Mirrors–light–atoms entanglement in ring optomechanical cavity

Oumayma El Bir · Morad El Baz

Received: 13 February 2023 / Accepted: 11 August 2023 / Published online: 5 September 2023
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2023

Abstract
The present paper illustrates the realization of an atom-optomechanical system where an atomic ensemble is confined in a ring optomechanical cavity consisting of a fixed mirror and two movable ones. An analysis of the dynamics and the linearization of the equations allows us to derive the multimode covariance matrix. Under realistic experimental conditions, we numerically simulate the steady-state bipartite and tripartite continuous variable entanglement using the logarithmic negativity and analyze the shared entanglement in the multimode system. The introduction of the atomic medium is responsible for a more expansive plateau of entanglement indicating its robustness against temperature-induced decoherence effects.

Keywords Ring cavity · Optomechanical systems · Quantum entanglement · Atom medium · Continuous variable systems

1 Introduction
Cavity optomechanics has witnessed a rapidly growing advance with the goal of controlling the interaction between electromagnetic radiation and nanomechanical motion [1–4]. Optomechanical systems can be coherently manipulated by pumping the cavity with an external laser field which leads to an optomechanical coupling between the cavity mode and the mechanical oscillator resulting in various interesting applications [5–9].

In this context, both experimental and theoretical efforts have shown that these systems can offer an exclusive platform to explore the phenomenon of entanglement between mechanical resonators, atoms and optical cavity fields in different ways...
For instance, generation of micro–macro-entanglement \cite{11} and generation of stationary optomechanical entanglement between a mechanical oscillator and its measurement apparatus were proposed in \cite{12}. One can also achieve an entanglement between two movable mirrors in the same optomechanical cavity as in \cite{13}, entangling a nanomechanical oscillator with a Cooper-pair box \cite{14} and entangling two mirrors in a ring cavity by using a phase-sensitive feedback loop \cite{15}. Recently, non-reciprocal entanglement in cavity optomechanics has attracted great interest \cite{16, 17} as well as entangling light and mechanical modes via polarization control \cite{18}.

Another concept, widely exploited in coupled optomechanical systems, is aimed at shedding light on the possibilities of constructing entangled states between the macroscopic and subatomic realms \cite{19–21}. This represents a novel type of hybrid systems, whereby an atomic ensemble is placed inside a cavity with a moving mechanical object \cite{22–24}. These atomic cavity optomechanical systems circumvent the difficulties in controlling quantum properties against decoherence and thus providing a bridge between the quantum and classical worlds \cite{25}. A recent experiment \cite{26} achieved such an entanglement between a mechanical oscillator and atomic spins located at a one-meter distance at room temperature.

To explore the impact of the atomic ensemble on the entanglement between mechanical resonators and the optical field, we study quantum entanglement between the atomic ensemble and the vibrating mirrors. For this reason, we use a compact ring optomechanical cavity, specifically designed to facilitate the integration of the atomic ensemble. The ring cavity is composed of a fixed mirror, two movable ones and an atomic ensemble of two-level atoms placed inside the cavity. The latter is pumped with a coherent laser source and squeezed vacuum source. The atoms respond to the vacuum light and the electromagnetic fields by spontaneously emitting photons, resulting in an increase in the number of photons within the cavity. Moreover, the mechanical displacement of the two movable mirrors is enhanced due to the force exerted by the higher number of photons on their surfaces, leading to a robust optomechanical entanglement against environmental fluctuations. The exchange of energy between atoms, photons and phonons offers a good platform to study the properties of tripartite entanglement in this type of cavities, which is our main contribution of this paper.

We focus on enhancing bipartite and tripartite entanglement by utilizing various processes to enhance the number of photons in the current system. Namely, we study the impact of injecting both vacuum and laser sources into an atomic-ring cavity. Furthermore, the cavity’s geometry comprising two movable mirrors, exhibits advantageous behavior toward the optomechanical coupling. Consequently, we establish that when both vacuum and laser sources are injected into an optimal cavity containing an atomic medium, it essentially forms a well-fortified environment to generate multipartite entanglement \cite{21, 27}.

The remainder of this paper is organized as follows. In Sect. 2, we present the theoretical model of the hybrid optomechanical system and establish the Hamiltonian, obtain the system dynamics, by solving the differential equations of motion and obtain the set of quantum Langevin equations of the system. In Sect. 3, we simulate the
stationary bipartite and tripartite entanglement by introducing logarithmic negativity and exploit this result to investigate the sharing structure of tripartite entanglement in such states. In Sect. 4, we conclude our results and summarize with some perspectives of the results.

2 System and model

2.1 System

The model studied here is an atom-optomechanical system, made of a ring cavity with length $L$, and an atomic ensemble of two-level atoms placed inside the cavity. This latter is driven by a squeezed light source with frequency $\omega_S$ and a coherent laser source with strength $E_L$; the setup is schematically shown in Fig. 1. The optical cavity is composed of two movable mirrors perfectly transmitting and a fixed one partially transmitting in a triangular design. We consider the two movable mirrors as quantum harmonic oscillators with effective masses, respectively, $m_1$, $m_2$, and frequencies $\omega_{m_1}$ and $\omega_{m_2}$, respectively. The cavity field is coupled to the motion of the two mechanical oscillators via the radiation pressure force. The total Hamiltonian of the coupled optomechanical system can be expressed as: $H = H_0 + H_{\text{interaction}} + H_{\text{Drive}}$, which will be well explained in the next section.

This setup is an upgraded version of the ring optomechanical cavity originally introduced in [28], in which the interaction with an atomic ensemble is considered. Importantly, systems based on atom-optomechanical models are experimentally fea-
sible [19, 29, 30]. It is worth mentioning that the proposal of manipulating two mechanical motions within atom-optomechanical cavity has already been introduced in [31], where the authors propose an atom-optomechanical system consisting of a fixed mirror and two charged nanomechanical oscillators coupled through the coulomb interaction in a linear design. However, the coulomb interaction between charged objects can generate repulsive or attractive forces [32], which may cause unwanted noise that can impact the system’s performance. To overcome this issue, we adopt a ring configuration to facilitate the generation of entanglement, that can be strengthened by the displacement of the mirror and an increase of the photon number.

