Gallai–Ramsey Numbers Involving a Rainbow 4-Path

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Abstract
Given two non-empty graphs $G$, $H$ and a positive integer $k$, the Gallai–Ramsey number $gr_k(G : H)$ is defined as the minimum integer $N$ such that for all $n \geq N$, every $k$-edge-coloring of $K_n$ contains either a rainbow colored copy of $G$ or a monochromatic copy of $H$. In this paper, we got some exact values or bounds for $gr_k(P_5 : H)$ ($k \geq 3$) if $H$ is a general graph or a star with extra independent edges or a pineapple.

Keywords Ramsey theory · Gallai–Ramsey number · Pineapple · Star with extra independent edges

Mathematics Subject Classification 05D10 · 05C15

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1 Introduction

All graphs considered are finite, simple and undirected. We follow the notation and terminology of Bondy [1]. Let $V(G)$, $E(G)$, $e(G)$, $\delta(G)$ be the vertex set, edge set, size, minimum degree of graph $G$, respectively. We use $G - X$ to denote the subgraph of $G$ obtained by removing all the vertices of $X$ together with the edges incident with them from $G$; similarly, we use $G \setminus M$ to denote the subgraph of $G$ obtained by removing all the edges of $M$ from $G$. The union $G \cup H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We call the path $P_{t+1}$, of order $t + 1$ and having $t$ edges, a $t$-path. A complete graph is a graph in which every pair of vertices are adjacent, and a complete graph on $n$ vertices is denoted by $K_n$. Let $[a, b]$ be the interval from $a$ to $b$.

An $k$-edge-coloring is exact if all colors are used at least once. In this work, we only consider exact edge-colorings of graphs. An edge colored graph is called rainbow if all edges have different colors and monochromatic if all edges have a single color.

1.1 Classical Ramsey Number

Ramsey theory were introduced in 1930; see [23]. The main subject of the theory are complete graphs whose subgraphs can have some regular properties.

Definition 1 Given $k$ graphs $H_1$, $H_2$, ..., $H_k$, the Ramsey number $R(H_1, H_2, \ldots, H_k)$ is defined as the minimum number of vertices $n$ needed so that every $k$-edge-coloring of $K_n$ contains a monochromatic $H_i$, where $1 \leq i \leq n$.

If $H_1 = H_2 = \cdots = H_k$, then we write the Ramsey number as $R_k(H)$. If $H_i = K_{k_i}$ ($1 \leq i \leq k$), then we use the abbreviation $R(k_1, \ldots, k_k)$.

Ramsey number has its applications on the fields of communications, information retrieval in computer science, and decision-making; see [24, 25] for examples. We refer the interested reader to [22] for a dynamic survey of small Ramsey numbers.

1.2 Gallai–Ramsey Number

Edge colorings of complete graphs that contain no rainbow triangle have very interesting and somewhat surprising structure. In 1967, Gallai [8] first examined this structure under the guise of transitive orientations of graphs and it can also be traced back to [2]. For this reason, colored complete graphs containing no rainbow triangle are called Gallai colorings. Gallai’s result was restated in [11] in the terminology of graphs. For the following statement, a trivial partition is a partition into only one part.

Theorem 1.1 [2, 8, 11] In any coloring of a complete graph containing no rainbow triangle, there exists a nontrivial partition of the vertices (called a Gallai partition), say $H_1, H_2, \ldots, H_t$, satisfying the following two conditions.

(a) The number of colors on the edges among $H_1, H_2, \ldots, H_t$ are at most two.

(b) For each pair of parts $H_i, H_j$ ($1 \leq i \neq j \leq t$), all the edges between $H_i$ and $H_j$ receive the same color.
The induced subgraph of a Gallai colored complete graph constructed by selecting a single vertex from each part of a Gallai partition is called the reduced graph. By Theorem 1.1, the reduced graph is a 2-colored complete graph. This kind of restriction on the distribution of colors has led to a variety of interesting works like [9, 13].

**Definition 2** Given two graphs $G$ and $H$, the general $k$-colored Gallai–Ramsey number $gr_k(G : H)$ is defined to be the minimum integer $m$ such that every $k$-coloring of the complete graph on $m$ vertices contains either a rainbow copy of $G$ or a monochromatic copy of $H$.

With the additional restriction of forbidding the rainbow copy of $G$, it is clear that $gr_k(G : H) \leq R_k(H)$ for any $G$. Till now, most work focuses on the case $G = K_3$; see [3, 4, 7, 11–13, 18–21, 28, 31]. For more details on the Gallai-Ramsey theory, we refer to the book [16] and papers [6, 17].

### 1.3 Structural Theorems and Main Results

Thomason and Wagner [27] obtained the structural theorems for $P_4$ and $P_5$.

**Theorem 1.2** [27] Let $K_{n,n} \geq 4$, be edge colored so that it contains no rainbow $3$-path $P_4$. Then one of the following holds:

(a) at most two colors are used;

(b) $n = 4$ and three colors are used, each color forming a $1$-factor.

In an edge-colored graph, define $V^{(j)}$ as the set of vertices with at least one incident edge in color $j$ and denote $E^{(j)}$ to be the set of edges of color $j$ for a given color $j$.

**Theorem 1.3** [27] Let $K_{n,n} \geq 5$, be edge colored so that it contains no rainbow $4$-path $P_5$. Then, after renumbering the colors, one of the following must hold:

(a) at most three colors are used;

(b) color $1$ is dominant, meaning that the sets $V^{(j)}$, $j \geq 2$, are disjoint;

(c) $K_n - a$ is monochromatic for some vertex $a$;

(d) there are three vertices $a, b, c$ such that $E^{(2)} = \{ab\}, E^{(3)} = \{ac\}, E^{(4)}$ contains $bc$ plus perhaps some edges incident with $a$, and every other edge is in $E^{(1)}$;

(e) there are four vertices $a, b, c, d$ such that $\{ab\} \subseteq E^{(2)} \subseteq \{ab, cd\}, E^{(3)} = \{ac, bd\}, E^{(4)} = \{ad, bc\}$ and every other edge is in $E^{(1)}$;

(f) $n = 5$, $V(K_n) = \{a, b, c, d, e\}$, $E^{(1)} = \{ad, ae, bc\}$, $E^{(2)} = \{bd, be, ac\}$, $E^{(3)} = \{cd, ce, ab\}$ and $E^{(4)} = \{de\}$.

Li et al. [15] got some exact values and bounds of $gr_k(P_5 : K_t)$, and investigated the edge-colorings of complete graphs and complete bipartite graphs without rainbow $4$-path and $5$-path. Fujita and Magnant [5] got the structural theorem for $G = S_t^+$ like Theorem 1.1. Li and Wang [14] studied the monochromatic stars in rainbow $K_3$-free and $S_t^+$-free colorings.

We now give the definitions of two graph classes.

The graph $S_t'$ is obtained from a star of order $t$ by adding an extra $r$ independent edges between the leaves of the so that there are $r$ triangles and $t - 2r - 1$ pendent...
edges in $S'_r$. For $r = 0$ we obtain $S'_r = K_{1,t-1}$, which are called stars. For $r = \frac{t-1}{2}$, if $t$ is odd we obtain $S'_r = F_{\frac{t-1}{2}}$, which are called fans.

A pineapple $PA_{t,\omega}$ is a graph obtained from the complete graph $K_\omega$ by attaching $t - \omega$ pendent vertices to the same vertices of $K_\omega$, we suppose that $t \geq \omega + 1$.

In Sect. 2, we get some exact values or bounds for $gr_k(P_5 : H)$, where $H$ is a general graph. In Sect. 3, we obtain some results when $H = S'_r$, where $t \geq 2r + 2$ and $r \geq 1$. We also get some results in Sect. 4 when $H$ is a pineapple.

## 2 General Results

In this section, we assume that $H$ is a graph of order $t$.

**Theorem 2.1** For two integers $k$, $t$ with $k \geq 7$ and $k \geq t + 1$, we have

$$gr_k(P_5 : H) = \left\lceil \frac{1 + \sqrt{1 + 8k}}{2} \right\rceil.$$  

**Proof** Let $N_k$ be an integer with $N_k = \left\lceil \frac{1 + \sqrt{1 + 8k}}{2} \right\rceil$. For the lower bound, if there is a $k$-edge-coloring $\chi$ of a complete graph $K_{N_k - 1}$, then $k \leq \left\lceil \frac{N_k - 1}{2} \right\rceil$, contradicting with $N_k = \left\lceil \frac{1 + \sqrt{1 + 8k}}{2} \right\rceil$. It follows that $gr_k(P_5 : H) \geq N_k$.

