GORENSTEIN HOMOLOGICAL INVARIANT PROPERTIES UNDER
FROBENIUS EXTENSIONS

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Abstract. We prove that for a Frobenius extension, a module over the extension
ring is Gorenstein projective if and only if its underlying module over the base
ring is Gorenstein projective. For a separable Frobenius extension between Artin
algebras, we obtain that the extension algebra is CM-finite (resp. CM-free) if and
only if so is the base algebra. Furthermore, we prove that the representation
dimension of Artin algebras is invariant under separable Frobenius extensions.

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ules, representation dimension

1. INTRODUCTION

Frobenius extensions are firstly introduced by Kasch in [17] as a generalization of
Frobenius algebra. The fundamental example is the group algebras induced by a finite
index subgroup. After that, some natural generalizations of different kinds of Frobenius
extensions are defined by Nakayama-Tzuzuku, Müller and Morita [20, 21, 22]. There are
other examples of Frobenius extensions including Hopf subalgebras, finite extensions of
enveloping algebras of Lie super-algebra and finite extensions of enveloping algebras of Lie
coloralgebras etc [10, 26].

Separable extensions are defined firstly by Hirata and Sugano in [12] as a generalization
of separable algebras, and they made a thorough study of these connection with Galois
theory for noncommutative rings and generalizations of Azumaya algebras. Separable
extensions are closely related to Frobenius extensions. In fact, Frobenius extensions is
a generalization of Frobenius algebras, while a separable algebra is a Frobenius algebra
if it is finite projective over the base ring. A ring extension that is both a separable
extension and a Frobenius extension is called a separable Frobenius extension. Sugano
proved that the central projective separable extensions are Frobenius extensions in [27].
More examples of separable Frobenius extensions can be found in Example 2.4. We refer
to a lecture due to Kadison [16] for more details.

Recall from [7] that an $A$-module $G$ is called Gorenstein projective if there is an exact
sequence of projective $A$-modules

$$
P := \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots
$$

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in Mod-$A$ which is still exact after applying the functor $\text{Hom}_A(\cdot, Q)$ for any projective $A$-module $Q$ and $G = \text{Ker}(P_0 \to P^0)$. In this case, the sequence $P$ is called a complete projective resolution of $G$. It is well-known that the notion of Gorenstein projective modules is a natural extension of totally reflexive modules to unnecessarily finitely generated modules (See [7]). The subcategory of all Gorenstein projective $A$-modules (resp. all finitely generated Gorenstein projective $A$-modules) is denoted by $\mathcal{GP}(A)$ (resp. $\mathcal{FGP}(A)$). The Gorenstein projective dimension of an $A$-module $M$, denoted by $\text{Gpd}(M)$, is defined as $\text{Gpd}(M) = \inf \{ n \mid \exists \text{ an exact sequence } 0 \to G_n \to \cdots G_1 \to G_0 \to M \to 0 \text{ with } G_i \text{ Gorenstein projective} \}$. We set $\text{Gpd}(M) = \infty$ if there is no such a Gorenstein projective resolution. As a hot topic of Gorenstein homological algebra, Gorenstein projectivity of modules and Gorenstein projective dimension of modules are studied widely (see [5, 13, 19, 24]).

The first motivation of this paper is inspired by the question in [4]. In [4], Chen introduced a generalization of Frobenius extensions, called the totally reflexive extension, and he proved that totally reflexivity of modules is preserved under this kind of ring extension. At the end of [4], Chen pointed out a question: is this true for Gorenstein projectivity in general?

In [25], Ren gave a partial answer for the above question. He proved that for a Frobenius extension, if a module over an extension ring is Gorenstein projective then it is also Gorenstein projective as a module over the base ring. Furthermore, the converse holds if the ring extension is a separable extension at the same time (see [25, Theorem A]). The first main result in this paper extends the result in [25]. See Theorem 3.2.

**Theorem A.** Let $A/S$ be a Frobenius extension of rings and $M$ a right $A$-module. Then $M_A$ is a Gorenstein projective module if and only if $M_S$ is a Gorenstein projective module.

Furthermore, we prove the following statements.

**Theorem B.** Let $A$ be an algebra over a commutative Artin ring $S$. If $A/S$ is a separable Frobenius extension, then we have

1. $A$ is CM-finite if and only if so is $S$;
2. $A$ is CM-free if and only if so is $S$.

It is well-known that determining the representation type of an algebra is fundamental and important in representation theory of Artin algebra. Auslander proved that there exists a 1-1 correspondence between the Morita equivalent classes of Artin algebras of finite representation type and that of Artin algebras with global dimension at most 2 and with dominant dimension at least 2 (see [1]). Motivated by this correspondence, Auslander introduced the notion of the representation dimension of Artin algebras. He proved that an Artin algebra is of finite representation type if and only if its representation dimension is at most 2. In this sense, the representation dimension of an Artin algebra is regarded as a trial to give a reasonable way of measuring homologically how far an Artin algebra is from being of finite representation type. Guo proved in [11] that the representation dimension of an Artin algebra is invariant under stable equivalences.

The second motivation of this paper comes from the results in [14]. In [14], Huang and Sun proved that the representation dimension of Artin algebras is preserved under excellent extensions. It follows from [14, Lemma 4.7] that an excellent extension is a Frobenius extension. And an excellent extension is a separable Frobenius extension if the
extension ring is commutative. But a separable Frobenius extension is not an excellent extension in general, see Example [2,4] for more details. As a separable Frobenius extension version of Theorem 4.8 in [14], we prove the following

**Theorem C.** Let $S$ be a commutative Artin ring and $A$ an $S$-algebra. If $A$ is a separable Frobenius extension of $S$, then $\text{rep.dim}(A) = \text{rep.dim}(S)$.

