FREE CHOOSABILITY OF THE CYCLE

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Abstract. A graph \( G \) is free \((a, b)\)-choosable if for any vertex \( v \) with \( b \) colors assigned and for any list of colors of size \( a \) associated with each vertex \( u \neq v \), the coloring can be completed by choosing for \( u \) a subset of \( b \) colors such that adjacent vertices are colored with disjoint color sets. In this note, a necessary and sufficient condition for a cycle to be free \((a, b)\)-choosable is given. As a corollary, some choosability results are derived for graphs in which cycles are connected by a tree structure.

1. Introduction

For a graph \( G \), we denote its vertex set by \( V(G) \) and edge set by \( E(G) \). A color-list \( L \) of a graph \( G \) is an assignment of lists of integers (colors) to the vertices of \( G \). For an integer \( a \), an \( a \)-color-list \( L \) of \( G \) is a color-list such that \( |L(v)| = a \) for any \( v \in V(G) \).

In 1996, Voigt considered the following problem: let \( G \) be a graph and \( L \) a color-list and assume that an arbitrary vertex \( v \in V(G) \) is precolored by a color \( f \in L(v) \). Under which hypothesis is it always possible to complete this precoloring to a proper color-list coloring? This question leads to the concept of free choosability introduced by Voigt [8].

Formally, for a graph \( G \), integers \( a, b \) and an \( a \)-color-list \( L \) of \( G \), an \((L, b)\)-coloring of \( G \) is a mapping \( c \) that associates to each vertex \( u \) a subset \( c(u) \) of \( L(u) \) such that \( |c(u)| = b \) and \( c(u) \cap c(v) = \emptyset \) for any edge \( uv \in E(G) \).

The graph \( G \) is \((a, b)\)-choosable if for any \( a \)-color-list \( L \) of \( G \), there exists an \((L, b)\)-coloring. Moreover, \( G \) is free \((a, b)\)-choosable if for any \( a \)-color-list \( L \), any vertex \( v \) and any set \( c_0 \subset L(v) \) of \( b \) colors, there exists an \((L, b)\)-coloring \( c \) such that \( c(v) = c_0 \).

As shown by Voigt [8], there are examples of graphs \( G \) that are \((a, b)\)-choosable but not free \((a, b)\)-choosable. Some related recent results concern defective free choosability of planar graphs [6]. We investigate, in the next section, the free-choosability of the first interesting case, namely the cycle. We derive a necessary and sufficient condition for a cycle to be free \((a, b)\)-choosable (Theorem 4). In order to get a concise statement, we introduce the free-choice ratio of a graph, in the same way that Alon, Tuza and Voigt [1] introduced the choice ratio (which is equal to the so-called fractional chromatic number).

For any real \( x \), let \( \text{FCH}(x) \) be the set of graphs \( G \) which are free \((a, b)\)-choosable for all \( a, b \) such that \( \frac{a}{b} \geq x \):

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\begin{align*}
\text{FCH}(x) &= \{G \mid \forall \frac{a}{b} \geq x, \ G \text{ is free } (a, b)\text{-choosable}\}. \\
\text{Moreover, we can define the free-choice ratio fchr}(G) \text{ of a graph } G \text{ by:}
\end{align*}
\begin{align*}
fchr(G) := \inf \{ \frac{a}{b} \mid G \text{ is free } (a, b)\text{-choosable}\}. \\
\text{Remark 1. \ Erdős, Rubin and Taylor have asked \cite{3} the following question: \ If } G \text{ is } (a, b)\text{-choosable, and } \frac{a}{b} > \frac{d}{e}, \text{ does it imply that } G \text{ is } (c, d)\text{-choosable? \ Gutner and Tarsi have shown \cite{5} that the answer is negative in general. \ If we consider the analogue question for the free choosability, then Theorem \cite{4} implies that it is true for the cycle.}
\end{align*}

The path \( P_{n+1} \) of length \( n \) is the graph with vertex set \( V = \{v_0, v_1, \ldots, v_n\} \) and edge set \( E = \bigcup_{i=0}^{n-1} \{v_iv_{i+1}\} \). The cycle \( C_n \) of length \( n \) is the graph with vertex set \( V = \{v_0, \ldots, v_{n-1}\} \) and edge set \( E = \bigcup_{i=0}^{n-1} \{v_iv_{i+1(\text{mod} \ n)}\} \). To simplify the notation, for a color-list \( L \) of \( P_n \) or \( C_n \), we let \( L(i) = L(v_i) \) and \( c(i) = c(v_i) \).

