Force-Free Interactions and Nondispersive Phase Shifts in Interferometry

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Abstract

Zeilinger’s observation that phenomena of the Aharonov-Bohm type lead to nondispersive, i.e. energy-independent, phase shifts in interferometers is generalized in a new proof which shows that the precise condition for nondispersivity is a force-free interaction. The converse theorem is disproved by a conceptual counter example. Applications to several nondispersive interference phenomena are reviewed briefly. Those fall into two classes which are objectively distinct from each other in that in the first class phase shifts depend only on the topology of the interfering beam paths, while in the second class force-free physical interactions take place at identifiable points along the path. Apparent disagreements in the literature about the topological nature of the phenomena in the second class stem from differing definitions.

I. INTRODUCTION

A. Zeilinger [1] observed that a nondispersive, i.e. energy independent, phase shift provides an operational signature of what he called Aharonov-Bohm effects in neutron interferometry. In his words,

"...the significant operational feature of the AB effect is that there exists no observable defined on either beam separately which is influenced by the electric or magnetic field. This is related to the property that the AB phase shifts are nondispersive. Only a constant overall phase change for the wave packet arises..."

That result has proved to be useful to experimenters in confirming the results of theories involving effects of the AB type and it has also been used as a test in characterizing such effects.

Zeilinger proved his nondispersivity theorem by listing the calculated phase shifts in each of what he called the generalized Aharonov-Bohm experiments with neutrons and observing that they were all nondispersive.

My purpose here is to expand on Zeilinger’s observation in several ways, to give a brief review of its application to interferometry measurements on neutrons and other particles,

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and to clarify the relations between nondispersivity, force-free motion, and what have come to be called topological effects in quantum mechanics. Section II presents a new proof from the Schroedinger equation that force-free interactions imply nondispersive phase shifts. This new proof is more general than the original one in that it allows for a wide class of nonlocal interactions, and it demonstrates that an objective sufficient condition for nondispersivity is that the motion of the particles should be force free. In Section III, I show by counter example that the converse theorem is not true; a nondispersive phase shift does not imply force-free motion in the absence of other assumptions about the interactions. Section IV contains a brief review of the application of the nondispersivity theorem to idealized versions of various kinds of interferometry experiments. In Section V, I separate the force-free interactions into two classes according to the presence or absence of Maxwell fields or other scattering interactions acting on the particles and I try to clarify the relation between nondispersive phase shifts, force-free motion, and topological interactions.

II. ZEILINGER’S NONDISPERSIVITY THEOREM

Suppose that charged or neutral particles, in a single spin state for simplicity, traverse a Mach-Zehnder interferometer (Fig. 1) and the relative phase of the two partial waves is measured by observing the intensities of the emergent beams, O and H. For present purposes, it is adequate to describe the motion of the particles in either arm through a wave function in one dimension, subject to the Hamiltonian

\[
H = \frac{p^2}{2m} + V. \tag{2.1}
\]

Here \( V \) represents a general interaction, possibly time dependent and possibly momentum dependent or otherwise nonlocal, but confined at all times to some interaction zone \( 0 \leq x \leq \ell \).

The interaction is taken to be entirely elastic; no particle exchanges energy with the applied field represented by \( V \). For convenience, I postulate temporarily that no particles are reflected from the interaction zone. That restriction is justified below. Then plane-wave particles incident from the left with momentum \( p = \hbar k \) will emerge on the right in a plane wave state having the same amplitude and the same momentum \( p \), but possibly shifted in phase by some amount \( \delta(k) \).

\[
\phi_k(x,t) = \begin{cases} 
\exp\left\{i(kx - \omega t)\right\} & \text{for } x < 0 \\
\exp\left\{i(kx - \omega t + \delta(k))\right\} & \text{for } x > \ell,
\end{cases}
\tag{2.2}
\]

where \( \omega = \hbar k^2/2m \).

A wave packet incident from the left at time zero,

\[
\psi(x,0) = \int dk \chi(k) \exp\{ikx\} \tag{2.3}
\]

will emerge at a later time \( T \) when it has completely cleared the interaction zone as

\[
\psi(x,T) = \int dk \chi(k) \exp\{i(kx - \omega T + i\delta(k))\}. \tag{2.4}
\]
The expectation of $x$ at time $T$ is given by

$$
\langle x \rangle_T = \int \int \int dxdkdk' \chi(k')^* x \chi(k) \exp \left\{ -i(k'x - \omega'T) - i\delta(k') \right\}
$$

$$
\times \exp \left\{ i(kx - \omega T) + i\delta(k) \right\}
$$

$$
= \int dk \chi(k)^* \exp \left\{ i\omega T - i\delta(k) \right\} \left( i \frac{\partial}{\partial k} \right) \left[ \chi(k) \exp \left\{ (-i\omega T) + i\delta(k) \right\} \right]
$$

(2.5)

