Ghost-Free $F(R)$ Gravity with Lagrange Multiplier Constraint

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We propose two new versions of ghost-free generalized $F(R)$ gravity with Lagrange multiplier constraint. The first version of such theory for a particular degenerate choice of the Lagrange multiplier, corresponds to mimetic $F(R)$ gravity. The second version of such theory is just the Jordan frame description of mimetic gravity with potential. As we demonstrate, it is possible to realize several cosmological scenarios in such theory. In particular, de Sitter solutions may also be found.

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I. INTRODUCTION

Two of the most difficult to explain mysteries, that still haunt modern theoretical cosmology, are the dark energy and dark matter problems. The dark energy problem is related to the late-time acceleration that the Universe experiences at present time, which was firstly observed in the late 90’s, while the dark matter problem is a bit older, and various proposals have appeared in the theoretical physics literature, that may model dark matter in an adequate way. The most well-known description of dark matter is given by particle physics, however modified gravity (for review, see [1–6]) has appealing properties and even dark matter can occur in a geometric way. A recent description of dark matter by using a geometric approach, was offered by mimetic gravity [7], which firstly appeared in [8] and later was developed for cosmological purposes in [9–27], and also cosmological and astrophysical applications can be found in [28–32]. In Ref. [12], the Einstein-Hilbert mimetic theoretical framework was extended in the $F(R)$ mimetic gravity with Lagrange multiplier and potential. The mimetic $F(R)$ gravity theory has many appealing properties and it is possible to describe in a unified way inflation and the late-time acceleration era [12–15].

The purpose of this paper is to investigate certain theoretical problems of generalized Lagrange-multiplier $F(R)$ gravity, with the mimetic $F(R)$ gravity belonging to some subcases of the generalized Lagrange-multiplier $F(R)$ gravity. Specifically, we shall be interested in the appearance of ghosts in this kind of theories, and we discuss the theoretical techniques that eliminate the ghosts. It is known the mimetic $F(R)$ gravity theories with Lagrange multiplier, always contain ghosts, thus making the theory less appealing. However, as we demonstrate, the ghost may be eliminated by appropriately modifying the gravitational action. Particularly, we shall present two wide classes of generalized $F(R)$ gravity, in which the ghosts are absent. In the context of these cosmologies it is possible to realize various cosmological scenarios, and we present some concrete examples.

This paper is organized as follows: In section II, we demonstrate in detail how the ghost fields occur in the mimetic $F(R)$ gravity with Lagrange multiplier, which is a degenerated subcase of the generalized $F(R)$ gravity with Lagrange multipliers. In section III we present a general class of $F(R)$ gravities with Lagrange multiplier, which are free of ghosts. In section III, another ghost-free class of models is introduced, and several cosmological realizations are presented. Finally, the conclusions follow in the end of the paper.

Before we proceed, let us briefly mention the geometrical background which shall be assumed in this paper, which
is a flat Friedmann-Robertson-Walker (FRW) geometric background, with line element,
\[ ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2, \]
with \( a(t) \) being the scale factor of the Universe. Also, the connection is assumed to be a metric compatible, torsion-less and symmetric connection, the Levi-Civita connection.

