Topology change in (2+1)-dimensional gravity with non-Abelian Higgs field

Alexander I. Nesterov

Abstract

We study topology change in (2+1)D gravity coupling with non-Abelian SO(2, 1) Higgs field from the point of view of Morse theory. It is shown that the Higgs potential can be identified as a Morse function. The critical points of the latter (i.e. loci of change of the spacetime topology) coincide with zeros of the Higgs field. In these critical points two-dimensional metric becomes degenerate, but the curvature remains bounded.

PACS number(s): 04.20.Gz, 02.40.-k

1 Introduction

It is known [1] that if one has any (time orientable) Lorentzian metric which interpolates between two compact spacelike surfaces of different topology, then there must exist either closed timelike curves or singularities and from the Tipler’s theorem [2] it follows that the spacetime which interpolates between topologically different boundaries cannot be compact, in other words, metric must be ill-defined somewhere.

What kind of singularities is it necessary to change topology? Horowitz [3] has shown that if the metric becomes degenerate on a set of measure zero, but the curvature remains bounded, there exist smooth solutions of Einstein’s equations in which the topology of space changes.

We explore the topology change in (2+1)D gravity coupling with non-Abelian SO(2, 1) Higgs field from the point of view of Morse theory. Our starting point is the first-order Palatini action in terms of a triad $e^a_{\mu}$, Lorentz connection $\omega^a_{\mu}b$ and a real scalar field $\phi^a$ (section 2). In section 3 we consider the ground state solutions of the field equations. Finally, in section 4 we show that the Higgs potential can be identified as a Morse function and the critical points of latter (i.e. loci of change of the spacetime topology) coincide with zeros of the Higgs field.

2 (2+1)-D Palatini action with matter coupling

In the Palatini formalism the fundamental variables are the base 1-forms $e^a$ and the Lorentz connection $\omega^a_{\mu}b$. These variables are assumed to satisfy the following relations

$$g^{\mu\nu}e^a_{\mu}e^b_{\nu} = \eta^{ab}, \quad e^a_{\mu}e^b_{\nu}\eta^{ab} = g_{\mu\nu}, \quad \omega_{ab\mu} = -\omega_{ba\mu},$$

*Departamento de Física, Universidad de Guadalajara, Guadalajara, Jalisco, México. E-mail: nesterov@cencar.udg.mx
where $\eta^{ab}$ is the constant Minkowski metric and $g_{\mu\nu}$ is a spacetime metric ($\mu, \nu$ are spacetime indices and $a, b$ internal indices, both running from 0 to 2; we assume signature being $(-, +, +)$ and define the Ricci tensor as $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$).\(^1\)

The Einstein–Hilbert action

$$I_g = \int \sqrt{-g} R \, d^3x$$

in the Palatini formalism is given by [3, 4, 5]

$$I_g = \int e^a \wedge R^{bc} \epsilon_{abc}, \quad (1)$$

where $R = d\omega + \omega \wedge \omega$ is the curvature form of the three-dimensional Lorentz connection and $\epsilon_{abc}$ is the constant antisymmetric tensor.

Varying with respect to $e^a_\mu$ one gets the field equations

$$\epsilon_{abc} \epsilon_{\mu\nu\rho} R^{bc}_{\nu\rho} = 0, \quad (2)$$

which are exactly the vacuum Einstein equations, $\sqrt{-g}G^{\mu\nu} = \sqrt{-g}G^{\mu\nu} e^a_\mu = 0$ ($G^{\mu\nu}$ being the Einstein tensor).

Variation with respect to $\omega^a_\mu$ leads to the equations

$$D_\mu e^a_\nu = 0 \quad (3)$$

which just determine the torsion–free connection and are equivalent to the first Cartan structure equations $de^a + \omega^a_\mu \wedge e^b = 0$.

We consider gravity coupled to a real matter field $\phi$,

$$\mathcal{L}_m = -\sqrt{-g} \left( \frac{1}{2} D_\mu \phi \cdot D^\mu \phi + V(\phi) \right), \quad (4)$$

where

$$V(\phi) = \frac{\gamma}{4} \left( \phi \cdot \phi + \frac{\mu}{\gamma} \right)^2$$

is the Higgs potential, $\phi \cdot \phi = \eta_{ab} \phi^a \phi^b$ and $\sqrt{-g} = \det ||e^a_\mu||$.

