Holomorphic bundles for higher dimensional gauge theory

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Abstract
Motivated by gauge theory under special holonomy, we present techniques to produce holomorphic bundles over certain non-compact 3-folds, called building blocks, satisfying a stability condition ‘at infinity’. Such bundles are known to parametrize solutions of the Yang–Mills equation over the $G_2$-manifolds obtained from asymptotically cylindrical Calabi–Yau 3-folds studied by Kovalev, Haskins et al. and Corti et al. The most important tool is a generalization of Hoppe’s stability criterion to holomorphic bundles over smooth projective varieties $X$ with $\text{Pic} \, X \simeq \mathbb{Z}^l$, a result which may be of independent interest. Finally, we apply monads to produce a prototypical model of the curvature blow-up phenomenon along a sequence of asymptotically stable bundles degenerating into a torsion-free sheaf.

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1. Introduction

This paper presents cohomological methods to construct and study examples of holomorphic bundles, over certain Fano 3-folds, having the property of asymptotic stability (see below) on a distinguished anti-canonical divisor, said to be ‘at infinity’ for geometrical reasons. It is based on the theory of instanton monads developed by several authors in the past three decades, in particular [8, 9]. Our motivation comes from gauge theory in higher dimensions, especially the interplay between Yang–Mills theory over Calabi–Yau 3-folds and $G_2$-manifolds.

An important method to produce examples of compact 7-manifolds with holonomy exactly $G_2$ is the twisted connected sum construction [3, 10, 11]. It consists of gluing a pair of asymptotically cylindrical (ACyl) Calabi–Yau 3-folds obtained from certain smooth projective 3-folds called building blocks. A building block $(Z, D)$ [3, Definition 3.5] is given by a projective morphism $f: Z \to \mathbb{P}^1$ such that $D := f^{-1}(\infty)$ is a smooth anti-canonical $K3$ surface, under certain mild topological assumptions; in particular, $D$ has trivial normal bundle. Choosing a convenient Kähler structure on $Z$, one can make $W := Z \setminus D$ into an ACyl Calabi–Yau 3-fold, that is, a non-compact Calabi–Yau with a tubular end modelled on $\mathbb{R}_+ \times S^1 \times D$. Then $S^1 \times W$ is an ACyl $G_2$-manifold with a tubular end modelled on $\mathbb{R}_+ \times T^2 \times D$ [4, 6].

Received 27 July 2015; revised 8 October 2016; published online 17 January 2017.

2010 Mathematics Subject Classification 14D21, 14F05 (primary), 53C07 (secondary).

M. Jardim was partially supported by CNPq grant 303332/2014-0 and FAPESP grant 2014/14743-8. G. Menet was supported by FAPESP grant 2014/05733-9. D. M. Prata was supported by FAPESP grant 2011/21398-7. H. N. Sá Earp was supported by FAPESP grant 2009/10067-0 and is partially supported by CNPq grant 312390/2014-9 and FAPESP grant 2014/24727-0.
In particular, several topological types of building blocks can be obtained as \( Z = \text{Bl}_\ell X \) by blowing up Fano 3-folds \( X \) along a self-intersection curve \( \ell \in |D \cdot D| \) for \( D \in |\mathcal{O}_X| \); these will be the varieties of interest in the present paper. It is reasonable to expect similar methods to work on more general building blocks (for example, blow-ups of the so-called weak Fanos), which will be addressed in the future. For a more detailed exposition of building blocks, we suggest the Introduction section of \([18]\) and references therein.

From the point of view of gauge theory, the last named author established the existence of Hermitian Yang–Mills (HYM) connections over ACyl Calabi–Yau 3-folds \([16, 17]\). The concept of asymptotic stability emerges as the natural boundary condition for that analytical problem. Let \( z = e^{-s + i\alpha} \) be the holomorphic coordinate along the tubular end, and denote by \( D_z \) the corresponding \( K3 \) fibre near infinity. A bundle \( E \to W \) is called asymptotically stable (or stable at infinity) if it is the restriction of an indecomposable holomorphic vector bundle \( E \to Z \) such that \( E|_D \) is stable (hence, also \( E|_{D_z} \) for small \( |z| < \delta \)). Such a bundle admits a smooth Hermitian reference metric \( H_0 \), with the property that \( H_0|_{D_z} \) are the corresponding HYM metrics on \( E|_{D_z} \), for \( 0 \leq |z| < \delta \), and which has ‘finite energy’, in a suitable sense.

Our crucial motivation is the fact that, given an asymptotically stable bundle with reference metric \((E, H_0)\), there exists a non-trivial smooth solution to the \( G_2 \)-instanton equation on \( p_1^*E \to W \times S^1 \) \([17 \text{, Theorem } 58]\). Our central aim therefore is to construct explicit examples of such asymptotically stable bundles; actually, it suffices to construct bundles \( E \to X \) over the original Fano 3-fold with \( E|_D \) stable, since stability pulls back under the proper transform \( Z \to X \).

It should be noted that, under certain rigidity and transversality assumptions, such solutions can be glued, according to the twisted connected sum, to produce a \( G_2 \)-instanton over the resulting compact 7-manifold with holonomy \( G_2 \) \([18]\). Thus, transversal matching pairs of asymptotically stable bundles parametrize (some) solutions to the corresponding 7-dimensional Yang–Mills equation. However, the matching problem for the tubular \( G_2 \)-instantons obtained with our methods is a non-trivial matter to be addressed in future work.

Outline

This article is organized as follows. In Section \( 2 \), we review standard stability theory and we establish a generalization of the so-called Hoppe criterion, which gives sufficient conditions for stability of a bundle over a projective variety with finitely generated Picard group in terms of the vanishing of certain cohomologies. In Section \( 3 \), we construct bundles over various types of building blocks of Picard rank 1. In Section \( 4 \), we use the famous Hartshorne–Serre correspondence to construct examples over polycyclic Fano 3-folds, and apply the generalized stability criterion to establish their asymptotic stability. Finally, Section \( 5 \) has a somewhat different vein, illustrating a convenient use of instanton monads to model degenerations of asymptotically stable bundles into torsion-free sheaves, with an explicit calculation of the curvature blow-up rate for a natural choice of metrics.

2. The generalized Hoppe criterion

Let \( X \) be a non-singular projective variety, that is, a non-singular, projective, integral, separated Noetherian scheme of finite type over the field of complex numbers. Let \( L \) be an ample line bundle over \( X \), and set \( n := \dim X \).

