Tree amplitudes of noncommutative $U(N)$ Yang–Mills theory

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Abstract

Following the spirit of the $S$-matrix program, we propose a modified Britto–Cachazo–Feng–Witten recursion relation for tree amplitudes of noncommutative $U(N)$ Yang–Mills theory. Starting from three-point amplitudes, one can use this modified BCFW recursion relation to compute or analyze color-ordered tree amplitudes without relying on any detailed information of noncommutative Yang–Mills theory. After clarifying the color structure of noncommutative tree amplitudes, we write down the noncommutative analogies of Kleiss–Kuijf and Bern–Carrasco–Johansson relations for color-ordered tree amplitudes and prove them using the modified BCFW recursion relation. This checks the consistency of the relation.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum field theory has been proved to be an efficient way to describe the world, yet one still does not fully understand it. Toward the understanding of quantum field theory, many approaches have been suggested, and one of them is the $S$-matrix program [1]. In the framework of the $S$-matrix program, one tries to study quantum field theory without relying on any detailed information but some general principles, such as Lorentz invariance, locality, gauge symmetry, etc. Due to the appearance of the on-shell BCFW recursion relation [2, 3], it is now possible and much easier to analyze or calculate scattering amplitudes in the framework of the $S$-matrix program. Together with the unitarity method [4–7], one can compute one-loop amplitudes and tree amplitudes efficiently.

In conventional field theory with gauge symmetry, the full scattering amplitude can be expressed as a sum of color-ordered amplitudes, known as color decomposition [8–11]. Thus
in order to compute the full amplitude, one just needs to calculate color-ordered amplitudes, which are often much easier to deal with. There are totally \((n-1)!\) color-ordered amplitudes for \(n\)-point scattering amplitudes, but thanks to some relations of amplitudes, we do not need to calculate all of these \((n-1)!\) ones. There is so-called \(U(1)\)-decoupling relation which states that certain linear combination of partial amplitudes must vanish. Besides this, there are also Kleiss–Kuijf relations [12] which will reduce independent amplitudes to \((n-2)!\) and Bern–Carrasco–Johansson relations [13] which will further reduce independent amplitudes to \((n-3)!\). Then it is possible to compute these independent amplitudes starting from three-point amplitudes by the BCFW recursion relation, and three-point amplitudes can be determined through the argument based on general principles of field theory and the three-point kinematics [14–16]. What is more, all the above-mentioned relations of amplitudes can be proved using BCFW recursion relation\(^3\) [24] (see further generalization and discussion [25, 26]). This again shows the power of the BCFW recursion relation in the \(S\)-matrix program.

After these achievements of the BCFW recursion relation in conventional field theory, it is natural to think whether this powerful on-shell recursion relation can be applied to nonlocal theories; one of which we are interested in is noncommutative field theory [27–30] (also review papers [31, 32] and references there in). Noncommutative field theory is a modification of field theory obtained by taking the position coordinates to be noncommutative variables, i.e. the coordinates satisfy

\[
[x^\mu, x^\nu] = i \theta^{\mu\nu},
\]

where \(\theta^{\mu\nu}\) is a constant antisymmetric tensor of dimension (length)\(^2\). This will in turn modify the Lagrangian of field theory and further bring modification to Feynman rules. It has been shown that in noncommutative \(U(N)\) Yang–Mills theory, Feynman rules for propagators stay the same, while those for vertices are changed [33, 34]. Thus the analytic structure of noncommutative amplitudes is very different from that of conventional field theory. Besides ordinary singularities from propagators, there are also, even at the tree level, so-called essential singularities in the complex plane [35]. These essential singularities would disable the application of the original BCFW recursion relation in noncommutative field theory. Due to a nice property that at the tree level noncommutative amplitudes can be expressed as ordinary amplitudes multiplied by additional phase factors [31, 36–41], in [35] the author argued that one can obtain ordinary amplitudes using the BCFW recursion relation of conventional field theory and then multiply them by corresponding phase factors to get noncommutative amplitudes, thus avoiding the effect of essential singularities. In this paper, by removing essential singularities from amplitudes of noncommutative \(U(N)\) field theory, we introduce a modified BCFW recursion relation which can be applied directly to noncommutative amplitudes. Using this modified BCFW recursion relation, it is possible to construct any tree amplitudes of noncommutative \(U(N)\) Yang–Mills theory from three-point noncommutative amplitudes.

Since the BCFW recursion relation is often applied to color-ordered amplitudes, we also investigate color structures of vertices of noncommutative \(U(N)\) Yang–Mills theory. The color algebra of noncommutative \(U(N)\) Yang–Mills theory has been discussed in many papers, and Feynman rules for propagators and vertices have also been worked out [31, 33, 34]. By working out color structures of vertices we show in detail how to decompose the full noncommutative amplitude into color-ordered noncommutative amplitudes. Based on the color structure, we are able to discuss nontrivial relations among these amplitudes. It will be shown that after

\(^3\) The KK relations have been proved by the field theory method in [17]. Both KK and BCJ relations have been proved by the string theory method in [18, 19]. An extension of BCJ relations to matter fields can be found in [20]. See further works [21–23].
modifications, \( U(1) \)-decoupling, KK and BCJ relations can also be held for noncommutative amplitudes, and all these relations can be proved by the modified BCFW recursion relation for noncommutative \( U(N) \) Yang–Mills theory. We should emphasize that all these things, such as computing tree level noncommutative amplitudes and proving nontrivial relations, can be done in the framework of the \( S \)-matrix program, by using only the noncommutative BCFW recursion relation and three-point amplitudes. This beautifully illustrates the idea of the \( S \)-matrix program.

This paper is organized as follows. In section 2, we will briefly review the color algebra of noncommutative \( U(N) \) Yang–Mills theory, and discuss color structures of three-gluon and four-gluon vertices. These lead to the color decomposition of noncommutative amplitudes. We will also present two useful relations considering cyclic permutation and reflection of color ordering. In section 3, we discuss the validation of the BCFW recursion relation in noncommutative theory, and suggest one modified BCFW recursion relation for noncommutative amplitudes. We also discuss three-point amplitudes and show one simple example to verify the BCFW recursion relation and three-point amplitudes of noncommutative theory. In section 4, we write down noncommutative analogies of KK and BCJ relations and prove them through the BCFW recursion relation of noncommutative \( U(N) \) Yang–Mills theory, which shows the validation of the modified relation in more general cases. In the last section some general discussions and conclusions will be offered.

