ζ−function and heat kernel formulae

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Abstract
We present a systematic study of asymptotic behavior of (generalised) ζ−functions and heat kernels used in noncommutative geometry and clarify their connections with Dixmier traces. We strengthen and complete a number of results from the recent literature and answer (in the affirmative) the question raised by M. Benameur and T. Fack \cite{1}.

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1. Introduction

The interplay between Dixmier traces, ζ−functions and heat kernel formulae is a cornerstone of noncommutative geometry \cite{8}. These formulae are widely used in physical applications. To define these objects, let us fix a Hilbert space $H$ and let $B(H)$ be the algebra of all bounded operators on $H$ with its standard trace $\text{Tr}$. Let $A$ and $B$ be positive operators from $B(H)$. Consider the following $[0, \infty]$-valued functions

\begin{align*}
& t \rightarrow \frac{1}{t} \text{Tr}(A^{1+1/t}), \quad t \rightarrow \frac{1}{t} \text{Tr}(A^{1+1/t}B)) \\
& t \rightarrow \frac{1}{t} \text{Tr}(\exp(-(tA)^{-q})), \quad t \rightarrow \frac{1}{t} \text{Tr}(\exp(-(tA)^{-q})B).
\end{align*}

When these functions are finitely valued, they are frequently referred to as ζ−functions and heat kernel functions associated with the operators $A$ and $B$.

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When these functions are bounded, a particular interest is attached to their asymptotic behavior when \( t \to \infty \), which is usually measured with the help of some generalised limit \( \gamma : L_\infty(0, \infty) \to \mathbb{R} \) yielding the following functionals

\[
\zeta_\gamma(A) := \gamma(\frac{1}{t} \text{Tr}(A^{1+1/t})), \quad \zeta_\gamma(B)(A) := \gamma(\frac{1}{t} \text{Tr}(A^{1+1/t}B))
\]

and,

\[
\varphi_\gamma(A) := \gamma(\frac{1}{t} \text{Tr}(\exp(-(tA)^{-q}))), \quad \varphi_\gamma(B)(A) := \gamma(\frac{1}{t} \text{Tr}(\exp(-(tA)^{-q}))B).
\]

A natural class of operators for which the formulae (1) and (3) are well defined (respectively, (2) and (4)) is given by the set \( M_{1, \infty} \), \( L_{1, \infty} \) of compact operators from \( B(H) \). More precisely, denote by \( \mu_n(T), n \in \mathbb{N} \), the singular values of a compact operator \( T \) (the singular values are the eigenvalues of the operator \( |T| = (T^*T)^{1/2} \) arranged with multiplicity in decreasing order, \( [23, \S 1] \)). Then

\[
M_{1, \infty} := M_{1, \infty}(H) = \{ T : \sup_{n \in \mathbb{N}} \frac{1}{\log(n+1)} \sum_{k=1}^{n} \mu_k(T) < \infty \}.
\]

defines a Banach ideal of compact operators. We set

\[
L_{1, \infty} := \{ T \in M_{1, \infty} : \exists C > 0 \text{ such that } \mu_n(A) \leq C/n, \ n \geq 1 \}.
\]

It is important to observe that the subset \( L_{1, \infty} \) is not dense in \( M_{1, \infty} \) (see e.g. \[18\]). It should also be pointed out that our notation here differs from that used in \[8\].

It follows from \[6, Theorem 4.5\] that the functions defined in (1) are bounded if and only if \( A \in M_{1, \infty} \). It also follows from \[6\] and \[4\] that the functions defined in (2) are bounded if and only if \( A \in L_{1, \infty} \). In fact the last result is a strong motivation to consider the following modification of formulae (2). Let us consider a Cesaro operator on \( L_\infty(0, \infty) \) given by

\[
(Mx)(t) = \frac{1}{\log(t)} \int_{1}^{t} x(s) \frac{ds}{s}, \quad t \in (0, \infty).
\]

