POINTLIKE SETS WITH RESPECT TO ER

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Abstract. We show that pointlike sets are decidable for the pseudovariety of finite semigroups whose idempotent-generated subsemigroup is $R$-trivial. Notably, our proof is constructive: we provide an explicit relational morphism which computes the ER-pointlike subsets of a given finite semigroup.

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1. Introduction

Let $V$ be a pseudovariety. A non-empty subset $X$ of a finite semigroup $S$ is said to be $V$-pointlike if for any relational morphism $\rho : S \rightarrow V$ with $V \in V$ there exists $v \in V$ for which $X \subseteq (v)^{-1} \rho^{-1}$. If there is an algorithm which produces the $V$-pointlike subsets of any finite semigroup given as input, then $V$ is said to have decidable pointlikes. For background on pointlike sets, see [6, 2].

The main result of this paper (Theorem 8.7) is that the pseudovariety

$$ER = \{ S \in \text{FinSgp} \mid \langle E(S) \rangle \in R \}$$

has decidable pointlikes, where $R$ is the pseudovariety of $R$-trivial semigroups. Moreover, this result is proven constructively; that is, for any finite semigroup $S$, we can explicitly construct a relational morphism from $S$ to a member of $ER$ which computes the ER-pointlike subsets of $S$.

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1.1. **Organization of paper.** Section 2 covers various preliminary notions and establishes basic notation. Basic facts about ER—particularly relating to group kernels and the type-II partition—are covered in Section 3.

Section 4 provides an overview of various key notions from the authors’ framework for pointlike sets—which was developed in [2]—and establishes the language used in the rest of the paper. This framework is particularized in Section 5 to define a candidate definition for ER-pointlikes and to prove that said candidate provides a lower bound.

Section 6 establishes a number of preliminary results which are necessary for the construction defined in Section 7. The primary data of said construction—and the central “conceptual contribution” of the paper—is an automaton whose transition semigroup is shown in Section 8 to belong to ER.\(^1\) This establishes by way of the previous two sections that our lower bound for ER-pointlikes is an upper bound as well, which in turn establishes the main result (Theorem 8.7).

2. **Preliminaries**

2.1. **Familiarity with finite semigroup theory and basic category theory** is assumed; for reference, the reader is directed to [3] (for finite semigroup theory) and [5] (for category theory).

2.2. **Notation.** Let \(S\) be a finite semigroup.
   
   * Write \(S^1\) for the semigroup obtained by adjoining a new element \(I\) to \(S\) and defining \(xI = lx = x\) for all \(x \in S\).
   * Let \(E(S)\) denote the set of idempotents of \(S\).
   * Given \(x \in S\), let \(x^{\omega}\) denote the unique idempotent generated by \(x\).
   * Green’s equivalence relations are denoted by \(R, L, H, J\); moreover, the various Green’s equivalence classes of \(x \in S\) are denoted by \(R_x, L_x, H_x, J_x\), respectively.

2.3. **Partial transformation semigroups.** A (finite) partial transformation semigroup, which we will abbreviate as PTS, is a pair \((Q, S)\) consisting of a finite set \(Q\) and a finite semigroup \(S\) which acts on the right of \(Q\) by partial functions. If \(q \ast s\) is undefined for some \(q \in Q\) and some \(s \in S\), we will write \(q \ast s = \emptyset\).

   A PTS morphism \((\zeta, \varphi) : (Q, S) \rightarrow (P, T)\) is given by a pair
   
   \[ \zeta : Q \longrightarrow P \quad \text{and} \quad \varphi : S \longrightarrow T \]

   where \(\zeta\) is a set function and \(\varphi\) is a morphism of semigroups such that
   
   \[ (q)\zeta \ast (s)\varphi = (q \ast s)\zeta \]

   for all \(q \in Q\) and all \(s \in S\) for which \(q \ast s \neq \emptyset\).

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\(^1\)This automaton—which the first author has nicknamed **PFFL** (*Permute First, Fall Later*)—could be seen as an evolution of a construction used in [4].
A congruence on a PTS \((Q, S)\) is an equivalence relation \(\equiv\) on \(Q\) such that
\[
q_1 \equiv q_2 \implies q_1 * s \equiv q_2 * s
\]
for all \(q_1, q_2 \in Q\) and all \(s \in S\) for which both \(q_1 * s\) and \(q_2 * s\) are defined. Such a congruence \(\equiv\) on \((Q, S)\) induces a quotient PTS
\[
(Q, S)/\equiv = (Q/\equiv, S)
\]
wherein the action is given by
\[
[q]_{\equiv} * s = \begin{cases} [q * s]_{\equiv}, & \text{if } q' * s \neq \emptyset \text{ for some } q' \equiv q; \\ \emptyset, & \text{otherwise;} \end{cases}
\]
for each \(q \in Q\) and each \(s \in S\).

A PTS \((Q, S)\) is said to be injective if \(S\) acts on \(Q\) by partial injections. Moreover, a congruence \(\equiv\) on a PTS \((Q, S)\) is said to be injective if \((Q, S)/\equiv\) is injective.

A (finite) transformation semigroup is a PTS \((Q, S)\) wherein the action of \(S\) on \(Q\) is by fully defined functions on \(Q\). If a transformation semigroup \((Q, S)\) satisfies the condition that
\[
q * s_1 = q * s_2 \text{ for all } q \in Q \implies s_1 = s_2
\]
for all \(s_1, s_2 \in S\), then \((Q, S)\) is said to be faithful.

