A Recommender System Based on a Double Feature Allocation Model

Qiaohui Lin and Peter Müller

Abstract

A collaborative filtering recommender system predicts user preferences by discovering common features among users and items. We implement such inference using a Bayesian double feature allocation model, that is, a model for random pairs of subsets. We use an Indian buffet process (IBP) to link users and items to features. Here a feature is a subset of users and a matching subset of items. By training feature-specific rating effects, we predict ratings. We use MovieLens Data to demonstrate posterior inference in the model and prediction of user preferences for unseen items compared to items they have previously rated.

Part of the implementation is a novel semi-consensus Monte Carlo method to accommodate large numbers of users and items, as is typical for related applications. The proposed approach implements parallel posterior sampling in multiple shards of users while sharing item-related global parameters across shards.

1 Introduction

We develop a nonparametric Bayesian model-based approach to collaborative filtering for random subsets of items and users. The main contributions are the construction of a suitable prior for pairs of subsets of items and users (features), the possibility to report coherent inference on such features, and a consensus Monte Carlo approach to allow practical implementation of posterior inference.

Collaborative filtering refers to recommender systems that predict personalized user preferences for products (i.e., ratings, rankings) by discovering similarity patterns among users and items, and make corresponding recommendations (Sarwar et al. [2001], Schafer et al. [2007], Koren and Bell [2015]). It has been widely adopted by e-commerce websites and online streaming services. The Netflix Prize since 2006 has encouraged more progress in this field.
The Netflix data (Marlin [2004], Koren and Bell [2015], Sedhain et al. [2015], Liang et al. [2018]) is a widely used benchmark dataset used in collaborative filtering research. It is a sparse matrix with 463435 rows for users, 17769 columns for items (movies) and 56.9 million entries based on ratings between 1999 and 2005. On average a movie has 5600 ratings and a user rates 208 movies (Koren and Bell [2015]). Thus the density of the matrix is as low as 0.69% (Liang et al. [2018]).

Collaborative filtering methods have progressed from naive nearest neighbor methods, to well-adopted matrix factorization methods, to probabilistic Bayesian models with latent factors and, more recently, to generative models. Nearest neighbour methods (Schafer et al. [2007], Koren and Bell [2015]) are intuitively appealing. The idea is that the rating $r_{ui}$ of user $u$ for item $i$ is likely to be close to the ratings $r_{u'i}$ of similar users $u'$ for the same item, or the ratings $r_{ui'}$ of similar items $i'$ by the same user $u$. The key here is to measure the similarity of users (or items). The simplest measure uses the correlation coefficient. Using a similarity measure, we can identify the $k$ items that are most similar to $i$ rated by $u$. Denote the set of such items by $S_k(i;u)$. We can then predict $r_{ui}$ using a weighted average of the ratings of items in $S_k(i;u)$ that are rated by user $u$, using weights proportional to the respective similarities, i.e, the more similar the neighbour is, the more weight it gets.

Assume now that ratings are metric, facilitating the use of one of the most popular methods in collaborative filtering based on matrix factorization. The $m \times n$ rating matrix $R = [r_{ui}]$ with $m$ users and $n$ movies can be written as the product of a $k \times m$ matrix $P$ and $k \times n$ matrix $Q$, i.e, $R = P^T Q$. Here, $k$ is the number of latent factors $z$, for example, movie genres. Then the $P$ matrix can be interpreted as representing the preferences of users for the $k$ genres and the $Q$ matrix can be interpreted as a classification of the movies relative to these genres. The rating $r_{ui}$ for a specific movie and user then becomes $r_{ui} = p_u^T q_i$, using $p_u$ and $q_i$ to denote the corresponding columns of $P$ and $Q$, respectively.

The problem then is to find preferences $P$ and $Q$ to best predict observed ratings $R$ by $\hat{R} = P^T Q$, while controlling the number of the latent factors (the rank of $P$ and $Q$). Thus the objective is

$$
\min_{Q,P} \sum_{u,i} (r_{ui} - p_u^T q_i)^2 + \lambda (\sum_i ||q_i||^2 + \sum_u ||p_u||^2).
$$

(1)

The parameter $\lambda$ is determined by cross validation. Minimization is performed by stochastic gradient descent. A popular implementation is reported in Funk [2006] and Paterek [2007].

Matrix factorization captures the latent pattern in users and items. A convenient stochastic
gradient descent made it a winner of the Netflix prize. An additional advantage is the ease of incorporating temporal dynamics. However, the point estimation of the prediction comes without an uncertainty measure and thus cannot serve a more complicated goal of filtering and understanding user behavior, one of the reasons why we introduce probabilistic models in the next section.

2 Probabilistic Models

Probabilistic matrix factorization [Mnih and Salakhutdinov, 2008] interprets (1) from a probabilistic perspective. They show that minimizing the sum of square errors while penalizing their Frobenius norm is equivalent to maximizing the log posterior in a probabilistic model with spherical Gaussian priors.

As before, define $R = [r_{ui}]$ as the $(m \times n)$ rating matrix, and $P$ and $Q$ are $(k \times m)$ and $(k \times n)$ the low rank latent feature-user and feature-item matrices. As before let $p_u$ and $q_i$ denote column $u$ and $i$ of $P$ and $Q$, respectively, and let $N(x \mid m, V)$ denote a normal p.d.f for random variable $x$ with moments $m$ and $V$. We assume

$$p(R \mid P, Q, \sigma^2) = \prod_{u=1}^{m} \prod_{i=1}^{n} [N(r_{ui} \mid p_u^T q_i, \sigma^2)]^{I_{ui}}.$$ 

with $I_{ui} = 1$ when user $u$ has rated movie $i$ and $I_{ui} = 0$ otherwise. The model is completed with zero-mean spherical Gaussian priors on $P$ and $Q$:

$$p(P \mid \sigma^2_P) = \prod_{u=1}^{m} N(p_u \mid 0, \sigma^2_P I), \quad p(Q \mid \sigma^2_Q) = \prod_{i=1}^{n} N(q_i \mid 0, \sigma^2_Q I)$$

