PREPERIODIC POINTS OF POLYNOMIALS OVER GLOBAL FIELDS

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Abstract. Given a global field $K$ and a polynomial $\phi$ defined over $K$ of degree at least two, Morton and Silverman conjectured in 1994 that the number of $K$-rational preperiodic points of $\phi$ is bounded in terms of only the degree of $K$ and the degree of $\phi$. In 1997, for quadratic polynomials over $K = \mathbb{Q}$, Call and Goldstine proved a bound which was exponential in $s$, the number of primes of bad reduction of $\phi$. By careful analysis of the filled Julia sets at each prime, we present an improved bound on the order of $s \log s$. Our bound applies to polynomials of any degree (at least two) over any global field $K$.

Let $K$ be a field, and let $\phi \in K(z)$ be a rational function. Let $\phi^n$ denote the $n^{th}$ iterate of $\phi$ under composition; that is, $\phi^0$ is the identity function, and for $n \geq 1$, $\phi^n = \phi \circ \phi^{n-1}$. We will study the dynamics $\phi$ on the projective line $\mathbb{P}^1(K)$. In particular, we say a point $x$ is preperiodic under $\phi$ if there are integers $n > m \geq 0$ such that $\phi^m(x) = \phi^n(x)$. The point $y = \phi^m(x)$ satisfies $\phi^{n-m}(y) = y$ and is said to be periodic, as its iterates will forever cycle through the same finite sequence of values. Note that $x \in \mathbb{P}^1(K)$ is preperiodic if and only if its orbit $\{\phi^n(x) : n \geq 0\}$ is finite.

For example, let $K = \mathbb{Q}$ and $\phi(z) = z^2 - 29/16$. Then $\{5/4, -1/4, -7/4\}$ forms a periodic cycle (of period 3), and $-5/4, 1/4, 7/4$, and $\pm 3/4$ each land on this cycle after one or two iterations. In addition, the point $\infty$ is of course fixed. These nine $\mathbb{Q}$-rational points are all preperiodic under $\phi$. Meanwhile, it is not difficult to see that no other point in $\mathbb{P}^1(\mathbb{Q})$ is preperiodic by showing that the denominator of a rational preperiodic point must be 4, and that the absolute value must be less than 2.

In general, for any global field $K$, any dimension $N \geq 1$, and any morphism $\phi : \mathbb{P}^N \to \mathbb{P}^N$ over $K$ of degree at least two, Northcott proved in 1950 that the number of $K$-rational preperiodic points of $\phi$ is finite [25]. More precisely, he showed that the preperiodic points form a set of bounded arithmetic height. Years later, by analogy with the Theorems of Mazur [19] and Merel [20] on $K$-rational torsion of elliptic curves, Morton and Silverman proposed the following Conjecture [23].

Uniform Boundedness Conjecture. (Morton and Silverman, 1994)

Given integers $D, N \geq 1$ and $d \geq 2$, there is a constant $\kappa = \kappa(D, N, d)$ with the following property. Let $K$ be a number field with $[K : \mathbb{Q}] = D$, and let $\phi : \mathbb{P}^N \to \mathbb{P}^N$ be a morphism of degree $d$ defined over $K$. Then $\phi$ has at most $\kappa$ preperiodic points in $\mathbb{P}^N(K)$.

The analogy between preperiodic points and torsion comes from the fact that the torsion points of an elliptic curve $E$ are precisely the preperiodic points of the multiplication-by-two map $[2] : E \to E$. In fact, taking $x$-coordinates, the map $[2]$ induces a rational function

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(known to dynamicists as a Lattès map) $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 4 whose preperiodic points are precisely the $x$-coordinates of the torsion points of $E$. Thus, Merel’s Theorem would follow as a simple corollary of the Morton and Silverman Conjecture for $N = 1$ and $d = 4$. More generally, Fakhruddin has shown [12] that the full Morton and Silverman Conjecture for $D = 1$ would imply uniform boundedness of torsion for abelian varieties.

The Conjecture seems to be very far from a proof. However, there is growing evidence that it is valid, at least in the simplest case, that $K = \mathbb{Q}$, $N = 1$, and $\phi$ is a polynomial of degree 2. (The problem then reduces to considering $\phi_c(z) = z^2 + c$, with $c \in \mathbb{Q}$.) In particular, the computations in [22] and [13] show that $\phi_c$ never has a rational point of period 4 or 5 (respectively). Moreover, Poonen showed in 1998 that if $\phi_c$ never has rational periodic points of period greater than 5, then it never has more than 9 rational preperiodic points [23]. (That bound, if true, would be sharp, in light of the $c = -29/16$ example above.) Those results all considered moduli spaces, for fixed $n > m \geq 0$, of pairs $(c,x)$ such that $\phi^n_c(x) = \phi^m_c(x)$, giving curves analogous to modular curves, but with no known structure to take the place of a Hecke ring. Instead, the theorems were proven by delicate ad hoc computations on the particular curves that arose.

Other researchers have found bounds for the longest possible period of a $K$-rational periodic point by analyzing at a prime of “good reduction” (see Definition 2.1 below); see, for example, [11, 23, 24, 26, 27, 35]. If $s$ is the total number of primes of “bad reduction,” then these results lead to bounds on the order of at least $d^{4s d}$ for the number of $K$-rational periodic points (cf. Corollary B of [23], for example). Moreover, these results say virtually nothing about the more general preperiodic points.

A different strategy (for the family $\phi_c(z) = z^2 + c$ for $c \in \mathbb{Q}$) appeared in a 1997 theorem of Call and Goldstine [9]. They showed that $\phi_c$ has at most $1 + 2^{s+2}$ rational preperiodic points, where $s$ is the number of primes of bad reduction. In their argument, they analyzed the dynamics at the primes of bad, not good, reduction. In particular, they studied the filled Julia set $K_v$ (see Definition 2.2 below) at a bad prime $v$. All preperiodic points except $\infty$ sit inside $K_v$, which in turn lies in a union of two $v$-adic disks, each of volume 1. For good $v$, a single disk of volume 1 suffices. (A slightly different condition holds at 2 and at the archimedean (i.e., standard) absolute value, of course.) The bound of $O(2^s)$ then follows naturally by anadelic argument.

In this paper, we will use a more detailed analysis of the filled Julia sets to improve the above bounds substantially. We will work only with polynomials and only in dimension 1, but we will allow arbitrary degree $d \geq 2$ and we will work over an arbitrary global field $K$. Following Call and Goldstine, we will study dynamics over a global field $K$ by considering each of the associated complete valued fields $\mathbb{C}_v$. Thus, we will end up studying both complex dynamics (if $\mathbb{C}_v \cong \mathbb{C}$) and the newer realm of $p$-adic and non-archimedean dynamics. We refer the reader to [4, 10, 21] for expositions on complex dynamics, and to [5, 6, 8, 17, 30, 31, 33] for papers exploring various aspects of $p$-adic dynamics.

Our main result is the following.

**Main Theorem.** Let $K$ be a global field, let $\phi \in K[z]$ be a polynomial of degree $d \geq 2$, and let $s$ be the number of bad primes of $\phi$ in $K$. Then the number of preperiodic points of $\phi$ in $\mathbb{P}^1(K)$ is at most $O(s \log s)$.

A more precise statement appears in Theorem 7.1. The big-$O$ constant is essentially $(d^2 - 2d + 2)/\log d$ for large $s$. 
The idea of the proof is to consider, for a given prime $v$, the product

$$P_v = \prod_{i \neq j} |x_i - x_j|_v,$$

where $\{x_1, \ldots, x_N\}$ are finite $K$-rational preperiodic points of $\phi$. (The product $P_v$ is related to transfinite diameters and capacities, as discussed in Section 4.) If $r_v$ is the diameter of the filled Julia set $\mathcal{K}_v$, then $P_v \leq r_v^{N(N-1)}$ naively; but in Lemmas 3.4.a and 4.1, we obtain $P_v \leq r_v^{(d-1)N\log_d N}$ (with some correction factors for $v$ archimedean).

The key, however, is our treatment of the prime $w$ with filled Julia set $\mathcal{K}_w$ of the largest diameter. We partition $\mathcal{K}_w$ into two pieces, and we show in Lemmas 3.4.b, 5.1, and 6.3 that the corresponding product $P_w$ on each piece satisfies $P_w \leq r_w^{(d-1)N\log_d N - AN + B}$ for certain simple constants $A$ and $B$. The product $P$ of all the $P_v$’s, restricted to preperiodic points in the given piece of $\mathcal{K}_w$, is then bounded by $r_w^{(d-1)NE}$, where $E = s \log_d N - AN + B$. For $N$ slightly larger than $(1/A)s \log_d N$ (see Lemma 3.5), we get $E < 0$, so that $P < 1$, which contradicts the product formula for the global field $K$. Thus, we get a bound of about $(1/A)s \log_d N$ on each piece; summing the two bounds gives the Theorem.

Of course, the details are complicated. In Sections 1 and 2, we will set terminology and recall fundamental facts concerning local and global fields, bad primes, and filled Julia sets. In Section 3, we will introduce notation for certain expressions that will arise later, and we will bound these expressions in a series of technical but completely elementary Lemmas. In Section 4, we will discuss transfinite diameters and prove our first nontrivial bound for $P_v$, for general bad primes. In Sections 5 and 6, we will describe the partition of the filled Julia set at a bad prime. Finally, in Section 7, we will state Theorem 7.1 and combine all the results from the preceding sections to prove it.

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### 1. Global Fields and Local Fields

In this section we present the necessary fundamentals from the theory of local and global fields. We also set some notational conventions for this paper. Although this material is well known to number theorists, we present it for the convenience of dynamicists. See Section B.1 of [16] or Section 4.4 of [29] for more details on global fields and sets of absolute values; see [14, 18] for expositions concerning the local fields $\mathbb{C}_v$.

#### 1.1. Global fields and absolute values

Throughout this paper, $K$ will denote a global field. That is, $K$ is either a number field (i.e., a finite extension of $\mathbb{Q}$) or a function field over a finite field (i.e., a finite extension of $\mathbb{F}_p(T)$ for some prime $p$). We will write $M_K$ for the set of standard absolute values on $K$. That is, $M_K$ consists of functions $|\cdot|_v : K \to \mathbb{R}$ satisfying $|x|_v \geq 0$ (with equality if and only if $x = 0$), $|xy|_v = |x|_v |y|_v$, and $|x + y|_v \leq |x|_v + |y|_v$, for all $x, y \in K$. (We will frequently abuse notation and write $v \in M_K$ when our meaning is clear.) Moreover, the absolute values in $M_K$ are chosen to satisfy a product formula, which
is to say that for each \( v \in M_K \), there is an integer \( n_v \geq 1 \) such that for all \( x \in K^\times \),

\[
\prod_{v \in M_K} |x|_v^{n_v} = 1.
\]

(Implicit in the product formula is the fact that for any \( x \in K^\times \), we have \( |x|_v = 1 \) for all but finitely many \( v \in M_K \).

All but finitely many \( v \in M_K \) satisfy the ultrametric triangle inequality

\[
|x + y|_v \leq \max\{|x|_v, |y|_v\}.
\]

(Note that this means \( |n|_v \leq 1 \) for all integers \( n \in \mathbb{Z} \)). Such \( v \) are called non-archimedean absolute values; the finitely many exceptions are called archimedean absolute values. If \( K \)

is a function field, then all absolute values are non-archimedean. If \( K \) is a number field, then there are archimedean absolute values, each of which, when restricted to \( \mathbb{Q} \), is the familiar absolute value \( |\cdot| \), commonly denoted \( |\cdot|_{\infty} \).

In fact, if \( K \) is a number field, then

\[
\sum_{v \in M_K, \ v \text{ archimedean}} n_v = [K:Q].
\]

Meanwhile, the non-archimedean absolute values in \( M_K \) correspond to prime ideals of the ring of integers of \( K \). For this reason, we frequently refer to the absolute values \( v \in M_K \) as primes of \( K \), even when \( v \) is archimedean.

If \( v \) is non-archimedean, then \( |K^\times|_v \) is a discrete subset of \( \mathbb{R} \), and we say that \( v \) is a discrete valuation on \( K \). In that case, let \( \varepsilon \in (0,1) \) be the largest absolute value less than 1 attained in \( |K^\times|_v \), and choose \( \pi_v \in K \) with \( |\pi_v|_v = \varepsilon \). Then \( \pi_v \) is called a uniformizer of \( K \) at \( v \), and we have \( |K^\times|_v = \{\varepsilon_m : m \in \mathbb{Z}\} \). Moreover, if \( K \) is a number field, then

\[
|\pi_v|_{v^{-n_v} = p^f}
\]

for some prime number \( p \in \mathbb{Z} \) and some positive integer \( f \), and \( |\cdot|_v \) restricted to \( \mathbb{Q} \) is the usual \( p \)-adic absolute value on \( \mathbb{Q} \). In this case, we say that \( v \) lies above the prime number \( p \).

1.2. Local fields. For each \( v \in M_K \), we can form the completion \( K_v \) (often called the local field at \( v \)) of \( K \) with respect to \( |\cdot|_v \). We write \( \mathbb{C}_v \) for the completion of an algebraic closure \( \overline{K}_v \) of \( K_v \). (The absolute value \( v \) extends in a unique way to \( \overline{K}_v \) and hence to \( \mathbb{C}_v \).) The field \( \mathbb{C}_v \) is then a complete and algebraically closed field. If \( v \) is archimedean, then \( K_v \) is isomorphic to \( \mathbb{R} \) (in which case we call \( v \) a real prime) or to \( \mathbb{C} \) (in which case we call \( v \) a complex prime), and \( \mathbb{C}_v \cong \mathbb{C} \). We will henceforth avoid the notation \( K_v \), as we will soon introduce the notation \( K_v \) to denote a completely different object in Section 2.

