Non-central moderate deviations for compound fractional Poisson processes

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Abstract

The term moderate deviations is often used in the literature to mean a class of large deviation principles that, in some sense, fill the gap between a convergence in probability to zero (governed by a large deviation principle) and a weak convergence to a centered Normal distribution. We talk about non-central moderate deviations when the weak convergence is towards a non-Gaussian distribution. In this paper we study non-central moderate deviations for compound fractional Poisson processes with light-tailed jumps.

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1 Introduction

The theory of large deviations gives an asymptotic computation of small probabilities on exponential scale (see [5] as a reference of this topic) and the basic definition of this theory is the large deviation principle. A large deviation principle provides some asymptotic bounds for a family of probability measures on the same topological space; these bounds are expressed in terms of a speed function (that tends to infinity) and a nonnegative lower semicontinuous rate function defined on the topological space.

The term moderate deviations is used for a class of large deviation principles which fill the gap between a convergence to a constant (at least in probability) and governed by a large deviation principle, and an asymptotic normality result. A more precise description is given in the following claim, where $t \to \infty$.

Claim 1.1. We have a family of $\mathbb{R}^h$-valued random variables $\{C_t : t > 0\}$ that converges (at least in probability) to the origin $0 \in \mathbb{R}^h$, and satisfies the large deviation principle with speed $v_t \to \infty$ and rate function $I_{LD}$; moreover $\{\sqrt{v_t}C_t : t > 0\}$ converges weakly to the centered Normal distribution with covariance matrix $\Sigma$. Then, for every family of positive numbers $\{a_t : t > 0\}$ such that $a_t \to 0$ and $v_ta_t \to \infty$,

$$a_t \to 0 \text{ and } v_ta_t \to \infty,$$

the family of random variables $\{\sqrt{a_tv_tC_t} : t > 0\}$ satisfies the large deviation principle with speed $1/a_t$ and a rate function $I_{MD}$ defined by

$$I_{MD}(x) := \sup_{\theta \in \mathbb{R}^h} \left\{ \langle \theta, x \rangle - \frac{1}{2} \langle \theta, \Sigma \theta \rangle \right\} \text{ for all } x \in \mathbb{R}^h;$$

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moreover we typically have $I_{LD}(x) = I_{MD}(x) = 0$ if and only if $x = 0$, and $I_{MD}$ behaves as $I_{LD}$ locally around 0.

Here we recall a well-known prototype example with a discrete parameter $n$ in place of $t$. We set

$$C_n := \frac{X_1 + \cdots + X_n}{n} \text{ for all } n \geq 1,$$

where $\{X_n : n \geq 1\}$ is a sequence of i.i.d. $\mathbb{R}^h$-valued centered random variables with finite covariance matrix $\Sigma$. For simplicity we also assume that $\mathbb{E}[e^{\theta X_1}]$ is finite if $\theta$ belongs to a neighborhood of 0. Then $\{C_n : n \geq 1\}$ converges to 0 by the law of large numbers, and satisfies the large deviation principle with speed $v_n = n$ and rate function $I_{LD}$ defined by

$$I_{LD}(x) := \sup_{\theta \in \mathbb{R}^h} \left\{ \langle \theta, x \rangle - \log \mathbb{E} \left[ e^{\langle \theta, X_1 \rangle} \right] \right\} \text{ for all } x \in \mathbb{R}^h$$

(see e.g. Cramér Theorem, i.e. Theorem 2.2.30 in [5]). The weak convergence of $\{\sqrt{n}C_n : n \geq 1\}$ is a consequence the central limit theorem. Finally, for every sequence of positive numbers $\{a_n : n \geq 1\}$ such that (1) holds with $v_n = n$ (actually we mean a version of (1) for a discrete index $n$ in place of $t$), $\{\sqrt{a_n n} C_n : n \geq 1\}$ satisfies the LDP with rate function $I_{MD}$ defined above by Theorem 3.7.1 in [5]. Moreover, if $\Sigma$ is invertible, we have

$$I_{MD}(x) = \frac{1}{2} \langle x, \Sigma^{-1} x \rangle$$

and the Hessian matrix of $I_{MD}(x)$ at $x = 0$ is equal to $\Sigma^{-1}$.