The paper is organized as follows. In Section 2, we give some notations in our terminology and some preliminary results which are used in this paper. In Section 3, we prove the main result: Theorem A and B, see Theorem [3,2] and [3,7] respectively. In Section 4, the representation dimension under separable Frobenius extension of Artin algebras are studied, the main result Theorem C is proved.

Throughout this paper, all rings are associative rings with identity and all modules are unital right modules unless stated otherwise. Let $A$ be a ring. We denote the category of all right $A$-module (resp. finitely generated right $A$-module) by $\text{Mod-}A$ (resp. $\text{mod-}A$).

2. **Preliminaries**

A ring extension $A/S$ is a ring homomorphism $S \xrightarrow{i} A$. A ring extension is an algebra if $S$ is commutative and $i$ factor as the composition $S \rightarrow Z(A) \hookrightarrow A$ where $Z(A)$ is the center of $A$. The natural bimodule structure of $SA_S$ is given by $s \cdot a \cdot s' := l(s) \cdot a \cdot l(s')$. Similarly, we can get some other module structures, for example $A_S$, $S_A$, and $AAS$, etc.

For any ring extension $A/S$, there is a restriction functor $R : \text{Mod-}A \rightarrow \text{Mod-S}$ which sends $M_A \mapsto M_S$, given by $m \cdot s := m \cdot l(s)$. In the opposite direction, there are two natural functors as follows:

1. $T = - \otimes_S A_A : \text{Mod-S} \rightarrow \text{Mod-}A$ is given by $M_S \mapsto M \otimes_S A_A$.
2. $H = \text{Hom}_S(AAS, -) : \text{Mod-S} \rightarrow \text{Mod-A}$ is given by $M_S \mapsto \text{Hom}_S(AAS, M_S)$.

It is easy to check that both $(T, R)$ and $(R, H)$ are adjoint pairs.

**Definition 2.1.** ([16] Theorem 1.2) A ring extension $A/S$ is a Frobenius extension, provided that one of the following equivalent conditions holds:

1. The functors $T$ and $H$ are naturally equivalent.
2. $SA_A \cong \text{Hom}_S(AAS, SS_S)$ and $AS$ is finitely generated projective.
3. $SA_A \cong \text{Hom}_S(SA_A, SS_S)$ and $SA$ is finitely generated projective.
4. There exists $E \in \text{Hom}_{SS_S}(A, S)$, $x_i, y_i \in A$ such that $\forall a \in A$, one has $\sum x_i E(y_i, a) = a$ and $\sum E(ax_i) y_i = a$.

**Definition 2.2.** A ring extension $A/S$ is a separable extension if and only if

$$\mu : A \otimes_S A \rightarrow A, \quad a \otimes b \mapsto ab,$$

is a split epimorphism of $A$-$A$-bimodules. If a ring extension $A/S$ is both a Frobenius extension and a separable extension, then it is called a separable Frobenius extension.

Let $A/S$ be a ring extension and $M \in \text{Mod-A}$. Then $M_S$ is a right $S$-module. There is a natural surjective map $\pi : M \otimes_S A \rightarrow M$ given by $m \otimes a \mapsto ma$ for any $m \in M$ and $a \in A$. It is easy to check that $\pi$ is split as a homomorphism of $S$-modules, we denoted by $M_S | M \otimes_S A_S$ that $M$ is a direct summand of $M \otimes_S A$ as right $S$-modules. However, $\pi$ is not split as an $A$-homomorphism in general. The following lemma comes from [25], which is analogous to the results in [23] for separable algebras over commutative rings.
Lemma 2.3. ([15] Lemma 2.9) Let $A/S$ be a ring extension. The following are equivalent:

1. $A/S$ is a separable extension.
2. For any $A$-$A$-bimodule $M$, $M \otimes_S A \to M$ is a split epimorphism of $A$-$A$-bimodules.
3. There exists an element $e \in A \otimes_S A$, such that $\mu(e) = 1_A$ and $ae = ea$ for any $a \in A$.

Example 2.4. (1) ([15] Example 2.10) For any finite group $G$, $ZG/\mathbb{Z}$ is a separable Frobenius extension. Indeed, let $e = \frac{1}{|G|} \sum_{g \in G} g \otimes \mathbb{Z}g^{-1} \in ZG \otimes \mathbb{Z}ZG$, where $|G|$ is the order of $G$. Then $e$ satisfies the condition (3) of the above lemma.

(2) ([16] Example 2.7 and 2.14) Let $K$ be a field and $A = M_4(K)$. Let $S$ be the subalgebra of $A$ with $K$-basis consisting of the idempotents and matrix units,

$$e_1 = e_{11} + e_{14}, e_2 = e_{22} + e_{33}, e_{21}, e_{31}, e_{41}, e_{42}, e_{43}.$$ Then $A/S$ is a separable Frobenius extension. But since $A_S$ is not free as a right $S$-module, $A/S$ is not an excellent extension.

3. Let $S$ be a commutative algebra and $A$ an Azumaya algebra over $S$. Then $A/S$ is a separable Frobenius extension. See [15] Chapter 5 for more details. We note that $A$ is not free over $S$, so $A/S$ is not an excellent extension in general.

(4) Every strongly separable extension (see [15] Definition 3.1) is a separable Frobenius extension. Specific examples of strongly separable extensions can be found in [15].

