Abstract

We give a simple, closed-form formula, what we call the Deflation Identity, for converting any 2-qubit circuit with exactly two controlled-U’s (and some 1-qubit rotations) into an equivalent circuit with just two CNOTs (and some 1-qubit rotations). We also give two interesting applications of the Deflation Identity; one to “opening and closing a breach” in a quantum circuit, the other to the CS decomposition of a 2-qubit operator.

1 Introduction and Motivation

In 1995, Ref.[1] showed that any controlled-U gate can be expressed with just 2 CNOTs (and some 1-qubit rotations); i.e.,

\[ \begin{array}{c}
\text{\ include image } \\
\end{array} \] \quad (1),

where the empty boxes represent 2-dimensional unitary matrices.

In 2003, Vidal and Dawson[2] showed that any 2-qubit operator can be expressed by a circuit containing just 3 CNOTs (and some 1-qubit rotations).

In light of the Vidal-Dawson result, a quantum circuit that contains exactly two controlled-U’s does not require 4 CNOTs (i.e., twice the number of CNOTs required by one controlled-U) to express; rather, it can be expressed with just 3
CNOTs. Actually, it turns out that even 3 CNOTs is more than what is needed. In this paper we show that any quantum circuit with exactly two controlled-U’s can be expressed with just 2 CNOTs; i.e., we show that

\[
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{circuit1.png}}
\end{array}
\]

and

\[
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{circuit2.png}}
\end{array}
\]

In this paper, we give a simple, closed-form formula, what we call the Deflation Identity, for performing the circuit conversions illustrated by Eq. (2). We also give two interesting applications of the Deflation Identity; one to “opening and closing a breach” in a quantum circuit, the other to the CS decomposition of a 2-qubit operator.

Note that I am including with the paper an Octave/Matlab m-file that checks the Deflation Identity numerically.

2 Notation

In this section, we will define some notation that is used throughout this paper. For additional information about our notation, see Ref.[3].

As usual, \( \mathbb{R}, \mathbb{C} \) will stand for the real and complex numbers, respectively. For any complex matrix \( A \), the symbols \( A^*, A^T, A^\dagger \) will stand for the complex conjugate, transpose, and Hermitian conjugate, respectively, of \( A \). (Hermitian conjugate a.k.a. conjugate transpose and adjoint)

The Pauli matrices are defined by:

\[
\begin{align*}
\sigma_X &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma_Y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\sigma_Z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
\]

They satisfy

\[
\sigma_X \sigma_Y = -\sigma_Y \sigma_X = i \sigma_Z,
\]

and the two other equations obtained from this one by permuting the indices \( (X,Y,Z) \) cyclically. We will also have occasion to use the operators \( n = \frac{1}{2}(1 - \sigma_Z), \quad \overline{n} = 1 - n \) and \( \vec{\sigma} = (\sigma_X, \sigma_Y, \sigma_Z) \). For any \( \vec{\theta}, \vec{a} \in \mathbb{R}^3 \),

\[
e^{-i \vec{\theta} \cdot \vec{\sigma}}(\vec{\sigma} \cdot \vec{a})e^{i \vec{\theta} \cdot \vec{\sigma}} = \vec{\sigma} \cdot \vec{b},
\]

where \( \vec{b} \) is the vector obtained by rotating \( \vec{a} \) by an angle \( |\vec{\theta}| \), with respect to an axis parallel to \( \vec{\theta} \). (The sense of the rotation is determined by the right hand rule, meaning that if the thumb of your right hand points in the direction \( \vec{\theta} \), then your other right hand fingers point from \( \vec{a} \) to \( \vec{b} \).)
In our quantum circuits, \( \overbrace{}^{D} \) will represent a 2 dimensional unitary matrix, whereas \( \overbrace{R_{D}}^{D} \) with \( D = X, Y, Z \) will represent a 1-qubit rotation \( e^{i\theta \sigma D} \) in the \( D \) direction.

