Irregular matrix model with $\mathcal{W}$ symmetry

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Abstract

We present an irregular matrix model which has $\mathcal{W}_3$ and Virasoro symmetry. The irregular matrix model is obtained using the colliding limit of the Toda field theories and produces the inner product between irregular modules of $\mathcal{W}_3$ symmetry. We evaluate the partition function using the flow equation which is the realization of Virasoro and $\mathcal{W}$ symmetry.

Keywords: conformal symmetry, $\mathcal{W}$ symmetry, random matrix model, irregular conformal state

1. Introduction

The irregular matrix model [1] is obtained by the colliding limit of the $\beta$-deformed Penner-type matrix model [2, 3] which is equivalent to the regular conformal block of primary fields. The colliding limit [4] is the fusion of primary fields at one point with their Virasoro momenta infinite and results in a non-trivial limit. In particular, there appears an irregular module of rank $n$ which is defined as the eigenvector of positive Virasoro generators $L_0, L_{n+1}, \ldots, L_{2n}$. The irregular module of rank 1 can be constructed as the combination of primary and descendant states [5]. However, for ranks greater than 1 the irregular module is not simple to construct because one needs to take account of the consistency condition [6] for the non-negative Virasoro generators such as $L_0, L_1, \ldots, L_{n-1}$. The consistency condition is not easy to manipulate algebraically. One may avoid this difficulty if one uses the irregular matrix model and its conformal symmetry. In our previous papers, we investigated the case with Virasoro symmetry [1, 6–8]. We may extend this idea to $\mathcal{W}$ symmetry by considering a multi-matrix model. $\mathcal{W}$ symmetry was previously used to construct a multi-matrix model of polynomial type in [9]. In this paper, we are going to construct the irregular matrix model with $\mathcal{W}$ symmetry using the colliding limit of the Toda field theory and present how to find the partition function using the $\mathcal{W}$ symmetry.

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Toda field theory is a generalization of the Liouville field theory and contains not only Virasoro symmetry but also higher spin symmetries which are summarized in terms of symmetry. Two-dimensional Toda field theory associated with simple Lie algebra with rank $k$ is defined by the Lagrangian

$$L = \frac{1}{8\pi} (\partial_{\nu} \vec{\varphi})^2 + \mu \sum_{i=1}^{k} \varepsilon^{b_{ij} \vec{\varphi}}$$  \hspace{1cm} (1.1)$$

where $\varepsilon^{1, \cdots, k}$ are the simple roots of the Lie algebra. The bosonic vector field $\vec{\varphi}$ has $k$ independent components. The Toda field theory is conformal provided there is a background charge $\vec{Q} = (b + 1/b) \vec{\rho}$ with $\rho$ the Weyl vector, half of the sum of all positive roots. In this case, the conformal dimension of the exponential terms is 1 and the central charge of the system is $c = k + 12\vec{Q}^2$.

The Toda field theory has $k$ holomorphic symmetry currents $W_2$, $W_3$, $\cdots$, $W_{k+1}$, where $W_2$ is identified as the Virasoro current. In this paper we concentrate on the colliding limit of the $A_2$ Toda field theory, construct the $A_2$ irregular matrix model and find the partition function using the Virasoro and $W_3$ symmetry. The generalization is straightforward.

The paper is organized as follows. Section 2 is devoted to the irregular matrix model. Starting with $A_2$ Toda field theory, we obtain the irregular matrix model using the colliding limit. In section 3, we investigate the $W$ symmetry in detail. The explicit representation of the $W$ symmetry is given as the differential operator of the potential variables. For concreteness and comparison, we will use the representation as in the colliding limit of the Penner-type two-matrix model. For the vertex primary field $V_\rho(z_{\rho}) = \exp^{\varphi(z_{\rho})}$ has the conformal dimension $\Delta_\rho = \partial_{\rho} \left( \vec{Q} - \frac{1}{2} \partial_{\rho} \right)$. We consider the $(n + 2)$-primary field correlation and put the conformal block (the holomorphic part of the correlation) in terms of the Selberg integral representation using the screening operator (exponential terms in the Toda Lagrangian). One may put the fields, say, one at the infinity $z_{n+1} \to \infty$, one at the origin $z_0 = 0$, and the rest at $z_a$ ($a = 1, \cdots, n$). In this case, the conformal block has the form $\left( \prod_{0 \leq k < \ell \leq n} (z_\rho - z_\ell)^{-\Delta_\rho + \Delta_\rho} \right) \times Z_\beta$ if one normalizes the result so that the infinite factor $z_{n+1}$ scales away. The front multiplicative factor is from the free correlation between the primary fields and the rest $Z_\beta$ is due to the correlation between the screenings and also with the primaries;

$$Z_\beta \equiv \oint \prod_{i=1}^{N_1} dx_i \prod_{j=1}^{N_2} dy_j \Delta(x_1)^{2\beta} \Delta(y_1)^{2\beta} \Delta(x, y)^{-\beta} e^{\beta [\sum_{i} \ell_i(x_i) + \sum_{j} \ell_j(y_j)]}$$  \hspace{1cm} (2.1)$$

where $\Delta(x) = \prod_{i<k} (x_i - x_k)$ and $\Delta(x, y) = \prod_{i<j} (x_i - y_j)$ are the Vandermonde determinants which come from the correlation between screening terms. We put $\beta = -b^2$ to make
the partition function similar to the $\beta$-deformed matrix model. $N_1$ and $N_2$ are the number of screening terms with the root vectors $e_1^\ast$ and $e_2^\ast$, respectively. The number satisfies the neutrality condition \[ \sum_{i=0}^{n} x_i + \alpha_{\infty} + b \sum_{k=1}^{2} N_k e_k^* = \bar{G}. \] (2.2)

The potentials $V_1$ and $V_2$ appear from the correlation between the screening and primary vertex operators. If one parametrizes $\beta_a = \alpha_a/\sqrt{3} (1, 1, -2) + \beta_a (1, -1, 0)$, one has the Penner-type potential
\[
\frac{1}{\hbar} V_1(x) = -\sum_{a=0}^{n} \beta_a \log(x - z_a), \quad \frac{1}{\hbar} V_2(y) = -\sum_{a=0}^{n} \frac{1}{2} (\sqrt{3} \alpha_a - \beta_a) \log(y - z_a).
\] (2.3)

