A "black hole" solution of scalar field theory

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Abstract

A black hole-like solution to a toy model scalar field action for the pion is presented. It has a "horizon" that traps pion field information (it cannot escape in finite time) and is thermal, with a finite and calculable temperature, \( T = \sqrt{2}/(\pi \alpha) \), with \( \alpha \) a coupling parameter. The action is a scalar DBI action (D-brane action), coupled to a particular fixed source term. The DBI action by itself has "catenoid" solutions that have horizons with infinite temperature and trap only high energy information. It is also proven that the unique scalar action that admits thermal horizons is of DBI type at leading order, making the D-brane action special. The existence of this "pionless hole" solution means that apparent information loss is not a feature of gravity theories (via black holes), but even a simple scalar theory can exhibit it. This is as it should, since the "pionless hole" is a toy model for the AdS-CFT dual to a black hole.

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1 Introduction

Ever since the original paper of Hawking [1] showing that black holes radiate thermally with a temperature dependent on the properties of the horizon (through the surface gravity at the horizon, $k$), people have been struggling to understand how this is possible. The presence of the horizon that traps information and radiates thermally seems to be in contradiction with quantum mechanics: the collapse of a pure quantum mechanical state into a black hole and its subsequent radiation as a mixed thermal state violates unitarity. It was realized that this is not merely a problem having to do with the unknown quantum gravity governing the singularity, but a paradox: the thermal property (and information loss) is associated with the horizon, which has no strong curvatures, so quantum gravity should not be needed. Within string theory, the belief is that there should be no information loss. Indeed, the entropy of extremal black holes has been counted by realizing them as D-brane states [2, 3], and near-extremal black holes were shown to radiate thermally as the corresponding D-brane system, with the outgoing information encoded in ”greybody factors” [4]. One might argue that these resolutions of the paradox of black hole thermality involve quantum gravity, but perhaps the essential ingredient is rather a new formalism allowing for the possibility of thermal radiation (though not blackbody!) from a pure quantum system.

In this paper I will not attempt to find a solution to the paradox, but rather to sharpen the question. I will show that the same properties usually associated with black holes are present already in scalar field theory, and thus the resolution of the paradox cannot be just the existence of an unknown quantum gravity. Indeed, one certainly understands quantum field theory, and a correct treatment should find no breakdown of unitarity. And yet we will find a solution that has a horizon that traps scalar field information (it cannot escape in a finite time) and radiates thermally at a finite temperature just like a black hole. This is no coincidence, as the model we use is a toy model for the AdS-CFT dual of a black hole. In a series of papers [5, 6, 7, 8, 9, 10] the high energy small angle scattering (fixed $t$, $s \to \infty$) in QCD was analyzed using AdS-CFT [11] a la Polchinski-Strassler [12], and it was found that black hole production dominates in the dual. In particular, it was argued that for the collisions observed at RHIC, the fireballs produced are dual to (analogs of) black holes living on an IR brane [8] (a related proposal was put forth in [13] where a black hole was considered to be dual to a large N gauge theory ”plasma ball”). The pion field action dominates the physics and is dual to the action of a D-brane, thus the pion field should have an action $S = \int d^4x \sqrt{1 + (\partial \phi)^2}$, as already used by Heisenberg [14, 6] to obtain the saturation of the Froissart bound [15].

In [9] it was argued that the ”catenoid” solution of the scalar DBI action [16, 17, 18] (itself similar to the Blon solution of the usual electromagnetic Born-Infeld action [19]) should correspond to the black hole in the dual via AdS-CFT. It was however found that even though the solution has a thermal horizon, similar to Unruh’s hydrodynamics ”dumb holes” [20], the temperature is infinite, and only the high energy modes are trapped at the horizon (see [21] for extensions and realizations of the ”dumb hole” idea in condensed matter systems). Moreover, it was assumed there that the nucleons themselves should be solitonic solutions of a correct Born-Infeld-type pion field action, Skyrmion-like, so it was assumed
there should be no need to write a coupling of the pions to a nucleon source for the "catenoid" solution. I will review the pure DBI action case in section 2.

In this paper, we will instead start with the DBI pion action coupled to a nucleon source, \( \int \phi \bar{N} N \), with the nucleon source \( \bar{N} N \) being replaced by an ad-hoc fixed spread-out distribution, \( \alpha/r^2 \). In section 3, I will show that this is enough to make the temperature finite, and trap all information in the pion field at the horizon of the solution. This is then a toy model for the dual of the black hole, as the particular source term \( \alpha/r^2 \) has no good physical justification other than it works, but one should take it as a proof of principle. I will however prove in section 4 that the most general scalar field action that has a thermal horizon analog to Unruh’s case is near the horizon of DBI type plus corrections, and the only correction that gives a finite temperature without any extra assumptions about the high energy behaviour is a coupling to a source term that decays faster than \( 1/r \). For completeness, I will also sketch in section 5 the steps of Hawking’s derivation of the temperature within the context of the DBI action.

The existence of this solution within scalar field theory means that perhaps one also needs a new formalism to deal with the possibility of thermal emission from a pure quantum state, in quantum field theory as well as in gravity. And in any case, the information loss paradox is as paradoxical in quantum field theory as it is in gravity.

