High fidelity state reconstruction of a qubit via dynamics of a dissipative resonator

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Using a dissipative quantum Rabi model, we study the dynamics of a slow qubit coupled to a fast quantum resonator embedded in a viscous fluid from weak to strong and ultra-strong coupling regimes. Solving the quantum Heisenberg equations of motion, perturbative in the internal coupling between qubit and resonator, we derive functional relationships directly linking the qubit coordinates on the Bloch sphere to resonator observables. We then perform accurate Matrix Product State simulations using the time dependent variational principle, and compare our results both with the analytical solutions obtained via the Heisenberg equations of motion, and with numerical solutions of a Lindblad master equation, perturbative in the external coupling between resonator and environment. The aim is to extend the standard qubit readout, limited to a closed system for a qubit in a state along one quantization axis, to the case of a spin directed along an arbitrary direction as a part of an open system. We demonstrate that a qubit state pointing to generic coordinates on the Bloch sphere can be fully reconstructed, up to the strong coupling regime with a damped resonator. Indeed, by monitoring the qubit fidelity with respect to free limit, we point out that, up to the strong coupling regime, the qubit state accurately fulfills the functional relationships of the Heisenberg equations of motion. Interestingly, a weak to intermediate resonator coupling to the bath is able to anticipate the state reconstruction of the qubit with respect to the same procedure for the closed system. Moreover, we find a parameter regime in which, even if the standard qubit readout fails, the resonator is still able to monitor the qubit dynamics. Finally, in the ultra-strong coupling regime, non-Markovian effects become robust and the dynamics of qubit and resonator are inextricably entangled making the qubit state reconstruction difficult. Our work is not only interesting from a fundamental point of view, but also for quantum control issues, aimed to provide information about system dynamics through measurements.

I. INTRODUCTION

The interaction between matter and electromagnetic fields is one of the most fundamental processes occurring in nature. This interaction, specified to quantized electromagnetic fields, is the central focus of investigation in cavity quantum electrodynamics (QED) and in circuit QED, with artificial atoms and on-chip resonators. This research field is really important for both atomic physics and quantum optics. The quantum Rabi model describes the most used setup, in which a two-level system (qubit) is coupled to a single-mode resonator (cavity) field. This model gives a mathematical framework for various quantum phenomena, inter alia, nonclassical-state generation, quantum state transfer and topological physics using photons. In particular, it is more general than the Jaynes-Cummings model, since the Rabi model explicitly includes the counter-rotating terms in the qubit-resonator interaction. Hence, it is crucial for describing strong and ultra-strong coupling regimes in which the qubit-cavity coupling becomes about an order of magnitude smaller than the characteristic frequency of both qubit and cavity. These models help to interpret the experiments that have reached the strong coupling regime for various QED setups and the ultra-strong one in superconducting QED. In the latter case the coupling between artificial atoms and resonators is obtained, for instance, by inductively coupling a flux qubit and an LC oscillator via Josephson junctions. The ultra-strong coupling regimes of light-matter interaction will keep on expanding at the frontier of quantum optics and quantum physics. Thus, topics related to these regimes will remain a prominent field in the foreseeable future. The other issue to address for a realistic description of such systems is assessing quantitatively the role of dissipation and decoherence. Coupled qubit-resonator systems are currently employed in circuit QED setups since the qubit readout through the Stark dynamical shift of the resonator frequency provides information on the spin component along the quantization axis. This procedure is strictly valid only in the weak coupling limit between qubit and resonator and when interactions with the environment are neglected. Actually, in the weak coupling regime, the closed Rabi model reduces to the closed Jaynes-Cummings model, for which it is possible to show that the splitting of the energy eigenvalues depends on qubit state, allowing its readout through the resonator state. It is still unclear if efficient qubit readout is possible also when the inevitable coupling to the environment is considered in the qubit-resonator model. Recently, the readout procedure has been generalized in some experiments showing that it is possible to follow simultaneously non-commuting spin components.
of a qubit through resonator observables. In particular, in Ref. 14, the procedure for a full state reconstruction of a superconducting qubit has been improved. These recent experiments have inspired the theoretical work presented in this paper. We will discuss in the concluding section IV how results exposed in this paper can be compared to those obtained in the experiments\textsuperscript{12–14}.

In this paper, we make an accurate theoretical analysis of static and dynamics properties of the open quantum Rabi system, from the weak up to the ultra-strong coupling regimes. The cavity frequency is greater than the qubit one, so that the system is in the anti-adiabatic regime. We use both numerically exact and approximate methods for solving the dynamics with the aim of establishing the validity limits of different methods\textsuperscript{15,16}. Firstly, we solve the Heisenberg equations of motions (HEM) in the weak coupling limit between the qubit and the cavity, in the presence of Ohmic bath. These allow us to derive analytical functional relationships directly linking cavity and qubit coordinates. Hence, to test the validity of the derived relationships, we perform extensive Matrix Product State (MPS) simulations. We further corroborate our MPS numerical results by solving Lindblad master equations (LME) for the coupled qubit-cavity system. Only MPS simulations provide correct results in every parameter region, being variationally exact in both the qubit-cavity coupling and coupling to the environment. Indeed, LME is obtained in the weak coupling limit between the cavity and the environment and so it is valid under the Markovian approximation\textsuperscript{11}. Therefore, in the following, LME results help us assess the non-Markovianity of the solutions obtained with MPS numerics. Up to strong coupling regime, we retrieve the typical qubit readout by making a Fourier transform of the average excitation number of the oscillator for a qubit in the up state. Then, we extend the readout to a generic superposition of up and down qubit states including the effects of the bath. Furthermore, we analyse the resonator dynamics in order to reconstruct the qubit state during its evolution. In fact, in addition to the number of bosons, we compute the mean values of the oscillator coordinates. We find that a full qubit state reconstruction can be achieved from weak to strong coupling regime if the resonator is not strongly damped. We quantify the quality of the quantum state reconstruction by monitoring the qubit fidelity with respect to free limit. We show that, up to the strong coupling regime, we can infer accurately the qubit state from the functional relationships of the Heisenberg equations of motion. Interestingly, in the strong coupling regime, when the qubit-cavity coupling is not too strong, the standard readout fails to provide the z spin component, but a full state reconstruction is still possible by following the resonator coordinates and excitation number over time. Quite surprisingly, results reveal that dissipation and decoherence can facilitate the prediction of qubit dynamics in the parameter regime from weak to strong coupling.

Finally, in the ultra-strong coupling regime, the qubit dynamics cannot be separated from the oscillator evolution due to an enhancement of their entanglement. As a result, the qubit state reconstruction from the resonator dynamics becomes prohibitive. The Wigner quasi-probability distribution for the resonator shows that it is in a damped momentum squeezed state, whose squeezing parameter depends on both the internal and the external couplings. Moreover, from the comparison with MPS simulations, we observe how Lindblad equation fails to provide correct results due to the non-Markovian effects of the interaction.

The results presented in this work are not limited to the anti-adiabatic regime. Actually, we will show how a full quantum state reconstruction can be extended to a parameter regime where the qubit energy scale is of the order of that of the oscillator in the weak to strong coupling regime.

The present work can be useful for better understanding the dissipative quantum Rabi model from a fundamental point of view. Furthermore, it can help to address quantum control issues, which are aimed to gain information about system dynamics through measurements enabling more powerful performance in computing, sensing, and metrology.

The paper is organised as follows: in Sec. II we will describe the dissipative quantum Rabi model and the methods used to solve the dynamics of this open quantum system; in Sec. III, we will discuss our results for three regimes of parameters: weak, strong, and ultra-strong coupling, which are summarized in the Fig. 12 in the concluding section IV.

