SMALL OVERLAP MONOIDS II: AUTOMATIC STRUCTURES AND NORMAL FORMS

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Abstract. We show that any finite monoid or semigroup presentation satisfying the small overlap condition $C(4)$ has word problem which is a deterministic rational relation. It follows that the set of lexicographically minimal words forms a regular language of normal forms, and that these normal forms can be computed in linear time. We also deduce that $C(4)$ monoids and semigroups are rational (in the sense of Sakarovitch), asynchronous automatic, and word hyperbolic (in the sense of Duncan and Gilman). From this it follows that $C(4)$ monoids satisfy analogues of Kleene’s theorem, and admit decision algorithms for the rational subset and finitely generated submonoid membership problems. We also prove some automata-theoretic results which may be of independent interest.

1. Introduction

Small overlap conditions are natural combinatorial conditions on monoid and semigroup presentations, which serve to limit the complexity of derivation sequences between equivalent words. They are the natural semigroup-theoretic analogues of the small cancellation conditions extensively employed in combinatorial and geometric group theory [15]. It has long been known that monoids with presentations satisfying the condition $C(3)$ have decidable word problem [8, 17, 18]; recent research of the author [13] has shown that the slightly stronger condition $C(4)$ implies that the word problem is solvable in linear time on a 2-tape Turing machine.

In this paper, we take an automatic-theoretic approach to the study of small overlap semigroups and monoids. Our main result is that the word problem for any $C(4)$ monoid or semigroup presentation is a deterministic rational relation (and moreover, effectively computable as such). It follows from results of automata theory [11, 12] that the set of all words which are lexicographically minimal in their equivalence classes forms a regular language of normal forms, and that a normal form for any element can be computed in linear time. We are also able to deduce that every monoid or semigroup admitting a presentation satisfying the condition $C(4)$ is rational (in the sense of Sakarovitch [19]) and hence also asynchronous automatic, and word hyperbolic (in the sense of Duncan and Gilman [3]). Another consequence is that $C(4)$ monoids satisfy an analogue of Kleene’s theorem (see for example [10]): their rational subsets coincide with their recognisable
subsets. It follows also that membership is uniformly decidable for rational subsets, and hence also for finitely generated submonoids, of such monoids.

In addition to this introduction, this article comprises four sections. Section 2 briefly reviews the definitions of monoid and semigroup presentations, and of small overlap conditions. Section 3 contains some purely automata-theoretic results which will be used to establish our main results, and may be of some independent interest. In Section 4 we combine the results of the previous section with those of [13] to prove our main theorem. Finally, in Section 5 we deduce some consequences.

2. Preliminaries

In this section we briefly recall the key definitions of semigroup and monoid presentations and of small overlap conditions, which will be used in the rest of this paper.

Let $A$ be a finite alphabet (set of symbols). A word over $A$ is a finite sequence of zero or more elements from $A$. The set of all words over $A$ is denoted $A^*$; under the operation of concatenation it forms a monoid, called the free monoid on $A$. The length of a word $w \in A^*$ is denoted $|w|$. The unique empty word of length 0 is denoted $\varepsilon$; it forms the identity element of the monoid $A^*$. The set $A^* \setminus \{\varepsilon\}$ of non-empty words forms a subsemigroup of $A^*$, called the free semigroup on $A$ and denoted $A^+$. For $k \in \mathbb{N}$ we write $A_k$, $A^k_\leq$ and $A^k_<$ to denote the set of words in $A^*$ of length respectively exactly $k$, less than or equal to $k$, and strictly less than $k$. If $w \in A^*$ is a word, we write $w^R$ to denote the reverse of $w$, that is, the word composed of the letters of $w$ written in reverse order.

A finite monoid presentation $\langle A \mid R \rangle$ consists of a finite alphabet $A$ (the letters of which are called generators), together with a finite set $R \subseteq A^* \times A^*$ of pairs of words (called relations). We say that $u, v \in A^*$ are one-step equivalent if $u = axb$ and $v = ayb$ for some possibly empty words $a, b \in A^*$ and relation $(x, y) \in R$ or $(y, x) \in R$. We say that $u$ and $v$ are equivalent, and write $u \equiv_R v$ or just $u \equiv v$, if there is a finite sequence of words beginning with $u$ and ending with $v$, each term of which but the last is one-step equivalent to its successor. Equivalence is clearly an equivalence relation; in fact it is the least equivalence relation containing $R$ and compatible with the multiplication in $A^*$. We write $\overline{u}$ for the equivalence class of a word $u \in A^*$. The equivalence classes form a monoid with multiplication well-defined by $\overline{u} \cdot \overline{v} = \overline{uv}$; this is called the monoid presented by the presentation.

The word problem for a (fixed) monoid presentation $\langle A \mid R \rangle$ is the algorithmic problem of, given as input two words $u, v \in A^*$, deciding whether $u \equiv_R v$.

Definitions corresponding to all of those above can also be made for semigroups (without necessarily an identity element), by taking $A^+$ in place of $A^*$ (in all places except the definition of one-step equivalence, where $a$ and $b$ must still be allowed to be empty).

Now suppose we have a fixed monoid or semigroup presentation $\langle A \mid R \rangle$. We begin by recalling some basic definitions from the theory of small overlap conditions [8,17]. A relation word is a word which appears as one side of a relation in $R$. A piece is a word which appears more than once as a factor.
in the relations, either as a factor of two different relation words, or as a
factor of the same relation word in two different (but possibly overlapping)
places. Let \( m \in \mathbb{N} \) be a positive integer. The presentation is said to satisfy
\( C(m) \) if no relation word can be written as a product of \( m \) pieces. Thus \( C(1) \) says that no relation word is empty (which in the
semigroup case is a trivial requirement); \( C(2) \) says that no relation word is a
factor of another.

Retaining our fixed presentation, we now recall some more specialist termi-
nology from [13]. For each relation word \( R \), let \( X_R \) and \( Z_R \) denote respectively
the longest prefix of \( R \) which is a piece, and the longest suffix of \( R \)
which is a piece. If the presentation satisfies \( C(3) \) then \( R \) cannot be written
as a product of two pieces, so this prefix and suffix cannot meet; thus, \( R \) ad-
mits a factorisation \( X_R Y_R Z_R \) for some non-empty word \( Y_R \). If moreover the
presentation satisfies the stronger condition \( C(4) \) then \( R \) cannot be written
as a product of three pieces, so \( Y_R \) is not a piece. The converse also holds: a
\( C(3) \) presentation such that no \( Y_R \) is a piece is a \( C(4) \) presentation. We call
\( X_R, Y_R \) and \( Z_R \) the maximal piece prefix, the middle word and the maximal
piece suffix respectively of \( R \).