## 2 Review of the DBI action case

This section is a review of parts of [9]. The DBI action (with delta function source)

\[
S = \beta^{-2} \int d^4 x \left[ \sqrt{1 + \beta^2 (\partial_\mu \phi)^2} - 1 \right] + \int d^4 x \phi (\bar{C} \delta(r)) \tag{2.1}
\]

has the "catenoid" solution [16, 17, 18]

\[
\phi(r) = \bar{C} \int_0^\infty \frac{dx}{\sqrt{x^4 - \beta^2 \bar{C}^2}} \tag{2.2}
\]

that has an apparent singularity at \( r_0 = \sqrt{\beta \bar{C}} \), where \( \phi'(r) \to \infty \), but \( \phi(r) \) is finite. This apparent singularity acts as the horizon of a black hole. The above action has a delta function source needed to obtain the catenoid solution, however a proper, SU(2)-invariant action for the pion, defined in eq.7.2 of [9] does not need a singular source for a solution with horizon, and in the presence of appropriate higher order corrections has a topological soliton solution representing nucleons, like the skyrmion. In most of the following, we will put \( \beta = 1 \) for simplicity. Note that \( \bar{C} \) is a scalar field charge (at infinity, the field is \( \phi \sim \bar{C}/r \)), quantized in the quantum theory. We also note that, since the solution never reaches \( r = 0 \), the source term could be argued to be unnecessary, but we kept it to point out that the solution is not solitonic in nature.

The above scalar DBI action was found by Heisenberg [14] to be needed to obtain the saturation of the Froissart unitarity bound, \( \sigma(s) = (\pi/m_r^2) \ln^2 s/s_0 \). He has found that in the high energy limit (fixed \( t, s \to \infty \)) of hadron scattering, the hadrons are irrelevant, and
the effective interaction is between pion field shockwaves sourced by the hadrons. He needed
the above DBI action to saturate the bound. More precisely, the action had a pion mass
term, $m_\pi^2 \phi^2$ inside the square root (and no source term), but we will neglect the mass term
in the following.

The fluctuation equation around the catenoid solution, for $\phi = \Phi(r) + \delta \Phi$ ($\Phi(r)$ is the
catenoid) is

$$
- \frac{1}{\sqrt{1 + (\nabla \Phi)^2}} \partial_t \delta \Phi + \frac{1}{\sqrt{1 + (\nabla \Phi)^2}} (\delta^{ij} - \frac{\partial^i \Phi}{\sqrt{1 + (\nabla \Phi)^2}} \frac{\partial^j \Phi}{\sqrt{1 + (\nabla \Phi)^2}}) \partial_j \delta \Phi = 0 \quad (2.3)
$$

The ratio of the coefficients of $\partial_t \delta \Phi \partial^i \delta \Phi$ and $\partial^2 \delta \Phi$ tends to zero at the horizon, where
$\Phi'(r) \to \infty$. This is suggest why we called the apparent singularity a horizon, and also
suggests a comparison with Unruh’s ”dumb holes” [20] in ultrasonic hydrodynamic fluid
flow (“sonic booms”). In that case, the hydrodynamic flow with speed $v$ is irrotational
($\nabla \times v = 0$), so is described by a potential $\Phi$ by $v = \nabla \Phi$, with fluctuation equation

$$
\frac{1}{\rho}(\frac{d}{dt} + \nabla \cdot \nabla + (\nabla \cdot v))\frac{\rho}{c^2}(\frac{d}{dt} + \nabla \cdot v)\delta \Phi - \frac{1}{\rho} \nabla (\rho \nabla \delta \Phi) = 0 \quad (2.4)
$$

which is exactly the equation of motion for a scalar field in a curved spacetime, $\partial_\mu \sqrt{g} g^\mu\nu \partial_\nu \delta \Phi = 0$ if the metric is given by

$$
\sqrt{g} g^\mu\nu = \rho \left( \frac{1}{c^2} \frac{\rho v^i v^j / c^2 - \delta_{ij}}{\rho v^i v^j / c^2 - \delta_{ij}} \right) \quad (2.5)
$$

After finding $g_{\mu\nu}$ in 4d and defining a new time coordinate by $d\tau = dt + v^i dx^i / (c^2 - v^2)$, we get the line element

$$
ds^2 = \frac{\rho}{c^2} [(c^2 - v^2) d\tau^2 - \frac{v^i v^j}{c^2 - v^2} dx^i dx^j] \quad (2.6)
$$

or, in the case of a radial flow

$$
ds^2 = \frac{\rho}{c^2} [(1 - v^2 / c^2) c^2 d\tau^2 - \frac{dr^2}{1 - v^2 / c^2} - r^2 d\Omega^2] \quad (2.7)
$$

In the new coordinates, the scalar wave equation is

$$
\partial_0 \frac{\rho / c^2}{1 - v^2 / c^2} \partial_0 \delta \Phi + \partial_i \rho (\frac{v^i v^j}{c^2 - v^2} - \delta_{ij}) \partial_j \delta \Phi = 0 \quad (2.8)
$$

We see that this is the same equation as (2.3) if we make the identifications

$$
c^2 = 1 + (\nabla \Phi)^2, \quad \rho = \frac{1}{\sqrt{1 + (\nabla \Phi)^2}}; \quad v^i = \partial^i \Phi \quad (2.9)
$$

where the last identification means that indeed $\Phi$ acts as the fluid potential.
As Unruh described, for the existence of a thermal horizon, all we need is that the propagation of the scalar fluctuation $\delta \Phi$ obeys the wave equation in a black hole metric, as found above (both for the ”dumb holes” and for the catenoid), and that as usual, one has a nonzero surface gravity at the horizon for that metric. Indeed, that was the only assumption in the calculation of Hawking of the black hole temperature [1]. For completeness we will sketch a few of the steps of that calculation in section 5, but we can now just take Hawking’s result for the temperature, $T = k/(2\pi)$, where $k$ is the surface gravity at the horizon.

