Littlewood-Paley characterization for $Q_\alpha(\mathbb{R}^n)$ spaces

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March, 2009

2000 Mathematics Subject Classification. 42B25

Keywords and phrases. $Q_\alpha$ spaces, Campanato spaces, Littlewood Paley characterization.

Abstract. In Baraka’s paper [2], he obtained the Littlewood-Paley characterization of Campanato spaces $L^{2,\lambda}$ and introduced $L^{p,\lambda,s}$ spaces. He showed that $L^{2,\lambda} = (-\Delta)^{-\frac{s}{2}}L^{2,\lambda}$ for $0 \leq \lambda < n + 2$. In [7], by using the properties of fractional Carleson measures, J Xiao proved that for $n \geq 2$, $0 < \alpha < 1$. $(-\Delta)^{-\alpha}L^{2,n-2\alpha}$ is essential the $Q_\alpha(\mathbb{R}^n)$ spaces which were introduced in [4]. Then we could conclude that $Q_\alpha(\mathbb{R}^n) = L^{2,n-2\alpha,\alpha}$ for $0 < \alpha < 1$. In fact, this result could be also obtained directly by using the method in [2]. In this paper, We proved this result in the spirit of [2]. This paper could be considered as the supplement of Baraka’s work [2].

1 Introduction

The $Q_\alpha$ spaces were first introduced in [1] as a proper subspace of BMOA defined by means of modified Garcia norm. In [5], authors showed that: Let $\alpha \in (0,1)$, an analytic function $f$ in the Hardy space $H^1$ on the unit disc belongs to $Q_\alpha$, if and only if its boundary values on the unit circle $T$ satisfies:

$$\sup_I |I|^{-\alpha} \int_I \int_I \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^{2-\alpha}} d\theta d\varphi < \infty.$$  \hspace{1cm} (1.1)

Where the supremum is taken over all subarcs $I \subset T$. In [4], the $Q_\alpha$ was extended to Euclidean space $\mathbb{R}^n (n \geq 2)$. They gave the definition of this kind of space as follows: For $\alpha \in (-\infty, +\infty)$, $f \in Q_\alpha(\mathbb{R}^n)$ if and only if

$$\|f\|_{Q_\alpha} \triangleq \left[ \sup_I l(I)^{2\alpha - n} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x - y|^{2\alpha + n}} dx dy \right]^\frac{1}{2} < \infty.$$ 

Here $I \subset \mathbb{R}^n$ be a cube with the edge parallel to the coordinate axes, and let $l(I)$ be the length of $I$. The supremum is taken over all cubs $I \subset \mathbb{R}^n$. There are systematic research of $Q_\alpha(\mathbb{R}^n)$ in [4].
In [4], we have known that if $\alpha < 0$, $Q_\alpha = BMO$. And if $\alpha \geq 1$, $Q_\alpha = \{\text{constants}\}$. We have also known ([7] theorem 1.2 (1))

$$Q_\alpha(\mathbb{R}^n) = (-\triangle)^{-\frac{\alpha}{2}}L^{2,n-2\alpha}$$

for the nontrivial case $\alpha \in (0, 1)$. $L^{2,n-2\alpha}$ denote the Campanato spaces:

$$L^{2,n-2\alpha} \triangleq \sup_I (\text{sup}l(I))^{2\alpha-n} \int_I |f(x) - f_I|^2 dx \frac{1}{2} < \infty.$$  

Combining this result with ([2], theorem 10). We can immediately obtain:

$$Q_\alpha(\mathbb{R}^n) = L^{2,n-2\alpha,\alpha}$$

The Littlewood-Paley characterization is now clear by the $L^{2,n-2\alpha,\alpha}$ definition ([2], definition 2):

$$\|f\|_{L^{2,n-2\alpha,\alpha}} \triangleq \sup_I \left( \frac{1}{|I|^{1-n}} \sum_{j \geq -\log_2 l(I)} 2^{2\alpha j} \|\Delta_j f\|_{L^2(I)}^2 \right)^{\frac{1}{2}} \quad (1.2)$$

In this paper we present an alternative proof of the result. Unlike J. Xiao’s arguments, which make a systematic research of fractional Carleson measures [3]. Our methods are in the spirit of [2]. We directly prove the Littlewood-Paley characterization from (1.1) which is the definition of $Q_\alpha(\mathbb{R}^n)$.

