Regularity/Controllability/Observability of an NDS with Descriptor Form Subsystems and Generalized LFTs

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Abstract—This paper investigates regularity, controllability and observability for a networked dynamic system (NDS) with its subsystems being described in a descriptor form and system matrices of each subsystem being represented by a generalized linear fractional transformation (LFT) of its parameters. Except a well-posedness condition, any other constraints are put on either parameters or connections of a subsystem. Based on the Kronecker canonical form of a matrix pencil, some matrix rank based necessary and sufficient conditions are established respectively for the regularity and complete controllability/observability of the NDS, in which the associated matrix depends affinely on both subsystem parameters and subsystem connections. These conditions keep the property that all the involved numerical computations are performed on each subsystem independently, which is attractive in the analysis and synthesis of a large scale NDS. Moreover, some explicit and easily checkable requirements are derived for both subsystem dynamics and subsystem parameters with which a controllable/observable NDS can be constructed more easily.

Index Terms—controllability, descriptor system, first principle parameter, generalized LFT, networked dynamic system, observability, regularity.

I. INTRODUCTION

Controllability and observability are extensively regarded as essential requirements for a system to work properly. Various system performances, such as minimizing the tracking error of a servo system, the existence of an optimal control parameter, generalized LFT, networked dynamic system, observability, regularity.

While various results have been obtained for controllability/observability of an NDS, many theoretical issues still require further efforts, which include influences from subsystem dynamics, subsystem connections, etc., to the controllability/observability of the whole system. Another challenging issue is computational costs and numerical stability [2, 3, 23].

In the description of plant dynamics, descriptor systems have proved to be an appropriate model. It is widely believed that compared with the extensively adopted state space model, a descriptor system is more suitable in keeping structural information of plant dynamics. Like the state space model, this model has also been adopted in many fields including engineering, economy, biology, etc., and has attracted extensive research attentions [5, 7, 12]. In addition, it has been argued in [9] that a generalized linear fractional transformation (LFT) is more efficient in describing dependence of system matrices on its parameters, in the sense that the associated matrices have a lower dimension.

In this paper, we investigate regularity and controllability/observability for NDSs in which each subsystem is described by a descriptor form like model, while its system matrices are described by a generalized LFT. Results of [20, 22, 23] have been extended, in which each subsystem is described by a state space like model and for each subsystem,
all of its system matrices are assumed to be known or to be described an LFT. In this investigation, except that the NDS is required to be well-posed, including the whole system and each of its subsystems, there are neither any other restrictions on a subsystem first principle parameter, nor any other restrictions on an element of the SCM. Using the Kronecker canonical form (KCF) of a matrix pencil, several rank based conditions are established respectively for the regularity of the NDS and its complete controllability/observability. In these conditions, the associated matrix affinely depends on both the subsystem connection matrix (SCM) and a matrix formed by the parameters of each subsystem, which agrees well with that of \cite{24, 22, 23}. These conditions keep the properties of the results reported in \cite{24, 22, 23} that in obtaining the associated matrices, all the required numerical computations are performed on each subsystem independently. This is quite attractive, as it means that the associated condition verification is scalable for an NDS constructed from a large number of subsystems. In addition, this explicit relation between the condition related matrix and subsystem parameters/connections may be helpful in system topology design and parameter selections, as well as subsystem dynamics selections. Moreover, some explicit conditions are obtained for both the dynamics and parameters of a subsystem with which a completely controllable/observable NDS can be constructed more easily. These conditions can be verified for each subsystem separately, which makes them convenient in the design of a large scale NDS.

The outline of this paper is as follows. At first, in Section II, a descriptor form like model is given for an NDS, together with some preliminary results. Regularity, controllability and observability of an NDS are investigated in Section III under the assumption that the parameters of each subsystem are known. These results are extended in Section IV to an NDS, in which system matrices of each subsystem depends on its parameters through a generalized LFT. Finally, some concluding remarks are given in Section V in which several further issues are discussed. Four appendices are included to give proofs of some technical results.

The following notation and symbols are adopted. $C$ and $R^n$ stand respectively for the set consisting of complex numbers and the $n$ dimensional real Euclidean space. $\text{det} (\cdot)$ represents the determinant of a square matrix, $\text{null} (\cdot)$ and $\text{span} (\cdot)$ the (right) null space of a matrix and the space spanned by the columns of a matrix. $\text{diag}\{X_i\}_{i=1}^{|\mathcal{J}|}$ denotes a block diagonal matrix with its $i$-th diagonal block being $X_i$, while $\text{col}\{X_i\}_{i=1}^{|\mathcal{J}|}$ the vector/matrix stacked by $X_i$'s with its $i$-th row block vector/matrix being $X_i$. For a $m \times n$ dimensional matrix $A$, $A(1 : k)$ stands for the matrix consisting of its first $k$ columns with $k$ satisfying $1 \leq k \leq n$, while $A(\mathcal{J})$ the matrix consisting of its columns indexed by the elements of the set $\mathcal{J}$ with $\mathcal{J} \subset \{1, 2, \cdots, n\}$. $\theta_m$ and $\theta_m \times n$ represent respectively the $m$ dimensional zero column vector and the $m \times n$ dimensional zero matrix. The subscript is usually omitted if it does not lead to confusions. The superscript $T$ and $H$ are used to denote respectively the transpose and the conjugate transpose of a matrix/vector. A matrix valued function is said to be full normal column/row rank, if there exists a value for its variable(s), at which the value of the matrix is of full column/row rank.

II. SYSTEM DESCRIPTION AND SOME PRELIMINARIES

In many actual engineering problems, an NDS is constructed from subsystems with distinguished input-output relations. A possible approach to describe the dynamics of a general linear time invariant (LTI) NDS is to represent the dynamics of each subsystem using an ordinary model, divide the inputs/outputs of each subsystem into external and internal ones, express subsystem interactions through connecting internal outputs of a subsystem to internal inputs of some other subsystems. This approach has been adopted in \cite{24, 22, 23} in which the dynamics of each subsystem is described by a state space model. To reflect structure information of each subsystem in an NDS more appropriately, a descriptor form is adopted in this paper in its dynamics description. In addition, in order to express the dependence of the system matrices of a subsystem on its first principle parameters, it is assumed that each of their elements is a function of these parameters. More precisely, the following model is used in this paper to describe the dynamics of the $i$-th subsystem $\Sigma_i$ of an NDS $\Sigma$ composing of $N$ subsystems,

$$
\begin{bmatrix}
E(i, p_i)x(t + 1, i) \\
y(t, i)
\end{bmatrix} =
\begin{bmatrix}
A_{xx}(i, p_i) & A_{xy}(i, p_i) & B_x(i, p_i) \\
A_{yx}(i, p_i) & A_{yy}(i, p_i) & B_y(i, p_i) \\
C_x(i, p_i) & C_y(i, p_i) & D_u(i, p_i)
\end{bmatrix}
\begin{bmatrix}
x(t, i) \\
v(t, i) \\
u(t, i)
\end{bmatrix}
$$

(1)

Here, $p_i$ stands for the vector that consists of all the parameters in the subsystem $\Sigma_i$, which may be the masses, spring and damper coefficients of a mechanical system, concentrations and reaction ratios of a chemical/biological process, resistors, inductor and capacitor coefficients of an electronic/electrical system, etc. These parameters are usually called a first principle parameter (FPP) as they can be selected or adjusted in designing an actual system. On the other hand, $t$ represents the temporal variable, $x(t, i)$ its state vector, $u(t, i)$ and $y(t, i)$ respectively its external input and output vectors, $v(t, i)$ and $z(t, i)$ respectively its internal input and output vectors which denote signals received from other subsystems and signals transmitted to other subsystems.

To emphasize the simultaneous existence of both external and internal inputs/outputs in the above description, it is called a descriptor form like model throughout this paper.

Define vectors $v(t)$ and $z(t)$ respectively as $v(t) = \text{col}\{v(t, i)\}_{i=1}^N$, $z(t) = \text{col}\{z(t, i)\}_{i=1}^N$. It is assumed in this paper that the interactions among subsystems of the NDS $\Sigma$ are described by

$$
v(t) = \Phi z(t)
$$

(2)

The matrix $\Phi$ is called the subsystem connection matrix (SCM), which describes influences between different subsystems of an NDS. A graph can be assigned to an NDS when each subsystem is regarded as a node and each nonzero element in the SCM $\Phi$ is regarded as an edge. This graph is
usually referred as the structure or topology of the associated NDS.

