Celestial Mellin amplitude

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ABSTRACT: Celestial holography provides a promising avenue to studying bulk scattering in flat spacetime from the perspective of boundary celestial conformal field theory (CCFT). A key ingredient in connecting the two sides is the celestial amplitude, which is given by the Mellin transform of momentum space scattering amplitude in energy. As such, celestial amplitudes can be identified with the correlation functions in celestial conformal field theory. In this paper, we introduce the further notion of celestial Mellin amplitude, which is given by the Mellin transform of celestial amplitude in coordinate. For technical reasons, we focus on the celestial Mellin amplitudes for scalar fields in three dimensional flat spacetime dual to 1D CCFT, and discuss the celestial Mellin block expansion. In particular, the poles of the celestial Mellin amplitude encode the scaling dimensions of the possible exchanged operators, while the residues there are related to the OPE coefficient squares in a linear and explicit way. We also compare the celestial Mellin amplitudes with the coefficient functions which can be obtained using inversion formulae. Finally, we make some comments about the possible generalizations of celestial Mellin amplitudes to higher dimensions.

KEYWORDS: Conformal and W Symmetry, Gauge-Gravity Correspondence, Scattering Amplitudes

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### 1 Introduction

Over the past few years, there has been some dramatic progress in studying holography for quantum gravity in the absence of cosmological constant. Such a new program of holographic duality is dubbed celestial holography. The crucial point of celestial holography stems from the fact that the Lorentz group in \( d \)-dimensional Minkowski spacetime \( \text{SO}(d-1,1) \) can be identified with the conformal group in \( (d-2) \) dimensions. Moreover, near the null boundary of flat spacetime, the symmetry group is even enhanced to the infinite dimensional asymptotic symmetry, called Bondi-van der Burg-Metzner-Sachs (BMS) symmetry [1, 2]. As shown in [3, 4], the infinite dimensional BMS symmetry can be used to understand the soft theorems of graviton scattering amplitudes. Since then, many interesting results related to the infrared structure of gravity were discovered, see [5] for a review. Furthermore, more and more non-trivial evidence has been accumulated, which indicates that the \( d \)-dimensional quantum gravity in flat spacetime can be described in terms of \( (d-2) \)-dimensional conformal field theory, called celestial conformal field theory.
(CCFT) living at null infinity. Just like the well-known AdS/CFT correspondence, such a kind of relation between bulk gravity and boundary field theory is reminiscent of the principle of holography, and is referred to as celestial holography. The connection between the bulk and boundary can be understood more directly by introducing the so-called celestial amplitude [6, 7], which is nothing but the Mellin transform of the momentum space scattering amplitude. Consequently, one can regard the celestial amplitudes as the correlation functions in CCFT. Nowadays, celestial holography has become a very rich and interesting subject. See [8, 9] for a review.

As such, understanding quantum gravity in flat Minkowski spacetime boils down to the study of celestial conformal field theory. Compared to the standard CFT, CCFT shares lots of similarities but also features many peculiarities. Little is known about the fundamental axioms for defining CCFT, and most of the understanding of CCFT is based on symmetry. For example, the notion of unitarity in CCFT is not clear. This should inherit from the unitarity of bulk quantum gravity, and is very different from unitarity in conventional CFTs. As a result, the scaling dimension of operators in CCFT can be even complex, whose implication is still vague. Nevertheless, one may hope to borrow some techniques from standard CFT, and try to understand some aspects of CCFT. A notable success is the operator product expansion (OPE) in CCFT, which arises from the collinear limit of bulk scattering amplitudes [10–13]. This enables us to understand the short distance behaviors and operator algebras in CCFT. Moreover, it also helps us to reveal the hidden new structure of CCFT, like the \( w_{1+\infty} \) symmetry [14–17] which is difficult to anticipate using standard techniques.

A natural question is how far can we go by importing the techniques from standard CFTs? In particular, recently there is a huge amount of progress in studying standard CFTs both analytically and numerically, under the name of conformal bootstrap. Many interesting and powerful techniques are developed. One of them is the Mellin amplitude [18]. The Mellin techniques are very useful in bootstrapping CFTs both perturbatively and non-perturbatively. The Mellin amplitude also offers an excellent tool in studying holographic CFTs and thus the corresponding AdS quantum gravity [19]. It can be regarded as the natural AdS analogue of flat space scattering amplitude, and the latter can be recovered from Mellin amplitude in a suitable limit [20]. The Mellin amplitude has simple poles corresponding to the exchanged operators, and the crossing symmetry of the CFT correlator maps to the amplitude crossing symmetry in Mellin space. A very successful application of Mellin techniques is the calculation of holographic correlators in \( AdS_5 \times S^5 \), which is simple when written in the Mellin basis, and can be bootstrapped based on general consistency conditions and maximal supersymmetry without worrying about the complicated interactions of bulk supergravity [21].

Considering the powerful role of Mellin space in AdS/CFT, in this paper, we would like to import the techniques of Mellin amplitudes to celestial holography. More precisely, we will introduce the notion of celestial Mellin amplitude. Recall that starting with momentum space scattering amplitudes, one can perform the Mellin transformations in energies and obtain the so-called celestial amplitudes. We will also refer to celestial amplitudes as celestial correlators, because they are indeed identified with the correlators in CCFT. As in
standard CFT, we can further consider the Mellin amplitudes for the celestial correlators in CCFT, and this naturally leads to the definition of celestial Mellin amplitudes. The relationship of various kinds of amplitudes in celestial holography is then illustrated below:

Although the Mellin amplitudes have been well studied and understood in standard CFTs, the implementation of Mellin amplitudes in CCFTs suffers various problems. A notable feature of CCFT correlators is that they come with some kinematical constraints arising from momentum conservation. For example, in four and higher dimensional bulk spacetime, the four-point correlators always contain a delta function, which enforces the positions of four operators to lie on the intersection of the celestial sphere and the scattering plane. This delta function makes the analytic structures of correlation functions obscure and is absent in standard CFTs. Therefore it is not obvious to what extent it is meaningful to directly apply the techniques of Mellin amplitude in standard CFT. Instead, here we will be mainly focusing on the three dimensional spacetime where the CCFT lives on the one-dimensional celestial circle. The restriction to three dimensions then avoids the problem of kinematical constraint, and the delta function is absent in the four-point celestial correlators of 1D CCFT. As a result, it seems that we can then apply the standard Mellin techniques to our one dimensional CCFT. However, the standard Mellin amplitude is normally used for CFTs in high enough dimensions, and seems to suffer from some subtleties in one dimension due to additional kinematical constraints.\(^1\) To further avoid these subtleties, we will use the Mellin techniques specially designed for 1D CFT in [23]. In such a case, the Mellin amplitude is literally given by the Mellin transform in the coordinate of 1D CFT correlators. This is the strategy that we will adopt in this paper.

To be even more precise and for simplicity, in this paper, we will be mainly focusing on the celestial Mellin amplitudes corresponding to the scattering of four identical scalar particles in three dimensional spacetime. The celestial Mellin amplitude is given by the double Mellin transformations of momentum space scattering amplitude, one in energy (2.12) and the other in coordinate (2.27). We will also consider the celestial Mellin block expansion, namely the decomposition of celestial Mellin amplitudes into the sum of Mellin blocks for different exchanged operators. A very nice feature of the celestial Mellin blocks is that they are by construction crossing symmetric. It turns out that the celestial Mellin amplitudes enable us to perform such an expansion easily. In particular, the poles of celestial Mellin amplitudes encode the scaling dimensions of possible exchange operators, while the residues there are related to the OPE coefficients in a specific way. We derive the explicit formula (2.56) to compute the OPE coefficients directly from the residues of celestial Mellin amplitude at various poles. The resulting celestial Mellin block expansion agrees with the

\(^1\)Nevertheless, [22] implemented such a kind of Mellin technique in bootstrapping 1D CFT.
conformal block expansion for 1D CCFT. We will also compare the celestial Mellin amplitudes with the coefficient functions which can be derived from Euclidean or Lorentzian inversion formulae [24, 25]. They are related by integral transformation (2.65). The poles of the coefficient functions give precisely the scaling dimensions of exchange operators, while the residues there are precisely the OPE coefficient squares. However, coefficient functions are usually quite complicated and difficult to compute explicitly. This is in sharp contrast with celestial Mellin amplitudes which are much easier to calculate. We will demonstrate all these features in a very concrete example of scalar fields in 3D.

Although we are mainly focusing on the celestial Mellin amplitude in 3D, we will also explore several possible generalizations to higher dimensions. As we discussed before, the main subtlety comes from the delta function constraint in the four-point correlators of CCFT. In the simplest scenario, one may ignore this problem and blindly apply the Mellin techniques in standard CFT. A better way to avoid the delta function is to perform the shadow or light transform on the CCFT correlators, and then study their corresponding Mellin amplitudes. The benefit of performing shadow or light transforms enables us to remove the delta functions, thus making the CCFT correlator better behaved. Finally, we may regard the four-point correlators in CCFT as one dimensional defect correlators, although the defect is actually trivial whose only role is restricting the positions of operators. On such a one dimensional defect CFT, we can then apply the Mellin transform directly, just like the case of 3D bulk dual to 1D CCFT. All these different approaches have their pros and cons, and they are supposed to be related in a non-trivial way. We will not study their connections in detail here, but leave this important question to the future.

The rest of the paper is organized as follows. In section 2, we introduce the notion of celestial Mellin amplitude in 3D and discuss its block expansion. In section 3, we consider a very explicit example of scalar fields in 3D, and compute its celestial Mellin amplitude as well as block expansion. We will also calculate the coefficient functions in order to compare with the celestial Mellin amplitude. In section 4, we briefly comment on the generalization of celestial Mellin amplitude to higher dimensional spacetime. In section 5, we conclude and discuss some possible future directions. In appendix A, we prove a claim about the inverse of an infinite dimensional matrix, relating residues and OPE coefficients. In appendix B, we include many technical details of the coefficient functions derived from Euclidean and Lorentzian inversion formulae.

2 Celestial Mellin amplitude in 3D

In this section, we introduce the notion of celestial Mellin amplitudes in 3D. We will first review the celestial amplitudes, and then define the celestial Mellin amplitudes by performing another Mellin transformation in coordinate. We will also discuss the Mellin block expansions of celestial Mellin amplitudes, and derive the key formula (2.60) relating the OPE coefficients and residues of celestial Mellin amplitudes. We will also relate the celestial Mellin amplitudes with the coefficient functions computed by inversion formulae. Finally, we will describe the Regge behavior of celestial Mellin amplitudes.

Lorentzian inversion formulae is another powerful CFT technique and also plays an important role in studying holographic CFTs. It would be very interesting to investigate its application in celestial holography.
2.1 Celestial amplitude

Let us first introduce the celestial amplitudes. For our purpose, we will be mainly considering the massless scalar fields in (2+1)D. The starting point originates from the fact that the Lorentz group in (2+1)D coincides with the conformal group in 1D, namely $\text{SO}(2,1) \simeq \text{SL}(2, \mathbb{R})$. To make the relation manifest, we can parameterize the momentum of each particle as follows:

$$p^\mu = \epsilon \omega q^\mu(x), \quad q^\mu = (1 + x^2, 2x, 1 - x^2), \quad q \cdot q = 0.$$ \hspace{1cm} (2.1)

where $\omega > 0$ is the energy along null direction $q^\mu$, and $\epsilon$ labels whether the particle is incoming ($\epsilon = -1$) or outgoing ($\epsilon = +1$). Here $x \in \mathbb{R}$ is just the (stereographic) coordinate on the celestial circle. It is easy to check that performing a global conformal transformation $\text{SL}(2, \mathbb{R})$ on $x$ together with a corresponding transformation on $\omega$ leads to a bulk Lorentz transformation $\text{SO}(2,1)$ on $p^\mu$.

In general, the scattering amplitude of $n$ massless scalar fields in (2+1)D can be written as

$$A(p_i) = A(p_i) \delta^3 \left( \sum_{i=1}^{n} p_i \right).$$ \hspace{1cm} (2.2)

The delta function in (2.2) enforces the momentum conservation, making the translational symmetry manifest.

