Ergodic Theorem for Stabilization of a Hyperbolic PDE Inspired by Age-Structured Chemostat

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December 14, 2014

Abstract

We study a feedback stabilization problem for a first-order hyperbolic partial differential equation. The problem is inspired by the stabilization of equilibrium age profiles for an age-structured chemostat, using the dilution rate as the control. Two distinguishing features of the problem are that (a) the PDE has a multiplicative (instead of an additive) input and (b) the state is fed back to the inlet boundary. We provide a sampled-data feedback that ensures stabilization under arbitrarily sparse sampling and that satisfies input constraints. Our chemostat feedback does not require measurement of the age profile, nor does it require exact knowledge of the model.

Key Words: bioreactor, hyperbolic partial differential equation, sampled control, stabilization.

1 Introduction

Age-structured models have been used in mathematical biology and mathematical demography for a long time. Models of age-structured populations are either discrete (resulting in Leslie matrix models) or continuous (which produce the McKendrick-von Foerster equation); see [4, 5]. Age-structured models have also been used in mathematical economics and environmental engineering [3]. The study of continuous age-structured models has focused on two different research directions, namely, the study of the dynamics of age-structured models, and optimal control problems. Optimal control problems for age-structured models have been used in [3, 21] and optimality conditions were derived in [2, 8]. The ergodicity problem has been studied in many works (such as [10, 11]) and many results have been

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*The work of Malisoff and Krstic was supported by US National Science Foundation Grants 1408295 and 1408376, respectively.

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presented for one or multiple continuous age-structured population models; see for instance [22].

Since the McKendrick-von Foerster equation (which is also called the Lotka-von Foerster equation) is a first-order hyperbolic partial differential equation (PDE), it is reasonable to expect recent results on controlling hyperbolic PDEs (such as [11, 16, 7, 13, 14, 23]) to apply to age-structured models. However, much of this literature uses boundary controls. For chemostat models (where the dilution rate is the most commonly used control [9, 15, 18]), it is natural to consider stabilization problems where the dilution rate is used to stabilize a specific age profile. The dilution rate control enters into the PDE directly, not at the boundary. Chemostats form the foundation of much current research in bioengineering, ecology, and population biology, and are important in biotechnological processes such as waste water treatment plants [19, 20]. Stability properties for the dynamics of chemostats have been well studied; see [17, 18] and the references therein.

In this paper, we study a simplified age-structured chemostat model without an equation for the substrate concentration, i.e., we consider the substrate concentration to be constant. This is justified in the important case where the inlet concentration of the substrate is used to control the substrate concentration (or the substrate concentration is slowly varying). Two distinguishing features of the control problem we consider are (a) that the PDE has a multiplicative (instead of an additive) input and (b) that the state is fed back to the inlet boundary; see the system dynamics (1)-(2) below. Moreover, we do not require that we know the age profile of the microorganism. Only the value of a linear functional of the state profile (i.e., an output) is known at certain times (namely, the sampling times); see (5) below. The controller determines the value of the dilution rate, and its key feature is the use of the natural logarithm of the output value. See (7) for our control formula. It is a sampled-data feedback for stabilizing an arbitrary positive valued age profile, i.e., only sampled measurements are required, and the control is constant between sampling times. The feedback is valued in a pre-specified bounded interval, to incorporate input constraints.

Other key novel features of our work relative to the existing control literature for hyperbolic PDEs are that we achieve global exponential stabilization for all positive valued initial age distributions, with arbitrarily sparse sampling, and that we do not require exact model knowledge. Our key stability estimate is in terms of the sup norm of the logarithmic deviation of the state profile from the equilibrium age profile; see (9). The proof of our main result uses the strong ergodic theorem and the connection between hyperbolic PDEs and integral delay equations (IDEs) from our prior work [13]. To our knowledge, this is the first time that the ergodic theorem has been used to solve a control problem. Our simulations show good performance of our control under three operating conditions, and so support our view that our work would be useful for industrial applications.

**Definitions and Notation**

We use the following notation. Let $I \subseteq \mathbb{R}$ be any interval and $\Omega \subseteq \mathbb{R}^n$ be any set. Let $C^0(I; \Omega)$ be the class of all continuous functions $f : I \rightarrow \Omega$, and $C^k(I; \Omega)$ for any integer $k \geq 1$ be the class of all functions in $C^0(I; \Omega)$ all of whose partial derivatives up through order $k$ exist and are continuous on $I$. Let $L^\infty(I; \Omega)$ be the equivalence classes of all essentially bounded Lebesgue measurable functions $f : I \rightarrow \Omega$ with norm $||f||_\infty = \text{ess sup}_{a \in I} |f(a)|$. Let
\(L^1(\mathcal{I}; \Omega)\) be the equivalence classes of measurable functions \(f : \mathcal{I} \to \Omega\) for which \(\|f\|_1 < \infty\), where \(\|f\|_1 = \int_{\mathcal{I}} |f(t)| dt\). For each \(x \in \mathbb{R}\), let \([x]\) be the integer part of \(x\), i.e., the largest integer \(p\) such that \(p \leq x\). We let \(K_\infty\) denote the set of all strictly increasing unbounded continuous functions \(\kappa : [0, \infty) \to [0, \infty)\) such that \(\kappa(0) = 0\).

For any subset \(S \subseteq \mathbb{R}\) and any \(A > 0\), we let \(PC^1([0, A]; S)\) denote the class of all continuous functions \(z : [0, A] \to S\) for which there exists a finite (or empty) subset \(B\) subset of \((0, A)\) such that: (i) the derivative \((dz/da)\)(a) exists at every point in \((0, A) \setminus B\) and is a continuous function on \((0, A) \setminus B\), (ii) all right and left limits of \((dz/da)\)(a) when \(a\) tends to a point in the set \(B \cup \{0, A\}\) exist and are finite. Given any subset \(S \subseteq \mathbb{R}\), we let \(PC^0(\mathcal{I}; S)\) denote the set of all piecewise continuous functions, i.e., the set of all functions \(u : \mathcal{I} \to \mathbb{R}\) for which there exists a (possibly empty) set \(B \subseteq \mathcal{I}\) such that: (i) \(u\) is continuous on \(\mathcal{I} \setminus B\), (ii) the intersection of every bounded subset of \(\mathcal{I}\) with \(B\) is finite (or empty), and (iii) all right and left limits of \(u(t)\) when \(t\) tends to a point (from the right or from the left) in the set \(B\) exist and are finite.

## 2 Main Result for Controlled Age-Structured Model

### 2.1 Statement of Problem and Theorem

We consider the age-structured chemostat model given by

\[
\frac{\partial f}{\partial t}(t,a) + \frac{\partial f}{\partial a}(t,a) = -(\mu(a) + D(t))f(t,a)
\]

for all \((t, a) \in (0, \infty) \times (0, A)\) and

\[
f(t,0) = \int_0^A k(a)f(t,a) da \quad \text{for all } t \geq 0 ,
\]

where \(A > 0\) is any constant, \(\mu : [0, A] \to [0, \infty)\) and \(k : [0, A] \to [0, \infty)\) are continuous functions, and we assume that \(\int_0^A k(a)da > 0\). The system (1)-(2) is a continuous age-structured model of a population in a chemostat. The boundary condition (2) is the renewal condition, which determines the number of newborn individuals \(f(t,0)\) at each time \(t \geq 0\), \(A > 0\) is the maximum reproductive age, \(\mu\) is the mortality function, \(f\) is the density of the population of age \(a \in [0, A]\) at time \(t \geq 0\), and \(k\) is the birth modulus. Given any constants \(D_{\min} > 0\) and \(D_{\max} > D_{\min}\), the variable \(D \in PC^0([0, \infty); [D_{\min}, D_{\max}])\) is called the dilution rate and is the control. Physically meaningful solutions of (1)-(2) are those satisfying \(f(t,a) \geq 0\) for all \((t,a) \in [0, \infty) \times [0, A]\).

We assume that there is a constant \(D^* \in (D_{\min}, D_{\max})\) such that

\[
1 = \int_0^A \left\{ k(a) \exp \left( -D^*a - \int_0^a \mu(s)ds \right) \right\} da ,
\]

which is the Lotka-Sharpe equation [4]. Our assumption that there is a constant \(D^* \in (D_{\min}, D_{\max})\) satisfying (3) is necessary for the existence of a non-zero equilibrium point for (1)-(2). In fact, for any constant \(M > 0\), any function of the form

\[
f^*(a) = M \exp \left( -D^*a - \int_0^a \mu(s)ds \right)
\]

(4)
for all \( a \in [0, A] \) is an equilibrium point for (1)-(2). Therefore, there are a continuum of equilibria. This implies that the dynamics (1)-(2) cannot be made open-loop asymptotically stable to an equilibrium with the constant control \( D(t) = D^* \), which is another motivation for our globally exponentially stabilizing feedback control design; see Section 2.2 for more discussion on the equilibria.

It is natural to try to design a dilution rate controller \( D \) for (1) that is a function of values of the densities of the newborn individuals, i.e., (2). However, such measurements may not be easy to obtain in practice. On the other hand, it is often possible to find a continuous function \( p : [0, A] \to [0, \infty) \) such that we can measure

\[
y(t) = \int_0^A p(a)f(t, a)\,da
\]

at each time \( t \geq 0 \). For instance, the case \( p(a) \equiv 1 \) corresponds to measuring the concentration of the microorganisms. Given any desired positive constant lower and upper bounds \( D_{\text{min}} > 0 \) and \( D_{\text{max}} > D_{\text{min}} \) for the controller, any constant \( T > 0 \), and any desired reference profile \( y^* \) for any \( M > 0 \), and setting

\[
y^* = \int_0^A p(a)f^*(a)\,da
\]

we can prove that our age-structured chemostat dynamics (1)-(2), in closed loop with the piecewise defined control defined by

\[
D(t) = \max\left\{D_{\text{min}}, \min\left\{D_{\text{max}}, D^* + T^{-1}\ln\left(\frac{y(iT)}{y^*}\right)\right\}\right\}
\]

for all \( t \in [iT, (i+1)T) \) and all integers \( i \geq 0 \), satisfies a uniform global asymptotic stability estimate for all initial functions \( f_0 \in PC^1([0, A]; (0, \infty)) \) satisfying

\[
f_0(0) = \int_0^A k(a)f_0(a)\,da,
\]

which means that we require that \( f(0, a) = f_0(a) \) for all \( a \in [0, A] \). Our main theorem is:

**Theorem 1.** Let \( A, T, D_{\text{min}}, \) and \( D_{\text{max}} \) be any positive constants, with \( D_{\text{max}} > D_{\text{min}} \). Let \( \mu : [0, A] \to [0, \infty) \) and \( k : [0, A] \to [0, \infty) \) be any continuous functions, and assume that \( k \in PC^1([0, A]; [0, \infty)) \) and is not the zero function. Let \( p : [0, A] \to [0, \infty) \) be any continuous function such that \( \int_0^A p(a)\,da > 0 \), and \( D^* \in (D_{\text{min}}, D_{\text{max}}) \) be any constant satisfying the Lotka-Sharpe condition (3). Then there exist a constant \( \sigma > 0 \) and a function \( \kappa \in K_{\infty} \) such that for each function \( f_0 \in PC^1([0, A]; (0, \infty)) \) satisfying (8), the unique solution of (1)-(2) in closed loop with (7) with the initial condition \( f_0 \) satisfies

\[
\max_{0 \leq a \leq A} \ln \left(\frac{f(t, a)}{f^*(a)}\right) \leq \exp(-\sigma t) \kappa \left(\max_{0 \leq a \leq A} \ln \left(\frac{f_0(a)}{f^*(a)}\right)\right)
\]

for all \( t \geq 0 \).
2.2 Discussion on Theorem \( \text{[1]} \)

We discuss the structure of the feedback control, as well as several key features that distinguish our controller analysis from existing results.

The tracking error norm \(|\ln (f(t,a)/f^*(a))|\) in the statement of the theorem is motivated by the fact that \([1]-[2]\) has the restricted state space \( X = \{ f \in PC^1([0,A] ; (0,\infty)) : f(0) = \int_0^A k(a)f(a)da \} \). In fact, our logarithmic transformation \( x(t,a) = \ln (f(t,a)/f^*(a)) \) for \((t,a) \in [0,\infty) \times [0,A]\) produces a control system whose state \( x_t \) takes values in \( \mathbb{R} \) and has equilibrium \( x = 0 \), which is the usual setting for hyperbolic PDEs. When \( M = 1 \), the state space in the new variables is

\[
\left\{ x \in PC^1([0,A]; \mathbb{R}) : \exp(x(0)) = \int_0^A k(a)f^*(a)\exp(x(a))da \right\}.
\]

The nonlinear character of the control problem we consider here is illustrated by the nonlinearity \( \exp \) in \((10)\).