II. EXISTENCE OF GHOST FIELDS IN MIMETIC LAGRANGE MULTIPLIER \( F(R) \) GRAVITY: A SHORT REVIEW

In this section, we shall demonstrate that ghost fields exist in the Lagrange multiplier \( F(R) \) gravity, following Ref. [35]. The action of the mimetic \( F(R) \) gravity with Lagrange multiplier is,
\[ S_{F(R)} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( F(R) + \lambda (\partial_\mu \phi \partial^\mu \phi + 1) \right), \]
where \( \lambda \) is the Lagrange multiplier and \( \phi \) is a scalar field. Let us now demonstrate how ghost fields may emerge in such a theory. The action of the mimetic \( F(R) \) model, can be rewritten by introducing the auxiliary field \( A \), as follows,
\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( F'(A) (R - A) + F(A) + \lambda (\partial_\mu \phi \partial^\mu \phi + 1) \right). \]
Upon variation of the action with respect to \( A \), we obtain the relation \( A = R \). Substituting \( A = R \) into the action, we can reproduce the action in Eq. (2). Furthermore, upon conformally transforming the metric in the following way,
\[ g_{\mu\nu} \rightarrow e^\sigma g_{\mu\nu}, \quad \sigma = -\ln F'(A), \]
the Einstein frame action is obtained,
\[ S_E = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R - \frac{3}{2} g^{\rho\sigma} \partial_\rho \sigma \partial_\sigma \sigma - V(\sigma) + \lambda \left( e^\sigma \partial_\mu \phi \partial^\mu \phi + e^{2\sigma} \right) \right), \]
\[ V(\sigma) = e^\sigma g(e^{-\sigma}) - e^{2\sigma} f(g(e^{-\sigma})) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2}. \]
In the above action, \( g(e^{-\sigma}) \) is given by solving the equation \( \sigma = -\ln (1 + f'(A)) = -\ln F'(A) \) as \( A = g(e^{-\sigma}) \). Upon variation of the action with respect to \( \lambda \), we obtain
\[ \sigma = \ln (-\partial_\mu \phi \partial^\mu \phi). \]
By substituting Eq. (3) in the action of Eq. (2), we obtain,
\[ S_E = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R - \frac{3}{2} (-\partial_\mu \phi \partial^\mu \phi)^{-2} g^{\rho\sigma} \partial_\rho (-\partial_\mu \phi \partial^\mu \phi) \partial_\sigma (-\partial_\mu \phi \partial^\mu \phi) \right. \]
\[ \left. - V(\sigma) \right). \]
Due to the fact that higher derivative terms occur, the existence of ghost fields is unavoidable.

In order to see this in a more rigid and formal way, we shall investigate the perturbations around the flat Minkowski spacetime. The existence of a Minkowski solution in Eq. (2) requires that the following condition holds true,
\[ V(\sigma) = V'(\sigma) = 0, \]
which implies,
\[ F(A) = 0, \]
and in effect, the solution is given by
\[ \lambda = 0, \quad \phi = t. \]
We consider the following perturbation,

$$\phi = t + \varphi,$$

and we expand the action with respect to $\varphi$. Then we find the following Lagrangian,

$$\mathcal{L}_\varphi = \frac{3}{2} (-\partial_\mu \phi \partial^\mu \phi)^2 g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi \partial_\sigma (-\partial_\mu \phi \partial^\mu \phi) - V(\sigma) \sim -6 \partial_\mu (\partial_\sigma \varphi) \partial^\mu (\partial_\sigma \varphi) - \alpha (\partial_\varphi)^2.$$  \hfill (12)

where $\alpha$ stands for,

$$\alpha \equiv \frac{F''''(0)}{2F''(0)^2} - \frac{6}{F''(0)}.$$  \hfill (13)

Since we are interested in the existence of ghost fields, we neglect the spatial derivatives in the above Lagrangian, and by introducing a new scalar field $\eta$, we rewrite the Lagrangian $\mathcal{L}_\varphi$ as follows,

$$\mathcal{L}_\varphi \sim -6 \partial_\eta \eta \partial_\varphi - \alpha (\partial_\varphi)^2 - 6 \lambda^2.$$  \hfill (14)

The matrix $M$ which is composed by the coefficients of the kinetic terms in the Lagrangian (14), are given by,

$$M = \begin{pmatrix} -\alpha & -3 \\ -3 & 0 \end{pmatrix}. $$  \hfill (15)

Due to the fact that the determinant of the matrix $M$ is negative $\det M = -9$, one of the two eigenvalues is always positive, but the other is always negative, and therefore the ghost fields definitely exist in the theory.

