I. INTRODUCTION

The Wigner function [1–4] provides a particularly useful visual representation of the state of a bosonic single mode quantum system as a real valued function on the two-dimensional system phase space. Integrating the Wigner function with respect to the system position coordinate \( x \) gives the marginal probability density in the momentum coordinate \( p \) (and vice versa); more generally, integrating the Wigner function with respect to any quadrature phase coordinate \( X_1 = x \cos(\theta) + p \sin(\theta) \) gives the marginal probability density in the complementary quadrature phase coordinate \( X_2 = -x \sin(\theta) + p \cos(\theta) \). In terms of the Wigner function, the quantum expectation value of a (Weyl ordered) observable \( A(x, p) \) is evaluated in exactly the same way as for the corresponding classical system described by a phase space probability distribution function. Furthermore, the master equation that describes the open quantum system dynamics gets mapped to a partial differential equation for the Wigner function dynamics that closely resembles the Fokker Planck equation for the classical system statistical dynamics.

Given the close resemblance between the Wigner function representation of the quantum system dynamical equations and the corresponding classical statistical dynamical equations, the Wigner function has helped provide an understanding of how classical dynamics arises by approximation from the underlying quantum dynamics [5–14]. Of particular interest in this respect are nonlinear single mode systems such as the driven, damped Duffing oscillator. A number of investigations have employed the Wigner function representation to explore the resulting quantum phase space dynamics in parameter regimes where the corresponding classical nonlinear dynamics exhibits, for example, bistability or chaos [7, 9, 11–13].

By varying the system damping and noise (diffusion) due to coupling to the environment, the quantum to classical transition can be explored in a controllable and visually direct way by comparing the corresponding quantum Wigner function phase space and classical phase space pictures.

However, the Wigner function can take negative values and so is not a true probability distribution, despite the properties mentioned above. The presence of regions in phase space where the Wigner function is negative is conventionally interpreted as a signature of nonclassicality in the quantum state; With the exception of Gaussian (i.e., coherent and squeezed) states, all pure states have negative-valued Wigner function regions and hence are nonclassical [15].

A sought after goal for such bosonic mode systems is to generate macroscopic quantum states that are stable over long times against the decohering effects of the environment – possibly even under ‘warm’ conditions where the environment temperature is large compared to the characteristic frequency scale of the system dynamics. Stabilized macroscopic quantum states are useful not only for quantum information processing applications, but also for fundamental explorations, especially concerning how macroscopic a quantum state can be in the presence of unavoidable decohering environments.

A sought after goal for such bosonic mode systems is to generate macroscopic quantum states that are stable over long times against the decohering effects of the environment – possibly even under ‘warm’ conditions where the environment temperature is large compared to the characteristic frequency scale of the system dynamics. Stabilized macroscopic quantum states are useful not only for quantum information processing applications, but also for fundamental explorations, especially concerning how macroscopic a quantum state can be in the presence of unavoidable decohering environments.

By ‘macroscopic’, we mean that the average photon or phonon number of the stabilized system quantum state is large, while by ‘quantum’ we mean that the Wigner function representation of the state has significant negative regions in the system phase space. How large can a negative Wigner valued region be? Two classic theorems that can be easily generalized to mixed states establish that the Wigner function is generally bounded in magnitude by \((\pi \hbar)^{-1} = 2/\hbar \) [34], while the area of a given negative Wigner valued region can exceed \( \hbar \), but where at least one of the dimensions must be of order \( \sqrt{\hbar} \) or smaller [35]. For example, the \( N = 2 \) harmonic oscillator Fock state has a single negative annular Wigner function region of area \( \approx 4.44 \hbar \) and radial width \( \approx 0.765\sqrt{\hbar} \).
Approaches to stabilizing quantum states involve measurement feedback to control the quantum system dynamics [39], as well as so-called autonomous methods that do not require measurement feedback control. The latter typically involve ‘reservoir engineering’, where the effective system-environment interaction is tailored in such a way as to evolve the system into a quantum state as well as to protect the state from the decohering effects of the environment [57, 42].

Another approach to autonomously generating quantum states exploits the nonlinearities in the closed bosonic mode system dynamics – equivalently anharmonicities in the system Hamiltonian. The presence of anharmonicities can cause initial Gaussian states with associated positive Wigner functions to evolve into non-classical states with associated negative valued Wigner functions (see, e.g., Ref. [13]). In terms of the quantum Fokker Planck dynamical equations for the Wigner function, the root cause of such evolution is the presence of a third or higher order position derivative term involving the system potential energy. Only when the potential energy is anharmonic is this term present and without this term, the Wigner function dynamical equation coincides with the classical Fokker Planck equation.