If \( v \) is non-archimedean, then \( \mathbb{C}_v \) is not locally compact, but it has other convenient properties not shared by \( \mathbb{C} \). In particular, the disk \( \mathcal{O}_v = \{c \in \mathbb{C}_v : |c|_v \leq 1\} \) forms a ring, called the ring of integers, which has a unique maximal ideal \( \mathcal{M}_v = \{c \in \mathbb{C}_v : |c|_v < 1\} \). The quotient \( k_v = \mathcal{O}_v/\mathcal{M}_v \) is called the residue field of \( \mathbb{C}_v \). The natural reduction map from \( \mathcal{O}_v \) to \( k_v \), sending \( a \in \mathcal{O} \) to \( \overline{a} = a + \mathcal{M}_v \in k_v \), will be used to define good and bad reduction of a polynomial in Definition 2.1 below; but after proving a few simple Lemmas about good and bad reduction, we will not need to refer to \( \mathcal{O}_v, \mathcal{M}_v \), or \( k_v \) again.

1.3. Disks. Let \( \mathbb{C}_v \) be a complete and algebraically closed field with absolute value \( v \). Given \( a \in \mathbb{C}_v \) and \( r > 0 \), we write

\[
\overline{D}(a, r) = \{x \in \mathbb{C}_v : |x - a|_v \leq r\} \quad \text{and} \quad D(a, r) = \{x \in \mathbb{C}_v : |x - a|_v < r\}
\]
for the closed and open disks, respectively, of radius \( r \) centered at \( a \). Note our convention that all disks have positive radius.

If \( v \) is non-archimedean and \( U \subseteq \mathbb{C}_v \) is a disk, then the radius of \( U \) is unique; it is the same as the diameter of the set \( U \) viewed as a metric space. However, any point \( b \in U \) is a center. That is, if \( |b - a|_v \leq r \), then \( \overline{D}(a, r) = \overline{D}(b, r) \), and similarly for open disks. It follows that two disks intersect if and only if one contains the other. In addition, all disks are both open and closed as topological sets; however, open disks and closed disks can still behave differently in other ways.

Still assuming that \( v \) is non-archimedean, the set \( |\mathbb{C}_v^\infty|_v \) of absolute values actually attained by elements of \( \mathbb{C}_v^\infty \) is usually not all of \((0, \infty)\). As a result, if \( r \in (0, \infty) \setminus |\mathbb{C}_v^\infty|_v \), then \( D(a, r) = \overline{D}(a, r) \) for any \( a \in \mathbb{C}_v \). However, if \( r \in |\mathbb{C}_v^\infty|_v \), then \( D(a, r) \subseteq \overline{D}(a, r) \).

2. Bad Reduction and Filled Julia Sets

The following definition originally appeared in [23]. We have modified it slightly so that “bad reduction” now means not potentially good, as opposed to not good.

**Definition 2.1.** Let \( \mathbb{C}_v \) be a complete, algebraically closed non-archimedean field with absolute value \( | \cdot |_v \), ring of integers \( \mathcal{O}_v = \{ c \in \mathbb{C}_v : |c|_v \leq 1 \} \), and residue field \( k_v \). Let \( \phi(z) \in \mathbb{C}_v(z) \) be a rational function with homogenous presentation

\[
\phi([x, y]) = [f(x, y), g(x, y)],
\]

where \( f, g \in \mathcal{O}_v[x, y] \) are relatively prime homogeneous polynomials of degree \( d = \deg \phi \), and at least one coefficient of \( f \) or \( g \) has absolute value 1. We say that \( \phi \) has good reduction at \( v \) if \( \overline{f} \) and \( \overline{g} \) have no common zeros in \( k_v \times k_v \) besides \( (x, y) = (0, 0) \). We say that \( \phi \) has potentially good reduction at \( v \) if there is some linear fractional transformation \( h \in \text{PGL}(2, \mathbb{C}_v) \) such that \( h^{-1} \circ \phi \circ h \) has good reduction. If \( \phi \) does not have potentially good reduction, we say it has bad reduction at \( v \).

Naturally, for \( f[x, y] = \sum_{i=0}^{d} a_i x^i y^{d-i} \), the reduction \( \overline{f}[x, y] \) in Definition 2.1 means \( \sum_{i=0}^{d} \overline{a_i} x^i y^{d-i} \). By convention, if \( \mathbb{C}_v \cong \mathbb{C} \) is archimedean, we declare all rational functions in \( \mathbb{C}_v(z) \) to have bad reduction.

In this paper, we will consider only polynomial functions \( \phi \) of degree at least 2; that is, \( \phi(z) = a_d z^d + \cdots + a_0 \), where \( d \geq 2 \), \( a_i \in \mathbb{C}_v \), and \( a_d \neq 0 \). If \( \mathbb{C}_v \) is non-archimedean, then, it is easy to check that \( \phi \) has good reduction if and only if \( |a_i|_v \leq 1 \) for all \( i \) and \( |a_d|_v = 1 \). In particular, by the product formula, if \( \phi \in K[z] \) for a global field \( K \), then there can be only finitely many primes \( v \in M_K \) at which \( \phi \) has bad reduction.

The main focus of our investigation will be filled Julia sets, which are standard objects of study in complex dynamics. The motivating idea is that for a polynomial \( \phi \), any point \( x \) of large enough absolute value will be sucked out to the attracting fixed point at \( \infty \) under iteration; thus, all of the interesting dynamics involves points that do not escape to \( \infty \) under iteration. Since we will be interested in both archimedean and non-archimedean fields, we state the definition here more generally.

**Definition 2.2.** Let \( \mathbb{C}_v \) be a complete, algebraically closed field with absolute value \( | \cdot |_v \), and let \( \phi(z) \in \mathbb{C}_v[z] \) be a polynomial of degree \( d \geq 2 \). The filled Julia set of \( \phi \) at \( v \) is

\[
\mathcal{K}_v = \{ x \in \mathbb{C}_v : \{|\phi^n(x)|_v\}_{n \geq 1} \text{ is bounded} \}.
\]
We use the notation $K_v$ rather than $K_\phi$ because in this paper, we will consider the polynomial $\phi \in K[z]$ to be fixed, and we will study its filled Julia sets at various different primes $v$ of $K$.

Non-archimedean filled Julia sets have been studied in Section 5 of [3], for example. It is important to note that while complex filled Julia sets are always compact, their non-archimedean counterparts are not usually compact. Fortunately, this technicality will not be an obstacle for our investigations.

We note four fundamental properties of filled Julia sets. First, $K_v$ is invariant under $\phi$; that is, $\phi^{-1}(K_v) = \phi(K_v) = K_v$. Second, all the finite preperiodic points of $\phi$ (that is, all the preperiodic points in $\mathbb{P}^1(C_v)$ other than the fixed point at $\infty$) are contained in $K_v$. Third, if $U_0$ is a disk containing $K_v$, then

$$K_v = \bigcap_{n \geq 0} \phi^{-n}(U_0).$$

Finally, if the polynomial $\phi \in C_v[z]$ has good reduction, then $K_v = \overline{D}(0,1)$.

Filled Julia sets have been studied extensively in the archimedean case $C_v = \mathbb{C}$. If $\phi_d(z) = z^d$, then the (complex) filled Julia $K$ of $\phi_d$ is simply the closed unit disk $\overline{D}(0,1)$. Meanwhile, since the degree $d$ Chebyshev polynomial $\psi_d$ satisfies $\psi \circ h = h \circ \phi$, where $h(z) = z + 1/z$, it follows that the complex filled Julia set of $\psi_d$ is the interval $[-2,2]$ in the real line. These two examples are misleadingly simple, however; most filled Julia sets are complicated fractal sets. For example, it is well known that for $|c| > 2$, the filled Julia set of $\phi(z) = z^2 + c$ is homeomorphic to the Cantor set. For many more complex examples (often in the form of the Julia set, which is the boundary of the filled Julia set), see [4, 10, 21].

There are not as many examples of non-archimedean filled Julia sets in the literature. For the convenience of the reader, we present a few here. More examples may be found in [6, 30].

**Example 2.3.** Given $C_v$ non-archimedean and $d \geq 2$, fix $c \in C_v$, and consider $\phi(z) = z^d - z^{d-1}c$. Assume for convenience that $|d-1|_v = 1$. If $|c|_v \leq 1$, then $\phi$ has good reduction, and hence $K_v = \overline{D}(0,1)$. Thus, we consider $|c|_v > 1$; let $r = |c|_v$ and $U = \overline{D}(0,r)$. Note that for $|x|_v > r$, we have $|\phi(x)|_v = |x|^2$, so that $\phi^n(x) \to \infty$. That is, $K_v \subseteq U_0$.

The set $\phi^{-1}(0)$ consists of 0 and $d-1$ other points, all distance $r$ from one another. Using standard mapping properties of non-archimedean polynomials (see, for example, Section 2 of [7]), it is not hard to show that $\phi^{-1}(U_0)$ consists of $d$ disks of radius $r^{2-d}$, each centered at one of the points of $\phi^{-1}(0)$. Moreover, each of these smaller disks maps one-to-one and onto $U_0$, and in fact $\phi$ multiplies distances by a factor of $r^{d-1} = |\phi'(0)|_v$ on each smaller disk. It follows that $U_n = \phi^{-n}(U_0)$ is a union of $d^n$ disks, each of radius $r^{1-(d-1)n}$. (The sets $U_n$ are nested so that each disk of $U_{n+1}$ contains exactly $d$ disks of $U_n$, arranged so that any two are the maximal distance $r^{1-(d-1)n}$ apart.) Since the radii of the disks approach zero, it is easy to verify that $K_v = \bigcap U_n$ is homeomorphic to a Cantor set on $d$ intervals.

**Example 2.4.** Given $C_v$ non-archimedean and $d \geq 3$, fix $a \in C_v$, and consider $\phi(z) = z^d - az^{d-1}$. Again, assume for convenience that $|d-1|_v = 1$. If $|a|_v \leq 1$, then $\phi$ has good reduction, and hence $K_v = \overline{D}(0,1)$. As in the previous example, then, we consider $|a|_v > 1$.

Let $r = |a|_v$; once again, we have $K_v \subseteq U_0 = \overline{D}(0,r)$.

This time, however, $\phi^{-1}(U_0)$ consists of only two disks. One, $W_1 = \overline{D}(a,r^{-(d-1)})$, is small and maps one-to-one onto $U_0$; but the other, $W_2 = \overline{D}(0,1)$, is comparatively large, and it
Moreover, because the unique smallest disk containing $K_v$ reasonably well. In particular, $\phi^{-2}(U_0)$ consists of $d + 2$ disks: two inside $W_1$ (one is a preimage of $W_1$, and the other is a preimage of $W_2$), and $d$ inside $W_2$ ($d − 1$ are preimages of $W_1$, and the last is a $(d − 1)$-to-one preimage of $W_2$). If we continue to take preimages, then each $U_n = \phi^{-n}(U_0)$ will be a union of disks. To compute $U_{n+1}$ from $U_n$, observe that each disk of $U_n$ will have one preimage inside $W_1$ and (with one exception) $d − 1$ disks inside $W_2$. The one exception is the unique disk of $U_n$ containing $0$; it has only one preimage inside $W_2$, and that preimage maps $(d − 1)$-to-one onto it. Ultimately $K_v = \bigcap U_n$ will consist of the disk $V$ and all of its (ininitely many) preimages together with a vaguely Cantor-like set at which the preimages of $V$ accumulate.

Thus, in contrast with Example 2.3, $K_v$ is neither a disk nor compact. In general, the filled Julia set of a polynomial of bad reduction over $\mathbb{C}_v$ will look something like this one. However, the dynamics can be even more complicated when there are regions on which $\phi$ maps $n$-to-1 for some integer $n$ divisible by $p$, the characteristic of the residue field $k_v$.

The preceding comments and examples made frequent reference to disks $U_0$ containing $K_v$. The smallest such disk will be of particular importance to us. The following Lemma shows the existence of the smallest disk and gives a partial characterization of it.

**Lemma 2.5.** Let $\mathbb{C}_v$ be a complete, algebraically closed field with absolute value $|·|_v$. Let $\phi \in \mathbb{C}_v[z]$ be a polynomial of degree $d \geq 2$ with lead coefficient $a_d \in \mathbb{C}_v$. Denote by $K_v$ the filled Julia set of $\phi$ in $\mathbb{C}_v$. Then:

a. There is a unique smallest disk $U_0 \subseteq \mathbb{C}_v$ which contains $K_v$.

b. $U_0$ is a closed disk of some radius $r'_v \in |\mathbb{C}_v^\times|_v$, with $r'_v \geq |a_d|_v^{-1/(d−1)}$.

c. If $|·|_v$ is non-archimedean, then $\phi$ has potentially good reduction if and only if $r'_v = |a_d|_v^{-1/(d−1)}$.

**Proof.** Choose $\alpha \in \mathbb{C}_v$ such that $\alpha^{d−1} = a_d$, and let $\psi(z) = \alpha \phi(\alpha^{-1}z)$, which is a monic polynomial with filled Julia set $\alpha K_v$. Given the scaling factors of $|a_d|_v^{-1/(d−1)} = |\alpha|_v^{-1}$ in parts (b) and (c), we may assume without loss that $a_d = 1$.

If $\mathbb{C}_v$ is archimedean, then $\mathbb{C}_v \cong \mathbb{C}$. In that case, it is well known that $K_v$ is a compact set in the plane. (See, for example, Lemma 9.4 of [21].) Since $K_v$ is bounded, there is a unique smallest disk containing $K_v$. (See, for example, Exercise 3 in Appendix I of [34].) Moreover, because $K_v$ is compact, this disk must be closed, and so we denote it $D(b, r_v)$.

It is well known that the filled Julia set of a monic polynomial over $\mathbb{C}$ has capacity 1; see, for example, Theorem 4.1 of [3]. If $r_v < 1$, then $K_v$ would fit inside a disk of radius strictly smaller than 1. However, the capacity of a disk in the plane is exactly its radius; as a result, the capacity of $K_v$ would be strictly smaller than 1, which is a contradiction. Therefore, $r_v \geq 1$, proving the Lemma in the archimedean case. (See Remark 2.6 below for an alternate proof not using capacity theory.)