2.4. Activators. Let \(J\) be a \(J\)-class of a finite semigroup \(S\). The set
\[
\{ a \in S^I \mid J(a) \cap J \neq \emptyset \}
\]
is a union of \(J\)-classes of \(S^I\) which contains a unique \(\leq_J\)-minimal \(J\)-class. Said minimal \(J\)-class—which is always regular—is called the right activator of \(J\) and will be denoted by \(\text{RA}(J)\). Note that \(J\) is regular if and only if \(\text{RA}(J) = J\).

2.5. Lemma. Let \(J\) be a \(J\)-class of a finite semigroup \(S\). For each \(x \in J\), there exists an element \(t \in \text{RA}(J)\) such that
\[
(1) \ xt = x; \\
(2) \ xs <_R x \text{ if and only if } ts <_R t \text{ for any } s \in S; \text{ and} \\
(3) \text{ left multiplication by } x \text{ defines a surjective function from the } R\text{-class of } t \text{ onto the } R\text{-class of } x.
\]
In fact, \(t\) may be chosen to be an idempotent.

Proof. See [4, Lemma 2.9]. \qed

2.6. Given \(x \in S\), let \(\mathcal{F}_x\) denote the set of elements in \(\text{RA}(J_x)\) which satisfy the claims of Lemma 2.5. Note that if \(x\) is regular then \(\mathcal{F}_x\) is the set of idempotents which are \(L\)-equivalent to \(x\).
2.7. **Relational morphisms.** A relational morphism \( \rho : S \to T \) is an equivalence class of spans in the category of finite semigroups of the form

\[
\begin{array}{ccc}
\cdot & \longrightarrow & T \\
\downarrow & & \\
S
\end{array}
\]

where the map to \( S \) is a regular epimorphism,\(^2\) and where two such spans are equivalent if the natural maps from each apex to \( S \times T \) have the same image.

2.8. **Pseudovarieties.** A pseudovariety is a class of finite semigroups which is closed under taking subsemigroups, quotients, and finite products of its members.

2.9. **Power semigroups.** Given a finite semigroup \( S \), let \( \mathcal{P}_1(S) \) denote the semigroup of non-empty subsets of \( S \) under the inherited operation given by

\[
X \cdot Y = \{xy \mid x \in X, y \in Y\}
\]

for all non-empty subsets \( X \) and \( Y \) of \( S \); also, let \( \text{sing}(S) \) denote the subsemigroup of \( \mathcal{P}_1(S) \) consisting of the singletons.

A morphism \( \varphi : S \to T \) extends to a morphism

\[
\hat{\varphi} : \mathcal{P}_1(S) \longrightarrow \mathcal{P}_1(T) \quad \text{given by} \quad (X)\hat{\varphi} = \{(x)\varphi \mid x \in X\}.
\]

Equipping the object map \( \mathcal{P}_1 \) with this action on morphisms yields a functor

\[
\mathcal{P}_1 : \text{FinSgp} \longrightarrow \text{FinSgp}
\]

which creates monomorphisms, regular epimorphisms, and isomorphisms.

2.10. **Pointlikes.** Given a finite semigroup \( S \) and a pseudovariety \( \mathcal{V} \), a non-empty subset \( X \subseteq S \) is said to be \( \mathcal{V} \)-pointlike if for any relational morphism of the form \( \rho : S \to \mathcal{V} \) with \( \mathcal{V} \in \mathcal{V} \) there exists some element \( v \in \mathcal{V} \) for which \( X \subseteq (v)\rho^{-1} \).

The set of \( \mathcal{V} \)-pointlike subsets of \( S \) is denoted by \( \mathcal{P}_V(S) \), and is in fact a subsemigroup of \( \mathcal{P}_1(S) \) which contains \( \text{sing}(S) \) and which is closed under taking non-empty subsets of its members. Equipping this object map with the action on morphisms sending \( \varphi : S \to T \) to the evident restriction of the extension described in 2.9 yields a subfunctor

\[
\mathcal{P}_V : \text{FinSgp} \longrightarrow \text{FinSgp}
\]

of \( \mathcal{P}_1 \) with the property that a finite semigroup \( S \) belongs to \( \mathcal{V} \) if and only if \( \mathcal{P}_V(S) = \text{sing}(S) \).\(^3\) Pointlike functors also create monomorphisms, regular epimorphisms, and isomorphisms.

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\(^2\)Here and throughout, “regular epimorphism” means “surjective homomorphism”.

\(^3\)The notation for \( \mathcal{P}_1 \) is due to it being the pointlikes functor for the trivial pseudovariety \( \mathcal{1} \).
3. Group kernels and the type-II partition

3.1. Group kernels. Let S be a finite semigroup. Recall that the group kernel of S is the subsemigroup \( K_G(S) \) consisting of those elements which are always contained in the inverse image of the identity under any relational morphism from S to a finite group. That is, \( x \in K_G(S) \) if and only if \( x \in (1_G)p^{-1} \) for any relational morphism \( p : S \to G \) with \( G \in G \).

It is well-known that \( K_G(S) \) is the smallest subsemigroup of S for which \( E(S) \subseteq K_G(S) \) and such that if \( s \in K_G(S) \) and \( x, y \in S \) with \( xyx = x \), then \( xsy, ysx \in K_G(S) \) as well.

3.2. Lemma. The object map \( K_G \) is a functor when equipped with the evident restriction action on morphisms. Moreover, \( K_G \) preserves regular epimorphisms.

Proof. See [3, Proposition 4.12.6].

3.3. Definition. The (right-sided) type-II partition on S is defined by

\[ x \equiv II y \iff xa = y \text{ and }yb = x \text{ for some } a, b \in K_G(S) \]

for all \( x, y \in S \). Let \( [x]_II \) denote the type-II equivalence class of \( x \in S \).

