Decaying neutrinos: The long way to isotropy

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We investigate a scenario in which neutrinos are coupled to a pseudoscalar degree of freedom $\varphi$ and where decays $\nu_1 \rightarrow \nu_2 + \varphi$ and inverse decays are the responsible mechanism for obtaining equilibrium. In this context we discuss the implication of the invisible neutrino decay on the neutrino-pseudoscalar coupling constant and the neutrino lifetime. Assuming the realistic scenario of a thermal background of neutrinos and pseudoscalar we update the bound on the (off-diagonal) neutrino-pseudoscalar coupling constant to $g < 2.6 \times 10^{-13}$ and the bound on the neutrino lifetime to $\tau < 1 \times 10^{13}$ s. Furthermore we confirm analytically that kinetic equilibrium is delayed by two Lorentz $\gamma$–factors – one for time dilation of the (decaying) neutrino lifetime and one from the opening angle. We have also confirmed this behavior numerically.

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I. INTRODUCTION

The possibility for neutrino interactions beyond the standard model has been studied in many contexts over the years. One particularly simple possibility is that neutrinos couple to a new pseudoscalar degree of freedom, as is, for example, the case in Majoron models - see the following references for previous discussions about the dynamics of the strong neutrino-pseudoscalar coupling and its astrophysical implications [1–10].

Astrophysics provides fairly stringent constraints on such couplings. For example SN1987A provides a bound on the dimensionless coupling constant of order $10^{-7} \lesssim g \lesssim 10^{-5}$ [11–14] by requiring that the neutrino signal should not be significantly shortened.

In the same way there are two cosmological bounds on $g$. First, the value of $g$ should not be large enough that pseudoscalars are fully thermalized before big bang nucleosynthesis. This leads to $g \lesssim 10^{-5}$. Second, a significant value of $g$ will make neutrinos self-interacting in the late universe and prevent neutrino free-streaming. This possibility has been discussed a number of times in the literature (see [15–19]).

The effect on cosmological observables such as the cosmic microwave background (CMB) spectrum were studied in [15–17, 19–23]); particularly it was found that although models with no neutrino free-streaming can mimic the matter power spectrum of $\Lambda$CDM models, they produce a distinct signature in the CMB spectrum which is much harder to reproduce. The feature arises because neutrinos act as a source term for photon perturbations. If there is no free-streaming, the source term is stronger and consequently the CMB anisotropy is increased for all scales inside the particle horizon at recombination. On the other hand, there is no effect on larger scales.

This distinct signature has been used to constrain models without neutrino free-streaming and in [20, 23] it was used to constrain the corresponding neutrino-pseudoscalar coupling parameters.

However, for decays and inverse decays the interaction was treated in a somewhat simplified manner in the sense that the momentum equilibration rate was assumed to be roughly $\Gamma^* \sim 1/(\gamma^2 \tau)$, where $\tau$ is the rest-frame lifetime and $\gamma \sim E_\nu/m_\nu$ is the Lorentz boost factor.

In this paper we wish to check this assumption in an explicit way with a realistic setup. The possible departure from the simple relation $\Gamma^* \sim 1/(\gamma^2 \tau)$ is something which is highly relevant for parameter estimations - such as placing bounds on the neutrino-pseudoscalar coupling and hence also for constraining the neutrino lifetime. Furthermore it is something which needs to be taken into account in numerical studies in which we allow for nonstandard neutrino interactions. One particular area where detailed knowledge of the interaction would be useful is the search for the cosmic neutrino background [24–27].

One comment is in order here: The motivation for looking at decays and inverse decays rather than various scattering processes involving neutrinos and pseudoscalars ($\nu\nu \rightarrow \varphi\varphi, \varphi\varphi \rightarrow \nu\nu, \nu\varphi \rightarrow \nu\varphi$) is that the probabilities of these

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scattering processes are proportional to $g^4$, where $g$ is the neutrino-pseudoscalar coupling constant. The probability of the decay $\nu_1 \rightarrow \nu_2 + \varphi$, on the other hand, is proportional to $g^2$. Consequently, at small values of $g$ the decay actually dominates over the scattering processes and allows us to put severe constraints on $g$.

