A criterion for existence of Néron models of jacobians

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Abstract

Néron models of abelian varieties do not necessarily exist if the base $S$ has dimension higher than 1. We introduce a new condition, called toric additivity, on a family of smooth curves having nodal reduction over a normal crossing divisor $D \subset S$. The condition is necessary and sufficient for existence of a Néron model of the jacobian of the family; it depends only on the Betti numbers of the dual graphs of the fibres of the family, or on the toric ranks of the fibres of the jacobian.

Introduction

Toric additivity

Consider an abelian variety $A$ over a number field $K$ with ring of integers $S$. In general, there may not exist an abelian scheme $A/S$ extending $A$. However, it was proved by A. Néron and M. Raynaud that $A$ admits a canonical model $\mathcal{N}/S$, satisfying a number of good properties. Among these: it is a smooth, separated group scheme, and every $K$-valued point of $A$ extends uniquely to a section $S \to \mathcal{N}$. Such a model is called a Néron model for $A$ (see definition 3.1).

It is natural to ask whether an abelian variety always admits a Néron model if the base $S$ is, instead of a Dedekind scheme, a regular scheme of dimension higher than 1.

In this paper, we focus on the case of jacobians of curves. We work over a regular base $S$, and consider a nodal curve $C/S$, smooth over the complement of a normal crossing divisor $D \subset S$. We introduce a new condition on $C/S,$
smooth-local on $S$, called \textit{toric additivity} (definition 2.6). Roughly, the family $\mathcal{C}/S$ is toric-additive at a geometric point $s$ of $S$ if the toric rank of the jacobian of the fibre over $s$ is equal to the sum of the toric ranks of the jacobian over the generic points of the components of the boundary divisor passing through $s$. More precisely, let $V$ be the spectrum of the étale local ring at $s$, $D_1, \ldots, D_n$ the irreducible components of $D \cap V$, $\zeta_i$ the generic point of $D_i$; there is an inequality
\[ \dim T \leq \dim T_1 + \ldots + \dim T_n \]
where $T$ is the maximal torus contained in $\text{Pic}^0_{C_s/s}$, and $T_i$ is the maximal torus contained in the fibre $\text{Pic}^0_{C_{\zeta_i}/\zeta_i}$. We say that $\mathcal{C}/S$ is \textit{toric-additive at} $s$ if the inequality above is actually an equality.

The toric rank of the jacobian of a nodal curve (over an algebraically closed field) can also be interpreted as the first Betti number of the dual graph of the curve. Therefore, an equivalent definition of toric-additivity at $s$ is
\[ h_1(\Gamma) = h_1(\Gamma_1) + \ldots + h_1(\Gamma_n) \]
where $\Gamma$ is the dual graph of the nodal curve $C_s/s$ and $\Gamma_i$ is the dual graph of the geometric fibre $C_{\zeta_i}$. 

Our main result is the following:

\textbf{Theorem 0.1.} [theorem 4.13] Suppose $S$ is a regular scheme, $D$ a normal crossing divisor, $\mathcal{C}/S$ a nodal curve, smooth over $U = S \setminus D$. Write $J$ for the relative jacobian $\text{Pic}^0_{\mathcal{C}_U/U}$.

\begin{itemize}
  \item[a)] If $\mathcal{C}/S$ is toric-additive, then $J$ admits a Néron model over $S$.
  \item[b)] If moreover $S$ is an excellent scheme, the converse is also true.
\end{itemize}

The fact that the condition of toric additivity can be formulated merely in terms of the generalized jacobian $\text{Pic}^0_{\mathcal{C}/S}$ – the unique semi-abelian model of $\text{Pic}^0_{\mathcal{C}_U/U}$, functorial with respect to $S$ – is particularly convenient: it has the consequence that toric additivity (and, in view of theorem 0.1, the property of existence of a Néron model for the jacobian) is stable with respect to various types of base change. For example, the property is stable under any morphism $f: T \to S$ such that $f^{-1}D$ is still a normal crossing divisor, and such that étale locally $f$ induces a bijection between the components of $f^{-1}D$ and of $D$ (see lemma 2.10).

Also, replacing the curve $\mathcal{C}/S$ by another nodal family $\mathcal{C}'/S$ via a blow-up of the total space $\mathcal{C}$ does not affect the property of toric additivity (lemma 2.14), since the blow-up induces the identity at the level of $\text{Pic}^0$. 

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Relation with Holmes’ alignment condition

The question of existence of Néron models of jacobians over higher dimensional bases was first raised in [Hol17], where Holmes gave it a negative answer: for a regular base $S$, and a nodal curve $C/S$ smooth over an open dense $U \subset S$, he showed that the jacobian $\text{Pic}_U^0$ does not always admit a Néron model. He actually showed that existence of Néron models of jacobians depends on a rather restrictive combinatorial condition, called alignment, on the dual graphs of the fibres of $C/S$ endowed with a certain labelling of the edges.

**Theorem 0.2** ([Hol17], theorem 5.16, theorem 5.2). Suppose $S$ is a regular scheme, $U \subseteq S$ an open dense, $C/S$ a nodal curve, smooth over $U$. Let $J = \text{Pic}_U^0$ be the relative jacobian.

1. if $J$ admits a Néron model over $S$, then $C/S$ is aligned;
2. if moreover the total space $C$ is regular, and $C/S$ is aligned, then $J$ admits a Néron model over $S$.

In part ii), the hypothesis that the total space $C$ is regular is essential. Holmes’ work left open the question of whether the jacobians of non-regular, aligned nodal curves admit a Néron model. Even in the apparently tame case of example 0.3 below it was not known what to expect.

**Example 0.3.** Let $k$ be a field, $S = \text{Spec} \ k[[u, v]]$, $U = S \setminus \{uv = 0\}$. Let $E/S$ be the family of nodal curves of arithmetic genus one

$$y^2 - x^3 - x^2 - uv = 0$$

which is smooth over $U$, and aligned. The total space $E$ is not regular, as the point $x = 0, y = 0, u = 0, v = 0$ is singular, and there exists no nodal model $E'/S$ of $E_U$ with regular total space $E'$.

Toric additivity is a stronger condition than alignment (lemma 4.4); and it turns out to be equivalent to it in the case where the total space $C$ is regular (lemma 4.5). It can be easily checked that the curve $E/S$ of example 0.3 is not toric-additive (as in this case $h_1(\Gamma) = 1 < 1 + 1 = h_1(\Gamma_1) + h_1(\Gamma_2)$), hence the jacobian of $E_U$ does not admit a Néron model over $S$.

The proof of theorem 0.1

The strategy of proof follows these lines: we show in lemma 4.9, lemma 4.10 and lemma 4.11 that if the hypotheses of a) or b) are satisfied, there exists a
blow-up $C' \rightarrow C$, such that $C'/S$ is still a nodal curve, smooth over $U$, and $C'$ is regular. Lemma 4.11 shows in particular that the curve of example 0.3 does not admit a Néron model. As the properties of admitting a Néron model or of being toric-additive are not affected by desingularization, we have reduced to the case where the relative curve has regular total space. In this case, it can be shown that alignment and toric additivity are equivalent, and we apply theorem 0.2 to reach the conclusion.

Outline

In section 1, we define the objects with which we work - mainly nodal curves, their dual graphs, their jacobians. In section 2, we define toric additivity (definition 2.6). In order to do this, we introduce the concept of purity map between character groups (definition 2.1). In section 3, we define Néron models (definition 3.1); after listing some of their properties, we look at their group of connected components and define a purity map (eq. (3.3)) also for connected components. In section 4, we recall the definition of alignment (definition 4.1) and compare it to toric additivity. Then we prove the key lemmas from which theorem 4.13 follows.

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1 Preliminaries

1.1 Nodal curves

Definition 1.1. A curve $C$ over an algebraically closed field $k$ is a proper morphism of schemes $C \to \text{Spec } k$, such that $C$ is connected and its irreducible components have dimension 1. A curve $C/k$ is called nodal if for every non-smooth point $p \in C$ there is an isomorphism of $k$-algebras $\hat{O}_{C,p} \to k[[x,y]]/xy$.

For a general base scheme $S$, a nodal curve $f : C \to S$ is a proper, flat morphism of finite presentation, such that for each geometric point $s$ of $S$ the fibre $C_s$ is a nodal curve.

We will denote by $C^{ns}$ the subset of $C$ of points at which $f$ is not smooth. Seeing $C^{ns}$ as the closed subscheme defined by the first Fitting ideal of $\Omega^1_{C/S}$, we have for a nodal curve $C/S$ that $C^{ns}/S$ is finite, unramified and of finite presentation.

The local structure of nodal curves is described by the following lemma from [Hol17].

Lemma 1.2 ([Hol17], Prop.2.5). Let $S$ be locally noetherian, $f : C \to S$ be nodal, and $p$ a geometric point of $C^{ns}$ lying over a geometric point $s$ of $S$. We have:

i) there is an isomorphism

$$\hat{O}_{C,p}^{sh} \simeq \hat{O}_{S,s}^{sh}[[x,y]]/xy - \alpha$$

for some element $\alpha$ in the maximal ideal of the completion $\hat{O}_{S,s}^{sh}$;

ii) the element $\alpha$ is in general not unique, but the ideal $(\alpha) \subset \hat{O}_{S,s}^{sh}$ is. Moreover, the ideal is the image in $\hat{O}_{S,s}^{sh}$ of a unique principal ideal $I \subset O_{S,s}^{sh}$, which we call thickness of $p$.

Definition 1.3. In the hypothesis of lemma [1.2] we call the ideal $I \subset O_{S,s}^{sh}$ the thickness of $p$.

We remark that, for $S$ regular at $s$, $C$ is regular at $p$ if and only if the thickness of $p$ is generated by a regular parameter of the regular ring $O_{S,s}^{sh}$. 
Split singularities

Let $k$ be a field (not necessarily algebraically closed), $C/k$ a nodal curve, $n: C' \to C$ its normalization. Following [Liu02, 10.3.8], we say that $p \in C'^{ns}$ is a split ordinary double point if its preimage $n^{-1}(p)$ consists of $k$-valued points. This implies in particular that $p$ is $k$-valued. Moreover, if $p$ belongs to two or more components of $C$, then it belongs to exactly two components $Z_1, Z_2$; these are smooth at $p$ and meet transversally ([Liu02, 10.3.11]). We say that $C/k$ has split singularities if every $p \in C^{ns}$ is a split ordinary double point.

Remark 1.4. A nodal curve $C/k$ attains split singularities after a finite separable extension $k \to k'$, by [Liu02, 10.3.7 b)].

Remark 1.5. A nodal curve with split singularities has irreducible components that are geometrically irreducible. Indeed, either $C/k$ is smooth, in which case it is geometrically connected and therefore geometrically irreducible; or every irreducible component of the normalization of $C$ contains a $k$-rational point and is therefore geometrically irreducible.

Lemma 1.6. Let $C \to S$ be a nodal curve and $s \in S$ such that $C_s$ has split singularities. Let $p$ be a geometric point of $C_s$. Then the thickness $I$ of $p$ is generated by an element of the Zariski-local ring $\mathcal{O}_{S,s}$.

Proof. The morphism $f: C^{ns} \to S$ is finite unramified. Because $C_s$ has split singularities, we see by [Sta16, TAG 04DG], that there exists an open neighbourhood $U$ of $s$ such that $f^{-1}(U) \to U$ is a disjoint union of closed immersions. In particular, $C^{ns} \to S$ is a closed immersion at $p$, and to it we can associate an ideal $J$ in the Zariski-local ring $\mathcal{O}_{S,s}$. We see (for example by [Hol17, proof of part 2 of Prop. 2.5]) that $I$ is the image of $J$ in $\mathcal{O}_{S,s}^{sh}$; and moreover, since $\mathcal{O}_{S,s} \to \mathcal{O}_{S,s}^{sh}$ is faithfully flat and $I$ is principal, $J$ is principal as well, which completes the proof.