Let $\psi(x)$ be a Schwartz function. supp$\psi(x) = \{x \in \mathbb{R}^n : \frac{1}{2} \leq |x| \leq 2\}$ is compact and $\sum \psi_j(x) \equiv 1$. We define the Littlewood-Paley operator by

$$\Delta_j(f)(x) = \psi_j * f(x)$$

where

$$\psi_j(x) = 2^{jn} \psi(2^j x)$$

In this paper, we study the case $f \in S'/\mathcal{P}$. The homogeneous decomposition of $f$ is given by the formula

$$f = \sum_{j \in \mathbb{Z}} \Delta_j(f)(x)$$

We denote $A \lesssim B$ if $A \leq C(n, \alpha)B$. And define $A \approx B$ if $A \leq C(n, \alpha)B$ and $B \leq C(n, \alpha)A$. We have the following main result.

**Main Theorem** Let $f \in L^2(\mathbb{R}^n)$, $0 < \alpha < 1$. We have the Littlewood-Paley characterization of $Q_\alpha(\mathbb{R}^n)$:

$$\|f\|_{Q_\alpha} \approx \sup_I \left( \frac{1}{|I|^{1-n}} \sum_{j \in \mathbb{Z}} 2^{2\alpha j} \|\Delta_j f\|_{L^2(I)}^2 \right)^{\frac{1}{2}} \quad (1.3)$$

The main theorem essentially contains two statements as follows:

If $f \in L^{2,n-2\alpha,\alpha}$ then $\|f\|_{Q_\alpha} \lesssim \|f\|_{L^{2,n-2\alpha,\alpha}}$;

If $f \in Q_\alpha(\mathbb{R}^n)$, then $\|f\|_{L^{2,n-2\alpha,\alpha}} \lesssim \|f\|_{Q_\alpha}$.

**Remark** From the main theorem, we get the relationship between $Q_\alpha$ spaces and Morrey type Besov spaces: In [6], authors introduced a kind of Morrey type Besov spaces:

$$\|f\|_{MB^\alpha_{2,n-2\alpha}} \triangleq \left( \sum_{j \in \mathbb{Z}} \left( \sup_I \frac{1}{|I|^{\frac{n}{2}}} \int_I (2^{2\alpha j} |\Delta_j f|)^q dx \right)^\frac{r}{q} \right)^\frac{1}{r} < \infty$$

We immediately have the embedding property: $MB^\alpha_{2,n-2\alpha} \subset Q_\alpha$ for $0 < \alpha < 1$. 

\[2\]
2 Preliminary Lemmas

The proof of the main theorem relies on following lemmas. To start with, we introduce some notations: Let $I$ be the any fixed cub in $\mathbb{R}^n$ with the edge parallel to the coordinate axes. We let $D_k(I), k \geq 0$, denote the set of the $2^{kn}$ subcubes of edge length $2^{-k}l(I)$ obtained by $k$ successive bipartition of each edge of $I$. We define $D(I)$ be the set of all the dyadic subcubes of $I$. Let $a > 0$ be a fixed number. We assume $aI$ be the dilation cube with the same center of $I$, and its length is $al(I)$.

Lemma 2.1 Let $-1 < \alpha \leq \frac{n}{2}$. Then we have quasi-norm $\|f\|_{\mathcal{L}^{2,n-2\alpha}}$ is well-defined.

Proof: As for another bump test function, we have the expression

$$\|f\|_{\mathcal{L}^{2,n-2\alpha}} = \sup_I \left( \frac{1}{|I|^{1-\frac{2\alpha}{2}}} \sum_{j \geq -\log_2 l(I)} 2^{2\alpha j} \|\Delta_j f\|_{L^2(I)}^2 \right)^{\frac{1}{2}}$$

We let $f = (-\Delta)^{-\frac{\alpha}{2}}g$. By the proof of Lemma 24 in [2], we have known that ([2], (22))

$$\frac{1}{|I|^{1-\frac{2\alpha}{2}}} \sum_{j \geq -\log_2 l(I)} 2^{2\alpha j} \|(-\Delta)^{-\frac{\alpha}{2}} \Delta_j g\|_{L^2(I)}^2 \lesssim \|g\|_{\mathcal{L}^{2,n-2\alpha}}^2$$

for any fixed cube $I \subset \mathbb{R}^n$.

Because of proposition 8 in [2], $\mathcal{L}^{2,n-2\alpha} = \mathcal{L}^{2,n-2\alpha}$ is Campanato space and thus well defined. We have

$$\|g\|_{\mathcal{L}^{2,n-2\alpha}} \lesssim \sup_I \left( \frac{1}{|I|^{1-\frac{2\alpha}{2}}} \sum_{j \geq -\log_2 l(I)} \|\Delta_j g\|_{L^2(I)}^2 \right)^{\frac{1}{2}}$$

Then

$$\|f\|_{\mathcal{L}^{2,n-2\alpha}} \lesssim \|f\|_{\mathcal{L}^{2,n-2\alpha}}$$

by lemma 24 in [2].

