Velocity distribution function and effective constant restitution coefficient for granular gas of viscoelastic particles

Awadhesh Kumar Dubey, Anna Bodrova, Sanjay Puri, and Nikolai Brilliantov

1School of Physical Sciences, Jawaharlal Nehru University, New Delhi, 110067, India.
2Faculty of Physics, M.V. Lomonosov Moscow State University, Moscow, 119991, Russia
3Department of Mathematics, University of Leicester, Leicester LE1 7RH, United Kingdom

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We perform large-scale event-driven Molecular dynamics (MD) simulations for granular gases of particles interacting with the impact-velocity dependent restitution coefficient $\varepsilon(v_{\text{imp}})$. We use $\varepsilon(v_{\text{imp}})$ as it follows from the simplest first-principle collision model of viscoelastic spheres. Both cases of force-free and uniformly heated gases are studied. We formulate a simplified model of an effective constant restitution coefficient $\varepsilon_{\text{eff}}$, which depends on a current granular temperature and compute $\varepsilon_{\text{eff}}$, using the kinetic theory. We develop a theory of the velocity distribution function for driven gases of viscoelastic particles and analyze evolution of granular temperature and of the Sonine coefficients, which characterize the form of the velocity distribution function. We observe that for not large dissipation the simulation results are in an excellent agreement with the theory for both, homogeneous cooling state and uniformly heated gases. At the same time a noticeable discrepancy between the theory and MD results for the Sonine coefficients is detected for large dissipation. We analyze the accuracy of the simplified model, based on the effective restitution coefficient $\varepsilon_{\text{eff}}$ and conclude that this model can accurately describe granular temperature. It provides also an acceptable accuracy for the velocity distribution function for small dissipation, but fails when dissipation is large.

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I. INTRODUCTION

Granular materials are abundant in nature; possible examples range from sands and powders on Earth to more dilute systems, termed as granular gases, in space. Astrophysical objects, like planetary rings or dust clouds are typical examples of granular gases. Depending on the average kinetic energy of individual grains, granular systems can be in a solid, liquid or gaseous state. In the present study we address a low-density granular gas – a system comprised of macroscopically large number of macroscopic grains that moveballistically between inelastic pair-wise collisions. The inelastic nature of inter-particle collisions is the main feature which distinguishes a granular gas from a molecular gas and gives rise to many astonishing properties of dissipative gases. If no external forces act on the system it evolves freely and permanently cools down. During the first stage of a granular gas evolution its density remains uniform and no macroscopic fluxes present. This state, which is called a homogeneous cooling state (HCS), is, however, unstable with respect to fluctuations and in later times clusters and vortices develop. The system then continue to lose its energy in the inhomogeneous cooling state (ICS).

If energy is injected into a granular gas to compensate its losses in dissipative collisions, the system settles into a nonequilibrium steady-state. There are different ways to put energy into the system: via shearing, vibration or rotating of the walls of a container, applying electrostatic or magnetic forces, etc. In planetary rings the energy losses due to particles’ collisions are replenished by gravitational interactions. To describe theoretically the injection of energy into a system a few types of thermostat, which mimics an action of external forces, have been proposed. Here we exploit a white-noise thermostat, where all particles are heated uniformly and independently (see the next sections for more detail).

Energy dissipation in a pair-wise collision is quantified by the restitution coefficient $\varepsilon$, $\varepsilon = \left| \frac{v_{12} \cdot e}{(v_{12} \cdot e)} \right|$, where $v_{12} = v_2' - v_1'$ and $v_{12} = v_2 - v_1$ are the relative velocities of two particles after and before a collision, and $e$ is a unit vector connecting their centers at the collision instant. The post-collision velocities are related to the pre-collision velocities $v_1$ and $v_2$ as follows, e.g.: $v_{1/2}' = v_{1/2} \mp \frac{1 + \varepsilon}{2} (v_{12} \cdot e) e$.

To date, most studies of the HCS and ICS have focused on the case of a constant restitution coefficient. This assumption contradicts, however, experimental observations, along with basic mechanical laws which indicate that $\varepsilon$ does depend on the impact velocity. This dependence may be obtained by solving the equations of motion for colliding particles with the explicit account for the dissipative forces acting between the grains. The simplest first-principle model of inelastic collisions assumes viscoelastic properties of particles’ material, which results in viscoelastic inter-particle...
force\[37\] and finally in the restitution coefficient\[36,39\]:
\[
\varepsilon = 1 - C_1\kappa^{2/5}\sqrt{v_{12}\cdot\epsilon} + C_2\kappa^{4/5}\left|v_{12}\cdot\epsilon\right|^{2/5} + \ldots .
\]
(3)
Here, the numerical coefficients are \(C_1 \simeq 1.15\) and \(C_2 \simeq 0.798\). The elastic constant
\[
\kappa = \left(\frac{3}{2}\right)^{3/2} \frac{Y\sqrt{\sigma}}{m(1 - \nu^2)}
\]
(4)
is a function of the Young’s modulus \(Y\), Poisson ratio \(\nu\), mass \(m\) and diameter \(\sigma\) of the particles; the constant \(A\) quantifies the viscous properties of the particles’ material. While the last equation is valid for not very large inelasticity, in the recent work of Schwager and Poschel\[40\] an expression for \(\varepsilon\), which is valid for high inelasticity and accounts for the delayed recovery of the shape of colliding particles has been derived:
\[
\varepsilon = 1 + \sum_{k=1}^{N_x} \frac{h_k}{2} \frac{\delta^{k/2}}{2^{k/20}} \left|\left|c_{12} \cdot \epsilon\right|\right|^{k/10},
\]
(5)
where \(h_1 = 0\), \(h_2 = - C_1\), \(h_3 = 0\), \(h_4 = C_2\), and other numerical coefficients up to \(N_x = 20\) are given in Ref.\[40\]. In the latter equation \(u(t) = T(t)/T_0\) (\(T_0 = T(0)\)), with the usual definition of temperature \(T(t)\),
\[
\frac{3}{2} nT(t) = \int dv \frac{mv^2}{2} f(v, t).
\]
(6)
Here \(n\) is the number density of the gas, \(v\) is a particle velocity, \(f(v, t)\) – the time-dependent velocity distribution function and \(c_{12}\) is the dimensionless relative velocity, defined as \(c_{12} = v_{12}/v_T\), where \(v_T(t) = \sqrt{2T(t)/m}\) is the thermal velocity. The dissipation constant \(\delta\) in Eq.\[3\] reads,
\[
\delta = A\kappa^{2/5} \left(\frac{T_0}{m}\right)^{1/10},
\]
(7)
so that the first few terms in Eq.\[3\] are identical to those in Eq.\[3\]. Note that \(\delta = 0\) corresponds to an elastic system, \(\varepsilon = 1\) and dissipation in the system increases with increasing \(\delta\). Furthermore, the restitution coefficient for all collisions increases and tends to unity, \(\varepsilon \to 1\), when the (reduced) temperature decreases, that is, when a granular gas cools down.