2.2 The Hamiltonian

The Hamiltonian of the system is given by $H = H_0 + H_{\text{Interaction}} + H_{\text{Drive}}$, with

$$H_0 = \hbar \omega_r a^\dagger a + \frac{\hbar \omega_{m1}}{2}(q_1^2 + p_1^2) + \frac{\hbar \omega_{m2}}{2}(q_2^2 + p_2^2) + \frac{\hbar \omega_a}{2} S_z,$$

$$H_{\text{Interaction}} = \hbar g (S_+ a + S_- a^\dagger) + \hbar G_0 a^\dagger a \cos^2(\theta/2)(q_1 - q_2),$$

$$H_{\text{Drive}} = i \hbar E_L \left(a^\dagger e^{-i\omega_L t} - a e^{i\omega_L t}\right).$$

$H_0$ represents the free Hamiltonian, the first term of which describes the cavity mode with the resonance frequency $\omega_r$, $a^\dagger$ and $a$ are the photon creation and annihilation operators of the optical field, with $[a, a^\dagger] = 1$. $q_j$ and $p_j (j = 1, 2)$ describe the dimensionless position and momentum operators of the two mechanical resonators, satisfying $[q_j, p_k] = i \delta_{jk}$. The last term describes the atomic ensemble composed of $N_a$ two-level atoms, where the frequency $\omega_a$ is the transition between the ground state $|g\rangle$ and the excited state $|e\rangle$, having, respectively, the following energies $E_g = -\frac{\hbar \omega_a}{2}$ and $E_e = \frac{\hbar \omega_a}{2}$. $S_z$ and $S_\pm$ are the spin operators $S_{z,\pm} = \sum_{i=1}^{N_a} \sigma^i_{z,\pm}$, where $\sigma^i_{z,\pm}$ are the Pauli matrices defined by $\sigma^i_z = |e^{(i)(i)}\rangle e|$, $\sigma^i_+ = |e^{(i)(i)}\rangle g|$ and $\sigma^i_- = |g^{(i)(i)}\rangle e|$ and satisfying the commutation relations $[\sigma^i_+, \sigma^j_-] = \sigma^i_z$ and $[\sigma^i_+, \sigma^j_+] = \pm 2 \sigma^i_-$.

$H_{\text{Interaction}}$ describes the coupling Hamiltonian, where the atoms interact with the two mechanical oscillators via the coupling with the intracavity field, which is called the atom–cavity coupling coefficient and expressed as $g = \mu \sqrt{\frac{\omega_c}{2\epsilon_0 V}}$ with $\mu$ being the dipole moment of the atomic transition, $V$ is the volume of the cavity and $\epsilon_0$ is the vacuum permittivity and $\hbar$ is Planck constant. The second term takes into account the interaction of the two mechanical resonators motion with the electromagnetic field confined in the cavity due to the radiation pressure force, $G_0$ is the optomechanical coupling coefficient, the angle between the incident and the reflected light on the surfaces of the movable mirrors, $\theta$ (see Fig. 1).

The part $H_{\text{Drive}}$ represents the drive laser input, where $E_L = \sqrt{\frac{P \omega_L}{\kappa \hbar}}$ with $P$, $\omega_L$ being, respectively, the power and the frequency of the driven laser, $\kappa$ is the decay rate of the optical cavity.
Working in the rotating frame at the input laser frequency $\omega_L$, the Hamiltonian of the system simplifies to

$$
H = \hbar \Delta_r a^\dagger a + \frac{\hbar \omega_{m1}}{2} (q_1^2 + p_1^2) + \frac{\hbar \omega_{m2}}{2} (q_2^2 + p_2^2) + \hbar \Delta_a c^\dagger c + \hbar G_a (c^\dagger a + ca^\dagger)
+ \hbar G_0 a^\dagger a \cos^2(\theta/2)(q_1 - q_2) + i \hbar E_L (a^\dagger - a),
$$

(4)

with $\Delta_r = \omega_r - \omega_L$ and $\Delta_a = \omega_a - \omega_L$ are, respectively, the cavity mode and atomic detuning. For simplification, we choose the low atomic excitation limit, i.e., when the atoms are initially in the ground state and the average number of photons is much smaller in the excited state, so that $S_Z \approx <S_Z> \approx -N_a$. In addition to that, we suppose the excitation probability of a single atom to be small. In this limit, the atomic polarization can be defined in terms of the bosonic annihilation and creation operators $c = \frac{S}{\sqrt{|<S_Z>|}}$, which satisfy the bosonic commutation relations $[c, c^\dagger] = 1$ [33]. We define $G_a = g \sqrt{N_a}$, the atom–cavity coupling strength.

### 2.3 The quantum Langevin equations

For a clear analysis of the dynamics of the system, we determine the Heisenberg equations of motion which can be obtained from the Hamiltonian (4). Taking into consideration the effects of noise and the dissipation terms leads to the following Heisenberg-Langevin equations [34]

$$
\dot{q}_1 = \omega_m p_1,
\dot{q}_2 = \omega_m p_2,
\dot{p}_1 = -\omega_m q_1 - G_0 a^\dagger a \cos^2(\theta/2) - \gamma_m p_1 + f_1,
\dot{p}_2 = -\omega_m q_2 + G_0 a^\dagger a \cos^2(\theta/2) - \gamma_m p_2 + f_2,
\dot{a} = -(\kappa + i \Delta_r) a - i G_a c - i G_0 \cos^2(\theta/2)(q_1 - q_2) a + E_L + \sqrt{2 \kappa \alpha_{in}},
\dot{c} = -iG_a a + \sqrt{2 \gamma_a \epsilon_{in}},
$$

(5)

where for the sake of simplicity and without loss of generality, we chose $\omega_{m1} = \omega_{m2} = \omega_m$, $\gamma_{m1} = \gamma_{m2} = \gamma_m$, with $\gamma_m$ being the mechanical damping rate and $\gamma_a$ the decay rate of the atomic excited level. In Eq. (5), we defined $G_0$ as $G_0 = (\omega_L)^2 \sqrt{\frac{\hbar}{m_{m0}}}$ while $f_1$ and $f_2$ are the Brownian noise operators with zero mean values. We choose the quality factor $Q_i = \frac{\omega_{m_i}}{\gamma_{m_i}} \gg 1$ meaning that one can assume the mechanical baths to be Markovian. Accordingly, the $f_i (i = 1, 2)$ operator’s nonzero correlation functions $[35, 36]$ are given by

$$
(f_i(t) f_i(t')) + (f_i(t') f_i(t))/2 \approx \gamma_m (2n_{th_i} + 1) \delta(t - t'),
$$

(6)

with $n_{th_i} = 1/(e^{\frac{\hbar \omega_{m_i}}{k_B T_i}}).$ being the $i^{th}$ thermal photon number and $k_B$ is the Boltzmann constant. The noise operator corresponding to the atomic ensemble with zero
mean value, $c_{in}$, appearing in (5), satisfy the non-vanishing correlations function $\langle c_{in}(t)c_{in}^\dagger(t') \rangle = \delta(t - t')$ [34].

