Carries, Shuffling and An Amazing Matrix

Persi Diaconis*       Jason Fulman†

Version of June 17, 2008

Abstract

The number of “carries” when \( n \) random integers are added forms a Markov chain \([23]\). We show that this Markov chain has the same transition matrix as the descent process when a deck of \( n \) cards is repeatedly riffle shuffled. This gives new results for the statistics of carries and shuffling.

1 Introduction

In a wonderful article in this monthly, John Holte \([23]\) found fascinating mathematics in the usual process of “carries” when adding integers. His article reminded us of the mathematics of shuffling cards. This connection is developed below.

Consider adding two 50-digit binary numbers:

\[
\begin{array}{llllllllllllllllllllllllllllll}
1 & 11111 & 11100 & 01110 & 01000 & 00001 & 00111 & 10111 & 00000 & 01111 & 1110011011001001101111001100001001111011100110000100111101110010001001
\end{array}
\]

For this example, \( 28/50=56\% \) of the columns have a carry of 1. Holte shows that if the binary digits are chosen at random, uniformly, in the limit 50\% of all the carries are zero. This holds no matter what the base. More generally, if \( n \) integers (base \( b \)) are produced by choosing their digits uniformly at random in \( \{0, 1, \ldots, b-1\} \), the sequence of carries \( \kappa_0 = 0, \kappa_1, \kappa_2, \ldots \) is a Markov chain taking values in \( \{0, 1, 2, \ldots, n-1\} \). Holte begins by deriving the transition matrix between successive carries \( \kappa, \kappa' \).

\((H1)\) \( P(i, j) = \)

\[
P(\kappa' = j | \kappa = i) = P\{jb \leq i + X_1 + \cdots + X_n \leq (j + 1)b - 1\}
= \frac{1}{b^n} \sum_{r=0}^{j-\lfloor i/b \rfloor} (-1)^r \binom{n+1}{r} \binom{n-1-i+(j+1-r)b}{n}
\]

Here, \( 0 \leq i, j \leq n - 1 \) and \( X_1, X_2, \ldots, X_n \) are independent and uniformly distributed on \( \{0, 1, \ldots, b-1\} \).

\((H2)\) When \( b = 2 \), for any \( n \), the transition matrix is

\[
P(i, j) = \frac{1}{2^n} \cdot \binom{n+1}{2j-i+1} \quad 0 \leq i, j \leq n - 1.
\]

*Departments of Mathematics and Statistics, Stanford University
†Department of Mathematics, University of Southern California; fulman@usc.edu
For $n = 3$, for all $b$

$$P(i, j) = \frac{1}{6b^2} \begin{pmatrix} b^2 + 3b + 2 & 4b^2 - 4 & b^2 - 3b + 2 \\ b^2 - 1 & 4b^2 + 2 & b^2 - 1 \\ b^2 - 3b + 2 & 4b^2 - 4 & b^2 + 3b + 2 \end{pmatrix}. $$

These are the “amazing matrices” of Holte’s title. Among many things, Holte shows

(H4) The matrix $P(i, j)$ of (H1) has stationary vector $\pi_n(j)$ (left eigenvector with eigenvalue 1) independent of the base $b$:

$$\pi_n(j) = \frac{A(n, j)}{n!}$$

with $A(n, j)$ the Eulerian number. This may be defined as

(H4') $A(n, j)$ is the number of permutations in the symmetric group $S_n$ with $j$-descents. Recall that $\sigma \in S_n$ has a descent at $i$ if $\sigma(i + 1) < \sigma(i)$. So $51324$ has two descents.

(H4'') $A(n, j)$ is the coefficient of $x^{j+1}$ in the polynomial $p_n(x)$ where

$$\sum_{i=0}^{\infty} i^n x^i = \frac{p_n(x)}{(1-x)^{n+1}}.$$

(H4''') $A(n, j) = \sum_{\ell=0}^{j} (-1)^{\ell+1} \binom{n+1}{\ell} (j + 1 - \ell)^n.$

Definition (H4') is most relevant to the present paper. (H4'') is equivalent to Worpitzky’s identity. It has many proofs and appearances, e.g., to juggling sequences [11]. Finally, (H4''') goes back to Euler. An elementary development of these ideas is in [12].

When $n = 2$, $A(2, 0) = A(2, 1) = 1$, thus $\pi_2(0) = \pi_2(1) = 1/2$ is the limiting frequency of carries when two long integers are added. When $n = 3$, $A(3, 0) = 1$, $A(3, 1) = 4$, $A(3, 2) = 1$, giving $\pi_3(0) = 1/6$, $\pi_3(1) = 2/3$, $\pi_3(2) = 1/6$.

Holte further shows

(H5) The matrix $P(i, j)$ of (H1) has eigenvalues $1, 1/b, 1/b^2, \ldots, 1/b^{n-1}$ with explicitly computable eigenvectors independent of $b$.

(H6) Let $P_b$ denote the matrix in (H1). Then for all real $a, b$

$$P_a P_b = P_{ab}.$$ 

When we saw properties (H4), (H5), (H6), we hollered “Wait, this is all about shuffling cards!” Knowledgeable readers may well think, “For these two guys, everything is about shuffling cards.” While there is some truth to these thoughts, we justify our claim in the next section. Following this we show how the connection between carries and shuffling contributes to each subject. The rate of convergence of the Markov chain (H1) to the stationary distribution $\pi_n$ is given in Section 4: the argument shows that the matrix $P$ is totally positive of order 2. Finally, we show how the same matrix occurs in taking sections of generating functions [9], discuss carries for multiplication, and describe another “amazing matrix”.