Variation of the total action

$$I = \frac{1}{2\kappa} I_g + I_m$$

yields the field equations

$$\frac{1}{\sqrt{-g}} D_\mu (\sqrt{-g}D^\mu \phi^a) = \frac{\delta V}{\delta \phi_a}, \quad \epsilon^{\mu\nu\lambda} D_\nu e^a_\lambda = \kappa \sqrt{-g} \epsilon_{abc} \phi_b D^\mu \phi_c,$$

$$\epsilon_{abc} \epsilon_{\mu\nu\rho} R^{bc}_{\nu\rho} = 4\kappa \mathcal{J}^\mu_\alpha, \quad (5)$$

where the current density $\mathcal{J}^\mu_\alpha$ is given by $\mathcal{J}^\mu_\alpha = \sqrt{-g} T^{\mu\nu} e_{\alpha\nu}$ and $T^{\mu\nu}$ is the energy-momentum tensor:

$$T_{\mu\nu} = D_\mu \phi \cdot D_\nu \phi - g_{\mu\nu} \left( \frac{1}{2} D_\lambda \phi \cdot D^\lambda \phi + V(\phi) \right).$$

\(^1\)For any internal vector $\rho^a$ on the spacetime the covariant derivative $D_\mu$ is defined by $D_\mu \rho^a = \rho^a_{\cdot \mu} + \omega^a_{\cdot \mu} \rho^\mu$. On external vectors it reduces to partial derivative, $D_\mu v^\nu = v^\nu_{\cdot \mu}$, and $D_\mu e^a_\nu = e^a_{\cdot \nu} + \omega^a_{\cdot \mu} e^\mu_\nu = \Gamma^a_{\mu\nu} e^\mu_\nu$. 

2
3 Ground state solutions

Let us consider solutions of the field equations in the neighborhood of the spacelike surface $\Sigma_g$ of genus $g \ (g > 1)$. The metric of spacetime can be written as
\[
ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt),
\]
where $N, N^i$ and $g_{ij}$ are the lapse function, shift vector and spatial metric induced on the spacelike hypersurface $\Sigma_g$, respectively. It is convenient to introduce a Gaussian normal coordinate system, which is specified by the following conditions: $N = 1, \ N^i = 0$.

It is well known that one can always choose a conformally flat metric for the 2-surface,
\[
g_{ij} = \Omega \delta_{ij},
\]
where $\Omega$ is the strictly positive conformal factor. Setting $\Omega = e^{f}$, we find
\[
ds^2 = -dt^2 + 2e^{f} dz d\bar{z},
\]
$z = (x + iy)/\sqrt{2}, \ \bar{z} = (x - iy)/\sqrt{2}$ being the complex coordinates and $f = f(t, z, \bar{z})$. For torsion–free connection field equations \[1\] are written as
\[
\frac{1}{2}\Delta(f + f) + \left(\frac{1}{2} D_0 \phi \cdot D_0 \phi - e^{-(f + f)} D_z \phi \cdot D_z \phi + V(\phi) - |\dot{f}|^2\right)e^{f + f} = 0,
\]
\[
\dot{f}_z - D_0 \phi \cdot D_z \phi = 0, \quad \left(e^{f}\right)\cdot + e^{f}\left(\frac{1}{2} D_0 \phi \cdot D_0 \phi - V(\phi)\right) = 0,
\]
\[
\epsilon_{abc} \phi^b D_z \phi^c = 0, \quad D_z \phi \cdot D_z \phi = 0,
\]
\[
e^{-\left(f + f\right)}(D_z D_z \phi^a + D_z D_z \phi^a) = \dot{\phi}_a^\alpha - \dot{\phi}_a^\alpha (\dot{f} + \dot{f}) - \phi_0 (\mu + \gamma \phi \cdot \phi) = 0,
\]
where $\Delta = 2\partial_z \partial_{\bar{z}}$ is a two-dimensional Laplacian (a dot denotes partial derivative with respect to $t$, $f_z = \partial_z f$, and we set $\kappa = 1$).

Further it is convenient to introduce a complex scalar field $\phi = (\phi^0, \varphi, \bar{\varphi})$ defined by
\[
\varphi^0 = \phi^0, \ \varphi = (1/\sqrt{2})(\phi^1 + i\phi^2), \ \bar{\varphi} = (1/\sqrt{2})(\phi^1 - i\phi^2).
\]
The covariant derivative of $\phi$ is given by
\[
D_z \varphi = \varphi_z - iA_z \varphi + A_0^z \phi^0, \quad D_z \bar{\varphi} = \bar{\varphi}_z + iA_z \bar{\varphi}, \quad D_z \phi^0 = \phi^0_z + A_0^z \varphi,
\]
\[
D_z \varphi = \bar{D}_z \bar{\varphi}, \quad D_z \phi^0 = \bar{D}_z \bar{\phi}^0, \quad D_z \varphi = \bar{D}_z \bar{\varphi},
\]
where $A_z = i\dot{f}_z, \ A_{\bar{z}} = \bar{A}_z, \ A_0^z = \dot{f} e^{f}$.