II. DISSIPATIVE QUANTUM RABI MODEL AND SOLUTION METHODS

In this section, we describe the quantum Rabi model, where a quantized single-mode field is coupled to a two-level system (TLS) through a quantum dipolar light-matter interaction:

\[ H_{\text{closed}} = \frac{\Delta}{2} \sigma_z + \omega_0 a^\dagger a + g \sigma_x (a + a^\dagger), \]

where \( \Delta \) is the tunneling matrix element, \( a (a^\dagger) \) is the annihilation (creation) operator for the oscillator with frequency \( \omega_0 \), \( g \) stands for the strength of the qubit-cavity coupling and \( \sigma_z \) and \( \sigma_x \) are the canonical Pauli matrices. The effects of decoherence and dissipation induced by the electromagnetic environment are taken into account via a linear coupling à la Caldeira-Leggett to a collection of
$N$ independent bosonic modes at zero temperature:

$$\mathcal{H}_{\text{bath}} = \sum_{j=1}^{N} \left[ \omega_j a_j^\dagger a_j - (a + a^\dagger) |c_j\rangle \langle a_j + a_j^\dagger| + \frac{x_0^2}{2} M_j \omega_j^2 \right].$$  \hspace{1cm} (2)

The bosonic modes have frequencies $\omega_j^2 = k_j/M_j$, coordinates and momenta given by $x_j$ and $p_j$, respectively; furthermore $x_0$ denotes the position operator of the oscillator with mass $m$ and frequency $\omega_0$: $x_0 = \sqrt{\hbar/2m\omega_0}(a + a^\dagger)$. Units are such that $\hbar = k_B = 1$. The coupling constants to the bath are $|c_j| = \sqrt{\frac{k_j}{2m\omega_0}}$. Moreover, we have neglected the energy shift $\sum_{j=1}^{N} \omega_j / 2$, which does not affect the dynamics. The last term ensures that the total energy is bounded from below, irrespective of the strength of the coupling to the bath. Hence, we can rewrite the Hamiltonian of the system plus the environment, by defining a renormalized oscillator frequency $\tilde{\omega}_0 = \sqrt{\omega_0^2 + \sum_{j=1}^{N} M_j \omega_j^2 / m}$ and renormalized coupling strengths $\tilde{g} = g \sqrt{\frac{\omega_0}{\tilde{\omega}_0}}$, between qubit and resonator, and $|\tilde{g}_j| = \frac{k_j}{4m\omega_0}$, between oscillators and each bath bosonic mode. The total Hamiltonian then reads:

$$\mathcal{H} = \frac{\Delta}{2} \sigma_z + \tilde{\omega}_0 b^\dagger b + \tilde{g} \sigma_x (b + b^\dagger)$$

$$+ \sum_{j=1}^{N} \omega_j a_j^\dagger a_j - (b + b^\dagger) \sum_{j=1}^{N} \left[ |\tilde{g}_j| (a_j + a_j^\dagger) \right],$$  \hspace{1cm} (3)

where $b (b^\dagger)$ is the annihilation (creation) operator for the renormalized oscillator with frequency $\tilde{\omega}_0$ and coordinates $x = \sqrt{\frac{1}{2m\omega_0}}(b + b^\dagger)$. The bath is then represented by an Ohmic spectral density: $J(\omega) = \sum_{j=1}^{N} |\tilde{g}_j|^2 \delta(\omega - \omega_j) = \frac{g^2}{4} \omega \Theta(\omega, -\omega)$, $\omega_c$ is the cutoff frequency, and $\theta(x)$ Heaviside function. Here the dimensionless parameter $\alpha$ measures the strength of the resonator-bath coupling (see Fig. 1).

This model can also be mapped to a cavity of frequency $\Delta$ coupled through $g$ to a cavity of frequency $\omega_0$, in turn coupled to a bath via a coupling strength $\alpha$. In particular, we are interested in the cases that allow to restore and extend the typical qubit readout. Moreover, thanks to the MPS simulations, we will be able to verify the functional relationships provided by HEM, useful to unveil the regimes in which the three methods provide a correct description.

In the following subsections we are going to show the approaches useful to describe the time-evolution of the systems of interest (qubit and oscillator) when coupled with the environment through the cavity. First, we will find the solution of the HEM for position and momentum of the cavity, perturbative in the coupling strength $g$. Then, in the second subsection, we present the dynamic simulations through MPS method. In Appendix, we expose the solution of global LME for the composite system qubit plus oscillator, that we will use to compare with HEM and MPS results. We underline the regimes in which the three methods provide a correct description.

### A. Perturbative solution of HEM

This approach is possible since our bath is made up of bosonic modes linearly coupled to the system made of qubit plus resonator. Hence, we can exactly

![FIG. 1. Dissipative quantum Rabi model described by the Hamiltonian in Eq. (1) for a qubit of frequency $\Delta$ coupled through $g$ to a cavity of frequency $\omega_0$, in turn coupled to a bath via a coupling strength $\alpha$.](image-url)
eliminate the dynamics of the bath from HEM for the variables pertaining to the reduced system (qubit and oscillator). Starting from the Hamiltonian $H$ in Eq. (3) and evaluating the HEM $\hat{A}(t) = i [H, A(t)]$ for a generic observable $A$ belonging to the qubit-resonator-bath system, we find:

$$\begin{cases}
\dot{x}(t) = \frac{p(t)}{m} \\
\dot{p}(t) = -M_0 \omega_0^2 x(t) + \lambda_j x(t)
\end{cases}
$$

(6)

The first system of equations refers to the bath positions and momenta (Eq. (6)), while the second one is for the oscillator observables (Eq. (7)). The coefficients that appear in both the systems are $\lambda_j = -\hat{g}_j \sqrt{2m\omega_0^2 \sqrt{2M_0}} \omega_j$. These differential equations are coupled, such that we can express for example $x(t)$ from the first system as a function of the $x_j(t)$ and substitute it in the second one. Therefore, we obtain an equation for $x(t)$ (and similarly for $p(t)$) as follows:

$$\dot{x}(t) + \omega_0^2 x(t) + \hat{g} \sigma_x(t) \sqrt{2\omega_0} + \frac{d}{dt} \int_0^t ds \gamma(t-s)x(s) = \frac{B(t)}{m}.
$$

(8)

It may be viewed as the quantum analogue of a classical stochastic differential equation\(^1\), involving a damping kernel

$$\gamma(t-s) = 4\omega_0 \int_0^\infty dw \frac{J(\omega)}{\omega} \cos[\omega (t-s)]
$$

(9)

and a force operator

$$B(t) = \sum_{j=1}^N \frac{\lambda_j}{\sqrt{2M_0 \omega_j}} \left(a_j e^{-i\omega_j t} + a_j^\dagger e^{i\omega_j t}\right),
$$

(10)

that on average becomes a stochastic force, whose statistics depends on the initial bath states distribution. In our case, when the Ohmic spectral density has an infinite cutoff, $\omega_c \to \infty$, we get $\gamma(t-s) \to 4\gamma \delta(t-s)$, where $\gamma = \frac{\pi}{2} \omega_0$ is the cavity decaying rate. In all the approaches used in this work, we consider the limit where $\omega_c$ is the largest frequency scale. This approximation allows us to simplify the derivative of the integral with the damping kernel to $2\gamma \dot{x}(t)$. Thus, we perform the average over the initial distribution $\rho(0)$ and find the HEM for the oscillator coordinates:

$$\langle \dot{x}(t) \rangle + \omega_0^2 \langle x(t) \rangle + 2\gamma \langle \dot{x}(t) \rangle + \hat{g} \langle \sigma_x(t) \rangle \sqrt{2\omega_0} = 0
$$

(11)

$$\langle \dot{p}(t) \rangle + \omega_0^2 \langle p(t) \rangle + 2\gamma \langle \dot{p}(t) \rangle - \hat{g} \langle \sigma_y(t) \rangle \sqrt{2\omega_0 m} = 0,
$$

