Abstract—A message passing algorithm (MP) is derived for recovering a dense subgraph within a graph generated by a variation of the Barabási-Albert preferential attachment model. The estimator is assumed to know the arrival times, order of attachment, of the vertices. The derivation of the algorithm is based on belief propagation under an independence assumption. Two precursors to the message passing algorithm are analyzed: the first is a degree thresholding (DT) algorithm and the second is an algorithm based on the arrival times of the children (C) of a given vertex, where the children of a given vertex are the vertices that attached to it. C significantly outperforms DT, showing it is beneficial to know the arrival times of the children, beyond simply knowing the number of them. It is shown that for a fixed fraction of vertices in the community, β, fixed number of new edges per arriving vertex, m, and fixed affinity between vertices in the community, ρ, the fraction of label errors for either of the algorithms DT or C, or converges as T → ∞.

I. INTRODUCTION
Community detection, a form of unsupervised learning, is the task of identifying dense subgraphs within a large graph. For surveys of recent work, see [1]–[3]. In this paper, we focus on the case when there is only a single dense cluster (community), and perform the task of distinguishing the vertices in the community from the outliers. Community detection is often studied in the context of a generative random graph model, of which the stochastic block model is the most popular. The model specifies how the labels of the vertices are chosen, and how the edges are placed, given the labels. The task of community detection then becomes an inference problem; the vertex labels are the parameters to be inferred, and the graph structure is the data. The advantage of a generative model is that it helps in the design of algorithms for community detection.

This paper is the first to consider the problem of recovering a community in a preferential attachment graph, along the lines of the Barabási-Albert model. A variation of the Barabási-Albert was introduced in [4] to include community structure. In this paper we use the special case of the model in [4] for a single dense community. The graphical model used in this paper is described in Section [4].

The algorithm we focus on is message passing. Algorithms that are precursors to message passing, in which the membership of a vertex is estimated from its radius one neighborhood in the graph, are also discussed. The algorithm is closest in spirit to that in the papers [5], [6]. Message passing algorithms are local algorithms; vertices in the graph pass messages to each of their neighbors, in an iterative fashion. The messages in every iteration are computed on the basis of messages in the previous iteration. In the context of community detection, a vertex in the community is expected to have a larger number of neighbors than an outlier. Thus, the neighborhood of a vertex conveys some information about its label. A quantitative estimate of this information is the belief (aposteriori probability) ratio of belonging to the community versus being an outlier.

A much better estimate of a vertex’s label could potentially be obtained if the labels of all other vertices were known. Since this information is not known, the idea of message passing algorithms is to have vertices simultaneously updating their beliefs.

The main similarity between the preferential attachment model with communities and the stochastic block model is that both produce locally tree-like graphs. However, the probabilities of edges existing are more complicated for preferential attachment models. To proceed to develop the message passing algorithm, we invoke an independence assumption that is suggested by asymptotic results in [4]. This approach is tantamount to constructing a belief propagation algorithm for a graphical model that captures the asymptotic distribution of neighborhood structure for the preferential attachment graphs. Section [4] examines the problem of estimating the community membership of a single vertex based on its neighborhood or on its number of neighbors, and some performance analysis is given. A key conclusion of that section is that, for the purpose of estimating the community membership of a single vertex, knowing the neighborhood of the vertex in the graph is significantly more informative than knowing the degree of the vertex. The message passing algorithm is discussed in Section [4].

II. BARABÁSI–ALBERT PREFERENTIAL ATTACHMENT MODEL WITH A PLANTED DENSE COMMUNITY
The model used in this paper is a special case of the model introduced in [4]. It has parameters m, ρ, and β, where m is the out-degree of each vertex, ρ is the probability a vertex is in the planted community, and β > 1 is the affinity between two vertices in the community. The model describes a directed random graph that evolves with time, hence it can be viewed as a sequence of graphs, (Gt = (Vt, Et) : t ≥ 1). The set of
vertices for graph $G_t$ is $V_t = \{t\} = \{1, \ldots, t\}$ for each $t \geq 1$. Each vertex, $t$, is assigned a label $\ell_t \in \{0, 1\}$, which indicates whether the vertex is a member of the planted community ($\ell_t = 1$) or not ($\ell_t = 0$). The vertex labels, $(\ell_t : t \geq 1)$, are assumed to be independent with $P\{\ell_t = 1\} = \rho$ for all $t$. The edge set $E_1$ consists of $m$ directed self-loops from vertex 1. Given the labels and the edge set $E_t$, the edge set $E_{t+1}$ is obtained by adding $m$ edges to $E_t$. The $m$ added edges are outgoing from vertex $t+1$, and each is incident to a vertex in $[t]$, selected by sampling with replacement, using the following attachment distribution over $[t]$. If $\ell_t \neq 0$, the attachment distribution is proportional to the degrees of the vertices in $[t]$. If $\ell_t = 0$, the attachment distribution is modified; the degrees of vertices in the community are multiplied by $\beta$ before normalization. The vertices an arriving vertex attaches to are called parents of the vertex, and vertices that later attach to it are called children of the vertex.

We view each edge as consisting of two half-edges. Each incident to a vertex, obtained by cutting an edge in half. There are $2mt$ half edges in $G_t$. Suppose each half edge inherits the label from the vertex it is incident to. By [4], the fraction of half-edges in $G_t$ with label 1, $\eta_t$, converges to $\eta^*$ almost surely, where $\eta^*$ is the solution in $[0,1]$ of the quadratic equation

$$(1+\rho)(\beta-1)s^2 + (1+2\rho-2\beta \rho)s - \rho = 0.$$ 

The fraction $\eta_t$ is important because $2mt(\eta_t \beta + 1 - \eta_t)$ is the denominator when the attachment distribution is calculated for vertex $t+1$ given $\ell_{t+1} = 1$.