Now, we would like to have manifest Lorentz symmetry. This can be achieved by introducing the so-called celestial amplitude, which is given by Mellin transforming the momentum space scattering amplitude [6, 7]

$$G(\Delta_i, x_i) = \prod_{i=1}^{n} \int_0^\infty d\omega_i \omega_i^{\Delta_i - 1} A\left(p_j^\mu = \epsilon_j \omega_j q^\mu(x_j)\right).$$ \hspace{1cm} (2.3)

The nice property of celestial amplitude is that it makes the Lorentz symmetry manifest. More specifically, the Lorentz transformation in the (2+1)D bulk induces a conformal transformation on the celestial circle, under which the celestial amplitude transforms as

$$G\left(\Delta_i, \frac{ax_i + b}{cx_i + d}\right) = \prod_j (cx_j + d)^{2\Delta_j} G(\Delta_i, x_i), \quad ad - bc = 1,$$

which is nothing but the transformation rule of CFT correlators. Therefore the celestial amplitudes can be regarded as the correlation functions of some operators in CCFT

$$G(\Delta_i, x_i) = \langle O_{\Delta_1}^{\epsilon_1}(x_1) \cdots O_{\Delta_4}^{\epsilon_4}(x_4) \rangle_{\text{CCFT}}.$$ \hspace{1cm} (2.5)

In particular, $\Delta_i$ is just the scaling dimension of operator in CCFT.

Let us now specialize the discussion to the case of four particle scattering. In this case, it is convenient to define the Mandelstam variables

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 + p_3)^2, \quad u = -(p_1 + p_4)^2,$$ \hspace{1cm} (2.6)

We use the signature $(- + +)$. 

\[\text{JHEP10(2022)042}\]
which are subject to the condition $s + t + u = 0$ since all the external particles are massless. Then the momentum space amplitude can be written in terms of Mandelstam variables

$$A(p_i) = A(s, t)\delta^3 \left(\sum_{i=1}^{4} p_i\right).$$

(2.7)

In order to compute the celestial amplitude, we would like to simplify (2.3). The delta function of momentum conservation in (2.7) enables us to perform the three among four integrals in (2.3), enforcing the constraint

$$\frac{\omega_2}{\omega_1} = -\frac{\epsilon_2 x_{13} x_{14}}{\epsilon_1 x_{23} x_{24}}, \quad \frac{\omega_3}{\omega_1} = \frac{\epsilon_3 x_{12} x_{14}}{\epsilon_1 x_{23} x_{34}}, \quad \frac{\omega_4}{\omega_1} = -\frac{\epsilon_4 x_{12} x_{13}}{\epsilon_1 x_{24} x_{34}}.$$  

(2.8)

where $x_{ij} = x_i - x_j$. As a result, we can express

$$s = -4x_{12}^2 x_{13} x_{14} x_{23} x_{24}, \quad t = -\frac{1}{z} s, \quad u = \frac{1 - z}{z} s,$$

(2.9)

where we introduce the cross-ratio

$$z = \frac{x_{12} x_{34}}{x_{13} x_{24}} \in \mathbb{R}.$$  

(2.10)

For simplicity, let us further assume that all the conformal dimensions are the same, $\Delta_i = \Delta_\phi$. Then the celestial amplitude has the following structure [26]

$$G(x_i) = \frac{1}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}} G(z),$$  

(2.11)

where

$$G(z) = 2^{-4\Delta_\phi+1} \frac{|z|}{\sqrt{|z-1|}} \int_0^\infty d\omega \omega^{4\Delta_\phi-4} A \left(\epsilon_1 \epsilon_2 \omega^2, -\frac{1}{z} \epsilon_1 \epsilon_2 \omega^2\right).$$  

(2.12)

One important feature of four-point celestial amplitude in (2+1)D is the absence of delta function, which is inevitable in higher dimensions due to kinematic constraints.

In the four particle scattering, there are various channels depending on which particle is incoming or outgoing. In different channels, the function $G(z)$ takes different forms, so we can write

$$G(z) = \begin{cases} 
G^(-)(z), & z \in (-\infty, 0), 
14 \leftrightarrow 23 \text{ channel}, \\
G^{(0)}(z), & z \in (0, 1), 
13 \leftrightarrow 24 \text{ channel}, \\
G^{(+)}(z), & z \in (1, \infty), 
12 \leftrightarrow 34 \text{ channel},
\end{cases}$$  

(2.13)

where we use arrows to distinguish incoming and outgoing particles; for example $12 \rightarrow 34$ indicates that 1, 2 are incoming, while 3, 4 are outgoing, thus in this case $\epsilon_1 = \epsilon_2 = -\epsilon_3 = -\epsilon_4 = -1$.

For the scattering of four identical particles, we have crossing symmetry which relates scattering amplitudes in different channels:

$$A(s, t) = A(t, s) = A(u, t), \quad s + t + u = 0.$$  

(2.14)
Using eq. (2.12), one can show that the crossing equations above lead to the following identities [26]

\[
G^{(-)}(z) = G^{(0)}\left(\frac{z}{z-1}\right), \quad z \in (-\infty, 0),
\]

(2.15)

\[
G^{(+)}(z) = z^{2\Delta_\phi}G^{(0)}\left(\frac{1}{z}\right), \quad z \in (1, \infty),
\]

(2.16)

\[
G^{(0)}(z) = \left(\frac{z}{1-z}\right)^{2\Delta_\phi}G^{(0)}(1-z), \quad z \in (0, 1).
\]

(2.17)

These are nothing but the equations arising from the Bose symmetry in 1D CFT by exchanging $1 \leftrightarrow 2, 2 \leftrightarrow 3, 2 \leftrightarrow 4$, respectively [27]. These equations also allow us to find the full profile of $G(z)$ once we know $G^{(0)}(z)$ in the range of $0 < z < 1$.

### 2.2 Celestial Mellin amplitude

In this subsection, we introduce the celestial Mellin amplitude by performing a further Mellin transform in coordinate.

As we discussed before, the most interesting feature of celestial amplitudes is that they can be regarded as the correlators in CCFT. In the special case of four point celestial amplitude (2.12), we can thus identify it as the correlation function of four identical scalar operators with dimension $\Delta_\phi$ in CCFT:

\[
G(x_i) = \langle \phi(x_1) \cdots \phi(x_4) \rangle_{\text{CCFT}} = \frac{1}{|x_{12}|^{2\Delta_\phi} |x_{34}|^{2\Delta_\phi}} G(z).
\]

(2.18)

The function $G(z)$ satisfies (2.15)–(2.17). Note that non-trivial 1D CFT is always non-local. Nevertheless, in the conventional situation, the 1D CFT is assumed to be unitary. For CCFT arising from celestial holography, it is not unitary anymore in the standard sense. For example, the scaling dimension of operators in CCFT can be an arbitrary complex number.

In the following discussion, we will be mostly focusing on the $13 \leftrightarrow 24$ channel with $0 < z < 1$. Starting with $G^{(0)}(z)$ in the range $0 < z < 1$, one can analytically continue the function $G^{(0)}(z)$ to the complex value. Then one can show that, at least for the standard 1D unitary CFT, the function $G^{(0)}(z)$ is holomorphic on $\mathbb{C} \setminus \{(-\infty, 0) \cup (1, \infty)\}$, where $(-\infty, 0)$ and $(1, \infty)$ are the branch cuts. For simplicity of notation, we will just write $G^{(0)}(z)$ as $G(z)$ below, unless it causes confusion.

Then we have the crossing equation (2.17), arising from the symmetry of exchanging of $x_1$ and $x_3$:

\[
z^{-2\Delta_\phi}G(z) = (1-z)^{-2\Delta_\phi}G(1-z), \quad z \in (0, 1).
\]

(2.19)

\[\text{Rigorously speaking, the four scalar operators are not identical, because some are incoming, some are outgoing. However, this turns out to be irrelevant in the discussion below, so we will ignore this minor difference.}

\[\text{1D CFT can arise from higher dimensional CFTs by restricting to the 1D defect, namely setting } z = \bar{z}, \text{ or appear on the boundary of } AdS_2.
\]

\[\text{However, it is worth emphasizing that there should a notion of unitarity induced from the unitarity in the bulk quantum fields. A better understanding of such a kind of unitarity is crucial in celestial holography.}
Now we would like to discuss the conformal block expansions of four-point CFT correlators. Recall that for 1D CFT, the $sl(2, \mathbb{R})$ global conformal block reads

$$G_\Delta(z) = z^{\Delta} \mathcal{F}_1^{\Delta \Delta \Delta \Delta; 2\Delta, 2\Delta; -z} .$$

(2.20)

Then in the limit $z \to 0$, namely $x_1 \to x_2$, we can consider the $s$-channel conformal block expansion

$$G(z) = \sum_{O \in \phi \times \phi} (c_{\phi \phi})^2 G_{\Delta O}(z) ,$$

(2.21)

where we sum over all exchanged operators $O$ which appear in the OPE of $\phi \times \phi$.

As reviewed in the introduction, for any CFT correlators, we can define the corresponding Mellin amplitudes [18]. In the particular case of 1D CFT considered here, we will apply the Mellin techniques following the prescription in [23].

For this aim, let us first change variable from $z$ to $t$ via the following simple transformation:

$$z = \frac{t}{1+t} \in [0, 1], \quad t = \frac{z}{1-z} \in [0, \infty) .$$

(2.22)

In the new variable, the four point function becomes

$$\hat{G}(t) = G\left(z = \frac{t}{1+t}\right) ,$$

(2.23)

and the crossing equation (2.19) reduces to

$$\hat{G}(t) = t^{2\Delta_\phi} \hat{G}\left(\frac{1}{t}\right) .$$

(2.24)

The conformal block expansion is now

$$\hat{G}(t) = \sum_{O \in \phi \times \phi} (c_{\phi \phi})^2 \hat{G}_{\Delta O}(t) ,$$

(2.25)

where\(^7\)

$$\hat{G}_\Delta(t) = G_\Delta\left(z = \frac{t}{1+t}\right) = t^{\Delta} \mathcal{F}_1^{\Delta \Delta \Delta \Delta; 2\Delta, 2\Delta; -t} .$$

(2.26)

We can now define the celestial Mellin amplitude for the celestial amplitude/correlator (2.18) by performing another Mellin transformation in coordinate [23]

$$\mathcal{M}(s) = \int_0^\infty dt \hat{G}(t) t^{-1-s} .$$

(2.27)

This is the main object we would like to study in this paper to understand celestial holography. Note that, the celestial Mellin amplitude is obtained from momentum space amplitudes by performing two times of Mellin transforms (2.12) and (2.27), one in energy and the other in coordinate.

\(^7\)Here we used the identity

$$\mathcal{F}_1(a, b; c; z) = (1 - z)^{-a} \mathcal{F}_1\left( a, c - b; c; \frac{z}{z-1} \right) .$$

(2.22)
To have a meaningful Mellin transform, we need to discuss the convergence of integral in (2.27). We assume that \( \hat{G}(t) \) is well behaved for \( t \in (0, \infty) \) and the only possible divergence comes from \( t \to 0 \) and \( t \to \infty \). Let us first consider the behavior near \( t = 0 \). As one can see from (2.25) and (2.26), the leading power is \( \hat{G}(t) \sim t^{\Delta_{\text{lightest}}} \), where \( \Delta_{\text{lightest}} \) is the dimension of the “lightest” exchanged operator, namely \( \text{Re} \Delta_{\text{lightest}} = \min_{\mathcal{O} \in \phi \times \phi} \text{Re} \Delta_{\mathcal{O}} \). Therefore, the requirement of convergent integral near 0 leads to the condition

\[
\text{Re}(2\Delta_{\phi} - \Delta_{\text{lightest}}) < \text{Re}s < \text{Re} \Delta_{\text{lightest}}.
\]

(2.28)

So to have a well-defined Mellin transform (2.27) for some value of \( s \), we must require \( \text{Re} \Delta_{\text{lightest}} > \text{Re} \Delta_{\phi} \). Equivalently, this means that all exchanged operators should satisfy:

\[
\forall \mathcal{O} \in \phi \times \phi, \quad \text{Re} \Delta_{\mathcal{O}} > \text{Re} \Delta_{\phi}.
\]

(2.29)

Throughout this paper, for simplicity, we will assume the condition (2.29) is always satisfied unless specified explicitly. Physically, we can imagine that \( \Delta_{\phi} \) is a free parameter in the complex plane. If the physical exchanged operators have scaling dimensions bounded from below, then we expect there always exists a range of \( \Delta_{\phi} \) such that (2.29) holds (at least when the bound is independent of \( \Delta_{\phi} \)). We can discuss the celestial Mellin amplitude for this range of \( \Delta_{\phi} \), and then analytically continue the final results to the whole complex plane for \( \Delta_{\phi} \). In case the condition (2.29) is violated, it is also possible to do Mellin transform by performing some subtractions and analytic continuations. This has been considered in [23, 28].

Given the celestial Mellin amplitude, we can also compute the celestial amplitude/correlator by performing the inverse Mellin transform

\[
\hat{G}(t) = \int_C \frac{ds}{2\pi i} \mathcal{M}(s) t^s,
\]

(2.30)

for a suitable choice of contour \( C \) from \(-i\infty\) to \(i\infty\). Since we assume the condition (2.29), any contour is valid as long as \( \text{Re}(2\Delta_{\phi} - \Delta_{\text{lightest}}) < \text{Re}s < \text{Re} \Delta_{\text{lightest}} \). In case this assumption (2.29) does not hold, one needs to deform the counter following the prescription in [23].