Let $C$ be a subcategory of mod-$A$ and $M \in \text{mod}-A$. A homomorphism $f : C \to M$ in mod-$A$ is called a right $C$-approximation of $M$ if $C \in C$ and the sequence $\text{Hom}_A(-, C) \xrightarrow{\sim, f} \text{Hom}_A(-, M) \to 0$ is exact in $C$. We say that an exact sequence $0 \to C_n \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} M \to 0$ in mod-$A$ is an $n$-$C$-resolution of $M$ if $C_i \in C$ for any $0 \leq i \leq n$, and the sequence

$$0 \to \text{Hom}_A(-, C_n) \xrightarrow{\sim, f_n} \text{Hom}_A(-, C_{n-1}) \xrightarrow{\sim, f_{n-1}} \cdots \xrightarrow{\sim, f_1} \text{Hom}_A(-, C_0) \xrightarrow{\sim, f_0} \text{Hom}_A(-, M) \to 0$$

is exact in $C$ (See [2]).

We denote by $\text{add}M$ the full subcategory of mod-$A$ consisting of all modules isomorphic to direct summands of finite direct sums of copies of $M$, and denoted by $\text{Gen}M$ the full subcategory of mod-$A$ consisting of all modules $X$ such that there exists an epimorphism $M_0 \to X$ with $M_0 \in \text{add}M$. The following lemma comes from [14].

Lemma 2.5. ([14] Lemma 4.2) Let $A$ be an Artin algebra and $M, X \in \text{mod}-A$. If $X = X_1 \oplus X_2 \in \text{Gen}M$ has an $n$-$\text{add}M$-resolution:

$$0 \to M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} M_0 \xrightarrow{f_0} X \to 0,$$

then $X_1$ has an $n$-$\text{add}M$-resolution:

$$0 \to M_n' \xrightarrow{f_n'} M_{n-1}' \xrightarrow{f_{n-1}'} \cdots \xrightarrow{f_1'} M_0' \xrightarrow{f_0'} X_1 \to 0.$$
Let $A$ be an Artin algebra and $M \in \text{mod-}A$. Recall that $M$ is called a generator-cogenerator for $\text{mod-}A$ if every indecomposable projective module and also every indecomposable injective module in $\text{mod-}A$ are in $\text{add}M$. The following lemma comes from [9].

**Lemma 2.6.** ([9] Lemma 2.1) Let $A$ be an Artin algebra and $M$ a generator-cogenerator for $\text{mod-}A$. Then the following statements are equivalent for any $n \geq 3$.

1. Any indecomposable module in $\text{mod-}A$ has an $(n-2)$-$\text{add}M$-resolution;
2. $\text{gl.dimEnd}(M) \leq n$.

### 3. Gorenstein projective dimensions under Frobenius extensions

In this section, we will prove that for a Frobenius extension, a module over the extension ring is Gorenstein projective if and only if its underlying module over the base ring is Gorenstein projective. Moreover, we obtain some homological properties, including the Gorenstein global dimension of rings and the CM-finiteness and CM-freeness of Artin algebras, are invariant under separable Frobenius extensions.

For a ring (or an algebra) $A$, we denote by $\mathcal{P}(A)$ the full subcategory of $\text{Mod-}A$ consisting of all projective right $A$-modules. For a right $A$-module $M$, we denote the projective dimension of $M$ by $\text{pd}(M)$. In order to prove that the Gorenstein projectivity of modules is preserved under Frobenius extensions, we need the following lemma.

**Lemma 3.1.** ([25] Lemma 2.3) Let $A/S$ be a Frobenius extension of rings and $M$ a right $A$-module. If the underlying module $M_S$ is Gorenstein projective, then the following statements hold.

1. For any projective right $A$-module $P$ and any $i \geq 1$, $\text{Ext}_A^i(M, P) = 0$;
2. $M \otimes_S A_A \cong \text{Hom}_S(A_A, M)$ is a Gorenstein projective right $A$-module.

**Theorem 3.2.** Let $A/S$ be a Frobenius extension of rings and $M$ a right $A$-module. Then $M_A$ is a Gorenstein projective module if and only if $M_S$ is a Gorenstein projective module.

**Proof.** The necessity holds by Lemma 2.2 in [25]. For the sake of completeness, we give the proof as follows. Assume that $M_A$ is Gorenstein projective as a right $A$-module. There exists a complete projective resolution $P := \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ in $\text{mod-}A$ such that $\text{Hom}_A(P, Q)$ is also exact for any projective $A$-module $Q$ and $M = \text{Im}(P_0 \to P^0)$. By the assumption, $A_S$ is finitely generated projective as a right $S$-module. Hence $(P_i)_S \cong P_i \otimes_A A_S$ is projective as an $S$-module for any $i$. For any projective $S$-module $L$, we have $\text{Hom}_S(A_A, L) \cong L \otimes_S A_A$ is also projective as a right $A$-module. Hence $\text{Hom}_S(P, L) \cong \text{Hom}_S(P \otimes_A A_S, L) \cong \text{Hom}_A(P, \text{Hom}_S(A_A, L))$ is exact. So $P \cong P \otimes_A A_S$ in $\text{mod-}S$ is a complete projective resolution of $M_S$ and therefore $M_S$ is a Gorenstein projective module.

