ON DISCRETE SUBGROUPS OF AUTOMORPHISM OF $\mathbb{P}^2_\mathbb{C}$

ANGEL CANO

Abstract. In this work we study discrete subgroups $\Gamma$ of $\text{PSL}_2(\mathbb{C})$ and some of their basic properties. We show that if there is a region of “discontinuity” of the action of $\Gamma$ on $\mathbb{P}^2_\mathbb{C}$ which contains $\Gamma$-cocompact components, then the group is either elementary, affine or fuchsian. Moreover, there is a largest open set on which $\Gamma$ acts properly discontinuously.

Introduction

The classical kleinian groups are discrete subgroups of $\text{PSL}_2(\mathbb{C})$ that act on the Riemann sphere $S^2 \cong \mathbb{P}^1_\mathbb{C}$ with non-empty region of discontinuity. Their study has played a major role in several areas of mathematics for a long time. More recently, there has been interest in studying generalizations of these groups to higher dimensional projective spaces. Of particular interest are the results of W. Goldman, R. Schwartz, M. Kapovich, N. Gusevskii and others about discrete subgroups of $\text{PU}(2,1) \subset \text{PSL}_3(\mathbb{C})$. These are groups of automorphisms of $\mathbb{P}^2_\mathbb{C}$ that preserve an open ball, which serves as model for complex hyperbolic geometry, and so they are analogous to the classical fuchsian groups.

More generally, in [18] the authors introduce the concept of a complex Kleinian group, which means a discrete subgroup $\Gamma$ of some $\text{PSL}_{n+1}(\mathbb{C})$ acting on $\mathbb{P}^n_\mathbb{C}$ with non-empty region of discontinuity (in the sense of Kulkarni, see [9]), and in a couple of subsequent articles ([19], [20]) they study some interesting families of such groups acting on odd-dimensional projective spaces. Here we look at the case of groups acting on $\mathbb{P}^2_\mathbb{C}$, continuing the work begun in [2, 12, 13].

An important problem in the theory of higher dimensional Kleinian groups is to provide a “nice” definition of the limit set. One of the first attempts in this direction is due to R. S. Kulkarni in [9] (see also definition 1.1), where the author proposes a notion of limit set -that we denote $\Lambda_{Kul}(\Gamma)$-, which seems more appropriate for this setting than taking the complement of the cluster points of the orbits (see proposition 1.5). Yet, Kulkarni’s region of discontinuity $\Omega_{Kul}(\Gamma)$ -which is the complement of $\Lambda_{Kul}(\Gamma)$- still has few things that may produce certain discomfort. For example the limit set is not monotone (see remark 8.4), $\Omega_{Kul}(\Gamma)$ is not a maximal set where the action is properly discontinuously (see corollary 4.8) and the region of equicontinuity could be empty even in the case when $\Omega_{Kul}(\Gamma) \neq \emptyset$ (see remark 7.2). For this reason, in this article, we consider subgroups $\Gamma$ of $\text{PSL}_3(\mathbb{C})$ acting on $\mathbb{P}^2_\mathbb{C}$ in such a way that there is an invariant open set $\Omega$ which is non-empty, where $\Gamma$ acts properly discontinuously and where the quotient $\Omega/\Gamma$ contains at least one connected component which is compact. In this case $\Gamma$ will be called a Quasi Co-compact kleinian group over $\Omega$. Results of Inoue, Klingler, Kobayashi, Ochiai

1991 Mathematics Subject Classification. Primary 37F99, 32Q, 32M Secondary 30F40, 20H10, 57M60, 53C.
and others (see Theorem [1.8]) assert that for each compact complex surface which admits a projective structure, the holonomy is either fuchsian or affine and the image of the developing map is one of the following: $\mathbb{P}^2_\mathbb{C}$, $\mathbb{H}_2^2$, $\mathbb{C}^2$, $\mathbb{C}^2 - \{0\}$, $\mathbb{C} \times \mathbb{C}^*$, $\mathbb{C}^* \times \mathbb{C}^*$, $\mathbb{H} \times \mathbb{C}$, $D \times \mathbb{C}^*$, where $D$ is a hyperbolic domain of the extended complex plane $\mathbb{P}^1_\mathbb{C}$, and also provide a complete classification of the complex manifold that admit a projective structure. Here we translate such results to the case of quasi co-compact kleinian groups and show as a main result that for any of such groups they admit a largest open set on which the group acts properly discontinuously (see Theorems [0.1, 0.2, 0.4] and remark [0.5] below).

Given a discrete subgroup $\Gamma \subset PSL_3(\mathbb{C})$ we will say that $\Gamma$ is fuchsian if it is conjugate to a subgroup of $PU(2,1)$, and we say that $\Gamma$ is affine if it is conjugate to a subgroup of affine automorphisms of $\mathbb{C}^2$, i.e., a group of automorphisms of $\mathbb{P}^2_\mathbb{C}$ having an invariant line. We say that $\Gamma \subset PSL_3(\mathbb{C})$ is elementary over $\Omega$ if $\Gamma$ acts properly discontinuously on $\Omega$ and $\mathbb{P}^2_\mathbb{C} - \Omega$ is the union of a non-empty finite set of lines and a finite set of points. Also, $\Gamma$ is called controllable if there is a line $\ell$ and a point $p \notin \ell$ which are invariant under the action of $\Gamma$. The group $\Gamma |_{\ell} = G$ is called the control group and $K = \{ h \in \Gamma : h(x) = x \text{ for all } x \in \ell \}$ the kernel of $\Gamma$ (see subsection 4.3 for an example). Clearly controllable groups provide special kinds of affine groups. We prove:

**Theorem 0.1.** Let $\Gamma \leq PSL_3(\mathbb{C})$ be a quasi co-compact group over $\Omega$, then $\Gamma$ is elementary, fuchsian or affine.

Set (see also section [1]):

\[ \mathbb{C}^2 = \{ [z; w; 1] : z, w \in \mathbb{C} \}; \]

\[ Af(\mathbb{C}^2) = \{ g \in PSL_3(\mathbb{C}) : g(\mathbb{C}^2) = \mathbb{C}^2 \} \]

\[ Af_2 = \{ g \in PSL_3(\mathbb{C}) : g(\mathbb{C} \times \mathbb{C}^*) = \mathbb{C} \times \mathbb{C}^* \}; \quad Af_3 = \{ g \in PSL_3(\mathbb{C}) : g(\mathbb{C}^* \times \mathbb{C}^*) = \mathbb{C}^* \times \mathbb{C}^* \} \]

\[ \text{Sol}_2^4 = \left\{ \begin{pmatrix} \lambda & 0 & a \\ 0 & |\lambda|^{-2} & b \\ 0 & 0 & 1 \end{pmatrix} : (\lambda, a, b) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{R} \right\} \];

\[ \text{Sol}_1^3 = \left\{ \begin{pmatrix} \epsilon & a & b \\ 0 & \alpha & c \\ 0 & 0 & 1 \end{pmatrix} : \alpha, a, b, c \in \mathbb{R}, \alpha > 0, \epsilon = \pm 1 \right\} ; \]

\[ \text{Sol}_4^4 = \left\{ \begin{pmatrix} 1 & a & b + i \log \alpha \\ 0 & \alpha & c \\ 0 & 0 & 1 \end{pmatrix} : \alpha, a, b, c \in \mathbb{R}, \alpha > 0 \right\} \]

\[ A_1 = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} : (a, b) \in \mathbb{C}^* \times \mathbb{C} \right\} ; \quad A_2 = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} : (a, b) \in \mathbb{C}^* \times \mathbb{C} \right\} \]

Then we prove:

**Theorem 0.2.** Let $\Gamma \leq PSL_3(\mathbb{C})$ be a quasi co-compact and elementary group over $\Omega$ such that $\Omega/\Gamma$ is not a Hopf surface. Then $\Omega_{\text{Kul}}(\Gamma)$ is the largest open set on which $\Gamma$ acts properly discontinuously, in particular $\Omega$ is contained in $\Omega_{\text{Kul}}(\Gamma)$. Moreover, up to projective equivalence, one of the following assertions applies:

1. $\Omega_{\text{Kul}}(\Gamma) = \mathbb{C}^2$, $\Gamma$ is affine and a finite extension of a unipotent group. And $\Omega_{\text{Kul}}(\Gamma)/\Gamma$ is a finite covering (possibly ramified) of a surface biholomorphic to a complex torus or a primary Kodaira surface.
(2) $\Omega_{Kul}(\Gamma) = \mathbb{C} \times \mathbb{C}^*$, $\Gamma$ is a finite extension of a group isomorphic to $\mathbb{Z}^3 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ which is contained in $A_1$ or $A_2$. Moreover, $\Omega_{Kul}(\Gamma)/\Gamma$ is a finite covering (possibly ramified) of a surface biholomorphic to a complex torus.

(3) $\Omega_{Kul}(\Gamma) = \mathbb{C}^* \times \mathbb{C}^*$, $\Gamma$ is a finite extension of a group isomorphic to $\mathbb{Z}^2$ and where each element is a diagonal matrix. Moreover, $\Omega_{Kul}(\Gamma)/\Gamma$ is a finite covering (possibly ramified) of a surface biholomorphic to a complex torus.

Remark 0.3. If $\Omega/\Gamma$ is a Hopf surface then by Theorem 1.18 and corollary 4.14 it follows that $\Gamma$ is virtually a cyclic group generated by an affine contraction $\gamma$ and in such case the maximal domains of discontinuity are described by corollary 4.18.

Theorem 0.4. Let $\Gamma \leq PSL_3(\mathbb{C})$ be a quasi co-compact, affine and non-elementary group over $\Omega$, then $\Omega_{Kul}(\Gamma)$ is the largest open set on which $\Gamma$ acts properly discontinuously, in particular $\Omega$ is contained in $\Omega_{Kul}(\Gamma)$. Moreover, up to projective equivalence, one of the following assertions applies:

(1) $\Omega_{Kul}(\Gamma) = \mathbb{C} \times (\mathbb{H}^- \cup \mathbb{H}^+)$, $\Gamma$ is an extension of order at most 2 (i.e., the index $[\Gamma:Isot(\mathbb{C} \times \mathbb{H}, \Gamma)] \leq 2$) and $Isot(\mathbb{C} \times \mathbb{H}, \Gamma)$ is a torsion free group of $Sol_0^4$ or $Sol_1^4$ or $Sol_2^4$. In addition, $\Omega_{Kul}(\Gamma)/\Gamma$ is equal to $M$ or $M \sqcup M$ where $M = (\mathbb{C} \times \mathbb{H})/Isot(\mathbb{C} \times \mathbb{H}, \Gamma)$ is an Inoue Surface.

(2) $\Omega_{Kul}(\Gamma) = D \times \mathbb{C}^*$ where $D \subset \mathbb{P}_1^c$ is the discontinuity region of a quasi co-compact group of $Möb(\mathbb{C})$, $\Gamma$ is a controllable group with quasi co-compact control group and infinite kernel. In addition, $\Omega_{Kul}(\Gamma)/\Gamma = \bigsqcup_{i \in I} N_i$ where $I$ is at most countable, $N_i$ are orbifolds whose universal covering orbifold is biholomorphic to $\mathbb{H} \times \mathbb{C}$ and every compact connected component is a finite covering (possibly ramified) of an elliptic affine surface.

Remark 0.5. If $\Gamma \leq PSL_3(\mathbb{C})$ is a co-compact fuchsian group then by Theorem 1.14 one has that $\Omega_{Kul}(\Gamma) = \mathbb{H}^2_3$ is the largest open set on which $\Gamma$ acts properly discontinuously.

This paper is organized as follows: in section 4 we review some facts about group actions, projective geometry, geometric orbifolds and introduce some terminology and notations. In section 2 we provide a lemma that connects the notions of developing pairs and groups acting properly discontinuously over domains. In section 3 we show that the complement of the equicontinuity region and the limit set in the sense of Greenberg coincide for non-discrete groups of $PSL_2(\mathbb{C})$. The main reason to include this material here lies in the fact that the understanding of the dynamics of controllable groups (see subsection 4.3), is "reduced" to the study of certain Möbius groups acting on a projective line (see lemma 4.12), some of which could be non-discrete (see example 4.11); in these conditions, by looking at the equicontinuity region of such groups, it is possible to describe the equicontinuity domain of the corresponding controllable group.

The basic properties of groups acting properly discontinuously over domains are given in section 4. Using the results of the previous sections, we give in section 5 a complete proof of Theorem 0.1 our main result. Section 6 deals with some technical aspects on finite orbifold maps over Inoue and elliptic affine surfaces, to be used in the proofs of the consequences of Theorem 0.1 which are given in section 7.

Finally, in section 8 we construct an example of a kissing Schottky group which is not fuchsian, affine nor elementary. This shows that Theorem 0.1 fails if we drop
the condition of being quasi co-compact. We also give in this section an example of a quasi co-compact group which is elementary but not affine. In both cases we prove the corresponding statements by using results of the previous sections to study the limit sets and discontinuity regions of the corresponding groups.

1. Preliminaries

1.1. Projective Geometry. We recall that the complex projective plane \( \mathbb{P}^2 \) is \((\mathbb{C}^3 - \{0\})/\mathbb{C}^* \), where \( \mathbb{C}^* \) acts on \( \mathbb{C}^3 - \{0\} \) by the usual scalar multiplication. This is a compact connected complex 2-dimensional riemannian manifold, naturally equipped with the Fubini-Study metric.

Let \( [ \cdot ]_2 : \mathbb{C}^3 - \{0\} \rightarrow \mathbb{P}^2_\mathbb{C} \) be the quotient map. If \( \beta = \{ e_1, e_2, e_3 \} \) is the standard basis of \( \mathbb{C}^3 \), we will write \([e_j]_2 = e_j\) and if \( w = (w_1, w_2, w_3) \in \mathbb{C}^3 - \{0\} \) then we will write \([w]_2 = [w_1; w_2; w_3]\). Also, \( \ell \subset \mathbb{P}^2_\mathbb{C} \) is said to be a complex line if \([\ell]_2^{-1} \cup \{0\} \) is a complex linear subspace of dimension 2. Given \( p, q \in \mathbb{P}^2_\mathbb{C} \) distinct points, there is a unique complex line passing through \( p \) and \( q \), such line will be denoted by \( \overline{pq} \). Moreover, if \( \ell_1, \ell_2 \) are different complex lines then it is verified that \( \ell_1 \cap \ell_2 \) contains exactly one point.

Taking \( \mathbb{H}^2_\mathbb{C} = \{ [a; b; c] \in \mathbb{P}^2_\mathbb{C} : |a| + |b| < |c| \} \), it is not hard to show that \( \mathbb{H}^2_\mathbb{C} \) is biholomorphic to the unitary ball in \( \mathbb{C}^2 \), \( \partial \mathbb{H}^2_\mathbb{C} \) is diffeomorphic to the 3-sphere and for each point in \( \partial \mathbb{H}^2_\mathbb{C} \) there is exactly one complex line tangent to \( \partial \mathbb{H}^2_\mathbb{C} \) passing through \( p \).

Consider the action of \( \mathbb{Z}_3 \) (viewed as the cubic roots of the unity) on \( SL_3(\mathbb{C}) \) given by the usual scalar multiplication, then \( PSL_3(\mathbb{C}) = SL_3(\mathbb{C})/\mathbb{Z}_3 \) is a Lie group whose elements are called projective transformations. Let \( [[\cdot]]_2 : SL_3(\mathbb{C}) \rightarrow PSL_3(\mathbb{C}) \) be the quotient map, \( \gamma \in PSL_3(\mathbb{C}) \) and \( \tilde{\gamma} \in GL_3(\mathbb{C}) \), we will say that \( \tilde{\gamma} \) is a lift of \( \gamma \) if \( [[\text{Det}(\gamma)^{-1/3\tilde{\gamma}}]]_2 = \gamma \). Also, \( PSL_3(\mathbb{C}) \) is a Lie group that acts transitively, effectively and by biholomorphisms on \( \mathbb{P}^2_\mathbb{C} \) by \([\gamma]_2([w]_2) = [\gamma(w)]_2\), where \( w \in \mathbb{C}^3 - \{0\} \) and \( \gamma \in SL_3(\mathbb{C}) \). Also it is possible to show that projective transformations take complex lines into complex lines.

1.2. Group Actions. Let \( G \) be a group acting on a space \( X \), \( g \in G \) and \( A \subset X \) a subset. We define \( \text{Isot}(A, G) = \{ g \in G : g(A) = A \} \), by \( GA \) we will denote the orbit of \( A \) under \( G \) and by \( \text{Fix}(g) \) the set of fixed points of \( g \). We will say that the action of \( G \) is locally faithful, if whenever \( f, g \in G \) agree in some open set, then it is verified that \( f = g \) on \( X \).

**Definition 1.1.** Let \( \Gamma \leq PSL_3(\mathbb{C}) \) be a subgroup. We define (following Kulkarni, see [2]):

1. \( L_0(\Gamma) \) as the closure of the points in \( \mathbb{P}^2_\mathbb{C} \) with infinite isotropy group.
2. \( L_1(\Gamma) \) as the closure of the set of cluster points of \( \Gamma z \) where \( z \) runs over \( \mathbb{P}^2_\mathbb{C} - L_0(\Gamma) \). Recall that \( q \) is a cluster point for \( \Gamma K \), where \( K \subset \mathbb{P}^2_\mathbb{C} \) is a non-empty set, if there is a sequence \( (k_m)_{m \in \mathbb{N}} \subset K \) and a sequence of distinct elements \( (\gamma_m)_{m \in \mathbb{N}} \subset \Gamma \) such that \( \gamma_m(k_m) \xrightarrow{m \to \infty} q \).
3. \( L_2(\Gamma) \) as the closure of cluster points of \( \Gamma K \) where \( K \) runs over all the compact sets in \( \mathbb{P}^2_\mathbb{C} - (L_0(\Gamma) \cup L_1(\Gamma)) \).
4. The Limit Set in the sense of Kulkarni for \( \Gamma \) is defined as:
   \[ \Lambda_{Kul}(\Gamma) = L_0(\Gamma) \cup L_1(\Gamma) \cup L_2(\Gamma). \]
(5) The Discontinuity Region in the sense of Kulkarni of $\Gamma$ is defined as:

$$\Omega_{Kul}(\Gamma) = P^2_C - \Lambda_{Kul}(\Gamma).$$

We will say that $\Gamma$ is a Complex Kleinian Group if $\Omega_{Kul}(\Gamma) \neq \emptyset$, see [13].

One has the following three results that we will use later.

**Theorem 1.2** (J. P. Navarrete, see [13]). Let $\gamma \in PSL_3(\mathbb{C})$ and $\tilde{\gamma} \in SL_3(\mathbb{C})$ be a lift of $\gamma$. The limit set in the sense of Kulkarni for the cyclic group generated by $\gamma$ (denoted $<\gamma>$), in terms of the Jordan’s normal form of $\tilde{\gamma}$, is given by:

| Normal Form of $\tilde{\gamma}$ | Condition over the $\lambda_i$’s | $L_0(<\gamma>$) | $L_1(<\gamma>$) | $L_2(<\gamma>$) |
|-------------------------------|---------------------------------|----------------|----------------|----------------|
| $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ | $\lambda_1^0 = \lambda_2^0 = \lambda_3^0 = 1$ for some $n.$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ | | | \begin{align*}
&= e_1, e_2, e_3: \\
&= e_1, e_2, e_3 \\
&= e_1, e_2, e_3
\end{align*} |
| | | | \begin{align*}
&= \tilde{e}_1, e_2, e_3: \\
&= \tilde{e}_1, e_2, e_3 \\
&= \tilde{e}_1, e_2, e_3
\end{align*} |
| | | | \begin{align*}
&= \tilde{e}_1, e_2, e_3: \\
&= \tilde{e}_1, e_2, e_3 \\
&= \tilde{e}_1, e_2, e_3
\end{align*} |

We recall that the limit set of $\Gamma \leq PU(2,1)$ in the sense of Chen-Greenberg, denoted $\Lambda_{CG}(\Gamma)$, is the set of cluster points of an orbit $\Gamma z$, where $z$ is any point in $\mathbb{H}_2^2$. And it satisfies (see [3]):

**Theorem 1.3** (Chen-Greenberg). Let $\Gamma \leq PU(2,1)$ be a discrete group, then it is verified that:

1. Let $Card(A)$ denote the number of elements in a set $A$, then $Card(\Lambda_{CG}(\Gamma))$ is $0$, $1$, $2$, or $\infty$.
2. If $Card(\Lambda_{CG}(\Gamma)) = \infty$, then $\overline{\Gamma z} = \Lambda_{CG}(\Gamma)$ for every $z \in \Lambda_{CG}(\Gamma)$.

One has the following Theorem of [12].

**Theorem 1.4** (J. P. Navarrete). Let $\Gamma \leq PU(2,1)$ be a discrete group. One has:

1. $\Lambda_{Kul}(\Gamma) = \bigcup_{p \in \Lambda_{CG}(\Gamma)} T_p$, where $T_p$ is the line tangent to $\mathbb{H}_2^2$ at $p$.
2. If $Card(\Lambda_{CG}(\Gamma)) = \infty$, then $\Omega_{Kul}(\Gamma)$ is the largest open set on which $\Gamma$ acts properly discontinuously.