If \( R \) is a relation word we write \( \overline{R} \) for the (necessarily unique, as a result of the
small overlap condition) word such that \((R, \overline{R})\) or \((\overline{R}, R)\) is a relation in
the presentation. We write \( \overline{X_R}, \overline{Y_R} \) and \( \overline{Z_R} \) for \( X_{\overline{R}}, Y_{\overline{R}} \) and \( Z_{\overline{R}} \) respectively.
(This is an abuse of notation since, for example, the word \( \overline{X_R} \) may be a
maximal piece prefix of two distinct relation words, but we shall be careful
to ensure that the meaning is clear from the context.)

A relation prefix of a word is a prefix which admits a (necessarily unique,
respectively of some relation word \( XY \) \( X \) and \( Y \) are the maximal piece prefix
and middle word respectively of some relation word \( XYZ \). An overlap prefix (of length \( n \) )
of a word \( u \) is a relation prefix which admits an (again necessarily unique)
factorisation of the form \( bX_1Y'_1X_2Y'_2 \ldots X_nY_n \) where

- \( n \geq 1 \);
- \( bX_1Y'_1X_2Y'_2 \ldots X_nY_n \) has no factor of the form \( X_0Y_0 \), where \( X_0 \) and
  \( Y_0 \) are the maximal piece prefix and middle word respectively of some
  relation word, beginning before the end of the prefix \( b \);
- for each \( 1 \leq i \leq n \), \( R_i = X_iY_iZ_i \) is a relation word with \( X_i \) and \( Z_i \)
  the maximal piece prefix and suffix respectively; and
- for each \( 1 \leq i < n \), \( Y'_i \) is a proper, non-empty prefix of \( Y_i \).

Let \( u \in A^* \) be a word and let \( p \) be a piece. We say that \( u \) is \( p \)-active if
\( pu \) has a relation prefix \( aXY \) with \( |a| < |p| \), and \( p \)-inactive otherwise.

We now recall some basic definitions from automata theory. If \( A \) is an
alphabet, we denote by \( A^\$ \) the alphabet \( A \cup \{\$\} \) where \( \$ \) is a new symbol
not in \( A \). The symbol \( \$ \) will be used as an end-marker for certain types of
automata. If \( R \subseteq A_1^* \times A_2^* \) is a relation, we denote by \( R^\$ \) the set
\[
R^\$ = R (\$, \$) = \{(u\$, v\$) \mid (u, v) \in R\} \subseteq A_1^\$ \times A_2^\$ \subseteq (A_1^\$ )^* \times (A_2^\$ )^*.
\]

A rational transducer from an alphabet \( A_1 \) to an alphabet \( A_2 \) is a finite
directed graph with edges labelled by elements of \( A_1^* \times A_2^* \), together with a
distinguished initial vertex and a set of distinguished terminal vertices. The
labelling of edges extends to a labelling of paths via the multiplication in the direct product monoid $A_1^* \times A_2^*$. A pair $(u, v) \in A_1^* \times A_2^*$ is accepted by the transducer if it labels some path from the initial vertex to a terminal vertex. The relation accepted by the transducer is the set of all pairs accepted. A relation accepted by some transducer is called a rational relation or rational transduction. Transductions, which were introduced in [4], are of fundamental importance in the theory of formal languages and automata; a detailed study can be found in [4].

A deterministic 2-tape finite automaton consists of two alphabets $A_1$ and $A_2$, a finite state set $Q$ partitioned into two disjoint subsets $Q_1$ and $Q_2$ with a distinguished initial state and set of distinguished terminal states, and for each $i = 1, 2$ a partial function

$$\delta_i : Q_i \times A_i^* \to Q.$$ 

Let $\rightarrow$ be the smallest binary relation on $A_1^* \times A_2^* \times Q$ such that

- $(au, v, p) \rightarrow (u, v, q)$ for all $a \in A_1$, $u \in A_1^* \times A_2^*$, $v \in A_2^*$, $p \in Q_1$, $q \in Q$ such that $\delta_1(p, a)$ is defined and equal to $q$; and
- $(u, bv, p) \rightarrow (u, v, q)$ for all $b \in A_2$, $u \in A_1^* \times A_2^*$, $v \in A_2^*$, $p \in Q_2$, $q \in Q$ such that $\delta_2(p, b)$ is defined and equal to $q$;

and let $\rightarrow^*$ be the reflexive, transitive closure of $\rightarrow$. We say that a pair $(u, v) \in A_1 \times A_2$ is accepted by the automaton if there exists an initial state $q_0$ and a terminal state $q_1$ such that that $(u, v, q_0) \rightarrow^* (\epsilon, \epsilon, q_1)$. Once again, the relation accepted by the automaton is the set of all pairs accepted.

A relation is called a deterministic rational relation if it is accepted by a deterministic 2-tape automaton, and a reverse deterministic rational relation if the relation

$$\{(u^R, v^R) \mid (u, v) \in R\}$$

is accepted by a deterministic 2-tape automaton. In general, a deterministic rational relation need not be reverse deterministic rational [5, Theorem 1]. Every [reverse] deterministic rational relation is accepted by a transducer [5] and so is indeed a rational relation. The following elementary proposition gives a partial converse to this statement; the general idea is well known but the precise formulation we need does not seem to have appeared in the literature, so for completeness we give an outline proof.

**Proposition 1.** Let $R \subseteq A_1^* \times A_2^*$ be a relation and suppose $R^S$ is accepted by a transducer with the property that for every state $q$, one of the following (mutually exclusive) conditions holds:

1. $q$ has an edge leaving it, and every edge leaving $q$ has the form $(a, \epsilon)$ for some $a \in A_1^*$, and there is at most one such edge for each $a \in A_1^*$;
2. $q$ has an edge leaving it, and every edge leaving $q$ has the form $(\epsilon, a)$ for some $a \in A_2^*$, and there is at most one such edge for each $a \in A_2^*$;
3. there are no edges leaving $q$;
4. there is exactly one edge leaving $q$, and that edge has label $(\epsilon, \epsilon)$.

Then $R$ is accepted by a deterministic 2-tape automaton.

**Proof.** Let $M$ be the transducer accepting $R^S$ with the given property, and let $Q$ be the state set of $M$. Notice that for each state $q$, there is at most
one state, which we call $\mathcal{7}$, with the property that there is a path from $q$ to $q'$ labelled $(\epsilon, \epsilon)$ and $\mathcal{7}$ satisfies condition (i) or (ii) in the statement of the proposition. Since (i) and (ii) are mutually exclusive, we may choose a partition $Q = Q_1 \cup Q_2$ of $Q$ into disjoint subsets such that for every $q \in Q$ with $\mathcal{7}$ defined we have that $\mathcal{7}$ satisfies condition (i) if and only if $q \in Q_1$, and similarly $\mathcal{7}$ satisfies condition (ii) if and only if $q \in Q_2$. (States $q$ for which $\mathcal{7}$ is not defined may be assigned arbitrarily to either $Q_1$ or $Q_2$).