If $\mathbb{C}_v$ is non-archimedean, then let $b \in \mathbb{C}_v$ be a fixed point of $\phi$. (Such $b$ exists because $\phi(z)−z$ is a polynomial of degree $d \geq 2$.) Clearly $b \in K_v$. By the coordinate change $z \mapsto z + b$, we may assume that $b = 0$. Write $\phi(z) = z^d + a_{d−1}z^{d−1} + \cdots + a_1z$. Let $\tilde{r}_v = \max\{|a_i|_v^{1/(d−i)} : i = 1, \ldots, d−1\}$ and $r_v = \max\{\tilde{r}_v, 1\}$; note that $r_v \in |\mathbb{C}_v^\times|_v$.

If $r_v = 1$, then $\phi$ is a monic polynomial with coefficients in $\mathcal{O}_v$. Hence, $\phi$ has good reduction; with $U_0 = D(0, 1)$, the Lemma follows.
If $r_v > 1$, then the Newton polygon (see Section 6.5 of [14] or Section IV.3 of [18]) for the equation $\phi(z) = 0$ shows that there is some $c \in \mathbb{C}_v$ with $|c|_v = r_v$ and $\phi(c) = 0$. In particular, any disk containing $\mathcal{K}_v$ must contain $\overline{D}(0, r_v)$.

Moreover, if $|z|_v > r_v$, then the $z^d$ term has larger absolute value than any other term of $\phi(z)$, so that $|\phi(z)|_v = |z|^d$. By induction, $|\phi^n(z)|_v = |z|^{d^n}$ for all $n \geq 1$. It follows that $\mathcal{K}_v \subseteq \overline{D}(0, r_v)$. By the previous paragraph, $\overline{D}(0, r_v)$ is the smallest disk containing $\mathcal{K}_v$.

It only remains to show that if $r_v > 1$, then $\phi$ cannot have potentially good reduction. However, after our coordinate changes, $\phi$ is a monic polynomial without constant term. The assumption that $r_v > 1$ means that $|a_i|_v > 1$ for some coefficient $a_i$ of $\phi$. Thus, by Corollary 4.6 of [5], $\phi$ cannot have good reduction even after a change of coordinates. □

**Remark 2.6.** The fact that $r_v \geq 1$ in the archimedean case can also be proven directly, without reference to the power of capacity theory. The following alternate argument was suggested to the author by Laura DeMarco.

Suppose that $\mathcal{K}_v \subseteq \overline{D}(a, r_v)$ for some $r_v < 1$; let $s = (1 + r_v)/2$. Since $\phi$ is a (monic) polynomial of degree at least 2, there is some radius $R > 1$ such that for all $z \in \mathbb{C}$ with $|z - a| > R$, we have $|\phi(z) - a| > R$.

Let $A$ denote the annulus $\{z \in \mathbb{C} : s \leq |z - a| \leq R\}$. Every point of $A$ is attracted to $\infty$ under iteration of $\phi$. Since $A$ is compact, there is some $n \geq 1$ such that $f(z) = \phi^n(z) - a$ has $|f(z)| > 1$ for all $z \in A$. Note that all $d^n$ zeros of $f$ lie in $D(a, s)$.

Let $g(z) = (z - a)^{d^n} - f(z)$, which is a polynomial of degree strictly less than $d^n$. However, for all $z \in \mathbb{C}$ with $|z - a| = s$, we have

$$|f(z) + g(z)| = |z - a|^{d^n} - s^{d^n} < 1 < |f(z)|.$$  

By Rouche’s Theorem (noting that $g(z) \neq 0$ for $|z - a| = s$), $f$ and $g$ have the same number of zeros in $D(a, s)$, counting multiplicity. That is a contradiction; thus, $r_v \geq 1$.

The next two Lemmas give slightly more detailed information about the filled Julia set for a polynomial of bad reduction over a non-archimedean field.

**Lemma 2.7.** With notation as in Lemma 2.5, suppose that $\mathbb{C}_v$ is non-archimedean and $r'_v > |a_d|_v^{1/(d-1)}$. Then $\phi^{-1}(U_0)$ is a disjoint union of closed disks $D_1, \ldots, D_\ell \subseteq U_0$, where $2 \leq \ell \leq d$. Moreover, there are positive integers $d_1, \ldots, d_\ell$ with $d_1 + \cdots + d_\ell = d$ such that for each $i = 1, \ldots, \ell$, $\phi$ maps $D_i$ $d_i$-to-1 onto $U_0$. That is, $\phi(D_i) = U_0$, and every point $U_0$ has exactly $d_i$ pre-images in $D_i$, counting multiplicity.

**Proof.** As in the previous proof, we may assume that $\phi(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_1z$, that $r'_v = \max\{|a_i|_v^{1/(d-i)} : 1 \leq i \leq d-1\} > 1$, and that $U_0 = \overline{D}(0, r'_v)$. As observed in that proof, $|\phi(z)|_v > |z|_v$ for $|z|_v > r'_v$. Thus, $\phi^{-1}(U_0) \subseteq U_0$. Moreover, $\phi(U_0) \supseteq U_0$.

We now construct the disks $D_1, \ldots, D_\ell$ inductively. For each $i = 1, 2, \ldots$, suppose we already have $D_1, \ldots, D_{i-1}$, and choose $b_i \in \phi^{-1}(U_0) \setminus (D_1 \cup \cdots \cup D_{i-1})$. (If this is not possible, then skip to the next paragraph.) By Lemmas 2.3 and 2.6 of [7], there is a unique disk $D_i$ containing $b_i$ which maps onto $U_0$, and this disk must be closed. Since $D_j$ was also unique for each $j < i$, the new disk $D_i$ must be disjoint from $D_j$. In addition, by Lemma 2.2 of [7], $\phi$ maps $D_i$ $d_i$-to-1 onto $U_0$, for some integer $d_i \geq 1$.

This process must stop with $\ell \leq d$, because $\phi^{-1}(0)$ consists of exactly $d$ points, counting multiplicity, and since each $d_i \geq 1$, at least one must be contained in each $D_i$. In fact, counting pre-images of 0 also shows that $d_1 + \cdots + d_\ell = d$. 
Finally, suppose that \( \ell = 1 \); that is, \( \phi^{-1}(U_0) \) is a single disk \( D_1 \subseteq U_0 \). However, \( K_v = \phi^{-1}(K_v) \subseteq D_1 \), contradicting the assumption that \( U_0 \) is the smallest disk containing \( K_v \). Thus, we must have \( \ell \geq 2 \).

**Lemma 2.8.** Let \( K \) be a field with a discrete valuation \( v \), and let \( \pi_v \in K \) be a uniformizer at \( v \). Let \( \mathbb{C}_v \) be the completion of an algebraic closure of \( K \).

Let \( \phi(z) = a_d z^d + \cdots + a_0 \in K[z] \) be a polynomial of degree \( d \geq 2 \). Denote by \( K_v \) the filled Julia set of \( \phi \) in \( \mathbb{C}_v \), and let \( r'_v > 0 \) be the radius of the smallest disk in \( \mathbb{C}_v \) containing \( K_v \). Suppose that \( r'_v > |a_d|^{-1/(d-1)} \).

If \( K_v \cap K \neq \emptyset \), then

\[
|a_d|^{1/(d-1)} r'_v \geq \begin{cases} |\pi_v|^{-1} & \text{if } d = 2, \\ |\pi_v|^{-1/[(d-1)(d-2)]} & \text{if } d \geq 3. \end{cases}
\]

**Proof.** Given \( b \in K_v \cap K \), we may replace \( \phi \) by \( \phi(z + b) - b \in K[z] \), which is a polynomial with the same degree and lead coefficient as \( \phi \), but with filled Julia set translated by \( -b \). In particular, the radius \( r'_v \) is preserved; so we may assume without loss that \( 0 \in K_v \).

As in the proof of Lemma 2.5, choose \( \alpha \in \mathbb{C}_v \) such that \( \alpha^{d-1} = a_d \), and let

\[
\psi(z) = \alpha \phi(\alpha^{-1}z) = z^d + \sum_{i=0}^{d-1} \alpha^{1-i} a_i z^i.
\]

Then \( \psi \) is a monic polynomial with filled Julia set \( K'_v = \alpha K_v \); however, \( \psi \) may not be defined over \( K \). Still, the radius \( r_v \) of the smallest disk containing \( K'_v \) satisfies \( r_v > 1 \), by hypothesis.

Let \( j \) be the largest integer between 0 and \( d - 1 \) that maximizes \( \lambda_j = |\alpha^{1-j} a_j|^{1/(d-j)} \). Note that \( \lambda_j > 1 \); for if not, then \( \psi \) has good reduction, contradicting Lemma 2.5.c. The Newton polygon for the equation \( \psi(z) = 0 \) shows that there is some \( \beta \in \mathbb{C}_v \) with \( |\psi(\beta)| = 0 \) and \( |\beta|_v = \lambda_j \). We have \( 0, \beta \in K'_v \); hence, \( r_v \geq \lambda_j \).

If \( j = 0 \), then a simple induction shows that \( |\psi^n(0)|_v = |\alpha a_0|^{-1} \) for \( n \geq 1 \). Since \( |\alpha a_0|_v > 1 \), this contradicts the hypothesis that \( 0 \in K'_v \).

Thus, \( 1 \leq j \leq d - 1 \), and we write \( |a_d|_v = |\pi_v|^e_1 \) and \( |a_j|_v = |\pi_v|^e_2 \); note that \( e_1, e_2 \in \mathbb{Z} \).

Our assumptions say that

\[
r_v \geq \lambda_j = |\alpha^{1-j} a_j|^{1/(d-j)} = |\pi_v|^{f} > 1, \quad \text{where} \quad f = \frac{1}{d-j} \left( \frac{(1-j)e_1}{d-1} + e_2 \right) < 0.
\]

If \( j = 1 \), then \( f = e_2/(d-1) \leq -1/(d-1) \), which proves the Lemma for \( d = 2 \). If \( 2 \leq j \leq d - 1 \), then \( f \leq -1/[(d-1)(d-j)] \leq -1/[(d-1)(d-2)] \), and we are done.

**Remark 2.9.** The bounds of Lemma 2.8 are sharp. Indeed, if \( d = 2 \), then the polynomial \( \phi(z) = z^2 - \pi_v^{-1}z \) has 0, \( \pi_v^{-1} \in K_v \). Because \( \phi^n(z) \to \infty \) for \( |z|_v > |\pi_v|^{-1} \), the smallest disk containing \( K_v \) is \( D(0,|\pi_v|^{-1}) \), so that \( r_v = |\pi_v|^{-1} \).

Similarly, if \( d \geq 3 \), then the polynomial \( \phi(z) = \pi_v^d z^d - \pi_v z^2 \) has 0 \( \in K_v \). Choosing a \((d-1)\)-st root of \( \pi_v \), we get an associated monic conjugate \( \psi(z) = z^d - \pi_v^{-1/(d-1)} z^2 \), from which it is easy to compute \( |\pi_v|^{1/(d-1)} r'_v = r_v = |\pi_v|^{-1/[(d-1)(d-2)]} \).
3. Elementary Computations

We will now define and bound certain integer quantities that will appear as exponents in the rest of the paper. The reader is encouraged to read the statements of Definition 3.1, Lemma 3.2, and Lemma 3.3 but to skip the proofs, which are tedious but completely elementary, until after seeing their use in Theorem 7.1.

We will write \( \log_d x \) to denote the logarithm of \( x \) to base \( d \).

**Definition 3.1.** Let \( N \geq 0 \) and \( d \geq 2 \) be integers. We define \( E(N, d) \) to be twice the sum of all base-\( d \) coefficients of all integers from \( 0 \) to \( N - 1 \). That is,

\[
E(N, d) = 2 \sum_{j=0}^{N-1} e(j, d), \quad \text{where} \quad e \left( \sum_{i=0}^{M} c_i d^i, d \right) = \sum_{i=0}^{M} c_i,
\]

for \( c_i \in \{0, 1, \ldots, d - 1\} \).

Moreover, if \( m \) is an integer satisfying \( 1 \leq m \leq d \), we may write \( N = c_0 + mk \) for unique integers \( c_0 \in \{0, 1, \ldots, m - 1\} \) and \( k \geq 0 \). We then define

\[
e(N, m, d) = c_0 + e(k, d) - (d - m)k \quad \text{and} \quad f(N, m, d) = c_0 + e(k, d),
\]

and

\[
E(N, m, d) = 2 \sum_{j=0}^{N-1} e(j, m, d) \quad \text{and} \quad F(N, m, d) = 2 \sum_{j=0}^{N-1} f(j, m, d).
\]

We declare \( E(N, d) = E(N, m, d) = F(N, m, d) = 0 \) for \( N \leq 1 \). Clearly, \( E(N, d) \) and \( F(N, m, d) \) are always positive for \( N \geq 1 \); but for \( N \) large and \( m < d \), \( E(N, m, d) \) is negative. Note that \( e(N, d, d) = f(N, d, d) = f(N, 1, d) = e(N, d) \), and therefore

\[
E(N, d, d) = F(N, d, d) = F(N, 1, d) = E(N, d).
\]

We will need the following two auxiliary Lemmas.

**Lemma 3.2.** Let \( N, m, d \) be integers satisfying \( N \geq 1 \), \( d \geq 2 \), and \( 1 \leq m \leq d \). Write \( N = c + mk \) with \( 0 \leq c \leq m - 1 \) and \( k \geq 0 \). Then:

a. \( F(N, m, d) = (m - c)E(k, d) + cE(k + 1, d) + (m - 1)N - c(m - c) \).

b. \( E(N, m, d) = F(N, m, d) - \frac{(d - m)}{[N^2 - mN + c(m - c)]} \).

c. If \( N \leq m \), then \( E(N, m, d) = F(N, m, d) = N(N - 1) \).

