Lecture 5

Constrains, Degree’s of freedom and generalized coordinates
Constrains

Motion of particle not always remains free but often is subjected to given conditions.

A particle is bound to move along the circumference of an ellipse in XZ plane.

At all position of the particle, it is bound to obey the condition \( \frac{x^2}{a^2} + \frac{z^2}{b^2} = 1 \)

Constrains: Condition or restrictions imposed on motion of particle/particles
Classification of constrains

- **Holonomic Constrains:** Expressible in terms of equation involving coordinates and time (may or may not present),

  \[ f(q_1, \ldots, q_n, t) = 0; \]  
  where \( q_i \) are the instantaneous coordinates

- **Non-holonomic constrains:** Constrains which are not holonomic

  Two types of constrains are there in this category

  (i) *Equations involving velocities:*  
  \[ f(q_1, \ldots, \dot{q}_1, \ldots, \dot{q}_n, t) = 0, \]  
  (& those cannot be **reduced** to the holonomic form!).

  (ii) Constraints as *in-equalities,*
  An example,  
  \[ f(q_1, \ldots, q_n, t) < 0 \]

  In both type of constrains (holonomic/non-holonomic) time may or may not be present explicitly.
Pendulum

Constrain equations

\[ x^2 + y^2 = l^2 \]

\[ x = \sqrt{l^2 - y^2} \]

One can not change \( x \) independently, any change in \( x \) will automatically change \( y \).

\( x, y \) are not independent due to presence of constrains

Independent coordinates: If you fix all but one coordinate and still have a continuous range of movement in the free coordinate.

If you fix \( y_1 \), leaving \( x_1 \) free, then there is no continuous range of \( x_1 \) possible. In fact in this case there will not be any motion if you fix \( y_1 \).
If you choose $\theta$ as the only coordinate, it can represent entire motion of the bob in XY plane.

In this problem, only one coordinate $\theta$ is sufficient which is sole independent coordinate.

**Degree of Freedom (DOF):** no of independent coordinate required to represent the entire motion = $3 \times (\text{no of particles}) - \text{no. of constrains} = 3 - 2 = 1$

In this case no. of particle = 1
No. of constrains = 2 \[x^2 + y^2 = l^2 \text{ and } z = 0\]

DOF = 1; Generalized Coordinate = $\theta$
Degree’s of freedom (DOF): No. of independent coordinates required to completely specify the dynamics of particles/system of particles is known as degree’s of freedom.

Degree’s of freedom =

\[ 3 \times (\text{no. of particles}) - (\text{No. of holonomic constrains}) \]

\[ = 3N - k \]

Where

\( N = \text{No. of particles} \)
\( k = \text{No. of constrains} \).
Particle moving along a line (say X-axis)

Constrain equations
\[ y = 0; z = 0 \]

DOF = 1; GC = x

General form of these constrain equations, \( f(q_1, \ldots, q_n) = 0 \)

A particle is moving along a straight wire, making an angle with x-axis.

Constrain equations
\[ y = x \tan(\theta); \]
\[ z = 0 \]

DOF = 1; GC = x or y

Atwood’s machine

Constrain equations
\[ z_1 + z_2 + \pi a = l \]
\[ x_1 = 0; y_1 = 0 \]
\[ x_2 = 0; y_2 = 0 \]

DOF = 1;
GC = z_1 or z_2
Pendulum of varying length!

Pendulum with stretchable string, the bob is constrain to move in a plane

The length of the string is changing with time \( l(t) \) and is known.

General form of these constrain equations \( f(q_1, \ldots, q_n, t) = 0 \)

Constrain equations

\[
\begin{align*}
x^2 + y^2 &= l^2(t) \\
z &= 0
\end{align*}
\]

DOF = 1; GC = \( \theta \)
Non-holonomic constraint

Gas molecules confined within a spherical container of radius $R$

Constrain condition $r_i \leq R$

Inequality!
Rolling Constraint

Rolling of a disc without slipping

\[ \mathbf{v} = R \dot{\theta} \]
\[ dx = Rd\theta \]

\[ x - R\theta = x_0 \quad \text{(constraint relation)} \]

DOF = 1; GC = \theta

Other Constraints:
\[ y = 0; z = R; \varphi = 0; \psi = 0; \]
More complicated constraint

Speed,
\[ v = R \dot{\phi} \]

\[ \dot{x} = v \sin \theta = R \dot{\phi} \sin \theta \]
\[ \dot{y} = -v \cos \theta = -R \dot{\phi} \cos \theta \]

Velocity dependence that can’t be integrated out! Non-holonomic!
To describe the motion double pendulum in XY plane, one needs four coordinates \((x_1, y_1, x_2, y_2)\) in Cartesian coordinate system.

The Cartesian coordinates are **not independent**, they are related by constrain equations

\[
x_1^2 + y_1^2 = l_1^2
\]

\[
(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2
\]

If you fix \(y_1, x_2, y_2\) leaving \(x_1\) free, then there is no continuous range of \(x_1\) possible. In fact in this case there will not be any motion by fixing three coordinates leaving one as free.
If you choose $\theta_1$ and $\theta_2$ as the coordinates, then they can adequately describe the motion of double pendulum at any instant. (they are complete)

No. of constrains $= 4$

$z_1 = 0; z_2 = 0$;

$x_1^2 + y_1^2 = l_1^2$;

$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$

**DOF:** No. of independent coordinates required to completely specify the motion

$= 3 \times (no. \ of \ particles) - (No. \ of \ constrains)$

$= 3 \times 2 - 4 = 2$

**Generalizer coordinates:** $\theta_1$ and $\theta_2$
Generalized coordinate?

- Generalized coordinate
  - non necessarily a distance
  - Not necessarily an angle.
  - Not necessarily belong to a particular coordinate system!
    (Cartesian, Cylindrical, Polar or Spherical polar)

Let’s check an example to clarify the above mentioned points

- \((x, \theta)\) are the independent generalized coordinates.
  (Check the independence)

- Generalized coordinates
  \(x \rightarrow \text{distance}\)
  \(\theta \rightarrow \text{Angle}\)
  Not belong to any specific coordinates system (mixed up)

A pendulum is attached with an linearly oscillating particle
Generalized coordinates properties

- $q_j \rightarrow$ To be generalized coordinates
  They must be
  - Must be independent
  - Must be complete
  - System must be holonomic

- **Meaning of Complete**: Capable to describe the system configuration at times. In other word, capable of locating all parts at all times.

- Generalized coordinates
  - Not necessarily Cartesian
  - Not necessarily any specific coordinate system
Rigid body has six degrees of freedom
Thus six generalized coordinates are necessary to specify the dynamics of rigid body

3 translational DOF for the center of Mass + 3 rotational degree of freedom about the center of mass = 6 generalized coordinates

Translational degree of freedom of CM: \((x, y, z)\)
Three rotational degree of freedom about CM: \((\varphi, \theta, \psi)\)

In case of only translation (motion of CM), a rigid body can be accounted as point particle during estimating the number degree of freedom
Degree’s of freedom = No. of independent coordinates required to completely specify particles configuration at all times (generalized coordinates) = $3N - k$
Where $N \rightarrow$ no. of particles
$k \rightarrow$ no. of holonomic, constrains

- Choice of generalized coordinates is not unique but no. must be equal to degree’s of freedom.
Question please