A MODULAR SYMBOL WITH VALUES IN CUSP FORMS

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ABSTRACT. In [B-G1] and [B-G2], Borisov and Gunnells constructed for each level \((N > 1)\) and for each weight \((k \geq 2)\) a modular symbol with values in \(S_k(\Gamma_1(N))\) using products of Eisenstein series.

In this paper we generalize this result by producing a modular symbol (for \(GL_2(\mathbb{Q})\)) with values in locally constant distributions on \(M_2(\mathbb{Q})\) taking values in the space of cuspidal power series in two variables (see Definition 5).

We can recover the above cited result by restricting to a principal open invariant for the action of \(\Gamma_1(N)\) and to the homogeneous degree \(k - 2\) part of the power series.

We should also mention that Colmez [Col] constructs similar distributions \((zEis(k, j))\).

The modification in the definition of such distributions allow us to observe further relations among these distributions (Manin Relations) which in turn makes possible the existence of our construction. In the last section we exhibit some instances of these relations for the full modular group.

INTRODUCTION

Our main result is the following theorem (see the Definitions 5, 6 and 7):

**Theorem 0.1.** There exists a unique modular symbol \(\Phi \in Symb_{GL_2(\mathbb{Q})}(D_{naive}(M_2(\mathbb{Q}), \tilde{S}_2))\) such that if \(\mu := \Phi(D_{\infty})\), we have:

\[
\mu(U) := H(E_{U_1}(\tau, X) \cdot E_{U_2}(\tau, Y))
\]

As a corollary we have the following:

**Theorem 0.2.** Let \(\Phi^{B-G}_{k,N} : \Delta_0 \to S_k(\Gamma_1(N))[X, Y]^{(k-2)}\) be the map defined by:

\[
\Phi^{B-G}_{k,N}(D) := \Phi(D)(U_1(N))^{(k-2)},
\]

where \(U_1(N) := \{\gamma \in M_2(\mathbb{Q}) | \gamma \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1/N \end{pmatrix} \mod M_2(\mathbb{Z})\}\) and for a power series \(F(X, Y)\), the homogeneous part of \(F\) of degree \(k - 2\) is denoted by \(F^{(k-2)}(X, Y)\).

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Then $\Phi_{k,N}^{B-G}$ is a modular symbol for $\Gamma_1(N)$, and coincide with the modular symbol defined by Borisov and Gunnells in [B-G1] for $k = 2$ and in [B-G2] for $k \geq 3$.

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1. Elliptic Functions

We refer to [Cha] and [Scho] for details about this section. In what follows we fix $\tau \in \mathcal{H}$ and we don’t mention it in the notations. For example, the derivatives involved are with respect to the (elliptic) variable $z$, not with respect to the (modular) variable $\tau$.

Definition 1. We let $\wp(z)$ and $\sigma(z)$ be the Weierstrass $\wp$ respectively $\sigma$-function. Let also $\theta(z)$ be the usual theta function ([Cha] V.1.):

$$\theta(z) := \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{(2n+1)^2}{8}} e^{(2n+1)\pi iz}$$

We have the following important lemma:

Lemma 1.1. We have the following identities:

1. $\sigma(z) = \theta(z) \cdot \frac{e^{\pi i z^2}}{\theta'(0)}$

2. $\wp(z_1) - \wp(z_2) = -\frac{\sigma(z_1 + z_2)\sigma(z_1 - z_2)}{\sigma^2(z_1)\sigma^2(z_2)} = -\frac{\theta(z_1 + z_2)\theta(z_1 - z_2)}{\theta^2(z_1)\theta^2(z_2)} \theta'(0)^2$.

$\varphi'(z_1) - \varphi'(z_2) = -2(\varphi(z_1) - \varphi(z_2)) \cdot (\partial_z \log \theta(z_1) + \partial_z \log \theta(z_2) - \partial_z \log \theta(z_1 + z_2))$

Proof. The first identity is from [Cha] V.2. Theorem 2 together with identity (2.1), same chapter cited above.

For the second identity, we consider the first identity as function in $z_1$ and we apply $\partial_z \log$ and we use the well known fact that $\partial_z \log(fg) = \partial_z \log(f) + \partial_z \log(g)$. We get:

$$\frac{\wp'(z_1)}{\wp(z_1) - \wp(z_2)} = -2\partial_z \log \theta(z_1) + \partial_z \log \theta(z_1 + z_2) + \partial_z \log \theta(z_1 - z_2).$$

We have by symmetry (or by a similar argument):

$$\frac{\wp'(z_2)}{\wp(z_2) - \wp(z_1)} = -2\partial_z \log \theta(z_2) + \partial_z \log \theta(z_1 + z_2) + \partial_z \log \theta(z_2 - z_1).$$
Since $\theta$ is an odd function in $z$, $\partial_z \log \theta$ will be also odd.

We add the above relations and we get (2).

Now we are able to prove the following theorem:

**Theorem 1.2.** We have the following identity:

$$\wp(z_1) + \wp(z_2) + \wp(z_1 + z_2) = \left(\partial_z \log \theta(z_1) + \partial_z \log \theta(z_2) - \partial_z \log \theta(z_1 + z_2)\right)^2$$

**Proof.** We use the famous addition formula for $\wp$-function:

$$\wp(z_1 + z_2) = \frac{1}{4} \frac{(\wp'(z_1) - \wp'(z_2))^2}{(\wp(z_1) - \wp(z_2))^2} - \wp(z_1) - \wp(z_2).$$

Then, use the previous lemma. □

**Corollary 1.3.** For any complex numbers $z_1, z_2, z_3$ such that $z_1 + z_2 + z_3 = 0$, we have:

$$\wp(z_1) + \wp(z_2) + \wp(z_3) = \left(\partial_z \log \theta(z_1) + \partial_z \log \theta(z_2) + \partial_z \log \theta(z_3)\right)^2$$

**Proof.** The proof is clear from the previous theorem because $z_3 = -(z_1 + z_2)$ and we know that $\wp$ is an even function while $\partial_z \log \theta$ is an odd function.

As a remark, in [Cha], in the Notes on Chapter IV is mentioned that Frobenius and Stickelberger have a quite similar formula in terms of Weierstrass $\zeta$-function:

$$\left(\zeta(x) + \zeta(y) + \zeta(z)\right)^2 + \zeta'(x) + \zeta'(y) + \zeta'(z) = 0,$$

whenever $x + y + z = 0$. One can observe that this relation is equivalent with the above relation since $\zeta' = -\wp$ and $\zeta$ differ from $\partial_z \log \theta$ by a linear term (i.e. $2\eta_1 \cdot z$). Therefore either we can consider Corollary 1.4 as a consequence of Frobenius-Stickelberger relation, or with the above proof one can consider it as a reproof of the Frobenius-Stickelberger relation. □

1.1. *q*-Expansions. We use the following notations:

1. For $z \in \mathbb{C}$, we use $q_z := e^{2\pi iz}$.

2. The variable $\tau \in \mathcal{H}$ is always the modular variable and $q := q_\tau$ respectively $q_N := q^{1/N}$. 


(3) \( \zeta_N := e^{2\pi i/N} \) be the \( N \)-th root of unity such that \( \zeta_N = q_z \big|_{z=1/N} \).

