INFLUENCE OF HYDRODYNAMIC FLUCTUATIONS ON THE 
PHASE TRANSITION IN THE E AND F MODELS OF CRITICAL 
DYNAMICS

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We use the renormalization group method to study the E model of critical dynamics in the presence of 
velocity fluctuations arising in accordance with the stochastic Navier–Stokes equation. Using the Martin– 
Siggia–Rose theorem, we obtain a field theory model that allows a perturbative renormalization group 
analysis. By direct power counting and an analysis of ultraviolet divergences, we show that the model 
is multiplicatively renormalizable, and we use a two-parameter expansion in $\epsilon$ and $\delta$ to calculate the 
renormalization constants. Here, $\epsilon$ is the deviation from the critical dimension four, and $\delta$ is the deviation from the Kolmogorov regime. We present the results of the one-loop approximation and part of the fixed- 
point structure. We briefly discuss the possible effect of velocity fluctuations on the large-scale behavior 
of the model.

Keywords: Bose condensation, F model, renormalization group, anomalous scaling exponent, critical 
dynamics

1. Introduction

Bose condensation is an important physical phenomenon observed nowadays not only in the superflu- 
idity of liquid helium but also in the condensation of inert gases [1]. According to [2], the critical dynamics 
near such a phase transition can be described using the F model. This model was analyzed in [3] using the 
renormalization group (RG) approach. It was shown that in the critical region, the F model is equivalent 
to the E model (according to the standard terminology introduced in [2]).

Both the E and F dynamical models of critical dynamics are free from hydrodynamic modes because 
the velocity field turns out to be infrared (IR) irrelevant in the critical range. Therefore, the critical 
exponent (e.g., for the viscosity) is still unknown, although the viscosity vanishes during the considered 
phase transition and manifests the features of an order parameter. Moreover, the problem of the influence 
of turbulence on the phase transition into the superfluid state remains unsolved.

A stochastic equation for the critical dynamics of a Bose system in the presence of a random velocity 
field was proposed in [4]. Such a modification of the E model leads to some deviations from the standard 
field theory approach, and we adopt it. We here continue the investigation begun in [4]. Our aim is to 
study different scaling regimes of the proposed model.

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This paper is structured as follows. We begin by analyzing the field theory formulation of the model and its renormalization (see Sec. 2). In Sec. 3, we present some interesting details of the one-loop calculation and give the relations between the renormalization constants. In Sec. 4, we analyze the fixed points and their IR-stable regions and give the results of the one-loop calculations of the RG functions. In Sec. 5, we present brief conclusions.

2. Field theory formulation of the model

The stochastic equations of Bose-like systems can be described in the vicinity of their critical points \[4\] by the equation

\[
\partial_t \psi + \partial_i (v_i \psi) = \lambda (1 + i b) \left[ \partial^2 \psi - \frac{g_1}{3} (\psi^+ \psi) \psi + g_2 m \psi \right] +
\]

\[
+ i \lambda g_3 [g_2 \psi^+ \psi - m + h] + f \psi^+ \]

(1)

and by the analogous equation for the complex conjugate field $\psi^+$. The fields $\psi$ and $\psi^+$ represent order parameters (averages of field operators of Bose particles). The field $m$ is a linear combination of internal energy and density \[2\] and is related to fluctuations of the temperature of the considered system; its evolution is described by

\[
\partial_t m + \partial_i (v_i m) = -\lambda u \partial^2 [g_2 \psi^+ \psi - m + h] + i \lambda g_3 m + f_m.
\]

The field $v$ is the fluctuating velocity field (transverse due to the incompressibility) and behaves according to

\[
\partial_t v + \partial_i (v_i v) = \nu \Delta v - \psi^+ \partial \left[ \partial^2 \psi - \frac{g_1}{3} (\psi^+ \psi) \psi + g_2 m \psi \right] -
\]

\[
- \psi \partial \left[ \partial^2 \psi^+ - \frac{g_1}{3} (\psi^+ \psi) \psi^+ + g_2 m \psi^+ \right] - m \partial [g_2 \psi^+ \psi - m + h] + f_v.
\]

(2)