We now define a new deterministic 2-tape automaton $N$ as follows. The two tape alphabets of $N$ are $A_1$ and $A_2$. The state set of $N$ is the state set $Q$ of $M$ partitioned into the subsets $Q_1$ and $Q_2$ constructed above. The initial state of $N$ is the initial state of $M$. The terminal states of $N$ consist of all states $p \in Q$ such that $M$ has a path from $p$ to a terminal state with label $(\epsilon, \epsilon)$. For each $a \in A_1^*$, $p \in Q_1$ and $q \in Q$ we set $\delta_1(p, a) = q$ if and only if $\mathcal{7}$ is defined and $M$ has an edge from $\mathcal{7}$ to $q$ with label $(a, \epsilon)$. Similarly, for each $a \in A_2^*$, $p \in Q_2$ and $q \in Q$ we set $\delta_2(p, a) = q$ if and only if $\mathcal{7}$ is defined and $M$ has an edge from $\mathcal{7}$ to $q$ with label $(\epsilon, a)$. It follows directly from the criteria on the automata that each $\delta_i$ is a well-defined partial function from $Q_i \times A_i^*$ to $Q$.

It is now a routine matter to verify that the deterministic 2-tape automaton $N$ accepts a pair $(u, v)$ if and only if $M$ accepts $(u$, $v)$. $\square$

3. Prefix-Rewriting Automata

In this section, we study a type of automaton called a 2-tape prefix-rewriting automaton. We show that any relation accepted by a deterministic 2-tape prefix-rewriting automaton with a certain property called bounded expansion is a deterministic rational relation. In Section 4 we shall apply this result to show that the word problem for a $C(4)$ monoid presentation is a deterministic rational relation.

Let $k \in \mathbb{N}$ and $A_1$ and $A_2$ be finite alphabets. A $k$-prefix-rewriting automaton from $A_1$ to $A_2$ is a finite directed graph with edges labelled by elements of

$$\left( (A_1^k \times A_1^{<k}) \cup (A_1^{<k} \times A_1^{k}) \right) \times \left( (A_2^k \times A_2^{<k}) \cup (A_2^{<k} \times A_2^{k}) \right),$$

together with a distinguished initial vertex and a set of distinguished terminal vertices. Given such an automaton with vertex set $Q$, we define a binary relation $\to$ on $A_1^* \times A_2^* \times Q$ by

$$(u_1$, $v_1$, $q_1) \to (u_2$, $v_2$, $q_2)$$

if and only if there exist words $x_1$, $x_2$, $y_1$, $y_2$, $u'$ and $v'$ in the appropriate alphabets such that

$$u_1 = x_1 u', \ u_2 = x_2 u', \ v_1 = y_1 v', \ v_2 = y_2 v'$$

and $(x_1, x_2, y_1, y_2)$ labels an edge from $q_1$ to $q_2$. If this holds we say that the edge $e$ is applicable in the configuration $(u_1$, $v_1$, $q_1)$. We call the automaton deterministic if in each configuration $(u, v, q) \in A_1^* \times A_2^* \times Q$ there is at most one edge applicable.

Let $\to^*$ denote the reflexive, transitive closure of the relation $\to$. We say that a pair $(u, v) \in A_1^* \times A_2^*$ is accepted by the automaton if there exists a
terminal state $q_1$ such that
\[(u\$, v\$, q_0) \rightarrow^* (\$, \$, q_1)\]

where $q_0$ is the initial state. As usual, the relation accepted by the automaton is the set of all pairs in $A_1^* \times A_2^*$ which are accepted by the automaton.

Intuitively, a 2-tape prefix-rewriting automaton is very similar to a 2-pushdown automaton; the only essential difference is that we allow both stacks to be initialised with non-empty words, and view the automaton as accepting pairs of words and defining a relation instead of a language. As one might expect, such automata are extremely powerful, being easily seen to accept in particular any relation of the form $L \times \{\epsilon\}$ where $L$ is a recursively enumerable language. However, we shall be interested in a more restricted class of such automata. We say that a prefix-rewriting automaton has bounded expansion if there exists a constant $b \in \mathbb{N}$ such that whenever
\[(u_1, v_1, q_1) \rightarrow^* (u_2, v_2, q_2)\]

we have $|u_2| \leq |u_1| + b$ and $|v_2| \leq |v_2| + b$. We call such a value of $b$ an expansion bound for the automaton.

Note that the bounded expansion condition places a requirement on the contents of each store independently. This contrasts with the shrinking and length-reducing conditions on 2-pushdown automata, used to describe growing context-sensitive and Church-Rosser languages [2], where a restriction is applied to the total size of the 2 stores considered together. It transpires that our condition is a very strong one, in that a relation accepted by a prefix-rewriting automaton with bounded expansion is necessarily rational.

**Theorem 1.** Any relation accepted by a (deterministic) 2-tape prefix-rewriting automaton with bounded expansion is a (deterministic) rational transduction. Moreover, given a (deterministic) 2-tape prefix-rewriting automaton and an expansion bound for it, one can effectively construct a (deterministic) transducer recognising the same relation.

**Proof.** Let $M$ be a 2-tape $k$-prefix-rewriting automaton with bounded expansion accepting a relation $R \subseteq A_1^* \times A_2^*$, and let $b \in \mathbb{N}$ be an expansion bound for $M$. We construct from $M$ a finite transducer $N$ which simulates $M$ and so accepts $R\$$. Intuitively, the new transducer will read $u$ and $v$, buffering at least the first $k$ characters of each in the finite state control. Prefix-modification can thus be simulated by modifying only the contents of the finite state control. Since a prefix-rewriting automaton can replace a prefix with a longer one, it may be necessary to store more than $k$ characters of each word in the finite state control, but the expansion bound serves to ensure that a buffer of some fixed size (namely $k + b$) will always suffice.

Formally, for $i = 1, 2$ we let $C_i = A_i^{\leq k + b} \cup A_i^{k + b}\$ and let $B_i$ be the set of all words $x \in C_i$ such that either $|x| \geq k$ or the final letter of $x$ is $\$.

(Intuitively, $C_i$ will be the set of all possible states for the buffer on tape $i$, while $B_i$ will be the set of “adequately populated” buffer states in which it is not immediately necessary to read any more of the input word.)

We construct a transducer $N$ as follows. The state set of $N$ is $C_1 \times C_2 \times Q$ where $Q$ is the state set of $M$. The initial state is $(\epsilon, \epsilon, q_0)$ where $q_0$ is the
initial state of M. The terminal states are those of the form \((\$, \$, q)\) with q a terminal state of M. The edges are as follows:

1. For every \(x \in C_1, y \in C_2\) with \(x \notin B_1\), every \(a \in A_1^S\) such that 
   \(xa \in C_1\) and every state \(q\), there is an edge from \((x, y, q)\) to \((xa, y, q)\)
   with label \((a, \epsilon)\);

2. For every \(x \in C_1, y \in C_2\) with \(x \in B_1\) but \(y \notin B_2\), every \(a \in A_2^S\)
   such that \(ya \in C_2\) and every state \(q\), there is an edge from \((x, y, q)\)
   to \((xa, y, q)\) with label \((\epsilon, a)\);

3. For each edge in M from p to q with label \((u_1, u_2, v_1, v_2)\) and each \(x', y'\) such that 
   \(u_1x' \in B_1\) and \(v_1y' \in B_2\), there is an edge from 
   \((u_1x', v_1y', p)\) to \((u_2x', v_2y', q)\) with label \((\epsilon, \epsilon)\) provided 
   \(u_2x' \in C_1\) and \(v_2y' \in C_2\).