3.4. It is easy to see that \( \equiv II \) is contained in \( R \). Given an \( R \)-class \( R \) of S, consider the PTS \( (R, S) \). The relation \( \equiv II \) is a congruence on \( (R, S) \)—that is, given \( x, y \in R \) with \( x \equiv II y \), then \( xs \equiv II ys \) for any \( s \in S \) such that both \( xs \) and \( ys \) are defined. Let \( (R/II, S) \) denote the quotient of \( (R, S) \) by \( \equiv II \). Crucially, \( (R/II, S) \) is an injective PTS—in fact, \( \equiv II \) is the minimal injective congruence on \( (R, S) \).

3.5. Lemma. Let \( R \) be an \( R \)-class of a finite semigroup \( S \). Then \( \equiv II \) is the minimal injective congruence on \( (R, S) \).

Proof. See [7].

3.6. Lemma. If \( R \) is an \( R \)-class of \( S \), then

1. \( K_G(S) \cap R \) is either empty or a II-class of \( R \), and
2. any \( a \in K_G(S) \) acts as a partial identity in \( (R/II, S) \).

Proof. Straightforward.

3.7. Lemma. Let \( x \in S \) and let \( t \in \mathcal{T}_x \). Then

1. \( x \cdot [t]_II = [x]_II \);  
2. if \( s \not\in R \) then \( x \cdot [s]_II = [xs]_II \); and
3. there is a PTS morphism

\[ (x \cdot (-), 1_S) : (R_t/II, S) \to (R_x/II, S), \]

sending \( [s]_II \in R_t/II \) to \( [xs]_II \) and acting as identity on S.

Proof. Straightforward.
3.8. **Lemma.** A finite semigroup $S$ belongs to $\text{ER}$ if and only if $(R, S)$ is an injective PTS for every $R$-class $R$ of $S$. Moreover,

$$\text{ER} = R * G = R \, \, \text{ref} \, \, G.$$  

*Proof.* See [3, Theorem 4.8.3]. \qed

4. General theory of pointlike sets

4.1. In this section we will briefly cover key aspects of the authors’ “general theory of pointlike sets” which provides the framework for our work here. The treatment here is incomplete and proofs are omitted; for further details, see [2].

4.2. **Semigroup complexes.** Let $S$ be a finite semigroup. An $S$-complex is a subsemigroup $K \subseteq \mathcal{P}_1(S)$ which

1. contains $\text{sing}(S)$, and which
2. is closed under taking non-empty subsets of its members, i.e., if $X \in K$ then any $Y \in \mathcal{P}_1(S)$ for which $Y \subseteq X$ also belongs to $K$.

The set of $S$-complexes—denoted by $\hat{\Delta}_S$—is a complete lattice wherein the order is inclusion, the top and bottom are $\mathcal{P}_1(S)$ and $\text{sing}(S)$ respectively, the meet is intersection, and the join is given by

$$K_1 \lor K_2 = \{ X \in \mathcal{P}_1(S) \mid \exists \bar{X} \in \langle K_1 \cup K_2 \rangle \}$$

for any $K_1, K_2 \in \hat{\Delta}_S$.

4.3. **Definition.** A **modulus** $\Lambda$ is a rule which assigns to each finite semigroup $S$ a set $\Lambda_S \subseteq \mathcal{P}_1(S)$ in a manner which satisfies the following axioms.

1. If $\varphi : S \rightarrow T$ is a morphism, then for any $X \in \Lambda_S$ there exists some $\bar{X} \in \Lambda_T$ such that $(\bar{X}) \varphi \subseteq \bar{X}$.
2. If $\varphi : S \rightarrow T$ is a regular epimorphism, then for any $Y \in \Lambda_T$ there exists some $\bar{Y} \in \Lambda_S$ such that $(\bar{Y}) \varphi = Y$.

When defining moduli, we will generally write

$$\Lambda = [S \vdash \Lambda_S],$$

to mean “$\Lambda$ is the rule which assigns $\Lambda_S$ to a given finite semigroup $S$”.

4.4. **Constructing lower bounds for pointlikes.** Given a modulus $\Lambda$, the $\Lambda$-**construct** of a finite semigroup $S$ is the $S$-complex defined by

$$\mathcal{C}_\Lambda(S) = \bigcap \{ K \in \hat{\Delta}_S \mid \text{if } \bar{X} \in \Lambda_{\bar{X}}, \text{ then } \bigcup \bar{X} \in \bar{K} \};$$

that is, $\mathcal{C}_\Lambda(S)$ is the minimal $S$-complex closed under unioning subsets assigned to it by the modulus $\Lambda$. Equipping the object map $\mathcal{C}_\Lambda$ with the action on morphisms sending $\varphi : S \rightarrow T$ to the extension

$$\bar{\varphi} : \mathcal{C}_\Lambda(S) \rightarrow \mathcal{C}_\Lambda(T) \quad \text{given by} \quad (X) \bar{\varphi} = (x) \varphi \mid x \in X$$
yields a functor which, moreover, admits a monad structure \((\mathcal{C}_\Lambda, \sigma_\Lambda, \mu_\Lambda)\), where the components of the unit \(\sigma_\Lambda : \mathbf{1}_{\text{FinSgp}} \Rightarrow \mathcal{C}_\Lambda\) are the singleton embeddings
\[
\sigma_\Lambda, S = (-) : S \hookrightarrow \mathcal{C}_\Lambda(S) \quad \text{given by} \quad x \mapsto \{x\}
\]
and the components of the multiplication \(\mu_\Lambda : \mathcal{C}_\Lambda^2 \Rightarrow \mathcal{C}_\Lambda\) are the union maps
\[
\mu_\Lambda, S = \bigcup(-) : \mathcal{C}_\Lambda^2(S) \longrightarrow \mathcal{C}_\Lambda(S) \quad \text{given by} \quad X \mapsto \bigcup_{X \in X} X
\]
for every finite semigroup \(S\).