Since we are interested in the celestial correlators in CCFT, we can find a more direct expression of celestial Mellin amplitudes in terms of momentum space amplitudes. Inserting (2.12) into (2.27), we obtain

\[
\mathcal{M}(s) = 16^{-\Delta_{\phi}} \int_{-\infty}^{0} ds \int_{-\infty}^{\infty} dt \frac{(-s)^{2\Delta_{\phi} - \frac{3}{2}} (s + t)^{\Delta_{\phi} - \frac{3}{2}}}{\sqrt{t}} A(s, t)
\]

(2.31)

\[
= 16^{-\Delta_{\phi}} \int_{-\infty}^{0} ds \int_{-\infty}^{\infty} du \frac{(-s)^{2\Delta_{\phi} - \frac{3}{2}} (-u)^{\Delta_{\phi} - \frac{3}{2}}}{\sqrt{-s - u}} A(s, u),
\]

(2.32)

---

*Here we consider \( \text{Re} \Delta_{\mathcal{O}} \) instead of \( \Delta_{\mathcal{O}} \) because the operators in CCFT generally have complex scaling dimensions.
where we used \( s + t + u = 0 \) and \( A(s, t) = A(s, u) \) (2.14). So the celestial Mellin amplitude looks like, but not exactly, the double Mellin transforms of scattering amplitude in two Mandelstam variables.

2.3 Mellin block expansion

In this subsection, we would like to discuss the Mellin block expansion of celestial Mellin amplitudes. Some of the results here have been discussed in [23], but we will derive the key and new formula (2.56) which relates the OPE coefficients and residues of celestial Mellin amplitudes.

Essentially, we would like to perform the Mellin transform (2.27) on both sides of (2.25) and then obtain the block expansion for celestial Mellin amplitude. However, we will do it in an indirect but illuminating way.

Using crossing symmetry (2.24), it is easy to see that

\[
M(s) = \int_0^\infty dt \hat{G}(1/t)t^{2\Delta \phi - 1 - s} = \int_0^\infty dr \hat{G}(r)r^{s - 2\Delta \phi - 1} = M(2\Delta \phi - s). \tag{2.33}
\]

This is the crossing symmetry for celestial Mellin amplitudes. It can also be seen obviously from (2.32) by exchanging \( s \leftrightarrow u \) and using \( A(s, u) = A(u, s) \).

This further suggests us to split the integration range in (2.27) into two parts and consider

\[
M(s) = \int_1^0 dt \hat{G}(t)t^{-1-s} + \int_1^\infty dt \hat{G}(t)t^{-1-s} = \int_0^1 dt \hat{G}(t)t^{-1-s} + \int_0^1 dr \hat{G}(r)r^{s - 2\Delta \phi - 1}. \tag{2.34}
\]

Therefore, we can rewrite the celestial Mellin amplitude as

\[
M(s) = M_1(s) + M_2(2\Delta \phi - s), \tag{2.35}
\]

where the half Mellin amplitude is

\[
M_1(s) = \int_0^1 dt \hat{G}(t)t^{-1-s}. \tag{2.36}
\]

The representation in (2.35) has the big advantage that it makes the crossing symmetry (2.33) manifest.

We can now plug in the conformal block expansion (2.25) into the (2.36). In particular, the conformal block itself (2.26) gives rise to the half Mellin block

\[
B_\Delta(s) = \int_0^1 dt \hat{G}_\Delta(t)t^{-1-s} = {3F_2}(\Delta, \Delta, \Delta - s; 2\Delta, \Delta - s + 1; -1) \frac{\Delta - s}{\Delta}, \tag{2.37}
\]

where the integral is performed on condition that \( \text{Re} s < \text{Re} \Delta \).

As a result, we arrive at the block expansion of celestial Mellin amplitude

\[
M(s) = \sum_\Delta C_\Delta M_\Delta(s) = \sum_\Delta C_\Delta {3F_2}(\Delta, \Delta, \Delta - s; 2\Delta, \Delta - s + 1; -1) \frac{\Delta - s}{\Delta} \bigg\{ s \to 2\Delta \phi - s \bigg\}, \tag{2.38}
\]
where we sum over all exchanged operators $O_\Delta \in \phi \times \phi$ and $C_\Delta \equiv (c_{\phi\phi}O_\Delta)^2$. Here the Mellin block is given by

$$M_\Delta(s) = B_\Delta(s) + B_\Delta(2\Delta_\phi - s) .$$

(2.39)

The nice property of Mellin blocks is that by construction they are manifestly crossing symmetric

$$M_\Delta(s) = M_\Delta(2\Delta_\phi - s) .$$

(2.40)

From (2.38), (2.37), it is obvious to see that $s = \Delta$ is the pole of celestial Mellin amplitude. This is an interesting feature. However, a crucial point is that there are also other poles in (2.38). To illustrate this, we first consider the case $\Delta = 1$. Then (2.37) takes the following explicit form

$$B_1(s) = \frac{\psi(0)}{2s} (1 - \frac{s}{2}) + \frac{-\psi(0)}{2s} (\frac{1}{2} - \frac{s}{2}) + \gamma_E + \psi(0) \left( \frac{1}{2} \right) ,$$

(2.41)

where $\psi(0) = \Gamma'(z)/\Gamma(z)$ is the polygamma function, and $\gamma_E = 0.577215 \cdots$ is the Euler-Gamma constant. It is easy to show that this function $B_1(s)$ has poles at all positive integers:

$$\lim_{s \to n} B_1(s) \sim \frac{(-1)^n/n}{s - n} , \quad n = 1, 2, 3, \cdots .$$

(2.42)

Actually, we can obtain similar explicit expressions for arbitrary $B_\Delta$ just by employing the defining series of generalized hypergeometric functions:

$$\sum_{n=0}^\infty \frac{(a_1)_n \cdots (a_p)_n x^n}{(b_1)_n \cdots (b_q)_n n!} ,$$

$$n \leq 0$$

(2.43)

Note that the infinite series would terminate and reduce to a polynomial if $a_i \in \mathbb{Z}_{\leq 0}$ is a non-positive integer.

Using (2.43), we then find the following expansion for the half Mellin block (2.37)

$$B_\Delta(s) = \sum_{n=0}^\infty (-1)^n \frac{(\Delta)^2_n}{n!(2\Delta)_n} \frac{1}{\Delta + n - s} ,$$

(2.44)

and it has simple poles at $s = \Delta + k$ with residues

$$\text{Res}_{s=\Delta+k} B_\Delta(s) = (-1)^{k+1} \frac{(\Delta)^2_k}{k!(2\Delta)_k} , \quad k = 0, 1, 2 \cdots .$$

(2.45)

Setting $\Delta = 1$ leads to the equation (2.42), as expected.

Collecting all results together, we obtain the following equation for celestial Mellin amplitude

$$M(s) = \sum_{\Delta} C_\Delta M_\Delta(s) = \sum_{\Delta} \sum_{n=0}^\infty \frac{C_\Delta}{\Delta + n - s} \frac{(-1)^n(\Delta)^2_n}{n!(2\Delta)_n} + \left\{ s \to 2\Delta_\phi - s \right\} .$$

(2.46)
So celestial Mellin amplitudes have only simple poles, whose positions and residues encode the information of OPE data.

Let us consider the residue at the pole \( s = s_* \). Note that due to crossing symmetry \((2.33)\), a pole at \( s_* \) implies a mirror pole at \( 2\Delta_\phi - s_* \) with opposite residue. Therefore, without loss of generality, we can assume that \( \Re s_* > \Re \Delta_\phi \). The residue at the pole \( s_* \) is then given by

\[
\text{Res}_{s=s_*} \mathcal{M}(s) = \sum_{\Delta} \sum_{n=0}^{\infty} \frac{C_\Delta (-1)^{n+1}(s_* - n)^2}{n!(2s_* - 2n)_n} \Theta(\Delta = s_* - n) - \left\{ s_* \to 2\Delta_\phi - s_* \right\} \quad (2.47)
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(s_* - n)^2}{n!(2s_* - 2n)_n} \sum_{\Delta_\phi = s_* - n} (c_{\phi\phi\phi})^2, \quad (2.48)
\]

where the step function \( \Theta = 1 \) if \( \Delta = s_* - n \) is satisfied and vanishes otherwise. Note that in the second equality, we dropped the mirror contribution from the second term in \((2.47)\) because there the operators have dimension \( \Re \Delta = \Re(2\Delta_\phi - s_* - n) \leq \Re(2\Delta_\phi - s_*) < \Re \Delta_\phi \), thus violating our assumption \((2.29)\).

Obviously, for the consideration of pole at \( s_* \), the residue only gets contribution from operators whose scaling dimensions differ from \( s_* \) by an integer. Together with our assumption \((2.29)\), this implies that all the operators \( \mathcal{O}_l \) contributing to the residue have scaling dimension of the form \( \Delta_l = \Delta_0 + l \) with \( l \in \mathbb{N} \). Obviously, \( s_* \) can only differ from \( \Delta_0 \) by an integer, so we can take \( s_* = \Delta_0 + k \) with \( k \in \mathbb{N} \). These considerations then lead to

\[
\text{Res}_{s=\Delta_0+k} \mathcal{M}(s) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(\Delta_0 + k - n)^2}{n!(2\Delta_0 + 2k - 2n)_n} \sum_{\Delta_\phi = \Delta_0 + k - n} (c_{\phi\phi\phi\phi})^2 \quad (2.49)
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(\Delta_0 + k - n)^2}{n!(2\Delta_0 + 2k - 2n)_n} (c_{\phi\phi\phi\phi})^2, \quad (2.50)
\]

\[
= \sum_{l=0}^{k} \frac{(-1)^{k-l+1}(\Delta_0 + l)^2}{(k - l)!(2\Delta_0 + 2k)_{k-l}} (c_{\phi\phi\phi\phi})^2. \quad (2.51)
\]

This equation takes the form of \( F_k = \sum_{l=0}^{k} M_{kl} C_l \), where

\[
F_k = \text{Res}_{s=\Delta_0+k} \mathcal{M}(s), \quad C_l = (c_{\phi\phi\phi\phi})^2, \quad M_{kl} = \frac{(-1)^{k-l+1}(\Delta_0 + l)^2}{(k - l)!(2\Delta_0 + 2k)_{k-l}}. \quad (2.52)
\]

They can be further regarded as the entries of infinite dimensional vectors \( F, C \) and matrix \( M \), respectively. Then \((2.51)\) can be compactly written as \( F = MC \). Once we know the celestial Mellin amplitude, we can compute the residues at all the poles, namely the vector \( F \). Then the OPE coefficients are encoded in the vector \( C = M^{-1}F = NF \), where \( N \) is the inverse matrix of \( M \). We can find the matrix \( M \) and inverse explicitly at the first few
orders:\(^9\)

\[
M = \begin{pmatrix}
-1 & 0 & 0 & \cdots \\
\frac{\Delta_0}{2} & -1 & 0 & \cdots \\
-\frac{\Delta_0(\Delta_0+1)^2}{8\Delta_0+4} & \frac{\Delta_0+1}{2} & -1 & \cdots \\
& & & \ddots
\end{pmatrix}, \quad N = M^{-1} = \begin{pmatrix}
-1 & 0 & 0 & \cdots \\
-\frac{\Delta_0}{2} & -1 & 0 & \cdots \\
-\frac{\Delta_0^2(\Delta_0+1)}{8\Delta_0+4} & -\frac{\Delta_0+1}{2} & -1 & \cdots \\
& & & \ddots
\end{pmatrix}.
\]

(2.53)

Note that both \(M\) and its inverse \(N\) are lower triangular matrices. It is natural to ask whether one can find the analytic closed expression for \(N\) explicitly. Indeed, after some trial and error, we find the explicit expression for the inverse:

\[
N_{kl} = \frac{(-1)^{-\left\lfloor \frac{k-l}{2} \right\rfloor}2^{-\left\lfloor \frac{k-l-2}{2} \right\rfloor} - \left\lfloor \frac{k-l}{2} \right\rfloor}{(2\Delta_0 + 2k - 3)(k-l)!} \frac{\Gamma(\Delta_0 + k) \Gamma(\Delta_0 + l + \left\lfloor \frac{k-l}{2} \right\rfloor)}{\Gamma(\Delta_0 + l)^2 \left( -\Delta_0 - k + \frac{3}{2} \right)^{\left\lfloor \frac{k-l}{2} \right\rfloor}}.
\]

(2.54)

where the floor function \(\lfloor x \rfloor\) gives the greatest integer less than or equal to \(x\). Surprisingly, we find that the expression can be simplified dramatically

\[
N_{kl} = -\frac{\Gamma(\Delta_0 + k)^2\Gamma(2\Delta_0 + k + l - 1)}{\Gamma(2\Delta_0 + 2k - 1)\Gamma(k - l + 1)\Gamma(\Delta_0 + l)^2}.
\]

(2.55)

Furthermore, one can show that this matrix \(N\) is indeed the inverse of \(M\) in (2.52). We present the proof in appendix A.

