The polytopologies of transfinite provability logic

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Abstract

Provability logics are modal or polymodal systems designed for modeling the behavior of Gödel’s provability predicate and its natural extensions. If Λ is any ordinal, the Gödel-Löb calculus GLP Λ contains one modality [λ] for each λ < Λ, representing provability predicates of increasing strength. GLP ω has no Kripke models, but it is sound and complete for its topological semantics, as was shown by Icard for the variable-free fragment and more recently by Beklemishev and Gabelaia for the full logic.

In this paper we generalize Beklemishev and Gabelaia’s result to GLP Λ for countable Λ. We also introduce provability ambiances, which are topological models where valuations of formulas are restricted. With this we show completeness of GLP Λ for the class of provability ambiances based on Icard polytopologies.

1 Introduction

Provability logic interprets modal operators as provability predicates in order to study the structure of formal theories, reading the modal formula □φ as the theory T proves φ. In [19], Solovay proved that if T is able to do a reasonable amount of arithmetic, the set of validities over the unimodal language is given by the Gödel-Löb logic GL, written GLP 1 in the current paper’s notation. This logic may also be interpreted over scattered spaces (where every non-empty subset has an isolated point), thus giving provability a surprising connection to topology. However, in practice these semantics are somewhat heavy-handed for such a logic, which already has finite Kripke models based on transitive, well-founded frames [15].

For Japaridze’s polymodal provability logic, the story is not as simple. It is an extension of GL known as GLP or, in our notation, GLP ω [15]. Here one considers countably many provability modalities [n], for n < ω. The formula [n]φ could be interpreted (for example) as φ is derivable using ω-rules of depth at most n. There is great interest in GLP since these logics are quite powerful.

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and useful; Beklemishev has shown how GLP can be used to perform ordinal analysis of Peano Arithmetic and its natural subtheories [1].

However, the logic is no longer as easy to work with as in the unimodal case. As we shall discuss later, it has no non-trivial Kripke frames. Thus the topological interpretation of the logic gives a reasonable alternative, but even then we do not get an immediate solution to the problem. In fact, the existence of so-called canonical ordinal models for these theories goes well beyond ZFC, as shown by Blass [8], Beklemishev [5] and in recent unpublished work by Bagaria.

There are, however, polytopologies based on ordinals for which GLP = GLP_ω is sound and complete, as shown by Beklemishev and Gabelaia [5]. The proof of this difficult result requires some heavy machinery including Zorn’s lemma, so the resulting spaces are non-constructive. There are also simpler spaces which provide semantics for the closed fragment, where no free variables occur; these were introduced by Icard [12] and are closely tied to Ignatiev’s Kripke model for the same fragment [14].

Our goal is to show how the constructions from [5] may be extended to the logics GLP_Λ, where Λ is an arbitrary ordinal. Here, one has transfinitely many provability operators, which as in the case of GLP_ω represent derivability in stronger and stronger theories. Indeed, Beklemishev and Gabelaia’s techniques carry over smoothly to the transfinite setting, and rather than give a new, self-contained completeness proof, we shall state the necessary results from [5] without proof in order to focus on applying these techniques beyond ω. A key point is the computation of the higher-order rank functions, which give us upper and lower bounds on the ordinals we need in order to build models. We shall also show how the use of non-constructive topologies may be circumvented and replaced by Icard topologies by passing to a more general class of models called ambiances.

**Layout.** In Section 2 we give a quick overview of the logics GLP_Λ, and Section 3 reviews topological semantics. Section 4 then states some basic facts about ordinal arithmetic that we shall need.

Section 5 introduces the most important functions in the study of GLP-spaces, ranks and d-maps. Then, Section 6 discusses Icard ambiances and Section 7 simple ambiances, the minimal structures in our framework.

After this, Section 8 discusses Beklemishev-Gabelaia spaces, which are particularly well-behaved GLP-spaces. In Section 9 we discuss and construct reductive functions, an important type of d-map, and Section 10 establishes a series of operations on ambiances which are used for constructing models.

We then go on to review the logic J in Section 11 which is a key ingredient in the completeness proof presented in Section 12. Finally, Section 13 uses worms, which are special variable-free formulas related to ordinals, to give a lower bound on the rank of models.
Given any ordinal \( \Lambda \), we can define a provability logic with modalities in \( \Lambda \). Formulas of the language \( L_\Lambda \) are built from \( \top \) and a countable set of propositional variables \( P \) using Boolean connectives \( \neg, \land, \lor, \to \) and a modality \( [\xi] \) for each \( \xi < \Lambda \). As is customary, we use \( \langle \xi \rangle \) as a shorthand for \( \neg [\xi] \neg \).

The logic \( \text{GLP}_\Lambda \) is then given by the following rules and axioms:

1. all propositional tautologies,
2. \( [\xi](\phi \to \psi) \to ([\xi]\phi \to [\xi]\psi) \) for all \( \xi < \Lambda \),
3. \( [\xi]([\xi]\phi \to \phi) \to [\xi]\phi \) for all \( \xi < \Lambda \),
4. \( [\xi]\phi \to [\xi]\phi \) for \( \xi < \zeta < \Lambda \),
5. \( \langle \xi \rangle \phi \to [\xi] \langle \xi \rangle \phi \) for all \( \xi < \zeta < \Lambda \),
6. modus ponens and
7. necessitation for each \( [\xi] \).

Note that the unimodal \( \text{GLP}_1 \) is the standard Gödel-Löb logic \( \text{GL} \). Let us write \( \text{sub}(\phi) \) for the set of subformulas of \( \phi \). Then, we say that \( \lambda \) appears in \( \phi \) if there is some formula \( \psi \) such that \( [\lambda]\psi \in \text{sub}(\phi) \). It is evident that only finitely many ordinals may appear in any formula \( \phi \); sometimes it is convenient to ignore all other ordinals. To this end we define the condensation of \( \phi \) as follows:

**Definition 2.1.** Given a formula \( \phi \in L_\Lambda \) such that \( \lambda_0 < \lambda_1 < \ldots < \lambda_{N-1} \) are the ordinals appearing in \( \phi \), we define a formula \( \phi^c \) (the condensation of \( \phi \)) as the result of replacing every operator \( [\lambda_n] \) in \( \phi \) by \( [n] \).

As it turns out, the formula \( \phi^c \) is derivable if and only if \( \phi \) is. One direction, which we will not need in this paper, is non-trivial and proven in [4]; the other is quite straightforward and will be used later.

**Lemma 2.1.** If \( \phi \) is a formula such that there are \( N \) ordinals appearing in \( \phi \) then \( \text{GLP}_N \vdash \phi^c \) implies that \( \text{GLP}_\Lambda \vdash \phi \).

This fact may be proven by uniformly substituting \( [\lambda_n] \) for \( [n] \) in a derivation of \( \phi^c \); we omit the details. Condensations will allow us to focus only on ‘relevant’ ordinals when analyzing formulas.

We shall also work with Kripke semantics. A *Kripke frame* is a structure \( \mathcal{F} = \langle W, \{R_n\}_{n<\mathbb{N}} \rangle \), where \( W \) is a set and \( \{R_n\}_{n<\mathbb{N}} \) a family of binary relations.
on $W$. A valuation on $\mathfrak{F}$ is a function $[\cdot] : L_\Lambda \to \mathcal{P}(W)$ such that

$$
[\bot] = \emptyset
$$

$$
[\neg \phi] = W \setminus [\phi]
$$

$$
[\phi \land \psi] = [\phi] \cap [\psi]
$$

$$
[(n) \phi] = R_n^{-1} [\phi]
$$

A Kripke model is a Kripke frame equipped with a valuation $[\cdot]$. Note that propositional variables may be assigned arbitrary subsets of $W$. If $M = (\mathfrak{F}, [\cdot])$ is a model, we may write $(M, x) \models \psi$ instead of $x \in [\psi]$. As usual, $\phi$ is satisfied on $M$ if $[\phi] \neq \emptyset$, and true on $M$ if $[\phi] = W$. It is valid on a frame $\mathfrak{F}$ if it is true on every model based on $\mathfrak{F}$.

It is well-known that Löb’s axiom is valid on $\mathfrak{F}$ whenever $R_n^{-1}$ is well-founded and transitive, in which case we denote it by $<_n$. However, constructing models of $\text{GLP}_\Lambda$ is substantially more difficult than constructing models of GL: the full logic $\text{GLP}_\Lambda$ cannot be sound and complete with respect to any class of Kripke frames. Indeed, let $\mathfrak{F} = (W, <_\xi)$ be a polymodal frame.

Then, it is not too hard to check that

1. Löb’s axiom $[\xi](\xi \phi \to \phi) \to [\xi] \phi$ is valid if and only if $<_\xi$ is well-founded and transitive,

2. the axiom $[\xi] \phi \to [\zeta] \phi$ for $\xi \leq \zeta$ is valid if and only if, whenever $w <_\zeta v$, then $w <_\xi v$, and

3. $[\langle n \rangle \phi] \to [\langle n \xi \rangle] \phi$ for $\xi < \zeta$ is valid if, whenever $v <_\zeta w$, $u <_\xi w$ and $\xi < \zeta$, then $u <_\xi v$.

Suppose that for $\xi < \zeta$, there are two worlds such that $w <_\zeta v$. Then from 2 we see that $w <_\xi v$, while from 3 this implies that $w <_\xi w$. But this clearly violates 1. Hence if $\mathfrak{F} \models \text{GLP}$, it follows that all accessibility relations (except possibly $<_0$) are empty.

This observation makes the topological completeness of $\text{GLP}_\omega$ established in [5] particularly surprising. Moreover, as we shall see, the techniques introduced there readily extend to the transfinite. To show this, let us begin by reviewing the topological semantics of provability logic.

## 3 Topological semantics

Recall that a topological space is a pair $X = (X, \mathcal{T})$ where $\mathcal{T} \subseteq \mathcal{P}(X)$ is a family of sets called ‘open’ such that

1. $\emptyset, X \in \mathcal{T}$

2. if $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$ and
3. if $U \subseteq T$ then $\bigcup U \in T$.

Given $A \subseteq X$ and $x \in A$, we say $x$ is a limit point of $A$ if, for all $U \in T$ such that $x \in U$, we have that $(A \setminus \{x\}) \cap U \neq \emptyset$. We denote the set of limit points of $A$ by $dA$, and call it the ‘derived set’ of $A$. We can define topological semantics for modal logic by interpreting Boolean operators in the usual way and setting

$$[\Diamond \psi]_X = d[\psi]_X.$$

In order to interpret provability logic, we will need to consider scattered spaces. A topological space $(X, T)$ is scattered if every non-empty subset $A$ of $X$ has an isolated point; that is, there exist $x \in A$ and a neighborhood $U$ of $x$ (i.e., $x \in U \in T$) such that $U \cap A = \{x\}$.

Many interesting examples of scattered spaces come from ordinals. The simplest is the initial segment topology. If $\Theta$ is an ordinal, we use $\Theta_0$ to denote the structure $(\Theta, T)$, where $T$ consists of all downward-closed subsets of $\Theta$. It is very easy to check that $\Theta_0$ is a scattered topological space, for if $A \subseteq \Theta$ is non-empty, then the least element of $A$ is isolated in $A$.

A second important example is the interval topology. This is generated by all intervals on $\Theta$ of the form $[0, \beta]$ or $(\alpha, \beta]$. The interval topology extends the initial segment topology, and it is straightforward to check that if $T$ is scattered and $T'$ is any refinement of $T$ (i.e., $T \subseteq T'$), then $T'$ is scattered as well. We will denote $\Theta$ equipped with the interval topology by $\Theta_1$.

Now, in order to interpret GLP$_\Lambda$ for $\Lambda > 1$, we need to consider polytopological spaces. A polytopological space is a structure $\mathfrak{X} = \langle X, \langle T_\lambda \rangle_{\lambda < \Lambda} \rangle$, where $\Lambda$ is an ordinal and each $T_\lambda$ is a topology. The derived set operator corresponding to $T_\lambda$ shall be denoted $d_\lambda$. We may also write $\mathfrak{X}_\lambda$ instead of $\langle X, T_\lambda \rangle$.

There are Kripke-incomplete modal logics which nevertheless are complete for general Kripke frames, which are Kripke frames where valuations are restricted to a special algebra of sets. A similar idea will prove useful in order to give constructive semantics of GLP$_\Lambda$. The following definition describes the algebras we shall use:

**Definition 3.1 (d-algebra).** A d-algebra over a polytopological space $\mathfrak{X} = \langle X, \langle T_\lambda \rangle_{\lambda < \Lambda} \rangle$ is a collection of sets $A \subseteq \mathcal{P}(X)$ which form a Boolean algebra under the standard set-theoretic operations and such that, whenever $\lambda < \Lambda$ and $S \in A$ it follows that $d_\lambda S \in A$.

Below we introduce ambiances, which will be the basis of our semantics; they are a slight generalization of polytopological models, which correspond to the special case where $A = \mathcal{P}(X)$.

**Definition 3.2 (Ambiance).** An ambiance is a structure

$$\mathfrak{A} = \langle X, \mathcal{T}, A \rangle$$

consisting of a polytopological space equipped with a d-algebra $A$. 

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If $\vec{T} = \langle T_\lambda \rangle_{\lambda < \Lambda}$, we may also say $x$ is a $\Lambda$-ambiance. The operator $d_\lambda$ will be used to interpret $\langle \lambda \rangle$:

**Definition 3.3.** Let $x = \langle X, \vec{T}, A \rangle$ be a $\Lambda$-ambiance.

A valuation on $x$ is a function $\llbracket \cdot \rrbracket : L_\Lambda \to A$ defined as in the case of Kripke semantics except that

$$\llbracket (\lambda) \phi \rrbracket = d_\lambda \llbracket \phi \rrbracket .$$

A polytopological model is an ambiance equipped with a valuation.

Let us check the conditions under which $GLP_\Lambda$ is sound for a given polytopological space.

**Lemma 3.1.** Let $\Lambda$ be an ordinal and $x = \langle X, \vec{T}, A \rangle$ be an ambiance.

Then,

1. Löb’s axiom $[\xi](\xi \phi \to \phi) \to [\xi] \phi$ is valid on $x$ whenever $\langle X, T_\xi \rangle$ is scattered,
2. the axiom $[\xi] \phi \to [\zeta] \phi$ for $\xi \leq \zeta$ is valid whenever $T_\xi \subseteq T_\zeta$ and
3. $[\xi] \phi \to [\zeta] (\xi) \phi$ for $\xi < \zeta$ is valid if $d_\xi A \in T_\zeta$ whenever $A \in A$.

**Proof.** See, for example, [5].

An ambiance satisfying the above properties will be called a provability ambiance.

When referring to topologies, we use the words extension and refinement indistinctly. We may also speak of refinements of spaces rather than refinements of topologies: $\langle X', T' \rangle$ is a refinement of $\langle X, T \rangle$ if $X = X'$ and $T \subseteq T'$. Thus the condition for $[\zeta] \phi$ can be rewritten as “$T_\zeta$ is a refinement of $T_\xi$”.

Conditions 2 and 3 suggest a very natural candidate for $T_{\xi+1}$ whenever $T_\xi$ is given; namely, the least topology that will satisfy all axioms.

**Definition 3.4 (d$^A$T).** Given a $x = \langle X, T \rangle$ and a $d$-algebra $A$ on $x$, we define $d^A T$ to be the topology on $X$ generated by

$$T \cup \{dS : S \in A\}.$$

We will denote $\langle X, d^A T \rangle$ by $d^A x$.

As in [6], we shall write $x^+$ instead of $d^{P(X)} x$. The above definition suggests natural candidate topologies for $T_\lambda$, at least for successor $\lambda$. For limit $\lambda$ we need to consider joins of topologies.

If $\vec{T} = T_{\xi < \lambda}$ is an increasing sequence of topologies, then $U = \bigcup_{\xi < \lambda} T$ is typically not a topology. Although it is always closed under finite unions and intersections, it need not be closed under arbitrary unions. However, $\vec{T}$ does generate a least topology $J = \bigcup_{\xi < \lambda} T$ (its ‘join’) containing all $T_\xi$, by closing $U$ under arbitrary unions. The elements of $J$ are then of the form $\bigcup_{\lambda < \lambda} U_\lambda$ with $U_\lambda \in T_\lambda$. In other words, $U$ forms a basis for $J$, so that $O$ is open in $J$ if and only if for every $x \in O$ there are $\lambda < \Lambda$ and $V \in T_\lambda$ such that $x \in V \subseteq O$.

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Due to the monotonicity axiom, this is the least topology we can choose at limit stages:

**Definition 3.5.** Given an ambiance \( X = (X, \vec{T}, \mathcal{A}) \) and an ordinal \( \xi > 0 \), define \( T_\xi^- \) to be:

- \( d^A T_\zeta \) if \( \xi = \zeta + 1 \)
- \( \bigcup_{\zeta < \xi} T_\zeta \) if \( \xi \) is a limit ordinal.

A naïve strategy for building models of GLP consists of always choosing the least possible topology at each stage; a structure \( X = (\Theta, \langle T_\lambda \rangle_{\lambda < \Lambda}) \) is a canonical ordinal model if \( X_0 = \Theta_1 \) and for all \( \lambda < \Lambda \), \( T_\lambda = T_\lambda^- \). In a canonical ordinal model the topology \( T_{\xi+1} = T_\xi^+ \) is usually much bigger than \( T_\xi \), since we are adding many new closed sets as opens. For example:

**Lemma 3.2.** Given an ordinal \( \Theta \), \( \Theta_0^+ = \Theta_1 \).

The reader may wish to prove this directly as an exercise; we will not give such a proof as it is a special case of Lemma [6.4]. After this the topologies increase very quickly; if \( \Theta \) is any countable ordinal and \( T \) is the interval topology, then \( T^+ \) is discrete. Moreover, the question of whether GLP\(_2\) is complete for its class of canonical ordinal models is independent of ZFC [8, 3]. In recent unpublished work, Bagaria has characterized non-trivial ordinal models for GLP\(_n\) in terms of large cardinals.

Thus, making the topologies as small as possible at each step is not the best strategy, so it is convenient to consider other alternatives. In [5], Beklemishev and Gabelaia realized the highly unintuitive fact that if we make each topology *as large as possible* then subsequent topologies become much smaller! Thus they obtain spaces where \( T_n \supseteq T_n^- \) for each \( n > 1 \). As we shall see, this idea readily extends beyond GLP\(_\omega\); perhaps the most technically challenging aspect of such an extension lies in the new computations with ordinals that arise.

**4 Operations on ordinals**

Before continuing, let us give a brief review of some notions of ordinal arithmetic as well as some useful functions in the study of provability logic. We skip most proofs; for further details on ordinal arithmetic, we refer the reader to a text such as [17], while the material on hyperexponentials and hyperlogarithms is treated in detail in [11].

We assume familiarity with ordinal sums, products and exponents. We shall also use the following operations:

**Lemma 4.1.**

1. Whenever \( \zeta < \xi \), there exists a unique ordinal \( \eta \) such that \( \zeta + \eta = \xi \). We will denote this unique \( \eta \) by \( -\zeta + \xi \).
2. Given \( \xi > 0 \), there exist ordinals \( \alpha, \beta \) such that \( \xi = \alpha + \omega^\beta \). The value of \( \beta \) is uniquely defined. We will denote this unique \( \beta \) by \( \ell_\xi \).

In previous work my colleague Joost Joosten and I realized that there were some particularly useful functions that arise when studying provability logics. They are hyperexponentials and hyperlogarithms, and are a form of transfinite iteration of the functions \( -1 + \omega^\xi \) and \( \ell \), respectively. These iterations have been used in [10] for describing well-orders in the Japardize algebra and in [9] for defining models of the variable-free fragment of GLP\(_\Lambda\). They will be essential in defining our semantics. We give only a very brief overview, but [11] gives a thorough and detailed presentation.

We shall denote the class of all ordinals by \( \text{On} \) and the class of limit ordinals by \( \text{Lim} \).

**Definition 4.1.** Let \( e(\xi) = -1 + \omega^\xi \). Then, we define the hyperexponentials \( \langle e^\xi \rangle_{\xi \in \text{On}} \) as the unique family of normal\(^1\) functions such that

1. \( e^1 = e \)
2. \( e^{\alpha + \beta} = e^\alpha e^\beta \) for all ordinals \( \alpha, \beta \)
3. \( \langle e^\xi \rangle_{\xi \in \text{On}} \) is pointwise minimal amongst all families of normal functions satisfying the above clause\(^2\).

It is not obvious that such a family of functions exists, but a detailed construction is given in [11], where the following is also proven:

**Proposition 4.1 (Properties of hyperexponentials).** The family of functions \( \langle e^\xi \rangle_{\xi \in \text{On}} \) has the following properties:

1. \( e^0 \) is the identity,
2. \( e^\xi 0 = 0 \) for all \( \xi \),
3. given \( \xi \in \text{On} \) and \( \lambda \in \text{Lim} \), \( e^\xi \lambda = \lim_{\eta \to \lambda} e^\xi \eta \) and
4. if \( \lambda \in \text{Lim} \) and \( \vartheta \in \text{On} \), \( e^\lambda (\vartheta + 1) = \lim_{\eta \to \lambda} e^\eta (e^\lambda (\vartheta) + 1) \).

