Wavefunction of a Black Hole and the Dynamical Origin of Entropy

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Abstract

Recently [1, 2] it was proposed to explain the dynamical origin of the entropy of a black hole by identifying its dynamical degrees of freedom with states of quantum fields propagating in the black-hole’s interior. The present paper contains the further development of this approach. The no-boundary proposal (analogous to the Hartle-Hawking no-boundary proposal in quantum cosmology) is put forward for defining the wave function of a black hole. This wave function is a functional on the configuration space of physical fields (including the gravitational one) on the three-dimensional space with the Einstein-Rosen bridge topology. It is shown that in the limit of small perturbations on the Kruskal background geometry the no-boundary wave function coincides with the Hartle-Hawking vacuum state. The invariant definition of inside and outside modes is proposed. The density matrix describing the internal state of a black hole is obtained by averaging over the outside modes. This density matrix is used to define the entropy of a black hole, which is to be divergent. It is argued that the quantum fluctuations of the horizon which are internally present in the proposed formalism may give the necessary cut-off and provide a black hole with the finite entropy.

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1. Introduction

According to the thermodynamical analogy in black hole physics, the entropy of a black hole is defined as $S^H = A^H/(4l_P^2)$, where $A^H$ is the area of a black hole surface and $l_P = m_P^{-1} = (\hbar G/c^3)^{1/2}$ is the Planck length [3, 4]. The Hawking discovery [5, 6] of the black hole thermal radiation confirmed the status of thermal properties of a black hole. Four laws of black hole physics formulated in [7] show that a black hole can be considered as a thermodynamical system and the entropy of a black hole plays essentially the same role as the entropy in the 'usual' physics, e.g., it shows up to which extent the energy contained in a black hole can be used to produce work. The generalized second law (i.e. the statement that the sum $S = S^H + S^m$ of a black hole entropy and the entropy $S^m$ of the outside matter cannot decrease) implies that in the case when a black hole is part of a thermodynamical system the thermodynamical laws will be self-consistent only if the black hole entropy is considered on equal footing with the entropy of the 'usual' matter [3, 4, 8] (see also [9, 10, 11, 12] and references therein). Gedanken experiment proposed by York [13] in which a black hole is located inside a heated cavity gives a nice example showing that such parameters of a black hole as a heat capacity and entropy have a well defined physical meaning.

The formal derivation of thermal properties of a black hole is usually performed in the framework of the Euclidean approach initiated by Gibbons and Hawking [14, 15]. It implies the existence of the thermodynamical ensemble of black holes characterized by the canonical partition function at finite temperature $T = 1/\beta$

$$Z(\beta) = \text{Tr} e^{-\beta \hat{H}}, \quad (1.1)$$

where $\hat{H}$ is the Hamiltonian of the full gravitational system. The known functional representation of finite temperature field theory in terms of the Euclidean quantum theory, directly extrapolated to quantum gravity, allows one to rewrite (1.1) as a Euclidean path integral over 4-geometries and matter fields. The evaluation of this integral by the steepest descent method determines $Z(\beta)$ and, in particular, gives $T = 1/8\pi M$. A refined version of this approach which emphasizes the role of boundary conditions was developed in [16, 17, 18, 19]. In the framework of this approach the construction of the microcanonical partition function within the Lorentzian context was analysed and
a revised issue of stability for the gravitational ensemble at a given temperature and
given boundary quasilocal characteristics was given.

Although the Euclidean approach allows to obtain the correct value for the black
hole entropy, it does not elucidate the number of questions. Mainly this is a question
of the origin of the thermodynamical partition (1.1) which is assumed to be given for
granted. In other words, it does not specify the physical degrees of freedom inaccessible
for observation for an external observer, their tracing out in the pure quantum state
of the whole gravitational system leading to the loss of information, emergence of
entropy and the density matrix corresponding to (1.1). In particular, the conventional
Euclidean approach to gravitational thermodynamics simply does not leave room for
a black hole interior, for it is located completely outside of the real Euclidean section
of the complex Schwarzschild geometry.

Despite some promising attempts [4, 9, 13, 20, 21, 22], the dynamical (statistical
mechanical) origin of a black hole entropy has not been well understood. According to
the ‘standard’ interpretation, the entropy of a black hole is considered as a logarithm
of the number of distinct ways that the hole might have been made [9, 21]. This
interpretation is not satisfactory. Soon after the black hole formation neither external
nor internal observer can see or affect these states and hence it does not make sense
to interpret them as usual dynamical degrees of freedom which specify the state of
the system at the chosen moment of time. The problem of relation of the black hole
entropy to the loss of information about the initial state of a collapsing body is a part
of very important problem of information loss in the black hole evaporation [23]. We
shall not consider the problem of information loss in the present paper and restrict
ourselves to the problem of dynamical origin of a black hole entropy.

York [13] proposed to identify the dynamical degrees of freedom of a black hole
with its quasi-normal modes. But the entropy of the quasinormal modes excited at a
given moment of time is much smaller $S^H = A^H/(4l_0^2)$. In order to obtain the required
large number ‘t Hooft [22] proposed a ”brick wall model”. In this approach the entropy
of a black hole is identified with the entropy of a thermal gas located outside a black
hole and supported in equilibrium by a heated wall located at small distance outside
the horizon. The value of the gap parameter in this model is chosen by equating the
entropy of the gas outside the wall to the entropy of a black hole. The relation of the "brick wall" model to the results obtained from the first principles remains unclear.

Recently [1, 2] a new approach to the problem of black hole entropy was proposed. According to this approach the dynamical degrees of freedom of a black hole are identified with those modes of physical fields which are located inside the horizon and which cannot be seen by a distant observer. It was shown that the main contribution to the entropy is given by thermally excited ‘invisible’ modes propagating in the close vicinity of the horizon. The so defined one-loop entropy of a black hole is formally divergent and requires a cut-off [26]. This divergence is caused by a sharp boundary of the invisible region and it arises already in the similar flat spacetime calculations [29, 30]. The natural cut-off arises because of the quantum fluctuations of the horizon. A calculation based on a simple estimate of the horizon fluctuations [1, 2] yields a value of the entropy which is in good agreement with the usually adopted value $A_H/(4l_P^2)$.

There are two important problems which naturally arise in connection with these results. 1) How to generalize the calculation of the entropy in order to include the quantum fluctuations of the horizon in a self-consistent way? 2) How to combine the developed approach with the calculations of the black hole entropy based on the Euclidean space approach?

It looks like that it is impossible to solve these problems without developing the quantum scheme which includes the quantization of the gravitational field. In this paper we present an approach which might be regarded as an attempt to fill the gaps in the theoretical foundation of black-hole thermodynamics. It consists of i) the proposal for the pure quantum state of the black hole, ii) the invariant dynamical criterion for the separation of its quantum degrees of freedom into observable ones and those inaccessible for an exterior observer and iii) averaging over the latter variables which leads to the density matrix of a black hole and the dynamical origin of its entropy. We also briefly discuss the recently proposed idea [28] that the problem of the entropy is related to the problem of renormalization of the gravitational constant.
2. Dynamical Degrees of Freedom of a Black Hole

The object we are interested in is a black hole which arises as a result of the gravitational collapse. For simplicity we assume that a black hole is non-rotating and spherically symmetric. Denote by $\Sigma_0$ a spacelike or null global Cauchy surface and denote by $\partial B$ the intersection of a surface $\Sigma_0$ with the event horizon $H^+$ of the black hole. The state of our system (a black hole and fields in its vicinity) can be characterized by giving the values of gravitational and other fields on a chosen surface $\Sigma_0$. It is evident that the states of the gravitational and other fields located inside $\partial B$ have no influence on the further evolution of the black hole exterior. For states of particles and fields which fall into the a black hole from exterior region the energy $E$ defined by means of the timelike Killing vector $\xi$ is always positive. (For particles $E \equiv -\xi^\mu p_\mu$, where $p^\mu$ is its momentum.) Besides these states inside the black hole there exist states with negative total energy $E < 0$. Such states, located inside the black hole at $\Sigma$, will be considered as its internal degrees of freedom [31].

The study of internal degrees of freedom of a black hole is complicated because a surface $\Sigma$ crosses the singularity. There exist more convenient approach which greatly simplifies the consideration. A lone black hole at late time (i.e. long after a black hole formation) is almost stationary, i.e., its state can be described as the classical static (Kruskal) metric and small perturbations (fields excitation) propagating on this background. Analytical continuations of a static black hole solution defines maximally extended solution which is known as eternal black hole metric. If $\Sigma_0$ is chosen at late time one can also trace back in time all the fields excitations present in the vicinity of $\Sigma_0$ so that the problem of specifying the states of a black hole can be reformulated as a problem for an eternal black hole. Technically the latter is much simpler, so that we use this approach.

The Kruskal metric for eternal black hole reads

$$ds^2 = -\frac{32M^3}{r} \exp \left[ -\left( \frac{r}{2M} - 1 \right) \right] dUdV + r^2 d\Omega^2;$$  \hspace{1cm} (2.1)

$$UV = \left( 1 - \frac{r}{2M} \right) \exp \left( \frac{r}{2M} - 1 \right).$$  \hspace{1cm} (2.2)

Denote by $\Sigma$ a global Cauchy surface defined by the equation $U + V = 0$. It has wormhole topology $R \times S^2$. This is a well-known Einstein-Rosen bridge connecting two
asymptotically flat three-dimensional spaces. The discrete isometry $U \rightarrow -U, V \rightarrow -V$ transforms the surface $\Sigma$ onto itself, so that one asymptotically flat region (say $\Sigma_+$) is mapped onto another (say $\Sigma_-$). Localized states with $E < 0$ being traced back in time in the Kruskal geometry cross $\Sigma_-$, while states with $E > 0$ cross $\Sigma_+$. A remarkable property of the Kruskal-Schwarzschild metric (2.2) is that it can be considered as a real Lorentzian-signature section of the complex manifold parametrized by the real radial $r$, $0 \leq r < \infty$, and complex time $z$ coordinates:

$$z = \tau + i t,$$  \hspace{1cm} (2.3)

$$U = -\left(\frac{r}{2M} - 1\right)^{1/2} \exp\left\{\frac{1}{2} \left(\frac{r}{2M} - 1\right) + \frac{i}{4M} \left(\frac{r}{2M} - 1\right)^{-1} - 2\pi M \right\},$$  \hspace{1cm} (2.4)

$$V = \left(\frac{r}{2M} - 1\right)^{1/2} \exp\left\{\frac{1}{2} \left(\frac{r}{2M} - 1\right) - \frac{i}{4M} \left(\frac{r}{2M} - 1\right)^{-1} - 2\pi M \right\}. $$  \hspace{1cm} (2.5)

Sectors $R_+$ and $R_-$ of the Kruskal metric are generated by the following segments in the complex plane of $z$

$$R_+: z = 2\pi M + it, \ -\infty < t < \infty, $$

$$R_-: z = -2\pi M + it, \ -\infty < t < \infty, $$  \hspace{1cm} (2.6)

and analytically joined by the real Euclidean section $E$

$$E: z = \tau, \ -2\pi M \leq \tau \leq 2\pi M. $$  \hspace{1cm} (2.7)

Here $t$ is a usual time-like Killing coordinate in the Schwarzschild metric, while $\tau$ is its Euclidean analogue playing the role of the angular coordinate in the Gibbons-Hawking black hole instanton periodic with the period $\beta = 8\pi M$:

$$ds_E^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. $$  \hspace{1cm} (2.8)

The Euclidean section (2.7) represents a half-period part of this instanton, the boundary $\Sigma_+ \cup \Sigma_-$ of which at $\tau_\pm = \pm 2\pi M$ represents the Einstein-Rosen bridge of the above type. At this boundary the Euclidean section analytically matches with the Lorentzian sectors $R_+$ and $R_-$ on the Penrose diagram of the Kruskal metric.