Compared with the subsystem model adopted in [20, 22, 23], it is clear that each of its system matrices in the above model, that is, $A_{s}(t), B_{s}(t), C_{s}(t)$ with $s, # = x, u, v, y$ or $z$, as well as the matrices $E(i)$ and $D_{m}(i)$, is a matrix valued function of the parameter vector $p_{i}$. This reflects the fact that in an actual system, elements of its system matrices are usually not independent of each other, and some of them can not be changed in system designs. It can therefore be declared that this model is more convenient in investigating influences of system parameters on the behaviors of a dynamic plant.

Obviously, the aforementioned model is also applicable to situations in which we are only interested in the influences from part of the subsystem FPPs on the performances of the whole NDS. This can be simply done through fixing all other FPPs to a particular numerical value.

The following assumptions are adopted throughout this paper for the NDS $\Sigma$.

- The dimensions of the vectors $u(t, i), v(t, i), x(t, i)$ and $z(t, i)$ are respectively $m_{ui}, m_{vi}, m_{xi}, m_{zi}$.
- Each subsystem $\Sigma_{i}, i = 1, 2, \cdots , N$, is well posed.
- The whole NDS $\Sigma$ is well posed.

Note that the first assumption is only for indicating the size of the involved vectors. On the other hand, well-posedness is an essential requirement for a system to work properly [12, 19, 23]. It appears safe to declare that all the above three assumptions must be satisfied for a practical system. Therefore, the adopted assumptions seem not very restrictive in actual applications.

Using these symbols, define integers $M_{xi}, M_{vi}, M_{xi}$ and $M_{vi}$ as $M_{xi} = \sum_{k=1}^{N} m_{sk}, M_{vi} = \sum_{k=1}^{N} m_{vk},$ and $M_{xi} = M_{vi} = 0$ when $i = 1$, $M_{xi} = \sum_{k=1}^{N} m_{sk}, M_{vi} = \sum_{k=1}^{N} m_{vk}$ when $2 \leq i \leq N$. Obviously, the SCM $\Phi$ is a $M_{x} \times M_{v}$ dimensional real matrix. These definitions are adopted throughout the rest of this paper.

The following results on a matrix pencil are required in deriving a computationally checkable necessary and sufficient condition for the regularity, controllability or observability of the aforementioned NDS, which can be found in many references, for example, [1, 11, 12].

For two arbitrary $m \times n$ dimensional real matrices $G$ and $H$, a first degree matrix valued polynomial $\Psi(\lambda) = \lambda G + H$ is called a matrix pencil. When $m = n$ and $\det(\Psi(\lambda)) \neq 0$, this matrix pencil is called regular. A regular matrix pencil is called strictly regular if both the associated matrix $G$ and the associated matrix $H$ are invertible. A matrix pencil $\Psi(\lambda)$ is said to be strictly equivalent to the matrix pencil $\Psi(\lambda)$, if there exist two invertible real matrices $U$ and $V$ satisfying $\Psi(\lambda) = U \Psi(\lambda)V$.

Given a positive integer $m$, two $m \times m$ matrix pencils $K_{m}(\lambda)$ and $N_{m}(\lambda)$, a $m \times (m + 1)$ matrix pencil $L_{m}(\lambda)$, as well as a $(m + 1) \times m$ matrix pencil $J_{m}(\lambda)$, are defined respectively as follows.

$$K_{m}(\lambda) = \lambda I_{m} + \begin{bmatrix} 0 & I_{m-1} \end{bmatrix}, \quad N_{m}(\lambda) = \lambda \begin{bmatrix} 0 & I_{m-1} \end{bmatrix} + I_{m}\quad (3)$$

$$L_{m}(\lambda) = \begin{bmatrix} K_{m}(\lambda) & 0 \\ 0 & 1 \end{bmatrix}, \quad J_{m}(\lambda) = \begin{bmatrix} K_{m}^{T}(\lambda) & 0 \\ 0 & 1 \end{bmatrix}\quad (4)$$

Obviously, $J_{m}(\lambda) = L_{m}^{T}(\lambda)$. In the following analysis of the regularity, controllability and observability of the NDS $\Sigma$, however, the roles of these two matrix pencils are completely different. To emphasize these differences, as well as to have a clear presentation, it appears better to adopt different symbols for these two matrix pencils. Moreover, when $m = 0$, $L_{m}(\lambda)$ is a $0 \times 1$ zero matrix, while $J_{m}(\lambda)$ is a $1 \times 0$ zero matrix.

From the definitions of the matrix pencils $K_{m}(\lambda)$, $N_{m}(\lambda)$, $L_{m}(\lambda)$ and $J_{m}(\lambda)$, the following characteristics can be straightforwardly established for their rank and the associated null spaces. The details of the proof are omitted due to their obviousness, but can be found in [22].

**Lemma 1:** For any positive integer $m$, the matrix pencils defined in Equations (3) and (4) respectively have the following properties.

- A $m \times m$ dimensional strictly regular matrix pencil $H_{m}(\lambda)$ is rank deficient only at some isolated values of the complex variable $\lambda$ which are different from zero. Moreover, the number of these values is equal to $m$.
- The matrix pencil $N_{m}(\lambda)$ is always of full rank (FR).
- The matrix pencil $J_{m}(\lambda)$ is always of full column rank (FCR).
- The matrix pencil $K_{m}(\lambda)$ is singular only at $\lambda = 0$. Moreover,

$$\text{Null } \{K_{m}(0)\} = \text{Span } \left\{ \begin{bmatrix} 1, 0, \cdots, 0 \end{bmatrix}^{T} \right\}_{m-1 \text{ copies}}$$

- The matrix pencil $L_{m}(\lambda)$ is not of FCR at an arbitrary complex $\lambda$. Moreover,

$$\text{Null } \{L_{m}(\lambda)\} = \text{Span } \left\{ 1, (-\lambda)^{j} \right\}_{j=1}^{m}$$

It is well known that any matrix pencil is strictly equivalent to a block diagonal form with its diagonal blocks being strictly regular, or in the form of the matrix pencils $K_{m}(\lambda)$, $N_{m}(\lambda)$, $L_{m}(\lambda)$ and $J_{m}(\lambda)$, which is extensively called the Kronecker canonical form (KCF). More precisely, we have the following results [8, 11].

**Lemma 2:** For any two $m \times n$ dimensional real matrices $G$ and $H$, there exist some unique nonnegative integers $\mu$, $a$, $b$, $c$ and $d$, some unique positive integers $\xi_{j}$ and $\eta_{j}$, some unique nonnegative integers $s_{j}$ and $r_{j}$, as well as a strictly regular $\mu \times \mu$ dimensional matrix pencil $H_{\mu}(\lambda)$, such that the matrix pencil $\Psi(\lambda) = \lambda G + H$ is strictly equivalent to a block diagonal form $\Psi(\lambda)$ with the following definition.

$$\Psi(\lambda) = \text{diag} \{H_{\mu}(\lambda), K_{\xi_{j}}(\lambda)\}_{j=1}^{s_{j}}, L_{\eta_{j}}(\lambda)\}_{j=1}^{r_{j}}, N_{\xi_{j}}(\lambda)\}_{j=1}^{s_{j}}, J_{\eta_{j}}(\lambda)\}_{j=1}^{r_{j}}\quad (5)$$

Descriptor systems are extensively utilized in describing
input-output relations of a dynamic plant. It is believed that compared with a state space model, a descriptor system is more suitable in describing system constraints and keeping system structures fixed. More precisely, if the input-output relations of an LTI plant can be described by the following equations,
\[ Ex(t + 1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) \] (6)
in which \( A, B, C, D \) and \( E \) are constant real matrices with consistent dimensions, then this plant is called a descriptor system. When a descriptor system is regular, unique-
\[ \text{Lemma 3: Assume that the descriptor system of Equation (6) is regular. Then it is completely observable if and only if the following two conditions are satisfied simultaneously,} \]
- the matrix \[ \begin{bmatrix} E & C \end{bmatrix} \] is of FCR;
- the matrix pencil \[ \lambda E - A \]
is of FCR at every \( \lambda \in \mathbb{C} \).
Moreover, it is completely controllable if and only if the following two conditions are satisfied simultaneously,
- the matrix \[ \begin{bmatrix} E & B \end{bmatrix} \] is of FRR;
- the matrix pencil \[ \lambda E - A \]
is of FRR at every \( \lambda \in \mathbb{C} \).

Obviously, similar to a state space model, complete controllability and complete observability of a descriptor system are dual to each other.

