New informations on the structure of the functional codes defined by forms of degree $h$ on non-degenerate Hermitian varieties in $\mathbb{P}^n(F_q)$

Frédéric A. B. Edoukou*, San Ling*, Chaoping Xing*
Division of Mathematical Sciences,
Nanyang Technological University,
21 Nanyang Link, Singapore 637371.
E.mail : {abfedoukou, lingsan, xingcp}@ntu.edu.sg

Abstract

We study the functional codes of order $h$ defined by G. Lachaud on $X \subset \mathbb{P}^n(F_q)$ a non-degenerate Hermitian variety. We give a condition of divisibility of the weights of the codewords. For $X$ a non-degenerate Hermitian surface, we list the first five weights and the corresponding codewords and give a positive answer on a conjecture formulated on this question. The paper ends with a conjecture on the minimum distance and the distribution of the codewords of the first $2h+1$ weights of the functional codes for the functional codes of order $h$ on $X \subset \mathbb{P}^n(F_q)$ a non-singular Hermitian variety.

Keywords: functional codes, Hermitian surface, Hermitian variety, weight.

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1 Introduction

Let $X$ be a projective algebraic variety over the finite field $F_q$. The functional codes $C_h(X)$ defined by evaluating the polynomials functions over the rational points of the algebraic variety $X$ have been studied in general

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way by several authors. The works of Goppa on codes constructed on the non-singular Hermitian curves inspired also several authors mainly I. M. Chakravarti and his group at the university of North-Carolina, to generalize theses codes to the non-singular Hermitian surfaces. For those who first of all, want to have a very readable treatment of the works Goppa on Hermitian curves but have a limited understanding of algebraic geometry the book of W. Cary and V. Pless [2, pp.526-544] can be examined. Some interesting results have been obtained in the case of the non-singular Hermitian surfaces by I. M. Chakravarti’s group [3]. Their works were mainly done on the fields $\mathbb{F}_4$ of order four. Therefore computer programs have been used to find all the structure of the codes. In 1991, A. B. Sørensen, in his Ph. D. Thesis [19, pp.7-9] recovered a part of their results by tools of algebraic geometry and finite geometry, and try also to study the codes $C_h(X)$ over the non-singular Hermitian surface by increasing the degree of the polynomial functions and the order of the fields. In 2007, in his Ph.D. Thesis [5] the first author of this paper continue the works of A. B. Sørensen and solved part of A. B. Sørensen’s conjecture formulated in the past years on Hermitian surface. He generalized the results to the code $C_2(X)$ constructed on the non-singular Hermitian solid (non-singular Hermitian varieties of dimension three) where he found a structure on the repartition of the first five weights, their frequency and a divisibility condition for all the weights of the code $C_2(X)$. He also stated two conjectures one on the minimum weight codewords of the codes $C_2(X)$ constructed on $X : x_{t+1}^0 + x_{t+1}^1 + x_{t+1}^2 + x_{t+1}^3 + x_{t+1}^4 = 0$ where $h \leq t$, and the second on the repartition of the first five weights of the codes $C_2(X)$ defined on the non-singular Hermitian variety $X : x_{t+1}^0 + x_{t+1}^1 + ... + x_{t+1}^{n-1} + x_{t+1}^n = 0$ in $\mathbb{P}^n(\mathbb{F}_q)$. The second conjecture formulated by him has been solved recently by A. Hallez and L. Storme under the condition that $n < O(t^2)$. Under this restrictive condition we can also remark a divisibility condition of the first five weights which was not mentioned explicitly in their paper [9, p.9].