Using the Jacobi triple product formula (\[Cha\] V.6.4.) we have the following:

**Lemma 1.4.** We have for any \( z \in \mathbb{C} \setminus \mathbb{Z} \):

\[
\partial_z \log \theta(z, \tau) = \pi i \frac{q_z + 1}{q_z - 1} - 2\pi i \sum_{n=1}^{\infty} \frac{q^n q_z}{1 - q^n q_z} - \frac{q^n q_z^{-1}}{1 - q^n q_z^{-1}}.
\]

Moreover, if \( \tau \pm z \in \mathcal{H} \), then

\[
\partial_z \log \theta(z, \tau) = \pi i \frac{q_z + 1}{q_z - 1} - 2\pi i \sum_{n,m \geq 1} q^{nm}(q_z^m - q_z^{-m}).
\]

We can easily then deduce the \( q \)-expansion of \( \partial_z \log \theta \) evaluated in torsion points:

**Corollary 1.5.** Let \( N > 1 \) be a positive integer and \( a = (a_1, a_2) \in \mathbb{Z}^2 \) such that

1. \( |a_1| < N \)
2. \( a \notin N\mathbb{Z}^2 \).

Then:

\[
\partial_z \log \theta\left(\frac{a_1 \tau + a_2}{N}, \tau\right) = -2\pi i \sum_{n \geq 1} \alpha_n q_N^n + \pi i \left\{ \begin{array}{ll}
\frac{\zeta_N^{a_2+1}}{\zeta_N^{a_2-1}} & \text{if } a_1 = 0 \\
-\operatorname{sgn}(a_1) & \text{if } a_1, \neq 0
\end{array} \right.
\]

where

\[
\alpha_n = \alpha_n(1, N, a) := \sum_{d \mid n; n/d \equiv A_1(N)} \zeta_N^{a_2 d} - \sum_{d \mid n; n/d \equiv -A_1(N)} \zeta_N^{-a_2 d}
\]

2. **Kronecker-Eisenstein Series**

Consider the Kronecker-Eisenstein series for \( k \geq 1, \phi_k : \mathbb{C} \times \mathcal{H} \times \mathbb{C} \longrightarrow \mathbb{C} \):

\[
\phi_k(z, \tau; s) := \sum_{\omega \in L_\tau} \left( (z + \omega)^{-k} |z + \omega|^{-s} \right)
\]

where \( L_\tau := \mathbb{Z}\tau + \mathbb{Z} \) and, in the sum, the term corresponding to \( \omega = -z \) if \( z \in L_\tau \) is omitted.

It is well known that \( \phi_k \) is analytic in \( s \) for \( \Re(s) > -k+2 \) and admits a meromorphic continuation in the whole \( s \)-plane and at \( s = 0 \) is holomorphic. We set:

\[
\phi_k(z, \tau) := \phi_k(z, \tau; 0).
\]

Also we can observe that

\[
\partial_z \phi_k(z, \tau) = -k \phi_{k+1}(z, \tau)
\]
2.1. **Evaluation in torsion points.** For \( a \in \mathbb{Q}^2 \) we set \( \tau_a := a_1 \tau + a_2 \). These are the division points of the lattice \( L_\tau \).

We have the following:

**Theorem 2.1.** For \( k \geq 1 \) and \( a \in \mathbb{Q}^2 \setminus \mathbb{Z}^2 \), set \( E_{k,a}(\tau) := \phi_k(\tau_a, \tau) \). Then

1. For any \( \gamma \in \text{SL}_2(\mathbb{Z}) \) we have:

\[
E_{k,a}(\gamma \tau) \cdot j(\gamma, \tau)^{-k} = E_{k,\gamma \tau a}(\tau).
\]

2. For any \( k \neq 2 \), \( E_{k,a} \) are holomorphic in \( \tau \).

For \( k = 2 \), \( \wp_a := E_{2,a} - E_{2,0} = \wp(\tau_a, \tau) \) is holomorphic in \( \tau \), where

\[
E_{2,0}(\tau) := 2(2\pi i)^2 \cdot \left( \frac{-1}{4\pi i(\tau - \bar{\tau})} - \frac{1}{24} + \sum_{n \geq 1} \sigma_1(n)q^n \right)
\]

(3) For \( a \in \mathbb{Q}^2 \), a positive integer \( N \) is called a level for \( a \) if \( N \cdot a \in \mathbb{Z}^2 \). For a minimal level, we write \( l(a) = N \).

Then for each integer \( N > 1 \) we have that the set \( \{E_{k,a}\}_{l(a)=N} \) generate the space of Eisenstein series of weight \( k \) for \( \Gamma(N) \). For \( k = 2 \) we take \( \wp_a \) instead.

For a proof see [Scho], also [Col].

We have a very important link between \( \phi_1(z, \tau) \) and \( \partial_z \log \theta(z, \tau) \), see [Wei] and [deS]:

**Theorem 2.2.** We have:

\[
\phi_1(z, \tau) = \partial_z \log \theta(z, \tau) + 2\pi i \frac{z - \bar{z}}{\tau - \bar{\tau}}
\]

Consequently, we can deduce the following corollary:

**Corollary 2.3.** For any complex numbers such that \( z_1 + z_2 + z_3 = 0 \) we have:

\[
(\phi_1(z_1, \tau) + \phi_1(z_2, \tau) + \phi_1(z_3, \tau))^2 = \wp(z_1, \tau) + \wp(z_2, \tau) + \wp(z_3, \tau).
\]

**Proof.** We have:

\[
(\phi_1(z_1, \tau) + \phi_1(z_2, \tau) + \phi_1(z_3, \tau))^2 =
\]

\[
= \left( \partial_z \log \theta(z_1) + \partial_z \log \theta(z_2) + \partial_z \log \theta(z_3) + 2\pi i \frac{z_1 - \bar{z}_1 + z_2 - \bar{z}_2 + z_3 - \bar{z}_3}{\tau - \bar{\tau}} \right)^2 =
\]

\[
= (\partial_z \log \theta(z_1) + \partial_z \log \theta(z_2) + \partial_z \log \theta(z_3))^2 =
\]

\[
= \wp(z_1, \tau) + \wp(z_2, \tau) + \wp(z_3, \tau).
\]
Where for the last equality we used Corollary 1.3.