The following results are useful in the this study about regularity, complete controllability/observability of the NDS \( \Sigma \), which are well known in matrix analysis [8, 10].

\[ \text{Lemma 4: Partition a matrix } M = \begin{bmatrix} M_1 & M_2 \end{bmatrix}. \text{ Let } M^\dagger \text{ denote a matrix consisting of column vectors that are independent of each other and span the null space of the submatrix } M_j, j = 1, 2. \text{ Then} \]
\[ \text{null}(M) = M_1^\perp \text{null}(M_2M_1^\dagger) = M_2^\perp \text{null}(M_1M_2^\perp) \]

**Proof:** Assume that \( \alpha \in \text{null}(M) \). Then
\[ M_1 \alpha = 0, \quad M_2 \alpha = 0 \] (9)
The first equality in the above equation means that there exists a vector \( \xi \), such that \( \alpha = M_1^\perp \xi \). Substitute this relation into the second equality of the above equation, we have that
\[ M_2M_1^\perp \xi = 0 \] (10)
That is, \( \xi \in \text{null}(M_2M_1^\perp) \). Hence, \( \alpha \in M_1^\perp \text{null}(M_2M_1^\perp) \).

On the contrary, assume that \( \alpha \in M_1^\perp \text{null}(M_2M_1^\perp) \). Then there is a vector \( \xi \) satisfying simultaneously
\[ M_2M_1^\perp \xi = 0, \quad \alpha = M_1^\perp \xi \] (11)
Therefore
\[ M \alpha = \begin{bmatrix} M_1M_1^\perp \xi \\ M_2M_1^\perp \xi \end{bmatrix} = 0 \] (12)
That is, \( \alpha \in \text{null}(M) \).

The second equality of the lemma can be proved similarly.}

This completes the proof. \[ \Box \]
III. CONNOLLABILITY/OBSERVABILITY OF THE NDS

For brevity, let $p$ denote the vector $\text{col}\{p_i|_{i=1}^{N}\}$. Moreover, for $\# = x, y, \text{ or } z$, define vector $\#(t)$ as $\#(t) = \text{col}\{\#(t, i)|_{i=1}^{N}\}$. Furthermore, define matrices $D_u(p)$ and $E(p)$ respectively as $D_u(p) = \text{diag}[D_u(i, p_i)|_{i=1}^{N}]$ and $E(p) = \text{diag}[E(i, p_i)|_{i=1}^{N}]$. In addition, define matrices $A_{\#(p)}, B_u(p)$ and $C_\#(p)$ with $\#, x, y, \text{ or } z$, respectively, as $A_{\#(p)} = \text{diag}[A_{\#}(i, p_i)|_{i=1}^{N}], B_u(p) = \text{diag}[B_u(i, p_i)|_{i=1}^{N}], C_\#(p) = \text{diag}[C_\#(i, p_i)|_{i=1}^{N}]$.

Using these symbols, the dynamics of all the subsystems of the NDS $\Sigma$ can be compactly expressed as

$$\begin{bmatrix}
E(p)x(t + 1) \\
z(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
A_{xx}(p) & A_{xv}(p) & B_x(p) \\
A_{xv}(p) & A_v(p) & B_v(p) \\
C_x(p) & C_v(p) & D_u(p)
\end{bmatrix}
\begin{bmatrix}
x(t) \\
v(t) \\
u(t)
\end{bmatrix}$$

Combining this equation with Equation (2), a descriptor form can be obtained for the dynamics of the NDS $\Sigma$ that has completely the same form as that of Equation (3), in which the matrices $A, B, C, D$ and $E$ are respectively replaced by the matrices $A(p), B(p), C(p), D(p)$ and $E(p)$. Here, the matrices $A(p), B(p), C(p)$ and $D(p)$ are defined respectively as

$$\begin{align*}
A(p) &= B(p) \\
C(p) &= D(p)
\end{align*}$$

(13)

With these expressions, a necessary and sufficient condition is obtained for the regularity of the NDS $\Sigma$. Its proof is given in Appendix A.

**Theorem 1:** Assume that the NDS $\Sigma$, as well as all of its subsystems $\Sigma_i|_{i=1}^{N}$, are well-posed. Let $A_r$ be a set consisting of $M_1 + 1$ arbitrary but distinguished complex numbers. Then the NDS $\Sigma$ is regular if and only if there exists a $\lambda_0 \in A_r$ at which the matrix pencil $\Theta(\lambda)$ which is defined as follows, is of FCR,

$$\Theta(\lambda) = \begin{bmatrix}
\lambda E(p) - A_{xx}(p) & -A_{xv}(p) \\
-\Phi A_{xx}(p) & I_{M_x} - \Phi A_{xv}(p)
\end{bmatrix}$$

(15)

From the lumped descriptor form representation for the input-output relations of the NDS described by Equations (1) and (2), the following results have been established for its complete observability. The essential ideas behind the derivations are similar to those of [20, 22]. That is, exploiting the LFT structure of the system matrices.

**Theorem 2:** Define a matrix pencil $\Xi^{|\Sigma|}(\lambda)$ and a matrix $\Xi^{|\Sigma|}$ respectively as

$$\Xi^{|\Sigma|}(\lambda) = \begin{bmatrix}
\lambda E(p) - A_{xx}(p) & -A_{xv}(p) \\
-\Phi A_{xx}(p) & I_{M_x} - \Phi A_{xv}(p)
\end{bmatrix}$$

(16)

$$\Xi^{|\Sigma|} = \begin{bmatrix}
E(p) & 0 \\
-\Phi A_{xx}(p) & I_{M_x} - \Phi A_{xv}(p)
\end{bmatrix}$$

(17)

Assume that the NDS $\Sigma$, as well as all of its subsystems $\Sigma_i|_{i=1}^{N}$, are well-posed. Then, this NDS is completely observable if and only if the following two conditions are satisfied simultaneously,

- at every complex scalar $\lambda$, the matrix pencil $\Xi^{|\Sigma|}(\lambda)$ is of FCR;
- the matrix $\Xi^{|\Sigma|}$ is of FCR.

A proof of this theorem is given in Appendix B. From Lemma 3 it is clear that complete controllability of a descriptor system is dual to its completely observability, which is well known in the analysis and synthesis of descriptor systems. On the basis of this relation, the following results are immediately obtained for the complete controllability of the NDS $\Sigma$. The proof is omitted due to its obviousness.

**Corollary 1:** Define a matrix pencil $\Xi^{(c)}(\lambda)$ and a matrix $\Xi^{(c)}$ respectively as

$$\Xi^{(c)}(\lambda) = \begin{bmatrix}
\lambda E(p) - A_{xx}(p) & -B_x(p) & -A_{xv}(p) \Phi \\
-\lambda A_{xx}(p) & -B_v(p) & I_{M_x} - A_{xv}(p) \Phi \\
0 & -B_v(p) & I_{M_v} - A_{xv}(p) \Phi
\end{bmatrix}$$

(18)

$$\Xi^{(c)} = \begin{bmatrix}
E(p) & -B_x(p) & -A_{xv}(p) \Phi \\
0 & -B_v(p) & I_{M_v} - A_{xv}(p) \Phi
\end{bmatrix}$$

(19)

Assume that the NDS $\Sigma$, as well as all of its subsystems $\Sigma_i|_{i=1}^{N}$, are well-posed. Then, this NDS is completely controllable if and only if the following two conditions are satisfied simultaneously,

- at every complex scalar $\lambda$, the matrix pencil $\Xi^{(c)}(\lambda)$ is of FRR;
- the matrix $\Xi^{(c)}$ is of FRR.

Theorem 1 gives a necessary and sufficient condition for the regularity of the NDS $\Sigma$, while Theorem 2 and Corollary 1 respectively a necessary and sufficient condition for its complete observability and complete controllability. However, these conditions are still not computationally feasible, noting that in these conditions, the rank must be checked for the matrix pencil $\Xi^{|\Sigma|}(\lambda)$ or the matrix pencil $\Xi^{(c)}(\lambda)$ at infinitely many complex $\lambda$ that is computationally prohibitive.

On the other hand, when a large scale NDS is under investigation, the dimension of the matrix pencil $\Theta(\lambda)$ is generally high at any $\lambda \in \mathbb{C}$. This is not computationally attractive.

When the parameters of each subsystem are known, all the matrices involved in Theorems 1 and 2, as well as Corollary 1, that is, the matrices $A_{xx}(p), A_{xv}(p)$, etc., are known. To develop a computationally feasible condition from the above results for the complete observability of the NDS $\Sigma$ under this situation, assume that for each $i = 1, 2, \ldots, N$,

$$\text{Null } ([C_x(i) \ C_v(i)]) = \text{span } \begin{bmatrix}
N_x(i) \\
N_v(i)
\end{bmatrix}$$

(20)

Here, the matrix $\text{col}\{N_x(i), N_v(i)\}$ is assumed to be of FCR, while the matrices $N_x(i)$ and $N_v(i)$ are assumed to have a dimension compatible with those of the matrices $C_x(i)$ and $C_v(i)$.