We talk about non-central moderate deviations when we have a situation similar to the one in Claim 1.1 and the weak convergence is towards a non-Gaussian distribution. Some univariate examples are presented in [8] and in some references cited therein. An example with multivariate random variables can be found in [11] (see Section 3), even if in that case the weak convergence is trivial because one has a family of identically distributed random variables.

In this paper we consider a compound fractional Poisson process $\{S_{\nu,\lambda}(t) : t \geq 0\}$ with light tailed jumps described in the next Condition 1.1.

**Condition 1.1.** Let $\{S_{\nu,\lambda}(t) : t \geq 0\}$ be defined by

$$S_{\nu,\lambda}(t) := \sum_{k=1}^{N_{\nu,\lambda}(t)} X_k$$

where $\{X_n : n \geq 1\}$ is a sequence of i.i.d. real random variables such that $\mathbb{E}[e^{\theta X_1}]$ is finite if $\theta$ belongs to a neighborhood of 0 (i.e. the light tail case), and $\{N_{\nu,\lambda}(t) : t \geq 0\}$ is a time fractional Poisson process with $\nu \in (0,1)$, independent of $\{X_n : n \geq 1\}$. In particular the random variables $\{X_n : n \geq 1\}$ have (common) finite mean and variance, and we set

$$\mu := \mathbb{E}[X_1] \text{ and } \sigma^2 := \text{Var}[X_1].$$

There are several references on fractional Poisson process; here we recall [3], [4] and [14]; in particular we recall that we refer to the time fractional Poisson process (for the space and space-time fractional Poisson process see [17]). Some properties of the process $\{N_{\nu,\lambda}(t) : t \geq 0\}$ will be recalled in Section 2.

Our aim is to prove non-central moderate deviations for $C_t := \frac{S_{\nu,\lambda}(t)}{t}$. More precisely we bear in mind what we said in Claim 1.1 and we mean the following three statements.
The family of random variables \( \left\{ \frac{S_{\nu, \lambda}(t)}{t} : t > 0 \right\} \) satisfies the large deviation principle with speed \( v_t = t \) and a rate function \( I_{LD} \) which does not depend on \( \mu \) (see Proposition 3.1). Note that we have \( I_{LD}(x) = 0 \) if and only if \( x = 0 \); so \( \frac{S_{\nu, \lambda}(t)}{t} \) converges to zero (as \( t \to \infty \)) at least in probability.

For \( \alpha(\nu) := \begin{cases} 1 - \nu/2 & \text{if } \mu = 0 \\ 1 - \nu & \text{if } \mu \neq 0, \end{cases} \) the family of random variables \( \left\{ \frac{(a_t)^{\alpha(\nu)} S_{\nu, \lambda}(t)}{t} : t > 0 \right\} \) converges weakly toward some non-degenerate and non-Gaussian distribution (see Proposition 3.2).

For every sequence of positive numbers \( \left\{ a_t : t > 0 \right\} \) such that (1) holds, the family of random variables \( \left\{ \frac{(a_t)^{\alpha(\nu)} S_{\nu, \lambda}(t)}{t} : t > 0 \right\} \) satisfies the large deviation principle with speed \( 1/a_t \) and a rate function \( I_{MD} \), which uniquely vanishes at zero (see Proposition 3.3).

So, in some sense, we have two non-central moderate deviation results concerning the cases \( \mu = 0 \) and \( \mu \neq 0 \); however these two results share a common underlying large deviation principle for the convergence in probability to zero which does not depend on \( \mu \).

We conclude with the outline of the paper. In Section 2 we recall some preliminaries on large deviations and on (possibly compound) fractional Poisson process. In Section 3 we prove the results. We conclude with a brief discussion of the heavy tail case (we refer to the terminology in Condition 1.1) in Section 4.