As a result, we find that the OPE coefficients are given by

\[
(c_{\phi\phi\mathcal{O}_k})^2 = \sum_{l=0}^{k} N_{kl} \text{Res}_{s=\Delta_0+l} \mathcal{M}(s)
\]

\[
= -\frac{\Gamma(\Delta_0 + k)^2}{\Gamma(2\Delta_0 + 2k - 1)} \sum_{l=0}^{k} \frac{\Gamma(2\Delta_0 + k + l - 1)}{\Gamma(k - l + 1)\Gamma(\Delta_0 + l)^2} \text{Res}_{s=\Delta_0+l} \mathcal{M}(s).
\]

(2.56)

This gives the explicit formula to compute the OPE coefficients, once we know the residues of celestial Mellin amplitudes.

### 2.4 Relation to coefficient functions from inversion formulae

In the previous subsection, we have shown that the poles and residues of celestial Mellin amplitudes just encode the information of OPE data in CCFT. This is reminiscent of the coefficient functions in CFT which can be calculated using inversion formulae. In this subsection, we would like to relate the celestial Mellin amplitudes with coefficient functions.

For 1D conformal group SL(2, \(\mathbb{R}\)), the complete set of wave function includes both the principal continuous series \(\Delta \in \frac{1}{2} + i\mathbb{R}_+\) and discrete series \(\Delta \in 2\mathbb{Z}_{>0}\); the latter is absent

\(^{9}\)Note that although \(M\) is an infinite dimensional matrix, its inverse is actually easy to compute because \(M\) is lower triangular. We can write \(M = -(I + T)\) where \(I\) is the identity matrix, then \(M^{-1} = -\sum_{l=0}^{\infty} (-1)^l T^l\). More explicitly \((M^{-1})_{kl} = -\delta_{kl} + \sum_{j_1 > j_2 \cdots > j_l > 1} (-1)^l T_{kj_1} T_{j_1 j_2} \cdots T_{j_{l-1}}\).
in higher dimensions. Given a four-point correlator $G(z)$ in 1D CFT, one can define the following coefficient functions for the principal and discrete series conformal partial waves

$$I_\Delta = \int_{-\infty}^{\infty} dz \, z^{-2} \Psi_\Delta(z) \, G(z), \quad \text{for } \Delta = \frac{1}{2} + ir, \quad r \in \mathbb{R}_+, \quad (2.57)$$

$$\tilde{I}_m = \int_{-\infty}^{\infty} dz \, z^{-2} \Psi_m(z) \, G(z), \quad \text{for } m \in 2\mathbb{Z}_{>0}. \quad (2.58)$$

These are the Euclidean inversion formulae. Here $\Psi_\Delta$ is the conformal partial wave which we review in appendix B, see (B.1). The set of $\Psi_\Delta$ including both principal and discrete series forms a complete and orthogonal basis of wave functions for the conformal group $\text{SL}(2, \mathbb{R})$.

In terms of the coefficient functions $I_\Delta, \tilde{I}_m$, the four-point function can be expanded in the complete set and then rewritten as follows [27]

$$G(z) = \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \frac{d\Delta}{2\pi i} \frac{I_\Delta}{2K_\Delta} \, G_\Delta(z) + \sum_{m \in 2\mathbb{Z}_{>0}} \frac{\Gamma(m)^2}{2\pi^2 \Gamma(2m-1)} \tilde{I}_m G_m(z), \quad 0 < z < 1. \quad (2.59)$$

We can deform the contour toward the right half plane and pick up various poles. Compared to (2.21), we have

$$(c_{\phi\phi})^2 = -\text{Res}_{\Delta = \Delta_0} \frac{I_\Delta}{2K_\Delta} + \sum_{m \in 2\mathbb{Z}_{>0}} \frac{\Gamma(m)^2}{2\pi^2 \Gamma(2m-1)} \tilde{I}_m \delta_{m, \Delta_0}. \quad (2.60)$$

Therefore, up to the contribution from discrete series, the position of simple poles give precisely the scaling dimensions of exchanged operators, and the residues there are just the OPE coefficient squares.

In addition to the Euclidean inversion formulae, one can also use Lorentzian inversion formulae to compute $I_\Delta, \tilde{I}_m$. The Lorentzian inversion formula for the discrete series coefficient function is [25]

$$\tilde{I}_m = \frac{4\Gamma(m)^2}{\Gamma(2m)} \int_0^1 dz \, z^{-2} G_m(z) \text{dDisc}[G(z)], \quad m \in 2\mathbb{Z}_{>0}, \quad (2.61)$$

where $\text{dDisc}[G(z)]$ is the double discontinuity which we review in (B.18). The above formula is valid for any physical four-point function satisfying $G(z) = G\left(\frac{z}{z+1}\right)$. This formula provides a particular analytic continuation of $\tilde{I}_\Delta$ beyond the discrete series $\Delta \in 2\mathbb{Z}_{>0}$. However, the resulting analytically continued $\tilde{I}_\Delta$ in general does not necessarily agree with the principal series coefficient function $I_\Delta$.

There is also a Lorentzian inversion formula for the principal series coefficient function [27]

$$I_\Delta = 2 \int_0^1 dz \, z^{-2} H_\Delta(z) \text{dDisc}[G(z)]. \quad (2.62)$$

Here the inversion kernel $H_\Delta(z)$ needs to satisfy various properties. The closed explicit expression of $H_\Delta(z)$ is in general not known, except for some special cases.

Since both coefficient functions $I_\Delta, \tilde{I}_m$ and celestial Mellin amplitudes encode the OPE data in terms of their simple poles and residues, we expect that they should be related to
each other. Very straightforwardly, we can apply Mellin transform (2.27) on both sides of (2.59), and find

\[
\mathcal{M}(s) = \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \frac{d\Delta}{2\pi i} \frac{I_\Delta}{2 K_\Delta} \frac{\Gamma(s)^2 \Gamma(2\Delta) \Gamma(\Delta - s)}{\Gamma(\Delta)^2 \Gamma(\Delta + s)} + \sum_{n=1}^{\infty} \frac{\Gamma(s)^2 \Gamma(4n) \Gamma(2n - s)}{2\pi^2 \Gamma(4n - 1) \Gamma(2n + s)} \tilde{I}_{2n},
\]

(2.63)

where we used the following Mellin transform of conformal block

\[
\hat{G}_\Delta(t) \xrightarrow{(2.27)} \frac{\Gamma(s)^2 \Gamma(2\Delta) \Gamma(\Delta - s)}{\Gamma(\Delta)^2 \Gamma(\Delta + s)}, \quad \text{Re } \Delta > \text{Re } s > 0.
\]

Alternatively, we can split the Mellin integral (2.27) into two regions following (2.34), and derive the following illuminating and manifestly crossing-symmetric equation

\[
\mathcal{M}(s) = \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \frac{d\Delta}{2\pi i} \frac{I_\Delta}{2 K_\Delta} M_\Delta(s) + \sum_{m \in 2Z_{>0}} \frac{\Gamma(m)^2}{2\pi^2 \Gamma(2m - 1)} \tilde{I}_m M_m(s),
\]

(2.65)

where \( M_\Delta \) is the Mellin block (2.39), (2.37). From (2.44) and (2.39), it is easy to see that a pole \( \Delta^* \) in \( I_\Delta/K_\Delta \) induces an infinite tower of poles \( \Delta^* + k \) together with their mirrors \( 2\Delta_\phi - \Delta^* - k \) in \( \mathcal{M}(s) \), with \( k \in \mathbb{N} \). There are also additional poles at \( m + k \) and \( 2\Delta_\phi - m - k \) coming from discrete series, with \( k \in \mathbb{N}, m \in 2Z_{>0} \). Note that due to possible cancellation between principal and discrete series, some of the poles at positive even integers may be absent.

2.5 Regge behavior

In this subsection, we discuss the behavior of celestial Mellin amplitudes in the Regge limit, which is a very interesting limit in standard unitary CFT and has a close connection with chaos.

In 1D CFT, we have the \( s \)-channel limit, \( x_1 \to x_2 \), and the \( t \)-channel limit \( x_2 \to x_3 \). But strictly speaking, there is no notion of \( u \)-channel limit as it is impossible to bring \( x_1 \) close to \( x_3 \) without crossing \( x_2 \). However, from higher dimensions, one can infer that the \( u \)-channel Regge limit is \( z \to i\infty \), more precisely arbitrary points at infinity on the upper or lower half complex plane excluding the real axis. For concreteness, we consider \( z = \frac{1}{2} + i\xi \) with \( \xi \to \infty \). Then one can show that in standard unitary 1D CFTs, the four-point functions are bounded in the Regge limit [27]

\[
\left| \tilde{G}\left( \frac{1}{2} + i\xi \right) \right| < \infty \quad \text{as} \quad \xi \to \infty,
\]

(2.66)

where \( \tilde{G}(z) = z^{-2\Delta_\phi} G(z) = \tilde{G}(1 - z) \).

We would like to translate (2.66) into Mellin amplitude. For convenience, we normalize the Mellin amplitude following [23]

\[
\mathcal{M}(s) = \frac{\mathcal{M}(s)}{\Gamma(s) \Gamma(2\Delta_\phi - s)}.
\]

(2.67)
If the Regge limit boundedness (2.66) holds, then the normalized Mellin amplitude satisfies [23]

\[ \mathcal{M}(s) = \mathcal{O}(|s|^0), \quad |s| \to \infty. \]  

(2.68)

Since we are interested in 1D CCFTs which are not standard unitary CFTs, there is no guarantee for the Regge boundedness behavior. Actually, the meaning of Regge limit is even not clear at all, and it would be very interesting to understand further. Nevertheless, one can show that if there exits \( n \) such that

\[ G(z) = \mathcal{O}(z^{2\Delta_\phi + n}), \quad z \to \frac{1}{2} + i\infty, \]  

(2.69)

then the normalized Mellin amplitude behaves as

\[ \mathcal{M}(s) = \mathcal{O}(|s|^n), \quad |s| \to \infty. \]  

(2.70)

3 An example

In this section, we illustrate many features of celestial Mellin amplitude in a concrete scalar model in 3D. We will first present the model, compute the celestial amplitude and its conformal block expansion. Then we will calculate the celestial Mellin amplitude and its Mellin block expansion, which is shown to agree with the conformal block expansion. Finally, we will use inversion formulæ to compute the coefficient functions in order to compare them with celestial Mellin amplitude.

3.1 A simple model in 3D

Let us consider a simple model in (2+1)D consisting of one real massless field \( \phi \) and one real massive scalar field \( \sigma \) with mass \( m \). They interact via the cubic vertex \( g\phi^2\sigma \). This model was studied in [26]. It is easy to compute the tree-level four massless particle scattering amplitude due to the massive particle exchange:

\[ A(s, t) = -\frac{g^2}{s - m^2} - \frac{g^2}{t - m^2} - \frac{g^2}{u - m^2}, \]  

(3.1)

where we have ignored the \( i\epsilon \) factor. Plugging into (2.12), one can then obtain the corresponding celestial amplitude [26]

\[ G(z) = G^{(0)}(z) = \mathcal{N} \frac{z}{\sqrt{1 - z}} \left[ 1 + e^{\pi i\alpha} z^\alpha + \left( \frac{z}{1 - z} \right)^\alpha \right], \quad 0 < z < 1, \]  

(3.2)

with

\[ \alpha = 2\Delta_\phi - \frac{3}{2}, \quad \mathcal{N} = \frac{g^2 \pi m^{4\Delta_\phi - 5}}{2^4 \Delta_\phi \cos(2\pi \Delta_\phi)}. \]  

(3.3)

Here we have taken 1,3 incoming, 2,4 outgoing. Note that there is no delta function because we are considering scattering in three dimensional spacetime. For simplicity, we will set \( \mathcal{N} = 1 \).

\(^{10}\)Here one needs to assume the absence of Stokes phenomena, otherwise the bound is \( \mathcal{M}(c + i\eta) = \mathcal{O}(|\eta|^0) \), as \( \eta \to \infty \) for proper \( c \) to ensure convergence.
3.2 Conformal block expansion

We would like to perform the conformal block expansion for the celestial correlator (3.2). Such a decomposition has been studied for a similar model in [29].

