Musicological, computational, and conceptual aspects of first-species counterpoint theory

Juan Sebastián Arias-Valero a*, Octavio Alberto Agustín-Aquino b, and Emilio Lluis-Puebla a

aDepartamento de Matemáticas, Universidad Nacional Autónoma de México, Ciudad de México, México; bInstituto de Física y Matemáticas, Universidad Tecnológica de la Mixteca, Huajuapan de León, México

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We re-create the essential results of a 1989 unpublished article by Mazzola and Muzzulini that contains the musicological aspects of a first-species counterpoint model. We include a summary of the mathematical counterpoint theory and several variations of the model that offer different perspectives on Mazzola’s original principles.

Keywords: Counterpoint; rings; modules; combinatorics

1. Introduction

The original idea of this article was to communicate the musicological results from Mazzola and Muzzulini (1990), which presents the results of a model for first-species counterpoint, based on ring and module theory, but was not published. This kind of counterpoint is the simplest one and the didactic basis of Renaissance counterpoint, as taught by Fux, Mann, and Edmunds (1965). The model was introduced by Mazzola et al. (1989) for $\mathbb{Z}_{12}$; a ring that can be used to model the algebraic behavior of the twelve intervals between tones in the chromatic scale of Western musical tradition. Then, the model was re-exposed with some additional computational results by Hichert in Mazzola (2002, Part VII). Further generalizations to the case when the rings are of the form $\mathbb{Z}_n$ were considered in Agustín-Aquino (2011), Junod (2010) and then included in a collaborative compendium of mathematical counterpoint theory and its computational aspects (Agustín-Aquino, Junod, and Mazzola 2015). The motivation for such a generalization to $\mathbb{Z}_n$ was the existence of microtonal scales with more than twelve tones, which have been used for making real music (Agustín-Aquino 2015).

However, the understanding of Mazzola and Muzzulini (1990) is unavoidably linked to that of the very model, so in this paper we expose a summary of it and the musicological, computer-aided, study of its results. We do not reduce this study to a mere exposition of the previous results but propose several variations of the model with the due justifications. The intention of these variations is not to destroy the original model, but to go deeper into some of its basic principles.

*Corresponding author. Email: jsariasv1@gmail.com

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Regarding other mathematical models of counterpoint, Mazzola’s school has an important counterpart in D. Tymoczko. Tymoczko’s model (Tymoczko 2011, Appendix) is based on orbifolds and is oriented towards voice leading. Moreover, it works by means of a geometric reading of the usual counterpoint rules, in contrast to the predictive character of Mazzola’s model, which follows the principle that these rules obey mathematical relations based on musical symmetries. There has been a polemic around Mazzola’s and Tymoczko’s models. Tymoczko’s initial critique can be found in Tymoczko (2011) and a response by Mazzola and the second author can be found in Agustín-Aquino and Mazzola (2011). Two of Tymoczko’s concerns just were that the musicological study (Mazzola and Muzzulini 1990) was not available and that there were some progressions forbidden by the model but good to Fux. Likely, this article can clarify those and other possible concerns.

To be more specific, in this article, we expose Mazzola’s first-species counterpoint model, highlighting that its power relies on the prediction of the prohibitions of parallel fifths and some tritones. Besides this vindication of the original theory, we furthermore point out some possible gray zones in there, related to the choice of dual numbers as a model of counterpoint intervals and the reduction modulo 12, which could lead to alternative models (Sections 5.1 and 7).

We organize this article as follows. First, in Section 2, we codify Fux’s strict style following two standard sources (Fux, Mann, and Edmunds 1965; Jeppesen 1992). This is a descriptive model, which helps to make a mathematical taxonomy of progressions into inadmissible, bad, and good ones, but hardly explains anything on the nature of the rules. Then, based on Mazzola (2007, Chapitre III), in Sections 3 and 4, we expose Mazzola’s model both in informal and formal synthetic ways. The quantitative computational results are also included. In Section 5, we reduce, modulo 12, the strict style to obtain the reduced strict style. This procedure helps to compare the original phenomenon to the model, whose base is the integers modulo 12. Here, we note that there are some new ambiguous progressions that probably do not deserve the defined names inadmissible, bad, or good. In Section 6, we compare the reduced strict style to the model. Based on the conceptual and structural understanding of the model recorded in the previous sections, in Section 8, we propose an alternative model that replaces the dual-numbers product (infinitesimals one) by the integer-modulo-12 product. The alternative to the dual numbers ring is also isomorphic to the product $\mathbb{Z}_{12} \times \mathbb{Z}_{12}$, and, therefore, could be suitable for further categorical generalizations. The quantitative results of the alternative model are very similar to those of the original one, and slightly improve them regarding the predictions of inadmissible or bad progressions in the reduced strict style. Also, in Sections 9 and 10, we propose two variations of the model that go deeper into the principle of local characterization of deformed consonances/dissonances. These variations become the same and, again, they slightly improve the prediction of inadmissible or bad progressions. Finally, we provide conclusions from our study and point out some directions for further research.

2. The strict style

The simplest case of first-species counterpoint consists of two voices, cantus firmus and discantus, whose notes occur simultaneously and have the same duration. Thus, given a note in the cantus firmus, the corresponding one in the discantus is determined by the interval between them. Each interval is a consonance and the composition must satisfy certain rules. See Figure 1.

The following is the codification of the strict style given in Mazzola and Muzzulini (1990), which is based on Tittel (1959). Here, for accessibility of the bibliographical sources, we justify each rule following the standard counterpoint books (Fux, Mann, and Edmunds 1965; Jeppesen 1992).
Figure 1. Some features of a first-species counterpoint example given by Fux, Mann, and Edmunds (1965, 29), which is in the Dorian mode. The integer 7 is the interval between the notes d and a, corresponding to 2 and 9, respectively. The linear polynomial $2 + 7x$ denotes the associated contrapuntal interval. The transition from a bar to the next is a progression.