Lemma 2.2 Let $\alpha > 0$. We have another quasi-norm definition of $\mathcal{L}^{2,n-2\alpha}$ as follows:

$$\|f\|_{\mathcal{L}^{2,n-2\alpha}} = \sup_I \sum_{k \geq 0} 2^{(2\alpha - n)k} \sum_{J \in D_k(I)} \frac{1}{|J|} \sum_{j \geq -\log_2 l(J)} \|\Delta_j f\|_{L^2(J)}^2$$

Proof: For a fixed $I \subset \mathbb{R}^n$,

$$\sum_{k \geq 0} 2^{(2\alpha - n)k} \sum_{J \in D_k(I)} \frac{1}{|J|} \sum_{j \geq -\log_2 l(J)} \|\Delta_j f\|_{L^2(J)}^2$$

$$= \sum_{k \geq 0} \frac{1}{|I|^{2\alpha k}} \sum_{j \geq k - \log_2 l(I)} \|\Delta_j f\|_{L^2(I)}^2$$

If $f \in \mathcal{L}^{2,n-2\alpha}$. By Fubini theorem, we exchange the order of summation of above identity as follows:

$$\sum_{k \geq 0} \frac{1}{|I|^{2\alpha k}} \sum_{j \geq k - \log_2 l(I)} \|\Delta_j f\|_{L^2(J)}^2 = \frac{1}{|I|} \sum_{j \geq -\log_2 l(I)} \left( \sum_{k = 0}^{j + \log_2 l(I)} 2^{2\alpha k} \right) \|\Delta_j f\|_{L^2(I)}^2$$

$$\approx \frac{1}{|I|^{1-\frac{\alpha}{2}}} \sum_{j \geq -\log_2 l(I)} 2^{2\alpha j} \|\Delta_j f\|_{L^2(I)}^2$$

Then

$$\sum_{k \geq 0} 2^{(2\alpha - n)k} \sum_{J \in D_k(I)} \frac{1}{|J|} \sum_{j \geq -\log_2 l(J)} \|\Delta_j f\|_{L^2(J)}^2 = \frac{1}{|I|^{1-\frac{\alpha}{2}}} \sum_{j \geq -\log_2 l(I)} 2^{2\alpha j} \|\Delta_j f\|_{L^2(I)}^2 < \infty$$

(2.1)
On the other hand. If
\[
\sup_I \sum_{k \geq 0} 2^{(2\alpha-n)k} \sum_{J \in D_k(I)} \frac{1}{|J|^2} \sum_{j \geq -\log_2 l(J)} \|\Delta_j f\|^2_{L^2(J)} < \infty
\]
We have (2.1) is also valid by Fubini theorem. Then we complete the proof.

**Lemma 2.3** Let \( m \geq 2, \alpha > -\frac{n}{2} \). We have
\[
\sum_{k \geq 0} 2^{(2\alpha-n)k} \sum_{J \in D_k(I)} \frac{1}{|J|^2} \int_{mJ} \int_{mJ} |f(x) - f(y)|^2 dxdy \lesssim m^{2\alpha+2n} \|f\|^2_{L^2_a}
\]
for any fixed cube \( I \subset \mathbb{R}^n \).

