Universal Factor Graphs

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Abstract. The factor graph of an instance of a symmetric constraint satisfaction problem on \( n \) Boolean variables and \( m \) constraints (CSPs such as \( k \)-SAT, \( k \)-AND, \( k \)-LIN) is a bipartite graph describing which variables appear in which constraints. The factor graph describes the instance up to the polarity of the variables, and hence there are up to \( 2^{km} \) instances of the CSP that share the same factor graph. It is well known that factor graphs with certain structural properties make the underlying CSP easier to either solve exactly (e.g., for tree structures) or approximately (e.g., for planar structures). We are interested in the following question: is there a factor graph for which if one can solve every instance of the CSP with this particular factor graph, then one can solve every instance of the CSP regardless of the factor graph (and similarly, for approximation)? We call such a factor graph \emph{universal}. As one needs different factor graphs for different values of \( n \) and \( m \), this gives rise to the notion of a family of universal factor graphs.

We initiate a systematic study of universal factor graphs, and present some results for max-\( k \)SAT. Our work has connections with the notion of preprocessing as previously studied for closest codeword and closest lattice-vector problems, with proofs for the PCP theorem, and with tests for the long code. Many questions remain open.

1 Introduction

A constraint satisfaction problem (CSP) has a set of \( n \) variables and a set of \( m \) constraints (also referred to as clauses, or factors). Every constraint involves a subset of the variables, and is satisfied by some assignments to the variables and not satisfied by others. An instance of a CSP is satisfiable if there is an assignment to the variables that satisfies all constraints. When variables are Boolean and constraints are symmetric a constraint is fully specified by the set of literals that it contains (where a literal is either a variable or its negation), and is satisfied if and only if the appropriate number of literals is set to true (e.g., at least one for \( \text{SAT} \), an odd number for \( \text{XOR} \), all for \( \text{AND} \), the majority for \( \text{MAJ} \), and at least one but not all for \( \text{NAE} \)). To simplify the presentation, we shall consider in this paper CSPs that are Boolean and symmetric, though
we remark that much of what we discuss can be extended to non-Boolean and non-symmetric CSPs.

The factor graph of an instance of a CSP is a bipartite graph. Vertices on one side represent the variables, vertices on the other side represent the constraints (also known as factors), and edges connect constraints to the variables that they contain. For Boolean symmetric CSPs, a factor graph together with a labeling of the edges with ±1 (indicating whether the corresponding variable has positive or negative polarity in the underlying clause) completely specifies an instance of the CSP. Without the edge labels, there are many instances of the CSP that share the same factor graph and differ only in the polarity of the variables.

As is well known, deciding satisfiability for CSPs is NP-hard for a large class of predicates (including, SAT, MAJ and NAE). See [24] for a complete classification. Here we shall consider NP-hard CSPs. The research question that motivates our current paper is to understand what are the obstacles for obtaining efficient algorithms for solving CSPs. Specifically, are algorithms having trouble in “understanding” the structure of the factor graph, and this translates to difficulties in solving the underlying CSP? Alternatively, are the computational difficulties a result of the combinatorial richness of the polarities?

The structure of the factor graph may cause the underlying CSP instance to be easy. For example, if the factor graph is a tree (or more generally, of bounded treewidth), then the underlying CSP instance can be solved in polynomial time (by dynamic programming). Our research question (once properly formalized) can be viewed as asking whether in other cases, the structure of the factor graph might be the major contributing factor to making a CSP hard.

The playing field of our research agenda is greatly enriched once optimization versions of CSPs are considered, namely max-CSP: find an assignment to the variables that satisfies as many constraints as possible. As is well known, even some polynomial time solvable CSPs (such as XOR, or 2SAT) become NP-hard when their optimization version is considered. See [8] for a classification. A standard way of dealing with NP-hard max-CSP instances is via approximation algorithms that in polynomial time find an assignment that is guaranteed to satisfy a number of constraints that is at least \( \rho \) times the maximum number of constraints that can be satisfied, for some \( 0 < \rho < 1 \). For many CSPs, the best possible \( \rho \) is known, in the sense that the approximation ratios provided by known approximation algorithms are matched by hardness of approximation results that show that better approximation ratios would imply that \( \text{P}=\text{NP} \). For example, \( \rho = 7/8 \) is a tight approximation threshold for max-3SAT [12]. Moreover, for all CSPs, an algorithm (based on semidefinite programming) with the optimal approximation ratio is given by Raghavendra [24], assuming the Unique Games Conjecture of Khot [15]. However, despite the optimality of this algorithm, it is difficult to figure out which approximation ratio it guarantees, and consequently there are CSPs for which the value of this threshold is not known. (And of course, if the Unique Games Conjecture is false then the approximation ratio implied by this algorithm need not be tight.)
Our research agenda naturally extends to max-CSP. One may ask whether approximation algorithms are having trouble in “understanding” the structure of the factor graph, and whether this translates to difficulties in approximating the underlying CSP. Moreover, now the question acquires also a quantitative aspect, and one may ask to what extent does the factor graph contribute to the approximation difficulty. For example, if algorithms had no difficulty in “understanding” factor graphs, could the approximation ratio for max-3SAT be improved from $7/8$ to $8/9$?

As in the case of tree factor graphs for decision versions, there are known families of factor graphs (such as planar graphs, or more generally, families of graphs excluding a fixed minor) on which the underlying CSP instance has improved approximation ratios, or even a PTAS ($\rho > 1 - \epsilon$ for every $\epsilon > 0$). On the other hand, it appears that for some CSPs, almost every factor graph is difficult. For example, there is no known approximation algorithm that runs in polynomial time on random 3CNF formulas (with say $m = n \log n$ constraints) and approximates max-3SAT within a ratio better than $7/8$. This suggests (though does not prove) that there is no need for clever design of the factor graph in order to make the underlying CSP instance difficult – almost any factor graph would do.

In contrast, for unique games (which is a special family of CSPs with two non-Boolean variables per constraint), the approximation ratios achievable on random factor graphs [4] are much better than those currently known to be achievable on arbitrary factor graphs. (Technically, the graphs considered by Arora et al. [4] have variables as vertices and constraints as edges, but there is a one-to-one correspondence between such graphs and factor graphs.) The same holds for some other classes of graphs [25,17]. Can we (and should we) identify more factor graphs on which unique games are easy? Is there a “universal” graph (e.g., a generalized Kneser constraint graph?) such that if unique games are easy on it, then the Unique Games Conjecture is false? Such questions lead naturally to the notion that we call here universal factor graphs.

1.1 Preprocessing

How can we provide evidence that algorithms for max-3SAT should be spending substantial time in analyzing the factor graph? Here is a possible formal approach. Reveal the input instance in two stages. In the first stage, only the factor graph is revealed. At this point the algorithm is allowed to run for arbitrary time and record (in polynomial space) whatever information about the factor graph that it may hope to find useful (e.g., an optimal tree decomposition of the factor graph, or a minimum dominating set in the factor graph, both of which are pieces of information that take exponential time to compute). Thereafter the polarities of the variables are revealed. At this stage the algorithm has only polynomial time, and it needs to find an optimal solution to the max-3SAT instance. If there is a combination of algorithms (unbounded time for stage 1, polynomial time for stage 2) that can do this on every instance, this establishes that a good understanding of the factor graph suffices for solving 3SAT instances.
If this cannot be done, this establishes that at least some substantial portion of the running time is a result of the combinatorial richness of space of possibilities for polarities of the variables. Refined versions of the preprocessing approach either require less of the stage 2 algorithm (finding nearly optimal solutions rather than optimal ones) or give it extra power (allow subexponential time), and may lead to a more quantitative understanding of the value of preprocessing.

To derive positive results in this model, it suffices to provide the respective algorithms and their analysis. But how does one provide negative results? This is where the notion of universal factor graphs comes in. Informally, these are factor graphs on which preprocessing is unlikely to help, because if it does, then all instances (regardless of their factor graph) can be solved even without preprocessing.

1.2 Universal Factor Graphs

We consider infinite families of factor graphs. Basically, for every value of \( N, M > 0 \), a family includes at most one factor graph with \( N \) variables and \( M \) constraints. However, for convenience in intended future uses, members of the family are indexed by two auxiliary indices that are called \( n \) and \( m \). Definition 1 does not exclude the possibility that several factor graphs in the family share the same values of \( N \) and \( M \), but their number is upper bounded by some polynomial in \( N + M \).

**Definition 1.** Consider an arbitrary CSP with \( k \) variables per-constraint. For integers \( n > 0 \) and \( 0 < m \leq 2^k \binom{n}{k} \), let \( N(n, m) \) and \( M(n, m) \) be two functions, each lower bounded by \( n \) and upper bounded by a polynomial in \( n + m \). A family of factor graphs associates with each pair of values of \( n \) and \( m \) a factor graph with \( N(n, m) \) variables and \( M(n, m) \) constraints. The family is uniform if there is an algorithm running in time polynomial in \( n + m \) that given \( n, m \) produces the associated factor graph.

Every member of a family of factor graphs for a \( k \)-CSP can give rise to \( 2^{kM} \) instances of the CSP, depending on how one sets the polarities of the variables in the constraints. Given any such instance as input, we shall consider computational tasks such as satisfiability (find a satisfying assignment if one exists), optimization (find an assignment satisfying as many clauses as possible) and approximation (get close to optimal).

The algorithms that perform the above tasks will be limited in their running times. In this work, we shall be interested in two classes of running times. One is the standard polynomial time (P) notion, which in our case will mean polynomial in \( (N + M) \). The other is subexponential time, (SUBEXP) which in this paper is taken to mean time \( 2^{O(N^{1-\epsilon})} \) for some \( \epsilon > 0 \).

Recall that in computational complexity theory, one distinguishes between uniform models of computation (such as Turing machines) and non-uniform models (such as families of circuits). This distinction is relevant in our context. The notion of preprocessing the factor graph can be captured by allowing for non-uniform algorithms. Hence we shall be dealing with the complexity classes P/poly,
SUBEXP/poly and SUBEXP/subexp (the parameters /poly and /subexp correspond to the length of advice that the preprocessing stage is allowed to record). For simplicity in our presentation, in each of our definitions below we shall specify one particular complexity class (either P/poly or SUBEXP/poly), but we note that our results extend to other complexity classes as well (such as P instead of P/poly, or SUBEXP/subexp instead of SUBEXP/poly).