Maximizing log $p(P, Q \mid R, \sigma^2, \sigma^2_P, \sigma^2_Q)$ under this model is equivalent to minimizing the sum of squares error with quadratic regularization, as in

$$E = \frac{1}{2} \sum_{u=1}^{m} \sum_{i=1}^{n} I_{ui} (r_{ui} - p_u^T q_i)^2 + \frac{\lambda_P}{2} \sum_{u=1}^{m} \|p_u\|^2 + \frac{\lambda_Q}{2} \sum_{i=1}^{n} \|q_i\|^2,$$  

(2) 

where $\lambda_P = \frac{\sigma^2}{\sigma^2_P}$, $\lambda_Q = \frac{\sigma^2}{\sigma^2_Q}$.

A minibatch gradient descent is used to find the optimal $P$ and $Q$. The optimization defines an extension of the classic SVD model, where the modified SVD is defined as the MAP estimate, and the classic SVD is a special case where prior variance goes to infinity.

Mnih and Salakhutdinov [2008] also discussed other constraints to be allowed onto the model. For all proposed models, instead of adopting a full Bayesian approach and leading to a MCMC
posterior simulation, the authors used stochastic gradient descent. This approach limits meaningful posterior inference for hyperparameters and uncertainty quantification of the MAP, but on the other hand has vastly decreased the computational cost.

Some probabilistic models proposed in the recent literature build on the Bayesian Mallows Model (Liu et al. [2019b], Liu et al. [2019a], Vitelli et al. [2017]). Different from other models, Mallows model treats the response variables as ordinal rankings. For example, Liu et al. [2019b] work with the ranking on $n$ items for a user, $R_u = \{R_u1, R_u2, \ldots, R_un\}$, $R ui \in \{1, \ldots, n\}$ and $u = 1, \ldots, m$. Mallows model is a probability model on the space $P_n$ of permutations of $n$ items.

A basic model uses a latent consensus ranking $\rho \in P_n$ to define,

$$p(R_u = R \mid \alpha, \rho) \propto \exp \left( -\frac{\alpha}{n} d(R, \rho) \right) ,$$

(3)

where $\alpha$ is a scale parameter and $d(R, \rho) = \sum_{i=1}^{n} |R_i - \rho_i|$ is a distance between $R$ and $\rho$. The normalizing constant $Z_n(\alpha, \rho)$ in (3) is usually not analytically tractable. Vitelli et al. [2017], for example, use instead importance sampling and Metropolis-Hastings posterior simulation schemes.

In a more complicated scenario where users are not homogeneous, $m$ users are arranged into $C$ clusters; each cluster has its own common consensus $\rho_c$. Latent cluster membership indicator $z_u$ assign user $u$ to cluster $z_u$. The model is then

$$p(R_1, \ldots, R_m \mid z_1, \ldots, z_m, \alpha_c, \rho_c; c = 1, \ldots, C) = \prod_{u=1}^{m} [Z_n(\alpha_{z_u})]^{-1} \exp \left\{ -\frac{\alpha_{z_u}}{n} d(R_u, \rho_{z_u}) \right\} .$$

(4)

with exponential priors on $\alpha_c$, uniform prior on $\rho_c$. The prior for the cluster assignments $z_u$, $u = 1, \ldots, m$ is $p(z_1, \ldots, z_u \mid \tau_1, \ldots, \tau_C) = \prod_{u=1}^{m} \tau_{z_u}$ with a Dirichlet prior on $\tau$.

Assume now that a given user $u$ has rated only $m_u < m$ (instead of all $m$) items and the objective is to make $L$ recommendations. This is equivalent to inferring the unseen items with the $L$ highest rankings. Let $H_u$ denote the top $L$ rankings different from the rankings of the observed items. To find the top $L$ items for user $u$, we evaluate the posterior probability

$$p_{iu} = p(R_{ui} \in H_u \mid data)$$

for all unrated items $i$ (skipping details of how model (4) is modified to allow for the observation of $m_u < m$ items only). The strength of Mallows model is the use of a distribution of rankings, allowing inference beyond point estimation. The main limitation is the need for computation-expensive posterior MCMC simulation, which is not suitable for large data sets on sequential updating.
Some approaches to collaborative filtering are based on LDA (latent Dirichlet allocation) type models. For example, the User Rating Profile model (URP) proposed in Marlin [2004] represents each user as a mixture as a user attitudes, and the mixture proportions are distributed according to a Dirichlet random variable. For any user \( u \) we introduce a set of latent item-specific attitudes \( Z_{ui} \). Here \( Z_{ui} \) is a user attitude that determines the rating of \( i \). Next, let \( \beta_{viz} = P(r_{ui} = v \mid Z_{ui} = z) \) denote item-specific rating probabilities. Like in the Latent Dirichlet Allocation (LDA) model, \( \theta_u \) is a Dirichlet random variable with parameter \( \alpha \), and \( p(Z = z) = \theta_{uz} \).

A user profile is thus \( r_u = [r_{u1}, \ldots, r_{un}] \) with a sampling model \( p(r_u \mid Z_u, \beta) \) determined by the described hierarchical model. However, the posterior distribution for \( \theta \) and \( Z_u \) is intractable.

Another line of research are based on the use neural networks or, more generally, generative models for collaborative filtering [Sedhain et al., 2015, Li et al., 2016, He et al., 2017, Liang et al., 2018]. Variational Autoencoder is one of the popular methods.