In the above equation, the dependence of the matrices $C_x(i)$, etc. on the parameter vector $p_i$ is omitted to simplify expressions. This omission is adopted in the rest of this section.

From Lemma 3, it can be declared that for each $i \in \{1, 2, \ldots, N\}$, there exist two invertible constant real matrices.
in which \( \bar{\lambda} \) also holds for the set \( \lambda \). In other words, in order to construct a completely whole complex plane. This means that under such a situation, to Appendix C.

Theorem 3. More precisely, the null space of the matrix \( \Lambda_{||}(\lambda_{0})E(p) - A_{xx}(p) - A_{xy}(p) \) can be constructed from each subsystem individually with a given \( \lambda_{0} \in C \) and a given set of subsystem parameters \( \{p_{i}\}_{i=1}^{\infty} \). With this null space and Lemma 4 a necessary and sufficient condition can be obtained for the invariance of the matrix \( \Theta(\lambda_{0}) \), which is similar to that of the aforementioned theorem. On the other hand, on the basis of Lemma 6 the null space of the matrix

\[
\begin{bmatrix}
  E(p) & 0 \\
  -C_{x}(p) & -C_{v}(p)
\end{bmatrix}
\]

can also be constructed from each subsystem independently. By means of this null space and Lemma 4 a necessary and sufficient condition can be obtained for the matrix \( \Xi_{0}(\lambda) \) being of FCR, which is again similar to that of the aforementioned theorem. The details are omitted due to space considerations.

By means of the same token, computationally feasible conditions can also be obtained for the complete controllability of the NDS \( \Sigma \), provided that the parameter vector is known for each of its subsystems. The associated conclusions and derivations are not included for their obviousness and space considerations.

IV. DEPENDENCE OF SYSTEM OBSERVABILITY ON SUBSYSTEM PARAMETERS

The previous section has made it clear that in order to construct a completely controllable/observable NDS, characteristics of its subsystems are quite important. Particularly, if a subsystem is not selected appropriately, it will lead to an infinite number of constraints on the SCM of the NDS. While it is still impossible to investigate the influence of a subsystem parameter on the regularity/controllability/observability of the whole NDS if this parameter affects a subsystem matrix in the way of an arbitrary function, it is assumed throughout this section that

\[
\begin{bmatrix}
  E(i, p_{i}) & A_{xx}(i, p_{i}) & A_{xy}(i, p_{i}) & C_{x}(i, p_{i}) & C_{v}(i, p_{i})
\end{bmatrix}
\]

\[
\begin{bmatrix}
  F_{1}(i) \\
  F_{3}(i)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \Xi_{0}(\lambda) \\
  \Xi_{0}(\lambda)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  T_{1}(i) \\
  T_{3}(i)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  P_{1}(i) \Phi(i) \\
  P_{2}(i) \Phi(i)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  M(i) - P_{1}(i)H(i) \Sigma_{0}(i) \\
  M(i) - P_{2}(i)S(i) \Sigma_{0}(i)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  P_{1}(i) \Phi(i) \\
  P_{2}(i) \Phi(i)
\end{bmatrix}
\]

In the above equations, the matrices \( P_{1}(i) \) and \( P_{2}(i) \) consisting of elements that are constantly equal to zero and
elements that can be expressed as a function of the elements of the parameter vector \( p_i \). The matrices \( G(i), H(i), S(i), K(i), M(i), N(i) \), together with the matrices \( F_j(i) \) and the matrices \( J_j(i) \) with \( j = 1, 2, 3, 4 \), are matrices reflecting how these FPPs affect the system matrices of this subsystem.

These matrices, together with the matrices \( E^{[0]}(i), A_{\#}(i) \) and \( C_{\#}(i) \) with \(*, \# = x, v \) or \( z \), are in general known and can not be selected or adjusted in system designs, as they reflect the physical, chemical, electrical principles, etc. governing the dynamics of this subsystem, such as the Kirchhoff's current law, Newton's mechanics, etc.

The expressions of Equations (23) and (24) are essentially in the form of

\[
\Psi_{22} + \Psi_{21}[\Xi - \Delta \Psi_{11}]^{-1}\Delta \Psi_{12}
\]

A transformation in this form with a fixed matrix \( \Xi \) and some fixed matrices \( \Psi_{ij} \) is called a generalized LFT of the matrix \( \Delta \), which is originally introduced in [9] for parametric uncertainty descriptions. It is argued there that although both an LFT and a generalized LFT are capable of expressing an arbitrary rational function, a generalized LFT usually has a lower matrix dimension than an LFT in the associated expressions. As computational cost is one of the most essential issues in the analysis and synthesis of a large scale NDS, rather than LFTs which are adopted in (23), it is the generalized LFT that is adopted in this paper in describing the dependence of system matrices of a subsystem in the NDS \( \Sigma \) on its parameters.

In the above description, the matrices \( P_1(i) \) and \( P_2(i) \) consist of fixed zero elements and elements which are from the set consisting of all the FPPs of the subsystem \( \Sigma_i, i = 1, 2, \cdots, N \). In some situations, it may be more convenient to use a simple function of some FPPs, such as the reciprocal of a FPP, the product of several FPPs, etc. These transformations do not affect results of this paper, provided that the corresponding global transformation is a bijective mapping. To avoid an awkward presentation, these elements are called pseudo first principle parameters (PFFP) in this paper, and are usually assumed to be algebraically independent of each other.

Note that in the adopted model, the matrix \( E(i, p_i) \) also depends on the subsystem parameter vector \( p_i \). This situation happens in actual applications [11], and disables the approach adopted in (23), in which the inputs and outputs of each subsystem, as well as the SCM of the NDS, are augmented such that the parameters of each subsystem are included in an augmented SCM, and the resulted NDS takes the same form as an NDS without any parameters or with each parameter being prescribed.

Define matrices \( P_1, P_2, G, K, M \) and \( N \) respectively as \( P_1 = \text{diag}[P_1(i)]_{N=1}^N \), \( P_2 = \text{diag}[P_2(i)]_{N=1}^N \), \( G = \text{diag}(G(i)]_{N=1}^N \), \( K = \text{diag}[K(i)]_{N=1}^N \), \( M = \text{diag}(M(i)]_{N=1}^N \) and \( N = \text{diag}(N(i)]_{N=1}^N \). Moreover, define matrices \( F_j \) with \( j = 1, 2, 3, 4 \) and matrices \( J_j \) with \( j = 1, 2, 3, 4 \) as \( F_j = \text{diag}[F_j(i)]_{N=1}^N \) and \( J_j = \text{diag}[J_j(i)]_{N=1}^N \). In addition, define matrices \( A_{\#}(i) \) and \( C_{\#}(i) \) with \(*, \# = x, v \) or \( z \), respectively as \( A_{\#}(i) = \text{diag}\{A_{\#}(i)]_{N=1}^N \}, C_{\#}(i) = \text{diag}(C_{\#}(i)]_{N=1}^N \). If for each \( i = 1, 2, \cdots, N \), the system matrices of the subsystem \( \Sigma_i \) depend on its parameters through a way expressed by Equations (23) and (24), then the following results can be obtained, while their proof is given in Appendix D.

**Theorem 4:** Define a matrix pencil \( \Xi^{[0]}(\lambda) \) as

\[
\Xi^{[0]}(\lambda) = \begin{bmatrix}
\lambda E^{[0]}(i) - A_{xx}(i) & \lambda F_1(i) - A_{xv}(i) & -A_{xv}(i) & -J_1 \\
-A_{xv}(i) & -C_{xx}(i) & -C_{xv}(i) & -J_2 \\
-P_1(i) G(i) & M(i) - P_2(i) H(i) & 0 & 0 \\
0 & 0 & -P_2(i) K(i) & N(i) - P_2(i) S(i) \\
-\Phi A_{xx}(i) & -\Phi F_4(i) & z v_{i} & I M_{4x} - \Phi A_{xv}(i) - \Phi J_3(i)
\end{bmatrix}
\]

Assume that the NDS \( \Sigma \), as well as all of its subsystems \( \Sigma_i \), \( i = 1, 2, \cdots, N \), are well-posed. Then, at any \( \lambda \in C \), the matrix \( \Xi^{[0]}(\lambda) \) of Equation (16) is of FCR, if and only if the matrix pencil \( \Xi^{[0]}(\lambda) \) holds this property.

