36, 9]). In [20] it is proved that on an Ahlfors regular metric spaces, an $L^2$-Poincaré inequality and a heat semigroup type curvature condition are sufficient to guarantee Lipschitz continuity of Cheeger-harmonic functions. Later, a quantitative gradient estimate for Cheeger-harmonic functions was given in [16]. The main aim of this paper is to relax the Ahlfors regularity in [20, 16] to doubling of the measure. Besides this, gradient estimates for a class of non-linear Poisson type equations are also given.

Let $(X, d, \mu)$ be a complete, pathwise connected metric space, equipped with a locally doubling measure $\mu$, i.e., for each $R_0 > 0$, there exists a constant $C_d(R_0)$ such that for each $0 < r < R_0/2$ and all $x$,

$$\mu(B(x, 2r)) \leq C_d(R_0)\mu(B(x, r)).$$

(1.1)

We then call the measure locally $Q$-doubling for some $Q > 0$, if for each $R_0 > 0$, there exists a constant $C_Q(R_0)$ such that such that for every $x \in X$ and all $0 < r < R \leq R_0$, it holds

$$\mu(B(x, R)) \leq C_Q(R_0)\left(\frac{R}{r}\right)^Q \mu(B(x, r)).$$

(1.2)

We say that $\mu$ is globally $Q$-doubling if the above holds with a constant that is independent of $R_0$.

Throughout the paper, we additionally require that $(X, d, \mu)$ is stochastically complete (see Section 2 below). The requirement comes from the technique used in the proof, and does not look...
like a very natural condition; on the other hand, it is satisfied on metric spaces with (Lott-Sturm-Villani) finite dimensional Ricci curvature bounded from below.

An $L^2$-Poincaré inequality is needed. Precisely, we assume that $(X, d, \mu)$ supports a local weak $L^2$-Poincaré inequality, i.e., for each $R_0 > 0$, there exists $C_P(R_0) > 0$ such that for all Lipschitz functions $u$ and each ball $B(x, r) = B(x, r)$ with $r < R_0$,

\[
\int_{B(x,r)} |u - u_B| d\mu \leq C_P(R_0) r \left( \int_{B(x,2r)} [\text{Lip } u]^2 d\mu \right)^{1/2},
\]

where and in what follows, $u_B = \int_B u d\mu = \mu(B)^{-1} \int_B u d\mu$, and

\[
\text{Lip } u(x) = \limsup_{r \to 0} \sup_{d(x,y) \leq r} \frac{|u(x) - u(y)|}{r}.
\]

We then say that $(X, d, \mu)$ supports a uniform weak $L^2$-Poincaré inequality, if (1.3) holds with a uniform constant $C_P$ for all $R_0 > 0$. According to [17] the Poincaré inequality here coincides with the one for all measurable functions and their upper gradients, as introduced in [14].

For a domain $\Omega \subset X$, the Sobolev space $H^{1,2}(\Omega)$ is defined to be the completion of all locally Lipschitz continuous functions $u$ on $\Omega$ under the norm

\[
\|u\|_{H^{1,2}(\Omega)} := \|u\|_{L^2(\Omega)} + \|\text{Lip } u\|_{L^2(\Omega)}.
\]

An important fact to us is that for each $u \in H^{1,2}(\Omega)$ we can assign a (Cheeger) derivative $Du$ by [8]. This derivative operator is linear, satisfies the Leibniz rule, and there is an inner product norm that is comparable to our original norm: for a locally Lipschitz function $u$, $Du \cdot Du$ is uniformly almost everywhere comparable to the square of the local Lipschitz constant $\text{Lip } u$, see Section 2 below. Notice that in many concrete settings, one can make a different choice of an operator that satisfies the above conditions. We call any operator $D$ that has the above properties a Cheeger derivative operator.

We next define the Cheeger-Laplace equation on $(X, d, \mu)$. For a domain $\Omega$, we say that $u \in H^{1,2}(\Omega)$ is a solution to the equation $Lu = g(x, u)$ in $\Omega$, if

\[
-\int_{\Omega} Du(x) \cdot D\phi(x) \, d\mu(x) = \int_{\Omega} g(x, u) \phi(x) \, d\mu(x)
\]

holds for all Lipschitz functions $\phi$ with compact support in $\Omega$, where $g(x, t)$ is a measurable function defined on $X \times \mathbb{R}$ and continuous with respect to the variable $t$. If $Lu = 0$ in $\Omega$, then we say that $u$ is Cheeger-harmonic in $\Omega$.

The above Dirichlet problem and related parabolic equations have been widely studied; see [5, 6, 21, 28, 29, 30, 31] for instance. According to [28, 31], the doubling condition and validity of an $L^2$-Poincaré inequality are equivalent to a parabolic Harnack inequality, which further implies an elliptic Harnack inequality and hence the Hölder continuity of harmonic functions (see [5, 31]).

However, Lipschitz regularity does not follow from doubling and Poincaré inequality, see the examples from the introduction of [20]. Thus, some additional requirement is needed for Lipschitz regularity of solutions.
Cheeger-harmonic functions

For $f$ and $g$ in $H^{1,2}(X)$, define the bilinear form $\mathcal{E}$ by

$$\mathcal{E}(f, g) = \int_X Df(x) \cdot Dg(x) \, d\mu(x).$$

Corresponding to such a form there exists an infinitesimal generator $A$ which acts on a dense subspace $D(A)$ of $H^{1,2}(X)$, and there is a semigroup $\{T_t\}_{t \geq 0}$ generated by $A$; see Section 2 below.

We say that $(X, D, \mu)$ satisfies heat semigroup curvature condition for our Cheeger derivative operator $D$, if there exists a nonnegative and nondecreasing function $c_\kappa(t)$ on $(0, \infty)$ such that for each $0 < t < T$ and every $g \in H^{1,2}(X)$, it holds

$$(1.4) \quad T_t(g^2)(x) - [T_t(g)(x)]^2 \leq (2t + c_\kappa(t)r^2)T_t(Dg)^2.$$  

Let us state the first gradient estimate.

**Theorem 1.1.** Let $(X, d)$ be a stochastically complete metric space with a locally $Q$-doubling measure $\mu$, $Q \in (1, \infty)$. Assume that $(X, d, \mu)$ supports a local weak $L^2$-Poincaré inequality and the heat semigroup curvature condition (1.4).

Let $u$ be a solution to the equation $Lu = -\lambda u$ in $2B$, where $B = B(y_0, R)$ and $\lambda \in L^\infty(2B)$. Then there exists $C = C(Q, C_Q(2R), C_P(2R), \|\lambda\|_{L^\infty(2B)R^2})$ such that

$$||Du||_{L^\infty(B)} \leq C \left(\frac{1}{R} + \sqrt{c_\kappa(R^2)}\right) \int_{2B} |u| \, d\mu.$$  

The above estimate in particular implies that Cheeger-harmonic functions are locally Lipschitz continuous under the above assumptions.

