A CHARACTERIZATION
OF SIERPİŃSKI CARPET RATIONAL MAPS

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Abstract. In this paper we prove that a postcritically finite rational map with
non-empty Fatou set is Thurston equivalent to an expanding Thurston map if
and only if its Julia set is homeomorphic to the standard Sierpiński carpet.

1. Introduction. In complex dynamics a central theme is to understand the global
dynamics of the postcritically finite rational maps (see Section 2.2 for its definition).
In the case of postcritically finite polynomials, Douady and Hubbard have intro-
duced the so-called Hubbard trees which capture their dynamical features [8]. But
for general rational maps, as far as we know, the overall understanding has remained
sketchy and unsatisfying (see e.g. [3, 4, 6]).

When ignoring the complex structure, we consider a postcritically finite rational
map as a postcritically finite branched covering of the sphere $S^2$. Such maps are
called Thurston maps. Recently, M. Bonk-D. Meyer [1], D. Meyer [15, 16], Z. Li [13]
et al. studied the dynamics of a kind of Thurston maps, called expanding Thurston
maps (see Definition 2.2), from the aspects of combinatorics, topology, geometry
and ergodic theory.

Thus, if we can establish a relation between expanding Thurston maps and some
class of rational maps (in the dynamical sense), one may, at least in principle,
apply the methods and results used for expanding Thurston maps to the study
of the corresponding rational maps. In the level of topological conjugacy, it was
shown in [1, Proposition 2.3] that a rational map topologically conjugates to an
expanding Thurston map if and only if its Julia set is the Riemann sphere. In
a weaker sense, Thurston introduced an equivalence relation among all Thurston
maps, called Thurston equivalence (see Definition 3.5). Then a natural question is:

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What kind of postcritically finite rational maps with non-empty Fatou sets are Thurston equivalent to expanding Thurston maps?

Our answer to this question is as follows.

**Theorem 1.1.** A postcritically finite rational map with non-empty Fatou set is Thurston equivalent to an expanding Thurston map if and only if its Julia set is homeomorphic to the standard Sierpiński carpet.

To verify this theorem, we will recall some basic definitions and results in Section 2, and prove a series of lemmas about homotopy and isotopy in Section 3 and Section 4. The detailed proof of Theorem 1.1 is left in Section 4.

We will end the introduction with two remarks.

1. There are many examples of postcritically finite rational maps with Julia sets homeomorphic to the standard Sierpiński carpet (see e.g. [20, Appendix] and [22]), and these Julia sets are quasisymmetric rigid [2] and admit Markov partitions [11]. Conjecturally, the components of these rational maps are relatively compact in the space of rational functions up to Möbius conjugation. [17, Question 5.3]

2. In the proof of the main theorem, we use the following trick: we first construct a homotopy \( H : S^2 \times I \to S^2 \) rel. \( P \) between two homeomorphisms, and then modify it to an isotopy relative to \( P \), where \( P \) is a finite subset of \( S^2 \). We emphasize that this result is generally false; see Section 5 for a counterexample and detailed discussion.

2. **Preliminaries.** Here we present some notations and elementary background that will be used in this paper. More details can be found in [1, 5, 19].

2.1. **Notations.** The 2-sphere is denoted by \( S^2 \), the Riemann sphere by \( \overline{\mathbb{C}} \) and the open/closed unit disk by \( \mathbb{D}/\partial \mathbb{D} \). The closure and interior of a subset \( K \subset S^2 \) is denoted by \( \text{cl}(K) \) and \( \text{int}(K) \) respectively. The family of all connected components of \( K \) is denoted by \( \text{Comp}(K) \). The spherical metric on \( S^2 \) is \( \sigma = \frac{2|dz|}{1+|z|^2} \). The set of critical points of a branched covering \( F \) is denoted by \( \text{crit}(F) \) and the set of postcritical points by \( \text{post}(F) \). The Julia set of a rational map \( f \) will be denoted by \( J_f \); the Fatou set is \( F_f \).

2.2. **Expanding Thurston maps.** Let \( F : X \to Y \) be a continuous map between two domains \( X, Y \subset S^2 \). The map \( F \) is called a branched covering if for each point \( q \in Y \), there exist an open neighborhood \( V \) with the following property: for some index set \( I \neq \emptyset \) we can write \( F^{-1}(V) \) as a disjoint union

\[
F^{-1}(V) = \bigcup_{i \in I} U_i
\]

for open sets \( U_i \subset X \) such that \( U_i \) contains precisely one point \( p_i \in F^{-1}(q) \). Moreover, we require that for each \( i \in I \) the exist an integer \( d_i \geq 1 \) and orientation-preserving homeomorphisms \( \phi_i : U_i \to \mathbb{D} \) and \( \psi_i : V \to \mathbb{D} \) with \( \phi_i(p) = \psi_i(q) = 0 \) such that \( \psi_i \circ F \circ \phi_i^{-1}(z) = z^{d_i} \) for all \( z \in \mathbb{D} \).

The integer \( d_i \) is uniquely determined by \( p = p_i \) and called the local degree or multiplicity of \( F \) at \( p \), denoted by \( \text{deg}_F(p) \). A point \( c \) with \( \text{deg}_F(c) \geq 2 \) is called a critical point of \( F \), and its image \( F(c) \) is called a critical value. Furthermore, if \( X \) is connected and \( F \) is proper, i.e., \( F^{-1}(K) \) is compact for any compact set \( K \subset Y \), then the cardinality of \( F^{-1}(q) \) counting with multiplicity is finite and constant for all \( q \in Y \). This number is called the (global) degree of \( F \), denoted by \( \text{deg}(F) \). A branched covering without critical points is called a covering.
Let $F : S^2 \to S^2$ be a branched covering. The set of critical points of $F$ is denoted by $\text{crit}(F)$, and the postcritical set $\text{post}(F)$ is defined as

$$\text{post}(F) := \bigcup_{n \geq 1} F^n(\text{crit}(F)).$$

The map $F$ is called postcritically finite if $\# \text{post}(F) < \infty$. We mention the following fact for future reference, which is a consequence of the definitions and of the Riemann-Hurwitz formula.

**Lemma 2.1.** Let $F : S^2 \to S^2$ be a branched covering and $V \subset S^2$ be a Jordan domain that contains at most a single critical value $p$. Then every component $U$ of $F^{-1}(V)$ is a Jordan domain and $F : U \to V$ is a branched covering of degree $\delta$ with only one possible critical point $c$, where $\delta = \deg_F(c)$ and $c$ is the unique preimage of $p$ in $U$.

**Definition 2.2.** A Thurston map is an orientation-preserving, postcritically finite, branched covering of $S^2$. We fix a base metric $\rho$ on $S^2$ that induces the standard topology on $S^2$. Consider a Jordan curve $C \supset \text{post}(f)$. The Thurston map $F$ is called expanding if

$$\text{mesh } F^{-n}(C) \to 0 \text{ as } n \to \infty,$$

where $\text{mesh } F^{-n}(C)$ denotes the maximal diameter of a component of $S^2 \setminus F^{-n}(C)$.

It was shown in [1] that the expansionary property is independent of the choice of the Jordan curve $C$ ([1 Lemma 6.1]), and the base metric $\rho$ on $S^2$ as long as it induces the standard topology on $S^2$ ([1 Proposition 6.3]).

