Bargmann Representation of Spin Chains

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Abstract. Spin chain Hamiltonians can be written in terms of complex differential operators using the Bargmann representation of the Jordan-Schwinger map. In this case, the eigenfunctions are expressed as the product of orthonormal monomials of the phase-space coordinates $z_i = Q_i + iP_i$ in the complex plane. Furthermore, the series constructed from each phase-space coordinate converges uniformly in any compact domain of the complex plane. Formulating spin chains with respect to the phase-space coordinates helps in discussing their classical limit and in the calculations of quasi-probability distributions.
1 Introduction

Phase-space formulation of quantum theory appears to be very effective in studying some aspects related to probability distributions in quantum optics [1]. Furthermore, the general shape of quantum mechanics formalism becomes similar to classical mechanics when we formulate it using complex canonical coordinates. This formulation is very suitable for discussing the classical limit of quantum theory [2]. One related interesting formulation of quantum mechanics is the holomorphic representation introduced by Bargmann and Segal in 60s[3–5]. The main idea behind this construction is to represent wave functions as monomials in the complex plane i.e. \( \psi_n(z) = \frac{z^n}{\sqrt{n!}} \) not as matrices. Here \( z \) is a complex phase-space variable i.e. \( z = \hat{Q} + i \hat{P} \), where \( \hat{Q} \) and \( \hat{P} \) are the configuration and momentum space operators respectively. In this case, the raising and lowering operators are \( a^\dagger = z \) and \( a = \frac{\partial}{\partial z} \) respectively\(^1\). Thus \( [\frac{\partial}{\partial z}, z] = 1 \) has the same structure similar to the commutator of \( a \) and \( a^\dagger \). This assignment of raising and lowering operators was originally proposed by Fock [6]. Moreover, the series \( \psi_n(z) = \sum c_n \frac{z^n}{\sqrt{n!}} \) converges uniformly in any compact domain of the complex \( z \)-plane since \( \sum_{n=0}^\infty |c_n|^2 = 1 \) [7–9]. Possible applications of Bargmann representation in various physical systems can be found in [10–15].

2 Bargmann Representation

**Definition:** The Bargmann or Segal-Bargmann spaces \( \mathcal{H}L^2(\mathbb{C}^d, \mu) \) are spaces of holomorphic functions with Gaussian integration measure \( \mu = (\pi)^{-d} e^{-|z|^2} \) such that the inner-product endowed with this space is [3–5, 8]

\[
\langle f | g \rangle_\mu = (\pi)^{-d} \int_{\mathbb{C}^d} \overline{f}(z) \ g(z) e^{-|z|^2} \, dz, \quad (2.1)
\]

\(^1\)Since for any given arbitrary complex function \( f(z) \) we have

\[
\left[ \frac{\partial}{\partial z}, z \right] f = \frac{\partial}{\partial z} (zf) - z \frac{\partial f}{\partial z} = f \quad (1.1)
\]
where \(|z|^2 = |z_1|^2 + \cdots + |z_d|^2\).

Any entire function \(f(z)\) in \(\mathcal{H}L^2(\mathbb{C}^d, \mu)\) obeys the following square-integrability condition

\[
||f||^2 := \langle f|f \rangle_\mu = (\pi)^{-d} \int_{\mathbb{C}^d} |f(z)|^2 e^{-|z|^2} \, dz < \infty, \tag{2.2}
\]

where \(d\) is the 2\(d\)-dimensional Lebesgue measure in \(\mathbb{C}^d\). The Bargmann space \(\mathcal{H}L^2(\mathbb{C}^d, \mu)\) is in fact a Hilbert space as shown by Bargmann in [3]. Using the inner-product defined in 2.1, we can prove that both \(\overline{z}\) and \(\frac{\partial}{\partial z}\) have the same effect i.e. \(\langle \frac{\partial f}{\partial z}, g \rangle_\mu = \langle f, \overline{z}g \rangle_\mu [5]\). We define the inverse Segal-Bargmann transform as a unitary map \(B^{-1} : \mathcal{H}L^2(\mathbb{C}^d, \mu) \to L^2(\mathbb{R}^d, dx) [5]\)

\[
B^{-1} f(x) = \int_\mathbb{C} \exp\left[-(\overline{z} \cdot \overline{z} - 2\sqrt{2}\overline{z} \cdot x + x \cdot x)/2\right] f(z) e^{-|z|^2} \, dz. \tag{2.3}
\]

