ON A GENERAL ANALYTICAL FORMULA FOR
$U_q(su(3))$-CLEBSCH-GORDAN COEFFICIENTS

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Abstract
We present the projection operator method in combination with the Wigner-Racah
 calculus of the subalgebra $U_q(su(2))$ for calculation of Clebsch-Gordan coefficients
 (CGCs) of the quantum algebra $U_q(su(3))$. The key formulas of the method are couplings
 of the tensor and projection operators and also a tensor form for the projection
 operator of $U_q(su(3))$. We obtain a very compact general analytical formula for the
 $U_q(su(3))$ CGCs in terms of the $U_q(su(2))$ Wigner 3nj-symbols.

1 Introduction
It is well known that the Clebsch-Gordan coefficients (CGCs) of the unitary Lie algebra
 $u(n)$ $(su(n))$ have numerous applications in various fields of theoretical and mathematical
 physics. For example, many algebraic models of nuclear theory (interacting boson model
 (IBM), Elliott $su(3)$ model, $su(4)$ supermultiplet scheme of Wigner, the shell model, and
 so on) demand the CGCs for $su(6)$, $su(5)$, $su(3)$, $su(4)$ and $su(n)$. Analogously, in quark
 models of hadrons we need the CGCs of $su(3)$, $su(4)$, etc. The theory of the $su(n)$ CGCs
 is connected with the theory of special functions, combinatorial analysis, topology, etc.

There are several methods for the calculation of CGCs of $su(n)$ ($u(n)$) and other Lie algebras:
 recursion method; method of employment of explicit bases of irreducible representations;
 method of generating invariants; method of tensor operators, where the Wigner-Eckart
 theorem is used; projection operator method; coherent state method; combined methods.

It is well known that the method of projection operators for usual (non-quantized) Lie
 algebras [1, 2] and superalgebras [2] is powerful and universal method for a solution of many
problems in the representation theory. In particular, the method allows to develop the
detailed theory of Clebsch-Gordan coefficients and another elements of Wigner-Racah calculus
(including compact analytic formulas of these elements and their symmetry properties) and so on. It is evident that the projection operators of quantum groups play the same role in their representation theory.

In this paper we present the projection operator method in combination with the Wigner-
Racah calculus of the subalgebra $U_q(sl(2))$ for calculation of CGCs of the quantum
algebra $U_q(sl(3))$. The key formulas of the method are couplings of the tensor and projection
operators and also a tensor form for the projection operator of $U_q(sl(3))$. It should be noted
that the first application of this method was for the $su(3)$ case in \cite{Racah}. Some simple elements
of this approach were also used in \cite{Racah} for the $U_q(sl(n))$ case. Also, the coherent state method
in combination with the Wigner-Racah calculus was applied in \cite{Racah} for $u(n)$.

\section{Gelfand-Tsetlin basis}

Let $\Pi := \{\alpha_1, \alpha_2\}$ be a system of simple roots of the Lie algebra $sl(3) (= sl(3, C) \simeq A_2)$,
which endowed with the following scalar product: $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2$, $(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1) = -1$. The root system $\Delta_+$ of $sl(3)$ consists of the roots $\alpha_1, \alpha_1 + \alpha_2, \alpha_2$. The quantum Hopf algebra
$U_q(sl(3))$ is generated by the Chevalley elements $e_{\pm \alpha_i}$, $e_{\alpha_i}$ $(i = 1, 2)$ with the relations:
\begin{equation}
q^{h_{\alpha_i}}q^{-h_{\alpha_i}} = q^{-h_{\alpha_i}}q^{h_{\alpha_i}} = 1, \quad q^{h_{\alpha_i}}q^{h_{\alpha_j}} = q^{h_{\alpha_j}}q^{h_{\alpha_i}}, \quad q^{h_{\alpha_i}}e_{\alpha_j}q^{-h_{\alpha_i}} = q^{(\alpha_i, \alpha_j)}e_{\alpha_j},
\end{equation}
where $[e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij}[h_{\alpha_i}]$, $[[e_{\pm \alpha_i}, e_{\pm \alpha_j}], e_{\pm \alpha_k}] = 0$ for $|i-j| = 1$.

