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On exact solutions of the Dirac equation in a homogeneous magnetic field in the Riemann spherical space

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There are constructed exact solutions of the quantum-mechanical Dirac equation for a spin S=1/2 particle in the space of constant positive curvature, spherical Riemann space, in presence of an external magnetic field, analogue of the homogeneous magnetic field in the Minkowski space. A generalized formula for energy levels, describing quantization of the motion of the particle in magnetic field on the background of the Riemann space geometry, is obtained.

1. Introduction

The quantization of a quantum-mechanical particle in the homogeneous magnetic field belongs to classical problems in physics [1, 2, 3]. In [4, 5, 6], exact solutions for a scalar particle in extended problem, particle in external magnetic field on the background of Lobachevsky $H_3$ and Riemann $S_3$ spatial geometries were found. A corresponding system in the frames of classical mechanics was examined in [7, 8, 9]. In the present paper, we consider a similar problem for a particle with spin 1/2 described by Dirac equation in Lobachevsky space in presence of the external magnetic field.

2. Cylindric coordinates ant the Dirac equation in spherical space $H_3 S_3$

In the spherical Riemann space $S_3$, let us use an extended cylindric coordinates

$$
\left\{ \begin{array}{c}
\rho \\
\phi \\
\psi \\
r \\
\phi \\
\end{array} \right. =
\left\{ \begin{array}{c}
-r \\
\phi \\
\psi \\
\phi \\
\phi \\
\end{array} \right.
$$

the curvature radius $\rho$ is taken as a unit of the length. An analogue of usual homogeneous magnetic field is defined as [1, 5, 6]

$$
A_\phi = -2B \sin^2 \frac{r}{2} = B (\cos r - 1).
$$

To coordinates (1) there corresponds the tetrad

$$
e^a_\beta(x) =
\left\{ \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \cos^{-1} z & 0 & 0 \\
0 & 0 & \cos^{-1} z \sin^{-1} r & 0 \\
0 & 0 & 0 & 1 \\
\end{array} \right.
$$

Christoffel symbols $\Gamma^r_{jk}$ and Ricci rotation coefficients $\gamma_{abc}$ are

$$
\Gamma^r_{jk} =
\left\{ \begin{array}{ccc}
0 & 0 & -\tan z \\
0 & -\sin r \cos r & 0 \\
-\tan z & 0 & 0 \\
\end{array} \right.
$$

$$
\Gamma^\phi_{jk} =
\left\{ \begin{array}{ccc}
0 & \cot g r & 0 \\
\cot g r & 0 & -\tan z \\
0 & -\tan z & 0 \\
\end{array} \right.
$$
A general covariant Dirac equation (for more detail see [10]) takes the form

\[
\left[ i\gamma^0 \partial_t + \frac{i\gamma^1}{\cos z} \left( \partial_r + \frac{1}{2 \tan r} \right) + \gamma^2 \frac{i\partial_\phi - eB(\cos r - 1)}{\cos z \sin r} + i\gamma^3 (\partial_z - \tan z) - M \right] \Psi = 0 .
\]  

(4)

With the substitution \( \Psi = \varphi / (\sqrt{\sin r} \cos) \) eq. (4) becomes simpler

\[
\left[ i\gamma^1 \frac{\partial}{\partial r} + \gamma^2 \frac{i\partial_\phi - eB(\cos r - 1)}{\sin r} + \cos z \left( i\gamma^0 \frac{\partial}{\partial t} + i\gamma^3 \frac{\partial}{\partial z} - M \right) \right] \varphi = 0 .
\]  

(5)

Solutions of this equation will be searched in the form

\[
\varphi = e^{-i\epsilon t} e^{i m \phi} \left| \begin{array}{c} f_1(r, z) \\ f_2(r, z) \\ f_3(r, z) \\ f_4(r, z) \end{array} \right| ;
\]

so that

\[
\left[ i\gamma^1 \frac{\partial}{\partial r} - \mu(r) \gamma^2 + \cos z \left( e\gamma^0 + i\gamma^3 \frac{\partial}{\partial z} - M \right) \right] \left| \begin{array}{c} f_1(r, z) \\ f_2(r, z) \\ f_3(r, z) \\ f_4(r, z) \end{array} \right| = 0 ,
\]  