(12)

where the third terms are friction forces and the fourth ones are “external” forces due to the interaction with the qubit. Clearly the solution of these coupled differential equations ($\langle x(t) \rangle = \langle \frac{p(t)}{m} \rangle$) depends on the time evolution of the mean values of the qubit Pauli matrices. A way to find an analytical solution in the anti-adiabatic regime ($\Delta \ll \omega_0$) is to consider the ratio between the qubit-oscillator coupling and the resonator frequency, $\hat{g}/\omega_0$ as a weak perturbation. By neglecting terms greater than or equal to the first order in $\hat{g}/\omega_0$ in the resonator equations, the HEM for the Pauli matrices can be solved at zero order in $\hat{g}/\omega_0$, describing the well-known Rabi oscillations for a free evolving qubit. Apart from a decaying homogeneous solution on a transient time of the order of $\gamma^{-1}$, the solutions for the mean dimensionless oscillator position $\dot{x} = x\sqrt{2m\omega_0}$ and momentum $\dot{p} = p\sqrt{2/(m\omega_0)}$ are the following:

$$\langle \dot{x}(t) \rangle \to -2\omega_0 k \langle \sigma_x(t) \rangle
$$

(13)

$$\langle \dot{p}(t) \rangle \to 2\Delta k \langle \sigma_y(t) \rangle,
$$

(14)

where the coefficient $k$ is

$$k = \frac{\hat{g}(\omega_0^2 - \Delta^2)}{\Delta^4 + \omega_0^4 + 4\gamma^2 \Delta^2 - 2\Delta^2 \omega_0^2}.
$$

(15)

At the zero-order in the coupling $\hat{g}$, the mean Pauli matrices are those of a free spin

$$\langle \sigma_x(t) \rangle = \langle \sigma_x(0) \rangle \cos(\Delta t) - \langle \sigma_y(0) \rangle \sin(\Delta t)
$$

(16)

$$\langle \sigma_y(t) \rangle = \langle \sigma_y(0) \rangle \cos(\Delta t) + \langle \sigma_x(0) \rangle \sin(\Delta t)
$$

(17)

$$\langle \sigma_z(t) \rangle = \langle \sigma_z(0) \rangle.
$$

(18)

In Sec. III, we will show that, in the weak coupling regime, this treatment of the mean values is sufficient to get accurate results.

In the next section, thanks to the MPS simulations, we will show that Eqs. (13-14) are valid up to the strong coupling regime. Moreover, we will show that the main effects of the non-zero $\hat{g}$ are to introduce a renormalized frequency ($\Delta_x$ and $\Delta_y$ for $\langle \sigma_x(t) \rangle$ and $\langle \sigma_y(t) \rangle$ respectively) and a decay rate ($\kappa_x$ and $\kappa_y$ for $\langle \sigma_x(t) \rangle$ and $\langle \sigma_y(t) \rangle$ respectively) in Eqs. (16-17):

$$\langle \sigma_x(t) \rangle = \left(\langle \sigma_x(0) \rangle \cos(\Delta_x t) - \langle \sigma_y(0) \rangle \sin(\Delta_x t)\right) e^{-\kappa_x t}
$$

(19)

$$\langle \sigma_y(t) \rangle = \left(\langle \sigma_y(0) \rangle \cos(\Delta_y t) + \langle \sigma_x(0) \rangle \sin(\Delta_y t)\right) e^{-\kappa_y t},
$$

(20)

while for the $z$ component, a good fit is obtained by using a linear regression with slope $a_z$ ($< 0$) and intercept $b_z$:

$$\langle \sigma_z(t) \rangle = a_z t + b_z.
$$

(21)

We will show that the relationships above fit very well the results obtained with MPS approach in Sec. III. There remains the problem of reconstructing the $z$-component of the qubit trajectory on the Bloch sphere.
An approximate time evolution can be written starting from the average excitation number for the oscillator in terms of its coordinates and their standard deviations $\delta \hat{x}$ and $\delta \hat{p}$. By using Eqs. (13-14), one obtains:

$$
\langle \hat{n}(t) \rangle = \tilde{\omega}_0^2 k^2 (1 - \langle \sigma_z(t) \rangle^2) + k^2 (\tilde{\omega}_0^2 - \Delta^2) \langle \sigma_y(t) \rangle^2 
+ \frac{1}{4} \left( (\delta \hat{x}(t))^2 + (\delta \hat{p}(t))^2 \right) - \frac{1}{2} - \omega_o^2 \tilde{\delta}_o^2
.$$  \hspace{1cm} (22)

Since $\langle \sigma_y(t) \rangle$ can be derived from previous Eqs. (13-14), we get access to $\langle \sigma_z(t) \rangle$ through the resonator observables over time. In particular, as found in the next section, the third term in Eq. (22) is only a small correction, not strongly dependent on time. Moreover, the mean number of bosons depends on the square of spin components. Therefore, this quantity provides the same results if the qubit is initially either in the up or down state.

In the case of a qubit state initially up along z direction, Eqs. (16-17), but also Eqs. (19-20), vanish, therefore, Eq. (22) provides a direct link between the average excitation number and the z component of the spin. This is equivalent to the standard qubit readout. On the other hand, if the qubit is initially in a linear combination of up and down states along z direction, Eqs. (13-14) are different from zero, hence the mean number of bosons in Eq. (22) displays a more complex time behaviour.

One of the aims of this paper is to extend the readout to the case of a qubit initially in a linear combination of up and down states along z direction, Eqs. (13-14-22) are the functional relationships linking qubit and oscillator observables which will guide us in the remaining part of the paper. These equations explicitly show how it is possible with proper tuning to follow the qubit state through the oscillator time evolution. We will follow essentially this procedure in the next section: we determine oscillator observables from independent HEM MPS simulations, then we will use the functional HEM Eqs. (13-14-22) to fit the MPS data for the oscillator observables by using MPS results for the spin components. In particular, the parameters of Eqs. (19-20) are obtained from MPS simulations. In the next section, we will call this procedure as $HEM + fit$. We will find that, up to the strong coupling regime, the agreement is excellent, implying that, by inverting Eqs. (13-14-22), the oscillator observables provide access to all three spin components, therefore to qubit state reconstruction.

1. Some remarks on the perturbative analytical solution

Let us have a closer look to the solution of the HEM derived above. We stress two points which will be relevant in the next sections:

1) dissipation and decoherence induced by the bath on the oscillator are not detrimental to the full qubit state reconstruction;

2) in the anti-adiabatic regime, with increasing oscillator-qubit coupling, the oscillator is in a well approximated squeezed state, depending on the qubit state, with a time dependent squeezing parameter $r(\alpha, \tilde{\gamma})$.

The first consideration is due to the fact that the homogeneous solution of the HME is proportional to the factor $e^{-\gamma t}$, hence, for $\gamma \rightarrow \infty$ it vanishes. We expect that as the coupling to the bath increases, the qubit state reconstruction can start earlier, on a time scale of the order of $\tau \approx 3/\gamma$.

For the second point, it is useful to consider a standard canonical transformation for the cavity at time $t$ such that the new annihilation operator is $c = b + \frac{\tilde{\gamma}}{\omega_o} \sigma_x$ and therefore the new oscillator Hamiltonian is $\mathcal{H}_o = \tilde{\omega}_0 c^\dagger c - \frac{\tilde{\gamma}^2}{\omega_o}$. We obtain just the Hamiltonian of a coherent state centred at $-\frac{\tilde{\gamma}}{\omega_o} \sigma_x$. Within the anti-adiabatic regime, the qubit is much slower than the cavity, therefore it can be followed by the oscillator in a coherent state moving according to the $\langle \sigma_z(t) \rangle$ of the qubit. Indeed, the HEM for qubit and oscillator are coupled so that $\sigma_x$ can be written as a term to the zero-order in $\tilde{g}$ plus a term which depends on $\tilde{g}^2 \tilde{x}$. As a consequence, a term to the third order in $\tilde{g}$ proportional to $\tilde{x}^2$ appears in the oscillator Hamiltonian, indicative of a squeezed state, with squeezing parameter $r(\alpha, \tilde{g})$. In particular, being the coupling to the qubit in the oscillator position, we expect a momentum squeezed state.