The set of points of a modulus \(\Lambda\), which is defined by
\[
pt[\Lambda] = \{S \in \text{FinSgp} \mid \Lambda_S \subseteq \text{sing}(S)\},
\]
is a pseudovariety (see \([2, \text{Proposition 9.7}]\)) with the additional property that \(S \in pt[\Lambda]\) if and only if \(\mathcal{C}_\Lambda(S) = \text{sing}(S)\).

This concept’s utility comes from \([2, \text{Theorem 9.12}]\), which states that if \(\Lambda\) is a modulus with \(pt[\Lambda] = V\), then \(\mathcal{C}_\Lambda(S) \subseteq \mathcal{P}_V(S)\) for all \(S \in \text{FinSgp}\).

4.5. Notation. If \(\Lambda\) is a modulus and \(V\) is a pseudovariety, we write \(\mathcal{C}_\Lambda \subseteq \mathcal{P}_V\) to indicate that \(\mathcal{C}_\Lambda(S) \subseteq \mathcal{P}_V(S)\) for all finite semigroups \(S\), and we write \(\mathcal{P}_V \subseteq \mathcal{C}_\Lambda\) to mean the evident analogous statement.

5. Lower bound

5.1. Modulus. Define a modulus \(\Lambda_{\text{ER}}\) by
\[
\Lambda_{\text{ER}} = [S \vdash [e]_\Pi \mid e \in E(S)]
\]
and let \(\mathcal{C}_{\text{ER}}(S)\) denote the \(\Lambda_{\text{ER}}\)-construct \((4.4)\) of a given finite semigroup \(S\).

5.2. Lemma. The rule \(\Lambda_{\text{ER}}\) is a modulus.

Proof. Let \(S\) be a finite semigroup. Lemma 3.6 implies that \([e]_\Pi = \mathcal{K}_G(S) \cap R_e\) for any \(e \in E(S)\), from which it follows by basic stuff that \([e]_\Pi\) is a regular \(\mathcal{K}\)-class of \(\mathcal{K}_G(S)\). Since \(\mathcal{K}_G\) is an endofunctor which preserves regular epimorphisms by Lemma 3.2, the required axioms are easily verified. \(\square\)

5.3. Proposition. There is an equality \(\text{ER} = pt[\Lambda_{\text{ER}}]\), and thus \(\mathcal{C}_{\text{ER}} \subseteq \mathcal{P}_{\text{ER}}\).

Proof. Let \(S\) be a finite semigroup. By Lemma 3.8, \(S \in \text{ER}\) if and only if the PTS \((R, S)\) is injective for every \(\mathcal{K}\)-class \(R\) of \(S\). Since \(\equiv_\Pi\) is the minimal injective congruence on each \((R, S)\) by Lemma 3.5, this condition is equivalent to the condition that \(\equiv_\Pi\) is the identity relation on all of \(S\).

It is clear that if \([x]_\Pi = [x]\) for all \(x \in S\) then \(S \in pt[\Lambda_{\text{ER}}]\), from which it follows that \(\text{ER} \subseteq pt[\Lambda_{\text{ER}}]\).

For the converse, suppose that \(S \in pt[\Lambda_{\text{ER}}]\). Given \(x \in S\), Lemma 3.7 states that there exists \(e \in E(S)\) such that \(x \cdot [e]_\Pi = [x]_\Pi\). But ex hypothesis \([e]_\Pi = (e)\), and hence \([x]_\Pi = [x]\) as well. Thus \(pt[\Lambda_{\text{ER}}] \subseteq \text{ER}\) and the desired equality holds.

The conclusion then follows from \([2, \text{Theorem 9.12}]\) (as discussed in 4.4). \(\square\)
6. Type-II blocks of pointlike sets

6.1. For the remainder of the paper, fix a finite semigroup $S$.

6.2. **Definition.** Define a map $\beta : \mathcal{C}_{ER}(S) \rightarrow \mathcal{C}_{ER}(S)$ by setting

$$(X)\beta = \bigcup [X]_{II}$$

for each $X \in \mathcal{C}_{ER}(S)$.

6.3. **Lemma.** The semigroup $\mathcal{C}_{ER}(S)$ is closed under the action of $\beta$.

*Proof.* Let $X \in \mathcal{C}_{ER}(S)$. By Lemma 3.7 there is an idempotent $E \in \mathcal{C}_{ER}(S)$ such that $[X]_{II} = X \cdot [E]_{II}$. But $\bigcup [E]_{II} = (E)\beta$ is an element of $\mathcal{C}_{ER}(S)$ by the definition of $\Lambda_{ER}$, and thus $(X)\beta = X \cdot (E)\beta$ belongs to $\mathcal{C}_{ER}(S)$ as well. \( \square \)

6.4. **Lemma.** If $X \in \mathcal{C}_{ER}(S)$, then $X \subseteq (X)\beta$ and $(X)\beta \leq \mathcal{R} X$. In particular, if $A \in \mathcal{F}_X$ then $(X)\beta = X \cdot (A)\beta$.