**Proposition 1.5.** Let $\Gamma$ be a complex kleinian group. The following properties are valid (see [13]):
Moreover, \( x \) map \( f \) component \( V \)

\( \) below commutes \( \phi \) whenever for a manifold, these charts must satisfy a certain compatibility condition. Namely, \( U \) precisely structure) where \( \Gamma \)

\( \) Hausdorff space \( \phi \) consists of an open set \( \tilde{x} \subset X, G \) and \( y \)

\( \) unique group morphism \( X, G \)

\( \) was introduced by W. P. Thurston. Let \( G \) be a Lie group acting effectively, transitively and locally faithfully on a smooth manifold \( X \). An \((X, G)\)-orbifold \( O \) is a Hausdorff space \( X_O \) with a countable basis and some additional structure. More precisely \( O \) is covered by an atlas \( \{ \tilde{U}_i, \Gamma_i, \phi_i, U_i \}_{i \in I} \) of folding charts each of which consist of an open set \( \tilde{U}_i \subset X \), a finite group \( \Gamma_i \leq G \) that leaves invariant \( \tilde{U}_i \), an open set \( U_i \subset O \) and a homeomorphism \( \phi_i : U_i \to \tilde{U}_i/\Gamma_i \) called folding map. As for a manifold, these charts must satisfy a certain compatibility condition. Namely, whenever \( U_i \subset U_j \), there is a group morphism \( f_{ij} : \Gamma_i \hookrightarrow \Gamma_j \) and \( \hat{\phi}_{ij} \in G \) with \( \hat{\phi}_{ij}(U_i) \subset U_j \) and \( \hat{\phi}_{ij}(\gamma x) = f_{ij}(\gamma)\phi_{ij}(x) \) for every \( \gamma \in \Gamma_i \), such that the diagram below commutes

\[
\begin{array}{ccc}
  \tilde{U}_i & \xrightarrow{\phi_i^{-1}} & U_i \\
  \downarrow{\hat{\phi}_{ij}} & & \downarrow{\phi_{ij}} \\
  \tilde{U}_j & \xrightarrow{f_{ij}(\Gamma_i)} & U_j/\Gamma_j \\
  & \downarrow{\phi_{ij}^{-1}} & \\
  & & U_j.
\end{array}
\]

A point \( y \in X_O \) is called singular if there is a folding map of \( x \) say \( \phi : U_i \to \tilde{U}_i/\Gamma_i \) (that is \( x \in U_i \)) such that there is \( z \in \tilde{U}_i \) that verify \( \Gamma_i z = \phi(x) \) and \( \text{Isot}(y, \Gamma_i) \) is non trivial. The set \( \Sigma_O = \{ y \in X_O : y \text{ is a singular point} \} \) is called the Singular Locus of \( O \). We shall say that \( O \) is an \((X, G)\)-manifold if \( \Sigma_O = \emptyset \).

Let \( M, N \) be two \((X, G)\)-orbifolds. A continuous function \( f : X_M \to X_N \) is called an \((X, G)\)-map if for each point \( y \in N \), a folding map for \( x, \phi_i : U_i \to \tilde{U}_i/\Gamma_i \), and \( y \in f^{-1}(x) \), there is a folding map of \( y, \phi_j : U_j \to \tilde{U}_j/\Gamma_j \), and \( \psi \in G \) with \( \psi(U_j) \subset U_i \), inducing \( f \) equivariant with respect to a morphism \( \psi : \Gamma_j \to \Gamma_i \). The map \( f \) is called an \((X, G)\)-equivalence if \( f \) is bijective and \( f, f^{-1} \) are \((X, G)\)-maps. Moreover, \( f \) is called an \((X, G)\)-covering orbifold map if \( f \) is a surjective \((X, G)\)-map such that each point \( x \in N \) has a folding map \( \phi_j : U_j \to \tilde{U}_j/\Gamma_j \) so that each component \( V_i \) of \( f^{-1}U_j \) has an homeomorphism \( \phi_i : V_i \to \tilde{U}_j/\Gamma_i \) (in the orbifold structure) where \( \Gamma_i \leq \Gamma_j \). We require that the quotient map \( U_j \to V_i \) induced by \( \phi_i \) composed with \( f \) should be the quotient map \( \tilde{U}_j \to U_j \) induced by \( \phi \). Also, if \( p : M \to N \) is a covering orbifold map with \( M \) a manifold, we will say that \( p \) is a ramified (respectively unramified, finite) covering if \( \Sigma_N \neq \emptyset \) (respectively \( \Sigma_N = \emptyset \), \( p^{-1}(y) \) is finite for every \( y \in N \)).

If \( M \) is a simply connected \((X, G)\)-manifold, then it is possible to show that there is an \((X, G)\)-map \( D : M \to X \). Moreover, if \( D : M \to X \) is any other \((X, G)\)-map, then there is a unique \( g \in G \) such that \( D = g \circ D \). Also one has that there is a unique group morphism \( \mathcal{H}_D : \text{Aut}(X_G)(M) \to G \), where \( \text{Aut}(X_G)(M) \) denotes the group of \((X, G)\)-equivalences of \( M \), that verify \( D \circ g = \mathcal{H}_D(g) \circ D \). Trivially, if \( D \) is another \((X, G)\)-map, then there is a unique \( g \in G \) such that \( D = g \circ D \) and
\( \mathcal{H}_D = g \circ \mathcal{H}_D \circ g^{-1} \). Every \((X,G)\)-map \( D : M \to X \) will be called developing map and the respective \( \mathcal{H}_D \) the holonomy morphism associated to \( D \).

Given an \((X,G)\)-orbifold \( M \), we will say that \( M \) is a good orbifold (respectively a very good orbifold) if there exist a covering orbifold map \( p : \tilde{M} \to M \) such that \( \tilde{M} \) is an \((X,G)\)-manifold (respectively a compact \((X,G)\)-manifold).

**Theorem 1.6** (Thurston [4]). If \( M \) is an \((X,G)\)-orbifold then \( M \) is a good orbifold. Moreover, there is a simply connected manifold \( \tilde{M} \) and a \((X,G)\)-covering orbifold map, with the following property: If \( q : O \to M \) is another \((X,G)\)-covering orbifold map, \( * \in X_M \), \( *' \in X_{\tilde{M}} \) and \( z \in X_O \) satisfy \( q(z) = p(*') = * \), then there is a \((X,G)\)-covering orbifold map \( p' : \tilde{M} \to O \) such that \( p = q \circ p' \) and \( p'(*)' = z \).

It is natural to call \( \tilde{M} \) the universal covering orbifold of \( M \), \( p \) the universal covering orbifold map and \( \pi_1^{Orb}(M) = \{ g \in Aut_{(X,G)}(M) : p \circ g = p \} \) the fundamental orbifold group of \( M \). Also, if \( D : \tilde{M} \to X \) is a developing map and \( \mathcal{H} \) the holonomy morphism associated to \( D \), then \( (D, \mathcal{H} \vert_{\pi_1^{Orb}}) \) will be called the developing pair associated to \( M \).

**Corollary 1.7.** Let \( M \) be a compact \((\mathbb{P}^2_C, PSL_3(\mathbb{C}))\)-orbifold, then \( M \) is a very good orbifold.

**Proof.** Let \( D \) be any riemannian metric on the orbifold \( M \) and let \( \tilde{D} \) be the pullback of \( D \) to the universal covering orbifold \( \tilde{M} \), then \( \tilde{D} \) induces a metric \( d \) on \( M \) compatible with the topology. Since \( M \) is compact, one can show that \((\tilde{M}, d)\) is a length space geodesically connected, geodesically complete and such that \( \pi_1^{Orb}(M) \) acts as subgroup of isometries, see [1] [16]. Thus \( \pi_1^{Orb}(M) \) is finitely generated, see [16]. Now, if \( (D, \mathcal{H}) \) is a developing pair for \( M \), by Selberg’s lemma (see [16]), \( \mathcal{H}(\pi_1^{Orb}(M)) \) has a normal subgroup \( \tilde{H} \) torsion free with finite index. Thus \( \tilde{H} = \mathcal{H}^{-1}(H) \) is a normal subgroup of \( \pi_1^{Orb}(M) \) with finite index and we claim that that \( \tilde{H} \) acts freely on \( \tilde{M} \). Otherwise, let \( x \in M \) such that \( Isot(x, \tilde{H}) \) is non trivial, thus \( Isot(x, \tilde{H}) \subset Ker(\mathcal{H}) \) and there is an open neighborhood \( W \) of \( x \) which is \( Isot(x, \tilde{H}) \)-invariant and such that \( D \vert_W \) is injective, hence \( D(g(z)) = D(z) \) with \( g(z) \in W \) for all \( z \in W \) and any \( g \in Isot(x, \tilde{H}) \), that is \( Isot(x, \tilde{H}) = \{ Id \} \), which is a contradiction. Therefore \( N = \tilde{M}/\tilde{H} \) is a \((\mathbb{P}^2_C, PSL_3(\mathbb{C}))\)-manifold and \( \Gamma = \pi_1^{Orb}(M)/\tilde{H} \) is a finite group that acts on \( N \) by \( \tilde{H}g(\tilde{H}x) = (gx) \) trivially \( \Gamma \leq Aut(\mathbb{P}^2, PSL_3(\mathbb{C}))(N) \). Let \( \varphi : \tilde{M}/\pi_1^{Orb}(M) \to N/\Gamma \) be defined by \( \varphi(\pi_1^{Orb}(M)x) = \Gamma(\tilde{H}x) \), then \( \varphi \) is an \((\mathbb{P}^2_C, PSL_3(\mathbb{C}))\)-equivalence. And since \( Card(\Gamma) < \infty \) is finite and \( N/\Gamma \) compact we conclude that \( N \) is compact. \[ \square \]

From corollary 1.7 we see that the results of Inoue, Kobayashi, Klingler, Mok, Ochiai, Yeung et al for compact \((\mathbb{P}^2_C, PSL_3(\mathbb{C}))\)-manifolds (see [17] [8]), can be extended to compact \((\mathbb{P}^2_C, PSL_3(\mathbb{C}))\)-orbifolds as follows:

**Theorem 1.8.** Let \( M \) be a compact \((\mathbb{P}^2_C, PSL_3(\mathbb{C}))\)-orbifold, then \( M \) is of one of the following 8 types.

1. \( \tilde{M} = \mathbb{P}^2_C \). In this case \( D = Id \) and \( \mathcal{H}(\pi_1^{Orb}(M)) \) is finite.
2. \( \tilde{M} = \mathbb{H}^2_C \). In this case \( D = Id \) and \( \mathcal{H}(\pi_1^{Orb}(M)) \leq PU(2,1) \).
3. \( \tilde{M} = \mathbb{C}^2 - \{0\} \). In this case \( D = Id \). \( \mathcal{H}(\pi_1^{Orb}(M)) \leq Af(\mathbb{C}^2) \) contains a cyclic group of finite index generated by a contraction and \( M \) is a finite covering (possibly ramified) of a Hopf surface.
(4) \( \mathcal{D}(\tilde{M}) = \mathbb{C}^* \times \mathbb{C}^* \). In this case \( \mathcal{H}(\pi_1^{\text{orb}}(M)) \leq Af_3(\mathbb{C}^2) \) and \( M \) is a finite covering (possibly ramified) of a surface biholomorphic to a complex torus. Moreover, if \( \Sigma_M = \emptyset \) one can say that \( \mathcal{H}(\pi_1(M)) \leq Af(\mathbb{C}^2) \).

(5) \( \mathcal{D}(M) = \mathbb{C} \times \mathbb{C}^* \). In this case \( \mathcal{H}(\pi_1^{\text{orb}}(M)) \leq Af_2(\mathbb{C}^2) \) has a subgroup of finite index contained on \( A_1 \) or \( A_2 \) and \( M \) is a finite covering (possibly ramified) of a complex surface biholomorphic to a torus. Moreover, if \( \Sigma_M = \emptyset \) one can say that \( \mathcal{H}(\pi_1(M)) \leq Af(\mathbb{C}^2) \).

(6) \( \mathcal{D}(M) = \mathbb{C}^2 \). In this case \( \mathcal{H}(\pi_1^{\text{orb}}(M)) \leq Af(\mathbb{C}^2) \) contains an unipotent subgroup of finite index and \( M \) is a finite covering (possibly ramified) of a surface biholomorphic to a complex torus or a primary Kodaira surface.

(7) \( \tilde{M} = \mathbb{C} \times \mathbb{H} = \mathcal{D}(M) \). In this case \( M \) is a finite covering (possibly ramified) of an Inoue surface. Moreover, if \( \Sigma_M = \emptyset \) we can say that \( M \) is an Inoue surface and \( \mathcal{H}(\pi_1(M)) \leq Af(\mathbb{C}^2) \) is a torsion free group contained on \( \text{Sol}_0^3 \) or \( \text{Sol}_1^3 \).

(8) \( \tilde{M} \) is biholomorphic to \( \mathbb{C} \times \mathbb{H} \). In this case there are \( A, B : \mathbb{H} \longrightarrow \mathbb{C} \) holomorphic maps and \( \mu \in \mathbb{C}^* \) such that \( \mathcal{D}(\tilde{M})(w, z) = (A(z)e^{w\mu}, B(z)e^{w\mu}) \), \( \mathcal{H}(\pi_1^{\text{orb}}(M)) \leq Af(\mathbb{C}^2) \), \( \pi_1^{\text{orb}}(M) \leq \text{Bihol}(\mathbb{C} \times \mathbb{H}) \) contains a subgroup of finite index \( \Xi \) which admits the presentation:

\[
< a_1, b_1, \ldots, a_g, b_g, c, d : c, d \text{ in the center and } \Pi_{i=1}^g [a_i, b_i] = c^r >
\]

where \( 2 \leq g, r \in \mathbb{N} \) and \( M \) is a finite covering (possibly ramified) of an elliptic affine surface.

2. Projective Structures and Projective Groups

Lemma 2.1. Let \( \Gamma \leq PSL_3(\mathbb{C}) \) be a group acting properly discontinuously over a non-empty, \( \Gamma \)-invariant domain \( \Omega \), then there is a developing pair \((\mathcal{D}, \mathcal{H})\) for \( M = \Omega/\Gamma \) such that \( \mathcal{D}(\tilde{M}) = \Omega \) and \( \mathcal{H}(\pi_1^{\text{orb}}(M)) = \Gamma \).

Proof. Step 1. -Construction of \( \mathcal{D} \). Let \( P : \tilde{M} \longrightarrow M \) be the universal covering orbifold map, \( q : \Omega \longrightarrow M \) be the quotient map, \( m \in X_M - \Sigma_M, \tilde{m} \in \tilde{M} \) and \( x \in \Omega \) such that \( P(\tilde{m}) = q(x) = m \). By Theorem 1.6, there is a \((\mathbb{P}^2, PSL_3(\mathbb{C}))\)-covering map \( \tilde{D} : (\tilde{M}, \tilde{m}) \longrightarrow (\Omega, x) \) such that \( q \circ \tilde{D} = P \).

Let \( i : \Omega \longrightarrow \mathbb{P}^2 \) be the inclusion map, then \( \mathcal{D} = i \circ \tilde{D} \) is an \((\mathbb{P}^2, PSL_3(\mathbb{C}))\)-map.

Step 2. -Construction of \( \mathcal{H} \). Let \( g \in \pi_1^{\text{orb}}(M) \). Since \( q(\mathcal{D}(g(\tilde{m}))) = q(x) \), we deduce that there is \( \hat{g} \in \Gamma \) such that \( \hat{g}(x) = \mathcal{D}(g(\tilde{m})) \). Since \( Isol(x, \Gamma) \) is trivial we conclude that \( \hat{g} \) is unique. Define \( \mathcal{H} : \pi_1^{\text{orb}}(M) \longrightarrow \Gamma \) by \( \mathcal{H}(g) = \hat{g} \).

Step 3. -\( \mathcal{D} \circ g = \mathcal{H}(g) \circ \mathcal{D} \). Let \( g \in \pi_1^{\text{orb}}(M) \). By Theorem 1.6 there is a \((\mathbb{P}^2, PSL_3(\mathbb{C}))\)-map \( S : (\tilde{M}, \tilde{m}) \longrightarrow (\tilde{M}, \tilde{m}) \) such that \( \mathcal{D} \circ g \circ S = \mathcal{H}(g) \circ \tilde{D} \). Hence,
the following diagram commutes:

$$\begin{array}{c}
(\tilde{M}, \tilde{m}) \\ P \\
\downarrow \\
(M, m) \\ q \\
\downarrow \\
(\Omega, x) \\ q \\
\downarrow \\
(\tilde{M}, g(\tilde{m})) \\
\downarrow \\
(\Omega, g(x)).
\end{array}$$

\[ \text{By Theorem 1.6 we conclude that } S = \text{Id}_{\tilde{M}}. \]

Step 4. \( \text{H is a group morphism- Let } g, h \in \pi_1^{Orb}(M), \text{ then } H(g) \circ H(h)(x) = H(g)(D(h(\tilde{m}))) = D(g(h(\tilde{m}))). \text{ That is } H(g \circ h) = H(g) \circ H(h). \]

Step 5. \( \text{-D(\pi_1^{Orb}(M)) = } \Gamma. \text{ Let } g \in \Gamma \text{ and } z \in \tilde{M} \text{ such that } D(z) = g(x). \text{ By Theorem 1.6 there is an } (\mathbb{P}_2^2, PSL_3(\mathbb{C}))\text{-equivalence } \hat{g} : (M, \tilde{m}) \longrightarrow (M, z) \text{ such that } D \circ \hat{g} = g \circ D. \text{ Thus the following diagram commutes:} \]

$$\begin{array}{c}
(\tilde{M}, \tilde{m}) \\ \hat{g} \\
\downarrow \\
(M, m) \\ \downarrow \\
(\Omega, x) \\ \downarrow \\
(\Omega, \hat{D}(g(x))).
\end{array}$$

That is, \( \hat{g} \in \pi_1^{Orb}(M). \)

**Corollary 2.2.** Let \( \Gamma, \Omega, \tilde{m}, x, m, P, q, D \text{ and } H \) be as in lemma 2.1, then we have the following exact sequence of groups:

$$0 \longrightarrow \pi_1(\Omega) \longrightarrow i \longrightarrow \pi_1^{Orb}(M) \xrightarrow{H} \Gamma \longrightarrow 0,$$

where \( i \) is the inclusion induced by the following commutative diagram:

$$\begin{array}{c}
\tilde{M} \\ \downarrow \\
\tilde{M} \\
\downarrow \\
\Omega \\
\downarrow \\
M.
\end{array}$$

**Proof.** Since \( H \) is an epimorphism we only have to show that \( \text{Ker}(H) = \pi_1(\Omega). \)

Step i. \( \pi_1(\Omega) \subset \text{Ker}(H). \text{ Let } g \in \pi_1(\Omega), \text{ then } D(g(\tilde{m})) = D(\tilde{m}) = \text{Id}(x). \text{ By the definition of } H \text{ we conclude that } \pi_1(\Omega) \subset \text{Ker}(H). \)
Step ii. \( \text{Ker}(\mathcal{H}) \subset \pi_1(\Omega) \). Let \( g \in \text{Ker}(\mathcal{H}) \), then \( D \circ g = D \). Since \( D \) is covering and \( \hat{M} \) is simply connected the result follows.

3. On the Equicontinuity Region for Subgroups of \( PSL_2(\mathbb{C}) \)

As we will see in lemma 4.12, an information which is necessary for the description of the dynamics of lines in the case of control groups, is the description of the equicontinuity region of non-discrete groups of Möbius transformations. Thus, in the following subsection we will focus on the description of such sets.

Consider the following definition:

**Definition 3.1.** Consider the usual identification of \( PSL_2(\mathbb{C}) \) with the isometry group of the hyperbolic 3-space \( \mathbb{H}^3 \) and the respective identification of \( \mathbb{P}^1\mathbb{C} \) with the sphere at infinity \( E \) of \( \mathbb{H}^3 \), then for every group \( \Gamma \leq PSL_2(\mathbb{C}) \) its limit set in the sense of Greenberg (see \([5]\)), denoted \( \mathcal{L}(\Gamma) \), is defined to be the intersection of \( E \) with the set of accumulation points of any orbit in \( \mathbb{H}^3 \).

The main purpose of this section is to show that the complement of \( \mathcal{L}(\Gamma) \) coincides with the region of equicontinuity of \( \Gamma \), and if \( \Gamma \) is non-discrete, \( \text{Card}(\mathcal{L}(\Gamma)) \geq 2 \) and its equicontinuity region is non empty, then \( \mathcal{L}(\Gamma) \) is a circle. The main tools to prove these statements are the following results due to Greenberg (see Theorem 1 and Proposition 12 in \([5]\)):

**Theorem 3.2.** Let \( G \) be a connected Lie subgroup of the group \( PSL_2(\mathbb{C}) \) and consider the usual identification of \( \mathbb{P}^1\mathbb{C} \) with the sphere at infinity of the hyperbolic space \( \mathbb{H}^3 \), then one of the following assertions is satisfied:

1. The elements of \( G \) have a common fixed point in \( \mathbb{H}^3 \) while \( G \) itself is conjugate to a Lie subgroup of \( O(3) \).
2. The elements of \( G \) have a common fixed point in \( \mathbb{P}^1\mathbb{C} \).
3. There exists a hyperbolic straight line \( \ell \subset \mathbb{H}^3 \) such that \( \ell \) is \( G \)-invariant.
4. There exists a hyperbolic plane \( L \subset \mathbb{H}^3 \) such that \( L \) is \( G \)-invariant.
5. \( G = PSL_2(\mathbb{C}) \).

**Theorem 3.3.** Let \( \Gamma \leq PSL_3(\mathbb{C}) \) be a subgroup with \( \text{Card}(\mathcal{L}(\Gamma)) \geq 2 \), then \( \mathcal{L}(\Gamma) \) is the closure of the loxodromic fixed points.

3.1. Basic Definitions and Examples.

**Definition 3.4.** Given \( \Gamma \leq PSL_2(\mathbb{C}) \) an arbitrary subgroup then, its equicontinuity region, denoted \( \text{Eq}(\Gamma) \), is defined to be the set of points \( z \in \mathbb{P}^1\mathbb{C} \) for which there is an open neighborhood \( U \) of \( z \) such that \( \Gamma|_U \) is a normal family.

**Remark 3.5.** Clearly the following properties hold:

1. \( \text{Fix}(\gamma) \subset \mathbb{P}^1\mathbb{C} - \text{Eq}(\Gamma) \) if \( \gamma \in \Gamma \) is non elliptic.
2. \( \text{Eq}(\Gamma) = \text{Eq}(\overline{\Gamma}) \).
3. \( \text{Eq}(\Gamma) \) is an open \( \Gamma \)-invariant set.

