BOUNDS ON THE VOLUME ENTROPY AND SIMPLICIAL VOLUME IN RICCI CURVATURE $L^p$-BOUNDED FROM BELOW

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Abstract. Let $(M,g)$ be a compact manifold with Ricci curvature almost bounded from below and $\pi: \bar{M} \to M$ be a normal, Riemannian cover. We show that, for any nonnegative function $f$ on $M$, the means of $f \circ \pi$ on the geodesic balls of $\bar{M}$ are comparable to the mean of $f$ on $M$. Combined with logarithmic volume estimates, this implies bounds on several topological invariants (volume entropy, simplicial volume, first Betti number and presentations of the fundamental group) in Ricci curvature $L^p$-bounded from below.

1. Introduction

We say that a compact manifold has Ricci curvature $L^p$ bounded from below by $k$ (denoted $R.L^p.b.$ subsequently) when the following quantity is small

$$\|\rho_k\|_p = \left( \frac{1}{\Vol M} \int_M \rho_k^p \right)^{\frac{1}{p}},$$

where $k$ is a real, $\rho_k = (\text{Ric} - k(n - 1))_-$, $\text{Ric}(x)$ is the least eigenvalue of the Ricci tensor $\text{Ric}$ at $x$ and $f_- = \max(-f,0)$. Our purpose is to study some topological constraints under such control of the curvature.

On manifolds whose Ricci curvature is bounded from below in the usual sense, topological bounds are often derived from geometric comparison theorems on the volume of geodesic balls or spheres applied to some Riemannian covers (see for instance the proofs of the bounds on the first betti number $\beta_1$ or the simplicial volume due to M. Gromov [5, 4] or the bounds on the fundamental group due to M. Anderson [1]).

There are many extensions of the classical volume comparison theorems to manifolds with $R.L^p.b.$ (see for instance [3, 11, 8, 9, 7, 2]), unfortunately, since these $L^p$ lower bounds on the Ricci curvature were not known to be preserved under non-finite Riemannian coverings, these volume estimates led to somewhat unsatisfactory topological bounds (for instance, the simplicial volume bound $\lambda$ a la Gromov in [3] depends also on a Ricci $L^p$ lower bound of the universal covering).

S. Gallot developped in [3] an alternative approach based on M"{o}ser iterations to bound the particular class of harmonic topological invariants (which are those which, as the Betti numbers or the $A$-genus, are associated to the spectral multiplicities of natural Schr"{o}dinger operators). Since these bounds depend only on some Sobolev constant of the manifold itself, the technic does not rely on a control of the Ricci curvature on a covering space. These bounds were, until [2], the only topological bound known on manifolds with $R.L^p.b.$.

In [2], we developed a technic based on the construction of Dirichlet domains with multiplicity in the universal cover (see Section 2 of this paper for more detail) and volume comparison results for star-shaped domains, to extend the Myers’s Finiteness of the fundamental group on manifolds with almost positive Ricci curvature. This was the first (nonharmonic) bound on the topology of a manifold with $R.L^p.b.$ Since, Z. Hu and S.

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Xu [6] have used the same technic to get some extensions of the Anderson’s bound on the presentation of the \( \pi_1 \) and of the Milnor’s polynomial growth of the \( \pi_1 \). They also recover the bound on the first betti number due to S. Gallot in R.Lp.b. [3] by using the schema of proof due to M. Gromov in Ricci curvature bounded from below [5].

Actually, the technic developped in [2] implies the following lemma, where \( \pi : (\tilde{M}, \tilde{g}) \to (M, g) \) is a normal Riemannian cover and \( A_k(r) \) is the volume of a geodesic ball of radius \( r \) on a space form of sectional curvature \( k \).