Note that the matrix pencil \( \Xi^{[0]}(\lambda) \) of Equation (25) has a similar structure as the matrix \( \Xi^{[0]}(\lambda) \) of Equation (16) with a prescribed parameter vector \( p \). Through the adoption of the null space of the matrix \( C_{xx}(i) F_3(i) C_{xv}(i) J_2(i) \) and utilization of Lemmas 2 and 4, a rank condition similar to that of Theorem 3 can be established for verifying whether or not the matrix pencil \( \Xi^{[0]}(\lambda) \) is of FCR at each \( \lambda \in C \). In this condition, the associated matrix depends affinely on both the SCM \( \Phi \) and the system parameter matrices \( P_1(i) \) and \( P_2(i) \).

More precisely, for each \( i = 1, 2, \cdots, N \), assume that

\[
\text{Null}\left(\begin{bmatrix} C_{xx}^{[0]}(i) & F_3(i) & C_{xv}^{[0]}(i) & J_2(i) \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} N_{xx}^{[0]}(i) \\
N_{xv}^{[0]}(i) \\
N_{v}^{[0]}(i) \\
N_{j}^{[0]}(i) \end{bmatrix}\right)
\]

(26)

Here, the matrices \( N_{xx}^{[0]}(i), N_{xv}^{[0]}(i), N_{v}^{[0]}(i) \) and \( N_{j}^{[0]}(i) \) are selected such that their dimensions are compatible respectively with those of the matrices \( C_{xx}(i), F_3(i), C_{xv}(i) \) and \( J_2(i) \), while the matrix \( \text{col}\left\{ N_{xx}^{[0]}(i), N_{xv}^{[0]}(i), N_{v}^{[0]}(i), N_{j}^{[0]}(i) \right\} \) is of FCR. Existence of these matrices is guaranteed by matrix theories [8][10].

From Lemma 4 for each \( i \in \{1, 2, \cdots, N\} \), there exist two invertible constant real matrices \( U^{[0]}(i) \) and \( V^{[0]}(i) \), some unique nonnegative integers \( a^{[0]}(i), b^{[0]}(i), c^{[0]}(i), d^{[0]}(i) \), some unique positive integers \( \xi^{[0]}(i), \eta^{[0]}(i) \) and \( \gamma^{[0]}(i) \), some unique nonnegative integers \( \kappa^{[0]}(i) \) and \( \rho^{[0]}(i) \), as well as a strictly regular \( \mu^{[0]}(i) \times \mu^{[0]}(i) \) dimensional matrix pencil \( H^{[0]}(\lambda) \), such that

\[
\lambda^{[0]}(i) N_{xx}^{[0]}(i) + F_1(i) N_{xv}^{[0]}(i) + A_{xx}^{[0]}(i) N_{v}^{[0]}(i) + A_{xv}^{[0]}(i) N_{j}^{[0]}(i) + F_2(i) N_{t}^{[0]}(i) + J_1(i) N_{j}^{[0]}(i) + J_2(i) N_{t}^{[0]}(i)
\]

\[
= U^{[0]}(i) \lambda^{[0]}(i) K_{\xi^{[0]}(i)}(\lambda) L_{\eta^{[0]}(i)}(\lambda) J_{\gamma^{[0]}(i)}(\lambda) \times V^{[0]}(i)
\]

(27)

For each \( i = 1, 2, \cdots, N \), let \( m^{[0]}(i) \) denote \( \mu^{[0]}(i) \) and \( \sum_{j=1}^{n^{[0]}(i)} \xi^{[0]}(i) + \sum_{j=1}^{n^{[0]}(i)} \kappa^{[0]}(i) \). Moreover, let \( V_1^{-1}(m^{[0]}(i)) \) represent the matrix constructed from the first \( m^{[0]}(i) \) columns
of the inverse of the matrix $V^{[0]}(i)$. Furthermore, let $\Lambda^{[0]}(i)$ stand for the set of complex numbers at which the following matrix pencil is not of FCR,

$$\text{diag}\left\{ H^{[0]}(i)(\lambda), K^{[0]}(i)(\lambda), L^{[0]}(i)(\lambda) \right\}_{\lambda=1}^{N}$$

Define a set $\Lambda^{[0]}$ as

$$\Lambda^{[0]} = \bigcup_{i=1}^{N} \Lambda^{[0]}(i)$$

For each $i \in \{1, 2, \cdots, N\}$ and each $\lambda_0 \in \Lambda^{[0]}$, let $N^{[0]}(\lambda_0, i)$ denote a matrix of FCR that spans the null space of the following matrix,

$$\text{diag}\left\{ H^{[0]}(i)(\lambda_0), K^{[0]}(i)(\lambda_0), L^{[0]}(i)(\lambda_0) \right\}_{\lambda=1}^{N}$$

Using these symbols, through similar arguments as those in the proof of Theorem 3, the following results can be established from Theorem 4 as well as Lemmas 3 and Theorem 5. Their proof is omitted due to the obviousness.

**Theorem 5:** For each $\lambda_0 \in \Lambda^{[0]}$ and each $i \in \{1, 2, \cdots, N\}$, denote the matrix $Y_i^{[0]}(\lambda_0) = N^{[0]}(\lambda_0, i)$ by $N^{[0]}(\lambda_0, i)$. Moreover, for each $\lambda_0 \in \Lambda^{[0]}$, define matrices $X_i^{[0]}(\lambda_0)$ and $Y_i^{[0]}(\lambda_0)$ with $i = 1, 2, 3$, respectively as

$$X_1^{[0]}(\lambda_0) = \text{diag}\left\{ M(i)N_i(i)N^{[0]}(\lambda_0, i) \right\}_{i=1}^{N}$$

$$Y_1^{[0]}(\lambda_0) = \text{diag}\left\{ (G(i)N_i(i)^* + H(i)N_i(i))N^{[0]}(\lambda_0, i) \right\}_{i=1}^{N}$$

$$X_2^{[0]}(\lambda_0) = \text{diag}\left\{ N(i)N_i(i)N^{[0]}(\lambda_0, i) \right\}_{i=1}^{N}$$

$$Y_2^{[0]}(\lambda_0) = \text{diag}\left\{ K(i)N_i(i)^* + S(i)N_j(i)N^{[0]}(\lambda_0, i) \right\}_{i=1}^{N}$$

$$X_3^{[0]}(\lambda_0) = \text{diag}\left\{ N_j(i)^*N_i(i)^*N^{[0]}(\lambda_0, i) + F_4(i)\times N_i(i) \right\}_{i=1}^{N}$$

Then, the matrix pencil $\Xi^{[0]}(\lambda_0)$ of Theorem 4 is of FCR at each $\lambda_0 \in \mathbb{C}$, if and only if at each $\lambda_0 \in \Lambda^{[0]}$, the following matrix is of FCR,

$$\begin{bmatrix}
X_1^{[0]}(\lambda_0) - P_1Y_1^{[0]}(\lambda_0) \\
X_2^{[0]}(\lambda_0) - P_2Y_2^{[0]}(\lambda_0) \\
X_3^{[0]}(\lambda_0) - \Phi Y_3^{[0]}(\lambda_0)
\end{bmatrix}$$

From the definitions of the matrices $X_i^{[0]}(\lambda_0)$, it is clear that each of them can be obtained through calculations with each subsystem independently. This property of the condition makes it attractive in the analysis and synthesis of a large scale NDS.

Using similar arguments as those in the proof of Theorem 3, it can be proved that the matrix pencil $\Theta(\lambda)$ of Equation 15 is of FCR at a particular $\lambda_0 \in \mathbb{C}$, if and only if the following matrix $\Theta_p(\lambda)$ satisfies this requirement,

$$\Theta_p(\lambda) = \begin{bmatrix}
\lambda_0E^{[0]} - A^{[0]}_x & \lambda_0F_1 - F_2 & -A^{[0]}_y & -J_1 \\
-P_1G & M - P_1H & 0 & 0 \\
0 & 0 & -P_2K & N - P_2S \\
-\Phi A^{[0]}_y & -\Phi F_1 & I_{M_x} - \Phi A^{[0]}_x & -\Phi J_3
\end{bmatrix}$$

Moreover, the matrix $\Xi^{[\infty]}$ of Equation 17 is of FCR, if and only if the matrix $\Xi^{[\infty]}_{\Sigma,i} \Theta_p$ meets this condition which is defined as follows,

$$\Xi^{[\infty]}_{\Sigma,i} = \begin{bmatrix}
E^{[0]} & F_1 & 0 & 0 \\
-C^{[0]}_x & -F_3 & 0 & -J_2 \\
-P_1G & M - P_1H & 0 & 0 \\
0 & 0 & -P_2K & N - P_2S \\
-\Phi A^{[0]}_x & -\Phi F_1 & I_{M_x} - \Phi A^{[0]}_x & -\Phi J_3
\end{bmatrix}$$

Moreover, equivalent conditions in the form of Equation 29 can also be derived respectively for the matrix $\Theta_p(\lambda)$ to be of FCR and the matrix $\Xi^{[\infty]}_{\Sigma,i}$ to be of FCR. The details of the derivations are omitted due to their straightforwardness.