To determine the propagation of small perturbations on the background of an eternal black hole one must specify initial data at $\Sigma_0$. It is evident that the data at $\Sigma_-$
(the part of $\Sigma_0$ lying inside the black hole) do not influence the black hole exterior. That is why these data can be identified with internal degrees of freedom of a black hole. The gravitational perturbations can be dealt with in the same manner. Our main idea can be described as follows. Fix a three-dimensional manifold with a wormhole topology $R \times S^2$ and consider any three-dimensional metrics on it which posses two asymptotically flat regions. Consider also configuration of matter fields on this manifold. This space of 3-geometries and matter fields will be considered as a configuration space for our problem. We introduce a wave function of a black hole as a functional on this configuration space. It should be stressed that the metric and fields at the 'internal' part $\Sigma_-$ of space are to be considered as defining the internal state of a black hole and hence they will be identified with its internal degrees of freedom.

Our proposal for the quantum state of a black hole is a "no-boundary" wavefunction of 3-geometry and matter fields on such a surface $\Sigma = R \times S^2$ given by the Euclidean path integral of Hartle and Hawking over 4-geometries and spacetime matter-field configurations bounded by $\Sigma$ and 4-dimensional asymptotically flat and empty infinity.

Obviously, the above picture is only an illustration of the general method we shall propose here. In the full quantum gravity incorporating the coupling of matter with the gravitational field (what is usually called a self-consistent back reaction of quantized matter on semiclassical background) many features of the Schwarzschild solution do not persist. There is no Killing symmetries, the very notion of the bifurcation surface of the Killing horizon separating physical variables into observable and unobservable ones does not exist and should be dynamically determined on the ground of some invariant criterion. In this paper we propose such a criterion which allows one to formulate the notion of the black hole horizon subject to quantum vibration (the horizon *zitterbewegung*) and to calculate its quantum dispersion. The latter quantity is very important in gravitational thermodynamics [13], for it, apparently, provides a self-consistent high energy cutoff for the one-loop entropy [1, 2].

It should be emphasized that the quantum state of the black hole we advocate here is merely a *proposal*, and we must verify its validity by comparing its consequences with the known properties of the conventional gravitational thermodynamics. For this purpose we first show that, semiclassically, this state generates the black-hole Hartle-
Hawking vacuum [32] for the particle excitations of all spins (including graviton) and produces by the procedure of the above type the thermal density matrix with the temperature $T = 1/8\pi M$. We then estimate the dominant contribution to the one-loop entropy of the black hole, which proves to be divergent in the vicinity of the horizon. This means that one-loop effects must necessarily be included in the consideration of the self-consistent gravitational thermodynamics.

3. No-boundary wavefunction of a black hole

The no-boundary wavefunction was first proposed by Hartle and Hawking [33, 34] in the context of quantum cosmology as a path integral

$$\Psi(\tilde{g}(\mathbf{x}), \varphi(\mathbf{x})) = \int D\tilde{g} D\phi e^{-I[\tilde{g}, \phi]}$$

(3.1)

of the exponentiated gravitational action $I[\tilde{g}, \phi]$ over Euclidean 4-geometries and matter-field configurations on a compact spacetime $M$ with a boundary $\partial M$. The integration variables are subject to the conditions $(\tilde{g}(\mathbf{x}), \varphi(\mathbf{x})), \mathbf{x} \in \partial M$, – the collection of 3-geometry and boundary matter fields on $\partial M$, which are just the argument of the wavefunction (3.1).

This construction was also applied in the asymptotically-flat case [35] when $M$ represents a noncompact 4-dimensional half-space whose boundary consists of two components, $\partial M = \mathbb{R}^3 \cup \partial M_\infty$: infinite 3-dimensional hyperplane $\mathbb{R}^3$ carrying the field argument of the wavefunction and the 4-dimensional asymptotically-flat and empty infinity $\partial M_\infty$. The latter is a singular component of the spacetime boundary and its boundary conditions are in certain sense trivial and do not enter the argument of the wavefunction.

We propose the quantum state of a black hole which is a modification of this asymptotically-flat no-boundary wavefunction of Hartle. It is given by eq.(3.1) where the total boundary

$$\partial M = \Sigma \cup \partial M_\infty$$

(3.2)

has instead of the hyperplane above the hypersurface with the topology of the Einstein-
Rosen bridge

$$\Sigma = R \times S^2$$ \hfill (3.3)

connecting two asymptotically flat 3-dimensional spaces.

The construction (3.1) - (3.3) forms a topological part of the definition for the no-boundary wavefunction. Apart from that the expression (3.1) signifies nothing unless we specify the meaning of the integration measure $\mathcal{I}_g, \phi$. We also need to determine the physical inner product with respect to which one can calculate the expectation values and matrix elements for a given wavefunction. In the context of the Lorentzian spacetime the problems have a solution which is based on the quantization of true physical variables [36, 37, 38] and can be constructively realized at least within the semiclassical loop expansion [39]. This quantization leads to the standard Faddeev-Popov integration measure [40] in the functional integral (3.1) and to its analogue in the physical inner product for the wavefunction $\Psi^{(3g(x), \varphi(x))}$ in the representation of local spatial 3-metric tensor and matter fields. The measure in this physical inner product is nontrivial. It is roughly the Faddeev-Popov measure in the configuration space of fields taken on a single spatial surface of the spacetime. The measure incorporates the gauge fixing procedure and effectively restricts the integration to the subset of true configuration-space coordinates among the dynamically redundant set $^{(3g(x), \varphi(x))}$ [38, 39]:

$$^{3}g(x), \varphi(x) \rightarrow \varphi = (g^T(x), \varphi(x)).$$ \hfill (3.4)

The geometrical content of the local gravitational variables can be very different depending on the choice of gauge and generally represents certain two dynamically independent degrees of freedom $g^T(x)$ per spatial point. They originate from solving the gravitational constraints and imposed gauge conditions for the original gravitational phase-space variables $^{3}g(x), ^3p(x)$ in terms of $g^T(x)$ and physical conjugated momenta $p_T(x)$.

\footnote{Complicated gauge conditions can generally mix the original gravitational variables with matter ones, but here we disregard this possibility and consider only the case of (3.4) when the gravitational physical degrees of freedom are disentangled from the gravitational sector of the theory. Still, in view of this fact, we use the neutral symbol $\varphi$ to denote the full set of physical configuration coordinates without emphasizing their metric or matter content.}
The wavefunction can be constructed directly in the representation of physical variables (3.4), \( \Psi(\varphi) \). In this representation the physical inner product has a trivial form
\[
<\Psi_1 | \Psi_2> = \int d\varphi \; \Psi^*_1(\varphi) \Psi_2(\varphi),
\]
that provides the unitary dynamics of \( \Psi(\varphi) = \Psi(\varphi, t) \) with the physical Hamiltonian whose functional form arises from the ADM reduction (3.4) \(^2\). For this reason, we shall formulate our proposal (3.1) - (3.3) for the wavefunction of a black hole in the representation of physical variables \(^3\). In this representation the wave function of a black hole is given by the path integral of the form (3.1), but with the physical configuration coordinates (3.4) fixed at \( \partial M \) instead of the 3-metric components of the dynamically redundant set \( (\g^0(x), \varphi(x)) \)
\[
\Psi(\varphi) = \int_{\phi|\Sigma = \varphi} D\phi \; e^{-I[\phi]}.
\]
Here the integration goes over those spacetime histories of physical ADM fields \( \phi = \phi(x) \) that generate the Euclidean 4-geometries asymptotically flat at the infinity \( \partial M_\infty \) of spacetime and match \( \varphi \) on its ”dynamically active” boundary (3.3). \( I[\phi] \) is the Lagrangian gravitational action in terms of these fields. The integration measure \( D\phi \) involves the local functional measure \([42]\) the structure of which is not very important for our purposes.

As it was mentioned above, the nature of physical degrees of freedom depends on the choice of gauge in the ADM reduction procedure. To effectively operate with the physical wavefunction, we have to fix this gauge and perform the reduction (3.4). Here we use a York gauge \([43]\) which consists of the condition
\[
\text{tr} \; \g^0(x) \equiv \g^{ab}(x) p_{ab}(x) = 0,
\]
selecting a spacetime foliation by minimal surfaces (of vanishing mean extrinsic curvature \( \text{tr} K(x) = 0 \)), and some other three conditions fixing the coordinatization of these

\(^2\)The unitary map between the Dirac-Wheeler-DeWitt wavefunctions \( \Psi(\g^0(x), \varphi(x)) \) and wavefunctions of true physical variables \( \Psi(\varphi, t) \) is discussed in much detail in \([39]\) both at the level of the path integral and operatorial quantizations.

\(^3\)In the cosmological context the no-boundary wavefunction in such a representation was considered in \([41]\) and also constructed as a unifying link between the Lorentzian and Euclidean quantum gravity theories in \([42]\).
surfaces. A distinguished nature of this gauge consists in the fact that, in contrast to a majority of other gauges, it does not suffer from the problem of Gribov copies invalidating the physical reduction when the latter is considered globally in phase space of the theory \(^4\). This property of the York gauge follows from a strong theorem of [44] on the uniqueness of a solution of the Lichnerowicz equation for the conformal factor in the conformal decomposition of a 3-metric [43], provided positive-energy condition holds for matter fields.

As known [43, 45], the physical degrees of freedom in the York gauge can be represented by the two variables characterizing the conformally-invariant part of the 3-metric \( \tilde{g}_{ab}(x) \) (in some gauge fixing of the 3-dimensional spatial diffeomorphisms) and the conjugated transverse traceless momenta \( \tilde{p}_{ab}(x) \), while the conformal mode \( \Phi(x) \) of the full 3-metric

\[
g_{ab}(x) = \Phi^4(x) \tilde{g}_{ab}(x) \tag{3.8}
\]

follows from the solution of the Lichnerowicz equation which is just the Hamiltonian gravitational constraint rewritten in the conformal decomposition of the above type

\[
(\tilde{\Delta} - \frac{1}{8} \tilde{\mathcal{R}}) \Phi + \frac{1}{8} C \Phi^{-7} + 2\pi \tilde{T}_a^* \Phi^{-3} = 0, \tag{3.9}
\]

\[
C \equiv \tilde{p}^a_b \tilde{p}^b_a / \tilde{g} \tag{3.10}
\]

Here \( \tilde{T}_a^* = \Phi^{-8} \tilde{T}_a^* \) is a conformally rescaled energy – Hamiltonian density – of matter fields and tilde denotes the quantities calculated in the conformal metric \( \tilde{g}_{ab} \). In the linearized approximation the physical gravitational variables in the York gauge are the transverse-traceless part of the linear excitations \( h_{ab} \) and their conjugated transverse-traceless momenta \(^6\).

\[
(g^T, p_T) = (h^T_{ab}, \tilde{p}^ab_T), \tag{3.11}
\]

\(^4\)This property actually poses a dilemma of York gauges versus the third quantization of gravity, a strong motivation for the latter being rooted in the problem of Gribov copies problem in quantum gravity theory (see a discussion in [39]).