**2.3. Partitions of $S^2$ induced by Thurston maps.** Let $F : S^2 \to S^2$ be a Thurston map and fix a Jordan curve $C \subset S^2$ with $\text{post}(F) \subset C$. The closure of one of the two components of $S^2 \setminus C$ is called a 0-tile (relative to $(F,C)$). Similarly, we call the closure of one component of $S^2 \setminus F^{-n}(C)$ an $n$-tile (for any $n \geq 0$). The set of all $n$-tiles is denoted by $X^n(C)$. For any $n$-tile $X$, the set $F^n(X) = X^0$ is a 0-tile and

$$F^n : X \to X^0 \quad \text{is a homeomorphism,}$$

see [1 Proposition 5.17]. This means in particular that each $n$-tile is a closed Jordan domain. The definition of “expansion” implies that $n$-tiles become arbitrarily small. Clearly, for each $n \geq 0$, all $n$-tiles relative to $(F,C)$ form a partition of $S^2$.

**2.4. Postcritically finite Sierpiński carpet rational map.** Let $f$ be a postcritically finite rational map. It was known that $\mathcal{J}_f$ is connected and locally connected ([18] [19]). The connectedness implies that each Fatou component is simply connected; and the local connectedness implies the following lemma (see [24] Theorem VI.4.4)).

**Lemma 2.3.** For any $\epsilon > 0$, there are finitely many Fatou components $U$ with $\text{diam}_r(U) \geq \epsilon$.

The following lemma is a well-known result. Refer to [14] Section 12.1] for the proof.

**Lemma 2.4** (Shrinking Lemma). Let $f$ be a postcritically finite rational map with $\deg(f) \geq 2$, and $U$ be any domain in $\mathbb{C}$ such that for any $n \geq 1$ and any component $U_n$ of $f^{-n}(U)$, the number of the critical points (counting with multiplicity) of $f^n : U_n \to U$ is bounded above by a constant. Then for any compact set $B \subset U$, the maximum of the spherical diameters of all components of $f^{-n}(B)$ converges to 0 as $n \to \infty$. 


By Boettcher’s theorem, there exists a family of tuples \( \{(U, \eta_U)\}_{U \in \text{Comp}(\mathcal{F}_f)} \) with \( \eta_U : \mathbb{D} \to U \) conformal maps such that \( f \circ \eta_U (z) = \eta_U (f(z)) \), \( d_U := \deg(f|_U) \), for all \( z \in \mathbb{D} \). Since the Julia set \( \mathcal{J}_f \) is locally connected, it follows from Carathéodory’s theorem that the conformal map \( \eta_U \) extends to a continuous and surjective map \( \eta_U : \overline{\mathbb{D}} \to \overline{U} \). An internal ray of \( U \) is the image \( \eta_U ([0, 1] \theta) \) for unit number \( \theta \in \partial \mathbb{D} \).

Note that internal rays are mapped to internal rays under \( f \).

A set \( S \subset \mathbb{C} \) is called a (Sierpiński) carpet if it is homeomorphic to the standard Sierpiński carpet. By Whyburn’s characterization [24], a set \( S \subset \mathbb{C} \) is a carpet if and only if it can be written as \( S = \bigcup_{n \geq 1} D_n \), where all \( D_n \) are Jordan domains with pairwise disjoint closures, such that the interior of \( S \) is empty and the spherical diameters \( \text{diam}_S (D_n) \to 0 \) as \( n \to \infty \).

We say a postcritically finite rational map is a postcritically finite carpet rational map if its Julia set is a carpet. This means that each Fatou component is a Jordan domain and distinct components of the Fatou set have disjoint closures. Furthermore, the boundary of a component of the Fatou set cannot contain postcritical points.

3. Homotopy, isotopy and Thurston equivalent.

**Definition 3.1** (Relative homotopy and isotopy). Let \( X, Y \) be topological spaces and \( A \) be a subset of \( X \) (maybe empty). Let \( \phi, \psi \) be continuous maps from \( X \) to \( Y \). We say that \( \phi \) and \( \psi \) are homotopic rel. \( A \) if there exists a continuous map \( H : X \times [0, 1] \to Y \), called a homotopy rel. \( A \), such that

\[
H(x, 0) = \phi(x), \quad H(x, 1) = \psi(x) \quad \forall x \in X, \quad H(x, t) = \phi(x) \quad \forall x \in A, \quad \forall t \in [0, 1].
\]

If the map \( H|_{X \times t} : X \to Y \) is a homeomorphism for each \( t \in [0, 1] \), we call \( H \) an isotopy rel. \( A \).

Let \( H : X \times [0, 1] \to Y \) be a homotopy. For simplicity, we usually denote the map \( H|_{X \times t} : X \to Y \) by \( H_t : X \to Y \), where \( t \in [0, 1] \).

**Lemma 3.2.** (1) Let \( I = [0, 1] \) and \( \phi : I \to I \) be a continuous map with \( \phi(0) = 0 \) and \( \phi(1) = 1 \). Then there is a homotopy \( H : I \times I \to I \) rel. \( \{0, 1\} \) from \( \text{id}_I \) to \( \phi \) such that \( H(\text{int}(I), t) \subseteq \text{int}(I) \forall t \in [0, 1] \).

(2) Let \( A \) be a subset of the unit circle and \( \phi : \partial \mathbb{D} \to \partial \mathbb{D} \) be a continuous map. Let \( h : \partial \mathbb{D} \times I \to \partial \mathbb{D} \) be a homotopy rel. \( A \) such that \( h_0 = \text{id}_{\partial \mathbb{D}} \) and \( h_1 = \phi|_{\partial \mathbb{D}} \). Then \( h \) can be extended to a homotopy \( H : \overline{\mathbb{D}} \times I \to \overline{\mathbb{D}} \) rel. \( A \) from \( \text{id}_{\overline{\mathbb{D}}} \) to \( \phi \) such that \( H|_{\partial \mathbb{D} \times I} = h \) and \( H(\mathbb{D}, t) \subseteq \mathbb{D}, \forall t \in [0, 1] \).

**Proof.** (1) The map \( H : I \times I \to I \) defined by

\[
H(s, t) = ts + (1 - t)\phi(s).
\]

is as required.

(2) We obtain the desired homotopy by a small change of the “Alexander trick”. Define the extended homotopy \( H : \overline{\mathbb{D}} \times I \to \overline{\mathbb{D}} \) by

\[
H(z, t) = \begin{cases} 
  t \cdot \phi(z/t), & 0 \leq |z| < t; \\
  t \cdot \phi(z/|z|) \cdot (1 - \frac{|z| - t}{1 - t}) + h(z/|z|, t) \cdot \frac{|z| - t}{1 - t}, & t \leq |z| \leq 1.
\end{cases}
\]

One can check that \( H \) satisfies the required conditions. \( \square \)
Remark 1. Lemma 3.2(2) can be seen as a generalization of Alexander Lemma: if $\phi : \overline{D} \to \overline{D}$ is a homeomorphism with $\phi|_{\partial D \cup \{0\}}$ equal to identity, then $\phi$ is isotopic to $\text{id}_\overline{D}$ rel. $\partial D \cup \{0\}$.

A classical result about the modification of homotopy to isotopy in surfaces is due to D. B. A. Epstein [9, Theorem 6.4]; see also [10, Theorem 1.12]. Throughout this paper, a surface $S$ refers to a topological space obtained from an orientable closed surface by removing $b \geq 0$ open disks and $n \geq 0$ points with disjoint closures, and its boundary $\partial S$ is defined as the union of the circles which bound the disks removed.

**Theorem 3.3** (Epstein). Let $S$ be a surface and $h$ be an orientation preserving homeomorphism of $S$ homotopic to $\text{id}_S$ rel. $\partial S$. Then they are isotopic rel. $\partial S$.