This paper is organized as follows: In Sec. II we look at the setup with a gas consisting of two neutrino species and a pseudoscalar – a gas that only has decays and inverse decays to obtain equilibrium. We argue for the $1/\gamma$ in the decay rate. In Sec. III we look at an initial situation of a standing wave of the heavy neutrinos and no other particles. In Sec. IV we introduce thermal distributions of the light neutrino and of pseudoscalars into a thermal background while keeping the initial conditions for the heavy neutrino. Furthermore we discuss the implication of the decay on the neutrino-pseudoscalar coupling and on the neutrino lifetime. We present numerical results in Sec. V. Finally we have a conclusion and an appendix concerning calculations for the numerical implementation of the system.

II. THERMALIZATION OF A GAS WITH ONLY DECAYS AND INVERSE DECAYS

Thermalization of a gas by decay and inverse decay is a nontrivial process because of phase space limitation. As long as one of the involved particles can interact with an external heat bath, it is in principle possible to thermalise the gas provided that the interaction rate is sufficiently fast. This is, for instance, the case with thermal leptogenesis in which the decay products are thermalized by SM gauge interactions. However, for the case studied here this is not true. The weak interactions are far too weak to maintain equilibrium at the eV temperatures considered here. In this case full thermal equilibrium can never be achieved.

The standard case usually studied, for example, in the case of thermal leptogenesis is a spatially homogeneous gas in which interactions drive the distribution toward thermal equilibrium (see e.g. [28, 29]). However, from the point of view of structure formation and more specifically free-streaming the important point is the rate of directional momentum transfer between species. For example, Thompson scattering is inefficient for maintaining energy equilibration between electrons and photons, but very efficient for exchanging momentum between the two species. This can be seen from the simple relations $|\Delta E_\gamma/E_\gamma| \sim E_\gamma/m_e$ and $|\Delta p/p| \sim 1$ in a single scattering event. Therefore Thompson scattering is very efficient for driving the acoustic baryon-photon oscillations prior to recombination. However, for a gas with only decays and inverse decays momentum transfer is even more inefficient than energy transfer. Roughly the energy transfer time scale is given by the decay rate $\Gamma = 1/(\gamma \tau)$, i.e. the usual Lorentz suppressed rest-frame decay rate. However, in the lab frame the decay products are emitted in a cone of opening angle $1/\gamma$ relative to the direction of momentum of the parent particle. Therefore, in a single decay the momentum direction is changed by only $|\Delta p/p| \sim 1/\gamma$. This finally means that the rate of momentum change in the gas is roughly $1/(\gamma \tau)$; i.e. for relativistic decays it is highly suppressed, and even suppressed relative to the energy exchange rate.

Let us begin with the Lagrangian for a generic pseudoscalar neutrino interaction

$$\mathcal{L} = -i \sum_{j,k} g_{jk} \bar{\nu}_j \gamma_5 \nu_k.$$  \hspace{1cm} (1)

We will consider only two neutrinos, one we consider to be heavy ($\nu_1$ [or just 1 for convenience] with mass $m_1$), a massless neutrino ($\nu_2$ [or just 2 for convenience]) and a massless pseudoscalar ($\varphi$). Thus we drop the index of $g (g \equiv g_{1,2})$ and the Lagrangian becomes

$$\mathcal{L} = -ig \varphi (\bar{\nu}_1 \gamma_5 \nu_2 + \bar{\nu}_2 \gamma_5 \nu_1).$$  \hspace{1cm} (2)

In the following section we derive the specific Boltzmann collision integrals relevant for decays and inverse decays in an inhomogeneous gas.