The conformal block expansion can be done with the help of the following identity [30]:

$$\frac{z^p}{(1-z)^q} = \sum_{n=0}^{\infty} C_{p,q}(n) G_{p+n}(z), \quad 0 < z < 1,$$

(3.4)

where \( G_\Delta \) is the conformal block in (2.20) and

$$C_{p,q}(n) = \frac{(p)_n^2}{n!(2p+n-1)_n} F_2(p-q, 2p+n-1, -n; p, p; 1), \quad (q)_n = \frac{\Gamma(q+n)}{\Gamma(q)}.$$  (3.5)

Using this identity (3.4), it is easy to find the conformal block expansion of celestial amplitude (3.2)

$$G(z) = \sum_{k=0}^{\infty} \frac{16^k \Gamma \left( k + \frac{1}{2} \right)^2}{\Gamma \left( \frac{1}{2} - k \right)^2 \Gamma(4k+1)} G_{2k+1}(z) + \sum_{n=0}^{\infty} \left( 1 + (-1)^n e^{\pi i \alpha} \right) C_{1+\alpha, \frac{1}{2}+\alpha}(n) G_{n+1+\alpha}(z),$$

(3.6)

where we used the property

$$C_{1, \frac{1}{2}}(2k+1) = 0, \quad C_{1+\alpha, \frac{1}{2}+\alpha}(n) = (-1)^n C_{1+\alpha, \frac{1}{2}+\alpha}.$$  (3.7)

Then it is easy to read off the scaling dimension of exchanged operators

$$\Delta_{O} = 2n + 1, \quad \Delta_{O} = n + 2\Delta_\phi - \frac{1}{2}, \quad n \in \mathbb{N}.$$  (3.8)

Note that operators with positive even integer scaling dimensions are absent in (3.8). Furthermore, in the conformal block expansion (3.6), the squared OPE coefficients are complex (unless \( \text{Re} \alpha \in \mathbb{Z} \)), which is thus a signature of non-unitarity.

3.3 Celestial Mellin amplitude

We can calculate the Mellin transform (2.27) of celestial correlator (3.2) and obtain the following celestial Mellin amplitude

$$\mathcal{M}(s) = \frac{\Gamma(1-s) \Gamma \left( s - \frac{1}{2} \right)}{\sqrt{\pi}} + \frac{\Gamma(s+1-2\Delta_\phi) \Gamma \left( 2\Delta_\phi - s - \frac{1}{2} \right)}{\sqrt{\pi}} + i e^{2\pi i \Delta_\phi} \frac{\Gamma \left( 2\Delta_\phi - s - \frac{1}{2} \right)}{\Gamma(2\Delta_\phi - 1)},$$

(3.9)

provided that \( \frac{1}{2} < \text{Re} s \), \( \text{Re}(2\Delta_\phi - s) < 1 \), and hence \( \frac{1}{2} < \text{Re} \Delta_\phi < 1 \). Without further justification, we will analytic continue the results on the complex plane, so (3.9) will be assumed to be valid for all \( s \in \mathbb{C} \). In the above evaluation, we have used the following equation for Mellin transform

$$\frac{z^p}{(1-z)^q} = t^p(1+t)^{q-p} \quad \overset{(2.27)}{\rightarrow} \quad \frac{\Gamma(p-s) \Gamma(s-q)}{\Gamma(p-q)}, \quad \text{Re} p > \text{Re} s > \text{Re} q.$$  (3.10)
Alternatively, the celestial Mellin amplitude can also be obtained directly from momentum space amplitude. Inserting (3.1) into (2.32), we get exactly (3.9), including the factor \( N \) in (3.3).

Now we would like to see the behaviors of celestial amplitude (3.2) and celestial Mellin amplitude (3.9) in the Regge limit discussed in section 2.5. For the celestial amplitude (3.2), the Regge behavior (2.69) is easy to see

\[
G(z) \sim z^{1/2}, \quad z^{2\Delta - 1}, \quad \text{as} \quad z \to i\infty. \quad (3.11)
\]

For the (normalized) celestial Mellin amplitude (3.9), the Regge behavior (2.70) is also easy to find

\[
\mathcal{M}(s) \equiv \frac{\mathcal{M}(s)}{\Gamma(s)\Gamma(2\Delta - s)} \sim s^{1/2 - 2\Delta - 1}, \quad s \to \infty. \quad (3.12)
\]

Therefore, we see that the (normalized) celestial Mellin amplitude does not satisfy the Regge boundedness condition (2.68), unless \( \text{Re} \Delta > 1/4 \).

### 3.4 Mellin block expansion

We can easily check the crossing invariance \( \mathcal{M}(s) = \mathcal{M}(2\Delta - s) \) of celestial Mellin amplitude (3.9). Consequently, the pole \( s_* \) and its mirror \( 2\Delta - s_* \) always appear together in pairs.

From (3.9), it is easy to read off all the simple poles

\[
s = k + 1, \quad -k + \frac{1}{2}, \quad 2\Delta - k + 1, \quad 2\Delta - \left( -k + \frac{1}{2} \right), \quad k = 0, 1, 2, \ldots \quad (3.13)
\]

For simplicity, we assume that \( \Delta \) is generic, so all these poles do not coincide with each other.

Here we find the poles of celestial Mellin amplitude, but our actual interest is the operator content in CCFT. Note that not all the poles of celestial Mellin amplitude correspond to the scaling dimension of physical exchange operators. First of all, due to crossing symmetry, the poles at \( s = \Delta \) and \( s = 2\Delta - \Delta \) actually come from the same physical operators (2.46). Secondly, it is possible that the physical exchange operator are actually not present at the pole of celestial Mellin amplitude due to the nontrivial relation (2.56).

\[11\text{When performing the integrals, the following two identities turn out to be useful}
\[
\int_0^\infty dSdU \frac{S^aU^b}{\sqrt{S + U}} \frac{1}{S + M} = \frac{M^{1/2 + a + b} \sqrt{\pi}}{\cos(\pi(a + b))} \Gamma\left( -\frac{1}{2} - b \right) \Gamma(1 + b),
\]
\[
\int_0^\infty dSdU \frac{S^aU^b}{\sqrt{S + U}} \frac{1}{S + U + M} = \int_0^\infty dX \int_0^X \frac{x^{a+b+1/2} x^{a}(1-x)^b}{X + M} \frac{1}{\cos(\pi(a + b))} \Gamma(1 + a)\Gamma(1 + b) \Gamma(2 + a + b). \]

\[12\text{Here we use the Stirling’s approximation for Gamma function, } \Gamma(x) \sim \sqrt{2\pi} x^{x - 1/2} e^{-x} \left( 1 + O(1/x) \right), \text{ as } |x| \to \infty \text{ at fixed } |\text{arg}(x)| < \pi. \text{ Thus up to an order one factor, we have } \Gamma(a + x)\Gamma(b - x) \sim x^{a+b-1}.
\]
between OPE coefficients and the residues of the celestial Mellin amplitude. In that case, the residues of the celestial Mellin amplitude contrive in a tricky way, leading to vanishing OPE coefficients in (2.56). Nevertheless, the poles of the celestial Mellin amplitude give the maximal possible set of scaling dimension of operators. If we further assume (2.29), we should then only keep those poles with \( \text{Re} s > \text{Re} \Delta_\phi \), hence resolving the first ambiguity. In addition, the formula (2.56) allows us to compute the OPE coefficients explicitly from residues of the celestial Mellin amplitude. All operators with non-vanishing OPE coefficients then correspond to the physical exchange operators.

Focusing on the problem at hand (3.13) and requiring \( \text{Re} s > \text{Re} \Delta_\phi \), we naturally find the subset

\[
s = k + 1, \quad 2\Delta_\phi + k - \frac{1}{2}, \quad k = 0, 1, 2, \cdots .
\]

(3.14)

Compared to (3.8), we see that operators with even scaling dimension \( s \in 2\mathbb{Z}_{>0} \) are actually absent. This is due to the vanishing OPE coefficient (3.7). Below, we will show that we can compute explicitly all the OPE coefficients from the celestial Mellin amplitude using the formula (2.56). The result agrees with (3.6), (3.7) obtained from conformal block expansion. This then automatically implies that those operators with \( \Delta \in 2\mathbb{Z}_{>0} \) are actually absent.

Let us now compute the OPE coefficients from celestial Mellin amplitude (3.9). Using the following formula

\[
\lim_{x \to k} \frac{\Gamma(x + m)}{(x - k)(x - m - k)!} = \frac{(-1)^{m-k}}{k!} \quad \text{for} \quad k + m = 0, -1, -2, \cdots ,
\]

(3.15)

we find the residues of celestial Mellin amplitude (3.9)

\[
\begin{align*}
\text{Res}_{s=k+1} \mathcal{M}(s) &= - \text{Res}_{s=2\Delta_\phi-(k+1)} \mathcal{M}(s) = \frac{(-1)^{k+1} \Gamma(k + \frac{1}{2})}{\sqrt{\pi} k!}, \\
\text{Res}_{s=-k+\frac{1}{2}} \mathcal{M}(s) &= - \text{Res}_{s=2\Delta_\phi-(-k+\frac{1}{2})} \mathcal{M}(s) \\
&= \frac{(-1)^k \Gamma(k + \frac{1}{2})}{\sqrt{\pi} k!} + \frac{i e^{2\pi i \Delta_\phi} (-1)^k \Gamma(2\Delta_\phi - 1)}{k! (2\Delta_\phi - 1)}. 
\end{align*}
\]

(3.16)- (3.18)

With these results, we can compute the OPE coefficients using the formula (2.56). As one can see from (3.14), these two infinite towers of poles take the form \( \Delta_0 + k \), with \( \Delta_0 = 1 \) and \( \Delta_0 = 2\Delta_\phi - \frac{1}{2} \) respectively. At very low orders, one can check explicitly that the resulting OPE coefficients computed by (2.56) from celestial Mellin amplitude are consistent with that obtained from conformal block expansion in (3.6). In fact, we can prove the full equivalence analytically. Since the conformal block expansion (3.6) just follows from the identity (3.4) and (3.5), we only need to show the general case in (3.4), whose corresponding “celestial Mellin amplitude” is given in (3.10). The OPE coefficients
can be then computed using (2.56):^{13}

\[
(c_{\phi\phi}O_k)^2 = \sum_{l=0}^{k} N_{kl} \text{Res}_{s=p+l} \frac{\Gamma(p-s)\Gamma(s-q)}{\Gamma(p-q)} 
\]

\[
= -\frac{\Gamma(p+k)^2}{\Gamma(2p+2k-1)} \sum_{l=0}^{k} \frac{\Gamma(2p+k+l-1)}{\Gamma(k-l+1)\Gamma(p+l)^2} (-1)^{k+1}\frac{\Gamma(p-q+k)}{k!\Gamma(p-q)} 
\]

\[
= \sum_{l=0}^{k} \frac{(-1)^l\Gamma(k+p)^2\Gamma(k+l+2p-1)\Gamma(l+p-q)}{l!\Gamma(k-l+1)\Gamma(2k+2p-1)\Gamma(l+p)^2\Gamma(p-q)},
\]

(3.22)

where we set $\Delta_0 = p$. This indeed equals to the coefficient $C_{p,q}(k)$ in (3.5), once we plug in the defining series (2.43) of $3F_2$ into (3.5), which actually reduces to a finite sum, and compare each summand term by term with (3.22).

As a consequence, we obtain the following Mellin block expansion of celestial Mellin amplitude (3.9)

\[
\mathcal{M}(s) = \sum_{k=0}^{\infty} \frac{16^k\Gamma\left(k + \frac{1}{2}\right)^2}{\Gamma\left(\frac{1}{2} - k\right)^2\Gamma(4k+1)} M_{2k+1}(s) + \sum_{n=0}^{\infty} \left(1 + (-1)^n e^{\pi i \alpha}C_{1,\alpha,\frac{3}{2}+\alpha}(n)\right) M_{n+1+\alpha}(s),
\]

(3.23)

which is consistent with conformal block expansion (3.6), and indicates that operators with $\Delta \in 2\mathbb{Z}_{\geq 0}$ are indeed absent.

### 3.5 Coefficient functions

As we discussed before, the CFT data are equivalently encoded in the coefficient functions $I_{\Delta}, \tilde{I}_m$ which can be computed from inversion formulae. Given the celestial correlator (3.2), we would like to perform the computation and find the explicit expressions of $I_{\Delta}, \tilde{I}_m$.^{14} We leave the technical details to appendix B, and only quote the final results here.