The strict style consists of the following data:

- **Pitch space.** We will work with the usual group $\mathbb{Z}_{12}$ of pitch classes of the equal-tempered scale. We denote by $\pi$ the natural projection from $\mathbb{Z}$ to $\mathbb{Z}_{12}$.
- **Intervals.** The intervals between pitch classes and pitches also correspond to the groups $\mathbb{Z}_{12}$ and $\mathbb{Z}$, since they are just differences.
- **Diatonic scale.** The basic pattern of modes is the subset $X$ of $\mathbb{Z}_{12}$, where $X = \{0, 2, 4, 5, 7, 9, 11\}$, and its extension $\pi^{-1}(X)$ to $\mathbb{Z}$.
- **Contrapuntal intervals.** Each interval between a given note $c$ in the cantus firmus and its discantus $d$ corresponds uniquely to a linear polynomial $c + (d - c)x$. This expression is equivalent to giving the two voices explicitly as a pair $(c, d)$, but our intention is to stress the role of intervals. In fact, Mazzola’s model is based on a mathematical characterization of consonant and dissonant intervals (Section 3.3) and the product in rings of linear polynomials, like dual numbers (Section 4), is valuable to the subsequent theory; see also Section 8. We denote by $P_1$ the additive Abelian group of contrapuntal intervals.
- **Consonances and dissonances.** We define consonances (unison, thirds, fifth, sixths) and dissonances (seconds, fourths, sevenths) by $K = \{0, 3, 4, 7, 8, 9\}$ and $D = \{1, 2, 5, 6, 10, 11\}$, respectively. Thus, $(K, D)$ is a partition of the intervals group $\mathbb{Z}_{12}$.

The consonance/dissonance partition of $\mathbb{Z}$ is naturally induced by $(K, D)$, namely $(\pi^{-1}(K), \pi^{-1}(D))$. The set $\pi^{-1}(K)$ consist of all possibly compound intervals that are consonances up to the octave, which are useful in instrumental counterpoint without vocal range restrictions. In turn, by classifying contrapuntal intervals $c + (d - c)x$ according to whether $d - c$ is a consonance or a dissonance, we have a partition of the contrapuntal intervals group $P_1$ into contrapuntal consonances and dissonances.

A composition of first-species counterpoint is a finite sequence $\xi_1, \xi_2, \ldots, \xi_n$ of contrapuntal intervals essentially satisfying the following rules.

### 2.1. Preliminary rules

- Each contrapuntal interval is a contrapuntal consonance (Fux, Mann, and Edmunds 1965, 27).
- We work with consonances up to the tenth (Jeppesen 1992, 112, 5), that is, for each contrapuntal interval $c + kx$, $k \in \{0, 3, 4, 7, 8, 9, 12, 15, 16\}$.

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1 For simplicity, we assume that it extends endlessly in both directions. The choice of this particular scale is only illustrative but irrelevant. It could equally denote a scale in a different tuning.
Both \(c\) (cantus firmus) and \(d\) (discantus \(c + k\)) of each contrapuntal interval \(c + kx\) are in the diatonic scale.\(^2\)

- Given a progression \((c + kx, c' + k'x)\), from a contrapuntal consonance to its successor in a composition, the maximum change of a voice is an octave (Jeppesen 1992, 109). Formally, \(|c' - c| \leq 12\) and \(|d' - d| \leq 12\), where \(d = c + k\) and \(d' = c' + k'\).

### 2.2. Progression rules

According to Mazzola and Muzzulini (1990), Fux, Mann, and Edmunds (1965) and Jeppesen (1992), we divide progressions \((c + kx, c' + k'x)\) into three categories.

- **Inadmissible:**
  - **Unison repetitions** (Jeppesen 1992, 112, 3). Parallel perfect consonances (Fux, Mann, and Edmunds 1965, 22). Those progressions with \(k \in \{0, 7, 12\}\), \(k = k'\), and \(c \neq c'\). Hidden parallel perfect consonances (Fux, Mann, and Edmunds 1965, 22). Those satisfying \(k' \in \{0, 7, 12\}\), \((d' - d)(c' - c) > 0\), and \(k \neq k'\). Tritones (Fux, Mann, and Edmunds 1965, 35). They satisfy \(|c' - c| = 6\) or \(|d' - d| = 6\). Too large skips (Fux, Mann, and Edmunds 1965, 27, Footnote 1). If \(7 < |c' - c| < 12\) or \(7 < |d' - d| < 12\). The octave is regarded as a sort of repetition (Jeppesen 1992, 112, 7) and is accepted.

- **Bad:** Those that are not inadmissible but fall into some of the following cases. Bad progressions are not strictly avoided.
  - **Imperfect consonances by similar skips**\(^3\) (Jeppesen 1992, 112, 7). This means that \(k' \in \{3, 4, 8, 9, 15, 16\}\), \((d' - d)(c' - c) > 0\), \(|c' - c| > 2\), \(|d' - d| > 2\), and \(5 < |c' - c| < 12\) or \(5 < |d' - d| < 12\).
  - Hidden tritones (Fux, Mann, and Edmunds 1965, 35, Footnote 9). If \(d' - c \equiv 6 \pmod{12}\) or \(d - c' \equiv 6 \pmod{12}\).

- **Good:** All other progressions.

### 2.3. Strict style modulo translation

Since all rules can be expressed in terms of intervals, they are invariant under translation of the progressions. Thus, to understand them, it is enough to study the representatives under translation of the form

\[
(0 + kx, c' + k'x).
\]

See Table 1 for the tally of all progressions and their types. These outcomes were obtained with Code 1 in the Online Supplement.

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\(^2\) Moreover, the choice of a distinguished element \(t\) in \(X\) determines a mode with tonic \(t\), where the composition occurs. Dorian, Phrygian, Mixolydian, Eolian, and Ionian modes are determined by the tonics \(2, 4, 7, 9,\) and \(0\), respectively. The tonic is essentially stressed in the first and last contrapuntal intervals of the composition (Fux, Mann, and Edmunds 1965, 31), but the role of modes in progressions is secondary since the latter do not depend on a particular note (Section 2.3). See Jeppesen (1992, 59–82) for a musical introduction to modes.