The use of variational autoencoder models originated from Kingma and Welling [2013]. To apply it to the collaborative filtering, following Liang et al. [2018], we assume the hidden factors are latent variables \( z_u \), the number of clicks of a user to all items is \( x_u = [x_{u1}, \ldots, x_{un}] \), \( f_{\theta}(z_u) \) is a non-linear function to produce a probability distribution of \( \pi(z_u) \), and \( x_u \) is multinomial with total number of clicks \( N_u \) and probability vector \( \pi(z_u) \).

\[
\begin{align*}
  z_u &\sim N(0, I_k), \\
  \pi(z_u) &\propto \exp[f_{\theta}(z_u)], \\
  x_u &\sim \text{Mult}(N_u, \pi(z_u)),
\end{align*}
\]

One common approach is to use variational inference to approximate the intractable posterior \( p(z_u \mid x_u) \) with a variational approximate distribution \( q(z_u) \). Full algorithm can be seen in Liang et al. [2018] and Kingma and Welling [2013].

Autorec model (Sedhain et al. [2015]) is another version of this model applied to rating instead of clicking numbers. Generative Models with variational inference is an efficient scalable inference well suited when the dataset is large in size and new users’ data keep flowing in. The problem is the lack of explainibility, the absence of uncertainty quantification, and the difficulty to fit the model to a more complicated goal, such as a specific criterion for recommendation diversity and accuracy tradeoff.

Below we introduce a model that builds on these approaches, aiming to (i) include learning on underlying structure that determines user preferences, and (ii) still allows (approximate) full
posterior inference. The underlying structure that can be discovered by the proposed model is an extension of the clusters that feature in Mallows model by adding matching subsets of items.

## 3 Double Feature Allocation Model

We introduce an alternative generative model for user ratings $r_{ui}$ of users $u = 1, \ldots, m$ for items $i = 1, \ldots, n$, using notation as before. We assume ordinal ratings $r_{ui} \in \{1, \ldots, 5\}$. The inference goal is to predict user preferences for un-rated movies by discovering similarity patterns among users and movies.

The proposed model construction is guided by symmetry assumptions with respect to items and users. That is, the probability model should be invariant with respect to arbitrary permutation of user indices, and/or permutations of movie indices. Models with such structure are also known as separately exchangeable, and the rating matrix can be characterized as

$$r_{1:m,1:n} \overset{d}{=} r_{\pi_1(1:m), \pi_2(1:n)}$$

for separate permutations $\pi_1$ and $\pi_2$ of rows and columns, respectively. Here $X \overset{d}{=} Y$ indicates equality in distribution for two random variables.

A double feature allocation model is a model for random pairs of subsets, first proposed in Ni et al. [2019b]. Our model is similar but different from the Coupled Indian Buffet Process Model proposed by Chatzis [2012]. Chatzis [2012] used two independent Indian Buffet process (IBP) for two separate feature allocations of users and items. While in our case, a feature is a subset of users together with a matching subset of items. We use an Indian buffet process (IBP) prior to link users and items to features. Suppose there are $K$ features. We use an $(m \times K)$ binary matrix $A$ to link users to features, with $A_{uk} = 1$ indicating that user $u$ is in feature $k$. Another $(n \times K)$ binary matrix $B$ links items to features, with $B_{ik} = 1$ meaning that movie $i$ is in feature $k$. Figure 1 is a stylized representation of the double feature allocation model. Same colored block indicate one feature $k$, a subset of users and a matching subset of items.

The IBP prior on $A$ [Griffiths and Ghahramani, 2011] is defined as follows. The model includes an unknown number $K$ of features, and can be written as

$$p(A) = \frac{\lambda^K \exp(-\lambda H)}{K!} \prod_{k=1}^{K} \frac{\Gamma(m_k)\Gamma(m - m_k + 1)}{\Gamma(m + 1)}$$
where $p(A)$ is without order of columns, $\lambda$ is a fixed hyperparameter (concentration parameter), $H$ is the harmonic number $H = \sum_{u=1}^{m} 1/u$, $m_k$ is the sum of column $k$, $m_k = \sum_{u=1}^{m} A_{uk}$. The number of features $K$ is random and unbounded, and features are exchangeable a priori. The IBP is easiest described as a generative model building up $A$ row by row, starting with $u = 1$, and adding columns of $A$, i.e., features, and indexing features by appearance. Let $K_u$ denote the number of features that are introduced after the first $u$ users, starting with $K_0 = 0$, and let $m_{u,k} = \sum_{v=1}^{u} A_{vi}$ denote the cardinality of feature $k$ among the first $u$ users, $k = 1, \ldots, K_u$. Considering the respective next user $u$ we then proceed as follows. First we decide inclusion into one of the existing features, $k = 1, \ldots, K_u-1$, with probability $p(A_{uk} = 1 \mid A_{1\ldots u-1,1\ldots K_u-1}) = m_{u-1,k}/u$ Then we add a Poisson random number $K^+_u \sim \text{Poi}(\alpha/u)$ new features with $A_{uk} = 1$, $k = K_{u-1}, \ldots, K_{u-1} + K^+_u$ and increment $K_u = K_{u-1} + K^+_u$. Implicit in the construction is a constraint of all zeroes in the right upper corner of $A$, i.e., $A_{vk} = 0$, $k > K_u$ and $v < u$. We remove the constraint of indexing items by appearance by using a final step of randomly permuting the final $K = K_n$ columns. In practice, for the purpose of prior sampling, we only need the conditional probability of $A_{uk} = 1$,

$$p(A_{uk} = 1 | A_{-u,k}) = m_{-u,k}/m,$$

where $A_{-u,k}$ is the $k$th column of $A$ excluding $A_{uk}$ and $m_{-u,k}$ is the sum of column $k$ of $A$ excluding $A_{uk}$.

Given $A$, the item-feature matrix $B$ inherits $K$ features from $A$. For simplicity, we assume
independent Bernoullis:

\[ p(B_{ik} = 1 | A) = p, \]

with the prior parameter \( p \) usually chosen to be a small number to control the number of features a movie can be in. Such parsimony ensures that features do not share too many common movies and preserve their differences. In posterior sampling, \( p \) and the dimension of \( B \) will be updated each time after we update \( A \).