The irregular matrix model is obtained if one uses the colliding limit similar to the Liouville case \[1, 4\]: $a, \beta_a \to \infty$ and $z_a \to 0$ so that $c_k \equiv \sum_{\alpha=0}^{n} a_\alpha z_{\alpha}^k$ and $b_k \equiv \sum_{\alpha=0}^{n} \beta_{\alpha} z_{\alpha}^k$ with $k = 0, 1, \cdots, n$ are finite. In this case, one has the $A_2$ irregular matrix model $Z_n$ with rank $n$ whose potential is given in terms of logarithmic and finite number of inverse powers
\[
\frac{1}{\hbar} V_1(z) = -b_0 \log z + \sum_{k=1}^{n} c_k z^k, \quad \frac{1}{\hbar} V_2(z) = -\bar{c}_0 \log z + \sum_{k=1}^{n} \bar{c}_k z^k.
\] (2.4)

where we use the short-hand notation $\bar{c}_0 = (\sqrt{3} c_0 - b_0)/2$. In the following, we equally put the irregular matrix model using the new parameter $\hbar \equiv -2i g$ so that
\[
Z_n = \int \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} dx_i dy_j \Delta(x)^{2\beta} \Delta(y)^{2\beta} \Delta(x, y)^{-\beta} e^{-\frac{\hbar}{\beta} \sum_{i=1}^{N_1} V(x_i) + \sum_{j=1}^{N_2} V(y_j)}.
\] (2.5)

3. $\mathcal{W}$ symmetries of the irregular matrix

3.1. Loop equations

The irregular matrix model (2.5) has two loop equations. One equation is given as the quadratic equation of the resolvent with its multipoint defined as
\[
R_{K_1, \cdots, K_n}(z_1; \cdots; z_n) = \beta \left( \frac{g}{\sqrt{\beta}} \right)^{2-s} \left( \sum_{i=1}^{N_1} 1/z_{i; K_i} - \sum_{i=1}^{N_2} 1/z_{i; K_i} \right)_{\text{connected}}.
\] (3.1)

Denoting $l_{i, 1} = x_i, l_{i, 2} = y_j$, one has
\[
R_0(z)^2 + R_1(z) R_2(z) - V_1(z) R_1(z) - V_2(z) R_2(z) + \frac{\hbar Q}{2} (R_1'(z) + R_2'(z)) = \frac{f_1(z) + f_2(z)}{4},
\] (3.2)

where the quantum corrections $f_1, f_2$ are defined as $f_1(z) = 4 g \sqrt{\beta} \sum_{i=1}^{N_1} \left( \frac{-V_1(x_i) + V_1(x_i)}{z_i - x_i} \right)$ and $f_2(z) = 4 g \sqrt{\beta} \sum_{i=1}^{N_2} \left( \frac{-V_2(y_j) + V_2(y_j)}{z - y_j} \right)$. Here $\langle \cdots \rangle$ denotes the expectation value within the matrix integral. This loop equation is obtained if one performs the conformal transformation of the integration variables $x_i \to x_i + \varepsilon/(x_i - z)$ and $y_j \to y_j + \varepsilon/(y_j - z)$. 


The other loop equation is given as a cubic equation [12–14]:

\[
0 = -R_1^2 R_2 + R_1 R_2^2 - V'_1 \left( R_1^2 + V'_1 R_1 - \frac{f_1}{4} \right) \\
+ \frac{h Q}{4} \left[ 3 (V'_2 R_2^2 - V'_1 R_1^2) + R_1 R_2^2 - R'_1 R_2 \\
+ 2 (R_2 R'_2 - R'_1 R'_1) + V_2'' R_2 - V_1'' R_1 + \frac{f'_1 - f'_2}{4} \right] \\
+ \frac{h^2 Q^2}{8} (R''_1 - R''_1) + \frac{h^2}{4} [V'_1 R_{1;1} - V'_2 R_{2;2} \\
+ R_{1;1} R_2 - R_{2;2} R_1 - 2 R_{1;2} (R_2 - R_1)] \\
+ \frac{h^3 Q}{16} \left[ R'_{1;1} - R'_{2;2} + \lim_{\varepsilon \to 0} \left( \frac{\partial}{\partial \varepsilon} R_{1;2}(z, \varepsilon) - \frac{\partial}{\partial \varepsilon} R_{1;2}(z, \varepsilon) \right) \right] \\
+ \frac{h^4}{16} (R_{1;2} - R_{2;1}),
\]

(3.3)

where \( g_1(z) = 4 g^2 \beta \sum_{i,j} \left\langle \frac{V_{1}(x_i) - V_{1}(x_j)}{(z - x_i)(z - x_j)} \right\rangle \) and \( g_2(z) = 4 g^2 \beta \sum_{i,j} \left\langle \frac{V_{2}(x_i) - V_{2}(x_j)}{(z - y_i)(z - y_j)} \right\rangle \). This is obtained after varying the integration variables \( x_i \to x_i + \sum_{j=1}^{N_i} \frac{\varepsilon_i}{(x_i - z)(x_i - y_j)} \)

and \( y_j \to y_j + \sum_{i=1}^{N_j} \frac{-\varepsilon_j}{(z - y_j)(z - x_i)} \).