**Proof:** If \( m \geq 2 \), We also adopt the idea of lemma 5.3 in [3] but need more complexity techniques. Observe that
\[
\sum_{k \geq 0} 2^{(2\alpha-n)k} \sum_{J \in D_k(I)} \frac{1}{|J|^2} \int_{mJ} \int_{mJ} |f(x) - f(y)|^2 dydx
\]
\[
= l(I)^{2n-2} \int_{mI} \int_{mI} k(x,y)|f(x) - f(y)|^2 dxdy
\]
And we have the following identity:
\[
k(x, y) = \sum_{J \in D(I)} \frac{\chi_{mJ(x)} \chi_{mJ(y)}}{l(J)^{2\alpha+n}}
\]
We let \( \Gamma \triangleq \{ J \in D(I) : x, y \in mJ \} \). Then we get the alternate expression of \( k(x, y) \):
\[
k(x, y) = \sum_{J \in \Gamma} \frac{1}{l(J)^{2\alpha+n}}
\]
It is crucial to estimate the magnitude of \( k(x, y) \).
To begin with, we give a definition of **allowed cubes**: Let \( J \) be an allowed cube if there is no such dyadic subcube \( J' \subset J \), such that \( J' \in \Gamma \). We note \( \Gamma^a \) be the set of allowed cubes.
We immediately conclude that all the allowed cubes disjoint each other.
We assert
\[
k(x, y) = \sum_{J \in \Gamma} \frac{1}{l(J)^{2\alpha+n}} \approx \sum_{J \in \Gamma^a} \frac{1}{l(J)^{2\alpha+n}}
\]
We now prove(2.3): First, it is trivial
\[
\sum_{J \in \Gamma} \frac{1}{l(J)^{2\alpha+n}} \geq \sum_{J \in \Gamma^a} \frac{1}{l(J)^{2\alpha+n}}.
\]
For any \( J \in \Gamma \), there exists only one sequence of dyadic cubes \( J_k (k = 1, ..., ) \), such that \( J \subset J_1 \subset J_2 \subset ... \), and \( J_k \in \Gamma \). We define a partial order "\(<" : J_1 < J_2 \) if and only if \( J_1 \subset J_2 \). Notice that \( \Gamma^a \) essentially correspond the equivalent class of \( \Gamma \). We denote \( T_{J_0} \) be the tree which contains \( J_0 \). We have the covering property:
\[
\bigcup_{J \in \Gamma} J \subset \bigcup_{J_0 \in \Gamma^a} \bigcup_{J_1 \in T_{J_0}} J_1
\]
By \( \alpha > -\frac{n}{2} \) we have following estimate:
\[
\sum_{J \in \Gamma} \frac{1}{l(J)^{2\alpha+n}} \leq \sum_{J_0 \in \Gamma^a} \sum_{J_1 \in T_{J_0}} \frac{1}{l(J)^{2\alpha+n}} \leq C(n, \alpha) \sum_{J_0 \in \Gamma^a} \frac{1}{l(J)^{2\alpha+n}}.
\]
This indicate (2.3) is valid.
Having established (2.3), we turn to estimate the magnitude of \( k(x, y) \). We denote an initial cube \( I_0 \) with the edge parallel to the coordinate axes and contains \( x, y \). The \( I_0 \) is fixed and set its length \( l(I_0) = \sqrt{n}|x - y| \). Here \( I_0 \) does not necessary belongs to \( D(I) \). We define a sequence of cubes \( I_k \) \((k = 0, 1, 2, \ldots)\) such that \( I_k = 2^k I_0 \).

Then we split \( \Gamma^n \) into two kinds of sets. First, we let

\[
\Gamma^{(1)}_0 \triangleq \{ J \in \Gamma^n : J \cap I_0 \neq \emptyset, J \subset I_1 \}.
\]

When \( k \geq 1 \), we define the following first kind of sets inductively:

\[
\Gamma^{(1)}_k \triangleq \{ J \in \Gamma^n : J \cap I_k \neq \emptyset, J \subset I_{k+1}, J \cap \cup_{j=0}^{k-1} I_j = \emptyset \}.
\]

We get first kind of sets by induction.

The second kind of sets are the complement of the first kind of sets counterpart. We construct these sets as follows: Let

\[
\Gamma^{(2)}_0 \triangleq \{ J \in \Gamma^n : J \cap I_0 \neq \emptyset, J \not\subset I_1 \},
\]

and also define:

\[
\Gamma^{(2)}_k \triangleq \{ J \in \Gamma^n : J \cap I_k \neq \emptyset, J \not\subset I_{k+1}, J \cap \cup_{j=0}^{k-1} I_j = \emptyset \}.
\]

The second kind of sets then given by induction.

We can immediately deduce

\[
\Gamma^n = \bigcup_{k \geq 0} \Gamma^{(1)}_k \bigcup \Gamma^{(2)}_k.
\]

By (2.3),

\[
k(x, y) \leq \sum_{k \geq 0} \left( \sum_{J \in \Gamma^{(1)}_k} \frac{1}{l(J)^{2\alpha+n}} + \sum_{J \in \Gamma^{(2)}_k} \frac{1}{l(J)^{2\alpha+n}} \right) = I + II. \tag{2.4}
\]

The estimate of \( I \):

For any cube \( J \in \Gamma^{(1)}_k \), let \( l_j \triangleq \min\{l(J) : J \in \Gamma^{(1)}_j\} \), \((j \geq 1)\). By geometric properties, and its definition, we know that the segment \([x, y]\) should be contained in \( mJ \). By definition of \( \Gamma^{(1)}_k \), we have \( \sqrt{n}l_0 \geq m^{-1}|x - y| \), and also \( \sqrt{n}l_1 \geq m^{-1}|x - y| \). Also, we know that \( mJ \) intersects the area of \( I_k \cap I_0 \) for \( k \geq 2 \). (See figure 1) Then we have

\[
ml_k \geq \frac{1}{2}(l(I_{k-1}) - l(I_0)) = \frac{2^{k-1} - 1}{2}l(I_0)
\]