Conversely, if $M_S$ is Gorenstein projective as a right $S$-module. Then we have $\text{Ext}_A^i(M, P) = 0$ for any projective $A$-module $P$ and any $i \geq 1$ and $M \otimes_S A_A \cong \text{Hom}_S(A_A, M)$ are Gorenstein projective as right $A$-modules by Lemma 3.1. We need to construct a complete projective resolution of $M_A$ in $\text{mod-}A$. 

Since \( M \otimes_S A_A \) is Gorenstein projective, there exists a short exact sequence \( 0 \rightarrow M \otimes_S A_A \xrightarrow{f} P_0 \rightarrow G \rightarrow 0 \) in Mod-\( A \) such that \( P_0 \in \mathcal{P}(A) \) and \( G \in G\mathcal{P}(A) \). Applying the restriction functor, we get the exact sequence \( 0 \rightarrow M \otimes_S A_S \xrightarrow{f} P_0 \rightarrow G \rightarrow 0 \) in Mod-\( S \) such that \( P_0 \in \mathcal{P}(S) \) and \( G, M \otimes_S A \in G\mathcal{P}(S) \) by the necessity. Let \( Q \) be any projective right \( S \)-module, and \( g : M \otimes_S A_S \rightarrow Q \) any \( S \)-homomorphism.

\[
\begin{tikzpicture}
    \node (A) at (0,0) {$M \otimes_S A$};
    \node (B) at (3,0) {$P_0$};
    \node (C) at (6,0) {$G$};
    \node (D) at (9,0) {$0$};
    \node (E) at (0,3) {$Q$};
    \draw[->] (A) -- (B) node[pos=0.5,above] {$f$};
    \draw[->] (B) -- (C) node[pos=0.5,above] {$g$};
    \draw[->] (E) -- (A) node[pos=0.5,left] {$h$};
\end{tikzpicture}
\]

There exists an \( S \)-homomorphism \( h : P_0 \rightarrow Q \) such that \( hf = g \) since \( G \) is Gorenstein projective as an \( S \)-module.

Note that there are two maps \( i : M \rightarrow \text{Hom}_S(A, M) \) given by \( i(m)(a) = ma \) and \( \pi : M \otimes_S A \rightarrow M \) given by \( \pi(m \otimes a) = ma \) for any \( m \in M \) and \( a \in A \). Since \( M \otimes_S A_A \cong \text{Hom}_S(A, A, M) \), there is an \( A \)-homomorphism, which still denoted it by \( i, i : M \rightarrow M \otimes_S A \) and it is split as a homomorphism of \( S \)-modules. It follows that we get an exact sequence \( 0 \rightarrow M_A \xrightarrow{f_i} P_0 \rightarrow G_0 \rightarrow 0 \) in Mod-\( A \) where \( G_0 = \text{Coker}(fi) \). Applying the restriction functor, we have an exact sequence \( 0 \rightarrow M_S \xrightarrow{f_i} P_0 \rightarrow G_0 \rightarrow 0 \) in Mod-\( S \) such that \( M_S \in G\mathcal{P}(S) \) (since \( M_S \mid M \otimes_S A_S \) and \( M \otimes_S A_S \) is Gorenstein projective) and \( P_0 \in \mathcal{P}(S) \). We claim that \( G_0 \in G\mathcal{P}(S) \).

Let \( Q \) be any projective \( S \)-module and \( \alpha : M \rightarrow Q \) any \( S \)-homomorphism. Then \( \alpha \pi \) is an \( S \)-homomorphism from \( M \otimes_S A \) to \( Q \). Hence, there exists an \( S \)-homomorphism \( \beta : P_0 \rightarrow Q \) such that \( \beta f = \alpha \pi \). And so \( \beta(fi) = \alpha(\pi i) = \alpha \).

\[
\begin{tikzpicture}
    \node (A) at (0,0) {$M \otimes_S A$};
    \node (B) at (3,0) {$P_0$};
    \node (C) at (6,0) {$G_0$};
    \node (D) at (9,0) {$0$};
    \draw[->] (A) -- (B) node[pos=0.5,above] {$f$};
    \draw[->] (B) -- (C) node[pos=0.5,above] {$\pi$};
    \draw[->] (C) -- (D) node[pos=0.5,above] {$\alpha$};
    \draw[->] (A) -- (D) node[pos=0.5,left] {$g\beta$};
\end{tikzpicture}
\]

It follows from the exact sequence
\[
0 \rightarrow \text{Hom}_S(G_0, Q) \rightarrow \text{Hom}_S(P_0, Q) \rightarrow \text{Hom}_S(M, Q) \rightarrow \text{Ext}^1_S(G_0, Q) \rightarrow 0
\]
that \( \text{Ext}^1_S(G_0, Q) = 0 \). Hence \( G_0 \) is Gorenstein projective as an \( S \)-module by [13] Corollary 2.11.

Note that \( G_0 \) is Gorenstein projective as a right \( S \)-module, then \( \text{Ext}^i_A(G_0, P) = 0 \) for any projective \( A \)-module \( P \) and any \( i \geq 1 \) and \( G_0 \otimes_S A_A \cong \text{Hom}_S(A, A, G_0) \) is Gorenstein projective as a right \( A \)-module by Lemma [13.1]. Repeating this process, we get an exact sequence \( Q : 0 \rightarrow M \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \cdots \) in Mod-\( A \) with \( P_i \) projective and \( \text{Hom}_A(Q, P) \) is also exact for any projective right \( A \)-module \( P \). Linking up the projective resolution of \( M_A \) and \( Q \), we get a complete projective resolution of \( M_A \). The proof is completed. \( \square \)
A functor $F : C \to D$ is called a Frobenius functor if there exists a functor $G : D \to C$ such that both $(F, G)$ and $(G, F)$ are adjoint pairs (see [20]). By Definition 2.1, we know that the functors $- \otimes_S A_A \cong \text{Hom}_S(A_A S, -)$ induced by the Frobenius bimodule (see [10] Definition 2.1) $A_A S$ are Frobenius functors.

