LIMIT THEORY FOR SOME POSITIVE STATIONARY PROCESSES WITH INFINITE MEAN

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Abstract. We prove distributional limit theorems and one-sided laws of the iterated logarithm for a class of positive, mixing, stationary, stochastic processes which contains those obtained from non-integrable observables over certain piecewise expanding maps. This is done by extending Darling-Kac theory to a suitable family of infinite measure preserving transformations.

Overview. We prove limit theorems for positive, stationary, processes (SPs) with infinite mean satisfying mixing conditions which occur naturally in certain dynamical systems: Stable limit theorems (SLTs) for certain $\vartheta$-mixing SPs and One-sided laws of the iterated logarithm (LILs) for certain $\psi^*$-mixing SPs (definitions below).

The method of proof is by inversion which is done by first building a Kakutani tower over the generating probability preserving transformation, using the time zero observation as height function.

The mixing properties of the stationary process ensure that the resulting infinite measure preserving transformation is weakly pointwise dual ergodic, which allows us to develop a generalized Darling Kac theory for ergodic sums of this system. The results for the original stochastic process then follow by a standard inversion argument.

We illustrate both the finite-measure and the infinite-measure results by applying them to certain one-dimensional dynamical systems.

1. Definitions and Preliminaries

Stationary processes. We are going to consider partial sums of ergodic $\mathbb{R}_+$-valued stationary processes $(\xi_n)_{n \geq 0}$ with $\mathbb{E}(\xi_n) = \infty$. Such a process can always be represented as $\xi_n = \varphi \circ S^n$, where $S$ is a measure preserving transformation (m.p.t.) on a probability space $(\Omega, \mathcal{A}, P)$, and $\varphi : \Omega \to \mathbb{R}_+$ is measurable with $\mathbb{E}(\varphi) = \infty$ (and w.l.o.g.

Key words and phrases. infinite invariant measure, transfer operator, infinite ergodic theory, Darling-Kac theorem, pointwise dual ergodic, mixing coefficient, stable limit, one-sided law of iterated logarithm.

2000 Mathematics Subject Classification: 60F (37A40, 60G10).
$\mathcal{A} = \sigma\{\varphi \circ S^n : n \geq 0\}$). Due to non-integrability, it will suffice to restrict attention to the $\mathbb{N}$-valued case, as by the ergodic theorem the partial sums of the fractional parts are asymptotically negligible compared to the partial sums of the process. If indeed $\varphi : \Omega \to \mathbb{N}$, we let $\alpha = \alpha_{\varphi} = \{[\varphi = l] : l \in \mathbb{N}\}$, and thus obtain a probability preserving fibred system $(\Omega, \mathcal{A}, P, S, \alpha)$ in the sense of the following definition.

**Nonsingular transformations and Fibred systems.** A measurable map $S$ on a $\sigma$-finite space $(\Omega, \mathcal{A}, m)$ is called nonsingular if $m \circ S^{-1} \ll m$. Its transfer operator (w.r.t. $m$) is the positive linear map $\widehat{S} : L^1(m) \to L^1(m)$ defined by

$$\int_{\mathcal{A}} \widehat{S} f dm = \int_{S^{-1}\mathcal{A}} f dm \quad (f \in L^1(m), A \in \mathcal{A}).$$

A fibred or piecewise (pcw) invertible system is a quintuple $(\Omega, \mathcal{A}, m, S, \alpha)$ where $S$ is a nonsingular transformation on $(\Omega, \mathcal{A}, m)$, and $\alpha \subset \mathcal{A}$ is a countable, unilateral generator so that the restriction $S : a \to Sa$ is invertible, non-singular on each $a \in \alpha$. In this case, for every $k \geq 1$, $(\Omega, \mathcal{A}, m, S^k, \alpha_k)$ is a fibred system, where $\alpha_k := \bigvee_{j=0}^{k-1} S^{-j}\alpha$.

The transfer operator of $(\Omega, \mathcal{A}, m, S, \alpha)$ can be represented as

$$\widehat{S} f = \sum_{a \in \alpha} 1_{Sa}v'_a(f \circ v_a),$$

where $v_a : Sa \to a$ denotes the inverse of $S : a \to Sa$, and $v'_a := \frac{dm_{v_a}}{dm}$. If $m$ actually is an $S$-invariant probability measure, the system is called probability preserving, and we write $P := m$.

**Mixing.** We let $\mathcal{P}(\Omega, \mathcal{A})$ denote the collection of probability measures on $(\Omega, \mathcal{A})$, and call $\mu \in \mathcal{P}(\Omega, \mathcal{A})$ equivalent to $P$, $\mu \sim P$ if $\mu \ll P \ll \mu$. The probability preserving fibred system $(\Omega, \mathcal{A}, P, S, \alpha)$ is called

- $\vartheta_{\mu}$-mixing (for some $\mu \sim P$) if $\vartheta_{\mu}(n) \to 0$, where
  $$\vartheta_{\mu}(n) := \sup \left\{ \frac{|P(A \cap S^{-n}B) - P(A)P(B)|}{\mu(B)} : k \geq 1, A \in \sigma(\alpha_k), B \in \mathcal{A} \right\};$$

- reverse $\phi$-mixing if $\phi^{-}(n) \to 0$, where
  $$\phi_{-}(n) := \sup \left\{ \frac{|P(A \cap S^{-n}B) - P(A)P(B)|}{P(B)} : k \geq 1, A \in \sigma(\alpha_k), B \in \mathcal{A} \right\};$$

- $\psi^*$-mixing if $\psi^*(n) \to 1$, where
  $$\psi^*(n) := \sup \left\{ \frac{P(A \cap S^{-n}B)}{P(A)P(B)} : k \geq 1, A \in \sigma(\alpha_k), B \in \mathcal{A} \right\};$$
• \(\psi\)-mixing if \(\psi(n) \to 0\), where
\[
\psi(n) := \sup \left\{ \frac{\mathbb{P}(A \cap S^{-(n+k)} B) - \mathbb{P}(A) \mathbb{P}(B)}{\mathbb{P}(A) \mathbb{P}(B)} : k \geq 1, A \in \sigma(\alpha_k), B \in \mathcal{A} \right\};
\]
and continued fraction mixing if, in addition to \(\psi\)-mixing, \(\psi(1) < \infty\).

**Remark 1.**

a) As shown in [Br1], \(\psi^*(1) < \infty\) implies \(\psi^*\)-mixing. Elementary computation shows that \(\varphi - (n) \leq \psi^*(n) - 1\) so \(\psi^*\)-mixing entails reverse \(\phi\)-mixing. Note that \(\psi^*(1) \leq 1 + \psi(1)\). For examples with \(\psi^*(1) < \infty\) which are not \(\psi\)-mixing, see chapter 5 in [Br2].

b) Note that \(\vartheta P \equiv \varphi\). In \(\S 6\), we consider a class of interval maps (weakly mixing RU maps) for which \(\vartheta \mu(n) \to 0\) exponentially. For these interval maps (as shown in [AN]) \(\psi^*(1) < \infty\) implies continued fraction mixing (see \(\S 6\)).

**Strong distributional convergence and Limit laws.** For \((X, \mathcal{B}, \mu)\) a \(\sigma\)-finite measure space, \(F_n : X \to [0, \infty] \) measurable, and \(Y \geq 0\) a random variable, we say that \((F_n)\) **converges strongly in distribution** to \(Y\), written

\[
F_n \xrightarrow{\text{d}} Y,
\]
if it converges in law with respect to all absolutely continuous probabilities, that is, if

\[
\int_X g(F_n) dP \xrightarrow{n \to \infty} \mathbb{E}(g(Y)) \quad \forall \ g \in \mathcal{C}([0, \infty]), \ P \in \mathcal{P}(X, \mathcal{B}), \ P \ll \mu.
\]

For \(\gamma \in [0, 1]\) we let \(Y_\gamma \geq 0\) denote a random variable which has the **normalized Mittag-Leffler distribution of order \(\gamma\)**, that is, \(\mathbb{E}(Y_\gamma^p) = \frac{p^{\gamma} \Gamma(1 + \gamma)}{\Gamma(1 + p\gamma)}\) for \(p \geq 0\). Evidently \(Y_1 \equiv 1\), and \(Y_0\) has exponential distribution. Also, \(Y_1\) is the absolute value of a centered Gaussian random variable.

For \(\gamma \in (0, 1]\), the variable \(Z_\gamma := Y_{-\frac{1}{\gamma}}\) then has a **positive \(\gamma\)-stable distribution** with \(\mathbb{E}(e^{-tZ_\gamma}) = \exp(-\Gamma(1 + \gamma)t^\gamma)\) for \(t > 0\).

### 2. Results on stationary processes

In the statements below, \((\Omega, \mathcal{A}, \mathbb{P}, S, \alpha)\) is a probability preserving fibred system, and \(\varphi : \Omega \to \mathbb{N}\) is \(\alpha\)-measurable. We let

\[
\varphi_n := \sum_{k=1}^{n} \varphi \circ S^k, \quad n \geq 0,
\]
denote the partial sums of the stationary process \((\xi_n)_{n \geq 0} = (\varphi \circ S^n)_{n \geq 0}\), and define

\[
a(n) := \sum_{k=1}^{n} P([\varphi_k \leq n]), \quad n \geq 0.
\]

In order to establish our results, we'll need to assume that the growth of \(a(n)\) is adapted to the decay of the mixing coefficients of the process. The main condition is as follows although we need a stronger version (2.6) in Theorem 2.3 (below).

**Definition of Adaptedness.** Let \(\tau(n) \downarrow 0\). We'll say that the increasing sequence \((a(n))_{n \in \mathbb{N}}\) is adapted to \((\tau(n))_{n \in \mathbb{N}}\) if

\[
\frac{n \tau(\delta a(n))}{a(n)} \to 0 \quad \text{for all } \delta > 0.
\]

Note that if \((a(n))_{n \in \mathbb{N}}\) is regularly varying with positive index, then it is adapted to \((\tau(n))_{n \in \mathbb{N}}\) as soon as (2.1) holds for one \(\delta > 0\).

Our first result is a distributional limit theorem. In the barely infinite measure case \((\gamma = 1)\) it comes with an associated a.e. result. For \(\gamma \in (0,1)\), corresponding statements will be established under stronger assumptions in Theorem 2.3 below.

**Theorem 2.1.** Suppose that \((\Omega, \mathcal{A}, P, S, \alpha)\) is a \(\vartheta\mu\)-mixing probability preserving fibred system, and that \(\varphi : \Omega \to \mathbb{N}\) is \(\alpha\)-measurable. Let \(a(n)\) be \(\gamma\)-regularly varying with \(\gamma \in (0,1]\) and adapted to \((\vartheta\mu(n))_{n \in \mathbb{N}}\).

(a) (Stable limit theorem)

\[
\frac{\varphi_n}{b(n)} \xrightarrow{\text{d}} Z_{\gamma},
\]

where \(b\) is asymptotically inverse to \(a\) in that \(b(a(n)) \sim a(b(n)) \sim n\) (and hence \(\frac{1}{\gamma}\)-regularly varying).

(b) (One-sided law of the iterated logarithm for \(\gamma = 1\))

If, in addition, \(\gamma = 1\) and \(b(n/\log \log n) \log \log n \sim b(n)\), then

\[
\lim_{n \to \infty} \frac{\varphi_n}{b(n)} = 1 \quad \text{a.s.}
\]

**Remark 2.**

a) Theorem 2.1(a) was established for \(\varphi\)-mixing processes in [S] and for continued fraction mixing processes in [D] (see also [A3]).

b) The functional version of (a) is also valid, and can be proved using a straightforward, appropriate adaptation of [B].

c) Theorem 2.1(b) was established for \(\psi\)-mixing processes in [AD1].
The results mentioned in Remark a) also compute the \( a(n) \) from the marginal distributions, for which additional “close correlation” assumptions such as \( \psi^*(1) < \infty \) are required. We now show how to determine the asymptotics of \( a(n) \) from the marginal distributions under the weaker close correlation condition (2.4) (but we still use the stronger \( \psi^*(1) < \infty \) in Theorem 2.3 below).

**Theorem 2.2 (Identifying the normalization).** Let \((\Omega, \mathcal{A}, P, S, \alpha)\) be a \( \psi \)-mixing probability preserving fibred system, that \( \varphi : \Omega \to \mathbb{N} \) is \( \alpha \)-measurable, and that there exists some \( \Psi \in L^1(P)_+ \) such that

\[
S(\varphi \wedge n) \leq \Psi \mathbb{E}(\varphi \wedge n) \quad \forall \ n \geq 1.
\]

Assume that \( A \) is strictly increasing, regularly varying with index \( \gamma \in (0, 1] \), and adapted to \((\vartheta(n))_{n \in \mathbb{N}}\), then

\[
\mathbb{E}(\varphi \wedge n) \sim \frac{n}{\Gamma(2-\gamma)\Gamma(1+\gamma)A(n)}
\]

implies

\[
a(n) \sim A(n).
\]

Finally, under stronger assumptions, we establish the following point-wise result, where \( C_\gamma := K^{-1/\gamma}_\gamma \) with \( K_\gamma := \frac{\Gamma(1+\gamma)}{(1-\gamma)^{1/\gamma}} \) for \( \gamma \in (0, 1) \).

