FRACTIONAL LAPLACIANS : A SHORT SURVEY

MAHA DAoud
Department of Mathematics and Computer Science
Faculty of sciences Ain Chock
B.P. 5366 Maarif, Casablanca, Morocco

EL HAJ LAAMRI*
Institut Elie Cartan de Lorraine
Université de Lorraine
B.P. 239, Vandoeuvre-lès-Nancy, France

Abstract. This paper describes the state of the art and gives a survey of the wide literature published in the last years on the fractional Laplacian. We will first recall some definitions of this operator in \( \mathbb{R}^N \) and its main properties. Then, we will introduce the four main operators often used in the case of a bounded domain ; and we will give several simple and significant examples to highlight the difference between these four operators. Also we give a rather long list of references : it is certainly not exhaustive but hopefully rich enough to track most connected results. We hope that this short survey will be useful for young researchers of all ages who wish to have a quick idea of the fractional Laplacian(s).

1. Introduction. The theory of fractional powers of operators on Banach spaces is a classical topic in mathematical analysis and Probability. It was developed over the last century, we refer the interested reader for instance to [9, 13, 57, 58] and the references therein. In the last years, this theory has gained a higher interest due to the increasing need for modeling in many scientific fields. In fact, many models issued from Biology, Ecology, Finance, etc, lead to nonlinear problems driven by fractional Laplace-type operators.

The basic example of these operators is the fractional Laplacian. In contrast to the Laplacian which is a local operator, the fractional Laplacian is a paradigm of the vast family of nonlocal operators, and this has immediate consequence in the formulation of basic questions such as the Poisson problem. There is an extensive literature devoted to this topic, contributions include [39, 12, 87, 22, 80, 25, 26, 43, 42, 52, 62] and the references therein. Needless to say, these references do not exhaust the rich literature on the subject.

From a probabilistic point of view, the fractional Laplacian is related to anomalous diffusion, that employs fractional order derivatives ; it represents the infinitesimal generator of Lévy stable diffusion process, as it is well known in the theory of probability. For more details, see for instance [14, 72, 52, 78, 31] and the references

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* Corresponding author : El Haj Laamri.
therein. On the other hand, it is possible to obtain a fractional heat equation as a limit of a random walk with long jumps, we refer the interested reader to the excellent note [86].

As is well known, there are several ways of defining this operator in the whole space \( \mathbb{R}^N \). As far as we know, there are ten definitions in the literature. It turns out that all these definitions are equivalent (see for instance [59]). On the other hand, several different fractional Laplacians can be defined on an open subset \( \Omega \neq \mathbb{R}^N \). These alternatives correspond to different ways of taking into account the information from the boundary and the exterior of domain.

The first goal of this work is to present an up-to-date survey on the fractional Laplacian in \( \mathbb{R}^N \), and on the different fractional Laplacians on an open subset \( \Omega \neq \mathbb{R}^N \). More precisely, we will present four fractional Laplacians. Two of them are well known and have been widely studied, while the other two have only recently been introduced in the literature. The second goal is to highlight the difference between these four operators. To this end, we will first give some simple and significant examples to illustrate this difference; then we will investigate it numerically. For the sake of simplicity, we will only consider homogeneous Dirichlet boundary-value problems, which will allow us to focus on the main features and subtleties of the theory.

Let us emphasize that the originality of this work does not lie so much in the content, for which we have drawn inspiration from existing literature — notably that cited in the bibliography — as in the presentation and choice of the themes. We do not claim to make significant additions to a theory that has changed little for several years; but we justify our presentation as a new and quick insight for new readers who wish to have a global and quick understanding of the subject before going deeper.

The rest of paper is composed of four sections. Section 2 is devoted to a review of several (and equivalent) definitions of the fractional Laplacian \((-\Delta)^s\) in \( \mathbb{R}^N \) and its main properties. For the sake of brevity and clarity of exposition, we have chosen to treat only the case \( 0 < s < 1 \). In section 3, we briefly recall some properties of fractional Sobolev spaces on an open bounded subset \( \Omega \) of \( \mathbb{R}^N \). In Section 4, we expose in some detail four different “fractional Laplacians” : regional (in Subsection 4.1), spectral (in Subsection 4.2), restricted (in Subsection 4.4) and peridynamic (in Subsection 4.5). Moreover, Subsections 4.3 and 4.6 are devoted to highlighting the differences between these four operators by suitable one-dimensional examples.

Needless to say, it is never pleasant for a mathematician not to give the demonstrations. Nevertheless, in order to keep this paper reasonably short, we will mainly state and comment results without reproducing the proofs. Accordingly, we have tried to give precise references that allow the reader to find them. Hence our rather long bibliography: it is certainly not exhaustive, but hopefully rich enough to track most relevant results.

**Notations**: Let \( N \) be a positive integer, \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N, \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N \) and \( \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N \).

Throughout this paper, we denote by:

- \( \Omega \subset \mathbb{R}^N \) an open, bounded with smooth boundary;
- \( D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} \) where \(|\alpha| := \alpha_1 + \cdots + \alpha_N\);
- \( x^\alpha := x_1^{\alpha_1} \cdots x_N^{\alpha_N} \).
\begin{itemize}
  \item \((x | \xi) := \sum_{k=1}^{N} x_k \xi_k\) the euclidean inner product of \(\mathbb{R}^N\);
  \item \(|x|^2 := \sum_{k=1}^{N} x_k^2\);
  \item \(\Delta u = \sum_{k=1}^{N} \frac{\partial^2 u}{\partial x_k^2}\);
  \item \(|u|_{L^p(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} |u(x)|^p dx \right)^{\frac{1}{p}}\) where \(p \in [1, +\infty)\);
  \item \(B(x, r) := \{y \in \mathbb{R}^N ; \|x - y\| < r\}\) where \(r \in (0, +\infty)\);
  \item \(C_0(\mathbb{R}^N) := \{u \in C(\mathbb{R}^N) ; u(x) \to 0 \text{ as } \|x\| \to +\infty\}\);
  \item \(C_c(\mathbb{R}^N) := \{u \in C_c(\mathbb{R}^N) ; u \text{ with compact support }\}\);
  \item \(C_c(\Omega) := \{u \in C_c(\mathbb{R}^N) ; u \text{ with compact support in } \Omega\}\);
  \item \(\mathcal{S}(\mathbb{R}^N) := \{\varphi \in C_c(\mathbb{R}^N) ; \forall \alpha, \beta \in \mathbb{N}^N, \sup_{x \in \mathbb{R}^N} |x^\beta D^\alpha \varphi(x)| < +\infty\}\);
  \item \(\mathcal{F}(u)(\xi) := \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-i(\xi \cdot x)} u(x) dx\) the Fourier transform of \(u\);
  \item \(\mathcal{F}^{-1}u(x) := \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{i(\xi \cdot x)} u(\xi) d\xi\);
  \item \(\Gamma(z) = \int_{0}^{+\infty} t^{z-1} e^{-t} dt, \ z \in (0, +\infty), \text{ the Gamma function.}\)
\end{itemize}

2. Fractional Laplacian operator in \(\mathbb{R}^N\). Let \(s \in (0, 1)\). In this section, we will introduce the fractional Laplacian \((-\Delta)^s\) in the whole-space \(\mathbb{R}^N\) and its basic properties. To the best of our knowledge, there are ten equivalent definitions of the fractional Laplacian in the whole-space \(\mathbb{R}^N\), see e.g. \cite{59}. Here, we will give only five definitions, namely the most frequently used.

2.1. Fractional Sobolev spaces \(H^s(\mathbb{R}^N)\). As will appear later (see Proposition 2.6 below), the fractional Sobolev space \(H^s(\mathbb{R}^N)\) is the right framework to define the fractional Laplacian in \(\mathbb{R}^N\). For the sake of completeness and the ease of reader, we now recall two equivalent definitions of this space.