In this paper, we shall focus on the lowest order of \( \hbar \) (also putting \( Q = 0 \)). Then, the two equations can be put in a more compact form if we define new notations as \( R = R_1 - R_2/2 \).

\[
\hat{R} = \sqrt{3} R_2/2, \quad \partial \phi_2 = V'_1 \quad \text{and} \quad \partial \phi_1 = \frac{1}{\sqrt{3}} (2 V'_2 + V'_1).
\]

\[
(2 \hat{R} - \partial \phi_2)^2 + (2 \hat{R} - \partial \phi_1)^2 = -\hbar^2 \xi_2
\]

\[
(2 \hat{R} - \partial \phi_1)^3 - 3 (2 \hat{R} - \partial \phi_2)(2 \hat{R} - \partial \phi_2)^2 = -\hbar^3 \xi_3
\]

(3.4) (3.5)

where \( \xi_2 \) and \( \xi_3 \) are given explicitly as

\[
-\hbar^2 \xi_2 = (\partial \phi_2)^2 + (\partial \phi_1)^2 + f_1 + f_2
\]

\[
-\hbar^3 \xi_3 = -(\partial \phi_1)^3 + 3 (\partial \phi_2)^2 \partial \phi_1 + 3 (\partial \phi_1 - \sqrt{3} \partial \phi_2)(f_1 + f_2) + 3 \sqrt{3} \partial \phi_2 f_1
\]

\[
- \frac{3}{2} (3 \partial \phi_1 - \sqrt{3} \partial \phi_2) f_2 - 3 \sqrt{3} (g_1 - g_2).
\]

(3.6) (3.7)

\( \xi_2 \) and \( \xi_3 \) look very complicated but turn out to be the representations of the \( W_2 \), \( W_3 \) currents, respectively. This is investigated in the following two subsections.

3.2. Virasoro current

One may use the explicit form of the potential (2.4) to put \( \partial \phi_1 = -\hbar \sum_{k=0}^{n} \epsilon_k z^k \), \( \partial \phi_2 = -\hbar \sum_{k=0}^{n} b_k z^{k+1} \) and

\[
\partial \phi = -\hbar \sum_{k=0}^{n} b_k/z^{k+1}.
\]
where we use the translation invariance to put \( \frac{\partial}{\partial z} = 0 \). In addition, the Virasoro differential operator is generalized from the one-matrix model, which has the form

\[
\mathcal{L}_k = \frac{1}{z_{k+2}} \mathcal{Z}_n
\]

where \( \mathcal{Z}_n \) is a constant

\[
A_k = -\sum_{\ell=0}^{k} (b_\ell b_{k-\ell} + c_\ell c_{k-\ell}).
\]

We use the convention that \( b_\ell = 0 = c_\ell \) when \( \ell \) does not belong to the element set \( \{0, 1, \ldots, n\} \). The explicit form \( \xi_2 \) is identified with the expectation value of Virasoro current

\[
\xi_2 = \frac{\langle \Delta(T(z))|I_n\rangle}{\Delta|I_n\rangle}, \quad T(z) = \sum_{k\in\mathbb{Z}} L_k z^{-k},
\]

if one recalls that the irregular module \( |I_n\rangle \) of rank \( n \) has the eigenvalue \( A_k \) of \( L_k \) when \( n \leq k \leq 2n \) and \( 0 \) when \( k > 2n \). In addition, this identification is consistent with the Virasoro generator representation for the parameter set \( \{b_\ell, c_\ell\} \ 0 \leq \ell \leq n \) as the differential operator \( \mathcal{L}_k = A_k + v_k \) which is the right action of the irregular module: \( L_k|I_n\rangle = \mathcal{L}_k|I_n\rangle \):

\[
\mathcal{L}_k = \begin{cases} 
0, & 2n < k \\
A_k, & n \leq k \leq 2n \\
A_k + v_k, & 0 \leq k \leq n - 1.
\end{cases}
\]

3.3. \( W_3 \) current

Let us rewrite \( \xi_2 \) using the parameters of the potential. First note that

\[
g_1(z) - g_2(z) = -h \sum_{k=-2}^{n-2} \frac{h^2 b^2}{z_{k+3}} \left( \sum_{i,j} \frac{1}{x_i - y_j} \sum_{r=0}^{n-k} \left( b_{r+k} x_r + c_{r+k} y_r \right) \right)
\]

To put the expectation values as the derivatives of the partition function, we use two identities. One is

\[
4 g^2 \beta \left( \sum_{i,j} \frac{1}{x_i - y_j} \right) = 4 g^2 \beta \sum_{m=1}^{r} \left( \sum_{i,j} \frac{1}{x_i^m \ y_j^{r+1-m}} \right) + 4 g \sqrt{3} \left( \sum_{j} \frac{V_j(y_j)}{y_j^r} \right)
\]

which is obtained if one changes the integration variable \( y_j \rightarrow y_j + \varepsilon / y_j^r \) in the partition function \( \mathcal{Z}_n \). The other is the same as (3.14) with the exchange of \( x_i \leftrightarrow y_j \) and \( v_2 \rightarrow v_3 \) due to the obvious symmetry; one may obtain the identity using the integration variable change \( x_i \rightarrow x_i + \varepsilon / x_i^r \).
Using the above two identities (3.14) and its companion, one finds

\[
g_3(z) - g_2(z) = \sum_{k=2}^{n-2} \sum_{r=2}^{n-k} b_{r+k} \left( \frac{2}{(r+k)!} \sum_{m=1}^{r-1} \left( \sum_{i,k} x_i \sum_{x_{r+1-m}} 1 \right) - 2b^2 \sum_{p=0}^{n} \left( \sum_{i} \frac{b_p}{x_{r+p}} \right) \right) - \tilde{c}_{r+k} \left( \frac{2}{(r+k)!} \sum_{m=1}^{r-1} \left( \sum_{j|,l|} y_j \sum_{y_{r+1-m}} 1 \right) - 2b^2 \sum_{p=0}^{n} \left( \sum_{j} \frac{\tilde{c}_p}{y_{j+p}} \right) \right)\]

(3.15)

Note that in the last term there appear the expectation values of the inverse power of \(y_j\) up to \((2n + 2)\) and the term higher than \(n\) is not directly obtained from the derivative of the rank \(n\) partition function. To avoid this difficulty, one may use an extended partition function which is partition function with higher rank, i.e. the partition function with the potential with the parameter \(c_k, b_k\) with \(k\) up to \(2n + 2\). In this one has the explicit form of \(x_3\)

\[
\xi_3 = \sum_{k=2}^{n-2} \sum_{r=2}^{n-k} M_k \frac{1}{(r+k)!} \sum_{m=1}^{r-1} \left( \sum_{i,k} x_i \sum_{x_{r+1-m}} 1 \right) - 2b^2 \sum_{p=0}^{n} \left( \sum_{i} \frac{b_p}{x_{r+p}} \right) - \tilde{c}_{r+k} \left( \frac{2}{(r+k)!} \sum_{m=1}^{r-1} \left( \sum_{j|,l|} y_j \sum_{y_{r+1-m}} 1 \right) - 2b^2 \sum_{p=0}^{n} \left( \sum_{j} \frac{\tilde{c}_p}{y_{j+p}} \right) \right)\]