**Proof.** Writing an arbitrary integer \( j \geq 0 \) as \( j = i + m\ell \) for \( 0 \leq i \leq m - 1 \), we compute

\[
F(N, m, d) = 2 \sum_{j=0}^{N-1} f(j, m, d) = 2 \sum_{i=0}^{c-1} \sum_{\ell=0}^{k} f(i + m\ell, m, d) + 2 \sum_{i=c}^{m-1} \sum_{\ell=0}^{k-1} f(i + m\ell, m, d)
\]

\[
= 2 \sum_{i=0}^{c-1} \sum_{\ell=0}^{k} (i + e(\ell, d)) + 2 \sum_{i=c}^{m-1} \sum_{\ell=0}^{k-1} (i + e(\ell, d))
\]

\[
= \sum_{i=0}^{c-1} [2(k+1)i + E(k+1, d)] + \sum_{i=c}^{m-1} [2ki + E(k, d)]
\]

\[
= cE(k+1, d) + (m-c)E(k, d) + (k+1)c(c-1) + km(m-1) - kc(c-1).
\]
Part (a) of the Lemma now follows by rewriting the last three terms as
\[ c(c - 1) + mk(m - 1) = c(c - m) + (c + mk)(m - 1) = (m - 1)N - c(m - c). \]

Next, we compute
\[
E(N, m, d) = 2 \sum_{j=0}^{N-1} e(j, m, d) = 2 \sum_{j=0}^{N-1} f(j, m, d) - 2(d-m) \left[ m \sum_{\ell=0}^{k-1} \ell + \sum_{j=0}^{c-1} k \right]
\]
\[ = F(N, m, d) - k(d-m)[m(k-1) + 2c] \]
Writing \( k = (N-c)/m \), the last term becomes
\[-\left(\frac{N-c}{m}\right)(d-m)(N+c-m) = -\left(\frac{d-m}{m}\right)[N^2 - mN + c(m-c)], \]
proving part (b). Finally, part (c) is immediate from the fact that \( e(j, m, d) = f(j, m, d) = j \) for \( 0 \leq j \leq m - 1 \).

**Lemma 3.3.** Let \( N, m, d \) be integers satisfying \( N \geq 1, d \geq 2, \) and \( 1 \leq m \leq d \). Write \( N = c + mk \) with \( 0 \leq c \leq m - 1 \) and \( k \geq 0 \). Then:

a. \((m-c) \log_d \left( \frac{mk}{N} \right) + c \log_d \left( \frac{mk + m}{N} \right) \leq 0. \]

b. If \( N \geq d \), then \((d-1) \log_d \left( \frac{mk + m}{N} \right) - (m-c) \leq 0. \]

**Proof.** The function \( \log_d(x) \) is of course concave down. Letting \( x_1 = mk/N \) and \( x_2 = (mk + m)/N \), then, we have \( x_1 \leq 1 < x_2 \), and therefore \( \log_d(1) \geq L(1) \), where
\[ L(x) = \frac{1}{x_2 - x_1} \left[ (x_2 - x) \log_d(x_1) + (x - x_1) \log_d(x_2) \right] \]
is the line through \((x_1, \log_d(x_1)) \) and \((x_2, \log_d(x_2)) \). That is,
\[ 0 \geq \frac{1}{m} \left[ (m-c) \log_d \left( \frac{mk}{N} \right) + c \log_d \left( \frac{mk + m}{N} \right) \right], \]
proving part (a). For part (b), we have
\[(d-1) \log_d \left( \frac{mk + m}{N} \right) = (d-1) \frac{\log d}{\log d} \cdot \log \left( 1 + \frac{m-c}{N} \right) \leq (d-1) \frac{(m-c)}{N} \]
However, \( \log d = -\log[1 - (d-1)/d] \geq (d-1)/d \), and since \( N \geq d \),
\[(d-1) \log_d \left( \frac{mk + m}{N} \right) \leq (d-1) \cdot \frac{d}{d-1} \cdot \frac{m-c}{N} = \frac{d}{N}(m-c) \leq (m-c). \]

**Lemma 3.4.** Let \( N, m, d \) be integers satisfying \( N \geq 1, d \geq 2, \) and \( 1 \leq m \leq d - 1 \). Then:

a. \( E(N, d) \leq (d-1)N \log_d N \), with equality if \( N \) is a power of \( d \).

b. \( E(N, m, d) \leq (d-1)N \left[ \log_d N + 1 - \log_d m - \frac{(d-m)}{m(d-1)}N \right] \), with equality if \( N/m \) is a power of \( d \).

c. \( F(N, m, d) \leq (d-1)N \log_d N \).

d. For \( N \geq m \), \( F(N, m, d) \leq (d-1)N \left[ \log_d N - \log_d m + \frac{m-1}{d-1} \right] \), with equality if \( N/m \) is a power of \( d \).
Proof. Fix $d \geq 2$. If $N = 1$, then both sides of part (a) are clearly zero. If $2 \leq N \leq d$, then $E(N, d) = N(N - 1)$ by Lemma 3.2.1 with $m = d$ and equation (3). Because $(\log x)/(x - 1)$ is a decreasing function of the real variable $x > 1$, we have $(\log d)/(d - 1) \leq (\log N)/(N - 1)$, with equality for $N = d$. Part (a) then follows for $1 \leq N \leq d$.

For $N \geq d + 1$, we proceed by induction on $N$, assuming part (a) holds for all positive integers up to $N - 1$. Write $N = c + dk$, where $0 \leq c \leq d - 1$, so that $1 \leq k \leq N - 2$. By Lemma 3.2.1 (with $m = d$) and equation (3), we have

$$E(N, d) = (d - c)E(k, d) + cE(k + 1, d) + (d - 1)N - c(d - c)$$

$$\leq (d - c)(d - 1)k \log_d k + c(d - 1)(k + 1) \log_d (k + 1) + (d - 1)N - c(d - c)$$

$$= (d - c)(d - 1)k \log_d (dk) + c(d - 1)(k + 1) \log_d (dk + d) - c(d - c)$$

where the final equality is because $N = (d - c)k + c(k + 1)$, and the inequality (which is equality if $N$ is a power of $d$) is by the inductive hypothesis, since $k, c - 1 \leq N - 1$. More generally, adding and subtracting $(d - 1)N \log_d N$, we have

$$E(N, d) \leq (d - 1)N \log_d N + (d - c)(d - 1)k \log_d \left(\frac{dk}{N}\right)$$

$$+ c(d - 1)(k + 1) \log_d \left(\frac{dk + d}{N}\right) - c(d - c)$$

$$= (d - 1)N \log_d N + c\left[(d - 1) \log_d \left(\frac{dk + d}{N}\right) - (d - c)\right]$$

$$+ (d - 1)k \left[(d - c) \log_d \left(\frac{dk}{N}\right) + c \log_d \left(\frac{dk + d}{N}\right)\right]$$

By Lemma 3.3 with $m = d$, the quantities in brackets are nonpositive, and part (a) follows.

If $m = 1$, then parts (c–d) are immediate from part (a) and equation (3). Moreover, by Lemma 3.2.1 (with $m = 1$) and part (a),

$$E(N, 1, d) = E(N, d) - (d - 1)N(N - 1) \leq (d - 1)N[\log_d N + 1 - N],$$

with equality if $N$ is a power of $d$. This is exactly part (b) for $m = 1$. Thus, we may assume for the remainder of the proof that $2 \leq m \leq d$.

We now turn to part (d). If $N = m$, then by Lemma 3.2.1, we have $F(m, m, d) \leq m(m - 1)$, which exactly equals the desired right hand side. For $N \geq m + 1$, write $N = c + mk$, where $k \geq 1$ and $0 \leq c \leq m - 1$. By Lemma 3.2.1,

$$F(N, m, d) = (m - c)E(k, d) + cE(k + 1, d) + (m - 1)N - c(m - c).$$

If $m + 1 \leq N \leq d - 1$, then $k \leq d - 1$, so that by Lemma 3.2.1, equation (1) becomes

$$F(N, m, d) = (m - c)k(k - 1) + ck(k + 1) + (m - 1)N - c(m - c)$$

$$= mk^2 - mk + c(k + 1)(m - c) = (m + k - 1)N - (m - c)(k + c)$$

$$= m^{-1} \left[(N + m^2 - c - m)N - (m - c)(N - c + cm)\right]$$

$$= m^{-1} \left[(N + m^2 - 2m)N - c(m - c)(m + 1)\right] \leq N \left[(N/m) + m - 2\right],$$

where we have substituted $k = (N - c)/m$ along the way. Thus, we must show

$$N \left[(N/m) + m - 2\right] \leq N \left[(d - 1) \log_d (N/m) + m - 1\right].$$
Equivalently, we must show
\[
\log d \leq \log \left( \frac{N}{m} \right) - 1 + \frac{\log \left( \frac{N}{m} \right)}{\frac{N}{m} - 1},
\]
which is true because \((\log x)/(x - 1)\) is a decreasing function of \(x > 1\), and \(1 < N/m < d\).

If \(N \geq d\) in part (d), then \(k \geq 1\). Applying part (a) to equation (4), we obtain
\[
(F(N, m, d) \leq (m - c)(d - 1)k \log_d k + c(d - 1)(k + 1) \log_d (k + 1)
\]
\[+ (m - 1)N - c(m - c),
\]
with equality if \(c = 0\) and \(k\) is a power of \(d\), whence we immediately obtain the statement of the Lemma for \(N = md^k\). More generally, (5) becomes
\[
F(N, m, d) \leq (d - 1)N \left( \log_d \frac{N}{m} + \frac{m - 1}{d - 1} \right) + (m - c)(d - 1)k \log_d \left( \frac{mk}{N} \right)
\]
\[+ c(d - 1)(k + 1) \log_d \left( \frac{mk + m}{N} \right) - c(m - c)
\]
\[= (d - 1)N \left( \log_d \frac{N}{m} + \frac{m - 1}{d - 1} \right) + c \left[ (d - 1) \log_d \left( \frac{mk + m}{N} \right) - (m - c) \right]
\]
\[+ (d - 1)k \left[ (m - c) \log_d \left( \frac{mk}{N} \right) + c \log_d \left( \frac{mk + m}{N} \right) \right].
\]

Part (d) now follows from Lemma 3.3 as before.

For part (c), if \(1 \leq N \leq m\), then by part (a) and Lemma 3.2.c,
\[
F(N, m, d) = N(N - 1) = E(N, d) \leq (d - 1)N \log_d N,
\]
as desired. The remaining case, that \(N \geq m\), will follow from part (d) provided
\[
m - 1 \leq (d - 1) \log_d m.
\]
However, this is the same as showing that \(\log d/(d - 1) \leq \log m/(m - 1)\), which once again follows from the fact that \((\log x)/(x - 1)\) is decreasing for \(x > 1\).

Last, we turn to part (b). If \(1 \leq N \leq m\), then \(E(N, m, d) = N(N - 1)\) by Lemma 3.2.c. Thus, we wish to show that
\[
N - 1 \leq (d - 1) \left( 1 + \log_d \frac{N}{m} \right) - (d - m) \frac{N}{m},
\]
which is to say
\[
\frac{dN}{m} - 1 \leq (d - 1) \log_d \left( \frac{dN}{m} \right),
\]
with equality when \(N = m\). Yet again, this inequality follows immediately from the facts that \((\log x)/(x - 1)\) is decreasing for \(x > 1\) and that \(1 \leq dN/m \leq d\).

It only remains to consider \(N \geq m + 1\). Writing \(N = c + mk\), where \(k \geq 1\) and \(0 \leq c \leq m - 1\), and invoking Lemma 3.2.b, we have
\[
E(N, m, d) = F(N, m, d) - \frac{(d - m)}{m} [N^2 - mN + c(m - c)]
\]
\[\leq F(N, m, d) - \frac{(d - m)}{m} N^2 + (d - m)N,
\]
with equality for \( c = 0 \). By part (d), we obtain

\[
E(N, m, d) \leq (d - 1)N \left[ \log_d N - \log_d m + \frac{m - 1}{d - 1} \right] - \frac{(d - m)}{m}N^2 + (d - m)N
\]

\[
= (d - 1)N \left[ \log_d N - \log_d m + 1 - \frac{d - m}{m(d - 1)}N \right],
\]

with equality if \( N \) is of the form \( N = md^k \).

Besides the preceding integer quantities and their bounds, we will need the following bound involving a certain family of real-valued functions.

**Lemma 3.5.** Let \( d \geq 2 \) be an integer, and let \( A, B, t \) be positive real numbers such that

\[
(d - 1)A \geq d^{B-1} \quad \text{and} \quad t \geq 1.
\]

Define \( \eta : (0, \infty) \to \mathbb{R} \) by

\[
\eta(x) = t \log_d x - Ax + B.
\]

Set the real number \( M(A, B, t) \) to be

\[
M(A, B, t) = \frac{t}{A} (\log_d t + \log_d (\max\{1, \log_d t\}) + 3).
\]

Then \( \eta(x) < 0 \) for all \( x \geq M(A, B, t) \).

**Proof.** By differentiating, we see that \( \eta \) is decreasing for \( x \geq t/(A \log d) \), and hence for \( x \geq M(A, B, t) \). Thus, it suffices to show that \( \eta(M(A, B, t)) < 0 \).

First, suppose that \( t < d \). Then

\[
\eta(M(A, B, t)) = t \log_d t + t \log_d [A^{-1}(\log_d t + 3)] - t \log_d t - 3t + B
\]

\[
= t \log_d [A^{-1}d^{B-3}(\log_d t + 3)] - B(t - 1) \leq t \log_d [(d - 1)(\log_d t + 3)/d^2],
\]

where the inequality is because \( A^{-1}d^{B-1} \leq (d - 1) \) and \( B > 0 \), by hypothesis. Since \( t < d \), the quantity inside square brackets is strictly less than \( 4(d - 1)/d^2 \leq 1 \). Thus, \( \eta(M(A, B, t)) < t \log_d(1) = 0 \), and we are done.