Edges of types (1) and (2) serve simply to read the input words into the buffers until each contains sufficient data (at least \(k\) letters or the entire of the input if this is less), while edges of type (3) simulate the transitions of the prefix-rewriting automaton M by operating only on the buffers.

Notice that once the transducer reaches a state in \(A_1^{<k+b} \times C_2 \times Q\) (that is, one where the first buffer content contains the symbol \$), it will always remain in such a state, and will never again read from the first input word. Similarly, once it reaches a state in \(C_1 \times A_2^{<k+b} \times Q\) it will always remain in such a state and will never again read from the second input word. Noting also that all the terminal states lie in both of these sets, it follows that all pairs accepted by the transducer lie in \(A_1^S \times A_2^S\).

We say that a configuration \((u_1, v_1, q_1)\) has expansion bound \((c, d) \in \mathbb{N} \times \mathbb{N}\) if whenever \((u_1, v_1, q_1) \rightarrow^* (u_2, v_2, q_2)\) we have 
\[|u_2| \leq |u_1| + c \quad \text{and} \quad |v_2| \leq |v_1| + d.\]
Note that the expansion bound condition on the automaton means that \((b, b)\) is an expansion bound for every configuration. We shall need the following lemma.

**Lemma 1.** Suppose \((u_1, v_1, q_1) \rightarrow^* (u_2, v_2, q_2)\) in the prefix-rewriting automaton M. Suppose further than \((u_1, v_1, q_1)\) has expansion bound \((c_1, d_1)\)
and that \(u_1 = s_1s_1', v_1 = t_1t_1'\) where \(|s_1| \leq k + b - c_1\) and \(|t_1| \leq k + b - d_1\).
Then there exist factorisations \(u_2 = s_2s_2'\) and \(v_2 = t_2t_2'\) and an expansion bound \((c_2, d_2)\) for \((u_2, v_2, q_2)\) such that 
\(|s_2| \leq k + b - c_2, |t_2| \leq k + b - d_2\)
and the transducer \(N\) has a path from \((s_1, t_1, q_1)\) to \((s_2, t_2, q_2)\) with label \((g, h)\)
where \(s_1' = gs_2\) and \(t_1' = ht_2'\).

**Proof.** We use induction on the number of steps in the transition sequence
from from \((u_1, v_1, q_1)\) to \((u_2, v_2, q_2)\). Clearly if \((u_1, v_1, q_1) = (u_2, v_2, q_2)\)
 it suffices to take 
\(s_2 = s_1, s_2' = s_1', t_2 = t_1, t_2' = t_1', c_2 = c_1, d_2 = d_1\)
and \(g = h = \epsilon\).

Next we consider one-step case, that is, the case in which \((u_1, v_1, q_1) \rightarrow (u_2, v_2, q_2)\).
Let \(g\) be the shortest prefix of \(s_1'\) such that \(s_1g \in B_1\); similarly, let \(h\) be the shortest prefix of \(t_1'\) such that \(t_1h \in B_2\).
It follows easily from the definition that our transducer \(N\) has a path from \((s_1, t_1, q_1)\) to \((s_1g, t_1h, q_1)\)
with label \((g, h)\).

Now since \((u_1, v_1, q_1) \rightarrow (u_2, v_2, q_2)\), by definition there exist words \(x_1, x_2, y_1, y_2, u'\) and \(v'\)
such that \(u_1 = x_1u', u_2 = x_2u', v_1 = y_1v', v_2 = y_2v'\)
and \((x_1, x_2, y_1, y_2)\) labels an edge from \(q_1\) to \(q_2\). Since \(|x_1|, |y_1| \leq k\) we have
that $x_1$ and $y_1$ are prefixes of $s_1g$ and $t_1h$ respectively, say $s_1g = x_1x'$ and $t_1h = y_1y'$. But now by the definition of our transducer, there is an edge from $(s_1g, x_1x', t_1h, y_1y', q_1)$ to $(x_2x', y_2y', q_2)$ with label $(\epsilon, \epsilon)$. Thus, setting $s_2 = x_2x'$ and $t_2 = y_2y'$ and defining $s_2'$ and $t_2'$ accordingly, we obtain a path from $(s_1, t_1, q_1)$ to $(s_2, t_2, q_2)$ with label $(g, h)$.

Now we have

$$x_2x' s_2' = s_2 s_2' = u_2 = x_2u'$$

so cancelling on the left we obtain $u' = x's_2'$. But now

$$s_1s_1' = u_1 = x_1u' = x_1x's_2' = s_1gs_2'$$

so cancelling again yields $s_1' = gs_2'$ as claimed. An entirely similar argument shows that $t_1' = ht_2'$.

Next, notice that we have $|u_1| - |u_2| = |s_1| - |s_2|$ and similarly $|v_1| - |v_2| = |s_1| - |s_2|$. Set $c_2 = c_1 + |s_1| - |s_2|$ and $d_2 = d_1 + |t_1| - |t_2|$. Clearly since any state derivable from $(u_2, v_2, q_2)$ is also derivable from $(u_1, v_1, q_1)$, it is readily verified that $(c_2, d_2)$ is an expansion bound for $(u_2, v_2, q_2)$. But now we have

$$|s_2| = |s_1| + c_1 - c_2 \leq (k + b - c_1) + c_1 - c_2 = k + b - c_2$$

and similarly $|t_2| \leq k + b - d_2$ as required to complete the proof of the lemma in the one-step case.

The inductive argument for the general case is now straightforward. □

Now if $(u, v)$ is accepted by the prefix-rewriting automaton then by definition we have $(u$, $v$, $q_0) \rightarrow^* (\$, $\$, $q_t)$ where $q_0$ is the initial state and $q_t$ is some terminal state. Since the automaton has expansion bound $b$, the state $(u$, $v$, $q_0)$ has expansion bound $(b, b)$. So taking $u_1 = u$, $v_1 = v$, $q_1 = q_0$, $q_2 = q_1$, $c_1 = d_1 = b$, $s_1 = t_1 = \epsilon$, $s_1' = u$ and $s_2' = v$ and applying Lemma 1 our transducer has a path from $(\epsilon, \epsilon, q_0)$ to $(s_2, t_2, q_t)$ with label $(g, h)$ where $s_2 s_2' = t_2 t_2' = \$, $u = s_1' = gs_2'$ and $v = t_1' = ht_2'$.