Let \( \sigma_{X_{\mu}} \) for \( \mu \in \{0,1,2,3\} \) be defined by \( \sigma_{X_{0}} = \sigma_{1} = I_{2} \), where \( I_{2} \) is the 2 dimensional identity matrix, \( \sigma_{X_{1}} = \sigma_{X} \), \( \sigma_{X_{2}} = \sigma_{Y} \), and \( \sigma_{X_{3}} = \sigma_{Z} \). Now define

\[
\sigma_{X_{\mu}X_{\nu}} = \sigma_{X_{\mu}} \otimes \sigma_{X_{\nu}}
\]

for \( \mu, \nu \in \{0,1,2,3\} \). For example, \( \sigma_{XY} = \sigma_{X} \otimes \sigma_{Y} \) and \( \sigma_{1X} = I_{2} \otimes \sigma_{X} \). The matrices \( \sigma_{X_{\mu}X_{\nu}} \) satisfy

\[
\sigma_{XX} \sigma_{YY} = \sigma_{YY} \sigma_{XX} = -\sigma_{ZZ},
\]

and the two other equations obtained from this one by permuting the indices \((X,Y,Z)\) cyclically.

We will also have occasion to use the exchange operator for two qubits, defined by

\[
\updownarrow = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(If the rows and columns of this matrix are labelled by \((a_{1}, a_{0}) = 00, 01, 10, 11\), where \( a_{0} \) labels the state of qubit 0 and \( a_{1} \) that of qubit 1, then \( \updownarrow \) indeed exchanges the two qubits.) A useful identity involving \( \updownarrow \) is:

\[
\begin{pmatrix}
a & 0 & b & 0 \\
0 & \alpha & 0 & \beta \\
c & 0 & d & 0 \\
0 & \gamma & 0 & \delta
\end{pmatrix} = \updownarrow
\begin{pmatrix}
a & b & 0 & 0 \\
c & d & 0 & 0 \\
0 & 0 & \alpha & \beta \\
0 & 0 & \gamma & \delta
\end{pmatrix} \updownarrow.
\]

Define

\[
\mathcal{M} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & i \\
0 & i & 1 & 0 \\
0 & i & -1 & 0 \\
1 & 0 & 0 & -i
\end{pmatrix}.
\]

It is easy to check that \( \mathcal{M} \) is a unitary matrix. The columns of \( \mathcal{M} \) are an orthonormal basis, often called the “magic basis” in the quantum computing literature. (That’s why we have chosen to call this matrix \( \mathcal{M} \), because of the “m” in magic).

In this paper, we often need to find the outcome of applying a similarity transformation (i.e., a change of basis) \( \mathcal{M}^{\dagger}(\cdot)\mathcal{M} \) (or \( \mathcal{M}(\cdot)\mathcal{M}^{\dagger} \)) to a matrix \( X \in \mathbb{C}^{4 \times 4} \). Since \( X \) can always be expressed as a linear combination of the \( \sigma_{X_{\mu}X_{\nu}} \), it is useful
to know the outcomes $\mathcal{M}^\dagger(\sigma_{X\mu}X\nu)\mathcal{M}$ (or $\mathcal{M}(\sigma_{X\mu}X\nu)\mathcal{M}^\dagger$) for $\mu, \nu \in \{0, 1, 2, 3\}$. One finds the following two tables:

$$
\mathcal{M}^\dagger(A \otimes B)\mathcal{M} =
\begin{array}{c|ccc}
\text{ } & B & \rightarrow \\
\hline
1 & \sigma_X & \sigma_Y & \sigma_Z \\
\sigma_X & -\sigma_{1Y} & \sigma_{YZ} & -\sigma_{YX} \\
\downarrow \sigma_Y & -\sigma_{1X} & \sigma_{XY} & -\sigma_{1Z} & \sigma_{1X} \\
\sigma_Z & -\sigma_{XY} & \sigma_{X1} & \sigma_{ZX} & \sigma_{ZZ} \\
\end{array}, \tag{11}
$$

and

$$
\mathcal{M}(A \otimes B)\mathcal{M}^\dagger =
\begin{array}{c|ccc}
\text{ } & B & \rightarrow \\
\hline
1 & \sigma_X & \sigma_Y & \sigma_Z \\
\sigma_X & -\sigma_{1Z} & -\sigma_{1X} & -\sigma_{YZ} \\
\downarrow \sigma_Y & -\sigma_{1X} & \sigma_{XY} & -\sigma_{1Z} & \sigma_{1Y} \\
\sigma_Z & \sigma_{XX} & \sigma_{XY} & -\sigma_{X1} & \sigma_{ZZ} \\
\end{array}, \tag{12}
$$