For example, in the case of the driven Duffing oscillator with \(x^4\) anharmonicity in the potential energy and the resulting coexistence of bistable large and small amplitude oscillatory solutions for the classical dynamics, an initial coherent state will transiently evolve into a Schrödinger cat-like state where the Wigner function displays a sequence of alternating negative and positive regions in between the corresponding large and small amplitude positive Wigner function peaks [15], in a classically chaotic regime, an initial coherent state will spread out in phase space, exhibiting a complex inference pattern of positive and sub-\(\hbar\) (sub-Planckian) scale negative Wigner function regions [7]. However, depending on the environment temperature, such non-classical features will typically diffuse away for the usual device system-environment couplings, leaving a long time steady state that is closely approximated by the corresponding classical system Fokker-Planck equation.

Nevertheless, the question is still largely unresolved as to whether it might be possible to stabilize quantum states largely with anharmonicities alone. In particular, for certain anharmonicity types and drives (whether externally or internally generated by the system dynamics), we may be able to prepare and maintain quantum states with significant associated negative Wigner function regions, despite the counteracting decoherence effects of environmental noise. Recent relevant developments in superconducting microwave resonator (as well as coupled nanomechanical resonator) circuits involving embedded Josephson junction elements provide strong motivation for pursuing this question [43, 51]. In particular, the Josephson elements can induce strong effective anharmonicities in the microwave mode Hamiltonian, as well as internally generated drive tones through the AC-Josephson effect. One consequence is lasing-like behavior [46], with the continuous, stimulated emission of amplitude-squeezed microwave photons [47].

With the above motivations in mind, in the present work we will extend the Wigner formulation of the open quantum system dynamics and take into account also the so-called Wigner function flow vector field (or ‘Wigner flow’ in short) [52, 53]. The Wigner flow allows a particularly concise reformulation of the quantum Fokker Planck equation as a standard continuity equation, equating (via the familiar Gauss’s theorem of vector calculus) the rate of change of the net Wigner quasiprobability within some two dimensional region to net Wigner flows normal to the boundary enclosing the region; loosely speaking, the phase space flow picture is somewhat analogous to a system involving a distribution of positive and negative charges (e.g. electrons and holes) that can be annihilated and created, albeit described by a quite different quantum statistical dynamics.

The potential advantage of bringing the Wigner flows into play is that they can give a strong visual representation of how non-classical states form through the system Hamiltonian anharmonicity, as well as diffuse away due to the environment. By exploring the relative contributions to the net Wigner flow across the boundary of a given negative region arising from the system Hamiltonian anharmonicity and from the interactions with the environment, we may be able to improve our understanding of how to ‘engineer’ system Hamiltonian anharmonicities and drive stones so as to stabilize macroscopic bosonic quantum states in the face of environmental noise. The present work gives some initial steps in this direction.

In Sec. II we introduce the Wigner flow formulation of the quantum Fokker-Planck equation, giving as specific system examples the harmonic oscillator and additively driven Duffing oscillator. In Sec. III we present our numerical results, which include simulations of the open harmonic and Duffing oscillator Wigner function and flows. Section IV discusses our numerical results and in particular gives some initial steps towards an understanding of how negative Wigner function regions can form and diffuse away from a Wigner flow perspective. In Sec. V we give some concluding remarks.

II. THE WIGNER FLOW

For a one dimensional particle with Hamiltonian \(H = p^2/(2m) + m\omega_0^2 x^2/2 + V(x, t)\), where \(V(x, t)\) is the (time dependent) anharmonic potential energy, a possible Lindblad master equation that describes the quantum dynamics of the system state characterized by density matrix \(\rho(t)\) interacting weakly with an oscillator bath can be written as follows:

\[
\frac{d\rho}{dt} = -\frac{i}{\hbar} [H, \rho] + \frac{\gamma}{2} (\hat{n} + 1) (2a \rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a) + \frac{\gamma}{2} \hat{n} (2a^\dagger \rho a - a a^\dagger \rho - \rho a a^\dagger),
\]  

(1)
where $\gamma$ is the system energy damping rate, $\bar{n} = (e^{\hbar \omega_0/(k_B T)} - 1)^{-1}$ is the Bose-Einstein thermal average occupation number of the temperature $T$ bath at the characteristic frequency $\omega_0$ of the system Hamiltonian. Strickly speaking, the master equation (1) is valid to a good approximation provided the system-environment interaction is weak: $\gamma \ll \omega_0$, the temperature is in the range $\hbar \gamma \ll k_B T \ll \hbar \omega_0$, and the anharmonic potential is sufficiently weak [54]. However, as is frequent practice, we will assume that the master equation still gives reasonable open system quantum dynamics even for larger temperatures $k_B T \sim \hbar \omega_0$.