Note that the IBP \( p(A) \) together with \( p(B | A) \) define a joint model \( p(A, B) \), and therefore also imply a marginal \( p(B) \). The current model does not imply a marginal IBP prior for \( p(B) \). If a more symmetric construction with an IBP marginal prior on \( B \) were desired, it could be easily achieved. Let \( p_{\text{IBP}}(B) \) denote an IBP prior on a random binary matrix \( B \), including the number of columns, \( K_B \), and let \( p_{\text{IBP}}(B | K_B) \) denote the conditional distribution of \( B \), conditional on the number of columns equal to \( K_B \) under the IBP. Then using \( p(B | A) = p(B | K_B = K_A) \) would by construction deliver \( p(B) = \text{IBP} \), marginally.

Finally, we complete the inference model with a sampling model for the observed ratings. The model links features to ratings by introducing probabilities for \( r_{ui} \) conditional on the currently imputed features of which user \( u \) and items \( i \) are members.

We consider a baseline \( b_0 = 2.5 \) (between the extremes 1 and 5). Each feature \( k \) of which user \( u \) and movie \( i \) are a member adds an adjustment \( \theta_k \) to this baseline. For example, feature \( k \) might be a pair of subsets of comedy movies and comedy fans. Then \( \theta_k \) would be a positive increment from baseline. Similarly, if feature \( k \) is a pair of subsets of comedy movies and comedy haters, then \( \theta_k \) should be a negative adjustment. We allow an item and a user to be in multiple subsets with features having aggregative influences.

We also include global parameters \( \rho_i \) for each movie regardless of feature allocation. The parameter \( \rho_i \) has an interpretation as overall mean rating for movie \( i \) (on the ordinal probit scale). This movie-specific offset reflects if a movie is generally popular and well-received among audiences or vice versa. We complete the prior model with independent priors for \( \theta_k \) and \( \rho_i \), assuming \( \theta_k \sim h_\theta \) and \( \rho_i \sim h_\rho \), i.i.d.

The sampling model is then defined as an ordinal probit model including the described feature-specific and item-specific parameters. Denote by \( Z_{ui} \) a latent continuous probit score for the rating of user \( u \) for movie \( i \), denote by \( K^*_{ui} = \{ k : A_{uk} = B_{ki} = 1 \} \) the set of features that include both,
user $u$ and movie $i$. We use an inverse-gamma hyperprior on $\tau^2$, and normal priors for both $\theta$ and $\rho$. We assume

$$Z_{ui} \mid A, B, \theta, \tau \sim N(b_0 + \sum_{k \in K^*} \theta_k + \rho_i, \tau^2).$$

The probit scores are linked with the data in the usual ordinal probit model as

$$r_{ui} = \begin{cases} 
1, & Z_{ui} \leq 1 \\
2, & x - 1 < Z_{ui} \leq x \\
3, & Z_{ui} > 4
\end{cases} \quad (6)$$

4 Posterior Inference

4.1 Posterior Sampling Algorithm

We implement posterior inference using MCMC posterior simulation. Let $\omega = (A, B, \theta, \tau, \rho)$ denote the currently computed parameters. See the appendix for transition probabilities to update $B$, $\theta$, $\rho$ and $\tau$. Only the transition probability for $A$ requires more discussion.

Denote the number of iteration as superscript $(t)$, denote the $u$th row of $A$ and $R$ as $A_u$ and $R_u$, respectively, and denote the $i$th column of $R$ as $R_i$. The following three steps define a reversible jump transition probability for $A_u$.

Below, let $(\tilde{A}, \tilde{\theta})$ denote proposed new values for $A$, $\theta$, and let $\tilde{\omega} = (\tilde{A}, B, \tilde{\theta}, \tau, \rho)$. The transition probability to update $A_u$ is defined as follows.

1. For all the $k$ with $m_{-u,k} \neq 0$, update $p(A_{uk} = x \mid \cdot) \propto \frac{m_{-u,k}}{m} p(R_u \mid A_{uk} = x, A_{-u,k}, \theta, \rho, \tau, B), x = \{0, 1\}$.

2. Reversible jump proposal. We refer to all features with $m_{-u,k} = 0$ as singular features. W.l.o.g. assume $k = 1, \ldots, K_0$ are not singular, and $k = K_0 + 1, \ldots, K$ are the singular features. We create a proposal by first dropping all singular features, i.e., retaining in $\tilde{A}$ only the first $K_0$ columns of $A$, proposing $\tilde{A} = A[\cdot, (1, \ldots, K_0)]$. Next we propose $\tilde{K}_u^+ \sim \text{Pois}(\lambda/n)$ new (singular) features, together with (new) feature-specific parameters $\tilde{\theta}_k \sim h_0, k = K_0 + 1, \ldots, K_0 + \tilde{K}_u^+$. We add the new features to $\tilde{A}$ with $\tilde{A}_{uk} = 1$ and $\tilde{A}_{vk} = 0$ for $k = K_0 + 1, \ldots, K_0 + \tilde{K}_u^+$ and $v \neq u$.

3. Metropolis-Hastings acceptance probability. We denote with $S = \{K_0 + 1, \ldots, K\}$ the indices
of the singular features, and find
\[ \alpha = \frac{p(R_u \mid \tilde{\omega}) \cdot p(\tilde{\theta}_S \mid K_u^+ \mid \frac{1}{n}) \cdot p(\theta_S \mid K_u^+ \mid \frac{1}{n})}{p(\tilde{\omega}) \cdot p(\tilde{\theta}_S \mid K_u^+ \mid \frac{1}{n}) \cdot p(\theta_S \mid K_u^+ \mid \frac{1}{n})} = \frac{p(R_u \mid \tilde{\omega})}{p(R_u \mid \omega)} \]

With probability \( \min(1, \alpha) \), we accept the proposal \( \tilde{A}, \tilde{\theta} \) and set \( A = \tilde{A}, \theta = \tilde{\theta} \). Otherwise we keep \( A, \theta \) unchanged.

