On the Dirac quantization condition.

G. I. Poulis\(^1\) and P.J. Mulders\(^{1,2}\)

\(^1\)National Institute for Nuclear Physics and High–Energy Physics (NIKHEF)
P.O. Box 41882, NL-1009 DB Amsterdam, the Netherlands

and

\(^2\)Department of Physics and Astronomy, Free University
De Boelelaan 1081, NL-1081 HV Amsterdam, the Netherlands

Abstract: We revisit the Dirac quantization condition for string–like and string–less (but multi–valued) magnetic monopole potentials. In doing so we allow for an \textit{a priori} different coupling \(\hat{e}\) associated with the longitudinal components of the gauge potential. By imposing physical criteria in the choice of the longitudinal–transverse decomposition we show that—in contrast to some recent claims—the “unphysical” coupling \(\hat{e}\) does not appear in the quantization condition.

As first discussed by Dirac in his seminal paper \(^{[1]}\) the existence of magnetic monopoles with magnetic charge \(g\) implies quantization of the electric charge \(e\) according to

\[
eg \in Z , \quad (1)
\]

The consistency of \(U(1)\) gauge theory in the presence of such magnetic monopoles has been recently questioned by He, Qiu and Tze \(^{[2,3]}\) who have proposed a generalized formulation of QED where they allow for two different coupling constants \(e\) and \(\hat{e}\) associated, respectively, with the transverse (physical) and longitudinal (unphysical) components of the gauge field. By considering both string–type and string–less (see below) monopole potentials they have argued that the conventional quantization condition, Eq. (1), is replaced by one where the unphysical coupling, \(\hat{e}\), enters, \textit{i.e.},

\[
\hat{e} \in Z . \quad (2)
\]

Thus, they conclude, since in the above equation a physical coupling, \(g\), is constrained by an unphysical one, \(\hat{e}\), the only viable scenario is that the monopole charge \(g\) has to be zero, unless one enforces \(e = \hat{e}\) which they view as unacceptable since no physical process would ever “see” \(\hat{e}\). However, it is very hard to reconcile such a statement with the prolific work on monopoles in lattice–regularized field theory: monopole potentials have been shown not only to be present but also to drive confinement in the strong coupling regime, both analytically \(^{[4]}\) and in Monte Carlo simulations \(^{[5]}\) and their non–observation is understood as a \textit{dynamical} effect, namely, exponential vanishing of the monopole density as one approaches the continuum limit, whereas the arguments quoted by the authors are essentially \textit{symmetry} arguments that would apply to both strong and weak couplings. Moreover, it is not clear whether one can call \(\hat{e}\) “unphysical” for processes involving \textit{virtual} photons (electron–hadron scattering, \(e^+ - e^-\) annihilation \textit{etc.}) where the longitudinal components of the gauge field (and therefore \(\hat{e}\) itself) do contribute to the \(S\)-matrix. Notwithstanding this observation we shall leave it aside and restrict ourselves to the discussion of how to \textit{properly} derive the quantization condition in this generalized version of \(U(1)\) gauge theory. In a first attempt to examine the arguments of the above authors we have pointed out \(^{[6]}\) that arriving to Eq. (2) instead of the conventional quantization condition, Eq. (1), depends crucially on the way the decomposition of the gauge field in longitudinal and transverse pieces is carried out. In particular, we noticed that the Dirac string–type solutions are always divergenceless. Thus, one could take them to be purely transverse and therefore there would be no term coupled to the unphysical coupling \(\hat{e}\): this coupling (whether unphysical or not) need not enter the quantization condition. However, the property of being divergenceless is not true for string–less, multi–valued potentials which provide a monopole field as well (see below). In this note we wish to examine all these types of monopole potentials in a unified framework. As a warm–up we will rederive the quantization condition for the following three (static) monopole magnetic potentials:

\[
\begin{align*}
A_\pm &= -\frac{g}{r} \left( \frac{\cos \theta \pm 1}{\sin \theta} \right) \hat{\phi} \\
A_{NS} &= -\frac{g}{r} \sin \theta \hat{\phi}. 
\end{align*}
\]

The first two correspond to potentials with Dirac strings in the \(\pm \hat{z}\) axis. Although initially formulated in terms of potentials with string–type singularities, magnetic monopole fields can be generated by non–singular, multi–valued magnetic potentials. The most well known type of those is the Wu and Yang construction \(^{[8]}\), where one defines a \textit{locally} non–singular potential which is gauge equivalent to a string–type one \(^{[9]}\). A less familiar monopole potential is the above \(A_{NS}\) potential \(^{[10]}\) which is explicitly \textit{multi–valued} due to the presence of \(\phi\), but \textit{non–singular} (except at the origin of course) and is again related via a gauge transformation \(^{[1]}\) to the string–type potentials. We derive a quantization condition (amongst other ways, see for example \(^{[11]}\)) by requiring that the phase factor for a charged particle’s wavefunction, is unobservable, that is

\[
\exp\{e \oint_\Gamma A \cdot dl\} = \exp(2in\pi) = 1, n \in Z , \quad (4)
\]
where $\Gamma$ is any closed loop shrunk to zero (that is, surrounding a minimal surface with zero area).