$\langle x \rangle_0$ is the same, but with 0 substituted for $T$ and for the phase shift $\delta(k)$.

$$
\langle x \rangle_T = i \int \chi(k)^* \frac{d\chi}{dk} dk + T \int |\chi(k)|^2 \frac{d\omega}{dk} dk + \int |\chi(k)|^2 \frac{d\delta}{dk} dk
$$

$$
= \langle x \rangle_0 + \langle v \rangle T + \int |\chi(k)|^2 \frac{d\delta}{dk} dk,
$$

(2.6)

where $\langle v \rangle$, the expectation of the group velocity, is given by

$$
\langle v \rangle = \frac{d}{dt} \langle x \rangle = \left\langle \frac{d\omega}{dk} \right\rangle.
$$

(2.7)

Ehrenfest’s theorem tells us that if the interaction is force free, then

$$
\langle x \rangle_T = \langle x \rangle_0 + \langle v \rangle T.
$$

(2.8)

Since the last term in Eq. (2.6) must then vanish for all wave packets $\chi(k)$, it follows that

$$
\frac{d\delta}{dk} = 0
$$

(2.9)

for all $k$. That result is Zeilinger’s nondispersivity theorem. The proof given here assumed no reflected wave, but that condition in fact follows from Ehrenfest’s theorem in the absence of any force on the particle; the expectation of the momentum cannot be conserved in the presence of a reflected wave unless the transmitted wave has greater kinetic energy than the incident wave, contrary to the assumption of no energy transfer.

Ehrenfest’s theorem does not require that the force $F(x,t)$ obeys $F = 0$ for all $x$ and $t$ as an operator equation, only that

$$
F(x,t)\psi(x,t) = 0
$$

(2.10)

for the wave packets in the interferometer. Therefore, the same is true of the nondispersivity theorem. The particle may traverse regions where, for example, a potential gradient existed at earlier times, as long as the potential is spatially uniform while the particle is present.

\[1\] In fact, the particle must traverse such a region to acquire a phase shift in the absence any force on the particle.\[2\]
III. THE CONVERSE THEOREM

The converse theorem, that a nondispersive phase shift at all energies implies a force-free interaction, does not follow from the Schroedinger equation, and it is in fact false. For a simple example to show that nonvanishing forces can cause a nondispersive phase shift, consider a nuclear phase shifter of the kind commonly used in neutron interferometry. A slab of some material is placed in one beam to alter its phase relative to the other beam. The material in the phase shifter is characterized by an energy-dependent index of refraction \( \eta(k) \) or equivalently by an energy-dependent interaction term in the Hamiltonian given by

\[
V = \frac{\hbar^2 k^2}{2m} \left( 1 - \eta(k)^2 \right)
\]

(3.1)

for values of \( x \) within the slab, and by \( V = 0 \) elsewhere. For this illustration I suppose that \( V \) is positive and \( \eta < 1 \) so that no bound states will be introduced.

The phase shift in a slab of thickness \( b \),

\[
\delta = kb \left( \eta(k) - 1 \right)
\]

(3.2)

can in principle be made nondispersive by using a material with index of refraction given by

\[
n(k) = 1 + \frac{\delta}{kb}
\]

(3.3)

with constant negative \( \delta \). However, the neutron does experience forces when it enters and exits the material.

A uniform slab whose index of refraction is given by Eq. (3.3) is far from being the only example of a nondispersive interaction that does exert forces, but physically they are all rather contrived.

IV. APPLICATIONS OF THE NONDISPERSIVITY THEOREM

The nondispersivity theorem has been applied to five distinct kinds of interferometry experiments performed or proposed:

A. The Magnetic Aharonov-Bohm Effect

In an idealized interferometric version of the magnetic Aharonov-Bohm (AB) effect \[3\] the current in a solenoid placed between the two arms of the interferometer in Fig. 1 is the source of a vector potential \( A \). Electrons in the interferometer are exposed to \( A \) but they are subjected to no force because they are confined to regions where the magnetic field \( B \) vanishes. The nondispersivity theorem can be applied directly to that case by choosing the gauge so that \( A \) vanishes outside of the interaction zone. Then

\[
V = \frac{e}{mc} p \cdot A
\]

(4.1)

and the phase shift is given by
\[ \delta = \int V \, dt = \frac{e}{\hbar c} \int A \cdot dx = \frac{e\Phi}{\hbar c}, \quad (4.2) \]

\( \Phi \) being the magnetic flux through the solenoid. Experiments equivalent to this have been reported \[4 – 7\]. The most precise \[7\] agreed with the calculated phase shifts to better than one percent.