\[ \square \]

3. TAYLOR SERIES

We make the following definition:

**Definition 2.** For a function \( h : \mathbb{C} \times \mathcal{H} \to \mathbb{C} \) which is \( C^\infty \) in the first variable, and for \( z_0 \in \mathbb{C} \) we define \( T a y_{z_0} \) as a power series with coefficients functions on \( \mathcal{H} \) by:

\[
T a y_{z_0} h(\tau, X) := \sum_{n \geq 0} \frac{\partial^n}{\partial z_0^n} h(z_0, \tau) \cdot \frac{X^n}{n!}.
\]

For \( a \in \mathbb{Q}^2 \setminus \mathbb{Z}^2 \), we set:

\[
E_a(\tau, X) := T a y_{a_1\tau + a_2} \phi_1(\tau, X) = \sum_{n \geq 0} E_{n+1,a}(\tau)(-X)^n.
\]

We are now able to deduce the following key result:

**Theorem 3.1.** For any complex numbers \( z_1, z_2, z_3 \in \mathbb{C} \) such that \( z_1 + z_2 + z_3 = 0 \) we have:

\[
(T a y_{z_1} \phi_1(\tau, X) + T a y_{z_2} \phi_1(\tau, Y) + T a y_{z_3} \phi_1(\tau, -X - Y))^2 =
T a y_{z_1} \psi(\tau, X) + T a y_{z_2} \psi(\tau, Y) + T a y_{z_3} \psi(\tau, -X - Y).
\]

**Proof.** The above identity is true if and only if the identity is true for \( X = u \in \mathbb{C}, Y = v \in \mathbb{C} \) such that the corresponding power series converge as complex functions.

But \( T a y_{z_0} h(\tau, u) = h(z_0 + u, \tau) \) in this case and the identity is provided by the Corollary 2.3. \[ \square \]

**Definition 3.** For a power series \( F = \sum_{n=0}^{\infty} f_n X^n \) we define:

\[
Int_X F := \sum_{n=0}^{\infty} f_n X^{n+1}.
\]

We have also the following lemma:

**Lemma 3.2.** Let \( f, g : \mathbb{C} \times \mathcal{H} \to \mathbb{C} \) two functions. Then the following are true:

1. \( T a y_{z_0} f(\tau, X) + T a y_{z_0} g(\tau, X) = T a y_{z_0} (f + g)(\tau, X). \)

2. \( T a y_{z_0} f(\tau, X) \cdot T a y_{z_0} g(\tau, X) = T a y_{z_0} (fg)(\tau, X). \)
\[(Tay_{z_0}f)(\tau, X) = f(z_0, \tau) + Int_X Tay_{z_0}(\partial_z f)(\tau, X).\]

\[Tay_{z_0}(\partial_\tau f)(\tau, X) = \partial_\tau Tay_{z_0} f(\tau, X) - \frac{\partial z_0}{\partial \tau} \cdot Tay_{z_0}(\partial_z f)(\tau, X).\]

**Proof.** For the first identity, we just use the additivity property of partial differentiation.

For the second, we can either proceed as above and observe that a trivial identity is achieved at the level of functions (when we replace \(X\) by a complex number \(u\) small enough). Or, we just write down the definitions and use Leibnitz rule

\[\partial^n fg = \sum_{k=0}^{n} \binom{n}{k} \partial^k f \cdot \partial^{n-k} g.\]

The third identity is merely a rewriting of the definition for both sides.

For the fourth identity, we have to mention that even though the variables \(z\) and \(\tau\) are viewed as independent variables, however, we can specialize \(z\) in a number depending on \(\tau\) (for ex. \(z_0 = \frac{a\tau+b}{N}\)). This formula explains the relation between the Taylor series under the derivation by \(\tau\) before and after taking the specialization.

The relation results simply because \(\frac{\partial}{\partial \tau}(f(z_0(\tau), \tau)) = (\partial_z f)(z_0(\tau), \tau) \cdot \frac{\partial z_0(\tau)}{\partial \tau} + (\partial_\tau f)(z_0(\tau), \tau)\)

\[\square\]

4. **(Nearly Holomorphic) Modular power series-Cuspidal power series**

For a number \(k\) we set the following operators action on the space of \(C^\infty\)-class of functions on \(\mathcal{H}\):

\[\delta_k = \frac{\partial}{\partial \tau} + \frac{k}{\tau - \bar{\tau}}\]

\[\varepsilon = (\tau - \bar{\tau})^2 \cdot \frac{\partial}{\partial \tau} \]

We follow [Hid] (Ch.10) and define the following:

**Definition 4.** For a congruence subgroup \(\Gamma\), and an integer \(r \geq 0\), we define the space of nearly holomorphic modular forms \(N^r_k(\Gamma)\) to be the space of \(C^\infty\)-functions \(f\) on \(\mathcal{H}\) with the following properties:

(N1) \(f\) is slowly increasing (at the cusps).

(N2) \(\varepsilon^r f = 0\).

(N3) \(f|k\gamma = f\) for all \(\gamma \in \Gamma\).
Of course, inside $\mathcal{N}_k^r(\Gamma)$ sits $\mathcal{N}_k^0(\Gamma)$ which is the space of holomorphic modular forms $M_k(\Gamma)$.

We also have a projection map $H : \mathcal{N}_k^r(\Gamma) \longrightarrow M_k(\Gamma)$, called holomorphic projection.

The maps $\delta_k$ and $\varepsilon$ (called shifting and lowering (weight) operators) are acting as follows:

$$\delta_k : \mathcal{N}_k^r(\Gamma) \longrightarrow \mathcal{N}_{k+2}^{r+1}(\Gamma)$$

$$\varepsilon : \mathcal{N}_{k+2}^{r+1}(\Gamma) \longrightarrow \mathcal{N}_k^r(\Gamma)$$

We also have obvious maps:

$$\mathcal{N}_{k_1}^{r_1}(\Gamma) \times \mathcal{N}_{k_2}^{r_2}(\Gamma) \longrightarrow \mathcal{N}_{k_1+k_2}^{r_1+r_2}(\Gamma)$$

$$(f, g) \longrightarrow f \cdot g$$

**Definition 5.** For a congruence subgroup $\Gamma$, and $m \geq 1$ we consider the following spaces:

$$\tilde{M}_m(\Gamma) := \left\{ h(\tau; X, Y) = \sum_{i,j \geq 0} h_{i,j}(\tau)X^iY^j \mid h_{i,j} \in M_{i+j+m}(\Gamma) \right\}.$$ 

$$\tilde{\mathcal{N}}_m^r(\Gamma) := \left\{ h(\tau; X, Y) = \sum_{i,j \geq 0} h_{i,j}(\tau)X^iY^j \mid h_{i,j} \in \mathcal{N}_{i+j+m}^r(\Gamma) \right\}.$$ 

Similarly we define $\tilde{\mathcal{E}}_m(\Gamma)$ and $\tilde{\mathcal{S}}_m(\Gamma)$.