Let us revisit an example from [20]. Consider the metric space $(\Omega, d)$ with $\Omega = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ and $d$ the Euclidean metric. Let $w(x, y) = \sqrt{|x|}$. Set $d\mu = w \, dx$. Then $(\Omega, d, \mu)$ supports an $L^2$-Poincaré inequality and $\mu$ is a doubling measure. The function $u(x, y) = \text{sgn}(x) \sqrt{|y|}$ is harmonic in $\Omega$, but it is not locally Lipschitz in $\Omega$. It was understood in [20] that in order to deduce Lipschitz regularity, the doubling condition should be strengthened to Ahlfors regularity. According to Theorem 1.1, the reason that the Lipschitz regularity of Cheeger-harmonic functions fails is due to lack of lower curvature bounds rather than to lack of Ahlfors regularity.

We have the following gradient estimates for positive Cheeger-harmonic functions.

**Theorem 1.2.** Let $(X, d)$ be a stochastically complete metric space with a locally $Q$-doubling measure $\mu$, $Q \in (1, \infty)$. Assume that $(X, d, \mu)$ supports a local weak $L^2$-Poincaré inequality and the heat semigroup curvature condition (1.4). Let $u$ be a positive Cheeger-harmonic function in $2B$, where $B = B(y_0, R)$.

(i) There exists $C = C(Q, C_Q(2R), C_P(2R)) > 0$ such that for almost every $x \in B$, it holds

$$\frac{|Du(x)|}{u(x)} \leq C \left(\frac{\sqrt{c_\kappa(R^2)} + 1}{R}\right);$$

(ii) If $c_\kappa(1) > 0$, then there exists a fixed constant $C = C(Q, C_Q(1), C_P(1))$ such that for almost every $x \in B$, it holds

$$\frac{|Du(x)|}{u(x)} \leq C \left(\frac{\sqrt{c_\kappa(1)} + 1}{R}\right).$$
Examples that satisfy assumptions in the above theorems were discussed in [20, 16]. Here we point out that, as a consequence of relaxing the Ahlfors regularity from [20, 16], the assumptions are satisfied on finite dimensional Riemannian manifolds with Ricci curvature bounded from below, weighted Riemannian manifold with Bakry-Emery’s curvature bounded from below, as well as compact Alexandrov spaces with curvature bounded from below; see [3, 4, 9, 12, 11].

Notice that Zhang and Zhu [35] have proved Yau’s gradient estimate on Alexandrov spaces with a new Ricci curvature condition (see [34]).

On a complete metric space satisfying Lott-Sturm-Villani’s curvature condition CD(K, N) for some $K \in \mathbb{R}$ and $N \in (1, \infty)$ (see [24, 33], in [24] only CD(0, N) condition is introduced), Sturm [33, Corollary 2.4] shows that a local doubling condition holds, and a global doubling condition holds if $K \geq 0$. Moreover, it is proved by Rajala [25, 26] that a local weak $L^2$-Poincaré inequality holds on them, and a uniform $L^2$-Poincaré inequality holds if $K \geq 0$.

However, as $CD(K, N)$ conditions include the Finsler geometry, it is not known if the heat semigroup curvature condition holds under them. Recently, Ambrosio et. al. [2] (see also [1]) introduced a Riemannian Ricci curvature condition $RCD(K, \infty)$ on metric spaces, under which Bakry-Emery’s curvature condition holds (see [2, Theorem 6.2]) for the minimal weak upper gradient. The heat semigroup curvature condition then holds under $RCD(K, \infty)$ conditions via an argument of Bakry [3].

Consequently, the gradient estimates in Theorems 1.1 and 1.2 apply on metric spaces satisfying both $CD(K, N)$ and $RCD(K, \infty)$.

The paper is organized as follows. In Section 2, we give some basic notation and notions for Cheeger derivatives, Dirichlet forms and heat kernels. Section 3 is devoted to establishing gradient estimates for equations of type $Lu = g(x, u)$ with the assumption that $g(x, u)$ is bounded. The main results are proved in Section 4.

Finally, we make some conventions. Throughout the paper, we denote by $C, c$ positive constants which are independent of the main parameters, but which may vary from line to line. The symbol $B(x, R)$ denotes an open ball with center $x$ and radius $R$, and $CB(x, R) = B(x, CR)$.

2 Preliminaries

2.1 Cheeger Derivative in metric measure spaces

The following result due to Cheeger [8] gives us a derivative operator on metric measure spaces.

**Theorem 2.1.** Assume that $(X, \mu)$ supports a local weak $L^2$-Poincaré inequality and that $\mu$ is doubling. Then there exists $N > 0$, depending only on the doubling constant and the constants in the Poincaré inequality, such that the following holds. There exists a countable collection of measurable sets $U_\alpha$, $\mu(U_\alpha) > 0$ for all $\alpha$, and Lipschitz functions $X_1^\alpha, \ldots, X_{k(\alpha)}^\alpha : U_\alpha \to \mathbb{R}$, with $1 \leq k(\alpha) \leq N$ such that $\mu(X \setminus \cup_{\alpha=1}^{\infty} U_\alpha) = 0$, and for all $\alpha$ the following holds: for $f : X \to \mathbb{R}$ Lipschitz, there exist $V_\alpha(f) \subseteq U_\alpha$ such that $\mu(U_\alpha \setminus V_\alpha(f)) = 0$, and Borel functions $b_1^\alpha(x, f), \ldots, b_{k(\alpha)}^\alpha(x, f)$ of class $L^\infty$ such that if $x \in V_\alpha(f)$, then

$$\text{Lip}(f - a_1 X_1^\alpha - \cdots - a_{k(\alpha)} X_{k(\alpha)}^\alpha)(x) = 0$$
if and only if \((a_1, \cdots, a_{k(\alpha)}) = (b_1^\alpha(x, f), \cdots, b_{k(\alpha)}^\alpha(x, f))\). Moreover, for almost every \(x \in U_{a_1} \cap U_{a_2}\), the “coordinate functions” \(X_i^{a_1}\) are linear combinations of the \(X_i^{a_2}\)’s.

By Theorem 2.1, for each Lipschitz function \(u\) we can assign a Cheeger derivative \(Du\), and for each locally Lipschitz \(f\), \(\text{Lip } f\) is are comparable to \(|Du|\) almost everywhere.

By [27] and [8], the Sobolev spaces \(H^{1,2}(X)\) are isometrically equivalent to the Newtonian Sobolev spaces \(N^{1,2}(X)\) defined in [27]. For a domain \(\Omega \subset X\), following [19], we define the Sobolev space with zero boundary values \(H^{1,2}_0(\Omega)\) to be the space of those \(u \in H^{1,2}(X)\) for which \(u|_{X \setminus \Omega}\) vanishes except a set of 2-capacity zero. A useful fact is that the Cheeger derivative satisfies the Leibniz rule, i.e., for all \(u, v \in H^{1,2}(X)\),

\[
D(uv)(x) = u(x)Dv(x) + v(x)Du(x).
\]

### 2.2 Dirichlet forms and heat kernels

In this subsection, we recall the Dirichlet forms and heat kernels from [5, 29, 30, 31]. Define the bilinear form \(\mathcal{E}\) by

\[
\mathcal{E}(f, g) = \int_X Df(x) \cdot Dg(x) \, d\mu(x)
\]

with the domain \(D(\mathcal{E}) = H^{1,2}(X)\). It is easy to see that \(\mathcal{E}\) is symmetric and closed. Corresponding to such a form there exists an infinitesimal generator \(A\) which acts on a dense subspace \(D(A)\) of \(H^{1,2}(X)\) so that for all \(f \in D(A)\) and each \(g \in H^{1,2}(X)\),

\[
\int_X g(x)Af(x) \, d\mu(x) = -\mathcal{E}(g, f).
\]

From [20], we have the following Leibniz rule for Dirichlet forms.