The Wigner function representation of the quantum state $\rho(t)$ as a real-valued function on phase space can be defined as [54]

$$W(x, p, t) = \frac{1}{\pi \hbar} \int_{-\infty}^{+\infty} dy e^{-2i p y / \hbar} (x + y | \rho(t) | x - y)$$

$$= \frac{1}{\pi \hbar} \int_{-\infty}^{+\infty} dp' e^{+2i p' x / \hbar} (p + p' | \rho(t) | p - p').$$

(2)

Expressing the master equation (1) in terms of the Wigner function (2), we obtain the so-called quantum Fokker-Planck equation

$$\frac{\partial W}{\partial t} = -\frac{p}{m} \frac{\partial W}{\partial x} + \left( m \omega_0 x + \frac{\partial V}{\partial x} \right) \frac{\partial W}{\partial p}$$

$$+ \sum_{n \geq 1} \frac{(-1)^n (\hbar/2)^{n+1}}{(2n+1)!} \frac{\partial^{2n+1} V}{\partial p^{2n+1}} \frac{\partial W}{\partial p}$$

$$+ \frac{\gamma}{2} \frac{\partial}{\partial x} \left[ xW + h \left( \bar{n} + \frac{1}{2} \right) \frac{1}{m \omega_0} \frac{\partial W}{\partial x} \right]$$

$$+ \frac{\gamma}{2} \frac{\partial}{\partial p} \left[ pW + h \left( \bar{n} + \frac{1}{2} \right) m \omega_0 \frac{\partial W}{\partial p} \right].$$

(3)

The Wigner vector fields for the system (52, 53) and environment are, respectively

$$J_{\text{sys}} = \left( \begin{array}{c} \frac{\hbar W}{m} \\ -\sum_{n=0}^{\infty} \frac{(-1)^n (\hbar/2)^{2n}}{(2n+1)!} \frac{\partial V}{\partial p} \partial x \partial (\partial^2 x + 1) W \end{array} \right)$$

(4)

and

$$J_{\text{env}} = \gamma \left( \begin{array}{c} xW + h \left( \bar{n} + \frac{1}{2} \right) m \omega_0 \partial x W \\ pW + h \left( \bar{n} + \frac{1}{2} \right) m \omega_0 \partial p W \end{array} \right),$$

(5)

where the first row is the position $x$ component and the second row is momentum $p$ component of the flow vector, and where we have used the shorthand notation $V' = x \frac{\partial V}{\partial x} + V$, $\partial x \equiv \frac{\partial}{\partial x}$, and $\partial p \equiv \frac{\partial}{\partial p}$. The environment flow can be further decomposed as a sum of damping and diffusion contributions: $J_{\text{env}} = J_{\text{damp}} + J_{\text{diff}}$, where

$$J_{\text{damp}} = -\gamma \left( \begin{array}{c} xW \\ pW \end{array} \right)$$

(6)

and

$$J_{\text{diff}} = \frac{\hbar}{2} \left( \begin{array}{c} (m \omega_0)^{-1} \partial x W \\ m \omega_0 \partial p W \end{array} \right).$$

(7)

In terms of the system and environment flows, the master equation for the Wigner function (3) takes the concise form of a continuity equation:

$$\frac{\partial W}{\partial t} + \nabla \cdot J = 0,$$

(8)

where $J = J_{\text{sys}} + J_{\text{env}}$ and $\nabla = (\partial_x, \partial_p)$.

The driven Duffing oscillator is characterized by the anharmonic + additive driving potential

$$V(x, t) = \lambda x^3 + \frac{\hbar^2 \lambda}{4} x \frac{\partial^2}{\partial p^2} W,$$

(9)

where the parameter $\lambda$ gives the strength of the anharmonic potential, the parameter $F$ gives the strength of the time-dependent sinusoidal drive, and $\omega_d$ is the drive frequency. Substituting Eq. (9) into Eq. (4), we obtain for the driven Duffing oscillator system Wigner flow:

$$J_{\text{Duff}} = \left( \begin{array}{c} \frac{\hbar W}{m} \\ -m \omega_0 x W - m \omega_0 x^2 W \end{array} \right).$$

(10)

For the harmonic oscillator the system Wigner flow simplifies to

$$J_{\text{HO}} = \left( \begin{array}{c} \frac{\hbar W}{m} \\ -m \omega_0 x W \end{array} \right).$$

(11)

In the numerical investigations below, it is convenient to work in terms of dimensionless forms of the Wigner function and flow. In terms of the length unit $x_0 = \sqrt{\hbar/(m \omega_0)}$ and time unit $t_0 = \omega_0^{-1}$, we transform the various coordinates and parameters into dimensionless form as follows: $\tilde{x} = x/x_0$, $\tilde{p} = p/(m \omega_0 x_0)$, $\tilde{F} = x_0 F/(\hbar \omega_0)$, $\tilde{\lambda} = \lambda x_0^3/(\hbar \omega_0)$, $\tilde{\gamma} = \gamma/\omega_0$, $\tilde{\omega}_d = \omega_d/\omega_0$, and $\tilde{t} = \omega_0 t$, where the tilde denotes the dimensionless form. The dimensionless form for the Wigner function is

$$\tilde{W} = \frac{\hbar W}{m},$$

$$\tilde{J} = \frac{\hbar J}{m},$$

$$\frac{\partial \tilde{W}}{\partial \tilde{t}} + \nabla \cdot \tilde{J} = 0,$$

where $\tilde{J} = \tilde{J}_{\text{Duff}} + \tilde{J}_{\text{env}}$, with

$$\tilde{J}_{\text{Duff}} = \left( \begin{array}{c} \frac{\tilde{p} \tilde{W}}{m} \\ -\tilde{x} + \tilde{F} \cos(\tilde{\omega}_d \tilde{t}) - \frac{\tilde{\lambda} \tilde{x}^3}{4} + \frac{\tilde{x} \tilde{p}^2}{\hbar} \tilde{W} \end{array} \right).$$

