Towards a diagrammatic derivation of the Veneziano-Yankielowicz-Taylor superpotential

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Abstract

We show how it is possible to integrate out chiral matter fields in $\mathcal{N} = 1$ supersymmetric theories and in this way derive in a simple diagrammatic way the $N_f S \log S - S \log \det X$ part of the Veneziano-Yankielowicz-Taylor superpotential.
Introduction

The recent renewed interest in the calculation of the glueball superpotential via matrix models [1] has led to an understanding of how to extract the non-logarithmic part of these superpotentials by ordinary diagrammatic methods [2]. Just as the matrix models in the applications to non-critical strings and 2d quantum gravity were convenient tools for solving specific combinatorial problems: the summation over all “triangulated” worldsheets with given weights, we understand now that the matrix model in the Dijkgraaf-Vafa (DV) context is an effective way of summing a set of ordinary Feynman graphs which by the magic of supersymmetry can be combined in such a way that they have no space-time dependence.

However, we are still left without a simple diagrammatic derivation of the logarithmic part of the glueball superpotential, the so called Veneziano-Yankielowicz-Taylor superpotential. This effective Lagrangian was originally derived for a pure $\mathcal{N} = 1$ $U(N_c)$ gauge theory by Veneziano and Yankielowicz [3] by anomaly matching and, by the same method, generalized to a $U(N_c)$ theory with $N_f$ flavors in the fundamental representation by Taylor, Veneziano and Yankielowicz [4]. It is given by

$$W_{\text{eff}}^{VYT}(S, X) = W_{\text{eff}}^{VY}(S) + W_{\text{eff}}^{\text{matter}}(S, X)$$

(1)

where $W_{\text{eff}}^{VY}(S)$ is the pure gauge part

$$W_{\text{eff}}^{VY}(S) = -N_c S \log \frac{S}{\Lambda^3}$$

(2)

while $W_{\text{eff}}^{\text{matter}}(S, X)$ denotes the part coming from $N_f$ flavors in the fundamental representation:

$$W_{\text{eff}}^{\text{matter}}(S, X) = N_f S \log \frac{S}{\Lambda^3} - S \log \frac{\det X}{\Lambda^2}.$$  

(3)

In the above formulas $S$ denotes the composite chiral superfield $W_\alpha^2/32\pi^2$ and $X = QQ$ is the $(N_f \times N_f)$ mesonic superfield, $Q$ being the chiral matter field. In (2) and (3) $\Lambda$ is an UV cut off. Usually this UV cut off is replaced by a renormalization group invariant scale $\Lambda_M$ by use of the one-loop renormalization group:

$$\Lambda_M = \Lambda e^{-\frac{S_x^2}{(3N_c-N_f)g^2}}.$$  

(4)
The beautiful derivation of (1)-(3) by anomaly matching has always been somewhat antagonizing since a clear diagrammatic understanding is missing. It is summarized in the following citation from [5]: “Its [i.e. (1)-(4)] only raison d'être is the explicit realization of the anomalous and non-anomalous symmetries of SUSY gluodynamics ....”.

In this letter we point out that there exists a simple diagrammatic derivation of (3). The derivation is inspired by diagrammatic techniques used in [2] and the observation that the DV-matrix models techniques could be extended to cover the case of superpotentials depending on mesonic superfields by considering the constrained (Wishart) matrix integrals [6]

\[ \int DQD\tilde{Q} \delta(\tilde{Q}Q - X) = \frac{(2\pi)^{N(N+1)/2}}{\prod_{j=N-N_f+1}^{N}(j-1)!} (\det X)^{N-N_f} \quad (5) \]

and taking the large \( N \) limit.

**Perturbative considerations**

The matter contribution to the effective superpotential was shown in [2] to arise from the path integral

\[ \int DQD\tilde{Q} e^{i d^4x d^2\theta \left(-\frac{1}{2}\tilde{Q}(\Box-i\partial\alpha)Q+W_{tree}(\tilde{Q},Q)\right)} \quad (6) \]

where \( W^\alpha \) is an external field and \( \partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} \). If the quarks are massive (\( W_{tree} = m\tilde{Q}Q \)) then the above path integral reduces to a functional determinant which can be easily evaluated using the Schwinger representation:

\[ \frac{1}{2} \int_\frac{1}{\Lambda}^\infty \frac{ds}{s} \int \frac{d^4p}{(2\pi)^4} \int d^2\pi_\alpha \exp \left(-s(p^2 + W^\alpha \pi_\alpha + m)\right) \quad (7) \]