Second, if \( t \geq d \), then by a similar computation,

\[
\eta(M(A, B, t)) = t \log_d [A^{-1}d^{B-3}u^{-1}(u + \log_d u + 3)] - B(t - 1)
\]

\[
< t \log_d [(d - 1)(u + \log_d u + 3)/(d^2 u)]
\]

where \( u = \log_d t \). Writing \( H(u) = (d - 1)(u + \log_d u + 3)/(d^2 u) \), it suffices to show that \( H(u) \leq 1 \) for \( u \geq 1 \). Differentiating, it is easy to see that \( H \) is decreasing for such \( u \). Since \( H(1) = 4(d - 2)/d^2 \leq 1 \), we are done.

\[\square\]

4. **Transfinite Diameters and Bad Primes**

Given a metric space \( X \) and an integer \( N \geq 2 \), the \( N^{th} \) diameter of \( X \) is defined to be

\[
d_N(X) = \sup_{x_1, \ldots, x_N \in X} \prod_{i \neq j} d_X(x_i, x_j)^{1/[N(N-1)]},
\]
which measures the maximal average distance between any two of \( N \) points in \( X \). (See [15], for example, for a computation of the \( N^{th} \) diameter of the interval \([0, 1]\).) This quantity is usually used to define the *transfinite diameter* of \( X \),

\[
d(X) = \lim_{N \to \infty} d_N(X),
\]

which converges because \( \{d_N(X)\}_{N \geq 2} \) is a decreasing sequence. If \( X \) is a nice enough (e.g., compact) subset of a valued field, then the transfinite diameter coincides with the Chebyshev constant and the logarithmic capacity of \( X \); see Section 5.4 of [1], or Chapters 3 and 4 of [32]. Baker and Hsia used this equality in [3] to compute the transfinite diameter of filled Julia sets of polynomials, even when those sets were not compact. (Their result of \( |a_d|_v^{-1/(d-1)} \), where \( d \) is the degree and \( a_d \) the lead coefficient of the polynomial, was already well known for \( \mathbb{C}_v = \mathbb{C} \).) See [32] for more on transfinite diameters and capacities in \( \mathbb{C}_v \).

However, in this paper we will be interested in the \( N^{th} \) diameters \( d_N(X) \) themselves, rather than the transfinite diameter. In particular, the following Lemma contains our main bound for \( d_N(K_v)^N(N-1) \), where \( K_v \) is the filled Julia set of a polynomial \( \phi \in \mathbb{C}_v[z] \). The proof uses an estimate involving van der Monde determinants similar to a bound that appears in the proof of Lemme 5.4.2 in [11].

**Lemma 4.1.** Let \( \mathbb{C}_v \) be a complete, algebraically closed field with absolute value \( |\cdot|_v \). Let \( \phi \in \mathbb{C}_v[z] \) be a polynomial of degree \( d \geq 2 \) with lead coefficient \( a_d \in \mathbb{C}_v \). Denote by \( K_v \) the filled Julia set of \( \phi \) in \( \mathbb{C}_v \), and let \( r'_v \) be the radius of the smallest disk that contains \( K_v \).

Set \( r_v = |a_d|_v^{1/(d-1)} r'_v \).

Then for any integer \( N \geq 2 \) and any set \( \{x_1, \ldots, x_N\} \subseteq K_v \) of \( N \) points in \( K_v \),

\[
\prod_{i \neq j} |x_i - x_j|_v \leq |a_d|_v^{-N(N-1)/(d-1)} \max\{1, |N|_v^N\} r_v^{E(N,d)},
\]

where \( E(N,d) \) is twice the sum of all base-\( d \) coefficients of all integers from 0 to \( N - 1 \), as in Definition [11]

**Proof.** Choose \( \alpha \in \mathbb{C}_v \) such that \( \alpha^{d-1} = a_d \), and let \( \psi(z) = \alpha \phi(\alpha^{-1} z) \). Then \( \psi \) is a monic polynomial with filled Julia set \( K'_v = \alpha K_v \), and the smallest disk containing \( K'_v \) has radius \( r_v \). If the Lemma holds for \( \psi \), then for \( x_1, \ldots, x_N \in K_v \), we have \( \alpha x_i \in K'_v \), and therefore

\[
\prod_{i \neq j} |x_i - x_j|_v = |\alpha|^{-N(N-1)} \prod_{i \neq j} |\alpha x_i - \alpha x_j|_v \leq |\alpha|^{-N(N-1)} \max\{1, |N|_v^N\} r_v^{E(N,d)},
\]

as desired. Thus, it suffices to prove the Lemma in the case that \( \phi \) is monic.

We will now construct a sequence \( \{f_j\}_{j=1}^\infty \) of monic polynomials over \( \mathbb{C}_v \) such that each \( f_j \) has degree \( j \) and such that \( |f_j(x)|_v \) is not especially large for any \( x \in K_v \).

First, let \( D(a, r_v) \) be the smallest disk containing \( K_v \), where \( a \in \mathbb{C}_v \) and \( r_v \) is as in the statement of the Lemma. For any integer \( j \geq 0 \) written in base-\( d \) notation as

\[
j = c_0 + c_1 d + c_2 d^2 + \cdots + c_M d^M,
\]

with \( c_i \in \{0, 1, \ldots, d-1\} \), define

\[
f_j(z) = \prod_{i=0}^M [\phi^i(z) - a]^{c_i}.
\]
Clearly, $f_j$ is monic of degree $j$. Moreover, for $x \in K_v$, we have $\phi^j(x) \in K_v$, and therefore
\[ |f_j(x)|_v \leq \prod_{i=0}^{M} r^{e_i}_v = r^{e(j,d)}_v, \]
where $e(j,d)$ is as in Definition 3.1.

Given $x_1, \ldots, x_N \in K_v$, denote by $V(x_1, \ldots, x_N)$ the corresponding van der Monde matrix (i.e., the $N \times N$ matrix with $(i,j)$ entry $x_i^j$). Recall that
\[ \prod_{i \neq j} |x_i - x_j|_v = |\det V(x_1, \ldots, x_N)|^2_v. \]

Because $f_{N-1}$ is monic, we may replace the last column of the matrix by a column with entry $f_{N-1}(x_i)$ in the $i$th row, without changing the determinant. We may then replace the second to last column by a column with entry $f_{N-2}(x_i)$ in the $i$th row, and so on. Thus, if we denote by $A(x_1, \ldots, x_N)$ the matrix with $(i,j)$ entry $f_{j-1}(x_i)$, then
\[ \det V(x_1, \ldots, x_N) = \det A(x_1, \ldots, x_N). \]

If $C_v = \mathbb{C}$ is archimedean, then by Hadamard’s inequality applied to the columns of $A$,
\[ |\det A(x_1, \ldots, x_N)|^2 \leq \prod_{j=0}^{N-1} (|f_j(x_1)|^2 + \cdots + |f_j(x_N)|^2) \leq \prod_{j=0}^{N-1} N r^{2e(j,d)}_v = N^N r^E(N,d)_v. \]

Similarly, if $C_v$ is non-archimedean, then by the non-archimedean version of Hadamard’s inequality (see, for example, [11, Preuve du Lemme 5.3.4]), we have
\[ |\det A(x_1, \ldots, x_N)|^2 \leq \prod_{j=0}^{N-1} \max_{i=1,\ldots,N} |f_j(x_i)|^2 \leq \prod_{j=0}^{N-1} r^{2e(j,d)}_v = r^E(N,d)_v. \]

\[ \square \]

**Remark 4.2.** We can recover the Baker and Hsia bound $d(K_v) \leq |a_d|_v^{-1/(d-1)}$ immediately from Lemmas 4.1 and 3.4.a. (The opposite inequality is more subtle, however.)

**Remark 4.3.** There are many cases for which the bound of Lemma 4.1 is sharp. In particular, for non-archimedean $v$, degree $d \geq 2$ with $|d-1|_v = 1$, and $c \in \overline{C}_v$ with $|c| > 1$, recall that the function $\phi(z) = z^d - c^{d-1}z$ of Example 2.3 has $K_v$ homeomorphic to a Cantor set on $d$ pieces. For arbitrary $N \geq 2$, one can distribute $N$ points in $K_v$ in the following way. Write $N = \sum_{i=0}^{M} c_i d^i$, and put $c_M$ points in each of the $d^M$ pieces at level $M$, maximally far apart in each piece; then put $c_{M-1}$ in each of the $d^{M-1}$ pieces at level $M-1$, each as far as possible from the existing points; and so on. Keeping track of the radii of the disks at each level, one can show that $\prod_{i \neq j} |x_i - x_j|_v = r^{E(N,d)}_v$ exactly.

In many other cases, however, the bound is not quite sharp, though it appears to be approximately the right order of magnitude. In the archimedean case, of course, the Hadamard inequality introduces some error. Still, the greater factor seems to be the choice of the monic polynomial $f_j$. When $j$ is a power of $d$, computations suggest that our choice of $f_j$ is very close to sharp, if not actually sharp. However, when $j$ is not a power of $d$, our construction of $f_j$ as a product of smaller factors is in general not optimal, even in the non-archimedean setting. For example, if $\phi(z) = z^3 - az^2$ is the map of Example 2.4 (non-archimedean, with $d = 3$, $|a|_v > 1$, and $|2|_v = 1$), then the function $f_6(z) = (\phi(z))^2$ of the proof has $|f_6(z)|_v$ growing as large as $r^2$ on $K_v$; but the function $\tilde{f}_{6}(z) = (\phi(z)) \cdot (\phi(z) - a)$
Moreover, since the equation of which lie in \( \psi \) some point \( X \) \( m \) integers the radius \( \prod \) is a center. Pick smaller than the bound of Lemma 4.1. We begin with non-archimedean primes.

Denote by \( \phi \) that contains \( K \). As in the proof of Lemma 4.1, we may assume that \( \phi \) is simply the interval \([−2, 2]\) in the real line. For \( j \geq 1 \), the \( j \)th Chebyshev polynomial \( \psi_j \) has \( |\psi_j| \leq 2 \) on \( K_v \), as compared with the proof’s bound of \( 2^{0 + c_1 + \cdots} \) for \( |f_j| \).

In general, however, knowing nothing about the polynomial other than its degree and the radius \( r_v \), we cannot substantially improve on Lemma 4.1.

5. A Partition of the Filled Julia Set: Non-archimedean Case

The key to the Main Theorem, as described in the introduction, is to divide the filled Julia set at a particular bad prime into two smaller pieces \( X_1 \) and \( X_2 \). As a result, the product \( \prod_{i \neq j} |x_i - x_j|_v \), when restricted to \( \{x_i\} \subseteq X_k \) (for fixed \( k = 1, 2 \)), will be substantially smaller than the bound of Lemma 4.1. We begin with non-archimedean primes.

**Lemma 5.1.** Let \( \mathbb{C}_v \) be a complete, algebraically closed field with non-archimedean absolute value \( |·|_v \). Let \( \phi \in \mathbb{C}_v[z] \) be a polynomial of degree \( d \geq 2 \) with lead coefficient \( a_d \in \mathbb{C}_v \). Denote by \( K_v \) the filled Julia set of \( \phi \) in \( \mathbb{C}_v \), and let \( r_v \) be the radius of the smallest disk \( U_0 \) that contains \( K_v \). Set \( r_v = |a_d|^{1/(d-1)} r_v \), and suppose that \( r_v > 1 \).

Then there are disjoint sets \( X_1, X_2 \subseteq K_v \) and positive integers \( m_1, m_2 \) with the properties that \( X_1 \cup X_2 = K_v \), that \( m_1 + m_2 = d \), that for \( k = 1, 2 \), \( \phi : X_k \rightarrow K_v \) is \( m_k \)-to-1, and that for \( k = 1, 2 \), for any integer \( N \geq 2 \), and for any set \( \{x_1, \ldots, x_N\} \subseteq X_k \) of \( N \) points in \( X_k \),

\[
\prod_{i \neq j} |x_i - x_j|_v \leq |a_d|^{-N(N-1)/(d-1)} r_v^{E(N,m_k,d)},
\]

where \( E(N,m_k,d) \) is as in Definition 3.7.

**Proof.** As in the proof of Lemma 4.1 we may assume that \( \phi \) is monic.

By Lemma 2.5, \( U_0 \) is a closed disk of radius \( r_v \in |\mathbb{C}^*_v|_v \). We may write \( U_0 = \overline{D}(a, r_v) \) for some point \( a \in K_v \), since \( K_v \) is nonempty, and since any point of a non-archimedean disk is a center. Pick \( b \in \phi^{-1}(a) \). Note that \( b \in K_v \subseteq U_0 \).

Write \( U_1 = \phi^{-1}(U_0) \). By Lemma 2.7, \( U_1 = D_1 \cup \cdots \cup D_\ell \) for some disjoint closed disks \( \{D_i\} \), with \( 2 \leq \ell \leq d \). Moreover, \( \phi : D_i \rightarrow U_0 \) maps \( d_i \)-to-one for some positive integers \( \{d_i\} \) with \( d_1 + \cdots + d_\ell = d \). Define

\[
W_1 = \{x \in U_1 : |x - b|_v < r_v\}, \quad \text{and} \quad W_2 = U_1 \setminus W_1,
\]

so that \( W_1 \cap W_2 = \emptyset \) and \( W_1 \cup W_2 = U_1 \). If \( W_2 = \emptyset \), then \( K_v \subseteq D(b, r_v) \subseteq U_0 \), contradicting the minimality of \( U_0 \). (The second inclusion is strict because \( r_v \in |\mathbb{C}^*_v|_v \).) Thus, since \( b \in W_1 \), both \( W_1 \) and \( W_2 \) are nonempty.

Furthermore, \( W_1 \) and \( W_2 \) are both finite unions of disks \( D_i \) above. Hence, there are integers \( m_1, m_2 \geq 1 \) so that each \( W_k \) maps \( m_k \)-to-one onto \( U_0 \), with \( m_1 + m_2 = d \). Let \( X_k = W_k \cap K_v \) for \( k = 1, 2 \). Since \( \phi^{-1}(K_v) = K_v \), \( \phi \) must map \( X_k \) \( m_k \)-to-one onto \( K_v \).

For any integer \( i \geq 1 \), observe that the polynomial \( \phi^i(z) - a \) is monic of degree \( d^i \). Moreover, since the equation \( \phi^{-1}(z) = a \) has exactly \( d^{-1} \) roots (counting multiplicity), all of which lie in \( U_0 \), it follows that \( \phi^i(z) = a \) has \( m_1 d_1^i \) roots in \( W_1 \) and \( m_2 d_2^i \) roots in \( W_2 \), counting multiplicity. Thus, we may write

\[
\phi^i(z) - a = g_i(z)h_i(z)
\]
where \( g_i \) is monic of degree \( m_1d^{i-1} \) with all its roots in \( W_1 \), and \( h_i \) is monic of degree \( m_2d^{i-1} \) with all its roots in \( W_2 \). In addition, define \( g_0(z) = h_0(z) = z - a \).