Another kind of noise affecting the system, is the input squeezed vacuum noise operator $a_{in}$, that are fully characterized by the nonzero correlations functions [37]:

$$\langle \delta a_{in}(t)\delta a_{in}(t') \rangle = N \delta(t - t'),$$
$$\langle \delta a_{in}(t)\delta a_{in}(t') \rangle = (N + 1) \delta(t - t'),$$
$$\langle \delta a_{in}(t)\delta a_{in}(t') \rangle = Me^{-i\omega_m(t+t')} \delta(t - t'),$$
$$\langle \delta a_{in}^\dagger(t)\delta a_{in}^\dagger(t') \rangle = M^* e^{i\omega_m(t+t')} \delta(t - t'),$$

where $M = \sinh r \cosh re^{i\phi}$ and $N = \sinh^2 r$, $r$ and $\phi$ being, respectively, the strength squeezing parameter and phase of the squeezed vacuum light.

2.4 Linearization of the quantum Langevin equations

To analyze the dynamics of the coupled system, we begin by linearizing the quantum Langevin Eqs. (5). In fact, these latter are in general non-solvable analytically. So one needs to expand each Heisenberg operator as a sum of its classical steady state value plus an additional operator of fluctuation with zero-mean value [38]:

$$a = a_s + \delta a, \quad q = q_s + \delta q, \quad p = p_s + \delta p, \quad c = c_s + \delta c.$$  \hspace{1cm} (8)

The corresponding steady-state values read

$$p_1^s = 0,$$
$$p_2^s = 0,$$
$$q_1^s = -\frac{G_0 \cos^2(\theta/2)|a_s|^2}{\omega_m},$$
$$q_2^s = \frac{G_0 \cos^2(\theta/2)|a_s|^2}{\omega_m},$$
$$c^s = \frac{-iG_a a^s}{(\gamma_a + i \Delta_a)},$$
$$a^s = \frac{E_L}{\kappa + i\Delta + \frac{G_0^2}{\gamma_a + i \Delta_a}},$$  \hspace{1cm} (9)

where $\Delta = \Delta_r + G_0 \cos^2(\theta/2)(q_1^s - q_2^s)$.

Inserting Eq. (8) in (5), and introducing $\delta X = \frac{\delta a + \delta a^\dagger}{\sqrt{2}}, \delta Y = \frac{\delta a - \delta a^\dagger}{i\sqrt{2}}, \delta X = \frac{\delta c + \delta c^\dagger}{\sqrt{2}},$

$\delta y = \frac{\delta c - \delta c^\dagger}{i\sqrt{2}}, X_{in} = \frac{a_{in} + a_{in}^\dagger}{\sqrt{2}}, \quad Y_{in} = \frac{a_{in} - a_{in}^\dagger}{i\sqrt{2}}, \quad x_{in} = \frac{c_{in} + c_{in}^\dagger}{\sqrt{2}}, \quad y_{in} = \frac{c_{in} - c_{in}^\dagger}{i\sqrt{2}}$ allows to
obtain the following linearized Langevin equations:

\[
\begin{align*}
\delta q_1 &= \omega_m \delta p_1, \\
\delta \dot{p}_1 &= -\omega_m \delta q_1 - \gamma_m \delta p_1 - G \cos^2(\theta/2)\delta X + f_1, \\
\delta q_2 &= \omega_m \delta p_2, \\
\delta \dot{p}_2 &= -\omega_m \delta q_2 - \gamma_m \delta p_2 + G \cos^2(\theta/2)\delta X + f_2, \\
\delta \dot{X} &= -\kappa \delta X + \Delta \delta Y + G_a \delta y + \sqrt{2\kappa} \delta X_{in}, \\
\delta \dot{Y} &= -G \cos^2(\theta/2)\delta q_1 + G \cos^2(\theta/2)\delta q_2 - \Delta \delta X - \kappa \delta Y - G_a \delta x + \sqrt{2\kappa} \delta Y_{in}, \\
\delta \dot{x} &= G_a \delta Y - \gamma_a \delta x + \Delta_a \delta y + \sqrt{2\gamma_a} \delta x_{in}, \\
\delta \dot{y} &= -G_a \delta X - \gamma_a \delta y - \Delta_a \delta x + \sqrt{2\gamma_a} \delta y_{in}.
\end{align*}
\]

with \( G = \sqrt{2} G_0 a^z \). The resulting evolution equations of motion for the fluctuations in (10) can be rewritten in the matrix form

\[
\dot{u}(t) = Au(t) + n(t),
\]

where \( u(t) \) and \( n(t) \) are, respectively, the column vector of the fluctuations and the column vector of noise operators, the transpose of which are, respectively, given by

\[
\begin{align*}
u^T(\infty) &= (\delta q_1(\infty), \delta p_1(\infty), \delta q_2(\infty), \delta p_2(\infty), \delta X(\infty), \delta Y(\infty), \delta x(\infty), \delta y(\infty)) \\
n^T(t) &= (0, f_1, 0, f_2, \sqrt{2\kappa} X_{in}, \sqrt{2\kappa} Y_{in}, \sqrt{2\gamma_a} x_{in}, \sqrt{2\gamma_a} y_{in}).
\end{align*}
\]

The drift matrix \( A \) reads

\[
A = \begin{pmatrix}
0 & \omega_m & 0 & 0 & 0 & 0 & 0 & 0 \\
-\omega_m & -\gamma_m & 0 & 0 & 0 & -G \cos^2(\theta/2) & 0 & 0 \\
0 & 0 & -\omega_m & -\gamma_m & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\kappa & \Delta & 0 \\
-2G \cos^2(\theta/2) & 0 & G \cos^2(\theta/2) & 0 & 0 & -\Delta & -\kappa & -G_a \\
0 & 0 & 0 & 0 & 0 & G_a & -\gamma_a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\gamma_a & -\Delta_a \\
0 & 0 & 0 & 0 & 0 & 0 & -G_a & 0
\end{pmatrix}
\]

The solution of the differential Eq. (11) is \( u(t) = Y(t)u(0) + \int_0^t dx \dot{Y}(x) \eta(t-x) \), with \( Y(t) = \exp At \).

### 2.5 Covariance matrix

The covariance matrix \( V \) of the system can be obtained using the following Lyapunov equation [39, 40]:

\[
AV + VA^T = -D,
\]

\[ \text{Springer} \]
where \( D \) is a diagonal matrix that represents the noise correlations. It is given by

\[
D = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \gamma_m(2n_{th} + 1) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma_m(2n_{th} + 1) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2\kappa(\Re\{M\} + N + \frac{1}{2}) & 2\kappa\Im\{M\} & 0 & 0 \\
0 & 0 & 0 & 2\kappa\Im\{M\} & 0 & 0 & 0 & \gamma_a \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma_a & 0 & 0 \\
\end{pmatrix}
\]  

(15)

The covariance matrix \( V \) can be written in a block form:

\[
V = \begin{pmatrix}
V_m^1 & V_{m1m2} & V_{m1op} & V_{m1a} \\
V_{m1m2}^T & V_m^2 & V_{m2op} & V_{m2a} \\
V_{m1op}^T & V_{m2op}^T & V_{op} & V_{opa} \\
V_{m1a}^T & V_{m2a}^T & V_{opa}^T & V_a \\
\end{pmatrix},
\]  

(16)

where \( V_a, V_m^j \) (\( j = 1, 2 \)) and \( V_{op} \) are the covariance matrices of the atomic mode, the \( (j = 1, 2) \) mechanical mode and the optical mode, respectively.