Our developments do not exhaust the material in Holte’s article, which we enthusiastically recommend. A “higher math” perspective on arithmetic carries as cocycles [24] suggests many further projects. We have tried to keep the presentation elementary, and mention the (more technical) companion paper [15] which analyzes the carries chain using symmetric function theory and gives analogs of our main results for other Coxeter groups.
2 Shuffling Cards

How many times should a deck of \( n \) cards be riffle shuffled to thoroughly mix it? For an introduction to this subject, see [2, 27]. The main theoretical developments are in [5, 17] with further developments in [19, 20]. A survey of the many connections and developments is in [14]. The basic shuffling mechanism was suggested by [21]. It gives a realistic mathematical model for the usual method of riffle shuffling \( n \) cards:

- Cut off \( C \) cards with probability \( \binom{n}{C}/2^n \), \( 0 \leq C \leq n \).
- Shuffle the two parts of the deck according to the following rule: if at some stage there are \( A \) cards in one part and \( B \) cards in the other part, drop the next card from the bottom of the first part with probability \( A/(A + B) \) and from the bottom of the second part with probability \( B/(A + B) \).
- Continue until all cards are dropped.

Let \( Q(\sigma) \) be the probability of generating the permutation \( \sigma \) after one shuffle, starting from the identity. Repeated shuffling is modeled by convolution:

\[
Q^2(\sigma) = \sum_{\eta} Q(\eta)Q(\sigma\eta^{-1}), \quad Q^h(\sigma) = \sum Q^{h-1}(\eta)Q(\sigma\eta^{-1}).
\]  

(1)

Thus to be at \( \sigma \) after two shuffles, the first shuffle goes to some permutation \( \eta \) and the second must be to \( \sigma\eta^{-1} \). The uniform distribution is \( U(\sigma) = 1/n! \). Standard theory shows that \( Q^h(\sigma) \to U(\sigma) \) as \( h \to \infty \). (2)

The references above give useful rates for the convergence in (2) showing that it takes \( h = 3/2 \log_2 n + c \) to get \( 2^{-c} \) close to random. When \( n = 52 \), this becomes \( h \approx 7 \) shuffles.

To explain the connection with carries, it is useful to have a second description of shuffling. Consider dropping \( n \) points uniformly at random into \([0, 1]\). Label these points in order \( x(1) \leq x(2) \cdots \leq x(n) \). The Bakers transformation \( x \mapsto 2x \mod 1 \) maps \([0, 1]\) into itself and permutes the points. Let \( \sigma \) be the induced permutation. As shown in [3], the chance of \( \sigma \) is exactly \( Q(\sigma) \). A natural generalization of this shuffling scheme to \( "b\)-shuffles" is induced from \( x \mapsto bx \mod 1 \) with \( b \) fixed in \( \{1, 2, 3, \cdots\} \). Thus ordinary riffle shuffles are 2-shuffles and a 3-shuffle results from dividing the deck into three piles and dropping cards sequentially from the bottom of each pile with probability proportional to packet size.

Let \( Q_b(\sigma) \) be the probability of \( \sigma \) after a \( b \)-shuffle. From this geometric description,

\[
Q_a * Q_b = Q_{ab}.
\]  

(3)

The Gilbert–Shannon–Reeds measure is \( Q_2 \) in this notation and we see that \( Q_b^2 = Q_{2b} \). Thus to study repeated shuffled, we need only understand a single \( b \)-shuffle. A main result of [3] is a simple formula:

\[
Q_b(\sigma) = \binom{n+b-r}{n} b^n.
\]  

(4)
Here $r = r(\sigma) = 1 + \#\{\text{descents in } (\sigma^{-1})\}$.

In addition to the similarities between (H6) and [3], [5] and [22] proved that the eigenvalues of the Markov chain induced by $Q_b$ are $1, 1/b, 1/b^2, \cdots, 1/b^{n-1}$. This and the appearance of descents convinced us that there must be an intimate connection between carries and shuffling. The main result of this article makes this precise.

**Theorem 2.1.** The number of descents in successive $b$-shuffles of $n$ cards forms a Markov chain on $\{0, 1, \cdots, n-1\}$ with transition matrix $P(i, j)$ of (H1).

### 3 Bijective Methods

First we describe some notation to be used throughout. The number of descents of a permutation $\tau$ is denoted by $d(\tau)$. Label the columns of the $n$ numbers to be added mod $b$ by $C_1, C_2, C_3, \cdots$ where $C_1$ is the right-most column.

The main purpose of this section is to give a bijective proof of the following theorem, which implies Theorem 2.1 from the introduction.

**Theorem 3.1.** Let $\kappa_j$ denote the amount carried from column $j$ to column $j+1$ when $n$ length $m$ numbers are added mod $b$. Let $\tau_j$ be the permutation obtained after the iteration of $j$ $b$-shuffles, started at the identity. Then

$$P(\kappa_1 = i_1, \cdots, \kappa_m = i_m) = P(d(\tau_1) = i_1, \cdots, d(\tau_m) = i_m)$$

for all values of $i_1, \cdots, i_m$.

In preparation for the proof, some notation and lemmas will be needed.

**Lemma 3.2.** Let $\kappa(C\_j \cdots C\_1)$ denote the amount carried from column $j$ to column $j+1$ when the corresponding $j$-tuples are added (adding consecutive $j$-tuples one at a time rather than adding a column at a time). Then $\kappa(C\_j \cdots C\_1) = \kappa_j$.

**Proof.** This is clear since in calculating the carry to column $j+1$ it is irrelevant how one adds the numbers in the preceding columns. ■

Given a length $n$ list of $j$-tuples of numbers mod $b$, one says that the list has a descent at position $i$ if the $i+1$st $j$-tuple is smaller than the $i$th $j$-tuple. For example the following 3-tuples of mod 3 numbers:

\[
\begin{array}{ccc}
0 & 1 & 2 \\
1 & 0 & 1 \\
2 & 2 & 0 \\
1 & 0 & 1 \\
0 & 2 & 0 \\
2 & 1 & 1 \\
\end{array}
\]

has a descent at position 3 since 220 is greater than 101, and a descent at position 4 since 101 is greater than 020.