The harmonic oscillator Hamiltonian \(\hat{H} = \hbar \omega (\hat{a}^\dagger \hat{a} + \frac{1}{2})\) assumes the following form in the Bargmann representation

\[
\hat{H} = \hbar \omega \left( z \frac{d}{dz} + \frac{1}{2} \right) \tag{2.4}
\]

In this case, the orthonormal eigenfunctions are \(\{z^n/\sqrt{n!}\}\) and the orthonormality relation is

\[
\int_\mathbb{C} e^{-|z|^2} z^n z^m \, dz = n! \pi \delta_{mn}. \tag{2.5}
\]

and the corresponding energy eigenvalues are

\[
\hat{H}|n\rangle = \hbar \omega \left( z \frac{d}{dz} + \frac{1}{2} \right) \frac{z^n}{\sqrt{n!}} = \hbar \omega \left( n + \frac{1}{2} \right) \frac{z^n}{\sqrt{n!}} = \hbar \omega \left( n + \frac{1}{2} \right) |n\rangle. \tag{2.6}
\]

The ground-state wave function in the Bargmann representation is simply \(\psi_0(z) = 1\) while in coordinate representation it is \(\psi_0(x) = (\frac{m \omega}{2\hbar})^{1/4} e^{-m \omega x^2/2\hbar}\). Generally, the wave functions of quantum harmonic oscillator in Bargmann representation \(\psi_n(z) = \frac{z^n}{\sqrt{n!}}\) correspond to \(\psi_n(x) = \frac{1}{\sqrt{2^n n!}} (\frac{m \omega}{2\hbar})^{1/4} e^{-m \omega x^2/2\hbar} H_n(\sqrt{\frac{m \omega}{\hbar}} x)\) where \(H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n}(e^{-y^2})\) is the Hermite polynomials in coordinate representation.

Since \(\{z^n\}\) forms an orthogonal basis, their magnitude carries no information. Therefore, only the exponents \(n\) are physically important. Thus, we can impose conditions on \(z_i\) so that we can safely make Taylor expansion of the function \(f(z_i)\) valid and use it as energy eigenstates. Many famous functions in mathematics give series expansion in terms of monomials \(z_i\) and their powers [16]. Thus we may extend energy eigenfunctions to include these as possible solutions. For example, consider the series expansion of the exponential function \(e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{\psi(z)}{\sqrt{n!}} = \sum_{n=0}^{\infty} \phi(z)\) as possible energy eigenstates where \(\psi(z) = \frac{z^n}{\sqrt{n!}}\) and \(\phi(z) = \frac{z^n}{n!}\). Then the expectation
value $\sum_{n=0}^{\infty} n! \langle \phi(z) | \hat{h} \omega \left( z \frac{d}{dz} + \frac{1}{2} \right) | \phi(z) \rangle_n$ gives the summation of energy eigenvalues of the harmonic oscillator $E = \left( \frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \ldots \right)$ as expected. Similarly, considering the series expansion of the functions $\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$ and $\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ give $E = \left( \frac{3}{2} + \frac{7}{2} + \frac{9}{2} + \ldots \right)$ and $E = \left( \frac{1}{2} + \frac{5}{2} + \frac{11}{2} + \ldots \right)$ respectively using proper normalization constants (namely multiplying the expectation value of the Hamiltonian by $2n!$ for $\cosh(z)$ and $(2n+1)!$ for $\sinh(z)$). One could consider other functions and combinations. For example, the series expansion of $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$ or $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$. The expectation value of the Hamiltonian with a proper normalization gives the same overall result as previous series expansion of $\sinh(z)$ and $\cosh(z)$ respectively. The minus sign in front of some monomials can be absorbed automatically and has no effect on the energy eigenvalues, consider for example the state $|\xi(z)\rangle = -z$, $\hat{H}|\xi\rangle = -\hbar\omega \left( z \frac{d}{dz} + \frac{1}{2} \right) z = \frac{3}{2}\hbar\omega (-z) = \frac{3}{2}\hbar\omega |\xi\rangle$ (2.7)