Now either $s_2 = \epsilon$ and $s_2' = \$, or $s_2 = \$ and $s_2' = \$. In the latter case we have $g = u\$. In the former case we have $g = u$ and there is clearly an edge from $(s_2, t_2, q_t)$ to $(s_2$, $\$, $t_2, q_t)$ labelled $(\$, $\$, $\$), so in either case there is a path from $(\epsilon, \epsilon, q_0)$ to $(\$, $\$, $q_t)$ with label $(u\$, $v\$). A similar argument deals with the case that $h = v$, showing that in all cases there is a path from the start state $(\epsilon, \epsilon, q_0)$ to the terminal state $(\$, $\$, $q_t)$ with label $(u\$, $v\$). Thus, the transducer $N$ accepts $(u\$, $v\$) as required.

Conversely, suppose $(u\$, $v\$) is accepted by our transducer. Then there must be a path $\pi$ from $(\epsilon, \epsilon, q_0)$ to $(\$, $\$, $q_t)$ for some initial state $q_0$ and terminal state $q_t$. Now clearly $\pi$ admits a unique decomposition of the form

$$\pi = \lambda_0 \rho_1 \lambda_1 \rho_2 \ldots \rho_n \lambda_n$$

where each $\rho_i$ is a single edge of type (3) and each $\lambda_i$ is a (possibly empty) path consisting entirely of edges of types (1) and (2). Clearly each $\rho_i$ has label $(\epsilon, \epsilon)$. Suppose each $\lambda_i$ has label $(u_i, v_i)$; then clearly $u\$ = $u_0 u_1 \ldots u_n$ and $v\$ = $v_0 v_1 \ldots v_n$. Suppose that for $0 \leq i \leq n$, after traversing the initial segment of the path $\pi$ up to and including $\lambda_i$, the automaton is in configuration $(x_i, y_i, q_i)$. Notice that, since the paths $\lambda_i$ do not change the state component, $q_0$ is consistent with its use above, and in particular is an
initial state in the prefix-rewriting automaton $M$. Similarly, $q_n = q_t$ is a terminal state of $M$. Now for $0 \leq i \leq n$ define

\[ c_i = x_i u_{i+1} u_{i+2} \ldots u_n \text{ and } d_i = y_i v_{i+1} v_{i+2} \ldots v_n. \]

Clearly we have that $x_0 = u_0$ and $y_0 = v_0$, from which it follows that $c_0 = u \$ \text{ and } d_0 = v \$. We also have $x_n = y_n = \$ \text{ so that } c_n = d_n = \$. 

Now it is straightforward to see that for $1 \leq i \leq n$ we have

\[ (c_{i-1}, d_{i-1}, q_{i-1}) \rightarrow (c_i, d_i, q_i) \]

so that

\[ (u \$, v \$, q_0) = (c_0, d_0, q_0) \rightarrow^* (c_n, d_n, q_n) = (\$, \$, q_t). \]

which by definition means that $(u, v)$ is accepted by the 2-tape prefix-rewriting automaton $M$. This completes the proof that the transducer $N$ accepts the relation $R \$. It is easy to show that for any relation $T$, $T$ is a rational relation if and only if $T \$ \text{ is a rational relation, so this suffices to prove that } R \text{ is a rational relation.} 

Finally, suppose that the original prefix-rewriting automaton $M$ is deterministic. We claim that the transducer $N$ which we have constructed to accept $R \$ \text{satisfies the conditions of Proposition[1] from which it will follow that } R \text{ is a deterministic rational relation, as required.} 

To this end, consider a state $(x, y, q)$ in $N$. If $x \notin B_1$ then it follows immediately from the definition that all out-edges have labels of the form $(a, \epsilon)$ with $a \in A_1$ and that there is exactly one such for each $a \in A_1$ so that condition (i) holds. Similarly, if $x \in B_1$ but $y \notin B_2$ then all out-edges have labels of the form $(\epsilon, a)$ and there is exactly one such for each $a \in A_2$ so condition (ii) holds.

Finally, suppose $x \in B_1$ and $y \in B_2$. From the definition of $N$, any edge leaving $(x, y, p)$ must have label $(\epsilon, \epsilon)$. If there were more than one such edge, then each would correspond to a different possible transition in $M$ from the state $(x, y, p)$; but by the determinism assumption on $M$ there can only be one such transition, so this would give a contradiction. Thus we deduce that there is at most one such edge, so that either condition (iii) or condition (iv) holds. This completes the proof.

We emphasise that Theorem[1] does not give a means to effectively construct a transducer for a relation $R$ starting only from a 2-tape prefix-rewriting automaton with bounded expansion which accept $R$. The construction in the proof makes explicit use of the expansion bound for the prefix-rewriting automaton, and it is not clear that one can effectively compute an expansion bound from the automaton, even given the knowledge that such a bound exists.

4. Automata for the Word Problem in Small Overlap Monoids

The aim of this section is to show that the word problem for any $C(4)$ monoid must be a deterministic rational relation. Throughout this section, we fix a monoid presentation $\langle A \mid R \rangle$ satisfying the condition $C(4)$.

In [13] we presented an efficient recursive algorithm which can be used to solve the word problem for such a presentation. For ease of reference the algorithm is reproduced in Figure 1. It takes as input a piece of the
WP-PREFIX \((u, v, p)\)

1. if \(u = \epsilon\) or \(v = \epsilon\)
   2. then if \(u = \epsilon\) and \(v = \epsilon\) and \(p = \epsilon\)
      3. then return \(\text{Yes}\)
   4. else return \(\text{No}\)
5. elseif \(u\) does not have the form \(XY u'\) with \(XY\) a clean overlap prefix
6. then if \(u\) and \(v\) begin with different letters
      7. then return \(\text{No}\)
   8. elseif \(p \neq \epsilon\) and \(u\) and \(p\) begin with different letters
      9. then return \(\text{No}\)
   10. else
       11. \(u \leftarrow u\) with first letter deleted
       12. \(v \leftarrow v\) with first letter deleted
       13. if \(p \neq \epsilon\)
           14. then \(p \leftarrow p\) with first letter deleted
       15. return WP-PREFIX \((u, v, p)\)
16. else
17. let \(X, Y, u'\) be such that \(u = XY u'\)
18. if \(p\) is a prefix of neither \(X\) nor \(Y\)
      19. then return \(\text{No}\)
   20. elseif \(u = XY Z u''\) and \(v = XY Z v''\)
      21. then if \(u''\) is \(Z\)-active
          22. then return WP-PREFIX \((Z u'', Z v'', \epsilon)\)
      23. else return WP-PREFIX \((Z u'', Z v'', \epsilon)\)
   24. elseif \(u = XY u'\) and \(v = XY v'\)
      25. then if \(p\) is a prefix of \(X\)
          26. then return WP-PREFIX \((u', v', \epsilon)\)
      27. else return WP-PREFIX \((u', v', Z)\)
   28. elseif \(u = XY Z u''\) and \(v = \overline{XY} Z v''\)
      29. then if \(u''\) is \(Z\)-active
          30. then return WP-PREFIX \((Z u'', Z v'', \epsilon)\)
      31. else return WP-PREFIX \((Z u'', Z v'', \epsilon)\)
   32. elseif \(u = XY u'\) and \(v = \overline{XY} Z v''\)
      33. then return WP-PREFIX \((u', Z v'', \epsilon)\)
   34. elseif \(u = XY Z u''\) and \(v = \overline{XY} v'\)
      35. then return WP-PREFIX \((Z u'', v', \epsilon)\)
   36. elseif \(u = XY u'\) and \(v = \overline{XY} v'\)
      37. then let \(z\) be the maximal common suffix of \(Z\) and \(\overline{Z}\)
          38. let \(z_1\) be such that \(Z = z_1 z\)
          39. let \(z_2\) be such that \(\overline{Z} = z_2 z\)
          40. if \(u'\) does not begin with \(z_1\) or \(v'\) does not begin with \(z_2\);
              41. then return \(\text{NO}\)
          42. elseif \(u''\) be such that \(u' := z_1 u''\)
              43. let \(v''\) be such that \(v' := z_2 v''\);
              44. return WP-PREFIX \((u'', v'', z)\)