In this work we will show that for some uniform families of factor graphs solving satisfiability or approximation tasks are hard. These families of factor graphs will be referred to as universal, and with slight abuse of terminology, individual factor graphs within these families will be referred to as universal factor graphs. The hardness results will be proved under some complexity assumption. If the complexity assumption is widely believed, such as that NP is not contained in P/poly, then the universal factor graphs support the view that the complexity of the underlying CSP cannot be attributed entirely to the factor graph and is at least partly due to the polarities of the variables, because the nonuniform algorithms could preprocess the factor graph for arbitrary time prior to receiving the polarities of the variables. If the complexity assumption is not as widely believed (such as the Unique Games Conjecture), the interpretation of these hardness result can be that if one wishes to refute the complexity assumption, it would suffice to design algorithms that are specifically tailored to work on instances with factor graphs as in the universal family.

We now present formal definitions that are tailored to match those results that we can prove in this paper. It is straightforward to adapt these definitions to other variations as well.

**Definition 2.** For a given CSP, a uniform family of factor graphs is P-universal if there is no P/poly algorithm for instances of the CSP with factor graphs from this family, unless NP is contained in P/poly.

**Definition 3.** For a given CSP, a uniform family of factor graphs is subexp-universal if there is no SUBEXP/poly algorithm for instances of the CSP with factor graphs from this family, unless there is a SUBEXP/poly algorithm for all instances of the CSP.

**Definition 4.** For a given CSP and $0 < \rho < 1$, a uniform family of factor graphs is $\rho$-universal if there is no P/poly approximation algorithm with approximation ratio better than $\rho$ on the instances of the CSP with factor graphs from this family, unless NP is contained in P/poly. This notion is referred to as threshold-universal. If $\rho$ is equal to the best approximation ratio known for the underlying CSP, we will refer to this as a tight threshold. When we do not wish to specify a particular value for $\rho$, we call the family APX-universal. A variation on $\rho$-universality is $(c,s)$-universality with $0 < s < c \leq 1$, where instead of approximation within a ratio of $\rho$, one considers distinguishing between instances with at least a $c$-fraction of the clauses being satisfiable, and instances with at most $s$-fraction being satisfiable. For a CSP for which the decision variant is NP-hard (e.g. 3SAT), $\rho$-universality will be taken to mean $(1,\rho)$-universal.
More generally, for optimization versions we shall allow vertices (representing constraints) of universal factor graphs to have nonnegative weights, thus representing instances in which one wishes to find an assignment that maximizes the weight (rather than the number) of satisfied constraints. As the weights will be fixed (independently of the subsequent polarities given to variables), this is in essence a condensed representation of an unweighted universal factor graph (which can be obtained by duplicating each vertex a number of times proportional to its weight, rounded to the nearest integer – details omitted).

1.3 Some Research Goals

The notion of universal factor graphs opens up many research directions that we find interesting. In our current work we attempt to answer questions such as: Does 3SAT have P-universal factor graphs? Subexp-universal factor graphs? Does max-3SAT have APX-universal factor graphs? Does max-3SAT have 7/8-universal factor graphs? These questions are part of a wider research agenda that concerns questions such as: Do all CSPs have tight threshold-universal factor graphs? Which CSPs do not have tight threshold-universal factor graphs? Other questions of interest include: How do universal factor graphs look like? Can knowledge of their structure help us either in designing new algorithms, or in reductions that prove new hardness results?

1.4 Related Work

There has been work showing that CSPs on particular factor graphs are NP-hard, and using such results to help in reductions establishing further NP-hardness results. For example, it is known that 3SAT is NP-hard even when the factor graph is planar [18], and this was used (for example) in showing that minimum-length rectangular partitioning of a rectilinear polygon (with holes) is NP-hard [19]. Our notion of universal factor graphs is stronger as it requires at most one particular factor graph for each instance size, rather than a whole family of factor graphs (e.g., the $n$ by $n$ grid, rather than all planar graphs).

A line of work that closely relates to our research agenda is that of preprocessing for NP-hard problems. As the universal factor graph is fixed, one may consider preprocessing it for arbitrary (exponential) time in order to produce a polynomial size “advice”, prior to getting the polarities of the variables. Preprocessing was extensively studied for some NP-hard problems, and hardness results in the context of preprocessing amount to designing instances that are universal (in our terminology). Naor and Bruck [7] show that the nearest code word problem remains NP-hard even when the code can be preprocessed. Nearest lattice vector (CVP) when the lattice can be preprocessed was shown to be NP-hard and APX-hard by Feige and Micciancio [10]. The tightest hardness results for lattice problems with preprocessing currently known are by Khot et al. [16]. An earlier work by Alekhnovich et al. [2] has some partial overlap with our current work, because it uses PCP theory and in the process gives hardness
of approximation results with preprocessing for additional problems. See more details in Section 2.2.

The above results on coding and lattice problems with preprocessing are motivated by the fact that in these problems, it is indeed often the case that part of the input is fixed in advance (the code, or a basis for the lattice), and part of the input (a noisy word that one wishes to decode, or vector for which one wishes to find the closest lattice point) is a query that is received only later. Moreover, multiple queries are expected to be received on the same fixed input. In these cases it really makes sense to invest much time in preprocessing the fixed part of the input, if this later helps answering the multiple queries more quickly. In contrast, our notion of universal factor graphs is independent of such practical concerns. Our motivation is to understand the source of difficulties in solving NP-hard problems. In particular, it is irrelevant to us whether there really is any real life situation in which one receives the factor graph of a 3CNF formula in advance, and then is asked a sequence of queries about it, each time with different polarities of the variables.

Is it at all plausible that preprocessing can help? For lattice problems, this indeed appears to be the case. There are no known approximation algorithms with subexponential ratios for CVP, but if preprocessing is allowed, then polynomial approximation ratios are known (by using an exponential time preprocessing procedure that derives a so called reduced basis of the lattice). For CSPs, the authors are aware of only much weaker evidence that preprocessing may help. This relates to the case that polarities of variables are random rather than arbitrary.

There is a refutation algorithm that is poly-time on random 3CNF formulas with more than \( n^{1.5} \) clauses. The obstacle to extending this to lower density of \( n^{1.4} \) is graph-theoretic: if one knew how to efficiently find certain substructures in the factor graphs (that almost surely exist), this would suffice \cite{11}. Preprocessing the factor graph would allow finding these structures. Hence at these densities, random factor graphs are not expected to be universal (with respect to random polarities).

In the current paper we consider arbitrary polarities for the variables rather than random polarities. Nevertheless, we remark that the case of random polarities is also well motivated, and related to possible cryptographic application. See \cite{3} as an example showing how results from \cite{11} can be used in a proposal of new public key cryptographic primitives.

More generally, cryptography offers many examples where preprocessing is believed to help (it will lead to the discovery of a so called trapdoor that would make solving future instances easy), but as this typically relates to computational problems that are believed not to be NP-hard, further discussion of this is omitted from the current manuscript.

1.5 Our Results

The first theorem is based on a straightforward reduction and we have no doubt that it was previously known (perhaps using different terminology).
Theorem 1. There are $P$-universal factor graphs for $3SAT$.

For the $P$-universal factor graphs constructed by our proof for Theorem 1 an algorithm running in time $2^{N^{1/\epsilon}}$ on instances of the universal family would correspond to time $2^{n^{3/\epsilon}}$ on general instances. Hence they are not subexp-universal. The next theorem addresses this issue.

Theorem 2. There are subexp-universal factor graphs for $3SAT$.

We would have liked to prove that there are $7/8$-universal factor graphs for max-$3SAT$, matching the tight threshold of approximability for max-$3SAT$. However, we only managed to prove weaker bounds.

Theorem 3. There are $77/80$-universal factor graphs for max-$3SAT$.

Is there any CSP for which we can obtain tight threshold-universal families? We do not know, but we do have almost tight results.

Theorem 4. For every $\epsilon > 0$ there is an integer $k$ for which there is a family of factor graphs that are $(1 - (1 - \epsilon)2^{-k})$-universal for max-$E_k SAT$.

Theorem 4 in nearly tight because every instance of max-$E_k SAT$ is $(1 - 2^{-k})$-satisfiable, and consequently there are several algorithms with a $(1 - 2^{-k})$ approximation ratio. To actually get tight results we would need to switch the order of quantifiers in Theorem 4 (show that for some $k$ the result holds for every $\epsilon$), but doing so remains an open question.

Using the techniques developed in our work and known reductions among CSPs one can obtain APX-universal factor graphs for additional CSPs. In particular, we derive APX-universal factor graphs for max-$2LIN$, thus illustrating that for approximating unique games (max-$2LIN$ is a unique game) at least part of the difficulty comes from the polarities of variables rather than from the structure of the factor graph. See Appendix F.

2 Overview of proofs

At a high level, to show that a factor graph is universal, one shows that any other factor graph (of the appropriate size) can be reduced to it. The details of how this is done depend on the context.

The proof of Theorem 1 appears in Appendix A. It is elementary and can serve as an introduction to some of the more complicated proofs that follow.

2.1 Subexp-Universal Families

Our proof of Theorem 2 combines two ingredients. One is a variation on a result of Impagliazzo et al. [14] (see Lemma 2 in Appendix B). It can be leveraged to show that for the purpose of constructing subexp-universal factor graphs it suffices to consider $3CNF$ instances with a linear number of clauses.

The other ingredient is a reduction with a tighter connection between $n + m$ and $N$ compared to the one used in our proof of Theorem 1.
Lemma 1. There is a factor graph with $N = O(m \log m \log n)$ variables that is P-universal with respect to 3SAT instances with $n$ variables and $m$ clauses.

Our proof of Lemma 1 makes use of oblivious sorting networks (specifically, the one of Ajtai et al. [1]).

