K-TWISTED EQUIVARIANT K-THEORY
FOR SU(N)

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Abstract

We present a version of twisted equivariant K-theory-K-twisted equivariant K-theory, and use Grothendieck differentials to compute the K-twisted equivariant K-theory of simple simply connected Lie groups. We did the calculation explicitly for SU(N) explicitly. The basic idea is to interpret an equivariant gerbe as an element of equivariant K-theory of degree 1.

1 Introduction

Let G be a finite dimensional simple Lie group, a classical question related to it is to understand the space Hom(π, G)/G, where π is a finitely presented group. This space Hom(π, G)/G is the moduli space of flat connections on a principal G-bundle on a manifold with fundamental group π. Because of Atiyah and Segal’s result [2], and the fact that K-theory is defined for a large class of geometric objects including usual topological spaces and non-commutative ones, our first approach is to study the equivariant K-theory of Hom(π, G). We get the answer for the case π = Z, i.e. the equivariant K-theory KG∗(G) ≅ Ω∗R(G)/Z[8] the algebra

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of Grothendieck differentials of the representation ring $R(G)$ of $G$ over $\mathbb{Z}$ when $G$ is compact and the fundamental group is torsion free (the general situation is still open). This is the origin of our project about Grothendieck differentials in $K$-theory.

We get interested in twisted $K$-theory because of Freed-Hopkins-Teleman’s result on twisted equivariant $K$-theory and Verlinde algebras [9], [10], [11], that is for a Lie group $G$, the Verlinde algebra $V_k(G)$ at level $k$ is twisted equivariant $K$-theory of $G$ (with adjoint action) at particular degree. Unfortunately, they didn’t publish their proof yet. The main idea for this paper is to use Grothendieck differentials to give a partial proof of their result, and supply a candidate for the geometric definition of twisted $K$-theory.

The first question we need solve is to find a good geometric model for twisted equivariant $K$-theory. Let $\mathbb{H}$ be a infinite dimensional separable Hilbert space, and $U = U(\mathbb{H})$ be the set of unitary operators on $\mathbb{H}$, we know that $U$ is contractible. The group $U$ has a natural subgroup $\{e^{i\theta}I\}$ which is isomorphic to $S^1$, let us denote the quotient group by $PU$. For a topological space $X$, in principle, a twistor is a principal $PU$-bundle over $X$, thus an element in $H^3(X, \mathbb{Z})$. Naturally a geometric realization of $H^3(X, \mathbb{Z})$ elements is needed. We already have a geometric realization of $H^3$ classes, i.e., gerbes [6]. Based on the idea of gerbes, there are some other geometric realizations, like bundle gerbes [17] or central extensions of groupoids [5] [19]. All these involves infinite dimensional objects. We are more interested in finite dimensional realization of gerbes, like [15] [12]. But how can we do twisting with gerbes? As far as I know, there is no clean geometric definition for twisted $K$-theory. The equivariant situation is more subtle, in this case, whether to use equivariant gerbes [7] to do twisting is questionable.

We present a solution to these questions in nice situation. We study the twisted equivariant $K$-theory of $G$ (with adjoint action). In this case, we interpret an equivariant gerbe as an element of $K^1$, then based on this $K^1$ element, we give an intuitive definition of $K$-twisted equivariant $K$-theory. The paper is basically two parts. In the first part, we prove that an element in the equivariant cohomology $H^3_G(G)$ can be interpreted as an element of $K^1_G(G)$, and in the second part, we use the definition we give to do calculation for $SU(N)$ explicitly (in fact we did the calculation for classical groups, but for simplicity and to demonstrate the idea, we just present the case for $SU(N)$).
2 \text{ K-Twisted K-theory}

In the section, we first discuss the general picture of twisted \textit{K} theory and then present our definition for \textit{K}-twisted (equivariant) \textit{K}-theory.

\textit{K}-theory is a generalized cohomology theory \cite{1}. For a paracompact topological space \textit{X}, \textit{K}*(\textit{X}) has several equivalent definitions:

1. Geometric definition: equivalence classes of complex of vector bundles over \textit{X}.

2. Homotopic definition: Homotopy classes of maps: \([X,Fred],[X,Fred_{as}]\), where \(Fred\) and \(Fred_{as}\) are the set of Fredholm operators and self-adjoint operators in \(\mathbb{H}\).

3. Algebraic definition: \textit{K}-theory of \(C^\ast\)-algebra \(C_0(\textit{X})\).

Based on the homotopic definition of \textit{K}-theory, the general picture of the twisted \(\textit{K}\)-theory can be as follows. If we have a principal \(PU\)-bundle \textit{P} over \textit{X}, notice there are natural actions of \(PU\) on \(Fred\) and \(Fred_{as}\), we can form the spaces \(P \times_{PU} Fred = (P \times Fred)/PU\) and \(P \times_{PU} Fred_{as}\), which are fiber bundles over \textit{X}, then we can define the twisted \textit{K}-theory as the homotopy classes of sections of these two bundles. There are general definitions of twisted \textit{K}-theory from point of view of \(C^\ast\)-algebra, see \cite{16}, or \cite{19} for the equivariant cases for detail.