**Lemma 1.1.** Let \( n \geq 2 \) be an integer and \( p > n/2 \) be a real. There exists a constant \( \zeta(p, n) > 0 \) such that if \((M^n, g)\) satisfies \( \text{Diam} M \leq D \leq R/3 \) and \( D^2 \| \rho_k \|_p \leq \left( \frac{A_k(D)}{A_k(2D)} \right)^{\frac{p}{p-1}} \zeta(p, n) \) (for some \( k \leq 0 \) and \( R > 0 \)) then for any nonegative function \( f \) on \( M \) and any \( x \in \tilde{M} \), we have that

\[
\frac{A_k(D)}{2A_k(2D)} \times \frac{1}{\text{Vol} M} \int_M f \leq \frac{1}{\text{Vol} B(\tilde{x}, R)} \int_{B(\tilde{x}, R)} f \circ \pi \leq \frac{2A_k(2D)}{A_k(D)} \times \frac{1}{\text{Vol} M} \int_M f.
\]

Applied to \( f = \rho_k \), it implies that any normal Riemannian cover of a compact manifold with R.Lp.b. is also with R.Lp.b. (where in the noncompact case, we say that \((M^n, g)\) is R.Lp.b. if for a given \( R \), \( \sup_{x \in M} \int_{B(x, R)} \rho_k^p / \text{Vol} B(x, R) \) is small). Lemma 1.1, combined with a some bounds on the exponential growth of the volume of balls in R.Lp.b. (see Theorem 3.2), allows to extend the Gromov’s volume entropy and simplicial volume bounds to manifold with R.Lp.b.

1.1. **Bounds on the volume entropy.** For any manifold \((M^n, g)\), we note \((\tilde{M}, \tilde{g})\) the Riemannian universal cover. The volume entropy of \((M^n, g)\) is defined by

\[
\text{Ent}(M) = \sup_{\tilde{x} \in \tilde{M}} \inf_{R > 0} \frac{\text{Vol} S(\tilde{x}, R)}{\text{Vol} B(\tilde{x}, R)}
\]

We set \( A_k(R) \) (resp. \( L_k(R) \)) the volume of a geodesic ball (resp. a geodesic sphere) of radius \( R \) in a space form of constant curvature \( k \).

**Theorem 1.2.** Let \( n \geq 2 \) be an integer and \( p > n/2 \), \( k \leq 0 \) be some reals. There exist constant \( \zeta(p, n) > 0 \) and \( C(p, n) > 0 \) such that if \((M^n, g)\) satisfies \( \text{Diam}(M) \leq D \) and \( D^2 \| \rho_k \|_p \leq \left( \frac{A_k(D)}{A_k(2D)} \right)^{\frac{p}{p-1}} \zeta(p, n) \) then for any \( k' \leq 0 \), we have that

\[
\text{Ent}(M) \leq \inf_{R \geq 3D} \frac{L_k(R)}{A_k(R)} \left( 1 + C(p, n) \left( \frac{2A_k(2D)}{A_k(D)} \right)^{\frac{1}{p-1}} \left( R^2 \| \rho_k \|_p \right)^{\frac{p}{2p-1}} \right),
\]

where \( C(p, n) = \left( \frac{2p-1}{2(p-n)^p/((2p)^{p-1})} \right)^{\frac{p}{2p-1}} \).

In the particular case \( k' = 0 \), we can compute the minimizing \( R \) and get the following corollary.

**Corollary 1.3.** Let \( n \geq 2 \) be an integer and \( p > n/2 \), \( k \leq 0 \) be some reals. There exist some constants \( \zeta(p, n) > 0 \) and \( C(p, n) > 0 \) such that if \((M^n, g)\) satisfies \( \text{Diam}(M) \leq D \) and \( D^2 \| \rho_k \|_p \leq \left( \frac{A_k(D)}{A_k(2D)} \right)^{\frac{p}{p-1}} \zeta(p, n) \) then

\[
\text{Ent}(M) \leq C(p, n) \left( \frac{2A_k(2D)}{A_k(D)} \right)^{\frac{1}{p}} \left( \frac{1}{\text{Vol} M} \int_M \text{Ric}^p \right)^{\frac{1}{p}},
\]

where \( C(p, n) = (n-1)^{\frac{p}{2p-1}} \left( \frac{2p-1}{(2p-n)^p/((2p)^{p-1})} \right)^{\frac{p}{2p-1}} \left( (2p-1)^{\frac{2}{p}} + (2p-1)^{\frac{1}{p}} \right) \).

**Remark 1.4.** The constant \( C(p, n) \) tends to \( \sqrt{n-1} \) when \( p \) tends to \( +\infty \), and so we recover the usual bound \( \text{Ent}(M) \leq (n-1)^{\sqrt{-k}} \) when \( \text{Ric} \geq k(n-1) \).
In the case $k' = k$ we have the following corollary.

