Abstract

We consider a one dimensional interacting particle system which describes the effective interface dynamics of the two dimensional Toom model at low temperature and noise. We prove a number of basic properties of this model. First we consider the dynamics on a half open finite interval $[1, N)$, bounding the mixing time from above by $2N$. Then we consider the model defined on the integers. Due to infinite range interaction, this is a non-Feller process that we can define starting from product Bernoulli measures with density $p \in (0, 1)$, but not from arbitrary measures. We show, under a modest technical condition, that the only possible invariant measures are those product Bernoulli measures. We further show that the unique stationary measure on $[-k, \infty)$ converges weakly to a product Bernoulli measure on $\mathbb{Z}$ as $k \to \infty$.

1 Introduction

In this paper, we consider a peculiar interacting particle system originally introduced in [4] to describe the effective dynamics of the interface between two phases in Toom’s Model (also known as the North-East, or North-East-Center, model) in the limit of weak noise. We recall here (see [11] for more details) that Toom’s model is a discrete time probabilistic cellular automata on $\mathbb{Z}^2$ in which the spin configurations $\sigma_t \in \{-1, 1\}^{\mathbb{Z}^2}$ are updated according to the rule

$$
\sigma_{t+1}(i,j) = \begin{cases} 
\text{sign}(\sigma_t(i,j+1) + \sigma_t(i+1,j) + \sigma_t(i,j)) & \text{with probability } 1 - p - q \\
+1 & \text{with probability } p \\
-1 & \text{with probability } q.
\end{cases}
$$


The parameters $p, q$ represent noise in the update scheme. What is important for us is that for $p$ and $q$ small enough, the system has two stationary states, one with mostly $+1$'s and the other with mostly $-1$'s.

One may impose an interface between these two phases by setting the model up in the third quadrant of $\mathbb{Z}^2$ and fixing boundary conditions for $\sigma_t(j,0) = +1$ and $\sigma_t(0,j) = -1$ for all $j < 0$ and for all $t$.

If $p = q = 0$, all "up-right" paths from $(-\infty, -\infty)$ to $(0, 0)$ define stable configurations (with $+1$ above and $-1$ below the path) for the deterministic dynamics and one may ask how these interfaces fluctuate for $p, q \neq 0$ but small. After some heuristic considerations, [4] suggests the following effective description of the dynamics at weak noise. From now on, when needed, we'll refer to these dynamics as the Toom Interface (on $\mathbb{N}, \mathbb{Z}$ or other intervals $I$, but for now take $\mathbb{N}$ for definiteness). First of all, the interfaces are encoded by spin configurations $\sigma := (\sigma(x))_{x \in \mathbb{N}} \in \Omega[\mathbb{N}]$, where $\Omega[\mathbb{N}] := \{-1, 1\}^\mathbb{N}$. Second, to model the limit of weak noise and to make the model a bit more tractable the dynamics is described by a continuous time Markov chain. Each $\pm 1$ particle is equipped with an exponential rate $\lambda \pm$ clock. We will assume throughout that $\lambda_+, \lambda_- > 0$ and that $\lambda_+ + \lambda_- = 1$, the latter being just a choice of unit of time. When the clock rings for a particle of fixed sign, the particle exchanges positions with the first particle to its right of opposite sign. Here and below, we'll refer to this process as $\sigma_t := (\sigma_t(x))_{x \in \mathbb{N}}$. For later reference, we note here that one can just as well define the process on $I = [x, \infty)$, i.e. with state space $\Omega[I] = \{-1, 1\}^I$. Up to a shift of the semi axis by $x$ units, this is the same.

One remarkable feature of this model is that the restriction from $\Omega[\mathbb{N}]$ to the first $N$ vertices $\Omega[N] := \{-1, 1\}^{|N|}$ is itself a Markov chain; the dynamics is the same unless a clock rings for a spin in the last block of constant sign in $[1, N]$. For updates of spins in the last block, the dynamics reduces to single vertex spin flips. As a result, the restricted chain is irreducible on $\Omega[N]$ and has a unique stationary measure $\pi_N$. As the sequence of measures $(\pi_N)_{N \in \mathbb{N}}$ is consistent, this in turn implies that the full chain has a unique invariant measure on $\Omega[\mathbb{N}]$, $\pi_\infty$, which restricts to $\pi_N$ on $\Omega[N]$.

Very little is understood rigorously about the behavior of either $\pi_\infty$ or the process $\sigma_t$, though the papers [2, 4, 5] contain a number of interesting conjectures, heuristics and numerics. The first paper on the subject [4] studies the Markov chain defined above as a model describing fluctuations via kinetic roughening, the height function $h_x(\sigma_t)$ being defined by $h_x(\sigma_t) = \sum_{i=1}^x \sigma_t(i)$. The striking observation there is that if $\lambda_+ = 1/2$, the statistical properties of the model cannot be in the class governed by the conventional KPZ equation: in this case the process $h_x(\sigma_t)$ is distributionally invariant under global spin flip.

One way to understand this at a heuristic level follows the work of Kardar, Parisi and Zhang and guess the behavior of $h(\sigma_t)$ in the appropriate scaling limit. The process
should satisfy the SPDE
\[ \partial_t h = \kappa \Delta h + W(t, x) + a(\nabla h)^2 + b(\nabla h)^3 \ldots, \]
where \( W \) is a space time white noise and the last set of terms make explicit the possible dependence on the gradient of \( h \). If \( \lambda_+ \neq \lambda_- \), one concludes that only the quadratic term is relevant using scaling theory [6]. However, if \( \lambda_+ = \frac{1}{2} \), \( h \) and \( -h \) are identically distributed, which forces \( a = 0 \) in any putative scaling limit. The extent to which the third order term is relevant is an intriguing open question. It is marginal in the renormalization group sense, and as such [4, 5] argue against its appearance for the scaling limits of microscopic models. The situation here is analogous to the expected relationship between the scaling limit of the Ising model in 4 dimensions and the putative \( \phi^4 \) field theory.

The simplest manifestation of the above discussion appears in the study of the variance, under \( \pi_{\infty} \), of the sum of the first \( L \) spins. Numerics, Renormalization group calculations and heuristics [4, 10] suggest that
\[ \text{Var}_{\pi_{\infty}}(\sum_{x=1}^{L} \sigma_x) \sim \begin{cases} L^{2/3} & \text{if } \lambda_+ \neq \frac{1}{2}, \\ L^{1/2} \log L^{1/4} & \text{if } \lambda_+ = \frac{1}{2}. \end{cases} \]

Another idea introduced in [4] relates the stationary state \( \pi_{\infty} \) with the behavior of density fluctuations of the model defined on \( \mathbb{Z} \). We refer the interested reader to the cited papers for more detail.

With this background in mind, our paper constitutes the first rigorous analysis of the Toom interface, though the results fall short of answering the most intriguing questions raised in [4]. Let us now present our main findings. We recall that the total variation distance between two measures \( \mu, \nu \) on a finite sample space \( \Omega \) is defined as
\[ \|\mu - \nu\| := \frac{1}{2} \sum_{\sigma \in \Omega} |\mu(\sigma) - \nu(\sigma)| \]
Abusing notation slightly, we also use \( \sigma_t \) to denote the restriction of the chain to \( \Omega[N] \) and let \( \mu_{N,t}^\xi \) denote the distribution of \( \sigma_t \) when starting from the initial configuration \( \xi \in \Omega[N] \). Recall that the mixing time of \( \sigma_t \) is defined as
\[ \tau_{\text{mix}}(N) := \inf \left\{ t : \max_{\xi} \|\mu_{N,t}^\xi - \pi_N\| < \frac{1}{2} \right\} \]
Our first result is as follows.

**Theorem 1.1.** For all \( N \in \mathbb{N} \),
\[ \tau_{\text{mix}}(N) \leq 2(\lambda_+ + \lambda_-)N \]
It is natural to study the behavior of $\pi_\infty$ in the bulk, far to the right of 0: for any spin configuration $\sigma$, with domain $D \subset \mathbb{Z}$, let $\tau_x \cdot \sigma$ denote the spin configuration with domain $D + x$ defined by $(\tau_x \cdot \sigma)(y) = \sigma(y - x)$ and let denote the induced map on the space of probability measures by $\tau_x^*$. One can guess that the influence of the boundary at 0 falls off and that therefore the shifted measure $\tau_x^*\pi_\infty$ converges to an invariant measure on $\Omega[\mathbb{Z}] := \{-1, 1\}^\mathbb{Z}$. We will prove, under some mild conditions, that the product Bernoulli measures Ber$_p$ with $p \in (0, 1)$ are the only invariant measures on Ber$_p$, see next section. This motivates

**Theorem 1.2.** Consider $(\tau_{-k}^*\pi_\infty)_{k \in \mathbb{N}}$ as a sequence of probability measures on $\Omega[\mathbb{Z}]$. Then this sequence converges weakly, as $k \to \infty$, to the product Bernoulli measure Ber$_p$ with

$$
\left(\frac{1 - p}{p}\right)^2 = \frac{\lambda_+}{\lambda_-}
$$

We derive other noteworthy results below, in particular Theorems 2.3 and 2.5. These show, respectively, exponential decay of correlations in the product Bernoulli measures and that (up to an certain integrability condition) the product Bernoullis are the only invariant measures for the process defined on $\mathbb{Z}$. However, such results presuppose that the dynamics may actually be defined on $\mathbb{Z}$, and this requires a fair amount of technical discussion to set up. Thus we have chosen to defer the statements Theorems 2.3 and 2.5 to the next section.

The main issue in defining the dynamics on $\Omega[\mathbb{Z}]$ is that the process cannot be constructed by the usual machinery of interacting particle systems since it is non-Feller. Thus part of the subject of § 2 is to formulate a workable notion of the dynamics on $\Omega[\mathbb{Z}]$, see Definition 2.4. Further, in § 2.2 we verify that a dynamics started from Ber$_p$ exists for which the conditions of Definition 2.4 hold. Proofs for the assertions of § 2.2 are deferred to the end of the paper, § 6. The remainder of the paper can be read largely independently from them.

The main thrust of § 2 is not this construction however. Rather, it is that we may, starting from various initial conditions, define a natural coupling of the dynamics in time. This coupling is central to all our results on the model. With this coupling in hand, we proceed to the justification of our main results. The most accessible is the proof of Theorem 1.1 appearing in § 2.4 Item 1. § 3 is devoted to a proof of the aforementioned Theorem 2.5 while § 4 gives a proof of Theorem 1.2. Except for some notation set out at the beginning of § 3 these latter two sections may be read independently of one another. § 5 is devoted to proofs of several technical propositions and lemmas used in §§ 3 and 4.

Let us also mention a second paper in preparation [3]. In that paper, we prove various functional CLTs for additive functionals of local observables, local currents, tagged particles and the like. Combining the results of that paper with the present paper, we
are in fact able to derive the bound
\[ \text{Var}_{\pi_\infty} \left( \sum_{x=1}^{L} \sigma_x \right) \lesssim L. \]

Going beyond this bound probably requires a new idea beyond the technology developed here and in [3].

2 The Main Coupling and Dynamics on \( \Omega[\mathbb{Z}] \)

The heuristic given in the run-up to theorem Theorem 1.2 presupposes that the dynamics may actually be defined on \( \mathbb{Z} \). This is a nontrivial issue as as the process does not have a finite interaction range – arbitrarily distant parts of the configuration on the negative axis can influence the local jump rate – and hence the standard Hille-Yosida construction is not applicable. This is not merely a technical issue. The Toom model on \( \Omega[\mathbb{Z}] \) cannot have the Feller property. As far as we know, only a few non-Feller interacting particle systems have been constructed, most of them relying on a monotonicity property that is missing here, see e.g. [9, 7]. However, we do have at our disposal candidate invariant measures, namely the product Bernoulli’s, and we will exploit this in our construction, which is somewhat analogous to the proof of existence of infinite volume dynamics for Hamiltonian systems [1]. However, even if this problem were absent, we still want to present the Toom process in a more complicated way than was given in § 1. The advantage of this new presentation is that various useful couplings can be constructed naturally.

Rather than thinking of \( \lambda_\pm \)-Poisson clocks as being attached to particles, we will consider a sequence of i.i.d. rate one Poisson point processes \( (N_x(t))_{x \in \mathbb{Z}} \) associated with vertices \( x \in \mathbb{Z} \). Besides these Poisson point processes, the sample space on which our coupling is defined supports a two dimensional array of of i.i.d Uniform \([0, 1]\) variables \( (U_{x,j})_{x \in \mathbb{Z}, j \in \mathbb{N}} \). Let \( (\Omega, \mathcal{P}; \mathcal{B}_\Omega) \) denote the probability space which supports all these variables. We define the full sample space \( \Sigma := \Omega[\mathbb{Z}] \times \Omega \) with \( \Omega[\mathbb{Z}] \) containing the initial configuration \( \sigma_0 \), so that, at least intuitively, an element of \( \Sigma \) contains all information to describe the dynamics of any given initial condition for all times. Define the filtration of sigma algebras \( (\mathcal{F}_t)_{t \in \mathbb{R}^+} \) on \( \Sigma \) by
\[ \mathcal{F}_t = \sigma \left( N_x(s) : s \leq t; U_{x,k} : k \leq N_x(t-) ; \sigma_0 \in \Omega[\mathbb{Z}] \right). \]

The sigma-algebra is the product \( \mathcal{B}_\mathbb{Z} \times \mathcal{B}_\Omega \) with \( \mathcal{B}_\mathbb{Z} \) the \( \sigma \)-algebra generated by the product topology on \( \Omega[\mathbb{Z}] \). The probability measure on \( \Sigma \) is the product \( \mathbf{P}_{\text{Ber}_p} := \text{Ber}_p \times \mathbf{P} \) with \( \text{Ber}_p \) on \( \Omega[\mathbb{Z}] \) the product Bernoulli measure with \( \text{Ber}_p(\sigma(x) = 1) = p \) for all \( x \in \mathbb{Z} \), with \( p \in (0, 1) \).

Our goal in upcoming subsections is to use the above to recast the one-sided processes and give meaning to the process on \( \mathbb{Z} \). We write \( D(\mathbb{R}^+) = D_{\Omega[\mathbb{Z}]}(\mathbb{R}^+) \) for the Skorohod
space of càdlàg paths. We have taken \( Z \) instead of \( N \), but for the moment we still deal with the process on \( N \), such that \( \sigma(x), x \leq 0 \) is frozen.

### 2.1 Construction on \( N \)

Let us begin by recasting the process on \( N \). We define the evolution \((\sigma_t)_{t \in \mathbb{R}^+} \in D_{\Omega[Z]}(\mathbb{R}^+)\) as given deterministically by a measurable function \( G^0 : \Omega[Z] \times \Omega \to D(\mathbb{R}^+) \)

\[
(\sigma_t)_{t \in \mathbb{R}^+} = G^0(\sigma_0, \omega),
\]

with \( \sigma_t(x) = \sigma_0(x) \) for all \( x \leq 0 \).

