SEMICLASSICAL MECHANICS OF CONSTRAINED SYSTEMS

O. Yu. Shvedov

Sub-Dept. of Quantum Statistics and Field Theory,
Department of Physics, Moscow State University
Vorobievy gory, Moscow 119899, Russia

Abstract

Semiclassical mechanics of systems with first-class constraints is developed. Starting from the quantum theory, one investigates such objects as semiclassical states and observables, semiclassical inner product, semiclassical gauge transformations and evolution. In ordinary quantum mechanics, there are a lot of semiclassical substitutions to the Schrodinger equation (not only the WKB-ansatz). All of them can be viewed as "composed semiclassical states" being infinite superpositions of wave packets with minimal uncertainties of coordinates and momenta ("elementary semiclassical states"). An elementary semiclassical state is specified by a set (X,f) of classical variables X (phase, coordinates, momenta) and quantum function f ("shape of the wave packet" or "quantum state in the background X"). A notion of an elementary semiclassical state can be generalized to the constrained systems, provided that one uses the refined algebraic quantization approach based on modifying the inner product rather than on imposing the constrained conditions on physical states. The inner product of physical states is evaluated. It is obtained that classical part of X the semiclassical state should belong to the constrained surface; otherwise, the semiclassical state (X,f) will have zero norm for all f. Even under classical constraint conditions, the semiclassical inner product is degenerate. It can be obtained from the refined algebraic quantization prescription by a linearization procedure. One should factorize then the space of semiclassical states. Semiclassical gauge transformations and evolution of semiclassical states are studied. The correspondence with semiclassical Dirac approach is discussed.

PACS: 03.65.Sq, 11.15.Kc, 11.30.Ly, 11.15.Ha.

Keywords: constrained systems, semiclassical approximation, Dirac quantization, refined algebraic quantization, group averaging, inner product, observable, state, structure functions, open gauge algebra

shvedov@qs.phys.msu.su
Constrained systems are widely investigated in modern physics. Gauge field theories, quantum gravity and supergravity, string and superstring theories are examples of systems with constraints. Only few models are exactly solvable; realistic physical theories require approximate methods. A perturbation theory is one of such techniques, it is usually applied to constrained field systems such as gauge theories.

Another important technique of quantum mechanics and field theory is a semiclassical approximation. Soliton quantization \[1, 2\], quantum field theory in a strong external background classical field \[3\] or in curved space-time \[4\], the one-loop approximation \[5, 6\], the time-dependent Hartree-Fock approximation \[5, 7\] and the Gaussian approximation \[8\] may be viewed as examples of application of semiclassical conceptions.

There are different ways of semiclassical investigation of quantum systems. Some of them are based on studying the asymptotic behavior of physical quantities as the parameter of the semiclassical approximation ("Planck constant") tends to zero.

The simplest Ehrenfest semiclassical method is based on writing down the equations for average values of semiclassical observables. Then one makes an assumption that quantum averages tend to classical and finds that the quantum wave packet can move only along the classical trajectory. Although the formal classical equations are obtained, the problem of semiclassical evolution of shape of the wave packet cannot be resolved within the Ehrenfest approach.

Many physical quantities can be expressed via the functional integrals. Calculating such integrals via the stationary-point (or the saddle-point) technique, one evaluates these quantities in the semiclassical approximation.

Another group of semiclassical approaches is based on direct substitution of hypothetical approximate wave functions to the Schrödinger equation. An important advantage of such approaches is that the accuracy of the semiclassical approximation can be estimated mathematically \[9, 10\], at least for finite-dimensional quantum mechanical systems. Moreover, one can justify the Ehrenfest conjecture that semiclassical wave packets are transformed to semiclassical under evolution. The behavior of the shape of the wave packet can be also investigated \[10\]. It is also possible to say what are semiclassical states and observables.

One can try to apply semiclassical methods to several formulations of quantum theory of systems with constraints. One can use the original Dirac approach \[11\] and consider states to satisfy not only the evolution equation but also additional constraint conditions. The most difficult problem of this quantization is construction of the inner product. One can impose additional gauge conditions \[12, 13\] but this approach is gauge-dependent, especially for the case of Gribov copies problem \[14, 15\].

The BRST-BFV approach \[16, 17\] based on extension of the phase space allows us to overcome difficulties of the Dirac approach and construct a manifestly covariant formulation of nonabelian gauge theories. However, the inner product is indefinite, so that space of physical states should be specified by imposing the BRST-BFV condition on physical states and factorization of state space.

An alternative way to develop the quantum theory is to use the conception of refined algebraic quantization \[18, 19, 20\] (related ideas were used in the projection operator approach \[21\]) and modify the inner product instead of imposing the constraint conditions on states. This gives a prescription for the inner product in the Dirac approach without introducing indefinite inner product spaces. The refined algebraic quantization approach seems to be the most suitable for developing the semiclassical technique. It is used in the present paper.

The purposes of this paper are:

(i) to clarify a notion of a semiclassical state (space of quantum states and classical phase space are well-known notions; the correspondence between them is not evident);

(ii) to show that there are equivalent semiclassical states, to investigate an analog of a notion of a gauge transformation for semiclassical mechanics;
(iii) to investigate semiclassical observables applied to semiclassical states;
(iv) to study semiclassical transformations ("time evolution") of semiclassical states;
(v) to investigate a role of the superposition principle in semiclassical mechanics;
(vi) to study the correspondence between the developed semiclassical approach, Ehrenfest and WKB-
method.

This paper is organized as follows. Section 2 is devoted to investigation of a notion of a semiclassical
state. In ordinary quantum mechanics, there are a lot of semiclassical substitutions that approximately
satisfy the Schrodinger equation in the semiclassical approximation. It is discussed in appendix B
that all these substitutions (including WKB-type wave functions) can be presented as superpositions of
wave-packet wave functions which can be viewed as ”elementary” semiclassical states specified by a set
of classical variables (coordinates, momenta, phase) and quantum function being a shape of the wave
packet. Constrained systems can be quantized in different ways. It is the refined algebraic quantization
approach that allows us to introduce elementary semiclassical states. The inner product of them is
evaluated in section 2. It appears to be degenerate, so that it is necessary to factorize the space of
semiclassical states: two states are called equivalent if their difference is of zero norm. One can also
perform gauge transformations of semiclassical states. They are discussed in section 3. Sections 4 and
5 deal with semiclassical observables and evolution. In section 6 composed semiclassical states being
superpositions of wave packets are considered. The relationship between refined algebraic quantization
and Dirac approach is also discussed in section 6. Section 7 contains concluding remarks.

2  Semiclassical states

The purpose of this section is to specify semiclassical states for constrained systems. First of all, let us
specify the dependence of constraints on the small parameter of semiclassical expansion.

2.1  Small parameter

1. It is known from quantum mechanics that semiclassical methods can be applied for such equations
that the coefficients of the derivative operators are small, of the order \( O(h) \) as the small parameter \( h \)
of the semiclassical expansion tends to zero, \( h \to 0 \), while the coefficients of the multiplication operators
are of the order \( O(1) \), i.e.

\[
\frac{i\hbar}{\partial t} = H(X, -i\hbar \frac{\partial}{\partial X})\psi, \quad X \in \mathbb{R}^n
\]

For quantum-field-type equations, it is convenient to rescale (cf.[2]) the argument \( X \),

\[
X = \sqrt{\hbar} q,
\]

so that the Schrodinger equation takes the following form

\[
i \frac{\partial \psi}{\partial t} = \frac{1}{\hbar} H(\sqrt{\hbar} q, \sqrt{\hbar} p)\psi
\]

with

\[
\hat{p} = -i \frac{\partial}{\partial q}, \quad \hat{q} = q, \quad [\hat{p}, \hat{q}] = -i.
\]

Thus, the semiclassical conception can be applied even if the commutator between coordinates and
momenta is not small, but semiclassical observables depend on the small parameter \( h \) in an unusual
way. It is the quantum operator

\[
\hat{H} = \frac{1}{\hbar} H(\sqrt{\hbar} q, \sqrt{\hbar} p)
\]
that corresponds to the classical observable $H(Q, P)$. One should specify the operator ordering in eq. (2.3). For the definiteness, choose the Weyl quantization of coordinate and momentum operators (see Appendix A). An advantage of Weyl quantization is that real classical observables correspond to Hermitian operators.

2. Consider the constrained system with the first-class classical constraints:

$$\Lambda_a(Q, P), \quad a = 1, M, \quad P, Q \in \mathbb{R}^n.$$ 

The main requirement is that the Poisson bracket

$$\{\Lambda_a, \Lambda_b\} = \frac{\partial \Lambda_a}{\partial P} \frac{\partial \Lambda_b}{\partial Q} - \frac{\partial \Lambda_a}{\partial Q} \frac{\partial \Lambda_b}{\partial P}$$

vanishes on the constraint surface $\Lambda_a(Q, P) = 0$. The quantum constraints should depend on the small parameter $\hbar$ according to formula (2.3). However, the ”quantum” corrections can be also nontrivial, so that in general one can expect the following dependence of quantum constraints on the small parameter:

$$\hat{\Lambda}_a = \frac{1}{\hbar} \Lambda_a(\sqrt{\hbar}q, \sqrt{\hbar}p) + \Lambda_a^{(1)}(\sqrt{\hbar}q, \sqrt{\hbar}p) + ...$$ (2.4)

The simplest case is abelian, when the quantum constraints are Hermitian and commute each other not only on the constraint surface,

$$[\hat{\Lambda}_a, \hat{\Lambda}_b] = 0.$$ 

Consider the quantum constrained system in the refined algebraic quantization approach [18] for the case of continuous spectrum of $\hat{\Lambda}_a$. The constraints are taken into account as follows. The inner product of the wave functions $\Phi_1(q)$ and $\Phi_2(q)$ is introduced as

$$<\Phi_1, \Phi_2> = \left(\Phi_1, \prod_{a=1}^{M} \delta(\hat{\Lambda}_a)\Phi_2\right) = \int dq\Phi_1^\ast(q) \prod_{a=1}^{M} \delta(\hat{\Lambda}_a)\Phi_2(q).$$ (2.5)

Since the inner product (2.3) is degenerate, the obtained inner product space should be factorized: states with zero norm are set to be equivalent to zero. The corresponding factorspace should be completed in order to obtain a Hilbert space.

3. The nonabelian closed algebra case can be considered with the help of the group averaging prescription [19]. Let $\hat{\Lambda}_a$ be Hermitian quantum constraints satisfying the commutation relations

$$[\hat{\Lambda}_a; \hat{\Lambda}_b] = if_{ab}^{c} \hat{\Lambda}_c,$$

where $f_{ab}^{c}$ are structure constants of the Lie algebra. Consider the corresponding representation of the Lie group of the form

$$T(\exp(i\mu^a L_a)) = \exp(i\mu^a \hat{\Lambda}_a),$$

where $L_a$ are generators of the Lie algebra with structure constants $f_{ab}^{c}$. exp is an exponential mapping between algebra and group. The inner product is introduced as follows [19],

$$(\Phi_1, \int d_R g (det Adg)^{-1/2} T(g) \Phi_2).$$ (2.6)

Here $d_R g$ is a right-invariant Haar measure on the Lie group. Eq.(2.3) is a partial case of (2.6).

Note that eq.(2.6) can be rewritten as follows. Consider the modified non-Hermitian constraints

$$\tilde{\Lambda}_a = \hat{\Lambda}_a - \frac{i}{2} f_{ab}^{c} \hat{\Lambda}_c$$
obeying the same commutation relations

\[
[\hat{\Lambda}_a; \hat{\Lambda}_b] = i f^c_{ab} \hat{\Lambda}_c,
\]

Formula (2.6) will be rewritten then as

\[
(\Phi_1, \int d_R g \exp(i\mu^a \hat{\Lambda}_a) \Phi_2).
\]

4. The case of constrained algebra with structure functions

\[
[\hat{\Lambda}_a; \hat{\Lambda}_b] = i \hat{\Lambda}_c \hat{U}_{ab}^c,
\]

where \(\hat{U}_{ab}^c\) is more complicated [22]. One should use the relationship between Dirac and BRST-BFV approaches and make use of the BRST-BFV inner product [23]. Introduce additional Grassmannian variables \(\Pi_a, \Pi^a, a = 1, M\). The quantum constrained system with an open algebra is specified by the B-charge

\[
\hat{\Omega}_0 = \sum_{n=1}^{\infty} \hat{\Omega}^{nb_1...b_n-1}_{a_1...a_n} \Pi_{b_1}...\Pi_{b_n-1} \frac{\partial}{\partial \Pi_{a_1}}...\frac{\partial}{\partial \Pi_{a_n}}
\]

being a Hermitian nilpotent operator, \(\hat{\Omega}_0^+ = \hat{\Omega}_0, \hat{\Omega}_0 \hat{\Omega}_0 = 0\). The first component of the B-charge can be identified with the quantum constraint,

\[
\hat{\Omega}_0^1 = \hat{\Lambda}_a.
\]

The semiclassical structure of the B-charge should be as follows:

\[
\hat{\Omega}_0 \sim \Omega_0(\sqrt{h} \hat{q}, \sqrt{h} \hat{p}, \sqrt{h} \Pi, \sqrt{h} \frac{\partial}{\partial \Pi}).
\]

If the quantum constraints depend on the small parameter according to eq.(2.3), the higher order structure functions \(\hat{\Omega}^n\) should depend on \(h\) as follows,

\[
\hat{\Omega}^{nb_1...b_n-1}_{a_1...a_n} = h^{n-2} \Omega^{nb_1...b_n-1}_{a_1...a_n}(\sqrt{h} \hat{q}, \sqrt{h} \hat{p}) + h^{n-1} \Omega^{n(1)b_1...b_n-1}_{a_1...a_n}(\sqrt{h} \hat{q}, \sqrt{h} \hat{p}) + ...
\]

The classical structure functions \(\Omega^n\) were constructed in [12] from the relation \(\{\Omega_0, \Omega_0\} = 0\). Higher quantum corrections \(\Omega^{n(1)}, \cdots\) can be calculated in analogous way. Let us investigate the corollaries of the conditions \(\hat{\Omega}_0^+ = \hat{\Omega}_0\) and \(\hat{\Omega}_0 \hat{\Omega}_0 = 0\) for the operators \(\hat{\Lambda}_a\) and \(\hat{\Omega}^{2b}_{a_1a_2}\). Property \(\hat{\Omega}_0 \hat{\Omega}_0 = 0\) implies that

\[
[\hat{\Lambda}_{a_1}; \hat{\Lambda}_{a_2}] + 2 \Lambda_b \hat{\Omega}^{2b}_{a_1a_2} = 0,
\]

so that the operator \(\hat{\Omega}^{2b}_{a_1a_2}\) is related to the structure functions \(\hat{U}^b_{a_1a_2}\) entering to eq.(2.8) as follows,

\[
\hat{\Omega}^{2b}_{a_1a_2} = -\frac{i}{2} \hat{U}^b_{a_1a_2}.
\]

We also see that the structure functions should depend on \(h\) as

\[
\hat{U}^b_{a_1a_2} = U^b_{a_1a_2}(\sqrt{h} \hat{q}, \sqrt{h} \hat{p}) + h U^{(1)b}_{a_1a_2}(\sqrt{h} \hat{q}, \sqrt{h} \hat{p}) + ...
\]

Since the Weyl symbol of the commutator is proportional to the Poisson bracket of the operators (see Appendix A), in the leading order in \(h\) one finds

\[
\{\Lambda_a; \Lambda_b\} = -\Lambda_c U^c_{ab}.
\]
Property $\hat{\Omega}_0^+ = \hat{\Omega}_0$ implies that

$$
(\Omega^{2a}_{\alpha_1\alpha_2})^* = -\Omega^{2b}_{\alpha_1\alpha_2}, \quad \Lambda_a^* = \Lambda_a, \quad \Lambda_a^{(1)*} - \Lambda_a^{(1)} = 2\Omega^{2b}_{\alpha_1\alpha_2}.
$$

(2.13)

We see that the classical constraints are real, while the quantum corrections $\Lambda_a^{(1)}$ have nontrivial imaginary part,

$$
\Lambda_a^{(1)} = Re\Lambda_a^{(1)} + \frac{i}{2} U_{aa}^d.
$$

(2.14)

The inner product of states is written as [22]

$$
< \Phi_1, \Phi_2 > = (\Phi_1, \int \prod_{a=1}^M d\mu_a d\Pi_a e^{-\Pi_a\Pi_a + i\mu_a [\Pi_a ; \hat{\Omega}_0]} \Phi_2).
$$

(2.15)

Formula (2.15) coincides with (2.7) for the closed-algebra case.

### 2.2 "Elementary" semiclassical states for constrained systems and their inner product

The most popular semiclassical approach to quantum mechanics is the WKB-approach based on substitution of rapidly oscillating wave function to the Schrödinger equation and estimation of the accuracy. However, there exist other types of wave functions which approximately satisfy the Schrödinger equation (appendix B). Such semiclassical solutions can be somehow classified with the help of the Maslov theory of Lagrangian manifolds with complex germ [10]. It happens, however, that one can consider first the wave packet solutions of the Schrödinger equation such that uncertainties of coordinates and momenta are of orders $O(\sqrt{\hbar})$ or after rescaling (2.2) $O(1)$ (contrary to $O(1/\sqrt{\hbar})$ in the WKB-approach). Other semiclassical wave functions (including WKB) can be viewed as superpositions of wave packets. Thus, wave packet states may be considered as "elementary" semiclassical states, while other semiclassical wave functions are "composed" states to be considered in section 6.

In the notations (2.2), elementary semiclassical state corresponds to the wave function

$$
\Psi(q) = ce^{iS} e^{iP(q\sqrt{\hbar} - Q)} f(q - Q/\sqrt{\hbar}).
$$

(2.16)

It is specified by classical variables ($S \in \mathbb{R}$, $P \in \mathbb{R}^n$, $Q \in \mathbb{R}^n$) and "quantum" function $f$ which is smooth and rapidly damps at the infinity.

Let us investigate the inner product of semiclassical states.

#### 2.2.1 Abelian case

Formula (2.3) can be rewritten as

$$
< \Phi, \Phi > = \int \prod_{a=1}^M d\mu_a (\Phi, e^{i\mu_a \hat{\Lambda}_a} \Phi)
$$

(2.17)

It is necessary to calculate the wave function

$$
\Phi^\tau = e^{-i\tau \mu_a \hat{\Lambda}_a} \Phi
$$

as $h \to 0$ at $\tau = -1$. Note that it satisfies the Schrödinger-type equation

$$
i \frac{\partial \Phi^\tau}{\partial \tau} = \mu_a \hat{\Lambda}_a \Phi^\tau = \left[ \frac{1}{\hbar} \mu_a \Lambda_a (\sqrt{\hbar}\hat{q}, \sqrt{\hbar}\hat{p}) + \mu_a \Lambda_a^{(1)} (\sqrt{\hbar}\hat{q}, \sqrt{\hbar}\hat{p}) + ... \right] \Phi^\tau
$$

(2.18)
and initial condition (2.16). Let us look for the asymptotic solution in the following form

$$\Phi^\tau(q) = c e^{\tau^2 S^\tau} e^{i \tau P^\tau(\sqrt{\hbar} Q^\tau - q^\tau)} f^\tau(q - \frac{Q^\tau}{\sqrt{\hbar}})$$

Substituting expression (2.19) to eq.(2.18), we find:

$$\left[ -\frac{1}{\hbar} (\dot{S}^\tau - P^\tau \dot{Q}^\tau) - \frac{1}{\sqrt{\hbar}} \dot{P}^\tau \xi + i \frac{\partial}{\partial \tau} - i \sqrt{\hbar} \dot{Q}^\tau \frac{\partial}{\partial \xi} \right] f^\tau(\xi) = \frac{1}{\hbar} \mu_a \Lambda_a(Q^\tau + \sqrt{\hbar} \xi, P^\tau - i \sqrt{\hbar} \frac{\partial}{\partial \xi}) f^\tau(\xi) + \ldots$$

where $\xi = q - Q^\tau / \sqrt{\hbar}$. It is shown in Appendix A that the operator $\Lambda_a(Q + \sqrt{\hbar} \xi, P - i \sqrt{\hbar} \frac{\partial}{\partial \xi})$ is expanded in $\sqrt{\hbar}$ as

$$\Lambda_a(Q + \sqrt{\hbar} \xi, P - i \sqrt{\hbar} \frac{\partial}{\partial \xi}) = \Lambda_a(Q, P) + \sqrt{\hbar} (\Xi \Lambda_a)(Q, P) + \frac{\hbar}{2} (\Xi^2 \Lambda_a)(Q, P) + \ldots$$

where

$$\Xi = \xi \frac{\partial}{\partial Q} + \frac{1}{i} \frac{\partial}{\partial \xi} \frac{\partial}{\partial P}.$$ 

The terms of the order $O(\hbar^{-1})$ in eq.(2.20) give us an equation on the phase factor $S^\tau$

$$\dot{S}^\tau = P^\tau \dot{Q}^\tau - \mu_a \Lambda_a(Q^\tau, P^\tau)$$

We see that $S^\tau$ is the action on the classical trajectory.

The terms of the order $O(\hbar^{-1/2})$ lead to classical equations

$$\dot{Q}^\tau = \mu_a \frac{\partial \Lambda_a}{\partial P}(Q^\tau, P^\tau), \quad \dot{P}^\tau = -\mu_a \frac{\partial \Lambda_a}{\partial Q}(Q^\tau, P^\tau),$$

Under conditions (2.21) and (2.22), we find in the leading order in $\hbar$ the following equation on $f$

$$i \frac{\partial f^\tau(\xi)}{\partial \tau} = \left[ \frac{1}{2} \mu_a (\Xi^2 \Lambda_a)(Q^\tau, P^\tau) + \mu_a \Lambda_a^{(1)}(Q^\tau, P^\tau) \right] f^\tau(\xi)$$

with the quadratic Hamiltonian.

Let us substitute the wave function (2.19) at $\tau = -1$ to formula (2.17). First of all, notice that the inner product ($\Phi, \Phi^\tau$) is not exponentially small only if

$$|P^\tau - P| \leq O(\sqrt{\hbar}) \quad |Q^\tau - Q| \leq O(\sqrt{\hbar}).$$

Namely, the wave function $\Phi^\tau(q)$ is not exponentially small only if $q - Q^\tau / \sqrt{\hbar} = O(1)$, so that $\Phi^*(q) \Phi^\tau(q)$ will be not exponentially small if

$$q - Q^\tau / \sqrt{\hbar} = O(1), \quad q - Q / \sqrt{\hbar} = O(1).$$

Therefore, $|Q^\tau - Q|$ should be $\leq O(\sqrt{\hbar})$. If $|P^\tau - P| > O(\sqrt{\hbar})$, the integral $\int dq \Phi^*(q) \Phi^\tau(q)$ will contain rapidly oscillating factor and be then exponentially small.