Proposition 3.3. Let $A/S$ be a Frobenius extension of rings. Then the Frobenius bimodule $A_A S$ induces a Frobenius functor from $\mathcal{GP}(A)$ to $\mathcal{GP}(S)$.

Proof. Put that $T = - \otimes_S A_A : \text{Mod-}S \to \text{Mod-}A$ given by $M_S \mapsto M \otimes_S A_A$. Then $T \cong H = \text{Hom}_S(A_A S, -)$ is a Frobenius functor with the restriction functor $R$ as a left and right adjoint at the same time.

By Lemma 3.1 and Theorem 3.2 we get $T |_{\mathcal{GP}(S)} \subseteq \mathcal{GP}(A)$ and $R |_{\mathcal{GP}(A)} \subseteq \mathcal{GP}(S)$, respectively. Hence $T$ is a Frobenius functor from $\mathcal{GP}(A)$ to $\mathcal{GP}(S)$. □

Proposition 3.4. Let $A/S$ be a Frobenius extension of rings and $M$ a right $A$-module. Then $\text{Gpd}(M_S) \leq \text{Gpd}(M_A)$. Moreover, if $A/S$ is separable, then $\text{Gpd}(M_S) = \text{Gpd}(M_A)$.

Proof. Without loss of the generality, we assume that $\text{Gpd}(M_A) = n < \infty$. There is a Gorenstein projective resolution of $M_A$ as follows

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

in $\text{Mod-}A$ with $G_i$ Gorenstein projective for any $0 \leq i \leq n$. Applying the restriction functor, we get the following exact sequence

$$0 \to (G_n)_S \to (G_{n-1})_S \to \cdots \to (G_1)_S \to (G_0)_S \to M_S \to 0$$

in $\text{Mod-}S$ with $(G_i)_S \cong G_i \otimes_A A_S$ Gorenstein projective as a right $S$-module for $0 \leq i \leq n$ by Theorem 3.2. Hence $\text{Gpd}(M_S) \leq n = \text{Gpd}(M_A)$.

Furthermore, if $A/S$ is separable, then $M_A | M \otimes_S A_A$ by Lemma 2.3. Without loss of the generality, we assume that $\text{Gpd}(M_S) = m \leq \infty$. There exists a Gorenstein projective resolution of $M_S$ as follows

$$0 \to L_m \to L_{m-1} \to \cdots \to L_1 \to L_0 \to M \to 0$$

in $\text{Mod-}S$ such that $L_i$ is Gorenstein projective as a right $S$-module for any $0 \leq i \leq m$. Applying the functor $- \otimes_S A_A$, we get the following exact sequence

$$0 \to L_m \otimes_S A_A \to L_{m-1} \otimes_S A_A \to \cdots \to L_1 \otimes_S A_A \to L_0 \otimes_S A_A \to M \otimes_S A_A \to 0$$

in $\text{Mod-}A$ and $L_i \otimes_S A_A$ is Gorenstein projective as a right $A$-module for $0 \leq i \leq m$ by Lemma 3.1. And so $\text{Gpd}(M \otimes_S A_A) \leq m$. Hence, by the [13] Corollary 2.11, $\text{Gpd}(M_A) \leq m = \text{Gpd}(M_S)$. □

Recall that the right Gorenstein global dimension of a ring $A$, denoted by $r.\text{Ggl.dim}(A)$, defined as $r.\text{Ggl.dim}(A) = \sup\{\text{Gpd}(M_A) | M \text{ is a right } A\text{-module}\}$ (see [6]). It follows from Proposition 3.4 that the right Gorenstein global dimension of rings is preserved under separable Frobenius extensions.

Theorem 3.5. Let $A/S$ be a separable Frobenius extension of rings. Then $r.\text{Ggl.dim}(A) = r.\text{Ggl.dim}(S)$. 

Proof. It is easy to see \( r.\text{Gldim}(A) \leq r.\text{Gldim}(S) \) by Proposition 3.4. Let \( M_S \) be any \( S \)-module. Also by Proposition 3.4, \( \text{Gpd}(M \otimes_S A_S) \leq \text{Gpd}(M \otimes_S A_A) \). Since \( M_S | M \otimes_S A_S, \text{Gpd}(M_S) \leq \text{Gpd}(M \otimes_S A_A) \) by [13] Proposition 2.19. It follows that \( r.\text{Gldim}(S) \leq r.\text{Gldim}(A) \). \( \square \)

Recall from [3] that an Artin algebra \( A \) is Cohen-Macaulay finite, or simply, CM-finite, if \( A \) has only finitely many isomorphism classes of indecomposable finitely generated Gorenstein projective modules in \( \text{mod-}A \). It is easy to see that \( A \) is CM-finite if and only if there exists a module \( G \in \text{mod-}A \) such that \( \mathcal{FGP}(A) = \text{add}G \). Clearly, \( A \) is CM-finite if \( A \) is of finite representation type. Some other examples of CM-finite algebra can be found in [24]. An Artin algebra \( A \) is called Cohen-Macaulay free, or simply, CM-free, if any Gorenstein projective module in \( \text{mod-}A \) is projective. It is well-known that \( \mathcal{GP}(A) = \mathcal{P}(A) \) if \( \text{gl.dim}(A) < \infty \), and \( A \) is CM-free if \( \text{gl.dim}(A) < \infty \).

