Optimal work extraction and thermodynamics of quantum measurements and correlations

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We analyze the role of indirect quantum measurements in work extraction from quantum systems in nonequilibrium states. In particular, we focus on the work that can be obtained by exploiting the correlations shared between the system of interest and an additional ancilla, where measurement backaction introduces a nontrivial thermodynamic tradeoff. We present optimal state-dependent protocols for extracting work from both classical and quantum correlations, the latter being measured by discord. We show that, while the work content of classical correlations can be fully extracted by performing local operations on the system of interest, the amount of work related to quantum discord requires a specific driving protocol that includes interaction between system and ancilla.

PACS numbers: 05.70.Ln, 05.70.-a 05.30.-d 03.67.-a 42.50.Dv

I. INTRODUCTION

One of the aims of quantum thermodynamics [1, 2] is the precise identification of the role of genuine quantum resources, such as coherence [3], correlations [4] or squeezing [5], both in the performance of thermodynamic tasks by nanomachines [6–8], and, on a more fundamental ground, in the description of finite-time non-equilibrium thermodynamic processes [9–11]. In this context, interest has been raised toward the process of work extraction from quantum systems [12, 13], and on its enhancement in feedback protocols [14]. Although entanglement generation is not essential for optimal work extraction from quantum batteries [15], it can, nevertheless, be exploited to increase the amount of extractable work [16–18]. Quantum discord can enter prominently in enhancing the performance of Maxwell demons [19], heat engines [20] and work extraction protocols [21] as well, and both entanglement and discord have been shown to play a role in the work gain obtained thanks to a feedback enhanced extraction protocol [22]. However, obtaining quantitative connections between the extracted work and both classical and quantum correlations has been shown very challenging, since these may depend on the allowed operations [18].

The prototypical scenario considers a cyclic unitary transformations to extract work from a quantum system $S$ [12]; this case may be extended by considering that $S$ shares correlations with an ancilla $A$ [22]. Moreover, we can generalize the unitary transformation setting by including thermalization processes, whose role, as we will shortly see, may be beneficial. The scope of this letter is to obtain quantitative relations between the optimal work gain and classical and quantum correlations in this general framework of work extraction, where access is allowed to a thermal reservoir [23, 24], and feedback is provided by a measurement performed on $A$. We will show that a tight link exists between the optimal work gain obtainable in presence of feedback and the classical correlations. Turning to the role of quantum correlations, we will introduce a work contribution due to quantum discord, arguing that it cannot be extracted in a feedback enforced protocol, as it is unavoidably lost after a local measurement is performed. However, an improved protocol can be designed, in which the work content of quantum discord can be possibly extracted before the measurement and the feedback enhanced protocol are performed. In doing this, we will elucidate the role and the energetic value of both classical and quantum correlations and, at the same time, discuss the energetic cost of the measurement that is necessary to provide feedback. Contrary to the classical case, quantum measurements introduce a tradeoff between the gain in extractable work due to the measurement-induced local entropy reduction, and its loss due to correlations erosion. Despite this, we will show that the total amount of work extractable with the generalized protocol can overcome the one obtained without measurement and feedback, provided optimal measurements are performed.

II. SETUP

We consider work extraction from a quantum system with Hamiltonian $H_S$ in an arbitrary nonequilibrium state $\rho_S$. In the most simple situation, only unitary operations are allowed, where the Hamiltonian changes according to some cyclic protocol, in which $H_S$ is the same Hamiltonian before and after the operation. In such a case, the maximum work that can be extracted from an initial (non-passive) state $\rho_S$ is the so-called ergotropy $W^S$ [12]. This framework can be naturally extended, with a performance enhancement, by including also non-unitary transformations. In particular, if, in addition to
unitary cyclic driving, contact with a thermal reservoir is also allowed, the system entropy may vary during the protocol, and this can be used to increase the extracted work. In this case, the maximum amount of extractable work is given by the difference in nonequilibrium free energy between the state \( \rho_S \) and the thermal equilibrium state at the reservoir inverse temperature \( \beta = 1/k_B T \) [23, 24]

\[
W_{\text{ext}} \leq W_{S,\beta} = F_{\beta}(\rho_S) - F_{\beta}(\rho_S^\beta) = k_B T D(\rho_S || \rho_S^\beta),
\]