We evaluate fitted mean ratings as \( \bar{r}_{ui} = \int p(r_{ui}^{mis} = x \mid \omega) p(\omega \mid R_{obs}) d\omega \approx \frac{1}{T} \sum_{t=1}^{T} p(r_{ui}^{mis} = x \mid \omega^t) \)
and predict unseen ratings for items \( i \) with \( I_{ui} = 0 \) by maximizing
\[ \hat{r}_{ui} = \arg\max_{x \in \{1, 2, 3, 4, 5\}} \bar{r}_{ui} \quad (7) \]

### 4.2 A Consensus Monte Carlo Method for Large Number of Users

The described posterior simulation can be computationally costly. Large recommender systems usually have at least thousands, or millions of users, render the usual MCMC impractical. Thus we use the idea of Consensus Monte Carlo [Scott et al., 2016], to split data into shards, run MCMC one each shard in parallel on different machines, and then reconcile posterior inference from the shards into a reconstruction of posterior inference under the full data. Ni et al. [2019a] applied Consensus Monte Carlo to Bayesian nonparametric models on clustering and feature allocation. Their method relies on shared data points (anchor points) across shards, and they merge random subsets (clusters or features) on different machines based on the number of common anchor points in the two sets. This strategy, though appealing, does not work in our case.

Using a set of users as common anchors to define a criterion for merging features would ignore the possibility of different sets of movies being paired with these users in different shards. And similarly for using sets of movies only. A practicable implementation would need to use a criterion based on shared users, shared movies and similar imputed feature-specific effects \( \theta_k \). A related criterion to merge subsets would require several ad-hoc choices and tuning parameters. In simulations we found that the involved approximations left the joint posterior reconstruction of little practical value. Instead we propose an alternative strategy based on Consensus Monte Carlo for global parameters, but keeping inference for random subsets local to each shard. We refer to this strategy as "semi-local Consensus Monte Carlo".

We use shards that split the data by subsets of users. Let \( s = 1, \ldots, S \) index the shards, and let \( \bigcup_{s=1}^{S} U_i = \{1, \ldots, m\} \) denote the split of users into the \( S \) shards. Also, let \( R_s \) denote the data for
the users in shard $s$. In our model, the random features that are imputed under posterior inference under shard $s$ naturally include only users from $U_s$, making the features local parameters, while the only global parameters are $\rho_i$, $i = 1, \ldots, n$. We use a CMC approximation of the joint posterior for the global parameter $\rho$ as

$$p(\rho | R) \approx \prod_{s=1}^{s} p(R_s | \rho) p(\rho)^{1/S}.$$  

The nature of the approximation is to assume independence of the marginal distribution of global parameters across shards. Conditional on a posterior sample $\tilde{\rho} \sim p(\rho | R)$ we then use shard-specific posterior samples of the shard-specific parameters. The latter is implemented by selecting stored posterior Monte Carlo samples $(A, B, \theta, \rho, \tau)$ with $|\rho - \tilde{\rho}| < \epsilon$. Here $A$ refers to users in shard $s$ only, and $B$ and $\theta$ are linked with the subsets that are represented by $A$, making $(A, B, \theta)$ parameters that are local to each shard only.

There remains the step of creating a Monte Carlo sample for $\rho \sim p(\rho | R)$. For this we use Consensus Monte Carlo for approximate normal posterior distributions. Let $(\mu_s, \sigma^2_s)$ denote posterior mean and standard deviation of $\rho_i$ in shard $s$ (for a movie $i$). We approximate $p(\rho | R) \approx N(\mu, \sigma^2)$ with $1/\sigma^2 = 1/\sigma^2_0 + \sum_s 1/\sigma^2_s$ and $\mu = (\mu_0/\sigma^2_0 + \sum_s \mu_s/\sigma^2_s)/(1/\sigma^2_0 + \sum_s 1/\sigma^2_s)$, where $(\mu_0, \sigma^2_0)$ are the prior moments for $\rho$. We call this strategy semi-local Consensus Monte Carlo.

The full algorithm is now, run MCMC described in Section 4.1 on each separate shards, store $A$, $B$, $\theta$, $\tau$, $\rho$, every 5 iterations. Merging the shard-specific posterior distributions for $\rho$ across shards as described, we get an approximate global posterior from which we then generate a posterior Monte Carlo draw $\tilde{\rho}$. In each shard, for each stored iteration $A$, $B$, $\theta$, $\tau$, resample $\tilde{\rho}$ from the global distributions, filter the iterations where the stored shard-posterior $\rho$ is close to resampled global $\tilde{\rho}$ and make predictions based on $A$, $B$, $\theta$, $\tau$ and $\tilde{\rho}$ in these iterations. Note that the filtering step within the shard is to ensure the closeness of shard posterior MCMC and global posterior MCMC and to not totally lose the dependence of $\tilde{\rho}$ and $(A, B, \theta, \tau)$ in the iterations we use for prediction.

Consensus Monte Carlo Algorithm

1. Separate users into S shards, keep entire list of movies.

2. In each shard, carry out MCMC simulation for $A$, $B$, $\theta$, $\tau$, $\rho$ according to the transition probability in Section 4.1 and appendix. Store after thinning.
3. Merge the shard-specific posterior distributions for $\rho$, and derive an approximate global posterior distribution for $\rho$ by aggregating shard precision.

4. In each shard, for each stored iteration, resampling $\tilde{\rho}$ from the (approximate) global posterior, keep those iterations for which $|\rho - \tilde{\rho}| < \epsilon$, perform prediction using $A$, $B$, $\theta$, $\tau$ and resampled $\tilde{\rho}$ in those iterations.