The random forces $f_i, i \in \{\psi^+, \psi, m, v\}$ are assumed to be Gaussian random variables with zero means and the correlators $D_i$:

\[
D_\psi(p, t, t') = \lambda \delta(t - t'), \quad D_m(p, t, t') = \lambda u \nu^2 \delta(t - t'), \quad D_v(p, t, t') = g_4 \nu^3 p^\delta \delta(t - t').
\]

To analyze the model, we use dimensional regularization (see below) around its critical dimension four with the standard $\epsilon$-expansion (where $\epsilon$ is defined by $d = 4 - \epsilon$). The parameter $\delta$ measures the deviation from the Kolmogorov regime, i.e., the value $\delta = -3$ (and $\epsilon = 1$) corresponds to the inclusion of equilibrium fluctuations of velocity, and $\delta = 4$ ($\epsilon = 1$) defines the regime of developed turbulence \[5\]–\[7\]. We note that Eq. (2) is the stochastic Navier–Stokes equation with added terms ensuring the existence of an equilibrium statistical limit for the proposed model. An important physical fact is that only the noise $D_v$ determines which specific hydrodynamic regime is realized.

Our considerations are based on a modification of the $E$ model, not only because it is relatively simple but also because it was shown in \[3\] that this model corresponds to a stable IR-scaling regime in the $F$ model \[2\]. The standard Martin–Siggia–Rose formalism \[8\] for system (1) leads to a field theory action of
the form

\[
S = 2\lambda\psi^+\psi' - \lambda u m' \partial^2 m' + v' D_\nu v' + \\
+ \psi^+ \left\{ - \partial_i \psi - \partial_i (v_1 \psi) + \lambda \left[ \partial^2 \psi - \frac{g_1}{3} (\psi^+ \psi) \right] + i \lambda g_3 \psi [-m + h] \right\} + \\
+ \psi' \left\{ - \partial_i \psi' - \partial_i (v_1 \psi') + \lambda \left[ \partial^2 \psi' - \frac{g_1}{3} (\psi^+ \psi) \right] - i \lambda g_3 \psi' [-m + h] \right\} + \\
+ m' \left\{ - \partial_i m - \partial_i (v_1 m) - \lambda u \partial^2 [-m + h] + i \lambda g_5 [\psi^+ \partial^2 \psi - \psi \partial^2 \psi'] \right\} + \\
+ v' \left\{ - \partial_i v + \nu \Delta v - \partial_i (v_1 v) \right\},
\]

(3)

where integrations over the space–time \((t, x)\) and summations over repeated vector indices are understood. The terms

\[
v' \left\{ - \psi^+ \partial \left[ \partial^2 \psi - \frac{g_1}{3} (\psi^+ \psi) \right] - \psi \partial \left[ \partial^2 \psi' - \frac{g_1}{3} (\psi^+ \psi) \right] - m \partial [-m + h] \right\}
\]

are not included in action (3), because it can be shown that they are IR-irrelevant.

The renormalization of the proposed model was described in detail in [4]. In the renormalization analysis, the following properties of the model must be applied:

- Galilean invariance is present;
- nonlocal counterterms of the type \(v' D_\nu v'\) are absent;
- the dimensionless constant \(\nu\) is expressed in the form \(\nu = u_1 \lambda\) with \(u_1\) and is considered a new charge of the model with its own renormalization constant;
- counterterms of the types \(v' \partial_i v\) and \(v' (v \partial v)\) are absent, as is usual in developed turbulence; and
- the derivative in interaction terms \(\phi' \partial_i (v_1 \phi)\) can always be transferred to the field \(\phi'\) or \(\phi\) using the partial integration method.

In the studied model, the connection with statistics is violated (because the form of the correlator \(D_v\) changes). Nevertheless, it was shown that the multiplicative renormalization can be recovered by adding one new charge at the interaction \(g_5 m' (\psi^+ \partial^2 \psi - \psi \partial^2 \psi')\), i.e., its nonrenormalized action is related to the renormalized action by the usual multiplicative relations for the fields and parameters:

\[
S_R(\varphi) = S(Z_\varphi \varphi), \quad Z_\varphi \varphi = \{ Z_\psi \psi, Z_\psi \psi', Z_{\psi^+} \psi^+, Z_{\psi^+} \psi^{+'}, Z_m m, Z_m m', Z_v v, Z_v v' \},
\]

\[
\lambda_0 = \lambda Z_\lambda, \quad u_0 = u Z_u, \quad u_{10} = u_1 Z_{u_1},
\]

