SOME IDENTITIES INVOLVING ASSOCIATED SEQUENCES OF SPECIAL POLYNOMIALS

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Abstract. In this paper, we study some properties of associated sequences of special polynomials. From the properties of associated sequences of polynomials, we derive some interesting identities of special polynomials.

1. Introduction

For \( r \in \mathbb{R} \), the Bernoulli polynomials of order \( r \) are defined by the generating function to be

\[
\left( \frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad (r \in \mathbb{R}),
\]

(1.1)

In the special case, \( x = 0 \), \( B_n^{(r)}(0) = B_n^{(r)} \) are called the \( n \)-th Bernoulli numbers of order \( r \). It is also well known that the Euler polynomials of order \( r \) are defined by

\[
\left( \frac{2}{e^t + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}, \quad (r \in \mathbb{R}),
\]

(1.2)

Let \( x = 0 \). Then \( E_n^{(r)}(0) = E_n^{(r)} \) are called the \( n \)-th Euler numbers of order \( r \).

Let \( F \) be the set of all formal power series in the variable \( t \) over \( \mathbb{C} \) with

\[
F = \left\{ f(t) = \sum_{k=0}^{\infty} a_k t^k \mid a_k \in \mathbb{C} \right\}.
\]

(1.3)

Let \( \mathbb{P} \) be the algebra of polynomials in the variable \( x \) over \( \mathbb{C} \) and \( \mathbb{P}^* \) be the vector space of all linear functionals on \( \mathbb{P} \). The action of the linear functional \( L \) on a polynomial \( p(x) \) is defined by \( \langle L \mid p(x) \rangle \) and the vector space structure on \( \mathbb{P}^* \) is derived by \( \langle L + M \mid p(x) \rangle = \langle L \mid p(x) \rangle + \langle M \mid p(x) \rangle \), \( \langle cL \mid p(x) \rangle = c \langle L \mid p(x) \rangle \), where \( c \) is a complex constant.

For \( f(t) = \sum_{k=0}^{\infty} a_k t^k \in F \), we define a linear functional on \( \mathbb{P} \) by setting

\[
\langle f(t) \mid x^n \rangle = a_n, \quad (n \geq 0), \quad (\text{see } [3,8,17]).
\]

(1.4)

By (1.3) and (1.4), we get

\[
\langle t^k \mid x^n \rangle = n! \delta_{n,k}, \quad (n,k \geq 0), \quad (\text{see } [4,5,7,10,17,18]),
\]

(1.5)

where \( \delta_{n,k} \) is the Kronecker symbol.

Let \( f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L \mid x^k \rangle}{k!} t^k \). Then, by (1.5), we get \( \langle f_L(t) \mid x^n \rangle = \langle L \mid x^n \rangle \). So, we see that \( f_L(xt) = L \). The map \( L \mapsto f_L(t) \) is a vector space isomorphism from \( \mathbb{P}^* \)
onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ is thought of as both a formal power series and a linear functional (see [4, 8, 17, 19]). We call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra (see [4, 10, 17]).

The order $o(f(t))$ of the non-zero power series $f(t)$ is the smallest integer $k$ for which the coefficient of $t^k$ does not vanish. If $o(f(t)) = 1$, then $f(t)$ is called a delta series and if $o(f(t)) = 0$, then $f(t)$ is called an invertible series. Let $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $S_n(x)$ of polynomials such that

$$\langle g(t)f(t)^k | S_n(x) \rangle = n! \delta_{n,k} \text{ where } n, k \geq 0.$$ 

The sequence $S_n(x)$ is called Sheffer sequence for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$. If $S_n(x) \sim (1, f(t))$, then $S_n(x)$ is called the associated sequence for $f(t)$. By (1.5), we see that $\langle e^{yt} p(x) \rangle = p(y)$.

Let $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t) | x^k \rangle}{k!} t^k, \quad p(x) = \sum_{k=0}^{\infty} \frac{\langle k | p(x) \rangle}{k!} x^k, \text{ (see [10,17]),}$$

and

$$\langle f_1(t)f_2(t)\cdots f_m(t) | x^n \rangle = \sum_{i_1 + \cdots + i_m = n} \binom{n}{i_1, \ldots, i_m} \langle f_1(t) | x^{i_1} \rangle \cdots \langle f_m(t) | x^{i_m} \rangle,$$

where $f_1(t), f_2(t), \ldots, f_m(t) \in \mathcal{F}$, (see [9, 10, 17]). From (1.6), we have

$$p^{(k)}(0) = \langle t^k | p(x) \rangle, \quad \langle 1 \mid p^{(k)}(x) \rangle = p^{(k)}(0).$$

Thus, by (1.8), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad (k \geq 0), \quad \text{ (see [17]).}$$

For $S_n(x) \sim (g(t), f(t))$, we have the following equations:

$$S_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_k(y) S_{n-k}(x), \quad \text{ where } p_k(y) = g(t) S_k(y),$$

and

$$\frac{1}{g(f(t))} e^{y \tilde{f}(t)} = \sum_{k=0}^{\infty} \binom{k}{k} \frac{S_k(y)}{k!} t^k, \text{ for all } y \in \mathbb{C} \text{ (see [10, 15, 16, 17, 18]),}$$

where $\tilde{f}(t)$ is the compositional inverse of $f(t)$.