2. The color structure

2.1. Color algebra and color structure of vertices

Let us consider scattering amplitudes of noncommutative \( U(N) \) Yang–Mills theory [31, 33, 34]. It is known that the \( U(N) \) group can be expressed as the direct product of the \( SU(N) \) and \( U(1) \) groups. Let \( t^a \) be the generator of the \( SU(N) \) group, where \( a \) takes the value from 1 to \( N^2 - 1 \), and \( t^a \) is normalized as
\[
\text{tr}(t^a t^b) = \frac{1}{2} \delta^{ab}.
\]
Furthermore, these generators satisfy
\[
[t^a, t^b] = i f^{abc} t^c, \quad [t^a, t^b] = \frac{1}{N} \delta^{ab} + d^{abc} t^c;
\]
thus, we define the structure constant \( f^{abc} \) and another completely symmetric tensor \( d^{abc} \). We also introduce \( t^0 \) as the generator of the \( U(1) \) group, and for normalization we set
\[
t^0 = \frac{1}{\sqrt{2N}} I_{N \times N}.
\]
These generators together lead to the generators of the \( U(N) \) group. If we denote \( t^A \) as the generator of the \( U(N) \) group, where \( A \) takes the value from 0 to \( N^2 - 1 \), then these generators satisfy the normalization condition
\[
\text{tr}(t^A t^B) = \frac{1}{2} \delta^{AB},
\]
and other two relations
\[
[t^A, t^B] = i f^{ABC} t^C, \quad [t^A, t^B] = d^{ABC} t^C.
\]
One should note that the \( SU(N) \) generators \( t^a \) are traceless, while \( U(N) \) generators \( t^A \) do not strictly hold this property.

What is the difference between color structures of noncommutative field theory and conventional field theory? This can be seen from three-point and four-point vertices in both
expressed using the \( U \) theories. In conventional Yang–Mills theory only \( f^{abc} \) is assigned to the three-point vertex, and from color algebra we have \[ f^{abc} \sim \text{tr}(t^at^bt^c) - \text{tr}(t^at^ct^b). \] But in noncommutative Yang–Mills theory both \( f^{ABC} \) and \( d^{ABC} \) would be present in one vertex. This difference will introduce additional phase factors in color-ordered amplitudes of noncommutative field theory. More concretely, we know that there are three kinds of interactions for the three-gluon vertex, namely \( SU(N) - SU(N) - SU(N), SU(N) - SU(N) - U(1) \) and \( U(1) - U(1) - U(1) \) interactions \[ [33]. \] All these three kinds of interactions can be formally expressed using the \( U(N) \) generators. We will use the notation that \( k_i \times k_j = k^l_i \theta_{ij} k_j^l \), then from Feynman rules we know that the structure constant \( f^{ABC} \) and the symmetric tensor \( d^{ABC} \) appear as \[ 33] \]

\[ f^{ABC} \cos \left(-\frac{1}{2}k_A \times k_B \right) - d^{ABC} \sin \left(-\frac{1}{2}k_A \times k_B \right) \]

in each three-gluon vertex. Using (2.5) we can rewrite \( f^{ABC} \) and \( d^{ABC} \) as the traces of the \( U(N) \) generators, i.e. we have \[ \cos \left(-\frac{1}{2}k_A \times k_B \right) \text{tr}[t^A t^B t^C] + i \sin \left(-\frac{1}{2}k_A \times k_B \right) \text{tr}[t^A t^B t^C] \]

assigned to each three-gluon vertex. Using Euler’s formula \( e^{ix} = \cos x + i \sin x \), we can reorganize the former result as \[ e^{-\frac{1}{2}k_A \times k_B} \text{tr}(t^A t^B t^C) - e^{-\frac{1}{2}k_A \times k_B} \text{tr}(t^B t^A t^C). \]

This differs from the result of conventional Yang–Mills theory with a phase factor. In double-line notation, we can diagrammatically express the color structure of the three-gluon vertex as in figure 1. Similar difference exists in the four-gluon vertex, where from Feynman rules we can see that for each such four-gluon vertex the color generators appear as \[ 33] \]

\[ \text{tr}\left[ -i \cos \left(-\frac{1}{2}k_A \times k_B \right) [t^A, t^B] + \sin \left(-\frac{1}{2}k_A \times k_B \right) [t^A, t^B] \right] \times \left[ -i \cos \left(-\frac{1}{2}k_C \times k_D \right) [t^C, t^D] + \sin \left(-\frac{1}{2}k_C \times k_D \right) [t^C, t^D] \right]. \]

With the same trick we can reorganize this expression as \[ -e^{-\frac{1}{2}k_A \times k_B} \text{tr}(t^A t^B t^C t^D) + e^{-\frac{1}{2}k_A \times k_B} \text{tr}(t^A t^B t^C t^D) \]

\[ + e^{i(k_A \times k_B - k_C \times k_D)} \text{tr}(t^A t^B t^C t^D) - e^{i(k_A \times k_B + k_C \times k_D)} \text{tr}(t^B t^A t^C t^D), \]

and we can further use momentum conservation \( k_D = -(k_A + k_B + k_C) \) for the first and third phase factors and \( k_C = -(k_A + k_B + k_D) \) for the second and fourth phase factors to express
them in a more systematic way. Combined with the property that $\theta^{\mu\nu}$ is antisymmetric, we have the following result:

\[-e^{-\frac{i}{2}(k_A \times k_B + k_A \times k_C + k_B \times k_C)} \text{tr}(t^A t^B t^C t^D) + e^{-\frac{i}{2}(k_A \times k_B + k_A \times k_D + k_B \times k_D)} \text{tr}(t^A t^B t^D t^C)\]
\[+ e^{-\frac{i}{2}(k_B \times k_A + k_B \times k_C + k_A \times k_C)} \text{tr}(t^B t^A t^C t^D) - e^{-\frac{i}{2}(k_B \times k_A + k_B \times k_D + k_A \times k_D)} \text{tr}(t^B t^A t^D t^C).\]

(2.11)

We see again that phase factors arise before color-ordered trace strings and the value of phase depends on the color order. In double-line notation this color structure of the four-gluon vertex can be expressed as shown in figure 2.

There is another important relation between $t^A$ strings of the $U(N)$ group. While some strings of $t^A$ are terminated by fundamental indices, we have

\[\langle t^A \rangle_1^i \langle t^A \rangle_2^j = \delta_1^2 \delta_i^j.\]

(2.12)

This relation enables us to express tree-level amplitudes of many gluons, which are constructed by three-gluon and four-gluon vertices, into the sum of color-ordered amplitudes.

2.2. Color decomposition of noncommutative tree amplitudes

All these above observations lead to the consequence that one can write down the $n$-point tree amplitude as the sum of single-trace terms, i.e. color-ordered amplitudes, with additional phase factors. This is known as the color decomposition of noncommutative scattering amplitudes. More explicitly, we have [8–11, 35]

\[A_{\text{tree}}^{\text{NC}}(\{k_i, \lambda_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n / Z_n} \text{tr}(t^{k_{a_1}} \ldots t^{k_{a_n}}) A_{\text{tree}}^{\text{NC}}(\sigma(1^a_1), \ldots, \sigma(n^a_n)),\]

(2.13)

where we have assumed that coupling constants of various vertices containing $SU(N)$ gluons and $U(1)$ gluons are the same and equal to $g$, $k_i$ and $\lambda_i$ are the gluon momenta and helicities, and $A_{\text{tree}}^{\text{NC}}$ are partial amplitudes of noncommutative $U(N)$ Yang–Mills theory which contain all the kinematic information and phase factors. $S_n$ is the set of all permutations of $n$ particles and $Z_n$ is the subset of cyclic permutations, which preserves the trace and should be eluded from summation in the case of over counting. $A_{\text{tree}}^{\text{NC}}$ can be expressed explicitly as [36–41, 31]

\[A_{\text{NC}}(k_1, \ldots, k_n) = A_C(k_1, \ldots, k_n) \phi(k_1, \ldots, k_n),\]

(2.14)
with the phase factor

$$\phi(k_1, \ldots, k_n) = \exp \left[ -\frac{i}{2} \sum_{1 \leq i < j \leq n} k_i \times k_j \right].$$

(2.15)

Since we will discuss only tree-level amplitudes in this paper, the superscript of ‘tree’ has been thrown away for simplicity. The multiplication in the phase factor is defined as

$$k_i \times k_j = k_i^\mu \theta_{\mu\nu} k_j^\nu,$$

and $A_C$ is color-ordered amplitude of conventional Yang–Mills theory.