Consider vertex $\tau$ for some $\tau \geq 2$. Let $Y_t$ denote the degree of that vertex at time $t$, so $(Y_t : t \geq \tau)$ denotes the evolution of the vertex degree, starting from $Y_\tau = m$. Based on the almost sure convergence $\eta_t \to \eta^*$, and on the fact that if $\tau$ is large it is unlikely for the degree of vertex $\tau$ to increase by more than one at a time, we consider the process $\tilde{Y} = (\tilde{Y}_t : t \geq \tau)$, which approximates $Y$. The initial value is $\tilde{Y}_\tau = m$. The distribution of $\tilde{Y}$, given the labels $(\ell_t : t' \geq \tau)$, is Markov with transition probabilities given by:

$$(\tilde{Y}_{t+1} | \ell_{t+1} = u, \ell_t = v) \overset{d}{=} \tilde{Y}_t + \text{Ber}\left(\frac{\tilde{Y}_t \theta^*_{u,v}}{t}\right),$$

where $d$ denotes equality of distribution, $\text{Ber}(p)$ represents a Bernoulli random variable with parameter $p$, and

$$\theta^*_0,0 = \theta^*_{0,1} = \frac{1}{2}$$

$$\theta^*_1,0 = \frac{1}{2(\eta^* \beta + 1 - \eta^*)}, \quad \theta^*_1,1 = \frac{\beta}{2(\eta^* \beta + 1 - \eta^*)}.$$ 

By using the law of total probability and the given distribution $\rho$ of $\ell_{t+1}$, it follows from (1) that the distribution of $\tilde{Y}$, given only the label $\ell_{\tau}$, is also Markov, with transition probabilities given by:

$$(\tilde{Y}_{t+1} | \tilde{Y}_t, \ell_{\tau} = v) \overset{d}{=} \tilde{Y}_t + \text{Ber}\left(\frac{\tilde{Y}_t \theta^*_{u,v}}{t}\right),$$

where, for $v \in \{0,1\}$,

$$\theta^*_{u,v} = \rho \theta^*_{u,v} + (1 - \rho) \theta^*_{u,v}.$$ 

As $t$ increases, the process $\tilde{Y}$ tends to slow down. Given $\ell_{\tau} = v$, $\tilde{Y}$ can be well approximated as $(\tilde{Y}_t : t \geq \tau) \approx (Z_{m(t/\tau)} : t \geq \tau)$, with the birth rate parameter, $\vartheta$, of $Z_t$ set to $\vartheta = \theta^*_v$. Here, $Z = (Z_s : s \geq 0)$ is a continuous time pure birth Markov process with initial state $Z_0 = m$ and birth rate $Z_1 \vartheta$ at time $s$. The process $Z$ represents the total number of individuals in a continuous time branching process beginning with $m$ individuals activated at time 0, such that each individual spawns another at rate $\vartheta$. It can be shown that for $s$ fixed, the distribution of $Z_s$ is the same as the sum of $m$ independent, geometrically distributed random variables with parameter $e^{-\vartheta s}$. In other words, $Z_s$ has the negative binomial distribution with parameters $(m, e^{-\vartheta})$.

**Remark 1.** (i) Knowing $G_T$ is equivalent to knowing the indices of the vertices and the undirected graph induced by dropping the orientations of the edges of $G_T$

(ii) The estimators considered in this paper are assumed to know the order of arrival of the vertices (which we take to be specified by the indices for brevity) and the parameters $m$, $\beta$ and $\rho$. We conjecture that, assuming the order of arrival of the vertices is known, the parameters $m$, $\beta$ and $\rho$ can be accurately estimated with high probability from a realization of the graph for sufficiently large $T$, if $\rho$ is in a closed interval $(0,1)$ and $\beta$ is in a closed interval $(1,\infty)$. Indeed, the parameter $m$ is directly observable. Results in [4] identify the limiting distributions of vertex degree for given vertex index and label, and also the empirical distribution of vertex degrees for given label. Hence estimation of $\rho$ and $\beta$ reduces to a problem of estimating the parameters of a distribution that is known to be the mixture of two distributions with parameters determined by $\rho$ and $\beta$.

(iii) If the indices of the vertices are not known and only the undirected version of the graph is given, it may be possible to estimate the indices if $m$ is sufficiently large. Such problem has been explored recently for the classical Barabási-Albert model [7], but we don’t pursue it here for the variation with a planted community.

### III. Hypothesis Testing for $Z$

**A. Inference based on indices of children (C)**

Let $1 < < \tau < T$ be integers. As noted above, given $\ell_{\tau} = v$ for $v \in \{0,1\}$, $(\tilde{Y}_t : t \geq \tau) \approx (Z_{m(t/\tau)} : t \geq \tau)$, where the parameter $\vartheta$ of $Z_t$ is $\theta^*_v$. Thus, the problem of inferring $\ell_{\tau}$ from the indices of its children is approximately the same as the following hypothesis testing problem for the birth rate parameter for the process $(Z_s : 0 \leq s \leq \ln(T/\tau))$:

$H_1 : \vartheta = \theta^*_1 \text{ vs. } H_0 : \vartheta = \theta^*_0.$

Scaling time by a constant factor is equivalent to changing the birth rate parameter of $Z$ by the same constant. So letting $\lambda = \theta^*_0/\theta^*_1$, (so $\lambda < 1$), the hypothesis testing problem is approximately equivalent to the following: Observe $Z_{[0,\lambda]} =$
(Z_s : 0 \leq s \leq \bar{s})$, where $\bar{s} = \theta_1^* \ln(T/\tau)$, and decide between the following two hypotheses:

- $H_1 : Z_{[0,\bar{s}]}$ has birth rate 1
- $H_0 : Z_{[0,\bar{s}]}$ has birth rate $\lambda$

Prior probabilities $\pi_1 = \rho$ and $\pi_0 = 1 - \rho$ are assumed, so it makes sense to seek a decision rule to minimize the average error probability, $p_e \triangleq \pi_1 p_{e1} + \pi_0 p_{e0}$, where $p_{e,v}$ is the conditional probability of error given $H_v$ is true. By a central result of Bayesian decision theory, the optimal decision rule is the MAP estimator, which can be expressed either in terms of the log belief ratio $\xi$ or log likelihood ratio $\Lambda$:

$$\hat{\xi}_{\text{MAP}} = 1(\xi > 0) = 1(\Lambda > \log(\pi_1/\pi_2))$$

$$\xi = \log \frac{P\{H_1|Z_{[0,\bar{s}]}\}}{P\{H_0|Z_{[0,\bar{s}]}\}} \quad \Lambda = \log \frac{f(Z_{[0,\bar{s}]}|H_1)}{f(Z_{[0,\bar{s}]}|H_0)}$$

Since the inter-jump periods are independent (exponential) random variables, the likelihood of observing the entire process is simply the product of the likelihoods of the observed inter-jump periods, with an additional factor of the likelihood of not seeing a jump in the last interval. Suppose one observes $n$ jumps at times $s_1, \ldots, s_n$. The log likelihood ratio for observing this is (letting $s_0 = 0$):

$$\Lambda = \log \frac{\prod_{i=0}^{n-1} (m + i)e^{-(m+i)(s_{i+1} - s_i)}}{\prod_{i=0}^{n-1} \lambda(m + i)e^{-\lambda(m+i)(s_{i+1} - s_i)} e^{-(n+m)(\bar{s} - s_n)}}$$

$$= \log \frac{\lambda^n e^{-\lambda((n+m)\bar{s} - \sum_{i=1}^{n} s_i)}}{\prod_{i=0}^{n-1} \lambda m e^{-\lambda((n+m)s - \sum_{i=1}^{n} s_i)}}$$

$$= n \log(1/\lambda) - (m\bar{s} + \sum_{i=1}^{n} (\bar{s} - s_i))(1 - \lambda)$$