Let $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, g(t))$. Then the transfer formula for associated sequence implies that, for $n \in \mathbb{N}$,

$$q_n(x) = x \left( \frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x), \quad \text{ (see [11, 17, 22]).}$$

Now we introduce several important sequences which are used to derive our results in this paper (see [10, 11, 17]):

(The Poisson-Charlier sequences)

$$C_n(x; a) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a^{-k} (x)_k \sim \left( e^{a(e^t-1)}, a(e^t-1) \right),$$
where $a \neq 0$, $(x)_n = x(x-1) \cdots (x-n+1)$,

$$
\sum_{k=0}^{n} C_n(k; a) \frac{k^t}{k!} = \left( \frac{t-a}{a} \right)^n, \quad (a \neq 0), \quad n \in \mathbb{N} \cup \{0\}, \quad (1.13)
$$

(The Abel sequences)

$$
A_n(x; b) = x(x-bn)^{n-1} \sim (1, te^{bt}), \quad (b \neq 0), \quad (1.14)
$$

(The Mittag-Leffler sequences)

$$
M_n(x) = \sum_{k=0}^{n} \binom{n}{k} (n-k)2^k(x)_k \sim \left( 1, \frac{e^t-1}{e^t+1} \right), \quad (1.15)
$$

(The exponential sequences)

$$
\phi_n(x) = \sum_{k=0}^{n} S_2(n, k)x^k \sim (1, \log(1+t)), \quad (1.16)
$$

and

(The Laguerre sequences)

$$
L_n(x) = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{n!}{k!} (-x)^k \sim \left( 1, \frac{t}{t-1} \right), \quad (1.17)
$$

In this paper, we study some properties of associated sequences of special polynomials. From the properties of associated sequences of special polynomials, we derive some interesting identities involving associated sequences of special polynomials.

2. Associated sequences of special polynomials.

As is well known, the Bessel differential equation is given by

$$
x^2y'' + 2(x+1)y' + n(n+1)y = 0, \quad (\text{see} \ [1,2]). \quad (2.1)
$$

From (2.1), we have the solution of (2.1) as follows:

$$
y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!} \left( \frac{x}{2} \right)^k, \quad (\text{see} \ [1,2]). \quad (2.2)
$$

Let us consider the following associated sequences:

$$
p_n(x) \sim \left( 1, t - \frac{t^2}{2} \right), \quad x^n \sim (1, t), \quad (\text{see} \ [1,2,10,17]). \quad (2.3)
$$
From (1.12) and (2.3), for $n \in \mathbb{N}$, we have
\[
p_n(x) = x \left( \frac{t}{t - \frac{t^2}{2}} \right)^n x^{-1} x^n = x \left( \frac{1 - t}{2} \right)^n x^{n-1}
\]
\[
= x \sum_{k=0}^{\infty} \binom{-n}{k} (-1)^k \left( \frac{t}{2} \right)^k x^{n-1}
\]
\[
= x \sum_{k=0}^{n-1} \binom{n + k - 1}{k} \left( \frac{1}{2} \right)^k (n - 1)_k x^{n-1-k}
\]
\[
= x \sum_{k=0}^{n-1} \frac{(n + k - 1)!}{k!(n - 1 - k)!} \left( \frac{1}{2} \right)^k x^{n-k}
\]
\[
= \sum_{k=1}^{n} \frac{(2n - k - 1)!}{(n - k)!(k - 1)!} \left( \frac{1}{2} \right)^{n-k} x^k.
\]
By (2.2) and (2.4), we get
\[
p_n(x) = x^n y_{n-1} \left( \frac{1}{x} \right) \sim \left( 1, t - \frac{t^2}{2} \right).
\]
From (1.11) and (2.5), we can derive the following generating function of $p_n(x)$:
\[
\sum_{k=0}^{\infty} p_k(x) \frac{t^k}{k!} = e^{x \left( 1 - (1-2t)^{\frac{1}{2}} \right)},
\]
and, by (1.10), we get
\[
(x + y)^n y_{n-1} \left( \frac{1}{x + y} \right) = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} y_{k-1} \left( \frac{1}{x} \right) y_{n-k-1} \left( \frac{1}{y} \right).
\]
By (1.12) and (2.3), we get
\[
x^n = x \left( \frac{t - \frac{t^2}{2}}{t} \right)^n x^{-1} p_n(x) = x \left( \frac{t - \frac{2}{2}}{2} \right)^n x^{-1} p_n(x).
\]
Thus, by (1.13), (2.4) and (2.8), we get
\[
(-1)^n x^{n-1} = \left( \frac{t - \frac{2}{2}}{2} \right)^2 x^{-1} p_n(x) = \sum_{k=0}^{\infty} C_n(k; 2) \frac{t^k}{k!} e^{-t} (x^{-1} p_n(x))
\]
\[
= \sum_{k=0}^{n-1} C_n(k; 2) \frac{t^k}{k!} (x - 1)^{-1} p_n(x - 1)
\]
\[
= \sum_{k=0}^{n-1} C_n(k; 2) \frac{t^k}{k!} \sum_{l=1}^{n} \frac{(2n - l - 1)!}{(l-1)!(n-l)!} \left( \frac{1}{2} \right)^{n-l} (x - 1)^{-1-l}
\]
\[
= \sum_{k=0}^{n-1} C_n(k; 2) \sum_{l=k+1}^{n} \frac{(2n - l - 1)!}{(l-1)!(n-l)!} \left( \frac{1}{2} \right)^{n-l} \frac{1}{k!} (x - 1)^{-1-l-k}
\]
\[
= \sum_{m=0}^{n-1} \sum_{k=0}^{m-1} C_n(k; 2) \left( \binom{m+k}{k} \frac{(2n - m - k - 2)!}{(m+k)!(n - m - k - 1)!} \frac{1}{2} \right)^{n-m-1} (x - 1)^m.
\]
From (2.8), we have

\[ x^{n-1} = \left(1 - \frac{t}{2}\right)^{n-1}p_n(x) = \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{t}{2}\right)^k x^{-1} p_n(x) \]

\[ = \sum_{k=0}^{n-1} \sum_{l=k+1}^{n} \binom{n}{k} (l-1)k \frac{(2n-l-1)!}{(l-1)!(n-l)!} (-1)^k \left(\frac{1}{2}\right)^{n+k-l} x^{l-1-k} \]

\[ = \sum_{m=0}^{n-1} \sum_{k=0}^{n-m-1} \binom{n}{k} (m+k) \frac{(2n-m-k-2)!}{(m+k)!(n-m-1-k)!} \left(\frac{1}{2}\right)^{n-m-1} x^m. \tag{2.10} \]

Therefore, by (2.9) and (2.10), we obtain the following theorem.