It follows from \[6\] and \[4\] that the functions

\[
M(t \to \frac{1}{t} \text{Tr}(\exp(-(tA)^{-q}))), \quad M(t \to \frac{1}{t} \text{Tr}(\exp(-(tA)^{-q}))B)
\]

are bounded if and only if \( A \in M_{1, \infty} \). Therefore, for a given generalised limit \( \omega \), let us set

\[
\omega' := \omega \circ M
\]

and instead of the functions given in (4) consider the functions

\[
\xi_\omega(A) := \omega'\left(\frac{1}{t} \text{Tr}(\exp(-(tA)^{-q}))\right), \quad \xi_\omega(B)(A) := \omega'\left(\frac{1}{t} \text{Tr}(\exp(-(tA)^{-q}))B\right).
\]

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The class of dilation invariant states $\omega'$ as above was introduced by A. Connes (see [8]) and it is natural to refer to this class as "Connes states". We prove in section 5 that if $\omega$ in (7) is dilation invariant, then $\xi_{\omega}$ is a linear functional on $M_{1,\infty}$. In fact, we also show in Proposition 18 that if $\omega$ in (7) is such that $\xi_{\omega}$ is linear on $M_{1,\infty}$, then necessarily there exists a dilation invariant generalised limit $\omega_0$ such that $\xi_{\omega} = \xi_{\omega_0}$.

There is a deep reason to require that the functionals $\xi_{\omega}$ and $\xi_{\gamma}$ be defined on $M_{1,\infty}$ and be linear (and thus, by implication, to consider Connes states). Important formulae in noncommutative geometry [8] and its semifinite counterpart [5, 7, 1, 6, 4] then connect these functionals with Dixmier traces on $M_{1,\infty}$. Recall that in [9], J. Dixmier constructed a non-normal semifinite trace (a Dixmier trace) on $B(H)$ using the weight

$$Tr_{\omega}(T) := \omega \left( \left\{ \frac{1}{\log(1+n)} \sum_{k=1}^{n} \mu_k(T) \right\}_{n=1}^{\infty} \right) \quad T > 0,$$

where $\omega$ is a dilation invariant state on $L_\infty(0,\infty)$.

The interplay between positive functionals $Tr_{\omega}$, $\zeta$, and $\xi_{\omega}$ on $M_{1,\infty}$ makes an important chapter in noncommutative geometry and has been treated (among many other papers) in [8, 5, 7, 1, 6, 22, 4, 24]. We now list a few most important known results concerning this interplay and explain our contribution to this topic.

In [5], the equality

$$Tr_{\omega}(AB) = (\omega \circ \log)(\frac{1}{t} \tau(A^{1+1/t}B)) = \zeta_{\omega \circ \log,B}(A), \quad 0 \leq A \in M_{1,\infty}$$

(10)

was established for every $B \in B(H)$ under very restrictive conditions on $\omega$. These conditions are dilation invariance for both $\omega$ and $\omega \circ \log$ and $M$–invariance of $\omega$. In [6], for the special case $B = 1$, the assumption that $\omega$ is $M$–invariant has been removed. However, the case of an arbitrary $B$ appears to be inaccessible by the methods in that article. In Section 4, we prove the general result which implies, in particular, that the equality (10) holds without requiring $M$–invariance of $\omega$.

In [1], the equality

$$\omega(\frac{1}{t} \tau(exp((-tA)^{-q})B)) = \Gamma(1 + \frac{1}{q}) \tau_{\omega}(AB)$$

(11)

was established under the same conditions on $\omega$ and $\omega \circ \log$ as above. In [24], in the special case $B = 1$ the equality (11) was established under the assumption that $\omega$ is $M$–invariant. However, again the case of an arbitrary $B$ appears to be inaccessible by the methods in that article. Here, we are able to treat the case of a general operator $B$.

In [1], a more general approach to the heat kernel formulae is suggested. It consists of replacing the function $t \rightarrow \exp(t^{-q})$ with an arbitrary function $f$.
from the Schwartz class. The following equality was proved in [1]

$$\omega(\frac{1}{t} \tau(f(tA)B)) = \int_0^\infty f(\frac{1}{s})ds \cdot \tau(AB)$$

(12)

for $A \in L_{1,\infty}$ and $M$–invariant $\omega$.