where \( \rho_S^\beta = e^{-\beta H_S}/Z_S \) is the equilibrium state, while \( D(\rho || \sigma) = \text{Tr}[\rho \log \rho - \log \sigma] \) stands for the quantum relative entropy [25, 26]. The nonequilibrium free energy for a system in state \( \rho_S \), with Hamiltonian \( H_S \), and with respect to a thermal bath at temperature \( T \) is defined as

\[
F_{\beta}(\rho_S) = \text{Tr}[H_S \rho_S] - k_B T S(\rho_S),
\]

where \( S(\rho) = -\text{Tr}[\rho \log \rho] \) is the von Neumann entropy, and where, for thermal states \( F_{\beta}(\rho_S^\beta) = -k_B T \ln Z_S \) reduces to the Helmholtz free energy. Equality in Eq. (1) may be obtained by implementing an operationally reversible isothermal process [13, 23, 24]. This is made up of two steps: first, a sudden quench is performed, in which the Hamiltonian \( H_S \) is changed into \( H_{\rho_S} = -k_B T \ln \rho_S \); then, a quasi-static isothermal transformation follows, during which the Hamiltonian turns back to \( H_S \), while the system is kept in contact with the heat bath. In this second step, the system always stays in equilibrium with the reservoir, ending up in the state \( \rho_S^\beta \) [23, 24]. Such an isothermal transformation can be constructed by means of an infinite sequence of quantum maps acting over infinitesimal time-steps (the demonstration is left to appendix A). This optimal isothermal work extraction procedure always outperforms cyclic unitary protocols: independently of the temperature, one can show that the decrease in free energy is larger than the entropy production, \( W_{S,\beta} \geq W_S, \forall \beta \), where the equality is achieved only when the temperature verifies \( S(\rho_S^\beta) = S(\rho_S) \) (the proof is given in appendix B). Notice that the presence of the environment plays here a constructive role, increasing our ability to extract work.

In the following, we will extend the optimal isothermal protocol to the case in which the system of interest \( S \) is prepared in a joint state \( \rho_{SA} \) with an uncoupled ancillary system \( A \), with which it may share classical and/or quantum correlations. Specifically, the total amount of correlations between the two parts can be measured by the quantum mutual information \( I(\rho_{SA}) = D(\rho_{SA} || (\rho_S \otimes \rho_A)) \geq 0 \), where \( \rho_S = \text{Tr}_A[\rho_{SA}] \) and \( \rho_A = \text{Tr}_S[\rho_{SA}] \) are the marginal (reduced) states.

We will first show that the amount of work extractable from the system of interest increases when some information is provided after a measurement is performed on the ancilla. In this way, we provide a generalization to the case of general entropy-changing transformations of the result of Ref. [22] for the ergotropy.

To start with, let us define \( W_{S|\Pi_A}^\beta \) as the maximum amount of work extractable from \( S \) by exploiting the feedback obtained from a measurement performed on \( A \). In particular, we consider a projective measurement, described by the set of projectors \( \Pi_A = \{\Pi_A^k\} \) for \( k = 1, ..., d_A \). After optimizing the extracted work over all possible sets of projectors \( \Pi_A \) (that is, over all possible measurements on \( A \)), we define \( W_{S|A}^\beta = \max_{\Pi_A} W_{S|\Pi_A}^\beta \) (see Fig. 1).

The measurement affects both the ancilla and the system state. In particular, if the outcome \( k \) occurs (with probability \( p_k = \text{Tr}[\Pi_A^k \rho_{SA}] \)), then the state of the system is updated to

\[
\rho_S \rightarrow \rho_{S|\Pi_A^k} = \text{Tr}_A[(I_S \otimes \Pi_A^k)\rho_{SA}(I_S \otimes \Pi_A^k)]/p_k,
\]

\( I_S \) being the identity operator for \( S \).

For an initially uncorrelated \( SA \) state, that is, for \( \rho_{SA} = \rho_S \otimes \rho_A \) — or equivalently \( I(\rho_{SA}) = 0 \) — measurements on the ancilla do not induce any change in the system state.

III. EXTRACTING WORK FROM CLASSICAL CORRELATIONS

In the optimal isothermal protocol discussed above, the maximum work that can be extracted from average from \( S \) is increased at best by the amount of classical correlations initially present in the state \( \rho_{SA} \).

