Bivariate Gompertz generator of distributions: statistical properties and estimation with application to model football data

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Submitted: 12 February 2019; Revised: 22 January 2020; Accepted: 24 January 2020

Abstract: In this paper, the bivariate extension of the so-called Gompertz-G family was introduced and studied in detail. Marshall and Olkin shock model was used to build the proposed bivariate family. The new family was constructed from three independent Gompertz-H families using a minimisation process. Some of its statistical properties such as joint probability density function, coefficient of median correlation, moments, product moment, covariance, conditional probability density function, joint reliability function, stress-strength reliability and joint reversed (hazard) rate function were derived. After introducing the general class, three special models of the new family were discussed. Maximum likelihood method was used to estimate the family parameters. A simulation study was carried out to examine the bias and mean square error of the maximum likelihood estimators. Finally, the importance of the proposed bivariate family was illustrated by means of real dataset, and it was found that the proposed model provides better fit than other well-known models in the statistical literature such as bivariate Gompertz, bivariate generalized Gompertz, bivariate Gumbel Gompertz and bivariate exponentiated Weibull-Gompertz.

Keywords: Bivariate distributions, Gompertz-H family of distributions, Marshall-Olkin shock model, maximum likelihood method.

INTRODUCTION

Several classes of distributions have been developed and applied to describe various phenomena in different areas such as engineering, biological studies, economics, actuarial, environmental, lifetime analysis and Olympic games, among others.

However, in many applied areas such as lifetime analysis, describing the pattern of adult deaths, Olympic games and insurance, there is a clear need for extended forms of these classes to model such data. For this reason, many classes have been proposed and studied in statistical literature, for example, transformed-transformer (T-X) family by Alzaatreh et al. (2013); generating T-Y family by Aljarrah et al. (2014); exponentiated half-logistic family by Cordeiro et al. (2014); Kumaraswamy Marshall-Olkin family by Alizadeh et al. (2015); a new Weibull-G family by Tahir et al. (2016); Gompertz-G family by Alizadeh et al. (2017) and its discrete version by Eliwa et al. (2020a); exponentiated Gompertz generated family by Cordeiro et al. (2016); odd Chen-G family by El-Morshedy et al. (2020a); exponentiated odd Chen-G
family by Eliwa et al. (2020b); odd flexible Weibull-H family by El-Morshedy and Eliwa (2019); odd log-logistic Lindley-G family by Alizadeh et al. (2020) and discrete Gompertz family by Eliwa et al. (2020a), among others.

In many practical situations, it is important to consider different bivariate families that could be used to model bivariate data. The bivariate data could be exchange rates in two time periods, strength components, results of two teams in Olympic games etc. Therefore, many bivariate distributions are proposed in literature, for example, bivariate generalised exponential distribution by Kundu and Gupta (2009); bivariate generalised linear failure rate distribution by Sarhan et al. (2011) Marshall-Olkin bivariate Weibull distribution by Kundu and Gupta (2013); bivariate Kumaraswamy distribution by Barreto-Souza and Lemonte (2013); bivariate exponential distribution by Balakrishnan and Shiji (2014); bivariate exponentiated generalised Weibull-Gompertz distribution by El-Bassiouny et al. (2016); bivariate exponentiated modified Weibull extension distribution by El-Gohary et al. (2016); bivariate exponentiated extended Weibull family by Roozegar and Jafari (2016); bivariate inverse Weibull distribution by Hiba (2016); bivariate exponentiated discrete Weibull distribution by El-Morshedy et al. (2020c); bivariate exponentiated generalised linear exponential distribution by Ibrahim et al. (2019); bivariate Gumbel-G family by Eliwa and El-Morshedy (2019); univariate and multivariate generalized slash student distribution by El-Bassiouny and El-Morshedy (2015), univariate and multivariate double slash distribution by El-Morshedy et al. (2020c), bivariate inverse Weibull distribution by Eliwa and El-Morshedy (2020a), bivariate odd Weibull-G family by Eliwa and El-Morshedy (2020b), bivariate Burr X generator by El-Morshedy et al. (2020c), among others. However, in many practical situations, classical bivariate distributions do not provide adequate fits to real data. Therefore, there has been an increased interest in developing more flexible distributions. Thus, in this paper, we introduce a flexible bivariate family based on Marshall-Olkin shock model (Marshall & Olkin, 1967), in the so-called bivariate Gompertz-H (BG-H) family.

Alizadeh et al. (2017) proposed and studied a flexible univariate family of distributions, in the so-called Gompertz-G (Go-H) family. The random variable $Y$ is said to have Go-H family if its CDF is given by

$$V_{Go-H}(y; \theta, \alpha, \eta) = 1 - e^\theta \left\{ \frac{1}{\bar{F}(y; \eta)} \right\}^{-\alpha}; \quad y \geq 0, \quad \text{(01)}$$

where $\theta > 0$ and $\alpha > 0$ are two additional shape parameters, and $\eta$ is a vector of parameters \( (1 \times k; \ k = 1, 2, 3, \ldots) \). Also, the $H(y; \eta) = 1 - \bar{F}(y; \eta)$ is the baseline CDF depending on a parameter vector $\eta > 0$. The survival function of the random variable $Y$ is given by

$$V_{Go-H}(y; \theta, \alpha, \eta) = e^\theta \left\{ \frac{1}{\bar{F}(y; \eta)} \right\}^{-\alpha}; \quad y \geq 0. \quad \text{(02)}$$

The probability density function (PDF) corresponding to equation (1) is given by

$$v_{Go-H}(y; \theta, \alpha, \eta) = \theta h(y; \eta) \left( \frac{1}{\bar{F}(y; \eta)} \right)^{(\alpha+1)} e^\theta \left\{ \frac{1}{\bar{F}(y; \eta)} \right\}^{-\alpha}; \quad y \geq 0, \quad \text{(03)}$$

where $h(y; \eta)$ is the baseline PDF. The main reasons for introducing this family are:

1. The joint CDFs and joint PDFs should preferably have a closed form representation; at least numerical evaluation should be possible.
2. This class of distributions is an important model that can be used in a variety of problems for modelling bivariate lifetime data.
3. This class contains several special bivariate models.