**Theorem 2.3 (The one-sided law of the iterated logarithm).** Suppose that \((\Omega, \mathcal{A}, P, S, \alpha)\) is a \( \psi^* \)-mixing probability preserving fibred system with \( \psi^*(1) < \infty \), and that \( \varphi : \Omega \to \mathbb{N} \) is \( \alpha \)-measurable.

If \( a(n) \) is \( \gamma \)-regularly varying for some \( \gamma \in (0, 1) \), and

\[
\frac{n \varphi - (\delta a(a(n)))}{a(n)} \to 0 \quad \text{for some} \ \delta > 0,
\]

then, for any sequence \((\tau(n))\) with \( \tau(n) \uparrow \) and \( \tau(n)/n \downarrow \) as \( n \to \infty \),

\[
\lim_{n \to \infty} \frac{\varphi_n}{b(n/\tau(n))\tau(n)} \geq C_\gamma \quad \text{a.s.} \quad \text{if} \ \sum_{n=1}^{\infty} \frac{1}{n} e^{-\beta \tau(n)} < \infty \ \forall \beta > 1,
\]

\[
\lim_{n \to \infty} \frac{\varphi_n}{b(n/\tau(n))\tau(n)} \leq C_\gamma \quad \text{a.s.} \quad \text{if} \ \sum_{n=1}^{\infty} \frac{1}{n} e^{-r \tau(n)} = \infty \ \forall r < 1,
\]

and

\[
\lim_{n \to \infty} \frac{\varphi_n}{b(n/\log \log(n))\log \log(n)} = C_\gamma \quad \text{a.s.}
\]
Remark 3. The conclusion of Theorem 2.3 was established for iid processes in [W], and for ψ-mixing processes in [AD2]. Our proof of Theorem 2.3 is by establishing the conditions needed for the methods of [AD2]. Therefore the functional version also follows as in [AD3].

Inversion: Kakutani towers and Return time processes.

Our results will be established using the well-known technique of “inverting” corresponding results for infinite measure preserving transformations, the connection being established via the following concept. The Kakutani tower of \((\Omega, \mathcal{A}, P, S, \varphi)\) is the object \((X, \mathcal{B}, m, T)\) with \((X, \mathcal{B}, m)\) the \(\sigma\)-finite space defined by

- \(X := \bigcup_{n \geq 1} [\varphi \geq n] \times \{n\}\),
- \(\mathcal{B} := \{\bigcup_{n \geq 1} B_n \times \{n\} : B_n \in \mathcal{A} \cap [\varphi \geq n] \forall n \geq 1\}\),
- \(m(A \times \{n\}) := P(A)\),

and \(T : X \to X\) is the map given by

\[
T(x, n) := \begin{cases} 
(x, n + 1) & \text{if } \varphi(x) > n, \\
(Sx, 1) & \text{if } \varphi(x) = n.
\end{cases}
\]

It follows that \((X, \mathcal{B}, m, T)\) is a conservative, measure preserving transformation which is ergodic iff \((\Omega, \mathcal{A}, P, S)\) is ergodic.

This “tower building process” is reversible. Given a conservative ergodic measure preserving system \((X, \mathcal{B}, m, T)\) we define the return time process of \(T\) on \(\Omega \in \mathcal{F} := \{B \in \mathcal{B} : 0 < m(B) < \infty\}\) as the \(\mathbb{N}\)-valued stationary process \((\varphi_{\Omega} \circ T^n\Omega)_{n \geq 0}\) on \((\Omega, \mathcal{B} \cap \Omega, m_\Omega, )\), where

- \(\varphi_{\Omega}(x) := \min \{n \geq 1 : T^n x \in \Omega\}\),
- \(T\Omega(x) := T^{\varphi(x)}(x)\), and
- \(m_{\Omega}(A) := m(A \cap \Omega)/m(\Omega)\).

It follows that the Kakutani tower of \((\Omega, \mathcal{B} \cap \Omega, m_\Omega, T_\Omega, \varphi_{\Omega})\) is a factor of \((X, \mathcal{B}, m', T)\) where \(m' = \frac{1}{m(\Omega)} m\) (and an isomorph in case \(T\) is invertible).

Now set \(\varphi_{j} := \sum_{i=0}^{j-1} \varphi_{\Omega} \circ T_{\Omega}^i\), which is the time of the \(n\)th return to \(\Omega\). It is straightforward to check that these are dual to the occupation times of \(\Omega\), \(S_{n}(1\Omega) := \sum_{k=0}^{n-1} 1_{\Omega} \circ T^k\) in that

\[
S_{n}(1\Omega) \leq j \quad \text{iff} \quad \varphi_{j} \geq n.
\]

This entails, via routine arguments, that various properties of \((\varphi_{j})_{j \geq 1}\) are equivalent to corresponding properties of \((S_{n}(1\Omega))_{n \geq 1}\). Specifically, suppose that \(a(n)\) is \(\gamma\)-regularly varying with \(\gamma \in (0, 1]\), and let \(b\) be
(asymptotically) inverse to $a$. Then, for $Y \geq 0$ a random variable,
\begin{equation}
\frac{1}{a(n)} S_n(1_\Omega) \overset{b}{\to} m(\Omega) Y \quad \text{iff} \quad \frac{\varphi_n}{b(n)} \overset{b}{\to} \left(\frac{1}{m(\Omega)}\right)^{\frac{1}{n}},
\end{equation}
and
\begin{equation}
\lim_{n \to \infty} \frac{1}{a(n)} S_n(1_\Omega) \overset{a.e.}{=} m(\Omega) \quad \text{iff} \quad \lim_{n \to \infty} \frac{\varphi_n}{b(n)} \overset{a.e.}{=} \left(\frac{1}{m(\Omega)}\right)^{\frac{1}{n}}.
\end{equation}

3. **Weak pointwise dual ergodic measure preserving transformations**

In this section, we consider the properties of infinite ergodic systems needed in the proofs of the results of the previous section.

**Weak, pointwise dual ergodicity.** Let $T$ be a conservative, ergodic, measure preserving transformation (not necessarily invertible) on the $\sigma$-finite space $(X, B, m)$, and $\hat{T} : L^1(m) \to L^1(m)$ its transfer operator, which naturally extends to all non-negative measurable functions. Invariance of $m$ means that $\hat{T}1_X = 1_X$, and since $T$ is c.e., any measurable $g : X \to [0, \infty)$ which is subinvariant, $\hat{T}g \leq g$, is, in fact, constant. Hurewicz’s ratio ergodic theorem (Theorem 2.2.1 of [A1]), guarantees that
\begin{equation}
\sum_{k=0}^{n-1} \hat{T}^k f \overset{m}{\to} m(f) \quad \text{a.e. on } X
\end{equation}
for all $f, g \in L^1_+(m) := \{ f \in L^1(m) : f \geq 0 \text{ and } m(f) > 0 \}$. (Due to conservativity, $\sum_{k=0}^{n-1} \hat{T}^k f \to \infty$ a.e. for such $f$.)

Throughout, convergence in measure, $\overset{m}{\to}$, for our $\sigma$-finite measure $m$, is understood to mean convergence in measure, $\overset{\nu}{\to}$, for every finite $\nu \ll m$ (or, equivalently, for all $\nu = m_A$ with $A \in \mathcal{F}$).

The c.e.m.p.t. $(X, B, m, T)$ will be called *weakly pointwise dual ergodic* if there exist constants $a_n = a_n(T) > 0$, $n \geq 1$, such that
\begin{equation}
\frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k f \overset{m}{\to} \int_X f \, dm \quad \text{for } f \in L^1_+(m),
\end{equation}
and
\begin{equation}
\lim_{n \to \infty} \frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k f = \int_X f \, dm \quad \text{a.e. on } X \quad \text{for } f \in L^1_+(m).
\end{equation}
This generalizes the notion of pointwise dual ergodicity (cf. §3.7 of [A1], or [A2]), which requires \( a_n = a_n(T) > 0 \) such that

\[
\frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k f \rightarrow \int_X f \, dm \quad \text{a.e. on } X \quad \text{for } f \in L^1_+(m).
\]

\[ \tag{3.4} \]

**Remark 4.** No invertible c.e.m.p.t. \((X, \mathcal{B}, m, T)\) with \( m(X) = \infty \) is pointwise dual ergodic. (Since in this case \( \hat{T} f = f \circ T^{-1} \), so that (3.4) would give a pointwise ergodic theorem with normalizing constants \( a_n \) for \( T^{-1} \), which is impossible, see §2.4 of [A1].) However, invertible systems can still be weakly pointwise dual ergodic.

For a concrete example, let \( T : (0, 1) \to (0, 1) \) be given by \( T x := \frac{x}{1-x} \) for \( x < \frac{1}{2} \), and \( T x := 2x - 1 \) for \( x > \frac{1}{2} \), which is conservative ergodic w.r.t. the invariant measure \( m \) with density \( \frac{1}{x} \), and define \( a_n := n/\log n \). By the Darling-Kac theorem for pointwise dual ergodic transformations, (see [DK], [A2], §3.7 of [A1], or [Z3]),

\[
\frac{1}{a_n} \sum_{k=0}^{n-1} f \circ T^k \rightarrow \int_X f \, dm \quad \text{a.e. on } X \quad \text{for } f \in L^1_+(m),
\]

and according to Proposition 2 of [AD1],

\[
\lim_{n \to \infty} \frac{1}{a_n} \sum_{k=0}^{n-1} f \circ T^k = \int_X f \, dm \quad \text{a.e. on } X \quad \text{for } f \in L^1_+(m).
\]

It is easily seen that these carry over to the natural extension \( T_* \) of \( T \). Therefore, the invertible c.e.m.p.t. \( T_*^{-1} \), with transfer operator \( f \mapsto f \circ T_* \) is weakly pointwise dual ergodic (and hence so is \( T_* \)).

Conditions of this flavour can be exploited most efficiently if one succeeds in identifying special sets on which there is additional control on the convergence. Recall (cf. [A1], [T2]) that \( A \in \mathcal{F} = \{ B \in \mathcal{B} : 0 < m(B) < \infty \} \) is called a uniform set (for \( f \in L^1_+(m) \)), written \( A \in \mathcal{U}(T) \), if there are \( a_n = a_n(T) > 0 \), s.t. (mod \( m \))

\[
\sup_A \left| \frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k f - m(f) \right| \rightarrow 0,
\]

and, more specifically, a Darling-Kac set, denoted \( A \in \mathcal{DK}(T) \), if

\[
\sup_A \left| \frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k 1_A - m(A) \right| \rightarrow 0.
\]

We now define \( A \in \mathcal{F} \) to be a limited set, \( A \in \mathcal{L}(T) \), if there exist
constants $a_n = a_n(T) > 0$ such that

$$
(3.5) \quad \frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k 1_A \xrightarrow{m_A} \frac{m_A}{\int_X f \, dm},
$$

and (mod $m$)

$$
(3.6) \quad \sup_A \frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k 1_A \xrightarrow{n \to \infty} m(A).
$$

In each case the return sequence $(a_n(T))_{n \geq 1}$, which is determined up to asymptotic equivalence and satisfies $a_{n+1}(T) \sim a_n(T) \to \infty$, can (and will) be taken non-decreasing. Due to (3.1) it does not depend on $f$ or $A$. It is easy to see that defining

$$
(a_n(A) := \sum_{k=0}^{n-1} \frac{m(A \cap T^{-k} A)}{m(A)^2} \quad \text{for } A \in \mathcal{F},
$$

we have

$$
(3.7) \quad a_n(T) \xrightarrow{n \to \infty} a_n(A) \quad \text{for } A \in \mathcal{L}(T).
$$

The existence of uniform sets is equivalent to pointwise dual ergodicity (but we do not know if $\mathcal{DK}(T) \neq \emptyset$ for every pointwise dual ergodic transformation). Similarly, weak pointwise dual ergodicity is equivalent to the existence of limited sets, as we have

**Proposition 3.1 (Limited sets and weak pde from local behaviour).** Let $T$ be a conservative ergodic m.p.t. on $(X, \mathcal{B}, m)$.

**a)** Suppose there are $A \in \mathcal{F}$, $f \in L_1^+(m)$, and constants $a_n > 0$, $n \geq 1$, such that

$$
(3.8) \quad \frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k f \xrightarrow{m_A} \int_X f dm,
$$

and

$$
(3.9) \quad \lim_{n \to \infty} \frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k f = \int_X f dm \quad \text{a.e. on } A.
$$

Then, for every $\eta > 0$, $T$ possesses a limited set $A' \in \mathcal{F}_A$ with return sequence $(a_n)_{n \geq 1}$ and $m(A') \geq m(A) - \eta$.