The purpose of this paper is to give some news informations on the structure of the codes $C_h(X)$. The paper has been organized as follows. First we recall some generalities on the number of solutions of a family of polynomials over a finite field. Secondly we give the information on the divisibility condition which should be respect by all the weights of the codes $C_h(X)$ constructed on the non-singular Hermitian variety $X : x_{t+1}^0 + x_{t+1}^1 + ... + x_{t+1}^{n-1} + x_{t+1}^n = 0$ in $\mathbb{P}^n(\mathbb{F}_q)$ with $q = t^2$ ($t$ is a prime power) even if, theses weights are not computed explicitly. Thirdly by using this result on the divisibility condition, we also solved the conjecture formulated in [5, p.55], [7, p.113] on the fourth and fifth weights of the codes $C_h(X)$ defined on the non-singular
Hermitian surface $\mathcal{X} : x_{t+1}^{t+1} + x_{t+1}^{t+1} + x_{t+1}^{t+1} + x_{t+1}^{t+1} = 0$ in $\mathbb{P}^3(\mathbb{F}_q)$ with $q = t^2$ ($t$ is a prime power).

The paper ends with a conjecture on the minimum distance and the distribution of the first $2h + 1$ weights of the code $C_h(\mathcal{X})$ where $\mathcal{X} \subset \mathbb{P}^n(\mathbb{F}_q)$ is the non-singular Hermitian variety.

2 Generalities

We denote by $\mathbb{F}_q$ the field with $q$ elements. Let $V = A^{n+1}(\mathbb{F}_q)$ be the affine space of dimension $n + 1$ over $\mathbb{F}_q$ and $\mathbb{P}^n(\mathbb{F}_q) = \Pi_n$ the corresponding projective space. Then

$$\pi_n = \# \mathbb{P}^n(\mathbb{F}_q) = q^n + q^{n-1} + \ldots + 1.$$  

We denote by $W_i$ the set of points with homogeneous coordinates $(x_0 : \ldots : x_n) \in \mathbb{P}^n(\mathbb{F}_q)$ such that $x_j = 0$ for $j < i$ and $x_i \neq 0$. The family $\{W_i\}_{0 \leq i \leq n}$ is a partition of $\mathbb{P}^n(\mathbb{F}_q)$. We use the term forms of degree $h$ to describe homogeneous polynomials $f$ of degree $h$, and $V = Z(f)$ (the zeros of $f$ in the projective space $\mathbb{P}^n(\mathbb{F}_q)$) is a hypersurface of degree $h$. Let $\mathcal{F}_h$ be the vector space of forms of degree $h$ in $V$. For any polynomial $f \in \mathcal{F}_h$ and any point $P \in \mathbb{P}^n(\mathbb{F}_q)$ we define

$$f(P) = f(x_0, \ldots, x_n)/x_i^h \quad \text{with} \quad P = (x_0 : \ldots : x_n) \in W_i$$

**Theorem 2.1** (Tsfasman-Serre-Sørensen) ([18, p.351], [19, chp.2, pp.7-10])

Let $f(x_0, \ldots, x_n)$ be a homogeneous polynomial in $n + 1$ variables with coefficients in $\mathbb{F}_q$ and degree $h \leq q$. Then the number of zeros of $f$ in $\mathbb{P}^n(\mathbb{F}_q)$ satisfies:

$$\# Z(f)(\mathbb{F}_q) \leq hq^{n-1} + \pi_{n-2}.$$  

This upper bound is attained when $Z(f)$ is a union of $h$ hyperplanes passing through a common linear space of codimension 2.

**Theorem 2.2** (Ax-Katz) [13, p.485] Let $\mathbb{F}_q$ be a finite fields of characteristic $p$, having $q = p^a$ elements. Let $N(S,T,f)$ defined as the number of points of $V(S,T,f)$ with values in $\mathbb{F}_q$ and $\lambda(S,T,f)$ defined as the least non-negative integer which is greater than

$$\text{Card}(S) - \sum d_i \sup (d_i).$$

Then $N(S,T,f) \equiv 0$ modulo $q^{\lambda(S,T,f)}$.  