### 3 Angular Momentum Operators in The Bargmann Space

Consider a pair of uncoupled harmonic oscillators with annihilation operators $a$ and $b$ respectively. The commutation relations are

$$[a, a^\dagger] = [b, b^\dagger] = 1, \quad [a, a] = [a^\dagger, b] = [a, b^\dagger] = [b, b] = 0. \quad (3.1)$$

We define the angular momentum operators as

$$J_1 = \frac{\hbar}{2} (a^\dagger b + b^\dagger a), \quad (3.2)$$
$$J_2 = \frac{\hbar}{2i} (a^\dagger b - b^\dagger a), \quad (3.3)$$
$$J_3 = \frac{\hbar}{2} (a^\dagger a - b^\dagger b). \quad (3.4)$$

It can be shown in a straightforward manner taking into account 3.1 that such construction obeys the canonical commutation relations for angular momentum

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k, \quad [J_i, J^2] = 0, \quad (3.5)$$

where $i = 1, 2, 3$ and $J^2$ is the squared total angular momentum operator (also known as the Casimir element or invariant). The total number operator is

$$N = N_a + N_b = a^\dagger a + b^\dagger b, \quad (3.6)$$

with integer eigenvalues $n, n_a, n_b = 0, 1, 2, \ldots$. In terms of number operators we may write

$$J^2 = \frac{\hbar^2 N}{2} \left( \frac{N}{2} + 1 \right), \quad J_3 = \hbar \left( \frac{N_a - N_b}{2} \right) \quad (3.7)$$
Comparing previous relations with
\[ J_3|j, m\rangle = \hbar m|j, m\rangle, \]  
\[ J^2|j, m\rangle = \hbar^2 j(j + 1)|j, m\rangle, \]  
we set \( j = \frac{n_a + n_b}{2} \) and \( m = \frac{n_a - n_b}{2} \).

For a given entire function \( f_{\alpha,\beta}(z, w) = \frac{z^\alpha w^\beta}{\sqrt{\alpha! \beta!}} \), we define the angular momentum operators in the two-dimensional Bargmann space \( \mathcal{H}L^2(\mathbb{C}^2, \mu) \) by writing the Jordan-Schwinger map in the holomorphic representation as \[ \hat{J}_1 = \frac{\hbar}{2} \left( z \frac{\partial}{\partial w} + w \frac{\partial}{\partial z} \right), \]  
\[ \hat{J}_2 = \frac{\hbar}{2i} \left( z \frac{\partial}{\partial w} - w \frac{\partial}{\partial z} \right), \]  
\[ \hat{J}_3 = \frac{\hbar}{2} \left( z \frac{\partial}{\partial z} - w \frac{\partial}{\partial w} \right). \]  

These operators belong to the \( SU(2) \) Lie algebra and obey the following commutation relations \[ [\hat{J}_i, \hat{J}_j] = i\hbar \varepsilon_{ijk} \hat{J}_k \] since the only non-trivial commutators between \( z, w \) and their partial derivatives are \[ \left[ \frac{\partial}{\partial z}, z \right] = \left[ \frac{\partial}{\partial w}, w \right] = 1 \]. \( \varepsilon_{ijk} \) is the well-known Levi-Civita totally antisymmetric symbol. With this construction, we can express the non-Hermitian raising and lowering operators as \( \hat{J}_+ = \hat{J}_1 + i\hat{J}_2 = \hbar z \frac{\partial}{\partial w} \) and \( \hat{J}_- = \hat{J}_1 - i\hat{J}_2 = \hbar w \frac{\partial}{\partial z} \).

The inner product in the two-dimensional Bargmann space \( \mathcal{H}L^2(\mathbb{C}^2, \mu) \) is
\[ \langle f_{\alpha',\beta'}(z, w)|f_{\alpha,\beta}(z, w)\rangle_\mu = \frac{1}{\pi^2} \int dz \, dw \exp[-|z|^2 - |w|^2] f_{\alpha',\beta'}(z, w)f_{\alpha,\beta}(z, w) = \delta_{\alpha',\alpha}\delta_{\beta',\beta}. \]  