Consider the special case where the function \( p \) in our output \((5)\) is the birth modulus \( k \). Then our output is the density \([2]\) of newborn individuals. Hence, our theorem is a general result on output feedback control that includes the special case where the density of newborn individuals is being measured. For our infinite dimensional systems, the state is the function \( f_t \), i.e., \( (f_t)(a) = f(t,a) \) for all \( a \in [0,A] \). Therefore, even if \( p = k \), our output feedback is not a state feedback. The motivation for selecting a specific \( M > 0 \) so that the desired equilibrium is \([4]\) is that for the complete chemostat (which also has an equation for the substrate), the selected \( M > 0 \) maximizes the yield, in the context of anaerobic digestion; see \([12\text{ Section 2.4}]\).

While there is no explicit Lyapunov functional in our proof of Theorem \([1]\) the function \( V_0(t) = |\ln (f(t,0)/f^*(0))| \) acts as a Lyapunov-Razumikhin functional. This can be seen by showing that \( v_0(t) = \ln (f(t,0)/f^*(0)) \) satisfies a suitable IDE; see \([15\text{, 16}]\), and \([33]\). The function \( V_0 \) is a Lyapunov-Razumikhin functional for an IDE, instead of a Lyapunov functional for a PDE. Our theorem provides bounds for solutions of the PDE in terms of the history of \( V_0 \), because

\[
|\ln (f(t,a)/f^*(a))| \leq V_0(t-a) + \int_{t-a}^t |D(s) - D^*| ds
\]

for all \( t \geq A \) and \( a \in [0,A] \); see \([34]\) below. See also \([13]\), which also uses functions of the form \( V_0(t) = |\ln (f(t,0)/f^*(0))| \) and which then builds a Lyapunov-Krasovskii functional of the form \( W(t) = \max \{ \exp(ps)V_0(t-s) : s \in [0,A] \} \) for a suitable \( p \in \mathbb{R} \); see the proof of \([13\text{ Theorem 2.6}]\). In the original coordinates, this functional is \( Q(t) = \max \{ \exp(pa)q(a) : a \in [0,A] \} \) for a suitable function \( q \).

As noted in the introduction, to our knowledge Theorem \([1]\) is the first application of an ergodic theorem to solve a feedback stabilization problem. The proof of Theorem \([1]\) has several steps. First, we use the ergodic theorem and a transformation to an IDE to obtain estimates for \( \ln (y(iT)/y^*) \). The control is designed such that \( y(t) \to y^* \) as \( t \to \infty \). In the next step, we produce an estimate relating \( \ln (y(iT)/y^*) \) and \( \ln (f(t,0)/f^*(0)) \), which we use in the third step to show that \( f(t,0) \to f^*(0) \) as \( t \to \infty \). In the next step, we use our IDE
transformation to relate \( \ln(f(t,a)/f^*(a)) \) for all \( a \in [0, A] \) to \( \ln(f(t,0)/f^*(0)) \). Finally, we show that \( f(t,a) \to f^*(a) \) for all \( a \in [0, A] \). Our proof requires several lemmas, which we turn to next.

3 Background: Uncontrolled Age-Structured Models

We review the needed background from [10] on uncontrolled chemostats, and other material from [16], which we use to prove Theorem 1 below. Let \( A > 0 \) be any constant, let \( \mu : [0, A] \to [0, \infty) \) and \( k : [0, A] \to [0, \infty) \) be any continuous functions, and assume that \( \int_0^A k(a)da > 0 \). Consider the initial value problem given by the two equations

\[
\frac{\partial z}{\partial t}(t,a) + \frac{\partial z}{\partial a}(t,a) = -\mu(a)z(t,a) \quad (12)
\]

for all \((t,a) \in (0, \infty) \times (0, A)\) and

\[
z(t,0) = \int_0^A k(a)z(t,a)da \quad (13)
\]

for all \( t \geq 0 \), with an initial condition \( z(0,a) = z_0(a) \) for all \( a \in [0, A] \). System (12)-(13) is a continuous age-structured model of a population in a closed ecosystem with no control, where \( \mu \) is the mortality function, \( z \) is the density of the population of age \( a \in [0, A] \) at time \( t \geq 0 \), and \( k \) is the birth modulus. Physically meaningful solutions are those satisfying \( z(t,a) \geq 0 \) for all \((t,a) \in [0, \infty) \times [0, A]\). The following existence and uniqueness result follows from [10] Proposition 2.4 and [16] Theorems 1.3-1.4:

**Lemma 1.** For each absolutely continuous function \( z_0 \in C^0([0, A]; \mathbb{R}) \) such that \( z_0(0) = \int_0^A k(a)z_0(a)da \), there is a unique function \( z : [0, \infty) \times [0, A] \to \mathbb{R} \) that satisfies: (a) For each \( t \geq 0 \), the function \( z_t \) defined by \( z_t(a) = z(t,a) \) for \( a \in [0, A] \) is in \( L^1([0, A]; \mathbb{R}) \), (b) the function \( \Phi : [0, \infty) \to L^1([0, A]; \mathbb{R}) \) defined by \( \Phi(t) = z_t \) is continuously differentiable, (c) for each \( t \geq 0 \), the function \( z_t \in L^1([0, A]; \mathbb{R}) \) is absolutely continuous and satisfies \( z_t(0) = \int_0^A k(a)z_t(a)da \), and (d) equation (12) holds for almost all \( t > 0 \) and \( a \in (0, A) \). Moreover, if \( z_0(a) \geq 0 \) for all \( a \in [0, A] \), then \( z(t,a) \geq 0 \) holds for all \((t,a) \in [0, \infty) \times [0, A]\). □

We refer to \( z \) or the function \( \Phi \) from Lemma 1 as the solution of (12)-(13). We also use:

**Lemma 2.** If \( k \in PC^1([0, A]; [0, \infty)) \), then for every \( z_0 \in PC^1([0, A]; \mathbb{R}) \) satisfying

\[
z_0(0) = \int_0^A k(a)z_0(a)da, \quad (14)
\]

the function \( z : [0, \infty) \times [0, A] \to \mathbb{R} \) from Lemma 1 is \( C^1 \) on

\[
S = \{(t,a) \in (0, \infty) \times (0, A) : t - a \not\in B \cup \{0, A\}\},
\]

where \( B \) is the finite (or empty) set where the derivative of \( z_0 \) is not defined, and it satisfies (12) on \( S \) and equation (13) for all \( t \geq 0 \). Also,

\[
z(t,a) = \exp\left(-\int_0^a \mu(s)ds\right)v(t-a) \quad (15)
\]
holds for all \((t, a) \in [0, \infty) \times [0, A]\), where \(v \in C^0([0, A]; \mathbb{R}) \cap C^1 ((0, \infty); \mathbb{R})\) solves
\[
v(t) = \int_0^A k(a) \exp \left( - \int_0^a \mu(s) ds \right) v(t - a) da
\]
for all \(t \geq 0\) for the initial condition \(v(-a) = \exp \left( \int_0^a \mu(s) ds \right) z_0(a)\) for all \(a \in (0, A]\).

In the context of Lemma 2, the function \(z_0\) is of class \(PC^1\) for every \(t \geq 0\). The solution of (16) is found by differentiating both sides of (16) with respect to \(t\), then applying integration by parts on the interval \([0, A]\), and then solving
\[
\dot{v}(t) = \tilde{k}(0)v(t) - \tilde{k}(A)v(t - A) + \int_0^A \frac{d\tilde{k}}{da}(a)v(t - a) da,
\]
where \(\tilde{k}(a) = k(a) \exp \left( - \int_0^a \mu(s) ds \right)\). Recalling that \(\int_0^A k(a) da > 0\), we also define the continuous functional \(P : L^1([0, A]; \mathbb{R}) \to \mathbb{R}\) by
\[
P(z_0) = \frac{\int_0^A z_0(a)( \int_0^a k(s) \exp \left( \int_0^s (\mu(l) + D^*) dl \right) ds) da}{\int_0^A ak(a) \exp \left( - \int_0^a (\mu(l) + D^*) dl \right) da}
\]
where \(D^*\) is from the Lotka-Sharpe condition (3). Recall the following strong ergodicity result, which follows from [11, Section 3]:

**Lemma 3.** There are constants \(\varepsilon > 0\) and \(K \geq 1\) such that for every absolutely continuous function \(z_0 \in C^0([0, A]; \mathbb{R})\) that satisfies and (14), the solution of (12)-(13) satisfies
\[
\int_0^A \exp(J(a)) |z(t, a) - \exp(D^*(t - a) - J(a)) P(z_0)| da \leq K \exp((D^* - \varepsilon)t) \int_0^A \exp(J(a)) |z_0(a)| da \quad \text{for all} \quad t \geq 0,
\]
where \(J(a) = \int_0^a \mu(s) ds\).

Lemma 3 follows by choosing \((S(t)z_0)(a) = z_t(a) = z(t, a)\) for all \(a \in [0, A]\) as the semigroup in [11]. If \(k \in PC^1([0, A]; [0, \infty))\), then for every \(z_0 \in PC^1([0, A]; \mathbb{R})\) that satisfies (14), we define
\[
\phi(t) = \exp(-D^*t) v(t) - P(z_0) \quad \text{for all} \quad t \geq -A,
\]
where \(v \in C^0([-A, \infty); \mathbb{R})\) is the solution of (16) with the initial condition
\[
v(-a) = \exp \left( \int_0^a \mu(s) ds \right) z_0(a)
\]
for all \(a \in (0, A]\). Then \(\phi \in C^0((-A, 0]; \mathbb{R})\). Also, (15) and (19) give
\[
\int_0^A \exp(-D^*a) |\phi(t - a)| da \leq K \exp(-\varepsilon t) \int_0^A \exp \left( \int_0^a \mu(s) ds \right) |z_0(a)| da
\]
for all \(t \geq 0\). Therefore, by setting
\[
C = \max_{0 \leq a \leq A} k(a) \exp \left( - \int_0^a \mu(s) ds \right),
\]

it follows from (16) and (3) that

\[ |\phi(t)| = \left| \int_0^A k(a) \exp \left( -D^* a - \int_0^a \mu(s) ds \right) \phi(t - a) da \right| \]
\[ \leq C \int_0^A \exp(-D^* a) |\phi(t - a)| da \leq KC \exp( -\varepsilon t ) \int_0^A \exp( \int_0^a \mu(s) ds ) |\phi(t - a)| da \]

holds for all \( t \geq 0 \).

4 Key Lemma

Our proof of Theorem 1 will also use the following key lemma, which follows from our recent results in [13]:

Lemma 4. Let \( G \in C^0([0, A]; [0, \infty)) \), set \( L = \int_0^A G(a) da \), and let \( \Delta \in (0, A) \) be any constant such that

\[ \int_0^\Delta G(a) da < 1. \]

If \( L > 1 \), then for each \( x_0 \in L^\infty([-A, 0); \mathbb{R}) \), the solution \( x \in L^\infty_{loc}([-A, \infty); \mathbb{R}) \) of

\[ x(t) = \int_0^A G(a) x(t - a) da \]

with the initial condition \( x(a) = x_0(a) \) for \( a \in [-A, 0) \) satisfies

\[ \min\{a_1, a_1 b^{1+t/h}\} \leq \inf_{-A \leq a < 0} x(t + a) \leq \sup_{-A \leq a < 0} x(t + a) \leq \max\{a_2, a_2 b^{1+t/h}\} \]

for all \( t \geq 0 \), where

\[ a_1 = \inf_{-A \leq a < 0} x_0(a), \quad a_2 = \sup_{-A \leq a < 0} x_0(a), \]
\[ c = \int_0^\Delta G(a) da, \quad b = \frac{c}{c^{-1} - c^{-1}}, \]

and \( h = \min\{\Delta, A - \Delta\} \).

For the proof of Lemma 4 see Appendix A.1. A consequence of Lemma 4 is that for every \( z_0 \in PC^1([0, A]; \mathbb{R}) \) satisfying \( z_0(0) = \int_0^A k(a) z_0(a) da \) and \( z_0(a) > 0 \) for all \( a \in [0, A] \), the solution of (12)-(13) satisfies \( z(t, a) > 0 \) for all \((t, a) \in [0, \infty) \times [0, A] \). To see why, note that if

\[ \int_0^A k(a) \exp\left( -\int_0^a \mu(s) ds \right) da \geq 1, \]

then we may apply Lemma 4 to (16), by choosing \( G(a) = k(a) \exp(-\int_0^a \mu(s) ds) \). If

\[ \int_0^A k(a) \exp\left( -\int_0^a \mu(s) ds \right) da < 1, \]
then set \( x(t) = \exp(pt)v(t) \) for all \( t \geq -A \) for any constant \( p > 0 \), where \( v \) satisfies (16). Then
\[
x(t) = \int_0^t k(a) \exp \left( p \int_0^a \mu(s)ds \right) x(t - a)da \text{ for all } t \geq 0
\]
and \( \int_0^t k(a) \exp \left( p \int_0^a \mu(s)ds \right) da \geq 1 \) when \( p \) is large enough, since \( \int_0^t k(a)da > 0 \).