Figure 1. Algorithm for the word problem of a \(C(4)\) presentation
presentation $p \in A^*$ and two words $u, v \in A^*$ and outputs \text{YES} if $u \equiv v$ and $p$ is a possible prefix of $u$ (and hence also of $v$). Otherwise it outputs \text{NO}. In particular, if $p = \epsilon$ then the algorithm outputs \text{YES} if $u \equiv v$ and \text{NO} if $u \not\equiv v$, thus solving the word problem for the presentation. See [13, Lemma 5] and [13, Lemma 6] for proofs of correctness and termination respectively.

The proof strategy for our main result is to show that this algorithm can be implemented on a deterministic 2-tape prefix-rewriting automaton with bounded expansion. The results of Section 3 then allow us to conclude that the word problem is a deterministic rational relation.

**Theorem 2.** Let $\langle A \mid R \rangle$ be a finite monoid presentation satisfying the small overlap condition $C(4)$. Then the relation

$$\{(u, v) \in A^* \times A^* \mid u \equiv v\}$$

is deterministic rational and reverse deterministic rational. Moreover, one can, starting from the presentation, effectively compute 2-tape deterministic automata recognising this relation and its reverse.

**Proof.** Let $k$ be twice the maximum length of a relation word in the presentation. We construct a deterministic 2-tape $k$-prefix-rewriting automaton recognising the desired relation, and an expansion bound for this automaton. By Theorem 1 this suffices to show that the given relation is deterministic rational and that a 2-tape deterministic automaton for it can be effectively constructed. Since the $C(4)$ condition on the presentation is entirely left-right symmetric, the claim regarding the reverse relation also follows.

Let $P$ be the set of all pieces of the presentation $\langle A \mid R \rangle$, and let $+$ be a new symbol not in $P$. Recall that $\epsilon$ is by definition a piece of every presentation, so certainly $\epsilon \in P$. Let $W = A^k \cup A^{<k}\$. We define a 2-tape prefix-rewriting automaton with

- state set $P \cup \{+\}$;
- initial state $\epsilon$;
- unique terminal state $+$;

and edges defined as follows.

- **(A)** an edge from $\epsilon$ to + labelled $(\$, $\$, $\$, $\$).
- **(B)** for every $u \in W$ with $u \neq \$ and such that $u$ has no clean overlap prefix of the form $XY$, and every $v \in W$ such that $v \neq \$ and $u$ and $v$ begin with the same letter, a transition from $p$ to $p'$ labelled $(u, u', v, v')$ where $u'$, $v'$ and $p'$ are obtained from $u$, $v$ and $p$ respectively by deleting the first letter.

In addition for every $p \in P$ and $u, v \in W$ such that $u$ has a clean overlap prefix (say $XY$) and $p$ is a prefix of either $X$ or $Y$ or both, the automaton may have an edge from $p$ to another state in $P$ as follows:

- **(C1)** If $u = XYZu''$, $v = XYZv''$ and $u''$ is $Z$-active, the automaton has an edge from $p$ to $\epsilon$ labelled $(u, Zu'', v, Zv'')$.
- **(C2)** If $u = XYZu''$, $v = XYZv''$ and $u''$ is not $Z$-active, the automaton has an edge from $p$ to $\epsilon$ labelled $(u, Zu'', v, Zv'')$. 


(C3) If \( u = XYu' \), \( v = XYv' \), \( u \) and \( v \) do not both have \( XYZ \) as a prefix, and \( p \) is a prefix of \( X \), the automaton has an edge from \( p \) to \( \epsilon \) labelled 
\((u, u', v, v')\).

(C4) If \( u = XYu' \), \( v = XYv' \), \( u \) and \( v \) do not both have \( XYZ \) as a prefix, and \( p \) is not a prefix of \( X \), the automaton has an edge from \( p \) to \( Z \) 
with label \((u, u', v, v')\).

(C5) If \( u = XYZu'' \), \( v = XYZv'' \) and \( u'' \) is \( Z \)-active, the automaton has an edge from \( p \) to \( \epsilon \) labelled 
\((u, Zu'', v, Zv'')\).

(C6) If \( u = XYZu'' \), \( v = XYZv'' \) and \( u'' \) is not \( Z \)-active, the automaton has an edge from \( p \) to \( \epsilon \) labelled 
\((u, Zu'', v, Zv'')\).

(C7) If \( u = XYu' \), \( v = ZYv'' \) and \( u \) does not have \( XYZ \) as a prefix, 
the automaton has an edge from \( p \) to \( \epsilon \) labelled 
\((u, u', v, Zv'')\).

(C8) If \( u = XYZu'' \), \( v = XYZu' \) and \( v \) does not have \( XYZ \) as a prefix, 
the automaton has an edge from \( p \) to \( \epsilon \) labelled 
\((u, Zu'', v, v')\).

(C9) If \( u = XYu' \), \( v = ZYv' \), \( u \) does not begin with \( XYZ \), \( v \) does not 
begin with \( XY \), \( Z \) is the maximum common 
suffix of \( Z \) and \( Z' \), \( Z = z_1z \), \( Z' = z_2z \), \( u' = z_1u'' \), \( v' = z_2v'' \), the automaton has an edge 
from \( p \) to \( \epsilon \) labelled 
\((u, u'', v, v')\).

First, notice that this automaton is deterministic. Indeed, all edges leaving 
given vertex \( p \in P \) have labels of the form \((u, x, v, y)\) with \( u, v \in W \). Notice that no member of the set \( W \) is a prefix of another; it follows that no 
word has two distinct words in \( W \) as prefixes, which means that the choice 
of prefixes \( u \) and \( v \) to act on is uniquely determined by the configuration in 
which the action is to be applied. Now it can be verified by examination 
that the various conditions on \( u, v \) and \( p \) which result in the inclusion of an 
edge from \( p \) with label of the form \((u, x, v, y)\) are mutually 
exclusive, so that there is at most one such edge, and hence at most one transition applicable 
in any given configuration.