Another similarity transformation that we shall often encounter in his paper is $\vec{\sigma}(\cdot)\vec{\sigma}$ (and its twin $\vec{\sigma}(\cdot)\vec{\sigma}$). One finds

$$
\vec{\sigma}(A \otimes B)\vec{\sigma} =
\begin{array}{c|ccc}
\text{ } & B & \rightarrow \\
\hline
1 & \sigma_X & \sigma_Y & \sigma_Z \\
\sigma_X & \sigma_{1X} & \sigma_{YZ} & \sigma_{ZZ} \\
\downarrow \sigma_Y & \sigma_{XX} & \sigma_{X1} & \sigma_{Y1} & \sigma_{YY} \\
\sigma_Z & \sigma_{X1} & \sigma_{XZ} & \sigma_{YY} & \sigma_{XY} \\
\end{array}, \tag{13}
$$

and

$$
\vec{\sigma}(A \otimes B)\vec{\sigma} =
\begin{array}{c|ccc}
\text{ } & B & \rightarrow \\
\hline
1 & \sigma_X & \sigma_Y & \sigma_Z \\
\sigma_X & \sigma_{XX} & \sigma_{XY} & \sigma_{1Z} \\
\downarrow \sigma_Y & \sigma_{XY} & \sigma_{XY} & \sigma_{1Y} & \sigma_{XZ} \\
\sigma_Z & \sigma_{XY} & \sigma_{X1} & \sigma_{1Y} & \sigma_{ZZ} \\
\end{array}. \tag{14}
$$

### 3 Statement and Proof of Deflation Identity

In this section, we will present Theorem 1 which is referred to in this paper as the Deflation Identity. We will also present a simple generalization, given by Theorem 2 of the Deflation Identity.
Theorem 1 For any real numbers \( \theta_L, \beta, \beta' \) and \( \theta_R \), if

\[
U = e^{i\theta_L \sigma_Z} e^{i\beta \sigma_Y} e^{i\theta_R \sigma_Z} e^{i\beta' \sigma_Y},
\]

then

\[
U = e^{i\frac{\theta_L}{2} \sigma_Z} e^{i\gamma_L \sigma_Y} e^{i\mu \sigma_Z} e^{i\gamma_R \sigma_Y} e^{i\frac{\theta_R}{2} \sigma_Z} e^{i\gamma'_L \sigma_Y} e^{i\mu' \sigma_X} e^{i\gamma'_R \sigma_Y},
\]

where the real numbers \( \gamma_L, \gamma'_L, \mu, \mu', \gamma_R, \gamma'_R \) are defined as follows. Let

\[
x_{1\pm} = \cos\left(\frac{\theta_L + \theta_R}{2}\right) \cos(-\beta' \pm \beta),
\]

\[
x_{2\pm} = \cos\left(\frac{\theta_L - \theta_R}{2}\right) \sin(-\beta' \pm \beta),
\]

and

\[
p_{\pm} = \sqrt{x_{1\pm}^2 + x_{2\pm}^2}.
\]

Suppose \( s \in \{+, -\} \). If \( p_s = 0 \), then \( \xi_s = 0 \). Otherwise,

\[
\cos \xi_s = \frac{x_{1\pm}}{p_{\pm}}, \quad \sin \xi_s = \frac{x_{2\pm}}{p_{\pm}}.
\]

Let

\[
y_{1\pm} = \mp \sin\left(\frac{\theta_L - \theta_R}{2}\right) \sin(-\beta' \pm \beta),
\]

\[
y_{2\pm} = \mp \sin\left(\frac{\theta_L + \theta_R}{2}\right) \cos(-\beta' \pm \beta),
\]

and

\[
q_{\pm} = \sqrt{y_{1\pm}^2 + y_{2\pm}^2}.
\]