The variation of any overall quantity $Q$ can be calculated from the distribution functions:

$$\frac{\partial Q_{\text{tot}}}{\partial t} = \sum_i \int \frac{d^3p_1}{(2\pi)^32E_1} \frac{d^3p_2}{(2\pi)^32E_2} \frac{d^3p_\varphi}{(2\pi)^32E_\varphi} (2\pi)^4 \delta^4(p_1 - p_2 - p_\varphi)|M|^2$$

$$\left[f_2 f_\varphi (1 - f_1) - f_1 (1 - f_2)(1 + f_\varphi)\right] Q_i S_i$$  \hspace{1cm} (3)

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1 The coupling structure could in principle be derivative instead of pseudoscalar. However, this point makes no difference to the discussion here since we study only decays and inverse decays. One could also choose a scalar coupling - it would only lead to a very small difference, which, in fact, is removed completely in the approximation where the lighter neutrino is massless.
where \( S_1 = 1 \) and \( S_2 = S_\phi = -1 \).

**III. TOTAL TRANSVERSE MOMENTUM - WITH NO BACKGROUND**

We want to calculate the initial transverse momentum when we start with a standing wave of 1’s. The distribution functions are

\[
\begin{align*}
    f_1 &= n_1 \left( \delta^3(p_1^\mu - p_0^\mu) + \delta^3(p_1^\nu + p_0^\nu) \right) \\
    f_2 &= f_\phi = 0.
\end{align*}
\]

(4)

where \( p_0^\mu \) is the momentum of the standing wave (and \( E_0 \) will be the corresponding on-shell energy). Since the two terms initially contribute equally, we are free to change to one beam instead

\[
\begin{align*}
    f_1 &= n_1 \delta^3(p_1^\mu - p_0^\mu) \\
    f_2 &= f_\phi = 0.
\end{align*}
\]

(5)

This will give the same result.

We calculate the matrix element. From tracing and averaging over incoming and summing over outgoing spins and assuming the masses of the neutrinos \( m_1 = m, m_2 = 0 \):

\[
|M|^2 = 2g^2(p_1 \cdot p_2 - m_1 m_2) = g^2 m^2.
\]

(6)

So we find

\[
\frac{\partial Q_{tot}}{\partial t} \text{Volume} = - \sum_i \int \frac{d^3p_1}{(2\pi)^32E_1} \frac{d^3p_2}{(2\pi)^32E_2} \frac{d^3p_\phi}{(2\pi)^32E_\phi} (2\pi)^4\delta^4(p_1 - p_2 - p_\phi) g^2 m^2 n_1 \delta^3(p_1^\mu - p_0^\mu) Q_i S_i
\]

\[
= \frac{g^2 m^2 n_1}{8(2\pi)^3} \sum_i \int \frac{d^3p_1 d^3p_2 d^3p_\phi}{E_1 E_2 E_\phi} g^4(p_1 - p_2 - p_\phi) \delta^3(p_1^\mu - p_0^\mu) Q_i S_i.
\]

What we want to find is a measure of the transverse momentum created in the very beginning. Obviously, there is no momentum if we just sum over the transverse momentum vectors, so we sum the magnitudes of created transverse momenta instead. This means putting \(-Q_i S_i = |\vec{p}_i \times \hat{p}_0|\). Thus we have

\[
\frac{\partial |\vec{P}_{\perp}|}{\partial t} \text{Volume} = \frac{g^2 m^2 n_1}{8(2\pi)^3} \int \frac{d^3p_1 d^3p_2 d^3p_\phi}{E_1 E_2 E_\phi} g^4(p_1 - p_2 - p_\phi) \delta^3(p_1^\mu - p_0^\mu)
\]

\[
\left( |\vec{p}_1 \times \hat{p}_0| + |\vec{p}_2 \times \hat{p}_0| + |\vec{p}_\phi \times \hat{p}_0| \right).
\]

(7)

After integration we find

\[
\frac{\partial |\vec{P}_{\perp}|}{\partial t} \text{Volume} = \frac{g^2 m^2 n_1}{64(2\pi)^4 E_0}.
\]

Inserting \( g^2 m = \frac{16\pi}{\tau} \) where \( \tau \) is the 1-lifetime, and \( \frac{m}{E_0} = \frac{1}{\gamma} \) our final result is

\[
\frac{\partial |\vec{P}_{\perp}|}{\partial t} \text{Volume} = \frac{n_1 E_0}{8(2\pi)^3 \tau \gamma^2} \times \frac{1}{\tau \gamma^2},
\]

(8)

as expected.