The notion of waterfall color-list was introduced in \cite{2} to obtain choosability results on the weighted path and then used to prove the \((5m, 2m)\)-choosability of triangle-free induced subgraphs of the triangular lattice. We recall one of the results from \cite{2} that will be used in this note, with the function \( \text{Even} \) being defined for any real \( x \) by: \( \text{Even}(x) \) is the smallest even integer \( p \) such that \( p \geq x \).

**Proposition 2 (\cite{2}).** Let \( L \) be a color-list of \( P_{n+1} \) such that \( |L(0)| = |L(n)| = b, \) and \( |L(i)| = a = 2b + e \) for all \( i \in \{1, \ldots, n-1\} \) (with \( e > 0 \)).

If \( n \geq \text{Even}\left(\frac{2b}{e}\right) \) then \( P_{n+1} \) is \((L, b)\)-colorable.

For example, let \( P_{n+1} \) be the path of length \( n \) with a color-list \( L \) such that \( |L(0)| = |L(n)| = 4, \) and \( |L(i)| = 9 \) for all \( i \in \{1, \ldots, n-1\} \). Then the previous proposition tells us that we can find an \((L, 4)\)-coloring of \( P_{n+1} \) whenever \( n \geq 8 \). In other words, if \( n \geq 8 \), we can choose 4 colors on each vertex such that adjacent vertices receive disjoint colors sets. If \( |L(i)| = 11 \) for all \( i \in \{1, \ldots, n-1\} \), then \( P_{n+1} \) is \((L, 4)\)-colorable whenever \( n \geq 4 \). On the other side, there are color-lists \( L \) for which \( P_{n+1} \) is not \((L, b)\)-colorable.

\section{Free choosability of the cycle}

We begin with a negative result for the even-length cycle:

**Lemma 3.** For any integers \( a, b, p \) such that \( p \geq 2, \) and \( \frac{a}{b} < 2 + \frac{1}{p} \), the cycle \( C_{2p} \) is not free \((a, b)\)-choosable.

\textbf{Proof.} We construct a counterexample for the free-choosability of \( C_{2p} \): let \( L \) be the \( a \)-color-list of \( C_{2p} \) such that

\begin{align*}
L(i) &= \begin{cases} 
\{1, \ldots, a\}, & \text{if } i \in \{0, 1\}; \\
\{\frac{i-1}{2}a + 1, \ldots, \frac{i-1}{2}a + a\}, & \text{if } i \neq 2p - 1 \text{ is odd}; \\
\{b + \frac{i-2}{2}a + 1, \ldots, b + \frac{i-2}{2}a + (\frac{i-2}{2} + 1)a\}, & \text{if } i \text{ is even and } i \neq 0; \\
\{1, \ldots, b, 1 + (p - 1)a, \ldots, 1 + pa - b - 1\}, & \text{if } i = 2p - 1.
\end{cases}
\end{align*}
The cycle $C_6$, along with a 9-color-list $L$ for which there is no $(L,4)$-coloring $c$ such that $c(v_0) = \{1, 2, 3, 4\}$.

If we choose $c_0 = \{1, \ldots, b\} \subset L(0)$, we can check that there does not exist an $(L, b)$-coloring of $C_{2p}$ such that $c(0) = c_0$, so $C_{2p}$ is not free $(a, b)$-choosable. See Figure 1 for an illustration when $p = 3$, $a = 9$ and $b = 4$. □

Now, if $\lfloor x \rfloor$ denotes the greatest integer less than or equal to the real $x$, we can state:

**Theorem 4.** For the cycle $C_n$ of length $n$,

$C_n \in \text{FCH} \left( 2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1} \right)$.

Moreover, we have:

$f\text{chr}(C_n) = 2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}$.