**Example 3.6.** Let \( g \in \text{Mob}(\hat{\mathbb{C}}) \) be defined by \( g(z) = -z \) and set \( \text{Mob}(\mathbb{R}) = \langle PSL_2(\mathbb{R}), g \rangle \), then one has that \( \text{Eq}(\text{Mob}(\mathbb{R})) = \mathbb{C} - \mathbb{R} = \mathbb{P}^1\mathbb{C} - \mathcal{L}(\Gamma) \).

**Example 3.7.** Let \( h \in \text{Mob}(\hat{\mathbb{C}}) \) be defined by \( h(z) = z^{-1} \) and set \( \text{Rot}_\infty = \{ T(z) = az : a \in \mathbb{S}^1 \} \) and \( \text{Dih}_\infty = \langle \text{Rot}_\infty, h \rangle \), then \( \text{Eq}(\text{Rot}_\infty) = \text{Eq}(\text{Dih}_\infty) = \mathbb{P}^1\mathbb{C} - \mathcal{L}(\Gamma) = \mathbb{P}^1\mathbb{C} \).
Example 3.8. Set \( Epa(C) = \{ \gamma(z) \in M\text{ob}(\mathbb{P}_1^1) : \gamma(\infty) = \infty \text{ and } 0 \leq tr^2(\gamma) \leq 4 \} \), then \( Eq(Epa(C)) = \mathbb{P}_1^1 \setminus \Sigma(\Gamma) = \mathbb{C} \).

Example 3.9. Define \( M\text{ob}(\mathbb{C}^*) = \langle \{ T(z) = az : a \in \mathbb{R}^* \}, Dih_\infty \rangle \), then \( Eq(M\text{ob}(\mathbb{C}^*)) = \mathbb{P}_1^1 \setminus \Sigma(\Gamma) = \mathbb{C}^* \).

3.1.1. The \( Cr \) Group. Since the following example is a little bit more sophisticated we have created this paragraph. Whose main purpose is to describe its geometry as well as some of its basical properties.

Let \( p < 0 \) and \( \tau_p(z) = \frac{z - p}{z^2 - 1} \). Define

\[
Cr(p) = \langle Rot_\infty, \tau_p^{-1} Rot_\infty \tau_p \rangle,
\]

then the following properties hold:

Properties 3.10. \( \text{(1)} \) For each \( z \in \mathbb{P}_1^1 \) one has that \( Cr(-1)z = \mathbb{P}_1^1 \).

\( \text{(2)} \)

\[
Cr(-1) = \left\{ \frac{az - c}{cz + a} \in M\text{ob}(\hat{\mathbb{C}}) : |a|^2 + |c|^2 = 1 \right\}
\]

In other words \( Cr(-1) \) is diffeomorphic to \( \mathbb{P}_1^1 \).

\( \text{(3)} \) \( Cr(-1) \) is a purely elliptic group.

\( \text{(4)} \) \( Eq(Cr(-1)) = \mathbb{P}_1^1 \).

\( \text{(5)} \) The limit set in the sense of Greenberg for \( Cr(-1) \) is empty.

\( \text{(6)} \) For each \( p < 0 \) there are \( z_p \in \mathbb{C} \) and \( \gamma_p \in Cr(p) \) such that \( \langle Rot_\infty, \gamma_p \rangle = Cr(p) \) and \( Fix(\gamma_p) = \{ z_p, -z_p \} \).

\( \text{(7)} \) For each \( p < 0 \) one has that \( Cr(-1) \) is conjugate to \( Cr(p) \).

Before we prove the preceding properties we will state and prove the following technical lemma:

Lemma 3.11. Let \( \gamma(z) = \frac{az + b}{cz + d} \), with \( ad - bc = 1 \), be an elliptic element such that \( Fix(\gamma) = \{ 1, p \} \subset \overline{\mathbb{D}} \). The following properties hold:

\( \text{(1)} \) \( a = d \) if and only if \( p \in \mathbb{R} \).

\( \text{(2)} \) If \( p \in \mathbb{R} \) then: \( |a| = 1 \) if and only if \( p = 0 \).

\( \text{(3)} \) If \( p \in \mathbb{R} \) then: \( |a| < 1 \) if and only if \( p < 0 \).

Proof. First at all, since \( \gamma \) is elliptic we can ensure that there is \( \lambda = e^{\pi i \theta} \) such that:

\[
\gamma(z) = \frac{\lambda z + \lambda^{-1} \lambda}{\lambda^{-1} z + \lambda}.
\]

\( \text{(1)} \) By equation \( 3.1 \) we have that:

\[
\text{Re}(a - d) = \frac{4Im(\lambda)Im(p)}{|p - 1|^2}.
\]

This implies the result.

\( \text{(2)} \) and \( \text{(3)} \) By equation \( 3.2 \) we have that \( |a| \leq 1 \) is equivalent to \( |p\lambda - \lambda| \leq |p - 1| \) which is equivalent to \( 4pIm(\lambda) \leq 0 \), whic proves the statement. ■

Proof of properties 3.10

\( \text{(1)} \) Let \( z \in \mathbb{P}_1^1 \), by \( 3.1 \) we have that:

\[
i |z| \in \tau_1^{-1} Rot_\infty \tau^{-1}(0) = \left\{ \frac{i \frac{\sin(\pi \theta)}{\cos(\pi \theta)} : \theta \in \mathbb{R} \} = i\mathbb{R} \cup \infty.\right\}
\]
Then $<\gamma_1> = \text{Rot}_{\infty}$ and $<\gamma_2> = \tau_{-1}^{-1} \text{Rot}_{\infty} \tau_{-1}$. To establish the contention
$\subseteq$ will be enough to show it for elements in $<\gamma_1, \gamma_2>$, which we will do by
induction on the length of the reduced words (recall that $w = w_{m-n}^* \cdots \overline{w_2^*} \overline{w_1^*} \in \Gamma$

And an easy calculation shows:

$$w(z) = \frac{(a - \overline{d})z - (a\overline{d} + bc)}{(d\overline{a} + bc)z + ab - dc}$$

Which concludes this part of the proof.

Let $\tau(z) = \frac{az - \overline{c}}{cz + \overline{a}}$, with $|a| = 1$, and observe that the set of fixed points
for $\gamma$ is given by:

$$p_\pm = \frac{Im(a) \pm \sqrt{1 - Re^2(a)}}{c}$$

Clearly we have $p_+\overline{p_-} = -1$. In particular we have shown that for every element
$\gamma \in Cr(-1)$ there is $z \in \mathbb{C}^*$ such that $Fix(\gamma) = \{z, -\overline{z}^{-1}\}$. On the other hand by

There is an element $\gamma_0 \in Cr(-1)$ such that $\gamma_0(\infty) = p_+$. Since $Fix(\gamma_0 \gamma_1 \gamma_0^{-1}) = \{\gamma_0(0), \gamma_0(\infty)\}$, it follows that $\gamma_0(0) = p_-$ and in consequence $\gamma_0 \tau \gamma_0 \in \text{Rot}_{\infty}$, which concludes the proof.

This follows immediately from [2]

Let $(\gamma_m)_{m \in \mathbb{N}} \subset Cr(-1)$ be a sequence. By part [2] of the present lemma there are sequences $(a_m)_{m \in \mathbb{N}}$, $(c_m)_{m \in \mathbb{N}} \subset \mathbb{C}$ such that $|a_m|^2 + |c_m|^2 = 1$

Hence there are subsequences $(b_m)_{m \in \mathbb{N}} \subset (a_m)_{m \in \mathbb{N}}$, $(d_m)_{m \in \mathbb{N}} \subset (c_m)_{m \in \mathbb{N}}$ and complex numbers $b, d \in \mathbb{C}$ such that $b_m \xrightarrow{m \to \infty} b$, $d_m \xrightarrow{m \to \infty} d$. Thus:

$$\frac{b_m z - d_m}{d_m z + b_m} \xrightarrow{m \to \infty} \frac{b z - d}{dz + b}$$

uniformly on $\mathbb{C}$.

Which concludes the proof.

By Theorem 3.2 and part (2) of the present proof, one has that $Cr(-1)$ is
conjugated to a subgroup of $O(3)$; hence $\mathcal{L}(Cr(-1))$ is empty.

Let $a, c : (0, 1) \to \mathbb{C}$ given by:

$$a(x) = \frac{x(p - 1) - i(p + 1)\sqrt{1 - x^2}}{p - 1}, \quad c(x) = \frac{-2i\sqrt{1 - x^2}}{p - 1}.$$
Then by equation\[8.1\] we can ensure that \( \phi : (0, 1) \to \tau_{-1}^{1} Rot_{\infty} \) given by:

\[
\phi(x)(z) = \frac{a(x)z + pc(x)}{c(x)z + a(x)}
\]
defines a local chart. Moreover one has:

\[
Re\left( \frac{d\phi(x)}{dz}(q) \right) = x, \text{ for each } q \in Fix(\phi(x)).
\]

Now, since \( \tau_{x}(z) = \frac{|a(x)|^{2}z}{a(z)} \in Cr(p) \) we conclude:

\[
\gamma_{x}(z) = \frac{|a(x)|z + \frac{|a(x)|pc(x)}{a(x)}}{c(x)a(x)} + |a(x)| \in Cr(p).
\]

Set \( Fix(\gamma_{x}) = \{ z_{x}, w_{x} \} \), then by a simple calculation we can show:

\[
z_{x} = \frac{i |a(x)| \sqrt{1 - |a(x)|^{2}}}{c(x)a(x)},
\]

\[
w_{x} = -z_{x},
\]

\[
Re(\gamma'_{x}(z_{x})) = Re(\gamma'_{x}(w_{x})) = \frac{-8px^{2} + p^{2} + 6p + 1}{(p - 1)^{2}}.
\]

Define

\[
f : (0, 1) \to (-1, 1);
\]

\[
f(x) = \frac{-8px^{2} + p^{2} + 6p + 1}{(p - 1)^{2}}
\]

and \( Ur = \{ x \in (0, 1) : x + i\sqrt{1 - x^{2}} \text{ is a root of the unity} \} \), then by a simple inspection we conclude that \( f \) injective and \( Ur \) is a countable set. Thus one has that \( Ur \cup f^{-1}(Ur) \) is a countable set. To conclude, take \( r_{0} \in (0, 1) - (Ur \cup f^{-1}(Ur)) \) and set \( \gamma_{p} = \gamma_{r_{0}}. \)

Let \( p < 0 \). By part \( 4 \) of the present we there is \( \gamma_{p} \in Cr(p) \) such that \( < Rot_{\infty}; \gamma_{p} > = Cr(p) \) and the set \( Fix(\gamma_{p}) = \{ z_{p}, w_{p} \} \) satisfy \( w_{p} = -z_{p} \). Set \( \kappa(z) = w_{p}z \), then \( \kappa^{-1} Rot_{\infty} \kappa = Rot_{\infty} \) and \( < \kappa^{-1} \gamma_{p} \kappa > = \tau_{-1}^{-1} Rot_{\infty} \tau_{-1}. \) Thus \( \kappa^{-1} Cr(p) \kappa = Cr(-1). \)

From now on \( Cr(-1) \) will be denoted by \( Cr \).

### 3.2. Elementary Groups.

The main goal of this subsection is to show that every non discrete group with non-empty equicontinuity region is a subgroup of: \( Dih_{\infty}, Epa(\mathbb{C}), Cr, M\tilde{o}b(\mathbb{R}), M\tilde{o}b(\mathbb{C}^+) \) (see subsection \( 3.1 \)). Our first step to show this assertion is to classify the groups with an equicontinuity set such that its complement contains at most 2 points (such groups will be called \textit{elementary} as in the discrete case). To do this we will use the following technical lemmas:

**Lemma 3.12.** Let \( \gamma_{1}, \gamma_{2} \in PSL_{2}(\mathbb{C}) \) be elliptic elements with infinite order and set \( Fix(\gamma_{1}) = \{ z_{1}, z_{2} \}, Fix(\gamma_{2}) = \{ w_{1}, w_{2} \}. \) If the cross ratio satisfies \( [z_{1}; z_{2}; w_{1}; w_{2}] \in \mathbb{C} - (\mathbb{R}^{-} \cup \{ 0 \}) \), then \( < \gamma_{1}, \gamma_{2} > \) contains a loxodromic element.
\textbf{Proof.} Since \textit{Möbius} transformations preserve the cross ratio we can assume that \( Fix(\gamma_1) = \{0, \infty\} \) and \( Fix(\gamma_2) = \{1, p\} \) with \( p \in \mathbb{C} - (\mathbb{R} \cup \{0\}) \). Hence there are \( \theta \in \mathbb{R} - \mathbb{Q}, a, b, c, d \in \mathbb{C} \) such that \( ad - bc = 1, \gamma_1(z) = \lambda z \) and
\[
\gamma_2(z) = \frac{az + b}{cz + d},
\]
where \( \lambda = e^{i\pi \theta} \). Without loss of generality we can assume that \( a \neq 0 \). Now, let \( (a_m)_{m \in \mathbb{N}} \subset (m)_{m \in \mathbb{N}} \) be a subsequence such that \( \lambda^{m} \rightarrow \frac{|a|}{a} \), then:
\[
\gamma_1^{m} \gamma_2 \rightarrow_{m \rightarrow \infty} f(z) = \frac{|a|}{a} \left( 1 + \frac{ad}{|a|^2} \right)^2 \text{ uniformly on } \mathbb{P}^1_{\mathbb{C}}.
\]
If \( f \) is loxodromic the result follows easily, thus we may assume that \( f \) is not loxodromic. In virtue of lemma 3.11 and since:
\[
Tr^2(f) = |a|^2 \left( 1 + \frac{ad}{|a|^2} \right)^2
\]
we deduce that \( r = da^{-1} \in \mathbb{R} - \{1\} \). On the other hand, since \( 0 \leq Tr^2(\gamma_1) = (a + d)^2 \), we conclude \( a + d \in \mathbb{R} \) and in consequence \( 0 = Im(1 - r)Im(a) \) which implies \( a, d \in \mathbb{R} \). Define \( H : S^1 \rightarrow \mathbb{R} \) by \( H(z) = (a^2 - d^2)Im(z) \) and observe that \( H(\lambda^n) = Im(Tr^2(\gamma^n \gamma_2)) \). To conclude observe that \( H(i) = -d + a \) and \( \{e^{i\pi |n|} : n \in \mathbb{Z}\} = S^1 \). \( \blacksquare \)

We will use also the following well know results, see [10] p. 11, 12 and 19:

\textbf{Lemma 3.13.} \textit{Two non trivial elements} \( f, g \in PSL_2(\mathbb{C}) \) \textit{commute if and only if either they have exactly the same fixed point set, or each is elliptic or order 2, and each interchanges the fixed points of the other.}

\textbf{Lemma 3.14.} \textit{Let} \( f, g \in PSL_2(\mathbb{C}) \) \textit{be such that} \( f \) \textit{has exactly two fixed points and} \( f, g \) \textit{share exactly one fixed point, then} \( fgf^{-1}g^{-1} \) \textit{is parabolic. Moreover, is} \( f \) \textit{is loxodromic, then there is sequence} \( (\gamma_m)_{m \in \mathbb{N}} \) \textit{contained in} \( \subset f, g > \) \textit{of distinct parabolic elements such that} \( \gamma_m \rightarrow_{m \rightarrow \infty} \text{Id uniformly on} \mathbb{P}^1_{\mathbb{C}} \).}

\textbf{Corollary 3.15.} \textit{Let} \( \Gamma \leq PSL_2(\mathbb{C}) \) \textit{be a subgroup, then:}

(1) \( Eq(\Gamma) = \mathbb{P}^1_{\mathbb{C}} \) \textit{if and only if} \( \Gamma \) \textit{is finite or is conjugate to subgroup of} \( \text{Dih}_{\infty} \) \textit{or} \( \text{Cr} \), \textit{where} \( \text{Dih}_{\infty} \) \textit{and} \( \text{Cr} \) \textit{are as in examples} 3.7 \textit{and} 3.13 \textit{respectively}.

(2) \( Eq(\Gamma) \) \textit{is} \( \mathbb{C} \) \textit{up to a projective transformation, if and only if} \( \Gamma \) \textit{is conjugate to a subgroup} \( \Gamma_0 \) \textit{of} \( \text{Epa}(\mathbb{C}) \) \textit{such that} \( \Gamma_0 \) \textit{contains a parabolic element, where} \( \text{Epa}(\mathbb{C}) \) \textit{is as in example} 3.8.

(3) \( Eq(\Gamma) \) \textit{is, up to projective transformation,} \( \mathbb{C}^* \) \textit{if and only if} \( \Gamma \) \textit{is conjugate to a subgroup} \( \Gamma_0 \) \textit{of} \( \text{Mob}(\mathbb{C}^*) \) \textit{such that} \( \Gamma_0 \) \textit{contains a loxodromic element.}

\textbf{Proof.} [10] \textit{Let} \( \Gamma \) \textit{be an infinite group. By the remark 3.15 we must only consider the following cases:}

\textit{Case 1.} \( o(\gamma) < \infty \) \textit{for all} \( \gamma \in \Gamma - \{\text{Id}\} \). \textit{Under this hypothesis, Selberg’s lemma and the classification of the finite groups of} \( PSL_2(\mathbb{C}) \), \textit{see} [10], \textit{we deduce that} \( B = \langle A \rangle : A \) \textit{is a non-empty finite subset of} \( \Gamma \) \textit{is an infinite set where each element is either a cyclic or a dihedral group. From this and lemma 3.13 it is easily deduced that} \( \Gamma \) \textit{is conjugated to a subgroup of} \( \text{Dih}_{\infty} \).
Case 2. $\Gamma$ contains an element $\gamma_1$ with infinite order. By lemma 3.12 and lemma 3.13 we deduce that in case of $Fix(\gamma) = Fix(\gamma_1)$ for each element $\gamma \in \Gamma$ with $o(\gamma) = \infty$ it is deduced that $\gamma(Fix(\gamma_1)) = Fix(\gamma_1)$ for each $\gamma \in \Gamma$ with $o(\gamma) < \infty$. Hence $\Gamma$ is conjugate to a subgroup of $\text{Dih}_\infty$. Thus we may assume that there is an element $\gamma_2$ with infinite order and such that $Fix(\gamma_1) \neq Fix(\gamma_2)$. Thus from remark 3.3 and lemmas 3.11, 3.12 we can assume that $Fix(\gamma_2) = \{0, \infty\}$, $Fix(\gamma_2) = \{1, p\}$ where $p < 0$. Thus $\gamma_1, \gamma_2 > = Cr(p)$. By (1) of properties 3.10, (2) of 3.10 and lemma 3.14 we deduce that there is $z \in \mathbb{C}$ such that $Fix(\gamma_3) = \{z, \overline{z}^{-1}\}$. Thus from the proof of (1) of properties 3.10, 3.11 we deduce that $\gamma_3 \in Cr$. That is, we have shown that $\Gamma$ is conjugate to a subgroup of $Cr$.

(2) and (3) follow easily from Remark 3.5.

From the proof of corollary 3.15 one has:

**Corollary 3.16.** Let $\Gamma \leq PSL_2(\mathbb{C})$ be an infinite closed group, then $\Gamma$ is purely elliptic if and only if $Eq(\Gamma) = P^1_{\mathbb{C}}.$

### 3.3. Non-elementary Groups.

**Corollary 3.17.** Let $\Gamma \leq PSL_2(\mathbb{C})$ be a group and $H \subset \Gamma$ an infinite normal subgroup, such that $\text{Card}(\mathcal{S}(H)) = 2, 0$ and $H$ is not conjugate to a subgroup of $Cr$, then $\Gamma$ is elementary.

**Proof.** By the proof of corollary 3.15 we deduce that there is an $H$-invariant set $\mathcal{P}$ with $\text{Card}(\mathcal{P}) = 2$ and with the following property: If $\mathcal{R}$ is another finite $H$-invariant set then $\mathcal{R} \subset \mathcal{P}$. Since $H$ is a normal subgroup it is deduced that $Hg(\mathcal{P}) = g(\mathcal{P})$ for all $g \in \Gamma$. Which implies that $\mathcal{P} = g(\mathcal{P})$.

**Lemma 3.18.** Let $\gamma_1, \gamma_2 \in PSL_2(\mathbb{C})$ be parabolic elements such that $Fix(\gamma_1) \cap Fix(\gamma_2) = \emptyset$, then $< \gamma_1, \gamma_2 >$ contains a loxodromic element.

**Proof.** Without loss of generality we can assume that $\gamma_1(z) = z + \alpha$ and $\gamma_2(z) = \frac{\overline{z}}{m + 1}$ for some $\alpha, \beta \in \mathbb{C}^*$, then $\text{Tr}^2(\gamma_2 \gamma_1) = (m \overline{\alpha} + 2)^2$ as $m \to \infty$. That is $\gamma_2 \gamma_1$ is loxodromic for $m$ large.

**Corollary 3.19.** Let $\Gamma \leq PSL_2(\mathbb{C})$ be a non-elementary subgroup, then $\Gamma$ contains a loxodromic element.

**Proof.** Assume that $\Gamma$ does not contain loxodromic elements, then by corollary 3.10 and remark 3.5 we deduce that $\overline{\Gamma}$ contains a parabolic element $\gamma_0$. Assume that $\infty$ is the unique fixed point of $\gamma_0$. By lemma 3.18 we conclude that $\overline{\Gamma} \infty = \infty$. In consequence every element in $\Gamma$ has the form $az + b$ with $|a| = 1$. That is $\Gamma \leq Epa(\mathbb{C})$. Which is a contradiction by (2) of corollary 3.15.

From this corollary and standard arguments, see [10], we can show the following corollaries:

**Corollary 3.20.** Let $\Gamma \leq PSL_2(\mathbb{C})$ be a non elementary group, then $P^1_{\mathbb{C}} - Eq(\Gamma)$ is the closure of the loxodromic fixed points.