The quantity $(m\bar{s} + \sum_{i=1}^{n} (\bar{s} - s_i))$ is the area under the trajectory of $Z_{[0,\bar{s}]}$, which we denote by $A_s$. Moreover, $n + m$ is the value of $Z_s$. So the log-likelihood ratio is given by:

$$\Lambda = (Z_s - m) \log(1/\lambda) - (A_s)(1 - \lambda), \quad (3)$$

which is a linear combination of $Z_s - m$ and $A_s$. Thus, the MAP decision rule has a simple form. Let $f_Z^D(\rho, \lambda, m, \bar{s})$ denote the average error probability $p_e$ for the MAP decision rule based on observation of $Z_{[0,\bar{s}]}$.

There is apparently no closed form expression for the distribution of $\Lambda$, so computation of $f_Z^D(\rho, \lambda, m, \bar{s})$ may require Monte Carlo simulation or some other numerical method. A closed form expression for the moment generating function of $\Lambda$ is given in the following lemma, and it can be used to either bound the probability of error or to accelerate its estimation by importance sampling.

**Proposition 1.** The joint moment generating function of $Z_s$ and $A_s$ is given as follows, where $\mathbb{E}_{\lambda,m}$ $[\cdot]$ denotes expectation assuming the parameters of $Z$ are $\lambda, m$:

$$\psi_{\lambda,m}(u, v, s) \triangleq \mathbb{E}_{\lambda,m} \left[ e^{uZ_s + vA_s} \right] = \left( \frac{e^{(v-\lambda)s + u}}{1 + \lambda \frac{e^{(v-\lambda)s}}{\lambda}} \right)^m$$

**Proposition 1** can be used to bound $p_e$ as follows. By a standard result in the theory of binary hypothesis testing (due to [8], stated without proof in [9], proved in special case $\pi_1 = \pi_0 = 0.5$ in [10], and same proof easily extends to general case), the probability of error for the MAP decision rule is bounded by

$$\pi_1 \pi_0 \rho_B^2 \leq p_e \leq \sqrt{\pi_1 \pi_0 \rho_B}, \quad (5)$$

where the Bhattacharyya coefficient (or Hellinger integral) $\rho_B$ is defined by $\rho_B = \mathbb{E} \left[ e^{u/2}|H_0\right]$, and $\pi_1$ and $\pi_0$ are the prior probabilities on the hypotheses. The lemma implies

$$\mathbb{E}_{\lambda,m} \left[ e^{u(Z_s - m) + vA_s} \right] = \psi_{\lambda,m}(u, v, s)\rho_u$$

$$= \left( \frac{e^{(v-\lambda)s}}{1 + \lambda \frac{e^{(v-\lambda)s}}{\lambda}} \right)^m$$

so by (3) and the definition of $\rho_B$, we substitute $u = \frac{1}{2} \log \left( \frac{\lambda}{1 - \lambda} \right)$, $v = -\frac{1}{2}(1 - \lambda)$ to yield (writing $\rho_{B,\lambda,m}$ to make explicit the dependence of $\rho_B$ on the parameters):

$$\rho_{B,\lambda,m} = \left( \frac{e^{(1 + \lambda)s}}{1 - \lambda \frac{e^{(1 + \lambda)s}}{\lambda}} \right)^m$$

Using this expression for $\rho_{B,\lambda,m}$ in (5) provides upper and lower bounds on $p_e = f_Z^D(\rho, \lambda, m, \bar{s})$.

**B. Inference based on number children (aka degree thresholding (DT))**

It is interesting to compare the error probability of this optimal scheme with that obtained simply by degree thresholding, which amounts to hypothesis testing on the basis of the random variable $Z_s$ alone. As noted earlier, if the parameters for $Z$ are $\lambda, m$, for any time $s$, $Z_s \sim \text{NegBinom}(m, e^{-\lambda s})$. Thus we have the following hypothesis testing problem:

$$\tilde{H}_1 : Z_s \sim \text{NegBinom}(m, e^{-s})$$

$$\tilde{H}_0 : Z_s \sim \text{NegBinom}(m, e^{-\lambda s})$$

It is easily shown that the log-likelihood ratio for the above problem is given by:

$$\tilde{\Lambda} = (Z_s - m) \log \frac{1 - e^{-s}}{1 - e^{-\lambda s}} - m\bar{s}(1 - \lambda) \quad (6)$$

For this problem the exact minimum probability of error, achieved by the MAP rule, which we denote by $f_Z^{DT}(\rho, \lambda, m, \bar{s})$, can be readily calculated numerically. The
probability of error can also be bounded by \( \tilde{p} \). The Bhattacharyya coefficient is given by:

\[
\tilde{\rho}_{B,\lambda,m} = \left( \frac{e^{-\lambda s}}{1 - \sqrt{(1-e^{-\lambda s})(1-e^{-s})}} \right)^m.
\]  

Some numerical results are shown in Figures 1 and 2. The figures show \( p_e \) vs. \( \lambda \) for \( s = 5 \) and \( s = 10 \) respectively, assuming the uniform prior, \( \rho = 0.5 \) and \( m = 1 \). For the estimator based on \( Z_s \) alone, numerical calculation of \( p_e \) and computation by simulation are shown. For the estimator based on \( Z_{[0,s]} \), the Bhattacharyya upper bound on \( p_e \) and computation by simulation are shown. For the larger \( s \), there is a substantial improvement for using \( Z_{[0,s]} \) instead of only the final value, \( Z_s \). The Bhattacharyya upper bound is not very tight for \( s = 5 \) but is close to estimate \( p_e \) for \( s = 10 \).

C. Back to the community recovery problem

Consider the community recovery problem for \( m, \rho, \) and \( \beta \) fixed, and large \( T \), and let \( \epsilon \) be an arbitrary small constant. The problem of recovering \( \ell_T \) for some vertex \( \tau \) with \( \epsilon T \leq \tau \leq T \) from \( G_T \) using children (C) (respectively, degree thresholding (DT)) is asymptotically equivalent to the hypothesis testing problem for \( Z_{[0,s]} \) (respectively, \( Z_s \)) with the same parameters \( m \) and \( \rho \), and with \( \lambda = \theta^*_0/\theta^*_1 \) and \( s = \theta^*_1 \ln(T/\tau) \). Both \( \theta^*_0 \) and \( \theta^*_1 \), and hence \( \lambda \), are determined by \( \rho \) and \( \beta \) as explained in Section III. This leads to the following proposition, proved in Appendix B using results on coupling of \( Y, \bar{Y} \) and \( Z \).