### 2.3. Cyclic permutation and reflection of color order

Tree amplitudes of conventional field theory possess some nice properties which can be deduced directly from their color structure, such as color cyclic relation and reflection relation. In noncommutative $U(N)$ Yang–Mills theory, these relations might not be trivially held, since we should also consider the effect of the phase factor after cyclic permutation or reflection. Let us first discuss cyclic permutation. The phase factor is invariant under the cyclic permutation of gluons [31], which can be seen directly from its definition, i.e. if we relabel $k_i$ as $k_{i+1}$ and $k_n$ as $k_1$, then

$$\sum_{1 \leq i < j \leq n} k_i \times k_j \rightarrow \sum_{2 \leq i < j \leq n} k_i \times k_j = \sum_{2 \leq i < j \leq n} k_i \times k_j + \sum_{2 \leq i < n} k_i \times k_n,$$

(2.16)

since $\sum_{2 \leq i < n} k_i = -(k_1 + k_n)$ and $k_n = -\sum_{1 \leq j < n} k_j$, and note that $k_i \times k_i = 0$, the second term of (2.16) is just $\sum_{1 < j < n} k_1 \times k_j$. After combining this with the first term we reproduced the original phase factor, and thus proved the cyclic property of the phase factor. In this case, noncommutative tree amplitudes follow the same cyclic relation as amplitudes of conventional Yang–Mills theory, i.e.

$$A_{NC}(k_1, k_2, \ldots, k_n) = A_{NC}(k_n, k_1, \ldots, k_{n-1}).$$

(2.17)

Next, let us discuss the color reflection relation of noncommutative Yang–Mills theory. When the color order is reversed, from the definition of the phase factor we have

$$\phi(k_n, \ldots, k_1) = \exp \left[ -\frac{i}{2} \sum_{n \geq i > j \geq 1} k_i \times k_j \right] = \exp \left[ \frac{i}{2} \sum_{2 \leq j < i \leq n} k_j \times k_i \right],$$

(2.18)

where in the second step we used the antisymmetric property of $\theta_{\mu\nu}$, so that $k_i \times k_j = -k_j \times k_i$.

The result is nothing but $\phi^{-1}(k_2, \ldots, k_n, k_1)$, which we have shown before. Then using the property that the phase factor is invariant under cyclic permutation, we obtain

$$\phi(k_n, \ldots, k_1) = \phi^{-1}(k_1, \ldots, k_n).$$

(2.19)

This is the color reflection relation of the phase factor. Using relation (2.14) and the known color reflection relation of conventional Yang–Mills theory, we can deduce the color reflection relation of noncommutative $U(N)$ Yang–Mills theory, i.e.

$$A_{NC}(k_n, \ldots, k_1) = (-1)^nA_{NC}(k_1, \ldots, k_n)\phi^{-2}(k_1, \ldots, k_n).$$

(2.20)

### 3. BCFW recursion relation

Since any amplitudes can be decomposed into color-ordered amplitudes, we could only focus on these color-ordered amplitudes. Is there an efficient method to compute these amplitudes?
Figure 3. Phase factor of one single amplitude equals to the product of phase factors of two sub-amplitudes.

Or is there a way to analyze these color-ordered amplitudes without knowing their explicit expressions? We know that the BCFW recursion relation can be served as a good answer to these problems. In this section, we will try to extend the BCFW recursion relation to noncommutative $U(N)$ Yang–Mills theory.

3.1. BCFW recursion relation for noncommutative $U(N)$ Yang–Mills theory

We have already seen from (2.14) that tree-level amplitudes of noncommutative Yang–Mills theory differ from amplitudes of conventional Yang–Mills theory in additional phase factors [36–41, 31], so when discussing the validation of the BCFW recursion relation of noncommutative Yang–Mills theory, we should focus on the discussion of the phase factor. Let us first consider the following property of the phase factor [36]:

$$\phi(k_1, \ldots, k_n) = \phi(k_1, \ldots, k_m, P)\phi(-P, k_{m+1}, \ldots, k_n), \quad (3.1)$$

which is nothing but the relation of phase factors shown in figure 3. In other words, the product of the two phase factors of sub-amplitudes, which are connected by a propagator, equals the phase factor of a single amplitude on the left-hand side of figure 3. All external momenta are going outward so that momentum conservation takes the form $\sum_{1 \leq i \leq n} k_i = 0$, and the momentum of the propagator is $P = -\sum_{1 \leq i \leq m} k_i$. To prove this property, let us write down the phase factor as

$$\phi(k_1, \ldots, k_n) = \exp\left[ -\frac{i}{2} \sum_{1 \leq i < j \leq n} k_i \times k_j \right] = \exp\left[ -\frac{i}{2} \sum_{1 \leq i < j \leq n} k_i \times k_j \right]$$

$$= \exp\left[ -\frac{i}{2} \sum_{1 \leq i < j \leq m} k_i \times k_j - \frac{i}{2} \sum_{m+1 \leq i < j \leq n} k_i \times k_j - \frac{i}{2} \sum_{m+1 \leq i < j \leq n} k_i \times k_j \right].$$

The first line is true because $\sum_{1 \leq i \leq n} k_i \times k_n = -k_n \times k_n = 0$. Let us deal with those three terms in the second line. The first term is $\phi(k_1, \ldots, k_m, P)$ by definition. The second term is zero since $\sum_{1 \leq i \leq m} k_i = -P$ and $\sum_{m+1 \leq i \leq n} k_j = P$, so the summation is equal to $-P \times P = 0$. The third term is $\phi(k_{m+1}, \ldots, k_n, -P)$ by definition, and because of the cyclic relation of the phase factor this is just $\phi(-P, k_{m+1}, \ldots, k_n)$. This proved relation (3.1). Note that we use only momentum conservation and antisymmetric property of $\theta^{\mu\nu}$ through the proof and the BCFW deformation of two selected momenta will always keep momentum conversation, so the above argument is also held when momenta are shifted.
Then let us discuss BCFW shifting of $n$-point noncommutative amplitude. We pick up two momenta $k_i$ and $k_j$ and take the following shifting:

$$
\hat{k}_i = k_i - z\hat{q}, \quad \hat{k}_j = k_j + z\hat{q}.
$$

(3.2)