One may ask when marking a finite set $P$ in the interior of surface $S$ (defined as in the theorem above), are two orientation preserving homeomorphisms of $S$ that are homotopic rel. $\partial S \cup P$ still isotopic rel. $\partial S \cup P'$? The answer is NO in general. Since the original homotopies may cross the marked points. Here is a simple counterexample:

Choose $S := \overline{D}$ the closed unit disk and marked set $P := \{z_1, z_2\} \subseteq \overline{D}$. Let $h$ be a Dehn twist on $\overline{D}$ along a Jordan curve surrounding $P$. It is known that $h$ is not isotopic to $\text{id}_\overline{D}$ rel. $\partial \overline{D} \cup P$. But $H(z, t) = tz + (1 - t)h(z), t \in [0, 1], z \in \overline{D}$ is a homotopy rel. $\partial \overline{D} \cup P$ between $h$ and $\text{id}_\overline{D}$. A similar counterexample can be given on $S^2$ with at least four marked points.

So, in general, two homeomorphisms of an orientable surface homotopic relative to marked points are not necessarily isotopic relative to the marked points. However, if the homotopy is well chosen, the conclusion holds. We leave the proof of the following theorem to the appendix.

**Theorem 3.4**. Let $P$ be a finite set in the interior of a surface $S$. Let $H : S \times I \to S$ rel. $P \cup \partial S$ be a homeomorphism such that $H_0 = \text{id}_S$ and $h := H_1$ is an orientation preserving homeomorphism. For each $p \in P$, set $K(p, H) := \text{cl}((\cup_{t \in [0, 1]} H_t^{-1}(p))$. If each $K(p, H)$ is contained in the interior of a closed topological disk $D_p$ with $D_p \subseteq \text{int}(S)$ and $D_p \cap D_{p'} = \emptyset$ for any $p \neq p' \in P$, then $h$ is isotopic to $\text{id}_S$ rel. $P \cup \partial S$.

At the end of this section, we introduce the concept of Thurston equivalence.

**Definition 3.5** (Thurston equivalent). Two Thurston maps $F, G$ on $S^2$ are said to be Thurston equivalent if there exist homeomorphisms $\psi, \phi : S^2 \to S^2$ that are isotopic rel. post$(F)$ and satisfy $G \circ \psi = \phi \circ F$, that is, the following diagram commutes:

\[
\begin{array}{ccc}
S^2 & \xrightarrow{F} & S^2 \\
\downarrow \psi & & \downarrow \phi \\
S^2 & \xrightarrow{G} & S^2
\end{array}
\]

4. A characterization of carpet rational maps. The objective of this section is to prove Theorem 1.1. We will first summarize the idea (Section 4.1) and then give the detailed proof (Sections 4.2, 4.3 and 4.4).
4.1. The outline of the proof. For the necessity, let \( f \) be a postcritically finite rational map, \( F \) an expanding Thurston map and \( \phi_0, \phi_1 \) two homeomorphisms on \( S^2 \) such that \( \phi_0 \circ f = F \circ \phi_1 \) and \( \phi_0 \) is isotopic to \( \phi_1 \) rel. post(\( f \)). By repeatedly using the isotopy lifting theorem, we obtain a sequence of homeomorphisms \( \{\phi_n\} \) such that \( \phi_n \circ f = F \circ \phi_{n+1} \) and \( \phi_n \) is isotopic to \( \phi_{n+1} \) rel. \( f^{-n} \) (post(\( f \))) for all \( n \geq 0 \). This sequence of homeomorphisms converges to a semi-conjugacy \( h \) from \( f: \mathbb{C} \to \mathbb{C} \) to \( F: S^2 \to S^2 \) by the expansionary property of \( F \). With the properties of this semi-conjugacy \( h \), we can prove that the Julia set of \( f \) is a Sierpiński carpet.

The sufficiency proceeds as follows. Suppose that \( f \) has Sierpiński carpet Julia set. By collapsing the closure of each Fatou component to a point, we obtain the quotient map \( \pi: \mathbb{C} \to S^2 \), by which the rational map \( f \) descends to an expanding Thurston map. This yields a semi-conjugacy \( \pi \) from the rational map \( f \) to an expanding Thurston map \( F \). We can carefully choose a homeomorphism \( \psi \) in the homotopy class of \( \pi \) rel. post(\( f \)) such that \( \psi \) has a lift \( \phi \) along \( f \) and \( F \), i.e., \( F \circ \psi = \phi \circ f \) on \( \mathbb{C} \), and the homeomorphism \( \phi \) is homotopic to \( \pi \) rel. post(\( f \)). We then get a homotopy rel. post(\( f \)) between \( \psi \) and \( \phi \) by concatenating the homotopy between \( \psi, \pi \) and that between \( \pi, \phi \). This homotopy turn out to satisfy the properties of Theorem 3.4. It follows that \( \phi \) and \( \psi \) are isotopic rel. post(\( f \)).

4.2. Regulated curves for postcritically finite carpet rational maps. Let \( f \) be a postcritically finite rational map with Sierpiński carpet Julia set.

Definition 4.1. An arc or a Jordan curve in \( \mathbb{C} \) is called regulated (with respect to \( f \)) if its intersection with the closure of each Fatou component is either empty or a connected set, i.e., one point or one arc.

The objective of this part is to construct a regulated Jordan curve passing though given finitely many points (Lemma 4.3 below). It will be used in the proof of Lemma 4.4 and Theorem 1.1. The base of the construction is the following Moore’s Theorem.

Lemma 4.2 (Moore [21]). Let \( \equiv \) be an equivalence relation on a 2-sphere \( S^2 \) satisfying

1. it is closed as a subset of \( S^2 \times S^2 \) equipped with the product topology;
2. it is not trivial, meaning that there are at least two distinct equivalence classes;
3. each equivalence class is a compact connected set;
4. the complementary component of each equivalence class is connected.

Then the quotient space \( S^2/\equiv \) is homeomorphic to \( S^2 \).

Define an equivalence relation on \( \hat{\mathbb{C}} \) by \( z \sim w \) if \( z = w \) or if \( z, w \) belong to the closure of a common Fatou component. Since \( J_f \) is a Sierpiński carpet it follows that distinct Fatou components have disjoint closures. So \( \sim \) is indeed an equivalence relation.

We claim that the equivalence relation \( \sim \) satisfies the 4 properties in Lemma 4.2. Clearly, the properties (2), (3), (4) holds. To check the property (1), it suffices to show that given two convergent sequences \( (z_n)_{n \geq 1} \) and \( (w_n)_{n \geq 1} \) in \( \hat{\mathbb{C}} \) with \( z_n \sim w_n \) for all \( n \geq 1 \) it follows that \( \lim z_n \sim \lim w_n \). This is clear in the case when for sufficiently large \( n \) the points \( z_n \) and \( w_n \) are contained in some fixed equivalence class, since each equivalence class is compact. Otherwise, we may assume that for distinct \( n, m \geq 1 \) the points \( z_n \) and \( z_m \) are contained in distinct equivalence classes. In this case, the diameter of the equivalence class containing \( z_n \) becomes arbitrarily small as \( n \to \infty \) by Lemma 2.3 then we have \( \lim z_n = \lim w_n \). Thus \( \sim \) is closed.
Using Lemma 4.2 the quotient space 

$$\mathbb{C}/\sim = \{[z] : [z] \text{ is the } \sim\text{-equivalence class of } z, z \in \mathbb{C}\}$$

with the quotient topology is homeomorphic to $S^2$. We identify $\mathbb{C}/\sim$ with $S^2$ so that the quotient map can be written as the continuous map

$$\pi : \mathbb{C} \rightarrow S^2.$$ (3)

**Lemma 4.3.** Let $f$ be a postcritically finite rational map with Sierpiński carpet Julia set.

1. Let $\tilde{P} := \{\tilde{p}_1, \cdots, \tilde{p}_N\}$ be a finite set in $\mathbb{C}$ with $\tilde{P} \cap \partial U = \emptyset$ and the cardinality 

$$\#(\tilde{P} \cap U) \leq 1 \text{ for each Fatou component } U.$$ Then there exists a regulated Jordan curve $\tilde{C} \subset \mathbb{C}$ which contains the set $\tilde{P}$.