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Let $\mathcal{X} \subset \mathbb{P}^n(\mathbb{F}_q)$ an algebraic variety and $\#\mathcal{X}(\mathbb{F}_q)$ the number of rational points of $\mathcal{X}$ over $\mathbb{F}_q$. The code $C_h(\mathcal{X})$ is the image of the linear map $c : \mathcal{F}_h \rightarrow \mathbb{P}^{\#\mathcal{X}(\mathbb{F}_q)}_q$, defined by $c(f) = (c_x(f))_{x \in \mathcal{X}}$, where $c_x(f) = f(x_0, ..., x_n)/x_i^h$ with $x = (x_0 : ... : x_n) \in W_i$. The length of $C_h(\mathcal{X})$ is equal to $\#\mathcal{X}(\mathbb{F}_q)$. The dimension of $C_h(\mathcal{X})$ is equal to $\dim \mathcal{F}_h - \dim \ker c$. Therefore, when $c$ is injective we get:

$$\dim C_h(\mathcal{X}) = \binom{n + h}{h}. \quad (1)$$

The minimum distance of $C_h(\mathcal{X})$ is equal to the minimum over all $f$ of $\#\mathcal{X}(\mathbb{F}_q) - \#\mathcal{X}(\mathbb{F}_q)_Z(f)$.  

### 3 Divisibility condition on the weights of the codes $C_h(\mathcal{X})$ defined on the non-degenerate Hermitian variety

In this section $\mathbb{F}_q$ denotes the field with $q$ elements, where $q = t^2$ and $\mathcal{X}$ denotes the non-degenerate (i.e. non-singular) Hermitian variety of $\mathbb{P}^n(\mathbb{F}_q)$ of equation $\mathcal{X} : x_0^{t+1} + x_1^{t+1} + ... + x_{n-1}^{t+1} + x_n^{t+1} = 0$.

In [1, p.1175] R. C. Bose and I. M. Chakravarti proved the following result:

**Theorem 3.1** Let $\tilde{\mathcal{X}} \subset \mathbb{P}^n(\mathbb{F}_q)$ be a non-degenerate Hermitian variety. Then,

$$\#\tilde{\mathcal{X}}(\mathbb{F}_q) = [t^{n+1} - (-1)^{n+1}][t^n - (-1)^n]/(t^2 - 1) \quad (2)$$

#### 3.1 Transformation of a non-degenerate Hermitian variety to a quadric variety

Here we will give an important result on the code defined on the Hermitian variety in the $\mathbb{P}^n(\mathbb{F}_q)$. Let $\mathcal{X} : x_0^{t+1} + x_1^{t+1} + ... + x_{n-1}^{t+1} + x_n^{t+1} = 0$ be the non-degenerate Hermitian variety over the field $\mathbb{F}_q$. If $\phi$ is an Hermitian form in $n+1$ variables with $n \geq 3$ over the field $\mathbb{F}_q = \mathbb{F}_{t^2}$ defining the Hermitian variety $\mathcal{X}$, we have:

$$\phi(x_0, ..., x_n) = x_0^{t+1} + ... + x_n^{t+1}$$

Let us denote by $\alpha$ an element of $\mathbb{F}_q$ which is not in $\mathbb{F}_t$. One has

$$\mathbb{F}_q = \mathbb{F}_t \oplus \alpha \mathbb{F}_t.$$
Thus we can decompose every element $x$ of $\mathbb{F}_q$

$$x = y + \alpha z.$$ 

The conjugaison is given by $x \mapsto x^t$ and transform $\alpha$ in $\overline{\alpha}$. Therefore $x^t = y + \overline{\alpha}z$, and therefore,

$$x^{t+1} = (y + \alpha z)(y + \overline{\alpha}z) = y^2 + (\alpha + \overline{\alpha})yz + \alpha \overline{\alpha}z^2.$$ 

From a result of R. C. Bose and I. M. Chakravarti [1, p. 1163], we know that the sum $\alpha + \overline{\alpha}$ as the product $\alpha \overline{\alpha}$ belong together in $\mathbb{F}_t$. Therefore the form $\phi$ is now a form of degree 2 in $2(n + 1)$ variables over the subfield $\mathbb{F}_t$. And its new equation becomes:

$$\phi(y_0, z_0, \ldots, y_n, z_n) = y_0^2 + (\alpha + \overline{\alpha})y_0z_0 + \alpha \overline{\alpha}z_0^2 + \ldots + y_n^2 + (\alpha + \overline{\alpha})y_nz_n + \alpha \overline{\alpha}z_n^2.$$ 

### 3.2 Structure of the weights of the codewords

Let us recall an important propriety on a codes which can be found in [2, pp. 10-11]. It is the propriety of divisor of a code. Divisible codes, are interesting because many optimal codes exhibit nontrivial divisibility.

