(2+1) Cosmology with a general scalar field

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Abstract. Einstein’s equations minimally coupled to a general scalar field are completely integrated for the cases of the two spatially homogeneous models in three dimensions. Then, Dirac’s method for canonical quantization is used in order to derive the corresponding Wheeler-DeWitt equations, which we try to solve for particular configurations.

1. Introduction
In 2+1 cosmology there exist two spatially homogeneous models, the Abelian and the non Abelian, the names coming from the algebras formed by the infinitesimal generators of the isometry groups of motions. In this paper, we will consider a scalar field \( \Phi \) being minimally coupled to the gravitational field and having an arbitrary self-interaction potential \( V(\Phi) \). At first, we shall treat the two models mentioned above at a classical level: we will derive the non-linear Klein-Gordon plus the Einstein’s equations and show that they can be fully integrated. Subsequently, we will continue with quantizing these cosmological models and deduce the Wheeler-DeWitt equation for each of the two cases. As far as the Abelian model is concerned, a solution will be presented for a potential of the form \( V(\Phi) = \kappa e^{\lambda \Phi} \).

2. Classical case
Since the scalar field is minimally coupled to gravity, we can write down the action

\[
S = \int (\mathcal{L}_g + \mathcal{L}_\Phi) \, d^3x
\]  

(1)
consisting of two parts: (a) the gravitation part \( \mathcal{L}_g = -\frac{1}{2} \sqrt{-g} R \) and (b) the matter part \( \mathcal{L}_\Phi = \frac{\sqrt{-g}}{2} (g^{\mu \nu} \partial_\mu \Phi \partial_\nu \Phi + V(\Phi)) \), where \( g_{\mu \nu} \) is the spacetime metric, \( g \) its determinant and \( R \) the Ricci scalar. The units used throughout this paper are \( 8\pi G = c = 1 \).

Varying the action with respect to the two fields \( g_{\mu \nu} \) and \( \Phi \), we get the following equations of motion:

\[
\mathcal{G}_{\mu \nu} = T_{\mu \nu} \quad (2)
\]

\[
\partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu \Phi) - \sqrt{-g} V'(\Phi) = 0 \quad (3)
\]

As we see, (2) is Einstein’s equation, with \( \mathcal{G}_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \) being Einstein’s tensor and

\[
T_{\mu \nu} = 2 \sqrt{-g} \delta S / \delta g_{\mu \nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu \nu} \left( g^{\alpha \lambda} \partial_\alpha \Phi \partial_\lambda \Phi + 2 V(\Phi) \right) \quad (4)
\]

the energy momentum tensor coming from the matter part of action (1). Since we will deal only with spatially homogeneous models, we assume in the following analysis the matter field \( \Phi \) to be a function of time only.

As we know, for spatially homogeneous manifolds [1], we can bring the line element in the form:

\[
ds^2 = [N^\alpha(t)N_\alpha(t) - N^2(t)]dt^2 + 2N_\alpha(t)\sigma^\alpha_i(x)dx^i dt + \gamma_{\alpha\beta}(t)\sigma^\alpha_i(x)\sigma^\beta_j(x)dx^i dx^j \quad (5)
\]

with \( N^\alpha(t) \) being the shift vector, \( N(t) \) the lapse function, \( \gamma_{\alpha\beta} \) the space metric and \( \sigma^\alpha_i(x) \) the duals of the invariant basis, which must obey the relation

\[
\sigma^\alpha_i,\beta(x) - \sigma^\alpha_j,\beta(x) = C^\alpha_{\beta\gamma} \sigma^\gamma_i(x) (6)
\]

where \( C^\alpha_{\beta\gamma} \) are the structure constants of the Lie algebra that is satisfied by the infinitesimal generators of the isometry group of motions. As far as the indices are concerned, Greek letters count the different one forms, while Latin letters are world indices (both range from 1 to 2).

The advantage of writing down the line element as in (5), is that the space metric \( \gamma_{\alpha\beta} \) depends only on \( t \).