\(^3\) It is a weak form of the combination of the rules Parallel imperfect consonances and Hidden parallel imperfect consonances.
3. The principles of Mazzola’s model

The following are the conceptual principles of the model; the mathematical details are in Section 4.

3.1. Counterpoint rules obey mathematical laws based on symmetries

Rather than an acoustic or psychological theory, counterpoint is a composition theory with its own logic. The fourth is an acoustic consonance, but it is a dissonance in counterpoint. Similarly, to justify the tritone prohibitions by saying that these intervals are hard to sing is not satisfactory (Mazzola 2007, 75) given the versatility of instrumental counterpoint.

On the other hand, symmetries (for example, transpositions and inversions) are the natural operations that occur in counterpoint, so they are a reasonable basis for a model. Formally, symmetries are affine automorphisms of rings; see Section 4 for details.

3.2. We restrict to the intervals modulo octave

The model does not intend to explain Fux counterpoint rules, but their reduction modulo octave, in the sense of Section 5. This choice is just a simplification procedure.

3.3. The division of intervals into consonances and dissonances has a mathematical conceptual characterization

The partition of the twelve intervals modulo octave into consonances (unison, thirds, fifth, and sixths) and dissonances (seconds, fourth, sevenths) has a unique symmetry that interchanges them. This fundamental property, discovered by Mazzola, characterizes this partition together with a monoid property for consonances (Mazzola 2007, 76).

3.4. The discantus is a tangential alteration of the cantus firmus

The dual numbers ring consists of all linear polynomials \( c + d\epsilon \) in the indeterminate \( \epsilon \) with coefficients in \( \mathbb{Z}_{12} \) subject to the relation \( \epsilon^2 = 0 \). These polynomials represent contrapuntal intervals (Section 2) modulo octave and \( d\epsilon \) represents an infinitesimal tangent.

Certainly, from a geometric point of view, there is an analogy of the intervals group \( \mathbb{Z}_{12} \) with a differentiable manifold, with associated tangent spaces, given the characterization of this Abelian group as the product \( \mathbb{Z}_3 \times \mathbb{Z}_4 \), which can be interpreted as a torus of thirds (Mazzola 2007, Section 12.1).

Table 1. Tally of the strict-style progressions modulo translation and their types.

| Type                                      | Count |
|-------------------------------------------|-------|
| 1057 prog.                                |       |
| 671 inadmissible                          |       |
| 22 parallel fifths                        |       |
| 49 parallel octaves and unisons           |       |
| 88 hidden fifths                          |       |
| 128 hidden octaves and unisons            |       |
| 170 tritones                              |       |
| 434 too large skips                       |       |
| 64 bad                                    |       |
| 38 imperfect consonances by similar skips |       |
| 26 hidden tritones                        |       |
| 322 good                                  |       |
On the other hand, the discantus is a variation or alteration of the cantus firmus, according to Mazzola (2007, 73) and Mazzola et al. (1989, 549), and alterations are tangents (Mazzola 2002, Section 7.5). Thus, dual numbers, which are the natural tangents in algebraic geometry (Hartshorne 1977, 80), suitably model this situation.

This point of view of alterations as tangents is likely inspired by infinitesimal deformations in algebraic geometry (Hartshorne 1977, Example 9.13.1), which are related to dual numbers.

From a musical perspective, Mazzola et al. (1989, 558) requires the characterizing relation $\epsilon^2 = 0$ because it entails two voices and it is consistent with two competing interpretations for three voices (cantus firmus, discantus, and countertenor), namely the English theory, which regards intervals only in relation to the cantus firmus, and the continental theory, which considers all intervallic relationships between the voices. More specifically, in the continental theory, we would consider the polynomial $a + b\epsilon + c\epsilon^2$, where $a$ is the cantus firmus and $b$ and $c$ the intervals between the cantus firmus and the discantus and between the discantus and the countertenor, respectively. On the other hand, we would have $a + b\epsilon$ and $a + (b + c)\epsilon$ in the English theory.

In Section 8, we explore the ring $\mathbb{Z}_{12} \times \mathbb{Z}_{12}$ as an alternative to the dual numbers.

3.5. Alternation and deformation

Contrapuntal tension is not only vertical (between cantus firmus and discantus) but horizontal (between an interval and its successor), as reflected by the division of consonances into perfect (unison/octave and fifth) and imperfect ones (thirds and sixths). This suggests a trace of dissonance inside consonance. These ideas are inspired by the work of Klaus-Jürgen Sachs as explained in Mazzola (2007, Section 14.1) and Mazzola (2002, 646).

To translate the idea of horizontal tension in the model, we consider progressions of consonances (interval to successor) that can be regarded as symmetry-deformed progressions from a dissonance to a consonance (Mazzola 2002, Section 31.1). This alternation (or contrast) process resembles the musical concept of the resolution of dissonances into consonances.

4. Summary of mathematical counterpoint theory

The following is a synthesis of a more robust counterpoint theory (Arias-Valero, Agustín-Aquino, and Lluis-Puebla 2021). We start with the following basic data.

- The ring $\mathbb{Z}_{12}$, which corresponds to the twelve intervals (up to the octave).
- The group of symmetries of $\mathbb{Z}_{12}$, denoted by $\text{Sym}(\mathbb{Z}_{12})$, which consists of all affine automorphisms of the form $e^t b : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12} : r \mapsto br + a$, where $b \in \mathbb{Z}_{12}^* = \{1, 5, 7, 11\}$. The symmetries of the form $e^t 1$, or $e^t$ for short, called translations, correspond to transpositions in music.
- The partition $\{K, D\}$ of $\mathbb{Z}_{12}$ into consonances $K$ and dissonances $D$, where $K = \{0, 3, 4, 7, 8, 9\}$ and $D = \{1, 2, 5, 6, 10, 11\}$. Mazzola’s fundamental observation is that $e^{25i}$ is the unique $p \in \text{Sym}(\mathbb{Z}_{12})$ such that $p(K) = D$.
- The dual numbers ring $\mathbb{Z}_{12}[\epsilon]$, which models contrapuntal intervals modulo octave. It consists of all linear polynomials $a + b\epsilon$ subject to the relation $\epsilon^2 = 0$. Formally, it is the quotient ring $\mathbb{Z}_{12}[x]/(x^2)$, where $\epsilon$ is the class of $x$.
- The group of symmetries of $\mathbb{Z}_{12}[\epsilon]$, denoted by $\text{Sym}(\mathbb{Z}_{12}[\epsilon])$, which consists of all affine automorphisms of the form $e^{u+ve}(c + d\epsilon) : \mathbb{Z}_{12}[\epsilon] \rightarrow \mathbb{Z}_{12}[\epsilon] : x + y\epsilon \mapsto (c + d\epsilon)(x + y\epsilon) + (u + v\epsilon)$, where $c \in \mathbb{Z}_{12}^*$. 
The induced partition \{K[\epsilon], D[\epsilon]\} of \(\mathbb{Z}_{12}[\epsilon]\) into contrapuntal consonances \(K[\epsilon]\) and contrapuntal dissonances \(D[\epsilon]\), where