For a static, spherically symmetric solution with only $g_{rr}(r)$ and $g_{tt}(r)$ nontrivial (and possibly an $r$-dependent conformal factor for the sphere metric), one can easily calculate that

$$ (2k)^2 = \lim_{\text{horizon}} \frac{g_{rr}}{g_{tt}} (\partial_r g_{tt})^2 \quad (2.10) $$

For a Schwarzschild black hole, $g_{rr} = g_{tt}$ and we calculate that $k = 1/(4MG)$, as known. For the “dumb hole”, using the above map to a curved spacetime (2.5) we get that [20]

$$ (2k)^2 = \{ \frac{1}{\rho} \partial_r [\rho c(1 - v^2/c^2)] \}^2 \bigg|_{v=c} \Rightarrow T = \frac{1}{4\pi} \frac{1}{\rho} \partial_r [\rho c(1 - v^2/c^2)] \bigg|_{v=c} \quad (2.11) $$

In Unruh’s case, where $\rho$ and $c$ are nonzero and finite at the horizon, one gets $T = (dv/dr|_{v=c})/(2\pi)$, but in our case that is not true.

For the fluctuation around the catenoid, one gets

$$ 2k = \sqrt{1 + \Phi^2} \frac{d}{dr} \left[ \frac{1}{1 + \Phi^2} \right] \bigg|_{r=r_0} \quad (2.12) $$

and putting the explicit form of $\Phi(r)$, we get

$$ T = \sqrt{\frac{r_0}{4(r-r_0)}} \frac{1}{\pi r_0} \quad (2.13) $$

thus the temperature is infinite.

Therefore the horizon is thermal (though of infinite temperature in this case), but another property that makes the black hole horizon special (and is related to its apparent lack of unitarity) is the trapping of information. All information cannot come out of the horizon: it takes an infinite time for a geodesic (massless particle) to come out of it due to the infinite gravitational redshift at the horizon.

The scalar field theory is in flat space, so obviously some information can travel to and from the horizon in finite time. But the essential point is only what happens to scalar field information. And we know that light in a medium for instance can be slowed down even to observable speeds. So we have to see what happens to the propagation of information, defined by the characteristic surface, or in other words to the phase and group velocities of propagation, $c_{ph} = \omega/k$ and $c_{gr} = d\omega/dk$. And because the equation of motion of scalar field fluctuation is the same as the one in a black hole background, the characteristic surface and phase and group velocities are the same as for motion in a black hole background. The time
delay at the horizon is defined by $ds^2 = 0$ in the equivalent black hole background, so by (in hydrodynamics variables)

$$\int dt = \int_{\text{horizon}}^{\text{horizon}} \frac{dr}{c(1 - v^2/c^2)}$$

(2.14)

and an infinite time delay at the horizon means $d(c(1 - v^2/c^2))/dr$ finite at the horizon, which is almost the same as the condition for finite temperature found above, that $d(\rho c(1 - v^2/c^2))/(\rho dr)$ is finite at the horizon. For the catenoid solution, neither is true, so we have an infinite temperature and a finite time delay. However, calculating the phase and group velocities one finds

$$c_{ph}^2 = \frac{\omega^2}{k^2} = \frac{r^4 - r_0^4}{r^4} + \frac{6i r_0^4}{k r^5} \equiv a + \frac{b}{k} \to \frac{6}{k r_0} \quad \text{as } r \to r_0$$

$$c_{gr} = \frac{d \omega}{dk} = \frac{a + b/(2k)}{a + b/k} \to \frac{1}{2} \sqrt{\frac{b}{k}} \to \frac{1}{2} \sqrt{\frac{6}{k r_0}} \quad \text{as } r \to r_0$$

(2.15)

so at least high energy modes do get an infinite time delay in the limit. Moreover, one easily sees that the condition that all modes get zero phase and group velocities at the horizon is the same as the condition of infinite time delay from the equivalent black hole metric.

3 Toy model

In this section we will describe a toy model that has both infinite time delay for information and finite temperature. We will use a spread out (hadron) source for the pion DBI action, of the type $\int d^4x \phi(\alpha/r^2)$ instead of the delta function source used above. This is an ad-hoc procedure, thus we will get a toy model, but it describes well the presence of the hadron sourcing the pion field.

Thus we start with the action

$$S = \beta^{-2} \int d^4x [\sqrt{1 + \beta^2 (\partial_\mu \phi)^2} - 1] + \int d^4x \phi(\alpha/r^2)$$

(3.1)

and set $\beta = 1$ as above for simplicity. The equation of motion for a radial solution is

$$\frac{d}{dr} \left( \frac{r^2 \phi'}{\sqrt{1 + \phi'^2}} \right) = \alpha$$

(3.2)

with the solution

$$\phi(r) = \int_r^\infty dx \frac{\bar{C} + \alpha x}{\sqrt{x^4 - (\bar{C} + \alpha x)}}$$

(3.3)

The position of the horizon, where $\phi' \to \infty$, is where the function $f(r) = r^4 - (\bar{C} + \alpha r)^2$ reaches a zero. For small enough $\alpha$ (with respect to $\sqrt{\bar{C}}$), the function has a single zero, that is just a perturbation of the $\alpha = 0$ result $r_0 = \sqrt{\bar{C}}$. For negative and large enough $\alpha$, there are 2 more zeroes, larger than the first. Thus in between, we have a case where there are only 2 zeroes, the smaller one being a perturbation of the $\alpha = 0$ case, and the larger one
is a local minimum of $f(r)$, thus locally $f(r) \simeq f_0(r - r_0)^2$. It will be clear a posteriori why we are interested in this solution.