**Corollary 1.5.** Let $n \geq 2$ be an integer and $p > n/2$, $k \leq 0$ and $\epsilon > 0$ be some reals. There exists a constant $\zeta(p, n, k, \epsilon) > 0$ such that if $(M^n, g)$ satisfies $\text{Diam}(M)^2 \|\rho_k\|_p \leq \zeta(p, n, k, \epsilon)$ then we have that

$$
\text{Ent}(M) \leq (n - 1)(\sqrt{-k} + \epsilon).
$$

**Remark 1.6.** Note that our assumption is on the Ricci curvature of $M$ itself and not on the Ricci curvature of the universal cover, as it is the case in the Gallot’s bound on the Volume entropy [3].

1.2. Bounds on the simplicial volume. Let $c$ a closed $l$-chain of $M$ and $||[c]|| = \inf \{\sum_i |\alpha_i|/|\sum_i \alpha_i c_i| = [c]\}$, where the $c_i$ are elementary simplices. When $c$ is the fundamental class of $M$ then $||[c]||$ is called the simplicial volume of $M$ and denoted $\text{Vol}_s(M)$.

The bounds on the simplicial volume or norms of homology-classes by the the volume entropy due to M. Gromov [4] give the following corollaries.

**Corollary 1.7.** Let $n \geq 2$ be an integer and $p > n/2$, $k \leq 0$ be some reals. There exist some constants $\zeta(p, n) > 0$ and $C(p, n) > 0$ such that if $(M^n, g)$ satisfies $\text{Diam}(M) \leq D$ and $D^2 \|\rho_k\|_p \leq \left(\frac{A_k(D)}{A_k(2D)}\right)^{2p-1} \zeta(p, n)$ then

$$
\text{Vol}_s(M) \leq C(p, n) \left(\frac{2A_k(2D)}{A_k(D)}\right)^{\frac{2}{n}} \left(\frac{1}{\text{Vol} M} \int_M (\text{Ric}^p)^{\frac{2}{n}} \text{Vol}(M),
$$

and for any closed $l$-form we have that

$$
||[c]|| \leq C(p, n) \left(\frac{2A_k(2D)}{A_k(D)}\right)^{\frac{2}{n}} \left(\frac{1}{\text{Vol} M} \int_M (\text{Ric}^p)^{\frac{2}{n}} \text{Vol}_l(c).
$$

**Corollary 1.8.** Let $n \geq 2$ be an integer and $p > n/2$, $k \leq 0$ and $\epsilon > 0$ be some reals. There exists a constant $\zeta(p, n, k, \epsilon) > 0$ such that if $(M^n, g)$ satisfies $\text{Diam}(M)^2 \|\rho_k\|_p \leq \zeta(p, n, k, \epsilon)$ then we have that

$$
\text{Vol}_s(M) \leq \left(\frac{\Gamma(n)}{\sqrt{\pi} \Gamma\left(\frac{n+1}{2}\right)}\right)^n n!(n - 1)^n (\sqrt{-k} + \epsilon)^n \text{Vol}(M)
$$

and for any closed $l$-form we have that

$$
||[c]|| \leq \ell! (n - 1)^l (\sqrt{-k} + \epsilon)^l \text{Vol}_l(c).
$$

1.3. Ollier bounds on the topology. Using Lemma 1.1, we can also prove the following generalization of a result due to M. Anderson.

**Theorem 1.9.** Let $n \geq 2$ be an integer and $p > n/2$, $k \leq 0$, and $D, V > 0$ be some reals. There exist some constants $\zeta(p, n, k, D, V) > 0$, $L(p, n, k, D, V) > 0$ and $N(p, n, k, D, V)$ such that if $(M^n, g)$ satisfies $\text{Diam}(M) \leq D$, $\text{Vol}(M) \geq V$ and $\int_M \rho_k^p \leq \zeta(p, n, k, D, V)$ then any subgroup $\Gamma \subset \pi_1(M)$ generated by elements of length less than $L(p, n, k, D, V)$ has order less than $N(p, n, k, D, V)$.

Note that for $k > 0$, we show in [2] that if $\|\rho_k\|_p \leq \zeta(p, n, k)$ then $\text{Vol}(M) \leq 2 \text{Vol}\mathbb{S}^n$ and $\text{Diam}(M) \leq 2\pi$ and so these bounds on $\pi_1(M)$ become trivial in the case $k > 0$ and do not need an a priori bound on the diameter.