**Lemma 2.1.** If \(u, v \in H^{1,2}(X)\), and \(\phi \in H^{1,2}(X)\) is a bounded Lipschitz function, then

\[
\mathcal{E}(\phi, uv) = \mathcal{E}(\phi u, v) + \mathcal{E}(\phi v, u) - 2 \int_X \phi Du(x) \cdot Dv(x) \, d\mu(x).
\]

Moreover, if \(u, v \in D(A)\), then we can unambiguously define the measure \(A(uv)\) by setting

\[
A(uv) = uAv + vAu + 2Du \cdot Dv.
\]

Associated with the Dirichlet form \(\mathcal{E}\), there is a semigroup \(\{T_t\}_{t \geq 0}\), acting on \(L^2(X)\). Moreover, there is a heat kernel \(p\) on \(X\), which is a measurable function on \(\mathbb{R} \times X \times X\) and satisfies

\[
T_t f(x) = \int_X f(y)p(t, x, y) \, d\mu(y)
\]

for every \(f \in L^2(X, \mu)\) and all \(t \geq 0\), and \(p(t, x, y) = 0\) for every \(t < 0\). Under the facts that the measure on \(X\) is locally doubling and a local \(L^2\)-Poincaré inequality holds, Sturm ([30, 31]) proved a Gaussian estimate for the heat kernel, i.e., for each \(t < R^2\) and all \(x, y \in X\),

\[
C^{-1} \mu(B(x, \sqrt{t}))^{-1/2} \mu(B(y, \sqrt{t}))^{-1/2} \exp \left\{ -\frac{d(x, y)^2}{C_2 t} \right\}
\]
As for every \( t > 0 \), then (2.1) can be written as
\[
(2.2) \quad C^{-1} \mu(B(x, \sqrt{t}))^{-1} \exp\left\{ -\frac{d(x, y)^2}{C_1 t} \right\} \leq p(t, x, y) \leq C \mu(B(x, \sqrt{t}))^{-1} \exp\left\{ -\frac{d(x, y)^2}{C_1 t} \right\}.
\]

Moreover, if the measure is globally doubling and a uniform \( L^2 \)-Poincaré inequality holds, the estimates (2.1) and (2.2) then hold for every \( t > 0 \) and all \( x, y \in X \).

By the assumption that the metric space is stochastically complete, we know that the heat kernel is a probability measure, i.e., for each \( x \in X \) and \( t > 0 \),
\[
T_t 1(x) = \int_X p(t, x, y) \, d\mu(y) = 1.
\]

Notice that heat kernel is a probability measure if the measure \( \mu \) on a ball \( B(x, r) \) does not grow faster than \( e^{c r^2} \) (see [29]).

The following lemma is essentially a Caccioppoli type inequality for heat equations.

**Lemma 2.2.** There exist \( c, C > 0 \) such that for every \( x \in X \),
\[
\int_0^s \int_{B(x, 2R) \setminus B(x, R)} |D_y p(t, x, y)|^2 \, d\mu(y) \, dt \leq C \mu(B(x, R))^{-1} e^{-cR^2/s},
\]
whenever \( R > 0 \) and \( s \in (0, R^2] \).

**Proof.** Let \( x \in X \) be fixed and
\[
\phi_x(y) := \min \left\{ 1, \frac{1}{R} \operatorname{dist}(y, X \setminus B(x, 3R)), \frac{2}{R} \operatorname{dist}(y, B(x, R/2)) \right\}
\]
for every \( y \in X \). Then \( |D \phi_x(y)| \leq C/R \) and we have
\[
\int_0^s \int_X |D_y p(t, x, y)|^2 \, d\mu(y) \, dt = \int_0^s \int_X |D_y p(t, x, y)|^2 \phi_x^2(y) \, d\mu(y) \, dt \geq \frac{1}{2} \int_0^s \int_X \left\{ |D_y p(t, x, y)|^2 \phi_x^2(y) - 4p(t, x, y)^2 |D \phi_x(y)|^2 \right\} \, d\mu(y) \, dt \geq \frac{1}{2} \int_0^s \int_{B(x, 2R) \setminus B(x, R)} |D_y p(t, x, y)|^2 \, d\mu(y) \, dt - \frac{C}{R^2} \int_0^s \int_{B(x, 3R) \setminus B(x, R/2)} p(t, x, y)^2 \, d\mu(y) \, dt.
\]

As for every \( y \in \operatorname{supp} \phi_x \), \( d(x, y) < 3R \). By using the doubling condition, (2.2) and (2.3), we further deduce that
\[
\frac{C}{R^2} \int_0^s \int_{B(x, 3R) \setminus B(x, R/2)} p(t, x, y)^2 \, d\mu(y) \, dt
\]
Cheeger-harmonic functions

\[ \int_0^s \int_{B(x,3R)} \int_{B(x,2R)} \mu(B(x, \sqrt{t}))^{-1} \exp \left\{ -\frac{d(x,y)^2}{C_1 t} \right\} p(t,x,y) \, d\mu(y) \, dt \]

\[ \leq \frac{C}{R^2} \int_0^s \mu(B(x,R))^{-1} \frac{R^2}{c} \exp \left\{ -\frac{R^2}{ct} \right\} \int_0^s p(t,x,y) \, d\mu(y) \, dt \]

\[ \leq \frac{C}{R^2} \mu(B(x,R))^{-1} \exp \left\{ -\frac{R^2}{cs} \right\} \int_0^s dt \leq C \mu(B(x,R))^{-1} \exp \left\{ -\frac{R^2}{cs} \right\}. \]

On the other hand, notice that \( \phi_x^2(y) = 0 \) on \( B(x,R/2) \). By using the property of heat semigroup, we have

\[ \int_0^s \int_X D_y^2 p(t,x,y) \cdot D_y^2 (p(t,x,y)\phi_x^2(y)) \, d\mu(y) \, dt \]

\[ = - \int_0^s \int_X \frac{\partial}{\partial t} p(t,x,y) p(t,x,y) \phi_x^2(y) \, d\mu(y) \, dt = - \frac{1}{2} \int_X p(s,x,y) \phi_x^2(y) \, d\mu(y) \leq 0 \]

Combining the above estimates, we see that

\[ \int_0^s \int_{B(x,2R) \setminus B(x,R)} |D_y^2 p(t,x,y)|^2 \, d\mu(y) \, dt \]

\[ \leq \frac{C}{R^2} \int_0^s \int_{B(x,3R) \setminus B(x,2R)} p(t,x,y)^2 \, d\mu(y) \, dt + \int_0^s \int_X D_y^2 p(t,x,y) \cdot D_y^2 (p(t,x,y)\phi_x^2(y)) \, d\mu(y) \, dt \]

\[ \leq C \mu(B(x,R))^{-1} \exp \left\{ -\frac{R^2}{cs} \right\}, \]

which proves the lemma. \( \square \)

3 From Hölder to Lipschitz

The main aim of this section is to prove the following result.

**Theorem 3.1.** Let \((X,d)\) be a stochastically complete metric space with a locally \(Q\)-doubling measure \(\mu\), \(Q \in (1,\infty)\). Assume that \((X,d,\mu)\) supports a local weak \(L^2\)-Poincaré inequality and the heat semigroup curvature condition (1.4).