The \( L \)-degree of a coherent sheaf \( E \to X \) is defined as usual by

\[
\deg_L E := c_1(E) \cdot L^{n-1},
\]
and, setting \( r := \text{rk}(E) \), the \( L \)-slope of \( E \) is
\[
\mu_L(E) := \frac{\deg_L E}{r}.
\]
Then \( E \) is (semi)stable if, for every proper coherent subsheaf \( F \subset E \) such that \( E/F \) is torsion-free, one has
\[
\mu_L(F) < \mu_L(E).
\]
If \( E \) is locally free, in order to test for stability it suffices to consider reflexive subsheaves \( F \subset E \).

2.1. Hoppe’s criterion over cyclic varieties
Suppose further that \( \text{Pic}(X) \simeq \mathbb{Z} \); such varieties are called cyclic. Given a locally free sheaf (or, equivalently, a holomorphic vector bundle) \( E \to X \) as above, there is a unique integer \( k_E \) such that
\[
-r + 1 \leq c_1(E(-k_E)) \leq 0.
\]
Setting \( E_{\text{norm}} := E(-k_E) \), we say that \( E \) is normalized if \( E = E_{\text{norm}} \). Then one has the following stability criterion [7, Lemma 2.6]:

**Proposition 1** (Hoppe criterion). Let \( E \) be a rank \( r \) holomorphic vector bundle over a cyclic projective variety \( X \). If \( H^0((\wedge^q E)_{\text{norm}}) = 0 \) for \( 1 \leq q \leq r - 1 \), then \( E \) is stable. If \( H^0((\wedge^q E)_{\text{norm}}(-1)) = 0 \) for \( 1 \leq q \leq r - 1 \), then \( E \) is semistable.

Note: In general, the converse of the previous statement is false. Take, for instance, the nullcorrelation bundle \( N \to \mathbb{P}^{2k+1} \) with \( k \geq 2 \), that is, the stable rank \( 2k \) bundle [5, Theorem 1.4] given by the following exact sequence:
\[
0 \longrightarrow \mathcal{O}_{\mathbb{P}^{2k+1}}(-1) \longrightarrow \Omega^1_{\mathbb{P}^{2k+1}}(1) \longrightarrow N \longrightarrow 0.
\]
As shown in [1, Lemma 1.10], for each \( 0 \leq j \leq k - 1 \), the exterior product \( \Lambda^{2j}N \) has the trivial line bundle \( \mathcal{O}_{\mathbb{P}^{2k+1}} \) as a direct summand, hence a non-trivial section.

Hoppe’s original result is proved only for the case \( X = \mathbb{P}^n \), but the proof generalizes easily for cyclic varieties. Conversely, for rank 2 bundles, stability is, in fact, equivalent to the non-existence of holomorphic sections:

**Criterion 2.** If \( F \) is a rank 2 holomorphic vector bundle over a cyclic variety \( X \), then \( F \) is stable if and only if \( h^0(F_{\text{norm}}) = 0 \).

Compare the above with [15, Lemma 1.2.5, p. 165] for the case \( X = \mathbb{P}^n \); the proof for \( X \) cyclic is the same. A similar necessary and sufficient criterion for rank 3 bundles can also be found in [15, Lemma 1.2.6b, p. 167]. In any case, Hoppe’s approach has been, up until now, the most general framework for co-homological stability criteria.

2.2. Hoppe’s criterion over polycyclic varieties
We now present a generalization of the Hoppe stability criterion for the much larger class of polycyclic varieties with \( \text{Pic}(X) = \mathbb{Z}\ell \). Given a divisor \( B \) on \( X \), we define, for convenience,
\[
\delta_L(B) := \deg_L \mathcal{O}_X(B).
\]
THEOREM 3 (Generalized Hoppe criterion). Let $G \to X$ be a holomorphic vector bundle of rank $r \geq 2$ over a polycyclic variety $X$ equipped with a polarization $L$; if

$$H^0(X, (\wedge^s G) \otimes \mathcal{O}_X(B)) = 0$$

for all $B \in \text{Pic}(X)$ and all $s \in \{1, \ldots, (r - 1)\}$ such that

$$\delta_L(B) \leq -s\mu_L(G),$$

then $G$ is (semi-)stable.

Conversely, if $G$ is (semi-)stable, then

$$H^0(X, G \otimes \mathcal{O}_X(B)) = 0, \forall B \in \text{Pic}(X) \text{ such that } \delta_L(B) \leq -\mu_L(G).$$

Proof. First, assume that there is $F \hookrightarrow G$ a torsion free subsheaf of rank $s$, such that $\det F = \mathcal{O}_X(B)$. The inclusion induces a map $\wedge^s F \hookrightarrow \wedge^s G$ and so

$$H^0(X, (\wedge^s G) \otimes \mathcal{O}_X(-B)) \neq 0.$$

By hypothesis, we have

$$\delta_L(-B) > -s\mu_L(G),$$

hence

$$\mu_L(F) = \frac{\delta_L(B)}{s} < \mu_L(G),$$

and therefore $F$ is not a destabilising sheaf.

Conversely, assume that $H^0(X, G \otimes \mathcal{O}_X(B))$ has a section with

$$\delta_L(B) \leq -\mu_L(G).$$

This induces a map $\mathcal{O}_X(B) \hookrightarrow G$ with

$$\mu_L(\mathcal{O}_X(B)) \geq \mu_L(G),$$

so $G$ is not (semi-)stable. □

COROLLARY 4. Let $G \to X$ be a holomorphic vector bundle of rank 2 over a polycyclic variety, and let $L$ be a polarization on $X$. The bundle $G$ is (semi)-stable if and only if

$$H^0(X, G \otimes \mathcal{O}_X(B)) = 0$$

for every $B \in \text{Pic}(X)$ such that

$$\delta_L(B) \leq -\mu_L(G).$$

3. Asymptotic stability over cyclic Fano-type building blocks

The existence of asymptotically stable bundles is a natural and important question, since they parametrize solutions to the HYM equation and hence $G_2$-instantons. Let us observe form the outset that an affirmative example, if somewhat exotic, is known from Mukai’s study of certain prime subvarieties $X_{22} \subset \mathbb{P}^{13}$; such spaces come equipped with a rank 2 holomorphic bundle which is indeed asymptotically stable [13, 14]. We propose a much more general setup to obtain examples over a larger class of building blocks. Let us begin by recalling their precise definition [3, Definition 3.5].
Definition 5. A building block is a non-singular algebraic 3-fold $Z$ together with a projective morphism $f: Z \to \mathbb{P}^1$ satisfying the following assumptions.