Suppose \( s \in \{+, -\} \). If \( q_s = 0 \), then \( \eta_s = 0 \). Otherwise,

\[
\cos \eta_s = \frac{y_{1\pm}}{q_{\pm}}, \quad \sin \eta_s = \frac{y_{2\pm}}{q_{\pm}}.
\]

The \( \gamma \)'s are defined by

\[
\gamma_L = \frac{1}{4}(\eta_+ - \eta_- + \xi_+ - \xi_-),
\]
\[ \gamma'_L = \frac{1}{4}(-\eta_+ - \eta_- - \xi_+ - \xi_- + \pi), \quad (25b) \]
\[ \gamma_R = \frac{1}{4}(-\eta_+ + \eta_- + \xi_+ - \xi_-), \quad (25c) \]
and
\[ \gamma'_R = \frac{1}{4}(+\eta_+ + \eta_- - \xi_+ - \xi_- - \pi). \quad (25d) \]

The \( \mu \)'s are defined by
\[ \cos \mu_\pm = p_\pm, \quad \sin \mu_\pm = q_\pm, \quad (26) \]
and
\[ \mu = \frac{\mu_+ - \mu_-}{2}, \quad \mu' = \frac{\mu_+ + \mu_-}{2}. \quad (27) \]

**proof:**

Note that
\[ e^{i\theta \sigma Z} e^{i\theta \sigma Z} = e^{i\theta \sigma Z(0)n(1)} = e^{i\theta (\sigma_1Z - \sigma_2Z)}. \quad (28) \]

For the matrix \( U \) given by Eq.(15), we can define a matrix \( V \) by:
\[ U = e^{i\frac{\theta_L}{2} \sigma_1Z} V e^{i\frac{\theta_R}{2} \sigma_1Z}. \quad (29) \]
Thus, \( V \) is merely \( U \) with the local \( \sigma_1Z \) rotations on both ends removed:
\[ V = e^{-i\frac{\theta_L}{2} \sigma_2Z} e^{i(\beta' \sigma Y_1 + \beta \sigma Y_1)} e^{-i\frac{\theta_R}{2} \sigma_2Z}. \quad (30) \]

Let
\[ \tilde{V} = M^\dagger V M. \quad (31) \]

Applying the table given in Eq.(11) to Eq.(30) yields:
\[ \tilde{V} = e^{-i\frac{\theta_L}{2} \sigma_2Z} e^{i(-\beta' \sigma Y_1 + \beta \sigma Y_2)} e^{-i\frac{\theta_R}{2} \sigma_2Z}. \quad (32) \]

We want to consider the real and imaginary parts of \( \tilde{V} \), defined by:
\[ \tilde{V}_1 = \frac{\tilde{V} + (\tilde{V})^*}{2}, \quad \tilde{V}_2 = \frac{\tilde{V} - (\tilde{V})^*}{2i}. \quad (33) \]

Using Eqs.(32) and (33), we can find more explicit expressions for \( \tilde{V}_1 \) and \( \tilde{V}_2 \). If
\[ C_L = \cos\left(\frac{\theta_L}{2}\right), \quad S_L = \sin\left(\frac{\theta_L}{2}\right), \quad \Lambda = -\beta' \sigma_1Y + \beta \sigma_2Y, \quad (34) \]

6
these explicit expressions for $\tilde{V}_1$ and $\tilde{V}_2$ are:

$$\tilde{V}_1 = \uparrow [CL_CRe^{i\Lambda} - SL_SRe^{-i\Lambda}] \downarrow,$$  \hspace{1cm} (35)

and

$$\tilde{V}_2 = \uparrow (-\sigma ZZ)[SL_CRe^{i\Lambda} - CL_SRe^{-i\Lambda}] \downarrow.$$ \hspace{1cm} (36)

Now observe that $\tilde{V}_1$ and $\tilde{V}_2$ are both of the following form: for $j = 1, 2$,

$$\tilde{V}_j = \uparrow (A_{j+} + 0 0 A_{j-}) \downarrow.$$ \hspace{1cm} (37)