More details on those two ingredients and how they are combined to prove Theorem 2 appear in appendix B.

2.2 Threshold-Universal Families

For our proof of Theorem 3 we use a notion that we call a factor graph preserving reduction (FGPR). It is an algorithm that transforms a source 3CNF instance $f_s$ to a target 3CNF instance $f_t$. The transformation has the following properties:

1. Polynomiality. The transformation algorithm runs in polynomial time (in the size of $f_s$). Consequently, the size of $f_t$ is polynomial in the size of $f_s$.
2. Faithfulness. If $f_s$ is satisfiable, so is $f_t$, and vice versa.
3. Factor graph preserving. Any two instances $f_s$ and $f'_s$ with the same factor graph are reduced to two instances $f_t$ and $f'_t$ that have the same factor graph.

To be useful for our purposes, we would like the FGPR to also have a gap amplification aspect. Namely, if $f_s$ is not satisfiable, then the fraction of clauses satisfiable in $f_t$ is smaller than the fraction of clauses satisfiable in $f_s$.

Theorem 3 will be broken into two sub-theorems, each of which is proved using FGPRs.

Theorem 5. There are APX-universal factor graphs for max-3SAT.

Theorem 6. There is a reduction from APX-universal factor graphs for max-3SAT to $77/80$-universal ones.

The proof of Theorem 5 strongly relates to the work of Alekhnovich et al. [2]. As explained in Section 1.4 in that work various APX-hardness results with preprocessing were obtained. Among them, there were APX-hardness results with preprocessing for certain CSPs (satisfying quadratic equations). It is not difficult to use these results in order to obtain APX-universal factor graphs for max-3SAT. However, we present an alternative proof because [2] claims the relevant theorem without providing a proof. Our proof is patterned after a proof of the PCP theorem due to Dinur [9].

Recall that Dinur’s proof is based on a sequence of gap amplification steps. However, some of these transformations are not factor graph preserving. Our proof performs a sequence of gap amplifying FGPRs, starting with the outcome of Theorem 1 and eventually proving Theorem 5. Every FGPR is based on

1 Quoting from [2]: “The proof of this theorem, which is a laborious and an almost exact mimic of the proof of the PCP Theorem, is beyond the scope of this version of the paper.” A subsequent paper [15] that extends [2] no longer uses this theorem, and hence does not contain the proof either.
modifying Dinur’s proof (or more exactly, on modifying a variation on Dinur’s
proof that is given in [21]). The modifications are related to those discussed
below for the long code (though our proof for Theorem 5 uses a quadratic code
rather than the long code).

The proof of Theorem 6 involves an FGPR from APX-universal factor graphs
for max-3SAT to 77/80-universal ones. Our proof is based on a modification of
the proof of Bellare et al. [6], and consequently obtains the same hardness
ratio of 77/80. The main difficulty we encounter is the following. Tight or nearly
tight hardness of approximation results use the so called long code. A major reason
why it is used is that its high redundancy allows one to replace explicit queries
that check whether an underlying predicate is satisfied by an implicit operation
(referred to as folding) that allows one to avoid making these queries. The only
queries that need to be made are those that check whether the encoding is really
(close to) a long code. The saving in queries translates to stronger hardness of
approximation results. The problem with folding is that it is sensitive to the
predicate that needs to be checked, and a change in the predicate (e.g., changing
the polarity of a single variable in a 3SAT clause) changes the folding. As a
result, query locations change, and the resulting reduction is not an FGPR. To
overcome this problem we introduce a notion of oblivious folding of the long code,
which does allow us to eventually obtain an FGPR. We remark that it was not
a-priori obvious that a construct such as oblivious folding should exist at all. In
particular, tight hardness of approximation results for 3SAT by Hastad [12] use
a notion related to folding but somewhat stronger, that is called conditioning
of the long code. We were unable to find an “oblivious” version of conditioning
that can replace the conditioning used by Hastad, and consequently we do not
know if 7/8-universal factor graphs for 3SAT exist.

For the full proofs of Theorems 5 and 6 see appendices C and D.

2.3 Threshold-Universal Families with Nearly Tight Bounds

Recall that the prefix E (for exact) in EkSAT indicates that every clause in the
CNF formula contains exactly k literals (rather than at most) and no two literals
in a clause correspond to the same variable. It is not difficult to see that the proof
of Theorem 3 in fact gives E3-CNF formulas, and not just 3CNF formulas (and
even if not, there are simple FGPRs from max-3SAT to max-E3SAT, with only
a bounded loss in the approximation ratio). Our proof of Theorem 4 is based
on a direct reduction from instances of max-E3SAT to instances of max-EkSAT.
This reduction has the property that mere APX-hardness of max-E3SAT suffices
in order to get nearly tight hardness of approximation ratios for the resulting
max-EkSAT instances, if k is sufficiently large.

Proof. Theorem 3 implies that there is a (1 – γ)-universal family of factor graphs
for E3-CNF formulas, for some 0 < γ < 1/4. We shall use this in an FGPR to prove
Theorem 4. For simplicity of the presentation we shall describe our reduction as
a reduction from a single E3-CNF formula φ_3 to a single Ek-CNF formula φ_k.
As the factor graph resulting for $\phi_k$ will be independent of polarities of variables in $\phi_3$, this will be an FGPR.

Let $\phi_3$ be an E3-CNF formula with $n$ variables and $m$ clauses for which one wants to distinguish between the case that it is satisfiable and the case that it is at most $(1 - \gamma)$-satisfiable. Formula $\phi_k$ will be obtained from a combination of $2^q$ auxiliary E$k$-CNF formulas called $\psi_i$, for $0 \leq i \leq 2^q - 1$. Let $q = k - 3$. Introduce $q$ fresh variables $y_1, \ldots, y_q$, and 3 fresh variables $z_1, z_2, z_3$. Formula $\psi_0$ is obtained from $\phi_3$ by adding the $y$ variables (all in negative polarity) to each clause of $\phi_3$. As to the other formulas indexed by $i \geq 1$, each such formula $\psi_i$ has eight clauses, where each clause contains the variables $y_1, \ldots y_q, z_1, z_2, z_3$. Excluding the all negative polarity combination, there are $2^q - 1$ remaining combinations of polarities for the $q$ variables of type $y$. Each such combination of polarities will be associated with the clauses of one $\psi_i$ for $i \geq 1$. One may think of the binary representation of $i$ as specifying the polarity of the $y$ variables in clauses of $\psi_i$, where if the $j$'th bit of $i$ is 0 then $y_j$ is negative, and if the $j$'th bit of $i$ is 1 then $y_j$ is positive. As to the $z$ variables, there are 8 possible combinations of polarities. Within a formula $\psi_i$ there are 8 clauses, and each of them has a different combination of polarities for the $z$ variables.

The formula $\phi_k$ will be a weighted mixture of the $\psi_i$ (see Appendix regarding an unweighted version). Formula $\psi_0$ is taken with weight $\frac{1}{8\gamma}$ (which is larger than 1 because $\gamma < \frac{1}{8}$), spreading this weight equally among its $m$ clauses. Each of the other $\psi_i$ is taken with weight 1, spreading the weight equally among its 8 clauses. The total weight of $\phi_k$ is $2^q - 1 + \frac{1}{8\gamma}$.

If $\phi_3$ is satisfiable, so is $\phi_k$: an assignment to the original variables of $\phi_3$ that satisfies $\phi_3$ also satisfies $\psi_0$, and assigning true to all $y$ variables satisfies all $\psi_i$ for $i \geq 1$. If $\phi_3$ is only $1 - \gamma$ satisfiable then the weight of unsatisfied clauses in $\phi_k$ is at least $\frac{1}{8\gamma}$: if all variables $y$ are assigned true, this results from $\psi_0$, and in all other cases, this results from one of the other $\psi_i$.

The total weight of $\phi_k$ is $W = 2^q - 1 + \frac{1}{8\gamma}$, and for $q$ satisfying $2^q \geq \frac{1}{\epsilon} \left( \frac{1}{8\gamma} - 1 \right)$ we have that $W \leq \frac{2^q}{1-\epsilon}$ which implies that $\frac{1}{8} \geq \frac{W(1-\epsilon)}{2^q}$. Hence $\phi_k$ is at most $\left( 1 - \frac{(1-\epsilon)}{2^q} \right)$-satisfiable, as desired.

\[ \square \]

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A Polytime-Universal Families

Proof of Theorem \[\text{A}\]

*Proof.* We design a universal factor graph for 3CNF formulas that have $n$ variables and any number of clauses. For simplicity of presentation, we use the following convention. A clause is a tuple of three variables that need not be distinct, and the polarities of the variables. Two clauses may not have the same
tuple, though two clauses may have the same set of variables if the order in which they appear in the respective tuples is different. Hence there are exactly \( n^3 \) possible tuples, and the number of clauses satisfies \( m \leq n^3 \).

The universal 3CNF formula is constructed as follows. Write down all \( n^3 \) possible tuples. For every tuple \( T_i \) introduce an auxiliary variable \( z_i \). For each tuple \( T_i \) introduce two clauses as in the following example. If \( T_i = x_1, x_2, x_3 \) then the two clauses are \((x_1 \lor x_2 \lor z_i)\) and \((x_3 \lor z_i \lor z_i)\). This gives a formula \( F \) with \( N = n + n^3 \) variables and \( M = 2n^3 \) clauses.

Every 3CNF instance \( f \) with \( n \) variables can be embedded in \( F \), by appropriately setting only the polarities of variables. For a given tuple \( T_i \), if no clause with this tuple is in \( f \), then in \( F \) give all occurrences of \( z_i \) positive polarity. By setting \( z_i \) to true this corresponding tuple drops also from \( F \). But if a clause with tuple \( T_i \) appears in \( f \) (there can be at most one such clause), then do as in the following example. If the clause is \((x_1 \lor \overline{x_2} \lor x_3)\) then in \( F \) set the polarities of the clauses derived from \( T_i \) to \((x_1 \lor \overline{x_2} \lor \overline{z_i})\) and \((x_3 \lor z_i \lor z_i)\). Any assignment that satisfies these two clauses in \( F \) satisfies also the original clause in \( f \).