**METHODOLOGY: BGO-H FAMILY**

A random vector $(Y_1, Y_2)$ follows the Marshall-Olkin shock model ↔ there exist three independent random variables $Z_1, Z_2$, and $Z_3$ such that $(Y_1 = \max(Z_1, Z_2))$ and $(Y_2 = \max(Z_2, Z_3))$ or $(Y_1 = \min(Z_1, Z_2))$ and $(Y_2 = \min(Z_2, Z_3))$. The proposed BGo-H family is constructed from three independent Go-H families using a minimisation process. Assume three mutually independent random variables $Z_i \sim Go - H(\theta_i, \alpha, \eta)$, such that $i = 1, 2, 3$. Define $Y_1 = \min\{Z_1, Z_3\}$ and $Y_2 = \min\{Z_2, Z_3\}$. So, the bivariate vector $(Y_1, Y_2)$ has the BGo-H family with parameter vector $\mathbf{\eta} = (\theta_1, \theta_2, \theta_3, \alpha, \eta)$. The joint survival function of $(Y_1, Y_2)$ is given as follows

$$V_{Y_1,Y_2}(y_1, y_2) = P\left[ \min\{Z_1, Z_3\} > y_1, \min\{Z_2, Z_3\} > y_2 \right]$$

$$= P\left[ Z_1 > y_1, Z_2 > y_2, Z_3 > \max\{y_1, y_2\} \right]$$

$$= \prod_{i=1}^{3} e^\theta \left\{ \frac{1}{\bar{F}(y_i; \eta)} \right\}^{-\alpha}, \quad \text{(04)}$$

where $y_3 = \max\{y_1, y_2\}$. Equation (4) can be written as follows
Bivariate gompertz generator of distributions

\[ \mathbb{V}(y_1, y_2) = \begin{cases} \mathbb{V}_{Go-H} \left( y_1; \theta_1, \alpha, \eta \right) \times \mathbb{V}_{Go-H} \left( y_2; \theta_2 + \theta_3, \alpha, \eta \right) ; y_1 \leq y_2 \\ \mathbb{V}_{Go-H} \left( y_1; \theta_1 + \theta_3, \alpha, \eta \right) \times \mathbb{V}_{Go-H} \left( y_2; \theta_2, \alpha, \eta \right) ; y_1 > y_2 \end{cases} \]  

\hfill (05)

Moreover, we can get the joint PDF of \((Y_1, Y_2)\) as follows:

\[ v_{Y_1,Y_2}(y_1, y_2) = \begin{cases} v_1(y_1, y_2) ; & 0 < y_1 < y_2 < \infty \\ v_2(y_1, y_2) ; & 0 < y_2 < y_1 < \infty \\ v_3(y) ; & 0 < y_1 = y_2 = y < \infty, \end{cases} \]

\hfill (06)

where

\[ v_1(y_1, y_2) = \mathbb{V}_{Go-H} \left( y_1; \theta_1, \alpha, \eta \right) \times \mathbb{V}_{Go-H} \left( y_2; \theta_2 + \theta_3, \alpha, \eta \right), \]

\[ v_2(y_1, y_2) = \mathbb{V}_{Go-H} \left( y_1; \theta_1 + \theta_3, \alpha, \eta \right) \times \mathbb{V}_{Go-H} \left( y_2; \theta_2, \alpha, \eta \right), \]

and

\[ v_3(y) = \frac{\theta_3}{\theta_1 + \theta_2 + \theta_3} \mathbb{V}_{Go-H} \left( y; \theta_1 + \theta_2 + \theta_3, \alpha, \eta \right). \]

The expressions \(v_i(y_1, y_2), i = 1, 2\) can be obtained by differentiating the equation (4) with respect to \(y_i, i = 1, 2\). However, we can use the following fact to get \(v_3(y)\)

\[ \int_0^{\infty} v_3(y) \, dy = 1 - \int_0^{\infty} \int_0^{y_1} v_1(y_1, y_2) \, dy_1 \, dy_2 - \int_0^{\infty} \int_0^{y_2} v_2(y_1, y_2) \, dy_2 \, dy_1. \]  

\hfill (07)

On the other hand, the marginal survival functions for the BG-H family can be represented as follows

\[ \mathbb{V}_{Y_i}(y_i) = P \{ \min \{ Z_i, Z_3 \} > y_i \} \]

\[ = \mathbb{V}_{Go-H} \left( y_i; \theta_i + \theta_3, \alpha, \eta \right), i = 1, 2. \]

\hfill (08)

So, we can get the marginal PDFs for the BG-H family as follows:

\[ v_{Y_i}(y_i) = v_{Go-H} \left( y_i; \theta_i + \theta_3, \alpha, \eta \right); i = 1, 2. \]

\hfill (09)

Using the power series for the exponential function and the generalized binomial theorem, we nd the marginal PDFs for the BG-H family can be expressed as an ininite linear combination of exponentiated-H (exp-H) density functions as follows:

\[ v_{Y_i}(y_i) = \sum_{l=0}^{\infty} \psi_{i,l+1}^{(i)} P_{l+1}(y_i; \eta), \]

\hfill (10)

where

\[ P_{l+1}(y_i; \eta) = (l + 1)h(y_i; \eta)H(y_i; \eta)^l, \]

\hfill (11)

represents the PDF of the exp-H family with power parameter \((l + 1), \psi_{0,l}^{(i)} = 1 - \psi_0, \psi_{l,i}^{(i)} = -\psi_{l,i}^{(i)} \) for \(l = 1, 2, 3, \ldots\), \(\psi_{l,i}^{(i)} = \sum_{m=0}^{\infty} \sum_{j=0}^{m} \Omega^{(i)}_{m,j,l} \) and

\[ \Omega^{(i)}_{m,j,l} = \frac{(-1)^{m+l}}{m!} \left( \frac{\theta_1 + \theta_3}{\alpha} \right)^m \left( \begin{array}{c} m \\ l \end{array} \right) \left( \begin{array}{c} m \\ j \end{array} \right). \]

\hfill (12)

Assume \((Y_1, Y_2)\) be a two dimensional random variable with joint survival function \(\mathbb{V}_{Y_1,Y_2}(y_1, y_2)\), and the marginal survival functions are \(\mathbb{V}_{Y_1}(y_1)\) and \(\mathbb{V}_{Y_2}(y_2)\), then the joint CDF is given by

\[ V_{Y_1,Y_2}(y_1, y_2) = 1 - \mathbb{V}_{Y_1}(y_1) - \mathbb{V}_{Y_2}(y_2) + \mathbb{V}_{Y_1,Y_2}(y_1, y_2) \]

\[ = 1 - e^{-\frac{(y_1 + y_2)}{\alpha}} \left( 1 - [\mathbb{H}(y_1; \eta)]^{-1} \right) \]

\[ - e^{-\frac{(y_1 + y_2)}{\alpha}} \left( 1 - [\mathbb{H}(y_2; \eta)]^{-1} \right) \]

\[ + \sum_{i=1}^{3} e^{-y_i} \left( 1 - [\mathbb{H}(y_i; \eta)]^{-1} \right). \]

\hfill (13)

If the bivariate vector \((Y_1, Y_2)\) has the BG-H family, then the distributions of \(S = \max \{ Y_1, Y_2 \} \) and \(T = \min \{ Y_1, Y_2 \} \) can be represented as

\[ V_{S}(t) = \prod_{i=1}^{3} V_{Go-H} \left( t; \theta_i, \alpha, \eta \right) \]

\hfill (14)

and

\[ V_{T}(t) = 1 - \prod_{i=1}^{3} V_{Go-H} \left( t; \theta_i, \alpha, \eta \right), \]