**Example 4.1.** We have that \( e(0) = -1 + \omega^0 = -1 + 1 = 0 \) and \( e(1) = -1 + \omega^1 = \omega \). Then, \( e^2 1 = ee1 = e(\omega) = -1 + \omega^\omega = \omega^\omega \), and continuing in this fashion one sees that

\[
e^n 1 = \omega_{\omega}^{\cdots}.
\]

We know from Proposition 4.1 that \( e^\omega 0 = 0 \), and in view of Proposition 4.1 we have

\[
e^\omega 1 = \lim_{n \to \omega} e^n (e^\omega (0) + 1) = \lim_{n \to \omega} e^n 1 = \lim_{n \to \omega} \omega_{\omega}^{\cdots}.
\]

\(^1\)That is, strictly increasing and continuous.
\(^2\)That is, if \( \langle g^\xi \rangle_{\xi \in \text{On}} \) is a family of functions satisfying conditions 1 and 2, then for all ordinals \( \xi, \zeta \), \( e^\xi \zeta \leq g^\zeta \xi \).
usually denoted $\varepsilon_0$. Meanwhile

$$e^{\omega^2} = \lim_{n \to \omega} e^n(\varepsilon_0 + 1) = \lim_{n \to \omega} \omega^{\varepsilon_0+1} = \varepsilon_1,$$

and more generally $\varepsilon_n = e^{\omega}(n + 1)$.

Finally, by Proposition 4.1.3 we see that

$$e^{\omega \cdot \omega} = \lim_{n \to \omega} e^{\omega n} = \lim_{n \to \omega} \varepsilon_n = \varepsilon_\omega.$$

This can be generalized to obtain $\varepsilon_\xi = e^{\omega}(1 + \xi)$ for every ordinal $\xi$.

Closely related to hyperexponentials are hyperlogarithms. Below, an initial function is one mapping initial segments to initial segments.

**Definition 4.2 (Hyperlogarithms).** We define the sequence $\langle \ell_\xi \rangle_{\xi \in \text{On}}$ to be the unique family of initial functions such that

1. $\ell_1 = \ell$,
2. $\ell^{\alpha+\beta} = \ell^\beta \ell^\alpha$ for all ordinals $\alpha, \beta$,
3. $\langle \ell_\xi \rangle_{\xi \in \text{On}}$ is pointwise maximal among all families of functions satisfying the above clauses.

The following properties of hyperlogarithms will be used throughout the text and are not too difficult to check:

**Proposition 4.2.** The hyperlogarithms $\langle \ell_\xi \rangle_{\xi \in \text{On}}$ have the following properties:

1. $\ell_0$ is the identity,
2. If $\alpha, \delta > 0$ and $\gamma$ is any ordinal, then $\ell^\alpha(\gamma + \delta) = \ell^{\alpha \delta}$.
3. For any ordinal $\gamma$, the sequence $\langle \ell^\xi \gamma \rangle_{\xi \in \text{On}}$ is non-increasing.

Observe that hyperexponentials are typically not surjective, hence not right-invertible. However, they are injective, thus left-invertible, and hyperlogarithms provide particularly well-behaved left inverses.

**Lemma 4.2.** If $\xi < \zeta$, then $\ell^\xi e^\zeta = e^{-\xi + \zeta}$ and $\ell^\xi e^\xi = e^{-\xi + \zeta}$.

Further, whenever $\alpha < \ell^\xi \beta$, it follows that $\ell^\xi \alpha < \beta$.

We may also use the contrapositive form of the above, that is, whenever $\beta \leq \ell^\xi \alpha$, then $\ell^\xi \alpha \leq \beta$. Note that it also follows from this that when $\beta < \ell^\xi \alpha$, then $\ell^\xi \alpha < \beta$, for if we had $\ell^\xi \alpha = \beta$ then also $\ell^\xi e^\xi \alpha = \alpha = \ell^\xi \beta$. 

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Example 4.2. Let us compute the sequence \((\ell^\xi \gamma)_{\xi \in \Omega_0}\) for \(\gamma = \varepsilon_{\omega \cdot 3} + \varepsilon_{\omega \cdot 2}\). First we have that \(\ell^0 \gamma = \gamma\) since \(\ell^0\) is the identity.

The \(\varepsilon\)-numbers have the property that they are fixed under the map \(\xi \mapsto \omega^\xi\), so we may also write \(\gamma\) as \(\varepsilon_{\omega \cdot 3} + \omega^\omega - 2\). In view of this, \(\ell^1 \gamma = \ell \gamma = \varepsilon_{\omega \cdot 2}\). After this, \(\ell^2 \gamma = \ell \ell \gamma = \ell \varepsilon_{\omega \cdot 2} = \varepsilon_{\omega \cdot 2}\), and continuing inductively we see that \(\ell^n \gamma = \varepsilon_{\omega \cdot 2}\) for all \(n < \omega\).

To go beyond \(\omega\) and in view of Example 4.1, we may write \(\gamma\) as \(\varepsilon^\omega (\omega \cdot 3) + \varepsilon^\omega (\omega \cdot 2)\), so that \(\ell \gamma = \varepsilon^\omega (\omega \cdot 2)\). Therefore, by Lemma 4.3, \(\ell^2 \gamma = \ell^\omega \varepsilon^\omega (\omega \cdot 2) = \omega \cdot 2\). Then, \(\ell^\omega + 1 \gamma = \ell^\omega \gamma = 1\), since \(\omega \cdot 2 = \omega + \omega^1\), and thus \(\ell^{\omega + 1} \gamma = \ell \omega + 1 \gamma = \ell 1 = 0\), since \(1 = \omega^0\). From here on we obtain \(\ell^\xi \gamma = 0\) for all \(\xi > \omega + 1\).

In summary, the sequence \((\ell^\xi \gamma)_{\xi \in \Omega_0}\) has the following form:

\[
\begin{array}{ccccccc}
\varepsilon_{\omega \cdot 3} + \varepsilon_{\omega \cdot 2}, & \varepsilon_{\omega \cdot 2}, & \varepsilon_{\omega \cdot 2}, & \omega \cdot 2, & 1 \cdot \omega, & 0, 0, \ldots \\
\xi = 0 & 1 \leq \xi < \omega & \xi = \omega & \xi = \omega + 1 & \xi > \omega + 1 \\
\end{array}
\]

There is a close relation between the iterates \(\varepsilon^\omega \xi\) and Veblen functions; this is also described in detail in \([11]\). For example:

Lemma 4.3. An ordinal \(\xi\) lies in the range of \(\varepsilon^\omega\) if and only if, for all \(\delta < \gamma\), we have that \(\xi = \varepsilon^\delta \xi\). In particular, \(\varepsilon^{\omega + 1}\) enumerates the fixpoints of \(\varepsilon^\omega\).

This has some consequences which will prove to be very useful to us:

Lemma 4.4. If \(\Lambda = \alpha + \omega^\beta\) is a limit ordinal and \(\xi\) is any ordinal, then there exists \(\lambda < \Lambda\) such that \(\ell^\vartheta \xi = \varepsilon^{\omega^\vartheta} \ell^\lambda \xi\) for all \(\vartheta \in [\lambda, \Lambda)\).

Proof. Since \(\ell^\vartheta \xi\) is non-increasing on \(\vartheta\), there must be some \(\lambda < \Lambda\) such that \(\ell^\vartheta \xi = \ell^\lambda \xi\) for all \(\vartheta \in [\lambda, \Lambda)\); clearly we may pick \(\lambda > \alpha\). Observe then that for all \(\delta < \beta\) we have that

\[
\varepsilon^{\delta} \ell^\lambda \xi = \ell^\lambda + \omega^{\delta} \xi = \ell^\lambda \xi,
\]

so that by Lemma 4.2, \(\ell^\vartheta \xi \geq \varepsilon^{\omega^\vartheta} \ell^\lambda \xi\); since \(\varepsilon^{\omega^\vartheta}\) is normal this means that \(\ell^\lambda \xi = \varepsilon^{\omega^\vartheta} \ell^\lambda \xi\). By Lemma 4.3 we have that \(\ell^\lambda \xi = \varepsilon^{\omega^\vartheta} \eta\) for some \(\eta > 0\), and applying \(\ell^{\omega^\vartheta}\) on both sides we obtain

\[
\ell^\Lambda \xi = \ell^{\lambda + \omega^\vartheta} \xi = \ell^{\omega^\vartheta} \ell^\lambda \xi = \eta.
\]

Thus for \(\vartheta \in [\lambda, \Lambda)\) we have that

\[
\ell^\vartheta \xi = \ell^\lambda \xi = \ell^\omega^\vartheta \eta = \varepsilon^{\omega^\vartheta} \ell^\lambda \xi,
\]

as needed.

\(\square\)

Lemma 4.5. Suppose that for ordinals \(\vartheta, \gamma\) and additively indecomposable \(\Lambda\) we have that \(\vartheta \in (e^\Lambda \gamma, e^\Lambda (\gamma + 1))\).

Then, there exists \(\lambda < \Lambda\) such that \(\ell^\lambda \vartheta \leq \ell^\lambda \gamma\).

Proof. By Lemma 4.4 we have that \(\ell^\lambda \vartheta = e^\lambda \ell^\lambda \vartheta\) for some \(\lambda < \Lambda\). Since \(\ell^\lambda \vartheta \leq \vartheta < e^\Lambda (\gamma + 1)\) and \(e^\Lambda\) is normal, we must have \(\ell^\lambda \vartheta \leq \gamma\) and thus \(\ell^\lambda \vartheta \leq \ell^\lambda \gamma\). \(\square\)
To conclude this section, let us discuss simple functions. We will often be faced with families of inequalities of the form \( \{ \ell^{\alpha_n} \xi > \beta_n \}_{n<N} \), and need to describe the ordinals \( \xi \) satisfying such constraints. Simple functions will be used to gather such inequalities into a single object. They will play a crucial role throughout the paper, as they provide a convenient, flexible tool for solving many of the problems that will arise later.

**Definition 4.3.** A simple function is a partial function \( s : \Lambda \rightarrow \Theta \) with finite domain, where \( \Theta, \Lambda \) are ordinals. We denote the domain of \( s \) by \( \text{dom}(s) \).

If \( r, s \) are simple functions, we define \( r \sqcup s \) to be the simple function with domain \( \text{dom}(r) \cup \text{dom}(s) \) given by

\[
    r \sqcup s(\lambda) = \begin{cases} 
        r(\lambda) & \text{if } \lambda \in \text{dom}(r) \setminus \text{dom}(s), \\
        s(\lambda) & \text{if } \lambda \in \text{dom}(s) \setminus \text{dom}(r), \\
        \max\{r(\lambda), s(\lambda)\} & \text{if } \lambda \in \text{dom}(r) \cap \text{dom}(s).
    \end{cases}
\]

We write \( s \sqsubseteq \alpha \) if for \( \lambda = \max(\text{dom}(s)) \) we have that \( s(\lambda) \leq \ell^\alpha \alpha \) and for all \( \xi \in \text{dom}(s) \setminus \{\lambda\}, s(\xi) < \ell^\xi \alpha \). If moreover \( s(\lambda) < \ell^\alpha \alpha \), we instead write \( s \sqsubseteq \alpha \). If \( \text{dom}(s) = \emptyset \), we also set \( s \sqsubseteq \alpha \) and \( s \sqsubseteq \alpha \) for all \( \alpha \). Given a simple function \( s \), we define \( [s] \) to be the least ordinal \( \sigma \) such that \( s \sqsubseteq \sigma \). Note that if \( \text{dom}(s) = \emptyset \), then \( [s] = 0 \).

The following lemma originally appeared in [9] in a different presentation.

**Lemma 4.6.** Let \( r \) be a simple function with non-empty domain and \( \lambda = \max(\text{dom}(r)) \).

Then,

1. the ordinal \([r]\) is defined and \( \ell^\lambda [r] = r(\lambda) \), and
2. whenever \( r \sqsubseteq \alpha \) and \( \xi \leq \lambda \), we have that \( \ell^\xi [r] \leq \ell^\xi \alpha \).

**Proof.** It will be convenient for our proof to define \( \text{dom}^+(r) = \{0\} \cup \text{dom}(r) \). We will proceed to construct an ordinal \( \vartheta \) such that \( r \sqsubseteq \vartheta \) and Claims \([1][2]\) hold.

Let us use \( \# S \) to denote the cardinality of the set \( S \) and work by induction on \( \# \text{dom}^+(r) \). The base case, where \( \text{dom}^+(r) = \{0\} \), is trivial, as \( r \sqsubseteq \vartheta \) becomes \( r(0) \leq \vartheta \), and clearly \( \vartheta = r(0) \) satisfies the required properties.

For the inductive step, let \( \eta = \min(\text{dom}^+(r) \setminus \{0\}) \) and consider \( \tilde{r} \) given by \( \tilde{r}(\xi) = r(\eta + \xi) \) whenever the latter is defined, so that \( \tilde{r} \) is just \( r \) ‘shifted’ by \( \eta \).

By induction hypothesis \( \tilde{\vartheta} = [\tilde{r}] \) is defined and satisfies both claims.

To find \( \vartheta = [r] \), define \( \gamma = 0 \) if \( 0 \notin \text{dom}(r) \) and \( \gamma = r(0) + 1 \) otherwise, and set \( \vartheta = \gamma + e^\eta \tilde{\vartheta} \). Let us begin by showing that \( r \sqsubseteq \vartheta \). First note that \( r(0) < \vartheta \) in the case that \( 0 \in \text{dom}(r) \). Meanwhile, for \( \xi \in (0, \lambda) \cap \text{dom}(r) \) we have that

\[
    r(\xi) = \tilde{r}(\eta + \xi) < \ell^{-\eta} \xi \tilde{\vartheta} = \ell^\xi e^\eta \tilde{\vartheta} = \ell^\xi \gamma + e^\eta \tilde{\vartheta} = \ell^\xi \vartheta.
\]

A similar argument shows that \( \ell^\lambda \vartheta = r(\lambda) \) for \( \lambda = \max(\text{dom}(r)) \), thus establishing that \( r \sqsubseteq \vartheta \).
It remains to check that if \( r \subseteq \alpha \), then Claim 2 is satisfied by \( \vartheta \). If \( \vartheta = 0 \) then this is obvious, for if \( 0 \notin \text{dom}(r) \) then \( \vartheta = e^{\eta} \vartheta = 0 \), and if \( 0 \in \text{dom}(r) \) then \( \vartheta = r(0) + 1 \) and therefore \( \ell^0 \vartheta = r(0) + 1 \leq \alpha \), whereas for \( \xi > 0 \) we have that \( \ell^\xi \vartheta = 0 \leq \ell^\xi \alpha \). Hence we may assume \( \vartheta > 0 \).

Pick \( \xi \leq \max(\text{dom}(r)) \). If \( \xi \geq \eta \), observe that \( r \subseteq \ell^\eta(\alpha) \) and thus, by our induction hypothesis,

\[
\ell^\xi \vartheta = \ell^{\xi-\eta} \ell^\eta \vartheta = \ell^{\xi-\eta} \vartheta \leq \ell^{\xi-\eta} \ell^\eta \vartheta = \ell^\xi \alpha.
\]

If \( \xi \in (0, \eta) \), then \( \ell^{\xi+\eta} \ell^\xi \alpha = \ell^\eta \alpha \geq \vartheta \). It follows by Lemma 4.2 that \( \ell^\xi \alpha \geq e^{\xi+\eta} \vartheta \). But then, as we are assuming \( \vartheta > 0 \) we obtain that

\[
\ell^\xi \vartheta = \ell^\xi (\gamma + e^{\eta} \vartheta) = \ell^\xi e^{\eta} \vartheta = e^{-\xi+\eta} \vartheta,
\]

and thus \( \ell^\xi \vartheta \leq \ell^\xi \alpha \).

Finally, we must see that \( \vartheta \leq \alpha \). Since \( r \subseteq \alpha \), we have \( \alpha = \gamma + \delta \) for some \( \delta \geq 0 \); but from the assumption that \( \vartheta > 0 \) we have that \( \ell^\eta \alpha > 0 \) and thus we must have \( \delta > 0 \) (for \( \ell^\gamma \alpha = 0 \)). Now, \( \ell^\eta \delta = \ell^\eta \alpha \geq \vartheta \), so once again by Lemma 4.2 we obtain \( \delta \geq e^\eta \vartheta \) and thus \( \alpha \geq \gamma + e^\eta \vartheta = \vartheta \).

Thus we may set \( [r] = \vartheta \) and obtain all the desired properties.

As a variant, we may be interested in the least \( \vartheta \) such that \( r \subseteq \vartheta \). We may construct it as follows: let \( r' \) be equal to \( r \) except that, for \( \lambda = \max(\text{dom}(r)) \), we set \( r'(\lambda) = r(\lambda) + 1 \). Then, it is straightforward to check that \( [r'] = \vartheta \).

**Example 4.3.** Consider the simple sequence with \( r(0) = \varepsilon_0, r(\omega) = \omega^2, r(\omega + 1) = 2 \), and undefined elsewhere. Let us compute \( \vartheta = [r] \).

Since \( \omega + 1 \) is the greatest element of \( \text{dom}(r) \) we must have \( \ell^{\omega+1} \vartheta = 2 \), and hence \( \ell^{\omega} \vartheta \) is of the form \( \alpha + \omega^2 \). Now, we cannot take \( \alpha = 0 \), or else we would not have \( \ell^{\omega} \vartheta > r(\omega) = \omega^2 \), and the least value of \( \alpha \) we could take to obtain a strictly larger value is \( \omega^2 \). Thus \( \ell^{\omega} \vartheta = \omega^2 + \omega^2 \).

In view of Lemma 4.2, the least value of \( \vartheta \) that satisfies this is \( e^{\omega}(\omega^2 + \omega^2) \). Moreover, we already have \( e^{\omega}(\omega^2 + \omega^2) > e^{\omega+1} \varepsilon_0 = r(0) \). Thus we may set \( \vartheta = e^{\omega}(\omega^2 + \omega^2) = e^{\omega+1} \varepsilon_0 = \varepsilon_0 = [r] \).

**5 Ranks and \( d \)-maps**

In this section we shall consider some fundamental concepts in the study of scattered spaces. We omit the proofs of those results which may already be found in [5].

Given a topological space \( (X, \mathcal{T}) \), we may iterate the corresponding derived set operator \( d : \mathcal{P}(X) \rightarrow \mathcal{P}(X) \) via the following recursion:

1. \( d^0 A = A \)
2. \( d^{\xi+1} A = dd^\xi A \) for all \( \xi \in \text{On} \)
3. \( d^\lambda A = \bigcap_{\xi < \lambda} d^\xi A \) for \( \lambda \in \text{Lim} \).

If \( X \) is scattered and \( d^\xi A \neq \emptyset \), then \( d^\xi A \) contains an isolated point \( x \) and thus \( x \notin dd^\xi A = d^{\xi+1} A \). In particular, \( d^\xi X \supseteq d^{\xi+1} X \) provided \( d^\xi X \neq \emptyset \).

Thus if \( \#\xi > \#X \) (recall that we use \# to denote cardinality), \( d^\xi X = \emptyset \), which means that for any point \( x \in X \) there is some ordinal such that \( x \notin d^\xi X \).

This motivates our following definition:

**Definition 5.1 (Rank).** If \( \mathfrak{X} = (X, T) \) is a scattered space and \( x \in X \), we define the rank of \( x \), denoted \( p(x) \), to be the least ordinal \( \alpha \) such that \( x \notin d^{\alpha+1} X \).

We define \( p(\mathfrak{X}) = \sup_{x \in X} (p(x) + 1) \). This is the Cantor-Bendixson rank of \( \mathfrak{X} \).

Many times it will turn out that ranks are not too difficult to compute; the following lemma gives an example of this.

**Lemma 5.1.** Given an ordinal \( \Theta \), let \( p_0 \) be the rank function on \( \Theta_0 \) and \( p_1 \) the rank function on \( \Theta_1 \).

Then, for all \( \xi < \Theta \), \( p_0(\xi) = \xi \) while \( p_1(\xi) = \ell\xi \).

These equalities have already appeared in [5], and it is an instructive exercise to prove them directly (by induction on \( \xi \)). However, we shall not provide such a proof, as they are instances of the more general Corollary 6.1 that we will give later.

**Example 5.1.** Let \( \mathfrak{X} = [0, \omega^\omega] \) with the interval topology. Then, every ordinal that is either zero or a successor is isolated; \( \{0\} \) is open as is \( \{\xi + 1\} = (\xi, \xi + 2) \) for all \( \xi \). Meanwhile, limit ordinals are not isolated; for example, any neighborhood of \( \omega \) contains an interval \( (N, \omega + 1) \) and hence a point \( N + 1 \neq \omega \). It follows that \( d\{0, \omega^\omega\} \) is the set of limit ordinals below \( \omega^\omega + 1 \).

Now, \( \omega \) is isolated in \( d\{0, \omega^\omega\} \) since we have removed all natural numbers, so that, for example, \((0, \omega + 1) \cap d\{0, \omega^\omega\} = \{\omega\} \). The same situation occurs for any ordinal of the form \( \gamma + \omega \). However, \( \omega^2 \) is not isolated in \( d\{0, \omega^\omega\} \), as any neighborhood of \( \omega^2 \) contains all elements of the form \( \omega \cdot N \) for \( N \) large enough. More generally, no ordinal of the form \( \gamma + \omega^n \) is isolated in \( d\{0, \omega^\omega\} = \{\omega\} \) if \( n \geq 2 \), and \( d\{0, \omega^\omega\} \) is the set of all ordinals \( \xi \) below \( \omega^\omega + 1 \) such that \( \ell\xi \geq 2 \).

This analysis could be carried further to see that \( d^\omega\{0, \omega^\omega\} \) contains exactly those elements \( \xi \) with \( \ell\xi \geq n \). It follows that \( d^\omega\{0, \omega^\omega\} = \{\omega^\omega\} \), and thus \( \omega^\omega \) is isolated in \( d^\omega\{0, \omega^\omega\} \), which means that it does not belong to \( d^{\omega+1}\{0, \omega^\omega\} \) and \( p_\omega \omega = \omega \). We conclude that \( p(\mathfrak{X}) = \omega + 1 \).

As the derived set operator is central to the semantics of \( \text{GLP}_\Lambda \), we need to focus on those operators that preserve it. Of course, \( d \) is homeomorphism-invariant, but this class of maps is too restrictive. Meanwhile, if \( f \) is merely continuous and open, it is not generally the case that \( df = d \). As a simple counterexample, consider the ordinals 1 and \( \omega + 1 \) equipped with the interval topology, and let \( f : \omega + 1 \to 1 \) be the map that is identically zero. Of course,
this is the only function between the two spaces. Further, it is easily checked to be continuous and open, yet \(d(\omega + 1) = \{\omega\}\) while \(d1 = \emptyset\).