*Proof.* It is obvious that $X \subseteq (X)\beta$. As for the second claim, Lemmas 2.5 and 3.7 guarantee the existence of some $A \in \mathcal{F}_X$ such that $[X]_{II} = X \cdot [A]_{II}$, from which it follows that $(X)\beta = X \cdot (A)\beta$. \( \square \)

6.5. **Lemma.** Let $E \in \mathcal{C}_{ER}(S)$ be an idempotent. Then

1. $(E)\beta$ is aperiodic, that is, $(E)\beta)^{\omega+1} = (E)\beta)^{\omega}$; and
2. if $(E)\beta)^{2} \mathcal{H} (E)\beta$ then $(E)\beta)^{2} = (E)\beta$.

*Proof.*

1. Note that $E \cdot (E)\beta = (E)\beta$ and $E \subseteq (E)\beta$. Hence

$$(E)\beta = E \cdot (E)\beta \subseteq (E)\beta \cdot (E)\beta;$$

and, more generally, if $(E)\beta)^{k-1} \subseteq (E)\beta)^{k}$ then

$$(E)\beta)^{k} = (E)\beta)^{k-1} \cdot (E)\beta \subseteq (E)\beta)^{k} \cdot (E)\beta = (E)\beta)^{k+1}.$$  

Hence, given a number $p$ such that $(E)\beta)^{\omega+p} = (E)\beta)^{\omega}$, one has that

$$(E)\beta)^{\omega} \subseteq (E)\beta)^{\omega+1} \subseteq \cdots \subseteq (E)\beta)^{\omega+p-1} \subseteq (E)\beta)^{\omega+p} = (E)\beta)^{\omega},$$

from which we conclude that $(E)\beta)^{\omega+1} = (E)\beta)^{\omega}$.

2. Since $(E)\beta$ is aperiodic, the $\mathcal{H}$-class of $(E)\beta)^{\omega}$ is a single point. Hence if $(E)\beta)^{2} \mathcal{H} (E)\beta$, then $(E)\beta \mathcal{H} (E)\beta)^{\omega}$ and so $(E)\beta)^{2} = (E)\beta$.

\( \square \)

6.6. For each $X \in \mathcal{C}_{ER}(S)$, choose an idempotent $E_X \in \mathcal{F}_X$ such that if $F$ is an idempotent then $E_F = F$.

6.7. **Definition.** Define a map

$$\psi : \mathcal{C}_{ER}(S) \longrightarrow \mathcal{C}_{ER}(S) \quad \text{given by} \quad (X)\psi = X \cdot (E_X \beta)^{\omega}.$$  

Note that if $E$ is an idempotent then $(E)\psi = (E)\beta)^{\omega}$. 
6.8. Lemma. Let \( X \in C_{ER}(S) \). Then

- (1) \((X)\psi \leq X\);
- (2) \((X)\psi = (X)\beta \cdot (E_X)\psi \) and \( X \subseteq (X)\psi \);
- (3) \(\psi\) is aperiodic, that is, \(\psi^{\omega+1} = \psi^{\omega}\);
- (4) if \( X \not\subseteq (X)\psi \) then \( X \equiv (X)\psi \); and
- (5) \((X)\beta = X\) if and only if \((X)\psi = X\).

Proof.

(1) Obvious.

(2) Since \((E_X)\beta^{\omega+1} = (E_X)\beta^{\omega}\) by Lemma 6.5,

\[
(X)\psi = X \cdot (E_X)\beta^{\omega} = X \cdot (E_X)\beta \cdot (E_X)\beta^{\omega} = (X)\beta \cdot (E_X)\beta^{\omega}.
\]

It follows that

\[
X = X \cdot E_X \subseteq (X)\beta \cdot (E_X)\beta^{\omega} = (X)\psi
\]

since \( X \subseteq (X)\beta \) and \( E_X \subseteq (E_X)\beta^{\omega} \).

(3) Straightforward.

(4) Since \((E_X)\psi = (E_X)\beta^{\omega}\) is an idempotent, it belongs to \( K_{G}(C_{ER}(S)) \). Hence if \((X)\psi \not\subseteq E_X\beta \) then

\[
[(X)\psi]_{II} = [X \cdot (E_X)\psi]_{II} = [X]_{II} \cdot (E_X)\psi = [X]_{II}
\]

since members of \( K_{G}(C_{ER}(S)) \) act as identity on II-classes when defined.

(5) For the “only if” direction, observe that

\[
(X)\beta = X \cdot (E_X)\beta = X \implies X \cdot (E_X)\beta^{k} = X
\]

for all \( k \geq 1 \), and hence \((X)\psi = X \cdot (E_X)\beta^{\omega} = X\).

As for the “if” direction, it follows from Lemma 2.5 that if \((X)\psi = X\) then \((E_X)\psi \not\subseteq E_X\beta \). Consequently,

\[
(E_X)\psi = (E_X)\beta^{\omega} = (E_X)\beta^{2} = (E_X)\beta
\]

by way of Lemma 6.5. Therefore

\[
(X)\beta = X \cdot (E_X)\beta = X \cdot (E_X)\psi = (X)\psi = X,
\]

at which point all desired claims have been established.

6.9. Definition. For each \( X \in C_{ER}(S) \), let \( \overline{X} = (X)\psi^{\omega} \).

6.10. Fixed point sets. Let \( F \) denote the set of fixed points of \( \psi \); that is, let

\[
F = \{ X \in C_{ER}(S) \mid (X)\psi = X \} = \{ \overline{X} \mid X \in C_{ER}(S) \},
\]

which is also the set of fixed points of \( \beta \). Moreover, define

\[
B = \{ [X]_{II} \mid X \in F \} = \{ \pi \in C_{ER}(S)/II \mid \bigcup \pi \in \pi \}.
\]
6.11. Lemma. Let \( X \in F \). If \( Y \in \mathcal{C}_{ER}(S) \) for which \( (XY)\psi \not\preceq XY \not\preceq X \), then it follows that \( [XY] \not\equiv [X] + Y \in \mathcal{B} \).