Let us explicitly calculate the expectation values of the operators \( \hat{J}_3 \) (or \( \hat{J}_z \) as called in most of references), \( \hat{J}_2 \) and \( \hat{J}_\pm \) using the basis \( f_{\alpha,\beta}(z, w) \), we find
\[ \langle f_{\alpha',\beta'}|\hat{J}_3|f_{\alpha,\beta}\rangle_\mu = \frac{\hbar}{2} (\alpha - \beta) \delta_{\alpha',\alpha}\delta_{\beta',\beta}, \]  
\[ \langle f_{\alpha',\beta'}|\hat{J}_2|f_{\alpha,\beta}\rangle_\mu = \hbar^2 \left( \frac{\alpha + \beta}{2} \right) \left( \frac{\alpha + \beta}{2} + 1 \right) \delta_{\alpha',\alpha}\delta_{\beta',\beta}, \]  
\[ \langle f_{\alpha',\beta'}|\hat{J}_+|f_{\alpha,\beta}\rangle_\mu = \hbar \sqrt{\alpha(\alpha + 1)\beta} \delta_{\alpha',\alpha+1}\delta_{\beta',\beta-1}, \]  
\[ \langle f_{\alpha',\beta'}|\hat{J}_-|f_{\alpha,\beta}\rangle_\mu = \hbar \sqrt{\alpha(\beta + 1)} \delta_{\alpha',\alpha-1}\delta_{\beta',\beta+1}. \]  

In contrast to the standard treatment of angular momentum mentioned for example in [17], we can directly compute the expectation values of operators \( \hat{J}_{1,2} \) without invoking \( \hat{J}_\pm \) and \( \hat{J}_3 \) in our calculations. We obtain
\[ \langle f_{\alpha', \beta'} | \hat{J}_1 | f_{\alpha, \beta} \rangle = \frac{\hbar}{2} \left( \sqrt{(\alpha + 1)\beta} \delta_{\alpha', \alpha+1} \delta_{\beta', \beta-1} + \sqrt{\alpha(\beta + 1)} \delta_{\alpha', \alpha-1} \delta_{\beta', \beta+1} \right), \quad (3.18) \]

\[ \langle f_{\alpha', \beta'} | \hat{J}_2 | f_{\alpha, \beta} \rangle = \frac{\hbar}{2} \left( \sqrt{(\alpha + 1)\beta} \delta_{\alpha', \alpha+1} \delta_{\beta', \beta-1} - \sqrt{\alpha(\beta + 1)} \delta_{\alpha', \alpha-1} \delta_{\beta', \beta+1} \right). \quad (3.19) \]

Comparing \( \hat{J}_3 \) and \( \hat{J}_2 \) with the standard relations written in \( |jm\rangle \) basis i.e. \( \hat{J}_1 |jm\rangle = m \hbar |jm\rangle \) and \( \hat{J}_2 |jm\rangle = \hbar^2 j(j+1) |jm\rangle \), we find

\[ j = \frac{\alpha + \beta}{2}, \quad m = \frac{\alpha - \beta}{2}. \quad (3.20) \]

For spin-1/2 particles we have \( j = 1/2 \), this implies either \( \alpha = 1, \beta = 0 \) which corresponds to \( m = 1/2 \) and \( f_{10} = z \) or \( \alpha = 0, \beta = 1 \) which gives \( m = -1/2 \) and \( f_{01} = w \). Analogously for \( j = 1 \) we have three different case I) \( \alpha = 1, \beta = 1 \) which corresponds to \( m = 0 \) and \( f_{11} = zw \), II) \( \alpha = 2, \beta = 0 \) with \( m = 1 \) and \( f_{20} = \frac{z^2}{\sqrt{2}} \), finally III) when \( \alpha = 0, \beta = 2 \) and this gives \( m = -1, f_{02} = \frac{w^2}{\sqrt{2}} \). The computation for \( j = N \) is straightforward, one should take into account that we have \( 2j + 1 \) number of possible states. For \( j = N \), where \( N \) is a positive integer (Bosons), \( m = 0 \) is equivalent to \( \alpha = \beta \) and in this case the energy eigenfunctions can be written as \( \frac{(zw)^n}{n!} \). Note that the series expansion of \( e^{zw} = \sum_{n=0}^{\infty} \frac{(zw)^n}{n!} \) gives all the eigenfunctions with \( m = 0 \) and \( j \) running from 0 to \( \infty \).