\hfill (15)

respectively.
Different statistical properties

**BGo-H family using Marshall-Olkin copula properties**

In this section, we find that the BGo-H family has both a singular part along the line $y_1 = y_2$ with weight $\theta_1 + \theta_2 + \theta_3$ and an absolute continuous part on $0 < y_1 \neq y_2 < \infty$ with weight $\theta_1 + \theta_2 + \theta_3$, similar to Marshall and Olkin's bivariate exponential model. Moreover, the BGo-H family can be obtained by using the Marshall-Olkin copula with the marginals as the Go-H families as follows: for $B^* : [0,1] \times [0,1] \to [0,1]$, we get

$$V_{Y_1,Y_2}(y_1, y_2) = B^* (V_{Y_1}(y_1), V_{Y_2}(y_2)); \text{ for all } (y_1, y_2) \in R^2.$$  \hspace{1cm} \text{...}(16)

where

$$B^*_{\tau_1, \tau_2} (D_1, D_2) = D_1^{\frac{1-\tau_1}{\tau_2}} D_2^{\frac{1-\tau_2}{\tau_1}} \max \{D_1^{\tau_1}, D_2^{\tau_2}\}, \quad \text{for } 0 < \tau_1, \tau_2 < 1,$$

$$\tau_1 = \frac{\theta_1}{\theta_1 + \theta_2 + \theta_3} \quad \text{and} \quad \tau_2 = \frac{\theta_2}{\theta_1 + \theta_2 + \theta_3}.$$  \hspace{1cm} \text{...}(17)

For more details around Marshall-Olkin copula properties see, Nelsen, 1999. Also, we find that

$$B^*(D_1, D_2) \geq D_1^* D_2^* \quad \text{for all } D_1^*, D_2^* \in [0,1]^2.$$  \hspace{1cm} \text{...}(18)

So, if $(Y_1, Y_2)$ follow the BGo-H family, then they are positive quadrant dependent.

**Coefficient of median correlation**

Assume $D_{Y_1}$ and $D_{Y_2}$ denote the median of $Y_1$ and $Y_2$, respectively. If $Y_1 \sim G_0 - H (\theta_1 + \theta_2, \alpha, \eta)$ and $Y_2 \sim G_0 - H (\theta_2 + \theta_3, \alpha, \eta)$, then

$$D_{Y_i} = H^{-1} \left\{ 1 - \left( 1 - \frac{\alpha}{\theta_1 + \theta_2} \log [ 1 - U ] \right)^{\frac{1}{\alpha}} \right\}; \quad i = 1, 2,$$

$$\quad \text{...}(19)$$

where $U$ has a uniform $U(0, 1)$ distribution, and $H^{-1}(.)$ represents the baseline quantile function. Domma (2010) presented the median correlation coefficient $D_{Y_1,Y_2}$ as a form $D_{Y_1,Y_2} = 4W_{Y_1,Y_2} (D_{Y_1}, D_{Y_2}) - 1$. So, the coefficient of median correlation between $Y_1$ and $Y_2$ is given as follows:

**Moments, product moment and covariance**

In this section, we derive the $r$th moment, the $n$th central moment and the $s$th incomplete moment of $Y_i$, when $Y_i \sim G_0 - H (\theta_i + \theta_2, \alpha, \eta)$, such that $i = 1, 2$. Also, we present the product moment, covariance and the $\text{Var} (Y_1 + Y_2)$ of the bivariate distribution $(Y_1, Y_2)$. The $r$th moment of $Y_i$, say $W^{(r)}_i$, can be expressed as follows $W^{(r)}_i = \mathbb{E} (Y_i^r) = \int_0^\infty y_i^r f_i(y_i|\eta)dy_i$, using equation (10), we get

$$W^{(r)}_i = \sum_{l=0}^{\infty} \Psi_{l+1}^{(i)} \mathbb{E} (X_{i,l+1}^r), \quad \text{...}(20)$$

where $X_{i,l+1}^r; i = 1, 2$ be a random variables having the exp-H CDF with power parameter $(l+1)$. The moments of the exp-H distributions are given by Nadarajah and Kotz (2006). Setting $r = 1, 2$ in equation (20), we get the mean $(W^{(1)}_i)$ and the variance $(\sigma^2_{W_i})$ of $Y_i, i = 1, 2$ as

$W^{(1)}_i = \sum_{l=0}^{\infty} \Psi_{l+1}^{(i)} \mathbb{E} (X_{i,l+1}^1),$  \hspace{1cm} \text{...}(21)

and

$$\sigma^2_{W_i} = \sum_{l=0}^{\infty} \Psi_{l+1}^{(i)} \mathbb{E} (X_{i,l+1}^2) - \left[ \mathbb{E} (W^{(1)}_i) \right]^2.$$  \hspace{1cm} \text{...}(22)

, respectively. Furthermore, the $n$th central moment of $Y_i$, say $L^{(n)}_i$, is given by

$$L^{(n)}_i = \sum_{r=0}^{\infty} \sum_{l=0}^{\infty} \left( \frac{n}{r} \right)^{n-r} \Psi_{l+1}^{(i)} \mathbb{E} (X_{i,l+1}^r); \quad i = 1, 2.$$  \hspace{1cm} \text{...}(23)

On the other hand, the incomplete moments are very important, which the main applications of the first incomplete moment are related to the mean deviations, Bonferroni and Lorenz curves. These curves are very useful in demography, economics, medicine, insurance and reliability. The $s$th incomplete moment of $Y_i$, say

June 2020 \hspace{1cm} Journal of the National Science Foundation of Sri Lanka 48(2)
\( \Lambda^{(s)}_{i}(t_i) \) can be expressed as follows:

\[
\Lambda^{(s)}_{i}(t_i) = \int_{0}^{t_i} y_i^{a} \, v(y_i) \, dy_i = \sum_{i=0}^{\infty} \Phi_{i+1}^{(s)}(t_i) , \quad i = 1, 2, \quad \ldots \tag{24}
\]

where \( \Phi_{i+1}^{(s)}(t_i) = \int_{0}^{t_i} y_i^{a} \, P_{i+1}(y_i; \eta) \, dy_i \). So, the mean deviations about the mean and the median are given by \( \lambda^* = 2W_{1}^{(1)}(V_{1}^{(1)}) - 2W_{1}^{(1)}(W_{1}^{(1)}) \) and \( \omega^* = W_{1}^{(1)}(D_{1}) \), respectively. Moreover, the product moment, say \( E(Y_1 Y_2^*) \), can be represented as

\[
E(Y_1 Y_2^*) = \int_{0}^{\infty} \int_{0}^{\infty} y_1 y_2^* v_1(y_1, y_2) dy_1 dy_2 + \int_{0}^{\infty} \int_{0}^{\infty} y_1 y_2^* v_2(y_1, y_2) dy_2 dy_1 + \int_{0}^{\infty} y^2 v_0(y) \, dy \\
= \sum_{i=0}^{\infty} \left[ \Phi_{i+1}^{(2)}(\theta_1) \Delta_{(1)}^{(r)} + \Phi_{i+1}^{(1)}(\theta_2) \Delta_{(1)}^{(r)} \right] + \frac{\theta_3}{\theta_1 + \theta_2 + \theta_3} \sum_{i=0}^{\infty} \Phi_{i+1}^{(2)}(\theta_1 + \theta_2 + \theta_3) \Phi_{i+1}^{(2)}, \quad \ldots \tag{25}
\]

where \( \Phi_{i+1}(\xi) = -\sum_{m=0}^{\infty} \sum_{o=0}^{m} \Omega_{m,o,j} \Phi_{i+1}(\xi) \) for \( (t^* = 1, 2, 3, \ldots) \),