2. If $\tilde{C}$ is a regulated Jordan curve with $\tilde{V}_0, \tilde{V}_1$ the two components of $\mathbb{C} \setminus \tilde{C}$, then $C := \pi(\tilde{C})$ is a Jordan curve, and $\pi(\operatorname{cl}(\tilde{V}_0)), \pi(\operatorname{cl}(\tilde{V}_1))$ are the closures of the components of $S^2 \setminus C$.

**Proof.** (1) Recall that $\pi : \mathbb{C} \rightarrow S^2$ is the quotient map of $\sim$ given in (3). We set

$$E = \{\pi(U) \mid U : \text{component of } \mathcal{F}_f\} \text{ and } P = \{p_k := \pi(\tilde{p}_k) \mid 1 \leq k \leq N\}.$$ We first claim that there exist closed disk neighborhoods $D_k$ for each point $p_k \in P$ such that they are pairwise disjoint and their boundaries avoid $E$. To see this, notice that $\{S_{r_k} := \{x \in S^2 \mid \sigma(x, p_k) = r\}\}_{r>0}$ is an uncountable family of pairwise disjoint sets and $E$ is countable. So we choose sufficiently small $r_k$ for each $k \in \{1, \ldots, N\}$ such that $S_{r_k} \cap E = \emptyset$ and $S_{r_i} \cap S_{r_j} = \emptyset$ ($i \neq j$). The neighborhoods $D_k$ defined as $D_k := \{x \in S^2 \mid \sigma(x, p_k) \leq r_k\}$ satisfy the requirement of the claim.

We then claim that there are pairwise disjoint (closed) arcs $\gamma_1, \ldots, \gamma_N \subset S^2$ avoiding $E$ such that the interior of $\gamma_k$ (the open arc without the endpoints) is disjoint with $\bigcup_{1 \leq j \leq N} D_j$ and $\gamma_k$ joins $D_k, D_{k+1}$ for each $k \in \{1, \ldots, N\}$ (with $D_{N+1} := D_1$). Indeed, it is easy to find a sequence of pairwise disjoint arcs $e_1, \ldots, e_N \subset S^2$ such that each $e_k$ joins $D_k$ and $D_{k+1}$, and the interior of $e_k$ is disjoint with $\bigcup_{1 \leq j \leq N} D_j$. Moreover, for each $k$ we choose a topological quadrilateral $Q_k = Q_k(a_k, b_k, c_k, d_k)$, i.e., the closed topological disk with $a_k, b_k, c_k, d_k$ in its boundary in the counterclockwise direction, such that

- $Q_1, \ldots, Q_N$ are pairwise disjoint and their interiors are contained in $S^2 \setminus \bigcup_{1 \leq j \leq N} D_j$;
- the interior of each $e_k$ is contained in the interior of $Q_k$;
- the four edges of $Q_k$, i.e., the closures of the components of $\partial Q_k \setminus \{a_k, b_k, c_k, d_k\}$, satisfy that $e(a_k, b_k) \subset \partial D_k, e(c_k, d_k) \subset \partial D_{k+1},$ and $e(a_k, d_k), e(b_k, c_k)$ join $D_k, D_{k+1}$.

We denote by $R = R(1-i, 1+i, i, -i)$ the rectangle with vertices $1-i, 1+i, i, -i$. For each $k \in \{1, \ldots, N\}$, let $h_k : Q_k \rightarrow R$ be a homeomorphism with $h_k(a_k) = 1-i, h(b_k) = 1+i, h(c_k) = i, h(d_k) = -i$ and $h_k(e_k) = [0, 1]$, the unit interval. Since $R$ contains an uncountable family of pairwise disjoint horizontal intervals of length 1 and $h_k(Q_k \cap E)$ is countable, we may choose a horizontal interval $l_k \subset R$ of length 1 avoiding $h_k(E)$. The preimage $h_k^{-1}(l_k)$ is then an arc joining $D_k$ and $D_{k+1}$ and avoiding $E$, which is denoted by $\gamma_k$. The arcs $\gamma_1, \ldots, \gamma_N$ satisfy the requirements in the claim. For each $k \in \{1, \ldots, N\}$, we denote $u_k$ and $v_k$ the intersection points of $D_k$ with $\gamma_{k-1}$ and $\gamma_k$ respectively.
By the first claim, the compact sets \( \tilde{S}_k := \pi^{-1}(\partial D_k), k = 1, \ldots, N, \) are pairwise disjoint and contained in \( \overline{C} \setminus \cup_{U \in \text{Comp}(f)} \text{cl}(U) \). It follows that each \( \pi|_{\tilde{S}_k} : \tilde{S}_k \to \partial D_k \) is a homeomorphism, and hence \( \tilde{S}_k \) is a Jordan curve. Note that \( \pi^{-1}(\text{int}(D_k)) \) is the component of \( S^2 \setminus \tilde{S}_k \) containing \( \tilde{p}_k \), then the sets \( \tilde{D}_k := \pi^{-1}(D_k), k = 1, \ldots, N, \) are pairwise disjoint closed disks with \( \tilde{P} \cap \tilde{D}_k = \{ p_k \} \) and \( \partial \tilde{D}_k = \tilde{S}_k \). By the second claim and a similar argument, we have that \( \tilde{\gamma}_k := \pi^{-1}(\gamma_k), k = 1, \ldots, N, \) are pairwise disjoint arcs in \( \overline{C} \setminus \cup_{U \in \text{Comp}(f)} \text{cl}(U) \), satisfying that their interiors are contained in \( S^2 \setminus \cup_{1 \leq j \leq N} \partial D_j \), and the intersection points of \( \tilde{\gamma}_k \) with \( \tilde{D}_k \) and \( \tilde{D}_{k+1} \) are \( \tilde{v}_k := \pi^{-1}(u_k) \) and \( \tilde{u}_{k+1} := \pi^{-1}(u_{k+1}) \) respectively.

As all \( \tilde{\gamma}_k \) are contained in \( \overline{C} \setminus \cup_{U \in \mathcal{F}} \text{cl}(U) \), they are then regulated. Therefore, to obtain a regulated Jordan curve containing \( \tilde{P} \), it is enough to select a regulated arc \( \tilde{\alpha}_k \) in each \( \tilde{D}_k \) that passes through the point \( \tilde{p}_k \) and joins the points \( \tilde{u}_k, \tilde{v}_k \in \partial \tilde{D}_k \).

This can be easily done if one notices that each \( J_k := \tilde{D}_k \cap J_f \) is a Sierpiński carpet and it is mapped onto the standard carpet by a self-homeomorphism of \( S^2 \). Finally, the set \( \tilde{C} := (\cup_{k=1}^N \tilde{\gamma}_k) \cup (\cup_{k=1}^N \tilde{\alpha}_k) \) is a regulated Jordan curve containing the set \( \tilde{P} \).

(2) By the definition of the regulated Jordan curves, the fiber \( (\pi|_{\tilde{C}})^{-1}(y) \) for each \( y \in C \) is one point or one arc on \( \tilde{C} \). It follows from [11, Lemma 13.30] that \( C \) is a Jordan curve.