We will now use the polynomials \( g_i \) to compute the bounds given in the Lemma for \( X_1 \); the proof for \( X_2 \) is similar, using \( h_i \). To simplify notation, write \( X = X_1 \) and \( m = m_1 \).

For any integer \( j \geq 0 \), write \( j = c_0 + mk \), and write \( k \) in base-\( d \) notation, so that

\[
j = c_0 + m(c_1 + c_2d + c_3d^2 + \cdots + c_md^{M-1}),
\]

with \( c_0 \in \{0, 1, \ldots, m - 1\} \), and with \( c_i \in \{0, 1, \ldots, d-1\} \) for \( i \geq 1 \). Define

\[
f_j(z) = \prod_{i=0}^{M} [g_i(z)]^{c_i}.
\]

Clearly, \( f_j \) is monic of degree \( j \). Meanwhile, for \( x \in X \) and \( i \geq 1 \), observe that \( \phi^i(x) \in K_v \), and therefore \( |\phi^i(x) - a| \leq r_v \). On the other hand, all roots of \( h_i \) lie in \( W_2 \), which is distance \( r_v \) from \( x \); therefore, \( |h_i(x)| = r_v^{(d-m)d^{-1}} \). It follows that

\[
|g_i(x)| \leq r_v^{1-(d-m)d^{-1}}
\]

for all \( i \geq 1 \). In addition, since \( X \subseteq U_0 \), we have \( |g_0(x)| \leq r_v \). Thus,

\[
|f_j(x)| < r_v^{c_0} \prod_{i=1}^{M} r_v^{c_i(1-(d-m)d^{-1})} = r_v^e,
\]

where \( e = e(j, m, d) \) in the notation of Definition 3.1.

By the same van der Monde determinant argument as in the proof of Lemma 4.1, it follows that if \( N \geq 2 \) and \( x_1, \ldots, x_N \in X \), then

\[
\prod_{i \neq j} |x_i - x_j| \leq r_v^{E(N, m, d)}. \tag*{\square}
\]

**Remark 5.2.** In some cases, \( K_v \) splits into more than two pieces, each much smaller than the \( X_1, X_2 \) of Lemma 5.1. For example, the filled Julia set of the map \( \phi(z) = z^2 - c \) of Example 2.3 breaks naturally into \( d \) pieces. Adapting the method of the Lemma for each piece, we could ultimately replace the coefficient \( d^2 - 2d + 2 \) in Theorem 7.1 by \( d \).

However, as previously noted, most polynomials are not so simple. Indeed, the filled Julia set of \( \phi(z) = z^2 - az^{-1} \) from Example 2.4 splits into only two pieces. (Of course, if we take a higher preimage \( U_n \) in that example, we get more than two pieces; but because of the large radii, there appears to be no improvement gained by using \( n > 1 \).) Even an application of the arguments of Remark 4.3 would result in only a slight decrease in the coefficient of \( N \log_d N \) in the exponent (cf. Lemma 3.2(b)). Unfortunately, a real improvement would require an increase in the size of the (negative) coefficient of \( N^2 \), not the \( N \log_d N \) term.

6. A Partition of the Filled Julia Set: Archimedean Case

The final tool needed for Theorem 7.1 is an archimedean analogue of Lemma 5.1. Roughly the same argument works, but only if the diameter of the filled Julia set \( K \) is large enough. This phenomenon is familiar to complex dynamicists. For example, given \( \phi(z) = z^2 + c \in \mathbb{C}[z] \), if the diameter of \( K \) is small, then \( c \) lies in the Mandelbrot set, in which case \( K \) is connected. However, once the diameter is large enough, \( c \) leaves the Mandelbrot set and \( K \) becomes disconnected. In fact, as the diameter grows, the various pieces of \( K \) shrink.
We begin with the following preliminary result.

**Lemma 6.1.** Let \( \phi \in \mathbb{C}[z] \) be a polynomial of degree \( d \geq 2 \) with lead coefficient \( a_d \in \mathbb{C} \). Denote by \( K \) the filled Julia set of \( \phi \) in \( \mathbb{C} \), and let \( U_0 = \overline{D}(a, r') \) be the smallest disk that contains \( K \). Set \( r = |a_d|^{1/(d-1)} r' \), and suppose that
\[
 r > \begin{cases} 
 3, & \text{if } d = 2, \text{ or} \\
 2 + \sqrt{3}, & \text{if } d \geq 3. 
\end{cases}
\]

Then \( K \) is contained in the union of \( d \) open disks of radius \( |a_d|^{-1/(d-1)} \).

**Proof.** As in the proof of Lemma 4.1, we may assume that \( \phi \) is monic. Denote by \( b_1, \ldots, b_d \) the (possibly repeated) roots of \( \phi(z) = a \), and let \( \overline{D}(c, s) \) be the smallest disk containing \( b_1, \ldots, b_d \). (Here, we break our convention and allow \( s = 0 \) if \( b_1 = \cdots = b_d \).) Because \( K \) is not contained in \( D(c, r) \), there must be some \( y_0 \in K \) such that \( |y_0 - c| \geq r \).

Let \( Y = \overline{D}(c, s) \cap D(y_0, |y_0 - c|) \). We claim that \( Y \) is contained in a disk of radius strictly less than \( s \) (or that \( Y \) is empty, if \( s = 0 \)). Indeed, if \( |y_0 - c| < s \), then \( Y \subseteq D(y_0, |y_0 - c|) \) trivially. Otherwise, \( |y_0 - c| \geq s \), and since the center \( c \) of the first disk lies on the boundary of the second, the intersection \( Y \) is contained in a strictly smaller disk. (For example, center the new disk at the midpoint of the two intersection points of the two boundary circles.)

By the minimality of \( s \), then, not all of \( b_1, \ldots, b_d \) can be in \( Y \). Thus, there is some \( 1 \leq i \leq d \) such that \( |y_0 - b_i| \geq |y_0 - c| \geq r \). Without loss, assume that \( |y_0 - b_1| \geq r \).

For all \( i \geq 2 \), we have \( |y_0 - b_i| \geq r - s \), because \( b_i \in \overline{D}(c, s) \). Since \( y_0 \in K \), we have \( \phi(y_0) \in K \), and therefore \( |\phi(y_0) - a| \leq r \). If \( r - s \geq 0 \), then, we have
\[
 r \geq |\phi(y_0) - a| = \prod_{i=1}^{d} |y_0 - b_i| = |y_0 - b_1| \cdot \prod_{i=2}^{d} |y_0 - b_i| \geq r \cdot (r - s)^{d-1},
\]
from which we obtain \( r - s \leq 1 \). Regardless of the sign of \( r - s \), then, we have \( s \geq r - 1 \).

Re-index so that \( b_1 \) and \( b_d \) are distance \( \max\{|b_i - b_j|\} \) apart, and so that for all \( i = 1, \ldots, d - 1 \), we have \( |b_{i+1} - b_1| \geq |b_i - b_1| \). Thus, the \( \{b_i\} \) are ordered by their distance from \( b_1 \). Moreover, \( |b_d - b_1| \geq \sqrt{3}s \geq \sqrt{3}(r - 1) \); see, for example, Exercise 6.1.

If \( d = 2 \), we can improve this lower bound. In that case, the smallest disk containing \( b_1 \) and \( b_2 \) is the closed disk centered at \((b_1 + b_2)/2\) of radius \( |b_1 - b_2|/2 \). That is, \( s = |b_1 - b_2|/2 \). It follows that \( |b_1 - b_2| = 2s \geq 2(r - 1) \).

For all degrees \( d \geq 2 \), we have \( r > 3 \), so that \( s > 2 \), and therefore the two disks \( \overline{D}(b_1, 1) \) and \( \overline{D}(b_d, 1) \) are disjoint. Moreover, as \( y \) ranges through \( \mathbb{C} \setminus [D(b_1, 1) \cup D(b_d, 1)] \), the minimum value of \( |y - b_1| \cdot |y - b_d| \) is \( |b_1 - b_d| - 1 \), attained at only two points, namely the point on the boundary of each disk closest to the other disk.

Let \( U_1 = \phi^{-1}(U_0) \). Since \( K = \phi^{-1}(K) \subseteq U_1 \), it suffices to show that
\[
 U_1 \subseteq \bigcup_{i=1}^{d} D(b_i, 1). 
\]

If not, then there is some \( y \in U_1 \setminus \bigcup D(b_i, 1) \). If \( d \geq 3 \), then by the above computations, we have \( |y - b_1| \cdot |y - b_d| \geq \sqrt{3}(r - 1) - 1 \). Since \( \phi(y) \in U_0 \), we obtain
\[
 r \geq |\phi(y) - a| = \prod_{i=1}^{d} |y - b_i| \geq (\sqrt{3}(r - 1) - 1) \prod_{i=2}^{d-1} |y - b_i| \geq (\sqrt{3}(r - 1) - 1),
\]
that the lower bound \( \phi(y) - a \geq 2 \), contradicting the hypothesis that \( r > 2 + \sqrt{3} \). Similarly, if \( d = 2 \), then \( r \geq |\phi(y) - a| = |y - b_1| \cdot |y - b_2| > 2(r - 1) - 1 = 2r - 3 \), contradicting the hypothesis that \( r > 3 \), and proving the Lemma.

\[ \square \]

**Remark 6.2.** Because \( K_v \) is compact for archimedean \( v \), the conclusion of Lemma 6.1 implies that \( K \) is in fact contained in \( d \) closed disks of radius strictly less than \( |a_d|^{-1/(d-1)} \). This fact will be useful in Cases 2 and 3 of the proof of Theorem 7.1.

We are now prepared to present our archimedean version of Lemma 5.1.

**Lemma 6.3.** Let \( \phi \in \mathbb{C}[z] \) be a polynomial of degree \( d \geq 2 \) with lead coefficient \( a_d \in \mathbb{C} \). Denote by \( K \) the filled Julia set of \( \phi \) in \( \mathbb{C} \), and let \( r' \) be the radius of the smallest disk \( U_0 \) that contains \( K \). Set \( r = |a_d|^{-1/(d-1)}r' \) and

\[
C_d = d^{-(d-2)/(d-1)} \leq \min \left\{ \frac{1}{d-1}, \frac{12}{d-1} \right\}.
\]

Suppose that

\[
r \geq \begin{cases} 
4, & \text{if } d = 2 \\
\frac{\sqrt{3} + 2(d - 1)}{\sqrt{3} - (d - 1)C_d}, & \text{if } d \geq 3.
\end{cases}
\]

Then there are disjoint sets \( X_1, X_2 \subseteq K \) and positive integers \( m_1, m_2 \) with the properties that \( X_1 \cup X_2 = K \), that \( m_1 + m_2 = d \), that for \( k = 1, 2 \), \( \phi : X_k \rightarrow K_v \) is \( m_k \)-to-1, and that for \( k = 1, 2 \), for any integer \( N \geq 2 \), and for any set \( \{x_1, \ldots, x_N\} \subseteq X_k \) of \( N \) points in \( X_k \),

\[
\prod_{i \neq j} |x_i - x_j| \leq N^N |a_d|^{-N(\sqrt{3} - (d - 1)C_d)} (C_d)^{E(N, m_k, d)} (C_d r)^{E(N, m_k, d)},
\]

where \( E(N, m_k, d) \) and \( F(N, m_k, d) \) are as in Definition 5.1.

**Proof.** As in the proof of Lemma 5.1, we may assume that \( \phi \) is monic. It is easy to check that \( C_d \leq \min\{1, 2, \frac{12}{(d-1)}\} \) (the closest approach for \( d \geq 3 \) occurs at \( d = 5 \)), and that the lower bound \( (\sqrt{3} + 2(d - 1))/(\sqrt{3} - (d - 1)C_d) \) (respectively, 4) for \( r \) is greater than \( 2 + \sqrt{3} \) (respectively, 3), so that we may invoke Lemma 6.1.

Write \( U_0 = \overline{D}(a, r) \), and define and order \( b_1, \ldots, b_d \) as in the proof of Lemma 6.1 so that \( |b_1 - b_d| \geq \sqrt{3}(r - 1) \) (or \( |b_1 - b_d| \geq 2(r - 1) \), if \( d = 2 \)).

If \( d \geq 3 \), observe that for some \( m = 1, \ldots, d - 1 \), we have

\[
|b_{m+1} - b_1| \geq |b_m - b_1| + 2 + C_d r.
\]

For if not, then

\[
\sqrt{3}(r - 1) \leq |b_d - b_1| < (d - 1)[2 + C_d r] = 2(d - 1) + (d - 1)C_d r,
\]

so that \( \sqrt{3} - (d - 1)C_d r < \sqrt{3} + 2(d - 1) \), contradicting the hypotheses.

If \( d = 2 \), we have \( |b_2 - b_1| \geq 2r - 2 \geq 2 + r \), since \( r \geq 4 \). Let \( m = 1 \) in this case.

Let \( U_1 = \phi^{-1}(U_0) \), and set \( W_1 = \overline{D}(b_1, |b_m - b_1| + 1) \cap U_1 \) and \( W_2 = U_1 \setminus W_1 \). Observe that \( \text{dist}(W_1, W_2) \geq C_d r \). Indeed, if \( y_1 \in W_1 \) and \( y_2 \in W_2 \), then \( y_1 \in \overline{D}(b_1, 1) \) and \( y_2 \in \overline{D}(b_1, 1) \) for some \( 1 \leq i \leq m \) and some \( m + 1 \leq j \leq d \); therefore

\[
|y_2 - y_1| \geq |b_j - b_1| - |b_i - b_1| - 2 \geq |b_{m+1} - b_1| - |b_m - b_1| - 2 \geq C_d r.
\]
Since $W_1$ contains $m$ preimages of $a$ and $W_2$ contains the other $d - m$, it follows that $\phi$ maps $W_1$ $m$-to-1 onto the connected set $U_0$, and it maps $W_2$ $(d - m)$-to-1 onto $U_0$.

