Theory of Recurrent Neural Network with Common Synaptic Inputs

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We discuss the effects of common synaptic inputs in a recurrent neural network. Because of the effects of these common synaptic inputs, the correlation between neural inputs cannot be ignored, and thus the network exhibits sample dependence. Networks of this type do not have well-defined thermodynamic limits, and self-averaging breaks down. We therefore need to develop a suitable theory without relying on these common properties. While the effects of the common synaptic inputs have been analyzed in layered neural networks, it was apparently difficult to analyze these effects in recurrent neural networks due to feedback connections. We investigated a sequential associative memory model as an example of recurrent networks and succeeded in deriving a macroscopic dynamical description as a recurrence relation form of a probability density function.

KEYWORDS: common synaptic inputs, recurrent neural networks, probability density function, correlated firings, sample dependence

1. Introduction

Synfire chains, namely, synchronous firings of neurons, can be observed in the brain.\textsuperscript{1} Diesmann \textit{et al.}\textsuperscript{2} and Câteau and Fukai\textsuperscript{3} discussed conditions for propagating the synchronous firings between layers in layered neural networks, while Amari \textit{et al.} considered common synaptic inputs to neurons in the layered neural networks and discussed correlated firings of neurons.\textsuperscript{4} These studies are based on theoretical models, and the biological structure of the synfire chains or the common synaptic inputs remain to be elucidated. In order to understand the structure, theoretical models must be analyzed. We therefore discuss the effects of the common synaptic inputs on an associative memory model from a theoretical viewpoint. In order to analyze these effects, the structure of our model is simple, unlike that for synfire chain models.

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Using the common synaptic inputs, the sums of inputs to neurons are correlated. The firings of the neurons are, therefore, also correlated, and with an infinite number of neurons there is no thermodynamic limit and sample dependence appears.\(^4,5\) All solvable models, including neural networks, that have been discussed in the statistical mechanics literature have been analyzed by applying the independence of units or neurons at the thermodynamic limit. There are few theoretical approaches, however, that address sample dependence. Yamana and Okada\(^6\) introduced uniform common synaptic inputs that depend on preneurons to the layered associative memory model, and was able to derive the probability density function (PDF) for its macroscopic states. This PDF allows for the analysis of dynamics with sample dependence in the layered associative memory model.

In the layered associative memory model, since synaptic connections within a layer are independent of each other, no correlation occurs between common synaptic inputs on different layers. However, in recurrent neural networks that contain feedback connections, correlations between common synaptic inputs at different times cannot be ignored. Theoretical analysis in such cases might be rendered difficult, and in fact it is indeed hard to analyze qualitatively the effect of common synaptic inputs in an autoassociative memory model.\(^7\)

In recurrent neural networks, correlated connections can also be found in asymmetric synaptic connections,\(^7,8\) e.g.,

\[
J_{ij} \propto \sum_{\mu} (\xi_{i}^{\mu} + a) (\xi_{j}^{\mu} + b) = \sum_{\mu} \xi_{i}^{\mu} \xi_{j}^{\mu} + a \xi_{j}^{\mu} + b \xi_{i}^{\mu} + ab, \tag{1}
\]

where \(\xi^{\mu} = (\xi_{1}^{\mu}, \cdots, \xi_{N}^{\mu})^T\) represents the \(\mu\)th memory pattern. The terms \(a \xi_{j}^{\mu}\) and \(b \xi_{i}^{\mu}\) indicate the connections that depend on pre- and postneurons, respectively. These terms can be considered to be noise to neurons.\(^7\) Since the preneuron-dependent connection leads to correlated firings of neurons, we take particular note of the term \(a \xi_{j}^{\mu}\). Moreover, we reduce this term to one that is independent of the memory patterns, \(w_{j}\).