In [1, p.51], M. Benameur and T. Fack have asked whether the result above continues to stand without the $M$–invariance assumption on $\omega$. In Theorem 49 below, we answer this question affirmatively for a much larger class of functions than the Schwartz class and for any $A \in M_{1,\infty}$.

Finally, it is important to emphasize the connection between our results with the theory of fully symmetric functionals. Recall that a linear positive functional $\varphi : M_{1,\infty} \to \mathbb{C}$ is called fully symmetric if $\varphi(B) \leq \varphi(A)$ for every positive $A, B \in M_{1,\infty}$ such that $B \ll A$. The latter symbol means that

$$\sum_{k=1}^n \mu_k(B) \leq \sum_{k=1}^n \mu_k(A), \ \forall n \in \mathbb{N}.$$ 

It is obvious that every Dixmier trace $\text{Tr}_\omega$ is a fully symmetric functional. However, the fact that every fully symmetric functional coincides with a Dixmier trace is far from being trivial (see [11] and Theorem 1 below). It is therefore quite natural to ask whether a similar result holds for the sets of all linear positive functionals on $M_{1,\infty}$ formed by the $\xi_\omega$ and $\zeta_\gamma$ respectively. To this end, we establish results somewhat similar to those of [11]. Firstly, in Theorem 22 we prove that if $\omega$ in (7) is dilation invariant, then the functional $\xi_\omega$ extends to a fully symmetric functional on $M_{1,\infty}$. Secondly, in Theorem 31 we show that in fact every normalized fully symmetric functional on $M_{1,\infty}$ coincides with some $\xi_\omega$, where $\omega$ is dilation invariant. Thus, in view of [11], we can conclude that the set $\{\text{Tr}_\omega : \omega \text{ is a dilation invariant generalised limit}\}$ coincides with the set $\{\xi_\omega : \omega \text{ is a dilation invariant generalised limit}\}$ (up to a norming constant). At the same time, a natural question, namely, whether the equality

$$\xi_\omega = \Gamma(1 + \frac{1}{q})\text{Tr}_\omega$$

holds for every dilation invariant generalised limit $\omega$ is answered in the negative in Theorem 37.

Finally, we note that the question on the relationship between the sets $\{\text{Tr}_\omega : \omega \text{ is a dilation invariant generalised limit}\}$, $\{\zeta_\gamma : \gamma \text{ is a generalised limit}\}$ and $\{\xi_\omega : \omega \text{ is a dilation invariant generalised limit}\}$ remains open.

2. Definitions and notations

The theory of singular traces on operator ideals rests on some classical analysis which we now review for completeness.

As usual, $L_\infty(0, \infty)$ is the set of all bounded Lebesgue measurable functions on the semi-axis equipped with the uniform norm $\| \cdot \|$. Given a function
\( x \in L_\infty(0, \infty) \), one defines its decreasing rearrangement \( \mu(x) = \mu(\cdot, x) \) by the formula (see e.g. [17])

\[
\mu(t, x) = \inf\{s \geq 0 : m(\{|x| > s\}) \leq t\}.
\]

Let \( H \) be a Hilbert space and let \( B(H) \) be the algebra of all bounded operators on \( H \) equipped with the uniform norm \( \| \cdot \| \). Let \( \mathcal{N} \subset B(H) \) be a semi-finite von Neumann algebra with a fixed faithful and normal semi-finite trace \( \tau \). For every \( A \in \mathcal{N} \), the generalised singular value function \( \mu(A) = \mu(\cdot, A) \) is defined by the formula (see e.g. [14])

\[
\mu(t, A) = \inf\{\|Ap\| : \tau(1 - p) \leq t\}.
\]

If, in particular, \( \mathcal{N} = B(H) \), then \( \mu(A) \) is a step function and, therefore, can be identified with the sequence \( \{\mu(n, A)\}_{n \geq 0} \) of singular numbers of the operators \( A \) (the singular values are the eigenvalues of the operator \( |A| = (A^*A)^{1/2} \) arranged with multiplicity in decreasing order).