It has been shown in the literature [2], that by the use of coordinate transformations which do not “destroy” the manifest homogeneity of space, we can “gauge” transform the different parts of line element (5) in the following manner:

\[
\tilde{\gamma}_{\alpha\beta}(t) = \Lambda^\alpha_{\beta}(t) N^\alpha(t) \gamma_{\alpha\beta}(t) \quad (7)
\]

\[
\tilde{N}(t) = N(t) \quad (8)
\]

\[
\tilde{N}_\alpha(t) = \Lambda^\alpha_\beta(t) [N_\beta(t) + P^\alpha(t) \gamma_{\alpha\beta}(t)] \quad (9)
\]

where \( \Lambda^\alpha_{\beta}(t) \) and \( P^\alpha(t) \) are functions of time and satisfy

\[
\Lambda^\alpha_{\beta}(t) C^\mu_{\beta\gamma} = \Lambda^\rho_{\beta}(t) \Lambda^\alpha_\rho(t) C^\mu_{\rho\gamma} \quad (10)
\]

\[
2 P^\alpha(t) C^\alpha_{\beta\gamma} \Lambda^\beta_\gamma(t) = \dot{\Lambda}^\alpha_\beta(t) \quad (11)
\]

(the dot means differentiation with respect to time).

With the help of the above transformations, and the freedom to arbitrary redefine time, we can:
Choose the lapse function to be \( N(t) = \sqrt{\gamma} \), \( \gamma \) being the determinant of the space metric. A choice that simplifies both equations of motion (2), (3).

Set the shift vector to zero \( N^\alpha(t) = 0 \) by the right choice of \( P^\rho(t) \).

Diagonalise the space metric with the help of matrices \( \Lambda^\mu_\nu(t) \)

\[
\gamma_{\mu\nu} = \begin{pmatrix} a^2(t) & 0 \\ 0 & b^2(t) \end{pmatrix}
\]  

In two dimensions there exist two algebras, one is the abelian and the other the non abelian. In the first, all structure constants are zero, while in the latter all are zero but one. We will proceed now to examining the induced models by these two algebras separately.

2.1. Abelian model \( (C^\alpha_{\beta\gamma} = 0) \)

It is easy to check that in this case, the invariant one forms are

\[
\sigma^\alpha_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

From (5) we derive the metric

\[
g_{\mu\nu} = \begin{pmatrix} -a^2(t)b^2(t) & 0 & 0 \\ 0 & a^2(t) & 0 \\ 0 & 0 & b^2(t) \end{pmatrix}
\]

The non zero components of Einstein’s equations are:

\[
a^2(t)b^2(t)V(\Phi) - \frac{\dot{a}(t)b(t)}{a(t)b(t)} + \frac{\dot{\Phi}^2(t)}{2} = 0 \tag{13}
\]

\[
a^2(t)b^2(t)V(\Phi) + \frac{\dot{a}(t)b(t)}{a(t)b(t)} + \frac{b^2(t)}{b(t)} - \frac{\dot{b}(t)}{b(t)} = 0 \tag{14}
\]

\[
a^2(t)b^2(t)V(\Phi) + \frac{\dot{a}(t)b(t)}{a(t)b(t)} + \frac{a^2(t)}{a(t)} - \frac{\dot{a}(t)}{a(t)} = 0 \tag{15}
\]

By subtracting the last two equations, we get a relation between the two scale factors \( a(t) \) and \( b(t) \).

\[
b(t) = a(t)e^{\lambda t} \tag{16}
\]

Using now the equation of motion for the matter field (3), we get

\[
a(t) = \left( -\frac{\dddot{\Phi}(t)e^{-2\lambda}}{V'(\Phi)} \right)^{1/4} \tag{17}
\]

With the help of relations (16) and (17) we can transform the quadratic constraint (13) into a third order ordinary differential equation in the single dependent variable \( \Phi \).

\[
\left( \frac{\dddot{\Phi}}{\Phi} - \frac{\dot{\Phi}V''(\Phi)}{V'(\Phi)} \right)^2 + 16\frac{\dot{\Phi}V'(\Phi)}{V'(\Phi)} - 8\dot{\Phi}^2 - 4\lambda^2 = 0 \tag{18}
\]
Equation (18) can be significantly simplified if we define a new function of time \( \chi(t) \) [3], as the ratio

\[
\chi(t) \equiv -\frac{\Phi'}{V'(\Phi)}
\]  

(19)

then (18) becomes an ODE of the first order

\[
\left(\frac{\dot{\chi}}{\chi}\right)^2 - 8f(\chi) - 4\lambda^2 = 0
\]  

(20)

where \( f(\chi) \) is defined as

\[
f(\chi) \equiv \Phi^2 + 2\chi V(\Phi)
\]  