Given a length $n$ list of $j$-tuples of numbers mod $b$, one says that the list has a carry at position $i$ if the addition of the $i+1$st $j$-tuple on the list to the sum of the first $i$ $j$-tuples increases the amount that would be carried to the $j+1$st column.
(it might seem more natural to say that the carry is at position $i + 1$, but our convention will be useful). For example the following 3-tuples of mod 3 numbers:

\[
\begin{align*}
&0 \ 1 \ 2 \\
&0 \ 1 \ 2 \\
&1 \ 1 \ 2 \\
&1 \ 1 \ 1 \\
&2 \ 1 \ 2 \\
&1 \ 2 \ 1 \\
\end{align*}
\]

has a carry at positions 3 and 4. Indeed $(0, 1, 2) + (0, 1, 2) = (1, 0, 1)$ which doesn’t create a carry. Adding $(1, 1, 2)$ gives $(2, 2, 0)$ which still doesn’t create a carry. Adding $(2, 1, 2)$ gives $(0, 2, 0)$ with a carry, so there is a carry at position 4. Finally adding $(1, 2, 1)$ gives $(2, 1, 1)$, which doesn’t create a carry.

For what follows we use a bijection, which we call the bar map, on sets of $j$ column vectors having length $n$ and entries in $0, 1, \cdots, b - 1$. Given $C_j \cdots C_1$, then $\overline{C_j \cdots C_1}$ is defined as follows: the $i$th $j$-tuple of $\overline{C_j \cdots C_1}$ consists of the right-most $j$ coordinates of the mod $b$ sum of the first $i$ $j$-tuples of $C_j \cdots C_1$. For example,

\[
\begin{align*}
C_3C_2C_1 &= \\
&0 \ 1 \ 2 \\
&0 \ 1 \ 2 \\
&1 \ 1 \ 2 \\
&1 \ 1 \ 1 \\
&2 \ 1 \ 2 \\
&1 \ 2 \ 1
\end{align*}
\]

\[
\begin{align*}
\Rightarrow \overline{C_3C_2C_1} &= \\
&0 \ 1 \ 2 \\
&1 \ 0 \ 1 \\
&1 \ 0 \ 1 \\
&0 \ 2 \ 0 \\
&2 \ 1 \ 1
\end{align*}
\]

Indeed $012 + 012 = 101$ giving the second line of $\overline{C_3C_2C_1}$. Then $101 + 112 = 220$ giving the third line, and $220 + 111 = 101$ (retaining only the last 3 coordinates), giving the fourth line, etc. One can easily invert the bar map, so it is a bijection.

The following lemma is immediate from these definitions.

**Lemma 3.3.** $\overline{C_j \cdots C_1}$ has a descent at position $i$ if and only if $C_j \cdots C_1$ has a carry at position $i$.

Given a length $n$ collection of $j$-tuples of numbers mod $b$, we define an associated permutation $\pi$ by labeling the $j$-tuples from lexicographically smallest to largest (considering the higher up $j$-tuple to be smaller in case of ties). For example with $n = 6, j = 2, b = 3$, one would have

\[
\begin{align*}
\pi &= \\
&\left( \begin{array}{c}
1 \ 2 \\
2 \ 1 \\
1 \ 0 \\
0 \ 1 \\
0 \ 0 \\
2 \ 1
\end{array} \right)
\end{align*}
\]

\[
\begin{array}{c|c|c|c}
& 4 & 5 & 6 \\
1 & 2 & 3 & \\
2 & 1 & & \\
0 & 1 & & \\
0 & 0 & & \\
2 & 1 & & \\
\end{array}
\]

since (0, 0) is the smallest, followed by (0, 1), (1, 0), (1, 2), then the uppermost copy of (2, 1) and finally the lowermost copy of (2, 1). Note that we use the standard convention for writing permutations, i.e. $1 \mapsto 4, 2 \mapsto 5, \text{ etc.}$ We mention that this construction appears in the theory of inverse riffle shuffling $[5]$.

**Lemma 3.4.** $\overline{C_j \cdots C_1}$ has a descent at position $i$ if and only if the associated permutation $\pi(\overline{C_j \cdots C_1})$ has a descent at position $i$. 

5
Proof. This is immediate from the definition of $\pi$. ■

To proceed define a second bijection, called the star map, on sets of $j$ column vectors having length $n$ and entries in $0, 1, \ldots, b - 1$. This sends column vectors $A_j \cdots A_1$ to $(A_j \cdots A_1)^*$ defined as follows. The right-most column of $(A_j \cdots A_1)^*$ is $A_1$. The second column in $(A_j \cdots A_1)^*$ is obtained by putting the entries of $A_2$ in the order specified by the permutation corresponding to right-most column of $(A_j \cdots A_1)^*$ (which is $A_1$). Then the third column in $(A_j \cdots A_1)^*$ is obtained by putting the entries of $A_3$ in the order specified by the permutation corresponding to the two right-most columns of $(A_j \cdots A_1)^*$, and so on.

For example,

$$A_3 A_2 A_1 = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \rightarrow (A_3 A_2 A_1)^* = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

Indeed, the right-most column of $(A_3 A_2 A_1)^*$ is $A_1$. The second column of $(A_3 A_2 A_1)^*$ is obtained by taking the entries of $A_2$ (namely 2, 2, 0, 0, 1, 1) and putting the 2 next to the smallest element of $A_1$ (so the highest 0), then the second 2 next to the 2nd smallest element (so the second 0), then the 0 next to the 3rd smallest element (so the highest 1), then the second 0 next to the 4th smallest element (so the second 1), then the 1 next to the 5th smallest element (so the third 1), and finally the second 1 next to the 6th smallest element (so the only 2), giving

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \\ 0 & 1 \\ 2 & 0 \\ 1 & 1 \end{pmatrix}$$

Then the third column from of $(A_3 A_2 A_1)^*$ is obtained by taking the entries of $A_3$ (namely 1, 1, 2, 0, 2, 0) and putting the 1 next to the smallest pair (so the highest (0, 1)), then putting the second 1 next to the 2nd smallest pair (so the second (0, 1)), then the 2 next to the third smallest pair (1, 1), then the 0 next to the fourth smallest pair (1, 2), then the second 2 next to the fifth smallest pair (the highest (2, 0)), and finally the second 0 next to the sixth smallest pair (the second (2, 0)).