Since all of the cubes in \( \Gamma^{(1)}_k \) contained in \( I_{k+1} \). We could calculate the number of elements in \( \Gamma^{(1)}_k \):

\[
\#\Gamma^{(1)}_k \leq \frac{l(I_{k+1})^n}{l_k^n} \leq C_1(n)m^n
\]

Thus the estimate of \( I \) is clear:

\[
I = \sum_{k \geq 0} \sum_{J \in \Gamma^{(1)}_k} \frac{1}{l(J)^{2\alpha+n}} \leq C_1(n)m^{2\alpha+2n} \sum_{k \geq 0} 2^{-2\alpha k - nk} |x - y|^{-2\alpha - n}
\]

Because \( \alpha > -\frac{n}{2} \). We could deduce

\[
I \leq n_1^{2\alpha+2n} |x - y|^{-2\alpha - n}. \tag{2.5}
\]

The estimate of \( II \):

For each \( J \in \Gamma^{(2)}_k \), notice that all of \( J \) intersect the area of \( I_{k+1} \cap I_0 \). We have \( l(J) \geq \frac{1}{2}(2^{k+1} - 2^k)l(I_0) \). The cross-sections \( R_k \) are rectangles have the mini-length greater than \( 2^{k-1}l(I_0) \), or at least contain a rectangle which has the mini-length greater than \( 2^{k-1}l(I_0) \). Also, \( J \in \Gamma^{(2)}_k \) disjoint
each other and therefore all of \( R_k \) are disjoint each other as well. (See figure 2) We immediately
obtain the number of elements in \( \Gamma_k^{(2)} \) satisfies:
\[
\# \Gamma_k^{(2)} \leq \max \left\{ \frac{|I_{k+1} \cap I_k^c|}{|R_k|} : R_k = J \cap I_{k+1} \cap I_k^c, J \in \Gamma_k^{(2)} \right\} \leq C_2(n).
\]
Thus we have the estimate of \( \Pi \):
\[
\Pi = \sum_{k \geq 0} \sum_{J \in \Gamma_k^{(2)}} \frac{1}{(J)^{2a+n}} \leq \sum_{k \geq 0} C_2(n)2^{-nk(2a+n)}|x-y|^{-2a-n}.
\]
Because \( \alpha > -\frac{3}{2} \). We have proved following estimate:
\[
\Pi \lesssim |x-y|^{-2a-n}. \quad (2.6)
\]
Combining estimates (2.3)(2.4)(2.5)(2.6), we get the desired conclusion by (1.1). Notice that if there
exists some \( k \) or \( j \) (\( j = 0, 1 \)) such that \( \Gamma_k^{(j)} = \emptyset \). It will lead the (2.4) be a lacunary series, and this
do not effect the correctness of the results. We then complete the proof of Lemma 2.4.

3 Proof of the main theorem

In the following discussion, all of the cube \( I \subset \mathbb{R}^n \) have the parallel to the coordinate axes edges.

The proof of statement: ”If \( f \in \mathcal{L}^{2,n-2a,\alpha} \) then \( \|f\|_{Q_\alpha} \lesssim \|f\|_{\mathcal{L}^{2,n-2a,\alpha}} \)”:

For \( f \in S'/P \), and for a fixed cube \( I \), we decompose \( f \) as follows:
\[
f = \sum_{j \in \mathbb{Z}} \Delta_j(f)(x) = \sum_{j < -\log_2 l(I)} \Delta_j(f)(x) + \sum_{j \geq -\log_2 l(I)} \Delta_j(f)(x)
\]
Then we have
\[
l(I)^{2a-n} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{2a+n}} dxdy 
\lesssim l(I)^{2a-n} \int_I \int_I \left| \sum_{j < -\log_2 l(I)} \Delta_j(f)(x) - \sum_{j \geq -\log_2 l(I)} \Delta_j(f)(y) \right|^2 \frac{|x-y|^{-2a-n}}{dxdy}
\]
\[
+ l(I)^{2a-n} \int_I \int_I \left| \sum_{j \geq -\log_2 l(I)} \Delta_j(f)(x) - \sum_{j < -\log_2 l(I)} \Delta_j(f)(y) \right|^2 \frac{|x-y|^{-2a-n}}{dx dy}
\]
\[
\triangleq \Pi + IV \quad (3.1)
\]

The estimate of \( \Pi \): In [2], we have known
\[
\sum_{j < -\log_2 l(I)} \max_{x \in I} |\partial_x \Delta_j f(x)| \leq \|f\|_{BMO(I)} l(I)^{-1}
\]
Combining the trivial property \( \mathcal{L}^{2,n-2a,\alpha} \subset BMO \) and the fact \( \alpha \in (0, 1) \). We have
\[
\Pi \leq \|f\|_{BMO(I)}^2 l(I)^{2a-n-2} \int_I \int_I |x-y|^{-2a-n} dxdy \lesssim \|f\|^2_{\mathcal{L}^{2,n-2a,\alpha}} \quad (3.2)
\]