To show this, we start by considering that, if the ancilla is subjected to the projective measurement introduced above, when outcome \( k \) is obtained, \( S \) suffers from the back-action corresponding to Eq. (3), and this amounts to a change in the system’s free energy

\[
\Delta F_{S|\Pi_A^k}^\beta = F_{\beta}(\rho_{S|\Pi_A^k}) - F_{\beta}(\rho_S).
\]

The outcome \( k \) being known, we may adapt the optimal isothermal protocol introduced above, which now will depend on \( k \): in the first step, a \( k \)-dependent sudden quench of the Hamiltonian is performed, with \( H_S \rightarrow S

FIG. 1. (color online) Starting from \( \rho_{SA} \), work can be extracted from \( S \) either by a direct isothermal protocol [path (a)], or by first performing a measurement on \( A \), and then applying an outcome dependent isothermal protocol [path (b)].
\[ H_{S|\Pi^k_A} = -k_B T \ln \rho_{S|\Pi^k_A} \], which is then quasi-statically brought back to \( H_S \) while in contact with the thermal reservoir. This process requires precise knowledge of the state \( \rho_{S|\Pi^k_A} \), which in turn implies knowing the initial state \( \rho_{SA} \) and the set of projectors \( \pi_A \).

With the same argument recalled above, one may conclude that the maximum amount of work extractable from the state \( \rho_{S|\Pi^k_A} \) is given by \( \mathcal{W}^3_{S|\Pi^k_A} = \mathcal{F}_3(\rho_{S|\Pi^k_A}) - \mathcal{F}_3(\rho_S^3) \). The average work extracted after many repetitions of this process is then,

\[
\mathcal{W}^3_{S|\pi_A} = \sum_k p_k \mathcal{W}^3_{S|\Pi^k_A} = \sum_k p_k \Delta \mathcal{F}^3_{S|\Pi^k_A} + \mathcal{W}^3_S, \tag{5}
\]

where, in the last equality, we used Eqs. (4) and (1). This implies an average increase of the work extracted during the process by an amount

\[
\Delta \mathcal{W}^3_{S|\pi_A} = \sum_k p_k \Delta \mathcal{F}^3_{S|\Pi^k_A} = k_B T [S(\rho_S) - \sum_k p_k S(\rho_{S|\Pi^k_A})] = k_B T J(\rho_{SA})_{\pi_A} \geq 0, \tag{6}
\]

where \( \Delta \mathcal{W}^3_{S|\pi_A} = \mathcal{W}^3_{S|\pi_A} - \mathcal{W}^3_S \) is the gain in extractable work, and where, in the last equality, we denoted as

\[ J(\rho_{SA})_{\pi_A} = S(\rho_S) - \sum_k p_k S(\rho_{S|\Pi^k_A}) \]

the mutual information on subsystem \( S \) extracted by local measurements on \( A \), using the set of projectors \( \pi_A = \{\Pi^k_A\} \). The same quantity, \( J(\rho_{SA})_{\pi_A} \), has been obtained for feedback controlled protocols in Ref. [29]. The inequality in (6) follows directly from concavity of von Neumann entropy and implies that an average enhancement in the extracted work is found for any measurement. No gain is obtained only if \( \rho_{SA} \) is factorized; while, if \( S \) and \( A \) are correlated to some extent, the extractable work can increase thanks to the feedback coming from the knowledge of the measurement outcome. Intuitively, this is due to the fact that a measurement can increase the free energy of \( S \) [24]. A sketch of the protocol and of this result is given in the upper panel of Fig. 1.

If we now maximize Eq. (6) over all sets of projectors \( \pi_A \), we obtain that the maximum enhancement in work extraction \( \Delta \mathcal{W}^3_{S|A} = \max_{\pi_A} \Delta \mathcal{W}^3_{S|\pi_A} \) reads

\[
\Delta \mathcal{W}^3_{S|A} = \max_{\pi_A} J(\rho_{SA})_{\pi_A} = k_B T J(\rho_{SA}), \tag{7}
\]

where \( J(\rho_{SA}) \) quantifies the classical correlations in state \( \rho_{SA} \), as defined in Refs. [27, 28]. Eq. (7) is the first of our main results; it tells us that the gain in the work extracted from \( S \) obtained by use of the feedback protocol in which \( A \) is measured, is due to (and upper bounded by) the classical correlations shared by \( S \) and \( A \).