(21)

By differentiating (21) with respect to \( t \) and using (19) we get

\[
\frac{df(\chi(t))}{dt} = 2\dot{\chi}V(t)
\]

also from (21) we have

\[
\frac{d\Phi}{dt} = \pm\sqrt{f(\chi(t)) - 2\chi(t)V(t)}
\]

The above system of differential equations can be brought in a closed form, and thus fully integrated, if we change the time variable from \( t \) to \( \chi \). This action yields the following differential equations

\[
\frac{df}{d\chi} = 2V(\chi)
\]

\[
\frac{d\Phi}{d\chi} = \pm\sqrt{f(\chi) - 2\chi V(\chi)}
\]

which upon integration give:

\[
f(\chi) = 2 \int V(\chi)d\chi + C_1
\]

\[
\Phi(\chi) = \pm \int \frac{1}{\chi} \sqrt{\frac{f(\chi) - 2\chi V(\chi)}{8f(\chi) + 4\lambda^2}} d\chi + C_2
\]

With the help of (20) we can find the relation between the old time \( t \) and the new time \( \chi \)

\[
t(\chi) = \pm \int \frac{1}{\chi \sqrt{8f(\chi) + 4\lambda^2}} d\chi + C_3
\]

At this point we are ready to write the new line element as

\[
ds^2 = -\frac{d\chi^2}{\chi (8f(\chi) + 4\lambda^2)} + \sqrt{\chi} e^{\left(-\lambda \int \frac{1}{x \sqrt{8f(x) + 4\lambda^2}} dx\right)} dx^2 + \sqrt{\chi} e^{\left(\lambda \int \frac{1}{x \sqrt{8f(x) + 4\lambda^2}} dx\right)} dy^2
\]

With the above procedure, we managed to acquire the space of solutions for the abelian case, without having to make any choice that would restrict the potential to be a specified function of \( \chi \). Next, we apply an analogous reasoning for the non abelian case.
2.2. Non abelian model \((C_{12}^1 = -C_{21}^1 = 1, \text{ all others zero})\)

The invariant one forms are

\[
\sigma_{i}^{a} = \begin{pmatrix} e^{-y} & 0 \\ 0 & 1 \end{pmatrix}
\]

the metric

\[
g_{\mu\nu} = \begin{pmatrix} -a^2(t)b^2(t) & 0 & 0 \\ 0 & a^2(t)e^{-2y} & 0 \\ 0 & 0 & b^2(t) \end{pmatrix}
\]

We have four non zero components of Einstein’s equation

\[
4a^2(t) + a^2(t)b^2(t)V(\Phi) - \frac{\dot{a}(t)b(t)}{a(t)b(t)} + \frac{\dot{\Phi}^2(t)}{2} = 0 \tag{22}
\]

\[
a^2(t)b^2(t)V(\Phi) + \frac{\dot{a}(t)b(t)}{a(t)b(t)} + \frac{\dot{\Phi}^2(t)}{2} - \frac{\dot{b}(t)}{b(t)} = 0 \tag{23}
\]

\[
a^2(t)b^2(t)V(\Phi) + \frac{\dot{a}(t)b(t)}{a(t)b(t)} + \frac{\dot{\Phi}^2(t)}{2} - \frac{\ddot{a}(t)}{a(t)} = 0 \tag{24}
\]

\[-\frac{\ddot{a}}{a} + \frac{b}{\dot{b}} = 0 \tag{25}\]

The last one suggests that \(a(t) = \sigma b(t)\) where \(\sigma = \text{constant}\) (from now on we shall consider for simplicity \(\sigma = 1\), since its value makes no difference in the procedure that follows). The quadratic constraint (22) can now be written:

\[
4a^2(t) + a^4(t)V(\Phi) - \frac{\dot{a}^2(t)}{a^2(t)} + \frac{\dot{\Phi}^2(t)}{2} = 0 \tag{26}
\]