\(^5\)In the geometrically invariant language, the physical content of \( \tilde{g}_{ab} \) can be described by the conformally invariant transverse-traceless tensor of York \( \beta^{ab} \) [43].

\(^6\)We assume that, without loosing the generality, the spatial gauge conditions fixing the coordinatization of metrically perturbed \( \Sigma \) can be chosen as transversality of \( h_{ab} \). The variables \( (h^T_{ab}, \tilde{p}^ab_T) \) are conformally related to their tilded conformally-invariant analogues the transversality and tracelessness of which holds with respect to \( \tilde{g}_{ab} \).
In a semiclassical approximation the wave function of a black hole

$$\Psi(\varphi) = P e^{-I[\phi(\varphi)]}$$  \hspace{1cm} (3.12)

is dominated by the classical action at the extremal of equations of motion $\phi(\varphi)$ subject to boundary conditions $\varphi$ on $\Sigma$. It also includes the preexponential factor $P$ accumulating the result of integration over quantum field deviations from the extremal. The physical variables $\varphi$ given by eqs.(3.4) and (3.11) are treated by perturbations and the Euclidean action $I[\phi(\varphi)]$ is to be expanded in powers of $\varphi$. To obtain the lowest-order term $I[\phi(0)]$, notice that the boundary 3-geometry on $\Sigma$ has, in virtue of (3.9) a conformal factor satisfying the homogeneous conformally-invariant equation in three dimensions. As shown in Appendix A, it gives for asymptotically flat boundary conditions exactly the spherically symmetric metric of the Einstein-Rosen bridge, characterized by a single constant – the mass $M$ of the black hole. The extremal of the Euclidean vacuum Einstein equations $\phi(0)$ satisfies asymptotically flat boundary conditions at $\partial M_\infty$. The corresponding solution is just one half of the Schwarzschild gravitational instanton of mass $M$ with the four-dimensional metric (2.8) for $-2\pi M \leq \tau \leq 2\pi M$. The classical action on this half of instanton reduces to the contribution of the surface term at $\partial M_\infty$ of the classical Einstein gravitational action

$$I[\phi(0)] = \frac{1}{8\pi} \int_{\partial M_\infty} K \sqrt{h} d^3 x = 2\pi M^2. \hspace{1cm} (3.13)$$

The expansion of $I[\phi(\varphi)]$ in powers of $\varphi$ on the background of $\phi(0)$ shows that the linear-order term vanishes due to the equations of motion for the background and the vanishing of the extrinsic curvature of $\Sigma$ (the latter property guarantees the absence of the surface terms). Therefore the leading contribution to the semiclassical wavefunction (3.12) takes the form

$$\Psi(\varphi, M) = P e^{-2\pi M^2 - I_2[\phi(\varphi)]}, \hspace{1cm} (3.14)$$

where $I_2[\phi(\varphi)]$ is a quadratic term of the action in the linearized physical fields (3.4) and (3.11).

Thus, our no-boundary wavefunction of a black hole turns out to be a functional of the local gravitational and matter degrees of freedom $\varphi(x)$, parametrized by a global
variable – the gravitational mass of the Einstein-Rosen bridge $M$. Obviously, if we include $M$ into the configuration space of the black hole, the dependence of the wavefunction on it will describe the probability distribution of black holes with different masses in this quantum state. A naive inclusion of $M$ into the ADM phase space of the theory in the York gauge does not seem to be fully consistent. However, it was recently performed in more general context by K.Kuchar [46] who persuasively advocated that $M$ has a conjugated momentum $P_M$, so that $(M, P_M)$ can be a subject to standard canonical quantization and incorporate as their quantum state an arbitrary function of the black-hole mass $M$. Thus, the proposed $M$-dependent no-boundary wavefunction can be regarded as a first example of such a quantum state of a black hole (or, more precisely, of the quantum Einstein-Rosen bridge) \(^7\). In what follows, however, we shall consider $M$ as an external parameter not entering the argument of the wavefunction and, correspondingly, excluded from the phase space and the Hilbert space of the theory. Therefore, up to $M$-dependent normalization, the semiclassical wavefunction of the black hole will be dominated by its $\exp(-I_2[\phi(\varphi)])$ part, describing the dynamics of local degrees of freedom. In the next section we show that it represents their Hartle-Hawking vacuum on the background of the Kruskal-Schwarzschild geometry.

4. Hartle-Hawking vacuum state

We demonstrate now the calculation of (3.14) and its vacuum properties on a simple example of a scalar field $\phi(x) = \phi(\tau, \mathbf{x})$ with the quadratic action

$$\frac{1}{2} \int d^4x \, g^{1/2}(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \xi R \phi^2). \quad (4.1)$$

The generalization to fields of higher spins in the quadratic approximation is obvious. This action generates on the Euclidean section (2.7) with the metric (2.8) the linear

\(^7\)The variables $(M, P_M)$ of Ref.[46] have the nature of angle-action variables in their canonical action. The variable $M$ plays the role of the positive conserved energy and the "angle" $P_M$ linearly grows in time with the speed determined by the way the observer anchors the spacetime foliation at spatial infinity with his physical clock. This means that, strictly speaking, their quantum state can not be absolutely arbitrary function of $M$ and, in particular, it can generate the discrete spectrum of masses for the quantum Einstein-Rosen bridge. [8, 48, 49].
equations of motion
\[
\left\{-g^{1/2}g^{\tau\tau} \frac{d^2}{d\tau^2} \partial_a g^{1/2}g^{ab}\partial_b \right\} \phi(\tau, x) = 0, \quad x = x^a, \quad a = 1, 2, 3,
\]
(4.2)
which must be solved subject boundary conditions \( \varphi = \varphi(x) \) on its boundary \( \Sigma \) to give the extremal \( \phi_\ast(\varphi) \) of eq.(3.14). On the Schwarzschild background with \( R = 0 \) the nonminimal interaction does not contribute to the equations. In what follows we denote the boundary fields on the two asymptotically flat parts of the Einstein-Rosen bridge \( \Sigma_\pm \) by \( \varphi_\pm \)
\[
\phi(x) \bigg|_{\Sigma_\pm} \equiv \phi(\pm \beta/4, x) = \varphi_\pm(x).
\]
(4.3)
With this notation the solution to (4.2) can be written as a decomposition
\[
\phi_\ast(\tau, x) = \sum_{\lambda} \left\{ \varphi_{\lambda,+}u_{\lambda,-}(\tau, x) + \varphi_{\lambda,-}u_{\lambda,+}(\tau, x) \right\}
\]
(4.4)
in the basis functions of this equation
\[
u_{\lambda,\pm}(\tau, x) = \frac{\sinh(\beta/4 \mp \tau)}{\sinh(\beta/2)} R_{\omega lm A}(x), \quad \lambda = (\omega, l, m, A),
\]
(4.5)
containing the set of spatial harmonics \( R_{\omega lm A}(x) \) — eigenfunctions of the following eigenvalue problem
\[
\partial_a \left(g^{1/2}g^{ab}\partial_b R_{\omega lm A}(x)\right) = -g^{\tau\tau}g^{1/2}\omega^2 R_{\omega lm A}(x)
\]
(4.6)
originating from the separation of variables in (4.2). These eigenfunctions are enumerated by a set of continuous \( \omega > 0 \) and discrete \( (l, m, A) \) labels, among which \( l \) and \( m \) are the usual quantum numbers of spherical harmonics and the label \( A = 1, 2 \) is responsible for two possible directions of propagation along the radial coordinate. As shown in Appendix B, these spatial harmonics can be chosen real. They are required to be regular at the horizon \( r = 2M \) and the spatial infinity, have a positive definite spectrum \( \omega^2 > 0 \) and satisfy the orthonormality and completeness conditions
\[
\int d^3x g^{\tau\tau}g^{1/2}R_\lambda(x) R_{\lambda'}(x) = \delta_{\lambda\lambda'}, \quad (4.7)
\]
\[
\sum_{\lambda} R_\lambda(x) R_\lambda(x') = \delta(x - x') \frac{g^{\tau\tau}g^{1/2}}{g^{\tau\tau}g^{1/2}}.
\]
(4.8)
Here, as in (4.5), we use a condensed notation $\lambda$ for the full collection of quantum numbers, the summation over which implies the following measure

$$\sum_\lambda (...) \equiv \int_0^\infty d\omega \sum_{l,m,A} (...) \delta_{\lambda\lambda'} \equiv \delta(\omega - \omega') \delta_{mm'} \delta_{AA'} \quad (4.9)$$

In view of these relations the coefficients $\varphi_{\lambda,\pm}$ in (4.4) are just the decomposition coefficients of the fields (4.3) in the basis of spatial harmonics

$$\varphi_{\pm}(x) = \sum_\lambda \varphi_{\lambda,\pm} R_\lambda(x). \quad (4.10)$$

Substituting (4.4) into (4.1), integrating by parts with respect to Euclidean time and taking into account the equations of motion (4.2), one finds that the Euclidean action reduces to the following quadratic form in $\varphi_{\pm}$ (cf. a similar derivation in [52]):

$$I^2(\varphi_{+}, \varphi_{-}) = \frac{1}{2} \frac{1}{\sqrt{2}} \sum_\lambda \left\{ \frac{\omega_\lambda \cosh(\beta \omega_\lambda/2)}{\sinh(\beta \omega_\lambda/2)} \left( \varphi^2_{\lambda,+} + \varphi^2_{\lambda,-} \right) - \frac{2 \omega_\lambda}{\sinh(\beta \omega_\lambda/2)} \varphi_{\lambda,+} \varphi_{\lambda,-} \right\} \quad (4.11)$$

that can be diagonalized by the following reparametrization to new variables $f_{\lambda,\pm}$

$$\varphi_{\pm} = \frac{f_{\pm} \pm f_{-}}{\sqrt{2}}, \quad (4.12)$$

$$I^2(\varphi_{+}, \varphi_{-}) = I^2(f_{+}, f_{-}) = \frac{1}{2} \frac{1}{\sqrt{2}} \sum_\lambda \omega_\lambda \left\{ \tanh(\beta \omega_\lambda/4) f^2_{\lambda,+} + \frac{1}{\tanh(\beta \omega_\lambda/4)} f^2_{\lambda,-} \right\}. \quad (4.13)$$

