Mixed-dimensional linked models of diffusion: mathematical analysis and stable finite element discretizations

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1 Introduction

Diffusion problems of interest in Applied Math and Engineering can be studied with ordinary differential equations (ODEs) or partial differential equations (PDEs). In the former, the model describes the evolution of the total or lumped value of the unknown variable. It could be, for example, the total amount of mass in a reservoir, or the total number of infected people in a susceptible population. In contrast, the unknown that is modeled with (initial) boundary-value problems involving partial differential equations is a field, where the spatial distribution is of interest and geometry plays therefore a more delicate role.

Remarkably, model reduction techniques can be used to define a hierarchy of models for a single physical phenomenon, resulting in related problems in one, two, or three spatial dimensions. In solid mechanics, for example, bodies can be modeled with the partial differential equations of three-dimensional elasticity, but also the equations of two-dimensional plates when the body is thin, or even the ordinary differential equations of beams when the body has two characteristic dimensions much smaller than the remaining one.

It sometimes occurs that it proves convenient to combine models of different order, such as, for example, when combining beams and solids \cite{1}, ground and subsurface hydraulic flow \cite{2}, arteries and the heart \cite{3}, etc. In these situations, always motivated by a reduction of complexity or computational cost, each of the connected models might be well-known, but linking them poses difficulties. In particular, the well-posedness of the joint problem is a delicate matter: even when each individual model is described with well-posed equations, one still needs to prove that the connection does not spoil this property. Hence, links between models of different nature are of theoretical as well as practical interest.
The connection of minimization problems is conveniently obtained with Lagrange multipliers [4]. More precisely, if two different models are considered whose independent solutions are the minimizers of corresponding energy functionals, the linked model can always be formulated as the saddle point solution of the sum of the two original potentials plus a linking constraint. The mathematical aspects of saddle point problems in Hilbert spaces are well-understood by now, as well as their stable approximation with Galerkin-type methods [5, 6, 7]. The current interest remains, thus, in formulating new links for different models and proving that the resulting coupled problem is stable, as well as its discretization.

In this article we analyze the coupled solution of two diffusive models, the first one defined over a continuum –and thus described by a partial differential equation– and the second one defined over a curve –and described with an ordinary differential equation. A prototypical example of the problem of interest in this article is thermal equilibrium. Its most general description in a three-dimensional body employs Poisson’s equation, the paradigmatic elliptic model. In addition, when a body is slender, its thermal equilibrium might be described by a second-order ODE. In practical applications, however, we might be interested in modeling the temperature of a part that is best described as a conductive body with a wire that connects some regions. While one could model the ensemble as a continuum, it proves more convenient –especially when using numerical discretizations– to use different models for the bulk and the wire. This is a relevant problem, for example, in the thermal design of printed circuit boards (PCBs) where the thermal conductivity of the electronic components and their (thin) connections are very different, and also their geometries.

Given its practical and theoretical interests, the numerical approximation of problems with mixed-dimensionality has been studied before (e.g., [8, 9, 10, 11]). In particular, some works have studied the approximation of mixed-dimensional problems where the low-dimension model is embedded within the high-dimensional one, with coupling fluxes between them [8]. Here, in contrast, we are only interested in situations where the low-dimensional model couples disjoint subsets of the high-dimensional body. Each of these problems is interesting and motivated by different applications, demanding hence different analyses.

In the current work, we study linked formulations of bulk and one-dimensional diffusive models, aiming at providing a rigorous footing for the family of mixed-dimensional problems alluded to above. The results presented herein address first the continuum model, i.e., the well-posedness of the mixed problem that appears when the bulk and one-dimensional problems are linked. Once this problem is studied using standard tools, we formulate finite elements of mixed-dimensional diffusion problems and analyze also their well-posedness. In contrast to elliptic problems, finite element discretizations of mixed problems do not inherit the well-posedness from the continuum counterpart and a different stability analysis has to be performed. In this work, we show the unconditional stability of mixed-dimensional problems by proving a discrete inf-sup condition. These theoretical findings are then illustrated with numerical examples. The main result of this work is thus that standard, linked mixed-dimensional finite elements of diffusive problems are unconditionally stable and convergent.

As explained, the main motivation for this work is modeling mixed-dimensional, coupled diffusion problems. As we show, however, the ideas of the proposed
coupling can be used in the solution of diffusive problems with missing data or partial information. An elegant solution to this seemingly unrelated problem—closer to data science than mechanics—can be obtained in a straightforward manner employing the formulation presented in this work.

The article is structured as follows. In Section 2 we describe the mathematical formulation of the mixed-dimensional problems. The theoretical investigations that prove the well-posedness of the joint problem are presented in Section 3. Then, in Section 4, we introduce the finite element discretization of the coupled problem and prove that the well-posedness of the continuum problem is inherited by the discrete problem. In Section 5, we demonstrate the applicability of the previously introduced concepts and methods for several problems that are of practical interest. Finally, we summarize our outcomes and discuss potential future work in Section 6.

2 Mixed-dimensional Poisson Problems

In this section, we define the diffusive problems whose analysis and approximated solution are the central topic in this article. The first part of this boundary-value problem describes the stationary solution of a transport problem in a continuum, and is ubiquitous in Mathematical Physics. For concreteness, in the following, we use the language of thermal analysis assuming that the unknown field is the temperature in a body. Throughout the article, all the equations could be reinterpreted in terms, for example, of mass concentration or electrical charge. The second type of problems refers to the temperature distribution on a one-dimensional wire whose solution is given by a second order ordinary differential equation. Finally, we describe a joint solution of these two problems when they are solved simultaneously, that is, when we consider a body in thermal equilibrium with two or more disjoint subsets connected by a thermal wire.