We also define $	ilde{M}_m := \bigcup_{\Gamma} \tilde{M}_m(\Gamma)$, $\tilde{\mathcal{E}}_m := \bigcup_{\Gamma} \tilde{\mathcal{E}}_m(\Gamma)$, $\tilde{\mathcal{S}}_m := \bigcup_{\Gamma} \tilde{\mathcal{S}}_m(\Gamma)$, as well as $\tilde{\mathcal{N}}_m^r := \bigcup_{\Gamma} \tilde{\mathcal{N}}_m^r(\Gamma)$.

We have the following identifications:

**Proposition 4.1.** We have:

$$\tilde{\mathcal{S}}_m(\Gamma) = \tilde{M}_m(\Gamma)/\tilde{\mathcal{E}}_m(\Gamma); \quad \tilde{\mathcal{S}}_m = \tilde{M}_m/\tilde{\mathcal{E}}_m$$

We need another important map, which is the holomorphic projection for nearly holomorphic modular power series:

$$H : \tilde{\mathcal{N}}_m^r \longrightarrow \tilde{M}_m$$

Several observations we can make:
Observation 4.2. (1) There exist well defined maps given by multiplication:

\[
\begin{align*}
\tilde{\mathcal{N}}^{r_1}_{m_1} \times \tilde{\mathcal{N}}^{r_2}_{m_2} & \longrightarrow \tilde{\mathcal{N}}^{r_1+r_2}_{m_1+m_2} \\
\tilde{\mathcal{M}}^{m}_{m_1} \times \tilde{\mathcal{M}}^{m}_{m_2} & \longrightarrow \tilde{\mathcal{M}}^{m}_{m_1+m_2} \\
\tilde{\mathcal{S}}^{m}_{m_1} \times \tilde{\mathcal{S}}^{m}_{m_2} & \longrightarrow \tilde{\mathcal{S}}^{m}_{m_1+m_2} \\
\tilde{\mathcal{E}}^{m}_{m_1} \times \tilde{\mathcal{E}}^{m}_{m_2} & \longrightarrow \tilde{\mathcal{E}}^{m}_{m_1+m_2} \\
(F, G) & \longrightarrow F \cdot G
\end{align*}
\]

(2) We let $\text{GL}_2^+(\mathbb{Q})$ act (on the left) on all of the above spaces (say $\tilde{\mathcal{N}}^r_m$) by the following:

\[
(\gamma \cdot m F)(\tau; X, Y) := F(\gamma^t \tau; \frac{X}{j(\gamma^t, \tau)}, \frac{Y}{j(\gamma^t, \tau)}) \cdot j(\gamma^t, \tau)^{-m}
\]

(3) $E_a \in \tilde{\mathcal{N}}^1_1$ for all $a \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$.

(4) Moreover, $\text{Tay}_{a, \phi_m}(\tau, uX + vY) \in \tilde{\mathcal{N}}^1_m$ and if $m \geq 3$, $\text{Tay}_{a, \phi_m}(\tau, uX + vY) \in \tilde{\mathcal{E}}_m$ for all $a \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ and for all $u, v \in \mathbb{C}$.

(5) $\text{Tay}_{a, \phi}(\tau, uX + vY) \in \tilde{\mathcal{E}}_2$ for all $a \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ and for all $u, v \in \mathbb{C}$.

(6) For $F = \sum_{i,j} f_{i,j} X^i Y^j \in \tilde{\mathcal{N}}^r_m$ we write $\delta F := \sum_{i,j} \delta_{i+j+m} f_{i,j} X^i Y^j \in \tilde{\mathcal{N}}^{r+1}_{m+2}$.

We have $H(\delta F) = 0$

\[
H(F) = F \iff F \in \tilde{\mathcal{M}}_m
\]

We will dedicate the next section in proving the following theorem:

**Theorem 4.3.** We have

\[
H(E_a^2) \in \tilde{\mathcal{E}}_2 \quad \forall a \in \mathbb{Q}^2 \setminus \mathbb{Z}^2.
\]

5. Proof of Theorem 4.3.

We start with the following lemma:

**Lemma 5.1.** We have:

\[
\left( \partial_{z} \log \theta(z, \tau) \right)^2 = \phi_2(z, \tau) + \frac{2\pi i}{\tau - \bar{\tau}} + 4\pi i \cdot \partial_{\tau} \log \theta(z, \tau).
\]

**Proof.** We start from the relation between $\phi_1$ and $\partial_z \log \theta$ mentioned in Theorem 2.2.:

\[
\phi_1(z, \tau) = \partial_z \log \theta(z, \tau) + \frac{2\pi i}{\tau - \bar{\tau}}\frac{z - \bar{z}}{\tau - \bar{\tau}}
\]
We take the derivative with respect to $z$ in the above relation and we get:

$$-\phi_2(z, \tau) = \frac{2\pi i}{\tau - \bar{\tau}} + \partial^2_z \log \theta(z, \tau) = \frac{2\pi i}{\tau - \bar{\tau}} + \partial_z \frac{\partial_z \theta(z, \tau)}{\theta(z, \tau)} =$$

$$\begin{align*}
&= \frac{2\pi i}{\tau - \bar{\tau}} + \frac{\partial^2 \theta(z, \tau) \cdot \theta(z, \tau) - (\partial_z \theta(z, \tau))^2}{(\theta(z, \tau))^2} = \\
&= \frac{2\pi i}{\tau - \bar{\tau}} + \frac{\partial^2 \theta(z, \tau)}{\theta(z, \tau)} - \left(\frac{\partial_z \theta(z, \tau)}{\theta(z, \tau)}\right)^2
\end{align*}$$

To finish the proof we just need to use the famous differential equation (heat equation) for which $\theta$ is a solution: $\partial^2_z \theta = 4\pi i \cdot \partial_\tau \log \theta(z, \tau)$. \hfill \Box

We can rewrite the above formula as:

**Lemma 5.2.** Let $A(z, \tau) := \frac{z - \bar{z}}{\tau - \bar{\tau}}$. We have:

$$(\phi_1(z, \tau))^2 = \phi_2(z, \tau) + \frac{2\pi i}{\tau - \bar{\tau}} + 4\pi i A(z, \tau) \cdot \phi_1(z, \tau) - (2\pi i A(z, \tau))^2 + 4\pi i \cdot \partial_\tau \log \theta(z, \tau).$$