(i) The anti-canonical class $-K_Z \in H^2(Z, \mathbb{Z})$ is primitive.

(ii) $D = f^{-1}(\infty)$ is a non-singular K3 surface and $D \sim -K_Z$.

In addition, identify $H^2(D, \mathbb{Z})$ with the K3 lattice $L$ (that is, choose a marking for $D$), and let $N$ denote the image of $H^2(Z, \mathbb{Z}) \to H^2(D, \mathbb{Z})$.

(iii) The inclusion $N \hookrightarrow L$ is primitive.

(iv) The groups $H^3(Z, \mathbb{Z})$ and $H^4(Z, \mathbb{Z})$ are torsion-free.

The building blocks considered in this paper are given by [3, Definition 3.15].

Proposition 6. Let $X$ be a Fano 3-fold, $|D_0, D_\infty| \subset |-K_X|$ a generic pencil with (smooth) base locus $\ell$, $D \in |D_0, D_\infty|$ generic, and $Z$ the blow-up of $X$ at $\ell$. Then, $D$ is a smooth K3 surface, its proper transform in $Z$ is isomorphic to $D$ and $(Z, D)$ is a building block. Such building blocks are called of Fano type, while $X$ is referred to as the underlying Fano 3-fold.

Throughout this section, $X$ will denote the Fano 3-fold underlying a building block $(Z, D)$ constructed as above.

Remark 7. Let $(Z, D)$ be a building block, then we say that a bundle $E$ on $Z$ is asymptotically stable if $E|_D$ is stable. Similarly, if $X$ is a Fano 3-fold, $F$ a bundle on $X$ and $D \in |-K_X|$ a smooth K3 surface, we say that $F$ is asymptotically stable if $F|_D$ is stable.

We remark that in order to provide asymptotically stable bundles on a given building block, it is enough to construct asymptotically stable bundles on its underlying Fano 3-fold. Indeed, let $X$ be a Fano 3-fold, $|D_0, D_\infty| \subset |-K_X|$ a generic pencil with (smooth) base locus $\ell$, $D \in |D_0, D_\infty|$ generic, and $r: Z \to X$ the blow-up of $X$ at $\ell$. If $F$ is an asymptotically stable bundle on $X$, then $r^*F$ is an asymptotically stable bundle on $Z$. This is the strategy we adopt below.

3.1. Stable bundles from monad constructions

Let us briefly review one of the main techniques to produce holomorphic bundles with prescribed topology and stability properties. This technology will then be mildly adapted for our main purpose of constructing asymptotically stable bundles.

A **monad** on $X$ is a complex of locally free sheaves such that $\beta$ is locally right-invertible and $\alpha$ is locally left-invertible. The (locally free) sheaf $E := \ker \beta/\text{im} \alpha$ is called the cohomology of $M_\bullet$. Monads are a valuable tool in the theory of sheaves over projective varieties and have been studied by many authors over the past four decades. We will be fundamentally interested in the so-called linear sheaves on $X$, that is, coherent sheaves (of rank $r$) which can be obtained as the cohomology of a linear monad of the form

$$
M_\bullet: \quad M_0 \xrightarrow{\alpha} M_1 \xrightarrow{\beta} M_2
$$

such that $\beta$ is locally right-invertible and $\alpha$ is locally left-invertible. The (locally free) sheaf $E := \ker \beta/\text{im} \alpha$ is called the cohomology of $M_\bullet$. Monads are a valuable tool in the theory of sheaves over projective varieties and have been studied by many authors over the past four decades. We will be fundamentally interested in the so-called linear sheaves on $X$, that is, coherent sheaves (of rank $r$) which can be obtained as the cohomology of a linear monad of the form

$$
0 \longrightarrow \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus r+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \longrightarrow 0.
$$

The paper [9] is of particular interest to us, since the first-named author constructed examples of stable linear sheaves over cyclic varieties of dimension 3. More precisely, the following result is proved, as an application of the Hoppe criterion (Proposition 1):
Theorem 8. Let $X$ be a cyclic non-singular complex projective 3-fold with fundamental class $h := c_1(O_X(1))$, and $c \geq 1$ an integer; then a linear monad of the form
\[
0 \longrightarrow O_X(-1)^{\oplus c} \overset{\alpha}{\longrightarrow} O_X^{\oplus 2+2c} \overset{\beta}{\longrightarrow} O_X(1)^{\oplus c} \longrightarrow 0
\] (3)
has the following properties:

(i) the kernel $K := \ker \beta$ is a stable bundle with
\[
\text{rk}(K) = c + 2, \quad c_1(K) = -c.h, \quad c_2(K) = \frac{1}{2}(c^2 + c).h^2,
\]
(ii) the linear sheaf $E := \ker \beta / \text{im} \alpha$ is a stable bundle with
\[
\text{rk}(E) = 2, \quad c_1(E) = 0, \quad c_2(E) = c \cdot h^2.
\]

A simple example of a linear monad of the form (3) with $c = 1$ can be given as follows. Let $X$ be a non-singular hypersurface of degree $d$ in $\mathbb{P}^4$ not containing the point $[0:0:0:0:1]$. Then the complex
\[
0 \longrightarrow O_X(-1) \overset{\alpha}{\longrightarrow} O_X^{\oplus 4} \overset{\beta}{\longrightarrow} O_X(1) \longrightarrow 0
\] (4)
given in homogeneous coordinates $[x_0 : x_1 : x_2 : x_3 : x_4]$ of $\mathbb{P}^4$ by
\[
\alpha = \begin{pmatrix}
  x_3 \\
  x_2 \\
  -x_1 \\
  -x_0
\end{pmatrix}
\quad \text{and} \quad
\beta = (x_0 \ x_1 \ x_2 \ x_3)
\]
is a monad on $X$, since $\alpha$ is injective and $\beta$ is surjective at every point of $X$. Examples with higher values for $c$ may be found in [9, Section 3].

3.2. Asymptotically stable monad cohomologies

Let $(Z, D)$ be a building block of Fano type, with $X$ being the underlying Fano 3-fold. Suppose further that $X$ is cyclic and that $D \in |-K_X|$ is a generic cyclic anti-canonical divisor, so that
\[
\text{Pic } X \simeq Z \simeq \text{Pic } D.
\]

Then, we can assign a monad construction with stable cohomology bundle $E$, say, to a large number of such underlying Fano 3-folds, for instance:

(i) $X = \mathbb{P}^3$;
(ii) $X \xrightarrow{2:1} \mathbb{P}^3$;
(iii) $X \subset \mathbb{P}^4$, hypersurface of degree 2, 3 or 4.