(14)

and

$$\tilde{J}_{\text{env}} = \tilde{J}_{\text{damp}} + \tilde{J}_{\text{diff}},$$

(15)
with
\[ \mathbf{J}_{\text{damp}} = -\frac{\tilde{\gamma}}{2} \begin{pmatrix} \tilde{x}\tilde{W} \\ \tilde{p}\tilde{W} \end{pmatrix} \] (16)

and
\[ \mathbf{J}_{\text{diff}} = -\frac{\tilde{\gamma}}{2} \begin{pmatrix} \tilde{n} + \frac{1}{2} \\ \tilde{n} + \frac{1}{2} \end{pmatrix} \begin{pmatrix} \partial_\tilde{x}\tilde{W} \\ \partial_\tilde{p}\tilde{W} \end{pmatrix}. \] (17)

For the harmonic oscillator, we have for the dimensionless flow: \( \mathbf{J} = \mathbf{J}_{\text{HO}} + \mathbf{J}_{\text{env}} \), with
\[ \mathbf{J}_{\text{HO}} = \begin{pmatrix} \tilde{p}\tilde{W} \\ -\tilde{x}\tilde{W} \end{pmatrix} \] (18)

and \( \mathbf{J}_{\text{env}} \) given by Eq. (15).

From now on, we drop the tildes for notational convenience, the dimensionless form of the parameters and coordinates understood.

III. NUMERICAL RESULTS

In this section we present the results of our numerical solutions to the Wigner function \( W \) and associated flow vector field \( \mathbf{J} \) for the undriven, open harmonic and driven Duffing oscillator systems. This involves first solving the Lindblad master equation (1) for the system density matrix \( \rho(t) \) using QuTiP \cite{45} and then evaluating the Wigner function and flows in terms of the density matrix; the source code can be obtained from Ref. \cite{56}. Complete videos of each simulation can be viewed at Ref. \cite{57}. While the Wigner function time dependence for the open harmonic oscillator system can be determined analytically \cite{58, 59}, we nevertheless solve the harmonic oscillator master equation numerically as a check on the validity of our code.

In the following Wigner function plots, regions color-coded blue correspond to positive Wigner function value, red regions correspond to negative Wigner function value, while the local color density gives a measure of the Wigner function magnitude.

A. Harmonic Oscillator

Figure 1 shows snapshots of the evolving Wigner function and associated flow \( \mathbf{J} = \mathbf{J}_{\text{HO}} + \mathbf{J}_{\text{env}} \) for the harmonic oscillator initially in an \( N = 2 \) Fock state \cite{52} and in the presence of a zero temperature bath; the damping rate is chosen to be \( \gamma = 0.01 \). A unit area square corresponding to Planck’s constant \( \hbar \) in our dimensionless units is indicated at the bottom right of each figure to give the scale; while the arrow legend at the top left of each figure indicates the scale for the flow vector field. The snapshot times are given in multiples of the free oscillation period \( \tau = 2\pi/\omega_0 = 2\pi t_0 \). Figure 2 shows the same evolving Wigner function snapshots as in Fig. 1 but with just the environmental diffusion flow \( \mathbf{J}_{\text{diff}} \) indicated.

Figures 3 and 4 show snapshots of the evolving Wigner function and associated full and environmental diffusion flows, respectively, for the harmonic oscillator in an initial superposition of coherent states separated by \( x = 4 \); the damping rate is \( \gamma = 0.01 \) and the bath temperature \( T = 0 \).

In the final indicated snapshots corresponding to \( t = 100\tau \) [Figs. 3(c)], the Wigner function and flows hardly change between subsequent snapshots separated by a free oscillation period (see also the strobe videos \cite{57}), indicating that the system dynamics has reached a steady state to a good approximation. This is to be expected given that \( \gamma t = 2\pi \), i.e., final the snapshot time is approximately six times longer than the harmonic oscillator relaxation time.

B. Duffing Oscillator

We now turn to the numerical solution of the Wigner function and associated flow vector field for the damped, driven Duffing oscillator. We choose the dimensionless Duffing oscillator parameter values \( \lambda = 0.05 \) (anharmonic strength), \( \omega_d = 1.09 \) (drive frequency), \( F = 0.092 \) (drive strength), and \( \gamma = 0.01 \) (damping rate). These parameter values result in the classical Duffing oscillator exhibiting bistability for the steady state dynamics at zero temperature, corresponding to coexisting small and large amplitude oscillations. For the above parameter choices, these small and large steady state amplitudes are 0.52 and 2.46, respectively.