\( \Omega_{m,o,j} (\xi) = \left( \frac{1}{m+1} \right) \left( \frac{m+1}{m} \right) \left( \frac{m+1}{m} \right) \left( \frac{m+1}{m} \right), \quad \Delta_{(i)}^{(r)} = \int_{0}^{\infty} y^r \Phi_{i+1}(y_i) P_{i+1}(y_i; \eta) \, dy_i ; \quad i = 1, 2, \quad \phi_{i+1}(y_i) = \int_{0}^{\infty} y^r \Phi_{i+1}(y_i) R_{i+1}(y_i; \eta) \, dy_i ; \quad i = 1, 2, \quad \phi_{i+1}^{(2)} = \int_{0}^{\infty} y^2 \phi_{i+1}(y_i) \, dy_i .

So, by using equations (20) and (25) when \( r = 1 \), we get

\[
\text{Cov}(Y_1, Y_2) = \sum_{i,j=0}^{\infty} \left[ \Phi_{i+1}^{(2)}(\theta_1) \Delta_{(2)}^{(r)} + \Phi_{i+1}^{(1)}(\theta_2) \Delta_{(1)}^{(r)} \right] + \frac{\theta_3}{\theta_1 + \theta_2 + \theta_3} \sum_{i=0}^{\infty} \Phi_{i+1}^{(2)}(\theta_1 + \theta_2 + \theta_3) \Phi_{i+1}^{(2)} \\
- \sum_{i=0}^{\infty} \Phi_{i+1}^{(2)}(Y_1^{1} Y_{2,i+1}) \sum_{i=0}^{\infty} \Phi_{i+1}^{(2)}(E(Y_2^{1} Y_{2,i+1}), \quad \text{where Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1) E(Y_2). Thus, we can compute the variance of \( Y_1 + Y_2 \) as follows

\[
\text{Var}(Y_1 + Y_2) = \sum_{i=1}^{2} \sigma_i^2 + \text{Cov}(Y_1, Y_2). \] If the two random variables \( Y_1 \) and \( Y_2 \) are uncorrelated \( \rightarrow \text{Cov}(Y_1, Y_2) = 0 \) and \( \text{Var}(Y_1 + Y_2) = \sum_{i=1}^{2} \sigma_i^2 \).

\textbf{Conditional PDFs}

The conditional PDF of \( Y_i \) given \( Y_j = y_j, (i, j = 1, 2, i \neq j) \), is given by

\[
v_{Y_i | Y_j}(y_i | y_j) = \frac{v_{Y_i Y_j}(y_i, y_j)}{v_{Y_j}(y_j)} = \frac{v_{Y_i Y_j}(y_i, y_j)}{v_{Y_j}(y_j)} = \left\{ \begin{array}{ll}
0 & y_i < y_j < \infty \\
0 & y_j < y_i < \infty \\
0 & y_i = y_j < \infty, \quad \ldots (26)
\end{array} \right.
\]

where

\[
v_{Y_i Y_j}(y_i, y_j) = \theta_3 h(y_i; \eta) [\overline{P}(y_i; \eta)]^{-(\alpha+1)} e^{-\frac{\theta_3}{\theta_1 + \theta_2 + \theta_3} \left\{ 1 - [\overline{P}(y_i; \eta)]^{-\alpha} \right\}},
\]

\[
v_{Y_i | Y_j}(y_i | y_j) = \left( \frac{\theta_3}{\theta_1 + \theta_2 + \theta_3} e^{-\frac{\theta_3}{\theta_1 + \theta_2} \left\{ 1 - [\overline{P}(y_i; \eta)]^{-\alpha} \right\}} \right)
\]

and

\[
v_{Y_i Y_j}(y_i, y_j) = \left( \frac{\theta_3}{\theta_1 + \theta_2 + \theta_3} e^{-\frac{\theta_3}{\theta_1 + \theta_2 + \theta_3} \left\{ 1 - [\overline{P}(y_i; \eta)]^{-\alpha} \right\}} \right).
\]

Equation (26) can be obtained by substituting from equations (6) and (9) in the relation

\[
v_{Y_i | Y_j}(y_i | y_j) = \frac{v_{Y_i, Y_j}(y_i, y_j)}{v_{Y_j}(y_j)} , \quad (i \neq j = 1, 2).
\]

\textbf{Stress-strength reliability function}

There are appliances, which survive due to their strength. These appliances receive a certain level of stress (load). The load may be defined as temperature, environment, mechanical load, and electric current, etc. However, if a higher level of load is applied, then their strength is unable to sustain and they break down. Let \( Y_1 \sim \text{Go} - \text{H} (\theta_1 + \theta_2, \alpha, \eta) \) be a random variable representing the stress, and \( Y_2 \sim \text{Go} - \text{H} (\theta_2 + \theta_3, \alpha, \eta) \) be a random variable representing the strength, then the reliability function \( R_* \) is given as follows:

\[
R_* = P(Y_1 < Y_2) = \frac{\theta_2 + \theta_3}{\theta_1 + \theta_2 + 2\theta_3} , \quad \ldots (27)
\]
Similarly, if \( Y_2 \) is a random variable representing the stress, and \( Y_1 \) is a random variable representing the strength, then the reliability function \( R^* \) is
\[
R^* = P[Y_2 < Y_1] = \frac{\beta_1 + \beta_3}{\beta_1 + \beta_2 + 2\beta_3}.
\]  
(28)

It is clear that the stress-strength model does not depend on the baseline function \( H(y_1; \eta) \).