To this end, we need to consider \(d\)-maps. Recall that a space is discrete if every subset is open. If \(\mathcal{X}\) and \(\mathcal{Y} = (Y, S)\) are topological spaces, a map \(f : \mathcal{X} \to \mathcal{Y}\) is pointwise discrete if \(f^{-1}(y)\) is discrete for all \(y \in Y\).

**Definition 5.2 (\(d\)-map).** Given topological spaces \(\mathcal{X} = (X, T)\) and \(\mathcal{Y} = (Y, S)\), a \(d\)-map from \(\mathcal{X}\) to \(\mathcal{Y}\) is a function \(f : \mathcal{X} \to \mathcal{Y}\) which is continuous, open, and pointwise discrete.

The property of being pointwise discrete is equivalent to the apparently stronger condition that \(f^{-1}A\) is discrete whenever \(A\) is (see [4]). With this observation one readily obtains the following:

**Lemma 5.2.** The composition of \(d\)-maps is a \(d\)-map.

As an important example, the rank function itself is a \(d\)-map, even a “canonical” \(d\)-map in a certain sense:

**Lemma 5.3.** Given a scattered space \(\mathcal{X}\), the rank function \(\rho : \mathcal{X} \to \rho(\mathcal{X})_0\) is a \(d\)-map. Moreover, if \(f : \mathcal{X} \to \Theta_0\) is a \(d\)-map, it follows that \(f = \rho\).

**Proof.** See [5, Lemma 3.3].

Thus \(\Theta_0\) may be seen as a final object in the category of scattered spaces with Cantor-Bendixon rank at most \(\Theta\) and \(d\)-maps as morphisms. One immediate consequence is that \(d\)-maps are rank-preserving. To be precise, if \(\mathcal{X}, \mathcal{Y}\) are scattered spaces with rank-functions \(\rho_\mathcal{X}, \rho_\mathcal{Y}\) and \(f : \mathcal{X} \to \mathcal{Y}\), we say \(f\) is rank-preserving if \(\rho_\mathcal{X} = \rho_\mathcal{Y} f\).

**Lemma 5.4.** If \(\mathcal{X}, \mathcal{Y}\) are scattered spaces and \(f : \mathcal{X} \to \mathcal{Y}\) is a \(d\)-map, then \(f\) is rank-preserving.

**Proof.** Let \(\rho_\mathcal{X}\) and \(\rho_\mathcal{Y}\) be the respective rank functions. The maps \(\rho_\mathcal{Y} f : \mathcal{X} \to \rho_\mathcal{Y}(\mathcal{Y})\) and \(\rho_\mathcal{X} : \mathcal{X} \to \rho_\mathcal{X}(\mathcal{X})\) are both \(d\)-maps by Lemmas 5.2 and 5.3; hence also by Lemma 5.3, they must be equal.

**Lemma 5.5.** Given a scattered space \((X, T)\) with rank function \(\rho\) and \(x \in X\), we have that if \(V\) is any neighborhood of \(x\), then \(\rho(V \setminus \{x\}) \supseteq [0, \rho(x))\).

Moreover, there is a neighborhood \(U\) of \(x\) with \(\rho(U \setminus \{x\}) = [0, \rho(x))\).

**Proof.** By Lemma 5.3, \(\rho\) is a \(d\)-map, hence continuous and open. It follows that if \(V\) is any neighborhood of \(x\), then \(\rho(V)\) is a neighborhood of \(\rho x\), i.e. an initial segment containing \([0, \rho x]\). It follows that \([0, \rho x] \subseteq \rho(V \setminus \{x\})\).

Now, since \(x\) is isolated in \(\rho^0(x)X\), there must be a neighborhood \(U\) of \(x\) such that \(U \cap \rho^0(x)X = \{x\}\), and hence \(\rho(U \setminus \{x\}) \subseteq [0, \rho x)\). By the previous claim, we in fact get \(\rho(U \setminus \{x\}) = [0, \rho x)\), as desired.

---

3We write \(f : \mathcal{X} \to \mathcal{Y}\) instead of \(f : X \to Y\) when the specific topologies are relevant.
The next result shows that it is particularly easy to compute the limit points of sets that are rank-determined; that is, sets such that whenever \( x \in A \) and \( \rho(x) = \rho(y) \), then \( y \in A \).

**Lemma 5.6.** If \( \mathfrak{X} = \langle X, T \rangle \) is a scattered space with rank function \( \rho \) and \( S \) a set of ordinals then \( dp^{-1}S \) is the set of all \( x \in X \) such that \( \rho x > \min S \).

**Proof.** This is a direct consequence of Lemma 5.5. Indeed, if \( \rho x \leq \min S \) then there is a neighborhood \( U \) of \( x \) such that \( \rho(U \setminus \{x\}) = [0, \rho(x)) \), and hence \( x \notin dp^{-1}S \); meanwhile, if \( \rho(x) > \min S \), then given a neighborhood \( U \) of \( x \) we have, once again by Lemma 5.5, that \( \rho(U \setminus \{x\}) \supseteq [0, \rho x) \), hence it must contain \( \min S \). We conclude that \( U \setminus \{x\} \) contains a point in \( \rho^{-1}S \) different from \( x \), and since \( U \) is arbitrary, \( x \in dp^{-1}S \). \( \square \)

There is one more extension of a scattered topology that will be useful to consider.

**Definition 5.3.** Given a topological space \( \mathfrak{X} = \langle X, T \rangle \), let \( \check{T} \) be the topology generated by \( T \) and all sets of the form \( d\xi+1X \) such that \( \xi \in \text{On} \). Then, define \( \check{\mathfrak{X}} = \langle X, \check{T} \rangle \).

The following claim is a modification of a result in [5]:

**Lemma 5.7.** If \( \mathfrak{X}, \mathfrak{Y} \) are scattered spaces and \( f : \mathfrak{X} \to \mathfrak{Y} \) is a \( d \)-map then \( f : \check{\mathfrak{X}} \to \check{\mathfrak{Y}} \) is also a \( d \)-map.

**Proof.** Let \( \mathfrak{X} = \langle X, T \rangle \) and \( \mathfrak{Y} = \langle Y, S \rangle \). Obviously \( f \) is pointwise discrete as a map from \( \mathfrak{X} \) to \( \mathfrak{Y} \). Let us show that \( f : \check{\mathfrak{X}} \to \check{\mathfrak{Y}} \) is continuous. Suppose that \( V \cap d\xi+1Y \) is an \( \tilde{S} \)-open set. Since \( f \) is rank-preserving by Lemma 5.4 we have that \( f^{-1}(d\xi+1Y) = d\xi+1X \), and thus \( f^{-1}(V \cap d\xi+1X) = f^{-1}(V) \cap d\xi+1X \), which is \( \check{T} \)-open.

The argument that \( f : \check{\mathfrak{X}} \to \check{\mathfrak{Y}} \) is open is very similar. Let \( U \cap d\xi+1X \) be a \( \check{T} \)-open set. Once again, we have that \( f^{-1}(d\xi+1Y) = d\xi+1X \), and thus \( f(U \cap d\xi+1X) = f(U) \cap d\xi+1Y \), which is \( \check{S} \)-open. \( \square \)

The above result will be useful in extending constructions to successor modalities. We will also need the following lemma in order to deal with limit modalities:

**Lemma 5.8.** Let \( \mathfrak{X} = \langle X, \check{T} \rangle \) and \( \mathfrak{Y} = \langle Y, \check{S} \rangle \) be \( \lambda \)-polytopologies such that both \( \check{T} \) and \( \check{S} \) are increasing.

If \( \lambda \in \text{Lim} \) and \( f : \mathfrak{X}_\xi \to \mathfrak{Y}_\xi \) is a \( d \)-map for all \( \xi < \lambda \) then

\[
f : \bigcup_{\xi<\lambda} \check{T}_\xi \to \bigcup_{\xi<\lambda} \check{S}_\xi
\]

is a \( d \)-map.
Proof. Let $\mathcal{T}_\lambda = \bigcup_{\xi<\lambda} \mathcal{T}_\xi$ and $\mathcal{S}_\lambda = \bigcup_{\xi<\lambda} \mathcal{S}_\xi$. It is obvious that $f$ is pointwise discrete as $f^{-1}(y)$ is already $0$-discrete in $\mathcal{T}_0$ and thus also $\lambda$-discrete\footnote{We often index topological properties by the topology they refer to, i.e. $0$-discrete means discrete in $\mathcal{T}_0$.}.

To check that $f$ is open, suppose that $U \subseteq X$ is open in $\mathcal{T}_\lambda$, so that $U = \bigcup_{\lambda<\Lambda} U_\lambda$. Then, $f(U) = \bigcup_{\lambda<\Lambda} f(U_\lambda)$, which is open in $\mathcal{S}_\lambda$ as it is a union of open sets. Similarly, if $V \in \mathcal{S}_\lambda$ and $V = \bigcup_{\lambda<\Lambda} V_\lambda$ then $f^{-1}(V) = \bigcup_{\lambda<\Lambda} f^{-1}(V_\lambda) \in \mathcal{T}_\lambda$ and $f$ is continuous. \hfill $\square$

6 Icard ambiances

In this section we shall discuss Icard topologies, originally introduced in \cite{Icard} for $GLP_\omega$ and generalized to arbitrary $GLP_\Lambda$ in \cite{Icard2}.

Let $I_0$ be the initial segment topology, and for $0<\lambda<\Lambda$ define a topology $I_\lambda$ on $\Theta$ by setting, for $\lambda < \Lambda$, $I_\lambda$ to be the topology generated by sets of the form $(\alpha,\beta]_\xi = \{\vartheta : \alpha < \ell^\xi \vartheta \leq \beta\}$ or of the form $[0,\beta]_\xi = \{\vartheta : \ell^\xi \vartheta \leq \beta\}$ for some $\alpha < \beta \leq \Theta$ and $\xi < \lambda$. For uniformity, we may write $[0,\beta]_\xi$ as $(-1,\beta]_\xi$, and thus we may assume all intervals to be open on the left.

We will call the resulting polytopological space $I_\Theta^\Lambda$. We will denote the derived-set operator with respect to $I_\lambda$ by $i_\lambda$ and the ordinal $\Theta$ equipped with $I_\lambda$ by $\Theta_\lambda$; note that there is no clash in notation in the cases $\lambda = 0, 1$ as the Icard topologies coincide with the initial segment and interval topologies, respectively. When we need to be more specific, we will write $I_\Theta^\Lambda$ or $i_\Theta^\Lambda$ to indicate that the underlying set is $\Theta$.

Recall that if $r$ is a simple function and $\alpha$ is an ordinal, we write $r \sqsubseteq \alpha$ if $r(\xi) < \ell^\xi \alpha$ for all $\xi \in \text{dom}(r)$ and $r(\lambda) \leq \ell^\lambda \alpha$. If $r$ is a simple function such that $r \sqsubseteq \alpha$, then we can associate an Icard-neighborhood of $\alpha$ to $r$. Namely, define

$$B_r(\alpha) = \bigcap_{\xi \in \text{dom}(r)} (r(\xi), \ell^\xi \alpha]_\xi.$$ 

Note that if $\lambda > \max(\text{dom}(r))$, then $B_r(\alpha)$ is $\lambda$-open. This will give us a useful way to describe “small” neighborhoods of $\alpha$. To be precise, given a topological space $(X, \mathcal{T})$ and $x \in X$, say a family of open sets $\mathcal{N}$ is a neighborhood base for $x$ if $x \in U$ for all $U \in \mathcal{N}$ and, given any neighborhood $V$ of $x$, there is $V' \in \mathcal{N}$ with $V' \subseteq V$.

**Lemma 6.1.** If $\Theta$ is any ordinal and $\xi < \Theta$ is any ordinal such that $\ell^\xi \xi > 0$, then the sets of the form $B_r(\xi)$ with $\text{dom}(r) \subseteq \lambda$ form a neighborhood base for $\xi$.

\[16\]
Proof. Every neighborhood of $\xi$ contains a set of the form $V = \cap_{k<K} (\alpha_k, \beta_k)_{\sigma_k}$ with $\xi \in V$ and all $\sigma_k < \lambda$. Since $\ell^\lambda \xi > 0$, so is $\ell^{\sigma_k} \xi$ for all $k$ and we may assume that all $\alpha_k$ are different from $-1$. We may also assume that all $\sigma_k$ are distinct, for if $\sigma_j = \sigma_k$ we have that $(\alpha_j, \beta_j)_{\sigma_j} \cap (\alpha_k, \beta_k)_{\sigma_k} = (\max\{\alpha_j, \alpha_k\}, \min\{\beta_j, \beta_k\})$. Thus we may define $r$ by $r(\sigma_k) = \alpha_k$, with $r(\xi)$ undefined elsewhere. Clearly, $B_r(\xi) \subseteq V$.

It is well-known that Icard spaces are not models of GLP, but as we shall see Icard ambiances are:

**Definition 6.1** (Icard ambiance). An Icard ambiance is a provability ambiance $X$ based on an Icard space.

Icard ambiances are rather nice to work with, since the topologies are all easy to describe. We will also use shifted Icard ambiances, based on $(I_{\lambda+1})_{\lambda < \Lambda}$: these are important as GLP$_1$ is incomplete for the initial segment topology. We will denote the shifted $\Lambda$-Icard space on $\Theta$ by $\hat{\Theta}^\Lambda_{\alpha}$.

The following useful property is a slight modification of a result from [9]:

**Lemma 6.2.** Given $\xi \leq \Theta$ and $\lambda < \Lambda$, there is an $I_\lambda$-neighborhood $U$ of $\xi$ such that whenever $\xi \neq \xi' \in U$, $\ell^\lambda \xi' < \ell^\lambda \xi$.

Proof. By induction on $\lambda$.

First assume that there is $\eta < \lambda$ with $\ell^\eta \xi = 0$. By induction there is an $\eta$-neighborhood $U$ of $\xi$ such that for all $\xi' \in U$ different from $\xi$, $\ell^\eta \xi' < \ell^\eta \xi$, which clearly implies that $U = \{\xi\}$. Since $U$ is also a $\lambda$-neighborhood of $\xi$, the result follows.

Now suppose that $\ell^\eta \xi > 0$ whenever $\eta < \lambda$ and consider two subcases. If $\lambda = \alpha + 1$, we have by induction hypothesis that there is an $\alpha$-neighborhood $V$ of $\xi$ such that whenever $\zeta \neq \xi$ in $V$ we have $\ell^\alpha \zeta < \ell^\alpha \xi$. Write $\ell^\alpha \xi$ as $\eta + \omega^\beta$ and consider the $\lambda$-neighborhood $U = V \cap (\eta, \ell^\alpha \xi]$. If $\beta = 0$, then $U = \{\xi\}$, for any other point $\zeta \in V$ satisfies $\ell^\alpha \zeta \leq \eta$ and thus does not belong to $U$. If $\beta > 0$, then $\omega^\beta = e\beta$. Suppose that $\zeta \neq \xi$ belongs to $U$. We have that $\zeta = \eta + \delta$ for some $\delta \in (0, \omega^\beta]$ while from $\zeta \in V$ we obtain $\delta < \omega^\beta = e\beta$. We then have by Lemma 4.2 that $\ell\delta < \beta$ and thus $\ell^\lambda \zeta = \ell\delta \zeta < \beta = \ell^\lambda \xi$.

Finally, if $\lambda$ is a limit ordinal, use Lemma 4.4 to find $\alpha < \lambda$ and $\rho > 0$ such that $\ell^\alpha \xi = e^{\omega^\rho} \ell^\lambda \xi$. By induction hypothesis, there is an $\alpha$-neighborhood $U$ of $\xi$ such that whenever $\zeta \neq \xi$ in $U$ we have $\ell^\alpha \zeta < \ell^\alpha \xi$. Then, $U$ already satisfies the desired properties; for indeed, since $\ell^\alpha \zeta < \ell^\alpha \xi = e^{\omega^\rho} \ell^\lambda \xi$ we also have, by Lemma 4.2 that $\ell^\lambda \zeta = \ell^\alpha + \omega^\rho \zeta = \ell^\omega \ell^\alpha \zeta < \ell^\lambda \xi$.

Constructing $d$-maps between Icard spaces will be crucial. Fortunately, hyperlogarithms already provide important examples. For simplicity, we shall henceforth write $\ell^{-\xi}$ instead of $(\ell^\xi)^{-1}$.

**Lemma 6.3.** If $\Theta, \xi, \zeta$ are ordinals, then $e^\xi : \Theta_{\xi+\zeta} \to \Theta_{\zeta}$ is a $d$-map.
Proof. That \(\ell^\xi\) is pointwise discrete is an immediate consequence of Lemma 6.2, so it remains to show that the maps are open and continuous.

Let us first consider the case when \(\zeta = 0\). Let \([0, \beta]_0\) be a 0-open set and \(\vartheta \in \ell^{-\xi}[0, \beta]_0\). Once again use Lemma 6.2 to find a \(\xi\)-neighborhood \(U\) of \(\vartheta\) such that for all \(\eta \in U\), \(\ell^\xi\eta \leq \beta\). But then, \(U \subseteq \ell^{-\xi}[0, \beta]_0\), and since all parameters were arbitrary we conclude that \(\ell^\xi : \Theta_\xi \to \Theta_0\) is continuous.

To see that it is open, let \(U = \bigcap_{n \leq N}(\alpha_n, \beta_n)_{\delta_n}\) be \(\xi\)-open and suppose that \(\vartheta \in U\). We claim that \([0, \ell^\xi\vartheta] \subseteq \ell^\xi U\). To see this, pick \(\eta \leq \ell^\xi\vartheta\). Define a simple function \(r\) with \(r(\delta_n) = \alpha_n\) and \(r(\xi) = \eta\). Then, by Lemma 4.6 \(\ell^\xi[r] = \eta\) while for all \(n \leq N\),

\[
\alpha_n < \ell^{\delta_n}[r] \leq \ell^{\delta_n}\vartheta \leq \beta_n.
\]

Thus \([\gamma] \in U\), so that \(\eta \in \ell^\xi U\). Since \(\eta\) was arbitrary, we conclude that \([0, \ell^\xi\vartheta]_0 \subseteq \ell^\xi U\), and thus \(\ell^\xi : \Theta_\xi \to \Theta_0\) is open, as claimed.

Now we must consider \(\zeta > 0\). To see that \(\ell^\xi : \Theta_{\xi+\zeta} \to \Theta_\xi\) is continuous, note that if \(\delta < \zeta\), \(\ell^\xi\gamma \in (\alpha, \beta)_{\delta}\) if and only if \(\ell^\xi\ell^\xi\gamma = \ell^{\xi+\delta}\gamma \in (\alpha, \beta)_{\delta}\), that is, \(\ell^{-\xi}(\alpha, \beta)_\delta = (\alpha, \beta)_{\xi+\delta}\), which is \((\xi + \zeta)\)-open.

Next, let us check that it is open. Suppose that \(\gamma \in U = \bigcap_{n \leq N}(\alpha_n, \beta_n)_{\delta_n}\), where \(\delta_n < \delta_{n+1} < \xi + \zeta\) and suppose that \(J \leq N\) is the largest index such that \(\delta_n < \xi\) for all \(n < J\). Consider the \(\zeta\)-neighborhood

\[
V = [0, \ell^\xi\gamma]_0 \cap \bigcap_{J \leq n < N} (\alpha_n, \ell^{\delta_n}\gamma)_{-\xi+\delta_n}
\]

of \(\ell^\xi\gamma\) and choose \(\eta \in V\). Define a simple function \(s\) by \(s(\delta_n) = \alpha_n\) for \(n < J\) and \(s(\xi) = \eta\), and undefined otherwise. By Lemma 4.6 we know that \(\alpha_n < [s] \leq \ell^{\delta_n}\gamma\) for all \(n < J\), while \(\ell^\xi[s] = \eta\) and for \(n \geq J\), \(\ell^{\delta_n}[s] = \ell^{-\xi+\delta_n}\eta \in (\alpha_n, \beta_n]\) if it exists. Thus \([s]\) is pointwise discrete is an immediate consequence of Lemma 6.2, so it remains to show that the maps are open and continuous.

To see that it is open, let \(U = \bigcap_{n \leq N}(\alpha_n, \beta_n)_{\delta_n}\) be \(\xi\)-open and suppose that \(\vartheta \in U\). We claim that \([0, \ell^\xi\vartheta] \subseteq \ell^\xi U\). To see this, pick \(\eta \leq \ell^\xi\vartheta\). Define a simple function \(r\) with \(r(\delta_n) = \alpha_n\) and \(r(\xi) = \eta\). Then, by Lemma 4.6 \(\ell^\xi[r] = \eta\) while for all \(n \leq N\),

\[
\alpha_n < \ell^{\delta_n}[r] \leq \ell^{\delta_n}\vartheta \leq \beta_n.
\]

Thus \([\gamma] \in U\), so that \(\eta \in \ell^\xi U\). Since \(\eta\) was arbitrary, we conclude that \([0, \ell^\xi\vartheta]_0 \subseteq \ell^\xi U\), and thus \(\ell^\xi : \Theta_\xi \to \Theta_0\) is open, as claimed.

An important corollary of this is the following:

**Corollary 6.1.** If \(\rho_\xi\) denotes the rank with respect to \(\mathcal{I}_\xi\), then \(\rho_\xi = \ell^\xi\).

**Proof.** Immediate from Lemma 6.3 with \(\zeta = 0\) and Lemma 5.3

With this we may also obtain another useful characterization of Icard topologies from [8]:

**Lemma 6.4.** Given ordinals \(\Theta, \xi\),

1. If \(\xi = \zeta + 1\) then \(\mathcal{I}_\xi = \mathcal{I}_\zeta\),

2. if \(\xi \in \text{Lim}\) then \(\mathcal{I}_\xi = \bigsqcup_{\zeta < \xi} \mathcal{I}_\zeta\).