Let $X_1 = W_1 \cap \mathcal{K}$, $X_2 = W_2 \cap \mathcal{K}$, $m_1 = m$, and $m_2 = d - m$. By the previous paragraph, $X_1$ and $X_2$ satisfy all of the mapping properties claimed in the Lemma. For any integer $i \geq 0$, define $g_i(z)$ and $h_i(z)$ as in the proof of Lemma 5.1. That is, for $i \geq 1$, write

$$\phi^i(z) - a = g_i(z)h_i(z),$$

where $g_i$ is a monic polynomial of degree $m_id^{i-1}$ with all of its roots in $W_1$, and $h_i$ is a monic polynomial of degree $m_2d^{i-1}$ with all of its roots in $W_2$. For $i = 0$, define $g_0(z) = h_0(z) = z - a$. We will now compute the bounds given in the Lemma for $X_1$; the proof for $X_2$ is similar. Write $X = X_1$ and $m = m_1$.

As in the proof of Lemma 5.1, we may write any integer $j \geq 0$ as

$$j = c_0 + m(c_1 + c_2d + c_3d^2 + \cdots + c_md^{M-1}),$$

with $c_0 \in \{0, 1, \ldots, m - 1\}$, and with $c_i \in \{0, 1, \ldots, d - 1\}$ for $i \geq 1$. Similarly, define

$$f_j(z) = \prod_{i=0}^M [g_i(z)]^{c_i},$$

which is clearly monic of degree $j$. As before, for any $x \in X$, we have $|\phi^j(x) - a| \leq r$. Similarly, the roots of $h_i$, which all lie in $W_2$ (for $i \geq 1$), are distance at least $Cdr$ from $x$ (which is worse than the Lemma 5.1 bound of $r$). Thus, $|h_0(x)| \geq (Cdr)^{(d-m)d^{i-1}}$, and hence

$$|g_0(x)| \leq r(Cdr)^{-(d-m)d^{i-1}} = C_d^{-1} (Cdr)^{1-(d-m)d^{i-1}}$$

for all $i \geq 1$. Moreover, since $x \in U_0$, we have $|g_0(x)| \leq r$. We obtain

$$|f_j(x)| \leq r^{c_0} \prod_{i=1}^M C_d^{-c_i} (Cdr)^{c_i(1-(d-m)d^{i-1})} = C_d^{-(c_0+c_1+\cdots+c_M)} (Cdr)^e,$$

where $e = e(j, m, d)$ in the notation of Definition 3.1. The Lemma then follows by the van der Monde determinant argument of the proof of Lemma 4.1. \hfill \Box

**Remark 6.4.** Later, in the proof of Theorem 7.1 we will consider the quantity $Cdr$, rather than the radius $r$, at the archimedean primes. It is easy to prove that the lower bound for $r$ given in Lemma 6.3 is guaranteed to hold provided $Cdr \geq 4 + \sqrt{3}$. (In fact, $4 + \sqrt{3}$ is the value of $C_d(\sqrt{3} + 2(d-1))/(\sqrt{3} - (d-1)C_d)$ at $d = 3$.) For $d = 2$, we also note the more obvious facts that $C_2 = 1$ and that the corresponding sufficient lower bound for $C_2r$ is 4.

**Remark 6.5.** The bounds in Lemma 6.1 and Lemma 6.3 are not sharp. Besides the fact that most of the comments from Remarks 4.3 and 5.2 apply here, our geometric arguments could also be improved. For example, in the proof of Lemma 6.1, if we considered $\overline{D(c,s)} \cap D(y_0,t)$ instead of $\overline{T(c,s)} \cap D(y_0,|y_0-c|)$, where $t = \sqrt{|y_0-c|^2 + s^2}$, we could show that some $b_i$ satisfies $|y_0 - b_i| \geq t$. Related arguments could show that two or more points $b_i, b_j$ must make the product $|y_0 - b_i| \cdot |y_0 - b_j|$ larger than we proved. Similarly, it should be possible to increase the $\sqrt{3}$ factor to something closer to 2 by considering the geometric arrangement of the $\{b_i\}$ more delicately.
At last, we are prepared to state and prove a precise version of the Main Theorem.

**Theorem 7.1.** Let $K$ be a global field, and let $\phi \in K[z]$ be a polynomial of degree $d \geq 2$. Let $s_\infty \geq 0$ be the number of archimedean primes of $K$, and let $s \geq s_\infty$ be the number of bad (i.e., not potentially good) primes of $\phi$ in $M_K$, including all archimedean primes.

If $K$ is a function field, let $q$ be the size of the smallest residue field of a prime $v \in M_K$. If $K$ is a number field, let $D = [K: \mathbb{Q}]$, and let

$$
\sigma = \begin{cases} 
7 & \text{if } d = 2, \\
2 \cdot 33^{(d-1)(d-2)} & \text{if } d \geq 3.
\end{cases}
$$

Set

$$
t = \begin{cases} 
s - s_\infty & \text{if } K \text{ is a number field and } s \leq \sigma D, \\
d \log d & \text{if } K \text{ is a number field and } s > \sigma D, \\
\frac{s}{2 \log 2} & \text{if } K \text{ is a function field},
\end{cases}
$$

and

$$
\beta = \begin{cases} 
9 & \text{if } K \text{ is a number field, } s \leq \sigma D, \text{ and } d = 2, \\
\max \{11, 2d\} & \text{if } K \text{ is a number field, } s \leq \sigma D, \text{ and } d \geq 3, \\
1 & \text{otherwise}.
\end{cases}
$$

Then $\phi$ has no more than $M + 1$ $K$-rational preperiodic points in $\mathbb{P}^1(K)$, where

$$
M = \begin{cases} 
q & \text{if } K \text{ is a function field and } s = 0, \\
\beta^D & \text{if } K \text{ is a number field and } s = s_\infty, \\
\beta^D(d^2 - 2d + 2)(t \log_d t + 3t) & \text{if } 0 < t < d, \\
\beta^D(d^2 - 2d + 2)(t \log_d t + t \log_d \log_d t + 3t) & \text{otherwise}.
\end{cases}
$$

**Proof.** For each prime $v \in M_K$, let $n_v \geq 1$ be the exponent so that the product formula holds for all $x \in K^\times$. Let $S$ be the (finite) set of primes of $K$ of bad reduction of $\phi$, including all the archimedean primes; that is, $\#S = s$. Let $a_d \in K$ be the lead coefficient of $\phi$. For each prime $v \in M_K$, let $K_v \subseteq \mathbb{C}_v$ denote the filled Julia set of $\phi$ in $\mathbb{C}_v$, let $r_v'$ be the radius of the smallest disk in $\mathbb{C}_v$ containing $K_v$, and let $r_v = |a_d|_{v}^{1/(d-1)} r_v'$.

For each non-archimedean prime $v$, let $R_v = r_v^{n_v}$. For each archimedean prime $v$, let $R_v = (C_d r_v)^{n_v}$, where $C_d = d^{-(d-2)/(d-1)} \leq 1$, as in the statement of Lemma 6.3. We consider four cases, some of which overlap with others.

**Case 0.** The simplest case is that $K$ is a function field and $S = \emptyset$; that is, there are no archimedean primes, and all primes have potentially good reduction. Let $w \in M_K$ be a prime whose residue field has only $q$ elements, and suppose that there are $q + 1$ distinct $K$-rational preperiodic points $\{x_1, \ldots, x_{q+1}\}$ besides the point at $\infty$.

By Lemma 2.3c, we have $|x_i - x_j|_v \leq |a_d|_{w}^{-1/(d-1)}$ for every $v \in M_K$ and every $i, j \in \{1, \ldots, n\}$. Moreover, by the pigeonhole principle, there must be some distinct $i, j \in \{1, \ldots, n\}$ such that $|x_i - x_j|_w < |a_d|_{w}^{-1/(d-1)}$. Hence,

$$
1 = \prod_{v \in M_K} |x_i - x_j|_v^{n_v} < \prod_{v \in M_K} \left[|a_d|_{v}^{-1/(d-1)}\right]^{n_v} = 1,
$$
which is a contradiction. Thus, there are at most \( q \) finite \( K \)-rational preperiodic points.

**Case 1.** Choose \( w \in M_K \) such that \( R_w \geq R_v \) for all \( v \in M_K \). (Such \( w \) exists because \( R_v = 1 \) for all but finitely many \( v \in M_K \).) In this main case, we suppose that:

- \( R_w > 1 \).
- If \( K \) is a number field, then \( R_w \geq 4 \) and \( s - s_\infty \geq 1 \).
- If \( w \) is archimedean, then the lower bounds of Lemma 6.3 hold for \( r_w \).

In particular, we may choose integers \( m_1, m_2 \) and sets \( X_1, X_2 \subseteq K_w \) for \( \phi \) according to Lemma 5.1 (if \( w \) is non-archimedean) or Lemma 6.3 (if \( w \) is archimedean).

For each index \( k = 1, 2 \), set

\[
A_k = \frac{d - m_k}{m_k(d - 1)}, \quad B_k = 1 - \log_d m_k, \quad \text{and} \quad N_k = M(A_k, B_k, t),
\]

where \( M(\cdot, \cdot, \cdot) \) is as in Lemma 6.3 and where \( t \) is as in the statement of the Theorem. We claim that there are fewer than \( N_k \) \( K \)-rational preperiodic points in \( X_k \).

To prove the claim, fix \( k = 1, 2 \), and let \( m = m_k \), \( A = A_k \), \( B = B_k \), and \( N = N_k \).

Suppose there are \( N \) distinct \( K \)-rational preperiodic points \( x_1, \ldots, x_N \) in \( X_k \). Then by the product formula applied to both \( \prod_{i \neq j} (x_i - x_j) \) and \( a_d \),

\[
1 = \prod_{v \in M_K} \left| \prod_{i \neq j} (x_i - x_j) \right|_{v}^{n_v} = \prod_{v \in M_K} \left[ a_d v^{N(N-1)/(d-1)} \prod_{i \neq j} \left| x_i - x_j \right|_v \right]^{n_v}
\]

(7)

\[
\leq \prod_{v \in S} \left[ a_d v^{N(N-1)/(d-1)} \prod_{i \neq j} \left| x_i - x_j \right|_v \right]^{n_v},
\]

where the inequality is because \( |x - y|_v \leq |a_d|_v^{1/(d-1)} \) for all \( v \in M_K \setminus S \) and \( x, y \in K_v \), by Lemma 2.5c, and because \( x_1, \ldots, x_N \in K_v \) for every \( v \in M_K \).

If \( w \) is non-archimedean, then by Lemma 4.1 and Lemma 5.1 (7) becomes

\[
1 \leq N^{DN} R_w^{n_w E(N,m,d)} \prod_{v \in S \setminus \{w\}} r_v^{n_v E(N,d)} = N^{DN} C_d^{-(d-1)E(N,d)} R_w^{E(N,m,d)} \prod_{v \in S \setminus \{w\}} R_v^{E(N,d)},
\]

where we set \( D = 0 \) if \( K \) is a function field. (The appearance of \( D \) in the exponent comes from equation (2).) Because \( R_w \geq R_v \) and \( E(N,d) \geq 0 \), we can replace each \( R_v \) by \( R_w \), and because \( R_w, C_d^{-1} \geq 1 \), we can apply Lemma 3.4a–b to obtain

(8)

\[
1 \leq N^{DN} C_d^{-(d-1)DN \log_d N} R_w^{(d-1)N[s \log_d N - AN + B]}.
\]

Similarly, if \( w \) is archimedean, then by Lemma 4.1 and Lemma 6.3 (7) becomes

\[
1 \leq N^{DN} C_d^{-(n_w F(N,M,d) - n_w F(N,m,d))} \prod_{v \in S \setminus \{w\}} r_v^{n_v E(N,d)}
\]

\[
= N^{DN} C_d^{-(D-n_w)E(N,d) - n_w F(N,m,d)} R_w^{E(N,m,d)} \prod_{v \in S \setminus \{w\}} R_v^{E(N,d)}.
\]

Replacing each \( R_v \) by \( R_w \) as before, and applying Lemma 3.4a–c, we obtain exactly inequality (8) once more.

Meanwhile, we compute

(9)

\[
N^{DN} C_d^{-(d-1)DN \log_d N} = d^{DN \log_d N} d^{(d-2)(DN \log_d N)} = d^{(d-1)DN \log_d N}.
\]
If $K$ is a number field, our assumption that $R_w \geq 4$ means that $d \leq R_w^{1/\log d}$. Combining (8) and (9), then, we obtain

$$1 \leq R_w^{(d-1)N[t \log_d N - AN + B]},$$

where $t = s + D \log d/(2 \log 2)$, as in the statement of the Theorem. The same inequality follows for function fields with $t = s$, since $D = 0$ in that case. By our definitions of $A$, $B$, and $t$, the hypotheses of Lemma 3.3 hold. Thus, by that Lemma and our choice of $N$, we have $t \log_d N - AN + B < 0$, so that $1 < 1$, which is a contradiction, proving the claim that there are fewer than $N_k$ $K$-rational preperiodic points in $X_k$. (However, since $N_k$ need not be an integer, we cannot claim that there are at most $N_k - 1$ such points.)

The total number of finite $K$-rational preperiodic points is the number in $X_1$ plus the number in $X_2$. That is, there are fewer than $N_1 + N_2$ such points. That upper bound is

$$N_1 + N_2 = M(A_1, B_1, t) + M(A_2, B_2, t).$$

From the definition of $M(A, B, t)$ in Lemma 3.3 it is each to check that, as $m_1$ varies from 1 to $d-1$, the largest value of $N_1 + N_2$ in equation (11) is attained at $m_1 = 1$ and $m_2 = d-1$ (or vice versa). In that case, the bound is

$$N_1 + N_2 = (d^2 - 2d + 2)(t \log_d t + t \log_d (\max\{1, \log_d t\}) + 3t).$$

Adding 1 for the point at infinity, we obtain the bound stated in the Theorem, with $\beta = 1$.

**Case 2.** Next, suppose that $K$ is a number field and $d = 2$. Write $S_\infty$ for the set of archimedean primes of $M_K$, and let $s_\infty = |S_\infty|$. We will remove the archimedean primes from the picture by covering the filled Julia set at each such prime $v \in S_\infty$ by at most $9^{n_v}$ disks of diameter less than $|a_d|^{-1}$. To simplify notation, let $K'_v = a_d K_v$; we wish to cover $K'_v$ by disks of diameter less than 1.