The motivation for this type of continuous/discrete solutions comes from practical modelling problems in which high conductive, yet slender, connectors (wires) are embedded in bulk bodies. Whereas all bodies are obviously three-dimensional, it proves convenient to combine models of different dimensions for the sake of efficiency. Later, corresponding discretization should take into account this disparity in the geometrical and mathematical description but ensure the well-posedness of the joint formulation.

2.1 The Poisson Problem in a Continuum Domain

The continuum body where we would like to study its temperature distribution is a bounded open set $\Omega \subset \mathbb{R}^d$ with boundary denoted as $\partial \Omega$. For simplicity, in what follows, we will restrict to $d = 2$, but no fundamental problem arises in the three-dimensional case. The temperature in this body is a field $\phi \in H_0^1(\Omega)$, the Hilbert space of (Lebesgue) square-integrable functions with square-integrable weak first derivatives and vanishing trace at the boundary. If $f \in H^{-1}(\Omega)$ is a known field of heat supply defined on the dual of $H_0^1(\Omega)$ then the stationary Poisson problem can be written in its standard strong form

$$-\kappa \Delta \phi = f \quad \text{in } \Omega ,$$
$$\phi = 0 \quad \text{on } \partial \Omega .$$

(1)
Here $\kappa$ is the thermal conductivity of the medium and will be assumed to be constant, for simplicity, and $\Delta$ is the Laplacian operator. See, e.g., [12] for a detailed description of this canonical elliptic boundary-value problem.

With a view to the analysis and discretization of Eq. (1), we rewrite the Poisson problem in weak form. For that we define $U \equiv H^1_0(\Omega)$ and recall that its norm is given, for any $\phi \in U$ by the standard expression

$$\|\phi\|_{U} := \left( \|\phi\|_{L^2(\Omega)}^2 + \ell^2 \|\nabla\phi\|_{L^2(\Omega)}^2 \right)^{1/2},$$  \hspace{1cm} (2)

where $\ell$ is the characteristic length of the problem (for example, the diameter of $\Omega$) and $\nabla$ denotes the gradient operator. Then, the weak form of Poisson’s problem consists in finding $\phi \in U$ such that

$$a_{\Omega}(\phi, \psi) = f_{\Omega}(\psi) \hspace{1cm} (3)$$

for all $\psi \in U$, where

$$a_{\Omega}(\phi, \psi) := \int_{\Omega} \nabla\phi \cdot \kappa \nabla\psi \, dV,$$

$$f_{\Omega}(\psi) := \int_{\Omega} f \cdot \psi \, dA,$$  \hspace{1cm} (4)

are, respectively, a bilinear and a linear form on $U$.

2.2 The Poisson Problem on a Curve

We describe next the stationary heat problem for a one-dimensional body, which is referred to throughout as a wire. The governing equations of this problem can be derived from the statement (1) of the Poisson problem, simply by assuming that the body $\Omega$ has a prismatic shape and the temperature field is constant in all the points of a cross section. Details of this projection are omitted and the final form of the equation is given.

Consider a wire of length $L$ with temperature $\theta : [0, L] \to \mathbb{R}$. When the wire is in thermal equilibrium, the temperature must satisfy

$$\tilde{\kappa} \frac{d^2 \theta}{dx^2} = r,$$  \hspace{1cm} (5)

where $r$ is the heat supply per unit length and $\tilde{\kappa}$ is the cross-sectional conductivity. The problem, however, is not well-posed unless we append suitable boundary conditions. Rather, the temperature is only defined modulo an affine function.

As before, we are interested in the weak formulation of this problem. Postponing for the moment the issue of the uniqueness in the solution, we can introduce the function space $W = H^1(0, L)$, analogous to the solution space for the body but without the trace constraint, and look for solutions $\theta \in W$ such that

$$a_{L}(\theta, \eta) = f_{L}(\eta),$$  \hspace{1cm} (6)

for all $\eta \in W$, where

$$a_{L}(\theta, \eta) := \int_0^L \tilde{\kappa} \, \theta' \eta' \, dx,$$

$$f_{L}(\eta) := \int_0^L r \, \eta \, dx.$$  \hspace{1cm} (7)
For future reference, we recall that for every function $\theta \in \mathcal{W}$, its norm is

$$
\|\theta\|_{\mathcal{W}} := \left( \|\theta\|^2_{L^2(0,L)} + L^2 \|\theta'\|^2_{L^2(0,L)} \right)^{1/2}.
$$

(8)

2.3 Linked Formulation

We study next a joint problem consisting of a body $\Omega$ with a temperature field $\phi$ satisfying the problem (3), where additionally we identify two disjoint non-empty regions $B_1, B_2 \subseteq \Omega$ whose temperatures are connected by means of a thermally conductive wire. This wire is such that the temperature at each of its ends coincides with the mean temperature of the regions $B_1, B_2$, respectively. See Fig. 1 for an illustration of the linked problem. We note that, alternatively, we could have considered two disjoint conductive bodies $\Omega_1$ and $\Omega_2$ that are in thermal equilibrium while a wire connects regions $B_\alpha \subset \Omega_\alpha$, with $\alpha = 1, 2$. The analysis of both of the problems described is analogous and, for conciseness, we focus on the former.