Even if quantum correlations do not contribute to Eq. (7), this does not imply that they do not play any role, as we will see in the remainder of this letter. First of all, our result gets a clear physical interpretation if discord is used to understand it. By definition, quantum discord gives the amount of correlations present in a bipartite quantum state, which cannot be accessed by local measurements on one party [27]

\[
\mathcal{D}(\rho_{SA}) \equiv I(\rho_{SA}) - J(\rho_{SA}) \geq 0. \tag{8}
\]

Therefore, intuition dictates that as long as this information is not available from measuring the ancilla \( A \), it cannot be used in any way to improve our ability of extracting work from \( S \). More precisely, the work extractable from the whole \( SA \) system in the state \( \rho_{SA} \) is given by the free-energy difference between this state and the thermal reference one,

\[
\mathcal{W}^3_{SA}(\rho_{SA}) = \mathcal{F}_3(\rho_{SA}) - \mathcal{F}_3(\rho_S^3 \otimes \rho_A^3) = \mathcal{W}^3_S(\rho_S) + \mathcal{W}^3_A(\rho_A) + k_B T I(\rho_{SA}), \tag{9}
\]

where \( \mathcal{W}^3_A(\rho_A) = \mathcal{F}_3(\rho_A) - \mathcal{F}_3(\rho_A^3) \geq 0 \) is the work locally extractable from the ancilla \( A \) without using measurements. From Eq. (7), it then follows that the work extractable from \( S \) through the optimal isothermal protocol supplemented by the feedback scheme, \( \mathcal{W}^3_{SA} \), plus the work extractable from \( \rho_A \), can never exceed \( \mathcal{W}^3_{SA} \);

\[
\mathcal{W}^3_{SA} + \mathcal{W}^3_A(\rho_A) = \mathcal{W}^3_{SA}(\rho_{SA}) - k_B T \mathcal{D}(\rho_{SA}). \tag{10}
\]

Equation (10) has a clear interpretation. The intrinsic irreversibility of the measurement process destroys the quantum correlations present in state \( \rho_{SA} \), as measured by quantum discord. As a consequence, the work extractable from system and ancilla decreases by an amount \( k_B T \mathcal{D}(\rho_{SA}) \), which corresponds to the work value of quantum correlations in the state \( \rho_{SA} \). Result (7) is then an exact expression stressing the deep link between work and knowledge. This interpretation of the role of discord agrees with that provided in Refs. [19], when comparing local and global Maxwell demon-like configurations.

\section*{IV. THERMODYNAMIC TRADEOFF OF QUANTUM MEASUREMENT}

In the above discussion, we naively summed up the two extractable works obtained i) from \( S \), with the optimal protocol including feedback, and, separately, ii) from \( A \). Although providing a nice interpretation for the work content of quantum discord, this does not properly take into account the measurement back-action on \( A \), as \( \mathcal{W}^3_A \) would be the work extractable from \( A \) if no measurement had been performed. In fact, the projective measurement, in the first stage of the feedback scheme, modifies the whole \( SA \)-state. After the \( k \)-th outcome, one has

\[
\rho_{SA} \rightarrow \rho_{S|\Pi^k_A} \otimes \Pi^k_A, \tag{11}
\]

Then, one may ask how the work extracted from \( SA \) in presence of the feedback gets modified and wether it can in fact surpass \( \mathcal{W}^3_{SA} \) in Eq. (9). To answer this question,
we consider the gain in work extraction obtained from the true post-measurement state, with respect to $W^\beta_{SA}$, i.e. $\Delta W^\beta_{SA|\pi_A} = \sum_k p_k F_\beta(\rho_{S|\pi_A} \otimes \Pi^k_A) - F_\beta(\rho_{SA})$.

To perform a proper energy balance in presence of the measurement process, we should also consider its work cost. If $H_A$ is the Hamiltonian of the ancilla, and if $\rho_{A|\pi_A} = \sum_k p_k \Pi^k_A$ is its unconditional, post-measurement state, then the cost $C(\pi_A) \equiv \text{Tr}[H_A(\rho_{A|\pi_A} - \rho_A)]$ corresponds to the work needed to perform the measurement $\pi_A$. It vanishes as soon as measurements are performed in the energy eigenbasis, $[\Pi^k_A, H_A] = 0$, or when energy-less ancillas are considered ($H_A \propto 1_A$). More importantly, if the optimal set of projectors $\pi^\text{opt}_A$ is taken, which maximizes the extracted classical information in Eq. (7), we have

$$\Delta W^\beta_{SA|\pi^\text{opt}_A} - C(\pi^\text{opt}_A) = k_B T [S(\rho_{SA}) - \sum_k p_k S(\rho_{S|\pi^k_A})]$$

$$= k_B T [S(\rho_A) - D(\rho_{SA})] \geq 0, \quad (12)$$

where the final inequality in Eq. (12) follows from the fact that discord is always bounded from above by the entropy of the measured system [30].