**Proposition 2.** Let \( \hat{p}^{(C)}_{\ell_T} \) denote the fraction of errors for recovery of the labels of \( G_T \) using the decision rule based on children of each vertex. Then,

\[
\hat{p}^{(C)}_{\ell_T} \xrightarrow{T \to \infty} \int_0^1 f_Z^C(r; \rho, \theta^*_0/\theta^*_1, m, \theta^*_1 \ln(1/r)) dr,
\]

where the convergence is in probability. The same result holds with \( C \) replaced by DT.

We conjecture that a similar limit exists for label recovery using the message passing (MP) algorithm described in the next section.

IV. THE MESSAGE PASSING ALGORITHM

In this section, we specify how algorithm C (the MAP rule given children) can be extended to a message passing algorithm. We describe the algorithm for the special case \( m = 1 \), and from that it is easy to write the equations for general fixed \( m \geq 1 \). The absence of loops for \( m = 1 \) (except for the self loop at vertex 1, which is ignored) implies that in that case, the message passing algorithm can be implemented as a single-pass algorithm. One message is sent along each edge of the graph in each direction. We specify for each vertex and each given edge incident to the vertex, what the message sent is as a function of messages received on the other edges incident to the vertex, and state how the messages are used to produce an estimate for the vertex’s label, once messages are received from all neighbors. The derivation of the algorithm is provided in the appendix. For \( m \geq 2 \) the same formulas for message updates are used, but the message passing graph has cycles, so the MP algorithm becomes loopy message passing.

Throughout the remainder of this section, let \((V, E)\) be a fixed instance of the random graph, \((V_T, E_T)\). The message passing algorithm is run on this graph, with the aim of calculating \( \Lambda_T \triangleq \log P(L_T = 1 | E_T = 1) / \log P(E_T = 1 | \theta_0 = \theta_1), \) and hence \( \tau_{MAP} = 1(\Lambda_T > \log((1-\rho)/\rho)) \), for \( 1 \leq \tau \leq T \). The first step towards this is to obtain a formula for the log-likelihood ratio of observing one’s children. The set of children of \( \tau \), denoted by \( \partial \tau \), is a random quantity whose distribution is governed by the degree-
growth process \((Y_t : t \in [\tau,T])\). Using the approximation \((Y_t) \sim (\bar{Z}_t)_t\) discussed in Section \([4]\), we find:

\[
\log \frac{\mathbb{P}\{\partial \tau = \{t_1, \ldots, t_n\}|\ell_\tau = 1\}}{\mathbb{P}\{\partial \tau = \{t_1, \ldots, t_n\}|\ell_\tau = 0\}} \equiv (\theta_1^\tau - \theta_0^\tau) \left( \log \frac{\tau}{T} + \sum_{t \in \partial \tau} \log \frac{t}{T} \right) + n \log \frac{\theta_1^\tau}{\theta_0^\tau} \quad (8)
\]

With appropriate assumptions, the joint probability of observing the set of children of multiple vertices can also be efficiently computed. This allows us to compute the desired quantity, \(\Lambda_\tau\), because the entire graph can be described by specifying the set of children of all vertices. The first assumption is regarding how the distribution of \(\partial \tau\) changes, given the label of another vertex. Briefly, by the definition of the \(\bar{Y}\),

\[
\mathbb{P}\{\partial \tau = \{t_1, \ldots, t_n\}|\ell_\tau = v, \ell_{\partial \tau} = u\} = \left\{ \begin{array}{ll}
\mathbb{P}\{\partial \tau = \{t_1, \ldots, t_n\}|\ell_\tau = v\} \theta_\tau^u/v \theta_\tau^v & \text{if } t \in \partial \tau \\
\mathbb{P}\{\partial \tau = \{t_1, \ldots, t_n\}|\ell_\tau = v\} & \text{if } t \in \partial \tau
\end{array} \right.
\]

The second assumption is regarding the joint distribution of degree-growth processes. Observing the degree-growth process of one vertex \(\tau\) changes the distribution of another vertex \(\tau'\) in one of two possible ways. Firstly, the children of the first vertex cannot be the children of the other (if \(m = 1\)). However, it can be shown that the change in distribution due to this effect is insignificant. Secondly, observing the degree-growth process gives us some information about the label of each vertex. If one vertex appears as a child of the other (say \(\tau' \in \partial \tau\)), the probability of the given observation is affected; else it is insignificant. Secondly, observing the degree-growth process gives us some information about the label of each vertex. If one vertex appears as a child of the other (say \(\tau' \in \partial \tau\)), the probability of the given observation is affected; else it is insignificant. Secondly, observing the degree-growth process gives us some information about the label of each vertex. If one vertex appears as a child of the other (say \(\tau' \in \partial \tau\)), the probability of the given observation is affected; else it is insignificant. Secondly, observing the degree-growth process gives us some information about the label of each vertex. If one vertex appears as a child of the other (say \(\tau' \in \partial \tau\)), the probability of the given observation is affected; else it is insignificant. Secondly, observing the degree-growth process gives us some information about the label of each vertex. If one vertex appears as a child of the other (say \(\tau' \in \partial \tau\)), the probability of the given observation is affected; else it is insignificant. Secondly, observing the degree-growth process gives us some information about the label of each vertex. If one vertex appears as a child of the other (say \(\tau' \in \partial \tau\)), the probability of the given observation is affected; else it is insignificant.
APPENDIX A

PROOF OF PROPOSITION[1]

The process $Z$ with parameters $\lambda, m$ represents the total population of a branching process starting with $m$ root individuals at time 0, such that each individual in the population spawns new individuals at rate $\lambda$. And $A_i$ represents the sum of the lifetimes, truncated at time $\bar{s}$, of all the individuals in the population. The joint distribution of $(Z, A)$ with parameters $\lambda, m$ is the same as the distribution of the sum of $m$ independent versions of $(Z, A)$ with parameters $\lambda, 1$, Hence, it suffices to prove the lemma for $m = 1$.