The strategy consists in adapting the linear monads from Theorem 8 to obtain instanton bundles over $X$. However, when generalising results from projective spaces to wider classes of projective varieties, one often encounters difficulties because $\mathbb{P}^n$ does not ‘have as much cohomology’. This principle has the following precise significance for us:

Definition 9. A line bundle $L \to X$ over a projective variety of dimension $n$ is said to be without intermediate cohomology (WIC) if
\[
H^i(L^\otimes k) = 0, \quad \forall \begin{cases} 
i = 1, \ldots, n - 1 \\ k \in \mathbb{Z} \end{cases}.
\]
A projective variety $X$ is said to be WIC if the line bundle $O_X(1)$ is WIC.
Remark 10. It is not difficult to see, using Kodaira’s vanishing theorem, that if $X$ is cyclic and Fano, then the positive generator of Pic $X$, denoted by $\mathcal{O}_X(1)$, is WIC.

Note that complete intersection subvarieties of dimension at least 3 in $\mathbb{P}^n$, $n \geq 4$, are cyclic and WIC.

Then, indeed, our instanton linear sheaves are asymptotically stable bundles:

Proposition 11. Let $X$ be a non-singular, cyclic Fano 3-fold, and let $D \subset X$ be a cyclic anti-canonical divisor. If $E \to X$ arises from an instanton monad of the form (3), then $E$ is an asymptotically stable bundle.

The rest of this subsection is devoted to the proof of Proposition 11. First of all, since $E$ can be obtained as the cohomology of a monad of the form (3), Theorem 8 guarantees that it is stable. Denote by $\mathcal{O}_X(1)$ the polarization, $\sigma_D \in H^0(K_X^{-1})$ the section cutting out $D$, $d := \deg D$ its degree and $r_{X,D}$ the restriction map; so, the restriction sequence reads

$$0 \longrightarrow E(-d) \xrightarrow{\sigma_D} E \xrightarrow{r_{X,D}} E|_D \longrightarrow 0. \quad (5)$$

Moreover, setting $K := \ker \beta$ and twisting the monad by $\mathcal{O}_X(-d)$, the relevant data fit in the following canonical diagram:

Note that $E|_D$ is a rank 2 bundle with $c_1(E|_D) = 0$, thus $E|_D = (E|_D)_{\text{norm}}$. In view of Criterion 2 and since $h^0(E) = 0$, it follows from the second row of the canonical diagram that it suffices to check the vanishing at

$$h^1(E(-d)) = 0.$$ 

Note that $h^0(\mathcal{O}_X(-k)) = 0$ and, since $X$ is WIC (cf. Remark 10), $h^2(\mathcal{O}_X(k)) = 0$ for all $k \in \mathbb{Z}$. It follows from the first row in the canonical diagram that $h^1(K(-d)) = 0$. Finally, from the column of the canonical diagram, we have $h^1(E(-d)) = 0$, which concludes the proof.

In particular, Proposition 11 gives many examples of rank 2 asymptotically stable bundles over varieties such as (i) and (iii) on the previous page. The same examples can also be pulled back to double covers of type (ii).
4. Asymptotically stable Harshorne–Serre bundles

Let $X$ be a complex manifold and $Y \subset X$ be a codimension 2 local complete intersection subscheme. A bundle $E \to X$ of rank $r$ will be called a Hartshorne–Serre bundle obtained from $Y$ if there exists a line bundle $L$ with an exact sequence:

$$0 \to \mathcal{O}_X \to E \to \mathcal{I}_Y \otimes \mathcal{L} \to 0,$$

where $\mathcal{I}_Y$ is the ideal sheaf of $Y$ in $X$. Heuristically, we think about $E$ as a global extension of the normal bundle of $Y$, in a sense which we will now make precise for the case of rank 2.

The following instance of [2, Theorem 1] gives sufficient conditions for the existence of Hartshorne–Serre bundles; to the interested reader, we strongly recommend the provided reference for a detailed and user-friendly exposition of that construction.

**Theorem 12** (Hartshorne–Serre construction in rank 2). Let $Y \subset X$ be a local complete intersection subscheme of codimension 2 in a smooth algebraic variety. If there exists a line bundle $L \to X$ such that $H^2(X, L^*) = 0$, and $\wedge^2 N_{Y/X} = \mathcal{L}|_Y$, then there exists a rank 2 Hartshorne–Serre bundle obtained from $Y$ such that

(i) $\wedge^2 E = L$,

(ii) $E$ has a global section whose vanishing locus is $Y$.

4.1. Generalized Hoppe criterion for the Harshorne–Serre construction

We will produce asymptotically stable examples over certain Fano-type building blocks as Hartshorne–Serre bundles. In order to check their stability, the following adapted version of the polycyclic criterion of Theorem 3 will be instrumental:

**Proposition 13.** Let $X$ be a polycyclic complex manifold endowed with a polarization $L$. Let $E$ be a rank 2 Hartshorne–Serre bundle obtained from a codimension 2 complete intersection subscheme $Y$. Then $E$ is stable (respectively, semistable) if

(i) $\mu_L(E) > 0$,

(ii) for every effective divisor $S$ with $\delta_L(S) \leq \mu_L(E)$, $Y$ is not contained in $S$.

**Proof.** We have the following exact sequence:

$$0 \to \mathcal{O}_X \to E \to \mathcal{I}_Y \otimes \mathcal{L} \to 0.$$

To apply Theorem 3, we tensorize by $\mathcal{O}_X(B)$ with $\delta_L(B) \leq -\mu_L(E)$ to obtain:

$$0 \to \mathcal{O}_X(B) \to E \otimes \mathcal{O}_X(B) \to \mathcal{I}_Y \otimes \mathcal{L} \otimes \mathcal{O}_X(B) \to 0,$$

which induces an exact sequence on cohomology:

$$0 \to H^0(X, \mathcal{O}_X(B)) \to H^0(X, E \otimes \mathcal{O}_X(B)) \to H^0(X, \mathcal{I}_Y \otimes \mathcal{L} \otimes \mathcal{O}_X(B)).$$