Let \(u\) be a solution to the equation \(Lu = g(x,u)\) in \(\Omega \subset X\) with \(g(x,u) \in L^\infty_{\text{loc}}(\Omega)\). Then \(u\) is locally Lipschitz in \(\Omega\). More precisely, for each ball \(B = B(y_0, R)\) with \(8B \subset \subset \Omega\) there exists \(C = C(Q, C_\Omega(8R), C_\Omega(8R))\) such that

\[ \|Du\|_{L^\infty(B)} \leq C \left( \frac{1}{R} + \sqrt{c(x,R^2)} \right) \|u\|_{L^\infty(8B)} + R^2 \|g(\cdot, u)\|_{L^\infty(8B)}. \]

Throughout this section, we will always let the assumptions and notions be the same as in Theorem 3.1 unless otherwise stated. Moreover, we always let \(\psi\) be a cut-off function, which is Lipschitz and \(\psi = 1\) on \(B(y_0, 2R)\), \(\text{supp} \psi \subset B(y_0, 4R)\) and \(|D\psi| \leq \frac{C}{R}\).

The following functional is the main tool for us; see [7, 20, 16]. Let \(x_0 \in B = B(y_0, R)\). For all \(t \in (0, R^2)\), define

\[ J(x_0, t) := \frac{1}{t} \int_0^t \int_X |D(\psi u)(x)|^2 p(s,x_0,x) \, d\mu(x) \, ds. \]
Lemma 3.1. The solution $u$ is locally Hölder continuous in $\Omega$. More precisely, there exists $\gamma \in (0, 1)$ such that for almost all $x, y \in 2B = B(y_0, 2R)$, it holds

$$|u(x) - u(y)| \leq C \left( \|u\|_{L^\infty(4B)} + R^2 \|g(\cdot, u)\|_{L^\infty(4B)} \right) \frac{d(x, y)^\gamma}{R},$$

where $C = C(Q, C_Q(4R), C_P(4R))$.

Proof. As $g(x, u) \in L^\infty(4B)$, from [5], there exists $v \in H^{1, 2}_0(4B)$ such that $Lv = g(x, u) \in 4B$. By [5], we see that

$$\|v\|_{L^\infty(4B)} \leq CR^2 \|g(\cdot, u)\|_{L^\infty(4B)},$$

and there is $\gamma_1 \in (0, 1)$, independent of $u, g, B$, such that for almost all $x, y \in 2B$,

$$|v(x) - v(y)| \leq CR^2 \|g(\cdot, u)\|_{L^\infty(4B)} \left( \frac{d(x, y)}{R} \right)^{\gamma_1}.$$

Moreover, since $u - v$ is harmonic in $4B$, we deduce from [5, corollary 1.2] that

$$|(u - v)(x) - (u - v)(y)| \leq C \left( \int_{4B} |u - v|\, d\mu \right) \left( \frac{d(x, y)}{R} \right)^{\gamma_2} \leq C \left( \|u\|_{L^\infty(4B)} + R^2 \|g(\cdot, u)\|_{L^\infty(4B)} \right) \left( \frac{d(x, y)}{R} \right)^{\gamma_2},$$

for some $\gamma_2 \in (0, 1)$. By letting $\gamma = \min(\gamma_1, \gamma_2)$, we complete the proof. \qed

Lemma 3.2. There exists $C = C(Q, C_Q(4R), C_P(4R)) > 0$ such that for almost all $x_0 \in B$, $x \in 2B$ and all $t \in (0, R^2)$, it holds

$$|u\psi(x) - T_t(u\psi)(x_0)| \leq C \left( \|u\|_{L^\infty(4B)} + R^2 \|g(\cdot, u)\|_{L^\infty(4B)} \right) R^{-\gamma}(d(x, x_0)^\gamma + t^{\gamma/2}),$$

and

$$T_t(|u\psi(\cdot) - (u\psi)(x_0)|)(x_0) \leq C \left( \|u\|_{L^\infty(4B)} + R^2 \|g(\cdot, u)\|_{L^\infty(4B)} \right) R^{-\gamma}t^{\gamma/2},$$

Proof. Let $C(u, g) = \|u\|_{L^\infty(4B)} + R^2 \|g(\cdot, u)\|_{L^\infty(4B)}$. From the previous lemma, we see that for almost all $x_0, x \in 2B$,

$$|u(x) - u(x_0)| \leq CC(u, g) \left( \frac{d(x, x_0)}{R} \right)^\gamma.$$
Cheeger-harmonic functions

\[ +e^{-eR^2/|t|} \|u\|_{L^\infty(4B)} \int_{X \backslash 2B} \frac{1}{\mu(B(x_0, \sqrt{t}))} \frac{1}{2} \mu(B(y, \sqrt{t})) \frac{1}{e} e^{-eR^2/|t|} \, d\mu(y) \]
\[ \leq C \left[ C(u, g) R^{-\gamma/2} + \|u\|_{L^\infty(4B)} e^{-eR^2/|t|} \right] \int_X p(l, t, x_0, y) \, d\mu(y) \]
\[ \leq CC(u, g) \left[ R^{-\gamma/2} \right]. \]

where \( l = \frac{2C}{C^2} \), which proves the second inequality and implies that

\[ |u(x) \psi(x) - T_t(u \psi)(x_0)| \leq |u(x) \psi(x) - u(x_0) \psi(x_0)| + |T_t(u(x_0) \psi)(x_0)| - T_t(u \psi)(x_0) \]
\[ \leq CC(u, g) \left[ R^{-\gamma}(d(x, x_0)^2 + t^{\gamma/2}) \right]. \]

The proof is then completed. \( \Box \)

**Proposition 3.1.** There exists \( C = C(Q, C_F(8R), C_Q(8R)) > 0 \) such that for almost every \( x_0 \in B \)

\[ J(x_0, R^2) \leq C \left[ \frac{1}{R^2} \|u\|^2_{L^\infty(8B)} + R^2 \|g(\cdot, u)\|^2_{L^2(8B)} \right]. \]

**Proof.** For each \( 0 < \epsilon < t \leq R^2 \), set

\[ J_\epsilon(x_0, t) := \frac{1}{t} \int_\epsilon^t \int_X |D(u \psi)(x)|^2 p(s, x_0, x) \, d\mu(x) \, ds. \]

Notice that

\[ |D(u \psi)|^2 = \frac{1}{2} A(u \psi)^2 - A(u \psi)T_s(u \psi)(x_0) - [u \psi - T_s(u \psi)(x_0)](\psi Au + uA\psi + 2Du \cdot D\psi) \]

in the weak sense of measures. Thus, for each \( 0 < \epsilon < t \) we have

\[ tJ_\epsilon(x_0, t) = \frac{1}{2} \int_\epsilon^t \int_X [A((u \psi)^2)(x) - 2A(u \psi)(x)T_s(u \psi)(x_0)] p(s, x_0, x) \, d\mu(x) \, ds \]
\[ - \int_\epsilon^t \int_X [(u \psi)(x) - T_s(u \psi)(x_0)](\psi Au + uA\psi + 2Du \cdot D\psi) p(s, x_0, x) \, d\mu(x) \, ds. \]