**Theorem 2.1.** For \( n \in \mathbb{N} \), we have

\[ (-1)^n x^{n-1} = \sum_{m=0}^{n-1} \sum_{k=0}^{n-m-1} C_n(k;2) \left(\frac{m+k}{k}\right) \frac{(2n-m-k-2)!}{(m+k)!(n-m-1-k)!} \left(\frac{1}{2}\right)^{n-m-k-1} (x-1)^m. \]

Moreover,

\[ \sum_{k=0}^{n-m-1} (-1)^k \left(\frac{1}{2}\right)^{n-m-1} \binom{n}{k} \frac{(2n-m-k-2)!}{m!(n-m-1-k)!} = 0, \]

where \( 0 \leq m \leq n-2 \).

Let us consider the following associated sequences:

\[ p_n(x) \sim \left(1, te^{(e^t-1)}\right), \ c \neq 0, \ A_n(x; b) = x(x - bn)^{n-1} \sim \left(1, te^b\right), \ b \neq 0. \tag{2.11} \]

By (2.12) and (2.11), we get

\[ p_n(x) = x \left(\frac{t}{te^{(e^t-1)}}\right)^n x^{-1} x^n = x \sum_{k=0}^{\infty} \frac{(-ne)^k}{k!} (e^t - 1)^k x^n. \tag{2.12} \]

We recall that Newton’s difference operator \( \Delta \) is defined by \( \Delta f(x) = f(x+1) - f(x) \). For \( n \in \mathbb{N} \), we easily see that

\[ \Delta^n p(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} p(x+k). \tag{2.13} \]

By (2.13), we get

\[ (e^t - 1)^k p(x) = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} e^t p(x) = \sum_{l=0}^{k} \binom{k}{l} (-1)^{l-k} p(x+l) = \Delta^k p(x). \tag{2.14} \]

In particular, if we take \( p(x) = x^{n-1} \), then we have

\[ (e^t - 1)^k x^{n-1} = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (x+j)^{n-1}. \tag{2.15} \]
Lemma 2.2. Therefore, by (2.16), we obtain the following lemma.

From (2.12) and (2.15), we have

\[ A_n(x; b) = x(x - bn)^{n-1} = x \left( \frac{t^e^{(c'-1)}}{t^e^{(c)}} \right)^n x^{-1} p_n(x) \]

(2.17)

where \( a_n^{(\beta)}(x) \sim ((1-t)^{-\beta}, \log(1-t)) \) is the actuarial polynomial with the generating function given by

\[ \sum_{l=0}^{\infty} \frac{a_l^{(\beta)}(x)}{l!} t^l = e^{\beta t + x(1-e')} \]

By Lemma 222 and (2.17), we get

\[ A_n(x; b) = x \left\{ \sum_{m=0}^{\infty} \sum_{l_1 + \ldots + l_n = m} \left( \frac{m!}{l_1! \ldots l_n!} \right) a_l^{(b)}(-c) \ldots a_n^{(b)}(-c) \right\} \]

\[ \times \left\{ \sum_{k=0}^{n-1} \sum_{j=0}^{k} \frac{(-1)^j (nc)^k}{k!} \left( \frac{k}{j} \right) (x + j)^{n-1} \right\} \]

(2.18)

Therefore, by (2.18), we obtain the following theorem.

Theorem 2.3. For \( n \geq 1, b \neq 0, c \neq 0 \), we have

\[ A_n(x; b) \]

(2.19)

For (2.17), we note that

\[ A_n(x; b) = x \left( e^{e^{(c'-1)}} \right)^n e^{-nbt} x^{-1} p_n(x) \]

\[ = x \left( e^{e^{(c'-1)}} \right)^n (x - nb)^{-1} p_n(x - nb). \]
By (1.16) and Lemma 2.2, we easily see that the generating function of exponential sequences is given by
\[ \sum_{k=0}^{\infty} \phi_k(x) \frac{t^k}{k!} = e^{x(e^t-1)}. \]  
(2.20)

From (2.19) and (2.20), we have
\[ A_n(x; b) = \sum_{m=0}^{\infty} \sum_{l_1+\ldots+l_n=m} \binom{m}{l_1, \ldots, l_n} \left( \prod_{j=1}^{n} \phi_j(c) \right) \frac{t^m}{m!} (x - nb)^{-1} p_n(x - nb) \]
\[ = \sum_{m=0}^{\infty} \sum_{l_1+\ldots+l_n=m} \binom{m}{l_1, \ldots, l_n} \phi_j(c) \frac{t^m}{m!} \sum_{k=0}^{n-1} \sum_{j=0}^{k} \frac{(-1)^j (nc)^k}{k!} \binom{k}{j} (x - nb + j)^{n-1} \]
\[ = \sum_{m=0}^{n-1} \sum_{l_1+\ldots+l_n=m} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \binom{n-1}{m} \binom{m}{l_1, \ldots, l_n} \phi_j(c) \frac{t^m}{m!} \frac{(-1)^j (nc)^k}{k!} (x - nb + j)^{n-m-1}. \]
(2.21)

Therefore, by (2.21), we obtain the following corollary.