**b)** If $T$ has a limited set, then it is weakly pointwise dual ergodic.
Proof \ a) Let $B_0 := A$. By Hurewicz's ratio ergodic theorem we may assume w.l.o.g. that $f = 1_{B_0}$, and given any set $B_j \in \mathcal{F} \cap A$ with $m(B_j) > m(A) - \eta$, we also have $a_n^{-1} \sum_{k=0}^{n-1} \widehat{T}^k 1_{B_j} \to m(B_j)$ a.e. on $A$. Egorov's theorem then provides us with some $B_{j+1} \in \mathcal{F} \cap B_j$ such that $m(B_{j+1}) > m(A) - \eta$, and

$$
\frac{1}{a_n} \sum_{k=0}^{n-1} \widehat{T}^k 1_{B_j} \to m(B_j) \quad \text{uniformly on } B_{j+1}.
$$

Using this to inductively define a decreasing sequence $(B_j)_{j \geq 0}$ in $\mathcal{F}$, we obtain a set $A' := \bigcap_{j \geq 0} B_j$ with $m(A') \geq m(A) - \eta$. Given $\epsilon > 0$ choose $j$ with $m(B_j) < m(A') + \epsilon/2$, then

$$
\sup_{A'} \frac{1}{a_n} \sum_{k=0}^{n-1} \widehat{T}^k 1_{A'} \leq \sup_{B_{j+1}} \frac{1}{a_n} \sum_{k=0}^{n-1} \widehat{T}^k 1_{B_j} < m(B_j) + \frac{\epsilon}{2} \quad \text{for } n \geq n_j(\epsilon).
$$

This gives the required control from above. Since, by (3.1), (3.8) also holds with $f = 1_{A'}$, we see that $A'$ is indeed a limited set.

b) Now start from the assumption that $T$ has a limited set $A$. Due to Hurewicz's theorem, weak pde follows as soon as we check the defining conditions (3.2) and (3.3) for $f = 1_A$, i.e. we have to prove

$$
(3.10) \quad \frac{1}{a_n} \sum_{k=0}^{n-1} \widehat{T}^k 1_A \xrightarrow{n \to \infty} m(A),
$$

and

$$
(3.11) \quad \bar{f} := \lim_{n \to \infty} \frac{1}{a_n} \sum_{k=0}^{n-1} \widehat{T}^k 1_A = m(A) \quad \text{a.e. on } X.
$$

Letting $A_n := A \cap [\varphi_A > n]$, $n \geq 0$, we have, by routine arguments,

$$
(3.12) \quad \sum_{n \geq 0} \widehat{T}^n 1_{A_n} = 1_X,
$$

and decomposing, for any $n \geq 0$, $A$ according to the time of the last return before time $n$, $A = A_n \cup \bigcup_{k=0}^{n-1} A \cap T^{-k}(A_n)$ (disjoint), we find that for $N \geq 0$,

$$
(3.13) \quad \sum_{n=0}^{N} \widehat{T}^n 1_A = \sum_{k=0}^{N-1} \widehat{T}^k \left( 1_{A_k} \sum_{j=0}^{N-k} \widehat{T}^j 1_A \right) + \sum_{n=0}^{N} \widehat{T}^n 1_{A_n}.
$$
Since $A \in \mathcal{L}(T)$, there is some $M \in (0, \infty)$ such that

$$
\frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k 1_A \leq M \quad \text{on } A \text{ for } n \geq 1.
$$

With (3.14) and (3.12) providing bounds for the sums in (3.13), we get

$$
\frac{1}{a_N} \sum_{n=0}^{N-1} \hat{T}^n 1_A \leq M' \cdot 1_X \quad \text{for } N \geq 1,
$$

where $M' := M + 1/a(1) \in (0, \infty)$. Consequently, $\bar{f} \leq M' \cdot 1_X$. By a canonical version of Fatou’s lemma for positive operators, we have $\hat{T} \bar{f} \geq \bar{f}$. Hence $g := M' \cdot 1_X - \bar{f} \geq 0$ is subinvariant, $\hat{T} g \leq g$. As $T$ is conservative ergodic, this implies that $g$ is constant a.e., and hence so is $\bar{f}$. In view of (3.5), this yields (3.11).

To finally prove convergence in measure on $X$, it suffices to check it on each of the sets $T^{-l}A$, $l \geq 0$, since these cover $X$. But for each $l$,

$$
\int_{T^{-l}A} \left( \frac{1}{a_n} \sum_{k=0}^{n-1} \hat{T}^k 1_A \right) dm = \frac{a_{n+l}(A) - a_l(A)}{a_n} m(A) \rightarrow m(A).
$$

Together with (3.15) and (3.11) this entails $a_n^{-1} \sum_{k=0}^{n-1} \hat{T}^k 1_A \overset{m_{T^{-l}A}}{\rightarrow} m(A)$, and hence (3.10).

The notion of return sequences has originally been introduced in the context of rationally ergodic transformations (cf. §3.3 of [A1]). The present use of the term is justified by

**Proposition 3.2 (Weak pde implies rational ergodicity).** Let $T$ be a weakly pointwise dual ergodic c.e.m.p.t. on $(X, \mathcal{B}, m)$. Then every limited set $A$ satisfies a Rényi inequality, meaning that there is some $M = M(A) \in (0, \infty)$ such that

$$
\int_A (S_n(1_A))^2 dm \leq M \left( \int_A S_n(1_A) dm \right)^2 \quad \text{for } n \geq 1.
$$

In particular, $T$ is rationally ergodic with $(a_n(T))_{n \geq 1}$ a return sequence in the sense of [A1].

**Proof** Same as the proof of Proposition 3.7.1 in [A1], using (3.7) and the existence of limited sets established above. □
(Weak) pointwise dual ergodicity and special sets for Kakutani towers. Kakutani towers above $\vartheta_\mu$-mixing systems satisfying our adaptedness conditions are weakly pointwise dual ergodic. In the presence of regular variation, $\vartheta_\mu$-mixing with sufficiently fast rate implies pointwise dual ergodicity.

Recall that if $(X, B, m, T)$ is a c.e.m.p.t. and $\Omega \in B$, $m(\Omega) = 1$, has return time $\varphi = \varphi_\Omega$, and $\varphi_j := \sum_{i=0}^{j-1} \varphi \circ T_i$, then, for $n \geq 1$,

\begin{equation}
(3.16) \quad a_n(\Omega) = \sum_{k=1}^n m(\Omega \cap T^{-k}\Omega) = \sum_{j=1}^n m(\Omega \cap [\varphi_j \leq n]).
\end{equation}

**Theorem 3.1 ((Weak) pde via $\vartheta_\mu$-mixing return processes).** Let $(X, B, m, T)$ be a c.e.m.p.t. and suppose that $\Omega \in B$, $m(\Omega) = 1$ has a countable partition $\alpha \subset B \cap \Omega$ such that $\varphi = \varphi_\Omega$ is $\alpha$-measurable and that $(\Omega, B \cap \Omega, m, T_\Omega, \alpha)$ is $\vartheta_\mu$-mixing for some $\mu \sim m_\Omega$.

(a) If the $a_n := a_n(\Omega)$ is adapted to $(\vartheta_\mu(n))_{n\in\mathbb{N}}$, then $(X, B, m, T)$ is weakly pointwise dual ergodic with $a_n(T) \sim a_n$, and

$\forall \epsilon > 0 \exists N_\epsilon$ such that $\forall n \geq N_\epsilon$,

\begin{equation}
(3.17) \quad \frac{1}{a_n} \sum_{k=1}^n \hat{T}^k 1_\Omega - 1 \leq \epsilon(1 + \frac{d\mu}{dP}) \text{ a.e. on } \Omega.
\end{equation}

In particular, if $\|\frac{d\mu}{dP}\|_\infty < \infty$, then $\Omega \in \mathcal{L}(T)$.

(b) If $(a_n)_{n \geq 1}$ is $\gamma$-regularly varying for some $\gamma \in (0,1]$ and

$\vartheta_\mu(n) = O(\frac{1}{n^{1-\gamma}})$ for some $r > \frac{1}{\gamma} - 1$,

then $a_n := a_n(\Omega)$ is adapted to $(\vartheta_\mu(n))_{n\in\mathbb{N}}$, and $(X, B, m, T)$ is pointwise dual ergodic.

(c) If $(a_n)_{n \geq 1}$ is $\gamma$-regularly varying for some $\gamma \in (0,1]$, $\psi^* (1) < \infty$, and

\begin{equation}
(3.18) \quad \frac{n\vartheta_\mu(\delta a(n))}{a(n)} \xrightarrow{n \to \infty} 0 \quad \text{for all } \delta > 0,
\end{equation}

then $a_n := a_n(\Omega)$ is adapted to $(\vartheta_\mu(n))_{n\in\mathbb{N}}$ is satisfied, and $(X, B, m, T)$ is pointwise dual ergodic. Moreover, $\forall \epsilon > 0 \exists N_\epsilon$ such that $\forall n \geq N_\epsilon$,

\begin{equation}
(3.19) \quad \left| \frac{1}{a_n} \sum_{k=1}^n \hat{T}^k 1_\Omega - 1 \right| \leq \epsilon(1 + \frac{d\mu}{dP}) \text{ a.e. on } \Omega.
\end{equation}

In particular, if $\|\frac{d\mu}{dP}\|_\infty < \infty$, then $\Omega \in \mathcal{DK}(T)$. 
Proof We write \((\Omega, A, P, S, \alpha) := (\Omega, B \cap \Omega, m, T, \alpha)\). Note first that \(\vartheta_{\mu}\)-mixing implies, for \(j, p \geq 1\) and \(A \in \alpha_j\),

\[
\hat{S}^{j+p}1_A \leq P(A) + \vartheta_{\mu}(p) \frac{d\mu}{dP} \quad \text{a.e. on } \Omega,
\]
as is easily seen by integrating over arbitrary \(B \in B \cap \Omega\). We claim that for \(n, p \geq 1\),

\[
\hat{T}_n := \sum_{k=1}^{n} \hat{T}^k1_{\Omega} \leq p + a_n + n\vartheta_{\mu}(p) \frac{d\mu}{dP} \quad \text{a.e. on } \Omega.
\]

To see this, observe that \(\hat{T}^k1_{\Omega} = \sum_{j \geq 1} \hat{S}^j1_{[\varphi_j=k]} \) a.e. on \(\Omega\), and hence

\[
\hat{T}_n = \sum_{j=1}^{n} \hat{S}^j1_{[\varphi_j \leq n]} \leq \sum_{j=1}^{n+p} \hat{S}^j1_{[\varphi_j \leq n]}
\]

\[
\leq p + \sum_{j=1}^{n} \hat{S}^{j+p}1_{[\varphi_j+p \leq n]} \leq p + \sum_{j=1}^{n} \hat{S}^{j+p}1_{[\varphi_j \leq n]}
\]

\[
\leq p + \sum_{j=1}^{n} \left(P([\varphi_j \leq n]) + \vartheta_{\mu}(p) \frac{d\mu}{dP}\right)
\]

\[
= p + a_n + n\vartheta_{\mu}(p) \frac{d\mu}{dP} \quad \text{a.e. on } \Omega,
\]
since \([\varphi_j \leq n] \in \alpha_j\).

Proof of (a): To establish (3.17), we let \(p = p_n, \epsilon := \lceil \epsilon a_n \rceil\) for \(\epsilon > 0\). By adaptedness we can choose \(N_\epsilon\) such that \(n\vartheta_{\mu}(p_n, \epsilon) < \epsilon a_n\) whenever \(n \geq N_\epsilon\). Then by (3.21),

\[
\hat{T}_n \leq a_n(1 + \epsilon(1 + \frac{d\mu}{dP})) \quad \text{a.e. on } \Omega \text{ for } n \geq N_\epsilon,
\]
as required.

To prove weak pointwise dual ergodicity, we will use part a) of Proposition 3.1. Write \(R_n := \hat{T}_n/a_n\) and observe first that by (3.17),

\[
\lim_{n \to \infty} R_n \leq 1 \quad \text{a.e. on } \Omega.
\]

In fact, (3.17) shows that for any \(M > 0\), the estimate (3.23) holds uniformly on \(\Omega \cap [\frac{d\mu}{dP} \leq M]\). Consequently,

\[
P([R_n \geq 1 + \epsilon']) \longrightarrow 0 \quad \text{for every } \epsilon' > 0.
\]

To obtain the analogous statement for convergence from below, we observe that for \(t \in (0, 1)\),

\[
R_n \leq 1 + \epsilon(1 + \frac{d\mu}{dP}) \quad \text{a.e. on } \Omega \implies P([R_n \leq t]) \leq \frac{2\epsilon}{1+\epsilon-t},
\]
since \((\mathbb{E} \text{ denoting expectation w.r.t. } P)\)
\[
1 = \mathbb{E}(R_n) = \mathbb{E}(R_n 1_{[R_n > t]}) + \mathbb{E}(R_n 1_{[R_n \leq t]}) \\
\leq (1 + \epsilon) P([R_n > t]) + \epsilon \mu([R_n > t]) + tP([R_n \leq t]) \\
\leq 1 + 2\epsilon - (1 + \epsilon - t)P([R_n \leq t]) .
\]
Now fix \(\epsilon \in (0, 1)\) and take \(N_{\epsilon}\) as in (3.22). For \(t := 1 - \sqrt{\epsilon} \in (0, 1)\), observation (3.25) then yields
\[
P([R_n \leq 1 - \sqrt{\epsilon}]) \leq 2\sqrt{\epsilon} \quad \text{for } n \geq N_{\epsilon}.
\]
This readily implies
\[
P([R_n \leq 1 - \epsilon']) \xrightarrow{n \to \infty} 0 \quad \text{for every } \epsilon' > 0,
\]
which, together with (3.24), gives (3.8). Combined with (3.23) the latter yields (3.9). (There is some subsequence \(n_k \to \infty\) such that \(R_{n_k} \to 1\) a.e. for \(P\).)