The addition of quantum angular momentum using holomorphic representation can be done in a systematic way by adding monomials without worrying about the dimensionality of tensor product of single-particle angular momentum subspaces. For simplicity we consider a two-particle system. The total angular momentum operator is

\[ \hat{J} = \hat{J}_1 + \hat{J}_2 \quad (3.21) \]

where 1 and 2 labels the two particles. We assume the joint two-particle system basis to be \( |f_{\alpha_1, \alpha_2, \beta_1, \beta_2}\rangle = \frac{z^{\alpha_1}}{\sqrt{\alpha_1!}} \frac{z^{\alpha_2}}{\sqrt{\alpha_2!}} \frac{w^{\beta_1}}{\sqrt{\beta_1!}} \frac{w^{\beta_2}}{\sqrt{\beta_2!}} |1_1, \beta_1\rangle |1_2, \beta_2\rangle \), then the operators \( \hat{J}_2 \) and \( \hat{J}_3 \) are

\[ \hat{J}^2 |f_{\alpha_1, \alpha_2, \beta_1, \beta_2}\rangle = \left( \hat{J}_1^2 + \hat{J}_2^2 \right) |f_{\alpha_1, \alpha_2, \beta_1, \beta_2}\rangle = \hat{J}_1^2 |f_{\alpha_1, \alpha_2, \beta_1, \beta_2}\rangle + \hat{J}_2^2 |f_{\alpha_1, \alpha_2, \beta_1, \beta_2}\rangle \]

\[ = \hbar^2 \left( \frac{\alpha_1 + \beta_1}{2} \right) \left( \frac{\alpha_1 + \beta_1}{2} + 1 \right) |f_{\alpha_1, \beta_1}\rangle + \hbar^2 \left( \frac{\alpha_2 + \beta_2}{2} \right) \left( \frac{\alpha_2 + \beta_2}{2} + 1 \right) |f_{\alpha_2, \beta_2}\rangle \]

\[ = \hbar^2 \left( \frac{\alpha + \beta}{2} \right) \left( \frac{\alpha + \beta}{2} + 1 \right) |f_{\alpha_1, \alpha_2, \beta_1, \beta_2}\rangle, \quad (3.22) \]

\[ \hat{J}_3 |f_{\alpha_1, \alpha_2, \beta_1, \beta_2}\rangle = \hbar \frac{\alpha - \beta}{2} |f_{\alpha_1, \alpha_2, \beta_1, \beta_2}\rangle, \quad (3.23) \]
where \( \alpha = \alpha_1 + \alpha_2 \) and \( \beta = \beta_1 + \beta_2 \).

The generalization to \( N \)-particle system is very natural in the analytical approach. The previous relations 3.22 and 3.23 generalize to

\[ \hat{J}^2 |f_{\alpha_1..\alpha_N,\beta_1..\beta_N} \rangle = \hbar^2 (\alpha + \beta) \left( \frac{\alpha + \beta}{2} + 1 \right) |f_{\alpha_1..\alpha_N,\beta_1..\beta_N} \rangle, \quad (3.24) \]

\[ \hat{J}_3 |f_{\alpha_1..\alpha_N,\beta_1..\beta_N} \rangle = \hbar \left( \frac{\alpha - \beta}{2} \right) |f_{\alpha_1..\alpha_N,\beta_1..\beta_N} \rangle, \quad (3.25) \]

where here \( \alpha = \alpha_1 + \cdots + \alpha_N \) and \( \beta = \beta_1 + \cdots + \beta_N \).

Having established the angular momentum operators and their main addition relations in the holomorphic representation, it is a straightforward procedure to reproduce all known formulas related to quantum angular momentum such as the Clebsch-Gordon coefficients, Wigner 3-\( j \), 6-\( j \) and Racah symbols by means of analytical functions and their partial derivatives[18]. Furthermore using the holomorphic representation of Jordan-Schwinger mapping we may find the holomorphic representation of spin chains [19, 20] and topologically protected magnetic solitons such as magnetic skyrmions [21–23].