Let \( x \neq y \) belong to a component of \( S^2 \setminus \tilde{C} \). Then \( \pi^{-1}(x) \) and \( \pi^{-1}(y) \) are contained in a common component of \( \overline{C} \setminus \tilde{C} \). Otherwise, we pick an arc \( \gamma \) in \( S^2 \setminus C \) joining \( x \) and \( y \). By [3, Lemma 3.1], the set \( \pi^{-1}(\gamma) \) is a continuum containing \( \pi^{-1}(x) \) and \( \pi^{-1}(y) \), and hence intersects \( \tilde{C} \). Consequently, we get \( \gamma \cap C \neq \emptyset \), a contradiction. By this fact, we can label the two components of \( S^2 \setminus C \) by \( V_0 \) and \( V_1 \) such that \( \pi^{-1}(V_0) \subset \tilde{V}_0 \) and \( \pi^{-1}(V_1) \subset \tilde{V}_1 \). It implies that \( \pi(\text{cl}(V_i)) \subset \text{cl}(\tilde{V}_i) \) for \( i = 0, 1 \). Since \( \pi \) is surjective, we have \( \pi(\text{cl}(V_0)) = \text{cl}(\tilde{V}_0) \) and \( \pi(\text{cl}(V_1)) = \text{cl}(\tilde{V}_1) \). \( \square \)

### 4.3. The expanding quotient

In this part, we will show that any postcritically finite carpet rational map is semi-conjugated to an expanding Thurston map (Proposition 1). This result was obtained in [11, Theorem 5.1]. Using a regulated Jordan curve constructed in Lemma 4.3(1) with \( \tilde{P} = \text{post}(f) \), we give an alternative approach for the paper to be self-contained.

**Proposition 1.** Let \( f \) be any postcritically finite carpet rational map, and \( \pi : \overline{C} \to S^2 \) the quotient map given in (3). Then there exists an expanding Thurston map \( F \) such that \( \pi \circ f = F \circ \pi \).

Note that the equivalence relation \( \sim \) defined in the last subsection is \( f \)-invariant, i.e., \( z \sim w \Rightarrow f(z) \sim f(w) \), then the map \( f \) descends to a map \( F \) defined by \( F(x) = \pi \circ f \circ \pi^{-1}(x) \) for all \( x \in S^2 \), that is, the commutative diagram holds:

\[
\begin{array}{ccc}
\overline{C} & \xrightarrow{f} & \overline{C} \\
\pi \downarrow & & \pi \downarrow \\
S^2 & \xrightarrow{F} & S^2
\end{array}
\]  

Furthermore, the relation \( \sim \) is also strongly invariant, i.e., the image of any equivalence class is an equivalence class, i.e., \( f([z]) = [f(z)] \) for any \( z \in \overline{C} \). The purpose of this condition is explained by the following lemma, see [11 Corollary 13.3] for a proof.
Lemma 4.4. If $F$ is obtained as a quotient of a Thurston map $f$ as in (1) by a strongly invariant equivalence relation, then $F$ is a Thurston map. Moreover, $\text{post}(F) = \pi(\text{post}(f))$ and $\deg(F) = \deg(f)$.

Applying Lemma 4.3(1) to the case of $\tilde{P} := \text{post}(f)$, we obtain a regulated Jordan curve passing through $\text{post}(f)$. Fix this curve and denote it by $C_f$. By Lemma 4.3(2), the set $C_F := \pi(C_f)$ is a Jordan curve in $S^2$ containing $\text{post}(F)$. We denote by $X^n(C_f)$ and $X^n(C_F)$ the sets of $n$-tiles relative to $(f, C_f)$ and $(F, C_F)$ respectively. The following lemma implies a correspondence between them.

Lemma 4.5. For any $n \geq 0$, the map $\Psi_n : X^n(C_f) \to X^n(C_F)$ defined by $\Psi_n(\tilde{X}^n) = \pi(\tilde{X}^n)$ is well-defined and one to one.

Proof. For any $n$-tile $\tilde{X}^n \in X^n(C_f)$, the restriction $f^n : \tilde{X}^n \to \tilde{X}^0 \in X^0(C_f)$ is a homeomorphism by (1). It follows that $\partial \tilde{X}^n$ is a regulated Jordan curve as well. Note that $f^n$ and $F^n$ are branched covering of degree $d^n$, then the cardinalities of $X^n(C_f)$ and $X^n(C_F)$ are both $2d^n$. We label the $n$-tiles relative to $(F, C_F)$ by $X^n_1, \ldots, X^n_{2d^n}$. With the equation $F^n \circ \pi = \pi \circ f^n$, we get $\pi(F^n(\tilde{C}_F)) = F^n(\tilde{C}_F)$.

An argument similar to the one used in the proof of Lemma 4.3(2) shows that, for each $X^n_k \in X^n(C_F)$, the set $\pi^{-1}(\text{int}(X^n_k))$ is contained in a unique $n$-tile in $X^n(C_f)$, denoted by $\tilde{X}^n_k$. This fact gives an one to one correspondence between $X^n(C_f)$ and $X^n(C_F)$ by $\tilde{X}^n_k \mapsto X^n_k = \pi(\tilde{X}^n_k)$. The lemma is proved.

Let $K_0$ be the union of all postcritical Fatou components, i.e., the Fatou components containing the postcritical points of $f$, and set $K_n := f^{-n}(K_0)$ for all $n \geq 0$.

Lemma 4.6. The maximum of the spherical diameters of all $\tilde{X}^n \setminus K_n$ with $\tilde{X}^n \in X^n(C_f)$ converges to 0 as $n \to \infty$.

Proof. We just need to prove that for any family $\mathcal{X} = \{\tilde{X}^n\}_{n \geq 1}$ with $\tilde{X}^n \in X^n(C_f)$, the spherical diameters of $\tilde{X}^n \setminus K_n$ converge to 0 as $n \to \infty$. By choosing a subsequence, we can further assume that $f^n(\tilde{X}^n) = \tilde{X}^0 \in X^0(C_f)$ for all $n \geq 1$. For simplicity, we set post$_f(f) := \text{post}(f) \cap J_f$.

For each point $p \in \text{post}_f(f)$ there exists an open neighborhood $W_p \subset \tilde{C} \setminus \text{cl}(K_0)$ of $p$ such that $p$ is the unique postcritical point in $W_p$ and these $W_p$ are pairwise disjoint. Let $n \geq 1$, $p \in \text{post}_f(f)$ and $W_p$ be any component of $f^{-1}(W_p)$. By Lemma 2.1, the map $f^n : W_p^n \to W_p^n$ has a single possible critical point $p_n$ which is the unique $n$-th preimage of $p$ in $W_p$. Note that the orbit of $p_n$ does not contain periodic critical points (since $p_n \in J_f$), and hence visits each critical point at most once. It follows that $\deg_{f^n}(p_n) \leq \prod_{c \in \text{crit}(f)} \deg_f(c)$. Consequently, all $W_p$ satisfy the known condition of Shrinking Lemma 2.4. For each $p \in \text{post}_f(f)$, let $B_p \subset W_p$ be a compact set with $p \in \text{int}(B_p)$. By Lemma 2.4, the maximum of the spherical diameters of all components of $f^{-n}(B_p)$ converges to 0 as $n \to \infty$.