Cancelling the constant and linear terms in $r - r_0$, we find this solution to be $\hat{C} = -\alpha^2/4, r_0 = \alpha/2$ for $\alpha > 0$ or $\hat{C} = \alpha^2/4, r_0 = -\alpha/2$ if $\alpha < 0$. We will consider the case $\alpha > 0$ in the following. Then near $r = r_0$, we have $\phi' \simeq r_0/(\sqrt{2}(r - r_0))$, or

$$\phi \simeq (\phi_0 +) \frac{\alpha}{2\sqrt{2}} \ln(r - r_0) \quad (3.4)$$

where the constant term will become subleading in the $r \to r_0$ limit.

Now we can repeat the calculation of the previous section, and the fluctuation equation in this background will be unmodified (since the $\alpha$ term is a source, linear in $\phi$). Then the fluctuation equation (2.3), as well as the identifications in (2.9) remain the same. The expression for the surface gravity at the horizon in the equivalent black hole metric in term of $\Phi(r)$ remains the same as well, as in (2.12), just that now the function $\Phi(r)$ is different. Plugging in $\Phi(r)$ near the horizon, we get

$$T = \frac{\sqrt{2}}{\pi \alpha} \quad (3.5)$$

so the temperature is now finite! We also see why we wanted $f(r) \simeq f_0(r - r_0)^2$ near the horizon, since it gives $\Phi \propto \ln(r - r_0)$, which is the only $\Phi(r)$ that makes the surface gravity in (2.12) finite. Note that if we restore the dependence on $\beta$, $\phi$ near the horizon remains the same, but $T$ gets multiplied by $1/\beta$.

We now also have an infinite time delay for scalar field information coming to or from the horizon, since now

$$\frac{d}{dr}[c(1 - \frac{v^2}{c^2})]_{r=r_0} = \frac{d}{dr}(\frac{1}{\sqrt{1 + \Phi^2}})|_{r=r_0} \simeq \frac{2\sqrt{2}}{\alpha} \quad (3.6)$$

and as a result the time delay at the horizon goes like

$$t \sim \frac{\alpha}{2\sqrt{2}} \ln(r - r_0) \quad (3.7)$$

thus is infinite.

In order to obtain the phase and group velocities, we substitute $\delta \Phi = A \exp(i(\omega t - kr))$ (spherical waves) in the perturbation equation (2.3), obtaining near the horizon

$$\omega^2 \simeq \frac{1}{1 + \Phi^2}(k^2 - 3ik\frac{\Phi'\Phi''}{1 + \Phi^2}) \simeq \frac{8(r - r_0)}{\alpha^2}[(r - r_0)k^2 + 3ik] \quad (3.8)$$

and then we get

$$c^2_{ph} = \frac{\omega^2}{k^2} \simeq \frac{8(r - r_0)}{\alpha^2}[r - r_0 + \frac{3i}{k}] \to 0; \quad c_{gr} = \frac{d\omega}{dk} \simeq \frac{\sqrt{i}}{\alpha} \sqrt{\frac{6(r - r_0)}{k}} \to 0 \quad (3.9)$$

so both go to zero at the horizon for all $k$.

Since this solution acts like a black hole with respect to perturbations in the pion field, we will call it a ”pionless hole”.
4 Uniqueness of the scalar theory

Let us now try to understand the generality of the above solution, with thermal horizons and information trapping, within scalar field theory.

Consider a relativistic lagrangean that depends both on $\phi$ and on its derivatives, through the combination $(\partial_\mu \phi)^2$, i.e. $\mathcal{L}(\phi, (\partial_\mu \phi)^2) = \mathcal{L}(\phi, -\phi^2 + (\nabla \phi)^2)$. Also consider a static, spherically symmetric solution $\Phi_0(r)$. The fluctuation equation in the background of this solution will be

$$\left\{ \frac{\delta^2 \mathcal{L}}{\delta \Phi^2} \left( \frac{d^2}{dt^2} \delta \Phi \right) + \nabla_j \left[ \frac{\delta^2 \mathcal{L}}{\delta \Phi \delta \nabla_j \Phi} \right] \nabla_i \delta \Phi - \frac{\delta^2 \mathcal{L}}{\delta \Phi^2} \delta \Phi \right\}_{\Phi = \Phi_0(r)} = 0 \quad (4.1)$$

We see that again, except for the last, $\Phi$-dependent term, the fluctuation equation looks the same as the hydrodynamic one of Unruh (2.8)!