This is done as follows: For \( x \geq 1 \), at each arrival \( t^*_n \) of the process \( N_x(t) \), we sample the uniform variable \( U_{x,N_x}(t^*_n) \). Let \( z(x) \) denote the first vertex to the right of \( x \) such that \( \sigma_t - \sigma_t^*(x) \neq \sigma_t - \sigma_t^*(z(x)) \). The rule for the update is as follows. Suppose that \( \sigma(t_n^*) - \sigma_t(x) = +1 \). If \( U_{x,N_x}(t^*_n) < \lambda_+ \), then we exchange the spins at \( x \) and \( z(x) \), otherwise we do nothing. If instead \( \sigma(t_n^*) - \sigma_t(x) = -1 \), the exchange takes place if and only if \( U_{x,N_x}(t^*_n) > \lambda_+ \). If \( x \leq 0 \) we do nothing.

It is easy to verify that, \( P \)-almost surely, \( \sigma_t \in D \) using the property that the restriction to \( \Omega[N] \) is a Markov process, as remarked in § 1. In the above construction, the process was trivial for \( x \leq 0 \), but in the same way, we can of course define it in the domain \([−L, \infty)\) instead of \( N \), and this dynamics is denoted by the function \( G^L(\sigma_0, \omega) \) that we also abbreviate simply as \( \sigma^L := G^L(\sigma_0, \omega) \) Note that

\[
\sigma^L_t(x) = \sigma^L_t'(x), \quad \text{for all } x < −\max(L, L').
\]

For future purposes, we note that the constructed processes (depending on \( L \)) satisfy the following SDE:

\[
d\sigma^L_t(x) = \sum_{\eta = \pm 1} (2\eta) \sum_{-L \leq y < x} \chi^\eta_{[y,x-1]}(\sigma^L_{t^-})\chi^\eta_{x}(\sigma^L_{t^-}) dN_{y,\eta}(t)
\]

where we used the notation \([a, b]\) for the discrete interval \( \{a, a + 1, \ldots, b\} \subset \mathbb{N} \) and the shorthand

\[
\chi^\eta_I(\sigma) = \chi[\forall x \in I : \sigma(x) = \eta]
\]

and where \( N_{x,\eta}(t) \) are Poisson processes with intensity \( \lambda_\eta \), obtained by thinning \( N_x(t) \):

\[
N_{x,\eta}(t) = \sum_{j: \tau_j \leq t} \chi[U_{x,j} \leq \lambda_+]
\]

\[
N_{x,\eta}(t) = \sum_{j: \tau_j \leq t} \chi[U_{x,j} > \lambda_+]
\]

with \( \tau_j, j = 1, 2, \ldots \) the ordered jump times of \( N_x \). Note that \( N_x(t) = N_{x,\eta}(t) + N_{x,-}(t) \).
2.2 Construction of Processes on $\mathbb{Z}$

To define the dynamics on the full line $\mathbb{Z}$, we recall the path-valued functions $\sigma^L = G^L$ on $\Sigma$ defined above. We write $\sigma^L_{[0,t]}$ for the restriction of $\sigma^L$ to the space $D[0,t)$. We first show the following:

**Proposition 2.1.** For any $t \geq 0$, the almost sure limit on $(P_{\text{Ber}_p}, \Sigma)$

$$\sigma_{[0,t]} := \lim_{L} \sigma^L_{[0,t]}$$

exists.

The processes $\sigma_{[0,t]}$ are consistent in the sense that, if $t' > t$, then the restriction of $\sigma_{[0,t']}^L$ to $D[0,t)$ is $\sigma_{[0,t]}$, almost surely. Therefore we can define the variable $(\sigma_t)_{t \in \mathbb{R}^+}$ whose finite-time restrictions are given by $\sigma_{[0,t]}$.

The above construction, and the fact that the processes $\sigma^L = G^L(\sigma(0), \omega)$ are Markov w.r.t. $\mathcal{F}_t$, imply (see §6) the following:

**Proposition 2.2.** $\sigma_t$ is a stationary Markov process w.r.t. the filtration $\mathcal{F}_t$, with invariant distribution $\text{Ber}_p$. It satisfies the SDE (1) with $L = \infty$, in the sense that for any $t_2 > t_1 \geq 0$ and $x \in \mathbb{Z}$,

$$\sigma_{t_2}(x) - \sigma_{t_1}(x) = \sum_{\eta = \pm 1} (2\eta) \sum_{y < x} \int_{t_1}^{t_2} \chi^n_{[y,x-1]}(\sigma_t) \chi^n_{x}(\sigma_t) dN_{y,\eta}(t), \quad \text{a.s.} \quad (3)$$

By a straightforward calculation and Markov’s inequality, the right hand side of (3) is almost surely absolutely summable provided that

$$\sup_{t_1 \leq s \leq t_2} E_{\text{Ber}_p}(l_x(\sigma_s)) < \infty$$

where $l_x(\sigma)$ denotes the cardinality of the block of spins to the left of $x$, starting from $x - 1$, more precisely, $l_x(\sigma) := \max\{n : \sigma(y) = \sigma(x-1) \forall y \text{ such that } x-n \leq y \leq x-1\}$. By stationarity, $E_{\text{Ber}_p}(l_x(\sigma_s)) = \text{Ber}_p(l_x)$. The latter is finite, so that (3) indeed makes sense.

One of the results on this process that follow easily from our construction is the following bound on decay of correlations in time. We say that $f$ on $\Omega[\mathbb{Z}]$ has support $S$ (Notation: $\text{Supp} f = S$) if $S$ is the smallest set such that $f$ is a function of $\sigma(x)$, $x \in S$.

**Theorem 2.3.** Let $f, g$ be local functions with $\text{Ber}_p(f) = \text{Ber}_p(g) = 0$ and $\text{Ber}_p(f^2) = \text{Ber}_p(g^2) = 1$. Then

$$E_{\text{Ber}_p}(f(\sigma_0)g(\sigma_t)) \leq Ce^{r-c't}, \quad (4)$$

with $r$ the length of the smallest interval containing both $\text{Supp} f$ and $\text{Supp} g$, and $C, c$ only dependent on $\lambda_{\pm}$.
From now on, constants $C, c$ throughout the paper will be allowed to depend on $p, \lambda_{\pm}$ without further mention.

2.3 Definition Invariant Measures on $\mathbb{Z}$

In the previous section, we defined the dynamics started from the Bernoulli measures $\text{Ber}_p$ and we argued that these measures are stationary. We now want to rule out other stationary measures $\mu$. As stressed previously however, the process is not defined started from an arbitrary configuration. It is therefore not a priori clear how to formulate their definition. Our classification of invariant measures thus involves some assumptions regarding the sense in which the dynamics started from $\mu$ exists.

Let $\mu$ be a probability measure on $\Omega\setminus \mathbb{Z}$ and let $\mathbf{P}_\mu = \mu \times \mathbf{P}$, a probability measure on $\Sigma$. Let $\sigma^L_t$ stand for the process where the configuration on sites $x < -L$ has been frozen (cfr. § 2.1).

**Definition 2.4.** We say that $\mu$ is invariant for the dynamics on $\mathbb{Z}$ if there is an $\mathbf{P}_\mu$-a.s. defined random variable $\sigma_t$ in $D[0, \infty)$, i.e. a càdlàg process, such that:

1. (Stationarity) $\sigma_t$ is $\mu$-distributed for any $t$.

2. (SDE is satisfied) $\mathbf{P}_\mu$-a.s., the right hand side of (3) is absolutely summable, and the equality (3) holds for any $x$ and $t_1 < t_2$.

3. (Finite speed of information propagation). For all $t_* > 0$ and all finite sets $S \subset \mathbb{Z}$,

\[ \sup_{t \in [0, t_*], x \in S} |\sigma_t(x) - \sigma^L_t(x)| \text{ converges to } 0 \text{ in distribution as } L \to \infty. \quad (5) \]

With this definition in hand, we classify invariant measures provided that these putative measures satisfy some regularity assumptions. The following theorem was conjectured in [4]. To state the regularity assumption, let $l_y(\sigma)$ and $r_y(\sigma)$ denote the cardinality of the block of spins to the left of $y$, starting from $y - 1$ and to the right of $y$, starting from $y + 1$, in particular $l_y, r_y \geq 1$.

**Theorem 2.5.** Let $\mu$ be a probability measure on $\Omega[\mathbb{Z}]$ that is invariant for the dynamics on $\mathbb{Z}$ in the sense of Definition 2.4. If the integrability condition

\[ \sup_{x \in \mathbb{Z}} \mu[(l_x r_x)^{1+\epsilon}] < \infty, \]

holds for some $\epsilon > 0$, then $\mu$ is a mixture of product Bernoulli measures. Furthermore, if we also assume that $\mu$ is translation invariant, then the same conclusion holds under the weaker integrability condition

\[ \mu[l_x] = \mu[l_0] < \infty. \]
2.4 Applications of the coupling

All our results rely on the fact that the different processes we consider can be coupled together in a natural way. We elucidate this by giving some key examples.

1. **Mixing Time Bounds on** $\Omega[\mathbb{N}]$: For each $\xi \in \Omega[\mathbb{N}]$, the coupling construction yields a process $\sigma^\xi_t = \sigma^\xi_t(\omega)$ $P$-a.s. that is a version of the process on $\Omega[\mathbb{N}]$ started from the point measure $\delta_\xi$. This immediately implies Theorem 1.1:

**Proof of Theorem 1.1:** One crucial observation regarding the dynamics of our coupling that it "pushes discrepancies to the right". Let $\tau_1$ be the first arrival on site 1 and for all $j \geq 2$ let $\tau_j$ be the first arrival on site $j$ after $\tau_{j-1}$. Because we are looking at the process with a wall to the left of site 1, once $\tau_1$ occurs, the value of $\sigma^\xi_1(t)$ is independent of $\xi$ for all $t > \tau_1$. By induction, the same is true for all $\{(j, \tau_j) : j \leq N\}$. Therefore, if we define

$$\tau_{\text{couple}} = \inf\left\{t : \sigma^\xi_1(t) = \sigma^\xi_2(t) \quad \forall x \leq N, \forall \xi_i \in \Omega[\mathbb{N}]\right\},$$

then $\tau_{\text{couple}} \leq \tau_N$. It is easy to see that

$$\tau_{\text{mix}} \leq \inf\left\{t : P(\tau_{\text{couple}} \geq t) \leq \frac{1}{2}\right\}.$$

The theorem is then proved by observing that $\tau_N$ is the time it takes for the $N$’th arrival of a Poisson point process which has rate 1.

2. **Classification of Invariant Measures on** $\Omega[\mathbb{Z}]$: To show that $\text{Ber}_p$ are, up to our regularity assumptions, the only invariant probability measures on $\Omega[\mathbb{Z}]$ (i.e. Theorem 2.5) we adopt a strategy inspired by the solution of the corresponding problem for exclusion processes (see [8]).

Let $\mu$ be a stationary measure in the sense of Definition 2.4. We consider the sample space $\Sigma_2 = \Omega[\mathbb{Z}] \times \Omega[\mathbb{Z}] \times \Omega$ and, if $\nu$ is a probability measure on $\Omega[\mathbb{Z}] \times \Omega[\mathbb{Z}]$, we shall write $P_\nu = \nu \times P$ (a probability measure on $\Sigma_2$). Often, $\nu$ will be such that $\text{Ber}_p, \mu$ are its marginals of $\nu$. By the above construction of stationary processes on $\Sigma = \Omega[\mathbb{Z}] \times \Omega$, we have processes $\sigma^1_t, \sigma^2_t$ defined $P_\nu$ a.s. as random variables on $\Sigma_2$, taking the initial values $(\sigma^1_0, \sigma^2_0)$ from the first/second factor of $\Omega[\mathbb{Z}] \times \Omega[\mathbb{Z}]$.

The main technical lemma needed to prove the classification is the following:

**Lemma 2.6.** Let $\nu$ on $\Omega[\mathbb{Z}] \times \Omega[\mathbb{Z}]$ be an invariant distribution for the process $(\sigma^1_t, \sigma^2_t)$, then

$$\nu(\sigma^1 \leq \sigma^2 \text{ or } \sigma^1 \leq \sigma^2) = 1$$
As explained in the beginning of § 3, Theorem 2.5 then follows from this Lemma by constructing a coupling of a fixed stationary measure $\mu$ to the family $\{\text{Ber}_p : p \in (0,1)\}$.

3. **Exponential decay of temporal correlations** The coupling naturally yields decay of correlations, i.e. Theorem 2.3. Take two local functions $f, g$ as in Theorem 2.3 and, for concreteness, say that $\text{Supp } f \cup \text{Supp } g \subset [0,r]$. We define the measure $\nu$ on $\sigma = (\sigma^1, \sigma^2) \in \Omega[Z]^2$ as follows:

(a) Both $\sigma^1$ and $\sigma^2$ are $\text{Ber}_p$-distributed.
(b) For $x < 0$, $\sigma^1(x) = \sigma^2(x)$.
(c) For $x \geq 0$, $\sigma^1(x)$ is independent of $\sigma^2$ and idem with $1 \leftrightarrow 2$.

and we consider the coupled process $\sigma_t = (\sigma^1_t, \sigma^2_t)$ defined as above, in Item 2, but now started from $\nu$.

Let $X(\sigma_t)$ denote the position of the "left most discrepancy" of the configuration $\sigma_t$, i.e. $X(\sigma_t) := \min\{x : \sigma^1_t(x) \neq \sigma^2_t(x)\}$. We are interested in $X(\sigma_t)$ for the following reason. Since the support of $f$ lies in $[0, \infty)$, $f(\sigma_t^0)$ is independent of $\sigma^2_t$ and therefore of $\sigma^2_t \forall t \in \mathbb{R}_+$. Moreover, if $X(\sigma_t)$ is to the right of $r$, then $g(\sigma^1_t) = g(\sigma^2_t)$. Since $f(\sigma^1_t)$ and $g(\sigma^2_t)$ are independent, we have

$$|E_\nu(f(\sigma^1_t)g(\sigma^1_t))| \leq P_\nu(X(\sigma_t) - X(\sigma_0) < r),$$

where we also used $\text{Ber}_p(f^2) = \text{Ber}_p(g^2) = 1$ and invariance of $\text{Ber}_p$. Finally observe that

$$X(\sigma_s) - X(\sigma_0)$$

is naturally coupled to a Poisson process $N(t)$ with rate $\min(\lambda_\pm)$ so that

$$X(\sigma_t) - X(\sigma_0) \geq N(t)$$

The theorem now follows from a large deviation estimate on the Poisson process and the fact that the process has stationary $\text{Ber}_p$-marginals: $E_\nu(f(\sigma^1_0)g(\sigma^1_t)) = E_{\text{Ber}_p}(f(\sigma_0)g(\sigma_t))$.