Equivalently, \( \mu(A) \) can be defined in terms of the distribution function \( d_A \). That is, setting \( d_A(s) := \tau(e^{|A|}(s, \infty)) \), \( s \geq 0 \), we obtain

\[
\mu(t, A) = \inf\{s : d_A(s) \geq t\}, \quad t > 0.
\]

Here, \( e^{|A|} \) denotes the spectral measure of the operator \( |A| \).

The following formula follows directly from the von Neumann definition of trace (see the definition at [20, Definition 15.1.1])

\[
\tau(f(A)) = -\int_0^\infty f(\lambda)dd_A(\lambda). \tag{13}
\]

Using the Jordan decomposition, every operator \( A \in B(H) \) can be uniquely written as

\[
A = (\Re(A)_+ - \Re(A)_-) + i(\Im(A)_+ - \Im(A)_-).
\]

Here, \( \Re(A) := \frac{1}{2}(A + A^*) \) (respectively, \( \Im(A) := \frac{1}{2i}(A - A^*) \)) for any operator \( A \in B(H) \) and \( B_{+} = Be^{B}(0, \infty) \) (respectively, \( B_{-} = Be^{B}(-\infty, 0) \)) for any self-adjoint operator \( B \in B(H) \). Recall that \( \Re A, \Im A \in \mathcal{N} \) for every \( A \in \mathcal{N} \) and \( B_{+}, B_{-} \in \mathcal{N} \) for every self-adjoint \( B \in \mathcal{N} \).

Let \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) be an increasing concave function such that \( \psi(t) = O(t) \) as \( t \to 0 \). The Marcinkiewicz function space \( M_\psi \) (see e.g. [17]) consists of all \( x \in L_\infty(0, \infty) \) satisfying

\[
\|x\|_{M_\psi} := \sup_{t > 0} \frac{1}{\psi(t)} \int_0^t \mu(s, x)ds < \infty.
\]

The Marcinkiewicz operator space \( M_\psi := M_\psi(\mathcal{N}, \tau) \) (see e.g. [5, 8]) consists of all \( A \in \mathcal{N} \) satisfying

\[
\|A\|_{M_\psi} := \sup_{t > 0} \frac{1}{\psi(t)} \int_0^t \mu(s, A)ds < \infty.
\]
We are especially interested in Marcinkiewicz spaces $\mathcal{M}_{1,\infty}$ and $\mathcal{M}_{1,1}$ that arise when $\psi(t) = \log(1+t)$, $t \geq 0$. In the literature, the ideal $\mathcal{M}_{1,\infty}$ is sometimes referred to as the Dixmier ideal. We recommend the recent paper of A. Pietsch, discussing the origin of $\mathcal{M}_{1,\infty}$ in mathematics.

For $s > 0$, dilation operators $\sigma_s : L_\infty \to L_\infty$ are defined by the formula $(\sigma_s x)(t) = x(t/s)$. Clearly, $\sigma_s : \mathcal{M}_{1,\infty} \to \mathcal{M}_{1,\infty}$ (see also [1] Theorem II.4.4).

Further, we need to recall the important notion of Hardy-Littlewood majorization. Let $A, B \in \mathcal{N}$. $B$ is said to be majorized by $A$ and written $B \prec \prec A$ if and only if

$$\int_0^t \mu(s, B)ds \leq \int_0^t \mu(s, A)ds, \quad t \geq 0.$$  \hspace{1cm} (14)

We have (see [14])

$$A + B \prec \prec \mu(A) + \mu(B) \prec \prec 2\sigma_{1/2}\mu(A + B).$$  \hspace{1cm} (15)

One of the most widely used ideals in von Neumann algebras is

$$\mathcal{L}_p := \mathcal{L}_p(\mathcal{N}, \tau) = \{A \in \mathcal{N} : \|A\|_p := \tau(|A|^p)^{1/p} < \infty\}, \quad p \geq 1,$$

usually called the Schatten-von Neumann ideal of $p$-summable operators. Using Hardy-Littlewood majorization, it is very easy to see (e.g. [1], Lemma 2.1) that $\mathcal{M}_{1,\infty} \subset \mathcal{L}_p$ for all $p > 1$.

