EDGE CURRENTS AND
VERTEX OPERATORS
FOR CHERN-SIMONS GRAVITY

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ABSTRACT

We apply elementary canonical methods for the quantization of 2+1 dimensional gravity, where the dynamics is given by E. Witten’s ISO(2, 1) Chern-Simons action. As in a previous work, our approach does not involve choice of gauge or clever manipulations of functional integrals. Instead, we just require the Gauss law constraint for gravity to be first class and also to be everywhere differentiable. When the spatial slice is a disc, the gravitational fields can either be unconstrained or constrained at the boundary of the disc. The unconstrained fields correspond to edge currents which carry a representation of the ISO(2, 1) Kac-Moody algebra. Unitary representations for such an algebra have been found using the method of induced representations. In the case of constrained fields, we can classify all possible boundary conditions. For several different boundary conditions, the field content of the theory reduces precisely to that of 1+1 dimensional gravity theories. We extend the above formalism to include sources. The sources take into account self- interactions. This is done by punching holes in the disc, and erecting an ISO(2, 1) Kac-Moody algebra on the boundary of each hole. If the hole is originally sourceless, a source can be created via the action of a vertex operator $V$. We give an explicit expression for $V$. We shall show that when acting on the vacuum state, it creates particles with a discrete mass spectrum. The lowest mass particle induces a cylindrical space-time geometry, while higher mass particles give an $n$-fold covering of the cylinder. The vertex operator therefore creates cylindrical space-time geometries from the vacuum.
1. INTRODUCTION

Even though there are no gravitons in three dimensional gravity, the theory can have a rich structure when the topology of space-time is nontrivial. This is especially evident in the Chern-Simons formulation of the theory, proposed by E. Witten [1]. In this formulation, the action is written in terms of connection one forms $e^a$ and $\omega^a$ for the $ISO(2, 1)$ group. $ISO(2, 1)$ denotes the Poincare group in three dimensions. The components of $e^a$ are the dreibein fields from which one can construct the space-time metric, while $\omega^a$ are the spin connections from which one constructs the $SO(2, 1)$ curvature two form.

In Chern-Simons theory, when the space-time manifold is a disc $D \times \mathbb{R}^1$ ($\mathbb{R}^1$ accounting for time) it is possible to quantize the system in a manner which eliminates degrees of freedom from the interior of the disc. This also was proposed by E. Witten [2] and examined further by a number of authors [3-10]. (Elementary canonical methods were used in the approach of A. P. Balachandran and us [9, 10], and they did not require clever manipulations of functional integrals or the imposition of gauge constraints. We believe that the methods of refs. [9, 10] have the virtue of simplicity and we shall apply them here to the case of the Chern-Simons description of 2+1 gravity.) After eliminating the degrees of freedom from the interior of the disc, one is left with states associated with the boundary $\partial D$, the so-called “edge states”. In $U(1)$ Chern-Simons theory, these edge states play an important role in the description of the quantum Hall effect [11, 12]. There, they carry the unitary representations of $U(1)$ Kac-Moody algebras. For the case of the Chern-Simons description of 2+1 gravity, the analogous of the edge states carry unitary representations of the $ISO(2, 1)$ Kac-Moody algebra. $ISO(2, 1)$ is a noncompact semidirect product group. Highest weight representations [13], which are standardly employed for the purpose of finding unitary representations of Kac-Moody algebras associated with compact groups, are plagued with difficulties in this case [14]. Nevertheless, unitary representations have
been found for the ISO(2, 1) Kac-Moody algebra \[14\]. For them one applies the method of induced representations\[15\]. We shall review the construction here for completeness. As usual, the Kac-Moody algebra defines a conformal family. In analogy to the Sugawara construction, a set of Virasoro generators can be formed which are bilinear in the elements ISO(2, 1) algebra. These Virasoro generators were shown to have zero central charge \[14\].

In this article, we shall also apply the formalism of refs. \[9, 10\] to the case of Chern-Simons gravity on a disc with holes. The holes correspond to sources for the curvature and torsion two forms. In the limit where the holes shrink to points, the points can be viewed of as massive spinning point particles. As in previous treatments \[17, 18, 16, 14, 19\], the charges associated with these sources are just the particle momenta and angular momenta. Here however, more information is needed to give a complete description of the system. This is because to each hole we must assign an entire ISO(2, 1) Kac-Moody algebra. Thus each particle carries a unitary representation of the infinite dimensional algebra, and the Hilbert space consists of tensor products of such unitary representations. Unlike in traditional approaches, we thereby are able to regularize the system, taking into account self-interactions of the particles in a natural manner.

It is known how to construct the Fubini-Veneziano vertex operator in Chern-Simons theory \[10\]. This operators when acting on a “sourceless” state, creates a source with a discrete charge. We construct the analogue of the Fubini-Veneziano vertex operator for Chern-Simons gravity. \textit{When acting on a “sourceless” state, it creates a particle with a quantized mass.} The lowest mass that the particle can have is precisely that needed to make the space-time geometry around the particle that of a cylinder $\times \mathbb{R}^1$. The possibility of such a space-time geometry was mentioned in ref. \[16\]. The $n^{th}$ state has a mass of $n$-times the lowest mass, and it yields an $n$-fold covering of the cylinder.

The plan of this article is as follows: In Section 2, we apply the canonical methods of \[9\] to Chern-Simons gravity on a disc. The essential ingredient is that the Gauss law
constraint be differentiable, as well first class (in the sense of Dirac). The next step is finding a complete set of observables for the theory. By definition, they must have zero Poisson brackets with the first class constraints. We shall show that the algebra of observables is precisely the $ISO(2,1)$ Kac-Moody algebra of ref. [14]. In Section 2, we also construct the corresponding Virasoro generators. Unitary representations of the $ISO(2,1)$ Kac-Moody algebra, which describe the quantum theory for Chern-Simons gravity on a disc, are reviewed in Section 3 [14]. The quantum theory is extended to include sources in Section 4. In Section 5, we define the vertex operator for $2+1$ gravity, and we show how it can create cylindrical geometries.

Section 6 represents a departure from the previous sections and also from previous treatments of Chern-Simons theories on manifolds with boundaries. In those treatments, all the connection one forms are unconstrained at the boundaries. Then, in order to make the Gauss law constraint differentiable, it is necessary to impose boundary conditions on the test or smearing functions used in defining the Gauss law constraint. For Chern-Simons gravity, the alternative possibility exists of imposing conditions on the fields at the boundary, while relaxing the conditions on the test functions. In Section 6, we are able to classify all possible such boundary conditions on the fields and test functions consistent with the requirement that the Gauss law constraint be differentiable and, also, first class. One surprising result is that in many instances, the field content of the (unconstrained) connection forms remaining on the boundary are just those needed to define two dimensional gravity. We speculate that it may be possible to generate a variety of different two dimensional gravity theories starting from three dimensional ones.

Concluding remarks are made in Section 7, including comments on the spin-statistics theorem for particles in $2+1$ gravity.

2. THE CANONICAL FORMALISM ON A DISC
The ISO\((2,1)\) Chern-Simons action on the solid cylinder \(D \times \mathbb{R}^1\) is given by [1]

\[
S = \kappa \int_{D \times \mathbb{R}^1} e_a \wedge \left( d\omega^a + \frac{1}{2} \epsilon^{abc} \omega_b \wedge \omega_c \right),
\]

where \(e^a\) and \(\omega_a, a = 0, 1, 2\), are dreibein one forms and spin connection one forms, respectively. Together, \(e^a\) and \(\omega_a\) define the ISO\((2,1)\) connection one forms. The indices \(a, b, c, \ldots\) are raised and lowered using the Minkowski metric, \(\eta = \text{diag}(-1, 1, 1)\). \(\epsilon^{abc}\) is the Levi-Civita symbol with \(\epsilon_{012} = 1\). The constant \(\kappa\) is related to the three dimensional gravitational constant.

We now apply the canonical approach of ref. [9, 10] to the action (2.1). As time is indispensable in this approach, we arbitrarily choose a time function denoted henceforth by \(x^0\). Any constant \(x^0\) slice of the solid cylinder is then the disc \(D\) with coordinates \(x^1, x^2\). The phase space of the action \(S\) is spanned by \(\omega^a_i\) and \(e^a_i\), \(i, j = 1, 2\), the components of the one forms \(\omega^a\) and \(e^a\), respectively, on the \(x^0\) slice. (The components \(e^a_0\) and \(\omega^a_0\) do not occur as coordinates of phase space. This is because their conjugate momenta are weakly zero and first class, in the sense of Dirac.) \(\omega^a_i\) and \(e^a_i\) satisfy the equal time Poisson brackets (PB’s):

\[
\{\omega^a_i(x), \omega^b_j(y)\} = \{e^a_i(x), e^b_j(y)\} = 0, \\
\{\omega^a_i(x), e^b_j(y)\} = \frac{1}{\kappa} \epsilon_{ij} \eta^{ab} \delta^2(x - y),
\]

as well as the Gauss law constraint. Here, we define the two-indexed Levi-Civita symbol by \(\epsilon_{ij} = \epsilon_{0ij}\). To write the Gauss law constraint, we introduce test or smearing functions on \(D\). We denote them by \(\Lambda^{(0)} = \{\Lambda^{(0)}_a(x), a = 0, 1, 2\}\) and \(\Sigma^{(0)} = \{\Sigma^{(0)}_a(x), a = 0, 1, 2\}\). The Gauss law constraint is given by:

\[
g(\Lambda^{(0)}, \Sigma^{(0)}) = \frac{\kappa}{2} \int_D \left( \Lambda^{(0)}_a(x) R^a(x) + \Sigma^{(0)}_a(x) T^a(x) \right) \approx 0,
\]

where \(R^a\) and \(T^a\) are the SO\((2,1)\) curvature and the torsion two forms, respectively,

\[
R^a = d\omega^a + \frac{1}{2} \epsilon^{abc} \omega_b \wedge \omega_c, \quad T^a = de^a + \epsilon^{abc} e_b \wedge \omega_c,
\]
and \( \approx \) denotes weak equality in the sense of Dirac.

It remains to state the space \( \mathcal{T}^{(0)} \) of test functions, \( \Lambda^{(0)} \) and \( \Sigma^{(0)} \). According to ref. [9, 10], we require that the Gauss law constraint is differentiable, and also first class. Differentiability requires that the smearing functions be continuous on \( D \), and that the Gauss law can be relied upon to generate well defined canonical transformations on phase space. The Gauss law constraints further generate the ISO(2,1) gauge transformations, provided they are first class.