**Remark 1.10.** When $(M^n, g)$ is a manifold with $\text{Ric}^p$, then Lemma 1.1 gives some $L^p$ lower bounds on the Ricci curvature of the Riemannian, normal covers of $M$. However, in the extensions of the Bishop-Gromov theorem in $L^p$ quoted above we need to have

$$
\frac{1}{\text{Vol} B(x, R)} \int_{B(x, R)} (\text{Ric} - k)^p \leq C(p, n) R^{-2p}
$$
in order to bound $\text{Vol} B(x, R)/\text{Vol} B(x, r)$ from above (for any $r \leq R$). So, extensions in $\text{R.Lp.b}$ of topological bounds such as the Gromov’s bound on the first Betti number or the Anderson’s bounds on the fundamental group, where the Bishop-Gromov estimate is only needed for balls of radius comparable to the diameter of $M$, more or less readily follow from Lemma 1.1 and the already known estimates on the volume in $\text{R.Lp.b}$. On the contrary, when we need apply the Bishop-Gromov comparison on volume to some balls of arbitrary large radius (as in the Milnor’s polynomial growth of the fundamental group in nonnegative Ricci curvature or to get an optimal bound on the volume entropy as in Corollary 1.5) the volume estimates of [8, 9, 7, 2] cannot apply under the assumption $D^2\|\rho_k\|_p \leq C(p, n)$. To avoid this problem in their extension of the Milnor’s polynomial growth in [6], the authors assume a supplementary lower bound on the volume. But this is a strong assumption since then it only remains a finite number of possible $\pi_1(M)$, which gives a bound on the radius of the balls we have to consider in the universal cover. In this paper, we prove a logarithmic estimate on the volume of balls (see theorem 3.1) which apply for balls of arbitrary large radii.

1.4. Precompactness results. Eventually, as a by product of our logarithmic volume estimate, we get the following result.

**Theorem 1.11.** Let $n \geq 2$ be an integer, $p > n/2$, $D > 0$ be some reals and $f$ be any positive, locally bounded function on $[0, +\infty[$. The set of compact manifolds which satisfy for any $R > 0$

$$\sup_{x \in M} \int_{B(x, R)} (\text{Ric})^p \leq f(R), \quad \text{and} \quad \text{Diam}(M) \leq D$$

is precompact for the Gromov-Hausdorff distance.

**Remark 1.12.** Contrarily to the precompactness result of [9] we do not have to suppose $f$ sufficiently small. Of course this result is more interesting when $f$ is chosen non bounded near $0$ and $(\text{Ric})_-$ can be replaced by $\rho_k$.

The same result holds for complete manifolds and the pointed Gromov-Hausdorff topology.

**Theorem 1.13.** Let $n \geq 2$ be an integer, $p > n/2$ be a real and $f$ be any positive, locally bounded function on $[0, +\infty[$. If $(M_i, x_i, g_i)$ is a pointed sequence of manifolds which satisfy

$$\int_{B(x, R)} (\text{Ric})^p \leq f(R),$$

for any $R > 0$, then a subsequence converges to a length space in the pointed Gromov-Hausdorff topology.

Combined with Lemma 1.1, we get the following precompactness result.

**Theorem 1.14.** Let $n \geq 2$ be an integer, $k$ and $p > n/2$ be some reals. There exists a constant $\zeta(p, n, k) > 0$ such that if $(M_i, g_i)_{i \in \mathbb{N}}$ is a sequence of manifolds which satisfy

$$\int_M \rho_k^p \leq \zeta(p, n, k), \quad \pi_i : (\overline{M_i}, \bar{g}_i) \rightarrow (M_i, g_i)$$

is a normal, Riemannian cover and $\bar{x}_i \in \overline{M_i}$ for any $i \in \mathbb{N}$, then $(\overline{M_i}, \bar{x}_i, \bar{g}_i)$ admits a convergent subsequence for the Gromov-Hausdorff pointed topology.

As noticed in [9], the Bishop-Gromov inequality used below to prove theorem 1.11, can also used to get the following theorem.