(3.16)

where \(M_k\) is a constant

\[
M_k = \sum_{r+s+t=k} (3b_r b_s c_t - c_r c_s c_t)\]

(3.17)

The terms with inverse powers of \(1/z^k\) is carefully rewritten for \(-2 \leq k \leq n - 1\) by introducing the extended partition functions, \(Z_{n+1}, \cdots, Z_{2n+2}\). Furthermore, the differential operator \(\mu_k\) has the different form for the extended case:

\[
\mu_k = \begin{cases} 
\mu_k^0, & n \leq k \leq 2n - 1 \\
\mu_k^0 + \mu_k^1, & 2 \leq k \leq n - 1 
\end{cases}
\]

(3.18)

where

\[
\mu_k^0 = \sum_{0 \leq r+s \leq k} \frac{t}{2} \left( 6b_r c_s \frac{\partial}{\partial b_t} + 3b_r b_s \frac{\partial}{\partial c_t} - 3c_r c_s \frac{\partial}{\partial c_t} \right)
\]

\[
\mu_k^1 = \sum_{0 \leq r+s \leq k} \frac{s}{4} \left( 6b_r \frac{\partial}{\partial b_s} \frac{\partial}{\partial c_t} + 3c_r \frac{\partial}{\partial b_s} \frac{\partial}{\partial b_t} - 3c_r \frac{\partial}{\partial c_s} \frac{\partial}{\partial c_t} \right)
\]

(3.19)

We identify \(\xi_3\) with the expectation value of the \(\mathcal{W}_3\) current between a regular module \(|\Delta\rangle\) and an irregular module \(|I_n\rangle\) as

\[
\xi_3 = \frac{\langle \Delta | W(z) | I_n \rangle}{\langle \Delta | I_n \rangle}, \quad W(z) = \sum_{k \in \mathbb{Z}^{n+1}} W_k \to z^{n+1}.
\]

(3.20)

Note that the irregular module has the eigenvalue \(M_k\) of \(W_k\) when \(2n \leq k \leq 3n\). Higher mode \(W_k\) with \(k > 3n\) annihilates the irregular module. In this identification one has the representation of the \(\mathcal{W}_3\) current \(W_k|I_n\rangle = \omega_k|I_n\rangle\) where
\[
\omega_k = \begin{cases} 
0, & 3n < k \\
M_k, & 2n \leq k \leq 3n \\
M_k + \mu_k, & n \leq k \leq 2n - 1 \\
M_k + \mu_k^0 + \mu_k^2, & 0 \leq k \leq n - 1.
\end{cases}
\] (3.21)

Considering a negative mode acts on the regular module and vanishes, i.e. \(\langle \Delta|\omega_k|\ell_n\rangle = 0\) for \(k < 0\), one expects the mode with \(k = -1, -2\) in (3.16) to vanish. This is confirmed in the next subsection.

3.4. Consistency check of the representation

One can check the representation (3.12) and (3.21) is compatible with the commutation relation of the Virasoro and \(\psi_3\) modes [15]:

\[
[L_p, L_q] = (p - q)L_{p+q} + \frac{c}{12} (p^3 - p)\delta_{p,-q},
\] (3.22)

\[
[L_p, W_q] = (2p - q)W_{p+q},
\] (3.23)

\[
-\frac{2}{9} [W_p, W_q] = \frac{c}{3 \cdot 5!} (p^2 - 1)(p^2 - 4)p\delta_{p,-q} + \frac{16}{22 + 5c} (p - q)\Lambda_{p+q} + (p - q) \left( \frac{1}{15} (p + q + 2)(p + q + 3) - \frac{1}{6} (p + 2)(q + 2) \right) L_{p+q}
\] (3.24)

where\(^2\)

\[
\Lambda_p = \sum_{k=-\infty}^{\infty} : L_k L_{p-k} : + \frac{1}{5} x_p L_p,
\]

\[
x_{2\ell} = (\ell + 1)(\ell - 1), \quad x_{2\ell+1} = (2 + \ell)(1 - \ell)
\]

and the central charge \(c = 2 + 12Q^2\).

Note that the irregular module \(|\ell_n\rangle\) with rank \(n\) is defined as a simultaneous eigenstate of \(L_k\)'s and \(W_k\)'s:

\[
L_k|\ell_n\rangle = \Lambda_k|\ell_n\rangle, \quad n \leq k \leq 2n
\] (3.25)

\[
W_k|\ell_n\rangle = M_k|\ell_n\rangle, \quad 2n \leq k \leq 3n
\] (3.26)

where \(\Lambda_k\) and \(M_k\) are eigenvalues. In addition, one requires that the action of \(L_{k>2n}\) and \(W_{k>3n}\) vanish.

The eigenvalues are not enough to define the irregular module. One needs \(L_{k<n}\) and \(W_{k<2n}\). The Virasoro generator has the representation \(L_k\) for \(0 \leq k < n\) in (3.12) and \(\psi_3\) generator \(\omega_k\) for \(-2 \leq k < 2n\) in (3.21). However, this representation is not enough to check the commutation relation (3.24). This is because \(\Lambda_p\) contains the negative mode \(L_{k<0}\) of Virasoro generators and hence the non-negative modes are not closed by themselves in the commutation relation. On the other hand, one cannot obtain the information of the negative modes from equation (3.20) because the negative mode contribution vanishes in the expectation value. Therefore we need another way to find the negative mode representation.