**Corollary 3.21.** Let $\Gamma \leq PSL_2(\mathbb{C})$ be a non elementary subgroup and define $Ex(\Gamma) = \{z \in P^1_{\mathbb{C}} - Eq(\Gamma) : \overline{\Gamma}z \neq P^1_{\mathbb{C}} - Eq(\Gamma)\}$, then $Card(Ex(\Gamma)) = 0, 1$. 
Corollary 3.22. Let $\Gamma \leq PSL_2(\mathbb{C})$ be a subgroup and $C \neq Ex(\Gamma)$ a closed $\Gamma$-invariant set. Then $\Sigma(\Gamma) \subset C$.

Corollary 3.23. Let $\Gamma \leq PSL_2(\mathbb{C})$ be a group, then $Eq(\Gamma) = \mathbb{P}^1_{\mathbb{C}} - \Sigma(\Gamma)$.

Lemma 3.24. Let $\Gamma \leq PSL_2(\mathbb{C})$ be a purely parabolic closed Lie group with $dim_{\mathbb{R}}(\Gamma) = 1$, and $\gamma_1\gamma_2 \in PSL_2(\mathbb{C})$ be loxodromic elements such that $\Gamma(\infty) = \gamma_1(\infty) = \gamma_2(\infty) = \infty$ and $Fix(\gamma_2) \not\subset \Gamma(Fix(\gamma_1))$, then $\Gamma_0 = \{ \gamma \in \Gamma, \gamma_1, \gamma_2 : Tr^2(\gamma) = 4 \}$ is a lie group with $dim_{\mathbb{R}}(\Gamma) = 2$.

Proof. On the contrary assume that $dim_{\mathbb{R}}(\Gamma) = 1$. Without loss of generality we can assume that:

$$\gamma_1(z) = t^2 z;$$
$$\gamma_2(z) = a^2 z + ab;$$
$$\Gamma = \{ z + r : r \in \mathbb{R} \}.$$  

Hence $\gamma_2\Gamma\gamma_2^{-1} = \{ z + t^2 r : r \in \mathbb{R} \}$, $\gamma_1\Gamma\gamma_1^{-1} = \{ z + a^2 r : r \in \mathbb{R} \}$, which implies that $a^2, t^2 \in \mathbb{R}$. On the other hand, observe that for all $n \in \mathbb{Z}$ we have that:

$$\gamma_1^n \gamma_2^{-1} \gamma_1^{-1}(z) = z + ab(t^{2n} - 1).$$

Thus we conclude that $\{ z + abr : r \in \mathbb{R} \} < \Gamma, \gamma_1, \gamma_2 >$ and in consequence $ab(1 - a^2)^{-1} \in Fix(\gamma_2)$.}

Lemma 3.25. Let $\Gamma \leq PSL_2(\mathbb{C})$ be a connected Lie group, $g \in \Gamma$ a loxodromic element and $U$ be a neighborhood of $g$ such that $hgh^{-1}g^{-1} = Id$ for each $h \in U$, then $hgh^{-1}g^{-1} = Id$ for each $h \in \Gamma$.

Proposition 3.26. Let $\Gamma \leq PSL_2(\mathbb{C})$ be non-discrete, non-elementary group and $Eq(\Gamma) \neq \emptyset$, then $\Sigma(\Gamma)$ is a circle in $\mathbb{P}^1_{\mathbb{C}}$.

Proof. Since $PSL_2(\mathbb{C})$ is a Lie group we deduce that $\Gamma$ is a Lie group (see [22]). Let $H$ be the connected component of the identity in $\Gamma$, thus $H$ is a connected and normal subgroup of $\Gamma$. By theorem 3.21 (1) of properties 3.10 and corollaries 3.17 and 3.20 we have consider one of the following cases:

Case 1.- There is exactly one point $p \in \mathbb{P}^1_{\mathbb{C}}$ such that $Hp = p$. In this case since $H$ is a normal subgroup normal, by means of the argument used in the proof of 3.17 it is deduced that $\Gamma p = p$. By lemma 3.14 we conclude that there is a purely elliptic Lie group $K \leq H$ with $dim_{\mathbb{R}}(K) = 1$. Let $\gamma_0 \in \Gamma$ be a loxodromic element, then by lemma 3.23 we deduce that for each loxodromic element $\gamma \in \Gamma$ one has that $Fix(\gamma) \subset K(Fix(\gamma_0))$. Moreover by corollary 3.22 we conclude that $K(Fix(\gamma_0)) = \Sigma(\Gamma)$. The result now follows because $K(Fix(\gamma_0))$ is a circle in $\mathbb{P}^1_{\mathbb{C}}$.

Case 2.- $Card(\Sigma(H)) \geq 2$ and there is a circle $C$ such that $HC = C$. Assume that $C = \mathbb{R}$, then for each loxodromic element $\gamma \in H$ one has that $Fix(\gamma) \subset \mathbb{R}$. Let $\gamma_0 \in H$ be a loxodromic element such that $Fix(\gamma_0) \in \mathbb{R}$ and let $p_1, p_2$ be the fixed points of $\gamma_0$, then there exist a neighborhood $W \subset \mathbb{R}^{dim_{\mathbb{R}}(H)}$ of 0 and real analytic maps $a, b, c, d : W \to \mathbb{C}$ such that $\phi : W \to H$ defined by:

$$\phi(w)(z) = \frac{a(w)z + b(w)}{c(w) + d(w)}$$

is a chart of $\gamma_0$ (that is $\phi(0) = \gamma_0$). Set $F : W \times \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ defined by $F(w, z) = \phi(w)(z) - z$. Thus $\partial_i F(0, p_i) = g'(p_i) - 1 \neq 0$. By the implicit function theorem, there is a neighborhood $W_0 \subset W$ of 0, and continuous functions $\tau_i : W_0 \to \mathbb{C}$, $i = 1, 2$, such that $F(w, \tau_i(w)) = 0$ and $\tau_1(w) \neq \tau_2(w)$ for each $w \in W_0$. Hence
\{\tau_1(w), \tau_2(w)\} = Fix(\phi(w)) \text{ for all } w \in W_0. \text{ By lemma 3.25 we can assume that } \tau_1 \text{ is non constant and } \phi(W_0) \text{ contains only loxodromic elements. Thus } \tau_1(W_0) \subset \mathcal{L} \text{ and contains an open interval, which clearly implies that } \mathcal{L}(\Gamma) = \mathbb{R}. \]

We end this section with a easy consequence of the previous proposition

**Corollary 3.27.** Let \( \Gamma \leq PSL_2(\mathbb{C}) \) be a subgroup such that \( Eq(\Gamma) \neq \emptyset \). If \( (\gamma_n)_{n \in \mathbb{N}} \subset \Gamma \) and \( \gamma_n \xrightarrow{n \to \infty} g \text{ uniformly on compact sets of } Eq(\Gamma) \), then \( g \) is either a constant function \( c \in \mathcal{L}(\Gamma) \) or \( g \in PSL_2(\mathbb{C}) \) with \( \gamma_n \xrightarrow{n \to \infty} g \text{ uniformly on } \mathbb{P}^1. \)

4. **Basic Properties of Kleinian Groups**

4.1. **A Characterization of finite groups.** We will use the following theorem due to Jordan, see [15], Theorem 8.29:

**Theorem 4.1.** For any \( n \in \mathbb{N} \) there is an integer \( S(n) \) with the following property: let \( G \leq GL_n(\mathbb{C}) \) be any finite subgroup, then \( G \) admits an abelian normal subgroup \( N \) such that \( \text{Card}(G) \leq S(n)\text{Card}(N). \)

**Lemma 4.2.** Let \( G \) be a countable subgroup of \( GL_3(\mathbb{C}) \), then there is an infinite commutative subgroup \( N \) of \( G. \)

**Proof.** Assume that every element has finite order. By Selberg’s lemma we deduce that \( G \) has an infinite set of generators say \( \{\gamma_m\}_{m \in \mathbb{N}}. \) Define \( A_m = < \gamma_1 \cdots \gamma_m >, \) then \( A_m \) is finite and by Theorem 4.1 there is a normal commutative subgroup \( N(A_m) \) of \( A_m \) such that \( \text{Card}(A_m) \leq S(3)\text{Card}(N(A_m)). \) Assume without loss of generality that \( \text{Card}(A_m) = k_0\text{Card}(N(A_m)) \) for some \( k_0 \in \mathbb{N} \) and every index \( m. \) Set \( n_m = \max\{o(g) : g \in N(A_m)\}, \) here \( o(g) \) represents the order of \( g. \) Consider the following cases:

- **Case 1.** (\( n(m)_{m \in \mathbb{N}} \) unbounded) Assume that \( k_0n_j < n_{j+1} \) for all \( j. \) For each \( m \) let \( \gamma_m \in N(A_m) \) be such that \( o(\gamma_m) = n_m. \) Thus \( \gamma_{k_0} \in N(A_j) \) and \( \gamma_{k_0} \neq \gamma_{k_0}. \) Therefore \( \gamma_{k_0} : m \in \mathbb{N} \) is an infinite commutative subgroup of \( G. \)

- **Case 2.** (\( n(m)_{m \in \mathbb{N}} \) bounded) We may assume that \( n_m = c_0 \) for every index \( m. \) Let us construct the following sequence:

  1. **Step 1.** Assume that \( \text{Card}(N(A_1)) > k_0c_0. \) For each \( m > 1, \) consider the following well defined map \( \phi_{1,m} : N(A_1) \to A_m/N(A_m) \) given by \( l \mapsto N(A_m)/l. \) Since \( \text{Card}(N(A_1)) > \text{Card}(A_m/N(A_m)) = k_0, \) we deduce that \( \phi_{1,m} \) is not injective, and then there is an element \( w_1 \neq id \) and a subsequence \( (A_n)_{n \in \mathbb{N}} \subset (A_n)_{n \in \mathbb{N}} \) such that \( \text{Card}(N(B_1)) > k_0c_0^4 \) and \( w_1 \in \bigcap_{m \in \mathbb{N}} N(B_m). \)

  2. **Step 2.** For every \( m \in \mathbb{N} \) consider the well defined map \( \phi_{2,m} : N(B_1)/ < w_1 > \to B_m/N(B_m) \) given by \( < w_1 > l \mapsto N(B_m)/l. \) As in step 1 we can deduce that there is an element \( w_2 \) and a subsequence \( (C_n)_{n \in \mathbb{N}} \subset (C_n)_{n \in \mathbb{N}} \) such that \( \text{Card}(N(C_1)) > k_0c_0^4 \) and \( w_2 \in \bigcap_{m \in \mathbb{N}} N(C_m). \)

  3. **Step 3.** For \( m \in \mathbb{N} \) consider the well defined map \( \phi_{3,m} : N(C_1)/ < w_1, w_2 > \to C_m/N(C_m) \) given by \( < w_1, w_2 > l \mapsto N(C_m)/l. \) As in step 2 we deduce that there is an element \( w_3 \) and a subsequence \( (D_n)_{n \in \mathbb{N}} \subset (D_n)_{n \in \mathbb{N}} \) such that \( \text{Card}(N(A_1)) > k_0c_0^4 \) and \( w_3 \in \bigcap_{m \in \mathbb{N}} N(D_m). \)

Continuing this process ad infinitum we deduce that \( < w_m : m \in \mathbb{N} > \) is an infinite commutative subgroup. \( \blacksquare \)
Lemma 4.3. Let \( N \subset SL_3(\mathbb{C}) \) be a commutative subgroup where every element is diagonalizable, then there is \( \tau \in SL_3(\mathbb{C}) \) such that every element in \( \tau N \tau^{-1} \) is a diagonal matrix.

Proof. Let \( g = (\gamma_{ij})_{i,j=1,3} \in N - \{Id\} \), we may assume that \( g \) is a diagonal matrix with \( \gamma_{11} \neq \gamma_{33} \neq \gamma_{22} \). Now let \( h = (h_{ij})_{i,j=1,3} \in N - \{Id\} \) be a non diagonal matrix. By comparing the coefficients in the equation \( gh = h_gh \), we deduce \( h_{13} = h_{31} = h_{32} = 0 \) and \( \gamma_{11} = \gamma_{22} \). Set \( \hat{h} = \left( \begin{array}{ccc} h_{11} & h_{12} \\ h_{21} & h_{22} \end{array} \right) \), then there is \( k \in SL_2(\mathbb{C}) \) such that \( k^{-1}\hat{h}k \) is a diagonal matrix with distinct eigenvalues. Set \( C = \sqrt{\text{det}(K)^{-1}} \left( \begin{array}{cc} k & 0 \\ 0 & 1 \end{array} \right) \), then \( C^{-1}gC = g \). To conclude observe that comparing both sides of the equations \( xg = gx \), \( xc^{-1}hc = c^{-1}hcx \) for each \( x \in C^{-1}NC \) one can deduce that every element in \( C^{-1}NC \) is a diagonal matrix.

Corollary 4.4. Let \( \Gamma \leq PSL_3(\mathbb{C}) \) be a discrete infinite group, then there is an element \( \gamma \in \Gamma \) with infinite order.

Proof. By lemma 4.3, \( \Gamma \) contains an infinite commutative subgroup \( N \). If \( o(\gamma) < \infty \) for every \( \gamma \in N \), then every element in \( \Gamma \) has a lift which is a diagonal matrix. By lemma 4.3 there is an element \( \tau \in PSL_3(\mathbb{C}) \) such that \( \tau N \tau^{-1} \) is a group where every element is a diagonal matrix whose eigenvalues are roots of the unity. Thus \( N \) is non-discrete, which is a contradiction.

Now one has:

Corollary 4.5. Let \( \Gamma \leq PSL_3(\mathbb{C}) \) be a discrete group, then the following conditions are equivalent:

1. \( \Gamma \) is finite.
2. \( o(\gamma) \) is finite for all \( \gamma \in \Gamma \).
3. \( \Gamma \) acts properly discontinuously on \( \mathbb{P}_\mathbb{C}^2 \).

4.2. Complex lines and projective Groups. The following result can be proved by standard arguments, see [9].

Lemma 4.6. Let \( \Gamma \leq PSL_3(\mathbb{C}) \) be a group acting properly and discontinuously on \( \Omega \), then \( L_0(\Gamma) \cup L_1(\Gamma) \subset \mathbb{P}_\mathbb{C}^2 - \Omega \), and for every compact set \( K \subset \Omega \) one has that the set of cluster points of \( \Gamma K \) is contained in \( \mathbb{P}_\mathbb{C}^2 - \Omega \).

Lemma 4.7. Let \( \Gamma \leq PSL_3(\mathbb{C}) \) be a group acting properly discontinuously on \( \Omega \), then there is a complex line \( \ell \) such that \( \ell \subset \mathbb{P}_\mathbb{C}^2 - \Omega \).

Proof. By corollary 4.4 there is an element \( \gamma \in \Gamma \) with infinite order. Let \( \tilde{\gamma} \in SL_3(\mathbb{C}) \) be a lift of \( \gamma \). By the normal Jordan form theorem we only need to consider the following cases:

i) For all \( n \in \mathbb{N} \) one has that:

\[
\tilde{\gamma}^n = \left( \begin{array}{ccc} 1 & n & \frac{n(n-1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{array} \right).
\]

We claim that \( \bar{\epsilon_1, \epsilon_2} \subset \mathbb{P}_\mathbb{C}^2 - \Omega \). Otherwise there exists \( z \in \mathbb{C} \) such that \( [z; 1; 0] \in \Omega \). Let \( \epsilon \in \mathbb{C} \), then \( [z; 1; \frac{2(\epsilon - n)}{n(n-1)}] \xrightarrow{n \to \infty} [z; 1; 0] \). Thus for \( n(\epsilon) \) large \( (a_n(\epsilon) = \ldots \)
Let us assume that $|\lambda| < 1$. We claim that $\hat{e}_1, e_2 \subset \mathbb{P}_\mathbb{C}^2 - \Omega$ or $\hat{e}_3, e_3 \subset \mathbb{P}_\mathbb{C}^2 - \Omega$. If $\hat{e}_1, e_2 \not\subset \mathbb{P}_\mathbb{C}^2 - \Omega$, then there is $z \in \mathbb{C}^*$ such that $[z; 1; 0] \in \Omega$. Observe that for each $w \in \mathbb{C}^*$ we have $[wz; w; n \lambda^{3n-1}] \xrightarrow{n \to \infty} [z; 1; 0]$ and

$$\gamma^n \left( \left[ z; 1; \frac{n \lambda^{3n-1}}{w} \right] \right) = \left[ wz \lambda \frac{n}{n} + w; \frac{w \lambda}{n}; 1 \right] \xrightarrow{n \to \infty} [w; 0; 1],$$

for all $w \in \mathbb{C}^*$. As in the previous case we deduce that $\hat{e}_1, e_3 \subset \mathbb{P}_\mathbb{C}^2 - \Omega$.

If $\hat{\gamma}$ has another normal Jordan form, by Theorem 1.2, $L_0(\gamma) \cup L_1(\gamma)$ contains a complex line.

**Corollary 4.8.** Let $\gamma \in PSL_3(\mathbb{C})$ and $\hat{\gamma}$ be a lift of $\gamma$. The maximal open sets where $< \gamma >$ acts properly discontinuously, in terms of the Jordan's normal form of $\hat{\gamma}$, are given by:

| Normal Form of $\hat{\gamma}$ | Condition over the $\lambda$'s | Maximal Regions of Discontinuity |
|-------------------------------|-------------------------------|---------------------------------|
| $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ | $| \lambda_1 | < | \lambda_2 | < | \lambda_3 |$ | $\mathbb{P}_\mathbb{C}^2 \setminus (\hat{e}_1, e_2 \cup \{e_3\})$ |
| $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1^{-1} \end{pmatrix}$ | $| \lambda_1 | \neq 1$ | $\mathbb{P}_\mathbb{C}^2 \setminus (\hat{e}_1, e_2 \cup \{e_3\})$ |
| $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ | | $\Omega_{Kul}(\Gamma)$ |
Lemma 4.9. Let \( \gamma \in PSL_3(\mathbb{C}) - \{Id\} \) and \( \tilde{\gamma} \) be a lift of \( \gamma \). The set of invariant lines under \( \gamma \), in terms of Jordan form of \( \tilde{\gamma} \), is given by:

| Normal Form | Condition over the \( \lambda \)'s | Invariant Lines |
|------------|--------------------------------|-----------------|
| \[
\begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}
\] | \( \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1 \) | \( \langle \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_2 \rangle \) |
| \[
\begin{pmatrix}
\lambda_1 & 1 & 0 \\
0 & \lambda_1 & 0 \\
0 & 0 & \lambda_1^{-2}
\end{pmatrix}
\] | \( \lambda_1 \neq 1 \) | \( \langle \tilde{e}_1, \tilde{e}_3 \rangle \) |
| \[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\] | \( \lambda_1 = 1 \) | \( \langle \tilde{e}_1, \tilde{e}_2 \rangle \) |

Proof. By the normal Jordan form theorem we must consider the following cases:

Case 1. \( -\gamma \) is diagonalizable- In this case we will consider the following options:

Option 1. -Every eigenvalue has multiplicity 1- Let \( \{u, v, w\} \) be the set of fixed points of \( \gamma \) and \( \ell \) be an invariant line under \( \gamma \). We claim that \( \ell = \overrightarrow{u, v} \) or \( \ell = \overrightarrow{w, \tilde{\gamma}} \) or \( \ell = \overrightarrow{\tilde{\gamma}, \tilde{\gamma}} \). Assume, on the contrary, that \( \ell \) is different from these lines, then \( \ell \) contains exactly one fixed point, say \( u \). Let \( K = \overrightarrow{\tilde{\gamma}, \tilde{\gamma}} \). Then \( K \cap \ell = \ast \neq u \) and \( \ast \) is a fixed point. Which is a contradiction.

Option 2. -One eigenvalue as multiplicity 2-. By Theorem 1.2 we know that \( Fix(\gamma) = \overrightarrow{\tilde{e}_1, \tilde{e}_2} \cup \{\tilde{e}_3\} \). Thus every line that contains \( \tilde{e}_3 \) is invariant. Let \( \ell \) be an invariant line such that \( \tilde{e}_3 \notin \ell \) and \( c \in \ell \), then \( \gamma(c) = \gamma(\tilde{e}_3, c \cap \ell) = c \). Thus \( c \in \overrightarrow{\tilde{e}_1, \tilde{e}_2} \). In consequence \( \ell = \overrightarrow{\tilde{e}_1, \tilde{e}_2} \).

Case 2. \( -\gamma \) have at most two eigenvectors linearly independent- In this case \( \tilde{\gamma} \) has the following Jordan’s normal form:

\[
\begin{pmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 0 \\
0 & 0 & \lambda^{-2}
\end{pmatrix}
\]

Thus we must consider the options:

Option 1. \( -\lambda^3 \neq 1 \) - Let \( \ell \) be an invariant line. We claim that \( \ell = \overrightarrow{\tilde{e}_1, \tilde{e}_2} \) or \( \ell = \overrightarrow{\tilde{e}_1, \tilde{e}_3} \). Assume that \( \ell \) is different from these lines, then there is \( [w] = [z_1; z_2; z_3] \in \ell \) with \( z_2z_3 \neq 0 \). Consider the equation \( 0 = \alpha_1w + \alpha_2\gamma(w) + \alpha_3\gamma^2(w) \). One can check that this equation is equivalent to the system:

\[
\begin{align*}
\alpha_1 + \alpha_2\lambda^{-2} + \alpha_3\lambda^{-4} &= 0 \\
\alpha_1 + \alpha_2\lambda + \alpha_3\lambda^2 &= 0 \\
\alpha_2 + 2\alpha_3\lambda &= 0.
\end{align*}
\]

Since the determinant of the system is \( (\lambda^{-2} - \lambda)^2 \neq 0 \), the equation has only the trivial solution. Therefore \( [w]_2, [\tilde{\gamma}(w)]_2, [\tilde{\gamma}^2(w)]_2 \) are not contained in a complex line, which is a contradiction.