2 Preliminaries

In this section we recall some preliminaries on large deviations and on fractional processes.

2.1 On large deviations

We start with the definition of large deviation principle (see e.g. [5], pages 4-5). In view of what follows we present definitions and results for families of real random variables \( \left\{ Z_t : t > 0 \right\} \) defined on the same probability space \( (\Omega, F, P) \), and we consider \( t \to \infty \). A family of numbers \( \left\{ v_t : t > 0 \right\} \) such that \( v_t \to \infty \) (as \( t \to \infty \)) is called a speed function, and a lower semicontinuous function \( I : \mathbb{R} \to [0, \infty] \) is called a rate function. Then \( \left\{ Z_t : t > 0 \right\} \) satisfies the large deviation principle (LDP from now on) with speed \( v_t \) and a rate function \( I \) if

\[
\limsup_{t \to \infty} \frac{1}{v_t} \log P(Z_t \in C) \leq - \inf_{x \in C} I(x) \quad \text{for all closed sets } C,
\]

and

\[
\liminf_{t \to \infty} \frac{1}{v_t} \log P(Z_t \in O) \geq - \inf_{x \in O} I(x) \quad \text{for all open sets } O.
\]

The rate function \( I \) is said to be good if, for every \( \beta \geq 0 \), the level set \( \{ x \in \mathbb{R} : I(x) \leq \beta \} \) is compact. We also recall the following known result (see e.g. Theorem 2.3.6(c) in [5]).

Theorem 2.1 (Gärtner Ellis Theorem). Assume that, for all \( \theta \in \mathbb{R} \), there exists

\[
\Lambda(\theta) := \lim_{t \to \infty} \frac{1}{v_t} \log \mathbb{E} \left[ e^{\nu \theta Z_t} \right]
\]

as an extended real number; moreover assume that the origin \( \theta = 0 \) belongs to the interior of the set

\[
\mathcal{D}(\Lambda) := \{ \theta \in \mathbb{R} : \Lambda(\theta) < \infty \}.
\]
Furthermore let $\Lambda^*$ be the function defined by

$$
\Lambda^*(x) := \sup_{\theta \in \mathbb{R}} \{ \theta x - \Lambda(\theta) \}.
$$

Then, if $\Lambda$ is essentially smooth and lower semi-continuous, then $\{Z_t : t > 0\}$ satisfies the LDP with good rate function $\Lambda^*$.

We also recall (see e.g. Definition 2.3.5 in [5]) that $\Lambda$ is essentially smooth if the interior of $D(\Lambda)$ is non-empty, the function $\Lambda$ is differentiable throughout the interior of $D(\Lambda)$, and $\Lambda$ is steep, i.e. $|\Lambda'(\theta_n)| \to \infty$ whenever $\theta_n$ is a sequence of points in the interior of $D(\Lambda)$ which converge to a boundary point of $D(\Lambda)$.

2.2 On (possibly compound) fractional Poisson process

We start with the definition of the Mittag-Leffler function (see e.g. [10], eq. (3.1.1))

$$
E_{\nu}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\nu k + 1)}.
$$

It is known (see Proposition 3.6 in [10] for the case $\alpha \in (0, 2)$; indeed $\alpha$ in that reference coincides with $\nu$ in this paper) that we have

$$
E_{\nu}(x) \sim e^{x^{1/\nu}} \text{ as } x \to \infty
$$

and

$$
E_{\nu}(x) \to 0 \text{ as } x \to -\infty.
$$

Now we recall some moment generating functions which can be expressed in terms of the Mittag-Leffler function. If we consider the inverse of the stable subordinator $\{L_{\nu}(t) : t \geq 0\}$, then we have

$$
E[e^{\theta L_{\nu}(t)}] = E_{\nu}(\lambda t^{\nu}) \text{ for all } \theta \in \mathbb{R}.
$$

This formula appears in several references with $\theta \leq 0$ only; however this restriction is not needed because we can refer to the analytic continuation of the Laplace transform with complex argument.

