THE MOMENT PROBLEM ON CURVES WITH BUMPS

DAVID P KIMSEY AND MIHAI PUTINAR

Abstract. The power moments of a positive measure on the real line or the circle are characterized by the non-negativity of an infinite matrix, Hankel, respectively Toeplitz, attached to the data. Except some fortunate configurations, in higher dimensions there are no non-negativity criteria for the power moments of a measure to be supported by a prescribed closed set. We combine two well studied fortunate situations, specifically a class of curves in two dimensions classified by Scheiderer and Plaumann, and compact, basic semi-algebraic sets, with the aim at enlarging the realm of geometric shapes on which the power moment problem is accessible and solvable by non-negativity certificates.

1. Introduction

Throughout the present note \( \mathbb{R}[x_1, \ldots, x_d] \) denotes the ring of polynomials with real coefficients in \( d \) indeterminates. We adopt the standard notation

\[
x^{\gamma} = \prod_{j=1}^{d} x_j^{\gamma_j} \quad \text{and} \quad |x| := \sqrt{x_1^2 + \ldots + x_d^2},
\]

where \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) and \( \gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}_0^d \). The convex cone of polynomials \( p \in \mathbb{R}[x_1, \ldots, x_d] \) which can written as a sum of squares is \( \Sigma^2 \). The elements of \( \Sigma^2 \) represent universally non-negative polynomials. The real zero set of the ideal \( \mathcal{I} := (p_1, \ldots, p_k) \) generated by \( p_1, \ldots, p_k \) in \( \mathbb{R}[x_1, \ldots, x_d] \) is

\[
\mathcal{V}(\mathcal{I}) := \{ x \in \mathbb{R}^d : p_1(x) = \ldots = p_k(x) = 0 \}.
\]

Recalling some basic notions of real algebraic geometry is also in order. Specifically, for a finite subset \( R = \{ r_1, \ldots, r_k \} \subseteq \mathbb{R}[x_1, \ldots, x_d] \), we let \( Q_R \) stand for the quadratic module generated by \( R \):

\[
Q_R = \{ \sigma_0 + r_1 \sigma_1 + \ldots + r_k \sigma_k : \sigma_0, \ldots, \sigma_k \in \Sigma^2 \}.
\]

Also,

\[
K_Q := \{ x \in \mathbb{R}^d : r_j(x) \geq 0 \quad \text{for} \quad j = 1, \ldots, k \}
\]
is the common non-negativity set of elements of $Q = Q_R$. In general a quadratic module is a subset of the polynomial algebra closed under addition and multiplication by sums of squares, see [4].

Given a multisequence $s = (s_\gamma)_{\gamma \in \mathbb{N}_0^d}$ and a closed set $K \subseteq \mathbb{R}^d$, the full $K$-moment problem on $\mathbb{R}^d$ entails determining whether or not there exists a positive Borel measure $\mu$ on $\mathbb{R}^d$ such that

\begin{equation}
\int_{\mathbb{R}^d} x^\gamma d\mu(x) \quad \text{for} \quad \gamma \in \mathbb{N}_0^d
\end{equation}

and

\begin{equation}
supp \mu \subseteq K.
\end{equation}

If conditions (1.1) and (1.2) are satisfied, then we say that $s$ has a $K$-representing measure.

A multisequence $s = (s_\gamma)_{\gamma \in \mathbb{N}_0^d}$ is called positive definite if

$L_s(f) \geq 0 \quad \text{for} \quad f \in \Sigma^2.$

It is clear that Riesz-Haviland functional $L_s$ is non-negative on the quadratic module $Q$, whenever the moment problem with $s$ has a $K_Q$-representing measure, where

\[ K_Q = \{ x \in \mathbb{R}^d : r(x) \geq 0 \quad \text{for} \quad r \in Q \}. \]

Whether the converse is true is one of the central questions of multivariate moment problem theory, see [6, 9] for ample details. In this direction we recall a useful terminology. A quadratic module $Q$ is said to satisfy the strong moment property (SMP) if every $Q$-positive functional $L : \mathbb{R}[x_1, \ldots, x_d] \to \mathbb{R}$ is a moment functional with the additional requirement that the measure supported on $K_Q$, i.e., there exists a positive Borel measure $\mu$ supported by $K_Q$ such that

\[ L(f) = \int f(x) d\mu(x) \quad \text{for} \quad f \in \mathbb{R}[x_1, \ldots, x_d]. \]

When dropping the requirement $\text{supp}(\mu) \subseteq K_Q$ in the (SMP), we simply say that $Q$ possesses the moment property (MP). A quadratic module $Q$ is called archimedean if there exists a positive constant $C$ with the property $C - |x|^2 \in Q$. In this case $K_Q$ is compact and, by an observation of the second author, the module $Q$ has property (SMP), see again [6, 9] for details.