(4)

\[
g_{10} = g_{10} t Z_{g_1}, \quad g_{30} = g_{30} \mu^{t/2} Z_{g_3}, \quad g_{40} = g_{40} \mu^4 Z_{g_4}, \quad g_{50} = g_{50} \mu^{t/2} Z_{g_5}.
\]

The model is logarithmic for \(\epsilon = \delta = 0\) and the ultraviolet (UV) divergences are manifested in the form of poles in various linear combinations of \(\epsilon\) and \(\delta\) in dimensional regularization, which is very convenient for practical calculations [9], [10]. These divergences are eliminated by introducing the renormalization
constants. Their explicit form depends on the choice of the subtraction scheme. Of course, universal results are independent of the choice of the particular scheme. In the MS scheme, only UV-divergent terms are subtracted from the Feynman diagrams, and we use this scheme in our calculations. The facts indicated above indeed allow proving that the renormalized action has the same form as (3) and differs by the renormalized parameters and fields

$$Z_{\psi}, Z'_{\psi}, Z_{m}, Z_{m'}, Z_{g}, Z_{g1}, Z_{g3}, Z_{u}, Z_{u1}, Z_{\lambda}.$$  

The following relations must be satisfied for the renormalization constants of the fields in (3):

$$Z_{\psi}Z'_{\psi} = 1, \quad Z_{m}Z_{m'} = 1,$$

which are the consequences of the absence of the renormalization of the terms $m' \partial_t m$ and $v' \partial_t v$.

### 3. The UV renormalization

We can express the RG invariance [9] by the differential equation $D_{RG} W = 0$, where $W$ denotes either the connected or the one-particle irreducible Green’s function and the differential part of the RG operator is defined as

$$D_{RG} \equiv \mu \frac{\partial}{\partial \mu} \bigg|_0 = \mu \frac{\partial}{\partial \mu} + \sum g_i \beta_{g_i} \frac{\partial}{\partial g_i} - \sum \gamma_{\alpha} a \frac{\partial}{\partial a}. \quad \text{(5)}$$

The differentiation is performed at fixed values of the bare parameters, which is indicated by the subscript 0. The first summation is over the whole set of charges $g_i = \{g_1, g_3, g_4, g_5, u, u_1\}$, and second is over the set $\alpha = \lambda, h$. The RG functions $\beta_{g_i}$ and $\gamma_F$, $F = a, g_i$, are given by

$$\beta_{g_i} = \mu \frac{\partial g_i}{\partial \mu} \bigg|_0, \quad \gamma_F = \mu \frac{\partial \log Z_F}{\partial \mu} \bigg|_0. \quad \text{(6)}$$

The explicit form of the $\beta$-functions follows from this definition and relations (4). It is useful to rescale the coupling constants as

$$\frac{g_1}{8\pi^2} \rightarrow g_1, \quad \frac{g_4}{8\pi^2} \rightarrow g_4, \quad \frac{g_3}{\sqrt{8\pi^2}} \rightarrow g_3, \quad \frac{g_5}{\sqrt{8\pi^2}} \rightarrow g_5. \quad \text{(7)}$$

It can be seen from the perturbation expansion that the “real” coupling constants are the quadratic forms $g_3^2$ and $g_5^2$ and not simply $g_3$ and $g_5$, whence comes the square root for $g_3$ and $g_5$ in (7). This fact is also manifested in the fixed-point coordinates because there we expect that $g_3^2 \propto \epsilon$ and hence $g_3 \propto \sqrt{\epsilon}$ (the same also applies to the charge $g_5$). Using definitions (4) and (5), we can write $\beta$-functions (6) in the forms

$$\beta_{g_1} = g_1( - \epsilon - \gamma_{g_1} ), \quad \beta_{g_4} = g_4( - \delta + 3 \gamma_{g_4} ), \quad \beta_{u_1} = - u_1 \gamma_{u_1},$$

$$\beta_{g_3} = g_3 \left( - \frac{\epsilon}{2} - \gamma_{g_3} \right), \quad \beta_{g_5} = g_5 \left( - \frac{\epsilon}{2} - \gamma_{g_5} \right), \quad \beta_u = - u \gamma_u. \quad \text{(8)}$$