When interested in studying the behavior of only two subsystems (and their correlations, among other things), the global \( 8 \times 8 \) covariance matrix \( V \) (16) can be reduced to a \( 4 \times 4 \) submatrix \( V_S \), containing only the covariance matrices of the subsystems of interest:

\[
V_S = \begin{pmatrix}
A & C \\
C^T & B \\
\end{pmatrix},
\]  

(17)

with \( A \) and \( B \) being the \( 2 \times 2 \) covariance matrices describing the single modes and \( C \) the \( 2 \times 2 \) covariance matrix of the quantum correlations between the two subsystems.

### 3 Entanglement analysis

The system is comprised of four modes: atomic \((a)\), optical \((op)\) and the two mechanical modes \((m_1 \text{ and } m_2)\); it allows the study of various types of entanglements. Bipartite entanglement, which is generally the type that is thoroughly studied in the literature, can be investigated in this case, using any bi-partition of the system. Moreover, tripartite entanglement can also be discussed in detail using the different tri-partitions of the system. The benefit of such a general study is that the comparison of the different types of entanglement brings more insight into their general behavior and their mutual influence.

#### 3.1 Bipartite entanglement

To quantify the bipartite stationary entanglement between any two modes \( x \) and \( y \) \((x, y = a, m_1, m_2 \text{ or } op)\), we use the logarithmic negativity \( E_N \). It is defined for Gaussian continuous variable systems as [41, 42]
Mirrors–light–atoms entanglement in ring optomechanical... Page 9 of 18

Fig. 2 Plot of the bipartite entanglement $E_{m_1a}$, between $m_1$ and $a$, versus the thermal bath temperature $T (mK)$, for different values of the input field squeezing parameter $r$.

\[ E_N = \max\{0, -\ln 2\tilde{\eta}\}, \]  

(18)

with

\[ \tilde{\eta} = \min \left\{ \text{eig} \left| \bigoplus_{j=1}^{2} (\sigma_y) P V_S P \right| \right\}, \]  

(19)

where $\sigma_y$ is the $y$-Pauli matrix, $V_S$ is the $4 \times 4$ covariance matrix of the two subsystems and $P = \sigma_z \oplus 1$, with $\sigma_z$ being the $z$-Pauli matrix.

In order to evaluate numerically the logarithmic negativity, we choose the power of the driven laser $P = 35$ mW, the masses and the frequencies of two oscillators are, respectively, $m = 10$ ng and $\omega_m = 2\pi \times 10^7$ Hz, the laser wavelength is $\lambda = 1064$ nm, $\theta = \frac{\pi}{4}$, the cavity decay rate $\kappa = \pi \times 10^7$ Hz, the mechanical damping rate $\gamma_m = 2\pi \times 10^2$ Hz, the length of the cavity $L = 1$ mm, the phase of the squeezed vacuum light $\phi = 0$. We choose the parameter of the atoms to be $G_a = 12\pi \times 10^6$ Hz and $\gamma'_a = 2\pi \times 10^7$ Hz. In addition, we consider that the atoms are resonant: $\Delta_a = -\omega_m$. Some parameters are taken from the set of experiments [43–45].

It is important to note that the symmetry between the mechanical modes is reflected in the expressions of the logarithmic negativities involving these modes. For instance, the logarithmic negativity between any mechanical mode and the optical mode is the same, $E_{m_1\text{op}} = E_{m_2\text{op}}$, and the same with respect to the atomic mode, $E_{m_1a} = E_{m_2a}$. Next, we investigate the stationary entanglement as a function of the thermal bath’s temperature $T$.

Figure 2 shows the bipartite stationary entanglement $E_{m_1a}$, we choose $\Delta = \omega_m$ and numerically simulate the logarithmic negativity between the mechanical mode 1 and the atomic mode for different values of the squeezing parameter $r$. This will allow us to...
distinguish the impact of the atomic medium on hybrid optomechanical systems. It is worth noting that the atoms are indirectly coupled to the mechanical resonators through their common interaction with the input field. In fact, when atoms are present within the cavity, they have the ability to interact coherently with the electromagnetic field mode. They respond to the cavity field by spontaneously emitting photons, increasing their overall number in the cavity. The collisions of the photons on the surfaces of the movable mirrors generate a strong optomechanical coupling due to the radiation pressure force. Consequently, a stronger optomechanical coupling can be achieved by increasing the number of atoms resulting in a more resilient atom-mirror entanglement.

It is well known that the entanglement decreases with the effect of the environment’s temperature due to thermal fluctuations; this is confirmed in the behavior of $E_{m_1a}$ in Fig. 2. Interestingly, for such systems in the presence of the atomic medium, a collective interaction can emerge between atoms and the cavity field mode, giving rise to the phenomenon of superradiance [46], responsible for the creation of entangled states involving the atoms and cavity. Simultaneously, the optical field being coupled to the mechanical oscillator, the presence of the atomic ensemble enhances the cooperativity between the optical mode and the mechanical mode, enabling the generation of a significant amount of entanglement. In addition, the ring structure can minimize optical losses compared to other cavity geometries, leading to higher finesse and powerful interaction against decoherence as is clearly seen in the plateau observed for the entanglement at low temperatures. For that reason, adding an atomic ensemble to a cavity allows for a stronger optomechanical coupling. Besides this, the atom-field coupling strength and the excitation number have an important influence on the atomic effective damping rate of the mechanical resonators [47].

The numerical simulation results show that the progressive injection of the squeezed light increases the entanglement, and it becomes more robust against the environment temperature, e.g., for low temperatures, when $r = 0.2$, the numerical value of the entanglement is $E_{m_1a} = 0.3$, whereas, for $r = 1$, it is $E_{m_1a} = 0.56$. Moreover, it is noticeable that the entanglement in the case of $r = 1$ persists against the bath environment more than in the case where the parameter of squeezing $r = 0.2$. This explains the impact of the squeezed vacuum light on entanglement due to the light–matter interaction. Indeed, the increase of the photon number in the cavity leads to a stronger radiation pressure force and does enhance the quantum correlations transfer from squeezed light to the subsystems.