For any real prime $v \in S_\infty$, the set $K'_v$ is contained either in a single interval of length 6 or in two intervals of length less than 2, by Lemma 6.1 and Remark 6.2. (In fact, the bound of 6 could be reduced to 4, but we will not need that stronger bound here.) In particular, $K'_v$ is contained in a union of seven or fewer intervals of length strictly less than 1.

For a complex prime $v \in S_\infty$, the same Lemma implies that $K'_v$ is contained either in a single disk of radius 3 or in two disks of radius less than 1. Each disk of radius 1 can easily be covered by nine disks of diameter slightly less than 1. Similarly, the disk of radius 3 can be covered by a square of side length 6. That square can then be divided into 81 squares of side length 2/3, each of which fits inside a disk of diameter less than 1.

Scaling back by $|a_d|^{-1}$, then, we have at each archimedean prime $v \in S_\infty$ at most $9^{n_v}$ disks of diameter less than $|a_d|^{-1}$ which together cover $K_v$, as promised. In total, then, we have at most $9^D$ choices of one disk for each archimedean prime.

For any such choice $D = \{D_v : v \in S_\infty\}$ of one disk of diameter less than $|a_d|^{-1}$ for each archimedean prime $v$, let $\mathcal{P}_D$ denote the set of $K$-rational preperiodic points $x$ for which $x \in D_v$ for every $v \in S_\infty$. We will bound the size of $\mathcal{P}_D$.

If $S = S_\infty$, then each set $\mathcal{P}_D$ can contain at most one point. Indeed, if there were distinct points $x, y \in \mathcal{P}_D$, then

$$1 = \prod_{v \in M_K} |x - y|_v^{n_v} < \prod_{v \in M_K} [(a_d|^{-1})^{n_v} = 1,$
by Lemma 2.5c, with the strict inequality coming from the fact that the diameter at each archimedean prime is strictly less than $|a_d|_v^{-1}$. Since there are $9^D$ choices of $\mathbb{D}$, there are at most $9^D$ finite $K$-rational preperiodic points.

On the other hand, if $S \supseteq S_\infty$, then choose $w \in M_K \setminus S_\infty$ such that $R_w \geq R_v$ for all $v \in M_K \setminus S_\infty$. By Lemma 2.5c, $r_w > 1$, so that we may apply Lemma 5.1 at $w$.

Now fix $\mathbb{D}$ and follow the argument of Case 1, but restricted to $\{x_i\} \subseteq P_\mathbb{D}$. At each archimedean prime $v \in S_\infty$ we have $|x_i - x_j|_v \leq |a_d|_v^{-1}$. Therefore, by Lemmas 4.1, 5.1 and 3.4a–b, inequality (7) becomes

$$1 \leq R_w^{(d-1)N[t \log_d N-AN+B]},$$

where $t = s - s_\infty$. Following the rest of the argument of Case 1 (from inequality (10) on), and multiplying by $9^D$ (the number of choices $\mathbb{D}$), we obtain the desired bounds.

**Case 3.** If $K$ is a number field and $d \geq 3$, we proceed roughly as in Case 2. Again, write $S_\infty$ for the set of archimedean primes of $M_K$, let $s_\infty = |S_\infty|$, and let $\mathcal{K}_v = \alpha \mathcal{K}_v$, where $\beta^{-1} = a_d$. This time, we will cover $\mathcal{K}_v$ by at most $\beta^{n_v}$ disks of diameter less than 1, where $\beta = \max\{11, 2d\}$.

For a real prime $v \in S_\infty$, Lemma 5.1 and Remark 6.2 imply that $\mathcal{K}_v$ is contained either in a single interval of length $4 + 2\sqrt{3}$ or in $d$ intervals of length less than 2. In particular, $\mathcal{K}_v$ is contained in a union of $\max\{8, 2d\} \leq \beta$ or fewer intervals of length less than 1.

For a complex prime $v \in S_\infty$, the same Lemma implies that $\mathcal{K}_v$ is contained either in a single disk of radius $2 + \sqrt{3}$ or in $d$ disks of radius less than 1. As before, each disk of radius 1 can be covered by nine disks of diameter less than 1. Similarly, the disk of radius $2 + \sqrt{3}$ can be covered by a square of side length $4 + 2\sqrt{3}$. That square can be divided into 121 squares of side length $(4 + 2\sqrt{3})/11$, each of which fits inside a disk of diameter less than 1. (In fact, using a hexagonal tiling, one could cover the big disk by 84 disks of diameter less than 1, but the messy proof gives only a minor improvement over 121.) Thus, $\mathcal{K}_v$ can be covered by a union of $\max\{121, 9d\} \leq \beta^d$ disks of diameter less than 1.

The rest of Case 3 then follows Case 2, with $\beta^D$ in place of $9^D$. This completes our analysis of the four cases.

**Final step.** If $K$ is a function field, we are done; indeed, by Lemma 2.5c, Cases 0 and 1 cover all the possibilities.

If $K$ is a number field, we will now show that for $s > \sigma D$, we are automatically in Case 1. Because $n_v \leq 2$ for an archimedean prime $v$, and by Remark 6.4 we need only show there is some $w \in M_K$ such that $R_w \geq 2$ if $d = 2$, or such that $R_w \geq (4 + \sqrt{3})^2$ if $d \geq 3$.

From basic algebraic number theory, there are at most $D$ primes of $K$ above any given prime of $\mathbb{Q}$. Given an integer $m \geq 1$, let $p_m$ denote the $m$th prime in $\mathbb{Q}$. (That is, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and so on.) Thus, if $s - s_\infty > D(m - 1)$, there must be some $w \in S \setminus S_\infty$ lying above a prime $p \geq p_m$ of $\mathbb{Q}$. Since $D \geq s_\infty$, we get such a $w$ provided $s > mD$.

Meanwhile, by Lemma 2.8 given $w \in S \setminus S_\infty$ lying over a prime $p$ of $\mathbb{Q}$, we have

$$R_w \geq |\pi_w|_w^{-n_w} \geq p \quad \text{if} \quad d = 2,$$

or

$$R_w^{(d-1)(d-2)} \geq |\pi_w|_w^{-n_w} \geq p \quad \text{if} \quad d \geq 3,$$

where $\pi_w$ is a uniformizer at $w$. (The Lemma applies because if $K_w \cap K = \emptyset$, then there are no finite $K$-rational preperiodic points at all, and the conclusion of the Theorem is trivial.)
For $d = 2$, then, the condition $R_w \geq 16$ is guaranteed provided $s \geq 7D + 1$, since 17 is the seventh prime of $\mathbb{Q}$. Thus, $s > 7D = \sigma D$ suffices for $d = 2$.

For $d \geq 3$, the elementary estimate in Theorem 4.7 of [2] says that $p_m > (1/6)m \log m$ for any integer $m \geq 1$. It is easy to check that $m = \lfloor \sigma \rfloor$ satisfies $m \log m \geq 6(4+\sqrt{3})^{2(d-1)(d-2)}$, where $\sigma = 2 \cdot 33^{(d-1)(d-2)/((d-1)(d-2))}$ as in the statement of the Theorem. (The 33 appears because it is the smallest integer larger than $(4 + \sqrt{3})^2$.) Thus, $s > \sigma D$ implies $R_w \geq (4 + \sqrt{3})^2$, once again forcing Case 1.

**Remark 7.2.** If, for a given polynomial $\phi$, we know that we are in Case 1 (say, by inspection of the filled Julia set at one prime), then we can set $\beta = 1$ in the statement of the Theorem, even if $s \leq \sigma D$. In particular, for a fixed function $\phi$, the conditions of Case 1 are preserved if one passes to a finite extension of $K$. Thus, one would not have to worry about the growth of $s$ relative to $\sigma D$ as one traveled up a tower of number fields, even though one cannot expect $s$ to increase as fast as $\sigma D$ in general.

**Remark 7.3.** Our covering methods in Cases 2 and 3 are rather crude, and it should be possible to cover the filled Julia sets $K_v$ in these cases far more efficiently. For example, our use of disks of diameter 1 was rather simplistic. Instead, one could cover $K_v$ by larger sets $Y$ for which $\prod_{1 \leq i,j \leq L} |y_i - y_j|_v \leq 1$ for some fixed small integer $L$. Such modifications could lead to a substantial reduction in the coefficient $\beta^D$ in the final bound.

Even without any extra work, the coefficient can be improved in special cases. For example, if $K$ is a totally real number field, then the cutoff $\sigma$, which determines when $\beta$ drops to 1, would be much smaller, since we would only need $R_w \geq 4$ (if $d = 2$), or $R_w \geq 4 + \sqrt{3}$ (if $d \geq 3$) rather than $4^2$ or $(4 + \sqrt{3})^2$. Moreover, if $K$ is totally real and $d = 2$, then each archimedean filled Julia set is contained in a union of four intervals of length 1. (See, for example, Lemma 6.4 and Proposition 6.6 of [9].) Thus, the coefficient of $9^D$ that appears in Theorem 7.1 could be replaced by $4^D$, with one exception.

The one exception is if all non-archimedean primes have good reduction and the archimedean filled Julia set is an interval of length 4. This occurs for the Chebyshev polynomial $\phi(z) = z^2 - 2$, which has filled Julia set $[-2, 2]$. In this special case, after removing the points $\infty$ and 2, the rest of the preperiodic points can be covered by four half-open intervals of length 1 at each archimedean prime. Since there are no non-archimedean bad primes, we obtain a bound of $2 + 4^D$ for the total number of preperiodic points in $\mathbb{P}^1(K)$.

**Remark 7.4.** Another approach to finding a cutoff $\sigma$ which forces Case 1 would be to consider the set $T \subseteq S$ consisting of non-archimedean bad primes $v$ at which there are actually $K$-rational preperiodic points $x, y$ for which $|x - y|_v > 1$. For such primes, the exponent of $-1/((d-1)(d-2))$ in Lemma 2.8 could be improved to $-1/(d-1)$. Unfortunately, there may not be very many such primes. As a result, although the exponent of $(d-1)(d-2)$ in the definition of $\sigma$ could be improved to $(d-1)$, it would come at the expense of introducing a extra factor like $\beta^D$ into the formula for $\sigma$.

**Remark 7.5.** For large degrees $d$, one can obtain slightly smaller bounds by using more than one big bad prime $w$. There is, of course, a trade-off. While using $\ell \geq 2$ big primes $w$ ultimately increases the coefficient $A$ of $-N$ in the exponent of $\phi$, it also increases the number of pieces $\{X_k\}$ from 2 to $2^\ell$. It appears that the optimal number of such primes to use is $\ell \approx 2 \log_2(d-1)$. The improved bound for the number of rational preperiodic points would be roughly the old bound divided by $2 \log_2(d-1)$, for large $d$. However, the proof
would be vastly more complicated, especially dealing with the archimedean primes. The slight improvement seems not to be worth the increased difficulty, especially given that the resulting bound would still be very far from the conjectured uniform bound.

We close by presenting a slight strengthening of Theorem 7.1 in the simplest case.

**Example 7.6.** Let $K = \mathbb{Q}$ (so that $D = s_\infty = 1$, and $n_v = 1$ for all $v \in M_\mathbb{Q}$) and $d = 2$. That is, we wish to bound the number of rational preperiodic points of a quadratic polynomial $\phi \in \mathbb{Q}[z]$. It is of course well known that any such polynomial is conjugate over $\mathbb{Q}$ to one of the form $\phi_c(z) = z^2 + c$, with $c \in \mathbb{Q}$.

Let us suppose that $\phi_c$ has at least one preperiodic point in $\mathbb{Q}$. This supposition implies that $c = j/m^2$ for some relatively prime integers $j, m \in \mathbb{Z}$, and that $-\infty < c \leq 1/4$; see, for example, Proposition 6.7 of [9]. (One can also easily establish that $j$ must satisfy one of approximately $2^4$ congruences modulo $m$, but we do not need that here.) For non-archimedean primes $v$ of $\mathbb{Z}$, we have $R_v = |m|_v^{-1}$ if $v$ is odd, and $R_2 = \max\{|m/2|_2^{-1}, 1\}$. (Note that if $4 \nmid m$, then $\phi_c$ has good reduction at $v = 2$, after a change of coordinates.)

In addition, for $c < 0$, we have $R_\infty = (1 + \sqrt{1 - 4c})/2$.

We will be in Case 1 of the proof of Theorem 7.1 provided there is some prime $v$ with $R_v \geq 4$. Still assuming that there is at least one preperiodic point in $\mathbb{Q}$, Lemma 2.8 says that such a prime must exist unless the only bad primes are $\infty$, 2, and 3, and $R_2, R_3, R_\infty < 4$. By our characterization of $R_v$ above, this means that the denominator $m$ is a divisor of 12, and that $-12 < c \leq 1/4$. There are only finitely many rational numbers of the form $c = j/144$ between $-12$ and $1/4$, and a simple computer search shows none of the corresponding polynomials $\phi_c$ has more than eight preperiodic points in $\mathbb{Q}$. (For five such values of $c$, namely $-21/16$, $-29/16$, $-91/36$, $-133/144$, and $-1333/144$, there are exactly eight preperiodic points in $\mathbb{Q}$. Incidentally, there are infinitely many values $c \in \mathbb{Q}$ for which $\phi_c$ has at least eight preperiodic point in $\mathbb{Q}$, by Theorem 2 of [28].)

For all other $c$, we are in Case 1, which means $t = s + D \log d/(2 \log 2) = s + 1/2$, and $\beta = 1$. If $s = 1$, then only the archimedean prime is bad, and in light of Remark 7.3 there are at most five preperiodic points in $\mathbb{Q}$; in fact, there are at most four for $s = 1$ and $c \neq -2$. The only remaining possibility is that $s \geq 2$ and $\beta = 1$, in which case the number of preperiodic points in $\mathbb{Q}$ is at most

$$(2s + 1) \left[ \log_2(2s + 1) + \log_2(\log_2(2s + 1) - 1) + 2 \right].$$

Since this bound is greater than eight even for $s = 2$, it holds even without making the exceptions from the previous paragraph.

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