It suffices to show that $gr_k(P_5 : H) \leq N_k$. Let $\chi$ be any $k$-edge-coloring of $K_n$ ($n \geq N_k$) containing no rainbow copy of $P_5$. From Theorem 1.3, (b) or (c) is true. If (b) is true, let $V^{(2)}$, $V^{(3)}$, ..., $V^{(k)}$ be a partition of $V(K_n)$ such that there are only edges of color 1 or $i$ within $V^{(i)}$ for $2 \leq i \leq k$, and hence there are only edges of color 1 among the parts. Choose one vertex of $V^{(i)}$, say $v_i$. Then the subgraph induced by $\{v_2, v_3, \ldots, v_k\}$ is a complete graph $K_{k-1}$. For $k \geq t + 1$, there exists a monochromatic copy of $K_t$ colored by 1, and hence there is a monochromatic copy of $H$ colored by 1. If (c) is true, then there is a vertex $v$ such that $K_{n - v}$ is monochromatic and $n \geq k$. For $k \geq t + 1$ and $n \geq t + 1$, there is a monochromatic copy of $K_t$, and so we can find a monochromatic copy of $H$. □

**Theorem 2.2** Let $k$, $t$ be two integers with $k = 5, 6$, $k \geq t + 1$ and $t \geq 3$. Then $gr_k(P_5 : H) = 5$.

**Proof** For the lower bound, we first suppose $k = 5$. Let $G_1$ denote a colored complete graph $K_4$ with $V(K_4) = \{v_i \mid 1 \leq i \leq 4\}$ under the 5-edge-coloring $\chi$ such that $\chi(v_1v_2) = \chi(v_3v_4) = 1$, $\chi(v_1v_3) = 2$, $\chi(v_1v_4) = 3$, $\chi(v_2v_3) = 4$ and $\chi(v_2v_4) = 5$. Next, we suppose $k = 6$. Let $G_2$ denote a colored complete graph $K_4$ with $V(K_4) = \{v_i \mid 1 \leq i \leq 4\}$ under the 6-edge-coloring $\chi$ such that $\chi(v_1v_2) = 1$, $\chi(v_3v_4) = 2$, $\chi(v_1v_3) = 3$, $\chi(v_1v_4) = 4$, $\chi(v_2v_3) = 5$ and $\chi(v_2v_4) = 6$. Since both $G_1$ and $G_2$ contain neither a rainbow copy of $P_5$ nor a monochromatic copy of $H$, it follows that $gr_k(P_5 : H) \geq 5$ for $k = 5, 6$.

It suffices to show that $gr_k(P_5 : H) \leq 5$. Suppose $G$ is any $k$ ($k = 5, 6$)-edge-coloring of $K_n$ ($n \geq 5$) which contains no rainbow copy of $P_5$. From Theorem 1.3, (b) or (c) is true, and the proof is similar to Theorem 2.1. □
For $k = t$, we first show that $H$ is not a complete graph.

**Theorem 2.3** Let $k, t$ be two integers with $k \geq 5$ and $k = t$. Then
\[
\text{gr}_k (P_5 : H) = t + 1.
\]

**Proof** For the lower bound, from Theorem 2.1, let $G_3$ be a complete graph obtained from a $K_{t-1}$ with vertex set $\{u_1, u_2, \ldots, u_{t-1}\}$ colored with 1 by adding a new vertex $u$ and edges $u_iu (1 \leq i \leq t-1)$ colored by $i+1$. Note that there is neither a rainbow copy of $P_5$ nor a monochromatic copy of $H$. Thus $\text{gr}_k (P_5 : H) \geq t + 1$.

It suffices to show that $\text{gr}_k (P_5 : H) \leq t + 1$. Let $\chi$ be any $k$-edge-coloring of $K_n$ ($n \geq t + 1$) containing no rainbow copy of $P_5$. From Theorem 1.3, (b) or (c) is true. If (b) is true, choose one vertex of $V(i) (2 \leq i \leq k)$, say $v_i$. Then $\{v_2, v_3, \ldots, v_k\}$ induces a complete graph $K_{k-1}$. For $k \geq t$, there exists a monochromatic copy of $K_{t-1}$ with color 1. As $|V(i)| \geq 2$, we can choose another vertex of $V(2)$, say $u$. Note that $\{u, v_2, v_3, \ldots, v_k\}$ induces a monochromatic $K_{t-e}, e = uv_2$. Then there is a copy of $H$ with color 1. If (c) is true, there is a vertex $v$ such that $K_n - v$ is monochromatic. For $n \geq t + 1$, there is a monochromatic copy of $K_t$, we can find a monochromatic copy of $H$. \(\square\)

Next we obtain the result on $H$ is not a complete graph by the following lemma.

**Lemma 2.1** For $4 \leq k \leq a$, if $H_t$ is a graph of order $t$ and $K_a$ ($a \geq 3$) is the maximal clique of $H_t$, then $\text{gr}_k (P_5 : H_t) \geq (a-1)(t-1) + 1$.

**Proof** Let $G_4$ be a complete graph with $V(G_4) = U_2 \cup U_3 \cup \cdots \cup U_a$ such that the graph induced by $U_i$ ($2 \leq i \leq a$) is a complete graph $K_{t-1}$, the set $\{U_2, U_3, \ldots, U_a\} = X_2 \cup \cdots \cup X_k$ with each $K_{t-1}$ of $X_j$ ($2 \leq j \leq k$) colored by $j$ and $|X_j| = \lceil \frac{a-1}{k-1} \rceil$ or $\lfloor \frac{a-1}{k-1} \rfloor + 1$ ($2 \leq j \leq k$), $|X_2| + |X_3| + \cdots + |X_k| = a-1$, all edges between $U_i$ and $U_s$ ($i \neq s$) are colored by $1$. Thus $|V(G_4)| = (a-1)(t-1)$. It is clear that $G_4$ contains neither a rainbow copy of $P_5$ nor a monochromatic copy of $H_t$, and so $\text{gr}_k (P_5 : H_t) \geq (a-1)(t-1) + 1$. \(\square\)

Next, we suppose that $H$ is a complete graph $K_t$.

**Theorem 2.4** For two integers $k, t$ with $k \geq 5$ and $k = t$, $\text{gr}_k (P_5 : K_t) = (t-1)^2 + 1$.

**Proof** From Lemma 2.1, we have $\text{gr}_k (P_5 : H) \geq (t-1)^2 + 1$. It suffices to show that $\text{gr}_k (P_5 : H) \leq (t-1)^2 + 1$. Suppose that $\chi$ is any $k$-edge-coloring of $K_n$ ($n \geq (t-1)^2 + 1$) containing no rainbow copy of $P_5$. From Theorem 1.3, (b) or (c) is true. If (b) is true, we can choose one vertex of $V(i) (2 \leq i \leq k)$, say $v_i$, then $\{v_2, v_3, \ldots, v_k\}$ induces a monochromatic copy of $K_{t-1}$. If there is a vertex of $V(i)$ ($2 \leq i \leq k$), say $V(2)$ and $u_2 \in V(2)$, such that $\chi(u_2u_2) = 1$, then the graph induced by $\{u_2, v_3, \ldots, v_k\}$ is a monochromatic copy of $K_t$ colored by 1. If the graph induced by $V(i) (2 \leq i \leq k)$ contains no edges colored by 1, as $n \geq (t-1)^2 + 1$, there is $V(i) \geq t (2 \leq i \leq k)$, say $|V(2)| \geq t$. Thus the graph induced by $V(2)$ is a monochromatic copy of $K_t$ colored by 2. \(\square\)
Remark 2.1 For two integers $k, t$ with $k \geq 5$ and $k = t$, if $H$ is not a complete graph and $|V(H)| = t$, then $\text{gr}_k(P_5 : K_t) - \text{gr}_k(P_5 : H) = (t-1)^2 + 1 - (t+1) = (t-1)(t-2) - 1$ can be arbitrarily large.

From Theorems 2.1, 2.2, 2.3 and 2.4, we obtain the following corollary.

Corollary 2.5 For integers $k \geq 5$ and $k \geq t$,

$$
\text{gr}_k(P_5 : H) = \begin{cases} 
\max \left\{ \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, 5 \right\}, & k \geq t + 1; \\
t + 1, & k = t \text{ and } H \text{ is not a complete graph}; \\
(t-1)^2 + 1, & k = t \text{ and } H \text{ is a complete graph}.
\end{cases}
$$

The following theorem shows the result on the graph $H$ obtained from a complete graph $K_t$ by deleting a maximally matching $M$.