In Fig. 3, we plot the logarithmic negativity $E_{m_1op}$ which expresses the entanglement between the mechanical mode 1 and the optical mode. In the following, we analyze the influence injecting the squeezed light in the system. The red line representing the entanglement when there is no squeezing and only the laser field is injected ($r = 0$) shows that the entanglement vanishes around $T \simeq 0.35$ mk. However, the blue line representing the squeezed vacuum light with ($r = 0.1$) shows that the entanglement survives until $T \simeq 0.42$ mk. We remark that for low temperatures we have $E_{m_1op} = 0.14$ (when $r = 0$) and $E_{m_1op} = 0.17$ (when $r = 0.1$). A relatively significant entanglement is reached for a sufficiently large number of photons from the two sources, as they allow for a robust photon–phonon interaction via the radiation pressure. Indeed, the photons exert a small push on the surface of the movable mirror and change the cavity’s length, which in turn, modifies the intensity of the field and does
enhance the radiation pressure force allowing for a strong optomechanical coupling that optimizes entanglement [48].

Overall, the injection of laser and squeezed vacuum light that increases the number of photons, in conjunction with the atomic ensemble placed inside the cavity that scatter photons through stimulated re-emission is of crucial importance for the motion of the mechanical oscillator. It should be noted that quantum fluctuations occur within the cavity as a result of the interaction between the pumped photons and the photons scattered by the atom, while thermal fluctuations originating from the mechanical bath environment contribute to decoherence. Since any feasible system couples with its own environment to some extent, a large number of photons are needed to overcome the decohering effects of the quantum fluctuations and to strengthen the resulting entanglement.

### 3.2 Tripartite entanglement

In order to study the existence of genuine tripartite entanglement we adopt a quantitative measure of tripartite negativity [49–51], which for a tripartite system $(ABC)$ is given by

$$E_{ABC} = (E_{A|BC} E_{B|AC} E_{C|AB})^{1/3}.$$  

$E_{A|BC} = \max[0, -\ln 2\nu_{A|BC}]$ is the logarithmic negativity of the one mode-versus-two modes bipartitions in the system, where

$$\nu_{A|BC} = \min \left\{ \text{eig} | \oplus_{j=1}^{3} (-\sigma_{y})P_{A|BC} V_{3} P_{A|BC} \right\},$$

with $P_{A|BC} = \sigma_{z} \oplus 1 \oplus 1$, $P_{B|AC} = 1 \oplus \sigma_{z} \oplus 1$ and $P_{C|AB} = 1 \oplus 1 \oplus \sigma_{z}$.
Fig. 4 Effect of the normalized atomic detuning on the a tripartite logarithmic negativities $E_1$, $E_2$ and $E_3$ and b bipartite negativities ($E_{m_1m_2}$, $E_{m_1\text{op}}$, $E_{m_2\text{op}}$ and $E_{\text{op}\text{a}}$). In all the cases we choose $\Delta = \omega_m$, $P = 10$ mW, $r = 0.1$ and the temperature $T = 0.1$ mK.

are the matrices of the partial transposition of the tripartite covariance matrix, and $V_3$ is the $6 \times 6$ covariance matrix of the tripartite system. In our case, $A$, $B$ and $C$ can be one of the mechanical modes, optical mode or atomic mode. Moreover, for the sake of simplicity we will use the following compact notations $E_1 \equiv E_{m_1m_2a}$, $E_2 \equiv E_{a m_1\text{op}} = E_{a m_2\text{op}}$ and $E_3 \equiv E_{m_1m_2\text{op}}$.

We plot in Fig. 4a the tripartite entanglement as captured by the logarithmic negativity (20) versus the normalized atomic detuning. The three logarithmic negativities react quite differently, and broadly speaking, it depends on whether the atomic mode is involved or not. For instance, $E_1$ and $E_2$ have their maximum close to the region where $\Delta_a = -\omega_m$, whereas $E_3$ decreases in this interval and reaches its minimum in this region. It is remarkable that the tripartite entanglement between the optical mode, the atomic mode and the mechanical mode ($E_2$), is the one that remains significant in a broader interval. This is due to the absorption and remission of photons by the atoms. On the other hand, the more photons we have in the cavity, the more the photon–phonon interaction is enhanced via the radiation pressure force due to the effect of the vibrating mirror. Under the atom–photon–phonon interaction, not only the region of the effective detuning is wider, but also a significant entanglement is obtained. These collective bosonic modes form an optimal quantum tripartite system that ensures strong entanglement sharing, more easily realized and observed in experiment.

It is worth noticing that when $\Delta_a > 0$, both $E_2$ and $E_3$ asymptotically increase, while $E_1$ is negligible, i.e., the tripartite $\{m_1, m_2, a\}$ entanglement is not present in this area. This result might be interpreted as a result of the negative value of the effective atomic detuning being a convenient choice since it regulates the evolution of the atomic quadrature [52].

This collective tripartite behavior can be explained by the corresponding underlying bipartite entanglement shown in Fig. 4b. As a matter of fact, the behavior of $E_1 \equiv E_{m_1m_2a}$ in Fig. 4a is identical to that of $E_{m_1\text{op}}$ in Fig. 4b because the other bipartite entanglement $E_{m_1m_2}$ involved in $E_1$ is shown in Fig. 4b to be negligible. Similarly,
\( \mathcal{E}_3 \equiv \mathcal{E}_{m1m2op} \), relies on the mechanical-optical, entanglement, \( E_{m1op} \), and the negligible mechanical-mechanical entanglement, \( E_{m1m2} \). This results in the behavior of \( \mathcal{E}_3 \) resembling that of \( E_{m1op} \). In contrast, \( \mathcal{E}_2 \equiv \mathcal{E}_{m1aop} \), depends on three bipartite entanglements \( E_{m1op}, E_{m1a} \) and \( E_{op,a} \), the last one of which is shown to be negligible in Fig. 4b. \( \mathcal{E}_2 \) is thus mainly driven by \( E_{m1op} \) and \( E_{m1a} \); henceforth, the behavior of \( \mathcal{E}_2 \) in Fig. 4a is a hybrid of that of \( E_{m1op} \) and \( E_{m1a} \) in Fig. 4b.

To analyze the influence of the pumping power, we plot in Fig. 5 \( \mathcal{E}_2 \) and \( \mathcal{E}_3 \) as a function of the environment temperature \( T \), for different values of \( P \). With the increase of driving power, the values of logarithmic negativity increase and, relatively, resist better the environment-induced decoherence. We also find that, for \( P = 15 \text{ mW} \), \( \mathcal{E}_3 \) vanishes at \( T \simeq 0.65 \text{ mk} \), while \( \mathcal{E}_2 \) survives until \( T \simeq 1.2 \text{ mk} \), so \( \mathcal{E}_3 \) vanishes quicker than \( \mathcal{E}_2 \) due to the thermal environment. On the other hand, for \( P = 5 \text{ mW} \) and for a fixed temperature \( T = 0.27 \text{ mk} \), \( \mathcal{E}_3 = 0.048 \) while \( \mathcal{E}_2 = 0.13 \), which shows that, compared to \( \mathcal{E}_2 \), \( \mathcal{E}_3 \) requires a much higher value of power that increases the number of photons in the cavity and a very low number of thermal phonons, i.e., temperature.