The wavefunction (3.14) rewritten in the new representation (4.12) is a gaussian state which is obviously a vacuum

$$\Psi(\varphi_{+}, \varphi_{-}) = \bar{\Psi}(f_{+}, f_{-}) = Pe^{-I^2(f_{+}, f_{-})}, \quad (4.14)$$

$$\bar{a}_{\pm} \bar{\Psi}(f_{+}, f_{-}) = 0, \quad (4.15)$$

of the following creation-annihilation operators (we omit for brevity the label $\lambda$ in the definition of $\bar{a}_{\pm}$ below as well as in $\omega = \omega_\lambda$):

$$\bar{a}_{+} = \frac{1}{\sqrt{2}} \left[ \left( \omega \tanh(\beta \omega/4) \right)^{1/2} \frac{\partial}{\partial f_{+}} + \left( \omega \tanh(\beta \omega/4) \right)^{1/2} f_{+} \right], \quad (4.16)$$

$$\bar{a}^\dagger_{+} = \frac{1}{\sqrt{2}} \left[ -\left( \omega \tanh(\beta \omega/4) \right)^{-1/2} \frac{\partial}{\partial f_{+}} + \left( \omega \tanh(\beta \omega/4) \right)^{1/2} f_{+} \right],$$

$$\bar{a}_{-} = \frac{1}{\sqrt{2}} \left[ \left( \frac{1}{\omega} \tanh(\beta \omega/4) \right)^{1/2} \frac{\partial}{\partial f_{-}} + \left( \frac{1}{\omega} \tanh(\beta \omega/4) \right)^{-1/2} f_{-} \right], \quad (4.17)$$

$$\bar{a}^\dagger_{-} = \frac{1}{\sqrt{2}} \left[ -\left( \frac{1}{\omega} \tanh(\beta \omega/4) \right)^{-1/2} \frac{\partial}{\partial f_{-}} + \left( \frac{1}{\omega} \tanh(\beta \omega/4) \right)^{1/2} f_{-} \right],$$
subject to standard commutation relations

\[ [\hat{a}_{\lambda,\pm}, \hat{a}_{\lambda',\pm}^\dagger] = \delta_{\lambda\lambda'} \]  \hspace{1cm} (4.18)

(all the other commutators are vanishing). For our purposes another choice of creation-annihilation operators is more useful, differing from (4.16)-(4.17) by the linear transformation not mixing the positive and negative frequencies

\[ a_{\lambda,\pm} = \frac{\hat{a}_{\lambda,\pm} \pm \hat{a}_{\lambda,-}}{\sqrt{2}}, \quad a_{\lambda,\pm} \bar{\Psi} (f_+, f_-) = 0. \]  \hspace{1cm} (4.19)

To give a particle interpretation for the obtained vacuum state we must construct the propagating physical modes corresponding to \( a_{\lambda,\pm} \). For this purpose consider the \( \Sigma_{\pm} \) parts of \( \Sigma \) as the initial Cauchy surfaces in the right (\( R_+ \)) and left (\( R_- \)) wedges of the Lorentzian Kruskal-Schwarzschild spacetime. In these two causally disconnected regions lying to the future of \( \Sigma_{\pm} \) one can construct two scalar field theories with the Lagrangians – the Lorentzian versions of (4.1)

\[ L_{\pm} = \int_{\Sigma_{\pm}} d^3x L (\phi, \partial \phi) = \frac{1}{2} \sum_{\lambda} (\dot{\varphi}_{\lambda,\pm}^2 - \omega_{\lambda}^2 \varphi_{\lambda,\pm}^2), \]  \hspace{1cm} (4.20)

which take such a form provided the corresponding spacetime fields evolving correspondingly in \( R_+ \) and \( R_- \) are decomposed in spatial harmonics with time-dependent coefficients \( \varphi_{\lambda,\pm}(t) \), \( \dot{\varphi}_{\lambda,\pm} \equiv d\varphi_{\lambda,\pm}(t)/dt \). At the quantum level, in the coordinate representation of \( \varphi_{\lambda,\pm} \) the creation-annihilation operators \( b_{\lambda,\pm} \) of these two theories associated with positive-negative frequency decomposition in the Killing time \( t \) look as follows

\[ \sqrt{2} b_{\lambda,\pm} = \frac{1}{\sqrt{\omega}} \frac{\partial}{\partial \varphi_{\lambda,\pm}} + \sqrt{\omega} \varphi_{\lambda,\pm}, \]  \hspace{1cm} (4.21)

\[ \sqrt{2} b_{\lambda,\pm}^\dagger = -\frac{1}{\sqrt{\omega}} \frac{\partial}{\partial \varphi_{\lambda,\pm}} + \sqrt{\omega} \varphi_{\lambda,\pm} \]  \hspace{1cm} (4.22)

and correspond to the following choice of positive-frequency basis functions

\[ w_{\lambda,\pm}(x) \big|_{R_+} = e^{-i\omega_{\lambda} t} R_{\lambda}(x), \quad w_{\lambda,\pm}(x) \big|_{R_-} = 0 \]  \hspace{1cm} (4.23)

\[ w_{\lambda,\pm}(x) \big|_{R_-} = e^{i\omega_{\lambda} t} R_{\lambda}(x), \quad w_{\lambda,\pm}(x) \big|_{R_+} = 0 \]  \hspace{1cm} (4.24)

(one should remember that the Schwarzschild Killing time coordinate is past pointing in \( R_- \) and \( w_{\pm} \) by construction have zero initial data on \( \Sigma_{\pm} \)).
This is a matter of a simple algebra, using the reparametrization (4.12), to show that the operators (4.21) are related to (4.19) by a nontrivial Bogolyubov transformation which mixes the positive and negative frequencies

\[ b_\pm = \left(2 \sinh \frac{\beta \omega}{2}\right)^{-1/2} \left[ e^{\beta \omega/4} a_\pm + e^{-\beta \omega/4} a_\mp^* \right] \]  

and generates, in terms of \( w_\pm \), the basis functions \( v_{\lambda,\pm}(x) \) associated with the creation annihilation operators \( a_{\lambda,\pm} \) of our vacuum quantum state (4.19)

\[ v_\pm = \left(2 \sinh \frac{\beta \omega}{2}\right)^{-1/2} \left[ e^{\beta \omega/4} w_\pm + e^{-\beta \omega/4} w_\mp^* \right]. \]  

This is a well-known transformation relating the Killing vacua, \((b_{\lambda,\pm}, w_{\lambda,\pm}(x))\), in the right \((R_+)\) and left \((R_-)\) wedges of the Kruskal diagram to the Hartle-Hawking vacuum, \((a_{\lambda,\pm}, v_{\lambda,\pm}(x))\), of quantum fields on the maximally extended black hole spacetime [54]. The latter is defined by the condition that its basis functions \( v_{\lambda,\pm}(x) \) contain only positive frequencies with respect to affine parameter on both horizons of the black hole metric. This property follows from eqs.(4.23)-(4.24), (4.26) and the asymptotic behaviors of \( w_{\lambda,\pm}(x) \) at the horizon (see Appendix B)

\[ w_+(x) \big|_{R_+} = \begin{cases} C_{\text{past}} (-U)^{4M\omega i}, & x \to H_{\text{past}}^+, \\ C_{\text{future}} (V)^{-4M\omega i}, & x \to H_{\text{future}}^+, \end{cases} \]  

\[ w_-(x) \big|_{R_-} = \begin{cases} (C_{\text{past}})^* (U)^{-4M\omega i}, & x \to H_{\text{future}}^-, \\ (C_{\text{future}})^* (-V)^{4M\omega i}, & x \to H_{\text{past}}^-, \end{cases} \]  

where \( C_{\text{past}} \) and \( C_{\text{future}} \) are some complex coefficients and \( H_{\text{past}}^+ \) and \( H_{\text{future}}^+ \) are past and future horizons of the \( \pm \) wedges of the Kruskal diagram. Substituting these behaviors into (4.26) one finds

\[ v_+(x) \big|_{H_{\text{past}}^+ \cup H_{\text{future}}^+} = C_{\text{past}} \left(2 \sinh \frac{\beta \omega}{2}\right)^{-1/2} \left[ \theta(-U) e^{2\pi M\omega (-U)^4M\omega i} \right. \]

\[ + \left. \theta(U) e^{-2\pi M\omega U^{-4M\omega i}} \right] \]  

which is a basis function with the needed positive frequency behavior matching with the analyticity in the lower half of the complex \( U \)-plane [54]. The same proof holds for another horizon of the Kruskal diagram.

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8 The doubled set of field modes in Schwarzschild-Kruskal spacetime and their thermofield nature [50] was noticed by W.Israel [51], this observation being further developed within the context of the Euclidean path integral in [52] (see also [53]).
As it was mentioned above, similar considerations apply to fields of all possible spins. Thus, the proposed no-boundary wavefunction of a black hole represents the Hartle-Hawking vacuum state of linearized field excitations of all physical fields.

5. One-Loop Contribution to Entropy of a Black Hole

Now we return to the problem of a black hole entropy. According to the above procedure of separating the physical variables into observable and unobservable ones, the proposed wave generates the density matrix of a black hole interior as a functional trace

$$\rho(\phi_-, \phi_+) = \text{Tr}_+ |\Psi\rangle <\Psi| \equiv \int D\phi_+ \Psi^* (\phi_-, \phi_+) \Psi (\phi_-, \phi_+).$$

(5.1)

It gives rise to the entropy of the black hole

$$S = -\text{Tr} [\hat{\rho} \ln \hat{\rho}] = -\int D\phi <\phi | \hat{\rho} \ln \hat{\rho} | \phi>.$$  

(5.2)

Up to normalization, the wavefunction defined by the path integral over physical degrees of freedom (3.6) in the \(\tau\)-foliation of the Euclidean spacetime, \(-\beta/4 < \tau < \beta/4\), actually represents the heat kernel or the matrix element between the configurations \(\phi_-\) and \(\phi_+\) of the Euclidean ”evolution” operator \(\exp(-\beta \hat{H}/2)\)

$$\Psi(\phi_-, \phi_+) = \exp \left(\frac{\Gamma}{2}\right) <\phi_- | \exp (-\frac{\beta}{2} \hat{H}) | \phi_+> |_{\beta=8\pi M},$$

(5.3)

where \(\hat{H}\) is a physical Hamiltonian of the system. In the full nonperturbative treatment of the problem this Hamiltonian is a complicated functional of physical degrees of freedom, numerically coinciding with the ADM surface integral, while in the linearized approximation (relevant to the one-loop order of semiclassical expansion) it is just an additive sum of quadratic Hamiltonians of fields of all spins on the Schwarzschild-Kruskal background. In particular, for a scalar field, we shall consider here, it is the following expression generated by the Lagrangian (4.1)

$$H(\pi, \phi) = \frac{1}{2} \int d^3x \left[ \frac{1}{g^{\tau\tau}} g^{\pi^2} + g^{\tau\pi} \left[ g^{ab} \partial_a \phi \partial_b \phi + \xi R \phi^2 \right] \right].$$

(5.4)
Using the composition law, \( \int d\varphi_+ |\varphi_+ > < \varphi_+ | = \hat{1} \), we obtain from (5.3) the following representation for the density matrix

\[
\rho(\varphi_+', \varphi_-) = \exp (\Gamma) < \varphi_+' | \exp (-8\pi M \hat{H}) | \varphi_- >,
\]

(5.5)

where \( \Gamma \) is defined from the normalization conditions

\[
\text{Tr}_- \hat{\rho} = \int D\varphi_- \rho(\varphi_-, \varphi_-) = 1.
\]

(5.6)

In addition to the density matrix \( \hat{\rho} \) it is convenient also to define a more general object \( \hat{\rho}_\beta \) which depends on the arbitrary parameter \( \beta \) independent of the black hole mass

\[
\hat{\rho} = \hat{\rho}_\beta \Big|_{\beta = 8\pi M},
\]

\[
\rho_\beta(\varphi_+', \varphi_-) = < \varphi_+' | \hat{\rho}_\beta | \varphi_- >= \exp (\Gamma_\beta) < \varphi_+' | \exp (-\beta \hat{H}) | \varphi_- >.
\]

(5.7)

In the one-loop approximation we have

\[
< \varphi_-' | \exp (-\beta \hat{H}) | \varphi_- > = \left[ \det \frac{1}{2\pi} \frac{\partial^2 I(\varphi_+', \varphi_-)}{\partial \varphi_- \partial \varphi_-} \right]^{1/2} \exp [-I(\varphi_+', \varphi_-)],
\]