1.2 The relative Picard scheme

Let $S$ be a connected base scheme, and $C \to S$ a nodal curve of arithmetic genus $g$. We denote by $\text{Pic}^0_C$ the degree-zero relative Picard functor; it is constructed as the fppf-sheaf associated to the functor

$$P^0_C: \text{Sch} / S \to \text{Ab}$$

$$T \to S \mapsto \text{Pic}^0(C \times_S T)$$

where by definition $\text{Pic}^0(C \times_S T)$ is the group of isomorphism classes of invertible sheaves $\mathcal{L}$ on $C \times_S T$ such that, for every geometric point $t$ of $T$ and irreducible component $X$ of the fibre $C_t$, deg $\mathcal{L}|_X = 0$. 
It turns out that the degree-zero Picard functor \( \text{Pic}^0_{\mathcal{C}/S} \) of a nodal curve has an easy description if \( \mathcal{C}/S \) admits a section. In this case, it is given by

\[
\text{Pic}^0_{\mathcal{C}/S} : \text{Sch}/S \to \text{Ab} \\
T \to S \mapsto \frac{\text{Pic}^0(\mathcal{C} \times_S T)}{\text{Pic}(T)}
\]

If \( \mathcal{C}/S \) is a smooth curve, it is well known that \( \text{Pic}^0_{\mathcal{C}/S} \) is represented by an abelian scheme, called the jacobian of \( \mathcal{C}/S \). If \( \mathcal{C}/S \) is only nodal, then \( \text{Pic}^0_{\mathcal{C}/S} \) is represented by a semi-abelian scheme of relative dimension \( g \) ([BLR90, 9.4/1]); in particular, for every point \( s \in S \), there exists an exact sequence of fppf-sheaves

\[
0 \to T \to \text{Pic}^0_{\mathcal{C}_s/s} \to B \to 0 \quad (1.1)
\]

where \( B = \text{Pic}^0_{\mathcal{C}_s/s} \) is the jacobian of the normalization of \( \mathcal{C}_s \), hence an abelian variety, and \( T/k(s) \) is the biggest subtorus of \( \text{Pic}^0_{\mathcal{C}/s} \).

We call \( \mu(s) := \dim T \) and \( \alpha(s) := \dim B \) respectively the toric rank and abelian rank of \( \text{Pic}^0_{\mathcal{C}_s/s} \). These numbers are stable under base field extension.

The function \( \mu : S \to \mathbb{Z}_{\geq 0} \) which associates to a point \( s \) the toric rank of \( \text{Pic}^0_{\mathcal{C}_s/s} \) is upper semi-continuous ([FC90], Remark 2.4 Chapter I); defining analogously \( \alpha : S \to \mathbb{Z}_{\geq 0} \), we obtain that \( \mu + \alpha \) is constant and equal to \( g \).

**Definition 1.7.** Let \( C/k \) be a nodal curve over a field, and let \( \overline{k} \) be a separable closure. Let \( T/k \) be the torus of the exact sequence (1.1). The étale \( k \)-group scheme \( X := \text{Hom}_k(T, \mathbb{G}_m) \) is the datum of the free abelian group \( X(\overline{k}) = \text{Hom}_k(T_{\overline{k}}, \overline{k}^*) \) of rank \( \mu \), endowed with the continuous action of \( \text{Gal}(\overline{k}/k) \). It is called the character group of the torus \( T \).

If the torus \( T \) is split, for example if \( k \) is separably closed, then the action of \( \text{Gal}(\overline{k}/k) \) is trivial. In this case, \( X \) is a constant group scheme and we will simply see it as an abstract free abelian group of rank \( \mu \).

### 1.3 Dual graphs of nodal curves

We start by listing some graph-theoretical notions. In what follows, we will use the word *graph* to refer to a finite, connected, undirected graph \( G = (V, E) \).

A *path* on \( G \) is a walk on \( G \) in which all edges are distinct, and that never goes twice through the same vertex, except possibly for the first and last; a *cycle*
is a path that starts and ends at the same vertex. A *loop* is a cycle consisting of only one edge.

Let now $C$ be a curve with split singularities over a field $k$. We define the dual graph of $C$ as in [Liu02, 10.3.17]: it is the graph $\Gamma = (V, E)$ with $V = \{\text{irreducible components of } C\}$, $E = \{\text{singular points of } C\}$; the extremal vertices of an edge $p$ are the components containing $p$, which are indeed either one or two.

The following result gives a graph theoretic interpretation of the character group of a curve and its rank.

**Lemma 1.8.** Let $C/k$ be a nodal curve with split singularities.

- i) The torus $T$ of the exact sequence 1.1 is split.
- ii) Let $\Gamma$ the dual graph of $C$. The character group $X$ of $T$ is canonically identified with the first homology group $H_1(\Gamma, \mathbb{Z})$. In particular the first Betti number $h_1(\Gamma, \mathbb{Z})$ is equal to $\mu = \text{rk } X = \text{dim } T = \text{toric rank of Pic}^0_{C/k}$.

**Proof.** The normalization morphism and the structure morphisms fit into a commutative diagram

$$
\begin{array}{ccc}
\tilde{C} & \xrightarrow{\pi} & C \\
\downarrow{\tilde{p}} & & \downarrow{p} \\
\text{Spec } k & & 
\end{array}
$$

Consider the exact sequence

$$1 \to \mathcal{O}_C^\times \to \pi_* \mathcal{O}_{\tilde{C}}^\times \to \frac{\pi_* \mathcal{O}_{\tilde{C}}^\times}{\mathcal{O}_C^\times} \to 1. \quad (1.2)$$

It gives a long exact sequence

$$p_* \mathcal{O}_C^\times \to \tilde{p}_* \mathcal{O}_{\tilde{C}}^\times \to p_* \frac{\pi_* \mathcal{O}_{\tilde{C}}^\times}{\mathcal{O}_C^\times} \to R^1 p_* \mathcal{O}_C^\times \to R^1 p_*(\pi_* \mathcal{O}_{\tilde{C}}^\times)$$

We write $V = \{v_1, \ldots, v_n\}$ for the set of components of $\tilde{C}$ (the set of vertices of $\Gamma$) and $E = \{e_1, \ldots, e_r\}$ for the set of singular points of $C$ (the set of edges of $\Gamma$). The morphism $\mathcal{O}_C \to \pi_* \mathcal{O}_{\tilde{C}}$ is an isomorphism on the smooth locus and is given by the diagonal morphism $k \to k \oplus k$ at the singular points because $C$ has split singularities. We make a choice of orientation of the edges of the dual graph, which allows us to identify the quotient of the diagonal morphism
\( k \to k \oplus k \) above with the morphism \( k \oplus k \to k, (x,y) \to x - y \). Hence the term \( p_* \pi_* \mathcal{O}_C^\times / \mathcal{O}_C^\times \) is identified with \( \bigoplus E \mathbb{G}_{m,k} \). Moreover, \( R^1 p_* \mathcal{O}_C^\times = \text{Pic}^0_{C/k} \) and the natural morphism \( R^1 p_*(\pi_* \mathcal{O}_C^\times) \to R^1 \tilde{p}_* \mathcal{O}_C^\times = \text{Pic}^0_{\tilde{C}/k} \) is injective. We obtain an exact sequence

\[
1 \to \mathbb{G}_{m,k} \to \bigoplus \mathbb{G}_{m,k} \to \bigoplus \text{Pic}^0_{C/k} \to \text{Pic}^0_{\tilde{C}/k}.
\]

The first map is the diagonal; the second is given by tensoring with \( \mathbb{G}_{m,k} \) the linear map \( M: k^V \to k^E \) given by the incidence matrix of the oriented graph \( \Gamma \). It follows that the kernel of \( \text{Pic}^0_{C/k} \to \text{Pic}^0_{\tilde{C}/k} \) is the split torus \( \text{coker } M \otimes_k \mathbb{G}_{m,k} \).

Notice that its rank is indeed \( r - n + 1 = h_1(\Gamma, \mathbb{Z}) \). This proves i) and the second part of ii). For the remaining part of ii), apply \( \text{Hom}_k(-, \mathbb{G}_{m,k}) \) to the exact sequence

\[
\bigoplus \mathbb{G}_{m,k} \to \bigoplus \mathbb{G}_{m,k} \to T \to 0
\]

to obtain an exact sequence

\[
0 \to X \to \mathbb{Z}^E \to \mathbb{Z}^V
\]

the rightmost map being given by the transpose of the incidence matrix. This identifies \( X \) with \( H_1(\Gamma, \mathbb{Z}) \).

**Labelled dual graphs**

Given a nodal curve \( f: \mathcal{C} \to S \) and a point \( s \) of \( S \) such that \( \mathcal{C}_s \) has split singularities, we write \( \Gamma_s = (V_s, E_s) \) for the dual graph associated to the fibre \( \mathcal{C}_s \). Using the notation of [Hol17], we write \( L_s \) for the monoid of principal ideals of the (Zariski-)local ring \( \mathcal{O}_{S,s} \); then we let \( l_s: E_s \to L_s \) be the function associating to each edge of \( \Gamma_s \) the thickness of the corresponding singular point of \( \mathcal{C}_s \) (which indeed is an ideal of \( \mathcal{O}_{S,s} \), by lemma [1.6]).

**Definition 1.9.** The pair \((\Gamma_s, l_s)\) constructed above is called **labelled graph** of \( \mathcal{C} \to S \) at the point \( s \).

**1.4 The generization map**

Let \( \mathcal{C} \to S \) be a nodal curve over a regular, strictly local scheme. Let \( Z \) be a regular closed subscheme of \( S \), defined by an ideal \( a \subset \mathcal{O}(S) \) and with generic point \( \zeta \), and let \( s \) be the closed point of \( S \). Our aim is to compare the fibres \( \mathcal{C}_\zeta \) and \( \mathcal{C}_s \), in terms of their graphs and character groups.
**Lemma 1.10.** The fibre $C_\zeta$ has split singularities.

**Proof.** We drop the name $Z$ and simply assume that $\zeta$ is the generic point of $S$. The non-smooth locus $C'^{ns}$ is finite unramified over $S$, hence a disjoint union of closed subschemes of $S$. Let $X \subseteq C'^{ns}$ be the part consisting of sections $S \to C$. Notice that on the generic fibre $X_\zeta = C_\zeta^{ns}$.

We claim that the open subscheme $C \setminus X$ is normal. We will show it by using Serre’s criterion for normality ([Liu02, 8.2.23]). First, as $X$ has been removed, $C \setminus X$ is regular at its points of codimension 1. Condition $S_2$ follows from the fact that $C \setminus X$ is locally complete intersection over a regular, noetherian base, hence Cohen-Macaulay by [Liu02, 8.2.18]. This proves the claim.

Our next claim is that the normalization $\pi : C' \to C$ is finite and unramified. Since these are properties fpqc-local on the target, and since we already know that $\pi$ induces an isomorphism over $C \setminus X$, it is enough to check the claim over the completion of the strict henselization of geometric points of $X$. Let $x$ be such a geometric point lying over a geometric point $s$ in $S$. Then $x$ has thickness zero, hence $\hat{O}_{S,x}^{sh} \cong \hat{O}_{S,s}^{sh}[[u,v]]_{uv}$. The normalization morphism

$$\text{Spec } \hat{O}_{S,s}^{sh}[[u]] \times \hat{O}_{S,s}^{sh}[[v]] \to \text{Spec } \frac{\hat{O}_{S,s}^{sh}[[u,v]]}{uv}$$

is indeed finite and unramified, which proves the claim.

Now, let $Y$ be the preimage of $X$ via $\pi : C' \to C$. We have that $Y$ is finite, unramified over $X$, and in particular finite étale over $S$. Hence $Y$ is a disjoint union of sections $S \to C'$. The restriction of $\pi$ to the generic fibre, $\pi_\eta : C'_\eta \to C_\eta$, is a normalization morphism, and we see that the preimage $Y_\eta$ of $X_\eta = (C_\eta)^{ns}$ consists of $k(\eta)$-valued points, as we wished to show.

\[\square\]

**Construction 1.11.** We can therefore associate a labelled graph (definition [1.9]) $\Gamma_\zeta = (V_\zeta, E_\zeta)$ to $C_\zeta$. Every singular point of $C_\zeta$ specializes to a singular point of $C_s$ with thickness contained in the ideal $\mathfrak{a}$ of $Z$ in $O_{S,s}$. So if $(\Gamma_s = (V_s, E_s), l_s)$ is the labelled graph of $C_s$, we have an inclusion $E_\zeta \subseteq E_s$; and the labelled graph $\Gamma_\zeta$ is obtained from the labelled graph $\Gamma_s$ of the closed fibre by:

1. contracting all edges labelled by an ideal of $O_{S,s}$ not contained in $\mathfrak{a}$;

2. labelling edges in $\Gamma_\zeta$ by the image via the inclusion $O_{S,s} \to O_{S,\zeta}$ of the label of the corresponding edge in $\Gamma_s$.  