Moreover the fractional process $\{N_{\nu,\lambda}(t) : t \geq 0\}$ can be expressed as

$$
N_{\nu,\lambda}(t) = N_{1,\lambda}(L_{\nu}(t)) \text{ for all } t \geq 0,
$$

i.e. a time-changed standard Poisson process $\{N_{1,\lambda}(t) : t \geq 0\}$ with an independent inverse of the stable subordinator $\{L_{\nu}(t) : t \geq 0\}$ (see e.g. Theorem 2.2 in [14]; see also Remark 2.3 in the same article for other references with related results). Then it is easy to check that

$$
E[e^{\theta N_{\nu,\lambda}(t)}] = E_{\nu}(\lambda(e^\theta - 1)t^{\nu}) \text{ for all } \theta \in \mathbb{R}
$$

and, moreover,

$$
E[e^{\theta S_{\nu,\lambda}(t)}] = E_{\nu}(\lambda(E[e^{\theta Z_1}] - 1)t^{\nu}) \text{ for all } \theta \in \mathbb{R}.
$$

3 Results

We start with the common underlying LDP for the convergence in probability of $\left\{ \frac{S_{\nu,\lambda}(t)}{t} : t > 0 \right\}$ to zero which does not depend on $\mu$.  

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Proposition 3.1. Assume that Condition 1.1 holds. Moreover let \( \Lambda_{\nu,\lambda} \) be the function defined by
\[
\Lambda_{\nu,\lambda}(\theta) := \begin{cases} 
(\lambda(\mathbb{E}[e^{\theta X_1}] - 1))^{1/\nu} & \text{if } \mathbb{E}[e^{\theta X_1}] > 1 \\
0 & \text{if } \mathbb{E}[e^{\theta X_1}] \leq 1,
\end{cases}
\]
and assume that it is essentially smooth. Then \( \{S_{\nu,\lambda}(t) : t > 0\} \) satisfies the LDP with speed \( v_t = t \) and good rate function \( I_{LD} \) defined by
\[
I_{LD}(x) := \sup_{\theta \in \mathbb{R}} \{\theta x - \Lambda_{\nu,\lambda}(\theta)\}.
\]

Proof. The desired LDP can be derived by applying the Gärtner Ellis Theorem (i.e. Theorem 2.1); in fact we have
\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}[e^{\theta S_{\nu,\lambda}(t)}] = \Lambda_{\nu,\lambda}(\theta) \text{ for all } \theta \in \mathbb{R}
\]
by (5) and (3). \( \square \)

Remark 3.1. In Proposition 3.1, since \( \nu \in (0, 1) \), we have \( I_{LD} = 0 \) if and only if \( x = \Lambda'_{\nu,\lambda}(0) = 0 \) for every \( \mu \in \mathbb{R} \). On the other hand, if \( \nu = 1 \) (and if the function \( \Upsilon \) defined by
\[
\Upsilon_{\lambda}(\theta) := \begin{cases} 
\lambda(\mathbb{E}[e^{\theta X_1}] - 1) & \text{if } \mathbb{E}[e^{\theta X_1}] < \infty \\
0 & \text{otherwise}
\end{cases}
\]
is essentially smooth), it is well-known that \( \{S_{\nu,\lambda}(t) : t > 0\} \) satisfies the LDP with speed \( v_t = t \) and good rate function \( I_{LD} \) defined by
\[
I_{LD}(x) := \sup_{\theta \in \mathbb{R}} \{\theta x - \Upsilon_{\lambda}(\theta)\}.
\]
In such a case we have \( I_{LD}(x) = 0 \) if and only if \( \Upsilon'_{\lambda}(0) = \lambda \mu \).