Thus the condition for the existence of a horizon that traps information and maps to Unruh’s analysis as above is that the ratio of the coefficients of $\nabla_i \delta \nabla_i \delta \Phi$ and $d^2 \delta \Phi/dt^2$ goes to zero at the horizon. This is then

$$\frac{\delta^2 \mathcal{L}}{\delta \Phi^2} / \frac{\delta^2 \mathcal{L}}{\delta (\partial_\mu \phi)^2} |_{\Phi = \Phi_0(r)} \to 0 \quad (4.2)$$

at the horizon or, for a relativistic lagrangean,

$$\frac{\delta^2 \mathcal{L}}{\delta ((\partial_\mu \phi)^2)^2} / \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)^2} |_{\Phi = \Phi_0(r)} \to -\frac{1}{2\Phi^2} |_{\Phi = \Phi_0(r)} \quad (4.3)$$

at the horizon. If at the horizon $\Phi'_0(r)$ goes to infinity and dominates over $\Phi(r)$ (which seems to be the only way to satisfy the horizon condition, but it seems hard to prove) we can approximate the lagrangean on the solution near the horizon as $\mathcal{L} \sim ((\partial_\mu \phi)^2)^n$ and substituting in the above condition we get that $n=1/2$, thus the lagrangean on the solution, looks to leading order near the horizon like the DBI one! So the DBI action can only be corrected by terms that are subleading near the horizon, in order to have a horizon.

The identification that makes maps the fluctuation equation (4.1) to the one for the hydrodynamic potential in (2.8) is

$$\rho = 2 \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)^2} \quad \frac{c = \sqrt{1 + 2(\nabla \phi)^2(\frac{\delta^2 \mathcal{L}}{\delta ((\partial_\mu \phi)^2)^2})^2 / \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)^2}}}{v^i = \nabla^i \Phi \sqrt{1 + 2(\nabla \phi)^2(\frac{\delta^2 \mathcal{L}}{\delta ((\partial_\mu \phi)^2)^2})^2 / \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)^2}} \quad (4.4)$$

Of course, the fluctuation equation in (4.1) has an extra term (as already noted), which looks like a mass term for the scalar fluctuation in the equivalent black hole metric, with mass

$$m^2 = \left. \frac{\delta^2 \mathcal{L}}{\delta \Phi^2} \right|_{\Phi = \Phi_0(r)} \quad (4.5)$$
which however is not constant in the background of a general solution to a general lagrangean.

Applying the calculation of the temperature of the equivalent black hole metric in (2.11) to the identification in (4.4) we obtain the horizon temperature

\[ T = \frac{1}{4\pi} \frac{1}{2\delta L/\delta (\partial_\mu \phi)^2} d \left[ 2\delta L/\delta (\partial_\mu \phi)^2 \sqrt{1 + 2(\nabla \phi)^2 (\delta^2 L/\delta ((\partial_\mu \phi)^2)^2)/\delta (\partial_\mu \phi)^2} \right]_{\phi = \phi_0(r)} \]  

(4.6)

But we have already seen that near the horizon the leading piece of the lagrangean is \( (\partial_\mu \phi)^2 \). We will only analyze subleading pieces of the lagrangean that depend on \( \phi \) only, it seems hard to figure out how to obtain derivative corrections that are subleading to the DBI lagrangean. Thus we will analyze corrections of the type a) \( \sqrt{\delta (\partial_\mu \phi)^2 + f(\phi)} \), b) \( f(\phi) \sqrt{\delta (\partial_\mu \phi)^2 + 1 + f(\phi)} \), and we will take \( f(\phi) = \phi^n, e^{\alpha \phi}, \log \phi \).

a) \( L \sim \sqrt{\delta (\partial_\mu \phi)^2 + f(\phi)} \).

If \( f(\phi) \sim \alpha \phi^n \), we look for a solution that behaves near the horizon like \( \phi \sim \phi_0 + a(r - r_0)^{-\epsilon} \), and obtain from the equations of motion that \( \epsilon = 1/(n - 2) \) and \( a = -2/(\alpha r_0(n - 2))^{1/(n - 2)} \). We obtain then for the temperature

\[ T \propto \phi' \frac{d}{dr} \frac{f(\phi)}{\phi'^2} \propto \frac{1}{\sqrt{r - r_0}} \]  

(4.7)

which is infinite.

If \( f(\phi) \sim \alpha e^{M \phi} \), we see that we can understand it as a limit of large \( n \) of the previous case, so we try a solution that behaves near the horizon as \( \phi \sim \phi_0 + a \log(r - r_0) \) (note that we cannot try a power law solution, because it will then either dominate \( (\partial_\mu \phi)^2 \) in \( L \), or be completely negligible, depending on its sign). Substituting in the equations of motion however, we obtain a contradiction, so there is no such solution (or rather, no \( r_0 \neq 0 \) that acts like a horizon exists).

If \( f(\phi) \sim \alpha \log \phi \), near the horizon we obtain the equation \( 2\phi \phi''/r_0 + \alpha \phi \log \phi \phi'' = \alpha \phi'^2 \), which has as a solution \( \phi \propto \sqrt{r - r_0} \) (leading behaviour) which means that

\[ T \propto \phi' \frac{d}{dr} \frac{f(\phi)}{\phi'^2} \propto \frac{1}{\sqrt{r - r_0}} \]  

(4.8)

which is infinite. Note that this means \( \phi_0 = 0 \); if not, we obtain a contradiction.

b) \( L \sim \sqrt{\delta (\partial_\mu \phi)^2 + 1} f(\phi) \).