The star map is straightforward to invert (we leave this as an exercise to the reader), so it is a bijection.

The crucial property of the star map is given by the following lemma, the $j = 2$ case of which is essentially equivalent to the “$A^B&B$” formula in Section 9.4 of [27].

Lemma 3.5.

$$\pi(A_j) \cdots \pi(A_1) = \pi[(A_j \cdots A_1)^*],$$

where the product on the left is the usual multiplication of permutations.
As an illustration,

\[
A_3A_2A_1 = \begin{pmatrix}
1 & 2 & 2 \\
1 & 2 & 1 \\
2 & 0 & 0 \\
0 & 0 & 1 \\
2 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\]

yields the permutations

\[
\begin{align*}
\pi(A_3) & = 3 & \pi(A_2) & = 5 & \pi(A_1) & = 6 \\
\pi(A_3) & = 4 & \pi(A_2) & = 6 & \pi(A_1) & = 3 \\
\pi(A_3) & = 5 & \pi(A_2) & = 1 & \pi(A_1) & = 1 \\
\pi(A_3) & = 1 & \pi(A_2) & = 2 & \pi(A_1) & = 4 \\
\pi(A_3) & = 6 & \pi(A_2) & = 3 & \pi(A_1) & = 2 \\
\pi(A_3) & = 2 & \pi(A_2) & = 4 & \pi(A_1) & = 5
\end{align*}
\]

Also as calculated above,

\[
(A_3A_2A_1)^* = \begin{pmatrix}
0 & 1 & 2 \\
1 & 0 & 1 \\
2 & 2 & 0 \\
1 & 0 & 1 \\
0 & 2 & 0 \\
2 & 1 & 1
\end{pmatrix}
\]

which yields the permutations

\[
\begin{align*}
\pi[(A_3A_2A_1)^*] & = 1 & \pi[(A_2A_1)^*] & = 4 & \pi[(A_1)^*] & = 6 \\
\pi[(A_3A_2A_1)^*] & = 3 & \pi[(A_2A_1)^*] & = 1 & \pi[(A_1)^*] & = 3 \\
\pi[(A_3A_2A_1)^*] & = 6 & \pi[(A_2A_1)^*] & = 5 & \pi[(A_1)^*] & = 1 \\
\pi[(A_3A_2A_1)^*] & = 4 & \pi[(A_2A_1)^*] & = 2 & \pi[(A_1)^*] & = 4 \\
\pi[(A_3A_2A_1)^*] & = 2 & \pi[(A_2A_1)^*] & = 6 & \pi[(A_1)^*] & = 2 \\
\pi[(A_3A_2A_1)^*] & = 5 & \pi[(A_2A_1)^*] & = 3 & \pi[(A_1)^*] & = 5
\end{align*}
\]

\(\pi(A_3^*) = \pi(A_1), \pi[(A_2A_1)^*] = \pi(A_2)\pi(A_1),\) and \(\pi[(A_3A_2A_1)^*] = \pi(A_3)\pi(A_2)\pi(A_1),\)

and Lemma 3.5 gives that this happens in general.

**Proof of Lemma 3.5.** This is clear for \(j = 1,\) so consider \(j = 2.\) Then the claim is perhaps easiest to see using the theory of inverse riffle shuffles. Namely given a column of \(n\) numbers mod \(b,\) mark cards 1, \(\cdots, n\) with these numbers, then bring the cards labeled 0 to the top (cards higher up remaining higher up), then bring the cards labeled 1 just beneath them, and so on. For instance,

\[
\begin{pmatrix}
2 & 3 \\
1 & 5 \\
0 & 2 \\
1 & 4 \\
0 & 6 \\
1 & 1
\end{pmatrix}
\]
Note that (in the notation of the example) this is $\pi(A_1)^{-1}$. Now repeat this process, using the column
\[
\begin{array}{c}
2 \\
2 \\
0 \\
0 \\
1 \\
1 \\
\end{array}
\]
to label the cards, placing the labels just to the left of the digit already on each card. A moment’s thought shows that this is equivalent to a single process in which one labels the cards with pairs from $(A_2A_1)^*$. Thus $\pi[(A_2A_1)^*]^{-1} = \pi(A_1)^{-1}\pi(A_2)^{-1}$, so that $\pi[(A_2A_1)^*] = \pi(A_2)\pi(A_1)$. The reader desiring further discussion for the case of two columns is referred to Section 9.4 of the expository paper [27]. The argument for $j \geq 3$ is identical: just use the observation that iterating the procedure three times is equivalent to a single process in which one labels the cards with triples from $(A_3A_2A_1)^*$.

With the above preparations in hand, Theorem 3.1 can be proved.

**Proof of Theorem 3.1** To begin, note that
\[
\kappa_1 = i_1, \ldots, \kappa_m = i_m \iff \kappa(C_j \cdots C_1) = i_j \ (1 \leq j \leq m) \\
\iff d(C_j \cdots C_1) = i_j \ (1 \leq j \leq m) \\
\iff d(\pi(C_j \cdots C_1)) = i_j \ (1 \leq j \leq m).
\]
The first step used Lemma 3.2, the second step used Lemma 3.3 and the third step used Lemma 3.4.

Let $A_m \cdots A_1 = (C_m \cdots C_1)^{-*}$. Then $A_j \cdots A_1 = (C_j \cdots C_1)^{-*}$ for all $1 \leq j \leq m$, and Lemma 3.5 implies that
\[
d[\pi(A_j) \cdots \pi(A_1)] = d[\pi((A_j \cdots A_1)^*)] = d[\pi(C_j \cdots C_1)] = i_j
\]
for all $1 \leq j \leq m$. Now note that if $C_m \cdots C_1$ are chosen i.i.d. with entries uniform in $0, 1, \ldots, b-1$, then the same is true of $A_m \cdots A_1$ since the bar and star maps are both bijections. Note that each $\pi(A_i)$ has the distribution of a permutation after a b-shuffle, so one may take $\tau_j$ to be the product $\pi(A_j) \cdots \pi(A_1)$, and the theorem is proved. 