\[ Y[\epsilon] = \{r + k\epsilon | r \in \mathbb{Z}_{12} \text{ and } k \in Y\} \]

for \(Y = K, D\).

We consider pairs \((\xi, \eta)\) of contrapuntal consonances in \(K[\epsilon]\), which we call progressions, and aim to determine when they are valid for counterpoint. The three principles in Sections 4.1, 4.2, and 4.3 serve this purpose.

4.1. Alternation and repetition

The model deals with polarized progressions, that is, progressions \((\xi, \eta)\) such that \(\xi \in g(D[\epsilon])\) and \(\eta \in g(K[\epsilon])\) for some \(g \in \text{Sym}(\mathbb{Z}_{12}[\epsilon])\). This is just the formal statement of the alternation principle in Section 3.5. As proved in Arias-Valero, Agustín-Aquino, and Lluis-Puebla (2021, Proposition 4.2), polarized progressions are exactly all but repetitions, the latter being of the form \((\xi, \bar{\xi})\). Since the model selects among them those that are optimal in a musical sense (Definition 4.2), we say that the model does not decide on the nature of repetitions.

4.2. Local characterization of consonances and dissonances

Although alternation helps to regard \(\xi\) as a deformed dissonance and \(\eta\) as a deformed consonance, we should ensure that they behave as dissonances and consonances in a mathematical sense. Thus, we would want a property for the partition \(\{K[\epsilon], D[\epsilon]\}\), analogous to the uniqueness one of \((K, D)\), that defines contrapuntal consonances and dissonances.

The symmetry \(e^{2\epsilon}5\) of \(\mathbb{Z}_{12}[\epsilon]\) is a natural extension of \(e^5\) sending \(K[\epsilon]\) to \(D[\epsilon]\) since \(e^{2\epsilon}5\) is simple and acts on the interval part \(d\) of a dual number \(a + be\) just as \(e^5\). But in this case, it is not the only one sending \(\mathbb{Z}_{12}[\epsilon]\) to \(D[\epsilon]\). For example, \(e^{2\epsilon+15}\) also does.

However, we have the following local uniqueness property (Mazzola 2002, Proposition 51), which is a characteristic one (Mazzola 2007, Section 14.2) of the consonance/dissonance partition \(\{z + K\epsilon, z + D\epsilon\}\) of the fiber \(z + \mathbb{Z}_{12}\epsilon\), thus offering a local definition of contrapuntal consonance and dissonance.\(^4\)

**Theorem 4.1** For each cantus firmus note \(z \in \mathbb{Z}_{12}\), the involutive symmetry \(p^z[\epsilon]\), defined by \(p^z[\epsilon] = e^z \circ e^{2\epsilon}5 \circ e^{-z} = e^{8z + 2\epsilon}5\), is the only one in \(\text{Sym}(\mathbb{Z}_{12}[\epsilon])\) that

1. leaves invariant the fiber \(z + \mathbb{Z}_{12}\epsilon\) and
2. sends \(K[\epsilon]\) to \(D[\epsilon]\).

In particular,

\[ p^z[\epsilon](z + K\epsilon) = z + D\epsilon. \]

If we apply this property to the deformed partition \(\{g(K[\epsilon]), g(D[\epsilon])\}\), we obtain the following one.

- The symmetry \(p^z[\epsilon]\) is the only one \(p^z \in \text{Sym}(\mathbb{R}[\epsilon])\) that leaves invariant the fiber \(z + \mathbb{Z}_{12}\epsilon\) and sends \(g(K[\epsilon])\) to \(g(D[\epsilon])\).

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\(^4\) See Arias-Valero, Agustín-Aquino, and Lluis-Puebla (2021, Section 4.2) for the emergence of a similar characterizing property of the consonance/dissonance partition \(z + K\epsilon, z + D\epsilon\). The two properties are related in Arias-Valero, Agustín-Aquino, and Lluis-Puebla (2021, Section 10), where it is proved that they are equivalent for the deformed partition \(\{g(K[\epsilon]), g(D[\epsilon])\}\) at a given fiber (Arias-Valero, Agustín-Aquino, and Lluis-Puebla 2021, Theorem 10.6).
By Arias-Valero, Agustín-Aquino, and Lluis-Puebla (2021, Proposition 10.5), this condition is equivalent to the following equation.

\[ p^z[\epsilon](g(K[\epsilon])) = g(D[\epsilon]) \]  

(1)

Mazzola requires it for the cantus firmus \( z \) of \( \xi \), which ensures that \( \xi \) is a local dissonance.

4.3. Variety

In this model, the variety principle of counterpoint (Fux, Mann, and Edmunds 1965, 21) corresponds to the condition that there is a maximum of alternations from \( \xi \), that is, the cardinality of the set \( g(K[\epsilon]) \cap K[\epsilon] \), where the possible successors \( \eta \) of \( \xi \) are, is maximum among all \( g \in \text{Sym}(R[\epsilon]) \) such that 1. \( \xi \in g(D[\epsilon]) \cap K[\epsilon] \) (alternation) and 2. Equation (1) holds for the cantus firmus \( z \) of \( \xi \) (local dissonance). See Figure 2.