![Figure 1: Sketch of linked solid-wire problem setting.](image)

More precisely, let $\theta_1, \theta_2$ denote the mean temperature in the body regions $B_1, B_2$, that is,

$$
\theta_\alpha := \frac{1}{|B_\alpha|} \int_{B_\alpha} \phi \, dV,
$$

(9)

with $\alpha = 1, 2$, and $|B_\alpha|$ denoting the non-zero measure of the set. Then, the problem governing the temperature field on the wire can be fully described by the boundary value problem

$$
\kappa \frac{d^2 \theta}{dx^2} = 0 \quad \text{on } (0, L),
$$

(10a)

$$
\theta(0) = \theta_1 ,
$$

(10b)

$$
\theta(L) = \theta_2 .
$$

(10c)

If the scalars $\theta_1, \theta_2$ were known, the problem (10) would be standard and its well-posedness would require no further analysis. However, here we are interested in the situation where the values $\theta_1, \theta_2$ are not known a priori but rather, obtained through the averages (9) of the solution to problem (3).
It remains, thus, to formulate the coupled problem that includes the thermal equilibrium of the body and wire, as well as the link conditions. We will use Lagrange multipliers \((\lambda_1, \lambda_2) =: \lambda \in Q \equiv \mathbb{R}^2\) to enforce the two constraints \([10b]\) and \([10c]\), and we will use the notation \(\| \cdot \|_Q\) to indicate the Euclidean norm in \(\mathbb{R}^2\). For convenience, we introduce the product space \(V = U \times W\) with norm
\[
\| (\phi, \theta) \|_V := (\| \phi \|_U^2 + \ell^{d-1} \| \theta \|_W^2)^{\frac{1}{2}},
\]
for all \((\phi, \theta) \in V\). On this space we can define the bilinear form \(a(\cdot, \cdot) : V \times V \to \mathbb{R}\) and linear form \(f : V \to \mathbb{R}\) as
\[
a(\phi, \theta; \psi, \delta) := a_\Omega(\phi, \psi) + a_L(\theta, \delta),
\]
\[
f(\psi, \delta) := f_\Omega(\psi) + f_L(\delta) \in V.
\]
for all \((\phi, \theta)\) and \((\psi, \delta)\) in \(V\).

The joint equilibrium of the solid and the wire then results from the saddle point of the Lagrangian \(L : V \times Q \to \mathbb{R}\):
\[
L(\phi, \theta, \lambda_1, \lambda_2) := \frac{1}{2} a(\phi, \theta; \phi, \theta) - f(\phi, \theta) + \lambda_1 \left( \theta(0) - \frac{1}{|B_1|} \int_{B_1} \phi \, dV \right) + \lambda_2 \left( \theta(L) - \frac{1}{|B_2|} \int_{B_2} \phi \, dV \right).
\]
Hence, we aim to solve the following problem
\[
(\phi, \theta, \lambda_1, \lambda_2)^* = \arg \inf_{\phi, \theta} \sup_{\lambda} L(\phi, \theta, \lambda_1, \lambda_2)
\]
where the Lagrangian as in Eq. \([13]\). The optimality conditions of the functional \(L\) are satisfied by the functions \((\phi, \theta, \lambda_1, \lambda_2) \in V \times Q\) such that
\[
a(\phi, \theta; \psi, \delta) + b(\psi, \delta; \lambda_1, \lambda_2) = f(\psi, \delta),
\]
\[
b(\phi, \theta; \Gamma_1, \Gamma_2) = 0,
\]
for all \((\psi, \delta, \Gamma_1, \Gamma_2) \in V \times Q\) with
\[
b(\phi, \theta; \Gamma_1, \Gamma_2) := \Gamma_1 \left( \theta(0) - \frac{1}{|B_1|} \int_{B_1} \phi \, dV \right) + \Gamma_2 \left( \theta(L) - \frac{1}{|B_2|} \int_{B_2} \phi \, dV \right).
\]
The bilinear form \(a(\cdot, \cdot)\) defines a linear continuous operator \(A : V \to V'\) by the relation
\[
\langle Au, v \rangle_{V' \times V} = a(u, v).
\]
for all \(u = (\phi, \theta) \in V, v = (\psi, \delta) \in V\). Also, the bilinear form \(b(\cdot, \cdot)\) on \(V \times Q\) defines a linear operator \(B : V \to Q'\) with transpose \(B^T : Q \to V'\) by
\[
\langle Bv, \Gamma \rangle_{Q' \times Q} = \langle v, B^T \Gamma \rangle_{V \times V'} = b(v, \Gamma),
\]
for all \(v \in V, \Gamma \in Q\). Employing these definitions, Eq. \([16]\) can be alternatively rewritten as:
\[
Au + B^T \lambda = f \quad \text{in } V',
\]
\[
Bu = 0 \quad \text{in } Q'.
\]
This last expression is the standard form of a mixed problem [6]. The solvability of this problem depends on conditions over the bilinear forms \(a(\cdot, \cdot)\) and \(b(\cdot, \cdot)\) as well as properties of the spaces where they are defined on. In particular, often the properties of the linear operator \(B\) are delicate to ascertain.

Remarks 1. Two modifications of problem (16) are interesting in their own right:

1. One could consider that the one-dimensional diffusive mechanism connects, instead of disjoint subsets \(B_1, B_2 \subset \Omega\), the boundary of two different bodies, or parts of them. The description of this modified problem and the functional setting are slightly different than the one presented up to here, since the new problem will be formulated in terms of the traces of functions.

2. Second, we could consider a situation where there is no connecting one-dimensional diffusive wire between regions of the body but only that the average temperature on a measurable set \(S \subset \Omega\) is known to have a fixed value \(\bar{\theta}\). In this simple case we will be left with Poisson’s problem with a constraint. The problem will still be of the form (16), more precisely, 

\[
\begin{align*}
a_{\Omega}(\phi, \psi) + \bar{b}(\psi, \gamma) &= f_{\Omega}(\psi), \\
b(\phi, \eta) &= 0,
\end{align*}
\]

with \(\bar{b}(\cdot, \cdot)\) on \(V \times \mathbb{R}\) defined as

\[
\bar{b}(\phi, \eta) := \eta \left( \bar{\theta} - \frac{1}{|S|} \int_S \phi \, dV \right).
\]

3 Analysis

In this section, we study the well-posedness of problem (16). According to the standard theory of mixed problems [5-7], we need to show that the bilinear forms \(a(\cdot, \cdot)\) and \(b(\cdot, \cdot)\) are continuous, that \(a(\cdot, \cdot)\) is elliptic on the kernel of the operator \(B\) and that a certain inf-sup condition, to be defined later, also holds for \(b(\cdot, \cdot)\).