Several cases should be considered. Here we investigate the simplest ”general position” or ”free” case (taking place in QED, Yang-Mills theories), when the action of the gauge group on the classical phase space is free, i.e. the stationary subgroup of any point is trivial. This means that $Q^\tau = Q$, $P^\tau = P$ only if $\tau = 0$. Nonfree case will be briefly discussed in section 7.

Conditions (2.24) are satisfied in the free case only if $\tau = O(\sqrt{\hbar})$. It is convenient then to perform a substitution

$$\mu_a \Rightarrow \mu_a \sqrt{\hbar}$$

6
for eq. (2.17). After substitution $q - Q^0 / \sqrt{\hbar} = \xi$ formula (2.17) is taken to the form

$$< \Phi, \Phi > = h^{M/2} |c|^2 \int d\mu d\xi e^{i(S - \sqrt{\hbar} - S^0 + P - \sqrt{\hbar}(Q^0 - Q - \sqrt{\hbar}))} e_{\sqrt{\hbar}}(P - \sqrt{\hbar} - P^0) f^*(\xi) f(-\sqrt{\hbar}(\xi - \frac{Q - \sqrt{\hbar} - Q^0}{\sqrt{\hbar}}))$$

(2.25)

Since

$$S - \sqrt{\hbar} = S^0 + P - \sqrt{\hbar}(Q^0 - Q - \sqrt{\hbar}) = -\sqrt{\hbar} (\dot{S}^0 - P^0 \dot{Q}^0) + \frac{h}{2} (\dot{S}^0 - P^0 \dot{Q}^0) - h \dot{P}^0 \dot{Q}^0 + o(h),$$

expression (2.25) is taken to the form

$$< \Phi, \Phi > \approx h^{M/2} |c|^2 \int d\mu d\xi e^{-i(S^0 - P^0 \dot{Q}^0)} e^{i(S^0 - P^0 \dot{Q}^0 - 2P^0 Q^0)} e^{-iP^0 \dot{Q}^0} f^*(\xi) f(\xi + \dot{Q}^0).$$

(2.26)

We see that if

$$\dot{S}^0 - P^0 \dot{Q}^0 = -\mu_a \Lambda_a(Q^0, P^0) \neq 0,$$

the integrand entering to eq. (2.26) contains a rapidly oscillating function. Therefore, integral (2.26) is exponentially small. We see that the wave function (2.16) $\Phi$ is a nontrivial semiclassical state, $< \Phi, \Phi > \neq 0$, only if

$$\Lambda_a(Q, P) = 0.$$  

(2.27)

This means that the classical state should belong to the constraint surface.

Differentiating eq. (2.21) with respect to $\tau$ at $\tau = 0$, we find

$$\ddot{S}^0 - P^0 \dot{Q}^0 = \dot{P}^0 \dot{Q}^0 - \frac{d}{d\tau}|_{\tau=0} \mu_a \Lambda_a(Q^\tau, P^\tau) = \dot{P}^0 \dot{Q}^0.$$  

Making use of the Baker-Hausdorff formula, we simplify eq. (2.20) under condition (2.27),

$$< \Phi, \Phi > \approx h^{M/2} |c|^2 \int d\mu d\xi f^*(\xi) e^{iP^0 \dot{Q}^0} f(\xi).$$

It follows from Hamiltonian equations (2.22) that the inner product of semiclassical states is

$$< \Phi, \Phi > \approx h^{M/2} |c|^2 \int d\xi d\mu f^*(\xi) e^{i \mu_a \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \mu} + \frac{\partial}{\partial \mu^a} \right)} f(\xi) = h^{M/2} |c|^2 \left( f, \prod_{a=1}^M \left( 2 \pi \delta(\Xi \Lambda_a) \right) f \right)$$

(2.28)

under condition (2.27).

We see that the normalization factor $|c|$ should be of the order $h^{-M/4}$ in order to make the norm of state (2.16) to be of the order $O(1)$. We also notice that the semiclassical inner product is obtained from the quantum formula (2.3) by linearization procedure: the constraint operators

$$\Lambda_a(Q + \sqrt{\hbar} \xi, P + \frac{1}{i} \sqrt{\hbar} \frac{\partial}{\partial \xi})$$

should be linearized. It has been understood that an important condition of validity of linearization prescription is free action of classical gauge group.

### 2.2.2 Nonabelian and open-algebra case

Let us investigate the inner product (2.7) for the nonabelian gauge group in the semiclassical approximation. The free case only will be considered. The integral in the inner product (2.7) is not exponentially small only if $\mu = O(\sqrt{\hbar})$. After rescaling $\mu \Rightarrow \mu \sqrt{\hbar}$ formula (2.7) takes the form

$$\hbar^{M/2} \int d\mu J(\mu \sqrt{\hbar})(\Phi^0, \Phi^{-v})$$
where $\Phi^\tau$ is a solution of the Cauchy problem for eq. (2.18) with initial condition $\Phi^0 = \Phi$ (2.16). The Jacobian $J(\mu)$ normalized as $J(0) = 1$ is defined from formula $dRg = \mu J(\mu)$. Analogously to the previous subsubsection, we obtain condition (2.27) and formula (2.28).

The open-algebra case can be investigated in the same way. Consider the wave function

$$\Phi^\tau(q, \Pi, \Pi) = \exp\left\{\tau \Pi_a \Pi^a - i\tau \mu_a [\Pi_a, \hat{\Omega}_0]_+ \right\} \Phi(q). \tag{2.29}$$

obeying the equation

$$i \frac{\partial \Phi^\tau}{\partial \tau} = \left[\mu_a \hat{\Lambda}_a + \sum_{n=2}^\infty n \hat{\Omega}^{a_1 \ldots a_n} \Pi_{a_1} \ldots \Pi_{a_n} \frac{\partial}{\partial \Pi_{a_1}} \ldots \frac{\partial}{\partial \Pi_{a_n}} \right] \Phi^\tau$$

and initial condition (2.16). Let us look for the solution of the Cauchy problem in the form

$$\Phi^\tau(q, \Pi, \Pi) = c e^{i \frac{\xi}{\sqrt{\hbar}} S^\tau} e^{\frac{i}{\sqrt{\hbar}} (P^\tau q - Q^\tau)} f^\tau(q - \frac{Q^\tau}{\sqrt{\hbar}}, \Pi, \Pi). \tag{2.30}$$

The integrand in the inner product (2.17) is not exponentially small only if $\mu = O(\sqrt{\hbar})$. After rescaling $\mu \Rightarrow \mu \sqrt{\hbar}$, we obtain conditions (2.27). The inner product takes the form

$$< \Phi, \Phi > \propto h^{M/2} |c|^2 \int \Pi^a \Pi = 1 d\mu_a d\Pi_a d\Pi^a (f, \Pi)^M | \Pi_a^a - i(\hat{F} + \hat{\Omega}^a + \hat{\Omega}^a) f \rangle.$$

We also obtain formula (2.28).

We see that the obtained expression for the semiclassical inner product is valid for the nonabelian and open algebra cases. We also see that the nontriviality of condition (2.27) is also valid for such cases.

### 2.3 Semiclassical bundle

We see that a semiclassical wave function (2.16) is specified if:

(i) a set $X = (S, P, Q)$ ("classical state") satisfying the requirement (2.27) is specified;

(ii) a smooth rapidly damping at the infinity function $f \in S(R^n)$ is specified.

Semiclassical wave functions will be denoted as $(X, f)$. The inner product of semiclassical wave functions is introduced as

$$<(X, f_1); (X, f_2)> = (f_1, \Pi = 1 \frac{M}{a=1} 2\pi \delta([\Xi \Lambda_a]_a(X)) f_2), \tag{2.31}$$

provided that classical gauge group is free. Note that according to Appendix A the operators $\Xi \Lambda_a$ commute each other on the constraint surface (2.27), since

$$[\Xi \Lambda_a, \Xi \Lambda_b] = -i\{\Lambda_a, \Lambda_b\} = iU^{\mu}_{ab} \Lambda_c = 0.$$
By \( F_X^0 \) we denote the inner product space of complex functions \( f \in \mathcal{S}(\mathbb{R}^n) \) with the inner product (2.31). Since it is degenerate, one should factorize the corresponding pre-Hilbert space as follows: two functions \( f_1 \) and \( f_2 \) are called equivalent, \( f_1 \equiv f_2 \), if their difference \( \phi = f_1 - f_2 \) has zero norm,

\[
(\phi, \prod_{a=1}^{M} 2\pi \delta((\Xi\Lambda_a)(X))\phi) = 0.
\]

For example, consider the wave function \( \phi \) of the form

\[
\phi = (\Xi\Lambda_a)(X)\chi^a.
\]

It has zero norm, so that the transformation

\[
f \to f + (\Xi\Lambda_a)(X)\chi^a
\]

(2.32)
takes the semiclassical wave function \((X, f)\) to the equivalent semiclassical wave function \((X, f + (\Xi\Lambda_a)(X)\chi^a)\). Since the classical state \( X \) does not vary during the transformation (2.32), it can be called as a "small" gauge transformation of the semiclassical wave function. "Large" gauge transformations varying the classical state \( X \) will be considered in the next section.

By \( F_X = F_X^0 / \sim \) we denote the factorspace of equivalence classes \([f]\). Let the Hilbert space \( F_X \) be a completeness of the pre-Hilbert space \( F_X \).

We see that it is more correct to consider "elementary" semiclassical states as pairs \((X, f)\), \( f \in F_X \) rather that pairs \((X, f)\). The set of all "elementary" semiclassical states can be viewed as a bundle. The base of the bundle is \( \mathcal{X} = \{(S, P, Q)|\Lambda_a(Q, P) = 0\} \), the fibres are Hilbert spaces \( F_X \). Such a bundle was called semiclassical in [24, 25]. "Elementary" semiclassical states are then points on the semiclassical bundle.

### 3 Gauge equivalent semiclassical states

#### 3.1 "Small" and "large" gauge transformations

Property of gauge invariance plays an important role in classical mechanics of constrained systems. This property means that classical constraints generate gauge transformations on classical phase space. Classical states that belong to one orbit of gauge transformation are called equivalent.

An analogous property takes place for the semiclassical theory as well. Namely, the state \( \hat{\Lambda}_aX^a \) has zero norm \([19, 22]\). Let

\[
X^a(q) = e^{i\frac{S}{\sqrt{\hbar}}} e^{i\frac{P(q\sqrt{\hbar} - Q)}{\sqrt{\hbar}}} \chi^a(q - Q/\sqrt{\hbar}).
\]

The wave function \( h^{-1/2}\hat{\Lambda}_aX^a \) has the following form in the leading order in \( \sqrt{\hbar} \):

\[
h^{-1/2}\hat{\Lambda}_aX^a = e^{i\frac{S}{\sqrt{\hbar}}} e^{i\frac{P(q\sqrt{\hbar} - Q)}{\sqrt{\hbar}}} (\Xi\Lambda_a\chi^a)(q - Q/\sqrt{\hbar}),
\]

since \( \Lambda_a(Q, P) = 0 \). We have obtained the "small" gauge transformation (2.32).

To obtain a "large" gauge transformation, note that semiclassical wave functions

\[
\Phi \quad \text{and} \quad e^{-i\tau\mu_0\hat{\Lambda}_a}\Phi
\]

should be called gauge-equivalent. However, the wave function \( \Phi^\tau = e^{-i\tau\mu_0\hat{\Lambda}_a}\Phi \) has been already calculated in the semiclassical approximation. It has the form (2.19), where \( S^\tau \) satisfies eq.(2.21), \( P^\tau, Q^\tau \) obey the classical Hamiltonian equations (2.22), while \( f^\tau \) is a solution of eq.(2.23).
By $\lambda_{\mu\tau}$ we denote the mapping taking $X = (S, P, Q)$ to $X^\tau = (S^\tau, P^\tau, Q^\tau)$ which is a classical gauge transformation. By $V^0(\lambda_{\mu\tau}X \leftarrow X) : \mathcal{F}^0_X \rightarrow \mathcal{F}^0_{\lambda_{\mu\tau}X}$ we denote the operator taking the initial condition for the Cauchy problem \( (2.23) \) to the solution of the Cauchy problem,

$$f^\tau = V^0(\lambda_{\mu\tau}X \leftarrow X)f^0.$$ 

The semiclassical wave functions

$$(X, f) \quad \text{and} \quad (\lambda_{\mu\tau}X, V^0(\lambda_{\mu\tau}X \leftarrow X)f)$$

are gauge-equivalent then. This is a ”large” gauge transformation. Obviously, it conserves the conditions $\Lambda_c(X) = 0$, since $\{\Lambda_a; \Lambda_c\} = 0$ on the constraint surface.

### 3.2 Unitarity problem

Let us show that semiclassical gauge transformation conserves the inner product \( (2.31) \). First, consider the commutator between operators $i\frac{\partial}{\partial \tau} - \mu_a \left[ \frac{1}{2}(\Xi^2 \Lambda_a)(Q^\tau, P^\tau) + \Lambda^{(1)}_a(Q^\tau, P^\tau) \right]$ and $\Xi \Lambda_b(Q^\tau, P^\tau)$. Making use of results of Appendix A, we find that it can be presented as

$$\left[ i\frac{\partial}{\partial \tau} - \mu_a \left[ \frac{1}{2}(\Xi^2 \Lambda_a)(Q^\tau, P^\tau) + \Lambda^{(1)}_a(Q^\tau, P^\tau) \right] ; \Xi \Lambda_b(Q^\tau, P^\tau) \right] = i(\Xi \{\mu_a \Lambda_a, \Lambda_b\})(Q^\tau, P^\tau). \quad (3.1)$$

It follows from eq.\((2.12)\) that

$$i\Xi \{\mu_a \Lambda_a; \Lambda_b\} = -i\Xi(\Lambda_c U_{ab}^c) = -i U_{ab}^c \Xi \Lambda_c \quad (3.2)$$

on the constraint surface $\Lambda_c = 0$. Thus, the commutator \((3.1)\) takes the form

$$\left[ i\frac{\partial}{\partial \tau} - \mu_a \left[ \frac{1}{2}(\Xi^2 \Lambda_a)(Q^\tau, P^\tau) + \Lambda^{(1)}_a(Q^\tau, P^\tau) \right] ; \Xi \Lambda_b(Q^\tau, P^\tau) \right] = -i U_{ab}^c (Q^\tau, P^\tau)(X_1 \Lambda_c)(Q^\tau, P^\tau). \quad (3.3)$$

Let $f_1^\tau$ and $f_2^\tau$ be solutions of eq.\((2.23)\). The time derivative of their inner product $<f_1^\tau, f_2^\tau>$ \( (2.31) \) can be written as

$$i \frac{\partial}{\partial \tau} <f_1^\tau, f_2^\tau> = (-i \dot{f}_1^\tau, \Pi_a(2\pi \delta(\Xi \Lambda_a)(Q^\tau, P^\tau))f_2^\tau) + (f_1^\tau, \Pi_a(2\pi \delta(\Xi \Lambda_a)(Q^\tau, P^\tau))i \dot{f}_2^\tau)$$

$$+ (f_1^\tau, i \frac{\partial}{\partial \tau} \{\Pi_a(2\pi \delta(\Xi \Lambda_a)(Q^\tau, P^\tau))\} f_2^\tau) \quad (3.4)$$

Making use of equation of motion \((2.24)\). Taking into account relation \((2.14)\), we take expression \((3.4)\) to the form

$$i \frac{\partial}{\partial \tau} <f_1^\tau, f_2^\tau> = i \mu_a U_{ba}^c(Q^\tau, P^\tau) <f_1^\tau, f_2^\tau> + (f_1^\tau, \left[ i \frac{\partial}{\partial \tau} - \mu_a \frac{1}{2}(\Xi^2 \Lambda_a)(Q^\tau, P^\tau), \int d\rho e^{i \rho_0(\Xi \Lambda_a)(Q^\tau, P^\tau)} \right] f_2^\tau) \quad (3.5)$$

Commutation relation \((3.3)\) implies the following commutation rule

$$e^{-i \rho_0(\Xi \Lambda_a)(Q^\tau, P^\tau)} \left\{ i \frac{\partial}{\partial \tau} - \mu_a \frac{1}{2}(\Xi^2 \Lambda_a)(Q^\tau, P^\tau) \right\} e^{i \rho_0(\Xi \Lambda_a)(Q^\tau, P^\tau)} = i \frac{\partial}{\partial \tau} - \mu_a \frac{1}{2}(\Xi^2 \Lambda_a)(Q^\tau, P^\tau)$$

$$- i \rho_0 \left[ (\Xi \Lambda_b)(Q^\tau, P^\tau); i \frac{\partial}{\partial \tau} - \mu_a \frac{1}{2}(\Xi^2 \Lambda_a)(Q^\tau, P^\tau) \right]$$

Higher order terms will vanish since $[\Xi \Lambda_a; \Xi \Lambda_b] = -i \{\Lambda_a, \Lambda_b\} = 0$ on the constraint surface. Therefore,

$$\left[ i \frac{\partial}{\partial \tau} - \mu_a \frac{1}{2}(\Xi^2 \Lambda_a)(Q^\tau, P^\tau); e^{i \rho_0(\Xi \Lambda_a)(Q^\tau, P^\tau)} \right] = -i \mu_a U_{ab}^c(Q^\tau, P^\tau) \rho_b \frac{\partial}{\partial \rho_c} e^{i \rho_0(\Xi \Lambda_a)(Q^\tau, P^\tau)}$$
Integrating this expression over $\rho$ by parts, we find that
\[
\left[i \frac{\partial}{\partial \tau} - \mu_a \frac{1}{2} (\Xi^2 \Lambda_a)(Q^r, P^r); \int d\rho e^{i\phi(\Xi \Lambda_a)(Q^r, P^r)} \right] = i \mu_a U_{ab}^r (Q^r, P^r) \int d\rho e^{i\phi(\Xi \Lambda_a)(Q^r, P^r)}
\]
Substituting this result to eq.(3.5), we obtain that "large" gauge transformations conserve the inner product,
\[
< f_1^r, f_2^r > = \text{const.}
\]
This implies that zero-norm semiclassical states are taken to zero-norm states. Therefore, one can correctly define the operators $V(\lambda_{\mu \tau} X \leftarrow X) : \mathcal{F}_X \rightarrow \mathcal{F}_{\lambda_{\mu \tau} X}$ by the formula
\[
V(\lambda_{\mu \tau} X \leftarrow X)[f] = [V^0(\lambda_{\mu \tau} X \leftarrow X)f].
\]
The operators $V(\lambda_{\mu \tau} X \leftarrow X)$ are also unitary. They can be uniquely extended to the completion $\overline{\mathcal{F}}_X$,
\[
\overline{V}(\lambda_{\mu \tau} X \leftarrow X) : \overline{\mathcal{F}}_X \rightarrow \overline{\mathcal{F}}_{\lambda_{\mu \tau} X}.
\]
Thus, the semiclassical states
\[
(X, \overline{f}) \quad \text{and} \quad (\lambda_{\mu \tau} X, \overline{V}(\lambda_{\mu \tau} X \leftarrow X)\overline{f})
\]
are equivalent. Gauge transformations appears to be morphisms of the semiclassical bundle.

### 3.3 Quasigroup properties

Let us show that composition of gauge equivalence transformations is a gauge equivalence transformation, i.e. for any $X \in \mathcal{X}$ and sufficiently small $\mu_1, \mu_2$ there exist $\mu_3(\mu_1, \mu_2, X)$ such that
\[
(\lambda_{\mu_1} \lambda_{\mu_2} X, \nabla_{\mu_1} (\lambda_{\mu_1} \lambda_{\mu_2} X \leftarrow \lambda_{\mu_2} X) \nabla_{\mu_2} (\lambda_{\mu_2} X \leftarrow X)\overline{f}) = (\lambda_{\mu_3} X, \overline{V}(\lambda_{\mu_3} X \leftarrow X)\overline{f}), \quad \overline{f} \in \overline{\mathcal{F}}_X.
\]
This means that set of gauge transformations form a local Batalin quasigroup [26].

First of all, investigate the classical gauge transformations.

1. Introduce the first-order differential operators $\delta_a$ from the relations:
\[
\mu^a (\delta_a f)(X) = \left. \frac{d}{d\tau} \right|_{\tau=0} f(\lambda_{\mu \tau} X). \tag{3.6}
\]
It follows from definition of $\lambda_{\mu \tau}$ that
\[
\delta_a = P \frac{\partial \Lambda_a}{\partial P} \frac{\partial}{\partial S} + \frac{\partial \Lambda_a}{\partial P} \frac{\partial}{\partial Q} - \frac{\partial \Lambda_a}{\partial Q} \frac{\partial}{\partial P}
\]
The operators $\delta_a$ satisfy the following commutation relations
\[
[\delta_a; \delta_b] = P \frac{\partial \Lambda_a}{\partial P} \{ \Lambda_a; \Lambda_b \} \frac{\partial}{\partial S} + \frac{\partial \{ \Lambda_a; \Lambda_b \}}{\partial P} \frac{\partial}{\partial Q} - \frac{\partial \{ \Lambda_a; \Lambda_b \}}{\partial Q} \frac{\partial}{\partial P}
\]
since $\{ \Lambda_a; \Lambda_b \} = 0$ on the constraint surface. It follows from eq.(2.12) that
\[
[\delta_a; \delta_b] = -U_{ab}^c (Q, P) \delta_c.
\]
This means that the operators $\delta_a$ form a Batalin quasialgebra [26].
It is shown in [26] that the quasialgebra property implies the quasigroup property: for all $X \in \mathcal{X}$ and sufficiently small $\mu_1, \mu_2$

$$\lambda_{\mu_1} \lambda_{\mu_2} X = \lambda_{\mu_3(\mu_1, \mu_2, X)} X$$  \hspace{1cm} (3.7)

for some $\mu_3$.