Figure 5 shows snapshots of the evolving Wigner function and associated flow \( \mathbf{J} = \mathbf{J}_{\text{Diff}} + \mathbf{J}_{\text{env}} \) for the Duffing oscillator initially in an undisplaced coherent state and in the presence of a zero temperature bath: the snapshot times are given in multiples of the drive period \( \tau_d = 2\pi/\omega_d \). Figure 6 shows the same evolving Wigner function snapshots as in Fig. 5 but with just the environmental diffusion flow \( \mathbf{J}_{\text{diff}} \) indicated. Figures 7 and 8 show snapshots of the evolving Wigner function together with the full flow and environmental diffusion flows, respectively, but at nonzero temperature \( T = 2\hbar \omega_0/k_B \).

In the final indicated snapshots corresponding to \( t = 300\tau_d \) [Figs. 8(c)], the Wigner function and flows hardly change between subsequent snapshots separated by a drive period (see also the strobe videos \cite{57}); these final snapshots should therefore correspond pretty accurately to the long time limit steady state Wigner function and flows.

IV. DISCUSSION

Common to the harmonic and Duffing oscillator quantum dynamics indicated in Figs. 1, 3, 5 and 7, the direction of the flow \( \mathbf{J} \) in the regions of positive-valued Wigner function is clockwise about the phase space origin, just as is the case for an evolving classical probabil-
FIG. 1. Snapshots of evolving harmonic oscillator Wigner function and associated flow vector field $J = J_{\text{HO}} + J_{\text{env}}$ for an initial $N = 2$ Fock state with damping rate $\gamma = 0.01$ and bath temperature $T = 0$.

FIG. 2. Snapshots of evolving harmonic oscillator Wigner function and associated environmental diffusion flow vector field $J_{\text{diff}}$ for an initial $N = 2$ Fock state with damping rate $\gamma = 0.01$ and bath temperature $T = 0$.

ity density that results from solving the corresponding classical Fokker-Planck equation for some initial probability distribution; for the harmonic oscillator system, the Wigner flow continuity equation \[\text{(8)}\] coincides with the classical, Brownian motion Fokker-Planck equation, while for the Duffing oscillator the Wigner flow continuity equation \[\text{(8)}\] differs from the classical Fokker-Planck equation only in the presence of the system quantum flow term \((0, \lambda x \partial_x^2 W/4)\) [see Eq. \[\text{(14)}\]]. In contrast, the flow direction in the regions of negative-valued Wigner function is \textit{counterclockwise}, i.e., in the opposite direction to the corresponding classical flow [52, 53, 60].

In Figs. 2, 4, 6, and 8 we can see that for any negative-valued Wigner function region, the diffusion contribution to the environmental flow $J_{\text{diff}}$ is always directed inwards on the boundary of the negative region, with the result that the environmental diffusion flow acts to destroy negative regions. This is just the process of decoherence viewed in terms of the Wigner flow.

In order to gain a better understanding of how regions where the Wigner function is negative initially form, are stabilized, or eventually disappear, let us suppose that the Wigner function at some given time instant $t$ is negative in certain regions of phase space. This is the case for the initial Fock state and coherent state superposition examples considered above (see Figs. 1-4), while for the Duffing oscillator, we see that negative Wigner function regions are generated through the dynamics (Figs. 5-8). Consider a particular negative region with phase space area $A(t)$ and boundary $\partial A(t)$, where the indicated $t$-dependence accounts for the fact that the negative region evolves in time. In particular, the boundary is defined by $W(x, p, t)|_{\partial A(t)} = 0$. A measure of the degree of negativity of the region is given by the negative 'volume'
under the integral \( \int_{A(t)} dxdp \) \( W(x,p,t) \). From Eqs. (13)-(17) and Gauss’s theorem, the time rate of change of this negative volume is

\[
\frac{d}{dt} \int_{A(t)} dxdp W(x,p,t) = \frac{\lambda}{4} \int_{\partial A(t)} ds \mathbf{n} \cdot (0, -x) \frac{\partial^2 W}{\partial p^2} + \frac{\gamma}{2} \left( n + \frac{1}{2} \right) \int_{\partial A(t)} ds \mathbf{n} \cdot \nabla W,
\]

(19)

where we have used the fact that the Wigner function vanishes on the boundary \( \partial A(t) \), \( s \) parametrizes the boundary curve, and \( \mathbf{n} \) is the unit vector outwards normal to the curve.

For the harmonic oscillator system, the first term on the right hand side of the equals sign in Eq. (19) vanishes (since \( \lambda = 0 \)) and the rate of change of the region negativity is affected solely by the environmental diffusion flow (17). Since the Wigner function is by definition negative on the interior region and positive on at least the immediate exterior region of the boundary \( \partial A(t) \), the gradient \( \nabla W \) points outwards so that \( \mathbf{n} \cdot \nabla W \geq 0 \) everywhere on the boundary. Therefore, for the harmonic oscillator we have that

\[ \frac{d}{dt} \int_{A(t)} dxdp W(x,p,t) \geq 0 \]

(20)

and we thus see that the size of the negative regions always decreases with time at a rate governed by the environmental diffusion flow.