**Joint hazard rate function and its marginal functions**

Let \( (Y_1, Y_2) \) be a two dimensional random variable with joint PDF \( v_{Y_1,Y_2}(y_1,y_2) \) and joint reliability function \( \overline{V}_{Y_1,Y_2}(y_1,y_2) \). Basu (1971) defined the bivariate hazard rate (BHR) function, say \( h_{Y_1,Y_2}(y_1,y_2) \), as follows:
\[
h_{Y_1,Y_2}(y_1,y_2) = \frac{v_{Y_1,Y_2}(y_1,y_2)}{\overline{V}_{Y_1,Y_2}(y_1,y_2)}.
\]
So, if the random vector \( (Y_1, Y_2) \) has the BGo-H family, then the BHR function is given by
\[
h_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 
  h_{y_1}(y_1, y_2) & \text{if } 0 < y_1 < y_2 < \infty \\
  h_{y_2}(y_1, y_2) & \text{if } 0 < y_2 < y_1 < \infty \\
  h_{y_3}(y_1, y_2) & \text{if } y_1 = y_2 = y < \infty,
\end{cases}
\]
(29)

where
\[
h_{y_1}(y_1, y_2) = \theta_1(\theta_2 + \theta_3)h(y_1; \eta)h(y_2; \eta) \left\{ \overline{H}(y_1; \eta) \right\}^{-(\alpha + 1)},
\]
\[
h_{y_2}(y_1, y_2) = \theta_2(\theta_1 + \theta_3)h(y_1; \eta)h(y_2; \eta) \left\{ \overline{H}(y_1; \eta) \right\}^{-(\alpha + 1)},
\]
and
\[
h_{y_3}(y_1, y_2) = \theta_3h(y_1; \eta) \left\{ \overline{H}(y_1; \eta) \right\}^{-(\alpha + 1)}.
\]

The marginal hazard rate (HR) functions \( h_{y_i}(y_i) \), \( i = 1, 2 \) of the BGo-H family can be represented by
\[
h_{y_i}(y_i) = (\theta_i + \theta_3)h(y_i; \eta) \left\{ \overline{H}(y_i; \eta) \right\}^{-(\alpha + 1)}; i = 1, 2.
\]
(30)

Furthermore, the joint reliability function of \( (Y_1, Y_2) \) can be represented in terms of the HR functions as follows:
\[
\overline{V}_{Y_1,Y_2}(y_1,y_2) = \exp \left\{ - \int_0^{y_1} \zeta_1(v, \infty) dv - \int_0^{y_2} \zeta_2(y_1, v) dv \right\}
\]
or
\[
\overline{V}_{Y_1,Y_2}(y_1,y_2) = \exp \left\{ - \int_0^{y_1} \zeta_1(v, \infty) dv - \int_0^{y_2} \zeta_2(y_1, v) dv \right\}
\]
where \( \zeta_1(y_1, \infty) = h_{y_1}(y_1) \) and \( \zeta_2(\infty, y_2) = h_{y_2}(y_2) \) are the marginal HR functions of \( Y_1 \) and \( Y_2 \), respectively. Further, if \( h_{y_2}(y_1, y_2) = h_{y_1}(y_1) \times h_{y_2}(y_2) \) the variables \( Y_1 \) and \( Y_2 \) are independent.

**Joint reversed hazard rate function and its marginal functions**

Bismi (2005) defined the bivariate reversed hazard rate (BHRH) function as a scalar, given by
\[
rh_{Y_1,Y_2}(y_1, y_2) = \frac{v_{Y_1,Y_2}(y_1,y_2)}{\overline{V}_{Y_1,Y_2}(y_1,y_2)}.
\]
So, if the random vector \( (Y_1, Y_2) \) has the BGo-H family, then
\[
rh_{Y_1,Y_2}(y_1, y_2) = \begin{cases} 
  rh_{y_1}(y_1, y_2) & \text{if } 0 < y_1 < y_2 < \infty \\
  rh_{y_2}(y_1, y_2) & \text{if } 0 < y_2 < y_1 < \infty \\
  rh_{y_3}(y) & \text{if } y_1 = y_2 = y < \infty,
\end{cases}
\]
(31)

where
\[
rh_{y_1}(y_1, y_2) = \theta_1(\theta_2 + \theta_3)h(y_1; \eta)h(y_2; \eta) \left\{ \overline{H}(y_1; \eta) \right\}^{-(\alpha + 1)},
\]
\[
rh_{y_2}(y_1, y_2) = \theta_2(\theta_1 + \theta_3)h(y_1; \eta)h(y_2; \eta) \left\{ \overline{H}(y_1; \eta) \right\}^{-(\alpha + 1)},
\]
and
\[
rh_{y_3}(y) = \theta_3h(y; \eta) \left\{ \overline{H}(y; \eta) \right\}^{-(\alpha + 1)}.
\]

The marginal reversed hazard rate (RHR) functions \( rh_{y_i}(y_i) \), \( i = 1, 2 \) of the BGo-H family can be expressed as follows:
\[
rh_{y_i}(y_i) = \frac{\theta_i}{\theta_1 + \theta_2 + \theta_3}v_{Go-H}(y; \theta_1 + \theta_2 + \theta_3, \alpha, \eta) \times \\
\left[ 1 - \overline{V}_{Y_1}(y_1; \theta_1 + \theta_3, \alpha, \eta) \right]^{-1} \\
- \overline{V}_{Y_1}(y_1; \theta_2 + \theta_3, \alpha, \eta),
\]
and
\[
rh_{y_3}(y) = \theta_3h(y; \eta) \left\{ \overline{H}(y; \eta) \right\}^{-(\alpha + 1)}.
\]

The marginal reversed hazard rate (RHR) functions \( rh_{y_i}(y_i) \), \( i = 1, 2 \) of the BGo-H family can be expressed as follows:
Bivariate gompertz generator of distributions

Let \( H(y; a, b) = \frac{1}{1 + e^{-ax}} \), for \( a, b > 0 \), be the CDF of the log-logistic distribution, then the joint survival of the BGoLLD is given by
\[
\mathcal{V}_{Y_1, Y_2}(y_1, y_2) = \sum_{i=1}^{\infty} e^{-\left[1 - \left(\frac{y_1}{y_1} + \frac{y_2}{y_2}\right)\right]} \cdot \left(1 - e^{-ax}\right)^{a-1} \cdot \left(1 - e^{-by}\right)^{b-1}.
\]...(33)

Figure 1 shows the joint PDF, BHR function and the BRHR function of the BGoLLD for the parameters \( \theta_1 = \theta_2 = \theta_3 = 3, \alpha = b = 3 \) and \( a = 50 \).

**Bivariate Gompertz-Frechet distribution (BGoFD)**

Let \( H(y; a, b) = e^{-\left(\frac{y}{a}\right)^b} \), for \( a, b, y > 0 \), be the CDF of the Frechet distribution, then the joint survival of the BGoFD is given by
\[
\mathcal{V}_{Y_1, Y_2}(y_1, y_2) = \prod_{i=1}^{\infty} e^{-\left(1 - e^{-\left(\frac{y_1}{y_1}\right)^b}\right)} \cdot \left(1 - e^{-ax}\right)^{a-1} \cdot \left(1 - e^{-by}\right)^{b-1}.
\]...(34)

Figure 2 shows the joint PDF, BHR function and the BRHR function of the BGoFD for the parameters \( \theta_1 = \theta_2 = \theta_3 = 3, \alpha = 2, b = 0.5 \) and \( a = 50 \).