First, we define the fractional Sobolev space \(H^s(\mathbb{R}^N)\) through the so-called Gagliardo semi-norm :

\[
[u]_{H^s(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{\|x - y\|^{N+2s}} dy dx \right)^{\frac{1}{2}}
\]

Then,

\[
H^s(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) ; [u]_{H^s(\mathbb{R}^N)} < +\infty\}.
\]

For \(u, v \in H^s(\mathbb{R}^N)\), we set

\[
\langle u, v \rangle_{H^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} u(x)v(x) dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{\|x - y\|^{N+2s}} dy dx
\]

One can easily check that \(\langle \cdot, \cdot \rangle_{H^s(\mathbb{R}^N)}\) is an inner product on \(H^s(\mathbb{R}^N)\) and that the space \(H^s(\mathbb{R}^N)\) endowed with this inner product is a Hilbert space.
Second, the space $H^s(\mathbb{R}^N)$ can be defined alternatively via the Fourier transform. We denote

$$\hat{H}^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + \|\xi\|^{2s}) |\mathcal{F}u(\xi)|^2 d\xi < +\infty \right\}.$$ 

The fractional Sobolev space $H^s(\mathbb{R}^N)$ coincides with $\hat{H}^s(\mathbb{R}^N)$ (see for instance [39, Proposition 3.4] or [12, Corollary 1.15]).

2.2. Definitions of fractional Laplacian and main properties. In this paragraph, we will give some equivalent definitions of fractional Laplacian in $\mathbb{R}^N$ and its main properties.

2.2.1. First definition. The first definition of the fractional Laplacian $(-\Delta)^s$ ($s \in (0,1)$) is expressed via the Fourier transform.

Let $u \in \mathcal{S}(\mathbb{R}^N)$. We have that $\mathcal{F}u \in \mathcal{S}(\mathbb{R}^N)$, but the function $\xi \mapsto \|\xi\|^{2s} \mathcal{F}u(\xi) \notin \mathcal{S}(\mathbb{R}^N)$ because $\|\xi\|^{2s}$ creates a singularity at $\xi = 0$. On the other hand, $\|\xi\|^{2s} \mathcal{F}u(\xi) \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, so we can use the inverse Fourier transform. Therefore, we have the following definition:

For any $u \in \mathcal{S}(\mathbb{R}^N)$, we define the operator $(-\Delta)^s : \mathcal{S}(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ by

$$(-\Delta)^s u(x) = \mathcal{F}^{-1}(\|\xi\|^{2s} \mathcal{F}u(\xi))(x) \quad \forall x \in \mathbb{R}^N,$$

(2.1)

This operator is called the fractional Laplacian of order $s$.

First properties:
Now, we give some elementary properties of the fractional Laplacian (see for instance [80]). Before this, let us recall that $\mathcal{S}(\mathbb{R}^N) \subset C_0(\mathbb{R}^N)$ and the inclusion is strict.

Let $u \in \mathcal{S}(\mathbb{R}^N)$, we have:

• $(-\Delta)^0 u = u$;
• $(-\Delta)^1 u = -\Delta u$;
• for any $0 < s_1, s_2 < 1$, $(-\Delta)^{s_1} \circ (-\Delta)^{s_2} u = (-\Delta)^{s_2} \circ (-\Delta)^{s_1} u$;
• for any multi-index $\alpha \in \mathbb{N}^N$, $D^\alpha (-\Delta)^s u = (-\Delta)^s D^\alpha u$. In particular, $(-\Delta)^s u \in C^\infty(\mathbb{R}^N)$.

2.2.2. Second definition. The second definition of the fractional Laplacian is often used in the literature. It is also the most adequate to define the regional fractional Laplacian (see Definition 4.1) and the restricted fractional Laplacian (see Definition 4.6) in the case of a bounded subset of $\mathbb{R}^N$.

For any $u \in \mathcal{S}(\mathbb{R}^N)$,

$$(-\Delta)^s u(x) := C(N, s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B(x, \varepsilon)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy \quad \forall x \in \mathbb{R}^N,$$

(2.2)

where

$$C(N, s) := \frac{s 4^s \Gamma(s + \frac{N}{2})}{\pi^{\frac{N}{2}} \Gamma(1 - s)}.$$

(2.3)
**Remark 2.1.** Let us give some light on the normalizing constant $C(N,s)$ given by (2.3).

(i) $C(N,s)$ is chosen so that the four definitions (2.1), (2.2), (2.7) and (2.8) are equivalent and the following two identities:

$$\lim_{s \to 0^+} (-\Delta)^s u = u \quad \text{and} \quad \lim_{s \to 1^-} (-\Delta)^s u = -\Delta u$$

hold (see Proposition 2.1).

(ii) Moreover

$$C(N,s) \sim s(1-s) \quad \text{as} \quad s \to 0^+ \quad \text{and} \quad s \to 1^-.$$  

For more details on the asymptotic behavior of $C(N,s)$, the interested reader is referred to [39, Section 4] (see also [12, Section 1.3.1]).

We recall that by definition:

$$\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{||x - y||^{N+2s}} \, dy := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B(x,\varepsilon)} \frac{u(x) - u(y)}{||x - y||^{N+2s}} \, dy.$$  

In other words, P.V. stands for the "Cauchy principal value". Since then, we can write

$$(-\Delta)^s u(x) := C(N,s) \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{||x - y||^{N+2s}} \, dy \quad \forall x \in \mathbb{R}^N. \quad (2.5)$$

The following theorem (see for instance [80, Theorem 1]) shows that we can do without using the "Cauchy principal value" in (2.5) if $s \in (0, \frac{1}{2})$.

**Theorem 2.1.** Let $u \in \mathcal{S}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. The integral

$$\int_{\mathbb{R}^N} \frac{u(x) - u(y)}{||x - y||^{N+2s}} \, dy \quad (2.6)$$

is absolutely convergent if and only if $s \in (0, \frac{1}{2})$.

So we have

$$(-\Delta)^s u(x) = C(N,s) \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{||x - y||^{N+2s}} \, dy,$$

for $s \in (0, \frac{1}{2})$.

2.2.3. **Third definition.** The third definition is given in the form of a well-defined integral on $\mathbb{R}^N$ as follows:

For any $u \in \mathcal{S}(\mathbb{R}^N)$, we have

$$(-\Delta)^s u(x) := -\frac{1}{2} C(N,s) \int_{\mathbb{R}^N} \frac{u(x + y) + u(x - y) - 2u(x)}{||y||^{N+2s}} \, dy, \quad \forall x \in \mathbb{R}^N. \quad (2.7)$$

It is worth to point out that the Definitions 2.1, 2.2 and 2.7 are equivalent. For a proof, see for instance [39, section 3] or [12, subsection 1.3].

2.2.4. **Fourth definition.** Another definition of the fractional Laplacian comes from the theory of semigroups in $\mathbb{R}^N \times [0, +\infty)$. This definition is classical (see, e.g., [9, 57, 58, 89]) and it is more useful to study partial differential equations.

For any $u \in \mathcal{S}(\mathbb{R}^N)$, we have

$$(-\Delta)^s u(x) = \frac{s}{\Gamma(1-s)} \int_0^{+\infty} (u(x) - w(x,t)) \, \frac{dt}{t^{1+s}}, \quad (2.8)$$

where $w(x,t)$ is the solution to the following problem

\[ \begin{cases} \partial_t w - \Delta w = 0 & \text{in } \mathbb{R}^N \times [0, +\infty), \\ w(x,0) = u(x). \end{cases} \]
All the definitions above are equivalent. For a proof of the equivalence between
2.2 and 2.8, see for instance [81, Lemma 2.1].