We are more interested in a geometric picture of twisted \textit{K}-theory, and if possible, a definition with finite dimensional objects.

The twistor, i.e., the principal \(PU\)-bundle over \textit{X} is classified by \(H^1(\textit{X},PU)\). The exact sequence of groups 1 \(\rightarrow S^1 \rightarrow U \rightarrow PU \rightarrow 1\) implies that \(PU\) is a model for \(BS^1\), the classifying space of \(S^1\). Thus \(H^1(\textit{X},PU) \cong H^2(\textit{X},S^1) \cong H^3(\textit{X},\mathbb{Z})\), So the twistor is classified by \(H^3(\textit{X},\mathbb{Z})\). The geometric construction of a class in \(H^3(\textit{X},\mathbb{Z})\) is a gerbe \cite{6}. In brief, we use a gerbe to do twisted \textit{K}-theory.

One might hope to use vector bundles to construct the twisted \textit{K}-theory geometrically. This is succeeded only in case that the twistor is a torsion element in \(H^3(\textit{X},\mathbb{Z})\) \cite{4}. In this case the twisted \textit{K}-theory is the Grothendieck group of the category of twisted bundles. The essential problem is the non-existence of finite dimensional twisted bundles in general.

The geometric picture for the twisted equivariant \textit{K}-theory is more subtle. Let \textit{G} be a topological group, \textit{X} be a \textit{G}-space, the equivariant \textit{K}-theory \(K^*_G(\textit{X})\) can be defined in the similar ways \cite{18}. The question in this case is what kind of twistor we can use. The natural generalization of non-equivariant case is the elements in \(H^3_G(X)\), the 3rd degree equivariant cohomology, in other words equivariant gerbes. But there is some problem if we use it to a geometric approach. The reason is that an element of \(H^3_G(X)\) is an object on \(EG \times_G X\), not exactly an equivariant object on
There is a question just like non-equivariant case, what kind geometric objects we can use, again the non-existence of twisted equivariant bundle is a problem.

There is another point of view for the whole picture. Let $X$ be a finite dimensional object, for example, a finite dimensional manifold, then the Chern character $\text{ch}: K^1(X) \otimes \mathbb{Q} \cong H^{\text{odd}}(X, \mathbb{Q})$. So up to $\mathbb{Z}$-torsion, an element in $H^3(X, \mathbb{Z})$ can be viewed as an element in $K^1(X)$. This simple observation suggests the following intuitive definition of $K$-twisted $K$-theory.

**DEFINITION 2.1** Let $X$ be a topological space, and $\alpha \in K^1(X)$, the $K$-twisted $K$-theory $\alpha K^\ast(X)$ is the homology of the following complex,

$$\cdots \xrightarrow{\wedge \alpha} K^0(X) \xrightarrow{\wedge \alpha} K^1(X) \xrightarrow{\wedge \alpha} K^0(X) \xrightarrow{\wedge \alpha} \cdots$$

The desired properties of twisted $K$-theory are obvious from this Definition. This definition should agree with the homotopic definition in case $\alpha$ is a non-torsion element in $H^3(X, \mathbb{Z})$, and there should be a more general geometric definition of twisted $K$-theory which generalizes this definition and twisted bundle in the torsion case. We are working on this topic.

This definition can be easily generalized to the equivariant case,

**DEFINITION 2.2** Let $X$ be a topological space, $G$ be a compact topological group acting on $X$, and $\alpha \in K^1_G(X)$, the $K$-twisted $K$-theory $\alpha K^\ast_G(X)$ is the homology of the following complex,

$$\cdots \xrightarrow{\wedge \alpha} K^0_G(X) \xrightarrow{\wedge \alpha} K^1_G(X) \xrightarrow{\wedge \alpha} K^0_G(X) \xrightarrow{\wedge \alpha} \cdots$$