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We also obtain a natural commutative diagram

\[
\begin{array}{c}
\mathbb{Z}^{E_s} \longrightarrow \mathbb{Z}^{V_s} \\
\downarrow \quad \downarrow \\
\mathbb{Z}^{E_\zeta} \longrightarrow \mathbb{Z}^{V_\zeta}
\end{array}
\]  (1.3)

The horizontal maps are the usual boundary maps, given by the incidence matrices of \(\Gamma_s\) and \(\Gamma_\zeta\). The left vertical map is induced by the inclusion \(E_\zeta \subseteq E_s\). For the right vertical map, notice that we have a natural surjective map \(\text{gen}_V : V_s \to V_\zeta\); then, the image of a vertex labelling \(\varphi\) via \(Z^{V_\zeta} \to Z^{V_\zeta}\) is the vertex labelling \(\varphi'(v) = \sum_{w \in \text{gen}_V^{-1}(v)} \varphi(w)\).

Taking kernels of the horizontal maps in diagram 1.3, we obtain a generization map on homology

\[
H_1(\Gamma_s, \mathbb{Z}) \to H_1(\Gamma_\zeta, \mathbb{Z})  
\]  (1.4)

Consider now the semi-abelian scheme \(\text{Pic}^0_C / S\), and its restriction \(P := \text{Pic}^0_{C_s / Z}\) to \(Z\). Let \(T'\) be the maximal torus contained in \(P_\zeta\), which is split by lemma 1.8 and \(T\) be the maximal torus contained in \(P_s\). Let \(X'\) and \(X\) be the respective character groups. The closure of \(T'\) inside \(P\) is a split subtorus \(T' \subseteq P\) by [FC90] Prop. 2.9 Chapter 1. In particular, the restriction \(T'_s\) is a subtorus of \(T\), having character group \(X'\). The inclusion \(T'_s \subseteq T\) induces a surjective homomorphism of free abelian groups

\[
X \to X' 
\]  (1.5)

called as well generization map of characters groups. Remember that by lemma 1.8, \(X = H_1(\Gamma_s, \mathbb{Z})\) and \(X' = H_1(\Gamma_\zeta, \mathbb{Z})\).

**Lemma 1.12.** The two generization maps \(H_1(\Gamma_s) \to H_1(\Gamma_\zeta)\) and \(X \to X'\) are the same map.

**Proof.** We write \(\mathcal{C}/Z\) for the restriction of the curve to \(Z\). Let \(\pi : \tilde{\mathcal{C}} \to \mathcal{C}\) be the normalization. The restriction of \(\pi\) to the generic fibre \(\pi_\zeta : \tilde{\mathcal{C}}_\zeta \to \mathcal{C}_\zeta\) is also a normalization morphism. On the other hand the restriction to \(s\), \(\pi_s : \tilde{\mathcal{C}}_s \to \mathcal{C}_s\) is only the partial normalization at the subset of singular points \(E_\zeta \subseteq E_s\). Normalizing the remaining nodes we find the normalization \(\pi^0 : C^0 \to \mathcal{C}_s\). We call \(p : \mathcal{C}_s \to \text{Spec} k\), \(q : \tilde{\mathcal{C}}_s \to \text{Spec} k\), \(r : C^0 \to \text{Spec} k\) the three structure morphisms.

As in eq. (1.2), the torus \(T'\) fits into an exact sequence

\[
0 \to \mathbb{G}_m,\zeta \to \bigoplus_{V_\zeta} \mathbb{G}_m,\zeta \to \bigoplus_{E_\zeta} \mathbb{G}_m,\zeta \to T' \to 0
\]
which prolongs to an exact sequence

$$0 \to \mathbb{G}_{m,Z} \to \bigoplus_{V_\zeta} \mathbb{G}_{m,Z} \to \bigoplus_{E_\zeta} \mathbb{G}_{m,Z} \to T' \to 0$$

We have a commutative diagram with exact rows

$$\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{O}_{\mathcal{C}_s}^\times & \longrightarrow & (\pi_* \mathcal{O}_{\mathcal{C}}^\times)_s & \longrightarrow & \left(\frac{\pi_* \mathcal{O}_{\mathcal{C}}^\times}{\mathcal{O}_{\mathcal{C}_s}^\times}\right)_s & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \mathcal{O}_{\mathcal{C}_s}^\times & \longrightarrow & \pi_* \mathcal{O}_{\mathcal{C}_s}^\times & \longrightarrow & \pi_* \mathcal{O}_{\mathcal{C}_s}^\times & \longrightarrow & 0.
\end{array}$$

Applying the functor $p_*$, we obtain a commutative diagram with exact rows

$$\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{G}_{m,s} & \longrightarrow & \bigoplus_{V_\zeta} \mathbb{G}_{m,s} & \longrightarrow & \bigoplus_{E_\zeta} \mathbb{G}_{m,s} & \longrightarrow & T' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \mathbb{G}_{m,s} & \longrightarrow & \bigoplus_{V_s} \mathbb{G}_{m,s} & \longrightarrow & \bigoplus_{E_s} \mathbb{G}_{m,s} & \longrightarrow & T & \longrightarrow & 0.
\end{array} \quad (1.6)$$

The leftmost vertical map is the identity; the next vertical map is the natural map $q_* \mathcal{O}_{\mathcal{C}_s}^\times \to r_* \mathcal{O}_{\mathcal{C}_s}^\times$. Therefore it is simply the map induced by gen: $V_s \to V_\zeta$. With a similar reasoning one deduces that the next vertical map is obtained by the inclusion $E_\zeta \to E_s$. Finally, the rightmost vertical map is the inclusion of split tori found previously.

Now, applying $\text{Hom}_s(-, \mathbb{G}_{m,s})$, we obtain the commutative diagram with exact rows

$$\begin{array}{cccccc}
0 & \longrightarrow & X & \longrightarrow & \mathbb{Z}^{E_s} & \longrightarrow & \mathbb{Z}^{V_s} & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & X' & \longrightarrow & \mathbb{Z}^{E_\zeta} & \longrightarrow & \mathbb{Z}^{V_\zeta} \\
& & & & & & &
\end{array} \quad (1.7)$$

where the first vertical map is the generization map $X \to X'$ and the rightmost square is diagram \red{1.3}.

\hfill \Box
2 Toric additivity

2.1 The purity map

We consider a regular, strictly local base scheme $S$ with a normal crossing divisor $D = D_1 \cup D_2 \cup \ldots \cup D_n \subset S$ for some $n \geq 0$. We let $t_1, \ldots, t_n \in \mathcal{O}_{S,s}$ be the functions cutting out the components $D_1, \ldots, D_n$. We also let $C/S$ be a nodal curve such that the base change $C_U/U$ to $U = S \setminus D$ is smooth.

Write $\zeta_1, \ldots, \zeta_n$ for the generic points of the irreducible components $D_1, \ldots, D_n$, and $C_1, \ldots, C_n$ for the fibres over each of them. Each $C_i$ has split singularities, hence by lemma 1.8 we can attach to it a dual graph $\Gamma_i$, a split torus $T_i \subseteq \operatorname{Pic}^0_{C_i}$ of dimension $\mu_i = h_1(\Gamma_i, \mathbb{Z})$, and a free abelian group $X_i = H_1(\Gamma_i, \mathbb{Z})$ of rank $\mu_i$. Similarly, we let $s \in S$ be the closed point, with residue field $k$, and $C/k$ be the fibre, with dual graph $\Gamma$, (split) torus $T$ of dimension $\mu = h_1(\Gamma, \mathbb{Z})$, character group $X = H_1(\Gamma, \mathbb{Z})$ of rank $\mu$.

For every $i = 1, \ldots, n$ there is a generization map of character groups $X \rightarrow X_i$; it is natural to define the following:

**Definition 2.1.** We call purity map the group homomorphism

$$X \rightarrow X_1 \oplus X_2 \oplus \ldots X_n$$

induced by the generization maps $X \rightarrow X_i$. It may be seen as a homomorphism of homology groups

$$H_1(\Gamma, \mathbb{Z}) \rightarrow H_1(\Gamma_1, \mathbb{Z}) \oplus H_1(\Gamma_2, \mathbb{Z}) \oplus \ldots \oplus H_1(\Gamma_n, \mathbb{Z}).$$

**Lemma 2.2.** The purity map $X \rightarrow X_1 \oplus \ldots \oplus X_n$ is injective.

**Proof.** Every edge of $\Gamma$ is labelled by a principal ideal generated by a product of powers of $t_1, t_2, \ldots, t_n$. Hence every edge is preserved in at least one contraction $\Gamma_i$, as one can see from construction 1.11. It immediately follows that the canonical morphism $H_1(\Gamma, \mathbb{Z}) \rightarrow H_1(\Gamma_1, \mathbb{Z}) \oplus \ldots \oplus H_1(\Gamma_n, \mathbb{Z})$ is injective.

**Corollary 2.3.** We have the inequalities of toric ranks

$$\mu \leq \mu_1 + \mu_2 + \ldots + \mu_n.$$  \hspace{1cm} (2.3)

and of Betti numbers

$$h_1(\Gamma, \mathbb{Z}) \leq h_1(\Gamma_1, \mathbb{Z}) + h_1(\Gamma_2, \mathbb{Z}) + \ldots + h_1(\Gamma_n, \mathbb{Z}).$$  \hspace{1cm} (2.4)

**Proof.** Follows by taking ranks in the injective homomorphism $X \rightarrow X_1 \oplus X_2 \oplus \ldots X_n$. 

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Lemma 2.4. The purity map $X \to X_1 \oplus X_2 \oplus \ldots \oplus X_n$ has torsion-free cokernel.

Proof. We have a commutative diagram with injective arrows

$$
\begin{array}{ccc}
H_1(\Gamma, \mathbb{Z}) & \xrightarrow{\alpha} & \bigoplus H_1(\Gamma_i, \mathbb{Z}) \\
\gamma & & \delta \\
\mathbb{Z}^{E} & \xrightarrow{\beta} & \bigoplus \mathbb{Z}^{E_i}
\end{array}
$$

and we want to show that coker $\alpha$ is torsion-free. Clearly coker $\beta$ is torsion-free. Moreover, coker $\gamma$ is contained in $\mathbb{Z}^V$ and is therefore torsion-free. By the Snake Lemma, the kernel of the induced map coker $\alpha \to$ coker $\beta$ is contained in coker $\gamma$, hence is torsion-free. It follows that coker $\alpha$ is torsion-free. □

Corollary 2.5. The purity map $X \to X_1 \oplus X_2 \oplus \ldots \oplus X_n$ is an isomorphism if and only if the rank of $X$ is equal to the rank of $X_1 \oplus \ldots \oplus X_n$, that is, if and only if $\mu = \mu_1 + \ldots + \mu_n$.