For ground state defined as $D_\mu \phi^a = 0 \ (\delta V/\delta \phi_a = 0)$ the system of field equations \[7\]–\[10\] reduces to
\[
\Delta (f + f) - 2(V(\phi) - |\dot{f}|^2)e^{f + f} = 0, \quad (12)
\]
\[
\dot{f}_z = 0, \ \dot{f} + \dot{f}^2 - V(\phi) = 0, \quad (13)
\]
\[
\phi_0 (\mu + \gamma \phi \cdot \phi) = 0, \quad (14)
\]
\[
D_z \varphi = 0, \quad D_z \varphi = 0, \quad D_z \phi^0 = 0. \quad (15)
\]

Eq.\[14\] yields two cases:

A. $\phi \cdot \phi = -\frac{\mu}{\gamma} \implies V = 0, \quad \text{B. } \phi^a = 0 \implies V = V_0 = \frac{\mu^2}{4\gamma}$. 

3
Case A. $\phi \cdot \phi = -\mu / \gamma$, $V = 0$

The solution of field equations is given by

$$
\varphi^0 = \sqrt{\frac{\mu}{\gamma}} \left( 1 + |F(z)|^2 \right), \quad \varphi = \sqrt{\frac{\mu}{\gamma}} \left( \frac{2F(z)|F(z)_z|}{F(z)_z (1 - |F(z)|^2)} \right) e^{i(\gamma^n \bar{\xi}_n + \gamma^n \xi_n)},
$$

$$
f = \ln \left( \frac{\sqrt{2}|F(z)| |F(z)_z|}{(1 - |F(z)|^2)} \right) + i(\gamma^n \bar{\omega}_n + \gamma^n \omega_n),
$$

(16)

where $F(z)$ is an arbitrary function satisfying $|F(z)| < 1$, $\gamma^n$ is a complex $g$-component vector,

$$
\xi_n = \int_{z_0}^{z} \omega_n,
$$

and $\omega_n = \omega_n dz$ is a holomorphic harmonic 1-form with the standard normalization

$$
\int_{\alpha_m} \omega_n = \delta_{mn}, \quad \int_{\beta_m} \omega_n = \tau_{mn},
$$

$\alpha_m, \beta_m$ being a canonical homology basis (or closed loops around handles on $\Sigma_g$) and imaginary part of $\tau$ a positive matrix.

To summarize, the spacetime metric is given by

$$
ds^2 = -dt^2 + t^2 \left( \frac{4|dF|^2}{(1 - |F(z)|^2)^2} \right).
$$

(17)

It is well known that (17) is a Poincaré metric for the Riemann surface $\Sigma_g$ ($g > 1$). Changing $t \to 2/\tau$ one can easily reduce this metric to the vacuum solution [9].

Case B. $\phi^a = 0$, $V = V_0 = \mu^2 / 4\gamma$

In this case one obtains

$$
f = \ln \left( \frac{\sqrt{2} \sinh(\lambda t)|F(z)_z|}{\lambda (1 - |F(z)|^2)} \right),
$$

(18)

where $\lambda = \sqrt{V_0}$. This leads to the following expression for the spacetime metric

$$
ds^2 = -dt^2 + \left( \frac{\sinh(\lambda t)}{\lambda} \right)^2 \left( \frac{4|dF|^2}{(1 - |F(z)|^2)^2} \right).
$$

(19)

In the limit $\lambda \to 0$ this metric reduces to (17).

We note that the energy-momentum tensor $T_{\mu\nu} = -g_{\mu\nu} V_0$, and the obtained solution can be interpreted as a vacuum solution with the cosmological constant $\Lambda = \kappa V_0$.

4 Topology change. Application of Morse theory

To make the description self-consistent it worth outlining some basic facts of the Morse theory [6, 7, 8]. A smooth function $F(x)$ on the manifold $\mathcal{M}$ is called Morse function if all critical points of this function are non degenerate ones. It means that in the critical points $\partial F/\partial x^a = 0$, but
corresponding to the critical point of index 2. Then the equation point is described by the Morse function point of index 1 to the Morse function is determined by $F$ here the critical level is determined by $F$ from $\Sigma$ be a two-sphere $S$ sphere with two holes, is a differentiable manifold $S$ sphere with one handle, or in other words a torus. It is easy to construct the reverse operation of $F$ a Morse function and $M$ a manifold, $\lambda$ is called a degenerate critical point, then $M$ and obtain two-sphere from a torus by a spherical modification.