So for the remainder of this proof suppose $m = 1$; there is a single root individual. Suppose there are $n(\bar{s})$ children of the root individual, produced at times $R_1, \ldots, R_{n(\bar{s})}$. Then

$$Z_{\bar{s}} = 1 + \sum_{l=1}^{n(\bar{s})} Z_{\bar{s}-R_l}^{\bar{s}}$$

(15)

$$A_{\bar{s}} = \bar{s} + \sum_{l=1}^{n(\bar{s})} A_{\bar{s}-R_l}^{\bar{s}}$$

(16)

where $Z_{\bar{s}-R_l}$ denotes the total subpopulation of the $l^{th}$ child of the root, $\bar{s} - R_l$ time units after the birth of the $l^{th}$ child, and $A_{\bar{s}-R_l}$ is the associated sum of lifetimes of that subpopulation, truncated $\bar{s} - R_l$ time units after the birth of the $l^{th}$ child (i.e. truncated at time $\bar{s}$). The processes $Z_1, A_1$ are independent and have the same distribution as $(Z, A)$. The variables $R_1, \ldots, R_{n(\bar{s})}$ are the points of a Poisson process of rate $\lambda$. Therefore,

$$e^{uZ_{\bar{s}} + vA_{\bar{s}}} = e^{u + vs} \prod_{l=1}^{n(\bar{s})} \exp(uZ_{\bar{s}-R_l} + vA_{\bar{s}-R_l}),$$

which after taking expectations yields

$$\psi_{\lambda,1}(u, v, \bar{s}) = e^{u + vs} \mathbb{E}_{\lambda,1} \left[ \prod_{l=1}^{n(\bar{s})} \exp(uZ_{\bar{s}-R_l}^{\bar{s}} + vA_{\bar{s}-R_l}^{\bar{s}}) \right].$$

Since $n(\bar{s})$ is a Poisson($\lambda$) random variable, and, given $n(\bar{s})$, $R_1, \ldots, R_{n(\bar{s})}$ are distributed uniformly on $[0, \bar{s}]$, the above expectation can be simplified by first conditioning on $n(\bar{s})$, and then summing over all possible values of $n(\bar{s})$ (tower property).

$$\psi_{\lambda,1}(u, v, \bar{s}) = e^{u + vs} \sum_{k=0}^{\infty} \frac{e^{-\lambda\bar{s}}(\lambda\bar{s})^k}{k!} \mathbb{E}_{\lambda,1} \left[ \prod_{j=1}^{k} e^{uZ_{\bar{s}-(\bar{s}-R_l)}^{\bar{s}-(\bar{s}-R_l)}} \right]$$

$$= e^{u + vs} \sum_{k=0}^{\infty} \frac{e^{-\lambda\bar{s}}(\lambda\bar{s})^k}{k!} \left( \frac{1}{\bar{s}} \int_{0}^{\bar{s}} \psi_{\lambda}(u, v, \tau) d\tau \right)^k$$

(17)

In the above step, the expectation of the product is the same as the product of the expectations, because the variables $Z_l(\bar{s} - R_l), A_l(\bar{s} - R_l), l = 1, \ldots, k$ are independent of each other. Moreover, the expectation of each of the $k$ terms is identical. Denoting $F(s) \triangleq \int_{0}^{s} \psi_{\lambda,1}(u, v, \tau) d\tau$, we can write (17) as

$$\hat{F}(\bar{s}) = e^{u + vs} e^{-\lambda\bar{s}} e^{\lambda F(\bar{s})}$$

$$\frac{d}{ds} \left( e^{-\lambda F(s)} \right) = -\lambda e^{(v-\lambda)\bar{s}}; \quad F(0) = 0$$

$$e^{-\lambda F(s)} = 1 - \lambda e^u \int_{0}^{\bar{s}} e^{(v-\lambda)s} ds$$

$$= 1 + \lambda e^u \frac{1}{v - \lambda} \left( 1 - e^{(v-\lambda)\bar{s}} \right)$$

$$F(s) = -\frac{1}{\lambda} \log \left( 1 + \lambda e^u \frac{1}{v - \lambda} \left( 1 - e^{(v-\lambda)\bar{s}} \right) \right)$$

(18)

Finally, using $\psi_{\lambda,1}(u, v, \bar{s}) = \hat{F}(\bar{s})$ yields (4) for $m = 1$, and the proof is complete.

APPENDIX B

PROOF OF PROPOSITION[2]

It is shown in (4) that as $\tau, T \to \infty$ such that $T > \tau$ and $T/\tau$ is bounded, for $v \in \{0,1\}$, the following convergence in total variation distance holds for any rate parameter $\nu$, and hence for either hypothesis (i.e. $H_1 : v = 1$ vs. $H_0 : v = 0$):

$$d_{TV}(Y_{[\tau,T]}(\nu, T), Y_{[\tau,T]}(\theta_0^*)) \to 0.$$ 

This implies the difference between the error probability for the MAP estimator based on $\hat{Y}$, when applied to the original process $Y$, and the MAP estimator of the same decision rule based on $\hat{Y}$, converges to zero. The convergence is uniform for $T/\tau$ bounded.

Secondly, we produce below a deterministic mapping from $Z_{[0,\bar{s}]}$ to $Y_{[\tau,T]}$ that does not depend on the parameter value $\lambda$ such that $d_{TV}(Y_{[\tau,T]}(\bar{s}, \theta_0^*), \bar{s}, \tau)) \to 0$ and such that the mapping, thought of as a quantizer, becomes arbitrarily fine as $\tau, T \to \infty$ with $T/\tau$ bounded. Therefore, the minimum error probability for the binary hypothesis testing problem for $Z$ is a lower bound for the error probability for $\hat{Y}$, and asymptotically the error probabilities for all three are the same. The key step for the construction is the following lemma, where $\hat{L}$ represents the holding time of $\hat{Y}$ in a state $k$ if $\Delta = \theta_{k0}^*$, and $U$ represents the holding time of $Z$ in state $k$ if $\bar{Z}$ has rate parameter $\theta_{k0}^*$.
Lemma 1. Let $t$ be a positive integer and let $\triangle > 0$. Let $L$ denote a random variable such that
\begin{align}
P\{L = 1\} &= \frac{\triangle}{t} \\
P\{L = i\} &= \left(1 - \frac{\triangle}{t}\right) \left(1 - \frac{\triangle}{t+1}\right) \cdots \left(1 - \frac{\triangle}{t+i-2}\right) \frac{\triangle}{t+i-1}
\end{align}
for $i \geq 2$, and let $U$ be an exponentially distributed parameter of rate $\triangle$. Let $g(t,u) = [te^u]$. In other words, $g(t,u)$ is truncated rounded up to the next integer. Note that $g$ does not depend on $\triangle$. Then $d_{TV}(g(t,U), L) \leq \frac{1+\triangle}{t}$.