To calculate the renormalization constants in the MS scheme [9], we must extract the UV-divergent terms (poles in $\epsilon$ and $\delta$ in our case) from the Feynman diagram expansion of the corresponding one-particle-irreducible functions for the given term in action (3). We can write these functions schematically in the
frequency–momentum representation as

\[ \Gamma_{\psi^+\psi'} = 2\lambda Z_1 + \]

\[ \Gamma_{m'm'} = 2\lambda p^2 Z_2 + \]

\[ \Gamma_{\psi^+\psi} = i\omega Z_3 - \lambda p^2 Z_4 + \]

\[ \Gamma_{\psi^+\psi + \psi'\psi} = -\frac{2\lambda g_1 \mu'}{3} Z_5 + \]

\[ \Gamma_{\psi^+\psi m} = -i\lambda g_3 \mu'/2 Z_6 + \]

\[ \Gamma_{m'm'} = -\lambda p^2 Z_7 + \]

\[ \Gamma_{m'\psi + \psi} = -i\lambda g_5 \mu'/2 Z_8 + \]

\[ \Gamma_{\psi'\psi} = -i\nu^2 Z_9 + \]

(9)

where the solid nonorientable lines denote the legs formed of \( \psi \) fields. The line with an arrow denotes the response field \( \psi' \), the lines with a cross denote the complex-conjugate fields (i.e., \( \psi^+ \) or \( \psi'^+ \)), the wavy lines denote the fields \( m \) and \( m' \), and the dashed lines denote the field \( v' \) (line with arrow) and the field \( v \). Shaded blobs represent all possible one-loop one-irreducible Feynman diagrams for the given function.

Renormalization constants (9) are related to the renormalization constants of the parameters and fields (4) via the relations

\[ Z_1 = Z_\lambda Z_{\psi^+\psi'}, \quad Z_2 = Z_\lambda Z_u Z_{m'}^2, \quad Z_3 = Z_{\psi^+\psi} = Z_{\psi'^+\psi' Z_v}, \]

\[ Z_3^* = Z_{\psi'\psi} Z_{\psi'^+} = Z_{\psi'\psi} Z_{\psi'^+} Z_v, \quad Z_4 = Z_{\psi^+\psi' Z_\lambda Z_{\psi}}, \quad Z_4^* = Z_{\psi'\lambda Z_{\psi^+}}, \]

\[ Z_5 = Z_{\psi^+Z_{g_1} Z_\lambda Z_{\psi'} + Z_{\psi}}, \quad Z_5^* = Z_{\psi'Z_{g_1} Z_\lambda Z_{\psi'}^2 Z_{\psi}}, \quad Z_6 = Z_{\psi^+Z_{\lambda g_3} Z_{\psi} Z_m}, \]

\[ Z_6^* = Z_{\psi Z_{g_3} Z_{g_1} Z_{\psi} Z_{m}}, \quad Z_7 = Z_{m'} Z_{\lambda Z_u Z_m}, \quad Z_8 = Z_{m'} Z_{\lambda Z_{g_3} Z_{\psi} Z_{\psi}}, \]

\[ Z_9 = Z_{\psi'Z_v Z_v}. \]

From these relations, we can easily obtain

\[ Z_\lambda = Z_4 Z_3^{-1}, \quad Z_u = Z_7 Z_3 Z_4^{-1}, \quad Z_{m'} = Z_2^{1/2} Z_7^{-1/2}, \quad Z_m = Z_2^{-1/2} Z_7^{1/2}, \]
We can thus obtain the anomalous dimensions \( \gamma \) directly from the knowledge of the renormalization constants \( Z_1-Z_9 \), and in the one-loop approximation, we obtain the results

\[
\gamma_\lambda = \frac{3g_4u_1^2}{8(1 + u_1)} + \frac{g_5^2}{(1 + u)^3} + \frac{g_3g_5u(2 + u)}{(1 + u)^3},
\]

\[
\gamma_u = -\frac{g_3^2}{(1 + u)^3} - \frac{g_3g_5u(3u^2 + 3u^3 - 1)}{2u(1 + u)^3} + \frac{3g_4u_1^2(1 + u_1 - uu_1 - u^2)}{8u(1 + u_1)(u + u_1)},
\]