**Proof of (b):** It is easily seen that the present assumptions entail adaptedness. As in part (a) this implies (3.23).

To check pointwise dual ergodicity, it remains to show that
\[
\lim_{n \to \infty} R_n \geq 1 \quad \text{a.e. on } \Omega.
\]
(Proposition 3.7.5 of [A1] ensures that a.e. convergence on some \(A \in \mathcal{F}\) suffices). To this end, choose \(c \geq 1\) such that \(\vartheta_\mu(n) \leq c/n^s\) for all \(n \geq 1\). There exist \(s > 0\) and \(N_0\) so that
\[
a(n) > \frac{1}{n^{s+1}} \quad \forall \ n > N_0.
\]
Choosing \(p = n^{\frac{1}{s+1}}\) in (3.21) we have
\[
R_n = \frac{\hat{T}_n}{a(n)} \leq 1 + \frac{1}{n^s} \left(1 + \frac{dn}{dP}\right) \quad \text{on } \Omega \ \forall \ n \geq N_0.
\]
Due to (3.25), we then see that for \(t \in (0, 1)\),
\[
q_t(n) := P([R_n \leq t]) \leq \frac{2c}{n} \cdot (1 - t + \frac{c}{n^s})^{-1} \leq \frac{2c}{(1-t)n^s} \quad \text{for } n \geq N_0.
\]
Since, for all \(t\) and \(\lambda > 1\), \(\sum_{n \geq 1} q_t([\lambda^n]) < \infty\), BCL now implies
\[
\lim_{n \to \infty} R_{[\lambda^n]} \geq 1 \quad \text{a.s. on } \Omega \text{ for all } \lambda > 1.
\]
To finally prove convergence (3.26) of the full sequence, fix any \(\lambda > 1\) and choose integers \(\kappa_n(\lambda) \to \infty\) so that \([\lambda^{\kappa_n(\lambda)}] \leq n \leq [\lambda^{\kappa_n(\lambda)+1}]\). Then regular variation of \((a_n)_{n \geq 1}\) yields
\[
\frac{\hat{T}_n}{a_n} \geq \frac{\hat{T}_{[\lambda^{\kappa_n(\lambda)}]}}{a[\lambda^{\kappa_n(\lambda)+1}]} \sim \frac{1}{\lambda^n} \frac{\hat{T}_{[\lambda^{\kappa_n(\lambda)}]}}{a[\lambda^{\kappa_n(\lambda)}]} \quad \text{a.e. on } \Omega \text{ as } n \to \infty.
\]
In view of (3.27), we thus have \( \lim_{n \to \infty} R_n \geq \lambda^{-\gamma} \) a.s. on \( \Omega \) for all \( \lambda > 1 \), and (3.26) follows.

Proof of (c): Note first that adaptedness, and hence (3.17) holds. Thus, to prove (3.19), it suffices to check that for \( n \geq N_\varepsilon \),

\[
\frac{1}{a(n)} \sum_{k=1}^{n} \hat{T}_k^\tau 1_{\Omega} \geq 1 - \varepsilon (1 + \frac{d\mu}{dP}) \quad \text{a.e. on } \Omega.
\]

We show first that \( \forall p, q, n \in \mathbb{N} \) with \( n \geq q \),

\[
\hat{T}_n \geq a(n) - p - n \vartheta \mu(p) \frac{d\mu}{dP} - \psi^*(1)^2 (P(\varphi_p > q)a(n) + q)
\]
a.e. on \( \Omega \). To see this, observe that

\[
\hat{T}_n = \sum_{j=1}^{n} \hat{S}_j^\tau 1_{[\varphi_j \leq n]}
\]

\[
\geq \sum_{j=1}^{n} \hat{S}_j^\tau 1_{[\varphi_j+p \leq n]} - p
\]

\[
= \sum_{j=1}^{n} \hat{S}_j^\tau 1_{[\varphi_j \leq n]} - \sum_{j=1}^{n} \hat{S}_j^\tau 1_{[\varphi_j \leq n < \varphi_j+p]} - p
\]

\[
=: \Sigma_1 - \Sigma_2 - p
\]

Now, because \([\varphi_j \leq n] \in \alpha_j\), we have

\[
\Sigma_1 \geq \sum_{j=1}^{n} (P(\varphi_j \leq n) - \vartheta \mu(p) \frac{d\mu}{dP}) = a(n) - n \vartheta \mu(p) \frac{d\mu}{dP}.
\]

On the other hand,

\[
\Sigma_2 \leq \psi^*(1) \sum_{j=1}^{n} P(\varphi_j \leq n < \varphi_{j+p})
\]

\[
= \psi^*(1) \sum_{j=1}^{n} \sum_{\ell=j}^{n} P(\varphi_j = \ell, \, \varphi_p \circ S^j > n - \ell)
\]

\[
\leq \psi^*(1)^2 \sum_{j=1}^{n} \sum_{\ell=j}^{n} P(\varphi_j = \ell) P(\varphi_p > n - \ell)
\]

\[
=: \psi^*(1)^2 (\Sigma_2' + \Sigma_2'')
\]
with

\[\Sigma_2' := \sum_{j=1}^{n} \sum_{\ell=j}^{n-q} P(\varphi_j = \ell) P(\varphi_p > n - \ell)\]

\[\leq P(\varphi_p > q) \sum_{j=1}^{n} \sum_{\ell=j}^{n-q} P(\varphi_j = \ell)\]

\[\leq P(\varphi_p > q) \sum_{j=1}^{n} P(\varphi_j \leq n)\]

\[= P(\varphi_p > q) a(n);\]

and

\[\Sigma_2'' := \sum_{j=1}^{n} \sum_{\ell=n-q+1}^{n} P(\varphi_j = \ell) P(\varphi_p > n - \ell)\]

\[\leq \sum_{j=1}^{n} \sum_{\ell=n-q+1}^{n} P(\varphi_j = \ell)\]

\[= \sum_{j=1}^{n} P(n - q < \varphi_j \leq n)\]

\[= \sum_{k=n-q+1}^{n} m(\Omega \cap T^{-k}\Omega) \leq q.\]

Putting this together establishes (3.29).

To finally check (3.28) for some given \(\epsilon > 0\), choose

\[\delta \in (0, \frac{\epsilon}{3\psi^*(1)^2})\]

for which \(\Pr(Z_\gamma > \frac{1}{\delta}) < \frac{\epsilon}{3\psi^*(1)^2}\),

and let

\[p_n = p_{n,\epsilon} := [\delta^2 a(a(n))], \quad q_n = q_{n,\epsilon} := [\delta a(n)].\]

Then \(a^{-1}(p_n) \sim \delta^2 a(n)\), and \(\frac{q_n}{a^{-1}(p_n)} \xrightarrow{n \to \infty} \frac{1}{\delta}\). Under the present assumptions, the stable limit theorem, Theorem 2.1, applies to our \((\varphi_n)\), as its proof below does not depend on part c) of Theorem 3.1. Therefore,

\[P(\varphi_p > q_n) \xrightarrow{n \to \infty} \Pr(Z_\gamma > \frac{1}{\delta}) < \frac{\epsilon}{3\psi^*(1)^2}.\]

Now choose \(N_\epsilon\) so large that, for all \(n \geq N_\epsilon\),

\[p_n < \frac{\epsilon}{3} a(n), \quad n\vartheta(\mu(p_n)) \leq \epsilon a(n) \quad \text{and} \quad P(\varphi_p > q_n) < \frac{\epsilon}{3\psi^*(1)^2}.\]
Then, using (3.29), we find indeed that for \( n \geq N_\epsilon \),
\[
\hat{T}_n \geq a(n) - p_n - n \vartheta_\mu(p_n) \frac{d\mu}{dP} - \psi^*(1)^2(P(\varphi > q_n))a(n) + q_n
\]
\[
\geq a(n) - \frac{\epsilon}{3} a(n) - \epsilon a(n) \frac{d\mu}{dP} - \frac{2}{3} a(n)
\]
\[
= (1 - \epsilon(1 + \frac{d\mu}{dP}))a(n)
\]
a.e. on \( \Omega \), which is (3.28). By (3.17) and (3.28),
\[
\frac{1}{a(n)} \sum_{k=1}^{n} \hat{T}_k 1_\Omega \xrightarrow{n \to \infty} 1 \quad \text{a.e. on } \Omega,
\]
so that \( T \) is pointwise dual ergodic (Proposition 3.7.5 of [A1] again). This convergence is in fact uniform on each \( \Omega \cap \{ \frac{d\mu}{dP} \leq M \} \), \( M > 0 \), whence the assertion about Darling-Kac sets. \( \square \)

4. MOMENT SETS AND THE STABLE LIMIT THEOREM

**Darling-Kac theorem and stable limits.** The statement of the stable limit theorem announced above is dual to a Darling-Kac type limit theorem for the Kakutani tower, which we now establish in the setup of weakly pointwise dual ergodic systems. It quantifies “return rates” and determines the limit distribution of occupation times \( S_n(1_A) = \sum_{k=0}^{n-1} 1_A \circ T^k \) of sets \( A \) of finite measure:

**Theorem 4.1 (Darling-Kac theorem for weakly pde systems).**
Let \( T \) be a weakly pointwise dual ergodic c.e.m.p.t. on \((X, \mathcal{B}, m)\). If its return sequence \((a_n(T))_{n \geq 1}\) is regularly varying of index \( \alpha \in [0, 1] \), then
\[
\frac{S_n(f)}{a_n(T)} \xrightarrow{n \to \infty} m(f) Y_\gamma \quad \text{for } f \in L^1_+(m),
\]
where \( Y_\gamma \) has the normalised Mittag-Leffler distribution of order \( \gamma \).

The stable limit theorem follows easily from this:

**Proof of Theorem 2.1(a).** Let \((X, \mathcal{B}, m, T)\) be the Kakutani tower of \((\Omega, \mathcal{A}, P, S, \varphi)\), then \((X, \mathcal{B}, m, T)\) is weakly pointwise dual ergodic with return sequence \( a(n) \). By the Darling-Kac Theorem,
\[
\frac{1}{a(n)} S_n(1_\Omega) \xrightarrow{\text{d}} Y_\gamma,
\]
and we need only recall (2.11). \( \square \)
Moment sets. The proof of the Darling-Kac theorem identifies sets for which the asymptotics of the moments of the occupation time distributions can be understood. Let \((X, \mathcal{B}, m, T)\) be a conservative, ergodic, measure preserving transformation. For \(A \in \mathcal{F} = \{ F \in \mathcal{B}, 0 < m(F) < \infty \}\), recall that

\[
a_n(A) = \sum_{k=0}^{n-1} \frac{m(A \cap T^{-k} A)}{m(A)^p}, \quad \text{and set} \quad u_A(\lambda) := \sum_{n=0}^{\infty} e^{-\lambda n} \frac{m(A \cap T^{-n} A)}{m(A)^p}.
\]

The set \(A \in \mathcal{F}\) is called a moment set for \(T\) if for all \(p \in \mathbb{N}\),

\[
\sum_{n=0}^{\infty} e^{-\lambda n} \int_A S_n(1_A)^p dm \sim p! m(A)^p \frac{u_A(\lambda)^p}{\lambda} \quad \text{as} \quad \lambda \to 0.
\]

**Remark 5.** a) If \(m(X) < \infty\) then (by the ergodic theorem) every \(A \in \mathcal{B}\) is a moment set.

b) Any conservative, ergodic, measure preserving transformation with moment sets is rationally ergodic. Thus, for example, a squashable conservative, ergodic, measure preserving transformation (which is not rationally ergodic, see [A1]) has no moment sets.

c) We extend both result and method from pointwise dual ergodic systems (as in [A1]) to weakly pointwise dual ergodic situations. A similar approach was used in [T2] to prove an arcsine-type limit theorem for pointwise dual ergodic maps. We do not know if the latter result generalizes accordingly.

**Theorem 4.2 (Moment set theorem).** Suppose that \(T\) is weakly pointwise dual ergodic, and that \(A \in \mathcal{F}\) is a limited set. Then \(A\) is a moment set for \(T\).