By varying \( g(\Lambda^{(0)}, \Sigma^{(0)}) \) with respect to \( \omega^a \) and \( e^a \), we obtain

\[
\frac{2}{\kappa} \delta g(\Lambda^{(0)}, \Sigma^{(0)}) = - \int_D \left( d\Lambda^{(0)}_a - \epsilon_{abc} \Lambda^{(0)b}_c \omega^c - \epsilon_{abc} \Sigma^{(0)b}_c \right) \wedge \delta \omega^a \\
- \int_D \left( d\Sigma^{(0)}_a - \epsilon_{abc} \Sigma^{(0)b}_c \right) \wedge \delta e^a + \int_{\partial D} \left( \Lambda^{(0)a}_a \delta \omega^a + \Sigma^{(0)}_a \delta e^a \right). \tag{2.5}
\]

By definition, \( g(\Lambda^{(0)}, \Sigma^{(0)}) \) is differentiable in \( e^a \) and \( \omega^a \) only if the boundary term, \( \int_{\partial D} \left( \Lambda^{(0)a}_a \delta \omega^a + \Sigma^{(0)}_a \delta e^a \right) \), is zero. If we do not wish to constrain the phase space by legislating that \( \delta e^a \) and \( \delta \omega^a \) be zero on \( \partial D \) to achieve this goal, we are led to the following conditions on the test functions \( \Lambda^{(0)} \) and \( \Sigma^{(0)} \) in \( \mathcal{T}^{(0)} \):

\[
\Lambda^{(0)} \mid_{\partial D} = 0, \quad \Sigma^{(0)} \mid_{\partial D} = 0. \tag{2.6}
\]

With these conditions, the Poisson brackets of \( g(\Lambda^{(0)}, \Sigma^{(0)}) \) with \( \omega^a \) and \( e^a \) are given by

\[
\{ g(\Lambda^{(0)}, \Sigma^{(0)}), \omega^a_i(x) \} = \frac{1}{2} \left( \partial_i \Sigma^{(0)}_a - \epsilon_{abc} \Sigma^{(0)}_b \omega^c \right)(x), \\
\{ g(\Lambda^{(0)}, \Sigma^{(0)}), e^a_i(x) \} = \frac{1}{2} \left( \partial_i \Lambda^{(0)}_a - \epsilon_{abc} \Lambda^{(0)}_b \omega^c - \epsilon_{abc} \Sigma^{(0)}_b \bar{e}^c \right)(x), \tag{2.7}
\]

from which we can compute the Poisson brackets of the \( g(\Lambda^{(0)}, \Sigma^{(0)}) \)'s

\[
\{ g(\Lambda^{(0)}, \Sigma^{(0)}), g(\Lambda'^{(0)}, \Sigma'^{(0)}) \} = g(\Lambda'^{(0)}, \Sigma'^{(0)}) + \frac{\kappa}{4} \int_{\partial D} \left( \Sigma'^{(0)}_a d\Lambda'^{(0)}_a + \Lambda'^{(0)}_a d\Sigma'^{(0)}_a \right) \\
+ \frac{\kappa}{2} \int_{\partial D} \left( \Sigma'^{(0)}_a e^a + \Lambda'^{(0)}_a \omega^a \right). \tag{2.8}
\]
where
\[ \Lambda_a^{(0)} = \frac{1}{2} \epsilon_{abc} \left( \Sigma_b^{(0)k} \Lambda_c^{(0)c} + \Lambda_b^{(0)b} \Sigma_c^{(0)c} \right), \]
\[ \Sigma_a^{(0)} = \frac{1}{2} \epsilon_{abc} \Sigma_b^{(0)k} \Sigma_c^{(0)c}. \] (2.9)

The boundary terms in eq. (2.8) vanish upon imposing the conditions (2.6) on all the test functions, and hence \( g(\Lambda^{(0)}, \Sigma^{(0)}) \approx 0 \) are first class constraints.

Next we write down the observables of the theory. By definition, they have zero Poisson brackets with the first class constraints, and hence are first class variables. Here, they are of the form
\[ q(\Lambda, \Sigma) = -\frac{k}{2} \int_D \left( d\Lambda_a \wedge \omega^a + d\Sigma_a \wedge e^a - \frac{1}{2} \epsilon_{abc} \Lambda^a \omega^b \wedge \omega^c - \epsilon_{abc} \Sigma^a e^b \wedge \omega^c \right), \] (2.10)
where in this case, the test functions \( \Lambda |_{\partial D} \) and \( \Sigma |_{\partial D} \) are not necessarily zero. We shall identify \( q(\Lambda, \Sigma) \) with \( q(\Lambda', \Sigma') \) if the boundary values of \( \Lambda \) and \( \Sigma \) agree with those of \( \Lambda' \) and \( \Sigma' \), i.e. \( \Lambda |_{\partial D} = \Lambda' |_{\partial D} \) and \( \Sigma |_{\partial D} = \Sigma' |_{\partial D} \). For then \( q(\Lambda, \Sigma) - q(\Lambda', \Sigma') = g(\Lambda - \Lambda', \Sigma - \Sigma') \approx 0 \). The \( q(\Lambda, \Sigma) \)'s are differentiable with respect to \( \omega^a \) and \( e^a \) for arbitrary test functions \( \Lambda \) and \( \Sigma \). To see that they correspond to first class variables, we compute their Poisson brackets with \( g(\Lambda^{(0)}, \Sigma^{(0)}) \):
\[ \{ g(\Lambda^{(0)}, \Sigma^{(0)}), q(\Lambda, \Sigma) \} = g(\Lambda'^{(0)}, \Sigma'^{(0)}) + \frac{k}{4} \int_{\partial D} (\Sigma^{(0)a} d\Lambda_a + \Lambda^{(0)a} d\Sigma_a) \]
\[ \approx 0, \] (2.11)
where
\[ \Lambda_a'^{(0)} = \frac{1}{2} \epsilon_{abc} \left( \Sigma_b'^{(0)k} \Lambda_c'^{(0)c} + \Lambda_b'^{(0)b} \Sigma_c'^{(0)c} \right), \]
\[ \Sigma_a'^{(0)} = \frac{1}{2} \epsilon_{abc} \Sigma_b'^{(0)k} \Sigma_c'^{(0)c}. \] (2.12)

The Poisson brackets of the \( q(\Lambda, \Sigma) \)'s with themselves are
\[ \{ q(\Lambda, \Sigma), q(\Lambda', \Sigma') \} = q(\Lambda'', \Sigma'') + \frac{k}{4} \int_{\partial D} (\Sigma^a d\Lambda'_a + \Lambda^a d\Sigma'_a), \] (2.13)
where
\[
\Lambda''_a = \frac{1}{2} \epsilon_{abc} \left( \Sigma^b \Lambda'^c + \Lambda^b \Sigma'^c \right),
\]
\[
\Sigma''_a = \frac{1}{2} \epsilon_{abc} \Sigma^b \Sigma'^c.
\]\quad (2.14)

It is known that the observables of Chern-Simons theory written on a disc are elements of a Kac-Moody algebra associated with some Kac-Moody group. The Kac-Moody group is an extension of the finite dimensional group $G$ in which the gauge transformations of Chern-Simons theory are defined. Here $G = ISO(2,1)$, so it follows that the observables $q(\Lambda, \Sigma)$ must generate the $ISO(2,1)$ Kac-Moody group. The brackets (2.13) define the $ISO(2,1)$ Kac-Moody algebra.

If we like, we can replace the elements $q(\Lambda, \Sigma)$ of the algebra by another set of variables $P^a(\psi)$ and $J^a(\psi)$ which were used in ref. [14]. [These variables were interpreted, respectively, as momentum and angular momentum current densities on the boundary $\partial D$.] For this, we introduce the test functions $\Xi^{(\psi,a)} = \{ \Xi^{(\psi,a)}_b(x) \}$ whose values on the boundary $\partial D$ correspond to a delta function centered around the point $\psi$ on $\partial D$. More precisely, let us introduce polar coordinates $r, \theta$ on $D$ (with $r = r_0$ defining the boundary $\partial D$, and $\theta$ parametrizing $\partial D$). Then the boundary value of $\Xi^{(\psi,a)}_b(r, \theta)$ is given by
\[
\Xi^{(\psi,a)}_b(r_0, \theta) = \delta^a_b \delta(\psi - \theta).
\]

We can now define $P^a(\psi)$ and $J^a(\psi)$ as follows:
\[
P^a(\psi) = 2 \, q(\Xi^{(\psi,a)}_b, 0) \quad \text{and} \quad J^a(\psi) = -2 \, q(0, \Xi^{(\psi,a)}_b).
\]\quad (2.15)

$P^a(\psi)$ and $J^a(\psi)$ can be used as a basis for the set of observables $\{q(\Lambda, \Sigma)\}$. To write down the completeness relation, let us again utilize polar coordinates $r, \theta$ on $D$ (with $r = r_0$ defining the boundary $\partial D$). Then
\[
q(\Lambda, \Sigma) \approx \frac{1}{2} \int_0^{2\pi} d\psi \left( \Lambda^a(r_0, \psi) P_a(\psi) - \Sigma^a(r_0, \psi) J_a(\psi) \right)
\]\quad (2.16)
To prove this relation, let us write $P^a(\psi)$ and $J^a(\psi)$ in polar coordinates:

\begin{align*}
P^a(\psi) &= -\kappa \omega^a_\theta(r_0, \psi) + \kappa \int_D \Xi_{b}^{(\psi,a)} R^b, \quad (2.17) \\
J^a(\psi) &= \kappa e^a_\theta(r_0, \psi) - \kappa \int_D \Xi_{b}^{(\psi,a)} T^b, \quad (2.18)
\end{align*}

where $\omega^a = \omega^a_r dr + \omega^a_\theta d\theta$ and $e^a = e^a_r dr + e^a_\theta d\theta$. Then the right hand side of eq. (2.16) can be written

\begin{align*}
-\frac{\kappa}{2} \int_0^{2\pi} d\psi \left( \Lambda_a(r_0, \psi) \omega^a_\theta(r_0, \psi) + \Sigma_a(r_0, \psi) e^a_\theta(r_0, \psi) \right) \\
+ \frac{\kappa}{2} \int_0^{2\pi} d\psi \int_D \Xi_{b}^{(\psi,a)} \left( \Lambda_a(r_0, \psi) R^b + \Sigma_a(r_0, \psi) T^b \right), \quad (2.19)
\end{align*}

while the left hand side of eq. (2.16) can be written

\begin{align*}
-\frac{\kappa}{2} \int_{\partial D} \left( \Lambda_a \omega^a + \Sigma_a e^a(x) \right) + \frac{\kappa}{2} \int_D \left( \Lambda_a R^a + \Sigma_a T^a \right). \quad (2.20)
\end{align*}

The difference of these two expressions is

\begin{align*}
\frac{\kappa}{2} \int_D \left\{ \Lambda_a - \int_0^{2\pi} d\psi \Xi_{a}^{(\psi,b)} \Lambda_b(r_0, \psi) \right\} R^a + \frac{\kappa}{2} \int_D \left\{ \Sigma_a - \int_0^{2\pi} d\psi \Xi_{a}^{(\psi,b)} \Sigma_b(r_0, \psi) \right\} T^a. \quad (2.21)
\end{align*}

Finally, we note that the boundary value of the functions in braces \{\} is zero. These functions are therefore defined in the text function space $T^{(0)}$, and hence the expression (2.21) corresponds to $g$ in the Gauss law constraint. It must vanish (weakly), proving (2.16).

