SHARP BOUNDS ON THE HEIGHT OF K-SEMISTABLE TORIC FANO VARIETIES

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Abstract. Inspired by Fujita’s algebro-geometric result that complex projective space has maximal degree among all K-semistable Fano varieties, we conjecture that the height of a K-semistable metrized arithmetic Fano variety $X$ of relative dimension $n$ is maximal when $X$ is the projective space over the integers, endowed with the Fubini-Study metric. Our main result establishes the conjecture for the canonical integral model of a toric Fano variety when $n \leq 6$ (the extension to higher dimensions is conditioned on a conjectural “gap hypothesis” for the degree). Translated into toric Kähler geometry this result yields a sharp lower bound on a toric invariant introduced by Donaldson, defined as the minimum of the toric Mabuchi functional. We furthermore reformulate our conjecture as an optimal lower bound on Odaka’s modular height. Along the way we show how to control the height of the canonical toric model $X$, with respect to the Kähler-Einstein metric, by the algebro-geometric degree of $X$ in any dimension $n$. Logarithmic generalizations of the results are also given.

1. Introduction

1.1. The height of K-semistable Fano varieties. Let $(X, \mathcal{L})$ be a polarized arithmetic variety, i.e., a normal projective flat scheme $X$ over $\mathbb{Z}$ of relative dimension $n$, endowed with a relatively ample line bundle $\mathcal{L}$. We will denote by $X$ the $n$-dimensional complex projective algebraic variety consisting of the complex points of $X$ and by $L$ the ample line bundle over $X$ induced by $\mathcal{L}$. In other words, $L$ is the restriction to the complexification $X$ of the generic fiber $X_\mathbb{Q}$ of the fibration $X \to \text{Spec } \mathbb{Z}$. Throughout the paper the complex projective variety $X$ will be assumed to be normal.

A central role in arithmetic and diophantine geometry is played by the height of $(X, \mathcal{L})$, which is defined with respect to a continuous metric $\|\cdot\|$ on $L$. This is an arithmetic analog of the algebro-geometric degree of $(X, \mathcal{L})$, i.e., the top intersection number $L^n$ on $X$. The height of $(X, \mathcal{L}, \|\cdot\|)$, also known as Falting’s height, is defined as the $(n+1)$-fold arithmetic intersection number of the metrized line bundle $(\mathcal{L}, \|\cdot\|)$ on $X$, introduced by Gillet-Soulé in the context of Arakelov geometry. Alternatively, it may be defined in terms of the arithmetic Hilbert-Samuel theorem (see Section 1.1). We recall that in Arakelov geometry the metric $\|\cdot\|$ on $L$ plays the role of a “compactification” of the fibration $X \to \text{Spec } \mathbb{Z}$. Accordingly, a metrized line bundle $(\mathcal{L}, \|\cdot\|)$ is usually denoted by $\mathcal{L}$. The definition of height naturally extends to any $\mathbb{Q}$-line bundle $\mathcal{L}$, using homogeneity.

In contrast to the algebro-geometric degree of $L$ the height of $\mathcal{L}$ can rarely be computed explicitly and all one can hope for, in general, is explicit bounds on the height. When $\mathcal{L}$ is the relative canonical line bundle, that we shall denote by $\mathcal{K}_X$ and
such conjectural upper bounds are motivated by the Bogolomov-Miyaoka-Yau inequality on $X$ and imply, in particular, the effective Mordell conjecture \[65, 70\]. Here we shall be concerned with the opposite situation where $L$ is the relative anti-canonical line bundle that we denote by $-K_X$, using additive notation for tensor products. In particular, $X$ is assumed to be a Fano variety; a normal variety whose canonical line bundle $-K_X$ defines an ample $\mathbb{Q}$-line bundle. As shown in \[8\] in the toric case and then \[37\] in general,

\[
(-K_X)^n \leq (-K_{\mathbb{P}^n})^n
\]

under the assumption that $X$ is $K$-semistable. Moreover, equality holds iff $X = \mathbb{P}^n$ \[49\]. In contrast, when $X$ is not K-semistable the degree $(-K_X)^n$ gets arbitrarily large in any given dimension $n$, for singular $X$ (see \[28\] Ex 4.2 for a simple two-dimensional toric example). The notion of K-stability first arose in the context of the Yau-Tian-Donaldson conjecture for Fano manifolds, saying that a Fano manifold admits a Kähler-Einstein metric if and only if it is K-polystable \[74, 31\]. The conjecture was settled in \[24\] and very recently also established for singular Fano varieties \[45, 51\]. From a purely algebro-geometric perspective K-stability can be viewed as a limiting form of Chow and Hilbert-Mumford stability \[67\], that enables a good theory of moduli spaces. Indeed, there exists a finite type Artin stack parametrizing all K-semistable Fano varieties of a given dimension and degree $(-K_X)^n$, which admits a morphism to a proper projective moduli spaces parametrizing the K-polystable Fano varieties (see the survey \[78\] and \[51\] where the properness was settled).

Is there an arithmetic analog of the inequality (1.1)? More precisely, it seems natural to ask if, under appropriate assumptions, the height $(-K_X)^{n+1}$ is bounded from above by the height $(-K_{\mathbb{P}^n})^{n+1}$ of the relative anti-canonical line bundle on the projective space $\mathbb{P}^n$ over the integers, endowed with its standard Kähler-Einstein metric (the Fubini-Study metric)? This would yield an explicit bound on the height $(-K_X)^{n+1}$, since the height of Fubini-Study metric on projective space was explicitly calculated in \[42, \S 5.4\], giving, after volume-normalization,

\[
(-K_{\mathbb{P}^n})^{n+1} = \frac{1}{2} (n+1)^n \left( (n+1) \sum_{k=1}^{n} k^{-1} - n + \log \left( \frac{n}{\pi n} \right) \right).
\]

If such a bound is to hold one needs, however, to impose a normalization condition on the metric on $-K_X$. Indeed, $\mathcal{L}^{n+1}$ is additively equivariant with respect to scalings of the metric. Accordingly, henceforth, the metric $\|\cdot\|$ on $-K_X$ will be assumed to be volume-normalized in the sense that the corresponding volume form on $X$ has total unit volume. As it turns out, the supremum of the height $(-K_X)^{n+1}$ over all volume-normalized metrics on $-K_X$ with positive curvature current is finite if and only if $X$ is K-semistable (Theorem \[2.4\]). It seems thus natural make the following conjecture (under appropriate assumptions on $X$):

**Conjecture 1.1.** Let $X$ be arithmetic variety whose relative anti-canonical line bundle defines a relatively ample $\mathbb{Q}$-line bundle over $X$. The following height inequality holds for any volume-normalized continuous metric on $-K_X$ with positive curvature current if $X$ is K-semistable:

\[
(-K_X)^{n+1} \leq \frac{(-K_{\mathbb{P}^n})^{n+1}}{2},
\]
where $-K_{\mathbb{P}^2}$ is endowed with the volume normalized Fubini-Study metric. Moreover, equality holds if and only if $X = \mathbb{P}^2$ and the metric is Kähler-Einstein, i.e. coincides with the Fubini-Study metric, modulo the action of an automorphism.

More generally, when $\mathbb{Z}$ is replaced by the ring of integers of a number field $F$, i.e. a finite field extension $F$ of $\mathbb{Q}$, the height $(-K_X)^{n+1}$ should be divided by the degree $[F : \mathbb{Q}]$. But, for simplicity, we will focus on the case when $F = \mathbb{Q}$ (see Section 2.1 for a very general formulation of the previous conjecture). The converse “only if” statement to the previous conjecture does hold, as a consequence of Theorem 2.4. Our main result concerns the case when $X$ is toric and $\mathcal{X}$ is its canonical toric integral model (see [54, Section 2] and [20, Def 3.5.6]):

**Theorem 1.2.** Let $X$ be an $n$–dimensional K-semistable toric Fano variety and denote by $\mathcal{X}$ its canonical integral model. Then the previous conjecture holds under anyone of the following conditions:

- $n \leq 6$ and $X$ is $\mathbb{Q}$–factorial (equivalently, $X$ is non-singular or has abelian quotient singularities)
- $X$ is not Gorenstein or has some abelian quotient singularity

Note when $n = 2$ any toric variety is, in fact, $\mathbb{Q}$–factorial. More generally, we will show that the curvature assumption may be dispensed with if the height $(-K_X)^{n+1}$ is replaced by the $\chi$–arithmetic volume $\hat{\text{vol}}_{\chi}(-K_X)$ of $-K_X$ (whose definition is recalled in Section 2.2.1). We expect that the maximum of $\hat{\text{vol}}_{\chi}(-K_X)$ over all integral models $(\mathcal{X}, -K_X)$ of a given toric Fano variety $(X, -K_X)$ is attained at the canonical model $X$ featuring in the previous theorem. In other words, we expect that the previous theorem implies the previous conjecture for any integral model $(\mathcal{X}, -K_X)$ of a given toric Fano variety $(X, -K_X)$. This expectation is inspired by a conjecture of Odaka discussed in Section 1.4 below.

The key ingredient in the proof of Theorem 1.2 is the following bound estimating the arithmetic volume $\hat{\text{vol}}_{\chi}(-K_X)$ of any volume-normalized metric on $-K_X$ in terms of the algebro-geometric volume $\text{vol}(X)$ (Prop 3.7):

$$(1.3) \quad \hat{\text{vol}}_{\chi}(-K_X) \leq -\frac{1}{2} \text{vol}(X) \log \left( \frac{\text{vol}(X)}{(2\pi^2)^n} \right) \text{vol}(X) := (-K_X)^n/n!$$

Since $\text{vol}(X)$ is maximal for $X = \mathbb{P}^n$ the right hand side above is bounded by a constant $C_n$ only depending on the dimension $n$. Under the “gap hypothesis” that $\mathbb{P}^{n-1} \times \mathbb{P}^1$ has the second largest volume among all $n$–dimensional K-semistable $X$ we show that the bound (1.3) implies Conjecture 1.1 for the canonical integral model $\mathcal{X}$ of a toric Fano variety $X$. The proof of Theorem 1.2 is concluded by verifying the gap hypothesis under the conditions in Theorem 1.2. But we do expect that the gap hypothesis above holds for any toric Fano variety (see Section 3.2.1).

In Section 4 we extend the previous theorem to the setting of log Fano pairs $(X, \Delta)$ up to relative dimension $n \leq 3$.

1.2. **The height of toric Kähler-Einstein metrics.** In the toric case, $X$ is K-semistable if and only if it admits a toric Kähler-Einstein metric [76, 6], i.e. a toric continuous metric on $-K_X$ whose curvature form defines a Kähler metric with constant positive Ricci curvature on the regular locus of $X$. Moreover, in general, any volume-normalized Kähler-Einstein metric maximizes $(-K_X)^{n+1}$ and thus the inequality in the previous theorem is equivalent to the corresponding inequality for
the volume-normalized toric Kähler-Einstein metric on \(-K_X\). The special role of the Kähler-Einstein condition in arithmetic (Arakelov) geometry, as an analog of the minimality of \(X\) over Spec \(\mathbb{Z}\), was emphasized already in the early days of Arakelov geometry by Manin [55]. However, it is rare that the Kähler-Einstein metric or the corresponding height \((-K_X)^{n+1}\) can be explicitly computed. In fact, in the toric case this seems to be the case only when \(X\) is a product of projective spaces, while there is an infinite number of \(K\)-semistable \(\mathbb{Q}\)-factorial toric Fano varieties in any given dimension \(n\) (see, for instance, the family of examples in Example 3.1).

The following complement to the general upper bound 1.3 yields a rather precise control on the height \((-K_X)^{n+1}\), with respect to a Kähler-Einstein metric.

**Theorem 1.3.** Let \(X\) be an \(n\)-dimensional toric Fano variety and denote by \(X\) its canonical integral model. Then the height \((-K_X)^{n+1}\) of any volume-normalized Kähler-Einstein metric satisfies

\[
\frac{(n+1)!}{2} \log \left( \frac{n!m_n \pi^n}{\text{vol}(X)} \right) \leq (-K_X)^{n+1} \leq \frac{(n+1)!}{2} \log \left( \frac{(2\pi)^n \pi^n}{\text{vol}(X)} \right)
\]

where \(m_n\) denotes the largest lower bound on the Mahler volume of a convex body. As a consequence, \((-K_X)^{n+1} > 0\).

The constant \(m_n\) in the previous theorem is the largest constant satisfying

\[
m_n \leq \text{vol}(P) \text{vol}(P^*),
\]

where \(P^*\) denotes the polar dual of any given convex body \(P\) containing the origin in its interior (the role of \(P\) in the present setting is played by the moment polytope of \(X\)). According to Mahler’s conjecture, the constant \(m_n\) is equal to \((n+1)^{n+1}/(n!)^2\) (which is realized for a simplex \(P\)). The case \(n = 2\) was settled in [53], but for our purposes the following general bound from [41] will be enough:

\[
m_n \geq \left( \frac{\pi}{2e} \right)^{n-1} (n+1)^{n+1}/(n!)^2,
\]

which implies the strict positivity of \((-K_X)^{n+1}\). In the light of this positivity it seems natural to ask if any \(K\)-semistable Fano variety \(X\) defined over \(\mathbb{Q}\) admits an integral model \(X\) and a volume-normalized metric on \(-K_X\) with positive curvature such that \((-K_X)^{n+1} > 0\)?

Before continuing we note that combining the previous theorem with the upper bound 1.1 yields the following universal bounds:

**Corollary 1.4.** Let \(X\) be an \(n\)-dimensional toric Fano variety and denote by \(X\) its canonical integral model. Then the height \((-K_X)^{n+1}\) of any volume-normalized Kähler-Einstein metrics satisfies the following universal bounds

\[
0 < (-K_X)^{n+1} \leq n(n+1)^{n+1}/(2! \log \left( \frac{2\pi^2 n!}{n+1} \right))
\]

Incidentally, the upper bound above is related to a question posed in [59], asking whether \((-K_X)^{n+1}\) is bounded from above by a universal constant \(C_n\), under the assumption that \(X\) be non-singular and \(-K_X\) be relatively ample. This is a stronger
1.3. Donaldson’s toric invariant. Let now \((X, L)\) be a polarized complex projective manifold. A prominent role in Kähler geometry is played by Mabuchi’s K-energy functional \(\mathcal{M}^{[22]}\), defined on the space \(\mathcal{H}(X, L)\) of all smooth metrics \(\|\cdot\|\) on \(L\) with positive curvature. Its critical points are the metrics whose curvature form \(\omega\) define a Kähler metric on \(X\) with constant scalar curvature. The precise definition of \(\mathcal{M}\) is recalled in Section 5.1. Since the definition of \(\mathcal{M}\) only involves its differential, the functional \(\mathcal{M}\) is only defined up to addition by a real constant. However, when \((X, L)\) is toric Donaldson \([31]\) exploited the toric structure to define the Mabuchi functional \(\mathcal{M}\) as a canonical functional on toric metrics:

\[
\mathcal{M}_L := \int_{\partial P} u d\sigma - a \int_P u dx - \int_P \log \det(\nabla^2 u) dx, \quad a := \int_{\partial P} d\sigma / \int_P dx
\]

where \(P\) is the moment polytope in \(\mathbb{R}^n\) corresponding to the polarized toric manifold \((X, L)\), whose boundary \(\partial P\) comes with a measure \(d\sigma\) induced by Lebesgue measure \(dx\) on \(\mathbb{R}^n\) and \(u\) is the smooth bounded convex function on \(P\) corresponding to a toric metric on \(L\) under Legendre transformation (see Section 3.1.2). In particular, in the last section of \([31]\) Donaldson introduced an invariant of a polarized toric manifold \((X, L)\), defined as the infimum of the toric Mabuchi functional \(\mathcal{M}_L\) defined by formula \((1.4)\) Here we show that Theorem 1.2 implies that when \(X\) is a Fano variety and \(L = -K_X\) a slight perturbation of Donaldson’s invariant is minimal when \(X\) is complex projective space, under the conditions on \(X\) appearing in Theorem 1.2.

**Theorem 1.5.** Let \(X\) be a K-semistable toric Fano variety of dimension \(n\), satisfying the conditions in Theorem 1.2. Then the invariant

\[
X \mapsto \inf_{\mathcal{H}(X, -K_X)} \mathcal{M}_{-K_X} - \frac{(-K_X)^n}{n!} \log \left( \frac{(-K_X)^n}{n!} \right)
\]

is minimal when \(X = \mathbb{P}^n\) (and only then), where the inf is attained at the metric on \(-K_{\mathbb{P}^n}\) induced by the Fubini-Study metric.

In the previous theorem the Fano variety \(X\) is allowed to be singular. The Mabuchi functional for singular general Fano varieties was introduced in \([30, 44]\) and Donaldson’s formula \([14]\) was extended to singular toric Fano varieties in \([6]\). In general, for Fano varieties the Mabuchi functional \(\mathcal{M}\) is bounded from below iff \(X\) is K-semistable \([44]\) (see the discussion following Theorem 2.4).

1.4. The arithmetic K-energy and Odaka’s modular height. For a general polarized manifold \((X, L)\) the infimum of \(\mathcal{M}\) is not canonically defined (since \(\mathcal{M}\) is only defined up to addition by a constant). But to any given integral model \((\mathcal{X}, \mathcal{L})\) of a polarized complex variety \((X, L)\) one may, as shown by Odaka \([63]\), attach a particular Mabuchi functional \(\mathcal{M}_{(\mathcal{X}, \mathcal{L})}\) which (up to a multiplicative normalization) is given as the following sum of arithmetic intersection numbers:

\[
\frac{1}{2} \mathcal{M}_{(\mathcal{X}, \mathcal{L})}(\mathcal{L}) := \frac{a}{(n + 1)!} \mathcal{L}^{n+1} - \frac{1}{n!} (-K_X) \cdot \mathcal{L}^n, \quad a = -n(K_X \cdot L^{n-1})/L^n
\]
where, as in the previous section, $\hat{L}$ denotes the metrized line bundle $(L, \|\cdot\|)$. In the definition of the second arithmetic intersection number above one also needs to endow $-K_X$ with a metric and one is confronted with two different natural choices: either the metric induced by the volume form $\omega^n/n!$ of the Kähler metric $\omega$ defined by the curvature form of $(L, \|\cdot\|)$ or the normalized volume form $\omega^n/L^n$ (which has unit total volume). The first choice is the one adopted in [63] and we show that when $X$ is a toric Fano variety and $(X, L)$ is the canonical integral model of $(X, L)$ this choice coincides with Donaldson’s one (formula (1.3)). However, for our purposes the second volume-normalized choice turns out to be the appropriate one, as it yields the shift by the logarithm of $(−K_X)^n$ appearing in Theorem 1.6.

\[
\mathcal{M}(X, -K_X) = M_{-K_X} \cdot \frac{(-K_X)^n}{n!} \log \left( \frac{(-K_X)^n}{n!} \right) 
\]

(Prop 6.2). The point is that with this choice the following formula holds in the arithmetic setting:

\[
\sup \left( \frac{(-K_X)^n}{(n+1)!} \right) = -\inf_{\mathcal{H}(X, -K_X)} \mathcal{M}(X, -K_X)
\]

where the sup ranges over all volume-normalized metrics in $\mathcal{H}(X, -K_X)$ (see Prop 6.3). As a consequence, Conjecture 1.1 is equivalent to the inequality

\[
\inf_{\mathcal{H}(X, -K_X)} \mathcal{M}(X, -K_X) \geq \inf_{\mathcal{H}(p^n, -K_{p^n})} \mathcal{M}(p^n, \ldots).
\]

Theorem 1.5 thus follows from Theorem 1.2.