The above implies that the factor graph of \( F \) is polytime-universal (for 3SAT with \( n \) variables). Any algorithm that decides satisfiability for formulas whose factor graph is that of \( F \) can be used to decide satisfiability of any 3CNF formula with \( n \) variables, by following the above embedding.

\[ \square \]

**B Subexponential-Universal Factor Graphs**

In this section we prove Theorem 2. Recall that this involves proving Lemma 2 and Lemma 1 and then combining them appropriately.

**Lemma 2.** Given a 3CNF formula \( \varphi \) with \( n \) variables and any number of clauses (at most \( O(n^3) \) as clauses may be assumed to be distinct), for every \( 0 < \epsilon < 1/10 \) there is an algorithm that runs in time \( 2^{O(n^{1-\epsilon})} \) and produces at most \( 2^{n^{1-\epsilon}} \) new 3CNF formulas, each with \( n \) variables and at most \( O(n^{3/2} \log n^2) \) clauses, such that \( \varphi \) is satisfiable iff at least one of the new formulas is satisfiable.

**Lemma 2** follows by substituting \( \epsilon(n) = n^{-\epsilon} \) in the following lemma.

**Lemma 3.** Given a \( k \)-CNF formula, \( \varphi \), with \( n \) variables, \( 0 < \epsilon(n) < 1 \), and \( \alpha(n) \) with \( \frac{\alpha(n)}{\log 4\alpha(n)} > 4k2^{k-1} \epsilon^{-1}(n) \), there is an algorithm that produces at most \( 2^{\epsilon(n)n} k \)-CNF formulas, each with at most \( nk(4\alpha(n))^{2k-2} \) clauses and \( n \) variables in time \( 2^{\epsilon(n)n}(4\alpha(n))^{2k-2} \) poly \( (n) \) such that \( \varphi \) is satisfiable iff at least one of the outputted formulas is satisfiable.

**Proof.** A similar statement was proved by Impagliazzo et al. [14] with constant \( \epsilon \) and \( \alpha \), and the same proof works when they are not constant. \[ \square \]

We now prove Lemma 1.
Proof. We first construct a nondeterministic circuit that receives as input an E3CNF formula with \( n \) variables and \( m \) clauses and outputs 1 if the formula is satisfiable. We wish to keep the circuit small, of size \( O(m \log m \log n) \). For this reason, the nondeterministic aspect of the circuit will not be a guess of the assignment to the variables (which amounts to \( n \) nondeterministic guesses), but rather a selection of one index per clause (hence \( m \) nondeterministic guesses, each among three possibilities), indicating a literal that satisfies this clause. The consistency of all these selections (namely, not selecting a variable in one clause and its negation in a different clause) will be checked using a circuit that mimics an oblivious sorting network. All selected literals will be sorted, implying that if there is a variable who was selected both positively and negatively, these two contradicting selections will “meet” during the sorting processes and the inconsistency will be detected. As there are oblivious sorting networks that sort \( m \) numbers using \( O(m \log m) \) comparisons [1], the size of the circuit will remain bounded by \( O(m \log m \log n) \) (the extra \( \log n \) term comes from the fact that it takes \( \log n \) bits to specify each of the sorted numbers). Such a nondeterministic circuit outputs 1 if the formula is satisfiable: a consistent selection of literals can always be completed to a satisfying assignment (by giving arbitrary values to variables for which no occurrence of their literals was selected), whereas given a satisfying assignment a consistent selection is obtained by selecting the first satisfied literal in every clause.

We now provide more details on the construction of the circuit. The circuit takes \( O(3m \log n) \) input bits. \( O(\log n) \) bits are used to represent each literal, with the least significant bit used to indicate if the literal is negated or not. In addition, there are 2 nondeterministic input bits per clause, used to select one of the three literals in the clause. The selection of literals can be done by using \( O(\log n) \) 3-to-1 multiplexers. The selected literals are sorted using a sorting network of size \( O(m \log m) \) (see [1,20]), where each comparison is done by adding the representation of one literal to the two’s complement of the representation of the other literal, using \( O(\log n) \) adders, and the most significant bit of the result determines the output of the comparison. The literals are switched or not, depending on the result of the comparison, using \( O(\log n) \) 2-to-1 multiplexers. Lastly, when all literals are sorted, each consecutive pair (with overlapping pairs) is checked that it does not contain the representation of a variable and its negation (all but last bit equal, using \( O(\log n) \) gates).

Given an E3CNF formula \( \varphi \) with \( m \) clauses and \( n \) variables, we use the circuit described above, to construct a 3-CNF formula \( \Phi_{\varphi} \) that is satisfiable iff \( \varphi \) is satisfiable. Additionally, if \( \psi \) is another E3CNF formula with \( m \) clauses and \( n \) variables, the factor graph of \( \Phi_{\psi} \) is also the factor graph of \( \Phi_{\varphi} \). For every input of a gate and for the output of the circuit there will be a variable. Note that the output of every gate is an input of some other gate or the output of the circuit. Each gate contributes a bounded number of clauses to \( \Phi_{\varphi} \) that encode the requirement that the output of the gate is correct. For example, a NAND gate with inputs \( x, y \) and output \( z \) would contribute the clauses \( \bar{x} \lor \bar{y} \lor \bar{z}, \bar{x} \lor y \lor z \),
This ensures that a satisfying assignment to $\Phi_\varphi$ is a valid calculation of the circuit. That is, every variable has the value passed to or from each gate.

Let $o$ be the variable representing the output of the circuit. The clause $o$ is also added to $\Phi_\varphi$. This ensures that an assignment satisfies the formula iff the output of the circuit is 1.

Let $i_1, \ldots, i_{3m(\lceil \log_2 n \rceil + 1)}$ be the input bits of the circuit. For every $j$, either the clause $i_j$ or the clause $\bar{i}_j$ is added to $\Phi_\varphi$, depending on the representation of $\varphi$. This ensures that a satisfying assignment to $\Phi_\varphi$ has the representation of $\varphi$ in the input of the circuit. Note that the only difference between $\Phi_\varphi$ and $\Phi_\psi$ is in the polarity of these clauses.

A standard transformation can be used to transform the formula from 3CNF to E3CNF.

If $\Phi_\varphi$ is satisfiable, the variables representing the selector bits prove that the circuit can be made to output 1, when $\varphi$ is given as input. If there are selector bits that make the circuit output 1 on $\varphi$, setting each variable to its respective input/output in the circuit, shows that $\Phi_\varphi$ is satisfiable.


\begin{proof}

For simplicity of the presentation, we omit the $O$ notation in the expressions that we derive.

Assume that for some $0 < \delta < 1/10$ a hypothetical algorithm $H$ can solve any instance on the universal factor graphs of Lemma [1] in time $2^{N^{1-\delta}}$. Consider now an arbitrary 3SAT instance with $n$ variables. For $\epsilon = \delta/3$, use Lemma [2] to create $2^{n^{1-\epsilon}}$ new 3CNF formulas with at most $n^{1+2\epsilon}(\log n)^2$ clauses. Use Lemma [1] to reduce every such 3CNF instance to an instance on a universal factor graph with $N = n^{1+2\epsilon}(\log n)^4$ variables. Use algorithm $H$ to solve these instances, thus obtaining the solution to the original 3SAT instance. The choice of $\epsilon = \delta/3$ implies that this whole procedure takes time roughly $2^{n^{1-\epsilon}}$.

\end{proof}

\section{APX-Universal Factor Graphs}

In this section we prove Theorem [5].

Any universal factor graph for 3SAT (e.g., the result of Theorem [1]) is $(1 - \frac{1}{m})$-universal, where $m$ is the number of clauses. In order to create a $(1-\epsilon)$-universal factor graph (for some $\epsilon > 0$) the instances will go through an iterative process, increasing the worst case unsatisfiability of the formulas by a factor of 2, while increasing the size of the factor graph by a constant factor. The construction is based on the combinatorial method to prove the PCP theorem by Dinur [9], and closely follows the proof of Radhakrishnan and Sudan [21]. Familiarity with these earlier proofs (an overview of which can be found in [21]) can aide the reader in following our proof.
C.1 Definitions

Definition 5. A constraint satisfaction problem \((CSP)\) has the form \(P = (V, \Sigma, C)\), where \(V\) is the set of variables, \(\Sigma\) is the alphabet, and \(C\) is the set of constraints. A constraint is \(c = \langle U, f \rangle\), where \(U \subset V\) and \(f : \Sigma^U \rightarrow \{0, 1\}\). An assignment is a function \(a : V \rightarrow \Sigma\), giving each variable a value. Given an assignment \(a\), a constraint \(c = \langle U, f \rangle\) is said to be satisfied by the assignment (usually, the assignment will be implied from the context) if \(f(a|_U) = 1\), otherwise it is unsatisfied by the assignment.

Given a constraint satisfaction problem \(P\), \(\text{UNSAT}(P)\) is the minimal fraction of constraints (over all assignments) that are unsatisfied. The size of \(P\) is \(|P| = |V| + |C|\).

Definition 6. A constraint hypergraph \(H = (V, E, \Sigma, C)\) is an alternative definition of a CSP, where \(V, \Sigma, C\) are as in the definition of a CSP, and for every constraint \(c = \langle U, f \rangle\), \(U \in E\).

The structure of a constraint hypergraph \(H = (V, E, \Sigma, C)\) is the hypergraph \((V, E)\).

The rank of \(H\) is the maximal cardinality of an edge.

In the special case where all sets in \(E\) have cardinality 2, \(H\) is a constraint graph.

The following definition adds parametrization to the earlier definition of FGPR given in Section 2.2.