In this paper, we discuss a sequential associative memory model that is also a recurrent neural network.\(^9-13\) The associative memory model stores memory patterns in the synaptic connections; that is, the synaptic connections are not uniform, but they do have a structure. Moreover, the synaptic connections are time invariant, unlike those for layered networks. We found, however, that time correlations of states in this model can be ignored when a memory pattern is retrieved, since the model retrieves a different pattern sequentially each time. The common synaptic inputs at different times can, therefore, be assumed to be independent. Under this consideration, we have succeeded in deriving a recurrence relation form of the probability density function for macroscopic states in the sequential associative memory model.
2. Sequential Associative Memory Model

Consider a sequential associative memory model consisting of \( N \) units or neurons. The state of the units takes \( x_i^t = \pm 1 \) and is updated synchronously by

\[
x_i^{t+1} = F \left( \sum_{j=1}^{N} J_{ij} x_j^t \right),
\]

where the output function is \( F(h) = \text{sgn}(h) \), and \( J_{ij} \) is a synaptic connection from the \( j \)th neuron to the \( i \)th neuron, and given by

\[
J_{ij} = \frac{1}{N} \sum_{\mu=0}^{p-1} \xi_{i}^{\mu+1} \xi_{j}^{\mu} + w_j,
\]

where \( \xi^p = \xi^0 \). The first term on the rhs represents the coupling as in the existing sequential associative memory model.\(^{9-13}\) It stores \( p \) random patterns \( \xi^t = (\xi_1^t, \cdots, \xi_N^t)^T \) so as to retrieve the patterns as \( \xi^0 \rightarrow \xi^1 \rightarrow \cdots \xi^{p-1} \rightarrow \xi^0 \) sequentially. The second term on the rhs, \( w_j \), represents preneuron-dependent coupling. From eqs. (2) and (3), we obtain

\[
x_i^{t+1} = F \left( \frac{1}{N} \sum_{\mu=0}^{p-1} \sum_{j=1}^{N} \xi_{i}^{\mu+1} \xi_{j}^{\mu} x_j^t + \sum_{j=1}^{N} w_j x_j^t \right).
\]

Let the second term on the rhs be

\[
\eta_t = \sum_{j=1}^{N} w_j x_j^t.
\]

We call \( \eta_t \) the common synaptic input, since it is independent of index \( i \) and affects all neurons equally. Of course one can consider \( \eta_t \) to be an external input coming from the outside system, which might be independent of preneurons \( x_j^t \). In order to analyze the dynamics theoretically, we will assume that the coupling \( w_j \) obeys the Gaussian distribution with \( \mathcal{N}(0, \delta^2/N) \). Therefore, the common synaptic inputs act like noise in this case.

The number of neurons is given by \( p = \alpha N \). We call \( \alpha \) the loading rate. Each component of the memory patterns is assumed to be an independent random variable that takes a value of either \(+1\) or \(-1\) according to the probability

\[
\text{Prob}[\xi_i^\mu = \pm 1] = \frac{1}{2}.
\]

We define the overlap by the direction cosine between the state \( x_i^t \) and the retrieval pattern \( \xi_i^t \) at time \( t \),

\[
m_t = \frac{1}{N} \sum_{i=1}^{N} \xi_i^t x_i^t,
\]

and determine the initial state \( x_i^0 \) according to the probability distribution

\[
\text{Prob}[x_i^0 = \pm 1] = \frac{1 \pm m_0 \xi_i^0}{2}.
\]
Therefore, the overlap between the pattern $\xi^0$ and the initial state $x^0$ is $m_0$. The network state $x^t$ at time $t$ is expected to be near the pattern $\xi^t$ when the initial overlap $m_0$ is large and the loading rate is under its storage capacity.

3. Theory

3.1 Macroscopic state

Let us derive the macroscopic state equations in the case of dynamics with sample dependence. From eqs. (4)–(7), we obtain

$$x_{i}^{t+1} = F(\xi_{i}^{t+1}m_{t} + z_{i}^{t} + \eta_{t}),$$

$$z_{i}^{t} = \frac{1}{N} \sum_{\mu \neq t}^{N} \xi_{i}^{\mu+1}\xi_{j}^{\mu}x_{j}^{t},$$

where $z_{i}^{t}$ is a crosstalk noise term. We assume that the crosstalk noise obeys the Gaussian distribution with mean 0 and variance $\sigma_{t}^{2}$ according to the statistical neurodynamics.$^{9,14-16}$