(6)

where \( \mu(r) = [m - eB(\cosh r - 1)] / \sinh r \). Taking the Dirac matrices in spinor basis, we get radial equations for \( f_a(t, z) \)

\[
\left( \frac{\partial}{\partial r} + \mu \right) f_4 + \cos z \left( \frac{\partial}{\partial z} f_3 + i \cos z \left( e f_3 - M f_1 \right) \right) = 0 ,
\]

\[
\left( \frac{\partial}{\partial r} - \mu \right) f_3 - \cos z \left( \frac{\partial}{\partial z} f_4 + i \cos z \left( e f_1 - M f_2 \right) \right) = 0 ,
\]

\[
\left( \frac{\partial}{\partial r} + \mu \right) f_2 + \cos z \left( \frac{\partial}{\partial z} f_1 - i \cos z \left( e f_1 - M f_3 \right) \right) = 0 ,
\]

\[
\left( \frac{\partial}{\partial r} - \mu \right) f_1 - \cos z \left( \frac{\partial}{\partial z} f_2 - i \cos z \left( e f_2 - M f_4 \right) \right) = 0 .
\]  

(7)

With linear restriction \( f_3 = Af_1, f_4 = Af_2 \), where

\[
\epsilon - \frac{M}{A} = -\epsilon + MA \quad \implies \quad A = A_{1,2} = \frac{\epsilon \pm p}{M} , \quad (p = +\sqrt{\epsilon^2 - M^2})
\]  

(8)

eqs. (7) give

\[
\left( \frac{\partial}{\partial r} + \mu \right) f_2 + \cos z \left( \frac{\partial}{\partial z} f_1 + i \cos z \left( -\epsilon + MA \right) \right) f_1 = 0 ,
\]

\[
\left( \frac{\partial}{\partial r} - \mu \right) f_1 - \cos z \left( \frac{\partial}{\partial z} f_2 + i \cos z \left( -\epsilon + MA \right) \right) f_2 = 0 .
\]  

(9)
Thus, we have two possibilities

\[ A = (\epsilon + p)/M, \]

\[ (\partial/\partial r + \mu) f_2 + \cos z (\partial/\partial z + i p) f_1 = 0, \quad (\partial/\partial r - \mu) f_1 - \cos z (\partial/\partial z - i p) f_2 = 0; \quad (10) \]

\[ A = (\epsilon - p)/M, \]

\[ (\partial/\partial r + \mu) f_2 + \cos z (\partial/\partial z - i p) f_1 = 0, \quad (\partial/\partial r - \mu) f_1 - \cos z (\partial/\partial z + i p) f_2 = 0. \quad (11) \]

For definiteness, let us consider the system (10) (transition to the case (11) is performed by the formal change \( p \rightarrow -p \)). Let us search solutions in the form

\[ f_1 = Z_1(z) R_1(r), \quad f_2 = Z_2(z) R_2(r). \]

Introducing the separating constant \( \lambda \)

\[ \cos z (\frac{d}{dz} + i p) Z_1 = \lambda Z_2, \quad \cos z (\frac{d}{dz} - i p) Z_2 = \lambda Z_1, \quad (12) \]

we get the radial system

\[ (\partial/\partial r + \mu) R_2 + \lambda R_1 = 0, \quad (\partial/\partial r - \mu) R_1 - \lambda R_2 = 0. \quad (13) \]

### 3. Solution of the equation in \( z \)-variable

From (12) it follows the second-order differential equation for \( Z_1(z) \)

\[ \left( \frac{d^2}{dz^2} - \frac{\sin z}{\cos z} \frac{d}{dz} + \frac{p^2 - i p \sin z}{\cos^2 z} - \frac{\lambda^2}{\cos^2 z} \right) Z_1 = 0. \quad (14) \]

In a new variable \( y = (1 + i \tan z)/2 \), eq. (14) gives

\[ \left[ 4y(1-y) \frac{d^2}{dy^2} + 2(1-2y) \frac{d}{dy} - p^2 \left( \frac{1}{1-y} + \frac{1}{y} \right) + p \left( \frac{1}{1-y} - \frac{1}{y} \right) + 4\lambda^2 \right] Z_1 = 0. \quad (15) \]