Amplitudes of conventional Yang–Mills theory are often written down in spinor formalism so that they have a more compact and elegant form (see [11] and references therein). In the spinor formalism $\hat{q}$ is often taken as $\hat{q}_{\alpha\dot{\alpha}} = \lambda_{\alpha\dot{\alpha}}\lambda_{\dot{\alpha}\alpha}$, so that $\hat{q}^2 = \hat{q} \cdot k_i = \hat{q} \cdot k_j = 0$, and $\hat{k}_i, \hat{k}_j$ are on-shell. We can always change $\hat{q}_{\alpha\dot{\alpha}}$ back to $\hat{q}^\mu$ with $\sigma^{\mu\nu}$ matrix. Tree amplitudes of noncommutative Yang–Mills theory contain two parts, one is the phase factor and the other is the amplitude of conventional field theory. Since here we want to discuss the effect of the phase factor after BCFW shifting, we will assume that for the amplitude of conventional Yang–Mills theory this shifting always leads to the correct boundary condition [42–45]. Then after taking $(k_i, k_j)$-shifting, the phase factor becomes

$$
\phi(k_1, \ldots, \hat{k}_i, \ldots, \hat{k}_j, \ldots, k_0) = \phi(k_1, \ldots, k_0)\varphi(z),
$$

(3.3)

where

$$
\varphi(z) = \exp \left[-\frac{z}{2} \left( k_i + 2 \sum_{i<j} k_i + k_j \right) \times \hat{q} \right].
$$

(3.4)

We see that the phase factor is not invariant after $(k_i, k_j)$-shifting and $z$-dependence enters into the phase factor through $\varphi(z)$. It is obvious that $\varphi(z)$ would not be equal to 1 if $z \neq 0$, since $(k_i + 2 \sum_{i<j} k_i + k_j) \times \hat{q}$ does not necessary vanish. In fact, the result of $(k_i + 2 \sum_{i<j} k_i + k_j) \times \hat{q}$ depends not only on the way of shifting but also on the value of $\theta^{\mu\nu}$. Is there a suitably chosen $\hat{q}$ that satisfies both requirements of BCFW deformation and the vanishing of above result? It seems impossible in four-dimensional spacetime. $\hat{q}$ is chosen that $\hat{q} \cdot k_i = \hat{q} \cdot k_j = \hat{q}^2 = 0$, and these requirements are satisfied only when the auxiliary momentum $\hat{q}$ is complex [42]. More explicitly, since $k_i$ and $k_j$ are massless, we could choose a suitable frame so that $k_i = (E, E, 0, 0)$ and $k_j = (E, -E, 0, 0)$, and assume $\hat{q} = (E_q, q_x, q_y, q_z)$. Two equations $k_1 \cdot \hat{q} = k_j \cdot \hat{q} = 0$ will determine two components of $\hat{q}$, and in the chosen frame we have $E_q = q_x = 0$. The massless condition of $\hat{q}$ gives one more constraint on $\hat{q}$ and we have $q_x^2 + q_y^2 = 0$. Then the requirement of $(k_i + 2 \sum_{i<j} k_i + k_j) \times \hat{q} = 0$ adds one more linear constraint on $\hat{q}$ so that $a\hat{q}_x + b\hat{q}_y = 0$, and $a$ and $b$ are certain real constants. The solution of these two constraints is $q_x = q_y = 0$. In this case, the auxiliary momentum $\hat{q}$ is just a null vector, so we see that there are no non-trivial solutions of $\hat{q}$ that satisfy all these requirements.

What is the problem if the phase factor after shifting is not equal to the phase factors that are not being shifted? Generally speaking, we would expect that BCFW recursion relation of noncommutative $U(N)$ Yang–Mills theory takes the same form as conventional Yang–Mills theory, i.e.

$$
A_{\text{NC}}(k_1, \ldots, k_n) = \sum_{P_{ab}} A_{\text{NC}}^h(\ldots, \hat{k}_i(z_{ab}), \ldots, \hat{P}_{\dot{a}\dot{b}}(z_{ab})) \frac{1}{P_{ab}} A_{\text{NC}}^h(-\hat{P}_{\dot{a}\dot{b}}(z_{ab}), \ldots, \hat{k}_j(z_{ab}), \ldots),
$$

(3.5)

where summation is over all possible helicities and propagators, and $z_{ab}$ is the solution of $P_{\dot{a}\dot{b}}(z_{ab}) = 0$. This is obviously not true from (2.14). The phase factor for the left-hand side of (3.5) is $\phi(k_1, \ldots, k_n)$. Each term of the BCFW expansion on the right-hand side of (3.5) has a phase factor $\phi(k_1, \ldots, \hat{k}_i(z_{ab}), \ldots, \hat{k}_j(z_{ab}), \ldots, k_0)$, after using the cyclic relation of the phase factor and (3.1). So the phase factor on the left-hand side of (3.5) is not equal to the phase factors on the right-hand side of (3.5), and even phase factors on the right-hand side are not equal to each other.
In order to write down a suitable BCFW recursion relation for noncommutative $U(N)$ Yang–Mills theory, let us recall the supersymmetric BCFW recursion relation \[15, 46–48\]. We are forced to take \((k_i, k_j)\)-shifting so that energy–momentum conservation is satisfied after shifting, and in supersymmetric BCFW recursion relation we are forced to take one more Grassmann variable \(\eta\)-shifting so that super-energy–momentum conservation is satisfied. In the noncommutative case, the one that must be kept invariant is the phase factor, so it is reasonable to consider taking some kind of phase deformation. It is also natural to think about this problem from the singularity point of view. In tree amplitudes of noncommutative $U(N)$ Yang–Mills theory, there are so-called essential singularities from phase factors and ordinary singularities from propagators. We could eliminate essential singularities by multiplying one additional phase factor, so that there are only ordinary singularities from propagators, and the BCFW recursion relation is valid. More specifically, we have

\[
\begin{align*}
A_{NC}(k_1, \ldots, k_n) &\phi^{-1}(k_1, \ldots, k_n) \\
&= \sum_{P_{ab}} \left[ A_{NC}^h(\ldots, \hat{k}_i(z_{ab}), \ldots, \hat{P}_{ab}(z_{ab})) \phi^{-1}(\ldots, \hat{k}_i(z_{ab}), \ldots, \hat{P}_{ab}(z_{ab})) \\
&\quad \times \frac{1}{P_{ab}^2} A_{NC}^{-h}(-\hat{P}_{ab}(z_{ab}) \ldots, \hat{k}_j(z_{ab}) \ldots) \phi^{-1}(-\hat{P}_{ab}(z_{ab}) \ldots, \hat{k}_j(z_{ab}) \ldots) \right]. \\
\end{align*}
\]

(3.6)

The above recursion relation can be further simplified by multiplying \(\phi(k_1, \ldots, k_n)\) on both the left-hand and right-hand sides. Using (3.1) and the cyclic relation of the phase factor, we have

\[
\begin{align*}
\phi^{-1}(\ldots, \hat{k}_i(z_{ab}), \ldots, \hat{P}_{ab}(z_{ab})) &\phi^{-1}(-\hat{P}_{ab}(z_{ab}) \ldots, \hat{k}_j(z_{ab}) \ldots) \\
&= \phi^{-1}(k_1, \ldots, \hat{k}_i(z_{ab}), \ldots, \hat{k}_j(z_{ab}) \ldots, k_n), \\
\end{align*}
\]