**Theorem 1.15.** Let $n \geq 2$ be an integer, $p > n/2$, $\nu, \Lambda, D > 0$ be some reals and $f : [0, +\infty[ \rightarrow [0, +\infty[$ be any continuous function. The set of compact manifolds which
satisfy for any $R > 0$

$$\sup_{x \in M} \int_{B(x,R)} \frac{|\text{Ric}|^p}{\text{Vol} B(x,R)} \leq f(R), \quad \text{Diam}(M) \leq D, \quad \text{Vol} M \geq v,$$

$$\int_M \|R\|^p \leq \Lambda.$$ forms a precompact set in the $C^\alpha$ topology, for any $\alpha < 2 - \frac{n}{p}$.

1.5. Remarks about our assumptions. In [3], S.Gallot constructs (for any $p > n/2$) a sequence $(M_k, g_k)$ of manifolds (example A.3 of the appendix) which satisfies

$$\text{Diam}(g_k) \leq D, \quad D^2\|\rho_0(g_k)\|_p \leq C(p, n), \quad \text{Vol}_p(M_k) \rightarrow +\infty, \quad b_1(M_k) \rightarrow +\infty.$$ Hence, if $D^2\|\rho_v\|_p$ is not smaller than an universal constant, we can not bound the first Betti number, the volume entropy nor the simplicial volume.

Similary, Examples A.2. of [3] shows that Theorems 1.2, 1.9 and Corollaries 1.3 and 1.5 are not valid for $p \leq n/2$ (even if we replace $\|\rho_0\|_{n/2}$ by $\int_M |\text{Ric}|^{n/2}$). Eventually, we have proved in [2] (Prop 9.3) that any compact manifold admit a metric with $\|\rho_0\|_{n/2}$ as small as we want, and so we infer a bound on the simplicial volume from a pinching on $\|\rho_0\|_{n/2}$. Note that a still open conjecture due to M.Gromov is that $\text{Vol}_p(M)$ should be bounded by a function of $\int_M |\text{Ric}|^{n/2}$.

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2. PROOF OF LEMMA 1.1

The proof of Lemma 1.1 relies on geometric group action arguments and some volume estimates in Ricci curvature $L^p$ bounded from below. We first recall some notations and results of [2].

Notations. We denote by $U_x$ the injectivity domain at $x \in M$ and we identify points of $U_x \setminus \{0_x\}$ with their polar coordinates $(r, v)$ in $\mathbb{R}_+^n \times S_z^{n-1}$ (where $S_z^{n-1}$ is the set of normal vectors at $x$). We note $d_{r,v}$ the Riemannian measure and set $\exp_x^* v_g = \theta(r, v) dr dv$, where $dr$ and $dv$ are the canonical measures of $S_z^{n-1}$ and $\mathbb{R}_+^n$, and we extend $\theta$ by $0$ to $\mathbb{R}_+^n \times S_z^{n-1}$.

We denote by $h(r, v)$ the mean curvature at $\exp_x^* (r v)$ of the sphere centered at $x$ and of radius $r$ and we set $\psi(r, v) = \left(\frac{(n-1)\sqrt{\text{cosh}(\sqrt{r^2 - k})} - h(r, v)}{\sinh(\sqrt{r^2 - k})}\right)_+$. Given $T$, a subset of $M$ star-shaped at $x$, let $A_T(r)$ be the volume of $B(x, r) \cap T$ and $L_T(r)$ be the $n-1$-volume of $(rS_z^{n-1}) \cap U_x \cap T_x$. We have $L_T(r) = \int_0^r S_z^{n-1} \mathbb{I}_T \theta(r, v) dv$ and $A_T(r) = \int_0^r L_T(t) dt$. We set $L_k$, $A_k$ the corresponding functions on the space form of sectional curvature $k$ and $s_k(t) = \sinh(\sqrt{-kt})/\sqrt{-k}$ if $k < 0$, $s_0(t) = t$, $s_k(t) = \sin(\sqrt{kt})/\sqrt{k}$ if $k > 0$.

Lemma 2.1. $L_T$ is a right continuous, left lower semi-continuous function. $A_T$ is a continuous, right differentiable function of derivative $L_T$. Moreover, the function

$$f(r) = \frac{L_T(r)}{s_k(r)^{n-1}} - \int_0^r \int_{S_z^{n-1}} \frac{\mathbb{I}_T \psi(r, \theta) dv}{s_k(s)^{n-1}} ds$$

is decreasing on $\mathbb{R}_+^n$.

Moreover, the mean curvature can be controlled by integral of the Ricci curvature thanks to the following Lemma (see Lemma 4.1 in [2]).