Option 2. \( -\lambda = 1 \) - Let \( \ell \) be an invariant line. Assume that \( \ell \neq \overrightarrow{\tilde{e}_1, \tilde{e}_2} \) and \( \ell \neq \overrightarrow{\tilde{e}_1, \tilde{e}_3} \). Thus there is a point \( [w] = [z_1; z_2; z_3] \in \ell \) such that \( z_2z_3 \neq 0 \). Consider
the equation \( 0 = \alpha_1 w + \alpha_2 \gamma(w) + \alpha_3 e_1 \), then it is not hard to check that such equation is equivalent to the system:

\[
\begin{align*}
\alpha_2 z_2 + \alpha_3 &= 0 \\
\alpha_1 + \alpha_2 &= 0.
\end{align*}
\]

Such system has the non-trivial solutions \( \alpha_1 = z_2^{-1}, \alpha_2 = -z_2^{-1}, \alpha_3 = 1 \). Therefore \([w]_2, [\gamma(w)]_2, e_1 \) lies in the same line. Since \( \ell \) is invariant we conclude that \( e_1 \in \ell \).

Case 3.- \( \gamma \) has the normal form:

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}.
\]

Let \( \ell \) be an invariant line and assume that \( \ell \neq \vec{e}_1, \vec{e}_2 \), then there exist \([w]_2 = [z_1, z_2, z_3]_2 \in \ell \) with \( z_3 \neq 0 \). Since the equation \( \alpha_1 w + \alpha_2 \gamma(w) + \alpha_3 \gamma^2(w) = 0 \) is equivalent to the system:

\[
\begin{align*}
\alpha_1 + \alpha_2 + \alpha_3 &= 0 \\
\alpha_2 + 2\alpha_3 &= 0 \\
\alpha_3 &= 0,
\end{align*}
\]

and such system has only the trivial solutions, we conclude that \([w]_2, [\gamma(w)]_2, [\gamma^2(w)]_2 \) are not contained in a complex line, which contradicts the initial assumption.

4.3. Controllable Groups. Let us begin with some examples

**Example 4.10.** Suspension with a group.- Let \( \Gamma \leq PSL_2(\mathbb{C}) \) be a discrete group with non-empty discontinuity region, \( G \leq \mathbb{C}^* \) be a discrete group and \( i : SL_2(\mathbb{C}) \to SL_3(\mathbb{C}) \) be the inclusion given by: \( i(h) = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \). The suspension of \( \Gamma \) with respect to \( G \), denoted \( Susp(\Gamma, G) \), is defined by:

\[
Susp(\Gamma, G) = \left\{ i(h) : h \in \left[\Gamma\right]^{-1}_2, \left\{ \begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g^{-2} \end{pmatrix} : g \in G \right\} \right\}.
\]

Observe that in case of \( G = \{\pm 1\} = \mathbb{Z}_2 \), \( Susp(\Gamma, \mathbb{Z}_2) \) coincides with the double suspension of \( \Gamma \) defined in [13], and when \( \left[\Gamma\right]^{-1}_2 \) contains a subgroup \( \tilde{\Gamma} \) for which \( \left[\tilde{\Gamma}\right]_2 = \Gamma \), then \( i(\tilde{\Gamma}) \) coincides with the suspension of \( \Gamma \) defined in [18]. Moreover, taking \( \Lambda(\Gamma) \) on \( \vec{e}_1, \vec{e}_2 \) as the usual limit set of the action of \( Susp(\Gamma, G) \) on \( \vec{e}_1, \vec{e}_2 \) (which is "identical" to the action of \( \Gamma \) on \( \mathbb{P}^1_\mathbb{C} \)), one can show, by the same arguments used in [13], the following:

\[
\Lambda_{Kul}(Susp(\Gamma, G)) = \begin{cases} 
\bigcup_{p \in \Lambda(\Gamma)} \overline{p, \vec{e}_3} & \text{If } G \text{ is finite}, \\
\bigcup_{p \in \Lambda(\Gamma)} \overline{p, \vec{e}_3} \cup \overline{\vec{e}_1, \vec{e}_2} & \text{If } G \text{ is infinite}.
\end{cases}
\]

**Example 4.11.** Fundamental groups of Inoue surfaces (see [23]).

(1) \( S_M \) family.- Let \( M \in SL_3(\mathbb{Z}) \) having eigenvalues \( \alpha, \beta, \overline{\beta} \) with \( \alpha > 1, \beta \neq \overline{\beta} \). Choose a real eigenvector \( (a_1, a_2, a_3) \) belonging to \( \alpha \) and an eigenvector \( (b_1, b_2, b_3) \) belonging to \( \beta \). Now let \( G_M \) be the group of automorphisms of \( \mathbb{H} \times \mathbb{C} \) generated by:

\[
\begin{align*}
\gamma_0(w, z) &= (\alpha w, \beta z), \\
\gamma_i(w, z) &= (w + a_i, z + b_i) \quad i = 1, 2, 3.
\end{align*}
\]
Then $G_M \leq \text{Sol}_1^2$, $G_M$ acts properly discontinuously on $\mathbb{H} \times \mathbb{C}$ and $(\mathbb{H} \times \mathbb{C})/G_M$ is a compact surface.

(2) $S_N^3$ family.- Let $N \in SL_2(\mathbb{Z})$ having real eigenvalues $\alpha, \alpha^{-1}$ with corresponding real eigenvectors $(a_1, a_2), (b_1, b_2)$. Choose a non-zero integer $r$, a complex number $t$ and complex number $c_1, c_2$ satisfying an integrability condition to be made precise. Define the automorphisms of $\mathbb{H} \times \mathbb{C}$ generated by:

$$\gamma_0(w, z) = (\alpha w, \beta z + t),$$
$$\gamma_i(w, z) = (w + a_i, z + b_i w + c_i) \text{ for } i = 1, 2, 3,$$
$$\gamma_3(w, z) = (w, z + r^{-1}(b_1 a_2 - b_2 a_1)).$$

Then these generate a group $G_M \leq \text{Sol}_1^2$ and $G_M$ acts properly discontinuously, freely and with compact quotient on $\mathbb{H} \times \mathbb{C}$.

(3) $S_N^3$ family.- This family is defined modifying the above as follows, let $N \in GL_2(\mathbb{Z})$ with real eigenvalues $\alpha, -\alpha^{-1}$. The rest as above except that we do no choose a $t$, but define instead $\gamma_0(w, z) = (\alpha w, -z)$.

Clearly each of these groups has a fixed point $p \in \mathbb{P}^2$ and the dynamics of any complex line that contains $p$ is governed by a non-discrete fuchsian group (for a precise argument see the proof of (1) of Theorem [1.1]).

With these examples in mind the following proposition is natural:

**Lemma 4.12.** Let $p \in \mathbb{P}^2$, $\Gamma \leq \text{PSL}_3(\mathbb{C})$ with $\Gamma \cap p = p$ and $\ell$ a complex line not containing $p$. Define $\Pi = \Pi_{p, \ell} : \Gamma \rightarrow \text{Bihol}(\ell)$ given by $\Pi(g)(x) = \pi(g(x))$ where $\pi = \pi_{p, \ell} : \mathbb{P}^2 \mathbb{C} - \{p\} \rightarrow \ell$ is given by $\pi(x) = \overline{x - p} \cap \ell$ then:

1. $\pi$ is a holomorphic function.
2. $\Pi$ is a group morphism.
3. If $\text{Ker}(\Pi)$ is finite and $\Pi(\Gamma)$ is discrete, then $\Gamma$ acts properly discontinuously on $\Omega = \bigcup_{z \in \Pi(\Gamma)} \overline{z - p} - \{p\}$. Here $\Omega(\Pi(\Gamma))$ denotes the discontinuity set of $\Pi(\Gamma)$.
4. If $\Gamma$ is discrete, $\Pi(\Gamma)$ is non-discrete and $\ell$ is invariant, then $\Gamma$ acts properly discontinuously on $\Omega = \bigcup_{z \in \Pi(\Gamma)} \overline{z - p} - (\ell \cup \{p\})$.

**Proof.** 1. If $p = e_3$ and $\ell = \overline{\tau_1, \tau_2}$, then $\pi_{e_3, \tau_1, \tau_2}([z; w; x]) = [z; w; 0]$, which is a holomorphic function. If this is not the case, take $g \in PSL_3(\mathbb{C})$ such that $g(p) = e_3$ and $g(\ell) = \overline{\tau_1, \tau_2}$. Now one can check the following identities:

$$\overline{h(x), p} = \Pi_{p, \ell}(\overline{h(x), p});$$
$$\overline{h(x), p} = h(\overline{x, p})$$
where $h \in \Gamma$ and $x \in \ell$.

To finish observe that:

$$\pi_{p, \ell}(x) = g^{-1}(\pi_{e_3, \overline{\tau_1, \tau_2}}(g(x)))$$

2. Step 1.- $\Pi(g \circ h) = \Pi(g) \circ \Pi(h)$.- From equation [4.2] we deduce:

$$\Pi_{p, \ell}(g) \circ \Pi_{p, \ell}(h)(x) = \overline{\Pi(g)(h(x)), p} \cap \ell = \Pi_{p, \ell}(g \circ h)(x).$$

Step 2.- $\Pi(\Gamma) \subset \text{Bihol}(\ell)$.- It is enough to observe that $\Pi_{p, \ell}(g) = \pi_{p, \ell} \circ g |_{\ell}$.

3. Let $K \subset \Omega$ be a compact set. Define $K(\Gamma) = \{\gamma \in \Gamma : g(K) \cap K \neq \emptyset\}$ and assume that $K(\Gamma)$ is infinite. Let $(\gamma_n)_{n \in \mathbb{N}}$ be an enumeration of $K(\Gamma)$. Since
Ker(\(\Pi\)) is finite, there is a \(k_1 \in \mathbb{N}\) such that \(\Pi(\gamma_n) \neq \Pi(\gamma_1)\) whenever \(k_1 \leq n\). Repeating the same argument for \(\gamma_k\), we deduce that there exist \(k_2 < k_3\) such that \(\pi(\gamma_n) \neq \Pi(\gamma_k)\) for \(k_2 \leq n\). By a recursive argument we deduce that there is a subsequence \((h_n)_{n \in \mathbb{N}}\) of \((\gamma_n)_{n \in \mathbb{N}}\) such that \(\Pi(h_n) \neq \Pi(h_k)\) if \(n \neq k\). Therefore \(\{\Pi(h_n) : n \in \mathbb{N}\} \subset \langle g \in \Pi(\Gamma) : g(\pi(K)) \cap \pi(K) \neq \emptyset \rangle\), which is a contradiction.

4. Take \(p = e_3\), \(l = e_1, e_2\) and assume that the action is not properly discontinuous, then there are \(k = [z; h, w], q \in \Omega\), \((k_n)_{n \in \mathbb{N}} \subset \Omega\) and \((\gamma_n = (\gamma_{ij}^{(n)})_{i,j=1,3})_{n \in \mathbb{N}} \subset \Gamma\) a sequence of distinct elements such that \(k_n \xrightarrow[n \to \infty]{} k\) and \(\gamma_n(k_n) \xrightarrow[n \to \infty]{} q\). By corollary [3.27] we can assume that there is a holomorphic map \(f : Eq(\Pi(\Gamma)) \to Eq(\Pi(\Gamma))\) such that \(\Pi(\gamma_n) \xrightarrow[n \to \infty]{} f\) uniformly on compact sets of \(Eq(\Pi(\Gamma))\). Moreover, \(f \in Bihol(\ell)\) or \(f\) is a constant function \(c \in \partial(\Pi(\Gamma))\). Since \(\pi(\gamma_n(k_n))\) tends to \(\pi(q) \in Eq(\Pi(\Gamma))\) as \(n\) tends to \(\infty\), we conclude that \(f\) is non-constant.

Thus \(f[z; w; 0] = [\gamma_{11}z + \gamma_{12}w; \gamma_{21}z + \gamma_{22}w; 0]\) with \(\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21} = 1\). Since \(\gamma_{13}^{(n)} = \gamma_{23}^{(n)} = \gamma_{31}^{(n)} = \gamma_{32}^{(n)} = 0\), \(\gamma_{33}^{(n)}(\gamma_{11}^{(n)}\gamma_{22}^{(n)} - \gamma_{12}^{(n)}\gamma_{21}^{(n)}) = 1\) we can assume that \(\gamma_{ij}^{(n)} \xrightarrow[n \to \infty]{} \gamma_{ij}\). In consequence we can assume that there is \(\gamma_{33}^{(n)} \in \mathbb{C}^*\) such that \(\gamma_{33}^{(n)} \xrightarrow[n \to \infty]{} 0\) or \(\gamma_{33}^{(n)} \xrightarrow[n \to \infty]{} \infty\). Which implies \(\gamma_n(k_n) \xrightarrow[n \to \infty]{} [\gamma_{11}z + \gamma_{12}h; \gamma_{21}z + \gamma_{22}h; 0]\) or \(\gamma_n(k_n) \xrightarrow[n \to \infty]{} [0; 0; 1]\). Which is a contradiction. Therefore

\[
\begin{pmatrix}
\gamma_{11}^{(n)} & \gamma_{12}^{(n)} & 0 \\
\gamma_{21}^{(n)} & \gamma_{22}^{(n)} & 0 \\
0 & 0 & \gamma_{33}^{(n)}
\end{pmatrix} \xrightarrow[n \to \infty]{}
\begin{pmatrix}
\gamma_{11}\sqrt{\gamma_{33}}^{-1} & \gamma_{12}\sqrt{\gamma_{33}}^{-1} & 0 \\
\gamma_{21}\sqrt{\gamma_{33}}^{-1} & \gamma_{22}\sqrt{\gamma_{33}}^{-1} & 0 \\
0 & 0 & \gamma_{33}
\end{pmatrix} \in PSL_3(\mathbb{C}).
\]

Which is a contradiction since \(\Gamma\) is discrete. ■

4.4. Quasi Co-compact Groups.

Lemma 4.13. Let \(\Gamma \leq PSL_3(\mathbb{C})\) be a quasi co-compact group over a domain \(\Omega\), then \(\Gamma\) is finitely generated and has a fundamental region \(D\) such that \(D \subset \Omega\) is compact.

Proof. By the proof of corollary [17] we deduce that \(\Gamma\) is finitely generated and that there is a metric \(d\) on \(\Omega\) such that \((\Omega, d)\) is geodesically complete, geodesically connected and \(\Gamma\) is a subgroup of isometries with respect to \(d\). Now, since \(\Omega/\Gamma\) is compact one has (see [16]) that the Dirichlet region for \(\Gamma\) is a fundamental region for \(\Gamma\) with compact closure on \(\Omega\). ■

Corollary 4.14. Let \(\Gamma \leq PSL_3(\mathbb{C})\) be a quasi co-compact infinite group over a domain \(\Omega\), and let \(\mathbb{H} = \{[0; z; 1] \mid Re(z) > 0\}, \mathbb{R} = \{[0; z; 1] \mid Im(z) = 0\}, \infty = e_2, \mathbb{R} = \mathbb{R} \cup \{\infty\}\), then, up to projective equivalence, one of the following facts hold:

1. \(\Omega = \mathbb{H}^2;\)
2. \(\Omega = \mathbb{H}^2 - \{e_1, e_2 \cup \{e_3\}\};\)
3. \(\Omega = \mathbb{H}^2 - \{e_1, e_2 \cup e_3, e_3 \cup e_5, e_2\};\)
4. \(\Omega = \mathbb{H}^2 - \{e_1, e_2 \cup e_1, e_3\};\)
5. \(\Omega = \mathbb{H}^2 - \{e_1, e_2\};\)
6. \(\Omega = \bigcup_{z \in \mathbb{H}} [z, e_1 - \{e_1\};\)
The action of $\Gamma_0$ one of the following cases must occur: 

$\text{H}0$ cluster point of $\Gamma$ lines and $D \times \text{H}$ discontinuously.

Proof. By Theorems 1.6, 1.8 and corollary 2.1 cases 1-6 correspond to $\Omega = \mathbb{H}^2$, $C^2 - \{0\}$, $C^* \times C^*$, $C^* \times C$, $C^2$, $C \times \mathbb{H}$. In the last case, the developing map $D : C \times \mathbb{H} \rightarrow \mathbb{P}_C^2$ is given by $(w, z) \mapsto [A(z) : B(z) ; e^{-\mu w}]$ where $\mu \in C^*$ and $A, B : \mathbb{H} \rightarrow C$ are holomorphic maps. Since $D$ is a local homomorphism we deduce that $D(C \times \{x\})$ contains more than one point and $D(C \times \mathbb{H})$ is open. Therefore $A, B$ do not have a common zero and $A, B$ are not constant. In consequence $D = \{(A(z) : B(z) ; 0) : z \in \mathbb{H}\}$ is a hyperbolic domain on $\tilde{e}_1, \tilde{e}_2$ which trivially satisfy $D(C \times \mathbb{H}) = (\bigcup_{z \in D} \tilde{z}, \tilde{e}_3 - \{e_3\}) - \tilde{e}_1, \tilde{e}_2$.

By standard arguments we can show the following (see [9]):

**Corollary 4.15.** Let $\Gamma \leq \text{PSL}_3(\mathbb{C})$ be a group acting properly discontinuously on a domain $\Omega$, then:

1. If $\Gamma$ is a quasi co-compact group over $\Omega$ and $R$ is a fundamental region for $\Gamma$, then every point $p \in \partial \Omega$ is a cluster point of $\Gamma R$.
2. If $\Omega_0 \subset \Omega$ is a domain and $\Omega_0 / \text{Isot}(\Omega_0, \Gamma)$ is compact, then $\Omega = \Omega_0$.
3. If $\Omega_0$ is a connected component of $\Omega$, then $\Omega_0 / \text{Isot}(\Omega_0, \Gamma) = \Gamma(\Omega_0) / \Gamma$.

5. **Proof of the Main Theorem**

Proof of Theorem 4.14. Let $p : \Omega \rightarrow \Omega / \Gamma$ be the quotient map and $M \subset \Omega / \Gamma$ be a compact connected component, then $\Omega = p^{-1}(M)$ has the form $\Gamma_0$, where $\Omega_0$ is a connected component of $\Omega$. Set $\Gamma_0 = \text{Isot}(\Omega_0, \Gamma)$. By lemma 2.1 there is a developing pair $(D, \mathcal{H})$ such that $D(M) = \Omega_0$ and $\mathcal{H}(\pi_1^{Orb}(M)) = \Gamma_0$. By corollary 4.13 one of the following cases must occur:

Case 1. $\Omega_0 = \mathbb{H}^2$. By (2) of corollary 4.15 and Theorem 1.4 we deduce that $\Gamma$ is fuchsian and $\mathbb{H}^2 \subset \Omega = \Omega_{Kul}(\Gamma)$ is the largest open set on which $\Gamma$ acts properly discontinuously.

Case 2. $\Omega_0 = C^2 - \{0\}$, $\Omega_0 = C^* \times C^* \subset C^2$, $C \times C^2$. Trivially in any of these cases $\Gamma$ is an elementary group.

Case 3. $\Omega_0 = \bigcup_{z \in \mathbb{H}} \tilde{e}_1, \tilde{e}_2 - \{e_1\}$. Let $\ell_1, \ell_2 \subset \partial \Omega_0 \subset \mathbb{P}_C^2 - \Omega$ be distinct complex lines and $g \in \Gamma$, then $\{e_1\} = \ell_1 \cap \ell_2$ and $g(\ell_1), g(\ell_2) \subset \mathbb{P}_C^2 - \Omega$ are different complex lines. Since $g(\ell_1) \cap \bigcup_{z \in \mathbb{H}} \tilde{e}_1, \tilde{e}_2 \neq \emptyset$ we deduce $\{e_1\} \subset g(\ell_1)$. In consequence $g(e_1) = e_1$. Therefore we can speak of $\Pi = \Pi_{p \ell}$ where $p = e_1$ and $\ell = \tilde{e}_2, e_3$. Now, let $p_1, p_2, p_3 \in \mathbb{R} \subset \ell$ be distinct elements and $K$ a fundamental region for the action of $\Gamma_0$ on $\Omega_0$. By (1) of corollary 4.15 we know that $p_i, 1 \leq i \leq 3$, is a cluster point of $\Gamma_0 K$. Hence $p_i, 1 \leq i \leq 3$, is a cluster point of $P(\Pi_0)(\pi(K))$. Thus by proposition 5.26 corollaries 5.21, 5.24 we deduce that $\Pi(\Gamma_0)$ is non-discrete and $\mathcal{L}(\Pi(\Gamma_0)) = \mathbb{R}$. Since $\Pi(\Gamma(\Omega))$ is an open set which omits $\mathbb{R}$ and contains $\mathbb{H}$ we conclude that $\mathcal{L}(\Pi(\Gamma)) \neq \ell$. Finally, since $\mathcal{L}(\Pi(\Gamma_0)) \subset \mathcal{L}(\Pi(\Gamma))$ with $\mathcal{L}(\Pi(\Gamma_0))$ is a circle in $\ell$, by proposition 5.26 we conclude $\mathcal{L}(\Pi(\Gamma)) = \mathbb{R}$. Moreover by (7) of Theorem 1.8 and corollary 5.21 one has that $\Pi(\Gamma) \infty = \infty$, which ends this part.

Case 4. $\Omega_0 = \bigcup_{z \in D} \tilde{z}, e_1 - (\Omega \cup \{e_1\})$. Let $\gamma \in \Gamma$. Since $\Omega$ is hyperbolic, we can assume that there exist $e_1, e_2, e_3, e_4$ distinct complex lines such that $e_1 \neq e_2$ and $\gamma(e_1) \neq e_3$. Let $z_1, z_2 \in \Omega$ then $\xi_{ij} = \tilde{z}_i, e_1 \cap \gamma(\ell_j) \subset \mathbb{P}_C^2 - \Omega$. From this $\xi_{ij} \in \{e_1, z_j\}$. Thus $e_1 \in \gamma(\ell_j) (i = 1, 2)$. Therefore $\gamma(e_1) = e_1$. 