Theorem 2.6 For two integers $k, t$ with $\left\lceil \frac{t+1}{2} \right\rceil \leq k \leq t - 1$ and $k \geq 5$, if $H$ is a graph obtained from a complete graph $K_t$ by deleting a maximally matching $M$, then

$$
\text{gr}_k(P_5 : H) = \max \left\{ \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, t + 1 \right\}.
$$

Proof From Theorem 2.3, we have $\text{gr}_k(P_5 : H) \geq \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil$. Let $G_5$ be a complete graph obtained from a $K_{t-1}$ with vertex set $\{w_1, w_2, \ldots, w_{t-1}\}$ colored by 1 by adding a new vertex $w$ and edge set $\{w_i w | 1 \leq i \leq t - 1\} = W_1 \cup W_2 \cup \cdots \cup W_k$ with $|W_i| = \lfloor \frac{t-1}{k-1} \rfloor + 1$, $|W_1| + \cdots + |W_k| = t - 1$. Clearly, $G_5$ contains neither a rainbow copy of $P_5$ nor a monochromatic copy of $H$, and hence $\text{gr}_k(P_5 : H) \geq t + 1$. So $\text{gr}_k(P_5 : H) \geq \max \left\{ \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, t + 1 \right\}$.

It suffices to show that $\text{gr}_k(P_5 : H) \leq \max \left\{ \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, t + 1 \right\}$. Let $N$ be an integer with $N = \max \left\{ \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, t + 1 \right\}$. Suppose that $\chi$ is any $k$-edge-coloring of $K_n$ ($n \geq N$) containing no rainbow copy of $P_5$. From Theorem 1.3, (b) or (c) is true. If (b) is true, then for $|V(i)| \geq 2$ ($2 \leq i \leq k$), we choose two vertices from $V(i)$ ($2 \leq i \leq k$), say $u_i, v_i$. Then $\{u_2, v_2, u_3, v_3, \ldots, u_k, v_k\}$ induced a monochromatic copy of graph $K_{2k-2} \setminus M$ with color 1, where $M$ is a maximally matching of $K_{2k-2}$. Since $2k - 2 \geq t$, it follows that there is a copy of graph $H$ with color 1. If (c) is true, then there is a vertex $v$ such that $K_n - v$ is monochromatic. For $n \geq t + 1$, there is a monochromatic $K_t$, we can find a monochromatic copy of $H$. □

The lower and upper bounds of $\text{gr}_k(P_5 : H)$ on $\Delta(H)$ is shown in the following theorem.

Theorem 2.7 Let $k, t, p, q$ be four positive integers with $5 \leq k \leq t - 1$. If $\Delta(H) - 1 = p(k - 2) + q, q \in \{0, 1, \ldots, k - 3\}$ and $R_2(H) \geq t + 1$, then

$$
\max \{\Delta(H) + p, t + 1\} \leq \text{gr}_k(P_5 : H) \leq R_2(H).
$$
Proof We first show that $\text{gr}_k(P_5 : H) \geq \max \{\Delta(H) + p, t + 1\}$. Let $G_6$ be a complete graph with $V(G_6) = U_2 \cup U_3 \cup \cdots \cup U_k$ such that the graph induced by $U_i (2 \leq i \leq k)$ is a monochromatic graph with color $i$, and all edges between $U_i$ and $U_j (i \neq j)$ are colored by 1, $|U_i| = p + 1 (2 \leq i \leq q + 1), |U_i| = p (q + 2 \leq i \leq k)$. Thus $|V(G_6)| = (k - 1)p + q$. Choose any $k - 2$ elements from $\{U_2, U_3, \ldots, U_k\}$, say $U_2, \ldots, U_{k-1}$. Then $|U_2| + \cdots + |U_{k-1}| \leq \Delta(H) - 1$. For any vertex of $U_i$, say $u_i$ ($2 \leq i \leq k$), the degree of $u_i$ in $G_6$ is at most $\Delta(H) - 1$. Note that both $G_5$ and $G_6$ have neither a rainbow copy of $P_5$ nor a monochromatic copy of $H$. So $\text{gr}_k(P_5 : H) \geq \max \{(k - 1)p + q + 1, t + 1\}$.

It suffices to show that $\text{gr}_k(P_5 : H) \leq R_2(H)$. Let $\chi$ be any $k$-edge-coloring of $K_n$ ($n \geq R(H)$) containing no rainbow copy of $P_5$. From Theorem 1.3, (b) or (c) is true. We first consider (b) is true. Let $V^{(2)}, V^{(3)}, \ldots, V^{(k)}$ be a partition of $V(K_n)$ such that there are only edges of color 1 or $i$ within $V^{(i)}$ for $2 \leq i \leq k$, and there are only edges of color 1 between the parts. Now we recolor the edges of $K_n$ to make a 2-edge coloring of $K_n$ such that all edges with color $i$ ($3 \leq i \leq k$) of $V^{(i)}$ ($3 \leq i \leq k$) are changed to color 2. Let $F$ denote the resulting graph. Since $|V(F)| = n \geq R_2(H)$, it follows that $K_n$ must contain a monochromatic copy of $H$. If (c) is true, as $R_2(H) \geq t + 1$, there is a vertex $v$ such that $K_{R_2(H)} - v$ is monochromatic, say color 1, and so there is a monochromatic copy of $H$. \Box

3 Results for the Rainbow 4-Path and Monochromatic $S'_r$

From the result on general results in Sect. 2, we investigate the case $3 \leq k \leq t - 1$ for the graph $S'_r$. First we consider $5 \leq k \leq t - 1$.

Theorem 3.1 Let $k, r, t$ be three integers with $5 \leq k \leq t - 1$ and $1 \leq r \leq k - 2$. Then

$$\text{gr}_k(P_5 : S'_r) = \max \{t + p - 1, t + 1\},$$

where $t - 2 = p(k - 2) + q, q \in \{0, 1, \ldots, k - 3\}$.

Proof From Theorem 2.7, we have $\text{gr}_k(P_5 : S'_r) \geq \max \{t + p - 1, t + 1\}$. It suffices to show that $\text{gr}_k(P_5 : S'_r) \leq \max \{t + p - 1, t + 1\}$. Let $N = \max \{t + p - 1, t + 1\}$. Suppose $G$ is any $k$-edge-coloring of $K_n$ ($n \geq N$) which contains no rainbow copy of $P_5$. From Theorem 1.3, (b) or (c) is true. We first consider (b) is true. As $N \geq t + p - 1$, there must be $k - 2$ elements in $\{V^{(2)}, V^{(3)}, \ldots, V^{(k)}\}$, say $V^{(2)}, V^{(3)}, \ldots, V^{(k-1)}$, such that $|V^{(2)}| + \cdots + |V^{(k-1)}| \geq t - 1$. Choose one vertex $u$ of $V^{(k)}$, and $\ell_i$ ($\ell_i \geq 2, \sum_{2 \leq i \leq k-1} \ell_i = t - 1$) vertices of $V^{(i)}$ ($2 \leq i \leq t - 1$), and $|V^{(i)}| \geq 2$ ($2 \leq i \leq k$), $\Delta(S'_r) = t - 1$, the graph induced by these vertices contains a monochromatic copy of $S'_r$. If (c) is true, as $n \geq t + 1$, there is a vertex $v$ such that $K_n - v$ is monochromatic, say color 1, and hence there is a monochromatic copy of $S'_r$. \Box

Next, we show the result on the case $k = 4$ for $S'_r$. ☒ Springer
Theorem 3.2 Let $k$, $t$, $r$ be three integers with $k = 4$, $t \geq 6$ and $r = 1, 2$. Then

$$\text{gr}_4(P_5 : S_r^t) = t + p - 1,$$

where $t - 2 = 2p + q$ and $q \in \{0, 1\}$.

Proof For the lower bound, let $F_1$ be a 4-edge-coloring complete graph with $V(F_1) = U_2 \cup U_3 \cup U_4$, and the graph induced by $U_2$, $U_3$, $U_4$ are $K_{p+1}$ if $q = 1$ or $K_p$ if $q = 0$ colored by 2, $K_p$ colored by 3, $K_p$ colored by 4 respectively, and there are only edges of color 1 between the parts. Then $|V(F_1)| = 3p + q$. Let $F_2$ be a complete graph obtained from a $K_{t-1}$ with vertex set $\{w_1, w_2, \ldots, w_{t-1}\}$ colored by 1 by adding a new vertex $w$ and edge set $\{w_iw, 1 \leq i \leq t - 1\} = W_1 \cup W_2 \cup W_3 \cup W_4$ with $|W_i| = \lceil \frac{t-1}{2} \rceil$ or $\lfloor \frac{t-1}{2} \rfloor + 1$, $|W_1| + |W_2| + |W_3| + |W_4| = t - 1$. Note that both $F_1$ and $F_2$ contain neither a rainbow copy of $P_5$ nor a monochromatic copy of $S_r^t$. So $\text{gr}_k(P_5 : S_r^t) \geq t + p - 1$.