To find the negative modes we consider the identity (3.14) and its companion for the extended partition function \(Z_{n-k}\) for \(k < -1:\)

\(^2\) If one rescales \(W_p\) as \(\frac{1}{\sqrt{c}} W_p\), then the algebra reduces to the original Fateev and Zamolodchikov convention [15].
\( (v_k^0 + v_k^o) Z_{n-k} = 0, \quad v_k^o \equiv - \sum_{-r+s+k} \frac{r s t}{4} \left( \frac{\partial}{\partial b_{r+s} b_{r+s}} + \frac{\partial}{\partial c_{r+s} c_{r+s}} \right) \) (3.27)

The invariant property of the partition function shows that one has \( L_k \) with \( v_k = v_k^0 + v_k^o \) for \( k < -1 \) with the extended set of parameters \( \{ b_{k>0}, c_{k>0} \} \) when necessary.

Likewise for \( \mathcal{Y} \), one has the desired identity if one considers the transformation \( x_j \rightarrow x_j + \sum_{i}^{N_j} \frac{\varepsilon_i}{(v_i - \gamma_j v_i)} \) and \( y_j \rightarrow y_j + \sum_{i}^{N_j} \frac{\varepsilon_i}{(v_i - \gamma_j v_i)} \) to obtain

\((\mu_k^0 + \mu_k^o) Z_{2n-k} = 0 \quad \text{for} \quad k = -1, -2 \quad (3.28)\)
\((\mu_k^0 + \mu_k^o + \mu_k^o) Z_{2n-k} = 0 \quad \text{for} \quad k \leq -3, \quad (3.29)\)

where

\[ \mu_k^o \equiv - \sum_{-r+s+k} \frac{r s t}{8} \left( 3 \frac{\partial}{\partial b_{r+s} b_{r+s}} - \frac{\partial}{\partial c_{r+s} c_{r+s}} \right) \] (3.30)

Equation (3.28) shows that one has \( \mu_k Z_{2n-k} = 0 \) for \( k = -1, -2 \), or \( \langle \Delta \mu_k \rangle L_n = 0 \) as asserted in sec 3.3. The invariant property of the partition function (3.29) allows one to put

\( \mu = \mu_k^0 + \mu_k^o + \mu_k^o \) for \( k \leq -3 \).

In fact, the negative mode representation is easily understood if one notes that this representation is found from the coherent state representation of Heisenberg algebra whose positive mode \( a_k \) has the eigenvalues \( b_k \) or \( c_k \) when \( k > 0 \). In the coherent state representation, the negative mode is given as the differential operator \( - \frac{\partial}{\partial b_k} \) when \( k > 0 \).

This is the reason why one- and two-derivative terms appear in the negative Virasoro mode representation while one-, two- and three-derivative terms appear in the negative mode representation as appeared in [9].

4. Irregular partition function

4.1. Differential equations

The loop equations (3.4) and (3.5) give a series of differential equations for the partition function \( Z_n \). Large \( z \) expansion gives \( 2n \)-differential equations:

\[-v_k \log Z_n = d_k, \quad 0 \leq k \leq n - 1 \quad (4.1)\]
\[-\mu_k \log Z_n = e_k, \quad n \leq k \leq 2n - 1 \quad (4.2)\]

where \( v_k \) and \( \mu_k \) are differential operators defined in (3.12), (3.21). One may also find differential equations corresponding to \( \mu_k \) for \( k < n \). However, this equation contains the extended set of parameters \( \{ b_{k>0}, c_{k>0} \} \) and is redundant since we have \( 2n \)-equations (4.1), (4.2) which will completely fix the partition function. \( d_k \) and \( e_k \) in (4.1) and (4.2) are given as

\[ d_k = A_k + \sum_{r+s=k} \left( d_r^e d_s^e + d_r^f d_s^f \right), \quad e_k = M_k + \sum_{r+s=k} \left( d_r^e d_s^e - 3d_r^b d_s^b d_r^f d_s^f \right) \]
\[ d_r^f = \frac{\sqrt{3}}{2} \left( \sum_{j}^{N_r} x_j^f \right) + c_r, \quad d_r^b = \left( \sum_{j}^{N_r} x_j^f \right) - \frac{1}{2} \left( \sum_{j}^{N_r} x_j^f \right) + b_r. \] (4.3)

Here we use the convention of the coefficients \( b_k = c_k = 0 \) when \( k > n \). Note that \( d_k \) and \( e_k \) in (4.3) are unknown except \( d_0 \) since they are given in terms of expectation values. One has to find these expectation values as the function of the coefficients of the potential explicitly. This
can be done by finding the filling fraction with the resolvents $R_1(z)$ and $R_2(z)$ across a given branch cut. This idea is used to evaluate the partition function in section 4.3.

### 4.2. Loop equations and spectral curve

The resolvents $R_1$ and $R_2$ have the asymmetric role in the loop equation. This is because we put the loop equation asymmetrically. One may formulate the loop equation in a symmetric way. Suppose we introduce

$$u_1(z) := R_1(z) + t_1(z), \quad t_1(z) = -\frac{2V'_1(z) + V'_2(z)}{3}, \quad (4.4)$$

$$u_2(z) := -R_2(z) + t_2(z), \quad t_2(z) = \frac{V'_1(z) + 2V'_2(z)}{3} \quad (4.5)$$

and $u_0(z) := -u_1(z) - u_2(z)$. Then, using the two loop equations (3.4) and (3.5) one finds a spectral curve in a cubic form [12–14, 16]

$$\prod_{i=0}^2 (u - u_i(z)) = u^3 + \frac{h^2 \xi_2(z)}{4} u - \frac{h^3 \xi_3(z)}{12\sqrt{3}} = 0. \quad (4.6)$$

This shows that $u_2$ is the solution of the cubic equation which is exactly the same form as the loop equation given in (3.4) and (3.5). The spectral curve also demonstrates that $u_1$ and $u_0$ are two other solutions. Therefore, $u_1$ and $u_0$ should respect the same loop equation (3.4) and (3.5).

To understand the three branches $u_0$, $u_1$, $u_2$, we first the case with no quantum correction, i.e. $f_1 = f_2 = g_1 = g_2 = 0$. In this case $\xi_2$ and $\xi_3$ are given in terms of the potentials only and the three roots $u_1$, $u_2$, $u_0$ are reduced to $t_1$, $t_2$ and $t_0(z) := -t_1(z) - t_2(z)$ defined in (4.4), (4.5). The classical spectral curve has common points. For example, when $t_0(z) = t_1(z)$ one finds $V'_1(z) = 0$. The stationary point of the potential $V_1$ corresponds to the double points, the intersect of the branches $t_0$ and $t_1$. Likewise, $t_0(z) = t_2(z)$ corresponds to the case $V'_2(z) = 0$, and $t_1(z) = t_2(z)$ to $V'_1(z) + V'_2(z) = 0$. This shows that the three classical branches are connected to each other at the stationary points of the potentials.