(7) There exist $D \subset \tilde{e}_1, \tilde{e}_2$ a a hyperbolic domain such that:

$$\Omega = \bigcup_{z \in D} \tilde{z}, e_3 - (\{e_3\} \cup \tilde{e}_1, \tilde{e}_2).$$
Moreover, we have shown: if $\ell$ is a line contained in $\mathbb{P}_C^2 - \Omega$ and $e_1 \notin \ell$, then $\ell = \overline{e_3, e_2}$. Now we can consider $\Pi = \Pi_{p, \ell}$, where $p = e_1$ and $\ell = \overline{e_3, e_2}$. Consider the following cases:

Case 1. $\Pi(\Gamma_0)$ is non-discrete- In this case $\overline{e_3, e_2} \subset \mathbb{P}_C^2 - \Omega$ and $e_1 \notin \gamma(\overline{e_3, e_2}) \subset \mathbb{P}_C^2 - \Omega$ for every $g \in \Gamma$. That is $\Gamma_0 \overline{e_3, e_2} = \overline{e_3, e_2}$.

Case 2. $\Pi(\Gamma_0)$ is discrete- By (3) of lemma 4.12 and corollary 4.5 we deduce that $\text{Ker}(\Pi)$ is infinite. By corollary 4.4 there is $\gamma \in \text{Ker}(\Pi)$ with infinite order. Since $\gamma(\ell \cup \{e_1\}) = \ell \cup \{e_1\}$ we conclude that:

$$\gamma = \begin{pmatrix} a^{-2} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \quad \text{where} \quad |a| \neq 1.$$ 

This implies that $e_1 \notin \overline{e_3, e_2} \subset L_0(\Gamma) \subset \mathbb{P}_C^2 - \Omega$ and therefore $\Gamma_0 \overline{e_3, e_2} = \overline{e_3, e_2}$. 

6. Technicalities on Inoue and Uniformizable Elliptic Affine Surfaces

6.1. On Finite Covering Orbifolds Maps with Base an Inoue Surface.

Trough this subsection $\Omega_0, \ell, \Pi, \pi, \Gamma_0$ will be taken as is the case $\Omega_0 = \mathbb{C} \times \mathbb{H}$ of the proof of Theorem 0.1. Also, let $\tilde{\Gamma}_0 \leq \Gamma_0$ be a normal and torsion free subgroup with finite index, then we prove:

**Lemma 6.1.** Either $\text{Ker}(\Pi |_{\tilde{\Gamma}_0})$ is trivial or every element $\gamma \in \text{Ker}(\Pi |_{\tilde{\Gamma}_0})$ has a lift $\tilde{\gamma}$, such that:

$$\tilde{\gamma} = \begin{pmatrix} 1 & 0 & \tau(\gamma) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\tau(\gamma) \neq 0$.

**Proof.** If $\text{Ker}(\Pi |_{\tilde{\Gamma}_0})$ is non-trivial, then $\text{Ker}(\Pi |_{\tilde{\Gamma}_0})$ is infinite, then by corollary 4.4 there is an element $\gamma_0$ with infinite order, let $\tilde{\gamma}_0$ be a lift of $\gamma_0$. By lemma 2.4 we may assume that $\tilde{\Gamma}_0 \leq \text{Sol}^{0}_{i_1}$ or $\tilde{\Gamma}_0 \leq \text{Sol}^{i} 0_{i}^4$ or $\tilde{\Gamma}_0 \leq \text{Sol}^{i} 0_{i}^4$. In any case there are $\epsilon = e^{2\pi i \theta}, \beta, \ell \in \mathbb{C}$, such that for all $n \in \mathbb{N}$ one has:

$$\tilde{\gamma}_0^n = \begin{pmatrix} e^n & \beta \sum_{j=0}^{n-1} \epsilon^j & \ell \sum_{j=0}^{n-1} \epsilon^j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Since $o(\gamma_0) = \infty$ we conclude that $\epsilon = 1$ or $\theta \in \mathbb{R} - \mathbb{Q}$. If $\theta \in \mathbb{R} - \mathbb{Q}$ it is clear that $\tilde{\gamma}_0$ is diagonalizable with unitary eigenvalues, then by Theorem 1.2 $L_0(\gamma_0) \cup L_0(\gamma_0) = \mathbb{P}_C^2$ which is a contradiction. Thus $\epsilon = 1$.

Now consider the case $\tilde{\Gamma}_0 \leq \text{Sol}^{0}_{i}$ (respectively $\tilde{\Gamma}_0 \leq \text{Sol}^{1}_{i} 0$). Since $\Pi(\Gamma_0)$ contains hyperbolic elements (see the proof of Theorem 0.1) there is $\tau = [(\tau_{i,j})^3_{i,j=1}]_{2} \in \tilde{\Gamma}_0$ such that $\tau_{22} > 1$ and $\tau_{11} = 1$. To conclude, observe that:

$$(\tau_{i,j})^n_{i,j=1} = (\gamma_{i,j})_{i,j=1}^{-n} = \begin{pmatrix} 1 & \tau_{22}^{-n} \beta & \ell - \tau_{23} \tau_{22}^{-1} \beta \sum_{j=0}^{n-1} \tau_{22}^{-j} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and when $n$ tends to $\infty$ the right hand term tends to:
Let $\gamma_0 \in PSL_3(\mathbb{C})$ be an element with finite order such that $\gamma_0(\mathbb{C} \times \mathbb{H}) = \mathbb{C} \times \mathbb{H}$ and $\pi_1(M)$ be the fundamental group of an Inoue surface. If $\pi_1(M)$ is a normal subgroup of $G = \langle \gamma_0, \pi_1(M) \rangle$, one can show:

1. $\gamma_0 \in Ker(\pi_1(M))$.
2. $Fix(\gamma_0) = \ell_{\gamma_0} \cup \{e_1\}$, where $\ell_{\gamma_0}$ is a complex line not containing $e_1$.
3. $Ker(\pi_1(M))$ is infinite.

**Proof.** Since $\gamma_0$ has finite order we conclude that $\pi_1(M)$ is a subgroup of $G$ with finite index. From the case $\gamma_0 = \mathbb{C} \times \mathbb{H}$ of the proof of Theorem [2.3] we deduce that $\pi(\gamma_0)\mathbb{H} = \mathbb{H}$ and $\pi(\gamma_0)(\mathbb{H}) = \mathbb{H}$. Thus $\Pi(\gamma_0) = Id$. Hence there are $\vartheta = e^{\tau_\rho i\theta}$ with $\theta \in \mathbb{Q} - \mathbb{Z}$, $\kappa, \sigma \in \mathbb{C}$ such that $\gamma_0$ has a lift $\tilde{\gamma}_0$ given by:

$$
\tilde{\gamma}_0 = \begin{pmatrix}
\vartheta & \kappa & \sigma \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

Therefore $\gamma_0 \in Ker(\Pi)$ and $Fix(\gamma_0) = \ell_{\gamma_0} \cup \{e_1\}$, where $\ell_{\gamma_0}$ is a complex line that does not contain $e_1$. We notice that every element $h \in G$ with $o(h) < \infty$ is in $Ker(\Pi|_{\gamma_0})$.

Assume on the contrary, that $Ker(\Pi|_{\pi_1(M)})$ is finite. Set $\Gamma = \langle \{ \gamma \in G : o(\gamma) < \infty \} \rangle$, then $\Gamma$ is a subgroup of $Ker(\Pi)$. Since $[G;\pi_1(M)] < \infty$, we conclude that $\Gamma$ is finite. Set $C_T(\gamma_0) = \{ h \in G : hg = gh \}$, then $C_T(\gamma_0)$ is a subgroup of $G$ with $[G;C_T(\gamma_0)] < \infty$, for otherwise, let $(\gamma_i) \subset G$ be such that $C_T(\gamma_0)\gamma_i = C_T(\gamma_0)\gamma_j$ if $i \neq j$. Hence $\gamma_i \gamma^{-1}_j \gamma_i \gamma^{-1}_j = \gamma^{-1}_i \gamma_0 \gamma_j \gamma^{-1}_j \gamma_0 \gamma_j = \gamma^{-1}_i \gamma_0 \gamma_0 \gamma_j \gamma_0 \gamma_j$. This is a contradiction. Thus $[\Pi(G);\Pi(C_T(\gamma_0))] < \infty$. Now, since $\Pi(G)$ is non-discrete we deduce that $\Pi(C_T(\gamma_0))$ is non-discrete. Thus we have that $\ell_{\gamma_0}$ is $C_T(\gamma_0)$-invariant and $\ell_{\gamma_0} \subset L_0(G) \cup L_1(G)$. On the other hand, one has $\ell_{\gamma_0} \cap (\bigcup_{e \in \mathbb{H}} \mathbb{Z} \mathbb{Z}, e \{e_1\}) \neq \emptyset$, which is a contradiction.

**Lemma 6.3.** Let $M$ be an Inoue surface and $g : M \to M$ an $(\mathbb{P}_\mathbb{C}, PSL_3(\mathbb{C}))$-equivalence, different from the identity, with at least one fixed point. Then $g$ has infinite order.

**Proof.** Assume, on the contrary, that there is $g : M \to M$ an $(\mathbb{P}_\mathbb{C}, PSL_3(\mathbb{C}))$-equivalence, different from the identity with finite order and with at least one fixed point $z \in M$. Let $P : \mathbb{C} \times \mathbb{H} \to M$ be the universal covering map and $x \in P^{-1}(z)$. By the standard lifting lemma for covering maps, there is a homeomorphism $\hat{g} : \mathbb{C} \times \mathbb{H} \to \mathbb{C} \times \mathbb{H}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{C} \times \mathbb{H}, x & \xrightarrow{\hat{g}} & (\mathbb{C} \times \mathbb{H}, x) \\
P & \downarrow & P \\
(M, z) & \xrightarrow{g} & (M, z).
\end{array}
$$
Since \( P, g \) are \((\mathbb{P}_3^2, \text{PSL}_3(\mathbb{C}))\)-maps we deduce that \( \hat{g} \) is an \((\mathbb{P}_3^2, \text{PSL}_3(\mathbb{C}))\)-map. Since \( \mathbb{C} \times \mathbb{R} \) has the projective structure induced by the natural inclusion we conclude that \( \hat{g} \) is the restriction of an element \( \gamma_g \in \text{PSL}_3(\mathbb{C}) \), moreover \( \gamma_g \) has a lift \( \hat{\gamma}_g \) given by:

\[
\hat{\gamma}_g = \begin{pmatrix}
\vartheta & \kappa & \sigma \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Now by diagram 6.2 we can ensure that \( \hat{g} \) has finite order and \( \pi_1(M) \) is a normal subgroup with finite index of \( \Gamma = \langle \gamma_g, \pi_1(M) \rangle \). By lemma 6.2 we have that \( \text{Ker}(\pi_1(M)) \) is infinite.

We have the following possibilities:

Possibility 1. \( \pi_1(M) \leq \text{Sol}_3^4 \) or \( \pi_1(M) \leq \text{Sol}_1^4 \). In this case, we claim that for each \( \tau = \left[[(\tau_{ij})]_{i,j=1}^3\right]_2 \in \pi_1(M) \) one has:

\[
\tau_{12} = \kappa \frac{\tau_{22} - 1}{1 - \vartheta}.
\]

Let \( h = \tau \gamma_g \gamma_{12} \gamma_{12}^{-1} \in \text{Ker}(\pi_1(M)) \), then \( h \) has a lift \( \hat{h} \) given by:

\[
\hat{h} = \begin{pmatrix}
1 & \kappa \tau_{22}^{-1}(1 - \vartheta) + \kappa \\
0 & 1 \\
0 & 0
\end{pmatrix}.
\]

Since \( [\Gamma; \pi_1(M)] < \infty \), there is \( n \in \mathbb{N} \) such that \( h^n \in \text{Ker}(\pi_1(M)) \). By lemma 6.1 this implies \( \tau_{12} = \kappa(\tau_{22} - 1)/(1 - \vartheta) \) which proves our claim.

Now, since \( \pi_1(M) \) is non-discrete, it is followed that there is a sequence \( (\phi_n = [[(\phi_{i,j}^{(n)})]_{i,j=1}^3]_{2} \in \pi_1(M) \) such that:

1. \( \Pi(\phi_n)(x) = \phi_{22}^{(n)} x + \phi_{23}^{(n)} \) is a sequence of distinct elements;
2. \( \phi_{22}^{(n)} \xrightarrow{n \to \infty} 1; \)
3. \( \phi_{23}^{(n)} \xrightarrow{n \to \infty} 0; \)
4. \( \text{Im}(\phi_{13}^{(n)}) \xrightarrow{n \to \infty} 0 \), this fact is attained, since \( \phi_{13}^{(n)} \in \mathbb{R} \) in the case \( \pi_1(M) \leq \text{Sol}_3^4 \) and \( \text{Im}(\phi_{13}^{(n)}) = \text{log}(\phi_{22}^{(n)}) \) in the case \( \pi_1(M) \leq \text{Sol}_1^4 \).

Now, let \( \phi = [[[\phi_{ij}]_{i,j=1,3}]_2 \in \text{Ker}(\pi_1(M)) \) be an element with infinite order. By lemma 6.1 we have \( \phi_{13} \in \mathbb{R} - \{0\} \). Thus we can assume that there is \( (l_n) \in \mathbb{N} \subset \mathbb{Z} \) and \( c \in \mathbb{R} \) such that \( l_n \phi_{13} + \text{Re}(\phi_{13}^{(n)}) \xrightarrow{n \to \infty} c \). Hence lemma 6.1 implies:

\[
(\phi_{ij})^{(n)} = \begin{pmatrix}
1 & \kappa \phi_{22}^{(n)} - 1/(1 - \vartheta) & \phi_{13} \phi_{22}^{(n)} + \phi_{13}^{(n)} \\
0 & \phi_{22}^{(n)} & \phi_{23}^{(n)} \\
0 & 0 & 1
\end{pmatrix}.
\]

and when \( n \) tends to \( \infty \) the right hand term tends to:

\[
\begin{pmatrix}
1 & 0 & c \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Which is a contradiction since \( \Gamma \) is discrete.

Option 2. \( \pi_1(M) \leq \text{Sol}_0^4 \). By \( \Pi \) of lemma 6.1 and corollary 6.4 we can ensure that \( \Pi(\pi_1(M)) \) contains hyperbolic elements. So there is a \( \tau = [[[\tau_{ij}]_{i,j=1,3}]_2 \in \pi_1(M) \) such that...
\[ \pi_1(M) \] such that \( \tau_{22} < 1 \). Let \( \gamma = [[(\gamma_{ij})_{i,j=1,3}]_2] \in Ker(\Pi|_{\pi_1(M)}) \) be an element with infinite order. By lemma 5.12 we deduce that \( \Pi(\Gamma_0) \) is infinite.

This is a contradiction since \( \Gamma \) is discrete.

Thus we have improved \((7)\) of Theorem 1.8 as follows:

**Corollary 6.4.** Let \( M \) be a compact \((\mathbb{P}^2_\mathbb{C}, PSL_3(\mathbb{C}))\)-orbifold, for which there is a developing map \( D \) such that \( D(M) = \mathbb{H} \times \mathbb{C} \). Then \( M \) is an Inoue Surface and the singular locus of \( M \) is empty.

6.2. On Orbifolds Covered by Uniformizable Elliptic Affine Surfaces.
Throughout this subsection \( \Omega_0, \ell, \Pi, \pi, \Gamma_0 \) will be taken as in the case \( \Omega_0 = \Omega \times \mathbb{C}^* \) of the proof of theorem 0.1. Also, let us define some terms. Let \( \hat{M} \) be the universal covering space of \( M = \Omega_0/\Gamma_0, \pi_1(M) \) be the fundamental group of \( M \) and \( (D, H) \) be a developing pair for \( M \). By \([5]\) of Theorem 1.8 we know:

1. \( \hat{M} \) is biholomorphic to \( \mathbb{C} \times \mathbb{H} \).
2. There are \( h_1, h_2 : \mathbb{H} \to \mathbb{C} \) holomorphic maps and \( \mu \in \mathbb{C}^* \) such that
   \[ D(M)(w, z) = [A(z)e^{w\mu}; B(z)e^{w_{\mu}}; 1]. \]
3. For every \( \vartheta \in \pi_1(M) \) one has:
   \[ \vartheta(z, w) = (h_{\vartheta}(w)z + \gamma_{\vartheta}(w), \omega_{\vartheta}(w)), \]
   where \( \omega_{\vartheta} \in PSL_2(\mathbb{R}) \) and \( h_{\vartheta}, \gamma_{\vartheta} : \mathbb{H} \to \mathbb{C} \) are holomorphic maps such that \( h_{\vartheta}(w) \neq w \) for all \( w \in \mathbb{H} \).
4. \( \pi_1(M) \) contains a subgroup \( \Xi \) of finite index whose center \( Zen(\Xi) \) contains a free abelian subgroup of rank 2 with generators \( c, d \).

Define \( P_2 : \pi_1(M) \to PSL_2(\mathbb{R}) \) by \( P_2(\vartheta) = \omega_{\vartheta} \) and observe that \( P_2 \) is a group morphism. Then we have:

**Lemma 6.5.** \( Ker(\Pi|_{\Gamma_0}) \) is infinite.

**Proof.** Let us proceed by contradiction. Without lost of generality assume that \( \Gamma_0 \) is torsion free, then \( Ker(\Pi|_{\Gamma_0}) \) is trivial. By \([1]\) of corollary 4.13 and \([3]\) of lemma 4.12 we deduce that \( \Pi(\Gamma_0) \) is non-discrete. Moreover, we have the following properties:

Property 1.-o(\(\Pi(\mathcal{H}(c)))\), o(\(\Pi(\mathcal{H}(d))) < \infty\): If this is not the case, observe that \( \Pi(\mathcal{H}(c)), \Pi(\mathcal{H}(d)) \in Zenn(\Pi(\mathcal{H}(\Xi))) \). That is \( Fix(g) = Fix(h) \) for all \( g, h \in \Pi(\mathcal{H}(\Xi)) \). By \([1]\) of lemma 4.12 and \([2]\) of corollary 4.13 this implies \( \Omega_0 = \mathbb{C}^* - \{0\} \) or \( \Omega_0 = \mathbb{C}^* \times \mathbb{C}^* \) or \( \Omega_0 = \mathbb{C}^* \times \mathbb{C} \) which is a contradiction.

Property 2.-o(P_2,c), o(P_2,d) < \infty: In other case \( P_2(\Xi) \) is commutative. Taking \( g \in \Pi(\mathcal{H}(\Xi)) \) and \( [k_1; k_2; 0] \in \pi(\Omega_0) \), it follows that there is \( \gamma \in \Xi \) and \( w \in \mathbb{H} \) such that \( \Pi(\mathcal{H}(\gamma)) = g \) and \( [h_1(w); h_2(w); 0] = [k_1; k_2; 0] \). Hence
\[
g[k_1; k_2; 0] = [h_1(P_2(\gamma)(w)); h_2(P_2(\gamma)(w)); 0],
\]
which implies that there is a group morphism \( \mathcal{H} : P_2(\Xi) \to \Pi(\mathcal{H}(\Xi)) \) such that the following diagram commutes:
We deduce that $K \in \mathbb{C}$ following cases:

\[ \Gamma \text{ discontinuously.} \]

Let $\Gamma$ be a torsion free subgroup $\Gamma$ of $\mathbb{C}$ with finite index and such that $\Gamma \cap \mathbb{Z}$ is a normal torsion free subgroup $\Gamma$. Thus $\Omega_0$ is either $\mathbb{C}^2 - \{0\}$, or $\mathbb{C} \times \mathbb{C}$, or $\mathbb{C} \times \mathbb{C}$, which is a contradiction.

Now, let $l \in \mathbb{N}$ be such that $c^l, d^l \in Ker(\Pi \circ \mathcal{H}) \cap Ker(P_2)$. Then

\[ e(c^l, w) = (h, c^l)(w + \gamma_c(w), w); \]

Since $c^l, d^l$ do not have fixed points, we conclude that $h, c^l = h, d^l = 1$. And since $\mathcal{H}(c^l) = \mathcal{H}(d^l) = Id$, we deduce

\[ [h_1(w); h_2(w); e^{-\mu_2}] = [h_1(w); h_2(w); e^{-\mu(z + \gamma_c(w))}]. \]

Hence $e^{nh_c} = e^{nh_d} = 1$. Thus there are $k, n \in Z - \{0\}$ such that:

\[ h_c = 2\pi ik\mu^{-1}, h_d = 2\pi in\mu^{-1}, \]

which implies $e^{hk} = d^n$, which is a contradiction. 

7. Proof of Theorems 0.2 and 0.4

In this section we use our main result, Theorem 0.1, to prove Theorems 0.2 and 0.4. We will use also the following lemma (see proposition 2 of [21]):

**Lemma 7.1.** Let $F$ be a free abelian group acting on $\mathbb{C}^2$ freely and properly discontinuously. If the rank of $F$ is less than or equal to 3, then the quotient space of $\mathbb{C}^2/F$ cannot be compact.

**Proof of Theorem 0.2:** By the proof of the theorem 0.1 we must consider the following cases:

Case 1. $\Omega_0 = \mathbb{C} \times \mathbb{C}$. In this case $\Gamma$ and $\Omega_0$ have the following properties.

Property 1. $\Omega_0$ is the largest open set on which $\Gamma$ acts properly discontinuously. Let $\Gamma_i = \text{Isot}(\xi_1, e_{i1}, \Gamma) \cap \text{Isot}(\xi_3, e_{i2}, \Gamma)$ and $\mathcal{H}_i = \text{Isot}(\xi_3, e_{i3}, \Gamma) \cap \text{Isot}(\xi_3, e_{i3}, \Gamma)$. Since $\mathbb{C} \times \mathbb{C}$ is $\Gamma$-invariant, we deduce that $\Gamma_i$ is a normal subgroup of $\Gamma$ and $\Gamma_0 = \Gamma_i$ is a subgroup of $\Gamma_i$. By Selberg’s lemma and lemma 4.13 there is a normal torsion free subgroup $\Gamma_0 \leq \Gamma$ with finite index. Thus $\Omega_0/\Gamma_0$ is a compact manifold and $\Gamma_0 e_1 = e_1 (1 \leq j \leq 3)$. Let $\Pi_i = \mathbb{C}, e_{ij}, e_{3k}$ where $j, k \in \{1, 2, 3\} - \{i\}$. By corollary 4.11 and lemma 4.12 we deduce that $\text{Ker}(\Pi_i)$ is infinite or non-discrete. Hence $\bar{\xi}_1, e_{i2} \in \bar{\mathcal{H}}$ $\bar{\xi}_1, e_{i2} \subseteq \Omega_0(\Gamma) \cup L_1(\Gamma)$. Therefore $\Omega_0$ is the largest open set on which $\Gamma$ acts properly discontinuously.