Assumption (i) implies $\delta_L(B) < 0$, so $H^0(X, \mathcal{O}_X(B)) = 0$. As to the vanishing of the right-hand term, let $B' := B + c_1(\mathcal{L}) = B + c_1(E)$, so that $\delta_L(B') \leq \mu_L(E)$. By assumption (ii), there is no effective divisor of class $B'$ containing $Y$, and we claim therefore

$$H^0(X, \mathcal{I}_Y \otimes \mathcal{L} \otimes \mathcal{O}_X(B)) = 0.$$

Indeed, considering the exact sequence

$$0 \to \mathcal{I}_Y \otimes \mathcal{L} \otimes \mathcal{O}_X(B) \to \mathcal{L} \otimes \mathcal{O}_X(B) \to \mathcal{L} \otimes \mathcal{O}_X(B)|_Y \to 0,$$
a global section of $I_Y \otimes L \otimes \mathcal{O}_X(B)$ provides a global section of $L \otimes \mathcal{O}_X(B)$, which is trivial on $Y$. □

**Remark 14.** When the Picard rank of $X$ is large, there is a helpful technique to find necessary conditions on the class of an effective divisor $S$ so that

$$\delta_L(S) \leq \mu_L(E).$$

Suppose first that $X$ is a surface. Expressing $[S]$ in terms of the basis of Pic $X$, we obtain constraints in inequality form. As an obvious starter, we have:

$$\delta_L(S) > 0. \quad (6)$$

Now let $C$ be curve in $X$ such that $\delta_L(S) < \delta_L(C)$. Then the following sequence is exact:

$$0 \to H^0(X, \mathcal{O}_X(S - C)) \to H^0(X, \mathcal{O}_X(S)) \to H^0(X, \mathcal{O}_X(S)|_C). \quad (7)$$

By assumption $H^0(X, \mathcal{O}_X(S - C)) = 0$, and since $H^0(X, \mathcal{O}_X(S)) \neq 0$, it follows that $H^0(X, \mathcal{O}_X(S)|_C) \neq 0$. Hence, $\deg \mathcal{O}_X(S)|_C \geq 0$, so:

$$S \cdot C \geq 0. \quad (8)$$

The technique also works when $\dim X > 2$. Let $C$ be an effective divisor in $X$ such that

(i) $\delta_L(S) < \delta_L(C)$;

(ii) there exists an ample class $A$ of $C$ such that $A = A|_C$ with $A$ being an ample class on $X$.

Then, as before, we have the exact sequence (7), from which it follows that $H^0(X, \mathcal{O}_X(S)|_C) \neq 0$. As a consequence, either $S|_C$ is an effective divisor on $C$ or it vanishes. Hence, by the Nakai–Moishezon criterion, $A^{n-2} \cdot S|_C \geq 0$, that is to say $S \cdot C \cdot A^{n-2} \geq 0$.

4.2. The blow-up of $\mathbb{P}^3$ along a plane conic

We show applications of this adapted criterion by providing two asymptotically stable bundles on the particular Fano 3-fold listed no. 30 in [12]. One could apply the same method to obtain more examples over other building blocks.

Let $r: X \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along a plane conic $C$ and $D \in |-K_X|$ be a generic smooth K3 surface, corresponding to the proper transform of a quartic in $\mathbb{P}^3$ containing $C$. We denote by $H = r^*(H_0)$ the pullback of the hyperplane $H_0 \subset \mathbb{P}^3$, by $\tilde{C}$ the exceptional divisor, by $h = r^*(h_0)$ the pullback of a line $h_0 \subset \mathbb{P}^3$ and by $l$ a fibre of $\tilde{C} \to C$.

**Lemma 15.** The Picard lattice of $D$ is spanned by the hyperplane class $A$ and the intersection with the exceptional divisor, which is isomorphic to $C$ in $D$. The intersection form in this basis is

$$\begin{pmatrix} 4 & 2 \\ 2 & -2 \end{pmatrix}.$$ 

**Proof.** The curve $C$ has genus 0 in $D$, hence $C^2 = -2$. Let $j: D \hookrightarrow \mathbb{P}^3$ be the inclusion map. We have $A = j^*(H_0)$, hence $A^2 = j^*(H_0^2) = j^*(H_0^2) \cdot D$. Then the projection formula yields:

$$A^2 = H_0^2 \cdot AH_0 = 4.$$ 

Similarly, $A \cdot C = j^*(H_0) \cdot C$ implies:

$$A \cdot C = H_0 \cdot 2h_0 = 2. \quad \square$$
In each of the following examples, we verify the hypotheses of Theorem 12 for the existence of a Hartshorne–Serre bundle, then we check its asymptotic stability using Proposition 13 on its restriction to the K3 surface $D$.

4.2.1. First example. We apply Theorem 12 to $X$ as above, with $Y = l$ and $L = \mathcal{O}_X(\tilde{C})$.

**Proposition 16.** Let $X \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along a plane conic $C$, $D \in |−K_X|$ be a generic smooth K3 surface, corresponding to the proper transform of a quartic in $\mathbb{P}^3$ containing $C$, and $l \subset X$ a fibre of the exceptional divisor $\tilde{C} \to C$; then there exists a rank 2 Hartshorne–Serre bundle $E \to X$ obtained from $l$ such that:

(i) $c_1(E) = \tilde{C}$,
(ii) $c_2(E) = [l]$,
(iii) $E|_D$ is $A$-stable.

**Existence of the Hartshorne–Serre bundle**

**Lemma 17.** In the hypotheses of Proposition 16, $H^2(X, \mathcal{O}_X(−\tilde{C})) = 0$.

**Proof.** We obtain, from the short exact sequence,

$$0 \to \mathcal{O}_X(−\tilde{C}) \to \mathcal{O}_X \to \mathcal{O}_{\tilde{C}} \to 0$$

the following exact sequence on cohomology:

$$H^1(\tilde{C}, \mathcal{O}_{\tilde{C}}) \to H^2(X, \mathcal{O}_X(−\tilde{C})) \to H^2(\tilde{C}, \mathcal{O}_{\tilde{C}}).$$

(9)

Since $\tilde{C}$ is a ruled surface, we have $H^2(\tilde{C}, \mathcal{O}_{\tilde{C}}) = 0$, and $H^1(\tilde{C}, \mathcal{O}_{\tilde{C}}) = H^1(C, \mathcal{O}_C)$. However, $H^1(C, \mathcal{O}_C) = 0$ because $C$ is a conic. □

**Lemma 18.** In the hypotheses of Proposition 16, $\mathcal{O}_X(\tilde{C})|_l = \wedge^2 \mathcal{N}_l/X$.

**Proof.** Since the claim concerns line bundles on a rational curve, it is enough to show that $c_1(\mathcal{O}_X(\tilde{C})|_l) = c_1(\wedge^2 \mathcal{N}_l/X)$. On the one hand, we have $c_1(\mathcal{O}_X(\tilde{C})|_l) = −1$, because $\tilde{C} \cdot l = −1$. On the other hand, one can prove that $\wedge^2 \mathcal{N}_l/X \simeq \mathcal{O}_l(−1)$ by a local argument, on a neighbourhood of the line $l$; however, here goes a shorter proof using intersection theory in $X$.

