On the existence of a solution of one variational inequality of the nonlinear filtration theory

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Abstract. A theorem on the existence of a generalized solution of one variational inequality describing the process of nonlinear nonstationary filtration of a liquid in a porous medium with the condition of one-way permeability on a part of the boundary is proved. The case is considered, in which the Kirchhoff transformation used in the determination of a generalized solution maps the real axis to the semi-axis bounded from below. In investigating the solvability of the resulting variational inequality with a lower-bound constraint on the solution, an auxiliary problem with no constraints is constructed. It is proved that any solution of the auxiliary problem is a solution of the problem studied in the paper. The solvability of the auxiliary problem is established by means of using the semidiscretization method with a penalty and the Galerkin method.

1. Introduction

A problem of nonlinear nonstationary filtration is considered in the bounded domain \( \Omega \in \mathbb{R}^n (n \geq 1) \) with the boundary \( \Gamma \).

\[
m \frac{\partial s(p)}{\partial t} + \text{div} \left( -b(s(p))(\nabla p - \rho g) \right) = f(s(p)).
\]

Here \( p \) is pore pressure; \( q \) is filtration velocity; \( m \) is porosity of the medium; \( s(p) \) is saturation; \( f(s(p)) \) is a given function; \( b(s(p)) \) is relative phase permeability of the medium; \( \rho \) is liquid density; \( g \) is gravitational force vector.

It is assumed that on a part of \( \Gamma \) (we call it \( \Gamma_1 \)), a condition of one-way permeability is valid, while on the part \( \Gamma_2 = \Gamma \setminus \Gamma_1 \), a Dirichlet condition can be imposed.

It is noteworthy that equation (1) is of the variable type: it is elliptic in the saturated filtration region (at \( p \geq 0 \)), as in this case \( s(p) = 1 \), and it is parabolic at \( p < 0 \).

In [1], a method is developed for investigating the solvability of initial-boundary value problems of this type using the Kirchhoff transformation:
\[ \mathcal{G}(p) = \int_{0}^{\frac{p}{b(s(\xi))}} d\xi \]  

(2)

From an unknown function \( p(x,t) \), we pass on to the function \( u(x,t) = \mathcal{A}(p(x,t)) \), satisfying the following equation:

\[ m \frac{\partial \tilde{\varphi}(u)}{\partial t} - \text{div}(\nabla u - b(\tilde{\varphi}(u))\rho g) = f(\tilde{\varphi}(u)), \quad \tilde{\varphi}(u) = s\left(\mathcal{G}^{-1}(u)\right) \]

(3)

The generalized solvability of the problem obtained as a result of this transformation is investigated in detail in [1] under the condition that \( \mathcal{G} \) is a one-to-one mapping of the real axis \( R \) on \( R \). From the viewpoint of applications, the latter assumption is very restrictive, since

\[ \mathcal{G}(-\infty) = \int_{0}^{b(s(\xi))} d\xi, \]  

(4)

and for a very wide class of the functions \( s \) and \( b \), this integral is convergent.

Let us demonstrate this in the dependences traditionally used in applied studies:

\[ s(p) = \begin{cases} (1-p)^{-\alpha}, & p < 0 \\ 1, & p \geq 0 \end{cases}, \quad b(s) = s^{\beta}, \quad \alpha > 0, \quad \beta > 0, \quad \alpha\beta > 1. \]  

(5)

It is easy to verify that in this case \( \mathcal{G}(p) \in [-\gamma, +\infty) \forall p \in R \); herewith, \( \gamma = \int_{-\infty}^{0} b(s(\xi)) d\xi = (\alpha\beta - 1)^{-1} \). In problems of the filtration theory, the parameter \( \gamma \) is called the macroscopic capillary length; in [2], it is established that the parameter is bounded.

The purpose of the present paper is to prove the existence of a generalized solution of the variational inequality for equation (1) under more general assumptions on the functions \( s(p) \) and \( b(s) \), which, in particular, also admit a dependence of the form (6).