Property 2. $\Gamma$ contains a subgroup $\Gamma_0$ with finite index, isomorphic to $\mathbb{Z}^2$ and where every element is a diagonal matrix. Let $\Gamma_0 \leq \Gamma$ be a torsion free subgroup with finite index and such that $\bar{\xi}_1, e_{i2}, e_{i3}, e_{i3}, e_{i3}, e_{i3}$ are $\Gamma_0$-invariant and $(\mathcal{D}, \mathcal{H})$ be a developing pair for $\Omega_0/\Gamma_0$. By lemma 2.11 we can assume that $\mathcal{D}(M) = \Omega_0$ and $\Gamma_0 = \mathcal{H}(\pi_1^{orb}(M))$. By (4) of Theorem 1.8 we can assume that $\Omega_0/\Gamma_0$ is a complex torus. Since $\pi_1(M) = \mathbb{Z}^4$ and $\mathcal{H} : \pi_1(\Omega_0) \rightarrow \Gamma_0$ is an epimorphism, we conclude
that $\Gamma$ contains a free abelian subgroup $\hat{\Gamma}$ of finite index and rank $k = 0, 1, 2, 3, 4$. Thus one has that $H^{-1}(\hat{\Gamma})$ is a free abelian group of rank 4, see [10]. Let $(\hat{D}, \hat{H})$ be the developing pair for $\hat{M} = \Omega_0/\hat{\Gamma}$ given by lemma 2.1 then, by corollary 2.2 we have the following exact sequence of groups:

\[
0 \xrightarrow{} \mathbb{Z}^2 = \pi_1(\Omega_0) \xrightarrow{\hat{q}_*} \mathbb{Z}^4 = \pi_1(\hat{M}) \xrightarrow{\hat{H}} \mathbb{Z}^k = \hat{\Gamma} \xrightarrow{} 0
\]

where $\hat{q}_*$ is the group morphism induced by the quotient map $\hat{q} : \Omega_0 \twoheadrightarrow \hat{M}$. Since $(\mathbb{Z})_1$ is a sequence of free abelian groups, we deduce that $\mathbb{Z}^4 = \mathbb{Z}^2 \oplus \mathbb{Z}^k$ (see [17]) and in consequence $k = 2$.

Case 2. $-\Omega_0 = \mathbb{C} \times \mathbb{C}^*$- As in the previous case we can show that there is a subgroup $\Gamma_0 \leq \Gamma$ with finite index, isomorphic to $\mathbb{Z}^3$ and $\Gamma_0 \leq A_1$ or $\Gamma_0 \leq A_2$. Without loss of generality assume that $\Gamma_0 \leq A_2$, then $\bar{e}_1, \bar{e}_2$ and $\bar{e}_1, \bar{e}_2$ are $\Gamma_0$-invariant. Let $D_i : \Gamma_0 \rightarrow \text{Bihol}(\bar{e}_1, \bar{e}_2)$ ($i = 2, 3$) defined by $D_i(\gamma) = \gamma \mid_{\bar{e}_1, \bar{e}_2}$, which are group morphisms. For simplicity we will assume that $\ker(D_2)$ and $\ker(D_3)$ are trivial. Hence $D_2(\Gamma_0)$ and $D_3(\Gamma_0)$ are isomorphic to $\mathbb{Z}^3$. On the other hand observe that:

\[
D_2(\gamma_{ij})_{i,j=1,3}(z) = z + g_{21}^{-1} \gamma_{12};
\]

\[
D_3(\gamma_{ij})_{i,j=1,3}(z) = \gamma_{11} z.
\]

Therefore $D_2(\Gamma_0)$ is isomorphic to an additive subgroup of $\mathbb{C}$ and $D_3(\Gamma_0)$ is isomorphic to a multiplicative subgroup of $\mathbb{C}^*$. Thus $D_2(\Gamma_0)$ and $D_3(\Gamma_0)$ are non-discrete groups, see [16]. Therefore $\bar{e}_1, \bar{e}_2 \cup \subset L_0(\Gamma) \cup L_1(\Gamma)$, concluding the proof in this case.

Case 3. $-\Omega_0 = \mathbb{C}^2$- By Selberg’s lemma, there is normal torsion free subgroup $\Gamma_1 \leq \Gamma$ with finite index. Since $\bar{e}_1, \bar{e}_2$ is $\Gamma_1$-invariant, we deduce that $D : \Gamma \rightarrow \text{Bihol}(\bar{e}_1, \bar{e}_2)$ given by $D(\gamma) = \gamma \mid_{\bar{e}_1, \bar{e}_2}$ is a group morphism. Observe that if $\ker(D)$ is non-trivial or $D(\Gamma)$ is non-discrete, we will have that $\bar{e}_1, \bar{e}_2 \subset L_0(\Gamma) \cup L_1(\Gamma)$, by this reason we will assume that $\ker(D)$ is trivial and $D(\Gamma)$ is discrete, thus $D$ is an isomorphism and every element in $\Gamma_1$ is unipotent (see [9] of Theorem 1.2), so we conclude that every element in $\Gamma_1 - \{\text{Id}\}$ has a lift with the following normal Jordan form:

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}.
\]

By lemma 4.9 we deduce that $D(\Gamma_1)$ contains only parabolic elements. Since $D(\Gamma_1)$ is discrete we conclude that $D(\Gamma_1)$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. Since $D$ is an isomorphism, by lemma 4.2 we conclude that $\Omega_0/\Gamma_1$ is non-compact, which is a contradiction. Therefore $\Omega_0$ is the largest open set on which $\Gamma$ acts properly discontinuously. \black

Proof of Theorem 0.3: Take $\Omega_0, \Gamma_0$ as in Theorem 0.1 Consider the following cases:

Case 1. $\Omega_0 = \mathbb{C} \times \mathbb{H}$. Take $\Pi, \pi$ and $\ell$ as in Theorem 0.1. Let $[z; r; 1] \in \partial \Omega_0 - \{e_1\}$, then $r \in \mathbb{R}$ and $z \in \mathbb{C}$. By (1) of corollary 3.15 there is a sequence $(\gamma_n = [(\phi_{ij}^{(n)}), i,j=1])_{n \in \mathbb{N}} \subset \Gamma_0$ of distinct elements and $(k_n = [z_n; w_n; 1])_{n \in \mathbb{N}} \subset \Omega_0$ such that:

1. $z_n \xrightarrow[n \to \infty]{} z_0 \in \mathbb{C}$;
2. $w_n \xrightarrow[n \to \infty]{} w_0 \in \mathbb{H}$;
(3) $\gamma_n(k_n) \xrightarrow{n \to \infty} [z; r; 1]$. 

Since $(\Pi(\gamma_n))_{n \in \mathbb{N}} \subset PSL_2(\mathbb{R})$ and $\Pi(\gamma_n)(\pi(k_n)) \xrightarrow{n \to \infty} r$ we can assume that $\Pi(\gamma_n) \xrightarrow{n \to \infty} r$ uniformly on compact sets of $\pi(\Omega_0)$. Therefore

$$\phi_{22}^{(n)} \xrightarrow{n \to \infty} 0;$$

$$\phi_{23}^{(n)} \xrightarrow{n \to \infty} r.$$

Consider the following options:

Option 1. $\Gamma_0 \leq SL_0^3$. In this case, by proposition 9.1 of [23] there is $M \in SL_3(\mathbb{Z})$ with eigenvalues $\beta, \overline{\beta}, |\beta|^{-2}, \beta \neq \overline{\beta}$, a real eigenvector $(a_1, a_2, a_3)$ belonging to $|\beta|^{-2}$ and an eigenvector $(b_1, b_2, b_3)$ belonging to $\beta$, such that $\Gamma_0 = \langle \gamma_0, \gamma_1, \gamma_2, \gamma_3 \rangle$, where $\gamma_i$ has a lift $\tilde{\gamma}_i$ given by:

$$\tilde{\gamma}_0 = \begin{pmatrix} \beta & 0 & 0 \\ 0 & |\beta|^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}; \tilde{\gamma}_i = \begin{pmatrix} 1 & b_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_i \\ 0 \end{pmatrix}; 1 \leq i \leq 3.$$

Let $i \in \{1, 2, 3\}$ be such that $b_i \neq 0$. Set $h_{i,m} = (\tilde{\gamma}_0^n \tilde{\gamma}_i^{-n})^m$. Then

$$h_{k,m} \circ h_{i,n} = \begin{pmatrix} 1 & b_i(n\beta^j + m\beta^k) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_i(n \beta^{-2l} + m \beta^{-2k}) \\ 0 \end{pmatrix}; k, l \in \mathbb{N}, n, m \in \mathbb{Z}.$$

Let $n_0 \in \mathbb{N}$ be such that

$$|b_i([r_{n_0}]\beta^{n_0} + [s_{n_0}]\beta^{n_0+1}) - z + z_0| < \epsilon,$$

Define

$$\tau_n = [[[\tau_{ij}^{(n)}]]]_{i,j=1,3} = \gamma_n \circ [h_{n_0,[r_{n_0}]}]_2 \circ [h_{n_0+1,[r_{n_0}]}]_2,$$

and

$$\tilde{k}_n = h_{n_0+1,[r_{n_0}]}(h_{n_0,[r_{n_0}]}(k_n)) = [[\tilde{z}_n; \tilde{w}_n; 1]],$$

then:

$$\tilde{k}_n = [z_n + b_i([r_{n_0}]\beta^{n_0} + [s_{n_0}]\beta^{n_0+1}); w_n + a_i([r_{n_0}] \beta^{-2n_0} + [s_{n_0}] \beta^{-2(n_0+1)})].$$

Thus $\tau_n(\tilde{k}_n) \xrightarrow{n \to \infty} [z; r; 1]$ and

$$\tilde{z}_n \xrightarrow{n \to \infty} z_0 + b_i([r_{n_0}]\beta^{n_0} + [s_{n_0}]\beta^{n_0+1})$$

$$\tilde{w}_n \xrightarrow{n \to \infty} w_0 + a_i([r_{n_0}] \beta^{-2n_0} + [s_{n_0}] \beta^{-2(n_0+1)}).$$

From here we can deduce that

$$\tau_{22}^{(n)}(1 - \tau_{11}^{(n)})^{-1}; \tau_{23}^{(n)}(1 - \tau_{22}^{(n)})^{-1}; 1] \in Fix(\tau_n),$$

Therefore,
and
\[ p_n \xrightarrow{n \to \infty} [z_0 + b_i([r_n]_0) + [s_n]_\infty) + r; 1] \in W_0 \cap L_0(\Gamma). \]

Which ends the proof in this case.

Option 2.- $\Gamma_0 \leq Sol^4_\mathcal{I}$. In this case $\phi_{11}^{(n)} = \pm 1$ and
\[ \gamma_n(k_n) = [\pm z_n + \phi_{12}^{(n)} w_n + \phi_{13}^{(n)} + \phi_{22}^{(n)} w_n + \phi_{23}^{(n)}; 1]. \]

Thus $\phi_{12}^{(n)} w_n + \phi_{13}^{(n)} \xrightarrow{n \to \infty} z \mp z_0$. Now we claim that $(\phi_{12}^{(n)})_{n \in \mathbb{N}}$ and $(\phi_{13}^{(n)})_{n \in \mathbb{N}}$ are bounded. Otherwise consider the following possibilities:

Possibility 1. -$(\phi_{12}^{(n)})_{n \in \mathbb{N}}$ and $(\phi_{13}^{(n)})_{n \in \mathbb{N}}$ are unbounded. In this case we may assume that $\phi_{12}^{(n)}, \phi_{13}^{(n)} \xrightarrow{n \to \infty} \infty$, then $(\phi_{12}^{(n)} (\phi_{13}^{(n)})^{-1})_{n \in \mathbb{N}}$ or $(\phi_{12}^{(n)})^{-1} (\phi_{13}^{(n)})_{n \in \mathbb{N}}$ are bounded. Assume without loss of generality that $\phi_{12}^{(n)} (\phi_{13}^{(n)})^{-1} \xrightarrow{n \to \infty} c \in \mathbb{C}$. If $cw_n + 1 \neq 0$ we can deduce that:
\[ \gamma_n(k_n) = [\pm z_n + \phi_{12}^{(n)} w_n + \phi_{13}^{(n)} + \phi_{22}^{(n)} w_n + \phi_{23}^{(n)}; 1] \xrightarrow{n \to \infty} [cw_n + 1; 0; 0]. \]

Since this is not the case, we conclude $c \neq 0$ and $\phi_{13}^{(n)} (\phi_{12}^{(n)})^{-1} \xrightarrow{n \to \infty} - w_0$. Thus $\text{Im} (\phi_{13}^{(n)} (\phi_{12}^{(n)})^{-1}) < 0$ for $n$ large, which is a contradiction since $\text{Im} (\phi_{13}^{(n)} (\phi_{12}^{(n)})^{-1}) = 0$ for all $n \in \mathbb{N}$.

Possibility 2. -$(\phi_{12}^{(n)})_{n \in \mathbb{N}}$ bounded and $(\phi_{13}^{(n)})_{n \in \mathbb{N}}$ unbounded. We may assume that $\phi_{12}^{(n)} \xrightarrow{n \to \infty} \phi_{12} \in \mathbb{C}$ and $\phi_{13}^{(n)} \xrightarrow{n \to \infty} \infty$, then
\[ \gamma_n(k_n) = [\pm z_n + \phi_{12}^{(n)} w_n + \phi_{13}^{(n)} + \phi_{22}^{(n)} w_n + \phi_{23}^{(n)}; 1] \xrightarrow{n \to \infty} [1; 0; 0]. \]

Which is not possible.

Using the same arguments we deduce that the case $(\phi_{12}^{(n)})_{n \in \mathbb{N}}$ unbounded and $(\phi_{13}^{(n)})_{n \in \mathbb{N}}$ bounded is not possible. Hence, by the preceding analysis, we can assume that $\phi_{12}^{(n)} \xrightarrow{n \to \infty} \phi_{12} \in \mathbb{C}$ and $\phi_{13}^{(n)} \xrightarrow{n \to \infty} \phi_{13} \in \mathbb{C}$. This implies:
\[ \gamma_n[z - \phi_{13} - \phi_{12} ; 1] = [z + i(\phi_{12} - \phi_{13}) + \phi_{22} + \phi_{23}; 1] \xrightarrow{n \to \infty} [z; r; 1]. \]

That is $[z; r; 1] \in L_0(\Gamma)$.

Option 3.- $\Gamma_0 \leq Sol^4_\mathcal{I}$. By proposition 9.1 in [23] we know that there is $\gamma \in \text{Ker}(\Pi | \Gamma_0)$ non-trivial and by lemma 6.1 we know that $\gamma$ has a lift $\bar{\gamma}$ given by:
\[ \bar{\gamma} = \begin{pmatrix} 1 & 0 & \gamma_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

Let $((\gamma_{1j}^{(n)})_{i,j=1,3})_{n \in \mathbb{N}} \subset \Gamma_0$ be a sequence such that $(\Pi(\gamma_n))_{n \in \mathbb{N}}$ is a sequence of distinct elements and $\Pi(\gamma_n) \xrightarrow{n \to \infty} \text{Id}$, then $\gamma_{22}^{(n)} \xrightarrow{n \to \infty} 1, \gamma_{23}^{(n)} \xrightarrow{n \to \infty} 0$ and $\text{Im}(\gamma_{13}^{(n)}) \xrightarrow{n \to \infty} 0$. We can assume that there is $(l_n)_{n \in \mathbb{N}} \subset \mathbb{Z}$ such that $\gamma_{13}^{(n)} + l_n \gamma_{13} \xrightarrow{n \to \infty} c \in \mathbb{C}$. One has:
\[ \gamma^{(n)} \gamma_n[z; 0; 1] = [z + \gamma_{13}^{(n)} + l_n \gamma_{13} ; \gamma_{23}^{(n)} ; 1] \xrightarrow{n \to \infty} [z + c; 0; 1]. \]
Therefore \( \bar{e}_1, \bar{e}_2 \subset L_0(\Gamma) \). To conclude, observe that \( \gamma_n(\bar{e}_1, \bar{e}_2) \overset{n \to \infty}{\longrightarrow} e_1, [0; r, 1] \).

Case 2.-\( \Omega_0 = \Omega \times \mathbb{C}^* \). Take \( \Pi, \pi \) and \( l \) as in the proof of case \( \Omega_0 = \Omega \times \mathbb{C}^* \) of Theorem 7.1. Then \( \Pi(\Gamma) \) has the following properties:

Property 1.-\( \Pi(\Gamma) \) is discrete. If this is not the case, then there is a sequence \((\gamma_n = [[(\gamma_{ij})^n_{i,j=1}]])_n \subset \Gamma \) such that \( (\Pi(\gamma_n))_n \) is a sequence of distinct elements that verify \( \Pi(\gamma_n) \overset{n \to \infty}{\longrightarrow} \text{Id} \). This implies:

\[
\sqrt{\gamma_{11}^{(n)}} \gamma_{12}^{(n)} \rightarrow \infty \quad 0
\]

By lemma 6.3 there is \( \gamma_0 = [[(\gamma_{ij})_{i,j=1,3}]]_2 \in \text{Ker}(\Gamma) \) an element with infinite order. Thus we can assume that there is \( (\lambda_n)_n \subset \mathbb{Z} \) such that \( \gamma_{11}^{(n)} \gamma_{33}^{(n)} \overset{n \to \infty}{\longrightarrow} h^2 \in \mathbb{C}^* \).

Which is a contradiction since \( \Gamma \) is discrete.

Property 2.- \( \Pi(\Gamma_0) \) acts properly discontinuously on \( \pi(\Omega_0) \). Since \( \partial(\pi(\Omega_0)) \) is closed and \( \Pi(\Gamma_0) \)-invariant, we have \( \Lambda(\Pi(\Gamma_0)) \subset \partial(\pi(\Omega_0)) \). Hence \( \pi(\Omega_0) \subset \Omega(\Pi(\Gamma_0)) \).

Property 3.- \( \pi(\Omega_0)/\Pi(\Gamma_0) \) is a compact orbifold. Let \( R \subset \mathbb{C} \) be a fundamental domain for the action of \( \Gamma_0 \) on \( \Omega_0 \), then \( \pi(R) = \pi(\Gamma_0) \subset \pi(\Omega_0) \) is compact. Now the assertion follows easily.

Property 4.- \( \bigcup_{\omega \in \Pi(\Gamma)} \bar{w} \cdot e_3 \sim \{e_3 \cup \bar{e}_1, \bar{e}_2\} \) is the largest open set on which \( \Gamma \) acts properly discontinuously. Without loss of generality we assume that \( [1; 1; 0], [1; 0; 0] \) and \( [0; 1; 0] \) are in \( \Lambda(\Pi(\Gamma_0)) \). We will show that \( l = [1; 1; 0], e_3 \subset \mathbb{C}^2 - \Omega \). Let \( [1; 1; z] \in l \) since \( \text{Ker}(\Pi) \) is infinite we can assume that \( z \neq 0 \). Let \( (\gamma_n)_n \subset \Gamma \), where each element has the lift \( (a_{ij}^{(n)})_{i,j=1,3} \), be such that \( (\Pi(\gamma_n))_n \subset \mathbb{C}^2 - \Omega \) is a sequence of distinct elements with \( \Pi(\gamma_n) \overset{n \to \infty}{\longrightarrow} [1; 1; 0] \) uniformly on compact sets of \( \pi(\Omega_0) \).

Set \( [z_0; 1] \in \pi(\Omega_0) \). We can assume that there is \( l_n \in \mathbb{Z} \), \( \gamma_n = [[(\gamma_{ij})_{i,j=1,3}]]_2 \in \text{Ker}(\Gamma) \) an element with infinite order such that: \( |\gamma_{11}| > 1 \) and:

\[
\frac{a_{33}^{(n)} \gamma_{11}^{(n)} c_{33}^{(n)}}{a_{21}^{(n)} z_0 + a_{22}^{(n)}} \overset{n \to \infty}{\longrightarrow} c \in \mathbb{C}^*.
\]

By lemma 4.6 the following convergence ends the proof

\[
\gamma^{-l(n)} \gamma_n [z_0; 1; z^{-1}] = [\pi(\gamma_n)(z_0); 1; \frac{a_{33}^{(n)} \gamma_{11}^{(n)} e^{-z}}{a_{21}^{(n)} z_0 + a_{22}^{(n)}}] \overset{n \to \infty}{\longrightarrow} [1; 1; z].
\]

**Remark 7.2.** Observe that the previous discussion implies that for fundamental groups \( \pi_1(M) \) of Inoue surfaces that satisfy \( \pi_1(M) \leq \text{Sol}_0^2 \), one has \( \text{Eq}(\pi_1(M)) = \emptyset \).
8. Examples

**Example 8.1.** A Kissing Schottky Group. A group \( \Gamma \leq PSL_{n+1}(\mathbb{C}) \) is called a Schottky (respectively kissing Schottky) group (see [11, 14]) if there exist a natural number \( g \geq 2 \), elements \( \gamma_1, \ldots, \gamma_g \in \Gamma \) and pairwise disjoint open sets \( R_1, \ldots, R_g, S_1, \ldots, S_g \), such that each of these open sets is the interior of its closure, the closures of the 2\( g \) open sets are pairwise disjoint (respectively \( \bigcup_{j=1}^{g} R_j \cup S_j \neq \mathbb{P}_C^n \)), \( \gamma_j(R_j) = \mathbb{P}_C^n - S_j \) and \( \Gamma = \langle \gamma_1, \ldots, \gamma_g \rangle \). We refer to [14, 20] for explicit examples of Schottky groups acting on \( \mathbb{P}_C^{2n+1} \).