In view of Theorem 3.6.4 of [A1], the existence of limited sets established in Proposition 3.1 above, and Karamata’s Tauberian theorem (cf. p. 116 of [A1] or Theorem 1.7.1 of [BGT]), Theorem 4.1 above is an immediate consequence of this result. To prove the Moment set theorem, we need the following observation:

**Lemma 4.1 (Convergence of Laplace transforms).** Suppose that \(T\) is a c.e.m.p.t. on \((X, \mathcal{B}, m)\), weakly pointwise dual ergodic with \(A\) a limited set. Then

\[
R_A(\lambda) := \frac{1}{u_A(\lambda)} \sum_{n=0}^{\infty} e^{-\lambda n} \hat{T}^n 1_A \frac{m_A}{\lambda \downarrow 0} m(A),
\]

and

\[
\lim_{\lambda \downarrow 0} R_A(\lambda) = m(A) \quad \text{a.e. on} \ A,
\]
as well as

\[
\lim_{\lambda \downarrow 0} \sup_A R_A(\lambda) = m(A) \quad \text{a.e. on } A.
\]

Also, each sequence \( \lambda_i \downarrow 0 \) contains a subsequence \( \lambda_i' \downarrow 0 \) for which

\[
R_A(\lambda_i') \xrightarrow{i \to \infty} m(A) \quad \text{a.e. on } A.
\]

Moreover, every \( B \in \mathcal{F}, \ B \subset A \), satisfies \( a_n(A) \sim a_n(B) \) as \( n \to \infty \), and hence, for \( \lambda \downarrow 0 \),

\[
u_B(\lambda) \sim \nu_A(\lambda), \quad \frac{R_A(\lambda)}{R_B(\lambda)} \to \frac{m(A)}{m(B)} \quad \text{a.e. on } X.
\]

Proof Multiplying numerator and denominator by \((1 - e^{-\lambda})\), we get

\[
0 \leq R_A(\lambda) = \frac{\sum_{n=0}^{\infty} e^{-\lambda n} \sum_{k=0}^{n-1} \hat{T}^k 1_A}{\sum_{n=0}^{\infty} e^{-\lambda n} a_n(A)}
\]

for \( \lambda > 0 \). As \( A \) is a limited set, and \( \hat{T}^k 1_A \leq 1_X \), we therefore see that (since \( \sum_{k=0}^{\infty} \hat{T}^k 1_A = \infty \) a.e.),

\[
\lim_{\lambda \downarrow 0} \sup_A R_A(\lambda) \leq m(A).
\]

On the other hand, monotone convergence ensures that \( \int R_A(\lambda) \, dm_A = m(A) \) for all \( \lambda > 0 \). Together with \( 4.7 \) this yields \( 4.1 \), and combining \( 4.7 \) with \( 4.1 \) proves \( 4.2 \). Together with \( 4.7 \) the latter gives \( 4.3 \). It is a standard fact from integration theory that sequences which converge in probability contain a.e. convergent subsequences, whence \( 4.4 \).

Fix \( B \in \mathcal{F}, \ B \subset A \). We have \( a_n(B) = m(B)^{-2} a_n(A) \int_B g_n(B) \, dm \) where \( g_n(B) := a_n(A)^{-1} \sum_{k=0}^{n-1} \hat{T}^k 1_B m_B \) by weak pointwise dual ergodicity. Since \( 0 \leq \hat{T}^k 1_B \leq \hat{T}^k 1_A \) with \( A \) a limited set, we see that \( \sup_n \sup_A g_n(B) < \infty \). Therefore, \( \int_B g_n(B) \, dm \to m(B)^2 \), and hence \( a_n(A) \sim a_n(B) \). It is then immediate that \( u_A(\lambda) \sim u_B(\lambda) \) since \( \sum_{n\geq0} a_n(A) = \infty \). Now, expanding by \((1 - e^{-\lambda})\) as above, we get

\[
\frac{R_A(\lambda)}{R_B(\lambda)} = \frac{u_B(\lambda) \sum_{n=0}^{\infty} e^{-\lambda n} \sum_{k=0}^{n-1} \hat{T}^k 1_A}{u_A(\lambda) \sum_{n=0}^{\infty} e^{-\lambda n} \sum_{k=0}^{n-1} \hat{T}^k 1_B} \to 1 \cdot \frac{m(A)}{m(B)} \quad \text{a.e. on } X.
\]
by Hurewicz’ theorem (3.1) since \( \sum_{k=0}^{\infty} \hat{T}^k 1_A = \infty \) a.e. This proves (4.5), and (4.6) follows at once. \( \square \)

We are now ready for the

**Proof of the Moment set theorem (i)** We amend the argument given in the proof of Theorem 3.7.2 in [A1], using the same combinatorial decomposition

\[
S_n(1_A)^p = \sum_{q=1}^{p} \gamma_p(q) a(q, n),
\]

where, for \( n, p \in \mathbb{N} \), \( a(p, n) : X \to \mathbb{Z} \) is defined by \( a(0, n)(x) := 1 \), and \( a(p + 1, n)(x) := \sum_{k=1}^{n} 1_A(T^k x) a(p, n-1)(T^k x) \), while \( \gamma_1(q) := \delta_{1,q} \), and \( \gamma_{p+1}(q) := q(\gamma_p(q) + \gamma_p(q-1)) \). In particular, \( \gamma_p(p) = p! \).

Proving that \( A \) is a moment set reduces to showing that for \( p \geq 0 \),

\[
u_p(\lambda) := \sum_{n=0}^{\infty} e^{-\lambda n} \int_A a(p, n) dm \sim \frac{m(A)^{p+1} u_A(\lambda)^p}{\lambda} \] as \( \lambda \to 0 \) \( \forall p \geq 0 \).

This is because \( p! \nu_p(\lambda) \), the \( q = p \) term of the Laplace transform of the sum in (4.8), dominates the \( q < p \) terms. Indeed, as we now check by induction on \( p \),

\[
u_p(\lambda) = O\left( \frac{u_A(\lambda)^p}{\lambda} \right) \text{ as } \lambda \to 0 \ \forall p \geq 0.
\]

For \( p = 0 \) this is evident. More precisely, we have

\[
u_0(\lambda) = \frac{m(A)}{1 - e^{-\lambda}} \sim \frac{m(A)}{\lambda} \text{ as } \lambda \to 0.
\]

To pass from \( p - 1 \) to \( p \), use the recursive relation

\[
u_p(\lambda) = \sum_{n=0}^{\infty} e^{-\lambda n} \sum_{k=1}^{n} \int_A (1_A a(p - 1, n - k)) \circ T^k dm
\]

\[
= \sum_{n=0}^{\infty} e^{-\lambda n} \sum_{k=1}^{n} \int_A \hat{T}^k 1_A a(p - 1, n - k) dm
\]

\[
= \int_A \left( \sum_{k=1}^{\infty} e^{-\lambda k} \hat{T}^k 1_A \right) \left( \sum_{n=0}^{\infty} e^{-\lambda n} a(p - 1, n) \right) dm,
\]

and combine it with Lemma 4.1. This proves (4.10) and hence suffi-
ciency of (4.9).
Since the $p = 0$ case of (4.9) is trivially fulfilled, we can establish (4.9) by proving

$$\lim_{\lambda \to 0} \frac{\lambda u_p(\lambda)}{u_A(\lambda)^p} \geq m(A)^{p+1} \quad \forall \ p \in \mathbb{N}$$

and

$$\limsup_{\lambda \to 0} \frac{\lambda u_p(\lambda)}{u_A(\lambda)^p} \leq m(A)^{p+1} \quad \forall \ p \in \mathbb{N}.$$  

(ii) Fix any $p \geq 1$. To prove (4.12) by contradiction, suppose that it is violated. Then there are some $\epsilon > 0$ and $\lambda_i \downarrow 0$ such that

$$\lambda_i u_p(\lambda_i) < (1 - \epsilon)^{p+2} m(A)^{p+1} u_A(\lambda_i)^p \quad \text{for } i \geq 1.$$ 

Let $(\lambda'_i)$ be a subsequence of $(\lambda_i)$ as in Lemma 4.1, so that

$$R_{B}(\lambda'_i) \xrightarrow{i \to \infty} m(B) \quad \text{a.e. on } A$$

for all $B \in \mathcal{F}$, $B \subset A$. We claim that there are measurable sets $A = A_0 \supset A_1 \supset \ldots \supset A_p$ with $m(A_j) > (1 - \epsilon) m(A)$ and

$$R_{A_j}(\lambda'_i) = \frac{1}{u_{A_j}(\lambda'_i)} \sum_{k \geq 1} e^{-\lambda'_i k} \tilde{T}^k 1_{A_j} \geq (1 - \epsilon) m(A_j) \quad \text{on } A_{j+1} \text{ for } i \geq \ell_j,$$

with $(\ell_j)_{j \geq 1}$ increasing in $\mathbb{N}$.

To see this, start with $A_0 = A$ and consider (4.15) with $B = A_0$. By Egorov’s theorem, there is some $A_1 \in \mathcal{B} \cap A_0$ with $m(A_1) > (1 - \epsilon) m(A)$ such that this convergence is uniform on $A_1$. Therefore, there exists a suitable $\ell_1$. If $A = A_0 \supset A_1 \supset \ldots \supset A_j$ have been constructed, consider (4.13) with $B = A_j$. By Egorov’s theorem, there is some $A_{j+1} \in \mathcal{B} \cap A_j$ with $m(A_{j+1}) > (1 - \epsilon) m(A)$ such that this convergence is uniform on $A_{j+1}$. Therefore, there exists a suitable $\ell_{j+1}$. 

We now find, for any \( j \in \{0, 1, \ldots, p - 1\} \) and \( i > \ell_j \), that

\[
\int_{A_j} \left( \sum_{n=0}^{\infty} e^{-\lambda'n}a(p - j, n) \right) \, dm
\]

\[
= \sum_{n=0}^{\infty} e^{-\lambda'n} \sum_{k=1}^{n} \int_{A_j} (1_A a(p - j - 1, n - k)) \circ T^k \, dm
\]

\[
= \sum_{n=0}^{\infty} e^{-\lambda'n} \sum_{k=1}^{n} \int_{A} \hat{T}^k 1_{A_j} a(p - j - 1, n - k) \, dm
\]

\[
= \int_{A} \left( \sum_{k=1}^{\infty} e^{-\lambda'k} \hat{T}^k 1_{A_j} \right) \left( \sum_{n=0}^{\infty} e^{-\lambda'n}a(p - j - 1, n) \right) \, dm
\]

\[
> \int_{A_{j+1}} \left( \sum_{k=1}^{\infty} e^{-\lambda'k} \hat{T}^k 1_{A_j} \right) \left( \sum_{n=0}^{\infty} e^{-\lambda'n}a(p - j - 1, n) \right) \, dm
\]

Putting these together, we obtain that for \( i > \ell_{p-1} \),

\[
u_p(\lambda') > ((1 - \epsilon)m(A))^p \left( \prod_{j=0}^{p-1} u_{A_j}(\lambda') \right) \int_{A_p} \left( \sum_{n=0}^{\infty} e^{-\lambda'n}a(0, n) \right) \, dm
\]

\[
> ((1 - \epsilon)m(A))^{p+1} \left( \prod_{j=0}^{p-1} u_{A_j}(\lambda') \right) \frac{1}{\lambda'},
\]

where the last step is immediate from \( a(0, n) = 1 \). Since \( u_{A_j}(\lambda) \sim u_A(\lambda) \) by Lemma \[4.1\] this contradicts \[4.14\], and thus establishes \[4.12\].

(iii) To prove \[4.13\], fix any \( p \geq 1 \), and \( \epsilon > 0 \). In view of \[4.3\] there is some \( \lambda' > 0 \) such that \( R_A(\lambda) < (1 + \epsilon)m(A) \) on \( A \) for \( \lambda < \lambda' \). For such \( \lambda \) we therefore find

\[
u_p(\lambda) = \int_{A} u_A(\lambda) R_A(\lambda) \left( \sum_{n=0}^{\infty} e^{-\lambda n}a(p - 1, n) \right) \, dm
\]

\[
\leq (1 + \epsilon)m(A)u_A(\lambda) \cdot u_{p-1}(\lambda)
\]

\[
\vdots
\]

\[
\leq ((1 + \epsilon)m(A)u_A(\lambda))^p \cdot u_0(\lambda),
\]
and our claim is immediate from (4.11).