In the following, for simplicity, let us assume that the conductivities \(\kappa\) and \(\bar{\kappa}\) are constant and positive. The first step of the analysis is to study the continuity of all the linear and bilinear forms appearing in the problem statement (16). This is trivial and we summarize all the results, without proof, in the following theorem:

**Theorem 3.1.** The bilinear forms \(a_{\Omega}(\cdot, \cdot)\), \(a_{L}(\cdot, \cdot)\), and \(a(\cdot, \cdot)\) are continuous in their corresponding spaces of definition, i.e.,

\[
\begin{align*}
|a_{\Omega}(\phi, \psi)| &\leq c_{\Omega} \|\phi\|_U \|\psi\|_U, \\
|a_{L}(\theta, \delta)| &\leq c_{L} \|\theta\|_W \|\delta\|_W, \\
|a(\phi; \theta; \psi; \delta)| &\leq c \|(\psi, \delta)\|_V,
\end{align*}
\]

for all \(\phi, \psi \in U\), \(\theta, \delta \in W\) and some generic positive constants \(c_{\Omega}, c_{L}\) and \(c\).

Next, we show the continuity of \(b(\cdot, \cdot)\). Since this bilinear form is non-standard, we provide the full proof of the result.
Theorem 3.2. The bilinear form $b$ is continuous on $(\mathcal{V} \times Q)$, i.e.

$$|b(\phi, \theta; \lambda)| \leq c \| (\phi, \theta) \|_\mathcal{V} \| \lambda \|_Q$$

(24)

for all $(\phi, \theta; \lambda) \in \mathcal{V} \times Q$ and some constant $c > 0$.

Proof. Since $H^1(0, L) \hookrightarrow C^0[0, L]$, we can use the mean value theorem to determine that there exists $m \in [0, L]$ such that

$$\theta(m) = \frac{1}{L} \int_0^L \theta(x) \, dx .$$

(25)

Then, by the fundamental theorem of calculus

$$\theta(L) = \theta(m) + \int_m^L \theta'(x) \, dx .$$

Hence,

$$\| \theta(L) \| \leq \frac{1}{L} \int_0^L | \theta(x) | \, dx + \int_m^L | \theta'(x) | \, dx$$

$$\leq L^{-1/2} \| \theta \|_{L^2(0, L)} + L^{1/2} \| \theta' \|_{L^2(0, L)}$$

$$= L^{-1/2} \| \theta \|_{H^1(0, L)} .$$

(27)

Similarly, the bound $|\theta(0)| \leq L^{-1/2} \| \theta \|_{H^1(0, L)}$ also holds. Using these results, we proceed to bound from above the bilinear form $b(\cdot, \cdot)$:

$$|b(\phi, \theta; \lambda)| = \left| \lambda_1 \left( \frac{1}{|B_1|} \int_{B_1} \phi \, dV - \theta(0) \right) + \lambda_2 \left( \frac{1}{|B_2|} \int_{B_2} \phi \, dV - \theta(L) \right) \right|$$

$$\leq |\lambda_1| \left( \frac{1}{|B_1|} \int_{B_1} |\phi| \, dV + |\theta(0)| \right) + |\lambda_2| \left( \frac{1}{|B_2|} \int_{B_2} |\phi| \, dV + |\theta(L)| \right)$$

$$\leq C \| \lambda \|_2 \left( \frac{1}{|\Omega|} \int_{\Omega} |\phi| \, dV + L^{-1/2} \| \theta \|_{H^1(0, L)} \right)$$

$$\leq C \| \lambda \|_2 \left( \epsilon^{-d/2} \| \phi \|_{H^1(\Omega)} + L^{-1/2} \| \theta \|_{H^1(0, L)} \right)$$

$$\leq C \epsilon^{-d/2} \| \lambda \|_Q \| (\phi, \theta) \|_\mathcal{V} .$$

where, throughout the proof, $C$ denotes a constant whose value might change from one step to another and $d$ denotes the dimension of the characteristic length corresponding to $\Omega$. 

Next, we have to ensure ellipticity on ker $B$, the kernel of the operator $B$. By definition, this set is

$$\text{ker } B := \{ (\phi, \theta) \in \mathcal{V} : b(\phi, \theta; \lambda) = 0, \quad \forall \lambda \in Q \} .$$

(28)

Elements $(\phi, \theta) \in \mathcal{V}$ in ker $B$ verify

$$\lambda_1 \left( \frac{1}{|B_1|} \int_{B_1} \phi \, dV - \theta(0) \right) + \lambda_2 \left( \frac{1}{|B_2|} \int_{B_2} \phi \, dV - \theta(L) \right) = 0 .$$

(29)
for all pairs \((\lambda_1, \lambda_2) \in Q\). Since the two Lagrange multipliers are independent, it must hold that
\[
\ker B = \left\{ (\phi, \theta) \in V : \frac{1}{|B_1|} \int_{B_1} \phi \, dV = \theta(0) \text{ and } \frac{1}{|B_2|} \int_{B_2} \phi \, dV = \theta(L) \right\}.
\] (30)

As advanced above, for the well-posedness of the mixed problem, it suffices that the bilinear form \(a(\cdot, \cdot)\) be elliptic on \(\ker B \subset V\), as shown in the following theorem:

**Theorem 3.3.** The bilinear form \(a(\cdot, \cdot)\) is elliptic on \(\ker B\), i.e., there exists a constant \(\bar{\alpha} > 0\) such that
\[
a(\phi, \theta; \phi, \theta) \geq \bar{\alpha} \| (\phi, \theta) \|^2_V,
\] (31)
for all \((\phi, \theta) \in \ker B\).