It suffices to show that $\text{gr}_k(P_5 : S_r^t) \leq t + p - 1$. Suppose $G$ is any 4-edge-coloring of $K_n$ where $n \geq t + p - 1$ which contains no rainbow copy of $P_5$. From Theorem 1.3, (b) or (c) or (d) or (e) is true. Suppose that (b) is true. Let $V^{(2)}$, $V^{(3)}$, $V^{(4)}$ be a partition of $V(G)$ such that there are only edges of color 1 or i within $V^{(i)}$ for $2 \leq i \leq 4$. Then there are only edges with color 1 among the parts. Since $n \geq t + p - 1$, it follows that there exist two elements in $\{V^{(2)}, V^{(3)}, V^{(4)}\}$, say $V^{(2)}$, $V^{(3)}$, such that $|V^{(2)}| + |V^{(3)}| \geq t - 1$, otherwise $n < t$, a contradiction. Choose one vertex of $V^{(4)}$, say $v$, $\ell_1$ ($\ell_1 \geq 2$) vertices of $V^{(2)}$, say $\{u_1, \ldots, u_{\ell_1}\}$ and $\ell_2$ ($\ell_2 \geq 2$) vertices of $V^{(3)}$, say $\{w_1, \ldots, w_{\ell_2}\}$, $\ell_1 + \ell_2 = t - 1$. Then the subgraph induced by $\{v, u_1, \ldots, u_{\ell_1}, w_1, \ldots, w_{\ell_2}\}$ contains a monochromatic copy of $S_r^t$. Suppose that (e) is true. Since $n \geq t + p - 1$, $t - 2 = 2p + q$ and $q \in \{0, 1\}$, it follows that $n \geq t + 1$, and hence there is a vertex $v$ such that $K_n - v$ is monochromatic, say color 1, and hence there is a monochromatic copy of $S_r^t$. Suppose that (d) or (e) is true. Since $t \geq 6$ and $n \geq 7$, it follows that there is a monochromatic copy of $S_r^t$. \hfill \Box

Lemma 3.1 $\text{gr}_4(P_5 : S_4^1) = 6$.

Proof Let $F_3$ be a colored complete graph $K_5$ with $V(K_5) = \{v_i | 1 \leq i \leq 5\}$ under the 4-edge-coloring $\chi$ such that $\chi(v_1v_4) = \chi(v_1v_5) = \chi(v_2v_3) = 1$, $\chi(v_1v_3) = \chi(v_2v_4) = \chi(v_2v_5) = 2$, $\chi(v_1v_2) = \chi(v_3v_4) = \chi(v_3v_5) = 3$, and $\chi(v_4v_5) = 4$. Since there is neither a rainbow copy of $P_5$ nor a monochromatic copy of $S_4^1$ under the coloring $\chi$, it follows that $\text{gr}_4(P_5 : S_4^1) \geq 6$. It suffices to show $\text{gr}_4(P_5 : S_4^1) \leq 6$. Suppose $G$ is any 4-edge-coloring of $K_n$ where $n \geq 6$ containing no rainbow copy of $P_5$. From Theorem 1.3, (b) or (c) or (d) or (e) is true. If (b) is true, then there is a monochromatic copy of $S_4^1$ with color 1. If (c) is true, then $K_n - v$ is monochromatic, and hence there is a monochromatic copy of $S_4^1$. If (d) or (e) is true, there is a monochromatic copy of $S_4^1$ with color 1. \hfill \Box

Lemma 3.2 $\text{gr}_4(P_5 : S_5^1) = 6$.

Proof Since $F_3$ contains neither a rainbow copy of $P_5$ nor a monochromatic copy of $S_5^1$ under the coloring $\chi$, it follows that $\text{gr}_4(P_5 : S_5^1) \geq 6$. It suffices to show
\( \text{gr}_4(P_5 : S_i^1) \leq 6 \). Suppose that \( G \) is any 4-edge-coloring of \( K_n \) where \( n \geq 6 \) containing no rainbow copy of \( P_5 \). From Theorem 1.3, (b) or (c) or (d) or (e) is true. If (b) is true, then there is a monochromatic copy of \( S_i^1 \) with color 1. If (c) is true, then \( K_n - v \) is monochromatic, and hence there is a monochromatic copy of \( S_i^1 \). If (d) or (e) is true, we can find a monochromatic copy of \( S_i^1 \) with color 1. \( \square \)

The following corollary follows from Theorem 3.2, Lemmas 3.1 and 3.2.

**Corollary 3.3** Let \( k, \ t, \ r \) be three integers with \( k = 4 \), \( t \geq 6 \) and \( r = 1, 2 \). Then

\[
\text{gr}_4(P_5 : S_i^t) = \begin{cases} 
6, & t = 4, 5; \\
t + p - 1, & t \geq 6.
\end{cases}
\]

**Theorem 3.4** If \( r \geq 3 \) and \( t \) is odd, then

\[
\text{gr}_4(P_5 : S_i^t) = \begin{cases} 
\frac{3r-5}{2}, & 3 \leq r \leq \left\lceil \frac{t-1}{4} \right\rceil; \\
t + 2r - 2, & \left\lceil \frac{t-1}{4} \right\rceil \leq r \leq \left\lfloor \frac{t-3}{2} \right\rfloor.
\end{cases}
\]

**Proof** Suppose \( 3 \leq r \leq \left\lceil \frac{t-1}{4} \right\rceil \). For the lower bound, let \( F_4 \) be a 4-edge-coloring complete graph with \( V(F_4) = U_2 \cup U_3 \cup U_4 \) and the graph induced by \( U_2 \) is a complete graph \( K_{\frac{t-1}{4}} \) colored by 2, and the graph induced by \( U_i \) \((i = 3, 4)\) is a complete graph \( K_{\frac{t-3}{2}} \) colored by \( i \), and there are only edges of color 1 between the parts, and so \( |V(G_1)| = \frac{3r-7}{2} \). Since \( F_4 \) contains neither a rainbow copy of \( P_5 \) nor a monochromatic copy of \( S_i^t \), it follows that \( \text{gr}_4(P_5 : S_i^t) \geq \frac{3r-5}{2} \).

It suffices to show that \( \text{gr}_4(P_5 : S_i^t) \leq \frac{3r-5}{2} \). Suppose that \( G \) is any 4-edge-coloring of \( K_n \) where \( n \geq \frac{3r-5}{2} \) containing no rainbow copy of \( P_5 \). From Theorem 1.3, (b) or (c) or (d) or (e) is true. If (b) is true, let \( V^{(2)}, V^{(3)}, V^{(4)} \) be a partition of \( V(G) \) such that there are only edges of color 1 or 2 within \( V^{(i)} \) for \( 2 \leq i \leq 4 \), and there are only edges of color 1 between the parts. Without loss of generality, suppose \( |V^{(2)}| \geq |V^{(3)}| \geq |V^{(4)}| \). If \( |V^{(3)}| \leq r - 1 \), then \( |V^{(4)}| \leq r - 1 \) and \( |V^{(2)}| \geq \frac{3r-5}{2} - 2(r - 1) = \frac{3r-4r-1}{2} \).

**Claim 1** The subgraph induced by edges with color 1 of \( V^{(2)} \) contains a maximally matching with at most \( r - 3 \) edges.

**Proof** Let \( M \) be the maximally matching of the subgraph induced by edges colored by 1 of \( V^{(2)} \). Suppose \( |M| \geq r - 2 \), and \( |V^{(3)}| \geq 2 \), \( |V^{(4)}| \geq 2 \). We can find a monochromatic copy of \( S_i^t \) colored by 1, a contradiction. \( \square \)

From Claim 1, there exists no edges colored by 1 within \( V^{(2)} \) by deleting at most \( 2r - 6 \) vertices, and so \( |V^{(2)}| - 2(r - 3) \geq \left\lceil \frac{3r-4r-1}{2} \right\rceil - (2r - 6) = \left\lceil \frac{3r-8r+11}{2} \right\rceil \). Since \( r \leq \left\lceil \frac{t-1}{4} \right\rceil \) and \( \left\lceil \frac{3r-8r+11}{2} \right\rceil \geq 2r + 7 \), it follows that there is a monochromatic copy of \( K_{2r+7} \) colored by 2. For \( |V^{(2)}| \geq \frac{3r-4r-1}{2} \geq t \) and \( r \leq \left\lceil \frac{t-1}{4} \right\rceil \), we can find a monochromatic copy of \( S_i^t \) colored by 2 within \( V^{(2)} \). Thus we can assume that \( |V^{(3)}| \geq r \), and \( |V^{(2)}| \geq |V^{(3)}| \geq r \).
Claim 2 \(|V(2)| + |V(3)| \geq t - 1\).