2. Let us justify the formula

$$\nabla_{\mu_1}(\lambda_{\mu_1} \lambda_{\mu_2} X \leftarrow \lambda_{\mu_2} X) \nabla_{\mu_2}(\lambda_{\mu_2} X \leftarrow X) = \nabla_{\mu_3}(\lambda_{\mu_3} X \leftarrow X),$$  \hspace{1cm} (3.8)

provided that $\lambda_{\mu_1} \lambda_{\mu_2} X = \lambda_{\mu_3} X$. Remind that the operator $V^0_{\mu}(\lambda_{\mu} X \leftarrow X) \equiv V^0_{\mu}[X]$ is defined as follows. Let $V^0_{\mu}[X]$ be the operator taking initial condition for the cauchy problem for equation

$$i\frac{\partial f}{\partial \tau} = \mu^a H_a(\lambda_{\mu} X) f^\tau$$  \hspace{1cm} (3.9)

to the solution of the Cauchy problem. Here

$$H_a(Q, P) = \frac{1}{2} \Xi_2 \Lambda_a(Q, P) + \Lambda_a^{(1)}(Q, P).$$

Since the operator $V^0_{\mu}[X]$ conserves the norm and takes zero-norm states to zero-norm states, one can consider the operator $V_{\mu}(\lambda_{\mu} X \leftarrow X)$ being also isometric. It can be extended to $\mathcal{F}^0_{X/\sim}: \mathcal{F}_X \rightarrow \mathcal{F}_{\lambda_{\mu} X/\sim}.$

3. First of all, investigate corollaries of eq.(3.8) for infinitesimal operators $H_a(X)$.

Let us consider the composition of transformation $\lambda_{\mu} \lambda_{\nu} \lambda_{-\mu}$. For some function $\rho(\tau, X)$, one locally has

$$\lambda_{\mu} \lambda_{\nu} \lambda_{-\mu} X = \lambda_{\rho(\tau, X)} X.$$

Denote

$$(W_{\mu} f)(X) = f(\lambda_{\mu} X).$$

One has then

$$\frac{d}{dt}|_{t=0}(W_{-\mu} W_{\nu} W_{\mu} F)(X) = \frac{d}{d\tau}|_{\tau=0}(W_{\rho(\tau, X)} f)(X).$$

In the leading order in $\tau$, one has

$$\rho(\tau, X) \sim \tau^a \tau$$

for some $\tau^a$. It follows from definition (3.9) of the operator $\delta_a$ that

$$W_{-\mu} \nu^a \delta_a W_{\mu} = \tau^b \delta_b.$$  \hspace{1cm} (3.10)

where $\tau^b$ linearly depends on $\nu^a$. Denote the corresponding matrix of linear transformation as $(Ad_{X \mu})^b_a$, so that

$$\tau^b = (Ad_{X \mu})^b_a \nu^a$$  \hspace{1cm} (3.11)

and

$$\rho(\tau, X) \sim (Ad_{X \mu})^b_a \nu^a \tau.$$

Denote $\lambda_{-\mu} X = Y$.

If the property (3.8) is satisfied, then

$$< f, V^0_{\mu}(\lambda_{\mu} \lambda_{\nu} \lambda_{-\mu} Y \leftarrow \lambda_{\nu} Y) V^0_{\nu}(\lambda_{\nu} Y \leftarrow Y) g > =$$

$$< f, V^0_{\rho(\tau, \lambda, \mu)}(\lambda_{\rho(\tau, \lambda, \mu) Y} \leftarrow \lambda_{\rho(\tau, \lambda, \mu)} Y) V^0_{\mu}(\lambda_{\mu} Y \leftarrow Y) g >$$  \hspace{1cm} (3.12)
for \( f \in \mathcal{F}_\lambda^0 \), \( g \in \mathcal{F}_Y^0 \). Consider this identity as \( \tau \to 0 \). In the leading order in \( \tau \), this relation is trivial. Consider the first nontrivial order. One has:

\[
V_\mu^0(\lambda_\mu \lambda_\nu Y \leftarrow \lambda_\nu Y) \sim V_\mu^0[Y] + \tau (\nu^c \delta_a V_\mu)[Y];
\]

\[
V_{\nu \tau}^0(\lambda_\nu Y \leftarrow Y) \sim 1 - i\tau \nu^b H_a(Y);
\]

\[
V^0_{\nu(\tau_\lambda_\nu Y)}(\lambda_{\nu(\tau_\lambda_\nu Y)} \lambda_\mu Y \leftarrow \lambda_\mu Y) \sim 1 - i\tau \nu^b (Ad_{\lambda_\nu Y} \mu_a^b) H_b(\lambda_\mu Y).
\]

Combining the terms of the order \( O(\tau) \) in eq.(3.12), one obtains:

\[
<f, ((\delta_a V_\mu^0)[Y] - iV_\mu^0[Y] H_a(Y))g >= -i(Ad_{\lambda_\nu Y} \mu_a^b) < f, H_b(\lambda_\mu Y) V_\mu^0[Y] g > .
\]

(3.13)

Property (3.13) is very important.

4. Consider the substitution \( \mu \to \mu t \) and let \( t \to 0 \). First, obtain an equation for \( (Ad_{\lambda_\nu Y} \mu t_a^b) \).

Relations (3.10) and (3.11) can be presented as

\[
W_{\mu t} \delta_a W_{-\mu} = (Ad_X(-\mu t))_a^b \delta_b.
\]

It follows from definition (3.6) that

\[
\frac{d}{dt} W_{\mu t} = \mu^b \delta_b W_{\mu t} = W_{\mu t} \mu^b \delta_b.
\]

Therefore,

\[
\frac{d}{dt} (Ad_X(-\mu t))_a^b \delta_b = W_{\mu t} [\mu^b \delta_b; \delta_a] W_{-\mu} = \mu^b U_{ab}^c (\lambda_\mu X) W_\mu \delta_c W_{-\mu} = \mu^b U_{ab}^c (\lambda_\mu X) (Ad_X(-\mu t))_a^d \delta_d
\]

and

\[
\frac{d}{dt} (Ad_X(-\mu t))_a^d = \mu^b U_{ab}^c (\lambda_\mu X) (Ad_X(-\mu t))_c^d
\]

(cf. [26]).

Making use of eqs.(3.10) and (3.11) twice, one finds

\[
\delta_a = W_\mu (Ad_X \mu_a^b) W_{-\mu} W_\mu \delta_b W_{-\mu} = (Ad_{\lambda_\mu X})_a^b (Ad_X(-\mu t))_b^c \delta_c,
\]

so that

\[
(Ad_{\lambda_\nu Y} \mu t_a^b) = ((Ad_X(-\mu t))^{-1})_a^b \sim \delta_a^b + t \mu^d U_{ad}^b(Y).
\]

Furthermore,

\[
(\delta_a V_\mu)[Y] \sim -it \mu^b \delta_a H_b(Y); \quad -iV_\mu[Y] H_a(Y) \sim -iH_a(Y) - t \mu^b H_b(Y) H_a(Y),
\]

while the right-hand side of eq.(3.13) reads

\[
<f, [-iH_a(Y) - it \mu^d U_{ad}^b(Y) H_b(Y) - it \mu^c \delta_c H_c(Y) - t \mu^c H_a(Y) H_c(Y)] g > .
\]

We see that the property (3.13) implies the following algebraic relation:

\[
<f, ([H_a(Y), H_b(Y)] - i \delta_a H_b(Y) + i \delta_b H_a(Y)) g > = iU_{ab}^c(Y) < f, H_c(Y) g >
\]

or

\[
<f, [H_a(Y) - i \delta_a; H_b(Y) - i \delta_b] g > = iU_{ab}^c(Y) < f, (H_c(Y) - i \delta_c) g >
\]

(3.16)

for \( f, g \in \mathcal{F}_Y^0 \).

5. Let us show that the algebraic property (3.17) implies the group property (3.8). First of all, let us obtain eq.(3.13) from eq.(3.17).
**Proposition 3.1.** Let property \((3.17)\) be satisfied. Then eq.\((3.13)\) is also satisfied.

**Proof.** One should check that

\[
< f, (-i(V^0_{\mu t}[Y])^+(Ad_{\lambda t Y} Y t)^b_a H_b(\lambda t Y)V^0_{\mu t}[Y] - (V^0_{\mu t}[Y])^+(\delta a V^0_{\mu t}))[Y] > = -i < f, H_a(Y)g >
\]

for all \(f, g \in \mathcal{F}_Y^0\).

For \(t = 0\), eq.\((3.18)\) is satisfied. Consider the \(t\)-derivative of the left-hand side of eq.\((3.18)\). First, eq.\((3.9)\) implies that

\[
i \frac{\partial}{\partial t} V^0_{\mu t}[Y] = \mu^a H_a(\lambda t Y)V^0_{\mu t}[Y],
\]

while the operator \((\delta a V^0_{\mu t})[Y]\) satisfies the following equation:

\[
v^0 \frac{\partial}{\partial t} (\delta a V^0_{\mu t})[Y] = \mu b \frac{\partial}{\partial t} |_{\alpha = 0} H_b(\lambda_{t Y} \nu a Y)V^0_{\mu t}[Y] + \mu^b H_b(\lambda t Y)\nu^a (\delta a V^0_{\mu t})[Y]
\]

Therefore,

\[
\frac{d}{dt} [(V^0_{\mu t}[Y])^+(\delta a V^0_{\mu t})[Y]] = -i (V^0_{\mu t}[Y])^+(\delta a W_{\mu t} H_b) (Y) V^0_{\mu t}[Y].
\]

Furthermore, eq.\((3.15)\) implies the following relation:

\[
\frac{d}{dt} Ad_{\lambda t Y} Y t = \frac{d}{dt} (Ad_Y(-\mu t))^{-1} = -(Ad_Y(-\mu t))^{-1} \frac{d}{dt} (Ad_Y(-\mu t))(Ad_Y(-\mu t))^{-1}
\]

Eq. \((3.14)\) implies that

\[
\frac{d}{dt} (Ad_{\lambda t Y} Y t)^0_\nu = -(Ad_{\lambda t Y} Y t)^a_\mu d U^b_{cd}(\lambda t Y).
\]

Combining all the terms, we find that the derivative of the left-hand side of eq.\((3.18)\) is

\[
< f, (V^0_{\mu t}[Y])^+ \{ -(Ad_{\lambda t Y} Y t)^0_\nu H_b(\lambda t Y) \nu^c H_\nu(\lambda t Y) - i \mu^c \delta a H_b(\lambda t Y) (Ad_{\lambda t Y} Y t)^0_\nu + i (Ad_{\lambda t Y} Y t)^0_\nu \nu^a W_{\mu t} H_b(\lambda t Y) + i (W_{\mu t} \delta a W_{\mu t} H_b) (\lambda t Y) \} (V^0_{\mu t}[Y])g >
\]

However, it follows from eq.\((3.17)\) that this expression vanishes. Proposition is justified.

**Proposition 3.2.** The following property is satisfied:

\[
(\delta a V^0_{\mu t})[Y] = -i \int_0^t d \tau V^0_{\mu t'}(\lambda t Y \leftarrow \lambda t' Y))(\nu^b(\delta a W_{\mu t} H_b)(Y) V^0_{\mu t'}(\lambda t Y \leftarrow Y)).
\]

This property is a direct corollary of eq.\((3.19)\).

Let \(\mu(\alpha)\) be a smooth curve. Note that for arbitrary function \(F(X)\) the derivative \(\frac{d}{d\alpha} F(\lambda \mu(\alpha) X)\) can be presented as a linear combination of operators \(\delta a\),

\[
\frac{d}{d\alpha} F(\lambda \mu(\alpha) X) = \rho^a(\alpha, X) \delta a F(\lambda \mu(\alpha) X).
\]

**Proposition 3.3.** For all \(f \in \mathcal{F}^0_{\mu(\alpha) X}, g \in \mathcal{F}^0_{X}\)

\[
< f, i \frac{d}{d\alpha} V^0_{\mu(\alpha)}[X]g > = < f, \rho^a(\alpha, X) H_a(\lambda \mu(\alpha) X) V^0_{\mu(\alpha)}[X]g >
\]

**Proof.** Consider the operator

\[
(V^0_{\mu(\alpha)}[X])^{-1} \frac{d}{d\alpha} V^0_{\mu(\alpha)}[X]
\]
Its time derivative has the form

\[(V_{\mu(\alpha)}^0[X])^{-1} \frac{d}{d\alpha}; \mu^a(\alpha) H_b(\lambda_{\mu(\alpha)} X)] V_{\mu(\alpha)}^0[X]\]

Therefore,

\[(V_{\mu(\alpha)}^0[X])^{-1} \frac{d}{d\alpha} V_{\mu(\alpha)}^0[X] = \int_0^t d\gamma V_{-\gamma\mu(\alpha)}^0(X \leftrightarrow \lambda_{\gamma\mu(\alpha)} X)[\frac{d}{d\alpha}; \mu^a(\alpha) H_a(\lambda_{\mu(\alpha)} X)] V_{\mu(\alpha)}^0(\lambda_{\gamma\mu(\alpha)} X \leftrightarrow X).\]

Thus, eq. (3.22) is equivalent to

\[< f, \rho^a(\alpha, X) H_a(\lambda_{\mu(\alpha)} X) g, \geq< f, \int_0^t d\gamma V_{(1-\gamma)\mu(\alpha)}^0(\lambda_{\gamma\mu(\alpha)} X \leftrightarrow \lambda_{\gamma\mu(\alpha)} X) \frac{d}{d\alpha}; \mu^a(\alpha) H_a(\lambda_{\mu(\alpha)} X)[V_{(1-\gamma)\mu(\alpha)}^0(\lambda_{\gamma\mu(\alpha)} X \leftrightarrow \lambda_{\mu(\alpha)} X)](3.23)\]

for all \(f, g \in \mathcal{F}_X^0.\)

Making use of the justified relation (3.13), one takes the right-hand side of eq. (3.23) to the form

\[\int_0^1 d\gamma d\alpha \frac{d}{d\alpha}(Ad_{\lambda_{\gamma}(\gamma(\mu))})_a b H_b(\lambda_{\gamma\mu(\alpha)} X) - i \int_0^1 d\gamma (\delta_a V_{(1-\gamma)\mu(\alpha)}^0[\lambda_{\gamma\mu(\alpha)} X] V_{(1-\gamma)\mu(\alpha)}^0(\lambda_{\gamma\mu(\alpha)} X \leftrightarrow X) \frac{d}{d\alpha}; \mu^a(\alpha) H_a(\lambda_{\mu(\alpha)} X)] V_{(1-\gamma)\mu(\alpha)}^0(\lambda_{\gamma\mu(\alpha)} X \leftrightarrow \lambda_{\mu(\alpha)} X).\]

(3.24)

The second term can be rewritten with the help of proposition 3.2 as

\[- \int_0^1 d\gamma \int_0^{1-\tau} d\tau V_{(1-\gamma)\mu(\alpha)}^0[\lambda_{\gamma\mu(\alpha)} X] \mu^b(\delta_a W_{\tau\mu} H_b(\lambda_{\gamma\mu(\alpha)} X) V_{(1-\gamma)\mu(\alpha)}^0(\lambda_{\gamma\mu(\alpha)} X) \frac{d}{d\alpha}; \mu^a]\]

where substitution \(\gamma \rightarrow \tau\) has been made. Let us perform a shift of integration variable \(\tau, \tau = \gamma - \tau\), so that the integral will be transformed as

\[- \int_0^1 d\gamma \int_0^\tau d\tau V_{(1-\gamma)\mu(\alpha)}^0[\lambda_{\gamma\mu(\alpha)} X] \mu^b(\delta_a W_{\gamma\mu} H_b(\lambda_{\gamma\mu(\alpha)} X) V_{(1-\gamma)\mu(\alpha)}^0(\lambda_{\gamma\mu(\alpha)} X) \frac{d}{d\alpha}; \mu^a]\]

We see that the sum of the second and the third term will vanish if

\[\int_0^\gamma d\tau \mu^b(\delta_a W_{\tau\mu} H_b(\lambda_{\gamma-\tau\mu(\alpha)} X) \frac{d}{d\alpha}; \mu^a) = \mu^b(\delta_a \frac{d}{d\alpha} H_b(\lambda_{\gamma\mu(\alpha)} X).\]

(3.25)

Here the substitution \(\tau \leftrightarrow \gamma - \tau\) is performed. The first term of the sum (3.24) coincides with the left-hand side of eq. (3.23) if

\[\rho^b(\alpha, X) = \int_0^1 d\gamma \frac{d}{d\alpha}; (Ad_{\lambda_{\gamma}(\gamma(\mu))})_a b.\]

(3.26)

Definition (3.22) of \(\rho^b\) implies that relation (3.26) is equivalent to

\[\frac{d}{d\alpha} f(\lambda_{\mu(\alpha)} X) = \int_0^\gamma d\tau \frac{d}{d\alpha}; (Ad_{\lambda_{\gamma\mu(\gamma)(\mu))})_a b(\delta_b f)(\lambda_{\mu(\alpha)} X)\]

(3.27)

for arbitrary function \(f\). Eq. (3.27) is a corollary of the more general statement

\[\frac{d}{d\alpha} f(\lambda_{\gamma\mu(\alpha)} X) = \int_0^\gamma (\delta_a W_{\tau\mu} f)(\lambda_{\gamma-\tau\mu(\alpha)} X) \frac{d}{d\alpha}; \mu^a\]

(3.28)

provided that \(\mu^b H_b = f\). Since

\[(\delta_a W_{\tau\mu} f)(\lambda_{\gamma-\tau\mu(\alpha)} X) = (W_{\tau\mu} \delta_a W_{\tau\mu} f)(\lambda_{\mu(\alpha)} X) = (Ad_{\lambda_{\tau\mu(\alpha)} X})(\delta_b f)(\lambda_{\mu(\alpha)} X),\]
equation (3.27) is a corollary of equation (3.28). To check relation (3.28), note that

\[ \frac{d}{d\gamma} W_{-\gamma} \frac{d}{d\alpha} W_{\gamma} = W_{-\gamma} \frac{d}{d\alpha} \delta \gamma W_{\gamma}. \]

Therefore,

\[ W_{-\gamma} \frac{d}{d\alpha} W_{\gamma} = \int_0^\gamma d\gamma W_{-\gamma} \frac{d}{d\alpha} \delta \gamma W_{\gamma} \]

and

\[ \frac{d}{d\alpha} (W_{\gamma} f)(X) = \int_0^\gamma d\gamma W_{(\gamma - \gamma)\mu} \frac{d}{d\alpha} \delta \gamma W_{\gamma}. \]

We obtain equation (3.28). Proposition is proved.

**Proposition 3.4.** Property (3.8) is satisfied.

**Proof.** Let \( \mu(\tau) \) be such a function that

\[ \lambda_{\mu(\tau)} X = \lambda_{\mu_1 \tau} \lambda_{\mu_2} X. \tag{3.29} \]

Let us show that

\[ <f, (V_{\mu_1 \tau} (\lambda_{\mu_1 \tau} \lambda_{\mu_2} X) \leftrightarrow \lambda_{\mu_2} X)) + V^0(\lambda_{\mu(\tau)} X) \leftrightarrow X)g >= <f, V_{\mu_2} (\lambda_{\mu_2} X \leftrightarrow X))g >. \tag{3.30} \]

For \( \tau = 0 \), property (3.30) is obviously satisfied. The \( \tau \)-derivative of the left-hand side vanishes because of proposition 3.3 and property \( \rho^a = \mu^a \). Proposition is proved.

6. Thus, we have understood that property (3.17) is a necessary and sufficient condition for satisfying the group relation. Let us check equation (3.17) for Weyl quantization. Relation (3.17) can be rewritten as

\[ <f, [i\delta \Lambda_a - \frac{1}{2} \Sigma^2 \Lambda_a - \Lambda^{(1)}_a; i\delta \Lambda_b - \frac{1}{2} \Sigma^2 \Lambda_b - \Lambda^{(1)}_b]g >= -iU_{ab}^c <f, (i\delta \Lambda_c - \frac{1}{2} \Sigma^2 \Lambda_c - \Lambda^{(1)}_c)g >. \tag{3.31} \]

It follows from the results of Appendix A that

\[ [i\delta \Lambda_a - \frac{1}{2} \Sigma^2 \Lambda_a; i\delta \Lambda_b - \frac{1}{2} \Sigma^2 \Lambda_b] = i\delta \{\Lambda_a; \Lambda_b\} = i\delta \{\Lambda^{(1)}_a; \Lambda^{(1)}_b\}. \tag{3.32} \]

Since \( <f, \Sigma \Lambda_a \Sigma > = 0 \), while \( \delta \Lambda_B = \{A, B\} \), making use of property (3.2), one takes property (3.31) to the following form

\[ \frac{1}{2} \{\Xi U_{ab}^c; \Xi \Lambda_a\} = i\{\Lambda_a; \Lambda^{(1)}_a\} = i\delta \{\Lambda^{(1)}_a; \Lambda^{(1)}_b\}. \tag{3.32} \]

It follows from equation (2.14) that equation (3.32) is satisfied if and only if

\[ \{\Lambda_a; Re \Lambda^{(1)}_b\} + \{Re \Lambda^{(1)}_a; \Lambda_b\} = U_{ab}^c Re \Lambda^{(1)}_b; \tag{3.33} \]

\[ \frac{1}{2} \{U_{ac}^b \Lambda_c\} + \frac{1}{2} \{\Lambda_a; U_{ab}^c\} + \frac{1}{2} \{U_{da}^d \Lambda_b\} = \frac{1}{2} U_{ab}^c U_{de}. \tag{3.34} \]

Equation (3.33) is a restriction on the real part of quantum correction to constraint. We see that if the Weyl quantization is used, the leading-order semiclassical purposes no quantum corrections to the real part are necessary: the case \( Re \Lambda^{(1)}_a = 0 \) is possible.

Let us show that equation (3.34) is automatically satisfied. Consider the Jacobi identity

\[ [[[\delta_a; \delta_b]; \delta_c] + [[[\delta_b; \delta_c]; \delta_a] + [[[\delta_c; \delta_a]; \delta_b] = 0. \]

Making use of equation (3.2), we find that

\[ [U_{ac}^b \delta_d; \delta_c] + [U_{bc}^d \delta_d; \delta_a] + [U_{ca}^d \delta_d; \delta_b] = 0. \]
Applying eq. (3.2) once again, one takes the Jacobi identity to the form
\[
\{\Lambda^c; U^{e}_{ab}\} \delta^a_d + \{\Lambda^a; U^{e}_{bc}\} \delta^d_b + U^{e}_{ab} U^{e}_{dc} \delta^c_d + U^{e}_{ab} U^{e}_{da} \delta^d_a + U^{e}_{ca} U^{e}_{db} \delta^c_b = 0.
\]
Since vector fields \(\delta^a_c\) are linearly independent (the gauge group is assumed to act free on the phase space), one has
\[
\{\Lambda^c; U^{e}_{ab}\} + \{\Lambda^a; U^{e}_{bc}\} + \{\Lambda^b; U^{e}_{ca}\} + U^{e}_{ab} U^{e}_{dc} + U^{e}_{bc} U^{e}_{da} + U^{e}_{ca} U^{e}_{db} = 0.
\]
Consider the partial trace; let \(c = e\) and sum over \(e\). We obtain that two last terms cancel each other and
\[
\{\Lambda^c; U^{e}_{ab}\} + \{\Lambda^a; U^{e}_{bc}\} + \{\Lambda^b; U^{e}_{ca}\} + U^{e}_{ab} U^{e}_{dc} = 0.
\]
This relation coincides with (3.34).