### 1.4.1. Odaka’s modular height.

Let $(X_F, L_F)$ be an $n$-dimensional polarized variety defined over a number field $F$. In [63] Odaka introduced the following invariant of $(X_F, L_F)$, dubbed the intrinsic $K$-modular height of $(X_F, L_F)$:

\[
h(X_F, L_F) = \inf_{(\mathcal{X}, \mathcal{L})} \mathcal{M}_{(\mathcal{X}, \mathcal{L})},
\]

where $(\mathcal{X}, \mathcal{L})$ is a model of $(X_F, L_F)$ over the rings of integers $\mathcal{O}_{F'}$ of a finite field extension $F'$ of $F$ and $\mathcal{M}_{(\mathcal{X}, \mathcal{L})}$ now denotes the arithmetic $K$-energy (1.5) divided by the degree $[F' : \mathbb{Q}]$. In contrast to [63], we will employ the volume-normalized metric on $-K_X$ in the definition of $\mathcal{M}_{(\mathcal{X}, \mathcal{L})}$, discussed in the previous section. As shown in [15] for a polarized abelian variety $(X_F, L_F)$, Odaka’s modular height $h(X_F, L_F)$ essentially coincides with Faltings’ modular height of $(X_K, L_K)$ ([33] (the infimum over $(\mathcal{X}, \mathcal{L})$ is attained at the Neron model). Furthermore, as explained in [63], $h(X_F, L_F)$ can be viewed as a “large rank limit” of Bost’s and Zhang’s intrinsic heights appearing in [15] [10] [80], where the role of $K$-semistability is played by Chow semistability (see formula (1.7)). We propose the following

**Conjecture 1.6.** Let $X_{\mathbb{Q}}$ be a Fano variety defined over $\mathbb{Q}$. Then Odaka’s modular invariant $h(X_{\mathbb{Q}}, -K_{X_{\mathbb{Q}}})$, normalized as above, is minimal when $X_{\mathbb{Q}} = \mathbb{P}^n_{\mathbb{Q}}$. 

According to a conjecture of Odaka [64] any globally $K$-semistable integral model $(\mathcal{X}, -K_\mathcal{X})$ of $(X, -K_X)$ minimizes $\mathcal{M}_{(\mathcal{X}, \mathcal{L})}$ over all models $(\mathcal{X}, \mathcal{L})$ (the function field analog of this minimization property is established in [14]). Global $K$-semistability means that all the fibers of $\mathcal{X} \to \text{Spec} \mathcal{O}_F$ are $K$-semistable. In other words, in addition to the $K$-semistability of the generic fiber $X_F$ this means that the variety $X_{F_{\mathbb{Q}}}$ over the finite field $F_{\mathbb{Q}}$, corresponding to the integral model $\mathcal{X}$, is $K$-semistable.
for any prime $p$. For example, as pointed out to us by Odaka the canonical model $X$ of a K-semistable toric Fano variety $X_Q$ appearing in Theorem 1.2 is globally K-semistable. Thus if Odaka’s minimization conjecture holds, then Theorem 1.2 implies Conjecture 1.6 for any toric Fano variety $X_Q$ satisfying the conditions in Theorem 1.2. Anyhow, the positivity statement in Theorem 1.3 implies that the modular invariant $h(X_Q, -K_{X_Q})$ is negative for any toric Fano variety $X_Q$.

1.5. Organization. In Section 2 we start by recalling the complex-geometric and arithmetic setup before proving Theorem 2.4, relating upper bounds on the height of Fano varieties to K-semistability. The proof leverages an arithmetic analog of the Ding functional. In Section 3 we specialize to the toric situation and prove the sharp height inequality in Theorem 1.2 stated in the introduction and the height bounds for Kähler-Einstein metrics in Theorem 1.3. In Section 4 the more general setting of log Fano pairs is considered. We then go on, in Section 5, to deduce Theorem 1.6 concerning the sharp lower bounds on Donaldson’s toric Mabuchi functional. In Section 6 Donaldson’s functional is related to Odaka’s arithmetic Mabuchi functional, which, in turn is related to the arithmetic Ding functional. In the last section we make a comparison with the function field case and speculate on a generalization of Conjecture 1.1, comparing with previous work of Bost and Zhang.

We have made an effort to make the paper readable for the reader with a back- ground in arithmetic geometry, as well as for the complex geometrists, by including most of the background material needed for the proofs of the main results.

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2. Heights, arithmetic volumes and K-stability of Fano varieties

In this section we show, in particular, that the height of a polarized integral model $(X, \mathcal{L})$ of a Fano manifold $(X, -K_X)$ is bounded from above - as the metric on $\mathcal{L}$ ranges over all volume-normalized metrics with positive curvature current - if and only if $(X, -K_X)$ is K-semistable (Theorem 2.4). See also [53] for further connections between K-stability of polarized varieties $(X, L)$ and arithmetic geometry. The main new feature here, compared to [53], is that we leverage an arithmetic version of the Ding functional in Kähler geometry, while [53] considers an arithmetic version of the Mabuchi functional (the two functionals are compared in Section 6).

2.1. Complex geometric setup. Throughout the paper $X$ will denote a compact connected complex normal variety, assumed to be $\mathbb{Q}$--Gorenstein. This means that the canonical divisor $K_X$ on $X$ is defined as a $\mathbb{Q}$--line bundle: there exists some positive integer $m$ and a line bundle on $X$ whose restriction to the regular locus $X_{reg}$ of $X$ coincides with the $m$-th tensor power of $K_{X_{reg}}$, i.e. the top exterior power of the cotangent bundle of $X_{reg}$. We will use additive notation for tensor powers of line bundles.
2.1.1. Metrics on line bundles. Let \((X, L)\) be a polarized complex projective variety i.e. a complex normal variety \(X\) endowed with an ample line bundle \(L\). We will use additive notation for metrics on \(L\). This means that we identify a continuous Hermitian metric \(\|\cdot\|\) on \(L\) with a collection of continuous local functions \(\phi_U\) associated to a given covering of \(X\) by open subsets \(U\) and trivializing holomorphic sections \(e_U\) of \(L \to U\):

\[
\phi_U := -\log(\|e_U\|^2),
\]

which defines a function on \(U\). Of course, the functions \(\phi_U\) on \(U\) do not glue to define a global function on \(X\), but the current \(dd^c \phi_U := i/2\pi \partial \bar{\partial} \phi_U\) is globally well-defined and coincides with the normalized curvature current of \(\|\cdot\|\) (the normalization ensures that the corresponding cohomology class represents the first Chern class \(c_1(L)\) of \(L\) in the integral lattice of \(H^2(X, \mathbb{R})\)). Accordingly, as is customary, we will symbolically denote by \(\phi\) a given continuous Hermitian metric on \(L\) and by \(dd^c \phi\) its curvature current. The space of all continuous metrics \(\phi\) on \(L\) will be denoted by \(C^0(L)\). We will denote by \(C^0(L) \cap \text{PSH}(L)\) the space of all continuous metrics on \(L\) whose curvature current is positive, \(dd^c \phi \geq 0\) (which means that \(\phi_U\) is plurisubharmonic, or psh, for short). Then the exterior powers of \(dd^c \phi\) are defined using the local pluripotential theory of Bedford-Taylor [9]. The volume of an ample line bundle \(L\) may be defined by

\[
\text{vol}(L) := \lim_{k \to \infty} k^{-n} \dim H^0(X, L^\otimes k) = \frac{1}{n!} L^n = \frac{1}{n!} \int_X (dd^c \phi)^n
\]

using in the second equality the Hilbert-Samuel theorem and where \(\phi\) denotes any element in \(C^0(L) \cap \text{PSH}(L)\).

More generally, metrics \(\phi\) are defined for a \(\mathbb{Q}\)-line bundle \(L\) : if \(mL\) is a bona fide line bundle, for \(m \in \mathbb{Z}^+\), then \(m\phi\) is a bona fide metric on \(mL\).

Remark 2.1. The normalization of \(\phi_U\) used here coincides with the one in [5, 6], but it is twice the one employed in [9].

2.1.2. Metrics on \(-K_X\) vs volume forms on \(X\). First consider the case when \(X\) is smooth. Then any smooth metric \(\|\cdot\|\) on \(-K_X\) corresponds to a volume form on \(X\), defined as follows. Given local holomorphic coordinates \(z\) on \(U \subset X\) denote by \(e_U\) the corresponding trivialization of \(-K_X\), i.e. \(e_U = \partial/\partial z_1 \wedge \cdots \wedge \partial/\partial z_n\). The metric on \(-K_X\) induces, as in the previous section, a function \(\phi_U\) on \(U\) and the volume form in question is locally defined by

\[
e^{-\phi_U} \left( i/2 \right)^n dz \wedge d\bar{z}, \quad dz := dz_1 \wedge \cdots \wedge dz_n,
\]

on \(U\), which glues to define a global volume form on \(X\). In other words, \(e^{-\phi_U}\) is the density of the volume form with respect to the local Euclidean volume form. Accordingly, we will simply denote the volume form in question by \(e^{-\phi}\), abusing notation slightly. Conversely, if \(dV\) is a volume form on \(X\) the corresponding metric on \(X\) may be expressed as

\[
\|\alpha\|^2 = \frac{dV}{(i/2)^n \alpha^* \wedge \alpha^*}
\]
2.2.2. **Remark**

The Donaldson-Futaki invariant $DF(\mathcal{X}, \mathcal{L})$ is defined as the restriction to the smooth locus of the Donaldson-Futaki invariant of a continuous metric on $\mathcal{X}$. When $\mathcal{X}$ is singular, we may identify the fiber of $\mathcal{X}$ over $1 \in \mathbb{C}$ with the smooth locus of $\mathcal{X}$ equipped with a continuous metric.

For any local holomorphic section $\alpha$ of $-K_X$, where $\alpha^*$ denotes the corresponding dual section of $K_X$, identified with a local holomorphic top form on $X$. More generally, a continuous metric on $-K_X$ corresponds to a continuous volume form (i.e. a measure with strictly positive continuous local densities).

When $X$ is singular any continuous metric $\phi$ on $-K_X$ induces a measure on $X$, symbolically denoted by $e^{-\phi}$, defined as before on the regular locus $X_{\text{reg}}$ of $X$ and then extended by zero to all of $X$. We will say that a measure $d\nu$ on $X$ is a continuous volume form if it corresponds to a continuous metric on $-K_X$. A Fano variety has log terminal singularities if and only if it admits a continuous volume form $d\nu$ with finite total volume [7, Section 3.1].

2.1.3. **K-semistability.** We briefly recall the notion of K-semistability (see [31, 67, 77, 61] for more background). A polarized complex projective variety $(X, L)$ is said to be K-semistable if, for example, the Donaldson-Futaki invariant $DF(\mathcal{X}, \mathcal{L})$ of any test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, L)$ is non-negative. A test configuration $(\mathcal{X}, \mathcal{L})$ is defined as a $\mathbb{C}^*$-equivariant normal model for $(X, L)$ over the complex affine line $\mathbb{C}$. More precisely, $\mathcal{X}$ is a normal complex variety endowed with a $\mathbb{C}^*$-action $\rho$, a $\mathbb{C}^*$-equivariant holomorphic projection $\pi$ to $\mathbb{C}$ and a relatively ample $\mathbb{C}^*$-equivariant $\mathbb{Q}$-line bundle $\mathcal{L}$ (endowed with a lift of $\rho$):

$$\pi: \mathcal{X} \rightarrow \mathbb{C}, \quad \mathcal{L} \rightarrow \mathcal{X}, \quad \rho: \mathcal{X} \times \mathbb{C}^* \rightarrow \mathcal{X}$$

such that the fiber of $\mathcal{X}$ over $1 \in \mathbb{C}$ is equal to $(X, L)$. Its Donaldson-Futaki invariant $DF(\mathcal{X}, \mathcal{L}) \in \mathbb{R}$ may be defined as a normalized limit, as $k \rightarrow \infty$, of Chow weights of a sequence of one-parameter subgroups of $GL(H^0(X, kL))$ induced by $(\mathcal{X}, \mathcal{L})$ (in the sense of Geometric Invariant Theory). As a consequence, $(X, L)$ is K-semistable if, for example, $(X, kL)$ is Chow semi-stable, for $k$ sufficiently large [67]. However, for the purpose of the present paper it will be more convenient to employ the intersection-theoretic formula for $DF(\mathcal{X}, \mathcal{L})$ established in [77, 61]:

$$DF(\mathcal{X}, \mathcal{L}) = \frac{a}{(n+1)!} \overline{\mathcal{L}}^{n+1} + \frac{1}{n!} \mathcal{K}_{\mathcal{X}/\mathbb{P}^1} \cdot \overline{\mathcal{L}}^{n}, \quad a = -n(K_X \cdot L^{n-1})/L^n$$

where $\overline{\mathcal{L}}$ denotes the $\mathbb{C}^*$-equivariant extension of $\mathcal{L}$ to the $\mathbb{C}^*$-equivariant compactification $\overline{\mathcal{X}}$ of $\mathcal{X}$ over $\mathbb{P}^1$ and $\mathcal{K}_{\mathcal{X}/\mathbb{P}^1}$ denotes the relative canonical divisor.

**Remark 2.2.** Usually the definition of $DF(\mathcal{X}, \mathcal{L})$ involves a factor of $1/L^n$, but the present definition will be more convenient here (since the factor $L^n$ is positive it does not alter the definition of K-stability). It is made so that $DF(\mathcal{X}, \mathcal{L}) = \overline{\mathcal{L}}^{n+1}$ when $\mathcal{L} = -\mathcal{K}_{\mathcal{X}/\mathbb{P}^1}$.

2.2. **Arithmetic setup.** Let $(\mathcal{X}, \mathcal{L})$ be a polarized arithmetic variety i.e. a normal projective flat scheme $\mathcal{X} \rightarrow \text{Spec} \ Z$ of relative dimension $n$, endowed with a relatively ample line bundle $\mathcal{L}$. We will denote by $X$ the $n$-dimensional complex projective algebraic variety consisting of the complex points of $\mathcal{X}$, which is assumed to be connected and normal and by $L$ the ample line bundle over $X$ induced by $\mathcal{L}$. This means that $L$ is the restriction to the the complexification $X$ of the generic fiber $X_{\mathbb{Q}}$ of the fibration $\mathcal{X} \rightarrow \text{Spec} \ Z$. We will then say that $(\mathcal{X}, \mathcal{L})$ is a model for $(X, L)$ (abusing terminology slightly; strictly speaking $(\mathcal{X}, \mathcal{L})$ is a model for $(X_{\mathbb{Q}}, L_{\mathbb{Q}})$ over $\text{Spec} \ Z$). For any positive integer $k$ we may identify the $\mathbb{Z}$–module...
\[ H^0(\mathcal{X}, k\mathcal{L}) \text{ with a lattice in } H^0(\mathcal{X}, k\mathcal{L}): \]
\[ H^0(\mathcal{X}, k\mathcal{L}) \otimes \mathbb{C} = H^0(\mathcal{X}, k\mathcal{L}). \]

By definition a \textit{metrized line bundle} \( \mathcal{L}^{\varphi} \) is a line bundle \( \mathcal{L} \rightarrow \mathcal{X} \) such that the corresponding line bundle \( \mathcal{L} \rightarrow \mathcal{X} \) is endowed with a metric \( \| \cdot \| \). We will use the additive notation \( \phi \) for metrics \( \| \cdot \| \) on \( L \) discussed in the previous section:

\[ \mathcal{L} := (\mathcal{L}, \phi) \]

\[ \text{2.2.1. The } \chi-\text{arithmetic volume, heights and arithmetic intersection numbers.} \]