Proof. Bijecitivity of the purity map implies equality of ranks. If, on the other hand, $\mu = \mu_1 + \ldots + \mu_n$, then the injective map $X \to X_1 \oplus \ldots \oplus X_n$ has a finite cokernel $F$. By lemma 2.4, $F$ is also torsion-free, hence it is zero. □

2.2 Definition of toric-additive curve

Definition 2.6 (Local definition of toric additivity). We say that the nodal curve $C/S$ is toric-additive if the following equivalent conditions are satisfied:

i) the purity map (definition 2.1) of character groups

$$X \to X_1 \oplus X_2 \oplus \ldots X_n$$

is an isomorphism;

ii) the purity map of homology groups

$$H_1(\Gamma, \mathbb{Z}) \to H_1(\Gamma_1, \mathbb{Z}) \oplus H_1(\Gamma_2, \mathbb{Z}) \oplus \ldots \oplus H_1(\Gamma_n, \mathbb{Z})$$

is an isomorphism;

iii) $\mu = \mu_1 + \mu_2 + \ldots + \mu_n$;

iv) $h_1(\Gamma, \mathbb{Z}) = h_1(\Gamma_1, \mathbb{Z}) + h_1(\Gamma_2, \mathbb{Z}) + \ldots + h_1(\Gamma_n, \mathbb{Z})$. 


Remark 2.7. If $n = 1$, then $\mathcal{C}/S$ is automatically toric-additive. Indeed, $\mu_1 = \text{rk} X_1 \leq \text{rk} X = \mu$ (by upper semi-continuity of the toric rank, or because the graph $\Gamma_1$ is a contraction of $\Gamma$), but at the same time eq. (2.3) tells us $\mu \leq \mu_1$.

In the trivial case $n = 0$, $\mathcal{C}/S$ is smooth, and is toric-additive, as the character group $X$ is zero and the empty sum is zero.

Lemma 2.8. Suppose that $\mathcal{C}/S$ is toric-additive and that $t$ is a geometric point of $S$. Let $T = \text{Spec} \mathcal{O}^{sh}_{S,t}$ be the strict henselization and $\mathcal{C}_T/T$ the base change. Then $\mathcal{C}_T/T$ is still toric-additive.

Proof. Without loss of generality, we assume that $t$ is a geometric point of $D_1, D_2, \ldots, D_m$, for some integer $0 \leq m \leq n$. We let $\Gamma'$ be the graph of the fibre $\mathcal{C}_t$ and $X'$ the associated group of characters. We obtain a purity map $X' \to X_1 \oplus \ldots \oplus X_m$. The isomorphism $X \to X_1 \oplus \ldots \oplus X_n$ factors as

$$X \to X' \oplus X_{m+1} \oplus X_{m+2} \oplus \ldots \oplus X_n \to X_1 \oplus X_2 \oplus \ldots \oplus X_n$$

where the first arrow is a sum of generization maps, and the second arrow is the direct sum of the purity map $X' \to X_1 \oplus X_2 \oplus \ldots \oplus X_m$ and of the identity morphisms $X_i \to X_i$, $m+1 \leq i \leq n$. Hence the second arrow is both injective and surjective, and it follows that $X' \to X_1 \oplus \ldots \oplus X_m$ is an isomorphism.

We now drop the hypothesis that $S$ is strictly local, and give a global definition of toric additivity. Let $S$ be regular, with a normal crossing divisor $D \subset S$ and let $\mathcal{C}/S$ be a nodal curve, smooth over $U = S \setminus D$.

Definition 2.9 (Global definition of toric additivity). We say that $\mathcal{C}/S$ is toric-additive at a geometric point $s$ of $S$ if the base change to the strict henselization $\text{Spec} \mathcal{O}^{sh}_{S,s}$ is toric-additive as in definition 2.6.

We say that $\mathcal{C}/S$ is toric-additive if it is toric-additive at all geometric points $s$ of $S$. In view of lemma 2.8, this definition is consistent with definition 2.6.

2.3 Base change properties

The functoriality of $\text{Pic}^0_{\mathcal{C}/S}$ allows us to show that toric additivity behaves well with respect to a quite general class of base change.

Lemma 2.10. Let $S$ be a regular scheme, $D \subset S$ a normal crossing divisor and $\mathcal{C}/S$ a nodal curve smooth over $U = S \setminus D$. Let $T$ be another regular scheme, and $f : T \to S$ a morphism such that $E := (D \times_S T)_{\text{red}} \subset T$ is a
normal crossing divisor. Let \( t \) be a geometric point of \( T \), lying over some geometric point \( s \) of \( S \), and let \( f': T' \to S' \) be the morphism between the strict henselizations. Suppose that \( f' \) satisfies the following assumption:

\((\ast)\) For every irreducible component \( E_i \) of \( E \cap T' \), the image via \( f' \) is an irreducible component \( D_j \) of \( D \cap S' \), and the induced function \( \{E_i\}_i \to \{D_j\}_j \) is a bijection between the set of irreducible components of \( E \cap T' \) and the set of irreducible components of \( D \cap S' \).

Then \( C/S \) is toric-additive at \( s \) if and only if the base-change \( C_T/T \) is toric-additive at \( t \).

**Proof.** The morphism \( f': T' \to S' \) induces a bijection between the generic points \( \xi_1, \ldots, \xi_n \) of \( E \cap T' \) and the generic points \( \zeta_1, \ldots, \zeta_n \) of \( D \cap S' \). Clearly \( \mu(t) = \mu(s) \) and \( \mu(\xi_i) = \mu(\zeta_i) \), and the statement follows. \( \square \)

**Corollary 2.11.** Toric additivity is local on the target for the smooth topology.

**Proof.** Let \( f': T' \to S' \) be the induced morphism between strict henselizations and let \( D_i \) be any component of \( D \times_S S' \). The induced morphism from \( E_i := D_i \times_S T' \) to \( D_i \) is flat and formally smooth. Moreover, \( D_i \) is regular. Now we follow the proof of [Liu02, theorem 4.3.36] to prove that \( E_i \) is also regular: let \( x \in E_i \) and \( y = f'(x) \). Let \( m = \dim \mathcal{O}_{E_i,x} \) and \( n = \dim \mathcal{O}_{D_i,y} \). By flatness we can apply [Liu02, theorem 4.3.12] and find that \( \dim \mathcal{O}_{(E_i)_y,x} = m - n \). By [GD67, 1, 19.6.5], \((E_i)_y \) is regular at \( y \), hence \( \mathcal{O}_{(E_i)_y,x} \) is generated by \( m - n \) elements \( b'_{n+1}, \ldots, b'_m \). Each of these is the image of an element \( b_i \) in the maximal ideal of \( \mathcal{O}_{E_i,x} \). If we let \( a_1, \ldots, a_m \) be generators of the maximal ideal of \( \mathcal{O}_{D_i,y} \), the \( m \) elements \( a_1, \ldots, a_n, b_{n+1}, \ldots, b_m \) generate \( \mathcal{O}_{E_i,x} \), which is therefore regular.

Finally, as \( E_i \) is regular and connected (every connected component of \( E_i \) contains the closed point of \( T' \)), it is irreducible. This shows that the function from the set of components of \( D \times_S T' \) to the set of components of \( D \) is a bijection and we apply lemma 2.10 \( \square \)

**Example 2.12.** Let \( R \) be a regular, noetherian local ring, with a system of regular parameters \( t_1, t_2, \ldots, t_n \in R \). Let \( S = \text{Spec} \ R \) and \( D = \{ t_1 \cdot t_2 \cdots t_n = 0 \} \). Let \( C/S \) be a nodal curve, smooth over \( U = S \setminus D \).

Let \( a_1, a_2, \ldots, a_n > 0 \) be integers and let \( T \) be the spectrum of the finite locally-free \( R \)-algebra

\[
\frac{R[u_1, \ldots, u_n]}{u_1^{a_1} - t_1, \ldots, u_n^{a_n} - t_n}
\]

Then \( T \) is a regular scheme, \( E = \{ u_1 \cdot u_2 \cdots u_n \} \) is a normal crossing divisor, and the components of \( E \) are in bijection with those of \( D \).
By lemma 2.10, $C/S$ is toric-additive if and only if $C_T/T$ is toric-additive.

**Lemma 2.13.** Toric additivity is an open condition on $S$.

*Proof.* Suppose that $C/S$ is toric-additive at a geometric point $s$. By corollary 2.11 it is enough to show that $C/S$ is toric-additive on an étale neighbourhood of $s$, since étale morphisms are open. We choose an étale neighbourhood of finite type $W \rightarrow S$ of $s$ such that $D_W = D \times_S W$ is a strict normal crossing divisor and such that $s$ belongs to all irreducible components $D_1, \ldots, D_n$ of $D_W$.

Let $t$ be another geometric point of $W$; we want to show that $C_W/W$ is toric-additive at $t$. This is true if $t \notin D_W$, so we may assume without loss of generality that $t$ belongs to $D_1, \ldots, D_m$ for some $1 \leq m \leq n$. Let $\zeta$ be a geometric point lying over the generic point of $D_1 \cap D_2 \cap \ldots \cap D_m$; write $W_\zeta, W_t, W_s$ for the spectra of the strict henselizations of $W$ at $\zeta, t, s$ respectively. The morphism $W_\zeta \rightarrow W$ factors via $W_s$; hence, by lemma 2.8, $C_W/W$ is toric-additive at $\zeta$. We also have a natural map $W_\zeta \rightarrow W_t$.

Now, we look at the graph $\Gamma_t$ of $C_t$: none of its edges are contracted in the graph $\Gamma_\zeta$ of $C_\zeta$. Hence the generization map $H_1(\Gamma_t, \mathbb{Z}) \rightarrow H_1(\Gamma_\zeta, \mathbb{Z})$ is an isomorphism. Composing it with the purity map $H_1(\Gamma_\zeta, \mathbb{Z}) \rightarrow \bigoplus_{i=1}^m H_1(\Gamma_i, \mathbb{Z})$, we find the purity map for $C_t$. As $C_{W_\zeta}/W_\zeta$ is toric-additive, this latter purity map is an isomorphism, hence $C_{W_t}/W_t$ is toric-additive, which completes the proof.

**Lemma 2.14.** Let $\pi : C \rightarrow C'$ be a proper birational morphism between nodal curves $C/S$ and $C'/S$ smooth over $U$. Then $C/S$ is toric-additive if and only if $C'$ is.

*Proof.* As $C'/U$ is smooth, the restriction of $\pi$ to $U$ is an isomorphism. By definition, toric additivity of $C/S$ depends only on the semi-abelian scheme $\text{Pic}^0_{C/S}$. By [Del85, Théorème pag.132], semi-abelian extensions of $\text{Pic}^0_{C'/U}$ are unique up to unique isomorphism. Hence the morphism $\text{Pic}^0_{C'/S} \rightarrow \text{Pic}^0_{C/S}$ induced by $\pi$ is an isomorphism, which completes the proof.

### 3 Néron models

#### 3.1 The definition of Néron model

Let $S$ be any scheme, $U \subset S$ an open and $A/U$ an abelian scheme.
Definition 3.1. A Néron model for $A$ over $S$ is a smooth, separated algebraic space $\mathcal{N}/S$ of finite type, together with an isomorphism $\mathcal{N} \times_S U \to A$, satisfying the following universal property: for every smooth morphism of schemes $T \to S$ and $U$-morphism $f: T \times_S U \to A$, there exists a unique morphism $g: T \to \mathcal{N}$ such that $g|_U = f$.

It follows immediately from the definition that a Néron model is unique up to unique isomorphism; moreover, applying its defining universal property to the morphisms $m: A \times_U A \to A, i: A \to A$, and $0_A: U \to A$ defining the group structure of $A$, we see that $\mathcal{N}/S$ inherits from $A$ a unique $S$-group-space structure.

3.2 Base change properties

We proceed to analyse how Néron models behave under different types of base change. In general, the property of being a Néron model is not stable under arbitrary base change. However, we have that:

Lemma 3.2. Let $\mathcal{N}/S$ be a Néron model for $A/U$; let $S' \to S$ be a smooth morphism and $U' = U \times_S S'$. Then the base change $\mathcal{N} \times_S S'$ is a Néron model of $A_{U'}$.

Proof. Let $X \to S'$ be a smooth scheme with a morphism $f: X_{U'} \to A_{U'}$; by composition with the smooth morphism $S' \to S$ we obtain a smooth scheme $X \to S$ and a map $X \times_S U \to A_U$, which extends uniquely to an $S$-morphism $X \to \mathcal{N}$. This is the datum of an $S'$-morphism $X \to \mathcal{N} \times_S S'$ extending $f$. □

Lemma 3.3. Let $\mathcal{N}/S$ be a smooth, separated algebraic space of finite type with an isomorphism $\mathcal{N} \times_S U \to A$. Let $S' \to S$ be a faithfully flat morphism and write $U' = U \times_S S'$. If $\mathcal{N} \times_S S'$ is a Néron model of $A \times_U U'$, then $\mathcal{N}/S$ is a Néron model of $A$.