B. Matrix Product States simulations

It is well known that three main issues complicate the study of the dynamics of open quantum systems:

- no exact analytical solution generally exists;
- perturbation theory fails at strong coupling between the system and the bath;
- size of the Hilbert space grows exponentially in the system size, making exact diagonalization prohibitive.

For these reasons, accurate and efficient numerical studies are needed.

We solve the Hamiltonian in Eq. (3) using the time-dependent MPS simulations. In particular, we adopt the star geometry depicted in Fig. 2 to describe the long-range interactions between the qubit, the cavity mode, and the bath modes.

Because of the long-range character of the interactions, we adopt two different methods for the solution of the time-dependent Schrödinger equation.

The first one was developed in Ref. 19 and consists in a first order approximation of the unitary time-evolution operator in terms of a Matrix Product Operator (MPO). This method, to which we refer to $W^f$ in the following,
FIG. 2. MPS chain of sites representing the Hamiltonian in Eq. (3). The first site is occupied by the qubit, the second one by the oscillator with Hilbert space of dimension $N_o$ and the sites from the third to the $(N + 2)_{\text{th}}$ form the Ohmic bath of bosonic modes, each with Hilbert space of dimension $N_{\text{ph}}$. Qubit and oscillator are coupled via $\tilde{g}$, while the oscillator and the $N$ bath modes are coupled by long range interactions of strength $\tilde{g}_j$.

has an error per site which diverges with the system size $L$, while giving a time-step error of $O(dt^2)$.

The second method we use is the time-dependent variational principle (TDVP)\textsuperscript{20–22}, where the time-dependent Schrödinger equation is projected to the tangent space of the MPS manifold of fixed bond dimension at the current time. In this work we employ the two-site TDVP (2TDVP in Ref. 22) using the second order integrator by sweeping left-right-left with half time step $dt/2$. The main advantage of this method is that it has a smaller time-step error $O(dt^3)$, and its accuracy is controlled only by the MPS bond dimension and the threshold to terminate the Krylov series. In this work, we stop the Krylov vectors recurrence when the total contribution of two consecutive vectors to the matrix exponential is less than $10^{-12}$.

In the star-geometry considered in this work, depicted in Fig. 2, we have placed the qubit on the first site, on the second one the resonator with Hilbert space of dimension $N_o$ and on the remaining sites the collection of $N$ bosonic modes of the bath, each with Hilbert space of dimension $N_{\text{ph}}$. The coupling between qubit and oscillator is $\tilde{g}$, while the oscillator and the bath experiment long-range interactions with couplings $\tilde{g}_j$. Using the ITensor library\textsuperscript{23}, we start the time-evolution from a product state with the cavity and the bath modes in their vacuum state (zero temperature), while the qubit is placed in a generic point on the surface of the Bloch sphere. In order to reduce the simulation time and simultaneously reach a longer dynamics, we tested the $W$\textsuperscript{T} and TDVP methods against each other on the exactly solvable closed model. We observed that the TDVP method reproduces the analytical exact solution by using a time step two or three orders of magnitude larger than that needed by $W$\textsuperscript{T}. Moreover, as expected, $W$\textsuperscript{T} has shown a much smaller accuracy than the TDVP method. Thus, we decided to use the TDVP method for our simulations in the presence of the interaction with the bath modes. An important observation about our numerical simulations is in order: we did not use more sophisticated approaches like local basis optimization\textsuperscript{24–29}. We instead have converged our simulations in the number of Fock states in the cavity and bath modes, finding the best compromise between the smallest bond dimension and longest simulation times. Our truncation error has kept below $10^{-12}$ requiring a maximum bond dimension of $D_{\text{max}} = 20$. At the same time, this optimal maximum bond dimension has allowed us to reach a time size for the simulations of $t_{\text{final}} = 35/\Delta$, about 5 periods of the qubit. All the optimal parameters used in the MPS simulations are specified in the captions of the figures of the section III.

A similar convergence analysis has already been done in previous works for the applyMPO method and for one\textsuperscript{30} or more\textsuperscript{31} qubits interacting with a bath.

We finally note that, in the star geometry, one could also adopt the TEBD method with swap gates. It was recently shown, however, in Ref. 32, that it usually requires larger bond dimensions compared to 2TDVP, despite giving smaller accumulated errors for long time evolutions.

In the next section, we compare the results of our MPS simulations against the perturbative solutions shown in the previous sections. Indeed, we have used our MPS simulations to explore the parameter regions where LME or HEM are expected to fail.

III. COMPARISON OF DIFFERENT METHODS

In this section we showcase our results by applying the previous methods to the dynamics of our model, in the anti-adiabatic regime ($\Delta \ll \tilde{\omega}_0$), from the weak to
the ultra-strong coupling regime. Specifically, up to the strong coupling regime, for weak to intermediate couplings of the resonator to the bath, we find that the resonator allows the typical qubit readout (up or down initial qubit state). Furthermore, through the time evolution of resonator observables, starting from a generic qubit initial state, we reconstruct the qubit state by evaluating the qubit fidelity in time with respect to its free evolution (see Fig. 12 for the sum up of the results). For parameters from the strong to the ultra-strong coupling regime, the interactions (between them and with the bath) are so important that the qubit behaviour differs from the free one as a result of the resonator back-action upon the qubit. However, our analysis allows us to go beyond the Lindblad approximation, observing non-Markovian features of the dynamics, even through the Wigner quasi-probability distribution for the resonator state.

This section is organised as follows. In Subsec. III A we will show how the state reconstruction of the qubit works in the weak coupling regime for the three techniques (HEM, LME, MPS); we evaluate the qubit fidelity, and clarify in what sense dissipation and decoherence are useful for the qubit state reconstruction. In Subsec. III B we will identify the range of bath-resonator couplings for which a typical readout is feasible in the strong coupling regime, for both the up state and the generic one; moreover, we will continue the analysis for the qubit state reconstruction. In Subsec. III C, finally, we will discuss the features of the dynamics in the ultra-strong coupling regime, where entanglement and non-Markovianity are non-negligible and neither qubit readout nor state reconstruction are achievable.

A. Weak coupling regime

We set the parameters such that the qubit frequency $\Delta$ is taken as unity, hence, for $\omega_0 = 10\Delta$, $g = 0.1\Delta$ and $\alpha = 0.05$, $g/\Delta \leq \Gamma/\Delta \cap \Gamma/\Delta \leq 0.1\omega_0/\Delta$, with $\Gamma = \omega_0\pi/2$ the bare resonator decay rate, involving that the quantum Rabi system is in the weak coupling regime\(^33\) (Fig. 12). We remark that, in the limit of small $\alpha$, the agreement between HME, LME and MPS results is excellent. In this subsection, we set the coupling $\alpha = 0.05$, an intermediate value of the resonator-bath coupling, since it allows to understand the general method used in the next subsections.

We present our implementation of qubit state reconstruction, by following the oscillator state. Starting from the state $|0\rangle \otimes |\uparrow\rangle$, with $|\uparrow\rangle = (\cos(\theta/2) |\uparrow\rangle + \sin(\theta/2) e^{-i\phi} |\downarrow\rangle)$, where $\theta = 0.3\pi$ and $\phi = 1.2\pi$, and an empty bath, we use MPS and the complete solutions of HEM (only particular solutions in Eqs. (13-14)). As shown in Fig. 3, we find a good agreement between the two approaches. The cavity starts following the qubit after $\tau \approx 3/\gamma \approx 3.12/\Delta$. The agreement is good for the oscillator coordinates, plotted in the panels (a) and (b) of Fig. 3. There is a small difference in the amplitude of the oscillations which is not quite well reproduced by the HEM approach, because it is proportional to the qubit-resonator coupling constant. For the number, instead, panel (c) of Fig. 3 shows an excellent agreement. We notice how the number becomes constant, only shifted from 0 to 0.0257, after a rapid oscillation.