Lemma 2.2. Let $p > n/2$, $r > 0$ be some reals. We have

$$\psi_k^{2p-1}(r, v) \theta(r, v) \leq (2p-1)^p \left(\frac{n-1}{2p-n}\right)^{p-1} \int_0^r \rho_k^p(t, v) \theta(t, v) dt$$

for all normal vector $v \in S_z^{n-1}$. Note that this inequality holds with $\theta$ replaced by $\mathbb{I}_{[0,s]} \theta$. 
Remark 2.3. Lemma 2.2 holds in dimension 2 with $p = 1$ and the constant before $\int_0^r \rho_k^p \theta$ equal to 1.

Proof of Lemma 1.1. We set $s_k(t) = \frac{\sinh(\sqrt{k}t)}{\sqrt{k}}$ for $k < 0$ and $s_0(t) = t$. By geometric action group arguments, we prove in [2] that there exists a subset $T \subset M$ such that

- $B(\bar{x}, R) \subset T \subset B(\bar{x}, R + D)$,
- $T$ is star shaped at $\bar{x}$,
- if $T$ is the deck transformation group of $\pi : M \to T$, then $x \mapsto \#(T \cap \pi^{-1}(x))$ is constant on $M$, and so we have for any function $f$ on $M$
\[
\frac{1}{\text{Vol} T} \int_T (f \circ \pi) \, dv_g = \frac{1}{\text{Vol} M} \int_M f \, dv_g.
\]

By Lemma 2.1, we have for any $R - D \leq t \leq r \leq R + D$, that
\[
s_k^{n-1}(t)L_T(r) \leq s_k^{n-1}(r)L_T(t) + s_k^{n-1}(r) \int_t^r \int_{S^2_{2k}} \mathbb{1}_{T_k} \psi \, dv \, ds
\]
\[
\leq s_k^{n-1}(r)L_T(t) + s_k^{n-1}(r)(A_T(r) - A_T(t))\frac{n-1}{2(n-1)} \left( \int_t^r \int_{S^2_{2k}} \mathbb{1}_{T_k} \psi \, dv \, ds \right)^{\frac{1}{n-1}}
\]
\[
\leq s_k^{n-1}(r)L_T(t) + C_1(p, n)s_k^{n-1}(r)A_T(R + D)D^{\frac{n}{2(n-1)}} \left( \int_T \rho_k^p \frac{dv}{\text{Vol} T} \right)^{\frac{1}{n-1}},
\]
where we have used the Hölder inequality and Lemma 2.2, and where $C_1(p, n)$ is equal to $\left( \frac{2(2p-1)(n-1)^{p-1}}{(2p-1)(n-1)} \right)^{\frac{1}{n-1}}$. By integration of this inequality between $R - D$ and $R$ with respect to $t$ and between $R$ and $R + D$ with respect to $r$, we get that
\[
(A_T(R + D) - A_T(R)) \int_{R-D}^R s_k^{n-1} \leq A_T(R) \int_{R}^{R+D} s_k^{n-1} + C_1(p, n)A_T(R + D)(D^2\|\rho_k\|_p)^{\frac{p}{2(n-1)}} \int_{R}^{R+D} s_k^{n-1},
\]
which implies that
\[
\left( \int_{R-D}^R s_k^{n-1} - C_1(p, n)(D^2\|\rho_k\|_p)^{\frac{p}{2(n-1)}} \int_{R}^{R+D} s_k^{n-1} \right) \frac{A_T(R + D)}{A_T(R)} \leq \int_{R-D}^R s_k^{n-1}.
\]
So there exists $\zeta(p, n)$ such that if $D^2\|\rho_k\|_p \leq \zeta(p, n)\left[ \frac{A_T(D)}{A_k(2D)} \right]^{\frac{p}{2(n-1)}}$ then $A_T(R + D) / A_T(R) \leq \frac{2A_k(2D)}{A_k(D)}$. But we have $A_T(R + D) = \text{Vol} T$ and $A_T(R) = \text{Vol} B(\bar{x}, R)$, and so
\[
\frac{1}{\text{Vol} B(\bar{x}, R)} \int_{B(\bar{x}, R)} (f \circ \pi) \leq \frac{A_T(R + D)}{A_T(R)} \int_T (f \circ \pi) \frac{dv}{\text{Vol} T} \leq \frac{2A_k(2D)}{A_k(D)} \int_{M} (f \circ \pi) \frac{dv}{\text{Vol} M}.
\]
To prove the inverse estimate, just consider $T'$ the star shaped subset associated to $B(\bar{x}, R - D)$. Then we have
\[
\frac{1}{\text{Vol} B(\bar{x}, R)} \int_{B(\bar{x}, R)} (f \circ \pi) \geq \frac{f_{T'}(f \circ \pi) \text{Vol} B(\bar{x}, R - D)}{\text{Vol} T'} \frac{\text{Vol} T}{\text{Vol} B(\bar{x}, R)}
\]
and the quotient $\frac{\text{Vol} B(\bar{x}, R - D)}{\text{Vol} B(\bar{x}, R)}$ can be bounded from below as above.