5 Simulation

We implement the proposed scheme in R. The code is available in author’s github. We use this to set up the simulations. In the simulation study, we generate a $100 \times 150$ rating matrix of $m = 100$ users and $n = 150$ movies from our model. We first simulate $A$ under the IBP prior with hyperparameter $\lambda = 3$. The (random) number of columns of $A$ determines the number of features $K$. Next we simulate a $n \times K$ binary matrix $B$ with $p$, using independent Bernoulli draws with success probability $0.2$. For user $u$ and movie $i$, identify the subset of features $K^*_{ui} = k : A_{uk} = B_{ki} = 1$. With $b_0 = 2.5$, $\theta_{1:K} \sim N(0, 2)$ and $\tau = 0.25$, the latent probit score for rating, $Z_{ui}$ is simulated from

$$Z_{ui} | A, B, \theta, \tau \sim N(b_0 + \sum_{k \in K^*_{ui}} \theta_k, \tau^2).$$

Note here we did not introduce global parameter $\rho_i$ for movies as the simulation data size is small and we are not using Consensus Monte Carlo to split users into shards. Thus a global parameter for movies is optional but not necessary in this example. The rating is generated from the probit score as in Eq 6.

We randomly split the simulated data into 80% and 20% for training and testing. We use the MCMC algorithm proposed in Section 4.1, implementing 10000 iterations conditional on the training data. We evaluate an estimated user-feature relationships $\hat{A}$ following Ni et al. [2019b]. We first calculate the maximum a posteriori (MAP) estimate $\hat{K}$ from the marginal posterior distribution of $K$. Conditional on $\hat{K}$, we follow Dahl [2006] and compute a point estimate $\hat{A}$ as

$$\hat{A} = \arg\min_{A'} \int d(A, A') dp(A | Z, R, \hat{K}),$$
where $d = \min_{\pi} H(A, \pi(A'))$, denotes the minimum Hamming distance between binary matrices $A$ and $A'$ over greedy searches of permutations of $\pi(A')$. In Figure 2, we show the estimated user-feature relationships $\hat{A}$, and conditional on $\hat{A}$, the point estimates for movie-feature relationships $\hat{B}$ and the rating adjustment for each feature $\hat{\theta}$, versus the true values $A, B, \theta$ used in simulation.

We predict rating for training and testing data by maximizing posterior predictive probability in Eq 7. The predicted rating for the training data set is 69.12% correct, meaning that in 69.12% of the cases the predicted rating exactly matches the recorded data. The same for the test data was 66.89%.

We compare to inference under the matrix factorization method as described in Section 2. We use the R package recosystem. Rank $k$ and sparsity parameters in Eq 2 are tuned by cross validation. After training, we find a test RMSE of 0.67 and prediction accuracy for the test data set of 60.80%.

Figure 3 shows the boxplots of users’ prediction accuracy for our methods and Matrix Factorization. The left boxplot shows all users’ accuracy of predicting ratings (level 1-5) of unseen movies. The right boxplot shows all users’ accuracy of predicting the top 10 unseen movies. Our method has a better simulation results in both boxplots.
Figure 3: Boxplots of prediction accuracy for our method (Bayesian Double Feature Allocation, BDFA) to Matrix Factorization (MF). Left shows boxplot of all users’ accuracy of predicting ratings (level 1-5) of unseen movies. Right shows boxplots of all users’ accuracy of predicting the top 10 unseen movies.

6 Preference Prediction with the MovieLens Data

We use the Movielens dataset \(^1\) with movie ratings from 6040 users. We clean the data as in Vitelli et al. [2017], to keep results comparable. We keep the 200 most rated movies and users who rated more than three movies, which yields to a 6040 × 200 rating matrix \(R\). Our goal include traditional goals for recommender systems, such as predicting individual user’s ratings to unseen movies, find the top-rated movies across all users, and a pairwise preference prediction goal similar to Vitelli et al. [2017]. The latter goal predicts the preference of unseen movies to rated movies for each user. Here preference is defined as the following. Denote the predicted rating of user \(u\) for movie \(i\) as \(\hat{R}_{ui}\), estimated as described in Section 3. We say that user \(u\) strictly prefers movie \(i\) to \(i'\), and write \(i \succ i'\) for user \(u\), if \(\hat{R}_{ui} > \hat{R}_{ui'}\).

In the 6040 × 200 rating matrix, only 24.7% entries are observed. We conduct a pairwise preference prediction test using our model. For each user, we randomly select one movie that he/she has rated as test, and train the model based on the remaining data for that user. We compare the user’s preference of this test movie to other (rated) movies based on the predicted rating for the test movie to the observed ratings in the training set. This pairwise preference test is also comparable to Vitelli et al. [2017].

We apply a double IBP prior described in Section 3 initialized with \(\lambda = 3\), base line \(\theta = 2.5\), non-baseline \(\theta\)s drawn from \(N(0, \sigma_0^2)\), \(\sigma_0 = 2\), \(\tau\) drawn from an inverse-gamma distribution with location \(^{1}\)https://grouplens.org/datasets/
and scale parameters \((5, 1)\). We initialize the number of features \(K\) from matrix factorization algorithm result.

Using the concensus Monte Carlo algorithm, we divided the 6040 users into 15 shards and implement MCMC posterior simulation for each shard separately in parallel. Each shard with 400 users and 200 movies stabilizes at around 39 to 43 features. As described in Section 4, we merge inference across shards by defining the approximate global posterior for \(\rho\). For simplicity, we set the prior precision for \(\rho\) to 0, i.e., \(\sigma_0 = \infty\) and simplify the merge step to evaluating \(\mu = \bar{\mu}_s\) as an unweighted average across shards. The after-merge prediction follows from re-sampling \(\rho\) from global posterior and previously stored MCMC draws of other variables.