The Poisson bracket algebra of $P_a$ and $J_a$ is easily obtained from eqs. (2.13) and (2.14). We find:

\begin{align*}
\{P_a(\psi), P_b(\psi')\} &= 0, \quad (2.22) \\
\{J_a(\psi), J_b(\psi')\} &= -\epsilon_{abc} J^c(\psi) \delta(\psi - \psi'), \quad (2.23) \\
\{J_a(\psi), P_b(\psi')\} &= -\epsilon_{abc} P^c(\psi) \delta(\psi - \psi') - \kappa \eta_{ab} \partial_\psi \delta(\psi - \psi'). \quad (2.24)
\end{align*}

This algebra is identical to that found in ref. [14]. The second term on the left hand side of eq. (2.24) defines the central extension to the Poincare loop group algebra.
We next construct the diffeomorphism generators $\ell$ of $D$. Since the action is diffeomorphism invariant, we know that they are associated with first class constraints, and hence should vanish weakly. Also, as before, we shall require differentiability. Following ref. [9, 10], we can write

$$\ell(v^{(0)}) = -\kappa \int_D v^{(0)i}(e^a_i R_a + \omega^a_i T_a) , \quad v^{(0)i} |_{\partial D} = 0 ,$$

which is weakly zero. The restriction of the vector valued function $v^{(0)}$ on the boundary insures that $\ell(v^{(0)})$ is differentiable with respect to $e^a$ and $\omega^a$. As a result, $\ell(v^{(0)})$ generates transformations which vanish on $\partial D$.

Eq. (2.25) can be generalized in order to obtain nontrivial transformations on the boundary. For this we replace $v^{(0)}$ by $v$ and set

$$\ell(v) = -\kappa \int_D v^i (e^a_i R_a + \omega^a_i T_a) + \frac{\kappa}{2} \int_D d(v^i e^a_i \omega_a + v^i \omega^a_i e_a) .$$

The requirement of differentiability can now be satisfied if we impose the following boundary conditions for the vector valued function $v^i$:

$$v^i |_{\partial D} = \epsilon(\theta) \left( \frac{\partial x^i}{\partial \theta} \right) |_{\partial D} ,$$

where $x_i$ denote the space coordinates, $\theta$ is an angular coordinate parametrizing $\partial D$ and $\epsilon(\theta)$ is an arbitrary function. To show that $\ell(v)$ is first class, we can take its Poisson bracket with the Gauss law constraint

$$\{ \ell(v), g(\Lambda^{(0)}, \Sigma^{(0)}) \} = g(\mathcal{L}_v \Lambda^{(0)}, \mathcal{L}_v \Sigma^{(0)}) \approx 0 ,$$

where $\mathcal{L}_v \Lambda^{(0)}$ is the Lie derivative of the scalar field $\Lambda^{(0)}$ with respect to $v$ and is defined by $\mathcal{L}_v \Lambda^{(0)} = v^i \partial_j \Lambda^{(0)}$. It also of course follows that $\ell(v^{(0)})$ is first class. Similar Poisson brackets are obtained for $\ell(v)$ with $q(\Lambda, \Sigma)$

$$\{ \ell(v), q(\Lambda, \Sigma) \} = q(\mathcal{L}_v \Lambda, \mathcal{L}_v \Sigma) .$$
while the Poisson bracket of two \( \ell \)'s gives the usual Virasoro algebra
\[
\{ \ell(v), \ell(v') \} = \ell(\mathcal{L}_v v') .
\] (2.30)

Now \( \mathcal{L}_v v' \) denotes the Lie derivative of the vector field \( v' \) with respect to the vector field \( v \) and is given by \( (\mathcal{L}_v v')^i = v^j \partial_j v'^i - v'^j \partial_j v^i \).

As usual, the Virasoro generators can be expressed (weakly) in terms of a product of Kac-Moody generators. The Sugawara construction for \( ISO(2, 1) \) is as follows:
\[
\ell(v) \approx -\frac{1}{\kappa} \int_{\partial D} d\psi \, \epsilon(\psi) P_a(\psi) J^a(\psi) .
\] (2.31)

The proof of (2.31) is essentially the same as in ref. [9]. For this, we can again use polar coordinates \( r, \theta \) on \( D \) (with \( r = r_0 \) defining the boundary \( \partial D \)). From the two expressions (2.15) for \( P^a \) and \( J^a \), the right hand side of eq. (2.31) is weakly equal to
\[
\kappa \int_{\partial D} d\psi \, \epsilon(\psi) \left[ \omega^a \theta e^a \theta(r_0, \psi) - \int_D \Xi^{(\psi, a)}(R^b e_a^b(r_0, \psi) + T^b \omega^a_b(r_0, \psi)) \right] .
\] (2.32)

In the above, we have dropped terms quadratic in the curvature \( R \) and the torsion \( T \) due to the Gauss law constraint. In this regard, the relevant test function \( \Sigma^{(0)} \) (or \( \Lambda^{(0)} \)) for the Gauss law constraint (2.3), involves \( R \) (or \( T \)) itself.

Concerning the left hand side of eq. (2.31), if we substitute the boundary value of \( v^i \) given in (2.27), into eq. (2.26) we get
\[
\kappa \int_{\partial D} d\psi \, \epsilon(\psi) \left[ \omega^a \theta e^a \theta(r_0, \psi) - \int_D \Xi^{(\psi, a)} e_i^a R_a + \omega^a_i T_a \right] .
\] (2.33)

In comparing (2.32) with (2.33), we find their difference to be equal to
\[
\kappa \int_D \left\{ v^i e^a_i - \int_{\partial D} d\psi \, \epsilon(\psi) \Xi^{(\psi, a)} e_i^a \theta b(r_0, \psi) \right\} R_a
\]
\[
+ \kappa \int_D \left\{ v^i \omega^a_i - \int_{\partial D} d\psi \, \epsilon(\psi) \Xi^{(\psi, a)} \omega_i^a \theta b(r_0, \psi) \right\} T_a .
\] (2.34)

Finally, we note that the boundary value of the functions in braces \( \{ \} \) is zero. These functions are therefore defined in the space \( T^{(0)} \), and hence the expression (2.34) corresponds to \( g \) the Gauss law constraint. It must vanish (weakly), proving (2.31).
3. THE QUANTUM THEORY

To quantize the system described in the previous section, it is sufficient to find unitary representations of the $ISO(2, 1)$ Kac-Moody algebra. This was already done in ref. [14]. We review it here for completeness.

In the previous section, we found that $P_a(\psi)$ and $J_a(\psi)$ formed a basis for the algebra. To quantize, we replace the phase space variables $P_a(\psi)$ and $J_a(\psi)$ by quantum operators $\hat{P}_a(\psi)$ and $\hat{J}_a(\psi)$. We replace Poisson brackets (2.22-24) by $-i$ times the commutator bracket, thereby obtaining the quantum version of the Kac-Moody algebra:

\[ \left[ \hat{P}_a(\psi), \hat{P}_b(\psi) \right] = 0, \tag{3.1} \]
\[ \left[ \hat{J}_a(\psi), \hat{J}_b(\psi') \right] = -i\epsilon_{abc} \hat{J}_c(\psi)\delta(\psi - \psi'), \tag{3.2} \]
\[ \left[ \hat{J}_a(\psi), \hat{P}_b(\psi') \right] = -i\epsilon_{abc} \hat{P}_c(\psi)\delta(\psi - \psi') - \kappa\eta_{ab}\partial_\psi\delta(\psi - \psi'). \tag{3.3} \]

$P_a(\psi)$ and $J_a(\psi)$ commute with $g(\Lambda(0), \Sigma(0))$, the quantum analogue of $g(\Lambda(0), \Sigma(0))$. $ISO(2, 1)$ gauge invariance in the quantum theory follows by requiring that $g(\Lambda(0), \Sigma(0))$ annihilates the states of the Hilbert space.

The Hilbert space consists of unitary representations of the Kac-Moody algebra eqs. (3.1-3) Highest weight constructions are standardly employed for the purpose of finding unitary representations of Kac-Moody algebras [13]. However, this procedure is complicated, at best, if the underlying group generated by the charges is noncompact. Here, the underlying group generated by the charges $\int_0^{2\pi} d\psi \hat{P}_a(\psi)$ and $\int_0^{2\pi} d\psi \hat{J}_a(\psi)$ is $ISO(2, 1)$. It is not only noncompact, but also a semidirect product group. A highest weight construction for this algebra was attempted in ref. [14], but it failed to give a nontrivial representation of the operators $\hat{P}_a(\psi)$ and $\hat{J}_a(\psi)$.

On the other hand, an alternative approach was given in ref. [14] which did yield nontrivial unitary representations for the algebra of $\hat{P}_a(\psi)$ and $\hat{J}_a(\psi)$. It employed the
method of induced representations \[15\], which is commonly used for finding unitary representations of semidirect product groups. We describe it below.

The method utilizes the fact that the momentum density operators $P_a(\psi)$ commute and hence are simultaneously diagonalizable. States in the Hilbert space can therefore be labeled by the eigenvalues $P_a(\psi)$ of $P_a(\psi)$. To construct a certain unitary representation, we first pick a “standard” set of eigenvalues, which we denote by $\hat{P} = \{\hat{P}_a(\psi)\}$, belonging to that representation. Let $|\hat{P}, 1, \alpha >$ be its corresponding eigenvector. Thus

$$P_a(\psi)|\hat{P}, 1, \alpha >= \hat{P}_a(\psi)|\hat{P}, 1, \alpha > . \quad (3.4)$$

We will see that the states of a given representation are not completely labeled by the eigenvalues of $P_a(\psi)$, hence the need for a degeneracy index $\alpha$ in the state vector $|\hat{P}, 1, \alpha >$.

In order to obtain the remaining states of the representation, let us introduce the matrix $M = M_{ab}(\psi)$, which denotes a local Lorentz transformation, and further, let $U(M)$ be a unitary representation of $M$. We want to express the latter as an exponential of the angular momentum operators $J_a(\psi)$. For then, the commutation relation (3.1-3) leads to the following operator equation

$$U(M)^{-1}P_a(\psi)U(M) = M_{ab}(\psi)P^b(\psi) - \frac{\kappa}{2} \epsilon_{abc}[\partial_\psi MM^{-1}]^{bc}(\psi) . \quad (3.5)$$

The first term in on the right-hand-side of eq. (3.5) represents a local Lorentz transformation of the momentum operator, while the second term results from the central term in the commutation relation (3.3).

Now the remaining states of the representation are obtained by having $U(M)$ act on the “standard state” $|\hat{P}, 1, \alpha >$. From eq. (3.5), the momentum density eigenvalues of the resulting states will be of the form

$$P_a(\psi) = M_{ab}(\psi)\hat{P}^b(\psi) - \frac{\kappa}{2} \epsilon_{abc}[\partial_\psi MM^{-1}]^{bc}(\psi) . \quad (3.6)$$
The set of all such $P = \{P_a(\psi)\}$ defines an orbit in the space of momentum densities. We can label the orbit by the point $\hat{P} = \{\hat{P}_a(\psi)\}$ in momentum density space through which it passes.

A state with a given momentum density eigenvalue $P$ on any particular orbit is defined only up to the action of the little group $G_P$ of $P$, i.e. the subset of local Lorentz transformations that leaves the momentum density eigenvalue unchanged. The little groups $G_P$ for all points $P$ on any particular orbit are isomorphic. Let $G_{\hat{P}} = \{\hat{M}\}$ be the little group associated with the “standard” eigenvalue $\hat{P}$. Then from eq. (3.6), $\hat{M}$ satisfies

$$\hat{P}_a(\psi) = \hat{M}_{ab}(\psi)\hat{P}_b(\psi) - \frac{\kappa}{2}\epsilon_{abc}[\partial_\psi \hat{M}\hat{M}^{-1}]^{bc}(\psi).$$

(3.7)

When $U(\hat{M})$ acts on $|\hat{P}, 1, \alpha>$ it can only change the degeneracy index $\alpha$. Thus

$$U(\hat{M})|\hat{P}, 1, \alpha> = D_{\alpha\beta}(\hat{M})|\hat{P}, 1, \beta>,$$

(3.8)

where we define $D_{\alpha\beta}(\hat{M})$ to be some unitary representation of the little group $G_{\hat{P}}$.