With substitution \( Z_1 = y^A(1-y)^C Z(y) \) we arrive at the equation

\[ 4y(1-y) \frac{d^2 Z}{dz^2} + 4 \left[ 2A + \frac{1}{2} - (2A + 2C + 1) y \right] \frac{dZ}{dz} + \]

\[ + \left[ \frac{2A(2A-1) - p(p+1)}{y} + \frac{2C(2C-1) - p(p-1)}{1-y} - 4(A+C)^2 + 4\lambda^2 \right] Z = 0. \]

Requiring

\[ A = -\frac{p}{2}, \quad p + \frac{1}{2}, \quad C = \frac{p}{2}, \quad \frac{1-p}{2}, \quad (16) \]

for \( Z_1 \) we get an equation of hypergeometric type

\[ y(1-y) \frac{d^2 Z}{dz^2} + \left[ 2A + \frac{1}{2} - (2A + 2C + 1) y \right] \frac{dZ}{dz} - [(A+C)^2 - \lambda^2] Z = 0, \]
where

\[\alpha = \lambda + A + C, \quad \beta = -\lambda + A + C, \quad \gamma = 2A + \frac{1}{2},\]

\[Z_1 = \left(\frac{e^{iz}}{\cos z}\right)^A \left(\frac{e^{-iz}}{\cos z}\right)^C F(\alpha, \beta, \gamma; \frac{e^{iz}}{2\cos z}).\]  \hspace{1cm} (17)

There arise four possibility depending on the choice of \(A\) and \(C\). For definiteness let us suppose that \(\lambda > 0\) which does not affect generality of consideration:

Variant 1, \(A = \frac{p+1}{2}, \quad C = \frac{1-p}{2}, \quad A + C = 1, \quad A - C = p,\)

\[\beta = -N, \quad \Rightarrow \quad \text{spectrum} \quad \lambda = 1 + N, \quad N = 0, 1, 2, ...\]

\[\alpha = \lambda + 1, \quad \gamma = p + \frac{3}{2}, \quad Z_1 = \frac{e^{ipz}}{\cos z} F(\lambda + 1, -N, p + \frac{3}{2}; \frac{e^{iz}}{2\cos z}).\]  \hspace{1cm} (18)

Variant 2, \(A = -\frac{p}{2}, \quad C = \frac{p}{2}, \quad A + C = 0, \quad A - C = -p,\)

\[\beta = -N, \quad \Rightarrow \quad \text{spectrum} \quad \lambda = N, \quad N = 0, 1, 2, ...\]

\[\alpha = +N, \quad \gamma = -p + \frac{1}{2}, \quad Z_1 = e^{-ipz} F(\lambda, -N, -p + \frac{1}{2}; \frac{e^{iz}}{2\cos z}).\]  \hspace{1cm} (19)

In these two cases, we do not have found quantization for \(p\), instead we have some quantization rules for \(\lambda\). However, quite different quantization rules (as expected) will follow from consideration of the differential equation in \(r\)-variable. By this reason, these cases 1 and 2 will not be considered in the following.

Variant 3, \(A = \frac{p+1}{2}, \quad C = \frac{p+1}{2}, \quad A + C = p + 1/2, \quad A - C = 1/2,\)

\[\beta = -N \quad \Rightarrow \quad \text{spectrum} \quad p = +\sqrt{\epsilon^2 - M^2} = \lambda - (N + 1/2), \quad N = 0, 1, 2, ..., N_{\text{max}}\]

\[\alpha = \lambda + p + \frac{1}{2}, \quad \gamma = p + \frac{3}{2}, \quad Z_1 = \frac{e^{iz/2}}{(\cos z)^{p+1/2}} F(\lambda + p + \frac{1}{2}, -N, p + \frac{3}{2}; \frac{e^{iz}}{2\cos z}).\]  \hspace{1cm} (20)

Variant 4, \(A = -\frac{p}{2}, \quad C = \frac{1-p}{2}, \quad A + C = -p + 1/2, \quad A - C = -1/2,\)

\[\alpha = -N, \quad \Rightarrow \quad \text{spectrum} \quad p = \sqrt{\epsilon^2 - M^2} = \lambda + (N + 1/2),\]

\[\beta = -\lambda - p + \frac{1}{2}, \quad \gamma = -p + \frac{1}{2}, \quad Z_1 = \frac{e^{-iz/2}}{(\cos z)^{-p+1/2}} F(\lambda - p + \frac{1}{2}, -N, -p + \frac{1}{2}; \frac{e^{iz}}{2\cos z}).\]  \hspace{1cm} (21)