Proof. We first show that $\mathcal{N}/S$ satisfies the universal property of Néron models when the smooth morphism $T \to S$ is the identity. So, let $f: U \to A$ be a section of $A/U$. To show that $f$ extends to a section $S \to \mathcal{N}$ we only need to check that the schematic closure $X$ of $f(U)$ inside $\mathcal{N}$ is faithfully flat over $S$: indeed, $X \to S$ is birational and separated; if it is also flat and surjective it is automatically an isomorphism. Now, by base change of $f$ we get a closed immersion $f': U' \to A \times_U U'$, which extends to a section $g': S' \to \mathcal{N} \times_S S'$ by hypothesis. The schematic image $g'(S')$ is necessarily the schematic closure of $f'(U')$ inside $\mathcal{N} \times_S S'$; since taking the schematic closure commutes with
faithfully flat base change, we have \( g'(S') = X \times_S S' \). We deduce that \( X \to S \) is faithfully flat, as its base change via \( S' \to S \) is such. Hence \( f: U \to A \) extends to a section \( g: S \to \mathcal{N} \). The uniqueness of the extension is a consequence of the separatedness of \( \mathcal{N} \).

Next, let \( T \to S \) be smooth and let \( f: T_U \to A \). In order to extend \( f \) to a morphism \( g: T \to \mathcal{N} \), it is enough to show that \( \mathcal{N} \times_S T \) satisfies the extension property for sections \( T_U \to A \times_U T_U \). By the previous paragraph, it is enough to know that \( (\mathcal{N} \times_S T) \times_S S' = (\mathcal{N} \times_S S') \times_S T \) is a Néron model of \( (A \times_U T_U) \times_U U' \). This is true by lemma 3.2, concluding the proof.

**Lemma 3.4.** Let \( A/U \) be abelian, \( f: S' \to S \) a smooth surjective morphism, \( U' = U \times_S S' \), and \( \mathcal{N}'/S' \) a Néron model of \( A \times_S S' \). Then there exists a Néron model \( \mathcal{N}/S \) for \( A \).

**Proof.** Write \( S'' := S' \times_S S' \), \( p_1, p_2: S'' \to S' \) for the two projections and \( q: S'' \to S \) for \( f \circ p_1 = f \circ p_2 \). By lemma 3.2 both \( p_1^* \mathcal{N} \) and \( p_2^* \mathcal{N} \) are Néron models of \( q^* A \). By the uniqueness of Néron models, we obtain a descent datum for \( \mathcal{N}' \) along \( S' \to S \). Effectiveness of descent data for algebraic spaces (Sta16 TAG 0ADV) yields a smooth, separated algebraic space \( \mathcal{N}/S \) of finite type. By lemma 3.3 this is a Néron model for \( A/U \).

Although Néron models are not stable under base change (not even flat), they are preserved by localizations, as we see in the following lemma:

**Lemma 3.5.** Assume \( S \) is locally noetherian. Let \( s \) be a point (resp. geometric point) of \( S \) and \( \tilde{S} \) the spectrum of the localization (resp. strict henselization) at \( s \). Suppose that \( \mathcal{N}/S \) is a Néron model for \( A/U \). Then \( \mathcal{N} \times_S \tilde{S} \) is a Néron model for \( A \times_U \tilde{U} \), where \( \tilde{U} = \tilde{S} \times_S U \).

**Proof.** Let \( \tilde{Y} \to \tilde{S} \) be a smooth scheme and \( \tilde{f}: \tilde{Y}_{\tilde{U}} \to A_{\tilde{U}} \) a morphism. We may assume that \( \tilde{Y} \) is of finite type over \( \tilde{S} \), hence of finite presentation. By GD67 3, 8.8.2] there exist an open neighbourhood (resp. étale neighbourhood) \( S' \) of \( s \), a scheme \( Y' \to S' \) restricting to \( \tilde{Y} \) over \( \tilde{S} \), and a \( (U \times_S S') \)-morphism \( f': Y' \times_{S'} (U \times_S S') \to \mathcal{N} \times_S (U \times_S S') \) restricting to \( \tilde{f} \) on \( \tilde{U} \). By lemma 3.2 \( \mathcal{N} \times_S S' \) is a Néron model of \( \mathcal{N} \times_S (U \times_S S') \), hence we get a unique extension \( g': Y' \to \mathcal{N} \times_S S' \) of \( f' \). The base-change of \( g' \) via \( \tilde{S} \to S' \) gives us the required unique extension of \( f \).

**Proposition 3.6.** Assume that \( S \) is regular. If \( A/S \) is an an abelian algebraic space, then it is a Néron model of its restriction \( A \times_S U \).

**Proof.** Using lemma 3.3 we may assume that \( S \) is strictly local and that \( A/S \) is a scheme. We identify \( A \) with its double dual \( A'' = \text{Pic}^0_{A/S} \). Now let
Let $T \to S$ be smooth and $f : T_U \to A_U$. Then $f$ corresponds to an element of $A_U(T_U) = \text{Pic}^0_{A_U/S}(T_U) = \text{Pic}^0(\mathfrak{A}'_{T_U})/\text{Pic}^0(T_U)$. Let $\mathcal{L}_U$ be an invertible sheaf with fibres of degree 0 on $\mathfrak{A}'_{T_U}$ mapping to $f$ in $A_U(T_U)$. As $\mathfrak{A}'_T$ is regular, $\mathcal{L}_U$ extends to an invertible sheaf of degree 0 on $\mathfrak{A}'_T$, which yields a $T$-point of $A'' = A$ extending $f$. The uniqueness of the extension follows from the separatedness of $A/S$.

We conclude the subsection by stating the main theorem about Néron models in the case where the base $S$ is of dimension 1.

**Theorem 3.7** ([BLR90], 1.4/3). Let $S$ be a connected Dedekind scheme with fraction field $K$ and let $A/K$ be an abelian variety. Then there exists a Néron model $\mathcal{N}$ over $S$ for $A/K$.

### 3.3 The group of connected components of a Néron model

Since we are mainly interested in the Néron model of the jacobian of a smooth curve $C_U/U$ extending to a nodal curve $C/S$, we will now work under the following hypotheses: $S$ is a regular scheme, $U \subset S$ is an open dense, $A/U$ is an abelian scheme and $A/S$ is a semi-abelian scheme with $A \times_S U = A$.

The following result is extremely useful:

**Lemma 3.8.** Suppose that $A$ admits a Néron model $\mathcal{N}/S$. Then the canonical morphism $A \to \mathcal{N}$ coming from the universal property of definition 3.1 is an open immersion, and induces an isomorphism from $A$ to the fibrewise-connected component of identity $\mathcal{N}^0$.

**Proof.** The fact that $A \to \mathcal{N}$ is an open immersion follows from [GRR72] IX, Prop. 3.1.e]. For every point $s \in S$ of codimension 1, the restriction of $\mathcal{N}$ to the local ring $\mathcal{O}_{S,s}$ is the Néron model of its generic fibre, by lemma 3.5. It follows by [Ray70] XI, 1.15] that the induced morphism $A \to \mathcal{N}^0$ is an isomorphism.

We obtain an exact sequence of fppf-sheaves on $S$

$$0 \to A \to \mathcal{N} \to \Phi \to 0$$

where $\Phi$ is an étale quasi-finite (in general non-separated) group algebraic space over $S$, the group of connected components of $\mathcal{N}$.

In [GRR72] IX, 11] this group is studied in depth. Although only the case where $S$ is the spectrum of a discrete valuation ring is treated, most results
carry over to more general bases. In this subsection we describe some results about \( \Phi \) which are useful for us.

Let us assume that \( S \) is moreover strictly local, with closed point \( s \), residue field \( k \), and fraction field \( K \). We are interested in the finite abelian group \( \Phi_s(k) \). Let \( m \) be an integer; multiplication by \( m \) on \( A \) is a faithfully flat morphism; hence, restricting the exact sequence 3.1 to the closed fibre and taking \( m \)-torsion, we find an exact sequence

\[
0 \to A_s[m] \to \mathcal{N}_s[m] \to \Phi_s[m] \to 0.
\]

Taking \( k \)-valued points, we have

\[
\Phi_s[m](k) = \frac{\mathcal{N}_s[m](k)}{A_s[m](k)}.
\]

The group scheme \( A[m]/S \) is separated, quasi-finite and flat over an henselian local ring; hence it decomposes canonically into a disjoint union \( A[m] = A[m]^f \sqcup B \), with \( A[m]^f/S \) a finite flat group scheme, and \( B_s = \emptyset \). In particular, we find that \( A[m]^f_K(K) \subseteq A[m](K) \) is the subgroup of points extending to elements of \( \mathcal{N}(S) \) contained in \( A(S) \).

We can rewrite

\[
\Phi_s[m](k) = \frac{\mathcal{N}_s[m](k)}{A_s[m](k)} = \frac{\mathcal{N}[m](S)}{A[m](S)} = \frac{A[m]_K(K)}{A[m]^f_K(K)}
\]

where the second equality is due to the fact that \( S \) is henselian, and the third equality is due to the universal property of Néron models.

Now let \( X \) be the character group of the maximal subtorus \( T \subseteq A_s \). It is a free abelian group of rank \( \mu = \dim T \). We let \( X \) be the constant group scheme on \( S \) with value \( X \), and \( X_m := X \otimes \mathbb{Z}/m\mathbb{Z} \). By [GRR72, IX, 11.6.7] there is a canonical isomorphism

\[
X_{m,K} = \frac{A[m]^f_K}{A[m]^f_K(K)}.
\]

As \( X/S \) is constant, we have \( X \otimes \mathbb{Z}/m\mathbb{Z} = X_{m,K}(K) \). As the natural map

\[
\frac{A[m]_K(K)}{A[m]^f_K(K)} \to \frac{A[m]_K(K)}{A[m]^f_K(K)}
\]

is injective, we have obtained a canonical injective homomorphism

\[
\Phi_s[m](k) \hookrightarrow X \otimes \mathbb{Z}/m\mathbb{Z}
\]

and taking the colimit over \( m \in \mathbb{Z} \), a canonical injective homomorphism of abelian groups

\[
\Phi_s(k) \hookrightarrow X \otimes \mathbb{Q}/\mathbb{Z}.
\]

(3.2)
Let us return to the case where $C/S$ is a nodal curve, smooth over $U$, such that $\text{Pic}^0_{C_U/U}$ admits a Néron model $N/S$. In view of lemma 3.8, the fibrewise-connected component of identity of $N$ is canonically identified with $\text{Pic}^0_{C/U}$, and we have an exact sequence of fppf-shaves on $S$

\[ 0 \to \text{Pic}^0_{C/S} \to N \to \Phi \to 0 \]

We are going first of all to define a generization map for the group of components. Let $\zeta$ be any point of $S$ and let $Z \subseteq S$ be its schematic closure, which is still a strictly local scheme. Let $a \in \Phi_s(s)$ be a component of the closed fibre $N_s$. By Hensel’s lemma we can extend $a$ to a section $\alpha \in \Phi_Z(Z)$. Because $\Phi_Z/Z$ is étale, this extension is actually unique. By restriction we find $\alpha_\zeta \in \Phi_\zeta(\zeta)$. We have constructed a generization homomorphism of finite abelian groups

$\Phi_s(s) \to \Phi_\zeta(\zeta), a \mapsto \alpha_\zeta$.