Note that although we address here dry granular particles, they still can stick in collisions with a very small impact velocity. Namely, if the normal component of the relative velocity \(|v_{12}\cdot\epsilon|\) is smaller than the sticking threshold,
\[
g_{st} = \sqrt{\frac{4q_0}{m\varepsilon^2} \left(\frac{\pi^2\gamma^2D^2}{2}\right)^{1/3} \left(\frac{\sigma}{4}\right)^{4/3}},
\]
where \(D = 3(1 - \nu^2)/2Y\), \(q_0 \simeq 1.45\) and \(\gamma\) is the adhesion coefficient, the colliding particles cling together\[42,43\]; in this case the restitution coefficient drops to zero. In the present study we neglect these processes.

The velocity distribution function in granular gases usually deviates from a Maxwellian distribution\[18, 21, 44, 45\] and may be described by the Sonine-polynomial expansion\[8\]:
\[
f(v, t) = \frac{n}{v_T^3} f(c, t),
\]
\[
f(c, t) = \pi^{-3/2} \exp\left(-c^2\right) \left[1 + \sum_{p=1}^{\infty} a_p(t) S_p(c^2)\right]
\]
(8)
where \(c = v/v_T\) and the Sonine polynomials read for \(d = 3\):
\[
S_p(c^2) = \sum_{k=0}^{p} \frac{(-1)^k (1/2 + p)!}{(1/2 + k)!(p - k)!k!} c^{2k}.
\]
(9)
Hence evolution of \(f(v, t)\) is completely determined by the time dependence of the Sonine coefficients \(a_p(t)\). The first Sonine coefficient is trivial, \(a_1 = 0\), as it follows from the definition of temperature, e.g.\[8\]; in the elastic limit, that is, for a Maxwellian distribution, all Sonine coefficients are zero, \(a_p = 0\). The Sonine coefficients characterize the successive moments \(\langle c^{2k}\rangle\) of the velocity distribution function\[8\], so that the first few moments read,
\[
\langle c^2 \rangle = \frac{3}{2},
\]
\[
\langle c^4 \rangle = \frac{15}{4} (1 + a_2),
\]
\[
\langle c^6 \rangle = \frac{105}{8} (1 + 3a_2 - a_3).
\]

It has been shown recently that for a granular gas with a constant \(\varepsilon\) the expansion\[8\] converges for small and moderate dissipation, up to \(\varepsilon \gtrsim 0.6\), but breaks down otherwise\[46\]. For a moderate dissipation it is sufficient to consider only the first two non-trivial coefficients \(a_2\) and \(a_3\), which have been studied both analytically\[21, 44, 44, 47\] and by means of computer simulations\[15\].

The theory developed for a granular gas with a constant \(\varepsilon\) in a homogeneous cooling state predicts a rapid relaxation of \(a_p\) to a steady state values, which depend on \(\varepsilon\)\[21, 40\]. After a certain time, spatial structures, like vortices and clusters, spontaneously arise in such a granular gas\[12, 19\]. The velocity distribution function becomes effectively Gaussian, which happens due to an averaging over a large number of independent clusters, where particles’ velocities are approximately parallel; \(a_p\), however, show very strong fluctuations around zero mean values\[18, 19\]. In a granular gas of viscoelastic particles clusters and vortices appear only as transient phenomena\[50\]; moreover cluster formation is completely suppressed in a granular gas, orbiting a massive body in its gravitational potential – a typical example of such systems is a Kepler-disc in astrophysics\[29\].
In contrast to the case of a constant $\varepsilon$, evolution of the velocity distribution function in a gas of viscoelastic particles demonstrates a complicated non-monotonous time dependence \cite{51, 52}. Typically, the magnitudes of the Sonine coefficients first increase with time, reach maximal values, and finally tend to zero, so that asymptotically the Maxwellian distribution is recovered \cite{51, 52}. This simply follows from the fact that the collisions become more and more elastic as temperature decreases. Although the time-dependence of $a_2(t)$ and $a_3(t)$ in a force-free granular gas of viscoelastic particles has been studied theoretically in \cite{52}, the respective analysis for heated gases is still lacking. Moreover, evolution of the velocity distribution function and its properties in such gases have been never studied numerically, by means of MD, neither for force-free nor for driven systems. Hence, it is still not known, how accurate is the theory for the velocity distribution function in this case and what is the range of convergence of the Sonine series with respect to dissipation parameter $\delta$.

Another interesting question refers to the possibility of a simplified description of a granular gas of viscoelastic particles, using an effective constant restitution coefficient: While it is known that the model of a constant $\varepsilon$ fails qualitatively to describe a complicated evolution of such gas in a homogeneous cooling state \cite{50}, it is still not clear, how accurate could be the respective description for the case of a driven system. Also, it would be interesting to check, whether a model of "quasi-constant" restitution coefficient, where $\varepsilon$ does not depend on an impact velocity, but depends on current temperature, may be used for a force-free gas.

In the present study we perform thorough large-scale event-driven MD simulations of the velocity distribution function in a gas of viscoelastic particles, both in a homogeneous cooling state and under a uniform heating. We develop a theory for $f(v, t)$ for driven gases and observe that our MD results for $a_2(t)$ and $a_3(t)$ are in an excellent agreement with theoretical predictions of \cite{52} for the HCS and of the novel theory for driven systems. We report simulation results for the high-order Sonine coefficients $a_4$ and $a_5$, which may be hardly obtained from the kinetic theory. The analysis of these coefficients allows to assess the convergence of the Sonine series. We also analyze the model of "quasi-constant" restitution coefficient $\varepsilon_{\text{eff}}$ and show that it can be used for an accurate description of a granular gas if dissipation is not large.

The paper is organized as follows. In Sec. \textbf{III} we develop a theory for the velocity distribution function for uniformly heated gases, that is, we compute time dependent Sonine coefficients $a_2(t)$ and $a_3(t)$. In Sec. \textbf{IV} we derive an effective "quasi-constant" restitution coefficient $\varepsilon_{\text{eff}}$. In Sec. \textbf{V} we report our MD results for a force-free and heated gas of viscoelastic particles. In Sec. \textbf{VI} the theoretical predictions are compared with simulation results and the accuracy of the theory, based on $\varepsilon_{\text{eff}}$, is scrutinized. Finally, in Sec. \textbf{VII} we summarize our findings.