In ref. [4], it was shown that $G_{\hat{P}}$ is a finite dimensional group. We shall not repeat the proof here. More specifically, $G_{\hat{P}}$ is isomorphic to either $SO(2, 1)$, $U(1)$ or $R^1$. The group $SO(2, 1)$ results, for instance, when we set the standard momentum densities $\hat{P}_b(\psi) = 0$. In that case, eq. (3.7) shows that $\{\hat{M}\}$ is the set of all $\psi$-independent $SO(2, 1)$ group matrices. [Note in this case that the unitary representations $\{D_{\alpha\beta}\}$ are either trivial or infinite dimensional.] $U(1)$ results when we set $\hat{P}_b(\psi) = \delta^b_0 \times const$. Then $\{\hat{M}\}$ is the $SO(2)$ subset of $\psi$-independent $SO(2, 1)$ matrices. When we set $\hat{P}_b(\psi) = \delta^b_2 \times const$, $\{\hat{M}\}$ is an $SO(1, 1)$ subset of $\psi$-independent $SO(2, 1)$ matrices, and hence isomorphic to $R^1$.

In order to define a unique state with momentum density eigenvalues $P = \{P^b(\psi)\}$ on an orbit through $\hat{P} = \{\hat{P}^b(\psi)\}$ which are not equal to the “standard” eigenvalues, i.e. $P \neq \hat{P}$, let us define a unique transformation $M_P = M_P(\psi)$ which takes $\hat{P}$ to $P$. Thus

$$P_a = [M_P]_{ab}\hat{P}^b - \frac{\kappa}{2}\epsilon_{abc}[\partial_\psi M_P M_P^{-1}]^{bc}. \quad \text{[For the above example where } \hat{P}^b(\psi) = 0, \text{ yielding}$$
the little group $G_P = SO(2, 1)$, we can choose $M_P(\psi)$ such that it is the $SO(2, 1)$ identity element at $\psi = 0$. The unique state with momentum density eigenvalues $P^b(\psi)$, which we denote by $|\hat{P}, M_P, \alpha >$, can then be defined according to

$$|\hat{P}, M_P, \alpha > = U(M_P)|\hat{P}, 1, \alpha > .$$

(3.9)

Then $P_a(\psi)|\hat{P}, M_P, \alpha > = P_a(\psi)|\hat{P}, M_P, \alpha >$.

Now an arbitrary $M$ which takes $\hat{P}$ to $P$ can be written $M = M_P \hat{M}$, where $\hat{M} \in G_P$. When $U(M)$ acts on the state $|\hat{P}, 1, \alpha >$, we get

$$U(M)|\hat{P}, 1, \alpha > = U(M_P)U(\hat{M})|\hat{P}, 1, \alpha > = D_{\alpha \beta}(\hat{M})|\hat{P}, M_P, \beta > .$$

(3.10)

It remains to determine the action of an arbitrary $U(N)$ on an arbitrary state $|\hat{P}, M_P, \alpha >$. For this we define $\hat{N} = M_P^{-1}NM_P$ and $P'_a = N_{ab}P^b - \frac{\kappa}{2} \epsilon_{abc}[\partial_\psi NN^{-1}]^{bc}$. It follows that $\hat{N} \in G_P$ and

$$U(N)|\hat{P}, M_P, \alpha > = U(M_P')U(\hat{N})U(M_P)^{-1}|\hat{P}, M_P, \alpha >$$

$$= U(M_P')U(\hat{N})|\hat{P}, 1, \alpha >$$

$$= D_{\alpha \beta}(\hat{N})|\hat{P}, M_P', \beta > ,$$

(3.11)

where we have used eq’s (3.8) and (3.9).

To summarize, in analogy with the unitary representations of the Poincare group, unitary representations for $ISO(2, 1)$ Kac-Moody group can be specified by their orbits in momentum density space, along with representation $\{D_{\alpha \beta}\}$ of the little group $G_P$.

4. POINT SOURCES ON THE DISC

Here we consider introducing point sources on the disc. They are characterized, in general, by their momenta and angular momenta, which play the role of charges in the
Chern-Simons theory. ISO($2, 1$) Chern-Simons theory with sources has been treated by many authors. [5, 18, 19, 20, 21] We shall offer a new approach below.

The equations of motion in the absence of any sources imply that the curvature and torsion two forms, $R_a^\nu$ and $T_a^\nu$, vanish everywhere. On the other hand, in the presence of a point source with the space-time coordinates $z_\mu = z_\mu(\tau)$, where $\mu = 0, 1, 2$ and $\tau$ parametrizes the particle world line, this result is modified to

$$\kappa^2 \epsilon^{\mu\nu\lambda} R_{\nu\lambda}^a(x) = \int d\tau \, \delta^3(x - z(\tau)) p^a \dot{z}^\mu, \quad (4.1)$$

$$\kappa^2 \epsilon^{\mu
u\lambda} T_{\nu\lambda}^a(x) = \int d\tau \, \delta^3(x - z(\tau)) j^a \dot{z}^\mu. \quad (4.2)$$

Here $p^a = p^a(\tau)$ and $j^a = j^a(\tau)$ are the particle momenta and angular momenta, respectively, and $R_{\nu\lambda}^a(x)$ and $T_{\nu\lambda}^a(x)$ are the space-time components of $R^a$ and $T^a$. (For the moment we are not considering the effect of the disc boundary $\partial D$.)

Equations of motion for the particle degrees of freedom $p^a$ and $j^a$, are deduced from the Bianchi identities for the fields

$$dR^a + \epsilon^{abc} \omega_b^\mu(x) R_c^\mu = 0, \quad (4.3)$$

$$dT^a + \epsilon^{abc} \left( \omega_b^\mu(x) T_c^\mu + e_b^\mu(x) R_c^\mu \right) = 0. \quad (4.4)$$

Upon substituting (4.1) and (4.2) into (4.3) and (4.4), we then get the following equations for $p^a$ and $j^a$,

$$\dot{p}_a + \epsilon_{abc} \omega_b^\mu(z) p^c \dot{z}^\mu = 0, \quad (4.5)$$

$$\dot{j}_a + \epsilon_{abc} \left( \omega_b^\mu(z) j^c + e_b^\mu(z) p^c \right) \dot{z}^\mu = 0. \quad (4.6)$$

These equations can also be derived starting from an action principal [13, 5]. In generalizing to the case of more than one point particle, each particle would satisfy equations of motion analogous to (4.5) and (4.6).

$\omega_b^\mu(z)$ and $e_b^\mu(z)$ in equations (4.5) and (4.6) are components of the spin connections $\omega^b$ and dreibein one forms $e^b$, evaluated at the particle space-time position $z_\mu$. If the particle
is treated as a test particle these functions are well defined. However, in general, we must consider self-interactions of the particle. In that case, $\omega^b_\mu(z)$ and $e^b_\mu(z)$ are singular functions, i.e. $\omega^b = \omega^b_\mu(x)dx^\mu$ and $e^b = e^b_\mu(x)dx^\mu$ have no definite limit when $x$ approaches $z$. This singularity demands regularization.

Following ref. [10], a good way to regularize is to punch a hole $H$ containing $z$, and eventually to shrink the hole to the point $z$. Once this hole is made, the action is no longer defined on a disc $D$, but on $D \setminus H$, a disc with a hole. $D \setminus H$ has a new boundary $\partial H$ and it must be treated exactly like $\partial D$. Thus for instance, the Gauss law must be changed to

$$g(\Lambda^{(1)}, \Sigma^{(1)}) \approx 0, \quad (4.7)$$

where the new test functions $\Lambda^{(1)}$ and $\Sigma^{(1)}$ satisfy

$$\Lambda^{(1)}|_{\partial D} = \Lambda^{(1)}|_{\partial H} = 0, \quad \Sigma^{(1)}|_{\partial D} = \Sigma^{(1)}|_{\partial H} = 0. \quad (4.8)$$

There are now two ISO(2,1) Kac-Moody algebras in the theory, one each for each boundary, $\partial D$ and $\partial H$. The elements of the algebras consist of the observables $q(\Lambda, \Sigma)$. We can again use them to define momentum and angular momentum densities $P^{(A)a}(\psi)$ and $J^{(A)a}(\psi)$, $A = 0, 1$. $A = 0$ corresponds to currents on $\partial D$, while $A = 1$ corresponds to currents on $\partial H$. We define these currents in an analogous fashion to eq. (2.15). For this we introduce test functions $\Xi^{(\psi,a,A)} = \left\{ \Xi^{(\psi,a,A)}_b(x) \right\}$, such that

$$\Xi^{(\psi,a,0)}_b(\theta) = \left\{ \begin{array}{ll} \delta^b_0 \delta(\psi - \theta), & \text{on } \partial D, \\ 0, & \text{on } \partial H \end{array} \right.,$$

and

$$\Xi^{(\psi,a,1)}_b(\theta) = \left\{ \begin{array}{ll} 0, & \text{on } \partial D, \\ \delta^b_0 \delta(\psi - 2\pi + \theta), & \text{on } \partial H \end{array} \right..$$

As in Section 2, $\theta$ is to be interpreted as an angular coordinate. [The coordinates $\theta$ on both $\partial D$ and $\partial H$ increase, say, in the anticlockwise sense. Both $\theta$ and $\psi$ are assumed to run from 0 to $2\pi$.] We may now define $P^{(A)a}(\psi)$ and $J^{(A)a}(\psi)$ as follows:

$$P^{(A)a}(\psi) = 2 q(\Xi^{(\psi,a,A)}, 0) \quad \text{and} \quad J^{(A)a}(\psi) = -2 q(0, \Xi^{(\psi,a,A)}). \quad (4.9)$$
In the unregularized theory the particle dynamics is given in terms of charges \( p^a \) and \( j^a \), while here it is described by \( P^{(A)a}(\psi) \) and \( J^{(A)a}(\psi) \). In the process of regularizing the theory we have effectively enlarged the phase space of the particle. Since \( p^a \) and \( j^a \) correspond to the total momenta and angular momenta, these quantities should be identified with \( \int_0^{2\pi} d\psi \, P^{(A)a}(\psi) \) and \( \int_0^{2\pi} d\psi \, J^{(A)a}(\psi) \), respectively, in the regularized theory.

In the quantum theory, we replace \( P^{(A)a}(\psi) \) and \( J^{(A)a}(\psi) \) by quantum operators \( \mathbf{P}^{(A)a}(\psi) \) and \( \mathbf{J}^{(A)a}(\psi) \). The commutators of these operators define the direct sum of two \( ISO(2,1) \) Kac-Moody algebras. The Hilbert space is obtained by taking tensor products of two induced representations of the type discussed in Section 3. Unitary representations of the Hilbert space are now characterized by a pair of orbits in momentum density space. The two orbits, which we can label \( (0) \) and \( (1) \), pass through some “standard” points, \( \hat{P}^{(0)} \) and \( \hat{P}^{(1)} \). We can then define a “standard” eigenstate \( |\hat{P}^{(0)}, 1, \alpha > \otimes |\hat{P}^{(1)}, 1, \beta > \), associated with the “standard” eigenvalues \( \hat{P}^{(0)} \) and \( \hat{P}^{(1)} \). In addition to the orbits in momentum density space, the tensor product representations are labeled by unitary representations of the little groups of \( \hat{P}^{(0)} \) and \( \hat{P}^{(1)} \). Since \( \alpha \) and \( \beta \) are indices associated with different such unitary representations, and also, in general, different little groups, they may range over different values.