**Corollary 2.4.** For \( n \geq 1, b \neq 0, c \neq 0 \), we have
\[ A_n(x; b) = \sum_{m=0}^{n-1} \sum_{l_1+\ldots+l_n=m} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \binom{n-1}{m} \binom{m}{l_1, \ldots, l_n} \phi_j(c) \frac{t^m}{m!} \frac{(-1)^j (nc)^k}{k!} (x - nb + j)^{n-m-1}. \]

Note that \( x^n \sim (1, t) \). By (1.12), (1.13) and (1.17), we get
\[ L_n(x) = x \left( \frac{t}{t-1} \right)^n x^{-1} x^n = x(t-1)^n x^{n-1} \]
\[ = x \sum_{k=0}^{n-1} C_n(k; 1) \frac{t^k}{k!} e^{-t} x^{n-1} = x \sum_{k=0}^{n-1} C_n(k; 1) \frac{t^k}{k!} (x-1)^{n-1} \]
\[ = x \sum_{k=0}^{n-1} C_n(k; 1) \binom{n-1}{k} (x-1)^{n-1-k} = x \sum_{k=0}^{n-1} \binom{n-1}{k} C_n(n-1-k; 1)(x-1)^k. \]
(2.22)

Therefore, by (2.22), we obtain the following theorem.

**Theorem 2.5.** For \( n \geq 1 \), we have
\[ L_n(x) = x \sum_{k=0}^{n-1} \binom{n-1}{k} C_n(n-1-k; 1)(x-1)^k. \]

Mott considered the associated sequences for \( f(t) = \frac{2t}{1-t^2} \). That is, the Mott sequence is given by
\[ S_n(x) \sim \left( 1, \frac{-2t}{1-t^2} \right). \]  
(2.23)
From (2.23), we note that the generating function of Mott sequences is given by
\[
\sum_{k=0}^{\infty} S_k(x) \frac{t^k}{k!} = \exp \left( x \left( \frac{1 - \sqrt{1 + t^2}}{t} \right) \right).
\]

By (1.12), (1.17) and (2.23), we get
\[
S_n(x) = x \left( \frac{t+1}{2t} \right)^n x^{-1} L_n(-x) = x \left( \frac{t-1}{2} \right)^n x^{-1} L_n(-x)
\]
\[
= 2^{-n} x (t-1)^n x^{-1} L_n(-x) = 2^{-n} x \left( \sum_{k=0}^{n-1} C_n(k;1) \frac{t^k}{k!} e^{-t} \right) x^{-1} L_n(-x)
\]
\[
= 2^{-n} x \sum_{k=0}^{n-1} C_n(k;1) \frac{t^k}{k!} (x-1)^{-1} L_n(1-x)
\]
\[
= 2^{-n} x \sum_{k=0}^{n-1} C_n(k;1) \frac{1}{k!} \sum_{l=1}^{n} \binom{n-1}{l-1} \frac{n!}{l!} (x-1)^{l-1}
\]
\[
= \frac{n!}{2^n} \sum_{k=0}^{n-1} \sum_{l=1}^{n} \binom{n-1}{l-1} \frac{n!}{l!} \binom{n-1}{k} C_n(k;1) \frac{1}{l!} x(x-1)^{l-1-k}.
\]
(2.24)

Thus, by (2.24), we obtain the following lemma.

**Lemma 2.6.** For \( n \in \mathbb{N} \), let \( S_n(x) \sim \left( 1, \frac{2i}{1-e^t} \right) \). Then we have
\[
S_n(x) = \frac{n!}{2^n} \sum_{k=0}^{n-1} \sum_{l=1}^{n} \binom{n-1}{l-1} \frac{n!}{l!} \binom{n-1}{k} C_n(k;1) \frac{1}{l!} x(x-1)^{l-1-k}.
\]

As is known, we have
\[
x E_{n-1}^{(an)}(x) \sim \left( 1, t \left( \frac{e^t - 1}{t} \right)^a \right), \quad x E_{n-1}^{(bn)}(x) \sim \left( 1, t \left( \frac{e^t + 1}{2} \right)^b \right),
\]
(2.25)

where \( a, b \) are positive integers (see [10, 11, 17]). For \( n \geq 1 \), by (1.12) and (2.26), we get
\[
x E_{n-1}^{(bn)}(x) = x \left( \frac{t \left( \frac{e^t - 1}{t} \right)^a}{t \left( \frac{e^t + 1}{2} \right)^b} \right)^n x^{-1} E_{n-1}^{(an)}(x)
\]
\[
= x \left( \frac{e^t + 1}{2} \right)^{-bn} \left( \frac{e^t - 1}{t} \right)^{an} B_{n-1}^{(an)}(x).
\]
(2.26)

Thus, by (2.26), we get
\[
\left( \frac{e^t + 1}{2} \right)^{bn} E_{n-1}^{(bn)}(x) = \left( \frac{e^t - 1}{t} \right)^{an} B_{n-1}^{(an)}(x).
\]
(2.27)
LHS of (2.27) = \(2^{-bn} (e^t + 1)^{bn} E_{n-1}^{(bn)}(x) = 2^{-bn} \sum_{k=0}^{bn} \binom{bn}{k} e^{kt} E_{n-1}^{(bn)}(x)\)

\[= 2^{-bn} \sum_{k=0}^{bn} \binom{bn}{k} E_{n-1}^{(bn)}(x + k).\]  \hspace{1cm} (2.28)

RHS of (2.27) = \(\left(\frac{1}{t}\right)^{an} (an)! \sum_{l=an}^{n-1} S_2(l, an) \frac{t^l}{l!} B_{n-1}^{(an)}(x)\)

\[= \sum_{l=0}^{n-1} \frac{(an)!}{(l + an)!} S_2(l + an, an)(n - 1) B_{n-1}^{(an)}(x)\]

\[= (n - 1)! \sum_{l=0}^{n-1} \frac{(an)!}{(l + an)!} S_2(l + an, an) B_{n-1}^{(an)}(x), \]  \hspace{1cm} (2.29)

where \(S_2(n, k)\) is the Stirling number of the second kind. Therefore, by (2.28) and (2.29), we obtain the following theorem.