**Proof.** The second claim is immediate from the definitions, so we shall check only the first.

Here we note that \(\mathcal{I}_\xi\) is obtained from \(\mathcal{I}_\zeta\) by adding sets of the form \((\alpha, \beta]_\zeta\) as opens. In view of Lemma 6.2 we know that \([0, \beta]_\xi\) is always \(\zeta\)-open, so it suffices to prove that \((\alpha, \Theta]_\xi\) is \(\xi\)-open as well. But this follows from Corollary

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6.1 as for all $\delta$ we have that $\rho_{\zeta} \delta = \ell \zeta \delta$ and hence $\delta \in [\alpha, \Theta) \zeta$ if and only if $\rho_{\zeta} \delta \geq \alpha$, i.e. if $\delta \in i_{\zeta}^{\alpha} [0, \Theta)$. We conclude that $(\alpha, \Theta) \zeta = i_{\zeta}^{\alpha + 1} [0, \Theta)$, which by definition is an element of $\dot{I}_\zeta$.

This gives us one further result:

**Lemma 6.5.** If $\Theta, \Xi$ are ordinals and $f: \Theta_\alpha \rightarrow \Xi_\beta$ is a $d$-map then for all $\gamma$, $f: \Theta_{\alpha + \gamma} \rightarrow \Xi_{\beta + \gamma}$ is a $d$-map.

**Proof.** By a simple induction on $\gamma$. For successor $\gamma$ we use Lemma 6.4 together with Lemma 5.7; for limit $\gamma$ we use Lemma 5.8. \qed

### 7 The simple ambiance

It will be convenient to focus on a specific ambiance to get a feel for how these may be constructed. The simple ambiance we shall present here is not entirely central to our completeness proof since GLP is not complete for the class of simple ambiances, but simple sets nevertheless provide the appropriate semantics for the closed fragment GLP$^0_\Lambda$ of GLP$\Lambda$, where propositional variables may not occur (only $\top$).

It turns out that a set is simple if and only if it is definable by a closed formula; we will prove one implication later. With this equivalence in mind, the set of valid formulas of L$\Lambda$ over the class of simple ambiances is equal to the set of validities over the closed-fragment definable sets. This logic is described in [13, 16] and extends GLP$\Lambda$ by the axioms for linear frames.

**Definition 7.1.** A set $S \subseteq \Theta$ is simple if there exist natural numbers $N, M$ and ordinals $\alpha_{nm}, \beta_{nm}, \sigma_{nm}$ (with $\alpha_{nm}$ possibly equal to $-1$) such that

$$S = \bigcup_{n < N} \bigcap_{m < M} (\alpha_{nm}, \beta_{nm}]_{\sigma_{nm}}.$$

If all $\sigma_{nm} \leq \lambda$, we say $S$ is $\lambda$-simple.

It is an easy observation that all $\lambda$-simple sets are $\lambda$-open.

**Lemma 7.1.** If $S, T$ are simple sets, then $\Theta \setminus S$, $S \cap T$, $S \cup T$ and $i_\lambda S$ are simple sets; further, $i_\lambda S$ is $(\lambda + 1)$-open.

**Proof.** We focus on showing that $i_\lambda S$ is $(\lambda + 1)$-simple as the other properties use standard Boolean algebra manipulations.

Note that

$$i_\lambda \bigcup_{n < N} \bigcap_{m < M} (\alpha_{nm}, \beta_{nm}]_{\sigma_{nm}} = \bigcup_{n < N} i_\lambda \bigcap_{m < M} (\alpha_{nm}, \beta_{nm}]_{\sigma_{nm}},$$

so our claim will be established if we prove that $i_\lambda \bigcap_{k < K} (\alpha_k, \beta_k]_{\sigma_k}$ is always $(\lambda + 1)$-simple.
Thus we suppose that

\[ S = \bigcap_{k \leq K} (\alpha_k, \beta_k]_{\sigma_k}. \]

Assume that \( S \neq \emptyset \), since otherwise the claim is trivial given that \( i_\lambda \emptyset = \emptyset \), which is \((\lambda+1)\)-simple, and let \( \delta \in S \). Assume also that the \( \sigma_k \)'s are in increasing order and let \( H \) be the largest index such that \( \sigma_H < \lambda \). Let \( r \) be a simple function defined by \( r(\sigma_k) = \alpha_k \) for all \( k < K \), \( r(\sigma_K) = \alpha_K + 1 \) and let \( \alpha_* = \ell^\lambda [r] \).

We claim that

\[ i_\lambda \bigcap_{k \leq K} (\alpha_k, \beta_k]_{\sigma_k} = (\alpha_*, \Theta) \lambda \cap \bigcap_{k \leq H} (\alpha_k, \beta_k]_{\sigma_k}. \]

Let us begin by showing that the right-hand side is contained in the left. Let \( \xi \in (\alpha_*, \Theta) \lambda \cap \bigcap_{k \leq H} (\alpha_k, \beta_k]_{\sigma_k} \) and pick any \( \lambda \)-neighborhood \( U \) of \( \xi \); in view of Lemma 6.1, we may assume \( U \) is of the form \( B_t(\xi) \) for some simple function \( t \) with \( \text{dom}(t) \subseteq \lambda \). Let \( r' \) be a simple function which is equal to \( r \) on all \( \xi < \lambda \), but \( r'(\lambda) = \alpha_* \) and \( r' \) is undefined otherwise. Then define \( s = t \uplus r' \); we claim that \( \zeta = \lceil s \rceil \in B_t(\xi) \cap S \).

First note that, by Lemma 4.6, \( \zeta \in B_t(\xi) \). Further, also using Lemma 4.6, \( \alpha_k < \ell^\sigma_k \zeta \leq \ell^\sigma_k \xi \leq \beta_k \) for all \( k \leq H \), and \( \ell^\lambda \zeta = \alpha_* \) so that for \( k > H \) we see that

\[ \alpha_i < \ell^\sigma_k \lceil r \rceil = \ell^\sigma_k \zeta \leq \ell^\sigma_k \delta \leq \beta_i, \]

and \( \zeta \in S \).

Finally, note that \( \ell^\lambda \zeta = \alpha_* < \ell^\lambda \xi \), and therefore \( \zeta \neq \xi \). Since \( B_t(\xi) \) was arbitrary, we conclude that \( \xi \in i_\lambda S \).

Now let us show that the left-hand side is contained in the right-hand side. To do this, pick \( \xi \in i_\lambda S \). For each \( k \leq H \), it is easy to see that \( \xi \in (\alpha_k, \beta_k]_{\sigma_k} \); otherwise, \( [0, \alpha_k]_{\sigma_k} \cup (\beta_k, \Theta)_{\sigma_k} \) is a \( \lambda \)-neighborhood of \( \xi \) which does not intersect \( S \). Meanwhile, if \( \ell^\lambda \xi \leq \alpha_* \), by Lemma 6.2 there is a \( \lambda \)-neighborhood \( V \) of \( \xi \) such that if \( \zeta \neq \xi \) is contained in \( V \), then \( \ell^\lambda \zeta < \ell^\lambda \xi \). But by Lemma 4.6 if \( r \uplus \zeta \) then \( \ell^\lambda \zeta \geq \alpha_* \), which means that \( V \cap (S \setminus \{\xi\}) = \emptyset \).

With this we may prove the following:

**Theorem 7.1.** Every simple ambiance is an Icard ambiance and thus \( \text{GLP}_\Lambda \) is sound for the class of simple \( \Lambda \)-ambiances.

**Proof.** Icard polytopologies are clearly scattered and increasing, and simple sets form a provability ambiance due to Lemma 7.1.

We conclude by mentioning a result relating simple sets to closed formulas. The converse claim is also true, i.e. that every simple set may be defined by a closed formula, but we shall not go into details here.
Lemma 7.2. Given a closed formula $\phi$ and an Icard ambiance $X$ on an ordinal $\Theta$ with valuation $\llbracket \cdot \rrbracket$, $\llbracket \phi \rrbracket$ is a simple set.

Proof. By induction on the build of $\phi$ using Lemma 7.1 and the fact that $\llbracket \top \rrbracket = (-1, \Theta)_0$ is simple. \qed

8 Beklemishev-Gabelaia spaces

One key observation when constructing GLP-spaces is that the operation $\cdot^+$ is not monotone. Thus it is possible that $\mathcal{T}^+$ is discrete yet for some suitable refinement $\mathcal{T}'$ of $\mathcal{T}$, $(\mathcal{T}')^+$ is not. In this case, it will be useful to pass to such an extension. This idea is central to the completeness proof of [5].

It remains to define what refinements are ‘suitable’; these are given by the following definition.

Definition 8.1 (rank-preserving, limit-maximal refinement). Let $\mathcal{T} \subseteq \mathcal{T}'$ be two topologies on a set $X$. Assume $\langle X, \mathcal{T} \rangle$ is scattered (so that $\langle X, \mathcal{T}' \rangle$ is scattered as well). Let $\rho, \rho'$ be the respective rank functions.

Then, $\mathcal{T}'$ is a

1. rank-preserving refinement of $\mathcal{T}$ if $\rho = \rho'$;
2. limit-refinement of $\mathcal{T}$ if it is a rank-preserving refinement and, whenever $\rho(\xi) \not\in \text{Lim}$ and $U$ is any $\mathcal{T}'$-neighborhood of $\xi$, there is a $\mathcal{T}$-neighborhood $V$ of $\xi$ such that $V \subseteq U$;
3. limit-maximal refinement of $\mathcal{T}$ if there is no limit-refinement $\mathcal{T}''$ of $\mathcal{T}$ such that $\mathcal{T}' \subseteq \mathcal{T}''$.

Limit-maximal refinements are very useful for constructing GLP-spaces. The following results are proven in [5] and are crucial in the construction. Recall that given $X = \langle X, \mathcal{T} \rangle$, $\hat{\mathcal{T}}$ is the topology generated by $\mathcal{T}$ and all sets of the form $d^{\xi+1}X$.

Lemma 8.1. Let $X, Y$ be scattered spaces. Then,

1. There exists a limit-maximal refinement of $X$.
2. If $X$ is limit-maximal then $X^+ = \hat{X}$.
3. If $f : X \to Y$ is a d-map, $Y$ is limit-maximal and $X'$ is a limit-maximal refinement of $X$, then $f : X' \to Y$ is also a d-map.
4. If $f : X \to Y$ is a d-map and $Y'$ is a limit-maximal refinement of $Y$ then there is a limit-maximal refinement $X'$ of $X$ such that $f : X' \to Y'$ is also a d-map.

Proof. These claims are all proven in [5], where Item 1 is Lemma 4.4, Item 2 is Lemma 5.1, Item 3 is Lemma 4.6 and Item 4 Lemma 4.7. \qed
With this we have all the tools we need to construct Beklemishev-Gabelaia spaces. Below, we use $\mathcal{T}_\xi$ in the sense of Definition 8.1.

**Definition 8.2.** Let $\Theta$ be an ordinal.

A polytopology $(\mathcal{T}_\xi)_{\xi < \Lambda}$ on $\Theta$ is a Beklemishev-Gabelaia space (BG-space) if $\mathcal{T}_0$ is a limit-maximal refinement of the interval topology and for all $\xi$, $\mathcal{T}_\xi$ is a limit-maximal refinement of $\mathcal{T}_\xi$.

There is a close relationship between BG and Icard spaces:

**Lemma 8.2.** If $\mathcal{X}$ is a $\Lambda$-BG space with topologies $\mathcal{T}$ then for all $\lambda < \Lambda$, $\mathcal{T}_\lambda$ is a rank-preserving refinement of $\mathcal{I}_{1+\lambda}$.

**Proof.** Suppose that $\mathcal{X}$ is based on an ordinal $\Theta$. Let $\rho_\lambda$ be the rank-function with respect to $\mathcal{T}_\lambda$; in view of Theorem 6.1, the rank with respect to $\mathcal{I}_{1+\lambda}$ is $\ell^{1+\lambda}$. We proceed to prove that $\rho_\lambda = \ell^{1+\lambda}$ by induction on $\lambda$, with the base case $\lambda = 0$ being immediate from the definitions and Lemma 5.1. For $\lambda = \xi + 1$, we use Lemma 8.1.2 to see that $\mathcal{X}_\xi = \mathcal{X}_\xi$, and hence $\mathcal{T}_\lambda$ is rank-preserving over $\mathcal{T}_\xi$, so we may compute ranks over $\mathcal{T}_\xi$ instead of $\mathcal{T}_\lambda$.

By induction hypothesis, $\mathcal{T}_\xi$ is a rank-preserving refinement of $\mathcal{I}_{1+\xi}$. Since we also have $\mathcal{I}_{1+\lambda} = \mathcal{I}_{1+\xi}$, it readily follows that $\mathcal{T}_\lambda$ is a refinement of $\mathcal{I}_{1+\lambda}$. We use a second induction on $\vartheta$ to show that $\rho_\lambda \vartheta = \ell^{1+\lambda} \vartheta$ for $\vartheta < \Theta$; that is, assume that if $\vartheta' < \vartheta$ then $\rho_\lambda \vartheta' = \ell^{1+\lambda} \vartheta'$. Use Lemma 5.1 to find a $\mathcal{T}_\lambda$-neighborhood $U \subseteq [0, \vartheta]$ of $\vartheta$ with $\rho_\lambda(U) = [0, \rho_\lambda \vartheta)$, so that $U = V \cap (\alpha, \ell^{1+\xi} \vartheta)_{1+\xi}$ for some $V \subseteq \mathcal{T}_\xi$ and $\alpha < \Theta$. Let $\delta < \ell^{1+\lambda} \vartheta$. Then, there is $\gamma \in (\alpha, \ell^{1+\xi} \vartheta] \cap \rho_\lambda \vartheta = \ell^{1+\lambda} \vartheta$, since $\ell$ maps intervals to initial segments.

But $\gamma < \ell^{1+\xi} \vartheta$ and since by induction hypothesis $\mathcal{T}_\xi$ is rank-preserving over $\mathcal{I}_{1+\xi}$, there is $\eta \in V$ with $\rho_\xi \eta = \ell^{1+\xi} \eta = \gamma$. It follows that $\eta \in U$ and, by induction on $\eta < \vartheta$, $\rho_\lambda \eta = \ell^{1+\lambda} \eta$. Since $\rho_\lambda(U) = [0, \rho_\lambda \vartheta)$ we have that $\rho_\lambda \vartheta > \rho_\lambda \eta = \ell^{1+\lambda} \eta$. Since $\delta < \ell^{1+\lambda} \vartheta$ was arbitrary, we conclude that $\rho_\lambda \vartheta \geq \ell^{1+\lambda} \vartheta$. The inequality $\rho_\lambda \vartheta \leq \ell^{1+\lambda} \vartheta$ follows from the fact that $\mathcal{T}_\lambda$ refines $\mathcal{I}_{1+\lambda}$, and hence the two are equal.

If $\lambda$ is a limit ordinal, first note that $1 + \lambda = \lambda$, which will simplify some expressions. We have that $\mathcal{I}_1 = \bigsqcup_{\xi < \Lambda} \mathcal{T}_\xi$ whereas $\mathcal{T}_\lambda \supseteq \mathcal{T}_\xi = \bigsqcup_{\xi < \Lambda} \mathcal{T}_\xi$, so by induction hypothesis $\mathcal{T}_\lambda$ is a refinement of $\mathcal{I}_\Lambda$. It remains to show that it is rank-preserving.

Observe that $\mathcal{T}_\lambda$ is rank-preserving over $\mathcal{I}_{1+\xi}$. Pick any basic $\mathcal{T}_\lambda$-neighborhood $U$ of $\vartheta$, so that $U \subseteq \mathcal{T}_\xi$ for some $\xi < \Lambda$, and $\delta < \ell^{\lambda} \vartheta$. We may assume $U \subseteq [0, \vartheta]$. By induction on $\xi < \lambda$, $\rho_{\xi} U \supseteq [0, \ell^{1+\xi} \vartheta]$. From $\delta < \ell^{\lambda} \vartheta = \ell^{-(1+\xi)} \ell^{1+\xi} \vartheta$ and Lemma 4.2 we obtain $e^{-(1+\xi)+\lambda} \delta < \ell^{1+\xi} \vartheta$ and hence there is $\gamma \in U$ with $\rho_{\xi} \gamma = e^{-(1+\xi)+\lambda} \delta$. But then, $\rho_{\lambda} \gamma$.

Since $U$ was arbitrary it follows that $\rho_\lambda \vartheta \geq \ell^{1+\lambda} \vartheta$, and hence the two are equal. \[\square\]
Thus the analogue of Theorem 6.1 also holds for BG-spaces, although here we obtain \( p_{\xi} = \ell^{1+\xi} \). In fact, the above result motivates our focusing on rank-preserving extensions of Icard spaces. Both BG-spaces and shifted Icard ambiances are examples of regular polytopologies, in the sense of the following definition:

**Definition 8.3** (regular space). A \( \Lambda \)-space \( X \) with topologies \( \vec{T} \) is regular if for all \( \lambda < \Lambda \), \( T_\lambda \) is a limit-refinement of \( I_{1+\lambda} \).

It remains to show that BG-spaces actually exist. This can be done via a non-constructive proof:

**Lemma 8.3.** Given ordinals \( \Theta, \Lambda \), there exists a BG-space \( \langle \Theta, \langle T_\lambda \rangle_{\lambda < \Lambda} \rangle \).

**Proof.** By a straightforward induction on \( \Lambda \) using Lemma 8.1.1 to find a limit-maximal refinement of \( T_\Lambda^- \); we remark that such a refinement is found using Zorn’s lemma and hence the resulting space is not given constructively. The case for \( \Lambda = \omega \) was first proven in [5].

Below, a polytopology \( \langle X, \langle T_\lambda \rangle_{\lambda < \Lambda} \rangle \) is based on a topological space \( \langle X, T \rangle \) if \( T_0 \) is a limit-extension of \( T \).

**Lemma 8.4.** Suppose that \( \Theta \) is an ordinal, \( \exists \) a \( \Lambda \)-BG-space and \( f : \Theta_1 \rightarrow \Theta_0 \) a \( d \)-map.

Then, there exists a BG-space \( X \) based on \( \Theta_1 \) such that \( f : X_\lambda \rightarrow \Theta_0 \) is a \( d \)-map for all \( \lambda < \Lambda \).

**Proof.** We proceed by induction on \( \lambda \), assuming we have constructed \( T_\xi \) for \( \xi < \lambda \). If \( \lambda = 0 \), \( f : \Theta_1 \rightarrow \Theta_0 \) is a \( d \)-map by assumption so by Lemma 8.1.1 there is a limit-maximal extension \( X_0 \) of \( \Theta_1 \) such that \( f : X_0 \rightarrow \Theta_0 \) is a \( d \)-map.

If \( \lambda = \xi + 1 \) then by Lemma 8.1.2 \( X_\xi^+=X_\xi \) and \( T_\xi^+ = T_\xi \) so that by Lemma 5.7 \( f : X_\xi^+ \rightarrow \Theta_0 \) is a \( d \)-map. Then, once again by Lemma 8.1.1 we can extend \( X_\xi^+ \) to a limit-maximal space \( X_\lambda \), making \( f : X_\lambda \rightarrow \Theta_0 \) a \( d \)-map.

Finally, if \( \lambda \in \text{Lim} \), we proceed as above, using Lemma 5.8.

BG-spaces and Icard ambiances can sometimes be united into a single structure. We call these *idyllic ambiances*:

**Definition 8.4** (idyllic ambiance). A shifted Icard \( \Lambda \)-ambiance \( \mathfrak{X} = \langle \Theta, \vec{T}, A \rangle \) is idyllic if there is a BG polytopology on \( \mathfrak{X} \) with derived set operators \( d_\lambda \) such that, for all \( \lambda < \Lambda \), \( d_\lambda \upharpoonright A = i_{1+\lambda} \upharpoonright A \).

The purpose of these ambiances is to “kill two birds with one stone”, since any model based on an idyllic ambiance may be regarded both as an Icard model and a BG model. It will also be curious to observe that in our completeness proof, we shall construct BG-models and Icard models with the same valuations.
9 Reductive maps

A fundamental technique in building models of GLP consists of “pulling back” valuations from a previously constructed model using well-behaved maps. If \( X, Y \) are scattered polytopologies, \([\cdot ]_Y\) is a valuation on \( Y \) and \( f: X \to Y \), then we may define a new valuation \([\cdot ]_X\) by setting \([p]_X = f^{-1}[p]_Y\). Specifically, we want \( Y \) to be of the form \( \Xi_1 \) for some ordinal \( \Xi \), since then we can borrow from the completeness of GL for the class of ordinals with the interval topology (see [7]).

This idea is used in [5] to prove the topological completeness of GLP\( \omega \) using \( f = \ell \). One can generalize this to arbitrary GLP\( \Lambda \) using hyperlogarithms, but it does not lead to optimal results; because of this, in this section we broaden our arsenal of useful functions by introducing reductive maps.

One key property of hyperlogarithms is that they are determined by the lower hyperlogarithms; if \( \Lambda \) is additively indecomposable and \( \lambda < \Lambda \), then \( \ell^\Lambda \xi = \ell^\Lambda \zeta \) whenever \( \ell^\lambda \xi = \ell^\lambda \zeta \). To be precise, for functions \( f: X \to Y \) and \( g: X \to Z \), say \( f \) is \( g \)-determined if \( f(x) = f(y) \) whenever \( g(x) = g(y) \); then, it is straightforward to check that \( \ell^\Lambda \) is \( \ell^\lambda \)-determined when \( \lambda < \Lambda \). This property will be extremely useful later; in fact, even a weaker, local version will already turn out to be quite powerful. For this, if \( f, g \) are as above and \( T \) is a topology on \( X \), say \( f \) is locally \( g \)-determined if for every \( x \in X \) there is a neighborhood \( U \) of \( x \) such that \( f|_U \) is \( g|_U \)-determined. If \( g = \ell^\lambda \) we will say simply \( \lambda \)-determined instead of \( \ell^\lambda \)-determined.