By adjunction, we have

$$c_1(\wedge^2 \mathcal{N}_l/X) = c_1(\mathcal{N}_l/X) = c_1((T_X)|_l) - c_1(T_l)$$

$$= c_1(T_X) \cdot l - c_1(T_l).$$

We know that $c_1(T_X) = −c_1(\omega_X) = 4H − \tilde{C}$, so

$$c_1((T_X)|_l) = (4H − \tilde{C}) \cdot l = 1.$$ 

Moreover, since $l$ is a line, we have $c_1(\omega_l) = −2$, so

$$c_1(T_l) = 2,$$

which yields $c_1(\wedge^2 \mathcal{N}_l/X) = −1$, as desired. □
Stability of $E|_D$

Note that $E|_D$ is also a Hartshorne–Serre bundle obtained from the point $y := l \cap D$, with $c_1(E|_D) = C$. We apply the adapted stability criterion from Proposition 13 with the polarization $A$, for which $\mu_A(E) = \frac{A \cdot C}{2} = 1$. Since the intersection form on $D$ is even, there is no curve $S$ such that $\delta_A(S) \leq 1$, hence $E|_D$ is stable.

4.2.2. Second example. The next example is slightly more sophisticated. For $c_2(E)$ fixed, $A$-stability over $D$ depends on the choice of the generating subscheme $\mathcal{Y}$, which in our case is a curve. We consider $R$ the proper transform of a line $R_0$ in $\mathbb{P}^3$ which meets the conic $C$ in just one point:

$$[R] = h - [l].$$

(10)

Let $\mathcal{P}$ be the plane that contains the conic $C$. The intersection $\mathcal{P} \cap D$ in $\mathbb{P}^3$ is a plane curve of degree 4 containing $C$; hence, it is a reduced curve having $C$ as an irreducible component. Since $D$ is generic, the only other irreducible component is another smooth plane conic, which we denote by $C^\vee$.

Definition 19. We say that the line $R_0$ is generic if $R_0$ is not a totally tangent line to $D$ in $\mathbb{P}^3$ at a point of $C$, that is, $D \cap R_0$ does not consist of a single degree 4 point in $C$.

We now apply Theorem 12 with $Y = R$ and $L = H$.

Proposition 20. Let $X \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along a plane conic $C$, $D \in |K_X|$ be a generic smooth $K3$ surface, corresponding to the proper transform of a quartic in $\mathbb{P}^3$ containing $C$, $\mathcal{R}$ be the proper transform of a line $R_0$ in $\mathbb{P}^3$ which meets the conic $C$ in just one point and $H \subset X$ be the pullback of a hyperplane; then there exists a rank 2 Hartshorne–Serre bundle $E \to X$, obtained from $\mathcal{R}$, such that:

(i) $c_1(E) = H$,
(ii) $c_2(E) = [\mathcal{R}]$,
(iii) $E|_D$ is $A$-stable, if $R_0$ is generic, and $A$-semistable otherwise.

Existence of the Hartshorne–Serre bundle

Consider the exact sequence

$$0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_H \to 0,$$

where $H$ can be regarded as the blow-up of a plane in two points. It follows that

$$H^2(X, \mathcal{O}_X(-H)) = H^1(H, \mathcal{O}_H) = 0.$$  (11)

To use the Hartshorne–Serre construction, we also need the following lemma.

Lemma 21. We have $\mathcal{O}_X(H)|_\mathcal{R} = \wedge^2 N_{\mathcal{R}/X}$.

Proof. As in the proof of Lemma 18, it is enough to show that the Chern classes of both line bundles coincide. On the one hand, we have $c_1(\mathcal{O}_X(H)|_\mathcal{R}) = 1$, because $H \cdot \mathcal{R} = 1$. On the other hand, since $c_1(T_X) = 4H - \tilde{C}$, we have

$$c_1((T_X)|_\mathcal{R}) = (4H - \tilde{C}) \cdot \mathcal{R} = 3.$$

Since $\mathcal{R}$ is a line, $c_1(T_\mathcal{R}) = 2$. It follows that $c_1(\wedge^2 N_{\mathcal{R}/X}) = c_1((T_X)|_\mathcal{R}) - c_1(T_\mathcal{R}) = 1$.  \(\square\)
Stability of $E|_D$

The bundle $E|_D$ is also a Hartshorne–Serre bundle obtained from the intersection $\xi := R \cap D$, which is a discrete set. Moreover, $c_1(E|_D) = A$. Again, by the stability criterion of Proposition 13, with $\mu_A(E) = \frac{A^2}{2} = 2$, in order to prove stability (respectively, semistability), we have to check that $\xi$ is not in $S$ for any curve $S$ with $\delta_A(S) \leq 2$ (respectively, $\delta_A(S) < 2$).

For semistability, note that the intersection on $D$ is even, so the condition on $S$ is actually $\delta_A(S) \leq 0$. Thus, such $S$ cannot exist, because the degree of an effective divisor is always strictly positive.

Now we address strict stability. The bundle $E|_D$ is stable if $\xi$ is not contained in any curve $S$ with $\delta_A(S) \leq 2$. Since $\delta_A(S) > 0$ and the intersection on $D$ is even, it follows that $\delta_A(S) = 2$. Then, by projection formula, it follows that $S$ is a curve of degree 2 in $\mathbb{P}^3$. Since $D$ is generic, there are just two possibilities to be ruled out: either $S = C$ or $S = C^\vee$. So, for the first case, we must argue that $\xi \not\subset C$; for the second case, we must argue that $\xi \not\subset C^\vee$.

5. Degeneration of asymptotically stable bundles

We used monads to obtain examples of instanton bundles in Subsection 3.1. Another illustration of the usefulness of monads in gauge theory is the modelling of degenerating instanton sequences. Concretely, let us examine the task of producing a one-parameter family $\{E_\lambda\}_{\lambda > 0}$ of asymptotically stable locally free sheaves over some building block, say $X = \mathbb{P}^3$, such that $E_0 := \lim_{\lambda \to 0} E_\lambda$ is torsion-free but not locally free. It seems reasonable to expect that this principle may in the future contribute to shed some light on the currently open and very hard problem of moduli space compactification for instantons in higher dimensions.