On the other hand, for the Duffing oscillator system (\( \lambda \neq 0 \)), we see from Eq. (19) that the rate of change of the region negativity is governed by two flow contributions: the system quantum flow \( (0, \lambda x \partial^2 W/4) \) and the environmental diffusion flow (17). Since the environmental
diffusion destroys negative regions, the system quantum flow must therefore be responsible for the initial generation and possible eventual stabilization of negative regions in the steady state. In particular, for a growing negative region we necessarily require that

\[
\frac{\lambda}{4} \int_{\partial A(t)} ds \mathbf{n} \cdot (0, -x) \frac{\partial^2 W}{\partial p^2} < 0. \tag{21}
\]

If the term \(\partial^2_p W\) were a constant on the boundary \(\partial A(t)\) of a negative region, then the integral in Eq. (21) would simply vanish. Supposing that \(\partial^2_p W \mid_{\partial A(t)} \geq 0\) [i.e., \(W(x, p)\) is concave upwards with respect to its momentum dependence on the boundary \(\partial A(t)\)] and for positive anharmonic coupling strength \(\lambda > 0\), the term \(\partial^2_p W\) must therefore be larger on segments of the boundary where \(\mathbf{n} \cdot (0, -x) < 0\) for condition (21) to hold. Referring to Fig. 9, we thus see that negative regions will tend to form at the leading edges of the clockwise rotating Wigner function peaks. The instant at which the first negative region appears for our Duffing oscillator simulations is shown in Fig. 10, which appears to be in accord with the above prediction.

Eventually, the negative regions practically vanish even at zero temperature as is clear from Fig. 5(c). Because the chosen parameter values result in coexisting small and large amplitude stable oscillations for the classical dynamics, the Wigner function must correspondingly spread out through flow and quantum diffusion from its initially narrow and strongly peaked coherent state distribution [Fig. 5(a)]. As a result, the magnitude of the term \(\partial^2_p W\) must decrease overall, and with the small chosen anharmonic coupling strength value \(\lambda = 0.05\), the system quantum term is too weak to counter the deleteri-
**FIG. 7.** Snapshots of evolving Duffing oscillator Wigner function and associated flow vector field $J = J_{\text{Duff}} + J_{\text{env}}$ for an initial undisplaced coherent state; the damping rate $\gamma = 0.01$ and bath temperature $T = 2\hbar\omega_0/k_B$.

**FIG. 8.** Snapshots of evolving Duffing oscillator Wigner function and associated environmental diffusion flow vector field $J_{\text{diff}}$ for an initial undisplaced coherent state; the damping rate $\gamma = 0.01$ and bath temperature $T = 2\hbar\omega_0/k_B$.

ous effects of the diffusion term in Eq. (19) and stabilize sizable negative regions.

Although only small negative Wigner function regions remain in the steady state for the considered Duffing parameter values, under the right conditions it might be possible to generate and stabilize sizable negative Wigner function regions, perhaps even at non-zero temperatures. In particular, the form of the quantum term in Eq. (19) and the accompanying picture provided by Fig. 9 suggest that negative regions are favored by large magnitude anharmonic coupling, single amplitude oscillatory solutions where the corresponding rotating Wigner function remains narrow and strongly peaked so as to ensure that the term $\partial_p^2 W$ at the leading (trailing) edge for $\lambda > 0$ ($\lambda < 0$) is sufficiently large. One possible way to maintain large values for $\partial_p^2 W$ at the leading and trailing peak edges might be through the continuous squeezing of the momentum uncertainty noise [61].

**V. CONCLUSION**

In this present work, we extended the Wigner phase space formulation of open quantum system dynamics to include a description of the Wigner flow vector fields on phase space. This enables the quantum Fokker-Planck equation describing the Wigner function dynamics to be written in the concise form of a continuity equation. The evolving Wigner flows were investigated numerically for a harmonic oscillator and a driven Duffing oscillator in the bistable regime, the latter serving as an illustrative anharmonic system. Through the application of the two-dimensional Gauss’s theorem to boundary-enclosed, negative Wigner function regions on system phase space, we
FIG. 9. Simplified picture of negative-valued Wigner function regions indicated as red discs and the dominant positive-valued Wigner function regions indicated as blue semicircular arches. The indicated relative positioning of these regions in the $x > 0$ and $x < 0$ half-planes is dictated by the sign of the dot product $(0, -x) \cdot n$, where $n$ is the unit vector outwards normal to the negative region boundary circle $\partial A(t)$. In particular, $(0, -x) \cdot n < 0$ for the upper semicircle boundary segment in the $x > 0$ half-plane and $(0, -x) \cdot n < 0$ for the lower semicircle boundary segment in the $x < 0$ half-plane.