**Remark and example:** The above construction may appear complicated, but we mention that the star map (though useful in the proof) is not needed in order to go from the columns of numbers being added to the $\tau$’s. Indeed, from the proof of Theorem 3.1 one sees that the $\tau_j$’s can be defined by $\tau_j = \pi(C_j \cdots C_1)$. Thus in the running example,
\[
\begin{array}{cccc}
0 & 1 & 2 & \tau_3 \\
0 & 1 & 2 & \tau_2 \\
1 & 1 & 2 & \tau_1 \\
\end{array}
\]
\[
C_3C_2C_1 = \begin{pmatrix}
1 & 1 & 2 \\
1 & 1 & 1 \\
2 & 1 & 2 \\
1 & 2 & 1
\end{pmatrix} \implies \begin{pmatrix}
0 & 1 & 2 \\
1 & 0 & 1 \\
0 & 2 & 0 \\
2 & 1 & 1
\end{pmatrix} \implies \begin{pmatrix}
1 & 4 & 6 \\
3 & 1 & 3 \\
5 & 2 & 4 \\
5 & 3 & 5
\end{pmatrix}.
\]
Observe that $\kappa_1 = 3$, $\kappa_2 = 3$, $\kappa_3 = 2$, and that $d(\tau_1) = 3$, $d(\tau_2) = 3$, $d(\tau_3) = 2$ as claimed.

As a corollary of Theorem 3.1, we deduce that the descent process after riffle shuffles is Markov (usually, a function of a Markov chain is not Markov).

**Corollary 3.6.** Let a Markov chain on the symmetric group begin at the identity and proceed by successive independent $b$-shuffles. Then $d(\pi)$, the number of descents, forms a Markov chain.

**Proof.** This follows from Theorem 3.1 and the fact that the carries process is Markov.

4 Applications to the Carries Process

As in previous sections, let $\kappa_j$ be the amount carried from column $j$ to column $j+1$ when $n$ length-$m$ numbers are added mod $b$. Suppose throughout this section that the “digits” of these numbers are chosen uniformly and independently in $\{0, 1, \ldots, b-1\}$.

**Theorem 4.1.** For $1 \leq j \leq m$, the expected value of $\kappa_j$ is $\mu_j = \frac{n-1}{2} \left( 1 - \frac{1}{b^j} \right)$. The variance of $\kappa_j$ is $\sigma_j^2 = \frac{n+1}{12} \left( 1 - \frac{1}{b^m} \right)$. Normalized by its mean and variance, for large $n$, $\kappa_j$ has a limiting standard normal distribution.

**Proof.** From Lemma 3.3 of Section 3, $\kappa_j$ is distributed exactly like the number of descents among the $n$ rows of the right-most $j$ digits of the random array. The distribution of these descents is studied in [8] where they are shown to be a 2-dependent process with the required mean and variance. The central limit theorem for 2-dependent processes is classical [3].

**Remarks:**

1. Note that $\mu_j, \sigma_j^2$ are increasing to their limiting value $\frac{n-1}{2}, \frac{n+1}{12}$ as $j$ increases.

2. Let $S_m = \kappa_1 + \kappa_2 + \cdots + \kappa_m$ be the total number of carries. By linearity of expectation and Theorem 4.1, this has mean

$$\bar{\mu}_m = \frac{n-1}{2} \left( m - \frac{1}{b-1} \left( 1 - \frac{1}{b^m} \right) \right).$$

When $n = 2$, this was shown by Knuth [26, p. 278]. He also finds the variance of $S_m$ when $n = 2$. For fixed $n$ and $b$, the central limit theorem for finite state space Markov chains [4] shows that $S_m$, normalized by its mean and variance, has a standard normal limiting distribution.

3. The fine properties of the number of carries within a column is studied in [7] where it is shown to be a determinantal point process.

As shown above, the carries process $\kappa_j, 0 \leq j \leq m$ (with $\kappa_0 = 0$) is a Markov chain which has limiting stationary distribution $\pi(j) = A(n, j)/n!$. To study the rate of convergence to the limit we first prove a new property of the amazing matrix $P(i, j)$ of (H1). Recall that a matrix is totally positive of order two (TP2) if all the $2 \times 2$ minors are non-negative.

**Lemma 4.2.** For every $n$ and $b$, the matrix $P(i, j)$ of (H1) is TP2.
Proof. As noted on p. 140 of [23],

\[ P(i, j) = \frac{1}{b^n} \left[ x^{(j+1)b-i-1} \right] \left( \frac{1 - x^b}{1 - x} \right)^{n+1} \]

where \( [x^i] f(x) \) denotes the coefficient of \( x^i \) in a polynomial \( f(x) \). Thus the transpose of \( P \) is a submatrix of the matrix with \((i, j)\) coordinates \( [x^i - j] \cdot (1 - x^b)/(1 - x)^{n+1} \).

Since the product of \( T \) \( P_2 \) matrices is \( T \) \( P_2 \), it is enough to treat the case \( n = 0 \).

Now, the matrix is a lower triangular, \( n \times n \) matrix with ones down the diagonal, ones on the next lowest \( b - 1 \) diagonals and zeros elsewhere. For example, when \( n = 6, b = 3 \) the relevant matrix is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix}
\]

By inspection, 13 of the 16 possible \( 2 \times 2 \) matrices can occur as minors. The missing ones are

\[
\begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

these being the only ones with negative determinants.