**Theorem 2.7.** For \(n \geq 1, a, b \in \mathbb{N} \cup \{0\}\), we have

\[\sum_{k=0}^{bn} \binom{bn}{k} E_{n-1}^{(bn)}(x + k) = 2^{bn} (n - 1)! \sum_{l=0}^{n-1} \frac{(an)!}{(l + an)!} S_2(l + an, an) B_{n-1}^{(an)}(x).\]

The Pidduck sequences is given by

\[P_n(x) \sim \left(\frac{2}{e^t + 1}, \frac{e^t - 1}{e^t + 1}\right).\]  \hspace{1cm} (2.30)

From (2.30), we can derive the generating function of the Pidduck sequences as follows:

\[\sum_{k=0}^{\infty} P_k(x) \frac{t^k}{k!} = (1 - t)^{-1} \left(\frac{1 + t}{1 - t}\right)^x.\]  \hspace{1cm} (2.31)

Let \(S_n(x) \sim \left(1, \frac{2t}{e^t + 1}\right)\). Then, from (1.12), (1.15) and (2.30), we have

\[M_n(x) = \frac{2}{e^t + 1} P_n(x) = x \left(\frac{2t}{e^t + 1}\right)^n x^{-1} S_n(x)\]

\[= 2^n x \left(\frac{t}{e^t - 1}\right)^n x^{-1} S_n(x).\]  \hspace{1cm} (2.32)

By (1.12), we easily get

\[S_n(x) = x \left(\frac{2t}{e^t + 1}\right)^{-n} x^{-1} x^n = 2^{-n} x (e^t + 1)^n x^{-1}\]

\[= 2^{-n} x \sum_{j=0}^{n} \binom{n}{j} e^{jt} x^{n-1} = 2^{-n} x \sum_{j=0}^{n} \binom{n}{j} (x + j)^{n-1}.\]  \hspace{1cm} (2.33)
From (2.32) and (2.33), we have

$$M_n(x) = 2^n x \left( \frac{t}{e^t - 1} \right)^n \left( 2^{-n} \sum_{j=0}^{n} \binom{n}{j} (x+j)^{n-1} \right)$$

$$= \sum_{j=0}^{n} \binom{n}{j} x \left( \frac{t}{e^t - 1} \right)^n (x+j)^{n-1} = \sum_{j=0}^{n} \binom{n}{j} x B_{n-1}^{(n)}(x+j).$$

By (2.32) and (2.34), we get

$$P_n(x) = \frac{1}{2} (e^t + 1) M_n(x) = \frac{1}{2} (e^t + 1) \sum_{j=0}^{n} \binom{n}{j} x B_{n-1}^{(n)}(x+j)$$

$$= \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} \left\{ (x+1) B_{n-1}^{(n)}(x+1+j) + x B_{n-1}^{(n)}(x+j) \right\}$$

$$= \frac{1}{2} \sum_{j=1}^{n+1} \left\{ \binom{n}{j-1} (x+1) + \binom{n}{j} x \right\} B_{n-1}^{(n)}(x+j) + (x+1) B_{n-1}^{(n)}(x+n+1) + x B_{n-1}^{(n)}(x)$$

$$= \frac{1}{2} \sum_{j=0}^{n+1} \left\{ \binom{n+1}{j} x + \binom{n}{j-1} \right\} B_{n-1}^{(n)}(x+j).$$

Therefore, by (2.35), we obtain the following theorem.

**Theorem 2.8.** For \( n \geq 1 \), we have

$$P_n(x) = \frac{1}{2} \sum_{j=0}^{n+1} \left\{ \binom{n+1}{j} x + \binom{n}{j-1} \right\} B_{n-1}^{(n)}(x+j).$$

Let us consider the following two associated sequences:

$$S_n(x) \sim \left( 1, \frac{2t}{e^t+1} \right), \quad M_n(x) \sim \left( 1, \frac{e^t - 1}{e^t + 1} \right).$$

For \( n \geq 1 \), by (1.12), we get

$$S_n(x) = x \left( \frac{2t}{e^t + 1} \right)^n x^{-1} M_n(x)$$

$$= 2^{-n} x \left( \frac{e^t - 1}{t} \right)^n x^{-1} M_n(x).$$

(2.37)
By (2.34) and (2.37), we get

\[ S_n(x) = 2^{-n} x \frac{1}{l^n} n! \sum_{l=0}^{\infty} S_2(l, n) \frac{t^l}{l!} x^{-1} M_n(x) \]

\[ = 2^{-n} x \sum_{l=0}^{n-1} n! S_2(l + n, n) \frac{t^l}{(l + n)!} x^{-1} M_n(x) \]

\[ = 2^{-n} x \sum_{l=0}^{n-1} n! \frac{S_2(l + n, n)}{(l + n)!} \sum_{j=0}^{n} (n-1)_j B_{n-1-l}^{(n)}(x+j) \]

\[ = 2^{-n} x n!(n-1)! \sum_{l=0}^{n-1} \sum_{j=0}^{n} \frac{(n)_j S_2(l + n, n)}{(l + n)!(n-l-1)!} B_{n-1-l}^{(n)}(x+j). \]  

(2.38)

Therefore, by (2.33) and (2.36), we obtain the following theorem.