Problems of the solvability of variational inequalities for the saturated-unsaturated filtration problem are also discussed in [3, 4, 5, 6].

2. Statement of the problem

In the domain \( Q_{t} = \Omega \times (0,T] \), we consider equation (1) with boundary conditions of the kind:

\[ p(x,t)q_{\nu}(x,t) = 0, \quad p(x,t) \leq 0, \quad q_{\nu}(x,t) \geq 0 \quad x \in \Gamma_{1}; \quad p(x,t) = 0, \quad x \in \Gamma_{2}. \]  

(6)

Here \( q_{\nu}(x,t) = q \cdot \nu \), where \( \nu \) is external normal to the boundary \( \Gamma \); \( q \cdot \nu \) is scalar product of vectors \( q \) and \( \nu \) in the space \( R^{d} \). The initial conditions are set in the form:

\[ s(p(x,0)) = s(p_{0}(x)), \quad x \in \Omega, \]  

(7)

where \( p_{0} \) is a given function.

It is easy to see that the conditions (6), (7) generate for the function \( u(x,t) = \mathcal{A}(p(x,t)) \) the boundary and initial conditions of the kind:

\[ u(x,t)\tilde{q}_{\nu}(x,t) = 0, \quad u(x,t) \leq 0, \quad \tilde{q}_{\nu}(x,t) \geq 0 \quad (x,t) \in \Gamma_{1} \times (0,T]; \]

\[ u(x,t) = 0, \quad (x,t) \in \Gamma_{2} \times (0,T]; \]

\[ \tilde{\varphi}(u(x,0)) = \tilde{\varphi}(u_{0}(x)), \quad x \in \Omega, \]  

(8)

where \( \tilde{q} = - (\nabla u - b(\tilde{\varphi}(u))\rho g), \quad u_{0} = \mathcal{A}(p_{0}) \).

Let \( V \) be a functional subspace of the space \( W_{2}^{1}(\Omega) \) with a zero trace on \( \Gamma_{2} \),
\[ K = \{ u \in V : u(x) \leq 0 \text{ almost everywhere on } \Gamma_2 \}. \]

A norm in the space \( V \) is defined by the relation 
\[ \|v\| = \sum_{i=1}^{n} \left( \frac{\partial v}{\partial x_i} \right)^2 \, dx. \]

**Definition 1.** A generalized solution of the problem (3), (8), (9), (10) is a function \( u \) from \( L_2(0,T;K) \), for which the followings conditions
\[ \frac{\partial \tilde{\phi}(u)}{\partial t} \in L_2(0,T;V^*) \quad u(x,t) \geq -\gamma \text{ almost everywhere on } Q_T, \]

are satisfied and for an arbitrary function \( z \in L_2(0,T;K) \) the following inequality is met
\[
\begin{align*}
T &\int_{0}^{T} < m \frac{\partial \tilde{\phi}(u)}{\partial t}, z-u > dt + \int_{Q_T} \nabla u \cdot \nabla (z-u) \, dxdt \geq \\
&\geq \int_{Q_T} \{ b(\tilde{\phi}(u))(\rho g \cdot \nabla (z-u)) + f(\tilde{\phi}(u))(z-u) \} \, dxdt.
\end{align*}
\]

Here \( Q_T = (0,T) \times \Omega : V^* \) is a space conjugate to \( V \); \( < v, z > \) is a value of the functional \( v \in V^* \) on the element \( z \in V \).