(5.8)

where the Euclidean Hamilton-Jacobi function \( I(\varphi_+', \varphi_-) \) is given by the equation (4.11) which in the coordinate representation of a scalar field gives rise to the following kernel of the Van Vleck - Morette functional matrix

\[
\frac{\partial^2 I(\varphi_+', \varphi_-)}{\partial \varphi_- (x) \partial \varphi_- (y)} = g^{\tau \tau} g^{\frac{1}{2}} \frac{\hat{\omega}}{\sinh \beta \hat{\omega}} \delta(x - y)
\]

(5.9)

with the operator of frequency \( \hat{\omega} \) defined on a spatial 3-dimensional hypersurface as

\[
\hat{\omega} = \left[ -\frac{1}{g^{\tau \tau} g^{\frac{1}{2}}} \partial_a g^{\frac{1}{2}} g^{ab} \partial_b + \frac{1}{g^{\tau \tau} g^{\frac{1}{2}}} \xi R \right]^{\frac{1}{2}} ;
\]

(5.10)

\[
g \equiv \det g_{\mu \nu} = g^{\tau \tau} \det g_{ab} ;
\]

\[\mu, \nu, ... = 0, 1, 2, 3.\]

(5.11)

The normalization factor of eq.(5.8) is, therefore, given by the following functional determinant on the space of functions of three spatial coordinates

\[
\Gamma_\beta = -\ln \left[ \int D\varphi_- < \varphi_- | \exp (-\beta \hat{H}) | \varphi_- > \right] = -\frac{1}{2} \ln \det \left[ \frac{1}{2(\cosh \beta \hat{\omega} - 1)} \delta(x - y) \right].
\]

(5.12)

(5.13)
It is worth emphasizing that all the quantities and operators entering the WKB approximation of the wave function and density matrix depend on a 3-geometry of space and values of fields on it. The whole information about 4-dimensional manifold is contained in the interval $\beta$ of Euclidean time between the points with the same spatial coordinates $x$ of spacelike slices $\Sigma_+$ and $\Sigma_-$. In the case of the Schwarzschild black hole $\beta = 8\pi M$.

The density matrix $\hat{\rho}_\beta$ satisfies the equation

$$\frac{\partial \hat{\rho}_\beta}{\partial \beta} = \left( \frac{\partial \Gamma_\beta}{\partial \beta} - \hat{H} \right) \hat{\rho}_\beta.$$  \hspace{1cm} (5.14)

Using this relation one can easily show that the entropy of the system in question can be obtained from the effective action $\Gamma_\beta$

$$S = S_\beta \bigg|_{\beta=8\pi M},$$  \hspace{1cm} (5.15)

$$S_\beta \equiv -\text{Tr}[\hat{\rho}_\beta \ln \hat{\rho}_\beta] = -\text{Tr}[\left( \Gamma_\beta - \beta \hat{H} \right) \hat{\rho}_\beta] = \beta \frac{\partial \Gamma_\beta}{\partial \beta} - \Gamma_\beta.$$  \hspace{1cm} (5.16)

Note that it would be incorrect to differentiate directly $\Gamma$ over $M$ in order to obtain the entropy $S$, since the total effective action is an integral over the whole space and depends also on its geometry. The Hawking temperature $T_{BH} = 1/(8\pi M)$ depends both on the space-geometry and on $g_{\tau\tau}$ of the four-dimensional metric and hence operations of differentiation over $M$ and integration over volume do not commute in general case. In order to avoid this difficulty we introduced the generalized density matrix $\hat{\rho}_\beta$.

In order to calculate $\text{Tr} \ln$ entering the expression for the effective action, it is convenient to expand all the functions $\varphi(x)$ in terms of eigenfunctions $R_\lambda(x)$ of the operator $\hat{\omega}$

$$\varphi(x) = \sum \varphi_\lambda R_\lambda(x), \hspace{1cm} \hat{\omega}^2 R_\lambda(x) = \omega^2_\lambda R_\lambda(x),$$  \hspace{1cm} (5.17)

$$\delta(x - y) = \sum g^{\tau\tau} g^{\frac{1}{2}} R_\lambda(x) R_\lambda(y),$$  \hspace{1cm} (5.18)

Here $\sum_\lambda$ denotes the sum over all quantum numbers $\lambda$. Substitution of the expansion of $\delta$-function in terms of eigenfunctions of the operator $\hat{\omega}$ gives

$$S_\beta = \int dx \left( \frac{\beta}{\partial \beta} - 1 \right) \left[ \ln \left( \frac{\beta}{2} \hat{\omega}_x \delta(x - y) \right) \right]_{y=x}.$$  \hspace{1cm} (5.19)
\[\int dx \left[ \frac{\beta \omega_y}{2} \coth \frac{\beta \omega_y}{2} - \ln \left( 2 \sinh \frac{\beta \omega_y}{2} \right) \right] \times \sum_{\lambda} \left( g^{\tau \tau}(x)g^{\frac{1}{2}}(x)R_\lambda(x)R_\lambda(y) \right)_{y=x} \]

\[= \int dx g^{\tau \tau} g^{\frac{1}{2}} \sum_{\lambda} R_\lambda(x)^2 \left[ \frac{\beta \omega_\lambda}{2} \coth \frac{\beta \omega_\lambda}{2} - \ln \left( 2 \sinh \frac{\beta \omega_\lambda}{2} \right) \right].\]

Thus we have

\[S_\beta = \int dx \sum_{\lambda} \mu_\lambda(x)s(\beta \omega_\lambda), \quad (5.19)\]

Here

\[s(\beta \omega) = \frac{\beta \omega}{e^{\beta \omega} - 1} - \ln(1 - e^{-\beta \omega}) \quad (5.20)\]

is a well known expression for the entropy of single oscillator with the frequency \(\omega\) at temperature \(T = 1/\beta\) and

\[\mu_\lambda(x) = g^{\tau \tau} g^{\frac{1}{2}} R_\lambda(x)^2 \quad (5.21)\]

is a phase space density of quantum modes. In order to estimate the contribution of regions of space in the vicinity of the horizon into the entropy of black hole we should find an asymptotic solution for the mode functions \(R_\lambda(x)\) near the horizon. Eigen functions \(R_\lambda(x)\) for a massless scalar field in the Schwarzschild spacetime are of the form

\[R_\lambda(r, \Omega) = R_{\omega l}(r)Y^l_m(\Omega). \quad (5.22)\]

Here \(Y^l_m(\Omega) = \frac{1}{\sqrt{2\pi}} \epsilon^{im\phi} P^m_l(\cos \theta)\) are spherical functions and radial functions \(R_{\omega l}(r)\) are real and obey the equation

\[\left[ \frac{d}{dr} \left( r^2 - 2Mr \right) \frac{d}{dr} - l(l+1) + \omega^2 \frac{r^3}{r-2M} \right] R_{\omega l}(r) = 0. \quad (5.23)\]

The expression for an entropy of such a system takes the form

\[S = \int_{2M}^{\infty} dr r^2 (1 - \frac{2M}{r})^{-1} \int_0^{\infty} d\omega \sum_{l=0}^{\infty} (2l+1)R_{\omega l}(r)^2 s(8\pi M \omega). \quad (5.24)\]

Regular near the horizon solutions of this equation are normalized by the condition

\[\int_{2M}^{\infty} dr r^2 (1 - \frac{2M}{r})^{-1} R_{\omega l}(r)R_{\omega^l}(r) = \delta(\omega - \omega'). \quad (5.25)\]
The entropy of an oscillator with frequency $\omega$ exponentially decreases for frequencies much larger than the black hole temperature $T = 1/(8\pi M)$. In the vicinity of the horizon $|\xi - 1| \ll 1$ a regular solution for radial modes takes a simple form

$$R_{\omega l}(r) \simeq A(M, \omega, l)K_{i4M\omega} \left(\sqrt{2l(l+1)}\left(\frac{r}{M} - 2\right)\right). \quad (5.26)$$

The normalization factor $A(M, \omega, l)$ depends on the coefficient of penetration of modes through the potential barrier. All the modes in the range of frequencies in question and with angular quantum numbers $l \geq 3$ are trapped. For such modes the penetration coefficient is exponentially small and normalization factor does not depend on $l$. The larger $l$ the closer to the horizon a return point lies and the better approximation becomes. The evaluation of normalization factor gives

$$A(M, \omega, l) \simeq \left[\frac{2\omega \sinh 4\pi M\omega}{M\pi^2}\right]^{\frac{1}{2}}, \quad (5.27)$$

where the relation

$$\int_0^\infty dy \frac{1}{y} K_{ix}(y)K_{ix'}(y) = \frac{\pi^2}{2} \frac{1}{x \sinh(\pi x)} \delta(x - x') \quad (5.28)$$

was used. We also use the fact that the modified Bessel functions decrease very fast with increasing of their argument and, hence, with a good accuracy the integration along radius can be extended to infinity. One can see that the main contribution to the integral of entropy near the horizon comes from large $l$. Replacement in Eq.(5.24) of summation over $l$ by integration leads to the expression

$$\sum_{l=0}^\infty (2l + 1)R_{\omega l}(r)^2 \simeq 2 \int_0^\infty dll R_{\omega l}(r)^2 = \frac{4\omega^2}{\pi} \frac{M}{r - 2M}. \quad (5.29)$$

This asymptotic formula reproduces the result by Candelas and Howard [25] for a mode summation in Schwarzschild geometry. For evaluation of the integral (5.24) it is convenient to split the region of integration into two points. In first part $2M \leq r \leq r_0 = 3M$ we can use the above described approximation. Namely this contribution is related to the entropy of a black hole $S_H$. The integration over another region ($r \leq 3M$) formally diverges at $r \to \infty$. This divergence is simply connected with the fact that we consider black hole in equilibrium with infinite reservoir of thermal
radiation. But this equilibrium is unstable. In order to get stable equilibrium one needs to insert a black hole into a cavity of the size comparable with $r \approx 3M$. For such physical problem the second contribution (related to the entropy of a thermal gas far from the black hole) becomes negligibly small and we can simply omit this term [1, 2]. That is why we have

$$S_H = \frac{4M}{\pi} \int_{2M}^{r_0} \frac{dr}{(r - 2M)^2} \int_0^\infty d\omega \omega^2 s(8\pi M \omega)$$

$$\simeq \frac{512M^5}{\pi} \int_0^{z_0} \frac{dz}{z^3} \int_0^\infty d\omega \omega^2 s(8\pi M \omega)$$

$$= \frac{4M^2}{45} \int_0^{z_0} \frac{dz}{z^3},$$

where $z_0 = r_0 \sqrt{1 - \frac{2M}{r_0}} + M \ln\left[ \frac{r_0}{M} - 1 + \frac{r_0}{M} \sqrt{1 - \frac{2M}{r_0}} \right]$ is a proper distance from the horizon to the point $r_0$. This result shows that one-loop contribution to the entropy of black hole $S_H$ diverges near the horizon. The expression (5.30) gives the leading divergent term and reproduces the result by Frolov and Novikov [1, 2]. This divergence is physical and its origin does not depend on particular properties of quantum fields surrounding a black hole. The analogous divergence evidently occurs for higher spins. Hence quantum corrections can never be neglected in description of thermodynamical properties of black holes. It is worth to emphasize that shifting of a position of the horizon as a whole due to the back reaction effect of quantum fields on the geometry of black hole does not remove the divergence. Fluctuations of the horizon are to be taken into account to provide the necessary cutoff.