As the heat kernel is a solution to the heat equation \( Au = \frac{\partial}{\partial t} u \) on \( (\epsilon, t) \times X \) (see Sturm [30, proposition 2.3]), we further deduce that

\[ \int_\epsilon^t \int_X [A((u \psi)^2)(x) - 2A(u \psi)(x)T_s(u \psi)(x_0)] p(s, x_0, x) \, d\mu(x) \, ds \]
\[ = [T_t((u \psi)^2)(x_0) - T_\epsilon((u \psi)^2)(x_0)] - \int_\epsilon^t 2T_s(u \psi)(x_0)AT_s(u \psi)(x_0) \, ds \]
\[ = [T_t((u \psi)^2)(x_0) - T_\epsilon((u \psi)^2)(x_0)] - \int_\epsilon^t \frac{\partial}{\partial s}[T_s(u \psi)(x_0)]^2 \, ds \]
\[ = \left( T_t((u \psi)^2)(x_0) - [T_\epsilon(u \psi)(x_0)]^2 \right) - \left( T_\epsilon((u \psi)^2)(x_0) - [T_\epsilon(u \psi)(x_0)]^2 \right). \]
As the functions \((u\psi)(x) - T_s(u\psi)(x_0), p(s, x_0, x)\) and \(\psi\) are bounded functions with gradient in \(L^2(X)\), and \(\text{supp } \{|(u\psi)(x) - T_s(u\psi)(x_0)|\psi p(s, x_0, x)| \subset 4B\), we deduce from Lemma 3.2 that

\[
\left| \int_\varepsilon^T \int_X [(u\psi)(x) - T_s(u\psi)(x_0)]\psi Aup(s, x_0, x) \, d\mu(x) \, ds \right|
\leq \int_\varepsilon^T \int_X [(u\psi)(x) - T_s(u\psi)(x_0)]\psi g(x, u(x)) p(s, x_0, x) \, d\mu(x) \, ds
\leq \int_\varepsilon^T \int_X [(u\psi)(x) - T_s(u\psi)(x_0) - T_s(u\psi - (u\psi)(x_0))(x_0)]\psi g(x, u(x)) p(s, x_0, x) \, d\mu(x) \, ds
\leq \|\psi g(\cdot, u)\|_{L^\infty(4B)} \int_\varepsilon^T T_s(|u\psi - (u\psi)(x_0)|)(x_0) \, ds
\leq C\|g(\cdot, u)\|_{L^\infty(4B)} \left[ \|u\|_{L^\infty(4B)} + R^2\|g(\cdot, u)\|_{L^\infty(4B)} \right] \int_0^T R^{-\gamma} s^{\gamma/2} \, ds
\leq C\|g(\cdot, u)\|_{L^\infty(4B)} \left[ \|u\|_{L^\infty(4B)} + R^2\|g(\cdot, u)\|_{L^\infty(4B)} \right] R^{-\gamma} t^{1+\gamma/2}.
\]

The estimates for second and third terms in (3.3) follow from the following Lemma 3.4,

\[
\left| \int_\varepsilon^T \int_X [(u\psi)(x) - T_s(u\psi)(x_0)] p(s, x_0, x)[u(x)A\psi(x) + 2Du(x) \cdot D\psi(x)] \, d\mu(x) \, ds \right|
\leq C e^{-cR^2/t}\|u\|_{L^\infty(8B)} \left[ \|u\|_{L^\infty(8B)} + R^2\|g(\cdot, u)\|_{L^\infty(8B)} \right].
\tag{3.5}
\]

As the underlying space is stochastically complete, the Hölder inequality implies for each \(t > 0\),

\[T_s((u\psi)^2)(x_0) - [T_s(u\psi)(x_0)]^2 \geq 0.\]

Combining the above estimates, by (3.3), we obtain that

\[
J_\varepsilon(x_0, t) \leq \frac{1}{2t} \left| \int_\varepsilon^T \int_X [A((u\psi)^2)(x) - 2A(u\psi)(x)T_s(u\psi)(x_0)] p(s, x_0, x) \, d\mu(x) \, ds \right|
+ \frac{1}{t} \left| \int_\varepsilon^T \int_X [(u\psi)(x) - T_s(u\psi)(x_0)] [Au + uA\psi + 2Du \cdot D\psi]p(s, x_0, x) \, d\mu(x) \, ds \right|
\leq \frac{1}{2t} \left[ T_s((u\psi)^2)(x_0) - [T_s(u\psi)(x_0)]^2 \right] + \frac{1}{2t} \left( T_s((u\psi)^2)(x_0) - [T_s(u\psi)(x_0)]^2 \right)
+ C e^{-cR^2/t} \left( \|u\|_{L^\infty(8B)} + R^2\|u\|_{L^\infty(8B)} \|g(\cdot, u)\|_{L^\infty(8B)} \right)
+ C\|g(\cdot, u)\|_{L^\infty(8B)} \left( \|u\|_{L^\infty(4B)} + R^2\|g(\cdot, u)\|_{L^\infty(4B)} \right) R^{-\gamma} t^{\gamma/2}
\leq \frac{1}{2t} \left[ T_s((u\psi)^2)(x_0) - [T_s(u\psi)(x_0)]^2 \right] + \frac{1}{2t} \left( T_s((u\psi)^2)(x_0) - [T_s(u\psi)(x_0)]^2 \right)
+ CR^{-\gamma} t^{\gamma/2} \left( \frac{1}{R^2} \|u\|_{L^\infty(8B)}^2 + R^2\|g(\cdot, u)\|_{L^\infty(8B)}^2 \right).
\]

Finally, observe that for almost every \(x_0\), it holds

\[
\lim_{\varepsilon \to 0} \left( T_s((u\psi)^2)(x_0) - [T_s(u\psi)(x_0)]^2 \right) = (u\psi)(x_0)^2 - (u\psi)(x_0)^2 = 0;
\]
see also the following inequality (3.7). Hence, the monotone convergence theorem gives us
\[ J(x_0, t) = \lim_{\epsilon \to 0} J_\epsilon(x_0, t) \leq \frac{1}{2t} \left\{ T_\epsilon((u\psi)^2)(x_0) - [T_\epsilon(u\psi)(x_0)]^2 \right\} \]
(3.6)
\[ + CR^{-\gamma} r^{n/2} \left[ \frac{1}{R^2} \|u\|^2_{L^\infty(B)} + R^2 \|\gamma(\cdot, u)\|^2_{L^\infty(B)} \right]. \]
Letting \( t = R^2 \) completes the proof of Proposition 3.1. \( \square \)

**Remark 3.1.** In [20], it was proved that
\[ \int_0^t \int_X [A((u\psi)^2)(x) - 2A(u\psi)(x)T_\epsilon(u\psi)(x_0)] p(s, x_0, x) d\mu(x) d\mu(s) = T_\epsilon((u\psi)^2)(x_0) - [T_\epsilon(u\psi)(x_0)]^2, \]
which was also used in [16]. The proof of the equality needs a careful argument to deal with the singularity of \(-\frac{\partial}{\partial t} p + A_s p(\cdot, x_0, \cdot)\) at \((0, x_0, x_0)\); see [20, proposition 3.4]. As pointed out by Kell [18], an upper bound for measure of balls with small radius is needed in the proof, i.e., \( \mu(B(x, r)) \leq Cr \) for \( r < 1 \), see [20, p.160]. We do not know if this is true in our settings.