Remark 3.2. If we consider the case \( X_n = 1 \) for all \( n \geq 1 \), Proposition 3.1 yields the LDP for \( \{N_{\nu,\lambda}(t) : t > 0\} \) and the rate function is given by (7) with and \( \Lambda_{\nu,\lambda} \) in (6) reads
\[
\Lambda_{\nu,\lambda}(\theta) := \begin{cases} 
(\lambda(e^{\theta} - 1))^{1/\nu} & \text{if } \theta > 0 \\
0 & \text{if } \theta \leq 0.
\end{cases}
\]
So one can check that
\[
I_{LD}(x) := \begin{cases} 
\sup_{\theta > 0} \{\theta x - (\lambda(e^{\theta} - 1))^{1/\nu}\} & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
\infty & \text{if } x < 0.
\end{cases}
\]
The LDP for \( \{N_{\nu,\lambda}(t) : t > 0\} \) was already proved; see Propositions 3.1 and 3.2 in [2] with \( h = 1 \), where the rate function expression is slightly different, i.e.
\[
I_{LD}(x) := \begin{cases} 
x \sup_{\eta < 0} \left\{ \frac{\eta}{x} - \log \frac{\lambda}{\lambda + (-\eta)^{-\nu}} \right\} & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
\infty & \text{if } x < 0.
\end{cases}
\]
Actually the rate functions expressions coincide; in fact, if we set \( \theta = -\log \frac{\lambda}{\lambda + (-\eta)^{-\nu}} \), for \( x > 0 \) we have
\[
x \sup_{\eta < 0} \left\{ \frac{\eta}{x} - \log \frac{\lambda}{\lambda + (-\eta)^{-\nu}} \right\} = \sup_{\theta > 0} \{-(\lambda(e^{\theta} - 1))^{1/\nu} + x\theta\}.
\]
Remark 3.3. We recall that \( \{N_{\nu,\lambda}(t) : t \geq 0\} \) is a renewal process; so we have
\[
N_{\nu,\lambda}(t) := \sum_{k=1}^{\infty} 1_{\{T_1 + \cdots + T_n \leq t\}}
\]
for some the i.i.d. interarrival times \( \{T_n : n \geq 1\} \). Then, if we set
\[
\kappa(\eta) := \log \mathbb{E}[e^{\eta T_1}],
\]
in Remark 3.2 we have considered the equality \( \theta = -\kappa(\eta) \) for \( \eta \in (0, \infty) \). In conclusion we have
\[
x \sup_{\eta < 0} \left\{ \frac{\eta}{x} - \kappa(\eta) \right\} = \sup_{\theta > 0} \{\kappa^{-1}(-\theta) + x\theta\} =: \Psi^{*}_\kappa(x),
\]
where
\[
\Psi^{*}_\kappa(x) := \sup_{\theta > 0} \{x\theta - \Psi_\kappa(\theta)\} \quad \text{and} \quad \Psi_\kappa(\theta) := -\kappa^{-1}(-\theta),
\]
and this agrees with formulas (12)-(13) in [9].

Now we present the weak convergence results as \( t \to \infty \). For the sake of completeness we give a brief proof by taking the limit of the moment generating functions even if some of these results are known. For instance the convergence for \( \mu \neq 0 \) agrees with the weak convergence stated just after eq. (3.7) in [19] for a less general case (i.e. for the case \( X_n = 1 \) for all \( n \geq 1 \), and therefore for \( \{N_{\nu,\lambda}(t) : t \geq 0\} \) instead of \( \{S_{\nu,\lambda}(t) : t \geq 0\} \)). The convergence for \( \mu = 0 \) appears in [13] (Section II), in [15] (Theorem 4.2) and it is also cited in the introduction of [12]; however in those references the results are given for sample paths. Another recent weak convergence result with \( \mu = 0 \) appears in [16] (Proposition 2.1); they let \( \lambda \) go to infinity with \( t = 1 \), and they get the same limit distribution called Normal variance mixture.

Proposition 3.2. Assume that Condition [17] holds and let \( \alpha(\nu) \) be defined in [2]. Then:

- if \( \mu = 0 \), then \( \{e^{\alpha(\nu) S_{\nu,\lambda}(t)/t} : t > 0\} \) converges weakly to \( \sqrt{\lambda \sigma^2 L_{\nu}(1)} Z \), where \( Z \) is a standard Normal distributed random variable, and independent to \( L_{\nu}(1) \);

- if \( \mu \neq 0 \), then \( \{e^{\alpha(\nu) S_{\nu,\lambda}(t)/t} : t > 0\} \) converges weakly to \( \lambda \mu L_{\nu}(1) \).