**Proof.** Let $a, b$ be two integers such that $a/b \geq 2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}$. Let $L$ be a $a$-color-list of $C_n$. Without loss of generality, we can suppose that $v_0$ is the vertex chosen for the free-choosability and $c_0 \subset L(v_0)$ has $b$ elements. It remains to construct an $(L, b)$-coloring $c$ of $C_n$ such that $c(v_0) = c_0$. Hence we have to construct an $(L', b)$-coloring $c$ of $P_{n+1}$ such that $L'(0) = L'(n) = L_0$ and for all $i \in \{1, \ldots, n-1\}$, $L'(i) = L(v_i)$. We have $|L'(0)| = |L'(n)| = b$ and for all $i \in \{1, \ldots, n-1\}$, $|L'(i)| = a$. Since $a/b \geq 2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}$ and $e = a - 2b$, we get $e/b \geq \left\lfloor \frac{n}{2} \right\rfloor^{-1}$ hence $n \geq \text{Even}(2b/e)$. Using Proposition 2 we get:

$C_n \in \text{FCH}(2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1})$.

Hence, we have that $f\text{chr}(C_n) \leq 2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}$. Moreover, let us prove that $M = 2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}$ is reached. For $n$ odd, Voigt has proved [9] that the choice ratio $\text{chr}(C_n)$ of a cycle of odd length $n$ is exactly $M$. Hence $f\text{chr}(C_n) \geq \text{chr}(C_n) = M$, and the result is proved. For $n$ even, Lemma 3 asserts that $C_n$ is not free $(a, b)$-choosable for $\frac{a}{b} < 2 + \left\lfloor \frac{n}{2} \right\rfloor^{-1}$. □

**Remark 5.** In particular, the previous theorem implies that if $b, e, n$ are integers such that $n \geq \text{Even}(\frac{2b}{e})$, then the cycle $C_n$ of length $n$ is free $(2b + e, b)$-choosable.
In order to extend the result to other graphs than cycles, the following
simple proposition will be useful:

**Proposition 6.** Let \( a, b \) be integers with \( a \geq 2b \). Let \( G \) be a graph and \( G_v \) be the graph obtained by adding a leave \( v \) to any vertex of \( G \). Then \( G \) is free \((a,b)\)-choosable if and only if \( G_v \) is free \((a,b)\)-choosable.

**Proof.** Since the "only if" part holds trivially, let us prove the "if" part. Assume \( G \) is free \((a,b)\)-choosable and let \( L \) be a \( a \)-color-list of \( G \). Let \( v \) be a new vertex and let \( G_v \) be the graph obtained from \( G \) by adding the edge \( uv \), for some \( u \in V(G) \). Then any \((L,b)\)-coloring \( c \) of \( G \) can be extended to an \((L,b)\)-coloring of \( G_v \) by giving to \( v \) \( b \) colors from \( L(v) \setminus c(u) \) (\( |L(v) \setminus c(u)| \geq b \) since \( a \geq 2b \)). If \( v \) is colored with \( b \) colors from its list, then, since \( G \) is free \((a,b)\)-choosable, the coloring can be extended to an \((L,b)\)-coloring of \( G_v \) by first choosing for \( u \) a set of \( b \) colors from \( L(u) \setminus c(v) \).

Starting from a single edge and applying inductively Proposition 6 allows to obtain the following corollary:

**Corollary 7.** Let \( T \) be a tree of order \( n \geq 2 \). Then
\[
T \in \text{FCH}(2).
\]

Now, we can state the following:

**Proposition 8.** If \( G \) is a unicyclic graph with girth \( g \), then
\[
G \in \text{FCH}\left(2 + \left\lfloor \frac{g}{2} \right\rfloor^{-1}\right).
\]

**Proof.** Let \( a, b \) be two integers such that \( a/b \geq 2 + \left\lfloor \frac{a}{2} \right\rfloor^{-1} \), \( L \) be a \( a \)-color-list of \( G \), \( C = v_1, \ldots, v_g \) be the unique cycle (of length \( g \)) of \( G \) and \( T_i, i \in \{1, \ldots, g\} \), be the subtree of \( G \) rooted at vertex \( v_i \) of \( C \).

Let \( v \) be the vertex chosen for the free choosability and let \( c_0 \subset L(v) \) be a set of cardinality \( b \). If \( v \in C \), then by virtue of Theorem 3 there exists an \((L,b)\)-coloring \( c \) of \( C \) such that \( c(v) = c_0 \). This coloring can be easily extended to the whole graph by coloring the vertices of each tree \( T_i \) thanks to Corollary 7. If \( v \in T_i \) for some \( i \), \( 1 \leq i \leq g \), then Corollary 7 asserts that the coloring can be extended to \( T_i \). Then color \( C \) starting at vertex \( v_i \) by using Theorem 3. Finally, complete it by coloring each tree \( T_j \), \( 1 \leq j \neq i \leq g \).

\[ \square \]

## 3. Applications

As an example to the possible use of the results from Section 2, we begin with determining the free choosability of a binocular graph, i.e. two cycles linked by a path.