4.4. Admitted successors

Now we can define the admitted successor of a contrapuntal interval.

**Definition 4.2** A contrapuntal symmetry for a consonance \( \xi \in K[\epsilon] \), where \( \xi = z + k\epsilon \), is a symmetry \( g \) of \( \mathbb{Z}_{12}[\epsilon] \) such that

1. \( \xi \in g(D[\epsilon]) \),
2. the symmetry \( p^z[\epsilon] \) sends \( g(K[\epsilon]) \) to \( g(D[\epsilon]) \), and
3. the cardinality of \( g(K[\epsilon]) \cap K[\epsilon] \) is maximum among all \( g \) satisfying 1 and 2.

Note that the contrapuntal symmetry for a given consonance is not required to be unique.

An admitted successor of a consonance \( \xi \in K[\epsilon] \) is an element \( \eta \) of \( g(K[\epsilon]) \cap K[\epsilon] \) for some contrapuntal symmetry \( g \). See Figure 2.

If \( \eta \) is an admitted successor of \( \xi \), we say that the progression \((\xi, \eta)\) is allowed. If it does not happen and \((\xi, \eta)\) is polarized, we say that it is forbidden.

4.5. Admitted successors computation

Denote by \( H \) the group of all symmetries of \( \mathbb{Z}_{12}[\epsilon] \) of the form \( e^{\pi \epsilon (c + d\epsilon)} \).

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Figure 2. Here, \( g \) is a deformation symmetry and \( \eta \) an admitted successor of \( \xi \).
According to Arias-Valero, Agustín-Aquino, and Lluis-Puebla (2021), after several simplifications, the admitted successors of \( z + k \epsilon \in K[\epsilon] \) are the elements of the sets of the form

\[
\epsilon^v (h(K[\epsilon]) \cap K[\epsilon]),
\]

where \( h = \epsilon^v (c + d \epsilon) \in H \) and

1. \( v \in k - cD \),
2. \( 5v + 2 = c2 + v \), and
3. the value \( \rho \sum_{i=0}^{\rho-1} |K_i| \) (the cardinality of \( h(K[\epsilon]) \cap K[\epsilon] \)) is maximum among all \( h = \epsilon^v (c + d \epsilon) \in H \) satisfying 1 and 2, where \( K_i = \{ k \in K \mid k \equiv i \pmod{\rho} \} \), \( \rho = \gcd(d, 12) \), and \( \epsilon^v c \) is reduced modulo \( \rho \).

The symmetries \( h \) satisfying the previous conditions are also contrapuntal symmetries for \( k \epsilon \).

This means that, to compute the admitted successors of a consonance \( z + k \epsilon \), it is enough to do so for \( k \epsilon \), and then apply the translation \( \epsilon^v \) (Equation (2)). This agrees with the previous result that counterpoint rules do not depend on particular notes; see Section 2.3. For these reasons, from now on we assume that all progressions are of the form \( (k \epsilon, c' + k' \epsilon) \).

Finally, the counterpoint symmetries and admitted successors of \( k \epsilon \in K[\epsilon] \) can be computed with Hichert’s algorithm (Mazzola 2002, 653). This algorithm ranges over all symmetries \( h \) satisfying (1) and (2), and updates in each step the set of those whose associated cardinalities, according to (3), are maximum so far. Then, with a counterpoint symmetry at hand, we compute the associated successors set according to the formula

\[
\epsilon^v (c + d \epsilon)(K[\epsilon]) \cap K[\epsilon] = \bigcup_{r \in \mathbb{Z}_{12}} cr + ((cK + v + dr) \cap K)\epsilon.
\]

Note that the previous union is disjunct since the map \( r \mapsto cr \) is a permutation of \( \mathbb{Z}_{12} \).

Table 1 in the Online Supplement shows all counterpoint symmetries and successors. It was obtained with the Code 2 in the Online Supplement.

According to Code 3 in the Online Supplement, out of the 287 progressions that occur in the diatonic scale \( X \), 6 of them are repetitions (non-polarized), 250 are allowed, and the remaining 31 are forbidden. Next, we reduce the strict style so as to compare it with the model.

### 5. The reduced strict style

Here we reduce, modulo 12 (up to the octave), the strict style modulo translation studied in Section 2. We use the projection that sends a progression \( (kx, c' + k'x) \) in the strict style to \( (\pi(k)e, \pi(c') + \pi(k')e) \), where \( \pi : \mathbb{Z} \longrightarrow \mathbb{Z}_{12} \) is the natural projection.

We first note that this projection covers all progressions in a diatonic scale. In fact, each such progression \( (k\epsilon, c' + k'\epsilon) \), regarded as \( (kx, c' + k'x) \), satisfies all preliminary rules in Section 2.1, except perhaps the condition that the maximum change between the discantus notes is an octave. In such a case, we transpose the second interval an octave downwards and obtain an strict style progression that projects on \( (k\epsilon, c' + k'\epsilon) \).

Now we define the progression rules of the reduced strict style. According to Mazzola and Muzzulini (1990), a progression in a diatonic scale is

- **good**, if it is the projection of at least one good progression,
- **inadmissible**, if it is derived from nothing but inadmissible progressions, and
- **bad**, if it is the projection of at least one bad progression, but not of a good one.
This definition leads to the following characterization of these rules. Code 4 in the Online Supplement contains some computations involved.

### Inadmissible progressions

Under projection, unison repetitions remain unchanged and they are good, parallel unisons or octaves become parallel unisons or unison repetitions, parallel fifths become parallel fifths or fifth repetitions (good), and tritones become tritones possibly greater than the octave.

**Parallel unisons and fifths**

*Some parallel unisons remain inadmissible.* If the skip is the tritone, then it is inadmissible. Now, parallel unisons come from three other kinds of progressions: 1. unison to octave, 2. parallel octaves (inadmissible), and 3. octave to unison. The cases 1 or 3 are inadmissible if and only if \( c' \) or \( 12 - c' \) are greater than 7, that is, \( c' \in \{1, 2, 3, 4, 8, 9, 10, 11\} \). Otherwise, we have good progressions by contrary motion and skips not too large.