5. Wandering rates, return sequences and tails of marginals

Wandering rates. Suppose that \((X, \mathcal{B}, m, T)\) is a c.e.m.p.t. The wandering rate of the set \(A \in \mathcal{F}\) is the sequence given by \(L_A(n) := m(\bigcup_{k=0}^{n} T^{-k}A), n \geq 1\). Evidently,

\[
A, B \in \mathcal{F}, A \subset B \Rightarrow L_A(n) \leq L_B(n),
\]

and for \(N \geq 1\) fixed,

\[
L(\bigcup_{k=0}^{N} T^{-k}A)(n) = L_A(n + N) \sim L_A(n).
\]

Wandering rates are expectations of truncated return times,

\[
L_A(n) = \int_A (\varphi_A \wedge n) dm.
\]

Therefore, letting \(c_A(\lambda) := \int_A (1 - e^{-\lambda \varphi_A}) dm, \lambda > 0\), for \(A \in \mathcal{F}\), we have

\[
c_A(\lambda) = (1 - e^{-\lambda}) \sum_{n=0}^{\infty} e^{-\lambda n} m(A \cap [\varphi_A > n]) \sim \lambda^2 \sum_{n=0}^{\infty} e^{-\lambda n} L_A(n).
\]

Thus if \(L_A(n) \sim L_B(n)\) as \(n \to \infty\), then \(c_A(\lambda) \sim c_B(\lambda)\) as \(\lambda \searrow 0\). In fact, since \(L_A(n + 1) - L_A(n) \searrow 0\) for all \(A \in \mathcal{F}\), Korenblum’s ratio Tauberian theorem ([K], see also Theorem 2.10.1 of [BGT]) shows that the converse is also true, so that

\[
(5.1) \quad \text{for } A, B \in \mathcal{F}: \quad L_A(n) \sim_{n \to \infty} L_B(n) \iff c_A(\lambda) \sim_{\lambda \searrow 0} c_B(\lambda).
\]

The set \(A \in \mathcal{F}\) is said to have minimal wandering rate if \(L_B(n) \sim L_A(n)\) for all \(B \in \mathcal{F}, B \subseteq A\). In this case, \(\liminf_{n \to \infty} \frac{L_B(n)}{L_A(n)} \geq 1\) for all \(B \in \mathcal{F}\). Thus if \(A, B \in \mathcal{F}\) both have minimal wandering rate, then \(L_B(n) \sim L_A(n)\), which defines the wandering rate of \(T\), \((L_T(n))_{n \geq 1}\) up to asymptotic equivalence. There are sufficient conditions for \(A \in \mathcal{F}\) to have minimal wandering rate. By Proposition 3.2, Remark 3.6, and equation (2.3) of [TZ],

\[
\text{if } (\frac{\hat{L}_A(\varphi_A \wedge n)}{L_A(n)})_{n \geq 1} \text{ is uniformly integrable, then } A \text{ has minimal wandering rate.}
\]

(5.2)

Also, uniform sets are known to have minimal wandering rate, provided that the return sequence is regularly varying (Theorem 3.8.3 of...
In Theorem 5.1 below we remove the latter condition.

Minimal wandering rates determine the return sequence \((a_n(T))_{n \geq 1}\) of a weakly pointwise dual ergodic system \((X, B, m, T)\) by means of the asymptotic renewal equation. Assuming w.l.o.g. that \(a_n(T) = \sum_{j=0}^{n-1} u_n(T)\) with \(u_n(T) \geq 0\), we let

\[
u_T(\lambda) := \sum_{n=0}^{\infty} e^{-\lambda n} u_n(T), \quad \text{for } \lambda > 0.
\]

As a consequence of (3.7) we have \(u_T(\lambda) \sim u_A(\lambda)\) as \(\lambda \to 0^+\) for all \(A \in \mathcal{L}(T)\) (with \(u_A(\lambda)\) as in DK-section).

**Theorem 5.1 (Minimal wandering rates and the asymptotic renewal equation).** Suppose that \(T\) is weakly pointwise dual ergodic.

(i) If \(A \in \mathcal{F}\) has minimal wandering rate, then it satisfies the asymptotic renewal equation

\[
c_A(\lambda) \sim \frac{1}{u_T(\lambda)}.
\]

(ii) Uniform sets have minimal wandering rates.

**Proof** (i) By Proposition 3.1 there is some limited set \(B \in \mathcal{F}_A\). In view of statement a) in Lemma 4.1 any sequence decreasing to 0 contains a subsequence \((\lambda_j)_{j \geq 1}\) along which

\[
\frac{1}{u_B(\lambda_j)} \sum_{n=0}^{\infty} e^{-\lambda_j n} \hat{T}^n 1_B \xrightarrow{j \to \infty} m(B) \quad \text{a.e. on } B.
\]

Egorov’s theorem then gives us some \(C \in \mathcal{F}_B\) on which this convergence is in fact uniform, so that

\[
\int_C (1 - e^{-\lambda_j \varphi_C}) \sum_{n=0}^{\infty} e^{-\lambda_j n} \hat{T}^n 1_B dm \sim m(B) u_B(\lambda_j) c_C(\lambda_j).
\]

On the other hand, Lemma 3.8.4 in [A1] shows that

\[
\int_C (1 - e^{-\lambda \varphi_C}) \sum_{n=0}^{\infty} e^{-\lambda n} \hat{T}^n 1_B dm = \sum_{n=0}^{\infty} e^{-\lambda_n} \int_{C_n} 1_B dm \xrightarrow{\lambda \to 0} 1,
\]

where \(C_0 := C\) and \(C_n := T^{-n}C \setminus \bigcup_{k=0}^{n-1} T^{-k}C\) for \(n \geq 1\). Together, these prove \(c_C(\lambda_j) \sim 1/u_B(\lambda_j)\) as \(j \to \infty\). But as \(A\) has minimal wandering rate, (5.1) ensures that \(c_A(\lambda) \sim c_C(\lambda)\), and we end up with \(c_A(\lambda_j) \sim 1/u_B(\lambda_j) \sim 1/u_T(\lambda_j)\). Our claim follows since this can be
done inside any sequence of λ’s decreasing to 0.

(ii) Suppose that $A \in \mathcal{U}(T)$ is uniform for $f \in L^1_+ (m)$, with return sequence $(a_n)_{n \geq 1}$. Then, this is also true for all $B \in \mathcal{F}_A$. Thus, by the asymptotic renewal equation for uniform sets (cf. 3.8.6 of [A1]),

$$c_B(\lambda) \sim \frac{1}{u_T(\lambda)} \quad \text{for } B \in \mathcal{F}_A,$$

where $u_T(\lambda)$ does not depend on $B$. In particular, $c_B(\lambda) \sim c_A(\lambda)$ for all $B \in \mathcal{F}_A$, which, due to (5.1), shows that $A$ has minimal wandering rate. □

**Proof of Theorem 2.2.** Let $(X, \mathcal{B}, m, T)$ be the Kakutani tower of $(\Omega, \mathcal{A}, P, S, \varphi)$, then

$$L_\Omega(n) = \mathbb{E}(\varphi \wedge n) \sim \frac{n}{(1+\gamma)A(n)},$$

whence by Theorem 3.8.1 of [A1], for large $n$,

$$a(n) = \sum_{k=1}^n m(\Omega \cap Y^{-k}\Omega) \geq \frac{n}{2L_\Omega(n)} > \frac{1}{2} A(n).$$

Thus, for all $\epsilon > 0$,

$$\frac{n \theta_\mu(a(n))}{a(n)} < \frac{2n \theta_\mu((\epsilon/2)A(n))}{A(n)} \xrightarrow{n \to \infty} 0.$$

Now, by Theorem 5.1 (a), $T$ is weakly pointwise dual ergodic with return sequence $a(n)$, and in view of (2.4) and (5.2), $\Omega$ has minimal wandering rate. According to the asymptotic renewal equation of Theorem 5.1, $c_\Omega(\lambda) \sim \frac{1}{u(\lambda)} \xrightarrow{\lambda \to 0^+} 0$ whence by Karamata’s theorem $a(n) \sim A(n)$. □

6. **The one-sided law of the iterated logarithm**

The $\gamma = 1$ version of the law of the iterated logarithm follows immediately from the previous results.

**Proof of Theorem 2.1(b).** It has already been pointed out in [AD1] that (2.3) holds for positive stationary processes satisfying a weak law of large numbers provided that the corresponding infinite measure preserving Kakutani tower is weakly pointwise dual ergodic. The latter is immediate from Theorem 3.1 □

We now prove Theorem 2.3 by applying [AD2].
Proof of Theorem 2.3. We first show that under the present assumptions

\[ \sum_{n=1}^{\infty} \frac{\phi_-(a(n))}{n} < \infty. \]

To see this, note that \[ \phi_-(a(a(n))) \leq \frac{a(n)}{n} \] for large \( n \). Let \( b \) be asymptotically inverse to \( a \) in that \( b(a(n)) \sim a(b(n)) \sim n \), then \( b \) is \( \frac{1}{\gamma} \)-regularly varying, and for large \( N := a(a(n)) \) we have

\[ \phi_-(N) = \phi_-(a(a(n))) \leq \frac{a(n)}{n} = \frac{b(N)}{b(b(N))} = \frac{1}{c(N)} \]

where \( c(N) := \frac{b(b(N))}{b(N)} \) is \( (\frac{1}{\gamma} - \frac{1}{\gamma}) \)-regularly varying. Since \( Nc(N) \) is \( (\frac{1}{\gamma} - \frac{1}{\gamma} + 1) \)-regularly varying we indeed get \( \sum_{N=1}^{\infty} \frac{1}{Nc(N)} < \infty \).

As an immediate consequence of (6.1), \((\Omega, \mathcal{A}, P, S, \alpha)\) is strongly mixing from below as defined in [AD2].

Let \((X, \mathcal{B}, m, T)\) be the Kakutani tower of \((\Omega, \mathcal{A}, P, S, \varphi)\). Part c) of Theorem 3.1 with \( \mu = P \), ensures that \( T \) is pointwise dual ergodic and \( \Omega \in D\mathcal{K}(T) \).

The assumptions of Theorem 4 in [AD2] are now satisfied. Hence,

\[ \sum_{n=1}^{\infty} \frac{1}{n} e^{-\beta \tau(n)} < \infty \quad \forall \ \beta > 1 \quad \Longrightarrow \]

\[ \lim_{n \to \infty} \frac{1}{a(n/\tau(n))\tau(n)} S_n(f) \leq K_\gamma \int_X f \, d\mu \quad \text{a.e.} \ \forall f \in L^1_+ \]

and

\[ \sum_{n=1}^{\infty} \frac{1}{n} e^{-r \tau(n)} = \infty \quad \forall \ r < 1 \quad \Longrightarrow \]

\[ \lim_{n \to \infty} \frac{1}{a(n/\tau(n))\tau(n)} S_n(f) \geq K_\gamma \int_X f \, d\mu \quad \text{a.e.} \ \forall f \in L^1_+. \]

Using the inversion technique in §5 of [AD2], statements (a) and (b) of Theorem 2.3 follow, and (c) is a consequence of (a) and (b). \( \square \)

7. INTERVAL MAPS

A piecewise monotonic (increasing) map of the interval is a triple \((\Omega, S, \alpha)\) where \( \Omega \) is a bounded interval, \( \alpha \) is a finite or countable generating partition \((\text{mod } m := \text{Lebesgue measure})\) of \( \Omega \) into open intervals, and \( S : \Omega \to \Omega \) is a map such that the restriction \( S : A \to SA \) is an
(increasing) homeomorphism for each $A \in \alpha$ so that both $S : A \to SA$ and its inverse $v_A : SA \to A$ are absolutely continuous.

In this case, each iterate $(\Omega, S^k, \alpha_k), k \geq 1,$ is also piecewise monotonic (increasing), where $\alpha_k := \bigvee_{i=0}^{k-1} S^{-i} \alpha.$ Generalizing the above, we let, for $A \in \alpha_k,$ $v_A$ denote the inverse of $S^k : A \to S^k A,$ so that the transfer operator of $S$ (with respect to $m$) satisfies

$$\hat{S}^k f = \sum_{A \in \alpha_k} 1_{S^k A} v_A' \cdot (f \circ v_A),$$

where $v_A' := \frac{dm \circ v_A}{dm}.$

Consider the following properties for a piecewise monotonic map of the interval $(\Omega, S, \alpha):$

(A) Adler’s condition: for all $A \in \alpha,$ $S|_A$ extends to a $C^2$ diffeomorphism on $A,$ and $S''/(S')^2$ is bounded on $\Omega.$

(B) Big images: $\min_{A \in \alpha} m(SA) > 0.$

(R) Rychlik’s condition: $\sum_{A \in \alpha} \|1_{SA} v_A'\|_{\hat{BV}} := R < \infty.$

(U) Uniform expansion: $\inf |S'| > 1.$

Recall that (A) ensures $\frac{v_A''}{v_A'} \leq M < \infty,$ whence $v_A' = \epsilon M \frac{m(A)}{m(SA)}$ on $SA$ for all $A \in \alpha.$ In (R), the space $\hat{BV}$ is the subspace of those functions in $L^\infty(m)$ with a version in $BV,$ the space of functions of bounded variation. The norm $\| \cdot \|_{\hat{BV}}$ is defined by

$$\|f\|_{\hat{BV}} := \|f\|_\infty + \hat{\bigvee}_{\Omega} f,$$

where $\hat{\bigvee}_{\Omega} f := \inf\{ \bigvee_{\Omega} (f^*) : f^* = f \text{ m-a.e.} \}.$

Piecewise monotonic maps $(\Omega, S, \alpha)$ of the interval with properties $(P_1), \ldots, (P_N)$ will be called $P_1 \ldots P_N$ maps (eg ABU, RU maps).