Recall from standard sheaf theory that the singular locus of a linear monad (2) over $X = \mathbb{P}^n$ is the set

$$\Sigma = \{z \in \mathbb{P}^n \mid \ker \alpha(z) \neq \{0\}\},$$

where $\alpha(z)$ denotes the fibre map at the point $z \in \mathbb{P}^n$ of the morphism $\alpha$ appearing in (2). We know from [9, Proposition 4] that the degeneration of the associated linear sheaf $E$ is essentially determined by the topology of $\Sigma$:

**Criterion 22.** Let $E$ be a linear sheaf with singular locus $\Sigma$; then

1. $E$ is locally-free $\iff \Sigma = \emptyset$;
2. $E$ is reflexive $\iff \Sigma$ is a subvariety with $\text{codim} \Sigma \geq 3$;
3. $E$ is torsion-free $\iff \Sigma$ is a subvariety with $\text{codim} \Sigma \geq 2$.

This result is relevant for the following Proposition.

**Proposition 23.** Let $\{M_* (\lambda)\}_{\lambda > 0}$ be the family of monads over $\mathbb{P}^3$

$$M_* (\lambda) : 0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \overset{\alpha_\lambda}{\longrightarrow} \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \overset{\beta}{\longrightarrow} \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow 0$$
given in homogeneous coordinates \([z_1 : z_2 : z_3 : z_4]\) by
\[
\alpha_\lambda = (z_1 \quad z_2 \quad \lambda z_3 \quad \lambda z_4) \quad \text{and} \quad \beta = (-z_2 \quad z_1 \quad -z_4 \quad z_3).
\]

Then the corresponding family of linear sheaves \(\{E_\lambda\}_{\lambda > 0}\) has the following properties:

(i) each \(E_\lambda\) is a locally free sheaf with \(\text{rk}(E_\lambda) = 2\), \(c_1(E_\lambda) = 0\) and \(c_2(E_\lambda) = 1\);

(ii) each \(E_\lambda\) is asymptotically stable, that is, each \(E_\lambda|_D\) is stable over a fixed anti-canonical quartic \(D \in |O_{\mathbb{P}^3}(4)|\);

(iii) the limit \(E_0 := \lim_{\lambda \to 0} E_\lambda\) is a properly torsion-free (that is, not locally free), asymptotically semistable sheaf with singular locus \(\Sigma_0 = \{z_1 = z_2 = 0\}\);

(iv) with respect to the family of standard Hermitian metrics induced from \(\mathbb{C}^4\) on the elements of \(\{E_\lambda\}\), the corresponding curvatures \(F_\lambda\) blow up along \(\Sigma_0\) at the rate \(1/\lambda\) as \(\lambda \to 0\), in the sense that \(\lim_{\lambda \to 0}(\lambda^m \|F_\lambda\|_{L^1(\Sigma_0)}) = \infty\) if \(m < 1\), while \(\lambda\|F_\lambda\|_{L^1(\Sigma_0)}\) is bounded for small \(\lambda\).

From the viewpoint of gauge theory outlined in the Section 1, condition (ii) implies, by [17, Theorem 58], that each holomorphic bundle \(E_\lambda|_W\), with \(\lambda > 0\), over the ACyl Calabi–Yau 3-fold \(W := (\mathbb{B}_L, \mathbb{P}^3) \setminus D\), admits an HYM connection \(H_\lambda\) exponentially asymptotic to the anti-self-dual (ASD) instanton on \(E_\lambda|_D\) along the cylindrical end. In this context, however, the Hermitian metric on each \(E_\lambda\) whose curvature \(F_\lambda\) we study does not a priori correspond to the actual HYM solution. Indeed, such solutions in general are not known explicitly, and in the non-compact ACyl Calabi–Yau case, not even the Kähler structure can be written out in coordinates, let alone the HYM condition be explicitly verified. We consider instead standard Hermitian metrics induced from \(\mathbb{C}^4\), expecting that the method outlined in the proof of Proposition 23 can be used to show, in the future, that HYM solutions \(H_\lambda\) display some similar degeneration patterns, hence inform us about the structure of the tangent cone of the singular \(\lambda \to 0\) limit.

The rest of this section is devoted to establishing Proposition 23. Note that the above construction is really a proof of principle, indeed much more general topological types for \(E_\lambda\) can be arranged.

5.1. Properly torsion-free limit as \(\lambda \to 0\)

For \(\lambda > 0\), clearly \(\Sigma_\lambda = \emptyset\), hence the corresponding linear sheaf is locally free. It follows from Theorem 8 that \(E_\lambda\) is stable, while Proposition 11 shows that \(E_\lambda\) is asymptotically stable. This proves claims (i) and (ii) in Proposition 23.

The limit \(E_0\) is obviously still a linear sheaf. The new phenomenon is that
\[
\Sigma_0 = \{z_1 = z_2 = 0\} \subset \mathbb{P}^3
\]
is a curve; hence, \(E_0\) is a properly torsion-free sheaf. One can show that \(E_0\) is properly semistable, see [8, Proposition 14 and Example 4]. Its restriction to an anti-canonical divisor \(D\) is also properly torsion-free, since it intersects the singular locus at a single point.

Moreover, \(E_0|_D\) is properly semistable; indeed, the anti-canonical quartic has \(\text{Pic}(D) = \mathbb{Z}\), so suffices to check that \(h^0(E_0(-1)|_D) = h^0(E_0^N(-1)|_D) = 0\), cf. [8, Lemma 13]. We already know that \(h^0(E_0(-1)|_D) = 0\). Recall that \(K = \ker \beta\); since \(E_0^N\) is a subsheaf of \(K^*\), we conclude that \(h^0(K^*(-1)|_D) = 0\) implies \(h^0(E_0^N(-1)|_D) = 0\); to check that \(h^0(K^*(-1)|_D) = 0\), simply consider the restriction sequence
\[
0 \rightarrow K^*(-5) \rightarrow K^*(-1) \rightarrow K^*(-1)|_D \rightarrow 0
\]
and note that \(h^0(K^*(-1)) = h^1(K^*(-5)) = 0\), which follows from the sequence
\[
0 \rightarrow O_{\mathbb{P}^3}(-1) \overset{\beta^*}{\rightarrow} O_{\mathbb{P}^3}^{\mathbb{Z}^4} \rightarrow K^* \rightarrow 0.
\]
Therefore, we conclude that the limit sheaf $E_0$ is asymptotically (properly) semistable, which is claim (iii) in Proposition 23.