**Proof.** To start the proof, we need two preliminary results. First, using the weighted Young inequality we note that
\[
\| \theta - \theta(0) \|_{L^2(0, L)}^2 + \| \theta - \theta(L) \|_{L^2(0, L)}^2
\]
\[
= 2 \| \theta \|_{L^2(0, L)}^2 + L |\theta(0)|^2 + L |\theta(L)|^2 - 2 \int_0^L \theta(0) \theta \, dx - 2 \int_0^L \theta(L) \theta \, dx
\]
\[
\geq 2 \| \theta \|_{L^2(0, L)}^2 + L |\theta(0)|^2 + L |\theta(L)|^2
\]
\[
- c_0^2 \| \theta \|_{L^2(0, L)}^2 - \frac{L}{c_0^2} |\theta(0)|^2 - \frac{L}{c_L^2} |\theta(L)|^2
\]
\[
\geq (2 - c_0^2 - c_L^2) \| \theta \|_{L^2(0, L)}^2 + L \left( 1 - \frac{1}{c_0^2} \right) |\theta(0)|^2 + L \left( 1 - \frac{1}{c_L^2} \right) |\theta(L)|^2,
\] (32)

for arbitrary scalars \(c_0, c_L > 0\). In the kernel of \(B\), moreover, we have that
\[
|\theta(0)|^2 = \frac{1}{|B_1|} \left| \int_{B_1} \phi \, dV \right|^2 \leq C \ell^{-d} \| \phi \|_{H^1_0(\Omega)}^2,
\]
\[
|\theta(L)|^2 = \frac{1}{|B_2|} \left| \int_{B_2} \phi \, dV \right|^2 \leq C \ell^{-d} \| \phi \|_{H^1_0(\Omega)}^2.
\] (33)

Assuming that the following conditions hold
\[
1 - \frac{1}{c_0^2} \leq 0, \quad 1 - \frac{1}{c_L^2} \leq 0,
\] (34)
then, combining Eqs. (32) and (33), the following bound is obtained
\[
\| \theta - \theta(0) \|_{L^2(0, L)}^2 + \| \theta - \theta(L) \|_{L^2(0, L)}^2
\]
\[
\geq (2 - c_0^2 - c_L^2) \| \theta \|_{L^2(0, L)}^2 + C \ell^{1-d} \left( 2 - \frac{1}{c_0^2} - \frac{1}{c_L^2} \right) \| \phi \|_{H^1_0(\Omega)}^2.
\] (35)

Also, as in the proof of Theorem 3.2 we use the fundamental theorem of calculus to bound
\[
\| \theta' \|_{L^2(0, L)}^2 \geq C \ell^{-2} \| \theta - \theta(0) \|_{L^2(0, L)}^2,
\]
\[
\| \theta' \|_{L^2(0, L)}^2 \geq C \ell^{-2} \| \theta - \theta(L) \|_{L^2(0, L)}^2.
\] (36)
To finally prove the ellipticity of the bilinear form \( a(\cdot, \cdot) \) we use the definition \((12)\), the bounds \((35), \,(36)\) and the Poincaré inequality from where it follows that

\[
a(\phi, \theta; \phi, \theta) = \int_{\Omega} \kappa |\nabla \phi|^2 \, dV + \int_0^L \bar{\kappa} |\theta'|^2 \, dx
\]

\[
\geq C \kappa \ell^{-2} \|\phi\|^2_{H^1_0(\Omega)} + \frac{\bar{\kappa}}{2} \|\theta'\|^2_{L^2(0,L)} + \frac{\bar{\kappa}}{2} \|\theta''\|^2_{L^2(0,L)}
\]

\[
\geq C \kappa \ell^{-2} \|\phi\|^2_{H^1_0(\Omega)} + C \kappa \ell^{-1} \|\theta''\|^2_{L^2(0,L)}
\]

\[
+ C \kappa \ell^{-3} \left( \|\theta - \theta(0)\|_{L^2(0,L)} + \|\theta - \theta(L)\|_{L^2(0,L)} \right)
\]

\[
\geq C \kappa \ell^{-2} \|\phi\|^2_{H^1_0(\Omega)} + C \kappa \ell^{-1} \|\theta''\|^2_{L^2(0,L)}
\]

\[
+ C \kappa \ell^{-3} \left( 2 - c_0^2 - \frac{\ell^2}{L} \right) \|\theta'\|^2_{L^2(0,L)}
\]

\[
+ C \kappa \ell^{-2} \left( 2 - \frac{1}{c_0^2} - \frac{1}{\ell^2} \right) \|\theta''\|^2_{L^2(0,L)}
\]

where, as usual, \( C \) denotes a generic constant whose value might change in each inequality. Finally, we rewrite this bound as

\[
a(\phi, \theta; \phi, \theta) \geq C \kappa \ell^{-2} \left( 3 - \frac{1}{c_0^2} - \frac{1}{\ell^2} \right) \|\phi\|^2_{H^1_0(\Omega)}
\]

\[
+ C \kappa \ell^{-3} \left( 2 - c_0^2 - \frac{\ell^2}{L} \right) \|\theta'\|^2_{L^2(0,L)} + L^2 \|\theta''\|^2_{L^2(0,L)}
\]

\[
\geq C \kappa \ell^{-2} \left( \|\phi\|^2_{H^1_0(\Omega)} + \ell^{-1} \|\theta'\|^2_{L^2(0,L)} \right)
\]

\[
= C \kappa \ell^{-2} \|\phi, \theta\|^2_{V^0},
\]

as long as there exist \( c_0^2, \ell^2 \geq 0 \) that verify conditions \((34)\) as well as

\[
3 - \frac{1}{c_0^2} - \frac{1}{\ell^2} > 0, \quad 2 - c_0^2 - \frac{\ell^2}{L} > 0.
\]

These three conditions are satisfied, for example, by \( c_0^2 = \ell^2 = 4/5 \), completing the proof.

