Ultraweak excitations of the quantum vacuum as physical models of gravity

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Abstract
It has been argued by several authors that the spacetime curvature observed in gravitational fields, and the same idea of forms of physical equivalence different from the Lorentz group, might emerge from the dynamical properties of the physical flat-space vacuum in a suitable hydrodynamic limit. To explore this idea, one could start by representing the physical vacuum as a Bose condensate of elementary quanta and look for vacuum excitations that, on a coarse grained scale, resemble the Newtonian potential. In this way, it is relatively easy to match the weak-field limit of classical general relativity or of some of its possible variants. The idea that Bose condensates can provide various forms of gravitational dynamics is not new. Here, I want to emphasize some genuine quantum field theoretical aspects that can help to understand (i) why infinitesimally weak, $1/r$ interactions can indeed arise from the same physical vacuum of electroweak and strong interactions and (ii) why, on a coarse-grained scale, their dynamical effects can be reabsorbed into an effective curved metric structure.

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1. Introduction
The usual interpretation of phenomena in gravitational fields is in terms of a fundamentally curved spacetime. However, showing the consistency of this interpretation with some basic aspects of the quantum theory has been proven to be a very difficult problem. For this reason, one could try to explore a completely different approach where an effective curvature reflects long-wavelength distortions of the same physical, flat-space vacuum (compare, e.g., with the curvature of light in Euclidean space when propagating in a medium with variable density).

Looking at gravity in this perspective, it is useful to start by first exploring those systems (moving fluids, condensed matter systems with a refractive index, Bose–Einstein condensates, . . . ) that can simulate the effects of a genuine spacetime curvature. For these systems, at a fundamental level, spacetime is exactly flat. However, an effective curved metric
emerges when describing the propagation of low-energy fluctuations. Thus, this ‘emergent-gravity’ approach [1] explores the possibility that the type of description of classical general relativity (and of its possible variants) might be similar to hydrodynamics that, concentrating on the properties of matter at scales larger than the mean free path for the elementary constituents, is insensitive to the details of the underlying molecular dynamics. In some way, the same type of idea was also at the basis of the original ‘induced-gravity’ approach [2].

For a definite and simple example where one can clearly separate out the various aspects of the problem, let us consider Visser’s analogy [3] with a moving irrotational fluid, i.e. where the velocity field is the gradient of a scalar potential \( s(x) \). In this system, the underlying spacetime is exactly flat but the propagation of long-wavelength fluctuations is governed by a curved effective metric \( g_{\mu\nu}(x) \) determined at each spacetime point \( x \) by the physical parameters of the fluid (density, velocity and pressure). These can all be expressed in terms of \( s(x) \), and of its derivatives, through the hydrodynamical equations. In this example, there is a system, the fluid, whose elementary constituents are governed by some underlying molecular interactions that, on some scale, can be summarized in the value of a scalar field \( s(x) \). At some intermediate level, \( s(x) \) contains the relevant dynamical information. On the other hand, the effective metric tensor \( g_{\mu\nu} \) is a derived quantity that depends on \( x \) in some parametric form

\[
g_{\mu\nu}(x) = g_{\mu\nu}[s(x)].
\]

The fluid analogy is also interesting for another reason. According to present particle physics, the physical vacuum is not trivially empty but is filled by particle condensates that play a crucial role in many fundamental phenomena such as particle mass generation and quark confinement. In the physically relevant case of the standard model of electroweak interactions, this can be summarized by saying [4] that ‘What we experience as empty space is nothing but the configuration of the Higgs field that has the lowest possible energy. If we move from field jargon to particle jargon, this means that empty space is actually filled with Higgs particles. They have Bose condensed’. Thus, it becomes natural to represent the vacuum as a superfluid medium, a quantum liquid.

As pointed out by Volovik [5], in this representation, if the inducing-gravity scalar field \( s(x) \) were identified with some excitation of such a vacuum state, i.e., with a function that vanishes exactly in the unperturbed state, it would be easy to understand why there is no non-trivial curvature in the equilibrium state \( s(x) = 0 \) where any liquid is self-sustaining. Namely, in the ground state, spacetime would look exactly flat,

\[
g_{\mu\nu}[s = 0] = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1),
\]

just because the large condensation energy of the medium plays no role. In any liquid, in fact, curvature requires deviations from the equilibrium state. The same happens for a crystal at zero temperature where all lattice distortions vanish and electrons can propagate freely as in a perfect vacuum. In both cases, the large condensation energies of the liquid and of the crystal do not generate any curvature. This is the most dramatic difference from the standard approach where, in principle, all forms of energy gravitate and generate a curvature. This point of view represents the simplest and most intuitive solution of the so-called cosmological-constant problem found in connection with the energy of the quantum vacuum. In this sense, by exploring emergent-gravity approaches based on an underlying superfluid medium, one is also taking seriously Feynman’s indication: ‘...the first thing we should understand is how to formulate gravity so that it does not interact with the energy in the vacuum’ [6].

However, before starting with any analysis of the physical vacuum and of its excitations, other general observations are still required. A first observation is that, if really gravity were a long-wavelength modification of the physical condensed vacuum, explaining its characteristic features might require one to use all possible information. This includes basic elements of the
quantum theory, such as particle–wave duality, or other technical details of particle physics as with the origin of hadronic masses from the gluon and chiral condensates, the ‘triviality’ of contact pointlike interactions in (3 + 1) dimensions and so on. In this way, by putting together all available theoretical and experimental knowledge, aspects that are apparently unrelated can finally provide a single consistent framework.

A second observation is that, due to the complexity of the problem, it is unlikely to get at once a satisfactory description of all possible phenomena. Thus, even though a scalar field cannot account for all possible gravitational phenomena, one could, nevertheless, start with this model, such as when describing the longitudinal density fluctuations of a medium. If, as happens with many physical systems (elastic media [7], turbulent fluids [8, 9], superfluids [10], . . . ), one has also to describe transverse waves, at some later stage one can introduce a genuine vector field $V(x)$ (with $\nabla \cdot V = 0$) and replace the parametric dependence of the metric tensor with the more general structure

$$g_{\mu\nu}(x) = g_{\mu\nu}[s(x), V(x)].$$

This type of extension, by itself, would not pose particular conceptual problems.

A third observation is that, in the presence of a condensed vacuum, one is tacitly adopting a ‘Lorentzian’ perspective [12], namely where physical rods and clocks are held together by the same basic forces entering the structure of the underlying ‘ether’ (the physical vacuum). Thus the principle of relativity means that the measuring devices of moving observers are dynamically affected in such a way that their uniform motions with respect to the ether frame become undetectable. In this sense, one is naturally driven to consider the possible, coarse-grained forms of effective curved metric structures as originating from a redefinition of the basic spacetime units.

This aspect was well summarized by Atkinson [13] as follows: ‘It is possible, on the one hand, to postulate that the velocity of light is a universal constant, to define natural clocks and measuring rods as the standards by which space and time are to be judged and then to discover from measurement that spacetime is really non-Euclidean. Alternatively, one can define space as Euclidean and time as the same everywhere, and discover (from exactly the same measurements) how the velocity of light and natural clocks, rods and particle inertia really behave in the neighborhood of large masses’. Such a type of correspondence, in fact, is known to represent one of the possible ways to introduce the concept of curvature (see e.g. [14]).

By adopting this point of view, one could start by assuming that, under the influence of $s(x)$, any mass scale $M$ (or binding energy) might be replaced by an effective mass, say

$$M \rightarrow \frac{M}{\lambda(s)}.$$  

At the same time, if the scalar function $s(x)$ somehow described the density fluctuations of the vacuum medium, it would be natural to introduce, besides such a rescaling, a vacuum refractive index $N(s)$. By considering the vacuum as a ‘non-dispersive’ transparent medium, this could take into account the different geometrical constraints that are placed, on the wavelength of light of a given frequency, by the presence of these fluctuations. Equivalently, it could be due to the redefinition of the local vacuum energy and, with it, of the local dielectric constants

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1 For a definite realization of this idea see Puthoff’s [11] derivation of the ‘gravitomagnetic’ and ‘gravitoelectric’ fields, entering linearized general relativity, from the truncated hydrodynamical equations for a slightly compressible turbulent fluid. Puthoff’s equation (41) for the effective metric has exactly the same form as in equation (3).
as in the polarizable-vacuum approach of [15–18]. In any case, one is driven to consider the 
effective metric structure
\[ g_{\mu\nu}[s] \equiv \text{diag} \left( \frac{\lambda^2(s)}{N^2(s)}, -\lambda^2(s), -\lambda^2(s), -\lambda^2(s) \right) \]  
(5)
that reabsorbs the local, isotropic modifications of spacetime into its basic ingredients: the 
value of the speed of light and the spacetime units.