The above Eq. (12) is the second of our main results. It remarkably ensures that the amount of extractable work from system and ancilla do not decrease when using optimal quantum measurements and feedback in the work extraction process, even if the cost of the measurement is properly accounted for and subtracted. The interpretation of the two terms above becomes clear if one notices that the measurement induced free energy change can be written $\Delta F^\beta_A \equiv \sum_k p_k F_\beta(\Pi^k_A) - F_\beta(\rho_A) = C(\pi_A) + k_B T S(\rho_A)$. Thus, even if the quantum measurement produces a decrease in the extractable work of the composite system by an amount $k_B T D(\rho_{SA})$, corresponding to the loss of quantum discord, this is always (over-)compensated by an increase, $\Delta W^\beta_A = \Delta F^\beta_A$, of the work locally extractable from the ancillary system after the measurement. Indeed, this provides both a compensation for the measurement cost, as well as the extra work amount $k_B T S(\rho_A)$, exceeding the work value of discord. It is worth noticing here that if the optimal set of projectors $\pi^\text{opt}_A$ were not used, then $\Delta W^\beta_{SA|\pi^\text{opt}_A} \geq C(\pi^\text{opt}_A)$ cannot be ensured anymore, and the tradeoff between the gain in extractable work due to the measurement, and its reduction due to correlation erasure may give a detrimental result, implying that the direct work extraction from $\rho_{SA}$ (without using measurements) is the best option.

V. EXTRACTING WORK FROM QUANTUM CORRELATIONS

Finally, we are interested in the possibility of extracting the work content of quantum correlations as well, without renouncing to the benefits of the measurement.

This may seem impossible at first sight, as including projective quantum measurements will eventually produce the loss of discord in state $\rho_{SA}$, as we already discussed. We propose a protocol for which this can be circumvented extracting the work content of quantum correlations before the projective measurement is performed on the ancilla. This means including a new initial step $\rho_{SA} \rightarrow \rho'_{SA}$ in the extraction protocol, performed before measurement and isothermal driving, sketched as step (c) in Fig. 2. Such a step unavoidably requires interaction between system and ancilla.

Leaving considerations motivating the construction of this reversible sub-process to appendix C, we require a final state of the step with zero quantum correlations, but intact classical ones

$$\rho_{SA} = \sum_k p_k \rho_{S|\Pi^k_A} \otimes \Pi^k_A. \quad (13)$$

where, once again, the projectors $\Pi^k_A$ are taken from the optimal set $\pi^\text{opt}_A$. The step goes as follows: First we perform a sudden quench of the total Hamiltonian, so that $H_S + H_A \rightarrow H_{SA} \equiv -k_B T \ln \rho_{SA}$. Then, a quasi-static driving is applied to the compound system, transforming $H_{SA} \rightarrow H_{SA}' \equiv -k_B T \ln \rho'_{SA}$, which leads the compound system to end up in the state $\rho'_{SA}$, as it follows from the fact that $\rho'_{SA}$ is now the equilibrium state at temperature $T$ with respect to the Hamiltonian $H_{SA}'$. Finally, a second sudden change of the Hamiltonian is performed $H_{SA}' \rightarrow H_S + H_A$. The maximum work extractable in this reversible three-step process is, then,

$$W^\beta_{SA} - W^\beta_{SA}(\rho'_{SA}) = k_B T D(\rho_{SA}). \quad (14)$$

The full extraction of work is finally completed by applying the feedback enhanced protocol to $SA$ (see Fig. 2). Summing up all of the contributions, the maximum work extractable from $\rho_{SA}$ is obtained by adding the
work value of discord [Eq. (14)], the one extractable directly from \( \rho_{SA} \) [Eq. (9)], plus the entropic gain due to the measurement [Eq. (12)] applied to \( \rho_{SA}' \); that is,

\[
W_{SA|\pi_{A}^{opt}}^{\beta} (\rho_{SA}) = - \frac{1}{k_{B}} \ln \rho_{SA} - W_{SA}^{\beta} (\rho_{SA}) + 2 \left( W_{SA}^{\beta} (\rho_{SA}) + k_{B} T I (\rho_{AB}) \right) + 2 k_{B} T S (\rho_{A}).
\]