In order to prove that the CM-finiteness and CM-freeness of Artin algebras are preserved under separable Frobenius extensions, we need the following easy observation, which is maybe known.

Lemma 3.6. Let \( A \) be a separable Frobenius extension over a commutative Artin ring \( S \) and \( M \) a right \( A \)-module. Then we have

(1) \( \text{pd}(M_A) = \text{pd}(M_S) \);
(2) \( r.\text{gl.dim}(A) = r.\text{gl.dim}(S) \).

Proof. (1) Without loss of generality, we assume that \( \text{pd}(M_A) = n < \infty \) and \( N_S \) is any right \( S \)-module. Then \( \text{Ext}^{n+i}_S(M, N) \cong \text{Ext}^{n+i}_S(M \otimes_A A_S, N) \cong \text{Ext}^{n+i}_A(M, \text{Hom}_S(A_A, N)) = 0 \) for any \( i \geq 1 \). Hence \( \text{pd}(M_S) \leq n = \text{pd}(M_A) \). Conversely, we can assume that \( \text{pd}(M_S) = m < \infty \) and \( W_A \) is any right \( A \)-module. Since \( A/S \) is a separable extension, \( M_A | (M \otimes S A_A) \) by Lemma 2.3. It follows from \( \text{Ext}^{n+i}_A(M \otimes S A_A, W) \cong \text{Ext}^{n+i}_S(M_S, \text{Hom}_A(S A_A, W)) = 0 \) that \( \text{Ext}^{n+i}_A(M, W) = 0 \) for any \( i \geq 1 \). Thus \( \text{pd}(M_A) \leq m = \text{pd}(M_S) \).

(2) It follows from (1) that \( r.\text{gl.dim}(A) \leq r.\text{gl.dim}(S) \). Conversely, let \( X_S \) be any right \( S \)-module. Then \( X_S | X \otimes S A_S \). Since \( \text{pd}(X \otimes S A_A) = \text{pd}(X \otimes S A_A) \) by (1), \( \text{pd}(X_S) \leq \text{pd}(X \otimes S A_A) \). It follows that \( r.\text{gl.dim}(S) \leq r.\text{gl.dim}(A) \). \( \square \)

Theorem 3.7. Let \( A \) be an algebra over a commutative Artin ring \( S \). If \( A/S \) is a separable Frobenius extension, then we have

(1) \( A \) is CM-finite if and only if so is \( S \);
(2) \( A \) is CM-free if and only if so is \( S \).

Proof. (1) Assume that \( S \) is CM-finite, then there exists a module \( G \in \text{mod-}S \) such that \( \mathcal{FGP}(S) = \text{add}G \). We claim that \( \mathcal{FGP}(A) = \text{add}(G \otimes S A_A) \). By Lemma 3.1 \((G \otimes S A_A)_A\) is Gorenstein projective as a right \( A \)-module, and so \( \text{add}(G \otimes S A_A) \subseteq \mathcal{FGP}(A) \). Let \( M_A \) be any indecomposable Gorenstein projective \( A \)-module in \( \text{mod-}A \). By Proposition 3.2 \( M_S \) is also Gorenstein projective as a right \( S \)-module. There exists a positive integer \( n \) such that \( M_S | (G \otimes S A_A)^n \). Since \( M_A | M \otimes S A_A \) by Lemma 2.3 \( M_A | (G \otimes S A_A)^n \). Thus \( \mathcal{FGP}(A) \subseteq \text{add}(G \otimes S A_A) \).

Conversely, if \( A \) is CM-finite, then there exists a module \( W \in \text{mod-}A \) such that \( \mathcal{FGP}(A) = \text{add}W \). It suffices to prove that \( \mathcal{FGP}(S) = \text{add}W_S \). By Proposition 3.2 \( W_S \) is Gorenstein
projective as a right $S$-module. Hence $\text{add} W_S \subseteq \Phi GP(S)$. Let $X_S$ be any indecomposable Gorenstein projective $S$-module in $\text{mod-}S$. By Lemma 3.1, $X \otimes_S A$ is also Gorenstein projective as a right $A$-module, and so $X \otimes_S A \mid W^n_A$ for some positive integer $n$. Applying the restriction functor, we get $X \otimes_S A_S \mid W^n_S$. It follows from $X_S \mid X \otimes_S A_S$ that $X_S \mid W^n_S$. Hence $\Phi GP(S) \subseteq \text{add} T_S$.

(2) Assume that $S$ is CM-free and $G_A$ is any Gorenstein projective right $A$-module. By Proposition 3.2, $G_S$ is Gorenstein projective as a right $S$-module. Hence $G_S$ is projective by assumption. It follows from Lemma 3.6(1) that $G_A$ is also projective as a right $A$-module. Therefore $A$ is CM-free.

Conversely, if $A$ is CM-free and $Q_S$ is any Gorenstein projective right $S$-module. By Lemma 3.1, $Q \otimes_S A_A$ is Gorenstein projective as a right $A$-module. Hence $Q \otimes_S A_A$ is projective by assumption. It follows from Lemma 3.6(1) that $Q \otimes_S A_S$ is projective as a right $S$-module. Hence $Q_S$ is a projective right $S$-module as a direct summand of $Q \otimes_S A_S$. So $S$ is CM-free. □

4. Representation dimensions under separable Frobenius extensions

In this section, we will show that the representation dimension of Artin algebras is invariant under separable Frobenius extensions.