Working as before we define the same function \(\chi(t)\) as in (19) and use the equation of motion (3) to bring (26) in the form:

\[
\left(\frac{\dot{\chi}}{\chi}\right)^2 - 8f(\chi) - 64\sqrt{\chi} = 0 \tag{27}
\]

where \(f(\chi)\) is defined exactly as in (21). The resulting relations in this case are:

\[
f(\chi) = 2\int V(\chi)d\chi + C_1
\]

\[
\Phi(\chi) = \pm \int \frac{1}{\chi}\sqrt{\frac{f(\chi)-2\chi V(\chi)}{8f(\chi)+64\sqrt{\chi}}}d\chi + C_2
\]

\[
t(\chi) = \pm \int \frac{1}{\chi\sqrt{8f(\chi)+64\sqrt{\chi}}}d\chi + C_3
\]

and the line element

\[
ds^2 = -\frac{d\chi^2}{\chi(8f(\chi)+64\sqrt{\chi})} + \sqrt{\chi}e^{-2y}dx^2 + \sqrt{\chi}dy^2
\]

Thus, we have once more obtained the entire space of solutions for an arbitrary potential. At this point, one might notice that the choice of time, as defined in (19), might be problematic in some cases. For instance, when \(-\frac{\ddot{\Phi}}{V(\Phi)}=\text{constant}=c\). But then we can immediately deduce from (16) and (17) for the scale factors, that \(a(t) = (ce^{-2\lambda t})^{1/4}\) and \(b(t) = (ce^{2\lambda t})^{1/4}\). So, the line element for the abelian model is now: \(ds^2 = -cdt^2 + \sqrt{ce^{-\lambda t}}dx^2 + \sqrt{ce^{\lambda t}}dy^2\). Applying the same in the non abelian case, we see that the scale factors are constants \(a(t) = b(t) = c^{1/4}\) and the line element becomes: \(ds^2 = -cdt^2 + \sqrt{ce^{-2y}}dx^2 + \sqrt{cd}dy^2\). Another problematic situation arises when \(V'(\Phi) = 0\). But even when this happens, the equation of motion for the scalar field implies that \(\Phi = 0\), so the ratio in (19) can still be defined in the limit without any problem.
3. Quantization
We will now proceed, using Dirac’s method for canonically quantizing constrained systems ([4] [5]), to derive the Wheeler-DeWitt equation for each of the two models.

3.1. Abelian model
Starting from the action (1), we derive the Lagrangian

\[ L = \frac{\dot{a}(t)b(t)}{N(t)} - \frac{a(t)b(t)}{2N(t)} \dot{\Phi}^2 + a(t)b(t)N(t)V(\Phi) \]  

(28)

plus a term that is a total derivative of time, and thus can be dropped out since it plays no role in the dynamics. The momenta conjugate to \( a, b, N \) and \( \Phi \) are:

\begin{align*}
    P_a &= \frac{\partial L}{\partial \dot{a}} = \frac{\dot{b}}{N} \\
    P_b &= \frac{\partial L}{\partial \dot{b}} = \frac{\dot{a}}{N} \\
    P_\Phi &= \frac{\partial L}{\partial \dot{\Phi}} = -\frac{ab}{N} \dot{\Phi} \\
    P_N &= \frac{\partial L}{\partial \dot{N}} = 0
\end{align*}

As we expected by the form of (28) there exists only one primary constraint \( P_N \approx 0 \).

The canonical Hamiltonian is

\[ H_c = \dot{a}P_a + \dot{b}P_b + \dot{\Phi}P_\Phi - L = N(t)\mathcal{H}_c \]  

(29)

where

\[ \mathcal{H}_c = P_aP_b - \frac{P_\Phi^2}{2ab} - abV(\Phi) \]  

(30)

is the hamiltonian constraint. The total Hamiltonian can now be written as \( H_T = H_c + v(t)P_N \), with \( v(t) \) being an arbitrary function of time.

We know from the theory, that in order to stay in the physical space, the constraints must be preserved in time. Thus we have the consistency condition \( \{P_N, H_T\} = 0 \), which reveals \( \mathcal{H}_c \) as the only secondary, first class constraint \( \mathcal{H}_c \approx 0 \).

At this point, we might want to observe that there exists a conditional symmetry [6], that is, a quantity which is linear and homogeneous in the momenta and has a vanishing Poisson bracket with the Hamiltonian

\[ \{b(t)P_b - a(t)P_a, \mathcal{H}_c\} = 0 \]  

(31)

In constrained systems, the Poisson bracket needs only to be weakly equal to zero. However in our case (31) is a strong equality. We point out the existence of such a conditional symmetry, because later on it will help us to reduce the Wheeler-DeWitt equation. But let’s proceed now with the quantization.