### 3 The basic gerbe as an element of $K^1_G(G)$

Let $G$ be a $n$-dimensional compact simple simply-connected Lie group of rank $d$, $T$ be a maximal torus of $G$, and $W$ be the Weyl group of $G$ with respect to $T$. We use $R(G)$, $R(T)$ to denote the representation rings of $G$ and $T$ respectively. If $\chi_1, \chi_2, \ldots, \chi_d$ are the simple characters of $T$, then the character group $X^\ast(T) = \text{Hom}(T, S^1)$ is the free abelian group generated by $\chi_1, \chi_2, \ldots, \chi_d$, and the representation ring $R(T)$ is the group ring $\mathbb{Z}[X^\ast(T)] = \mathbb{Z}[\chi_1, \chi_2, \ldots, \chi_d, \chi_1^{-1}, \chi_2^{-1}, \ldots, \chi_d^{-1}]$. The Weyl group $W$ acts on $R(T)$, the invariant subalgebra $R(T)_W$ is the representation ring $R(G)$, which is a polynomial ring generated by “basic” representations $\rho_1, \rho_2, \ldots, \rho_d$ corresponding to a choice of a set of simple roots.
The cohomology of $T$ can be easily described in terms of these characters. The character $\chi_i : T \to S^1$ can be viewed as an element of $[X, S^1] \cong H^0(X, S^1) \cong H^1(X, \mathbb{Z})$, let us denote this element by $\eta_i$. By this way, we get a homomorphism of abelian groups $X^*(T) \to H^1(T, \mathbb{Z})$, and $H^*(T, \mathbb{Z}) \cong \wedge(\eta_1, \eta_2, \cdots, \eta_d)$.

The $K$-theory can be described in similar way. A character $\chi_i : T \to S^1 = U(1)$ defines a line bundle over the suspension of $T$, thus defines an element of $K^1(T)$, again we denote this element by $\eta_i$. Therefore we have a homomorphism between abelian groups $X^*(T) \to K^1(T)$, and $K^*(T) \cong \wedge(\eta_1, \eta_2, \cdots, \eta_d)$. In particular we see that there is an isomorphism $c : K^*(T) \cong H^*(T, \mathbb{Z})$, where the map is in fact the first chern class of bundles, and this map is equivariant under the action of Weyl group $W$.

Let $X$ be a paracompact space, $H$ be a compact topological group acting on $X$, then the equivariant cohomology is defined as $H^*_H(X) = H^*(EH \times_H X)$, where $EH \to BH$ is a universal principal $H$-bundle, $BH$ is a classifying space for $H$. In particular, $H^*_H(pt) = H^*(BH)$, and the bundle map $EH \times_H X \to BH$ give $H^*_H(X)$ a $H^*_H(pt)$-module structure.

In the case of the torus $T$, the coefficient ring $H^*(BT)$ can also be described in terms of the character group $X^*(T)$. For any character $\chi : T \to S^1$, it defines a line bundle $ET \times_T C\chi$ over $BT$, the first chern class of this bundle gives an abelian group homomorphism: $X^*(T) \to H^2(BT)$, this induces an isomorphism between $H^*(BT)$ and the symmetric algebra $S_T$ of $X^*(T)$. Notice that $H^*(BT)$ carries a natural action of the Weyl group $W$.

Let us consider $G$ as a $G$-space with adjoint action, it is well-known that $H^*_G(G, \mathbb{Z}) \cong \mathbb{Z}$, and the generator (up to sign) is called the basic (equivariant) gerbe. There are several ways to describe this gerbe [5] [15] [12], the main result of this section is to present another way to view this basic gerbe.

Let us recall two lemmas about equivariant $K$-theory and equivariant cohomology of $G$ [7] [8].

**Lemma 3.1** For a compact simple simply-connected Lie group $G$,

$$H^*_G(G) \cong (H^*(BT) \otimes H^*(T))^W$$

**Lemma 3.2** For a compact simple simply-connected Lie group $G$,

$$K^*_G(G) \cong (R(T) \otimes K^*(T))^W$$

**Proposition 3.3** For a compact simple simply-connected Lie group $G$, the basic equivariant gerbe can be viewed as an element of $K^1_H(G)$. 

5
Proof. By above lemmas,
\[ H^3_G(G) \cong (H^0(BT) \otimes H^3(T) \oplus H^2(BT) \otimes H^1(T))^W \]
\[ \subset (R(T) \otimes K^1(T))^W \cong K^1_G(G), \]
here, \( H^0(BT) \cong \mathbb{Z}, H^2(BT) \cong X^*(T) \) can be viewed as subset of \( R(T) \). \( \square \)

4 \( K \)-Twisted \( K \)-theory for \( SU(N) \)

In this section, we will use our definition of \( K \)-twisted equivariant \( K \)-theory and Grothendieck differentials to do the calculation for \( SU(N) \).