**Lemma 7.1.** Any ABU map is an RU map.
Proof. This is similar to Proposition 2 of [Z1]. For any piecewise monotonic map \((\Omega, S, \alpha)\), (A)\&(B) imply (R). Indeed,

\[
\sum_{A \in \alpha} \|1_{SA}v'_A\|_{B^V} \leq 3\|v'_A\|_{\infty} + \sqrt{\int_{SA} |v''_A|dm}
\]

\[
\leq \sum_{A \in \alpha} (3e^M m(A) + \int_{SA} |v''_A|dm)
\]

\[
\leq \sum_{A \in \alpha} (3e^M \frac{m(A)}{m(SA)} + M \int_{SA} v'_A dm)
\]

\[
\leq \sum_{A \in \alpha} (3e^M \frac{m(A)}{m(SA)} + Me^M m(A))
\]

\[
\leq M' \sum_{A \in \alpha} m(A) = M'. \quad \square
\]

Ergodic properties of Rychlik’s maps. Suppose that \((\Omega, S, \alpha)\) is a RU map, then, according to [R],

- \((\Omega, B, m, S, \alpha)\) is a fibred system where \(m\) is Lebesgue measure on \(\Omega\) and \(B\) denotes the Borel \(\sigma\)-field,
- the ergodic decomposition of \((\Omega, B, m, S)\) is finite,
- to each ergodic component there corresponds an absolutely continuous invariant probability, with density in \(BV\) and with respect to which \(S\) is isomorphic to the product of a finite permutation and a mixing RU map.

If \(S\) is weakly mixing (with respect to \(m\) in the sense that \(f : \Omega \to \mathbb{S}^1\) measurable, \(f \circ S = \lambda f\) a.e. where \(\lambda \in \mathbb{S}^1\) implies \(f\) constant), then there are constants \(K > 0, \theta \in (0, 1)\) such that

\[
\left\| \hat{S}^n f - \left( \int_{\Omega} f dm \right) h \right\|_{B^V} \leq K\theta^n \|f\|_{B^V}
\]

where \(h\) is the unique \(T\)-invariant probability density. In this case, let \(dP := hdm\) and \(\mu := m|_{h > 0}\). Then [AN] shows that the probability preserving fibred system

\[
(\Omega, B, P, S, \alpha) \text{ is exponentially } \vartheta_{\mu}\text{-mixing.}
\]

We next observe that \(\psi^*(N) < \infty\) already implies cf-mixing in the present context, provided \(h\) has a positive lower bound. (This shows that for such ABU maps, the conclusions of Theorem 2.3 already follow from earlier results for cf-mixing systems.)
Proposition 7.1 (cf-mixing ABU maps). Let \((\Omega, S, \alpha)\) be a weakly mixing ABU map with invariant density \(h\) bounded away from 0. If \(\exists N \geq 1\) such that \(\psi^*(N) < \infty\), then \((\Omega, S, \alpha)\) is cf-mixing.

Proof. Suppose that \(\eta \in (0, 1)\) satisfies \(\eta^\pm 1\), which we use to abbreviate \(\eta \leq h \leq \eta^{-1}\). A standard argument then shows that \(\sup_{n \geq 1} \sup_{\Omega} |(S^n)'^{(N)}|/(S^n)'^2 < \infty\), and we can also assume that

\[
v'_A = \eta^\pm 1 \frac{m(A)}{m(S^k A)} \text{ on } S^k A \quad \text{for all } A \in \alpha_k, k \geq 1.
\]

Let \(\hat{S}_P\) be the transfer operator with respect to the absolutely continuous invariant probability \(P\), then \(\hat{S}_P f = \hat{S}(hf)/h\), and therefore

\[
\hat{S}_P^n f = \eta^\pm 2 \hat{S}_n f \quad \text{for all } n \geq 1, f \in L_+^\infty.
\]

We now show that

\[
(7.2) \quad m(S^k A) \geq \Delta \quad \text{for all } A \in \alpha_k, k \geq 1, \quad \text{where } \Delta := \frac{\eta^6}{\psi^*(N)}.
\]

To this end, let \(B \subset \Omega\) be measurable with \(m(B) > 0\). Then

\[
\frac{P(A)}{m(S^k A)} \leq \eta^{-1} \frac{m(A)}{m(S^k A)}
\]

\[
\leq \eta^{-2} \frac{1}{m(S^{-N} B)} \int_{S^{-N} B} 1_{S^k A} v'_A dm
\]

\[
= \eta^{-2} \frac{1}{m(S^{-N} B)} \int_{S^{-N} B} \hat{S}^k 1_A dm
\]

\[
\leq \eta^{-6} \frac{1}{P(S^{-N} B)} \int_{S^{-N} B} \hat{S}^k 1_A dP
\]

\[
= \eta^{-6} \frac{1}{P(B)} P(A \cap S^{-N+k} B)
\]

\[
\leq \eta^{-6} \psi^*(N) P(A)
\]

whence (7.2), as claimed.

To complete the proof of the proposition, we can then proceed as in the proof of Theorem 1(b) in [AN].

By exponential \(\psi_\mu\)-mixing (7.1), Theorem 2.1 implies the general

Proposition 7.2 (Stable limit theorem for RU maps). Suppose that \((\Omega, S, \alpha)\) is a weakly mixing RU map with absolutely continuous invariant probability \(dP = hdm\). Let \(\varphi : \Omega \to \mathbb{N}\) be \(\alpha\)-measurable, and denote \(\varphi_n := \sum_{k=0}^{n-1} \varphi \circ S^k\).
If \( a(n) := \sum_{k=1}^{n} P(\lfloor \varphi_k \leq n \rfloor) \) is \( \gamma \)-regularly varying for \( \gamma \in (0, 1] \), with inverse \( b \), then
\[
\frac{\varphi_n}{b(n)} \xrightarrow{\text{d}} Z_\gamma.
\]

**Remark 6.** For the subfamily of those RU-maps \( S \) which satisfy (A) plus the finite image condition (F) which requires \( \{SA : A \in \alpha\} \) to be finite, more general stable limit theorems (for observables \( \varphi \) which need not have constant sign) follow from [ADSZ] (see the end of Section 5 there). These AFU maps occur as induced maps of the infinite measure preserving AFN maps studied in [Z1],[Z2] (generalizing [T1]). The final subsections below illustrate that the present results allow us to analyse, via weak pointwise dual ergodicity, infinite measure preserving interval maps more general than those of [Z2].

The asymptotic type. Next, we turn to the asymptotic identification, via Theorem 2.2 of the normalizing constants \( a(n) \) in this setup.

**Proposition 7.3 (Asymptotic type of ABU maps).** Suppose that \( (\Omega, S, \alpha) \) is a weakly mixing ABU map with absolutely continuous invariant probability \( dP = hdm \).

Suppose that \( \varphi : \Omega \to \mathbb{N} \) is \( \alpha \)-measurable and satisfies
\[
P(\lfloor \varphi \geq n \rfloor) \sim c \quad \text{as } n \to \infty,
\]
where \( A(t) \) is strictly increasing and regularly varying with index \( \gamma \in (0, 1] \). Then
\[
a(n) := \sum_{k=1}^{n} P(\lfloor \varphi_k \leq n \rfloor) \sim c^{-1} A(n).
\]

The main point is condition (2.4) of Theorem 2.2.

**Lemma 7.2.** Suppose that \( (\Omega, S, \alpha) \) is a weakly mixing ABU map with absolutely continuous invariant probability \( dP = hdm \). Suppose that \( \varphi : \Omega \to \mathbb{N} \) is \( \alpha \)-measurable and satisfies
\[
\int_{\Omega} \varphi \wedge n \, dm = O\left( \int_{\Omega} \varphi \wedge n \, dP \right) \quad \text{as } n \to \infty.
\]
Then there is some $\Psi \in L^1(P)$ such that

$$\hat{S}_P(\varphi \wedge n) \leq \Psi \int_{\Omega} \varphi \wedge n \, dP \quad \text{a.e. for all } n \geq 1.$$  

**Proof.** We first record a corresponding statement w.r.t. Lebesgue measure $m$,

(7.5) \quad \exists \tilde{M} > 0 \text{ so that } \hat{S}(\varphi \wedge n) \leq \tilde{M} \int_{\Omega} (\varphi \wedge n) \, dm \quad \forall n \geq 1.

Letting $F_n := \sum_{A \in \alpha} (\varphi(A) \wedge n) m(A) 1_{SA}$ and $M' := (\inf_{A \in \alpha} m(\varphi(A)))^{-1}$, we have indeed

$$\hat{S}(\varphi \wedge n) = \sum_{A \in \alpha} (\varphi(A) \wedge n) \hat{S} 1_A = \sum_{A \in \alpha} (\varphi(A) \wedge n) v'_A 1_{SA}$$

$$\leq e^{M} \sum_{A \in \alpha} (\varphi(A) \wedge n) \frac{m(A)}{m(\varphi(A))} 1_{SA} \leq M' e^M F_n.$$  

But $\|F_n\|_{\infty} \leq \sum_{A \in \alpha} (\varphi(A) \wedge n) m(A) = \int (\varphi \wedge n) \, dm$, whence (7.5). To deduce (2.4), note that

$$\hat{S}_P(\varphi \wedge n) = 1_{[h>0]} \frac{1}{h} \hat{S}(h(\varphi \wedge n))$$

$$\leq 1_{[h>0]} \|h\|_{\infty} \frac{1}{h} \hat{S}(\varphi \wedge n)$$

$$\leq 1_{[h>0]} \|h\|_{\infty} \frac{1}{h} \tilde{M} \int_{\Omega} (\varphi \wedge n) \, dm \quad \text{by (7.5)}$$

$$\sim 1_{[h>0]} \|h\|_{\infty} \frac{1}{h} c \tilde{M} \int_{\Omega} (\varphi \wedge n) \, dP$$

$$= \Psi \int_{\Omega} (\varphi \wedge n) \, dP,$$

where $\Psi := 1_{[h>0]} \|h\|_{\infty} \frac{1}{h} c M \in L^1(P)$ since $h \in L^1(m)$. \qed

**Proof of Proposition 7.3.** We are going to verify the conditions of Theorem 2.2. Note first that adaptedness follows from the other two by exponential $\varphi_\mu$-mixing (7.1). Next, condition (2.5) is immediate from (7.3), as

$$\int_{\Omega} \varphi \wedge n \, dP = \sum_{k=1}^{n} P([\varphi \geq k]) \quad \text{(7.3)} \quad \sim c \sum_{k=1}^{n} m([\varphi \geq k]) = c \int_{\Omega} \varphi \wedge n \, dm.$$  

To check condition (2.4), use this and the previous lemma. \qed
The common image property. Typically, for interval maps, one will first obtain information on \( [\varphi \geq n] \) in terms of Lebesgue measure \( m \). This needs to be combined with an analysis of \( h \) to yield information on \( P([\varphi \geq n]) \), and hence on \( \int \varphi \wedge n \, dP = \sum_{k=1}^{n} P([\varphi \geq k]) \). Here we discuss simple sufficient conditions which allow us to validate property (7.3) of Proposition 7.3 in this way.

Consider a piecewise increasing map \( (\Omega, S, \alpha) \), with \( \Omega = [\omega_l, \omega_r] \). We shall say that \( (\Omega, S, \alpha) \) has the common image property if \( \bigcap_{A \in \alpha} S A = (\omega_l, \omega_l + z_S) \) where \( z_S > 0 \). Evidently, this entails the big image property (B). Moreover, we find:

Lemma 7.3. Suppose that \( (\Omega, S, \alpha) \) is a piecewise increasing AU map with the common image property and an absolutely continuous invariant probability \( dP = h \, d m \) on \( \Omega = [\omega_l, \omega_r] \), then

(7.6) \( \text{essinf}_{[\omega_l, \omega_l + z_S]} h > 0 \).

Moreover, \( S \) is weakly mixing.

Proof. We assume w.l.o.g. that \( \Omega = [0, 1] \). Fix a version \( h \in BV \) of the invariant density and set

\( \mathcal{J} := \{ J \subset [0, 1] : J \text{ is a nonempty open interval with } \inf_J h > 0 \} \).

It is clear that \( \mathcal{J} \neq \emptyset \). We need to show that \( (0, z_S) \in \mathcal{J} \). Observe first that

(7.7) there exist \( J \in \mathcal{J} \) and \( A \in \alpha \) so that \( J \cap \partial A \neq \emptyset \).

To see this, suppose otherwise i.e. that \( \forall J \in \mathcal{J}, \exists A_J \in \alpha : J \subset A_J \). Then \( J \in \mathcal{J} \) implies \( S J \in \mathcal{J} \) since for \( x \in S J \subset S A_J \),

\[
\begin{align*}
h(x) &\geq v_{A_J}'(x) h(v_{A_J} x) \\
&\geq \text{const} \cdot m(A_J) \inf_J h > 0.
\end{align*}
\]

But then, for each \( k \geq 1 \), \( S^k J \subset A_k \) for some \( A_k \in \alpha \), an impossibility since this entails \( m(S^k J) \geq \lambda^k m(J) \to \infty \).

Due to (7.7), there are \( J \in \mathcal{J} \) and \( A = (u, v) \in \alpha \) such that \( u \in J \). Set \( J_0 := A \cap J = (u, w) \) with \( u < w \). It follows as above that \( I_0 := S J_0 \in \mathcal{J} \), and the common image property implies \( I_0 = (0, c) \) for some \( c > 0 \).

Note then that there exist some \( J' \in \mathcal{J} \) and \( A' \in \alpha \) such that \( J' \supset A' \): Unless \( I_0 \) contains some \( A' \), it is contained in a specific \( A' \in \alpha \), and by the special structure of our map there is some \( k \geq 1 \) for which \( S^k I \subset A' \subset S^{k+1} I \). By the argument proving (7.7), we have \( J' := S^{k+1} I \in \mathcal{J} \).