Our proof of Proposition 3.1 above avoids using this equality, and is more direct and easier.

To estimate the remaining term (3.5) in Proposition 3.1 we recall the Caccioppoli inequality, which follows by inserting a suitable test function into the equation.

**Lemma 3.3 (Caccioppoli inequality).** If \( Lu = g(\cdot, u) \) in \( B(y_0, R) \), then there exists \( C > 0 \) independent of \( Q, C_Q, C_P \) such that for every \( r < R \) it holds
\[ \int_{B(y_0, r)} |Du|^2 d\mu \leq \frac{C}{(R - r)^2} \int_{B(y_0, R)} |u|^2 d\mu + \int_{B(y_0, R)} |g(x, u(x))| \|u(x)\| d\mu(x). \]

**Lemma 3.4.** There exists \( C = C(Q, C_Q(8R), C_P(8R)) > 0 \) such that for almost all \( x_0 \in B = B(y_0, R), x \in 2B \) and all \( 0 < \epsilon < t \leq R^2 \), it holds
\[ \left| \int_{\epsilon}^t \int_X [A((u\psi)^2)(x) - T_\epsilon(u\psi)(x_0)] p(s, x_0, x)u(x)A\psi(x) + 2Du(x) \cdot D\psi(x) \right] d\mu(x) ds \]
\[ \leq C e^{\frac{\alpha^2}{C}} \|u\|_{L^\infty(B)} \left( \|u\|_{L^\infty(B)} + R^2 \|\gamma(\cdot, u)\|_{L^\infty(B)} \right). \]

**Proof.** Let \( w(x, s) = u(x)\psi(x) - T_\epsilon(u\psi)(x_0) \) and notice that
\[
\begin{align*}
w(x, s) & = u(x)\psi(x) - T_\epsilon(u\psi)(x_0) \\
& = [w(x, s)p(s, x_0, x)Du(x) - Du(x)u(x)p(s, x_0, x) - w(x, s)u(x)Dp(s, x_0, x)] \cdot D\psi(x) \\
& = -T_\epsilon(u\psi)(x_0)p(s, x_0, x)Du(x) \cdot D\psi(x) - |D\psi(x)|^2 u(x)^2 p(s, x_0, x) \\
& \quad - w(x, s)u(x)Dp(s, x_0, x) \cdot D\psi(x) \\
& =: H_1 + H_2 + H_3
\end{align*}
\]
in the weak sense of measures. By using the Caccioppoli inequality (Lemma 3.3) and the Hölder inequality, we obtain
\[ \left| \int_0^t \int_X H_1 d\mu(x) ds \right| \]
By (2.2), we have
\[
\begin{align*}
&\leq \int_\varepsilon^T \int_{B(\gamma_0,4R)\setminus B(\gamma_0,2R)} |T_s(u\psi)(x_0)| |Du(x)| |D\psi(x)| p(s, x_0, x) \, d\mu(x) \, ds \\
&\leq \frac{C}{R} \int_0^T \int_{B(\gamma_0,4R)\setminus B(\gamma_0,2R)} |Du(x)| \frac{R^Q}{s^{Q/2} \mu(B(x_0, R))} e^{-\frac{x_0^2}{ct}} \, d\mu(x) \, ds \\
&\leq \frac{C t}{R} \frac{1}{\mu(B(x_0, R))^{1/2}} e^{-\frac{x_0^2}{ct}} \int_{B(\gamma_0,4R)\setminus B(\gamma_0,2R)} |Du(x)| \, d\mu(x) \\
&\leq \frac{C t}{R} \frac{1}{\mu(B(x_0, R))^{1/2}} \int_{B(\gamma_0,4R)\setminus B(\gamma_0,2R)} |Du(x)| \, d\mu(x) \\
&\leq C e^{-\frac{x_0^2}{ct}} (|u|^2_{L^p(\gamma_0)} + R^2 |u|^2_{L^p(\gamma_0)} g(\cdot, u) |g(\cdot, u)|_{L^p(\gamma_0)}).
\end{align*}
\]

By (2.2), we have
\[
\left| \int_\varepsilon^T \int_X H_2 \, d\mu(x) \, ds \right| \leq \frac{C}{R^2} ||u||^2_{L^p(\gamma_0)} \int_0^T \int_{B(\gamma_0,4R)\setminus B(\gamma_0,2R)} \frac{1}{\mu(B(x_0, \sqrt{s}))} e^{-\frac{x_0^2}{ct}} \, d\mu(x) \, ds \\
\leq \frac{C}{R^2} ||u||^2_{L^p(\gamma_0)} \int_0^T \int_{B(\gamma_0,4R)\setminus B(\gamma_0,2R)} \frac{R^Q}{s^{Q/2} \mu(B(x_0, R))} e^{-\frac{x_0^2}{ct}} \, d\mu(x) \, ds \\
\leq C e^{-\frac{x_0^2}{ct}} ||u||^2_{L^p(\gamma_0)}.
\]

We use Lemma 2.2 and the Hölder inequality to estimate the last term,
\[
\left| \int_\varepsilon^T \int_X H_3 \, d\mu(x) \, ds \right| \\
\leq \frac{C}{R} \int_0^T \int_{B(\gamma_0,4R)\setminus B(\gamma_0,2R)} ||u(x)\psi(x) - T_s(u\psi)(x_0)|| |Dp(s, x_0, x)| \, d\mu(x) \, ds \\
\leq \frac{C}{R} ||u||^2_{L^p(\gamma_0)} \int_0^T \int_{B(\gamma_0,4R)\setminus B(\gamma_0,2R)} |Dp(s, x_0, x)| \, d\mu(x) \, ds \\
\leq \frac{C}{R} ||u||^2_{L^p(\gamma_0)} C \sqrt{t} \mu(B(x_0, R))^{1/2} \mu(B(x_0, R))^{-1/2} \exp \left\{ -\frac{R^2}{ct} \right\} \\
\leq C e^{-\frac{R^2}{ct}} ||u||^2_{L^p(\gamma_0)},
\]
which completes the proof.

\[\square\]

**Lemma 3.5.** There exists \( C = C(Q, C_p(4R), C_Q(4R)) > 0 \) such that almost all \( x_0 \in B = B(\gamma_0, R) \) it holds
\[
\int_0^{R^2} \left\{ T_t((u\psi)^2)(x_0) - \frac{1}{t} (T_t(u\psi)(x_0))^2 \right\} \, dt \leq C \left[ ||u||_{L^p(4B)} + R^2 ||g(\cdot, u)||_{L^p(4B)} \right]^2.
\]