6. Entropy and Effective Action

In previous section we used the proposal for a wave function of black hole in the calculation of an entropy of a scalar field in the vicinity of Schwarzschild black hole. Only the properties of 3-dimensional space and fields on it were used in the consideration. It would be interesting to compare this result with that of the 4-dimensional Euclidean action approach. This also allows one to generalize the result of section 5 to arbitrary static black holes.

Consider 4-dimensional Euclidean effective action $\Gamma_\beta$ for a conformal scalar field $\phi(\tau, \vec{x})$ with Hamiltonian Eq.(5.4) on a manifold periodic in Euclidean time with period
\( \beta \). Up to a contribution of a local functional measure it can be represented in the form

\[
\Gamma_{\beta} = \frac{1}{2} \text{Tr} \ln F + \delta^4(0) (\ldots), \tag{6.1}
\]

\[
F = -\Box + \frac{1}{6} R. \tag{6.2}
\]

This effective action and the corresponding free energy \( F_{\beta} = \Gamma_{\beta}/\beta \) have ultraviolet divergences. Note that, though the last term in Eq.(6.1) diverges, it is proportional to \( \beta \) and hence the free energy does not depend on \( \beta \) and its contribution into entropy vanishes. The same argument remains valid for all ultraviolet divergences in the effective action. Thus we have

\[
S = \beta^2 \frac{\partial}{\partial \beta} F_{\beta} = \beta^2 \frac{\partial}{\partial \beta} F_{\beta}^{\text{Ren}}. \tag{6.3}
\]

The effective action and thermodynamic potential for scalar fields at finite temperature in static curved spacetime were calculated by Dowker and Schofield [57]. It was proved that in the case of the conformal scalar field the effective actions in two conformally related spaces \( \bar{g}_{\mu\nu} = e^{-2\omega} g_{\mu\nu} \) are related to each other by the equation

\[
\Delta \Gamma[g, \omega] = \Gamma_{\beta}^{\text{Ren}}[\bar{g}] - \Gamma_{\beta}^{\text{Ren}}[g],
\]

\[
\Delta \Gamma[g, \omega] = -\frac{1}{2880\pi^2} \int_0^{\beta} d\tau \int d^3 x \left[ +3(\Box \omega)^2 - 4\omega^\sigma \omega_\sigma \Box \omega 
+ 2(\omega^\sigma \omega_\sigma)^2 - 2R_{\mu\nu} \omega^\mu \omega^\nu + \omega \left\{ R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - R_{\alpha\beta} R^{\alpha\beta} + \Box R \right\} \right], \tag{6.4}
\]

\[
\Gamma[\bar{g}] = \frac{1}{2} \text{Tr} \ln \bar{F}, \quad \bar{F} = -\Box + \frac{1}{6} \bar{R}, \quad \bar{g}_{\mu\nu} = e^{-2\omega} g_{\mu\nu}.
\]

The difference \( \Delta \Gamma[g, \omega] \) for two conformally related theories is proportional to \( \beta \) and hence does not contribute to the entropy. We apply these relations to the particular case of an ultrastatic metric \( \bar{g} \), i.e. when \( \omega \equiv \frac{1}{2} \ln g_{\tau\tau} \) and \( \omega_\mu = \nabla_\mu \omega \).

For the calculation of an effective action in the ultrastatic space \( \bar{g} \) it is convenient to apply the heat-kernel technique. In a proper time representation, the effective action of a scalar field on an ultrastatic 4-dimensional Euclidean manifold

\[
ds^2 = d\tau^2 + d\bar{l}^2, \tag{6.5}
\]

\[
d\bar{l}^2 = \frac{1}{(1 - 2M/r)^2} dr^2 + \frac{r^2}{1 - 2M/r} (d\theta^2 + \sin^2 \theta d\phi^2),
\]
which is periodic in $\tau$ with constant period $\beta$ takes the form

$$\Gamma[g] \equiv \Gamma_{\beta} = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \text{Tr} \bar{K}_{\beta}(s) .$$

(6.6)

The heat kernel $\bar{K}_{\beta}$ periodic in $\tau$ with a period $\beta$ is a solution of the problem

$$\frac{\partial}{\partial s} \bar{K}_{\beta}(s|\tau, x; \tau', x') = \bar{F} \bar{K}_{\beta}(s|\tau, x; \tau', x'),$$

$$\bar{K}_{\beta}(s|\tau, x; \tau', x') = \bar{K}_{\beta}(s|\tau + \beta, x; \tau', x'),$$

(6.7)

$$\bar{K}_{\beta}(0|\tau, x; \tau', x') = \delta(\tau - \tau')\delta(x - x').$$

It can be obtained by the method of images

$$\bar{K}_{\beta}(s|\tau, x; \tau', x') = \sum_{n=-\infty}^\infty (s|\tau, x; \tau' + \beta n, x')$$

(6.8)

from the nonperiodic heat kernel $\bar{K} = \bar{K}_\infty$ defined on a complete interval $-\infty < \tau, \tau' < \infty$. Due to the separation of variables in the operator $\bar{F} = -\partial^2/\partial \tau^2 - \bar{\Delta} + 1/6 \bar{R}$, the heat kernel $\bar{K}$ takes the form

$$\bar{K}(s|\tau, x; \tau', x') = (4\pi s)^{-\frac{1}{2}} \exp \left[-\frac{(\tau - \tau')^2}{4s}\right] \bar{K}^{\text{3d}}(s|x; x') ,$$

(6.9)

where $\bar{K}^{\text{3d}}(s|x; x')$ is 3-dimensional analogue of the heat kernel corresponding to the operator $-\bar{\Delta} + 1/6 \bar{R}$. From Eq.(6.8) and Eq.(6.9) we have

$$\bar{K}_{\beta}(s|\tau, x; \tau', x') = \theta_3 \left(-i\frac{(\tau - \tau')\beta}{4\pi s}, \exp \left[-\frac{\beta^2}{4s}\right]\right) \bar{K}(s|\tau, x; \tau', x') ,$$

(6.10)

where $\theta_3$ is a Riemann theta function

$$\theta_3(0, \exp[-b]) = \sum_{n=-\infty}^\infty \exp[-bn^2] .$$

(6.11)

The "zero-temperature" heat kernel $\bar{K}(s|\tau, x; \tau', x')$ can be expanded in nonlocal series in powers of curvatures of a spacetime $[58, 55]$. To calculate the effective action we need to know the trace of the heat kernel with coincident points $(\tau = \tau', x = x')$. In the notations of $[55]$ it reads

$$\text{Tr} \bar{K}_{\beta}(s|\tau, x; \tau, x) = \theta_3 \left(0, \exp \left[-\frac{\beta^2}{4s}\right]\right) \text{Tr} \bar{K}(s|\tau, x; \tau, x) ,$$

(6.12)

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\[
\text{Tr} \bar{K}(s|\tau, x; \tau, x) = \frac{1}{(4\pi s)^2} \int_0^\beta d\tau \int d^3x \sqrt{\bar{g}} \left\{ 1 + s\bar{P} + s^2 \left[ \bar{R}_{\mu\nu} f_1(-s \Box) \bar{R}^{\mu\nu} + \bar{R} f_2(-s \Box) \bar{R} \right] + P f_3(-s \Box) \bar{R} + \bar{P} f_4(-s \Box) \bar{P} \right\} + O(\text{Curvatures}^3),
\]

where \( f_i(-s \Box) \) are nonlocal form factors [55] and \( \bar{P} = 0 \) for conformal scalar field.

The first two terms in this expression are local. Nonlocalities appear only in quadratic and in higher orders in curvature terms.

We apply this formula to the Schwarzschild geometry. Near the horizon the corresponding ultrastatic metric Eq.(6.5) with a good accuracy describes the geometry of \( R^1 \times H^3 \) space. Where \( H^3 \) is a space of constant negative curvature

\[
ds_H^2 = d\tau^2 + dl^2 + (4M)^2 \sinh^2 \left[ \frac{l}{4M} \right] \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right). \tag{6.14}
\]

The difference between this metric and the metric (6.5)

\[
h^{\mu\nu}_i = \bar{g}^{\nu\sigma} [\bar{g}_{\mu\sigma} - g_{H\mu\sigma}] = \Delta(r) \text{diag}(0, 0, 1, 1) \tag{6.15}
\]

vanishes at the horizon as \( \sim r - 2M \). One can show that local invariants constructed from the tensors of the form \( \nabla_\alpha \cdots \nabla_\beta \nabla^\gamma \cdots \nabla^\delta h^{\mu\nu}_i = O(r - 2M) \) vanish at the horizon too. That is why all the invariants \( I \) constructed from the metric \( \bar{g} \) and curvature \( \bar{R} \) differ from the corresponding invariants \( I_H \) for the metric \( g_H \) by terms vanishing at the horizon \( I = I_H + O(r - 2M) \).

The heat kernel on spaces of constant negative curvature is known explicitly and for a conformally invariant scalar field it reads [59]

\[
\bar{K}_H(s|\tau, x; \tau', y) = \frac{1}{(4\pi s)^2} \frac{\sigma(x, y)}{4M \sinh \left[ \sigma(x, y)/4M \right]} \exp \left[ \frac{\tau^2 + \sigma^2(x, y)}{4s} \right] \sqrt{g_H}. \tag{6.17}
\]

Here \( \sigma(x, y) \) is the geodesic distance on the \( H^3 \) space section.

The relation between \( K \) and \( K_H \) for coincident points implies \( \bar{K}(s|\tau, x; \tau, x) = 1/(4\pi s)^2 + O(r - 2M) \) and we have

\[
\text{Tr} \bar{K}_\beta(s|\tau, x; \tau, x) = \frac{1}{(4\pi s)^2} \theta_3 \left( 0, \exp \left[ -\frac{\beta^2}{4s} \right] \right) \int_0^\beta d\tau \int d^3x \sqrt{\bar{g}}[1 + O(r - 2M)] \tag{6.18}
\]

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Composition of this expression, Eq.(6.4) and Eq.(6.12) gives for the free energy of a conformal scalar field the expression

\[ F^{\text{Ren}}_\beta - F^{\text{Ren}}_\infty = -\frac{\pi^2}{90} \int d\mathbf{x} g^{\frac{1}{2}} \left[ (g_{\tau\tau})^{-2} \frac{1}{\beta^4} - \left( \frac{1}{2\pi} \right)^4 (\omega^\sigma \omega_\sigma)^2 \right] + \ldots , \tag{6.19} \]

where the integral relation

\[ \int_0^\infty dx x^{a-1} [\theta_3(0, e^{-x}) - 1] = 2\Gamma(a)\zeta(2a) \quad \text{for} \ a = 2 \quad \frac{\pi^4}{45} \tag{6.20} \]

has been used and the physical metric \( g_{\mu\nu} \) has been restored. Dots designate terms which are less divergent or finite at the horizon. The corresponding entropy

\[ S = \beta^2 \frac{\partial}{\partial \beta} F^{\text{Ren}}_\beta = \beta^2 \frac{\partial}{\partial \beta} \left[ F^{\text{Ren}}_\beta - F^{\text{Ren}}_\infty \right] \]

\[ = \frac{2\pi^2}{45} \frac{1}{\beta^3} \int d\mathbf{x} (g_{\tau\tau})^{-2} g^{\frac{1}{2}} + \ldots , \tag{6.21} \]

calculated for a particular case of Schwarzschild black hole reproduces the result Eq.(5.30).