Based on the fact that complete controllability of a descriptor system is dual to its complete observability, similar results can be obtained for the verification of the complete controllability of the NDS $\Sigma$ under the situation that the system matrices of its subsystems are expressed through some generalized LFTs of their parameters.

On the other hand, as argued in the previous section, in order to reduce difficulties in constructing a completely observable NDS, it is helpful to select subsystems with each of them satisfying $c(i) = 0, i = 1, 2, \cdots, N$. Here, $c(i)$ stands for the number of the matrix pencils having the form of $L_s(\lambda)$ in the KCF of the matrix pencil $\lambda E(i, p_i) - [A_{xx}(i, p_i)N_{xx}(i, p_i) + A_{yx}(i, p_i)N_{yx}(i, p_i)]$, and its definition is given in Equation 21. Using similar arguments as those in the proofs of Theorems 4 and 5 the following corollary is obtained, which provides a necessary and sufficient condition for the aforementioned requirement that can be verified with each subsystem individually.

**Corollary 2:** Assume that the NDS $\Sigma$ and its subsystems are well-posed. Moreover, assume that the system matrices of its subsystems are expressed by Equations 23 and 24. Furthermore, assume that parameters of different subsystems are independent of each other. Let $\Lambda_0$ be a set consisting of $M_k + M_y + 1$ arbitrary but distinguished complex numbers. Then $c(i)$ of Equation 24 is equal to zero for each $i = 1, 2, \cdots, N$, if and only if one of the following two conditions are satisfied,

- $c^{[0]}(i)$ of Equations 27 is equal to zero for each $i = 1, 2, \cdots, N$;
- there exists a $\lambda_0 \in \Lambda_0$, such that the following matrix is of FCR for each $i = 1, 2, \cdots, N$,

$$\begin{bmatrix}
X_1^{[0]}(\lambda_0, i) - P_1Y_1^{[0]}(\lambda_0, i) \\
X_2^{[0]}(\lambda_0, i) - P_2Y_2^{[0]}(\lambda_0, i) \\
X_3^{[0]}(\lambda_0, i) - \Phi Y_3^{[0]}(\lambda_0, i)
\end{bmatrix}$$

Here, $X_i^{[0]}(\lambda_0, i)$ and $Y_i^{[0]}(\lambda_0, i)$ stand respectively for the $i$-th diagonal block of the matrix $X_i^{[0]}(\lambda_0)$ and that of the matrix
Y_j^{[i]}(\lambda_0), \text{ in which } j = 1, 2 \text{ and } i = 1, 2, \ldots, N.

It is worthwhile to mention that before combining together to form an NDS, each subsystem works independently. It appears safe to declare that the assumption adopted in the above corollary is reasonable that parameters of different subsystems are independent of each other.

Clearly, the 1st condition of Corollary \[2\] depends only on the principles that govern the movements of a subsystem and sensor positions. That is, it is independent of any subsystem parameter. This condition is expected to be helpful in subsystem dynamics selections and sensor placements that reduce difficulties in constructing a completely observable NDS.

On the other hand, the condition of Equation (32) provides some requirements on both the dynamics of a subsystem and its parameters. Satisfaction of this condition by each subsystem reduces significantly the number of constraints on the SCM of the NDS \(\Sigma\), and therefore may also greatly cut down difficulties in the construction of a completely observable NDS. It is believed that this condition can provide some useful guidelines in subsystem parameter selections, as well as in subsystem dynamics selections.

An attractive property of the conditions in Corollary \[2\] is that they can be verified for each subsystem individually.

Using the duality between complete observability and complete controllability of a descriptor system, similar requirements can be obtained for each subsystem with which a completely controllable NDS can be constructed more easily.

V. CONCLUDING REMARKS

This paper investigates regularity, complete controllability and complete observability for a networked dynamic system, in which each subsystem is described by a descriptor form, while its system matrices is represented by a generalized linear fractional transformation of its (pseudo) first principle parameters. Some matrix rank based necessary and sufficient conditions have been derived, in which the matrix depends affinely on both the subsystem connection matrix and subsystem parameters. These results extends those on the controllability/observability of an NDS with its subsystems being described by a state space model, and keep the attractive properties that all the involved calculations can be performed on each subsystem independently. In addition, these conditions also reveals some requirements on a subsystem from which a completely controllable/observable can be constructed more easily, which are expected to be helpful in subsystem designs and parameter selections.

As a further issue, it is interesting to see applicability of the obtained results to actuator/sensor placements for an NDS, as well as to quantitative measures of its controllability/observability.

APPENDIX A

PROOF OF THEOREM 1

When both the subsystems and the whole system of the NDS \(\Sigma\) are well-posed, direct matrix manipulations show that the matrix \(I_{M_x} - \Phi A_{xv}(p)\) is invertible \[20, 23\]. Hence

\[
\det \{\Theta(\lambda, p)\} = \det(I_{M_x} - \Phi A_{xv}(p)) \det(\lambda E(p) - [A_{xx}(p)
+ A_{xv}(p)I_{M_x} - \Phi A_{xv}(p)]^{-1}\Phi A_{xx})
= \det(I_{M_x} - \Phi A_{xv}(p)) \det(\lambda E(p) - A(p)) \quad (a.1)
\]

This means that at each \(\lambda\), the nonsingularity of the matrix pencil \(\Pi(\lambda)\) is equal to that of the matrix pencil \(\lambda E - A\).

Assume now that there is a \(\lambda_0\) belonging to the set \(\Lambda_\ast\) such that the matrix \(\Theta(\lambda_0)\) is of FCR. Then, the above arguments means that at this particular value, the matrix pencil \(\lambda E(p) - A(p)\) is invertible. The regularity of the NDS \(\Sigma\) follows from its definition for a descriptor system.

On the other hand, assume that for each \(\lambda_0 \in \Lambda_\ast\), the matrix pencil \(\Theta(\lambda)\) is rank deficient. Then, the equivalence in the nonsingularity between the matrix \(\Theta(\lambda_0)\) and the matrix \(\lambda_0 E(p) - A(p)\) implies that \(\det[\lambda_0 E(p) - A(p) = 0\) whenever \(\lambda_0\) is an element of the set \(\Lambda_\ast\). On the other hand, note that \(\det[\lambda E(p) - A(p)]\) is a polynomial of the variable \(\lambda\) with its degree not exceeding \(M_x\). This means that if \(\det[\lambda E(p) - A(p)]\) is not a zero polynomial, it has at most \(M_x\) roots. In addition, recall that the set \(\Lambda_\ast\) consists of \(M_x\) + 1 different elements. It can therefore be declared that \(\det[\Theta(\lambda_0)] = 0\) for each \(\lambda_0 \in \Lambda_\ast\) means that \(\det[\lambda E(p) - A(p)] \equiv 0\). Hence, the NDS \(\Sigma\) is not regular, according to the definition of a descriptor system.

The proof is now completed.

\[\diamondsuit\]

APPENDIX B

PROOF OF THEOREM 2

To shorten mathematical expressions in this proof, the dependence of a matrix on the parameter vector \(p\) is eliminated for each associated matrix. For example, the matrix \(A_{xx}(p)\) is written as \(A_{xx}\), etc. This elimination does not introduce any confusions in the following derivations.

Assume that the NDS \(\Sigma\) is completely observable. Then according to Lemma \[3\] and Equation (14), we have that for every \(\lambda \in C\) and every nonzero \(M_x\) dimensional complex vector \(\alpha\),

\[
\begin{bmatrix}
\lambda E - [A_{xx} + A_{xv}(I_{M_x} - \Phi A_{xv})^{-1}\Phi A_{xx}] \\
C_x + C_v(I_{M_x} - \Phi A_{xv})^{-1}\Phi A_{xx}
\end{bmatrix} \alpha \neq 0
\]

(a.2)

Now, assume that at a particular \(\lambda_0\), the matrix pencil \(\Xi^{[0]}(\lambda)\) is not of FCR. Then a nonzero \(M_x + M_v\) dimensional complex vector \(\xi\) exists satisfying

\[
M(\lambda_0)\xi = 0
\]

(a.3)

Partition the vector \(\xi\) as \(\xi = \text{col}\{\xi_1, \xi_2\}\) consistently with the matrix pencil \(\Xi^{[0]}(\lambda)\).