Finally, for the $A_{NS}$ potential we choose the following closed loop $\Gamma$ (Fig. 2): start at $(r, \theta, \phi) = (R, 0, 0)$ then go to $(R, \pi, 0)$ along the $\hat{\theta}$ direction ($\Gamma_1$), then to $(R, \pi, 2\pi)$ along $\hat{\phi}$ ($\Gamma_2$ above), then to $(R, 0, 2\pi)$ along $-\hat{\theta}$ ($\Gamma_3$), and finally clockwise (along $-\hat{\phi}$) back to $(R, 0, 0)$ ($\Gamma_4$). The only nonvanishing contribution comes from $\Gamma_3$ and equals $2\pi e g \int_0^\pi \sin \theta d\theta = 4\pi eg$ and we arrive again at the Dirac quantization condition, Eq. (1). We note that the loop $\Gamma$ constructed in this way, has several nice features. For single-valued potentials it reduces to $\Gamma_2 + \Gamma_4$, which depending on where one has a singularity reduces to just $\Gamma_2$ or $\Gamma_4$. For multi-valued potentials it picks up contributions through $\Gamma_1 + \Gamma_3$.

Let us now introduce the “generalized” covariant derivative used in Ref. [2]

\[ D_\mu = \partial_\mu - ieA_\mu - i\tilde{e}\tilde{A}_\mu . \] (5)

Here the gauge field $A_\mu$ is decomposed into transverse, $A_{\mu T} = T_{\mu\nu}A^\nu$, and longitudinal, $\tilde{A}_\mu = L_{\mu\nu}A^\nu$, components, coupled to charges $e$ and $\tilde{e}$, respectively; we employ the projectors $L_{\mu\nu} = \partial_\mu \partial_\nu - \partial^2 / \partial^2$ and $T_{\mu\nu} = g_{\mu\nu} - L_{\mu\nu}$. The longitudinal components do not enter the field strength tensor $F_{\mu\nu}$ and are unphysical. This theory is invariant under local $U(1)$ transformations

\[ cA_\mu(x) + \tilde{e}\tilde{A}_\mu(x) \rightarrow eA_\mu(x) + \tilde{e}\tilde{A}_\mu(x) - \partial_\mu\Omega(x) . \] (6)

By applying the projectors $L_{\mu\nu}$, $T_{\mu\nu}$ on both sides of (6) one obtains

\[ A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e}\partial_\mu\Omega(x) + \frac{1}{\tilde{e}}\partial_\mu \frac{1}{\partial^2}\partial^2\Omega(x) \]
\[ \tilde{A}_\mu(x) \rightarrow \tilde{A}_\mu(x) - \frac{1}{\tilde{e}}\partial_\mu \frac{1}{\partial^2}\partial^2\Omega(x) . \] (7)

Thus, in the static case, as noticed in [2], the transverse components are left invariant and only the longitudinal ones change:

\[ A(x) \rightarrow A(x) \]
\[ \tilde{A}(x) \rightarrow A(x) - \frac{1}{e}\nabla\Omega(x) , \] (8)

However, as pointed out in [2] this statement is ambiguous when the potential $A$ is divergenceless, for then $\partial^{-2}A_\mu$ cannot be defined. In fact, this is the case for all string–type potentials, since [2][2]

\[ \nabla \cdot A = 0, \text{ with } A(x) = - \int_\Gamma du \times B(r-u) . \] (9)

Here $B(x) = -g\nabla(1/4\pi r)$ and the Dirac string lies along a generic single–valued semi–infinite path $\Gamma$. But if the potential is divergenceless we can as well take the longitudinal part to be zero. Moreover, if we consider a (static) transformation $\Omega_{tr}$ that moves the Dirac string to lie along a different path $\Gamma'$ the above property guarantees...
that \( \partial^2 \Omega_{\Gamma, r'} = 0 \) and that under this gauge transformation the longitudinal component remains the same (trivially, since it is zero) while the transverse changes

\[
A(r) \rightarrow A(r) - \frac{1}{e} \hat{\nabla} \Omega(r) \quad A(r) \rightarrow \hat{A}(r) = 0 .
\]