In the arithmetic setup there are different analogs of the volume \( \text{vol}(L) \) of an ample line bundle \( L \). Here we shall focus on the one defined by the following asymptotic arithmetic Euler characteristic originating in [34] (called the \( \chi-\text{arithmetic volume} \) [20, 21] and the \textit{sectional capacity} in [69]):

\[ \hat{v}_{\chi}(\mathcal{L}) := \lim_{k \rightarrow \infty} k^{-(n+1)} \chi \mathcal{L}^2(\mathcal{L}), \quad \chi \mathcal{L}^2(\mathcal{L}) := -\log \sqrt{\det_{i,j \leq N} \langle s_i, s_j \rangle_{\mathcal{L}}} \]

where \( s_1, \ldots, s_{N_k} \) denote any \( N_k \) generators of the integral lattice \( H^0(\mathcal{X}, k\mathcal{L}) \) in the \( N_k \)-dimensional real vector space \( H^0(\mathcal{X}, k\mathcal{L}) \otimes \mathbb{R} \subset H^0(\mathcal{X}, k\mathcal{L}) \) and \( \langle s_i, s_j \rangle_{\mathcal{L}} \) denotes the Hermitian product on \( H^0(\mathcal{X}, k\mathcal{L}) \) induced from the metric \( \phi \) on \( L \) and a fixed measure \( dV \) on \( X \) satisfying the following property:

\[ \sup_X \| s_k \|^2 \leq C e^{k/C} \int_X \| s_k \|^2 dV, \quad \forall s_k \in H^0(\mathcal{X}, k\mathcal{L}) \]

for some positive constant \( C \). This property is known as the \textit{Bernstein-Markov property} in pluripotential theory [9] and generalizes Gromov’s inequality in Arakelov geometry [43]. It ensures that the limit \( \hat{v}_{\chi}(\mathcal{L}) \) is independent of the choice of \( dV \). Indeed, it ensures that \( \chi \mathcal{L}^2(\mathcal{L}) \) may as well be replaced by

\[ \chi \mathcal{L}^2(\mathcal{L}) := \log \text{Vol} \left\{ s_k \in H^0(\mathcal{X}, k\mathcal{L}) \otimes \mathbb{R} : \sup_X \| s_k \|_{\mathcal{L}} \leq 1 \right\}, \]

expressed in terms of the unique Haar measure \( \text{Vol} \) on \( H^0(\mathcal{X}, \mathcal{L}^{\otimes k}) \otimes \mathbb{R} \) which gives unit-volume to a fundamental domain of the lattice \( H^0(\mathcal{X}, \mathcal{L}^{\otimes k}) \) (see [9, Lemma 2.5]). If the metric on \( L \) has positive curvature current, then, by the arithmetic Hilbert-Samuel theorem [23, 79],

\[ \hat{v}_{\chi}(\mathcal{L}) = \frac{\mathcal{L}^{n+1}}{(n+1)!}, \]

where \( \mathcal{L}^{n+1} \) denotes the top arithmetic intersection number in the sense of Gillet-Soulé [42], which defines the \textit{height} of \( \mathcal{X} \) with respect to \( \mathcal{L} \) [35, 17]. For the purpose of the present paper formula \( (\ref{eq:2.8}) \) may be taken as the definition of \( \mathcal{L}^{n+1} \) (arithmetic intersections between general \( n+1 \) metrized line bundles could then be defined by polarization). More generally, \( \hat{v}_{\chi}(\mathcal{L}) \) is naturally defined for \( \mathbb{Q} \)-line bundles, since it is homogeneous with respect to tensor products of \( \mathcal{L} \):

\[ \hat{v}_{\chi}(m\mathcal{L}) = m^{n+1} \hat{v}_{\chi}(\mathcal{L}), \quad \text{if } m \in \mathbb{Z}_+. \]

Moreover, \( \hat{v}_{\chi}(\mathcal{L}) \) is additively equivariant with respect to scalings of the metric:

\[ \hat{v}_{\chi}(\mathcal{L}, \phi + c) = \hat{v}_{\chi}(\mathcal{L}) + \frac{c}{2} \text{vol}(L), \quad \text{if } c \in \mathbb{R}, \]
Theorem 2.4. K-semistability: bounds on the $\chi$ψ. In the toric case this implies that the Riemann-Roch theorem. This analogy is reinforced when the degree of the determinant of the cohomology, appearing in the Grothendieck-Riemann-Roch theorem [34] [35]. However, here it will be important to employ the $L^2$-metric, when the metric on $L$ is not assumed to have positive curvature current. In the complex-geometric framework functionals of the form [2.5] usually called Donaldson functionals, were introduced in [32], with respect to a given bases in $H^0(X, kL)$ (see also the more general setup in [9] where $dV$ is taken to be any measure satisfying the Bernstein-Markov property).

2.3. Upper bounds on the $\chi$–arithmetic volume vs K-semistability of Fano varieties. We are now ready to prove the following theorem, relating upper bounds on the $\chi$–arithmetic volume of a metrized integral model of $(X, -K_X)$ to K-semistability:

Theorem 2.4. Let $(X, \mathcal{L})$ be a polarized arithmetic variety such that $X$ is a Fano variety and $L = -K_X$. Then the following is equivalent:

1. $(X, -K_X)$ is K-semistable
2. The supremum of $\psi_{\mathcal{L}}(\mathcal{L}, \phi)$ over all continuous volume-normalized metrics $\phi$ on $-K_X$ is finite.
3. The supremum of $\psi_{\mathcal{L}}(\mathcal{L}, \phi)$ over all continuous volume-normalized metrics $\phi$ on $-K_X$, which are invariant under complex conjugation, is finite.

Before embarking on the proof we recall the definition of the Ding functional on $C^0(-K_X) \oplus \operatorname{PSH}(-K_X)$, introduced in [29], which depends on the choice of a reference metric $\psi_0$ in $C^0(-K_X) \oplus \operatorname{PSH}(-K_X)$:

$$
\mathcal{D}_{\psi_0}(\psi) := -\frac{1}{\det(-K_X)} \mathcal{E}_{\psi_0}(\psi) - \log \int_X e^{-\psi},
$$

(2.11)

where the functional $\mathcal{E}_{\psi_0}$ is a primitive of $(dd^c \psi)^n/n!$ (see formula 2.14). More generally, as shown in [14] $\mathcal{D}_{\psi_0}(\psi)$ can be extended to the space $\mathcal{E}^1(-K_X)$ of all metrics in $\operatorname{PSH}(-K_X)$ with finite energy and a finite energy metric $\psi$ minimizes $\mathcal{D}_{\psi_0}(\psi)$ iff $\psi$ is a Kähler-Einstein metric, i.e. $dd^c \psi$ defines a Kähler metric on the regular locus of $X$ with constant positive Ricci curvature. When $\psi$ is volume-normalized this equivalently means that

$$
\frac{(dd^c \psi)^n}{\det(-K_X)n!} = e^{-\psi}
$$
on the regular locus of $X$. The identity 2.8 was extended to finite energy metrics in [10]. But for our purposes it will be enough to work with continuous metrics.

Remark 2.5. In general, any Kähler-Einstein metric $\psi$ in $\mathcal{E}^1(-K_X)$ is locally bounded [7]. In the toric case this implies that $\psi$ is, in fact, continuous [26 Prop 4.1].

By introducing an arithmetic version of the Ding functional we show that item 2 in the previous theorem is equivalent to the the Ding functional $\mathcal{D}_{\psi_0}$ being bounded from below on $C^0(-K_X) \oplus \operatorname{PSH}(-K_X)$ (which is equivalent to lower boundedness of the Mabuchi functional; see [5,7]). By [41] this is equivalent to K-semistability as follows directly from the definition.

Remark 2.3. $\chi_{L^2}(k\mathcal{L})$ is the arithmetic degree $\deg \pi_*(k\mathcal{L})$ of the vector bundle $\pi_*(k\mathcal{L}) \to \operatorname{Spec}(\mathbb{Z})$, wrt the $L^2$-metric induced by $(\phi, dV)$. It is thus analogous to degree of the determinant of the cohomology, appearing in the Grothendieck-Riemann-Roch theorem. This analogy is reinforced when the $L^2$-metric is replaced by the Quillen metric, whose arithmetic degree is given by the arithmetic Riemann-Roch theorem [34] [35].
when $X$ is non-singular. In the proof of Theorem 2.4 we explain how to extend this result to general Fano varieties, leveraging the very recent solution of the Yau-Tian-Donaldson conjecture for singular Fano varieties [45, 51]. The equivalence with item 3 leverages the recent result [81].

**Remark 2.6.** The proof will also reveal that $(X, -K_X)$ is K-polystable iff the supremum in item 2 above is attained at some metric $\psi$ in $PSH(-K_X)$. In general, any such a maximizer is a Kähler-Einstein metric.

2.3.1. **Proof of Theorem 2.4.** We start with two lemmas. First, to a given continuous metric $\phi$ on $L$ we associate, following [9], a continuous psh metric $\psi$ on $L$ defined as the following point-wise envelope:

$$P\phi := \sup \{ \psi : \psi \text{psh, } \psi \leq \phi \}.$$  

**Remark 2.7.** More generally, when $L$ is big the envelope above has to be replaced by its upper semi-continuous regularization in order to obtain a psh metric. However, when $L$ is an ample line bundle over a normal variety $X$, as we assume here, the envelope $P\phi$ is already continuous (see [19, Lemma 7.9]).

**Lemma 2.8.** Assume that $L$ is relatively ample. Let $\phi$ be a continuous metric on $L$ with positive curvature current and $dV$ a measure on $X$ satisfying the Bernstein-Markov property 2.6. Then the arithmetic $\chi$-volume may be expressed as the following top arithmetic intersection number:

$$\hat{\text{vol}}_\chi (L, \phi) = \frac{(L, P\phi)^{n+1}}{(n+1)!}.$$  

**Proof.** First recall that the Bernstein-Markov property ensures that $\chi_{L^2(k\mathbb{Z})}$ appearing in the definition of $\hat{\text{vol}}_\chi$ may as well be replaced by $\chi_{L^\infty(k\mathbb{Z})}$, defined in formula 2.7 (see [9, Lemma 2.5]). Thus in the case when $\phi$ is psh the lemma follows from [79, Thm 1.4] (the latter proof reduces to the original arithmetic Hilbert-Samuel theorem in [43], where $X$ is assumed non-singular, using a perturbation argument on a resolution of $X$). In fact, the result [79, Thm 1.4] applies more generally when $L$ is merely assumed to be relatively nef over the closed points of $\text{Spec} \mathbb{Z}$. Next, the general case follows from the case when $\phi$ is psh (applied to $P\phi$) by the following simple observation:

$$\sup_X \|s\|_{\phi} = \sup_X \|s\|_{P\phi}, \text{ if } s \in H^0(X, kL),$$  

as follows directly from the definition (2.12) of $P\phi$ (see [9 Prop 1.8]).

In order to state the next lemma consider the following functional on $C^0(L) \cap PSH (L)$, defined with respect to a given reference $\psi_0 \in C^0(L) \cap PSH (L)$:

$$E_{\psi_0}(\psi) := \frac{1}{(n+1)!} \int_X (\psi - \psi_0) \sum_{j=0}^n (dd^c\psi)^n - j \wedge (dd^c\psi_0)^n - j$$  

Alternatively, the functional $E_{\psi_0}$ may be characterized as the primitive of the one-form on $C^0(L) \cap PSH (L)$ defined by the measure $(dd^c\psi)^n/n!$:

$$dE_{\psi_0}(\psi) = \frac{1}{n!} (dd^c\psi)^n, \quad E_{\psi_0}(\psi_0) = 0.$$  

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It follows directly from the definition of $E_{\psi_0}(\psi)$ and the classical Hilbert-Samuel formula \[2.2\] that

\[
E_{\psi_0}(\psi + c) = E_{\psi_0}(\psi) + c vol(L), \forall c \in \mathbb{R}.
\]

The following lemma is an arithmetic refinement of the previous formula:

**Lemma 2.9.** (change of metrics formula). For any two continuous metrics on $L$, which are invariant under complex conjugation,

\[
(2.16) \quad \hat{vol}_\chi(L, \phi_1) - \hat{vol}_\chi(L, \phi_2) = \frac{1}{2} (E_{\psi_0}(P\phi_1) - E_{\psi_0}(P\phi_2)).
\]

**Proof.** When $\phi_i$ are psh this is well-known and follows from basic properties of arithmetic intersection numbers; see formula 6.1 or [63, Prop 2.2]. Alternatively, the result follows from the previous lemma combined with [9]. To see this first note that when calculating the difference appearing in the lhs in formula 2.16 the bases $s_1, \ldots, s_N$ may be taken to be any basis in $H^0(X, kL)$. In particular, taking a basis which is orthonormal wrt an $L^2$-metric defined by the metric $\psi_0$ on $L$ and applying [9, Thm A (ii)] to the measure $dV$ yields the desired formula. In order to check that the multiplicative normalizations adopted here are compatible note that the scaling relations 2.10 and 2.15 are indeed compatible. $\square$

2.3.2. **Conclusion of the proof of Theorem 2.4.** Consider the following functional on the space $C^0(-K_X)$ of continuous metrics on $-K_X$

\[
(2.17) \quad D_Z(\phi) := -2 \frac{\hat{vol}_\chi(L, \phi)}{vol(-K_X)} - \log \int_X e^{-\phi}.
\]

Since this functional is invariant under scalings of the metric, $\phi \mapsto \phi + c$, the finiteness statement in the second point of the proposition amounts to showing that the infimum of $D_Z(\phi)$ over $C^0(-K_X)$ is finite. Now fix a continuous psh metric $\psi_0$ on $-K_X$ and consider the following extension of the Ding functional 2.11 to all of $C^0(-K_X)$:

\[
(2.18) \quad D_{\psi_0}(\phi) := -\frac{1}{vol(-K_X)} E_{\psi_0}(P\phi) - \log \int_X e^{-\phi}.
\]

Combining the previous two lemmas reveals that

\[
(2.19) \quad D_Z(\phi) = D_{\psi_0}(\phi) + C_0, \quad C_0 := -\frac{2 (L, \psi_0)^{n+1}}{vol(-K_X)(n+1)!}.
\]

Next, observe that

\[
(2.20) \quad \inf_{C^0(-K_X)} D_{\psi_0} = \inf_{C^0(-K_X) \cap PSH(-K_X)} D_{\psi_0}
\]

Indeed, this follows directly from the fact that the operator $\phi \mapsto P\phi$ from $C^0(L)$ to $C^0(L) \cap PSH(L)$ is increasing and satisfies $P^2 = P$.

"3" $\Rightarrow$ "1". Let us first recall how Item 2 implies Item 1. First Item 2 implies, thanks to the identities 2.19 and 2.20 that the infimum of $D_{\psi_0}$ over $C^0(-K_X) \cap PSH(-K_X)$ is finite. Thus it follows from results in [5] that $(X, -K_X)$ is K-semistable. Let us next show how to refine the proof in [5] to show the stronger statement "3" $\Rightarrow$ "1". More generally, we will show that when $X$ is defined over the real field $\mathbb{R}$ $X$ is K-semistable if the infimum of $D_{\psi_0}$ over the space
$C^0(-K_X) \cap \text{PSH}(-K_X)$ is finite, where $C^0(L)$ denotes the subspace of $C^0(L)$ consisting of metrics which are invariant under complex conjugation. To this end let us first summarize the main steps in the proof in [5]. First, a test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, -K_X)$ and a given metric $\phi$ for $-K_X$ in $C^0(-K_X) \cap \text{PSH}(-K_X)$ determines a ray $\phi_t$ in $\text{PSH}(-K_X)$ emanating from $\phi$ parametrized by $t \in [0, \infty[$ (i.e. $\phi_0 = \phi$). Using the notation in formula 2.20 the ray $\phi_t$ is defined by

$$\phi_{-\log |t|} = \rho(t)^*(\Phi_{|\mathcal{X}_t}), \quad \tau \in \mathbb{C}^*$$

where $\Phi$ is the $S^1$-invariant metric on the restriction of $\mathcal{L}$ to the inverse image $\pi^{-1}(\mathbb{D})$ in $\mathcal{X}$ of the unit-disc $\mathbb{D} \subset \mathbb{C}$ defined by

$$\Phi := \sup \{ \Psi : \Psi_{|x \to \infty} = \phi, \quad \Psi \in C^0(\mathcal{L}) \cap \text{PSH}(\mathcal{L}^{-1}(\mathbb{D})) \},$$

where we have used the $\mathbb{C}^*$-action $\rho$ to identify $X$ with $X_{\tau}$ for any $\tau$ in the unit-circle $\partial \mathbb{D}$. By [5, Thm 1.3]

$$\text{DF} (\mathcal{X}, \mathcal{L}) \geq \lim_{t \to \infty} \left( t^{-1} \mathcal{D}_{\phi_0}(\phi_t) \right).$$

When $\mathcal{D}_{\phi_0}(\phi_t)$ is bounded from below this means that $\text{DF} (\mathcal{X}, \mathcal{L}) \geq 0$, showing that $X$ is K-semistable. Now assume that $X$ is defined over the real field $\mathbb{R}$. Then it follows from [81, Thm 1.1] that in order to check K-semistability of $(X, -K_X)$ it is enough to consider test configurations $(\mathcal{X}, \mathcal{L})$ defined over $\mathbb{R}$. Thus, we just have to verify that for such test configurations, if the given metric $\phi$ is taken to be in $C^0(-K_X) \cap \text{PSH}(-K_X)$, then the ray $\phi_t$ remains in $C^0(-K_X) \cap \text{PSH}(-K_X)$, for all $t > 0$. Since $(\mathcal{X}, \mathcal{L})$ is defined over $\mathbb{R}$ there is a complex conjugation map $F$ from $\mathcal{X}$ to $\mathcal{X}$ (that lifts to $\mathcal{L}$) and thus it is enough to show that $F^* \phi = \phi$ implies that $F^* \Phi = \Phi$. But this follows from the definition 2.21 of $\Phi$ only using that $F^*$ preserves the psh property of a metric (as follows from a direct local calculation that reduces to the fact that the Laplacian $\partial \bar{\partial} \bar{z}$ in $\mathbb{C}$ is invariant under $z \mapsto \bar{z}$).