Assume now that $S$ has a normal crossing divisor $D = D_1 \cup D_2 \ldots \cup D_n \subset S$, with $U = S \setminus D$. Let $\zeta_1, \ldots, \zeta_n$ be the generic points of $D_1, \ldots, D_n$. We have seen in lemma 1.10 that $C_\zeta_i$ has split singularities and that therefore the maximal torus $T_i$ is split (lemma 1.8). This means that the absolute Galois group $\text{Gal}(\overline{\zeta_i}/\zeta_i)$ acts trivially on the character group $X_i$, and in particular on the subgroup $\Phi(\zeta_i)$ of $X_i \otimes \mathbb{Q}/\mathbb{Z}$. Hence $\Phi_{\zeta_i}/\zeta_i$ is a constant group scheme, completely determined by its group $\Phi_i = \Phi_{\zeta_i}(\zeta_i) = \Phi_{\zeta_i}(\overline{\zeta_i})$ of $\zeta_i$-valued points.

Putting together the generization homomorphisms $\Phi(s) \to \Phi_i$, we obtain a purity map of groups of components:

$\Phi(s) \to \Phi_1 \oplus \Phi_2 \oplus \ldots \oplus \Phi_n.$

(3.3)

Lemma 3.9. The group homomorphism

$\Phi(s) \to \Phi_1 \oplus \Phi_2 \oplus \ldots \oplus \Phi_n$

is injective.

Proof. The diagram

\[
\begin{array}{ccc}
\Phi(s) & \longrightarrow & \Phi_1 \oplus \ldots \oplus \Phi_n \\
\downarrow & & \downarrow \\
X \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & (X_1 \oplus \ldots \oplus X_n) \otimes \mathbb{Q}/\mathbb{Z}
\end{array}
\]

where the vertical maps are as in (3.2) and the horizontal maps are the purity homomorphisms3.3 and 2.1 is commutative. The vertical maps are injective;
moreover the purity map $X \to X_1 \oplus \ldots X_n$ is also injective (lemma 2.2), and has torsion-free cokernel (lemma 2.4), hence it is still injective when tensored with $\mathbb{Q}/\mathbb{Z}$. It follows that the top horizontal map is injective.

\[\square\]

4 Toric additivity as a criterion for existence of Néron models

4.1 Aligned curves

We start by recalling the definition of aligned nodal curve introduced in [Hol17].

Definition 4.1 ([Hol17], definition 2.11). Let $\mathcal{C} \to S$ be a nodal curve and $s$ a geometric point of $S$. Consider the base change $\mathcal{C}_S'/S'$ to the strict henselization $S'$ at $s$, and the labelled graph $(\Gamma, l)$ of definition 1.9. We say that $\mathcal{C}/S$ is aligned at $s$ if for every cycle $\gamma \subset \Gamma$ and every pair of edges $e, e'$ of $\gamma$, there exist integers $n, n'$ such that

$$l(e)^n = l(e')^{n'}.$$ 

We say that $\mathcal{C}/S$ is aligned if it is aligned at every geometric point of $S$.

The following is the main theorem of [Hol17], establishing the relation between alignment and existence of Néron models of jacobians:

Theorem 4.2 ([Hol17], theorem 5.16, theorem 6.2). Let $S$ be regular, $U \subset S$ a dense open, $f : \mathcal{C} \to S$ a nodal curve, with $f_U : \mathcal{C}_U \to U$ smooth.

i) If the jacobian $\text{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model over $S$, then $\mathcal{C}/S$ is aligned;

ii) if $\mathcal{C}$ is regular and $\mathcal{C}/S$ is aligned, then $\text{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model over $S$.

Remark 4.3. The proof of theorem 4.2 part ii) is constructive: for $\mathcal{C}/S$ aligned with $\mathcal{C}$ regular, the Néron model is the smooth, separated group-algebraic space of finite type

$$\mathcal{N} = \frac{\text{Pic}^0_{\mathcal{C}/S}}{E}$$

where:
• Pic\[^{[0]}\]_C/S is the smooth group-algebraic space representing the functor of
invertible sheaves on C/S with fibres of total degree zero; equivalently, it is the schematic closure of Pic\(^0\)_{C_U/U} inside Pic_C/S.

• E/S is the schematic closure of the unit section U → Pic\(^0\)_{C_U/U} inside Pic\[^{[0]}\]_C/S.

The fact that the quotient of fppf-sheaves N exists as an algebraic space is due to the fact that under the assumptions of theorem 4.2, part ii), E/S is a flat subgroup space of Pic\[^{[0]}\]_C/S. See theorem 6.2 of [Hol17] for more details.

In the case where C/S is smooth outside of a normal crossing divisor, we have the notion of toric additivity introduced in section 2, we are going to explore its relation with alignment.

Let’s consider then a regular base scheme S with a normal crossing divisor D ⊂ S, and a nodal curve C/S, such that the base change C_U/U to U = S \ D is smooth.

If S' → S is a strict henselization at some geometric point s of S, and D ∩ S' is given by regular parameters t_1, ..., t_n ∈ O(S'), then the thickness of any non-smooth point p ∈ C_s is generated by t^{m_1} \cdot \cdots \cdot t^{m_n} for some non-negative integers m_1, ..., m_n. In particular, C is regular at p if and only if its thickness is generated by t_i for some 1 ≤ i ≤ n.

**Lemma 4.4.** Suppose that C/S is toric-additive. Then it is aligned.

**Proof.** As both alignment and toric additivity are checked over the strict henselizations at geometric points of S, we may assume that S is strictly local, with \{D_i\}_{i=1,...,n} the components of the divisor D. Each of them is cut out by a regular element t_i ∈ O_S(S) and is itself a regular, strictly local scheme. Let ζ_i be the generic point of D_i. By lemma 1.10, the curve C_{ζ_i} has split singularities; its labelled graph (Γ_{ζ_i}, l_{ζ_i}) is obtained from (Γ_s, l_s) by contracting edges according to the procedure in section 1.3; that is, by contracting the edges with label generated by an element of O(S) invertible at the point ζ_i.

We have an injective homomorphism eq. (2.2)

\[ F: H_1(Γ_s, Z) → H_1(Γ_1, Z) ⊕ \cdots ⊕ H_1(Γ_n, Z). \]

Notice that, for a general graph G, the choice of an orientation of the edges allows to see a cycle on G as a labelling of the edges E → Z taking only values in \{-1, 0, 1\}, and in particular as an element of H_1(G, Z). Moreover, H_1(G, Z)
is generated by elements arising from cycles of $G$. Therefore we make a choice of orientation on $\Gamma_s$, which determines orientations of the $\Gamma_i$’s as well.

We only need to show that if $C/S$ is not aligned, then $F$ is not surjective.

Consider the commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H_1(\Gamma_s, \mathbb{Z}) & \rightarrow & \mathbb{Z}^E & \rightarrow & \mathbb{Z}^V \\
& & \downarrow F = F_1 \oplus \ldots \oplus F_n & & \downarrow & & \downarrow \\
0 & \rightarrow & \bigoplus_i H_1(\Gamma, \mathbb{Z}_i) & \rightarrow & \bigoplus_i \mathbb{Z}^{E_i} & \rightarrow & \bigoplus_i \mathbb{Z}^{V_i}.
\end{array}
\]

with maps as described in eq. (1.3). The central vertical map is injective. If we call $A = \ker(\text{im } \alpha \rightarrow \text{im } \beta)$, $B = \text{coker } F$, $C = \text{coker } (\mathbb{Z}^E \rightarrow \bigoplus_i \mathbb{Z}^{E_i})$, $D = \text{coker } (\mathbb{Z}^V \rightarrow \bigoplus_i \mathbb{Z}^{V_i})$, we have an exact sequence

\[0 \rightarrow A \rightarrow B \rightarrow C \rightarrow D\]

by the Snake Lemma.

Assume that $C/S$ is not aligned. This means that in the labelled graph $(\Gamma_s, l_s)$ over the closed point $s \in S$, there is a cycle $\gamma$ with two edges $e_1, e_2$ such that $l_s(e_1)^m \neq l_s(e_2)^{m'}$ for all $m, m'$ non-negative integers.

There are two possibilities:

Case 1: for every edge $e \in \gamma$ there exists an $i$ and an $m$ such that $l_s(e) = t_1^m$.

Assume this is the case. Then there are two adjacent edges $e_1, e_2$, such that (without loss of generality) the label of $e_1$ is a power of $t_1$ and the label of $e_2$ is a power of $t_2$. Let’s see $\gamma$ as an edge labelling in $H_1(\Gamma_s, \mathbb{Z})$, and let $\varphi \in \mathbb{Z}^E$ be the edge labelling described by

\[
\varphi(e) = \begin{cases} 
\gamma(e) & \text{if } l_s(e) \text{ is a power of } t_1 \\
0 & \text{otherwise.}
\end{cases}
\]

In particular $\varphi(e_1) = \pm 1$ and $\varphi(e_2) = 0$. Let $v \in V$ be a vertex lying on the cycle $\gamma$ and joining $e_1$ and $e_2$. Then $\alpha(\varphi)$ is non-zero at $v$. On the other hand, in $\Gamma_1$ all edges of $\gamma$ the label of which is not a power of $t_1$ are contracted; hence the image of $\varphi$ via the central vertical map is actually contained in $\bigoplus H_1(\Gamma_i, \mathbb{Z})$. In particular, by the commutativity of the diagram $\alpha(\varphi)$ is sent to zero by $\mathbb{Z}^V \rightarrow \bigoplus \mathbb{Z}^{V_i}$. But then $\alpha(\varphi)$ is a non-zero element of $A$, which implies that $B \neq 0$ and that $F$ is not surjective.
Case 2: there exists some edge $\bar{e}$ in $\gamma$ that is preserved in at least two of the contracted graphs $\Gamma_i$’s. We may assume these are $\Gamma_1$ and $\Gamma_2$. In this case, we define an element $\varphi = (\varphi_1, \ldots, \varphi_n) \in \bigoplus H_1(\Gamma_i, \mathbb{Z})$ as follows: $\varphi_1$ is the (non-trivial) cycle image of $\gamma$ in $\Gamma_1$; and for $2 \leq i \leq n$, $\varphi_i = 0$. Hence $\varphi$ is a non-zero element of $\bigoplus H_1(\Gamma_i, \mathbb{Z})$. Now let $\psi$ be any element of $H_1(\Gamma_s, \mathbb{Z})$; if $\psi(\bar{e}) = 0$ then $F_1(\psi) \neq \varphi_1$, and if $\psi(\bar{e}) \neq 0$, then $F_2(\psi) \neq \varphi_2$. Therefore $\varphi$ is not in the image of $F$, and the proof is finished.

\[ \square \]

**Lemma 4.5.** Suppose that the total space $\mathcal{C}$ is regular. Then $\mathcal{C}/S$ is aligned if and only if $\mathcal{C}/S$ is toric-additive.

**Proof.** Once again we may assume that $S$ is strictly local. Let $\Gamma_s = (V_s, E_s)$ be the dual graph of the fibre of $\mathcal{C}$ over the closed point $s \in S$, and $l_s: E_s \to L_s$ the labelling of the edges, taking value in the monoid $L_s$ of principal ideals of $\mathcal{O}_S(S)$. Since $\mathcal{C}$ is regular, the labels can only take the values $(t_1), \ldots, (t_n) \in L_s$. This means that $\mathcal{C}/S$ is aligned if and only if every cycle of $\Gamma$ has edges with the same label.

Now, let $\{D_i\}_{i=1,\ldots,n}$ be the components of the divisor $D$. Each of them is cut out by a regular element $t_i \in \mathcal{O}_S(S)$ and is itself a regular, strictly local scheme. Let $\zeta_i$ be the generic point of $D_i$. By lemma 1.10, the curve $\mathcal{C}_{\zeta_i}$ has split singularities; its labelled graph $(\Gamma_{\zeta_i}, l_{\zeta_i})$ is obtained from $(\Gamma_s, l_s)$ by contracting edges according to the procedure in section 1.3; that is, by contracting the edges with label different from $(t_i)$.

We have already proved in lemma 4.4 that toric additivity implies alignment. Assume that $\mathcal{C}/S$ is aligned; we want to show that the injective homomorphism

$$F: H_1(\Gamma_s, \mathbb{Z}) \to H_1(\Gamma_1, \mathbb{Z}) \oplus \ldots \oplus H_1(\Gamma_n, \mathbb{Z})$$

is surjective.