**III. GHOST-FREE GENERALIZED LAGRANGE MULTIPLIER F(R) GRAVITY: MODEL I**

As we explicitly demonstrated in the previous section, ghost fields appear in the model of Eq. (2), so in this section we shall consider a variant form of the model (2), in which no ghosts appear. The modified model has the following action,

$$S_{F(R)} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ F(R) + \lambda (\partial_\mu \phi \partial^\mu \phi + G(R)) \right\},$$

where $G(R)$ is an differentiable function of the scalar curvature $R$. As we did in the case of the action (3), we rewrite the action (12) as follows,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ (F'(A) + \lambda G'(A)) (R - A) + F(A) + \lambda (\partial_\mu \phi \partial^\mu \phi + G(A)) \right\}. $$ \hfill (17)

Again upon varying the action with respect to $A$, we obtain the equation $A = R$. Substituting $A = R$ in the action of Eq. (17), we reproduce the action of Eq. (16). Instead of (3), by using the following conformal transformation,

$$g_{\mu\nu} \rightarrow e^\sigma g_{\mu\nu}, \quad \sigma = - \ln (F'(A) + \lambda G'(A)), $$

we obtain the following Einstein frame action,

$$S_E = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ R - \frac{3}{2} g^{\rho\sigma} \partial_\rho \sigma \partial_\sigma \sigma - V(\sigma) + \lambda \left( e^\sigma \partial_\mu \phi \partial^\mu \phi + e^{2\sigma} G(A) \right) \right\}, $$

$$V(\sigma) = \frac{A}{F'(A)} - \frac{F(A)}{F''(A)^2}. $$ \hfill (19)

By using the second equation in (18), we may eliminate the function $\lambda$ as long as the condition $G'(A) \neq 0$ holds true, and we obtain,

$$S_E = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ R - \frac{3}{2} g^{\rho\sigma} \partial_\rho \sigma \partial_\sigma \sigma - V(A, \sigma) + \frac{e^{-\sigma} - F'(A)}{G'(A)} \left( e^\sigma \partial_\mu \phi \partial^\mu \phi + e^{2\sigma} G(A) \right) \right\}, $$

$$V(A, \sigma) = Ae^\sigma - F(A)e^{2\sigma}. $$ \hfill (20)
We should note that the model \( \text{(22)} \) corresponds to \( G(A) = 1 \) and therefore \( G'(A) = 0 \). By using the equation obtained when the action is varied with respect to \( A \),

\[
0 = \left( -\frac{F''(A)}{G'(A)} - \frac{(e^{-\sigma} - F'(A))G''(A)}{G'(A)^2} \right) \left( e^\sigma \partial_\mu \phi \partial^\mu \phi + e^{2\sigma} G(A) \right),
\]

we can find the function \( A \) as a function of \( \sigma \) and \( \partial_\mu \phi \partial^\mu \phi \) as \( A = A(\sigma, \partial_\mu \phi \partial^\mu \phi) \). In effect, Eq. \( \text{(20)} \) can be written as follows,

\[
S_E = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left[ R - \frac{3}{2} g^{\mu \sigma} \partial_\mu \sigma \partial_\nu \sigma - V(A(\sigma, \partial_\mu \phi \partial^\mu \phi), \sigma) + e^{-\sigma} - F'(A(\sigma, \partial_\mu \phi \partial^\mu \phi)) \left( e^\sigma \partial_\mu \phi \partial^\mu \phi + e^{2\sigma} G(A(\sigma, \partial_\mu \phi \partial^\mu \phi)) \right) \right].
\]

Although it is difficult to find the explicit form of \( A(\sigma, \partial_\mu \phi \partial^\mu \phi) \), the action \( \text{(22)} \) does not include any higher derivative terms, a result which is different from the one corresponding in the action of Eq. \( \text{(7)} \), as we showed in the previous section. Therefore the model of Eq. \( \text{(16)} \) is ghost free.

If \( \frac{F''(A)}{G'(A)} - \frac{(e^{-\sigma} - F'(A))G''(A)}{G'(A)^2} \neq 0 \), Eq. \( \text{(21)} \) gives,

\[
0 = e^\sigma \partial_\mu \phi \partial^\mu \phi + e^{2\sigma} G(A),
\]

which is nothing but the constraint equation in the Einstein frame given by the variation of \( \lambda \) in the Jordan frame action \( \text{(17)} \). The variations of the action \( \text{(20)} \) with respect to \( \sigma \), \( \phi \), and \( g_{\mu \nu} \) give

\[
0 = \frac{3}{2} \nabla_\mu \nabla^\mu \sigma - A e^\sigma + 2F(A)e^{2\sigma} - e^{-\sigma} \left( \frac{e^{\sigma} \partial_\mu \phi \partial^\mu \phi + e^{2\sigma} G(A)}{G'(A)} \right) + \frac{e^{-\sigma} - F'(A)}{G'(A)} \left( e^\sigma \partial_\mu \phi \partial^\mu \phi + e^{2\sigma} G(A) \right),
\]