If \( f(\phi) = \phi^n \), near the horizon we get the equation \( \phi'' + 2\phi'^3/r_0 = n \phi'^2/\phi \), with the solution \( \phi \sim \phi_0 + a \sqrt{r - r_0} \) as for the pure DBI case. Then the temperature is

\[ T \sim \phi' \frac{d}{dr} \frac{f(\phi)}{f(\phi) \phi'^2} \sim \frac{1}{\sqrt{r - r_0}} \rightarrow \infty \]  

(4.9)

so we have again infinite temperature.

If \( f(\phi) = e^{M \phi} \), near the horizon we get again \( r \phi'' + 2\phi'^3 = 0 \), with the solution \( \phi \sim \phi_0 + a \sqrt{r - r_0} \) as for the pure DBI case, and the temperature is again infinite as for the previous case.
If we have \( f(\phi) = \log \phi \), we get \( \phi'' + 2\phi'^3/r_0 = \phi'^2/(\phi \log \phi) \) with solution \( \phi = \phi_0 + a\sqrt{r-r_0} \) and again \( T \propto 1/\sqrt{r-r_0} \), thus infinite.

c) \( \mathcal{L} \sim \sqrt{(\partial_{\mu} \phi)^2} + 1 + f(\phi) \). The equation of motion is

\[
\frac{d}{dr} \left( r^2 \frac{\phi'}{\sqrt{1 + \phi'^2}} \right) = f'(\phi) \quad (4.10)
\]

The fluctuation equation in the background of a solution is the same as for the pure DBI case, with \( f''(\phi) \) acting as a mass term for the scalar fluctuation. Thus also the temperature is again

\[
T \propto \phi' \frac{1}{dr \phi'^2} \bigg|_{r=r_0} \quad (4.11)
\]

so a finite temperature can only be obtained if \( \phi' \propto 1/(r-r_0) \), so \( \phi \simeq \phi_0 + a \ln(r-r_0) \). On such a solution the left hand side of (4.11) near the horizon is approximately equal to \( 2r_0 \) (constant), so \( f'(\phi) \) should also be constant near the horizon. That excludes \( e^{M\phi} \), whereas for \( f(\phi) = \log \phi \) or \( f(\phi) = \phi^n \) with \( n > 1 \) it is only true for an intermediate regime, where \( \phi_0 \gg a \ln(r-r_0) \), but eventually it also becomes excluded.

In conclusion, the only possibility for a scalar field theory having a solution with a horizon with nonzero and finite temperature is the DBI action with a source coupling \( f(\phi, r) = \phi g(r) \), where \( g(r) \) is a spread out source. That satisfies the equation of motion to leading order, but in fact can fail at subleading orders.

The full solution is then

\[
\phi(r) = \int_r^{\infty} \frac{\bar{C} + (\int x^2 g(x)dx)}{\sqrt{x^4 - (\bar{C} + (\int x^2 g(x)dx)^2)}} \quad (4.12)
\]

and in order to have a horizon near which \( \phi'(r) \propto 1/(r-r_0) \) we see that we need the function \( f(x) \equiv x^4 - (\bar{C} + (\int x^2 g(x)dx)^2) \) to have a local minimum \( x_0 \) at \( f(x_0) = 0 \) for some value of \( \bar{C} \). It is easy to check that this is only possible if \( g(x) \) decays faster than \( 1/x \) at infinity.

Thus the toy model of the previous section is the unique scalar action that has a horizon with nonzero and finite temperature, the only modification allowed would be to change the shape of the spread out source \( g(r) \), with the only constraint to be that it needs to decay faster than \( 1/r \) at infinity. That is why we have called the model a toy model for the dual of a black hole, as it was not clear how to select the source \( g(r) \), otherwise it is fixed.

The uniqueness of the DBI action in this context recalls another such property, of the electromagnetic DBI action, which is the the unique nonlinear correction of the Maxwell theory that is both causal and has only one characteristic surface (generically, there are two).

It is also clear why a solution of scalar field theory that looks like a black hole was not considered before: there doesn’t seem to be another relativistic action that has such a solution!
Sketch of Hawking’s proof within scalar theory

In this paper I have mentioned that once we have the effective metric in which the perturbation of the scalar field moves, we can just follow Hawking’s derivation to the final result (as noted already by Unruh [20]), but it is perhaps instructive to elaborate a bit and address some objections one might have.

Throughout the paper I have used the map to black holes via Unruh’s hydrodynamic “dumb holes.” I have found the construction intuitive, especially in light of the fascinating identification of the scalar field Φ with the hydrodynamic potential. But all we needed actually to retain was the parametrization used in (2.6) of a potential black hole-type metric, giving the equation for a scalar field propagating in it, \( \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \Phi = 0 \), as in (2.8). Here \( \rho, v^i, c \) can be thought as just parameters of the effective metric.

It is important to stress that Hawking’s derivation does not use the dynamics of gravity (Einstein’s equation), only its geometry, i.e. the fact that the propagation of a scalar field occurs within a 4d black hole space, with the usual equation \( \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \Phi = 0 \). In fact, the expression of \( T \) in terms of the surface gravity at the horizon \( k \), shows that we could calculate the temperature for a scalar field propagating within a solution to gravity with any field equation, or even within a background that is not a solution to the gravity field equations. That is the analogy we are pursuing here, as the effective line element in (2.6) is actually defined by a solution to a scalar field theory, not by a solution to some gravity theory.