Both LME and MPS show the possibility to follow the qubit and are consistent with each other. They also confirm that the HEM solution can be improved by using the fit for the mean values of spin observables given in Eqs. (19-20). In fact, in panel (a) of Fig. 4, we notice this agreement for the time evolution of the oscillator position. For LME and HEM there is a difference with MPS in the amplitude of the oscillations due to a more accurate treatment of the coupling.

We assess the quality of the quantum state reconstruction by evaluating the qubit fidelity $F_q(t)$ with respect to
its free evolution, when decoupled from the cavity:

$$\rho_{\text{free}}(t) = \frac{1}{2} (I + \langle \hat{\sigma}_{\text{free}}(t) \rangle \cdot \hat{\sigma}_{\text{free}}),$$  \hspace{1cm} (23)

where the free Bloch vector describes the Rabi oscillations and it is a pure state. Thus, the qubit fidelity reads:

$$\begin{align*}
F_q(t) &= \left( \text{Tr} \sqrt{\rho_{\text{free}}(t) \rho(t) \sqrt{\rho_{\text{free}}(t)}} \right)^2 \\
&= \langle \psi_{\rho_{\text{free}}(t)} | \rho(t) | \psi_{\rho_{\text{free}}(t)} \rangle \\
&= \frac{1}{2} (1 + \langle \hat{\sigma}_{\text{free}}(t) \rangle \cdot \langle \hat{\sigma}(t) \rangle).
\end{align*}$$  \hspace{1cm} (24-26)

A fidelity above the 90 percent is assumed reliable. Indeed, in this coupling regime, the state reconstruction is robust, as shown in panel (b) of Fig. 4.

![FIG. 4. Cavity position in (a) and qubit fidelity in (b) as a function of time (in units of $1/\Delta$). In panel (a) the cavity position is computed with three methods: MPS simulations (blue solid line), LME solution (red dashed line) and HEM approach (green dotted line), improved by the fit given in Eqs. (19-20). Parameters: $\omega_0 = 10\Delta$, $\alpha = 0.05$, $\omega_c = 50\Delta$; for MPS simulations, $N_o = 10, N = 500, N_{ph} = 3, D_{max} = 20.$]

This panel also shows a comparison between MPS and Lindblad methods by following a linear fit for both: $F_q(t) \approx a_n - \kappa_n t$, where $n = M$ for fitting of MPS curve, $n = L$ for fitting of LME plot. The Lindblad average fidelity decreases with a slope smaller in absolute value than that computed with MPS. Anyway, both MPS and Lindblad solutions give a fidelity close to 100% up to $t \approx 5T$, where $T$ is the qubit period (see Fig. 12 and Table I in the concluding section IV).

With increasing $\alpha$, as expected, the HEM homogeneous solution effectively drops earlier since the transient $\tau \approx 3/\gamma$ gets reduced up to very strong couplings, when the interactions break the qubit-resonator coherence. Furthermore, starting from the vacuum state, the oscillator evolves in a momentum squeezed state, not so different from the vacuum, due to the low value of $\tilde{g}$, as underlined in II A 1. In particular, the figure Fig. 5 shows that the position uncertainty $\delta \hat{x}(t) > 1$, the momentum one is instead $\delta \hat{p}(t) < 1$, indication of a momentum squeezed state, so that the product $\delta \hat{x}(t)\delta \hat{p}(t) \geq 1$, as ruled by the uncertainty principle.

![FIG. 5. Cavity position uncertainty (blue solid line), momentum one (red dashed line) and uncertainty product (green dotted line) as a function of time (in units of $1/\Delta$) from MPS simulations. Parameters: $\omega_0 = 10\Delta, g = 0.1\Delta, \alpha = 0.05, \omega_c = 50\Delta$; for MPS simulations, $N_o = 10, N = 500, N_{ph} = 3, D_{max} = 20.$]

B. Strong coupling regime - non-perturbative state reconstruction

We choose again the qubit frequency $\Delta$ as unity and $\omega_0 = 10\Delta$, by tuning $g$ and $\alpha \in [0.01,0.1]$ such that $g/\Delta \geq \Gamma/\Delta \cap g/\Delta \leq 0.1\omega_0/\Delta \cap \Gamma/\Delta \leq 0.1\omega_0/\Delta$ or only $g/\Delta \leq 0.1\omega_0/\Delta$ if $\Gamma/\Delta \geq 0.1\omega_0/\Delta$ (see Fig. 12). Hence, the closed quantum Rabi system is in the strong coupling regime. At the same time, the ratio between the coupling and the sum of the qubit and oscillator frequencies of the closed system is $g/(\Delta + \omega_0) \approx 10^{-2}$, so that it is possible to rewrite the Hamiltonian of the Rabi model (Eq. (1)) as the Bloch-Siegert one

$$H_{BS} = \frac{\Delta}{2} \sigma_z + \omega_0 a^\dagger a + \omega_{BS} \left[ \sigma_z \left( n + \frac{1}{2} \right) - \frac{1}{2} \right]$$  \hspace{1cm} (27)

$$+ G(n)a^\dagger \sigma_- + a \sigma_+ G(n),$$  \hspace{1cm} (28)

with the Bloch-Siegert frequency $\omega_{BS} = g^2/(\Delta + \omega_0)$ and $G(n) = g[1 - n\omega_{BS}/(\Delta + \omega_0)]$. This Hamiltonian can be block diagonalised at fixed total number of excitations $N$ obtaining the eigenvalues

$$E_{N,\pm}^{BS} = (\Delta + \omega_0)N \pm \Gamma_N - gc,$$  \hspace{1cm} (29)

$$\Gamma_N = \sqrt{(\Delta - \omega_0 + 2g\epsilon N)^2 + g^2 N(1 - \epsilon^2 N)^2},$$  \hspace{1cm} (30)

where $\epsilon = g/(\Delta + \omega_0)$. As for the Jaynes-Cummings model, the splitting of the energy eigenvalues depends on the qubit state ($\pm$), which can be read through the resonator state.

Another way to determine the excitation energies of the closed system is to directly apply the stationary perturbation theory for low $g$ on the Hamiltonian in Eq. (1).
The unperturbed Hamiltonian reads:
\[ \mathcal{H}_0 \approx \pm \Delta / 2 \sigma_z + \omega_0 n \ | n, \pm \rangle \langle n, \pm | \]
so that the eigenvalues up to the second order in \( g \) are the following:
\[ E_{n, \pm} \approx \pm \Delta / 2 + \omega_0 n \pm g^2 \left( \frac{n + 1}{\Delta + \omega_0} + \frac{n}{\Delta - \omega_0} \right). \]
Hence the excitation energies can be easily evaluated as:
\[ E_u = E_{1,-} - E_{0,+}, \]
\[ E_d = E_{1,+} - E_{0,-}, \]
where \( E_u \) is the energy corresponding to the up state of the qubit, lesser than the cavity energy, and \( E_d \) that of the down one, greater than the cavity energy.
The excitation energies are obviously in agreement with the Bloch-Siegert Hamiltonian (Eq. (29)) approach. In fact, similarly to the Jaynes-Cummings model, the oscillator frequency \( \omega_0 \) is shifted depending on the qubit state as follows:
\[ \omega_0 \to \omega_0 \pm \left[ \Delta + g^2 \left( \frac{1}{\Delta + \omega_0} + \frac{1}{\Delta - \omega_0} \right) \right], \]
so that the \((-)\) sign refers to the excitation energy \( E_u^{BS} \), while the \((+)\) to \( E_d^{BS} \).

We first verify if the standard qubit readout through the dissipative cavity works also in this coupling regime. In particular, we perform the same analysis for a qubit in an up state \((|0\rangle \otimes |↑\rangle)\) or in a generic state \((|0\rangle \otimes |\gamma\rangle)\), and an empty cavity in contact with the bath at zero temperature. In order to locate the excitation energies of the open system and read the qubit state from the resonator, we analyse the peaks of the Fast Fourier Transform (FFT) of the mean oscillator number.