**Proof.** Let \( C(u, g) = ||u||_{L^p(4B)} + R^2 ||g(\cdot, u)||_{L^p(4B)} \). By Lemma 3.2, we deduce that
\[
\begin{align*}
&T_t((u\psi)^2)(x_0) - \frac{1}{t} (T_t(u\psi)(x_0))^2 \\
&= \int_{2B} (u\psi)(x) - T_t(u\psi)(x_0))^2 p(t, x_0, x) \, d\mu(x) + \int_{X\setminus 2B} (u\psi)(x) - T_t(u\psi)(x_0))^2 p(t, x_0, x) \, d\mu(x)
\end{align*}
\]
Cheeger-harmonic functions

\[
\begin{align*}
& \leq CC(u, g)^2 \int_{B} \frac{(d(x, x_0)^2 + t'^2/2)^2}{R^2 t \mu(B(x_0, \sqrt{t}))^{1/2} \mu(B(x, \sqrt{t}))^{1/2}} e^{-\frac{d(x, x_0)^2}{2c_1 t'}} e^{-\frac{d(x, x_0)^2}{2c_1 t'}} \, d\mu(x) \\
& + C||u||_{L^\infty(4B)}^2 \int_{X \setminus B} \frac{1}{\mu(B(x_0, \sqrt{t}))^{1/2} \mu(B(x, \sqrt{t}))^{1/2}} e^{-\frac{d(x, x_0)^2}{2c_1 t'}} e^{-\frac{d(x, x_0)^2}{2c_1 t'}} \, d\mu(x) \\
& \leq CC(u, g)^2 \frac{t^y}{R^2} \int_{2B} p(t, x_0, x) \, d\mu(x) + C e^{-tR^2/4} ||u||_{L^\infty(4B)}^2 \int_{X \setminus 2B} p(t, x_0, x) \, d\mu(x) \\
& (3.7) \quad \leq C \left[ ||u||_{L^\infty(4B)} + R^2 ||g(\cdot, u)||_{L^\infty(4B)} \right]^2 \frac{t^y}{R^2},
\end{align*}
\]

where \( l = \frac{2c_1}{c_2} \geq 2 \) and we used the doubling condition that \( \frac{1}{\mu(B(x_0, \sqrt{t}))} \leq C \frac{1}{\mu(B(x_0, l \sqrt{t}))} \). From this, we further conclude that

\[
\int_0^{R^2} \left[ T_t((u \psi)^2)(x_0) - [T_t(u \psi)(x_0)]^2 \right] \, dt \leq C \left[ ||u||_{L^\infty(4B)} + R^2 ||g(\cdot, u)||_{L^\infty(4B)} \right]^2,
\]

which completes the proof. \( \square \)

We remark here that Lemmas 3.1-3.5 and Proposition 3.1 only require a doubling condition and an \( L^2 \)-Poincaré inequality.

**Proof of Theorem 3.1.** Let us first estimate the derivative \( J'(x_0, t) = \frac{d}{dt} J(x_0, t) \). By (3.6), we deduce that

\[
\begin{align*}
\frac{d}{dt} J(x_0, t) &= -\frac{1}{R^2} J(x_0, t) + \frac{1}{t} \int_X |D(u \psi)(x)|^2 p(t, x_0, x) \, d\mu(x) \\
& \geq \frac{1}{t} \left( \int_X |D(u \psi)(x)|^2 p(t, x_0, x) \, d\mu(x) - \frac{1}{2t} \left[ T_t((u \psi)^2)(x_0) - [T_t(u \psi)(x_0)]^2 \right] \right) \\
& \quad - \frac{C t'^{y/2-1}}{R^2} \left[ \frac{1}{R^2} ||u||_{L^\infty(8B)}^2 + R^2 ||g(\cdot, u)||_{L^\infty(8B)}^2 \right].
\end{align*}
\]

For each fixed \( t \in (0, R^2) \), either

\[
\int_X |D(u \psi)(x)|^2 p(t, x_0, x) \, d\mu(x) \geq \frac{1}{2t} \left[ T_t((u \psi)^2)(x_0) - [T_t(u \psi)(x_0)]^2 \right]
\]

or

\[
\int_X |D(u \psi)(x)|^2 p(t, x_0, x) \, d\mu(x) \leq \frac{1}{2t} \left[ T_t((u \psi)^2)(x_0) - [T_t(u \psi)(x_0)]^2 \right].
\]

In the first case, we have

\[
(3.8) \quad \frac{d}{dt} J(x_0, t) \geq -\frac{C t'^{y/2-1}}{R^2} \left[ \frac{1}{R^2} ||u||_{L^\infty(8B)}^2 + R^2 ||g(\cdot, u)||_{L^\infty(8B)}^2 \right].
\]

In the second case, by the curvature condition (1.4) with \( T = R^2 \), we deduce that

\[
\frac{d}{dt} J(x_0, t) \geq -c_4(R^2) \int_X |D(u \psi)(x)|^2 p(t, x_0, x) \, d\mu(x)
\]
\[-\frac{C_{\gamma/2-1}}{R^\gamma} \left[ \frac{1}{R^2} ||u||^2_{L^\infty(B)} + R^2 ||g(x,u)||^2_{L^\infty(B)} \right] \]
\[ \geq -\frac{c_x(R^2)}{2t} \left\{ T_t((u\psi)^2)(x_0) - [T_t(u\psi)(x_0)]^2 \right\} \]
\[ - \frac{C_{\gamma/2-1}}{R^\gamma} \left[ \frac{1}{R^2} ||u||^2_{L^\infty(B)} + R^2 ||g(x,u)||^2_{L^\infty(B)} \right]. \tag{3.9} \]

From (3.8) and (3.9), we see that (3.9) holds in both cases. Integrating over \((0,R^2)\) and applying Lemma 3.5 we conclude that
\[
\int_0^{R^2} J'(x_0,t) \, dt \geq - \int_0^{R^2} c_x(R^2) \left\{ T_t((u\psi)^2)(x_0) - [T_t(u\psi)(x_0)]^2 \right\} \, dt 
+ C \left[ \frac{1}{R^2} ||u||^2_{L^\infty(B)} + R^2 ||g(x,u)||^2_{L^\infty(B)} \right] 
\geq - C \left( \frac{1}{R^2} + c_x(R^2) \right) \left[ ||u||^2_{L^\infty(B)} + R^4 ||g(x,u)||^2_{L^\infty(B)} \right],
\]
which completes the proof of Theorem 3.1. \hfill \Box

4 \hspace{1em} Proof of the main results

Recall that a Sobolev function \(u \in H^{1,2}(\Omega)\) is called non-negative sub-harmonic, \(Lu \geq 0\), in \(\Omega\) if \(u \geq 0\) on \(\Omega\), and
\[ -\int_{\Omega} Du(x) \cdot D\phi(x) \, d\mu(x) \geq 0, \quad \forall \phi \in H^{1,2}_0(\Omega) \text{ and } \phi \geq 0. \tag{4.1} \]

Similarly, \(u\) is called non-negative super-harmonic, \(Lu \leq 0\), in \(\Omega\) if \(u \geq 0\) on \(\Omega\), and
\[ -\int_{\Omega} Du(x) \cdot D\phi(x) \, d\mu(x) \leq 0, \quad \forall \phi \in H^{1,2}_0(\Omega) \text{ and } \phi \geq 0. \tag{4.2} \]