"1" $\Rightarrow$ "2". First recall that any K-semistable normal Fano variety (i.e. such that $(X, -K_X)$ is K-semistable) has log terminal singularities [60, Thm 1.3]. In the case that $X$ is non-singular it was shown in [43] that if $X$ is K-semistable, then the infimum of the Ding functional $\mathcal{D}_{\psi_0}$ over $C^0(-K_X) \cap \text{PSH}(-K_X)$ is finite. Thus, by formula 2.21 so is the infimum of $\mathcal{D}_{\psi_0}$ over $C^0(-K_X)$. The proof in [43] relied, in particular, on the resolution of the Yau-Tian-Donaldson conjecture in [24] for Fano manifolds. But thanks to the recent resolution of the Yau-Tian-Donaldson conjecture for singular Fano varieties the proof in [43] can be extended to singular Fano varieties, mutatis mutandis. We briefly summarize the argument, using Deligne pairings as in [5] (rather than the Bott-Chern classes used in [43]). The starting point is the result [48, Thm 1.3], saying that if $X$ is K-semistable then there exists a test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, -K_X)$ whose central fiber $X_0$ is given by a K-polystable Fano variety. More precisely, the test configuration is special in the sense that $\mathcal{L}$ is the relative anti-canonical line bundle. Since the central fiber $X_0$ of $\mathcal{X}$ is K-polystable it admits, by the solution of the Yau-Tian-Donaldson conjecture for singular Fano varieties [51] (building on [15]) a Kähler-Einstein metric $\phi_{KE}$. It thus follows from [7, Thm 4.8] that the Ding functional is bounded from below on $C^0(-K_{X_0}) \cap \text{PSH}(-K_{X_0})$. More precisely, its infimum is attained at the Kähler-Einstein metric $\phi_{KE}:

$$\inf_{C^0(-K_{X_0}) \cap \text{PSH}(-K_{X_0})} \mathcal{D} = \mathcal{D}(\phi_{KE}) > -\infty.$$
Now, given a metric $\phi$ in $C^0(-K_X)\cap PSH(-K_X)$ let $\Phi$ be the corresponding metric on $L' \to \pi^{-1}(\mathbb{D})$ defined by formula 2.21. It induces a metric on the $(n+1)$-fold Deligne pairing $\langle L', L', ..., L' \rangle \to \mathbb{D}$ that we denote by $\langle \Phi \rangle$ (see [5, Section 2.3]). Consider the corresponding twisted metric on $L' \to \pi^{-1}(\mathbb{D})$ defined by

$$- \langle \Phi \rangle - \log \int_{X'} e^{-\Phi} \cdot dx,$$

dubbed the Ding metric in [5]. Fixing a trivialization $S(\tau)$ of $\langle L', L', ..., L' \rangle \to \mathbb{D}$ we may identify this metric with a function $\psi(\tau)$ on $\mathbb{D}$:

$$\psi(\tau) := \log \left( \| S(\tau) \|_{\langle \Phi \rangle}^2 \right) - \log \int_{X'} e^{-\Phi} \cdot dx,$$

For a fixed $\tau$ this metric coincides with the Ding functional $D(\phi_\tau)$ up to an additive constant depending on $\tau$ (by the “change of metrics formula” for Deligne pairing; see [5, Section 2.3]). In particular, there exists $a \in \mathbb{R}$ such that

$$\psi(1) := D_{\phi_\psi}(\phi) + a, \quad \psi(0) \geq b := \log \left( \| S(0) \|_{\langle \phi_{K,E} \rangle}^2 \right) - \log \int_{X} e^{-\phi_{K,E}},$$

using 2.22 in the inequality. As shown in [5, Prop 3.5] $\psi(\tau)$ is subharmonic on $\mathbb{D}$ and the first term $\langle \Phi \rangle$ is continuous on $\mathbb{D}$ (as follows from [57, Thm A]; see the proof of [5, Prop 3.6]). Moreover, the second term is also continuous on $\mathbb{D}$, as shown when $X$ is non-singular in [44, Lemma 1.9] and in general in [46, Lemma 7.1]. As a consequence,

$$\psi(0) \leq \int_{\partial \mathbb{D}} \psi d\theta = \psi(1),$$

using that $\psi(\tau)$ is $S^1$-invariant in the last equality. Finally, invoking formula 2.23 shows that $D_{\phi_\psi}(\phi)$ is uniformly bounded from below, as desired.

3. Sharp height inequalities in the toric case

We now specialize to the case when $X$ is toric Fano variety and denote by $(X, -K)$ the canonical integral model of $(X, -K_X)$ over $\mathbb{Z}$ (see [54, Section 2] and [20, Def 3.5.6]).

3.1. The toric setup. We start by recalling the notation for toric metrics employed in [5] and the relation to the canonical toric integral model.

3.1.1. The moment polytope $P(L)$. Let $X$ be an $n$-dimensional complex projective toric variety, i.e. a complex projective variety endowed with the action of the $n$-dimensional complex torus $\mathbb{C}^n$ with an open dense orbit. We shall denote by $T_c$ the complex torus and by $T$ the real maximal compact subtorus of $T_c$, i.e. $T = (S^1)^n$. Let $L$ be a toric ample line, i.e. an ample line bundle over $X$ endowed with a $T_c$-action covering the action of $T$ on $X$. It induces a bounded convex polytope $P(L)$ in $\mathbb{R}^d$ with non-empty interior, defined as follows. Consider the induced action of the group $T_c$ on the space $H^0(X, kL)$ of global holomorphic sections of $kL \to X$ (for $k$ a given positive integer). Decomposing the action of $T_c$ according to the corresponding one-dimensional representations $e^\alpha$, labeled by $m \in \mathbb{Z}^n$:

$$H^0(X, kL) = \oplus_{m \in B_c} \mathbb{C} e^\alpha$$

(3.1)
the lattice polytope $P(X,L)$ may be defined as the convex hull of $k^{-1}B_k$ in $\mathbb{R}^n$. More generally, by homogeneity, $P(X,L)$ is defined for any ample $\mathbb{Q}$–line bundle.

In particular, if $X$ is Fano, then the polytope $P(-K_X)$ has vertices in $\mathbb{Q}^n$ and may represented as follows:

$$P(-K_X) = \{ p \in \mathbb{R}^n : \langle l_F, p \rangle \geq -1, \forall F \},$$

where $F$ ranges over all facets of $P(-K_X)$ and $l_F$ denotes the unique primitive element in $\mathbb{Z}^n$ which is an interior normal to the facet $F$. Conversely, any such polytope corresponds to a Fano variety $X$.

**Example 3.1.** When $X = \mathbb{P}^n$ the polytope $P(-K_X)$ is $(n+1)(\Sigma_n - (1,\ldots,1))$ where $\Sigma_n$ denotes the $n$–dimensional unit-simplex. An infinite family of two-dimensional toric Fano varieties $X_{p,q}$, parametrized by two prime numbers $p$ and $q$, is obtained by setting

$$P(-K_{X_{p,q}}) := \{ (x_1, x_2) : |x_1p + x_2q| \leq 1 \} \subset \mathbb{R}^2.$$  

In particular, vol $(-K_{X_{p,q}}) = 2/(pq)$ tends to zero when $pq$ tends to infinity.

**Remark 3.2.** From an invariant point of view, the real vector space $\mathbb{R}^n$ above arises as $M \otimes_\mathbb{Z} \mathbb{R}$, where $M$ is the lattice $\text{Hom}(T_c, \mathbb{C}^*)$ of characters of the group $T_c$ (cf. [27]).

3.1.2. *Logarithmic coordinates and the Legendre transform $\phi^*$ of a metric $\phi$ on $L$.*

Since $X$ is toric we can identify $T_c$ with its open orbit in $X$. Let $\text{Log}$ be the map from $T_c$ to $\mathbb{R}^n$ defined by

$$\text{Log}: T_c \rightarrow \mathbb{R}^n, \text{Log}(z) := x := (\log(|z_1|^2), \ldots, \log(|z_n|^2)).$$

The real compact torus $T$ acts transitively on its fibers. We will refer to $x$ as the (real) *logarithmic coordinates* on $T_c$. Let $L$ be a toric ample line bundle over $X$ and assume that $P$ contains the origin, $0 \in P$, and denote by $e^0$ the corresponding $T$–invariant element in $H^0(X, kL)$. Any continuous $T$–invariant metric $\|\cdot\|$ on $L$ induces a continuous function on $\mathbb{R}^n$ that we shall denote by $\phi(x)$, defined as

$$\phi(x) := - \log \left( \| e^0 \|^2(z) \right), \ z \in T_c \Subset X, \ x := \text{Log} \ z.$$  

Thus, in the present additive notation $\phi$ for metrics we have $\phi(x) = \phi_{U^0}(x)$, when $U = T_c$, abusing notation slightly. The Legendre transform of $\phi(x)$, which defines a lower-semicontinuous convex function on $\mathbb{R}^n$ (taking values in $]-\infty, \infty]$) will be denoted by $\phi^*$:

$$\phi^*(p) := \sup_{x \in \mathbb{R}^n} \langle p, x \rangle - \phi(x).$$

A $T$–invariant continuous metric $\psi$ on $L$ is psh iff the corresponding function $\psi(x)$ on $\mathbb{R}^n$ is convex (iff $\psi(x) = \psi^{**}(x)$). We will denote by $\psi_{P(L)}$ the unique continuous convex function on $\mathbb{R}^n$ whose Legendre transform is equal to 0 on $P(L)$ and equal to $\infty$ on the complement of $P(L)$:

$$\psi_{P(L)}(x) := \sup_{p \in \mathbb{R}^n} \langle p, x \rangle \quad (\psi_{P(L)}^* = 0 \text{ on } P, \ \psi_{P(L)}^* = \infty \text{ on } P^c)$$

It corresponds to a continuous psh metric on $L$ (see the proof of [16 Prop 3.3]) and it will be used as a canonical reference metric in the present toric setup. It follows that for any other continuous metric $\phi$ on $L$

$$\phi - \psi_{P(L)} \in L^\infty(\mathbb{R}^n), \ P = \{ \phi^* < \infty \}. $$
Remark 3.3. From an invariant point of view the logarithm coordinates take value in $N \otimes \mathbb{R}$, where $N$ is the lattice $\text{Hom}(\mathbb{C}^*, T_c)$ of one-parameter subgroups of $T_c$, i.e. the dual of the lattice $\text{Hom}(T_c, \mathbb{C}^*)$ of characters of $T_c$.

3.1.3. Pushing forward measures from $X$ to $\mathbb{R}^n$. For any $T$–invariant continuous psh metric $\psi$ on $L$ the push-forward of the measure $(dd^c \psi)^n/n!$ on $L$ under the map $\log$ is given by

$$\log \left( \frac{(dd^c \psi)^n}{n!} \right) = \det(\nabla^2 \phi) dx,$$

(since the integral along the $T^n$–fibers equals $(2\pi)^n$). The measure in the right hand side is defined in the weak sense of Alexandrov. Since the closure of the image of $\mathbb{R}^n$ under the sub-gradient map of $\phi(x)$ equals $P$ it follows that

$$\text{vol}(L) = \int_P dy := \text{Vol}(P).$$

Next consider the case when $L = -K_X$. Then

$$e_0 := z_1 \frac{\partial}{\partial z_1} \wedge \cdots \wedge z_n \frac{\partial}{\partial z_n}$$

defines a $T_c$–invariant global holomorphic section of $-K_X$, trivializing $-K_X$ over $U := \mathbb{C}^n$. We can thus identify a continuous metric $\phi$ on $-K_X$ with the corresponding function $\phi_U$ on $\mathbb{C}^n$ (formula 2.4) and volume form on $X$ (formula 2.3) expressed as follows on $\mathbb{C}^n$, with respect to the local holomorphic coordinate $\log z$:

$$e^{-\phi_u} \left( \frac{i}{2} \right)^n d(\log z_1) \wedge d(\log \overline{z}_1) \wedge \cdots \wedge d(\log z_1) \wedge d(\log \overline{z}_1)$$

symbolically denoted by $e^{-\phi}$. Using again that the integral along the $T^n$–fibers equals $(2\pi)^n$ yields

$$\int_X e^{-\phi} = \int_{\mathbb{R}^n} e^{-\phi(x)} dx.$$

3.1.4. $K$-semistability and toric Kähler-Einstein metrics. We recall the following result, which is a combination of the results [6, Thm 1.2] and [5, Cor 1.2] (which are formulated in terms of $T_c$–equivariant $K$-polystability and $K$-polystability, respectively).

**Proposition 3.4.** Let $X$ be a toric Fano variety. The following is equivalent:

- $X$ is $K$-semistable
- $X$ is $K$-polystable
- $X$ admits a $T$–invariant Kähler-Einstein metric
- The barycenter of $P(-K_X)$ coincides with the origin $0$.

3.1.5. The arithmetic $\chi$–volume of a toric metric. The following result is a special case of the main result of [20 Thm 3]:

**Proposition 3.5.** Let $L \rightarrow X$ be an ample toric line bundle and denote by $(X, L)$ its canonical toric model over $\mathbb{Z}$. Assume that $\phi$ is a continuous $T$–invariant metric on $L$. Then

$$2 \hat{\text{vol}}_X(L, \phi) = -\int_{P(L)} \phi^* d\lambda$$
In order to verify that all normalizations are consistent we provide, for the benefit of the reader an alternative proof of this result. First recall that the canonical integer model $(X, \mathcal{L})$ has the property that the integral lattice $H^0(X, k\mathcal{L}) + iH^0(X, k\mathcal{L})$ in $H^0(X, k\mathcal{L})$ is generated by the $T_c$--equivariant bases $e^m$ appearing in the decomposition. Next, note that this bases is orthonormal with respect to the $L^2$--norm on $H^0(X, k\mathcal{L})$ induced by the metric $\psi_{P(L)}$ (defined by formula 3.3) and the Haar measure on the unit-torus $T \subset X$. Since this measure has the Bernstein-Markov property Cor A] yields

$$\hat{2}\vol(L, \phi) = \mathcal{E}_{P(L)}(\phi).$$

But, by [6, Prop 2.9], the right hand side above coincides with the right hand side of the formula to be proved.

### 3.2. Proof of Theorem 1.2

Given a Fano variety $X$ we will denote by $\hat{\vol}(L, \phi)$ the arithmetic volume of the canonical integral model $(X, L)$ of $(X, -K_X)$ wrt a volume-normalized metric $\phi$ on $-K_X$. We will prove the following more general formulation of the inequality in Theorem 1.2:

$$\hat{\vol}_{\chi}(L, \phi) \leq \hat{\vol}_{\chi}(-K_X)$$

where the metric on $-K_{\mathbb{P}^n}$ is the one induced by the volume-normalized Fubini-Study metric.

A $T$--invariant continuous metric $\phi$ will, as above, be identified with a convex function $\phi(x)$ on $\mathbb{R}^n$. If $\phi$ is moreover volume-normalized Prop 3.5 gives

$$2\vol_{\chi}(L, \phi) / \vol(-K_X) = -D_Z(\phi) = -D_{\psi_p}(\phi) = -\int_{\mathbb{P}^n} e^{-\phi(x)} dx + n \log \pi,$$

where $D_Z(\phi)$ and $D_{\psi_p}(\phi)$ are the Ding type functionals defined by formula 2.17 and formula 2.18 respectively, and we have used formula 3.6.

We start by recording the following explicit formula for the arithmetic volume of projective space $\mathbb{P}^n$, endowed with a volume normalized Kähler-Einstein metric (which may be assumed to be the metric induced by the Fubini-Study metric).

**Lemma 3.6.** The following formulas holds for the metrics $\phi_{KE}$ on the anti-canonical line bundles of $\mathbb{P}^n$ induced by a volume normalized toric Kähler-Einstein metric:

$$X = \mathbb{P}^n \implies 2\vol_{\chi}(L, \phi_{KE}) = \frac{(n+1)^n}{n!} \left( (n+1) \sum_{k=1}^{n} k^{-1} - n + \log \left( \frac{\pi^n}{n!} \right) \right) > 0$$

**Proof.** First consider the case when $X = \mathbb{P}^n$, whose canonical integral model is given by $X = \mathbb{P}^n$. The canonical model of the anti-canonical line bundle of $\mathbb{P}^n$ is given by $O(1)^{\otimes n+1} \to \mathbb{P}^n$. As shown in [42, §5.4] (using the induction formula for the height; see also [71, Prop 3.10]) the height $h_{FS}$ of $O(1) \to \mathbb{P}^n$ endowed with the Fubini-Study metric $\phi_{FS}$ is given by

$$h_{FS} = \frac{1}{2} \sum_{k=1}^{n} \sum_{m=1}^{k} m^{-1}. $$
Since \((n+1)\phi_{FS}\) defines a Kähler-Einstein metric on \(-K_p\) and \(\pi^{-n} \int_p e^{-(n+1)\phi_{FS}} = 1/n!\) this gives
\[
2\widetilde{\text{vol}}_X(\mathcal{L}, \phi_{KE}) - n \log \pi = (n+1)^{n+1} \frac{h_{FS}}{(n+1)!} + \frac{(n+1)^n}{n!} \log \left( \frac{1}{n!} \right) = \frac{(n+1)^n}{n!} \left( h_{FS} + \log \left( \frac{1}{n!} \right) \right),
\]
using formula (2.8) in the first term, combined with the homogeneity property (2.9) and, in the second term, the scaling property (2.10). Rewriting the formula for \(h_{FS}\) above as a triangle sum and changing the order of summation then concludes the proof of the formula of the lemma. The last positivity statement will be shown in the course of the proof of Lemma 3.8.

The key ingredient in the proof of Theorem 1.2 is the following universal bound on the arithmetic volume, in terms of the ordinary volume:

**Proposition 3.7.** For any \(n\)-dimensional toric Fano variety \(X\) which is K-semistable, the following bound holds for any volume-normalized continuous metric \(\phi\) on \(-K_X\),
\[
2\widetilde{\text{vol}}_X(\mathcal{L}, \phi) \leq -\text{vol}(X) \log \left( \frac{\text{vol}(X)}{(2\pi^n)^n} \right) \text{vol}(X) := \text{vol}(-K_X).
\]

**Proof.** First recall that, as shown in the beginning of the proof of Theorem 2.4 it is equivalent to establish the upper bound for \(-D_{\phi_p}(\phi)\) when \(\phi\) is a continuous psh metric on \(L\). Since \(X\) is assumed K-semistable it follows from Prop 3.4 that \(X\) admits a \(T\)-invariant Kähler-Einstein metric. In general, a Kähler-Einstein metric \(\phi\) on \(-K_X\) minimizes the Ding functional \(D_{\psi_0}\) (7). Thus in the toric case the infimum of \(D_{\psi_0}\) coincides with the infimum over all continuous \(T\)-invariant psh metrics. As explained in Section 3.1.2 such a metric may be identified with a convex function \(\phi(x)\) on \(\mathbb{R}^n\) satisfying \(\phi - \psi_p \in L^\infty(\mathbb{R}^n)\). By formula (3.8) it will be enough to show that for such convex functions
\[
(3.9) \quad -\int_P \phi^* \frac{dy}{V} + \log \int_{\mathbb{R}^n} e^{-\phi(x)} dx \leq -\log V + n \log(2\pi), \quad V := \text{vol}(-K_X).
\]
Since 0 is contained in the interior of \(P\) the measure \(e^{-\phi} dx\) on \(\mathbb{R}^n\) has finite moments. Recall that, by Prop 3.4 the barycenter of \(P\) coincides with 0 \(\in \mathbb{R}^n\) and, as a consequence, the left hand side in inequality (3.9) is invariant under translations of \(\phi, \phi(x) \mapsto \phi(x + a)\) for any given \(a \in \mathbb{R}^n\) (6) Lemma 2.14]. As a consequence, in order to prove the inequality (3.9) we may as well assume that
\[
\int_{\mathbb{R}^d} x e^{-\phi} dx = 0.
\]
By the functional form Santaló’s inequality (2) Lemma 2.14] this implies that
\[
\int_{\mathbb{R}^n} e^{-\phi^*(y)} dy \cdot \int e^{-\phi(x)} dx \leq (2\pi)^n
\]
(where equality holds if \(\phi = \phi^*\) i.e. if \(\phi(x) = |x|^2/2\)). Moreover, by Jensen’s inequality
\[
-\int_P \phi^* d\lambda/V \leq \log \left( \int_P e^{-\phi^*(y)} dy/V \right) = \log \left( \int_{\mathbb{R}^n} e^{-\phi^*(y)} dy/V \right),
\]
using in the last equality that \(\phi^* = \infty\) on the complement of \(P\) (see formula 3.4). Combining the latter two inequalities yields the desired inequality 3.9. \(\square\)
Recall that $\mathbb{P}^n$ has maximal volume among all $K$-semistable $n$–dimensional Fano varieties (as shown in [8] in the toric case and in [37] in general). We next show that it will be enough to prove that, in the toric case, the next to largest volume is attained by $\mathbb{P}^{n-1} \times \mathbb{P}^1$.