It is worth emphasizing that as a result of the restoration of the physical metric

\[ \bar{g}_{\mu\nu} = e^{-2\omega} g_{\mu\nu} = \frac{1}{g_{\tau\tau}} g_{\mu\nu} , \quad \sqrt{\bar{g}} = e^{-\omega} \sqrt{g} = \frac{1}{(g_{\tau\tau})^{\frac{1}{2}}} \sqrt{g} , \tag{6.22} \]

\[ \bar{R}_\mu^\nu = e^{2\omega} \left[ R_\mu^\nu + 2 \omega^\nu_{\mu} + \Box \omega^{\nu}_{\mu} + 2 \omega^\nu_{\mu} \omega^\sigma_{\lambda} - 2 \omega^\sigma \omega_\sigma \delta_\mu^\nu \right] \]

in the general expansion Eq.(6.12) we get a nonlocal expansion of the effective action in terms of curvature, ”acceleration” \( \omega_\mu \) and their derivatives. One can use this effective action in order to get \( < T_{\mu\nu} >^{\text{Ren}} \). The action can be written in a completely invariant form if we substitute \( g_{\tau\tau} = g_{\mu\nu} \xi^\mu \xi_\nu \) and consider \( \xi^\mu \) as external field, which is fixed during the variations over \( g_{\mu\nu} \) and is taken to coincide with the Killing vector field after the variations were performed. An additional (external) vector field \( \xi \) in the effective action for thermal state is required because such a state is possible only in a stationary spacetime, i.e. the spacetime with additional geometric structure.

7. Conclusions

In conclusion we make some remarks concerning the obtained result. The proposed no-boundary anzatz for a wave function of a black hole appears to be a natural
approach, which in particular allows one to give a covariant description of its degrees of freedom. The quantum state of a black hole is characterized by an amplitude of different realizations of dynamical variables on the Einstein-Rosen bridge. For any particular realization one can define a Euclidean horizon by means of the following procedure. Take a two-sphere $S$ from the first non-trivial homotopic class $\pi_2$ and define $A[S]$ as the surface area of $S$. Because $S$ is non-contractable the functional $A[S]$ has a non-vanishing minimum. The corresponding sphere $S_0$ defines a position of the Euclidean horizon for a chosen realization. A surface $S_0$ changes from one realization to another. This dependence of $S_0$ on the realization can be interpreted as quantum fluctuations of the horizon. The effect of quantum fluctuation (zitterbewegung) of the horizon is important for the problem of entropy discussed in the paper.

The one-loop calculations of entropy of internal degrees of freedom of a black hole was shown to be divergent at the horizon. The divergency has a universal law near the horizon of a black hole for all fields (massless and massive, with and without spin). The divergences arise because in our one-loop approximation the background geometry (and hence the position of the Euclidean horizon) is fixed. Quantum fluctuations of the horizon result in its spreading. Due to spreading we cannot any more to split the states of quantum fields located inside the region of a fluctuating horizon into the ’visible’ and ’invisible’ ones. In other words for any chosen realization of a quantum field the splitting of states into internal and external states of a black hole depends on the realization. Averaging over different realization (which effectively takes into account the zitterbewegung of the horizon) may produce the required cut-off for the entropy.

It is expected that the quantum effects with the properly described fluctuations of the horizon must give the standard expression $A/4$ for the entropy of a black hole. It is important that this result must not depend on the number of fields and their properties. We should emphasize that in the framework of our approach the dynamical degrees of freedom of a black hole contribute to the entropy only on the one-loop level, while there is no tree-level contributions. The remarkable fact is that in the standart Euclidean action approach the ”correct” answer for the entropy ($A/4l_p^2$) is obtained by calculating the topological tree-level contribution into the Euclidean gravitational action. The
relation between "dynamical" and "topological" contributions to the entropy as well as the origin of the universality of the expression for the entropy of a black hole is a real puzzle.

Recently, an elegant proposal [28] has been given for a mechanism maintaining the exact relation between the black hole entropy and its horizon area on the nonperturbative level of quantum gravitational thermodynamics in the limit of very heavy black holes. Briefly it looks as follows. Suppose, we have the gravitational effective action of the theory \( \Gamma[g] \), possibly generated by the fundamental theory of (super)strings and, therefore, finite. It may have a very general structure about which only one assumption is made: it is supposed to be analytic in the curvature and free from the effective cosmological term (thus admitting the existence of the asymptotically flat solutions of effective Einstein equations)

\[
\Gamma[g] = \sum_{n=1}^{\infty} \int dx_1...dx_n \Gamma_n(x_1,...,x_n) R(x_1)...R(x_n). \quad (7.1)
\]

Here \( R(x) \) is a collective notation for the curvature and Ricci tensors and \( \Gamma_n(x_1,...,x_n) \) is a set of (generally nonlocal) form factors accumulating all the information about the quantum and statistical effects in the theory. Since these form factors represent the coordinate kernels of some nonlocal operators constructed of derivatives, the only covariant expression available for \( \Gamma_1(x) \) is just the local density

\[
\Gamma_1(x) = -\frac{1}{16\pi^2 l_{\text{eff}}^2} g^{1/2}(x) \quad (7.2)
\]

with a purely numerical coefficient which can be identified with the effective (renormalized) gravitational constant or Planck length \( l_{\text{eff}} \) (all the covariant derivatives in \( \Gamma_1 \) contract to form a total derivative which disappears when integrated over asymptotically flat spacetime).

According to eq. (5.16) the calculation of entropy involves the effective action \( \Gamma_\beta = \Gamma[g^\beta] \) calculated on the conical spacetime with metric \( g^\beta \) having a conical singularity with \( \beta \neq 8\pi M \). On such a manifold the curvature has a form

\[
R_\beta(x) = (\beta - 8\pi M) f(x) + R_{\text{reg}}(x), \quad (7.3)
\]

where \( R_{\text{reg}}(x) \) is a regular part of the curvature bounded by \( 1/M^2 \) and, therefore, negligible for heavy black holes \( M \to \infty \). The singular part caused by conical structure
for $\beta \neq 8\pi M$ involves the generalized function $f(x)$ which, when regulated, can be even nonsingular one, but having the compact support in the vicinity of the tip of the cone (black hole horizon) and satisfying the relation

$$\int dx\, g^{1/2}(x)\, f(x) = -8\pi M.$$  \hspace{0.5cm} (7.4)

Substituting the structure (7.3) into (7.1) and using (5.16) we immediately find that the entropy is entirely generated by the effective Einstein term of the action, because the expansion in powers of the curvature becomes the expansion in powers of the angle deficit ($\beta - 8\pi M$) of the conical manifold:

$$S = \left(\beta \frac{\partial}{\partial \beta} - 1\right) \Gamma_\beta = \beta \int dx\, \Gamma_1(x)\, R(x) = \frac{A}{4\ell_{\text{eff}}^2}. \hspace{0.5cm} (7.5)$$

The above arguments could have been even generalized to the case of the finite-mass black hole by noting that in asymptotically flat spacetime the actual expansion of the effective action can be performed in powers of the Ricci curvature $R_{\mu\nu}$ only \cite{55, 56}, for which $R_{\mu\nu\text{reg}}(x) \equiv 0$ in eq.(7.3). However there is a serious objection to this mechanism which apparently invalidates this proposal. If it were correct then the perturbative calculations of entropy would maintain the universal relation between the entropy and one quarter of the horizon area in units of the effective Planck length, the quantum corrections to the classical entropy being compensated by the simultaneous renormalization of this length. But this is definitely not the case for the dominant divergent contribution (6.21) obtained in the one-loop approximation. Indeed, as it follows from eq.(6.19), this contribution involves the invariant of the Killing vector field $x^\mu \xi_\mu = g_{\tau\tau}$. This invariant can be regarded as a restriction of some nonlocal functional of metric to the manifold with Killing symmetries. Killing field $\xi^\mu$ as a functional of the metric does not have a unique continuation off the symmetric (Killing) points in the configuration space of metric, but it is undoubtedly nonlocal and most likely has a structure of the solution of the Killing equation

$$\Box \xi^\mu + R^\mu_\nu \xi^\nu = 0 \hspace{0.5cm} (7.6)$$

as a functional of the metric and boundary conditions $\xi^\mu = \xi^\mu[g, \text{boundary data}]$. The boundary data is an inalienable part of the solution of (7.6), and this data is nontrivial
and nontrivially depends on $\beta$. This means, that iteratively solving the equation (7.6) we can obtain $\xi^\mu$ as a nonlocal expansion in curvatures, but the nontrivial dependence on $\beta$ will enter this functional through boundary conditions. Therefore, the dependence of $\Gamma_\beta$ on $\beta$ will be induced not only by the metric argument of $\Gamma[g]$: $\Gamma_\beta = \Gamma[g^\beta, \beta]$ ($\Gamma_n(x_1, \ldots x_n) \equiv \Gamma_n(\beta | x_1, \ldots x_n)$) and the above mechanism will break down, since the first-order term in $(\beta - 8\pi M)$ will no longer be generated by the Einstein term of the effective action.

Even if this specific mechanism proposed by Susskind does not work, there may be other solutions of the puzzle. But it looks like practically impossible to explain the huge entropy of black holes without relating it to the properties of vacuum in a strong gravitational field of a black hole and without identifying the dynamical degrees of freedom of a black holes with states of physical fields located inside a black hole.

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**A Lichnerowicz equation and the geometry of Einstein-Rosen bridge**

Here we show that the three-geometry of a spatial section on which we define the no-boundary wavefunction of true physical variables (3.11), $(g^T, p_T) = (h^T_{ab}, p^b_T, \text{matter variables})$ coincides with the geometry of the Einstein-Rosen bridge in the lowest order of the perturbation theory in $(g^T, p_T)$. This approximation corresponds to a ground state of physical excitations (of both matter and gravitational fields) on the spatial section with the topology (3.3).

Consider three-geometry $\tilde{g}_{ab}$ and define

$$\tilde{\beta}^{ab} = \epsilon^{aef} \nabla_e [\sqrt{\tilde{g}} (\tilde{R}^b_f - \frac{1}{4} \delta^b_f \tilde{R})]. \quad (A.1)$$
York [43] showed that $\tilde{\beta}^{ab}$ gives a pure spin-two representation of intrinsic geometry. Conditions $\tilde{\beta}^{ab} = 0$ together with $p_{ab} = 0$ specify the state where no dynamical gravitational perturbations are present. In the absence of matter the Lichnerowicz equation (3.9) reduces to the equation

$$^{3}R = 0. \quad (A.2)$$

Condition $\tilde{\beta}^{ab} = 0$ implies that the three-metric is conformally flat

$$dl^{2} = \Phi^{4}d\tilde{l}_{0}^{2} = \Phi^{4}(dx^{2} + dy^{2} + dz^{2}). \quad (A.3)$$

The Lichnerowicz equation (A.2) in this case is equivalent to the equation

$$\Delta \Phi = 0 \quad (A.4)$$

for the conformal factor $\Phi$. A solution which is regular everywhere is constant and the corresponding geometry is a flat three-dimensional space $R^3$. Non-trivial solutions have singularities. A solution with one simple pole generates a three-dimensional space $S^2 \times R^1$ with the Einstein-Rosen bridge geometry. We choose coordinates so that the pole is located at the origin of coordinates, then we have

$$\Phi = 1 + \frac{M}{2\rho}, \quad (A.5)$$

where $\rho^{2} = x^{2} + y^{2} + z^{2}$. For this conformal factor the metric $dl^2$ can be written as

$$dl^2 = \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2, \quad (A.6)$$

where $r \equiv \rho (1 + M/2\rho)^2$. A point $\rho = \infty$ corresponds to spatial infinity of $\Sigma_+$, while a point $\rho = 0$ corresponds to spatial infinity of $\Sigma_-$, the constant $M$ being the mass ($M_+ = M_- = M$). The important property of the obtained solution describing the state without excitation is that the corresponding three-metric is spherically-symmetric.