\[
\gamma_{\omega_1} = -\frac{g_3^2}{(1 + u)^3} - \frac{g_3g_5(2u + u_1)}{8(1 + u_1)},
\]

\[
\gamma_{g_1} = -\frac{3g_4u_1^2}{4(1 + u_1)} - \frac{g_5^2}{(1 + u)^3} - \frac{2g_3(1 + 3u + u^2)(g_3 - g_5)}{(1 + u)^3},
\]

\[
\gamma_{g_3} = -\frac{3g_4u_1^2}{8(1 + u_1)} - \frac{g_5^2}{(1 + u)^3} - \frac{g_3g_5(1 + 3u + 11u^2 + 5u^3)}{4u(1 + u)^3},
\]

\[
\gamma_{g_5} = -\frac{3g_4u_1^2(1 + 2u + 2u_1)}{8(1 + u_1)(u + u_1)} - \frac{g_3g_5(5u + 23u_1^2 + 9u^3 - 1)}{4u(1 + u)^3} + \frac{g_5^2(2 + 9u + 3u^2)}{2(1 + u)^3} - \frac{g_3^2}{4u},
\]

\[
\gamma_{\nu_1} = \frac{g_3g_5}{4u} - \frac{g_5^2}{4u}, \quad \gamma_{\nu_2} = -\frac{g_3g_5}{4u} + \frac{g_5^2}{4u},
\]

\[
\gamma_{\psi} = \gamma_{\psi^+} = \frac{3g_4u_1^2}{16(1 + u_1)} - \frac{g_3(g_3 - g_5)(2 + 4u + u^2)}{2(1 + u)^3},
\]

\[
\gamma_{\nu} = \gamma_{\psi^+} = -\frac{3g_4u_1^2}{16(1 + u_1)} + \frac{g_3(g_3 - g_5)u(2 + u)}{2(1 + u)^3}.
\]

We note that the limit case \( g_3 = g_5, g_4 = 0 \) agrees with the results for the \( E \) model without velocity fluctuations \([3], [11]\).

4. Scaling regimes and the fixed-point structure

Scaling regimes are associated with fixed points of the corresponding RG functions. The fixed points are defined as the points \( g^* = (g_1^*, g_3, g_4^*, g_5, u^*, u_1^*) \) at which all \( \beta \)-functions vanish simultaneously:

\[
\beta_{g_1}(g^*) = \beta_{g_3}(g^*) = \beta_{g_4}(g^*) = \beta_{g_5}(g^*) = \beta_u(g^*) = \beta_{u_1}(g^*) = 0.
\]

The type of the fixed point is determined by the eigenvalues of the matrix of its first derivatives \( \Omega = \{ \Omega_{ik} = \partial \beta_i / \partial g_k \} \), where \( \beta_i \) is the full set of \( \beta \)-functions and \( g_k \) is the full set of charges \( \{ g_1, g_3, g_4, g_5, u, u_1 \} \). The IR-asymptotic behavior is governed by IR-stable fixed points, for which all eigenvalues of \( \Omega \) are positive. Analysis of \( \beta \)-functions (8) reveals that there are several possible regimes in the case without thermal fluctuations, i.e., for \( g_3 = 0 \). The stable fixed points are listed in Table 1, and the unstable fixed points are listed in Table 2.
### Table 1

| Type of fixed point | FP-1 | FP-2 | FP-3 | FP-4 |
|---------------------|------|------|------|------|
| $g_1$               | 0    | 0    | $\frac{3\epsilon}{5}$ | $\frac{3\epsilon}{5}$ |
| $g_3$               | 0    | 0    | $\epsilon^{1/2}$ | $\epsilon^{1/2}$ |
| $g_4$               | 0    | $\frac{8\delta}{3}$ | 0    | $\frac{8\delta}{3}$ |
| $g_5$               | 0    | 0    | $\epsilon^{1/2}$ | $\epsilon^{1/2}$ |
| $u$                 | 0    | 1    | 1    | 1    |
| $u_1$               | 0    | $\frac{1+\sqrt{13}}{6}$ | 0    | 0    |

Stable fixed points.