2.2.5. Relation between the “classical” Laplacian and the fractional Lapla-
cian. The following proposition describes the relation between the “classical” Lapla-
cian and the fractional Laplacian (see, e.g., [22, Lemma 3.11] and [39, Proposition 4.4]).

Proposition 2.1. Let \( u \in \mathcal{S}(\mathbb{R}^N) \). Then

(i) \( \lim_{s \to 0^+} (\Delta)^s u = u \),

(ii) \( \lim_{s \to 1^-} (\Delta)^s u = -\Delta u \).

As we will see later, Proposition 2.1 holds for less regular functions, see Theorem
2.2 and Theorem 2.3 below. \( \square \)

2.2.6. Extensions for less regular functions. Definitions (2.1), (2.2), (2.7) and
(2.8) are valid for any \( u \in \mathcal{S}(\mathbb{R}^N) \). Actually, we can extend these definitions to
\( C^{0,\beta}(\mathbb{R}^N) \) and even to \( H^s(\mathbb{R}^N) \).

- Extension to \( C^{0,\beta}(\mathbb{R}^N) \) and some regularity results Let \( V \) be an open subset
  of \( \mathbb{R}^N \), \( k \in \mathbb{N} \) and \( \beta \in (0, 1) \). We denote by \( C^{0,\beta}(V) \) the space of \( \beta \)-Hölder continuous
  functions on \( V \), and by \( C^{k,\beta}(V) \) the space of functions of \( C^k(V) \) whose \( k \)th partial
  derivatives are Hölder continuous with exponent \( \beta \). Let us recall the seminorms :

\[
[u]_{C^{0,\beta}(V)} := \sup_{x, y \in V} \left( \frac{|u(x) - u(y)|}{|x - y|^{\beta}} \right)
\]

and

\[
[u]_{C^{k,\beta}(V)} := \sum_{|\alpha| = k} \sup_{x, y \in V} \left( \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^{\beta}} \right), \quad k > 0.
\]

Furthermore, let introduce the following space

\[
L^1_s(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : \int_{\mathbb{R}^N} \frac{|u(x)|}{1 + |x|^{N + 2s}} \, dx \leq +\infty \right\}.
\]

We have the following results whose proofs can be found in [77, Propositions 2.4, 2.5, 2.6 and 2.7] (see also [76] and [80, section 7]).

Proposition 2.2. Let \( u \in L^1_s(\mathbb{R}^N) \). Assume that \( u \in C^{0,2s+\varepsilon}(V) \) if \( s \in (0, \frac{1}{2}) \) or
\( u \in C^{1,2s+\varepsilon-1}(V) \) if \( s \in [\frac{1}{2}, 1) \), for some \( \varepsilon > 0 \). Then, \( (\Delta)^s u \) is a continuous
function in \( V \) and \( (\Delta)^s u(x) \) is given by the definitions above for every \( x \in V \).

Proposition 2.3. Let \( u \in C^{0,\beta}(\mathbb{R}^N) \) for \( \beta > 2s > 0 \). Then, \( (\Delta)^s u \in C^{0,\beta-2s}(\mathbb{R}^N) \)
and there exists \( C > 0 \) such that

\[
[(\Delta)^s u]_{C^{0,\beta-2s}(\mathbb{R}^N)} \leq C [u]_{C^{0,\beta}(\mathbb{R}^N)},
\]

where \( C \) depends only on \( \beta, s \) and \( N \).

Proposition 2.4. Let \( u \in C^{1,\beta}(\mathbb{R}^N) \).

(i) If \( \beta > 2s \), then \( (\Delta)^s u \in C^{1,\beta-2s}(\mathbb{R}^N) \) and there exists \( C > 0 \) such that

\[
[(\Delta)^s u]_{C^{1,\beta-2s}(\mathbb{R}^N)} \leq C [u]_{C^{1,\beta}(\mathbb{R}^N)},
\]

where \( C \) depends only on \( \beta, s \) and \( N \).
(ii) If \( \min(0, 2s - 1) < \beta < 2s \), then \((-\Delta)^s u \in C^{0, \beta-2s+1}(\mathbb{R}^N) \) and there exists \( C > 0 \) such that

\[
[(\Delta)^s u]_{C^{0, \beta-2s+1}(\mathbb{R}^N)} \leq C[u]_{C^{1, \beta}(\mathbb{R}^N)},
\]

where \( C \) depends only on \( \beta, s, \) and \( N \).

Proposition 2.5. Let \( u \in C^{k, \beta}(\mathbb{R}^N) \) such that \( k + \beta - 2s \) is not an integer. Then, \((-\Delta)^s u \in C^{\ell, \alpha}(\mathbb{R}^N) \) where \( \ell \) is the integer part of \( k + \beta - 2s \) and \( \alpha = k + \beta - 2s - \ell \).

Before ending this paragraph, it is worth to point out that Proposition 2.1 can be extended for less regular functions. More precisely, we have the following two results (see [80, Theorem 3] and [80, Theorem 4]).

Theorem 2.2. Let \( \mathbf{x} \in \mathbb{R}^N \) and \( u \in C^2(\overline{B(\mathbf{x}, 1)}) \cap L^\infty(\mathbb{R}^N) \). Then

\[
\lim_{s \to 1^-} (-\Delta)^s u(\mathbf{x}) = -\Delta u(\mathbf{x}).
\]

Theorem 2.3. Let \( \mathbf{x} \in \mathbb{R}^N \) and \( u \in C^{0, \beta}(\overline{B(\mathbf{x}, 1)}) \cap L^1_0(\mathbb{R}^N) \) where

\[
L^1_0(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable : } \int_{\mathbb{R}^N} \frac{|u(\mathbf{x})|}{1 + ||\mathbf{x}||^N} d\mathbf{x} < +\infty \right\}.
\]

Then

\[
\lim_{s \to 0^+} (-\Delta)^s u(\mathbf{x}) = u(\mathbf{x}).
\]

**Extension to \( H^s(\mathbb{R}^N) \)** Since the Schwartz space \( S(\mathbb{R}^N) \) is dense in \( H^s(\mathbb{R}^N) \), the fractional Laplacian operator \((-\Delta)^s\) can be extended by density to \( u \in H^s(\mathbb{R}^N) \). More precisely, we have the following relation (see, e.g., [39, Proposition 3.6], or [12, Proposition 1.18]).

Proposition 2.6. For any \( u \in H^s(\mathbb{R}^N) \),

\[
[u]_{H^s(\mathbb{R}^N)}^2 = 2C(N, s)^{-1} \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)}^2.
\]  

As mentioned before, the identity (2.9) shows that the space \( H^s(\mathbb{R}^N) \) is the right framework to define the fractional Laplacian \((-\Delta)^s\).

2.2.7. **Fifth definition.** An important tool in some works on nonlinear problems driven by fractional Laplacians (see, e.g., [76, 25, 77, 26]) is the following transformation: any fractional Laplacian can be characterized as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem. More precisely, the fractional Laplacian can be realized in a local manner through the boundary value problem

\[
\begin{align*}
\text{div} \left( y^a \nabla U \right) &= 0 & & & & \text{in} \quad \mathbb{R}^{N+1}_{+}, \\
U(x, 0) &= u(x) & & & & \text{on} \quad \partial \mathbb{R}^{N+1}_{+},
\end{align*}
\]

(2.10)

where \( \mathbb{R}^{N+1}_{+} := \{(x, y) \in \mathbb{R}^N \times \mathbb{R} : y > 0\} \) is the upper half-space and \( \partial \mathbb{R}^{N+1}_{+} := \mathbb{R}^N \).