The estimate of \( IV \):
First, we rewrite
\[
IV = l(I)^{2a-n} \int_{|y| \leq l(I)} \int_I \left| \sum_{j \geq -\log_2 l(I)} \Delta_j(f)(x) - \sum_{j < -\log_2 l(I)} \Delta_j(f)(y) \right|^2 \frac{|x+y|^{-2a-n}}{dy}
\]

6
The following arguments are rather standard as the proof of \( \|f\|_{L^2,\infty} \lesssim \|f\|_{L^{2,n-2 \alpha}} \) in [2], but need a slight modification.

There exists \( \hat{\theta}(\xi) \in C_0^\infty \) be a positive and radial function such that \( \hat{\theta}(x) \geq 1 \), for \( |x| \leq \frac{1}{2} \) and supported in \( \{ \xi \in \mathbb{R}^n : |\xi| \leq \frac{1}{2} \} \). We denote \( c(I) \) be the center of \( I \). Let

\[
\varphi_I(x) = l(I)^{\alpha-\gamma} \hat{\theta}(\pi \frac{x-c(I)}{l(I)})
\]

For this fixed cube \( I \), Schwartz function \( \varphi_I \) has the following properties:

\[
|\varphi_I(x)|^2 \geq CI(2^{\alpha} - n, x \in I
\]

\[
\text{supp} \varphi_I(\xi) \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq \frac{1}{2}l(I)^{-1} \}
\]

Then

\[
\mathbb{V} \lesssim \int |y| \leq l(I) \int_\mathbb{R}^n |\varphi_I(x)|^2 \sum_{j \geq \log_2 l(I)} \Delta_j(f)(x) - \sum_{j \geq \log_2 l(I)} \Delta_j(f)(x+y)|^2dx|y|^{-2\alpha-n}dy \tag{3.3}
\]

By Plancherel theorem,

\[
\int_\mathbb{R}^n |\varphi_I(x)|^2 \sum_{j \geq \log_2 l(I)} \Delta_j(f)(x) - \sum_{j \geq \log_2 l(I)} \Delta_j(f)(x+y)|^2dx
\]

\[
= \int_\mathbb{R}^n |\sum_{j \geq \log_2 l(I)} (\varphi_I(\xi) \ast \Delta_j(f)(\xi))|^2 |1 - e^{-2\pi i y \xi}|^2d\xi \tag{3.4}
\]

And because of \( |1 - e^{-2\pi i y \xi}| \leq \min\{2, C_{\mu_0} |y|^{\mu_0} |\xi|^{\mu_0} \} \). We note \( \mu_0 \) be a fixed positive number with \( \alpha < \mu_0 < 1 \). We have the fact

\[
\int |y| \leq l(I) \int_\mathbb{R}^n |\varphi_I(\xi) \ast \Delta_j(f)(\xi)|^2 \lesssim \int_\mathbb{R}^n |y|^{2\mu_0 - 2\alpha - n}dy + \int_{|y| \geq 1} |y|^{-2\alpha - n}dy \lesssim |\xi|^{2\alpha} \tag{3.5}
\]

We define another Littlewood-Paley operator:

\[
\hat{\Delta}_j(f)(\xi) = |2^{-j} \xi|^{\alpha} \hat{\psi}(2^{-j} \xi) \hat{f}(\xi) \tag{3.6}
\]

Because of the orthogonality property, we citing the following estimate in [2]

\[
| \sum_{j \geq \log_2 l(I)} (\varphi_I(\xi) \ast \hat{\Delta}_j(f)(\xi))|^2 \leq \sum_{j \geq \log_2 l(I)} |(\varphi_I(\xi) \ast \Delta_j(f)(\xi))|^2 \tag{3.7}
\]

Combining (3.3)(3.4)(3.5)(3.6)(3.7) as well as exchange the order of integration of (3.3), we have

\[
\mathbb{V} \lesssim \sum_{j \geq \log_2 l(I)} \int_\mathbb{R}^n |\varphi_I(\xi) \ast \Delta_j(f)(\xi)|^2 2^{2\alpha j}d\xi = \sum_{j \geq \log_2 l(I)} \int_\mathbb{R}^n \varphi_I(x) \Delta_j(f)(x)|^2 2^{2\alpha j}dx
\]

The following arguments are almost the same as in [2].