Let us note that the bilinear form \( a(\cdot, \cdot) \) is not elliptic in the whole space \( V \), because the bilinear form of the one-dimensional conductor, namely \( a_L(\cdot, \cdot) \) is not elliptic in the space \( W \), precisely due to the lack of Dirichlet boundary conditions in problem \((3)\).

Finally, to ensure that the mixed problem is well-posed, we also have to show that the following inf-sup condition holds for \( b(\cdot, \cdot) \).

**Theorem 3.4.** *Inf-sup condition. There exists a constant \( \beta > 0 \) such that*

\[
\sup_{(\phi, \theta) \in V} \frac{b(\phi, \theta; \lambda)}{\|\phi, \theta\|_V} \geq \beta \|\lambda\|_{Q \setminus \ker B^T}.
\]

**Proof.** Since \( B \) is an operator from \( V \) to \( Q' \), a finite dimensional space, its range is closed (Theorem 1.1. in \([6]\)). Hence, the inf-sup condition holds.

With Theorems **3.1, 3.2, 3.3 and 3.4**, it follows that problem \((16)\) is well-posed. Let us remark that, typically, in other constrained minimization problems, the proof of the inf-sup condition might be quite involved. In the problem discussed in this article, however, this proof is trivial.
4 Approximation of the Problem

To solve the saddle-point problem (16) we use a mixed finite element discretization. We now briefly recall the details of this method as it applies to the project at hand.

The first step in the finite element approximation of problem (16) is the definition of a finite dimensional subspace, $V_h \subset V$ of the infinite dimensional solution space. Moreover, we assume that all the finite element functions in $V_h$ are linear combinations of piecewise polynomials defined in their corresponding domains, namely, $\Omega$ or $[0, L]$. Since the space of the Lagrange multipliers is already a space of dimension 2 we do not need to introduce a subspace for it and we simply define $Q_h \equiv Q$.

The (mixed) Galerkin finite element method consists in finding $u_h = (\phi_h, \theta_h) \in V_h, \lambda_h \in Q_h$ such that

$$a(u_h, v_h) + b(v_h, \lambda_h) = f(v_h),$$
$$b(u_h, \Gamma_h) = 0,$$

holds for all $v_h = (\psi_h, \delta_h) \in V_h$ and $\Gamma_h \in Q_h$. In contrast with the finite element approximations of elliptic boundary value problems, the well-posedness of mixed methods such as (38) does not follow from the well-posedness of the corresponding continuous problem. Instead, a new analysis has to be carried out and, in particular, a discrete inf-sup condition needs to be proven. This is typically the most difficult ingredient of this analysis but, as we will show below, it is not the case for the problem at hand given the finite dimension of the space $Q$.

4.1 Analysis

The well-posedness of the discrete problem (38) can be established by following similar steps as in the analysis of the continuous problem and presented in Section 3. For that, we start by introducing $B_h$, the discrete counterpart of the operator $B$ used in Section 3, and now defined by the relation

$$\langle B_h v_h, \Gamma_h \rangle_{Q_h^\ast \times Q_h} = \langle v_h, B_h \Gamma_h \rangle_{V \times V} = b(v_h, \Gamma_h),$$

for all $v_h \in V_h$ and $\Gamma_h \in Q_h$. Since $B_h$ is surjective and $\ker B_h^T \subset \ker B^T$, $\ker B_h^T = \{0\}$ and $B_h = B|_{V_h}$. So, we end up in the special case of $\ker B_h \subset \ker B$. Hence, the ellipticity of $a(\cdot, \cdot)$ on $\ker B_h$ follows from the ellipticity of $a(\cdot, \cdot)$ on $\ker B$. As advanced, the key condition for the well-posedness of (38) is thus the discrete inf-sup condition. However, given that the range of $B_h$ is finite dimensional, it is closed and thus the discrete inf-sup condition holds. From Theorem 2.1. in [6] we obtain the following estimate

$$\|u - u_h\|_V + \|\lambda - \lambda_h\|_{Q \setminus \ker B^T} \leq c \left( \inf_{v_h \in V_h} \|u - v_h\|_V + \inf_{\lambda_h \in Q_h} \|\lambda - \lambda_h\|_Q \right),$$

where $c$ is a constant that depends on $a(\cdot, \cdot), b(\cdot, \cdot), \bar{\alpha}$ as in Theorem 3.3 and $\beta$ as in Theorem 3.4. Details can be found in [6].
5 Numerical Examples

In this section, we use first the mixed method (38) for two examples chosen to illustrate its ability to link regions of thermally conductive solids. Then, and to complete the section, we show that the ideas of the mixed-dimensional formulations can be exploited to solve one particular type of inference problems in diffusive situations.

5.1 Heat Flux between Dissimilar Regions

In this example, we choose a square solid with dimensions $2 \times 2$ where two subregions, the first with the shape of a circle, the second an ellipse, are linked through a conductive wire. In a Cartesian coordinate system located at the center of the square and with axes parallel to its edges, the circular region has center $(x,y) = (-0.625, -0.5625)$ and radius 0.5. The elliptical region is centered at $(x,y) = (0.375, 0.4375)$, and has horizontal and vertical semi-axes of lengths 0.175 and 0.3, respectively (see Fig. 2). The setting is very similar to the one we employed in Section 2.3 to describe the mixed-dimensional boundary value problem. See Fig. 1 for comparison.

The block has conductivity $\kappa = 100$ and the wire has a high conductivity of value $\bar{\kappa} = 10{,}000$. The temperature is constrained at the top and bottom edges to values of 500 and 300, respectively. If the problem had no thermal link, the thermal field would be linear with constant vertical heat flux. However, due to the presence of the linking wire, which is selected with a high conductivity, the thermal field is distorted. The elliptical region —closer to the hotter top— serves as a heat source for the circular region —this one closer to the cooler bottom edge. As a result, the temperature fields in the two connected regions are close to 400.