### B. The Electric Aharonov-Bohm Effect

The electric version of Aharonov-Bohm effect \[3\] is in principle a second direct example, although the practical obstacles to carrying out such an experiment are formidable. There, each arm of an electron interferometer is surrounded in the interaction zone by a cylindrical conducting shield. While the partial wave packets in both arms are inside their respective conducting cylinders, the two cylinders are caused to have a potential difference \( \Delta \phi(t) \). That leads to a phase difference

\[ \delta = \int V \, dt = \frac{e}{\hbar} \int \Delta \phi(t) \, dt, \quad (4.3) \]

again independently of the energy of the electrons.

### C. The Aharonov-Casher Effect

In an idealized interferometric version of the Aharonov-Casher (AC) effect \[8,9\] atoms or neutrons with magnetic moment \( \mu \) move in the \( x \) direction with their spins polarized in the \( z \) direction. In the interaction zone, they traverse an electric field of strength \( \pm E \) in the \( z \) direction, \( + \) in one arm of the interferometer and \( - \) in the other.

The interaction term in the Hamiltonian is given for nonrelativistic motion by

\[ V = \frac{\mu}{mc} \sigma \cdot p \times E = \pm \frac{\mu E}{mc} p. \quad (4.4) \]

That leads to force-free motion with phase shifts given by

\[ \delta_{\pm} = \int V_{\pm} \, dt = \pm \frac{\mu E}{c} \int \frac{p}{m} \, dt = \pm \frac{\mu E \ell}{c}. \quad (4.5) \]

The relative phase shift between the two arms is

\[ \delta = 2 E \ell \frac{\mu}{c}, \quad (4.6) \]

independently of the energy. This result has been confirmed in neutron interferometry experiments \[10,11\] and with high precision in atom interferometry experiments \[12,13\] that demonstrated the nondispersivity over a wide range of energies.
D. The Scalar Aharonov-Bohm Effect

In what has been called the scalar Aharonov-Bohm (SAB) effect, neutrons in the interaction zone traverse an external magnetic field $B(t)$. In the simplest case, the neutrons are polarized in the direction of $B$. Ideally, $B(t)$ should be turned on and then off while the neutrons wave packet is inside the interaction zone, to avoid forces due to grad $B$ at the edges of the field. The phase shift is then given by

$$\delta = \int V \, dt = -\frac{\mu}{\hbar} \int B(t) \, dt, \quad (4.7)$$

independently of the energy of the neutron. Experiments doing the practical equivalent of that have been reported, both with unpolarized \cite{14} and polarized \cite{15,16} neutrons, and did find the expected phase shifts. In neutron interferometry, the range of neutron energies is narrow, but the nondispersivity can nevertheless be useful because it guarantees that the phase shift is uniform across the wave packet.

E. The Force-Free Nuclear Phase Shifter

Conceptually, it is possible to create a phase shifter for neutron interferometry by having a gas cell in the interaction zone. Some gas is pumped into the cell and then pumped out again, all while the neutron’s wave packet is entirely within the interaction zone. This results in a time-dependent index of refraction, or equivalently a time-dependent force-free interaction potential $V$, and therefore a nondispersive phase shift analogous to the ones in the AC and SAB effects and sharing their Schröedinger equations under appropriate substitution in the interaction term $V$. The force-free nuclear phase shifter was introduced as a thought experiment \cite{17} to separate the question of force-free motion from issues involving electromagnetism, gauge fields, and torques on spinning particles.

V. CONCLUSIONS

Nondispersive phase shifts are a signature of force-free motion. A nondispersive phase shift at all energies is required for, but in principle does not imply, force-free motion. However, for motion under the influence of forces to exhibit a nondispersive phase shift at all energies seems improbable in a real case.

There are two objectively different classes of force-free interactions in interferometry, both nondispersive. In the first class are the magnetic and electric Aharonov-Bohm effects. There, the interaction is with a gauge field and a gauge transformation can move the location of the interaction from one arm of the interferometer to another.

The second class contains the AC and SAB effects and the force-free nuclear phase shifter. There the physics is significantly different \cite{17,18}. The interaction occurs in an objectively known place. In the case of AC and SAB there are torques whose expected values vanish but whose quantum fluctuations do not and there are observables, measurable in principle, that respond to those torques. In the nuclear phase shifter there is an ordinary potential energy.
A nondispersive phase shift has often been cited as evidence of a topological interaction. That is true if a topological interaction is defined to be a force-free one. A possibly unwanted consequence of that definition is that it includes the interaction with the nondispersive nuclear phase shifter as a topological interaction, but not the interaction with an ordinary nuclear phase shifter with very similar physical properties.

The AB effect is thought to be a topological phenomenon for the reasons given above and because it has an important relation to Berry’s geometrical phase [19], which is topological in nature. The literature contains many discussions or statements about the topological nature of the interactions in the second class [20]. Their lack of agreement appears to arise entirely from the use of different, sometimes unstated, definitions.

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FIGURES

FIG. 1. Mach-Zehnder interferometer shown with an interaction zone of length $\ell$ in each arm. The solid circle represents a solenoid used for the magnetic Aharonov-Bohm effect.