Also,
\[
\min_{0 \leq a \leq A} \left( \exp \left( D^*a + \int_0^a \mu(s)ds \right) z_0(a) \right) \leq \min_{-A \leq a \leq 0} \left( P(z_0) + \phi(t + a) \right)
\]
\[
\leq \max_{-A \leq a \leq 0} \left( P(z_0) + \phi(t + a) \right) \leq \max_{0 \leq a \leq A} \left( \exp \left( D^*a + \int_0^a \mu(s)ds \right) z_0(a) \right)
\]
for all \( t \geq 0 \). The inequalities (31) are obtained by using Lemma 4 with \( L = 1 \), \( x(t) \) from Lemma 4 taken to be \( \exp(-D^*t)v(t) \), and \( G(a) \) taken to be the integrand in curly braces in (3). Moreover, using our choice (4) of \( f^* \), our formula (15) for the solutions \( z(t,a) \) of the uncontrolled chemostat (12)-(13), our formula (20) for \( \phi(t) \), (31), and the fact that the solution of the controlled chemostat dynamics (11)-(2) with \( D(t) \equiv D^* \) and any initial condition \( f(0,a) = f_0(a) \) with \( f_0 \in PC^1([0,A]; \mathbb{R}) \) satisfies
\[
f(t,a) = \exp(-D^*t)z(t,a)
\]
for all \( (t,a) \in [0, \infty) \times [0, A] \) where \( z(t,a) \) is the solution of (12)-(13) for the initial condition \( z(0,a) = f_0(a) \), we obtain the following inequalities for all \( t \geq 0 \):
\[
\min_{a \in [a,A]}(f_0(a)/f_*(a)) \leq \min_{a \in [a,A]}(f(t,a)/f_*(a))
\]
\[
\leq \max_{a \in [a,A]}(f(t,a)/f_*(a)) \leq \min_{a \in [a,A]}(f_0(a)/f_*(a)).
\]

The preceding inequalities show that every equilibrium profile (4) for every choice of the constant \( M > 0 \) is stable. However, since every neighborhood of an equilibrium profile (in the \( L^1 \) norm or in the sup norm) contains infinitely many equilibria, each equilibrium profile is stable but not asymptotically stable (neutral stability).

5 Proof of Theorem 1

Existence and uniqueness of the solution of the closed-loop system (1)-(2) with the control (7) can be established by the method of steps, as follows. First notice that by Lemma 1 the solution \( z(t,a) \) of (12)-(13) with the control (7) and the initial condition \( z_0 = f_0 \) exists for all \( t \geq 0 \), and Lemmas 2 and 4 guarantee that \( z_t \) is of class \( PC^1([0,A]; (0,\infty)) \) for all \( t \geq 0 \). Assume that the solution of (11)-(2), in closed loop with (7), is defined on \([0,iT]\) for some non-negative integer \( i \) and that \( f_t \in PC^1([0,A]; (0,\infty)) \) for all \( t \in [0,iT] \). Then \( D(t) \) can be defined uniquely by (7) on \([iT,(i+1)T]\), and \( D \) is of class \( PC^0([0,(i+1)T]; [D_{\min}, D_{\max}]) \). Moreover, the solution \( f \) of (11)-(2) with the control (7) satisfies
\[
f(t,a) = \exp \left( - \int_0^t D(l)dl \right) z(t,a)
\]
for all $(t, a) \in [0, \infty) \times [0, A]$ wherever the solution $f$ is defined. Hence, we are in a position to uniquely define $f(t, a)$ on $[iT, (i + 1)T] \times [0, A]$. Notice that $f_i$ is of class $PC^1([0, A]; [0, \infty))$ for all $t$ in $[0, (i + 1)T]$. We can continue this process to conclude that the solution of (1)-(2) with the control (7) is defined for all $t \geq 0$ and satisfies $f_t \in PC^1([0, A]; [0, \infty))$ for all $t \geq 0$.

Using the fact that the solution of (1)-(2) with (7) satisfies (33) for all $(t, a) \in [0, \infty) \times [0, A]$, our choice (5) of the output gives

$$y(t) = \exp \left(-\int_0^t D(l) dl \right) \int_0^A p(a) z(t, a) da = \exp \left(-\int_0^t D(l) dl \right) \int_0^A p(a) \exp \left(-\int_0^a \mu(s) ds \right) v(t - a) da = \exp \left(D^* t - \int_0^t D(l) dl \right) \times \int_0^A p(a) \exp \left(-D^* a - \int_0^a \mu(s) ds \right) (P(f_0) + \phi(t - a)) da$$

for all $t \geq 0$, by our choices of $v$ and $\phi$ from (16) and (20), and the relationship (15) between $z$ and $v$. Using (4) and (6), we conclude that

$$y^* = \int_0^A p(a) f^*(a) da = M \beta, \text{ where}$$

$$\beta = \int_0^A p(a) \exp \left(-D^* a - \int_0^a \mu(s) ds \right) da .$$

Combining (34) and (35) gives the following for all $i \geq 0$ and $t \in [iT, (i + 1)T]$:

$$\ln \left(\frac{y(t)}{y^*} \right) = \ln \left(\frac{y(iT)}{y^*} \right) + D^*(t - iT) - \int_{iT}^t D(l) dl + \ln \left(\frac{P(f_0) + \int_0^A g(a) \phi(t - a) da}{P(f_0) + \int_0^A g(a) \phi(iT - a) da} \right)$$

for all $a \in [0, A]$ and

$$D_i = \max \left\{D_{\min}, \min \left\{D_{\max}, D^* + T^{-1} \ln \left(\frac{y(iT)}{y^*} \right) \right\} \right\}$$

for all integers $i \geq 0$, and where the second equality in (36) followed from the sampling structure of our controller (7). We now set

$$x(t) = \ln \left(\frac{y(t)}{y^*} \right), \; x_i = \ln \left(\frac{y(iT)}{y^*} \right), \text{ and}$$

$$u_i = \ln \left(\frac{P(f_0) + \int_0^A g(a) \phi((i + 1)T - a) da}{P(f_0) + \int_0^A g(a) \phi(iT - a) da} \right)$$

for all integers $i \geq 0$. Choosing the positive constant

$$\delta = \frac{1}{2} \min \left\{(D_{\max} - D^*)T, (D^* - D_{\min})T \right\},$$

we use the following claim:
Claim 1. The inequality
\[ |x_{i+1}| \leq |x_i| - \min \{|x_i|, 2\delta\} + |u_i| \] (41)
holds for all integers \(i \geq 0\).

\[ \square \]

For the proof of Claim 1, see Appendix A.2. We also require the following two claims, which we also prove in the appendices:

Claim 2. For all integers \(i \geq 0\), the inequalities
\[
x_i \geq \min \left\{ 0, x_0 + i \left( D^* - D_{\text{min}} \right) T \right\} + \min_{k=0, \ldots, i} \left( \ln \left( \frac{P(f_0) + \int_0^A g(a) \phi(kT-a) da}{P(f_0) + \int_0^A g(a) \phi(kT-a) da} \right) \right) \quad (42)
\]

\[
x_i \leq \max \left\{ 0, x_0 - i \left( D_{\text{max}} - D^* \right) T \right\} + \max_{k=0, \ldots, i} \left( \ln \left( \frac{P(f_0) + \int_0^A g(a) \phi(kT-a) da}{P(f_0) + \int_0^A g(a) \phi(kT-a) da} \right) \right) \quad (42)
\]

are satisfied.

\[ \square \]

Claim 3. The inequalities
\[
\min \left\{ 0, x(0) \right\} + \min_{k=0, \ldots, \lfloor t/T \rfloor} \left( \ln \left( \frac{P(f_0) + \int_0^A g(a) \phi(t-a) da}{P(f_0) + \int_0^A g(a) \phi(kT-a) da} \right) \right) \leq x(t) \leq \max \left\{ 0, x(0) \right\} + \max_{k=0, \ldots, \lfloor t/T \rfloor} \left( \ln \left( \frac{P(f_0) + \int_0^A g(a) \phi(t-a) da}{P(f_0) + \int_0^A g(a) \phi(kT-a) da} \right) \right) \quad (43)
\]

and
\[
\begin{align*}
|x(t)| & \leq |x(\lfloor t/T \rfloor)| + \left| \ln \left( \frac{P(f_0) + \int_0^A g(a) \phi(t-a) da}{P(f_0) + \int_0^A g(a) \phi(t/T - a) da} \right) \right| \quad (44)
\end{align*}
\]

hold for all \(t \geq 0\).

\[ \square \]

We can combine estimate (43) with our bounds (31) on \(P(z_0) + \phi(t + a)\), our choice (4) of \(f^*\), and our choices of \(g\) and \(\beta\) in (35) and (37) (which imply that \(\int_0^A g(a) da = 1\)) to get
\[
|x(t)| \leq |x_0| + \ln \left( \frac{\max_{0 \leq a \leq A} (f_0(a)/f^*(a))}{\min_{0 \leq a \leq A} (f_0(a)/f^*(a))} \right) \quad (45)
\]

for all \(t \geq 0\). The proof of (45) uses the fact that the upper and lower bounds in (31) are independent of \(t\). By (23) and (31), we have
\[
|\phi(t)| \leq K^* \exp(-\epsilon t) \int_0^A f_0(a) da, \quad (46)
\]

and where \(K\) and \(\epsilon\) are from Lemma 3 and \(C\) was defined in (22). Let \(j\) be the smallest integer in \([\lfloor A/T \rfloor + 1, \infty)\) such that
\[
K^* \|f_0\|_1 \exp(-\epsilon (jT - A)) \leq \frac{\exp(\delta) - 1}{\exp(\delta) + 1} P(f_0). \quad (47)
\]

where \(\delta\) is from (40). We need the following claim, which we prove in Appendix A.5:
Claim 4. For all integers $i \geq j$, we have
\[
|u_i| \leq \delta \quad \text{and} \quad |u_i| \leq \frac{K^* \|f_0\|_1 (\exp(\delta) + 1) \exp(\varepsilon A)}{P(f_0)} \exp(-\varepsilon iT).
\] (48)

Also,
\[
\left| \ln \left( \frac{P(f_0) + \int_{a=0}^{\infty} g(a) \phi(t-a) da}{P(f_0) + \int_{a=0}^{\infty} g(a) \phi(iT-t) da} \right) \right| \leq \frac{K^* \|f_0\|_1 (\exp(\delta) + 1) \exp(\varepsilon A)}{P(f_0)} \exp(-\varepsilon iT) \tag{49}
\]

holds for all integers $i \geq j$ and all $t \geq iT$. \hfill \Box

We next show that
\[
\exp(|x_{i+1}|) - 1 \leq \exp(-\delta) (\exp(|x_i|) - 1) + \exp(|u_i|) - 1 \tag{50}
\]
holds for all $i \geq j$. When $|x_i| \leq 2\delta$, we can use (41) to get $|x_{i+1}| \leq |u_i|$, which implies (50). On the other hand, when $|x_i| > 2\delta$, we conclude from (41) and Claim 1 that $|x_{i+1}| \leq |x_i| - 2\delta + |u_i|$. The previous inequality, in conjunction with the fact that $|u_i| \leq \delta$ for all $i \geq j$ (which follows from Claim 4) gives
\[
|x_{i+1}| \leq |x_i| - 2\delta + |u_i| - 1
\]
\[
= \exp(|u_i|) - 1 + \exp(|u_i| - 2\delta) (\exp(|x_i|) - 1 + 1 - \exp(2\delta))
\]
\[
\leq \exp(|u_i|) - 1 + \exp(|u_i| - 2\delta) (\exp(|x_i|) - 1)
\]
\[
\leq \exp(|u_i|) - 1 + \exp(-\delta) (\exp(|x_i|) - 1).
\]

Hence, (50) holds for all $i \geq j$. Using (50) and induction, it follows that
\[
\exp(|x_i|) - 1 \leq \exp(-\delta(i-j)) (\exp(|x_j|) - 1)
\]
\[
+ \sum_{l=j}^{i-1} \exp(-\delta(i-1-l)) (\exp(|u_l|) - 1)
\] (51)
holds for all integers $i > j$.