Thus, the algebraic property (3.31) is checked. We see that if the Weyl quantization is used, there are no quantum anomalies in the leading order of semiclassical theory.

On the other hand, it is known from QFT that quantum anomalies arise in the one-loop approximation. This exactly corresponds to the leading order of semiclassical approximation.

A possible source of QFT anomalies may be as follows. The Weyl quantization cannot be applied to QFT systems: one usually use Wick ordering. There are also divergent counterterms to the Lagrangian. This implies that the quantum correction \(\Lambda^{(1)}_e\) appears to be not only nonzero but also divergent. Eq. (3.33) is then a nontrivial relation providing the cancellation of quantum anomalies.

4 Semiclassical observables

1. In classical mechanics of constrained systems, observables are such functions \(O\) on classical phase space that commute with all the constraints on the constraint surface
\[
\{O, \Lambda^a\} = 0, \quad \text{provided that} \quad \Lambda^b = 0.
\]
Let us apply the corresponding quantum observable of the form
\[
h\hat{O} = O(\sqrt{\hbar}q, \sqrt{\hbar}p)
\]
to the wave packet (2.16). The result \(\hat{\Phi} = h\hat{O}\Phi\) will be of the analogous form
\[
\tilde{\Phi}(q) = ce^{\frac{1}{\hbar}s} e^{\pi P(q\sqrt{\hbar} - Q)} \tilde{f}(q - Q/\sqrt{\hbar})
\]
with
\[
\tilde{f}(\xi) = O(Q + \sqrt{\hbar}\xi, P - i\sqrt{\hbar} \frac{\partial}{\partial \xi})f(\xi).
\]
We see that a semiclassical wave function \((S, P, Q; f)\) is taken to \((S, P, Q; \tilde{f})\) with the same \((S, P, Q)\). In the leading order in \(h\) the function \(f\) is multiplied by the classical value \(O(Q, P)\) of the observable. In the next order, one has
\[
\tilde{f} = Of + \sqrt{\hbar} \Xi O f + O(h),
\]
see Appendix A. We see that the operator \(\Xi O\) is the first nontrivial contribution to the classical observable \(O\) and may be viewed as a semiclassical observable.

Note that zero-norm semiclassical wave functions are indeed taken to zero-norm states. Namely, let \(<f, f> = 0\). Then for all \(g\) from the domain of \((\Xi O)^+\)
\[
< g, \Xi O f > = (g, \prod_{a=1}^{M} (2\pi \delta(\Xi \Lambda^a)) \Xi O f) = ((\Xi O)^+ g, \prod_{a=1}^{M} (2\pi \delta(\Xi \Lambda^a)) f) = 0,
\]
since
\[ [ξΛ_α; ξO] = -i{Λ_α; O}. \]

Therefore, the operator \( ξO \) can be reduced to factorspace \( F_X = F^0_X / ∼ \). Any bounded function of this operator \( φ(ξO) \) can be extended to \( F_X \).

2. It happens that it is sufficient to specify an observable on the constraint surface \( Λ_b = 0 \) only in order to specify the operator \( ξO \) in \( F_X \).

**Proposition 4.1.** Let \( O(Q, P) = 0 \), provided that \( Λ_b(Q, P) = 0, b = 1, M \). Then \( ξO f \) has zero norm.

**Proof.** It follows from the condition of proposition that the linear form
\[ \frac{∂O}{∂P_s} δP_s + \frac{∂O}{∂Q_s} δQ_s \]  
vanishes provided that
\[ \frac{∂Λ_b}{∂P_s} δP_s + \frac{∂Λ_b}{∂Q_s} δQ_s = 0. \]

Formula (4.3) specifies \( 2n − M \)-dimensional subspace of \( 2n \)-dimensional space \( \{(δQ, δP)\} \), since the operators \( ξΛ_b \) are linearly independent because of free action of the gauge group. Choose such a basis \( \{(δQ^{(i)}, δP^{(i)}), i = 1, 2n\} \) that vectors \( \{(δQ^α, δP^α), α = M + 1, 2n\} \) satisfy condition (4.3). The first \( M \) vectors \( (δQ^{(a)}, δP^{(a)}), a = 1, M \) should be chosen in such a way that
\[ \frac{∂Λ_b}{∂P_s} δP_s + \frac{∂Λ_b}{∂Q_s} δQ_s = δ^α_a \]

Expand \( (δQ, δP) \) as a linear combination
\[ (δQ, δP) = \sum_{i=1}^{n} b_i (δQ^{(i)}, δP^{(i)}). \]

The linear form (4.2) can be presented as
\[ \sum_{i=1}^{n} A_i b_i. \]

Let \( b_α = 1 \) for some \( M + 1 ≤ α2n \) and other \( b_i = 0 \). Then expression (4.4) should vanish, so that \( A_α = 0, α = M + 1, 2n \). One has
\[ \frac{∂O}{∂P_s} δP_s + \frac{∂O}{∂Q_s} δQ_s = \sum_{a=1}^{M} A_a b_a = \sum_{a=1}^{M} A_a (\frac{∂Λ_a}{∂P_s} δP_s + \frac{∂Λ_a}{∂Q_s} δQ_s). \]

This means that
\[ ξO = \sum_{a=1}^{M} A_a ξΛ_a. \]

However, the states \( ξΛ_a f \) have zero norm. Proposition is proved.

3. Let us show that gauge-equivalent semiclassical states are taken by the operator \( O + \sqrt{h}ξO \) to gauge-equivalent. Let
\[ (S^0, P^0, Q^0, f^0) \sim (S^τ, P^τ, Q^τ : f^τ), \]

\( (P^τ, Q^τ) \) satisfy eq.(2.23), \( S^τ \) be of the form (2.21), \( f^τ \) be a solution of eq.(2.23). First, one should check that
\[ O(Q^τ, P^τ) = O(Q^0, P^0) \]

or \( \frac{d}{dτ} O(Q^τ, P^τ) = 0. \) However, the property is equivalent to \( \{O, Λ_a\} = 0 \) and therefore satisfied.
Let us check that states

\[(S^0, P^0, Q^0; \Xi f^0) \sim (S^\tau, P^\tau, Q^\tau; \Xi f^\tau)\]

are gauge-equivalent. Consider the gauge transformation \((S^\tau, P^\tau, Q^\tau; g^\tau)\) of the state \((S^0, P^0, Q^0; \Xi f^0)\).

\(g^\tau\) is a solution of the Cauchy problem for eq.(2.23) with the initial condition \(g^0 = \Xi f^0\). The difference \(\zeta^\tau = g^\tau - \Xi f^\tau\) satisfies the following equation

\[
(i\frac{d}{d\tau} - \mu^a[\frac{1}{2}\Xi^2\Lambda_a + \Lambda^{(1)}_a])\zeta^\tau = -[i\frac{d}{d\tau} - \frac{1}{2}\mu^a\Xi^2\Lambda_a; \Xi f]f^\tau.
\]

It follows from Appendix A that

\[
[i\frac{d}{d\tau} - \frac{1}{2}\mu^a\Xi^2\Lambda_a; \Xi f] = \Xi\{\mu^a\Lambda_a; O\}.
\]

Since \(\{\Lambda_a; O\} = 0\) on the constraint surface,

\[
\Xi\{\mu^a\Lambda_a; O\} = A^a\Xi\Lambda_a
\]

for some coefficients \(A^a\). Therefore, the wave function \(\zeta^\tau\) satisfies the following equation

\[
(i\frac{d}{d\tau} - \mu^a[\frac{1}{2}\Xi^2\Lambda_a + \Lambda^{(1)}_a])\zeta^\tau = \Xi\Lambda_a\chi^\tau
\]

for some \(\chi^\tau_a\). For the inner product \(<\zeta^\tau, \zeta^\tau>, \) analogously to eq.(3.6) one has

\[
i\frac{d}{d\tau} <\zeta^\tau, \zeta^\tau> = <\zeta^\tau, \Xi\Lambda_a\zeta^\tau> - <\Xi\Lambda_a\zeta^\tau, \zeta^\tau> = 0. \tag{4.7}
\]

Therefore, \(\zeta^\tau\) is a state of a zero norm. This means that wave functions \(g^\tau\) and \(\Xi f^\tau\) are equivalent. Thus, equivalent semiclassical states are indeed taken to equivalent by the semiclassical observable \(\Xi\).

4. It happens also that semiclassical observables \(\Xi\) possesses also the following geometric interpretation (cf.[24, 25]). By \(K_{S,P,Q}\) we denote the operator taking the function \(f\) to the wave function \(\Phi\)

\[
(K^h_{S,P,Q}f)(q) = \frac{1}{h^{M/4}}e^{i\sqrt{\hbar}S}e^{i\sqrt{\hbar}(Pq+Q-\frac{1}{2}\sqrt{\hbar}\delta Q)}f(q - Q/\sqrt{\hbar}). \tag{4.8}
\]

Consider the shift of classical variables of the order \(O(\hbar^{1/2})\), \(S \rightarrow S + \sqrt{\hbar}\delta S, \ P \rightarrow P + \sqrt{\hbar}\delta P, \ Q \rightarrow Q + \sqrt{\hbar}\delta Q\). The operator \(K_{S,P,Q}\) will transform then as follows,

\[
(K^h_{S+\sqrt{\hbar}\delta S, P+\sqrt{\hbar}\delta P, Q+\sqrt{\hbar}\delta Q}f = \text{const}K^h_{S,P,Q}e^{i\Omega[\delta P, \delta Q]}f,
\]

where \(\Omega[\delta P, \delta Q]\) is the following linear combination of coordinate and momenta operators,

\[
(\Omega[\delta P, \delta Q]f)(\xi) = [\delta P\xi - \delta Q\frac{1}{i}\frac{\partial}{\partial \xi}]f(\xi).
\]

\(\Omega\) is an operator-valued differential form: a tangent vector \((\delta P, \delta Q)\) to the phase space is mapped to an operator; the mapping is linear. The form \(\Omega\) is an important geometric characteristics of the operator \(K^h_{S,P,Q}\).

We see that linear combinations of coordinate and momentum operators can be expressed via the operators \(\Omega\). However, \(\delta P, \delta Q\) should obey additional restrictions since the mapping \(K^h_{S,P,Q}\) is defined only if \((P, Q)\) belongs to the constraint surface

\[
\Lambda_a(Q, P) = 0.
\]
This means that
\[
\frac{\partial \Lambda_a}{\partial Q} \delta Q + \frac{\partial \Lambda_a}{\partial P} \delta P = 0.
\] (4.9)

Under condition (4.9), the operator \( \Omega[\delta P, \delta Q] \) transforms zero-norm states to zero-norm states, since
\[
[\Omega(\delta P, \delta Q); \Xi \Lambda_a] = i \left( \frac{\partial \Lambda_a}{\partial Q} \delta Q + \frac{\partial \Lambda_a}{\partial P} \delta P \right) = 0.
\]

Let \( O \) be a classical observable. We see that the corresponding semiclassical observable \( \Xi O \) can be presented as
\[
\Xi O = \Omega[\frac{\partial O}{\partial Q} - \frac{\partial O}{\partial P}].
\]

Note that the tangent vectors
\[
\delta_o P = -\frac{\partial O}{\partial Q}; \quad \delta_o Q = \frac{\partial O}{\partial P}.
\]
corresponds to the Hamiltonian vector field \( \delta_o \) on the phase space which is generated by the classical observable \( O \). One therefore has
\[
\Xi O = -\Omega[\delta_o (P, Q)].
\] (4.10)

### 5 Semiclassical transformations

Quantum observables \( \hat{O} \) can be also viewed as generators of one-parametric transformation groups \( e^{-i\hat{O}t} \). Let us investigate their analogs in the semiclassical mechanics. Let \( \hat{O} \) depend on the small parameter \( h \) as
\[
\hat{O} = \frac{1}{h} O(\sqrt{h}\hat{q}, \sqrt{h}\hat{p}) + O_1(\sqrt{h}\hat{q}, \sqrt{h}\hat{p}) + ... \] (5.1)

It happens that unitary condition for the constrained systems make necessary adding a quantum correction \( O_1 \) with a nontrivial imaginary part.

Let us apply the operator \( e^{-i\hat{O}t} \) to the semiclassical wave function \( \Phi (2.16) \). Consider the wave function
\[
\Phi^t = e^{-i\hat{O}t} \Phi^0 = \Phi.
\] (5.2)

Analogously to section 2, substitution
\[
\Phi^t(q) = const e^{\frac{1}{\hbar} \hat{O}t} e^{\frac{i}{\hbar} \hat{P}t(q\sqrt{\hbar} - Q^t)} f^t(q - Q^t / \sqrt{\hbar})
\]
gives us in the leading order in \( h \) the following system of equations,
\[
\dot{S}^t = P^t \dot{Q}^t - O(Q^t, P^t); \quad \dot{Q}^t = \frac{\partial O}{\partial P}(Q^t, P^t); \quad \dot{P}^t = -\frac{\partial O}{\partial Q}(Q^t, P^t).
\] (5.3)

\[
i \dot{f}^t = \left[ \frac{1}{2} (\Xi^2 O)(Q^t, P^t) + O_1(Q^t, P^t) \right] f^t
\] (5.4)

Note that the classical trajectory \( Q^t, P^t \) lies on the constraint surface, provided that \( \{ O, \Lambda_a \} = 0 \) on this surface. Therefore, one can define the transformation \( u_t : \mathcal{X} \to \mathcal{X} \) taking the initial data \( (S^0, Q^0, P^0) \) for eqs.(5.3) to the solution \( (S^t, Q^t, P^t) \). By \( U^0_t(u_tX \leftarrow X) : \mathcal{F}^0_X \to \mathcal{F}^0_{u_tX} \) we denote the operator taking the initial wave function \( f^0 \) to the solution \( f^t \). Let us investigate the unitarity property of the infinitesimal operator.
5.1 Unitarity problem

One can check that the operator $U_t^0(u_t X \leftarrow X)$ conserves the norm analogously to section 3. Let us investigate the commutator

$$[i \frac{d}{dt} - \frac{1}{2}(\Xi^2 O)(u_t X); (\Xi \Lambda_b)(u_t X)]$$

which has the form

$$i\Xi \{O; \Lambda_b\}$$

according to appendix A. However, the Poisson bracket $\{O; \Lambda_b\}$ vanishes on the constraint surface. Therefore, proposition 4.1 implies that

$$\Xi \{O; \Lambda_b\} = A^b_a \Xi \Lambda^a_b,$$

where coefficient functions $A^b_a(X)$ are uniquely defined from the relation

$$\delta \{O; \Lambda_a\} = A^b_a \delta \Lambda^a_b. \quad (5.5)$$

Therefore,

$$[i \frac{d}{dt} - \frac{1}{2}(\Xi^2 O)(u_t X); e^{i\rho^a(\Xi \Lambda_a)(u_t X)}] = iA^b_a(u_t X) \rho^a \frac{\partial}{\partial \rho^b} e^{i\rho^c(\Xi \Lambda_c)(u_t X)}. \quad (5.6)$$

Let $f^t_1, f^t_2$ be solutions of eq. (5.2). Analogously to eq. (3.4), we find

$$i \frac{\partial}{\partial t} <f^t_1, f^t_2> = [O_1(u_t X) - O^*_1(u_t X)] <f^t_1, f^t_2> - iA^a_a(u_t X) <f^t_1, f^t_2>.$$

Thus, a semiclassical transformations conserves the norm of semiclassical state if and only if $O_1$ contains a nontrivial imaginary part

$$ImO_1(X) = \frac{1}{2} A^a_a(X). \quad (5.7)$$

Under this condition zero-norm states are taken to zero-norm states, so that the operators $U_t^0(u_t^0 X \leftarrow X)$ can be reduced to factorspace. Namely, introduce the unitary operators $U_t(u_t X \leftarrow X) : \mathcal{F}_X \rightarrow \mathcal{F}_{u^t X}$ by the following definition

$$U_t(u_t X \leftarrow X)[f^0] = [U_t^0(u_t X \leftarrow X)f^0], \quad f^0 \in \mathcal{F}^0_X \quad (5.8)$$

which is correct. Operators (5.8) can be extended to the completion of $\mathcal{F}_X$,

$$\overline{U}_t(u_t X \leftarrow X) : \overline{\mathcal{F}}_X \rightarrow \overline{\mathcal{F}}_{u^t X}.$$

We see that a semiclassical transformation may be viewed as an automorphism of the semiclassical bundle.

5.2 Gauge invariance

The purpose of this subsection is to show that gauge-equivalent states are taken to gauge-equivalent. It is convenient to introduce a notion of a pre- semiclassical bundle with base $\mathcal{X}$ and fibres $\mathcal{F}^0_X$ and consider the sections of this bundle. Remind that a section of a bundle is specified if for each $X \in \mathcal{X}$ a wave function $\chi^0_X \in \mathcal{F}^0_X$ is chosen; certain requirements on smoothness of dependence of $\chi^0_X$ on $X$ may be imposed.
A section $\chi^0$ of the pre-semiclassical bundle is called gauge-invariant if for all $\mu, X \in \mathcal{X}$ the property

$$\chi^0_{\lambda\mu X} \sim V^0_{\mu}(\lambda_{\mu} X \leftarrow X)\chi^0_X$$

is satisfied.

**Proposition 5.1.** Property (5.9) is equivalent to

$$< f^0, [i\delta_a - H_a(X)]\chi^0_X > = 0, \quad \text{for all } f^0 \in \mathcal{F}^0_X. \quad (5.10)$$

**Proof.** Property (5.9) can be rewritten as

$$< f^0, \chi^0_X - V^0_{\mu t}(X \leftarrow \lambda_{\mu t} X)\chi^0_X > = 0. \quad (5.11)$$

Consider the limit $t \to 0$. One has

$$V^0_{\mu t}(X \leftarrow \lambda_{\mu t} X) \sim 1 - i\mu_a t H_a(X) + o(t),$$

$$\chi^0_{\lambda_{\mu t} X} \sim \chi^0_X - t\mu_a \delta_a \chi^0_X + o(t).$$

We obtain relation (5.10).

Let us check the implication (5.10) $\Rightarrow$ (5.9). The wave function

$$\zeta_t = \chi^0_{\lambda_{\mu t} X} - V^0_{\mu t}(X \leftarrow \lambda_{\mu t} X)\chi^0_X$$

satisfies the equation

$$i \frac{d}{dt} \zeta_t = H_a(\lambda_{\mu t} X)\zeta_t + [i\mu_a \delta_a - H_a(\lambda_{\mu t} X)]\chi^0_{\lambda_{\mu t} X},$$

so that

$$i \frac{d}{dt} < \zeta_t, \zeta_t > = 0$$

analogously to (4.17). Proposition is proved.

**Proposition 5.2.** Let $f_1 \in \mathcal{F}^0_{X_1}, f_2 \in \mathcal{F}^0_{X_2}, f_1 \neq 0, f_2 \neq 0$. Then the semiclassical states $(X_1, f_1)$ and $(X_2, f_2)$ are gauge-equivalent if and only if for all gauge-invariant sections $\chi^0$ of the pre-semiclassical bundle the relation

$$< [\chi^0_{X_1}]; f_1 >= < [\chi^0_{X_2}]; f_2 >$$

is satisfied.

**Proof.** Let $(X_1, f_1) \sim (X_2, f_2)$, $\chi^0_X$ be a gauge-invariant section. Then

$$X_2 = \lambda_\mu X_1, \quad f_2 = V_\mu(X_2 \leftarrow X_1)f_1, \quad [\chi^0_{X_2}] = [V^0_\mu(X_2 \leftarrow X_1)\chi^0_{X_1}]$$

for some $\mu$. One should check that

$$< [\chi^0_{X_1}]; f_1 >= < [V^0_\mu(X_2 \leftarrow X_1)\chi^0_{X_1}]; V_\mu(X_2 \leftarrow X_1)f_1 >.$$

This is true because of unitarity of $V_\mu(X_2 \leftarrow X_1)$.

Let $(X_1, f_1) \not\sim (X_2, f_2)$. One should consider two cases.

1. $X_1 \not\sim X_2$.

Then $X_1, X_2$ belong to different gauge orbits. One therefore can choose such an invariant section $\chi^0$ that $\chi^0_{X_1} = F^0, \chi^0_{X_2} = 0$, where $f^0$ is an arbitrary wave function. Therefore,

$$< [f^0]; f_1 >= 0$$

for all $f^0$, so that $f_1 = 0$. 

22
2. \( X_1 \sim X_2 \), so that \( X_2 = \lambda_{|mu} X_1 \), but \( f_2 \neq V_\mu(X_2 \leftarrow X_1)f_1 \). Choose such an invariant section \( \chi^0 \) that \( \chi^0_2 = f^0 \) be an arbitrary nonzero wave function. One has

\[
[\chi^0_{X_2}] = [V^0_\mu(X_2 \leftarrow X_1)\chi^0_{X_1}].
\]

Therefore,

\[
< [\chi^0_{X_1}]; f_1 > = < [\chi^0_{X_2}]; V_\mu(X_2 \leftarrow X_1)f_1 > .
\]

For some \( f^0 \), one has

\[
< [f^0]; V_\mu(X_2 \leftarrow X_1)f_1 > \neq < [f^0]; f_1 > ,
\]

so that

\[
< [\chi^0_{X_1}]; f_1 > \neq < [\chi^0_{X_2}]; f_2 > .
\]

Proposition is proved.

Call the section \( \chi^t \) of the form

\[
\chi^t_{uiX} = U^0_t(u_tX \leftarrow X)\chi^0_X
\]

(5.13)
as a semiclassical transformation of the section \( \chi^0 \).