It is interesting that, independent of the specific underlying mechanisms that one can 
imagine to generate \( N(s) \) and \( \lambda(s) \), these two functions can further be related through general 
arguments that express the basic property of light of being, at the same time, a corpuscular and 
undulatory phenomenon. In flat space, this particle–wave ‘duality’ reflects the equivalence of 
the speed of light defined as a ‘particle’ velocity from the condition \( ds^2 = 0 \) with that obtained 
from the solutions of the D’Alembert wave equation \( \Box F = 0 \).

To consider the analogous situation in curved space, let us assume \( s = 0 \) at infinity (where 
\( \lambda = N = 1 \)) and consider a solution of the wave equation that describes asymptotically a 
monochromatic signal of definite frequency \( \omega \) and wave vector \( k \). By rewriting equation (5) 
as a general isotropic metric
\[ g_{\mu\nu} \equiv \text{diag}(A, -B, -B, -B) \]  
(6)
one may ask under which conditions the local speed of light \( \sqrt{AB} \), defined from the condition
\[ ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = 0, \]
agrees with that obtained from the covariant D’Alembert wave equation [19]
\[ \frac{1}{A} \frac{\partial^2 F}{\partial t^2} - \frac{1}{B} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) F = \frac{1}{\sqrt{AB}}(\nabla \sqrt{AB}) \cdot (\nabla F) = 0 \]  
(7)
or, by introducing the 3-vector \( g = \sqrt{AB}(\nabla \sqrt{AB}) \),
\[ \frac{1}{F} \frac{\partial^2 F}{\partial t^2} = A \frac{1}{B} \Delta F + \frac{1}{F} g \cdot (\nabla F). \]  
(8)
Thus, by identifying \( \frac{1}{F} \frac{\partial^2 F}{\partial t^2} \) as the local equivalent of \(-\omega^2 \) and \( \frac{1}{F} \Delta F \) as the corresponding one 
for \(-k^2 \), one finds that particle velocity and the ‘phase velocity’ \( \frac{1}{F} \) agree with each other only 
when \( g = 0 \), i.e., when \( AB \) is a constant. This product can be fixed to unity with flat-space 
boundary conditions at infinity and, therefore, the resulting value
\[ AB = 1 \]  
(9)
or, in our case
\[ N(s) = \lambda^2(s) \]  
(10)
can be considered a consistency requirement on the possible modifications of the underlying 
physical vacuum, if these modifications have to preserve, at least to some definite order in 
powers of \( s(x) \), the observed particle–wave duality which is intrinsic in the nature of light.

In a more technical language, one could say that this type of particle–wave duality, among 
all possible forms of the covariant D’Alembert wave equation \( (g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma) \)
\[ \Box F = g^{\alpha\beta} \frac{\partial^2 F}{\partial x^\alpha \partial x^\beta} - \Gamma^\nu_{\alpha\beta} \frac{\partial F}{\partial x^\nu} = 0 \]  
(11)
selects the harmonic-coordinate condition
\[ \Gamma^\nu_{\alpha\beta} = g^{\alpha\beta} \Gamma^\nu_{\alpha\beta} = 0. \]  
(12)
More generally, the special role of harmonic coordinates, when imposing flat-space boundary conditions at infinity, had been strongly emphasized by Fock [20]. In his view, one should not confuse the general covariance of a set of partial differential equations with the notion of physical equivalence that is used to formulate a principle of relativity. The former is a logical requirement in all cases where the coordinate system is not fixed in advance. The latter, on the other hand, is related to the existence of frames of reference of a certain class for which one can define the corresponding physical processes and, in the presence of a given gravitational field, depends on the specific boundary conditions. These boundary conditions cannot be given in a general-covariant form so that the fact that for determinateness Einstein’s ten equations must be supplemented by four additional conditions is itself evident. For physical equivalence, what really matters is not the possibility of an indeterminate formulation but just the opposite, namely a formulation that is as determinate as is allowed by the nature of the problem as, for instance, by the requirement of uniformity at infinity. In this sense, Fock concludes that the harmonic coordinates could be considered the analogue of the inertial systems when dealing with gravitational fields that vanish asymptotically. As such, it is not surprising that they may enjoy other special properties (e.g., particle–wave duality) that do not hold in other arbitrary coordinate systems. Here I want to emphasize that, in an emergent-gravity approach, the selection of this class of coordinates might be a natural consequence of the vacuum structure.

To consider in some more detail the issue of general covariance in this type of approach, let us start from Einstein’s original idea, namely to consider forms of physical equivalence that could be naturally described within a general-covariant formalism. To understand how these forms of physical equivalence could originate from the hydrodynamic limit of the same flat-space vacuum, let us tentatively assume that, on a coarse-grained scale, $s(x)$ resembles the Newtonian potential. Then, particle trajectories in this field would not depend on the particle mass thus providing a basic ingredient to represent dynamical effects as an overall modification of the spacetime geometry. At the same time, once $s(x)$ were coupling universally to the various forms of matter, there would be the possibility of establishing an analogy between the motion of a body in a gravitational field and the motion of a body, not subject to an external field, but viewed by a non-inertial reference frame. Therefore, it is this relation with the non-inertial forces that produces new forms of physical equivalence (i.e., different from the simplest uniform translational motions) and leads one naturally to adopt a general-covariant formulation. This type of derivation could even be pursued to argue that the parametric dependence of the effective metric tensor $g_{\mu\nu}(x) = g_{\mu\nu}[s(x)]$ should correspond to solving Einstein’s field equations with a suitable stress tensor that, in addition to the standard matter contributions, might also depend on the $s$-field. In this way, one could partially fill the conceptual gap with classical general relativity or with some of its possible variants.

For a definite realization of this idea, let us consider Yilmaz’ original approach [22]. In his view, the Newtonian potential that solves the Poisson equation in flat space for a given mass distribution (I set $c = 1$ and denote by $G_N$ the Newton constant)

$$s(x) = -G_N \sum_k \frac{M_k}{|x - x_k|}$$

is the true agent of gravity. On the other hand, the metric tensor

$$g_{\mu\nu}[s] = \text{diag}(e^{2s}, -e^{-2s}, -e^{-2s}, -e^{-2s})$$

2 '…the set of all transformations in any case includes those which correspond to all relative motions of three-dimensional systems of coordinates’ [21].
that solves Einstein’s field equations, with a stress tensor for the $s$-field $t^\mu_\nu(s) = -\partial^\mu s \partial_\nu s + 1/2 \delta^\mu_\nu \partial^\alpha s \partial_\alpha s$, is a derived quantity that depends on $s(x)$ in a parametric form. In spite of the obvious differences, in the one-body case and in the weak-field limit, one can expand the Schwarzschild metric of general relativity as

$$g_{00} = \left( \frac{1 + s/2}{1 - s/2} \right)^2 = 1 + 2s + 2s^2 + \ldots \sim e^{4s}$$  \hspace{1cm} (15)$$

$$g_{11} = g_{22} = g_{33} = -(1 - s/2)^4 = -(1 - 2s + \ldots) \sim -e^{-2s}$$  \hspace{1cm} (16)$$

and get the same agreement with experiments to the present level of accuracy.

Thus, to match both metric structures, it should be possible to show (i) that scalar excitations that resemble the Newtonian potential might indeed arise from the same physical vacuum of present particle physics, (ii) that under their influence one can expand the rescaling of equation (4) as

$$\frac{1}{\lambda(s)} = 1 + s + O(s^2).$$  \hspace{1cm} (17)$$

This leads to the line element

$$g_{\mu\nu}[s] \equiv \text{diag}((1 + 2s), -(1 - 2s), -(1 - 2s), -(1 - 2s))$$  \hspace{1cm} (18)$$

that agrees with the first approximation both in general relativity and in the Yilmaz approach.

The idea that Bose condensates can provide various forms of gravitational dynamics is not new (see e.g. [25, 26] and references quoted therein). In the following four sections I will illustrate the nature of the hydrodynamic limit and discuss some genuine quantum field theoretical aspects that could be useful to understand how an infinitesimally weak, long-range interaction such as the Newtonian potential could indeed arise from the same physical vacuum of electroweak and strong interactions.

It is interesting, however, that in principle, in some extreme situations, these ultraweak effects could become important. In fact, the ambiguity about the higher-order $O(s^2)$ effects means that, for large massive bodies, the energy content of the $s$-field could become so strong as to screen the Schwarzschild singularity expected for a single pointlike mass. These issues will be briefly addressed in the final section, together with a more general discussion of the virtues and present limitations of the approach.