**Definition 2.** The function \( p \) defined in \( Q_T \) is called a generalized solution of the problem (1), (6), (7), if
\[ \bar{s}(p) \in L_2(0,T;K), \quad \frac{\partial \bar{s}(p)}{\partial t} \in L_2(0,T;V^*) \quad s(p(x,0)) = s(p_0(x)) \text{ almost everywhere on } \Omega, \]

and for any function \( z \in L_2(0,T;K) \), the following inequality is met
\[
\begin{align*}
T &\int_{0}^{T} < m \frac{\partial \bar{s}(p)}{\partial t}, z-\bar{s}(p) > dt + \int_{Q_T} b(s(p)) (\nabla u - \rho g) \cdot \nabla (z-\bar{s}(p)) \, dxdt \geq \\
&\geq \int_{Q_T} f(s(p))(z-\bar{s}(p)) \, dxdt.
\end{align*}
\]

Hereinafter, it is assumed that for the functions \( s(p), b(s) \), the following conditions are satisfied:

A. The function \( s : R \to (0,1] \) is a continuous and nondecreasing function, such that \( s(p) = 1 \forall p \geq 0, \lim_{p \to -\infty} s(p) = 0 \).

A. The function \( b : [0,1] \to [0,1] \) is a continuous and nondecreasing function and \( b(0) = 0, b(1) = 1 \).

A. The mapping \( \bar{s} : R \to [-\gamma, +\infty), \infty < -\gamma < 0 \), defined by formula (2), is one-to-one, where the parameter \( -\gamma = \bar{s}(-\infty) \) is defined by the equality (4).

3. Existence theorems

**Theorem 1.** Let the functions \( s, b \) and \( \bar{s} \) satisfy the relations A, A, A and the function \( f(\xi) \) be continuous on \([0,1]\) and \( f(0) = 0 \). Then for any function \( u_0 \in K \), there exists at least one generalized solution of the problem (3), (8)-(10).
Proof. First, we note that the properties of the function $\tilde{\varphi}$ that follow from the conditions $A_1, A_2$ are as follows: domain of the function is the set $[-\gamma, +\infty)$, $\tilde{\varphi}(-\gamma) = 0$, $\tilde{\varphi}(\xi) = 1$ for nonnegative $\xi$; on the interval $[-\gamma, 0]$ $\tilde{\varphi}(\xi)$ is a continuous monotonically increasing function.

Hereinafter, it is taken that

$$\varphi(\xi) = \begin{cases} \tilde{\varphi}(\xi), & -\gamma \leq \xi < \infty, \\ 0, & -\infty < \xi < -\gamma. \end{cases}$$

(13)

Along with the problem (3), (8)-(10), we consider a problem hereinafter referred to as a predetermined problem, a generalized solution of which is determined using the following relations:

$$\frac{\partial \varphi(u)}{\partial t} \in L_2(0, T; V^*), \quad u \in L_2(0, T; K), \quad \varphi(u(x, 0)) = \varphi(u_0(x)) \quad \text{almost everywhere on } \Omega,$n
and for any function $z \in L_2(0, T; K)$, the following inequality is met

$$\int_0^T \int_{Q_T} \langle m \frac{\partial \varphi(u)}{\partial t} \rangle_{Q_T} z - u > dt + \int_0^T \int_{Q_T} \nabla u \cdot \nabla (z - u) \ dx dt \geq \int_0^T \int_{Q_T} \left[ b(\varphi(u)) (\rho \varphi \cdot \nabla (z - u)) + f(\varphi(u))(z - u) \right] \ dx dt.$$

(14)

It is easy to see that if the function $u$ is a solution of the predetermined problem, and, in addition, $u(x, t) \geq -\gamma$ for almost all $(x, t) \in Q_T$, then $\tilde{u}$ is a solution of the problem (3), (8)–(10).

The proving of the existence of a generalized solution of the predetermined problem is carried out with the use of the semidiscretization method with a penalty.

Let $\bar{\omega}_T = \{t = k \tau, k = 0, 1, ..., M, M \tau = T\}$, be a grid in the interval $[0, T]$, $\omega_T = \bar{\omega}_T \setminus \{0\}$.