**Bivariate Gompertz-Weibull distribution (BGoWD)**

Let \( H(y; a, b) = 1 - e^{-ay^b} \), for \( a, b, y > 0 \), be the CDF of the Weibull distribution, then the joint survival of the BGoWD is given by
\[
\mathcal{V}_{Y_1, Y_2}(y_1, y_2) = \prod_{i=1}^{\infty} e^{-\left[1 - \left(1 - e^{-ax}\right)^{-a}\right]} \cdot \left[1 - e^{-by}\right]^{-b}.
\]...(35)

Figure 3 shows the joint PDF, BHR function and the BRHR function of the BGoWD for the parameters \( \theta_1 = \theta_2 = \theta_3 = 3, \alpha = b = 0.5 \) and \( a = 0.8 \).

From Figures 1, 2 and 3, we note the BGo-H family presents different shapes of the joint PDF, BHR function and the BRHR function for different baseline CDF \( H(y; \eta) \).

**Maximum likelihood estimation (MLE)**

In this section, we estimate the unknown parameters of the BGo-H family using the maximum likelihood method. Suppose that \( \{(y_{11}, y_{21}), (y_{12}, y_{22}), \ldots, (y_{1n}, y_{2n})\} \) is a sample of size \( n \) from the BGo-H family.

We use the following notation \( I_1 = \{y_{1i} < y_{2i}\} \), \( I_2 = \{y_{1i} > y_{2i}\} \), \( I_3 = \{y_{1i} = y_{2i}\} \), \( I = I_1 \cup I_2 \cup I_3 \), \( |I_1| = n_1, |I_2| = n_2, |I_3| = n_3 \), and \( |I| = n_1 + n_2 + n_3 = n \).

Based on the observations, the likelihood function \( l(\theta) \) of this sample is
\[
l(\theta) = \prod_{i \in I_1} v_1(y_{1i}, y_{2i}) \prod_{i \in I_2} v_2(y_{1i}, y_{2i}) \prod_{i \in I_3} v_3(y_{1i}).
\]...(36)
Substituting equation (6) into equation (36), the log-likelihood function \( L(\boldsymbol{\theta}) \) can be written as

\[
L(\boldsymbol{\theta}) = -n_2 \ln \left[ \theta_3 \phi_1 + \sum_{i \in f_2} \ln \left( h(y_{1i}; \eta) \right) \right] - \theta_2 \sum_{i \in f_2} \left[ 1 - \left( \frac{\theta_3}{\phi_1} \frac{\eta}{y_{1i}} \right)^{-\alpha} \right] + \sum_{i \in f_2} \ln \left( h(y_{1i}; \eta) \right) - \left( \alpha + 1 \right) \sum_{i \in f_2} \ln \left( \frac{\phi_1}{\eta} \right) + \frac{\theta_2}{\alpha} \sum_{i \in f_2} \left[ 1 - \left( \frac{\theta_3}{\phi_1} \frac{\eta}{y_{1i}} \right)^{-\alpha} \right] - \left( \alpha + 1 \right) \sum_{i \in f_2} \ln \left( \frac{\phi_1}{\eta} \right) + \frac{\theta_2}{\alpha} \sum_{i \in f_2} \left[ 1 - \left( \frac{\theta_3}{\phi_1} \frac{\eta}{y_{1i}} \right)^{-\alpha} \right] + \frac{\theta_2}{\alpha} \sum_{i \in f_2} \left[ 1 - \left( \frac{\theta_3}{\phi_1} \frac{\eta}{y_{1i}} \right)^{-\alpha} \right] + \frac{\theta_2}{\alpha} \sum_{i \in f_2} \left[ 1 - \left( \frac{\theta_3}{\phi_1} \frac{\eta}{y_{1i}} \right)^{-\alpha} \right] + \sum_{i \in f_2} \ln \left( h(y_{1i}; \eta) \right) - \left( \alpha + 1 \right) \sum_{i \in f_2} \ln \left( \frac{\phi_1}{\eta} \right) + \frac{\theta_2}{\alpha} \sum_{i \in f_2} \left[ 1 - \left( \frac{\theta_3}{\phi_1} \frac{\eta}{y_{1i}} \right)^{-\alpha} \right] + \frac{\theta_2}{\alpha} \sum_{i \in f_2} \left[ 1 - \left( \frac{\theta_3}{\phi_1} \frac{\eta}{y_{1i}} \right)^{-\alpha} \right] + \sum_{i \in f_2} \ln \left( h(y_{1i}; \eta) \right) \]

The first partial derivatives of equation (37) with respect to \( \theta_1, \theta_2, \theta_3, \alpha \) and \( \eta_k \) \((k = 1, 2, 3, \ldots)\) are

\[
\frac{\partial L}{\partial \theta_1} = \frac{n_1}{\phi_1} + \frac{1}{\alpha} \sum_{i \in f_2} \left[ 1 - \left( \frac{\theta_3}{\phi_1} \frac{\eta}{y_{1i}} \right)^{-\alpha} \right] + \frac{n_2}{\phi_1 + \phi_3} + \frac{1}{\alpha} \sum_{i \in f_2} \left[ 1 - \left( \frac{\theta_3}{\phi_1} \frac{\eta}{y_{1i}} \right)^{-\alpha} \right] + \frac{1}{\alpha} \sum_{i \in f_2} \left[ 1 - \left( \frac{\theta_3}{\phi_1} \frac{\eta}{y_{1i}} \right)^{-\alpha} \right],
\]

\( \ldots(38) \)
\[
\frac{\partial L}{\partial \theta_3} = \frac{n_3}{\theta_3^2} + \frac{1}{\theta_3} \sum_{i \in I_3} \left[ 1 - \left( \frac{H(y_{i1}; \eta)}{H(y_{i1}; \eta)^{\alpha}} \right)^{-\alpha} \right] + \frac{n_2}{\theta_3} + \\
- \frac{1}{\alpha} \sum_{i \in I_3} \left[ 1 - \left( \frac{H(y_{i1}; \eta)}{H(y_{i1}; \eta)^{\alpha}} \right)^{-\alpha} \right] + \frac{1}{\alpha} \sum_{i \in I_3} \left[ 1 - \left( \frac{H(y_{i2}; \eta)}{H(y_{i2}; \eta)^{\alpha}} \right)^{-\alpha} \right],
\]
and
\[
\frac{\partial L}{\partial \alpha} = \sum_{i \in I_3} \ln \left( H(y_{i1}; \eta) \right) + \\
\frac{\theta_1 + \theta_2}{\alpha^2} \sum_{i \in I_3} \left[ \left( \frac{H(y_{i1}; \eta)}{H(y_{i1}; \eta)^{\alpha}} \right)^{-\alpha} \left\{ \alpha \ln \left( \frac{H(y_{i1}; \eta)}{H(y_{i1}; \eta)^{\alpha}} \right) + 1 \right\} - 1 \right] \\
- \sum_{i \in I_3} \ln \left( H(y_{i2}; \eta) \right) + \frac{\theta_3}{\alpha^2} \sum_{i \in I_3} \left[ \left( \frac{H(y_{i2}; \eta)}{H(y_{i2}; \eta)^{\alpha}} \right)^{-\alpha} \left\{ \alpha \ln \left( \frac{H(y_{i2}; \eta)}{H(y_{i2}; \eta)^{\alpha}} \right) + 1 \right\} - 1 \right],
\]
where \(\alpha\) is the derivative of the function \(A(.)\) with respect to \(\eta_k\). By equating the equations (38 – 42) by zeros, we get the non-linear normal equations. So, the solution has to be obtained numerically.