For the principal series, $I_{\Delta}$ is given by

\[
I_{\Delta} = J_{\Delta}\left(1, \frac{1}{2}\right) + J_{\Delta}\left(1 + \alpha, \frac{1}{2}\right) + J_{\Delta}\left(1 + \alpha, \frac{1}{2} + \alpha\right), \quad J_{\Delta}(p,q) = 2J_{\Delta}^+(p,q) + J_{\Delta}^-(p,q),
\]

(3.24)

^{13} Alternatively, from (2.51) we just need to show

\[
\sum_{l=0}^{k} \frac{(-1)^{k-l+1}(p+l)^2}{(k-l)!(2p+2k)_{k-l}} C_{p,q}(l) = \text{Res}_{s=p+k} \frac{\Gamma(p-s)\Gamma(s-q)}{\Gamma(p-q)} = \frac{(-1)^{k+1}\Gamma(p-q+k)}{k!\Gamma(p-q)}. \]

This can be proved by performing the Mellin transform on both sides of (3.4)

\[
\frac{\Gamma(p-s)\Gamma(s-q)}{\Gamma(p-q)} = \sum_{n=0}^{\infty} C_{p,q}(n) \frac{\Gamma(s)\Gamma(2(p+n))\Gamma(p+n-s)}{\Gamma(p+n)^2\Gamma(p+n+s)},
\]

(3.19)

and then taking residue at pole $s = p + k$.

^{14} In [29], the authors computed the coefficient functions of the imaginary part of a similar celestial amplitude. Here we will compute the coefficient functions $I_{\Delta}, \tilde{I}_m$ for the full celestial amplitude (3.2).
where the two functions $J_\Delta^0, J_\Delta^+$ take the following explicit forms:

$$J_\Delta^+(p,q) = 2\pi \csc(\pi \Delta) B\left(p - \frac{1}{2} - \alpha, -q + 1\right) \left[ 3F_2 \left( p - \frac{1}{2} - \alpha, 1 - \Delta, \Delta; 1, p + \frac{1}{2} - \alpha - q; 1 \right) + 3F_2 \left( -q + 1, 1 - \Delta, \Delta; 1, p + \frac{1}{2} - \alpha - q; 1 \right) \right],$$

(3.25)

$$J_\Delta^0(p,q) = \frac{(\sec(\pi \Delta) + 1) \Gamma(\Delta)^2 \Gamma(1 - q) \Gamma(p + \Delta - 1)}{\Gamma(2\Delta) \Gamma(p - q + \Delta)} 3F_2(\Delta, \Delta, -1 + p + \Delta; 2\Delta, p - q; 1) + \{\Delta \to 1 - \Delta\}.$$  

(3.26)

For the discrete series, $\tilde{I}_m$ is given by

$$\tilde{I}_m = \frac{\pi^{2m} \Gamma(m)^2}{\Gamma(m + \frac{1}{2})} \left[ 3F_2 \left( m, m, m; m + \frac{1}{2}, 2m; 1 \right) + 3F_2 \left( \alpha + m, m, m; \alpha + m + \frac{1}{2}, 2m; 1 \right) \frac{\Gamma(m + \frac{1}{2}) \Gamma(\alpha + m)}{\Gamma(m) \Gamma(\alpha + m + \frac{1}{2})} e^{i\pi\alpha} + 3F_2 \left( \alpha + m, m, m; m + \frac{1}{2}, 2m; 1 \right) \frac{\Gamma(\frac{1}{2} - \alpha) \Gamma(\alpha + m)}{\sqrt{\pi} \Gamma(m)} (\sin(\pi\alpha) + 1) \right].$$

(3.27)

We see that the coefficient functions corresponding to (3.2) are quite complicated. In contrast, the celestial Mellin amplitude (3.9) takes a much simpler form.

Despite the complicated expressions of the coefficient functions, we still would like to understand some of the structures. As discussed in appendix B, we find the following residue formula (B.17)

$$\text{Res}_{\Delta=p+n} \frac{J_\Delta}{2K_\Delta} = -c_{p,q}(n), \quad n \in \mathbb{N}.$$  

(3.28)

Together with (3.24), this gives rise to the conformal block expansion which is the same as (3.6).

Furthermore, we have verified the following formula (B.24)

$$\text{Res}_{\Delta=m} \frac{I_\Delta}{K_\Delta} = \frac{\Gamma(m)^2}{2\pi^2 \Gamma(2m - 1)} \tilde{I}_m, \quad m \in 2\mathbb{Z}_{>0}.$$  

(3.29)

This shows that there is a perfect cancellation between principal and discrete series at positive even integers. Using (2.60), we learn that operators with positive even integer scaling dimensions should be absent in exchange of $\phi$. This is again in agreement with (3.8).

4 Comments on higher-dimensional generalizations

In this section, we will explore several possible definitions of celestial Mellin amplitude in four and higher dimensions. As we discussed in the introduction, the main technical difficulty comes from the delta function in celestial correlators due to momentum conservation.
We will discuss how to overcome this subtlety. For simplicity, we will focus on the case of scalar fields in four dimensions.

Let us first review the Mellin technique in standard CFTs. In general, the Mellin amplitude for $n$-point scalar correlator in CFT is given by [18]

$$G(x_i) = \langle O_1(x_1) \cdots O_n(x_n) \rangle = \int \left[ d\gamma \right] \mathcal{M}(\gamma_{ij}) \prod_{1 \leq i < j \leq n} \Gamma(\gamma_{ij}) (x_{ij}^2)^{-\gamma_{ij}},$$

where

$$\sum_{j=1}^n \gamma_{ij} = 0, \quad \gamma_{ii} = -\Delta_i, \quad \gamma_{ij} = \gamma_{ji}.$$ (4.2)

Specializing to four scalar operators with the same conformal dimensions $\Delta$, the correlation function is encoded in a function of cross-ratios:

$$\langle O(x_1) \cdots O(x_4) \rangle = G(u, v) = \frac{G(u,v)}{(x_{13}^2 x_{24}^2)\Delta},$$

where the cross-ratios are given by

$$u = z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}.$$ (4.4)

The Mellin amplitude is then defined by

$$G(u, v) = \int_C ds dt \left( \frac{2\pi i}{4\Delta T_{3} - 4\sum_{i=1}^4 \Delta_i} \right) u^{-s} v^{-t} \Gamma(s)^2 \Gamma(t)^2 \Gamma(\Delta - s - t) \mathcal{M}(s, t),$$

for a properly chosen contour $C$. If the scalar operators are identical, we have the crossing symmetry $\mathcal{M}(s, t) = \mathcal{M}(t, s) = \mathcal{M}(s, \Delta - s - t)$.

We would like to generalize our previous discussions on celestial Mellin amplitude to higher dimensions. For simplicity, let us consider the four scalar scattering in 4D spacetime. As in the 3D case, the celestial amplitude is given by Mellin transform of the momentum space scattering amplitude (2.3). It can be simplified as

$$G(x_i, \bar{x}_i) = \langle O(x_1, \bar{x}_1) \cdots O(x_4, \bar{x}_4) \rangle = K(x_i, \bar{x}_i) \times G(z, \bar{z}),$$

where

$$K(x_i, \bar{x}_i) = \prod_{i<j} |x_{ij}|^{\Delta_T - \Delta_i - \Delta_j}, \quad \Delta_T = \sum_{i=1}^4 \Delta_i,$$ (4.7)

and

$$G(z, \bar{z}) = X(z, \bar{z}) G(z),$$ (4.8)

with

$$X(z, \bar{z}) = 2^{-\Delta_T + 2} \left| \frac{z}{\sqrt{1 - \bar{z}}} \right|^{\Delta_T} \delta(z - \bar{z}),$$ (4.9)

and

$$G(z) = \int_0^\infty d\omega \omega^{\Delta_T - 4} A\left( \epsilon_1 \epsilon_2 \omega^2, -\frac{1}{z} \epsilon_1 \epsilon_2 \omega^2 \right).$$ (4.10)
The notable feature in the above four point correlator is that in (4.9) we have a delta function which sets \( z = \bar{z} \). This comes from the momentum conservation which enforces the four scattering particles lie on a plane, whose intersection with celestial sphere restricts the position of four operator insertions.

We would like to define the celestial Mellin amplitude for the celestial correlator (4.6). The delta function we emphasize above poses some challenges for a straightforward generalization. Nevertheless, there are several options and we would like to discuss them below.

In the first approach, we just ignore the issue of delta function and just apply the Mellin techniques of standard CFT. So we can insert \( G = XG \) (4.8) into (4.5), and try to find the corresponding Mellin amplitude \( M(s,t) \). In particular, we must reproduce the delta function from \( M(s,t) \) using (4.5). This is very subtle and not obvious at all, due to the very singular behavior of delta function. Nevertheless, we need to mention that in the study of conformal block expansion \[31\], the authors indeed succeeded in reproducing such a kind of delta function.\(^15\)

In the second approach, we can remedy the singular delta function behavior by performing the light transformation \[32\] or shadow transformation \[33\] on some operators in (4.6). The integral transformation would remove the delta function in (4.9),\(^16\) and the corresponding four point function would be very well behaved. Then we can use the Mellin technique (4.5) to study the light/shadow transformed correlator.

In the final approach, we can restrict to the subspace \( z = \bar{z} \) of CCFT, which actually gives rise to a 1D defect CFT.\(^17\) The correlator in the defect CFT is then given by (4.8), (4.9) but with delta function removed, possibly up to some extra functors involving \( z \) which can be fixed by one dimensional \( SL(2,\mathbb{R}) \) conformal symmetry. Then we can define the corresponding celestial Mellin amplitude using (2.23) and (2.27), just like the case of 3D spacetime we discussed before.

These are the most natural three possible definitions of celestial Mellin amplitude in higher dimensions. The first approach is very subtle due to the delta function; the last approach seems to be the simplest but the physical meaning in terms of 4D physics is not clear. They are not independent, and should be related in some way because they originate from the same celestial amplitude (4.6). A detailed investigation of these problems is left to the future.

5 Conclusion

In this paper, we initiate the study of CCFT in Mellin representation. For technical reasons, we mainly focus on the 3D spacetime. We introduce the notion of celestial Mellin amplitude, which is given by Mellin transforming the celestial amplitude in coordinate (2.27). As a result, the celestial Mellin amplitude is the double Mellin transformations of momentum

\(^{15}\)In particular, \[31\] found a continuous family of intermediate exchanges of spinning light-ray states, which seem to be the crucial ingredient to reproduce the delta function.

\(^{16}\)It is worth mentioning that we can also remove the unwanted delta function by breaking the translational invariance, and then apply (4.5) directly on the celestial amplitude.

\(^{17}\)The defect is actually a bit trivial, because we are not really inserting any one dimensional defect into CCFT.
space scattering amplitude. The celestial Mellin amplitude is useful in understanding the
conformal block expansion in CCFT. More precisely, one can read off possible exchanged
operators from the position of poles, and the OPE coefficients from the residues at the poles
via equation (2.60). This formula is supposed to be also useful in the general study of 1D
CFTs and the corresponding AdS$_2$. For illustration, we consider a simple model of scalar
fields in 3D. We compute the celestial Mellin amplitude and its Mellin block expansion.
We also compare it with the coefficient functions calculated from inversion formulae.

There are many interesting questions which remain to be further explored. One of the
most important questions is to get a better understanding of celestial Mellin amplitudes
in four and higher dimensions. Although we provide several possible definitions in higher
dimensions, it remains to work out the technical details, in particular for the case with
spinning particles. After all, we would be finally interested in the graviton scattering in
spacetime. Nevertheless, a good starting point would be to consider the 4D analogue of
the example considered in section 3 and apply our various prescriptions in section 4.

Even for 3D spacetime, there are various generalizations to be further explored. One
straightforward direction is to consider the case with fermions or scalars with multiple
species. The other more non-trivial question is to generalize the celestial Mellin amplitudes
to higher points.

It would be also very interesting to understand the possible connections between ce-
lestial holography and AdS/CFT from Mellin space. On the one hand, we can obtain the
flat spacetime by taking the flat limit of AdS. On the other hand, we can also foliate part
of the flat Minkowski space in terms of AdS slices. So it is natural to ask whether there
are some connections between the Mellin amplitudes in AdS and the celestial Mellin am-
plitudes in flat spacetime. In [20], the author proposed a prescription to recover the flat
space scattering amplitude from the Mellin amplitude. It would be fantastic to explore
further along this direction and discover the connection between AdS/CFT and celestial
holography.

The Mellin representation is especially powerful in bootstrapping holographic CFTs
and the dual AdS quantum gravity. Here we introduce the concept of celestial Mellin
amplitude, aiming at generalizing the bootstrap philosophy to flat spacetime. As a starting
point, we may consider the following bootstrap question even in 3D spacetime. From the
EFT point of view, we can generalize the simple model in section 3 by including all different
higher derivative interactions, compute the scattering amplitude and map it to the celestial
basis. This helps us establish a dictionary between bulk interaction vertices and boundary
CCFT data. We can then use various consistency conditions to bootstrap CCFT and thus
the bulk couplings. We believe our celestial Mellin amplitude would be a powerful tool in
such a kind bootstrap question.