(15)

In particular, this implies that the process involving feedback from \( A \) helps in increasing the extractable work even in comparison with the optimal isothermal protocol (without feedback) applied to the whole \( SA \) system. This means that using a local quantum measurement may allow not only to extract full work associated to the total amount of correlations present in \( \rho_{AB} \), namely \( k_{B} T I (\rho_{AB}) \), but also an enhancement proportional to the entropy of the ancilla. The latter, eventually, may be lost in restoring the initial state of \( A \) [29].

In conclusion, we derived quantitative relations linking the optimal work extractable from bipartite quantum systems and their classical and quantum correlations, assessing both the role of thermal environments and quantum measurements. Moreover we proposed a protocol to extract the work associated to the presence of not only classical but also quantum correlations. Our results might be of interest in practical applications regarding the design of quantum batteries [31].

Acknowledgements – We thank useful comments from Juan M. R. Parrondo. The authors acknowledge support from the Horizon 2020 EU collaborative project QuProCS (Grant Agreement No. 641277). G. M. and R. Z. acknowledge MINECO/AEI/FEDER through projects NoMaQ FIS2014-60343-P and EPheQuCS FIS2016-78010-P.

Appendix A: Quasi-static isothermal processes with CPTP maps

Here we explicitly construct a concatenation of CPTP maps leading to a generic isothermal quasi-static process as needed for reversible work extraction in the protocols introduced in the main text. As mentioned there, an isothermal reversible process for a system with Hamiltonian \( H \) and density operator \( \rho_{0} \) can be constructed by a sudden quench, changing the Hamiltonian \( H \rightarrow H_{0} = - k_{B} T \ln \rho_{0} \) (while leaving the system in state \( \rho_{0} \) ) \( T \) being the temperature of the bath, followed by a quasi-static process where \( H_{0} \) is changed back to \( H \) and the system remains in equilibrium with the reservoir at any time, ending thus in \( \rho_{0} e^{- \beta H} / Z_{n} \). In order to build the map describing such a process, we take inspiration from Ref. [32], where isothermal processes are constructed by means of alternating infinitesimal adiabatic and isochoric processes in an infinite sequence. Similarly, we assume, here, an infinite sequence of maps \( \mathcal{E}_{1} \circ \mathcal{E}_{2} \circ \ldots \circ \mathcal{E}_{N} \), with \( N \rightarrow \infty \), each of them describing an infinitesimal time step of the dynamics, with \( \mathcal{E}_{n} (\rho_{n-1}) = \rho_{n} \), and the Hamiltonian changing as \( H_{n-1} \rightarrow H_{n} \), where we set \( H_{N} \equiv H \).

Let us decompose every CPTP map of the sequence in two steps

\[
\mathcal{E}_{n} (\rho) \equiv \mathcal{G}_{n} \circ \mathcal{U}_{n} (\rho),
\]

(A1)

where \( \mathcal{U}_{n} (\rho_{n-1}) = \rho_{n-1} \) is a unitary sudden quench of the system Hamiltonian, \( H_{n-1} \rightarrow H_{n} \), performed by the driving agent, followed by a instantaneous Gibbs-preserving map verifying \( \mathcal{G}_{n} (\frac{e^{- \beta_{H_{n-1}}}}{Z_{n-1}}) = \frac{e^{- \beta_{H_{n}}}}{Z_{n}} \), which describes the interaction with the environment. The Hamiltonian remains constant during this second step.