We choose an orientation of the edges of $\Gamma_s$, so we can associate to every cycle $\gamma$ in $\Gamma_s$ an element $\gamma \in H_1(\Gamma_s, \mathbb{Z})$.

Fix $1 \leq i \leq n$, let $\gamma$ be a cycle in $\Gamma_i$, and $(0, 0, \ldots, 0, \gamma, 0, \ldots, 0) \in \bigoplus H_1(\Gamma_i, \mathbb{Z})$ the corresponding element. The edges of $\gamma$, when seen in $\Gamma_s$, have label $(t_i)$. Moreover, they are adjacent, by the hypothesis of alignment. Hence they still form a cycle in $\Gamma_s$ which is sent to $(0, \ldots, 0, \gamma, 0, \ldots, 0)$ via the map $F$. We conclude that $F$ is surjective.

\[ \square \]
4.2 Toric additivity and desingularization of curves

Let $S$ be a regular base scheme $S$ with a normal crossing divisor $D \subset S$, and let $C/S$ be a nodal curve, such that the base change $C_U := S \setminus D$ is smooth.

In [dJ96, 3.6], it is proven that if $C \to S$ has split fibres, there exists a blow-up $\varphi : C' \to C$ such that $C' \to S$ is still a nodal curve, and $C'$ is regular. The condition of splitness implies that the irreducible components of the geometric fibres are smooth; or equivalently, that the dual graphs of the geometric fibres do not admit loops. We are going to introduce a condition on $C/S$, weaker than splitness, and show that a statement analogous to the one in [dJ96, 3.6] holds for curves satisfying this condition.

**Definition 4.6.** Let $C \to S$ be a nodal curve. We say that $C/S$ is disciplined if, for every geometric point $\overline{s}$ of $S$, and $p \in C_{\overline{s}}^{ns}$, at least one of the following is satisfied:

i) $p$ belongs to two irreducible components of $C_{\overline{s}}$;

ii) the thickness of $p$ is generated by a power of a regular parameter of $O_{S,\overline{s}}^{sh}$.

**Example 4.7.** Let $k$ be a field, $S = \text{Spec } k[[u,v]]$, $U = S \setminus \{uv = 0\}$. Let $E/S$ be the family of nodal curves of arithmetic genus one

$$y^2 - x^3 - x^2 - uv = 0$$

smooth over $U$.

The family $E/S$ is not disciplined: the point $p = (x = 0, y = 0, u = 0, v = 0)$ belongs to only one component of the fibre over $u = 0, v = 0$, and its thickness is the ideal $(uv)$.

We give first an auxiliary lemma:

**Lemma 4.8.** Hypothesis as in the beginning of the subsection. Suppose that $S$ is strictly local; write $D = D_1 \cup \ldots \cup D_n$ and write $\zeta_i$ for the generic point of $D_i$. Suppose that $C/S$ is disciplined. Let $p \in C_{\overline{s}}^{ns}$ be a non-smooth point of the fibre over the closed point, such that $p$ does not satisfy condition ii) of definition 4.6. Let $X_1, X_2$ be the distinct irreducible components of the closed fibre $C_{\overline{s}}$ containing $p$. Then there exists $i \in \{1, \ldots, n\}$ and $Y_1, Y_2$ irreducible components of $C_{\zeta_i}$ whose schematic closures $\overline{Y}_1, \overline{Y}_2 \subset C$ satisfy: $X_1 \not\subset \overline{Y}_2 \supset X_2$ and $X_2 \not\subset \overline{Y}_1 \supset X_1$.

**Proof.** Let $(\Gamma, l_s)$ be the labelled graph of $C_{\overline{s}}$. By hypothesis, the edge $e(p)$ corresponding to $p$ has distinct extremal vertices, $v_1$ and $v_2$, and label $t_1^{m_1} \cdot \ldots \cdot t_l^{m_l}$, with $2 \leq l \leq n$ and $m_1, \ldots, m_l \geq 1$. The fibres over the generic
points \( \zeta_1, \ldots, \zeta_n \) have split singularities by lemma 1.10, so we can consider their labelled graphs \((\Gamma_i, l_i)\). What we want to prove is that there exists \( i \in \{1, \ldots, l\} \) such that \( v_1 \) and \( v_2 \) are mapped to distinct vertices of \((\Gamma_i, l_i)\) via the procedure described in section 1.3.

Suppose the contrary. As \( e(p) \) is not contracted in any \( \Gamma_i, 1 \leq i \leq l \), we deduce that there exists a cycle \( \gamma \) in \( \Gamma_s \), containing \( e(p) \), such that for all \( 1 \leq i \leq l \), all edges \( e \neq e(p) \) of \( \gamma \) are contracted in \( \Gamma_i \). Let \( \zeta_{12} \) be the generic point of \( D_1 \cap D_2 \); all edges \( e \neq e(p) \) of \( \gamma \) are contracted in \( \Gamma_{12} \), the labelled graph of \( C_{\zeta_{12}} \), and in particular \( v_1 \) and \( v_2 \) are mapped to the same vertex. The edge \( e(p) \) is therefore mapped to a loop in \( \Gamma_{12} \), with label \( t_1^{m_1} t_2^{m_2} \). However, this contradicts the fact that \( C \to S \) is disciplined at \( \zeta_{12} \), and we have obtained a contradiction.

We introduce now some notation: given a scheme \( X \), we will denote by \( \text{Sing}(X) \subseteq X \) the set of points that are not regular. We say that the center of a blow-up \( \pi: Y \to X \) is the complement of the largest open \( U \subseteq X \) such that \( \pi^{-1}(U) \to U \) is an isomorphism.

**Lemma 4.9.** Hypotheses as in the beginning of the subsection. Suppose \( f: C \to S \) is disciplined. Then there is an \( \text{étale} \) surjective \( g: S' \to S \) and a blow-up \( \varphi: C' \to C \times_S S' \) such that

- the center of \( \varphi \) is contained in \( \text{Sing}(C \times_S S') \);
- \( C' \) is a nodal curve over \( S' \), smooth over \( g^{-1}(U) \);
- \( C' \) is regular.

**Proof.** First, notice that the order in which the blow-ups of the curve and the \( \text{étale} \) covers of the base are taken does not matter, as blowing-up commutes with \( \text{étale} \) base change. After replacing \( S \) by a suitable \( \text{étale} \) cover, we may assume that \( D \) is a strict normal crossing divisor. We can now apply [1.1.2] and assume that \( \text{Sing}(C) \subseteq C \) has codimension at least 3.

Next, we claim that there exists an \( \text{étale} \) cover \( S' \to S \), such that for every point \( s' \in S' \), the irreducible components of \( C_{s'} \) are geometrically irreducible. To prove the claim, let’s take \( s \in S \). Replacing \( S \) by an \( \text{étale} \) neighbourhood of \( s \) in \( S \) we may assume that \( C/S \) admits sections \( \sigma_1, \ldots, \sigma_r \) through the smooth locus of \( C/S \) and intersecting every irreducible component of the fibre of \( C_s \). Now, the sheaf \( F := \mathcal{O}(\sigma_1 + \ldots + \sigma_r) \) is ample over \( s \). Since ampleness is an open condition there exists \( U \subseteq S \) open neighbourhood of \( s \) where \( F \) is ample. Then for every point \( u \in U \), every irreducible component of \( C_u \) is met by a section \( \sigma_i \), and is therefore geometrically irreducible. This proves the claim.
Hence, replacing \( S \) by a suitable étale cover, we may assume that for every generic point \( \zeta \) of \( D \), the fibre \( C_\zeta \) has irreducible components that are geometrically irreducible.

Now, let \( E \) be an irreducible component of \( C_D = C \times_S D \) and let \( \pi : C' \to C \) be the blow-up of \( C \) along \( E \). If \( p \in E \) is a regular point of \( C \), \( f \) is an isomorphism at \( p \), because \( E \) is cut out by one equation. Otherwise, the completion of the strict henselization at (a geometric point lying over) \( p \) is of the form

\[
\hat{\mathcal{O}}_{\hat{C}, \hat{p}}^h \simeq \frac{\hat{O}_{\hat{S}, \hat{f}(p)}^h[[x, y]]}{xy - t_1^{m_1} \cdots t_l^{m_l}}
\]

with \( t_1, \ldots, t_n \) regular parameters cutting out \( D \), \( 1 \leq l \leq n \) and positive integers \( m_1, \ldots, m_l \). In fact, because the singular locus has codimension at least three, we have \( l \geq 2 \), and \( m_1 = \ldots = m_l = 1 \).

The ideal of the pullback of \( E \) to \( \hat{\mathcal{O}}_{\hat{C}, \hat{p}}^h \) is either \( (t_i) \) for some \( 1 \leq i \leq l \), or one between \( (x, t_i) \) and \( (y, t_i) \) for some \( 1 \leq i \leq l \). In the first case, \( \pi \) is an isomorphism at \( p \). In the second case, one can compute explicitly the blowing up of Spec \( \mathcal{O}_{\hat{C}, \hat{p}}^h \) at the ideal \( (x, t_i) \) (or \( (y, t_i) \)) and find that \( f' : C' \to S \) is still a nodal curve, disciplined, with \( \text{Sing}(C) \) of codimension at least three, and such that for every generic point \( \zeta \) of \( D \) the fibre \( C_\zeta \) has irreducible components that are geometrically irreducible. We omit the explicit computations.

Let \( Y \subset C \) be the center of \( \pi : C' \to C \). Then \( Y \) consists only of non-regular points, hence it has codimension at least 3. As \( f : C' \to S \) is a curve, the fibres of \( \pi \) have dimension at most 1, hence \( \pi^{-1}(Y) \) has codimension at least 2 in \( C \). It follows that there is a bijection between the irreducible components of \( C_D \) and \( C_D' \), given by taking the preimage under \( \pi \). Now, \( \pi^{-1}(E) \) is a divisor, and for any other irreducible component \( E' \) of \( C_D \) that is a divisor, \( \pi^{-1}(E') \) is also a divisor. We conclude that \( \pi^* : C^* \to C \), the composition of the blowing-ups of all irreducible component of \( C_D \), is such that every component of \( C^*_D \) is a divisor. Besides, as previously noticed, \( f^* : C^* \to S \) is a nodal curve, disciplined, and \( \text{Sing}(C^*) \) has codimension at least three.

Assume now by contradiction that \( \text{Sing}(C^*) \neq \emptyset \), and let \( p \in \text{Sing}(C^*) \). Then without loss of generality the thickness at \( p \) is \( (t_1 \cdot \ldots \cdot t_l) \) for some \( 2 \leq l \leq n \). Consider the base change \( C^*_T / T \), where \( T \) is the spectrum of some strict henselization at \( s = f^*(p) \). For every \( i \) let \( \xi_i \) be the generic point of \( D_i \cap T \). By lemma 4.8 for some \( i \in \{1, \ldots, l\} \), there are distinct components \( Y_1, Y_2 \) of \( C_{\xi_i}^* \) whose closure in \( C_{T \cap D_i}^* \) contain \( p \). Because the irreducible components of the fibre \( C_{\xi_i}^* \) are geometrically irreducible, there are components \( X_1, X_2 \) of \( C_{\xi_i}^* \) whose closures \( E_1, E_2 \) in \( C_{D_i}^* \) contain \( p \). But then \( E_1 \) and \( E_2 \) are given by \( (x, t_1) \) and \( (y, t_1) \) in \( \hat{\mathcal{O}}_{\hat{C}, \hat{p}}^h \). In particular, they are not divisors. This is a contradiction, and therefore \( \text{Sing}(C^*) = \emptyset \). \( \square \)
Lemma 4.10. Hypotheses as in the beginning of the subsection. Suppose that \( f : C \to S \) is toric-additive. Then \( C/S \) is disciplined.

Proof. We may assume that \( S \) is strictly local, with closed point \( s \), and with \( D \) given by a system of regular parameters \( t_1, \ldots, t_n \). Let \( p \in C_s^{ns} \), with thickness \( t_1^{m_1} \cdots t_l^{m_l} \) for some \( 1 \leq l \leq n \) and \( m_1, \ldots, m_l \geq 1 \). We have to show that if \( l \geq 2 \) then \( p \) lies on two components of \( C_s \).