**Proposition 5.3.** Semiclassical transformation takes any gauge-equivalent section to a gauge-invariant section if and only if

\[
(X_1, f_1) \sim (X_2, f_2) \Rightarrow (u_tX_1; U_t(u_tX_1 \leftarrow X_1)f_1) \sim (u_tX_2; U_t(u_tX_2 \leftarrow X_2)f_2).
\]

(5.14)

**Proof.** Let the semiclassical transformation take any gauge-invariant section to a gauge-invariant and \((X_1, f_1) \sim (X_2, f_2)\). Let us show

\[
(u_tX_1; U_t(u_tX_1 \leftarrow X_1)f_1) \sim (u_tX_2; U_t(u_tX_2 \leftarrow X_2)f_2)
\]

(5.15)

According to proposition 5.3, one should check that

\[
< [\chi^0_{uiX_1}]; U_t(u_tX_1 \leftarrow X_1)f_1 > = < [\chi^0_{uiX_2}]; U_t(u_tX_2 \leftarrow X_2)f_2 >
\]

(5.16)

for all gauge-invariant sections \( \chi^0 \). Property (5.16) is equivalent to

\[
< [U^0_{-t}(X_1 \leftarrow u_tX_1)\chi^0_{uiX_1}]; f_1 > = < [U^0_{-t}(X_2 \leftarrow u_tX_2)\chi^0_{uiX_2}]; f_2 >
\]

or

\[
< [\chi^0_{-t}]; f_1 > = < [\chi^0_{-t}]; f_2 > .
\]

(5.17)

However, the section \([\chi^0_{-t}]\) is gauge-invariant, while \((X_1; f_1) \sim (X_2; f_2)\). Thus, eq. (5.17) is satisfied. Property (5.13) is checked.

Let us suppose now that implication (5.14) takes place. Let \( \chi^0_X \) be a gauge-invariant section. Show that the section \( \chi^t_X \) is gauge-invariant, i.e.

\[
(X, [\chi^t_X]) \sim (\lambda_\mu X, [\chi^t_{\lambda_\mu X}])
\]

or

\[
(X, [U^0_t(X \leftarrow u_{-t}X)\chi^0_{uiX}]) \sim (\lambda_\mu X, [U^0_t(\lambda_\mu X \leftarrow u_{-t}\lambda_\mu X)\chi^0_{ui\lambda_\mu X}])
\]

(5.18)

However,

\[
(u_{-t}X, [\chi^0_{uiX}]) \sim (u_{-t}\lambda_\mu X, [\chi^0_{ui\lambda_\mu X}])
\]

since \( u_{-t}X \sim u_{-t}\lambda_\mu X \), while section \( \chi^0 \) is gauge-invariant, so that implication (5.14) implies (5.18). Proposition is proved.
To check that equivalent states are taken to equivalent by the semiclassical transformation, it is sufficient to show that gauge-invariant sections are taken to gauge-invariant, i.e.

\[ < f^0, [i\delta_a - H_a(X)]\chi_X^l > = 0 \quad \text{for all} \quad f^0. \]  

(5.19)

under condition (5.10). Denote

\[ \delta_O f(X) = \frac{d}{dt}|_{t=0} f(u_tX). \]  

(5.20)

**Proposition 5.4.** (5.10) \(\Rightarrow\) (5.14) if and only if

\[ < f, [i\delta_a - H_a(X); i\delta_O - \frac{1}{2}(\Xi^2O)(X) - O_1(X)]g > = -i < f, A_a^b(X)(i\delta_b - H_b(X))g >. \]  

(5.21)

where \(A_a^b(X)\) are defined from eq. (5.4).

**Proof.** Let property (5.19) be satisfied for all \(\chi_X^0\) obeying eq. (5.10). Let us obtain eq. (5.21). Relation (5.19) can be rewritten as

\[ < f^0, (i\delta_a - H_a(X))U^0_t(X \leftarrow u_{-t}X)\chi_{u_{-t}X}^0 = 0. \]  

(5.22)

Consider the limit \(t \to 0\). The leading order in \(t\) gives us trivial result (5.10). The next order leads us to the following relations,

\[ \chi_{u_{-t}X}^0 \sim \chi_X^0 - t\delta_O\chi_X^0 + o(t); \]

\[ U_t^0(X \leftarrow u_{-t}X) \sim 1 - it[\frac{1}{2}(\Xi^2O)(X) + O_1(X)] + o(t), \]

so that

\[ < f^0, (i\delta_a - H_a(X))(i\delta_O - \frac{1}{2}(\Xi^2O)(X) - O_1(X))\chi_X^0 >= 0. \]

On the other hand, for

\[ g_X^0 = (i\delta_a - H_a(X))\chi_X^0 \]

one has

\[ < f^0, (i\delta_O - \frac{1}{2}(\Xi^2O)(X) - O_1(X))g_X^0 >= \]

\[ (f^0, [\Pi_c(2\pi\delta(\Xi\Lambda_c(X)))); i\delta_O - \frac{1}{2}(\Xi^2O)(X) - O_1(X)]g_X^0) = \]

\[ i(f^0, [\Pi_c(2\pi\delta(\Xi\Lambda_c(X)))]iA_a^b(X)g_X^0) = 0, \]

here properties (5.10), (5.4) are used. Therefore,

\[ < f^0, [i\delta_a - H_a(X); i\delta_O - \frac{1}{2}(\Xi^2O)(X) - O_1(X)]\chi_X^0 >= 0. \]  

(5.23)

The commutator entering to this expression has the structure

\[ -\delta_{(A_a,O)} + B_a(X) \]  

(5.24)

for some operator function \(B_a(X)\). It follows from (5.3) that eq. (5.23) takes the form

\[ < f^0, (A_a^b(X)\delta_b + B_a(X))\chi_X^0 >= 0. \]

Eq. (5.11) implies that

\[ A_a^b(X)iH_b(X) = B_a(X) \]

in the weak sense. Thus, eq. (5.24) implies (5.21).

Let eq. (5.21) be satisfied. Check the property (5.19). Consider the wave function

\[ \zeta^a[t; X] = [U_t(u_tX \leftarrow X)]^{-1}\phi^a[t, u_tX]. \]  

(5.25)
Let us obtain an equation for $\zeta^a$. Notice that the wave function $\chi_{u,t,X}^t$ satisfies the equation

$$i \frac{d}{dt} \chi_{u,t,X}^t = \left[ \frac{1}{2} (\Xi^2 O)(u_t X) + O_1(u_t X) \right] \chi_{u,t,X}^t \tag{5.26}$$

being a corollary of relation (5.13). The left-hand side of eq. (5.26) can be presented as

$$(i \frac{\partial}{\partial t} + i \delta_O) \chi_Y^t \big|_{Y = u_t X},$$

so that

$$i \frac{\partial}{\partial t} \chi_Y^t = \left[ \frac{1}{2} (\Xi^2 O)(Y) + O_1(Y) - i \delta_O \right] \chi_Y^t.$$

Therefore, the function $\phi^a[t, u_t X]$ obeys the equation

$$i \frac{d}{dt} \phi^a[t, u_t X] = i \delta_O \phi^a[t, u_t X] + (i \delta_a - H_a(Y)) \left( \frac{1}{2} (\Xi^2 O)(Y) + O_1(Y) - i \delta_O \right) \big|_{Y = u_t X} \chi_{u,t,X}^t.$$

It follows from eq. (5.24) that it can be rewritten as

$$i \frac{d}{dt} \phi^a[t, u_t X] = O_2(u_t X) \phi^a[t, u_t X] + i A^b_a(u_t X) \phi^b[t, u_t X].$$

Thus,

$$i \frac{d}{dt} \zeta^a[t, u_t X] = [U_t(u_t X \leftarrow X)]^{-1} i A^b_a(u_t X) [U_t(u_t X \leftarrow X)] \zeta^b[t, X]. \tag{5.27}$$

The operator entering to the right-hand side is bounded, so that there exists a unique solution for the Cauchy problem for eq. (5.27), which can be presented as a strongly convergent series,

$$\zeta^a[t, X] = (T \exp \{- \int_0^t d\tau [U_t(u_t X \leftarrow X)]^{-1} A(u_t X) [U_t(u_t X \leftarrow X)] \}) \zeta^a[0, X].$$

Each term of the series has zero norm, so that $< \zeta^a, \zeta^a > = 0$. Proposition 5.4 is proved.

### 5.3 Check of infinitesimal properties

Let us check property (5.21) analogously to subsection 3.3. It follows from the results of appendix A that

$$[i \delta_a - \frac{1}{2} \Xi^2 \Lambda_a - \Lambda^a_1; i \delta_O - \frac{1}{2} \Xi^2 O - O_1] = i (i \delta_{(\Lambda_a; O)} - \frac{1}{2} \Xi^2 \{\Lambda_a; O\} + \delta_O \Lambda^a_1 - \delta_a O_1) \tag{5.28}$$

Let us first calculate the operator $\frac{1}{2} \Xi^2 \{\Lambda; O\}$. It follows from (5.3) that

$$\frac{\partial \{O; \Lambda_a\}}{\partial Q_j} = A^b_a \frac{\partial \Lambda_b}{\partial Q_j}; \quad \frac{\partial \{O; \Lambda_a\}}{\partial P_j} = A^b_a \frac{\partial \Lambda_b}{\partial P_j};$$

on the constraint surface. According to proposition 4.1 (formula (4)), one can write:

$$\frac{\partial^2 \{O; \Lambda_a\}}{\partial Q_i \partial Q_j} = \frac{\partial A^b_a}{\partial Q_i} \frac{\partial \Lambda_b}{\partial Q_j} + A^b_a \frac{\partial^2 \Lambda_b}{\partial Q_i \partial Q_j} + \lambda^a_{ij} \frac{\partial \Lambda_i}{\partial Q_j} + \lambda^a_{ij} \frac{\partial \Lambda_j}{\partial Q_i};$$

$$\frac{\partial^2 \{O; \Lambda_a\}}{\partial P_i \partial Q_j} = \frac{\partial A^b_a}{\partial P_i} \frac{\partial \Lambda_b}{\partial Q_j} + A^b_a \frac{\partial^2 \Lambda_b}{\partial P_i \partial Q_j} + \lambda^a_{ij} \frac{\partial \Lambda_i}{\partial Q_j} + \lambda^a_{ij} \frac{\partial \Lambda_j}{\partial P_i};$$

$$\frac{\partial^2 \{O; \Lambda_a\}}{\partial P_i \partial P_j} = \frac{\partial A^b_a}{\partial P_i} \frac{\partial \Lambda_b}{\partial P_j} + A^b_a \frac{\partial^2 \Lambda_b}{\partial P_i \partial P_j} + \lambda^a_{ij} \frac{\partial \Lambda_i}{\partial P_j} + \lambda^a_{ij} \frac{\partial \Lambda_j}{\partial P_i};$$
for some functions $\lambda^a_Q$, $\lambda^a_P$. Thus,

$$\Xi^2\{O; \Lambda_a\} = A^b_a \Xi^2 \Lambda_b + \Xi A^b_a \Xi \Lambda_b + \Xi \Lambda_c (\lambda^a_{ac} \xi_j + \lambda^a_{ac} \partial_{\xi_j}).$$

However, the matrix elements of the operator $\Xi \Lambda_c \hat{B}_c$ are zero for arbitrary $\hat{B}_c$. Thus, the operator $\Xi^2\{O, \Lambda_a\}$ is equivalent in sense of matrix elements to

$$\Xi^2\{O, \Lambda_a\} \sim A^b_a \Xi^2 \Lambda_b + [\Xi A^b_a; \Xi \Lambda_b].$$

Expression (5.28) can be transformed as

$$-iA^b_a (i \delta_b - \frac{1}{2} \Xi^2 \Lambda_b) + \frac{i}{2} [\Xi A^b_a; \Xi \Lambda_b] + i \delta_O \Lambda_a^{(1)} - i \delta_a O_1.$$

Eq.(5.21) is then satisfied if and only if

$$iA^b_a \Lambda_a^{(1)} = \frac{1}{2} \{A^b_a; \Lambda_b\} + i \{O; \Lambda_a^{(1)}\} - i \{\Lambda_a; O_1\}.$$ (5.29)

Decompose this relation into real and imaginary parts:

$$iA^b_a \text{Re} \Lambda_a^{(1)} = i \{O; \text{Re} \Lambda_a^{(1)}\} - i \{\Lambda_a; \text{Re} O_1\};$$

$$- \frac{1}{2} A^b_a U_{cb} = \frac{1}{2} \{A^b_a; \Lambda_b\} - \frac{1}{2} \{O; U_{ca}\} + \frac{1}{2} \{\Lambda_a; A_c^a\} = 0.$$ (5.31)

It happens that eq.(5.31) is automatically satisfied.

Namely, the Jacobi identity

$$[[\delta_a, \delta_b], \delta_O] + [[\delta_b, \delta_O], \delta_a] + [[\delta_O, \delta_a], \delta_b] = 0.$$

It can be rewritten as

$$U^c_a A^d_c \delta_d + \delta_O U^d_a \delta_d + A^d_a U^d_{ca} \delta_d + \delta_a A^d_c \delta_d - A^d_c U^d_{cb} \delta_d - \delta_b A^d_a \delta_d = 0.$$

Since $\delta_d$ are independent operators, one has

$$U^c_a A^d_c + A^c_a U^d_{ca} - A^c_a U^d_{cb} + \{O; U^d_{ab}\} + \{\Lambda_a; A^d_b\} - \{\Lambda_b; A^d_a\} = 0.$$

Multiplying this relation by the $\delta$-symbol $\delta^d_a$, we obtain eq.(5.31).

Thus, gauge-equivalent states are indeed taken to gauge-equivalent via the semiclassical transformation generated by the observable $\hat{O}$ under condition (5.30). If $\text{Re} \Lambda_a^{(1)} = 0$, one can choose $\text{Re} O_1 = 0$. Otherwise, the non-anomaly condition for $O_1$ (5.30) is obtained.

### 5.4 Relationship with BRST-BFV approach

Let us compare the obtained results (5.7), (5.29) with the definition of the observable in the BRST-BFV quantization. In this approach, a B-extension of an observable $\hat{O}$ is considered. It is looked for in the following form

$$\hat{O}_B = \hat{O} + \hat{O}^{nb_1...b_n} \prod_{b_1} ... \prod_{b_n} \frac{\partial}{\partial \Pi_{a_1} ... \partial \Pi_{a_n}} + ...$$

in such a way that

$$[\hat{O}_B; \hat{O}_0] = 0; \quad \hat{O}^+_B = \hat{O}_B.$$
The purpose of this subsection is to show that if the quantum observable \( \hat{O} \) possesses a B-extension, then relations (5.7) and (5.29) will be satisfied. We suppose that the B-extension depends on the small parameter \( h \) semiclassically (eq.(2.9)), so that the coefficient operators \( \hat{O}^n \) depend on \( h \) as

\[
\hat{O}^{nb_1\ldots b_n} = h^{n-1}O^{nb_1\ldots b_n}(\sqrt{h}q, \sqrt{h}p)h^nO^{0b_1\ldots b_n}(\sqrt{h}q, \sqrt{h}p) + ...
\]

Let us investigate first the condition \( [\hat{O}_B; \hat{\Omega}_0] = 0 \). One has

\[
\hat{O}_B\hat{\Omega}_0 = \hat{\Omega}_0^{1a} \frac{\partial}{\partial \Pi_a} + ...; \quad \hat{\Omega}_0\hat{O}_B = \hat{\Omega}_0^{1a} \hat{O}_a^{\dagger} \frac{\partial}{\partial \Pi_a} + \hat{\Omega}_0^{1a} \hat{\Lambda}_a \frac{\partial}{\partial \Pi_a} + ...
\]

where ... are terms containing \( \Pi_b \). The \( \Pi - \frac{\partial}{\partial \Pi} \) ordering is chosen. Thus, one has the following condition

\[
[\hat{O}; \hat{\Lambda}_a] = \hat{\Lambda}_a \hat{O}_a^{lb}.
\]

In the leading order of the semiclassical approximation results of appendix A imply that

\[
-i\{O; \Lambda_a\} = \Lambda_b O_a^{lb}; \quad \tag{5.32}
\]

\[
-i\{O_1; \Lambda_a\} + \{O; \Lambda_a^{(1)}\} = \Lambda_b^{(1)} O_a^{lb} + \Lambda_b O_a^{lb} - \frac{i}{2} \{\Lambda_b; O_a^{lb}\}. \tag{5.33}
\]

Let us consider the condition \( \hat{O}_B^* = \hat{O}_B \). In the leading orders in \( h \), this implies that

\[
O^* = O; \quad O_a^{lb*} = -O_a^{lb}; \quad O_1^* - O_1 + O_a^{1*} = 0. \tag{5.34}
\]

Comparison of formulas (5.32) and (5.5) gives us the following relation

\[
O_a^{lb} = -i A_a^b.
\]

which is valid on the constrained surface. Thus,

\[
ImO_1 = \frac{1}{2} A_a^a.
\]

On the constraint surface eq.(5.33) coincides with eq.(5.29). Thus, relations (5.7) and (5.29) are checked.

### 5.5 Equivalent observables

The constructed semiclassical transformation depends not only on values of the classical observable \( O \) on the constraint surface but also on the off-constraint-surface values. Namely, if one adds to the observable \( O \) terms \( \alpha^a \Lambda_a \), even the classical equations (5.3) will change. It happens, however, that classical states \( u_tX \) corresponding to transformations generated by observables \( \hat{O} \) and \( O + \alpha^a \Lambda_a \) are gauge-equivalent. The purpose of this subsection is to show that an analogous statement is valid also for the semiclassical transformation.

Let \( \hat{C} \) and \( \hat{B} \) be two semiclassical observables,

\[
\hat{C} = \frac{i}{h} C(\sqrt{h}q, \sqrt{h}p) + C_1(\sqrt{h}q, \sqrt{h}p) + ...; \quad \hat{B} = \frac{i}{h} B(\sqrt{h}q, \sqrt{h}p) + B_1(\sqrt{h}q, \sqrt{h}p) + ...;
\]

such that \( C = B \) and \( ReC_1 = ReB_1 \) on the constraint surface. Let

\[
(X, f^0) \mapsto (u_t[B]X, U_0^0(u_t[B]X \leftarrow X; B)f^0); \quad (X, f^0) \mapsto (u_t[C]X, U_0^0(u_t[B]X \leftarrow X; C)f^0)
\]

27
be semiclassical transformations generated by the observables $B$ and $C$.

**Proposition 5.5** The following relation is satisfied:

$$ (u_t[B]X, [u_t[B]X \leftarrow X; B] f^0]) \sim (u_t[C]X, [u_t[C]X \leftarrow X; C] f^0)). \tag{5.35} $$

To prove property (5.35), it is sufficient according to proposition 5.2 to check that for all gauge-invariant sections $\chi^0_X$ the property

$$ < \chi^0_{u_t[B]X}, U^0_t(u_t[B]X \leftarrow X; B) f^0 > = < \chi^0_{u_t[C]X}, U^0_t(u_t[C]X \leftarrow X; C) f^0 > \tag{5.36} $$

is satisfied. Relation (5.36) can be rewritten as

$$ < f^0; U^0_{-t}(X \leftarrow u_t[B]X; B) \chi^0_{u_t[B]X} > = < f^0; U^0_{-t}(X \leftarrow u_t[C]X; C) \chi^0_{u_t[C]X} > \tag{5.37} $$

Denote

$$ \chi^{-t}_{X}[B] = U^0_{-t}(X \leftarrow u_t[B]X; B) \chi^0_{u_t[B]X}, \quad \chi^{-t}_{X}[C] = U^0_{-t}(X \leftarrow u_t[C]X; C) \chi^0_{u_t[C]X}. $$

One should check that there is an equivalence,

$$ \chi^{-t}_{X}[B] \sim \chi^{-t}_{X}[C]. \tag{5.38} $$

Note that the considered sections obey the following equations,

$$ \frac{\partial}{\partial t} \chi^{-t}_{X}[B] = i(\frac{1}{2}(\Xi^2 B)(X) + B_1(X) - i\delta_B) \chi^{-t}_{X}[B]; $$
$$ \frac{\partial}{\partial t} \chi^{-t}_{X}[C] = i(\frac{1}{2}(\Xi^2 C)(X) + C_1(X) - i\delta_C) \chi^{-t}_{X}[C]. $$

Therefore, the difference

$$ \rho^{-t}_{X} = \chi^{-t}_{X}[B] - \chi^{-t}_{X}[C] $$

satisfies the following equation

$$ \frac{\partial}{\partial t} \rho^{-t}_{X} = i(\frac{1}{2}(\Xi^2 B)(X) + B_1(X) - i\delta_B) \rho^{-t}_{X} + \gamma^{-t}_{X} \tag{5.39} $$

with

$$ \gamma^{-t}_{X} = i(\frac{1}{2}(\Xi^2 O)(X) + O_1(X) - i\delta_O) \chi^{-t}_{X}[C]. $$

where $O = B - C$, so that $O = 0$ on the constrained surface. One should check that $\rho^{-t}_{X} \sim 0$ or

$$ < \rho^{-t}_{u_{-t}[B]X}; \rho^{-t}_{u_{-t}[B]X} > = 0. \tag{5.41} $$

The time derivative of the left-hand side of eq. (5.41) is

$$ i < \rho^{-t}_{u_{-t}[B]X}; \gamma^{-t}_{u_{-t}[B]X} > - i < \gamma^{-t}_{u_{-t}[B]X}; \rho^{-t}_{u_{-t}[B]X} >. $$

Since at the initial moment of time relation (5.41) is obviously satisfied, it is sufficient to check

$$ \gamma^{-t}_{X} \sim 0. $$

First of all, note that eq. (5.41) implies that

$$ \delta_O = A_a \delta_a; $$
or
\[
\frac{\partial O}{\partial P_i} = A_a \frac{\partial \Lambda_a}{\partial P_i}; \quad \frac{\partial O}{\partial Q_i} = A_a \frac{\partial \Lambda_a}{\partial Q_i}.
\]

Therefore, analogously to subsection 5.3, we obtain
\[
\Xi^2 O = A_a \Xi^2 \Lambda_a + [\Xi A_a; \Xi \Lambda_a] + \Xi \Lambda_a \hat{B}_a
\]
for some operator \(\hat{B}_a\). Therefore,
\[
\gamma^{-t} \sim i\left(\frac{1}{2} A_a(X)(\Xi^2 \Lambda_a)(X) - i\left\{A_a; \Lambda_a\right\}(X) + O_1(X) - iA_a(X)\delta_a\right)\chi^{-t}[C].
\]
Making use of eq.(5.19), we find that
\[
\gamma^{-t} \sim i\left(-\frac{i}{2}\left\{A_a; \Lambda_a\right\}(X) + O_1(X) - iA_a(X)\Lambda_a^{(1)}(X)\right)\chi^{-t}[C].
\]
Thus, it is sufficient to show that the relation
\[
\frac{1}{2}\left\{A_a; \Lambda_a\right\} + iO_1 + A_a\Lambda_a^{(1)} = 0 \quad (5.42)
\]
is satisfied. Let us decompose condition (5.42) into real and imaginary parts. Let us find coefficients in eq.(5.5). One has
\[
\left[\delta O; \delta \Lambda_a\right] = \left[\delta c U^d_{ca}; \delta_a\right] = -A_c U^d_{ca} \delta_d - \left\{\Lambda_a; A_c\right\} \delta_c,
\]
so that
\[
A'_d = -A_c U^d_{ca} - \left\{A_a; \Lambda_d\right\}
\]
and
\[
ImO_1 = -\frac{1}{2} A_c U^d_{ca} - \frac{1}{2}\left\{\Lambda_a; A_a\right\} = A_c \Lambda_a^{(1)} + \frac{1}{2}\left\{A_a; \Lambda_a\right\}.
\]
Eq.(5.42) takes the form \(ReO_1 = 0\). Therefore, under condition \(ReB_1 = ReC_1\) relation (5.35) is satisfied. Semiclassical transformations generated by observables \(B\) and \(C\) are gauge-equivalent.