A notation of a spherical modification can be illustrated by the following example. Let $M$ be a two-sphere $S^2$ and take zero-dimensional sphere $S^0 \subset S^2$. $S^0$ is a set of two points and has a neighborhood consisting of two disjoint disks, $U = S^0 \times D^2$. Evidently $S^2 - \text{Int } U$ is a sphere with two holes in it, the boundary being $S^0 \times S^1$. On the other hand, $S^0 \times S^1$ is also a boundary of the cylinder $E^2 \times S^1$. The union $S^2 - \text{Int } U$ and $E^2 \times S^1$, when ends of the cylinder are attached to the circumferences of the two holes, is a differentiable manifold $M'$ which is a sphere with one handle, or in other words a torus. It is easy to construct the reverse operation and obtain two-sphere from a torus by a spherical modification.

This situation is quite general. For instance, the Riemann surface $\Sigma_{g+1}$ can be obtained from $\Sigma_g$, where $g$ is the genus of $\Sigma_g$, by spherical modification corresponding near the critical point of index 1 to the Morse function

$$F(x, y, z) = c - z^2 + x^2 + y^2.$$ Here the critical level is determined by $F(x, y, z) = c$ and $\Sigma_g$ by $F(x, y, z) = c - \epsilon$, while $\Sigma_{g+1}$ is determined by $F(x, y, z) = c + \epsilon$. The inverse process $\Sigma_{g+1} \rightarrow \Sigma_g$, annihilation of a handle, is described by the Morse function

$$F(x, y, z) = c + z^2 - x^2 - y^2$$ corresponding to the critical point of index 2.

Assuming $\det|\partial \phi^\alpha / \partial x^\beta| \neq 0$, let us introduce a new coordinate system $\{X^i\}$:

$$X = \phi^1(x^\alpha), \quad Y = \phi^2(x^\alpha), \quad Z = \phi^0(x^\alpha).$$ Then the equation $V(X^i(x^\alpha)) = \text{const}$ determines embedding $\Sigma_g \rightarrow E^3$. Let $\phi^\alpha = 0$ in the point $p_0 \in \Sigma_g$. Near to $p_0$ the Morse function defined as $F(X^i) = V(\phi)$ is given by

$$F = F_0 + \frac{\mu}{2}( - Z^2 + X^2 + Y^2),$$ and $p_0$ is the critical point of index 1 if $\mu > 0$ (or index 2, if $\mu < 0$).
5 Concluding remarks

For the solutions obtained in the section 3 the critical points of the Higgs potential are in (A) degenerates critical points, in (B) nondegenerates ones. Thus, only in case B it is possible to identify the Higgs potential as a Morse function. The critical points \( p_i \) of the latter (i.e. loci of change of the spacetime topology) coincide with zeros of the Higgs field, \( \phi^a(p_i) = 0 \). From field equations it follows that near to these points the two-dimensional metric is \( g_{zz} \sim (\phi^1)^2 + (\phi^2)^2 \), but the curvature remains bounded.

The scenario of topology change can be as follows. Far from the critical points the spacetime metric takes the form (17) and the topology of spacetime is hold by the scalar field \( \phi \). Near the zeros of the Higgs field the spacetime metric is given by the expression (19). After topology change the spacetime metric takes the same form (19) but it is considered as the vacuum solution with the cosmological constant \( \Lambda = \kappa V_0 \). Thus the appearence of the cosmological constant is the result of topology change.

The relation between topology change and Morse theory found here can be seen as a more general phenomenon and expand to four-dimensional spacetime. This work is being developed.

Acknowledgments

This essay was selected for an Honorable Mention by the Gravity Research Foundation, 1996. The work was supported by CONACyT Grant 1626P-E.

References

[1] Geroch, R. P. (1967). *J. Math. Phys.* 8, 782.
[2] Tipler, F. (1977). *Ann. Phys., NY* 108, 1.
[3] Horowitz, G. T. (1991). *Class. Quantum Grav.* 8, 587.
[4] Ashtekar, A., and Romano J. D. (1989). *Phys. Lett. B* 229, 56.
[5] Unruh W. G., and Newbary P. (1993). *Preprint* [gr-qc/9307029](http://arxiv.org/abs/gr-qc/9307029), p. 29.
[6] Milnor, J. (1963). *Morse Theory* (Princeton University Press, Princeton, NJ).
[7] Wallace A. H. (1968). *Differential Topology: First steps* (Benjamin, N. Y.).
[8] Fomenko A. T. (1983). *Differential Geometry and Topology. Additional Chapters* (Moscow, Nauka) (In Russian).
[9] Hosoya A., and Nakao K. (1990). *Class. Quantum Grav.* 7, 163.