\[
0 = \nabla^\mu \left( \frac{\delta F - F'(A)}{G'(A)} \right) \partial_\mu \phi,
\]

\[
0 = -R_{\mu \nu} + \frac{1}{2} g_{\mu \nu} R + \frac{1}{2} \left\{ -\frac{3}{2} g^{\mu \sigma} \partial_\mu \sigma \partial_\nu \sigma - V(A, \sigma) + \frac{e^{-\sigma} - F'(A)}{G'(A)} \left( e^\sigma \partial_\mu \phi \partial^\mu \phi + e^{2\sigma} G(A) \right) \right\} g_{\mu \nu}
\]

\[
+ \frac{3}{2} \partial_\mu \sigma \partial_\nu \sigma - \frac{(e^{-\sigma} - F'(A)) e^\sigma}{G'(A)} \partial_\mu \phi \partial_\nu \phi,
\]

We now consider the condition that the flat Minkowski space-time becomes a solution. Because \( A \) is nothing but the scalar curvature in the Jordan frame, we require \( A = 0 \) and we also assume that \( \sigma \) is a constant and \( \phi \) only depends on time \( t \),

\[
A = 0, \quad \sigma = \sigma_0, \quad \phi = \phi(t).
\]

Then Eq. \( \text{(25)} \) is trivially satisfied and Eqs. \( \text{(23)}, \text{(24)}, \) and \( \text{(26)} \) reduce to the following forms,

\[
0 = -e^{\sigma_0} \dot{\phi}^2 + e^{2\sigma_0} G(0),
\]

\[
0 = 2F(0)e^{2\sigma_0} + \frac{e^{-\sigma_0} - F'(0)}{G'(0)} \left( -e^{\sigma_0} \dot{\phi}^2 + 2e^{2\sigma_0} G(0) \right),
\]

\[
0 = \frac{1}{2} F(0) e^{2\sigma_0} - \frac{e^{-\sigma_0} - F'(0)}{G'(0)} e^{\sigma_0} \dot{\phi}^2
\]

\[
0 = \frac{1}{2} F(0) e^{2\sigma_0}.
\]

Then by using \( \text{(29)} \), we find

\[
0 = F(0) = \left( e^{-\sigma_0} - F'(0) \right) G(0),
\]

Eq. \( \text{(20)} \) can be solved to give

\[
\phi = \phi_0 \pm 1 e^{\sigma_0} \sqrt{G(0)}.
\]
Here $\phi_0$ is a constant. In order to investigate if there is a ghost or not, we consider the perturbation from the flat Minkowski space-time. By using (33), we consider the case that $F(0) = G(0) = 0$ and the perturbation from the solution given by (28) and (34),

$$A = \delta A, \quad \sigma = \sigma_0 + \delta \sigma, \quad \phi = \phi_0 + \delta \phi.$$  \hfill (35)

Then the scalar part in the action (20) has the following form,

$$S_E = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ R - \frac{3}{2} \partial_\mu \delta \sigma \partial^\mu \delta \sigma - e^{\sigma_0} \delta A \delta \sigma + 2e^{2\sigma_0} F'(0) \delta A \delta \sigma \\
+ \frac{(e^{-\sigma_0} - F'(0)) e^{\sigma_0}}{G'(0)} \left( \partial_\mu \delta \phi \partial^\mu \delta \phi + \frac{1}{2} e^{2\sigma_0} G''(0) \delta A^2 + 2e^{2\sigma_0} G'(0) \delta \sigma \delta A \right) \\
+ \left( -e^{-\sigma_0} \delta A - F''(0) \delta A - \frac{(e^{-\sigma_0} - F'(0)) G''(0)}{G'(0)} \delta A \right) e^{2\sigma_0} \delta A \right\}. \hfill (36)$$

The equation given by the variation with respect to $\delta A$ gives $\delta A$ in terms of $\sigma$. Then by substituting the expression $\delta A = C \delta \sigma$ with a constant $C$, we obtain the mass term for $\delta \sigma$. The action (36) tells that as long as the following relation holds true,

$$\frac{e^{-\sigma_0} - F'(0)}{G'(0)} < 0,$$  \hfill (37)

the ghost does not appear.