**Definition 3.2 (2, pp. 10-11)** We say that a code $C$ (over any field) is divisible provided all codewords have weights divisible by an integer $\Delta > 1$. The code is said divisible by $\Delta$; $\Delta$ is called a divisor of the code $C$, and the largest such divisor is called the divisor of the code $C$.

To our knowledge, during the past forty years since the discovering of the notion of divisor of a code, the main achievement in the determination of a divisor of a code, has been done for two kind of codes: cyclic codes over prime fields by R. J. McEliece [16], Griesmer codes (codes meeting the Griesmer bound) over prime fields in the binary case by S. M. Dodunekov and N. L. Manev [4]. Making used of the divisibility criteria [21, p. 323] H. N. Ward [22, p. 80, pp. 84-87] extend the result to Griesmer codes over prime fields in the nonbinary case. This is the result on divisibility for Griesmer code.

**Theorem 3.3 (2, p. 86)** Let $C$ be a linear code over $\mathbb{F}_p$, where $p$ is a prime, which meets the Griesmer bound. Assume that $p^i|d(C)$ where $d(C)$ is the minimum distance of $C$, then $p^i$ is a divisor of the code $C$.

For further details in the study of the divisibility properties of codes, the survey paper of H. N. Ward [23] where he generalized the above theorem for Griesmer codes over the field $\mathbb{F}_q$ ($q = p^a$, $p$ is a prime, $a$ an integer) by a conjecture [23, p. 271] and the recent Ph. D. Thesis of X. Liu [15] where he gave bounds on dimension of divisible codes, can be excellent companions.
Theorem 3.4 Let \( n \) and \( h \) be two positive integers such that \( h \leq n \) and \( n = sh + r \) where \( 0 \leq r \leq h - 1 \). Let us consider the code \( C_h(X) \) defined on the non-singular Hermitian variety \( X: x_0^{t+1} + x_1^{t+1} + \ldots + x_{n-1}^{t+1} + x_n^{t+1} = 0 \) over the field \( \mathbb{F}_q \) \((q = t^2 \text{ and } t = p^a)\). Then \( \Delta = t^\lambda \) is the divisor of the code \( C_h(X) \) where:

\[
\lambda = \begin{cases} 
  n - 2 & \text{if } h = 2 \\
  2E(\frac{n}{h}) - 2 & \text{if } h \geq 3 \text{ and } r = 0 \\
  2E(\frac{n}{h}) - 1 & \text{if } h \geq 3 \text{ and } h = 2r \\
  2E(\frac{n}{h}) + E(\frac{2r}{h}) - 1 & \text{if } h \geq 3 \text{ and } h \neq 2r 
\end{cases}
\]

with \( E(x) \) equal the integer part of \( x \).

Proof: Let \( f \) be a form of degree \( h \) in \( n+1 \) variables over the field \( \mathbb{F}_q \). By using the above transform on the coefficients of \( f \) and the \( n+1 \) variables, \( f \) can newly be written as:

\[
f(x_0, x_1, \ldots, x_n) = f_0(y_0, z_0, \ldots, y_n, z_n) + \sum_{i=1}^{h} \alpha_i f_i(y_0, z_0, \ldots, y_n, z_n)
\]

where \( f_0, f_1, \ldots, f_h \) are homogeneous polynomials of degree \( h \) in \( 2(n+1) \) variables over the field \( \mathbb{F}_t \). By using again the above transform on the \( h-1 \) elements \( \alpha_2, \ldots, \alpha_h \) of \( \mathbb{F}_q \), we deduce that finally