5.6 Semiclassical transformations of semiclassical observables

It happens that the linear combinations of operators \(\xi\) and \(\partial/\partial \xi\) transform under time evolution in a simple way.

Let \(\delta P^t, \delta Q^t\) satisfy the variation system for eqs.(5.3):
\[
\begin{align*}
\frac{d}{dt}\delta Q^t &= \frac{\partial^2 O}{\partial P_0 \partial P}(Q^t, P^t)\delta P^t + \frac{\partial^2 O}{\partial Q_0 \partial P}(Q^t, P^t)\delta Q^t; \\
\frac{d}{dt}\delta P^t &= -\frac{\partial^2 O}{\partial Q_0 \partial P}(Q^t, P^t)\delta P^t - \frac{\partial^2 O}{\partial Q_0 \partial Q}(Q^t, P^t)\delta Q^t.
\end{align*}
\]

**Proposition 5.6.** The following identity is satisfied:
\[
\Omega[\delta P^t, \delta Q^t](u_t X \leftarrow X) = U^0_t(u_t X \leftarrow X)\Omega[\delta Q^0, \delta P^0]. \quad (5.43)
\]

**Proof.** Let \(f^t = U^0_t(u_t X \leftarrow X)f^0, g^0 = \Omega[\delta Q^0, \delta P^0]f^0, g^t = U^0_t(u_t X \leftarrow X)g^0\) is a solution of the Cauchy problem for the equation
\[
\frac{i}{\partial t}g^t = \left[\frac{1}{2}(\Xi^2 O)(u_t X) + O_1(u_t X)\right]g^t. \quad (5.44)
\]
One should check that \( g^t = \Omega[\delta q^t, \delta P^t]f^t \). This function indeed satisfies the initial condition, while

\[
[i \frac{\partial}{\partial t} - \frac{1}{2}(\Xi^2 O)(u_t X) - O_i(u_t X); \Omega[\delta Q^t, \delta P^t]] = \\
i \Omega[\frac{d}{dt}\delta Q^t - \frac{\partial}{\partial P^t}O(Q^t, P^t)\delta P^t - \frac{\partial}{\partial Q^t}O(Q^t, P^t)\delta Q^t; \frac{d}{dt}\delta P^t + \frac{\partial}{\partial Q^t}O(Q^t, P^t)\delta Q^t\delta Q^t] = 0,
\]

so that the function \( \Omega[\delta Q^t, \delta P^t]f^t \) obeys also eq.(5.44). Proposition is proved.

Property (5.43) which is satisfied not only for real \( \delta Q^t, \delta P^t \) but also for the complex \( \delta Q^t, \delta P^t \) is very useful for constructing the quasi-Gaussian solutions of eq.(5.4).

Namely, the Gaussian function

\[ f^0(\xi) = \text{const} e^{\frac{1}{4}\alpha_{ij}\xi_i\xi_j} \]

can be geometrically interpreted in terms of the Maslov complex germ. For the quantum mechanics, the corresponding theory was developed in [10]. It was generalized to the case of the abstract semiclassical mechanics in [25].

A Maslov complex germ is a \( n \)-dimensional plane in the complexified \( 2n \)-dimensional tangent space to the phase space,

\[ \mathcal{G}_\alpha = \{ (\delta Q, \delta P)|\delta P_i = \alpha_{ij}\delta Q_j \}. \]

One has

\[ (\delta Q, \delta P) \in \mathcal{G}_\alpha \iff \Omega[\delta Q, \delta P]f^0 = 0. \]

Moreover, the property \( \Omega[\delta X]f^0 = 0 \) for all \( \delta X \in \mathcal{G}_\alpha \) uniquely specify the wave function \( f^0 \) up to a multiplicative factor.

Let \( u_{st}[X] \) be a mapping of tangent spaces to the phase space. It follows from proposition 5.6 that

\[ \Omega[u_{st}[X]\delta X]f_t = 0, \quad \delta X \in \mathcal{G}_\alpha. \]

Thus, \( f_t \) is a Gaussian function, while the complex germ \( \mathcal{G}_{\alpha^t} \) is \( u_{st}[X]\mathcal{G}_{\alpha^0} \).

If the initial condition is a product of a polynomial by the exponent, it can be presented as a sum of functions

\[ \Omega[\delta X_1]...\Omega[\delta X_n]f_0, \]

so that at time moment \( t \) one obtains the function

\[ \Omega[\delta X_1^t]...\Omega[\delta X_n^t]f_t \]

with \( \delta X_i^t = u_{st}[X]\delta X_i \).

One can also develop the complex-germ theory for the function \( \prod_{\alpha}(2\pi\delta(\Xi\Lambda))f^0(\xi) \) which is also Gaussian (see [25] for details).

6 Composed semiclassical states

We have investigated the properties of the wave packet function (2.16) corresponding to an “elementary” semiclassical state. It is known from quantum mechanics (see appendix B) that infinite superpositions of states (2.16) may be also viewed as semiclassical solutions of the semiclassical equations. In particular, WKB-states can be obtained in such a way.

However, one should be careful: sometimes the superposition (B.8) vanishes up to \( O(h^\infty) \) and gives us therefore a trivial state \( \Psi = 0 \). There is also a gauge-like ambiguity even for theories without constraints: under certain transformations of \( g \) the function \( \Psi \) does not vary.
6.1 Superpositions of elementary semiclassical states

Let \( X(\alpha) = (S(\alpha), Q(\alpha), P(\alpha)) \), \( \alpha = (\alpha_1, ..., \alpha_k) \) be a \( k \)-dimensional surface embedded to the base of the semiclassical bundle \( X \). Let \( g(\alpha, \xi) \) be a function of the class \( \mathcal{S}(\mathbb{R}^{k+n}) \). Consider the superposition of the wave packets (2.13)

\[
\Phi(q) = c \int d\alpha e^{\frac{i}{\hbar}S(\alpha)} e^{\frac{i}{\hbar}P(\alpha)(q\sqrt{\hbar}-Q(\alpha))} g(\alpha, q - \frac{Q(\alpha)}{\sqrt{\hbar}}). \tag{6.1}
\]

### 6.1.1 Explicit form of the composed semiclassical state

One can calculate the integral (6.1) explicitly analogously to [29]. First, notice that the integral (6.1) is exponentially small if the distance between \( q \) and the surface \( Q(\alpha) \) is of the order \( O(\hbar^{-1/2}) \). A nontrivial result will be obtained only if this distance is of the order \( O(\hbar^{1/2}) \), i.e. for some \( \pi q = \hbar^{-1/2}Q(\pi) + \xi \), \( \xi = O(1) \). Consider the substitution \( \alpha = \tilde{\alpha} + \beta \sqrt{\hbar} \). The integral will be taken to the form:

\[
\chi h^{k/2} \frac{e^{\frac{i}{\hbar}S(\pi)}}{e^{\frac{i}{\hbar}P(\pi)(q\sqrt{\hbar}-Q(\pi))}} \int d\beta e^{\frac{i}{\hbar}\beta_j(\frac{\partial S}{\partial \alpha_j} - \frac{\partial Q}{\partial \alpha_j})} e^{\frac{i}{\hbar}\beta_j(\frac{\partial^2 S}{\partial \alpha_j^2} - \frac{\partial^2 Q}{\partial \alpha_j^2})} e^{i\beta_j \frac{\partial}{\partial \alpha_j} \xi} g(\alpha, \xi - \frac{\partial Q}{\partial \alpha_j} \beta_j), \tag{6.2}
\]

here the higher-order terms is \( \hbar \) are omitted. Notice that this integral contains a rapidly oscillating expression \( e^{\frac{i}{\hbar}\beta_j(\frac{\partial S}{\partial \alpha_j} - \frac{\partial Q}{\partial \alpha_j})} \). Therefore, the integral is exponentially small, except for the case:

\[
\frac{\partial S}{\partial \alpha_j} = \frac{\partial Q}{\partial \alpha_j}. \tag{6.3}
\]

This is the Maslov isotropic condition [11]. Under this requirement, one can simplify the integral (6.2):

\[
\chi h^{k/2} \frac{e^{\frac{i}{\hbar}S(\pi)}}{e^{\frac{i}{\hbar}P(\pi)(q\sqrt{\hbar}-Q(\pi))}} f(\alpha, q - \frac{Q(\pi)}{\sqrt{\hbar}}), \tag{6.4}
\]

with

\[
f(\alpha, \xi) = \int d\beta e^{i\beta_j(\frac{\partial P}{\partial \alpha_j} \xi - \frac{\partial P}{\partial \alpha_j} \frac{1}{\hbar} \frac{\partial}{\partial \xi})} g(\alpha, \xi) = \prod_{i=1}^{k} \delta(\frac{\partial P}{\partial \alpha_j} \xi - \frac{\partial P}{\partial \alpha_j} \frac{1}{\hbar} \frac{\partial}{\partial \xi}) g(\alpha, \xi).
\]

### 6.1.2 Semiclassical inner product

Investigate the inner product \( \langle \Phi, \Phi \rangle \). Let us make use of formula (2.13). The wave function (2.24) will have then the following form

\[
\Phi^\tau(q) = c \int d\alpha e^{\frac{i}{\hbar}S^\tau(\alpha)} e^{\frac{i}{\hbar}P^\tau(\alpha)(q\sqrt{\hbar}-Q^\tau(\alpha))} \chi^\tau(\alpha, q - \frac{Q^\tau(\alpha)}{\sqrt{\hbar}}, \Pi, \Pi).
\]

The functions \( S^\tau(\alpha) \), \( P^\tau(\alpha) \), \( Q^\tau(\alpha) \) satisfy system (2.21), (2.22), while \( \chi^\tau(\alpha, \xi, \Pi) \) = \( g(\alpha, \xi) \). For \( \tau = -1 \) denote \( S^\tau(\alpha) \equiv S(\alpha, \mu) \), \( P^\tau(\alpha) \equiv P(\alpha, \mu) \), \( Q^\tau(\alpha) \equiv Q(\alpha, \mu) \), \( \chi^\tau(\alpha, \xi, \Pi) \equiv \chi(\alpha, \mu, \xi, \Pi) \). Therefore, the wave function

\[
\Psi(q) = \int M d\mu_{\alpha} d\Pi_{\alpha} d\Pi^\alpha e^{-\frac{i}{\hbar}P_{\alpha} + i\mu_{\alpha}\Pi_{\alpha} + \Pi_{\alpha}^\alpha + \frac{i}{\hbar} \Phi(q).
\]

entering to the inner product (2.14) has the form analogous to (6.1):

\[
\Psi(q) = c \int d\alpha d\mu e^{\frac{i}{\hbar}S(\alpha, \mu)} e^{\frac{i}{\hbar}P(\alpha, \mu)(q\sqrt{\hbar}-Q(\alpha, \mu))} g(\alpha, \mu, q - \frac{Q(\alpha, \mu)}{\sqrt{\hbar}}). \tag{6.5}
\]
with
\[ g(\alpha, \mu, \xi) = \int \prod_{\alpha=1}^{M} d\Pi_\alpha dP^\alpha \chi(\alpha, \mu, \xi, \Pi, \Pi). \] (6.6)

This is a Dirac wave function corresponding to the state \([6.1]\) [22].

Let us suppose that the \(k\)-dimensional surface \((Q(\alpha), P(\alpha))\) contains no gauge-equivalent states. More precisely, we require that the manifold \((Q(\alpha, \mu), P(\alpha, \mu))\) is smooth, \(k + M\)-dimensional, without self-intersections. For such a case, consider the inner product
\[ < \Phi, \Phi > \equiv \langle \Phi, \Psi \rangle \]
which has the form
\[ \langle \Phi, \Phi > = |c|^2 \int d\alpha d\gamma d\mu e^{i(S(\gamma, \mu) - S(\alpha))} \times \int dqg^*(\alpha, q - \frac{Q(\gamma, \mu)}{\sqrt{\hbar}}) \frac{P(\gamma, \mu)(\sqrt{\hbar} - Q(\gamma, \mu))}{P(\alpha)(\sqrt{\hbar} - Q(\alpha))} g(\gamma, \mu, q - \frac{Q(\gamma, \mu)}{\sqrt{\hbar}}). \] (6.7)

Analogously to subsection 2.2, the integrand is exponentially small, except for the case \(|P(\alpha) - P(\gamma, \mu)| = O(\sqrt{\hbar}), |Q(\alpha) - Q(\gamma, \mu)| = O(\sqrt{\hbar})\). Therefore, only the domain \(|\gamma - \alpha| = O(\sqrt{\hbar}), |\mu| = O(\sqrt{\hbar})\) gives rise to a nontrivial contribution to the integral (6.7). Thus, one should perform a substitution
\[ \gamma = \alpha + \beta \sqrt{\hbar}, \quad \mu = \rho \sqrt{\hbar}, \quad q = Q(\alpha) + \xi \sqrt{\hbar}. \]

The inner product (6.7) takes the form
\[ < \Phi, \Phi > = |c|^2\hbar^{\frac{M + k}{2}} \int d\alpha d\beta d\rho \exp\left\{ i \left[ S(\alpha + \sqrt{\hbar}\beta, \rho \sqrt{\hbar}) - S(\alpha) - P(\alpha + \sqrt{\hbar}\beta, \rho \sqrt{\hbar}) - Q(\alpha) \right] \right\} \times \int d\xi g^*(\alpha, \xi) e^{\frac{i}{\hbar}(P(\alpha + \sqrt{\hbar}\beta, \rho \sqrt{\hbar}) - P(\alpha))} g(\alpha + \beta \sqrt{\hbar}, \rho \sqrt{\hbar}, \xi + \frac{Q(\alpha) - Q(\alpha + \sqrt{\hbar}\beta, \sqrt{\hbar}\rho)}{\sqrt{\hbar}}) \] (6.8)
the singular term entering to the exponent is
\[ \frac{i}{\sqrt{\hbar}} \left( \frac{\partial S}{\partial \alpha} \beta + \frac{\partial S}{\partial \mu} \beta - P \frac{\partial Q}{\partial \alpha} \beta - P \frac{\partial Q}{\partial \mu} \rho \right) \]
If it is nonzero, the integral (6.8) contains a rapidly oscillating factor and becomes therefore exponentially small. Thus, one should impose the condition (6.3) and the following requirement:
\[ \frac{\partial S(\alpha, 0)}{\partial \mu_a} = P \frac{\partial Q(\alpha, 0)}{\partial \mu_a}. \] (6.9)

However, relations (6.3) is automatically satisfied because of (2.21), provided that
\[ \Lambda_a(Q(\alpha), P(\alpha)) = 0. \] (6.10)
(cf. subsection 2.2).

If conditions (6.3) and (6.9) are satisfied, the integral (6.8) is of the order \(O(1)\), provided that the normalizing factor is chosen to be
\[ |c| = \hbar^{-\frac{M + k}{4}} \]
Consider the limit \(\hbar \to 0\). Notice that eq. (2.22) implies for \(\mu_a = 0\) that
\[ \frac{\partial Q}{\partial \mu_a} = -\frac{\partial \Lambda_a}{\partial P}; \quad \frac{\partial P}{\partial \mu_a} = \frac{\partial \Lambda_a}{\partial Q}. \] (6.11)

Differentiating eq. (2.21) with respect to \(\tau\), one finds
\[ \tilde{S}^0 - P^0 \tilde{Q}^0 = \tilde{P}^0 \tilde{Q}^0, \]
or
\[
\frac{\partial^2 S}{\partial \mu_a \partial \mu_b} - P \frac{\partial^2 Q}{\partial \mu_a \partial \mu_b} = \frac{\partial P}{\partial \mu_a} \frac{\partial Q}{\partial \mu_b} - \frac{\partial \Lambda_a}{\partial \mu_a} \frac{\partial \Lambda_b}{\partial \mu_b}
\]

Differentiating eq. (6.3) with respect to \(\alpha\), one finds:
\[
\frac{\partial^2 S}{\partial \alpha_i \partial \alpha_j} = \frac{\partial P}{\partial \alpha_i} \frac{\partial Q}{\partial \alpha_j} + P \frac{\partial^2 Q}{\partial \alpha_i \partial \alpha_j}.
\]

Furthermore, eq. (6.3) implies
\[
\frac{\partial^2 S}{\partial \alpha_i \partial \alpha_j} = \frac{\partial P}{\partial \alpha_i} \frac{\partial Q}{\partial \alpha_j} + P \frac{\partial^2 Q}{\partial \alpha_i \partial \alpha_j}.
\]

We see that
\[
\frac{\partial P}{\partial \alpha_i} \frac{\partial Q}{\partial \alpha_j} = \frac{\partial Q}{\partial \alpha_j} \frac{\partial P}{\partial \alpha_i}
\]

since \(\frac{\partial^2 S}{\partial \alpha_i \partial \alpha_j}\) should be symmetric.

Thus, the exponent under conditions (6.3), (6.9) takes the form
\[
i \left[ \frac{1}{2} \beta_i \left( \frac{\partial^2 S}{\partial \alpha_i \partial \alpha_j} - P \frac{\partial^2 Q}{\partial \alpha_i \partial \alpha_j} \right) - \beta_j \left( \frac{\partial^2 S}{\partial \alpha_i \partial \alpha_j} - P \frac{\partial^2 Q}{\partial \alpha_i \partial \alpha_j} \right) \right] \rho_a + \frac{1}{2} \rho_a \left( \frac{\partial^2 S}{\partial \mu_a \partial \mu_b} - P \frac{\partial^2 Q}{\partial \mu_a \partial \mu_b} \right) \rho_b.
\]

Under condition (6.10), making use of eq. (6.11), we obtain as \(h \to 0\) that
\[
< \Phi, \Phi > \to \int d\alpha d\beta dp e^{i \beta_i \frac{\partial P}{\partial \alpha_i} \rho_a - \frac{i}{2} \rho_a \frac{\partial Q}{\partial \alpha_i} \rho_a + \frac{1}{2} \rho_a \frac{\partial \Lambda_a}{\partial \alpha_i} \rho_a \frac{\partial \Lambda_b}{\partial \alpha_j} \rho_b - \frac{i}{2} \rho_a \frac{\partial \Lambda_a}{\partial \alpha_i} \rho_a \frac{\partial \Lambda_b}{\partial \alpha_j} \rho_b} \times (g, e^{i \left( \frac{\partial P}{\partial \alpha_i} \rho_a - \frac{\partial Q}{\partial \alpha_i} \rho_a \right) \phi g}.
\]

Making use of the Baker-Hausdorff formula and relation
\[
\frac{\partial \Lambda_a}{\partial P} \frac{\partial P}{\partial \alpha_i} + \frac{\partial \Lambda_a}{\partial Q} \frac{\partial Q}{\partial \alpha_i} = 0
\]

which is a corollary of property (6.10), one can simplify the expression for \(< \Phi, \Phi >\):
\[
< \Phi, \Phi > \approx \int d\alpha (g, \int d\beta dp e^{i \beta_i \frac{\partial P}{\partial \alpha_i} \rho_a - \frac{i}{2} \rho_a \frac{\partial Q}{\partial \alpha_i} \rho_a + \frac{1}{2} \rho_a \frac{\partial \Lambda_a}{\partial \alpha_i} \rho_a \frac{\partial \Lambda_b}{\partial \alpha_j} \rho_b - \frac{i}{2} \rho_a \frac{\partial \Lambda_a}{\partial \alpha_i} \rho_a \frac{\partial \Lambda_b}{\partial \alpha_j} \rho_b} \times (g, e^{i \left( \frac{\partial P}{\partial \alpha_i} \rho_a - \frac{\partial Q}{\partial \alpha_i} \rho_a \right) \phi g}.
\]

6.1.3 Definition of a composed semiclassical state

We see that the composed semiclassical wave function (6.1) is specified, if the following properties C1-C2 are satisfied.

C1. A manifold \(X(\alpha) \equiv (S(\alpha), Q(\alpha), P(\alpha))\), \(\alpha = (\alpha_1, ..., \alpha_k)\) is given. It should obey eq. (6.3) (the Maslov "isotropic condition") and belong to the base \(X\) of the semiclassical bundle (i.e. obey eq. (6.10)). The manifold \(X(\alpha, \mu) = \lambda(\mu) X(\alpha)\) should be smooth, its Q, P-component \((Q(\alpha, \mu), P(\alpha, \mu))\) should be a smooth \(k + M\)-dimensional manifold without self-intersections.

C2. A function \(g \in \mathcal{S}(\mathbb{R}^{k+n})\) is specified.

Note that the set of parameters \(\alpha\) may take values on a nontrivial manifold \(\Lambda^k\) (such as circle, torus etc.) rather than \(\mathbb{R}^k\).
The inner product of the composed semiclassical states is given by formula (6.12). All the operators \( \frac{\partial P}{\partial \alpha_i} \xi - \frac{\partial Q}{\partial \alpha_i} i \frac{\partial \xi}{\partial \alpha_i}, \) \( \Xi \lambda_a \) commute each other.