To the cases 3 and 4 there correspond different energy spectra:

\[3, \quad p = +\sqrt{\epsilon^2 - m^2} = \lambda - (N + 1/2);\]

\[4, \quad p = +\sqrt{\epsilon^2 - m^2} = \lambda + (N + 1/2).\]  \hspace{1cm} (22)
4. Solution of the equations in \( r \)-variable

From eqs. (13) it follows a second-order differential equation for \( R_1 \) (for brevity let \( eB \) be noted as \( B \))

\[
\frac{d^2 R_1}{dr^2} + \left[ \frac{m \cos r - B (\cos r - 1)}{\sin^2 r} - \frac{[m + B (\cos r - 1)]^2}{\sin^2 r} \right] R_1 = 0 .
\]

(23)

With a new variable \( y = (1 + \cos r)/2 \), eq. (24) reads

\[
y(1 - y)\frac{d^2 R_1}{dy^2} + \left( \frac{1}{2} - y \right) \frac{d R_1}{dy} - \left[ -\lambda^2 + \frac{m^2}{4} \left( \frac{1}{y} + \frac{1}{1 - y} \right) + \frac{m}{4} \left( \frac{1}{y} - \frac{1}{1 - y} \right) - \frac{mB}{y} - B^2 \left( 1 - \frac{1}{y} \right) - \frac{B}{2y} \right] R_1 = 0 .
\]

(24)

With the substitution \( R_1 = y^A(1 - y)^C R(y) \), we get

\[
y(1 - y)\frac{d^2 R}{dy^2} + \left[ 2A + \frac{1}{2} - (2A + 2C + 1) y \right] \frac{d R}{dy} + \left[ \frac{A^2 - A/2 - m^2/4 - m/4 + mB - B^2 + B/2}{y} \right.
\]

\[
+ \left. \frac{C^2 - C/2 - m^2/4 + m/4}{1 - y} - (A + C)^2 + \lambda^2 + B^2 \right] R = 0 .
\]

Requiring

\[
A = \frac{2B - m}{2}, \quad -2B + m + 1 \quad \frac{m}{2}, \quad C = \frac{m}{2}, \quad 1 - m \quad \frac{2}{2},
\]

we arrive at a differential equation of hypergeometric type

\[
y(1 - y)\frac{d^2 R}{dy^2} + \left[ 2A + \frac{1}{2} - (2A + 2C + 1) y \right] \frac{d R}{dy} - \left[ (A + C)^2 - \lambda^2 - B^2 \right] R = 0 .
\]

(26)

where

\[
\alpha = A + C - \sqrt{B^2 + \lambda^2}, \quad \beta = A + C + \sqrt{B^2 + \lambda^2}, \quad \gamma = 2A + \frac{1}{2},
\]

\[
R_1 = (1 + \cos r)^A \left( 1 - \cos r \right)^C F(\alpha, \beta, \gamma; \frac{1 + \cos r}{2}) .
\]

(27)

In order to have a finite solution at the origin \( r = 0 \) (that corresponds to the half-curve \( u_0 = + \cos z, u_3 = \sin z, u_1 = 0, u_2 = 0 \)) and at \( r = \pi \) (that corresponds to the other part of the curve, \( u_0 = - \cos z, u_3 = \sin z, u_1 = 0, u_2 = 0 \)), we must take positive values for \( A \) and \( C \):

\[
R_1 = (1 + \cos r)^A \left( 1 - \cos r \right)^C F(\alpha, \beta, \gamma; \frac{1 + \cos r}{2}), \quad A > 0, \quad C > 0 .
\]

(28)

Let us consider all the four variants (for definiteness let us assume that \( B > 0 \ )):

Variants 1,

\[
C = \frac{m}{2} > 0, \quad A = \frac{2B - m}{2} > 0, \quad C + A = +B .
\]

Variants 2,

\[
C = \frac{m}{2} > 0, \quad A = \frac{m + 1 - 2B}{2} > 0, \quad C + A = -B + m + \frac{1}{2} .
\]