We apply Lemma 3.2. several times to get:

**Lemma 5.3.** For any $a \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ we have:

$$(T\text{ay}_a \partial_\tau \log \theta)(\tau, X) = (\partial_\tau \log \theta)(\tau_a, \tau) + 2\pi i \frac{\tau_a - \bar{\tau}_a}{(\tau - \bar{\tau})^2} X + \pi i \frac{X^2}{(\tau - \bar{\tau})^2} + \partial_\tau \text{Int}_X E_a(\tau, X) +$$

$$+ \frac{\tau_a - \bar{\tau}_a}{\tau - \bar{\tau}} \cdot \text{Int}_X(T\text{ay}_a \phi_2)(\tau, X).$$

**Lemma 5.4.** We have the following relations:

1. $$4\pi i \frac{\tau_a - \bar{\tau}_a}{\tau - \bar{\tau}} \cdot E_a + 4\pi i \frac{\tau_a - \bar{\tau}_a}{\tau - \bar{\tau}} \cdot \text{Int}_X T\text{ay}_a \phi_2 = 4\pi i \frac{\tau_a - \bar{\tau}_a}{\tau - \bar{\tau}} \cdot \phi_1(\tau_a, \tau).$$

2. $$4\pi i \frac{X}{\tau - \bar{\tau}} \cdot E_a + 4\pi i \partial_\tau \text{Int}_X E_a = 4\pi i \delta \cdot \text{Int}_X E_a.$$

**Proof.** For the first relation, the right hand side is just the first term in the expansion of the first power series.

For $n \geq 1$, the coefficient of $X^n$ in the left hand side is:

$$4\pi i \frac{\tau_a - \bar{\tau}_a}{\tau - \bar{\tau}} \cdot ((-1)^n \phi_{n+1}(\tau_a, \tau) + (-1)^{n-1} \phi_{n-1}(\tau_a, \tau)) = 0.$$

Or, even simpler, use the relation 3.2.3 knowing that $\phi_2(z, \tau) = -\partial_z \phi_1(z, \tau)$. 

For the second relation, the coefficient of $X^n$ for $n \geq 1$ on the left hand side is:

$$4\pi i (-1)^{n-1} \frac{\phi_n(\tau_a, \tau)}{\tau - \bar{\tau}} + 4\pi i (-1)^{n-1} \frac{\partial_\tau \phi_n(\tau_a, \tau)}{n} = (4\pi i) (-1)^{n-1} \frac{\partial_\tau \cdot \phi_n(\tau_a, \tau)}{n}$$

The relation now is clear since $\sum_{n \geq 1} (-1)^{n-1} \phi_n(\tau_a, \tau) X^n = Int \mathcal{E}_a$. □

We get the following formula:

**Lemma 5.5.** For any $a \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ we have:

$$E_a^2 = E_{1,a}^2 - 2 \text{Int}_X(Tay_n \phi_3)(\tau, X) + 4\pi i \delta \cdot \text{Int}_X \mathcal{E}_a.$$  

Finally, the last ingredient is:

**Lemma 5.6.** For any $a \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, we have that:

$$E_{1,a}^2 \in \mathcal{E}_2(\Gamma(l(a))).$$

**Proof.** Since $a \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, we can always find $\gamma \in \text{SL}_2(\mathbb{Z})$ such that $\gamma a \equiv \left(\begin{smallmatrix} 0 \\ 1/l(a) \end{smallmatrix}\right) \mod \mathbb{Z}^2$. We have:

$$E_{1,a}^2 |_{2\gamma} = E_{1,a}^2 = E_{1,\left(\begin{smallmatrix} 0 \\ 1/l(a) \end{smallmatrix}\right)}^2,$$

where $N = l(a)$ is the minimal level of $a$.

Since the action of $\text{SL}_2(\mathbb{Z})$ preserve the space of Eisenstein series, we see that it is enough to prove our lemma when $a = \left(\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right)$ for some $N > 1$. But this is exactly Proposition 3.8. in [B-G1].

We have to warn the reader that they prove that $E_{1,\left(\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right)}^2$ is Eisenstein, which for weight two includes in their notation also the nonholomorphic Eisenstein $E_{2,\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)}$. But $E_{1,a}$ is holomorphic, therefore its square lands in the holomorphic part of the Eisenstein series space of weight 2. The authors in [B-G1] make this observation somewhere else, but is not included in that section. □

Returning to the proof of Theorem 4.3., we take the formula proved in Lemma 5.5. and apply Lemma 5.6. and the observations 4.2.4 and 4.2.6.

### 6. Naive Distributions on $M_2(\mathbb{Q})$

Let $M$ be any module. We consider the space $\mathcal{D}_{\text{naive}}(M_2(\mathbb{Q}), M)$ of locally constant compactly supported distributions on $M_2(\mathbb{Q})$ with values in $M$. We will call them $M$-valued naive distributions on $M_2(\mathbb{Q})$ since the topology on the target space is
considered the discrete topology.
The space of test function is generated by the characteristic functions of the open-compact subsets of $M_2(\mathbb{Q})$ of the form $\gamma + \alpha \cdot M_2(\mathbb{Z})$, with $\gamma \in M_2(\mathbb{Q})$ and $\alpha \in \mathbb{Q}^\times$.

Our interest is to study the situation when $M = \tilde{S}_2 \cong \tilde{M}_2/\tilde{E}_2$. We can define a $\text{GL}_2(\mathbb{Q})$ action on $\mathcal{D}_{\text{naive}}(M_2(\mathbb{Q}), M)$ by:

$$(\rho \mu)(U)(X, Y) := \det(\rho) \cdot \mu(U \rho)((XY) \rho),$$

**Definition 6.** For a compact-open set $V \subseteq \mathbb{Q}^2$ we define

$$V_\tau := \{ a_1 \tau + a_2 \mid a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in V \}.$$

the generalized Kronecker-Eisenstein series:

$$\phi_{n,V}(z, \tau; s) := \sum_{\omega \in V_\tau} (z + \omega)^n |z + \omega|^{-s},$$

where we omit the term $\omega = -z$ if $-z \in V_\tau$.

Since any compact-open is a finite union of cosets of $N\mathbb{Z}^2$ for some fixed $N$, each $\phi_{n,U}$ admits a meromorphic continuation in the whole $s$-plane and is holomorphic at $s = 0$.