Remark: When \( b = 2 \), the original \( P(i, j) = 2^{-n} \binom{n+1}{2j-i+1} \) is totally positive \( (TP_\infty) \). Indeed, \( P(i, j) = 2^{-n} [x^{2j-i+1}] (1 + x)^{n+1} \). Let \( i' = i + 1, j' = j + 1 \). This becomes \( 2^{-n} [x^{2j'-i'}] (1 + x)^{n+1} \). Each minor of this is a subminor of \( 2^{-n} [x^{j'-i'}] (1 + x)^{n+1} \). This is totally positive by the classification of Polya frequency sequences due to Schoenberg and Edrei ([25], Chap. 8).

Consider the basic transition matrix \( P(i, j) \) for general \( b, n \). This has stationary distribution \( \pi(j), 0 \leq j \leq n \), given in (H4). The carries Markov chain starts at 0 and the right-most carries tend to be smaller. This is seen in Theorem 4.1. It is natural to ask how far over one must go so that the carries process is stationary. If \( P^r(0, j) \) is the chance of carry \( j \) after \( r \) steps, we measure the approach to stationarity by separation

\[ \text{sep}(r) = \max_j \left[ 1 - \frac{P^r(0, j)}{\pi(j)} \right]. \]

Thus \( 0 \leq \text{sep}(r) \leq 1 \) and \( \text{sep}(r) \) is small provided \( P^r(0, j) \) is close to \( \pi(j) \) for all \( j \). See [2] or [14] for further properties of separation. The following theorem shows that convergence requires \( r = 2 \log_b n \).

**Theorem 4.3.** For any \( b \geq 2, n \geq 2 \), the transition matrix \( P(i, j) \) of (H1) satisfies

1. For all \( r \geq 0 \), the separation distance \( \text{sep}(r) \) of the carries chain after \( r \) steps (started at 0) is attained at the state \( j = n - 1 \).

2. For \( r = 2 \log_b (n) + \log_b (c) \),

\[ \text{sep}(r) \to 1 - e^{\frac{-c}{b}} \]

if \( c > 0 \) is fixed and \( n \to \infty \).
Proof. By Lemma 4.2 the matrix $P(i, j)$ is $T P_2$. Thus the matrix $P^*(i, j) := [P(j, i)\pi(j)]/\pi(i)$ is also $T P_2$, since every $2 \times 2$ minor of $P^*$ is a positive multiple of a $2 \times 2$ minor of $P$. Now consider the function $f_r(i) = P^r(0, i)/\pi(i)$. We claim that $P^* f_r = f_{r+1}$. Indeed,

$$[P^* f_r](i) = \sum_j P^*(i, j) f_r(j)$$
$$= \sum_j P^*(i, j) \frac{P^r(0, j)}{\pi(j)}$$
$$= \sum_j \frac{P(j, i)\pi(j)}{\pi(i)} \frac{P^r(0, j)}{\pi(j)}$$
$$= \frac{P^{r+1}(0, i)}{\pi(i)} .$$

Now the “variation-diminishing property” (p.22 of [25]) gives that if $f$ is monotone and $P^*$ is $T P_2$, then $P^* f$ is monotone. Since $f_0$ is monotone (the walk is started at 0), it follows that $f_r$ is monotone, i.e., that the separation distance $s(r)$ is attained at the state $n - 1$.

For the second assertion, note that by the relation between riffle shuffling and the carries chain in Theorem 3.1, $P^r(0, n-1)$ is equal to the chance of being at the unique permutation with $n - 1$ descents after $r$ iterations of a $b$-shuffle; by [5] this is $b^{-rn} \binom{b^r}{n}$.

Thus

$$\text{sep}(r) = 1 - \frac{P^r(0, n-1)}{\pi(n-1)}$$
$$= 1 - \prod_{i=1}^{n-1} \left( 1 - \frac{i}{b^r} \right)$$
$$= 1 - \exp \left( \sum_{i=1}^{n-1} \log \left( 1 - \frac{i}{b^r} \right) \right).$$

Letting $b^r = cn^2$ with $c > 0$ fixed, this becomes

$$1 - \exp \left( - \sum_{i=1}^{n} \frac{i}{cn^2} + O \left( \frac{i^2}{n^3} \right) \right) \sim 1 - e^{-\frac{1}{n}},$$

as $n \to \infty$. ■

Remark: It is known [2] that it takes $r = 2 \log_b n$ $b$-shuffles to make separation distance small on the symmetric group. Via Theorem 3.1 this shows $2 \log_b n$ steps suffice for the carries process. Of course, fewer steps might suffice but Theorem 4.3 shows the result is sharp for large $n$. In mild contrast, it is known [1, 5] that $(3/2) \log_2 n$ “ordinary” ($b = 2$) riffle shuffles are necessary and suffice for total variation convergence. We can show that for $b = 2$, $\log_2 n$ carry steps suffice for binary addition. Our argument uses the monotonicity proved above, the first eigenvector from [23], and Proposition 2.1 of [16]; for a second argument, using symmetric functions, see [15]. We do not know that this upper bound is sharp; the best total variation lower bound we have is $(1/2) \log_2 n$. 

11
5 Three Related Topics

The “amazing matrix” turns up in different contexts (sections of generating functions) in the work of Brenti–Welker [9]. There is an analog of carries for multiplication which has interesting structure. Finally, there are quite different amazing matrices having many of the same properties as Holte’s. These three topics are briefly developed in this section.

5.1 Sections of generating functions

Some natural sequences \(a_k, 0 \leq k < \infty\) have generating functions:

\[
\sum_{k=0}^{\infty} a_k x^k = \frac{h(x)}{(1-x)^{n+1}}
\]

with \(h(x) = h_0 + h_1 x + \cdots + h_{n+1} x^{n+1}\) a polynomial of degree at most \(n + 1\). For example, the generating function of \(a_k = k^n\) has this form with \(h(x)\) the Eulerian polynomials of \((H4)\). Rational generating functions characterize sequences \(\{a_h\}\) which satisfy a constant coefficient recurrence [28]. They arise naturally as the Hilbert series of graded algebras ([18], Chapter 10.4).