**Lemma 3.8.** For any $n$–dimensional toric Fano variety $X$ which is $K$-semistable

$$\text{vol}(X) \leq \text{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1) \implies \widehat{\text{vol}}_{\chi}(\mathcal{L}, \phi) < \widehat{\text{vol}}_{\chi}(-K_{\mathbb{P}^n})$$

where $-K_{\mathbb{P}^n}$ is endowed with the volume-normalized Fubini-Study metric.

**Proof.** First observe that the function of $\text{vol}(X)$ appearing in the rhs of the inequality in the previous proposition is increasing when $\text{vol}(X) \leq (2\pi^2)^n/e$. This bound is, in fact, satisfied for any $K$-semistable $X$. Indeed, by [8],

$$\text{vol}(X) \leq \text{vol}(\mathbb{P}^n) = \frac{(n+1)^n}{n!} < (2\pi^2)^n/e.$$ (using, in the last inequality a simple induction argument). Thus, by the previous proposition, it will be enough to show that

$$\text{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1) \log(\text{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)/(2\pi^2)^n) < 2\widehat{\text{vol}}_{\chi}(-K_{\mathbb{P}^n}).$$

for any $n \geq 2$. To this end first note that

$$-\text{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1) \log(\text{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)/(2\pi^2)^n) = -\frac{2n^{n-1}}{(n-1)!} \left(\log\left(\frac{2n^{n-1}}{(n-1)!}\right) - n \log(2\pi^2)\right)$$

We check that the inequality holds for $n = 2$ and with induction in mind we simplify the right hand side of (3.11) with $n + 1$ for $n$ and get

$$2\widehat{\text{vol}}_{\chi}(-K_{\mathbb{P}^n+1}) = -\frac{(n+2)^{n+1}}{(n+1)!}(n+1 - (n+2) \sum_{k=1}^{n+1} \frac{1}{k} + \log((n+1)!)) - (n+1) \log(\pi))$$

$$= -\frac{n+2}{n+1}^{n+1} \frac{(n+1)^n}{n!} \left((n - (n+1)) \sum_{k=1}^{n} \frac{1}{k} + \log(n!) - n \log(\pi)\right) +$$

$$\left(1 - (n+2) \sum_{k=1}^{n+1} \frac{1}{k} + (n+1) \sum_{k=1}^{n} \frac{1}{k} + \log(n+1) - \log(\pi)\right)$$

$$= \frac{n+2}{n+1}^{n+1} 2\widehat{\text{vol}}_{\chi}(-K_{\mathbb{P}^n}) - \frac{n+2}{n+1}^{n+1}(1 - \log(\pi) + \log(n+1) - \log(n^2))$$

Here we observe for later use that $\widehat{\text{vol}}_{\chi}(-K_{\mathbb{P}^n}) > 0 \forall n \geq 1$ by evaluating it at $n = 1$ and then using the above to perform induction and noting that

$$-(1 - \log(\pi) + \log(n+1) - \frac{n+2}{n+1}^{n+1} \sum_{k=1}^{n} \frac{1}{k}) > -(\log(\pi) + \log(2)) = \log\left(\frac{\pi}{2}\right) > 0$$

for $n \geq 1$. We have used that $-\log(n+1) + \sum_{k=1}^{n} \frac{1}{k}$ is increasing and can thus be estimated from below by putting $n = 1$. We also simplify the left hand side of (3.11)
\[ -\text{vol}(\mathbb{P}^n \times \mathbb{P}^1) \log(\text{vol}(\mathbb{P}^n \times \mathbb{P}^1)/(2\pi^2)^n) = - \frac{2(n+1)^n}{n!} \left( \log\left( \frac{2(n+1)^n}{n!} \right) - (n+1) \log(2\pi^2) \right) \]

\[ = - \frac{2(n+1)^n}{n!} \left( \log\left( \frac{2n^n}{n!} \right) - n \log(2\pi^2) \right) + (\log\left( \frac{n+1}{n} \right)^n - \log(2\pi^2)) \]

\[ = - \left( \frac{n+1}{n} \right)^n \text{vol}(\mathbb{P}^n \times \mathbb{P}^1) \log(\text{vol}(\mathbb{P}^n \times \mathbb{P}^1)/(2\pi^2)^n) \]

\[ - 2 \frac{(n+1)^n}{n!} \left( - \log\left( \frac{n+1}{n} \right)^n - \log(2\pi^2) \right). \]

Fix \( n \geq 2 \) and assume \( -\text{vol}(\mathbb{P}^n \times \mathbb{P}^1) \log(\text{vol}(\mathbb{P}^n \times \mathbb{P}^1)/(2\pi^2)^n) \leq 2\widehat{\text{vol}}_\chi \left(-\mathcal{K}_{\mathbb{P}^n} \right). \)

Define for brevity \( e_n = (1 + \frac{1}{n})^n \) and estimate

\[ 2\widehat{\text{vol}}_\chi \left(-\mathcal{K}_{\mathbb{P}^n} \right) - \left( -\text{vol}(\mathbb{P}^n \times \mathbb{P}^1) \log(\text{vol}(\mathbb{P}^n \times \mathbb{P}^1)/(2\pi^2)^n) \right) \]

\[ = e_{n+1} \text{vol}_\chi \left( -\mathcal{K}_{\mathbb{P}^n} \right) - (e_n \text{vol}(\mathbb{P}^n \times \mathbb{P}^1) \log(\text{vol}(\mathbb{P}^n \times \mathbb{P}^1)/(2\pi^2)^n) \right) \]

\[ + 2 \frac{(n+1)^n}{n!} \left( \log\left( \frac{(n+1)^n}{n!} \right) - \log(2\pi^2) \right) \]

\[ - \frac{(n+2)^{n+1}}{(n+1)!} \left( 1 - \log(\pi) + \log(n+1) - \frac{n+2}{n+1} - \sum_{k=1}^{n+1} \frac{1}{k} \right) \]

\[ > 2 \frac{(n+1)^n}{n!} \left( \log\left( \frac{(n+1)^n}{n!} \right) - \log(2\pi^2) \right) \]

\[ - \frac{(n+2)^{n+1}}{(n+1)!} \left( 1 - \log(\pi) + \log(n+1) - \frac{n+2}{n+1} - \sum_{k=1}^{n+1} \frac{1}{k} \right) \]

\[ = \frac{(n+2)^{n+1}}{(n+1)!} \left( \log(e_n) - (\log(2\pi^2) + \log(\pi) + \sum_{k=1}^{n+1} \frac{1}{k} - \log(n+1)) \right) \]

In the inequality above we have used \( \text{vol}_\chi \left(-\mathcal{K}_{\mathbb{P}^n} \right) > 0 \forall n \geq 1 \) and \( e_n < e_{n+1} \) and the induction hypothesis. Next check numerically that this last expression is positive for \( n = 2, 3 \). For \( n \geq 4 \) we have

\[ \frac{2}{e_n} (\log(e_n) - \log(2\pi^2)) + \log(\pi) + \frac{1}{n+1} + \sum_{k=1}^{n+1} \frac{1}{k} - \log(n+1) \]

\[ > \frac{2}{e_4} (\log(e_4) - \log(2\pi^2)) + \log(\pi) + \gamma > 0. \]
We used again that $e_n < e_{n+1}$ and the fact that $\sum_{k=1}^n \frac{1}{k} - \log(n) > \gamma$ [82], where $\gamma$ is the Euler-Mascheroni constant. The last inequality is checked numerically.

We expect that any K-semistable toric Fano variety $X$, not equal to $\mathbb{P}^n$, satisfies the volume bound in the previous lemma (see the following section). Here we will show that this is the case under the conditions of Theorem 1.2. First, the singular cases are handled using the following bound.

**Lemma 3.9.** Let $X$ be a singular K-semistable toric Fano variety. Then

$$\text{vol}(-K_X) \leq \frac{1}{2}(n+1)^n/n!$$

if anyone of the following conditions hold:

- $X$ is $\mathbb{Q}$-factorial (or equivalently, $X$ has abelian quotient singularities).
- $X$ is not Gorenstein

In particular, by the first point, when $n = 2$ this inequality holds for any singular K-semistable toric Fano variety $X$.

**Proof.** The result concerning the first point is the toric case of [49] Thm 3 concerning quotient singularities, but in the toric case it also follows from the proof of [8] Thm 1.2. For future reference we recall the argument in [8]. Let $P$ be a given polytope with rational vertices and represent $P$ as the intersection of hyperplanes $\{p \in \mathbb{R}^n : \langle \ell_F, p \rangle \geq -a_F\}$, where the index $F$ ranges over the facets of $P$, $l_F$ is a primitive vector in $\mathbb{Z}^n$ and $a_F$ is a non-zero positive numbers. In the present Fano case $a_F = 1$. Moreover, since $X$ is assumed to be $\mathbb{Q}$-factorial for any vertex $p_0$ of $P$ there are precisely $n$ facets $F_1, ..., F_n$ of $P$ intersecting $p_0$, numbered so that the corresponding normals define a positively oriented bases in $\mathbb{R}^n$ [27]. Fixing a vertex $p_0$ of $P$ we denote by $P'$ the image of $P$ under the map

$$p \mapsto \left( \frac{\langle \ell_{F_1}, p \rangle + a_{F_1}}{a_{F_1}}, ..., \frac{\langle \ell_{F_n}, p \rangle + a_{F_n}}{a_{F_n}} \right),$$

which is a polytope in $[0, \infty]^n$. Moreover, assuming that 0 is the barycenter of $P$ the barycenter of $P'$ is $(1, ..., 1)$. By [8] Thm 1.5 the volume $\text{Vol}(P')$ of any such polytope is maximal when $P'$ is $(n+1)$ times the unit-simplex in $[0, \infty]^n$ with vertex at $(0, ..., 0)$. Hence,

$$\text{Vol}(P') \leq (n+1)^n/n!, \quad \text{Vol}(P') = \frac{\delta}{a_{F_1} \cdots a_{F_n}} \text{Vol}(P)$$

where $\delta$ is the determinant of the map $p \mapsto (\langle \ell_{F_1}, p \rangle, ..., \langle \ell_{F_n}, p \rangle)$. Thus $\delta$ is a positive integer and $\delta = 1$ iff the map is invertible, i.e. if and only if $l_{F_1}, ..., l_{F_n}$ generate $\mathbb{Z}^n$, which is equivalent to the $T_e$-invariant neighbourhood $U_0$ corresponding to the vertex $p_0$ being biholomorphic to $\mathbb{C}^n$ [27]. Hence, if $X$ is singular (i.e. $X$ is not non-singular), then there must be some vertex $p_0$ with $\delta \geq 2$. Since $a_{F_1} = 1$ this concludes the proof.

To prove the second point we employ a similar argument. This time, for $X$ possibly not $\mathbb{Q}$-factorial, there might be more than $n$ facets intersecting a vertex $p_0$. Still, there are at least $n$ facets intersecting at $p_0$, and we can construct the map [8.12] by choosing any $n$ of them. Next note that if $\delta = 1$, the map and its inverse have integer coefficients (since $a_{F_1} = 1$ when $X$ is Fano) and since $p_0$ is mapped to 0, $p_0 \in \mathbb{Z}^n$. Since $p_0$ was arbitrary, it follows that $P$ is a lattice polytope and hence $X$ is Gorenstein. Thus $\delta \geq 2$ and we are done. \qed
The volume bound in the previous lemma implies the volume bound in Lemma 3.8 is satisfied:

\[(n + 1)^n \leq \frac{2n^{n-1}}{(n-1)!} \iff (1 + 1/n)^n \leq 4.\]

The lhs in the latter inequality increases to \(e\), which is, indeed, smaller than 4. This proves Theorem 1.2 in the singular cases. Finally, in the case that \(X\) is non-singular there are, for any given dimension \(n\) only a finite number of cases to check in order to verify the the volume bound in Lemma 3.8. When \(n \leq 6\) we may apply the database [58] of all non-singular Fano varieties of dimension \(n\). The condition that the barycenter of \(P\) vanishes, corresponds in the database to the condition “zero dual barycentre”. Adding the condition \((-K_X)^n \geq n!\text{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)\) the database only furnishes \(\mathbb{P}^n\) and \(\mathbb{P}^{n-1} \times \mathbb{P}^1\), as desired.

3.2.1. Remarks on the “gap hypothesis”. In order to extend the proof of Theorem 1.2 to any dimension \(n\) one would need to establish the following conjecture (established above under the conditions in Theorem 1.2):

**Conjecture 3.10.** (the “gap hypothesis”). For any \(n\)-dimensional toric K-semistable Fano manifold \(X\) different from \(\mathbb{P}^n\), \(\text{vol}(X) \leq \text{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)\).

This conjecture might even hold without the toric assumptions in any dimension. For example, when \(n = 3\) and \(X\) is non-singular it follows from the well-known classification of three dimensional Fano manifolds (see the “big table” in [1 Section 6]) that the only Fano manifolds \(X\), different from \(\mathbb{P}^3\), which do not satisfy the inequality in question are \(\mathbb{P}^3\) blown-up in one point and \(\mathbb{P}(O \oplus O(2))\). But both of these are K-unstable, i.e. they are not K-semistable. Indeed, these two Fano manifolds are toric and if they were K-semistable they would satisfy the gap hypothesis, by the toric case \((n \leq 6)\) applied to \(n = 3\). Let us also point out that in the toric case it is only \(\mathbb{P}^{n-1} \times \mathbb{P}^1\) that saturates the inequality in the “gap hypothesis” when \(n \leq 6\) and it seems thus natural to ask if this is also the case when \(n > 6\)? However, in the general non-toric case the inequality is also saturated by the non-singular quadratic hypersurface \(X_2\) in \(\mathbb{P}^{n+1}\), i.e. the base of the Ordinary Double Point (ODP). Moreover, as pointed out to us by Yuji Odaka, in the general case our “gap hypothesis” is reminiscent of the ODP-conjecture in [72], very recently settled in the toric case [56]. More precisely, in our setup, the ODP-conjecture implies that

\[(3.15) \quad \text{vol}(X) \leq \text{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)(n/I(X)),\]

where \(I(X)\) denotes the Fano index of \(X\) (i.e. largest positive integer such that \(K_X/I(X)\) is a line bundle). However, \(I(X) \leq n\) when \(X \neq \mathbb{P}^n\) (with equality iff \(X = X_2\)) and hence the inequality [4, 44] is weaker than our “gap hypothesis”.

3.3. Proof of Theorem 1.3. By Prop 3.7 it only remains to prove the lower bound. Using the notation in the proof of Prop 3.7 we have that, for any continuous convex function \(\psi\) on \(\mathbb{R}^n\) such that \(\psi - \psi_P\) is bounded,

\[2(-K_X)^n/\text{Vol}(-K_X) \geq -\int_P \psi^*dy/\text{Vol}(P) + \log \int_{\mathbb{R}^n} e^{-\psi} dx + n \log \pi\]

In particular, taking \(\psi = \psi_P\) the first term in the right hand side vanishes. Moreover,

\[I := \int_{\mathbb{R}^n} e^{-\psi_P} dx = n!\text{Vol}(P^*),\]
where $P^*$ denotes the polar dual of $P$, i.e. $P^*$ consists of all $x \in \mathbb{R}^n$ such that $x \cdot p \leq 1$ for all $p \in P$. Indeed,
\[
I = \int_{[0, \infty]} e^{-t} (\psi P), dx = \int e^{-t} dV(t) dt = \int e^{-t} V(t) dt = \int_0^\infty e^{-t} t^n dt \text{Vol}(P^*),
\]
where $V(t)$ is the Lebesgue volume of $\{ \psi_p < t \}$ i.e. of $tP^*$. Hence,
\[
2(-K_X)^n \geq \text{Vol}(P) \left( \log (n! \text{Vol}(P^*)) + n \log \pi \right).
\]
Since, by definition, $\text{Vol}(P^*) \text{Vol}(P) \geq m_n$ this concludes the proof of the lower bound in the theorem. Next, by \cite{[41]} Cor 1.8 (see also \cite{[12]})
\[
m_n \geq \left( \frac{\pi}{2e} \right)^{n-1} (n+1)^{n+1} / (2e)^{n-1} \geq \left( \frac{\pi}{2e} \right)^{n-1} \sigma_n,
\]
where $\sigma_n = \text{vol}(\mathbb{P}^n)$. Since $\text{Vol}(P) \leq \sigma_n$ (by \cite{[3.10]}, this means that
\[
n! \pi^n m_n \text{Vol}(P)^{-1} \geq n! \pi^n m_n \sigma_n^{-1} = \pi (\frac{\pi}{2e})^{n-1} (n+1) > 1
\]
proving the positivity in the theorem.