Proof. In both cases \( \mu = 0 \) and \( \mu \neq 0 \) we study the limit as \( t \to \infty \) of the moment generating functions. We take into account [13] for the expressions of the moment generating functions, and we take into account [3] when we take the limit.

If \( \mu = 0 \) we have
\[
\mathbb{E} \left[ e^{\theta \alpha(\nu) S_{\nu,\lambda}(t)/t} \right] = \mathbb{E} \left[ e^{\theta S_{\nu,\lambda}(t)/t} \right] = E_{\nu}(\lambda(\mathbb{E}[e^{\theta X_1 / t^\nu}] - 1)t^\nu)
\]
\[
= E_{\nu}\left( \lambda \left( 1 + \frac{\sigma^2 t^\nu}{2} + o\left(\frac{1}{t^\nu}\right) - 1\right) t^\nu \right) \to E_{\nu}\left( \frac{\lambda \sigma^2 t^\nu}{2} \right) \quad \text{for all} \, \theta \in \mathbb{R}.
\]
Thus the desired weak convergence is proved noting that (here we take into account [4])
\[
\mathbb{E} \left[ e^{\theta \sqrt{\lambda \sigma^2 L_{\nu}(1)} Z} \right] = \mathbb{E} \left[ e^{\theta S_{\nu,\lambda}(t)/t} \right] = E_{\nu}\left( \frac{\lambda \sigma^2 t^\nu}{2} \right) \quad \text{for all} \, \theta \in \mathbb{R}.
\]

If \( \mu \neq 0 \) we have
\[
\mathbb{E} \left[ e^{\theta \alpha(\nu) S_{\nu,\lambda}(t)/t} \right] = \mathbb{E} \left[ e^{\theta S_{\nu,\lambda}(t)/t} \right] = E_{\nu}(\lambda(\mathbb{E}[e^{\theta X_1 / t^\nu}] - 1)t^\nu)
\]
\[
= E_{\nu}\left( \lambda \left( 1 + \frac{\mu \theta}{t^\nu} + \frac{\sigma^2 t^\nu}{2} + o\left(\frac{1}{t^\nu}\right) - 1\right) t^\nu \right) \to E_{\nu}\left( \lambda \mu \theta \right) \quad \text{for all} \, \theta \in \mathbb{R}.
\]
Thus the desired weak convergence is proved by [4].
Now we present the non-central moderate deviation results.

**Proposition 3.3.** Assume that Condition 1.1 holds and let $\alpha(\nu)$ be defined in (2). Then, for every family of positive numbers $\{a_t : t > 0\}$ such that (1) holds, the family of random variables
\[
\left\{ \frac{e^{\theta x_t} \alpha(\nu) S_{x\lambda}(t)}{t} : t > 0 \right\}
\]
satisfies the LDP with speed $1/a_t$ and good rate function $I_{MD,\mu}$ defined by:

- If $\mu = 0$, $I_{MD,\mu}(x) := \left( (\nu/2)^{(2-\nu)/(2-\nu)} - (\nu/2)^{2/(2-\nu)} \right) \left( \frac{2x^2}{\lambda \sigma^2} \right)^{(2-\nu)/2}$.
- If $\mu > 0$, $I_{MD,\mu}(x) := \begin{cases} (\nu/(1-\nu) - 1/\nu) \left( \frac{x}{X^\mu} \right)^{(1-\nu)/\nu} & \text{if } x \geq 0 \\ \infty & \text{if } x < 0 \end{cases}$
- If $\mu < 0$, $I_{MD,\mu}(x) := \begin{cases} (\nu/(1-\nu) - 1/\nu) \left( \frac{x}{X^\mu} \right)^{(1-\nu)/\nu} & \text{if } x \leq 0 \\ \infty & \text{if } x > 0. \end{cases}$