(3.7)

and after using (3.3) we have

\[
\begin{align*}
\phi(k_1, \ldots, k_n) &\phi^{-1}(k_1, \ldots, \hat{k}_i(z_{ab}), \ldots, \hat{k}_j(z_{ab}) \ldots, k_n) = \varphi^{-1}(z_{ab}). \\
\end{align*}
\]

(3.8)

where \(\varphi(z)\) is defined in (3.4). Then we have

\[
\begin{align*}
A_{NC}(k_1, \ldots, k_n) &= \sum_{P_{ab}} A_{NC}^h(\ldots, \hat{k}_i(z_{ab}), \ldots, \hat{P}_{ab}(z_{ab})) \\
&\times \frac{\varphi^{-1}(z_{ab})}{P_{ab}^2} A_{NC}^{-h}(-\hat{P}_{ab}(z_{ab}) \ldots, \hat{k}_j(z_{ab}) \ldots). \\
\end{align*}
\]

(3.9)

Since essential singularities have been removed from amplitudes of noncommutative Yang–Mills theory in the BCFW recursion relation (3.9), there would be no problems considering singularities or large-\(z\) behavior of \(A_{NC}\) in the complex plane as long as the BCFW recursion relation of conventional field theory is held. Of course, (3.9) is not the best way to compute tree amplitudes of noncommutative $U(N)$ Yang–Mills theory. Thanks to relation (2.14), in order to get amplitudes of noncommutative Yang–Mills theory, we can simply obtain tree amplitudes of conventional field theory by an ordinary BCFW recursion relation and multiply them with corresponding phase factors. But the recursion relation (3.9) could be served as a good tool for noncommutative theory in the $S$-matrix program where, for example, one may want to prove some identities of amplitudes without knowing explicit expressions of these amplitudes, etc.
3.2. Three-point amplitudes

The on-shell BCFW recursion relation allows us to construct any tree amplitudes from three-point amplitudes theoretically. After writing down the BCFW recursion relation for noncommutative Yang–Mills theory, we further want to get noncommutative three-point amplitudes. Three-point amplitudes are fundamental amplitudes, but we know that there are no three-point scattering amplitudes when all three momenta are real [14–16]. In order to get three-point amplitudes theoretically, after writing down the BCFW recursion relation for noncommutative Yang–Mills theory, we further want to get noncommutative three-point amplitudes from three-point amplitudes. Let us consider the four-point amplitude of the helicity configuration $(++--)$. We will take $(k_1, k_4)$-shifting, i.e.,

$$\hat{k}_1 = k_1 - zq, \quad \hat{k}_4 = k_4 + zq.$$  

These forms are determined up to overall dimensionless constants constructed from momenta. In conventional field theory, we have an additional parameter $\theta^{\mu \nu}$, we have $k_1 \times k_2 = k_2 \times k_3 = k_3 \times k_1$; thus, the phase factor can be written in other equivalent forms. Different from conventional field theory, we have an additional parameter $\theta^{\mu \nu}$ in noncommutative Yang–Mills theory and this parameter enables us to construct dimensionless constant from momenta. For example, the multiplication $k_i \times k_j$ is dimensionless, and phase factors (3.11) are also dimensionless. Then forms of (3.11) are determined up to overall coupling constant and dimensionless constants constructed from $\theta^{\mu \nu}$. It would be an interesting work to discuss consistency conditions on the S-matrix of noncommutative field theory.

There is one property of three-point amplitudes we want to address. By direct verification we see that

$$A_{NC}(k_1, k_2, k_3) = -A_{NC}(k_3, k_2, k_1)\phi^{-2}(k_3, k_2, k_1).$$

This relation can be considered as a special case of the color reflection relation (2.20) when $n = 3$. It is simple to prove (2.20) by the BCFW recursion relation of noncommutative Yang–Mills theory starting from the three-point relation (3.12), and (3.12) will play an important role in the later demonstration of amplitude relations of noncommutative Yang–Mills theory.

3.3. A simple example: computing four-point amplitude $A_{NC}(1^+, 2^+, 3^-, 4^-)$

As a verification of these three-point amplitudes and noncommutative BCFW recursion relation, a simple example will be given below to show how to construct four-point amplitude from three-point amplitudes. Let us consider the four-point amplitude of the helicity configuration $(++--)$. We will take $(k_1, k_4)$-shifting, i.e.,

$$\hat{k}_1 = k_1 - zq, \quad \hat{k}_4 = k_4 + zq.$$
and in the spinor formalism we have \( \tilde{q} = |4⟩|1⟩ \). Consequently, we have shifted the holomorphic part of \( k_1 \) and the anti-holomorphic part of \( k_4 \). According to (3.9), we have

\[
A_{NC}(1^+, 2^+, 3^-, 4^-) = A_{NC}(\hat{1}^+, 2^+, -\hat{P}^-) \times \exp[\z_{14} \frac{1}{2} (k_1 + 2(k_2 + k_3 + k_4) \times \tilde{q})] s_{12} A_{NC}(\hat{P}^+, 3^-, 4^-),
\]

(3.14)

where \( \hat{P} = \hat{k}_1 + k_2 \) and \( z_{14} \) is the solution of \( \hat{P}^2 = 0 \) and \( s_{12} = (k_1 + k_2)^2 \) is the propagator. In order to get \( z_{14} \) we should solve the equation

\[
\hat{P}^2 = ⟨1 2 | [12] - z (2 4) [2 1] = 0,
\]

(3.15)

and the result is

\[
z_{14} = -\langle1 2 | \langle 2 4 \rangle.
\]

(3.16)

The three-point sub-amplitudes can be written down directly as

\[
A_{NC}(\hat{1}^+, 2^+, -\hat{P}^-) = e^{-\frac{i}{2} k_1 \times k_2} [12]^4 [12][2\hat{P}][\hat{P} 1],
\]

\[
A_{NC}(\hat{P}^+, 3^-, 4^-) = e^{-\frac{i}{2} k_4 \times k_3} [34]^4 (\langle\hat{P} 3 \rangle \langle34\rangle \langle4\hat{P}\rangle).
\]

Firstly, let us compute the phase factor on the right-hand side of (3.14), which is

\[
\exp \left[ -\frac{i}{2} (k_1 \times k_2 - z_{14} \tilde{q} \times k_2 - z_{14} (k_1 + 2(k_2 + k_3 + k_4) \times \tilde{q} + k_3 \times k_4 + z_{14} k_3 \times \tilde{q}) \right]
\]

\[
= \exp \left[ -\frac{i}{2} (k_1 \times k_2 + k_3 \times k_4 - z_{14} (k_1 + k_2 + k_3 + k_4) \times \tilde{q}) \right]
\]

\[
= \exp \left[ -\frac{i}{2} (k_1 \times k_2 + k_3 \times k_4) \right] \equiv \phi(1, 2, 3, 4),
\]

where we have used \( k_4 = -(k_1 + k_2 + k_3) \) and antisymmetry of \( \theta^{\mu\nu} \) in the second step. Then we want to compute

\[
\frac{[12]^4}{[12][2\hat{P}][\hat{P} 1]} \frac{1}{s_{12}} \frac{(34)^4}{\langle\hat{P} 3 \rangle \langle34\rangle \langle4\hat{P}\rangle} = \frac{[12]^3}{s_{12} [2\hat{P}][\hat{P} 3][1\hat{P}[4].
\]