To promote the canonical variables to operators we define:

\begin{align*}
    \hat{a} &= a & P_a &= -i \frac{\partial}{\partial a} \\
    \hat{b} &= b & P_b &= -i \frac{\partial}{\partial b} \\
    \hat{\Phi} &= \Phi & P_\Phi &= -i \frac{\partial}{\partial \Phi} \\
    \hat{N} &= N & P_N &= -i \frac{\partial}{\partial N}
\end{align*}
It is obvious that the above operators satisfy the canonical commutation relations \([\hat{x}, \hat{P}_x] = i\delta_{xx'}\).

We write the kinetical part of (30) as

\[
\frac{1}{2} \Gamma^{\mu\nu} \dot{P}_\mu \dot{P}_\nu
\]

with \(\Gamma^{\mu\nu}\) being the metric of the momenta space

\[
\Gamma^{\mu\nu} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1/a^2
\end{pmatrix}
\]  

(32)

At this stage we make use of the symmetry we saw earlier. In order to reduce the metric (32), we treat the symmetry (31) as an additional constraint. As we know from the theory, the action of the constraints on the wave function must be zero, so from (31)

\[
b(t) \frac{\partial \Psi}{\partial b} - a(t) \frac{\partial \Psi}{\partial a} = 0
\]

and by the method of the characteristics we get

\[
\Psi(a, b, \Phi, N) = \Psi(ab, \Phi, N) = \Psi(u, \Phi, N)
\]

where \(u(t) = a(t)b(t)\). Considering the primary constraint \(P_N \approx 0\), we conclude that the wave function \(\Psi\) cannot depend on \(N\) and that it can only be a function of two variables \(\Psi(u, \Phi)\), one of which is the product of the two scale factors and the other the scalar field. Computing the reduced metric gives

\[
\gamma^{\mu\nu} = \Gamma^{\rho\sigma} \frac{\partial w^\rho}{\partial z^\rho} \frac{\partial w^\nu}{\partial z^\sigma} = \begin{pmatrix}
2u(t) & 0 \\
0 & -\frac{1}{u(t)}
\end{pmatrix}
\]

\(w^\mu = (u, \Phi), \ z^\mu = (a, b, \Phi)\).

In order to surpass the factor ordering problem, we use the conformal Laplace-Beltrami operator to realize the kinetic part of (30). In two dimensions this operator assumes the form:

\[
\hat{H}_o = \frac{1}{2} \gamma^{-1/2} \hat{P}_\mu \gamma^{1/2} \gamma^{\nu\mu} \hat{P}_\nu
\]

and by adding the potential term we have the full Hamiltonian operator :

\[
\hat{H} = \frac{1}{2u} \frac{\partial^2}{\partial \Phi^2} - u \frac{\partial^2}{\partial u^2} - \frac{\partial}{\partial u} - uV(\Phi)
\]

Since the Hamiltonian is itself a constraint, its action on the wave function must be zero. Thus we have at last the Wheeler-DeWitt equation

\[
\hat{H}\Psi = 0 \Rightarrow \frac{1}{2u} \frac{\partial^2 \Psi}{\partial \Phi^2} - u \frac{\partial^2 \Psi}{\partial u^2} - \frac{\partial \Psi}{\partial u} - uV(\Phi)\Psi = 0
\]  

(33)

which we were able to solve only for a potential of the form \(V(\Phi) = \kappa e^{\lambda \Phi}\);
If \( \lambda \neq \pm 2\sqrt{2} \)

\[
\Psi(u, \Phi) = A \exp \left( \frac{CN(\sqrt{2} \lambda - 4)u^{1+\sqrt{2}\lambda/2}}{\lambda^2 - 8} e^{\frac{\lambda+2\sqrt{2}\lambda}{2}} + \frac{CN(\sqrt{2}\lambda + 4)u^{1-\sqrt{2}\lambda/2}}{C(\lambda^2 - 8)} e^{\frac{\lambda-2\sqrt{2}\lambda}{2}} \right) \tag{34}
\]

and if \( \lambda = \pm 2\sqrt{2} \)

\[
\Psi(u, \Phi) = Au^{-\frac{1}{2\sqrt{2}}} \exp \left( \frac{C Ku^2}{4} e^{\pm 2\sqrt{2}\Phi} \pm \sqrt{2C} \Phi \right) \tag{35}
\]

where \( u(t) = a(t)b(t) \) and \( A, C \) are just constants of integration.