The parameter \( a \) belongs to \((-1, 1)\) and is related to the power \( s \) of the fractional Laplacian \((-\Delta)^s\) by the formula \( a = 1 - 2s \in (-1, 1) \). We set

\[
\frac{\partial U}{\partial y^a} := - \lim_{y \to 0^+} y^a \partial_y U.
\]
The following formula relating the fractional Laplacian to the Dirichlet-to-Neumann operator for (2.10) has been proven in [25] (cf. also [19, 28, 82]):
\[ (-\Delta)^s u = d_s \frac{\partial U}{\partial \nu} \quad \text{in } \mathbb{R}^N, \]
where \( d_s = 2^{2s-1} \frac{\Gamma(s)}{\Gamma(1-s)} \) (see e.g. [24] for more details).

3. Fractional Sobolev spaces on an open bounded subset of \( \mathbb{R}^N \). In what follows, \( \Omega \) is an open bounded subset of \( \mathbb{R}^N \) with smooth boundary, and \( s \in (0,1) \).

As in the case of \( \mathbb{R}^N \), we start this paragraph by recalling one of the possible definitions of the classical fractional Sobolev spaces \( H^s(\Omega) \) and \( H_0^s(\Omega) \). Then we define new spaces \( X_0^s(\Omega) \), which are the appropriate functional framework to study Dirichlet boundary-value problems governed by fractional Laplacians.

3.1. Classical fractional Sobolev spaces \( H^s(\Omega) \) and \( H_0^s(\Omega) \).

First, we define the fractional Sobolev space \( H^s(\Omega) \) through the Gagliardo semi-norm:
\[ |u|_{H^s(\Omega)} := \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{\|x - y\|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}. \] (3.1)

Then,
\[ H^s(\Omega) := \{ u \in L^2(\Omega) : |u|_{H^s(\Omega)} < +\infty \}. \]

For \( u, v \in H^s(\Omega) \), we set
\[ \langle u, v \rangle_{H^s(\Omega)} := \int_{\Omega} u(x)v(x) \, dx + \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{\|x - y\|^{N+2s}} \, dx \, dy. \]

It is easy to check that \( \langle \cdot, \cdot \rangle_{H^s(\Omega)} \) is an inner product and the space \( H^s(\Omega) \) endowed with this inner product is a Hilbert space.

From now on, we denote
\[ \|u\|_{H^s(\Omega)} := \left( \int_{\Omega} |u(x)|^2 \, dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{\|x - y\|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}. \] (3.2)

Furthermore, we define \( H_0^s(\Omega) \) as the closure of \( C_\infty^\infty(\Omega) \) in \( H^s(\Omega) \).

It is worth to point out that the spaces \( H^s(\Omega) \) have no trace where \( s \in (0, \frac{1}{2}] \).

More precisely, we have the following result:

**Theorem 3.1.** [60, Theorem 11.1]
- If \( s \in (0, \frac{1}{2}] \), then \( H_0^s(\Omega) = H^s(\Omega) \).
- If \( s \in (\frac{1}{2}, 1) \), the inclusion \( H_0^s(\Omega) \subset H^s(\Omega) \) is strict.

The previous result holds for any \( p \in (1, +\infty) \), see [53, Theorem 1.4.2.4].

3.2. Other type of fractional Sobolev spaces \( X_0^s(\Omega) \). In order to study nonlocal problems governed by fractional Laplacians, we need appropriate fractional Sobolev spaces. These new spaces are inspired by (but not equivalent to) the fractional Sobolev spaces \( H^s(\Omega) \), in order to correctly encode the data of the homogeneous Dirichlet condition in the weak formulation.

For this purpose, let us introduce the following space
\[ X_0^s(\Omega) := \{ u \in H^s(\mathbb{R}^N) : u = 0 \ a.e. \text{ in } \mathbb{R}^N \setminus \Omega \} \]
endowed with the norm induced by \( \| \cdot \|_{H^s(\mathbb{R}^N)} \).
Let $u \in X^s_0(\Omega)$, we have

$$\|u\|_{X^s_0(\Omega)}^2 = \|u\|^2_{H^s(\mathbb{R}^N)}$$

$$= \|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{\|x - y\|^{N+2s}} \, dx \, dy.$$  

As $\|u\|^2_{L^2(\mathbb{R}^N)} = \|u\|^2_{L^2(\Omega)}$, then

$$\|u\|_{X^s_0(\Omega)}^2 = \|u\|^2_{L^2(\Omega)} + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{\|x - y\|^{N+2s}} \, dx \, dy. \quad (3.3)$$

We deduce from (3.3) that the norms $\| \cdot \|_{X^s_0(\Omega)}$ and $\| \cdot \|_{H^s(\Omega)}$ are not the same. Moreover, the norm $\| \cdot \|_{X^s_0(\Omega)}$ takes into account interaction between $\Omega$ and $\mathbb{R}^N \setminus \Omega$. So, the space $X^s_0(\Omega)$ is the appropriate space to deal with elliptic partial differential equations driven by fractional laplacian operators.

Now, for $u \in X^s_0(\Omega)$, we set

$$\|\cdot\|_{X^s_0(\Omega)} := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{\|x - y\|^{N+2s}} \, dx \, dy \right)^{1/2}.$$

We have the following result (see, e.g., [12, Lemma 1.29]) :

**Proposition 3.1.** The space $(X^s_0(\Omega), \langle \cdot, \cdot \rangle_{X^s_0(\Omega)})$ is a Hilbert space with the inner product

$$\langle u, v \rangle_{X^s_0(\Omega)} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{\|x - y\|^{N+2s}} \, dx \, dy \quad \forall u, v \in X^s_0(\Omega). \square$$

Moreover, the norms $\| \cdot \|_{X^s_0(\Omega)}$ and $\| \cdot \|_{X^s_0(\Omega)}$ are equivalent.

**Remark 3.1.** This space is also denoted $\tilde{H}^s(\Omega)$ or $H^s_{00}(\Omega)$ in the literature. As is well-known (see, e.g., [60, 53, 6]), it holds that

$$X^s_0(\Omega) = \begin{cases} H^s(\Omega) & \text{if } 0 < s < \frac{1}{2}, \\ H^s_0(\Omega) & \text{if } \frac{1}{2} < s < 1, \end{cases}$$

while in the limiting case $s = \frac{1}{2}$, it holds that :

$$X^{\frac{1}{2}}_0(\Omega) := \left\{ u \in H^{\frac{1}{2}}(\Omega) ; \int_{\Omega} \frac{u^2(x)}{\rho(x)} \, dx < +\infty \right\},$$

with $\rho(x) := \text{distance}(x, \partial \Omega)$.

In all cases, the canonical norm induced by the alternative definition is equivalent to the previous ones.

4. Fractional Laplacians on an open bounded domain. Throughout this paragraph, $\Omega$ is a bounded open subset of $\mathbb{R}^N$ with smooth boundary.

As we have seen in Section 2, the different definitions of the fractional Laplacian $(-\Delta)^s$ are equivalent when the operator acts in $\mathbb{R}^N$. Contrary to the case of whole space, several different fractional Laplacians $^1$ can be defined on a open bounded

$^1$Unfortunately, the mathematical community has not yet reached a unified terminology and standard notations for the fractional Laplacians on a bounded domain. The reader is strongly advised to check the definitions carefully in order to avoid any confusion.
subset of \( \mathbb{R}^N \). These alternatives correspond to different ways in which the information coming from the boundary and the exterior of domain is to be taken into account.

In this section, we will present four fractional Laplacians and compare their definitions, their probabilistic interpretations and their analytical properties.