Recall that a module $M \in \text{mod-}A$ is called an additive generator for $\text{mod-}A$ if any indecomposable module in $\text{mod-}A$ is in $\text{add} M$. Obviously, an Artin algebra $A$ is of finite representation type if and only if $A$ has an additive generator.

**Proposition 4.1.** Let $A$ be an algebra over a commutative Artin ring $S$. If $A/S$ is a separable Frobenius extension, then $A$ is of finite representation type if and only if so is $S$.

**Proof.** We assume that $A$ is of finite representation type. Then there is an additive generator $M_A$ for $\text{mod-}A$. We claim that $M_S \cong M \otimes_A A_S$ is an additive generator for $\text{mod-}S$. Let $X_S \in \text{mod-}S$ be any indecomposable right $S$-module. Then $X \otimes_S A_A \mid M^n_A$ for some positive integer $n$. Applying the restriction functor, we get $X \otimes_S A_S \mid M^n_S$. It follows from $X_S \mid X \otimes_S A_S$ that $X_S \mid M^n_S$. Thus, $M_S$ is an additive generator for $\text{mod-}S$.

Conversely, if $S$ is of finite representation type and $G_S$ is an additive generator for $\text{mod-}S$. We claim that $G \otimes_S A_A$ is an additive generator for $\text{mod-}A$. Let $N_A$ be any indecomposable right $A$-module in $\text{mod-}A$. Then $N_S \cong N \otimes_A A_S \mid G^n_S$ for some positive integer $m$. And we get $N \otimes_S A_A \mid (G \otimes_S A_A)^n_A$. It follows from $N_A \mid N \otimes_S A_A$ that $N_A \mid (G \otimes_S A_A)^n_A$. Therefore $G \otimes_S A_A$ is an additive generator for $\text{mod-}A$. □

As a homological dimension of measuring homologically how far an Artin algebra is from being of finite representation type, the representation dimension of an Artin algebra $A$, denoted by $\text{rep.dim}(A)$, was defined as follows in [1].

**Definition 4.2.** Let $A$ be an Artin algebra. The representation dimension $\text{rep.dim}(A)$ of $A$ is defined as $\inf \{ \text{gl.dimEnd}(M_A) \mid M$ is a generator-cogenerator for $\text{mod-}A \}$ if $A$ is non-semisimple; and $\text{rep.dim}(A) = 1$ if $A$ is semisimple.

Let $A$, $S$ be Artin algebras and $A/S$ a separable Frobenius extension. By Lemma 3.1(2), $\text{rep.dim}(A) = 1$ if and only if $\text{rep.dim}(S) = 1$. And it follows from Proposition 4.1.
that \( \text{rep.dim}(A) \leq 2 \) if and only if \( \text{rep.dim}(S) \leq 2 \). In general, we have the following main result of this section.

**Theorem 4.3.** Let \( S \) be a commutative Artin ring and \( A \) an \( S \)-algebra. If \( A \) is a separable Frobenius extension of \( S \), then \( \text{rep.dim}(A) = \text{rep.dim}(S) \).

**Proof.** By the definition of Frobenius extensions, \( A \) is an Artin algebra. Then the assertion holds true provided either \( \text{rep.dim}(A) \) or \( \text{rep.dim}(S) \) is at most 2 by Lemma 3.6(2) and Proposition 4.1.

Now assume that \( \text{rep.dim}(A) = n(\geq 3) \) and \( M_A \) is a generator-cogenerator for \( \text{mod-}A \) such that \( \text{gl.dim}(M_A) = n \). Let \( X \in \text{mod-}S \) be indecomposable. It follows from \( X_S \mid X \otimes_S A_S \) that it is easy to see \( M_S \) is a generator-cogenerator for \( \text{mod-}S \). Since \( X \otimes_S A \in \text{mod-}A \), by Lemma 2.6 we have the following exact sequence

\[
0 \to M_{n-2} \to M_{n-3} \to \cdots \to M_0 \to X \otimes_S A \to 0
\]

in \( \text{mod-}A \) (and hence in \( \text{mod-}S \)) with \( M_i \in \text{add}M_A \) such that

\[
0 \to \text{Hom}_A(M, M_{n-2}) \to \text{Hom}_A(M, M_{n-3}) \to \cdots
\]

\[
\to \text{Hom}_A(M, M_0) \to \text{Hom}_A(M, X \otimes_S A) \to 0
\]

is also exact. By assumption, \( A/S \) is a Frobenius extension, \( S \) is a Frobenius extension, \( A \) is a generator-cogenerator for \( \text{mod-}A \). Then we get the following exact sequence

\[
0 \to \text{Hom}_A(M, M_{n-2}) \otimes_S A \to \text{Hom}_A(M, M_{n-2}) \otimes_S A \to \cdots
\]

\[
\to \text{Hom}_A(M, M_0) \otimes_S A \to \text{Hom}_A(M, X \otimes_S A) \otimes_S A \to 0.
\]

(\*)