It is now an entirely routine matter to prove by induction that for every 
piece \( p \in A^* \) and words \( u, v \in A^* \) we have 
\[(u\$, v\$, p) \rightarrow^* (\$, \$, +)\] 
if and only if the algorithm outputs \textbf{YES}, that is, if and only if \( u \equiv v \) and \( p \) 
is a possible prefix of \( u \). Transitions of types B, C1, C2, C3, C4, C5, C6, C7, 
C8 and C9 correspond to the recursive calls at lines 15, 24, 25, 28, 29, 32, 33, 
35, 37, 46 respectively, while transition of type A corresponds to termination 
with the answer \textbf{YES} at line 3 of the algorithm. The conditions under which 
the algorithm terminates with the answer \textbf{NO} (at lines 4, 7, 9, 19, 21 and 
43) all correspond to non-terminal configurations of the automaton in which 
no transitions are applicable. It follows from \cite{13} Lemma 7 that the tests for 
clean overlap prefixes and \( Z \)-activity on the buffer contents are equivalent 
to performing the corresponding tests on the whole of the remaining input, 
as demanded by the algorithm.

In particular, we have 
\[(u\$, v\$, \epsilon) \rightarrow^* (\$, \$, +)\] 
if and only if \( u \equiv v \), as required to show that our prefix-rewriting automaton 
solves the word problem. It remains only to find an expansion bound for
the automaton. Let $b$ be the length of the longest relation word in the presentation $\langle A \mid R \rangle$.

Suppose $(u_0, v_0, q_0) \to^* (u_1, v_1, q_1)$ and suppose that $u_0 = z_0u_0'$ and $v_0 = z_0v_0'$ where $z_0$ is either a proper suffix of a relation word or the empty word.

We claim that there are factorisations $u_1 = z_1u_1'$ and $v_1 = z_1v_1'$ where $z_1$ is a proper suffix of relation word or the empty word, $|u_1'| \leq |u_0'|$ and $|v_1'| \leq |v_0'|$.

We consider first the one-step case, that is, where $(u_0, v_0, q_0) \to (u_1, v_1, q_1)$. Then the claim is clear, so suppose the transition is of type $C_1-C_9$. Then from the definitions of these transitions, we must have $u_0 = XYu'$ for some maximum piece prefix $X$ and middle word $Y$ of a relation word $XYZ$. Now $XY$ cannot be a piece, so it cannot be a prefix of $z_0$, which is a proper suffix of a relation word. Thus, we must have $|XY| > |z_0|$ and hence $|u'| < |u_0'|$.

Looking again at the definitions of the transitions, we see that $u_1$ and $v_1$ either

(i) are (not necessarily proper) suffixes of $u'$ and $v'$ respectively; or
(ii) have the form $u_1 = Zu''$ and $v_1 = Zu''$ where $u''$ and $v''$ are (not necessarily proper) suffixes of $u'$ and $v'$ respectively; or
(iii) have the form $u_1 = Zv''$ and $v_1 = Zv''$ where $u''$ and $v''$ are (not necessarily proper) suffixes of $u'$ and $v'$ respectively.

In case (i) it suffices to set $z_1 = \epsilon$ and $u'_1 = u_1$. In case (ii) it suffices to set $z_1 = Z$ and $u'_1 = u''$, noting that $Z$ must be a proper suffix of a relation word since is a maximal piece suffix of $XYZ$ and no relation word can be a piece.

It now follows easily by induction that the claim also holds when

$$(u_0, v_0, q_0) \to^* (u_1, v_1, q_1).$$

In particular, taking $z_0 = \epsilon$ and $u'_0 = u_0$ and then writing $u_1 = z_1u'_1$ as above we have

$$|u_1| = |z_1| + |u'_1| \leq |z_1| + |u'_0| = |z_1| + |u_0| \leq |u_0| + b$$

and similarly $|v_1| \leq |v_0| + b$, as required to show that the automaton has expansion bound $b$. \hfill \square

As an immediate corollary we obtain a corresponding statement for semigroups.

**Corollary 1.** Let $\langle A \mid R \rangle$ be a finite semigroup presentation satisfying the small overlap condition $C(4)$. Then the relation

$$\{(u, v) \in A^+ \times A^+ \mid u \equiv v\}$$

is deterministic rational and reverse deterministic rational. Moreover, one can, starting from the presentation, effectively compute 2-tape deterministic automata recognising this relation and its reverse.

**Proof.** Since the presentation has no empty relation words, the semigroup with presentation $\langle A \mid R \rangle$ arises as the subsemigroup of non-identity elements in the monoid with presentation $\langle A \mid R \rangle$. It follows that

$$\{(u, v) \in A^+ \times A^+ \mid u \equiv v\} = \{(u, v) \in A^* \times A^* \mid u \equiv v\} \setminus \{(\epsilon, \epsilon)\}.$$
Now it is easy to verify that a relation $R$ between free monoids is a deterministic rational relation only if $R \setminus \{(\epsilon, \epsilon)\}$ is a deterministic rational relation between free semigroups, so the result follows from Theorem 2.

5. Consequences

In this section we consider a number of interesting consequences and corollaries of Theorem 2. We begin with some terminology from language theory.

Let $A$ be a finite alphabet, and choose some arbitrary total order $\leq$ on the letters of $A$. Recall that the corresponding lexicographic order is an extension of this order to a total order $\leq_L$ on the free monoid $A^*$, defined inductively by $\epsilon \leq_L w$ for all $w$, and for all $x, y \in A$ and $u, v \in A^*$ we have $xu \leq_L yv$ if either $x \neq y$ and $x \leq y$, or $x = y$ and $u \leq_L v$. Lexicographic order is a total order but not (unless $|A| = 1$) a well-order, since it contains infinite descending chains such as $b, ab, aab, \ldots, a^i b, \ldots$

Hence, if $R$ is an equivalence relation on $A^*$ (even a rational one) there is no guarantee that every equivalence class of $R$ will contain a lexicographically minimal element. In the case that $R$ is locally finite (that is, each equivalence class is finite), however, every class must clearly contain a unique lexicographically minimal element, and the set of elements which are minimal in their class forms a cross-section of the relation, that is, a language of unique representatives for the equivalence classes of the relation; we shall call these representatives lexicographic normal forms. Remmers showed that if $\langle A \mid R \rangle$ is a C(3) monoid [semigroup] presentation then the corresponding equivalence relation on $A^*$ [respectively, $A^+$] is locally finite [8, 17]; it follows that every element of a C(3) monoid has a lexicographic normal form. Johnson [11, 12] showed that if $R$ is a deterministic rational locally finite equivalence relation then the function which maps each word to the corresponding lexicographic normal form can be computed by a deterministic transducer. Thus, we obtain the following corollary to Theorem 2.