Using our upper bounds (48) on $|u_i|$ from Claim 4 and the fact that $\exp(p) - 1 \leq p \exp(p)$ for all $p \geq 0$, we get $\exp(|u_i|) - 1 \leq \exp(\delta) |u_i|$ for all $i \geq j$, and also the following consequence of (51) for all $i > j$:
\[
\exp(|x_i|) - 1 \leq \exp(-\delta(i-j)) (\exp(|x_j|) - 1)
\]
\[
+ \frac{K^* \|f_0\|_1 \exp(\varepsilon A) \exp(\delta) + 1}{P(f_0)} \sum_{l=j}^{i-1} \exp(-\delta(i-2-l)) \exp(-\varepsilon lT).
\]

Since $x \leq \exp(x) - 1 \leq x \exp(x)$ holds for all $x \geq 0$, we conclude that the following holds for all $i > j$:
\[
|x_i| \leq \exp\left(\tilde{-\delta}(i-j)\right) \exp(|x_j|) |x_j|
\]
\[
+ \frac{K^* \|f_0\|_1 \exp(\varepsilon A + 2\delta) \exp(\delta) + 1}{P(f_0)} \sum_{l=j}^{i-1} \exp\left(-\tilde{-\delta}(i-l)\right) \exp(-\varepsilon lT),
\] (52)
where
\[ \tilde{\delta} = \min\{\delta, \varepsilon T\}. \] (53)

Since \( \tilde{\delta} \leq \varepsilon T \), it follows that \( \exp(-\tilde{\delta}(i - l)) \exp(-\varepsilon lT) \leq \exp(-\tilde{\delta} i) \) for all \( l = j, \ldots, i - 1 \) and thus (52) implies the following inequality for all \( i > j \):
\[ |x_i| \leq \exp \left( -\tilde{\delta}(i - j) \right) \exp(|x_j|) x_j \left( 1 + \frac{K^* \|f_0\|_1 \exp(\varepsilon A + 2\delta)(\exp(\delta) + 1)}{P(f_0)} \right) (i - j) \exp \left( -\tilde{\delta} i \right) \] (54)

Notice that (54) holds for \( i = j \) as well and consequently, (54) holds for all \( i \geq j \). Using (44), (59), (60), the conclusion (49) from Claim 4, and our choice (53) of \( \tilde{\delta} \), we obtain the following inequality for all \( i \geq j \):
\[ |x_i| \leq \frac{K^* \|f_0\|_1 \exp(-\varepsilon ((j - 1)T - A))}{P(f_0)} \exp \left( -\varepsilon (A + T) \right) > \frac{\exp(\delta) - 1}{\exp(\delta) + 1} \] (56)

In either case, we have
\[ \max \left\{ 1, \frac{K^* \|f_0\|_1}{P(f_0)} \right\} \exp(\delta) + 1 \] (57)

by our choice (53) of \( \tilde{\delta} \).

Using (55)-(57) combined with the fact that
\[ (i - j) \exp(-\tilde{\delta} i/2) \leq i \exp(-\tilde{\delta} i/2) \leq 2 \exp(-1)/\tilde{\delta} \] (58)

for all integers \( i \geq j \geq 0 \) (which follows because \( \exp(-r) \leq \exp(-1) \) for all \( r \geq 0 \)), we get the following inequality for all \( i \geq j \):
\[ |x_i| \leq \max \left\{ 1, \frac{K^* \|f_0\|_1}{P(f_0)} \right\} G \left( S(x_0, f_0) \ln(S(x_0, f_0)) + 1 \right) \exp(-\tilde{\delta} i/2), \] (59)

where
\[ G = \exp(\delta + 1) \exp(\varepsilon A) \max \left\{ \frac{\exp(\varepsilon T)}{\exp(\delta) - 1}, \frac{\exp(\varepsilon T)}{\exp(\delta) - 1}, 1 \right\} \] and
\[ S(x_0, f_0) = \frac{\max_{0 \leq a < A} f_0(a) / f^*(a)}{\min_{0 \leq a < A} f_0(a) / f^*(a)} \exp(|x_0|). \] (60)

Using (44), (59), (60), the conclusion (49) from Claim 4 and our choice (53) of \( \tilde{\delta} \), we obtain the following inequality for all \( t \geq jT \):
\[ |x(t)| \leq \max \left\{ 1, \frac{K^* \|f_0\|_1}{P(f_0)} \right\} G \left( S(x_0, f_0) \ln(S(x_0, f_0)) + 2 \right) \exp \left( -\frac{\tilde{\delta} \left\lceil \frac{t}{T} \right\rceil}{2} \right) \] (61)
Using (45), (57) and (60), we get the following for all $t \in [0, jT]$:

$$ |x(t)| \leq |x_0| + \ln \left( \frac{\max_{0 \leq a \leq A} (f_0(a)/f^*(a))}{\min_{0 \leq a \leq A} (f_0(a)/f^*(a))} \right) = \ln (S(x_0, f_0))$$

$$ \leq \exp \left( -\frac{\delta}{2} \left[ \frac{t}{T} \right] \right) \exp \left( \frac{\delta}{2} j \right) \ln (S(x_0, f_0))$$

$$ \leq \max \left\{ 1, \frac{K^* \| f_0 \|_1}{P(f_0)} \right\} \tilde{G} \exp \left( -\frac{\delta}{2} \left[ \frac{t}{T} \right] \right) S(x_0, f_0) \ln (S(x_0, f_0))$$  \hspace{1cm} (62)

Estimate (62) shows that inequality (61) holds for all $t \geq 0$.

Defining

$$ \sigma = \frac{\delta}{4T} \text{ and } \tilde{G} = G \exp \left( \frac{\delta}{2} \right) $$  \hspace{1cm} (63)

and using the fact that $\left[ \frac{t}{T} \right] \geq \frac{t}{T} - 1$, we can use (45), (60), and (62) to obtain the following for all $t \geq 0$:

$$ |x(t)| \leq \min \left\{ \max \left\{ 1, \frac{K^* \| f_0 \|_1}{P(f_0)} \right\} \tilde{G} \mathcal{N}(x_0, f_0) \exp (-2\sigma t), \ln (S(x_0, f_0)) \right\} $$

where $\mathcal{N}(x_0, f_0) = S(x_0, f_0) \ln (S(x_0, f_0)) + 2$. It now follows directly from the fact that $\min \{a, b\} \leq \sqrt{ab}$ for all $a \geq 0$ and $b \geq 0$ that

$$ |x(t)| \leq \left( \max \left\{ 1, \frac{K^* \| f_0 \|_1}{P(f_0)} \right\} \tilde{G} \mathcal{N}(x_0, f_0) \ln (S(x_0, f_0)) \right)^{1/2} \exp (-\sigma t) $$  \hspace{1cm} (64)

for all $t \geq 0$. Using (5) and (6), we get

$$ y^* \min_{0 \leq a \leq A} (f_0(a)/f^*(a)) \leq y(0) = \int_0^A p(a) f_0(a) da \leq y^* \max_{0 \leq a \leq A} (f_0(a)/f^*(a)) $$  \hspace{1cm} (65)

Using definition (39) and (65) then gives

$$ |x_0| \leq \ln \left( \max \left\{ \max_{0 \leq a \leq A} (f_0(a)/f^*(a)), \frac{1}{\min_{0 \leq a \leq A} (f_0(a)/f^*(a))} \right\} \right) $$  \hspace{1cm} (66)

Next, we define the functions

$$ Q(f_0) = \frac{\max_{0 \leq a \leq A} (f_0(a)/f^*(a))}{\min_{0 \leq a \leq A} (f_0(a)/f^*(a))} \max \left\{ \max_{0 \leq a \leq A} (f_0(a)/f^*(a)), \frac{1}{\min_{0 \leq a \leq A} (f_0(a)/f^*(a))} \right\} $$  \hspace{1cm} (67)

and

$$ R(f_0) = \left( \max \left\{ 1, \frac{K^* \| f_0 \|_1}{P(f_0)} \right\} \tilde{G} \left( Q(f_0) \ln (Q(f_0)) + 2 \right) \ln (Q(f_0)) \right)^{1/2} $$  \hspace{1cm} (68)

It follows from (64) and our formula for $S$ from (60) that the following holds:

$$ |x(t)| \leq R(f_0) \exp (-\sigma t) \text{ for all } t \geq 0 $$  \hspace{1cm} (69)
Using our formula for our control \( D(t) \), definition (39), and (69), we obtain

\[
|D(t) - D^*| \leq T^{-1} R(f_0) \exp \left( -\sigma T \left| \frac{t}{T} \right| \right) \leq T^{-1} R(f_0) \exp \left( -\sigma (t - T) \right)
\]

for all \( t \geq 0 \), because \( t/T \geq \left| \frac{t}{T} \right| - 1 \geq (t/T) - 1 \) for all \( t \geq 0 \). Also, our relationship (15) between \( v(t - a) \) and the classical solution, combined with our formula (20) for \( \phi(t) \) and our relationship (33) between \( f(t, a) \) and the solution \( z(t, a) \) for the corresponding uncontrolled dynamics give

\[
P(f_0) = \exp(-D^* t)v(t) - \phi(t)
\]

\[
= \exp(-D^* t)z(t, 0) - \phi(t) = \exp(-D^* t) \left( \int_0^t D(\ell) d\ell \right) f(t, 0) - \phi(t)
\]

for all \( t \geq 0 \). Hence, our output \( y(t) \) satisfies

\[
y(t) = \int_0^A p(a) f(t, a) da = \int_0^A p(a) \exp \left( - \int_0^t D(\ell) d\ell \right) z(t, a) da
\]

\[
= \int_0^A p(a) \exp \left( - \int_0^t D(\ell) d\ell - \int_0^a \mu(s) ds \right) v(t - a) da
\]

\[
= \int_0^A p(a) \exp \left( - \int_0^t D(\ell) d\ell - \int_0^a \mu(s) ds \right) (\phi(t - a) + P(f_0)) \exp(D^*(t - a)) da
\]

\[
= \int_0^A p(a) \exp \left( - \int_0^t D(\ell) d\ell - \int_0^a \mu(s) ds \right) (\phi(t - a) - \phi(t)) \exp(D^*(t - a)) da
\]

\[
+ \int_0^A p(a) \exp \left( - \int_0^t D(\ell) d\ell - \int_0^a \mu(s) ds \right) \exp \left( - D^* a + \int_0^t D(\ell) d\ell \right) f(t, 0) da
\]

\[
= \beta f(t, 0) + \beta \exp \left( D^* t - \int_0^t D(l) dl \right) \int_0^A g(a) (\phi(t - a) - \phi(t)) da
\]

for all \( t \geq 0 \), where we used the relationship (33) between \( z(t, a) \) and \( f(t, a) \), our choice (37) of \( g \), the relationship (15) between \( z(t, a) \) and \( v(t - a) \), our choice (20) of \( \phi \), our formula in (35) for \( \beta \), and the formula (71) for \( P(f_0) \). Dividing through by \( \beta M \) gives

\[
f(t, 0)/M = y(t)/y^* - M^{-1} \exp \left( D^* t - \int_0^t D(l) dl \right) \int_0^A g(a) (\phi(t - a) - \phi(t)) da
\]

Using (46), (70) and the fact that \( \int_0^A g(a) da = 1 \), we get this for all \( t \geq A \):

\[
\left| M^{-1} \exp \left( D^* t - \int_0^t D(l) dl \right) \int_0^A g(a) (\phi(t - a) - \phi(t)) da \right|
\]

\[
\leq 2 M^{-1} K^* \exp \left( -\varepsilon (t - A) \right) \frac{\| f_0 \|_1}{\| y^* \|_\infty} \exp \left( \frac{1}{\sigma T} R(f_0) \exp(\sigma T) \right)
\]

\[
\leq 2 K^* \exp \left( -\varepsilon (t - A) \right) \frac{\| f_0 \|_1}{\| y^* \|_\infty} \exp \left( \frac{1}{\sigma T} R(f_0) \exp(\sigma T) \right),
\]

since \( \| f^* \|_\infty \leq M \). Using (41), (31) with \( f_0 = z_0 \), and (70) gives

\[
\left| M^{-1} \exp \left( D^* t - \int_0^t D(l) dl \right) \int_0^A g(a) (\phi(t - a) - \phi(t)) da \right|
\]

\[
\leq \exp \left( \frac{1}{\sigma T} R(f_0) \exp(\sigma T) \right)
\]

\[
\times \left( \max_{0 \leq a \leq A} \left( f_0(a)/f^*(a) \right) - \min_{0 \leq a \leq A} \left( f_0(a)/f^*(a) \right) \right)
\]

for all \( t \geq 0 \) (by adding and subtracting \( P(z_0) \) in the integrand in (74)).
Combining (73) and (74), we obtain the following for all $t \geq 0$:

$$
M^{-1} \exp \left( D^*t - \int_0^t D(l)dl \right) \int_0^A g(a) (\phi(t - a) - \phi(t)) da
\leq \exp \left( \frac{1}{\sigma T} R(f_0) \exp(\sigma T) - \varepsilon(t - A) \right)
\times \max \left\{ \max_{0 \leq a \leq A} \left( f_0(a)/f^*(a) \right) - \min_{0 \leq a \leq A} \left( f_0(a)/f^*(a) \right), 2K^* \frac{\|f_0\|_1}{\|f^*\|_\infty} \right\}
$$

(75)

We next define the following two functions:

$$
V(f_0) = \exp \left( \frac{1}{\sigma T} R(f_0) \exp(\sigma T) \right) \left( H(f_0) \max \left\{ H(f_0), 2K^* \frac{\|f_0\|_1}{\|f^*\|_\infty} \right\} \right)^{1/2}
$$
and

$$
H(f_0) = \max_{0 \leq a \leq A} \left( f_0(a)/f^*(a) \right) - \min_{0 \leq a \leq A} \left( f_0(a)/f^*(a) \right)
$$

(76)

Combining (74) and (75), and using the fact that $\min\{a, b\} \leq \sqrt{ab}$ for all $a \geq 0$ and $b \geq 0$, gives

$$
M^{-1} \exp \left( D^*t - \int_0^t D(l)dl \right) \int_0^A g(a) (\phi(t - a) - \phi(t)) da
\leq V(f_0) \exp \left( -\varepsilon(t - A)/2 \right)
$$

(77)

for all $t \geq 0$.