Let $B_0 := \bigcup_{p \in \text{post}_f(f)} \text{int}(B_p)$ and $B_n := f^{-n}(B_0)$ for $n \geq 1$. For each $n \geq 0$ and $\tilde{X}^n \in \mathcal{X}$, we set $Y^n := \tilde{X}^n \setminus (K_n \cup B_n)$. Clearly, the set $Y^0$ is compact and disjoint with post$_f(f)$. There then exists an open disk neighborhood $V$ of $Y^0$ avoiding post$_f(f)$ since $X^0$ is a topological disk and its boundary contains post$_f(f)$. In this case, all components of $f^{-n}(V)$ avoid the critical points of $f^n$ and thus $V$ satisfies the known condition in Shrinking Lemma 2.4. On the other hand, the fact of $f^n(\tilde{X}^n) = \tilde{X}^0$ implies that $f^n(Y^0) = Y^0$. Then each $Y^n$ is contained in a component of $f^{-n}(Y^0)$. Using Lemma 2.4 again, the spherical diameters of $Y^n$ converge to 0 as $n \to \infty$. \[ \]
Since \( f : \tilde{X}^n \to \tilde{X}^0 \) is a homeomorphism, for each \( p \in \text{post}_f(f) \), there is a unique component of \( f^{-n}(B_p) \) intersecting \( \tilde{X}^n \), denoted by \( B^n_p \). Therefore each \( \tilde{X}^n \setminus \mathcal{K}_n \) is contained in the set \( Y^n \cup \cup_{p \in \text{post}_f(f)} B^n_p \). By the argument above, the spherical diameters of all \( Y^n \) and \( B^n_p \) converge to 0. It follows that

\[
\text{diam}_s(\tilde{X}^n \setminus \mathcal{K}_n) \leq \text{diam}_s(Y^n) + \sum_{p \in \text{post}_f(f)} \text{diam}_s(B^n_p) \to 0, \text{ as } n \to \infty.
\]

The lemma is proved. \( \square \)

**Proof of Proposition 7.** Let \( F \) be the map given in \([4]\). It is by Lemma 4.4 a Thurston map. To show the expansiory of \( F \), we only need to prove that the maximum of the spherical diameters of all \( X^n \in \mathbf{X}^n(C_F) \) converge to 0 as \( n \to 0 \).

Note that \( \pi \) is uniformly continuous and \( \pi(\tilde{X}^n \setminus \mathcal{K}_n) = \pi(\tilde{X}^n) \) for all \( \tilde{X}^n \in \mathbf{X}^n(C_f) \). Then the proposition follows immediately from Lemmas 4.5 and 4.6 \( \square \)

### 4.4. Proof of the main theorem.

**Proof of Theorem 1.1.** The proof follows the outline given in Section 4.1

We first prove the sufficiency. Let \( f \) be a postcritically finite rational map with Sierpiński carpet Julia set. From Section 4.3 we obtain a semi-conjugacy \( \pi \) from the rational map \( f \) to an expanding Thurston map \( F \), and two regulated Jordan curves \( C_f \) and \( C_F \).

We label the points in \( \text{post}(f) \cap C_f \) by \( \bar{x}_1, \ldots, \bar{x}_m, \bar{x}_{m+1} = \bar{x}_1 \) successively in the cyclic order, and denote by \( C_f(\bar{x}_i, \bar{x}_{i+1}) \) the closure of the connected component of \( C_f \setminus \{ \bar{x}_i, \bar{x}_{i+1} \} \) disjoint with \( \text{post}(f) \). Set \( x_i = \pi(\bar{x}_i) \) for all \( i \in \{1, \ldots, m+1\} \) and similarly define \( C_F(x_i, x_{i+1}) \). It is clear that \( C_F(x_i, x_{i+1}) = \pi(C_f(\bar{x}_i, \bar{x}_{i+1})) \).

Moreover, by Lemma 4.5 there is an one to one correspondence between \( \mathbf{X}^n(C_f) \) and \( \mathbf{X}^n(C_F) \), characterized by the map

\[
\mathbf{X}^n(C_f) \ni \tilde{X}^n_k \mapsto X^n_k := \pi(\tilde{X}^n_k) \in \mathbf{X}^n(C_F), \forall \ n \geq 0.
\]

Let \( \psi : C_f \to C_F \) be an orientation preserving homeomorphism such that \( \psi(\bar{x}_i) = x_i \) and \( \psi(C_F(\bar{x}_i, \bar{x}_{i+1})) = C_F(x_i, x_{i+1}) \) for all \( i \in \{1, \ldots, m\} \). There exists then a homotopy \( h^0 : C_f \times I \to C_F \) rel. \( \text{post}(f) \) from \( \psi \) to \( \pi|_{C_f} \) as described in Lemma 3.2 (1). We extend \( \psi \) to an orientation preserving homeomorphism of \( \overline{\mathbb{C}} \), also denoted by \( \psi \), with \( \psi(\tilde{X}^n_k) = X^n_k, k = 0, 1 \). It follows from Lemma 3.2 (2) that the homotopy \( h^0 \) can be extended to a homotopy \( H^0 : \overline{\mathbb{C}} \times I \to S^2 \) rel. \( \text{post}(f) \) from \( \psi \) to \( \pi \). By the property that \( \pi^{-1}(x) \) is either a point in \( J_F \) or the closure of a Fatou component, and the specific construction of the homotopy in \( \square \) in Lemma 3.2 we have that \( (H^0)^{-1}(x_i) = x_i, \forall t \in [0, 1) \) and \( (H^0)^{-1}(x_i) = [\bar{x}_i] \) for each \( x_i \in \text{post}(F) \).

We know that the Riemann sphere \( \overline{\mathbb{C}} \) and the 2-sphere \( S^2 \) admit a partition by the 1-tiles relative to \( (f, C_f) \) and \( (F, C_F) \) respectively, and the numbers of \( \mathbf{X}^1(C_f) \) and \( \mathbf{X}^1(C_F) \) are both \( 2d \). For each \( j \in \{1, \ldots, 2d\} \), we define a map

\[
\phi_j := (F|_{X^j_1})^{-1} \circ \psi \circ f|_{\tilde{X}^j_1} : \tilde{X}^j_1 \to X^j_1.
\]

It is a composition of three homeomorphisms, and hence a homeomorphism.

Globally there is a well-defined homeomorphism \( \phi : \overline{\mathbb{C}} \to S^2 \) with \( \phi(z) := \phi_j(z) \) if \( z \in \tilde{X}^j_1 \) such that on the whole sphere

\[
\psi \circ f = F \circ \phi.
\]
Note that all components of both sets $f^{-1}(\mathcal{C}_f) \setminus f^{-1}(\text{post}(f))$ and $F^{-1}(\mathcal{C}_F) \setminus F^{-1}(\text{post}(F))$ are open Jordan arcs. We denote by $\overline{\Gamma}$ and $\Gamma$ the two families of such arcs. Then the map $\phi$ and $\pi$ induces a one to one correspondence from $\overline{\Gamma}$ to $\Gamma$ by

$$\overline{\Gamma} \ni \overline{\gamma} \mapsto \gamma := \phi(\overline{\gamma}) = \pi(\overline{\gamma}) \in \Gamma.$$  

By Lemma 3.2.(1), there exists a homotopy $h_\gamma : \overline{\gamma} \times I \to \gamma$ rel. $\{\gamma(0), \gamma(1)\}$ from $\pi$ to $\phi$ such that $\gamma(\text{int}(\overline{\gamma}), t) \subseteq \text{int}(\gamma) \ \forall t \in (0, 1]$ for each $\overline{\gamma} \in \overline{\Gamma}$.

Pasting these homotopies $\{h_\gamma\}_{\gamma \in \Gamma}$ together, we obtain a specific homotopy $h^1 : f^{-1}(\mathcal{C}_f) \times I \to F^{-1}(\mathcal{C}_F)$ rel. $f^{-1}(\text{post}(f))$ from $\pi$ to $\phi$. According to Lemma 3.2.(2), $h^1$ can be extended to the interior of any 1-tiles. Finally we have a global homotopy $H^1 : \overline{\mathbb{C}} \times I \to S^2$ rel. $f^{-1}(\text{post}(f))$ from $\pi$ to $\phi$ such that $H^1|_{f^{-1}(\mathcal{C}_f) \times I} = h^1$.