A linear functional $\varphi : \mathcal{M}_{1,\infty} \to \mathbb{C}$ is said to be symmetric if $\varphi(B) = \varphi(A)$ for every positive $A, B \in \mathcal{M}_{1,\infty}$ such that $\mu(B) = \mu(A)$. A linear functional $\varphi : \mathcal{M}_{1,\infty} \to \mathbb{C}$ is said to be fully symmetric if $\varphi(B) \leq \varphi(A)$ for all $A, B \in \mathcal{M}_{1,\infty}^+$ such that $B \prec \prec A$ [10, 11, 12]. Every fully symmetric functional is symmetric and bounded. The converse fails [13].

A positive normalised linear functional $\gamma : L_\infty(0, \infty) \to \mathbb{R}$ is called a generalised limit if $\gamma(z) = 0$ for every $z \in L_\infty(0, \infty)$ such that $\lim_{t \to \infty} z(t) = 0$. A linear functional $\gamma : L_\infty(0, \infty) \to \mathbb{R}$ is called dilation invariant if $\gamma(\sigma_{s} z) = \gamma(z)$ for every $z \in L_\infty(0, \infty)$ and every $s > 0$.

Let $S \subseteq B(H)$. We denote by $S^+$ the set of all positive operators from $S$.

Let $\omega : L_\infty(0, \infty) \to \mathbb{R}$ be a dilation invariant generalised limit. Define a functional $\tau_\omega$ on $\mathcal{M}_{1,\infty}^+$ by the formula

$$\tau_\omega(A) = \omega\left(\frac{1}{\log(1+t)}\int_0^t \mu(s, A)ds\right).$$

The functional $\tau_\omega$ is additive and unitarily invariant on $\mathcal{M}_{1,\infty}^+$. Thus, $\tau_\omega$ extends to a fully symmetric functional on $\mathcal{M}_{1,\infty}$. One usually refers to it as to a Dixmier trace. We refer the reader to [2, 3, 4, 5, 6, 7, 8, 9] for details.

Further, we use the following properties of Dixmier traces. Let $A \in \mathcal{M}_{1,\infty}$ and let $B \in \mathcal{N}$. We have (see [8, 9])

$$\tau_\omega(AB) = \tau_\omega(BA).$$  \hspace{1cm} (16)

Suppose that $B > 0$. It follows from [10] that

$$\tau_\omega(AB) = \tau_\omega(B^{1/2}AB^{1/2}).$$  \hspace{1cm} (17)
Suppose that the trace \( \tau \) on the von Neumann algebra \( \mathcal{N} \) is infinite and the algebra \( \mathcal{N} \) is either diffuse (that is with no minimal projections) or else is \( B(H) \). Given any finite sequence \( \{A_n\} \) of operators, we can construct a sequence of operators \( \{B_n\} \) such that \( \mu(A_n) = \mu(B_n) \) for all \( n \)'s and \( B_nB_m = 0 \) for all \( n \neq m \). Further, we refer to any such sequence \( \{B_n\} \) as a "sequence of disjoint copies of \( \{A_n\} \)."

Cesaro operator \( M \) is defined on \( L_\infty(0, \infty) \) by the formula
\[
(Mx)(t) = \frac{1}{\log(t)} \int_1^t x(s) \frac{ds}{s}, \quad t \in (0, \infty).
\]

3. Preliminary important results

In this section, for the reader’s convenience, we collect a number of key known results, which will be used throughout this paper.

The following important theorem is proved in [19, Theorem 11] for general Marcinkiewicz spaces.