Here and elsewhere we use the standard notation $[a] := (q^a - q^{-a})/(q - q^{-1})$, and $[e_{\alpha_i}, e_{\beta_j}] := e_{\alpha_i}e_\beta - q^{(\alpha_i, \beta_j)}e_\beta e_{\alpha_i}$. The Hopf structure of $U_q(u(3))$ is given by
\begin{equation}
\Delta_q(h_{\alpha_i}) = h_{\alpha_i} \otimes 1 + 1 \otimes h_{\alpha_i}, \quad S_q(h_{\alpha_i}) = -h_{\alpha_i},
\end{equation}
\begin{equation}
\Delta_q(e_{\pm \alpha_i}) = e_{\pm \alpha_i} \otimes q^{\mp h_{\alpha_i}} + q^{\pm h_{\alpha_i}} \otimes e_{\pm \alpha_i}, \quad S_q(e_{\pm \alpha_i}) = -q^{\pm 1}e_{\pm \alpha_i}.
\end{equation}

For construction of the composite root vectors $e_{\pm (\alpha_1 + \alpha_2)}$ we fix the normal ordering in $\Delta_+$:
$\alpha_1, \alpha_1 + \alpha_2, \alpha_2$. According to this ordering we put
\begin{equation}
e_{\alpha_1 + \alpha_2} := [e_{\alpha_1}, e_{\alpha_2}]_{q^{-1}}, \quad e_{-\alpha_1 - \alpha_2} := [e_{-\alpha_2}, e_{-\alpha_1}]_q.
\end{equation}

Let us introduce another standard notations for the Cartan-Weyl generators:
\begin{align*}
e_{12} &:= e_{\alpha_1}, & e_{21} &:= e_{-\alpha_1}, & e_{11} - e_{22} &:= h_{\alpha_1}, \\
e_{23} &:= e_{\alpha_2}, & e_{32} &:= e_{-\alpha_2}, & e_{22} - e_{33} &:= h_{\alpha_2}, \\
e_{13} &:= e_{\alpha_1 + \alpha_2}, & e_{31} &:= e_{-\alpha_1 - \alpha_2}, & e_{11} - e_{33} &:= h_{\alpha_1} + h_{\alpha_2}.
\end{align*}

The explicit formula for the extremal projector for the quantum groups specialized to the
case of $U_q(sl(3))$ has the form
\begin{equation}
p(U_q(sl(3))) = p_{12}p_{13}p_{23},
\end{equation}
where the elements \( p_{ij} \) \((1 \leq i < j \leq 3)\) are given by
\[
p_{ij} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \varphi_{ij,n} e_{ij}^n e_{ji}^n, \quad \varphi_{ij,n} = q^{-(j-i-1)n} \left\{ \prod_{s=1}^{n} \left[ e_{ii} - e_{jj} + j - i + s \right] \right\}^{-1}.
\]
(2.6)

The extremal projector \( p := p(U_q(sl(3))) \) satisfies the relations:
\[
e_{ij} p = p e_{ji} = 0 \quad (i < j) , \quad p^2 = p .
\]
(2.7)

The quantum algebra \( U_q(su(3)) \) can be considered as the quantum algebra \( U_q(sl(3)) \) endowed with the additional Cartan involution \((^*)\):
\[
h_{\alpha_1}^* = h_{\alpha_1} , \quad e_{\pm \alpha_i}^* = e_{\mp \alpha_i} , \quad q^* = q \quad (\text{or} \quad \bar{q} := q^{-1}) .
\]
(2.8)

Let \((\lambda \mu)\) be a finite-dimensional irreducible representation (IR) of \( U_q(su(3)) \) with the highest weight \((\lambda \mu)\) \((\lambda \text{ and } \mu \text{ are nonnegative integers})\). The vector of the highest weight, denoted by the symbol \(|(\lambda \mu)\rangle\), satisfy the relations
\[
h_{\alpha_1}|(\lambda \mu)\rangle = \lambda |(\lambda \mu)\rangle , \quad h_{\alpha_2}|(\lambda \mu)\rangle = \mu |(\lambda \mu)\rangle , \quad e_{ij}|(\lambda \mu)\rangle = 0 \quad (i < j) .
\]
(2.9)

Labelling of another basis vectors in IR \((\lambda \mu)\) depends upon choice of subalgebras of \( U_q(su(3)) \) (or in another words, depends upon which reduction chain from \( U_q(su(3)) \) to subalgebras is chosen). Here we use the Gelfand-Tsetlin reduction chain:
\[
U_q(su(3)) \supset U_q(u_T(1)) \otimes U_q(su_T(2)) \supset U_q(u_T^0(1)) ,
\]
(2.10)
where the subalgebra \( U_q(su_T(2)) \) is generated by the elements
\[
T_+ := e_{23} , \quad T_- := e_{32} , \quad T_0 := \frac{1}{2}(e_{22} - e_{33}) ,
\]
(2.11)
the subalgebra \( U_q(u_T^0(1)) \) is generated by \( q^{T_0} \), and \( U_q(u_Y(1)) \) is generated by \( q^Y \) (In the classical (non-deformed) case in the elementary particle theory the subalgebra \( su_T(2) \) is called the T-spin algebra and the element \( Y \) is the hypercharge operator), where:
\[
Y = -\frac{1}{3}(2h_{\alpha_1} + h_{\alpha_2}) .
\]
(2.12)