As expected, we observe in Fig. 6 only a peak for the up state, with frequency smaller that the cavity one, and two peaks for the generic qubit state due to the presence of up (left peak) and down (right peak) contributions. With increasing the resonator-bath coupling \( \alpha \), we find that the peaks become less resolved making the qubit readout more difficult. In both panels of Fig. 6, there is a critical value \( \alpha_c \) of \( \alpha \), about 0.03, above which the qubit readout is not feasible. We see how the FFT below the critical value for the generic initial state has memory of the down contribution.

We then pursue the full qubit state reconstruction choosing the initial state as \(|0\rangle \otimes |\gamma\rangle\) analysing in detail the system behaviour for \( g = 0.6 \) (the same as in Fig. 6) and \( g = 1.0 \), with \( \alpha \in \{0.01, 0.1\} \). For \( \alpha = 0.01 \), results from LME and MPS are still in agreement and the full state reconstruction is possible (see Fig. 7).

![FFT of the mean cavity number](image)

**FIG. 6.** FFT of the mean cavity number \( \bar{n} \) computed at \( g = 0.6 \) as a function of frequency (in units of \( \Delta \)) through MPS simulations for three values of \( \alpha \) and for the two initial states: \((|↑\rangle \text{ left panel}), (|\gamma\rangle \text{ right panel})\). Parameters: \( \omega_0 = 10\Delta, g = 0.6\Delta, \alpha \in \{0.01, 0.03\}, \omega_c = 50\Delta \) for MPS simulations, \( N_o = 20, N = 500, N_{ph} = 3, D_{max} = 20 \).

We first verify if the standard qubit readout through the dissipative cavity works also in this coupling regime. In particular, we perform the same analysis for a qubit in an up state \((|0\rangle \otimes |↑\rangle)\) or in a generic state \((|0\rangle \otimes |\gamma\rangle)\), and an empty cavity in contact with the bath at zero temperature. In order to locate the excitation energies of the open system and read the qubit state from the resonator, we analyse the peaks of the Fast Fourier Transform (FFT) of the mean oscillator number.

![Cavity position](image)

**FIG. 7.** Cavity position (in panels (a) and (b)) and qubit fidelity (in panels (c) and (d)) as a function of time (in units of \( 1/\Delta \)) for \( g = 0.6 \) and \( g = 1.0 \) at \( \alpha = 0.01 \) using different methods: MPS simulations (blue solid line), the LME solution (red dashed line) and the HEM approach (green dotted line), improved by the fitting with spin observables obtained by MPS simulations as in Eqs. (19-20). Parameters: \( \Delta = 1, \omega_0 = 10\Delta, g = \{0.6\Delta, 1.0\Delta\}, \alpha = 0.01, \omega_c = 50\Delta \) for MPS simulations, \( N_o = 20, N = 500, N_{ph} = 3, D_{max} = 20 \).

At \( g = 0.6 \), the three approaches give the correct description, because the regime is still perturbative in \( g \) and the effect of the bath negligible. At \( g = 1.0 \), there is a difference in the amplitude of the oscillations of the cavity position obtained from HEM. Indeed, this solution is strictly valid only for small couplings \( g \), unable to significantly influence the qubit dynamics. For the fidelity \((c)\) and \((d)\) in Fig. 7 we compare MPS and Lindblad methods. For \( g = 0.6 \), they overlap almost completely with each other, ever above the 95% for 5
qubit periods. This robust fidelity together with the two peaks of the FFT in Fig. 6 shows how it is possible to still use the HEM approach for oscillator coordinates and energy to reconstruct the qubit state in time. Instead, for \( g = 1.0 \) the system ends up in the limiting case when the qubit state reconstruction is feasible for both LME and MPS \( (F_q(5T\Delta) \approx 90\%) \). Therefore, it is possible through the resonator to read the qubit state which is still close to the free behaviour.

For \( \alpha = 0.1 \), we find interesting behaviours with increasing \( g \). We start again from \( g = 0.6 \), which has been considered both in Fig. 6 and 7. We recall that, for \( \alpha = 0.1 \), the standard readout is not effective to get the spin component along the z-axis. Therefore, we investigate if a state reconstruction can still be achieved. As shown in Fig. 8, the time evolution of the resonator number agrees with that predicted from the HEM improved by the fitting of the spin z-component (Eq. (21)), while the LME remains perturbative and does not significantly change the state from the initial vacuum. For the position, instead, the agreement is good at \( g = 0.6 \) between LME and HEM, while there is a slightly different amplitude in the MPS position. In fact, LME fails at high coupling to the bath while HEM at high renormalization of the qubit dynamics. Thus, these perturbative approaches are unable to properly transfer the effect of the environment to the qubit through the interaction with the cavity. To understand this point, one can consider the mapped model in Eq. (4) where the effective spectral density is Ohmic at low frequencies with coupling proportional to \( g^2/\omega_0^2 \). For \( g = 1.0 \), we observe also a discrepancy between LME and HEM, due to the increased \( g \). Moreover, they are phase-shifted with respect to the MPS due to the fact that a high value of the coupling in the mapped model\(^{17} \) means a qubit strongly coupled to the effective bath.

The panels \((c) \) and \((f) \) of Fig. 8 show qubit fidelity for \( \alpha = 0.1 \), comparing MPS and LME methods. For \( g = 0.6 \), LME gives an incorrect behaviour, while the MPS simulations provide the limiting case \( F_q(5T\Delta) \approx 90\% \) in which the state reconstruction can be performed. Indeed, one can still determine \( x, y \) and \( z \) spin components from the behaviour of oscillator position, momentum and number as predicted by HEM solutions. In panel Fig. 8\((f) \), for \( g = 1.0 \), the difference between LME and MPS becomes large, indicative of the fact that in this regime LME fails, while for MPS \( F_q(5T\Delta) \approx 60\% \). In fact, LME would provide still a perturbative description, with an acceptable fidelity; MPS, however, shows a fidelity almost linearly decreasing so that it results \( F_q(5T\Delta) \approx 60\% \). Summarizing, we have found that for low \( \alpha \) in the strong coupling regime both the typical readout for an upper state and for a generic state can be achieved making the qubit state reconstruction possible. For increasing \( \alpha \), anyway, the readout is unachievable, while for \( g \leq 0.6 \) a state reconstruction is still attainable (see Fig. 12 and Table I in the concluding section IV).

In analogy with weak coupling regime discussed in the previous subsection, we find that the effect of the decoherence helps the qubit state reconstruction by reducing the initial transient and the oscillator is in a moving momentum squeezed state as pointed out in II A 1.

\[
\text{C. Ultra-strong coupling regime - interactions prevent state reconstruction}
\]

Lastly, with the qubit frequency \( \Delta \) taken as unity and \( \omega_0 = 10\Delta \), by focusing on \( g/\Delta = 5 \) and tuning \( \alpha \in \{0.01, 0.1\} \) such that \( g/\Delta \geq 0.1\omega_0/\Delta \) (see Fig. 12), the quantum Rabi system is in the ultra-strong-coupling regime.

As in the previous regime, if one chooses \( |0\rangle \otimes |\uparrow\rangle \) as initial state, for low values of \( \alpha \) it is possible to follow the peaks of the FFT of the mean resonator number. In Fig. 9 we see, as expected, that, for \( \alpha = 0.01 \), a peak is
prominent, while, for $\alpha \geq 0.1$, the FFT is almost flat. However, even for $\alpha = 0.01$, the frequency corresponding to the peak is no more in perfect agreement with the Bloch-Siegert shift given in Eq. (35) because the system is in the non-perturbative region of ultra-strong coupling regime ($g \approx \Delta + \omega_0$). Moreover, also the peak corresponding to the down contribution is emerging. Therefore, the qubit readout is not feasible because of the marked back-action of the resonator. Below we will show that, in this coupling regime, qubit and resonator are indeed strongly entangled.