2. Excitations of the scalar condensate

As anticipated, the fundamental phenomenon of symmetry breaking, which is believed to determine the physical vacuum of electroweak interactions, consists in the spontaneous creation from the empty vacuum of elementary spinless quanta and in their macroscopic occupation of the same quantum state. The translation from ‘field jargon to particle jargon’ can be obtained, for instance, along the lines of [27]. This amounts to establishing a well-defined functional relation $n = n(\phi^2)$ between the average density $n$ of scalar quanta in the $k = 0$ mode and the average value $\phi$ of the scalar field. Thus, Bose condensation is just

3 In this sense, Yilmaz’ original formulation could be considered the prototype of all emergent-gravity approaches based on a parametric dependence of the effective metric tensor on (scalar, vector, tensor, . . .) excitations of the flat-space vacuum. In the long-wavelength limit, these excitations determine self-consistently the effective metric through their contributions to the energy–momentum tensor.

4 For a discussion of this point, in the original Yilmaz theory, see [23]. Cosmological implications, in more recent formulations of the Yilmaz approach, are also discussed in [24].
a consequence of the shape of the effective potential of the theory whose stability requires values of $\phi$ such that $n \neq 0$.

In this framework, a simple unified picture of the underlying scalar system can be given in terms of two basic quantities, the particle density $n$ of the elementary condensing quanta and their scattering length $a$ that represents the quantum-mechanical analogue of the hard-sphere radius introduced in a classical description. In terms of these quantities, one finds the order of magnitude estimate \[ m_h \sim \sqrt{na}, \] 
with $m_h$ being the parameter associated with the massive excitations of the condensed vacuum, usually denoted as the massive Higgs boson. In this representation, the two quantities $n$ and $a$ can be combined to form a hierarchy of length scales
\[ a \ll \frac{1}{\sqrt{na}} \ll \frac{1}{na^2}. \]
that decouple for an infinitely dilute system where
\[ na^3 \to 0. \]
This hierarchical situation is obtained when approaching the continuum limit of quantum field theory where, due to the basic ‘triviality’ property of the underlying contact $\Phi^4$ theory in $3 + 1$ spacetime dimensions \[28\], the scattering length $a$ should vanish in units of the typical elementary particle length scale
\[ \xi_h = \frac{1}{m_h} \sim \frac{1}{\sqrt{na}}. \]
Thus, the ‘triviality’ limit, where $am_h \to 0$, can be simulated by an ultraviolet cutoff
\[ \Lambda \sim \frac{1}{a} \]
such that $m_h^2/\Lambda^2 \sim na^3 \to 0. \] \[5\] In the same limit, the mean free path for the elementary condensed quanta
\[ r_{\text{mfp}} \sim \frac{1}{na^2}, \]
diverges in units of $\xi_h = 1/\sqrt{na}$. For instance by choosing $a \sim 10^{-33}$ cm and a typical electroweak scale $1/m_h = \xi_h \sim 10^{-17}$ cm, one obtains $r_{\text{mfp}} \sim 10^{-1}$ cm. In this way the hydrodynamic limit of the system, expected for wavelengths larger than $r_{\text{mfp}}$, decouples from the scale $\xi_h$.

Equivalently the region in momentum space $|\mathbf{k}| < \delta$, where
\[ \delta \sim \frac{1}{r_{\text{mfp}}} \sim \frac{m_h^2}{\Lambda}, \]
vanishes in the continuum limit $\Lambda \to \infty$. In this sense, the hydrodynamic region could be considered one of the ‘re-entrant violations of special relativity in the low-energy corner’ mentioned by Volovik \[29\]. These characterize the condensed phase of quantum field theories and can be used to obtain infinitesimally weak interactions from those regions of the spectrum that become a zero-measure set in the continuum limit of the theory.

Now, to construct a model of the Newtonian potential in a Bose condensate of spinless quanta, one could start from the long-range, attractive $1/r$ potential derived by Ferrer
\[5\] Note that the average interparticle distance $d \sim n^{-1/3}$, although much larger than the scattering length $a$, is also much smaller than the length scale $\xi_h$. This means that the scalar condensate could be considered as infinitely dilute or as infinitely dense depending on the adopted unit of length. This type of situation is characteristic of a hierarchical system.
Grifols [30]. This is similar to the long-range attractive potential among electrons moving inside an ion lattice and can be qualitatively understood in terms of phonons, the quantized long wavelength oscillations of the condensate. Their result can be rephrased as follows. Let us consider a scalar field \( \Phi(x) \), whose elementary quanta have some mass \( m \), that interacts with an external density \( \rho(x) \) through the Lagrangian
\[
\mathcal{L}_{\text{int}} = g \rho(x) \Phi^2(x). \tag{26}
\]

Our problem is to evaluate the interaction energy of two density distributions, say \( \rho_1(r) \) and \( \rho_2(r) \), centered respectively around \( r_1 \) and \( r_2 \), in the limit of large spatial separations \( |r - r'| \sim |r_1 - r_2| \to \infty \). This interaction energy is (minus) the Fourier transform of the Feynman graph with a loop formed by two \( \Phi \) quanta that are exchanged between the sources. In the trivial empty vacuum of the \( \Phi \) field, i.e. when \( \langle \Phi \rangle = 0 \), this takes the form
\[
V_{\text{int}} = -g^2 \int \frac{\rho_1(r)\rho_2(r')e^{-2m|r-r'|}}{|r-r'|^3} \, d^3r \, d^3r'. \tag{27}
\]

However, if the \( \Phi \) quanta were Bose condensing below some critical temperature \( T_c \), i.e. when now the vacuum at \( T = 0 \) has \( \langle \Phi \rangle \neq 0 \), the same type of computation gives a very different interaction, namely
\[
V_{\text{int}} = -g^2 T_c^2 \int \frac{\rho_1(r)\rho_2(r')}{|r-r'|} \, d^3r \, d^3r'. \tag{28}
\]

Ferrer and Grifols explain that this happens because, when one of the two exchanged \( \Phi \) quanta is in the condensate, the other behaves as a massless scalar photon, that responsible for the Coulomb interaction. This means that the long-range \( 1/r \) interaction can be obtained by the replacement
\[
\Phi^2(x) = (\langle \Phi \rangle + \delta \Phi(x))^2 = \langle \Phi \rangle^2 + 2\langle \Phi \rangle \delta \Phi(x) + \delta \Phi^2(x) \tag{29}
\]
and picking up the crossed term \( 2\langle \Phi \rangle \delta \Phi(x) \). In the presence of a non-zero \( \langle \Phi \rangle \), the propagator of the fluctuation \( \delta \Phi(x) \) (the connected propagator) behaves as \( 1/p^2 \) for \( p_\mu \to 0 \).

In general, the connected propagator \( G(x - y) \) determines the fluctuation of the scalar field \( \delta \Phi(x) \), around the average value \( \langle \Phi \rangle \), that is produced by an external perturbation \( J(x) \) through the relation
\[
\delta \Phi(x) = \int d^4x' G(x - x')J(x'). \tag{30}
\]
Thus, for a pointlike, static source, say \( J(x) = J\delta^3(r) \), one finds
\[
\delta \Phi(r) = J D(r) \tag{31}
\]
with
\[
D(r) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot r} G(p, \, p_4 = 0) \tag{32}
\]
in terms of the Fourier transform \( G(p) \) in Euclidean space
\[
G(x) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} G(p), \tag{33}
\]
where \( p_\mu \equiv (p, \, p_4) \). Therefore, if one requires long-range modifications such that \( \delta \Phi \) remains sizeable at large distances from a perturbing source, the propagator has to become singular when \( p_\mu \to 0 \). For instance, in the case of a free massless scalar field, where \( G(p, \, p_4 = 0) = 1/p^2 \), one obtains a \( \delta \Phi \) that vanishes as \( 1/r \). On the other hand, in the case of a free massive scalar field where \( G(p, \, p_4 = 0) = 1/(p^2 + m^2_0) \), one finds the corresponding short-range Yukawa behavior \( e^{-m_0 r}/r \).
Thus the possibility of a long-range $1/r$ potential in the broken-symmetry vacuum depends crucially on the zero-momentum limit of the connected scalar propagator. At the same time, for any given strength of the source, the magnitude of $\delta/\Phi_1$ depends on the slope of $G(p)$. The standard unit normalization corresponds to genuine massless particles (e.g., photons) whose propagator behaves as $1/p^2$ in the whole range of momenta. However, a condensed vacuum can exhibit different types of excitations in different ranges of momenta. For instance, when $p_\mu \to 0$, the propagator could behave as $\zeta/p^2$, where $\zeta$ is some positive number, and approach the massive form at larger $|p|$.