Hawking’s derivation relies on the fact that the vacuum for incoming particles (on \( I^- \)) differs from the vacuum for outgoing particles (on \( I^+ \)), and there is a Bogoliubov transformation between them. Thus on \( I^- \) we have

\[
\phi = \sum_i (f_i a_i + \bar{f}_i a_i^+) \tag{5.1}
\]

where \( f_i \) are incoming eigenfunctions (on \( I^- \)) and the vacuum is defined by \( a_i |0_- > = 0 \), whereas on \( I^+ \) the expansion is

\[
\phi = \sum_i (p_i b_i + \bar{p}_i b_i^+ + q_i c_i + \bar{q}_i c_i^+) \tag{5.2}
\]

where \( p_i \) are purely outgoing eigenfunctions (with no Cauchy data on the event horizon) and \( q_i \) have no outgoing component (no Cauchy data on \( I^+ \)). There is a Bogoliubov transformation between the 2 expansions, giving

\[
p_i = \sum_j (\alpha_{ij} f_j + \beta_{ij} \bar{f}_j) \tag{5.3}
\]

with particle creation on \( I^+ \) in the vacuum state \( |0_- > \),

\[
< 0_- | b_i b_i^+ | 0_- > = \sum_j |\beta_{ij}|^2 \tag{5.4}
\]
So in order to calculate the (thermal) particle creation, the outgoing functions $p_i$ are continued back to $\mathcal{I}^-$, and expanded in incoming waves, finding the $\alpha_{ij}$ and $\beta_{ij}$'s and then deriving the particle creation $\langle 0_-|b^+_i b_i|0_-\rangle$.

The rest of the derivation involves this continuation of the outgoing wavefunctions, which is done in the region outside the horizon, so no continuation inside the horizon needs to be done. This is good, since in fact there does not seem to be possible to analytically continue inside the horizon in a physically meaningful way. This was analyzed in some detail for the catenoid in [9], and it was found that the only possible continuation was to another asymptotic region, which can be chosen to be a large, metastable bubble. This continuation is somewhat analogous to the Einstein-Rosen bridge for a Schwarzschild black hole, which connects two asymptotic regions through the black hole horizon (the Schwarzschild throat). Of course, the analogy is not perfect, as the Einstein-Rosen bridge is just another way of foliating the complete black hole 4d space in 3d slices, and therefore the bridge is not static, but gets created and collapses quickly, before it can be traversed. But all we need for the Hawking derivation is the 4d effective metric outside the horizon, and of course the gradient of the metric at the horizon, determined by the surface gravity $k$.

The expansion of $f_i$ and $p_i$ for the 4d Schwarzschild black hole behaves asymptotically as

$$f_{\omega lm} \sim \frac{1}{r^{\sqrt{\omega}}} F_\omega(r) e^{i\omega v} Y_{lm}(\theta, \phi)$$

$$p_{\omega lm} \sim \frac{1}{r^{\sqrt{\omega}}} P_\omega(r) e^{i\omega u} Y_{lm}(\theta, \phi)$$ (5.5)

and a similar formula will apply in our case. The only difference is in the definition of the lightcone expansion parameters $u$ and $v$, which for the 4d Schwarzschild black hole is

$$v = t + r_* = t + r + 2M \log \left| \frac{r}{2M} - 1 \right|$$

$$u = t - r_* = t - r - 2M \log \left| \frac{r}{2M} - 1 \right|$$ (5.6)

where $r_*$ are ”tortoise” coordinates that go to infinity at the horizon and measure the time delay for a geodesic going to the horizon ($u = $ constant is an incoming geodesic). Thus for the ”pionless hole” we will have

$$v = t + r_* \sim t + r + \frac{\alpha}{2\sqrt{2}} \ln(r - r_0)$$

$$u = t - r_* \sim t - r - \frac{\alpha}{2\sqrt{2}} \ln(r - r_0)$$ (5.7)

In order to estimate the form of the scattering part of $p_{\omega lm}$ on $\mathcal{I}^-$, near the event horizon at $v = v_0$, Hawking uses a trick by parallel transporting vectors defined on the event horizon and going to an auxiliary space that has also a past event horizon that intersects the actual future event horizon. The essential point is the fact that the affine parameter $\lambda$ parametrizing the event horizon is written as $\lambda = -Ce^{-ku}$ with $k$ the surface gravity at the horizon. Then the phase $e^{i\omega u}$ of $p_{\omega lm}$ is found to turn for the scattered part $p_{(2)}$ into

$$exp(-i\frac{\omega}{k} \log \left( \frac{v - v_0}{CD} \right)) \Rightarrow p^{(2)}_{\omega} \sim \frac{1}{r^{\sqrt{\omega}}} exp(-i\frac{\omega}{k} \log \left( \frac{v - v_0}{CD} \right))$$ (5.8)
on $\mathcal{I}^-$, where $C$ and $D$ are constants. Note that the only thing that was used was the existence of the effective metric for propagation of the scalar field, which possesses an $\mathcal{I}^-$ for incoming waves, an $\mathcal{I}^+$ for outgoing waves, and an event horizon with a nonzero and finite surface gravity $k$.

The last steps of the derivation involve calculating the decomposition of $p^{(2)}$ into $f_{\omega'}$ and $\tilde{f}_{\omega'}$, calculating the coefficients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ and obtaining a thermal spectrum with $T = k/(2\pi)$.