Definition 7. A \((\delta, d)\)-FGPR (Factor Graph Preserving Reduction) is a transformation of instances of one class of CSP to instances of another class of CSP with the additional requirements:

- If the factor graphs of \(A\) and \(B\) are equal, then the factor graphs of their transformations are equal.
- There is some constant \(\xi > 0\) such that if \(A\) is transformed to \(A'\) then:
  - if \(\text{UNSAT}(A) = 0\), then \(\text{UNSAT}(A') = 0\).
  - if \(\text{UNSAT}(A) \geq \epsilon\), then \(\text{UNSAT}(A') \geq \delta \min\{\epsilon, \xi\}\).
- \(d |A| \geq |A'|\)

For example, the standard reduction from 3SAT to E3SAT is a \((1, 4)\)-FGPR \((\frac{1}{2}, 2)\)-FGPR if all clauses have at least two distinct variables).

In order to create an APX-universal factor graph a \((\delta, d)\)-FGPR with \(\delta > 1\) will be constructed, using a composition of several FGPRs. In order to compose these FGPRs correctly, each will need to have additional properties.

It will convenient for us to represent constraints as polynomials. For example, a constraint of the form “the first bit in the representation of the variable \(x\) is equal to the second bit of the representation of the variable \(y\)” can be represented as requiring that polynomial \(x_1 + y_2\) be equal to 0 (where \(x_1\) refers to the value of the first bit in the assignment of \(x\), and \(y_2\) refers to the second bit of the assignment of \(y\)).
Definition 8. Let $\Sigma = \mathbb{F}_2^3$. A constraint $e$ is $m$-restricted if it can be represented as a set of up to $m$ polynomials of degree two, $\{P_i^e\}$, such that the constraint is satisfied iff all the polynomials are 0 (where the assignment of variables is treated as the values of $k$ Boolean variables). In such case we say that $e$ is associated with $\{P_i^e\}$. An $m$-restricted constraint (hyper)graph is a constraint (hyper)graph with only $m$-restricted constraints.

Definition 9. Two $m$-restricted constraint hypergraphs are close if they share the same factor graph and alphabet, and for every edge $e$ of the factor graph, if the constraint corresponding to $e$ is associated with $\{P_i^e\}$ then the constraint corresponding to $e$ in the other graph is associated with $\{P_i^e + b_i^e\}$, where $b_i^e \in \{0, 1\}$.

C.2 Changing the Representation

Lemma 4. There is an explicit $(\frac{1}{4}, 6)$-FGPR from 3-SAT to 2-restricted constraint graphs. Additionally, if $H, H'$ are created from $\varphi, \varphi'$, respectively, using the specified reduction and $\varphi, \varphi'$ have the same factor graph then $H$ and $H'$ are close.

Proof. Each vertex in the constraint graph will represent up to two literals plus an additional bit, so $\Sigma = \mathbb{F}_2^3$. For a vertex $w$, $A_i(w)$ will correspond to the $i$'th bit of the assignment of the vertex. The value of the first bits is intended to be the value of the represented literals, 0 if true, 1 if false (and the constraints will try to enforce that).

For every variable $v_i$ in the formula there will be a corresponding vertex $w_i$. For every clause of the form $x_i \lor x_j \lor x_k$ (where $x_\ell$ is $v_\ell$ or $\bar{v}_\ell$) there will be two vertices: $u_{ij}, u_{ik}$ with an edge between them. The constraint corresponding to the edge $(u_{ij}, u_{ik})$ expects the following two polynomials to be satisfied: $A_3(u_{ij}) = A_1(u_{ij}) A_2(u_{ij})$, (the last bit is true if one of the represented literals is true) and $A_3(u_{ij}) A_1(u_{ik}) = 0$ (the clause is satisfied). For every clause of the form $x_i \lor x_j$ there will be two vertices, $u_i$ and $u_j$ with an edge between them with the constraint $A(u_i) A(u_j) = 0$ (the clause is satisfied). For every clause of the form $x_i$ a self loop will be added to the vertex $w_i$ with constraint $A_1(v_i) = b$, where $b = 0$ if $x_\ell$ is $v_\ell$ and $b = 1$ otherwise.

In addition, consistency constraints will be added: $(u_{ij}, w_i), (u_{ij}, w_j), (u_k, w_k)$ with respective constraints $A_1(u_{ij}) - A_1(w_i) = b_i^v$, $A_2(u_{ij}) - A_1(w_j) = b_j^v$, $A_1(u_k) - A_1(w_k) = b_k^v$, where $b_i^v$ is 0 or 1, depending on whether the variable $v_\ell$ or its negation appear in the clause the first vertex of the edge corresponds to.

Every clause is responsible for the creation of at most two vertices and four edges. Every variable is responsible for the creation of one vertex. Thus $|H| \leq 6 |\varphi|$.

If UNSAT$(\varphi) = 0$, there is a assignment $a_i$ for each $v_i$ satisfying all clauses. setting $A_1(w_\ell)$ according to this assignment (0 if $a_\ell$ is true, 1 otherwise), and setting $A_1(u_k), A_1(u_{ij}), A_2(u_{ij})$ according to the value of the corresponding literal using the assignment will satisfy all edges of the graph.
If UNSAT \((G) < \epsilon\), the best assignment to vertices can be transformed into an assignment to variables. If an edge is unsatisfied, the clause that generated that edge is considered to be unsatisfied. All edges and variables generated by this clause will be removed. Repeating this for as long as unsatisfied edges remain, we are left with a completely satisfiable graph, with all consistency constraints holding. Thus, the assignment for the graph can be transformed to an assignment for the formula. Each unsatisfied edge may have caused a single clause to be unsatisfied, and since there are at most 4 times as many edges as there are clauses, UNSAT \((\varphi) < 4\epsilon\).

Lastly, it is immediate that all formulas that have the same factor graph generate close constraint graphs. \(\square\)

### C.3 Gap Amplification

**Definition 10.** An \((\eta, d)\)-expander is a regular graph \(G = (V, E)\) with degree \(d\) and for all \(S \subseteq V\) with \(|S| \leq \frac{|V|}{2}\), \(|\{ (u, v) \in E | u \in S, v \notin S \}| \geq \eta |S|\). An \((\eta, d)\)-expander is positive if every vertex has at least \(d^2\) self loops.

**Theorem 7** (See several constructions in [13]). There are \(\eta, d > 0\) such that positive \((\eta, d)\)-expander graphs exist on \(n\) vertices, for all \(n > 0\).

**Theorem 8.** There is a universal constant \(\alpha\) such that for every \(k, m, t \in \mathbb{N}\), there is an explicit \((\delta^*_t(t), c_1(t))\)-FGPR from \((m\text{-restricted})\) constraint graphs with alphabet \(F^2_k\) to \((O(mt)\text{-restricted})\) constraint graphs with alphabet \(F^2_{s(k,t)}\) with \(\delta^*_t(t) \geq \alpha t\).

Furthermore, every constraint of the produced constraint graph is a conjunction of \(O(t)\) constraints from the input graph and equality constraints, and the set of constraints only depends on the factor graph of the input.

Specifically, two \(m\text{-restricted close constraint graph are transformed into two close } O(mt)\text{-restricted constraint graphs.}

**Proof.** The proof closely follows the construction of the transformation in Section 5 in [21].

Let the input of the transformation be a graph \(G\). As in Lemma 5.3 in [21], a regular graph \(G_1\) is created from the graph \(G\). Each vertex \(u \in V\) is replaced by \(d_u\) \((u\text{'s degree})\) vertices, with an \((\eta, d)\)-expander embedded on them. In addition, each of the new vertices is connected to one of \(u\text{'s neighbors. The constraints on the expanders’ edges are equality constraints (polynomials of degree 1). UNSAT (G_1) \geq \delta_1 UNSAT (G), where } \delta_1 \text{ depends on } d \text{ and } \eta \text{ (which are constants). The proof for the last claim is contained in [21]. Also, If } G \text{ is satisfied it is immediate that } G_1 \text{ can be satisfied.}

As in Lemma 5.5 in [21], an expander \(G_2\) is created from the graph \(G_1\) by superimposing an \((\eta, d)\)-expander on the graph \(G_1\), with constraints that are always satisfied. Let \(d_0\) be the degree of \(G_1\). Then, immediately, \(G_2\) is a \((\eta, d + d_0)\) expander. By adding \(d + d_0\) self loops with constraints that are always satisfied (polynomials of degree 0), \(G_2\) is a positive \((\eta, 2d + 2d_0)\)-expander.
probability

bors of each vertex. The edges are intended to represent a simple random walk more than \( \emptyset \)/assignment \( A \) of vertices of distance at most \( t \) the number of vertices at distance at most \( t \) \([21]\). The value of \( a \) on the path (including continue to step \( k \) walk moves to the neighbor of \( v \)

Definition 11. The Hadamard code of a binary string \( x = x_1x_2\ldots x_t \in \{0,1\}^t \)
is the value of all linear functions \( \{0,1\}^t \to \{0,1\} \) on the bits of \( x \). Given some arbitrary ordering on the linear functions, the \( i \)'th bit of the Hadamard code of \( x \) is the value of \( x \) on the \( i \)'th function.

The quadratic code of a binary string \( x = x_1x_2\ldots x_t \in \{0,1\}^t \) is the value of all homogeneous quadratic functions \( \{0,1\}^t \to \{0,1\} \) on the bits of \( x \). Given some arbitrary ordering on the quadratic functions, the \( i \)'th bit of the quadratic code of \( x \) is the value of \( x \) on the \( i \)'th function.
Lemma 5. For every \( k,m \in \mathbb{N} \) there is an explicit \((\delta_2^2, c_2(k,m))\)-FGPR from close \( m\)-restricted constraint graphs with alphabet \( \mathbb{F}_2^k \) to close 1-restricted constraint hypergraphs with alphabet \( \mathbb{F}_2 \). Additionally, the constructed hypergraph has rank 4.