**Proof** Suppose that \(|V(2)| + |V(3)| \leq t - 2\), then \(|V(2)| + |V(4)| \leq t - 2\) and \(|V(3)| + |V(4)| \leq t - 2\), \(|V(2)| + |V(3)| + |V(4)| \leq \frac{3t-6}{2} < n\), a contradiction.

From Claim 2, choose one vertex of \(V(4)\), say \(v, \ell_1 (\ell_1 \geq r)\) vertices of \(V(2)\), say \(u_1, \ldots, u_{\ell_1}\) and \(\ell_2 (\ell_2 \geq r)\) vertices of \(V(3)\), say \(w_1, \ldots, w_{\ell_2}\), where \(\ell_1 + \ell_2 = t - 1\). Then the graph induced by \(\{v, u_1, \ldots, u_{\ell_1}, w_1, \ldots, w_{\ell_2}\}\) contains a monochromatic copy of \(S'_r\) colored by 1.

Since \(r \geq 3\), \(t\) is odd, and \(3 \leq r \leq \frac{t-1}{4}\), it follows that \(t \geq 13\), and hence \(n \geq \frac{3t-5}{2} \geq t + 4\). For \((b), (c), (d), (e)\), there is a monochromatic copy of \(S'_r\).

Suppose \(\frac{t-1}{4} < r \leq \frac{t-3}{2}\). For the lower bound, let \(F_5\) be a 4-edge-coloring complete graph with \(V(F_6) = U_2 \cup U_3 \cup U_4\) and the graph induced by \(U_2\) is a complete graph \(K_{r-1}\) colored by 2, and the graph induced by \(U_i (i = 3, 4)\) is a complete graph \(K_{r-1}\) colored by \(i\), and there are only edges of color 1 between the parts, so \(|V(F_5)| = t + 2r - 3\). Since \(F_5\) contains no rainbow copy of \(P_5\) and no monochromatic copy of \(S'_r\), it follows that \(gr_4(P_5 : S'_r) \geq t + 2r - 2\).

It suffices to show that \(gr_4(P_5 : S'_r) \leq t + 2r - 2\). Suppose \(G\) is any 4-edge-coloring of \(K_n\) where \(n \geq t + 2r - 2\) which contains no rainbow copy of \(P_5\). From Theorem 1.3, \((b)\) or \((c)\) or \((d)\) or \((e)\) is true. If \((b)\) is true, let \(V(2), V(3), V(4)\) be a partition of \(V(G)\) such that there are only edges of color 1 or 2 within \(V(i)\) for \(2 \leq i \leq 4\), and there are only edges of color 1 between the parts. Without loss of generality, suppose \(|V(2)| \geq |V(3)| \geq |V(4)|\). If \(2 \leq |V(3)| \leq \frac{t-2}{2}\), then \(|V(4)| \leq \frac{t-3}{4}\) and \(|V(2)| \geq t + 2r - 2 - 2 \left\lfloor \frac{t-4}{2} \right\rfloor \geq t + r + 2\). From Claim 1, the subgraph induced by edges colored by 1 of \(V_2\) has a maximally matching containing at most \(r - 3\) edges. Therefore there exists no edge colored by 1 within \(V(2)\) by deleting at most \(2r - 6\) vertices. Since \(|V(2)| - (2r - 6) \geq t + r + 2 - (2r - 6) = t + r + 8 \geq r + 11\) for \(\frac{t-1}{4} \leq r \leq \frac{t-3}{2}\), it follows that there is a monochromatic copy of \(K_r\). Since \(|V(2)| \geq t + 5\), it follows that there is a monochromatic copy of \(S'_r\) colored by 2.

If \(|V(2)| \leq |V(3)| \leq r - 1\), then \(|V(2)| \geq t + 2r - 2 - 2(r - 1) = t\). From Claim 1, the subgraph induced by edges colored by 1 of \(V(2)\) has a maximally matching containing at most \(r - |V(3)| - 1\) edges. Therefore, there exists no edge colored by 1 within \(V(2)\) by deleting at most \(2(r - |V(3)| - 1)\) vertices. Then \(|V(3)| - 2r - |V(3)| - 1 \geq t - 2r + 2|V(3)| + 2 \geq r + 1\) for \(|t - 4| \leq |V(3)| \leq r - 1\), and so there is a monochromatic copy of \(K_r\). As \(|V(2)| \geq t\), there is a monochromatic copy of \(S'_r\) colored by 2.

Thus we can assume that \(|V(3)| \leq r\), and \(|V(2)| \geq |V(3)| \geq r\). From Claim 2, \(|V(2)| \geq |V(3)| \geq t - 1\), choose one vertex of \(V(4)\), say \(v, \ell_1 (\ell_1 \geq r)\) vertices of \(V(2)\), say \(u_1, \ldots, u_{\ell_1}\) and \(\ell_2 (\ell_2 \geq r)\) vertices of \(V(3)\), say \(w_1, \ldots, w_{\ell_2}\). Then the graph induced by \(\{v, u_1, \ldots, u_{\ell_1}, w_1, \ldots, w_{\ell_2}\}\) contains a monochromatic copy of \(S'_r\) with color 1.

Suppose that \((c)\) is true. Since \(r \geq 3\) and \(n \geq t + 2r - 2 \geq t + 4\), it follows that there is a vertex \(v\) such that \(K_n - v\) is a complete graph colored by 1, and hence there is a monochromatic copy of \(S'_r\) colored by 1.

Suppose that \((d)\) is true. Since \(n \geq t + 2r - 2 \geq t + 4\), it follows that \(K_n - \{v_1, v_2, v_3\}\) is a complete graph with color 1, and hence there is a monochromatic copy of \(S'_r\) with color 1.
Suppose that \((e)\) is true. Since \(n \geq t + 2r - 2 \geq t + 4\), it follows that \(K_n - \{v_1, v_2, v_3, v_4\}\) is a complete graph colored by 1, and so there is a monochromatic copy of \(S'_t\) colored by 1. \(\Box\)

**Theorem 3.5** Let \(k, r, t\) be three integers with \(k = 4\) and \(r \geq 3\). If \(t\) is even, then

\[
gr_4(P_5 : S'_t) = \begin{cases} \left\lfloor \frac{3t-4}{2} \right\rfloor, & 3 \leq r \leq \left\lfloor \frac{t}{4} \right\rfloor; \\ t + 2r - 2, & \left\lfloor \frac{t}{4} \right\rfloor \leq r \leq \frac{t-2}{2}. \end{cases}
\]

**Proof** Suppose \(3 \leq r \leq \left\lfloor \frac{t}{4} \right\rfloor\). For the lower bound, let \(F_6\) be any 4-edge-coloring complete graph with \(V(F_6) = U_2 \cup U_3 \cup U_4\) and the graph induced by \(U_i\) \((2 \leq i \leq 4)\) is a complete graph \(K_{\frac{t-2}{2}}\) colored by \(i\), and there are only edges of color 1 between the parts, thus \(|V(F_6)| = \frac{3r-6}{2}\). Note that \(F_6\) contains no rainbow copy of \(P_5\) and no monochromatic copy of \(S'_t\). Thus \(gr_4(P_5 : S'_t) \geq \frac{3t-4}{2}\). For the upper bound, the proof is similar to Theorem 3.4.

Suppose \(\left\lfloor \frac{t}{4} \right\rfloor \leq r \leq \frac{t-2}{2}\). The proof is similar to Theorem 3.4. \(\Box\)

We obtain the result on the case \(k = 3\) for \(S'_t\) in the following lemmas.

**Lemma 3.3** \([22, 29, 30]\) \(R_2(K_3, K_5) = 14; R_3(S'_4) = 17; R_3(S'_5) = 21; R_3(S'_6) = 26.\)

**Lemma 3.4** \(\max\{5t - 4, 2R_2(S'_t) - 1\} \leq R_3(S'_t) \leq 3R_2(S'_t) + 6r - 6.\)

**Proof** For the lower bound, let \(F_7\) denote a 3-edge-coloring complete graph by making five copies of \(K_{t-1}\) colored by 1 and inserting edges of colors 2 and 3 between the copies to form a unique 2-edge-coloring \(K_5\) which contains no monochromatic triangle, \(|V(F_7)| = 5(t-1)\). Let \(F'\) be a 2-edge-coloring complete graph colored by 1 and 2 on \(R_2(S'_t) - 1\) vertices containing no monochromatic copy of \(S'_t\). We construct \(F_8\) by making two copies of \(F'\) and inserting all edges between the copies in color 3, \(|V(F_8)| = 2(R_2(S'_t) - 1)\). Note that both \(F_7\) and \(F_8\) contain no monochromatic copy of \(S'_t\), it follows that \(R_3(S'_t) \geq \max\{5t - 4, 2R_2(S'_t) - 1\}\).