\[
f(x_0, x_1, \ldots, x_n) = \tilde{f}_0(y_0, z_0, \ldots, y_n, z_n) + \alpha \tilde{f}_1(y_0, z_0, \ldots, y_n, z_n)
\]

where \( \tilde{f}_0 \) and \( \tilde{f}_1 \) are two homogeneous polynomials of degree \( h \) in \( 2(n+1) \) variables over the field \( \mathbb{F}_t \). Therefore the variety \( V \) defined by one equation on \( \mathbb{F}_q \), is defined by two equations on \( \mathbb{F}_t \) and the variety \( X \cap V \) is defined by the following system of three equations:

\[
\begin{align*}
  y_0^2 + (\alpha + \overline{\alpha})y_0z_0 + \alpha\overline{\alpha}z_0^2 + \ldots + y_n^2 + (\alpha + \overline{\alpha})y_nz_n + \alpha\overline{\alpha}z_n^2 &= 0 \\
  \tilde{f}_0(y_0, z_0, \ldots, y_n, z_n) &= 0 \\
  \tilde{f}_1(y_0, z_0, \ldots, y_n, z_n) &= 0
\end{align*}
\]

which are equations on \( \mathbb{F}_t \). By the theorem of Ax-Katz, the number of common zeros \( N \) of \( \tilde{f}_0(y_0, z_0, \ldots, y_n, z_n) \), \( \tilde{f}_1(y_0, z_0, \ldots, y_n, z_n) \) and \( \phi \) (in \( \mathbb{F}_t^{2(n+1)} \)) is divisible by \( t^{\lambda(S,T,f)} \) where \( \lambda(S,T,f) \) is the least non-negative integer such that
\[
\lambda(S, T, f) \geq \frac{2(n + 1) - (2h + 2)}{h} = 2\left(\frac{n}{h} - 1\right). \tag{3}
\]

If \( h = 2 \), then \( \lambda(S, T, f) = 2 \) and \( N \) is divisible by \( t^{n-2} \).

Let us suppose now \( h \geq 3 \) and \( n = ah + r \) where \( 0 \leq r \leq h - 1 \).

If \( r = 0 \), then \( N \) is divisible by \( t^{2E(\frac{n}{h}) - 2} \).

If \( r \neq 0 \), then \( N \) is divisible by
\[
\begin{cases}
    t^{2E(\frac{n}{h}) - 1} & \text{if } h = 2r \\
    t^{2E(\frac{n}{h}) + E(\frac{r}{h}) - 1} & \text{if } h \neq 2r
\end{cases}
\]

In fact, we have \( 1 \leq r \leq h - 1 \) therefore \( 0 < \frac{2r}{h} < 2 \). When \( h = 2r \), the least non-negative integer such that \( (3) \) is verified is \( 2E(\frac{n}{h}) + E(\frac{r}{h}) - 2 \). When \( h \neq 2r \), we have \( \frac{2r}{h} \in [0, 1] \cup [1, 2] \), therefore the least non-negative integer satisfying \( (3) \) is \( 2E(\frac{n}{h}) + E(\frac{r}{h}) + 1 - 2 \).

On the other hand, \( \tilde{f}_0(y_0, z_0, ..., y_n, z_n) \), \( \tilde{f}_1(y_0, z_0, ..., y_n, z_n) \) and \( \phi \) are homogeneous polynomials, therefore \( N - 1 \) is divisible by \( t - 1 \).

Let \( X \), \( V_1 \) and \( V_2 \) be the projective varieties associated respectively to the forms \( \phi \), \( \tilde{f}_0(y_0, z_0, ..., y_n, z_n) \), and \( \tilde{f}_1(y_0, z_0, ..., y_n, z_n) \), one has
\[
\#(X \cap V) = \#(X \cap V_0 \cap V_1) = \frac{N - 1}{t - 1}.
\]

Let \( M = \frac{N - 1}{t - 1} \), and \( \lambda \) as above, one has
\[
M = \frac{j t^\lambda - 1}{t - 1} = j' t^\lambda + \pi_{\lambda-1} \quad (\ast),
\]

where \( j \) and \( j' \) are non-null integers such that \( j = j'(t - 1) + 1 \).