Proof. Given an \( m\)-restricted constraint graph \( G = (V, E, \mathbb{F}_2^k, C) \), we construct a 1-restricted constraint hypergraph \( H = (V', E', \mathbb{F}_2, C') \). Following the construction of Section 6.3 in [21], for every \( v \in V \), and linear function \( L : \{0,1\}^k \to \{0,1\} \) create a vertex \( v(L) \) in \( V' \). For every \( e \in E \), for every homogeneous quadratic function \( q : \{0,1\}^{2k} \to \{0,1\} \) create a vertex \( e(q) \) in \( V' \). For every \( e \in E \), for every linear function \( L : \{0,1\}^{2k} \to \{0,1\} \) create a vertex \( e(L) \) in \( V' \). The value of the assignment \( A \) on vertex \( v \) is referred to as \( A(v) \).

For every \( e = (u,v) \in E \), \( L_1, L_2 : \{0,1\}^k \to \{0,1\} \), \( L_3, L_4 : \{0,1\}^{2k} \to \{0,1\} \) linear functions, \( q_1, q_2 : \{0,1\}^{2k} \to \{0,1\} \) homogeneous quadratic functions, \( w \in \{0,1\}^m \) there will be seven constraints and corresponding edges (with multiplicity):

1. A constraint that is satisfied iff \( A(u(L_1)) + A(u(L_2)) = A(u(L_1 + L_2)) \).
2. A constraint that is satisfied iff \( A(v(L_1)) + A(v(L_2)) = A(v(L_1 + L_2)) \).
3. A constraint that is satisfied iff \( A(e(L_3)) + A(e(L_4)) = A(e(L_3 + L_4)) \).
4. A constraint that is satisfied iff \( A(e(q_1)) + A(e(q_2)) = A(e(q_1 + q_2)) \).
5. Let \( L \) be the linear function given by \( L(X,Y) = L_1(X) + L_2(Y) \). There is a constraint that is satisfied iff \( A(u(L_1)) + A(v(L_2)) = A(e(L)) \).
6. A constraint that is satisfied iff \( A(e(L_3)) A(e(L_4)) = A(e(q_1 + L_3L_4)) - A(e(q_1)) \).
7. Let \( P \) be a homogeneous degree two polynomial and \( b \in \{0,1\} \) such that \( P = \Sigma u_i P_i + b \), where \( \{P_i\} \) is the set of polynomials associated with \( e \). There is a constraint that is satisfied iff \( A(e(q_1 + P)) - A(q_1) = b \).

There are \( 2^\ell \) linear functions on \( \ell \) bits. There are \( 2^{\alpha\ell} \) quadratic functions on \( \ell \) bits. Thus, there are \( 2^k |V| + (2^{2k} + 2^{k(2k-1)}) |E| \) vertices and \( 7 |E| \cdot 2^{m+6k+2k(2k-1)} \) edges. Hence \( |H| \leq 2^{O(k^2+m)} |G| \).

The satisfaction of every hyperedge depends only on the value of at most 4 vertices.

Given an assignment \( A \) satisfying \( G \), there is an assignment \( A' \) satisfying \( H \). For every \( i,v \), assign the \( i \)'th bit of the Hadamard code of \( A(v) \) to the vertex \( v(L_i) \). Note that this satisfies all constraints of type 1 and 2, since the Hadamard code is linear. For every \( i,e = (u,v) \) assign the \( i \)'th bit of the Hadamard code of \( A(u) \circ A(v) \) (the concatenation of the binary strings) to the vertex \( e(L_i) \). Note that this satisfies all constraints of type 3. Assign the \( i \)'th bit of the quadratic code of \( A(u) \circ A(v) \) to the vertex \( e(q_i) \). A quadratic code of any string \( x \) on the vertices \( e(q) \) (for all quadratic functions \( q \)) will pass the constraints of type 6, if the Hadamard code of \( x \) is on the vertices of \( e(L) \) (for all linear functions \( L \)), and this is the case for \( A' \). The constraints of type 5 check the consistency of the Hadamard code between vertices of \( e(L) \) and vertices corresponding to \( u \) and
v, and they are consistent in $A'$. Lastly, constraints of type 7 are satisfied, since the polynomials associated with edges of $G$ are all 0, when given assignment $A$. Since the code on $e(q)$ is linear, the check becomes $A'(e(P)) - b$, which, by definition of $A'$, is the value of $P - b$ on the assignment $A$, which is a sum of polynomials that are all 0.

If two constraint graphs $G$ and $G^*$ are close, then their transformation $H$ and $H^*$, only differ in the constraints of type 7, in the constant of the associated polynomials (actually linear functions).

The proof that the transformation decreases the fraction of unsatisfiable constraints by at most a constant factor is the same as in Lemma 6.11 in [21]. $\square$

\section*{C.5 Composition}

Firstly, we transform the hypergraph back to a 3CNF formula.\footnote{Lemma 6. There is an explicit $(\delta^*_3(h), c_3(h))$-FGPR from close 1-restricted constraint hypergraphs with alphabet $\mathbb{F}_2$ to 3-SAT, where $h$ the rank of the hypergraph.}

\begin{proof}
Given a 1-restricted constraint hypergraph $H = (V, E, \Sigma, C)$, a 3-CNF formula $\phi$ is defined.

For every vertex $v \in V$, there is a variable $v$. For every edge $e$, there is a variable $w_e$. For every edge $e$, the corresponding constraint $c_e$ is satisfied iff the quadratic polynomial $P_e + b_e$ is evaluated to 0, where $P_e$ is homogeneous. There is a set of at most $2^h$ 3-CNF clauses that is evaluated to true iff $P_e(u_1, u_2, \cdots, u_h) + w_e = 0$. Adding to this set the clause $w_e$ if $b_i = 1$ and the clause $\bar{w}_e$ otherwise, gives a set of at most $2^h + 1$ clauses that are all satisfied iff the constraint $c_e$ is satisfied (using the same assignment, omitting the variables of the form $w_e$).

Thus, we have that $|\phi| \leq (2^h + 1) |H|$. Also, if UNSAT ($\phi$) $\leq \epsilon$, then UNSAT ($H$) $\leq (2^h + 1) \epsilon$. Finally, it is immediate that transforming close 1-restricted constraint hypergraph the resulting 3CNF formulas all have the same factor graph. $\square$

Now we can compose all transformation to get the required FGPR.\footnote{Theorem 9. There is an explicit $(2, c_4)$-FGPR from 3-SAT to 3-SAT.}

\begin{proof}
Using lemma [H] theorem [8] lemma [5] and lemma [9] there is a

$$\left(\frac{1}{4} \alpha t \delta_2^* \delta_3^* \left(4, \frac{1}{2} \right), 6c_1 \left(t \right) c_2 \left(2^t(3,t), O \left(t \right) \right) c_3 \left(4, \frac{1}{2} \right) \right)$$

from 3-SAT to 3-SAT formulas, for all $t$ (it is easy to verify that the composition of these FGPRs is indeed an FGPR). Specifically, there is a constant $t$ to get a $(2, c_4(t))$-FGPR. $\square$

Starting with a universal factor graph for 3-SAT, $O \left(\log n \right)$ repetitions of Theorem 9 proves Theorem 5.
D Threshold-Universal Factor Graphs

D.1 Oblivious folding of the long code - an overview

A major ingredient in tight hardness of approximation results is the long code, introduced in [6]. We present here an overview of the difficulties involved in adapting hardness proofs based on the long code to the oblivious setting, and of our approach for handling these difficulties. A superficial familiarity with previous work should suffice in order to follow this overview – there is no need to know concepts related to the analysis of the long code, such as dictatorship tests and Fourier analysis.

For a given value of \( k \), the long code replaces a vector \( x \) of \( k \) original Boolean variables by a vector \( z \) of \( 2^2k \) new Boolean variables. The method of doing this can be visualized as follows. Consider a \( 2^k \) by \( 2^{2k} \) matrix LC (for long code). The rows are indexed by all \( 2^k \) possible assignments to the \( x \) variables. The columns are indexed by all possible Boolean functions. Namely, the column vectors are all possible \( 2^{2k} \) truth-tables for Boolean functions on \( k \) variables (equivalently, all possible column vectors of dimension \( 2^k \)). Every row of the matrix is then the long code of its corresponding assignment. The \( z \) variables are intended to correspond to the columns, and their values (as a vector) are intended to be equal to one row in the matrix (one long code). A verifier may perform various tests to verify (in a probabilistic sense) that this is indeed the case. Moreover, the verifier would like this row to correspond to an assignment to the \( x \) variables that satisfies some predicate \( h \). The concept of folding assists in these tests.

**Folding over true.** Columns in LC can be paired, where each column (say \( z_i \)) is paired with the column (say \( z_j \)) that is its complement. If \( z \) is indeed a long code, then in each such pair, one of the two corresponding variables is redundant and hence is dropped. For example, to read the value of \( z_j \), read the value of \( z_i \) and flip this value. After folding over true, half the number of \( z \) variables remain.

**Folding over \( h \).** Consider the truth table for \( h \) (as a column vector). Only certain rows of LC have a value of 1 in this vector, and these rows correspond to the assignments that satisfy the predicate \( h \). In these rows, the following holds. Consider an arbitrary variable \( z_i \) whose column corresponds to the function \( f \). Then its value is exactly the complement of the variable \( z_j \) whose column corresponds to the function \( f + h \) (addition modulo 2). Hence again columns can be paired (columns that differ by \( h \)), and one member from each pair can be dropped. After folding over \( h \), half the number of \( z \) variables remain. We remark that folding over true is a special case of folding over \( h \), for a trivial choice of \( h \) as the predicate that always accepts (its truth table is all 1).