It is interesting to note that maximal value of dimensionality \( k \) is \( n - M \). Namely, there are \( M + k \) tangent vectors to the phase space, 

\[
\delta P^{(i)} = \frac{\partial P}{\partial \alpha_i}, \quad \delta Q^{(i)} = \frac{\partial Q}{\partial \alpha_i}, \quad i = \overline{1,k}, \quad a = \overline{1,M}
\]

such that

\[
\delta P^{(a)} \delta Q^{(b)} - \delta Q^{(a)} \delta P^{(b)} = 0.
\]

The \( M + k \)-dimensional plane \( \text{span}\{\delta Q^{(a)}, \delta P^{(a)}\} \) is called isotropic. It is known \([10, 28]\) that maximal dimensionality of an isotropic plane is \( n \).

One can also notice that there is a gauge freedom in choosing \( g \). Namely, the transformation

\[
g \to (\Xi \Lambda_a) \chi^a + \Omega[\delta Q^{(i)}, \delta P^{(i)}] \eta^i
\]

takes a composed semiclassical state to equivalent.

Let us formulate a definition of a composed semiclassical state (cf.\([32]\)). First, introduce auxiliary notions. Let \( L_{k+M} \) be a \( k + M \)-dimensional isotropic plane with a measure \( d\sigma \) being invariant under shifts. Let \( (\delta Q^{(a)}, \delta P^{(a)}), \alpha = \overline{1,k+M} \) be a basis on \( L_{k+M} \). One can assign then coordinates \( \beta_1, ..., \beta_{k+M} \) to any element \( (\delta Q, \delta P) = \sum_{a=1}^{k+M} \beta_a (\delta Q^{(a)}, \delta P^{(a)}) \). The measure \( d\sigma \) is presented as \( ad\beta_1...d\beta_{k+M} \) for some constant \( a \). Consider the inner product

\[
< g_1, g_2 >_{L_{k+M}} = a \int d\beta (g_1, e^{i\beta_a \Omega(\delta Q^{(a)}, \delta P^{(a)})} g_2) = \int d\sigma (g_1, e^{i\Omega[\delta Q, \delta P]} g_2),
\]

\( g_1, g_2 \in \mathcal{S}^k(\mathbb{R}^{k+n}) \). This definition is invariant under change of basis. We say that \( g \overset{L_{k+M}}{\sim} 0 \) if \( < g, g >_{L_{k+M}} = 0 \). Denote \( \mathcal{F}(L_{k+M}) = \mathcal{S}^k(\mathbb{R}^n)/\sim \).

Let \( M^k \) be a \( k \)-dimensional manifold embedded to \( \mathcal{X}, \{X(\alpha), \alpha \in M^k\}, d\Sigma \) be a measure on \( M^k \). Let property C1 be satisfied. Define

\[
L_{k+M}(\alpha) = L_{k+M}(\alpha : M^k) = \text{span}\{(\delta Q^{(a)}, \delta P^{(a)})\}.
\]

for vectors \( (\delta Q^{(a)}, \delta P^{(a)}) \) of the form (6.13). Plane \( L_{k+M}(\alpha) \) does not depend on the particular choice of coordinates \( \alpha_1, ... \alpha_k \). Introduce the following measure \( d\sigma(\alpha) \) on \( L_{k+M}(\alpha) \):

\[
d\sigma(\alpha) = \frac{D\Sigma(\alpha)}{D\alpha} d\beta_1...d\beta_{k+M}.
\]

(6.15)

Here \( (\beta_1, ..., \beta_{k+M}) \) are coordinates on \( L_{k+M}(\alpha) \) which are introduced as

\[
(\delta Q, \delta P) = \sum_{\alpha} \beta_\alpha (\delta Q^{(a)}, \delta P^{(a)}).
\]

Definition (6.13) is invariant under change of coordinates \([32]\).

Introduce the Hilbert bundle \( \pi_{M^k} \) as follows. The base is the isotropic manifold \( M^k \). The fibre corresponding to the point \( \alpha \in M^k \) is \( \mathcal{H}_\alpha = \mathcal{F}(L_{k+M}(\alpha)) \). Composed semiclassical states are viewed as sections \( Z \) of \( \pi_{M^k} \) such that the inner product

\[
< (M^k, Z), (M^k, Z) > = \int_{M^k} d\Sigma < Z(\alpha), Z(\alpha) >_{L_{k+M}(\alpha)}.
\]

(6.16)

converges.
6.1.4 Semiclassical transformations of composed semiclassical states

Analogously to section 5, one can apply the operator $e^{-i\hat{O}t}$ to the composed semiclassical state for the quantum observable $\hat{O}$ (5.1). The wave function $\Phi^t = e^{-i\hat{O}t}\Phi$ will satisfy the Cauchy problem (5.2).

The substitution
\[
\Phi^t(q) = c \int d\alpha e^{i \frac{\hbar}{2} S^t(\alpha)} e^{i \frac{\hbar}{2} P^t(\alpha)(q - Q^t(\alpha))} g^t(\alpha, q - \frac{Q^t(\alpha)}{\sqrt{\hbar}})
\]
will give an asymptotic solution of the Cauchy problem (5.2), provided that $(S^t, Q^t, P^t)$ satisfies the Cauchy problem for eq.(5.3), while eq.(5.4) is valid for $g^t(\alpha, \xi)$. After differentiation of eq.(5.3) with respect to $\alpha$ we find that
\[
[i \frac{d}{dt} - \frac{1}{2}(\Xi^2 \Lambda_a)(Q^t, P^t) + \mu_a \Lambda_a^{(1)}(Q^t, P^t)] \phi^t = 0.
\]

Analogously to subsection 5.1, this implies that the inner product (6.12) (or (6.16)) conserves under time evolution.

Equivalence of composed semiclassical states is introduced in the same way as equivalence of elementary semiclassical states. Since expression (6.1) is a linear superposition of elementary semiclassical states, all the results concerning equivalence are valid for the composed semiclassical states as well.

6.2 Composed semiclassical states in the Dirac approach

Composed semiclassical states can be also investigated in the Dirac approach. Consider the wave function (6.5). Let us calculate the function $\chi^\tau$ being a solution of the Cauchy problem for eq.(2.30). Let us look for it in the following form:
\[
\chi^\tau(\alpha, \xi, \Pi, \bar{\Pi}) = \varphi^\tau(\alpha, \xi) \exp[-\bar{\Pi}_a M^a_b(\alpha, \tau) \Pi^b]
\]
(6.17)

Substituting expression (6.17) to eq.(2.30), one obtains that
\[
i \dot{\varphi}^\tau = \frac{1}{2} \mu_a (\Xi^2 \Lambda_a)(Q^\tau, P^\tau) + \mu_a \Lambda_a^{(1)}(Q^\tau, P^\tau) \varphi^\tau,
\]
while
\[
\dot{M}_b^a = -\delta_b^a - \mu_c U_{ac}(Q^\tau, P^\tau) M^d_b.
\]
(6.19)

Integrating function (6.17) over Grassmannian variables, we find
\[
g^\tau(\alpha, \xi) = \det M \varphi^\tau(\alpha, \xi).
\]
We are interested in $g^\tau$ at $\tau = -1$.

One can notice that for $X(\alpha, \mu) = (S(\alpha, \mu), Q(\alpha, \mu), P(\alpha, \mu))$ one has
\[
X(\alpha, \mu) = \lambda_{-\mu} X(\alpha),
\]
while
\[
\varphi^{-1}(\alpha, \xi) = V_{0, \mu}(\lambda_{-\mu} X \leftarrow X) g(\alpha).
\]

Let us investigate the matrix $M$. It follows from eq.(6.14) that eq.(6.19) can be rewritten as
\[
\frac{dM}{d\tau} + (Ad_X(-\mu \tau))^{-1} \frac{d}{d\tau} (Ad_X(-\mu \tau)) = -1,
\]so that
\[
Ad_X(-\mu \tau) M(\tau) = -\int_0^\tau d\tau' Ad_X(-\mu \tau').
\]
For the case \( \tau = -1 \), one takes this relation to the form
\[
M(-\mu, X) = (Ad_X \mu)^{-1} \int_0^1 d\tau Ad_X(\mu \tau). \tag{6.20}
\]
Eq. (6.20) can be simplified. First, notice
\[
(Ad_X \mu)^{-1} Ad_X(\mu(1 - \tau)) = Ad_{\lambda_{-\mu} X}(-\mu \tau),
\]
so that
\[
M(-\mu, X) = \int_0^1 d\tau Ad_{\lambda_{-\mu} X}(-\mu \tau).
\]
It follows from eq. (3.26) that
\[
M_a^b = \frac{\delta \sigma^a}{\delta \mu^b}. \tag{6.21}
\]
Here \( \delta \sigma \) and \( \delta \mu \) are related as follows,
\[
\lambda_{-\delta \sigma} \lambda_{-\mu} X = \lambda_{-\mu - \delta \rho} X.
\]
Thus,
\[
g(\alpha, \mu) = \text{det} M(-\mu, X(\alpha)) V_0^\mu(\lambda_{-\mu} X(\alpha) \leftarrow X(\alpha)) g(\alpha).
\]
The measure \( \text{det} M d\mu \) resembles the invariant measure on the quasigroup constructed in [26].

Analogously to expression (6.1), the wave function (6.5) will not vary in the leading order in \( h \) under transformation
\[
g \Rightarrow g + \Omega(\frac{\partial Q}{\partial \alpha}, \frac{\partial P}{\partial \alpha}) \chi^i + \Omega(\frac{\partial Q}{\partial \mu_a}, \frac{\partial P}{\partial \mu_a}) \zeta^a. \tag{6.22}
\]
Let us show that the Dirac wave function is invariant under gauge transformations of \( \Phi \) and satisfies the constraint conditions
\[
\hat{\Lambda}^+_a \Psi = 0. \tag{6.23}
\]
It is sufficient to check these properties for the case of elementary semiclassical states.

6.2.1 Invariance of Dirac wave function under gauge transformations

Let
\[
\Phi(q) = c(K^h_X g)(q) \equiv ce^{\frac{i}{\hbar} S} e^{\frac{i}{\hbar} P(q\sqrt{\hbar} - Q)} f(q - \frac{Q}{\sqrt{\hbar}})
\]
be a wave packet wave function for \( X = (S, Q, P) \). Then the Dirac function will be of the form
\[
\Psi = c \int d\mu K^h_{\lambda_{-\mu} X} V^0_{-\mu}(\lambda_{-\mu} X \leftarrow X) g \text{ det } M(-\mu, X). \tag{6.24}
\]

Consider the gauge transformation
\[
X \to \lambda_{-\mu} X; \quad g \to V^0_{-\mu}(\lambda_{-\mu} X \leftarrow X) g.
\]
For the transformed semiclassical state, the Dirac wave function will take the form
\[
\tilde{\Psi} = c \int d\mu K^h_{\lambda_{-\mu} \lambda_{-\nu} X} V^0_{-\mu}(\lambda_{-\mu} \lambda_{-\nu} X \leftarrow \lambda_{-\nu} X) V^0_{-\nu}(\lambda_{-\nu} X \leftarrow X) g \text{ det } M(-\mu, \lambda_{-\nu} X). \tag{6.25}
\]
Making use of the gauge function (6.22) (if necessary), one takes formula (6.23) to the form
\[
\tilde{\Psi} \simeq c \int d\mu K^h_{\lambda_{-\mu} X} V^0_{-\rho}(\lambda_{-\mu} X \leftarrow X) g \text{ det } M(-\mu, \lambda_{-\nu} X)). \tag{6.26}
\]
Here
\[ \lambda_{-\nu}X = \lambda_{-\mu} \lambda_{-\nu}X \]  \hspace{1cm} (6.27)
for \( \rho = \rho(\mu, \nu, X) \). Expressions (6.24) and (6.26) coincide, provided that
\[ \frac{D\mu}{D\rho} \det M(-\mu, \lambda_{-\nu}X) = \det M(-\rho, X). \]  \hspace{1cm} (6.28)
Relation (6.28) is a corollary of (6.21). Namely,
\[ \frac{D\mu}{D\rho} = \det \delta \mu \delta \rho; \quad \lambda_{-\nu} \lambda_{-\mu} X = \lambda_{-\rho} \lambda_{-\delta \mu} X; \]
\[ \det M(-\mu, \lambda_{-\nu}X) = \det \delta \delta \sigma \delta \mu; \quad \lambda_{-\nu} \lambda_{-\mu} X = \lambda_{-\delta \mu} \lambda_{-\nu} X. \]
Thus, eq. (6.28) is satisfied, and the Dirac wave function is invariant under gauge transformation.

6.2.2 Constraint conditions
To check relation (6.23), it is more convenient to justify that
\[ \exp[i\hat{\Lambda}_{\alpha}^{+} \nu_{\alpha}] \Psi = \Psi. \]  \hspace{1cm} (6.29)
The left-hand side of eq. (6.29)
\[ \tilde{\Psi} = c \int d\mu K_{\lambda_{-\nu} \lambda_{-\mu} X} V_{-\nu}^{0}(\lambda_{-\nu} \lambda_{-\mu} X \leftarrow \lambda_{-\mu} X) V_{\mu}^{0}(\lambda_{-\mu} X \leftarrow X) g \det M(-\mu, X) \times \exp[-i \int_{0}^{1} d\tau \nu_{\alpha} \{ \Lambda_{\alpha}^{(1)}(\lambda_{\nu \tau} \lambda_{-\mu} X) - \Lambda_{\alpha}^{(1)}(\lambda_{\nu \tau} \lambda_{-\mu} X) \}] \]
can be presented as
\[ \tilde{\Psi} = c \int d\rho \frac{D\mu}{D\rho} K_{\lambda_{-\nu} \lambda_{-\mu} X} V_{-\nu}^{0}(\lambda_{-\mu} X \leftarrow X) g \det M(-\mu, X) \exp[- \int_{0}^{1} d\tau \nu_{\alpha} U_{\alpha}^{d}(\lambda_{\nu \tau} \lambda_{-\mu} X)] \]  \hspace{1cm} (6.30)
Here
\[ \lambda_{-\nu} \lambda_{-\mu} X = \lambda_{-\rho} X \]  \hspace{1cm} (6.31)
and property (2.13) is taken into account. Moreover, eq. (3.14) implies that
\[ \exp[- \int_{0}^{1} d\tau \nu_{\alpha} U_{\alpha}^{d}(\lambda_{\nu \tau} \lambda_{-\mu} X)] = \exp[- \log \det Ad_{-\lambda_{-\rho} X}(\nu \tau)]^{-1} = (\det Ad_{-\lambda_{-\rho} X}(\nu))^{-1}. \]
Eqs. (6.30) and (6.24) coincide if
\[ \frac{D\mu}{D\rho} \det M(-\mu, X)(\det Ad_{-\lambda_{-\rho} X}(\nu))^{-1} = \det M(-\rho, X). \]  \hspace{1cm} (6.32)
Here \( \mu(\rho, \nu) \) is defined from eq. (6.31).
Moreover, eq. (6.32) is satisfied, since
\[ \frac{D\mu}{D\rho} = \det \delta \mu \delta \rho; \quad \lambda_{-\nu} \lambda_{-\mu} \lambda_{-\delta \mu} X = \lambda_{-\rho} \lambda_{-\delta \mu} X; \]
\[ \det M(-\mu, X) = \det \delta \sigma \delta \mu; \quad \lambda_{-\delta \sigma} \lambda_{-\mu} X = \lambda_{-\delta \mu} \lambda_{-\mu} X. \]
\[ (\det Ad_{-\lambda_{-\mu} X}(\nu))^{-1} = \det \delta \nu \delta \sigma; \quad \lambda_{-\nu} \lambda_{-\delta \sigma} \lambda_{-\mu} X = \lambda_{-\delta \nu} \lambda_{-\mu} \lambda_{-\mu} X; \]
\[ \det M(-\rho, X) = \det \delta \nu \delta \rho; \quad \lambda_{-\delta \nu} \lambda_{-\rho} X = \lambda_{-\rho} \lambda_{-\delta \nu} X. \]
Constraint (6.23) is checked.
7 Discussion

Let us discuss the obtained results.

We have started from the quantum theory of the system with $M$ first-class constraints depending on the small parameter of the semiclassical expansion $\hbar$ according to eq. (2.4). Notions of elementary and composed semiclassical states have been introduced.

There are different ways to quantize a constrained system. To investigate elementary semiclassical states, the most convenient quantization technique is the refined algebraic quantization approach. Elementary semiclassical states are specified by sets $(X, f)$, where $X = (S, P, Q)$ is a classical state belonging to the constraint surface $\Lambda_a(Q, P) = 0$, while $f$ is a quantum state in the external background $X$. The quantum wave function depends on $\hbar$ as $\hbar^{-1/2}$.

It has been shown that the condition $\Lambda_a(Q, P) = 0$ is very important. If it is not satisfied, the norm of elementary semiclassical state is exponentially small.

The inner product of semiclassical states has been calculated in the semiclassical approximation (eq. (2.31)). However, this formula is valid only if the linearized constraints $\Xi\Lambda_a$ (eq. (A.1)) are independent. The case of linearly dependent operators $\Xi\Lambda_a$ can be investigated as follows. One should choose such a basis in the Lie algebra of constraints that $\Xi\Lambda_A = 0$, $A = 1, D$, $0 < D \leq M$, while $\Xi\Lambda_{D+\alpha}$, $1 \leq \alpha \leq M - D$ are linearly independent. Then one writes down formula (2.7), rescale $\mu^{D+\alpha} \Rightarrow \sigma^\alpha \sqrt{\hbar}$, $\mu^A \Rightarrow \rho^A$. Analogously to subsection 2.2, one finds

$$
< \Phi, \Phi > \sim \hbar^{D/2} |c|^2 \int d\rho d\sigma J(\rho, 0)(f, e^{-i\rho^{A+1/2}\Xi^{2}\Lambda_A - i\sigma^\alpha \Xi^{D+\alpha} f}).
$$

We see that if some of linearized constraints vanish, one should take into account the quadratic part of them.

Since the inner product (2.31) has appeared to be degenerate, one should say that two semiclassical wave functions corresponding to the same $X$ are equivalent if their difference has zero norm (for example, of the form $(\Xi\Lambda_a)(X)\chi^\alpha$). Thus, it is more correct to say that elementary semiclassical states are specified by a set of a classical state $X$ and a class of equivalence $[f]$. Set of all elementary semiclassical states forms a semiclassical bundle with the base $\{(S, P, Q)|\Lambda_a(Q, P) = 0\}$ and fibres being spaces of states $[f]$. Elementary semiclassical states are then points on the semiclassical bundle.

An important property of theories of constrained systems is gauge invariance. In the refined algebraic quantization approach, this means that quantum states $\Phi$ and $e^{-i\tau^{a}\Lambda_{a}}\Phi$ are equivalent. For the elementary semiclassical state $\Phi$ specified by $(X, [f])$, the wave function $e^{-i\tau^{a}\Lambda_{a}}\Phi$ calculated explicitly in the semiclassical approximation has appeared to be an elementary semiclassical state $(\lambda_{\mu^{a}}X, V(\lambda_{\mu^{a}}X \leftarrow X)f)$. We see that gauge group acts on the semiclassical bundle, so that some of elementary semiclassical states are gauge-equivalent. The group and quasigroup properties of elementary semiclassical transformations have been investigated.

Elementary semiclassical states can be also investigated within the Dirac approach discussed in section 6. The wave function (6.21) is specified then by an $M$-dimensional surface on the semiclassical bundle. The surface can be interpreted as a gauge orbit.

Composed semiclassical states are introduced as superpositions (1.1) of elementary semiclassical states. In the refined algebraic quantization approach, they are specified by $k$-dimensional surfaces on the semiclassical bundle $(X(\alpha), g(\alpha))$ with $\alpha$ being a $k$-dimensional variable, $X(\alpha)$ be an $\alpha$-dependent classical state, $g(\alpha)$ be an $\alpha$-dependent quantum state in the external background $X(\alpha)$. The inner product of composed semiclassical states has been evaluated (eq. (6.12)). In the Dirac approach, the dimensionality of the surface embedded to the semiclassical bundle is $M + k$.

Evolution transformations of elementary and composed semiclassical states have been investigated, provided that the quantum Hamiltonian depends on the small parameter as $\hbar^{1/2}$. Semiclasical state $(X, f)$ is taken to $(u_{t}X, U_{t}(u_{t}X \leftarrow X)f)$. It has been also shown that gauge-equivalent semiclassical states are taken to gauge equivalent.
The obtained results can be used for finding the semiclassical spectrum of a semiclassical observable. One should consider such a semiclassical initial condition for the Schrödinger equation that conserves its form under time evolution. This means that manifold \((P_t(\alpha), Q_t(\alpha))\) should be gauge-equivalent to \((P(\alpha), Q(\alpha))\). Certain conditions on \(f^t\) can be also obtained. For example, one can choose a stationary point of a Hamiltonian system or a periodic trajectory \((P(t), Q(t))\) as an 0- or 1-dimensional isotropic manifold and obtain a static or periodic soliton quantization theory [1].

We have noticed that \(k\)-dimensional isotropic manifolds in the refined algebraic quantization theory correspond to \(k + M\)-dimensional isotropic manifolds in the Dirac approach. It is well-known [4] that maximal dimensionality of an isotropic manifold is \(n\); this corresponds to the semiclassical WKB theory. We see that WKB-method can be developed for the Dirac quantization approach only, while the wave packet method can be applied to the refined algebraic quantization only. Main formulas of the WKB theory in the Dirac approach may be obtained without using integral representation (6.24). Namely, the WKB wave function has the following explicit form according to subsubsection 6.1.1. If \(k + M = n\), in "general position" case for all \(q\) there exists such \(\alpha\) that \(q = h^{-1/2}Q(\alpha)\). Eq.(6.4) reads:

\[
\Psi(q) = \varphi(q\sqrt{h}) \exp\left(\frac{i}{\hbar}S(q\sqrt{h})\right)
\]

for some \(\varphi\) and \(S\). Eq.(6.3) implies that the isotropic manifold \((P(\alpha), Q(\alpha))\) coincides with

\[
\{(\frac{\partial S}{\partial x_i}, x_i) | x \in \mathbb{R}^n\}
\]

The Dirac condition (6.23) can be rewritten as

\[
\Lambda^\ast(X, \frac{\partial S}{\partial X} - ih\frac{\partial}{\partial X})\varphi(X) = 0.
\]

In the leading order in \(h\), one finds that the isotropic manifold lies on the constraint surface, the next-to-leading order gives us first-order equations on \(\varphi\).

However, the main difficulty of the WKB-approach is that the Cauchy problem Hamiltonian-Jacobi equation may have no solutions (this means that the projection of the isotropic manifold to the plane \(P = 0\) is not unique). For such cases, the more complicated semiclassical methods such as the Maslov canonical operator approach [9] should be used. One of possible formulations of this method is based on the integral representation (6.1).