If the spectral curve is deformed by $f_1$, $f_2$, $g_1$ and $g_2$, then the double point splits and forms a branch cut. As a result, $3n$ branch cuts appear on the complex plane $z$ when all the double points are distinct. Let us denote the branch cuts as $l_{(i)}^{(i)}$ connecting two branches $u_i$ and $u_0$ with $i = 1, \ldots, n$ (see figure 1).

One may count the number of eigenvalues of the potential (the number of integration variables) by taking the contour integral of the resolvent around the cut. Let us denote the contour around the cut $l_{(i)}^{(i)}$ as $A_{(i)}^{(i)}$ assuming the branch index $k \equiv 3$. One may find the filling fractions (number of eigenvalues) using $u_k$,

$$\frac{hb}{2} n_{k+1}^{(i)} := \frac{1}{2\pi i} \oint_{A_{(i)}^{(i)}} u_k(z) dz \quad (4.7)$$

or using $u_{k+1}$,

$$\frac{hb}{2} n_{k+1}^{(i)} := \frac{1}{2\pi i} \oint_{A_{(i)}^{(i)}} u_{k+1}(z) dz. \quad (4.8)$$

These two quantities add up to zero: $n_{k+1}^{(i)} + n_{k+1}^{(i)} = 0$ since

$$u_k(\lambda + i0) - u_k(\lambda - i0) = -(u_{k+1}(\lambda + i0) - u_{k+1}(\lambda - i0)) \quad \text{for} \quad \lambda \in l_{(i)}^{(i)} \quad (4.9)$$

This is due to the fact that $u_k + u_{k+1} + u_{k+2} = 0$ and $u_{k+2}$ is continuous on the cut $l_{(i)}^{(i)}$. Therefore, one may have $N = N_1 + N_2$ where
Note the minus sign in $N_2$ comes from the definition of $u_2$ in (4.5.).

4.3. Evaluation of the partition function

We evaluate the partition function of rank 1 explicitly in this subsection. The rank 1 partition function has two differential equations

$$
-v_0 \log Z_1 = d_0, \quad -\mu_1 \log Z_1 = e_1
$$

(4.11)

where $v_0 = b_1 \frac{\partial}{\partial b_1} + c_1 \frac{\partial}{\partial c_1}$, $\mu_1 = -3b_1c_1 \frac{\partial}{\partial b_1} - \frac{3}{2} (b_1^2 - c_1^2) \frac{\partial}{\partial c_1}$ and $d_0, e_1$ are defined in (4.3). Since $d_0$ is a constant

$$
d_0 = (bN_1)^2 - bN_1 bN_2 + (bN_2)^2 + 2b_0 bN_1 - b_0 bN_2 + \sqrt{3} c_0 bN_2
$$

(4.12)

one has the solution of the first equation in (4.11) of the form,

$$
\log Z_1 = -d_0 \log c_1 + H (t)
$$

(4.13)

where $t := b_1/c_1$ and $H(t)$ is a homogeneous solution to $v_0$.

Plugging this into the second one in (4.11), one obtains

$$
(3 - t^2) \frac{\partial H (t)}{\partial t} = \frac{2}{3} e_1 + (t^2 - 1) d_0.
$$

(4.14)

where $e_1$ is given in terms of the expectation values \( \sum_i x_i \) and \( \sum_j y_j \),

$$
e_1 = 3b_0^2 c_1 - 3c_0^2 c_1 + 6b_0 c_0 b_1 + \frac{3}{8} (2c_0 + \sqrt{3} N_2)^2 \left( 2c_1 + \sqrt{3} \left( \sum_j y_j \right) \right)
- \frac{3}{8} (2b_0 + 2N_1 - N_2) \left[ 2 \left( 2b_1 + 2 \left( \sum_i x_i \right) - \left( \sum_j y_j \right) \right) (2c_0 + \sqrt{3} N_2)
+ (2b_0 + 2N_1 - N_2) \left( 2c_1 + \sqrt{3} \left( \sum_j y_j \right) \right) \right].
$$

(4.15)
To solve (4.14), we need the explicit from of $\epsilon_1/c_1$ as the function of $t$. This can be done using the constraint of the filling fraction.

For rank 1, we have three cuts $I_{[01]}$, $I_{[12]}$ and $I_{[20]}$ whose classical double points are $-b_n/b_0$, $-h_n - i\sqrt{3}c_n/b_0 + 3c_n$ and $-h_n + i\sqrt{3}c_n/b_0 - 3c_n$, respectively. However, one filling fraction is enough, say

$$\frac{1}{2\pi i} \oint_{\gamma_{[01]}} u_1(z) dz = \frac{\hbar b}{2} n_{[01]}.$$ (4.16)

The branch $u_1$ in (4.6) is given as

$$\frac{1}{\hbar} u_1(z) = \frac{(\sqrt{3} + i\beta)P_2 + (\sqrt{3} - i\beta)(P_3 + \sqrt{P_2^2 - P_3^2})^{2/3}}{12(P_3 + \sqrt{P_2^2 - P_3^2})^{2/3}}$$ (4.17)

where $P_2$ and $P_3$ are polynomials given by

$$P_2(z) = (b_0^2 + c_0^2 + d_0)z^2 + 2(b_0b_1 + c_0c_1)z + b_1^2 + c_1^2,$$
$$P_3(z) = (c_0^2 - 3b_0^2c_0 + e_0)z^3 + (3c_0^2c_1 - 3b_0^2c_1 - 6b_0c_0b_1 + e_1)z^2 - 3(c_0b_1^2 + 2b_0b_1c_1 - c_0c_1^2)z + c_1^3 - 3b_0^2c_1.$$

Here $e_0$ is a constant, $e_0 = 3b_0^2c_0 - c_0^3 - 3\left(\frac{N_1 - N_2}{2} + b_0\right)^2\left(\frac{N_1 - N_2}{2} + c_0\right) + \left(\frac{\sqrt{3}}{2}N_2 + c_0\right)^3$. Six roots of $(P_3^2 - P_2^2)$ implies three branch cuts on the $z$-plane. (Note that a cubic branch cut whose branch points are roots of $(P_3 + \sqrt{P_2^2 - P_3^2})$ is not reduced to the classical point and no eigenvalues lie on it. Thus we don not need to take account of this cut.)