### II. VELOCITY DISTRIBUTION FUNCTION FOR UNIFORMLY HEATED GAS

Assuming the molecular chaos, which is an adequate hypothesis for dilute granular gases addressed here, e.g. \cite{53}, we can write the Boltzmann-Enskog (BE) equation, supplemented by the diffusive Fokker-Planck term, that mimics a uniform heating \cite{21}:

$$ \frac{\partial f(v, t)}{\partial t} = g_2(\sigma) I(f, f) + \frac{\varepsilon_0}{2} \frac{\partial^2}{\partial v^2} f(v, t). $$

(11)

Here $I(f, f)$ is the collision integral and $g_2(\sigma)$ is contact value of the pair correlation function that takes into account the excluded volume effects \cite{53}. For the scaling distribution function $\tilde{f}(c, t)$ [cf. Eq. \cite{53}], the above equation reads

$$ \frac{\partial \tilde{f}}{\partial t} - \frac{1}{v_T} \frac{d v_T}{d t} \left( 3 \tilde{f} + \frac{\partial \tilde{f}}{\partial c} \right) = g_2 \sigma^2 n v_T \tilde{I} + \frac{1}{v_T} \frac{\varepsilon_0^2}{2} \frac{\partial^2}{\partial c^2} \tilde{f}, $$

(12)

where $\tilde{I}(\tilde{f}, \tilde{f}) = \sigma^{-2} v_T^2 I(f, f)$ is the dimensionless collision integral, defined as

$$ \tilde{I}(\tilde{f}, \tilde{f}) = \int dc_2 \int dc \Theta(-c_1 \cdot e) | -c_1 \cdot e | \chi \tilde{f}(c_1''', t) \tilde{f}(c_2''', t) - \tilde{f}(c_1, t) \tilde{f}(c_2, t) $$

with $c_1'''$ and $c_2'''$ being the (reduced) pre-collision velocities in the so-called inverse collision, resulting in the post-collision velocities $c_1$ and $c_2$ \cite{50}. The Heaviside function $\Theta(-c_1 \cdot e)$ selects the approaching particles and the factor $\chi$ equals the product of the Jacobian of the transformation $(c_1''', c_2''') \rightarrow (c_1, c_2)$ and the ratio of the lengths of the collision cylinders of the inverse and the direct collisions \cite{53}:

$$ \chi = \frac{|c_1'''|}{|c_1|} \frac{D(c_1''', c_2''')}{|c_1||D(c_1, c_2)|}. $$

(13)

Now we substitute the Sonine-polynomial expansion for $\tilde{f}(c, t)$ [Eq. \cite{53}] into Eq. \cite{12} and neglect all terms with $p > 3$. Multiplying the resulting equation with $c_1^2$, $c_1^4$ and $c_1^6$, and integrating over $c_1$, we obtain

$$ \frac{\partial(c_1^2)}{\partial t} + \frac{1}{T} \frac{d T}{d t} (c_1^2) = -\sqrt{\frac{2 T}{m}} g_2(\sigma) \sigma^2 n \mu_2 + \frac{3 m \varepsilon_0^2}{2 T}, $$

(15)

$$ \frac{\partial(c_1^4)}{\partial t} + \frac{2}{T} \frac{d T}{d t} (c_1^4) = -\sqrt{\frac{2 T}{m}} g_2(\sigma) \sigma^2 n \mu_4 + 10(c_2^2 \varepsilon_0)^2 \frac{m \varepsilon_0^2}{2 T}, $$

$$ \frac{\partial(c_1^6)}{\partial t} + \frac{3}{T} \frac{d T}{d t} (c_1^6) = -\sqrt{\frac{2 T}{m}} g_2(\sigma) \sigma^2 n \mu_6 + 24(c_1^2 c_2^2 \varepsilon_0^2) \frac{m \varepsilon_0^2}{2 T}. $$

Here,

$$ \mu_p = -\int dc_1^p \tilde{I}(\tilde{f}, \tilde{f}) $$

(16)
is the \( p \)-th moment of the dimensionless collision integral. The moments \( \mu_p \) can be calculated analytically up to \( O(\delta^{10}) \) using a formula manipulation program as explained in detail in Ref. 8. They can be written as follows:

\[
\mu_p = \sum_{k=0}^{20} \left( M_k^{(p,0)} + M_k^{(p,2)} a_2 + M_k^{(p,3)} a_3 + M_k^{(p,22)} a_2^2 + M_k^{(p,33)} a_3^2 + M_k^{(p,23)} a_2 a_3 \right) \delta^{k/2} (2u)^{k/20}
\]

(17)

The numerical values of the coefficients \( M_k^{(p,i)} \) for \( \mu_p \) with \( p = 2, 4, 6 \) are given in Appendix B.

Let us define the dimensionless time \( \tau = t/T_c(0) \), where

\[
\tau_c^{-1}(t) = 4\sqrt{\pi} g_2(\sigma) \sigma^2 n \sqrt{\frac{T(t)}{m}}.
\]

(18)

Using Eqs. (10), which express the the moments of the reduced velocity \( \langle \omega(2k) \rangle \) in terms of the Sonine coefficients \( a_k \), we recast Eqs. (15), into the following form,

\[
du = -\sqrt{\frac{2}{3}} \mu_2 u^2 + K, \quad \frac{d \omega_2}{d \tau} = \frac{\sqrt{2}}{3\sqrt{\pi}} \mu_2 (1 + a_2) - \frac{\sqrt{2}}{15\sqrt{\pi}} \mu_4 \sqrt{u} - \frac{K \mu_2}{2u},
\]

(19)

\[
\frac{d \omega_3}{d \tau} = \frac{\sqrt{u}}{2\sqrt{\pi} \mu_2} (1 - a_2 - a_3) - \frac{\sqrt{3}}{5\sqrt{\pi}} \mu_4 \sqrt{u} + \frac{2\sqrt{2u}}{105\sqrt{\pi} \mu_6} - \frac{3a_3 K}{4u},
\]

(20)

where the moments \( \mu_p, p = 2, 4, 6 \) are the functions of the Sonine coefficients \( a_2, a_3 \) and known numerical constants \( M_k^{(p,i)} \), Eq. (17). The constant \( K \) reads,

\[
K = \frac{m^{3/2} \xi_0^2}{\sqrt{\pi} g_2(\sigma) \sigma^2 n T_0^{3/2}}.
\]

Eqs. (19) describe evolution of the temperature and the velocity distribution function for a heated gas of viscoelastic particles. For \( K = 0 \), that is, for \( \xi_0^2 = 0 \), Eqs. (19) reduce to corresponding equations of Ref. 52, which describe evolution of \( a_2 \) and \( a_3 \) in a force-free gas.