The discussed derivation of the semiclassical theory is based on the quantum theory: it was used as a starting point, then the notions of semiclassical states, observables and evolution was introduced. However, semiclassical mechanics can be also viewed as a first step of quantization of classical theory. Indeed, it has been shown that all objects of the semiclassical theory (inner product, gauge transformation, semiclassical observables, semiclassical transformation) are expressed in terms of classical variables.

Acknowledgments

The author is indebted to J.Klauder, D.Marolf and V.P.Maslov for helpful discussions. This work was supported by the Russian Foundation for Basic Research, projects 99-01-01198 and 01-01-06251.

A Some properties of Weyl quantization

Let us review some results of the theory of Weyl quantization (see, for example, [27, 28]).
There are different ways to define a notion of a function of operators $q$ and $-i\partial/\partial q$. One can put the coordinates to the right and momenta to the left and vice versa. An alternative way is to use the Weyl ordering. Let $f(q,p)$ be an arbitrary function of $q \in \mathbb{R}^n$, $p \in \mathbb{R}^n$. Consider its expansion as a Fourier integral,

$$f(q,p) = \int d\alpha d\beta \tilde{f}(\alpha, \beta)e^{i\alpha q + i\beta p}. \quad (A.1)$$

By definition, set

$$W[f] \equiv f(q, -i\frac{\partial}{\partial q}) = \int d\alpha d\beta \tilde{f}(\alpha, \beta)e^{i\alpha q + i\beta (-i\frac{\partial}{\partial q})}. \quad (A.2)$$

The operator $W[f]$ is called a Weyl quantization of the function $f$, while $f$ is called a Weyl symbol of the operator $W[f]$. The exponent $e^{i\alpha q + i\beta (-i\frac{\partial}{\partial q})}$ entering to eq. $(A.2)$ can be defined with the help of the Baker-Hausdorff formula,

$$(e^{i\alpha q + i\beta p})^\psi(x) = (e^{i\alpha}e^{i\beta p}e^{\frac{1}{2}i\beta q})^\psi(x) = e^{i\alpha(x + \frac{1}{2}p)}(e^{i\frac{1}{2}q})^\psi(x) = e^{i\alpha(x + \frac{1}{2}p)}\psi(x + \beta).$$

**Proposition A.1.** The Weyl symbol of the product $W[f]W[g]$ is

$$(f * g)(q, p) = \int \frac{dk_1dk_2dx_1dx_2}{(2\pi)^{2n}}f(q + x_1, p + \frac{k_2}{2})g(q + x_2, p - \frac{k_1}{2})e^{-ik_1x_1 - ik_2x_2} \quad (A.3)$$

**Proof.** One has

$$W[f]W[g] = \int d\alpha' d\beta' f(\alpha', \beta')g(\alpha', \beta')e^{i\alpha'q + i\beta'p}.$$ 

Making use of the Baker-Hausdorff formula, we take this relation to the form

$$W[f]W[g] = d\alpha''d\beta''d\alpha' d\beta' \tilde{f}(\alpha'', \beta'')\tilde{g}(\alpha', \beta')e^{i(\alpha'' + \alpha')q + i(\beta'' + \beta)p}e^{\frac{1}{2}(\alpha'' - \alpha')^2}$$

here the redefining $\alpha \to \alpha'$, $\beta \to \beta'$ is made. Therefore, the Fourier transformation of $f * g$ is

$$(f * g)(\alpha, \beta) = \int d\alpha' d\beta' \tilde{f}(\alpha - \alpha', \beta - \beta')\tilde{g}(\alpha', \beta')e^{\frac{1}{2}(\alpha - \alpha')^2}.$$ 

Making use of formula $(A.1)$ and analogous formula for the inverse Fourier transformation, we obtain relation $(A.3)$. Proposition is proved.

Let $f$ and $g$ depend on the small parameter $h$ as follows,

$$f = F(q\sqrt{h}, p\sqrt{h}), \quad g = G(q\sqrt{h}, p\sqrt{h}).$$

then

$$F(\tilde{q}\sqrt{h}, \tilde{p}\sqrt{h})G(\tilde{q}\sqrt{h}, \tilde{p}\sqrt{h}) = H_h(\tilde{q}\sqrt{h}, \tilde{p}\sqrt{h})$$

with

$$H_h(Q, P) = \int \frac{dk_1dk_2dx_1dx_2}{(2\pi)^{2n}}F(Q - \frac{\sqrt{h}}{2}x_1, P + \frac{\sqrt{h}k_2}{2})G(Q + \frac{\sqrt{h}x_2}{2}, P + \frac{\sqrt{h}k_1}{2})e^{-ik_1x_1 - ik_2x_2}.$$ 

Here the rescaling $x_1 \to -x_1/2$, $k_1 \to -2k_1$ is performed. Let us simplify this expression. One has

$$F(Q - \frac{\sqrt{h}}{2}x_1, P + \frac{\sqrt{h}k_2}{2})e^{-ik_1x_1 - ik_2x_2} = F(Q - \frac{i\sqrt{h}}{2}\frac{\partial}{\partial k_1}, P + \frac{i\sqrt{h}}{2}\frac{\partial}{\partial x_2})e^{-ik_1x_1 - ik_2x_2}.$$
Integrating exponent over $k_2$ and $x_1$, we obtain $\delta$-function $\delta(x_2)\delta(k_1)$. Integration by parts gives us the following expression

$$ H_h(Q, P) = F(Q + \frac{i\sqrt{h}}{2} \frac{\partial}{\partial k_1}; P - \frac{i\sqrt{h}}{2} \frac{\partial}{\partial x_2})G(Q + \sqrt{h}x_2; P + \sqrt{h}k_1)|_{x_2=0,k_1=0}. $$

We obtain the following proposition.

**Proposition A.2.** The following relation takes place:

$$ H_h = FG + i\hbar \left( \frac{\partial F}{\partial Q} \frac{\partial G}{\partial P} - \frac{\partial F}{\partial P} \frac{\partial G}{\partial Q} \right) + \hbar^2 \left( -\frac{1}{2} \frac{\partial^2 F}{\partial Q_{ij} \partial P_{ij}} - \frac{1}{2} \frac{\partial^2 G}{\partial Q_{ij} \partial P_{ij}} + \frac{\partial^2 F}{\partial Q_{ij} \partial P_{ij}} + \frac{\partial^2 G}{\partial Q_{ij} \partial P_{ij}} \right) + o(h^2). $$

Consider now the operator $F(Q + \xi \sqrt{h}, P + \frac{\sqrt{h}}{i} \frac{\partial}{\partial \xi})$ which can be defined as

$$ F(Q + \xi \sqrt{h}, P + \frac{\sqrt{h}}{i} \frac{\partial}{\partial \xi}) = \int d\alpha d\beta \tilde{F}(\alpha, \beta) e^{i\alpha(Q + \xi \sqrt{h}) + i\beta(P + \frac{\sqrt{h}}{i} \frac{\partial}{\partial \xi})}. $$

We obtain the following proposition.

**Proposition A.3.** The following relation is satisfied:

$$ F(Q + \xi \sqrt{h}, P + \frac{\sqrt{h}}{i} \frac{\partial}{\partial \xi}) = e^{\sqrt{h}(\xi \frac{\partial}{\partial Q} + \frac{1}{2i} \frac{\partial}{\partial \xi} \frac{\partial}{\partial P})} F(Q, P). \quad (A.4) $$

Let us expand the operator expression \((A.4)\) into a series. One has

$$ F(Q + \xi \sqrt{h}, P + \frac{\sqrt{h}}{i} \frac{\partial}{\partial \xi}) = F(Q, P) + \sqrt{h}(\Xi F)(Q, P) + \frac{h}{2}(\Xi^2 F)(Q, P) + \ldots $$

with

$$ \Xi = \xi \frac{\partial}{\partial Q} + \frac{1}{i} \frac{\partial}{\partial \xi} \frac{\partial}{\partial P}. $$

Explicitly, one has

$$ \frac{1}{2}(\Xi^2 F) = \frac{1}{2} \xi_i \frac{\partial^2 F}{\partial Q_i \partial Q_j} \xi_j + \frac{1}{2} \xi_i \frac{\partial^2 F}{\partial Q_i \partial P_j} (-i \frac{\partial}{\partial Q_j} + \frac{1}{2} (\frac{\partial}{\partial Q_j} \frac{\partial^2 F}{\partial P_j \partial Q_i} \xi_j + \frac{1}{2} (\frac{\partial}{\partial Q_j} \frac{\partial^2 F}{\partial P_j \partial Q_i} \xi_j + \frac{1}{2} (\frac{\partial}{\partial Q_j} \frac{\partial^2 F}{\partial P_j \partial Q_i} (-i \frac{\partial}{\partial Q_j})). \quad (A.5) $$

Let us investigate some properties of the operators $\Xi F$ and $\Xi^2 F$.

**Proposition A.4.** The following properties are obeyed:

$$ \Xi(AB) = \Xi A \cdot B + A \cdot \Xi B; $$

$$ \frac{1}{2} \Xi^2(AB) = \frac{1}{2} \Xi^2 A \cdot B + \frac{1}{2} \Xi A \cdot \Xi B + \frac{1}{2} \Xi B \cdot \Xi A + \frac{1}{2} A \cdot \Xi^2 B; $$

$$ [\Xi A; \Xi B] = -i\{A; B\} = -i\left( \frac{\partial A}{\partial P} \frac{\partial B}{\partial Q} - \frac{\partial A}{\partial Q} \frac{\partial B}{\partial P} \right). $$

The proof is by direct calculations.

For each function $A(Q, P)$ introduce the Hamiltonian vector field

$$ \delta_A = \frac{\partial A}{\partial P_i} \frac{\partial}{\partial Q_i} - \frac{\partial A}{\partial Q_i} \frac{\partial}{\partial P_i}. \quad (A.6) $$
Obviously,
\[ [\delta_A; \delta_B] = \delta_{\{A,B\}}. \]

Let us investigate properties of the operators
\[ i\delta_A - \frac{1}{2} \Xi^2 A. \]  
(A.7)

**Proposition A.5.** The operators \((A.7)\) satisfy the following properties,
\[ [i\delta_A - \frac{1}{2} \Xi^2 A; \Xi B] = i\Xi\{A; B\}; \]
\[ [i\delta_A - \frac{1}{2} \Xi^2 A; i\delta_B - \frac{1}{2} \Xi^2 B] = i(i\delta_{\{A,B\}} - \frac{1}{2} \Xi^2 \{A; B\}). \]

The proof is by direct usage of formulas \((A.3); (A.6)\).

**B Types of semiclassical wave functions**

The most popular semiclassical approach to quantum mechanics is the WKB-approach. It is the following. One considers the initial condition for the equation
\[ i\hbar \frac{\partial \Psi_t(X)}{\partial t} = h(X, -i\hbar \frac{\partial}{\partial X}) \Psi_t(X), \quad X \in \mathbb{R}^n, \]  
(B.1)

which depends on the small parameter \(h\) as follows,
\[ \Psi_0(X) = \varphi_0(X) e^{\frac{i}{\hbar} S_0(X)}, \]  
(B.2)

where \(S_0(X)\) is a real function. The WKB-result [9] is that the solution of the Cauchy problem at time moment \(t\) has also the form \((B.2)\) up to \(O(h)\),
\[ \Psi_t(X) = \varphi_t(X) e^{\frac{i}{\hbar} S_t(X)} + O(h). \]  
(B.4)

Equations for \(\varphi_t\) and \(S_t\) can be obtained.

However, we are not obliged to choose the initial condition for eq.\((B.1)\) in a form \((B.2)\). There are other substitutions to eq.\((B.2)\) that conserve their form under time evolution as \(h \to 0\). For example, consider the Maslov complex-WKB wave function [10],
\[ \Psi_0(X) = \text{const} e^{\frac{i}{\hbar} S_0(X)} e^{\frac{i}{\hbar} P_0(X - Q_0)} f_0\left(\frac{X - Q_0}{\sqrt{\hbar}}\right), \]  
(B.3)

corresponding to the wave packet with uncertainties of the coordinate and momentum of the order \(O(\sqrt{\hbar})\). Formula \((B.3)\) specifies the classical particle with classical coordinate \(Q_0\) and classical momentum \(P_0\). The function \(f_0\) specifies the shape of the wave packet: we see that semiclassical mechanics is indeed richer than classical since there are no analogs of \(f_0\) in classical mechanics.

It happens that the initial condition \((B.3)\) conserves its form under time evolution \([10],\)
\[ \Psi_t(X) \simeq \text{const} e^{\frac{i}{\hbar} S_t} e^{\frac{i}{\hbar} P_t(X - Q_t)} f_t\left(\frac{X - Q_t}{\sqrt{\hbar}}\right), \]  
(B.4)

up to \(O(\sqrt{\hbar})\). The phase factor \(S_t\) is the action along the classical trajectory, \(P_t, Q_t\) obey the classical Hamiltonian system. For the function \(f_t\) specifying time evolution of the form of the wave packet, the Schrodinger equation with a time-dependent quadratic Hamiltonian is obtained \([10], [29]\).
The wave function (B.2) rapidly oscillates with respect to all variables \(X \in \mathbb{R}^n\). The wave packet (B.3) rapidly damps at \(X - Q_0 \gg O(\sqrt{\hbar})\). One should come to the conclusion that there exists a wave function asymptotically satisfying eq.(B.1) which oscillates with respect to one group of variables and damps with respect to other variables. Construction of such states is given in the Maslov theory of Lagrangian manifolds with complex germ \([10]\). Let \(\alpha \in \mathbb{R}^k\), \((P(\alpha), Q(\alpha)) \in \mathbb{R}^{2n}\) be a \(k\)-dimensional surface in the \(2n\)-dimensional phase space, \(S(\alpha)\) be a real function, \(f(\alpha, \xi), \xi \in \mathbb{R}^n\) be a smooth function. Set \(\Psi(X)\) to be not exponentially small if and only if the distance between point \(X\) and surface \(Q(\alpha)\) is of the order \(\lesssim O(\sqrt{\hbar})\). Otherwise, set \(\Psi(X) \simeq 0\). If \(\min_\alpha |X - Q(\alpha)| = |X - Q(\bar{\alpha})| = O(\sqrt{\hbar})\), set

\[
\Psi(X) = \text{const} e^{\frac{i}{\hbar} S(\bar{\alpha})} e^{\frac{i}{\hbar} P(\bar{\alpha}) (X - Q(\bar{\alpha}))} f(\bar{\alpha}, \frac{X - Q(\bar{\alpha})}{\sqrt{\hbar}}) \tag{B.5}
\]

One can note that wave functions (B.2) and (B.3) are partial cases of the wave function (B.5). Namely, for \(k = 0\) the manifold \((P(\alpha), Q(\alpha))\) is a point, so that the function (B.3) coincides with (B.3). Let \(k = n\). If the surface \((P(\alpha), Q(\alpha))\) is in the general position, for \(X\) in some domain one has \(X = Q(\bar{\alpha})\) for some \(\bar{\alpha}\), so that \(\Psi(X) = \text{const} e^{\frac{i}{\hbar} S(\bar{\alpha})} f(\bar{\alpha}, 0)\) is a WKB-function.

The lack of formula (B.5) is that the dependence of \(\bar{\alpha}\) on \(X\) is implicit and too complicated. However, under certain conditions formula (B.5) is invariant if \(\bar{\alpha}\) is shifted by a quantity of the order \(O(\sqrt{\hbar})\). Such conditions are

\[
\frac{\partial S}{\partial \alpha_i} = P \frac{\partial Q}{\partial \alpha_i}; \tag{B.6}
\]

\[
(\xi \frac{\partial P}{\partial \alpha_i} - \frac{1}{i} \frac{\partial}{\partial \xi} \frac{\partial Q}{\partial \alpha_i}) f = 0. \tag{B.7}
\]

The form (B.5) of the semiclassical state appeared in the Maslov theory of Lagrangian manifolds with complex germ is not convenient for quantum field theory. It is much more suitable to consider the superposition

\[
\Psi(X) = \text{const} \int d\alpha e^{\frac{i}{\hbar} S(\alpha)} e^{\frac{i}{\hbar} P(\alpha) (X - Q(\alpha))} f(\alpha, \frac{X - Q(\alpha)}{\sqrt{\hbar}}) \tag{B.8}
\]

of states (B.2). Partial cases of such a superposition were considered in \([30]\); general case is investigated in \([29,31]\). The case of semiclassical mechanics in abstract spaces is considered in \([25]\).

It happens that the integral (B.8) approximately coincides with (B.5), provided that condition (B.4) is satisfied and

\[
f(\bar{\alpha}, \xi) = \prod_{s=1}^k (2\pi \delta(\frac{\partial P}{\partial \alpha_s} \xi - \frac{\partial Q}{\partial \alpha_s} \frac{1}{i} \frac{\partial}{\partial \xi})) g(\bar{\alpha}, \xi)
\]

Thus, condition (B.7) is automatically satisfied.

We see that the wave-packet wave function (B.4) may be viewed as "elementary" semiclassical states, while wave functions appeared in the theory of Lagrangian manifolds with complex germ (including WKB functions) can be considered as superpositions of "elementary" semiclassical states.
References

[1] R.Dashen, B.Hasslasher, A.Neveu, Phys. Rev. D10 (1974), 4114;
R.Rajaraman, "Solitons and Instantons. An Introduction to solitons and instantons in quantum
field theory", North-Holland,Amsterdam, Netherlands, 1982;
J. Coldstone, R.Jackiw, Phys.Rev. D11 (1975). 1486;
L.D.Faddeev, V.E.Korepin, Phys. Rep. 42 (1978) 1.

[2] R.Jackiw, Rev.Mod.Phys. 49 (1977), 681.

[3] A.A. Grib, S.G. Mamaev, V.M. Mostepanenko, "Vacuum Quantum Effects in Strong Fields",
Atomizdat, Moscow, 1988; Friedmann Laboratory Publishing, St. Petersburg 1994.

[4] N.D. Birrell, P.C.W. Davies, "Quantum Fields in Curved Space", Cambridge, UK: Univ. Pr. ,
1982.

[5] D.Boyanovsky, H.J. de Vega and R.Holman, Phys. Rev. D49 (1994), 2769;
D.Boyanovsky, H.J. de Vega, R.Holman, D.S.Lee and A.Singh, Phys. Rev. D51 (1995), 4419.

[6] Ju.Baacke, K.Heitmann and C.Pätzold, Phys. Rev. D55 (1997), 2320;
Ju.Baacke, K.Heitmann and C.Pätzold, Phys. Rev. D56 (1997), 6556.

[7] F.Cooper, E.Mottola, Phys.Rev. D36 (1987), 3114;
S.-Y.Pi, M.Samiullah, Phys. Rev. D36 (1987), 3128.

[8] R.Jackiw and A.Kerman, Phys.Lett. A71 (1979), 158;
F.Cooper, S.-Y.Pi and P.Stancioff, Phys. Rev. D34 (1986), 3831;
O.Eboli, R.Jackiw and S.-Y.Pi, Phys. Rev. D37 (1988), 3557;
O.Eboli, S.-Y.Pi and M.Samiullah, Ann. Phys. 193 (1989), 102.

[9] V.P.Maslov, "Perturbation Theory and Asymptotic Methods", Moscow, Moscow Univ. Press, 1965.

[10] V.P.Maslov, "Operational Methods", Moscow, Nauka, 1973; Mir Publishers, 1976;
V.P.Maslov, "The Complex-WKB Method for Nonlinear Equations", Moscow, Nauka, 1977.

[11] P.A.M.Dirac, Lectures on Quantum Mechanics, Yeshiva Univ., New York, 1965.

[12] M.Henneaux, Phys. Reports 126 (1985) 1.

[13] L.D.Faddeev, Teor. Mat. Fiz. 1 (1969) 1.

[14] V.N.Gribov, Nucl. Phys. B139 (1978) 1.

[15] S.V.Shabanov, Phys. Reports 326 (2000) 1.

[16] C. Becchi, A. Rouet and R. Stora, Phys. Lett. B52 (1974) 344;
C. Becchi, A. Rouet and R. Stora, Ann. Phys. 98 (1976) 287;
I.V. Tyutin, FIAN preprint 39 (1975).

[17] E.S.Fradkin and G.A.Vilkovisky, Phys. Lett. B55 (1975) 224;
I.A.Batalin and G.A.Vilkovisky, Phys. Lett. B69 (1977) 309;
T.Kugo and I.Ojima, Suppl. Prog. Theor. Phys., No 66 (1979) 1.

[18] A.Ashtekar, J.Lewandowski, D.Marolf, J.Mourao and T.Thiemann, J. Math. Phys. 36 (1995) 6456;
D.Marolf, gr-qc/9508013.
[19] D.Giulini and D.Marolf, Class. Quant. Grav. 16 (1999) 2489.

[20] D.Marolf, gr-qc/0011112.

[21] J.Klauder, Ann. Phys. 254 (1997) 419;
    J.Govaerts, J. Phys. A30 (1997) 603.

[22] O.Yu.Shvedov, hep-th/0107064.

[23] R.Marnelius and M.Orgen, Nucl. Phys. B351 (1991) 474;
    R.Marnelius, Nucl. Phys. B418 (1994) 353;
    I.A.Batalin and R.Marnelius, Nucl. Phys. B442 (1995) 669.

[24] O.Yu.Shvedov, Matematicheskie Zametki 65 (1999) 437.

[25] O.Yu.Shvedov, Matematicheskii Sbornik 190 (1999) N10, 123.

[26] I.A.Batalin, J. Math. Phys. 22 (1981), 1837.

[27] F.A.Berezin, Teor. Mat. Fiz. 6 (1971), 194.

[28] M.V.Karasev, V.P.Maslov, ”Nonlinear Poisson Bracket. Geometry and Quantization”, Moscow, Nauka, 1991.

[29] V.P.Maslov and O.Yu.Shvedov, ”The Complex Germ Method in Many-Particle Problem and Quantum Field Theory”, Moscow, Editorial URSS, 2000.

[30] M.M.Popov, Zapiski Nauchnogo Seminara LOMI, 104 (1981) 195;
    M.V.Karasev, Zapiski Nauchnogo Seminara LOMI, 172 (1989) 41;
    M.V.Karasev and Yu.M.Vorobiev, preprint ITP-90-85E, Kiev, 1990.

[31] V.P.Maslov, O.Yu.Shvedov Teor. Mat.Fiz. 104 (1995) 479.

[32] O.Yu.Shvedov, math-ph/0109016.