Variants 3,

\[
C = \frac{1 - m}{2} > 0, \quad A = \frac{m + 1 - 2B}{2} > 0, \quad C + A = 1 - B > 0 .
\]

Variants 4,

\[
C = \frac{1 - m}{2} > 0, \quad A = \frac{2B - m}{2} > 0, \quad C + A = B - m + \frac{1}{2} .
\]

(29)
To these there correspond the following solutions:

**Variant 1**, \(0 < m < 2B\),

\[
R_1 = (1 + \cos r)^{(2B-m)/2} (1 - \cos r)^{m/2} F(\alpha, \beta, \gamma; \frac{1 + \cos r}{2}),
\]

\[
\alpha = -n, \quad \beta = 2B + n, \quad \gamma = 2B - m - 1,
\]

spectrum \(\lambda^2 = 2Bn + n^2, \quad n = 0, 1, 2, \ldots\) \hspace{1cm} (30)

**Variant 2**, \(m > 0, \ m > 2B - 1\),

\[
R_1 = (1 + \cos r)^{(m+1-2B)/2} (1 - \cos r)^{m/2} F(\alpha, \beta, \gamma; \frac{1 + \cos r}{2}),
\]

\[
\alpha = -n, \quad n + m + 1/2 - B = \sqrt{B^2 + \lambda^2}, \quad \beta = n + 2m + 1 - 2B, \quad \gamma = -2B + m + 3/2.
\]

\[
\alpha = -n \quad \implies \quad \sqrt{B^2 + \lambda^2} = B + (m + n + \frac{1}{2}),
\]

spectrum \(\lambda^2 = -2B (m + n + \frac{1}{2}) + (n + m + \frac{1}{2})^2, \quad \lambda^2 > 0 \quad \implies \quad n + m + \frac{1}{2} > 2B\). \hspace{1cm} (31)

**Variant 3**, \(m < 1, \ m > 2B - 1, \ 0 < B < 1\),

\[
R_1 = (1 + \cos r)^{(m+1-2B)/2} (1 - \cos r)^{(m-1)/2} F(\alpha, \beta, \gamma; \frac{1 + \cos r}{2});
\]

\[
\alpha = -n, \quad 1 - B + n = \sqrt{B^2 + \lambda^2}, \quad \beta = 2 - 2B + n, \quad \gamma = -2B + m + 3/2.
\]

spectrum \(\lambda^2 = -2B (n + 1) + (n + 1)^2, \quad n + 1 > 2B\). \hspace{1cm} (32)

**Variant 4**, \(m < 1, \ m < 2B, \ m < B + 1/2\)

\[
R_1 = (1 + \cos r)^{(2B-m)/2} (1 - \cos r)^{(1-m)/2} F(\alpha, \beta, \gamma; \frac{1 + \cos r}{2}),
\]

\[
\alpha = -n, \quad B + n - m + \frac{1}{2} = \sqrt{B^2 + \lambda^2}, \quad \beta = 2B - 2m + 1 + n, \quad \gamma = 2B - m + \frac{1}{2}.
\]

spectrum \(\lambda^2 = 2B (n - m + \frac{1}{2}) + (n - m + \frac{1}{2})^2\). \hspace{1cm} (33)

In the end there should be given two clarifying additions. In fact, the above used relationship

\[-i\partial_\phi \Psi = m \Psi\] represents transformed from cartesian coordinates to cylindrical an eigen-value equation for the third projection of the the total angular momentum of the Dirac particle

\[
\hat{J}_3 \Psi_{\text{Cart}} = (-i \frac{\partial}{\partial \phi} + \Sigma_3) \Psi_{\text{Cart}} = m \Psi = m \Psi_{\text{Cart}}; \hspace{1cm} (34)
\]
this means that for the quantum number $m$ are permitted half-integer values $m = \pm \frac{1}{2}, \pm \frac{3}{2}, ...$.

When using ordinary units (in the system SI), we should change the symbol $B$ into $eB\rho^2/h$. Therefore, in the limit of vanishing curvature $\rho \to +\infty$, from the four noted classes of solutions only two of them remain:

1. $\rho \to +\infty \quad m = +1/2, +3/2, ...$; 
2. $\rho \to +\infty \quad m = -1/2, -3/2, ...$.

they coincide with known results for the energy spectrum for particle with spin 1/2 in Minkowski space.

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