The remaining states of the Hilbert space are gotten by acting on the standard state \( |\hat{P}^{(0)}, 1, \alpha > \otimes |\hat{P}^{(1)}, 1, \beta > \) with local Lorentz transformations \( \mathbf{U}^{(0)}(M_{P^{(0)}}) \) and \( \mathbf{U}^{(1)}(M_{P^{(1)}}) \) generated, respectively, by \( \mathbf{J}^{(0)a}(\psi) \) and \( \mathbf{J}^{(1)a}(\psi) \). A general tensor product state diagonal in \( \mathbf{P}^{(A)} \) is thus

\[
|\hat{P}^{(0)}, M_{P^{(0)}}, \alpha > \otimes |\hat{P}^{(1)}, M_{P^{(1)}}, \beta > = \mathbf{U}^{(0)}(M_{P^{(0)})} \mathbf{U}^{(1)}(M_{P^{(1)})} |\hat{P}^{(0)}, 1, \alpha > \otimes |\hat{P}^{(1)}, 1, \beta > , \tag{4.10}
\]

with \( P^{(A)} \) being the eigenvalues of \( \mathbf{P}^{(A)} \).

It is straightforward to generalize these results to the case of \( N \) sources or holes. In that case, there would \( N + 1 \) \( ISO(2,1) \) Kac-Moody algebras, one for each hole and one for
the boundary $\partial D$. The Hilbert space for the quantum theory is then obtained by taking
tensor products of $N+1$ induced representations of the type discussed in Section 3.

5. VERTEX OPERATORS

Vertex operators of conformal field theory have been utilized for the purpose of creating
sources in Chern-Simons theories. Here we write down the analogue of the vertex operator
for topological gravity. That operator is seen to create a source for the curvature two form
$R^a$. The source can be interpreted as a massive point particle in its rest frame. The mass
of the point particle determines the surrounding space-time metric. The vertex operator
for topological gravity therefore creates certain space-time geometries.

To proceed we shall closely follow ref. [10]. Unlike ref. [10], however, we do not deal
with highest weight representations, but rather with the induced representations discussed
previously.

As in the previous section, we shall examine the manifold $D \setminus H$ of a disc with a hole.
We eventually would like to take the limit where the hole is shrunk to a point $z$. The hole
or point is originally to be “sourceless” (with regards to both the $SO(2,1)$ curvature $R^a$
and the torsion $T^a$). It should then be described by a state $|0\rangle$ where the momentum
density associated with $\partial H$ vanishes, i.e. the eigenvalues of $P^{(1)a}(\psi)$ are zero. Let us call
this the “standard” state. Hence $\hat{P}^{(1)} = 0$, and

$$|0\rangle = |\hat{P}^{(0)}, 1, \alpha > \otimes |0, 1, \beta > .$$

(5.1)

$\hat{P}^{(0)}$ is the momentum density associated with the disc boundary $\partial D$, which we can
assume, in general, to be nonzero. $\beta$ labels unitary representations of the little group
$G_{\hat{P}^{(1)}=0}$, which is isomorphic to $SO(2,1)$. Here we shall assume the trivial representations
for $G_{\hat{P}^{(1)}=0}$, and thus $\beta$ is a trivial index which we set to zero. In this case, from eq. (3.8)

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we have
\[ U^{(1)}(\hat{M})|0> = |0> . \] (5.2)

\(U^{(1)}(\hat{M})\) is the unitary representation of \(\hat{M} \in G_{\hat{\rho}^{(1)}=0}\). (Our notation agrees with that in the previous section, where \(U^{(1)}\) acts on the second ket in the tensor product.) \(U^{(1)}(\hat{M})\) is generated by the angular momentum operators, \(\int_0^{2\pi} d\psi \ J^{(1)a}(\psi)\), associated with the hole. From eq. (5.2), we may then conclude that the angular momentum, as well as the momentum, are zero for the state \(|0>\). In the limit where the hole is shrunk to a point \(z\), the point has no mass or spin. Therefore, as desired, it is indeed “sourceless” with regards to both \(R^a\) and \(T^a\).

From ref. [10], the vertex operator can be written in terms of \(q\)’s (defined in Section 2), whose test functions are linear in the polar angle \(\theta\). \(\theta\), and hence the vertex operator, is well defined on \(D \setminus (H \cup L_0)\), where the line \(L_0\) from \(\partial D\) to \(\partial H\) has zero polar angle. Accordingly, we define the quantity \(\overline{q} = q(0, \Sigma)\), where
\[ \Sigma_a = 2\theta \delta_a^0 . \] (5.3)

As we shall see shortly, this particular choice for the test function satisfies a rather restrictive criterion that the vertex operator preserves the gauge invariance of states.

From eq. (5.3),
\[ \overline{q} = -\kappa \int_D \left( d\theta \wedge e^0 - \epsilon^{abc} \theta e_b \wedge \omega_c \right) . \] (5.4)

Its Poisson brackets with the observables \(q(\Lambda, \Sigma)\) are
\[ \{q(\Lambda, \Sigma), \overline{q}\} = q(\overline{\Lambda}, \overline{\Sigma}) + \frac{\kappa}{2} \int d\theta \left( \Lambda^0 |_{\partial D} - \Lambda^0 |_{\partial H} \right) , \] (5.5)

where
\[ \overline{\Lambda} = \epsilon^{a00} \theta \Lambda_b , \quad \overline{\Sigma} = \epsilon^{a00} \theta \Sigma_b . \]

In quantum theory, this leads to the following commutation relations with \(P^{(A)a}(\psi)\) and \(J^{(A)a}(\psi)\):
\[ [P^{(0)}_a(\psi), \overline{q}] = i\epsilon_{ab0} \psi P^{(0)b}(\psi) + i\kappa \delta_a^0 , \] (5.6)
\[ [\mathbf{P}_a^{(1)}(\psi), \overline{\mathbf{r}}] = i\epsilon_{ab0} \psi \mathbf{P}^{(1)b}(\psi) - i\kappa\delta_a^0, \]  
(5.7)

\[ [\mathbf{J}_a^{(0)}(\psi), \overline{\mathbf{r}}] = i\epsilon_{ab0} \psi \mathbf{J}^{(0)b}(\psi), \]  
(5.8)

\[ [\mathbf{J}_a^{(1)}(\psi), \overline{\mathbf{r}}] = i\epsilon_{ab0} \psi \mathbf{J}^{(1)b}(\psi). \]  
(5.9)

We now define the vertex operator \( V \) as follows:

\[ V = e^{in\overline{\mathbf{r}}}. \]  
(5.10)

From eq. (5.7), it creates a state \( V|0> \) which has a uniform "energy" density \( P_0^{(1)}(\psi) = n\kappa \) at the hole boundary \( \partial H \), i.e.

\[ P_0^{(1)}(\psi) \, V|0> = n\kappa \, V|0> \]  
(5.11)

The eigenvalues for the spatial-momentum density \( P_1^{(1)}(\psi) \) and \( P_2^{(1)}(\psi) \) are zero for \( V|0> \).

In the limit where \( H \) is shrunk to the point \( z \), the state \( V|0> \) describes a point particle located at \( z \) with three-momenta

\[ p_a = 2\pi n\kappa\delta_a^0. \]  
(5.12)

It therefore describes a particle with mass \( m = 2\pi n\kappa \) in its rest frame. We shall see that the requirement that the state \( V|0> \) be gauge invariant, in addition to fixing the form (5.3) of \( \Sigma_a \), forces \( n \) to be an integer. It then follows that the vertex operator can only create particles with a quantized mass spectrum, i.e.

\[ m = 2\pi\kappa \times \text{integer} \]  
(5.13)

The states \( |0> \) and \( V|0> \) belong to two different representations of the Kac-Moody algebras. This is evident because there exists no unitary transformation \( U^{(0)}(M^{(0)}) \, U^{(1)}(M^{(1)}) \) where \( M^{(A)} \) are defined in the loop group of \( SO(2,1) \), connecting the two states, and also because the little groups \( G_{P^{(1)}} \) associated with the two states are different. \( G_{P^{(1)}=0} \) is the little group for \( |0> \), while \( G_{P^{(1)}=n\kappa\delta_a^0} \) is the little group for \( V|0> \).
The former is isomorphic to $SO(2,1)$, and the latter is isomorphic to $SO(2)$. Further, $G^{(1)}_{P_a=\kappa\delta_0} \subset G_{P(1)=0}$.

As mentioned earlier, the state $|0>$ has zero angular momentum (associated with the hole). The same is true for the state $V|0>$. To prove this, let $\hat{M}_0$ be an element of the little group $G^{(1)}_{P_a=\kappa\delta_0}$. Then it is also an element of the little group $G_{P(1)=0}$. From eq. (5.2), $U^{(1)}(\hat{M}_0)$ acts trivially on $|0>$. From eq. (5.9), $U^{(1)}(\hat{M}_0)$ commutes with the vertex operator, since it is generated by the zero component of the angular momentum operator, $\int_{\psi}^{2\pi} d\psi \ J^{(1)}(\psi)$. Hence,

$$U^{(1)}(\hat{M}_0) V|0> = V U^{(1)}(\hat{M}_0)|0>= V|0>$$

The last equality follows from eq. (5.2). Thus upon assuming trivial representation of the little group for $|0>$, we end up with the trivial representation of the little group for $V|0>$. Therefore the rotation generator, i.e. zero component of angular momentum, is zero when acting on $V|0>$. Since the corresponding point particle is in its rest frame, we can conclude that it has no spin. Although the particle is a source of curvature, it is not a source for the torsion.

What space-time geometry is created through the action of the vertex operator? That is, what is the space-time metric associated with the point particle with three-momenta (5.12)?

For this, let us solve for the spin connection $\omega^a$ and the dreibein one form $e^a$, starting from the field equations (4.1) and (4.2). We can place the point particle at the origin of the coordinate system. Hence, $z^1 = z^2 = 0$. For time, we set $z^0 = x^0 = \tau$. Substituting eq. (5.12) into the field eq. (4.1), we find that the only non-zero component of the curvature is

$$R^0_{12}(x) = 2\pi n\delta^2(x).$$

(5.15)
A solution to (5.15) for the spin connections is just
\[ \omega^0 = n \theta , \]
with \( \omega^1 = \omega^2 = 0 \). We can solve for the dreibein one forms by setting the torsion equal to zero. (This is since the particle has no spin.) A solution consistent with (5.16) is
\[ e^1 = \frac{1}{r} \cos n \theta \, dr - n \sin n \theta \, d\theta , \]
\[ e^2 = \frac{1}{r} \sin n \theta \, dr + n \cos n \theta \, d\theta . \]
The space-space components of the metric tensor now easily follow. We find
\[ d\ell^2 = \frac{1}{r^2} dr^2 + n^2 d\theta^2 . \]
(5.19)
For \( n = 1 \), (5.19) is the invariant length in cylindrical space, with \( \ln r \) and \( \theta \) being the coordinates of the cylinder. Locally, a cylindrical space is recovered as well for \( n = 2, 3, \ldots \). This is clear if we replace \( n \theta \) by a new angle \( \theta' \). Globally, we get an \( n \)-fold covering of the cylinder.