Suppose we are interested in every \(r\)-th term \(\{a_{rk}\}, 0 \leq k < \infty\). It is not hard to see that

\[
\sum_{k=0}^{\infty} a_{rk} x^k = \frac{h^{<r>}(x)}{(1-x)^{n+1}}
\]

for another polynomial \(h^{<r>}(x)\) of degree at most \(n + 1\). Brenti and Welker [9] show that the \(i\)-th coefficient of \(h^{<r>}(x)\) satisfies

\[h^{<r>}_i = \sum_{j=0}^{n+1} C(i,j) h_j\]

with \(C\) an \((n+2) \times (n+2)\) matrix with \((i,j)\) entry \((0 \leq i, j \leq n + 1)\) equal to the number of solutions to \(a_1 + \cdots + a_{n+1} = ib - j\) where \(0 \leq a_i \leq b - 1\) are integers. The carries matrix is closely related to their matrix. Indeed, remove from \(C\) the \(i = 0, n + 1\) rows and the \(j = 0, n + 1\) columns. Let \(i' = i - 1, j' = j - 1\). This gives an \(n \times n\) matrix with \((i',j')\) entry \((0 \leq i', j' \leq n - 1)\) equal to the number of solutions to \(a_1 + \cdots + a_{n+1} = (i' + 1)b - (j' + 1)\) where \(0 \leq a_i \leq b - 1\) are integers. Multiplying by \(b^{-n}\) and taking transposes gives the carries matrix for mod \(b\) addition of \(n\) numbers (see the top of p. 140 of [23]). Brenti and Welker develop some properties of the transformation \(C\). We hope some of the facts from the present development (in particular the central limit theorems satisfied by the coefficients) will illuminate their algebraic applications.

5.2 Carries for multiplication

Consider the process of base \(b\) multiplication of a random number (digits chosen from the uniform distribution on \(\{0, 1, \ldots, b-1\}\)) by a fixed number \(k > 0\). We do not require that \(k\) is single-digit. Then there is a natural way to define a carries process, which is best defined by example. Let \(k = 26\) and consider multiplying 1423 by 26 base 10. The zeroth carry is defined as \(\kappa_0 = 0\). To compute the first carry, note that \(26 \times 3 = 78\), so \(\kappa_1 = 7\). Then \(\kappa_1 + 26 \times 2 = 59\), so \(\kappa_2 = 5\). Next \(\kappa_2 + 26 \times 4 = 109\), so \(\kappa_3 = 10\). Finally, \(\kappa_3 + 26 \times 1 = 36\), so \(\kappa_4 = 3\).
It is not difficult to see that the above process is a Markov chain on the state space \( \{0, 1, \cdots, k-1\} \). For example, if \( b=10 \) and \( k=7 \), the transition matrix is

\[
\begin{bmatrix}
2 & 1 & 2 & 1 & 2 & 1 & 1 \\
2 & 1 & 2 & 1 & 2 & 1 & 1 \\
2 & 1 & 1 & 2 & 1 & 2 & 1 \\
1 & 2 & 1 & 2 & 1 & 1 & 2 \\
1 & 2 & 1 & 1 & 2 & 1 & 2 \\
1 & 1 & 2 & 1 & 2 & 1 & 2 \\
1 & 1 & 2 & 1 & 2 & 1 & 2
\end{bmatrix}
\]

The matrix above \( K(i, j) \) does not have all eigenvalues real, but the following properties do hold in general:

- \( K(i, j) \) is doubly stochastic, meaning that every row and column sums to 1.
- \( K(i, j) \) is an generalized circulant matrix, meaning that each column is obtained from the previous column by shifting it downward by \( b \mod k \).
- Fix \( k \) and let \( K_a, K_b \) be the base \( a,b \) transition matrices for multiplication by \( k \). Then \( K_{ab} = K_aK_b \).

The first two properties are at the level of undergraduate exercises, and Chapter 5 of [13] is a useful reference for generalized circulants. The third property holds for the same reason that it does for Holte’s matrix (see the explanation on page 143 of [23]).

Since \( K \) is doubly stochastic, the carries chain for multiplication has the uniform distribution on \( \{0, 1, \cdots, k-1\} \) as its stationary distribution. Concerning convergence rates, one has the following simple upper bound for total variation distance.

**Proposition 5.1.** Let \( K_0^r \) denote the distribution of the carries chain for multiplication by \( k \) base \( b \) after \( r \) steps, started at the state 0. Let \( \pi \) denote the uniform distribution on \( \{0, 1, \cdots, k-1\} \). Then

\[
\frac{1}{2} \sum_{j=0}^{k-1} |K_0^r(j) - \pi(j)| \leq \frac{k}{2b^r}.
\]

**Proof.** Observe that

\[
K_0^r(j) = \frac{1}{b^r} |\{x : jb^r \leq kx < (j+1)b^r, 0 \leq x < b^r\}|.
\]

The number of integers \( x \) satisfying \( \frac{jb^r}{k} \leq x < \frac{(j+1)b^r}{k} \) is between \( \frac{b^r}{k} - 1 \) and \( \frac{b^r}{k} + 1 \). Hence |\( K_0^r(j) - \pi(j) \)| \( \leq \frac{1}{b^r} \), and the result follows by summing over \( j \).