By the theorem of Ax-Katz again, we get that the number of zeros of the polynomial \( \phi \) (in \( \mathbb{F}_t^{2(n+1)} \)) is divisible by \( t^n \) and therefore the number of zeros of \( \mathcal{X} \) in \( \mathbb{P}^{2n+1}(\mathbb{F}_t) \) is:
\[
\#\mathcal{X} = \frac{k t^n - 1}{t - 1} = k' t^n + \pi_{n-1} \quad (**),
\]

where \( k \) and \( k' \) are non-null integers such that \( k = k'(t - 1) + 1 \).

The weight of the codeword associated to the projective variety \( V \) defined by the form \( f \) is equal to:
\[
w = \#\mathcal{X} - \#(\mathcal{V} \cap \mathcal{X}) = \#\mathcal{X} - M, (***).
\]
Therefore, from (⋆), (⋆⋆), and (⋆ ⋆ ⋆) we deduce that
\[ w = k't^n + t^{n-1} + \cdots + t^{\lambda+1} + (1-j')t^\lambda. \]
Thus,
\[ w \equiv 0 (\text{mod. } t^\lambda). \]

**Remark 3.5** What is important in this paper is that our technique gives directly the divisor, of the code \( X \), without any knowledge of the minimum distance. In fact most of the results in the literature relate the determination of a divisor of a code to its minimum distance, in the particular cases of cyclic codes, Griesmer codes etc... Here we don’t need to know the minimum distance. We don’t need to know if the code is cyclic or attains the Griesmer bound, but we have a strong information (its divisor).

**Remark 3.6** In the case where \( h = 2 \), the first five weights of the code \( C_2(X) \) defined on the non-singular Hermitian variety have been determined by A. Hallez and L. Storme [9, p.9] in their two tables under the restrictive condition \( n < O(t^2) \). There are divisible by \( t^{n-2} \).

In the particular case where \( h \leq t \), by using Theorem 3.1 and Theorem 2.1 we deduce that the evaluation map \( e \) defining the code \( C_h(X) \) is injective. The length and the dimension of the code \( C_h(X) \) on the non-singular Hermitian variety are given respectively by Theorem 3.1 and relation (1). In the general case, a lower bound for the minimum distance of \( C_h(X) \) on the non-singular Hermitian variety has been given by F. Rodier [17, pp.207-208]. The result of Theorem 3.4 gives an improvement of this lower bound.

4 **The code \( C_2(X) \) defined on the Hermitian surface**

The code \( C_2(X) \) on the non-singular Hermitian surface \( X : x_0^{t+1} + x_1^{t+1} + x_2^{t+1} + x_3^{t+1} = 0 \) has been studied by the first author in [5, pp.14-18, pp.28-58], [6], [7]. He found the three first weights of the codewords and their frequency. Based on the result obtained in the intersection of quadric surfaces and the non-singular Hermitian surface, he formulated a conjecture on the fourth and fifth weight of this code. The resolution of this conjecture depended mainly on how we could improve the upper bound
\[ 2t^3 + 2t + 2 \]
found in [7, pp.107-108] for the number of intersection points in the section of the Hermitian surface and the elliptic quadric.

Now we will use a very simple technique to study it. From the works of R. C. Bose and I. M. Chakravarti [1, p.1179], we know that there are exactly $\alpha_t = (t^3 + 1)(t + 1)$ lines contained in the non-singular Hermitian surface $\mathcal{X} : x_0^{t+1} + x_1^{t+1} + x_2^{t+1} + x_3^{t+1} = 0$ and through each point of $\mathcal{X}$ it pass exactly $t + 1$ lines contained in $\mathcal{X}$. We also know from [10, p.123] that there is no line in the elliptic quadric surface. Therefore, every line of $\mathcal{X}$ intersects the elliptic quadric $\mathcal{E}_3$ in at most two points. Thus we deduce that,

$$\#(\mathcal{X} \cap \mathcal{E}_3) \leq \frac{2\alpha_t}{t + 1} = 2(t^3 + 1).$$

(4)

**Corollary 4.1** Let $C_2(\mathcal{X})$ be the functional code defined on the non-singular Hermitian surface $\mathcal{X} : x_0^{t+1} + x_1^{t+1} + x_2^{t+1} + x_3^{t+1} = 0$ over the field $\mathbb{F}_q$ ($q = t^2$ and $t = p^a$). Then $\Delta = t$ is the divisor of $C_2(\mathcal{X})$.