According to the properties of $H^0$ and $H^1$, the homotopy $H$ satisfies that $H^{-1}_t(x_i) = \overline{x}_i, \forall t \in [0, 1]\setminus\{1/2\}$ and $H^{-1}_1(x_i) = [\overline{x}_i]$ for each $x_i \in \text{post}(F)$. Consequently, each $K(\overline{x}_i, H) := \bigcup_{t \in [0, 1]} H^{-1}_t(x_i)$ is contained in the closed disk $[\overline{x}_i]$, and these disks are pairwise disjoint (since $f$ has a Sierpiński carpet Julia set). It means that the homotopy $H$ satisfies the conditions of Theorem 3.4 and hence the homeomorphisms $\psi$ and $\phi$ are isotopic rel. $\text{post}(f)$.

We now turn to the necessity. Let $f$ be a postcritically finite rational map with non-empty Fatou set. By Whyburn’s characterization (see Section 2.4) and Lemma 2.3, in order to show that $J_f$ is a Sierpiński carpet, we just need to prove that the closures of any two distinct Fatou components are disjoint and each Fatou component is a Jordan domain.

Suppose $f$ is Thurston equivalent to an expanding Thurston map $F$ via $h_0, h_1$. Using isotopy lifting theorem (see [1] Proposition 11.3] repeatedly, we obtain a sequence of homeomorphisms $\{h_n\}_{n \geq 0}$ such that $h_n \circ f = F \circ h_{n+1}$ and $h_n$ is isotopic to $h_{n+1}$ rel. $f^{-n}(\text{post}(f))$, i.e., the following diagram commutes.

$$\begin{array}{cccccccc}
\overline{\mathbb{C}} & \longrightarrow & \mathbb{C} & \longrightarrow & \overline{\mathbb{C}} & \longrightarrow & \mathbb{C} & \longrightarrow & \overline{\mathbb{C}} \\
\downarrow \cdots & & \downarrow f & & \downarrow f & & \downarrow f & & \downarrow f \\
S^2 & \longrightarrow & S^2 & \longrightarrow & S^2 & \longrightarrow & S^2 & \longrightarrow & S^2 \\
\big| & & \big| h_{n+1} & & \big| h_n & & \big| h_2 & & \big| h_1 \\
\big| h_0 & & & & & & & & \big| h_0 \\
\end{array}$$

Since $F$ is expanding, by [1] Lemma 11.4, the sequence of homeomorphisms $\{h_n\}_{n \geq 0}$ uniformly converges to a continuous map $h$ on $S^2$. We then get a semi-conjugacy $h$ from $f$ to $F$, i.e., $F \circ h = h \circ f$ on $\overline{\mathbb{C}}$. Besides, the restriction

$$h : \bigcup_{n \geq 0} f^{-n}(\text{post}(f)) \to \bigcup_{n \geq 0} F^{-n}(\text{post}(f))$$

is bijective. Because for all $k \geq n \geq 0$, $h_k = h_n : f^{-n}(\text{post}(f)) \to F^{-n}(\text{post}(F))$ is bijective.

We claim that $h(U) = h(c)$ is a singleton for each Fatou component $U$ with $c = c(U)$ the center of $U$. We first assume $f^p(U) = U$. Given a periodic internal
ray $\tilde{\gamma}$ in $U$ of period $pq$, we have $F^{npq} \circ h_{npq}(\tilde{\gamma}) = h_0 \circ f^{npq}(\tilde{\gamma}) = h_0(\tilde{\gamma})$ for any $n \geq 0$ by $\text{[4]}$. This means $\gamma_n := h_{npq}(\tilde{\gamma})$ is a lift of Jordan arc $\gamma_0$ under $F^n$. According to $\text{[4]}$ Lemma 8.8, there is a metric $\omega$ on $S^2$, called the visual metric of $F$, with expansion factor $\Lambda > 1$ such that

$$\text{diam}_\omega(\gamma_n) \leq A\Lambda^{-n},$$

where $A$ depends on $\gamma_0$ but not on $n$. Since $\{h_{npq}\}_{n \geq 0}$ converges uniformly to $h$, it follows that $\text{diam}_\omega(h(\tilde{\gamma})) = 0$ and hence $h(\tilde{\gamma}) = h(c)$ is a singleton. The argument above holds for any periodic internal rays of $U$, so $h$ sends every periodic internal ray of $U$ to $h(c)$. As the periodic internal rays are dense in $\overline{U}$, the claim then follows for periodic Fatou components. By Sullivan's non-wandering Fatou component theorem, we are left to deal with the case that $U$ is strictly pre-periodic. Assume that $f^k(U)$ is periodic. Since $F \circ h = f \circ f$ on the sphere, the set $F^k \circ h(\overline{U}) = h \circ f^k(\overline{U})$ is a singleton. It follows that $h(\overline{U})$ is contained in the finite set $F^{-k}(h \circ f^k(\overline{U}))$. The connectedness of $h(\overline{U})$ implies it is a singleton. Then we complete the proof of the claim. By this claim and $\text{(6)}$, the closures of distinct Fatou components of $f$ are pairwise disjoint.

It remains to show that each Fatou component is a Jordan domain. Without loss of generality, let $U$ be a fixed Fatou component. We argue by contradiction and assume that $U$ is not a Jordan domain. Note that $\partial U$ is locally connected. From the Böttcher’s theorem there exist two internal rays of $U$ landing at a common point in $\partial U$. The closure of their union is a Jordan curve bounding two domains $W_0$ and $W_1$ with $W_i \cap J_f \neq \emptyset$, $i \in \{0, 1\}$.

We claim that both of the domains $W_0, W_1$ contain some Fatou components. Otherwise, there is $i \in \{0, 1\}$ such that $f^n(W_i) \subseteq (U \cup J_f)$ for all $n \geq 0$. By the topological transitivity of the Julia set, the set $f^n(W_i)$ for sufficiently large $n$, hence $U \cup J_f$, covers $\overline{C}$ except at most two points (see $\text{[19]}$ Theorem 4.10). It means that $f$ has only one Fatou component $U$. By the claim above $h$ maps $\overline{U} = \overline{C}$ to one point. We then have $\text{post}(f) = \text{crit}(f) = f^{-1}(\text{post}(f)) = \{c(\overline{U})\}$ from $\text{(6)}$. One can easily get a contradiction by applying Riemann-Hurwitz formula to the branch covering $f: \overline{C} \to \overline{C}$.

Let $U_0$ and $U_1$ be the Fatou components contained in $W_0$ and $W_1$ respectively. By the discussion above, we have that the images $h(\overline{U}), h(\overline{U}_0)$ and $h(\overline{U}_1)$ are pairwise different points. Consequently, the set $h^{-1}(h(\overline{U}))$ contains the Jordan curve $\partial W_0 = \partial W_1 \subset \overline{U}$, and is disjoint with $U_0, U_1$. It implies that $S^2 \setminus h^{-1}(h(\overline{U}))$ is not connected. On the other hand, note that $h$ is the limit of a sequence of homeomorphisms of $S^2$. By $\text{[5]}$ Lemma 3.1, such a map $h$ has a property that $S^2 \setminus h^{-1}(x)$ is connected for any $x \in S^2$. It contradicts that $S^2 \setminus h^{-1}(h(\overline{U}))$ is not connected. The proof of the necessity is completed.

\[\square\]

Appendix.

