On the periodical solutions for single-dimensional gas dynamics

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Abstract. In the theory of one-dimensional non stationary adiabatic gas dynamics the motion of a perfect ideal gas with plane waves is considered. With the help of Lagrangian mass variable the equations can be reduced to the system with quadratic nonlinearity. The solutions for the motion law and pressure, in particular, are sought in the form of series in powers of the sine of time with coefficients depending on the mass. For the series construction it is necessary to define the first three coefficients according to initial conditions. All the other coefficients are calculated recurrently without solving any differential equations or integration. Numerical calculations of series coefficients also show a regular behavior of the constructed solutions under certain restrictions on the derivatives of predetermined functions. The advantage of this approach over the Fourier series expansions of motion law and pressure is precisely the finiteness of algebraic recurrence relations associated only with the computation of derivatives with respect to the mass. As a result a new class of gas dynamics solutions is obtained. The examples with no any singularity are presented. The way to generalize the constructed solution is discussed.

1. Construction of solutions
Let us consider one-dimensional non stationary adiabatic motion of a perfect ideal gas with plane waves. In the framework of corresponding theory [1, 2] we can write the system of gas dynamics equations using Lagrangian mass variable $m$ in the form

$$
x_{tt} + p_m = 0, \quad p_x + \gamma p p_{xt} = 0.
$$

(1.1)

Here $x$ is the denotation for the motion law, $p$ is the denotation for the pressure, $t$ means time and $\gamma$ is adiabatic index. The system is supposed to be dimensionless. Density $1/x_m$ is related to the pressure in adiabatic law.

We seek a solution of this system as the power series of the sine of time:

$$
x(m,t) = \sum_{n=0}^{\infty} x_n(m) \sin^n t, \quad p(m,t) = \sum_{n=0}^{\infty} p_n(m) \sin^n t.
$$

(1.2)

A constitution of expressions (1.2) in the system (1.1) brings us to the recurrent connections of the series coefficients.
These relations are valid for all the natural \( n \) except 1. The coefficient \( p_1 \) is found separately and has the form

\[
p_1(m) = -\gamma p_0(m) \frac{x_0'(m)}{x_0''(m)}.
\]  

It should be mentioned that coefficients \( x_0, x_1 \) and \( p_0 \) are known and defined by the initial conditions. They could be expressed in the form

\[
x_0(m) = x(m,0), \quad x_1(m) = x_1(m,0), \quad p_0(m) = p(m,0).
\]  

Thus, we can calculate all the series coefficients exactly via the finite number of known terms without solving any differential equations or integration. It is the most efficient advantage of this approach over the Fourier series expansion. As a result, we construct a new class of periodical gas dynamics solutions.

The described approach allows computational procedures that help to calculate the coefficients exactly. It could be provided using any computational package that supports symbolic transformations. We should define only the initial conditions. Boundary conditions could be found a posteriori.

On the practice we can’t use an infinite number of series coefficients, so it is necessary to make a stop on the certain step, to replace the series with partial sums of them, to cut the series. To control the precision of this procedure and at the same time to test the necessary condition for convergence of the series we can compare entropy that is calculated with the cut series of motion law and pressure with the exact value of entropy that is defined by the initial conditions (entropy is conserved in time in adiabatic processes [3, 4]). It is very useful to calculate the relative error

\[
r(m,t) = \left| \frac{px''_x - px'_x}{p_0x''_0} \right|,
\]  

\( p \) and \( x \) are the cut series here.

We can define boundary conditions, for example, as an impermeability through the movable walls (pistons) that are placed in the points \( m = 0 \) and \( m = 1 \). Motion laws of these pistons can be found a posteriori, i.e. after the calculating of solution is provided. They are described by the functions \( x(0, t) \) and \( x(1, t) \). In this sense we can assume that the problem is solved by semi-inverse method.

In the next section we show some examples of these solutions and discuss their properties. For all the calculations Maple computing environment was used.