Proof. Let $p$ denote the pmf of $L$ and $p'$ denote the pmf of $g(t,U)$. The pmfs can be factored as $p_i = a_i b_i$ and $p'_i = a'_i b'_i$ where, for $i \geq 1$,
\begin{align}
a_i &= \left(1 - \frac{\triangle}{t}\right) \left(1 - \frac{\triangle}{t+1}\right) \cdots \left(1 - \frac{\triangle}{t+i-2}\right) \\
b_i &= \frac{\triangle}{t+i-1} \\
a'_i &= e^{-\triangle \ln\left(\frac{i+1}{t+i}\right)} \\
b'_i &= 1 - e^{-\triangle \ln\left(\frac{i}{t+i+1}\right)}
\end{align}

By comparing their logarithms it can be shown that $a_i \leq a'_i$ for all $i \geq 1$. By comparing their derivatives with respect to $\triangle$ it can be shown that and $b_i \geq b'_i$ for all $i \geq 1$. Therefore,
\begin{align}
d_{TV}(p,p') &\leq \frac{1}{2} \sum_{i=1}^{\infty} |p_i - p'_i| = \sum_{i=1}^{\infty} (p_i - p'_i)_+ \\
&= \sum_{i=1}^{\infty} (a_i b_i - a'_i b'_i)_+ \leq \sum_{i=1}^{\infty} a_i (b_i - b'_i).
\end{align}

Note that with $x = \frac{1}{t+i-1}$,
\begin{align}
b_i - b'_i &= \triangle x - 1 + e^{-\triangle \ln(1+x)}, \\
d\left(\frac{b_i - b'_i}{dx}\right)_{x=0} &= 0 \quad \text{and} \quad d^2 \left(\frac{b_i - b'_i}{dx^2}\right) \leq \triangle (1 + \triangle) \quad \text{for} \quad x \geq 0.
\end{align}

Therefore, $b_i - b'_i \leq \frac{\triangle (1 + \triangle) x^2}{2} \leq b_i \frac{1+\triangle}{2t}$. Thus, using (21),
\begin{align}
d_{TV}(p,p') \leq \sum_{i=1}^{\infty} a_i b_i \frac{1+\triangle}{2t} = \left(\sum_{i=1}^{\infty} p_i\right) \frac{1+\triangle}{2t} = \frac{1+\triangle}{2t}.
\end{align}

Given observation of the process $Z_{[0,\ln(T/\tau)]}$, we map the holding times of the states of $Z$ into holding times for the process $\tilde{Y}$, such that if $U_m, U_{m+1}, \ldots$ are the successive holding times of $Z$ in states $m, m+1, \ldots$, then the holding times of $\tilde{Y}$ in those states are given by $R_m = g(U_m, \tau)$ and $R_{m+1} = g(U_{m+1}, \tau + R_m + \cdots + R_{m+1})$. Note that the mapping from $Z_{[0,\ln(T/\tau)]}$ does not depend on the rate parameter $\theta$. If hypothesis $H_1$ is true for $Z_{[0,\delta]}$, then Lemma [1] and the tightness of the distribution of $Z_{[0,\ln(T/\tau)]}$ for $T/\tau$ bounded implies that $d_{TV}(\mathcal{L}(\tilde{Y}_{[\tau,T]}|H_1), \mathcal{L}(\tilde{Y}_{[\tau,T]}|H_0)) \to 0$ under either hypothesis $H_0$ or $H_1$, assuming $\tau, T \to \infty$ such that $\tau \leq T$ and $T/\tau$ is bounded.

Convergence of the expected number of label errors follows:
\begin{align}
E \left[ p_{C}^{(C)} \right] &\to \int_0^T f_Z^*(p, \theta_1^*, \theta_1^*, m, \theta_1^* \ln(T/\tau)) d\tau \\
&= \int_0^1 f_Z^*(p, \theta_1^*, \theta_1^*, m, \theta_1^* \ln(1/r)) dr,
\end{align}

The last part of the proof is to show that the converge is true not only in mean, but also in probability. That follows by the same method used to establish convergence of the empirical distribution of degrees of $G_{\tau}$ in probability in [4]. The key step is a proof that the joint degree evolution processes $(Y^j)$ for a finite number $J$ of vertices (really only need to consider $J = 2$ here) are asymptotically independent, in the sense that the total variation distance to a process with independent vertex evolution converges to zero. That implies the error events for different labels are asymptotically uncorrelated, so convergence in probability to the mean follows by the Chebyshev inequality. The same proof works for $C$ replaced by $DT$.

**APPENDIX C**

**DERIVATION OF THE MESSAGE PASSING ALGORITHM**

The objective of the message passing algorithm is to compute $\Lambda_\tau$ for every vertex $\tau$ in the graph, where:
\begin{align}
\Lambda_\tau &= \log \frac{\mathbb{P}\{D_1|\ell_\tau = 1\}}{\mathbb{P}\{D_1|\ell_\tau = 0\}}.
\end{align}

The message passing algorithm is described in terms of the following four quantities, for vertices $\tau$ and $\tau_0$ such that $\tau$ is a child of $\tau_0$:
\begin{align}
\Lambda^p_\tau &= \log \frac{\mathbb{P}\{D_1 \setminus D_\tau|\ell_\tau = 1\}}{\mathbb{P}\{D_1 \setminus D_\tau|\ell_\tau = 0\}} \\
\Lambda^c_\tau &= \log \frac{\mathbb{P}\{D_\tau|\ell_\tau = 1\}}{\mathbb{P}\{D_\tau|\ell_\tau = 0\}} \\
\nu_{\tau \to \tau_0} &= \log \frac{\mathbb{P}\{D_1 \setminus D_\tau_\tau = 1\}}{\mathbb{P}\{D_1 \setminus D_\tau|\ell_\tau_0 = 0\}} \\
\nu_{\tau_0 \to \tau} &= \log \frac{\mathbb{P}\{D_1 \setminus D_\tau|\ell_\tau_0 = 0, \ell_\tau = 1\}}{\mathbb{P}\{D_1 \setminus D_\tau|\ell_\tau = 0, \ell_\tau_0 = 0\}}.
\end{align}