A classical theorem due to Hamburger (see [2, 3, 4] and [9] for a contemporary treatment) asserts that on the real line every positive definite sequence has the strong moment property. In equivalent terms, the non-negativity of the infinite Hankel matrix $(s_{k+n})_{k,n=0}^\infty$ is necessary and sufficient for $(s_n)_{n=0}^\infty$ to be the power moment sequence of a positive measure on $\mathbb{R}$.

Two dimensions are special, notably for allowing to extend similar sufficient positivity conditions for the solvability of the moment problem along
codimension-one unbounded varieties, that is real algebraic curves. Note
that if $q \in \mathbb{R}[x_1, x_2]$ is non-zero, than $\mathcal{V}(q)$ is a curve, or a set of real points.

One step further, we are seeking only reduced principal ideals, that is we
enforce that a polynomial $f$ vanishes on $\mathcal{V}(q)$ if and only if $f \in (q)$. This
happens if the factorization of $q$ into irreducible factors is square free and
each factor changes sign in $\mathbb{R}^2$. See [1] for a proof and the natural framework
for such a real Nullstellensatz. In this scenario we simply say that $(q)$ is a
real ideal. The main results of [7] and [5] may be combined to produce the
following theorem.

**Theorem 1.1** ([7], [5]). Let $(q)$ be a non-trivial, real principal ideal in
$\mathbb{R}[x_1, x_2]$. Then

$$(q) + \Sigma^2 = \{ p \in \mathbb{R}[x_1, \ldots, x_d] : p(x) \geq 0 \text{ for all } x \in \mathcal{V}(q) \}$$

if and only if the following conditions hold:

(i) All real singularities of $\mathcal{V}(q)$ are ordinary multiple points with inde-
pendent tangents.

(ii) All intersection points of $\mathcal{V}(q)$ are real.

(iii) All irreducible components of $\mathcal{V}(q)'$ (i.e., the union of all irreducible
components of $\mathcal{V}(q)$ that do not admit any non-constant bounded
polynomial functions) are non-singular and rational.

(iv) The configuration of all irreducible components of $\mathcal{V}(q)'$ contains no
loops.

In particular, the above result implies that the quadratic module $(q) + \Sigma^2$
has the strong moment property [7, 5]. The above result is in sharp contrast
to higher dimensional situations, where in general not every positive definite
functional along a variety is represented by integration against a positive
measure (see, [8] for details).

2. Main result

We consider the union of a curve which satisfies conditions (i)-(iv) in
Theorem 1.1 with a side (to become clear in an instant) of a truly compact
semi-algebraic set with the aim at providing positivity certificates for the
moment problem to be solvable on that prescribed support.

**Theorem 2.1.** Let $(q)$ be a non-trivial, real principal ideal of $\mathbb{R}[x_1, x_2]$ whose
zero set satisfies conditions (i)-(iv) in Theorem 1.1 and let $Q \subseteq \mathbb{R}[x_1, x_2]$ be
an archimedean quadratic module. Then the quadratic module $\Sigma^2 + qQ$ has
the strong moment property.

Before proving Theorem 2.1 we pause to note that the positivity set of
$\Sigma^2 + qQ$ is $\mathcal{V}(q) \cup [K_Q \cap \{ q > 0 \}]$. For instance, taking $q(x_1, x_2) = x_1$ and
$Q$ generated by $1 - x_1^2 - x_2^2$ one finds the positivity set of the composed
quadratic module to be the $x_2$-axis union with the half-disk $\{(x_1, x_2), x_1 \geq
0, x_1^2 + x_2^2 \leq 1 \}$. Whence the title of this note.
Figure 1. $Q = \{1 - (x_1 - 1)^2 - (x_2 - 2)^2\}$ and $q(x_1, x_2) = x_2 - x_1^2$

Figure 2. $Q = \{1 - x_1^2 - x_2^2\}$ and $q(x_1, x_2) = x_2(3x_1^2 - x_2^2)$

Proof of Theorem 2.1. We denote in short $x = (x_1, x_2)$. Let $L \in \mathbb{R}[x]'$ be a non-trivial linear functional which is non-negative on $\Sigma^2 + qQ$. We want to prove that $L$ is represented by integration against a positive measure supported by $\mathcal{V}(q) \cup [K_Q \cap \{q \geq 0\}]$. Since $L$ is non-zero, the Cauchy-Schwarz inequality

$$L(f)^2 \leq L(f^2)L(1), \quad f \in \mathbb{R}[x],$$

implies $L(1) > 0$. Below we will use repeatedly the observation that there are elements $f \in \mathbb{R}[x]$ with the property $L(f) > 0$.