**Parallel fifths are inadmissible.** They can only be the octave reduction of parallel fifths, which are inadmissible, because twelfths are not in the range of the strict style.

**Tritones**

*Tritones are inadmissible.* Tritones only come from tritones, which are inadmissible.

**Projected hidden parallelisms**

*Hidden parallel fifths from a sixth are inadmissible.* They are only the projection of progressions of the same kind.

The remaining cases that have no tritones are not inadmissible (Code 4).

**Too large skips under projection**

All cases fall into the previous inadmissible ones or are not inadmissible (Code 4).

### Bad progressions

Bad progressions in \( \mathbb{Z}_{12}[\epsilon] \) have to be projections of at least one bad progression in \( \mathbb{Z}[\epsilon] \).

**Projected imperfect consonances by similar skips**

*Here, the only bad progression is* \((0 + 7\epsilon, 5 + 9\epsilon)\); see Code 4. The other ones are good.

**Hidden tritones**

*Hidden tritones are bad.* They only come from hidden tritones, which are bad.

To sum up, in the reduced strict style, the inadmissible progressions are all tritones and, besides them, some parallel unisons, all parallel fifths, and all hidden parallel fifths from a sixth. Moreover, as noted in Mazzola and Muzzulini (1990), only the parallel fifths and tritone rules preserve their generality (unrestricted validity for all cases).

### 5.1. On semantics

However, out of the previous categories (*inadmissible, bad, good*), the most appropriate to provide a quantitative assessment of the agreement of the model with the original rules seem to be the *inadmissible* and *bad* ones. Certainly, an inadmissible progression only comes from inadmissible progressions so it is *unequivocally inadmissible* and bad progressions are optional, and unequivocally bad in this case (check). In contrast, good progressions also come from inadmissible or bad progressions, except for the four imperfect consonance repetitions, which only come from good ones, as we deduce from the following definition.

We have a refined semantics of the reduced strict style. We preserve the definition of inadmissible and bad progressions. We subclassify a remaining good progression as
Table 2. Tally of the reduced-strict-style progressions and their types.

| Total   | Inadmissible | Parallel Fifths* | Parallel Unisons | Hidden Fifths from a Sixth | Tritones* |
|---------|--------------|------------------|-----------------|---------------------------|-----------|
| 287 prog. | 74           | 10               | 9               | 13                        | 45        |
| 23 bad   | 1            | 1                |                 |                           |           |
| 190 good | 4            | 16               | 197             |                           |           |

Note: The symbol * refers to rules whose generality (validity for all cases) remains under projection.

Table 3. Inadmissible, bad, and good progressions (reduced strict style) versus allowed and forbidden ones (model).

| Allowed | Inadmissible | Bad | Good | Good* | Good-Good | Good-Bad | Ambiguous |
|---------|--------------|-----|------|-------|-----------|----------|-----------|
| 55      | 36           | 17  | 178  | 197   | 0         | 16       | 162       |
| 19      | 19           | 6   | 6    | 6     | 0         | 0        | 6         |
| 0       | 0            | 0   | 6    | 4     | 0         | 2        |           |

Note: The symbol * refers to rules whose generality remains under projection.

Table 4. Allowed and forbidden kinds of inadmissible and bad progressions.

| Parallel Fifths | Parallel Unisons | Hidden Fifths | Tritone | ($0 + 7\epsilon, 5 + 9\epsilon$) | Hidden Tritone |
|-----------------|------------------|---------------|---------|----------------------------------|---------------|
| Allowed         | 0                | 8             | 13      | yes                              | 16            |
| Forbidden       | 10               | 1             | 0       | no                               | 6             |

good-good, if it is derived from nothing but good progressions,
ambiguous, if it also comes from at least an inadmissible progression, and
good-bad, if it also comes from at least a bad progression but not an inadmissible one.

This definition offers a fairer classification: ambiguous progressions have not a defined value comparable to allowed and forbidden.

Table 2 contains the tally of all reduced-strict-style progressions and their types, according to Code 4 in the Online Supplement.

6. The reduced strict style and the model

Table 3 compares all types of strict-style progressions with allowed and forbidden ones, according to Code 5 in the Online Supplement.

Table 4 shows all allowed and forbidden kinds of inadmissible and bad progressions, according to Code 6 in the Online Supplement. We observe that the model predicts the parallel fifths prohibition and some tritone rules.

We can measure the agreement between the model and the reduced style by means of the number of matches and mismatches. The matches are all allowed/good, forbidden/inadmissible, and (possibly) forbidden/bad progressions. The mismatches are all forbidden/good and allowed/inadmissible ones. From Table 3, we observe that there are 203 matches, including forbidden/bad progressions, and 61 mismatches. If we only take into account rules whose generality remains under projection, we obtain 222 matches and 42 mismatches.
We define matches and mismatches in the refined semantics by replacing good progressions by good-good progressions in the previous paragraph. In the case of the previous model, there are 25 matches and 55 mismatches. But, since there are only six good-good progressions, these measures are not appropriate. For example, the trivial model that forbids all progressions, except the four imperfect consonance repetitions, has 101 matches and no mismatches – the best possible result. This means that the ambivalence of 186 good progressions (ambiguous and good-bad) does not allow appropriate semantics on reduced progressions that captures the essence of the original good ones. However, we use these measures to discard possible ambiguities.

Thus, a careful review of the semantics offers a different point of view on the results of the model. For instance, we observe that the twelve examples (other than repetitions) of progressions forbidden by Mazzola but not by Fux, according to Tymoczko (2011, Figure 3b), are the six forbidden/bad together with the six forbidden/good progressions in Table 3. The last six progressions are ambiguous according to the refined semantics, so the twelve examples are not actually wrong predictions of the model.

7. Some questions on the model

The conceptual review of the principles in Sections 3.4 and 4.2 leads to the following questions.