In the preceding analysis, we have ignored the boundary of the disc \( \partial D \). From eq. (5.6), the “energy” density eignevalue \( P^{(0)}_0(\psi) \) associated with the disc boundary \( \partial D \), has a contribution which is negative,
\[ P^{(0)}_0(\psi) \, V|0> = \left( \hat{P}^{(0)}_0 - n \kappa \right) \, V|0> . \]
(5.20)
It thus appears that an antiparticle gets created at \( \partial D \).

We now examine the question of gauge invariance, and explain why \( n \) must be an integer. Let us assume that the state \( |0> \) is gauge invariant. Here for a state to be gauge invariant it must be annihilated by \( g(\Lambda^{(1)}, \Sigma^{(1)}) \), the quantum analogue of \( g(\Lambda^{(1)}, \Sigma^{(1)}) \), where the test functions \( \Lambda^{(1)} \) and \( \Sigma^{(1)} \) are well defined on \( D \setminus H \) and satisfy eqs. (4.8). To check whether the state \( V|0> \) is gauge invariant, we first note the following property,
\[ V^{-1}g(\Lambda^{(1)}, \Sigma^{(1)})V = g(R_n \Lambda^{(1)}, R_n \Sigma^{(1)}) , \]
(5.21)
where $R_n$ is an $SO(2)$ rotation matrix, which in polar coordinates is given by

$$R_n(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos n\theta & -\sin n\theta \\ 0 & \sin n\theta & \cos n\theta \end{pmatrix}.$$  (5.22)

Now upon redefining the test functions $\Lambda^{(1)}$ and $\Sigma^{(1)}$, we have

$$g(\Lambda^{(1)}, \Sigma^{(1)}) V|0\rangle = V g(R_n^{-1}\Lambda^{(1)}, R_n^{-1}\Sigma^{(1)})|0\rangle$$  (5.23)

The right-hand-side of eq. (5.23) is zero only if $R_n^{-1}\Lambda^{(1)}$ and $R_n^{-1}\Sigma^{(1)}$ are well defined on $D \setminus H$, and satisfy eqs. (4.8). Only then are they suitable test functions for the Gauss law on $D \setminus H$. But this happens only when $R_n(\theta)$ is well defined on $D \setminus H$. This necessarily implies that $n$ must be an integer, thus leading to the quantized mass spectrum (5.13).

We see that although the test function $\Sigma_a$ used in defining $q$ is only defined on $D \setminus (H \cup L_0)$, the test functions relevant for the Gauss law may be defined on all of $D \setminus H$. Requiring the latter puts a severe restriction on the type of test functions we can use in defining $q$, as well as on the values for $n$. Basically, eq. (5.2) and smooth deformations of this function (keeping the values of $\Sigma_a$ at $\theta = 0$ and $2\pi$ fixed), are the only possible such test functions. (Actually, the values of $\Sigma_a$ at the end points can be changed with a corresponding change in $n$.)

In the above, we have examined the action of the vertex operator acting on the “sourceless” state $|0\rangle$. More generally, we can define the action of $V$ on any “standard” state $|\hat{P}^{(0)}, 1, \alpha > \otimes |\hat{P}^{(1)}, 1, \beta >$, according to

$$V |\hat{P}^{(0)}, 1, \alpha > \otimes |\hat{P}^{(1)}, 1, \beta > = U^{(0)}(R_n) U^{(1)}(R_n) |\hat{P}^{(0)}_a - n\kappa_1\delta_a, 1, \alpha > \otimes |\hat{P}^{(1)}_b + n\kappa_0\delta_b, 1, \beta >.$$  (5.24)

Eq. (5.24) fixes any ordering ambiguities inherent the definition of $V$. By acting on eq. (5.24) with unitary transformations $U^{(A)}(M_{P(A)})$ and using eq. (4.10), we then obtain the action of $V$ on any state in the representation.
In Section 2, we chose the variations of the fields $e^a$ and $\omega^a$ to be unconstrained at the boundary $\partial D$ of a disc. From eq. (2.5), conditions then had to be imposed on the test functions $\Lambda^{(0)}$ and $\Sigma^{(0)}$ in order for the Gauss law to be differentiable. Specifically, the test functions for the Gauss law had to vanish on $\partial D$. In this section, we shall examine what happens when the variations of $e^a$ and $\omega^a$ are restricted at the boundary $\partial D$ of a disc. From equations (2.17) and (2.18), this will also put restrictions on the dynamical properties of the current densities $P_a(\psi)$ and $J_a(\psi)$. [Here we do not attach an extra index $(A)$ to $P_a(\psi)$ and $J_a(\psi)$ since there is only one boundary $\partial D$.] For example, in one case that we shall study [case c)], the angular momentum current density $J_a(\psi)$ is required to vanish, leading to a torsionless theory.

The requirement that the Gauss law be differentiable will again impose certain conditions on the test functions $\Lambda^{(0)}$ and $\Sigma^{(0)}$, but now they will be less in number, as not all components of $\Lambda^{(0)}$ and $\Sigma^{(0)}$ need vanish at $\partial D$. From eq. (2.5), $\Lambda_a^{(0)}$, for a given $a$, need not be zero at $\partial D$, if the value of $\omega^a|_{\partial D}$ is fixed. Similarly, $\Sigma_a^{(0)}$ need not vanish at $\partial D$, if the value of $e^a|_{\partial D}$ is fixed.

We shall only consider restrictions on $e^a$ and $\omega^a$ at $\partial D$ such that the Gauss law constraints remain first class. We do this in order to preserve the full set of $ISO(2,1)$ gauge symmetries, which presumably a 2+1 dimensional topological theory of gravity should have. (In addition, there would be technical difficulties if some of the Gauss law constraints were second class. This is since it would then become necessary to compute Dirac brackets for the observables.) Below we shall classify the possible restrictions on $e^a$ and $\omega^a$ at $\partial D$.

First, let $ISO(2,1)$ denote the Lie algebra of $ISO(2,1)$, and let $t_a$ and $u_a$, $a = 0, 1, 2,$
be a basis for the algebra. The latter have the following Lie brackets:

\[
[u_a, u_b] = 0, \quad [u_a, t_b] = \epsilon_{abc} u^c, \quad [t_a, t_b] = \epsilon_{abc} t^c.
\]  

(6.1)

Next define a Lie algebra valued test function \( F(0) = \frac{1}{2} \left( \Lambda_a(0) u^a + \Sigma_a(0) t^a \right) \) for the Gauss law constraints. From eq. (2.8), the Gauss law constraints will be first class provided:

1) \( F(0)|_{\partial D} \) belongs to a subalgebra \( H \) of \( ISO(2,1) \).

2) \( \Lambda_a(0)|_{\partial D} \) and/or \( \Sigma_a(0)|_{\partial D} \) vanishes for all \( a \).

Ignoring boundary terms, condition 1) is needed so that the Poisson bracket of two Gauss law constraints equals a Gauss law constraint. The boundary terms in eq. (2.8) must vanish for the same reason. Condition 2) insures that the first such boundary term, i.e. \( \frac{\pi}{4} \int_{\partial D} \left( \Sigma(0)_a d\Lambda(0)_a + \Lambda(0)_a d\Sigma(0)_a \right) \), vanishes. From this condition it follows that the maximum dimension of the subalgebra \( H \) is 3. Condition 2) is sufficient but not necessary for the first boundary term in eq. (2.8) to vanish. The boundary term will also vanish for \( F(0)|_{\partial D} = \frac{1}{2} \left( \Lambda_0(0)|_{\partial D} u_+ + \Sigma_0(0)|_{\partial D} t_+ \right) \), when \( u_+ \) and \( v_+ \) are parallel light-like vectors, e.g. \( u_+ = u_0 + u_1 \) and \( v_+ = v_0 + v_1 \). In this example, \( u_+ \) and \( v_+ \) are a basis for \( H \). Then \( \text{dim}(H) = 2 \), so it is still true that \( \text{dim}(H) \leq 3 \). We shall study this exceptional case (case i)) at the end of this section.

We can classify all possible boundary conditions (consistent with \( g(\Lambda(0), \Sigma(0)) \approx 0 \) being first class) by the subalgebras \( H \) of \( ISO(2,1) \), or by the subgroups \( H \) of \( ISO(2,1) \).

For \( \text{dim}(H) = 3 \), we have:

a) \( H = ISO(2) \). For this case, \( F(0)|_{\partial D} = \frac{1}{2} \left( \Lambda_1(0)|_{\partial D} u^1 + \Lambda_2(0)|_{\partial D} u^2 + \Sigma_0(0)|_{\partial D} t^0 \right) \), and we fix the values of the forms \( e^0, \omega^1 \) and \( \omega^2 \) on \( \partial D \). In order for the last boundary term in eq. (2.8) to vanish it is necessary that we fix the values of \( \omega^1 \) and \( \omega^2 \) on \( \partial D \) to be zero.

b) \( H = ISO(1,1) \). For this case, \( F(0)|_{\partial D} = \frac{1}{2} \left( \Lambda_0(0)|_{\partial D} u^0 + \Lambda_1(0)|_{\partial D} u^1 + \Sigma_2(0)|_{\partial D} t^2 \right) \), and the forms \( e^2, \omega^0 \) and \( \omega^1 \) are held fixed on \( \partial D \). In order for the last boundary term in eq. (2.8) to vanish it is necessary that we fix the values of \( \omega^0 \) and \( \omega^1 \) on \( \partial D \) to be zero.
c) $H = \text{SO}(2,1)$. For this case, $\mathcal{F}^{(0)}|_{\partial D} = \frac{1}{2} \Sigma^{(0)}_a|_{\partial D} t^a$ and the forms $e^a$ are held fixed on $\partial D$. In order for the last boundary term in eq. (2.8) to vanish it is necessary that we fix the values of $e^a$ on $\partial D$ to be zero. Then from eq. (2.18), the angular momentum current density $J^a$ vanishes.

d) $H = T^3$ (The translation group in three dimensions). For this case, $\mathcal{F}^{(0)}|_{\partial D} = \frac{1}{2} \Lambda^{(0)}_a|_{\partial D} u^a$ and the forms $\omega^a$ are held fixed on $\partial D$. In this case there are no restrictions on the values of $\omega^a$ on $\partial D$.

When no conditions were placed on the values of $\omega^a$ and $e^a$ on $\partial D$, $q(\Lambda, \Sigma)$, or equivalently the $\text{ISO}(2,1)$ generators $P_a(\psi)$ and $J_a(\psi)$, were the observables of the theory. Let us see what happens to these quantities for the cases a-d).

a) Here $J^0$, $P^1$ and $P^2$ have zero Poisson brackets with $g(\Lambda^{(0)}, \Sigma^{(0)})$ and hence are gauge invariant. However they are non dynamical, because from eqs. (2.17) and (2.18), they are weakly equal to the boundary values of $\kappa e^0_\theta$, $-\kappa \omega^1_\theta$ and $-\kappa \omega^2_\theta$, respectively (the latter two values being zero).