4. The logarithmic setting

A log pair $(X, \Delta)$ is a normal complex projective variety $X$ together with an effective $\mathbb{Q}$–divisor $\Delta$ on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$–Cartier, i.e. defines a $\mathbb{Q}$–line bundle \cite{[40]}. In the logarithmic setting this bundle plays the role of the canonical line bundle and is thus called the log canonical line bundle. A log pair $(X, \Delta)$ is said to be a log Fano pair if $\Delta$ is effective and $-(K_X + \Delta) > 0$. The notion of K-stability extends to log Fano pairs and by the solution of the Yam-Tian-Donaldson conjecture a log Fano pair $(X, \Delta)$ is K-polystable if and only if it admits a Kähler-Einstein metric. By definition, this is a locally bounded psh metric $\phi$ on $-(K_X + \Delta)$ whose curvature form $dd^c \phi$ defines a Kähler metric with constant positive Ricci curvature on the complement of the support of $\Delta$ in the regular locus of $X$ \cite{[43, 54]}. In an arithmetic context log Fano varieties are discussed in, for example, \cite{[66, 36]}. Here we will say $(\mathcal{X}, \mathcal{D})$ is an arithmetic log Fano variety (or an integral model of a log Fano pair $(X, \Delta)$) if

- $\mathcal{X}$ is a normal projective scheme, which is flat over Spec($\mathbb{Z}$),
- $\mathcal{D}$ is an effective divisor on $\mathcal{X}$ whose irreducible components are flat over Spec($\mathbb{Z}$)
- the dual of the relative log canonical line bundle

$-K_{(\mathcal{X}, \mathcal{D})} := -(K_X + \mathcal{D})$

is a relatively ample $\mathbb{Q}$–line bundle.

**Remark 4.1.** If a log Fano variety $(X, \Delta)$ is K-semistable, then $(X, \Delta)$ has Kawamata Log Terminal (klt) singularities \cite{[18, Cor 9.6]}. When $X$ is non-singular and $\Delta$ has simple normal crossings this means that all the coefficients of $\Delta$ along its irreducible components are strictly smaller than 1.

Conjecture \cite{[14]} admits a natural extension to arithmetic log Fano varieties. To see this first recall that a continuous metric $\phi$ on $-(K_X + \Delta)$ naturally induces a measure on $X$, which puts no mass on the support $\text{supp(}\Delta)$ of $\Delta$. Indeed, on the complement $X - \text{supp(}\Delta)$ in $X$ of the support of $\Delta$ the $\mathbb{Q}$–line bundle the measure is simply defined as in Section 2.1.2 using that $-(K_X + \Delta)$ may be identified with
−K_X (since Δ has a canonical trivialization on X − supp(Δ)). A log pair (X, Δ) has klt singularities iff −(K_X + Δ) admits a continuous metric φ such that the corresponding measure on X has finite total mass [7 Section 3.1].

**Conjecture 4.2.** Let (X, D) be an arithmetic log Fano variety. Then the following height inequality holds for any volume-normalized continuous metric on −(K_X + Δ) with positive curvature current if X is K-semistable:

\[ (-K_{x,y})^{n+1} \leq (-K_{x,y})^{n+1}, \]

where −K_{x,y} is endowed with the volume normalized Fubini-Study metric. Moreover, equality holds if and only if (X, D) = (P^n, 0) and the metric is Kähler-Einstein, i.e. coincides with the Fubini-Study metric, modulo the action of an automorphism.

A log pair (X, D) is said to be toric if X and the divisor D are toric, i.e. if the \( \mathbb{Q} \)-divisor D is invariant under the torus action on X. Any toric log Fano variety admits a canonical integral model (X, D) which is log Fano (see [54, Section 2] and [20, Def 3.5.6]). One advantage of the logarithmic setup is that on any given toric Fano variety X there exist many toric \( \mathbb{Q} \)-divisors D such that −(K_X + Δ) is a K-semistable log Fano variety. The following result generalizes Theorem 1.2 when \( n \leq 3 \):

**Theorem 4.3.** Let (X, D) be the canonical integral model of a K-semistable toric log Fano variety (X, D). Conjecture 4.2 holds for (X, D) under anyone of the following conditions:

- n ≤ 3 and X is \( \mathbb{Q} \)-factorial (equivalently, X has at worst abelian quotient singularities)
- X is not Gorenstein or has some abelian quotient singularity

To prove this theorem we start by introducing some notation. Given a toric log Fano variety (X, D) set \( L = −(K_X + Δ) \) and denote by P the corresponding moment polytope in \( \mathbb{R}^n \). Then

\[ P = \{ p \in \mathbb{R}^n : \langle l_F, p \rangle \geq −a_F, \ \forall F \}, \]

where \( a_F \in [0, 1] \) (generalizing the Fano case in formula 3.2 see [4]). As shown in [5, 6] (X, Δ) is K-semistable iff 0 is the barycenter of P iff the log Ding functional \( D_{φ,P} \) is bounded from below. Moreover, the infimum of \( D_{φ,P} \) is attained at a \( T \)-invariant psh metric φ on L. We will identify the metric φ with a continuous convex function on \( \mathbb{R}^n \), as in Section 3.1.2. As in formula 3.3 we still have that

\[ D_{φ,P}(φ) = \int_P φ^*dy/V − \log \int_{\mathbb{R}^n} e^{−φ(x)}dx − n log π, \ V := vol(P) \]

(since the support of \( D \) is contained in the complement of \((\mathbb{C}^*)^n \) in X). Thus the inequality in Prop 3.7 generalizes to the canonical toric model \( L \) of L (which coincides with −K_{(X,D)}):

\[ 2vol_x(−K_{(X,D)}, φ) \leq vol(X, Δ) log \left( \frac{vol(X, Δ)}{(2π)^n} \right) \]

We will first prove Theorem 1.3 in the case that \( X = \mathbb{P}^n \), using the following lemma, formulated in terms of the divisor \( D_0 \) cut out by the \( T_e \)-invariant element
Then attained at

Denote by

Proof.

of

X

d dimensional K-semistable toric Fano variety X and denote by D₀ the standard anti-canonical divisor on X. Then

\[ \frac{(-K_X(1-t)D_0)^{n+1}/(n+1)!}{(-K_X + (1-t)D_0)^n/n!} = \frac{(-K_X)^{n+1}/(n+1)!}{(-K_X^n/n!} - \frac{1}{2} n \log t \]

with respect to the volume-normalized Kähler-Einstein metrics.

Proof. Denote by P the moment polytope corresponding to the toric Fano variety X. Then tP is the moment polytope corresponding to the log Fano variety (X, (1 − t)D₀) [6]. Denote by d(P) the infimum of the log Ding functional \( \mathcal{D}_{\psi,F} \) over all continuous psh metrics \( \phi \) on \( -K_X \) and by d(tP) the infimum of the log Ding functional \( \mathcal{D}_{\psi,tP} \) corresponding to \( (X, (1-t)D_0) \). The identity in the lemma is equivalent to the following relation

(4.3) \[ d(tP) = d(P) + \log(t^n) \]

To prove this relation first recall that the infimum of the log Ding functional may be restricted to \( T \)-invariant metrics, since the infimum is attained at a toric log Kähler-Einstein metric [6]. As in Section 3.1.2 we will identify a locally bounded psh metric on \( -K_X \) with a convex function on the interior of \( P \) and equals infinity on the complement of \( P \). The convex function corresponding to any locally bounded psh metric on \( -(K_X + \Delta) \) may thus be expressed as \( \phi(tx) \) for some convex function corresponding to a locally bounded psh metric on \( -K_X \). Indeed, under Legendre transformation

\[ (\phi(t\cdot))^*(p) = \phi^*(t^{-1}p). \]

Hence, by formula 3.3

\[ \mathcal{D}_{\psi,tP}(\phi(t)) + n \log \pi = \int_{t^P} \phi^*(t^{-1}p)dy/V(tP) - \log \int_{\mathbb{R}^n} e^{-\phi(tx)}dx = \]

\[ = \int_{tP} \phi^*(p)dy/V(P) - \log \int_{\mathbb{R}^n} e^{-\phi(x)}dx + n \log t =: \mathcal{D}_{\psi,P}(\phi) + n \log \pi \]

where the second equality results from making the change of variables \( p \to t^{-1}p \) and \( x \to tx \) in the integrals. Taking the inf over all \( \phi \) thus proves formula 4.3. □

Lemma 4.5. Let \( (X, \mathcal{D}) \) be a toric K-semistable log Fano variety such that \( X = \mathbb{P}^n_2 \). Then \( \frac{(-K_{(X,\mathcal{D})})^{n+1}}{(-K_{\mathbb{P}^n_2})^{n+1}} \leq \frac{(-K_{\mathbb{P}^n})^{n+1}}{(-K_{\mathbb{P}^n})^{n+1}} \) with equality iff \( \mathcal{D} = 0 \).

Proof. First observe that there exists \( t \in [0, 1] \) such that \( \mathcal{D} = (1-t)D_0 =: D_t \). This is a special case of [39, Cor. 1.6], which applies to \( \mathbb{P}^n \), in any dimension \( n \), using that toric log Fano varieties are never uniformly K-stable. It will thus be enough to show that \( t \to \left( \frac{(-K_{(X,\mathcal{D})})^{n+1}}{(-K_{\mathbb{P}^n_2},\mathcal{D}_t)^{n+1}} \right) \) is increasing on \( [0, 1] \) (and thus its maximum is attained at \( t = 1 \)). By the previous lemma

\[ 2 \left( \frac{(-K_{\mathbb{P}^n})^{n+1}/(n+1)!}{(-K_{\mathbb{P}^n})^{n/n!}} \right) = t^n \cdot 2 \left( \frac{(-K_{\mathbb{P}^n})^{n+1}/(n+1)!}{(-K_{\mathbb{P}^n})^{n/n!}} \right) - t^n \log(t^n). \]
Differentiating wrt \((t^n)\) reveals that the right hand side above is increasing with respect to \(t\) iff \(2 \frac{(-K_{P^n})^{n+1}/(n+1)!}{(-K_n)^{n}/n!} \geq 1\). The latter inequality is indeed satisfied, as follows from the explicit formula in Lemma 3.6. □

Combining the universal bound \([2,3]\) with Lemma \([3,3]\) all that remains to prove Theorem \([1,3]\) is to establish the “logarithmic gap hypothesis”

\[
(4.4) \quad \text{vol}(X, \Delta) \leq \text{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1).
\]

**Proposition 4.6.** The logarithmic gap hypothesis holds for all toric K-semistable log Fano varieties (manifolds) \((X, \Delta)\) such that \(X \neq \mathbb{P}^n\) iff the following bound holds for all Fano varieties (manifolds) \(X \neq \mathbb{P}^n\)

\[
(4.5) \quad S(X) \leq \text{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1) \quad S(X) := \sup \{ \text{vol}(-(K_X + \Delta)) : (X, \Delta) \text{ K-semistable} \}.
\]

The “logarithmic gap hypothesis” holds for all log Fano varieties \((X, \Delta)\) such that \(X\) is \(\mathbb{Q}\)-factorial and of dimension \(n \leq 3\) and for any dimensions \(n\) if \(X\) has some abelian quotient singularity or if \(X\) is not Gorenstein.

**Proof.** Since, trivially, \(\text{vol}(X, \Delta) \leq S(X)\) the first equivalence follows directly from the definitions. Next, let us show the last statement of the proposition, first assuming that \(X\) is singular, which means that the moment polytope \(P\) of \((X, \Delta)\) is “singular” in the sense that there exists a vertex of \(\partial P\) such that the corresponding primitive vectors \(l_{F_1}, \ldots, l_{F_n}\) do not generate \(\mathbb{Z}^n\). It follows from the proof of Lemma \([3,6]\) that

\[
\text{vol}(P) \leq \frac{1}{2}(n+1)^n/n! \leq \text{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)
\]

Indeed, since \(a_F \leq 1\) the first inequality follows from the inequality \([3,3]\) using that \(\delta \geq 2\), according to the singularity assumption on \(P\) (for the the second inequality see formula \([3,13]\)). All that remains is thus to show the bound \([4,5]\) for \(S(X)\) when \(n \leq 3\) and \(X\) is non-singular. First assume that \(n = 2\). This means, by classical classification results, that \(X\) is either \(\mathbb{P}^1 \times \mathbb{P}^1\) or the blow-up \(X^{(m)}\) of \(\mathbb{P}^2\) in \(m\) points for \(m \leq 3\). But \((-K_{X^{(m)}})^2 = (-K_{\mathbb{P}^2})^2 - m\) and thus \(\text{vol}(X) \leq 4 = \text{vol}(\mathbb{P}^{n-1} \times \mathbb{P}^1)\), proving the bound \([4,5]\). Finally, consider the case when \(n = 3\). Starting with the trivial bound \(\text{vol}(X, \Delta) \leq \text{vol}(X)\) it follows from the discussion after the statement of Conjecture \([3,10]\) that it is enough to show that the bound \([4,5]\) holds when \(X\) is \(\mathbb{P}^3\) blown-up in one point and \(\mathbb{P}(O \oplus O(2))\) (whose degrees are 58 and 62, respectively \([1\text{ Section 6}]\)). According to the following proposition the corresponding invariants \(S(X)n!\) are, approximately, given by 41.8 and 30.3, respectively, which are well below the degree 54 of \(\mathbb{P}^2 \times \mathbb{P}^3\), as desired. □

In the proof above we used the following result.

**Proposition 4.7.** After rounding to the nearest decimal place the invariant \(n!S(X)\) (formula \([4,4]\)) is given by 41.8 and 30.3 when \(X\) equals \(\mathbb{P}^3\) blown up in one point and \(\mathbb{P}(O \oplus O(2))\), respectively.

**Proof.** Given a convex subset \(P\) of \(\mathbb{R}^n\) let

\[
s(P) := \sup \{ \text{vol}(P_0) : P_0 \subset P, b_{P_0} = 0 \},
\]

where \(P_0\) is a closed subset of \(P\) with barycenter \(b_{P_0}\) at the origin. We will compute \(s(P)\) when \(P\) is the moment polytope of the manifolds \(X\) appearing in the proposition, showing at the same time that \(s(P) = S(X)\). The moment polytopes
P of both $\mathbb{P}^3$ blown up in one point and $\mathbb{P}(O \oplus O(2))$ are of the form a simplex, with a simplex subset removed, by chopping off a vertex (see ID 20 and ID7 in the database [58]). After a general linear transformation, they are of the form $(a\Delta_3 - 1) - (b\Delta_3 - 1)$ where $\Delta_3$ is the standard unit simplex in dimension three, $\mathbf{1}$ is the vector with all ones and $a$ and $b$ are positive real numbers. For $\mathbb{P}^3$ blown up in one point we can transform the moment polytope to $(4\Delta - 1)\setminus (2\Delta - 1)$ and for $\mathbb{P}(O \oplus O(2))$ we get $(5\Delta - 1)\setminus (\Delta - 1)$. In the first case, the linear transformation is unimodular, but in the second case the transformation has determinant 2. This will not matter when computing $s(P)$ as long as we correct for the non-unit determinant. Next we compute the barycenter $b_P$ in one point we can transform the moment polytope to

$\mathbf{1}$ is the vector with all ones and $a/(n + 1)$, and then scaling and linearity properties of the volumes times the barycenter. The barycenter of $(a\Delta_3 - 1)\setminus (b\Delta_3 - 1)$ is given by

$\frac{a^3/3((a/4 - 1) - b^3/3(4/4 - 1)) - 3}{a^3/3 - b^3/3} \mathbf{1}$. Next we use a general fact, to be proved in the lemma below, stating that the closed subset $P'$ of $P$ which maximizes volume, with the relaxed constraint

(4.6) $b_{P'} \cdot \mathbf{1} = 0$

is the one given by $P \cap H$ where $H$ is a half-space with normal $\mathbf{1}$. In our case, by symmetry, this $P'$ automatically satisfies the stronger constraint $b_{P'} = 0$. Moreover, since the boundary of $P \cap H$ is parallel to a facet of $P$ it corresponds to a divisor $\Delta$ on $X$ defining a log Fano pair $(X, \Delta)$. Thus $(X, \Delta)$ is also the K-semistable log Fano pair realizing the sup in the definition of $S(X)$, showing that $s(P) = S(X)$. We can find $H$ by imposing the constraint. We introduce the weight $w$ such that

$P \cap H = ((a - w)\Delta_3 - 1)\setminus (b\Delta_3 - 1)$.

From here it is clear that if $b_{P'} \cdot \mathbf{1} = 0$, then, in fact, the entire barycenter will vanish and the condition $b_{P'} \cdot \mathbf{1} = 0$ turns into the following fourth order polynomial equation for $w$:

$$(a - w)^3/3!((a - w)/4 - 1) - b^3/3(4/4 - 1) = 0.$$ 

The solution $w$ and the corresponding value $s(P)$ for $\mathbb{P}^3$ blown up in one point, is given by $w = \frac{2}{3}(5 - \frac{4}{\sqrt{19} - 3\sqrt{33}} - \sqrt{19} - 3\sqrt{33})$ and $\text{ns}(P) = \frac{1}{2}\text{vol}(P') = ((4 - w)^3 - 2^3) \approx 41.8$ and for $\mathbb{P}(O \oplus O(2))$, $w = (4 - \frac{4}{\sqrt{2} - \sqrt{2}} - \sqrt{2}(2 - \sqrt{3}))$ and $\text{ns}(P) = \frac{1}{2}\text{vol}(P') = 4 ((5 - w)^3 - 2^3) \approx 30.3$, where we have corrected for the non-unimodular transformation used in the second case. \qed

In the above proof we used the following

**Lemma 4.8.** Let $P$ be a closed subset of $\mathbb{R}^n$ with the origin as an interior point. Given $v \in \mathbb{R}^n$ assume that $\int_P x \cdot v > 0$. Then the maximum

$$\max_{Q \subset P: Q \cap \Delta = \emptyset, w \cdot d\lambda(x) = 0} \int_Q d\lambda$$

is attained at $Q = P \cap H$ with $H$ a closed half-space with outward pointing normal $v$. Here $d\lambda$ is Lebesgue measure.

**Proof.** Without loss of generality we can assume that $v = (0, \ldots, 0, 1)$. Denote by $(x_1, x_2, \ldots, x_{n-1}, y)$ the coordinates on $\mathbb{R}^n$. Since the origin is an interior point
of $P$ and $\int_P x \cdot v > 0$ there is a closed half-space $H$ as in the lemma satisfying $\int_{P \cap H} yd\lambda = 0$. Hence, any candidate $Q$ for the maximum in question satisfies $\int_{P \cap H} yd\lambda = \int_Q yd\lambda$. Subtracting the left hand side from the right hand side and vice versa yields $\int_{P \cap H \cup Q} yd\lambda = \int_{Q \cap P \cap H} yd\lambda$. Since $\sup_{Q \cap P \cap H} y \leq \inf_{P \cap H \cup Q} y$ it follows that $\text{vol}(P \cap H \cup Q) \geq \text{vol}(Q \cap P \cap H)$ which, in turn, implies that $\text{vol}(P \cap H) \geq \text{vol}(Q)$, as desired.