For pedagogical purposes, this section is composed of six subsections. In the first one, we introduce the \textit{regional} fractional Laplacian (\( \text{FL}_{\text{Reg}} \) for short) and give some examples. Subsection 2 is devoted to the \textit{spectral} fractional Laplacian (\( \text{FL}_{\text{Spec}} \)). In subsection 3, we highlight the difference between these two operators by comparing, in particular, their respective spectra. In Subsection 4, we introduce the so-called \textit{restricted} fractional Laplacian (\( \text{FL}_{\text{Rest}} \)), and we compare it with \( \text{FL}_{\text{Reg}} \). Then, we introduce the so-called \textit{peridynamic} fractional Laplacian (\( \text{FL}_{\text{Pery}} \)) in Subsection 5. In the last subsection, we give three examples in dimension 1 in order to illustrate the difference between these four operators.

In order to go deeper into the study of fractional Laplacian operators, we refer the reader, for instance, to [61, 10, 11]. Those papers give basic estimates and some regularity results for solutions to problems governed by \((-\Delta)^s\).

On the other hand, the paper [32] gives, through the theory of semigroups, a characterization of the realization in \( L^2(\Omega) \) of the fractional Laplacian with different exterior conditions.

### 4.1. \textbf{Regional Fractional Laplacian (\( \text{FL}_{\text{Reg}} \))}

In this subsection, we give the definition of the so-called \textit{regional fractional Laplacian} which we denote by \((-\Delta)^s_{\text{Reg}}\). We also give a few examples for illustration purposes.

**Definition** This operator acts on functions \( u \) defined in \( \Omega \) and extended by zero to \( \mathbb{R}^N \setminus \Omega \). So we can use Definition 2.2.

Let \( u \in X^s_0(\Omega) \). The \textit{regional fractional Laplacian} is defined by:

\[
(-\Delta)^s_{\text{Reg}} u(x) := C(N,s) P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{||x - y||^{N+2s}} dy.
\]  

(4.1)

**Remark 4.1.** From (4.1), we deduce

\[
(-\Delta)^s_{\text{Reg}} u(x) := C(N,s) \left[ P.V. \int_{\Omega} \frac{u(x) - u(y)}{||x - y||^{N+2s}} dy + \int_{\mathbb{R}^N \setminus \Omega} \frac{u(x) - u(y)}{||x - y||^{N+2s}} dy \right].
\]

As \( u = 0 \in \mathbb{R}^N \setminus \Omega \), we obtain

\[
(-\Delta)^s_{\text{Reg}} u(x) := C(N,s) \left[ P.V. \int_{\Omega} \frac{u(x) - u(y)}{||x - y||^{N+2s}} dy + \int_{\mathbb{R}^N \setminus \Omega} \frac{u(x)}{||x - y||^{N+2s}} dy \right].
\]

(4.2)

**Probabilistic interpretation:** The \textit{regional fractional Laplacian} is the infinitesimal generator of jump-type processes killed upon leaving the domain \( \Omega \). In this respect, the operator generates a killed subordinate Brownian motion (see for instance [14, 31, 51, 52, 79] and references therein). Indeed, this process describes trajectories of particles which are allowed to jump outside \( \Omega \) and then one subordinates these paths in \( \Omega \), \textit{i.e.}, one leaves marks only on the parts of paths inside the domain.
domain. The homogeneous Dirichlet exterior condition in $\mathbb{R}^N \setminus \Omega$ corresponds to this operation.

- **Examples** For the reader’s convenience, let us give some examples, (see [44]).

**Example 1. One-dimensional case :** $\Omega = (-1, 1)$.

| $u(x)$               | $(-\Delta)^s_{\text{Reg}} u(x)$ |
|----------------------|----------------------------------|
| $(1 - x^2)^{r+1}_+$  | 0                               |
| $(1 - x^2)^{r+1}$    | $\Gamma(2s + 1)$                |
| $(1 - x^2)^{r+1}$    | $(s + 1)\Gamma(2s + 1)(1 - (2s + 1)x^2)$ |
| $(1 - x^2)^{r+1}$    | $(s + 1)\Gamma(2s + 1)(1 - (2s + 1)x^2)$ |
| $x(1 - x^2)^{r+1}$   | 0                               |
| $x(1 - x^2)^{r+1}$   | $\Gamma(2s + 2)x$               |
| $x(1 - x^2)^{r+1}$   | $\frac{2s+2}{60}\Gamma(2s + 3)(3 - (2s + 3)x^2)x$ |
| $x(1 - x^2)^{r+1}$   | $\frac{2s+2}{60}\Gamma(2s + 3)(3 - (2s + 3)x^2)x$ |

Table 1. The regional fractional Laplacian of some functions that vanish in $\mathbb{R} \setminus (-1, 1)$.

**Example 2. Multidimensional case :** Let $N \geq 2$, $x = (x_1, x_2, ..., x_N) \in \mathbb{R}^N$ and $\Omega = B(0, 1)$ be the ball of center the origin and radius 1.

| $u(x)$               | $(-\Delta)^s_{\text{Reg}} u(x)$ |
|----------------------|----------------------------------|
| $(1 - |x|^2)^{r+1}_+$   | $4^s \Gamma(s + 1)\Gamma\left(\frac{2s + N}{2}\right)\Gamma\left(\frac{N}{2}\right)^{-1}$ |
| $(1 - |x|^2)^{r+1}_+$   | $4^s \Gamma(s + 2)\Gamma\left(\frac{2s + N}{2}\right)\Gamma\left(\frac{N}{2}\right)^{-1}(1 - (1 + \frac{2s}{N})||x||^2)$ |
| $(1 - |x|^2)^{r+1}_+$   | $4^s \Gamma(s + 1)\Gamma\left(\frac{2s + N}{2}\right)\Gamma\left(\frac{N}{2} + 1\right)^{-1} x_N$ |
| $(1 - |x|^2)^{r+1}_+$   | $4^s \Gamma(s + 2)\Gamma\left(\frac{2s + N}{2}\right)\Gamma\left(\frac{N}{2} + 1\right)^{-1}(1 - (1 + \frac{2s}{N+2})||x||^2)$ |

Table 2. The regional fractional Laplacian of some functions that vanish in $\mathbb{R}^N \setminus B(0, 1)$.

- **Spectrum of $\text{FL}_{\text{Reg}}$** It is shown (see, e.g., [12, Chapter 3]) that, thus defined, $(-\Delta)^s_{\text{Reg}} : X^0_0(\Omega) \rightarrow L^2(\Omega)$ is self-adjoint operator with a discrete spectrum and positive eigenvalues, more precisely

$$0 < \mu_{1,s} \leq \mu_{2,s} \leq \cdots \leq \mu_{k,s} \leq \mu_{k+1,s} \leq \cdots$$

From now on, we denote by $\{e^{R}_{k,s}\}_{k \geq 1}$ the corresponding eigenfunctions normalized in $L^2(\Omega)$ (i.e. $||e^{R}_{k,s}||_{L^2(\Omega)} = 1$).

It is worth to point out that the $\{e^{R}_{k,s}\}_{k \geq 1}$ are (only) Hölder continuous up the boundary, namely $e^{R}_{k,s} \in C^{0,s}(\Omega)$ (see, e.g., [31], [69]).

4.2. **Spectral fractional Laplacian (FL$_{\text{Spec}}$)**. In this subsection, we will give two (equivalent) definitions of the spectral fractional Laplacian and we denote it simply by $(-\Delta)^s_{\text{Spec}}$. The first one is given via the spectrum of the “classical” Laplacian on the same domain, see for instance [73], [12, Chapter 5],[31],[2],[26], [17] and the references therein. The second one is given by an integral formula via the semigroups, see e.g. [81], [87] and the references therein.