**Corollary 2.** Let $\langle A \mid R \rangle$ be a monoid presentation satisfying C(4) and suppose $A$ is equipped with a total order. Then the relation

$$\{(u, v) \in A^* \times A^* \mid u \equiv v \text{ and } v \text{ is a lexicographic normal form}\}$$

is a deterministic rational function.

The image of a rational function is always a regular language [1, Corollary II.4.2]) and deterministic rational functions can be computed in linear time Johnson [12, Theorem 5.1] so we have:

**Corollary 3.** Let $\langle A \mid R \rangle$ be a monoid presentation satisfying C(4) and suppose $A$ is equipped with a total order. Then the lexicographic normal forms comprise a regular language of unique representatives for elements of the monoid. Moreover, there is an algorithm which, given a word $w$ in $A^*$, computes in linear time the corresponding lexicographic normal form.

A monoid $M$ is called rational [19, 16] if there exists a finite generating set $A$ for $M$ and a regular cross-section $L \subseteq A^*$ for $M$ such that the normal forms in $L$ are computed by a transducer.
Corollary 4. Every monoid admitting a $C(4)$ presentation is rational.

Recall that the rational subsets of a monoid $M$ are those which can be obtained from finite subsets by the operations of union, product and submonoid generation (the “Kleene star” operation). If $M$ is generated by a finite subset $A$ then the rational subsets of $M$ are exactly the images in $M$ of regular languages over $A$, which means they have natural finite representations as finite automata over $A$. The recognisable subsets of $M$ are the homomorphic pre-images in $M$ of subsets of finite monoids. In the case that $M$ is a free monoid, the rational subsets are just the regular languages. Kleene’s Theorem asserts that the rational subsets of a free monoid (that is, the regular languages) coincide with the recognisable subsets \[10\]. More generally, a monoid in which the rational and recognisable subsets coincide is called a Kleene monoid, or sometimes is said to satisfy Kleene’s Theorem. Rational monoids were originally introduced in an attempt to obtain a concrete characterisation of Kleene monoids \[19\], and indeed every rational monoid is a Kleene monoid (although it transpires that the converse does not hold). Thus, we obtain:

Corollary 5 (Kleene’s Theorem for Small Overlap Monoids). Let $M$ be a monoid or semigroup admitting a $C(4)$ presentation, and $S$ a subset of $M$. Then $S$ is rational if and only if $S$ is recognisable.

Recall that a collection of subsets of some given base set is called a boolean algebra if it contains the empty set and is closed under union, intersection and complement. As another corollary of the rationality of $M$ we obtain the following fact about rational subsets of $M$.

Corollary 6. Let $M$ be a monoid admitting a $C(4)$ presentation $\langle A \mid R \rangle$. Then the rational subsets of $M$ form a boolean algebra. Moreover, if rational subsets of $M$ are represented by automata over $A$, then the operations of union, intersection and complement are effectively computable.

Proof. Let $\sigma : A^* \rightarrow M$ be the canonical morphism mapping $A^*$ onto $M$, and let

$$\rho = \{(u, v) \in A^* \times A^* \mid u \equiv v \text{ and } v \text{ is a lexicographic normal form}\}.$$  

Suppose $X, Y \in A^*$ are rational subsets, with say $X = \bar{X}\sigma$ and $Y = \bar{Y}\sigma$ where $\bar{X}, \bar{Y} \subseteq A^*$ are regular languages. Then using the facts that $A^*\rho$ contains a unique representative for every element and that $\rho\sigma = \sigma$, it is readily verified that $M \setminus X = (A^*\rho \setminus \bar{X}\rho)\sigma$, $X \cap Y = (\bar{X}\rho \cap \bar{Y}\rho)\sigma$ and $X \cup Y = (\bar{X}\rho \cup \bar{Y}\rho)\sigma$. The result now follows from the fact that regular languages in a free monoid form a boolean algebra with effectively computable operations. \[\square\]

Recall that the rational subset membership problem for a finitely generated monoid $M$ is the problem of deciding, given a rational subset of $M$ (represented by a finite automaton over some fixed generating set for $M$) and an element of $M$ (represented as a word over the same generating set), whether the given element belongs to the given subset. The decidability of this problem is independent of the chosen generating set \[14\], Corollary 3.4].
Corollary 7. Any monoid admitting a $C(4)$ presentation has decidable rational subset membership problem (and hence decidable submonoid membership problem).

Proof. Suppose $M$ has $C(4)$ presentation $\langle A \mid R \rangle$, and let $\sigma : A^* \to M$ be once again the canonical morphism. Suppose we are given a finite automaton recognising a language $\hat{X} \subseteq A^*$ (representing the rational subset $\hat{X}\sigma \subseteq M$) and a $w \in A^*$ (representing the element $w\sigma \in M$). Certainly we can compute from the latter a finite automaton recognising the singleton language $\{w\}$. Hence, by Corollary 6 we can compute a finite automaton recognising a language $\hat{Y} \subseteq A^*$ such that $\hat{Y}\sigma = \hat{X}\sigma \cap \{w\}\sigma$. But $w\sigma \in \hat{X}\sigma$ if and only if $\hat{X}\sigma \cap \{w\}\sigma$ is non-empty, so this reduces the problem to deciding emptiness of the regular language $\hat{Y}$; the latter is well known to be decidable. \qed

A monoid $M$ is called asynchronous automatic (see, for example, [9]) if there exists a finite generating set $A$ and a regular language $L \subseteq A^*$ such that $L$ contains a representative for every element of $M$, and the relation

$$\{(u, v) \in A^* \times A^* \mid ua \equiv v\}$$

is a rational transduction for each $a \in A$ and for $a = \epsilon$. It has been shown [9, Theorem 6.2] that rational monoids are asynchronous automatic, so we also obtain the following.

Corollary 8. Every monoid admitting a $C(4)$ presentation is asynchronous automatic.

We have already remarked that small overlap conditions are the natural semigroup-theoretic analogue of the small cancellation conditions extensively used in combinatorial group theory (see, for example, [15]). It is well known that a group admitting a finite presentation satisfying sufficiently strong small cancellation conditions is word hyperbolic in the sense of Gromov [7]. The usual geometric definition of a word hyperbolic group has no obvious counterpart for more general monoids or semigroups; however, Gilman [6] has given a language-theoretic characterisation of word hyperbolic groups. Specifically, he showed that a group is word hyperbolic if and only if it admits a finite generating set $A$ and a regular language $L \subseteq A^*$ containing a representative for every element of $M$ such that the multiplication table

$$\{u\#v\#w^R \mid uv \equiv w\}$$

is a context-free language, where $\#$ is a new symbol not in $A$. Motivated by this result, Duncan and Gilman [3] have suggested calling a monoid word hyperbolic if it satisfies this language-theoretic condition. Since every rational monoid is word hyperbolic [9, Theorem 6.3] we can deduce that every $C(4)$ monoid is word hyperbolic in this sense.

Corollary 9. Every monoid admitting a $C(4)$ presentation is word hyperbolic in the sense of Duncan and Gilman (and furthermore admits a hyperbolic structure with unique representatives).
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