The crucial issue in the bootstrap above is to find the consistency conditions of CCFT,
which are supposed to inherit from the bulk unitarity, locality, causality, etc. Translating
these universal bulk principles into CCFT is a vital question. Ultimately, we should be able
to bootstrap quantum gravity in flat spacetime from CCFT, just like the case of AdS/CFT.
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A A proof on the inverse of infinite dimensional matrices

In this appendix, we show that the two infinite dimensional matrices $M, N$ given in (2.52), (2.55) are indeed the inverse of each other.

By definition, we need to show that $MN = NM = 1$. So we should consider the product of two matrices

$$\sum_{l=k_2}^{k_1} N_{kl} M_{lk_2} = \frac{\Gamma(\Delta_0 + k_1)^2 \Gamma(2\Delta_0 + 2k_2)}{\Gamma(\Delta_0 + k_2)^2 \Gamma(2\Delta_0 + 2k_1 - 1)} \sum_{l=k_2}^{k_1} (-1)^{l-k_2} \frac{\Gamma(2\Delta_0 + k_1 + l - 1)}{(l-k_2)! \Gamma(k_2-l+1) \Gamma(2\Delta_0+k_2+l)}$$

(A.1)

$$= \frac{\Gamma(\Delta_0 + k_1)^2 \Gamma(2\Delta_0 + 2k_2)}{\Gamma(\Delta_0 + k_2)^2 \Gamma(2\Delta_0 + 2k_1 - 1)} \times \frac{\delta_{k_1 k_2}}{2\Delta_0 + 2k_2 - 1} = \delta_{k_1 k_2},$$

(A.2)

where we used the following identity which can be proved with the help of Mathematica:

$$\sum_{\ell=0}^k \frac{(-1)^\ell}{\ell!(k-\ell)!} \frac{\Gamma(S+\ell+k-1)}{\Gamma(S+\ell)} = \frac{1}{(k+S-1)\Gamma(1-k)\Gamma(1+k)} = \begin{cases} \frac{1}{S-1}, & k = 0, \\ 0, & k \in \mathbb{Z}_{>0} \end{cases}.$$

(A.3)

Similarly, we find that

$$\sum_{l=k_2}^{k_1} M_{kl} N_{lk_2} = \frac{\Gamma(\Delta_0 + k_1)^2}{\Gamma(\Delta_0 + k_2)^2 \Gamma(2\Delta_0 + 2k_1 - 1)} \times \Gamma(2\Delta_0 + k_1 + k_2) \delta_{k_1 k_2} = \delta_{k_1 k_2},$$

(A.4)

after using the identity

$$\sum_{\ell=0}^k \frac{(-1)^{k-\ell+1}}{\Gamma(\ell+1)\Gamma(k-\ell+1)\Gamma(S+\ell)} = \begin{cases} \Gamma(S-1), & k = 0, \\ 0, & k \in \mathbb{Z}_{>0} \end{cases}.$$

(A.5)

B More details on coefficient functions and inversion formulae

In section 2.4, we briefly review the inversion formulae which can be used to compute the coefficient functions. In this appendix, we provide more technical details and apply the inversion formula to our celestial correlator (3.2).

Let us first review the conformal partial wave which takes different forms in different regions [27]:

$$\Psi_\Delta(z) = \begin{cases} \Psi_\Delta^{(0)}(z), & z \in (0,1), \\ \Psi_\Delta^{(-)}(z) = \Psi_\Delta^{(0)} \left( \frac{1}{z} \right), & z \in (-\infty,0), \\ \Psi_\Delta^{(+)}(z) = \frac{1}{\pi} \left[ \Psi_\Delta^{(0)}(z + i\epsilon) + \Psi_\Delta^{(0)}(z - i\epsilon) \right], & z \in (1,\infty), \end{cases}$$

(B.1)
\[ \Psi_\Delta^{(0)}(z) = K_{1-\Delta}G_\Delta(z) + K_\Delta G_{1-\Delta}(z), \quad K_\Delta = \frac{\sqrt{\pi} \Gamma \left( \Delta - \frac{1}{2} \right) \Gamma \left( \frac{1-\Delta}{2} \right)^2}{\Gamma(1-\Delta) \Gamma \left( \frac{\Delta}{2} \right)^2}. \] (B.2)

Actually, \( \Psi_\Delta^{(+)}(z) \) also has various explicit forms [26, 27]

\[ \Psi_\Delta^{(+)}(z) = \frac{2\Gamma \left( \frac{1-\Delta}{2} \right) \Gamma \left( \frac{\Delta}{2} \right)}{\sqrt{\pi}} \, _2F_1 \left( \frac{1-\Delta}{2}, \frac{\Delta}{2}; \frac{1}{2}; \frac{(2-z)^2}{z^2} \right) \] (B.3)

\[ = 2\pi \csc(\pi \Delta) \left[ 2F_1 \left( 1-\Delta, \Delta; 1; \frac{1}{z} \right) + 2F_1 \left( 1-\Delta, \Delta; 1; \frac{z-1}{z} \right) \right]. \] (B.4)

We would like to use inversion formulae to compute \( I_\Delta, I_m \). For \( I_\Delta \), we use the Euclidean inversion formula (2.57). We can split the integration into three regions, and use the relations (B.1) and (2.15)–(2.17) of \( \Psi, G \) in these three regions. As a result, we can rewrite (2.57) as an integral just in one region \( 0 < z < 1 \):

\[ I_\Delta = \int_{-\infty}^{\infty} dz \, z^{-2} \Psi_\Delta(z) G(z) \]

\[ = \int_{-\infty}^{0} dz \, z^{-2} \Psi_\Delta^{(0)} \left( \frac{z}{z-1} \right) G^{(0)} \left( \frac{z}{z-1} \right) + \int_{0}^{1} dz \, z^{-2} \Psi_\Delta^{(0)}(z) G^{(0)}(z) \]

\[ + \int_{1}^{\infty} dz \, z^{-2} \Psi_\Delta^{(+)}(z) z^{2\Delta_\phi} G^{(0)} \left( \frac{1}{z} \right) \]

\[ = -\int_{1}^{0} dw \, w^{-2} \Psi_\Delta^{(0)}(w) G^{(0)}(w) + \int_{0}^{1} dz \, z^{-2} \Psi_\Delta^{(0)}(z) G^{(0)}(z) - \int_{1}^{0} dw \, \Psi_\Delta^{(+)}(1/w) w^{-2\Delta_\phi} G^{(0)}(w) \]

\[ = 2 \int_{0}^{1} dz \, z^{-2} G^{(0)}(z) \Psi_\Delta^{(0)}(z) + \int_{0}^{1} dz \, z^{-2} G^{(0)}(z) \Psi_\Delta^{(+)} \left( \frac{1}{z} \right) G^{(0)} \left( \frac{1}{z} \right) \]

\[ = 2I_\Delta^{\alpha} + I_\Delta^{\alpha}, \] (B.5)

where \( \alpha = 2\Delta_\phi - \frac{3}{2} \).

Before applying the above formula to (3.2), let us first consider some integrals:

\[ J_\Delta^{(0)}(p, q) = \int_{0}^{1} dz \, z^{-2} \Psi_\Delta^{(0)}(z) \frac{z^p}{(1-z)^q} \] (B.6)

\[ = K_{1-\Delta} \int_{0}^{1} dz \, G_\Delta(z) z^{p-2}(1-z)^{-q} + \left\{ \Delta \to 1-\Delta \right\} \]

\[ = \frac{\sqrt{\pi} \Gamma \left( \frac{1-\Delta}{2} \right) \Gamma \left( \frac{\Delta}{2} \right)^2}{\Gamma(\Delta) \Gamma \left( \frac{1-\Delta}{2} \right)^2} \frac{\Gamma(1-q) \Gamma(-1+p+\Delta)}{\Gamma(p-q+\Delta)} {}_3F_2 \left( \Delta, \Delta, -1+p+\Delta; 2\Delta, p-q; 1 \right) \]

\[ + \left\{ \Delta \to 1-\Delta \right\} \]

\[ = \frac{(\sec(\pi \Delta) + 1) \Gamma(\Delta)^2 \Gamma(1-q) \Gamma(p+\Delta) \Gamma(p-\Delta)}{\Gamma(2\Delta) \Gamma(p-q+\Delta)} {}_3F_2 \left( \Delta, \Delta, -1+p+\Delta; 2\Delta, p-q; 1 \right) \]

\[ + \frac{(\sec(\pi \Delta) + 1) \Gamma(1-\Delta)^2 \Gamma(1-q) \Gamma(p-\Delta)}{\Gamma(2-2\Delta) \Gamma(p-q+1-\Delta)} {}_3F_2 \left( 1-\Delta, 1-\Delta, p-\Delta; 2-2\Delta, p-q; 1 \right). \] (B.7)
In the above evaluations, we use the following formula

\[
\int_0^1 dz \, G_\Delta(z)z^a(1-z)^b = \frac{\Gamma(1 + b)\Gamma(1 + a + \Delta)}{\Gamma(2 + a + b + \Delta)} \, _3F_2(\Delta, \Delta, 1 + a + \Delta; 2\Delta, 2 + a + b; 1),
\]

(B.8)

which is valid for Re \( b > -1, \Re(a + \Delta) > -1 \).

Similarly, we can also consider

\[
\int_0^1 dz \, z^{-2}\Psi_\Delta^{(+)}(1/z) \frac{z^p}{(1-z)^q} = 2\pi \csc(\pi \Delta) \left[ \int_0^1 dz \, _2F_1(1 - \Delta, \Delta; 1; z)z^{p-2}(1-z)^{-q} 
+ \int_0^1 dz \, _2F_1(1 - \Delta, \Delta; 1; 1-z)z^{p-2}(1-z)^{-q} \right]
= 2\pi \csc(\pi \Delta) \left[ \int_0^1 dz \, _2F_1(1 - \Delta, \Delta; 1; z)z^{p-2}(1-z)^{-q} 
+ \int_0^1 dz \, _2F_1(1 - \Delta, \Delta; 1; z)(1-z)z^{p-2}z^{-q} \right]
= 2\pi \csc(\pi \Delta) B(p-1, -q+1) \left[ _3F_2(p-1, 1 - \Delta, \Delta; 1, p - q; 1) 
+ _3F_2(-q+1, 1 - \Delta, \Delta; 1, p - q; 1) \right],
\]

(B.9)

where we use the formula

\[
\int_0^1 dz \, z^a(1-z)^b \, _2F_1(1 - \Delta, \Delta; 1; z) = B(a + 1, b + 1) \, _3F_2(a + 1, 1 - \Delta, \Delta; 1, a + b + 2; 1), \\
\Re a, b > -1.
\]

(B.10)

Therefore we have

\[
J_\Delta^+(p,q) = \int_0^1 dz \, z^{-2}\Psi_\Delta^{(+)}(1/z) \frac{z^p}{(1-z)^q} z^{\frac{1}{2}-a}
= 2\pi \csc(\pi \Delta) B(p - \frac{1}{2} - \alpha, -q + 1) \left[ _3F_2\left(p - \frac{1}{2} - \alpha, 1 - \Delta, \Delta; 1, p + \frac{1}{2} - \alpha - q; 1\right) 
+ _3F_2\left(-q + 1, 1 - \Delta, \Delta; 1, p + \frac{1}{2} - \alpha - q; 1\right) \right].
\]

(B.11)

With the help of two integrals in (B.7), (B.12), it is straightforward to compute the principal series coefficient function \( I_\Delta \) for (3.2). The full result is given in (3.24).