**Theorem 2.9.** For \( n \geq 1 \), we have

\[ \sum_{j=0}^{n} \binom{n}{j} (x+j)^{n-1} = n!(n-1)! \sum_{l=0}^{n-1} \sum_{j=0}^{n} \frac{(n)_j S_2(l + n, n)}{(l + n)!(n-l-1)!} B_{n-1-l}^{(n)}(x+j). \]

Moreover,

\[ \sum_{j=0}^{n} \binom{n}{j} j^{n-1} = n!(n-1)! \sum_{l=0}^{n-1} \sum_{j=0}^{n} \frac{(n)_j S_2(l + n, n)}{(l + n)!(n-l-1)!} B_{n-1-l}^{(n)}(j). \]

By (1.15), we get

\[ x^{-1} M_n(x) = \sum_{k=1}^{n} \binom{n}{k} (n-1)_{n-k} 2^k \sum_{j=0}^{k-1} S_1(k-1, j)(x-1)^j, \]  

(2.39)

where \( S_1(k, j) \) is the Stirling number of the first kind. From (2.38) and (2.39), we can derive

\[ S_n(x) = 2^{-n} x \sum_{l=0}^{n-1} \frac{n!}{(l + n)!} S_2(l + n, n) t^l (x^{-1} M_n(x)) \]

\[ = 2^{-n} x \sum_{l=0}^{n-1} \frac{n!}{(l + n)!} S_2(l + n, n) t^l \sum_{k=1}^{n} \binom{n}{k} (n-1)_{n-k} 2^k \sum_{j=0}^{k-1} S_1(k-1, j)(x-1)^j \]

\[ = 2^{-n} x n!(n-1)! \sum_{j=0}^{n-1} \sum_{k=j+1}^{n} \binom{n}{j} 2^k S_2(l + n, n) S_1(k-1, j)(j)_l (x^{-1} (x-1)^j). \]  

(2.40)

Therefore, by (2.33) and (2.40), we obtain the following theorem.

**Theorem 2.10.** For \( n \geq 1 \), we have

\[ \sum_{j=0}^{n} \binom{n}{j} (x+j)^{n-1} = n!(n-1)! \sum_{l=0}^{n-j} \sum_{k=j+1}^{n} \binom{n}{k} 2^k S_2(l + n, n) S_1(k-1, j)(j)_l (x^{-1} (x-1)^j). \]
Remark. From (2.34), we note that
\[ x^{-1}M_n(x) = \sum_{k=0}^{n} \binom{n}{k} (x + k - 1)_{n-1}. \] (2.41)

By (2.38) and (2.41), we get
\[
S_n(x) = 2^{-n}x \sum_{l=0}^{n-1} \frac{n!}{(l+n)!} S_2(l+n, n) t^l (x^{-1}M_n(x)) \]
\[ = 2^{-n}x \sum_{l=0}^{n-1} \frac{n!}{(l+n)!} S_2(l+n, n) t^l \left( \sum_{k=0}^{n} \binom{n}{k} (x + k - 1)_{n-1} \right) \]
\[ = 2^{-n}x n! \sum_{j=0}^{n-1} \sum_{l=0}^{j} \binom{n}{l} S_2(l+n, n) S_1(n-1, j) (j!) (x + k - 1)^{-j-l} \]
\[ = 2^{-n}x n! \sum_{j=0}^{n-1} \sum_{l=0}^{j} \binom{n}{l} S_2(l+n, n) S_1(n-1, j) j! (l+n)! (j-l)! (x + k - 1)^{-j-l}. \] (2.42)

So, by (2.38) and (2.41), we get
\[
\sum_{j=0}^{n} \binom{n}{j} (x+j)^{n-1} = n! \sum_{j=0}^{n-1} \sum_{l=0}^{j} \binom{n}{l} S_2(l+n, n) S_1(n-1, j) j! (l+n)! (j-l)! (x + k - 1)^{-j-l}. \] (2.43)

The Narumi polynomials \( N_k^{(a)}(x) \) of order \( a \) is defined by the generating function to be
\[
\sum_{k=0}^{\infty} \frac{N_k^{(a)}(x)}{k!} t^k = \left( \frac{\log(1+t)}{t} \right)^a (1+t)^x. \] (2.44)

Thus, from (2.44), we see that
\[
N_n^{(a)}(x) \sim \left( \left( \frac{e^t - 1}{t} \right)^a, e^t - 1 \right), \text{ (see [17,18])}. \] (2.45)

In the special case, \( x = 0 \), \( N_k^{(a)}(0) = N_k^{(a)} \) are called the \( k \)-th Narumi numbers of order \( a \). If \( a = 1 \) in (2.45), then we will write \( N_n(x) \) and \( N_n \) for \( N_n^{(1)}(x) \) and \( N_n^{(1)} \).

By (1.12) and (1.16), we get
\[
\phi_n(x) = x \left( \frac{t}{\log(1+t)} \right)^n x^{-1}x^n = x \left( \frac{t}{\log(1+t)} \right)^n x^{n-1} \]
\[ = x \left( \sum_{k=0}^{\infty} \frac{N_k^{(-n)}(0)}{k!} t^k \right) x^{n-1} = x \sum_{k=0}^{n-1} \frac{N_k^{(-n)}(0)}{k!} (n-1)_k x^{n-1-k} \]
\[ = \sum_{k=0}^{n-1} \binom{n-1}{k} N_k^{(-n)} x^{n-k} = \sum_{k=1}^{n} \binom{n-1}{k-1} N_{n-k}^{(-n)} x^k. \] (2.46)

Therefore, by (1.16) and (2.46), we obtain the following lemma.