We define:

$$\phi_{n,V}(z, \tau) := \phi_{n,V}(z, \tau; s)|_{s=0}$$

We put

$$E_V(\tau, X) := \sum_{n \geq 0} (-1)^n \phi_{n+1,V}(0, \tau) \cdot X^n = \text{Taylor}_0 \phi_{1,V}(\tau, X)$$

We have the following:

**Lemma 6.1.** The following are true:

1. $E_{aV}(\tau, \alpha X) = \frac{1}{\alpha} \cdot E_V(\tau, X)$ for all $\alpha \in \mathbb{Q}^\times$.
2. $E_V(\tau, X) \in \tilde{N}_1^1$.
3. $E_{z_2}(\tau, X) = \left(\text{Taylor}_{z_0}(\phi_1(z, \tau) - \frac{1}{z})(\tau, X)\right)_{z_0 = 0}$
4. $E_{z_2}^2 = \text{Taylor}_0 \phi_{2,z_2} - 2 \text{Taylor}_0 \phi_{1,z_2}^2 + 4\pi i \cdot \delta \cdot \text{Int}_X E_{z_2} = -\partial_X E_{z_2}(X) - 2e_{z_2}(X) + 4\pi i \cdot \delta \cdot \text{Int}_X E_{z_2}(X) - 3e_{2,(3)}.$
Proof. For (1), we just unravel the definition. For (2) we take into account that any open compact is a finite union of translations of a multiple of \( \mathbb{Z}^2 \), then we use (1) and (3) together with 4.2.3. to finish the argument.

For (3), we unravel the definition for \( \phi_{1,\mathbb{Z}^2} \) together with the formula \( \partial_z \phi_n(z, \tau) = -n\phi_{n+1}(z, \tau) \) which is also true for the function \( \frac{1}{z} \) and also for \( \phi_{n,\mathbb{Z}^2} \).

For (4), we use Lemma 5.5 ( which is true for \( a \in \mathbb{R}^2 \setminus \mathbb{Z}^2 \) ) and pass to the limit and use the fact that \( E_{1,\mathbb{Z}^2} = 0 \).

\[ \square \]

For an open-compact \( U \in M_2(\mathbb{Q}) \) we define the two projections on \( \mathbb{Q}^2 \):

\[
U_1 := U \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad U_2 := U \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Let’s define the following element \( \mu \in D_{\text{naive}}(M_2(\mathbb{Q}), \tilde{S}_2) \):

\[
\mu(U) := H(E_{U_1}(\tau, X) \cdot E_{U_2}(\tau, Y))
\]

We need to make the remark that the way is defined, the distribution takes values in \( \tilde{M}_2 \). We may and will consider as being the element in \( \tilde{S}_2 \) corresponding to the class of that value.

We have the following:

**Theorem 6.2.** The following are true:

1. \( \mu \) is well defined, i.e. \( \mu \) satisfies the distribution relations.
2. \( \gamma \mu = \mu \) for all diagonal matrices \( \gamma = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in T_2(\mathbb{Q}) \), where \( T_2(\mathbb{Q}) \subseteq \text{GL}_2(\mathbb{Q}) \) is the two dimensional torus, i.e. the diagonal invertible matrices with rational entries.

Proof. For (1), we can see from the definition that the distribution relations are satisfied.

For (2) we have:

\[
(\gamma \cdot \mu)(U)(X,Y) = \det(\gamma) \cdot \mu(U\gamma)(aX,dY) = H(E_{(U\gamma)_1}(aX) \cdot E_{(U\gamma)_2}(dY)) \cdot ad =
\]

\[= H(E_{aU_1}(aX) \cdot a \cdot E_{dU_2}(dY) \cdot d)\]

By 6.1.1 we get the desired result. \( \square \)
7. Modular Symbols

We denote by $\Delta_0 := \text{Div}_0(\mathbb{P}^1(\mathbb{Q}))$ the group of degree 0-divisors on the projective line $\mathbb{P}^1(\mathbb{Q})$. Since $\text{GL}_2(\mathbb{Q})$ acts on $\mathbb{P}^1(\mathbb{Q})$ by linear fractional transformation, we let also $\text{GL}_2(\mathbb{Q})$ act on $\Delta_0$ by the induced action.

**Definition 7.** For any subgroup $\Gamma \leq \text{GL}_2(\mathbb{Q})$ and for any (left) $\Gamma$-module $M$ we define the space of $M$-valued modular symbols for $\Gamma$ to be the set:

$$Symb_\Gamma(M) := \text{Hom}_\Gamma(\Delta, M) = \{\phi : \Delta_0 \longrightarrow M \mid \phi(\gamma D) = \gamma \cdot \phi(D) \ \forall \gamma \in \Gamma\}.$$

Define the matrices $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ respectively $R := \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. Let also the main divisor $D_\infty := \{i\infty\} - \{0\}$.

We have the following theorem:

**Theorem 7.1.** Let $\phi \in Symb_{\text{GL}_2(\mathbb{Q})}(M)$. Then

1. $\phi(D_\infty) \in M^{T_2(\mathbb{Q})}$.

2. For an element $m \in M^{T_2(\mathbb{Q})}$ there exists a modular symbol (unique) such that $\phi(D_\infty) = m$ if and only if $m$ satisfies the following conditions:
   (Man1): $m + S \cdot m = 0$
   (Man2): $m + R \cdot m + R^2 \cdot m = 0$.

**Proof.** This is a well known theorem (Stevens). We just sketch the proof to give a partial insight of the theorem.

The first assertion results from the fact that $T_2(\mathbb{Q})$ preserves $D_\infty$. Since $\phi$ is a modular symbol for $\text{GL}_2(\mathbb{Q})$, we have:

$$\phi(D_\infty) = \phi(\gamma D_\infty) = \gamma \cdot \phi(D_\infty),$$

for all $\gamma \in T_2(\mathbb{Q})$. So, indeed $\phi(D_\infty) \in M^{T_2(\mathbb{Q})}$.

For the second assertion, we should observe that the conditions (they are called The Manin Relations) are necessary since $SD_\infty = -D_\infty$ and $D_\infty, RD_\infty$ and $R^2D_\infty$ represent the edges of the modular triangle with vertices in $\infty, 0, 1$. By the linearity
of the modular symbols we get that the sum of any modular symbol over a closed path is 0 (it represent the nil divisor).

For the sufficiency, we can see that any divisor can be written as a finite sum of \( \text{GL}_2(\mathbb{Q}) \)-translates of \( D_\infty \). The fact that \( m \in M^{T_2(\mathbb{Q})} \) guarantees that there is no ambiguity in defining \( \phi(\gamma D_\infty) := \gamma \cdot m \) for all \( \gamma \in \text{GL}_2(\mathbb{Q}) \). The fact that the conditions are enough was proved by Manin.