**Definition 9.1.** Suppose that \( \Theta, \Xi \) are ordinals. We say a \( d \)-map \( f: \Theta_{1+\Lambda} \to \Xi_1 \) is \( \Lambda \)-reductive if for all \( \lambda < 1 + \Lambda \), \( f \) is \( \mathcal{I}_\lambda \)-locally \( \lambda \)-determined.

In other words, given \( \vartheta < \Theta \) and \( \lambda < 1 + \Lambda \), there is a \( \lambda \)-neighborhood \( U \) of \( \vartheta \) such that if \( \xi, \zeta \in U \) and \( \ell^\Lambda \xi = \ell^\Lambda \zeta \), then \( f\xi = f\zeta \). As mentioned above, reductive maps generalize hyperlogarithms:

**Lemma 9.1.** Given ordinals \( \Theta \) and \( \Lambda > 0 \), the hyperlogarithm \( \ell^\Lambda: (e^{\Lambda+1}\Theta + 1)_{\Lambda+1} \to (\Theta + 1)_1 \) is \((-1 + \Lambda + 1)\)-reductive.

**Proof.** In view of Lemma 6.3, \( \ell^\Lambda \) is a \( d \)-map. Moreover, if \( \lambda < \Lambda + 1 \) we have that \( \ell^\Lambda = \ell^{-\lambda+\Lambda} \ell^\Lambda \), so indeed given \( \xi \) we have that \( \ell^\Lambda \) is \( \lambda \)-determined on all of \([0, \Theta]\), which is clearly \( \lambda \)-open.

Thus we have reductive maps when \( \Lambda \) is a successor, but for limit \( \Lambda \) we will need to look elsewhere. Later in this section we will construct \( d \)-maps between \((e^{\Lambda}\Theta + 1)_{\Lambda} \) and \((\Theta + 1)_1 \) when \( \Lambda \in \text{Lim} \), but first let us discuss some of the properties of reductive maps.

Reductive maps will be particularly important in the study of idyllic ambiences. Note that on such ambiences, \( d_\lambda \upharpoonright \mathcal{A} = i_{1+\lambda} \upharpoonright \mathcal{A} \) must hold only for a specific BG-topology. But there are sets \( S \) such that \( d_\lambda S = i_{1+\lambda} S \) whenever \( d_\lambda \) is based on any BG-polytopology. We will say such a set \( S \) is \( \lambda \)-absolute.
Definition 9.2. Let $\Theta$ be an ordinal. A set $A \subseteq \Theta$ is $\lambda$-absolute if for every $BG$ $(\lambda + 1)$-polytopology $\bar{T}$ on $\Theta$ we have that $d_{\lambda}A = i_{1+\lambda}A$.

A very easy example of a $\lambda$-absolute set is the empty set, since $d\emptyset = \emptyset$ no matter what topology $d$ is defined by. Lemma 5.6 gives us a way to construct more interesting $\lambda$-absolute sets, for given any set $S$ we know that $\ell^{-(1+\lambda)}S$ is $\lambda$-absolute. More generally, any reductive map gives rise to absolute sets:

Lemma 9.2. If $f$ is $\Lambda$-reductive, $\lambda < \Lambda$ and $A$ is any set, then $f^{-1}A$ is $\lambda$-absolute.

Proof. Let $f : \Theta \to \Xi$ be $\Lambda$-reductive, let $\lambda < \Lambda$ and let $(d_{\xi})_{\xi < \Lambda}$ be the derived-set operators for a $BG$ polytopology $\bar{T}$ on $\Theta$. Note that the rank function for $T_{\lambda}$ is given by $\ell^{1+\lambda}$.

Let $\vartheta < \Theta$ and pick a $\mathcal{I}_{1+\lambda}$-neighborhood $U$ of $\vartheta$ such that $f$ is $(1+\lambda)$-determined on $U$.

Meanwhile, letting $B = U \cap f^{-1}(A)$ we claim that

$$B = U \cap \ell^{-1}(1+\lambda)\ell^{1+\lambda}B.$$  

To see the left-to-right inclusion, observe that $B \subseteq \ell^{-(1+\lambda)}\ell^{1+\lambda}B$ and, since $B = U \cap f^{-1}(A)$, it follows that $B \subseteq U \cap \ell^{-(1+\lambda)}\ell^{1+\lambda}B$.

For the other inclusion, suppose that $\xi \in U \cap \ell^{-(1+\lambda)}\ell^{1+\lambda}B$. Then, $\ell^{1+\lambda}\xi \in \ell^{1+\lambda}B$, that is, there is $\xi' \in B$ such that $\ell^{1+\lambda}\xi = \ell^{1+\lambda}\xi'$. But since we have that $\xi, \xi' \in U$ and $f$ is $(1+\lambda)$-determined on $U$, it follows that $f\xi = f\xi'$. From $\xi' \in B$ we obtain $f\xi' \in A$ and thus $f\xi \in A$, i.e. $\xi \in U \cap f^{-1}A = B$, as desired.

Moreover, since $U$ is $\mathcal{I}_{1+\lambda}$-open, we have that it is $T_{\lambda}$-open as well and thus

$$d_{\lambda}B = U \cap d_{\lambda}\ell^{-1}(1+\lambda)\ell^{1+\lambda}B,$$

and similarly

$$i_{1+\lambda}B = U \cap i_{1+\lambda}\ell^{-1}(1+\lambda)\ell^{1+\lambda}B.$$  

Since $\ell^{1+\lambda}$ is the rank both on $T_{\lambda}$ and $\mathcal{I}_{1+\lambda}$, by Lemma 5.6 we see that

$$d_{\lambda}\ell^{-1}(1+\lambda)\ell^{1+\lambda}B = i_{1+\lambda}\ell^{-1}(1+\lambda)\ell^{1+\lambda}B$$

$$= \{ \vartheta \leq \Theta : \ell^{1+\lambda}\vartheta > \min \ell^{1+\lambda}B \},$$

from which we obtain that $\vartheta \in d_{\lambda}(U \cap f^{-1}(A))$ if and only if $\ell^{1+\lambda}\vartheta > \min \ell^{1+\lambda}B$ if and only if $\vartheta \in i_{1+\lambda}(U \cap f^{-1}(A))$. Since $\vartheta$ was arbitrary, $d_{\lambda}f^{-1}A = i_{1+\lambda}f^{-1}A$, as claimed.

Another nice property of reductive maps is that they behave well with respect to extensions of limit topologies:

Lemma 9.3. Suppose that that $\Theta, \Xi$ and $\Lambda$ are ordinals with $\Lambda \in \text{Lim}$, $f : (\Theta + 1)_{\Lambda} \to (\Xi + 1)_{1}$ is a $\Lambda$-reductive map and $(T_{\lambda})_{\lambda \leq \Lambda}$ is a regular polytopology on $\Theta + 1$ with $\mathcal{I}_{\lambda} = \bigcup_{\lambda < \Lambda} T_{\lambda}$. Let $\mathcal{K} = (\Theta + 1, \bar{T})$ be the resulting $(\Lambda + 1)$-space.

Then, $f : \mathcal{K}_{\Lambda} \to \Xi_{1}$ is a $d$-map.
Proof. Clearly \( f \) is continuous and pointwise discrete, so let us check that it is open. Suppose that \( U \) is a \( T_\Lambda \)-neighborhood of a point \( \vartheta < \Theta \), so that it is a \( T_\Lambda \)-neighborhood of \( \vartheta \) for some \( \lambda < \Lambda \). Pick an \( T_\Lambda \)-neighborhood \( D \) of \( \vartheta \) such that \( f \) is \( \ell^{1+\lambda} \)-determined on \( D \) and \( \ell^{1+\lambda}(D \setminus \{ \vartheta \}) = [0, \ell^{1+\lambda}\vartheta) \); the first condition can be met because \( f \) is \( \Lambda \)-reductive, the second by Lemmas 5.1 and 6.1.

We claim that \( f(U \cap D) = f(D) \). Indeed, if \( \zeta \in f(D) \), then \( \zeta = f(\delta) \) for some \( \delta \in D \). But since \( \rho_\lambda \vartheta = \ell^{1+\lambda} \vartheta \geq \rho_\lambda \delta \), there is some \( \delta' \in U \cap D \) with
\[
\ell^{1+\lambda} \delta' = \rho_\lambda \delta' = \ell^{1+\lambda} \delta,
\]
and hence \( f(\delta') = f(\delta) = \zeta \). Since \( \zeta \) was arbitrary, the claim follows.

Now, \( f \) is a \( d \)-map with respect to \( T_\Lambda \), so that \( f(D) \) (and hence \( f(U \cap D) \)) is open, as desired. \( \square \)

Not all reductive maps are given by hyperlogarithms. Let us now construct another interesting example. Here we will work with fundamental sequences; that is, we assume that to each (small enough) countable limit ordinal \( \xi \) we have assigned a sequence of ordinals \( \langle \xi[n] \rangle_{n<\omega} \) with the property that \( \xi[n] < \xi[n+1] \) for all \( n \) and \( \xi = \lim_{n \to \omega} \xi[n] \). If \( \xi = \xi + 1 \), we will define \( \zeta = \xi[n] \) for all \( n \).

Suppose that \( \Lambda \) is infinite and additively indecomposable. If \( \vartheta < e^\Lambda \Theta \) is any ordinal that is not in the range of \( e^\Lambda \), there exists a value of \( N \) such that \( \ell^{\Lambda[N]} \vartheta \leq e^\Lambda(\Theta[N]) \); if \( \Theta \) is a successor ordinal this is essentially Lemma 4.5, otherwise \( e^\Lambda \Theta = \lim_{n \to \omega} e^\Lambda(\Theta[n]) \) (since \( e^\Lambda \) is normal), and hence for some value of \( N \) we already have that \( \vartheta < e^\Lambda(\Theta[n]) \). We will denote the smallest such value of \( N \) by \( N_\Lambda^\Theta(\vartheta) \).

Definition 9.3. Given countable ordinals \( \Theta, \Lambda \) with fundamental sequences \( \langle \Theta[n] \rangle_{n<\omega}, \langle \Lambda[n] \rangle_{n<\omega} \) and \( \vartheta < e^\Lambda \Theta \), we define \( N = N_\Lambda^\Theta(\vartheta) \) to be the least natural number \( N \) such that \( \ell^{\Lambda[N]} \vartheta \leq e^\Lambda(\Theta[N]) \).

Example 9.1. Suppose that \( \Theta = 2 \), so that \( \Theta[n] = 1 \) for all \( n < \omega \), and \( \Lambda = \omega \). A standard fundamental sequence we may take for \( \Lambda = \omega = n \). Then, \( e^\Lambda \Theta = e_1 \), while \( e^\Lambda(\Theta[n]) = e_0 \) for all \( n < \omega \). Let \( \vartheta = e^\omega 1 + 1 \). Then, \( e^0 \vartheta = e^\omega 0 + 1 > e_0 = e^\Lambda \Theta[0] \), \( e^1 \vartheta = e_0 + 1 > e_0 = e^\Lambda \Theta[1] \), but \( e^2 \vartheta = e^\omega \vartheta = \ell(\epsilon_0 + 1) = 0 \). Thus, \( N_2^\omega (\omega^\omega + 1) = 2 \).

Meanwhile, if instead \( \vartheta = e_0 \) we already have \( e^0 e_0 = e_0 \leq e^\Lambda(\Theta[1]) \), so \( N_2^\omega (\epsilon_0) = 0 \).

Example 9.2. Now suppose that \( \Theta = \Lambda = \omega \), again with fundamental sequence \( \omega[n] = n \). Suppose further that \( \vartheta = \omega^{\omega^2 3} = e((e^{\omega^3} \cdot 3)) \).

Then, \( e^0 \vartheta = \omega^{\omega^2 3} > 0 = e^\omega(\Theta[0]) \), \( e^1 \vartheta = e_2 \cdot 3 > e_0 = e^\Lambda(\Theta[1]) \), and \( e^2 \vartheta = e_2 > e_1 = e^\Lambda(\Theta[2]) \). After this, \( e^n \vartheta = e_2 \) for all \( n \geq 2 \). However, \( \Theta[n] \) is still increasing, so that \( e^\Lambda(\Theta[3]) = e_2 \geq e^\Lambda \vartheta \), and thus \( N_\omega^\omega (\omega^{\omega^2 3}) = 3 \).

We will use the parameter \( N_\Lambda^\Theta(\vartheta) \) to “diagonalize” and in this way define a new family of reductive maps. This parameter will be used to partition \([0, e^\Lambda]\) into countably many sets.
Lemma 9.4. If $\Theta, \Lambda$ are ordinals such that $\Lambda$ is infinite and additively indecomposable then for every $N > 1$, the set

$$\Delta^\Theta_N = \{ \vartheta < e^\Lambda \Theta : N^\Theta_N(\vartheta) = N \}$$

is $(\Lambda[N] + 1)$-simple and $\Lambda[N]$-open.

Further,

$$\ell^\Lambda[N]\Delta^\Theta_N = [0, e^\Lambda(\Theta[N])].$$

(1)

Proof. Let $\alpha = e^\Lambda(\Theta[N])$ and $\beta = e^\Lambda(\Theta)$. We have that

$$\Delta^\Theta_N = \bigcap_{n < N} (\alpha, \beta)_{\Lambda[n]} \cap [0, \alpha]_{\Lambda[N]},$$

which is $(\Lambda[N] + 1)$-simple and, in view of Lemma 6.2, it is $\Lambda[N]$-open.

Now, let $\delta \leq \alpha$ and consider the simple function given by $r(\lambda[n]) = \alpha$ for $n < N$ and $r(\lambda[N + 1]) = \delta$. By Lemma 4.6, $[r] \in \Delta^\Theta_N$ and $\ell^\Lambda[N][r] = \delta$. Since $\delta \leq \alpha = e^\Lambda(\Theta[N])$ was arbitrary, we conclude that (1) holds.

Recall that a family of sets $U$ forms a neighborhood base for $x$ if every element of $U$ is a neighborhood of $x$ and for every neighborhood $V$ of $x$ there exists $U \in U$ such that $U \subseteq V$.

Lemma 9.5. Let $\Lambda$ be countable and additively indecomposable and $\Theta$ be any ordinal.

Then, the sets

$$D^\Theta_A[N] = (e^\Lambda(\Theta[N]), e^\Lambda(\Theta[\Lambda[N]])$$

form a $\Lambda$-neighborhood base for $e^\Lambda \Theta$.

Further, $D^\Theta_A[N] = \{e^\Lambda \Theta\} \cup \bigcup_{n > N} \Delta^\Theta_A[n]$.

Proof. Clearly $e^\Lambda \Theta \in D^\Theta_A[N]$ for all $N$ and $D^\Theta_A[N]$ is open.

Now, let $B_r(e^\Lambda \Theta)$ be any basic $\Lambda$-neighborhood of $e^\Lambda \Theta$; we need to find a smaller neighborhood of the form $D^\Theta_A[N]$. Let us consider two cases.

First assume $\Theta = \Theta' + 1$, so that $\Theta[n] = \Theta'$ for all $n$.

We have by Proposition 4.1.4 that

$$e^\Lambda \Theta = \lim_{n \to \omega} e^{\Lambda[n]}(e^{\Lambda \Theta'} + 1)$$

and hence for some value of $N$ we have that $e^{\Lambda[N]}(e^{\Lambda \Theta'} + 1) > r(\xi)$ for all $\xi$.

Let $M > N$ be large enough so that $\Lambda[M] > \xi + \Lambda[N]$ for all $\xi \in \text{dom}(r)$. Then, we claim that $D^\Theta_A[M] \subseteq B_r(\vartheta)$; for indeed, if $\xi \in \text{dom}(r)$ and $\delta \in D^\Theta_A[M]$ then

$$\ell^\Lambda[M][\delta] = e^{-\xi + \Lambda[M]} e^{\Lambda \Theta'} + 1$$

so that by Lemma 4.2

$$e^{\Lambda \Theta'} + 1 \geq e^{-\xi + \Lambda[M]}(e^{\Lambda(\Theta')} + 1) \geq e^{\Lambda[N]}(e^{\Lambda(\Theta')} + 1) \geq r(\xi).$$

Thus, $D^\Theta_A[M] \subseteq B_r(e^\Lambda \Theta)$.
If Θ is a limit ordinal, the argument is somewhat simpler: pick M so that Θ[M] > e^r(ξ) for all ξ ∈ dom(ξ). Then, it is easy to check that \(D^\Theta_\Lambda[M] \subseteq B_r(e^\Lambda \Theta)\).

Finally, to see that \(D^\Theta_\Lambda[N] = \{e^\Lambda \Theta \} \cup \bigcup_{n > N} \Delta^\Theta_\Lambda[n]\), note that every \(ξ < e^\Lambda \Theta\) lies in \(\Delta^\Theta_\Lambda[n]\) for some n, and n > N if and only if \(ξ \in D^\Theta_\Lambda[N]\). □

On \([0, e^\Theta]\) we shall consider a different class of neighborhoods. Define

\[σ^\Theta_\Lambda[N] = \sum_{n < N} (e^\Theta[n]) + 1\]

and \(Σ^\Theta_\Lambda[N] = [σ^\Theta_\Lambda[N], σ^\Theta_\Lambda[N] + 1]\).

Similarly, define \(S^\Theta_\Lambda[N] = [σ^\Theta_\Lambda[N], e^\Theta]\).

Then we have that:

**Lemma 9.6.** The sets \(\{Σ^\Theta_\Lambda[n] : n < ω\}\) form a partition of \((0, e^\Theta)\) into 1-open sets.

Further, the sets \(\{S^\Theta_\Lambda[n] : n < ω\}\) form a 1-neighborhood base for \(e^\Theta\), and

\[S^\Theta_\Lambda[N] = \{e(\Theta)\} \cup \bigcup_{n > N} Σ^\Theta_\Lambda[n].\]

**Proof.** Note that despite its formal appearance the set \(Σ^\Theta_\Lambda[N]\) is always open since \(σ^\Theta_\Lambda[N]\) is always a successor ordinal or zero. The rest of the claims are obvious if we show that \(⟨σ^\Theta_\Lambda[n]⟩_{n < ω}\) is unbounded in \(e^\Theta\).

Here we consider two cases; if \(Θ = Θ' + 1\), then \(σ^\Theta_\Lambda[n] = e(Θ')n\) for all \(n < ω\), and

\[e(Θ' + 1) = ω^{Θ' + 1} = \lim_{n→ω} (ω^{Θ'} + 1)n = \lim_{n→ω} (e(Θ') + 1)n.\]

But \((e(Θ') + 1)n = σ^\Theta_\Lambda[N]\).

Meanwhile, if \(Θ \in \text{Lim}\), then \(σ^\Theta_\Lambda[N + 1] = e(Θ[N]) + 1\) (all previous terms cancel) and \(e^\Theta = \lim_{n→ω} e(Θ[N]) + 1.\) □

With this we may define the following maps:

**Definition 9.4.** Given countable ordinals \(Λ, Θ\) such that \(Λ\) is infinite and additively indecomposable, we will define a function

\[r^\Theta_Λ : (e^Λ(Θ) + 1) → (e(Θ) + 1)\]

assuming \(r^\Theta_Λ\) is defined whenever \(Θ' < Θ\).

First define \(r^\Theta_Λ e^Λ = e^\Theta\).

Then, for \(ξ < e^Λ Θ\), set \(N = N^\Theta_Λ(ξ)\) and

\[r^\Theta_Λ ξ = σ^\Theta_Λ[N] + r^\Theta_Λ[N]e^Λ[N]ξ.\]

To illustrate the above recursion, let us show that \(r^\Theta_Λ 0\) is always zero.

**Lemma 9.7.** Given arbitrary \(Θ, Λ, r^\Theta_Λ 0 = 0.\)
Proof. Observe that $N^0(0)$ is always zero since $0 \leq e^\Lambda(0)$ independently of $\Theta$ or $\Lambda$, while $\sigma_0[0]$ is always zero as well since it is an empty sum. With this, we may proceed by induction on $\Theta$; for the base case, we see that $e^0\Lambda = 0$ so $r^0\Lambda[0] = e0$. For the inductive step we have that

$$r^\Theta\Lambda[0] = \sigma_\Theta[0] + r^\Theta[0] \Lambda[0] \equiv 0 + 0,$$

where we are using our induction hypothesis on $\Theta[0] < \Theta$. Thus $r^\Theta\Lambda[0] = 0$ for all $\Theta, \Lambda$, as claimed.

Example 9.3. In Example 9.1 we set $\Theta = 2$, $\Lambda = \omega$ and showed that $N^2_0(\vartheta) = 2$, where $\vartheta = 2^{\omega}\cdot 1$. Let us use this to compute $r^2_\omega\omega^{\vartheta + 1}$. Observe that $e(\Theta) = \omega^2$, while $e^\Lambda = e^\omega = \epsilon_1$, so $r^2_\omega : [0, \epsilon_1) \to [0, \omega^2]$.

First we must set $N = N^2_0(\vartheta) = 2$. Since $\Theta[0]$ is the constant $1$, we have that $\sigma_2[2] = e(2[0]) + 1 + e(2[1]) + 1 = \omega \cdot 2 + 1$.

Meanwhile, $\Lambda[2] = \omega[2] = 2$, so $e^\Lambda(\vartheta) = \ell^2(\vartheta + 1) = \ell(\epsilon_0 + 1) = 0$. Thus,

$$r^\Theta\Lambda[\vartheta] = \sigma_\Theta[N] + r^\Theta[N] \ell^\Lambda(\vartheta) = \omega \cdot 2 + 1 + r^1_\omega 0.$$

But by Lemma 9.7 we know that $r^1_\omega 0 = 0$, so $r^2_\omega\omega^{\vartheta + 1} = \omega \cdot 2 + 1$.