In fact, with just a slight variation of the argument above, any maximizer must be of the special form above and, in addition, assuming connectedness of $P$, the maximizer is unique. The proof of the previous proposition thus reveals that the unique toric divisor $\Delta$ on $X$ realizing the sup defining the invariant $S(X)$ is a multiple of the prime divisor $D_F$ defined by the zero-section of $P_1 = O(1) \rightarrow \mathbb{P}^2$ and hyperplane “at infinity” in $\mathbb{P}^3$ blown up at the origin in $\mathbb{C}^3 \subset \mathbb{P}^3$, respectively (i.e. the zero-section of $P_2 = O(1) \rightarrow \mathbb{P}^2$). A similar argument also applies when $X$ is the blow-up of $\mathbb{P}^2$ at the origin in $\mathbb{C}^2$ (i.e. the first Hirzebruch surface $P_3 = O(2) \rightarrow \mathbb{P}^2$). The unique maximizer for the invariant $S(X)$ is then a log Fano pair $(X, \Delta)$ for a multiple of the hyperplane divisor $D$ “at infinity” (i.e. the zero-section of $P_4 = O(2) \rightarrow \mathbb{P}^2$). Interestingly, this K-polystable log pair $(X, \Delta)$ was also singled out in [88 Cor 1.5] by the following rigidity property (answering a question of Cheltsov): it admits a rigid Kähler-Einstein metric in the sense that for any other multiple $cD$ the log pair $(X, cD)$ does not admit a Kähler-Einstein metric. The same rigidity property holds for the two three-dimensional log pairs discussed above (since there is a unique half-space $H$ satisfying the constraint in formula [140].

4.1. The height with respect to Kähler-Einstein metrics. Theorem [133] (and its corollary) generalizes directly to the case of log Fano pairs and their Kähler-Einstein metrics in any relative dimension $n$ (with the same proof, by letting $P$ be the moment polytope corresponding to $(X, \Delta)$):

$$\frac{1}{2} \text{vol}(X, \Delta) \log \left( \frac{n!m_n \pi^n}{\text{vol}(X, \Delta)} \right) \leq \frac{(-K_{(X, D)})^{n+1}}{(n+1)!} \leq \frac{1}{2} \text{vol}(X, \Delta) \log \left( \frac{(2\pi)^n \pi^n}{\text{vol}(X, \Delta)} \right)$$

Interestingly, Lemma [144] reveals that the the family of log Fano pairs $(X, D)$ appearing in the lemma may be explicitly expressed in terms of the algebro-geometric volume $\text{vol}(X, \Delta)$ in the same functional form as the one appearing in the previous upper and lower bounds:

$$\frac{(-K_{(X, D)})^{n+1}}{(n+1)!} = \frac{1}{2} \text{vol}(X, \Delta) \log \left( \frac{b e^{2a}}{\text{vol}(X, \Delta)} \right)$$

with $a := \frac{-K(X)^{n+1}/(n+1)!}{(-K_X)^{n}/n!}$ and $b = \text{vol}(X)$.

5. Sharp bounds on Donaldson’s toric Mabuchi functional

Let $(X, L)$ be a polarized complex manifold and denote by $\mathcal{H}(X, L)$ the space of all smooth metrics $\psi$ on $L$ whose curvature form $dd^c \psi$ is positive, $dd^c \psi > 0$.

5.1. The Mabuchi functional (recap). The Mabuchi functional $\mathcal{M}$ on $\mathcal{H}(X, L)$ is defined, up to addition by a constant, by declaring that its differential on $\mathcal{H}(X, L)$ at a given point $\psi$ is represented by the following measure on $X$:

$$(5.1) \quad d\mathcal{M}_\psi := (\mathcal{M}(\psi) + a) \frac{(dd^c \psi)^n}{n!}, \quad a := n(-K_X) \cdot L^{n-1}/L^n,$$
induced by $\psi$ in the Kähler form $(dd^c \psi)$, i.e. the trace of the Ricci curvature:
$$S(\psi) \frac{(dd^c \psi)^n}{n!} := \text{Ric}(dd^c \psi) \wedge \frac{(dd^c \psi)^{n-1}}{(n-1)!}.$$  

Recall that the Ricci curvature $\text{Ric}(dd^c \psi)$ of the Kähler form $dd^c \psi$ is the $(1,1)$-form defined as the curvature of the metric on $-K_X$ induced by the volume form of $dd^c \psi$. We have followed Donaldson’s multiplicative normalizations in [31] formula 3.2.1, which differ from the original definition in [32], where the measure $\frac{(dd^c \psi)^n}{n!}$ on $X$ is volume-normalized. At any rate, formula (5.1) only determines the Mabuchi functional $\mathcal{M}$ up to an additive constant.

5.1.1. *The case when $X$ is a Fano manifold and $L = -K_X$.* We now specialize to the case when $L = -K_X$ and note that a choice of reference metric $\psi_0$ in $\mathcal{C}^0(L) \cap \text{PSH}(L)$ induces a particular choice of Mabuchi functional, i.e. a functional whose differential satisfies formula (5.1) that we shall denote by $\mathcal{M}_{\psi_0}$. This is a consequence of the thermodynamical formalism introduced in [31], which expresses

$$(5.2) \quad \mathcal{M}_{\psi_0}(\psi) := \text{vol}(-K_X) F_{\psi_0}(MA(\psi)),$$

where $MA(\psi)$ is the probability measure on $X$ defined by the normalized volume form of the Kähler metric $dd^c \psi$:

$$(5.3) \quad MA(\psi) := \frac{1}{n!} \frac{(dd^c \psi)^n}{\text{vol}(L)}$$

and $F_{\psi_0}(\mu)$ denotes the free energy functional on the space $\mathcal{P}(X)$ of all probability measures on $X$, defined as follows:

$$(5.4) \quad F_{\psi_0}(\mu) := -E_{\psi_0}(\mu) + \text{Ent}_{dV_0}(\mu) \in ]-\infty, \infty]$$

Here $\text{Ent}_{dV_0}(\mu)$ denotes the entropy of $\mu$ relative to the volume form $dV_0$ on $X$ induced by $\psi_0$ (i.e. $dV_0 = e^{-\psi_0}$ in the notation of Section 2.1.2) defined by

$$\text{Ent}_{dV_0}(\mu) := \int \log \frac{\mu}{dV_0} \, d\mu$$

when $\mu \in L^1(X, dV_0)$ and otherwise $\text{Ent}_{dV_0}(\mu) := \infty$. Furthermore, $E_{\psi_0}(\mu)$ is the pluricomplex energy of $\mu$, relative to $\psi_0$, introduced in [31], which may be defined as a Legendre-Fenchel transform of the functional $\mathcal{E}_{\psi_0}/\text{vol}(L)$ (defined by formula 2.1.13). For our purposes it will be enough to define $E_{\psi_0}(\mu)$ when $\mu$ is of the form $\mu = MA(\psi)$ for $\psi$ in $\mathcal{C}^0(L) \cap \text{PSH}(L)$:

$$(5.5) \quad E_{\psi_0}(MA(\psi)) = \frac{\mathcal{E}_{\psi_0}(\psi)}{\text{vol}(L)} - \int_X (\psi - \psi_0) MA(\psi).$$

We recall that formula (5.2) follows readily from the fact that on the subspace of all volume forms $\mu$ in $\mathcal{P}(X)$ the differential of $E_{\psi_0}$ at $\mu \in \mathcal{P}(X)$ is represented by the function $\psi_0 - \psi_0$:

$$dE_{\psi_0}|_{\mu} = -(\psi_0 - \psi_0)$$

(this formula is dual to formula 2.1.14 in the sense of Legendre transforms; see [31]).

**Remark 5.1.** Formula (5.2) defines $\mathcal{M}_{\psi_0}(\psi)$ on the space $\mathcal{C}^0(L) \cap \text{PSH}(L)$ as a function taking values in $] - \infty, \infty]$. More generally, the functional $\mathcal{M}_{\psi_0}(\psi)$ is well-defined as soon as $E(MA(\psi)) < \infty$ (see [31] (7)). For $\psi$ smooth formula (5.2) is essentially equivalent to a formula for the Mabuchi functional appearing in [73] and [23].
5.1.2. The case when $X$ is a singular Fano variety. In the case when $X$ is a singular Fano variety we will denote by $\mathcal{H}(X, -K_X)$ the space of all continuous metrics $\psi$ on $L$ such that $\psi$ is smooth on the regular locus $X_{\text{reg}}$ of $X$ and $dd^c \psi > 0$ on $X_{\text{reg}}$.

5.2. Proof of Theorem 1.5. First recall the following basic inequality that holds on any Fano variety [7] Lemma 4.4:

$$ F_{\psi_0} (MA(\psi)) \geq \mathcal{D}_{\psi_0} (\psi) $$

as follows from the non-negativity of the relative entropy between two probability measures (or Jensen’s inequality). In fact, the following identity holds [7, Lemma 4.4]:

$$ \inf_{C^0(L) \cap \text{PSH}(L)} F_{\psi_0} (MA(\psi)) = \inf_{C^0(L) \cap \text{PSH}(L)} \mathcal{D}_{\psi_0} (\psi), $$

(the two infima above may, equivalently, be restricted to $\mathcal{H}(X, L)$; see the regularization result in [11]).

Combining Theorem 1.2 with the inequality 5.6 the proof is concluded by invoking the following formula relating $\mathcal{M}_{\psi_P}$ (where $\psi_P$ is the canonical toric reference defined by formula 3.3) to Donaldson’s toric Mabuchi functional

$$ \mathcal{M}_{-K_X}(\psi) := \int_{\partial P} \psi^* d\sigma - n \int_P \psi^* dx - \int_P \log \det (\nabla^2 \psi^*) dx, $$

where $\psi^*$ denotes the Legendre transform of the $T$–invariant metric $\psi \in \mathcal{H}(X, -K_X)$ and $d\sigma$ is the measure on $\partial P$, absolutely continuous wrt the $(n - 1)$–dimensional Lebesgue measure $d\lambda_{\partial P}$, defined by $d\sigma = d\lambda_{\partial P} / \|t_F\|$ on a facet $F$ of $\partial P$, where $\|t_F\|$ denotes the Euclidean norm of a primitive normal vector to $F$.

**Lemma 5.2.** Let $X$ be an $n$–dimensional toric Fano variety. The following identity holds on the space of all $T$–invariant metrics in $\mathcal{H}(X, -K_X)$:

$$ \mathcal{M}_{\psi_P} = \mathcal{M}_{-K_X} - \text{vol}(-K_X) \log \text{vol}(-K_X) $$

**Proof.** This formula is essentially the content of [6] Prop 4.6], but since the normalizations are a bit different we recall the proof. First identifying a toric metric $\psi$ with a convex function on $\mathbb{R}^n$ (as in Section 5.1.2) formula 5.2 combined with formula 5.5 yields

$$ \mathcal{M}_{\psi_P}(\psi) = -E_{\psi_P}(\psi) + \int (\psi - \psi_P)(dd^c \psi)^n / n! + \int_{\mathbb{R}^n} \log \left(\frac{MA(\psi)}{e^{-\psi^*} dx}\right) \text{vol}(-K_X) MA(\psi) = $$

$$ = \int_P \psi^* d\lambda + \int_P (dd^c \psi)^n / n! + \int_{\mathbb{R}^n} \log \det (\nabla^2 \psi) \det (\nabla^2 \psi^*) \text{vol}(-K_X) \log \text{vol}(-K_X). $$

By [6] Lemma 4.7] making the change of variables $y = \nabla \psi$ the second term above may be expressed

$$ \int_{\mathbb{R}^n} (dd^c \psi)^n / n! = \int_{\partial P} \psi^* d\sigma - (n + 1) \int u dp, $$

giving

$$ \mathcal{M}_{\psi_P}(\psi) = \int_{\partial P} \psi^* d\sigma - \int_P \psi^* d\lambda + \int_{\mathbb{R}^n} \log \det (\nabla^2 \psi) \det (\nabla^2 \psi^*) \text{vol}(-K_X) \log \text{vol}(-K_X). $$

Again making the change of variables $y = \nabla \psi$ in the remaining integral over $\mathbb{R}^n$ concludes the proof, using the standard relation $\det (\nabla^2 \psi) / (\nabla^2 \psi^*) (\nabla \psi) =$
(which follows from the fact that the map \( y \mapsto \nabla \psi^*(y) \) is the inverse of \( x \mapsto \nabla \psi(x) \)).

\[ 1 \]

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\[ (K, \omega) \] is a consequence of the restriction formula [17, Prop 2.3.1], as discussed in Section 4.4 this functional coincides, up to additive and multiplicative normalizations, with the arithmetic Mabuchi functional introduced in [63].

**Lemma 6.1.** The differential of the functional \( \psi \mapsto \mathcal{M}(X, L)(\mathcal{L}, \psi) \) on \( \mathcal{H}(X, L) \) satisfies the defining formula 5.1 of the Mabuchi functional.

**Proof.** As pointed out in [63] this formula can be deduced from the formula for the Mabuchi functional in [23, 22]. But for completeness and to check the normalizations we provide a simple direct proof. First recall the following property of arithmetic intersection numbers:

\[
\text{vol}(X) = \frac{1}{n!} \int_X \phi_0 \cdot \phi_1 \cdot \ldots \cdot \phi_n,
\]

where \( \phi_0 \) is the globally well-defined function on \( X \) defined by formula (6.1) when \( e_U \) is the standard global trivialization 1 of the trivial line bundle over \( X \), i.e. \( \phi_0/2 = -\log \|s\|_{\phi_0} \), where \( s \) is a global trivialization of \( L \). In particular, differentiating along a curve \( t \mapsto \psi_t \) in \( \mathcal{H}(X, L) \) and using the symmetry of arithmetic intersection numbers gives

\[
\frac{d}{dt} \left( \mathcal{L}, \psi_t \right)^{n+1} = (n + 1) \left( \mathcal{L}, \psi_t \right)^n \cdot \left( \mathcal{L}, \psi_t \right)^n = \frac{1}{2} \int_X \psi_t (dd^c \psi_t)^n
\]

where \( \psi_t \) is a globally well-defined function on \( X \) and can thus be identified with a metric on the trivial line bundle that we denote by \( \mathcal{L}_0 \). Likewise, denoting by \( \rho_t \) a local density for \( MA(\psi_t) \) with respect the Euclidean measure defined by local holomorphic coordinates,

\[
\frac{d}{dt} \left( \mathcal{K}_X, \log \rho_t \right) \left( \mathcal{L}, \psi_t \right)^n = (\mathcal{K}_X, \log \rho_t) n \left( \mathcal{L}, \psi_t \right)^n \cdot \left( \mathcal{L}, \psi_t \right)^n = \frac{1}{2} \int_X \psi_t (dd^c \psi_t)^n
\]

where we have used Leibniz rule. Applying formula (6.1) the second term above may, after multiplication by 2, be expressed as

\[
\int_X \frac{d}{dt} \log \rho_t (dd^c \psi_t)^n = n! \text{vol}(L) \int_X \frac{d}{dt} \log \rho_t \rho_t = n! \text{vol}(L) \frac{d}{dt} \int_X \rho_t = 0,
\]

using in the last equality that \( \int_X \rho_t = \text{vol}(L) \) for any \( t \). Likewise, applying formula (6.1) to the first term in formula (6.2) yields

\[
2 (\mathcal{K}_X, \log \rho_t) \left( \mathcal{L}, \frac{d\psi}{dt} \right) / n = \int_X \frac{d\psi}{dt} dd^c (\log \rho_t) \wedge (dd^c \psi_t)^n = - \int_X \frac{d\psi}{dt} \text{Ric}(dd^c \psi_t) \wedge (dd^c \psi_t)^n - 1.
\]
The following proposition relates the arithmetic Mabuchi functional $\mathcal{M}(X, -K_X)$ to Donaldson’s toric Mabuchi functional $\mathcal{M}_{-K_X}$ (formula (5.3)).

**Proposition 6.2.** Given a toric Fano variety $X$ denote by $\mathcal{X}$ its canonical integral model. Then the following formula holds for any $T$–invariant metric in $\mathcal{H}(X, -K_X)$:

$$\mathcal{M}(X, -K_X) = \mathcal{M}_{-K_X} - \log(-K_X) \log \mathrm{vol}(-K_X)$$

**Proof.** In this case $a = n$ and thus we can decompose $\frac{1}{2} \mathcal{M}(X, \mathcal{L})(\psi)$ as

$$\frac{1}{(n+1)!} c^n + \frac{1}{n!} (\mathcal{L} + K_X) \cdot c^n = - \frac{1}{(n+1)!} c^n + \frac{1}{2} \int \log(MA(\psi)) (dd^c \psi)^n / n!$$

where, in the last equality, we have exploited that $\mathcal{L} + K_X$ is trivial so that formula (6.1) applies. Applying formula (5.7) to the first term in the rhs above thus gives

$$\mathcal{M}(X, \mathcal{L})(\psi) := -\mathcal{E}_\psi(\psi) + \int \log(MA(\psi)) (dd^c \psi)^n / n! = \log(-K_X) \left( - \frac{1}{V(X)} \mathcal{E}_\psi(\psi) + (\psi - \psi_p, MA(\psi)) + \int \log(MA(\psi)) (dd^c \psi)^n / n! \right).$$

The rhs in the last equation above equals $\mathcal{M}_{\psi_p}(\psi)$ (by definition (5.2)). Invoking Lemma (5.2) thus concludes the proof.

Next, consider a general arithmetic variety $X$ whose relative anti-canonical line bundle $-K_X$ is defined as a $\mathbb{Q}$–line bundle and assumed relatively ample. Denote by $D_Z(\psi)$ the functional defined by formula (2.17) corresponding to the model $\mathcal{L} = -K_X$. In this arithmetic setup the following variants of the inequality (5.6) and the identity (5.7) hold.

**Proposition 6.3.** When $\mathcal{L} = -K_X$ and $\pi_* \mathcal{O}_X = \mathbb{Z}$ the following relations hold:

$$\mathcal{M}(X, -K_X) \geq \log(-K_X) D_Z$$

and

$$\inf_{c^0(\mathcal{L}) \cap \mathcal{PSH}(\mathcal{L})} \mathcal{M}(X, \mathcal{L}) = \log(-K_X) \inf_{c^0(\mathcal{L}) \cap \mathcal{PSH}(\mathcal{L})} D_Z.$$
Accordingly, expressing \((L, \psi)^{n+1} = (L_0, \psi_0)^{n+1} + (n+1)!E_\psi_0(\psi)/2\) (using Lemma 2.9) gives
\[
\frac{1}{2\text{vol}(-K_X)}M_{(X, -K_X)}(\psi) = \frac{1}{2}F_{\psi_0}(MA(\psi)) - \frac{1}{(n+1)!}(L, \psi_0)^{n+1},
\]
where \(F_{\psi_0}(\mu)\) is the free energy functional [7]. The proof is thus concluded by invoking the identity 5.7.