Saw that the formation and disappearance of the negative regions are governed solely by the so-called quantum flow due to the system anharmonicity and the diffusion flow across the negative region boundaries. By examining the form of these specific contributions to the total Wigner flow, we were able to gain some initial insights as to how negative regions form as a result of the system anharmonicity, as well as how they might be stabilized through the use of suitable system anharmonicities and drive tones.
FIG. 10. Snapshots showing the formation of the first negative Wigner region (appearing in the lower right quadrant) for the Duffing Oscillator at bath temperature $T = 0$. Wigner functions are scaled in order to resolve the initial negative region.
ACKNOWLEDGEMENTS

We thank Paul Nation for helpful discussions, as well as John Hudson and Susan Schwarz for their assistance with using the Dartmouth Discovery Cluster. This work was supported by the National Science Foundation under Grant No. DMR-1507383.

[1] E. Wigner, Phys. Rev. 40, 749 (1932).
[2] M. Hillery, R. F. O’Connell, M. O. Scully, and E. P. Wigner, Phys. Rep. 106, 121 (1984).
[3] W. B. Case, Am. J. Phys. 76, 937 (2008).
[4] T. L. Curtright, D. B. Fairlie, and C. K. Zachos, A Concise Treatment On Quantum Mechanics in Phase Space, (World Scientific, Singapore, 2014).
[5] W. H. Zurek and J. P. Paz, Phys. Rev. Lett. 72, 2508 (1994).
[6] S. Kohler, T. Dittrich, and P. Hänggi, Phys. Rev. E 55, 300 (1997).
[7] S. Habib, K. Shizume, and W. H. Zurek, Phys. Rev. Lett. 80, 4361 (1998).
[8] D. Monteloiva and J. P. Paz, Phys. Rev. E 64, 056238 (2001).
[9] S. Habib, K. Jacobs, H. Mabuchi, R. Ryne, K. Shizume, and B. Sundaram, Phys. Rev. Lett. 88, 040402 (2002).
[10] M. J. Everitt, T. D. Clark, P. B. Stiffell, J. F. Ralph, A. R. Bulsara, and C. J. Harland, Phys. Rev. E 72, 066209 (2005).
[11] M. I. Dykman, Phys. Rev. E 75, 011101 (2007).
[12] B. D. Greenbaum, S. Habib, K. Shizume, and B. Sundaram, Phys. Rev. E 76, 046215 (2007).
[13] I. Katz, R. Lifshitz, A. Retzker, and R. Straub, New. J. Phys. 10, 125023 (2008).
[14] M. Stobińska, G. J. Milburn, and K. Wódkiewicz, Phys. Rev. A 78, 013810 (2008).
[15] R. L. Hudson, Rep. Math. Phys. 6, 249 (1974).
[16] E. Bimbard, N. Jain, A. Macrae, and A. I. Lvovsky, Nat. Photonics 4, 243 (2010).
[17] J. Yoshikawa, K. Makino, S. Kurata, P. van Loock, and A. Furusawa, Phys. Rev. X 3, 041028 (2013).
[18] S. Delégise, I. Dotsenko, C. Sayrin, J. Bernu, M. Brune, J.-M. Raimond, and S. Haroche, Nature 455, 510 (2008).
[19] M. Hofheinz, H. Wang, M. Ansmann, R. C. Bialczak, E. Lucero, M. Neeley, A. D. O’Connell, D. Sank, J. Wenner, J. M. Martinis, and A. N. Cleland, Nature 459, 546 (2009).
[20] H. Wang, M. Hofheinz, M. Ansmann, R. C. Bialczak, E. Lucero, M. Neeley, A. D. O’Connell, D. Sank, M. Weides, J. Wenner, A. N. Cleland, and J. M. Martinis, Phys. Rev. Lett. 103, 200404 (2009).
[21] F. Mallet, M. A. Castellanos-Beltran, H. S. Ku, S. Glancy, E. Knill, K. D. Irwin, G. C. Hilton, L. R. Vale, and K. W. Lehnert, Phys. Rev. Lett. 106, 220502 (2011).
[22] C. Eichler, D. Bozyigit, C. Lang, L. Steffen, J. Fink, and A. Wallraff, Phys. Rev. Lett. 106, 220503 (2011).
[23] C. Eichler, D. Bozyigit, and A. Wallraff, Phys. Rev. A 86, 032106 (2012).
[24] Y. Shalibo, Y. Rofe, I. Barth, L. Friedland, R. Bialczack, J. M. Martinis, and N. Katz, Phys. Rev. Lett. 108, 037701 (2012).
[25] Y. Shalibo, R. Resh, O. Fogel, D. Shwa, Bialczack, J. M. Martinis, and N. Katz, Phys. Rev. Lett. 110, 100404 (2013).
[26] G. Kirchmair, B. Vlastakis, Z. Leghtas, S. E. Nigg, H. Paik, E. Ginossar, M. Mirrahimi, S. M. Girvin, and R. J. Schoelkopf, Nature 495, 205 (2013).
[27] B. Vlastakis, G. Kirchmair, Z. Leghtas, L. Frunzio, S. M. Girvin, M. Mirrahimi, M. H. Devoret, and R. J. Schoelkopf, Science 342, 607 (2013).
[28] C. Wang, Y. Y. Gao, P. Reinhold, R. W. Heeres, N. Ofek, K. Chou, C. Axline, M. Reagor, J. Blumoff, K. M. Sliwa, L. Frunzio, S. M. Girvin, L. Jiang, M. Mirrahimi, M. H. Devoret, and R. J. Schoelkopf, Science 352, 1087 (2016).
[29] S. Rips, M. Kifnner, I. Wilson-Rae, and M. J. Hartmann, New. J. Phys. 14, 023042 (2012).
[30] P. D. Nation, Phys. Rev. A 88, 053828 (2013).
[31] S. Rips, I. Wilson-Rae, and M. J. Hartmann, Phys. Rev. A 89, 013854 (2014).
[32] M. R. Vanner, I. Pikovski, and M. S. Kim, Ann. Phys. (Berlin) 527, 15 (2015).
[33] M. Abdi, P. Degenfeld-Schonburg, M. Sameti, C. Nataro, T. Byrnes, Phys. Rev. A 94, 063802 (2016).
[34] G. A. Baker Jr., Phys. Rev. 109, 2198 (1958).
[35] N. D. Cartwright, Physica 83 A, 210 (1976).
[36] C. Joana, P. van Loock, H. Deng, and T. Byrnes, Phys. Rev. A 94, 043802 (2016).
[37] J. F. Poyatos, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. 77, 4728 (1996).
[38] A. Sarlette, J. M. Raimond, M. Brune, and P. Rouchon, Phys. Rev. Lett. 107, 010402 (2011).
[39] Y. Lin, J. P. Gaebler, F. Reiter, T. R. Tan, R. Bowler, A. S. Sørensen, D. Leibfried, and D. J. Wineland, Nature 504, 415 (2013).
[40] S. Shankar, M. Hatridge, Z. Leghtas, K. M. Sliwa, A. Narla, U. Vool, S. M. Girvin, L. Frunzio, M. Mirrahimi, and M. H. Devoret, Nature 504, 419 (2013).
[41] A. Roy, Z. Leghtas, A. D. Stone, M. Devoret, and M. Mirrahimi, Phys. Rev. A 91, 013810 (2015).
[42] Z. Leghtas, S. Touzard, I. M. Pop, A. Kon, B. Vlastakis, A. Petrenko, K. M. Sliwa, A. Narla, S. Shankar, M. J. Hatridge, M. Reagor, L. Frunzio, R. J. Schoelkopf, M. Mirrahimi, and M. H. Devoret, Science 347, 853 (2015).
[43] F. Chen, A. J. Sirois, R. W. Simmonds, and A. J. Rimb erg, Appl. Phys. Lett. 98, 132509 (2011).
[44] M. P. Blencowe, A. D. Armour, and A. J. Rimb erg, in Fluctuating Nonlinear Oscillators: From Nanomechanics to Quantum Superconducting Circuits, ed. M. Dykman (Oxford University Press, Oxford, 2012).
[45] A. J. Rimberg, M. P. Blencowe, A. D. Armour, and P. D. Nation, New J. Phys. 16, 055008 (2014).