Again, the motivations for this idea originate from the representation of the broken-symmetry vacuum as a physical condensate and from the analogy with superfluid $^4$He, the physical system that is usually considered as a non-relativistic realization of $\Phi^4$ theory. In fact, as originally proposed by Landau [31], a superfluid should have two different energy branches, namely gapless density oscillations (phonons) and massive vortical excitations (rotons)\textsuperscript{6}. Experiments however have shown that these two different branches actually merge into a single energy spectrum, a sort of ‘hybrid’ that smoothly interpolates between the two different functional forms.

Analogously, in our case, the long-range components of $\delta/\Phi_1(x)$, for brevity denoted as $s(x)$, would totally be determined by the infrared part of the propagator. As such, their interactions, proportional to the $\zeta$ parameter, could be very different from the typical interaction strength of the short-wavelength fluctuations. The discussion of these other aspects will be presented in the following two sections.

3. The zero-momentum propagator in the broken phase

The long-range forces considered in the previous section originate from the macroscopic occupation of the same quantum state. As such, they are quite unrelated to the Goldstone phenomenon that characterizes the spontaneous breaking of continuous symmetries, and are solely determined by the condensing scalar field, i.e. from that field $\Phi$ which acquires a non-zero vacuum expectation value. For this reason, these long-range forces would also exist if spontaneous symmetry breaking were induced by a one-component field. Their ultimate origin has to be traced back to a peculiarity of the zero-momentum connected propagator for the $\Phi$-field: this is a \textit{two-valued} function [33]. Namely, its inverse, in addition to the standard massive solution $G_a^{-1}(p = 0) = m_a^2$, includes the value $G_b^{-1}(p = 0) = 0$ as in a massless theory. To show this, one can use different arguments.

Let us introduce preliminarily the semi-classical non-convex effective potential $V_{NC}(\phi) = V_{NC}(\phi)$ (NC = non-convex), as computed in the standard loop expansion. Let us also denote by $\phi = \pm v$ its absolute minima and by $m_b^2 = V_{NC}''(\pm v) > 0$ its quadratic shape at these minima. In full generality, one could first study the theory at an arbitrary value of $\phi$ and then take the $\phi \to \pm v$ limit afterward. In this case, for any $\phi \neq \pm v$, the diagrammatic representation of the connected propagator [33] requires one to first include the one-particle reducible tadpole graphs where zero-momentum propagator lines are attached to the one-point function $\Gamma_1(p = 0) = V_{NC}'(\phi)$. By implicitly assuming the regularity of the zero-momentum propagator, these graphs are usually ignored at $\phi = \pm v$ where $V_{NC}''(\pm v) = 0$. Thus, $G^{-1}(p)$ is identified with the 1PI two-point function $\Gamma_2(p)$, whose zero-momentum value $\Gamma_2(p = 0)$ is nothing but $V_{NC}''(\pm v)$, a positive-definite quantity. On the other hand, by allowing for a singular $G(p = 0)$ at $\phi = \pm v$, one is faced with a completely different diagrammatic

\textsuperscript{6} The analogy is consistent with the results of [32] where, by using quantum hydrodynamics, the mass parameter $m_h \sim \sqrt{\alpha}$ was shown to be proportional to the energy gap for vortex formation in a suitable superfluid medium possessing the same constituents and the same density as in the condensate picture of $\Phi^4$. 

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expansion, and thus the simple picture of the broken phase as a pure massive theory, based on the chain
\[ G^{-1}(p = 0) = \Gamma_2(p = 0) = V_{NC}^\prime (\pm v) = m_b^2 > 0, \]  
breaks down.

The existence of two solutions for \( G^{-1}(p = 0) \) can also be derived by evaluating in the saddle point approximation the generating functional \( W[J] \) for a constant source and taking the double limit where \( J \to \pm 0 \) and the spacetime volume \( \Omega \to \infty \) \([33]\). As such, the two solutions admit a geometrical interpretation in terms of left and right second derivatives of the Legendre-transformed, quantum effective potential. This is convex downward and is not an infinitely differentiable function when \( \Omega \to \infty \) \([34]\). For the convenience of the reader, this latter type of derivation will be reported below with several details that were not included in \([33]\).

To describe spontaneous symmetry breaking in full generality, let us consider a scalar field \( \Phi_1(x) \) which can interact with itself and with \( n \) other fields, say \( \Psi_1(x), \Psi_2(x), \ldots \Psi_n(x) \) according to some action
\[ S[\Phi; \Psi_1, \Psi_2, \ldots \Psi_n]. \]  
(35)

At this stage, both the nature of the \( \Psi \)-fields (scalar, fermion or gauge fields) and the structure of the action are completely arbitrary. For instance, for \( n = 0 \), the action could just describe a single scalar field and be invariant under the simple discrete reflection symmetry \( \Phi_1 \to -\Phi_1 \). Or, for some \( 0 < k \leq n \), \( \Psi_1, \Psi_2, \ldots \Psi_k \) could be \( k \) other scalar fields and the action be invariant for rotations in the \((k + 1)\)-dimensional space \( (\Phi, \Psi_1, \Psi_2, \ldots \Psi_k) \). Or, this type of global continuous symmetry could also be transformed into a local symmetry provided gauge fields (with corresponding gauge-fixing and gauge-compensating terms) were introduced.

In this framework, Green’s functions of the \( \Phi_1 \)-field are obtained from the generating functional in the presence of a source \( J(x) \)
\[ Z[J] = \int [d\Phi(x)] [d\Psi_1(x) d\Psi_2(x) \ldots d\Psi_n(x)] e^{\int d^4x \Phi(x) J(x) - S[\Phi; \Psi_1, \Psi_2, \ldots \Psi_n]} \]  
(36)

Before exploring the conditions for a non-zero \( \langle \Phi \rangle \), let us perform formally the functional integration on the \( \Psi \)-fields
\[ \int [d\Psi_1(x) d\Psi_2(x) \ldots d\Psi_n(x)] e^{-S[\Phi; \Psi_1, \Psi_2, \ldots \Psi_n]} \equiv e^{-S_{\text{eff}}[\Phi]} \]  
(37)

so that the generating functional can be expressed as in a one-component theory
\[ Z[J] = \int [d\Phi(x)] e^{\int d^4x \Phi(x) J(x) - S_{\text{eff}}[\Phi]} \]  
(38)

As mentioned above, the standard motivation to interpret the broken-symmetry phase as a pure massive theory derives from the chain relations \([34]\). These are obtained by assuming implicitly (i) the regularity of \( G(p = 0) \) and (ii) that the constant background \( \phi \) entering the expression for the full scalar field \( \Phi(x) \)
\[ \Phi(x) = \phi + h(x) \]  
(39)
can be kept ‘frozen’ at one of the two absolute minima \( \pm v \) of a non-convex effective potential \( V_{NC}(\phi) \). To check the validity of these two assumptions, let us first define the theory in some large 4-volume \( \Omega \). In this way, the possible spacetime averages of the field, namely
\[ \phi = \frac{1}{\Omega} \int d^4x \Phi(x) \]  
(40)
represent the zero-momentum mode of the scalar field and, as such, enter the full functional measure of the theory

\[ \int [d\Phi(x)] \cdots = \int_{-\infty}^{\infty} d\phi \int [dh(x)] \cdots \]  

(41)

In the above relation, the measure \([dh(x)]\) is overall quantum modes with \(p_\mu \neq 0\) that, for \(\Omega \to \infty\), will include arbitrarily small values of \(|p|\). According to the standard interpretation of the broken phase as a pure massive theory, the zero-momentum limit of the connected propagator should be uniquely determined, say

\[ \lim_{p_\mu \to 0} G(p) = \frac{1}{m_h^2}. \]  

(42)

To check this expectation, one can compute directly \(G(p = 0)\) from the generating functional in the presence of a constant source \(J\) and then send \(J \to 0\) and \(\Omega \to \infty\). If there are no subtleties associated with the infinite-volume limit, one should obtain the same result.