When a heated gas settles in a steady state, the granular temperature and the velocity distribution function become time-independent. Then Eqs. (19) yield the algebraic equations for the steady-state temperature and moments \( \mu_p, p = 2, 4, 6 \):

\[
T_{\text{s.s.}} = \left[ \frac{3}{2} \frac{2}{\sqrt{\pi}} \mu_2 g_2(\sigma) \sigma^2 n \right]^{2/3},
\]

(21)

\[
\mu_4 = 5\mu_2,
\]

(22)

\[
\mu_6 = 105 \frac{4}{\mu_2 (1 + a_2)}.
\]

(23)

Solving this system in the linear approximation with respect to the dissipation parameter \( \delta \), we obtain the steady state Sonine coefficients

\[
\begin{align*}
a_2^{\infty} &= -A_2 \delta, \quad A_2 = 2^{1/5} \frac{157}{500} \Gamma \left( \frac{21}{10} \right) C_1 \approx 0.435, \\
a_3^{\infty} &= -A_3 \delta, \quad A_3 = 2^{1/5} \frac{28}{500} \Gamma \left( \frac{21}{10} \right) C_1 \approx 0.078.
\end{align*}
\]

(24)

(25)

The above expressions for \( a_2 \) and \( a_3 \), together with Eq. (17) for \( \mu_2 \) yield the corresponding analytical expression for the steady state temperature \( T_{\text{s.s.}} \).

III. EFFECTIVE RESTITUTION COEFFICIENT

The expression [5] for the restitution coefficient gives this quantity for a collision with a particular impact velocity. A natural question arises, whether is it possible to define an average value of \( \varepsilon \), which characterizes the whole ensemble and may be exploited as an effective constant restitution coefficient \( \varepsilon_{\text{eff}} \) for all impacts; this however should depend on the current temperature of a granular gas, see Eq. (5). Since \( \varepsilon_{\text{eff}} \) describes the collisions, it is natural to define this as a "collision average", e.g. [41]:

\[
\varepsilon_{\text{eff}} = \frac{\int dv_1 dv_2 df_{f_1} f_{f_1} v_{12} \cdot e \{ (-v_{12} \cdot e)(v_{12} \cdot e) \}}{\int dv_1 dv_2 df_{f_1} f_{f_1} v_{12} \cdot e \{ (-v_{12} \cdot e) \}},
\]

where the velocity distribution functions \( f_{f_1} = f(v_1, t) \) and \( f_{f_2} = f(v_2, t) \) are given by Eq. (5). The integral in the above equation can be calculated up to \( O(\delta^{10}) \) using a formula manipulation program as explained in detail in Ref. 52. Neglecting in the Sonine expansion [5] for \( f_1 \) and \( f_2 \) the high-order coefficients \( a_p \) with \( p \geq 4 \), we obtain,

\[
\varepsilon_{\text{eff}}(t) = 1 + \sum_{k=1}^{N_v} \left[ \delta^{k/2} (2u)^{k/20} \times \right.
\]

\[
\left. B_k + B_k^{(2)} a_2 + B_k^{(3)} a_3 + B_k^{(22)} a_2^2 + B_k^{(23)} a_2 a_3 + B_k^{(33)} a_3^2 + B_k^{(23)} a_2 a_3 \right]
\]

\[
\left. \times B_k^{(2)} a_2 + B_k^{(3)} a_3 + B_k^{(22)} a_2^2 + B_k^{(33)} a_3^2 + B_k^{(23)} a_2 a_3 \right]
\]

(26)

with the numerical coefficients \( B_k \) and \( B_k^{(i)} \) given in the Appendix A. The above expression, although general, is rather involved. If we neglect the dependence of \( \varepsilon_{\text{eff}} \) on the Sonine coefficients (that is, perform the collision averaging of \( \varepsilon \) with a Maxwellian distribution), Eq. (26) reduces to:

\[
\varepsilon_{\text{eff}}(t) = 1 + \sum_{k=1}^{N_v} B_k \delta^{k/2} \left[ 2T(t)/T_0 \right]^{k/20},
\]

(27)

where the above coefficients \( B_k \) may be also written in a compact analytical form,

\[
B_k = \eta_k \frac{2^{1/5} \Gamma \left( \frac{21}{10} \right)}{2 + \frac{k}{20}} \left[ 2 + \frac{k}{20} \right].
\]

(28)
Here $\Gamma(x)$ is the Gamma-function. We wish to stress again, that in opposite to the impact-velocity dependent coefficient $\varepsilon$, Eq. (5), the above restitution coefficient $\varepsilon_{\text{eff}}$ is the same for all collisions. Its dependence on the dissipation parameter $\delta$ is shown in Fig. 1 for different $N_c$. As it follows from this figure the series (27) demonstrates an excellent convergence for $\delta \lesssim 0.5$. In what follows we will use the number of terms in the expansion (27) equal to $N_c = 20$, which guarantees an accurate description of the effective restitution coefficient $\varepsilon_{\text{eff}}$ for the dissipative parameter $\delta$ up to $\delta \lesssim 0.5$, that is, for a rather large dissipation.

The system of equations, Eqs. (19), may be also used for a constant restitution coefficient, or, equally for the quasi-constant coefficient $\varepsilon_{\text{eff}}$, Eq. (26), provided the respective moments $\mu_p$ are expressed in terms of $a_2, a_3$ and $\varepsilon_{\text{eff}}$, as it follows for the case of a constant $\varepsilon$ [46]. We write these in the following form,

$$\mu_2 = \frac{1}{16384} \sqrt{2\pi (1 - \varepsilon_{\text{eff}}^2)} (35a_2^2 + 144a_2^2)$$
$$+ 120a_2a_3 + 3072a_2 + 256a_3 + 16384)$$

$$\mu_4 = \frac{1}{32768} \sqrt{2\pi (1 + \varepsilon_{\text{eff}})} \sum_{k=0}^{3} \left( N_k^{(0,0)} + N_k^{(4,2)} a_2^2 + N_k^{(4,3)} a_3 + N_k^{(4,23)} a_2a_3 \right) \varepsilon_{\text{eff}}^k$$

$$\mu_6 = \frac{3}{262144} \sqrt{2\pi (1 + \varepsilon_{\text{eff}})} \sum_{k=0}^{5} \left( N_k^{(6,0)} + N_k^{(6,2)} a_2^2 + N_k^{(6,3)} a_3 + N_k^{(6,23)} a_2a_3 \right) \varepsilon_{\text{eff}}^k$$

with the coefficients $N_k^{(p,i)}$ for $p = 4, 6$ listed in the Appendix B.

Using Eqs. (19) and Eqs. (29) together with Eq. (26) one can find temperature of a gas and the Sonine coefficients $a_2$ and $a_3$. Ultimately, $\varepsilon_{\text{eff}}$ as a function of current temperature, as it follows from Eq. (26), may be found. This, however, is very close to the simplified form (27), so that the difference between the results, obtained with the use of the complete expression (26) and the simplified one, (27), can be hardly distinguished on the figures (See Fig. 2). Hence, for practical purposes one can safely use the simple expression (27) for the effective quasi-constant restitution coefficient.