**Proof:** It is a direct consequence of Theorem 3.4.

From Corollary 4.1 and the relation (4), we deduce a new upper bound on the number of points in the intersection of the non-singular Hermitian surface and the elliptic quadric:

$$\#(\mathcal{X} \cap \mathcal{E}_3) \leq s_4(t) = 2t^3 + 1.$$

(5)

**Theorem 4.2** The fourth weight is $w_4 = t^5 - t^3 + t^2$ and for $t \neq 2$ the corresponding codewords are given by quadrics which are union of two non-tangent planes to $\mathcal{X}$ and the line of intersection of the two planes meets $\mathcal{X}$ at a single point. There are exactly $\frac{1}{2}(t-1)(t^4 - 1)^2$ codewords of fourth weight.

The fifth weight is $w_5 = t^5 - t^3 + t^2 + t$ and for $t \neq 2, 3$ the corresponding codewords are given by quadrics which are union of two non-tangent planes to $\mathcal{X}$ such that the line of intersection of the two planes intersects $\mathcal{X}$ in $t + 1$ points. There are exactly $\frac{1}{2}(t-1)(t^3 + 1)(t^2 + 1)^2$ codewords of fifth weight.

**Proof:** From the results of tables 1, 2, 3 of paragraph 4.2 in [7, pp.111-112], and the improved upper bound (5) obtained on the number of points in the intersection of the Hermitian surface and the elliptic quadric, we deduce that $s_4(t)$ gives the fourth weight $w_4 = t^5 - t^3 + t^2$. From the results of tables 1, 2, 3 in [7, pp.111-112] and Corollary 4.1, we deduce that $s_5(t) = 2t^3 - t + 1$.
gives the fifth weight \( w_5 = t^5 - t^3 + t^2 + t \).

We will compute now the number of codewords of fourth and fifth weight. From the fundamental formula of Wan and Yang [20] or [12, Th. 23.4.3, pp.70-71] , we deduce that there are exactly

\[ N(\mathcal{L}; \mathcal{X}, \Pi_0 \mathcal{U}_0) = \frac{t(t^3 + 1)(t^4 - 1)}{(t + 1)} \quad (6) \]

lines \( \mathcal{L} \) intersecting \( \mathcal{X} \) at a single point \( \Pi_0 \) (i.e. a singular Hermitian variety \( \Pi_0 \mathcal{U}_0 \) of rank 1 in \( \mathbb{P}^1(\mathbb{F}_q) \)). We also know from [1. p.1179] that through the point \( \Pi_0 \), there pass exactly \( t + 1 \) lines which constitute the intersection with \( \mathcal{X} \) of the tangent plane at \( \Pi_0 \). Thus, among the \( q + 1 \) planes through the line \( \mathcal{L} \), we deduce that there is only one plane which is tangent to \( \mathcal{X} \). Therefore, through the \( q(q + 1)/2 \) pairs of planes through \( \mathcal{L} \) we have exactly

\[ \frac{q(q - 1)}{2} \quad (7) \]

pairs of planes non-tangent to \( \mathcal{X} \). From \( (6) \) and \( (7) \) we deduce that there are exactly \( (t^2 - 1).N(\mathcal{L}; \mathcal{X}, \Pi_0 \mathcal{U}_0) \cdot \frac{q(q - 1)}{2} \) codewords of fourth weight.