### Table 2

| Type of fixed point | FP-5 | FP-6 | FP-7 | FP-8 | FP-9 |
|---------------------|------|------|------|------|------|
| $g_1$               | 0    | $\frac{3\epsilon-2\delta}{5}$ | $\frac{3\epsilon-2\delta}{5}$ | 0    | 0    |
| $g_3$               | 0    | 0    | 0    | $\epsilon^{1/2}$ | $\epsilon^{1/2}$ |
| $g_4$               | $\frac{8\delta}{3}$ | $\frac{8\delta}{3}$ | $\frac{8\delta}{3}$ | 0    | $\frac{8\delta}{3}$ |
| $g_5$               | $\sqrt{\frac{2(-19+\sqrt{13})\delta+18\epsilon}{3}}$ | $\sqrt{\frac{2(-19+\sqrt{13})\delta+18\epsilon}{3}}$ | 0    | $\epsilon^{1/2}$ | $\epsilon^{1/2}$ |
| $u$                 | 1    | 1    | 1    | 1    | 1    |
| $u_1$               | $\frac{1+\sqrt{13}}{6}$ | $\frac{1+\sqrt{13}}{6}$ | $\frac{1+\sqrt{13}}{6}$ | 0    | 0    |

Unstable fixed points.

The trivial Gaussian-like fixed point FP-1 is IR-stable for $\epsilon < 0$ and $\delta < 0$ and corresponds to the model without any nontrivial interactions. The fixed point FP-2 is IR-stable in the region given by the inequalities $\delta > 0$ and $\delta > 3\epsilon/2$ and corresponds to the turbulent regime (because $g_4^* \neq 0$, $\delta = 4$, and $\gamma_\nu^* = \delta/3$). The fixed points FP-3 and FP-4 differ only by the value of the charge $g_4^*$. The hydrodynamic fluctuations of the velocity field are IR-irrelevant for FP-3 and relevant for FP-4. The fixed point FP-3 is stable in the region where $\delta < 0$ and $\epsilon > 0$, and FP-4 is stable for $0 < \delta < 3\epsilon/2$. Comparing FP-8 and FP-9 with their analogues FP-3 and FP-4, we can see that the absence of the interaction term $\psi^+\psi^+\psi^2$ leads to system instability. We expect that this behavior can be explained by the disordering effect due to thermal
fluctuations (the charge \( g_3 \neq 0 \)) because there are no other interactions between the relevant degrees of freedom (fields of the type \( \psi \)) that could stabilize the system.

Briefly examining the common properties of the fixed points FP-5–FP-7, we see that regardless of the presence of the interaction \( \psi^+\bar{\psi}^+\psi^2 \), velocity fluctuations destabilize the IR behavior.

The charges \( u \) and \( u_1 \) do not play the role of expansion parameters, and it therefore seems reasonable to consider specific limits as their values tend to infinity. We consider the case where \( u \to \infty \) in Table 3. To analyze this regime, we introduce the new variables \( w \equiv 1/u \), \( f_3 \equiv g_3^2/u \), and \( f_5 \equiv g_5^2/u \). Their \( \beta \)-functions have the forms \( \beta_w = w\gamma_u \), \( \beta_{f_3} = f_3[-\epsilon + \gamma_u - 2\gamma_{g_3}] \), and \( \beta_{f_5} = f_5[-\epsilon + \gamma_u - 2\gamma_{g_5}] \). The fixed point FP-1a

| Type of fixed point | FP-1a | FP-2a | FP-3a | FP-4a | FP-5a |
|---------------------|-------|-------|-------|-------|-------|
| \( g_1 \)           | 0     | \( \frac{3\epsilon}{5} \) | \( \frac{3\epsilon}{5} \) | \( \frac{3\epsilon - 2\delta}{5} \) | \( \frac{3\epsilon - 2\delta}{5} \) |
| \( f_3 \)           | 0     | \( \frac{2\epsilon}{3} \) | \( \frac{2\epsilon}{3} \) | 0     | 0     |
| \( f_5 \)           | 0     | \( \frac{2\epsilon}{3} \) | \( \frac{2\epsilon}{3} \) | 0     | \( 2\epsilon - 2\delta \) |
| \( g_4 \)           | 0     | 0     | \( \frac{8\delta}{3} \) | \( \frac{8\delta}{3} \) | \( \frac{8\delta}{3} \) |
| \( w \)             | 0     | 0     | 0     | 0     | 0     |
| \( u_1 \)           | 0     | 0     | 0     | \( 1 + \sqrt{13} \) | \( \frac{1 + \sqrt{13}}{6} \) |