- **First definition** Let $\{\lambda_k\}_{k \in \mathbb{N}}$, with $0 < \lambda_1 < \lambda_2 \leq \ldots$ be the divergent sequence of eigenvalues of the “classical” Laplacian $-\Delta$ in $\Omega$ with homogeneous Dirichlet
boundary data on ∂Ω, and \( \{e_k\}_{k \in \mathbb{N}^*} \) the corresponding eigenfunctions, that is,

\[
\begin{align*}
-\Delta e_k &= \lambda_k e_k \quad \text{in } \Omega, \\
 e_k &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

normalized in such a way that \( \|e_k\|_{L^2(\Omega)} = 1 \) (see, e.g., [56]).

Now let us introduce the following space defined by

\[
\tilde{H}^s(\Omega) := \left\{ u = \sum_{k=1}^{+\infty} u_k e_k \in L^2(\Omega) : \sum_{k=1}^{+\infty} \lambda_k^s u_k^2 < +\infty \right\}.
\]

(4.3)

We easily check that \((\tilde{H}^s(\Omega), \langle \cdot, \cdot \rangle_{\tilde{H}^s(\Omega)})\) is a Hilbert space, where

\[
\langle u, v \rangle_{\tilde{H}^s(\Omega)} := \sum_{k=1}^{+\infty} \lambda_k^s u_k v_k,
\]

for \( u = \sum_{k=1}^{+\infty} u_k e_k \) and \( v = \sum_{k=1}^{+\infty} v_k e_k \) belonging to \( \tilde{H}^s(\Omega) \). It turns out that \( \tilde{H}^s(\Omega) = X_s^0(\Omega) \), as stated in [17, section 3.1.3] and the references therein.

Now we are ready to state the definition. More precisely:

for \( u = \sum_{k=1}^{+\infty} u_k e_k \in X_s^0(\Omega) \), the spectral fractional Laplacian, denoted by \((-\Delta)^s_{\text{Spec}}\), is defined by

\[
(-\Delta)^s_{\text{Spec}} u(x) := \sum_{k=1}^{+\infty} u_k \lambda_k^s e_k(x),
\]

(4.4)

- **Spectrum of FL_{Spec}** Thanks to (4.4), the eigenvalues and the eigenfunctions of FL_{Spec} are, respectively, \( \lambda_k^s \) and \( e_k \) for all \( k \in \mathbb{N}^* \).

In order to avoid any confusion, from here on we denote by \( \{e_k^s, \lambda_k^s\}_{k \in \mathbb{N}^*} \) the eigenfunctions and the eigenvalues of \((-\Delta)^s_{\text{Spec}}\) on \( \Omega \) in the rest of this paper.

**Example 3.**

- Let \( \Omega = (0,1) \) and \( u(x) = \sin(\pi x) \). Then,

\[
(-\Delta)^s_{\text{Spec}} u(x) = \pi^{2s} \sin(\pi x).
\]

- Let \( \Omega = (-1,1) \) and \( u(x) = \sin \left( \frac{\pi(1+x)}{2} \right) \). Then,

\[
(-\Delta)^s_{\text{Spec}} u(x) = \left( \frac{\pi}{2} \right)^{2s} \sin \left( \frac{\pi(1+x)}{2} \right).
\]

- **Second definition** As in the case of \( \mathbb{R}^N \) (see Definition 2.8), the spectral fractional Laplacian can also be defined through the heat semigroup (see, e.g., [81] and the references therein). This technique has been efficiently used in recent research papers (see for instance [33, 34, 66]). More precisely, we have:

\[
(-\Delta)^s_{\text{Spec}} u(x) = \frac{s}{\Gamma(1-s)} \int_0^{+\infty} \left( e^{t \Delta} u(x) - e^{t \Delta} u(x) \right) \frac{dt}{t^{1+s}}, \quad x \in \Omega,
\]

(4.5)

where \( e^{t \Delta} u \) denotes the solution to the heat equation in \( \Omega \times [0, +\infty) \) with initial datum \( u \).

Definitions 4.4 and 4.5 are equivalent (see, e.g., [26, Lemma 2.2]).
• **Probabilistic interpretation**: Roughly speaking, the spectral fractional Laplacian generates a subordinate killed Brownian motion, i.e., the process that first kills Brownian motion in $\Omega$ and then subordinates it via a $s$-stable subordinator. The killed process is generated by the classical Laplacian $-\Delta$ with homogeneous Dirichlet boundary conditions, and the subordination provides its fractional power $(-\Delta)^{s}_{\text{Spec}}$. See, e.g., [14, 31, 51, 52, 54, 79] and the references therein.

4.3. **Comparison of the operators FL_{Spec} and FL_{Reg}**. The main goal of this paragraph is to show that the operators FL_{Spec} and FL_{Reg} are different.

**First**: As we have just seen in the previous two paragraphs, the definition itself of $(-\Delta)^{s}_{\text{Spec}}$ only depends on the domain $\Omega$ considered, while that of $(-\Delta)^{s}_{\text{Reg}}$ depends on the domain $\Omega$ and on $\mathbb{R}^N \setminus \Omega$ (see (4.2)).

**Second**: We state below three results showing that the spectra of $(-\Delta)^{s}_{\text{Spec}}$ and $(-\Delta)^{s}_{\text{Reg}}$ are indeed different. For more details, the interested reader is referred to [73] or [12, Chapter 5] and the references therein.

**Theorem 4.1.** [73, Theorem 1.1 and Theorem 1.2] The eigenfunctions $e_{k,s}^{R}$ are only Hölder continuous on $\Omega$. However, the eigenfunctions $e_{k,s}^{S}$ are smooth on $\Omega$ (at least $e_{k,s}^{S} \in C^\infty(\Omega) \cap C^2(\Omega)$).

Furthermore, the operators $(-\Delta)^{s}_{\text{Reg}}$ and $(-\Delta)^{s}_{\text{Spec}}$ have different eigenvalues. More precisely, we have these two following results:

**Theorem 4.2.** [73, Theorem 1] We recall that $\mu_{1,s}$ (resp. $\lambda_{1,s}$) is the first eigenvalue of FL_{Reg} (resp. FL_{Spec}). Then,

$$\mu_{1,s} < \lambda_{1,s}.$$ 

**Theorem 4.3.** [31, Theorem 4.5] There exists a constant $C = C(\Omega) > 0$ such that:

$$c\lambda_{k,s} \leq \mu_{k,s} \leq \lambda_{k,s}, \ \forall k \in \mathbb{N}^*, \ k \geq 2.$$ 

In particular, $C = \frac{1}{2}$ when $\Omega$ is convex.

We refer also to [63] and [64].

4.4. **Restricted fractional Laplacian (FL_{Rest})**. In this subsection, we will introduce another fractional Laplacian operator called restricted fractional Laplacian,\(^3\) which we denote $(-\Delta)^{s}_{\text{Rest}}$. It is defined for any $u \in X_{0}^s(\Omega)$ by the formula

$$(-\Delta)^{s}_{\text{Rest}}u(x) = C(N,s)P.V. \int_{\Omega} \frac{u(x) - u(y)}{\|x - y\|^{N + 2s}}dy, \quad x \in \Omega. \quad (4.6)$$

It goes without saying that the operators FL_{Rest} and FL_{Reg} are different from each other, even though their names and notations may be interchanged in some works. In what follows, we will highlight these differences:

**First**: Unlike the regional fractional Laplacian, the operator FL_{Rest} is generated by symmetric stable processes describing motions of random particles in $\Omega$ which are allowed to jump inside $\Omega$, but are either reflected in $\Omega$ or killed when they reach the boundary $\partial \Omega$ (see for instance [14, 30, 54, 79]).