In principle, the region and its center can lay anywhere in the domain. The centers of these regions determine the amount of temperature going into and coming out of the wire calculated as the average temperature in the corresponding region. To study the convergence of the formulation, we obtain six finite element solutions for this problem, with increasing resolution (see Fig. 2). In each of the solutions, the linked regions $B_1, B_2$ consist of those elements of the mesh whose centers fall inside the circle and ellipse, respectively. As can be observed in Fig. 2, coarse meshes do not represent accurately the linked regions (see the top two figures in Fig. 2). However, when the mesh is refined, both the circle as well as the ellipse are accurately approximated (see, again, Fig. 2). We note that, alternatively, we could have chosen to employ a mesh that was adapted to the linked regions, but the converged solutions would not differ.

5.2 Convergence of Increasingly Finer Connected Regions

This second example examines the classical Fourier problem and the relation of conductive links, of the type introduced in this work, with the continuum notion of flux. For that, we will consider a square domain of unit side, a material with conductivity $\kappa = 100$, and temperature boundary conditions described by parabolas at the four edges, all with their maximum values at the center of the edges and zero at the vertices. The maximum temperatures at each edge are 100 (bottom), 200 (right), 300 (top) and 400 (left).
Figure 2: Thermal link connecting dissimilar regions of circle and ellipse inside a block for six meshes with $8^2$, $16^2$, $32^2$, $64^2$, $128^2$, $256^2$ elements (top left to bottom right).
This problem can be solved with a classical numerical method, such as the finite element or finite volume method (see Fig. 4a for the finite element solution with a mesh of $512^2$ bilinear elements). However, here we solve the problem discretizing the domain into disjoint and independent square regions. These regions are connected with their neighbors through wires with conductivity $\kappa \cdot \Delta y$ (horizontal wires) and $\kappa \cdot \Delta x$ (vertical wires), where $\Delta x$ and $\Delta y$ are the lengths of the corresponding region in $x$ and $y$ direction, respectively. Then, this linked problem is solved using the formulation described in Section 4. Since we use only one finite element for each region, the method explained above is equivalent to a finite volume method where the fluxes are obtained solving the boundary value problem of the wires.

Fig. 3 shows the solutions obtained with an increasing number of independent regions. In addition to the thermal field, the wires are also drawn as well as their temperature field, using the same color scale as in the continuum regions. In the coarser solutions, one can clearly see the discontinuity of the thermal field across the boundaries of the regions since, as explained, they are independent meshes and do not share any node. Also, it is apparent in these coarser solutions that the thermal field in the wires differs from the continuum field at corresponding points.

Remarkably, as the size of the regions is reduced—and hence the length and conductivity of the wires—the connected solution (Fig. 3) resembles more closely the finite element solution (Fig. 4a) used as reference. This apparent convergence is verified by the results depicted in Fig. 4b. This figure shows the root-mean-square error of the temperature field of each linked solution compared with the reference finite element solution, and obtained as

$$e = \sqrt{\frac{1}{N} \sum_{i} (\phi_i - \phi_{FE}^i)^2},$$

where $N$ is the number of regions and $\phi_i$ refers to the temperature at the center of the $i$–th region, or element, respectively.

5.3 Using Linked models to Infer Diffusive Solutions

This final example studies the possibility of using the methods introduced in Section 4 to infer the optimal thermal field in a domain when only partial information is available. More precisely, we are interested in determining the most likely temperature field in a domain when there is information about the temperature on part of the boundary and the average temperature in some regions of the domain.

When studying a Fourier-type diffusive problem, one needs to know the Dirichlet and Neumann conditions in all of the boundary as well as the heat applied in the interior of the domain, see Eq. (1). The lack of either of these data renders the problem ill-posed and no solution can be analytically (nor numerically) obtained. Here we are interested in finding the optimal temperature field of a continuum domain, $\Omega$, where the Dirichlet boundary conditions are partially known but also the average temperature in some regions, $\{B_i\}_{i=1}^{N_{\text{regions}}}$ with $B_i \subseteq \Omega$. In particular, the heat supplied through the Neumann boundary and the interior of the domain is completely unknown. This
Figure 3: Temperature field in thermal problem solved with independent regions linked with wires. The linking wires are depicted as lines.
Finite element solution of the several regions and wires problem with a quadratic mesh of $512 \times 512$ elements. 

The solution to this optimization problem is the thermal field on the solid $\Omega$ with imposed mean-temperature constraints using degenerated wires that link the assumed temperature with the selected regions using the constraint given in Eq. (22).

To analyze the ability of problem (42) to infer an unknown thermal field, we consider a $2 \times 1$ rectangle with $\kappa = 100$. We compute a reference solution with boundary conditions and thermal loading depicted in Fig. 5a. In this figure, the temperature on the left and top edges corresponds to $\phi = 400$ and $\phi = 200$, respectively. Also a wavy heat supply is imposed in the whole domain of the form

$$h(x, y) = \begin{cases} 0, & \text{if } (x, y) \in C \\ \sin \left(\frac{4\pi(x-x_c)^2+(y-y_c)^2}{r^2}\right), & \text{otherwise} \end{cases},$$

where $C$ is the circle with center $(x, y) = (1.5, 0.25)$ and radius $r = 0.25$, where the coordinates refer to a Cartesian system located at the bottom left corner of the rectangle, with the $x, y$ axes parallel to the horizontal and vertical directions, respectively. Fig. 5b shows the solution of this boundary value problem (in
what follows, the reference problem) obtained with a finite element mesh of size $128 \times 64$.