It suffices to show that \(R_3(S'_t) \leq 3R_2(S'_t) + 6r - 6\). Suppose that \(G\) is any 3-edge-coloring of \(K_n\) \((n \geq 3R_2(S'_t) + 6r - 6)\) which is colored by 1, 2, 3. For any vertex \(v \in V(G)\), and \(n \geq 3R_2(S'_t) + 6r - 6\), there are at least \(R_2(S'_t) + 2r - 2\) edges incident with \(v\) colored by \(i\) \((i = 1, 2, 3)\), say 1. Without loss of generality, the end vertices of these edges except \(v\) are denoted by \(u_1, u_2, \ldots, u_{R_2(S'_t)+2r-2}\). Let \(G'\) be the subgraph induced by \(\{u_1, u_2, \ldots, u_{R_2(S'_t)+2r-2}\}\), and from Claim 1 of Theorem 3.4, the subgraph induced by edges colored by 1 of \(G'\) contains a maximal matching which has at most \(r - 1\) edges, otherwise there is a monochromatic copy of \(S'_t\) colored by 1. Thus \(G'\) contains no edge colored by 1 by deleting at most \(2(r - 1)\) vertices, and the resulting graph is denoted by \(R\). Since \(|V(R)| \geq |V(G')| - 2(r - 1) \geq R(S'_t)\), it follows that there must be a monochromatic copy of \(S'_t\) colored by 2 or 3, completing the proof. \(\Box\)

Finally, we show the result on \(S'_4, S'_5, S'_6\).
Theorem 3.6  For integers \( k \geq 3 \), we have

\[
\text{gr}_k(P_5 : S^1_4) = \begin{cases} 
17, & k = 3; \\
6, & k = 4; \\
5, & k = 5, 6; \\
\ell, & (\ell-1)/2 + 1 \leq k \leq \ell/2 \text{ and } \ell \geq 5.
\end{cases}
\]

Proof  If \( k = 3 \), then it follows from Lemma 3.3 that \( \text{gr}_3(P_5 : S^1_4) = R_3(S^1_4) = 17 \). If \( k = 4 \), then it follows from Lemma 3.1 that \( \text{gr}_4(P_5 : S^1_4) = 6 \).

Suppose that \( k = 5 \). Let \( F_5 \) be a colored complete graph \( K_4 \) with \( V(K_4) = \{ u_i \mid 1 \leq i \leq 4 \} \) under the 5-edge-coloring \( \chi \) such that \( \chi(v_2v_3) = \chi(v_2v_4) = 1, \chi(v_1v_2) = 2, \chi(v_1v_3) = 3, \chi(v_1v_4) = 4 \) and \( \chi(v_3v_4) = 5 \). Since there is neither a rainbow copy of \( P_5 \) nor a monochromatic copy of \( S^1_4 \) under the coloring \( \chi \), it follows that \( \text{gr}_5(P_5 : S^1_4) \geq 5 \). It suffices to show \( \text{gr}_5(P_5 : S^1_4) \leq 5 \). Let \( \chi \) be any 5-edge-coloring of \( K_n \) (\( n \geq 5 \)) containing no rainbow copy of \( P_5 \). From Theorem 1.3, (b) or (c) is true. If (b) is true, then \( \chi \) contains at most 3 colors, a contradiction. If (c) is true, then there exists a vertex \( v \) such that \( K_n - v \) is monochromatic, and hence there is a monochromatic copy of \( S^1_4 \).

Suppose \( k = 6 \). Let \( F_{10} \) be a colored complete graph \( K_4 \) with \( V(K_4) = \{ u_i \mid 1 \leq i \leq 4 \} \) under the 5-edge-coloring \( \chi \) such that \( \chi(v_2v_3) = 1, \chi(v_1v_2) = 2, \chi(v_1v_3) = 3, \chi(v_1v_4) = 4, \chi(v_3v_4) = 5 \) and \( \chi(v_2v_4) = 6 \). Since there is neither a rainbow copy of \( P_5 \) nor a monochromatic copy of \( S^1_4 \) under the coloring \( \chi \), it follows that \( \text{gr}_6(P_5 : S^1_4) \geq 5 \). It suffices to show \( \text{gr}_6(P_5 : S^1_4) \leq 5 \). Let \( \chi \) be a 6-edge-coloring of \( K_n \)(\( n \geq 5 \)) containing no rainbow copy of \( P_5 \). From Theorem 1.3, (b) or (c) is true. If (b) is true, then \( \chi \) contains at most 3 colors, a contradiction. If (c) is true, then there exists a vertex \( v \) such that \( K_n - v \) is monochromatic, and hence there is a monochromatic copy of \( S^1_4 \).

Suppose \( \ell \geq 5 \). If \( k \geq (\ell - 1)/2 + 1 \), there is no \( k \)-edge-coloring \( \chi \) of \( K_{\ell-1} \), and so \( \text{gr}_k(P_5 : S^1_4) \geq \ell \). It suffices to show \( \text{gr}_k(P_5 : S^1_4) \leq \ell \). Suppose that there is a coloring \( \chi \) of \( K_n \)(\( n \geq \ell \)) containing no rainbow copy of \( P_5 \). From Theorem 1.3, (b) or (c) is true. If (b) is true, then \( \ell \geq 2(k-1) > 2((\ell-1)/2) - 1 \), and hence \( \ell \leq 4 \), a contradiction. If (c) is true, then there is a vertex \( v \) such that \( K_n - v \) is monochromatic, and hence there is a monochromatic copy of \( S^1_4 \).

Theorem 3.7  For integers \( k \geq 3 \), we have

\[
\text{gr}_k(P_5 : S^1_5) = \begin{cases} 
21, & k = 3; \\
6, & k = 4, 5; \\
5, & k = 6; \\
\left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil, & k \geq 7.
\end{cases}
\]

Proof  If \( k = 3 \), then it follows from Lemma 3.3 that \( \text{gr}_3(P_5 : S^1_5) = R_3(S^1_5) = 21 \). If \( k = 4 \), then it follows from Lemma 3.1 that \( \text{gr}_4(P_5 : S^1_5) = 6 \).
Suppose \( k = 5 \). Let \( F_{11} \) be a colored complete graph obtained from a \( K_4 \) with vertex set \( \{u_1, u_2, u_3, u_4\} \) colored by 1 by adding a new vertex \( v \) and edges \( u_i \) \( v \) (\( 1 \leq i \leq 4 \)) colored by \( i + 1 \). Since there is neither a rainbow copy of \( P_5 \) nor a monochromatic copy of \( P_5 \) under the coloring, it follows that \( gr_5(P_5 : S^1_6) \geq 6 \). It suffices to show \( gr_5(P_5 : S^1_6) \leq 6 \). Let \( \chi \) be any 5-edge-coloring of \( K_n \) (\( n \geq 6 \)) containing no rainbow copy of \( P_5 \). From Theorem 1.3, \( (b) \) or \( (c) \) is true. If \( (b) \) is true, then \( \chi \) contains at most 4 colors, a contradiction. If \( (c) \) is true, then there exists a vertex \( v \) such that \( K_n - v \) is monochromatic, and hence there is a monochromatic copy of \( S^1_6 \).

For \( k = 6 \), the result follows from Theorem 2.2. For \( k \geq 7 \), the result follows from Theorem 2.1.

**Theorem 3.8** For integer \( k \geq 3 \), we have

\[
gr_k(P_5 : S^1_6) = \begin{cases} 
26, & k = 3; \\
7, & 4 \leq k \leq 6; \\
\left\lceil \frac{1 + \sqrt{1 + 8k}}{2} \right\rceil, & k \geq 7.
\end{cases}
\]

**Proof** If \( k = 3 \), then it follows from Lemma 3.3 that \( gr_3(P_5 : S^1_6) = R_3(S^1_6) = 26 \). If \( k = 4 \), then it follows from Theorem 3.2 that \( gr_4(P_5 : S^1_6) = 7 \). If \( k = 5 \), then it follows from Theorem 3.1 that \( gr_5(P_5 : S^1_6) = 7 \). If \( k = 6 \), then it follows from Theorem 2.3 that \( gr_6(P_5 : S^1_6) = 7 \). If \( k \geq 7 \), then it follows from Theorem 2.1 that \( gr_k(P_5 : S^1_6) = \left\lceil \frac{1 + \sqrt{1 + 8k}}{2} \right\rceil \), completing the proof.

From Theorems 2.1, 2.3, 3.1, 3.2 and Lemma 3.4, we can obtain the following result.