4. **Comparison of Half Line and Full Line Dynamics:** In § 4 we’ll prove Theorem 1.2. It is difficult to get much information about $\pi_\infty$. One way to do this is to compare the process $\sigma^1_t$ started from $\pi_\infty$ on $\mathbb{N}$ with the process $\sigma^2_t$ started from $\text{Ber}_p$ on $\mathbb{Z}$.

Similar to Item 2, let $\nu$ be a probability measure on $\Omega[\mathbb{N}] \times \Omega[\mathbb{Z}]$ stationary for the coupled process and whose marginals are $\pi_\infty, \text{Ber}_p$, respectively. There is necessarily a steady state discrepancy current which runs through the system. We will gain information on $\pi_\infty$ through the study of this current.
Further variations will be used throughout the rest of the text.

3 Invariant Measures on \(\mathbb{Z}\)

In this section we investigate invariant measures for the Toom interface on \(\mathbb{Z}\), using the coupling from § 2.

We formally define discrepancies. Let \(\sigma \in \Omega[\mathbb{Z}]^2\) and set

\[
D^\eta = D^\eta_\eta := \{ x \in \mathbb{Z} : \sigma^1(x) = \eta, \sigma^2(x) = -\eta \},
\]

\[
D = D_\sigma := D^+_\sigma \cup D^-_\sigma = \{ x \in \mathbb{Z} : \sigma^1(x) \neq \sigma^2(x) \}
\]

Rather than focusing on the discrepancies themselves, it is useful to restrict attention to the study of gaps between consecutive discrepancies of type \((+, -)\) and type \((-+, +)\), i.e. elements of \(D^+\) and \(D^-\), respectively. Let us, arbitrarily, call the first type discrepancies of signature \(+\) and the second discrepancies of signature \(-\). To keep track of ”interfaces” between the two types of discrepancy, let, for \(x \in D^\eta\),

\[
d(x) = \inf \{ y > x : y \in D \},
\]

\[
d_\partial(x) = \inf \{ y > x : y \in D^-\}\]

should such \(y\)'s exist and set \(d(x), d_\partial(x) = \infty\) otherwise. The set of interface discrepancies is then

\[
\mathcal{D} = \mathcal{D}_\sigma := \{ x \in D : d_\partial(x) < \infty \text{ and } d(x) = d_\partial(x) \}.
\]

Finally, for this proof and for future use, let

\[
\hat{r}_x = \max(r_x(\sigma^1), r_x(\sigma^2)), \quad \hat{l}_x = \max(l_x(\sigma^1), l_x(\sigma^2))
\]

with \(r_x, l_x\) as in Theorem 2.5.

**Proof of Lemma 2.6.** In general, our goal is to show that \(E_\nu[|\mathcal{D}|] = 0\) with \(\nu\) an invariant measure on \(\Omega[\mathbb{Z}]^2\) with first marginal \(\text{Ber}_p\) for some \(p \in (0, 1)\). Let us first sketch the proof in case \(\nu\) is translation invariant. We fix an interval \(I\). By stationarity, we are tempted to write

\[
0 = \partial_t E_\nu[|\mathcal{D} \cap I|] \leq E_\nu[\hat{l}_{\min I}] - \min(\lambda_\pm) E_\nu[|\mathcal{D}^1 \cap I|],
\]

(6) follows because the first term is an upper bound on the flow of discrepancies from \((-\infty, \min I - 1]\) into \(I\) and \((-\infty, \min I - 1]\) the second term lower bounds the annihilation rate inside \(I\). By hypothesis, \(\sup_x E_\nu[\hat{l}_x] < \infty\) so that these two inequalities together imply that \(E_\nu[|\mathcal{D}^1 \cap I|]\) is uniformly bounded in \(I\). In particular, if we assume \(\nu\) is translation invariant, this implies \(E_\nu[|\mathcal{D}^1|] = 0\). This argument can be iterated (considering the
discrepancies that can be promoted into $\mathcal{D}^1$ in one step, etc.) and eventually one then concludes that $E_\nu([\mathcal{D}]) = 0$. This argument is slightly formal because $|\mathcal{D} \cap I|$ is not a local function and therefore the inequality in (6) would need additional justification (see later). This could be easily remedied but anyhow the statement in the translation invariant case follows immediately from the upcoming Lemma 3.1, which we need for the general proof of Lemma 2.6.

Lemma 3.1. Let $\nu$ be invariant for the coupling and satisfy $\sup_x \nu(\hat{I}_x) < \infty$, then

$$\limsup_{|I| \to \infty} \frac{1}{|I|} E_\nu [|\mathcal{D} \cap I|] = 0.$$ \hspace{1cm} (7)

We postpone the proof of this lemma to § 5 and we now continue with the proof of Lemma 2.6. In § 6.5, we prove that for local $f$ and $\nu$ with sufficiently regular marginals (but not necessarily stationary), $E_\nu(f(\sigma_t))$ is differentiable in $t$ and the derivative is $E_\nu(Lf(\sigma_t))$ with $L$ the formal generator. Of course, if $\nu$ is stationary, as here, this derivative is zero. Let $\mathcal{D}_x = \{y \in \mathcal{D} : d_\theta(y) \leq x\}$, and note that $1\{y \in \mathcal{D}_x\}$ is a local function (its support is $[y, x]$) in contrast to $1\{y \in \mathcal{D}\}$. In what follows, let $x \in \mathbb{Z}$ and $h \in \mathbb{N}$ be fixed, and we omit them from the notation to avoid clutter. Set

$$\theta_\ell(y) = \begin{cases} 
1 & \text{for } y \in [x - h, x], \\
0 & \text{for } y \in (-\infty, -x - h - \ell], \\
\frac{j}{\ell} & \text{for } y = -x - h - j, 0 \leq j \leq \ell.
\end{cases}$$

Via an argument similar to the translation invariant case (but now with a local function on the left hand side)

$$0 = \partial_t E_\nu \left[ \sum_{0 \leq j \leq h + \ell} \theta_\ell(x - j) \chi(x - j \in \mathcal{D}_x) \right] \leq \sum_{j \geq 0} E_\nu \left[ Z_\ell(x - j) \right] - \min(\lambda_\pm) E_\nu \left[ X \right], \hspace{1cm} (8)$$

where

$$X := 1\{\exists y \in \mathcal{D}_x \cap [x - h + 1, x] : d_\theta(y) \text{ can leave } (-\infty, x) \text{ in one step}\} \hspace{1cm} (9)$$

$$Z_\ell(y) := [\theta_\ell(y + \hat{r}_y) - \theta_\ell(y)] \hat{y} \mathbf{1}\{y \in \mathcal{D}_x\} \hspace{1cm} (10)$$

To exploit (8), we bound $\sum_j E_\nu \left[ Z_\ell(x - j) \right]$ by splitting the sum over $j$. For $j < h$, $Z_\ell(x - j) = 0$. For $h \leq j \leq h + \ell$, we use $|\theta_\ell(y + \hat{r}_y) - \theta_\ell(y)| \leq \hat{r}_y/\ell$ and a Hölder
inequality (for any $1 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$) to get

$$
\sum_{h \leq j \leq h + \ell} E_{\nu} \left[ |Z_\ell(x - j)| \right] \leq \left( \frac{1}{\ell} \sum_{h \leq j \leq h + \ell} E_{\nu} \left[ (\hat{l}\hat{r})^q \right] \right)^{\frac{1}{q}} \left( \frac{1}{\ell} \sum_{h \leq j \leq h + \ell} \nu(j \in \mathcal{D}_x) \right)^{\frac{1}{p}} \tag{11}
$$

If we choose $q$ sufficiently close to 1, then the first factor is bounded by $C$ (uniformly in $\ell, h, x$) by the integrability assumption we placed on the second marginal of $\nu$. The second factor decays as $\ell \to \infty$ (uniformly in $h, x$) by Lemma 3.1 and the fact that $\mathcal{D}_x \cap I \subset \mathcal{D} \cap I$. For $j \geq h + \ell$, the sum is treated in a similar way and we conclude that

$$
\limsup_{\ell \to \infty} \sum_{j \geq 0} E_{\nu} \left[ |Z_\ell(x - j)| \right] = 0.
$$

Combining with (8), we conclude that $E_{\nu} [X] = 0$. Since this holds for all $x, h$, it follows that $E_{\nu} [\mathbb{D}] = 0$. \qed

Let us now explain how we arrive at Theorem 2.5 from Lemma 2.6.

**Proof of Theorem 2.5.** Let us fix a stationary measure $\mu$ as in the statement of Theorem 2.5. The heuristic idea is to construct a process $(\sigma_1, (\sigma(p))_{p \in [0,1]})$ such that:

(a) For any $p \in [0,1]$, the distribution of $\sigma(p)$ is $\text{Ber}_p$.

(b) The distribution of $\sigma_1$ is $\mu$.

(c) For any $p$: $\mathbb{P}(\sigma_1 \leq \sigma(p) \text{ or } \sigma_1 \geq \sigma(p)) = 1$.

(d) If $p' > p$, then $\sigma(p') \geq \sigma(p)$ a.s.

If we then define the random variable

$$
P := \sup \{ p \in \mathbb{Q} : \sigma_1 \geq \sigma(p) \} = \sup \{ p : \sigma_1 \geq \sigma(p) \}, \tag{12}
$$

it is tempting to believe that $\sigma_1 = \sigma(P)$, and that the distribution of $\sigma(P)$ is a mixture of product Bernoulli’s with mixing measure given by the distribution of $P$. To turn this into a rigorous proof is a bit delicate. What follows is one such implementation.

**Step 1:** Let $\mathcal{P} \subset (0,1)$ be a finite set. Then there exists a measure $\nu^P$ on configurations $(\sigma_1, (\sigma(p))_{p \in \mathcal{P}}) \in \Omega[\mathbb{Z}] \times (\times_{p \in \mathcal{P}} \Omega[\mathbb{Z}])$ satisfying the properties $a - d$ listed above for $p \in \mathcal{P}$. To see this, we start from an infinite array $(V_x)_{x \in \mathbb{Z}}$ of i.i.d. random variables uniformly distributed on $[0,1]$ and we set

$$
\sigma_0(p, x) := \begin{cases} 1 & V_x \geq p \\ -1 & V_x < p \end{cases}
$$

13
Then, the coupling construction gives us a process \((\sigma_t, \sigma_t(p))_{p \in P}\) started from \(\nu_0 = \mu \times (\times_p \text{Ber}_p)\) such that, at each \(t \geq 0\), the Properties \(a, b, d\) are satisfied. Therefore, any limit point \(\nu^*\) of the collection of time averaged measures \((1/T) \int_0^T dt\nu_t^P\) satisfies these properties as well. By Lemma 2.6, Property \(c\) will hold if \(\nu^*\) restricts to an invariant state of the coupling process for each pair \((\sigma_1, \sigma(p))\). Since our process does not have the Feller property, this is not automatic, but we prove it explicitly in § 6.4. Thus we may assume \(\nu^P\) satisfies Properties \(a - d\) in the remainder of the proof.

**Step 2**: Consider the measurable space \(\mathcal{M} := \Omega[\mathbb{Z}] \times (\times_{p \in Q} \Omega[\mathbb{Z}])\) equipped with its natural product sigma-field. Writing again \(\sigma^1\) for the variable in the first factor of the product space and \(\sigma(p)\) for the variables in the \(p\)-factors, we construct a measure \(\nu_\infty\) on \(\mathcal{M}\) which satisfies the properties \(a\)–\(d\) above, i.e., explicitly

(a) For any \(p\), the distribution of \(\sigma(p)\) is \(\text{Ber}_p\). Here, it is understood that \(\sigma(p) = \sigma(1)\) for \(p > 1\) and \(\sigma(p) = \sigma(0)\) for \(p < 0\).

(b) The distribution of \(\sigma^1\) is \(\mu\).

(c) For any \(p\): \(\mathbb{P}(\sigma^1 \leq \sigma(p) \text{ or } \sigma^1 \geq \sigma(p)) = 1\).

(d) If \(p' > p\), then \(\sigma(p') \geq \sigma(p)\) a.s.

This is achieved by considering an increasing sequence \((\mathcal{P}_n)\) of finite sets \(\mathcal{P}_n\) as above, such that \(\cup_n \mathcal{P}_n = Q\) and using the Kolmogorov extension theorem. We will not use any information about \(\nu_\infty\) apart from these properties. To avoid being repetitive we note here, and ask the reader to bear in mind, that all relevant assertions below should be understood as holding \(\nu_\infty\) a.s.

**Step 3**: We introduce also variables \(\sigma(p)\) for \(p \in [0, 1] \setminus Q\) such that Properties 1 and 4 above still hold. This is done by defining, for each \(x \in \mathbb{Z}\),

\[
W_x := \inf\{p \in Q : \sigma(x, p) = 1\}
\]

and, for any \(p\),

\[
\sigma(x, p) := \begin{cases} 
1 & \text{if } p \geq W_x, \\
-1 & \text{if } p < W_x.
\end{cases}
\]

For \(p \in Q\), this definition agrees with the original definition up to a set of measure 0, so we do not distinguish between them in our notation. It is straightforward to check that Properties \(a\),\(d\) still hold.

**Step 4**: Define the random variable

\[
P := \sup\{p \in Q : \sigma^1 \geq \sigma(p)\} = \sup\{p : \sigma^1 \geq \sigma(p)\} \tag{13}
\]

(the equality following by monotonicity).
Since \( \mathbf{P}(\sigma^1 \leq \sigma(p) \text{ or } \sigma^1 \geq \sigma(p)) = 1 \) for \( p \in \mathbf{Q} \), monotonicity implies
\[
P = \inf\{p \in \mathbf{Q} : \sigma^1 \leq \sigma(p)\} = \inf\{p : \sigma^1 \leq \sigma(p)\}. \tag{14}
\]

**Step 5:** Observe that \( \mathbf{P} \) is translation invariant under simultaneous shift factors of \( \mathcal{M} \) and recall that the product Bernoulli measures are extremal among all translation invariant probability measures on \( \Omega[\mathbb{Z}] \). Therefore, for any fixed \( q \), the distribution of \( \sigma(q) \) conditional on \( \mathbf{P} \) remains \( \text{Ber}_q \). Using this and monotonicity of \( \sigma(p) \) in \( p \), we claim that as \( \epsilon \to 0 \),

(a) \( \sigma(\mathbf{P} + \epsilon) - \sigma(\mathbf{P} - \epsilon) \) converges to 0 in distribution,

(b) \( \sigma(\mathbf{P} + \epsilon) - \sigma(\mathbf{P}) \) converges to 0 in distribution

and hence
\[
\sigma^1 = \sigma(\mathbf{P}).
\]

Finally we claim that

(c) The distribution of \( \sigma(\mathbf{P}) \) is a mixture of product Bernoulli’s with mixing measure given by the distribution of \( \mathbf{P} \).