Proof. It follows from claims (2) and (4) of Lemma 6.8 that
\[
(XY)\psi \not\preceq XY \not\preceq X \implies XY \not\equiv [XY] + \mathcal{E}_{\mathcal{B}}(XY) \beta
\]
since \( (XY)\psi = (XY)\beta \cdot (E_{XY}\beta)^{\omega} \) (and since idempotents act as partial identity on \( \Pi \)-blocks), and so \( [XY] \not\equiv [X] + Y \in \mathcal{B} \). \( \square \)

7. Automaton and flow

7.1. An automaton is the data of a tuple \( A = (\Sigma, Q, I, \circ) \), where

1. \( \Sigma \) is a finite set of input symbols,
2. \( Q \) is a finite set of states,
3. \( I \in Q \) is a distinguished initial state, and
4. \( (\circ \circ -) : Q \times \Sigma \longrightarrow Q \) is a set function called the transition function.

The transition semigroup of \( A \) is the semigroup \( \mathcal{T}_A \) which is generated by the functions \( (\circ \circ a) : Q \longrightarrow Q \) induced by each \( a \in \Sigma \).

7.2. Definition. A flow automaton is a triple \( (S, A, \Phi) \) where \( S \) is a finite semigroup, \( A = (S, Q, I, \circ) \) is an automaton, and \( \Phi \) (the nominal flow) is a set function
\[
\Phi : Q \setminus \{i\} \longrightarrow \mathcal{P}(S),
\]
such that
\[
s \in (I \circ s)\Phi \quad \text{and} \quad (q)\Phi \cdot \{s\} \subseteq (q \circ s)\Phi
\]
for all \( s \in S \) and all \( q \in Q \setminus \{i\} \).

7.3. The cover complex of a flow automaton \( (S, A, \Phi) \) is defined by
\[
\text{Cov}(S, A, \Phi) = \{ X \in \mathcal{P}(S) | X \subseteq (q)\Phi \text{ for some } q \in Q \setminus \{i\} \}.
\]

It is straightforward to see that
\[
\mathcal{P}_V(S) = \bigcap \{ \text{Cov}(S, A, \Phi) | \mathcal{T}_A \in V \}
\]
for any pseudovariety \( V \) (see [1, Proposition 2.5]).

7.4. We will prove that \( \mathcal{P}_V(S) \subseteq \mathcal{C}_{ER}(S) \) by defining a flow automaton
\[
(S, A(S), \Phi) \quad \text{where} \quad A(S) = (S, Q(S), I, \circ)
\]
such that \( \text{Cov}(S, A(S), \Phi) \subseteq \mathcal{C}_{ER}(S) \) and \( \mathcal{T}_{A(S)} \in \mathcal{E}_{ER} \).
7.5. **Local group actions.** Let \( R \) be an \( \mathcal{R} \)-class of \( \mathcal{C}_{\mathcal{E}R}(S) \), and consider the PTS \((R/\mathcal{I}, S)\) (which is isomorphic to \((R/\mathcal{I}, \text{sing}(S))\) in the evident manner). For each \( s \in S \), extend the partial injection \((-) \ast s\) to a permutation \( g_{(s, R)} \) on \( R/\mathcal{I} \), whose action is written as

\[
[X]_{\mathcal{I}} \xrightarrow{g_{(s, R)}} [X]_{\mathcal{I}} \circ g_{(s, R)}
\]

for each \( [X]_{\mathcal{I}} \in R/\mathcal{I} \), and which has the property that

\[
[X]_{\mathcal{I}} \ast s \neq \emptyset \implies [X]_{\mathcal{I}} \circ g_{(s, R)} = [X]_{\mathcal{I}} \ast s
\]

for all \( [X]_{\mathcal{I}} \in R/\mathcal{I} \). Let \( G_{\mathcal{R}} \) denote the group of permutations of \( R/\mathcal{I} \) generated by the various \( g_{(s, R)} \) as \( s \) ranges over \( S \).

7.6. **Global group actions.** Let \( \mathcal{R} \) denote the set of \( \mathcal{R} \)-classes of \( \mathcal{C}_{\mathcal{E}R}(S) \) which contain some member of \( \mathcal{F} \); that is,

\[
\mathcal{R} = \{ R_x \mid (X)\psi = X \}.
\]

For each \( s \in S \), let \( g_s = (g_{(s, R)})_{R \in \mathcal{R}} \). Moreover, let

\[
G = \{ g_s \mid s \in S \};
\]

that is, \( G \) is the subsemigroup of \( \prod_{R \in \mathcal{R}} G_R \) generated by the various \( \mathcal{R} \)-tuples \( g_s \) as \( s \) ranges over \( S \). The group \( G \) acts on \( \bigcup_{R \in \mathcal{R}} R/\mathcal{I} \) by

\[
[X]_{\mathcal{I}} \circ g = [X]_{\mathcal{I}} \circ g_{R_x}
\]

for all \( [X]_{\mathcal{I}} \in \bigcup_{R \in \mathcal{R}} R/\mathcal{I} \) and all \( g = (g_R)_{R \in \mathcal{R}} \in G \).

7.7. **States.** The state set of our automaton will be given by

\[
Q(S) = \{ (X, d, g) \in \mathcal{F} \times G \times G \mid [X]_{\mathcal{I}} \circ d^{-1} \in \mathcal{B} \} \cup \{ i \}.
\]

Moreover, define a map

\[
\| - \| : Q(S) \setminus \{ i \} \rightarrow \mathcal{C}_{\mathcal{E}R}(S) \text{ given by } \| X, d, g \| = \bigcup \left( [X]_{\mathcal{I}} \circ d^{-1} \right).
\]

Note that the definition of \( Q(S) \) guarantees that \( \| X, d, g \| \in \mathcal{F} \) always.