The metric (A.3) with (A.5) can be identically rewritten in the form in which both spatial infinities are represented in the completely symmetric way. To do this we remind that the flat metric is conformally related with a metric on a three-sphere $S^3$, so that we have

$$dl^2 = \tilde{\Phi}^4(d\chi^2 + \sin^2 \chi d\Omega^2) = \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2, \quad (A.7)$$

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where $\tilde{\Phi}_0$ is a solution of the conformal invariant equation on the three-sphere

$$(\tilde{\Delta} - \frac{1}{8} 3\tilde{R}) \tilde{\Phi}_0 = 0,$$  \hspace{1cm} (A.8)

which is of the form $\tilde{\Phi}_0 = \phi_0/\sin \chi$, with $\phi_0 = M^{1/2}[\sin (\chi/2) + \cos (\chi/2)]$.

In the presence of gravitational perturbations and matter the Lichnerowicz equation (3.9) reads

$$(\tilde{\Delta} - \frac{1}{8} 3\tilde{R}) \tilde{\Phi} = J,$$  \hspace{1cm} (A.9)

where the source $J$ in terms of conformally transformed variables looks as

$$J = -\frac{1}{8} (\tilde{g})^{-1} 3\tilde{\rho}^{ab} 3\tilde{\rho}_{ab} \Phi^{-7} - 2\pi \tilde{T}_* \Phi^{-3}. $$  \hspace{1cm} (A.10)

Denote by $G(x, x')$ the Green function defined as the solution of the equation

$$(\tilde{\Delta} - \frac{1}{8} 3\tilde{R}) G(x, x') = -3\delta(x, x'). $$  \hspace{1cm} (A.11)

The solution of the equation (A.9) can be presented in the form

$$\tilde{\Phi} = \tilde{\Phi}_0 + \int G(x, x') J(x') d\chi'. $$  \hspace{1cm} (A.12)

The solution $\tilde{\Phi}_0$ is invariant with respect to the reflection $\chi \to \pi - \chi$. In general case $J$ does not obeyes this property and the solution $\tilde{\Phi}$ is not invariant under reflection and asymptotic values of $M_+$ and $M_-$ are different. In order to illustrate this general property we consider here a simple case when $J$ is spherically symmetric.

We write $\tilde{\Phi}$ in the form $\tilde{\Phi} = \phi/\sin \chi$. The function $\phi$ obeys the equation

$$\frac{d^2 \phi}{d\chi^2} + \frac{1}{4} \phi = j \equiv J \sin \chi. $$  \hspace{1cm} (A.13)

and has a general solution in terms of the Green function $G(\chi, \chi')$:

$$\phi(\chi) = \phi_0(\chi) + \int_0^\pi G(\chi, \chi') j(\chi') d\chi', $$  \hspace{1cm} (A.14)

$$G(\chi, \chi') = -2 \{ \theta(\chi - \chi') \sin(\chi'/2) \cos(\chi'/2) + \theta(\chi' - \chi) \sin(\chi/2) \cos(\chi'/2) \}. $$  \hspace{1cm} (A.15)

The asymptotic masses at two spatial infinities are

$$M_+ = \phi \left. \frac{d\phi}{d\chi} \right|_{\chi=0}, \quad M_- = -\phi \left. \frac{d\phi}{d\chi} \right|_{\chi=\pi}. $$  \hspace{1cm} (A.16)
From this expression it follows that

$$\phi(0) = M^{1/2}, \quad \phi(\pi) = M^{1/2} \quad \phi'(0) = M^{1/2}/2 - \alpha, \quad \phi'(\pi) = -M^{1/2}/2 + \beta,$$

$$\alpha \equiv \int_0^\pi \cos(\chi'/2)j(\chi')d\chi', \quad \beta \equiv \int_0^\pi \sin(\chi'/2)j(\chi')d\chi',$$

whence

$$M_+ - M_- = 2M^{1/2}(\beta - \alpha).$$

This relation shows that in the general case the asymmetric distribution of matter on the Einstein-Rosen bridge results in different masses $M_+$ and $M_-$ at two asymptotic infinities. For a known distribution and fixed $M_+$ the value of $M_-$ can be obtained by solving of the Lichnerowicz equation.

\section*{B \ R-Modes}

In this appendix we construct the basis of positive frequency solutions

$$w_\lambda = \frac{1}{\sqrt{4\pi\omega}} \exp(-i\omega t)R_{\omega lm}(r, \vartheta, \phi)$$

for the scalar field in the exterior region $R_+$ of the eternal black hole, for which spatial functions $R_{\omega lm}$ are real ($R$-modes).

By using the separation of variables for the equation $\Box \varphi = 0$ we write

$$R_{\omega lm}(r, \vartheta, \phi) = R_{\omega l}(r)\hat{Y}_{lm}(\vartheta, \phi),$$

where

$$\hat{Y}_{lm}(\vartheta, \phi) = P^m_l(\vartheta) \left\{ \begin{array}{ll}
\frac{1}{\sqrt{2\pi}}, & m = 0, \\
\frac{1}{\sqrt{\pi}} \cos m\phi, & 0 < m \leq l; \\
\frac{1}{\sqrt{\pi}} \sin m\phi, & -l \leq m < 0.
\end{array} \right.$$

We choose the spherical harmonics $\hat{Y}_{lm}$ to be real so that the R-basis will be constructed if solutions $R_{\omega l}(r)$ of the radial equation (5.23) are chosen to be real. Denote $\hat{R}_{\omega l}(r) = rR_{\omega l}(r)$ then the radial equation reads

$$\frac{d^2\hat{R}_{\omega l}}{dr^2} + \left(\omega^2 - V_l\right)\hat{R}_{\omega l} = 0,$$
where \( r^* = r - 2M + 2M \ln [(r - 2M)/2M] \), and
\[
V_l = \left(1 - \frac{2M}{r}\right) \left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right].
\] (B.5)

For any two solutions of (B.4) the Wronskian \( W[f_1, f_2] \equiv f_1 \frac{df_2}{dr^*} - f_2 \frac{df_1}{dr^*} = \text{const} \).

Functions \( \hat{R}_{\omega l} \) have the asymptotics \( \exp(\pm i\omega r^*) \) at \( r^* \to -\infty \) and \( \exp(\pm i\omega r^*) \) at \( r^* \to -\infty \). We begin by defining so called \( UP \)-modes which are specified (for \( \omega > 0 \)) by the asymptotics
\[
\hat{R}_{\omega l}^{\text{up}}(r) = \begin{cases} e^{i\omega r^*} + r_{\omega l} e^{-i\omega r^*}, & r^* \to -\infty, \\ t_{\omega l} e^{i\omega r}, & r \to \infty. \end{cases}
\] (B.6)

By comparing the Wronskians at \( r^* = \pm \infty \) for \( \hat{R}_{\omega l}^{\text{up}} \) and its complex conjugated one gets the standard relations between reflection and absorption coefficients
\[
|r_{\omega l}|^2 + |t_{\omega l}|^2 = 1.
\] (B.7)

The coefficients of the radial equation are real. That is why \( \hat{R}_{\omega l}^{\text{down}}(r) \equiv \bar{\hat{R}}_{\omega l}^{\text{up}}(r) \) is again a solution. One has
\[
\bar{\hat{R}}_{\omega l}^{\text{up}}(r) = \hat{R}_{\omega l}^{\text{down}}(r),
\] (B.8)
so that \( \bar{r}_{\omega l} = r_{-\omega l} \) and \( \bar{t}_{\omega l} = t_{-\omega l} \). The \( Re \)- and \( Im \)-parts of \( \hat{R}_{\omega l}^{\text{up}}(r) \) (for \( \omega > 0 \)) can be used as real basic solutions. The problem is that the corresponding solutions \( w_{\lambda} \) do not possess the proper normalization conditions. Namely one has
\[
(w_{\omega l m}^{\text{up}}, w_{\omega l' m'}^{\text{up}}) = \delta(\omega - \omega') \delta_{l l'} \delta_{m m'},
\] (B.9)
\[
(w_{\omega l m}^{\text{down}}, w_{\omega l' m'}^{\text{down}}) = \delta(\omega - \omega') \delta_{l l'} \delta_{m m'},
\] (B.10)
\[
(w_{\omega l m}^{\text{up}}, w_{\omega l' m'}^{\text{up}}) = r_{\omega l} \delta(\omega - \omega') \delta_{l l'} \delta_{m m'}.
\] (B.11)

Here
\[
(f_1, f_2) = -i \int (\tilde{f}_2 f_{1, \mu} - \tilde{f}_2 f_{1, \mu}) d\sigma^\mu
\] (B.12)
is a scalar product in the space of solutions.

The proper normalization conditions can be satisfied by the following linear transformation of the basic functions
\[
\hat{R}_{\omega l}^{\text{up'}} = a_{\omega l} \hat{R}_{\omega l}^{\text{up}} + b_{\omega l} \hat{R}_{\omega l}^{\text{down}},
\] (B.13)
\[
\hat{R}_{\omega l'}^{\text{down'}} = b_{\omega l} \hat{R}_{\omega l}^{\text{up}} + a_{\omega l} \hat{R}_{\omega l}^{\text{down}},
\] (B.14)
where
\[ a_{\omega l} = \frac{\sqrt{1 + |t_{\omega l}|}}{\sqrt{2|t_{\omega l}|}}, \quad b_{\omega l} = -\frac{r_{\omega l}}{\sqrt{2|t_{\omega l}|\sqrt{1 + |t_{\omega l}|}}}. \]  
(B.15)

The following functions are real and for \( \omega > 0 \) form a proper normalized basis
\[ \hat{R}^{\text{real}}_{\omega l} = \frac{1}{\sqrt{2}}(\hat{R}^{\text{up}'\omega l} + \hat{R}^{\text{down}'\omega l}), \]  
(B.16)
\[ \hat{R}^{\text{real}}_{\omega l2} = \frac{1}{i\sqrt{2}}(\hat{R}^{\text{up}'\omega l} - \hat{R}^{\text{down}'\omega l}). \]  
(B.17)

To summarize we construct the basis \( \{w_\lambda\} \) (\( \omega > 0, \ A = 1, 2 \))
\[ w_\lambda = \frac{1}{\sqrt{4\pi\omega}} \exp(-i\omega t)R^{\text{real}}_{\omega l m A}(r, \vartheta, \phi), \]  
(B.18)
where \( R^{\text{real}}_{\omega l m A} \equiv r^{-1} \hat{R}^{\text{real}}_{\omega l A} \hat{Y}_{lm}(\vartheta, \phi) \) are real functions.
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