2. Examples of solutions
Let us define initial conditions (1.6) as

\[
x_0(m) = m, \quad x_1(m) = 0.1 \tanh(m - 1/2), \quad p_0(m) = 1.
\]
Actually we consider gas expansion in the tube. Let us assume that adiabatic index equals 1.4 and let us take the first 10 coefficients of the series according to the formulae (1.3)–(1.5). In this case we get the graphs of velocity, density, pressure and relative error (1.7) presented on figure 1.

Relative error is sufficiently small, the order is 0.07%. It says that series behave well and precision is very high even if we leave only ten members of the series. Gas expands regularly, there are no any singularity appears for all the time moments $t$ and for all the particles with Lagrangian variable $m$.

Figure 1. Surface graphs of velocity (a), density (b), pressure (c) and relative error (d) for initial conditions are given by expressions (2.1).

Another example is devoted to the evolution of the high pressure region. For this case let us write initial conditions (1.6) in the form

$$
\begin{align*}
    x_0(m) &= m, & x_1(m) &= 0, & p_0(m) &= 0.1 \exp \left(- \left[ m - 1/2 \right]^2 \right). \\

\end{align*}
$$

For the same adiabatic index and the number of calculating coefficients the presentation of the considered flow according to the formulae (1.3)–(1.5) is placed on figure 2.

Relative error (1.7) for this example has the order of 0.4%. As in the previous case maximum of the error is in the points where the cosine of time equals zero. There is also no any singularity for all the $t$ and $m$. 
Figure 2. Surface graphs of velocity (a), density (b), pressure (c) and relative error (d) for initial conditions are given by expressions (2.2).

We can combine these two solutions and make it more complex if consider density heterogeneity. For these reasons let us take coefficients (1.6) as the following functions

\[ x_0(m) = \arctan(m - 1/2), \quad x_1(m) = 0.1 \tanh(m - 1/2), \quad p_0(m) = 0.1 \exp\left( - [m - 1/2]^2 \right) \]. (2.3)

In this case we take only 8 coefficients of the series because of calculation complexity. Adiabatic index is the same as before. The solution is presented on figure 3. Relative error order equals 0.7% and is slightly greater than in previous cases because of complexity of the constructed solution. The oscillations are regular.

It should be mentioned that the order of numerical factors in formulae (2.1)–(2.3) such as 0.1 is important. For a great value of these factors the solution could behave bad and show a big relative error (1.7). Hence, the control of relative error is important feature of the constructed solutions realization in practice.
3. Generalization of solutions

Pythagorean trigonometric identity allows us to generalize the constructed series solution. It could be sought in the form

\[ x(m,t) = \sum_{n=0}^{\infty} x_{1n}(m) \sin^n t + \cos t \sum_{n=0}^{\infty} x_{2n}(m) \sin^n t, \]

\[ p(m,t) = \sum_{n=0}^{\infty} p_{1n}(m) \sin^n t + \cos t \sum_{n=0}^{\infty} p_{2n}(m) \sin^n t. \]

All the unknown coefficients can be found by substituting the expressions (3.1)–(3.2) in the system of equations (1.1). In this solution we should define six functions of mass \( x_{10}, x_{11}, p_{10}, x_{20}, x_{21} \) and \( p_{20} \). It is very important that the difference between the squares of the very first series coefficients of the motion law series must not be zero, i.e. \( x_{10}^2 \neq x_{20}^2 \). If this equity is not valid, the solution cannot be determined uniquely.

Figure 3. Surface graphs of velocity (a), density (b), pressure (c) and relative error (d) for initial conditions are given by expressions (2.3).
Solutions (1.2) and (3.1)–(3.2) represent Fourier series recombination that helps to solve nonlinear equations (1.1). It is possible to apply this idea to other physical systems whose equations have the appropriate form.

Acknowledgements
The work was supported by Russian Foundation for Basic Research (project No 17-01-00037).

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