7.8. **Updating maps.** For each \( s \in S \), define a map \( \lambda_s : Q(S) \setminus \{ i \} \rightarrow \mathcal{F} \times G \) by

\[
(X, d, g)\lambda_s = \begin{cases} (X, d), & \text{if } \| X, d, g \| \cdot \{ s \} \mathcal{R} \| X, d, g \| \text{ and } (X, d, g_{gs}) \in Q(S); \\ \| X, d, g \| \cdot \{ s \}, g_{gs} \), & \text{otherwise}; \end{cases}
\]

for each non-initial state \( (X, d, g) \in Q(S) \setminus \{ i \} \).
7.9. **Automaton action.** The action of \( s \in S \) in our automaton is given by

\[
1 \odot s = (\overline{s}, g_s, g_s) \quad \text{and} \quad (X, d, g) \odot s = ((X, d, g)\lambda_s, gg_s)
\]

at each non-initial state \((X, d, g) \in Q(S) \setminus \{i\}\).

7.10. **Lemma.** If \( q \in Q(S) \) and \( s \in S \) then \( q \odot s \in Q(S) \setminus \{i\}\).

**Proof.** Clearly \( q \odot s \neq i \) always. Observe that \( q \odot s \) is always of the form

\[
q \odot s = (X, d, gg_s)
\]

for some \( X \in F \) and some \( d, g \in G \). Notice that there are two possible cases.

1. In one case it is guaranteed that \( q \odot s = (X, d, gg_s) \in Q(S) \).
2. Otherwise one has that \( d = gg_s \), and so

\[
||q \odot s|| = \bigcup \left( [X]_I \oplus (gg_s)^{-1}(gg_s) \right) = \bigcup [X]_I = X.
\]

It follows immediately that \( q \odot s \in Q(S) \).

Since these are the only two possibilities, we are done. \( \square \)

7.11. **Flow automaton.** The data of our automaton \( A(S) \) is given by

\[
A(S) = (S, Q(S), 1, \odot)
\]

as defined thusfar in the section. Our flow \( \Phi \) will be the map

\[
\Phi = ||-|| : Q(S) \setminus \{i\} \rightarrow \mathcal{P}_I(S),
\]

whose image is clearly contained in \( \mathcal{C}_{ER}(S) \).

7.12. **Proposition.** The map \( \Phi \) is a flow, and so \((S, A(S), \Phi)\) is a flow automaton for which \( \text{Cov}(S, A(S), \Phi) \subseteq \mathcal{C}_{ER}(S) \).

**Proof.** Given \( s \in S \), we will consider ||q \odot s|| for each \( q \in Q(S) \). In the case where \( q = i \), the value of ||1 \odot s|| is equal to

\[
\left\| \left( \overline{s}, g_s, g_s \right) \right\| = \bigcup \left( [s]^I \oplus g_s^{-1}g_s \right) = \bigcup [s]^I = \overline{s},
\]

of which \( s \) is clearly an element.

The remaining two cases involve non-initial states \((X, d, g) \in Q(S) \setminus \{i\}\).

1. If \( ||X, d, g|| \cdot \{s\} \subseteq ||X, d, g|| \) and \((X, d, gg_s) \in Q(S) \), then

\[
(X, d, g) \odot s = (X, d, gg_s) \quad \text{and} \quad [X]^I \oplus d^{-1}gg_s = [X, d, g] \cdot \{s\}^I.
\]

From here, the computation

\[
||X, d, g \odot s|| = \bigcup \left( [X]^I \oplus d^{-1}gg_s \right) = \bigcup ||X, d, g \cdot \{s\}||
\]

yields the fact that \( ||X, d, g|| \cdot \{s\} \subseteq ||X, d, g \odot s|| \).
(2) In all remaining cases one has that
\[(X, d, g) \circ s = (||X, d, g|| \cdot \lambda s, gg_s, gg_s).\]

It follows from claim (2) of Lemma 6.8 that
\[||X, d, g|| \circ s = ||X, d, g|| \cdot \lambda s \supseteq ||X, d, g|| \cdot \lambda s.\]

Having covered all cases, we have established that \(\Phi\) is a flow. \(\square\)

8. Upper bound

8.1. Given a preordered set \(P\), let \(\mathcal{D}_P\) denote the set of (not necessarily monotone) functions \(f : P \to P\) which satisfy

\[(x)f = x \quad \text{or} \quad (x)f < x\]

for all \(x \in P\). Clearly \(\mathcal{D}_P\) is closed under composition and thus is a semigroup.

8.2. Lemma. If \(P\) is a preordered set, then \(\mathcal{D}_P\) is \(\mathcal{R}\)-trivial.

Proof. Let \(f, g, h \in \mathcal{D}_P\) and suppose that \(fgh = f\). If \(x \in P\), then

\[(x)f = (x)fgh \leq (x)fg \leq (x)f,\]

and hence \((x)fg = (x)f\) already. \(\square\)

8.3. Definition. Let \((F \times G)^\bullet = (F \times G) \cup \{\bullet\}\). Define a preorder \(\leq\) on \((F \times G)^\bullet\) by setting

1. \(\bullet \leq \bullet\) and \((X, d) \leq \bullet\) always; and
2. \((X_1, d_1) \leq (X_2, d_2)\) if and only if \(X_1 \leq_X X_2\).