IV. EVENT-DRIVEN SIMULATIONS

A. Force-free Gas

We perform event-driven MD simulations [52] of a granular gas of smooth viscoelastic particles with the restitution coefficient $\varepsilon$. The grains are modeled as identical hard spheres of mass $m = 1$ and diameter $\sigma = 1$. In event-driven simulations particles move freely between pairwise collisions, where the velocities of the grains are updated according to Eq. (2). We use the system of $N = 2.048 \times 10^6$ particles placed in a three-dimensional cubic box with the edge $L = 418$, which corresponds to the number density of $n = 0.028$. This number density is rather small, which saves allows to approximate the contact value of the correlation function by unity, $g_2(\sigma) \approx 1$. Periodic boundary conditions are applied in all three directions.

The system was initialized by placing randomly particles in the box. For initial particles’ velocities randomly
directed vectors of the same length have been assigned. The initial total momentum \( \sum m v_i(0) \) was in this case with a high accuracy zero. We start with \( \varepsilon = 1 \), which corresponds to elastic collisions, and allow the system to relax to a Maxwellian velocity distribution, which used to happen in a few collisions per particle. Hence, a homogeneous system with a Maxwellian distribution was an initial condition for each MD run; then we evolve the system with a velocity-dependent restitution coefficient \( \varepsilon \). In simulations we used the initial granular temperature \( T(0) = 400/3 \) with the corresponding initial mean collision time \( \tau_c(0) = 0.436 \). All statistical quantities presented here have been obtained as averages over 25 independent runs. We have confirmed that the system remains approximately homogeneous during our simulations and that the Molecular Chaos hypothesis holds true with a high degree of accuracy, as it used to be in dilute gases, e.g., \[54\].

In Fig. 2, we plot time dependence of the current average of the restitution coefficient; this was obtained, averaging \( \varepsilon \) over \((N/20)\) successive particles collisions. As it follows from the figure, the average restitution coefficient is in a good agreement with the theoretical prediction for the effective "quasi-constant" coefficient \( \varepsilon_{\text{eff}} \), Eq. (27), for both, freely evolving and heated granular gases.

In Fig. 3 we present simulation results for the time dependence of the reduced granular temperature \( u(\tau) \) on the dimensionless time \( \tau = t/\tau_c(0) \), expressed in terms of the initial mean collision time \( \tau_c(0) \), Eq. (15). In a HCS temperature decays asymptotically as the power-law \( u(\tau) \sim \tau^{-5/3} \) \[8, 51, 52\] while in a heated gas it relaxes to a steady-state value.

Next, we analyze evolution of the velocity distribution function, characterized by the time-dependent Sonine coefficients [cf. Eq. (8)]. In order to calculate \( a_2 \) and \( a_3 \) numerically, we compute the moments of the velocity distribution function \( \langle \phi^k \rangle \) and use Eqs. (16). In Fig. 4 we show evolution of \( a_2 \) and \( a_3 \) for a force-free gas for \( \delta = 0.15 \), while in Fig. 5 the according simulation data for the Sonine coefficients \( a_2, a_3, a_4 \) and \( a_5 \) for \( \delta = 0.1 \) and \( \delta = 0.3 \) are given.

### B. Uniformly heated gas

The equation of motion for \( i \)-th particle in a white-noise thermostat, that is, for the gas under a uniform heating, reads,

\[
 m \frac{dv_i}{dt} = F_i^{\text{coll}} + F_i^{\text{ext}}. \tag{30}
\]

Here \( F_i^{\text{coll}} \) is the force acting on \( i \)-th particle only in a pairwise collision with another gas particle, while \( F_i^{\text{ext}} \) is the external force acting from the thermostat. Since we use the event-driven simulations \[53\], we do not need to solve explicitly the equation of motion for colliding particles with the forces \( F_i^{\text{coll}} \). Instead, we directly apply the collision rules \[2\] and write the after-collision velocities, expressed in terms of the pre-collision ones with the restitution coefficient \( \varepsilon \), Eq. (5) \[8, 53\]. For the external force \( F_i^{\text{ext}} \) we use the model of a Gaussian white noise:

\[
 \langle F_i^{\text{ext}}(t) \rangle = 0, \quad \langle F_i^{\text{ext}}(t) F_j^{\text{ext}}(t') \rangle = m^2 \xi_0^2 \delta_{ij} \delta_{\alpha \beta} \delta(t - t'), \tag{31}
\]

where \( \alpha, \beta = x, y, z \), and \( \xi_0^2 \) characterizes the magnitude of the stochastic force.

We apply the algorithm suggested in Ref. \[51\] (see also \[21, 55\]): During the event-driven simulations, we heat the system after a time-step \( dt \) by adding to the velocity of each \( i \)-th particle a random increment, which mimics the heating by noise,

\[
 v_{i,\alpha}(t + dt) = v_{i,\alpha}(t) + \sqrt{r} \sqrt{dt} \varphi, \tag{32}
\]

where \( \alpha = x, y, z \) and \( \varphi \) is a random number uniformly distributed within the interval \([-0.5, 0.5]\), and \( r \) is the amplitude of the noise, \( r = 12 \xi_0 \). After the change in velocities, the system is transferred to the centre-of-mass frame,

\[
 v_i = v_i - \frac{1}{N} \sum_{i=1}^{N} v_i,
\]

to ensure the conservation of the momentum. In simulations we used \( r = 0.1 \) and \( dt = 0.1 \), which implies that the random kicking interval is small as compared to the mean free time. The number of particles and the system size were the same as in the force-free simulations; the initial conditions have been also prepared as in the force-free case. All statistical quantities were obtained as averages over 10 independent runs. The results are presented at Figs. 2-5 where we show evolution of the average restitution coefficient (Fig. 2), of temperature (Fig. 3), and of the Sonine coefficients (Figs. 4 and 5).

### V. RESULTS AND DISCUSSION

Let us compare the results of the MD simulations with the predictions of our theory. As we already mentioned, the current average of the restitution coefficient, obtained by a straightforward averaging of \( \varepsilon \) in successive collisions is in a good agreement with the theoretical value of the effective quasi-constant restitution coefficient \( \varepsilon_{\text{eff}} \), found as a collision average from the kinetic theory, Fig. 2. In Fig. 3 evolution of granular temperature is shown. As it follows from the figure, during the early stages of cooling, the temperature of a heated granular gas evolves in the same way as the respective force-free gas with the same \( \delta \). However, while the temperature of the heated gas settles at a steady-state value, the force-free gas continues to cool down. The results of our theory (numerical solution of system of differential equations \[19\], where moments of collisional integral \( \mu_p \) are given by Eqs. (17)) are in
The values of $a_2$ and $a_3$ in a heated and force-free gas are practically indistinguishable. At later times, the Sonine coefficients for the heated gas settle at the steady-state [cf. Eqs. (24)-(25)]. For the freely evolving gas they continue to decay and evolve as following: Starting from zero, $a_k(0) = 0$, $(k = 1, 2, \ldots)$, which corresponds to the initial Maxwellian distribution, $a_2$ and $a_3$ decrease, reach a minimum, and eventually return back to zero, see Fig. 4.