In order to ensure an isothermal reversible process, we need, for each CPTP map, that the change in the entropy of the system equals (minus) the heat introduced by the environment divided by \( k_{B} T \), that is

\[
\Delta S_{n} = S (\rho_{n}) - S (\rho_{n-1}) = \beta \text{Tr} [H_{n} (\rho_{n} - \rho_{n-1})] = - \beta Q_{n},
\]

(A2)

where only \( H_{n} \) appears in the expression above as the system only exchanges energy with the reservoir during the second step, \( \mathcal{G}_{n} (\rho_{n-1}) = \rho_{n} \). This would require that the system state is always close to the instantaneous equilibrium state for every step, \( \frac{e^{- \beta_{H_{n}}}}{Z_{n}} \). In the following we show that when assuming an infinitesimal change in the drive during any step

\[
H_{n} = H_{n-1} + \epsilon \Delta H_{n}
\]

(A3)

with \( \epsilon \ll 1 \), then the sequence of CPTP maps defined by Eq. (A1), verify the condition for reversibility, Eq. (A2), up to first order in \( \epsilon \), i.e. irreversibilities come only to order \( O (\epsilon^{2}) \).

We prove the above statement in two steps. First, we will show that if the state of the systems starts close to the equilibrium state before the map, then it remains close to the new equilibrium state after the application of the map. That is

\[
\rho_{n} = \mathcal{G}_{n} \left( \frac{e^{- \beta H_{n-1}}}{Z_{n-1}} + \epsilon \Delta \rho_{n-1} \right) = \frac{e^{- \beta H_{n}}}{Z_{n}} + \epsilon \Delta \rho_{n} + O (\epsilon^{2}),
\]

(A4)

where \( \Delta \rho_{n-1} \) and \( \Delta \rho_{n} \) are traceless operators. This ensures self-consistency of our construction. As a second step, we will prove that, since we may always rewrite \( \rho_{n} = \rho_{n-1} + \epsilon \sigma_{n} \) for a suitable traceless \( \sigma_{n} \) (in general \( \Delta \rho_{n} \neq \sigma_{n} \)), this implies Eq. (A2).

We start introducing Eq. (A3) into the left-hand-side of Eq. (A4), which by using linearity and expanding \( e^{\epsilon \Delta H_{n}} = 1 + \epsilon \Delta H_{n} + O (\epsilon^{2}) \), and \( Z_{n-1} = Z_{n} [1 + \epsilon \text{Tr} (\Delta H_{n}) + O (\epsilon^{2})] \) gives
Then, we obtain the eigenvalues and eigenvectors of $E$ during the first order in $\mathcal{O}$ of the instantaneous Hamiltonian $\epsilon$. On the other hand, we may obtain the same quantities to any temperature is allowed, the maximum extractable work is given by the nonequilibrium free energy with respect to the nonequilibrium state $\rho$ and the equilibrium state at the reservoir temperature, $\rho^\beta = e^{-\beta H}/Z$, where $\beta = 1/k_B T$ is the inverse temperature and $Z = \text{Tr}[e^{-\beta H}]$ the partition function [24]. That is
\begin{equation}
W^\beta = F_\beta(\rho) - F_\beta(\rho^\beta),
\end{equation}
where $F_\beta(\sigma) \equiv \text{Tr}[H\sigma] - k_B T S(\rho)$ is the definition of the nonequilibrium free energy with respect to $\beta$, $S(\sigma)$ denoting the von Neumann entropy.

On the other hand, when only unitary operations are available, the maximum extractable work is given by the ergotropy $\mathcal{W}$, which is always bounded by [12]
\begin{equation}
\mathcal{W} \leq \text{Tr}[H\rho] - \text{Tr}[H\rho^\beta] \leq \mathcal{W}_{\text{max}},
\end{equation}
where $\beta_\ast$ is that particular temperature which guarantees that entropy is unchanged: $S(\rho) = S(\rho^\beta)$. We notice that $\beta_\ast$ is obtained when minimizing the energy of the system for a fixed entropy.
In the following, we will prove that
\[ W^\beta \geq W_{\text{max}} \quad \forall \beta \in \mathbb{R}^+, \]  
and that the equality is reached if and only if \( \beta = \beta^*_\alpha \).