3. Logarithmic bounds on the volume of geodesic balls

We establish a new bound on the volume in $R.Lp.b.$ which, contrarily to other volume estimates in $R.Lp.b.$, does not require any smallness of $\|\rho_k\|_p$ to apply.
Theorem 3.1. Let \( n \geq 2 \) be an integer and \( p > n/2, \) \( k \leq 0 \) be some reals. There exists a constant \( C(p, n) > 0 \) such that for any \((M^n, g), \) any \( x \in M\) and any \( R > 0, \) we have

\[
\frac{Vol S(x, R)}{Vol B(x, R)} \leq \frac{C(p)}{A_k(R)} \left( 1 + C(p, n) \left( R^p \left\| \frac{r}{\rho} \right\|_{p, R}^{p} \right) \frac{1}{R^p} \right).
\]

Proof. We set \( \| \rho_k \|_{p, R} = \frac{\int_{B(x, R)} r^p}{\int_{B(x, R)} r^{p+1}}. \) As in the proof of Lemma 1.1, for any \( t \leq r \) we have that

\[
s^k_t^{-1}(t) L_T(r) \leq s^k_t^{-1}(r) L_T(t) + C(p, n) \frac{1}{r^{p+1}} s^k_t^{-1}(r) A_T(r) \frac{1}{r^{p+1}} \| \rho_k \|_{p, R}^{p} \frac{1}{R^p},
\]

where \( C(p, n) = (2p-1)C(p) \left( \frac{u-1}{2p-n} \right)^{p-1}. \) Integrating with respect to \( t \) between 0 and \( r, \) it gives us that

\[
\frac{L_T(r)}{A_T(r)} \leq \frac{L_k(r)}{A_k(r)} \left( 1 + C(p, n) \frac{1}{R^p} \right).
\]

To prove Theorems 1.1 and 1.13, we just have to use \( s^k_t \| \rho_0 \|_{p, s} \leq R^2 \sup_{[r, R]} f \) for any \( s \in [r, R] \) and integrate the inequality given by Theorem 3.1 between \( r \) and \( R, \) wich gives

\[
\sup_{x \in M} \frac{Vol B(x, R)}{Vol B(x, r)} \leq \Phi(n, r, R)
\]

and so implies the result by [5].

By Lemma 1.1 we get the following corollary

Corollary 3.2. Let \( n \geq 2 \) be an integer and \( p > n/2, \) \( k \leq 0 \) be some reals. There exist some constants \( \zeta(p, n) > 0 \) and \( C(p, n) > 0 \) such that if \((M^n, g)\) satisfies \( \text{Diam}(M) \leq D \) and \( D^2 \| \rho_k \|_{p} \leq \left( \frac{A_k(D)}{A_k(2D)} \right) \frac{2p-1}{p} \zeta(p, n) \) then for any normal cover \( \overline{M} \to M, \) any \( \hat{x} \in \overline{M}, \) any \( R > 3D \) and any \( k' \leq 0, \) we have that

\[
\frac{Vol S(\hat{x}, R)}{Vol B(\hat{x}, R)} \leq \frac{L_{k'}(R)}{A_{k'}(R)} \left( 1 + C(p, n) \left( \frac{2A_k(2D)}{A_k(D)} \right) \frac{1}{R^p} \right),
\]

Theorem 1.2 and its corollaries follow readily from Corollary 3.2. The bounds on the cohomology norms and simplicial volume follow from the following result

Theorem 3.3 (Gromov[5]).

\[
\| [c] \| \leq \ell (\text{Ent}(M))^t \text{Vol}_t(c),
\]

\[
\text{Vol}_t(M) = \| [M] \| \leq \left( \frac{\Gamma(\frac{3}{2})}{\sqrt{\pi} \Gamma(\frac{1}{2})} \right)^n \text{Vol}(\text{Ent}(M))^n \text{Vol}(M).
\]

Integrating the inequality of Corollary 3.2 with respect to \( R \) we get the following corollary.