The state $x_{j}^{t+1}$ is expanded in terms of $\xi_{j}^{\mu}$ as follows:

$$z_{i}^{t+1} = \frac{1}{N} \sum_{\mu \neq t+1}^{N} \xi_{i}^{\mu+1}\xi_{j}^{\mu}F(\xi_{j}^{t+1}m_{t} + z_{j}^{t} + \eta_{t}),$$

$$= \frac{1}{N} \sum_{\mu \neq t+1}^{N} \xi_{i}^{\mu+1}\xi_{j}^{\mu}x_{j}^{t+1,\mu} + \frac{1}{N} \sum_{\mu \neq t+1}^{N} \xi_{i}^{\mu+1}\xi_{k}^{\mu-1}x_{k}^{t}U_{t+1},$$

where

$$x_{j}^{t+1,\mu} = F\left(\frac{1}{N} \sum_{\nu \neq \mu}^{N} \xi_{j}^{\nu+1}\xi_{k}^{\nu}x_{k}^{t} + \eta_{t}\right),$$

$$U_{t+1} = \frac{1}{N} \sum_{j=1}^{N} F'(\xi_{j}^{t+1}m_{t} + z_{j}^{t} + \eta_{t}).$$

Therefore, the variance of the crosstalk noise becomes$^{11,13}$

$$\sigma_{t+1}^{2} = E\left[(z_{i}^{t+1})^{2}\right] = \alpha + U_{t+1}^{2}\sigma_{t}^{2}. \quad (15)$$

Let us next consider the initial state to be $x^{0} = \xi^{0}$. In this case, since the pattern $\xi^{t}$ is retrieved at time $t$ and the memory patterns are independent of each other, we can assume that $x^{t} \approx \xi^{t}$ and the state $x_{i}^{t}$ can become independent of each other. When the correlation between $w_{j}$ and $x_{j}^{t}$ can be neglected, the common synaptic inputs $\eta_{t}$ are independent with respect to time $t$. Therefore, $\eta_{t}$ are iid and they obey the Gaussian distribution with $N(0, \delta^{2})$. First of all, we will discuss the case in which $\eta_{t}$ is given. Here, $m_{t+1}$ and $\sigma_{t+1}$ can be represented as
functions of $m_t$, $\sigma_t$ and $\eta_t$:

$$m_{t+1}(m_t, \sigma_t, \eta_t) = \int D_\xi \langle \xi^{t+1} F (\xi^{t+1} m_t + \sigma_t z + \eta_t) \rangle_\xi,$$

$$= \frac{1}{2} \left[ \text{erf}(u) + \text{erf}(v) \right],$$

$$U_{t+1}(m_t, \sigma_t, \eta_t) = \int D_\xi \langle F' (\xi^{t+1} m_t + \sigma_t z + \eta_t) \rangle_\xi,$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \left[ \exp(-u^2) + \exp(-v^2) \right],$$

$$\sigma^2_{t+1}(m_t, \sigma_t, \eta_t) = \alpha + U^2_{t+1}(m_t, \sigma_t, \eta_t) \sigma^2_t(m_{t-1}, \sigma_{t-1}, \eta_{t-1}),$$

where $D_\xi = \frac{d}{d\xi} \exp \left(\frac{-\xi^2}{2}\right)$ and $u = (m_t + \eta_t)/\sqrt{2}\sigma$, $v = (m_t - \eta_t)/\sqrt{2}\sigma$; $\langle \cdot \rangle_\xi$ denotes the average over $\xi$.

3.2 Probability density function

From eqs. (17)–(19) we can evaluate the dynamics for various values of $\delta$. In the case of $\delta = 0$, the behavior is deterministic as with the existing sequential associative memory model. In the case of $\delta > 0$, the behavior changes drastically. Furthermore, $m_t$ and $\sigma_t$ are distributed, and these distributions are described as the probability density function $p(m_t, \sigma_t, \eta_t)$. As described above, since $m_t$ and $\sigma_t$ are independent of $\eta_t$, the probability density function is decoupled as

$$p(m_t, \sigma_t, \eta_t) = p(m_t, \sigma_t) p(\eta_t).$$

We can, therefore, obtain the PDF by

$$p(m_{t+1}, \sigma_{t+1}) = \int dm_t d\sigma_t d\eta_t p(m_t, \sigma_t) p(\eta_t) \times \delta(m_{t+1} - m_{t+1}(m_t, \sigma_t, \eta_t)) \delta(\sigma_{t+1} - \sigma_{t+1}(m_t, \sigma_t, \eta_t)),$$

where $\delta(\cdot)$ denotes the Dirac's delta function. The PDF of $p(\eta_t)$ is given by

$$p(\eta_t) = \frac{1}{\sqrt{2\pi}\delta} \exp \left(-\frac{\eta_t^2}{2\delta^2}\right).$$