If instead we set $\vartheta = \epsilon_0$, we obtain $N = N^2(\vartheta) = 0$. Then, $\sigma_2[0] = 0$ and

$$r^\Theta[N](\epsilon_0) = r^1_\omega (e\epsilon_1) = e1 = \omega,$

i.e. $r^2_\omega \epsilon_0 = \omega$.

We remark that, if $p_\omega, p$ represent the ranks with respect to $I_\omega$ and $I_1$, then

$$p_\omega \omega^{\vartheta + 1} = e^\omega \cdot 2 \cdot \vartheta + 1 = 0 = \ell(\omega \cdot 2 + 1) = p^2_\omega\omega^{\vartheta + 1},$$

while

$$p_\omega \epsilon_0 = e^\omega \epsilon_0 = 1 = \ell \omega = p^2_\omega \epsilon_0.$$

The above example suggests that $r^\Theta_\omega$ is rank-preserving. In Lemma 9.8 we will show that $r^\Theta_\Lambda$ is always a $d$-map, so in general we always have $p_\Lambda = p^\Theta_\Lambda$. In fact, these maps are $\Lambda$-reductive, but proving this will require several steps. Let us begin with a useful technical lemma.

Lemma 9.8. Given an ordinal $\Theta$ and a limit ordinal $\Lambda$,

1. $r^\Theta_\Lambda[0, e^\Lambda] = [0, e\Theta]$

2. If $N > 0$, $r^\Theta_\Lambda \Delta_\Lambda N = \Sigma_\Theta N$.

Proof. Assume both claims are true for $\Theta'$ when $\Theta' < \Theta$. Note that the first claim is trivial when $\Theta = 0$ because then both sides of the equality are the singleton $\{0\}$, so we may assume $\Theta > 0$. We shall begin by checking the inclusion

$$r^\Theta_\Lambda \Delta_\Lambda N \subseteq \Sigma_\Theta N.$$
For \( \vartheta \in \Delta_\Lambda^\Theta[N] \) we see that
\[
\tau_\Lambda^\Theta \vartheta = \sigma_\Theta[N] + r_\Lambda^\Theta[\ell^\Lambda[N]\vartheta] \in [\sigma_\Theta[N], \sigma_\Theta[N + 1]),
\]
where the last step uses Claim 1 by induction on \( \Theta[N] < \Theta \), given that \( \ell^\Lambda[N]\vartheta \in [0, e^\Lambda(\Theta[N])] \).

From this it easily follows that \( r_\Lambda^\Theta[0, e^\Lambda\Theta] \subseteq [0, e\Theta] \), since \( r_\Lambda^\Theta e^\Lambda\Theta = e\Theta \in [0, e\Theta] \), while for \( \vartheta < e^\Lambda\Theta \), we set \( N = N_\Lambda^\Theta(\vartheta) \) and by Claim 2 see that
\[
r_\Lambda^\Theta \vartheta \in \Sigma_\Theta[N] \subseteq [0, e\Theta].
\]

Now let us check that
\[
\Sigma_\Theta[N] \subseteq r_\Lambda^\Theta [\Delta_\Lambda^\Theta[N] - \vartheta] .
\]
Let \( \xi = \sigma_\Theta[N] + \xi' \) with \( \xi' \leq e(\Theta[N]) \). Then, using Claim 1 by induction on \( \Theta[N] < \Theta \), \( \xi' = r_\Lambda^\Theta[\gamma'] \) for some \( \gamma' \leq e^\Lambda(\Theta[N]) \); meanwhile, by Lemma 9.3, \( \ell^\Lambda[N]\Delta_\Lambda^\Theta[N] = [0, e^\Lambda(\Theta[N])] \), hence \( \gamma' = \ell^\Lambda[N]\gamma \) for some \( \gamma \in \Delta_\Lambda^\Theta[N] \). It immediately follows that \( \xi = r_\Lambda^\Theta\gamma \).

To check the remaining inclusion of Claim 1, consider \( \xi \in [0, e\Theta] \). If \( \xi = e\Theta \), then evidently \( \xi = r_\Lambda^\Theta e^\Lambda(\Theta) \). Otherwise, \( \xi \in [\sigma_\Theta[N], \sigma_\Theta[N + 1]) \) for some \( N \).

But by Claim 2,
\[
\xi \in r_\Lambda^\Theta [\Delta_\Lambda^\Theta[N] - \vartheta] = r_\Lambda^\Theta [0, e^\Lambda\Theta],
\]
as required. \( \square \)

**Lemma 9.9.** Given countable ordinals \( \Theta, \Lambda \) such that \( \Lambda \) is infinite and additively indecomposable,
\[
r_\Lambda^\Theta : (e^\Lambda(\Theta) + 1) \Lambda \to (e(\Theta) + 1)_1
\]
is an onto \( d \)-map.

**Proof.** Assume the claim is true for all \( \Theta' < \Theta \). Note that surjectivity is already proven in Lemma 9.8. Also, the case \( \Theta = 0 \) is trivial because then both sides of the equality are the singleton \( \{0\} \), so that we may assume \( \Theta > 0 \).

Pick \( \vartheta \leq e^\Lambda(\Theta) \). If \( \vartheta < e^\Lambda(\Theta) \), \( \vartheta \in \Delta_\Lambda^\Theta[N] \) for some \( N \), and since these sets are all open by Lemma 9.3, it suffices to observe that \( r_\Lambda^\Theta \vartheta \mid \Delta_\Lambda^\Theta[N] \) is a \( d \)-map. But it is equal to \( r_\Lambda^\Theta [\ell^\Lambda[N] e^\Lambda[N] \vartheta] \), which by induction on \( \Theta[N] < \Theta \) and Lemma 9.3 is a composition of \( d \)-maps. Hence it is open and continuous near \( \vartheta \).

Otherwise, \( \vartheta = e^\Lambda(\Theta) \). Here we claim that \( r_\Lambda^\Theta D_\Lambda^\Theta[N] = S_\Theta[N] \), from which openness and continuity are immediate.

We have that \( D_\Lambda^\Theta[N] = \{e^\Theta\} \cup \bigcup_{n > N} \Delta_\Lambda^\Theta[n] \), whereas \( S_\Theta[N] = \{e\Theta\} \cup \bigcup_{n > N} \sigma_\Theta[N] \), and thus
\[
r_\Lambda^\Theta D_\Lambda^\Theta[N] = r_\Lambda^\Theta \vartheta \bigcup \bigcup_{n > N} r_\Lambda^\Theta \Delta_\Lambda^\Theta[n] \]
\[
= \{e\Theta\} \cup \bigcup_{n > N} \Sigma_\Theta[N] \]
\[
= S_\Theta[N],
\]
30
where the second equality follows from Lemma 9.8.

To check that \( r^\Theta_\Lambda \) is pointwise discrete, pick \( \xi \leq e\Theta \). If \( \xi = e\Theta \) then \((r^\Theta_\Lambda)^{-1}\xi = \{e^\Lambda\Theta\}\), which is discrete.

Otherwise, \( \xi \in \Sigma_\Theta[N] \) for some \( N \) and thus \((r^\Theta_\Lambda)^{-1}\xi = \Delta^\Theta_\Lambda[N] \cap (r^\Theta_\Lambda)^{-1}\xi\), which is discrete as \( r^\Theta_\Lambda \mid \Delta^\Theta_\Lambda[N] \) is a d-map. \( \square \)

**Lemma 9.10.** For any additively indecomposable limit ordinal \( \Lambda \), the map \( r^\Theta_\Lambda \) is \( \Lambda \)-reductive.

**Proof.** By Lemma 9.9, it suffices to prove that \( r^\Theta_\Lambda \) is \( \lambda \)-locally \( \lambda \)-determined for all \( \lambda < 1 + \Lambda \); that is, that given \( \vartheta < e^\Lambda\Theta \) there is a \( \lambda \)-neighborhood \( U \) of \( \vartheta \) with \( r^\Theta_\Lambda \) \( \lambda \)-determined on \( U \).

Let \( M \) be the largest natural such that \( \Lambda[M] < \lambda \). We will consider two cases. First suppose that \( \vartheta \in D^\Theta_\Lambda[M] \). Note that

\[ D^\Theta_\Lambda[M + 1] = (e^\Lambda[M]\Theta[M]e^\Lambda\Theta][\Lambda[M] \]

is \( \lambda \)-open. Let us check that \( r^\Theta_\Lambda \) is \( \lambda \)-determined on \( D^\Theta_\Lambda[N + 1] \). Assume that \( \xi, \zeta \in D^\Theta_\Lambda[N + 1] \) and \( \ell^\lambda \xi = \ell^\lambda \zeta \). It readily follows that, for \( n > M \), \( \ell^\Lambda[n]\xi = \ell^\Lambda[n]\zeta \), and thus \( n := N^\Lambda_\Lambda[\xi] = N^\Lambda_\Lambda[\zeta] \), from which we obtain

\[ r^\Theta_\Lambda \xi = r^\Theta_\Lambda \zeta = \sigma_\Theta[n] + r^\Theta_\Lambda \ell^\Lambda[n] \xi. \]

Thus we see that \( r^\Theta_\Lambda \) is \( \lambda \)-determined on the \( \lambda \)-open set \( U = D^\Theta_\Lambda[M + 1] \).

Otherwise we have that \( \vartheta \in \Delta^\Theta_\Lambda[N] \) for some \( N \) such that \( \lambda \geq \Lambda[N] \). By induction on \( \Theta[N] < \Theta \) there is a \( (-\Lambda[N] + \lambda) \)-neighborhood \( U \subseteq [0, e^\Lambda(\Theta[N])] \)

of \( \ell^\Lambda[N]\vartheta \) such that \( r^\Theta_\Lambda \) \( (-\Lambda[N] + \lambda) \)-determined on \( U' \). By Lemma 9.3 \( \ell^\Lambda[N]U' \) is \( \lambda \)-open, so that \( U = \Delta^\Theta_\Lambda[N] \cap \ell^\Lambda[N]U' \) is \( \lambda \)-open as well. Moreover, \( r^\Theta_\Lambda \) is \( \lambda \)-determined on \( U \). To see this, assume that \( \xi, \zeta \in U \) and \( \ell^\lambda \xi = \ell^\lambda \zeta \); observe that the latter is equivalent to stating that

\[ \ell^\Lambda[N] + \lambda \ell^\Lambda[N] \xi = \ell^\Lambda[N] + \lambda \ell^\Lambda[N] \zeta. \]

It follows that

\[ r^\Theta_\Lambda \xi = \sigma[N] + r^\Theta_\Lambda \ell^\Lambda[N] \xi \]

\[ = \sigma[N] + r^\Theta_\Lambda \ell^\Lambda[N] \zeta \]

\[ = r^\Theta_\Lambda \xi, \]

where (2) follows from the fact that \( \ell^\Lambda[N] \xi, \ell^\Lambda[N] \zeta \in U' \) and \( r^\Theta_\Lambda \) \( (-\Lambda[N] + \lambda) \)-determined on \( U' \). \( \square \)

Let us conclude this section by extending \( r^\Theta_\Lambda \) to the case where \( \Lambda \) is not necessarily additively indecomposable.

**Theorem 9.1.** Given countable ordinals \( \Theta, \Lambda \), with \( \Lambda \geq 0 \) there exists a \( \Lambda \)-reductive surjection

\[ r^\Theta_\Lambda : (e^1 + \Lambda(\Theta) + 1)_{1 + \Lambda} \to (e(\Theta) + 1)_{1}. \]
Proof. We may set \( r_0^\Theta = \text{id} \), and if \( \Lambda + 1 \) is a successor, we may define \( r_{\Lambda+1}^\Theta = \ell^\Lambda \) in view of Lemma 9.1.

If \( \Lambda \) is infinite and additively indecomposable we have already defined \( r_\Lambda^\Theta \). Otherwise, if \( \Lambda = \alpha + \omega^\beta \) with \( \beta > 1 \) define \( r_\Lambda^\Theta = r_{\omega^\beta}^\Theta \ell^\alpha \), which is easily seen to be \( \Lambda \)-reductive.

### 10 Operations on ambiances

In this section we shall review some operations, many of which were introduced in [5], that may be used to construct new provability ambiances from existing ones. First, let us observe that ambiances may be “pulled back”.

If \( X \) is an \((\alpha + \beta)\)-BG space and \( Y \) is a \( \beta \)-BG space, \( f : X \to Y \) is an \( \alpha \)-lift if it is \( \alpha \)-reductive and, for all \( \delta < \beta \), \( f : X_{\alpha+\delta} \to Y_\delta \) is a \( d \)-map both with respect to the BG topologies and the shifted Icard topologies.

**Lemma 10.1.** Suppose that \( X \) is a \((\xi + \zeta)\)-BG-space and \( Y \) is an idyllic \( \zeta \)-ambiance with algebra \( A \). Suppose further that \( f : X \to Y \) is a \( \xi \)-lift. Then, \( f^{-1}A \) is an idyllic algebra on \( X \).

**Proof.** To see that \( d_{\xi+\delta} \upharpoonright f^{-1}A = i_{1+\xi+\delta} \upharpoonright f^{-1}A \), note that for \( A \in A \),

\[
d_{\xi+\delta}f^{-1}A = f^{-1}d_{\xi+\delta}A = f^{-1}i_{1+\xi+\delta}A = i_{1+\xi+\delta}f^{-1}A.
\]

That \( d_\delta \upharpoonright f^{-1}A = i_{1+\delta} \upharpoonright f^{-1}A \) for \( \delta < \alpha \) follows from Lemma 9.2 since \( f^{-1}A \) is always \( \delta \)-absolute.

One of the basic ways of including topological spaces into a larger space is by their topological sum. In the case of ordinal spaces, this is closely tied to the sum of ordinals, as has already been observed in [5].

**Definition 10.1.** Given families of sets \( A \subseteq \mathcal{P}(\Xi) \) and \( B \subseteq \mathcal{P}(\Theta) \), where \( \Xi, \Theta \) are ordinals, we define \( A \oplus B \) to be the family of subsets of \( \Xi + \Theta \) of the form

\[
S = S_0 \cup (\eta + S_1),
\]

with \( S_0 \in A \) and \( S_1 \in B \).

The following lemma is standard and easy to check:

**Lemma 10.2.** If \( \Xi, \Theta \) are ordinals and \( A \subseteq \mathcal{P}(\Xi) \), \( B \subseteq \mathcal{P}(\Theta) \) are topologies, then \( A \oplus B \) is a topology on \( \Xi + \Theta \).

If \( \Xi \) is a successor and both topologies are Icard or BG, then \( A \oplus B \) is Icard or BG, respectively.

In view of this we define, given \( \Lambda \)-polytopologies \( X = \langle \Xi, T \rangle \) and \( Y = \langle \Theta, S \rangle \), the sum

\[
X \oplus Y = \langle \Xi + \Theta, (T_\lambda \oplus S_\lambda)_{\lambda<\Lambda} \rangle.
\]

We may also apply the sum operation to \( d \)-algebras:
Lemma 10.3. Suppose $\mathfrak{X} = \langle \Xi + 1, \bar{T}, \mathcal{A} \rangle$ and $\mathfrak{Y} = \langle \Theta, \bar{S}, \mathcal{A} \rangle$ are ambiances. Then, $\mathfrak{X} \oplus \mathfrak{Y}$ equipped with $\mathcal{A} \oplus \mathcal{B}$ is also an ambiance. Further, if both ambiances are idyllic, then so is the corresponding sum.

We shall not present a proof, as this result is fairly obvious once we observe that all relevant operations may be carried out independently within the two disconnected subspaces.

The last major topological construction needed for the completeness proof is the notion of a $d$-product. Let $\mathfrak{X} = \langle [0, \Xi], \langle T_\lambda \rangle_{\lambda < \Lambda} \rangle$ and $\mathfrak{Y} = \langle [0, \Theta], \langle S_\lambda \rangle_{\lambda < \Lambda} \rangle$ be polytopologies and define $\Xi \otimes \Theta$ be its complement; we will call $\mathfrak{X} \otimes \mathfrak{Y}$ equipped with $\mathcal{A} \otimes \mathcal{B}$ an $\alpha$-product. Let $\frac{\mathfrak{X}}{\mathfrak{Y}} = \alpha$. Note that the term $\frac{\mathfrak{X}}{\mathfrak{Y}}$ is undefined but $\frac{\mathfrak{X}}{\mathfrak{Y}} = \frac{\mathfrak{X}}{\mathfrak{Y}}$ is the only element of $\mathfrak{X} \otimes \mathfrak{Y}$.

Example 10.1. Let $\Xi = \omega^2$ and $\Theta = \omega + 1$. Then, $\Xi \otimes \Theta = \omega^3 + \omega^2$. Elements of $G_1$ are those of the form $(1 + \Xi)\alpha$ with $\alpha \in \text{Lim}$, i.e. those of the form $\omega^3 \gamma$.

Let us compute some projections of specific elements.

- $3 \notin G_1$, and $3 = -1 + (1 + \omega^2)0 + 1 + 3$, so $\pi_0 3 = 3$ and $\pi_1 3 = -1 + 0 + 1 = 0$.
- $\omega^2 + 3 = -1 + (1 + \omega^2)1 + 1 + 2$ and hence $\pi_0 (\omega^2 + \omega) = 2$ whereas $\pi_1 \omega = -1 + 1 + 1 = 1$.
- $\omega^3 + \omega + 3 = -1 + (1 + \omega^2)\omega + 1 + \omega + 3$, so we have $\pi_0 (\omega^3 + \omega + 3) = \omega + 3$ and $\pi_1 (\omega^3 + \omega + 3) = -1 + \omega + 1 = \omega + 1$.
- $\omega^3 = (1 + \omega^2)\omega$ is the only element of $G_1$, so $\pi_0 (\omega^3)$ is undefined but $\pi_0 (\omega^3) = \omega$.

Although the projections are not injective, they have natural bijective restrictions.

Lemma 10.4. Let $\Theta, \Xi$ be ordinals. The restrictions $\pi_0 : \pi_1^{-1}(\alpha) \to \Xi + 1$ with $\alpha \in [0, \Theta] \setminus \text{Lim}$ and $\pi_1 : G_1 \to [0, \Theta] \cap \text{Lim}$ are strictly increasing and onto.
**Proof.** Pick \( \alpha \in [0, \Theta) \setminus \text{Lim} \). Let \( \beta = \alpha \) if \( \alpha \) is finite or \( \beta + 1 = \alpha \) if \( \alpha \) is infinite, and let \( \gamma = -1 + \beta + 1 \). By observation on the definition of \( \pi_1, \pi_1^{-1}\{\alpha\} \) is an interval of the form \( J_\alpha = [\gamma, \gamma + \Xi] \). On this interval, \( \pi_0(\xi) = -\gamma + \xi \), which is clearly increasing and onto \([0, \Xi]\).

For the second claim, given \( \xi \in [0, \Xi \otimes_d \Theta] \), \( \xi \in G_1 \) if and only if \( \pi_1 \xi \in \text{Lim} \), and indeed \( \pi_1 \upharpoonright G_1 \) is given by the map \( (1 + \Xi) \alpha \mapsto \alpha \) which is increasing and onto \([0, \Theta] \cap \text{Lim} \). \( \square \)

It will also be useful to see how the projections treat hyperlogarithms.

**Lemma 10.5.** Let \( \Xi, \Theta \) be ordinals, and write \( 1 + \Xi = \omega^\alpha + \beta \) with \( \beta < 1 + \Xi \). Let \( G_0, G_1 \) be the components of \([0, \Xi \otimes_d \Theta]\).

Then, for \( \xi \in G_0 \) we have that \( \ell \pi_0 \xi = \ell \xi \), whereas for \( \xi \in G_1 \) we have that \( \ell \xi = \alpha + \ell \pi_1 \xi \).

Moreover, for \( \lambda > 1 \), we have that \( \ell^\lambda \pi_0 \xi = \ell^\lambda \xi \), whereas for \( \xi \in G_1 \) we have that \( \ell^\lambda \xi = \ell^\lambda \pi_1 \xi \).

**Proof.** Note that the claim for \( \lambda > 1 \) is an immediate consequence of the first claim, so we focus on \( \lambda = 1 \).

An ordinal \( \xi \in G_0 \) is of the form \( -1 + (1 + \Xi) \alpha + 1 + \pi_0 \xi \), and hence \( \ell \xi = \ell \pi_0 \xi \); note that this holds even when \( \pi_0 \xi = 0 \), in which case \( \ell \xi = \ell \pi_0 \xi = 0 \). Any ordinal \( \xi \in G_1 \) is of the form \( (1 + \Xi) \omega(1 + \zeta) \), with \( \omega(1 + \zeta) \leq 1 + \Theta \). Write \( 1 + \zeta = \gamma + \omega^\delta \). Then, we see that

\[
\xi = (\omega^\alpha + \beta) \omega(\gamma + \omega^\delta) = \omega^{\alpha+1} \gamma + \omega^{\alpha+1+\delta}.
\]

Hence \( \ell \xi = \alpha + 1 + \delta \). However, \( \pi_1 \xi = \omega \gamma + \omega^{1+\delta} \), and thus \( \ell \pi_1 \xi = 1 + \delta \). The result follows. \( \square \)

We are now ready to define the \( d \)-product of spaces:

**Definition 10.2** (\( d \)-product). Let \( \mathfrak{X} = (\Xi+1, (T_\lambda)_{\lambda < \Lambda}) \) and \( \mathfrak{Y} = (\Theta+1, (S_\lambda)_{\lambda < \Lambda}) \) be polytopologies.

For \( \lambda < \Lambda \), define a topology \( \mathcal{O}_\lambda \) on \([0, \Xi \otimes_d \Theta]\) to be generated by sets of the forms

- \( \pi_0^{-1} U \cap \pi_1^{-1}\{\alpha\} \), where \( U \subseteq [0, \Xi] \) is \( \lambda \)-open and \( \alpha \leq \eta \) is finite or a successor, or
- \( \pi_1^{-1} U \), where \( U \subseteq [0, \Theta] \) is \( \lambda \)-open.