$A_{1\pm}$ is of the form

$$A_{1\pm} = x_{1\pm} + i\sigma_Y x_{2\pm} = p_\pm e^{i\sigma_Y \xi_\pm}.$$ \hspace{1cm} (38)

This last equation defines $x_{1\pm}, x_{2\pm}, p_\pm$ and $\xi_\pm$. If we define $\xi^L_\pm$ and $\xi^R_\pm$ to be arbitrary real numbers such that

$$\xi^L_\pm + \xi^R_\pm = \xi_\pm,$$ \hspace{1cm} (39)

then

$$A_{1\pm} = e^{i\sigma_Y \xi^L_\pm} p_\pm e^{i\sigma_Y \xi^R_\pm}.$$ \hspace{1cm} (40)

$A_{2\pm}$ is of the form

$$A_{2\pm} = y_{1\pm} \sigma_X + y_{2\pm} \sigma_Z = q_\pm e^{i\sigma_Y \eta_\pm} \sigma_X.$$ \hspace{1cm} (41)

This last equation defines $y_{1\pm}, y_{2\pm}, q_\pm$ and $\eta_\pm$. Define $\phi_\pm$ by

$$\phi_\pm - \frac{\pi}{2} = -\eta_\pm + \xi^L_\pm - \xi^R_\pm.$$ \hspace{1cm} (42)

Then

$$e^{-i\sigma_Y \xi^L_\pm} (A_{2\pm}) e^{-i\sigma_Y \xi^R_\pm} = e^{-i\sigma_Y \xi^L_\pm} (q_\pm e^{i\sigma_Y \eta_\pm} \sigma_X) e^{-i\sigma_Y \xi^R_\pm}$$ \hspace{1cm} (43a)

$$= q_\pm \sigma_X e^{i\sigma_Y (\phi_\pm - \frac{\pi}{2})}$$ \hspace{1cm} (43b)

$$= q_\pm [\sin(\phi_\pm) \sigma_X + \cos(\phi_\pm) \sigma_Z]$$ \hspace{1cm} (43c)

$$= e^{-i\sigma_Y \frac{\phi_\pm}{2}} q_\pm \sigma_Z e^{i\sigma_Y \frac{\phi_\pm}{2}}.$$ \hspace{1cm} (43d)

Inserting these explicit expressions for $A_{1\pm}$ and $A_{2\pm}$ into Eq. (37) yields

$$\tilde{V} = \uparrow \left[ \begin{array}{cc} e^{i\sigma_Y (\xi^L_+ + \phi_+)} & 0 \\ 0 & e^{i\sigma_Y (\xi^L_- + \phi_-)} \end{array} \right] \left[ \begin{array}{cc} e^{i\mu_+} \sigma_Z & 0 \\ 0 & e^{i\mu_-} \sigma_Z \end{array} \right] \left[ \begin{array}{cc} e^{i\sigma_Y (\xi^R_+ + \phi_+)} & 0 \\ 0 & e^{i\sigma_Y (\xi^R_- + \phi_-)} \end{array} \right] \downarrow.$$ \hspace{1cm} (44)
where
\[ e^{i\mu \sigma_Z} = p_\pm + i\sigma_Z q_\pm . \] (45)

Next we want to calculate the effect of a similarity transformation \( M \uparrow (\cdot) \uparrow M^\dagger \) on each of the 3 matrices that are being multiplied on the right hand side of Eq.(44). For any real numbers \( \alpha, \beta \),

\[
M \uparrow \begin{bmatrix} e^{iao_y} & 0 \\ 0 & e^{ib_y} \end{bmatrix} \uparrow M^\dagger = \\
= M \uparrow e^{iao_y (0) \mathbf{1} (1)} e^{ib_y (0) n (1)} \uparrow M^\dagger \\
= M \uparrow e^{i \frac{a}{2} (\sigma_1 Y + \sigma_Z Y)} e^{i \frac{b}{2} (\sigma_1 Y - \sigma_Z Y)} \uparrow M^\dagger \\
= e^{i \frac{a-b}{2} \sigma_1 Y_1} e^{i \frac{a+b}{2} \sigma_1 Y_1} \tag{46a}
\]