where we introduced an UV cut-off \( \Lambda \). Due to fermionic integrations the result is

\[ \frac{W^2}{32\pi^2} \int_\frac{1}{\Lambda}^\infty \frac{ds}{s} e^{-ms} \quad (8) \]

which reduces for large \( \Lambda \) to

\[ S \log \left(\frac{m}{\Lambda}\right) \quad (9) \]
At this stage one could integrate-in $X$ to obtain (3). However, as “integrating-in” is in fact an assumption and we would like to obtain the desired result perturbatively, or more precisely: diagrammatically. To this end we impose the *superspace* constraint

$$X = \tilde{Q}Q$$  \hspace{1cm} (10)

at the level of the path integral (6). This is done by introducing a Lagrange multiplier chiral superfield $\alpha$. Since the antichiral sector does not influence the chiral superpotentials, we will perform a trick analogous to [2] and introduce an antichiral partner $\bar{\alpha}$ with a tree level potential $M\bar{\alpha}^2$. Thus we have

$$\int d^4x d^4\theta \, \bar{\alpha}\alpha + \int d^4x d^2\theta \, M\bar{\alpha}^2.$$  \hspace{1cm} (11)

The path integral w.r.t. $\bar{\alpha}$ is Gaussian and yields (c.f. [2])

$$-\frac{1}{2M} \int d^4x d^2\theta \, \alpha\Box\alpha.$$  \hspace{1cm} (12)

The final path integral is

$$\int D\alpha D\tilde{Q}DQ \, e^{\int d^4x d^2\theta (\frac{1}{2}Q(\Box+nW^{\alpha}\partial_{\alpha})Q - \frac{1}{2}\alpha\Box\alpha - \alpha X + \alpha\tilde{Q}Q)},$$  \hspace{1cm} (13)

where we also took $W_{tree} = 0$ and fixed the auxiliary mass $M = 1$ (it will be clear from the arguments below that the result is independent of $M$).

This is no longer a free field theory, but nevertheless there are significant simplifications if we only want to extract the $\text{tr} \, W^2$ dependence. This implies that we must have two $W$ insertions per $\tilde{Q}Q$ loop. The integrals over the fermionic momenta thus force all graphs which contain an $\alpha$-line in a loop to vanish. Thus we are left with graphs coming from (13) which have the structure of $\tilde{Q}Q$ loops connected by at most one $\alpha$ propagator, and $\alpha$ propagators connected to the external field $X$ as shown in fig. 1.

Moreover, if the field $X$ contains a zero momentum component, which will generically be the case, the integrals will be dominated by this constant mode which forces the $\alpha$ propagators to be evaluated at zero momentum. Consequently we have to introduce an IR cut-off $\Lambda_{IR}$. Each 0-momentum $\alpha$ propagator will then just contribute a factor of $1/\Lambda_{IR}$. Thanks to the above property we may find the full $\tilde{Q}Q$ propagator in terms of the $\alpha$ 1-point function which we will denote by $F$:

$$\frac{1}{p^2 + W^{\alpha}\pi_{\alpha} + F}.$$  \hspace{1cm} (14)
Figure 1: Only tree level graphs survive, i.e. we are left with the the graphs shown in fig. 1c).

and the effective action will be given by the formula (7) with $m$ substituted by $F$:

$$S \log \det \frac{F}{\Lambda}$$  \hspace{1cm} (15)

It remains to determine $F$. The Schwinger-Dyson equation for $F$ is (see fig. 2)

$$F = -\frac{1}{\Lambda_{IR}}X + \frac{1}{\Lambda_{IR} F} S$$  \hspace{1cm} (16)

where we used

$$\int_0^\infty ds \int \frac{d^4 p}{(2\pi)^4} \int d^2 \pi \alpha \ e^{-s(p^2+W_{\alpha}^2+X_{\alpha}^2+F)} = \frac{S}{F}$$  \hspace{1cm} (17)

Eq. (16) is quadratic and has 2 solutions. Since the final result has to be IR finite, we will take the solution which has a finite limit as $\Lambda_{IR} \to 0$. Therefore

$$F = \frac{S}{X}$$  \hspace{1cm} (18)

and by substituting this back in (15) one obtains the desired result:

$$S \log \det \frac{SX^{-1}}{\Lambda}$$,  \hspace{1cm} (19)

or, in the case of $N_f$ flavors:

$$N_f S \log \frac{S}{\Lambda^3} - S \log \det \frac{X}{\Lambda^2}.$$  \hspace{1cm} (20)
Further examples

Exactly the same technique can be adapted to the theories studied in [7] where the matter effective superpotentials in terms of only mesonic fields are quite complex (see eqn. (1.1) in [7]) and follow from quite intricate physical analysis. However, as noted in [7] the superpotentials with both glueball fields and matter fields are simpler. The pure matter superpotentials can then be obtained by integrating out the glueball fields $S_i$.