The functional $h \mapsto L(qh)$ is non-negative for $h \in Q$ and $Q$ is an archimedean quadratic module, so there exists a positive measure supported $\nu$ by $K_Q$,
such that:

\[ L(qf) = \int_{K_Q} f \, d\nu, \quad f \in \mathbb{R}[x], \]

see [6, 9].

We claim that the measure \( \nu \) does not carry mass on the set \( \{q \leq 0\} \):

(2.1) \[ \nu(\{q \leq 0\}) = \emptyset. \]

The positivity of the functional \( L \) on squares yields

\[ L(\{tqg + f\}^2) \geq 0 \quad t \in \mathbb{R} \quad \text{and} \quad f, g \in \mathbb{R}[x]. \]

On the other hand,

\[ L(\{tqg + f\}^2) = t^2L(q^2g^2) + 2tL(qfg) + L(f^2), \]

hence

(2.2) \[ \left( \int_{K_R} f(x)g(x) \, d\nu(x) \right)^2 \leq \left( \int_{K_R} q(x)g(x)^2 \, d\nu(x) \right) L(f^2) \]

for \( f, g \in \mathbb{R}[x] \).

The non-negativity set \( K_Q \) is compact, therefore continuous functions on \( K_R \) can be uniformly approximated by polynomials. Moreover, continuous functions on \( K_Q \) are dense in \( L^2(\nu) \). That is

(2.3) \[ \left( \int_{K_R} f(x)\psi(x) \, d\nu(x) \right)^2 \leq \left( \int_{K_R} q(x)\psi(x)^2 \, d\nu(x) \right) L(f^2), \]

where \( f \in \mathbb{R}[x] \) and \( \psi \in L^2(\nu) \). If we let \( \chi = 1_{\mathcal{V}(I) \cap K_Q} \) denote the characteristic function of \( \mathcal{V}(I) \cap K_Q \) and \( \psi = \chi \), then (2.3) becomes

\[ \left( \int_{K_R} f(x)\chi(x) \, d\nu(x) \right)^2 \leq \left( \int_{K_R} q(x)\chi(x) \, d\nu(x) \right) L(f^2). \]

From \( q(x)\chi(x) = 0 \) we infer

\[ \int_{\mathcal{V}(q) \cap K_R} f(x) \, d\nu(x) = 0, \quad f \in \mathbb{R}[x]. \]

Choosing next \( \psi \) to be the characteristic function of a compact subset of the open set \( \{q < 0\} \), we find from (2.3):

\[ 0 \leq \left( \int_{K_R} q(x)\chi(x) \, d\nu(x) \right) L(f^2). \]

In particular,

\[ 0 \leq \int_{K_R} q(x)\chi(x) \, d\nu(x) \leq 0, \]

for every characteristic function of a compact subset of \( \{q < 0\} \). This proves (2.1).
Next we choose $\psi$ in (2.3) of the form $\psi = \frac{f}{q}\phi$, where $\phi = \phi^2$ is the characteristic function of a compact subset of $\{q > 0\}$. We find

$$\left( \int_{K_R} f(x) \left( \frac{f(x)}{q(x)} \right) \phi(x) \, d\nu(x) \right)^2 \leq \left( \int_{K_R} q(x) \left( \frac{f(x)^2}{q(x)^2} \right) \phi(x) \, d\nu(x) \right) L(f^2),$$

or equivalently, since $q > 0$ on the support of $\phi$:

$$\left( \int_{K_R} \frac{f(x)^2}{q(x)} \phi(x) \, d\nu(x) \right)^2 \leq \left( \int_{K_R} \frac{f(x)^2}{q(x)} \phi(x) \, d\nu(x) \right) L(f^2).$$

But $\left( \frac{f^2}{q} \right) \phi \geq 0$, hence

$$\int_{K_R} \frac{f(x)^2}{q(x)} \phi(x) \, d\nu(x) \leq L(f^2).$$

A monotonic sequence of such characteristic functions $\phi$ converging pointwise to the characteristic function of $\{q > 0\}$ implies $\frac{1}{q} \in L^1(\nu)$. Recall that $L(1) > 0$.