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2 In the numerical calculation we do not change to one beam

3 Had one chosen a scalar interaction, the \( m_1 m_2 \) term would change sign. However, since we set \( m_2 = 0 \) anyway, it makes no difference under this approximation.
IV. A MORE GENERAL CASE

In case of a thermal background of neutrinos and pseudoscalars we cannot use the simple approach specified by Eqs. 4 and 5. Hence we consider all the distribution functions

\[ f_2 f_\varphi (1 - f_1) - f_1 (1 - f_2)(1 - f_\varphi) = f_2 f_\varphi - f_1 (1 + f_\varphi - f_2). \]  

(9)

This complicates things a bit. But if we again take only the initial time it is solvable. The term we are discussing is \( |f_2 f_\varphi - f_1 (1 + f_\varphi - f_2)||\vec{p}_1 \times \hat{p}_0| \). We assume initial equilibrium densities of 2 and \( \varphi \), which using Boltzmann statistics means that the distribution functions of 2 and \( \varphi \) are identical. It also means that we can use \( f_2 f_\varphi = e^{-E_1/T}. \) Hence we can split the term in an \( f_1 \) part and an extra part. The \( f_1 \) part yields exactly the same result as in Sec. III, but the extra part is proportional to \( f_2 f_\varphi \) and yields the following:

\[ \frac{\partial |P_\perp|}{\partial t} \bigg|_{\text{Volume extra}} = \int \frac{d^3p_1}{(2\pi)^32E_1} \frac{d^3p_2}{(2\pi)^32E_2} \frac{d^3p_\varphi}{(2\pi)^32E_\varphi} (2\pi)^4 \delta^4(p_1 - p_2 - p_\varphi) g^2 m^2 \frac{1}{E_1/T} \left(|\vec{p}_1 \times \vec{p}_0| + |\vec{p}_2 \times \vec{p}_0| + |\vec{p}_\varphi \times \vec{p}_0|\right) \]

(10)

which reduces to

\[ \frac{\partial |P_\perp|}{\partial t} \bigg|_{\text{Volume extra}} = \frac{E_0 p_0}{4\pi^2 T^\gamma} \left( E_0 T e^{-m/T} + \int_0^\infty dE_1 e^{-E_1/T} p_1 \right) \]

(11)

with relativistic limit

\[ \frac{\partial |P_\perp|}{\partial t} \bigg|_{\text{Volume extra}} = \frac{E_0 p_0}{4\pi^2 T^\gamma} \left( E_0 T + 2T^2 \right). \]

(12)

Thus the final result is

\[ \frac{\partial |P_\perp|}{\partial t} \bigg|_{\text{Volume}} = \frac{E_0 n_1}{8(2\pi)^3 T^\gamma} + \frac{E_0 p_0}{4\pi^2 T^\gamma} \left( E_0 T e^{-m/T} + \int_0^\infty dE_1 e^{-E_1/T} p_1 \right) \]

(13)

with relativistic limit

\[ \frac{\partial |P_\perp|}{\partial t} \bigg|_{\text{Volume}} = \frac{E_0 n_1}{8(2\pi)^3 T^\gamma} + \frac{E_0 p_0}{4\pi^2 T^\gamma} \left( E_0 T + 2T^2 \right). \]

(14)

Hence, we are led to conclude that in the realistic scenario of having a background thermal distribution of light neutrinos as well as of pseudoscalars, we see a correction in the form of the second term in Eq. 14. This extra contribution is something which should be taken into account when putting bounds on the neutrino-pseudoscalar interaction.