In the case of the reduction chain (2.10) the basis vectors of IR \((\lambda \mu)\) are denoted by
\[
|(\lambda \mu)jtt_z\rangle .
\]
(2.13)

Here the set \( jtt_z \) characterize the hypercharge \( Y \) and the T-spin and its projection:
\[
q^{T_0} |(\lambda \mu)jtt_z\rangle = q^z |(\lambda \mu)jtt_z\rangle , \quad q^Y |(\lambda \mu)jtt_z\rangle = q^Y |(\lambda \mu)jtt_z\rangle ,
\]
\[
T_{\pm} |(\lambda \mu)jtt_z\rangle = \sqrt{[t \mp t_z][t \mp t_z + 1]} |(\lambda \mu)jtt_z \pm 1\rangle ,
\]
(2.14)
where the parameter $j$ is connected with the eigenvalue $y$ of the operator $Y$ as follows $y = -\frac{1}{3}(2\lambda + \mu) + 2j$. It is not hard to show that the orthonormalized vectors (2.13) can be represented in the following form

$$\langle (\lambda\mu)jt t_z \rangle = N_{jt}^{(\lambda\mu)} P^t_{ts:t} \bar{e}^{(j+\frac{1}{2}\mu-t)j\frac{1}{2}\mu+t} (\lambda\mu) h \rangle,$$

(2.15)

where $P^t_{ts:t}$ is the general projection operator of the quantum algebra $U_q(su_T(2))$ [4], and the normalizing factor $N_{jt}^{(\lambda\mu)}$ has the form

$$N_{jt}^{(\lambda\mu)} = \left( \frac{[\lambda+\frac{1}{2}\mu-j+t+1][(\lambda+\frac{1}{2}\mu-j-t)][(\lambda+\frac{1}{2}\mu+j+t+1)][(\lambda+\frac{1}{2}\mu-j+t)]}{q^{2j+\mu-2t} [\lambda][\mu][\lambda+\mu+1][j+\frac{1}{2}\mu-t][j-\frac{1}{2}\mu+t][2t+1]} \right)^{\frac{1}{2}}.$$

(2.16)

The quantum numbers $jt$ are taken all nonnegative integers and half-integers such that the sum $\frac{1}{2}\mu + j + t$ is an integer and they are subjected to the constraints:

$$\left\{ \begin{array}{l} \frac{1}{2}\mu + j - t \geq 0, \\ -\frac{1}{2}\mu + j + t \geq 0, \\ \frac{1}{2}\mu - j + t \geq 0, \\ \frac{1}{2}\mu + j + t \geq \lambda + \mu. \end{array} \right.$$

(2.17)

For every fixed $t$ the projection $t_z$ runs the values $t_z = -t, -t+1, \ldots, t-1, t$. These results can be obtained from the explicit form of the Gelfand-Tsetlin bases for the case $U_q(su(n))$ [4] specializing to the given case $U_q(su(3))$.

### 3 Couplings of tensor and projection operators

Let $\{R^{(q)}_{jz}\}$ be an irreducible tensor operator (ITO) of the rank $j$, that is $(2j+1)$-components $R^{(q)}_{jz}$ are transformed with respect to the $U_q(su_T(2))$ adjoint action as the $U_q(su_T(2))$ basis vectors $|jj_z\rangle$ of the spin $j$:

$$T_i \triangleright R^{(q)}_{jz} := (ad_q T_i) R^{(q)}_{jz} \equiv ((\text{id} \otimes S_q) \Delta_q(T_i)) \circ R^{(q)}_{jz} = \sum_{jj_z} \langle jj_z | T_i | jj_z \rangle R^{(q)}_{jj_z},$$

(3.1)

where $(a \otimes b) \circ x = axb$. The tensor operator of the type $\{R^{(q)}_{jz}\}$ will be also called the left irreducible tensor operators (LITO) because the generators $T_i$ ($i = \pm, 0$) act to the left-side of the components $R^{(q)}_{jz}$. (The given denotation of the ITOs is differed from one of the papers [3] by the replacement $q$ by $q^{-1}$). Following to the paper [3] we also introduce a right irreducible tensor operator (RITO) denoted by the tilde symbol $\{\tilde{R}^{(q)}_{jz}\}$, on which the $U_q(su_T(2))$ generators $T_i$ act on the right-side, namely

$$T_i \triangleright \tilde{R}^{(q)}_{jz} := (ad_q^* T_i) \tilde{R}^{(q)}_{jz} \equiv \tilde{R}^{(q)}_{jz} \circ ((\tilde{S}_q \otimes \text{id}) \tilde{\Delta}_q(T_i^*)) = \sum_{jj_z} \langle jj_z | T_i^* | jj_z \rangle \tilde{R}^{(q)}_{jj_z},$$