Lemma 2.11. For \( n, k \in \mathbb{N} \) with \( k \leq n \), we have
\[
S_2(n, k) = \binom{n-1}{k-1} N_{n-k}^{(-n)}. \]
By (1.12), (1.16) and (1.17), we get

\[
\phi_n(x) = x \left( \frac{t}{(1 + t) \log(1 + t)} \right)^n x^{-1} L_n(-x)
\]

\[
= x \left( \frac{t}{\log(1 + t)} \right)^n (1 + t)^{-n} \sum_{l=1}^{n} \frac{(n-1)!}{l!} x^{l-1}
\]

\[
= x \left( \sum_{k=0}^{n-1} \frac{N_k^{(-n)}(-n)}{k!} \right) \sum_{l=1}^{n} \frac{(n-1)!}{l!} (l-1) x^{l-1-k}
\]

\[
= n! \sum_{k=0}^{n-1} \sum_{l=k+1}^{n} \frac{(l-1)!}{l!} N_k^{(-n)}(-n) x^{l-k}
\]

\[
= n! \sum_{k=0}^{n-1} \sum_{m=1}^{n-k} \frac{(k+m-1)}{(k+m)!} N_k^{(-n)}(-n) x^m
\]

\[
= n! \sum_{m=1}^{n} \left( \sum_{k=0}^{n-m} \frac{(k+m-1)}{(k+m)!} N_k^{(-n)}(-n) \right) x^m.
\]

From (1.16) and (2.47), we have

\[
S_2(n, m) = n! \sum_{k=0}^{n-m} \frac{(n-1)}{(k+m)!} N_k^{(-n)}(-n),
\]

where \(1 \leq m \leq n\).

Therefore, by Lemma (2.11) and (2.48), we obtain the following theorem.

**Theorem 2.12.** For \(m, n \in \mathbb{N}\) with \(m \leq n\), we have

\[
\left( \frac{n-1}{m-1} \right) N_{n-m}^{(-n)} = n! \sum_{k=0}^{n-m} \frac{(k+m-1)}{(k+m)!} N_k^{(-n)}(-n).
\]

It is well known that

\[
\left( \frac{t}{\log(1 + t)} \right)^n (1 + t)^{-1} = \sum_{k=0}^{\infty} B_k^{(k-n+1)}(x) \frac{t^k}{k!}, \text{ (see [17]).}
\]

(2.49)

Thus, by (2.44) and (2.49), we get

\[
\sum_{k=0}^{\infty} B_k^{(k-n+1)}(x) \frac{t^k}{k!} = \left( \frac{t}{\log(1 + t)} \right)^n (1 + t)^{-1} = \sum_{k=0}^{\infty} N_k^{(-n)}(x-1) \frac{t^k}{k!}.
\]

(2.50)

By comparing the coefficients on the both sides of (2.50), we get

\[
B_k^{(k-n+1)}(x) = N_k^{(-n)}(x-1).
\]

(2.51)

Therefore, by (2.51), we obtain the following corollary.

**Corollary 2.13.** For \(m, n \in \mathbb{N}\) with \(m \leq n\), we have

\[
\left( \frac{n-1}{m-1} \right) B_{n-m}^{(-m+1)}(1) = n! \sum_{k=0}^{n-m} \frac{(k+m-1)}{(k+m)!} B_k^{(k-n+1)}(-n + 1).
\]
Let us consider the following associated sequence:

\[ S_n(x) \sim (1, t(1 + t)^n), \ a \neq 0. \] (2.52)

Then, by (1.12) and (2.52), we get

\[
S_n(x) = x \left( \frac{t}{t(1 + t)^n} \right)^n x^{-1} x^n = x(1 + t)^{-an} x^{n-1}
\]

\[
= x \sum_{l=0}^{n-1} \binom{-an}{n-l} t^l x^{n-1} = \sum_{l=0}^{n-1} \left( \frac{-an}{n-l} \right) (n-1)_l x^{n-l}
\] (2.53)

\[
= \sum_{l=1}^{n} \binom{-an}{n-l} (n-1)_{n-l} x^{l}.
\]

For \( n \geq 1 \), from (1.10) and (2.52), we have

\[
\phi_n(x) = x \left( \frac{t(1 + t)^n}{\log(1 + t)} \right)^n x^{-1} S_n(x)
\]

\[
= x \left( \frac{t}{\log(1 + t)} \right)^n (1 + t)^{an} x^{-1} S_n(x)
\] (2.54)

\[
= x \left( \sum_{k=0}^{n-1} \frac{N_k(-n)(an)}{k!} \right) x^{-1} S_n(x).
\]

By (2.53) and (2.54), we get

\[
\phi_n(x) = x \sum_{k=0}^{n-1} \frac{N_k(-n)(an)}{k!} \sum_{l=1}^{n} \frac{(-an)}{n-l} (n-1)_{n-l} t^k x^{l-1}
\]

\[
= x \sum_{k=0}^{n-1} \frac{N_k(-n)(an)}{k!} \sum_{l=k+1}^{n} \binom{-an}{n-l} (n-1)_{n-l} (l-1)_{k} x^{l-1-k}
\]

\[
= (n-1)! \sum_{k=0}^{n-1} \sum_{l=k+1}^{n} \binom{-an}{n-l} \frac{(l-1)!}{(l-1)!} N_k^{(-n)}(an) x^{l-k}
\]

\[
= (n-1)! \sum_{k=0}^{n-1} \sum_{l=1}^{n-k} \binom{-an}{n-k-m} \frac{(k+m-1)}{(k+m-1)!} N_k^{(-n)}(an) x^{m}
\]

\[
= (n-1)! \sum_{k=0}^{n-1} \sum_{m=1}^{n-k} \binom{-an}{n-k-m} \frac{(k+m-1)}{(k+m-1)!} B_k^{(k-n+1)}(an+1) x^{m}
\]

Therefore, by (1.10) and (2.55), we obtain the following theorem.