It is an intuitive fact that entanglement is highly sensitive to fluctuations, with the mechanical modes being much noisier and the fluctuations of the atomic mode being less noisy resulting from the collision between photons of the light source and the photons emitted by the atoms. This partly explains the reason of the fragility of \( \mathcal{E}_3 \) compared to \( \mathcal{E}_2 \) as the latter depends less on the much more fragile correlations involving the mechanical modes than the former. Moreover, we have an additional squeezed vacuum light, the injection of which reduces the fluctuations and results in stronger correlations when it is involved; this illustrates well the usefulness of squeezed light.

The system under consideration is an optical ring cavity, which is characterized by a high-quality factor and efficient coupling between the optical and mechanical modes since the use of two movable mirrors enhances the field-mirrors interaction. Indeed, the pumping power circulation inside the cavity amplifies its intensity and the displacement of the two movable mirrors do enhance the optomechanical coupling. Lengthening the cavity provides further enhancement of the interaction. A significant
value of the power increases the circulation inside the cavity, which in turn enhances the interaction and thus the entanglement, establishing the importance of the parameter of power as well.

Let us finally explore the effect of the atomic–field coupling strength $G_a$. For this purpose, we simulate the tripartite entanglement $\mathcal{E}_1$ and $\mathcal{E}_2$ versus the temperature for different values of $G_a$. It is observed that, for a fixed temperature, e.g., $T = 0.5 \text{ mK}$ as $G_a$ increases, both $\mathcal{E}_1$ and $\mathcal{E}_2$ increase, and in general, one can achieve significant amounts of tripartite entanglement for high couplings. This can, for instance, be achieved by increasing the atom numbers in the cavity and with a strong driven laser that increases the number of photons as explained previously, then a well-fortified interaction is achieved leading to a strong atom–field coupling. Indeed, for high-quality cavities such as ring cavities considerable coupling can be easily realized experimentally [53].

By comparing the plots 6a and b, it comes out that $\mathcal{E}_1$ is more resistant than $\mathcal{E}_2$ to the thermal-induced decoherence, e.g., when $G_a = 8\pi 10^6 \text{ Hz}$, $\mathcal{E}_2$ vanishes at $T \approx 1.3 \text{ mk}$ while $\mathcal{E}_1$ vanishes at $T \approx 1.5 \text{ mk}$ all the while maintaining a significant amount and plateauing over a longer interval. It is clearly seen that $\mathcal{E}_1$, i.e., the $\{m_1, m_2, a\}$ entanglement which is of interest in this paper is greater than $\mathcal{E}_2$, i.e., $\{a, m_1, \text{op}\}$ even though atoms and mirrors are only indirectly coupled in the Hamiltonian 4. As a matter of fact, $\mathcal{E}_1$ enhances at the expanse of the field-mirror interaction via the radiation pressure as in [21].

As explained in the previous sections, the ring geometry provides a closed-loop configuration for the light circulating within the cavity. This feedback mechanism allows for powerful interaction between the atomic ensemble and cavity field leading to a stronger coupling. As the atoms absorb energy from the cavity field, they can undergo spontaneous emission. This process involves the atom transitioning back to a lower energy state and releasing a photon into the cavity field. In addition to spontaneous emission, stimulated emission can also occur. If the atom is already in an excited state, an incoming photon from the cavity field can stimulate the atom to

![Fig. 6](image-url)
emit a photon that is in phase and coherent with the incoming photon. This allows to avoid a large number of photons interacting with the mechanical oscillators and get a stronger and more persistent correlation between optical and mechanical modes. This enhances $E_1$ and results in the large plateau being less affected by the temperature effects.

All these results confirm that a significant entanglement is established with a strong atom–field coupling and can be optimized by the optical field squeezing.

4 Conclusion

In this work, we have proposed a theoretical scheme for the study of steady-state bipartite and tripartite entanglement. The system consists of a ring cavity with a fixed mirror and two movable mirrors, as well as an atomic ensemble composed of two-level atoms that are confined within the cavity. This latter is pumped with a coherent laser source and a squeezed vacuum light allowing to enhance the quantum correlations. A proper analysis of the dynamics of the coupled system is carried out, allowing for the derivation of the set of quantum Langevin equations. These equations are then linearized to obtain the $8 \times 8$ steady state covariance matrix that is fully describing the hybrid optomechanical system (cavity field–atomic ensemble–two vibrating mirrors). The bipartite and tripartite logarithmic negativity is used to evaluate the entanglement of the multimode system.

An analysis of the bipartite and tripartite entanglement shared by the coupled system confirms the negative influence of the environment’s temperature on the different types of entanglement. Some types of entanglement are more resilient than others. In particular, the benefits of adding the atomic ensemble in the cavity are observed as it allows for a more resistant entanglement. In this regard, we have shown that a stronger atomic–field coupling allows for a better entanglement. On the other hand, with the increase of the power of the driving laser, significant entanglement and broader effective detuning region can be achieved. In addition to that, we have shown the needfulness of the squeezed light source since this latter allows for a strong optomechanical coupling. The combination of an atomic ensemble, strong optomechanical coupling and optimized cavity design makes the system less susceptible to thermal fluctuations. As a result, entanglement is preserved even at higher temperatures. In our case, we obtain a large plateau for entanglement against temperature. This is achieved because of the fact that the atoms can extract thermal energy from the mechanical oscillator. This energy transfer occurs when the atoms absorb phonons from the mechanical motion, effectively reducing temperature and cooling the mechanical oscillator [54, 55]. This process provides a strong and stable system against environment fluctuations.

Such a scheme will open new perspectives for the application of quantum teleportation in cavity optomechanics, the implementation of quantum memories [56] and quantum routers [57] for continuous variable quantum information processing. The enhancement of the entanglement in the cavity will prove critical for the cavity optomechanical sensing [58, 59]. The presence of strong multipartite entanglement can be exploited to realize teleportation in optomechanical ring cavities, similar to what was
achieved in [60]. The added advantage here is the ability to teleport quantum states between a multitude of modes.

The system can, nevertheless, still be enhanced by addressing some of its vulnerabilities. For instance, the entanglement can be suppressed or destroyed, due to effects such as the dark-mode induced by the coupling of vibrational modes to a common optical mode. In this regard, there exist proposals to create optomechanical entanglement by breaking the dark-mode through synthetic magnetism [61]. This proposition can be effectively incorporated in our system.