The final output is given by $\Lambda_\tau = \Lambda^p_\tau + \Lambda^c_\tau$. The above quantities are computed according to the following equations:
\begin{align}
\nu_{\tau \to \tau_0} &= (\theta_i^* - \theta_0^*) \left(\log \frac{\tau}{T} + \sum_{t \in \partial \tau} \log \frac{t}{T}\right) \\
&+ \sum_{t \in \partial \tau} \log \frac{2 \theta_i^* \frac{1}{T} e^{\theta_0^* - \theta_i^*} + 1}{2 \theta_i^* \frac{1}{T} e^{\theta_0^* - \theta_i^*} + 1} \\
\nu_{\tau_0 \to \tau} &= \log \frac{\theta_i^* \frac{1}{T} e^{\theta_0^* - \theta_i^*} + 1}{\theta_i^* \frac{1}{T} e^{\theta_0^* - \theta_i^*} + 1} + \log \theta_i^* \\
&+ (\theta_i^* - \theta_0^*) \left(\log \frac{\tau_0}{T} + \sum_{t \in \partial \tau_0} \log \frac{t}{T}\right).
\end{align}
\[
+ \sum_{\ell \in \partial \tau} \log \left( \frac{2 \theta_{1,1}^{\ell_1} \rho_1^{\ell_1 \tau_{1,1} \tau \to 0} + 1}{2 \theta_{1,0}^{\ell_1} \rho_1^{\ell_1 \tau_{1,0} \tau \to 0} + 1} \right) \quad (28)
\]

\[
\Lambda_{\tau}^{t} = \log \left( \frac{\beta}{1 - \rho} e^{\theta_{1,t}^{\tau \tau_{1,0} \tau \to 0} + 1} + \log 2 \theta_{1,0}^{t} \right) \quad (29)
\]

Each of the quantities defined in (22)-(26) involve the observation of the children set, \( \partial \tau \), for some set of vertices \( t \). The event \( D_1 \setminus D_\tau \) includes the observation of the children set of the parent of \( \tau \), but excludes the children set of \( \tau \) and any of its descendants. To simplify the joint distribution of children sets (equivalently, of degree-growth processes), we rely on certain independence assumptions. These assumptions are stated explicitly in the derivation that follows.

Consider the expression in (27). The principle idea behind this equation is that the likelihood of observing the descendants of \( \tau \) can be rewritten as the likelihood of observing \( D_1 \setminus D_\tau \) with some extra information about the labels of the children of \( \tau \) being provided by their descendants. This notion is made clear by rewriting the log-likelihood ratio of \( D_1 \setminus D_\tau \) in the following manner:

\[
\log \frac{P \{ D_1^{\tau}_{1,1} | \ell_{1,1} = v \}}{P \{ D_1^{\tau}_{1,1} | \ell_{1,1} = 0 \}} = (\theta_{1,1}^{\tau_{1,1}} - \theta_{0,1}^{\tau_{1,1}}) \left( \frac{\log \tau + \sum_{t \in \partial \tau} \log t} {\theta_{1,0}^{\tau_{1,0}}} + \sum_{t \in \partial \tau} \log \frac{\theta_{1,1}^{\tau_{1,1}}}{\theta_{0,1}^{\tau_{1,1}}} \right) \quad (30)
\]

\[
+ \sum_{t \in \partial \tau} \log \left( \frac{2 \theta_{1,1}^{\ell_1} \rho_1^{\ell_1 \tau_{1,1} \tau \to 0} + 1}{2 \theta_{1,0}^{\ell_1} \rho_1^{\ell_1 \tau_{1,0} \tau \to 0} + 1} \right) \quad (31)
\]

where the last step follows from the definitions:

\[
\theta_{1,0}^{t} = \theta_{0,1}^{t} = 1, \quad \theta_{1,v}^{t} = \theta_{0,v}^{t} + (1 - \rho) \theta_{0,v}^{1} \quad \text{for} \quad v \in \{0,1\}.
\]

Note the similarity between (27) and (31). The only difference between the two are the terms \( e^{\theta_{1,v} \tau_{1,v} \tau \to 0} \) for each \( \ell \), where, by the definition (25), \( e^{\theta_{1,v} \tau_{1,v} \tau \to 0} = \frac{P \{ D_1^{\tau}_{1,v} | \ell_{1,v} \}}{P \{ D_1^{\tau}_{1,v} \}} \). Continuing with the proof of (27), note that:

\[
P \{ D_1^{\tau}_{1} | \ell_{1} = v \} = P \{ D_1^{\tau}_{1}, D_{t_1}, \ldots, D_{t_n} | \ell_{1} = v \}
= \sum_{u_1, \ldots, u_n} P \{ \ell_{t_1} = u_1 \} \ldots P \{ \ell_{t_n} = u_n \}
\times P \{ D_1^{\tau}_{1}, D_{t_1}, \ldots, D_{t_n} | \ell_{1} = v, \ell_{t_1} = u_1, \ldots, \ell_{t_n} = u_n \}
= (a) \sum_{u_1, \ldots, u_n} P \{ D_1^{\tau}_{1} | \ell_{1} = v, \ell_{t_1} = u_1, \ldots, \ell_{t_n} = u_n \}
\prod_{i=1}^{n} P \{ D_{t_i} | \ell_{t_i} = u_i \} P \{ \ell_{t_i} = u_i \}
\]

\[
= (b) \sum_{u_1, \ldots, u_n} P \{ D_1^{\tau}_{1} | \ell_{1} = v \} \prod_{i=1}^{n} \frac{P \{ D_{t_i} | \ell_{t_i} = u_i \}}{P \{ \ell_{t_i} = u_i \}} \frac{\theta_{n+1,v}^{\tau_{n+1,v} \tau \to 0} \theta_{n,v}^{\tau_{n,v} \tau \to 0}}{\theta_{1,v}^{\tau_{1,v} \tau \to 0} \theta_{0,v}^{\tau_{0,v} \tau \to 0} + 1}
\]

\[
= \prod_{i=1}^{n} P \{ D_{t_i} | \ell_{t_i} = u_i \} \prod_{u \in \{0,1\}} \sum_{i=1}^{n} P \{ D_{t_i} | \ell_{t_i} = u | \} P \{ \ell_{t_i} = u \} \theta_{u,v}^{\tau_{u,v} \tau \to 0} \quad (32)
\]

In step (a), we make the assumption:

\[
P \{ D_1^{\tau}_{1}, D_{t_1}, \ldots, D_{t_n} | \ell_{1} = v, \ell_{t_1} = u_1, \ldots, \ell_{t_n} = u_n \}
= P \{ D_1^{\tau}_{1} | \ell_{1} = v, \ell_{t_1} = u_1, \ldots, \ell_{t_n} = u_n \} \prod_{i=1}^{n} P \{ D_{t_i} | \ell_{t_i} = u_i \},
\]

which amounts to assuming that, given the children of \( \tau \) and the labels of the children, their sets of descendants are conditionally independent. Step (b) follows from the expressions (11) and (12) for the transition probabilities of \( Y \) with and without conditioning on the label of the incoming vertex. Using (30) and (12), we write:

\[
\nu_{\tau \to 0} = \log \frac{P \{ D_1^{\tau}_{1} | \ell_{1} = 1 \}}{P \{ D_1^{\tau}_{1} | \ell_{1} = 0 \}}
= \log \frac{P \{ D_1^{\tau}_{1} | \ell_{1} = 1 \}}{P \{ D_1^{\tau}_{1} | \ell_{1} = 0 \} + n \log \theta_{1,0}^{\tau_{1,0}}}
+ \sum_{i=1}^{n} \log \frac{P \{ D_{t_i} | \ell_{t_i} = 1 \} \theta_{1,1}^{\tau_{1,1}} + P \{ D_{t_i} | \ell_{t_i} = 2 \} (1 - \rho) \theta_{0,1}^{\tau_{0,1}}}{P \{ D_{t_i} | \ell_{t_i} = 1 \} \theta_{1,1}^{\tau_{1,1}} + P \{ D_{t_i} | \ell_{t_i} = 2 \} (1 - \rho) \theta_{0,1}^{\tau_{0,1}}}
\]