By restricting to the case of a constant source \(J\) in the generating functional

\[ Z(J) = \int_{-\infty}^{+\infty} d\phi \exp(\Omega J \phi) \int [dh(x)] \exp(-S_{\text{eff}}[\phi + h]) \]  

(43)

one can compute the field expectation value

\[ \frac{1}{\Omega Z(J)} \frac{dZ}{dJ} = \langle \Phi \rangle_J = \phi(J) \]  

(44)

and the zero-momentum propagator

\[ \frac{1}{\Omega Z(J)} \frac{d^2 Z}{dJ^2} = \Omega \psi^2(J) + \frac{d\psi}{dJ}. \]  

(45)

To express the connected parts, it is convenient to introduce the generating functional for connected Green’s function

\[ W(J) = \Omega w(J) = \ln \frac{Z(J)}{Z(0)} \]  

(46)

from which one obtains the field expectation value

\[ \psi = \frac{dw}{dJ} \]  

(47)

and the zero-momentum connected propagator

\[ G(p = 0) = \frac{d^2 w}{dJ^2} = \frac{d\psi}{dJ}. \]  

(48)

In this framework, spontaneous symmetry breaking corresponds to a non-zero value of \(\psi\) in the double limit \(J \to 0\) and \(\Omega \to \infty\).

To study this limit, I will assume the standard condition for the occurrence of spontaneous symmetry breaking, namely that the result of the \(h\)-integration in equation (43) can be expressed formally, as in the loop expansion (see e.g. [35, 36]), in terms of some non-convex effective potential which has absolute minima for \(\phi \neq 0\)

\[ \int [dh(x)] \exp(-S_{\text{eff}}[\phi + h]) = \exp(-\Omega V_{\text{NC}}(\phi)) \]  

(49)

so that

\[ Z(J) = \int_{-\infty}^{+\infty} d\phi \exp(-\Omega (V_{\text{NC}}(\phi) - J\phi)). \]  

(50)
In the $\Omega \to \infty$ limit, $Z(J)$ can then be evaluated in a saddle-point approximation (see e.g. [36–39]). In this approximation, where the result is only expressed in terms of the absolute minima $\pm v$ of $V_{NC}(\phi)$ and of its quadratic shape there, say $V_{NC}'(\pm v) = m_h^2$, the relevant relation is

$$Z(J) \sim e^{\Omega/J^2(2\pi)^2} \cosh(\Omega J v)$$

(51)

up to a $J$-independent proportionality factor. One thus obtains

$$w(J) = \frac{J^2}{2m_h^2} + \frac{1}{\Omega} \ln \cosh(\Omega J v)$$

(52)

and

$$\varphi = \frac{dw}{dJ} = \frac{J}{m_h^2} + v \tanh(\Omega J v)$$

(53)

$$G(p = 0) = \frac{d^2w}{dJ^2} = \frac{1}{m_h^2} + \frac{\Omega v^2}{\cosh^2(\Omega J v)}$$

(54)

Since both $J$ and $\Omega$ are dimensioned quantities, it is convenient to introduce dimensionless variables

$$\epsilon = \frac{J}{m_h^2 v}$$

(55)

and

$$y = \Omega m_h^2 v^2$$

(56)

so that $\Omega J v = \epsilon y$. In this representation, one finds

$$\varphi = v(\epsilon + \tanh \epsilon y)$$

(57)

and

$$G(p = 0) = \frac{1}{m_h^2} [1 + y(1 - \tanh^2 \epsilon y)]$$

(58)

with the two limits $J \to \pm 0$ and $\Omega \to \infty$ corresponding to $\epsilon \to \pm 0$ and $y \to \infty$.

It is clear that, when $\epsilon \to 0$, any non-zero limit of $\varphi$ requires a non-zero limit of $\epsilon y$. Now, this limit can be finite or infinite. If it is finite, say $\epsilon y \to X_0$ with $|X_0| < +\infty$, one finds $|\varphi| = v |\tanh X_0| < v$ and an expectation value whose magnitude is reduced with respect to the putative value $v$. In the same range, the zero-momentum propagator $G(p = 0)$ diverges for any value of $\varphi$. These results can be rephrased by saying that, in the $\Omega \to \infty$ limit, both $J = J(\varphi)$ and $G^{-1}(p = 0)$ vanish in the open range $-v < \varphi < +v$.

On the other hand, if the double limit $\epsilon \to \pm 0$ and $y \to \infty$ corresponds to $\epsilon y \to \pm \infty$ one finds $\varphi \to \pm v$. In this case, there are two different possibilities for the zero-momentum propagator. Namely either $G(p = 0) \to 1/m_h^2$, if $y(1 - \tanh^2 \epsilon y) \to 0$, or $G(p = 0) \to \infty$, if $y(1 - \tanh^2 \epsilon y) \to \infty$. Again this can be rephrased by saying that, in the $\Omega \to \infty$ limit, if $J(\varphi)$ vanishes and $G(p = 0)$ becomes a double-valued function, then $\varphi$ tends to one of the absolute minima of the underlying non-convex potential.

As anticipated, the above results admit a simple geometrical interpretation in terms of the Legendre-transformed, quantum effective potential $V_{LT}(\varphi)$. This is defined through the relation

$$V_{LT}(\varphi) = J \varphi - w(J)$$

(59)
after inverting $J = J(\varphi)$ so that
\[
\frac{dV_{LT}(\varphi)}{d\varphi} = J(\varphi)
\]
(60)
and
\[
\frac{d^2V_{LT}(\varphi)}{d\varphi^2} = \frac{dJ}{d\varphi} = G^{-1}(p = 0).
\]
(61)
Therefore, by using equation (53) to define $J(\varphi)$, one obtains
\[
G^{-1}(p = 0) = \frac{m_h^2}{1 + y(1 - \tanh^2 \epsilon y)}.
\]
(62)
Then, the divergent value of $G(p = 0)$, found in connection with all values $|\varphi| < v$, corresponds to the inner region where the Legendre-transformed effective potential becomes exactly flat in the infinite-volume limit.

Analogously, the two possible solutions for $G(p = 0)$, when $\epsilon y \to \pm \infty$ and $\varphi \to \pm v$, correspond to compute left and right second derivatives of $V_{LT}(\varphi)$ at $\varphi = \pm v$. To this end, let us compute the right second derivative for $\varphi = v$, i.e. the value of $G^{-1}(p = 0)$ when $\varphi \to v^+$, and set $\varphi = v + |\Delta \varphi| > v$. From equation (57), one has $\epsilon \sim \frac{|\Delta \varphi|}{v}$ up to terms $O(\epsilon^{-v})$ that vanish exponentially when $y \to \infty$ and $\epsilon y \to \infty$. Thus, one finds
\[
y(1 - \tanh^2 \epsilon y) \to 0
\]
(63)
and
\[
G^{-1}(p = 0) \to m_h^2
\]
(64)
(analogous results hold for the left derivative at $\varphi = -v$, i.e. for $\varphi = -v - |\Delta \varphi| < -v$).

Let us now consider the left second derivative for $\varphi = v$, i.e. the value of $G^{-1}(p = 0)$ when $\varphi \rightarrow v^-$, and set $\varphi = v - |\Delta \varphi| < v$. Here, the situation is different since, now, one cannot set $\epsilon \sim -\frac{|\Delta \varphi|}{v}$ to solve equation (57). In fact, $\epsilon y$ has to be large and positive in order $\tanh \epsilon y$ to be a positive number slightly smaller than unity. Rather, one can use equation (57) to replace
\[
\tanh \epsilon y = 1 - \epsilon - \frac{|\Delta \varphi|}{v}
\]
(65)
and show that
\[
y[1 - \tanh^2 \epsilon y] = y \left[ 2\epsilon + 2\frac{|\Delta \varphi|}{v} + O(\epsilon^2) \right] > 2\epsilon y \to \infty.
\]
(66)
Then the left second derivative vanishes
\[
G^{-1}(p = 0) \sim \frac{m_h^2}{1 + 2\epsilon y} \to 0
\]
(67)
(analogous results hold for the right derivative at $\varphi = -v$, i.e. when $\varphi = -v + |\Delta \varphi| > -v$).

In conclusion, at the absolute minima $\pm v$ of the non-convex potential, $G^{-1}(p = 0)$ becomes a two-valued function in the infinite-volume limit of the theory.

Note that, independently of any specific calculation, the gapless solution $G^{-1}(p = 0) = 0$ is also needed for a consistent interpretation of symmetry breaking with a convex-downward quantum effective potential. In fact, if there were only massive excitations in the spectrum one expects a non-degenerate ground state and, therefore, an effective potential with only one minimum. Due to the underlying reflection symmetry of the theory, this unique minimum could only be $\varphi = 0$. On the other hand, if there were a gapless branch in the spectrum, by adding a sufficiently large number of these excitations in the zero 3-momentum mode,
one could construct new translational invariant states with different values of \( \varphi \) and the same energy. This construction could be done by respecting the underlying \( \varphi \rightarrow -\varphi \) symmetry, at least in some range of \(|\varphi|\), and leads to the type of degenerate ground state associated with a flat effective potential. As discussed in the following section, it is precisely this type of degeneracy that is responsible for the existence of infinitesimally weak long-range forces.