We denote the resulting space \( ([0, \Xi \otimes_d \Theta], (\mathcal{O}_\lambda)_{\lambda < \Lambda}) \) by \( \mathfrak{X} \otimes_d \mathfrak{Y} \).

This definition will be sufficient for our purposes, but the \( d \)-product of spaces is treated with much more generality and detail in [3], which gives it a slightly different presentation. There, the following properties are established:

**Lemma 10.6.** If \( \mathfrak{X} = ([0, \Xi], (T_\lambda)_{\lambda < \Lambda}) \) and \( \mathfrak{Y} = ([0, \Theta], (S_\lambda)_{\lambda < \Lambda}) \) are regular polytopologies and \( \mathfrak{Z} = \mathfrak{X} \otimes_d \mathfrak{Y} \), then

1. \( G_0 \) is 0-open and \( G_1 \) is 1-open,
2. For every $\xi \in [0, \Xi]$, $\pi_0^{-1}\xi$ is 0-dense in $G_1$.

3. $A \subseteq G_1$ is open in $(T_0 \otimes_d S_0) \upharpoonright G_1$ (i.e., in the subspace topology) if and only if $G_0 \cup A$ is open in $T_0 \otimes_d S_0$.

Proof. It is easy to see that $G_0$ is 0-open, since

$$G_0 = \bigcup_{\alpha \in [0, \Theta] \setminus \text{Lim}} \pi_0^{-1}\{0, \eta\} \cap \pi_1^{-1}\{\alpha\}. $$

$G_1$ is 1-open since $G_1 = \pi_1^{-1}(1, \Theta)$.

To see that $\pi_0^{-1}\xi$ is 0-dense in $G_1$, pick $\zeta \in G_1$. A basic neighborhood of $\zeta$ is of the form $\pi_1^{-1}U$ with $U \subseteq [0, \Theta]$ 0-open. By Lemma 5.3, $U$ contains an element $\delta$ with rank 0, i.e. zero or a successor. Then, by Lemma 10.4, $\pi_0\pi_1^{-1}\delta = [0, \Xi]$, i.e. $\xi \in \pi_0\pi_1^{-1}\delta$ or, equivalently, $\emptyset \neq \pi_0^{-1}\xi \cap \delta \subseteq \pi_1^{-1}U$. Since $B$ was arbitrary, the result follows.

For the third claim, note that $A$ is 0-open in $G_1$ if and only if there is a 0-open set $U$ such that $U \cap G_1 = A$. But then we have that $G_0 \cup U = G_0 \cup (G_0 \cap U) \cup (G_1 \cap U) = G_0 \cup A$, and since it is a union of opens it must be open. Conversely, if $G_0 \cup U$ is 0-open, then $(G_0 \cup A) \cap G_1 = A$ and hence $A$ is 0-open in the subspace topology.

Lemma 10.7. If $\mathfrak{X} = \langle [0, \eta], (\mathfrak{T}_\lambda)_{\lambda < \Lambda} \rangle$ and $\mathfrak{Y} = \langle [0, \Theta], (\mathfrak{S}_\lambda)_{\lambda < \Lambda} \rangle$ are regular polytopologies and $\mathfrak{Z} = \mathfrak{X} \otimes_d \mathfrak{Y}$, then

1. For all $\lambda < \Lambda$, $\pi_0: (3_\lambda \upharpoonright G_0) \to \mathfrak{X}_\lambda$ is a d-map.

2. For all $\lambda < \Lambda$ and $\alpha \in [0, \Theta] \setminus \text{Lim}$, $\pi_0: (3_\lambda \upharpoonright \pi_1^{-1}\alpha) \to \mathfrak{X}_\lambda$ is a homeomorphism.

3. $\pi_1$ is $\lambda$-continuous and $\lambda$-open.

4. $\pi_1 \upharpoonright G_1: (3_\lambda \upharpoonright G_1) \to (\mathfrak{Y}_\lambda \upharpoonright \text{Lim})$ is a homeomorphism.

Proof. Items 1 and 3 are proven by showing that images of basic opens are open, as are preimages of basic opens. We will not provide the details, but only show as an example that $\pi_0: (3_\lambda \upharpoonright G_0) \to \mathfrak{X}_\lambda$ is open. The $\lambda$-topology on $\mathfrak{Z}$ is generated by sets of the form $\pi_0^{-1}V \cap \pi_1^{-1}\alpha$ with $\alpha \in [0, \Theta] \setminus \text{Lim}$ and $V \subseteq [0, \Xi]$ $\lambda$-open, or $\pi_1^{-1}U$ with $U \subseteq [0, \Theta]$ $\lambda$-open. By Lemma 10.4, $\pi_0 \upharpoonright \pi_1^{-1}\alpha$ is a bijection and hence $\pi_0(\pi_0^{-1}V \cap \pi_1^{-1}\alpha) = V$, which is open. In the second case, $U$ either contains some $\alpha \in [0, \Theta] \setminus \text{Lim}$, or it does not. If it does, $\pi_0(U \cap G_0) = [0, \Theta]$, and if not, $\pi_0(U \cap G_0) = \emptyset$, both of which are open. Note that the fact that $\pi_0$ is pointwise discrete follows also by the injectivity of $\pi_0 \upharpoonright \pi_1^{-1}\alpha$, as given $\zeta \in \pi_0^{-1}\xi$, $\zeta$ is the only point of $\pi_0^{-1}\xi$ contained in the open set $\pi_0^{-1}[0, \Xi] \cap \pi_1^{-1}\pi_1\xi$.

Items 2 and 4 follow from Items 2 and 3, respectively, and Lemma 10.4, $\pi_0 \upharpoonright \pi_1^{-1}\alpha$ is a restriction of a continuous and open map to an open set, hence it is continuous and open, and as it is bijective, it is a homeomorphism. Since $G_1 = \pi_1^{-1}([0, \Theta] \cap \text{Lim})$, we have that $\pi_1 \upharpoonright G_1 = \pi_1 \upharpoonright \pi_1^{-1}([0, \Theta] \cap \text{Lim})$ is continuous and open, and hence, being bijective, a homeomorphism. □

In general, if $f: X \to Y$ is a continuous (open) function and $A \subseteq Y$, then $f \upharpoonright f^{-1}A$ is continuous (open).
Lemma 10.8. Suppose that \( X \) and \( Y \) are regular polytopologies and \( \mathfrak{I} = X \otimes_d Y \). If \( X, Y \) are both BG spaces, then so is \( \mathfrak{I} \). Similarly, if \( X, Y \) are shifted Icard spaces, then so is \( \mathfrak{I} \).

Proof. [5] Lemma 7.6] states that, if both \( X' \) and \( Y' \) are limit-maximal extensions of \( X \) and \( Y \), respectively, then \( X' \otimes_d Y' \) is a limit-maximal extension of \( X \otimes_d Y \). Hence in order to show that the \( d \)-product of BG-spaces is BG, it suffices to show that the \( d \)-product of their underlying Icard space is Icard.

So let \( \Xi, \Theta, \Lambda \) be ordinals, \( \Omega = \Xi \otimes_d \Theta \) and \( G_0, G_1 \) be the components of \( \Omega + 1 \) with projections \( \pi_0, \pi_1 \). Let \( \mathcal{O}_\lambda \otimes \mathcal{I}^\Theta_1 \otimes \mathcal{I}^\Xi_1 \) for each \( \lambda < \Theta \). We claim that \( \mathcal{O}_\lambda \otimes \mathcal{I}^\Xi_1 \) for all \( \lambda < \Lambda \).

We proceed by induction on \( \lambda \). The base case, when \( \lambda = 1 \), is proven in [5] Section 7]. For \( \lambda = \eta + 1 \) a successor, we have that \( \mathcal{I}^\Xi_1 = \mathcal{I}^\Xi_1 \otimes \mathcal{I}^{\Theta}_1 \). Thus we must show that \( \mathcal{O}_\eta \otimes \mathcal{I}^{\Omega+1}_1 \), under the hypothesis that \( \mathcal{O}_\eta \otimes \mathcal{I}^{\Omega+1}_1 \). To show that \( \mathcal{O}_\eta \subseteq \mathcal{I}^\Xi_1 \otimes \mathcal{I}^{\Theta}_1 \), it suffices to show that \( (\delta, \Omega) \subseteq \mathcal{I}^{\Xi+1}_1 \otimes \mathcal{I}^{\Theta+1}_1 \) for all \( \delta \).

First assume that \( \xi \in \mathcal{I}_G \cap (\delta, \Omega]_1 \). Then, by Lemma 10.3 we have that \( \ell^1 + \eta \xi = \ell^1 \eta + \xi \) and thus

\[
\xi \in \pi^{-1}_0(\delta, \Xi]_1 \cap \pi^{-1}_0 \pi_1 \xi \subseteq (\delta, \Omega + 1)_1.
\]

and \( \pi^{-1}_0(\delta, \Xi]_1 \cap \pi^{-1}_0 \pi_1 \xi \subseteq \mathcal{I}_G \). If \( \xi \in G_1 \), we consider two subcases. If \( \eta = 0 \) then write \( 1 + \Xi = \omega^\alpha + \beta \). Without loss of generality we may assume that \( \delta \geq \alpha \), for otherwise we have that \( (\alpha, \Omega + 1]_1 \subseteq (\delta, \Omega + 1)_1 \). Then, we have by Lemma 10.3 that \( (\delta, \Omega + 1)_1 = \pi^{-1}_1(-\alpha + \delta, \Theta + 1)_1 \). The case where \( \eta > 0 \) is similar, but here we have simply that

\[
\xi \in G_1 \cap \pi^{-1}_1(\delta, \Theta + 1)_1 \subseteq (\delta, \Omega + 1)_1.
\]

Since \( \Omega + 1 = G_0 \cup G_1 \), we conclude that \( (\delta, \Omega + 1)_1 \subseteq \mathcal{I}_G \).

For the other inclusion, we note that \( \mathcal{I}^{\Xi+1}_1 \otimes \mathcal{I}^{\Theta+1}_1 \) is generated by \( \mathcal{I}^{\Xi+1}_1 \otimes \mathcal{I}^{\Theta+1}_1 \) and sets of the form \( \pi^{-1}_0(\delta, \Xi]_1 \cap \pi^{-1}_1 \theta \) for \( \theta \in (0, \Theta] \setminus \text{Lim} \), or of the form \( \pi^{-1}_1(\delta, \Theta]_1 \). It remains to check that they are both open in \( \mathcal{O}_\eta \). But in the first case we have that

\[
\pi^{-1}_0(\delta, \Xi]_1 \cap \pi^{-1}_1 \theta = (\delta, \Omega]_1 \cap \pi^{-1}_1 \theta,
\]

which is an intersection of opens in \( \mathcal{O}_\eta \) and hence open, whereas for \( \xi \in \pi^{-1}_1(\delta, \Theta]_1 \) we have that, if \( \xi \in G_0 \), then \( \pi^{-1}_0 \pi^{-1}_1 \xi \subseteq \pi^{-1}_0 \pi^{-1}_1 \xi \subseteq (\delta, \Omega]_1 \) and \( \pi^{-1}_0 \pi^{-1}_1 \xi \subseteq \mathcal{O}_\eta \), whereas if \( \xi \in G_1 \), then

\[
G_1 \cap \pi^{-1}_1(\gamma + \delta, \Omega]_1 \subseteq \pi^{-1}_1(\delta, \Theta]_1,
\]

where \( \gamma = \alpha \) if \( \eta = 0 \) and \( \gamma = 0 \) if \( \eta > 0 \).

Finally, we consider the case where \( \lambda \in \text{Lim} \). But here it is very easy to check that

\[
\mathcal{O}_\lambda = \bigcup_{\eta < \lambda} \mathcal{I}^{\Xi+1}_1 \otimes \mathcal{I}^{\Theta+1}_1 = \bigcup_{\eta < \lambda} \mathcal{I}^{\Theta+1}_1 = \mathcal{I}^{\Theta+1}_1.
\]
With these ingredients, we define the \textit{d-product} of algebras:

**Definition 10.3 (d-product of algebras).** Given \(d\)-algebras \(A, B\) based on \(\Xi + 1, \Theta + 1\) and letting \(G_0, G_1\) be the components of \([0, \Xi \otimes_d \Theta]\), we define \(A \otimes_d B\) to be the algebra of all sets \(S\) of the form

\[
S = \pi_0^{-1}(S_0) \cup \pi_1^{-1}(S_1 \cap G_1),
\]

where \(S_0 \in A\) and \(S_1 \in B\).

Of course, we would like for the \(d\)-product of algebras to be itself a \(d\)-algebra. The next lemma will be useful in showing this.

**Lemma 10.9.** Let \(\mathfrak{X}, \mathfrak{Y}\) be polytopologies based on ordinals \(\Xi + 1, \Theta + 1\), respectively. Let \(d_{\xi}\) denote the \(\xi\)-derived set operator on \(\mathfrak{Y}\) and \(d'_{\xi}\) on \(\mathfrak{X} \otimes_d \mathfrak{Y}\).

Then, for any \(E \subseteq i_1[0, \Theta]\) and \(\lambda < \Lambda\), \(d'_{\xi} \pi_1^{-1} E = \pi_1^{-1} d_\lambda E\).

**Proof.** Assume first that \(\xi \in d'_{\xi} \pi_1^{-1}(E)\). Note that this immediately implies that \(\xi \not\in G_0\), since the latter is 0-open.

Then, for every \(\lambda\)-neighborhood \(U\) of \(\xi\) there is \(\zeta \neq \xi \in \pi_1^{-1}(E) \cap U\). Now, if \(V\) is a \(\lambda\)-neighborhood of \(\pi_1 \xi\) in \(\Theta + 1\), then \(\pi_1^{-1}(V)\) is a \(\lambda\)-neighborhood of \(\xi\), so that there is \(\zeta \neq \xi \in \pi_1^{-1}(E) \cap \pi_1^{-1}(V)\), hence \(\pi_1 \zeta \in E \cap V\). Further, by Lemma 10.3, \(\pi_1\) is injective on \(G_1 = \pi_1^{-1}(0, \Theta)\) so \(\pi_1 \zeta \neq \pi_1 \xi\). Since \(V\) was arbitrary, it follows that \(\pi_1 \xi \in d_\lambda E\).

Conversely, if \(\pi_1 \xi \in d_\lambda E\), then every \(\lambda\)-neighborhood \(V\) of \(\pi_1 \xi\) contains some \(\zeta \neq \xi \in E\). Consider a \(\lambda\)-neighborhood \(U = \pi_1^{-1} V\) of \(\xi\), where \(V \subseteq [0, \Theta]\) is \(\lambda\)-open. Since \(\pi_1 \xi \in d_\lambda E\) there is some \(\zeta \in E \cap V\) with \(\zeta \neq \xi\) and since \(\pi_1\) is onto we have that \(\zeta = \pi_1 \zeta'\) for some (unique) \(\zeta' \in G_1\) and thus \(\zeta' \in U \cap \pi_1^{-1} E\). Since \(U\) was arbitrary and \(\zeta' \neq \xi\), we conclude that \(\xi \in d'_{\xi} \pi_1^{-1} E\).

**Lemma 10.10.** Suppose \(\mathfrak{X} = \langle [0, \Xi], \mathfrak{T}, A \rangle\) and \(\mathfrak{Y} = \langle [0, \Theta], \mathfrak{S}, B \rangle\) are ambiences. Then, \(\mathfrak{X} \otimes_d \mathfrak{Y}\) equipped with \(A \otimes_d B\) is also an ambience. Further, if both ambiences are idyllic, then so is the corresponding product.

**Proof.** Let \(S = \pi_0^{-1}(S_0) \cup \pi_1^{-1}(S_1 \cap G_1)\) in \(A \otimes_d B\) and \(\lambda < \Lambda\). Let us check that \(d_\lambda S \in A \otimes_d B\).

If \(\lambda = 0\) and \(S_0 \neq \emptyset\), then by Lemma 10.6, \(\pi_0^{-1} S_0\) is 0-dense in \(G_1\) and thus \(G_1 \subseteq d_\lambda S_0\).

Meanwhile, \(G_0\) is 0-open and \(\pi_0\) is a \(d\)-map so for \(\xi \in G_0, \xi \in d_\lambda S\) if and only if \(\pi_0 \xi \in d_\lambda S_0\), and we conclude that

\[
d_\lambda S = \pi_0^{-1} d_\lambda S_0 \cup G_1 \in A \otimes_d B.
\]

By similar reasoning,

\[
i_1 S = \pi_0^{-1} i_1 S_0 \cup G_1 \in A \otimes_d B,
\]

and if the original structures are idyllic this is evidently equal to \(d_\lambda S\) as well.
Now suppose $\lambda = 0$ and $S_0 = \emptyset$. In this case, $S = \pi_1^{-1}(S_1 \cap G_1)$ and, by Lemma 10.11, $d_\lambda S = \pi_1^{-1}d_\lambda S_1 \in A \otimes d B$. If the original algebras were idyllic, we note further that

$$d_0 S = \pi_1^{-1}d_0 S_1 = \pi_1^{-1}i_1 S_1 = i_1 S.$$ 

Finally, if $\lambda > 0$, then both projections are $d$-maps with respect to the $\lambda$-topology and

$$d_\lambda S = d_\lambda\pi_0^{-1}S_0 \cup d_\lambda\pi_1^{-1}(S_1 \cap G_1) = \pi_0^{-1}d_\lambda S_0 \cup \pi_1^{-1}d_\lambda(S_1 \cap G_1),$$

with the analogous equalities holding for $i_{1+\lambda} S$, from which all required claims follow easily.

With this we conclude the topological constructions we shall need. Now, we turn to the last ingredient in the completeness proof: the modal logic $J$.

11 The logic $J$

As we have seen, $\text{GLP}_{\Lambda}$ has no non-trivial Kripke frames. In order to work around this issue, we pass to a weaker logic, Beklemishev’s $J$. This was introduced in [2] and here we only review the necessary results without proof. For this logic we shall only use modalities $n < \omega$ and replace Axiom 4 of $\text{GLP}_{\Lambda}$ by the two axioms

6. $[n] \phi \rightarrow [m][n] \phi$, for $n \leq m$ and

7. $[n] \phi \rightarrow [n][m] \phi$, for $n < m$.

The logic $J$ is sound and complete for the class of finite Kripke models $\langle W, (\prec_n)_{n < N}, \llbracket \cdot \rrbracket \rangle$ such that

1. the relations $\prec_n$ are transitive and well-founded,

2. if $n < m$ and $w \prec_m v$ then $\prec_n(w) = \prec_n(v)$ and

3. if $n < m$ then $w \prec_m v \prec_n u$ implies that $w \prec_n u$.

Here, $\prec_n(w) = \{ v : v \prec_n w \}$. It will also be convenient to define $w \prec_n v$ if for some $m \geq n$, $w \prec_m v$. Let $\sim_n$ denote the symmetric, transitive, reflexive closure of $\prec_n$ and let $[w]_n$ denote the equivalence class of $w$ under $\sim_n$. Define $[w]_{n+1} \prec_n [v]_{n+1}$ if there exist $w' \in [w]_{n+1}, v' \in [v]_{n+1}$ such that $w' \prec_n v'$.

Then, say $W$ is tree-like if

1. for each $w \in W$ and $n \leq N$, $[w]_n/\sim_{n+1}$ is a tree under $\prec_n$ and

2. if $[w]_{n+1} \prec_n [v]_{n+1}$ then $w \prec_n v$.

With this we may state the following completeness result from [2]:

**Lemma 11.1.** Any $J$-consistent formula can be satisfied on a finite, tree-like $J$-frame.
Thus if we can reduce $\text{GLP}_\Lambda$ to $J$, we will immediately obtain finite Kripke models. For this, given a formula $\phi$, let $N$ be the largest modality appearing in $\phi$ and define

$$M(\phi) = \bigwedge_{n \in \text{sub}(\phi)\ n < m \leq N} [n]\psi \rightarrow [m]\psi.$$ 

Then we set $M^+(\phi) = M(\phi) \land \bigwedge_{n < N} [n]M(\phi)$.

The following is also proven in [2]:

**Lemma 11.2.** For any formula $\phi \in L_\omega$, $\text{GLP}_\omega \vdash \phi$ if and only if $J \vdash M^+(\phi) \rightarrow \phi$.

To prove completeness, it then suffices to construct a $J$-model of a given formula and then “pull back” the valuations onto a topological model. Hence it is important to identify the appropriate maps for such pullbacks.

First observe that a partially ordered set $\langle W, < \rangle$ can be identified with a topological space by letting $U \subseteq W$ be open if, whenever $v < w$ and $w \in U$, it follows that $v \in U$. If $\mathfrak{M} = \langle W, \langle < \rangle_{n \leq N} \rangle$ is a $J$-frame, we will let $W_n$ be the topological space associated to $\langle W, < \rangle_n$. Below, we say $w \in W$ is a **hereditary $n$-root** if $w$ is $<k$-maximal for all $k \geq n$.

**Definition 11.1.** Let $\mathcal{X} = \langle X, (T_n)_{n \leq N} \rangle$ be a polytopological space and $\mathfrak{M} = \langle W, \langle < \rangle_{n \leq N} \rangle$ a $J$-frame.

A function $f : \mathcal{X} \rightarrow \mathfrak{M}$ is a $J$-map if

1. $f : X_N \rightarrow W_N$ is a $d$-map
2. $f : X_n \rightarrow W_n$ is open for $n < N$
3. if $n < N$ and $w$ is a hereditary $(n + 1)$-root then $f^{-1}(\ll_n(w))$ and $f^{-1}(\ll_n(w) \cup \{w\})$ are $n$-open and
4. if $n < N$ and $w$ is a hereditary $(n + 1)$-root then $f^{-1}(w)$ is $n$-discrete.