Proof of Theorem $\text{[3, 4]}$. Let $\gamma_p$ be the boundary of $D_p$ for each $p \in P$, and set $S_1 := S \setminus P$. Then $\partial S_1 = \partial S$. Since every $\gamma_p$ avoids the set $\cup_{p \in P} K(p, H)$, the homotopy $H$ induces a homotopic imbedding $H_{|\gamma_p \times I} : \gamma_p \times I \to S_1$ from $\gamma_p$ to $h(\gamma_p)$ for each $p \in P$. By $\text{[3]}$ Theorem 2.1, there exists an isotopy

$$\Phi : S_1 \times I \to S_1 \text{ rel. } \partial S_1$$

such that $\Phi_0 = \text{id}_{S_1}, \Phi_1|_{\gamma_p} = h|_{\gamma_p}$ for each $p \in P$, and $\Phi$ coincides with the identity outside a compact subset of $\text{int}(S_1)$. This means that we can view $\Phi$ as an isotopy
Φ : S × I → S rel. ∂S ∪ P by complementarily defining Φ(p, t) = p for every p ∈ P and t ∈ I. Thus, the homotopy H′ : S × I → S rel. ∂S ∪ P, defined as

\[
H'(z,t) = \begin{cases} 
H(z,1-2t) & 0 \leq t \leq 1/2, \\
Φ(z,2t-1) & 1/2 \leq t \leq 1,
\end{cases}
\]

joins h and Φ1, and K(p, H′) = K(p, H) for every p ∈ P. It follows that Ḥ := h⁻¹ ∘ H′ is a homotopy rel. ∂S ∪ P from idS to the homeomorphism ̃h := h⁻¹ ∘ Φ1, such that ̃h|γp = idγp for every p ∈ P and

\[
K(p, Ḥ) = K(p, H′) = K(p, H) \subseteq D_p \text{ for any } p \in P.
\] (7)

Since Φ1 ∼ idS rel. ∂S ∪ P, it remains to prove that ̃h ∼ idS rel. ∂S ∪ P. Hence, for simplicity, we may assume that the original h is identity when restricted on \(\cup_{p \in P} γ_p\), i.e., ̃h homo. idS rel. ∂S ∪ P and ̃h|γp = idγp for each p ∈ P.

We claim that h is homotopic to idS not only relative to ∂S ∪ P but also relative to ∂S ∪ P \(\cup \bigcup_{p \in P} γ_p\). To see this, we first decompose the surface S into the surfaces \(D_1, \ldots, D_p, \text{ and } M := S \setminus \bigcup_{p \in P} \text{int}(D_p)\). Note that each γp belongs to both \(D_p\) and M. For distinguishing, we denote the γp in \(D_p\) by γp⁻ and that in M by γp+. And a point \(ξ \in γ_p\) is represented by \(ξ^±\) in γp± respectively. We then paste each \(D_p\) with M by the annulus \(A_p := γ_p \times [-1, 1]\). Precisely, let \(≈\) be an equivalence relation on the disjoint union \((\bigcup_{p \in P} D_p) \cup (\bigcup_{p \in P} A_p) \cup M\) such that

\[
x ≈ y \text{ if and only if } \begin{cases} 
x = y \in (\bigcup_{p \in P} D_p) \cup (\bigcup_{p \in P} A_p) \cup M, \\
x = ξ^+ \in γ^+_p \subseteq M, y = (ξ, 1) \in A_p & \text{for all } ξ \in γ_p, p \in P, \\
x = ξ^- \in γ^-_p \subseteq D_p, y = (ξ, -1) \in A_p & \text{for all } ξ \in γ_p, p \in P.
\end{cases}
\]

Let φ denote the projection from \((\bigcup_{p \in P} D_p) \cup (\bigcup_{p \in P} A_p) \cup M\) to the quotient space \(S_b := (\bigcup_{p \in P} D_p \cup \bigcup_{p \in P} A_p \cup M)/≈\).

Then \(S_b\) is homeomorphic to S. One can construct a homeomorphism φ : S → S_b such that φ(x) := φ(x) if x avoids the ε-neighborhood of each γp (here we identify each \(D_p\) and M a sub-surface of S, hence any point \(x \in S \setminus (\bigcup_{p \in P} γ_p)\) is also considered in M or \(\bigcup_{p \in P} D_p\)), and φ(ξ) = φ(ξ × 0) for every ξ ∈ γp, p ∈ P.

On the other hand, we define a map \(H_b : S_b \times I \rightarrow S\) as

\[
H_b(x_b, t) := \begin{cases} 
H(x,t) & \text{if } x_b = φ(x) \text{ with } x \in (\bigcup_{p \in P} D_p) \cup M; \\
H(ξ, t|s) & \text{if } x_b = φ(x) \text{ with } x = (ξ, s) \in A_p \text{ for some } p \in P.
\end{cases}
\]

It is easy to check that \(H_b\) is a homotopy rel. φ(P) \(\cup (\bigcup_{p \in P} φ(γ_p) \times 0)) \cup ∂S_b\). Using the homeomorphism φ : S → S_b constructed above, we get a homotopy \(\tilde{H} : S \times I \rightarrow S\) rel. ∂S ∪ P \(\cup (\bigcup_{p \in P} γ_p)\) defined by

\[
\tilde{H}(x,t) := H_b(φ(x), t), \quad ∀x \in S, t \in [0, 1].
\]

By the definition of \(H_b\) and the property of φ, we also know that \(\tilde{H}(x,t) = H(x,t)\) for all \(t \in I\) if x avoids the ε-neighborhood of each γp; the map \(\tilde{H}|_{S × 0}\) (resp. \(\tilde{H}|_{S × 1}\)) is homotopic to idS (resp. h) rel. ∂S ∪ P \(\cup (\bigcup_{p \in P} γ_p)\); and the equation \(K(p, H) = K(p, Ḥ)\) holds for each \(p \in P\). It follows that

\[
\text{idS homo. } \tilde{H}|_{S × 0} \sim \tilde{H}|_{S × 1} \sim h \text{ rel. } ∂S ∪ P \cup (\bigcup_{p \in P} γ_p),
\]
and this homotopy $\tilde{H}'$ from $id_S$ to $h$ can be chosen such that $K(p, H) = K(p, \tilde{H}')$ for each $p \in P$. Then the proof of the claim is completed.

By this claim, we may assume that the original homotopy $H : S \times I \rightarrow S$ from $id_S$ to $h$ is relative to $\partial S \cup P \cup (\cup_{p \in P} \gamma_p)$. As previous, we decompose the surface $S$ into the surfaces $D_1, \ldots, D_p$ and $M := S \setminus \cup_{p \in P} int(D_p)$. Clearly $\partial M = \partial S \cup (\cup_{p \in P} \gamma_p)$.

On each closed disk $D_p$, according to Alexander Lemma (see Remark 1), we have

$$h|_{D_p} \sim id_{D_p} \text{ rel. } \{p\} \cup \gamma_p.$$  

To construct a homotopy in $M$, we define for each $p \in P$ a radial projection

$$\pi_p : D_p \setminus \{p\} \rightarrow \gamma_p \text{ } \gamma \mapsto \alpha_p^{-1}(\alpha_p(\gamma))/|\alpha_p(\gamma)|,$$

where $\alpha_p : D_p \rightarrow \mathbb{D}$ with $p \mapsto 0$ is a homeomorphism. Given any $x \in M$, since the curve $H(x, t), t \in I$ avoids $P$, one can define a map $\Pi : M \times I \rightarrow M$ by

$$\Pi(x, t) := \begin{cases} 
\pi_p \circ H(x, t), & \text{if } H(x, t) \in D_p \text{ for some } p \in P; \\
H(x, t), & \text{otherwise.}
\end{cases}$$

This map is continuous and satisfies that

- $\Pi_0 = id_M$ and $\Pi_1 = h|_M$,
- $\Pi(z, t) = z$ for $z \in \partial M, t \in [0, 1]$.

In other words, the map $\Pi : M \times I \rightarrow M$ is a homotopy rel. $\partial M$ from $id_M$ to $h|_M$. By Theorem 3.3, the homeomorphisms $id_M$ and $h|_M$ are also isotopic rel. $\partial M$.

Combining this fact with (8), we see that globally $id_S$ is isotopic to $h$ rel. $\partial S \cup P$. The theorem is then proved.

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