As the thermal velocity decreases during the gas cooling, the behavior of the system tends to that of a gas of elastic particles, that is, the velocity distribution tends asymptotically to a Maxwellian. For both, the force-free and heated gases, our theory perfectly agrees with the computer simulations.

To investigate the convergence of the Sonine-polynomial expansion on we analyze the high-order Sonine coefficients, $a_4$ and $a_5$. In Fig. 3(a) the simulation data for the time dependence of $a_2$, $a_3$, $a_4$, $a_5$ along with the analytical predictions for $a_2$ and $a_3$ are shown. As it may be seen from Fig. 3(a), $a_4$ and $a_5$ evolve in a similar fashion as $a_2$ and $a_3$, that is, their magnitudes first increase, reach maximum and relax back to zero. Certainly, this is not surprising, since it is a simple manifestation of the fact that the system starts from a Maxwellian velocity distribution due to initial conditions and returns to it later, when the collisions become elastic. Furthermore, $a_4$ is an order of magnitude smaller than $a_3$, while $a_5$ is $\sim 10^2$ times smaller than $a_2$. This confirms the convergence of the Sonine-polynomial expansion for the velocity distribution, at least for these values of $\delta$. In this figure we also show, in addition to the predictions of the complete theory, theoretical results for the case of an effective quasi-constant restitution coefficient $\varepsilon_{\text{eff}}(t)$. Again we see that the accuracy of the simplified theory, based on the effective $\varepsilon_{\text{eff}}(t)$, is surprisingly good.

In Fig. 4(b) evolution of the Sonine coefficients for the larger dissipation, $\delta = 0.3$ is illustrated. The behavior of the coefficients is qualitatively the same as for the smaller $\delta$. Moreover, the subsequent coefficients $a_k$ decrease by an order of magnitude with increasing $k$, which again indicates the convergence of the Sonine-polynomial expansion for this dissipation. At the same time a rather noticeable discrepancy between the kinetic theory and the MD results is obvious. In the earlier work [52], two of us have reported that the expansion for $\varepsilon$ with $N_v = 20$ does not yield an acceptable accuracy for the Sonine coefficients $a_2$ and $a_3$ for $\delta \geq 0.3$ (when the complete theory is used). The discrepancy between the theoretical curves and the simulation data, seen in Fig. 4(b), seemingly supports this theoretical conclusion.

Figure 5(c) demonstrates the convergence of the Sonine-polynomial expansion for heated granular gases. As in Fig. 5(a), we plot here the dependence of $a_2$, $a_3$, $a_4$ and $a_5$ on $\tau$ for $\delta = 0.1$. It may be seen that the successive Sonine coefficients decrease by an order of magnitude, which clearly indicates the convergence of this series. As it has been already mentioned in the discussion of Fig. 4 our complete theory is in a perfect agreement with the simulation data for both the homogeneous cooling state and uniformly heated gas. Moreover, the kinetic theory based on the effective quasi-constant restitution coefficient $\varepsilon_{\text{eff}}$ (numerical solution of Eqs. (19), where $\mu_p$ are given by Eqs. (17)) yields almost the same accuracy for the description of granular temperature as the complete theory.

In Fig. 4 evolution of the Sonine coefficients $a_2$ and $a_3$ for the force-free and heated granular gases with $\delta = 0.15$. The symbols correspond to simulation results, the lines to our theory (numerical solution of Eqs. (19), where $\mu_p$ are given by Eqs. (17)).
FIG. 5: Evolution of the Sonine coefficients $a_k$ ($k = 2, 3, 4, 5$) for a force-free (a,b) and uniformly heated (c, d) granular gas with $\delta = 0.1$ (a, c) and $\delta = 0.3$ (b,d). The symbols correspond to the MD simulation results and the thin black lines - to the theory (numerical solution of Eqs. (19), where $\mu_\rho$ are given by Eqs. (17)). Thick grey lines depict theoretical predictions (numerical solution of Eqs. (19), where $\mu_\rho$ are given by Eqs. (29)) for a granular gas with a “quasi-constant” restitution coefficient $\varepsilon_{\text{eff}}(t)$, Eq. 27.

with the simulation data for the heated gas, but here we also notice that the simplified model with a quasi-constant restitution coefficient $\varepsilon_{\text{eff}}$ gives rather accurate description of the evolution of $a_2$ and $a_3$ too.

Finally, in Fig. 5(d), we plot time dependence of $a_2$, $a_3$, $a_4$ and $a_5$ for a heated gas with relatively large dissipation, $\delta = 0.3$. In this case one can again see a noticeable discrepancy between our theory and the numerical results (cf. Fig. 5(b) for the freely-evolving gas). Nevertheless, smallness of the high-order coefficients $a_4$ and $a_5$ seemingly manifests the convergence of the Sonine series. Interestingly, although the simulation data deviate from the theoretical curves during the relaxation to the steady state, in the steady state itself, the numerical and theoretical values of the complete theory for the Sonine coefficients $a_2$ and $a_3$ are very close, see Fig. 5(d) for $\tau > 10^3$. At the same time the results of the simplified model of the quasi-constant restitution coefficient $\varepsilon_{\text{eff}}$ do not agree well with the MD results at the steady state for this dissipation.

VI. CONCLUSION

To date, most studies of granular gases have been focused on systems with a constant restitution coefficient, although the basic physics requires impact-velocity dependence of $\varepsilon$. Here we studied numerically, by means of Molecular Dynamics, and theoretically a granular gas of particles with the impact-velocity dependent restitution coefficient, as it follows from the simplest first-principle model of viscoelastic spheres. Since the impact-velocity dependence of $\varepsilon$ significantly complicates the description of a granular gas dynamics, we tried to develop a simplified collision model. Namely, we considered a model where $\varepsilon$ is the same for all collisions, but depends on the current value of the thermal velocity of the gas, or, equally, on granular temperature. Using the collision average of the kinetic theory, we obtained an explicit expression for this quasi-constant restitution coefficient $\varepsilon_{\text{eff}}[T(t)]$.

We explored evolution of granular temperature and the
velocity distribution function in large-scale MD simulations of force-free and uniformly heated gases and developed a theory of the velocity distribution function $f(v, t)$ for the case of driven gases. In the MD simulations we found a few first Sonine coefficients $a_2$, $a_3$, $a_4$, $a_5$, which quantify deviations of $f(v, t)$ from a Maxwellian distribution and noticed that the successive-order Sonine coefficients decrease by an order of magnitude with respect to the preceding ones. In other words, we observed that $a_3$ is approximately 10 times smaller than $a_2$, $a_4$ is 10 times smaller than $a_3$ and $a_5$ is 10 time smaller than $a_4$. This implies the convergence of the Sonine series for both heated and force-free gases in the range of dissipation studied.