**Proof.** For every $\mu \in \mathbb{R}$ we apply the Gärtner Ellis Theorem (Theorem 2.1). So we have to take
\[
\Lambda_{\nu,\lambda,\mu}(\theta) := \lim_{t \to \infty} \frac{1}{1/a_t} \log E \left[ e^{\theta x_t \alpha(\nu) S_{x\lambda}(t)} \right] 
\]
or equivalently
\[
\Lambda_{\nu,\lambda,\mu}(\theta) := \lim_{t \to \infty} a_t \log E \left[ \lambda \left( E \left[ e^{\theta (a_t)^{\nu} X_1} \right] - 1 \right) \right] 
\]
in particular we refer to (8) when we take the limit. Moreover, for every $\mu$, the function $\Lambda_{\nu,\lambda,\mu}$ satisfies the hypotheses of the Gärtner Ellis Theorem (this can be checked by considering the expressions below), and therefore the LDP holds with good rate function $I_{MD,\mu}$ defined by
\[
I_{MD,\mu}(x) := \sup_{\theta \in \mathbb{R}} \{ \theta x - \Lambda_{\nu,\lambda,\mu}(\theta) \}. \tag{8}
\]

Then, as we shall explain below, for every $\mu$ the rate function expression in (8) coincides with the rate function $I_{MD,\mu}$ in the statement.

If $\mu = 0$ we have
\[
\begin{align*}
& a_t \log E \nu \left( \lambda \left( E \left[ e^{\theta (a_t)^{\nu} X_1} \right] - 1 \right) \right) = a_t \log E \nu \left( \lambda \left( 1 + \frac{\theta^2 \sigma^2}{2(a_t)^{\nu}} + o \left( \frac{1}{(a_t)^{\nu}} \right) - 1 \right) t^\nu \right) \\
& = a_t \log E \nu \left( \lambda \left( \frac{\theta^2 \sigma^2}{2(a_t)^{\nu}} + o \left( \frac{1}{(a_t)^{\nu}} \right) \right) t^\nu \right) = a_t \log E \nu \left( \frac{\lambda \theta^2 \sigma^2}{a_t^\nu} + o \left( \frac{1}{(a_t)^{\nu}} \right) \right),
\end{align*}
\]
and therefore
\[
\lim_{t \to \infty} a_t \log E \nu \left( \lambda \left( E \left[ e^{\theta (a_t)^{\nu} X_1} \right] - 1 \right) \right) = \left( \frac{\lambda \theta^2 \sigma^2}{a_t^\nu} \right)^{1/\nu} := \Lambda_{\nu,\lambda,\mu}(\theta) \text{ for all } \theta \in \mathbb{R};
\]
thus the desired LDP holds with good rate function $I_{MD,\mu}$ defined by (8) which coincides with the rate function expression in the statement (indeed one can check that this supremum is attained at $\theta = \theta_x := \left( \frac{\theta^2 \sigma^2}{a_t^\nu} \right)^{1/(2-\nu)} \left( \frac{\nu^\nu (2-\nu)}{\lambda \sigma^2} \right)^{(2-\nu)/2}$).

If $\mu > 0$ we have
\[
\begin{align*}
& a_t \log E \nu \left( \lambda \left( E \left[ e^{\theta (a_t)^{\nu} X_1} \right] - 1 \right) \right) \\
& = a_t \log E \nu \left( \lambda \left( 1 + \frac{\theta \mu}{(a_t)^{\nu}} + o \left( \frac{1}{(a_t)^{\nu}} \right) - 1 \right) t^\nu \right) = a_t \log E \nu \left( \frac{\lambda \theta \mu}{a_t^\nu} + (a_t)^{\nu} o \left( \frac{1}{(a_t)^{\nu}} \right) \right),
\end{align*}
\]
and therefore
\[
\lim_{t \to \infty} a_t \log E_{\nu} \left( \lambda \left( \mathbb{E} \left[ e^{(\mu t)^{\theta - \sigma(\nu)} X_t} \right] - 1 \right) \right) = \begin{cases} 
(\lambda \theta \mu)^{1/\nu} & \text{if } \theta > 0 \\
0 & \text{if } \theta \leq 0
\end{cases} =: \Lambda_{\nu,\lambda,\mu}(\theta) \text{ for all } \theta \in \mathbb{R};
\]
thus the desired LDP holds with good rate function \( I_{\text{MD,}\mu} \) defined by \( 5 \) which coincides with the rate function expression in the statement (indeed one can check that this supremum is equal to infinity for \( x < 0 \) (by letting \( \theta \) go to \(-\infty\)), and it is attained at \( \theta = \theta_2 := (\nu x)^{\nu/(1-\nu)} / (\lambda \mu)^{\nu/(1-\nu)} \) for \( x \geq 0 \).