- Could the discantus be something not a tangential alteration of the cantus firmus?
- Could we require the local characterization condition on \( \{g(K[\epsilon]), g(D[\epsilon])\} \) for the cantus firmus of a possible successor \( \eta \)?
- More radically, could we require the local characterization on \( \{g(K[\epsilon]), g(D[\epsilon])\} \) for all fibers?

These questions inspire the following three variations of the model.

8. First variation of the model

Besides the uniqueness condition on the partition \( \{K, D\} \), the dual numbers structure on counterpoint intervals is the most important structural feature of Mazzola’s model. However, it is not utterly clear why we use the structure of the dual numbers ring to model intervals and not another one.

Another simple choice for a ring structure on counterpoint intervals is that induced by the product ring \( \mathbb{Z}_{12} \times \mathbb{Z}_{12} \), whose elements \( (c, d) \) can be regarded as a pair of cantus firmus and discantus notes that occur simultaneously. If we transfer its structure to contrapuntal intervals, under the map that sends \( (c, d) \) to \( c + (d - c)x \) (cantus firmus and interval with the discantus), we obtain (check) the structure of the quotient ring \( \mathbb{Z}_{12}[x] / (x^2 - x) \) – a variation of the dual numbers ring. In this case, we denote the class of \( x \) by \( , \) and the ring by \( \mathbb{Z}_{12}[\delta] \). The latter consists of linear polynomials \( a + b\delta \), where \( a, b \in \mathbb{Z}_{12} \) and \( \delta^2 = \delta \), that is, is idempotent.

The rings \( \mathbb{Z}_{12}[\epsilon] \) and \( \mathbb{Z}_{12}[\delta] \) are equal as additive groups but they differ in their multiplications. In \( \mathbb{Z}_{12}[\epsilon] \), \( d\epsilon d'\epsilon = dd'\epsilon^2 = 0 \) for all \( d, d' \in \mathbb{Z}_{12} \), so intervallic variations are regarded as infinitesimals, whereas in \( \mathbb{Z}_{12}[\delta] \), \( d\delta d'\delta = dd'\delta^2 = dd'\delta \), so these variations are just integers modulo 12.

According to Arias-Valero, Agustín-Aquino, and Lluis-Puebla (2021), like in the case of dual numbers, symmetries of \( \mathbb{Z}_{12}[\delta] \) are of the form \( e^{a + b\delta} (c + d\delta) \), but both \( c \) and \( c + d \) are in \( \mathbb{Z}_{12}^* \).

---

6 In other words, a strong attribute of being good is invisible or undetectable from the point of view of the reduced progressions.
now. In the same way, the extension of consonances and dissonances \( \{K[\delta], D[\delta]\} \) to counterpoint intervals can be defined and Theorem 4.1 remains valid by changing \( \epsilon \) for \( \delta \). The non-polarized symmetries, according to the definition in Section 4.1, are all repetitions together with all parallelisms by tritone leaps in this case. Definition 4.2 is the same because the motivations remain valid, except that we replace \( \epsilon \) by \( \delta \). Regarding Section 4.5, now \( H \) consists of all symmetries of \( \mathbb{Z}_{12}[\delta] \) of the form \( e^{\delta}(c + d\delta) \), which satisfy \( c, c + d \in \mathbb{Z}_{12}^* \). However, conditions 1 and 2 become

\[
(1') \ v \in k - (c + d)D \quad \text{and} \\
(2') \ 5v + 2 = (c + d)2 + v,
\]

whereas 3 remains unchanged. We compute a typical successors set with the formula:

\[
e^{\delta}(c + d\delta)(K[\delta]) \cap K[\delta] = \bigcup_{r \in \mathbb{Z}_{12}} cr + (((c + d)K + v + dr) \cap K)\delta.
\]

The results of this model, and their comparison to the strict reduced style, are summarized in Tables 5 and 6. Here, out of the 287 progressions that occur in a diatonic scale, 7 are non-polarized, 240 are allowed, and 40 are forbidden. These results correspond to Code 7 in the Online Supplement. We observe that the model predicts 21 inadmissible progressions. Actually, it predicts 18 out of the 19 inadmissible progressions of the classical model together with three new hidden fifths from a sixth. We lose the parallel unison by tritone skip, since it is non-polarized here. Regarding bad progressions, it predicts the same six hidden tritones. However, in this case, we have 198 matches and 65 mismatches with the reduced strict style. In the refined semantics the respective measures are 27 and 52.

### Table 5. Inadmissible, bad, and good progressions (reduced strict style) versus allowed and forbidden ones (first variation).

|       | inad. | bad  | good | good-good | good-bad | amb. |
|-------|-------|------|------|-----------|----------|------|
| Allowed | 52    | 17   | 171  | 0         | 15       | 156  |
| Forbidden | 21    | 6    | 13   | 0         | 1        | 12   |
| Non-polarized | 1    | 0    | 6    | 4         | 0        | 2    |

### Table 6. Allowed and forbidden kinds of inadmissible progressions (first variation).

|       | par. 5ths | par. un. | hid. 5ths | trit. | (0 + 7\delta, 5 + 9\delta) | hid. trit. |
|-------|-----------|----------|-----------|-------|---------------------------|-----------|
| allowed | 0         | 8        | 10        | 36    | yes                       | 16        |
| forbidden | 10       | 0        | 3         | 8     | no                        | 6         |
| non-polarized | 0     | 1        | 0         | 1     | no                        | 0         |

9. **Second variation of the model**

If we also require the local characterization on the fibers of the possible successors of a consonance, then we change condition (3) in Definition 4.2 for the following one, where \( P(z) \) denotes the property (2) in Definition 4.2.