and

\[
M \uparrow \begin{bmatrix} e^{iao_z} & 0 \\ 0 & e^{ib_z} \end{bmatrix} \uparrow M^\dagger = \\
= M \uparrow e^{iao_z (0) \mathbf{1} (1)} e^{ib_z (0) n (1)} \uparrow M^\dagger \\
= e^{i (\frac{a+b}{2}) \sigma_{XX}} e^{i (\frac{a-b}{2}) \sigma_{ZZ}} \tag{47a}
\]

Applying the identities given by Eqs.(46) and (47) to Eq.(44) yields

\[
U = e^{i \gamma L} \sigma_{1Z} M \tilde{V} M^\dagger e^{i \gamma R} \sigma_{1Z} = \\
= e^{i \frac{a-b}{2} \sigma_{1Z}} e^{2 (\gamma L \sigma_1 Y_1 + \gamma' L \sigma_1 Y_1)} e^{i (\mu' \sigma_{XX} + \mu \sigma_{ZZ})} e^{2 (\gamma R \sigma_1 Y_1 + \gamma' R \sigma_1 Y_1)} e^{i \frac{a-b}{2} \sigma_{1Z}} \tag{48a}
\]

A simple consequence of the table given in Eq.(13) is that:

\[
e^{i (\mu' \sigma_{XX} + \mu \sigma_{ZZ})} = e^{i \mu \sigma_Z} e^{i \mu' \sigma_Z} e^{i \gamma \sigma_Z} \tag{49}
\]

Eqs.(48) and (49) imply Eq.(16).
\[
\text{QED}
\]

Our next goal is to generalize the above theorem. But first, let us make some observations that will pave the way towards this goal.

As is well known and easily proven, for any \( A \in SU(2) \), one can find real numbers \( \alpha, \beta \) and \( \gamma \), such that:

\[
A = e^{i a \sigma_Z} e^{i b \sigma_Y} e^{i \gamma \sigma_Z} . \tag{50}
\]
A translation of this last equation into circuit language is:

\[ \begin{array}{l}
\square \quad = \quad \bigtriangleup \quad R_Z \quad - \quad R_Y \quad - \quad R_Z \quad . \\
\end{array} \tag{51} \]

Any \( A \in SU(2) \) can be diagonalized by a unitary matrix \( U \):

\[ A = U e^{iθσ_z} U^† . \tag{52} \]

Therefore,

\[ [A(0)]^{n(1)} = U(0)[e^{iθσ_z(0)}]^{n(1)} U(0)^† . \tag{53} \]

A translation of this last equation into circuit language is:

\[ \begin{array}{l}
\square \quad = \quad \bigtriangleup \quad R_Z \quad . \\
\end{array} \tag{54} \]

Since \( n = \frac{1}{2}(1 - σ_z) \),

\[ e^{iθσ_z(1)n(0)} = e^{i\frac{θ}{2}σ_z(1)[1 - σ_z(0)]} \]
\[ = e^{iθ(σ_z(1) - σ_z(0) + σ_z(0)[1 - σ_z(1)])} \]
\[ = e^{iθ[σ_z(1) - σ_z(0)]} e^{iθσ_z(0)n(1)} . \tag{55c} \]

A translation of this last equation into circuit language is:

\[ \begin{array}{l}
\bigtriangleup \quad R_Z \quad = \quad \bigtriangleup \quad R_Z \quad R_Z \quad . \\
\end{array} \tag{56} \]

We are now ready to prove a simple generalization of the Deflation Identity:

**Theorem 2**

\[ \begin{array}{l}
\square \quad \square \quad \square \quad \square \quad = \quad \square \quad \square \quad \square \quad \square \quad , \\
\end{array} \tag{57} \]

and

\[ \begin{array}{l}
\square \quad \square \quad \square \quad \square \quad = \quad \square \quad \square \quad \square \quad \square \quad . \\
\end{array} \tag{58} \]