Taken together, Claims (a) – (c) prove the statement of Theorem 2.5. Since verification of these claims are all similar, we give here a proof of (c) only.

To prove (c) it is enough to show that for any increasing local function \( f : \Omega[\mathbb{Z}] \to \mathbf{R} \),
\[
\mathbf{E}[f(\sigma(\mathbf{P}))] = \mathbf{E}[h(\mathbf{P})] \tag{15}
\]

where \( h : [0,1] \to \mathbf{R} \) is defined as \( h_p := \text{Ber}_p(f) \).

Starting from the left hand side, let \( I_j := ((j-1)/k, j/k) \). Then
\[
\mathbf{E}[f(\sigma(\mathbf{P}))|\mathbf{P}] = \sum_{j=1}^{k} \mathbf{1}\{\mathbf{P} \in I_j\} \mathbf{E}[f(\sigma(\mathbf{P}))|\mathbf{P}].
\]

Monotonicity implies
\[
\mathbf{E}[f(\sigma((j-1)/k))|\mathbf{P}] \leq \mathbf{E}[f(\sigma(\mathbf{P}))|\mathbf{P}] \leq \mathbf{E}[f(\sigma(j/k))|\mathbf{P}]. \tag{16}
\]
on the event \( \{\mathbf{P} \in I_j\} \). The extremality of the Bernoulli measures mentioned above implies
\[
\mathbf{E}[f(\sigma(j/k))|\mathbf{P}] = \mathbf{E}_{\text{Ber}_{j/k}}[f(\sigma)] = h(j/k). \tag{17}
\]

15
Now since $f$ is local $h$ is just a polynomial in $p$ and, in particular, is uniformly continuous on $[0, 1]$. Thus

$$
\int_{[0, 1]} dP(p) h(p) = \lim_{k \to \infty} \sum_{j=1}^{k} P(P \in I_j) h(j/k) = \lim_{k \to \infty} \sum_{j=1}^{k} P(P \in I_j) h((j - 1)/k).
$$

and Claim (c) follows from (16) and (17).

4 Proof of Theorem 1.2

In this section we consider the coupling process defined on the configuration space $\Omega[\mathbb{N}] \times \Omega[\mathbb{Z}]$. We will write $\sigma^1(x), \sigma^2(x)$ for the two configurations of $\sigma \in \Omega[\mathbb{N}] \times \Omega[\mathbb{Z}]$. We also use the notation set out at the beginning of § 3, with the understanding that all vertices $x \leq 0$ host discrepancies at all times.

Throughout this section, we let $p$ be the unique solution in $(0, 1)$ to the equation

$$
\left(\frac{1 - p}{p}\right)^2 = \frac{\lambda_+}{\lambda_-}
$$

Let $\nu$ be a probability measure on $\Omega[\mathbb{N}] \times \Omega[\mathbb{Z}]$ stationary for the coupling process and with respective marginals $\pi_\infty, \text{Ber}_p$. (There is in fact a unique such measure, though we don’t need to use this explicitly). From the following proposition, Theorem 1.2 follows easily.

**Proposition 4.1.** With $\nu$ as above,

$$
\lim_{x \to \infty} \nu(x \in D) = 0.
$$

Before discussing Proposition 4.1 further, let us show how it leads to the

**Proof of Theorem 1.2.** Consider a local function $f$ and recall the shifts $\tau_x$. By the definition of $D$;

$$
|E_{\pi_\infty}[f \circ \tau_x] - E_{\text{Ber}_p}[f \circ \tau_x]| \leq \|f\|_\infty \sum_{y \in x + \text{Supp } f} \nu(y \in D).
$$

By Proposition 4.1 the RHS tends to 0 as $x$ tends to infinity. But since $\text{Ber}_p$ is invariant under spatial shifts, this implies the push forward of $\pi_\infty$ by $\tau_x$ converges weakly to $\text{Ber}_p$ as $x$ tends to infinity.

Returning to the setup for Proposition 4.1, let us indicate how our choice of $p$ enters. We need a definition to do this. Fix a site $x \geq 1$ and consider the counting processes $H^y_\eta(t)$ that records, from time $t = 0$, the number of discrepancies of signature $\eta$ which
have jumped from \((-\infty, x)\) to \([x, \infty)\). There are some choices to make in this definition. Namely, it may be that a discrepancy gets annihilated in the jump in which it crosses \(x\). In this case we do count it in \(H^\eta_x(t)\). Let us further define
\[
K_x(t) = H^+_x(t) - H^-_x(t).
\]
A special role is played by these processes for \(x = 1\), since then they are in fact functions of \(\sigma^2\) only. Indeed, spin exchanges in \(\sigma^2\) across the bond \(\langle 01 \rangle\) are in one-to-one correspondence with the entrance of discrepancies into \([1, \infty)\). More precisely
\[
dH^\eta_1(t) = \sum_{y \leq 0} \chi^\eta_{[y,0]}(\sigma^2_{t-}) dN_{y,\eta}(t).
\]
(18)
and hence, by stationarity of \(\text{Ber}_\nu\)
\[
E_\nu(H^\eta_1(t)) = t\lambda_\eta \text{Ber}_\nu(l_1|\sigma_0 = \eta) \text{Ber}_\nu(\sigma_0 = \eta)
\]
(19)
By explicit calculation we check that \(E_\nu[K_1(t)] = 0\) only for the choice of \(p\) made above.
In order to explain the basic argument used to prove Proposition 4.1, let us introduce the process of annihilations at \(x\), \(A_x(t)\). This process counts the number of times, in the time interval \((0, t]\), that a discrepancy lands on a discrepancy of opposite signature at \(x\). Then the key relation which points the way forward is
\[
H^\eta_{x-1}(t) - H^\eta_x(t) = A_x(t) + \chi(x \in D^\eta(t)) - \chi(x \in D^\eta(0)).
\]
(20)
We define the annihilation rates by
\[
a_x := \lim_{t \to \infty} \frac{1}{t} E_\nu(A_x(t)),
\]
(21)
and the steady state currents via
\[
j^\eta_x := \lim_{t \to \infty} \frac{1}{t} E_\nu(H^\eta_x(t)), \quad \vec{j}_x := j^+_x + j^-_x
\]
Since \(\nu\) is stationary, both limits exist. Since \(\text{Ber}_\nu(l_1) < \infty\), we see that \(j^\eta_1 < \infty\) as well, cfr. (19).
Let us record a few simple observations: First of all,
\[
j^\eta_{x-1} - j^\eta_x = a_x.
\]
This immediately implies:

\[
\lim_{x \to \infty} j^\eta_x \text{ exists.} \tag{22}
\]

\[
 j^\eta_1 \geq \sum_{x=2}^{\infty} a_x. \tag{23}
\]

\[
\lim_{y \to \infty} \sum_{x>y} a_x = 0. \tag{24}
\]

It is more challenging to show that the steady state currents also decay.

**Lemma 4.2.** We have

\[
\lim_{x \to \infty} j^\eta_x = 0.
\]

Lemma 4.2 is the key to proving Proposition 4.1:

**Proof of Proposition 4.1.** Calculating the drift of the process \(H^\eta_x(t)\) we find that

\[
\mathbb{E}_\nu[H^\eta_x(t)] \geq \min(\lambda_{\pm})\nu(x \in D^\eta)t.
\]

Using the definition of \(j^\eta_x\) above and Lemma 4.2, the conclusion follows. \(\square\)

To prove Lemma 4.2, we split the process \(H^\eta_x(t)\) into its \(k\)-stretch contributions \((1/k)H^{\eta,k}_x(t)\). By definition, \((1/k)H^{\eta,k}_x(t)\) counts the number of stretches of exactly \(k\) consecutive \(\eta\)-discrepancies that have crossed from \((-\infty, x)\) to \([x, \infty)\) in \((0, t]\). Explicitly, let \((\tau_j, \eta_j)\) with \(j = 1, 2, \ldots\) be the increasing sequence of crossing times of discrepancies at \(x\) along with their associated signatures. Then

\[
(1/k)H^{\eta,k}_x(t) = \sum_{j:0<\tau_j\leq t} 1 \{\eta_j = \eta_{j+1} = \ldots = \eta_{j+k-1} = \eta \text{ and } \eta_{j-1} = \eta_{j+k} = -\eta\} \tag{25}
\]

(with the convention that the condition \(\eta_{j-1} = -\eta\) is also satisfied if \(j-1 = 0\)) The notation has been chosen so that \(H^\eta_x(t) \geq \sum_{k \geq 1} H^{\eta,k}_x(t)\) (equality would follow if we exclude current carried by \(\infty\) consecutive discrepancies of equal sign). Let further

\[
H^k_x(t) := \sum_{\eta=\pm 1} H^{\eta,k}_x(t), \quad H_x(t) := \sum_{\eta=\pm 1} H^\eta_x(t)
\]

\[
j^{\eta,k}_x := \lim_{t \to \infty} 1/t\mathbb{E}[H^{\eta,k}_x(t)], \quad j^{k}_x := j^{+,k}_x + j^{-,k}_x.
\]

Again, the limits exist by stationarity. The analysis proceeds by using two distinct mechanisms to bound \(j^k_x\), depending on whether \(k\) is small or large. Intuitively, if \(k\) is small and \(x\) is large, \(j^k_x\) should be small due to annihilation effects. This intuition leads
us in § 4.0.1 to prove the following lemma: Let \( q = \min(p, 1 - p) \) and denote

\[
\rho_x := \nu(x \in \mathcal{D}).
\]

We use the notation \( o_x(1) \) to denote an expression which tends to 0 as \( x \) tends to \( \infty \)

**Lemma 4.3.** Let \( k \in \mathbb{N} \) be fixed. There exists \( C > 0 \) depending only on \( \lambda_\pm \) and \( k \) such that for any \( x \in \mathbb{N} \),

\[
j^k_x \leq C \sum_{z=1}^{x-1} \sum_{y=z}^{x-1} |z - y|^k \min(q^{x-z}, \rho_y) + o_x(1). \tag{26}
\]

For \( k \) large, \( j^k_x \) is controlled by fluctuations of the process \( K_x(t) \) and, more conveniently, \( K_1(t) \). After time of order \( x \), all discrepancies which start in \([1, x]\) will be to the right of \( x \). So for time intervals \([s_0, t_0]\) with \( s_0 \gg x \), \( K_x(t_0) - K_x(s_0) \geq k \), implies \( K_1(b) - K_1(a) \geq k \) for some earlier time interval \([a, b]\). This follows because discrepancies only annihilate in pairs of opposite signatures. Thus, the occurrence of large \( k \)-stretches for any \( x \) can be related to a property of \( K_1 \), which we now and later abbreviate as \( K \).

For a partition \( P \) of \([0, t]\) into intervals, let

\[
\mathcal{K}_k(P) = \sum_{I \in P} |\Delta K(I)| 1 \{ |\Delta K(I)| \geq k \}
\]

where \( \Delta K(I) := K(\sup I) - K(\inf I) \), and

\[
\mathcal{K}^*(k, t) := \sup_{\{P\}} \mathcal{K}_k(P).
\]

Then, the above reasoning implies that

\[
H_x(t) - \sum_{k' < k} H_{x}^{k'}(t) \leq \mathcal{K}^*(k, t) + x + k, \tag{27}
\]

where \( x + k \) on the RHS accounts for possible initial \( k \)-stretches crossing \( x \) which cannot be completely attributed to fluctuations of \( K \).

It turns out that the following, proved in § 4.0.2, is enough for our purposes:

**Lemma 4.4.**

\[
\lim_{k} \sup_{t} \lim_{t} \sup \frac{1}{t} \mathbb{E}_{\text{Ber}_p} [\mathcal{K}^*(k, t)] = 0.
\]

Note that we have written \( \mathbb{E}_{\text{Ber}_p} \) instead of \( \mathbb{E}_\nu \) since \( K \) depends only on \( \sigma^2 \), cfr. (18).

Finally, we next need a lemma analogous to Lemma 3.1. We defer the proof to § 5.
Lemma 4.5.
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{x=1}^{N} \rho_x = 0 \]

With Lemmas 4.3, 4.4 and 4.5 in hand, we are able to prove Lemma 4.2.

Proof of Lemma 4.2. We shall argue that
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{x=1}^{N} j_x = 0 \]

Since \( j_x \) is decreasing in \( x \), this implies
\[ \lim_{x \to \infty} j_x = 0. \]

Taking the time-average of (27), we have
\[ j_x - \sum_{k' < k} j_{k'}^x \leq \lim \sup_{t} \frac{1}{t} E_{\text{Ber}} [K^*(k, t)] . \tag{28} \]

uniformly in \( x \).

By Lemma 4.4, for any \( \epsilon > 0 \) we can choose \( k_0 \) so that \( j_x - \sum_{k' < k_0} j_{k'}^x \leq \epsilon \) for all \( x \in \mathbb{N} \), so \( j_x \leq \sum_{k=1}^{k_0} j_{k}^x + \epsilon \). Using Lemma 4.3,
\[ j_x \leq \epsilon + o_x(1) + C(k_0) \sum_{z=1}^{x-1} \sum_{y=z}^{x-1} |z - y|^{k_0} \min(q^{x-z}, \rho_y). \]

Hence
\[ \frac{1}{N} \sum_{x=1}^{N} j_x \leq \epsilon + o_N(1) + C(k_0) \sum_{x=1}^{N} \sum_{z=1}^{x-1} \sum_{y=z}^{x-1} |z - y|^{k_0} \min(q^{x-z}, \rho_y). \]

Finally, we claim that for \( N \) sufficiently large, term I is less than \( \epsilon \). To see this observe that for any fixed \( m \)
\[ \sum_{z=1}^{x-1} \sum_{y=z}^{x-1} |z - y|^{k_0} \min(q^{x-z}, \rho_y) \leq C(k_0, m) \sum_{j=x-m}^{x-1} \rho_j + \sum_{z=1}^{x-m} (x-z)^{k_0+1} q^{x-z}. \]

Combined with Lemma 4.5, we thus obtain, for \( m \) and \( k_0 \) fixed
\[ I \leq o_N(1) + \sum_{z=1}^{x-m} (x-z)^{k_0+1} q^{x-z}. \]
The second term can be made as small as we like by choosing \( m \) large.

Thus given \( \epsilon \) and choosing \( m, k_0 \) large enough but fixed,

\[
\limsup_{N \to \infty} \frac{1}{N} \sum_{x=1}^{N} j_x \leq 2\epsilon.
\]

Since \( \epsilon \) was arbitrary and the LHS is independent of the choice of \( m, k_0 \), we conclude

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{x=1}^{N} j_x = 0.
\]

\( \square \)

The next two subsections address, respectively, the proofs of Lemma 4.3 and Lemma 4.4.

