Time-optimal trajectories for a trolley-like system with state constraint

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Abstract. We consider a time-optimal problem for trolley-like system (matheerial point with inertia) passage from start to zero position (i.e. origin) under bounded control (acceleration) and linear state constraint. We solve it with Pontryagin maximum principle for state-constrained problems usage. We construct optimal trajectories synthesis and show that it is not simple “cut” of classical trolleylike problem synthesis. State constraint generates “forbidden zone” bounded by state constraint boundary and half of parabola. We also shows that optimal trajectory (for selected range of state constraint line coefficient) has no more than three switching points (for the classical case, there is no more than one switching point). We analyse synthesis for different cases of parameters and demonstrates its evolution.

1. Introduction
Trolley-like problem is a well-known in optimal control theory and applications. Its goal is to move a matheerial point with coordinate \(x(t)\) and velocity \(y(t)\) from prescribed initial position \((x_0, y_0)\) to prescribed final position \((x_T, y_T)\) by acceleration control \(u(t)\) such that time \(T\) of motion is minimal:

\[
T \rightarrow \min_{u(t) \in U} \quad (1)
\]

According to the 2nd Newton law, the dynamical system is as follows:

\[
\begin{align*}
\dot{x} &= y, \quad x(0) = x_0, \quad x(T) = x_T, \\
\dot{y} &= u, \quad y(0) = y_0, \quad y(T) = y_T.
\end{align*}
\]

In the simplest case, \(x(t) \in \mathbb{R}^1\), \(y(t) \in \mathbb{R}^1\), \(u(t) \in \mathbb{R}^1\), and \(x_T = y_T = 0\), i.e. the goal is to stabilize system. The admissible constrols set \(U\) in the simplest case is \(u \in [-1, 1]\).

It is a well-known fact (see e.g. [1]) that in this case optimal control is of bang-bang form with no more than one switching (either from \(-1\) to \(1\) or vice versa) and geometrical structure of time-optimal trajectories is as shown on the Figure 1 with so-called “switching line” AOB generated by two parabolas

\[
x = -\frac{y^2}{2} \quad \text{and} \quad x = \frac{y^2}{2},
\]

corresponding to movement with \(u \equiv -1\) and \(u \equiv 1\), respectively.
In this work we modify original problem (1)-(2) by introducing linear state constraint of depth 1: we do not allow point \((x(t), y(t))\) of plane \(\mathbb{R}^2\) to lie below the line \(y = kx - b\) where \(k, b\) - real coefficients such that \(b > 0\) (to make the origin reachable).

![Figure 1](image)

**Figure 1.** Classic synthesis in trolley-like problem. AOB is switching line, any optimal trajectory has no more than one switchings.

The goal is to, at first, obtain form of optimal control and optimal synthesis for different collections of parameters values (and compare with classical case), and, at second, get the base for problem with \(x \in \mathbb{R}^2\) and \(y \in \mathbb{R}^2\), which is admissible to describe motion of motorized satellite antenna with two controllable axes (azimuth and elevation).

### 2. Problem statement

Thus, we consider the following optimal control problem:

\[
\begin{align*}
    \dot{x} &= y, \quad x(0) = x_0, \quad x(T) = 0, \\
    \dot{y} &= u, \quad y(0) = y_0, \quad y(T) = 0, \\
    y &\geq kx - b, \quad u \in [-1, 1], \quad T \rightarrow \min. 
\end{align*}
\]

(3)

### 3. Optimality conditions

If process \((t, x^*(t), y^*(t), u^*(t))\) delivers minimum in problem (3), then (see, e.g. \([1, 2, 3]\)) there exists non-zero collection of constants \(\alpha_0 \geq 0, \beta_1, ..., \beta_4\) and functions \(\psi_x, \psi_y, \dot{\mu}\), generating endpoint Lagrange function

\[
l = \beta_1 (x(0) - x_0) + \beta_2 x(T) + \beta_3 (y(0) - y_0) + \beta_4 y(T),
\]

(4)

Pontryagin function

\[
H = \psi_x y + \psi_y u
\]

(5)

and augmented Pontryahin function

\[
\overline{H} = \psi_x y + \psi_y u + \dot{\mu} (y - kx + b)
\]

(6)
such that the following conditions are satisfied:

(i) adjoint variables with transversality conditions

\[
\begin{align*}
\dot{\psi}_x &= -\frac{\partial H}{\partial x} = k\dot{\mu}, \\
\dot{\psi}_y &= -\frac{\partial H}{\partial y} = -\psi_x - \dot{\mu}, \\
\dot{\psi}_x(0) &= l'_x(0) = \beta_1, \\
\dot{\psi}_y(0) &= l'_y(0) = \beta_3,
\end{align*}
\]

(ii) nonnegativity condition

\[\dot{\mu} \geq 0\]

(iii) complementary slackness condition

\[\dot{\mu} (y - kx + b) = 0,\]

(iv) Energy conservation law

\[H = \psi_x y + \psi_y u \equiv \alpha_0,\]

(v) maximum condition

\[H \rightarrow \text{max} \Rightarrow u^* \in \text{Sign } \psi_y,\]

where

\[
\text{Sign } z = \begin{cases} 
1, & z > 0, \\
[-1, 1], & z = 0, \\
-1, & z < 0.
\end{cases}
\]

is standard multi-valued signum function.

4. Movement along state boundary

Let time interval be such that our point \((x(t), y(t))\) stays on state boundary. Thus, we get

\[y \equiv kx - b.\]

Differentiating the previous equation by \(t\) w.r.t. system (3), we get

\[u \equiv ky.\]

Thus, the following statement takes place.

**Statement 1.** State vector \((x(t), y(t))\) of system (3) can move along state boundary if and only if \(|y(t)| \leq \frac{1}{|k|}\)

Optimality conditions also get us the following important statement.

**Statement 2.** If process \((t, x^*(t), y^*(t), u^*(t))\) delivers minimum in problem (3), then corresponding optimal trajectory \((x^*(t), y^*(t))\) has no more than one state boundary arc.

These statements allow us to construct optimal synthesis, i.e. for any given initial point get optimal trajectory passing state vector to origin or say that there is no such a trajectory (corresponding points construct so-called “forbidden zone”).
5. Optimal synthesis in case $k > 0$
In this section, we represent structure of optimal synthesis depending on state constraint coefficients $k, b$.

At all figures presented in the following text, we use the following objects names:

- $P_T$ is parabola touching state constraint boundary line. It is generated by control $u = -1$ and has equation
  \[ x(y) = \frac{-y^2}{2} + \frac{b}{k} - \frac{1}{2k^2}, \]
- $P'_{-1}, P''_{-1}$ etc. are parabolas generated by the control $u = -1$,
- $P'_1, P''_1$ etc. are parabolas generated by the control $u = 1$,
- $P_B$ and $L_B$ are boundary of “admissible” zone generated by parabola
  \[ x(y) = \frac{-y^2}{2} + \frac{b}{k} \]
and state constraint boundary line
  \[ y = kx - b, \]
respectively,
- $S_1, S_2$ etc. are optimal trajectory switching points,
- $P_S$ is switching line part generated by parabola with $u = -1$.

Structure of synthesis depends on the following key points related location:

- Point $A = \left( -\frac{1}{k^2} + \frac{b}{k}, -\frac{1}{k} \right)$ where parabola $P_T$ generated by control $u = -1$ touches state constraint boundary (line $y = kx - b$), if such a touching takes place;
- Point $B = \left( \frac{1 - \sqrt{2bk + 1}}{k^2} + \frac{b}{k}, \frac{1 - \sqrt{2bk + 1}}{k^2} \right)$ which is intersection of state boundary and switching line;
- Point $C = \left( \frac{2bk - 1}{4k^2}, -\frac{\sqrt{2bk - 1}}{\sqrt{2k}} \right)$ which is intersection of touch parabola $P_T$ and switching line;
- Point $D = \left( \frac{b}{k}, 0 \right)$ which is intersection of state boundary and axis $x$.

Due to presence of terms $\sqrt{2bk + 1}$ and $\sqrt{2bk - 1}$, we have to deal with the following classification w.r.t. $2bk$ value.

5.1. Case $2bk < 3$
In case $2bk < 3$, the picture of optimal synthesis is shown at figure 2.
Figure 2. Synthesis for $2bk < 3$. Depending on $k, b$ values point $A$ has different sign of $x$–coordinate.

In this case, optimal trajectory has no more that two switchings. Points lying “above and right” from parabola $P_B$ and “below and right” from line $L_B$ are forbidden, i.e. one can not reach the origin from these points. What is more, point $D$ is forbidden too.