8.4. Given \(s \in S\) and \(g \in G\), extend the map \((-, g)\lambda_s\) to \((F \times G)^\bullet\) by setting

\((\bullet, g)\lambda_s = ([s], g_s);\)

and, for any \((X, d)\) such that \((X, d, g)\) does not belong to \(Q(S)\), setting

\[(X, d, g)\lambda_s = (X, d).\]

8.5. Lemma. The map \((-, g)\lambda_s\) belongs to \(\mathcal{D}_{(F \times G)^\bullet}\) for all \(s \in S\) and all \(g \in G\).

Proof. We begin by considering the “extended” cases defined in 8.4. First, since \(\bullet\) is strictly above all non-\(\bullet\) elements of \((F \times G)^\bullet\), it follows that \((\bullet, g)\lambda_s < \bullet\). Next, if \((X, d, g)\) is not a member of \(Q(S)\), then \((X, d, g)\lambda_s = (X, d)\).

We now move on to the cases where \((X, d, g) \in Q(S)\), of which there are three.

1. If \(||X, d, g|| \cdot \{s\} <_R ||X, d, g||\) and \((X, d, gg_s) \in Q(S)\), then \((X, d, g)\lambda_s = (X, d)\).
2. If \(||X, d, g|| \cdot \{s\} \leq_\mathcal{R} ||X, d, g||\), then
\[
||X, d, g|| \cdot \{s\} \leq_\mathcal{R} ||X, d, g|| \cdot \{s\} <_\mathcal{R} ||X, d, g|| \cdot \mathcal{R} X
\]

and therefore \((X, d, g)\lambda_s < (X, d)\).
(3) Finally, suppose that $\|X, d, g\| \cdot \{s\} \not\in X, d, g\|$ but $(X, d, gg_s)$ does not belong to $Q(S)$. In this situation, $\|X, d, g\| \in F$ but the II-block

$$[X]_I \oplus d^{-1} gg_s = \|X, d, g\|_I \oplus g_s = \|X, d, g\| \cdot \{s\}_I$$

does not belong to $B$. This implies via the contrapositive of Lemma 6.11 that

$$\|X, d, g\| \cdot \{s\} \not\in \|X, d, g\| \not\in X,$$

from which we conclude that $(X, d, g) \lambda_s < (X, d)$. Having verified the desired condition in all cases, we are done. $\square$

8.6. Proposition. The transition semigroup of $A(S)$ belongs to $\text{ER}$. 

Proof. Let $\mathcal{T}(S)$ denote the transition semigroup of $A(S)$. Moreover, for each $s \in S$, let $\tilde{s}$ denote the transformation $(-) \circ s \in \mathcal{T}(S)$, and note that $\mathcal{T}(S)$ is generated by the various $\tilde{s}$ as $s$ ranges over $S$.

We will define an embedding of (faithful) transformation semigroups

$$(\tilde{\zeta}, \varphi) : (Q(S), \mathcal{T}(S)) \longrightarrow ((F \times G)^*, D_{(F \times G)^*} \cdot (G, G)),$$

which will establish via Lemmas 8.2 and 3.8 that $\mathcal{T}(S) \in R \ast G = \text{ER}$.

The map $\tilde{\zeta} : Q(S) \rightarrow (F \times G)^* \ast G$ is given by

$$(1) \tilde{\zeta} = (\ast, 1_G) \quad \text{and} \quad (X, d, g) \tilde{\zeta} = (X, d, g).$$

It is clear that $\tilde{\zeta}$ is injective.

Next, we define the morphism $\varphi : \mathcal{T}(S) \rightarrow D_{(F \times G)^*} \cdot (G, G)$. To do so, it suffices to define $\tilde{\varphi}$ for all $s \in S$ since $\varphi$ is determined by its values on the generators of $\mathcal{T}(S)$. So, for each $s \in S$, define

$$(\tilde{s}) \varphi = (\tilde{\lambda}_s, g_s)$$

where the function $\tilde{\lambda}_s : G \rightarrow D_{(F \times G)^*}$ is given by

$$(g) \tilde{\lambda}_s = (-, g) \lambda_s : (F \times G)^* \rightarrow (F \times G)^*$$

at each group element $g \in G$. This is well-defined since each $(-, g) \lambda_s$ belongs to $D_{(F \times G)^*}$ by Lemma 8.5. Moreover, it is clear that $\varphi$ is injective.

Now, observe that

$$(1) \tilde{\zeta} \ast (\tilde{s}) \varphi = (\ast, 1_G) \ast (\tilde{\lambda}_s, g_s) = \left(\left(\tilde{s}, g_s, g_s\right)\right) = (1 \circ s) \tilde{\zeta}$$

and if $(X, d, g) \in Q(S) \setminus \{1\}$ then

$$(X, d, g) \tilde{\zeta} \ast (\tilde{s}) \varphi = ((X, d, g) \lambda_s, gg_s) = ((X, d, g) \circ s) \tilde{\zeta}.$$
8.7. **Theorem.** Pointlike sets are decidable for ER. In particular, $\mathcal{P}_{ER} = \mathcal{C}_{ER}$.

**Proof.** It was established in Proposition 5.3 that $\mathcal{C}_{ER} \subseteq \mathcal{P}_{ER}$. As for the other bound, Propositions 7.12 and 8.6 show that for each finite semigroup $S$ there exists a flow automaton whose cover complex is contained in $\mathcal{C}_{ER}(S)$ and whose transition semigroup belongs to ER. Considering this alongside 7.3 establishes that $\mathcal{P}_{ER} \subseteq \mathcal{C}_{ER}$. Thus $\mathcal{P}_{ER} = \mathcal{C}_{ER}$; and, since $\mathcal{C}_{ER}$ is computable, we conclude that ER has decidable pointlikes. $\square$

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