**Second**: Now, we will illustrate their difference analytically.

---

\(^3\)As already mentioned, there is no consensus in terminology. Indeed, the operator FL_{Rest} is sometimes called regional (see, e.g., [42, 62, 47] or censored. We have opted for ‘restricted’ because the values of $(-\Delta)^{s}_{\text{Rest}}u$ only depend on the values of $u$ in $\Omega$, not in the complementary.
Let \( u \) be a smooth function such that \( u(x) = 0 \) for \( x \in \mathbb{R} \setminus \Omega \). Let \( x \in \Omega \). Thanks to formula (4.2), we have

\[
(-\Delta)^s_{\text{Reg}} u(x) = C(N, s) \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy
\]

\[
= C(N, s) \left[ \text{P.V.} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy + \int_{\mathbb{R}^N \setminus \Omega} \frac{u(y)}{|x - y|^{N+2s}} dy \right]
\]

So

\[
D_1(x) = \left[ (-\Delta)^s_{\text{Reg}} - (-\Delta)^s_{\text{Rest}} \right] u(x) = u(x) \int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|x - y|^{N+2s}} dy \quad (4.7)
\]

Now, for the reader’s comfort, we explicitly calculate the difference (4.7) in the case \( \Omega = (-L, L) \) with \( L > 0 \). Then we have

\[
D_1 u(x) = C(1, s) \left( \int_{-\infty}^{-L} \frac{1}{|x - y|^{1+2s}} dy + \int_{L}^{+\infty} \frac{1}{|x - y|^{1+2s}} dy \right) u(x).
\]

Thus, for any \( x \in \Omega \),

\[
D_1 u(x) = \frac{C(1, s)}{2s} \left( \frac{1}{(x + L)^{2s}} + \frac{1}{(L - x)^{2s}} \right) u(x). \quad (4.8)
\]

As \( \lim_{s \to 1^-} C(1, s) = 0 \) (see (2.4)), we deduce from (4.8) that \( \lim_{s \to 1^-} D_1 u(x) = 0 \). Consequently, \( \text{FL}_{\text{Rest}} \) can be used to approximate \( \text{FL}_{\text{Reg}} \) as \( s \to 1^- \).

Applying again (2.4), then \( \lim_{s \to 0^+} \frac{C(1, s)}{2s} = \frac{1}{2} \). So \( \lim_{s \to 0^+} D_1 u(x) = u(x) \). Thus, when \( s \to 0^+ \), \( \text{FL}_{\text{Reg}} \) can be written as the sum of \( \text{FL}_{\text{Rest}} \) and the identity operator.

To end this sub-paragraph, it is worth pointing out that unlike the regional and spectral fractional Laplacians, the operator \( \text{FL}_{\text{Rest}} \) has not been studied much. For example, nothing is known about its eigenvalues and eigenfunctions. \( \square \)

4.5. Peridynamic fractional Laplacian (FL\text{pery}). The peridynamic model was originally proposed as a reformulation of the classical solid mechanics. We refer the interested reader to the pioneering article [75], as well as to [35, 55, 42, 7, 62] and references therein.

**Probabilistic interpretation** : Let \( \delta \in (0, +\infty) \). The peridynamic fractional Laplacian (FL\text{pery} in short) of horizon \( \delta \) can be seen as the infinitesimal generator of a symmetric 2s-stable Lévy process restricted to \( B(x, \delta)^4 \) for \( x \in \Omega \). For more details, see, e.g., [62], [42] and the references therein. The real number \( \delta \) is called the horizon parameter, and it is often used to measure the range of nonlocal interactions in many nonlocal models. Thence, this operator is sometimes called the “horizon-based nonlocal” operator.

**Notation** : In what follows, we will use the following notation \( (-\Delta)^{s, \delta}_{\text{pery}} \) or \( (-\Delta)^{s}_{\text{pery}} \) if there is no risk of confusion.

**Definition** Before stating the definition, we need to define the so-called interaction domain of horizon \( \delta \), defined by

\[
\Omega_{I, \delta} := \{ y \in \mathbb{R}^N \setminus \Omega : \| x - y \| < \delta, \text{ for } x \in \Omega \}.
\]

\( ^4 \)Let us recall that \( B(x, \delta) \) denotes the ball centered at point \( x \) with radius \( \delta \)
In other words, the domain $\Omega_{I,\delta}$ consists of those points outside of $\Omega$ that interact with points in $\Omega$. In fact, $(-\Delta)^{s,\delta}_{Pery}$ acts on functions $u$ defined in $\Omega$ extended by 0 to $\Omega_{I,\delta}$. More precisely, $(-\Delta)^{s,\delta}_{Pery}$ is defined by

$$(-\Delta)^{s,\delta}_{Pery} u(x) = C(N, s) P.V. \int_{B(x,\delta)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy. \quad (4.9)$$

**Comparison between FL\(_{\text{Reg}}\) and FL\(_{\text{Pery}}\)**

1) In the limiting case of $\delta \to \infty$, $\text{FL}_{\text{Pery}}$ coincides with $\text{FL}_{\text{Reg}}$, and thus the latter is often used to approximate $\text{FL}_{\text{Pery}}$ if the parameter $\delta$ is sufficiently large (see for example [35, 55]).

2) In order to illustrate the difference between these two operators, we will proceed as in the previous paragraph. For this, we choose a smooth function $u$ such that $u(x) = 0$ for $x \in \mathbb{R} \setminus \Omega$ with $\Omega = (-L, L)$. Here, we assume that the horizon size $\delta$ in (4.9) is large enough such that $\delta > 2L$. As is done in the previous paragraph, we compute their difference:

$$D_2 u(x) = (-\Delta)^{0,\delta}_{\text{Reg}} - (-\Delta)^{s,\delta}_{\text{Pery}} u(x)$$

$$= C(1, s) \left( P.V. \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+2s}} dy - P.V. \int_{x-\delta}^{x+\delta} \frac{u(x) - u(y)}{|x - y|^{1+2s}} dy \right)$$

$$= C(1, s) \left( \int_{-\infty}^{x-\delta} \frac{1}{|x - y|^{1+2s}} dy + \int_{x+\delta}^{\infty} \frac{1}{|x - y|^{1+2s}} dy \right) u(x)$$

So, for all $x \in \Omega$

$$D_2 u(x) = ((-\Delta)^{0,\delta}_{\text{Reg}} - (-\Delta)^{s,\delta}_{\text{Pery}}) u(x) = \frac{C(1, s)}{s \delta^{2s}} u(x), \quad (4.10)$$

We conclude that the difference of these two operators is of order $O(\frac{1}{\delta^{2s}})$ when $u(x)$ is uniformly bounded on $\Omega$, hence their difference vanishes as $\delta \to \infty$. On the other hand, the convergence of $\text{FL}_{\text{Pery}}$ to $\text{FL}_{\text{Reg}}$ as $\delta \to \infty$ depends on the power $s$, and it may degenerate rapidly for small $s$. Additionally, in the limiting case of $s \to 1^-$, the difference $D_2 u(x) \to 0$, because the coefficient $C(1, s) \to 0$ (see (2.4)).

4.6. **Examples.** After this brief review of pairwise differences between them, let us compare all four operators by applying them to the same functions on $\Omega = (-1, 1)$.

**Example 1:** Consider the function (see [42]) defined by

$$u(x) = \begin{cases} \sin \left( \frac{\pi(1+x)}{2} \right) & \text{if } x \in (-1, 1), \\ 0 & \text{otherwise}. \end{cases} \quad (4.11)$$

As already noticed (see Example 3, Paragraph 4.2), we have

$$(-\Delta)^{s}_{\text{Spec}} u(x) = \left( \frac{\pi}{2} \right)^{2s} \sin \left( \frac{\pi(1+x)}{2} \right) \quad \text{for } x \in \Omega,$$

However, it is not easy to calculate $(-\Delta)^{s}_{\text{Reg}} u$, $(-\Delta)^{s}_{\text{Rest}} u$ and $(-\Delta)^{s}_{\text{Pery}} u$, thus they are numerically computed using the finite difference method proposed in [40].