Since \( \hat{P} = |1⟩|1⟩ + |2⟩|2⟩ - z_{14} |4⟩|1⟩ \) it is easy to see that

\[
[1\hat{P}[4] = [12]|4⟩,
\]

and

\[
[2\hat{P}[3] = [21]|3⟩ - z_{14} [21] (4 3)
\]

\[
= [21] (\langle2 4 | (1 3) + ⟨1 2 | (4 3))
\]

\[
= [21] (2 3) (1 4),
\]

where we have used the Schouten identity in the second step. By writing \( s_{12} = ⟨12 | 12⟩ \) and substituting all the above results back into (3.14), we get the final result:

\[
A_{NC}(1^+, 2^+, 3^-, 4^-) = \frac{(34)^4}{(12) (23) (34) (41)} \phi(1, 2, 3, 4).
\]

(3.18)

This result is the same as the one obtained from (2.14).
4. Relations of amplitudes in noncommutative $U(N)$ Yang–Mills theory

We know that any tree amplitudes can be expressed as a sum of color-ordered amplitudes, and there is a so-called BCFW recursion relation which can be used to efficiently compute or analyze these amplitudes. But thanks to some nontrivial relations of amplitudes we do not need to calculate all these color-ordered amplitudes. In noncommutative $U(N)$ Yang–Mills theory, there are also analogies of these nontrivial relations. In this section, we will write down the analogies of KK and BCJ relations in noncommutative Yang–Mills theory and prove them with the BCFW recursion relation of noncommutative version recursively, starting from the verified three-point relations.

4.1. Noncommutative KK relations

General KK relations should also be modified to fit the requirement of noncommutative Yang–Mills theory, and we conclude that noncommutative KK relations take the form

$$A_{NC}(1, [\alpha_1, \ldots, \alpha_l], n, [\beta_1, \ldots, \beta_m]) = (-1)^m \sum_{\sigma \in \text{OP}([\alpha], [\beta^T])} A_{NC}(1, \sigma, n) \phi^{-1}(1, \sigma, n),$$

(4.1)

where OP([\alpha], [\beta^T]) is the ordered permutation between sets [\alpha] and [\beta^T], i.e. the permutation keeps the order of sets [\alpha] and [\beta^T]. [\beta^T] is the reverse ordering of [\beta].

We will use the noncommutative BCFW recursion relation to expand tree amplitudes on both sides of KK relations. It can be shown that each term on the left-hand side belongs to the right-hand side and both sides have the same number of terms. For the right-hand side of noncommutative KK relations, using the BCFW recursion relation and $(n, 1)$-shifting, we have

$$(-1)^m \sum_{\sigma \in \text{OP}([\alpha], [\beta^T])} A_{NC}(1, \sigma, n) \phi^{-1}(1, \sigma, n)$$

$$= (-1)^m \sum_{l+m \geq 1} \sum_{\hat{l} \geq m} A_{NC}(1, \sigma_1, \ldots, \sigma_l, \sigma_{l+1}, \ldots, \sigma_{l+m}, \hat{\sigma}) \phi^{-1}(\hat{\sigma}, \sigma).$$

(4.2)

The total number of terms in the above expansion is

$$\frac{(l+m)!}{l!m!} (l+m-1).$$

(4.3)

For the left-hand side of noncommutative KK relations, we would firstly deal with amplitudes of noncommutative Yang–Mills theory and phase factors separately, and then combine them to get the final results. For amplitudes on the left-hand side, we have

$$A_{NC}(1, [\alpha_1, \ldots, \alpha_l], n, [\beta_1, \ldots, \beta_m]) = \sum_{h, j=0}^{m-1} A_{NC}([\beta_{j+1}, \ldots, \beta_m], \hat{\sigma}, \hat{\sigma}') \times \frac{1}{P^2_{ij}} A_{NC}(\hat{\sigma}', [\alpha_1, \ldots, \alpha_l], \hat{\sigma}, [\beta_1, \ldots, \beta_j]) \phi^{-1}(z_{ij})$$

$$+ \sum_{h, j=0}^{m} A_{NC}([\beta_{j+1}, \ldots, \beta_m], \hat{\sigma}, [\alpha_1, \ldots, \alpha_l], \hat{\sigma}') \times \frac{1}{P^2_{ij}} A_{NC}(\hat{\sigma}', [\beta_1, \ldots, \beta_j]) \phi^{-1}(z_{ij})$$

$$+ \sum_{h, j=0}^{m} A_{NC}([\beta_{j+1}, \ldots, \beta_m], [\beta_1, \ldots, \beta_j], \hat{\sigma}) \times \frac{1}{P^2_{ij}} A_{NC}(\hat{\sigma}', [\alpha_1, \ldots, \alpha_l]) \phi^{-1}(z_{ij}).$$
and using relation (3.3) and the cyclic symmetry of phase factors, we have
\[
\phi^{-1}(1, \{\alpha_1, \ldots, \alpha_l\}) n, \{\beta_1, \ldots, \beta_m\}) = \phi^{-1}(1, \{\beta_j, \ldots, \beta_m\}, \mu) n, \{\alpha_1, \ldots, \alpha_l\})
\]
and
\[
\phi^{-1}(1, \{\alpha_1, \ldots, \alpha_l\}) n, \{\beta_1, \ldots, \beta_m\}) \phi^{-1}(z_0) = \phi^{-1}(1, \{\beta_j, \ldots, \beta_m\}, \mu, \{\alpha_1, \ldots, \alpha_l\}) \phi^{-1}(z_0)
\]
and
\[
\phi^{-1}(1, \{\alpha_1, \ldots, \alpha_l\}) n, \{\beta_1, \ldots, \beta_m\}) \phi^{-1}(z_{l,j}) = \phi^{-1}(1, \{\beta_j, \ldots, \beta_m\}, \mu, \{\alpha_1, \ldots, \alpha_l\}) \phi^{-1}(z_{l,j})
\]
and
\[
\phi^{-1}(1, \{\alpha_1, \ldots, \alpha_l\}) n, \{\beta_1, \ldots, \beta_m\}) \phi^{-1}(z_{l,j}) = \phi^{-1}(1, \{\beta_j, \ldots, \beta_m\}, \mu, \{\alpha_1, \ldots, \alpha_l\}) \phi^{-1}(z_{l,j})
\]
It is easy to check that these terms belong to the right-hand side of noncommutative KK relations (4.2). Using the same trick, it is easy to show that the last two terms on the right-hand side of equation (4.4) with their phase factors (4.7) and (4.8) also belong to the right-hand side of noncommutative KK relations (4.2). So in order to complete this proof, we want to show that the number of terms on both sides is equal. The total number of terms on the left-hand side is

\[
\sum_{j=0}^{m-1} C^j_{i+j} + \sum_{j=1}^{m} C^j_{i+m-j} + \sum_{j=0}^{l-1} \sum_{i=1}^{m} C^i_{m+i-j} C^j_{i+j-l}.
\]  

This number is the same as the total number (4.3) on the right-hand side of noncommutative KK relations, which can be checked in Mathematica.