We can make a rough estimate of the improvement on the bound of the decay coupling constant when taking Eq. 14 into account. First, we notice that in the presence of the first term on the right-hand side of Eq. 14 we are investigating the standard case which was previously studied in [20]. Here the naive decay rate \( \Gamma_{\text{decay}} = \frac{m}{4\pi} \) is translated into a transport rate \( \Gamma_{\text{transport}} = \Gamma_{\text{decay}} \left( \frac{m}{E} \right)^3 \), where the factor \( \left( \frac{m}{E} \right)^3 \) is due to three Lorentz gamma factors: The first one comes from transforming from the rest frame of the parent neutrino to the frame of the thermal medium. This will give us a decay rate in the frame of the thermal medium. The other two come from the following reason: The decay is isotropic in the rest frame of the parent neutrino; however, the decay products will have directions within an angle corresponding to a factor \( \gamma \). So, to randomize the direction of the original neutrino we must include another factor of \( \gamma \). In total when we transform from the medium frame decay rate to the relevant transport rate we get two factors of gamma. All in all we arrive at the desired expression

\[ \Gamma_{\text{transport}} = \Gamma_{\text{decay}} \left( \frac{m}{E} \right)^3. \]

(15)
To ensure that the neutrinos are still free-streaming at the time of photon decoupling as required by observations of the CMB [15, 23], we can then compare the transport rate with the expansion rate of the universe $H_{\text{dec}}$ at photon decoupling. The requirement for free-streaming is $\Gamma < H_{\text{dec}}$. This leads to the bound [20]

$$g < 0.61 \times 10^{-11} \left( \frac{50 \text{ meV}}{m_\nu} \right)^2 .$$

In the event of the decay taking place in a thermal distribution of light neutrinos and pseudoscalars we need to take the second term in Eq. 14 into account. Especially since at the time of photon decoupling, assuming a generic heavy neutrino mass of $m_\nu = 50 \text{ meV}$ and energy $E = 3T_{\nu,\text{dec}} \sim 3 \times 0.18 \text{ eV}$, we get $\gamma \sim \frac{E}{m_\nu} \sim 3.6$. Combined with the fact that for a relativistic species ($m_\nu < 3T_{\nu,\text{dec}}$) and for our relativistic heavy neutrino $n_\nu \sim T_{\nu,\text{dec}}^3$ up to factors of order unity, this means that the transport rate we should be comparing is rather

$$\Gamma_{\text{transport}} \sim \Gamma_{\text{decay}} \left( \frac{m}{E} \right)^3 + 16\pi \left( \frac{m}{E} \right)^2 ,$$

where the factor of $16\pi$ takes into account this missing factor in the denominator of the second term. If we translate into a bound on the decay coupling constant, this gives

$$g < 2.6 \times 10^{-13} \left( \frac{50 \text{ meV}}{m_\nu} \right)^{3/2} \left( \frac{T_{\nu}}{0.18 \text{ eV}} \right)^{-1/2} \left( 1 + 1.8 \times 10^{-3} \left( \frac{50 \text{ meV}}{m_\nu} \right) \left( \frac{T_{\nu}}{0.18 \text{ eV}} \right)^{-1} \right)^{-1/2} .$$

For $\gamma \sim 3.6$ with a neutrino mass $m_\nu = 50 \text{ meV}$ this translates simply into the bound

$$g < 2.6 \times 10^{-13} ,$$

i.e. an improvement of more than a factor of 10. Translating this into a limit on the neutrino lifetime in the restframe we get

$$\tau < 1 \times 10^{13} \text{ s} ,$$

hence there is still the possibility for the neutrino to be short lived when we let the decay take place in a thermal background.

**V. NUMERICAL RESULTS**

In the numerical implementation we had to change the setup a little. Because of problems with the bins, it was impossible to get the heavy neutrino to have vanishing momentum in the transverse direction. This could not be remedied by increasing the number of bins. Therefore we made a thermal distribution of 1’s around an average momentum. Specifically we chose a standard scenario with $m = 2T$, $\langle p_x \rangle = \pm 2T$ and made a thermal distribution around this. We chose the artificially high value $g = \frac{1}{\gamma_{\text{dec}}}$ (making $T = \frac{\gamma_{\text{dec}} E_0}{2}$) to let the code find equilibrium in a reasonable time. We checked that the code did reach equilibrium both under the initial condition of no light neutrinos from the beginning and under the assumption of an initial thermal distribution of light neutrinos. More on the numerics is provided in the Appendix VII.