Let $\Lambda : \mathbb{R}[x] \to \mathbb{R}$ denote the linear functional

$$\Lambda(f) = L(f) - \int_{K_R} f(x) \frac{f(x)}{q(x)} \, d\nu(x), \quad f \in \mathbb{R}[x].$$

We claim that

$$\Lambda(qf) = 0 \quad \text{for} \quad f \in \mathbb{R}[x]$$

and

$$\Lambda(f^2) \geq 0 \quad \text{for} \quad f \in \mathbb{R}[x].$$

Assertion (2.5) follows immediately from the definition of $\Lambda$. We will now verify (2.6). Given the $\nu$-integrability of $\frac{1}{q}$ we can choose $\psi = f/q$ in (2.3). This yields

$$\left( \int_{K_R} f(x) \frac{f(x)^2}{q(x)} \, d\nu(x) \right)^2 \leq \left( \int_{K_R} \frac{f(x)^2}{q(x)} \, d\nu(x) \right) L(f^2)$$

for $f \in \mathbb{R}[x]$. Since $\frac{f(x)^2}{q(x)} \geq 0$ on the support of the measure $\nu$, we find

$$L(f^2) \geq \Lambda(f^2) \quad \text{for} \quad f \in \mathbb{R}[x]$$

which is exactly (2.6).

Finally, because the ideal $(q)$ is real and its zero set satisfies conditions (i)-(iv) in Theorem 1.1, $\Lambda$ has a representing measure supported on $\mathcal{V}(q)$. This proves that the integration against the measure $\mu = \frac{1}{q} + \sigma$ represents the original functional $L$. Moreover, the support of $\mu$ in contained in the union of the supports of $\sigma$ and $\nu$, that is $\text{supp} \mu \subset \mathcal{V}(q) \cup [K_Q \cap \{q > 0\}]$. □
Given a bisequence $s = (s_\gamma)_{\gamma \in \mathbb{N}^2_0}$ and $p(x) = \sum_{0 \leq |\lambda| \leq n} p_\lambda x^\lambda \in \mathbb{R}[x_1, x_2]$, we shall let $p(E)s$ denote the bisequence given by

$$(p(E)s)(\gamma) := \sum_{0 \leq |\lambda| \leq n} q_\lambda s_{\lambda+\gamma}$$

**Corollary 2.2.** Let $s = (s_{(\gamma_1, \gamma_2)})_{(\gamma_1, \gamma_2) \in \mathbb{N}^2_0}$ be a positive definite bisequence and let $Q_R \subseteq \mathbb{R}[x_1, x_2]$ be an archimedean quadratic module, where $R = \{r_1, \ldots, r_m\} \subseteq \mathbb{R}[x_1, x_2]$. If there exists $q \in \mathbb{R}[x_1, x_2]$ such that $(q)$ is a non-trivial, real principal ideal of $\mathbb{R}[x_1, x_2]$ whose zero set satisfies conditions (i)-(iv) of Theorem 1.1 and

$$(2.7) \quad qr_j(E)s$$

is positive definite for $j = 1, \ldots, m$,

then $s$ has a representing measure $\mu$ with

$$\text{supp} \mu \subseteq V(q) \cup \{K_{Q_R} \cap \{q > 0\}\}.$$

Proof. Let $L_s : \mathbb{R}[x_1, x_2] \to \mathbb{R}$ denote the Riesz-Haviland functional with respect to $s$. Then, since $s$ is positive definite and we have a suitable $q \in \mathbb{R}[x_1, x_2]$ such that (2.7) is in force, we have

$$L_s(f + qg) \geq 0 \quad \text{for} \quad f + qg \in \Sigma^2 + qQ_R.$$ 

Thus, the desired conclusion follows immediately from Theorem 2.1. \qed

We add a few remarks on the above result and its proof.

(a) If the quadratic module in the statement of the theorem is finitely generated $Q = Q(r_1, \ldots, r_k)$, then the enhanced quadratic module which is shown to carry (SMP) is $Q(q, qr_1, \ldots, qr_k)$. Notice that the latter may not be archimedean, although $Q$ is.

(b) Changing the generator of the principal ideal $(q)$ will alter the outcome of the statement, for instance $-q$ instead of $q$ in the enhanced quadratic module will flip the “bumps” on the other side of the curve $V(q)$.

(c) The statement of Theorem 2.1 and Corollary 2.2 can be generalized to any number of variables, keeping $(q)$ a real ideal with its zero set hypersurface possessing the (SMP). This is the case for instance of a compact zero set $V(q)$. Indeed, if $V(q)$ is compact, then $\Sigma^2 + (q)$ is a quadratic module with (SMP) [9].

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School of Mathematics, Statistics and Physics, Newcastle University, Newcastle upon Tyne NE1 7RU UK
E-mail address: david.kimsey@ncl.ac.uk

Department of Mathematics, University of California Santa Barbara, Santa Barbara, CA 93106-3080 USA and School of Mathematics, Statistics and Physics, Newcastle University, Newcastle upon Tyne NE1 7RU UK
E-mail address: mihai.putinar@ncl.ac.uk