A revisit of circular hole problem by the equivalent inclusion method

R Zhang¹, Y H An¹, P Li¹, X N Zhang¹, and X Q Jin¹,²

¹State Key laboratory of Mechanical Transmissions, Chongqing University, Chongqing 400030, China
²College of Aerospace Engineering, Chongqing University, Chongqing, 400030, China

Corresponding Author’s E-mail: jinxq@cqu.edu.cn

Abstract. Stress and displacement analysis of an infinite extended isotropic elastic plane with a circular cavity is a classical elasticity problem. To solve this kind of problem, finite element method (FEM) and complex variable function are often used which have their pros and cons to some degree. In classical textbooks of elasticity, the closed-form solutions are usually given for the stresses, but the displacement solutions are seldom listed. In this paper, we revisit the circular hole problem via the equivalent inclusion method (EIM), and attempt to derive both the stress and displacement solutions simultaneously under the combined tensile and shear loading. With the knowledge of Eshelby’s inclusion in two dimensional and EIM, the solutions to be sought may be constructed in a straightforward and elegant manner. Several examples are examined to validate the present solutions, demonstrating that the EIM may provide a more concise and convenient way to solve the circular cavity problem.

1. Introduction

Structures with circular holes are very common in practical engineering, covering a lot of fields such as architectural engineering, water supply and drainage, aerospace materials, complex mechanical construction and biomedical materials. The corresponding elasticity solutions of the stress and displacement are essential to their engineering design. It is widely known that the cavities or pores in structure may cause crack or local stress concentration under external load, leading to severe deterioration of the mechanical properties of the engineering materials. The problem is also of theoretical interest, because studies on stress and displacement solutions of pore structure might be a corner-stone for other complex elasticity problems, such as notched circular openings in rocks, an infinite wedge with a circular cavity, elastic half plane and composite plate, micromechanical research of ceramic strength [1–4].

Generally, the stress and displacement in the vicinity of the cavities may exhibit strong effect of concentration, which are usually solved by either numerical or analytical method. The finite element method (FEM) is most widely used in the numerical method. FEM is powerful and cost-efficient; however, it has to solve the problem case by case, and tends to be difficult to achieve a systematic and exhaustive parametric study for design guidelines. Moreover, accurate determination of the concentration effects usually requires extremely fine meshes in the vicinity of the geometric discontinuities, such as pores or cracks. Sometimes may even cause inaccurate simulation of real
situation without initial knowledge. The complex variable function method is often used in analytical solution. Although this method can formulate the stresses or displacements effectively, it often requires profound knowledge of complex variable function, complicated calculations and well mathematical basis. This could be a challenge for general engineering and technical person.

An infinite plate containing a circular hole is a classical elasticity problem, and has been well-documented in many textbooks. However, most of the articles only concern about the stress solution, while the displacement results are less noted. This paper attempts to solve the circular hole problem by the equivalent inclusion method (EIM). A merit of the current work is that both the stresses and displacements are derived simultaneously in closed-form, facilitating a convenient and comprehensive reference for many engineering designs in practice.

2. Formulation

The equivalent inclusion method is a well-known method and has a significant influence on the development of micromechanics [5, 6]. In 1957, Eshelby [7] first proposed that perturbed field due to the presence of inhomogeneity can be simulated by the disturbance field of the appropriately distributed equivalent eigenstrain in an inclusion. This constitutes the essence of the well-known Eshelby’s equivalent inclusion method. When applying EIM to 2D circular cavity problem, the corresponding formulation may be greatly simplified since the elastic moduli of the inhomogeneity becomes zero, i.e. \( C_{ijkl}^* = 0 \) (Figure. 1).