This means that the calculated formulas cannot be verified explicitly, since we have no well-defined gamma factor. However, the factor of $\frac{1}{\tau_{\gamma}}$ can almost be found. First, the $\frac{1}{\tau}$ is, of course, trivial. If the coupling is weakened, the lifetime is correspondingly longer. We checked that our code yielded this result. The $\frac{1}{\gamma}$ for the time dilation of the neutrino lifetime is also quite trivial. We found this as well - by noticing the decreased numbers of 2’s produced in the very first step when we used an alternative scenario: $m = 2T$, $\langle p_x \rangle = \pm 4T$. Since we had thermal distributions rather than sharply defined momenta, we could not expect to find the exact relation between average $\gamma$s to be the same as the relation between the created particles.

However, the most interesting second $\frac{1}{\tau}$ can be illustrated by numerical plots. Fig. 1 shows the first distributions after the very first step, whereas Fig. 2 shows the distributions after the first step in the alternative scenario with a larger gamma factor. Two things are very important to notice. First, the created 2’s are indeed anisotropic. This is the effect of taking their isotropic distribution in the frame of the decaying particle 1 and making a Lorentz transformation to the (cosmic) laboratory frame. Second, the fact that the alternative scenario shows more anisotropy among the 2’s should make it clear that it must be a gamma factor. One could alter Eqs. (13), (14) to match the
FIG. 1: The situation just after the first step, $t = 9.95 \times 10^{-3} \tau$, in the standard scenario. The lines are contour plots of the distribution function $\log(f \times T)$ – as defined in the Appendix. $p_y$ means transverse momentum and is measured in units of bin length $dp = \frac{1}{3}$.

FIG. 2: The situation just after the first step, $t = 9.95 \times 10^{-3} \tau$, in the alternative scenario. The lines are contour plots of the distribution function $\log(f \times T)$ – as defined in the Appendix. $p_y$ means transverse momentum and is measured in units of bin length $dp = \frac{1}{3}$.

FIG. 3: The situation later, $t = 14.5 \tau$, in the standard scenario.

FIG. 4: The situation even later, $t = 1.40 \times 10^3 \tau$, in the standard scenario.

initial conditions for the numerics. But since Eqs. (13), (14) are complicated enough already, and since the numerics is not done to create new results but only to confirm the pattern of Eqs. (13), (14), which it does, we have chosen not to do so.

For completeness, Fig. 3 shows the development in the standard scenario at a later time (roughly 15 times the rest-frame lifetime – this is still an intermediate time due to the two gamma factors) – and Fig. 4 shows the standard scenario at an even later time (roughly 1400 times the rest frame life time), where equilibrium is almost reached.

One should note that distribution functions are defined according to the cylinder coordinates used in the code – see the Appendix VII [especially Eq. 24 which shows that the distribution function has dimension of time] for details. Also, one should note that $p_y$ in the plots are, in fact, the transverse momentum - not the momentum in one of the transverse directions. The unit of $p$ in the figures is $dp = \frac{\Delta x}{\tau} = \frac{T}{5}$ which is the distance between adjacent bins in momentum space.
VI. CONCLUSION

We have investigated a neutrino-pseudoscalar gas with only decays $\nu_1 \rightarrow \nu_2 + \varphi$ and inverse decays to obtain equilibrium. We started with an anisotropic distribution of $\nu_1$ and confirmed that kinetic equilibrium is delayed by two Lorentz $\gamma$–factors – one for time dilation of the heavy neutrino lifetime and one from the opening angle in the transformation of the isotropic distribution of the decay products in the rest frame of the decaying particle back to the (cosmic) laboratory frame. We found explicit analytical expressions for the rate of creation of transverse momentum – both in a case with no background of the decay products and in the case of thermal backgrounds and the ultrarelativistic limits hereof. We have confirmed this behavior in numerical simulations as well – though we had to make a thermal smear of the initial anisotropy, making the analytical and numerical results open to a qualitative, but not quantitative, comparison.

Furthermore we have obtained updated bounds on the neutrino-pseudoscalar coupling constant as well as on the neutrino lifetime in the realistic case of a thermal background of neutrinos and pseudoscalars.