Using (15), our formula [20] for $\phi(t)$, our bounds [31] with $z_0 = f_0$, and our relationship [33] between $f(t, a)$ and the solution $z(t, a)$ of the corresponding uncontrolled system, we obtain the following for all $t \geq 0$:

$$
\exp \left( -\int_0^t (D(l) - D^*)dl \right) \min_{0 \leq a \leq A} \left( f_0(a)/f^*(a) \right) \leq \frac{f(t, 0)}{M}
\leq \exp \left( -\int_0^t (D(l) - D^*)dl \right) \max_{0 \leq a \leq A} \left( f_0(a)/f^*(a) \right)
$$

(78)

Using (70) and (78), we obtain the following for all $t \geq 0$:

$$
\exp \left( -A(f_0) \right) \min_{0 \leq a \leq A} \left( f_0(a)/f^*(a) \right) \leq \frac{f(t, 0)}{M}
\leq \exp \left( \frac{1}{\sigma T} R(f_0) \exp(\sigma T) \right) \max_{0 \leq a \leq A} \left( f_0(a)/f^*(a) \right)
$$

(79)

where $A(f_0) = \frac{1}{\sigma T} R(f_0) \exp(\sigma T)$. Using definition [39] and [69] we get the following for all $t \geq 0$:

$$
\exp \left( -R(f_0) \exp(-\sigma t) \right) \leq \frac{y(t)}{y^*} \leq \exp \left( R(f_0) \exp(-\sigma t) \right)
$$

(80)

Combining (72), (77), (79), (80) and the fact that $\sigma \leq \varepsilon/2$ (which follows from our choice [63] and the fact that $\delta \leq \varepsilon T$), we obtain the following for all $t \geq 0$:

$$
\max \left\{ \exp \left( -A(f_0) \right) \min_{0 \leq a \leq A} \left( f_0(a)/f^*(a) \right), \mathcal{R}^2(t, f_0) \right\}
\leq \frac{f(t, 0)}{M}
\leq \min \left\{ \exp \left( A(f_0) \right) \max_{0 \leq a \leq A} \left( f_0(a)/f^*(a) \right), \mathcal{R}^2(t, f_0) \right\}
$$

(81)

where $\mathcal{R}^2(t, f_0) = \exp \left( R(f_0) \exp(-\sigma t) \right) + V(f_0) \exp \left( -\sigma(t - A) \right)$. 
Using the relationship (15) between the function $v$ and the uncontrolled solution $z(t,a)$ of (12)-(13) with the initial condition $z(0,a) = f_0(a)$ and (33), we obtain $v(t) = z(t,0)$, and therefore:

$$f(t,a) = \exp \left( - \int_0^t D(\ell) d\ell - \int_0^a \mu(s) ds \right) v(t-a)$$

$$= \exp \left( - \int_0^t D(\ell) d\ell - \int_0^a \mu(s) ds \right) z(t-a,0)$$

$$= \exp \left( - \int_0^t D(\ell) d\ell - \int_0^a \mu(s) ds \right) f(t-a,0) \exp \left( \int_{t-a}^t D(\ell) d\ell \right)$$

when $t \geq a$, and

$$f(t,a) = \exp \left( - \int_0^t D(\ell) d\ell - \int_0^a \mu(s) ds \right) \exp \left( \int_0^{a-t} \mu(s) ds \right) f_0(a-t)$$

when $t \in [0,a)$, since $v(t-a) = \exp(\int_0^{a-t} \mu(s) ds) f_0(a-t)$. Hence, we can use the formula (4) for $f^*$ to obtain

$$\frac{f(t,a)}{f^*(a)} = \exp \left( - \int_{t-a}^t (D(\ell) - D^*) d\ell \right) \frac{f(t-a,0)}{M}$$

(84)

for all $(t,a) \in [0,\infty) \times [0,A]$ such that $t \geq a$, and

$$\frac{f(t,a)}{f^*(a)} = \exp \left( - \int_0^t (D(\ell) - D^*) d\ell \right) \frac{f_0(a-t)}{f^*(a-t)}$$

(85)

for all $(t,a) \in [0,\infty) \times [0,A]$ such that $t < a$. We now define the functions

$$B_1(t,f_0) = \max \left\{ \exp \left( -C(t,f_0) \right) - \Xi(t,f_0), \right. \left. \exp \left( -\frac{1}{\sigma T} R(f_0) \exp(\sigma T) \right) \min_{0 \leq a \leq A} \left( f_0(a)/f^*(a) \right) \right\},$$

$$B_2(t,f_0) = \min \left\{ \exp \left( C(t,f_0) \right) + \Xi(t,f_0), \right. \left. \exp \left( \frac{1}{\sigma T} R(f_0) \exp(\sigma T) \right) \max_{0 \leq a \leq A} \left( f_0(a)/f^*(a) \right) \right\},$$

$$C(t,f_0) = R(f_0) \exp(-\sigma(t-A)), \quad \Xi(t,f_0) = V(f_0) \exp(-\sigma(t-2A)).$$

Combining (81), (84), (85), and (70) gives the following for all $t \geq 0$:

$$\exp \left( -AT^{-1}\sigma^{-1} \exp(-\sigma(t-T-A)) R(f_0) \right) B_1(t,f_0)$$

$$\leq \min_{0 \leq a \leq A} \left( f(t,a)/f^*(a) \right) \leq \max_{0 \leq a \leq A} \left( f(t,a)/f^*(a) \right)$$

$$\leq \exp \left( AT^{-1}\sigma^{-1} \exp(-\sigma(t-T-A)) R(f_0) \right) B_2(t,f_0)$$

(87)

We now set

$$w(t) = \max_{0 \leq a \leq A} \left| \ln \left( \frac{f(t,a)}{f^*(a)} \right) \right|,$$

$$w_1(t) = \max_{0 \leq a \leq A} \ln \left( \frac{f(t,a)}{f^*(a)} \right)$$

and

$$w_2(t) = \max_{0 \leq a \leq A} \ln \left( \frac{f^*(a)}{f(t,a)} \right)$$

(88)
for all $t \geq 0$. Clearly, definition (88) implies that

$$w(t) = \max_{0 \leq a \leq A} |\ln (f(t, a)/f^*(a))|$$

$$= \max_{0 \leq a \leq A} \left\{ \max_{0 \leq a \leq A} \{ \ln (f(t, a)/f^*(a)), \ln (f^*(a)/f(t, a)) \} \right\}$$

$$= \max \left\{ \max_{0 \leq a \leq A} \ln (f(t, a)/f^*(a)), \max_{0 \leq a \leq A} \ln (f^*(a)/f(t, a)) \right\}$$

$$= \ln \left( \max \left\{ \max_{0 \leq a \leq A} (f(t, a)/f^*(a)), \max_{0 \leq a \leq A} (f^*(a)/f(t, a)) \right\} \right)$$

for all $t \geq 0$, from which we get

$$\max_{0 \leq a \leq A} (f(t, a)/f^*(a)) \leq \exp(w(t))$$

and

$$\min_{0 \leq a \leq A} (f(t, a)/f^*(a)) \geq \exp(-w(t))$$

(90)

for all $t \geq 0$. Therefore, our definitions (18), (67), and (68) for $P$, $Q$, and $R$ give

$$P(f_0) \geq \exp(-w(0))P(f^*), \quad Q(f_0) \leq \exp(3w(0)), \quad R(f_0) \leq \tilde{Q}(w(0)),$$

$$\|f_0\|_1 \leq \exp(w(0)) \|f^*\|_1,$$

(91)

where

$$\tilde{Q}(s) = 3 \exp(3s) \left( \tilde{G} \max \left\{ 1, \frac{K^* \|f^*\|_1}{P(f^*)} \right\} \right)^{1/2} ((s + 1)s)^{1/2}$$

(92)

for all $s \geq 0$.

Also, our definition of $V(f_0)$ in (76) in conjunction with our bounds (90) and (91) give

$$V(f_0) \leq \tilde{P}(w(0)),$$

(93)

where

$$\tilde{P}(s) =$$

$$\exp \left( \frac{1}{\sigma T} \tilde{Q}(s) \exp(\sigma T) \right) \left( 2 \sinh(s) \max \left\{ 2 \sinh(s), 2K^* \frac{\|f^*\|_1}{\|f^*\|_\infty} \exp(s) \right\} \right)^{1/2}$$

(94)

for all $s \geq 0$. Also, our definitions of $w_1$ and $w_2$ in (88) in conjunction with estimate (87) and our bound on $R(f_0)$ in (91) give the following for all $t \geq 0$:

$$w_2(t) \leq AT^{-1}\sigma^{-1} \exp(-\sigma(t - T - A))\tilde{Q}(w(0)) + \ln \left( B_1^{-1}(t, f_0) \right)$$

and

$$w_1(t) \leq AT^{-1}\sigma^{-1} \exp(-\sigma(t - T - A))\tilde{Q}(w(0)) + \ln \left( B_2(t, f_0) \right)$$

(95)

Definitions (86) in conjunction with (90), our bounds on $R(f_0)$ and $V(f_0)$ in (91) and (93) and the facts that $\ln(a + b) \leq \ln(a) + a^{-1}b$ for all $a > 0$ and $b > 0$ and $\min\{a, b\} \leq \sqrt{ab}$ for
all \( a \geq 0 \) and \( b \geq 0 \), imply that

\[
\ln (B_2(t, f_0)) \leq \min \left\{ \frac{1}{\sigma_T} \exp(\sigma T) \tilde{Q}(w(0)) + w(0), \ln(\exp(\sigma T) + \Xi(t, f_0)) \right\}
\]

\[
\leq \min \left\{ \frac{1}{\sigma_T} \exp(\sigma T) \tilde{Q}(w(0)) + w(0), C(t, f_0) + \Xi(t, f_0) \exp(-C(t, f_0)) \right\}
\]

\[
\leq \min \left\{ \frac{1}{\sigma_T} \exp(\sigma T) \tilde{Q}(w(0)) + w(0), \exp(-\sigma(t - 2A)) (R(f_0) + V(f_0)) \right\}
\]

\[
\leq \exp(-\sigma(t - 2A)/2)
\]

\[
\leq \exp(\frac{1}{\sigma T} \exp(\sigma T) \tilde{Q}(w(0)) + w(0)) \left( \tilde{P}(w(0)) + \tilde{Q}(w(0)) \right)
\]

(96)

holds for all \( t \geq 0 \). Also, (86) in conjunction with (90) and our bound (91) on \( R(f_0) \) give

\[
\ln (B_1^{-1}(t, f_0)) \leq \frac{1}{\sigma_T} \tilde{Q}(w(0)) \exp(\sigma T) + w(0) \quad \text{for all} \quad t \geq 0
\]

(97)

and

\[
\ln (B_1^{-1}(t, f_0)) \leq - \ln \left( \exp(-C(t, f_0)) - \Xi(t, f_0) \right)
\]

(98)

for all \( t \geq 0 \) such that \( 1 > \exp(C(t, f_0)) \Xi(t, f_0) \). Using the facts that \( \ln(1 + x) \leq x \) and \( e^x - 1 \leq e^x x \) hold for all \( x \geq 0 \) and (98), we obtain the following for all \( t \geq 0 \) that satisfy \( 1 \geq 2 \exp(C(t, f_0)) \Xi(t, f_0) \):

\[
\ln \left( \frac{1}{\exp(-C(t, f_0)) - \Xi(t, f_0)} \right) \leq \ln \left( 1 + \frac{1 - \exp(-C(t, f_0)) + \Xi(t, f_0)}{\exp(-C(t, f_0)) - \Xi(t, f_0)} \right)
\]

\[
\leq \frac{\exp(C(t, f_0)) - 1 + \Xi(t, f_0) \exp(C(t, f_0))}{1 - \Xi(t, f_0) \exp(C(t, f_0))}
\]

\[
\leq 2 (\exp(C(t, f_0)) - 1 + \Xi(t, f_0) \exp(C(t, f_0)))
\]

\[
\leq 2 (C(t, f_0) + \Xi(t, f_0)) \exp(C(t, f_0))
\]

(99)