**Theorem 1.** Every fully symmetric functional on \( \mathcal{M}_{1,\infty} \) is a Dixmier trace.

The following theorem is an analog of Lidskii formula (see [23]) for Dixmier traces. It is proved in [24, Theorem 33] for a large subclass of Marcinkiewicz spaces which contains \( \mathcal{M}_{1,\infty} \).

**Theorem 2.** Let \( A \in \mathcal{M}_{1,\infty} \) and let \( \tau_\omega \) be an arbitrary Dixmier trace on \( \mathcal{M}_{1,\infty} \). We have
\[
\tau_\omega(A) = \omega \left( \frac{1}{\log(t)} \sum_{|\lambda| > \log(t)/t, \lambda \in \sigma(A)} \lambda \right).
\]

The following \( \omega \)-variant of the classical Karamata theorem is established in [5].

**Theorem 3.** Let \( \beta \) be a continuous increasing function. Set
\[
h(t) = \int_0^\infty e^{-(u/t)^q} d\beta(u).
\]
We have
\[
\omega\left(\frac{h(t)}{t}\right) = \Gamma(1 + \frac{1}{q})\omega\left(\frac{\beta(t)}{t}\right)
\]
for any dilation invariant generalised limit \( \omega \).

Consider the ideal \( K_N \) of \( \tau \)-compact operators in \( \mathcal{N} \) (that is the norm closed ideal generated by the projections \( E \in \mathcal{N} \) with \( \tau(E) < \infty \)). The following result is not new (see [12, Chapter II, Lemma 3.4]). We present a short proof for convenience of the reader.
Theorem 4. Let $A, B \in \mathcal{N}$ be positive $\tau-$compact operators. We have $B \prec \prec A$ if and only if
\[ \tau((B-t)e^B(t,\infty)) \leq \tau((A-t)e^A(t,\infty)), \quad \forall t > 0. \]  
\[ (18) \]

Proof. Fix $t > 0$. It follows from the definition of generalised singular value function that $\mu(Ae^A(t,\infty)) = \mu(A)\chi_{[0,d_A(t)]}$. Applying [14, Proposition 2.7] to the operator $Ae^A(t,\infty)$, we have
\[ \tau(Ae^A(t,\infty)) = \int_0^{d_A(t)} \mu(s,A)ds, \]
and hence
\[ \tau((A-t)e^A(t,\infty)) = \int_0^{d_A(t)} (\mu(s,A) - t)ds. \]  
\[ (19) \]
The function
\[ u \to \int_0^u (\mu(s,A) - t)ds \]
attempts its maximum at $u = d_A(t)$.
If $B \prec \prec A$, then
\[ \int_0^{d_B(t)} (\mu(s,B) - t)ds \leq \int_0^{d_B(t)} (\mu(s,A) - t)ds \leq \int_0^{d_A(t)} (\mu(s,A) - t)ds. \]
Inequality (18) follows now from (19).
Suppose now that (15) holds. Fix $u > 0$ and set $t = \mu(u,A)$. It follows that
\[ \int_0^u (\mu(s,B) - t)ds \leq \int_0^{d_B(t)} (\mu(s,B) - t)ds = \tau((B-t)e^B(t,\infty)) \leq \tau((A-t)e^A(t,\infty)) = \int_0^u (\mu(s,A) - t)ds. \]
Hence,
\[ \int_0^u \mu(s,B)ds \leq \int_0^u \mu(s,A)ds. \]
Since $u$ is arbitrary, we have $B \prec \prec A$. \hfill \square

4. $\zeta-$function formulae

We begin by showing that the functionals given in (3) are well defined on $\mathcal{M}_{1,\infty}^+$. 

Lemma 5. If $\gamma : L_\infty(0,\infty) \to \mathbb{R}$ is a generalised limit, then $\zeta_\gamma(A) < \infty$ and $\zeta_{\gamma,B}(A) < \infty$ for any $A \in \mathcal{M}_{1,\infty}^+$. 