### 4.0.1 Current Carried by Short Stretches; Proof of Lemma 4.2

**Proof of Lemma 4.3.** For concreteness, we deal below with \( j_x, k \), the argument for \( \eta = -1 \) is identical. We will need a few definitions and conventions. First, we will write \( D_t := D_{\sigma t}, D_\eta t := D_{\eta \sigma t}, \ldots \) to make the formulas lighter. We let \( \beta_j \) be the ordered jump times of \((1/k)H_x^{+,k}(t)\) and let \( r_j \) be the site at which the Poisson arrival occurs at \( \beta_j \), (thus causing the crossing of a discrepancy from \((-\infty, x)\) to \([x, \infty)\)). It follows that \( r_j < x \) and that \( \sigma^{(2)}_{\beta_j} \) takes a single value on \([r_j, x - 1] \setminus \mathcal{D}_{\beta_j}^+\), i.e. either all + or all −.

Let us define \( y_j \) as follows: Moving from \( x \) to the left, consider the discrepancies of \( \sigma_{\beta_j} \). If the first \( k + 1 \) have signature +, we let \( y_j \) be the position of the \((k + 1)\)'st of them. Otherwise, we set \( y_j = 0 \). In other words, \( y_j \neq 0 \) is defined by the conditions

\[
y_j \in \mathcal{D}_{\beta_j}^- \quad \text{and} \quad \mathcal{D}_{\beta_j}^- \cap [y_j, x - 1] = \emptyset \quad \text{and} \quad |\mathcal{D}_{\beta_j}^+ \cap [y_j, x - 1]| = k + 1.
\]

Finally, let

\[
z_j = \max(\mathcal{D}_{\beta_j}^- \cap [1, x - 1]),
\]

where the \( \max(\emptyset) := 0 \), and define the event \( B_j \) by

\[
B_j := \{ z_j \geq 1, z_j \in \mathcal{D}_{\beta_j}^- \}.
\]

Note in particular that if \( B_j \) occurs, the first discrepancy to the right of \( z_j \) has signature +.
For any $t$, we have
\[
1/kH_k^x(t) \leq \sum_{i=1,3} \sum_{j; \beta_j \leq t} 1\{E^i_j\} + \sum_{i=2,4} \sum_{j; \beta_j \leq t} \sum_{w \leq x} 1\{E^i_j(w)\}. 
\]
(29)

where
\[
E^1_j := B^c_j, \\
E^2_j(w) := \{r_j = w > y_j\} \cap B_j, \\
E^3_j := \{y_j > \lfloor \sqrt{x}\rfloor\}, \\
E^4_j(w) := \{0 < r_j = w \leq y_j \leq \lfloor \sqrt{x}\rfloor\}. 
\]

To check (29), note that for any fixed $j$, the sum on the right hand side covers the whole sample space $\Sigma_2$ and that the counting process on the left hand side increases by 1 at each jump time $\beta_j$.

Taking expectations and dividing by $t$ in (29) will give a bound on the current of $k$ stretches. To bound the expectation of the RHS, we observe first that
\[
E^1_j \subset \left\{ \sum_{y \geq x} [A_y(\beta_j-) - A_y(\beta_{j-1})] \geq 1 \right\} \cup \{z_j = 0\}, 
\]
(30)
\[
E^3_j \subset \left\{ \sum_{y \geq \lfloor \sqrt{x}\rfloor} [A_y(\beta_{j+1}-) - A_y(\beta_j)] \geq 1 \right\}. 
\]
(31)

This is seen as follows:

- (for $E^1_j$): Suppose $z_j \geq 1$. By definition, at $\beta_j$ – the first discrepancy to the left of $x$ does not belong to $\Sigma \cap D^+$ and $z_j \in D^+$, so the first discrepancy to the right of $x$ must be a +. Hence, there must have been at least one annihilation to the right of $x$ in the time-interval $[\beta_{j-1}, \beta_j]$.

- (for $E^3_j$): On this event, $z_j \geq 1$. By definition of $\beta_j$, we deduce that at least one annihilation event has to occur to the right of $z_j$ in the time-interval $[\beta_j, \beta_{j+1}]$.

Thus
\[
\lim_{t \to \infty} \frac{1}{t} \sum_{i=1,3} \sum_{j; \beta_j \leq t} E^i_j \leq q^x + \sum_{y \geq x} a_y + \sum_{y \geq \lfloor \sqrt{x}\rfloor} a_y = o(1). 
\]

where the second bound follows from (24).
Let us define

\[ A(w, S, s) := \{ \sigma_{s-}^{(2)} \text{ takes a single value on } [w, x - 1] \setminus S \} \]

\[ B(w, S, s) := \{ S = D_{s-}^{+} \cap [w, x - 1], \max S \in \mathcal{D}_{s-} \}. \]

Then we have

\[ E_{2j}^{2}(w) \subset \bigcup_{S \subset [w, x - 1]} \bigcap_{s \leq t} \left\{ A(w, S, \beta_{j}) \cap B(w, S, \beta_{j}) \right\} \]

where

\[ \nu_{w}(s) \text{ is the rate one Poisson process at } w \text{ defined in the coupling construction. Now the events } A(w, S, s), B(w, S, s) \text{ are in the sigma algebra generated by} \]

\[ (\sigma_{u})_{u<s}, \text{ hence the first inequality in} \]

\[ I \leq \int_{0}^{t} ds \mu_{w}(A(w, S, s) \cap B(w, S, s)) \leq t \min(q^{x-w-1}, \rho_{\max} t), \]

the second following by the upper bound \( \mu(A \cap B) \leq \min(\mu(A), \mu(B)) \), stationarity of \( \nu \), and properties of the Bernoulli measure. Reinstating in (32), we get

\[ \sum_{w=1}^{x-1} \nu_{w} \left[ \sum_{j=0}^{t} 1\{E_{2j}^{2}(w)\} \right] \leq C \sum_{w=1}^{x-1} \sum_{y=w}^{x-1} |w - y|^{k} \min(q^{x-w}, \rho_{y}). \]

The bound for the contribution from \( E_{2j}^{4}(w) \) proceeds in a similar manner. We obtain

\[ \sum_{w \leq x} \nu_{w} \left[ \sum_{j=0}^{t} 1\{E_{2j}^{4}(w)\} \right] \leq C \sum_{w \leq x} \sum_{S \subset [w, x - 1]} \left[ A(w, S, s) \right] \leq Ct e^{-cx} \]

(34)

using a simple large deviation bound for the Bernoulli measure. Hence, upon dividing by \( t \), this contribution is also \( o(1) \) as \( x \to \infty \).
4.0.2 Current carried by long stretches; Proof of Lemma 4.4

In this section we prove Lemma 4.4. Before beginning, we need two preliminary results which are interesting in their own right. To motivate the first statement, recall that the parameter \( p \) of the product Bernoulli used in the coupling is determined by the condition that the driving process for the entrance of discrepancies at 0, i.e. \( K(t) \) is centered, \( \mathbb{E}[K(t)] = 0 \). It is possible to show that \( K \) satisfies a functional CLT under proper rescaling, see [3], but to prove Lemma 4.4, the following diffusive bound suffices. Its proof is supplied in § 5.

**Lemma 4.6.** There exists \( C > 0 \) such that for all \( t \in \mathbb{R} \),

\[
\mathbb{E}_{\text{Ber}}[(K(t) - K(0))^2] \leq Ct.
\]

We need a bound on the total flux across a fixed vertex for the process on \( \Omega[Z] \). For \( I = [a, b] \) set \( \mathcal{N}(I) = (H_1(b) - H_1(a)) \) i.e. \( \mathcal{N}(I) \) is the total variation of \( K \) in \( I \).

**Lemma 4.7.** For sufficiently small \( \gamma \),

\[
\mathbb{E}_{\text{Ber}} \left[ \exp \frac{\gamma \mathcal{N}(I)}{1 + |I|} \right] \leq C, \quad \text{for any } I.
\]

For \( I \) small, the above does not lead to an effective bound for the probability of having a large flux. Hence we record a second bound:

**Lemma 4.8.** If \( |I| \in (0, 1), R \geq 2 \) and \( \epsilon > 0 \), there is a constant \( C = C(R, \epsilon) \) so that

\[
\mathbb{P}_{\text{Ber}}(\mathcal{N}(I) \geq R) \leq C|I|^{R-\epsilon}.
\]

The proofs these two lemmas are deferred to § 5, see also § 6.2 in which a similar (but weaker) a priori bound is needed to construct the process started from a product Bernoulli measure.

**Proof of Lemma 4.4.** The proof of this lemma relies on a separation of scales. Fix \( \alpha \in (0, 1) \) and recall that \( \Delta K(I) = K(\sup I) - K(\inf I) \). For each partition \( P \) of \( [0, t] \), let us split the sum \( \mathcal{K}_k(P) \) according to interval sizes:

\[
\mathcal{K}_k(P) = \sum_{I \in P: |I| \leq k^\alpha} |\Delta K(I)| \mathbb{1}\{|\Delta K(I)| \geq k\} + \sum_{I \in P: |I| > k^\alpha} |\Delta K(I)| \mathbb{1}\{|\Delta K(I)| \geq k\}
\]

(35)

We will use separate mechanisms to bound each of \( \mathcal{K}^1 \) and \( \mathcal{K}^2 \) uniformly in \( P \).

Let us first attend to \( \mathcal{K}^1(P) \). The idea here is simple: since a contributing interval \( I \) is small relative to \( k \), it necessitates too many Poisson arrivals, at least \( k \), in \( I \). This is
a rare event, and gets exponentially rarer as $|I| \to 0$. This fact allows us to handle all partitions simultaneously via a properly chosen infinite covering of $[0, t]$.

For each $j \in \mathbb{Z}$ let $P_j$ denote the partition of $[0, t]$ by dyadic intervals $[\ell 2^j, (\ell + 1)2^j]$ (except for the last interval perhaps). Let

$$X_j = 2 \sum_{I \in P_j} \mathcal{N}(I)1\{\mathcal{N}(I) \geq k/2\}.$$ 

Then for any $P$, exploiting that $\mathcal{N}(I)$ is the total variation of $K(I)$,

$$\sup_P K^1(P) \leq \sum_{j \leq \log_2(k^\alpha) + 1} X_j.$$ 

Taking expected values on both sides and using stationarity of increments

$$\frac{1}{t} \mathbb{E}_{Ber_p} \left[ \sup_P K^1(P) \right] \leq \sum_{j \leq \log_2(k^\alpha) + 1} 2^{1-j} \mathbb{E}_{Ber_p} \left[ \mathcal{N}([0, 2^j])1\{\mathcal{N}([0, 2^j]) \geq k\} \right]. \quad (36)$$

By the Cauchy-Schwarz inequality,

$$\mathbb{E}_{Ber_p} \left[ \mathcal{N}([0, 2^j])1\{\mathcal{N}([0, 2^j]) \geq k\} \right] \leq \sqrt{\mathbb{E}_{Ber_p} \left[ \mathcal{N}([0, 2^j]) \right] \mathbb{P}(\mathcal{N}([0, 2^j]) \geq k)}.$$ 

Using Lemma 4.7 to bound the first factor on the RHS and Lemma 4.8 to bound the second factor, we find that for $k$ sufficiently large, the RHS of (36) is summable and that, moreover, it tends to 0 as $k$ tends to $\infty$.

To bound $K^2(P)$ let us introduce a reference partition $P_0$ of $[0, t]$ consisting of intervals with end points from the set $I := \{\ell k^{2\alpha/3} : \ell \in \mathbb{Z}\}$. Let

$$I(P) = \left\{ x \in I : \exists [y, z] \in P \text{ with } \min(|x - y|, |x - z|) < k^{2\alpha/3} \text{ and } |y - z| > k^\alpha \right\}.$$ 

Using the shorthand notation $\mathcal{N}_x = \mathcal{N}[x - k^{2\alpha/3}, x + k^{2\alpha/3}]$ we have

$$K^2(P) \leq \sum_{I \in P_0} |\Delta K(I)| + 2 \sum_{x \in I} \mathcal{N}_x 1\{\mathcal{N}_x \geq k^{3\alpha/4}\} + 2 \sum_{x \in I(P)} \mathcal{N}_x 1\{\mathcal{N}_x \leq k^{3\alpha/4}\} \quad (37)$$

Term III is bounded by $4tk^{-\alpha}k^{3\alpha/4}$ since the total number of intervals $I \in P$ with $|I| \geq k^\alpha$ is at most $2tk^{-\alpha}$. For the other two terms, we take the expectation and use that they do not any longer depend on $P$. For II, the bound ??, yields

$$\mathbb{E}_{Ber_p}[II] \leq 2tk^{-\alpha} \times C(m)(k^{2\alpha/3}/k^{3\alpha/4})^m.$$
which vanishes as \( k \to \infty \) by choosing \( m \) sufficiently large. For the first term, we argue

\[
E_{\text{Ber}}[I] \leq tk^{-2\alpha/3}(E_{\text{Ber}}[|K(k^{2\alpha/3}) - K(0)|^2])^{1/2} \leq Ctk^{-\alpha/3}
\]  

(38)

where the first inequality is by stationarity and Cauchy-Schwarz, and the second follows from the diffusive moment estimate Lemma 4.6. Hence we have obtained

\[
\limsup_{k \to \infty} (1/t) \sup_{P} \mathcal{K}^2(P) = 0,
\]

Combining with the analogous bound on \( \mathcal{K}^1 \), the assertion of the lemma follows.

\[\square\]

5 Proofs for Lemmas 3.1, 4.5, 4.6, 4.7 and 4.8.

We begin this section by proving Lemmas 3.1 and 4.5. In §4, we defined the process \( A_x(t) \), which counts the number annihilations at \( x \) in \((0, t]\). We may also consider this process for the dynamics on \( \Omega \backslash \mathbb{Z}^2 \), as well as the counting processes \((1/k)H_{\eta}x(t)\) and the mean values \( a_x, j_{\eta}x \). It is important to note that, on \( \Omega \backslash \mathbb{Z}^2 \) as well, we have that \( \sum_{x \in \mathbb{Z}} a_x < \infty \), the argument is the same as the one leading to (24), the crux being that the finite moment assumption \( \sup_x E_{\nu}(l_x) < \infty \) on \( \nu \) implies that we have a finite mean current \( j_x \) from \(( -\infty, x)\) to \( [x, \infty) \), cfr. (18) and (19). Given this observation, the proof of Lemma 4.5 is the same as that of Lemma 3.1. For concreteness, we restrict ourselves to the

Proof of Lemma 4.5. Given the discrete interval \([a, b]\), let

\[
E_{a,b}(\sigma) := \{ a \in \mathcal{D}_\sigma, \ b = d_\beta(a) \}\]