The simplest case considered in [7] is a gauge theory with gauge group $SU(2)_1 \times SU(2)_2$, with a bifundamental matter field $Q$ in the $(2,2)$ representation. The natural gauge invariant matter superfield is

$$X = Q^2 \equiv \frac{1}{2} Q_{ab} Q_{cd} \varepsilon^{ac} \varepsilon^{bd}, \tag{21}$$

and the matter part of the superpotential $W_{eff}(S, X)$ is (eq. (4.19) in [7]):

$$(S_1 + S_2) \log \frac{S_1 + S_2}{X \Lambda} \tag{22}$$

We will now show that the expression (22) also follows from a diagrammatic reasoning.

Since for $SU(2)$ the fundamental and antifundamental representations are equivalent through $\tilde{Q}_a \equiv Q_a \varepsilon^{a'} a'$ the Lagrangian for the bifundamental fields takes the form:

$$Q_{a'b'} \varepsilon^{a’a} \varepsilon^{b'b}(\Box - iW_{ac}^{(1)} \partial_a - iW_{bd}^{(2)} \partial_d)Q_{cd} \tag{23}$$

Again we introduce a Lagrange multiplier superfield $\alpha$ enforcing the above constraint. We thus have

$$Q(C \otimes C)(\Box - W^{(1)}_a \otimes 1\pi_\alpha - 1 \otimes W^{(2)}_a \pi_\alpha + \frac{1}{2} \alpha)Q - \alpha X \tag{24}$$

where $C^{ab} \equiv \varepsilon^{ab}$. 

Figure 2: The Schwinger-Dyson equation for $F$. 
The analogue of formula (15) will then be

$$\frac{1}{2} 2^2 (S_1 + S_2) \log \left( \frac{F}{2\Lambda} \right)$$

(25)

where the 1/2 comes from the fact that we are dealing with a real representation, while the 2 comes from performing the trace over the trivial factor in $\mathcal{W}^{(2)} \otimes 1^2$. The Schwinger-Dyson equation for $F$ will then have the form

$$F = -\frac{1}{\Lambda_{IR}} X + \frac{1}{2} \frac{1}{\Lambda_{IR}} \frac{1}{2} (S_1 + S_2)$$

(26)

hence

$$F = \frac{2(S_1 + S_2)}{X}$$

(27)

Inserting $F$ into (25) reproduces precisely the nontrivial result (22).

Another example studied in [7] for the gauge group $SU(2)_1 \times SU(2)_2$ is matter $L_{\pm}$ in the $(1,2)$ representation. The classical D-flat direction is labeled by $Y = L_{\alpha \beta} L_{\beta \alpha} - \varepsilon_{\alpha \beta}$ and the matter contribution to $W^{VYT}_{eff}$ was found in [7] to be:

$$S_2 \log \frac{S_2}{Y \Lambda}.$$  

(28)

We can also reproduce this expression$^1$ by computing diagrammatically the contribution from the $L_{\pm}$ fields, starting with the Lagrangian

$$L(C \otimes 1)(\Box - \mathcal{W}^{(2)}_{\alpha} \otimes 1 \pi_{\alpha} + \alpha 1 \otimes C)L - \alpha Y,$$

(29)

where the second component in the tensor product is the flavor space.

**Discussion**

We have shown that it is possible to obtain the matter part of some generalized $W^{VYT}_{eff}(X, S)$ potentials by simple diagrammatic reasoning. It would be interesting to generalize the diagrammatic derivation to the gauge part of the Taylor-Veneziano-Yankielowicz superpotential. That would complete the diagrammatic derivation of the glueball superpotential.

$^1$Up to a trivial rescaling of $\Lambda$. Note that in our approach the definition of the UV cut-off $\Lambda$ (see e.g. (8)) is a matter of convention and may be modified.
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