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VII. APPENDIX: NUMERICS

In order to follow this numerically we notice that we have three particles in three momentum coordinates - that is, nine dimensions (no isotropy). We notice that when we assume Maxwell-Boltzmann statistics particles 2 and $\varphi$ behave alike. This means that if starting conditions are the same, we have to track only one of them. Even though there is not isotropy, azimuthal angles are arbitrary. So we end up with four dimensions, two particles, with a momentum in the initial beam direction and momentum in a transverse direction. So we want to integrate the remaining coordinates out. We start with

$$C[1] = \frac{1}{2E_1} \int \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_\varphi}{(2\pi)^3} \left(2\pi\right)^4 \delta^4(p_1 - p_2 - p_\varphi) g^2 m^2 (f_1 - f_2 f_\varphi)$$

$$= \frac{dp_\varphi dp_\varphi R dp_2 dp_2 R dp_2 R \delta(E_1 - p_2 - p_\varphi) \delta(p_{1x} - p_{2x} - p_{\varphi x})}{(2\pi)^2 E_1 E_2 E_\varphi}$$

$$(f_1 - f_2 f_\varphi) g^2 m^2 \int dp_\psi dp_\theta \delta(p_\psi) \delta(p_\theta), \quad (21)$$

where we have defined (including aligning the coordinate system with particle 1) momenta

$$p_1 = (p_{1x}, p_{R1}, 0), p_2 = (p_{2x}, p_{2R} \cos(\psi), p_{2R} \sin(\psi)), p_\varphi = (p_{\varphi x}, p_{R \varphi} \cos(\theta), p_{R \varphi} \sin(\theta))$$

After integration we find

$$C[1] = \frac{dp_\varphi dp_\varphi R dp_2 dp_2 R dp_2 R \delta(E_1 - p_2 - p_\varphi) \delta(p_{1x} - p_{2x} - p_{\varphi x}) g^2 m^2}{\pi^2 E_1 E_2 E_\varphi} \frac{1}{\sqrt{S}}$$

$$\frac{2p_{1R}^2}{\sqrt{S}} \frac{1}{\sqrt{4p_{1R}^2 p_{\varphi R}^2 - S + \sqrt{4p_{1R}^2 p_{\varphi R}^2 - S}}}, \quad (22)$$

where $S$ is given by

$$S = 4p_{1R}^2 p_{\varphi R}^2 - (p_{2R}^2 - p_{\varphi R}^2 - p_{1R}^2)^2.$$  

For numerical purposes, let us underline the formula in the way it should be implemented.

$$\frac{df(\tilde{p_i})}{dt} = C[i](\tilde{p_i}), \quad (23)$$
however the vectors will not be introduced. Rather we use
\[ \int_0^1 f(p)dp = \int_0^1 f(p_{ix}, p_{iR}, p_0)p_{iR}dp_0 = 2\pi p_{iR} \ast f(p_{ix}, p_{iR}, p_0) \equiv 2\pi p_{iR}f(p_{ix}, p_{iR}) \] (24)
and likewise for \( C[i] \). This means that the function \( \tilde{f} \) that we implement is of dimension \( E^{-1} \) and its derivative dimensionless. The equation implemented is thus
\[
\frac{d\tilde{f}(p_{ix}, p_{iR})}{dt} = \tilde{C}[1] = \frac{dp_{ix}dp_{ix}Rdp_{ix}Rdp_{ix}Rdp_{ix}R}{\pi^3 E_1 E_2 E_\varphi} \delta(E_1 - p_2 - p_\varphi) \\
\left[ \frac{\delta(p_{ix} - p_{2x} - p_{ix}) g^2 m^2}{\sqrt{S}} (f_1 - f_2 f_\varphi) \frac{p_{iR}}{\sqrt{S}} \right] \\
\sqrt{4p_{iR}p_{iR} - S} + \sqrt{4p_{iR}p_{iR} - S}.
\] (25)
The implementation of 2 is quite easy since for fixed momenta \( \vec{p}_1, \vec{p}_2, \vec{p}_\varphi \)
\[ C[1] = -C[2] \] (26)
or
\[ 2\pi p_{iR}\tilde{C}[1] = -2\pi p_{iR}\tilde{C}[2]. \] (27)