This equation should be contrasted with Eq. (8). In order to be more specific let us discuss the longitudinal–transverse decomposition for the potentials in Eq. (3). He, Qiu and Tze offer the following decomposition into transverse and longitudinal components:

\[
A^{(I)}_\Omega = \frac{g \cos \theta}{r} \hat{\phi}, \quad \hat{A}^{(I)}_\Omega = \frac{g}{r \sin \theta} \hat{\phi} .
\]

Thus, the transverse piece is the same for the two strings and the gauge transformation that maps one string solution to the other is of the type (8) with \( \Omega(r) = 2e \phi \). However, as we said above, \( \hat{\nabla}^2 \Omega = 0 \) and thus we could as well have the decomposition

\[
A^{(II)}_\Omega = -\frac{g \cos \theta}{r} \hat{\phi}, \quad \hat{A}^{(II)}_\Omega = 0 .
\]

Notice that the gauge transformation that connects the two strings is now of the type (10) with \( \Omega(r) = 2e \phi \) which is the same as above but with the physical coupling, \( e \), appearing instead of the unphysical one, \( \hat{e} \). What about the string–less potential \( A_{NS} \)? In this case we cannot take it to be purely transverse, since

\[
\nabla \cdot \frac{\sin \theta \hat{\phi}}{r} = 2 \cos \theta \hat{\phi} \neq 0 .
\]

It is straightforward to check that all of the following are legitimate longitudinal–transverse decompositions for \( A_{NS} \):

\[
A^{(I)}_{NS} = \frac{g \cos \theta}{r \sin \theta} \hat{\phi}, \quad \hat{A}^{(I)}_{NS} = \frac{g}{r \sin \theta} \hat{\phi} - \frac{g}{r} \sin \theta \hat{\phi} \hat{\theta} ,
\]

\[
A^{(II)}_{NS} = -\frac{g \cos \theta}{r \sin \theta} \hat{\phi}, \quad \hat{A}^{(II)}_{NS} = \frac{g}{r \sin \theta} \hat{\phi} + \frac{g}{r} \sin \theta \hat{\phi} \hat{\theta} .
\]

As before, decomposition (I) is the one used by He, Qiu and Tze. In the decompositions (II+, II-) we have used our experience from the string–type potentials above to move \( \hat{\nabla} \hat{\phi} = (1/r \sin \theta) \hat{\phi} \) terms so as to compensate for the singularity. For example, a singularity of the type \( (\cos \theta / r \sin \theta) \hat{\phi} \) which extends over the whole \( z \) axis can, by adding \( \pm \hat{\nabla} \hat{\phi} \), be reduced to the half \( z \) axis. That means that we introduce a decomposition that is different in various space regions (à la Wu and Yang), but the potential is multivalued anyway so that’s not a problem.

In Table 1 we show the results for the \( \Gamma_4 \) loop. We have made the choice \( A_{NS} = A_\perp \) for loop \( \Gamma_4 \), so as to keep the decomposition non–singular, in accordance with criterion (C).

| decomposition (I) | decomposition (II) |
|-------------------|-------------------|
| \( e \oint_{\Gamma_4} A \cdot dl + \hat{e} \oint_{\Gamma_4} \hat{A} \cdot dl \) | \( e \oint_{\Gamma_4} A \cdot dl + \hat{e} \oint_{\Gamma_4} \hat{A} \cdot dl \) |
| \( A_+ \) | +2\( \pi \)e +2\( \pi \)\hat{e} +4\( \pi \)e 0 |
| \( A_- \) | +2\( \pi \)e -2\( \pi \)\hat{e} 0 0 |
| \( A_{NS} \) | +2\( \pi \)e -2\( \pi \)\hat{e} 0 0 |

\[\text{Table 1: the r=fixed, } \theta \rightarrow 0 \text{ loop } \Gamma_4\]

\(^1\)they have both zero curl and zero divergence
Criterion (A) then implies the following constraints for the loop \( \Gamma_4 \) and decomposition (I) for the three potentials \( n \in \mathbb{Z} \) throughout:

\[
\mathbf{A}_+ : g(e + \tilde{e}) = n \Rightarrow \begin{cases} n = g = 0, & \text{if } \tilde{e} \text{ arbitrary} \\ eg = n/2, & \text{if } \tilde{e} = e \end{cases} \tag{16}
\]

\[
\mathbf{A}_- : g(e - \tilde{e}) = n \Rightarrow \begin{cases} n = g = 0, & \text{if } \tilde{e} \text{ arbitrary} \\ \text{no constraint}, & \text{if } \tilde{e} = e \end{cases}
\]

\[
\mathbf{A}_{NS} : g(e - \tilde{e}) = n \Rightarrow \begin{cases} n = g = 0, & \text{if } \tilde{e} \text{ arbitrary} \\ \text{no constraint}, & \text{if } \tilde{e} = e \end{cases}
\]

while for the same loop \( \Gamma_4 \) decomposition (II) implies

\[
\mathbf{A}_+ : ge = n/2
\]

\[
\mathbf{A}_- : \text{no constraint}
\]

\[
\mathbf{A}_{NS} : \text{no constraint .}
\]

A number of observations can be made here:

1. Decomposition (II) meets all criteria (A), (B) and (C). Moreover, the only constraint it leads to is the conventional quantization condition \( ge = n/2 \).