By varying the action (10) with respect to the function $\lambda$ and with respect to the scalar field $\phi$, we obtain the following equations,

$$0 = \partial_\mu \phi \partial^\mu \phi + G(R), \hfill (38)$$
$$0 = \nabla_\mu (\lambda \partial_\mu \phi), \hfill (39)$$

On the other hand, upon variation of the action with respect to the metric $g_{\mu\nu}$, we obtain,

$$0 = \frac{F(R)}{2} g_{\mu\nu} - (F'(R) + \lambda G'(R)) R_{\mu\nu} - \lambda \partial_\mu \phi \partial_\nu \phi + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2) (F'(R) + \lambda G'(R)). \hfill (40)$$

Let us now demonstrate how the gravitational equations of the model (10) become, when a specific cosmological background is considered. We assume that the geometric background is flat a FRW metric with line element of the form of Eq. (11), and also that the function $\lambda$ and also the scalar field $\phi$ depend only on the cosmic time $t$. In effect, the Eqs. (35) and (36), take the following form,

$$0 = -\dot{\phi}^2 + G(R), \quad 0 = \frac{d}{dt} \left( a^3 \lambda \dot{\phi} \right), \hfill (41)$$

which can be rewritten as follows,

$$\dot{\phi} = \pm \sqrt{G(R)}, \quad a^3 \lambda \dot{\phi} = C,$$  \hfill (42)

where $C$ is an integration constant. Also, the $(t, t)$ and $(i, j)$ components of Eq. (40) yield the following equations,

$$0 = -\frac{F(R)}{2} + 3 \left( \dot{H} + H^2 \right) \left( F'(R) \pm \frac{CG'(R)}{a^3 \sqrt{G(R)}} \right) \pm \frac{C \sqrt{G(R)}}{a^3} - 3H \frac{d}{dt} \left( F'(R) \pm \frac{CG'(R)}{a^3 \sqrt{G(R)}} \right), \hfill (43)$$
$$0 = \frac{F(R)}{2} - \left( \dot{H} + 3H^2 \right) \left( F'(R) \pm \frac{CG'(R)}{a^3 \sqrt{G(R)}} \right) + \left( \frac{d^2}{dt^2} + 2H \frac{d}{dt} \right) \left( F'(R) \pm \frac{CG'(R)}{a^3 \sqrt{G(R)}} \right). \hfill (44)$$

If we define a new quantity $J(R, a)$ as follows,

$$J(R, a) \equiv F(R) \pm \frac{2C \sqrt{G(R)}}{a^3},$$  \hfill (45)
Eq. (43) can be rewritten in the following form,

\[ 0 = - \frac{J(R, A)}{2} + 3 \left( \dot{H} + H^2 - 3H \frac{d}{dt} \right) \frac{\partial J(R, a)}{\partial R}. \]  

(46)

We should note that when \( C = 0 \), Eqs. (43) and (44) become identical to the equations of the standard \( F(R) \) gravity, which indicates that any solution of the standard \( F(R) \) gravity is also a solution of the model (16).

An analytic form for the \( F(R) \) and \( G(R) \) gravity, can be given if the de Sitter spacetime is considered, in which case \( H = H_0 \) and \( a = e^{H_0 t} \). Then Eqs. (43) and (44) can be cast in the following form,

\[ 0 = - \frac{F(R_0)}{2} + 3H_0^2 F'(R_0) \pm \frac{C}{a^3 \sqrt{G(R_0)}} (12H_0^2 G'(R_0) - G(R_0)), \]  

(47)

\[ 0 = \frac{F(R_0)}{2} - 3H_0^3 F'(R_0), \]  

(48)

where \( R_0 = 12H_0^2 \). Then in order for the solution describing the de Sitter space-time to exist, the functions \( F(R) \) and \( G(R) \) must simultaneously satisfy the following differential equations,

\[ 0 = 2F(R_0) - R_0 F'(R_0), \quad 0 = R_0 G'(R_0) - G(R_0). \]  

(49)

A special solution to the differential equations (49) is the following,

\[ F(R) = aR^2, \quad G(R) = \beta R, \]  

(50)

and both the differential equations (49) are satisfied. Note that other examples of such theory leading to de Sitter space maybe found.