To find the filling fraction perturbatively, we assume that $|t| \ll 1$. In this case, the cut $I_{[01]}$ is widely separated from the others. Rescaling the integration variable in the contour integral (4.16) with $b_1$, the cut $I_{[01]}$ has a finite width whereas other cuts shrink to a point as $t \to 0$. Thus, one can expand the $u_1(z)$ in (4.17) safely in powers of $t$ to get $(P_3^2 - P_2^2) = b_1^2 c_1^2 [Q_0 + tQ_1 + O(t^2)]$. However, the explicit form of $Q_i$ depends on the relative scales of $e_i$. The small $t$-expansion assumes that $|b_1| < |c_1|$. In addition, since $e_1$ is a quantum deformation, $|c_1| < |e_1|$ is not allowed. Therefore, we have two different relative scales: (I) $|e_1| \lesssim |b_1| \ll |c_1|$ and (II) $|b_1| \ll |e_1| \lesssim |c_1|$.

Let us first consider the case (I). In this case $\tilde{e}_1 := e_1/b_1$ is small. Assuming $\tilde{e}_1 = O(t^0)$ at most, one has

$$Q_0(z) = 3((3b_0^2 + d_0)z^2 + 6b_0z + 3)$$
$$Q_1(z) = z((3b_0^2c_0 + 12c_0d_0 - 2e_0)z^2 + (72b_0c_0 - 2\tilde{e}_1)z + 36c_0).$$ (4.18)

The filling fraction $n_{[01]}$ is given in powers of $t$,

$$bn_{[01]} = \frac{1}{\pi i} \oint_{\gamma_{[01]}} dz \left[ c_0 z + \frac{\sqrt{Q_0(z)/3}}{2\sqrt{3}z^2} - \frac{\tilde{e}_1 + e_0 - d_0\sqrt{Q_0(z)/3}}{6\sqrt{3}\sqrt{Q_0(z)/3}} + O(t^2) \right]$$
$$= -b_0 + \sqrt{b_0^2 + \frac{d_0}{3}} - \frac{(d_0 + 3b_0^2)e_0 - 3b_0c_0}{3\sqrt{3}(d_0 + 3b_0^2)^{3/2}} + O(t^2).$$ (4.19)

Noting $\tilde{e}_1 = O(t^0)$, one has to require the filling fraction $bn_{[01]} = -b_0 + \sqrt{b_0^2 + d_0/3} + t\Delta n_{[01]}$ so that $\Delta n_{[01]} = O(t^0)$. Since we regard $n_{[01]}$ as the controlling parameter, we force $\Delta n_{[01]} = 0$ so that
In this case, one has

\[\bar{e}_1 = \frac{3b_0e_0}{3b_0^2 + d_0} - \frac{3(d_0(d_0^3 + 2c_0d_0e_0 - e_0^2) + b_0^2(3d_0^3 + 6c_0d_0e_0 + 2e_0^2))}{4(3b_0^2 + d_0)^3} + \mathcal{O}(t^2)\]  

and finds the partition function from (4.14)

\[Z_1 = \mathcal{N} b_1 \frac{4\pi}{c_1} \frac{2\pi e_0}{e_N \varepsilon_0} T e^{\frac{3}{2} \phi_1} + \mathcal{O}(t^2)\]  

where \(\mathcal{N}\) is the normalization independent of \(t\).

For case (II), one finds a quite different structure from that of (I). Note that \(|b_1| < |e_1|\). This parameter domain allows one to put \(b_1 = 0\). In addition, \(b_1\) should be zero when the quantum correction \(e_1\) is zero. In this case, the double point of the classical branches of \(t_0\) and \(t_1\) does not exist on the finite domain of the complex plane since it is now given by the stationary point of \(V_1\), satisfying \(V_1'(z) = \frac{6}{2} = 0\). Considering this, one may force \(n_{1[0]} = 0\) which is true at the classical level. With \(b_1\) which can be 0, we had better introduce a new parameter \(\hat{e}_1 = e_1/c_1\) instead of \(\bar{e}_1\). Assuming \(\hat{e}_1 = \mathcal{O}(t^0)\) at most, one has

\[Q_0(z) = (9b_0^2 + 3d_0 - 2\hat{e}_1)z^2 + 18b_0z + 9\]
\[Q_1(z) = z((36b_0^2c_0 + 12c_0d_0 - 2e_0 - 6c_0\hat{e}_1)z^2 + 72b_0c_0z + 36c_0)\]

and the filling fraction has the form

\[bn_{1[0]} = \frac{1}{\pi i} \oint_{\mathcal{A}_{[01]}} dz \left[ \frac{\sqrt{3}c_0z + \sqrt{Q_0(z)}}{6c^2} + i \frac{9(c_0\hat{e}_1 - e_0)z + \sqrt{3}(3d_0 + \hat{e}_1)\sqrt{Q_0(z)}}{54\sqrt{Q_0(z)}} + \mathcal{O}(t^2) \right] \]

Putting \(n_{1[0]} = 0\), one finds

\[\hat{e}_1 = \frac{3}{2} d_0 + i \frac{2e_0 - 3c_0d_0}{2b_0} + \mathcal{O}(t^2)\]  

and the partition function from (4.14),

\[Z_1 = \hat{\mathcal{N}} c_1^{-d_0} e^{\frac{3}{2}\phi_1} + \mathcal{O}(t^2)\]  

with \(\hat{\mathcal{N}}\) a normalization independent of \(t\). In this evaluation, we assume that \(b_1 > 0\) and the vanishing filling fraction is obtained when \(b_0 > 0\). The same is true if one assume \(b_1 < 0\) with \(b_0 < 0\). The vanishing filling fraction condition holds as far as \(b_0b_1 > 0\), which is the case \(V_1\) having no stationary point in our complex domain. (Note that due to the logarithmic potential, we exclude the negative real axis in our complex domain.)