If \( \mu < 0 \) we can repeat the same computations presented for the case \( \mu > 0 \) but, when we take the limit as \( t \to \infty \), we have
\[
\Lambda_{\nu,\lambda,\mu}(\theta) := \begin{cases} 
(\lambda \theta \mu)^{1/\nu} & \text{if } \theta < 0 \\
0 & \text{if } \theta \geq 0
\end{cases} \quad \text{for all } \theta \in \mathbb{R};
\]
thus the desired LDP holds with good rate function \( I_{\text{MD,}\mu} \) defined by \( 5 \) which coincides with the rate function expression in the statement (indeed one can check that this supremum is equal to infinity for \( x > 0 \) (by letting \( \theta \) go to \(+\infty\)), and it is attained at \( \theta = \theta_2 := (\nu x)^{\nu/(1-\nu)} / (\lambda \mu)^{\nu/(1-\nu)} \) for \( x \leq 0 \).

**Remark 3.4.** The rate function \( I_{\text{MD,}\mu} \) for the case \( \mu < 0 \) can be computed by referring to the case \( \mu > 0 \); indeed we have
\[
I_{\text{MD,}\mu}(x) = I_{\text{MD,}-\mu}(-x) = \begin{cases} 
\nu^{\nu/(1-\nu)}(\nu/(1-\nu))^{1/(1-\nu)} & \text{if } -x \geq 0 \\
\infty & \text{if } -x < 0,
\end{cases}
\]
and we immediately recover the rate function in the statement of Proposition 3.3 for the case \( \mu < 0 \).

### 4 A brief discussion on the heavy tail case

One can wonder what happens if the random variables \( \{X_n : n \geq 1\} \) are not light-tailed distributed (see Condition 1.1). Obviously in this case we cannot apply the Gärtner Ellis Theorem, and we cannot say how to approach the problem.

Typically the results on the asymptotic behavior of (possibly compound) sums of i.i.d. heavy tailed distributed random variables are not formulated in terms of large deviation principles. However some references provide large deviation principles for sums of i.i.d. semi-exponential distributed random variables; in particular the heavy tailed Weibull distribution, i.e. the case with distribution function
\[
F(x) = 1 - e^{-(\lambda x)^r} \quad \text{for } x \geq 0,
\]
belongs to this class. Lemma 1 in [7] provides the LDP for empirical means (that reference also provides sample path versions of this result); the speed function is \( n^r \) (so it is a slower speed) and the rate function is not convex. In some sense the results in [7] reveal the bigger influence of extreme values on the partial sums, and the situation is very different from the light tail case.

Large deviations for compound Poisson sums of i.i.d. semi-exponential distributed random variables can be derived from Proposition 2.1 in [18] (a sample-path version of this result can be found in [6]) which concerns Poisson shot noise processes, and the result does not depend on the shot shape. The rate function keeps the properties cited above for the result in [7] case without compound sums. So one could expect that it is possible to prove a similar result for the compound fractional Poisson processes in this paper when \( \{X_n : n \geq 1\} \) are i.i.d. and semi-exponential distributed. In this case we should have the analogue of Proposition 3.1 in this paper, and this could be a starting point to obtain a non-central moderate deviations for a heavy tail case.
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