**Simulation results**

In this section, the MLE method is used to estimate the parameters \(a, b, \alpha, \theta_1, \theta_2\) and \(\theta_3\) of the BGoLDD. The population parameters are generated using software R package. The sampling distributions are obtained for different sample sizes \(n = [50; 250; 600; 1000]\) from \(N = 1000\) replications. This study presents an assessment of the properties of the MLE for the parameters in terms of variance (Var) and mean square error (MSE). The following algorithm shows how to generate data from the BGoLDD:

1. Generate \(U_1, U_2\) and \(U_3\) from \(U(0,1)\).
2. Compute \(Z_k = H^{-1} \left\{ 1 - \left( 1 - \frac{\alpha}{\alpha_{max}} \log [1 - U_k] \right)^{-\frac{1}{\alpha}} \right\}; \ k = 1, 2, 3.\)
3. Obtain \(Y_j = \min \{ Z_j, Z_3 \}; j = 1, 2.\)
The BGoLLD to analyse this data comparing with other famous bivariate models, such as Marshall-Olkin bivariate exponential (MOBE); bivariate generalised exponential (BGE); bivariate Gumbel exponential (BGuE); bivariate Burr X exponential (BBUXE); bivariate Weibull exponential (BWE); bivariate generalised linear failure rate (BGLFR); bivariate Weibull (BW); bivariate exponentiated Weibull (BEW); bivariate generalised power Weibull (BGPW); bivariate Gompertz (BGo); bivariate generalised Gompertz (BGGo); bivariate Gumbel Gompertz (BGuGo); bivariate Burr X Gompertz (BBUXGo); bivariate exponentiated Weibull Gompertz (BEWGo); bivariate exponentiated modified Weibull extension (BEMWEx), bivariate exponentiated log-logistic (BELL), and bivariate Kumaraswamy log-logistic (BKwLL) models. To make this comparison, we will use the log-likelihood values (L), Bayesian information criterion (BIC), Akaike information criterion (AIC), correct Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC). Figure 4 shows the data representation.

Table 1 lists the MLEs, Var and MSE values for the BGoLLD.

| n   | Parameter | Estimate | Var     | MSE      |
|-----|-----------|----------|---------|----------|
| 50  | a = 0.7   | 0.7654   | 0.07694 | 0.08121  |
|     | b = 0.9   | 0.9877   | 0.10317 | 0.11086  |
|     | α = 1.5   | 1.8015   | 0.35470 | 0.44560  |
|     | θ₁ = 1.6  | 1.7218   | 0.14329 | 0.15812  |
|     | θ₂ = 1.7  | 1.7598   | 0.07035 | 0.07392  |
|     | θ₃ = 1.8  | 1.8659   | 0.07752 | 0.08187  |
| 250 | a = 0.7   | 0.7501   | 0.05894 | 0.06444  |
|     | b = 0.9   | 0.9701   | 0.08247 | 0.08659  |
|     | α = 1.5   | 1.8001   | 0.35305 | 0.36738  |
|     | θ₁ = 1.6  | 1.7048   | 0.12329 | 0.12957  |
|     | θ₂ = 1.7  | 1.7265   | 0.03117 | 0.03697  |
|     | θ₃ = 1.8  | 1.8425   | 0.05051 | 0.05485  |
| 600 | a = 0.7   | 0.7200   | 0.02354 | 0.02394  |
|     | b = 0.9   | 0.9308   | 0.03623 | 0.03718  |
|     | α = 1.5   | 1.6187   | 0.13964 | 0.15373  |
|     | θ₁ = 1.6  | 1.6784   | 0.09223 | 0.09838  |
|     | θ₂ = 1.7  | 1.7015   | 0.00176 | 0.00176  |
|     | θ₃ = 1.8  | 1.8288   | 0.03388 | 0.03471  |
| 1000| a = 0.7   | 0.7002   | 0.00002 | 0.00023  |
|     | b = 0.9   | 0.9126   | 0.01480 | 0.01495  |
|     | α = 1.5   | 1.6014   | 0.11929 | 0.12957  |
|     | θ₁ = 1.6  | 1.6358   | 0.04211 | 0.04339  |
|     | θ₂ = 1.7  | 1.7004   | 0.00047 | 0.00048  |
|     | θ₃ = 1.8  | 1.8015   | 0.00175 | 0.00177  |

From Table 1, we note that the Var and the MSE are reduced as the sample size is increased. These results indicate that the BGoLLD works well under the situation where no censoring occurs, and the MLE is a good method to estimate the model parameters.

RESULTS AND DISCUSSION: REAL DATA ANALYSIS

This data represents football (soccer) data of the UEFA Champion’s League (Meintanis, 2007). We consider

Table 2: The -L, K-S and p values for Y₁, Y₂ and min(Y₁, Y₂)

| Model | -L  | K-S | p value | -L  | K-S | p value | -L  | K-S | p value |
|-------|-----|-----|---------|-----|-----|---------|-----|-----|---------|
| GoLL  | 161.9| 0.095| 0.889   | 162.6| 0.099| 0.864   | 157.808| 0.082| 0.966   |

Figure 4: The scatter plot for football data
Figure 5: Estimated CDFs for the marginal distributions