Suppose we put \(b_1 \equiv 0\) from the beginning and require the filling fraction \(n_{1[0]}\) to vanish. In this case, one finds that the partition function is simply reduced to \(Z_1 = \hat{\mathcal{N}} c_1^{-d_0}\) since \(t = 0\). In addition, the two differential operators \(\nu_0\) and \(\mu_1\) are not independent but has the relation \(\mu_1 = \frac{3}{2} c_1 \nu_0\). The case with \(b_1 = 0\) is identified in [10] with the semi-degenerate where
the regular module has the null vector at the first level [15, 17, 18]. In our approach, if the semi-degenerate module is put at infinity, we have the semi-degenerate $\langle \Delta \rangle$ in (3.20) whose conformal dimension is $\Delta = \tilde{a}_\infty \left( \tilde{Q} - \frac{1}{2} \tilde{a}_\infty \right)$ with $\tilde{a}_\infty = z \tilde{w}_2$, where $\tilde{w}_2$ is the fundamental weight, satisfies $\bar{\tilde{e}} \cdot \tilde{e} = \delta_{0i}$. In this case, we have $N_1 = 0$ as the semi-degenerate result.

5. Summary and discussion

We generalize the (Virasoro) irregular matrix model so that the model contains the Virasoro and $\mathcal{W}_3$ symmetry. This model is constructed using the colliding limit of $A_2$ Toda field theory. The symmetry of the irregular models is analyzed through loop equations which have quadratic and cubic forms. The spectral curve obtained corresponds to the Seiberg–Witten curve of the $SU(3)$ super-conformal linear quiver theory.

It should be emphasized the spectral curve (4.6) is enough to find the partition function of the irregular matrix model without evaluation of the functional integral or Selberg integral. Using the differential representation of the Virasoro and $\mathcal{W}_3$ symmetry generators we derive the differential equations for the partition function from the loop equations. The differential equations allow us to find the partition function of the irregular model. We present the explicit form of the representation and find the partition function to the lowest order of $h$ (corresponding to the large $N$ limit) for the non-trivial case (irregular module with rank 1). It is not difficult to find the partition function with rank greater than 1 if one uses the parameter scale as $\left| \frac{h_{k+1}}{h_k} \right| \ll \left| \frac{h_{k+1}}{h_{k-1}} \right| \ll \left| \frac{h_{k+1}}{h_{k-2}} \right|$ and $\left| \frac{h_{k+1}}{h_k} \right| \ll \left| \frac{h_{k+1}}{h_{k-1}} \right|$ as was used in Liouville case [1, 6].

Once the partition function is known, one may construct the irregular conformal block (ICB), noting that the partition function with appropriate potential is related to an inner product between an irregular module and regular/irregular modules. The simplest ICB, the inner product $\langle \mathcal{L}_n | \mathcal{L}_n \rangle$ is given in terms of an irregular matrix model. One may consider the irregular partition function $Z_{(m,n)}$ where an irregular module of rank $n$ lies at the origin and one with rank $m$ at infinity:

$$Z_{(m,n)} = \int \prod_{i=1}^{N_1} \prod_{j=1}^{N_j} dx_i dy_j \Delta(x)^{2j} \Delta(y)^{2j} \Delta(x, y)^{-\beta} e^{-\frac{\beta}{2} \left[ \sum_i \mathcal{V}^{(a_i)}(x_i) + \sum_j \mathcal{V}^{(a_j)}(y_j) \right]}.$$

$$\frac{V_1(z)}{h} = -b_0 \log z + \sum_{k=1}^{n} \frac{b_k}{k^2} + \sum_{k=1}^{m} \frac{k^2}{k^2} \ell,$$

$$\frac{V_2(z)}{h} = -\tilde{c}_0 \log z + \sum_{k=1}^{n} \frac{\tilde{c}_k}{k^{2k}} + \sum_{k=1}^{m} \frac{\tilde{c}_k}{k^{2k}} \ell.$$

There are a few subtle issues when one identifies this partition function with the inner product $\langle \mathcal{L}_n | \mathcal{L}_n \rangle$. One is the extra contribution due to the colliding limit: $\prod_{i=0}^{n} \left( 1 - z_i / z_0 \right)^{-\tilde{a}_i / k_3} \to e^{\zeta_{(m,n)}}$ with $\zeta_{(m,n)} = \sum_{k=1}^{\min(m,n)} (2b_k b_{-k} + c_k c_{-k}) / k$ as $z_0 \to 0$, $z_0 \to \infty$. This non-trivial contribution was considered in [7] for the $A_1$ case. The other is about the normalization for the case with rank greater than 1. One has to take account the normalization properly because of the consistency condition for the Virasoro and $\mathcal{W}$ symmetry [6]. This consideration leads to the ICB $\mathcal{F}_{(m,n)}$:

$$\mathcal{F}_{(m,n)} = \frac{e^{\zeta_{(m,n)}} Z_{(m,n)}}{Z_{(0,n)} Z_{(m,0)}}.$$


One may find the complete representation of the symmetry generators which has non-leading order of $\hbar$ with $Q = 0$ without difficulty considering the full equations given in (3.2), (3.3). In addition, the irregular matrix model is easy to generalize into the $A_k$ model since it is composed of a $k$-matrix potential along with the Vandermonde determinant whose power is given in terms of the Dynkin index. More than cubic power will appear in the loop equations. To control these complicated equations, one may resort to DDAHA as demonstrated in [19]. This will be presented elsewhere in the future.

Finally, the irregular matrix model is motivated by Argyres–Douglas theory [20, 21] in connection with AGT conjecture [22], which develops irregular punctures to the holomorphic one-form of the Hitchin system [23–25]. So far, the Virasoro case has been intensively investigated using the irregular matrix model. Extension to $W$ symmetry will provide a useful tool to study Argyres–Douglas theory corresponding to $SU(N)$ gauge theory.

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