### 3.2. Non abelian model

We work exactly in the same manner as in the abelian case. Firstly we start from the Lagrangian

\[
L = \frac{\dot{a}^2(t)}{N(t)} - \frac{a^2(t)}{2N(t)} \dot{\Phi}^2 + a^2(t)N(t)V(\Phi) + 4N(t)
\]

Again we have only one primary constraint \( P_N \approx 0 \). From the consistency condition that demands the Poisson bracket of the constraints with the Hamiltonian to vanish in time, we get for once more the Hamiltonian density to be the only secondary constraint. In this case:

\[
\mathcal{H}_c = \frac{1}{4} P_a^2 - \frac{1}{2a^2} P_{\Phi}^2 - a^2 V(\Phi) - 4 \tag{36}
\]

By the use of the conformal Laplace-Beltrami operator to express the kinetic part, we are able to write the Hamiltonian as

\[
\hat{H} = \frac{1}{2a^2} \frac{\partial^2}{\partial \Phi^2} - \frac{1}{4a^2} \frac{\partial^2}{\partial a^2} - \frac{1}{4a} \frac{\partial}{\partial a} - a^2 V(\Phi) - 4 \tag{37}
\]

So the corresponding Wheeler-DeWitt equation is

\[
\frac{1}{2} \frac{\partial^2 \Psi}{\partial \Phi^2} - \frac{a^2}{4a^2} \frac{\partial^2 \Psi}{\partial a^2} - \frac{a}{4a} \frac{\partial \Psi}{\partial a} - (a^4 V(\Phi) - 4a^2) \Psi = 0 \tag{38}
\]

which would be exactly the same as in the abelian case if not for the last term \(-4a^2 \Psi\), owing for the curvature of the spatial slice. It’s because of this “bad” term that we didn’t succeed in solving equation (38) even for an exponential form of the potential \( V(\Phi) \).

### 4. Energy

As we already stated, we dropped of the action terms which involve accelerations, by writing them as total time derivatives. At the classical level, these terms are considered as the energy of the gravitational field. We will see now if the same can be said at the quantum level as well. For the abelian model we dropped out the term: \(-\frac{d}{dt} \left( \frac{bP_b + aP_a}{N} \right)\). If we substitute the velocities with the momenta, we get

\[
-\frac{d}{dt} (bP_b + aP_a) = \frac{d\Omega}{dt} \tag{39}
\]
where $\Omega = -bP_b - aP_a$.

The time derivative of $\Omega$ is

$$\{\Omega, H_T\} = -2NP_aP_b + \frac{N}{ab}P_b^2 - 2abNV(\Phi)$$

(40)

but from (30) we have that: $-2abV(\Phi) = 2\mathcal{H}_c - 2P_aP_b + \frac{1}{ab}P_b^2$, so (40) becomes

$$\dot{\Omega} = 2\mathcal{H}_c - 4\mathcal{H}_0$$

(41)

where $\mathcal{H}_0$ is the kinetic part of the canonical hamiltonian. We already have expressed $\mathcal{H}_c$ and $\mathcal{H}_0$ as operators. The action of $\mathcal{H}_c$ on $\Psi$ is zero, thus we are left only with the action of $\mathcal{H}_0$. By virtue of the Wheeler-DeWitt equation we have

$$\hat{\Omega}\Psi = \hat{E}\Psi = -4NabV(\Phi)\Psi = -4\sqrt{-g}V(\Phi)\Psi$$

(42)

which means that the quantity $\hat{E}$, as we would expect from the energy, gives eigenstates of the wave-function. So, a term that has to do with the geometry of spacetime is miraculously connected via $\mathcal{H}_c = 0$ with the potential of the matter part. It is the quantum analog of the classical notion that matter energy is counterbalanced by the gravitational energy. For the non-abelian model, we deduce

$$\hat{E}\Psi = -4N (a^2V(\Phi) + 2) \Psi$$

(43)

5. Conclusion
We studied the two spatially homogeneous cosmological models that exist in three dimensions, together with a scalar field $\Phi$ minimally coupled to gravity. For the classical case, Einstein’s equations reduced to ODEs which could be fully integrated for the potential being an arbitrary function of $\Phi$. The same could not be done in the quantum case, where we deduced the corresponding Wheeler-DeWitt equations and found a solution just for the abelian model and under the assumption that $V(\Phi)$ is an exponential function of the matter field $\Phi$.

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