2. Decomposition (I) implies individual quantization conditions for \( e - \tilde{e} \) and \( e + \tilde{e} \). If we treat \( \tilde{e} \) as independent of \( e \) and subtract these constraints we get

\[
\text{Eq. (16) used in Eq. (13) to prove that QED is inconsistent with monopoles. However, we view this result as unphysical since for } \tilde{e} \text{ arbitrary decomposition (I) violates both (B) and (C) criteria. In particular it violates (B) by leading to a constraint (the second of Eqs. (16) above) coming from a potential, } \mathbf{A}_-, \text{ which is smooth and single–valued in the area of the loop } \Gamma_4. \text{ It also violates (C) by introducing singularities in the vicinity of loop } \Gamma_4 \text{ for the string–less potential } \mathbf{A}_{NS}.

Analogous remarks can be made for the \( \mathbf{A}_+ \) potential in the case of the \( \Gamma_2 \) loop (see Table 2). In this case we have chosen \( \mathbf{A}_{NS} = \mathbf{A}_+ \), in accordance with criterion (C).

| Table 3: the loop } \Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 |
|--------------------------------------------------|--------------------------------------------------|
| decomposition (I) | decomposition (II) |
| \( e \int_{\Gamma_1} \mathbf{A} \cdot \mathbf{dl} \) | \( e \int_{\Gamma_1} \tilde{\mathbf{A}} \cdot \mathbf{dl} \) |
| \( \mathbf{A}_+ \) | \(+4\pi e\gamma\) | \(0\) |
| \( \mathbf{A}_- \) | \(+4\pi e\gamma\) | \(0\) |
| \( \mathbf{A}_{NS} \) | \(+4\pi e\gamma\) | \(0\) |

We summarize: the fact that criterion (A) is applicable to any closed loop allow us to recover the conventional quantization condition \( ge = n/2 \) not only for string–type potentials where their divergenceless makes this result quite obvious but also when considering the string–less potential \( \mathbf{A}_{NS} \) which is not purely transverse. This result stems from the decomposition–independent \( \Gamma \) contour integrations in Table 3. Moreover, by imposing some physical criteria (B) and (C) for choosing a physical longitudinal–transverse decomposition we have shown that the freedom to move \( \nabla \phi \) parts between longitudinal and transverse pieces amounts to the following:

- The only quantization condition that can be obtained is the conventional one, Eq. (1).

Thus, even by treating the longitudinal coupling \( \tilde{e} \) as arbitrary (despite the questions this raises for virtual photons) we have shown that quantum electrodynamics is consistent with magnetic monopoles.

**Acknowledgements:** This work is supported in part by the Foundation for Fundamental Research on Matter (FOM) and the National organization for Scientific Research (NWO) as well as the Human Capital and Mobility Fellowship ERBCHBCT941430.
[1] P.A.M. Dirac, Proc. Poy. Soc. (London) Ser. A, 133, 60 (1931).
[2] H.-J. He, Z. Qiu and C.-H. Tze, Z. Phys. C 65, 175 (1994).
[3] H.-J. He, Z. Qiu and C.-He, Virginia Polytechnic Institute preprint VPI-HEP-93-14 (1993).
[4] H.-J. He and C.-H. Tze, Addendum to “Inconsistency of QED in the presence of Dirac Monopoles”, Z. Physik C (in press).
[5] G.I. Poulis and P.J. Mulders, NIKHEF preprint NIKHEF-94-P11, Z. Physik C (in press).
[6] T. Banks, R. Myerson and J. Kogut, Nucl. Phys. B 129 493 (1977); A.M. Polyakov in “Gauge Fields and Strings”, Harwood Academic Publishers, Chur, Switzerland (1987).
[7] T.A. DeGrand and D. Toussaint, Phys. Rev. D 22, 2478 (1980), H. D. Trottier and R. M. Woloshyn, Phys. Rev. D 48, 4450 (1993).
[8] T.T. Wu and C.N. Yang, Phys. Rev. Lett. 13, 380 (1964).
[9] Sidney Coleman, Appendix 5 in “Classical lumps and their quantum descendants”, page 260, Aspects of Symmetry, Cambridge University Press (1988).
[10] H.A. Cohen, Progr. Theor. Phys. Vol 50, No. 2, (1973).
[11] T.-P. Cheng and L.-F. Li in “Gauge Theory of Elementary Particle Physics”, Oxford University Press (1984).