(3.2)

where $x \circ (a \otimes b) = axb$, and $\tilde{\Delta}_q$ is the opposite coproduct ($\tilde{\Delta}_q = \Delta_q$) and $\tilde{S}_q$ is the corresponding antipode ($\tilde{S}_q = S_q$). It is not hard to verify that any LITO $\{R^{(q)}_{jz}\}$ is the
RITO \( \{ \tilde{R}^{(q)}_{j s} \} \): \( R^{(q)}_{j s} = (-1)^{i_s} q^{t_z} \tilde{R}^{(q)}_{j s} \). The projection operator set \( \{ P^{t}_{t_z:t_z'} \} \) for a fixed IR \( t \) and for various \( t_z \) and \( t_z' \) will be called the \( \textbf{P}^{t} \)-operator. It is not hard to see that the subset of the left components of this operator satisfy the relations for the LITO \( R^{(q)}_{j s} := \{ \tilde{R}^{(q)}_{j s} \} \) if we understand the action “\( \triangleright \)" of the generator \( T_{t_z} \) as the usual multiplication of the operators \( P^{t}_{t_z:t_z'} \) and the subset of the right components of the \( \textbf{P}^{t} \)-operator satisfy the relations for the RITO \( \tilde{R}^{(q)}_{j s} := \{ \tilde{R}^{(q)}_{j s} \} \) if we understand the action "\( \triangleleft \)" as the usual multiplication of the operators \( P^{t}_{t_z:t_z'} \) and \( T_{t_z} \):

\[
T_{t_z} \triangleright P^{t}_{t_z:t_z'} := T_{t_z} P^{t}_{t_z:t_z'} = \sum_{t_z'} \langle t_{t_z'} | T_{t_z} | t_{t_z} \rangle P^{t}_{t_z:t_z'},
\]

\[
T_{t_z} \triangleleft P^{t}_{t_z:t_z'} := P^{t}_{t_z:t_z'} T_{t_z} = \sum_{t_z'} \langle t_{t_z'} | T_{t_z} | t_{t_z} \rangle P^{t}_{t_z:t_z'},
\]

Using the \( U_q(su_T(2)) \) CGCs we can couple the LITO \( R^{(q)}_{j s} \) with the left components of the \( \textbf{P}^{t} \)-operator and the RITO \( \tilde{R}^{(q)}_{j s} \) with the right components of the \( \textbf{P}^{t} \)-operator:

\[
\text{P}^{t}_{t_z:t_z'} R^{(q)}_{j s} := \sum_{t_z'} \langle j j_z | t_{t_z'} t_{t_z} \rangle R^{(q)}_{j s} P^{t}_{t_z:t_z'},
\]

\[
\text{P}^{t}_{t_z:t_z'} \tilde{R}^{(q)}_{j s} := \sum_{t_z'} \langle j j_z | t_{t_z'} t_{t_z} \rangle \tilde{R}^{(q)}_{j s} P^{t}_{t_z:t_z'},
\]

Here the symbol \( \hat{\otimes} \) means that we first take the usual tensor product and then in a resulting expression we replace the tensor product by the usual operator product. It is not hard to show that the couplings (3.3) and (3.6) are connected as follows

\[
\text{R}^{j(q)}_{t_z:t_z'} := \sqrt{2t + 1} P^{t}_{t_z:t_z'} R^{j(q)}_{j s} = (-1)^{t_z} \sqrt{2t' + 1} \text{P}^{t}_{t_z:t_z'} \tilde{R}^{j(q)}_{j s}.
\]

Using (3.7) and an unitary relation of the \( U_q(su(2)) \) CGCs one can obtain the following useful permutation relations between the components of the tensors \( R^{j(q)}, \tilde{R}^{j(q)} \) and \( \text{P}^{t} \)-operator:

\[
R^{j(q)}_{j s} P^{t}_{t_z:t_z'} = \sum_{t_{t_z'}} (-1)^{1-t_z'} \sqrt{\frac{2t + 1}{2t' + 1}} \langle j j_z | t_{t_z'} t_{t_z} \rangle \text{P}^{t}_{t_z:t_z'} \tilde{R}^{j(q)}_{j s} P^{t}_{t_z:t_z'},
\]

\[
P^{t}_{t_z:t_z'} \tilde{R}^{j(q)}_{j s} = \sum_{t_{t_z'}} (-1)^{1-t_z'} \sqrt{\frac{2t + 1}{2t' + 1}} \langle j j_z | t_{t_z'} t_{t_z} \rangle \text{P}^{t}_{t_z:t_z'} R^{j(q)}_{j s} P^{t}_{t_z:t_z'}.
\]