Figure 1 below shows the curves of the four functions $(-\Delta)^{s}_{\text{Reg}} u$, $(-\Delta)^{s}_{\text{Rest}} u$, $(-\Delta)^{s}_{\text{Spec}} u$ and $(-\Delta)^{s}_{\text{Pery}} u$. We observe that:
— for all \( s \in (0, 1) \), \((-\Delta)^s_{\text{Spec}} u\) is proportional to the function \( u \) in \( \Omega \);
— as \( s \to 1^- \), \((-\Delta)^s_{\text{Reg}} u\), \((-\Delta)^s_{\text{Rest}} u\) and \((-\Delta)^s_{\text{Pery}} u\) converge to \(-\Delta u\);
— both \((-\Delta)^s_{\text{Rest}}\) and \((-\Delta)^s_{\text{Pery}}\) can be used to approximate \((-\Delta)^s_{\text{Reg}}\), if \( s \) is close to 1;
— for small \( s \), the values of \((-\Delta)^s_{\text{Rest}}\) are at variance with those of \((-\Delta)^s_{\text{Reg}}\).

However, \((-\Delta)^s_{\text{Pery}}\) can still provide a good approximation to \((-\Delta)^s_{\text{Reg}}\) by enlarging the horizon size \( \delta \).

**Example 2:** Consider the function (see [42]) defined by

\[
u(x) = \begin{cases} 
(1 - x^2)^{q+s} & \text{if } x \in (-1, 1), \\
0 & \text{otherwise}, 
\end{cases}
\]

where \( q \in \mathbb{N} \). In this case, \((-\Delta)^s_{\text{Reg}} u\) can be computed explicitly

\[
(-\Delta)^s_{\text{Reg}} u(x) = \frac{2^{2s} \Gamma \left( \frac{1+2s}{2} \right) \Gamma(s + q + 1)}{\sqrt{\pi} \Gamma(q + 1)} 2F_1 \left( \frac{1 + 2s}{2}, -q; \frac{1}{2}; x^2 \right) \quad \text{for } x \in \Omega,
\]

where \( 2F_1 \) denotes the Gauss hypergeometric function. Moreover, we can obtain the exact values of \((-\Delta)^s_{\text{Rest}} u\) and \((-\Delta)^s_{\text{Pery}} u\) by using their relation to \((-\Delta)^s_{\text{Reg}}\) in (4.8) and (4.10), respectively. For \((-\Delta)^s_{\text{Spec}}\), \((-\Delta)^s_{\text{Spec}} u\) is numerically computed.

![Figure 1](image-url)
by the finite difference method proposed in [41].

In Figure 2, the four functions \((-\Delta)^s_{\text{Spec}} u, (-\Delta)^s_{\text{Reg}} u, (-\Delta)^s_{\text{Pery}} u\) and \((-\Delta)^s_{\text{Rest}} u\) are compared. The function \(u\) is defined by (4.12) with \(q = 2\).

**Figure 2.** Comparison of the functions \((-\Delta)^s u\) where \(u\) is defined in (4.12) and \((-\Delta)^s\) represents \((-\Delta)^s_{\text{Spec}}\), \((-\Delta)^s_{\text{Reg}}\), \((-\Delta)^s_{\text{Pery}}\) or \((-\Delta)^s_{\text{Rest}}\) with \(\delta = 4\). The result for \(-\Delta\) (\(\ast\ast\ast\)) is included in the plot of \(s = 0.975\).

Figure 2 shows that the functions \((-\Delta)^s_{\text{Spec}} u, (-\Delta)^s_{\text{Rest}} u, (-\Delta)^s_{\text{Pery}} u\) and \((-\Delta)^s_{\text{Reg}} u\) exist on the closed domain \(\Omega\) for any \(s \in (0, 1)\), but their values are very different, especially for small \(s\). The values of \((-\Delta)^s_{\text{Spec}} u\) are always zero at boundary points.

Also, both \((-\Delta)^s_{\text{Rest}}\) and \((-\Delta)^s_{\text{Pery}}\) with relatively small \(\delta\) can provide a good approximation to \((-\Delta)^s_{\text{Reg}}\), if \(s\) is large. When \(s\) is small, however, \((-\Delta)^s_{\text{Pery}}\) can be still used to approximate \((-\Delta)^s_{\text{Reg}}\) with a large \(\delta\), but \((-\Delta)^s_{\text{Rest}}\) gives results starkly different from \((-\Delta)^s_{\text{Reg}}\). Finally, as \(s \to 1^-\), the differences between the four operators become negligible (see the right-bottom square), and they all converge to the Laplacian \(-\Delta\).

**Example 3:** We consider the following Poisson problem (see [42]):

\[
\begin{align*}
\begin{cases}
-\mathcal{L}u(x) = 1 & \text{if } x \in (-1, 1), \\
u(x) = 0 & \text{if } x \in \Sigma,
\end{cases}
\end{align*}
\]

(4.13)
where $\mathcal{L} = (-\Delta)^s_{\text{Reg}}$ and $\Sigma = (-\infty, -1] \cup [1, +\infty)$; $\mathcal{L} = (-\Delta)^s_{\text{Spec}}$ and $\Sigma = \{ -1, 1 \}$; $\mathcal{L} = (-\Delta)^s_{\text{Rest}}$ and $\Sigma = \{ -1, 1 \}$; or $\mathcal{L} = (-\Delta)^s_{\text{Pery}}$ and $\Sigma = \Omega_{I,\delta} := \{ y \in \mathbb{R} \setminus (-1, 1) ; |x-y| < \delta, \text{ for } x \in (-1, 1) \}$.

One can check that:

- if $\mathcal{L} = (-\Delta)^s_{\text{Reg}}$,
  $$u(x) = \frac{2^{-2s} \sqrt{\pi}}{\Gamma\left(\frac{1+s}{2}\right)\Gamma(1+s)} (1-x^2)^s \quad \text{for } x \in (-1, 1);$$

- if $\mathcal{L} = (-\Delta)^s_{\text{Spec}}$,
  $$u(x) = \sum_{k=0}^{+\infty} \frac{2(1-(-1)^k)}{k\pi} \left(\frac{k\pi}{2}\right)^{-2s} \sin\left(\frac{k\pi}{2}(1+x)\right) \quad \text{for } x \in (-1, 1).$$

On the other hand, the solutions of the nonlocal problem (4.13) with $\mathcal{L} = (-\Delta)^s_{\text{Rest}}$ or $(-\Delta)^s_{\text{Pery}}$ are not analytically known and will be numerically computed.

Figure 3. Comparison of the solution to (4.13) with $(-\Delta)^s_{\text{Spec}}$ (——), $(-\Delta)^s_{\text{Reg}}$ (---), $(-\Delta)^s_{\text{Rest}}$ (-----) or $(-\Delta)^s_{\text{Pery}}$ (-----) with $\delta = 4$. The result for $-\Delta$ (****) is included in the plot of $s = 0.975$.

Figure 3 shows the solutions to (4.13) with all four operators. In the case of $\mathcal{L}_{\text{Rest}}$, since the solution does not exist for $s \leq \frac{1}{2}$, we only present it for $s > \frac{1}{2}$.

Generally speaking, these solutions are significantly different. However, as $s \to 1^-$ they all converge to the function $u(x) = \frac{1}{2}(1-x^2)$, the solution to the classical Poisson equation:

$$-u''(x) = 1, \text{ if } x \in (-1, 1); \quad u(-1) = u(1) = 0.$$
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E-mail address: mahaadaoud@gmail.com
E-mail address: el-haj.laamri@univ-lorraine.fr