We are now able to prove our main result Theorem 0.1.:

**Theorem 0.1.** We already know by Theorem 6.1. that our distribution \( \mu \) is invariant under the action of the torus, so \( \mu \in \mathcal{D}_{\text{naive}}(M, \mathcal{S})^{T_2(\mathbb{Q})} \). By Theorem 7.1. it is enough to prove that \( \mu \) satisfies the two Manin conditions (Man1) and (Man2).

(Man1): We have:

\[
(S\mu)(U) = H(E(U,2)(Y) \cdot E(U,1)(2X)).
\]

It is clear that \((US)_1 = U_2\) and \((US)_2 = -U_1\). We get:

\[
(S\mu)(U)(X, Y) = H(E(U,2)(Y) \cdot E(U,1)(2X)) = H(-E(U,2)(Y) \cdot E(U,1)(2X)) = -\mu(U).
\]

(Man2): The relation we need to prove is:

\[
\mu(U)(X, Y) + \mu(U, R)(Y, -X - Y) + \mu(U, R^2)(-X - Y, X) = 0, \quad \forall U
\]

One should notice that by our convention, we only need to prove that the left hand side is in \( \tilde{\mathcal{E}}_2 \).

Since the relation is linear in \( U \) is enough if we prove it for open compacts of the form \( \gamma + \alpha M_2(\mathbb{Z}) \). Moreover the invariance under the torus allows us to consider only open compacts of the form \( U = \gamma + M_2(\mathbb{Z}) \), \( \gamma \in M_2(\mathbb{Q}) \).

There are three situation that we need to consider:

a) \( \gamma \in M_2(\mathbb{Z}) \)

b) \( \gamma \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2 \) but \( \gamma \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{Z}^2 \).

c) Both \( \gamma \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \gamma \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) are nonintegral.

We will consider first the third case c). Let \( a := \gamma \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( b := \gamma \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). So we need to prove that:

\[
H(E_a(X)E_b(Y) + E_{-a-b}(-X - Y)E_b(Y) + E_{-a-b}(-X - Y)E_a(X)) \in \tilde{\mathcal{E}}_2.
\]
But the power series before taking the holomorphic projection is:
\[
\frac{1}{2}(E_a(X) + E_b(Y) + E_{-a-b}(-X - Y))^2 - \frac{1}{2}(E_a^2(X) + E_b^2(Y) + E_{-a-b}^2(-X - Y))
\]

The first parenthesis is in $\tilde{E}_2$ by Theorem 3.1. and by Observation 4.2.5. Each term in the second parenthesis has the holomorphic projection in $\tilde{E}_2$ by Theorem 4.3.

Next we consider the first case $a).$ So we may assume that $U = M_2(\mathbb{Z}).$ For a shortcut, we put $F(X) := E_{z^2}(\tau, X)$.

So we need to prove that $H(F(X)F(Y) + F(Y)F(-X - Y) + F(-X - Y)F(X)) \in \tilde{E}_2$. After taking limit in the formulae 3.1. and 5.5. we get that before taking the holomorphic projection the power series is:
\[
\frac{1}{2} \left( \text{Tay} \theta \phi(z, \tau) \right) - \frac{1}{2} \left( \text{Tay} \theta \phi(z, \tau) \right) + \text{Tay} \theta \phi(z, \tau) - \frac{1}{2} \left( \text{Tay} \theta \phi(z, \tau) \right)
\]

In the first parenthesis we recognize the positive terms of Taylor expansion of the Weierstrass $\phi$-function. The coefficients are exactly the holomorphic Eisenstein series of level 1.

The second parenthesis has also holomorphic projection in $\tilde{E}_2$ by 6.1.4 and 4.2. The third term is also a power series with Eisenstein series coefficients. We just have to notice that indeed the power series is integral in $X$ and $Y$ since $F(X)$ is odd in $X$, so the first parenthesis of this last term is divisible by $X$ and by $Y$ and by $X + Y$.

At last, the remaining case $b),$ take $\gamma(1) =: a \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ and $\gamma(0) = 0$ we need to prove that:
\[
H(E_a(X)F(Y) + F(Y)E_{-a}(-X - Y) + E_{-a}(-X - Y)E_a(X)) \in \tilde{E}_2
\]

As before we split it in
\[
\frac{1}{2} \left( E_a(X) + F(Y) + E_{-a}(-X - Y) \right)^2 - \frac{1}{2} \left( E_a^2(X) + F^2(Y) + E_{-a}^2(-X - Y) \right)
\]

The last parenthesis has holomorphic projection as before (Theorem 4.3. and 6.1.4). The first parenthesis can be written as:
\[
\text{Tay}_{\tau_a} \phi(\tau, X) + \text{Tay} \theta \phi(z, \tau) - \frac{1}{z^2} \left( \text{Tay} \theta \phi(z, \tau) \right) + \text{Tay}_{\tau-a} \phi(\tau, -X - Y) - \frac{2E_a(X) + F(Y) + E_{-a}(-X - Y)}{Y}
\]
Same arguments as above prove that also in this case, the holomorphic projection of our left hand side of the second Manin relation is in $\tilde{E}_2$.

We now prove Theorem 0.2, which is the specialization of our construction in the attempt to recover the construction of Borisov and Gunnells in [B-G1] and [B-G2].

**Theorem 0.2.** First of all, we should explain the relation between the language of Modular symbols and Manin symbols. (The last ones are used in [B-G1] and [B-G2]).

**Definition 8.** Let $\Gamma$ be a congruence subgroup and let $M$ be a trivial $\Gamma$-module. We define $\text{Man}_\Gamma(M)$ to be the set of all maps $f: \Gamma\backslash\SL_2(\mathbb{Z}) \rightarrow M$ such that

1. $f(\hat{\gamma}) + f(\hat{\gamma}S) = 0$
2. $f(\hat{\gamma}) + g(\hat{\gamma}R) + f(\hat{\gamma}R^2) = 0$.

The elements of $\text{Man}_\Gamma(M)$ are called $M$-valued Manin symbols for $\Gamma$.

**Theorem 7.2.** In the situation as above, we have an isomorphism:

$$\text{Symb}_\Gamma(M) \rightarrow \text{Man}_\Gamma(M)$$

$$\phi \rightarrow f_\phi(\hat{\gamma}) := \phi(\gamma D_\infty).$$

**Observation 7.3.** For $\Gamma := \Gamma_1(N)$ there exists a bijection

$$\Gamma_1(N)\backslash\SL_2(\mathbb{Z}) \rightarrow (\mathbb{Z}/N \times \mathbb{Z}/N)^\prime$$

$$\hat{\gamma} \rightarrow (0 1)\gamma,$$

where $(\mathbb{Z}/N \times \mathbb{Z}/N)^\prime$ is the set of primitive vectors mod $N$ (of exact order $N$), i.e. $a = (a_1 \ a_2)$ such that $\gcd(a_1, a_2, N) = 1$.