The resulting average pairwise preference accuracy in each shard is 79.1% with standard deviation 0.012. Vitelli et al. [2017] report an accuracy of 79.6% for the pairwise comparison on the same data. Figure 4 (left panel) shows the before-merge individual shards MCMC pairwise predication accuracy and after-merge pairwise predication accuracy. After-merge prediction accuracy has a higher average and a smaller standard deviation among shards.

For exact predicted rating for unseen movies, on the test set within one star distance, our model has provided 88.5% accuracy. That is saying, we have on average 88.5% probability that our predicted rating is at or within in one star difference of the actual rating. Figure 4 (right panel) shows in each shard the prediction accuracy and its 95% Confidence Interval using binomial distribution variance and a normal approximation CI, \(\hat{p} - z_\alpha \sqrt{\frac{\hat{p}(1-\hat{p})}{n_s}}, \hat{p} + z_\alpha \sqrt{\frac{\hat{p}(1-\hat{p})}{n_s}}\), where \(\hat{p}\) denotes the shard accuracy, \(z_\alpha\) denotes \(1-\frac{\alpha}{2}\) quantile of a standard normal distribution, \(\alpha = 0.05\).
Figure 5: Three randomly selected users’ predicted expectation of probit scores for ratings of all movies (seen and unseen), sorted from high to low. The red line indicates the 95% credible intervals. The blue dots are true ratings in training and green dot is the seen test movie rating.

We also present individual predicted rating for users. Figure 5 shows three randomly selected user’s predicted ratings, 95% credible intervals for all movies (seen and unseen), sorted by predicted ratings, versus true seen ratings of train and test movies. Figure 6 shows in one randomly picked shard, all users’ predicted expectation of probit scores for test movies’ ratings versus observed test movies’ ratings.

For general rating of all movies, we show the top 10 rated movies across all shards, which is an analysis not available by Mallows Model in Vitelli et al. [2017]. Table 1 shows top 10 movies with highest posterior mean of merged $\rho$.

Finally, we discuss the computational complexity reduction from our Consensus Monte Carlo method and strategy to choose shard number $S$. Ghahramani and Griffiths [2006] has established that an Indian Buffet Process with $N$ data points and linear-Gaussian likelihood model has at least $O(N^3)$ computation complexity per iteration. To be specific, Doshi-Velez and Ghahramani [2009] pointed out that in each iteration for $N$ number of $D$-dimensional data points and a linear-Gaussian likelihood model, given an $N \times K$ feature assignment matrix in that iteration, the collapsed Gibbs sampler has complexity $O(N^3(K^2 + KD))$. In our double feature allocation model with $m \times K$
Figure 6: All users’ predicted expectation of probit scores for ratings of test movies versus observed ratings of test movies in one randomly picked shard.

| Movie Name                        | Genre                          |
|-----------------------------------|--------------------------------|
| 1 The Godfather (1972)            | Action/Crime/Drama             |
| 2 Schindler’s List (1993)         | Drama/War                      |
| 3 Star Wars Series                | Action/Adventure/Drama/Sci-Fi/War|
| 4 American Beauty (1999)          | Comedy/Drama                   |
| 5 The Usual Suspects (1995)       | Crime/Thriller                 |
| 6 Casablanca (1942)               | Drama/Romance/War              |
| 7 Raiders of the Lost Ark (1981)  | Action/Adventure               |
| 8 The Shawshank Redemption (1994) | Drama                          |
| 9 Pulp Fiction (1994)             | Crime/Drama                    |
| 10 Rear Window (1954)             | Mystery/Thriller               |

Table 1: Top 10 highest rated movies across shards, selected by highest posterior mean of merged $\rho$
matrix $A$ and $n \times K$ matrix $B$ ($m > n$) and a probit likelihood model will at least have complexity $O(m^3)$ each iteration. If we split the entire data set into $S$ shards, then for each shard we will have complexity of $O((m/S)^3)$ each iteration, (in total $O(S(m/S)^3)$ for all shards). With $m = 6000$ in Movielens data, we see in Figure 7 the complexity per iteration decreases and eventually stabilizes with $S$. We also note that more shards would result in fewer users in a shard and fewer ratings observed in each feature $k$ in the shard and thus a larger standard error for $\theta_k$. Consider the normal distribution of $\theta$, heuristically the standard error of $\theta_k$ would be proportional to $\frac{1}{n_{ks}}$ where $n_{ks}$ is the number of ratings observed in feature $k$ in the shard $s$. Following Korwar et al. [1973], the approximate number of features in shard $s$ is $\log(m_s n)$. Suppose $n_{ks}$ is proportional to number of users and movies in the shard and inversely proportional to number of features in shard $s$, i.e., $n_{ks} \propto m_s n (\log(m_s n))^{-1}$, then standard of error of $\theta_k$ is proportional to $((m_s \times n) / \log(m_s \times n))^{-1/2}$.

Figure 7 shows this trade-off of standard error of estimating $\theta_k$ increasing with number of shards $S$ while the computation cost per iteration decreasing with $S$. Figure 7 can serve as a eyeball guideline to pick suitable $S$.

## 7 Conclusion

We introduced a novel model-based approach to collaborative filtering, with the main features being full posterior inference with interpretable parameters and structure. Based on a generative model, our inference includes a full probabilistic description of any desired summary. For example, the same approach could be used for the performance of learners over problems in a large on-line course. Inference would allow to identify the subsets of similar learners and courses.
Limitations in the current implementation is the very approximate nature of the reconciliation of the shard-specific posterior distribution into a reconstructed joint posterior distribution. In particular, there is no borrowing of information about random subsets across shards. Such features could be added using, for example, common anchors in the Consensus Monte Carlo method. An important limitation is the lack of using any covariate information. For example, movies, or generally any items, have known characteristics like actors, length, origin, year etc. Such information could be used to include a rudimentary regression in the subset selection of the generative model.