Denote \( k \in \mathbb{Z}^n \), \( a_k = \max\{|\hat{\theta}(x)|^2 : |x-k| \leq \frac{1}{2} \} \). We let \( Q_k \) be the disjoint cubes in \( \mathbb{R}^n \) have the center at \( l(I)k \) with the length of \( l(I) \). Then \( Q_k (k \in \mathbb{Z}^n) \) become the partition of \( \mathbb{R}^n \). We have

\[
\mathbb{V} \lesssim \sum_{k \in \mathbb{Z}^n} a_k \sum_{j \geq \log_2 l(I)} \frac{1}{|l|^{1-\frac{\alpha}{2}}} \int_{Q_k} |\Delta_j(f)(x)|^2 2^{2\alpha j}dx
\]

By the property of Schwartz function and Lemma 2.1 we have

\[
\mathbb{V} \lesssim \sup_I \frac{1}{|l|^{1-\frac{\alpha}{2}}} \sum_{j \geq \log_2 l(I)} 2^{2\alpha j} \|\Delta_j f\|^2_{L^2(I)} \lesssim \|f\|^2_{L^{2,n-2\alpha}}
\]
Combining above estimate and (3.1)(3.2), we have
\[ l(I)2^{a-n} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x - y|^{2a+n}} dx dy \lesssim \|f\|_{L^{2,\alpha}}^2 \]
for any fixed cube \( I \).

By (1.1) we complete the proof of \( \|f\|_{Q_\alpha} \lesssim \|f\|_{L^{2,\alpha}} \).

The proof of statement: "If \( f \in Q_\alpha(\mathbb{R}^n) \), then \( \|f\|_{L^{2,\alpha}} \lesssim \|f\|_{Q_\alpha} \)"

To begin with, by lemma 2.2, it suffices to show
\[ \sum_{k \geq 0} 2^{(2a-n)k} \sum_{J \in D_k(I)} \frac{1}{|J|} \sum_{j \geq -\log_2 l(I)} \|\Delta_J f\|_{L^2(J)}^2 \lesssim \|f\|_{Q_\alpha}^2 \quad (3.8) \]
for any fixed cube \( I \).

For any fixed subcube \( J \subset I \), we have the decomposition of \( f \) related to \( J \) as follows: \( f = (f - f_{2J})\chi_{2J} + (f - f_{2J})\chi_{2J}^c + f_{2J} \). Then we have the following decomposition:

\[
\sum_{k \geq 0} 2^{(2a-n)k} \sum_{J \in D_k(I)} \frac{1}{|J|} \sum_{j \geq -\log_2 l(I)} \|\Delta_J f\|_{L^2(J)}^2
\]
\[ \lesssim \sum_{k \geq 0} 2^{(2a-n)k} \sum_{J \in D_k(I)} \frac{1}{|J|} \sum_{j \geq -\log_2 l(I)} \|\Delta_J (f - f_{2J})\chi_{2J}\|_{L^2(J)}^2
\]
\[ + \sum_{k \geq 0} 2^{(2a-n)k} \sum_{J \in D_k(I)} \frac{1}{|J|} \sum_{j \geq -\log_2 l(I)} \|\Delta_J (f - f_{2J})\chi_{2J}^c\|_{L^2(J)}^2
\]
\[ + \sum_{k \geq 0} 2^{(2a-n)k} \sum_{J \in D_k(I)} \frac{1}{|J|} \sum_{j \geq -\log_2 l(I)} \|\Delta_J f_{2J}\|_{L^2(J)}^2 \triangleq \mathcal{V} + \mathcal{VII} + \mathcal{VI} \]

It is obviously that \( f_{2J} \) is a constant and we have \( \Delta_J (f_{2J}) = 0 \) for all the \( j \in \mathbb{Z} \) and all the subcube \( J \subset I \). Then we have \( \mathcal{VII} = 0 \). In order to prove (3.8), we only need to demonstrate \( \mathcal{V} \lesssim \|f\|_{Q_\alpha}^2 \) and also \( \mathcal{VI} \leq \|f\|_{Q_\alpha}^2 \).

The estimate of \( \mathcal{V} \):

By Plancherel theorem, we have
\[ \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} |\Delta_j (f - f_{2J})\chi_{2J}|^2 dx = \|(f - f_{2J})\chi_{2J}\|_{L^2}^2. \]

We can deduce
\[
\mathcal{V} \leq \sum_{k \geq 0} 2^{(2a-n)k} \sum_{J \in D_k(I)} \frac{1}{|J|} \sum_{j \geq -\log_2 l(I)} \int_{\mathbb{R}^n} |\Delta_j (f - f_{2J})\chi_{2J}|^2 dx
\]
\[ = \sum_{k \geq 0} 2^{(2a-n)k} \sum_{J \in D_k(I)} \frac{1}{|J|} \int_{2J} |f - f_{2J}|^2 dx \]

Then \( \mathcal{V} \lesssim \|f\|_{Q_\alpha}^2 \) follows by (2.2) with the case of \( m = 2 \).