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \SL_2(\mathbb{Z})$ we have that $U_{1,\gamma}(N) := U_1(N) \cdot \gamma = \begin{pmatrix} 0 & 0 \\ c/N & d/N \end{pmatrix} + M_2(\mathbb{Z})$. Also notice that the projection of a power series to its $(k-2)$ homogenous part commutes with the action of $\SL_2(\mathbb{Z})$.

We have:

$$\Phi_{B-G}^{B_{k,N}}(\gamma D_\infty) = (\phi(\gamma D_\infty)(U_1(N)))^{(k-2)} = (\phi(D_\infty)(U_1(N)\gamma)((X \ Y)\gamma))^{(k-2)} =$$

$$= (\mu(U_{1,\gamma}(N))((X \ Y)\gamma))^{(k-2)} = H(E_{(c/N)}((X \ Y)\gamma \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \cdot E_{(d/N)}((X \ Y)\gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix}))^{(k-2)}.$$. 
For $k = 2$ we get:

$$
\Phi^B_G(\gamma D_\infty) = H(E_{1, (c/N)} \cdot E_{1, (d/N)}) = H(s_{c/N} \cdot s_{d/N}),
$$

where $s_{c/N}$ is $E_{1, (c/N)}$ in the notation of [B-G1].

Notice that the weight 1 Eisenstein series are holomorphic, therefore the holomorphic projection $H$ doesn’t change anything and that $E_{1, (0/N)} = 0$. So, $f_{\Phi^B_G((c \ d))} = s_{c/N} \cdot s_{d/N}$ for $c \neq 0$ and $d \neq 0$ and $f_{\Phi^B_G((c \ d))} = 0$ otherwise. This is exactly the construction in [B-G1].

For the higher weights, a similar analysis shows that the specialization of our modular symbol viewed as a Manin symbol coincide with the construction in [B-G2]. □

8. SOME IDENTITIES

We make some observation concerning the relations involving Eisenstein series for the full modular group.

**Definition 9.** We denote by $E_k$ to be the unique Eisenstein series for $SL_2(\mathbb{Z})$ of weight $k$. Therefore:

$$
E_k(\tau) := \phi_{k, \mathbb{Z}^2}(0, \tau) = \begin{cases} 
0 & \text{if } k \notin 2\mathbb{Z}^+ \\
\frac{2(2\pi i)^k}{(k-1)!} \left( -\frac{B_k}{2k} + \sum_{n \geq 1} \sigma_{k-1}(n)q^n \right) & \text{if } k \in 2\mathbb{Z}^+ \setminus \{2\} \\
\frac{2(2\pi i)^2}{4\pi^2(\tau - \bar{\tau})} - \frac{1}{24} + \sum_{n \geq 1} \sigma_1(n)q^n & \text{if } k = 2
\end{cases}
$$

Also, let as before $F(\tau, X) := E_{2\mathbb{Z}^2}(\tau, X)$.

We have the following:

**Proposition 8.1.** We have $F(X) = \sum (-1)^n E_{n+1} \cdot X^n$. The identity 6.1.4 can be rewritten as:

$$
\sum_{i+j=n} E_{i+1} \cdot E_{j+1} = (n+3) \cdot E_{n+2} - 4\pi i \frac{\delta_n}{n} E_n,
$$

for all $n \geq 2$.

**Corollary 8.2.** We have the following identities:

1. $5E_4 = E_2^2 + 4\pi i \frac{\delta_2}{2} E_2$. In particular:

$$
5\sigma_3(n) = (6n - 1)\sigma_1(n) + 12 \cdot \sum_{i+j=n} \sigma_1(i) \cdot \sigma_1(j)
$$
(2) $7E_6 = 2E_2E_4 + 4\pi i \frac{\delta_4}{\delta_6} E_4$. In particular:

$$21\sigma_5(n) = 10(3n - 1)\sigma_3(n) + 240 \cdot \sum_{i+j=n} \sigma_1(i) \cdot \sigma_3(j)$$

(3) $9E_8 = 2E_2E_6 + E_4^2 + 4\pi i \frac{\delta_6}{\delta_8} E_6$. In particular:

$$9\sigma_7(n) = 7(2n - 1)\sigma_5(n) + \frac{1}{15} (14\sigma_3(n) - 5\sigma_1(n)) + 56 \cdot \sum_{i+j=n} (3\sigma_1(i) \cdot \sigma_5(j) + 5\sigma_3(i) \cdot \sigma_3(j))$$

Another corollary is the following:

**Corollary 8.3.** The space of modular form for $\text{SL}_2(\mathbb{Z})$ is generated by $E_2$ in the sense that any modular form can be written uniquely as a "weighted-homogeneous" polynomial in the weight-liftings of $E_2$, $\delta^r E_2$.

Note that $P(X_0, X_1, \ldots, X_r)$ is considered weighted-homogeneous if $P(\alpha X_0, \alpha^2 X_1, \ldots, \alpha^{r+1} X_r) = \alpha^{mr} \cdot P(X_0, X_1, \ldots, X_r)$, for all $\alpha$ and where $m$ is the total degree of the polynomial which is $\deg_{X_0}(P)$. The proof of this corollary is obvious since we wrote down the formulae for $E_4$ and $E_6$. Since the space of modular forms for $\text{SL}_2(\mathbb{Z})$ is isomorphic with the space of polynomials in $E_4$ and $E_6$, we get our result.

As a remark, we prove in [Pa-Sp] that we can generate the space of modular forms for $\text{SL}_2(\mathbb{Z})$ just with linear combinations of products of Eisenstein series $E_i E_j$, and Eisenstein series $E_k$ for $i, j, k \geq 4$. In particular, we can write formulae similar with the the formulae in Corollary 8.2., without involving $\sigma_1(n)$. However, the above formulae have the advantage of being "closed".

Also, in the proof of the Manin relations for Theorem 0.1., we used a series of identities that in turn will lead us to deduce new identities involving divisor functions, and divisor functions with character. We exhibit here just one of them for the sake of keeping the paper compact. The formula used to prove the second Manin relation for the case $a)$ gives us:

**Proposition 8.4.** Let $P_n(X, Y) := X^n + Y^n + (-X - Y)^n$. Then:

$$\sum_{i+j=n} P_i(X,Y)P_j(X,Y)E_{i+1}E_{j+1} = \left( (n+1)P_n(X, Y) - \frac{P_{n+1}(X,Y)P_2(X,Y)}{XY(X+Y)} \right) E_{n+2},$$

for all $n > 0$. 

Note that this relations are "holomorphic" in the sense that all the terms involving $E_2$ don’t actually appear since $P_1(X,Y) = 0$.

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