Step (c) follows from the fact that \( \theta_{1,0}^{\tau_{1,0}} = \theta_{0,1}^{\tau_{0,1}} = 1/2 \), while the last step follows from the definition of \( \nu_{\tau \to 0} \). This concludes the derivation of (27).

We now turn to the derivation of (28) and (29). Using the same independence assumption as was used in (8) of the main paper to simplify the expression for \( P \{ D_1 \setminus D_\tau | \ell_{t_1} = u, \ell_{t_0} = v \} \):

\[
P \{ D_1 \setminus D_\tau | \ell_{t_1} = u, \ell_{t_0} = v \}
= P \{ D_1 \setminus D_\tau | \ell_{t_1} = u, \ell_{t_0} = v \} P \{ D_\tau | \ell_{t_1} = u, \ell_{t_0} = v \}
= P \{ D_1 \setminus D_\tau | \ell_{t_1} = u, \ell_{t_0} = v \} P \{ D_\tau | \ell_{t_1} = u, \ell_{t_0} = v \} \quad (33)
\]

The last step follows because \( \tau \) does not feature in \( D_1 \setminus D_\tau \), so knowing its label does not affect \( P \{ D_1 \setminus D_\tau | \ell_{t_1} = v \} \). Starting from the definition of \( \mu_{\tau \to 0} \) in (26), we have:

\[
\mu_{\tau \to 0} = \log \frac{P \{ D_1 \setminus D_\tau | \ell_{t_1} = 0, \ell_{t_0} = 1 \}}{P \{ D_1 \setminus D_\tau | \ell_{t_1} = 0, \ell_{t_0} = 0 \}}
= \log \frac{P \{ D_1 \setminus D_\tau | \ell_{t_1} = 0 \} P \{ D_\tau | \ell_{t_1} = 0, \ell_{t_0} = 1 \}}{P \{ D_1 \setminus D_\tau | \ell_{t_1} = 0 \} P \{ D_\tau | \ell_{t_1} = 0, \ell_{t_0} = 0 \}}
= \Lambda_{\tau}^{t} + \log \frac{P \{ D_\tau | \ell_{t_1} = 0, \ell_{t_0} = 1 \}}{P \{ D_\tau | \ell_{t_1} = 0, \ell_{t_0} = 0 \}} \quad (34)
\]
To complete (28), it remains to establish (29) and the following:

\[
\log \frac{\mathbb{P} \{ D_0 \setminus D_r | \ell_{\tau_0} = 1, \ell_r = 0 \}}{\mathbb{P} \{ D_0 \setminus D_r | \ell_{\tau_0} = 0, \ell_r = 0 \}} = (\theta_1^* - \theta_0^*) \left( \log \frac{\tau_0}{T} + \sum_{t \in \partial \tau_0} \log \frac{t}{T} \right) + \sum_{t \in \partial \tau_0 \setminus \{ \tau \}} \log \left( \frac{2\theta_{1,1}^* \frac{\rho}{1-\rho} e^{v_1 t + \rho} e^{v_1 t + \rho} + 1}{2\theta_{1,0,0}^* \frac{\rho}{1-\rho} e^{v_1 t + \rho} e^{v_1 t + \rho} + 1} \right). \tag{35} \]

(36)

We show (36) first. The expression for \( \mathbb{P} \{ D_0 \setminus D_r | \ell_r = u, \ell_{\tau_0} = v \} \) is very similar to the expression for \( \mathbb{P} \{ D_0 | \ell_{\tau_0} = u \} \). The latter was treated as a modification of the expression \( \mathbb{P} \{ D_1 | \ell_{\tau_0} = v \} \), with different beliefs about the children’s labels depending upon the children’s descendants. Here, we know exactly that the label of one of the children, \( \tau \). Following a derivation similar to the one in (32), we get:

\[
\mathbb{P} \{ D_0 \setminus D_r | \ell_r = u, \ell_{\tau_0} = v \} \mathbb{P} \{ D_1 | \ell_{\tau_0} = v \} \theta^*_{u,v} \times \prod_{t \in \partial \tau_0 \setminus \{ \tau \}} \left( \sum_{u' \in \{0,1\}} \mathbb{P} \{ D_t | \ell_t = u' \} \mathbb{P} \{ \ell_t = u' \} \theta^*_{u',v} \right) \tag{37} \]

From (37), (36) follows using the same steps as in the derivation of the expression for \( \nu_{\tau \to \tau_0} \).

Finally, we derive (29). Starting from the definition of \( \Lambda_r^p \) in (23), we have:

\[
\Lambda_r^p = \log \frac{\mathbb{P} \{ D_1 | D_r | \ell_r = 1 \}}{\mathbb{P} \{ D_1 | D_r | \ell_r = 0 \}} = \log \frac{\mathbb{P} \{ D_1 | D_r | \ell_r = 1, \ell_{\tau_0} = 1 \} \rho + \mathbb{P} \{ D_1 | D_r | \ell_r = 1, \ell_{\tau_0} = 0 \} (1-\rho)}{\mathbb{P} \{ D_1 | D_r | \ell_r = 0, \ell_{\tau_0} = 1 \} \rho + \mathbb{P} \{ D_1 | D_r | \ell_r = 0, \ell_{\tau_0} = 0 \} (1-\rho)} \tag{38} \]

Using \( \theta^*_{1,1} = \beta \theta^*_{1,0} \theta^*_{0,0} = 2 \theta^*_{1,0} \), and (37) yields:

\[
\mathbb{P} \{ D_1 | D_r | \ell_r = 1, \ell_{\tau_0} = 1 \} = \beta \mathbb{P} \{ D_1 | D_r | \ell_r = 0, \ell_{\tau_0} = 1 \} = \beta \mathbb{P} \{ D_1 | D_r | \ell_r = 0, \ell_{\tau_0} = 0 \} = 2 \theta^*_{1,0} \]

which together with (38) yields (29).