![Figure 1. The inhomogeneity problem](image)

Through the construction of stress equivalence, the homogeneous part \( \sigma_{ij}^0 \) Figure. 2(a) plus the inclusion part Figure. 2(b) in form of stress Eshelby tensor i.e. \( T_{ijkl}^* \varepsilon_{ij}^* \) based on our previous work [8]. Accordingly, the EIM of the circular cavity may be represented as:

\[
\sigma_{ij}^0 + T_{ijkl}^* \varepsilon_{ij}^* = 0
\]  

(1)

where \( \sigma_{ij}^0 \) is the uniform load applied at remote, also representing the homogeneous material solution in the absence of the cavity [9]. The influence coefficients \( T_{ijkl} \) are termed as the Eshelby tensor for interior stresses [10]. It is noted that the present EIM formulation are established for plane stress; however, the final results may be used to interpret the plane strain case as well [10].
2.1. Plane circular inclusion solution

For the current circular inclusion solution, both stresses and displacements can be derived from a degenerate case of a two-dimensional elliptical inclusion, by letting $a = b$, (cf. [8]). The detailed formulae for the stress and displacement results are given as follows in sequel.

2.1.1. Interior stress field solution. Two independent elastic constants, i.e. shear modulus, $\mu$, and the Kolosov constant $\kappa$, are chosen with $\mu = E / 2(1 + \nu)$, and for plane stress

$$\kappa = \frac{3 - \nu}{1 + \nu}$$ (2)

For any points $(x, y)$ inside the circular inclusion, i.e., $x^2 + y^2 < a^2$, where $a$ is the radius of the circular hole, the eigenstress components are

$$[\sigma'] = \begin{bmatrix} \sigma_x' \\ \sigma_y' \\ \tau_{xy}' \end{bmatrix} = [T][\epsilon'] = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_x' \\ \epsilon_y' \\ \gamma_{xy}' \end{bmatrix}$$ (3)

2.1.2. Exterior stress field solution. Although the exterior solution is more complex, it can also be expressed in explicit closed-form and the results are concisely listed here in detail. First of all, for any point $p(x, y)$ located outside the circle, i.e., $x^2 + y^2 > a^2$, the stress components of the exterior field are listed as follows in which $r = \left(x^2 + y^2\right)^{1/2}$,

$$[\sigma] = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{4\mu a^2}{(\kappa + 1)r^2} \begin{bmatrix} H_{11} & H_{12} & H_{14} \\ H_{21} & H_{22} & H_{24} \\ H_{41} & H_{42} & H_{44} \end{bmatrix} \begin{bmatrix} \epsilon_x' \\ \epsilon_y' \\ \gamma_{xy}' \end{bmatrix}$$ (4)

where,
\[ H_{14} = H_{41} = \sin \theta \cos \theta \left[ 1 - \frac{3a^2}{r^2} + \left( \frac{6a^2}{r^2} - 4 \right) \cos^2 \theta \right] \]
\[ H_{24} = H_{42} = \sin \theta \cos \theta \left[ 1 - \frac{3a^2}{r^2} + \left( \frac{6a^2}{r^2} - 4 \right) \sin^2 \theta \right] \]
\[ H_{11} = \frac{2 + 3 \rho^2}{4} + \cos^2 \theta \left[ 2 - \frac{6a^2}{r^2} + \left( \frac{6a^2}{r^2} - 4 \right) \cos^2 \theta \right] \]
\[ H_{22} = \frac{2 + 3 \rho^2}{4} + \sin^2 \theta \left[ 2 - \frac{6a^2}{r^2} + \left( \frac{6a^2}{r^2} - 4 \right) \sin^2 \theta \right] \]
\[ H_{12} = H_{21} = H_{44} = \frac{1}{2} - \frac{3a^2}{4r^2} + \left( \frac{6a^2}{r^2} - 4 \right) \sin^2 \theta \cos^2 \theta \]

Noting that all the \( H_{ij} \), \( (i, j = 1, 2, 3, 4) \), in Eq. (5) are independent of the elastic constants and the \([H]\) matrices are symmetric in Eq. (4). The normal and the shear components are decoupled in the interior field in contrast to the exterior field. The strain solution in exterior field may be obtained directly from the stress field through Hooke's law, and is therefore omitted here, but the interested readers are referred to Ref. [9] for detailed expressions.