For points lying above line $L_B$ and below the switching line optimal synthesis is simple cut of the classical one: its typical trajectories are $P'_1S_3O$ and $P''_1S_4O$ – trajectories of the original trolley problem with start at state constraint.

For points from set bounded by switching line and line $P''_{-1}BO$ the situation is the same: typical trajectory is $P''_{-1}S_2O$ - is trajectory of classical trolley problem.

But for points from the set generated by line $BDP_B$ the synthesis is as follows: we start with $u = -1$ and move along parabola (the typical one is $P''_{-1}S_1$), then moves along state boundary before point $B$ and then switch to the switching line and move to origin.

5.2. Limit case $2bk = 1$

Note that point $C$ may not exists. More precisely, with decreasing $2bk$, point $A$ moves left along state constraint, so we get the “limit” case when touch parabola $P_T$ intersects the origin, i.e. coincides switching line (more precisely, its part generated by control $u = -1$). In this “limit” case point $C$ coincides with the origin. This case is shown at figure 3.
Figure 3. “Limit” case: synthesis for $2bk = 1$. Points $O$ and $C$ coincide. If $2bk < 1$ then $P_T$ lies in left semi-plane and $C$ does not exist.

5.3. Case $2bk > 3$

If $2bk > 3$, then the picture of optimal synthesis is shown at figure 4.

Figure 4. Synthesis for $2bk > 3$. Trajectories segments in set $ABC$ are obtained i.e. as segments of parabola $P'_1S_1$. They are not include in trajectory with start segment $P''_{-1}S_1$ since arc $AC$ gives less time.
In this case, optimal trajectory has no more that three switchings. Set bounded by line $P_BDL_B$ is still forbidden, point $D$ is forbidden too..

For points lying above line $L_B$ and below the switching line optimal synthesis is simple cut of the classical one: its typical trajectories are $P'_1S_4O$ and $P''_1S_5O$ – trajectories of the original trolley problem with start at state constraint.

For points from set $ABC$ the situation is the same: trajectories are trajectories of classical trolley problem with possible start at line $AB$.

For points from set bounded by switching line and line $PTACO$ the situation is the same: typical trajectory are $P''_1S_2O$ and $P''_1S_3O$ - trajectories of classical trolley problem.

But for points from the set bounded by line $PTADP_B$ the synthesis is as follows: we start with $u = -1$ and move along parabola (the typical one is $P'_{-1}S_1$), then moves along state boundary before point $A$, the switch to segment $CA$ of parabola $PT$ and then switch to the switching line and move to origin.

5.4. Limit case $2bk = 3$

Finally, if $2bk = 3$, the previous synthesis reduces to one shown at figure 5. Here, points $A, B, C$ coincides, so set $ABC$ reduces to one point, and corresponding trajectories reduce to one trajectory $AO$ generated by constant control $u \equiv 1$.

![Figure 5](image.png)

**Figure 5.** Synthesis for $2bk = 3$. Points $A, B, C$ coincides.

6. Conclusion

In this paper, we consider a time-optimal problem for trolley-like system (material point with inertia) passage from start to zero position (i.e. origin) under bounded control (acceleration) and linear state constraint. We solve it with Pontryagin maximum principle for state-constrained problems usage. We construct optimal trajectories synthesis and show that it is not simple “cut” of classical trolley-like problem synthesis. State constraint generates “forbidden zone” bounded
by state constraint boundary and half of parabola. We also shows that optimal trajectory (for selected range of state constraint line coefficient) has no more than three switching points (for the classical case, there is no more than one switching point). We analyse synthesis for different cases of parameters and demonstrates its evolution.

Our following work includes, at first, investigation of case $k < 0$; at second investigation of $x(t), y(t), u(t) \in \mathbb{R}^2$; and then investigation of other state constraints (not linear, but “non-linear a little”) with other contact points and contact arcs structure (see, e.g., [4, 5, 6, 7]). We hope that obtained results allow us to investigate optimal control problems for motorized satellites antennas (one-dimensional case of $x(t), y(t), u(t)$ is applicable for one-motor antenna to move along selected arc)

Acknowledgments
This work is supported by the Russian Science Foundation under grant No. 19-71-00103

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This work is supported by the Russian Science Foundation under grant No. 19-71-00103

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