Lemma 4.1. Let \((X,d)\) be a metric space with a locally \(Q\)-doubling measure \(\mu, Q \in (1,\infty)\). Assume that \((X,d,\mu)\) supports a local weak \(L^2\)-Poincaré inequality. Let \(u\) be a solution to the equation \(Lu = -\lambda u\) in \(2B\), where \(B = B(y_0,2R)\) and \(\lambda \in L^\infty(2B)\). Then for each \(p > 0\), there exists \(C = C(Q,C_Q(2R),C_p(2R),p,||\lambda||_{L^\infty(2B)R^2})\) such that
\[
||u||_{L^\infty(B(y_0,R))} \leq C \left( \int_{B(y_0,2R)} |u|^p \, d\mu \right)^{1/p}. 
\]
Proof. From Lemma 3.3, we have the following Caccioppoli inequality
\[\int_{B(y_0,r_1)} |Du|^2 \, d\mu \leq \frac{C}{(r_2 - r_1)^2} \int_{B(y_0,r_2)} |u|^2 \, d\mu + \int_{B(y_0,r_2)} |(\lambda u)(x)||u(x)| \, d\mu(x)\]
\[\leq \frac{C(1 + ||\lambda||_{L^\infty(B_0,R^2)})}{(r_2 - r_1)^2} \int_{B(y_0,r_2)} |u|^2 \, d\mu,\]
for arbitrary \(0 < r_1 < r_2 \leq 2R\). By using the Caccioppoli inequality and the Sobolev inequality (see [13, 28, 31]), the proof follows via the Moser iterations technique and [5, Lemma 5.2]; see [22] for instance. We omit the details here.

\[\square\]

Remark 4.1. For \(Q \in (1, 2)\), we can have a better estimate by combining the Caccioppoli inequality and the Sobolev-Poincaré inequality from [13]. Precisely,
\[||u||_{L^\infty(B)} \leq ||u\phi||_{L^\infty(B(y_0,3R/2))} \leq \int_{B(y_0,3R/2)} |u| \, d\mu + CR \left( \int_{B(y_0,3R/2)} |D(u\phi)|^2 \, d\mu \right)^{1/2} \]
\[\leq C(1 + \sqrt{||\lambda||_{L^\infty(2B)}R^2}) \left( \int_{B(y_0,2R)} |u|^2 \, d\mu \right)^{1/2},\]
where \(\phi\) is a cut-off function that equals one on \(B(y_0, R)\) and is supported in \(B(y_0, 3R/2)\).

For \(Q \in (2, \infty)\), the Moser iteration would give us that
\[||u||_{L^\infty(B(y_0,R))} \leq C(Q, C_p(2R), C_Q(2R)) \left(1 + ||\lambda||_{L^\infty(2B)}R^2\right)^{Q/4} \left( \int_{B(y_0,2R)} |u|^2 \, d\mu \right)^{1/2} .\]
The above estimate implies that for the case \(Q = 2\), it holds
\[||u||_{L^\infty(B(y_0,R))} \leq C(Q, C_p(2R), C_Q(2R)) \left(1 + ||\lambda||_{L^\infty(2B)}R^2\right)^{\epsilon+1/2} \left( \int_{B(y_0,2R)} |u|^2 \, d\mu \right)^{1/2},\]
for arbitrary \(\epsilon > 0\).

We choose to avoid the precise dependence on \(\lambda\) mainly because our main aim in the paper is to give gradient estimates for harmonic functions, and also to avoid complicated calculations.

Lemma 4.2. Let \((X, d)\) be a metric space with a locally \(Q\)-doubling measure \(\mu\), \(Q \in (1, \infty)\). Assume that \((X, d, \mu)\) supports a local weak \(L^2\)-Poincaré inequality. Let \(u\) be a positive superharmonic function in \(B(y_0, 2R)\). Then there exists \(q > 0\) and \(C = C(Q, C_Q(2R), C_p(2R), q)\) such that
\[\left( \int_{B(y_0,2R)} u^q \, d\mu \right)^{1/q} \leq \inf_{x \in B(y_0,R)} u(x).
\]
Proof. From [5, Proposition 5.7], we see that there exists \(q > 0\) such that
\[\int_{B(y_0,2R)} u^q \, d\mu \int_{B(y_0,2R)} u^{-q} \, d\mu \leq C.
\]
Also from the Moser iteration, it holds
\[\left( \int_{B(y_0,2R)} u^{-q} \, d\mu \right)^{-1/q} \leq C\inf_{x \in B(y_0,R)} u(x);
\]
see [5, pp.162-163] for instance. The lemma follows from the above two inequalities. \(\square\)
Proof of Theorem 1.1. For each \( x \in B(y_0, R) \), we choose a ball \( B(x, R/16) \). Then for each \( z \in B(x, R) \), it holds
\[
d(z, y_0) \leq d(z, x) + d(x, y_0) < 2R,
\]
and hence \( B(x, R) \subset B(y_0, 2R) \). Lemma 4.1 and the doubling condition imply that
\[
\|u\|_{L^\infty(B(x, R/2))} \leq C \int_{B(x, R)} |u| \, d\mu \leq C \int_{B(y_0, 2R)} |u| \, d\mu.
\]
By Theorem 3.1, we obtain that
\[
\|[Du]\|_{L^\infty(B(x, R/16))} \leq C \left( \frac{1}{R} + \sqrt{c_s(R^2)} \right)^2 \left( \|u\|_{L^\infty(B(x, R/2))} + R^2 \|u\|_{L^\infty(B(x, R/2))} \right)
\leq C(Q, C_p(2R), C_Q(2R), \|\lambda\|_{L^\infty(2B)} R^2) \left( \frac{1}{R} + \sqrt{c_s(R^2)} \right) \int_{B(y_0, 2R)} |u| \, d\mu.
\]
Thus
\[
\|[Du]\|_{L^\infty(B(y_0, R))} \leq C(Q, C_p(2R), C_Q(2R), \|\lambda\|_{L^\infty(2B)} R^2) \left( \frac{1}{R} + \sqrt{c_s(R^2)} \right) \int_{B(y_0, 2R)} |u| \, d\mu,
\]
which proves Theorem 1.1. \( \square \)

Proof of Theorem 1.2. Let \( u \) be a positive harmonic function \( u \) on \( B(y_0, 2R) \). By Theorem 1.1 and Lemma 4.2, we have
\[
(4.3) \quad \|[Du]\|_{L^\infty(B(y_0, R))} \leq C(Q, C_Q(2R), C_p(2R)) \left( \sqrt{c_s(R^2)} + \frac{1}{R} \right) \inf_{x \in B(y_0, R)} u(x),
\]
which implies that for almost every \( x \in B(y_0, R) \),
\[
|Du(x)| \leq C(Q, C_Q(2R), C_p(2R)) \left( \sqrt{c_s(R^2)} + \frac{1}{R} \right) u(x),
\]
which proves (i).

Let \( c_s(1) > 0 \). If \( R < 1/2 \), then in the estimate we can always use \( C_Q(1), C_p(1), c_s(1) \) to replace \( C_Q(2R), C_p(2R), c_s(2R) \) and obtain
\[
|Du(x)| \leq C(Q, C_Q(1), C_p(1)) \left( \sqrt{c_s(1)} + \frac{1}{R} \right) u(x).
\]
If \( R > 1/2 \), then for each point \( x \in B(y_0, R) \), we choose a ball \( B(x, 1/8) \). Then \( B(x, 1/2) \subset B(y_0, 2R) \), by (4.3), we have
\[
\|[Du]\|_{L^\infty(B(x, 1/8))} \leq C(Q, C_Q(1), C_p(1)) \left( \sqrt{c_s(1)} + 1 \right) \inf_{y \in B(x, 1/8)} u(y),
\]
which implies that for almost every \( x \in B(y_0, R) \), it holds
\[
(4.4) \quad |Du(x)| \leq C(Q, C_Q(1), C_p(1)) \left( \sqrt{c_s(1)} + \frac{1}{R} \right) u(x),
\]
which completes the proof. \( \square \)
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