Note that there is nothing particular about $\mathbb{P}^3$ here. Similar families of monads can be constructed over a wide class of projective Fano 3-folds $X$, say, using $\mathcal{O}_X(1)$ and the embedding coordinates to form the maps.

5.2. Curvature blow-up along degenerating instanton sequences

An interesting feature of such explicit models is the quantitative description of the curvature blow-up rate as the family degenerates into the limiting torsion-free sheaf.

In order to check claim (iv) in Proposition 23, we set out to write explicitly the natural curvature associated to the holomorphic structure of a linear sheaf $E_\lambda$. Dualizing the map $\alpha_\lambda$ in $M_\bullet(\lambda)$, we have an 'Euler characteristic' map

$$R_\lambda := \beta \oplus \alpha_\lambda^\vee : \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}$$

given by

$$R_\lambda = \left(\begin{array}{cccc}
\bar{z}_1 & \bar{z}_2 & \lambda \bar{z}_3 & \lambda \bar{z}_4 \\
-\bar{z}_2 & \bar{z}_1 & -\bar{z}_4 & \bar{z}_3
\end{array}\right) \quad \text{thus} \quad R_\lambda^\vee = \left(\begin{array}{cccc}
z_1 & -\bar{z}_2 \\
z_2 & \bar{z}_1 \\
\lambda z_3 & -\bar{z}_4 \\
\lambda z_4 & \bar{z}_3
\end{array}\right).$$

Let $P_\lambda : \Gamma(\mathcal{O}_{\mathbb{P}^3}^{\oplus 4}) \to \Gamma(E_\lambda)$ denote the orthogonal projection onto $\ker R_\lambda$; this is given by

$$P_\lambda = 1 - R_\lambda^\vee D_\lambda R_\lambda,$$

with

$$D_\lambda := (R_\lambda R_\lambda^\vee)^{-1} = \begin{pmatrix}
\frac{1}{|\alpha_\lambda|^2} & 0 \\
0 & \frac{1}{|\beta|^2}
\end{pmatrix}. $$

The crucial observation is that, fixing a Hermitian metric (for example, the standard metric from $\mathbb{C}^4$), the middle cohomology bundle can be identified as

$$E_\lambda = \frac{\ker \beta}{\im \alpha_\lambda} \simeq \ker R_\lambda$$

and the induced Chern connection is given by $\nabla_\lambda = P_\lambda \circ d:\nabla_\lambda : \Gamma(E_\lambda) \to \Gamma(E_\lambda) \otimes \Omega^1$

so curvature is given by $F_\lambda = (P_\lambda d) \circ (P_\lambda d) = (P_\lambda \circ dP_\lambda) \circ d$.

We will now determine explicitly the curvature along the curve $\Sigma_0$. By direct differentiation, one has

$$-dP_\lambda = (dR_\lambda^\vee) D_\lambda R_\lambda + R_\lambda^\vee [D_\lambda \circ dR_\lambda + (dD_\lambda) \circ R_\lambda].$$

Specializing to the singular locus $\Sigma_0 = \{z_1 = z_2 = 0\}$, with $(z_3, z_4) \neq (0, 0)$, we have

$$R_\lambda|_{\Sigma_0} = \begin{pmatrix}
0 & 0 & \lambda z_3 & -\bar{z}_4 \\
0 & 0 & \lambda z_4 & \bar{z}_3
\end{pmatrix} \quad \text{and} \quad D_\lambda|_{\Sigma_0} = \frac{1}{|z_3|^2 + |z_4|^2} \begin{pmatrix}
\frac{1}{\lambda^2} & 0 \\
0 & 1
\end{pmatrix}. $$
so the projection restricted to $\Sigma_0$ takes the upper left $2 \times 2$ block diagonal form:

$$P_\lambda|_{\Sigma_0} = 1 - \frac{1}{|z_3|^2 + |z_4|^2} R_\lambda^\vee|_{\Sigma_0} \left( \begin{array}{cc} \frac{1}{\lambda^2} & 0 \\ 0 & 1 \end{array} \right) R_\lambda|_{\Sigma_0} = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right).$$

In particular, $(P_\lambda \circ R_\lambda^\vee)|_{\Sigma_0} = 0$, so the only non-trivial contribution to $(P_\lambda \circ dP_\lambda)|_{\Sigma_0}$ comes from the first summand in (12). Defining for convenience

$$\xi_{ij} := \bar{z}_i dz_j,$$

we have

$$[(dR_\lambda^\vee) D_\lambda R_\lambda]|_{\Sigma_0} = \frac{1}{|z_3|^2 + |z_4|^2} \left( \begin{array}{cc} dz_1 & -d\bar{z}_2 \\ d\bar{z}_1 & dz_2 \\ \lambda dz_3 & -d\bar{z}_4 \\ \lambda d\bar{z}_3 & dz_4 \end{array} \right) \left( \frac{1}{\lambda^2} \begin{array}{c} 0 \\ 0 \end{array} \right) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) = \left( \begin{array}{cc} 0 & 0 \\ 0 & \lambda \end{array} \right) \left( \begin{array}{c} \xi_{31} \\ \xi_{32} \end{array} \right).$$

We conclude that

$$-(P_\lambda \circ dP_\lambda)|_{\Sigma_0} = \frac{1}{|z_3|^2 + |z_4|^2} \left( \begin{array}{cc} 0 & -\bar{z}_2 \\ -z_2 & 0 \\ \xi_{31} & 0 \\ \xi_{32} & 0 \end{array} \right) \left( \begin{array}{c} 0 \\ 0 \end{array} \right) = \left( \begin{array}{cc} 0 & 0 \\ 0 & \lambda \end{array} \right) \left( \begin{array}{c} \xi_{31} \\ \xi_{32} \end{array} \right).$$

with $\xi_{ij}$ not all zero. Therefore, towards the limiting torsion-free sheaf $E_0$, curvature blows up along the curve $\Sigma_0$ as

$$\left| F_\lambda|_{\Sigma_0} \right| = \left| (P_\lambda|_{\Sigma_0} \circ dP_\lambda|_{\Sigma_0} \circ d) \right| \xrightarrow{\lambda \to 0} \frac{1}{\lambda} \to \infty.$$

**Acknowledgements.** We thank A. A. Henni and P. Coronica for comments and T. Walpuski for suggesting the topic of Section 5. The authors thank especially the referee for numerous contributions to the manuscript.

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