**Data availability** No data statement is available.

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

**References**

1. Caves, C.M.: Quantum-mechanical radiation-pressure fluctuations in an interferometer. Phys. Rev. Lett. **45**(2), 75 (1980)
2. Corbitt, T., Ottaway, D., Innerhofer, E., Pecl, J., Mavalvala, N.: Measurement of radiation-pressure-induced optomechanical dynamics in a suspended fabry-perot cavity. Phys. Rev. A **74**(2), 021802 (2006)
3. Kippenberg, T.J., Vahala, K.J.: Cavity opto-mechanics. Opt. Express **15**(25), 17172–17205 (2007)
4. Fabre, C., Pinard, M., Bourzeix, S., Heidmann, A., Giacobino, E., Reynaud, S.: Quantum-noise reduction using a cavity with a movable mirror. Phys. Rev. A **49**(2), 1337 (1994)
5. Genes, C., Mari, A., Vitali, D., Tombesi, P.: Quantum effects in optomechanical systems. Adv. At. Mol. Opt. Phys. **57**, 33–86 (2009)
6. Zhang, J., Peng, K., Braunstein, S.L.: Quantum-state transfer from light to macroscopic oscillators. Phys. Rev. A **68**(1), 013808 (2003)
7. Cao, C., Mi, S.-C., Gao, Y.-P., He, L.-Y., Yang, D., Tie-Jun, W., Ru, Z., Chuan, W.: Tunable high-order sideband spectra generation using a photonic molecule optomechanical system. Sci. Rep. **6**(1), 1–8 (2016)
8. Bo, L., Liu, X.-F., Gao, Y.-P., Cao, C., Wang, T.-J., Wang, C.: Berry phase in an anti-pt symmetric metal-semiconductor complex system. Opt. Express **27**(16), 22237–22245 (2019)
9. Cao, C., Mi, S.-C., Tie-Jun, W., Ru, Z., Chuan, W.: Optical high-order sideband comb generation in a photonic molecule optomechanical system. IEEE J. Quantum Electron. **52**(6), 1–5 (2016)
10. Cao, C., Zhang, L., Han, Y.-H., Yin, P.-P., Fan, L., Duan, Y.-W., Zhang, R.: Complete and faithful hyperentangled-bell-state analysis of photon systems using a failure-heralded and fidelity-robust quantum gate. Opt. Express **28**(3), 2857–2872 (2020)
11. Ghobadi, R., Kumar, S., Pepper, B., Bouwmeester, D., Ai, L., Christoph, S.: Optomechanical micro-macro entanglement. Phys. Rev. Lett. **112**(8), 080503 (2014)
12. Gut, C., Winkler, K., Hoelscher-Obermaier, J., Hofer, S.G., MoghadasNia, R., Walk, N., Steffens, A., Eisert, J., Wieczorek, W., Slater, J.A., et al.: Stationary optomechanical entanglement between a mechanical oscillator and its measurement apparatus. Phys. Rev. Res. **2**(3), 033244 (2020)
13. Li, J., Hou, B., Zhao, Y., Wei, L.: Enhanced entanglement between two movable mirrors in an optomechanical system with nonlinear media. Europhys. Lett. **110**(6), 64004 (2015)
14. Armour, A.D., Blencowe, M.P., Schwab, K.C.: Quantum dynamics of a cooper-pair box coupled to a micromechanical resonator. Phys. Rev. Lett. **88**, 148301 (2002)
15. Vitali, D., Mancini, S., Ribichini, L., Tombesi, P.: Macroscopic mechanical oscillators at the quantum limit through optomechanical cooling. JOSA B **20**(5), 1054–1065 (2003)
16. Jiao, Y.-F., Zhang, S.-D., Zhang, Y.-L., Miranowicz, A., Kuang, L.-M., Jing, H.: Nonreciprocal optomechanical entanglement against backscattering losses. Phys. Rev. Lett. **125**(14), 143605 (2020)
17. Jiao, Y.-F., Liu, J.-X., Li, Y., Yang, R., Kuang, L.-M., Jing, H.: Nonreciprocal enhancement of remote entanglement between nonidentical mechanical oscillators. Phys. Rev. Appl. 18(6), 064008 (2022)
18. Li, Y., Jiao, Y.-F., Liu, J.-X., Miranowicz, A., Zuo, Y.-L., Kuang, L.-M., Jing, H.: Vector optomechanical entanglement. Nanophotonics 11(1), 67–77 (2021)
19. Genes, C., Vitali, D., Tombesi, P.: Emergence of atom-light-mirror entanglement inside an optical cavity. Phys. Rev. A 77(5), 050307 (2008)
20. Bai, C.-H., Wang, D.-Y., Wang, H.-F., Zhu, A.-D., Zhang, S.: Robust entanglement between a movable mirror and atomic ensemble and entanglement transfer in coupled optomechanical system. Sci. Rep. 6(1), 1–11 (2016)
21. Barzanjeh, S., Naderi, M.H., Soltanolkotabi, M.: Steady-state entanglement and normal-mode splitting in an atom-assisted optomechanical system with intensity-dependent coupling. Phys. Rev. A 84(6), 063850 (2011)
22. Kanamoto, R., Meystre, P.: Optomechanics of ultracold atomic gases. Phys. Scr. 82(3), 038111 (2010)
23. Xihua, Y., Nicholas, B., Paul, D.L.: Quantum treatment of cavity-assisted entanglement of three-level atoms and two fields in an electromagnetically-induced-transparency configuration. Phys. Rev. A 105(2), 023711 (2022)
24. Ian, H., Gong, Z.R., Liu, Y.-X., Sun, C.P., Nori, F.: Cavity optomechanical coupling assisted by an atomic gas. Phys. Rev. A 78(1), 013824 (2008)
25. Yong-Chun, L., Yu-Wen, H., Chee, W.W., Yun-Feng, X.: Review of cavity optomechanical cooling. Chin. Phys. B 22(11), 114213 (2013)
26. Thomas, M.K., Baptiste, G., Chun, T.N., Gian-Luca, S., Klemens, H., Philipp, T.: Light-mediated strong coupling between a mechanical oscillator and atomic spins 1 meter apart. Science 369(6500), 174–179 (2020)
27. Yang, X., Liu, J., Yan, X., Xiao, M.: Enhanced multipartite entanglement via quantum coherence with an atom-assisted optomechanical system. J. Phys. B At. Mol. Opt. Phys. 51(20), 205501 (2018)
28. Huang, S., Agarwal, G.: Entangling nanomechanical oscillators in a cavity field by feeding squeezed light. New J. Phys. 11(10), 10344 (2009)
29. Ivan, S.G., Hansuek, L., Oskar, P., Kerry, J.V.: Phonon laser action in a tunable two-level system. Phys. Rev. Lett. 108(8), 083901 (2010)