### 4.2. Noncommutative BCJ relations

Finally, let us discuss BCJ relations of amplitudes of noncommutative U(N) Yang–Mills theory. In conventional gauge theory, it is argued that general BCJ relations can be derived from level-one BCJ relations, combined by KK and \( U(1) \)-decoupling relations [24, 25]. One can expect that this is also true in noncommutative Yang–Mills theory. In order to prove general noncommutative BCJ relations, we just need to prove level-one noncommutative BCJ relations. As we have done before, we will prove these relations recursively. The four-point BCJ relations can be checked directly, and if we suppose that all less than \( n \)-point noncommutative BCJ relations are correct, then we could prove that \( n \)-point noncommutative BCJ relations are also correct. The general expression of level-one noncommutative Yang–Mills theory is

\[
\sum_{i=3}^{n} A_{\text{NC}}(1, 3, \ldots, i-1, 2, i, \ldots, n) \phi^{-1}(1, 3, \ldots, i-1, 2, i, \ldots, n) \sum_{j=1}^{i} s_{2j} = 0. 
\]  

All noncommutative tree amplitudes in the above summation can be written down explicitly as

\[
\begin{align*}
A_{\text{NC}}(1, 2, 3, 4, \ldots, n-1, n), \\
A_{\text{NC}}(1, 3, 2, 4, \ldots, n-1, n), \\
\ldots, \\
A_{\text{NC}}(1, 3, 4, \ldots, 2, n-1, n), \\
A_{\text{NC}}(1, 3, 4, \ldots, n-1, 2, n).
\end{align*}
\]  

Using noncommutative BCFW recursion relation under \((n, 1)\)-shifting, all expansion terms in BCJ relations can be arranged as follows:

\[
\sum_{i=5}^{n} \sum_{k=3}^{n-i-2} A_{\text{NC}}(\hat{1}, 3, \ldots, k|k+1, \ldots, i-1, 2, i, \ldots, \hat{n}) \phi^{-1} \left( \sum_{j=1}^{n} s_{2j} + s_{2n} - s_{2\hat{n}} \right) 
\]  

(4.16)

\[
+ \sum_{k=3}^{n-1} A_{\text{NC}}(\hat{1}, 3, \ldots, k|2, k+1, \ldots, \hat{n}) \phi^{-1} \left( \sum_{j=1}^{n} s_{2j} + s_{2n} - s_{2\hat{n}} \right) 
\]  

(4.17)

\[
+ \sum_{i=3}^{n-2} \sum_{k=1}^{n-i-2} A_{\text{NC}}(\hat{1}, 3, \ldots, i-1, 2, i, \ldots, k|k+1, \ldots, \hat{n}) \phi^{-1} \left( \sum_{j=1}^{n} s_{2j} + s_{2n} - s_{2\hat{n}} \right) 
\]  

(4.18)

\[
+ \sum_{k=2}^{n-2} A_{\text{NC}}(\hat{1}, 3, \ldots, k, 2|k+1, \ldots, \hat{n}) \phi^{-1} \left( \sum_{j=1}^{n} s_{2j} + s_{2n} - s_{2\hat{n}} \right). 
\]  

(4.19)
In the above expression, for simplicity, we do not write down phase factors explicitly, but one should recall that momentum ordering of each phase factor is the same as their corresponding amplitude. One should also recall that $\phi^{-1}$ comes from the multiplication of two different phase factors: one phase factor $\varphi^{-1}(z)$ from the noncommutative BCFW recursion relation and the other phase factor from the modified BCJ relations (4.14) with all momenta unshifted; thus, according to (3.3) we know that momenta in $\phi^{-1}$ are shifted. In this subsection we will always take this abbreviation.

Note that momenta in sub-amplitudes are shifted; in order to use lower point BCJ relations, we should also shift momenta in the kinetic factors, i.e. we should replace $s_{2n}$ with $s_{2n} - s_{2h}$. Now let us consider the sum of first two terms in the above equation. Summation $\sum_{j=3}^{n} \sum_{k=3}^{n-2}$ in equation (4.16) can be replaced by $\sum_{k=3}^{n} \sum_{j=k+1}^{n-2}$ after changing the summation order. For a given $k$, terms involving $\sum_{j} s_{2j}$ can be calculated as

$$
\sum_{j=3}^{n} \sum_{k=3}^{n} \sum_{h} A_{NC}(1, 3, \ldots, k, \hat{P}_h^{(k)}) \phi^{-1} \frac{1}{P_k} A_{NC}(-\hat{P}_h^{(k)}, k + 1, \ldots, i - 1, 2, i, \ldots, \hat{n}) \phi^{-1} \sum_{j=1}^{\hat{n}} s_{2j} \nonumber
$$

$$
+ \sum_{h} A_{NC}(1, 3, \ldots, k, \hat{P}_h^{(k)}) \phi^{-1} \frac{1}{P_k} A_{NC}(-\hat{P}_h^{(k)}, 2, k + 1, \ldots, \hat{n}) \phi^{-1} \sum_{j=k+1}^{\hat{n}} s_{2j},
$$

(4.20)

where we should emphasize again that $\phi^{-1}$ are just abbreviations and momentum ordering of these phase factors are the same as those amplitudes just before them, and momenta are shifted. Using $(n-k+2)$-point noncommutative BCJ relations, it is easy to see that the sum of the above equation is zero. Then let us consider terms involving $s_{2n} - s_{2h}$, which are

$$
\sum_{k=3}^{n-2} \left[ \sum_{i=k+2}^{n} \sum_{h} A_{NC}(1, 3, \ldots, k, \hat{P}_h^{(k)}) \phi^{-1} \right.

\times \frac{1}{P_k} A_{NC}(-\hat{P}_h^{(k)}, k + 1, \ldots, i - 1, 2, i, \ldots, \hat{n}) \phi^{-1} (s_{2n} - s_{2h})

+ \sum_{h} A_{NC}(1, 3, \ldots, k, \hat{P}_h^{(k)}) \phi^{-1} \frac{1}{P_k} A_{NC}(-\hat{P}_h^{(k)}, 2, k + 1, \ldots, \hat{n}) \phi^{-1} (s_{2n} - s_{2h})

+ \sum_{h} A_{NC}(1, 3, \ldots, n - 1, \hat{P}_{2n}^{(h)}) \phi^{-1} \frac{1}{P_{2n}} A_{NC}(-\hat{P}_{2n}^{(h)}, 2, \hat{n}) \phi^{-1} (s_{2n} - s_{2h}).