Next note that that by our definitions (86) and our bounds on \( R(f_0) \) and \( V(f_0) \) in (91) and (93), the inequality \( 1 \geq 2 \exp(C(t, f_0)) \Xi(t, f_0) \) holds if

\[
1 \geq 2 \exp(\tilde{Q}(w(0)) \exp(2\sigma A)) \tilde{P}(w(0)) \exp(-\sigma(t - 2A)),
\]

which holds if

\[
0 \geq \ln(2 \exp(\tilde{Q}(w(0)) \exp(2\sigma A)) + \ln(\tilde{P}(w(0)) + 1) - \sigma(t - 2A).
\]

(100)

On the other hand, (100) holds if

\[
t \geq 2A + \sigma^{-1} \ln(\tilde{P}(w(0)) + 1) + \sigma^{-1} \ln(2 \exp(\tilde{Q}(w(0)) \exp(2\sigma A)) - 1).
\]
Consequently, we conclude from (99), (86), our bound on $R(f_0)$ from (91), and (93) that
\[
\ln \left( B_{1}^{-1}(t, f_0) \right) \leq 2 \exp \left( -\sigma (t - 2A) \right) \left( \hat{P}(w(0)) + \hat{Q}(w(0)) \right) \text{ for all } t \geq \bar{T}(w(0)),
\]
where $\bar{T}(s) = 2A + \sigma^{-1} \ln(\bar{P}(s) + 1) + \sigma^{-1} \ln(2 \exp(\bar{Q}(s) \exp(2\sigma A)))$ for all $s \geq 0$. Then (97) and (101) give:
\[
\ln \left( B_{1}^{-1}(t, f_0) \right) \leq \exp \left( -\sigma (t - \bar{T}(w(0))) \right) \times \max \left\{ w(0) + \frac{1}{\sigma T} \bar{Q}(w(0)) \exp(\sigma T), 2\bar{P}(w(0)) + 2\bar{Q}(w(0)) \right\}
\]
for all $t \geq 0$. Also, (89) and our definitions of $w_1$ and $w_2$ in (88) give $w(t) = \max \{ w_1(t), w_2(t) \}$ for all $t \geq 0$. Using (95), (96) and (102) and noting that (a) the functions $\bar{P}$ and $\bar{Q}$ in (94) and (92), respectively, are of class $K_\infty$ and (b) the function $\bar{T}$ is non-decreasing, we conclude that there is a function $\kappa \in K_\infty$ such that $w(t) \leq \exp \left( -\sigma t / 2 \right) \kappa (w(0))$ for all $t \geq 0$. Therefore, the theorem follows from our definition of $w(t)$ from (88).

6 Simulations

To demonstrate our control designs from Theorem 1, we carried out three simulations. In each simulation, we took the horizon $A = 2$, the constant mortality function $\mu(a) = \mu = 0.1$, $D^* = 1$, and the birth modulus
\[
k(a) = \begin{cases} ag, & a \in [0, 1] \\ (2 - a)g, & a \in [1, 2] \end{cases}, \text{ where} \]
\[
g = \frac{(\mu + D^*)^2}{(1-\exp(-\mu + D^*))^2} = 2.718728 .
\]
The constant $g$ is chosen such that the Lotka-Sharpe condition (3) holds with $D^* = 1$. The output is
\[
y(t) = \int_0^2 f(t, a) da
\]
which is the total concentration of the microorganism in the chemostat. Our objective is to stabilize the equilibrium profile
\[
f^*(a) = \exp \left( -(D^* + \mu)a \right), \quad a \in [0, 2].
\]
The equilibrium value of the output is
\[
y^* = \int_0^2 f^*(a) da = \frac{1-\exp(-2(D^*+\mu))}{D^*+\mu} = 0.808361.
\]
We tested the output feedback law
\[
D(t) = \max \{ D_{\min}, \min \{ D_{\max}, D^* + T^{-1} \ln \left( f(iT, 0)/f^*(0)) \right) \} \text{, } t \in [iT, (i+1)T)
\]
and the output feedback law

\[ D(t) = \max \{D_{\min}, \min \{D_{\max}, D^* + T^{-1} \ln (y(t)/y^*)\} \}, \quad t \in [iT, (i+1)T) \]  

(108)

where \(i = 0, 1, 2, \ldots\), and for both controllers we chose \(T = 0.4\), \(D_{\min} = 0.5\), and \(D_{\max} = 1.5\).

We took our initial conditions to have the form

\[ f_0(a) = b_0 - b_1 a + c \exp(-\theta a), \quad a \in [0, 2] \]  

(109)

where \(b_0\), \(c\), and \(\theta\) are positive parameters that we specify below, and where

\[ b_1 = g^{-1}(g - 1)b_0 + c \theta^2 (1 - \exp(-\theta))^2 - cg^{-1} \]  

(110)

is chosen so that \(f_0(0) = \int_0^2 k(a)f_0(a) \, da\) holds. We must also choose the parameters such that \(\min_{a \in [0, 2]} f_0(a) > 0\). We generated the simulations using a uniform grid of function values \(f(ih, jh)\) for \(j = 0, 1, \ldots, 50\) and \(i \geq 0\), where \(h = 0.04\). For \(i = 0\) we had \(f(0, jh) = f_0(jh)\) for \(j = 1, \ldots, 50\), where \(f_0\) is from (109).

We computed the integrals

\[ y(ih) = \int_0^2 f(ih, a) \, da \quad \text{and} \quad f(ih, 0) = \int_0^2 k(a)f(ih, a) \, da \]  

(111)

numerically for each \(i \geq 0\). Since we wanted the numerical integrator to evaluate the integrals (111) exactly for every \(i \geq 0\) when \(f(ih, a) = C \exp(\sigma a)\) for certain real constants \(C\) and \(\sigma\), we did not use a conventional numerical integration scheme, such as the trapezoid rule or Simpson’s rule. The reason we wanted to evaluate the integrals exactly when \(f(ih, a)\) is an exponential function is that the equilibrium profile (105) is an exponential function and we would like to avoid a steady-state error due to the error induced by the numerical integrator. To this end, we set

\[ \mathcal{L}(i, j, h) = \ln (f(ih, (j + 1)h)) - \ln (f(ih, jh)) \]

and \(\mathcal{I}(i) = \{j : f(ih, (j + 1)h) = f(ih, jh)\}\),

(112)

and we used the integration schemes

\[
\int_{jh}^{(j+1)h} f(ih, a) \, da \approx I_i(j) = \begin{cases} 
 h \mathcal{L}(i, j, h), & j \notin \mathcal{I}(i) \\
 hf(ih, jh), & j \in \mathcal{I}(i)
\end{cases}
\]  

(113)

for \(j = 2, 3, \ldots, 49\) and \(i \geq 0\), and

\[
\int_0^{2h} f(ih, a) \, da \approx I_i(1) = \begin{cases} 
 h \frac{f(ih, 2h) - f^2(ih, h)}{\ln(f(ih, 2h)) - \ln(f(ih, h))}, & \text{if } f(ih, h) \neq f(ih, 2h) \\
 2hf(ih, h), & \text{if } f(ih, h) = f(ih, 2h)
\end{cases}
\]  

(114)

and we set

\[
\int_{jh}^{(j+1)h} af(ih, a) \, da \approx J_i(j) = \begin{cases} 
 h^2 \left( f(ih, (j+1)h) + jf(ih, (j+1)h) - f(ih, jh) \right) - \frac{f(ih, (j+1)h) - f(ih, jh)}{\mathcal{L}(i, j, h)}, & j \notin \mathcal{I}(i) \\
 2jh^2 f(ih, jh), & j \in \mathcal{I}(i)
\end{cases}
\]  

(115)
for $j = 2, 3, ..., 24$ and $i \geq 0$, and
\[
\int_0^{2h} af(ih, a) da \approx J_i(1) = \begin{cases} 
\frac{2h^2 f(ih, 2h)}{\ln(f(ih, 2h)) - \ln(f(ih, h))}, & \text{if } f(ih, h) \neq f(ih, 2h) \\
2h^2 f(ih, 2h), & \text{if } f(ih, h) = f(ih, 2h)
\end{cases}
\] (116)

and
\[
\int_{jh}^{(j+1)h} (2 - a) f(ih, a) da \approx K_i(j) = \begin{cases} 
\frac{-h^2 f(ih, jh)}{L(i,j,h)} + \left(2 - jh + \frac{h}{L(i,j,h)}\right) \frac{h f(ih, jh)}{L(i,j,h)}, & j \notin I(i)
\end{cases}
\] (117)

for $j = 25, 26, ..., 49$ and $i \geq 0$.

The derivation of formulas (113)–(117) is based on the interpolation of
\[
\tilde{f}_j(a) = C_j \exp(\sigma_j a)
\] (118)
through the points $(jh, f(ih, jh))$ and $((j + 1)h, f(ih, (j + 1)h))$ for $j = 1, 2, ..., 49$. More specifically, we obtain the following for $j = 1, 2, ..., 49$:
\[
\sigma_j = h^{-1} \ln \left(\frac{f(ih, (j + 1)h)}{f(ih, jh)}\right) \quad \text{and} \quad C_j = f(ih, jh) \left(\frac{f(ih, (j + 1)h)}{f(ih, jh)}\right)^{-j}
\] (119)

Using this interpolation, the exact integration formulas are used. For example, for
\[
\int_{jh}^{(j+1)h} af(ih, a) da \quad \text{for } j = 2, 3, ..., 24,
\] (120)
we get the following when $\sigma_j = h^{-1} \ln \left(\frac{f(ih, (j + 1)h)}{f(ih, jh)}\right) \neq 0$:
\[
\int_{jh}^{(j+1)h} af(ih, a) da \approx \int_{jh}^{(j+1)h} a \tilde{f}_j(a) da = C_j \int_{jh}^{(j+1)h} a \exp(\sigma_j a) da
\]
\[
= C_j \sigma_j^{-1} \exp(\sigma_j jh) ((j + 1) \exp(\sigma_j h) - j) - C_j \sigma_j^{-2} (\exp(\sigma_j (j + 1)h) - \exp(\sigma_j jh))
\]

On the other hand, when $\sigma_j = h^{-1} \ln \left(\frac{f(ih, (j + 1)h)}{f(ih, jh)}\right) = 0$, we get
\[
\int_{jh}^{(j+1)h} af(ih, a) da \approx \int_{jh}^{(j+1)h} a \tilde{f}_j(a) da = C_j \int_{jh}^{(j+1)h} ada = \frac{h^2}{2} C_j (2j + 1).
\]

Combining the above formulas with the estimated values for $C_j$ and $\sigma_j$ in (119), we obtain formula (115). Similarly, we derive formulas (113), (114), (116), and (117). Notice that the formulas (113), (114), (115), (116), and (117) allow the numerical evaluation of the integrals (111) for every $i \geq 0$ without knowing $f(ih, 0)$. 
Since the time and space discretization steps are both $h$, we have the exact formula

$$f((i+1)h, jh) = f(ih, (j-1)h) \exp(-(\mu + D_i)h)$$

for $j = 1, 2, \ldots, 50$ and all $i \geq 0$, \hspace{1cm} (121)

where $D_i = D(ih)$. Therefore, we have the following algorithm for simulating the closed-loop system:

**Algorithm:** Given $f(ih, jh)$, for $j = 1, \ldots, 50$ and certain $i \geq 0$ do the following:

1. Calculate $f(ih, 0) \approx g \sum_{j=2}^{41} J_i(j) + g \sum_{j=25}^{49} K_i(j)$, where $J_i(j)$ and $K_i(j)$ are given by (115), (116), and (117).

2. Calculate $y(ih) \approx \sum_{j=2}^{49} I_i(j)$, where $I_i(j)$ is given by (113) and (114).

3. If $\frac{ih}{T}$ is an integer, then set $D_i = \max\{D_{\min}, \min\{D_{\max}, D^* + T^{-1} \ln(y(ih)/y^*)\}\}$; otherwise, set $D_i = D_{i-1}$.

4. Calculate $f((i+1)h, jh)$, for $j = 1, \ldots, 50$ using (121).

The above algorithm with obvious modifications was also used for the simulation of the open-loop system, and for the simulation of the closed-loop system under the output feedback law (108). We next present the results of our three simulations.

In our first simulation, we used the parameter values $b_0 = 0.2$, $b_1 = 0.15184212$, $c = 0.8$, and $\theta = 1$ in our initial conditions. In Figure 1, we plot the control values and the newborn individual values. We show the values for the open loop feedback $D(t) \equiv 1$, and for the state and output feedbacks from (107) and (108). Our simulation shows the efficacy of our control design.

In our second simulation, we changed the parameter values to $b_0 = 1$, $b_1 = 0.7592106$, $c = 4$, and $\theta = 1$ and plotted the same values as before, in Figure 2. The responses for the output feedback law (107) and the output feedback law (108) are almost identical.

This second simulation was made with an initial condition which is not close to the equilibrium profile (in the sense that it is an initial condition with very large initial population). The difference in the performance of the feedback controllers (107) and (108) cannot be distinguished.