Definition 3. The function $u_{\tau}(t)$ is called a semi-discrete solution of the predetermined problem, if $u_{\tau}(t) \in V$ for any $t \in \bar{\omega}_T$, $\varphi(u_{\tau}(x, 0)) = \varphi(u_0(x))$ almost everywhere in the domain $\Omega$ and for any function $z \in V$, the following equality is true

$$\int_\Omega \frac{\varphi(\hat{u}_{\tau}) - \varphi(u_{\tau})}{\tau} z \ dx + \int_\Omega \nabla \hat{u}_{\tau} \cdot \nabla z \ dx + \frac{1}{\varepsilon} \int_{\Gamma_1} \hat{u}_{\tau}^+ z \ dx =$$

$$= \int_\Omega \left[ b(\varphi(\hat{u}_{\tau}))(\rho \varphi \cdot \nabla z) + f(\varphi(\hat{u}_{\tau}))(z) \right] \ dx dt.$$

(15)

Here $\hat{u}_{\tau} = u_{\tau}(t + \tau)$, $\hat{u}_{\tau}^+ = (\hat{u}_{\tau} + \hat{u}_{\tau})/2$.

Lemma 1. For solving the semi-discrete problem (15), the following estimate is true

$$u_{\tau}(x, t) \geq -\gamma \quad \text{almost everywhere on } \Omega \quad \forall t \in \omega_T.$$

(16)

Proof. In the equality (15), let us select $z = (\gamma + \hat{u}_{\tau})^-$ and write it in the form

$$\int_\Omega \frac{\varphi(\hat{u}_{\tau})(\gamma + \hat{u}_{\tau})^-}{\tau} \ dx - \int_\Omega \varphi(u_{\tau})(\gamma + \hat{u}_{\tau})^- \ dx +$$

$$+ \int_\Omega \nabla \hat{u}_{\tau} \cdot \nabla (\gamma + \hat{u}_{\tau})^- \ dx + \frac{1}{\varepsilon} \int_{\Gamma_1} \hat{u}_{\tau}^+ (\gamma + \hat{u}_{\tau})^- \ dx =$$

$$= \int_\Omega \left[ b(\varphi(\hat{u}_{\tau}))(\rho \varphi \cdot \nabla (\gamma + \hat{u}_{\tau})^-) + f(\varphi(\hat{u}_{\tau}))(\gamma + \hat{u}_{\tau})^- \right] \ dx dt.$$

(17)
Here \( \xi^- = (|\xi| - \xi)/2 \). Since (see definition (13)) \( \phi(\xi) = 0 \) at \( \xi \leq -\gamma \), and the equality \( \psi(\xi) = 0 \) for the function \( \psi(\xi) = (\gamma + \xi)^- \) is satisfied for \( \xi \geq -\gamma \), then \( \phi(\xi)\psi(\xi) = \phi(\xi)(\gamma + \xi)^- = 0 \) for any values of \( \xi \). Similarly, we have \( \xi^+ = 0 \) at \( \xi \leq 0 \); therefore, the product \( \xi^+(\gamma + \xi)^- \) is equal to zero for any \( \xi \). Consequently, the first and fourth terms on the left-hand side of (17) are equal to zero. The right-hand side of (17) is also equal to zero because \( V(\gamma + \hat{u}_\tau)^- = 0 \) in those points, in which \( (\gamma + \hat{u}_\tau)^- = 0 \) (see, for example, [7, p. 45], [8, p. 150]), and from the conditions \( b(0) = 0 \) and \( f(0) = 0 \), it follows that \( b(\phi(\xi)) \) and \( f(\phi(\xi)) \) are not equal to zero only when \( \xi > -\gamma \).