References

[1] Y. Chikashige, R. N. Mohapatra and R. D. Peccei, Phys. Lett. B 98 (1981) 265.
[2] G. B. Gelmini and M. Roncadelli, Phys. Lett. B 99 (1981) 411.
[3] G. B. Gelmini, S. Nussinov and M. Roncadelli, Nucl. Phys. B 209 (1982) 157.
[4] J. Schechter and J. W. F. Valle, Phys. Rev. D 25 (1982) 774.
[5] G. Gelmini and E. Roulet, Rept. Prog. Phys. 58 (1995) 1207 [arXiv:hep-ph/9411278].
[6] E. W. Kolb and M. S. Turner, Phys. Lett. B 159 (1985) 102.
[7] A. Manohar, Phys. Lett. B 192 (1987) 217.
[8] D. A. Dicus, S. Nussinov, P. B. Pal and V. L. Teplitz, Phys. Lett. B 218 (1989) 84.
[9] R. V. Konoplich and M. Y. Khlopov, Sov. J. Nucl. Phys. (1988) V. 47, PP. 565-566.
[10] Z. G. Berezhiani and A. Y. Smirnov, Phys. Lett. B 220 (1989) 279.
[11] G. G. Raffelt, Stars As Laboratories For Fundamental Physics: The Astrophysics Of Neutrinos, Axions, And Other Weakly Interacting Particles, (University of Chicago Press, Chicago, 1996), ISBN 0-226-70271-5, p. 664.
[12] K. Choi and A. Santamaria, Phys. Rev. D 42, 293 (1990).
[13] Y. Farzan, Phys. Rev. D 67, 073015 (2003) [arXiv:hep-ph/0211375].
[14] M. Kachelriess, R. Tomas and J. W. F. Valle, Phys. Rev. D 62, 023004 (2000) [arXiv:hep-ph/0001039].
[15] S. Hannestad, JCAP 0502, 011 (2005) [arXiv:astro-ph/0411475].
[16] R. F. Sawyer, Phys. Rev. D 74, 043527 (2006) [arXiv:astro-ph/0601525].
[17] N. F. Bell, E. Pierpaoli and K. Sigurdson, Phys. Rev. D 73, 063523 (2006) [arXiv:astro-ph/0511410].
[18] G. Raffelt and J. Silk, Phys. Lett. B 192, 65 (1987).
[19] J. F. Beacom, N. F. Bell and S. Dodelson, Phys. Rev. Lett. 93, 121302 (2004) [arXiv:astro-ph/0404585].
[20] S. Hannestad and G. Raffelt, Phys. Rev. D 72, 103514 (2005) [arXiv:hep-ph/0509278].
[21] A. Friedland, K. M. Zurek and S. Bashinsky, arXiv:0704.3271 [astro-ph].
[22] M. Cirelli and A. Strumia, JCAP 0612, 013 (2006) [arXiv:astro-ph/0607086].
[23] A. Basboll, O. E. Bjaelde, S. Hannestad and G. G. Raffelt, Phys. Rev. D 79 (2009) 043512 [arXiv:0806.1735 [astro-ph]].
[24] F. de Bernardis, A. Melchiorri, L. Verde and R. Jimenez, JCAP 0803 (2008) 020 [arXiv:0707.4170 [astro-ph]].
[25] S. Hannestad and J. Brandbyge, JCAP 1003 (2010) 020 [arXiv:0910.4578 [astro-ph.CO]].
[26] A. Ringwald, Nucl. Phys. A 827 (2009) 501C [arXiv:0901.1529 [astro-ph]].
[27] F. De Bernardis, L. Pagano, P. Serra, A. Melchiorri and A. Cooray, JCAP 0806 (2008) 013 [arXiv:0804.1925 [astro-ph]].
[28] G. D. Starkman, N. Kaiser and R. A. Malaney, Astrophys. J. 434, 12 (1994) [arXiv:astro-ph/9312020].
[29] A. Basboll and S. Hannestad, JCAP 0701, 003 (2007) [arXiv:hep-ph/0609025].