Interesting applications arise in many other fields beyond marketing. For example, users could be HIV patients, items could be medications and outcomes could any ordinally reported health outcomes, for example mental health outcomes. Including important baseline covariates and classes of medications could then allow to recommend suitable treatment combinations for future patients.

Appendix

A Transition probabilities to update $B, \theta, \rho$ and $\tau$.

- Update $B|R, A, \theta, \rho, \tau$,
  \[ p(B_{ik} = x) \propto p(B_{ik} = x)p(r_{i} | B_{ik} = x, B_{i,k}, \theta, \rho, \tau, A), x = \{0, 1\} \]

- Sample auxiliary variable $Z|R, A, B, \theta, \rho, \tau$

  \[ p(Z_{ui} | \cdot) \sim \text{TruncatedNormal}(b_0 + \sum_{k \in K^*} \theta_k + \rho_i, \tau^2) \]

  with lower bound $r_{ui} - 1$ ($r_{ui} > 0$), upper bound $r_{ui}$ ($r_{ui} \leq 5$), $K^*$ is the set of $k$ where $A_{uk} = B_{ik} = 1$.

- Update $\tau|Z, A, B, \theta, \rho$ from conjugate Inverse-Gamma families.

- Update $\theta|\rho, Z, \tau, A, B$ and $\rho|\theta, Z, \tau, A, B$ from conjugate Normal families.
References

Sotirios P Chatzis. A coupled indian buffet process model for collaborative filtering. In *Asian Conference on Machine Learning*, pages 65–79. PMLR, 2012.

David B Dahl. Model-based clustering for expression data via a dirichlet process mixture model. *Bayesian inference for gene expression and proteomics*, 4:201–218, 2006.

Finale Doshi-Velez and Zoubin Ghahramani. Accelerated sampling for the indian buffet process. In *Proceedings of the 26th annual international conference on machine learning*, pages 273–280, 2009.

S. Funk. Netflix update: Try this at home. [http://sifter.org/~simon/journal/20061211.html](http://sifter.org/~simon/journal/20061211.html), 2006.

Zoubin Ghahramani and Thomas L Griffiths. Infinite latent feature models and the indian buffet process. In *Advances in neural information processing systems*, pages 475–482, 2006.

Thomas L Griffiths and Zoubin Ghahramani. The indian buffet process: An introduction and review. *Journal of Machine Learning Research*, 12(Apr):1185–1224, 2011.

X. He, L. Liao, H. Zhang, L. Nie, Hu., and T. Chua. Neural collaborative filtering. In *Proceedings of the 26th international conference on world wide web*, pages 173–182. International World Wide Web Conferences Steering Committee, 2017.

Diederik P Kingma and Max Welling. Auto-encoding variational bayes. *arXiv preprint arXiv:1312.6114*, 2013.

Y. Koren and R. Bell. Advances in collaborative filtering. In *Recommender systems handbook*, pages 77–118. Springer, 2015.

Ramesh M Korwar, Myles Hollander, et al. Contributions to the theory of dirichlet processes. *The Annals of Probability*, 1(4):705–711, 1973.

S. Li, A. Karatzoglou, and C. Gentile. Collaborative filtering bandits. In *Proceedings of the 39th International ACM SIGIR conference on Research and Development in Information Retrieval*, pages 539–548. ACM, 2016.
D. Liang, R. G. Krishnan, M. D. Hoffman, and T. Jebara. Variational autoencoders for collaborative filtering. In *Proceedings of the 2018 World Wide Web Conference*, pages 689–698. International World Wide Web Conferences Steering Committee, 2018.

Q. Liu, M. Crispino, I. Scheel, V. Vitelli, and A. Frigessi. Model-based learning from preference data. *Annual review of statistics and its application*, 6, 329-354, 2019a.

Q. Liu, A. H. Reiner, A. Frigessi, and I. Scheel. Diverse personalized recommendations with uncertainty from implicit preference data with the bayesian mallows model. *arXiv:1904.03099*, 2019b.

B. M Marlin. Modeling user rating profiles for collaborative filtering. In *Advances in neural information processing systems*, pages 627–634, 2004.

A. Mnih and R. R. Salakhutdinov. Probabilistic matrix factorization. In *Advances in neural information processing systems*, pages 1257–1264, 2008.

Yang Ni, Yuan Ji, and Peter Mueller. Consensus monte carlo for random subsets using shared anchors. *arXiv preprint arXiv:1906.12309*, 2019a.

Yang Ni, Peter Müller, and Yuan Ji. Bayesian double feature allocation for phenotyping with electronic health records. *Journal of the American Statistical Association*, pages 1–15, 2019b.

Arkadiusz Paterek. Improving regularized singular value decomposition for collaborative filtering. In *Proceedings of KDD cup and workshop*, volume 2007, pages 5–8, 2007.

Badrul Sarwar, George Karypis, Joseph Konstan, and John Riedl. Item-based collaborative filtering recommendation algorithms. In *Proceedings of the 10th international conference on World Wide Web*, pages 285–295, 2001.

J. B. Schafer, D. Frankowski, J. Herlocker, and S. Sen. Collaborative filtering recommender systems. In *The adaptive web*, pages 291–324. Springer, 2007.

Steven L Scott, Alexander W Blocker, Fernando V Bonassi, Hugh A Chipman, Edward I George, and Robert E McCulloch. Bayes and big data: The consensus monte carlo algorithm. *International Journal of Management Science and Engineering Management*, 11(2):78–88, 2016.
S. Sedhain, A. K. Menon, S. Sanner, and L. Xie. Autorec: Autoencoders meet collaborative filtering. In *Proceedings of the 24th International Conference on World Wide Web*, pages 111–112. ACM, 2015.

V. Vitelli, Ø. Sørensen, M. Crispino, A. Frigessi, and E. Arjas. Probabilistic preference learning with the mallows rank model. *Journal of Machine Learning Research*, 18, 158-1, 2017.