The remaining quantities $P^0$, $J^1$ and $J^2$ do not have zero Poisson brackets with $g(\Lambda^{(0)}, \Sigma^{(0)})$,

\[
\{g(\Lambda^{(0)}, \Sigma^{(0)}), P^0(\psi)\} = -\kappa \frac{1}{2} \partial_\psi \Sigma^{(0)0}(r_0, \psi), \tag{6.2}
\]

\[
\{g(\Lambda^{(0)}, \Sigma^{(0)}), J^1(\psi)\} = \kappa \frac{1}{2} \partial_\psi \Lambda^{(0)1}(r_0, \psi)
+ \frac{1}{2} \Sigma^{(0)}_0(r_0, \psi) J^2(\psi) + \frac{1}{2} \Lambda^{(0)}_2(r_0, \psi) P^0(\psi), \tag{6.3}
\]

\[
\{g(\Lambda^{(0)}, \Sigma^{(0)}), J^2(\psi)\} = \kappa \frac{1}{2} \partial_\psi \Lambda^{(0)2}(r_0, \psi)
- \frac{1}{2} \Sigma^{(0)}_0(r_0, \psi) J^1(\psi) - \frac{1}{2} \Lambda^{(0)}_1(r_0, \psi) P^0(\psi), \tag{6.4}
\]

where we have written $\Lambda^{(0)a}$ and $\Sigma^{(0)a}$ as functions of polar coordinates $r$ and $\theta$, with $r = r_0$ once again corresponding to the boundary. Equations (6.2-4) show that $P^0$, $J^1$ and $J^2$ are not observables. Instead, they transform under gauge transformations as components of an $H = \text{ISO}(2)$ connection one form.
To see this define one forms $\Omega$ and $E^i$, $i = 1, 2$ on a circle parametrized by $\psi$ as follows:

$$\Omega = -\frac{1}{\kappa} P^0(\psi) d\psi \quad \text{and} \quad E^i = \frac{1}{\kappa} J^i(\psi) d\psi .$$

(6.5)

$\Omega$ is an $SO(2)$ spin connection, and $E^i$ are zweibein one forms. Under infinitesimal gauge transformations generated by $g(\Lambda(0), \Sigma(0))$,

$$\delta \Omega = d\rho ,$$

(6.6)

$$\delta E^i = -\rho \epsilon^{ij} E_j + \epsilon^{ij} \lambda_j \Omega + d\lambda^i ,$$

(6.7)

where $\rho(\psi) = \frac{1}{2} \Sigma^{(0)0}(r_0, \psi)$ parametrize infinitesimal rotations, and $\lambda_i(\psi) = \frac{1}{2} \Lambda^{(0)}_i(r_0, \psi)$ parametrize infinitesimal translations in a plane. Eq. (6.6) leads to a nonlocal observable for this system. It is namely the integral of $\Omega$ around the boundary. Then from eq. (6.5), $\int_0^{2\pi} P^0(\psi) d\psi$, or the “energy” associated with the boundary, is an observable.

If one enlarges the underlying manifold (i.e., $\partial D$) on which $\Omega$ and $E^i$ are defined (say to $\partial D \times \mathbb{R}^1$, where $\mathbb{R}^1$ denotes the time variable), then it is possible to introduce an $SO(2)$ curvature two form

$$R^{(2)} = d\Omega ,$$

(6.8)

and a torsion two form

$$T^{(2)i} = dE^i + \epsilon^{ij} E_j \wedge \Omega$$

(6.9)

on the manifold. $R^{(2)}$ is invariant under local $ISO(2)$ transformations, while $\delta T^{(2)i} = \epsilon^{ij}(\rho T^{(2)}_j - \lambda_j R^{(2)})$. The two forms $R^{(2)}$ and $T^{(2)i}$ have been utilized previously in a theory of two dimensional gravity [22].

b) This case is essentially the same as a), except instead of (6.5), we define

$$\Omega = -\frac{1}{\kappa} P^2(\psi) d\psi \quad \text{and} \quad E^i = \frac{1}{\kappa} J^i(\psi) d\psi ,$$

(6.10)

where now $i = 0, 1$. Further, $\Omega$ is an $SO(1, 1)$ spin connection one form, and now $\rho$ is defined by $\rho(\psi) = \Sigma^{(0)2}(r_0, \psi)$ which parametrizes infinitesimal local Lorentz transformations. The integral of $\Omega$ is again an observable of the system, but here it corresponds to
the total momentum in the 2- direction. \( R^{(2)} = d\Omega \) is now the \( SO(1, 1) \) curvature two form.

c) In this case, the \( J^a \)'s have zero Poisson brackets with \( g(\Lambda^{(0)}, \Sigma^{(0)}) \), and hence are
gauge invariant. However from eq. (2.8) they are weakly equal to the boundary value of \( \kappa \epsilon^a_\theta \) which is zero.

The remaining variables \( P^a \) have non zero Poisson brackets with the Gauss law con-
straints,

\[
\{ g(\Lambda^{(0)}, \Sigma^{(0)}), P^a(\psi) \} = -\frac{1}{2} \epsilon^{abc} \Sigma^a_b(r_0, \psi) P^c(\psi) - \frac{\kappa}{2} \partial_\psi \Sigma^{(0)a}(r_0, \psi) ,
\]

(6.11)

and are therefore not gauge invariant. They transform as components of an \( H = SO(2, 1) \)
connection one form. The connection one form can be defined by

\[
\Omega^a = -\frac{1}{\kappa} P^a(\psi)d\psi .
\]

(6.12)

Then under gauge transformations

\[
\delta \Omega^a = -\epsilon^{abc} \rho^c \Omega_b + d\rho^a ,
\]

(6.13)

where \( \rho^a(\psi) = \frac{1}{2} \Sigma^{(0)a}(r_0, \psi) \) parametrize infinitesimal \( SO(2, 1) \) transformations.

If, as with case a), one enlarges the underlying manifold on which \( \Omega^a \) are defined to
\( \partial D \times \mathbb{R}^1 \), it is possible to introduce an \( SO(2, 1) \) curvature two form

\[
R^{(2)a} = d\Omega^a + \frac{1}{2} \epsilon^{abc} \Omega_b \Lambda \Omega_c ,
\]

(6.14)

which is gauge covariant, i.e. \( \delta R^{(2)a} = -\epsilon^{abc} \rho_b R^{(2)c} \). In this case, there is no torsion two form which we can define on the boundary.

d) Now the \( P^a \)'s have zero Poisson brackets with \( g(\Lambda^{(0)}, \Sigma^{(0)}) \), and hence are gauge
invariant. But they are weakly equal to the boundary values of \(-\kappa \omega^a_\theta \). They are then non
dynamical.
The $J^a$'s have non zero Poisson brackets with the Gauss law constraints,

$$\{g(\Lambda^{(0)}, \Sigma^{(0)}), J^a(\psi)\} = \frac{1}{2} \varepsilon^{abc} \Lambda^0_b(r_0, \psi) \, P_c(\psi) + \frac{\kappa}{2} \partial_\psi \Lambda^{(0)a}(r_0, \psi),$$

(6.15)

Now define the one forms

$$E^a = \frac{1}{\kappa} J^a(\psi) d\psi$$

and

$$\Omega^a = -\frac{1}{\kappa} P^a(\psi) d\psi .$$

(6.16)

Then under gauge transformations

$$\delta E^a = -\varepsilon^{abc} \lambda_b \Omega_c + d\lambda^a ,$$

(6.17)

where $\lambda^a(\psi) = \frac{1}{2} \Lambda^{(0)a}(r_0, \psi)$ parametrize local translations. If one enlarges the underlying manifold on which $E^a$ and $\Omega^a$ are defined to $\partial D \times \mathbb{R}^1$, one can define a torsion two form

$$T^{(2)a} = dE^a + \varepsilon^{abc} E_b \wedge \Omega_c .$$

(6.18)

Under a local translation $\delta T^{(2)a} = -\varepsilon^{abc} \lambda_b R^{(2)}_c$, where $R^{(2)}_a$ was defined in eq. (6.14). Here however it is non dynamical, since neither is $\Omega^a$.

In summary, we find that for dim ($H$) = 3 the field content of the theory consists to the set of connection one forms associated with gauge group $H$. These connection one forms are constructed from the (unconstrained) $P_a$’s and $J_a$’s. Two dimensional gravity theories can be formulated in terms of such fields [22].

Connection one forms for the gauge group $H$ can also be identified when dim($H$) < 3. For these cases, there also exist observable degrees of freedom amongst the $P_a$’s and $J_a$’s. The case of $H$ being the trivial group (containing just the identity) was discussed in the previous section, where it was found that all of the $P_a$’s and $J_a$’s were observable. When $H$ is not the trivial group, combinations of (unconstrained) $P_a$’s and $J_a$’s can be formed which are observables, which we shall show below. There are essentially five such cases to be considered. They are: e) $H = T^2$, f) $H = T^1$, g) $H = SO(1, 1)$, h) $H = SO(2)$ and i) $H = T^2$. We discuss these cases in what follows.
e) $H = T^2$. In this case, $F^{(0)}|_{\partial D} = \frac{1}{2} \Lambda^{(0)}_i |_{\partial D} u^i$ and we choose $i = 0, 1$. For $g(\Lambda^{(0)}, \Sigma^{(0)})$ to be differentiable the forms $\omega^i$ are held fixed on $\partial D$ and there is no restriction on their values coming from the requirement that $g(\Lambda^{(0)}, \Sigma^{(0)})$ is first class. As in case d), the $P^a$'s have zero Poisson brackets with $g(\Lambda^{(0)}, \Sigma^{(0)})$, and hence are gauge invariant. From eq. (2.17), they are weakly equal to the boundary values of $-\kappa \omega^a$. Then $P^2$ is a dynamical quantity, while $P^i$, $i = 0, 1$ are constrained.

The $J^a$'s, in general, have non zero Poisson brackets with the Gauss law constraints, as eq. (6.15) still applies (only with $\Lambda^{(0)}_2(r_0, \psi) = 0$). From $J^i$, $i = 0, 1$, we can construct the $T^2$ connection one forms $E^i$ according to eq. (6.16). Under local translations, $\delta E^i = -\epsilon^{ij} \lambda_j \Omega_2 + d\lambda^i$, where $\lambda^i(\psi) = \frac{1}{2} \Lambda^{(0)i}(r_0, \psi)$ and $\Omega_2$ is defined in eq. (6.16). If one again enlarges the underlying manifold on which $E^a$ and $\Omega^a$ are defined to $\partial D \times \mathbb{R}^1$, one can define the torsion two form eq. (6.18). Under the action of the two dimensional translation group $\delta T^{(2)i} = -\epsilon^{ij} \lambda_j R^{(2)}_a$, where $R^{(2)}_a$ was defined in eq. (6.14).