For the current example, we define the potentially linked regions as the ones obtained with a simple grid of the whole domain of $8 \times 8$ equal rectangles. We only employ as known boundary data the temperature on the left edge. Then, we analyze a sequence of discrete problems that employ an increasing large number of data concerning the average temperature in random regions. Denoting these regions as $B_i$ (see Eq. (42)), we proceed as follows: first, we solve the problem with known Dirichlet data, and a known average temperature $\theta_i$ on a random region $B_1$ which we can obtain from the exact solution; then, we select a second random region $B_2$, obtain its average temperature $\theta_2$ from the exact solution and solve the minimization problem with the two constraints. Proceeding in this fashion, we obtain the solutions illustrated in Fig. 6. This figure shows that when the only available data is the average temperature of one region and the Dirichlet boundary condition, the solution of minimal energy is far from the reference (see top figures in Fig. 6). As expected, the more data is available, the better the obtained solution. When the average temperatures are available in all the rectangular regions, the error is minimal and the solution of the optimization problem is close to the reference solution. Naturally, the small oscillations in the solution are not captured by the approximation because the available data do not have enough resolution. One might expect that in the limit, when the regions of known average become very small and cover the whole domain, the solution of the optimization problem converges to the exact solution.

Finally, in Fig. 7 the energy error $\varepsilon$—relative to the energy of the reference solution—, i.e.,

$$\varepsilon = \frac{|\mathcal{E}_{N_{\text{regions}}} - \mathcal{E}_{\text{FE}}|}{\mathcal{E}_{\text{FE}}}$$

for the calculated sequence is illustrated. The energy error depends on the random sequence of known data, but it should be always monotonically decreasing. Note that for this sequence, there is almost no improvement in the solution for the last 33 added data instances. The average temperature of the region added in the 32nd iteration makes the difference, reducing the energy error to approximately 20%. From that iteration, the solutions of the constrained optimization problems are very similar. (See Figs. 6h and 6j).

6 Conclusions and main results

In this paper, we studied mixed-dimensional linked models of diffusion from the theoretical and numerical points of view. Although we have described all the results and examples in the language of thermal problems, the scope of the article is more general and applies to any type of linear Fourier-type boundary value problem.

After introducing the setting for the linked formulations of bulk and one-dimensional diffusive models. We proved the well-posedness of the continuous linked problem in the corresponding functional spaces by resorting to the theory of constrained problems. Next, we proved that the well-posedness continues to hold for the approximate problem, i.e. the problem discretized via mixed finite elements. The main result of the article is thus an error estimate for the mixed finite element showing optimal convergence to the continuous solution.
(a) Heat supply vector field and boundary conditions.  
(b) Reference solution with a finite element mesh of size $128 \times 64$.

Figure 5: Reference boundary value problem solution of the inference example.

The previous theoretical findings were illustrated with three numerical examples. They show that the finite element method is always stable and that the links take care of the diffusive fluxes in a different scale than the continuum, but that in the limit both discrete and continuous mechanisms are equivalent. Finally, we showed that the ideas of the method can be used to carry out inference in the solution of diffusive problems with only partial information available about the average solution in subsets of the whole domain.

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References

[1] Romero I. Coupling nonlinear beams and continua: Variational principles and finite element approximations. *International Journal for Numerical Methods in Engineering* 2018; 114: 1192–1212. doi: 10.1002/nme.5782

[2] Furman A. Modeling Coupled Surface–Subsurface Flow Processes: A Review. *Vadose Zone Journal* 2008; 7(2): 741-756. doi: 10.2136/vzj2007.0065

[3] Moghadam ME, Vignon-Clementel IE, Figliola R, Marsden AL. A modular numerical method for implicit 0D/3D coupling in cardiovascular finite element simulations. *Journal of Computational Physics* 2013; 244: 63–79. doi: 10.1016/j.jcp.2012.07.035

[4] Boyd S, Vandenberghe L. *Convex Optimization*. Cambridge University Press. 2004.
Figure 6: Inference problem. Sequence of solutions with different number of available region data. Left: gray regions are the ones where the average temperature is available. Right: solution with the corresponding average temperatures and only left edge boundary condition known.
Figure 7: Energy error for the inference problem for different numbers of available region data.

[5] Roberts JE, Thomas JM. Mixed and hybrid methods. In: Ciarlet PG, Lions JL., eds. *Handbook of Numerical Analysis* Amsterdam: North-Holland. 1989 (pp. 523–639).

[6] Brezzi F, Fortin M. *Mixed and hybrid finite element methods*. Berlin: Springer. 1991.

[7] Auricchio F, Brezzi F, Lovadina C. Mixed finite element methods. In: Stein E, Borst dR, Hughes TJR., eds. *Encyclopedia of Computational Mechanics* Wiley & Sons. 2005.

[8] Laurino F, Zunino P. Derivation and analysis of coupled PDEs on manifolds with high dimensionality gap arising from topological model reduction. *ESAIM: Mathematical Modelling and Numerical Analysis* 2019; 53(6): 2047–2080. doi: 10.1051/m2an/2019042

[9] Steinbrecher I, Popp A, Meier C. Consistent coupling of positions and rotations for embedding 1D Cosserat beams into 3D solid volumes. *Computational Mechanics* 2021; 69(3): 701–732.

[10] Kuchta M, Laurino F, Mardal KA, Zunino P. Analysis and Approximation of Mixed-Dimensional PDEs on 3D-1D Domains Coupled with Lagrange Multipliers. *SIAM Journal on Numerical Analysis* 2021; 59(1): 558–582. doi: 10.1137/20M1329664

[11] Hagmeyer N, Mayr M, Steinbrecher I, Popp A. One-way coupled fluid–beam interaction: capturing the effect of embedded slender bodies on global fluid flow and vice versa. *Advanced Modeling and Simulation in Engineering Sciences* 2022; 9(1).

[12] Evans LC. *Partial differential equations*. AMS Press. 1999.