The estimate of \( \mathcal{VI} \):

To start with, we assume \( x \in J \). We give the following arguments:
\[
|\Delta_j (f - f_{2J})\chi_{2J}^c(x)| = \left| \int_{\mathbb{R}^n} \psi_j(x - y)(f(y) - f_{2J})\chi_{2J}^c(y) dy \right|
\]
\[ \leq \sum_{l \geq 1} \int_{2^{l+1}J \cap (2^l J)^c} |\psi_j(x - y)||f(y) - f_{2J}| dy \quad (3.9) \]
Since $\psi$ is a Schwartz function, then $\psi$ descend faster than any polynomial. Let $M > 2\alpha + n$ be a fixed large number. We have

$$|\psi_j(x - y)| \leq C_M 2^{jn} (1 + |x - y|2^j)^{-M-n}$$

(3.10)

Notice that $|x - y| \geq 2^{j-1}l(J)$. By (3.9)(3.10), we have the Littlewood-Paley operator could be controlled by the mean oscillation:

$$|\Delta_j(f - f_{2J})\chi_{(2J)^c}(x)| \leq 2^{-jM}l(J)^{-M} \sum_{l \geq 1} 2^{-lM}(|f - f_{2J}|)_{2(l+1),J}$$

(3.11)

We could also deduce the following estimate by Cauchy-Schwarz inequality

$$\sum_{l \geq 1} 2^{-lM}(|f - f_{2J}|)_{2(l+1),J} \leq \left( \sum_{l \geq 1} 2^{-lM} \right)^{1/2} \left( \sum_{l \geq 1} 2^{-lM} (|f - f_{2J}|)_{2(l+1),J}^2 \right)^{1/2}$$

(3.12)

Combining (3.11)(3.12) and using the Jensen inequality, we get the estimate of $\forall I$ as follows:

$$\forall I \lesssim 2^{2(2\alpha - n)k} \sum_{j \in D_k(l)} \sum_{l \geq 1} 2^{-lM} \left( \frac{1}{|2^{l+1}J|} \int_{2^{l+1}J} |f - f_{2J}| dy \right)^2$$



Using the growth estimate provided in Lemma 2.3. The above summation could be exchanged and we could obtain

$$\forall I \lesssim \sum_{l \geq 1} 2^{-l(M-2\alpha-n)} \|f\|_{Q_\alpha}^2 \lesssim \|f\|_{Q_\alpha}^2$$

This completes the proof.

4 Remark

In fact, we have known that $Q_\alpha(\mathbb{R}^n) \subset L^{2,n-2\alpha,\alpha}$ for $-\infty < \alpha < \infty$. But $L^{2,n-2\alpha,\alpha} \subset Q_\alpha(\mathbb{R}^n)$ probably no longer available for $\alpha \geq 1$. That means if $\alpha \geq 1$, $f \in L^{2,n-2\alpha,\alpha}$. Then we cannot deduce $f(x)$ is a constant function. At least, if we let $\alpha = 1$, $n = 2$. We could easily construct a non-constant Sobolev function $f(x)$, such that $\partial_2 f(x) \in L^2(\mathbb{R}^2)$. For example, let $f(x)$ be a non-constant Schwartz function. By ([2], theorem 10), we know that $f \in L^{2,0,1}$.

References

[1] R. Aulaskari, J. Xiao, R. Zhao, On subspaces and subsets of BMOA and UBC. Analysis, 15 (1995), 101-121.

[2] A. El. Baraka, Littlewood-Paley characterization for Campanato spaces. J. Function Spaces and Applications 4, No.2 (2006), 193-220.

[3] G. Dafni, J. Xiao Some new tent spaces and duality theorems for fractional Carleson measures and $Q_\alpha(\mathbb{R}^n)$. J. Funct. Anal., 208 (2004), 377-422.

[4] M. Essén, S. Janson, Lizhong. Peng, Jie. Xiao, Q Spaces of several real variables. Indiana Univ. Math. J. 49, No.2 (2000), 575-615.

[5] M. Essén, J. Xiao, Some results on $Q_p$ spaces, $0 < p < 1$. J. Reine Angew. Math. 485 (1997), 173-195.
[6] H. Kozono and M. Yamazaki, Semilinear heat equations and the Navier-Stokes equations with distributions in new function spaces as initial data. Comm. Partial Differential Equations, 19 (1994), 959-1014.

[7] J. Xiao, Homothetic variant of fractional Sobolev space with application to Navier-Stokes system. Dynamics of P.D.E. 4 (2007), 227-245.