For the initial state $|0\rangle \otimes |\uparrow\rangle$ and the bath empty, regardless of the value of $\alpha$, qubit state reconstruction is no more feasible (see Fig. 10). In particular, the agreement between different computational methods is good for $\alpha = 0.01$ at short times, while there is a dephasing in LME solution due to the fact that a high value of the coupling in the mapped model$^{17,37}$ means a qubit strongly coupled to the effective bath. For larger $\alpha$ also HEM fails because the fit is not enough to describe the qubit dynamics. In Fig. 10(c) and Fig. 10(d) we compare the fidelity for the MPS and LME approaches. For $\alpha = 0.01$, the results are very similar, oscillating around 0.5 due to the growing entanglement, while for $\alpha = 0.1$ they have an exponential decaying behaviour, with an overmodulation for MPS. Anyway, during the first 5 qubit periods, both MPS and LME solutions give a minimum fidelity close to 20%. We emphasize that a state reconstruction is still possible, at least for low $\alpha$ values, but what we are reconstructing is the qubit state as a part of the coupled system. As a consequence, it is very entangled, so its evolution is far from the free one and the fidelity is low (see Fig. 12 and Table I in the concluding section IV).

This set of parameters allows us to observe the moving of the squeezed state (being the amplitude of the displacement proportional to $g$) (Fig. 11). The position of the resonator oscillates around zero as shown in Fig. 10, while its momentum remains close to the zero. The cavity starts in the vacuum and evolves in a momentum squeezed state whose center oscillates with damping around zero until stopping at the origin of the quantum phase space, as underlined in II A 1.

**IV. DISCUSSION AND CONCLUSIONS**

In the present work, we have provided an exhaustive description of a method for reconstructing the qubit state through the dynamics of its coupled resonator. In particular, we have tested the functional relationships derived from the perturbative (in $g$) solutions of HEM, by simulating the quantum Rabi model with MPS numerical method. We have compared it also with a numerical solution of the LME, which is perturbative in the oscillator-bath coupling and unable to describe the non-Markovian effects of the bath. We have found the critical values of the coupling to the bath for the usual case of the qubit readout (up state) and for a generic qubit state since we have extended the procedure to an open system. Moreover, we have implemented full state reconstruction of the qubit, by computing the time evolution of the cavity observables. We have found the parameter regimes where
the procedure does not work, taking the qubit fidelity with respect to its free evolution as a measure of the goodness of the state reconstruction. Table I shows the qubit fidelity computed at time $t_{\text{final}} = 35/\Delta$ for weak and strong coupling regimes, while for the ultra-strong regime the minimum value is reached at time $t = 7/\Delta$ for the MPS method and $t = 19.5/\Delta$ for the LME one.

![Fig. 11](image_url) Reduced density matrix of the cavity computed through MPS simulations at 8 times $t\Delta$ during its evolution. From this, we obtain the Wigner quasi-probability distribution by using the proper function of the package QuTIP. Parameters: $\Delta = 1, \omega_0 = 10\Delta, g = 5\Delta, \alpha = 0.1, \omega_c = 50$; for MPS simulations $N_o = 20, N = 500, N_{\text{ph}} = 3, D_{\text{max}} = 20$.

![Fig. 12](image_url) Phase diagram of the quantum Rabi model for varying $\alpha$ and $g$. On the background there are the three regions of parameters analysed in the paper: weak coupling (light grey), strong coupling (grey) and ultra-strong coupling (black). The two methods under study are shown using different patterns: qubit readout with blue “+” and qubit state reconstruction with red “/”. The grey vertical line at $\alpha = 2/(10\pi)$ underlines that above this value the regime is again the strong coupling one because $\Gamma/\Delta > 0.1\omega_0/\Delta$. Parameters: $\Delta = 1, \omega_0 = 10\Delta, g \in [0, 2\Delta], \alpha \in [0, 2/(10\pi)], \omega_c = 50, N_o = 20, N = 500, N_{\text{ph}} = 3, D_{\text{max}} = 20$.

![Fig. 13](image_url) Von Neumann entropy $S_q = -\text{Tr}\{\rho_q \log \rho_q\}$ and the purity $P_q = 1 - \text{Tr}\{\rho_q^2\}$ are displayed for the three different regimes of parameters examined in the paper. We clearly observe how they increase towards the maximum values (dotted black lines $S_q^{\text{max}} = \log 2$ and $P_q^{\text{max}} = 1/2$, respectively) in the ultra-strong coupling regime. Actually, for the coupled system qubit-resonator, entanglement becomes bigger and bigger. This is the main reason why the qubit fidelity with respect to its free evolution becomes smaller and smaller.

![Fig. 14](image_url) The procedure does not work, taking the qubit fidelity with respect to its free evolution as a measure of the goodness of the state reconstruction. Table I shows the qubit fidelity computed at time $t_{\text{final}} = 35/\Delta$ for weak and strong coupling regimes, while for the ultra-strong regime the minimum value is reached at time $t = 7/\Delta$ for the MPS method and $t = 19.5/\Delta$ for the LME one.

### Table I. Minimum qubit fidelity $F_q(t)$ (%) for the three regimes analysed in the paper

| Method  | Weak       | Strong     | Ultra-strong |
|---------|------------|------------|--------------|
|         | $g = 0.1, \alpha = 0.05$ | $g = 0.6, \alpha = 0.1$ | $g = 5.0, \alpha = 0.1$ |
| MPS     | 99.8       | 61.2       | 20.4         |
| Lindblad| 99.9       | 88.0       | 18.7         |

In Fig. 12 we show the validity of the two methods analysed in detail in the paper, that are the standard qubit readout and the full qubit state reconstruction via oscillator dynamics in the three regimes: weak, strong and ultra-strong coupling. We stop the plot at $\alpha = 2/(10\pi)$ that corresponds to $\Gamma/\Delta = 0.1\omega_0/\Delta$. The latter indicates that the estimation of the perturbative decay rate $\Gamma$ for the definition of the parameter regions no longer holds. Hence, above this value of $\alpha$ the regime is again the strong coupling one. We emphasize that for internal coupling $g \leq 0.8\Delta$ the state reconstruction is possible for coupling strengths $\alpha$ bigger than those necessary for the standard readout. Furthermore, for increasing internal couplings, even if the LME solution would give a reliable qubit fidelity, regardless of the coupling to the bath, the MPS simulations have shown that it is already much lesser than the acceptance threshold of 90%. Nevertheless, the state reconstruction is still possible, but the interactions have modified significantly the qubit dynamics with respect to the free one. In the ultra-strong regime the fidelity oscillates around the value of 50%, that can be easily interpreted by looking at the qubit entanglement in time.
reconstruction can be obtained at any time by raw realizations. It is remarkable that a full quantum state is equivalent to the Lindblad approach for the qubit in types of jumps that can occur.

The method is also known as the method of quantum measurements. The unraveling of quantum trajectories has been used in order to follow the qubit dynamics. In fact, the functional relationships linking the qubit dynamics to the resonator evolution, that we have derived from HEM, bear some resemblance to those used in Ref. 14 to interpret measurements. The main difference is that in Ref. 14 the bath has not been explicitly considered, while, in this paper, we have exploited its effects on the qubit state reconstruction. Moreover, the LME used in this work takes qubit and resonator on the same footing. Finally, we have introduced dissipation and decoherence through the presence of an Ohmic bath, without describing a measurement apparatus. A generalization of our study to a more realistic model, including the measurement process description, deserves future investigations.

The study carried out in the present work can be extended to the systems where there are more qubits and more resonators. However, while these generalizations are straightforward in HME and LME approaches, they would be clearly more challenging within a MPS framework. Such architectures are typical of the recent noisy intermediate-scale quantum processors. Furthermore, the resonator could be investigated in more detail, since the interest about the bosonic behaviour, specially in circuit QED, is growing. In fact, quantum information can be encoded into subspaces of a bosonic superconducting cavity mode with long coherence time.