**Conditioning over \( h \).** Consider the rows correspond to the assignments that satisfy the predicate \( h \), and let \( r \) denote their number. The columns restricted to these rows can be partitioned into \( 2^r \) equivalence classes, based on equality. Only one member from each equivalence class needs to be retained, as its \( z \)-variable determines the value of the corresponding \( z \) variables for all other columns in its equivalence class. Hence after conditioning, the number of \( z \) variables that would remain is \( 2^r \).
Hardness of approximation results involve long code tests that query some of the \( z \) variables and determine whether their values are consistent with the assumption that all \( z \) variables form a valid long code. Basically, there are two versions of long code tests. One version (as in Bellare et al. [6]) involves decoding to long codes, in which if the \( z \) variables pass the test (with high probability) then the conclusion is that there are long codes that are close (in Hamming distance) to the vector of values of the \( z \) variables. (There might be more than one such long code, depending on the error probability that we allow in the long code test, but their number is small.) The other version which is the one that leads to tight hardness of approximation results (as in Hastad [12]) involves decoding to linear combinations, in which if the \( z \) variables pass the test (with high probability) then the conclusion is that there are words that are close to the vector of values of the \( z \) variables, where these words are linear combinations over a small number of long codes words. (Also here there might be several such words that are linear combinations.) Folding over \( h \) (and also conditioning over \( h \)) ensures that every decoded long code in the first version corresponds to an assignment to the \( x \) variables that satisfies \( h \). However, for the second version, there is a distinction between folding and conditioning. Conditioning over \( h \) ensures that every decoded linear combination is over long code words that correspond to assignments to the \( x \) variables that satisfy \( h \). Folding over \( h \) only ensures that at least one long code word does.

We remark that on top of folding (or conditioning) over \( h \), long code tests employ also folding over true (which regardless of other folding or conditioning operations, drops half of the previously remaining \( z \) variables). This is relevant to our discussion, as it illustrates the principle that two \( z \) variables can be deemed equivalent not only if they have equal values in all long codes of interest (e.g., all long codes that correspond to assignments to the \( x \) variables that satisfy the predicate \( h \)), but also if they have complementary values in all these long codes.

In order to obtain universal factor graphs, we need that even if \( h \) is changed due to a change of polarity of variables (e.g., from \( x_1 \lor x_2 \lor x_3 \) to \( x_1 \lor \overline{x}_2 \lor x_3 \)), the factor graph does not change. However, changing \( h \) changes the pairing in the folding and also changes the equivalence classes in the conditioning, and hence changes which \( z \) variables remain. As a consequence, the resulting factor graphs change.

Our solution to this problem is through the notion of an oblivious folding. Consider for example a predicate \( h \) corresponding to clause \( C_j = x_1 \lor x_2 \lor x_3 \). The long code associated with an assignment to the three variables \( x_1, x_2, x_3 \) will have \( 2^{2^3} \) variables (\( z \) variables). Folding over \( h \) will pair some of these variables. If polarities in \( C_j \) change (say, to \( x_1 \lor \overline{x}_2 \lor x_3 \)) we get a different predicate \( h' \) that leads to a different pairing. To overcome this problem, oblivious folding introduces auxiliary variables and corresponding new predicates. In the example above, this entails introducing three fresh auxiliary variables \( y_{j1}, y_{j2}, y_{j3} \) and replacing the clause \( C_j \) by the conjunction of four constraints, one that we call here a shadow constraint \( y_{j1} \lor y_{j2} \lor y_{j3} \), and three equality constraints \( y_{j1} = x_1, y_{j2} = x_2, y_{j3} = x_3 \). If polarities in \( C_j \) change as above, the shadow constraint
remains unchanged and only the equality constraints change (to $y_j \neq x_j$ in our example). Note that an equality constraint is simply negated when the corresponding $x$ variable is negated (equality changes to inequality). Now consider a long code associated with an assignment to six variables (namely, to the $x$ variables and the auxiliary $y$ variables). The number of $z$ variables is now $2^6$. Consider folding over an equality constraint $h'$. If the constraint is negated to give an inequality constraint $\overline{h'}$, then rather than change the pairing in the folding (to be folding over $h'$ rather than over $h'$), one can instead keep the same folding as for the original $h'$, but flip the nature of the relation between the two $z$ variables that form a pair (instead of requiring them to be different, requiring them to be the same).

In summary, our oblivious folding is performed as follows. Given a set of $x$ variables and a predicate $h$, we add auxiliary $y$ variables and express $h$ as the conjunction of a shadow predicate over the $y$ variables and equality constraints between the $x$ and $y$ variables. We consider the long code with respect to the combination of all variables. Thereafter we fold over all the new predicates one by one (the order does not matter), eventually giving a partition of the $z$ variables into equivalent classes. In fact, for the shadow constraint we could use conditioning rather than folding – the main aspect is that for each of the equality constraints we use folding. If the polarity of an $x$ variable changes, the partition of $z$ variables does not change. The only change is the relation among $z$ variables within the partition – two variables that were previously deemed equal might now be considered as negations of each other. As a result, when an $x$ variable is negated the factor graph over the $z$ variables remains unchanged, and only polarities of some of the $z$ variables change.

Our oblivious folding can be used in conjunction with the proof of Bellare et al [6] (after proper modifications), because our equivalence classes capture all equivalences used by (standard) folding over $h$. We do not know whether our oblivious folding can be used in conjunction with the proof of Hastad [12], because oblivious folding does not capture all equivalences captured by conditioning. For example, given a long code (of length $2^t$) for $t$ variables, conditioning over an equality constraint between two variables creates $2^{t-1}$ equivalence classes, whereas folding (which is an oblivious folding, since we only have an equality constraint) creates $2^{t-1}$ equivalence classes. As explained earlier, when decoding to a linear combination of long codes, we know that for each equality constraint at least one long code in the linear combination satisfies it. However, several equality constraints are introduced by our oblivious folding, and it might be the case that none of the long codes in the linear combination satisfy all of them.

### D.2 Proof of Theorem 6

Recall that the FGPR in Lemma 6 creates a 1-restricted constraint hypergraph of rank 4, such that the degree of every vertex depends only on the size of the alphabet and the degree of the constraint graph. Additionally, the FGPR in
Theorem 8 creates a bounded degree graph with alphabet that depends only on \( t \). Thus, it is possible to use the series of FGPRs in Theorem 9 without the FGPR in Lemma 4 to get a universal factor graph for 1-restricted constraint hypergraphs of rank 4, such that the degree of every vertex is bounded by a universal constant.

We now follow the proof of Bellare et al. [6] (with the modification of the folding) to show hardness of approximation within a factor of \( \frac{22}{25} \).

Given some fixed 1-restricted constraint hypergraph \( H \) with alphabet \( \{0, 1\} \), rank 4 and bounded degree, consider the two prover game \( H^n \): The outer verifier picks uniformly at random a set \( C \) of \( u \) hyperedges (constraints). For a constraint \( c \), let \( B_c \) be the set of bits the value of \( c \) depends on. For every \( c \in C \), the outer verifier picks uniformly at random a bit in \( B_c \). Call the set of selected bits \( B \). Then, the outer verifier gives \( P_1 \) the set \( C \) and gives \( P_2 \) the set \( B \). \( P_1 \) is expected to return a satisfying assignment to each edge. \( P_2 \) is expected to return an assignment to the set of bits, consistent with the assignment \( P_1 \) gives (the same assignment for the same bits). The outer verifier accepts if both conditions on the responses of the provers hold.

For \( u = 1 \), it is easy to see that the answers of \( P_2 \) define an assignment, \( a \), for the graph and the answers of \( P_1 \) define an assignment for each edge, separately. If an assignment for some edge is inconsistent with \( a \) and the verifier chose this edge, the probability that the verifier finds the inconsistency is at least \( \frac{1}{u} \) (the probability that \( P_2 \) is asked to reveal an inconsistent bit). Thus, if UNSAT \( (G) > \epsilon \), the verifier rejects with probability at least \( \frac{\epsilon}{u} \). It is immediate that if UNSAT \( (G) = 0 \), there is an assignment that makes the verifier always accept.

Using the Parallel Repetition Theorem 23, for every \( \epsilon \), if the original graph is at most \( 1 - \epsilon \) satisfiable, there is \( c_\epsilon > 0 \) such that the verifier \( V^u \) accepts with probability at most \( c_\epsilon^u \), for all \( u > 0 \). Again, if the graph is satisfiable, the verifier (after parallel repetition) can be made to always accept.

**Definition 12.** The string \( A \in \{0, 1\}^{F_n} \) (where \( F_n \) is the set of all functions from \( n \) bits to 1 bit) is the long code of a string \( x \in \{0, 1\}^n \) if for all \( f \in F_n \), \( A_f = f(x) \).

**Definition 13.** A \( A \in \{0, 1\}^{F_n} \) is said to be folded over \( (h, b) \) (\( h \in F_n, b \in \{0, 1\} \)), if for all \( f \in F_n \), \( A_{f+h} = A_f + b \).

Let \( h_1, \ldots, h_k \) be linearly independent functions and \( b_1, \ldots, b_k \) be bits. Let \( \prec \) be some total ordering of \( F_n \). Let \( \mu(f) \) be the minimal function among \( \{ f + \sum \sigma_i h_i | \sigma_i \in \{0, 1\} \} \), and let \( \sigma_i^f \) be such that \( \mu(f) = f + \sum \sigma_i^f h_i \). We say that \( B \in \{0, 1\}^{F_n} \) is the folding over \( (h_1, \ldots, h_k), (b_1, \ldots, b_k) \) of \( A \in \{0, 1\}^{F_n} \) if \( B_f = A_{\mu(f)} + \sum \sigma_i^f b_i \). Note that \( B \) is folded over \( (h_i, b_i) \).

It is possible to define folding over a set of linearly dependent functions, provided that the values of the bits \( \{b_i\} \) are consistent with the linear dependencies over the functions \( \{h_i\} \). We do not present such a definition, because it is not required for our proofs.
For our usage of folding, it suffices to fold over linearly independent functions.

In order to check the validity and consistency of the assignment using disjunctive clauses with 3 literals each, the provers are expected to give the long code for the assignment for the variables they are given (the sets $C$ and $B$ as before), and an inner verifier will check their answer. Let $A$ be the answer of $P_1$, $D$ the answer of $P_2$. Let $h_1, \ldots, h_u$ be the homogeneous part of the constraints in $C$ (the constraints are degree two polynomials). The verifier will access the folding of $A$ over $(1, h_1, \ldots, h_u)$, $(1, b_1, \ldots, b_u)$, where $b_1, \ldots, b_u$ are the constants of the respective constraints. Since every vertex appears a bounded number of times, independent of the size of the graph, for large enough hypergraphs the functions will be linearly independent with very high probability. If they are dependent, we can always accept while only slightly increasing the satisfiability of the game.