**proof:**

By virtue of Eqs.\( (51) \) and \( (52) \),

\[ \begin{array}{l}
\square \quad \square \quad \square \quad \square \quad = \quad \square \quad R_Z \quad \square \quad R_Y \quad \square \quad \square \quad R_Y \quad . \\
\end{array} \tag{59a} \]

\[ \begin{array}{l}
\square \quad \square \quad \square \quad \square \quad = \quad \square \quad R_Z \quad \square \quad R_Y \quad \square \quad R_Z \quad . \\
\end{array} \tag{59b} \]
Applying the Deflation Identity to the right hand side of the last equation establishes Eq. (57).

By virtue of Eqs. (54) and (56),

\[ R_Z = R_{Z} \]

(60)

Applying Eq. (57) to the right hand side of the last equation establishes Eq. (58).

QED

4 Two Applications of Deflation Identity

In this section, we will give two application of the Deflation Identity, one to “opening and closing a breach” in a quantum circuit, the other to the CS decomposition of a 2-qubit operator.

4.1 Opening and Closing a Breach

Once more unto the breach, dear friends, once more; Or close the wall up with our English dead! (from “King Henry V” by W. Shakespeare)

Theorem 3 (Closing a breach) If

\[ U = B A \]

(61)

then

\[ U = B A \]

(62)

proof:

\[ U = B A A \]

(63a)

\[ = B A A \sigma_X A \sigma_X \]

(63b)

An application of the Deflation Identity to the right hand side of the last equation establishes the theorem.

QED
The last theorem shows how one can “close a breach” within a quantum circuit, while simultaneously reducing the number of CNOTs in the circuit by 1. It is also possible to “open a breach” within a circuit; i.e., replace the circuit by an equivalent one that has a breach:

\[
\begin{array}{c}
\text{T} \\
\text{U} \\
= \\
\text{T'} \\
\text{U'}
\end{array}
\]  \hspace{1cm} (64)

In the above circuit, we assume that the total number of CNOTs within the subcircuits \( \text{U} \) and \( \text{T} \) does not increase when they are replaced by \( \text{U'} \) and \( \text{T'} \). A useful strategy for reducing the number of CNOTs in a circuit by 1 is to open a breach within the circuit without increasing its number of CNOTs, and then to close the breach using Theorem 3. In a future paper, we will say more about this strategy for reducing the number of CNOTs in a circuit. We will show that the strategy also works for N-qubit circuits with \( N > 2 \).

### 4.2 Expressing 2-qubit CSD as a circuit with 3 CNOTs

According to the CS decomposition [4][5], any 2-qubit unitary operation \( U \) can be expressed as:

\[
U = e^{i\alpha} \begin{bmatrix} e^{i\alpha L_0} & 0 \\ 0 & e^{-i\alpha L_1} \end{bmatrix} \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{bmatrix} e^{i\alpha R_0} & 0 \\ 0 & e^{-i\alpha R_1} \end{bmatrix},
\]  \hspace{1cm} (65)

where

\[
C = \text{diag}(\cos \theta_1, \cos \theta_2), \quad S = \text{diag}(\sin \theta_1, \sin \theta_2),
\]  \hspace{1cm} (66)

and \( \alpha, \alpha_L, \alpha_R, \theta_1, \theta_2 \in \mathbb{R} \), and \( L_0, L_1, R_0, R_1 \in SU(2) \). Our goal for this section is to express the right hand side of Eq.\( (65) \) as a circuit with just 3 CNOTs (and some 1-qubit rotations). Ref.\[2\] showed how, given any 2-qubit unitary operation \( U \), one can first perform a KAK1 decomposition \[3\] of \( U \), and then express the outcome as a 3 CNOT circuit. What we give below is an alternative method for expressing a 2-qubit unitary operation as a quantum circuit with 3 CNOTs. Our method is via the CS decomposition, rather than via KAK1.