**IV. GHOST-FREE GENERALIZED \( F(R) \) GRAVITY: MODEL II**

Another ghost-free model of generalized \( F(R) \) gravity, can be obtained in the Einstein frame, if the scalar fields \( \tilde{\lambda} \) and \( \phi \) are introduced in the Lagrangian as follows [9],

\[ S_E = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ R - \frac{3}{2} g^{\sigma \tau} \partial_{\sigma} \sigma \partial_{\tau} \sigma - V(\sigma) + \tilde{\lambda} (\partial_{\mu} \phi \partial^{\mu} \phi + 1) \right\}. \]  

(51)

By applying the inverse of the transformation (4), we obtain,

\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ F'(A) (R - A) + F(A) + \lambda (\partial_{\mu} \phi \partial^{\mu} \phi + F'(A)) \right\}, \]  

(52)

where \( \lambda = F'(A) \tilde{\lambda} \). Upon varying the action with respect to \( A \), we obtain the following equation,

\[ A = R + \lambda. \]  

(53)

Then by substituting Eq. (53) in the action (52), we obtain the following action,

\[ S_{F(R)} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ F(R + \lambda) + \lambda \partial_{\mu} \phi \partial^{\mu} \phi \right\}, \]  

(54)

which is the action of the mimetic \( F(R) \) gravity without ghost. If we further redefine \( \lambda \) as follows \( \lambda \to \lambda - R \), we obtain the following action,

\[ S_{F(R)} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ F(\lambda) + (\lambda - R) \partial_{\mu} \phi \partial^{\mu} \phi \right\}, \]  

(55)

If we assume that the leading order of \( F(\lambda) \) is linear,

\[ F(\lambda) = \lambda + O(\lambda^2), \]  

(56)
or equivalently,

\[ F(R + \lambda) = R + \lambda + O(R + \lambda^2), \tag{57} \]

the leading order in the action (54) is effectively the standard Einstein action with the mimetic constraint,

\[ S = \frac{1}{2\kappa^2} \int d^4x\sqrt{-g} \left\{ R + \lambda (\partial_\mu \phi \partial^\mu \phi + 1) + O(R + \lambda^2) \right\}. \tag{58} \]

The variation of the action (55) with respect to the scalar fields \( \lambda, \phi \) and also with respect to the metric \( g_{\mu\nu} \), gives the following equations,

\[ 0 = F'(\lambda) + \partial_\mu \phi \partial^\mu \phi, \tag{59} \]
\[ 0 = \nabla_\mu \left\{ (\lambda - R) \partial^\mu \phi \right\}, \tag{60} \]
\[ 0 = \frac{1}{2} \left\{ F(\lambda) + (\lambda - R) \partial_\mu \phi \partial^\mu \phi \right\} g_{\mu\nu} + R_{\mu\nu} \partial_\rho \phi \partial^\rho \phi - \nabla_\mu \nabla_\nu (\partial_\rho \phi \partial^\rho \phi) + g_{\mu\nu} \nabla^2 (\partial_\rho \phi \partial^\rho \phi). \tag{61} \]

By assuming a spatially flat FRW universe \(^1\) and also that \( \lambda \) and \( \phi \) depend only on the cosmic time coordinate \( t \), the above equations (59), (60), and (61) take the following form,

\[ 0 = F'(\lambda) - \dot{\phi}^2, \tag{62} \]
\[ 0 = \frac{d}{dt} \left\{ a^3 \left( \lambda - 6\dot{H} - 12H^2 \right) \dot{\phi} \right\}, \tag{63} \]
\[ 0 = -\frac{1}{2} \left\{ F(\lambda) - (\lambda - 6H^2) \dot{\phi}^2 \right\} - 3H \frac{d}{dt} \left( \dot{\phi}^2 \right). \tag{64} \]
\[ 0 = \frac{1}{2} \left\{ F(\lambda) - (\lambda - 4\dot{H} - 6H^2) \dot{\phi}^2 \right\} + \left( \frac{d^2}{dt^2} + 4H \frac{d}{dt} \right) \left( \dot{\phi}^2 \right). \tag{65} \]