We obtained theoretical predictions for $a_2$ and $a_3$ using the complete theory and the one based on an effective quasi-constant restitution coefficient $\varepsilon_{\text{eff}}$. We found that the results of our complete theory for granular temperature are in a very good agreement with the simulation data for both, force-free and heated gases. Furthermore, if dissipation is not large, $\delta = 0.1$ or $\delta = 0.15$, the dependence on time of the Sonine coefficient, obtained in simulations is also in an excellent agreement with the theoretical predictions of the complete theory for a homogeneous cooling state (previously existed theory) as well as for a uniform heating (the novel theory). At the same time for large dissipation, $\delta = 0.3$, a noticeable discrepancy between the theoretical and numerical curves is observed, especially for the homogeneous cooling; deviations for the uniform heating are also noticeable, but less pronounced. Moreover the steady-state values of $a_2$ and $a_3$ are given rather accurately by the new theory for heated gases.

We also analyzed the accuracy of the simplified model based on the effective quasi-constant restitution coefficient $\varepsilon_{\text{eff}}(t)$. We have observed that this model can accurately describe evolution of temperature for all studied dissipations and of the velocity distribution function, provided dissipation is not large, for both, force-free and heated gases. However, for large dissipation the simplified model allows only qualitative description of $f(v, t)$ and fails to provide a quantitative one. Still we wish to stress that the simple model, where $\varepsilon_{\text{eff}}(t)$ is determined by the granular temperature and material properties of the grains may be important for applications. Indeed, the description of a granular gas dynamics for the case of a constant restitution coefficient is significantly simpler than that of an impact-velocity dependent $\varepsilon$.

Appendix: Numerical coefficients

The effective restitution coefficient $\varepsilon_{\text{eff}}$ and moments of the dimensionless collision integral $\mu_p = -\int \delta c^p \tilde{I}(\tilde{f}, \tilde{f})$, where $p = 2, 4, 6$, can be calculated using a formula manipulation program as explained in Ref. [27]. Below we present coefficients $B_k^{(i)}$, which define $\varepsilon_{\text{eff}}$ [Eqs. (21) and (26)], numerical values of the coefficients $M_k^{(p,i)}$ for $\mu_p$ ($p = 2, 4, 6$) [Eqs. (17)] in the case of a granular gas of viscoelastic particles and the coefficients $N_k^{(p,i)}$ for $\mu_p$ ($p = 4, 6$) [Eqs. (20)] for a granular gas with a constant restitution coefficient.

| $k$ | $B_k$ | $B_k^{(2)}$ | $B_k^{(3)}$ | $B_k^{(22)}$ | $B_k^{(33)}$ | $B_k^{(23)}$ |
|-----|-------|-------------|-------------|-------------|-------------|-------------|
| 0   | -0.0629 | -0.0156   | -0.0146   | -0.0064    | 0.017     |               |
| 1   | 0     | 0          | 0          | 0          | 0          |               |
| 2   | -0.1196 | 0.07       | 0.016     | 0.0148     | 0.00616    | 0.0168     |
| 3   | 0     | 0          | 0          | 0          | 0          |               |
| 4   | 0.84  | -0.044     | -0.0086   | -0.00826   | -0.00326   | -0.0009    |
| 5   | 0.287 | -0.0135    | -0.00427  | -0.00237   | -0.00091   | -0.00267   |
| 6   | -0.518 | 0.0231    | 0.0046    | 0.0038     | 0.00142    | 0.0044     |
| 7   | -0.524 | 0.0167    | 0.0032    | 0.00258    | 0.000937   | 0.0027     |
| 8   | 0.408 | -0.00919   | -0.00188  | -0.00135   | -0.00047   | -0.00147   |
| 9   | 0.547 | -0.0065    | -0.0011   | -0.00087   | -0.0003    | -0.00089   |
| 10  | -0.184 | 0         | 0          | 0          | 0          |               |
| 11  | -0.479 | -0.0063    | -0.00099  | -0.000727  | -0.000235  | -0.00072   |
| 12  | -0.0586 | -0.00161  | -0.00024  | -0.00017   | -0.000054  | -0.00017   |
| 13  | 0.398 | 0.017      | 0.00243   | 0.00169    | 0.00051    | 0.0016     |
| 14  | 0.218 | 0.013      | 0.00175   | 0.00118    | 0.00035    | 0.0011     |
| 15  | -0.277 | -0.022     | -0.0027   | -0.00178   | -0.00051   | -0.0016    |
| 16  | -0.289 | -0.028     | -0.00329  | -0.00002   | -0.00058   | -0.0019    |
| 17  | 0.112 | 0.0133     | 0.00014   | 0.00089    | 0.00024    | 0.00079    |
| 18  | 0.308 | 0.043      | 0.0043    | 0.00258    | 0.00067    | 0.0022     |
| 19  | 0.0478 | 0.0078    | 0.00071   | 0.00042    | 0.0001     | 0.00035   |
| 20  | -0.276 | -0.0517    | -0.0043   | -0.0024    | -0.00059   | -0.002     |

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TABLE I: Numerical coefficients $B_k$ and $B_k^{(i)}$ for calculation of the effective quasi-constant restitution coefficient $\varepsilon_{\text{eff}}$ [Eqs. (20)–(27)].
TABLE II: Numerical coefficients for $\mu_2$ for a gas of viscoelastic particles [Eqs. (17)].

| $k$ | $M_k^{(2,0)}$ | $M_k^{(2,2)}$ | $M_k^{(2,3)}$ | $M_k^{(2,22)}$ | $M_k^{(2,33)}$ | $M_k^{(2,23)}$ |
|-----|---------------|---------------|---------------|---------------|---------------|---------------|
| 0   | 0             | 0             | 0             | 0             | 0             | 0             |
| 1   | 0             | 0             | 0             | 0             | 0             | 0             |
| 2   | 6.49          | 1.56          | 0.10          | 0.054         | 0.012         | 0.044         |
| 3   | 0             | 0             | 0             | 0             | 0             | 0             |
| 4   | -0.29         | -2.76         | -0.14         | -0.067        | -0.014        | -0.052        |
| 5   | -1.80         | -5.19         | -0.025        | -0.012        | -0.0023       | -0.0087       |
| 6   | 10.40         | 3.74          | 0.12          | 0.056         | 0.011         | 0.041         |
| 7   | 5.91          | 2.32          | 0.058         | 0.025         | 0.0047        | 0.018         |
| 8   | -10.44        | -4.46         | -0.074        | -0.031        | -0.0055       | -0.021        |
| 9   | -10.53        | -4.88         | -0.04         | -0.016        | -0.0028       | -0.011        |
| 10  | 8.77          | 4.39          | -6.28E-08     | 1.57E-07      | -6.91E-07     | -1.10E-06     |
| 11  | 14.13         | 7.60          | -0.063        | -0.023        | -0.0036       | -0.015        |
| 12  | -4.54         | -2.62         | 0.044         | 0.015         | 0.0023        | 0.0093        |
| 13  | -16.37        | -10.12        | 0.25          | 0.080         | 0.012         | 0.030         |
| 14  | -1.58         | -1.04         | 0.035         | 0.010         | 0.0015        | 0.0063        |
| 15  | 16.74         | 11.77         | -0.49         | -0.14         | -0.018        | -0.08         |
| 16  | 8.09          | 6.05          | -0.30         | -0.079        | -0.01         | -0.045        |
| 17  | -14.29        | -11.33        | 0.66          | 0.16          | 0.02          | 0.089         |
| 18  | -13.91        | -11.69        | 0.79          | 0.175         | 0.02          | 0.0335        |
| 19  | 8.82          | 7.83          | -0.59         | -0.12         | -0.013        | -0.063        |
| 20  | 18.13         | 16.99         | -1.42         | -0.27         | -0.028        | -0.13         |