**Theorem 2.14.** For \( m, n \in \mathbb{N} \) with \( m \leq n \), we have

\[
\binom{n-1}{m-1} B_{n-m}^{(-m+1)}(1) = (n-1)! \sum_{k=0}^{n-m} \frac{(-an)}{n-k-m} \frac{(k+m-1)}{(k+m-1)!} B_k^{(k-n+1)}(an+1).
\]
Remarks (I). For \( n \geq 1 \), we have

\[
x^n = x \left( \log(1 + t) \right)^n x^{-1} \phi_n(x)
\]

\[
= \sum_{m=1}^{n} \left\{ \sum_{k=0}^{n-m} \binom{k + m - 1}{k} S_2(n, k + m) N_k^{(n)} \right\} x^m.
\]  

(2.56)

By comparing the coefficients on the both sides of (2.56), we get

\[
\sum_{k=0}^{n-m} \binom{k + m - 1}{k} S_2(n, k + m) N_k^{(n)} = \delta_{m,n},
\]

(2.57)

where \( 1 \leq m \leq n \).

(II). For \( n \geq 1 \), we have

\[
L_n(-x) = x \left( \log(1 + t) \right)^n x^{-1} \phi_n(x)
\]

\[
= \sum_{m=1}^{n} \left\{ \sum_{k=0}^{n-m} \binom{k + m - 1}{k} S_2(n, k + m) N_k^{(n)} \right\} x^m.
\]  

(2.58)

By (1.17) and (2.58), we get

\[
\binom{n-1}{m-1} \frac{n!}{m!} = \sum_{k=0}^{n-m} \binom{k + m - 1}{k} S_2(n, k + m) N_k^{(n)}(n),
\]

(2.59)

where \( 1 \leq m \leq n \).

As is known, the Laguerre polynomials of order \( \alpha \) are given by Sheffer sequences to be

\[
L_n^{(\alpha)}(x) \sim \left( 1 - t \right)^{-\alpha - 1}, \frac{t}{t - 1}.
\]

(2.60)

Thus, by the definition of Sheffer sequence, we get

\[
\left\langle (1 + t)^{-\alpha - 1} \left( \frac{t}{t + 1} \right) \right\rangle^{k} L_n^{(\alpha)}(-x) = n! \delta_{n,k} (n, k \geq 0).
\]

(2.61)

By (2.61), we easily see that

\[
L_n(-x) = (1 + t)^{-\alpha - 1} L_n^{(\alpha)}(-x) \sim \left( 1, \frac{t}{1 + t} \right).
\]

(2.62)

From (2.58) and (2.62), we have

\[
(1 + t)^{-\alpha - 1} L_n^{(\alpha)}(-x) = L_n(-x)
\]

\[
= \sum_{m=1}^{n} \left\{ \sum_{k=0}^{n-m} \binom{k + m - 1}{k} S_2(n, k + m) N_k^{(n)} \right\} x^m.
\]  

(2.63)
Thus, by (2.63), we get
\[
L^{(\alpha)}_n(-x) = (1 + t)^{\alpha+1} \sum_{m=1}^{n} \left\{ \sum_{k=0}^{n-m} \binom{k + m - 1}{k} S_2(n, k + m) N_k^{(n)}(n) \right\} x^m
\]
\[
= \sum_{l=0}^{n} \left\{ \sum_{m=1}^{n} \sum_{k=0}^{n-m} \binom{k + m - 1}{k} (\alpha + 1) \binom{m - l}{m - l} (m)_{m-l} S_2(n, k + m) N_k^{(n)}(n) \right\} x^l.
\] (2.64)

It is known that
\[
L^{(\alpha)}_n(-x) = \sum_{l=0}^{n} \binom{n + \alpha}{n - l} \frac{n!}{l!} x^l.
\] (2.65)

By (2.64) and (2.65), we get
\[
\binom{n + \alpha}{n - l} \frac{n!}{l!} = \sum_{m=1}^{n-m} \binom{k + m - 1}{k} (\alpha + 1) \binom{m - l}{m - l} (m)_{m-l} S_2(n, k + m) N_k^{(n)}(n),
\]
where \(0 \leq l \leq n\).

Finally, we consider the following associated sequences:
\[
S_n(x) = \sum_{k=1}^{n} \binom{-an}{n-k} (n-1)_{n-k} x^k \sim (1, t(1 + t)^a), a \neq 0. \tag{2.66}
\]

Thus, by (1.12) and (2.66), we get
\[
S_n(x) = x \binom{\log(1 + t)}{t(1 + t)^a} x^{-1} \phi_n(x)
\]
\[
= x \binom{\log(1 + t)}{t} (1 + t)^{-an} x^{-1} \phi_n(x)
\]
\[
= x \sum_{k=0}^{n-1} N_k^{(n)}(-an) \frac{k!}{t} \sum_{l=0}^{n} S_2(n, l) x^{l-1}
\]
\[
= \sum_{m=1}^{n} \left\{ \sum_{k=0}^{n-m} \binom{k + m - 1}{k} S_2(n, k + m) N_k^{(n)}(-an) \right\} x^m.
\] (2.67)

From (2.66) and (2.67), we have
\[
\binom{-an}{n-m} (n-1)_{n-m} = \sum_{k=0}^{n-m} \binom{k + m - 1}{k} S_2(n, k + m) N_k^{(n)}(-an),
\]
where \(m, n \in \mathbb{N}\) with \(m \leq n\) and \(a \neq 0\).

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