Note that when we say that the verifier checks the bit $A_f$, it will actually access the bit $A_{\mu(f)}$ and will invert it depending on $(h_1, \ldots, h_u), (b_1, \ldots, b_u)$. $\mu(f)$ only depends on $(h_1, \ldots, h_u)$. Close constraint hypergraphs have the same constraints, up to a constant. So, if two constraint hypergraphs are close, the accessed bit will be the same and the only difference will be in whether the bit is negated or not.

The verifier will be modeled as a 3CNF formula, such that all close constraint hypergraphs are transformed into a formula with the same factor graph. The inner verifier will check one of the following four constraints:

1. For $f, g \in \mathcal{F}_u$ chosen uniformly, check that $A_f + A_g = A_{f+g}$. This test passes iff the four clauses $\overline{A_f} \lor \overline{A_g} \lor \overline{A_{f+g}}, A_f \lor \overline{A_g} \lor A_{f+g}, A_f \lor A_g \lor \overline{A_{f+g}}$ are satisfied.

2. For $f, g, h \in \mathcal{F}_u$ chosen uniformly, if $A_f = 0$, check that $A_{f+g} = A_h$. If $A_f = 1$, check that $A_{f+g+h} = A_h$. This test passes iff the four clauses $A_f \lor A_{f+g} \lor A_h, A_f \lor A_{f+g+h} \lor \overline{A_h}, \overline{A_f} \lor A_{f+g} \lor A_h, \overline{A_f} \lor A_{f+g+h} \lor \overline{A_h}$ are satisfied.

3. For $f \in \mathcal{F}_u$, $g' \in \mathcal{F}_u$ chosen uniformly, check that $D_{g'} = A_{g+f} + A_f$ (where $g$ is $g'$ extended to the domain of all $3u$ bits of $C$, such that $g$ does not depend on the bits in $C$ that are not in $B$). This test passes iff the four clauses $A_f \lor A_{g+f} \lor D_{g'}, A_f \lor A_{g+f} \lor D_{g'}, A_f \lor A_{g+f} \lor D_{g'}, A_f \lor A_{g+f} \lor D_{g'}$ are satisfied.

The verifier chooses which of the constraints to check with some probability that will be implied in the proof.

**Definition 14.** The distance between two strings $x, y \in \{0, 1\}^\ell$ is the fraction of coordinates in which they differ.

**Claim.** Let $E \in \{0, 1\}^{F_n}$ be a long code of some string $x \in \{0, 1\}^n$. Let $D \in \{0, 1\}^{F_n}$ be some string folded over $(h, b)$ such that $h(x) \neq b$. Then $D$ and $E$ are $\frac{1}{\ell}$-far.

**Proof.** $\alpha = \{f | E_f = D_f\}$. If $|\alpha| > \frac{1}{\ell} |\mathcal{F}_n|$, then there is $f$ such that $f, f+h \in \alpha$. $E_f = f(x), E_{f+h} = f(x) + h(x) = f(x) + b + 1$. However $D_{f+h} = D_f + b$, so it is impossible that $f, f+h \in \alpha$. \qed
Definition 15. Fix $\delta > 0$ arbitrarily small. We say that $A, D$ are $(C, B, \delta)$-solid if $A, D$ are $(\frac{1}{2} - \delta)$-close to some $\tilde{A}, \tilde{D}$, respectively, where $\tilde{A}$ is a long code of an assignment $a$ satisfying the edges in $C$, and $\tilde{D}$ is the long code of an assignment $d$ for $B$, such that $d$ is consistent with $a$. Usually, $\delta$ will be known from the context and will be omitted.

Claim. For a game $G^u$, let $E'$ be the set of random coins such that given the corresponding set $C, B$ to $P_1, P_2$ respectively, the answers $A, D$ are $(C, B)$-solid. Then, there are provers $P_1, P_2$ for the game $G^u$ and a set $E$ of random coins, $|E| \geq \frac{\delta^4}{16} |E'|$, such that $V_u$ will accept for all choices of random coins in $E$.

Proof. Consider a bipartite graph where the vertices of one side are all possible sets of $u$ constraints queried in $G^u$ and the other side is the set of all possible sets of variables queried. There is an edge between a set of constraints and a set of variables if there is a choice of random coins such that the sets are queried at the same time. Note that the graph produced can be seen as a constraint graph, representing the game $G^u$ with the property that an assignment satisfying $k$ edges will satisfy $V_u$ for $k$ choices of random coins.

If a string is $\frac{1}{2} - \delta$ close to a long code, then there are at most $4\delta^{-2}$ long codes close to it (Lemma 3.11 in [6]). From claim D.2 all of these long codes must satisfy the selected constraints.

For any edge $(C, B)$ and respective answers of $(P_1, P_2)$, with $(A, D)$ $(C, B)$-solid, there are $16\delta^{-4}$ possible choices of assignments for $C$ and $B$ (derived from the closest $4\delta^{-2}$ for each of them), with at least one choice satisfying the edge between them. Selecting an assignment randomly satisfies $16\delta^{-4}|E'|$ edges in expectation. Thus, there is an assignment for $G^u$ with the property that an assignment satisfying $k$ edges will satisfy $V_u$ for $k$ choices of random coins.

Corollary 1. If $G^u$ is at most $\epsilon$ satisfiable, then for any pair of provers at most $16\delta^{-4}\epsilon$ of the answers can be solid for their respective queries.

Given a query to the provers, let $A, D$ be the answers of $P_1, P_2$, respectively. Let $x$ be the distance of $A$ from the closest linear function, $\tilde{A}$.

The fraction of tests (of the first type) that will fail is lower bounded by the function (Lemma 3.15 in [6], also stated as Lemma 3.15 in [6])

$$\Delta(x) = \begin{cases} 
3x - 6x^2 & 0 \leq x \leq \frac{5}{128} \\
\frac{45}{128} x & \frac{5}{128} < x \leq \frac{5}{16} \\
\frac{45}{128} & \frac{45}{128} < x
\end{cases}$$

If $\tilde{A}$ is not a long code, then the fraction of tests that the second test will fail is at least $\frac{3}{2} (1 - 2x)$ (Lemma 3.19 in [6]).

If $\tilde{A}$ is a long code of $a$, $\tilde{D}$ is a long code of $d$, where $d$ is the restriction of $a$ on the respective bits, and $y$ is the distance between $D$ and $\tilde{D}$, then the fraction of tests that the third test will fail is at least $y (1 - 2x)$ (Lemma 3.21 in [6]).

Theorem 10. There is a $(\frac{77}{80} + \epsilon)$-universal factor graph for 3-SAT, for any $\epsilon > 0$. 

Proof. Let $G_n$ be the set of graphs of size $n$ produced by the transformation from lemma 4 activated on APX-universal factor graph generated in appendix C. Note that the degree of every vertex is bounded due to the transformation used to create the factor graph. The factor graph of the 3CNF formula checking the satisfiability of $G^u$ is the same, for all $G \in G_n$.

Suppose that $G$ cannot be completely satisfied. Then, the fraction of answers that can be solid to their respective queries is arbitrarily small (fixing small $\delta$, increasing $u$). Non solid answers fail some of the tests, so our goal is to increase the fraction of clauses that non solid answers cannot satisfy. If $A, D$ is the answer to the query $(C, B)$ which is not $(C, B)$-solid then there are several possibilities:

- $\tilde{A}$ is not a long code, then at least $n_1 \Delta(x) + \frac{3n_2}{8} (1 - 2x)$ tests fail.
- $\tilde{A}$ is the long code of $a$, but $D$ is at least $\frac{1}{2} - \delta$ far from a long code of the restriction of $a$, then at least $n_1 \Delta(x) + n_3 \left(\frac{1}{2} - \delta\right) (1 - 2x)$ tests fail.

Otherwise, $A, D$ is $(C, B)$-solid.

Let $n_1, n_2, n_3$ be the number of tests of each type for every query (theses numbers uniquely define the probability the inner verifier chooses which of the three tests to execute). The total number of clauses is $4 (n_1 + n_2 + n_3)$. If a test fails then at least a single clause cannot be satisfied (using the respective assignment). For $n_3 = \frac{3}{4} n_2$ all the fractions of failed tests (and the number of unsatisfied clauses) is the same for both cases ($\delta$ can be arbitrarily small). Let $k$ be the total number of clauses. We need to maximize the fraction of unsatisfied clauses, that is

$$\frac{n_1 \Delta(x) + \frac{3}{8} (k - 4n_1) (1 - 2x)}{k}$$

Then, the minimum must be at $x = 0$, $x = \frac{45}{128}$, or $x = \frac{1}{2}$, so $n_1 = n_3 = \frac{3k}{40}$, $n_2 = \frac{k}{10}$, which gives that at least a $\frac{3}{80}$ fraction of the clauses are unsatisfiable. \qed

E A Remark on Nearly Tight Thresholds

The following remark relates to the proof of Theorem 4.

Remark 1. If $\phi_3$ is unweighted, then weighted formula $\phi_k$ can easily be replaced by an unweighted formula without affecting the correctness of the proof. Scale all weights by a multiplicative constant so that clauses in $\psi_0$ have weight 1, and then clauses in $\psi_i$ for $i \geq 1$ have weight $\gamma m$. We may assume without loss of generality that $\gamma m$ is an integer. (Otherwise, decrease $\gamma$ by a little.) Duplicating each formula $\psi_i$ (for $i \geq 1$) $\gamma m$ times, each time using a fresh set of new three variables as the $z$ variables, gives the desired unweighted version of the formula $\phi_k$. 

Many known reductions are in fact FGPRs, and this gives universal factor graphs for additional CSPs. Examples include the standard reductions from max-3SAT to max-4NAE (adding a variable to all clauses), from max-4NAE to max-3NAE (break each clause into two clauses of two literals and add a new variable to one clause and its negation to the other) and from max-3NAE to max-2LIN (replace every 3NAE clause by three 2LIN clauses, each of which is the XOR of two literals from the NAE clause).