TABLE III: Numerical coefficients for $\mu_4$ for a gas of viscoelastic particles [Eqs. (17)].

| $k$ | $M_k^{(4,0)}$ | $M_k^{(4,2)}$ | $M_k^{(4,3)}$ | $M_k^{(4,22)}$ | $M_k^{(4,33)}$ | $M_k^{(4,23)}$ |
|-----|---------------|---------------|---------------|---------------|---------------|---------------|
| 0   | 0             | 10.03         | -2.51         | 0.31          | 0.049         | 0.16          |
| 1   | 0             | 0             | 0             | 0             | 0             | 0             |
| 2   | 36.32         | 46.85         | -6.50         | -0.29         | -0.015        | -0.14         |
| 3   | 0             | 0             | 0             | 0             | 0             | 0             |
| 4   | -71.50        | -100.66       | 16.09         | 0.59          | 0.029         | 0.25          |
| 5   | -10.36        | -15.39        | 2.66          | 0.077         | 0.0034        | 0.032         |
| 6   | 116.21        | 170.73        | -29.24        | -0.98         | -0.054        | -0.39         |
| 7   | 45.69         | 73.09         | -13.997       | -0.275        | -0.012        | -0.105        |
| 8   | -169.5        | -260.38       | 47.12         | 1.01          | 0.055         | 0.37          |
| 9   | -117.70       | -190.09       | 39.54         | 0.34          | 0.016         | 0.12          |
| 10  | 219.56        | 354.27        | -67.36        | 9.42E-07      | -9.59E-05     | 2.20E-06      |
| 11  | 234.61        | 406.45        | -85.03        | -0.94         | 0.042         | 0.30          |
| 12  | -240.62       | -406.38       | 79.92         | -2.49         | -0.11         | -0.75         |
| 13  | -398.72       | -742.58       | 157.86        | -6.21         | -0.248        | -1.76         |
| 14  | 196.27        | 345.6         | -67.8         | 5.33          | 0.22          | 1.42          |
| 15  | 597.98        | 1147.26       | -261.54       | 19.24         | 0.66          | 3.81          |
| 16  | -57.49        | -93.78        | 7.19          | -5.89         | -0.149        | -0.91         |
| 17  | -797.17       | -1618.27      | 386.31        | -43.42        | -1.25         | -9.41         |
| 18  | -208.15       | -471.79       | 132.09        | -11.56        | -0.26         | -2.31         |
| 19  | 935.74        | 2101.21       | -501.69       | 77.96         | 1.81          | 14.30         |
| 20  | 604.56        | 1387.68       | -379.52       | 55.88         | 1.1           | 9.31          |
TABLE IV: Numerical coefficients for $\mu_0$ for a gas of viscoelastic particles [Eqs. (14)].

| $k$ | $M_k^{(0)}$ | $M_k^{(2)}$ | $M_k^{(4)}$ | $M_k^{(6,22)}$ | $M_k^{(6,43)}$ | $M_k^{(6,23)}$ |
|-----|-------------|-------------|-------------|---------------|---------------|---------------|
| 0   | 112.80      | -84.60      | -4.93       | 0.022         | -0.82         |
| 1   | 0           | 0           | 0           | 0             | 0             |
| 2   | 209.94      | 653.78      | -245.02     | 16.75         | 0.26          | 2.23          |
| 3   | 0           | 0           | 0           | 0             | 0             |
| 4   | -525.04     | -1718.36    | 717.54      | -48.53        | -0.45         | -4.85         |
| 5   | -61.21      | 203.84      | 86.30       | -7.2          | -0.061        | -0.66         |
| 6   | 1097.657    | 3731.6      | -1607.5     | 133.88        | 0.64          | 8.93          |
| 7   | 338.22      | 1206.7      | -544.06     | 46.04         | 0.18          | 2.3           |
| 8   | -2042.4     | -7140.7     | 3126.2      | -347.5        | -0.75         | -11.58        |
| 9   | -1116.58    | -4131.0     | 1913.43     | -196.42       | -0.21         | -5.27         |
| 10  | 3412.       | -12219.7    | -5395.76    | 806.66        | -0.0022       | -0.0012       |
| 11  | 2856.1      | 10832.9     | -5076.7     | 683.4         | -0.047        | -11.39        |
| 12  | -3055.2     | -18485.3    | 8165.6      | -1634.9       | -3.29         | -54.5         |
| 13  | -6187.9     | -24012.3    | 11320       | -2006.6       | 5.30          | 100.33        |
| 14  | 6253.9      | 23886       | -10411.7    | 2818.3        | -10.83        | -187.9        |
| 15  | 11807       | 46930.9     | -22224.5    | 5002.8        | -21.37        | -421.9        |
| 16  | -6433.2     | -23805.8    | 9879.        | -386.7        | 21.14         | 386.7         |
| 17  | -20203.4    | -82418.7    | -39185.8    | -11160.6      | 60.27         | 1302.07       |
| 18  | 3253.3      | 10084       | -2516.6     | 3251.2        | -23.5         | -433.49       |
| 19  | 31188.6     | 130743      | -62400.8    | 21721.9       | -133.14       | -3258.3       |
| 20  | 5784.3      | 29433.95    | -17812.7    | 2307.9        | -6.45         | -384.65       |

TABLE V: Numerical coefficients for $\mu_4$ and $\mu_6$ for a gas with a constant restitution coefficient [Eqs. (29)].

| $k$ | $N_k^{(0)}$ | $N_k^{(2)}$ | $N_k^{(4)}$ | $N_k^{(6,22)}$ | $N_k^{(6,43)}$ | $N_k^{(6,23)}$ |
|-----|-------------|-------------|-------------|---------------|---------------|---------------|
| 0   | 154746      | 277504      | -46336      | 2192          | 455           | 1096          |
| 1   | -147456     | -211968     | 29052       | -144          | -135          | -72           |
| 2   | 32768       | 30720       | -2560       | -480          | -50           | -240          |
| 3   | -32768      | -30720      | 2560        | 480           | 50            | 240           |

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