In the above discussion we found that $P^2$ is an observable. A second observable can be obtained in this system if we set $\omega^i|_{\partial D} = 0$. This other observable is $J^2$, because $P^i$ are weakly zero, and consequently

$$\{g(\Lambda^{(0)}, \Sigma^{(0)}), J^2(\psi)\} = \frac{1}{2} \epsilon^{ij} \Lambda^{(0)}_i(r_0, \psi) P_j(\psi) \approx 0 .$$

The Poisson bracket between the two observables $P^2$ and $J^2$ is obtained from eq. (2.24),

$$\{P_2(\psi), J_2(\psi')\} = -\kappa \partial_\psi \delta(\psi - \psi') .$$

This relation defines an abelian Kac-Moody algebra, where the generators are $J^1 + P^1$ and $J^1 - P^1$.

If we had we chosen $i$ to take the values 1, 2, instead of 0, 1, then the above analysis would be the same, with perhaps the only difference being that the sign of the central term in eq. (6.20) would then be +.
f) $H = T^1$. We choose $F(0)|_{\partial D} = \frac{1}{2} \Lambda_2(0)|_{\partial D} u^2$, i.e. $H$ is the group of space translations. Now for $g(\Lambda(0), \Sigma(0))$ to be differentiable we need only require that $\omega^2$ be held fixed on $\partial D$. As before, there is no restriction on its value coming from the requirement that $g(\Lambda(0), \Sigma(0))$ be first class. Also as before, the $P^a$’s are gauge invariant. Here, however, $P^2$ is not dynamical, as it is weakly equal to the boundary value of $-\kappa e^2_\theta$, while $P^i$, $i = 0, 1$ are dynamical quantities, and hence observables. They have zero Poisson brackets with each other, and thus yield a trivial algebra.

The $J^a$’s are not gauge invariant. $J^i$, $i = 0, 1$, transforms according to

\[ \{ g(\Lambda(0), \Sigma(0)), J^i(\psi) \} = -\frac{1}{2} \epsilon^{ij} \Lambda_2(0)(r_0, \psi) P_j(\psi). \] (6.21)

It follows that the product $J^i(\psi) P_i(\psi)$ is gauge invariant and hence observable. From $J^2$, we can construct the $T^1$ connection one form $E^2$ as in eq. (6.16). Under local translations, $\delta E^2 = d\lambda^2$, where $\lambda^2(\psi) = \frac{1}{2} \Lambda(0)^2(r_0, \psi)$. It follows that the integral of $E^2$ around the boundary is an observable. It corresponds to the total angular momentum in the 2-direction.

After enlarging the underlying manifold to $\partial D \times \mathbb{R}^1$, one can define the torsion two form according to eq. (6.18). $T^{(2)2}$ is invariant under the action of the one dimensional translation group.

When $F(0)|_{\partial D} = \frac{1}{2} \Lambda_0(0)|_{\partial D} u^0$, $H$ is the group of time translations. In that case, $P^1$, $P^2$ and $J^1 P_1 + J^2 P_2$ are observables, while $E^0$ is the connection one form associated with $T^1$.

g) $H = SO(1, 1)$ . Here $F(0)|_{\partial D} = \frac{1}{2} \Sigma_2(0)|_{\partial D} t^2$, and $e^2$ is held fixed on $\partial D$. No restriction on the value of $e^2$ comes from the requirement that $g(\Lambda(0), \Sigma(0))$ is first class. Now, of all the $J^a$’s and $P^a$’s, only $J^2$ is gauge invariant. But it is not dynamical, as it is weakly equal to the boundary value of $\kappa e_\theta^2$. Under a local $SO(1, 1)$ gauge transformation, both $J^i$ and $P^i$, $i = 0, 1$ transform as two dimensional vectors. Thus although they are not observables, their magnitudes are. It is also possible to construct additional bilinears
from $J_i$ and $P^i$, namely $P_i(\psi) J_i(\psi)$ and $\epsilon^{ij} P_i(\psi) J_j(\psi)$, which are gauge invariant.

The connection one form for $SO(1,1)$ is $\Omega$ as defined in eq. (6.10). Under $SO(1,1)$
gauge transformations, $\delta \Omega = d\rho^2$, where $\rho^2(\psi) = \frac{1}{2} \Sigma^{(0)2}(r_0, \psi)$. Now the integral of $\Omega$, corresponding to the total momentum in the 2-direction, is an observable.

After enlarging the underlying manifold to $\partial D \times \mathbb{R}$, one can define $SO(1,1)$ curvature two form according to $R^{(2)2} = d\Omega^2$. It is invariant under the action of the local Lorentz group.

**h) $H=SO(2)$**. Now $\mathcal{F}^{(0)}|_{\partial D} = \frac{1}{2} \Sigma^{(0)}|_{\partial D} t^0$, and $e^0$ is held fixed on $\partial D$. The analysis here is the same as in case **g)**, only now $i$ takes values 1 and 2, and the index 2 is replaced everywhere by 0.

**i) $H=\mathbf{T}_2$**. Finally, we consider the exceptional case referred to earlier, where the
group $H$, which we denote by $T^2$, is generated by the parallel light-like vectors $u_+ = u_0 + u_1$
and $v_+ = v_0 + v_1$, and is thus abelian. We write, $\mathcal{F}^{(0)}|_{\partial D} = \frac{1}{2} \left( \Lambda^{(0)}|_{\partial D} u_+ + \Sigma^{(0)}|_{\partial D} t_+ \right)$,
with $e_+ = e_0 + e_1$ and $\omega_+ = \omega_0 + \omega_1$ held fixed on $\partial D$. $g(\Lambda^{(0)}, \Sigma^{(0)})$ is then differentiable, and no restrictions on the values of $e_+$ and $\omega_+$ come from the requirement that it be
first class. $J^- = J^0 - J^1$ and $P^- = P^0 - P^1$ are invariant under gauge transformations, but they are weakly equal to $\kappa e^-_\theta = \kappa (e^0_\theta - e^1_\theta)$ and $-\kappa \omega^-_\theta = -\kappa (\omega^0_\theta - \omega^1_\theta)$, respectively, and hence are not dynamical. The remaining $P^a$s and $J^a$s are, in general, not gauge invariant. However, $P^2$ will be gauge invariant if $\omega_+$ vanishes on the boundary, and, in
addition, $J^2$ will be gauge invariant if $e_+$ vanishes there as well. In the latter case, we
recover the abelian Kac-Moody algebra eq. (6.20) for the two observables.

The connection one forms for $T^2$ are $\Omega^+ = \Omega^0 + \Omega^1$ and $E^+ = E^0 + E^1$. Under
gauge transformations, $\delta \Omega^+ = 2\rho_- \Omega_2 + d\rho_-$ and $\delta E^+ = 2\lambda_- \Omega_2 + 2\rho_- E_2 + d\lambda_-$, where $\rho_-(\psi) = \frac{1}{2} \Sigma^{(0)}(r_0, \psi)$ and $\lambda_-(\psi) = \frac{1}{2} \Lambda^{(0)}(r_0, \psi)$ parametrize the $T^2$ transformations. The corresponding curvature and torsion two forms are $R^{(2)+}$ and $T^{(2)+}$. Under gauge trans-
formations, $\delta R^{(2)+} = 2\rho_- R_2^{(2)}$ and $\delta T^{(2)+} = 2\lambda_- T_2^{(2)}$. 

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7. CONCLUDING REMARKS

We believe that the formalism developed here for treating $2 + 1$ gravity on manifolds with boundaries has the virtue of simplicity. Now we outline some possible implications and extensions of our work.

The spin-statistics theorem. In Section 4, point particles were obtained by taking the zero size limit of holes on the disc. As a result, an entire $ISO(2,1)$ Kac-Moody algebra was associated with each point particle. It would be of interest to study the question of the spin-statistics theorem from this point of view. The spin-statistics theorem in $2 + 1$ gravity has been examined by a number of authors, from a number of points of view [20, 5, 21]. It was studied by two of us in ref. [5], but there, point particles were only labeled by their charges, the particle momenta $p^a$ and angular momenta $j^a$, and the very important question of self-interactions was neglected. With the formalism developed here, however, self-interactions of the particle are included in a natural way.

In ref. [10], the spin-statistics theorem was proved for particles in general Chern-Simons theories, where the particles are gotten by acting on the vacuum with the vertex operator. (The particles are therefore associated with the state $V|0>$.) It appears that this result can be lifted straightforwardly to apply to Chern-Simons gravity. In ref. [10], the Virasoro charge $L_0$ played the role of the rotation generator, as it was responsible for global deformations of the disc. The eigenvalue of $L_0$ was shown to be identical to the phase associated with a two-particle exchange, hence proving the spin-statistics connection.

But in Chern-Simons gravity, there is another kind of rotation generator, namely, the particle angular momentum operator $\int_0^{2\pi} d\psi J^0(\psi)$, which generates rotations in the
internal vector space. (Here we ignore the particle index \(A\).) A question then arises as to whether the spin-statistics theorem should be formulated in terms of \(L_0\) or the particle angular momentum. Happily, both of these operators are equivalent when acting on the state \(V|0\rangle\), and when \(n = 1\). To see this, let us promote the expression (2.31) for the Virasoro generators to an operator equation,

\[
L(v) = -\frac{1}{\kappa} \int_0^{2\pi} d\psi \epsilon(\psi) J_a(\psi) P^a(\psi).
\] (7.1)

With the ordering as shown in eq. (7.1), \(L(v)\) has a well defined action on the states \([14]\). In particular, when acting on the state \(V|0\rangle\), we have

\[
L(v) V|0\rangle = n \int_0^{2\pi} d\psi \epsilon(\psi) J^0(\psi) V|0\rangle,
\] (7.2)

where we have used eq. (5.11). The Virasoro charge \(L_0\) is obtained by setting \(\epsilon(\psi)\) in \(L(v)\) equal to a constant. The correct normalization is \(\epsilon(\psi) = 1\). So for the case \(n = 1\), \(L_0\) is equal to the angular momentum operator when acting on \(V|0\rangle\). (In Sec. 5, we assumed the trivial representation for the little group of \(|0\rangle\), and hence also for the little group of \(V|0\rangle\). Then both \(L_0\) and \(\int_0^{2\pi} d\psi J^0(\psi)\) are zero on \(V|0\rangle\). The assumption of triviality may however be dropped for the purpose of the discussion here.) For the case \(n > 1\), there is an \(n\) to 1 map from the set of rotations \(\{e^{i\theta L_0}\}\) to \(\{e^{i\theta \int_0^{2\pi} d\psi J^0(\psi)}\}\).

**Cylindrical space-time.** Although much is known about conical space-time, the space-time created by particles in \(2 + 1\) dimensions with mass \(m < 2\pi\kappa\), not much seems to be known about cylindrical space-times \([14]\). Once again, it is the latter that are created by our vertex operator and they are associated with particles whose masses are given in eq. (5.13). The scattering of such particles may be of interest, along with their possible relevance for cosmic strings.

**Alternative boundary conditions.** In Sec. 6, we classified all possible boundary conditions for \(2 + 1\) gravity on a disc. A rich structure was found, which in several instances, yielded the field content of \(1 + 1\) gravity. Thus it seemed possible to generate
the lower dimensional gravity theory. However, the dynamics of the lower dimensional theory seemed to be lacking. The questions then remain as to how to introduce dynamics in a natural manner, and further how to quantize the resulting theory.

If there is no unique way to treat the boundary of a disc, then there is also no unique way to treat the boundaries of holes on the disc. Since point particles, here, result from shrinking holes to points, there can be alternative descriptions of point sources as well. The classification of all such point sources which one can have in $2 + 1$ gravity should be of considerable interest.
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