From [20] or [12, Th. 23.4.3, pp.70-71] , we deduce that there are exactly

\[ N(\mathcal{L}; \mathcal{X}, \mathcal{U}_1) = \frac{t^4(t^3 + 1)(t^2 + 1)}{t + 1} \quad (8) \]

lines \( \mathcal{L} \) intersecting \( \mathcal{X} \) at \( t + 1 \) points (i.e. a non-singular Hermitian variety \( \mathcal{U}_1 \) in \( \mathbb{P}^1(\mathbb{F}_q) \) ). Here the \( q + 1 \) planes through the line \( \mathcal{L} \) are all non-tangent to \( \mathcal{X} \), because any line of a tangent plane to \( \mathcal{X} \) can not meet \( \mathcal{X} \) at \( t + 1 \) points. Therefore, through the \( q(q + 1)/2 \) pair of planes through \( \mathcal{L} \) we have exactly

\[ \frac{q(q + 1)}{2} \quad (9) \]

pair of planes non-tangent to \( \mathcal{X} \). From \( (8) \) and \( (9) \) we deduce that there are exactly \( (t^2 - 1).N(\mathcal{L}; \mathcal{X}, \mathcal{U}_1) \cdot \frac{q(q + 1)}{2} \) codewords of fifth weight.

**Remark 4.3** If \( w_i \ (1 \leq i \leq 5) \) are the first five weights of the code \( C_2(\mathcal{X}) \), then there exist degenerate quadrics \( Q \) reaching the Tsfasman-Serre-Sørensen’s upper bound for hypersurfaces (i.e. \( Q \) is a union of two distinct planes \( Q = H_1 \cup H_2 \)), giving codewords of weight \( w_i \). And for \( i > 5 \), there is no quadric which is a union of distinct planes, giving codewords of weight \( w_i \).  




5 Conjecture on the first $2h+1$ weights of the code $C_h(\mathcal{X})$ in $\mathbb{P}^n(\mathbb{F}_q)$

The author has also tried to generalize the study of the code $C_h(\mathcal{X})$ to $\mathcal{X}$ the non-degenerate Hermitian variety defined by $x_0^{t+1} + x_1^{t+1} + \ldots + x_{n-1}^{t+1} + x_n^{t+1} = 0$ in $\mathbb{P}^n(\mathbb{F}_q)$ ($q = t^2$) and conjecture that:

Conjecture

1 If $w_i$ ($1 \leq i \leq 2h+1$) are the first $2h+1$ weights of the code $C_2(\mathcal{X})$, then there exist degenerate hypersurface $\mathcal{V}$ reaching the Tsfasman-Serre-Sørensen’s upper bound for hypersurfaces (i.e. $\mathcal{V}$ is a union of $h$ distinct hyperplanes $\mathcal{V} = H_1 \cup \ldots \cup H_h$, meeting in a common linear space of codimension 2), giving codewords of weight $w_i$.

2 The minimum weight (i.e. $w_1$) codewords are only given by degenerate hypersurfaces which are union of $h$ distinct hyperplanes ($\mathcal{V} = H_1 \cup \ldots \cup H_h$) such that $(H_1 \cap \ldots \cap H_h) \cap \mathcal{X}$ is a non-singular Hermitian variety in $\mathbb{P}^{N-2}(\mathbb{F}_q)$ and:
   - If $n$ is even, the $h$ hyperplanes $H_1 \ldots, H_h$ are non-tangent to $\mathcal{X}$
   - If $n$ is odd, the $h$ hyperplanes $H_1, \ldots, H_h$ are tangent to $\mathcal{X}$.

3 For $i > 2h + 1$, there is no hypersurface which is a union of distinct hyperplanes, giving codewords of weight $w_i$.

Unfortunately no proof has been found yet.

Remark 5.1 The conjecture is true for $n = 3$ and $h = 2$ (see section 4 of this paper and [7, § 4.1-4.2]). The conjecture is also true in the case $n = 4$ and $h = 2$: Theorem 4.4 [8, p.143] gives the result. A. Hallez and L. Storme [9] have proved this conjecture in the case $h = 2$ under the condition that $n < O(t^2)$.

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