- The cardinality of \( \{z' + k'\epsilon \in g(K[\epsilon]) \cap K[\epsilon] | P(z')\} \) is maximum among all \( g \) satisfying (1) and (2).
Table 7. Inadmissible, bad, and good progressions (reduced strict style) versus allowed and forbidden ones (final variations).

|        | inad. | bad | good | good-good | good-bad | amb. |
|--------|-------|-----|------|-----------|----------|------|
| allowed| 53    | 15  | 167  | 0         | 12       | 155  |
| forbidden| 21   | 8   | 17   | 0         | 4        | 13   |
| repetitions| 0    | 0   | 6    | 4         | 0        | 2    |

Table 8. Allowed and forbidden kinds of inadmissible and bad progressions (final variations).

|        | par. 5ths | par. un. | hid. 5ths | trit. | (0 + 7\(\epsilon\), 5 + 9\(\epsilon\)) | hid. trit. |
|--------|------------|----------|-----------|-------|----------------------------------------|------------|
| allowed| 0          | 4        | 13        | 38    | yes                                    | 14         |
| forbidden| 10        | 5        | 0         | 7     | no                                     | 8          |

Remarkably, this variation is equivalent to the following one, as proved in Arias-Valero, Agustín-Aquino, and Lluis-Puebla (2021, Section 11.3).

10. Third variation of the model

If we require the local characterization property on the deformed partition for all fibers, which amounts to a *global condition*, then we change condition (2) in Definition 4.2 for the following one.

- For all \(z \in \mathbb{Z}_{12}\), \(P(z)\) holds.

According to Arias-Valero, Agustín-Aquino, and Lluis-Puebla (2021, Section 11.3), so as to compute the admitted successors according to this definition, it is enough to follow the procedure in Section 4.5 by adding to condition (2) the equation \(5d = d\). The final results of the second and third variations coincide and they are the same for the case of \(\mathbb{Z}_{12}[\delta]\) (Arias-Valero, Agustín-Aquino, and Lluis-Puebla 2021, Sections 11.2–11.3), except for the difference that the inadmissible parallel unison by tritone skip is forbidden in the dual numbers case but non-polarized in the case of \(\mathbb{Z}_{12}[\delta]\).

The comparison of these results with the reduced strict style are in Tables 7 and 8. They correspond to Code 8 in the Online Supplement. We only put the results for the case of \(\mathbb{Z}_{12}[\epsilon]\). Here, there are six repetitions, 46 forbidden progressions, and 235 allowed ones, that occur in a diatonic scale.

The most important feature of these results, regarding the original model, is the *prediction of new parallel unisons*. However, we have 196 matches and 70 mismatches. In the refined semantics, the respective measures are 29 and 53.

11. Conclusions

_Economy_

Essentially, we deduce all mathematical results of the theory from just a fact: the uniqueness property of the consonance/dissonance partition. In particular, we deduce the parallel fifths prohibition.
Table 9. Matches and mismatches of all models with the reduced strict style according to the original semantics and the refined semantics.

|           | matches | mismatches | matches (ref. sem.) | mismatches (ref. sem.) |
|-----------|---------|------------|---------------------|------------------------|
| classical | 203     | 61         | 25                  | 55                     |
| first var.| 198     | 65         | 27                  | 52                     |
| final var.| 196     | 70         | 29                  | 53                     |

Notes: Matches include all forbidden/inadmissible, forbidden/bad, and allowed/good progressions. Mismatches include all forbidden/good and allowed/inadmissible progressions. In the refined semantics, we change good progressions for good-good progressions, which do not come from inadmissible or bad progressions in the strict style.

**Generality and universality**

The models have a generalization to any (not necessarily commutative) ring (Arias-Valero, Agustín-Aquino, and Lluis-Puebla 2021), taking the place of $\mathbb{Z}_{12}$. This is just an expression of the original intention of establishing a universal counterpoint by detecting the essential features of the Renaissance incarnation. Thus, the model paves the way to many other forms of counterpoint, which opens up a lot of fields of musical experimentation.

**Quantitative interpretation of the results**

Table 9 shows the matches and mismatches of all models with the reduced strict style. There is a number of mismatches in all cases, which might be interpreted as a weakness of the model, but the number of matches is about three times greater.

The number of matches and mismatches can increase or decrease from the classical to the final variations according to the semantics used, but the differences do not exceed five units.

**Qualitative interpretation of the results**

Rather than a mere description of the Renaissance counterpoint rules (Section 2.3), we believe that the point is what mathematical counterpoint theory can explain about them. Mazzola’s model predicts the parallel fifths rule in its generality and some tritone prohibitions. Additionally, the first variation of the model predicts some hidden fifths, and the final variation predicts some parallel octaves.

11.1. **Recovery of the original phenomenon from the model**

We can characterize perfect consonances as 0 (which belongs to any ring) and the consonance with a general parallelism prohibition. The tritone can be characterized as the distinguished skip that makes a parallel unison forbidden. In the first variation, the tritone is the non-polarized skip.

In this way, we can recover the original rules by establishing as prohibitions the parallel and hidden perfect consonances, the distinguished skip, and the too large skips.

12. **Further developments**

**Structural approaches**

The mathematical counterpoint theory is an independent field of study. There are several open questions like whether there is an entirely structural proof of the Counterpoint theorem, which
establishes all counterpoint symmetries and admitted successors of a given consonance, beyond
algorithmic computations. Some recent advances in that direction, like the counting formulas
for successors sets cardinalities and a maximization criterion, can be found in Arias-Valero,
Agustín-Aquino, and Lluis-Puebla (2021).

Study of new counterpoint worlds

The musical study of these generalizations is an open field of research. It can start by establishing
rules based on the model, analogous to the original rules of counterpoint, following the procedure
suggested in Section 11.1. Then, composition processes should generate new music in these
worlds, as mentioned in Section 1. The counterpoint world induced by Scriabin’s mystic chord
has also been the subject of study (Agustín-Aquino and Mazzola 2019).

Musicology

An initial comparison of the model’s results with the Missa Papae Marcelli can be found in Nieto
(2010). The analysis of other works of Renaissance polyphony is also a pending task.

Remaining species

Although there are some advances regarding the second-species counterpoint (Agustín-Aquino
and Mazzola 2022), a mathematical theory of the remaining species, and the cases of three or
more voices, are not at hand.

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Online Supplement

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ORCID

Juan Sebastián Arias-Valero http://orcid.org/0000-0001-6812-7639
Octavio Alberto Agustín-Aquino http://orcid.org/0000-0002-0556-6236
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