Corollary 3.4. Let \( n \geq 2 \) be an integer and \( p > n/2, \) \( k \leq 0 \) be some reals. There exist some constants \( \zeta(p, n) > 0 \) and \( C(p, n) > 0 \) such that if \((M^n, g)\) satisfies \( \text{Diam}(M) \leq D \) and \( D^2 \| \rho_k \|_{p} \leq \left( \frac{A_k(D)}{A_k(2D)} \right) \frac{2p-1}{p} \zeta(p, n) \) then for any normal cover \( \overline{M} \to M, \) for any \( \hat{x} \in \overline{M}, \) for any \( R \geq r > 3D \) and any \( k' \leq 0, \) we have that

\[
\frac{Vol B(\hat{x}, R)}{Vol B(\hat{x}, r)} \leq \frac{A_{k'}(R)}{A_{k'}(r)} \left( 1 + C(p, n) \left( \frac{2A_k(2D)}{A_k(D)} \right) \frac{1}{R^p} \right),
\]

Eventually, combining Lemma 1.1 with the Bishop-Gromov inequalities of [2] or [9], we get the following corollary, which, combined with the previous corollary implies Theorem 1.14.
Corollary 3.5. Let \( n \geq 2 \) be an integer and \( p > n/2 \), \( k \leq 0 \), \( \alpha > 1 \) be some reals. There exist some constants \( \zeta(\alpha,p,n) > 0 \) and \( C(\alpha,p,n) > 0 \) such that if \( (M^n, g) \) satisfies \( \text{Diam}(M) \leq \delta \) and \( \delta^2 \| p_k \|_p \leq \zeta(\alpha,p,n) \) then for any normal cover \( \tilde{M} \to M \), for any \( \tilde{x} \in \tilde{M} \), for any \( \alpha D \geq R \geq \delta > 0 \), we have that
\[
\frac{\text{Vol} B(\tilde{x}, R)}{\text{Vol} B(\tilde{x}, r)} \leq \frac{A_k(R)}{A_k(r)} \left( 1 + C(\alpha,p,n)(D^2 \| p_k \|_p)^{\frac{1}{p}} \right).
\]

This last corollary also implies Theorem 1.9. Indeed, \( \| p_k \|_p^p \leq \frac{\zeta(p,n,k,V,D)}{V} \), hence for \( \zeta \) small enough, Corollary 3.5 applies for \( \alpha = 2 \). We set
\[
N(p,n,k,D,V) = \frac{A_k(2D)}{V} \left( 1 + C(2,p,n)(D^2 \| p_k \|_p)^{\frac{1}{p}} \right) + 1
\]
\[
L(p,n,k,D,V) = \frac{D}{N(p,n,k,V,D)}.
\]
Let \( \Gamma \) be a subgroup of \( \pi_1(M) \) generated by \( k \) elements \( (g_1, \cdots, g_k) \) of length less than \( L(p,n,k,V,D) \). Let \( (\tilde{M}, \tilde{g}) \) be the Riemannian, universal cover of \( (M, g) \), \( \tilde{x} \in \tilde{M} \) and \( F \) be the Dirichlet domain of \( \tilde{x} \). Then \( \text{Diam}(F) \leq \delta \) and if we set \( \Gamma(N) = \{ g \in \Gamma/|\gamma| \leq N \} \) then by Corollary 3.5 (with \( r = 0 \)), we have
\[
\#\Gamma(N) \text{ Vol } F = \text{ Vol } (\bigcup_{\gamma \in \Gamma(N)} \gamma F) \leq \text{ Vol } (\text{Vol } B(\tilde{x}, 2D)) \leq (N - 1)V < N \text{ Vol } M = N \text{ Vol } F
\]
hence \( \#\Gamma(N) < N \). We infer that \( \Gamma(N) = \Gamma \) and so \( k \leq N \), which gives the result.

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