We start by analyzing the quantity \( W^\beta \), to notice that
\[
W^\beta = \text{Tr}[H_\beta] - \text{Tr}[H_\beta^\beta] - k_B T[S(\rho) - S(\rho^\beta)] = W_{\text{max}} + \text{Tr}[H_\beta^\beta] - \text{Tr}[H_\beta] - k_B T[S(\rho) - S(\rho^\beta)],
\]  
where we simply added and subtracted \( \text{Tr}[H_\beta^\beta] \) and identified the maximum ergotropy from Eq. (B2). Now, we notice that, by definition, \( S(\rho) = S(\rho^\beta) \), so that Eq. (B4) can be rewritten as
\[
W^\beta = W_{\text{max}} + F_\beta(\rho^\beta) - F_\beta(\rho^\beta). \tag{B5}
\]
Finally, by noticing that \( F_\beta(\rho) - F_\beta(\rho^\beta) = k_B T D(\rho||\rho^\beta) \) for any initial state \( \rho \), we immediately obtain:
\[
W^\beta = W_{\text{max}} + k_B T D(\rho^\beta||\rho^\beta) \geq W_{\text{max}}, \tag{B6}
\]
where \( D(\rho||\sigma) = \text{Tr}[\rho (\ln \rho - \ln \sigma)] \) is the quantum relative entropy. Eq. (B3) then follows from the non-negativity of the quantum relative entropy, \( D(\rho||\sigma) \geq 0 \), and the fact that the latter is zero only for \( \rho = \sigma \) (Klein’s inequality [33]). Consequently, we have that \( W^\beta = W_{\text{max}} \) only when \( \beta = \beta^*_\alpha \), which completes the proof.

**Appendix C: Reversible work extraction from quantum correlations**

In the main text we propose a protocol for extracting work from quantum correlations based on the inclusion of a new initial step before projective measurements on the ancilla are performed. Here, we provide some intuitive considerations motivating the construction of this initial sub-process. First, it should be noticed that we are looking for a reversible process leaving the total system in a state with zero quantum correlations, but with the same classical correlations as \( \rho_{SA} \). As mentioned in the main text, this means requiring the final state after this initial sub-process to be
\[
\rho'_{SA} = \sum_k p_k \rho_S |\Pi^k_A \otimes \Pi^k_A, \tag{C1}
\]
with \( \Pi^k_A \) belonging to the optimal set of projectors.

In order to construct a reversible stroke between \( \rho_{SA} \) and \( \rho'_{SA} \), we may first imagine the two following steps: i) apply to the whole \( SA \) system an isothermal protocol that allows for the reversible conversion of \( \rho_{SA} \) into the thermal product state \( \rho^\beta_S \otimes \rho^\beta_A \) [henceforth extracting an amount of work \( W^\beta_{SA} \) as given in Eq. (9) of the main text]. This includes the sudden quench of the total Hamiltonian \( H_S + H_A \rightarrow H_{SA} \equiv -k_B T \ln \rho_{SA} \) and a quasi-static driving of the full system as explained in the first section of this Supplemental Material; ii) perform the reversible transformation of \( \rho^\beta_S \otimes \rho^\beta_A \) into \( \rho'_{SA} \) by simply inverting the isothermal protocol which should be applied for optimal work extraction from state \( \rho'_{SA} \) and leading to \( \rho^\beta_S \otimes \rho^\beta_A \). This means that we need to apply the quasi-static driving \( H_S + H_A \rightarrow H'_{SA} \equiv -k_B T \ln \rho'_{SA} \), followed by a final quench \( H'_{SA} \rightarrow H_S + H_A \). Notice that step ii) requires performing the work
\[
W_{\text{in}} = F_\beta(\rho'_{SA}) - F_\beta(\rho^\beta_S \otimes \rho^\beta_A) = W^\beta_S + W^\beta_A + k_B T F(\rho_{SA}). \tag{C2}
\]

From the above considerations, we find that the reversible stroke we are searching for can be obtained e.g. by combining steps i) and ii), since both of them are reversible and their combination leaves the system in \( \rho'_{SA} \). Moreover, we notice that they can be merged since i) ends with a quasi-static process leading to \( \rho^\beta_S \otimes \rho^\beta_A \) while ii) starts with a quasi-static process from the same state. As a result, we obtain the process introduced in the main text: A sudden quench of the total Hamiltonian, \( H_S + H_A \rightarrow H_{SA} \equiv -k_B T \ln \rho_{SA} \), followed by a quasi-static driving transforming \( H_{SA} \rightarrow H'_{SA} \), which leads the compound system to end up in the state \( \rho'_{SA} \), as it follows from the fact that \( \rho'_{SA} \) is now the equilibrium state at temperature \( T \) with respect to the Hamiltonian \( H'_{SA} \). Finally, a second sudden change of the Hamiltonian is performed \( H'_{SA} \rightarrow H_S + H_A \). Finally, we notice that, in order to achieve each of these three steps, one would need not only to control system and ancilla locally, but also to manipulate their interaction in a suitable way.

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