For integers \( m, n \) and \( p \) such that \( m, n \geq 3 \) and \( p \geq 0 \), the **binocular graph** \( \text{BG}(m, n, p) \) is the disjoint union of an \( m \)-cycle \( u_0, u_1, \ldots, u_{m-1} \) and of an \( n \)-cycle \( v_0, \ldots, v_{n-1} \) with vertices \( u_0 \) and \( v_0 \) linked by a path of length \( p \) given by \( u_0, x_1, \ldots, x_{p-1}, v_0 \). Note that if \( p = 0 \), then \( u_0 \) and \( v_0 \) are the same vertex.

**Proposition 9.** For any \( m \geq 3, n \geq 3 \) and \( p \geq 0 \),
\[
\text{BG}(m, n, p) \in \text{FCH} \left( 2 + \left\lfloor \frac{\min(m, n)}{2} \right\rfloor^{-1} \right).
\]
Proof. Assume without loss of generality that \( m \geq n \) and let \( a, b \) be integers such that \( \frac{a}{b} \geq 2 + \left\lfloor \frac{n}{2} \right\rfloor - 1 \). Let \( L \) be a \( a \)-color-list of \( BG(m, n, p) \). Let \( y \) be the vertex chosen for the free choosability and let \( c_0 \subset L(y) \) be a set of cardinality \( b \). If \( y \) lies on the \( m \)-cycle, then by virtue of Theorem 4, there exists an \((L, b)\)-coloring \( c \) of the \( m \)-cycle such that \( c(y) = c_0 \). By Corollary 7, this coloring can be extended to the vertices of the path. Now, it remains to color the vertices of the \( n \)-cycle, with \( v_0 \) being already colored. This can be done thanks to Theorem 4. If \( y \) lies on the \( n \)-cycle, the argument is similar. If \( y \in \{x_1, \ldots, x_{p-1}\} \), then the coloring can be extended to the whole path and the coloring of the \( m \)-cycle and \( n \)-cycle can be completed thanks to Theorem 4.

This method can be extended to prove similar results on graphs with more than two cycles, connected by a tree structure.

Define a tree of cycles to be a graph \( G \) such that all its cycles are disjoint and collapsing all vertices of each cycle of \( G \) produces a tree.

**Corollary 10.** Any tree of cycles of girth \( g \) is in \( \text{FCH}(2 + \left\lfloor \frac{g}{2} \right\rfloor - 1) \).

### 4. Algorithmic Considerations

Let \( n \geq 3 \) be an integer and let \( a, b \) be two integers such that \( \frac{a}{b} \geq 2 + \left\lfloor \frac{n}{2} \right\rfloor - 1 \). Let \( L \) be a \( a \)-color-list of \( C_n \).

As defined in \cite{2}, a waterfall list \( L \) of a path \( P_{n+1} \) of length \( n \) is a list \( L \) such that for all \( i, j \in \{0, \ldots, n\} \) with \( |i - j| \geq 2 \), we have \( L(i) \cap L(j) = \emptyset \). Let \( m = | \bigcup_{i=0}^{n} L(i) | \) be the total number of colors of the color-list \( L \).

The algorithm behind the proof of Proposition 2 consists in three steps: first, the transformation of the list \( L \) into a waterfall list \( L' \) by renaming some colors; second, the construction of the \((L', b)\)-coloring by coloring vertices from 0 to \( n - 1 \), giving to vertex \( i \) the first \( b \)-colors that are not used by the previous vertex; third, the backward transformation to obtain an \((L, b)\)-coloring from the \((L', b)\)-coloring by coming back to original colors and resolving color conflicts if any. It can be seen that the time complexity of the first step is \( O(mn) \); that of the second one is \( O(a^2 n) \) and that of the third one is \( O(\max(a, b^3)n) \). Therefore, the total running time for computing a free \((L, b)\)-coloring of the cycle \( C_n \) is \( O(\max(m, a^2, b^3)n) \).


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