At this stage, as recalled in the introduction I, we can thoroughly discuss the results of the experiments in Ref. 14. Indeed, in these experiments, a full qubit state reconstruction is realised by reading the quadratures of a coupled resonator via homodyne and heterodyne measurements. The unraveling of quantum trajectories has been used in order to follow the qubit dynamics. The method is also known as the method of quantum jumps, exactly the same as the Lindblad operators. The resulting master equation is then an appropriately weighted stochastic average over all of the different times at which the jumps could occur, and all of the different types of jumps that can occur. Hence, this approach is equivalent to the Lindblad approach for the qubit in the limit of a sufficiently large number of experimental realizations. It is remarkable that a full quantum state reconstruction can be obtained at any time by raw averaging measurement outcomes of many realizations of a single experiment despite the incompatibility of the three spin components of the qubit.

In our work the idea of how performing the qubit state reconstruction is similar to that in Ref. 14, that is to read the resonator dynamics. Hence, the functional relationships linking the qubit dynamics to the resonator evolution, that we have derived from HEM, bear some resemblance to those used in Ref. 14 to interpret measurements. The main difference is that in Ref. 14 the bath has not been explicitly considered, while, in this paper, we have exploited its effects on the qubit state reconstruction. Moreover, the LME used in this work takes qubit and resonator on the same footing. Finally, we have introduced dissipation and decoherence through the presence of an Ohmic bath, without describing a measurement apparatus. A generalization of our study to a more realistic model, including the measurement process description, deserves future investigations.

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1. X. Gu, A. F. Kockum, A. Miranowicz, Y.-x. Liu, and F. Nori, Phys. Rep. 718, 1 (2017).
2. I. Rabi, Phys. Rev. 49, 324 (1936).
3. E. T. Jaynes and F. W. Cummings, Proc. IEEE 51, 89 (1963).
4. D. Zueco, G. M. Reuther, S. Kohler, and P. Hänggi, Phys. Rev. A 80, 033846 (2009).
5. A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, R.-S. Huang, J. Majer, S. Kumar, S. M. Girvin, and R. J. Schoelkopf, Nat. 431, 162 (2004).
6. F. Yoshihara, T. Fuse, S. Ashhab, K. Kakuyanagi, S. Saito, and K. Semba, Nat. Phys. 13, 44 (2017).
7. P. Forn-Díez, L. Lamata, E. Rico, J. Kono, and E. Solano, RMP 90, 025005 (2019).
8. C. Perroni, D. Nimmo, and V. Cataudella, J. Condens. Matter Phys. 28, 373001 (2016).
9. A. Nocera, C. A. Ferreri, V. M. Ramaglia, and V. Cataudella, Beilstein J. Nanotechnol. 7, 439 (2016).
10. C. A. Perroni and G. Benenti, Advances in Thermoelectricity: Foundational Issues, Materials and Nanotechnology 207, 115 (2021).
11. A. M. Zagospkin, Quantum Engineering: Theory and Design of Quantum Coherent Structures (Cambridge University Press, 2011).
12. S. Hacohen-Gourgy, L. S. Martin, E. Flurin, V. V. Ra-...

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Appendix A: Lindblad master equation

The most straightforward approach to the dynamics of an open quantum system is based on the solution of a quantum master equation which generalises the Liouville - Von Neumann equation for the reduced density operator of the system of interest. In fact, when the system is isolated, described by a time-independent Hamiltonian, or closed, described by a time-dependent Hamiltonian (e.g. a driving field), its dynamics is fully characterised...
by the Liouville - Von Neumann equation:

\[
\frac{d\rho_I(t)}{dt} = -\frac{i}{\hbar} [H_I(t), \rho_I(t)] = L\rho_I(t), \quad (A1)
\]

written in the interaction picture. The subscript \( I \) stays for “interaction” and the whole system Hamiltonian is \( H(t) = H_0 + H_I(t) \), with \( H_0 \) the unperturbed energy of the separated quantum systems and \( H_I(t) \) the interaction between them. Moreover, \( L \) is the Liouville superoperator that acting on the density matrix gives its time evolution.

The famous Lindblad master equation\(^{42}\) (LME) is a Markovian quantum master equation that has the advantage of being easy to solve numerically through suitable tools, and analytically in some special situations. An open quantum system is said to be Markovian when its behaviour at a given time is independent of its behaviour in the past. The main drawback of Markovian framework is the lack of the back-action upon the system, and its behaviour at a given time is independent of its behaviour in the past. The main drawback of Markovian approximation, valid for systems weakly coupled to the environment\(^{43}\). In fact, according to this approximation, the environment relaxes before the system changes its state, \( \rho(t) = \rho_S(t) \otimes \rho_B(0) \) so that it cannot act on the system (Born approximation). Furthermore, \textit{Markov approximation}, i.e. \( \tau_R \gg \tau_B \), where \( \tau_R \) is the relaxation time of the system \( S \) and \( \tau_B \) is the correlations time of the bath, means that the dynamics over a time of the order of \( \tau_B \) cannot be resolved. The Born-Markov approximation does not warrant the complete positivity of the dynamical map. Therefore, we need to perform one further approximation, the \textit{secular or rotating wave approximation} (RWA). If \( \tau_S \) is the time scale of the system evolution and \( \tau_S \ll \tau_R \), the non-secular terms may be neglected, since they oscillate very rapidly during the time over which \( \rho_S \) varies appreciably. If we separate the Hermitian and non-Hermitian parts of the dynamics of the system, and we return to the Schrödinger picture diagonalising the matrix formed by the rate coefficients, we obtain the LME for \( \rho_S(t) \):

\[
\frac{d\rho_S}{dt} \equiv L\rho_S = -\frac{i}{\hbar} [H_S + H_{LS}, \rho_S] \\
+ \sum_{\omega,k} \gamma_k(\omega) \left( L_k(\omega)\rho_S L_k^\dagger(\omega) - \frac{1}{2} \left\{ L_k^\dagger(\omega)L_k(\omega), \rho_S \right\} \right), \quad (A2)
\]

where the first term is the unitary evolution with \( H_{LS} \) the Lamb and Stark shift Hamiltonian, whose role is to renormalize the system energy levels due to the interaction with the environment. The sum is the Dissipator with rates \( \gamma_k \) and \( L_k \) the Lindblad or jump operators.

When dealing with multipartite systems as in our Hamiltonian given in Eq. (3), the key point is how to eigendecompose the system observables occurring in the interaction Hamiltonian. One way (Local) consists in neglecting the “internal” interaction among the parties\(^{44}\). In this case the eigenstates are factorised and one can only add the dissipators pertaining the different operators in order to describe the dissipation. The other one (Global) instead is based on the use of the correct eigenstates of the Hamiltonian of the system surrounded by the bath. Moreover, it is known that the global approach provides an accurate description of the system evolution at long timescales\(^{45}\). Thus, in the paper we follow this approach, because we aim to properly describe the interaction between qubit and oscillator up to times of the order of some qubit periods. Hence, we diagonalise the Hamiltonian of qubit plus oscillator system (Eq. (3) without the bath) at each time step \( dt \), we eigendecompose the system observables and we evaluate the Lindbladian \( L \). Eq. (A2) for the reduced density matrix of the bipartite system \( \rho_S(t) \), vector in the Liouville space, is solved numerically through the very well-known fourth-order Runge-Kutta algorithm, derived from two different Taylor expansions of the dynamical variable \( \rho_S \) and its derivative\(^{16}\):

\[
\rho_S(t + dt) = \rho_S(t) + \frac{1}{6} (c_1 + 2c_2 + 2c_3 + c_4), \quad (A3a)
\]

\[
c_1 = dt \left( L(\rho_S(t)) \right), \\
c_2 = dt \left( L(\rho_S(t) + \frac{c_1}{2}) \right), \\
c_3 = dt \left( L(\rho_S(t) + \frac{c_2}{2}) \right), \\
c_4 = dt \left( L(\rho_S(t) + c_3) \right). \quad (A3b)
\]

We study the solution of LME in order to compare a Markovian description with other analytical and numerical methods, such as the MPS approach introduced in Sec. II B.