We can show that the monomials \( e_{21}^m e_{31}^m \) and \( e_{12}^m e_{13}^m \) are components of ITOs with respect to the adjoint action of the subalgebra \( U_q(su_T(2)) \):

\[
R^{j(q)}_{j s} = \sqrt{\frac{2j}{|j - j_z| j + j_z}} q^{2j^2 - j} e_{21}^j e_{31}^j e_{12}^{j-j_z} e_{13}^{j-j_z} q^{-j j_{01} - (j - j_z) T_0},
\]
With the help of (3.15) we easily find the action of the ITO (3.10) on the Gelfand-Tsetlin basis:

\[
R^{ij(q)}_{jz} = \sqrt{\frac{[2j]!}{[j-j_z]!(j+j_z)!}} e^{i_1-j_z} e^{i_2-j_z} e^{i_3-j_z} q^{-j\alpha_1-(j+j_z)T_0}.
\] (3.11)

where the generator \(e'_{13}\) is defined according to the inverse normal ordering: \(\alpha_2, \alpha_1+\alpha_2, \alpha_1,\) i.e. \(e'_{13} = [e_{23}, e_{12}]_{q^{-1}}.\) These ITOs have the remarkable properties: A result of the coupling of two ITOs of the type (3.10) or (3.11) is non-zero only for an irreducible component of the maximal rank, e.g.

\[
\{ R^{j(q)} \otimes R^{j'(q)} \}^{j''}_{j_j j_j'} = \delta_{j_j+j_j'} R^{j+j'(q)}_{j''}.
\] (3.12)

The property is also useful in applications: For ITOs of the type (3.12) the relation is valid

\[
R^{ij(q)}_{jz} \mathbf{R}^{j'(q)}_{ut; t' t''} = \sum_{v't'_z} \sqrt{\frac{[2t+1]}{[2t''+1]}} (jz z t t''_{t''})_q U(jj' t' t''; j+j't')_q R^{j+j'(q)}_{jz; v't'_z}. \] (3.13)

Here \(U(\ldots;\ldots)_q\) is recoupling coefficient which can be expressed via the stretched \(q\)-\(6j\)-symbols of \(U_q(su_T(2))\) [5]:

\[
U(jj' t' t''; j+j't')_q = (-1)^{j+j'+t'+t''} \sqrt{[2j+2j'+1][2t+1]} \left\{ \begin{array}{ccc} j & j' & j+j' \\ t' & t'' & t \end{array} \right\}_q. \] (3.14)

Using (3.9) and (3.10) we can present the basis vectors (2.13) in the form of

\[
| (\lambda\mu) j z t \rangle = \mathcal{F}_{\lambda \mu}^{-}(j z t) | (\lambda\mu) h \rangle = N^{(\lambda\mu)}_{jz} \; \mathbf{R}^{j(q)\dagger}_{zt; \frac{1}{2} \frac{1}{2} \frac{1}{2}} | (\lambda\mu) h \rangle. \] (3.15)

The normalizing factor \(N^{(\lambda\mu)}_{jz}\) is given by

\[
N^{(\lambda\mu)}_{jz} = (-1)^{2j} q^{(j+\frac{1}{2}\mu-t)(j-\frac{1}{2}\mu+t)+j\lambda+\frac{1}{2}(j+\frac{1}{2}\mu-t)-2j+2t-\frac{1}{2} \mu}
\]

\[
\times \sqrt{[j-\frac{1}{2} \mu + t] (j+\frac{1}{2} \mu + t)} \frac{[2j]!(\mu+1)}{[2j]!(\mu+1)} (j \frac{1}{2} \mu - t) (\frac{1}{2} \mu + \frac{1}{2} \mu)_q N^{(\lambda\mu)}_{jz}. \] (3.16)