2.2. The Displacement Eshelby Tensor

2.2.1. Exterior displacement field solution. The Voigt notation [10] is applied to represent the matrix form tensorial equations, e.g., \([u] = [u_1 \quad u_2 \quad u_3]^T\), \([\varepsilon^*] = [\varepsilon_{11}^* \quad \varepsilon_{22}^* \quad \varepsilon_{33}^* \quad 2\varepsilon_{12}^*]^T\), where the superscript \(T\) indicates the transpose.

\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
  u_3 \\
\end{bmatrix} =
\begin{bmatrix}
  W_{111} & W_{112} & W_{113} & W_{114} \\
  W_{211} & W_{222} & W_{223} & W_{224} \\
  W_{311} & W_{322} & W_{333} & W_{334} \\
  \end{bmatrix}
\begin{bmatrix}
  \varepsilon_{11}^* \\
  \varepsilon_{22}^* \\
  \varepsilon_{33}^* \\
  2\varepsilon_{12}^* \\
\end{bmatrix}
\]

in which,

\[
W_{111}(x) = \frac{2x}{\kappa + 1} \left[ (\kappa - 1) \frac{a^2}{4r^2} + \frac{3a^4}{4r^2} + \frac{a^2}{r^2} \left( 1 - \frac{a^2}{r^2} \right) \cos^2 \theta \right]
\]
\[
W_{112}(x) = \frac{2}{\kappa + 1} \left\{ \left[ (\kappa - 1) \frac{a^2}{4r^2} + \frac{a^4}{4r^2} \right] x + \left[ \frac{a^2}{r^2} \left( 1 - \frac{a^2}{r^2} \right) \sin \theta \cos \theta \right] \right\}
\]
\[
W_{122}(x) = \frac{2x}{\kappa + 1} \left[ (1 - \kappa) \frac{a^2}{4r^2} + \frac{a^4}{4r^2} + \frac{a^2}{r^2} \left( 1 - \frac{a^2}{r^2} \right) \sin^2 \theta \right]
\]
\[
W_{133}(x) = \frac{3 - \kappa}{2(\kappa + 1)} \frac{a^2}{r^2} x
\]

2.2.2. Interior displacement field solution. A benefit of the displacement solution is that the interior field solution may be immediately derived from the exterior one in a way of letting \( \lambda = 0 \) and
dropping the trigonometric terms. To be specific, the displacement Eshelby tensor inside the inclusion can be expressed as

\[
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} = \begin{bmatrix}
  W_{111} & W_{122} & W_{133} & W_{112} \\
  W_{211} & W_{222} & W_{233} & W_{212}
\end{bmatrix} \begin{bmatrix}
  \varepsilon_{11}^* \\
  \varepsilon_{22}^* \\
  \varepsilon_{33}^* \\
  2\varepsilon_{12}^*
\end{bmatrix}
\]

in which,

\[
\begin{align*}
  w_{111}(x) &= \frac{2 + \kappa}{2(\kappa+1)} x, & w_{222}(x) &= \frac{2 + \kappa}{2(\kappa+1)} y, & w_{112}(x) &= \frac{2 - \kappa}{2(\kappa+1)} x, & w_{211}(x) &= \frac{2 - \kappa}{2(\kappa+1)} y \\
  w_{133}(x) &= \frac{3 - \kappa}{2(\kappa+1)} x, & w_{233}(x) &= \frac{3 - \kappa}{2(\kappa+1)} y, & w_{112}(x) &= \frac{\kappa}{2(\kappa+1)} y, & w_{212}(x) &= \frac{\kappa}{2(\kappa+1)} x
\end{align*}
\]

3. EIM in circular cavity

3.1. Implementation and stress solution

For an infinite matrix in a state of uniaxial tensile loading together with a shear loading, the remote stresses are recorded in matrix form as \([\sigma_x^0, 0, \tau_{xy}^0]\). From Eqs. (1) and (3), the EIM is established as

\[
\begin{bmatrix}
  \sigma_x^0 \\
  0 \\
  \tau_{xy}^0
\end{bmatrix} - \begin{bmatrix}
  \frac{3}{\kappa_i + 1} & 1 & 0 & \varepsilon_{xy}^* \\
  1 & 3 & 0 & \varepsilon_{xy}^* \\
  0 & 0 & 1 & \gamma_{xy}^*
\end{bmatrix} = 0
\]