But then \( (0, z_S) \subset S A' \in \mathcal{J} \) as required.

Finally, in view of Lemma 7.1 and [R], \( S \) has only finitely many ergodic acims, and these have densities \( h_i \in BV \), which can be chosen to
be lower semicontinuous, so that the sets \([h_i > 0]\) are open and pairwise disjoint. However, by the above, each \([h_i > 0]\) contains \((0, z_S)\). Hence \(h\) and \(P\) are unique, meaning that \(S\) is ergodic. Moreover, the structural results of [R] also show that there is a finite tail decomposition \(h = \sum_{j=0}^{p-1} g_j\) with \(g_j \in BV\) and the \([g_j > 0]\) open and pairwise disjoint, such that \(S[g_j > 0] = [g_l > 0]\) a.e., \(l = j + 1 \mod p\), and \(S\) is weakly mixing iff \(p = 1\). Bounded variation of the \(g_j\) together with (7.6) now implies that (after renumbering the \(g_j\) if necessary) there is some \(y > 0\) for which \((0, y) \subseteq [g_0 > 0]\). However, there is at least one cylinder \(A = (a, b) \in \alpha\) with \(a < y\), and then \((a, c) := [g_0 > 0] \cap A\) is nonempty and open. Due to the common image property, \(S(a, c) \subseteq [g_1 > 0]\) has nonempty open intersection with \((0, y)\). Hence \([g_0 > 0] = [g_1 > 0]\), as these sets overlap. Therefore \(p = 1\). □

This immediately allows us to deal with situations in which \(\varphi\) only diverges at 0.

**Example 1.** Suppose that \(((0, 1], S, \alpha)\) is a p.c.w increasing AU map with the common image property and absolutely continuous invariant probability \(dP = hdm\). Suppose that \(\varphi : [0, 1] \to \mathbb{N}\) is \(\alpha\)-measurable and satisfies \([\varphi \geq n] = [0, y_n]\), where

\[
y_n \sim \frac{1}{\Gamma(1-\gamma)\Gamma(1+\gamma)A(n)}
\]

with \(A\) strictly increasing and regularly varying of index \(\gamma \in (0, 1]\). Then \(a(n) \sim A(n)\) as \(n \to \infty\).

Indeed, we need only check condition (7.3) of Proposition 7.3. Fixing a version \(h \in BV\) of the invariant density, Lemma 7.3 shows that \(\lim_{x \to 0+} h(x) =: h(0^+) > 0\). Whence \(P([\varphi \geq n]) = \int_{[\varphi \geq n]} h \, dm \sim h(0^+) \cdot m([\varphi \geq n])\), and our claim follows since, by Karamata’s theorem,

\[
\int_{\Omega} (\varphi \wedge n) \, dm \sim n \frac{1}{\Gamma(2-\gamma)\Gamma(1+\gamma)A(n)}.
\]

Next, we record a little preparatory observation which will enable us to also study functions \(\varphi\) which diverge at countably many points. (This will be the case for the return time functions of the null-recurrent maps studied in the final subsection below.)

**Lemma 7.4.** Let \(h : \Omega \to [0, \infty)\) have right-hand limits \(h(x^+)\) everywhere. Let \(x_j, y_{j,n} \in \Omega\), \(j, n \geq 0\), be such that for each \(n\) the sets \((x_j, x_j + y_{j,n})\) are pairwise disjoint, and suppose that there are \(s_j \in [0, \infty)\) with \(\sum_{j \geq 0} s_j < \infty\), and \(q_n \downarrow 0\) for which \(y_{j,n}/q_n \to s_j\) as
\[ n \to \infty, \text{ uniformly in } j. \text{ If } s_j h(x_j^+) > 0 \text{ for some } j, \text{ then} \]

\[ \sum_{j \geq 0} \int_{(x_j, x_j + y_{j,n})} h(x) \, dx \sim \left( \sum_{j \geq 0} s_j h(x_j^+) \right) q_n \text{ as } n \to \infty. \]

**Proof.** Assume w.l.o.g. that \( s_0 h(x_0^+) > 0 \), and take any \( \epsilon > 0 \). Choose \( n_1 \) so large that \( y_{j,n} \leq e^\epsilon s_j q_n \) for \( n \geq n_1 \) and all \( j \). Take \( J \geq 1 \) so large that \( (\sup h) \sum_{j > J} s_j < \epsilon \sum_{j \geq 0} s_j h(x_j^+) \). Next, there is some \( n_2 \geq n_1 \) such that for all \( n \geq n_2 \) and all \( j \leq J \),

\[ \sup_{(x_j, x_j + e^\epsilon s_j q_n)} h \leq e^\epsilon h(x_j^+) + \frac{\epsilon s_0 h(x_0^+)}{J + 1} \]

(\( h(x_j^+) \) need not be positive, but \( h(x_0^+) \) is). Then, for all \( n \geq n_2 \),

\[ \sum_{j \geq 0} \int_{(x_j, x_j + y_{j,n})} h(x) \, dx \leq e^\epsilon q_n \sum_{j \geq 0} s_j \sup_{(x_j, x_j + e^\epsilon s_j q_n)} h \]

\[ \leq e^\epsilon q_n \left( e^\epsilon \sum_{j = 0}^J s_j h(x_j^+) + \epsilon s_0 h(x_0^+) + \sup h \sum_{j > J} s_j \right) \]

\[ \leq e^\epsilon (e^\epsilon + 2\epsilon) \left( \sum_{j \geq 0} s_j h(x_j^+) \right) q_n. \]

As \( \epsilon > 0 \) was arbitrary, this proves one half of our claim. The corresponding estimate from below follows by similar but even simpler arguments which we omit. \( \square \)

We can now go beyond the scenario of Example [ ] above. Situations of the following type naturally occur in the study of interval maps with neutral fixed points, see below.

**Proposition 7.4** (\( \varphi \) with countably many singularities). Let \((\Omega, S, \alpha)\) be a weakly mixing pce increasing ABU map with absolutely continuous invariant probability \( dP = h \, dm \). Suppose that \( \varphi : \Omega \to \mathbb{N} \) is \( \alpha \)-measurable and such that for \( n \) sufficiently large, \( \lfloor \varphi \geq n \rfloor \) is a countable disjoint union of intervals \((x_j, x_j + y_{j,n})\) satisfying the assumptions of Lemma [ ] where

\[ q_n \sim \frac{1}{\Gamma(1 - \gamma)\Gamma(1 + \gamma)A(n)}. \]
with $A$ strictly increasing and regularly varying of index $\gamma \in (0, 1]$. Then

$$a(n) := \sum_{k=1}^{n} P(\lfloor \varphi k \leq n \rfloor) \sim \left( \sum_{j \geq 0} s_j h(x_j^+) \right) A(n).$$

Proof. It is clear that $m(\lfloor \varphi \geq n \rfloor) \sim (\sum_{j \geq 0} s_j) q_n$. According to Lemma 7.4,

$$P(\lfloor \varphi \geq n \rfloor) = \sum_{j \geq 0} \int_{(x_j, x_j+y_j, n)} h(x) \, dx \sim \left( \sum_{j \geq 0} s_j h(x_j^+) \right) q_n,$$

so that by Karamata’s theorem

$$\int_{\Omega} (\varphi \wedge n) dP \sim \frac{\left( \sum_{j \geq 0} s_j h(x_j^+) \right) n}{\Gamma(2-\gamma) \Gamma(1+\gamma) A(n)},$$

and Proposition 7.3 applies. \hfill \square

Some infinite measure preserving interval maps. We conclude with a class of infinite measure preserving interval maps $T$ with indifferent fixed point, which induce probability preserving maps $S$ of the above type. These $T$ do not, in general, belong to the family of AFN-maps studied in [Z2].

Proposition 7.5 (Maps with indifferent fixed points). Let $(X, T, \beta)$ be a pcw increasing A map on $X = [\eta_l, \eta_r]$ with the common image property which satisfies $\inf_{(\eta_l + \epsilon, \eta_r)} T' > 1$ for every $\epsilon \in (0, \eta_r - \eta_l)$.

Assume that $T$ possesses a leftmost cylinder $B_\epsilon = (\eta_l, \xi)$, and that $z_T := \inf_{B_\epsilon \in \beta} m(TB)$ satisfies $z_T > \xi - \eta_l$. Suppose that $T$ is convex near $\eta_l$, and satisfies, for some $\gamma \in (0, 1)$,

$$T(\eta_l + x) \sim \eta_l + x + \kappa x^{1+1/\gamma} + o(x^{1+1/\gamma}) \quad \text{as} \quad x \searrow 0. \quad (7.9)$$

Then $T$ is conservative ergodic with an infinite acim $m_T = h_T \, dm$, with $h_T$ bounded on each $(\eta_l + \epsilon, \eta_r)$. Moreover, $T$ is weakly pointwise dual ergodic and exhibits Darling-Kac asymptotics,

$$\frac{S_n(f)}{a(n)} \xrightarrow{n \to \infty} m_T(f) Y_\gamma \quad \text{for} \quad f \in L^1_+(m_T),$$

with return sequence satisfying $a(n) \sim c/n^\gamma$ for some $c > 0$.

Proof. Let $\Omega := [\xi, \eta_r]$, and consider the induced map $S = T_\Omega$ and the corresponding return time function $\varphi = \varphi_\Omega$. We are going to show that the induced system naturally comes as a pcw increasing map $(\Omega, S, \alpha)$,
which together with \( \varphi \) satisfies the assumptions of Proposition 7.4. Therefore Proposition 7.2 applies, which via (2.11) entails the DK-limit. Weak pointwise dual ergodicity implicit in the application of these propositions.

Note first that \( B_s = \bigcup_{n \geq 1} (\tau_{n+1}, \tau_n) \), where \( \tau_1 := \xi \) and \( \tau_{n+1} := w(\tau_n) \) with \( w := (T|_{B_s})^{-1} \) denoting the inverse of the leftmost branch of \( T \).

As a consequence of (7.9) we have (Corollary on p. 82 of [T1])

\[
(7.10) \quad q_n := \tau_n - \eta \sim (\kappa n/\gamma)^{\gamma} \quad \text{as } n \to \infty.
\]

Fix any \( B \in \beta \setminus \{ B_\ast \} \), and let \( B(k) := B \cap [\varphi = k], k \geq 1 \), which defines the cylinders of \( S \) inside \( B \). The induced map \( S \) is trivially p.w. increasing and satisfies (U). It also satisfies (A), which is checked by the same argument as in [T1] or [Z1], which we do not reproduce here.

Now \( SB(1) = TB(1) \cap \Omega \supset (\xi, z_T - \xi) \). For \( k \geq 2 \), \( TB(k) = (\tau_{k-1}, \tau_{k-2}) \), and hence \( SB(k) = T^kB(k) = T(\tau_1, \tau_0) \supset (\xi, z_T - \xi) \). Therefore \( S \) has the common image property.

Enumerating \( \beta \setminus \{ B_\ast \} = \{ B_0, B_1, \cdots \} \), we get

\[
[\varphi > n] \cap B_j = v_{B_j}(\eta, \tau_n) =: (x_j, x_j + y_{j,n})
\]

for suitable \( x_j, y_{j,n} \), where \( v_{B_j} := (T|_{B_j})^{-1} \). This collection of intervals now satisfies the assumptions of Lemma 7.3 with \( s_j := v_{B_j}(\beta_j) \) where \( \beta_j \) is the left endpoint of \( B_j \). Uniformity of \( y_{j,n}/q_n \to s_j \) in \( j \) is a consequence of the distortion control for (the first iterate of) \( T \) provided by condition (A), which also implies \( \sum s_j < \infty \). To see that \( s_j h(x_j^+) > 0 \) for some \( j \), use the common image property of \( S \) and Lemma 7.3 to obtain some \( z_S > 0 \) such that \( \inf_{(\xi, \xi + z_S)} h > 0 \). Now choose some \( j \) for which \( \beta_j \in (\xi, \xi + z_S) \). \( \square \)

**Example 2.** Fix \( \gamma \in (0, 1) \) and define \( F(x) := x(1 + x^{1/\gamma})/(1 - x^{1/\gamma}), \) \( x \in X := (0, 1) \). Let \( T : X \to X \) be of the form \( Tx = F(x) - F(\xi_n) \) for \( x \in (\xi_n, \xi_{n+1}) =: A_n \), where \( 0 = \xi_0 < \xi_1 < \ldots < \xi_n \) are such that \( T\xi_n^- \geq T\xi_{n+1}^- \). (It is easily seen that when \( \xi_0, \ldots, \xi_n \) with these properties have been chosen, there is a nondegenerate interval \( J_n \) of admissible choices of \( \xi_{n+1} \).) Then \( T \) satisfies the assumptions of the preceding proposition, but the finite image (\( F \)) is only fulfilled in the exceptional cases when \( T\xi_n = 1 \) for \( n \geq n_0 \) (i.e. when \( \xi_{n+1} \) is the right endpoint of \( J_n \) for \( n \geq n_0 \)). Therefore \( T \) does not in general belong to the AFN maps of [Z2].
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