Since \( \text{Hom}_A(S, A)(\cong - \otimes_A A_S), - \otimes_S A_A \) is an adjoint pair, for any \( N \in \text{mod-}A \) (also in \( \text{mod-}S \)) we get

\[
\text{Hom}_A(M, N) \otimes_S A \cong \text{Hom}_A(M, N \otimes_S A)
\]

\[
\cong \text{Hom}_S(\text{Hom}_A(A, M), N)
\]

\[
\cong \text{Hom}_S(M, N),
\]

where the first isomorphism comes from \( S \) Theorem 3.2.14] and the second isomorphism holds by the adjoint isomorphism. Hence from the exact sequence (\*), we get the following exact sequence

\[
0 \to \text{Hom}_S(M, M_{n-2}) \to \text{Hom}_S(M, M_{n-3}) \to \cdots
\]

\[
\to \text{Hom}_S(M, M_0) \to \text{Hom}_S(M, X \otimes_S A) \to 0.
\]

Thus \( X \otimes_S A \) as an \( S \)-module has an \( (n-2)\)-add\( M_S \)-resolution. Since \( X \mid (X \otimes_S A)_S \), \( X_S \) has an \( (n-2)\)-add\( M_S \)-resolution by Lemma 2.6 So \( \text{gl.dim}(M_S) \leq n \) by Lemma 2.6 and therefore \( \text{rep.dim}(S) \leq n \).

Conversely, assume that \( \text{rep.dim}(S) = m(\geq 3) \) and \( Q_S \) is a generator-cogenerator for \( \text{mod-}S \) such that \( \text{gl.dim}(Q_S) = m \). Since \( S \in \text{add}Q_S \), \( A_A \cong S \otimes_S A_A \in \text{add}(Q \otimes_S A)_A \).

It follows that \((Q \otimes_S A)_A \) is a generator for \( \text{mod-}A \). On the other hand, if \( Y \in \text{mod-}A \), then \( Y \) is also a right \( S \)-module. Hence there exists a positive integer \( t \) such that \( 0 \to Y_S \to Q_S^{(t)} \) is exact in \( \text{mod-}S \), and so \( 0 \to Y \otimes_S A \to (Q \otimes_S A)^{(t)} \) is exact in \( \text{mod-}A \). By the assumption, \( A/S \) is a separable Frobenius extension. So \( Y_A \mid (Y \otimes_S A)_A \), and hence \((Q \otimes_S A)_A \) is a cogenerator for \( \text{mod-}A \). Thus \((Q \otimes_S A)_A \) is a generator-cogenerator for \( \text{mod-}A \).
Let $V \in \text{mod-}A$ be indecomposable. By Lemma 2.6, $V_S$ has an $(m-2)$-add $Q_S$-resolution as an $S$-module

$$0 \rightarrow Q_{m-2} \rightarrow \cdots \rightarrow Q_0 \rightarrow V_S \rightarrow 0. $$

We claim that the following sequence

$$0 \rightarrow Q_{m-2} \otimes S A \rightarrow Q_{m-3} \otimes S A \rightarrow \cdots \rightarrow Q_0 \otimes S A \rightarrow V \otimes S A \rightarrow 0. $$

is an $(m-2)$-add $(Q \otimes S A)_A$-resolution of $(V \otimes S A)_A$.

It is easy to see that $Q_i \otimes S A \in \text{add}(Q \otimes S A)_A$. Let $K_i = \text{Ker} f_i$ for any $0 \leq i \leq m-2$ and $K_{-1} = V$. Because $S A$ is a finitely generated projective $S$-module, we have the following exact sequence

$$0 \rightarrow K_i \otimes S A \rightarrow Q_i \otimes S A \rightarrow K_{i-1} \otimes S A \rightarrow 0,$$

which is exact both as right $A$-modules and as right $S$-modules for any $0 \leq i \leq m-2$. So the sequence $(**)$ is exact in mod-$A$. On the other hand, we have the following sequence

$$0 \rightarrow \text{Hom}_S(Q, K_i) \rightarrow \text{Hom}_S(Q, Q_i) \rightarrow \text{Hom}_S(Q, K_{i-1}) \rightarrow 0$$

in mod-$S$, which induces the following exact sequence

$$0 \rightarrow \text{Hom}_S(Q, K_i) \otimes S A \rightarrow \text{Hom}_S(Q, Q_i) \otimes S A \rightarrow \text{Hom}_S(Q, K_{i-1}) \otimes S A \rightarrow 0$$

for any $0 \leq i \leq m-2$. By Lemma 3.2.4 in [3], $\text{Hom}_S(Q, L) \otimes S A \cong \text{Hom}_A(Q \otimes S A, L \otimes S A)$ for any $L \in \text{mod-}S$. Hence the sequence

$$0 \rightarrow \text{Hom}_A(Q \otimes S A, K_i \otimes S A) \rightarrow \text{Hom}_A(Q \otimes S A, Q_i \otimes S A) \rightarrow \text{Hom}_A(Q \otimes S A, K_{i-1} \otimes S A) \rightarrow 0$$

is also exact for any $0 \leq i \leq m-2$, which implies that the following sequence

$$0 \rightarrow \text{Hom}_A(Q \otimes S A, Q_{m-2} \otimes S A) \rightarrow \text{Hom}_A(Q \otimes S A, Q_{m-3} \otimes S A) \rightarrow \cdots$$

$$\text{Hom}_A(Q \otimes S A, Q_0 \otimes S A) \rightarrow \text{Hom}_A(Q \otimes S A, V \otimes S A) \rightarrow 0$$

is exact. The claim is proved.

Notice that $V_A \mid (V \otimes S A)_A$, so $V_A$ has an $(m-2)$-add $(Q \otimes S A)_A$-resolution by Lemma 2.6. Thus $\text{gl.dim End}(Q \otimes S A)_A \leq m$ by Lemma 2.6 and therefore $\text{rep.dim}(A) \leq m$. The proof is finished.

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