In other words, there is a boundary discrepancy at \( a \), and the first discrepancy to its right occurs at \( b \). Then Lemma 3.1 is a consequence of the following:

Lemma 5.1. Let \( \nu \) be a stationary probability measure for the coupling process on either \( \Omega[\mathbb{Z}]^2 \) or \( \Omega[\mathbb{N}] \times \Omega[\mathbb{Z}] \). Then for any \( a < b \), with \( a > 0 \) in the latter case, and \( t > 0 \),

\[
P_{\nu} \left[ \sum_{x=a}^{b} (A_x(t) - A_x(0)) \geq 1 \right] \geq c(t, |b - a|)
\]  

(39)

Let us finish the proof of Lemma 3.1 and then attend to this claim. Fix some arbitrary
\( t > 0 \), then (39) implies

\[
\frac{1}{N} \sum_{x \in [1,N]} \nu(x \in \mathcal{D}, d_\partial(x) - x \leq k) \leq c(t, k) \frac{1}{N} \sum_{x \in [1,N+k]} a_x = o_N(1) \tag{40}
\]

where we recall that \( d_\partial(x) \) denotes the first discrepancy (hence of opposite sign) to the right of \( x \) and that the RHS tends to \( 0 \) as \( N \) tends to \( \infty \) due to (24). On the other hand, for any \( k \),

\[
\frac{1}{N} \sum_{x \in [1,N]} \nu(x \in \mathcal{D}, d_\partial(x) - x \geq k) \leq 1/k.
\]

Lemma 4.5 thus follows from this and the fact that (40) holds for any fixed \( k \). \( \square \)

**Proof of Lemma 5.1.** The main observations we make are that arrivals at sites \( x > b \) do not harm us – they cannot move the discrepancy at \( b \) to the front – while arrivals at sites \( x < a \) only help us – they can push the discrepancy at \( a \) to the front of or possibly on top of \( b \) (such that an annihilation occurs). To arrive at an annihilation event, it suffices to have at least \( b - a \) Poisson arrivals at the location of the discrepancy which is at \( a \) at time \( 0 \) before any occur between the location of the discrepancy and \( b \). This yields the Lemma. \( \square \)

**Proof of Lemma 4.6.** By explicit calculation

\[
\mathbf{E}_{\text{Ber}_p} \left[ (K_1(t))^2 \right] = \mathbf{E}_{\text{Ber}_p} \left[ \int_0^t ds g(\sigma_s) \right] + \mathbf{E}_{\text{Ber}_p} \left[ \left( \int_0^t ds f(\sigma_s) \right)^2 \right]. \tag{41}
\]

where

\[
g = \sum_{y \leq 0, \eta = \pm 1} \lambda_\eta \chi_{[y, 0]}, \quad f = \sum_{y \leq 0, \eta = \pm 1} \eta \lambda_\eta \chi_{[y, 0]} \tag{42}
\]

The first term on the right hand side of (41) is bounded by \( Ct \), and the second term is by stationarity bounded by

\[
2t \int_0^\infty ds \mathbf{E}_{\text{Ber}_p}[f(\sigma_s)f(\sigma_0)].
\]

We shall show \( \mathbf{E}_{\text{Ber}_p}[f(\sigma_s)f(\sigma_0)] \) decays exponentially in \( s \) to complete the proof.

One slight complication is that the function \( f \) has unbounded support. To handle this issue, let \( f_n \) be the approximation to \( f \) above by restricting the sum over \( y \) to \( -n \leq y \leq 0 \). Then, by inspection,

\[
\mathbf{E}_{\text{Ber}_p}[|f - f_n|^2] \leq C e^{-cn}
\]
so that, for any \( n \), Cauchy-Schwarz yields

\[
|E_{\text{Ber}}[f(\sigma_s)f(\sigma_0)] - E_{\text{Ber}}[f_n(\sigma_s)f_n(\sigma_0)]| \leq Ce^{-cn/2}.
\]

The autocorrelation of \( f_n \) is handled by Theorem 2.3, with \( r = n \), so that

\[
|E_{\text{Ber}}[f(\sigma_t)f(\sigma_0)]| \leq C(e^{-cn/2} + e^{ct-n})
\]

and, choosing \( n = 2ct \), we get

\[
|E_{\text{Ber}}[f(\sigma_t)f(\sigma_0)]| \leq Ce^{-ct}.
\]

Lemma 4.6 follows as discussed above.

\( \Box \)

**Proof of Lemma 4.7.** Let us first recall from (18) that \( H_1(t) \) may be represented as a stochastic integral

\[ dH_1(t) = \sum_{\eta, y \leq 0} \chi_{\eta}(s) \, dN_{y, \eta}(t). \tag{43} \]

Restricting the above sum to \( y > -L \), we define variables \( H_L^1 \). They have a drift given by (recall \( \sum \lambda_\eta = 1 \))

\[ v_L^t := \sum_\eta \lambda_\eta \sum_{-L < y \leq 0} \int_0^t ds \chi_{\eta}(s) \leq \int_0^t ds l_1(s) \]

and by stationarity \( E_{\text{Ber}}(v_L^t) \leq t Ber_p(l_1) \leq Ct. \) Since \( E_{\text{Ber}}(H_L^1(t)) = E_{\text{Ber}}(v_L^t) \) and, for any \( t \), \( H_L^1(t) \) is non-decreasing in \( L \), we conclude that

\[ \lim_{L \to \infty} H_L^1(t) = H_1(t), \quad \text{a.s.} \]

and therefore also \( e^{\kappa H_L^1(t)} \to e^{\kappa H_1(t)} \) almost surely. Hence, by dominated convergence,

\[ E_{\text{Ber}}(e^{\kappa H_1(t)}) = \lim_L E_{\text{Ber}}(e^{\kappa H_L^1(t)}). \tag{44} \]

To estimate the right hand side, we first find an estimate on the exponential of \( v_L^t \). By Jensen inequality and stationarity, with \( \alpha > 0 \)

\[ E_{\text{Ber}}(e^{\alpha v_L^t}) \leq (1/t) \int_0^t ds E_{\text{Ber}}(e^{\alpha l_1(\sigma_s)}) = Ber_p[e^{\alpha l_1}] \tag{45} \]

which is finite for \( \alpha < c(p)/t \). Next, we verify by direct computation that, for any \( \kappa > 0 \),

\[ Z_\kappa^L(t) := \exp \{ \kappa H_L^1(t) - (e^\kappa - 1)v_L^t \} \]
is a martingale. To exploit this, let \( \kappa' := (1/2)(e^{2\kappa} - 1) \). Then we can write

\[
E_{\text{Ber}}[e^{\kappa H_1^L(t)}] = E_{\text{Ber}}[e^{\kappa H_1^L(t) - \kappa' v_t^L} \times e^{\kappa' v_t^L}].
\]

Applying the Cauchy-Schwarz inequality,

\[
E_{\text{Ber}}[e^{\kappa H_1^L(t)}] \leq E_{\text{Ber}}[Z_2^\kappa(t)]^{1/2}E_{\text{Ber}}[e^{2\kappa' v_t^L}]^{1/2} \leq E_{\text{Ber}}[e^{2\kappa' v_t^L}]^{1/2}
\]

where the second inequality follows because \( Z_\kappa(0) = 1 \) and \( Z_\kappa(t) \) is a martingale.

The lemma follows from (45) and (44) upon choosing \( \kappa \sim \gamma/(1 + t) \) for \( \gamma \) small enough.

Proof of Lemma 4.8. Let us consider the event

\[
F_T = \{ \exists s \in [0, T] : l_1(\sigma_s) \geq A|\log T| \}
\]

where the constant \( A > 0 \) is a parameter to be fixed at the end of the proof. A straightforward argument shows that for any \( \epsilon > 0 \),

\[
P_{\text{Ber}}(\{N(I) > R \} \cap F_T^c) \leq C(A, \epsilon)T^{R-\epsilon}.
\]

Hence it suffices to bound \( P_{\text{Ber}}(F_T) \).

Let \( M(\sigma) = \sum_{x=0}^{A|\log T|} \sigma(x) \). Then

\[
\text{Ber}(\{|M(\sigma_0)| > (1 + |p - 1/2|)A|\log T|\}) \leq C(p)e^{-c(p)A|\log T|}
\]

so that

\[
P_{\text{Ber}}(F_T) \leq P_{\text{Ber}}(F_T \cap \{|M(\sigma_0)| > (1 + |p - 1/2|)A|\log T|\}) + C(p)e^{-c(p)A|\log T|}.
\]

Finally we note that \( F_T \cap \{|M(\sigma_0)| > (1 + |p - 1/2|)A|\log T|\} \subseteq \{N(I) > A(1 - |p - 1/2|)|\log T|\} \). Hence Lemma 4.7 combined with Markov’s inequality implies

\[
P_{\text{Ber}}(F_T \cap \{|M(\sigma_0)| > (1 + |p - 1/2|)A|\log T|\}) \leq Ce^{-A(1-|p-1/2|)|\log T|}.
\]

Choosing \( A \) sufficiently large and combining the above estimates together finishes the proof.
6 Existence of Dynamics on $\Omega[\mathbb{Z}]$

6.1 The formal generator

First we need some notation. The flip operator $F_x$ at site $x$ acts on finite polynomials in $\sigma$ as

$$F_x(\sigma(x)p_x(\sigma)) = -\sigma(x)p_x(\sigma), \quad F_x(p_x(\sigma)) = p_x(\sigma),$$

where $p_x(\sigma)$ is a polynomial that does not depend on $\sigma(x)$. For a finite subset $S \subset \mathbb{Z}$ and $\hat{\sigma} \in \{-1,1\}^S$, we have the indicators

$$\chi_{\hat{\sigma}}^S = \chi[\sigma(x) = \hat{\sigma}(x) \forall x \in S]$$

and whenever $\hat{\sigma}$ is all 1 or all $-1$, then we simply write $\chi_{\hat{\sigma}}^+ S$ and $\chi_{\hat{\sigma}}^- S$. We also need the associated projectors –

$$P_{\hat{\sigma}}^S f(\sigma) = \chi_{\hat{\sigma}}^S(\sigma) f(\sigma).$$

Then the generator of the process on the domain $[-L, \infty)$ is formally defined as

$$\mathcal{L}_L = \sum_{L \leq x < y} (\mathcal{L}_{x,y,+} + \mathcal{L}_{x,y,-}) = \sum_{L \leq x < y} (\lambda_+ P_{[x,y]}^+ P_y^- + \lambda_- P_{[x,y]}^- P_y^+)(F_x F_y - 1)$$

One can immediately check that if $\text{Supp}(f) \subset [-L, R]$, then $\text{Supp}(\mathcal{L}_L f) \subset [-L, R]$, as well. This reflects the property, mentioned already in, that the restriction of the process to $x \leq R$ is Markov. Simple functional analysis implies then that the closure of $\mathcal{L}_L$ on the set of local $f$ is the generator of a strongly continuous semigroup on $C(\Omega[\mathbb{Z}])$, corresponding to the dynamics on $[-L, \infty)$, and that, for local $f$

$$e^{t\mathcal{L}_L} f = \sum_{m \geq 0} \frac{t^m}{m!} \mathcal{L}_L^m f. \quad (46)$$

We will need to consider also nonlocal functions that are not in $C(\Omega[\mathbb{Z}])$, but rather only in, say, $L^q(\Omega[\mathbb{Z}], \text{Ber}_p)$ for some $q > 0$. A prime example is the limit

$$\mathcal{L} f := \lim_L \mathcal{L}_L f, \quad \text{for local } f$$

We call $\mathcal{L}$ the formal generator of the dynamics on $\Omega[\mathbb{Z}]$. An important property, that can be checked by explicit computation, is that

$$\text{Ber}_p(\mathcal{L} f) = 0. \quad (47)$$

The following is our first technical result:

**Lemma 6.1.** Fix $p \in (0,1)$. There is a $\epsilon > 0$ and $\kappa > 0$ such that
1. For $f = \sum_{\ell} e^{\mathcal{L}_{[\ell,0]}} \xi_\eta^{\ell}$ with $\eta = \pm 1$,
\[
\sup_{0 \leq \ell \leq L} \sup \text{Ber}_p(e^{\mathcal{L}_L} f) \leq C(p). \tag{48}
\]

2. For local $f$,
\[
\lim_{L} \text{Ber}_p(e^{\mathcal{L}_L} f) = \text{Ber}_p(f)
\]

Proof. We start with 1). Let us first set
\[
\text{Proof.}
\]
where \(\bar{w} = (w_0, w_1, \ldots, w_m)\), we define the intervals
\[
I_j := \cup_{w_i; i < j} \text{Supp}(w_i), \quad \text{with } \text{Supp}(w_i) := [x_i, y_i]
\]
and we note that the support of \(\mathcal{L}_{w_j} \cdots \mathcal{L}_{w_1} \xi_\eta^{\ell,0} \) is contained in \(I_j\). Moreover,
\[
\mathcal{L}_{w_m} \cdots \mathcal{L}_{w_1} \xi_\eta^{\ell,0} = M(w) \chi_{\tilde{\sigma}}^{\tilde{\sigma}}, \tag{51}
\]
for some \(\tilde{\sigma} = \tilde{\sigma}(w) \in \{1, -1\}^I_m\) and numbers \(|M(w)| \leq 1\), i.e. the expression is either proportional to the indicator of a single configuration or zero. This is checked iteratively by using
\[
P_{x}^{\tilde{\sigma}} F_y (F_x F_y - 1) \chi_\tilde{\sigma} = \begin{cases} 
\chi_\tilde{\sigma}^{x,y} & \text{if } (\tilde{\sigma}_x, \tilde{\sigma}_y) = - (\eta_x, \eta_y) \\
- \chi_\tilde{\sigma}^{x,y} & \text{if } (\tilde{\sigma}_x, \tilde{\sigma}_y) = (\eta_x, \eta_y) \\
0 & \text{otherwise}
\end{cases}
\]
where \(\tilde{\sigma}^{x,y}\) is the configuration \(\tilde{\sigma}\) flipped at \(x\) and \(y\), and the fact that the projectors of the type \(P_{\tilde{\sigma}}\) act by multiplication with indicators. Furthermore, if \(M(w) \neq 0\), then for any \(0 < j \leq m\), one of the two following conditions (mutually exclusive) holds

1): \(v\)-case \(\text{Supp}(w_j) \cap I_j \neq \emptyset\) but \(\text{Supp}(w_j) \not\subset I_j\). We will call \(v_j\) the unique site in \(\{x_j, y_j\}\) that is not in \(I_j\) and we note that choosing \(v_j\) and \(\eta_j\) automatically fixes the other element of \(\{x_j, y_j\}\).

2): \(z\)-case \(\text{Supp}(w_j) \subset I_j\). In that case the choice of \(x_j, \eta_j\) uniquely fixes \(y_j\). To avoid confusion, we rename \(z_j := x_j\) in that case.