4. An infinitesimal \( 1/r \) long-range potential

The existence of two solutions for \( G^{-1}(p = 0) \), as deduced in section 3, is just a consequence of spontaneous symmetry breaking in the infinite-volume limit of the theory. As such, it does not imply any specific functional form of \( G(p) \). However, by exploiting the analogy with superfluid \( ^4\text{He} \), one expects the full \( G(p) \) to correspond to a suitable interpolation between gapless and massive solutions. It is interesting that such interpolation can also be deduced \[41\] by using the strong constraints on the possible structure of \( G(p) \) placed by the accepted ‘triviality’ \[28\] of the scalar self-interacting theories in four spacetime dimensions.

In fact, for the continuum theory, defined in the limit of infinite ultraviolet cutoff, ‘triviality’ dictates a Gaussian set of Green’s functions, no observable dynamics at any value of the 4-momentum \( p_\mu \neq 0 \) and the standard free-field type form \( G^{-1}(p) = (p^2 + m_\varphi^2) \). Still, consistently with these constraints, one cannot exclude a discontinuity of the truncated Green’s functions in the zero-measure, Lorentz-invariant subset \( p_\mu = 0 \). This plays a fundamental role in translational invariant vacua characterized by spacetime constant expectation values of local operators such as \( \langle \Phi \rangle \). Therefore, by accepting ‘triviality’, the only possible not-entirely-trivial continuum limit of the connected propagator has \( G^{-1}(p) = (p^2 + m_\varphi^2) \) for any \( p_\mu \neq 0 \) with the exception of a discontinuity at \( p_\mu = 0 \) where \( G^{-1}(p = 0) = 0 \).

Let us now consider the finite-cutoff theory. Here, in the presence of an ultraviolet cutoff \( \Lambda \), the distinction between \( p_\mu = 0 \) and \( p_\mu \neq 0 \) has no obvious meaning. One can always consider a whole set of ‘infinitesimal’ (but non-zero) momentum values, such as \(|p| \sim m_\varphi^2/\Lambda, |p| \sim m_\varphi^2/\Lambda^2, \ldots \) that however all approach the same \( p_\mu = 0 \) value when \( \Lambda \rightarrow \infty \). Therefore, in the cutoff theory, if one wants to obtain a continuum limit where \( G^{-1}(p = 0) = 0 \), at a certain point, i.e. for sufficiently small momenta, one should necessarily replace the standard massive form \( G^{-1}(p) \sim (p^2 + m_\varphi^2) \rightarrow m_\varphi^2 \) with some different behaviors for which \( G^{-1}(p) \rightarrow 0 \). For this reason, the sharp singularity of the continuum theory will be replaced by a smooth behavior in the cutoff theory. Then, even though the continuum theory has only massive, free-field excitations, the cutoff version would exhibit non-trivial qualitative differences, as weak long-range forces, that cannot be considered uninteresting perturbative corrections.

For a quantitative description, one can write the connected propagator of the cutoff theory in a general interpolating form, say \[41\]

\[
G^{-1}(p) = (p^2 + m_\varphi^2) f(p^2/\delta^2). \tag{68}
\]

In order to reproduce the mentioned continuum-limit behavior, the function \( f(p^2/\delta^2) \) has to refer to some infrared momentum scale \( \delta \neq 0 \) (with \( \delta/m_h \rightarrow 0 \) when \( m_h/\Lambda \rightarrow 0 \), as for \( \delta \sim m_h^2/\Lambda \)) in such a way that

\[
\lim_{\delta \rightarrow 0} f(p^2/\delta^2) = 1 \quad (p_\mu \neq 0) \tag{69}
\]

with the only exception

\[
\lim_{p_\mu \rightarrow 0} f(p^2/\delta^2) = 0 \tag{70}
\]

(think for instance of \( f(x) = \tanh(x) \), \( f(x) = 1 - \exp(-x) \), \( f(x) = x/(1 + x), \ldots \)).
As anticipated, to understand what kind of long-range effects in coordinate space are associated with such a propagator for the scalar field, one has to consider the standard Fourier transform of the zero-energy propagator $G(p, p^4 = 0)$

$$D(r) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{ipr}}{(p^2 + m^2_h)f(p^2/\delta^2)}$$

that in the case of a free massless scalar field, $G(p, p^4 = 0) = 1/p^2$, gives a 1/r potential.

Now, a straightforward replacement $f(p^2/\delta^2) = 1$ would produce the Yukawa potential $e^{-mhr}/r$. However, if we consider the finite-cutoff theory, we have to take into account the region $p^2 \ll \delta^2$ where the relevant limiting relation is rather

$$\lim_{p \to 0} f(p^2/\delta^2) = 0.$$ (72)

For this reason, since the dominant contribution for $r \to \infty$ comes from $p = 0$, where the denominator in (71) vanishes, there will be long-range forces. In this case, by expanding around $p = 0$ and using the Riemann–Lebesgue theorem on Fourier transforms [41], whatever the detailed form of $f(x)$ at intermediate $x$, the leading contribution at asymptotically large $r$ will be $1/r$. One thus obtains

$$\lim_{r \to \infty} D(r) = D_\infty(r) = \frac{\delta^2}{f'(0)m^2_h} \frac{1}{4\pi r}$$ (73)

all dependence on the interpolating function being contained in the factor $f'(0)$ expected to be $O(1)$. In this way, the same asymptotic $1/r$ trend as in [30] is obtained by using only general properties of the underlying quantum field theory.

To put some numbers (in units $\hbar = c = 1$), let us consider for definiteness the scenario $\delta \sim m^2_h/\Lambda$ which is motivated by the relations of section 2 namely $\Lambda \sim 1/a$, $m^2_h \sim na$ and the hydrodynamic-limit relation $\delta \sim 1/r_{mfp} \sim na^2$. By fixing the same values of section 2 namely $a \sim 10^{-33}$ cm, $1/m_h = \xi_h \sim 10^{-47}$ cm, and $r_{mfp} \sim 10^{-4}$ cm, let us consider the couplings of the singlet standard model Higgs boson. Two fermions $i$ and $j$, of masses $m_i$ and $m_j$, couple to it with strength $y_i = m_i/v$ and $y_j = m_j/v$ and thus feel the instantaneous potential

$$V(r) = -y_i y_j D(r).$$ (74)

From the previous analysis, besides the short-distance Yukawa potential governed by the Fermi constant $G_F \equiv 1/v^2$

$$V_{\text{Yukawa}}(r) = -\frac{G_F m_i m_j}{4\pi r} e^{-m_i r}$$ (75)

(that dominates for $r \lesssim 1/m_h$), they would feel the asymptotic potential associated with equation (73). This can be conveniently expressed as

$$\lim_{r \to \infty} V(r) = V_\infty(r) = -\frac{G_\infty m_i m_j}{r}$$ (76)

with the effective coupling

$$G_\infty = \frac{\delta^2}{4\pi f'(0)m^2_h} G_F \sim 10^{-33} G_F.$$ (77)

Strictly speaking, this asymptotic potential represents a ‘cutoff artifact’ since the continuum theory has only massive, free-field excitations, with the only exception of a discontinuity at $p^4 = 0$ where $G^{-1}(p) = 0$. At least, this is the only possible remnant of symmetry breaking allowed by exact Lorentz invariance and ‘triviality’. However in the cutoff theory, where one
expects a smooth behavior, the deviation from the massive form will necessarily extend, from the zero-measure set $p_\mu = 0$, to an infinitesimal momentum region $\delta$. It is this infinitesimal momentum region where the propagator will look like it does in a massless theory, producing the long-range $1/r$ potential of infinitesimal strength $\delta^2/m_0^2$.

5. The scalar coupling in the hydrodynamic limit

As discussed at the end of section 2 the infrared limit $p_\mu \to 0$ of the propagator determines the long-range scalar fluctuations that have been denoted as the $s$-field. In terms of this field, one can reformulate the two-body interaction

$$ V_{12}(r_1 - r_2) = -\frac{G_\infty M_1 M_2}{|r_1 - r_2|} $$

as

$$ V_{12}(r_1 - r_2) = M_1 s(r_1) $$

with

$$ s(r_1) = -\frac{G_\infty M_2}{|r_1 - r_2|} $$

or as

$$ V_{12}(r_1 - r_2) = M_2 s(r_2) $$

with

$$ s(r_2) = -\frac{G_\infty M_1}{|r_1 - r_2|}. $$

In this way, provided the Newton constant $G_N$ is identified with the infinitesimal coupling $G_\infty$, one gets the idea of the Newtonian potential as a fundamental excitation of the scalar condensate. The two equivalent ways to write the same two-body interaction, namely as particle 1 in the field generated by particle 2 or as particle 2 in the field of particle 1, would then express the identity of ‘active’ and ‘passive’ gravitational masses and their equality to the inertial mass generated by the vacuum structure.