Note that given any two \( 2 \times 2 \) matrices \( A, B \),

\[
\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = A(0)^{n(1)}B(0)^{n(1)} = A(0)[A(0)^\dagger B(0)]^{n(1)}.
\]  \hspace{1cm} (67)

A translation of this last equation into circuit language is:

\[
\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{array}{c}
A \\
\circ \\
B \\
= \begin{array}{c}
A \\
A^\dagger B \\
\end{array}
\end{array}
\]  \hspace{1cm} (68)
Therefore,\
\[
\begin{bmatrix}
e^{i\alpha L} L_0 & 0 \\
0 & e^{-i\alpha L} L_1
\end{bmatrix}
= e^{i\alpha L} \sigma_Z \begin{bmatrix}
L_0 & 0 \\\n0 & L_1
\end{bmatrix}.
\] (69)

Note also that\
\[
\begin{bmatrix}
C & S \\
-S & C
\end{bmatrix}
= \uparrow \begin{bmatrix}
e^{i\theta_1 \sigma_Y} & 0 \\
0 & e^{i\theta_2 \sigma_Y}
\end{bmatrix}
= \downarrow \begin{bmatrix}
e^{i\theta_1 \sigma_Y} & e^{i\theta_2 \sigma_Y}
\end{bmatrix}.
\] (70)

By virtue of Eqs. (69) and (70), we can express the CS Decomposition Eq. (65) in circuit language as follows:

\[
U = e^{i\alpha} \begin{array}{c}
L_0 \\
L_0 L_1 \\
\sigma_Z \end{array} = \begin{array}{c}
\sigma_Z R_1 R_0 \\
R_1 \sigma_Z \\
\end{array}.
\] (71)

Note that
\[
\begin{bmatrix}
e^{i\theta_1 \sigma_Y} & e^{i\theta_2 \sigma_Y}
\end{bmatrix}
= \begin{bmatrix}
e^{i\theta_1 + \theta_2 \sigma_Y} & e^{i\theta_2 - \theta_1 \sigma_Y}
\end{bmatrix},
\] (72)

a result which is easily proven by considering the two possible cases \( n(0) = 0, 1 \). Note that
\[
\begin{array}{c}
R_1 \sigma_Z \\
\sigma_Z R_1 \\
\end{array} = \begin{array}{c}
R_1 \sigma_Z \\
\sigma_Z R_1 \\
\end{array}.
\] (73)

We can always find 2-dimensional unitary matrices \( U_L \) and \( U_R \) such that:
\[
L_0^\dagger L_1 = U_L e^{i\lambda L \sigma_Z} U_L^\dagger, \quad \sigma_Z R_1 R_0^\dagger = U_R e^{i\lambda R \sigma_Z} U_R^\dagger.
\] (74)

Applying Eqs. (72), (73) and (74) to Eq. (71) yields
\[
U = e^{i\alpha} \begin{array}{c}
L_0 U_L \\
\sigma_Z \end{array} = \begin{array}{c}
\sigma_Z U_R \\
U_R \sigma_Z \\
\end{array}.
\] (75)

Note that
\[
[\sigma_Z(0)]_{n(1)} = (-i)^{n(1)} [i \sigma_Z(0)]_{n(1)} = e^{-i\pi} e^{i\pi \sigma_Z(1)} e^{i\pi \sigma_Z(0)n(1)}.
\] (76)
A translation of this last equation into circuit language is:

\[
\begin{align*}
(\frac{-i}{n}) & = e^{-i \frac{\pi}{4}} \\
& = e^{i \frac{\pi}{2}} \sigma
\end{align*}
\]  

(77)

Eq. (77) and the Deflation Identity, together, imply that

\[
R_Y R_Z = e^{-i \frac{\pi}{4}}
\]

(78)

We can reduce the right hand side of Eq. (75) to a quantum circuit containing just 3 CNOTs by applying the identity Eq. (78) twice:

\[
U = e^{i \alpha} R_Z R_Y R_Z R_Y R_Z R_Y R_Z
\]

(79a)

\[
= e^{i(\alpha - \frac{\pi}{4})} R_Z R_Y
\]

(79b)

\[
= e^{i(\alpha - \frac{\pi}{4})} R_Z R_Y R_Y
\]

(79c)

\[
= e^{i \alpha}
\]

(79d)

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