We combine the terms of $\eta_t$ into kernel function $K(m_{t+1}, \sigma_{t+1}; m_t, \sigma_t)$:

$$p(m_{t+1}, \sigma_{t+1}) = \int dm_t d\sigma_t p(m_t, \sigma_t) K(m_{t+1}, \sigma_{t+1}; m_t, \sigma_t),$$

$$K(m_{t+1}, \sigma_{t+1}; m_t, \sigma_t) = \int d\eta_t p(\eta_t) \delta(m_{t+1} - m_{t+1}(m_t, \sigma_t, \eta_t)) \times \delta(\sigma_{t+1} - \sigma_{t+1}(m_t, \sigma_t, \eta_t)).$$

The kernel function $K(m_{t+1}, \sigma_{t+1}; m_t, \sigma_t)$ can be evaluated analytically. Let $m^*_t, \sigma^*_t$ be the value of $m_t, \sigma_t$ satisfying eqs. (17)–(19). Then, $K(m_{t+1}, \sigma_{t+1}; m_t, \sigma_t)$ becomes

$$K(m_{t+1}, \sigma_{t+1}; m_t, \sigma_t) = \int d\eta_t p(\eta_t) \frac{2\pi\eta^2_t}{(u-v)^2} \frac{2\pi\alpha + (e^{-u^2} + e^{-v^2})}{(e^{-u^2} + e^{-v^2}) e^{-u^2-v^2}},$$

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where \( u = (m_t^* + \eta_t) / \sqrt{2\sigma_t^*} \) and \( v = (m_t^* - \eta_t) / \sqrt{2\sigma_t^*} \). Our PDF agrees with the PDF for the layered associative memory model obtained by Yamana and Okada.\(^6\)

4. Effect of Common Synaptic Inputs

We demonstrate the effect of the common synaptic inputs in our model with computer simulations. The effect of the inputs depends on the variance \( \delta^2 \). In the case of \( \delta = 0 \), the dynamical behaviors are uniquely determined according to the initial states. On the other hand, in the case of \( \delta > 0 \), since there exists the correlation between the inputs to neurons, sample dependence arises. That is, the dynamical behaviors are not determined according to the initial states, and the model either succeeds or fails to retrieve the memory pattern from the same initial state.

First, we show the time evolutions of overlap when there is no common synaptic input \( (\delta = 0) \). Figure 1 shows 30 samples of overlap \( m_t \) for initial overlaps \( m_0 = 0.30 \) and 0.45, where the loading rate is \( \alpha = 0.20 \) and the number of neurons is \( N = 5,000 \). While there are fluctuations in the overlaps, they are caused by a finite number of neurons; the larger the number of neurons is, the smaller the fluctuations are. Therefore, no sample dependence can be seen in the case of \( \delta = 0 \).

Next, we show the time evolutions of overlap for various \( \delta \) values in order to find the effect of the common synaptic inputs. Figure 2 shows the overlaps for \( \delta = 0.1, 0.2 \) and 0.3. We use an initial overlap of \( m_0 = 1 \) in order to discuss the stability of the memory state. For small \( \delta \) values, as in Figs. 2(a) and 2(b), the stored patterns can be stably retrieved. For \( \delta = 0.3 \) as in Fig. 2(c), however, the network gradually reaches away from the memory state in many of the samples.

Furthermore, we verify that a memory state is an attractor when the memory state is stable. Figure 3 shows the time evolutions of overlap for \( \delta = 0.20 \), where \( m_0 = 0.30 \) and 0.45,
Fig. 2. Time evolutions of overlap with common synaptic inputs for (a) $\delta = 0.1$, (b) $\delta = 0.2$, and (c) $\delta = 0.3$, where $\alpha = 0.20$ and $m_0 = 1.0$.