With the help of (3.13) we easily find the action of the ITO (3.10) on the Gelfand-Tsetlin basis:

\[
R^{ij(q)}_{jz} | (\lambda\mu) j z t \rangle = \sum_{v't'_z} (j_j' t'_z z t''_{t''})_q \langle (\lambda\mu) j'' t'' | R^{ij(q)}_{jj'} | (\lambda\mu) j t \rangle \langle (\lambda\mu) j'' t'' | (\lambda\mu) j z t'' \rangle, \] (3.17)

where

\[
\langle (\lambda\mu) j'' t'' | R^{ij(q)}_{jj'} | (\lambda\mu) j t \rangle_q = \delta_{j_j'; j_j'} \sqrt{\frac{[2t+1]}{[2t''+1]}} \frac{N^{(\lambda\mu)}_{jj'}}{N^{(\lambda\mu)}_{jj''}} U(jj' t' t''; \frac{1}{2} \mu; j' j t)_q. \] (3.18)
4 Tensor form of the projection operator

It is obvious that the extremal projector $p(U_q(\mathfrak{su}(3)))$ can be presented in the form

$$p(U_q(\mathfrak{su}(3))) = p(U_q(\mathfrak{su}_T(2)))(p_{12}p_{13})p(U_q(\mathfrak{su}_T(2))).$$  \hspace{1cm} (4.1)

Now we present the middle part of (4.1) in the terms of the $U_q(\mathfrak{su}_T(2))$ tensor operators (3.10) and (3.11). To this end, we substitute the explicit expression (2.6) for the factors $p_{12}$ and $p_{13}$, and combine monomials $e_{21}^m e_{31}^n$ and $e_{12}^n e_{31}^m$. After some summation manipulations we obtain the following expression for the extremal projection operator $p := p(U_q(\mathfrak{su}(3)))$ in terms of the tensor operators (3.10) and (3.11):

$$p = p(U_q(\mathfrak{su}_T(2))\left(\sum_{j\ell s} A_{j\ell s} \bar{R}^{j(q)}_{j\ell s} R^{j(q)}_{j\ell s}\right)p(U_q(\mathfrak{su}_T(2))).$$  \hspace{1cm} (4.2)

Here

$$A_{j\ell s} = \frac{(-1)^{3j}[\varphi_{12}\varphi_{12} + j + j_z - 1][\varphi_{13}]}{[2j]![\varphi_{12} + 2j][\varphi_{13} + j + j_z]} q^{4j^2 + 2j_h a_1 + 2(j + j_z)\gamma_0},$$  \hspace{1cm} (4.3)

where $\varphi_{i+1} := e_{i+1} - e_{i+1+1} + i$ ($i = 1, 2$). Below we assume that the $U_q(\mathfrak{su}(3))$ extremal projection operator $p$ acts in a weight space with the weight $(\lambda\mu)$ and in this case the symbol $p$ is supplied with the index $(\lambda\mu)$, $p^{(\lambda\mu)}$, and all the Cartan elements $h_\alpha$, on the right side of (4.2) are replaced by the corresponding weight components $\lambda$ and $\mu$. Now we multiple the projector $p^{(\lambda\mu)}$ from the left side by the lowering operator $J^{(\lambda\mu)}(jtt_z)$ and from the right side by the rising operator $(J^{(\lambda\mu)}(jtt_z))^*$, and by applying a relation of type (3.13) we finally find the tensor form of the general $U_q(\mathfrak{su}(3))$ projection operator:

$$B^{(\lambda\mu)}_{jtt_z,jtt_{t'}z'} = \sum_{j\ell s, t'q} B^{(\lambda\mu)}_{jtt_z,jtt_{t'}z'} R^{j(q)}_{j\ell s,t't'} R^{j(q)}_{jtt_z,t't'z'},$$  \hspace{1cm} (4.4)

were the coefficients $B^{(\lambda\mu)}_{jtt_z,jtt_{t'}z'}$ are given by

$$B^{(\lambda\mu)}_{jtt_{t'}z'} = \frac{(-1)^{2j + j' + j'' - t - t'} q^6[\lambda + 1][\mu + 1][\lambda + \mu + 2]_{[2j]!} [\lambda + \frac{1}{2}\mu - j - t]^{12j} [\lambda + \frac{1}{2}\mu - j' + t']^{12j} [\lambda + \frac{1}{2}\mu - j'' - t'' + j + j_z - 1][2j - t'' + 1]^{12j}}{[\lambda + \frac{1}{2}\mu - j - t][2j + 1][2j + t - 1][2j + t - 3][2j + t - 4][2j + 2j'' + 1][2j + 2j' + 1]} \frac{1}{2},$$  \hspace{1cm} (4.5)

$$\phi = \varphi(\lambda, \mu, j, t) + \varphi(\lambda, \mu, j', t') - 2\varphi(\lambda, \mu, j'', t'') + j''(4\lambda + 2\mu - 1) + 4t'' - 2\mu - 3j'.$$  \hspace{1cm} (4.6)

Here and elsewhere we use the notation $\varphi(\lambda, \mu, j, t) := \frac{1}{2}(\frac{3}{2}\mu + j - t)(\frac{3}{2}\mu + j + t - 3) + j(\lambda - 2j + 1)$.