**Theorem 3.9** For integer \( k \geq 3 \), \( t \geq 6 \), \( r = 1, 2 \), we have

\[
gr_k(P_5 : S^t_r) = \begin{cases} 
\left\lceil \max \left\{ 5t - 4, 2R(S^t_r) - 1 \right\} \right\rceil, & k = 3; \\
t + p - 1, & 4 \leq k \leq t - 1; \\
t + 1, & k = t; \\
\left\lceil \frac{1 + \sqrt{1 + 8k}}{2} \right\rceil, & k \geq t + 1.
\end{cases}
\]

**4 Results for the Rainbow 4-Path and Monochromatic Pineapples**

In this section, we will get some exact values or bounds for \( gr_k(P_5 : H) \) when \( H \) is a pineapple.

**Theorem 4.1** Let \( k, t, \omega \) be three integers with \( k = \omega \) and \( k \geq 4 \). Then

\[
gr_k(P_5 : PA_{t, \omega}) = (\omega - 1)(t - 1) + 1.
\]

**Proof** From Lemma 2.1, we have \( gr_k(P_5 : PA_{t, \omega}) \geq (\omega - 1)(t - 1) + 1 \). It suffices to show that \( gr_k(P_5 : PA_{t, \omega}) \leq (\omega - 1)(t - 1) + 1 \). Let \( G \) be any \( k \)-edge-coloring
of \( K_n \) where \( n \geq (\omega - 1)(t - 1) + 1 \) which contains no rainbow copy of \( P_5 \). From Theorem 1.3, (b) or (c) or (d) or (e) is true if \( k = 4 \), and (b) or (c) is true if \( k \geq 5 \).

For \( k \geq 4 \), suppose that (b) is true. Let \( V(2), V(3), \ldots, V(k) \) be a partition of \( V(G) \) such that there are only edges of color 1 or \( i \) within each \( V(i) \) for \( 2 \leq i \leq k \), and there are only edges with color 1 among the parts. Then there exists some \( V(i) (2 \leq i \leq k) \) with \( |V(i)| \geq t \), otherwise \( |V(2)| + |V(3)| + \cdots + |V(k)| \leq (\omega - 1)(t - 1) + 1 < n \), a contradiction. Without loss of generality, let \( |V(2)| \geq t \). Suppose that \( V(2) \) contains one edge with color 1, say \( v_2u_2 \). Choose one vertex of \( V(i) (2 \leq i \leq k) \), say \( v_i \). Then the subgraph induced by \( \{u_2, v_2, v_3, \ldots, v_k\} \) is a copy of \( K_n \) with color 1. Since \( |V(2)| \geq t \), it follows that there is a copy of \( PA_{t,4} \) with color 1. Therefore, \( V(2) \) contains no edges with color 1, and hence the subgraph induced by \( V(2) \) is a monochromatic copy of \( K_t \) with color 2, and so there is a monochromatic copy of \( PA_{t,\omega} \) with color 2.

Suppose that (c) is true. Since \( n \geq (\omega - 1)(t - 1) + 1 \geq t + 1 \) for \( \omega \geq 4 \) and \( t \geq 5 \), it follows that there is a vertex \( v \) such that \( K_n - v \) is a complete graph colored by 1, and hence there is a monochromatic copy of \( PA_{t,\omega} \) colored by 1.

For \( k = 4 \), then (d) or (e) is true. Suppose that (d) is true. Since \( n \geq 3t - 2 \geq t + 8 \), it follows that \( K_n - \{v_1, v_2, v_3\} \) is a complete graph colored by 1, and so there is a monochromatic copy of \( PA_{t,4} \) colored by 1. Suppose that (e) is true. Since \( n \geq 3t - 2 \geq t + 8 \), it follows that \( K_n - \{v_1, v_2, v_3, v_4\} \) is a complete graph colored by 1, and hence there is a monochromatic copy of \( PA_{t,4} \) colored by 1.

\[ \square \]

**Theorem 4.2** Let \( k, t, \omega \) be three integers with \( k = 4, \omega = 5 \) and \( t \geq 8 \). Then

\[ gr_4(P_5 : PA_{t,\omega}) = 4t - 3. \]

**Proof** From Lemma 2.1, we know that \( gr_4(P_5 : PA_{t,\omega}) \geq 4t - 3 \). It suffices to show that \( gr_4(P_5 : PA_{t,\omega}) \leq 4t - 3 \). Let \( G \) be any 4-edge-coloring of \( K_n \), where \( n \geq 4t - 3 \), which contains no rainbow copy of \( P_5 \). From Theorem 1.3, (b) or (c) or (d) or (e) is true. Suppose that (b) is true and \( |V(2)| \geq |V(3)| \geq |V(4)| \). To avoid a monochromatic copy of \( K_5 \) colored by 1, \( |V(G)| \geq \sum_{i=2}^{4} |V(i)| \leq 1 \). If \( |V(G)| \geq \sum_{i=2}^{4} |V(i)| = 0 \), then \( \sum_{i=2}^{4} |V(i)| \geq 4t - 3 \), and hence \( |V(2)| \geq |2t - 3| \), otherwise \( |V(2)| + |V(3)| + |V(4)| < 4t - 3 \), a contradiction.

**Claim 3** Both \( V(3) \) and \( V(4) \) contain no edges with color 1.

**Proof** Assume, to the contrary, that both \( V(3) \) and \( V(4) \) contain one edge with color 1, say \( v_3u_3 \) and \( v_4u_4 \). Choose one vertex of \( V(2) \), say \( v_2 \). Then the subgraph induced by \( \{v_2, v_3, u_3, v_4, u_4\} \) contains a copy of \( K_5 \) with color 1. Since \( |V(2)| \geq |2t - 3| \geq t + 1 \) for \( \omega = 5 \) and \( t \geq 8 \), it follows that there is a copy of \( PA_{t,5} \) with color 1. If either \( V(3) \) or \( V(4) \) contains one edge with color 1, say \( V(3) \) and \( v_3u_3 \) with color 1, then \( V(2) \) contains no edges with color 1, otherwise there is a copy of \( PA_{t,5} \) with color 1 by the above proof. Thus the subgraph induced by \( V(2) \) is monochromatic copy of complete graph with color 2, and hence \( |V(2)| \geq |2t - 3| \geq t + 1 \), and so there is a monochromatic copy of \( PA_{t,5} \) with color 2.

From Claim 3, both \( V(3) \) and \( V(4) \) contain no edges with color 1. Then \( |V(3)| \leq t - 1 \), \( |V(4)| \leq t - 1 \), otherwise there is a monochromatic copy of \( K_t \) colored by 3 or

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4. If $V^{(2)}$ contains three vertices which induce a monochromatic copy of $K_3$ with color 1, say $u_2, w_2, x_2$, choose one vertex $w_i$ of $V^{(i)}$ ($i = 3, 4$), then the subgraph induced by $\{u_2, w_2, x_2, w_3, w_4\}$ is a monochromatic copy of $K_5$ with color 1. Choose $t - 5$ vertices of $V^{(2)}$, say $y_1, \ldots, y_{t-5}$, then the subgraph induced by $\{u_2, w_2, x_2, w_3, y_1, \ldots, y_{t-5}\}$ contains a monochromatic copy of $PA_{t-5}$ colored by 1. It follows that the subgraph induced by color 1 within $V_2$ must not be $K_3$. Since $n \geq 4t - 3$, it follows that $|V^{(2)}| \geq 2t - 1 \geq t + 7 \geq 15$ for $t \geq 8$. From Lemma 3.3, $R_2(K_3, K_5) = 14$, and hence there is a monochromatic copy of $K_5$ with color 2 in $V^{(2)}$.

For any vertex $v \in V^{(2)}$, let $Q_i$ ($i = 1, 2$) be the set of vertices such that the edges from any vertex of $Q_i$ to $v$ are with color $i$.

**Claim 4** $|Q_1| \leq t - 1$.

**Proof** Assume, to the contrary, that $|Q_1| \geq t$. If the subgraph induced by $Q_1$ contains one edge with color 1, say $w_1w_2$, then $\{w_1, w_2, v\}$ induces a monochromatic copy of $K_3$ colored by 1, a contradiction. That means that $Q_1$ contains no edges colored by 1 and there is a monochromatic copy of $K_t$ colored by 2. So there is a monochromatic copy of $PA_{t-5}$ colored by 2.

From Claim 4, we have $|Q_2| \geq t - 1$, and hence $V^{(2)}$ contains a monochromatic copy of $PA_{t-5}$ colored by 2. If $|V(G)| - \sum_{i=2}^{4} |V^{(i)}| = 1$, then $V^{(2)}$ contains no edge colored by 1, otherwise there is a monochromatic copy of $K_5$. Since $\sum_{i=2}^{4} |V^{(i)}| \geq 4t - 4$, it follows that $|V^{(2)}| \geq \left\lceil \frac{4t - 4}{3} \right\rceil \geq t + 1$ for $t \geq 8$. Hence, $V^{(2)}$ contains a monochromatic copy of $PA_{t-5}$ colored by 2.