For an $N$-body system, one can use the relation

$$ \Delta \left( \frac{1}{r} \right) = -4\pi \delta^{(3)}(r) $$

that expresses the asymptotic propagator as Green’s function of the Poisson equation with a given mass density, i.e.

$$ \Delta s(x) = 4\pi G_N \sum M_n \delta^{(3)}(r - r_n). $$

Note that, by writing such a Poisson equation, one assumes that by adding more and more sources the resulting $s$-fields superpose linearly. This is only true if the residual self-interaction effects of the $s$-field are negligible. Formally, in the broken-symmetry phase of a $\Phi^4$ theory, these self-interaction effects start to $O(s^3)$ and might show up as $O(s^2)$ in the equations of motion. The fact that these terms can be neglected (in ordinary circumstances) is consistent with the idea that $s(x)$ represents an excitation of the vacuum in a ‘trivial’ theory, where physical states should exhibit no observable self-interaction.

Thus, the derivation of the Newtonian potential from the vacuum structure could be considered complete for all elementary particles that couple directly to the fundamental Higgs field. In the standard model, these elementary particles are the leptons and quarks and, for
them, the coupling to any scalar field can simply be considered a local redefinition of their mass parameters. To this end, one should start from the basic elementary Yukawa couplings

$$\mathcal{L}_{\text{Yukawa}} = -y_f \bar{\psi}_f \psi_f \Phi$$  \hspace{1cm} (85)

and decompose the full scalar field \(\Phi(x)\) in the sum of its vacuum expectation value \(\langle \Phi \rangle = v\), of the long-range component \(v_s(x)\) (for wave vectors \(k \lesssim \delta\)) and of the short-range part (for \(k > \delta\)). In this way, by defining \(m_f = y_f v\) (and dropping the short-wavelength components) one obtains trivially

$$\mathcal{L}_{\text{Yukawa}} = -m_f \bar{\psi}_f \psi_f (1 + s)$$  \hspace{1cm} (86)

so that a non-zero \(s\) amounts to rescaling all elementary fermion masses according to \(m_f \rightarrow m_f (1 + s)\).

A possible objection can arise when considering the hadronic states, because nucleons, nuclei etc., unlike leptons and quarks, have no tree-level coupling to the scalar field as in equation (85). Besides depending on the quark masses, their masses refer to other basic parameters of the vacuum, namely the gluon and the chiral condensate. In this case, why should one introduce the same overall type of mass rescaling?

To see this, let us first consider the unperturbed situation where \(s = 0\). In this limit, the mutual interactions among the various scalar, gluon and quark condensates give rise to suitable relations arising from the minimization of the overall energy density. One can express these relations as

$$\alpha_{\text{QCD}} \langle F^a_{\mu\nu} F^{a\mu\nu}_\alpha \rangle = c_1 v^4$$  \hspace{1cm} (87)

and

$$m^{(\alpha)}_q \langle \bar{\psi}_q \psi_q \rangle_\alpha = c_2 v^4,$$  \hspace{1cm} (88)

where \(c_1\) and \(c_2\) are dimensionless numbers and \(\langle \ldots \rangle_\alpha\) denotes the unperturbed vacuum expectation values for \(s = 0\).

Now, let us consider an external perturbation that induces long-wavelength oscillations of the scalar condensate so that \(v \rightarrow v(1 + s)\). Let us also assume that QCD has only short-range fluctuations whose wavelengths are much smaller than \(1/\delta\). In this situation, \(v(1 + s)\) can be considered to define a new local vacuum, whose variations occur over regions that are much larger than the QCD scale. To this new scalar field the quark masses and all relevant expectation values of the gluon and quark operators will (‘adiabatically’) be adjusted according to the new minimization relations

$$\alpha_{\text{QCD}} \langle F^a_{\mu\nu} F^{a\mu\nu}_\alpha \rangle = c_1 v^4 (1 + s)^4$$  \hspace{1cm} (89)

and

$$m_q \langle \tilde{\psi}_q \psi_q \rangle = c_2 v^4 (1 + s)^4.$$  \hspace{1cm} (90)

In this way, the overall result of a non-zero \(s\) is equivalent to a rescaling of the QCD scale parameter

$$\Lambda_{\text{QCD}} \rightarrow \Lambda_{\text{QCD}} (1 + s)$$  \hspace{1cm} (91)

of the chiral condensate

$$\langle \tilde{\psi}_q \psi_q \rangle \rightarrow \langle \tilde{\psi}_q \psi_q \rangle (1 + s)^3$$  \hspace{1cm} (92)

and of the quark masses

$$m_q \rightarrow m_q (1 + s).$$  \hspace{1cm} (93)

Therefore, any combination with the dimension of a mass, say \(\frac{\Lambda_{\text{QCD}}^4}{\langle \tilde{\psi}_q \psi_q \rangle}\) or \(\frac{\langle \tilde{\psi}_q \psi_q \rangle}{\Lambda_{\text{QCD}}^2}\), that is expressed in terms of the elementary quark masses as well as of the
expectation value of both quarks and gluon condensates, will also undergo the same rescaling $(1 + s)$. This holds for the nucleon mass $m_N$

$$ m_N \rightarrow m_N (1 + s) \quad (94) $$

for the mass of the nuclei and, more generally, for all mass parameters and binding energies that can be expressed in terms of the basic quantities of the theory, namely the masses of the elementary fermions and the vacuum condensates. This is due to the extremely large $s$-field wavelengths as compared to any elementary particle, or nuclear or atomic scale. Thus $s(x)$ couples universally to the various forms of matter and one obtains the same overall type of rescaling equation (17) foreseen in section 1.

Finally, let us consider the extension to the case of variable fields. To this end, one should take into account both the full $p^2$-dependence in the propagator and replace, in the basic coupling to the scalar field, the mass density with the trace of the energy–momentum tensor. This replacement can be understood by considering a fermion field $\psi_f$ that describes a sharply localized wave packet with momentum $p$ and velocity $v$, i.e. such that

$$ \int d^3x \langle \bar{\psi}_f \psi_f \rangle = \sqrt{p^2 + m_f^2} = \sqrt{1 - v^2}. \quad (95) $$

If $s(x)$ does not vary appreciably over the localization region, one obtains from equation (86) the classical action

$$ \int d^4x L_{\text{Yukawa}} = -m_f \int d\tau (1 + s(x)) \quad (96) $$

for a pointlike particle interacting with a scalar field $s(x)$. In the last expression $x_\mu = x_\mu(\tau)$ and $d\tau = dt \sqrt{1 - v^2}$ denote the proper-time element of the particle.

As for any linear coupling, this relation, which gives the action of a particle in an external $s$-field, can also be used to express the $s$-field that is generated by a given source. In this case, by using equation (68) to describe the full $p^2$-dependence of the propagator for $p_\mu \rightarrow 0$, taking the Fourier transform and identifying $G_\infty = G_N$, one obtains an equation of motion that is valid for large spacetime separation from the sources. For a many-particle system, this amounts to

$$ \Box s(x) = 4\pi G_N T^\mu_\mu (x), \quad (97) $$

where

$$ T^\mu_\mu (x) \equiv \sum_n M_n \sqrt{1 - v^2_n} \delta^3 (x - x_n(t)). \quad (98) $$

Note that the trace of the energy–momentum tensor can be considered the Lorentz-invariant density of inertia. In fact, as discussed by Dicke [17], when averaged over sufficiently long times (e.g., with respect to the atomic times), by the virial theorem [42], the spatial integral of $T^\mu_\mu$ represents the total energy of a bound system, i.e. includes the binding energy. Therefore, for microscopic systems whose components have large $v^2/c^2$ but very short periods, this definition becomes equivalent to the rest energy. On the other hand, for macroscopic systems, which have long periods but small $v^2/c^2$, the definition gets close to the mass density.\(^7\)

\(^7\) In lowest order, the source of the $s$-field is just the trace of the energy–momentum tensor. However, the effective metric (18) is not obtained from the flat-space metric through an overall conformal factor $e^{2s} \sim 1 + 2s + \cdots$. Thus, there is a basic difference with pure scalar theories of gravitation such as, for instance, Nordström gravity [43].