5 General form of Clebsch-Gordan coefficients

For convenience we introduce the short notations: $\Lambda := (\lambda\mu)$ and $\gamma := jtt_z$ and therefore the basis vector $|(\lambda\mu)jtt_z\rangle$ will be is denoted by $|\Lambda\gamma\rangle$. Let $\{|\Lambda_i\gamma_i\rangle\}$ be bases of two IRs $\Lambda_i$ ($i = 1, 2$)
1, 2). Then \{\langle \Lambda_1 \gamma_1 | \Lambda_2 \gamma_2 \rangle \} be a basis in the representation \(\Lambda_1 \otimes \Lambda_2\) of \(U_q(su(3)) \otimes U_q(su(3))\). In this representation there is an another coupled basis \(\langle \Lambda_1 \Lambda_2 : s \Lambda_3 \gamma_3 \rangle_q\) with respect to \(\Delta_q(U_q(su(3)))\) where the index \(s\) classifies multiple representations \(\Lambda_3\). We can expand the coupled basis in terms of the tensor uncoupled basis \{\langle \Lambda_1 \gamma_1 | \Lambda_2 \gamma_2 \rangle \}:

\[
\langle \Lambda_1 \Lambda_2 : s \Lambda_3 \gamma_3 \rangle_q = \sum_{\gamma_2} \langle \Lambda_1 \gamma_1 | \Lambda_2 \gamma_2 \rangle \langle s \Lambda_3 \gamma_3 \rangle q \langle \Lambda_1 \gamma_1 | \Lambda_2 \gamma_2 \rangle ,
\]

where the matrix element \(\langle \Lambda_1 \gamma_1 | \Lambda_2 \gamma_2 \rangle | s \Lambda_3 \gamma_3 \rangle_q\) is the Clebsch-Gordan coefficient of \(U_q(su(3))\). In just the same way as for the non-quantized Lie algebra \(su(3)\) (see \([3]\) we can show that any CGC of \(U_q(su(3))\) can be represented in terms of the linear combination of the matrix elements of the projection operator \([4, 4]\):

\[
\langle \Lambda_1 \gamma_1 | \Lambda_2 \gamma_2 | s \Lambda_3 \gamma_3 \rangle_q = \sum_{\gamma_2} C(\gamma_2) \langle \Lambda_1 \gamma_1 | \Lambda_2 \gamma_2 \rangle \langle s \Lambda_3 \gamma_3 \rangle q \langle \Lambda_1 \gamma_1 | \Lambda_2 \gamma_2 \rangle .
\]

Classification of multiple representations \(\Lambda_3\) in the representation \(\Lambda_1 \otimes \Lambda_2\) is special problem and we shall not touch it here. For the non-deformed algebra \(su(3)\) this problem was considered in details in \([3]\). Concerning of the matrix elements in the right-side of (5.2) we give here an explicit expression for the more general matrix element:

\[
\langle \Lambda_1 \gamma_1 | \Lambda_2 \gamma_2 | s \Lambda_3 \gamma_3 \rangle_q = C(\gamma_2) \langle \Lambda_1 \gamma_1 | \Lambda_2 \gamma_2 \rangle \langle \Lambda_3 \gamma_3 \rangle q \langle \Lambda_1 \gamma_1 | \Lambda_2 \gamma_2 \rangle .
\]

Using \([4, 4]\) and the Wigner-Racah calculus for the subalgebra \(U_q(su(2))\) \([3]\) (analogously to the non-quantized Lie algebra \(su(3)\) \([3]\) it is hard to obtain the following result:

\[
\langle \Lambda_1 \gamma_1 | \Lambda_2 \gamma_2 | \Delta_q(P_{\Lambda_3 \gamma_3}^\Lambda_1 \gamma_1) | \Lambda_1' \gamma_1' | \Lambda_2 \gamma_2' \rangle = (t_1 t_2 t_2 t_3 t_3) q \langle t_1' t_2' t_2' t_3' t_3' q \rangle.
\]