Schottky groups appear in the one dimensional complex case as planar covers of compact Riemann surfaces; in higher dimension they provide a “large” class of compact complex manifolds. One can show that Schottky groups are discrete, free in \( g \) generators and quasi co-compact. It is thus natural to ask whether one can get information about Schottky groups from our results in this article. Indeed, using Theorems 0.1, 0.2 and 0.4 one can easily show that there are no Schottky groups acting on \( \mathbb{P}_C^n \): a result already proved in [2] by a different method. In fact, there are no Schottky groups acting on any \( \mathbb{P}_C^n \), by [2]. Yet, in these dimensions one does have “Kissing Schottky Groups”; we will construct a family of such groups acting on \( \mathbb{P}_C^n \), and we study these groups using the results of this article.

Consider the M"{o}bius transformations given by:

\[
\begin{align*}
    m_1(z) &= \frac{(1+i)z-i}{iz+1-i}, \\
    m_2(z) &= \frac{(1-i)z-i}{iz+1+i}, \\
    m_3(z) &= \frac{3iz+10i}{iz+3i}.
\end{align*}
\]

It is not hard to check that

\[
\begin{align*}
    m_1(\mathbb{D}+1+i) &= \mathbb{P}_C^1 - \mathbb{D} + 1 - i; \\
    m_2(\mathbb{D} - 1 + i) &= \mathbb{P}_C^1 - \mathbb{D} - 1 - i; \\
    m_3(\mathbb{D} - 3) &= \mathbb{P}_C^1 - \mathbb{D} + 3.
\end{align*}
\]

Thus \( \Gamma_x = \langle m_1, m_2, m_3 \rangle \) is a Kissing Schottky group. Let \( \epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \in \mathbb{C}^* \times \mathbb{C}^2 \) and

\[
M_1 = \begin{pmatrix} -1 - i & 0 & 0 \\ -i & -1 + i & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad M_2 = \begin{pmatrix} 1 - i & -i & 0 \\ i & 1 + i & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad M_3 = \begin{pmatrix} 3\epsilon_1 & 10\epsilon_1 & 0 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 \\ \epsilon_1 \epsilon_2 & \epsilon_2 \epsilon_3 & \epsilon_1 \epsilon_2 \epsilon_3 \end{pmatrix}.
\]

**Lemma 8.2.** If \( P_*(\lambda) \) denotes the characteristic polynomial of \( M_3 \), then

\[
P_*(\lambda) = -(\lambda - \epsilon_1^{-2})(\lambda - i\epsilon_1(3 - \sqrt{10}))(\lambda - i\epsilon_1(\sqrt{10} + 3)).
\]

Taking \( \epsilon_1 = -(3 + \sqrt{10})^{1/3}e^{-i\pi(1+4\theta)/6} \) with \( \theta \in \mathbb{R} - \mathbb{Q} \), we deduce that:

\[
\begin{align*}
    (1) \beta &= \left\{ p_1 = \begin{pmatrix} -\sqrt{10} \\ 1 \\ k_c^- \end{pmatrix}, \quad p_2 = \begin{pmatrix} \sqrt{10} \\ 1 \\ k_c^+ \end{pmatrix}, \quad p_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ is an ordered basis of eigenvectors, where:}
\end{align*}
\]

\[
k_c^\pm = \frac{i(\pm\sqrt{10}\epsilon_2 + \epsilon_3)e^{i\pi(1+4\theta)/6}}{(3 + \sqrt{10})^{1/3}(3(1 - e^{2i\pi\theta}) - \sqrt{10}(\pm 1 - e^{2i\pi\theta}))}.
\]

And \( \{ \alpha_-, \alpha_+, e^{2\pi i\theta} \alpha_- \} \) are respective eigenvalues, where

\[
\alpha_\pm = \frac{-i(3 \pm \sqrt{10})(3 + \sqrt{10})^{1/3}}{e^{i\pi(1+4\theta)/6}}.
\]
(2) For every point \( x \in \mathbb{P}^2_{\mathbb{C}} - \{[p_1]_2, [p_2]_2 \cup [p_1]_2, e_3 \cup e_3, [p_2]_2 \} \) the set of cluster points of \( \{([-M]^{-n})_2(x)\}_{n \in \mathbb{N}} \) is contained in \([p_1]_2, e_3\) and is diffeomorphic to \( S^1 \).

**Proposition 8.3.** Let \( \Gamma_\epsilon = <[M_1]_2, [M_2]_2, [M_3]_2> \), then one has:

1. \( \Gamma_\epsilon \) is a Complex Kissing-Schottky group with 3 generators.
2. The discontinuity region in the sense of Kulkarni is the largest open set on which \( \Gamma_\epsilon \) acts properly discontinuously, and its complement is given by \( \Lambda_{Kul}(\Gamma_\epsilon) = \bigcup_{p \in \Lambda(\Gamma_\epsilon)} \tilde{p}, e_3 \).
3. For \( \epsilon_1 = -\sqrt{3} \), \( K_\epsilon \neq 0 \), one has that \( \Gamma_\epsilon \) is not topologically conjugate to an affine group.

**Proof.** First at all, let us take \( p = e_3, l = \tilde{e}_2, \tilde{e}_1, \Pi = \Pi_{p, l}, \pi = \pi_{p, l}, \) then \( \Pi([M_1]_2) = m_1, \Pi([M_2]_2) = m_2, \Pi([M_3]_2) = m_3 \). Now, consider the following properties of \( \Gamma_\epsilon \):

1. Consider the following disjoint family of open sets

   \[
   R_1 = \pi^{-1}(D + 1 + i); \quad S_1 = \pi^{-1}(D + 1 - i);
   \]

   \[
   R_2 = \pi^{-1}(D - 1 + i); \quad S_2 = \pi^{-1}(D - 1 - i);
   \]

   \[
   R_3 = \pi^{-1}(D - 3); \quad S_3 = \pi^{-1}(D + 3).
   \]

   Thus one has that:

   \[
   ([M_1]_2)_2(R_1) = \mathbb{P}^2_{\mathbb{C}} - \overline{S_1};
   \]

   \[
   ([M_2]_2)_2(R_2) = \mathbb{P}^2_{\mathbb{C}} - \overline{S_2};
   \]

   \[
   ([M_3]_2)_2(R_3) = \mathbb{P}^2_{\mathbb{C}} - \overline{S_3};
   \]

   \[
   \bigcup_{i=1}^3 R_i \cup \overline{S_i} \neq \mathbb{P}^2_{\mathbb{C}}.
   \]

   Therefore \( \Gamma_\epsilon \) is a Kissing-Schottky group with 3 generators and \( Ker(\Pi) \) is trivial.

2. Since \( tr^2(m_2) = 4 \) and \( det(M_2 + I_3) = 8 \) we deduce that \( M_2 \) has the following Jordan's normal form:

   \[
   \begin{pmatrix}
   1 & 1 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 1
   \end{pmatrix}
   \]

   Then there is a complex line \( \ell \) such that \( e_3 \in \ell = Fix([M_2]_2) \). Thus \( \pi(\ell - \{e_3\}) = Fix(m_2) \) by equation \( 1\) of \( 2 \), we conclude \( \Gamma_\epsilon = \bigcup_{p \in \Lambda(\Gamma_\epsilon)} \tilde{p}, e_3 \). Thus \( \bigcup_{p \in \Lambda(\Gamma_\epsilon)} \tilde{q}, e_3 \subset L_0(\Gamma_\epsilon) \). By lemma \( 1.2 \), \( \Gamma_\epsilon \) acts properly discontinuously on \( \mathbb{P}^2_{\mathbb{C}} - \bigcup_{p \in \Lambda(\Gamma_\epsilon)} \tilde{p}, e_3 \). By \( 1 \) of proposition \( 1.3 \) this implies that \( \Lambda_{Kul}(\Gamma_\epsilon) = \bigcup_{p \in \Lambda(\Gamma_\epsilon)} \tilde{q}, e_3 \).

3. \( \Gamma_\epsilon \) is not topologically conjugate to a fuchsian group- By lemma \( 8.2 \) and Theorem \( 1.2 \) we deduce that \( \Lambda_{Kul}([M_3]_2) = \{[p_1]_2, e_3 \cup \{[p_2]_2 \} \}. \) Theorem \( 1.4 \) implies that \( [[M_3]_2] \) cannot be topologically conjugate to an element of \( PU(2, 1) \).

- \( \Gamma_\epsilon \) is not topologically conjugate to an affine group- Assume, on the contrary, that there is a homeomorphism \( \phi : \mathbb{P}^2_{\mathbb{C}} \rightarrow \mathbb{P}^2_{\mathbb{C}} \) such that \( \phi^{-1}\Gamma_\epsilon \phi \leq PSL_3(\mathbb{C}) \) and \( \phi^{-1}\Gamma_\epsilon \phi(\ell) = \ell \) for some complex line \( \ell \), then \( \phi(\ell) \) is a 2-sphere, \( \Gamma_\epsilon \)-invariant.

If there is a point \( q \in \phi(\ell) \) such that \( q \notin \{[p_1]_2, [p_2]_2 \cup [p_2]_2, e_3 \cup [p_1]_2, e_3 \}. \) By \( 2 \) of lemma \( 8.2 \) the set of cluster points of \( \{([-M]^{-n})_2(q)\}_{n \in \mathbb{N}} \) is contained in \([p_1]_2, e_3 \) and is diffeomorphic to \( S^1 \). In consequence \( |\phi^{-1}([p_1]_2, e_3) \cap \ell| > 2 \). On the other
hand, since \( [p_1]_2, e_3 \subset L_0([M_1]_2) \), Theorem 1.2 implies that \( \phi^{-1}([p_1]_2, e_3) \) is a complex line. Thus \( \phi(\ell) = [p_1]_2, e_3 \), but this is a contradiction. Therefore \( \phi(\ell) \subset \overline{[p_1]_2, p_2}_2 \cup [p_2]_2, e_3 \cup [p_1]_2, e_3 \). Now, since \( \phi(\ell) \neq \{[p_1]_2, [p_2]_2, e_3\} \) is connected we conclude that \( \phi(\ell) \) is either \([p_1]_2, [p_2]_2, e_3\) or \([p_2]_2, e_3, [p_1]_2, e_3\). Since the closures of the orbits of \([p_2]_2, e_3\) and \([p_1]_2, e_3\) under \( \Gamma \) satisfy:

\[
\begin{align*}
\Gamma_e[p_1]_2, e_3 &= \Gamma_e[p_2]_2, e_3 = \bigcup_{q \in \Lambda(\Gamma_e)} q, e_3 \text{ with } Card(\Lambda(\Gamma_e)) > 2,
\end{align*}
\]

we deduce \( \phi(\ell) = [p_1]_2, [p_2]_2 \). On the other hand, one can easily check that \([1; 1; 0]\) and \([0; 0; 1]\) are the unique fixed points of \([M_1]_2\) and the respective Jordan’s normal of \( M_1 \) with respect to the ordered \( \{(1, 1, 0), e_2, e_3\} \) basis is:

\[
(8.1)
\begin{pmatrix}
-1 & 1 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

By Lemma 4.3 we conclude that \([1; 1; 10], e_2 = \tilde{e}_1, e_2 \) and \([1; 1; 0], e_3 \) are the unique invariant complex lines under \([M_1]_2\). Since \( \Gamma_{[1; 1; 0], e_3} = \bigcup_{\gamma \in \Gamma_e} \gamma(1), e_3 \) with \( card(\{\gamma(1) : \gamma \in \Gamma_e\}) = \infty \) we conclude \([p_1]_2, [p_2]_2 = \tilde{e}_1, e_2 \). From this and (1) of Lemma 8.2 we conclude \( k_+^e = k_-^e = 0 \), which is a contradiction.

Remark 8.4. By Theorem 1.2 one has that the limit set \( \Lambda_{Kul}(< [M_1]_2 >) \) is not a subset of \( \Lambda_{Kul}(\Gamma_e) \) for \( |e_1| > (\sqrt{10} + 3)^{1/3} \). This shows that the limit set in the sense of Kulkarni is not monotone.

Example 8.5. An elementary quasi co-compact group which is not affine.

Let \( M_a, B \in SL_3(\mathbb{C}) \) given by:

\[
M_a = \begin{pmatrix}
a & 0 & 0 \\
0 & a & 0 \\
0 & 0 & a^{-2}
\end{pmatrix}, B = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

where \( a \in \mathbb{C}^* \). Define \( \Gamma_a \) as the group generated by \([M_a]_2\) and \([B]_2\).

Lemma 8.6.

(1) \([B]_2(\tilde{e}_1, e_2) = \tilde{e}_3, e_2\), \([B]_2(\tilde{e}_3, e_2) = \tilde{e}_1, e_3\), \([B]_2(\tilde{e}_1, e_3) = \tilde{e}_2, e_2\).

(2)

\[
\Gamma_a = \left\{ B_2^k \begin{pmatrix}
a^{n_1-2n_2+n_3} & 0 & 0 \\
0 & a^{n_1+n_2-2n_3} & 0 \\
0 & 0 & a^{n_2+n_3-2n_1}
\end{pmatrix} : n_i \in \mathbb{Z} \text{ and } k = 0, 1, 2 \right\}
\]

(3) \( \tilde{\Gamma} = < [B^2M_aB]_2, [BM_aB^2]_2 > \), then \( \tilde{\Gamma}_a \) is a normal subgroup with index 3 such that \( o(\gamma) = \infty \) for all \( \gamma \in \tilde{\Gamma}_a - \{Id\} \) and \( o(\gamma) = 3 \) if \( \gamma \in \Gamma_a - \tilde{\Gamma}_a \).

Proof. An easy computation shows:

\[
M_a B = B \begin{pmatrix}
a & 0 & 0 \\
0 & a^{-2} & 0 \\
0 & 0 & a
\end{pmatrix}; M_a B^2 = B^2 \begin{pmatrix}
a^{-2} & 0 & 0 \\
0 & a & 0 \\
0 & 0 & a
\end{pmatrix}.
\]

By an inductive argument we deduce the result. \( \blacksquare \)
Proposition 8.7. The group $\Gamma_a$ is complex kleinian with limit set $\Lambda_{Kul}(\Gamma_a) = \hat{e_1}, e_2 \cup \hat{e_1}, e_3 \cup \hat{e_3}, e_2$. This group is not topologically conjugate to an affine group and the quotient of its discontinuity region, $\Omega_{Kul}(\Gamma)/\Gamma$, is a compact orbifold with non-empty singular locus.

Proof. Part 1.- $\Gamma_a$ is a complex kleinian group- By lemma [8.6] we have that $\hat{\Gamma}_a$ is a normal subgroup of $\Gamma_a$ with index 3, that $\hat{\Gamma}_a$ acts properly, discontinuously and freely on $\mathbb{C}^* \times \mathbb{C}^*$, the quotient $(\mathbb{C}^* \times \mathbb{C}^*)/\hat{\Gamma}_a$ is a complex torus and $L_0(\Gamma_a) = \hat{e_1}, e_2 \cup \hat{e_1}, e_3 \cup \hat{e_3}, e_2$. From this and the $\Gamma$-invariance of $\mathbb{C}^* \times \mathbb{C}^*$ we conclude that the set of cluster points of the orbit $\Gamma_a K$ is contained in $\hat{e_1}, e_2 \cup \hat{e_1}, e_3 \cup \hat{e_1}, e_3 = L_0(\Gamma_a)$ for every compact set $K \subset \mathbb{C}^* \times \mathbb{C}^*$. This, together with (4) of proposition 1.3 implies that $\Lambda_{Kul}(\Gamma_a) = \hat{e_1}, e_2 \cup \hat{e_3}, e_2 \cup \hat{e_1}, e_3$.

Part 2.- $\Gamma_a$ is not topologically conjugate to a fuchsian group- By Theorem 1.2 we know that $\Lambda_{Kul}([M_a]_2) = \hat{e_1}, e_2 \cup \{e_3\}$ and by Theorem 1.4 we see that $[M_a]_2$ cannot be topologically conjugate to an element of $PU(2,1)$.

Part 3.- $\Gamma_a$ is not topologically conjugate to an affine group- On the contrary, assume that there is a homeomorphism $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\hat{\Gamma} = \phi^{-1} \Gamma \phi \leq PSL_3(\mathbb{C})$ is an affine group. Since $\phi^{-1}(\hat{e_1}, e_2) \cup \phi^{-1}(e_1) = Fix(M_a)$, where $M_a = \phi^{-1}([M_a]_2)\phi$. By Theorem 1.2 we conclude that $\phi^{-1}(\hat{e_1}, e_2)$ is a complex line and $\phi^{-1}(e_1), \phi^{-1}(e_2), \phi^{-1}(e_3)$ are non-collinear points fixed by $M_a$. We can assume that $\phi(e_1) = e_1, \phi(e_2) = e_2, \phi(e_3) = e_3$ and in consequence $M_a$ and $B = \phi^{-1}([B]_2)\phi$ has lifts $\bar{M}_a$ and $\bar{B}$ given by:

$$(8.2) \quad M_a = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b^{-2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \lambda_3 \\ \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \end{pmatrix},$$

for some $\lambda_1, \lambda_2, \lambda_3, b \in \mathbb{C}^*$ that satisfy $\lambda_1 \lambda_2 \lambda_3 = 1$. Let $\ell$ be the invariant line under $\Gamma_a$, then $\ell$ is invariant under $M_a$. By equation (8.2) and lemma 4.9 we deduce $\ell \in \{\hat{e_3}, p \in \hat{e_1}, e_2 \} \cup \{\hat{e_1}, e_2\}$. Since $B(e_1) = e_2, B(e_2) = e_3$ and $B(e_3) = e_1$ we conclude that $e_1, e_2, e_3 \in l$, which is a contradiction.

Part 4.- $\Omega_{Kul}(\Gamma_a)/\Gamma_a$ is a compact orbifold with non-empty singular locus- For this it is enough to observe that $\Omega_{Kul}(\Gamma_a)/\Gamma_a = ((\mathbb{C}^* \times \mathbb{C}^*)/\hat{\Gamma}_a)/(\Gamma_a/\hat{\Gamma}_a)$ with $(\mathbb{C}^* \times \mathbb{C}^*)/\hat{\Gamma}_a$ compact, $\Gamma_a/\hat{\Gamma}_a = Z_3$ and $[B]_2 \in Isot([1;1;1], \Gamma_a)$ with $[1;1;1] \in \Omega_{Kul}(\Gamma_a)$.

Acknowledgments

The author would like to thank to the Instituto de Matemáticas, Unidad Cuernavaca, de la Universidad Nacional Autónoma de México for its hospitality during the writing of this paper; to Juan P. Navarrete for his valuable comments, and to José Seade for suggesting the problem studied in this article, for valuable discussions and his advice during the writing of this paper.

References

1. J. Borzellino, Riemannian Geometry of Orbifolds, Ph.D. Thesis UCLA, 1992.
2. A. Cano, Schottky Groups are not Realizable in $PSL(2n + 1, \mathbb{C})$, to appear in Bulletin of the Brazilian Mathematical Society.
3. S. S. Chen, L. Greenberg, Hyperbolic spaces, Contributions to analysis (a collection of papers dedicated to Lipman Bers), 49-87, Academic Press, New York, 1974.
4. S. Choi, *Geometric Structures on Orbifolds and Holonomy Representations*, Geometriae Dedicata, vol. 104 (2004), 161-199.
5. L. Greenberg, *Discrete Subgroups of the Lorentz Group*, Math. Scand., Vol. 10 (1962), 85-107.
6. M. Inoue, *On Surfaces of Class VIIo*, Invent. Math. 24 (1974), pp 269-310.
7. B. Klingler, *Structures Affines et Projectives sur les Surfaces Complexe*, Annales de L’Institut Fourier, Grenoble, vol. 48, 2 (1998), 441-447.
8. B. Klingler, *Un théorème de Rigidité Non-métrique pour les Varités Localement Symétriques Hermitiennes*, Comment. Math. Helv. 76 (2001), no. 2, 200-217.
9. R. S. Kulkarni, *Groups with Domains of Discontinuity*, Math. Ann. No 237, pp. 253-272 (1978).
10. B. Maskit, *Kleinian Groups*, Springer-Verlag, 1972.
11. D. Mumford, C. Series, D. Wright, *Indra’s Pearls*, Cambridge University Press, 2002.
12. J. P. Navarrete, *On the Limit Set of Discrete Groups of PU(2, 1)*, Geom. Dedicata, Vol. 122 (2006), No. 1, 1-13.
13. J. P. Navarrete, *The Trace Function and Complex Kleinian Groups in P^2C*, To appear in Int. Jour. of Math.
14. M. V. Nori, *The Schottky Groups in Higher Dimensions*, Contemporary Math., Vol. 58, Part I (1986), 195-197.
15. M. S. Raghunathan, *Discrete Groups of Lie Groups*, Springer-Verlag, 1972.
16. J. G. Ratcliffe, *Foundations of Hyperbolic Geometry*, Springer-Verlag, 1994.
17. J. J. Rotman, *An Introduction to the Theory of Groups*, Springer-Verlag, 1994.
18. J. Seade, A. Verjovsky, *Actions of Discrete Groups on Complex Projective Spaces*, Contemporary Math., Vol. 269 (2001), 155-178.
19. J. Seade, A. Verjovsky, *Higher Dimensional Kleinian groups*, Math. Ann., Vol. 322 (2002), 279-300.
20. J. Seade, A. Verjovsky, *Complex Schottky groups*, Geometric methods in dynamics. II. Astérisque No. 287 (2003), xx, 251-272.
21. T. Suwa, *Compact quotients of C^2 by Affine transformation Groups*, J. Diff. Geometry, 10 (1975), 239-252.
22. V. S. Varadarajan, *Lie groups, Lie Algebras and their Representations*, Springer-Verlag, 1974.
23. C. T. C. Wall, *Geometric Structures on Compact Complex Analytic Surfaces*, Topology Vol. 25, No. 2, pp. 119-156, 1986.