In our final simulation, we tested the robustness of the controller with respect to errors in the choice of $D^*$ being used in the controllers. We chose the values $b_0 = 0.2$, $b_1 = 0.15184212$, $c = 0.8$, and $\theta = 1$, but instead of (107) and (108), we applied the controllers which are defined for all $t \in [iT, (i+1)T)$ and for integers $i \geq 0$ by

$$D(t) = \max\{D_{\min}, \min\{D_{\max}, 0.7 + T^{-1} \ln(f(iT, 0)/f^*(0))\}\} \hspace{1cm} (122)$$

and

$$D(t) = \max\{D_{\min}, \min\{D_{\max}, 0.7 + T^{-1} \ln(y(iT)/y^*)\}\} \hspace{1cm} (123)$$
In both cases, we obtained \( \lim_{t \to +\infty} f(t, 0) = 1.1275 \) and

\[
\lim_{t \to +\infty} f(t, 0) = 1.1275 \quad \text{and} \quad \lim_{t \to +\infty} D(t) = D^* = 1.
\]

(124)

Hence, a \(-30\%\) error in \( D^* \) gave a \(+12.75\%\) steady-state deviation from the desired value of the newborn individuals. See Figure 3.

Notice that a constant error in \( D^* \) is equivalent to an error in the set point since we have:

\[
D(t) = \max\{D_{\min}, \min\{D_{\max}, 0.7 + T^{-1} \ln (f(iT, 0)/f^*(0))\}\}
\]

\[
= \max\{D_{\min}, \min\{D_{\max}, D^* + T^{-1} \ln (f(iT, 0)/f^*(0)) - T^{-1}0.12\}\}
\]

(125)

\[
= \max\{D_{\min}, \min\{D_{\max}, D^* + T^{-1} \ln (f(iT, 0)/(1.1275f^*(0)))\}\}
\]
Figure 2: Second simulation. The red line is for the feedbacks (107) and (108) and the black line is for the open-loop system with $D(t) \equiv 1$.

for the output feedback case (107) and

$$D(t) = \max \left\{ D_{\min}, \min \left\{ D_{\max}, 0.7 + T^{-1} \ln \left( \frac{y(iT)}{y^*} \right) \right\} \right\}$$

$$= \max \left\{ D_{\min}, \min \left\{ D_{\max}, D^* + T^{-1} \ln \left( \frac{y(iT)}{y^*} \right) - T^{-1}0.12 \right\} \right\}$$

$$= \max \left\{ D_{\min}, \min \left\{ D_{\max}, D^* + T^{-1} \ln \left( \frac{y(iT)}{(1.1275y^*)} \right) \right\} \right\}$$

for the output feedback case (108). An interesting feature of the closed-loop system is that

$$\lim_{t \to +\infty} D(t) = D^* = 1.$$  \hspace{1cm} (127)

It may be worth considering an adaptive strategy for the elimination of errors in $D^*$ (i.e., a hybrid strategy that adapts the applied value of $D^*$). We leave the search for such a strategy for future work.
Figure 3: Third simulation. The red line is for the output feedback (122) and the blue line is for the output feedback (123).

7 Conclusions

Chemostats play a vital role in biotechnological applications, such as the production of insulin and in waste water treatment plants. Age-structured chemostats produce challenging control problems for first-order hyperbolic PDEs that are beyond the scope of the existing controller methods for ODEs. We studied the problem of stabilizing an equilibrium age profile in an age-structured chemostat, using the dilution rate as the control. We built a sampled-data dilution rate feedback control law that ensures stability under arbitrary physically meaningful initial conditions and arbitrarily sparse sampling. Our control does not require measurement of the whole age profile, or exact model knowledge. The proposed feedback also applies under arbitrary input constraints. The proof of our main result is based on (a) the strong ergodic theorem and (b) our approach from [13] for transforming
a first-order hyperbolic PDE into an integral delay equation. Our simulations demonstrate the good performance of our controllers.

We hope to build on our research, in two ways. First, since the growth of the microorganism may sometimes depend on the concentration of a substrate, it would be useful to solve the stabilization problem for an enlarged system that has one PDE for the age distribution, coupled with one ODE for the substrate (as proposed in [22], in the context of studying limit cycles with constant dilution rates instead of a control). Second, it would be useful to extend our work to cases where the control is subject to uncertainties, and then seek generalizations of our exponential stability estimate such as input-to-state stability under input constraints and sampling. Finally, we hope to cover state constrained problems, which add the requirement that the states must stay in prescribed subsets of the state space for all nonnegative times, in addition to the nonnegativity requirements on the physical quantities.

Appendices

A.1 Proof of Lemma 4

Local existence and uniqueness for every initial condition $x_0 \in L^\infty([-A, 0); \mathbb{R})$ is guaranteed by [13, Theorem 2.1]. We define the functions

$$V_s(t) = \sup_{-A \leq a < 0} x(t + a) \quad \text{and} \quad W(t) = \inf_{-A \leq a < 0} x(t + a)$$

for all $t \geq 0$ for which the solution of (25) exists. Let $q > 0$ and $t \geq 0$ be sufficiently small such that the solution exists on $[t, t + q)$. From equation (25), we get

$$V_s(t + q) = \sup_{-A \leq a < 0} x(t + q + a) = \sup_{q - A \leq s < q} x(t + s)$$

$$= \max \left\{ \sup_{q - A \leq s < 0} x(t + s), \sup_{0 \leq s < q} x(t + s) \right\}$$

$$\leq \max \left\{ V_s(t), \sup_{0 \leq s < q} \left\{ \int_{-A}^{s} G(a) x(t + s - a) da \right\} \right\}$$

$$= \max \left\{ V_s(t), \sup_{0 \leq s < q} \left\{ \int_{-A}^{s} G(a) x(t + s - a) da + \int_{-A}^{s} G(a) x(t + s - a) da \right\} \right\}$$

$$\leq \max \left\{ V_s(t), \sup_{0 \leq s < q} \left\{ \sup_{-A \leq l < s - \Delta} \int_{-A}^{s} G(a) x(t + l) da + \sup_{-A \leq l < s - \Delta} \int_{-A}^{s} G(a) da \right\} \right\}$$

$$\leq \max \left\{ V_s(t), \sup_{-A \leq l < q - \Delta} \int_{-A}^{s} G(a) da + \sup_{-A \leq l < q - \Delta} \int_{-A}^{s} G(a) da \right\}.$$
which gives \( V_*(t + q) \leq \max\{bV_*(t), V_*(t)\} \) with \( b \) as in the statement of the lemma, by separately considering the two possible values for the maximum on the right side of (A.3). (We need the maximum \( \max\{bV_*(t), V_*(t)\} \) instead of just \( V_*(t) \), to allow the possibility that \( V_*(t) \) is nonpositive.)

Similarly, we can use the decomposition
\[
W(t + q) = \inf_{-A \leq a < 0} x(t + q + a) = \inf_{q - A \leq s < q} x(t + s)
\]
\[
= \min \left\{ \inf_{q - A \leq s < q} x(t + s), \inf_{0 \leq s < q} x(t + s) \right\},
\]
the definition of \( W(t) \), equation (25), and the fact that \( q \leq \min\{\Delta, A - \Delta\} \) to get
\[
W(t + q) \geq \min \left\{ W(t), \inf_{0 \leq s < q} \int_0^\Delta G(a)x(t + s - a)da \right\}
\]
\[
= \min \left\{ W(t), \inf_{0 \leq s < q} \left\{ \int_0^\Delta G(a)x(t + s - a)da + \int_0^\Delta G(a)x(t + s - a)da \right\} \right\}
\]
\[
\geq \min \left\{ W(t), \inf_{-A \leq l - q - \Delta} x(t + l) \int_0^\Delta G(a)da, \inf_{-\Delta \leq l < q} x(t + l) \int_0^\Delta G(a)da \right\}
\]
\[
\geq \min \left\{ W(t), W(t) \left( L - \int_0^\Delta G(a)da \right) + W(t + q) \int_0^\Delta G(a)da \right\},
\]
so (A.2) gives \( W(t + q) \geq \min\{bW(t), W(t)\} \). It follows from (A.3) that the solution of (25) is bounded on \([t, t + q]\) when \( q \leq \min\{\Delta, A - \Delta\} \). A standard contradiction argument in conjunction with [13, Theorem 2.1] implies that the solution exists for all \( t \geq 0 \). Using induction and (A.3), we can now show that
\[
\min\{bW(0), W(0)\} \leq W(ih) \leq V_*(ih) \leq \max\{bV_*(0), V_*(0)\} \quad (A.4)
\]
for all integers \( i \geq 0 \), where \( h = \min\{\Delta, A - \Delta\} \). Inequality (26) now follows from the definitions of \( V_* \) and \( W \) and (A.3) and (A.4), by choosing \( i \) such that \( i \leq t/h \). This proves the lemma.

### A.2 Proof of Claim 1

We distinguish between the following cases.

Case 1: \( D_{\text{min}} \leq D* + T^{-1}x_i \leq D_{\text{max}} \). In this case, our choices of the \( D_i \)'s and \( x_i \)'s imply that \( D_i = D* + T^{-1}x_i \). Using our expressions \( \{34\} \) for the output \( y(t) \), our choices of the \( D_i \)'s, the fact that \( D_i = D* + T^{-1}x_i \), and the fact \( \phi(t) \) and \( y(t) \) are continuous mappings, we get \( x_{i+1} = u_i \), which directly implies \( \{41\} \).
Case 2: $D_{\min} > D^* + T^{-1}x_i$. In this case, (38) and (39) give $D_i = D_{\min}$. Using the continuity of $\phi(t)$ and $y(t)$ and setting $t = (i + 1)T$ in (34), and using definition (38) and the fact that $D_i = D_{\min}$, we get

$$x_{i+1} = x_i + (D^* - D_{\min})T + u_i.$$ 

The inequality $D_{\min} > D^* + T^{-1}x_i$ implies that $0 > (D^* - D_{\min})T + x_i$ and $0 > x_i$. Using the previous inequalities, the equality $x_{i+1} = x_i + (D^* - D_{\min})T + u_i$, and our choice of $\delta$, we get

$$|x_{i+1}| \leq |x_i + (D^* - D_{\min})T| + |u_i| = -x_i - (D^* - D_{\min})T + |u_i| \leq |x_i| - 2\delta + |u_i|,$$

which again gives (41).

Case 3: $D^* + T^{-1}x_i > D_{\max}$. Then (38) and (39) give $D_i = D_{\max}$. Since $\phi$ and $y$ are continuous, we can set $t = (i + 1)T$ in (34), and use (38) and the fact that $D_i = D_{\min}$ to get $x_{i+1} = x_i - (D_{\max} - D^*)T + u_i$. The inequality $D^* + T^{-1}x_i > D_{\max}$ implies that $x_i - (D_{\max} - D^*)T > 0$ and $0 < x_i$. Then the equality $x_{i+1} = x_i - (D_{\max} - D^*)T + u_i$ give

$$|x_{i+1}| \leq |x_i - (D_{\max} - D^*)T| + |u_i| = x_i - (D_{\max} - D^*)T + |u_i| \leq |x_i| - 2\delta + |u_i|,$$

so (41) holds again. This proves Claim 1.

### A.3 Proof of Claim 2

The proof of (42) is by induction. First notice that both inequalities in (42) hold for $i = 0$. Next assume that (42) hold for certain integer $i \geq 0$. We consider three cases.

Case 1: $D_{\min} \leq D^* + T^{-1}x_i \leq D_{\max}$. In this case, our treatment of Case 1 in our proof of Claim 1 gives $x_{i+1} = u_i$. Consequently, our definition (39) gives

$$x_{i+1} = u_i = \ln \left( \frac{P(0) + \int_0^A g(a)\phi((i+1)T-a)da}{P(0) + \int_0^A g(a)\phi(iT-a)da} \right) \leq \max_{k=0,...,i+1} \left( \ln \left( \frac{P(0) + \int_0^A g(a)\phi((i+1)T-a)da}{P(0) + \int_0^A g(a)\phi(kT-a)da} \right) \right) \leq \max \{0, x_0 - (i + 1) (D_{\max} - D^*) T \} + \max_{k=0,...,i+1} \left( \ln \left( \frac{P(0) + \int_0^A g(a)\phi((i+1)T-a)da}{P(0) + \int_0^A g(a)\phi(kT-a)da} \right) \right),$$

which implies the second inequality in (42) with $i + 1$ in place of $i \geq 0$. Similarly, we obtain the first inequality in (42) with $i + 1$ in place of $i \geq 0$.