Then Equation (a.3) can be equivalently rewritten as

\[
\begin{align}
(\lambda_0 E - A_{xx})\xi_1 - A_{xv}\xi_2 &= 0 \quad (a.4) \\
C_x\xi_1 + C_v\xi_2 &= 0 \quad (a.5) \\
-\Phi A_{xx}\xi_1 + (I_{M_x} - \Phi A_{xv})\xi_2 &= 0 \quad (a.6)
\end{align}
\]

On the other hand, from the well-posedness assumptions on each subsystem and the whole system of the NDS \(\Sigma\), it can be directly proved that the matrix \(I_{M_v} - \Phi A_{xv}\) is invertible.
It can therefore be declared from Equation (a.6) and the assumption \( \xi \neq 0 \) that \( \xi_1 \neq 0 \) and
\[
\xi_2 = (I_{M_\nu} - \Phi A_{zv})^{-1} \Phi A_{zv} \xi_1
\]  
(a.7)

Combing Equations (a.4), (a.5) and (a.7) together, we further have that
\[
\left[ \begin{array}{c}
\lambda_0 E - [A_{xx} + A_{xv}(I_{M_\nu} - \Phi A_{zv})^{-1} \Phi A_{ax}] \\
C_x + C_v(I_{M_\nu} - \Phi A_{zv})^{-1} \Phi A_{ax}
\end{array} \right] \xi_1 = 0
\]
(a.8)
which is clearly in contradiction with Equation (a.2). Hence, the matrix pencil \( \Xi^{[0]}(\lambda) \) must be of FCR at each complex number \( \lambda \).

Now assume that the matrix \( \Xi_{\infty} \) is not of FCR. Similar arguments show that it will lead to the rank deficiency of the matrix \( \text{col}(E, A) \), which further results that the NDS \( \Sigma \) is not completely observable.

On the contrary, assume that the NDS \( \Sigma \) is not completely observable. Then according to Lemma 6 there exist a \( \lambda_0 \in C \) and a nonzero vector \( \alpha \) such that
\[
\left[ \begin{array}{c}
\lambda_0 E - [A_{xx} + A_{xv}(I_{M_\nu} - \Phi A_{zv})^{-1} \Phi A_{ax}] \\
C_x + C_v(I_{M_\nu} - \Phi A_{zv})^{-1} \Phi A_{ax}
\end{array} \right] \alpha = 0
\]  
(a.9)
or there is a nonzero vector \( \alpha \) such that
\[
\left[ \begin{array}{c}
E \\
C_x + C_v(I_{M_\nu} - \Phi A_{zv})^{-1} \Phi A_{ax}
\end{array} \right] \alpha = 0
\]  
(a.10)
Assume now that Equation (a.9) is satisfied. Define a vector \( \xi \) as
\[
\xi = \left[ \begin{array}{c}
I_{M_\nu} \\
(I_{M_\nu} - \Phi A_{zv})^{-1} \Phi A_{ax}
\end{array} \right] \alpha
\]
Then \( \xi \neq 0 \) and the following equality is obviously satisfied by this vector,
\[
[-\Phi A_{xx} I_{M_\nu} - \Phi A_{zv}] \xi = 0
\]  
(a.11)
Moreover, Equation (a.9) can be equivalently rewritten as
\[
\left[ \begin{array}{c}
\lambda_0 E - A_{xx} \\
-C_x
\end{array} \right] \xi = 0
\]  
(a.12)
It can therefore be declared from the definition of the matrix pencil \( \Xi^{[0]}(\lambda) \) that
\[
\Xi^{[0]}(\lambda_0) \xi = 0
\]  
(a.13)
That is, this matrix pencil is not of FCR for each \( \lambda \in C \).

Now assume that Equation (a.10) is satisfied. Similar arguments as those for the situation in which Equation (a.9) is satisfied show that, there exists a nonzero vector \( \xi \) satisfying
\[
\left[ \begin{array}{cccc}
E & 0 \\
-C_x & -C_v \\
-\Phi A_{ax} & I_{M_\nu} - \Phi A_{zv}
\end{array} \right] \xi = 0
\]  
(a.14)
This completes the proof.

### Appendix C

**Proof of Theorem 3**

For brevity, denote the matrix pencils
\[
\lambda E(i) - [A_{xx}(i) N_x(i) + A_{xv}(i) N_v(i)]
\]
\[
\text{diag} \{ \Pi_{\mu_1}(\lambda), K_{\xi_1}(\lambda) \}_{j=1}^{(i)}, L_{\xi_1}(\lambda) \}_{j=1}^{(i)}
\]
\[
N_{\eta_1}(\lambda) \}_{j=1}^{(i)}, J_{\rho_1}(\lambda) \}_{j=1}^{(i)}
\]
and
\[
\text{diag} \{ \Pi_{\mu_1}(\lambda), K_{\xi_1}(\lambda) \}_{j=1}^{(i)}, L_{\xi_1}(\lambda) \}_{j=1}^{(i)}
\]
respectively by \( \Pi(\lambda, i), \Psi(\lambda, i) \) and \( \Psi(\lambda, i) \), in which \( i \in \{1, 2, \ldots, N\} \).

Let \( \alpha \) be an arbitrary \( M_x + M_v \) dimensional real column vector. Partition this vector as \( \alpha = \text{col}\{\alpha_x(i)\}_{i=1}^{N_x}, \alpha_v(i)\}_{i=1}^{N_v} \), in which \( \alpha_x(i) \) belongs to \( R^{m_x \times r} \), while \( \alpha_v(i) \) belongs to \( R^{m_v \times r}, i = 1, 2, \ldots, N \). From the block diagonal structure of the matrix \( C_x \) and \( C_v \), it is immediate that
\[
[C_x C_v] \alpha = \left[ \begin{array}{cccc}
C_x(1) \alpha_x(1) + C_v(1) \alpha_v(1) \\
C_x(2) \alpha_x(2) + C_v(2) \alpha_v(2) \\
\vdots \\
C_x(N) \alpha_x(N) + C_v(N) \alpha_v(N)
\end{array} \right]
\]  
(a.15)
From this relation and the definitions of the matrices \( N_x(i) \) and \( N_v(i) \), as well as Lemma 5 direct algebraic operations show that
\[
\text{Null} \left[ \begin{array}{c}
C_x \\
C_v
\end{array} \right] = \text{Span} \left[ \begin{array}{c}
N_x \\
N_v
\end{array} \right]
\]  
(a.16)
in which
\[
N_x = \text{diag} \{ N_x(i)\}_{i=1}^{N_x}, \quad N_v = \text{diag} \{ N_v(i)\}_{i=1}^{N_v}
\]

On the basis of Equation (a.16) and Lemma 4 it is clear that at an arbitrary \( \lambda_0 \in C \), the value of the matrix pencil \( \Xi^{[0]}(\lambda) \) which is defined in Equation (13), that is, the matrix \( \Xi^{[0]}(\lambda_0) \), is of FCR, and if only if the following matrix is,
\[
\left[ \begin{array}{cccc}
\lambda_0 E - A_{xx} \\
-C_x \\
-\Phi A_{ax} & I_{M_\nu} - \Phi A_{zv}
\end{array} \right] \left[ \begin{array}{c}
N_x \\
N_v
\end{array} \right] = 1
\]
\[
\left[ \begin{array}{cccc}
\lambda_0 E N_x - [A_{xx} N_x + A_{xv} N_v] \\
N_v - \Phi A_{ax} N_v + A_{xv} N_v
\end{array} \right]
\]  
(a.17)
From the consistent block diagonal structures of the involved matrices, as well as Equation (21), we have that
\[
\lambda_0 E N_x - [A_{xx} N_x + A_{xv} N_v]
\]
\[
= \text{diag} \left\{ \Pi(\lambda_0, i)\}_{i=1}^{N_x} \right\}
\]
\[
= \text{diag} \left\{ U(i) \Psi(\lambda_0, i) V(i)\}_{i=1}^{N_x} \right\}
\]
\[
= \text{diag} \left\{ U(i)\}_{i=1}^{N_x} \right\} \text{diag} \left\{ \Psi(\lambda_0, i)\}_{i=1}^{N_x} \right\}
\]
\[
\text{diag} \left\{ U(i)\}_{i=1}^{N_x} \right\}
\]  
(a.18)
Recall that for each \( i = 1, 2, \ldots, N \), both the matrix \( U(i) \) and
the matrix $V(i)$ are invertible. We therefore have that,
\[
\lambda_0 E N_N - [A_{xx} N_x + A_{xz} N_x ] = \begin{bmatrix}
\lambda_0 E N_N - [A_{xx} N_x + A_{xz} N_x ] \\
N_N - \Phi [A_{xx} N_x + A_{xz} N_x ]
\end{bmatrix}
\]

Thus, there exist vectors $\lambda$ such that at a particular $\lambda_0$ and the associated matrix imply that
\[
\text{null} \left\{ \text{diag} \left\{ \Psi(\lambda_0, i) \right\}_{i=1}^N \right\} = \text{span} \left\{ \text{diag} \left\{ N(\lambda, i) \right\}_{i=1}^N \right\}
\]
In addition, it can be directly declared from Lemma 5 that the matrix $\text{diag} \left\{ N(\lambda, i) \right\}_{i=1}^N$ is of FCR. Based on this relation and Lemma 4 we have that the matrix of Equation (a.23) is of FCR, if and only if the following matrix is of FCR,
\[
\text{diag} \left\{ N(\lambda_0, i) \right\}_{i=1}^N
\]
which is obviously equal to the matrix $X(\lambda_0) - \Phi Y(\lambda_0)$. This completes the proof.