Note that the normal and shear eigenstrain components are decoupled. Consequently, the solution of the unknown equivalent eigenstrains in Eq. (10) is obtained after some algebraic simplification:

\[
\begin{align*}
  \varepsilon_x^* &= \frac{\kappa_i + 1}{8\mu_i} \begin{bmatrix}
  3 & 1 & 0
\end{bmatrix} \begin{bmatrix}
  \sigma_x^0 \\
  0 \\
  \tau_{xy}^0
\end{bmatrix} \\
  \gamma_{xy}^* &= \frac{\kappa_i + 1}{\mu_i} \tau_{xy}^0
\end{align*}
\]

The stress solution outside the circular hole may be expressed as

\[
[\sigma] = [\sigma^0] + [H][\varepsilon^*]
\]

Substituting Eq. (4) into (12) and after transforming it into the polar coordinates, one obtains the stress components as follows:

\[
\begin{align*}
  \sigma_r &= \frac{\sigma_x^0}{2} \left[ 1 - \frac{a^2}{r^2} \right] \left[ 1 + \cos 2\theta \left( 1 - \frac{3a^2}{r^2} \right) \right] + \tau_{xy}^0 \left( \frac{3a^2}{r^2} - 1 \right) \left( \frac{a^2}{r^2} - 1 \right) \sin 2\theta \\
  \sigma_\theta &= \frac{\sigma_x^0}{2} \left[ 1 + \frac{a^2}{r^2} - \left( \frac{3a^4}{r^4} + 1 \right) \cos 2\theta \right] - \tau_{xy}^0 \left( \frac{3a^4}{r^4} + 1 \right) \sin 2\theta \\
  \tau_{r\theta} &= \frac{3a^4 - 2a^2 r^2 - r^4}{2r^4} \left( \sigma_x^0 \sin 2\theta - 2\tau_{xy}^0 \cos 2\theta \right)
\end{align*}
\]
3.2. Displacement solution

The homogeneous strain field may be obtained by the given stress \( \sigma_0, 0, \tau_0 \) through Hooke’s law directly

\[
\varepsilon_x^0 = \frac{K_1 + 1}{8\mu_i} \sigma^0, \quad \varepsilon_y^0 = \frac{K_1 - 3}{8\mu_i} \sigma^0, \quad \gamma_{xy}^0 = \frac{1}{\mu_i} \tau^0
\]  

Hence, the displacement state of this part is

\[
u_x^0 = \sigma_x^0 \frac{K_1 + 1}{8\mu_i} x + \tau_{xy}^0 \frac{1}{2\mu_i} y
\nu_y^0 = \sigma_x^0 \frac{K_1 - 3}{8\mu_i} y + \tau_{xy}^0 \frac{1}{2\mu_i} x
\]  

The inclusion solution of the displacement may be obtained through Eqs. (6) and (11)

\[
u_x^* = -\frac{\rho^2}{4\mu_i} \sigma^0 x \left[ 2 \cos 2\theta (\rho^2 - 1) - \rho^2 - \kappa_i \right] + 4\tau^0 x \sin 2\theta (\rho^2 - 1) + 2\tau^0 y \left( 1 - \rho^2 - \kappa_i \right)
\nu_y^* = -\frac{\rho^2}{4\mu_i} \sigma^0 y \left[ 2 \cos 2\theta (\rho^2 - 1) + \rho^2 + \kappa_i - 2 \right] + 4\tau^0 y \sin 2\theta (\rho^2 - 1) + 2\tau^0 x \left( 1 - \rho^2 - \kappa_i \right)
\]  