Finally, let me mention another objection that one might have. In Hawking’s derivation it was essential that one uses the geometric optics approximation (treat the scalar field perturbation as a light ray) up to arbitrarily high energies, since close to the horizon, the rays (virtual modes) that scatter at $r=0$ experience an arbitrarily high blue shift. In fact, this is sometimes considered to be a possible way out of the information paradox (how quantum gravity becomes relevant after all). But note that the propagation of virtual modes has little reason to be influenced by high energy corrections. Unruh noted that in his case, the hydrodynamics equations are expected to be relevant only up to a small energy scale, so most of the paper [20] was devoted to the analysis of perturbations due to high energy corrections (to the effective action). Moreover, the ”dumb holes” were used to show that high energy perturbations are irrelevant to Hawking’s derivation.

In our case, we can treat the relativistic toy model as a good toy model up to arbitrarily high energies. The ”pionless hole” is a toy model for the AdS-CFT dual of a black hole, so the pion action need not be coupled to gravity (we are in the $M_P \to \infty$ limit), so from a purely theoretical standpoint we could consider it to be a good (effective) action valid up to arbitrarily high energies! Also, the DBI action itself arises as an effective action, with arbitrarily high number of derivatives, coming in string theory from summing string interactions on a D-brane (with $\beta \sim \alpha'$ fixed, $g_s \to 0, M_P \to \infty$), so again it has the right to be consider as a good (effective) action up to arbitrarily high energies. Yet another argument for that is the fact that Heisenberg used it for the $t$ fixed, $s \to \infty$ limit of hadron scattering, to obtain the saturation of the Froissart bound. Finally, note that in this case (unlike Hawking’s case) both the virtual mode and the background it moves in have the same origin: the DBI (effective) action, so there is no reason to postulate some unknown interaction between the virtual modes and the background.

6 Conclusions

In this paper I have presented a scalar field theory model that has a solution, the ”pionless hole”, that has the properties of a black hole: it has a horizon that traps all pion field information (it cannot escape in a finite time), and a finite temperature, $T = \sqrt{2}/(\pi\alpha)$. The model is the DBI action coupled to a fixed, spread out source term $g(r) = \alpha/r^2$, and is a toy model for the AdS-CFT dual to a black hole.

The DBI action itself produces a solution, the catenoid, that has a horizon with infinite temperature, and that only traps information in high energy modes. The DBI action arises as a good model for the fixed $t$, $s \to \infty$ behaviour of QCD (Heisenberg showed that it is needed to reproduce the saturation of the Froissart bound [14, 6]), and within AdS-CFT can
be used for the dual of a D-brane action, and was indeed expected to have an analog of a black hole as solution.

The particular source term $g(r)$ taken is an ad-hoc model representing the coupling to a nucleon (or hadron) source, $\int d^4x \phi \bar{NN}$, since for the high $s$ collisions, the only role of the hadrons is as sources for pion field shockwaves \[14\] \[6\]. I have shown that the most general scalar field action that gives a solution with a horizon that traps information and has finite temperature is of the type considered, except with a general source term $g(r)$ that decays faster than $1/r$ at infinity. One could presumably fix $g(r)$ on physical grounds, though that would have to be done in the future, and it is clear that it needs to decay faster than $1/r$, as that would correspond to a hadron spread out like a free scalar field. Since the action considered is the only one giving a black hole-like solution, it is perhaps not surprising that such a solution has not been thought of before.

The way the "pionless hole" traps information at the horizon is the same as a black hole does, namely the phase and group velocities, measuring the propagation of information in the nontrivial background, go to zero at the horizon. This is similar to the fact that light in a medium will propagate with a smaller velocity (even though we are in Minkowski space).

For completeness, I have also sketched Hawking’s derivation of the temperature within the context of the scalar field solution. I pointed out that only the geometry of gravity is needed (not the dynamics, defined by the Einstein equation), which we have in the form of the effective metric for propagation of perturbations in the scalar field. Moreover, we only need this metric outside the horizon, together with the fact that the surface gravity at the horizon is nonzero and finite. This is good, since there does not seem to be any physically meaningful way to analytically continue the solution inside the horizon. I also pointed out that, unlike Unruh’s case \[20\], we can consider the action to be a good (effective) action defined up to arbitrarily high energies (at least from a purely theoretical standpoint), as it is sometimes argued to be needed in Hawking’s derivation. Also, the common (DBI) origin of virtual modes and background makes irrelevant the postulation of an unknown interaction between the two (as is sometimes argued in gravity).

The implication of this paper for the black hole information paradox is that the resolution of the paradox cannot be the existence of an unknown theory of quantum gravity, but rather a new formalism allowing for the formation of an object that radiates thermally (but not blackbody!) from a pure quantum state. In other words, since the paradox appears in scalar field theory, it is as paradoxical here as in gravity, so its solution should not be quantum-gravitational (even if it might involve a theory that includes quantum gravity, like string theory).

In fact, it was argued in \[10\] \[9\] \[8\] that we already have experimental evidence for this apparent information loss in field theory: the "fireballs" observed at RHIC involve collisions of pure quantum states (2 nuclei, each with about 200 nucleons), with subsequent radiation of a mixed state (thermal radiation of tens of thousands of the lightest particles in the theory, the pions, together with the rest of the particles of the theory). In fact, the RHIC collisions were argued to be in the Froissart regime, in which the pion DBI action is relevant, so the "pionless hole" should in fact be a toy model for the fireballs observed at RHIC as well!

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