Here

\[
A = \frac{\frac{[2 t_1 + 1][2 t_2 + 1][2 j_1 + 1][2 j_2 + 1][2 j_3 + 1]}{[\lambda_1 + \frac{1}{2} j_1 - j_1 - t_1 + 1]}[\lambda_1 + \frac{1}{2} j_1 - j_1 - t_1 + 1][\lambda_2 + \frac{1}{2} j_2 - j_2 + t_2 + 1][\lambda_2 + \frac{1}{2} j_2 - j_2 - t_2 + 1][\lambda_3 + \frac{1}{2} j_3 - j_3 + t_3 + 1][\lambda_3 + \frac{1}{2} j_3 - j_3 - t_3 + 1]}{[\lambda_1 + \frac{1}{2} j_1 - j_1 - t_1 + 1][\lambda_2 + \frac{1}{2} j_2 - j_2 + t_2 + 1][\lambda_2 + \frac{1}{2} j_2 - j_2 - t_2 + 1][\lambda_3 + \frac{1}{2} j_3 - j_3 + t_3 + 1][\lambda_3 + \frac{1}{2} j_3 - j_3 - t_3 + 1]},
\]

\[
\times \frac{[2 t_1' + 1][2 t_2' + 1][2 j_1 + 1][2 j_2 + 1][2 j_3 + 1]}{[\lambda_1 + \frac{1}{2} j_1 - j_1' - t_1' + 1][\lambda_1 + \frac{1}{2} j_1 - j_1' - t_1' + 1][\lambda_2 + \frac{1}{2} j_2 - j_2' + t_2' + 1][\lambda_2 + \frac{1}{2} j_2 - j_2' - t_2' + 1][\lambda_3 + \frac{1}{2} j_3 - j_3' + t_3' + 1][\lambda_3 + \frac{1}{2} j_3 - j_3' - t_3' + 1]} \left(\frac{1}{2}\right)^{t_1' t_2' t_3'},
\]

\[
\times \frac{[2 t_1 + 1][2 t_2 + 1][2 j_1 + 1][2 j_2 + 1][2 j_3 + 1]}{[\lambda_1 + \frac{1}{2} j_1 - j_1 - t_1 + 1][\lambda_1 + \frac{1}{2} j_1 - j_1 - t_1 + 1][\lambda_2 + \frac{1}{2} j_2 - j_2 + t_2 + 1][\lambda_2 + \frac{1}{2} j_2 - j_2 - t_2 + 1][\lambda_3 + \frac{1}{2} j_3 - j_3 + t_3 + 1][\lambda_3 + \frac{1}{2} j_3 - j_3 - t_3 + 1]}^{\frac{1}{2}}.
\]
\[ C_{i_1i_2j_1j_2k_1k_2} = \frac{(-1)^{(j_1+j_2+j_3-j_4-j_5)}}{[2j_1][2j_2][2j_3][2j_4][2j_5]} q^\psi \frac{[2(j_1+j_2-j_3-j_4+j_5)+1][2(j_1+j_2-j_3-j_5+j_4)+1]}{[2j_1-2j_5][2j_2-2j_5][2j_3-2j_5][2j_4-2j_5]} \]

\[ \times \frac{[\lambda_1+\frac{1}{2}\mu_1-j_1-j_2+t_1^\prime+t_2^\prime+1][\lambda_1+\frac{1}{2}\mu_1-j_1-t_1^\prime+1][\lambda_2+\frac{1}{2}\mu_2-j_2-t_2^\prime+1][\lambda_2+\frac{1}{2}\mu_2-j_2-t_2^\prime+1][\lambda_3+\frac{1}{2}\mu_3+j_1+j_2-j_3-j_4-j_5-t_3^\prime+1]}{[\lambda_3+\frac{1}{2}\mu_3+j_1+j_2-j_3-j_4-j_5-t_3^\prime+1]}
\]

where \( \psi = \sum_{i=1}^{2} \left( 2\varphi(\lambda_i, \mu_i; j_i'', t_i') - \varphi(\lambda_i, \mu_i; j_i', t_i) - \varphi(\lambda_i, \mu_i; j_i', t_i) - t_i(t_i+1) - t_i(t_i'+1) \right)

- 2\varphi(\lambda_3, \mu_3; j_3'', t_3') + \varphi(\lambda_3, \mu_3, j_3, t_3) + \varphi(\lambda_3, \mu_3, j_3, t_3) + j_3''(4\lambda_3 + 2\mu_3 + 2) - 2t_3''(t_3''-1) - 2\mu_3

- (j_2-j_2''-2j_2')(2\lambda_1+\mu_1-6j_1''+4(j_1-j_1')(j_2-j_2') + 4(j_1''-j_1')(j_2''-j_2') - 2t_3''(t_3''-1) - 2\mu_3

- (j_3+j_3')(j_3''+j_3''+1), \quad j_3'' := j_1+j_2-j_3-j_1'-j_2' = j_1+j_2-j_3-j_1'-j_2' \]

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