Fixed points in the limit case \( u \to \infty \).

| Type of fixed point | FP-1b | FP-2b | FP-3b | FP-4b | FP-5b |
|---------------------|-------|-------|-------|-------|-------|
| \( g_1 \)           | 0     | 0     | \( \frac{3\epsilon}{5} \) | \( \frac{3(\epsilon - 2\delta)}{5} \) | \( \frac{3(\epsilon - 2\delta)}{5} \) |
| \( g_3 \)           | 0     | 0     | 0     | 0     | 0     |
| \( g_5 \)           | 0     | \( \sqrt{2\epsilon} \) | \( \sqrt{2\epsilon} \) | \( \sqrt{2(\epsilon - 4\delta)} \) | 0     |
| \( f_4 \)           | 0     | 0     | 0     | \( \frac{8\delta}{3} \) | \( \frac{8\delta}{3} \) |
| \( u \)             | 0     | 1     | 1     | 1     | 1     |
| \( w \)             | 0     | 0     | 0     | 0     | 0     |

Fixed points in the limit case \( u_1 \to \infty \).
| Type of fixed point | FP-1c | FP-2c | FP-3c | FP-4c | FP-5c |
|---------------------|-------|-------|-------|-------|-------|
| $g_1$               | 0     | 0     | $\frac{3e}{5}$ | $\frac{3(\epsilon - 2\delta)}{5}$ | $\frac{3(\epsilon - 2\delta)}{5}$ |
| $f_3$               | 0     | $\frac{2e}{3}$ | $\frac{2e}{3}$ | 0     | 0     |
| $f_5$               | 0     | $\frac{2e}{3}$ | $\frac{2e}{3}$ | 0     | $2(\epsilon - 3\delta)$ |
| $f_4$               | 0     | 0     | 0     | $\frac{8\delta}{3}$ | $\frac{8\delta}{3}$ |
| $w$                 | 0     | 0     | 0     | 0     | 0     |
| $w_1$               | 0     | 0     | 0     | 0     | 0     |

Fixed points in the limit case $u \to \infty$, $u_1 \to \infty$.

is Gaussian (free). The fixed points FP-2a and FP-3a differ only by the value of $g_4^*$. The fixed point FP-4a corresponds to the turbulent regime where the interaction $\psi^+\psi^+\psi^2$ is relevant. The last fixed point FP-5a is a case without thermal fluctuations ($f_5 = 0$).

We consider another limit case where $u_1 \to \infty$ in Table 4. In this case, we introduce the new variables $w_1 = 1/u_1$ and $f_4 = g_4 u_1$. The corresponding $\beta$-functions have the forms $\beta_{w_1} = u_1 \gamma_{u_1}$ and $\beta_{f_4} = f_4 \left[-\delta + 3\gamma_{u} - \gamma_{u_1}\right]$. From Table 4, we again see that the only difference between FP-2b and FP-3b is the charge $g_4^*$, and FP-4b corresponds to a turbulent regime. The fixed point FP-5b corresponds to a nontrivial IR-scaling regime without thermal fluctuations.

Finally, we analyze the case where both charges $u$ and $u_1$ tend to infinity simultaneously (see Table 5). In the FP-2c regime, the presence of the interaction $\psi^+\psi^+\psi^2$ is irrelevant, unlike in the FP-3c regime. The fixed point FP-4c corresponds to a turbulent regime with the interaction $\psi^+\psi^+\psi^2$, while that interaction is irrelevant in the FP-5c regime.

The last, most nontrivial case corresponds to the situation where all charges have nonzero values. But because the structure of the $\gamma$-functions is cumbersome, we have not yet found the coordinates of this fixed point and its region of stability. Of course, from other fixed points, we know where to expect such a stability region. In the near future, we hope to confirm our expectations by direct numerical calculations.

5. Conclusion

We have studied the $E$ model in the vicinity of the critical point of the phase transition from the normal to the superfluid phase with both critical and velocity fluctuations taken into account. We showed that the model can be made multiplicatively renormalizable by adding a new charge in the interaction part of the action. We calculated the renormalization constants and RG functions up to the first order (one-loop) in the perturbation theory and partly analyzed the fixed-point structure. Our main observation is that incorporating velocity fluctuations destabilizes the critical behavior.

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