[46] F. Chen, J. Li, A. D. Armour, E. Brahim, J. Stettenheim, A. J. Sirois, R. W. Simmonds, M. P. Blencowe, and A. J. Rimberg, Phys. Rev. B 90, 020506 (2014).

[47] A. D. Armour, M. P. Blencowe, E. Brahimi, and A. J. Rimberg, Phys. Rev. Lett. 111, 247001 (2013).

[48] V. Gramich, B. Kubala, S. Rohrer, and J. Ankerhold, Phys. Rev. Lett. 111, 247002 (2013).

[49] A. D. Armour, B. Kubala, and J. Ankerhold, Phys. Rev. B 91, 184508 (2015).

[50] J.-R. Souquet and A. A. Clerk, Phys. Rev. A 93, 060301 (2016).

[51] S. Dambach, B. Kubala, and J. Ankerhold, arXiv:1609.08990.

[52] H. Bauke and N. R. Itzhak, arXiv:1101.2683v1.

[53] O. Steuernagel, D. Kakofengitis, and G. Ritter, Phys. Rev. Lett. 110, 030401 (2013).

[54] F. Haake, H. Risken, C. Savage, and D. Walls, Phys. Rev. A 34, 3969.

[55] J. R. Johansson, P. D. Nation, and F. Nori, Comput. Phys. Commun. 183, 1760 (2012); 184, 1234 (2013).

[56] See ancillary files accompanying this arXiv submission.

[57] O. D. Friedman, Supplementary videos for The Wigner Flow for Open Quantum Systems, https://goo.gl/vUx7Xa.

[58] M. S. Kim and V. Bužek, Phys. Rev. A 46, 4239 (1992).

[59] J. P. Paz, S. Habib, and W. H. Zurek, Phys. Rev. D 47, 488 (1993).

[60] F. Albarelli, T. Guaita, and M. G. A. Paris, Int. J. Quantum. Inf. 14, 1650032 (2016).

[61] O. D. Friedman and M. P. Blencowe, unpublished (2017).