On the grounds of EIM, the resultant displacement outside the circular hole may be derived by superposing the inclusion solution (16) with the homogeneous material solution Eq. (15). After converting the final result into polar coordinates, one obtains

\[
u_x = \frac{1}{8\mu_i} \sigma_x^0 \left[ (\kappa_i - 1) r^2 + 2a^2 \right] + \frac{1}{4\mu_i} r \sigma_{xy}^0 \cos 2\theta + 2\tau_{xy}^0 \sin 2\theta \left[ r^4 - a^4 + a^2 r^2 (\kappa_i + 1) \right]
\nu_y = \frac{1}{4\mu_i} r^3 \left[ -\sigma_x^0 \sin 2\theta + 2\tau_{xy}^0 \cos 2\theta \right] \left[ r^4 + a^4 + a^2 r^2 (\kappa_i - 1) \right]
\]  

4. Results and discussions

Although the textbook [11] does not explicitly list the final solutions of the present elasticity problem, the exact solution may be derived from the tables given in [11], and the details are omitted here. On the basis of the closed-form solution in this paper, a double precision code is programmed by FORTRAN language to examine the variation trend along some target lines and verify the validity of the present work. In the benchmark study, \( v = 0.3 \) and all the three target lines are chosen along the radial direction, with \( \theta \) equals to \( 0, \pi/4, \) and \( \pi/2 \), respectively (Figure. 3).

The center of the circular hole is placed at the origin of the coordinate system, as shown in Figure. 3 Contrast between normalized stress and displacement along target lines are demonstrated in Figures. 4 and 5 correspondently. It is worth noting that the solutions of this paper are in perfect consistency with the exact solution. As can be seen from Figure. 4, stress concentration occurs at the boundary of the circular hole, when the points are far away from the boundary of the hole, the magnitude of stress components all decrease gradually to the applied stresses at remote. For displacement contrast in Figure. 5, we notice that the variations of the displacement along different radial directions show somewhat distinct patterns.
5. Conclusion
The present study revisits the circular cavity problem in an infinite elastic plane, and the equivalent inclusion method (EIM) is employed to obtain the elastic field solution under the combined tensile and shear loading. The disturbance solution due to the plane circular inclusion may be derived from a degenerate 2D case of an elliptical inclusion. With the help of the equivalent inclusion method, both the displacement and stress solutions are derived in closed-form. Benchmark examples are presented to validate the current solutions. The analytical solutions in this work may provide a convenient reference for engineering design.

6. References
[1] Exadaktylos G E, Liolios P A, and Stavropoulou M C 2003 Int. J. Solids Struct 40 1165-1187.
[2] Liu G, Ji B, Chen H, Liu D 2009 J. Appl. Mech 76 061008-1-9.
[3] Lu A, Zeng X, Xu Z 2016 Int. J. Rock Mech 89 34-42.
[4] Toubal L, Karama M, Lorrain B 2005 Compos. Struc 68 31-36.
[5] Jarali C S, Madhusudan M, Vidayashankar S, Raja S 2018 Compos. Part B-Eng 152 17-30.
[6] Lee J, Nozaki H, Lee H 2018 Compo. Interfaces 25 221-249.
[7] Eshelby J D, 1957 Proc. R. Soc 241 376-396.
[8] Jin X, Wang Z, Zhou Q, Keer L M, Wang Q, 2014 J. Elast 114 1-18.
[9] Mura T 1982 *Micromechanics of defects in solids. 2nd ed* (Dordrecht: Kluwer Academic Publishers).

[10] Jin X, Lyu D, Zhang X, Zhou Q, Wang Q, Keer L M, 2016 *J. Appl Mech* 83 121010-1-12.

[11] Barber J R 1992 *Elasticity* (Netherlands: Springer Dordrecht) p 749.

**Acknowledgments**

This work is supported by Fundamental Research Funds for the Central Universities (2018CDYJSY0055). The authors are grateful to the Graduate Research and Innovation Foundation of Chongqing, China (Grant Nos. CYB18020 and CYB17025). X.J. would acknowledge the support from Fundamental Research Funds for the Central Universities (No. 106112017CDI1328839), the National Natural Science Foundation of China (Grant Nos. 51475057 and 51875059), and the State Key Laboratory of Mechanical Transmissions through funding (SKLMT-ZZKT-2017M15).