Let us now estimate \(\text{Ber}_p(e^{\mathcal{L}_f})\). Note first that, for \(p \in (0, 1)\), \(\sup_{\tilde{\sigma}} \text{Ber}_p(\chi_\tilde{\sigma}^{\tilde{\sigma}}) \leq e^{-c(p)|S|}\)
and therefore (51) yields

$$|\text{Ber}_p(e^L f)| \leq \sum_m \frac{e^m}{m!} \sum_w |M(w)| \text{Ber}_p(\tilde{\xi}(w)) \leq \sum_m \frac{e^m}{m!} \sum_w \chi(M(w) \neq 0) |e^{-c|I_m|}| \quad (52)$$

As argued above, the sum over $w$ such that $M(w) \neq 0$ can be viewed as a sum over $m+1$ sites and $m+1$ signatures, and the sites are divided into two types $v$ and $z$. Let us now call $m_v, m_z$, the number of $v,z$-sites, respectively, with $m_v + m_z = m+1$, and $m_v \geq 1$ as we say (by convention) that $w_0$ gives rise to a $v$-site. Note also that $I_m$ is in fact only determined by the $v$-coordinates and $I_m = I_{m_v}$. The number of ways of choosing $m_v$ sites of type $v$ from $m+1$ sites is bounded by $\frac{m!}{m_v! m_z!}$. We get hence

$$|\text{Ber}_p(e^L f)| \leq \sum_{m_v \geq 1} \sum_{\underline{z}} \frac{(2e)^{m_v-1}}{m_v!} e^{-c|I_{m_v}|} e^{2\epsilon|I_{m_v}|} \quad (53)$$

where $\underline{z}, \underline{z}, \underline{w}$ stand for the arrays of the $v,z$-sites and signatures. The number of possibilities for any of the $z$ coordinates is bounded by $I_{m_v}$ and the number of possibilities for $\underline{w}$ is bounded by $2^m$, so we get

$$|\text{Ber}_p(e^L f)| \leq \sum_{m_v \geq 1} \sum_{\underline{z}} \frac{(2e)^{m_v-1}}{m_v!} e^{-c|I_{m_v}|} e^{2\epsilon|I_{m_v}|} \quad (54)$$

The sum $\sum_{\underline{z}}$ is now a sum over a sequence of $m_v$ sites such the $j$-th site is in the exterior of $I_j$, the set spanned by the sites up to $j-1$. The expression (54) is bounded by a universal constant $C$ for all sufficiently small $\epsilon$. Reinstating $\kappa$ in the argument and bounding $e^{\kappa \ell} \leq e^{\kappa |I_{m_v}|}$, the estimate is still valid provided $2\epsilon + \kappa - c < 0$.

To get 2), we first note that the expansion in 1) can be repeated for local $f$ by writing $f$ as a finite sum of indicators. As the power expansion of $e^{L_l} f$ is summable, uniformly in $L$, we invoke dominated convergence to take the limit $L \to \infty$ term-by term and to represent for each $m$

$$\lim_{L \to \infty} \lim_{L'} \lim_{L'} \text{Ber}_p(L_L L_{L'} f) = \lim_{L \to \infty} \text{Ber}_p(L L_{L'} f) = 0,$$

where the last equality is by (47), since $L_{L'}^{m-1} f$ is local. $\square$

### 6.2 Bound on the speed of information propagation for time $\epsilon$

We now derive the estimate on the time-integrated currents.

Given the process $\sigma^L_t$, let $J^L_{L'}(t)$ be the total number of particles (of any sign) that have crossed from $(-\infty, x)$ to $[x, \infty)$ in the time $[0, t)$. In terms of the underlying Poisson
point processes,
\[
J^L_x(t) = \sum_{\eta} \sum_{y=-L}^{x-1} \int_0^t \chi_{[y,x-1]}(\sigma_{s-}^L) dN_{y,\eta}(s)
\]  
(55)
if \(x \geq 1 - L\) (and this current is 0 otherwise).

**Proposition 6.2.** For all \(\epsilon\) sufficiently small,
\[
\sup_{t \leq \epsilon} \mathbb{E}_{\text{Ber}_p}[(J^L_x(t))^k] \leq C(p,k)\epsilon, \quad \text{for any } k \in \mathbb{N}.
\]  
(56)

**Proof.** Since all estimates are uniform in \(L\), we drop it from the notation. Let us abbreviate the integrand to \(g_{y,\eta}(s) := \chi_{[y,x-1]}(\sigma_{s-}^L)\). Then, by Ito calculus, we get
\[
\mathbb{E}_{\text{Ber}_p}[(J_x(t))^k] = \sum_{\eta,y} \lambda_\eta \int_0^t ds \sum_{l=1}^k \mathbb{E}_{\text{Ber}_p} \left[ g_{y,\eta}(s)(J_x(t-s))^{k-l} \right]
\]
and hence we get the upper bound (using that \(g_{y,\eta} = g_{y,\eta}\) and \(\sum_{y,\eta} g_{y,\eta} = l_x\) and that \(J_x^k(t)\) is non-decreasing in \(t\) and \(k\).
\[
\mathbb{E}_{\text{Ber}_p}[(J_x(t))^k] \leq \int_0^t ds \left( \mathbb{E}_{\text{Ber}_p}(l_x(s)) \right)^{1/2} k \left( \mathbb{E}_{\text{Ber}_p}[(J_x(t))^{2k}] \right)^{1/2}
\]
From Lemma 6.1, we know that \(\sup_{s \in [0,\epsilon]} \mathbb{E}_{\text{Ber}_p}(l_x(s)) < C\) and hence the claim of the proposition will follow if we prove
\[
\sup_{s \in [0,\epsilon]} \mathbb{E}_{\text{Ber}_p} \left[ (J_x(t))^k \right] \leq C(k).
\]  
(57)
To obtain this, we remark that Lemma 4.7 holds for the cutoff-dynamics as well (uniformly in the cutoff) with the interval \(I := [0,\epsilon]\) (the cutoff in the dynamics should not be confused with the cutoff \(L\) in Lemma 4.7 in the definition of the current) Indeed, in the proof of Lemma 4.7 we used stationarity only to simplify notation and we used the properties of the Bernoulli measure to bound (45), but this bound follows for the cutoff dynamics from Lemma 6.1. From Lemma 4.7, we then get the desired bound (57) and the proof is complete. \(\square\)

Recall the simultaneous coupling of the one-sided processes \((\sigma_t^L)_{L \geq 0}\) from § 2.1. Here is the key estimate that we will use in this section:

**Proposition 6.3.** For \(\epsilon\) sufficiently small, we have that for any \(k > 0\) and any \(L' > L\),
\[
\mathbb{P}_{\text{Ber}_p} \left( \sigma_t^{L'}(y) = \sigma_t^L(y) \text{ for all } y \geq 0, \text{ and } t \in [0,\epsilon] \right) = 1 - C(p,k)L^{-k}.
\]  
(58)
Proof. Fix $L, L'$ and consider the set of discrepancies determined by $\sigma := (\sigma^L, \sigma^{L'})$ at time $\epsilon$, $D_\epsilon = D_{\sigma_\epsilon}$. The claim amounts to proving that

$$P_{\text{Ber}_p}(\sup D_\epsilon > 0) \leq C(p, k) L^{-k}.$$ 

Let us first partition the interval $[-L, 0]$ into $n$ intervals $(I_j)_{j=1}^n$ (ordered from left to right, and $n$ even) such that their lengths are

$$L^{1/4} \leq |I_j| \leq 2L^{1/4}. $$

Under these constraints, $n = O(L^{3/4})$.

Let us first observe that

$$\{\sup D_\epsilon > 0\} \subset E_1 \cup E_2 \cup E_3$$

with

1. $E_1 = \{\sum_{\eta=\pm1} J^\eta_{-L}(\epsilon) \geq L^{1/8}\}$ with the currents $J^\eta_{-L}$ as defined above.
2. $E_2 = \cup_j E_{2,j}$ and

$$E_{2,j} = \{\exists t \in [0, \epsilon]: \sigma^{(1)}_t(x) \text{ has the same sign for all } x \in I_j \setminus S \text{ with } |S| \leq 2L^{1/8}\}$$

3. $E_3$ is the event that, during $[0, \epsilon)$, there has been a sequence of $n/2$ arrivals (we label them by the index $i$) such that arrival $i$ occurs before arrival $i+1$, and arrival $i$ occurs on one of the sites in $I_{2i-1} \cup I_{2i}$.

The inclusion (59) is explained as follows. First, if $E_1$ does not happen, then in particular it follows that the number of discrepancies present in $[-L, 0]$ at any time in $[0, \epsilon]$ is bounded by $L^{1/8}$. If neither $E_1$ nor $E_2$ happens, then we conclude that at each time in $[0, \epsilon)$ and in each box $I_j$, there are many particles of both signs present, both for $\sigma^L$ and $\sigma^{L'}$ (as they differ only on the discrepancy set $D$). We then see that for the propagation of the front of the discrepancy set (i.e. $\sup D$) to occur, we need the arrivals specified by event $E_3$.

It remains to show $P_{\text{Ber}_p}(E_i) \leq C(p, k) L^{-k}$ for $i = 1, 2, 3$. For $E_1$, this follows immediately by Proposition 6.2 and Markov’s inequality. For $E_3$, this follows by a simple consideration on the Poisson arrivals. Indeed, the number of needed arrivals is $n/2 = O(L^{3/4})$ and each arrival needs one of $O(L^{1/4})$ clocks to ring, so it occurs with a rate $O(L^{1/4})$. We see that $P_{\text{Ber}_p}(E_3) \leq e^{-c(p)L^{1/4}}$ follows by large deviation estimates.

We are left with $E_2$ which is the most involved estimate.
Let us first consider a single box $I_j$. We claim
\[ P_{\text{Ber}_p}(E_{2,j}) \leq C(p, k)L^{-k} \] (60)

Let
\[ m_{t,I_j} = \frac{1}{|I_j|} \sum_{x \in I_j} \sigma_t(x). \]

The idea leading to (60) is that, for $1 - |m|$ small enough, the event \{ $m_{t,I_j} \approx m$ \} is satisfied only if either the initial condition $|m_{0,I_j}|$ is atypically close to 1 or the integrated current across the left or right boundary of $I_j$ has to be at least $cL^{1/4}$. The former event has a probability of which is exponentially small in $L^{1/4}$ under a product Bernoulli initial condition (with $p \notin \{0, 1\}$) while the probability of a large integrated current across any edge $(x, x + 1)$ is estimated as
\[ P_{\text{Ber}_p} \left( \max_{y = \pm 1} J^y_x(\varepsilon) \geq M \right) \leq C(p, k)M^{-k} \] (61)

by Proposition 6.2 and the Markov inequality. Hence the lemma follows. To finish
\[ P_{\text{Ber}_p}(E_2) \leq \sum_j P_{\text{Ber}_p}(E_{2,j}) \leq C(p, k)L^{3/4-k}. \] (62)

and this concludes the proof since $k$ was arbitrary.

6.3 Proof of Proposition 2.1

We are now ready to construct the dynamics (for all time) of the Toom interface on $\mathbb{Z}$ when started from product Bernoulli initial conditions. We first construct the dynamics up to a small time $\varepsilon > 0$. Then, using stationarity of the dynamics throughout the time interval $[0, \varepsilon)$, we iterate the construction in the interval $[\varepsilon/2, 3\varepsilon/2)$. Proceeding by induction, we construct the dynamics for all time.

6.3.1 Local Existence

Note first that Proposition 6.3 holds if we replace the constraint $y \geq 0$ by $|y| \leq \ell$ for $\ell < \infty$, at the cost of replacing the constant $C(p)$ by $C(p, \ell)$. This is because the sequence of process $\sigma^L$ restricted to $[-x, \infty)$ may be viewed as a spatial shift the sequence of processes restricted to $[0, \infty)$. Thus, if
\[ E_{L,\ell} := \{ \sigma^{L+1}(y) = \sigma^L(y) \text{ for all } |y| \leq \ell \text{ and } t \in [0, \varepsilon] \}, \]
Proposition 6.3 applies and yields
\[ \mathbb{P}_{\text{Ber}_p}(E_{L,\ell}^c) \leq CL^{-2}. \]

Hence, the Borel-Cantelli Lemma implies
\[ \mathbb{P}_{\text{Ber}_p}(E_{L,\ell}^c \text{ i.o.}) = 0. \]

We conclude the complementary event, \( \{E_{L,\ell} \text{ eventually}\} \), has probability 1. Therefore, the sequence of path-valued random variables \((\sigma^L_x(x))_{s \leq \epsilon p, |x| \leq \ell} \) restricted to \([-\ell, \ell]\) becomes eventually constant as \( L \to \infty \) \( \mathbb{P}_{\text{Ber}_p} \) almost surely. This holds for any \( \ell \) and hence, after modifying by an event of measure 0, also for all \( \ell \) simultaneously. This means exactly that the sequence \( \sigma^L_{\leq \epsilon} \) converges almost surely in \( D[0, \epsilon) \). We call the limit point \( \sigma_{\leq \epsilon} \).

### 6.3.2 SDE for short times

We now check that the constructed process \( \sigma_{\leq \epsilon} \) satisfies the SDE (3), i.e. we check the validity of
\[ \mathbb{E}_{\text{Ber}_p}\left| \sigma_{t_2}(x) - \sigma_{t_1}(x) + I_{x,t_1,t_2}^L(\sigma) \right| = 0 \]
for \( 0 \leq t_1 < t_2 \leq \epsilon \), with \( I_{x,t_1,t_2}(\sigma) := I_{x,t_1,t_2,L=\infty}(\sigma) \) defined by
\[ I_{x,t_1,t_2,L}(\sigma) = \sum_{\eta=\pm 1} (2\eta) \int_{t_1}^{t_2} \sum_{-L<y<x} \chi_{[y,x-1]}(\sigma_{t_-}) \chi_{x}(\sigma_{t_-}) dN_{y,\eta}(t) \]

In what follows we simply write \( I^L \) and \( I := I_{L=\infty} \). We may bound
\[
|\sigma_{t_2}(x) - \sigma_{t_1}(x) - I(\sigma)| \leq \sum_{j=1,2} |\sigma_{t_j}(x) - \sigma^L_{t_j}(x)| + |\sigma^L_{t_2}(x) - \sigma^L_{t_1}(x) - I_L(\sigma^L)| \label{eq:63}
\]
\[
+ |I(\sigma) - I_L(\sigma)| + |I_L(\sigma) - I_L(\sigma^L)|.
\]

As \( L \to \infty \), the first term (for both values of \( j = 1, 2 \)) on the RHS vanishes in probability by the convergence of the processes. Next we observe that the second term on the RHS is 0 a.s. since \( \sigma^L \) satisfies the associated SDE (to see this simply, recall that \( \sigma^L \) restricted to \((-L,j)\) is a finite-state-space Markov process for all \( j > -L \). The third term depends only on the stationary process \( \sigma \). Using this it is straightforward to show that \( \mathbb{E}(|I(\sigma) - I_L(\sigma)|) \to 0 \) as \( L \to \infty \).