In effect, Eq. (63) can be integrated and it yields,

\[ C = a^3 \left( \lambda - 6\dot{H} - 12H^2 \right) \dot{\phi}, \tag{66} \]

with \( C \) being again a constant of the integration. By using Eq. (62), we may eliminate \( \dot{\phi} \) from Eqs. (64), (65), and (66) as follows,

\[ 0 = -\frac{1}{2} \left\{ F(\lambda) - (\lambda - 6H^2) F'(\lambda) \right\} - 3H \frac{dF'(\lambda)}{dt}. \tag{67} \]
\[ 0 = \frac{1}{2} \left\{ F(\lambda) - \left( \lambda - 4\dot{H} - 6H^2 \right) F'(\lambda) \right\} + \left( \frac{d^2 F'(\lambda)}{dt^2} + 4H \frac{dF'(\lambda)}{dt} \right), \tag{68} \]
\[ C^2 = a^6 \left( \lambda - 6\dot{H} - 12H^2 \right)^2 F'(\lambda). \tag{69} \]

By using Eqs. (67) and (68), we may eliminate \( F(\lambda) \) and we obtain,

\[ 0 = \frac{d^2F'(\lambda)}{dt^2} + H \frac{dF'(\lambda)}{dt} + 2\dot{H} F'(\lambda). \tag{70} \]

Let us now use the formalism we just presented in order to relaizd various cosmological scenarios. We shall assume that \( C = 0 \) in Eqs. (66) and (69). Then we obtain,

\[ \lambda = 6\dot{H} + 12H^2. \tag{71} \]

First we consider the power-law expansion of the Universe, in which case the Hubble rate is,

\[ a \propto t^{h_0} \text{ or } H = \frac{h_0}{t}. \tag{72} \]

Then Eq. (71) yields,

\[ \lambda = -6h_0 + 12h_0^2. \tag{73} \]
On the other hand, by assuming $F'(\lambda) \propto t^0$, Eq. (70) yields,

$$0 = f_0^2 + (h_0 - 1) f_0 - 2h_0,$$

which when solved yields,

$$f_0 = \frac{1 - h_0 \pm \sqrt{h_0^2 + 6h_0 + 1}}{2}.$$  \hspace{1cm} (75)

Therefore $F'(\lambda) \propto \lambda^{-\frac{3}{2}}$ and therefore we have,

$$F(\lambda) \propto \lambda^{1 - \frac{h_0}{2}} = \lambda^{\frac{1 + h_0 \mp \sqrt{h_0^2 + 6h_0 + 1}}{2}}.$$  \hspace{1cm} (76)

Now let us consider the de Sitter space-time, where $H$ is a constant $H = H_0$, by assuming $C = 0$, again. Then Eq. (64) gives,

$$\lambda = 12H_0^2.$$  \hspace{1cm} (77)

Since $\lambda$ is a constant, $F(\lambda)$ and $F'(\lambda)$ are also constants. Then Eq. (67) or Eq. (68) yields,

$$0 = F(\lambda) - (\lambda - 6H_0^2) F'(\lambda) = F(\lambda) - \frac{\lambda}{2} F'(\lambda),$$

which can be integrated and yields

$$F(\lambda) \propto \lambda^2.$$  \hspace{1cm} (79)

Hence, we demonstrated that even in this ghost-free model, various cosmological scenarios can be realized.

V. CONCLUSIONS

In this paper we presented two ghost-free generalized $F(R)$ gravity models, which are variants to mimetic $F(R)$ gravity models, without the ghost fields. After we discussed how ghosts may occur in the mimetic $F(R)$ gravity models, we presented the first model, which is a Lagrange multiplier $F(R)$ gravity model. Also we presented a second model of standard mimetic gravity for which the Lagrangian is considered in the Einstein and in the Jordan frame, and we demonstrated that several cosmological scenarios can be realized. In principle, other cosmological solutions maybe constructed, such as bouncing cosmologies or even alternative inflationary scenarios, and also singular inflationary scenarios. We shall address these issues in a future work.

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