Convergence rate lower bounds depend on the number theoretic relation of \( k \) and \( b \) in a complicated way. For instance if \( k = b \), the process is exactly random after 1 step.
5.3 Another amazing matrix

From one point of view, Holte’s amazing matrix exists because there is a “big” Markov chain on the symmetric group $S_n$ with eigenvalues $1, 1/b, 1/b^2, \cdots$ and a function $T : S_n \to \{0, 1, \cdots, n-1\}$ with image this very same Markov chain. Of course, the interpretation as “carries” remains amazing. There are many functions of the basic riffle shuffling Markov chain which remain Markov chains. Here is a simple one. Consider repeated shuffling of a deck of $n$ cards using the Gilbert–Shannon–Reed $b$-shuffles. The position of card labeled “one” gives a Markov chain on $\{1, 2, \cdots, n\}$. In [4] the transition matrix of this chain is shown to be

$$Q_b(i,j) = \frac{1}{b^n} \times$$

$$\sum_{h=1}^{b} \sum_{r=\ell}^{u} \left( \begin{array}{c}
(j-1) \\
r
\end{array} \right) \left( \begin{array}{c}
n-j \\
i-r-1
\end{array} \right) h^r (b-h)^{i-1-r} (h-1)^{j-1-r} (b-h+1)^{(n-j)-(i-r-1)}$$

where the inner sum is from $\ell = \max(0, (i+j)-(n+1))$ to $u = \min(i-1, j-1)$. For example, when $n = 2, 3$ the matrices are

$$Q_b = \begin{pmatrix}
\frac{1}{2b} & \frac{b+1}{b-1} & \frac{b-1}{b+1} \\
2(b-1) & 2(b^2-1) & 2(b-1)(2b+1) \\
2(b^2-1) & 2(b^2+2) & 2(b^2-1)
\end{pmatrix}.$$

The matrix $Q_b$ is shown to satisfy

- $Q_b$ has eigenvalues $1, 1/b, 1/b^2, \cdots, 1/b^{n-1}$.

- The eigenvectors of $Q_b$ do not depend on $b$; in particular, the stationary distribution is uniform: $\pi(i) = 1/n, 1 \leq i \leq n$.

- $Q_b Q_b = Q_{ab}$.

We guess that $Q_b$ has other nice properties and appearances.

Acknowledgments

We thank Alexi Borodin, Francesco Brenti, Jim Fill and Phil Hanlon for real help with this paper. The work of Fulman was supported by NSF grant DMS-0503901.

References

[1] D. Aldous, Random walk on finite groups and rapidly mixing Markov chains, Séminaire de Probabilités XVII. Lecture Notes in Math, 986 (1983), 243-297. Springer, New York.

[2] D. Aldous and P. Diaconis, Shuffling cards and stopping times, Amer. Math. Mon. 93 (1986) 333–348.

[3] T. W. Anderson, The Statistical Analysis of Time Series, John Wiley & Sons, New York, 1971.
[4] S. Asaf, P. Diaconis and K. Soundarajan, A rule of thumb for riffle shuffling, Preprint, Department of Statistics, Stanford University, Stanford, CA, 2008.

[5] D. Bayer and P. Diaconis, Trailing the dovetail shuffle to its lair, *Ann. Appl. Probab.* 2 (1992) 294–313.

[6] P. Billingsley, *Probability and Measure*, John Wiley and Sons, New York, 1986.

[7] A. Borodin, P. Diaconis and J. Fulman, On adding a list of numbers, Preprint, Department of Statistics, Stanford University, Stanford, CA, 2008a.

[8] A. Borodin, P. Diaconis and J. Fulman, Carries, descents and determinants, Preprint, Department of Statistics, Stanford University, Stanford, CA, 2008b.

[9] F. Brenti and V. Welker, The Veronese construction for formal power series and graded algebras, arXiv:0712.2645 (2007).

[10] K. S. Brown and P. Diaconis, Random walks and hyperplane arrangements, *Ann. Probab.* 26 (1998) 1813–1854.

[11] J. Buhler, D. Eisenbud, R. Graham and C. Wright, Juggling drops and descents, *Amer. Math. Mon.* 101 (1994) 507–519.

[12] L. Comtet, *Advanced combinatorics*, D. Reidel Publishing Co., Dordrecht, 1974.

[13] P. Davis, *Circulant matrices*, John Wiley & Sons, New York, 1979.

[14] P. Diaconis, Mathematical developments from the analysis of riffle-shuffling, in *Groups, Combinatorics and Geometry*, A. Ivanov, M. Liebeck and J. Saxl, eds, World Scientific, New Jersey, 2003, 73–97.

[15] P. Diaconis and J. Fulman, Carries and symmetric functions, Preprint, Department of Statistics, Stanford University, Stanford, CA, 2008.

[16] P. Diaconis, K. Khare and L. Saloff-Coste, Gibbs sampling, exponential families and coupling, Technical report, Department of Statistics, Stanford University, Stanford, CA, 2007.

[17] P. Diaconis, M. McGrath and J. Pitman, Riffle shuffles, cycles and descents, *Combinatorica* 15 (1995) 11–29.

[18] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Springer-Verlag, New York, 2004.

[19] J. Fulman, Applications of the Brauer complex: card shuffling, permutation statistics, and dynamical systems, *J. Algebra* 243 (2001) 96–122.

[20] J. Fulman, Applications of symmetric functions to cycle and increasing subsequence structure after shuffles, *J. Algebr. Comb.* 16 (2002) 165–194.

[21] E. Gilbert, Theory of shuffling, Technical Report, Bell Laboratories, 1955.

[22] P. Hanlon, The action of $S_n$ on the components of the decomposition of Hochschild homology, *Mich. Math. J.* 37 (1990) 105–124.

[23] J. Holte, Carries, combinatorics, and an amazing matrix, *Amer. Math. Mon.* 104 (1997) 138–149.
[24] D. Isaksen, A cohomological viewpoint on elementary school arithmetic, *Amer. Math. Mon.* **109** (2002) 796–805.

[25] S. Karlin, *Total Positivity*, Vol. 1, Stanford University Press, Stanford, CA, 1968.

[26] D. E. Knuth, *The Art of Computer Programming*, Vol. 2, 3rd ed., Addison-Wesley Professional, 1997.

[27] B. Mann, How many times should you shuffle a deck of cards?, *UMAP J.* 15 (1994) 303–332; reprinted in *Topics in contemporary probability and its applications*, 261–289, Probab. Stochastics Ser., Ed. J.L. Snell, CRC Press, Boca Raton, FL, 1995.

[28] R. P. Stanley, *Enumerative Combinatorics I*, 2nd ed., Cambridge University Press, Cambridge, 1997.