4 Spin Chains

Now, consider the general XYZ-spin chain Hamiltonian

\[
\hat{H}_{XYZ} = \sum_i \left( J_x \hat{S}_i^x \hat{S}_{i+1}^x + J_y \hat{S}_i^y \hat{S}_{i+1}^y + J_z \hat{S}_i^z \hat{S}_{i+1}^z \right),
\]

(4.1)

where \( i \) is the site index. Applying the transformations 3.10,3.11,3.12 into 4.1 give the holomorphic representation of the XYZ-spin chain,

\[
\hat{H}_{XYZ} = \frac{\hbar^2 J_x}{4} \sum_i \left( z_{i+1} \frac{\partial^2}{\partial w_i \partial w_{i+1}} + w_{i+1} \frac{\partial^2}{\partial z_i \partial w_{i+1}} + w_{i+1} z_i \frac{\partial^2}{\partial z_{i+1} \partial w_i} + w_i w_{i+1} \frac{\partial^2}{\partial z_i \partial z_{i+1}} \right) - \frac{\hbar^2 J_y}{4} \sum_i \left( z_{i+1} \frac{\partial^2}{\partial w_i \partial w_{i+1}} - w_{i+1} z_i \frac{\partial^2}{\partial z_i \partial w_{i+1}} - z_{i+1} w_i \frac{\partial^2}{\partial w_i \partial z_{i+1}} + w_{i+1} w_i \frac{\partial^2}{\partial z_i \partial z_{i+1}} \right)
\]

\[
+ \frac{\hbar^2 J_z}{4} \sum_i \left( z_{i+1} \frac{\partial^2}{\partial z_i \partial z_{i+1}} - 2 w_{i+1} z_i \frac{\partial^2}{\partial w_i \partial z_{i+1}} + w_{i+1} w_i \frac{\partial^2}{\partial w_i \partial w_{i+1}} \right)
\]

When \( J_x = J_y = J_z \), 4.2 reduces to the XXX Spin chain Hamiltonian,

\[
\hat{H}_{XXX} = \frac{\hbar^2 J_x}{4} \left( z_{i+1} \frac{\partial^2}{\partial z_i \partial z_{i+1}} + 2 w_{i+1} \frac{\partial^2}{\partial w_i \partial z_{i+1}} + w_{i+1} \frac{\partial^2}{\partial w_i \partial w_{i+1}} \right)
\]

(4.3)
with eigenfunctions written as

$$f^i_{\alpha\beta\gamma\delta} = \frac{z_i^\alpha z_{i+1}^\beta w_i^\gamma w_{i+1}^\delta}{\sqrt{\alpha!\beta!\gamma!\delta!}}.$$  \hspace{1cm} (4.4)

The expectation value of $XXX$ spin chain Hamiltonian is

$$\langle f^i_{\alpha'\beta'\gamma'\delta'}|\hat{H}_{XXX}|f^j_{\alpha\beta\gamma\delta}\rangle_\mu = \frac{\hbar^2 J_x (\alpha + \beta)}{4} \sum_i \langle f^i_{\alpha'\beta'\gamma'\delta'}|f^j_{\alpha\beta\gamma\delta}\rangle_\mu \delta_{ij} + \frac{\hbar^2 J_x \gamma \delta}{2} \sum_i \langle f^i_{\alpha'\beta'\gamma'\delta'}|f^j_{\alpha+1\beta-1\gamma+1\delta-1}\rangle_\mu \delta_{ij}$$ \hspace{1cm} (4.5)

where $\langle f^i_{\alpha'...}|f^j_{\kappa...}\rangle_\mu = \delta_{ij}\delta_{\alpha...}$. From the holomorphic representation of $\hat{H}$, we can compute the partition function from which we may determine thermodynamics quantities such as entropy and free energy...

\section{5 Conclusion}

In this note, we presented a general procedure for formulating spin chains in the space of holomorphic functions with Gaussian integration measure know as Bargmann representation. One advantage of the Bargmann representation is the analog of quantum and classical formalism as emphasized in [2]. Another privilege of the Bargmann representation stems from the fact that all operators are written in terms of the phase-space canonical coordinates. Since quasi-probability distributions such as the Glauber–Sudarshan $P$-representation and Husimi-Kano $Q$-representations are written in the phase-space. Thus, Bargmann representation is the natural home for studying these quantities which has deep applications in quantum optics.

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