**Example 11.1.** Consider a simple $J$-frame $\mathfrak{M}$ with three worlds, $u, v, w$, such that $u <_0 w$ and $v <_0 w$. Let us define a $J$-map $f : (\omega + 1) \rightarrow \mathfrak{M}$.

The worlds $u, v$ are isolated, and hence we must have that $\xi$ is isolated whenever $f(\xi) \in \{u, v\}$. It follows that $f^{-1}\{u, v\} = \{0, \omega\}$. Meanwhile, the only possible value for $f(\omega)$ is $w$ as it is the only point that is not isolated and $d$-maps preserve rank.

Now, notice that any neighborhood of $w$ contains both $u$ and $v$. Thus we will need for every neighborhood of $\omega$ to intersect both $f^{-1}(u)$ and $f^{-1}(v)$. A simple
way to achieve this is to let $f^{-1}(u)$ be the set of even numbers and $f^{-1}(v)$ the set of odds. Thus we may define

$$f(\xi) = \begin{cases} u & \text{if } \xi = 2n < \omega, \\ v & \text{if } \xi = 2n + 1 < \omega, \\ w & \text{if } \xi = \omega. \end{cases}$$

One can then easily check that the function $f$ thus defined is in fact a $J$-map.

With this we have the following, proven in [5]:

**Lemma 11.3.** If $\mathfrak{M} = \langle W, \langle <_n \rangle_{n \leq N}, \llbracket \cdot \rrbracket \rangle$ is a $J$-model such that $M^+(\varphi)$ is valid on $\mathfrak{M}$, $\mathfrak{X} = \langle X, \langle T_\xi \rangle_{\xi < A} \rangle_{n \leq N}$ is a GLP-space and $f : \mathfrak{X} \to \mathfrak{M}$ a $J$-map, then there is a valuation $\llbracket \cdot \rrbracket$ on $\mathfrak{X}$ such that $\llbracket \psi \rrbracket = f^{-1}(\llbracket \psi \rrbracket)$ for all $\psi \in \text{sub}(\varphi)$.

We conclude with a simple observation, also established in [5]:

**Lemma 11.4.** Let $X, \mathfrak{Y}$ be polytopological spaces and $W$ a $J$-frame.

Then, if $f : \mathfrak{X} \to \mathfrak{Y}$ is a d-map and $g : \mathfrak{Y} \to \mathfrak{W}$ is a $J$-map it follows that $gf$ is a $J$-map.

In the next section we shall exploit the completeness of the logic $J$ for finite frames together with Lemma 11.3 to construct GLP-ambiances satisfying any consistent formula.

12 Completeness

Given an increasing sequence of ordinals $\vec{\lambda} = \langle \lambda_n \rangle_{n \leq N}$, a polytopological space $\langle X, \langle T_\xi \rangle_{\xi < A} \rangle$ and a J-frame $\langle W, \langle <_n \rangle_{n \leq N} \rangle$, we will say a map $f : X \to W$ is a $\vec{\lambda}$-map if it is a $J$-map on $\langle X, \langle T_\lambda \rangle_{n \leq N} \rangle$. These maps will allow us to focus on finitely many modalities at one time in the completeness proof.

Say a $\vec{\lambda}$-map is suitable if it is a surjective function of the form $f : (\Theta + 1) \to W$, for the 0-root $w_0$ of $W$ we have $f^{-1}(w_0) = \{ \Theta \}$ and $\Theta$ lies in the range of $e$ (i.e., $\Theta = 0$ or it is infinite and additively indecomposable). We use $\text{hgt}(<_n)$ to denote the height of $<_n$, that is, the maximal $k$ such that there exist $w_0 <_n w_1 <_n \cdots <_n w_k$.

**Lemma 12.1.** Given a J-frame $\mathfrak{W} = \langle W, \langle <_n \rangle_{n \leq N} \rangle$ and ordinals $\vec{\lambda} = \langle \lambda_n \rangle_{n \leq N}$ all less than $\Lambda$, there exists an idyllic $\Lambda$-ambiance $\mathfrak{X}$ based on some $\Theta < e^{1+\Lambda}1$ and a suitable $\vec{\lambda}$-map $f : \mathfrak{X} \to \mathfrak{W}$.

**Proof.** We proceed as in the proof of an analogous result in [5].

Suppose $\langle W, \langle <_n \rangle_{n \leq N} \rangle$ is a J-frame and $\lambda_0, \ldots, \lambda_N$ are ordinals. We work by induction on $N$ with a secondary induction on $\text{hgt}(<_0)$ to construct a suitable $\vec{\lambda}$-map. Without loss of generality, we assume $\lambda_0 = 0$, for otherwise we can always let $<_0 = \emptyset$. 

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Case 1: \( N = 0 \). In this case it is known that there is an ordinal \( \Theta = e^\Theta' < \omega^\omega = e(\omega) \) and a suitable \( \ell \)-map \( g : (\Theta + 1)_1 \to W \) (see [7]). Note that since \( \Theta < \omega^\omega \), it has no points of limit rank and hence the interval topology is already limit-maximal so that \( P(\Theta + 1) \) is an idyllic algebra.

Case 2: \( \lambda^0 = 0 \). Here we have that \( <_0 = \emptyset \). For \( 0 < n \leq N \) let \( \lambda_n' = -\lambda_1 + \lambda_n \) and consider the J-frame \( (W, \langle <_{n+1} \rangle_{0 \leq n < N}) \), where \( \langle 0 \rangle \) has been removed. By induction on \( N \) we may assume there is an idyllic ambient \( X \) based on a polytopology \( T \) on an ordinal \( \Theta + 1 < e^{1+(-\lambda_1+\lambda)} \) and a suitable \( \lambda' \)-map \( g \) from \( X \) onto \( W \).

Let \( \Omega = e^{1+\lambda_1} \theta \) and use Lemma 8.3 to construct a BG-polytopology
\[
\mathcal{G} = (\Omega + 1, \langle S_\lambda \rangle_{\lambda < \lambda_1}.
\]
Note that from \( \Theta + 1 < e^{1+(-\lambda_1+\lambda)} \) we obtain \( f \theta < e^{-\lambda_1+\lambda} \) and thus
\[
e^{1+\lambda_1} \theta < e^{1+\lambda_1} e^{-\lambda_1+\lambda} + 1 = e^{1+\lambda_1} + 1.
\]

Then, by Theorem 9.1 there is a \( \Lambda \)-reductive map
\[
f = e^{f \theta} : (\Omega + 1)_{1+\lambda_1} \to (\Theta + 1)_1,
\]
so that by Lemma 9.3 \( f : \mathcal{G}_\lambda \to X_0 \) is a \( \lambda \)-map. By Lemma 8.1 there exists a BG-space \( \mathcal{G}_\lambda \) extending \( \mathcal{G}_\lambda \) such that \( f : \mathcal{G}_\lambda \to X_0 \) is a \( \lambda \)-map.

Hence we may use Lemma 8.3 to define BG-topologies \( \langle S_\zeta \rangle_{\lambda < \zeta < \lambda_1} \) on \( \Omega + 1 \) such that \( f : \mathcal{G}_\lambda \to X_\zeta \) is a \( \lambda \)-map for all \( \zeta \); in particular, for each \( n \in (0, N] \) we have that \( f : \mathcal{G}_\lambda \to X_\lambda' \) is a \( \lambda \)-map.

It follows that \( f : \mathcal{G} \to X \) is a \( \lambda \)-lift, and by Lemma 11.1 \( f^{-1}A \) is an idyllic \( \lambda \)-algebra. Further, by Lemma 11.4 we know that \( g f : \mathcal{G} \to W \) is a suitable \( \lambda \)-map, as needed.

Case 3: \( \lambda^0 = m > 0 \). Let \( w \) be the 0-root of \( W \) and \( w_0, \ldots, w_K \) be its \( <_0 \)-daughters which are hereditary 1-roots.

Let \( V = \{ w \} \cup \{ w \} \) and \( W_k = \langle 0, w_k \rangle \). Then we have, as in Case 2, an idyllic ambient \( X \) based on an ordinal \( \Theta + 1 < e^{1+\lambda_1} \) and by induction on \( N \) a \( \lambda' \)-map \( f \) from \( \Theta + 1 \) to \( V \), as well as for each \( k \leq K \) an idyllic ambient \( \mathcal{G}_k \) based on an ordinal \( \Xi_k + 1 < e^{1+\lambda_1} \) and a suitable \( \lambda \)-map \( f_k \) from \( \Xi_k + 1 \) onto \( W_k \).

Let \( \Xi = (\Xi_0 + 1) + (\Xi_1 + 1) + \ldots + (\Xi_K + 1) + (\Xi_{K+1} + 1) + \xi \). Define \( g : \mathcal{G} \to W \) by
\[
g((\Xi_0 + 1) + \ldots + (\Xi_{K-1} + 1) + (\Xi_K + 1) + (\Xi_{K+1} + 1) + \xi) = f_k(\xi)
\]
and \( h : \mathcal{G} \to W \) by
\[
h(\xi) = \begin{cases} 
g_0(\xi) & \text{if } \xi \in G_0, \\
g_1(\xi) & \text{otherwise.} \end{cases}
\]
Note that $\Lambda > 0$ so that $e^{1+\Lambda}1$ is closed under sums and products and thus $\Xi \otimes_d \Theta < e^{1+\Lambda}1$. Further, by Lemma 10.10 3 is an idyllic ambiance.

Now, let us check that $h$ satisfies the conditions of Definition 11.1.

1. We know that $\lambda_N > 0$, since $N > 0$. Thus $G_0, G_1$ are both $\lambda_N$-clopen and hence it is enough to check that $h: (\mathfrak{Z}_{\lambda_N} \mid G_j) \to (W, \langle <, K \rangle)$ is a $d$-map for $j = 0, 1$. But this is immediate from the assumption that $f, g$ are $d$-maps, as are the respective projections.

2. Let $\lambda = \lambda_n, U$ be $\lambda$-open and $v \in h(U)$. We have that
\[ h(U) = g\pi_0(G_0 \cap U) \cup f\pi_1(G_1 \cap U). \]
First note that $G_0 \cap U$ is $\lambda$-open since $G_0$ is $0$-open, hence $g\pi_0(G_0 \cap U)$ is also $\lambda$-open given that $g\pi_0$ is a composition of $\lambda$-open maps. Meanwhile, for $\lambda > 1$, $G_1$ is also $\lambda$-open from which it follows that $f\pi_1(G_1 \cap U)$ is $\lambda$-open as well.

It remains to check that $h(U)$ contains a 0-neighborhood around any $v \in f\pi_1(G_1 \cap U)$. So suppose $u <_0 v$. Since $g$ is onto $\bigcup W_n$, there is some $\delta \leq \Xi$ such that $g(\delta) = u$. By Lemma 10.4.2 and using the fact that $G_1 \cap U \neq \emptyset$ (otherwise $v$ would not exist), there is $\gamma \in U \cap \pi_0^{-1}\delta$. It follows that $h(\gamma) = g\pi_0(\gamma) = u$, as desired.

3. If $n > 0$ and $v$ is an $n + 1$ root the claim follows form the assumption that $f, g$ were already $J$-maps and the respective projections are $d$-maps. Meanwhile, if $n = 0$ and $v <_0 w$, we may use the fact that $g$ is a $(\lambda_n)_{1 \leq n \leq N}$-map, since here it follows that $h^{-1}(\ll_0(v)) \subseteq G_0$ so that
\[ h^{-1}(\ll_0(v)) = (g\pi_0)^{-1}(\ll_0(v)). \]
But $g^{-1}(\ll_0(v))$ is a 0-open subset of $\Xi$, hence $(g\pi_0)^{-1}(\ll_0(v))$ is open in $\mathfrak{Z}$. Similarly for $\{v\} \cup \ll_0(v)$.

If $v <_1 w$, then $<_0(v) = _<0(w)$, and thus $h^{-1}(<_0(v)) = G_0$. But then we see that
\[ h^{-1}(<_0(v)) = G_0 \cup \pi_1^{-1}f^{-1}(\ll_0(v)); \]
since by assumption $f$ is a $J$-map, $f^{-1}(\ll_0(v))$ is 0-open in $\mathfrak{Z} \mid G_1$, and therefore by Lemma 10.6.3 $G_0 \cup \pi_1^{-1}f^{-1}(\ll_0(v))$ is 0-open in the $d$-product $\mathfrak{Z}$. The argument for $\{v\} \cup \ll_0(v)$ is analogous.

Finally, note that $f^{-1}(w) = \{\Xi \otimes_d \Theta\}$ so that $f^{-1}(\ll_0(w)) = W \setminus \{w\} = [0, \Xi \otimes_d \Theta]$, which is open, as is $f^{-1}(\{w\} \cup \ll_0(w)) = [0, \Xi \otimes_d \Theta]$.

4. For $w$ we have that $h^{-1}(w) = \{\Xi \otimes_d \Theta\}$, which is obviously $\lambda_n$-discrete for any $n$. If $v <_0 w$ then $h^{-1}(v) = (g\pi_0)^{-1}(v)$ which is discrete, as $g\pi_0$ is a $J$-map.

If $v <_1 w$ then $h^{-1}(v) = (f\pi_1)^{-1}(v)$, which similarly must be discrete. \[ \square \]

With this, we are ready to state and prove our main result.
Theorem 12.1. GLP_\Lambda is complete for the class of idyllic ambiances based on some \( \Theta < e^{1+\Lambda_1} \).

Proof. Suppose that \( \phi \) is consistent over GLP_\Lambda. Then, by Lemma 2.1, \( \phi^c \) is consistent over GLP_\Lambda. By Lemma 11.2, \( M^+(\phi) \land \phi \) is consistent over J, and thus by Lemma 11.1 we have a tree-like J-model \( (W, \langle < \rangle_{n \leq \Lambda}, \llbracket \cdot \rrbracket) \) satisfying \( M^+(\phi) \land \phi \). We may then use Lemma 12.1 to find an idyllic ambiance \( X \) based on an ordinal \( \Theta < e^{\Lambda_\omega} \) and a surjective \( \vec{\lambda} \)-map \( f : X \to W \).

Then, by Lemma 11.3, there is a valuation on \( \Theta \) agreeing with \( f^{-1}[\cdot] \) on \( \text{sub}(\phi) \). Since \( f \) is surjective, \( f^{-1}[\phi] \neq \emptyset \), and thus \( \phi \) is satisfied on \( X \), as desired.

As corollaries we get a sequence of completeness results:

Corollary 12.1. GLP_\Lambda is complete for both the class of BG-spaces and the class of shifted Icard ambiances based on \( e^{1+\Lambda_1} \).

Further, the variable-free fragment GLP_\Lambda^0 is complete for the class of simple Icard ambiances based on \( e^{\Lambda_1} \).

Proof. Completeness for BG-spaces and shifted Icard ambiances is immediate from Theorem 12.1 as idyllic ambiances may be seen as either kind of structure.

Meanwhile, given any Icard ambiance based on an algebra \( A \) and satisfying a closed formula \( \phi \), we use Lemma 7.2 to note that all valuations of \( \phi \) and its subformulas are simple, and hence we obtain a simple ambiance satisfying \( \phi \) by replacing \( A \) by the class of simple sets.

This gives us an ambiance on \( \mathfrak{S}^{e^{1+\Lambda_1}}_\Lambda \), the shifted Icard space. To pass to an ambiance based on \( \mathfrak{S}^{\Lambda_1}_\Lambda \), note that \( \ell : \mathfrak{S}^{e^{1+\Lambda_1}}_\Lambda \to \mathfrak{S}^{\Lambda_1}_\Lambda \) is a d-map and thus by an easy induction preserves valuations of closed formulas.

The completeness result for simple ambiances is not new, as it was already proven by Icard for GLP_\omega in [12] and by Joosten and I for arbitrary GLP_\Lambda in [9]. However, the current argument is quite different from those used in previous works.

13 Worms and the lower bound

As it turns out, our bound of \( e^{1+\Lambda_1} \) is sharp. To show this, let us consider worms.

A worm is a formula of the form

\[ \langle \lambda_0 \rangle \ldots \langle \lambda_I \rangle \top. \]

These formulas correspond to iterated consistency statements, and indeed can be used to study the proof-theoretic strength of many theories related to Peano Arithmetic, as Beklemishev has shown [1].
Worms are well-ordered by their consistency strength. Let us denote the set of worms with entries less than $\Lambda$ by $W^\Lambda$; then, given worms $v, w \in W^\Lambda$, define $v \prec w$ if $GLP_\Lambda \vdash w \rightarrow \Diamond v$.

The relation $\prec$ we have just defined is a well-order $\upuparrows$. Thus we may compute the order-type of a worm $w \in W^\Lambda$:

$$o(w) = \sup_{v < w} (o(v) + 1).$$

**Lemma 13.1.** Let $w$ be a worm, $X = \langle X, \langle T_\lambda \rangle_{\lambda < \Lambda}, [\cdot] \rangle$ be a GLP-model and $x \in X$.

Then, if $x \in [w]$, it follows that $\rho(x) \geq o(w)$.

**Proof.** By induction on $o(w)$. For the base case, note that if $o(w) = 0$ then we vacuously have that $x \in [w] \implies \rho(x) \geq 0$.

For the inductive step, if $v \prec w$ and $U$ is any neighborhood of $x$, since $\vdash w \rightarrow \Diamond v$, it follows that there is $y \in U$ satisfying $v$. By induction on $v \prec w$, we have that $\rho(y) \geq o(v)$.

We then see that

$$\rho(x) \geq \sup_{v < w} (o(v) + 1) = o(w).$$

It will be convenient to review the calculus for computing $o$ that is given in [10]. First, if $v = \langle \xi_1 \rangle \ldots \langle \xi_N \rangle \uparrow$ and $w = \langle \zeta_1 \rangle \ldots \langle \zeta_M \rangle \uparrow$, define

$$v \Diamond w = \langle \xi_1 \rangle \ldots \langle \xi_N \rangle \langle 0 \rangle \langle \zeta_1 \rangle \ldots \langle \zeta_M \rangle \uparrow.$$

Further, if $\alpha$ is any ordinal, set

$$\alpha \uparrow w = \langle \alpha + \zeta_1 \rangle \ldots \langle \alpha + \zeta_M \rangle \uparrow.$$

**Lemma 13.2.** Let $v, w$ be worms and $\alpha$ an ordinal.

Then,

$$o(\uparrow) = 0$$

$$o(v \Diamond w) = o(w) + 1 + o(v)$$

$$o(\alpha \uparrow w) = e^\alpha o(w).$$

**Example 13.1.** Consider $w = \langle \omega + 1 \rangle \langle \omega \rangle \langle \omega + 1 \rangle \uparrow$; let us compute $o(w)$. Note that $w = \omega \uparrow v$, where $v = \langle 1 \rangle \langle 0 \rangle \langle 1 \rangle \uparrow$, so that $o(w) = e^\omega o(v)$. Now, $o(v) = o(\langle 1 \rangle \uparrow) + 1 + o(\langle 1 \rangle \uparrow)$, and we have that $o(\langle 1 \rangle \uparrow) = o(\langle 1 \rangle \uparrow) = e^0 (\langle 0 \rangle \uparrow) = e^0 (\omega + 1 + 0) = \omega$, so that $o(v) = \omega + 1 + \omega = \omega + \omega$ and $o(w) = e^{\omega + \omega} = \varepsilon_{\omega + \omega}$.

**Theorem 13.1.** GLP$_\Lambda$ is incomplete for the class of BG-spaces or Icard ambiances based on any fixed $\Theta < e^{1+\Lambda 1}$. 

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Proof. Note that $\ell(e^{1+\Lambda}1) = e^\Lambda 1$. Now, if GLP$_\Lambda$ is complete for the class of models based on $\Theta$, in particular any worm $w \in W^\Lambda$ must be satisfiable on one such model $X$, which implies by Lemma 13.2 that there must be $\vartheta \in \Theta$ with $\ell(\vartheta) \geq o(w)$. Thus it suffices to show that

$$\sup_{w \in W^\Lambda} o(w) \geq e^\Lambda 1.$$

To do this, first assume $\Lambda = \lambda + 1$. Then we have that

$$o(\langle \lambda \rangle^n \top) = e^\lambda o(\langle 0 \rangle^n \top) = e^\lambda n.$$

The last equality is obtained by repeated applications of (5). But then, by Proposition 4.1.3 we see that

$$\rho(X) \geq \lim_{n \to \omega} e^\lambda n = e^\lambda \omega = e^\lambda 1 = e^{\lambda+1} 1.$$

Meanwhile, if $\Lambda \in \text{Lim}$,

$$\rho(X) \geq \sup_{\lambda < \Lambda} o(\langle \lambda \rangle \top) = \sup_{\lambda < \Lambda} e^\lambda 1 = e^\Lambda 1.$$

In either case $\rho(X) = \sup_{\vartheta \in \Theta} \ell\vartheta \geq e^\Lambda 1$, from which it follows by Lemma 4.2 that $\Theta \geq e^{1+\Lambda} 1$. □

14 Concluding remarks

The goal of this paper was essentially to answer two main questions. The first is perhaps not so much Is GLP$_\Lambda$ complete for its topological semantics independently of $\Lambda$? as, rather, What is needed to construct topological models of GLP$_\Lambda$? For this we had to introduce several tools that were not required in the case $\Lambda = \omega$. Most notable is the use of hyperlogarithms and $-exponents, already employed in [9] to study models of the closed fragment, and the addition of new $d$-maps to our toolkit. Aside from possible connections to proof theory, these are novel constructions in scattered topology and might spark some independent interest.

The second question is, Are there good constructive semantics for GLP$_\Lambda$? Icard ambiances are a possible answer to this question. Not only are the topologies easily definable, unlike the non-constructive BG-topologies, but if one analyzes the proof of Lemma 12.1 all sets that appear in valuations are constructive as well. As such, Icard ambiances may be well-suited for applications in the proof theory of systems much stronger than Peano Arithmetic – perhaps the ultimate motivation for contemporary work in provability logic.

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