$\alpha = 0.20$ and $N = 5,000$. The figure shows 30 samples of different trials. Whereas the network reaches the nonretrieval state for all samples in the case of $m_0 = 0.30$, as in Fig. 3(a), it reaches either the retrieval or nonretrieval state depending on the samples in the case of $m_0 = 0.45$, as in Fig. 3(b). From these results, the memory state is the stationary state of the network, and
it has a finite basin of attraction. From the same initial state the network can reach different attractors by the common synaptic inputs, which means that sample dependence exists and that self-averaging breaks down.\textsuperscript{4,6}

5. Probability Distribution

From Figs. 2 and 3 we can see the sample dependence by the common synaptic inputs. We therefore need to discuss the distribution of macroscopic states instead of the behavior for each trial in order to analyze the behavior of the network; that is, we must discuss the probability distribution of overlaps. From eq. (23), the probability distribution at any time $t$ can be evaluated when the initial distribution $p(m_0, \sigma_0)$ is given. Here, we introduce the marginal probability distribution, $p(m_t)$, which is integrated with respect to $\sigma_t$:

$$p(m_t) = \int d\sigma_t p(m_t, \sigma_t).$$

We analyze the probability distribution of overlaps at time $t = 5, 30$ and $90$ in Fig. 3(b). Figure 4 shows the marginal probability distribution obtained by our theory and histograms obtained from the computer simulations. The lines denote the results obtained from eq. (26),
and the boxes denote the histograms for 1,000 samples obtained from the computer simulations ($N = 5,000$). In the cases of Figs. 4(a) and 4(b), the results obtained by the theory agree with those obtained by the computer simulations. On the other hand, in the case of Fig. 4(c), both results agree at $m_t \approx 1$, but the distribution by the computer simulations spreads at $m_t \approx 0$. In the nonretrieval state, the assumption of $x^t \approx \xi^t$ may not be satisfied, in which case the time correlation may not be ignored. We have, however, verified with the computer simulations that $x^t$ has no time correlation in the nonretrieval state. Figure 5 shows the time evolutions of overlap $m_t$ and the time correlation coefficient for an initial overlap $m_0 = 0.10$, where $\alpha = 0.20$, $\delta = 0.20$ and $N = 20,000$. The error bars and the line represent the average and standard deviation of the time correlation coefficients and the average of overlaps over 20 trials, respectively. The network state goes to the nonretrieval state, since the overlap $m_t$ becomes zero. The time correlation coefficients are calculated using the states $x_i^t$ and $x_i^{t+1}$. Since they are almost zero, $x^t$ has no time correlation.

Another possible source of disagreement in the nonretrieval case is in the Gaussian assumption of the crosstalk noise. Although in the sequential associative memory model without the common synaptic inputs the crosstalk noise obeys the Gaussian distribution even in the nonretrieval case,\textsuperscript{13} it is difficult to show that it is the Gaussian in the model with the common synaptic inputs. Therefore, we suppose that the fluctuation at $m_t \approx 0$ is caused by either the breakdown of the Gaussian assumption or the fact that there is a finite number of neurons, as is the case with $\delta = 0$ (Fig. 1).

6. Summary

Correlated firing such as that by synfire chains is a noticeable phenomenon. The mechanism will be elucidated by theoretical models in the future. We discussed the effects of the common synaptic inputs in a sequential associative memory model. In this model, correlated firing occurs because the input to each neuron has a correlation due to the common synaptic inputs; therefore, sample dependence exists. We verified the existence of sample dependence via computer simulations. In order to investigate the correlated firing, we need to analyze theoretically novel phenomena caused by the sample dependence. However, we were unable to use the independence of units or neurons at the thermodynamic limit. Moreover, in recurrent neural networks, theoretical treatment is much more difficult because of feedback connections. We therefore considered the sequential associative memory model, in which time correlation can be ignored, allowing us to derive a recurrence relation form of the PDF at the macroscopic state. The probability distributions obtained by our theory agree with those obtained by the computer simulations.

We analyzed the sequential associative memory model that had common synaptic inputs. However, it may be hard to rigorously analyze models such as autoassociative memory models since the time correlation cannot be neglected.
Fig. 4. Marginal probability distribution at (a) $t = 5$, (b) $t = 30$, and (c) $t = 90$, where $\alpha = 0.20$, $m_0 = 0.45$ and $\delta = 0.20$.

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Fig. 5. Time evolutions of overlap and time correlation coefficient, where $\alpha = 0.20, m_0 = 0.10, \delta = 0.20$. 

\[ \alpha = 0.20, \delta = 0.20, m_0 = 0.1 \]
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