Remark 6.4. When \(-K_X\) admits a Kähler-Einstein metric \(\phi_{KE}\) both the infima in the previous proposition are attained at \(\phi_{KE}\) [7]. The identity then follows directly from the Kähler-Einstein equation, giving \(MA(\phi_{KE}) = e^{-\phi_{KE}},\) when \(\phi_{KE}\) is volume-normalized.

In Section 7.1 the inequality in the previous proposition will be generalized to any model \((X', L')\) of \((X, -K_X)\) by introducing an arithmetic Ding functional \(D_{(X, L)}\), coincides with the functional \(D_Z\) under the conditions in the previous proposition.

7. Comparison with the function field case and outlook

The function field analog of the arithmetic setup in Conjecture 1.1 is a flat projective fibration
\[
\mathcal{X} \to \mathcal{B}
\]
of a normal complex projective variety \(\mathcal{X}\) to a non-singular complex projective curve \(\mathcal{B}\) with the property that the generic fiber is isomorphic to \(X\) and such that the relative anti-canonical divisor \(-\mathcal{K}_{\mathcal{X}/\mathcal{B}}\) defines a relatively ample \(\mathbb{Q}\)-line bundle. The analog of the inequality in Conjecture 1.1 does hold in this situation, but not the uniqueness statement. More precisely, if \((X, -K_X)\) is assumed K-semistable then
\[
(-\mathcal{K}_{\mathcal{X}/\mathcal{B}})^{n+1} \leq 0
\]
(a special case of this inequality appears in [62, Thm 4.1] and the general case follows from results in [14], as pointed out to us by Yuji Odaka). Equality holds for the trivial fibration \(\mathcal{X} = X \times \mathcal{B}\). In particular,
\[
(-\mathcal{K}_{\mathcal{X}/\mathcal{B}})^{n+1} \leq (-\mathcal{K}_{\mathbb{P}^n \times \mathcal{B}/\mathcal{B}})^{n+1} = 0
\]
which is the function field analog of the inequality in Conjecture 1.1. Note that when \(\mathcal{B} = \mathbb{P}^1\) and the \(\mathbb{C}^*\)-action on \(\mathbb{P}^1\) lifts to \(\mathcal{X}\), the inequality 7.1 follows directly from the definition of K-semistability.

In contrast to Conjecture 1.1 this means that in the function field case equality holds in the inequality 7.2 not only for \(\mathbb{P}^n \times \mathcal{B}\), but also for any product \(\mathcal{X} = X \times \mathcal{B}\). Hence, in the function field case, projective space plays no special role. Conversely, it should be stressed that the analog of the inequality 7.1 fails in the arithmetic situation (by the strict positivity in Lemma 3.6). Our general motivation for Conjecture 1.1 is rather that projective space maximizes the degree of \(-K_X\) [37], as well as a range of other positivity properties of \(-K_X\) (see, for example, the discussion and references in the introduction of [50]).
7.1. Outlook on a generalization of Conjecture 1.1. We conclude with some speculations on a more general formulation of Conjecture 1.1. Consider a Fano variety $X_F$ defined over a number field $F$, i.e., a field extension $F$ of $\mathbb{Q}$ of finite degree $[F : \mathbb{Q}]$. Let $(\mathcal{X}, \mathcal{L})$ be a normal irreducible model of $(X_F, -K_{X_F})$ over the ring of integers $\mathcal{O}_F$ of $F$. Let $\mathcal{K}_{X/\text{Spec} \mathcal{O}_F}$ be defined as a $\mathbb{Q}$-line bundle. We will denote by $\psi$ a collection of continuous psh $\psi_\sigma$ metrics on $-K_X$, as $\sigma$ ranges over all embeddings of the field $F$ into $\mathbb{C}$, where $X_\sigma$ denotes the corresponding complex projective varieties. To the model $(\mathcal{X}, \mathcal{L})$ we attach an arithmetic Ding functional, defined as follows. First consider a model $(\mathcal{X}, \mathcal{L})$ of $(X_F, -K_{X_F})$ such that $\mathcal{L} + \mathcal{K}_{X/\text{Spec} \mathcal{O}_F}$ defines a bona fide line bundle. Then

$$D_{(\mathcal{X}, \mathcal{L})}(\psi) := -\left(\frac{\mathcal{L}, \psi}{[F : \mathbb{Q}]}(n+1)(-K_X)^n + \frac{1}{[F : \mathbb{Q}]}\deg \pi_*(\mathcal{L} + \mathcal{K}_{X/\text{Spec} \mathcal{O}_F})\right),$$

where the second term above denotes the arithmetic (Arakelov) degree of the line bundle $\pi_*(\mathcal{L} + \mathcal{K}_{X/\text{Spec} \mathcal{O}_F}) \to \text{Spec} \mathcal{O}_F$, endowed with the $L^2$-metric induced by the metric $\psi$ on $\mathcal{L}$ (i.e., on $-K_X$). More generally, when $\mathcal{K}_{X/\text{Spec} \mathcal{O}_F}$ is merely defined as a $\mathbb{Q}$-line bundle we fix a positive integer $r$ such that $r(\mathcal{L} + \mathcal{K}_{X/\text{Spec} \mathcal{O}_F})$ is defined as a line bundle and replace $\deg \pi_*(\mathcal{L}, (\mathcal{L} + \mathcal{K}_{X/\text{Spec} \mathcal{O}_F}))$ with $r^{-1}\deg \pi_*(\mathcal{L}, (r(\mathcal{L} + \mathcal{K}_{X/\text{Spec} \mathcal{O}_F})))$, where now $\pi_*(r(\mathcal{L} + \mathcal{K}_{X/\text{Spec} \mathcal{O}_F}))$ is endowed with the $L^2/r$-metric induced by $\psi$. Concretely, this means that

$$(7.3) \quad 2r^{-1}\deg \pi_*(r(\mathcal{L} + \mathcal{K}_{X/\text{Spec} \mathcal{O}_F})) = -\sum_{\sigma} \log \int_{X_\sigma} \frac{|s_\sigma|^2/r e^{-\psi_\sigma}}{\sigma},$$

where $s_\sigma$ is a generator of the rank one $\mathcal{O}_F$-module $H^0(\mathcal{X}, r(\mathcal{L} + \mathcal{K}_{X/\text{Spec} \mathcal{O}_F}))$ and $|s_\sigma|^2/r e^{-\psi_\sigma}$ denotes corresponding measure on $X_\sigma$.

The functional $D_{(\mathcal{X}, \mathcal{L})}$ coincides with the functional $D_Z$, defined in formula (2.17) up to an additive constant. Indeed, replacing $s_\sigma$ with $1 \in H^0(X_\sigma, \mathbb{C})$ in formula (2.17) and applying the product formula in $\mathcal{O}_F$ gives

$$(7.4) \quad [F : \mathbb{Q}] D_{(\mathcal{X}, \mathcal{L})}(\psi) := D_Z(\psi) + \frac{1}{r} \log |p| \sum_p \text{ord}_p(1),$$

where $\text{ord}_p(1)$ denotes the order of vanishing at the closed point $p$ in $\text{Spec} \mathcal{O}_F$ of the rational section $"1"$ of the line bundle $\pi_*(r(\mathcal{L} + \mathcal{K}_{X/\text{Spec} \mathcal{O}_F})) \to \text{Spec} \mathcal{O}_F$ coinciding with $1 \in H^0(X_\mathbb{Q})$ on the generic fiber and $|p|$ denotes the norm of the ideal in $\mathcal{O}_F$ defined by $p$ (i.e., the cardinality of the corresponding residue field; in particular, $\log |p| \geq 0$).

Remark 7.1. The functional $D_{(\mathcal{X}, \mathcal{L})}(\psi)$ is the arithmetic analog of the degree of the Ding line bundle of a test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, -K_X)$ introduced in [5]. As shown in [38], Fano variety $X$ is K-semistable iff the degree of the Ding line bundle is non-negative for any test configuration $(\mathcal{X}, \mathcal{L})$.

Now consider the following invariant of the Fano variety $X_F$:

$$D(X_F) := (-K_X)^n \inf D_{(\mathcal{X}, \mathcal{L})},$$

where the the inf runs over all integral models $(\mathcal{X}, \mathcal{L})$ of $(X, -K_X)$ and metrics $\psi$ as above.

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Conjecture 7.2. Let $X_F$ be a $K$-semistable Fano variety defined over a number field $F$. Then

$$D(X_F) \geq (-K_{\mathbb{P}^n})^n D((\mathbb{P}^n_k, -K_{\mathbb{P}^n}))\psi_{FS},$$

where $\psi_{FS}$ denotes the volume-normalized Fubini-Study metric $\psi_{FS}$ on $-K_{\mathbb{P}^n}$. Equivalently, for any model $(X, \mathcal{L})$ and continuous psh metric $\psi$, normalized so that $\deg \pi_i H^0(X, (\mathcal{L} + K_{\mathcal{X}/\text{Spec } O_F})) = 0$,

$$\frac{1}{|F : \mathbb{Q}|} (\mathcal{L}, \psi)^{n+1} \leq (-K_{\mathbb{P}^n}, \psi_{FS})^{n+1}.$$

Moreover, equality holds if and only if $(X, \mathcal{L}) = (\mathbb{P}^n_k, -K_{\mathbb{P}^n})$ and $\psi$ coincides with $\psi_{FS}$, up to the action of an automorphism of $\mathbb{P}^n$.

For example, by formula [7.3] when $F = \mathbb{Q}$ and $\mathcal{L}$ equals $-K_{\mathcal{X}/\text{Spec } O_F}$ the second inequality in the previous conjecture specializes to Conjecture [7.4] if $\pi_*O_\mathcal{X}$ coincides with $O_F$ (which ensures that the integral lattice in $H^0(X, O_\mathcal{X})$ corresponding to $H^0(\pi_*O(X))$ is generated by the constant function 1 on $X$). This condition can always be achieved after a base change, using Stein factorization (thanks to Dennis Eriksson and Gerard Freixas i Montplet for pointing this out). We expect that any integral model $(X, \mathcal{L})$ which is globally $K$-semistable realizes the infimum defining the invariant $D(X_F)$, inspired by Odaka’s conjecture discussed in Section [1.6].

In general, the following inequality between the arithmetic Mabuchi functional and the arithmetic Ding functional holds, showing, in particular, that Conjecture 7.2 implies Conjecture 1.6 concerning Odaka’s modular invariant.

Proposition 7.3. Let $(X, \mathcal{L})$ be a normal irreducible model of $(X, -K_X)$ over $\text{Spec } O_F$ which is $\mathbb{Q}$-Gorenstein. Then

$$\mathcal{M}_{(X, \mathcal{L})}(\psi) \geq \text{vol}(-K_X) D((X, \mathcal{L})(\psi)$$

with equality iff $\psi$ is a Kähler-Einstein metric and $\mathcal{L} = -K_{\mathcal{X}/\text{Spec } O_F}$.

Proof. To simplify the notation we assume that $r = 1$ and $F = \mathbb{Q}$ (but the proof in the general case is essentially the same). Let $s$ be a generator of the rank one module $H^0(X, \mathcal{L} + K)$, where $K := K_{\mathcal{X}/\text{Spec } O_F}$. It follows directly from the definitions that we need to prove that

$$\frac{1}{n!L^n} (\mathcal{L} + K) \cdot \mathcal{L}^n - \deg \pi_*(\mathcal{L} + K_{\mathcal{X}/\text{Spec } O_F}) \geq 0$$

with equality iff the conditions in the proposition hold. After scaling the metric we may as well assume that $\deg \pi_*(\mathcal{L} + K_{\mathcal{X}/\text{Spec } O_F}) = 0$, i.e. that $|s|^2 e^{-\psi}$ is a probability measure on $X$. Then

$$\frac{1}{n!L^n} (\mathcal{L} + K) \cdot \mathcal{L}^n \geq \int_X \log \left( \frac{MA(\psi)}{|s|^2 e^{-\psi}} \right) MA(\psi) =: \text{Ent}_{|s|^2 e^{-\psi}} (MA(\psi))$$

Indeed, by the restriction formula for arithmetic intersection numbers [17, Prop 2.3.1]

$$\frac{1}{n!L^n} (\mathcal{L} + K) \cdot \mathcal{L}^n = \int_X \log \left( \frac{MA(\psi)}{|s|^2 e^{-\psi}} \right) MA(\psi) + (s = 0) \cdot \mathcal{L}^n,$$

where $(s = 0)$ denotes the subscheme of $\mathcal{X}$ cut out by $s$. The second term above is non-negative since $(s = 0)$ is supported on the closed fibers (by assumption $s$ is non-vanishing over the generic fiber). Moreover, since $\mathcal{L}$ is relatively ample the
term vanishes iff \( s \) is globally non-vanishing, i.e. \( \mathcal{L} + \mathcal{K} \) is trivial. Finally, the first term above is proportional to the relative entropy \( \text{Ent}_{\log e^\psi} (MA(\psi)) \) between the probability measures \( MA(\psi) \) and \( |s|^2 e^{-\psi} \) and thus non-negative and vanishes iff the two probability measures coincide, i.e. \( \psi \) is Kähler-Einstein. \( \square \)

7.2. Comparison with bounds on Bost-Zhang’s normalized heights. The arithmetic Ding functional \( \mathcal{D}(\mathcal{X}, \mathcal{L}) \) is reminiscent of Bost’s normalized height \( h_{\text{norm}} \), introduced in \([16]\) in the general setup of polarized variety \((X_F, L_F)\) defined over a number field \( F \):

\[
h_{\text{norm}}(\mathcal{L}, \psi) := \frac{(\mathcal{L}, \psi)^{n+1}}{[F : \mathbb{Q}](n+1)(L_F)^n} - \frac{1}{[F : \mathbb{Q}]N} \hat{\deg}_* (\mathcal{X}, \mathcal{L}),
\]

assuming that the rank \( N \) of the vector bundle \( \pi_* (\mathcal{X}, \mathcal{L}) \to \text{Spec} \mathcal{O}_F \) is non-zero and \( \pi_* (\mathcal{X}, \mathcal{L}) \) is endowed with the \( L^2 \)–norm induced by the continuous psh metrics \( \psi_* \) on \( L_\sigma \) and the volume forms \( MA(\psi_* \pi) \) on \( X_\sigma \) (defined by formula \([80]\)). When \( L_F \) is very ample it is shown in \([16]\) that the functional \( h_{\text{norm}}(\mathcal{L}, \cdot) \) is bounded from below iff the Chow point of \((X_F, L_F)\) is semistable wrt the action of the group \( GL(N, F) \) on the Chow variety (in the sense of Geometric Invariant Theory). More precisely, it is shown in \([16]\) that the semi-stability in question is equivalent to a lower bound on Bost’s intrinsic normalized height of \((X_F, L_F)\):

\[
\inf h_{\text{norm}} > -\infty
\]

where the infimum runs over all models \((\mathcal{X}, \mathcal{L})\) and metrics \( \psi \) as above. In fact, by \([16] \) Prop 2.1 and \([80]\) Thm 4.4] the Chow-semistability in question is equivalent to the following explicit lower bound:

\[
(7.6) \quad h_{\text{norm}}(\mathcal{L}, \psi) \geq -\frac{1}{2} \sum_{n=1}^{N+1} \sum_{m=1}^{n} \frac{1}{m} - \frac{1}{2} \log N
\]

(it is moreover conjectured in \([80]\) that the first term in the right hand side above may be replaced by 0).

In this setup the role of the normalization \( \hat{\deg}_* H^0(\mathcal{X}, (\mathcal{L} + \mathcal{K}_\mathcal{X}/\text{Spec} \mathcal{O}_F)) = 0 \) in Conjecture \([24]\) is thus played by the normalization \( \deg_* H^0(\mathcal{X}, \mathcal{L}) = 0 \). However, in contrast to Conjecture \([24]\) the lower bound \( (7.6) \) on \( h_{\text{norm}}(\mathcal{L}, \psi) \) corresponds to a lower bound on \( (\mathcal{L}, \psi)^{n+1} \) for any normalized metric. Note also that one virtue of the normalization condition in Conjecture \([24]\) is that it is comparatively explicit, since \( \pi_* (\mathcal{L} + \mathcal{K}_\mathcal{X}/\text{Spec} \mathcal{O}_F) \) has rank one (so that formula \([43]\) applies, showing that it is enough to assume that the volume forms \( |s|^2 e^{-\psi_*} \) on \( X_\sigma \) are normalized). Another advantage of this normalization condition is that it applies to any continuous metric \( \psi \) (at the price of replacing \( (\mathcal{L}, \psi)^{n+1} \) with the \( \chi \)–arithmetic volume of \( \mathcal{L} \), as in Theorem \([24]\).

Finally, we recall that when \( \mathcal{L} \) is replaced by \( k \mathcal{L} \) for a large positive integer \( k \) it follows from \([53]\) Thm 3.7] that there exists constants \( a \) and \( b \) (depending only on \((X_F, L_F))\) such that \( a > 0 \)

\[
(7.7) \quad \mathcal{M}_{(\mathcal{X}, \mathcal{L})}(\psi)/L^n = h_{\text{norm}}(k \mathcal{L}, \psi) - a \log N_k + b + o(1),
\]

as \( k \to \infty \), where \( N_k \) denotes the rank of \( \pi_* H^0(\mathcal{X}, k \mathcal{L}) \) which diverges as \( k \to \infty \). Unfortunately, the diverging term \( a \log N_k \) makes it impossible to infer lower bounds on \( \mathcal{M}_{(\mathcal{X}, \mathcal{L})}(\psi) \) from lower bounds on \( h_{\text{norm}}(k \mathcal{L}) \). Since \( \mathcal{M}_{(\mathcal{X}, \mathcal{L})}(\psi) \) coincides with
when $\mathcal{L}$ equals $-K_X/\text{Spec}O_F$, this means that Conjecture 7.2 cannot be deduced from bounds of the form $\mathcal{L} 7.4$ by letting $k$ (and hence $N$) tend to infinity.

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