Case 2: $D_{\min} > D^* + T^{-1}x_i$. Arguing as in our treatment of Case 2 in our proof of Claim 1, we get $x_{i+1} = x_i + (D^* - D_{\min})T + u_i$. Hence, (39) and (42) and the fact that $D^* \geq D_{\min}$
give
\[
x_{i+1} = x_i + (D^* - D_{\min}) T + u_i
\]
\[
\geq \min \left\{ 0, x_0 + i (D^* - D_{\min}) T \right\} + (D^* - D_{\min}) T
\]
\[
+ \min_{k=0, \ldots, i} \left( \ln \left( \frac{P_f a \phi (iT - a)}{P_f a \phi (kT - a)} \right) \right)
\]
\[
+ \ln \left( \frac{P_f a \phi (i+1)(T-a)}{P_f a \phi (kT-a)} \right)
\]
\[
\geq \min \left\{ (D^* - D_{\min}) T, x_0 + (i + 1) (D^* - D_{\min}) T \right\}
\]
\[
+ \min_{k=0, \ldots, i+1} \left( \ln \left( \frac{P_f a \phi (i+1)(T-a)}{P_f a \phi (kT-a)} \right) \right)
\]
which is the first inequality (42) with \( i + 1 \) in place of \( i \geq 0 \). Furthermore, \( 0 > (D^* - D_{\min}) T + x_i \). Combining the previous inequality with definition (39) and the fact that \( x_{i+1} = x_i + (D^* - D_{\min}) T + u_i \), we get
\[
x_{i+1} = x_i + (D^* - D_{\min}) T + u_i \leq u_i
\]
\[
\leq \max_{k=0, \ldots, i+1} \left( \ln \left( \frac{P_f a \phi (i+1)(T-a)}{P_f a \phi (kT-a)} \right) \right)
\]
\[
\leq \max \left\{ 0, x_0 - (i + 1) (D_{\max} - D^*) T \right\}
\]
\[
+ \max_{k=0, \ldots, i+1} \left( \ln \left( \frac{P_f a \phi (i+1)(T-a)}{P_f a \phi (kT-a)} \right) \right)
\]
which is the second inequality (42) with \( i + 1 \) in place of \( i \geq 0 \). Furthermore, \( 0 > (D^* - D_{\min}) T + x_i \). Combining the previous equality with definition (39) and inequality (42), we get
\[
x_{i+1} = x_i - (D_{\max} - D^*) T + u_i
\]
\[
\leq \max \left\{ 0, x_0 - i (D_{\max} - D^*) T \right\} - (D_{\max} - D^*) T
\]
\[
+ \max_{k=0, \ldots, i} \left( \ln \left( \frac{P_f a \phi (iT-a)}{P_f a \phi (kT-a)} \right) \right)
\]
\[
+ \ln \left( \frac{P_f a \phi (i+1)(T-a)}{P_f a \phi (kT-a)} \right)
\]
\[
\leq \max \left\{ - (D_{\max} - D^*) T, x_0 - (i + 1) (D_{\max} - D^*) T \right\}
\]
\[
+ \max_{k=0, \ldots, i} \left( \ln \left( \frac{P_f a \phi (i+1)(T-a)}{P_f a \phi (kT-a)} \right) \right)
\]
\[
\leq \max \left\{ 0, x_0 - (i + 1) (D_{\max} - D^*) T \right\}
\]
\[
+ \max_{k=0, \ldots, i+1} \left( \ln \left( \frac{P_f a \phi (i+1)(T-a)}{P_f a \phi (kT-a)} \right) \right),
\]
which is the second inequality (42) with \( i + 1 \) in place of \( i \geq 0 \). Furthermore, \( x_i - (D_{\max} - D^*) T > 0 \). Combining the previous inequality with definition (39) and the fact that \( x_{i+1} = \).
\[ x_i - (D_{\max} - D^*)T + u_i, \] we get
\[ x_{i+1} = x_i - (D_{\max} - D^*)T + u_i \]
\[ \geq u_i \geq \min_{k=0,...,i+1} \left( \ln \left( \frac{P(f_0) + \int_0^1 g(a)\phi((i+1)T-a)da}{P(f_0) + \int_0^1 g(a)\phi(kT-a)da} \right) \right) \]
\[ \geq \min \{ 0, x_0 + (i + 1)(D^* - D_{\min})T \} \]
\[ + \min_{k=0,...,i+1} \left( \ln \left( \frac{P(f_0) + \int_0^1 g(a)\phi((i+1)T-a)da}{P(f_0) + \int_0^1 g(a)\phi(kT-a)da} \right) \right) , \]
which is the first inequality (42) with \( i + 1 \) in place of \( i \geq 0 \). This proves Claim 2.

A.4 Proof of Claim 3

Our expressions (34) for the output give
\[ x(t) = x_i - (D_i - D^*)(t - iT) + \ln \left( \frac{P(f_0) + \int_0^1 g(a)\phi((t-a)da}{P(f_0) + \int_0^1 g(a)\phi(iT-a)da} \right) \quad (A.5) \]
for all \( t \geq 0 \), where \( i = \lfloor t/T \rfloor \). We again consider three cases.
Case 1: \( D_{\min} \leq D^* + T^{-1}x_i \leq D_{\max} \). In this case, (38) and (39) imply that \( D_i = D^* + T^{-1}x_i \).
Hence, (A.5) gives
\[ x(t) = (1 - (t - iT)T^{-1})x_i + \ln \left( \frac{P(f_0) + \int_0^1 g(a)\phi((t-a)da}{P(f_0) + \int_0^1 g(a)\phi(iT-a)da} \right) \]
for all \( t \geq 0 \). The above equality in conjunction with the fact that \( 0 \leq 1 - (t - iT)T^{-1} \leq 1 \) and inequality (42) gives estimates (43)-44.
Case 2: \( D_{\min} > D^* + T^{-1}x_i \). Now our definitions (38) and (39) give \( D_i = D_{\min} \). The inequality \( D_{\min} > D^* + T^{-1}x_i \) implies that \( 0 > (D^* - D_{\min})T + x_i \) and \( 0 > x_i \), which combined with (A.5) give:
\[ x_i + \ln \left( \frac{P(f_0) + \int_0^1 g(a)\phi((t-a)da}{P(f_0) + \int_0^1 g(a)\phi(iT-a)da} \right) \leq x(t) \leq \ln \left( \frac{P(f_0) + \int_0^1 g(a)\phi((t-a)da}{P(f_0) + \int_0^1 g(a)\phi(iT-a)da} \right) . \]
The above inequality in conjunction with inequalities (42) gives (43)-44.
Case 3: \( D^* + T^{-1}x_i > D_{\max} \). Definitions (38)-(39) imply that \( D_i = D_{\max} \). The inequality \( D^* + T^{-1}x_i > D_{\max} \) implies that \( x_i - (D_{\max} - D^*)T > 0 \) and \( 0 < x_i \), which combined with (A.5) gives
\[ \ln \left( \frac{P(f_0) + \int_0^1 g(a)\phi((t-a)da}{P(f_0) + \int_0^1 g(a)\phi(iT-a)da} \right) \leq x(t) \leq \ln \left( \frac{P(f_0) + \int_0^1 g(a)\phi((t-a)da}{P(f_0) + \int_0^1 g(a)\phi(iT-a)da} \right) , \]
which we can combine with (42) to get (43)-44. This proves Claim 3.

A.5 Proof of Claim 4

Since \( |\ln(x)| = \ln(\max\{x, x^{-1}\}) \) for all \( x > 0 \), we get
\[ \left| \ln \left( \frac{P(f_0) + \int_0^1 g(a)\phi((t-a)da}{P(f_0) + \int_0^1 g(a)\phi(iT-a)da} \right) \right| = \]
\[ \ln \left( \max \left\{ \frac{P(f_0) + \int_0^1 g(a)\phi((t-a)da}{P(f_0) + \int_0^1 g(a)\phi(iT-a)da} , \frac{P(f_0) + \int_0^1 g(a)\phi((t-a)da}{P(f_0) + \int_0^1 g(a)\phi(iT-a)da} \right\} . \]
and

\[ |u_i| = \ln \left( \max \left\{ \frac{P(f_0) + \int_0^1 g(a) \phi((i+1)T-a) \, da}{P(f_0) + \int_0^1 g(a) \phi(iT-a) \, da}, \frac{P(f_0) + \int_0^1 g(a) \phi(i+1)T-a) \, da}{P(f_0) + \int_0^1 g(a) \phi(iT-a) \, da} \right\} \right). \]  

(A.6)

On the other hand, we can use (47) to get the following for all \( i \geq j \):

\[ \frac{P(f_0) + h}{P(f_0) - h} \leq \exp(\delta), \]  

(A.7)

where \( h = K^* \| f_0 \|_1 \exp(-\varepsilon(iT - A)) \). Using our bound (46) on \( \phi \) and the fact \( j \geq [A/T] + 1 \) (which implies that \( jT \geq A \), i.e., \( iT - a \geq 0 \) for all \( a \in [0, A] \)), we get

\[ \frac{P(f_0) - K^* \| f_0 \|_1 \exp(-\varepsilon((i+1)T-A)) \int_0^1 g(a) \, da \leq P(f_0) + K^* \| f_0 \|_1 \exp(-\varepsilon(iT-A)) \int_0^1 g(a) \, da}{P(f_0) - K^* \| f_0 \|_1 \exp(-\varepsilon((i+1)T-A)) \int_0^1 g(a) \, da \leq P(f_0) + K^* \| f_0 \|_1 \exp(-\varepsilon(iT-A)) \int_0^1 g(a) \, da} \]

and

\[ \frac{P(f_0) - K^* \| f_0 \|_1 \exp(-\varepsilon(t-A)) \int_0^1 g(a) \, da \leq P(f_0) + K^* \| f_0 \|_1 \exp(-\varepsilon(t-A)) \int_0^1 g(a) \, da}{P(f_0) - K^* \| f_0 \|_1 \exp(-\varepsilon((i+1)T-A)) \int_0^1 g(a) \, da \leq P(f_0) + K^* \| f_0 \|_1 \exp(-\varepsilon((i+1)T-A)) \int_0^1 g(a) \, da} \]

for all \( t \geq iT \).

Our formulas (35) and (37) for \( \beta \) and \( g \) imply that \( \int_0^1 g(a) \, da = 1 \). Hence, the preceding inequalities give

\[ \frac{P(f_0) - h}{P(f_0) + h} \leq \frac{P(f_0) + \int_0^1 g(a) \phi((i+1)T-a) \, da}{P(f_0) + \int_0^1 g(a) \phi(iT-a) \, da} \leq \frac{P(f_0) + h}{P(f_0) - h} \]

and

\[ \frac{P(f_0) - h}{P(f_0) + h} \leq \frac{P(f_0) + \int_0^1 g(a) \phi(t-a) \, da}{P(f_0) + \int_0^1 g(a) \phi(iT-a) \, da} \leq \frac{P(f_0) + h}{P(f_0) - h} \]

for all \( t \geq iT \).

where \( h = K^* \| f_0 \|_1 \exp(-\varepsilon(iT - A)) \). Combining (A.6) and (A.8), we get:

\[ |u_i| \leq \ln \left( \frac{P(f_0) + h}{P(f_0) - h} \right) \]  

and

\[ \ln \left( \frac{P(f_0) + \int_0^1 g(a) \phi(t-a) \, da}{P(f_0) + \int_0^1 g(a) \phi(iT-a) \, da} \right) \leq \ln \left( \frac{P(f_0) + h}{P(f_0) - h} \right) \]  

for all \( t \geq iT \).

(A.9)

Using (A.7) and (A.9) we obtain the desired inequality \( |u_i| \leq \delta \) for all \( i \geq j \). Also, our assumptions on \( K \) and \( f_0 \) ensure that \( P(f_0) > 0 \).

Next, using (47) and (A.9), writing

\[ \frac{P(f_0) + h}{P(f_0) - h} = 1 + \frac{2h}{h - P(f_0)}, \]

(A.10)

and using the inequality \( \ln(1 + x) \leq x \) for all \( x \geq 0 \), we obtain:

\[ |u_i| \leq \frac{2h}{P(f_0) - h} \leq \frac{2h}{P(f_0) - \frac{\exp(\delta)-1}{\exp(\delta)}} P(f_0) = \frac{h(\exp(\delta)+1)}{P(f_0)} \]

and

\[ \ln \left( \frac{P(f_0) + \int_0^1 g(a) \phi(t-a) \, da}{P(f_0) + \int_0^1 g(a) \phi(iT-a) \, da} \right) \leq \frac{h(\exp(\delta)+1)}{P(f_0)} \]

for all \( t \geq iT \). (A.11)
Since $h = K^* \| f_0 \|_1 \exp(-\varepsilon (i T - A))$, we obtain

$$|u_i| \leq \frac{K^* \| f_0 \|_1 (\exp(\delta)+1) \exp(\varepsilon A)}{P(f_0)} \exp(-\varepsilon i T) \text{ for all } i \geq j \text{ and }$$

(A.12)

$$\left| \ln \left( \frac{P(f_0) + \int_{f_0}^1 g(a) \phi(t-a) \, da}{P(f_0) + \int f_0^1 g(a) \phi(i T - a) \, da} \right) \right| \leq \frac{K^* \| f_0 \|_1 (\exp(\delta)+1) \exp(\varepsilon A)}{P(f_0)} \exp (-\varepsilon i T)$$

(A.13)

for all $i \geq j$ and $t \geq i T$. This completes the proof of Claim 4.

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