Suppose that (c) or (d) or (e) is true. Since $n \geq 4t - 3 \geq t + 21$ for $t \geq 8$, it follows that there is a monochromatic copy of $K_{t+17}$.

**Theorem 4.3** Let $k, t, \omega$ be three integers with $k = 4$, $\omega = 5$ and $t = 6$. Then

$$\text{gr}_4(P_5 : PA_{6,5}) = 24.$$  

**Proof** For the lower bound, let $F_{12}$ be a complete graph with $V(F_{12}) = U_2 \cup U_3 \cup U_4$ such that the subgraph induced by $U_2$ is $K_{13}$ colored by 1 and 2 which contains neither a monochromatic copy of $K_3$ nor a monochromatic copy of $K_5$, and the subgraph induced by $U_i$ ($i = 3, 4$) is $K_5$ colored by $i$, and all edges between $U_i$ and $U_j$ ($i, j \in \{2, 3, 4\}, i \neq j$) are colored by 1. It is clear that $F_{12}$ contains neither a rainbow copy of $P_5$ nor a monochromatic copy of $PA_{6,5}$, and $\text{gr}_4(P_5 : PA_{6,5}) \geq 24$.

It suffices to show that $\text{gr}_k(P_5 : PA_{6,5}) \leq 24$. Let $G$ be any 4-edge-coloring of $K_n$, where $n \geq 24$, which contains no rainbow copy of $P_5$. From Theorem 1.3, (b) or (c) or (d) or (e) is true. By the proof of Theorem 4.2, we know that the upper bound holds.

**Theorem 4.4** Let $k, t, \omega$ be three integers with $k = 4$, $\omega = 5$ and $t = 7$. Then

$$\text{gr}_4(P_5 : PA_{7,5}) = 26.$$  

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Proof For the lower bound, let $F_{13}$ be a complete graph with $V(F_{12}) = U_2 \cup U_3 \cup U_4$. Then the subgraph induced by $U_2$ is $K_{13}$ colored by 1 and 2 which contains neither a monochromatic copy of $K_3$ nor a monochromatic copy of $K_5$, and the subgraph induced by $U_i$ ($i=3,4$) is $K_6$ colored by $i$, and all edges from $U_i$ to $U_j$ ($i,j \in \{2,3,4\}, i \neq j$) are colored by 1. It is clear that $F_{13}$ contains neither a rainbow copy of $P_5$ nor a monochromatic copy of $PA_{7,5}$, and hence $gr_d(P_5 : PA_{7,5}) \geq 26$.

It suffices to show that $gr_k(P_5 : PA_{7,5}) \leq 26$. Let $G$ be any 4-edge-coloring of $K_n$, where $n \geq 26$, which contains no rainbow copy of $P_5$. From Theorem 4.4, (b) or (c) or (d) or (e) is true. By the proof of Theorem 4.2, we know that the upper bound holds.

Theorem 4.5 Let $k, t, \omega$ be three integers with $k = 4$ and $\omega \geq 6$. Then

$$(\omega - 1)(t - 1) + 1 \leq gr_4(P_5 : PA_{t,\omega}) \leq 3R_2(PA_{t,\omega}) - 2.$$  

Proof From Lemma 2.1, we can obtain the lower bound. For the upper bound, let $G$ be any 4-edge-coloring of $K_n$, where $n \geq 3R_2(PA_{t,\omega}) - 2$, containing no rainbow copy of $P_5$. From Theorem 1.3, (b) or (c) or (d) or (e) is true. Suppose that (b) is true. Let $V(2), V(3), V(4)$ be a partition of $V(G)$ such that there are only edges of color 1 or $i$ within $V(i)$ for $2 \leq i \leq 4$, and there are only edges of color 1 between the parts. There exists some $V(i)$ ($2 \leq i \leq 4$) with $|V(i)| \geq \lceil \frac{3R_2(PA_{t,\omega}) - 2}{3} \rceil$, otherwise $|V(2)| + |V(3)| + |V(4)| < 3R_2(PA_{t,\omega}) - 2 \leq n$, a contradiction. Without loss of generality, let $|V(2)| \geq \lceil \frac{3R_2(PA_{t,\omega}) - 2}{3} \rceil \geq R_2(PA_{t,\omega})$. Then there is a monochromatic copy of $PA_{t,\omega}$ colored by 1 or 2. Since $R_2(PA_{t,\omega}) \geq t$, it follows that $3R_2(PA_{t,\omega}) - 2 > t + 4$ for $\omega \geq 6$ and $t \geq 7$. If (b) or (c) or (e) is true, then there is a monochromatic copy of $PA_{t,\omega}$. $\square$

Sah [26] obtained the following result.

Theorem 4.6 [26] There is an absolute constant $c > 0$ such that for $k \geq 3$,

$$R_2(k + 1) \leq \left(\frac{2k}{k}\right) e^{-c(log k)^2}.$$  

By the upper bound in Theorem 4.5, we can derive the following result.

Theorem 4.7 There is an absolute constant $c > 0$ such that for $\omega \geq 4$,

$$R_2(PA_{t,\omega}) \leq \left( \frac{2\omega - 2}{\omega - 1} \right) e^{-c \log^2(\omega - 1)} + (t - 2)(\omega - 1).$$  

Proof Let $n = \left( \frac{2\omega - 2}{\omega - 1} \right) e^{-c \log^2(\omega - 1)} + (t - 2)(\omega - 1)$. For any red/blue-edge-coloring of $K_n$, from Theorem 4.4, there is a monochromatic copy of $K_\omega$, say $K^1_\omega$. Without loss of generality, assume that $K^1_\omega$ is red. Choose one vertex in $K^1_\omega$, say $v_1$. Let $X_1$ be the set of vertices with red edges from $v_1$ to $K_n - K^1_\omega$. Then $|X_1| \leq t - \omega - 1$. By deleting the vertices of $K^1_\omega \cup X_1$, $K_n - K^1_\omega - X_1$ contains a red clique of order $\omega$, say $K^2_\omega$. Choose one vertex in $K^2_\omega$, say $v_2$. Let $X_2$ be the set of vertices with red edges from $v_2$ to $K_n - (K^1_\omega \cup K^2_\omega \cup X_1)$. Then $|X_2| \leq t - \omega - 1$. By deleting the vertices of
\[ K^2_\omega \cup X_2, \] one can see that \( K_n - (K^1_\omega \cup K^2_\omega \cup X_1 \cup X_2) \) contains a red clique of order \( \omega \), say \( K^3_\omega \).

Continue this process, \( K_n - (\bigcup_{i=1}^{\omega-2} K^i_\omega) - \bigcup_{i=1}^{\omega-2} X_i \) contains a red clique of order \( \omega \), say \( K^{\omega-1}_\omega \). Choose one vertex in \( K^{\omega-1}_\omega \), say \( v_{\omega-1} \). Let \( X_{\omega-1} \) be the set of vertices with red edges from \( v_{\omega-1} \) to \( K_n - (\bigcup_{i=1}^{\omega-1} K^i_\omega) - \bigcup_{i=1}^{\omega-2} X_i \). Then \( |X_{\omega-1}| \leq t - \omega - 1 \).

Choose one vertex in \( K_n - (\bigcup_{i=1}^{\omega-1} K^i_\omega) - \bigcup_{i=1}^{\omega-2} X_i \), say \( v_{\omega} \). Since the number of red edges from \( v_1 \) to \( K_n - K^1_\omega \) is at most \( t - \omega - 1 \), it follows that the number of blue edges from \( v_1 \) to \( K_n - K^1_\omega \) is at least \( n - t \), and hence there is a blue copy of \( PA_t,\omega \).

\( \Box \)

The following corollary is immediate.

**Corollary 4.8** Let \( k, t, \omega \) be three integers with \( k = 4 \) and \( \omega \geq 6 \). Then
\[
(\omega - 1)(t - 1) + 1 \leq \text{gr}_4(P_5 : PA_t,\omega) \leq 3 \left( \frac{2\omega - 2}{\omega - 1} \right) e^{-c \log^2(\omega - 1)} + (t - 2)(\omega - 1) - 2,
\]
where \( c > 0 \) is an absolute constant.

From Lemma 2.1, Theorems 2.7 and 4.7, the following corollary is true.

**Corollary 4.9** Let \( k, t, \omega \) be three integers with \( 5 \leq k \leq \omega - 1 \) and \( \omega \geq 6 \). Then
\[
(\omega - 1)(t - 1) + 1 \leq \text{gr}_k(P_5 : PA_t,\omega) \leq \left( \frac{2\omega - 2}{\omega - 1} \right) e^{-c \log^2(\omega - 1)} + (t - 2)(\omega - 1).
\]

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**Declarations**

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