Next, we would like to find the poles and residues of \( I_\Delta/K_\Delta = 2I_\Delta^+ / K_\Delta + I_\Delta^- / K_\Delta \), which encodes the OPE data (2.60). For simplicity, we will assume that \( p, q, \alpha \) are generic, in order to avoid the overlapping of poles.
Let us first consider the simple poles at $\Delta = p + n$ with $n \in \mathbb{N}$, which come from $\Gamma(p - \Delta)$ in (B.7). The residues there are given by

$$
\text{Res}_{\Delta=p+n} \frac{J_0^\Delta}{K_\Delta} = \frac{(-1)^n q \Gamma(-q)}{\Gamma(n+1)\Gamma(-n-q+1)} \times 3F_2 \left( -n-p+1, -n-p+1, -n; -n-q+1, -2(n+p-1); 1 \right) 
$$

$$
= \frac{(-1)^{3n+1} \Gamma(-2(n+p-1))\Gamma(n+p)^2}{\Gamma(n+1)\Gamma(p)^2\Gamma(-2n+2p+2)} \times 3F_2 \left( p-q, n+2p-1, -n; p, p; 1 \right) 
$$

$$
= -C_{p,q}(n),
$$

where $C_{p,q}(n)$ is given in (3.5). In the second equality above, we used the identity

$$
3F_2(-n-p+1, -n-p+1, -n; -n-q+1, -2n-2p+2; 1) = \frac{(-1)^{2n}\Gamma(-2(n+p-1))\Gamma(n+p)^2}{\Gamma(p)^2\Gamma(1-q)\Gamma(-2n+2p+2)} 3F_2(p-q, n+2p-1, -n; p, p; 1),
$$

which can be derived by combining [29, 34]

$$
3F_2(a, b, -n; e, f; 1) = (-1)^n \frac{\Gamma(f)\Gamma(b-f+1)}{\Gamma(f+n)\Gamma(b-f-n+1)} 3F_2(b, e-a, -n; e, b-f-n+1; 1),
$$

and

$$
3F_2(a, b, -n; e, f; 1) = \frac{(-1)^n\Gamma(e)\Gamma(f)\Gamma(a-f+1)\Gamma(b-f+1)}{\Gamma(e+n)\Gamma(f+n)\Gamma(a-f-n+1)\Gamma(b-f-n+1)} \times 3F_2(-f-n+1, a+b-e-f-n+1, -n; a-f-n+1, b-f-n+1; 1).
$$

For generic $p, q, \alpha$, it is easy to convince oneself that this type of poles at $\Delta = p + n$ does not appear in $J_\Delta^+(\Delta)$ (B.11). As a result, we find

$$
\text{Res}_{\Delta=p+n} \frac{J_\Delta(p, q)}{2K_\Delta} = \text{Res}_{\Delta=p+n} \frac{2I_{\Delta}^0(p, q) + J_\Delta^+(p, q)}{2K_\Delta} = -C_{p,q}(n).
$$

If we deform the contour in (2.59) to enclose such a pole, we find the OPE coefficients there (2.60), in agreement with (3.4). Note that the minus sign is due to the orientation of the contour.

From the formulae above, it appears that there may be also other poles at integers or half integers. Since we have reproduced (3.4), we expect that the residues at all the rest of poles should be cancelled. Showing this explicitly is tedious, so we will not try to do it here. Instead, we will try to reproduce the important results in (3.8) where all the positive even integral conformal dimensions are absent. This means that there is a complete cancellation between discrete and principal series. We will now try to show this explicitly.

For this aim, we need to know the discrete series coefficient function $\tilde{I}_n$. To have a non-trivial and interesting check, we will use the Lorentz inversion formula (2.61). For this purpose, let us first introduce the vital quantity, namely the double discontinuity, which is defined as [24, 27]

$$
d\text{Disc}[G(z)] = G(z) - \frac{G^+(z) + G^-(z)}{2} = G^0(z) - \frac{G^+(z+i\epsilon) + G^+(z-i\epsilon)}{2}, \quad z \in (0, 1),
$$

(B.18)
\( \text{where } G^{(0)}(z) \text{ is defined on } z \in (0, 1), \text{ analytically continuing to the upper and lower half complex plane; while } G^{(+)}(z) \text{ is given by (2.16) in terms of } G^{(0)}(z), \text{ defined on } \mathbb{C} \setminus (-\infty, 1) \text{ after analytic continuation. The meaning of } G^{(+)}(z \pm i \epsilon) \text{ is that we can start with } G^{(+)} \text{ defined on } (1, \infty) \text{ (using its relation (2.16) with } G^{(0)} \text{ defined on } (0, 1)), \text{ and then perform analytic continuation to the points } z \pm i \epsilon \text{ which sit above and below the branch cut } z \in (0, 1) \subset (-\infty, 1). \)

Following the definition, we have
\[
\text{dDisc}[z^a(1-z)^b] = z^a(1-z)^b - z^{2\Delta-a-b}(1-z)^b \cos(\pi b) .
\] (B.19)

Therefore, for the celestial correlator in (3.2), the double discontinuity is given by
\[
d\text{Disc}[G(z)] = \frac{z}{\sqrt{1-z}} \left[ 1 + e^{\pi i \alpha} z^\alpha + (1 + \sin \pi \alpha) \left( \frac{z}{1-z} \right)^\alpha \right] .
\] (B.20)

Applying the discrete series Lorentz inversion formula (2.61), we find
\[
\tilde{I}_m = \frac{4 \Gamma(m)^2}{\Gamma(2m)} \int_0^1 dz \, z^{-2} G_m(z) d\text{Disc}[G(z)]
\] (B.21)
\[
= \frac{4 \Gamma(m)^2}{\Gamma(2m)} \int_0^1 dz \, z^{-2} \frac{z}{\sqrt{1-z}} \left[ 1 + e^{\pi i \alpha} z^\alpha + (1 + \sin \pi \alpha) \left( \frac{z}{1-z} \right)^\alpha \right] G_m(z) \]
\[
= \frac{\pi^{3-2m} \Gamma(m)^2}{\Gamma \left( m + \frac{1}{2} \right)^2} \left[ _3 F_2 \left( m, m; m + \frac{1}{2}, 2m; 1 \right) \right.
\]
\[
+ \left. \frac{\pi^{3-2m} \Gamma(m)^2}{\Gamma \left( m + \frac{1}{2} \right)^2} \right] \left[ _3 F_2 \left( \alpha + m, m, m; \alpha + m + \frac{1}{2}, 2m; 1 \right) \frac{\Gamma \left( m + \frac{1}{2} \right) \Gamma(\alpha + m)}{\Gamma \left( m \right) \Gamma \left( \alpha + m + \frac{1}{2} \right)} e^{i \pi \alpha} \right.
\]
\[
+ \left. \frac{\pi^{3-2m} \Gamma(m)^2}{\Gamma \left( m + \frac{1}{2} \right)^2} \right] \left[ _3 F_2 \left( \alpha + m, m, m; \alpha + m + \frac{1}{2}, 2m; 1 \right) \frac{\Gamma \left( \frac{1}{2} - \alpha \right) \Gamma(\alpha + m)}{\sqrt{\pi} \Gamma(m)} \right] \right] ,
\] (B.22)

where we use (B.8) to perform the integral.

We want to show that the OPE coefficients at positive even positive integers vanish. This is equivalent to showing
\[
\text{Res}_{\Delta=m} \frac{I_\Delta}{K_\Delta} = \frac{\Gamma(m)^2}{2 \pi^2 \Gamma(2m-1)} \tilde{I}_m .
\] (B.24)

\( I_\Delta \) gets contribution from \( I^0_\Delta \) and \( I^+_\Delta \). For \( I^0_\Delta / K_\Delta \), there are simple poles at positive even integers because of the zeros of \( K_\Delta \) (B.2). Using (B.7), it is straightforward to calculate the residues of \( I^0_\Delta / K_\Delta \) at positive even integers; the final result is very similar to (B.23) except for an overall factor and the \( \sin \pi \alpha \) term. More precisely, we find the difference
\[
\text{Res}_{\Delta=m} \frac{I^+_\Delta}{K_\Delta} = \text{Res}_{\Delta=m} \frac{I^0_\Delta}{K_\Delta} - \text{Res}_{\Delta=m} \frac{2 I^0_\Delta}{K_\Delta} = \frac{\Gamma(m)^2}{2 \pi^2 \Gamma(2m-1)} \tilde{I}_m - \text{Res}_{\Delta=m} \frac{2 I^0_\Delta}{K_\Delta}
\] (B.25)
\[
= \frac{\pi^{3-2m} \Gamma(m)^2}{\Gamma \left( m + \frac{1}{2} \right)^2} \left[ _3 F_2 \left( \Delta, \Delta; 2\Delta, \Delta + \frac{1}{2}; 1 \right) \frac{\pi^{3/2} \Gamma \left( \Delta - \frac{1}{2} \right) \Gamma(\Delta) \left( \Delta + \frac{1}{2} \right) \Gamma \left( \Delta + \frac{1}{2} \right)}{\pi \Gamma(2\Delta) \Gamma(\Delta + \frac{1}{2})} \sin(\pi \alpha) .
\] (B.26)
To show (B.24) is true, we need to compute the residues of $I_\Delta^+/K_\Delta$ and prove the equation (B.26). We will show below that this is indeed the case. For the celestial correlator (3.2), we have

$$I_\Delta^+ = J_\Delta^+ \left( 1 + \alpha, \frac{1}{2} \right) + J_\Delta^+ \left( 1, \frac{1}{2} \right) + J_\Delta^+ \left( 1 + \alpha, \frac{1}{2} + \alpha \right).$$  \hspace{1cm} (B.27)

For the first term, it can be evaluated explicitly using (B.12)

$$J_\Delta^+ \left( 1 + \alpha, \frac{1}{2} \right) = \frac{2\pi^3 \csc(\pi\Delta)}{\left( 1 - \frac{3}{2} \right)^2 \Gamma \left( \frac{\Delta + 1}{2} \right)^2},$$  \hspace{1cm} (B.28)

which has zeros at positive even integer points. So $J_\Delta^+ (1 + \alpha, \frac{1}{2})/K_\Delta$ has no poles at positive even integers.

The rest two terms together combine to

$$J_\Delta^+ \left( 1, \frac{1}{2} \right) + J_\Delta^+ \left( 1 + \alpha, \frac{1}{2} + \alpha \right) = \frac{2\pi^3/2 \Gamma \left( \frac{1}{2} - \alpha \right) \csc(\pi\Delta)}{\Gamma(1 - \alpha)} \left[ 3F_2 \left( \frac{1}{2}, 1 - \Delta, \Delta; 1, 1 - \alpha; 1 \right) \right. $$

\hspace{2cm} \left. + 3F_2 \left( \frac{1}{2} - \alpha, 1 - \Delta, \Delta; 1, 1 - \alpha; 1 \right) \right], \hspace{1cm} (B.29)

Using identity (B.15), one can show that the two terms in the square bracket cancel for positive even integers $\Delta \in 2\mathbb{Z}_{>0}$. So near the positive even integers, we expect that the two terms in the square bracket of (B.29) have the following expansion

$$\left[ \cdots \right] = 0 + (\Delta - m)A_1 + \cdots, \quad m \in 2\mathbb{Z}_{>0}, \hspace{1cm} (B.30)$$

On the other hand, the overall coefficient in (B.29) can be expanded as

$$\frac{2\pi^3/2 \Gamma \left( \frac{1}{2} - \alpha \right) \csc(\pi\Delta)}{\Gamma(1 - \alpha)K_\Delta} = \frac{2^{3-2\Delta} \Gamma \left( \frac{1}{2} - \alpha \right) \Gamma(\Delta)}{\pi\Gamma(1 - \alpha)\Gamma \left( \Delta - \frac{1}{2} \right)} \frac{1}{(\Delta - m)^2} - \# \frac{1}{(\Delta - m)} + \cdots. \hspace{1cm} (B.31)$$

As a result, we find the residues at positive even integers are given by

$$\text{Res}_{\Delta=m} \frac{I_\Delta^+}{K_\Delta} = \frac{2^{3-2\Delta} \Gamma \left( \frac{1}{2} - \alpha \right) \Gamma(\Delta)}{\pi\Gamma(1 - \alpha)\Gamma \left( \Delta - \frac{1}{2} \right)} A_1. \hspace{1cm} (B.32)$$

If (B.24) or equivalently (B.26) holds, then we should have the following relation

$$\lim_{\Delta \to m} \frac{3F_2 \left( \frac{1}{2}, 1 - \Delta, \Delta; 1, 1 - \alpha; 1 \right) + 3F_2 \left( \frac{1}{2} - \alpha, 1 - \Delta, \Delta; 1, 1 - \alpha; 1 \right)}{\Delta - m}$$

\hspace{2cm} $$= \frac{\partial}{\partial \Delta} \left[ 3F_2 \left( \frac{1}{2}, 1 - \Delta, \Delta; 1, 1 - \alpha; 1 \right) + 3F_2 \left( \frac{1}{2} - \alpha, 1 - \Delta, \Delta; 1, 1 - \alpha; 1 \right) \right] \bigg|_{\Delta=m} \hspace{1cm} (B.33)$$

\hspace{2cm} $$= \frac{\sin(\pi\alpha)}{\Gamma(1 - \alpha)\Gamma(m)\Gamma(m + \frac{1}{2})} \Gamma(\alpha + m) \frac{m}{\sqrt{\Gamma(2m)\Gamma(m + \frac{1}{2})}} 3F_2 \left( m, m + \alpha + 2m, m + 1 \right), \quad m \in 2\mathbb{Z}_{>0}, \hspace{1cm} (B.34)$$
where we use the definition of $A_1 = \lim_{\Delta \to m}[\cdots]/(\Delta - m)$ in (B.30). We don’t have an analytic proof, but we have verified it numerically for a certain range of $\alpha$. To conclude, we have thus argued that (B.24) holds for the celestial correlator studied in this paper (3.2), implying a perfect cancellation between principal and discrete series at positive even integers. Therefore, the operators with positive even integer scaling dimensions are absent.

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