Figure 6: PP plots for the marginal distributions

Table 3: MLEs for models using football data

| Model   | $\hat{\theta}_1$ | $\hat{\theta}_2$ | $\hat{\theta}_3$ | $\hat{\theta}_4$ | $\hat{\theta}_5$ | $\hat{\theta}_6$ | $\hat{\theta}_7$ | $\hat{\theta}_8$ |
|---------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| MOBE    | 0.012             | 0.014             | 0.022             | –                 | –                 | –                 | –                 | –                 |
| BGE     | 1.553             | 0.499             | 1.156             | 0.039             | –                 | –                 | –                 | –                 |
| BGuE    | 2.678             | 0.962             | 2.065             | 5.011             | 4.081             | –                 | –                 | –                 |
| BBUXE   | 0.385             | 0.136             | 0.310             | 0.012             | –                 | –                 | –                 | –                 |
| BWE     | 0.135             | 0.302             | 0.265             | 0.025             | –                 | –                 | –                 | –                 |
| BGLFR   | 0.452             | 0.156             | 0.360             | 0.0002            | 0.0008            | –                 | –                 | –                 |
| BW      | 0.397             | 0.274             | 0.339             | 0.083             | –                 | –                 | –                 | –                 |
| BEW     | 1.227             | 0.382             | 0.661             | 0.012             | 1.268             | –                 | –                 | –                 |
| BGPW    | 3.229             | 1.983             | 4.084             | 0.037             | –                 | –                 | –                 | –                 |
| BGo     | 0.003             | 0.002             | 0.021             | 0.040             | –                 | –                 | –                 | –                 |
| BGGo    | 0.742             | 0.262             | 0.598             | 0.011             | 0.029             | –                 | –                 | –                 |
| BGuGo   | 0.578             | 0.204             | 0.475             | 0.009             | 0.047             | 2.278             | –                 | –                 |
| BBUXGo  | 0.132             | 0.187             | 0.201             | 0.006             | 0.015             | –                 | –                 | –                 |
| BEWGo   | 0.547             | 0.192             | 0.444             | 0.411             | 0.079             | 0.005             | 1.358             | –                 |
| REMWEx  | 0.167             | 0.061             | 0.139             | 85.918            | 4.505             | 0.025             | –                 | –                 |
| BELL    | 0.038             | 0.039             | 0.092             | 72.08             | 11.59             | –                 | –                 | –                 |
| BKwLL   | 24.39             | 21.17             | 46.06             | 396.0             | 11.21             | 0.156             | –                 | –                 |
| BGoLL   | 15.881            | 9.379             | 44.143            | 2903.789          | 1.076             | 144.350           | –                 | –                 |
Table 4: \( L \), AIC, CAIC, HQIC and BIC values for the models using football data

| Model  | \( L \)    | AIC    | CAIC   | BIC    | HQIC   |
|--------|-----------|--------|--------|--------|--------|
| MOBE   | –298.9    | 607.9  | 609.8  | 615.9  | 610.7  |
| BGE    | –299.9    | 607.7  | 608.9  | 614.2  | 609.9  |
| BGaE   | –297.8    | 605.6  | 607.5  | 613.6  | 608.4  |
| BBUXE  | –294.8    | 597.6  | 598.9  | 604.0  | 599.9  |
| BWE    | –291.1    | 592.3  | 594.2  | 600.3  | 595.1  |
| BGLFR  | –296.8    | 603.7  | 605.6  | 611.7  | 606.5  |
| BW     | –346.0    | 700.0  | 701.3  | 706.4  | 702.3  |
| BEW    | –298.9    | 607.9  | 609.8  | 615.9  | 610.7  |
| BGPW   | –344.8    | 697.5  | 698.8  | 703.9  | 699.8  |
| BGo    | –303.5    | 614.9  | 616.2  | 621.4  | 617.2  |
| BBGo   | –294.4    | 599.8  | 601.7  | 607.9  | 602.7  |
| BGuGo  | –294.2    | 600.5  | 603.3  | 610.1  | 603.9  |
| BBUxGo | –301.2    | 612.4  | 614.3  | 620.5  | 615.2  |
| BBWEx  | –294.1    | 600.3  | 603.1  | 609.9  | 603.7  |
| BELL   | –284.9    | 579.8  | 581.8  | 587.9  | 582.7  |
| BKwLL  | –283.9    | 579.9  | 582.7  | 589.6  | 583.3  |
| BGoll  | –272.8    | 557.6  | 560.4  | 567.3  | 561.0  |

Before trying to analyze the data using the BGoLLD, we fit at first the marginals \( Y_1 \) and \( Y_2 \) separately and the min\((Y_1, Y_2)\) on the UEFA Champion’s League data. The MLEs of the parameters \((\theta, \alpha, b, \alpha)\) of the corresponding Gompertz-log-logistic distribution (GoLLD) for \( Y_1, Y_2 \) and min\((Y_1, Y_2)\) are \((2.675, 137.3, 1.466, 6.059)\), \((2.874, 127.4, 1.142, 3.631)\) and \((3.469, 102.650, 2.674, 1.278)\), respectively. Table 2 reports \(-L\), Kolmogorov-Smirnov (K-S) distance and p values for \( Y_1, Y_2 \) and min\((Y_1, Y_2)\). Based on the p values, it is clear that the GoLLD fits the data for the marginals. Figures 5 and 6 show the estimated CDF and PP plots for real data, which support our results in Table 2.

Now, we fit the BGoLLD on this data. Tables 3 and 4 list the MLEs, \( L \), AIC, CAIC, HQIC and BIC values for the competitive models based on football data.

From Table 4, it is clear that the BGoLLD provides a better fit than the other competitive models, because it has the smallest value among \(-L\), AIC, CAIC, HQIC and BIC.

CONCLUSIONS

In this paper, we have presented a new flexible bivariate generator of distributions, in the so-called bivariate Gompertz-H (BGo-H) family, whose marginal distributions are Gompertz-H families. The joint CDF and joint PDF of the BGo-H family have simple forms, therefore, this new model can be easily used in practice for modelling bivariate data restricted in the interval \((0, \infty)\). Some statistical and mathematical properties of the new family have been studied. The simulation results have indicated that the MLE works quite satisfactorily and it can be used to compute the model parameters. Also, we have analysed a real dataset and showed through goodness-of-fit tests that the proposed family can be used for modelling the data considered herein.

A multivariate extension of the Gompertz-H family is presented as conclusion. Assume \( Z_1, Z_2, \ldots, Z_{n+1} \) be independent random variables with \( Z_i \sim \text{Go} - H (\theta_i, \alpha, \eta)\), such that \( i = 1, 2, \ldots, n + 1 \). Define \( Y_j = \min \{Z_j, Z_{n+1}\} \) for \( j = 1, 2, \ldots, n \). Hence, the joint survival function of \( Y_1, Y_2, \ldots, Y_n \) is given by

\[
\mathbb{V}(y_1, y_2, \ldots, y_n) = \mathbb{P}[Z_1 > y_1, Z_2 > y_2, \ldots, Z_n > y_n, Z_{n+1} > y] \\
= \mathbb{V}(y, \theta_{n+1}, \alpha, \eta) \prod_{j=1}^{n} \mathbb{V}(y_j, \theta_j, \alpha, \eta),
\]

for \((y_1, y_2, \ldots, y_n) \in (0, \infty)^n\), where \( y = \max \{y_1, y_2, \ldots, y_n\} \). Clearly, the BGo-H family arises from this multivariate Gompertz-H family by taking \( n = 2 \). In the future, we will discuss in detail the multivariate extension of the Gompertz-H family, because it has many applications in lifetime analysis, environmental, economics, engineering and medical sciences.

Acknowledgement

The authors extend their appreciation to the Deanship of Scientific Research at Majmaah University for funding this research under project number No. (RGP-2019-2).

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