Let us finish our verification by addressing the fourth term via a dominated convergence argument. Writing it explicitly as a sum over sites \( y \),
\[
|I_L(\sigma) - I_L(\sigma^L)| \leq \int dt \sum_{L<y<x} \mathbb{E}_{\text{Ber}_p}|f_y(\sigma^L_{t_-}) - f_y(\sigma_{t_-})| \label{eq:64}
\]
with
\[ f_y(\sigma) = \sum_{y < x, \eta = \pm 1} 2\lambda_\eta \chi_{[y, x-1]}(\sigma) \chi_{x}^{-\eta}(\sigma) \]

Note that we have the bound
\[ \int dt \sum_{L < y < x} \mathbb{E}_{\text{Ber}_p}(f_y(\sigma^L_t)) + \mathbb{E}_{\text{Ber}_p}(f_y(\sigma_t)) < C \] (65)

by Lemma 6.1. This bound holds uniformly in \( L \) and for \( 0 \leq t_1 < t_2 \leq \epsilon \) with \( \epsilon \) taken small enough independent of \( L \). On the other hand, for any \( y \) and \( t \), we have
\[ \mathbb{E}_{\text{Ber}_p}|f_y(\sigma^L_t) - f_y(\sigma_t)| \to 0, \quad L \to \infty \]

by the convergence of the processes. Therefore as \( L \to \infty \), (64) vanishes, as desired.

### 6.3.3 Global Existence

By Lemma 6.1 2), we know that the dynamics is stationary for \( t \in [0, \epsilon) \). In other words, Proposition 2.1 has been fully verified for \( t < \epsilon \). An alternative description of the result up to this point is as follows. There is a \( \mathbb{P}_{\text{Ber}_p} \)-a.s. defined measurable function \( G : (\sigma_0, \omega) \to G(\sigma_0, \omega) \) with values in \( D[0, \epsilon) \). Using part of the Fubini – Tonelli theorem, we obtain, for \( \text{Ber}_p \)-a.e. \( \sigma_0 \), a measurable function \( G(\sigma_0, \cdot) \). Choose a \( \epsilon/2 < \tau < \epsilon \), then \( \sigma_\tau \) is \( \mathbb{P}_{\text{Ber}_p} \)-distributed and hence \( \mathbb{P}_{\text{Ber}_p} \)-as, the concatenation of the paths
\[ G(\sigma_0, \omega)[0, \tau) \text{ and } G(\sigma_\tau, \cdot)[0, \tau), \quad \text{with } \sigma_\tau = (G(\sigma_0, \omega))_\tau \]

is a measurable function on \( \Sigma \) with values in \( D[0, 2\tau) \). It is straightforward to check that it satisfies all the required properties and that the construction can be iterated.

### 6.4 Existence of Invariant Measures for Couplings

The setup for this subsection is as follows. Let the measures \( \mu_1, \mu_2 \) on \( \Omega[\mathbb{Z}] \) be invariant for the dynamics on \( \mathbb{Z} \) in the sense of Definition 2.4 and denote by \( \sigma^1_t, \sigma^2_t \) the stationary processes. Given a measure \( \nu \) on \( \Omega[\mathbb{Z}] \times \Omega[\mathbb{Z}] \) with marginals \( \mu_1, \mu_2 \), let \( \mathbb{P}_\nu \) be a measure on the space \( D_{\Omega[\mathbb{Z}] \times \Omega[\mathbb{Z}]}(\mathbb{R}^+) \) with path marginals \( \sigma^1_t, \sigma^2_t \) such that the SDEs they satisfy (see (3)) are driven by the same Poisson point processes.

**Lemma 6.4.** There is a \( \nu_\infty \) on \( \Omega[\mathbb{Z}] \times \Omega[\mathbb{Z}] \) such that the coupling process \((\sigma^1_t, \sigma^2_t)\) started from \( \nu_\infty \) is stationary, i.e. \((\sigma^1_t, \sigma^2_t)\) is \( \nu_\infty \)-distributed for all \( t \).

To prove this we need the following preliminary result:
Lemma 6.5. There is a time $t_*$ such that for any small $\epsilon > 0$ and local $f$ there is an $L$ such that
\[ \sup_{t \leq t_*} P_{\nu}[f(\sigma^L_t) \neq f(\sigma_t)] \leq \epsilon \] (66)
for any $\nu$ with marginals $\mu_1, \mu_2$.

Proof. We have
\[ P_{\nu}[f(\sigma^L_t) \neq f(\sigma_t)] \leq P_{\nu}[f(\sigma^{1,L}_t) \neq f(\sigma^1_t)] + P_{\nu}[f(\sigma^{2,L}_t) \neq f(\sigma^2_t)] \]
\[ = P_{\mu_1}[f(\sigma^{1,L}_t) \neq f(\sigma^1_t)] + P_{\mu_2}[f(\sigma^{2,L}_t) \neq f(\sigma^2_t)]. \] (67)

Item 2) of Definition 2.4 assures that, if $t \leq \min(t_*(\mu_1), t_*(\mu_2))$, we can find $L$ such that both terms (67) are smaller than $\epsilon/2$. Indeed, since $f$ takes discrete values, $f(\sigma^{1,L}_t) - f(\sigma^1_t) \implies 0$ implies that the difference becomes 0 eventually. \( \Box \)

Proof of Lemma 6.4. Let $\nu$ be given in the first paragraph of this subsection. To arrive at a stationary measure, we attempt to mimic the standard proof for Feller processes. Consider the time averages
\[ \nu_n := \frac{1}{T_n} \int_0^{T_n} dt \nu_t, \quad \text{with} \quad \nu_0 := \mu_1 \times \mu_2. \] (68)

The sequence $T_n$ is increasing and chosen such that the $\nu_n$ converge weakly, and we call the limit point $\nu_\infty$. The tricky point is to establish that $\nu_\infty$ is invariant.

We wish to establish
\[ E_{\nu_\infty}[f(\sigma_t)] = \nu_\infty(f). \] (69)

Telescoping the difference of the two sides of this equality, we have
\[ E_{\nu_\infty}[f(\sigma_t)] - \nu_\infty(f) = (E_{\nu_\infty}[f(\sigma_t)] - E_{\nu_\infty}[f(\sigma^L_t)]) + (E_{\nu_\infty}[f(\sigma^L_t)] - E_{\nu_n}[f(\sigma^L_t)]) + (E_{\nu_n}[f(\sigma^L_t)] - E_{\nu_n}[f(\sigma_0)]) + (E_{\nu_n}[f(\sigma_0)] - E_{\nu_\infty}[f(\sigma_0)]). \] (70)

Note first of all that since Lemma 6.5 does not depend on $n$, the first and third differences can be made arbitrarily small by choosing $L$ sufficiently large. The fifth difference tends to 0 as $n$ tends to $\infty$ since $\nu_n - \nu_\infty \to 0$ weakly. To handle the second difference, note that if $L < \infty$ is fixed, then all processes involved are Feller, i.e. $e^{t\hat{L}}f \in C(\Omega[Z])$ with $\hat{L}$ the cut-off generator of the coupling process. Therefore the weak convergence of $\nu_n$ to $\nu_\infty$ implies
\[ E_{\nu_n}(f(\sigma^L_t)) - E_{\nu_\infty}(f(\sigma^L_t)) = (\nu_n - \nu_\infty)(e^{t\hat{L}}f) \to 0, \quad \text{as} \quad n \to \infty, \] (71)
Finally, inspection of (68) implies the fourth difference tends to 0 as $n$ tends to $\infty$. Since
the choice \( L \) is arbitrary on the RHS of (70), (69) follows.

\[\]  

6.5 Formal generators and derivatives

In this section, we show that our processes, even though non-Feller, have a formal generator that is related to derivatives of expectation values. Consider a process \( \sigma_t \) started from \( \mu \) that satisfies the SDE, in the sense of (3). Recall the formal generator \( L \) introduced in § 6.1

\[ L = \sum_{x<y} (\lambda_+ P^+_{x,y-1} P^-_{y-1} + \lambda_- P^-_{x,y-1} P^+_{y-1})(F_x F_y - 1) \]

Lemma 6.6. Assume}

\[ \sup_{t \leq t'} E_\mu(l_x(\sigma_t)) < \infty, \quad \text{for all } x \in \mathbb{Z}, t' \geq 0. \]  

Let \( f \) be a local function. Then \( t \mapsto E_\mu[f(\sigma_t)] \) is differentiable and

\[ \frac{d}{dt} E_\mu[f(\sigma_t)] = E_\mu[L f(\sigma_t)], \quad \text{and } |E_\mu[L f(\sigma_t)]| \leq C(f) \]

Proof. To get boundedness, note that contributions from the sum in (72) to \( L f \) vanish unless \( x \in \text{Supp} \ f \) or \( y \in \text{Supp} \ f \). Bounding these contributions leads to

\[ |E_\mu(L f(\sigma_t))| \leq C(f) \sum_{x \in \text{Supp} \ f} E_\mu(l_x(\sigma_t)) \]

which is finite by assumption. To get the differentiability, let us first consider \( f(\sigma) = \sigma(x) \). Then we write

\[ E_\mu(\sigma_{t_2}(x) - \sigma_{t_1}(x)) = E_\mu(I(\sigma)) = \sum_{y \leq x} \int_{t_1}^{t_2} dt E_\mu(f_y(\sigma_t)), \]

with \( I \) and \( f_y \) as in § 6.3.2. Using again the condition \( \sup_{t \leq t'} E_\mu(l_x(\sigma_t)) < \infty \), we get summability and integrability of the right hand side as in § 6.3.2. The existence of \( \lim_{t \to 0}(1/t) E_\mu(\sigma_{t_2}(x) - \sigma_{t_1}(x)) \) follows. General local functions follow by a simple generalization of this argument and the identification of the derivative with \( E_\mu(L f) \) is by inspection.

Now we consider the coupled process \( (\sigma^1_t, \sigma^2_t) \) on \( \Sigma' = \Omega[Z] \times \Omega[Z] \times \Omega \) started from a probability measure \( \nu \), such that \( \sigma^1_0 \) is \( \mu \)-distributed and \( \sigma^2_0 \) is \( \text{Ber}_p \)-distributed. We assume that the \( \sigma^1 \) process satisfies the SDE, in the sense of (3) and \( \sigma^2 \) of course satisfies it as well. The formal generator of the joint process is (cfr. § 6.1), with now \( \bar{P}^{\nu,\eta} \) the
and put $\xi$ case being a straightforward generalization. Let us abbreviate $\Delta L,j$ also $\Delta t \rightarrow f$ differentiability, we again consider the simplest functions $t$ of both marginals, i.e. sup $Boundedness follows as in the proof of Lemma 6.6, now using the integrability $\nu t \rightarrow \hat{\nu} \rightarrow t \rightarrow f$ L,$j$ and we express $\hat{E} t \rightarrow \hat{\nu} \rightarrow f$ 1 the conclusion that $\lim sup t \rightarrow \hat{\nu} \rightarrow f$ \hat{E} t \rightarrow \hat{\nu} \rightarrow f$ 1, which we show now Since $|\Delta t \rightarrow 0$ exists and converges, as $L \rightarrow \infty$, to $\hat{E} t \rightarrow \hat{\nu} \rightarrow f$ $\hat{\nu} \rightarrow f$. Hence it suffices to show that for the first and third terms we can obtain (for concreteness, consider the third term, as the first one is treated in the same way) $\lim sup t \rightarrow \hat{\nu} \rightarrow f$ \hat{E} t \rightarrow \hat{\nu} \rightarrow f$ 1$| \leq C(f)$ (75) with $\hat{\nu}$ the formal generator of the coupled process.

Proof. Boundedness follows as in the proof of Lemma 6.6, now using the integrability of both marginals, i.e. $\sup_{t' \leq t} \hat{E} \nu(l_x(l_t(\sigma_{t'}))) < \infty$ and $\sup_{t' \leq t} \hat{E} \nu(\mu_{t'}(l_x(\sigma_{t'}))) < \infty$. For the differentiability, we again consider the simplest functions $f(\sigma) = \sigma^1(x)\sigma^2(x)$, the general case being a straightforward generalization. Let us abbreviate $\Delta t \rightarrow f$ L,$j$ and also $\Delta t \rightarrow f$ L,$j$ = $\sigma_{t'}^1(x) - \sigma_{t'}^1(x)$ and $\Delta t \rightarrow f$ L,$j$ = $\sigma_{t'}^1(x) - \sigma_{t'}^1(x)$ with $L$ referring to the cutoff processes used also in § 6.3.2 and put $\xi_{t \rightarrow f}$ L,$j$ = $\Delta t \rightarrow f$ L,$j$ - $\Delta t \rightarrow f$ L,$j$. We now split

\[
\frac{1}{t} \hat{E} \nu[\Delta t \rightarrow f \Delta t \rightarrow f] = \frac{1}{t} \hat{E} \nu[\xi_{t \rightarrow f}^L \Delta t \rightarrow f] + \frac{1}{t} \hat{E} \nu[\Delta t \rightarrow f^L \Delta t \rightarrow f] + \frac{1}{t} \hat{E} \nu[\Delta t \rightarrow f^L \xi_{t \rightarrow f}^L]
\]

The second term on the right hand side concerns a Feller process and it is clear that the limit $t \rightarrow 0$ exists and converges, as $L \rightarrow \infty$, to $\hat{E} \nu(\hat{\nu}f)$. Hence it suffices to show that for the first and third terms we can obtain (for concreteness, consider the third term, as the first one is treated in the same way) $\lim sup_{t \rightarrow 0} \frac{1}{t} \hat{E} \nu[\xi_{t \rightarrow f}^L L,1] = 0$, which we show now Since $|\Delta t \rightarrow f|^1 \leq 4$, we can bound

\[
\frac{1}{t} \hat{E} \nu[\Delta t \rightarrow f^L \xi_{t \rightarrow f}^L L,1] \leq \frac{4}{t} \hat{E} \nu[||\xi_{t \rightarrow f}^L||].
\]

and we express $\hat{E} \nu[||\xi_{t \rightarrow f}^L||]$ in terms of $\hat{E} \nu$ of the stochastic integrals $I, I^L$ as in § 6.3.2 and the proof of Lemma Lemma 6.6. Using the integrability assumption on $\mu$, this leads to the conclusion that $\lim sup_{t \rightarrow 0} \frac{1}{t} \hat{E} \nu[||\xi_{t \rightarrow f}^L||] = 0$, which finishes the proof. \qed
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