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6. Summary and conclusions

Several authors (see e.g. the review in [1]) have pointed out interesting analogies between what we call ‘Einstein gravity’ and the hydrodynamic limit of many condensed matter systems in flat space. Although one does not expect to reproduce exactly the same features of classical general relativity, still there is some value in exploring this type of correspondence. In fact, beyond the simple level of an analogy, there might be a deeper significance whenever the properties of the hypothetical underlying medium could be matched with those of the physical vacuum of present particle physics. In this case, the vacuum condensates (Higgs, gluon, chiral, . . .) of electroweak and strong interactions, which play a crucial role for fundamental phenomena, such as particle mass generation and quark confinement, could also represent a ‘bridge’ from particle physics to gravity.

To explore this possibility, one could start on a general ground by first representing the physical flat-space vacuum as a medium in which there is a scalar fluctuation field \( s(x) \) that, on a coarse-grained scale, acts as in equation (4). As discussed in section 1 this leads naturally to the isotropic form of the effective metric

\[
\begin{align*}
g_{\mu\nu}[s] &= \text{diag} \left( \frac{1}{\lambda^2(s)}, -\lambda^2(s), -\lambda^2(s), -\lambda^2(s) \right) \\
&\equiv \text{diag} \left( (1+2s), -(1-2s), -(1-2s), -(1-2s) \right)
\end{align*}
\]

so that, by expanding in powers of \( s(x) \) with the natural condition \( \lambda(0) = 1 \) and choosing the units so that

\[
\frac{1}{\lambda(s)} = 1 + s + O(s^2)
\]

one arrives at the weak-field line element

\[
\begin{align*}
g_{\mu\nu}[s] &= \text{diag}((1+2s), -(1-2s), -(1-2s), -(1-2s))
\end{align*}
\]

Then, by comparing with experiments in weak gravitational field, one obtains a coarse-grained identification of the \( s \)-field with the Newtonian potential, i.e.

\[
s \sim U_N = -G_N \sum_i \frac{M_i}{|r - r_i|}
\]

and it becomes natural to ask whether the ultimate origin of such a scalar field \( s \sim U_N \) could be found in the presently accepted vacuum condensates of particle physics. While \( 1/r \) long-range potentials are indeed expected among bodies placed in a Bose condensate of spinless quanta [30], one should try to understand from first principles why the magnitude of the relevant \( 1/r \) potential is so small in units of the physical coupling strength set by the Fermi constant \( G_F \). To this end, one can use (i) the basic two-valued nature of the zero-momentum connected scalar propagator in the broken-symmetry phase and (ii) the ‘triviality’ of contact, pointlike interactions in (3 + 1) dimensions. These two requirements, implying that the far-infrared, gapless region of the spectrum of the broken phase has to become of zero measure in the continuum limit of the theory, could naturally explain the origin of a hydrodynamic coupling \( G_\infty \) that is infinitesimally weak in units of \( G_F \). If this coupling is identified with the Newton constant \( G_N \), one has a toy model of gravity, i.e. restricted to a definite class of phenomena, that could be used as a first approximation to reduce the present conceptual gap with particle physics.

At present, the approach has some interesting aspects but also several limitations. The interesting features consist in a potentially simple explanation of the hierarchical pattern of

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8 As mentioned in section 1, the idea that Bose condensates can provide various types of gravitational dynamics is not new. In particular, one is not bound to a long-range \( 1/r \) behavior and can also consider the possibility of short-range analogs of the Newtonian potential [26].
scales observed in nature. By describing the scalar condensate as a true physical medium, made up of elementary constituents, one discovers that the broken phase is much richer than what is usually believed. In fact, by approaching the continuum limit of the underlying scalar quantum field theory, one could easily handle a hierarchy of scales containing an extremely small scattering length \( a \sim 10^{-33} \text{ cm} \), a typical elementary-particle scale \( 1/\sqrt{na} \sim 10^{-17} \text{ cm} \) and a macroscopic length \( 1/na^2 \sim 10^{-1} \text{ cm} \) that can be used to mark the onset of a hydrodynamic regime.

At the same time, one can easily understand basic properties of the gravitational interaction. For instance, the exact equality of both passive gravitational mass and active gravitational mass with the same inertial mass generated by the vacuum structure is straightforward in this picture but is not so obvious in a pure geometrical formalism where inertial and gravitational masses could differ by a universal, but otherwise arbitrary, proportionality constant.

Finally, as in other frameworks based on an underlying superfluid vacuum [5], the relevant curvature effects will be orders of magnitude smaller than those expected by solving Einstein’s equations with the full energy–momentum tensor as a source term. In fact, by definition, the large condensation energy of the unperturbed vacuum does not play any observable role.

On the other hand, there are also several limitations. For instance the numerical relations have still a certain degree of uncertainty and I have been unable to determine exactly the value of the infinitesimal coupling \( G_\infty \) in equation (77). Even knowing the Higgs mass parameter \( m_h \), the interpolating function in the propagator (and thus the proportionality factor \( f'(0) \)) remains unknown since, to this end, one should solve a nonlinear integral equation. Analogous problems arise if one wants to fix the precise size of the infrared momentum scale \( \delta \) in terms of the mass parameter \( m_h \) and of the cutoff \( \Lambda \). Also in this case, it is not easy to improve the simple order of magnitude estimate \( \delta \sim m_h^2/\Lambda \) that is suggested by the identification of the length scale \( 1/\delta \) with the mean free path of the elementary constituents. To determine the precise values of the various parameters, the shape of the potential in the intermediate region and check the overall consistency of the picture with the precise measurements of gravity, at and below the millimeter scale [44], a new generation of lattice calculations would be important. As discussed in [41], in fact, precise measurements on large lattices of the slope of the scalar propagator in the infrared region could be used to extrapolate the trend to both the continuum and infinite-volume limit of the theory.

Another problem concerns the structure of the neglected \( \mathcal{O}(s^2) \) effects in \( \lambda(s) \). These arise from higher-order Feynman graphs where more and more scalar quanta are exchanged between two sources and interact with each other. In principle, one could try a direct re-summation of these higher-order, infrared effects analogously to the Bloch–Nordsieck exponentiation method in QED. However, re-summing infrared effects in self-interacting theories is not so simple and it will not be easy to find the answer in this way. At the same time, non-leading \( 1/r^2 \) effects are also expected from the higher-derivative expansion around \( p = 0 \) of the propagator entering the same one-boson exchange graph.

Now, as mentioned in section 1, in an emergent-gravity approach, it is natural to consider the hydrodynamic limit of the underlying superfluid vacuum (embodied in the peculiar aspects of the Newtonian potential) as the crucial ingredient that induces forms of physical equivalence requiring a general-covariant formulation. Therefore, the structure of these higher-order terms in the effective metric should correspond to a solution of Einstein’s field equations. However, the metric structure depends on the energy–momentum tensor that, in general, might depend on \( s(x) \). To better appreciate this point, let us first consider Einstein’s field equations for the weak-field metric when \( s \sim U_N \)

\[
g_{\mu\nu}[U_N] \equiv \text{diag}((1 + 2U_N), -(1 - 2U_N), -(1 - 2U_N), -(1 - 2U_N)). \quad (103)
\]
As discussed by Synge [45], this corresponds to bodies with energy density

\[ T_{00}^{\text{matter}} \sim \Delta U_N = \sum_i M_i \delta(3)(r - r_i) \]  

(104)

embedded in a medium with energy density

\[ T_{00}^{\text{medium}} \sim (\nabla U_N)^2. \]  

(105)

As mentioned in section 1, beyond this lowest order approximation, exact solutions have been obtained for the two cases of the Schwarzschild and Yilmaz metrics. They are both in agreement with the weak-field tests but differ non-trivially in the strong-field limit. The physical reason for this difference consists in the presence of an energy density \( E \) associated with the scalar field itself, in particular in the voids among the elementary constituents that exist inside massive bodies whereas the Schwarzschild metric corresponds to the limit \( E = 0 \).

Thus, it is understandable that, with different forms of such energy density, there could be an effective screening of the Schwarzschild singularity expected for a single pointlike mass.

This could have non-trivial phenomenological implications for astrophysics and even for cosmology. For instance, in the case of the Yilmaz metric, for a given equation of state for dense matter, stable stellar objects of larger mass might be allowed [46]. Analogously, since in the Yilmaz metric there is no limit to the gravitational redshift of light emitted by dense matter, one could find alternative explanations for the controversial huge quasar redshifts, a large part of which could be interpreted as being of gravitational (rather than cosmological) origin [48].

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