We transform the third term on the left-hand side of (17):

\[
\int_\Omega \nabla \hat{u}_\tau \cdot \nabla (\gamma + \hat{u}_\tau)^- \, dx = -\int_\Omega \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (\gamma + \hat{u}_\tau) \frac{\partial}{\partial x_i} (\gamma + \hat{u}_\tau)^- \, dx = -\left\| (\gamma + \hat{u}_\tau)^- \right\|^2_1. \tag{18}
\]

In view of the foregoing, we rewrite (17) in the form:

\[
-\left\| (\gamma + \hat{u}_\tau)^- \right\|^2_1 - \frac{m}{\tau} \int_\Omega \phi(u_\tau)(\gamma + \hat{u}_\tau)^- \, dx = 0. \tag{19}
\]

Both terms on the left-hand side of (19) are nonpositive, and the right-hand side is equal to zero; therefore it follows from (19) that \( (\gamma + \hat{u}_\tau(x,t))^- = 0 \) almost everywhere in \( \Omega \ \forall t \in \omega_{\tau} \). The estimate (16) is proved.

Using the Galerkin method, the existence of a semi-discrete solution is proved, and the following a priori estimates are established:

\[
\int_\Omega \Phi(u_\tau(t')) \, dx \leq C, \quad \sum_{t=0}^{t-\tau} \|\hat{u}_\tau(t)\|^2 \leq C, \quad \frac{1}{\varepsilon} \sum_{t=0}^{t-\tau} \|\hat{u}_\tau^+(t)\|^2_{L_2(\Gamma_1)} \leq C, \quad \forall t' \in \omega_{\tau}, \tag{20}
\]

\[
\sum_{t=0}^{t-\tau} \|\phi_t(u_\tau(t))\|_s \leq C \quad \forall t' \in \omega_{\tau}, \tag{21}
\]

\[
\frac{1}{k\tau} \sum_{t=0}^{T-k\tau} \int_\Omega (\phi(u_\tau(t + k\tau)) - \phi(u_\tau(t))) \, dx \leq C, \tag{22}
\]

where \( \Phi(\xi) = \int_0^\xi (\phi(\xi) - \phi(\xi)) \, \xi \), \( \xi \in R, \|u\|_s \) is a norm in the space \( V^* \).

The introduction of the function \( \phi(u) \) and obtaining for this function the estimates (20)-(22) is an important stage in proving the theorem, since these estimates provide (see [1]) the existence of a subsequence of functions \( u^\tau_\tau \) converging almost everywhere in \( Q_\tau \). In studying non-linear problems, the existence of such convergence is fundamental. In the last stage of the proving of Theorem 1, it is established that the limit of a sequence of piecewise constant fulfilments with respect to the variable \( \tau \) of solutions of a semi-discrete problem, singled out in the course of the proof, is a generalized solution of the predetermined problem.

**Theorem 2.** Let the functions \( s, b, \) and \( \mathcal{A} \) satisfy the conditions \( A_1, A_2, A_3 \), the function \( f \) be continuous on \([0,1]\) and \( f(0) = 0 \). Then for any function \( p_0 \) such that \( \mathcal{A}(p_0) \in V \), there exists at least one generalized solution of the problem (1), (6), (7).
References
[1] Alt H W, Luckhaus S and Visintin A 1984 Ann. Mat. Pura ed Appl 136 303–316
[2] Morel-Seytoux H J, Meyer P D, Nachabe M, Touma J, van Genuchten M T and Lenhard R J 1996 Water Resources Research 32 58 1251–1258
[3] Hornung U 1982 Manuscripta Math 39 155–172
[4] Kenmochi N and Pawlow I 1988 Japan J. Appl. Math 5 87–121
[5] Kroener D and Luckhaus S 1984 J. of differential equations 55 276–288
[6] Kubo M, Shirakawa R and Yamazaki N 2012 J. Math. Anal. Appl 387 490–511
[7] Karchevskii M M and Pavlova M F 2016 Uravneniya matematicheskoj fiziki. Dopolnitelnie glavi (St. Petersburg: Lan)
[8] Gilbarg D and Trudinger M 1989 Ellipticheskie differencialnie uravneniya s chastnimi proizvodnimi vtorogo porjadka (Moscow: Nauka)