Suppose by contradiction that \( l \geq 2 \) and that \( p \) lies on only one component of \( C_s \). The dual graph \( \Gamma \) over \( s \) has a loop \( L \) corresponding to \( p \), with label \( t_1^{m_1} \cdots t_l^{m_l} \). For \( 1 \leq i \leq n \) call \( \Gamma_i \) the dual graph of the fibre \( C_{\xi_i} \) over the generic point of \( D_i \). The loop \( L \) is preserved in the dual graphs \( \Gamma_i \) for \( 1 \leq i \leq l \). Let \( \Gamma' \) be the graph obtained by \( \Gamma \) by removing the loop \( L \), and define similarly \( \Gamma_i' \), \( 1 \leq i \leq l \).

Inequality 2.4 of corollary 2.3 says

\[
h_1(\Gamma', \mathbb{Z}) \leq \sum_{i=1}^{l} h_1(\Gamma_i', \mathbb{Z}) + \sum_{j=l+1}^{n} h_1(\Gamma_j, \mathbb{Z}).
\]

For every \( 1 \leq i \leq l \), \( h_1(\Gamma_i, \mathbb{Z}) = h_1(\Gamma_i', \mathbb{Z}) + 1 \). Since \( l \geq 2 \), we find that \( h_1(\Gamma, \mathbb{Z}) = h_1(\Gamma', \mathbb{Z}) + 1 < \sum_{i=1}^{n} h_1(\Gamma_i, \mathbb{Z}) \). This contradicts the fact that \( C/S \) is toric-additive. \( \square \)

4.3 Toric additivity and existence of Néron models

We consider again a regular base scheme \( S \) with a normal crossing divisor \( D \subset S \), and a nodal curve \( C/S \) such that the base change \( C_U/U := S \setminus D \) is smooth. Theorem 4.2 tells us that if \( \text{Pic}^0_{C_U/U} \) admits a Néron model over \( S \), then \( C/S \) is aligned. We show that being disciplined is also a necessary condition for existence of a Néron model.

Lemma 4.11. Assume that \( S \) is an excellent scheme. Suppose that \( C/S \) is such that \( \text{Pic}^0_{C_U/U} \) admits a Néron model \( \mathcal{N} \) over \( S \). Then \( C/S \) is disciplined.

Proof. We may assume that \( S \) is strictly henselian, with closed point \( s \) and residue field \( k = k(s) \). Assume by contradiction that \( C/S \) is not disciplined. Then there is some \( p \in C_s^{ns} \) that belongs to only one component \( X \) of \( C_s \), and such that its thickness is \( t_1^{m_1} \cdots t_l^{m_l} \) with \( m_i \geq 1 \) and \( 2 \leq l \leq n \). Let \( q \in C_s(k) \) be a smooth \( k \)-rational point belonging to the same component as \( p \). By Hensel’s lemma, there exists a section \( \sigma_q : S \to C \) through \( q \). We claim that the same is true for \( p \): let \( \tilde{S} \) be the spectrum of the completion of \( \mathcal{O}(S) \)
at its maximal ideal and consider the morphism
\[ W := \text{Spec} \, \hat{O}_{C, p} \cong \text{Spec} \frac{\mathcal{O}(\hat{S})[[x, y]]}{xy - t_1^{m_1} \cdots t_i^{m_l}} \to \hat{S}. \]

This has a section given by \( x = t_1^{m_1}, y = t_2^{m_2} \cdots t_i^{m_l} \). Composing the section with the canonical morphism \( W \to C \), gives a morphism \( \hat{\sigma}_p : \hat{S} \to C \) going through \( p \). Because \( S \) is excellent and henselian, it has the Artin approximation property, and there exists a section \( \sigma_p : S \to C \) which agrees with \( \hat{\sigma}_p \) when restricted to the closed point \( s \), hence going through \( p \).

We write \( F := I(\sigma_p) \otimes_{\mathcal{O}_C} \mathcal{O}(\sigma_q) \) for the coherent sheaf on \( C \) given by the tensor product of the ideal sheaf of \( \sigma_p \) with the invertible sheaf associated to the divisor \( \sigma_q \). It is what is called a torsion free, rank 1 sheaf in the literature: it is \( S \)-flat, its fibres are of rank 1 at the generic points of fibres of \( C \), and have no embedded points. Notice that \( F \) is not an invertible sheaf, as \( \dim_k(p) F \otimes k(p) = 2 \).

Let \( u_p \) and \( u_q \) be the restrictions of \( \sigma_p \) and \( \sigma_q \) to \( U \). They are \( U \)-points of the smooth curve \( C_U/U \); the restriction of \( F \) to \( U \) is the invertible sheaf \( F_U = \mathcal{O}_{C_U}(u_q - u_p) \). This is the datum of a \( U \)-point \( \alpha \) of \( \text{Pic}^0_{C_U/U} \): indeed, \( \text{Pic}(U) = 0 \) because \( \mathcal{O}(U) \) is a UFD, and \( C_U/U \) has a section, so \( \text{Pic}^0_{C_U/U}(U) = \text{Pic}^0(C_U) \).

By the definition of Néron model, there is a unique section \( \beta : S \to N \) with \( \beta_U = \alpha \). We write \( J \) for \( \text{Pic}^0_{C_U/S} \). As \( J \) is semi-abelian, the canonical open immersion \( J \to N \) identifies \( J \) with the fibrewise-connected component of identity \( N^0 \) (lemma 3.8). Write \( \zeta_i, i = 1 \ldots, n \) for the generic points of the divisors \( D_i \). Then \( S_i := \text{Spec} \mathcal{O}_{S, \zeta_i} \) is a trait, and the restriction \( N_{S_i} \) is a Néron model of its generic fibre. Therefore \( \alpha_K \) extends uniquely to a section \( \alpha_i : S_i \to N_{S_i} \). As \( F_{S_i} \) is an invertible sheaf of degree 0 on every irreducible component of \( C_{\zeta_i} \), \( F_{S_i} \) is a \( S_i \)-point of \( J_{S_i} \), and \( \alpha_i \) is given by \( F_{S_i} \). Therefore, the restriction of \( \alpha : S \to N \) to \( S_i \) factors through \( J = N^0 \) for every \( i = 1 \ldots, n \).

We denote now by \( \Phi/S \) the étale group space of connected components of \( N \). By lemma 3.9 the canonical morphism
\[ \Phi(s) \to \bigoplus_{i=1}^n \Phi(\zeta_i) \]
is injective. This implies that \( \alpha \) lands inside \( J = N^0 \), or in other words that \( F_U \) extends to an invertible sheaf \( \mathcal{L} \) on \( C \) such that \( \mathcal{L}_s \) is of degree 0 on every component.

Now, let \( Z \to S \) be a closed immersion, with \( Z \) a trait, such that the generic point \( \xi \) of \( Z \) lands into \( U \) (it is an easy check that such a closed immersion
exists). As $F_\xi$ and $L_\xi$ define the same point of $\text{Pic}_*^{0}C_\xi$, there are isomorphisms $\mu_\xi: F_\xi \to L_\xi$ and $\lambda_\xi: L_\xi \to F_\xi$. By the same argument as in [AKS0] 7.8, $\mu_\xi$ and $\lambda_\xi$ extend to morphisms $\mu: F_Z \to L_Z$ and $\lambda: L_Z \to F_Z$, which are non-zero on all fibres. Let’s look at the restrictions to the closed fibre, $\mu_s: F_s \to L_s$, $\lambda_s: L_s \to F_s$. We know that $F_s$ is trivial away from the component $X_s \subset C_s$.

So, if we write $Y$ for the closure in $C_s$ of the complement of $X_s$, we may restrict $\mu_s$ and $\lambda_s$ to $Y$ to get global sections $l$ and $l'$ of $L_Y$ and $L_Y'$ respectively. Now, if $l = 0$, then the restriction $\mu_X$ of $\mu_s$ to $X$ is non-zero, because $\mu_s$ is non-zero.

If $l \neq 0$, as $L_s$ is of degree zero on every component, we have $l(y) \notin m_yL_y$ for every $y \in Y$, and in particular for $y \in Y \cap X$. It follows that also in this case $\mu_X \neq 0$. We can apply the same argument to $l'$ and conclude that $\lambda_X \neq 0$. Then the compositions $\mu_X \circ \lambda_X: L_X \to L_X$ and $\lambda_X \circ \mu_X: F_X \to F_X$ are non-zero. As $\text{End}_{O_X}(F_X) = k = \text{End}_{O_X}(L_X)$, they are actually isomorphisms. It follows that $\mu_X: F_X \to L_X$ is an isomorphism. However, $\dim k(p)F_k(p) = 2$, while $L_X$ is an invertible sheaf. This gives us the required contradiction.

Example 4.12. Consider again the curve $E/S$ of example 4.7. The fibres of $E/S$ are geometrically irreducible, hence the family is aligned. As the total space $E$ is not regular, theorem 4.2 does not allow us to deduce the existence of a Néron model for $\text{Pic}^{0}_{E/\mathcal{U}}$ over $S$.

However we have seen that $E/S$ is not disciplined; hence by lemma 4.11 we know for certain that there exists no Néron model over $S$ for $E_U$.

Combining the previous lemmas of this section, we obtain the following theorem, which shows that toric additivity is a criterion for existence of Néron models of jacobians:

Theorem 4.13. Let $S$ be a regular scheme, $D$ a normal crossing divisor on $S$, $C \to S$ a nodal curve smooth over $U = S \setminus D$.

i) If $C/S$ is toric-additive, then $\text{Pic}^{0}_{C_\mathcal{U}/\mathcal{U}}$ admits a Néron model over $S$.

ii) If moreover $S$ is excellent, the converse is also true.

Proof. Whether we are in the hypotheses of i) and ii), we know by lemmas 4.10 and 4.11 above that $C/S$ is disciplined; hence by lemma 4.9 there exists an étale cover $g: S' \to S$ and a blow-up $\pi: C' \to C_{S'}$ which restricts to an isomorphism over $U' = U \times_S S'$, such that $C'$ is a regular nodal curve.

Assume that $\text{Pic}^{0}_{C/S}$ is toric-additive. To show the existence of a Néron model over $S$, it is enough to show it over $S'$, by lemma 3.4. The base change $C'_{S'}/S'$ is toric-additive by corollary 2.11. The blow-up $C'/S'$ is also toric-additive by
lemma 2.14. We can now apply lemma 4.5 and deduce that $C'/S'$ is aligned. Hence by theorem 4.2, we find that $\text{Pic}^0_{C'/U'}$ admits a Néron model over $S'$, proving i).

Now assume that $S$ is excellent and that $\text{Pic}^0_{C'/U'}$ admits a Néron model $\mathcal{N}$ over $S$. Then $\mathcal{N}' = \mathcal{N} \times_S S'$ is a Néron model for $\text{Pic}^0_{C'/U'}$ over $S'$, by lemma 3.2. Hence $C'/S'$ is aligned by theorem 4.2. As $C'$ is regular, we deduce by lemma 4.3 that $C'/S'$ is toric-additive. By lemma 2.14, so is $C_{S'/S'}$. As toric additivity descends along étale covers (corollary 2.11), $C/S$ is toric-additive.

**Corollary 4.14.** Let $S$ be an excellent, regular scheme, $D$ a codimension one regular subscheme of $S$. Let $C/S$ be a nodal curve, smooth over $U = S \setminus D$. Then $\text{Pic}^0_{C_U/U}$ admits a Néron model over $S$.

**Proof.** At the strict henselization of each geometric point $s$ of $S$, $D$ is irreducible. Hence by remark 2.7, $C/s$ is toric-additive at $s$.

**Corollary 4.15.** Let $S$ be an excellent, regular scheme, $D$ a normal crossing divisor on $S$, $C \to S$ a nodal curve smooth over $U = S \setminus D$. There exists a biggest open $V \subset S$ over which $\text{Pic}^0_{C_U/U}$ admits a Néron model.

**Proof.** By lemma 2.13, toric additivity is an open condition on $S$.

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