(4.21)

Using the cyclic symmetry and $(n-k+2)$-point noncommutative $U(1)$-decoupling relation, the above equation can be rewritten as

$$
- \sum_{k=3}^{n-2} \sum_{h} A_{NC}(1, 3, \ldots, k, \hat{P}_h^{(k)}) \phi^{-1} \frac{1}{P_k} A_{NC}(-\hat{P}_h^{(k)}, k + 1, \ldots, \hat{n}, 2) \phi^{-1} (s_{2n} - s_{2h})

- \sum_{h} A_{NC}(1, 3, \ldots, n - 1, \hat{P}_{2n}^{(h)}) \phi^{-1} \frac{1}{P_{2n}} A_{NC}(-\hat{P}_{2n}^{(h)}, \hat{n}, 2) \phi^{-1} (s_{2n} - s_{2h})

= - \sum_{k=3}^{n-1} \sum_{h} A_{NC}(1, 3, \ldots, k, \hat{P}_h^{(k)}) \phi^{-1}

\times \frac{1}{P_k} A_{NC}(-\hat{P}_h^{(k)}, k + 1, \ldots, \hat{n}, 2) \phi^{-1} (s_{2n} - s_{2h}).
$$

(4.22)
This is the final result of equation (4.16) plus (4.17). Using the same trick, we could systematically perform the calculation of the last two equations (4.18) and (4.19), which yields the result

\[-\sum_{k=3}^{n-1} \sum_{h} A_{NC}(2, \hat{1}, 3, \ldots, k-1, \hat{P}_k^{h}) \phi^{-1} \frac{1}{P_k^2} A_{NC}(-\hat{P}_k^{h}, k, \ldots, \hat{n}) \phi^{-1}(s_{2n} - s_{2\hat{n}}). \]  

(4.23)

Then the total sum of BCFW expansions of BCJ relations is

\[-\sum_{k=3}^{n-1} \sum_{\hat{n}} \left[ A_{NC}(\hat{1}, 3, \ldots, k, \hat{P}_k) \phi^{-1} \frac{1}{P_k^2} A_{NC}(-\hat{P}_k^{h}, k+1, \ldots, \hat{n}, 2) \phi^{-1}(s_{2n} - s_{2\hat{n}}) \right. \]

\[+ A_{NC}(2, \hat{1}, 3, \ldots, k-1, \hat{P}_k) \phi^{-1} \frac{1}{P_k^2} A_{NC}(-\hat{P}_k^{h}, k, \ldots, \hat{n}) \phi^{-1}(s_{2n} - s_{2\hat{n}}) \]  

(4.24)

What is the meaning of this result? To understand this problem, let us pay attention to phase factors. The phase factor for the first term of (4.24) is

\[\phi^{-1}(\hat{1}, 3, \ldots, k, \hat{P}_k) \phi^{-1}(-\hat{P}_k, k+1, \ldots, \hat{n}, 2) = \phi^{-1}(\hat{1}, 3, \ldots, \hat{n}, 2), \]  

(4.25)

and the phase factor for the second term of (4.24) is

\[\phi^{-1}(2, \hat{1}, 3, \ldots, k-1, \hat{P}_k) \phi^{-1}(-\hat{P}_k, k, \ldots, \hat{n}) = \phi^{-1}(2, \hat{1}, 3, \ldots, n-1, \hat{n}). \]  

(4.26)

Note that \(z_k\) in above phase factors are determined by equations of propagator \(\hat{P}_k = 0\). Then let us consider the integration

\[\oint \frac{dz}{z} A_{NC}(2, \hat{1}, 3, \ldots, n-1, \hat{n}) \phi^{-1}(2, \hat{1}, 3, \ldots, n-1, \hat{n}) s_{2\hat{n}}(z). \]  

(4.27)

Residue at pole \(z = 0\) yields

\[A_{NC}(2, 1, 3, \ldots, n) \phi^{-1}(2, 1, 3, \ldots, n)s_{2n}, \]  

(4.28)

where we can further use the BCFW recursion relation to expand \(A_{NC}\) and obtain

\[\sum_{k=3}^{n-1} \sum_{\hat{n}} \left[ A_{NC}(\hat{1}, 3, \ldots, k, \hat{P}_k) \phi^{-1}(z_k) \frac{1}{P_k^2} A_{NC}(-\hat{P}_k^{h}, k+1, \ldots, \hat{n}, 2) \phi^{-1}(2, 1, 3, \ldots, n)s_{2n} \right. \]

\[+ A_{NC}(2, \hat{1}, 3, \ldots, k-1, \hat{P}_k) \phi^{-1}(z_k) \frac{1}{P_k^2} A_{NC}(-\hat{P}_k^{h}, k, \ldots, \hat{n}) \phi^{-1}(2, 1, 3, \ldots, n)s_{2\hat{n}} \]  

(4.29)

The sum of residues at poles coming from propagators is

\[-\sum_{k=3}^{n-1} \sum_{\hat{n}} \left[ A_{NC}(\hat{1}, 3, \ldots, k, \hat{P}_k) \frac{1}{P_k^2} A_{NC}(-\hat{P}_k^{h}, k+1, \ldots, \hat{n}, 2) \phi^{-1}(2, \hat{1}, 3, \ldots, n-1, \hat{n})s_{2\hat{n}} \right. \]

\[+ A_{NC}(2, \hat{1}, 3, \ldots, k-1, \hat{P}_k) \frac{1}{P_k^2} A_{NC}(-\hat{P}_k^{h}, k, \ldots, \hat{n}) \phi^{-1}(2, \hat{1}, 3, \ldots, n-1, \hat{n})s_{2\hat{n}} \]  

(4.30)

So the final result of integration (4.27) differs from (4.24) in an overall minus sign. But we know that

\[A_{NC}(2, \hat{1}, 3, \ldots, n-1, \hat{n}) \phi^{-1}(2, \hat{1}, 3, \ldots, n-1, \hat{n}) = A_{C}(2, \hat{1}, 3, \ldots, n-1, \hat{n}), \]  

(4.31)

and because the shifted momenta \((1, n)\) are not adjacent in \(A_{C}(2, \hat{1}, 3, \ldots, n-1, \hat{n})\), this amplitude behaves as \(1/z^2\) in the boundary [15]. Thus, integration (4.27) is equal to zero, and so is the result (4.24). Thus, we proved the noncommutative \(n\)-point BCJ relations.
5. Conclusion

In this paper, we propose a modified BCFW recursion relation for noncommutative $U(N)$ Yang–Mills theory. By using the noncommutative BCFW recursion relation, we prove noncommutative analogies of KK and BCJ relations, which checks the consistency of it. We expect that similar modification to the supersymmetric BCFW recursion relation would also be valid when supersymmetry is considered. In [35] a basis for non-planar one-loop amplitudes of noncommutative $U(N)$ Yang–Mills theory has been proposed; it is also interesting to see if one can efficiently get coefficients of one-loop amplitudes by using the noncommutative BCFW recursion relation. The study of the BCFW recursion relation in nonlocal field theories might also deepen our understanding of field theory; thus, it is of interest to extend on-shell recursion method to other nonlocal theories.

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