Let us first recall some background of Grothendieck differentials. Let \( A \subset B \) be commutative rings. The algebra of Grothendieck differentials \( \Omega^*_B/A \) \[\text{[13]}\] is the differential graded \( A \)-algebra constructed as follows:

Let \( F \) be the free \( B \)-module generated by all elements in \( B \), to be clear, we use \( db \) to denote the generator corresponding to \( b \in B \), so
\[ F = \bigoplus_{b \in B} Bdb. \]
and let \( I \subset F \) be the \( B \)-submodule generated by
\[ \left\{ \begin{array}{l}
   da, \forall a \in A \\
   d(b_1 + b_2) - db_1 - db_2, \forall b_1, b_2 \in B \\
   d(b_1b_2) - b_1db_2 - b_2db_1, \forall b_1, b_2 \in B
\end{array} \right\}, \]
we then get the quotient \( B \)-module
\[ \Omega^*_B/A = F/I. \]

Let \( \Omega^0_B/A = B, \Omega^1_B/A = \Omega_B/A, \) and \( \Omega^p_B/A = \Lambda^p_B \Omega_B/A, p \). There is a differential:
\[ d : \Omega^p_B/A \rightarrow \Omega^{p+1}_B/A, \] which maps \( b \in B \) to \( db \), then
\[ \Omega^*_B/A = \bigoplus_{p=0}^{\infty} \Omega^p_B/A. \]
is the differential graded algebra of Grothendieck differentials of \( B \) over \( A \). It is the generalization of the algebra of differentials on affine spaces, for example, if \( B = A[x_1, \cdots, x_n] \), then \( \Omega^p_{A[x_1, \cdots, x_n]/A} = \bigoplus_{1 \leq i_1 < \cdots < i_p} A[x_1, \cdots, x_n]dx_{i_1} \wedge \cdots \wedge dx_{i_p}. \)
For any representation $\rho : G \to GL(V)$, it defines a vector bundle over the suspension of $G$, which is $G$-equivariant, so it defines an element $d\rho$ of $K^1_G(G)$. The main result in [8] is this defines an isomorphism $\Omega^*_R(G)/\mathbb{Z} \cong K^*_G(G)$, when $\pi_1(G)$ is torsion free.

This result applies to the case of a torus $T$. In terms of Grothendeick differentials, for any character $\chi_i$ of $T$, $d\chi_i = \chi_i \eta_i$, where $\eta_i$ is the $K$-theory element or cohomology element of $T$ constructed in the previous section, or in other words, $\eta_i = \frac{d\chi_i}{\chi_i}$.

In the case $G = SU(N)$, if we let $\rho_i$ be the $i$-th elementary symmetric polynomial in $\chi_1, \chi_2, \cdots, \chi_N$, then $R(G) = \mathbb{Z}[\rho_1, \rho_2, \cdots, \rho_{N-1}]$, and the equivariant $K$-theory is $K^*_G(G) = \wedge_{R(G)}(d\rho_1, d\rho_2, \cdots, d\rho_{N-1})$.

**PROPOSITION 4.1** For $SU(N)$, let $\delta$ be the basic gerbe, than as an element of $K^1_{SU(N)}(SU(N))$, is

$$\delta = \sum \chi_i \eta_i$$

$$n \delta = \sum \chi^{(n)}_i \eta_i$$

Let $\alpha = n \delta$, now we are going to calculate the $K$-twisted $K$-theory $^\alpha K^*_G(G)$ for $G = SU(N)$, we need a lemma.

**LEMMA 4.2** Let $\alpha = \sum \chi^{(n)}_i dx_i$ $n \ge 0$, then the following complex is exact except at the last spot:

$$0 \to \mathbb{Z}[x_1, x_2, \cdots, x_N] \xrightarrow{\wedge \alpha} \mathbb{Z}[x_1, x_2, \cdots, x_N]dx_i \wedge \alpha \xrightarrow{\wedge \alpha} \cdots \xrightarrow{\wedge \alpha} \mathbb{Z}[x_1, x_2, \cdots, x_N]dx_1 \cdots dx_N \xrightarrow{\wedge \alpha} 0$$

Now it is a standard calculation to get $K$-twisted equivariant $K$-theory, in particular,

**THEOREM 4.3** Let $\alpha = (N + k)\delta$, then $^\alpha K^N_{SU(N)}(SU(N))$ is the Verlinde algebra $V_k$ of $SU(N)$ at level $k$.

**Proof** By the above lemma and taking $W$-invariants, the non-trivial term of $^\alpha K^N_{SU(N)}(SU(N))$ only appears in degree $N$. If $\alpha = a_i d\rho_i$ (These $a_i \in R(SU(N))$ are classical functions, for example see [14]), then the $K$-twisted equivariant $K$-theory is $^\alpha K^N_{SU(N)}(SU(N)) = R(SU(N))d\rho_1d\rho_2 \cdots d\rho_{N-1}/(a_1, a_2, \cdots a_{N-1})d\rho_1d\rho_2 \cdots d\rho_{N-1} \cong R(SU(N))/(a_1, a_2, \cdots a_{N-1})$, which is the Verlinde algebra at level $k$.  

7
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