**APPENDIX D**

**PROOF OF THEOREM 4**

Assume that at a particular $\lambda_0 \in C$, the matrix pencil $\Xi(\lambda)$ is rank deficient. Then, there exist vectors $\alpha$ and $\beta$ such that at least one of them is nonzero and the following equation is satisfied,
\[
\begin{bmatrix}
\lambda_0 E(p) - A_{xx}(p) & -A_{xz}(p) \\
-C_x(p) & -C_z(p) \\
-\Phi A_{xx}(p) & I_M - \Phi A_{xz}(p)
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} = 0
\]
Defining vectors $\xi$ and $\eta$ respectively as
\[
\xi = (M - P_1 H)^{-1} P_1 G, \quad \eta = (N - P_2 H)^{-1} P_2 K
\]
Then, the vectors $\alpha$, $\beta$, $\xi$ and $\eta$ obviously satisfy
\[
\begin{bmatrix}
-\lambda_0 G M - P_1 H \\
-\lambda_0 N - P_2 S
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} = 0
\]
On the other hand, from the definitions of the associated matrices which are given by Equations (23) and (23), as well as the paragraph immediately before this theorem, Equation (a.24) can be rewritten as
\[
\begin{bmatrix}
\lambda_0 E(p) - A_{xx}(p) & -A_{xz}(p) \\
-C_x(p) & -C_z(p) \\
-\Phi A_{xx}(p) & I_M - \Phi A_{xz}(p)
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} = 0
\]
Combining Equations (a.25)-(a.27) together, the following equality is obtained,
\[
\begin{bmatrix}
\lambda_0 E(p) - A_{xx}(p) & -A_{xz}(p) \\
-C_x(p) & -C_z(p) \\
-\Phi A_{xx}(p) & I_M - \Phi A_{xz}(p)
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix} = 0
\]
Obviously, the vector $O_{(\alpha, \xi, \beta, \eta)}$ is not a zero vector. The definition of the matrix pencil $\Xi(\lambda)$ implies that the matrix $\Xi(\lambda)$ is not of FCR.

On the contrary, assume that the matrix pencil $\Xi(\lambda)$ is rank deficient at a particular complex $\lambda_0$. Similar arguments show that at this $\lambda_0$, the matrix pencil $\Xi(\lambda)$ is also rank deficient.

This completes the proof.

**APPENDIX E**

**PROOF OF COROLLARY 2**

It can be declared from Lemmas 2 and 4 that, the number $c(i)$ of Equation (21) is equal to zero for each $i = 1, 2, \cdots, N$, if and only if there exists a $\lambda_0 \in C$, such that the matrix pencil $\Xi(\lambda)$ is of FCR at this particular value. Here, the matrix pencil $\Xi(\lambda)$ is defined as
\[
\Xi(\lambda) = \begin{bmatrix}
\lambda E(p) - A_{xx}(p) & -A_{xz}(p) \\
-C_x(p) & -C_z(p) \\
-\Phi A_{xx}(p) & I_M - \Phi A_{xz}(p)
\end{bmatrix}
\]
That is, the matrix pencil $\Xi(\lambda)$ is of full normal column rank (FCSR).

Let $\Xi(\lambda_0)$ represent the following matrix
\[
\begin{bmatrix}
\Psi \Phi(\lambda_0, i) \right\}_{i=1}^N \\
M(i) N(i) - P_1 G(i) N(i) + H(i) N(i) \\
N(i) N(i) - P_2 K(i) N(i) + S(i) N(i)
\end{bmatrix}
\]
in which $\Psi(\lambda_0, i)$ with $1 \leq i \leq N$ stands for the matrix
\[
\begin{bmatrix}
H_{\mu(i)}(\lambda_0), K_{\xi(i)}(\lambda_0) |_{i=1}^{N(\lambda_0)}
\end{bmatrix}
\]
On the basis of the above observations, the assumption that the NDS $\Sigma$ and its subsystems are well-posed, as well as the assumption that the system matrices of its subsystems
can be expressed by Equations (23) and (24), using similar arguments as those in the proofs of Theorems 4 and 5 it can be proved that the matrix $\Xi^{[0]}(\lambda_0)$ is of FCR, if and only if the matrix $\Xi^{[0]}(\lambda_0)$ satisfies this requirement.

Assume now that $c(i) = 0$ for each $i = 1, 2, \ldots, N$. Then according to Lemma 1 the set $\Lambda^{[i]}$ consists of only a finite number of elements. This means that the set $C \setminus \Lambda^{[i]}$ is not empty. Moreover, for each $\lambda_0 \in C \setminus \Lambda^{[i]}$ and each $i = 1, 2, \ldots, N$, the matrix $\Psi^{[i]}(\lambda_0, i)$ is of FCR, which further leads to that the matrix $\text{diag}\left\{\Psi^{[i]}(\lambda_0, i)\right\}_{i=1}^{N}$ is of FCR. It can therefore be claimed from that Equation (30) that the matrix $\Xi^{[0]}(\lambda_0)$ is of FCR. Hence, the matrix pencil $\Xi^{[0]}(\lambda)$ is of FNCR, which is equivalent to that $c(i) = 0$ for each $i = 1, 2, \ldots, N$. 

Now, assume that there exists a $i \in \{1, 2, \ldots, N\}$, such that $c(i) > 0$. Then according to Lemma 1 the set $\Lambda^{[i]}$ is equal to the whole complex plane. That is, $\Lambda^{[i]} = C$. 

If there exists a $\lambda_0 \in \Lambda^{[i]}$, such that for each $i = 1, 2, \ldots, N$, the matrix of Equation (22) is of FCR, then from the definition of the matrices $N^{[0]}(\lambda_0, i)_{i=1}^{N}$ and Lemma 4 it can be claimed that the matrix $\Xi^{[0]}(\lambda_0)$, and hence the matrix $\Xi^{[0]}(\lambda)$, is of FCR. Therefore, $c(i) = 0$ for every $1 \leq i \leq N$. 

On the contrary, assume that for each $\lambda_0 \in \Lambda^{[i]}$, there is at least one $i \in \{1, 2, \ldots, N\}$, such that the matrix of Equation (22) is not of FCR. Then, the above arguments imply that for every $\lambda_0 \in \Lambda^{[i]}$, the matrix $\Xi^{[0]}(\lambda_0)$ is not of FCR. Hence, the determinant of every $(M_K + M_\nu) \times (M_K + M_\nu)$ dimensional submatrix of the matrix $\Xi^{[0]}(\lambda_0)$ is equal to zero. 

Note that each $(M_K + M_\nu) \times (M_K + M_\nu)$ dimensional submatrix of the matrix pencil $\Xi^{[0]}(\lambda)$ is still a matrix pencil. Moreover, its determinant is a polynomial of the variable $\lambda$ with a degree at most $M_K + M_\nu$. On the other hand, the set $\Lambda^{[i]}$ consists of $M_K + M_\nu + 1$ different elements. It can therefore be declared that if the 2nd condition of this corollary is not satisfied, then the determinant of every $(M_K + M_\nu) \times (M_K + M_\nu)$ dimensional submatrix of the matrix pencil $\Xi^{[0]}(\lambda)$ is a zero polynomial. Hence, it is rank deficient at each $\lambda \in C$. That is, the matrix pencil $\Xi^{[0]}(\lambda)$ is not of FCR at every $\lambda \in C$, which is equivalent to that this matrix pencil is not of FNCR.

This completes the proof. 

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