30. Chang, L., Jiang, X., Hua, S., Yang, C., Wen, J., Jiang, L., Li, G., Wang, G., Xiao, M.: Parity-time symmetry and variable optical isolation in active-passive-coupled microresonators. Nat. Photonics 8(7), 524–529 (2014)
31. Chen, Y., Chen, A.-X.: Robust mechanical entanglement in an atom-assisted hybrid optomechanical system. Quantum Inf. Process. 21(11), 370 (2022)
32. Michael, J.H., Fernando, G.S.L.B., Martin, B.P.: Strongly interacting polaritons in coupled arrays of cavities. Nat. Phys. 2(12), 849–855 (2006)
33. Holstein, T., Primakoff, H.L.: Field dependence of the intrinsic domain magnetization of a ferromagnet. Phys. Rev. 58(12), 1098 (1940)
34. Crispin, W.G., Peter, Z.: Quantum Noise. Springer, Berlin (2000)
35. Benguria, R., Kac, M.: Quantum Langevin equation. Phys. Rev. Lett. 46(1), 1 (1981)
36. Giovannietti, Vittorio, Vitali, David: Phase-noise measurement in a cavity with a movable mirror undergoing quantum Brownian motion. Phys. Rev. A 63(2), 023812 (2001)
37. Crispin, W.G.: Inhibition of atomic phase decays by squeezed light: A direct effect of squeezing. Phys. Rev. Lett. 56(18), 1917 (1986)
38. Daniel, F.W., Gerard, J.M.: Quantum Optics. Springer-Verlag, New York (1994)
39. Vitali, D., Gigan, S., Ferreira, A., Böhm, H., Tombesi, P., Guerreiro, A., Vedral, V., Zeilinger, A., Aspelmeyer, M.: Optomechanical entanglement between a movable mirror and a cavity field. Phys. Rev. Lett. 98(3), 030405 (2007)
40. David, L.E.: Stability theory [book reviews]. IEEE Trans. Autom. Control 41(3), 473 (1996)
41. Guifré, V., Reinhard, F.W.: Computable measure of entanglement. Phys. Rev. A 65(3), 032314 (2002)
42. Martin, B.P.: Logarithmic negativity: a full entanglement monotone that is not convex. Phys. Rev. Lett. 95(9), 090503 (2005)
43. Schwab, G., Hannes, B., Mauro, P., Florian, B., Gregor, L., Hertzberg, J.B., Keith, C.S., Dieter, B., Markus, A., Anton, Z.: Self-cooling of a micromirror by radiation pressure. Nature 444(7115), 67–70 (2006)
44. Olivier, A., Pierre-François, C., Briant, T., Pinard, M., Heidmann, A., Mackowski, J.-M., Christine, M., Pinard, L., François, O., Rousseau, L.: High-sensitivity optical monitoring of a micromechanical resonator with a quantum-limited optomechanical sensor. Phys. Rev. Lett. 97(13), 133601 (2006)
45. Thomas, C., Christopher, W., Timothy, B., David, O., Daniel, S., Nicolas, S., Stanley, W., Nergis, M.: Optical dilution and feedback cooling of a gram-scale oscillator to 69 mk. Phys. Rev. Lett. 99(16), 160801 (2007)
46. Thomas Kipf and girish Agarwal: Superradiance and collective gain in multimode optomechanics. Phys. Rev. A 90(5), 052303 (2014)
47. Zeng, W.,Nie, W., Li, L., Chen, A.: Ground-state cooling of a mechanical oscillator in a hybrid optomechanical system including an atomic ensemble. Sci. Rep. 7(1), 17258 (2017)
48. Aspelmeyer, Markus, Tobias, J.K., Florian, M.: Cavity optomechanics. Rev. Modern Phys. 86(4), 1391 (2014)
49. Anza, Fabio, Militello, Benedetto, Messina, Antonino: Tripartite thermal correlations in an inhomogeneous spin-star system. J. Phys. B At. Mol. Opt. Phys. 43(20), 205501 (2010)
50. Sabín, C., García-Alcaine, G.: A classification of entanglement in three-qubit systems. Eur. Phys. J. D 48, 435–442 (2008)
51. Buscemi, F., Bordone, P.: Measure of tripartite entanglement in bosonic and fermionic systems. Phys. Rev. A 84(2), 022303 (2011)
52. Gabriele, D.C., Mauro, P., Massimo Palma, G.: Entanglement detection in hybrid optomechanical systems. Phys. Rev. A 83(5), 052324 (2011)
53. Xiao, R.-J., Pan, G.-X., Zhou, L.: Multiple optomechanically induced transparency in a ring cavity optomechanical system assisted by atomic media. Int. J. Theor. Phys. 54, 3665–3675 (2015)
54. Ralf, R., Sungkun, H., Richard, A.N., Joshua, A.S., Juying, S., Alexander, G.K., Vikas, A., Markus, A., Simon, G.: Non-classical correlations between single photons and phonons from a mechanical oscillator. Nature 530(7590), 313–316 (2016)
55. Remi, R., Samuel, D., Stefan, W., Emanuel, G., Olivier, A., Albert, S., Tobias, J.K.: Optomechanical sideband cooling of a micromechanical oscillator close to the quantum ground state. Phys. Rev. A 83(6), 063835 (2011)
56. Maitre, X., Hagley, E., Nogues, G., Wunderlich, C., Goy, P., Brune, M., Raimond, J.M., Haroche, S.: Quantum memory with a single photon in a cavity. Phys. Rev. Lett. 79(4), 769 (1997)
57. Cao, C., Duan, Y.-W., Xi, C., Ru, Z., Tie-Jun, W., Wang, C.: Implementation of single-photon quantum routing and decoupling using a nitrogen-vacancy center and a whispering-gallery-mode resonator-waveguide system. Opt. Express 25(15), 16931–16946 (2017)
58. Li, B.-B., Lingfeng, O., Lei, Y., Liu, Y.-C.: Cavity optomechanical sensing. Nanophotonics 10(11), 2799–2832 (2021)
59. Fabienne, S., Sofia, Q., Alessio, S., André, X., Daniel, B., Dennis, R., David, E.B.: Optimal estimation with quantum optomechanical systems in the nonlinear regime. Phys. Rev. A 101(3), 033834 (2020)
60. Sebastian, G.H., Witlef, W., Markus, A., Klemens, H.: Quantum entanglement and teleportation in pulsed cavity optomechanics. Phys. Rev. A 84(5), 052327 (2011)
61. Lai, D.-G., Liao, J.-Q., Miranowicz, A., Nori, F.: Noise-tolerant optomechanical entanglement via synthetic magnetism. Phys. Rev. Lett. 129(6), 063602 (2022)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.