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\textbf{Proof.} It is clear that \( \mu(s,A) \ll (1+s)^{-1}\|A\|_{1,\infty} \). Therefore,
\[
\tau(A^{1+1/t}) \leq \|A\|_{1,\infty}^{1+1/t} \int_0^\infty \frac{dt}{(1+s)^{1+1/t}} = t\|A\|_{1,\infty}^{1+1/t}.
\]
Hence, \( \zeta_\gamma(A) \leq \|A\|_{1,\infty} \). It follows from
\[
\tau(A^{1+1/t}B) \leq \|B\|\tau(A^{1+1/t})
\]
that \( \zeta_\gamma,B(A) \leq \|B\|\zeta_\gamma(A) \).
\[ \square \]

\textbf{Remark 6.} Let \( x,y \in L_\infty(0,\infty) \). For any generalised limit \( \gamma \) such that \( \gamma(|x-1|) = 0 \), we have \( \gamma(xy) = \gamma(y) \). Indeed, \( |\gamma(xy-y)| \leq \gamma(|x-1|)|y| = 0 \).

\textbf{Lemma 7.} For any \( A,C \in M_{1,\infty}^+ \) we have
\[
\tau(A^{1+s}+C^{1+s}) \leq \tau((A+C)^{1+s}) \leq 2^s\tau(A^{1+s}+C^{1+s}), \quad s > 0.
\]

\textbf{Proof.} In the special case when \( \gamma = B(H) \), the first inequality can be found in \cite{10}, (2.9)). In the general case, it follows directly from Proposition 4.6(ii) of \cite{13} when \( f(u) = u^{1+s}, \, u > 0 \). The second inequality follows from the same proposition by setting there \( a = a^* = b = b^* = 2^{-1/2} \).
\[ \square \]

Let \( A \in M_{1,\infty} \). For a functional \( \zeta_\gamma \) defined on \( M_{1,\infty}^+ \) by \cite{3} (see Lemma \cite{4}), we set
\[
\zeta_\gamma(A) := (\zeta_\gamma(\Re(A)_+) - \zeta_\gamma(\Re(A)_-)) + i(\zeta_\gamma(\Im(A)_+) - \zeta_\gamma(\Im(A)_-)) \quad (20)
\]

The following theorem shows that functionals \( \zeta_\gamma \) defined by (20) are fully symmetric on \( M_{1,\infty} \).

\textbf{Theorem 8.} If \( \gamma : L_\infty(0,\infty) \to \mathbb{R} \) is a generalised limit, then \( \zeta_\gamma \) is a fully symmetric linear functional on \( M_{1,\infty} \).

\textbf{Proof.} To verify that \( \zeta_\gamma \) is linear, it is sufficient to check that \( \zeta_\gamma(A+C) = \zeta_\gamma(A) + \zeta_\gamma(C) \) for any \( A,C \in M_{1,\infty}^+ \). It follows from the left hand side inequality of Lemma \cite{4} that
\[
\zeta_\gamma(A+C) \geq \zeta_\gamma(A) + \zeta_\gamma(C).
\]
Noting that \( \gamma(|2^{1/t}-1|) = 0 \), it follows from the right hand side inequality of Lemma \cite{4} and Remark \cite{3} that
\[
\zeta_\gamma(A+C) \leq \zeta_\gamma(A) + \zeta_\gamma(C).
\]

Therefore, we have
\[
\zeta_\gamma(A+C) = \zeta_\gamma(A) + \zeta_\gamma(C).
\]

The homogeneity of \( \zeta_\gamma \) follows from Remark \cite{3}. Finally, if \( 0 \leq C \ll A \in M_{1,\infty}^+ \), then \( C,A \in M_{1,s}^+ \) and \( \tau(C^{1+s}) \leq \tau(A^{1+s}) \). Hence, \( \frac{1}{t}\tau(C^{1+1/t}) \leq \frac{1}{t}\tau(A^{1+1/t}) \) and so \( \zeta_\gamma(C) \leq \zeta_\gamma(A) \).
\[ \square \]
Let $B \in \mathcal{N}$. We extend the functional $\zeta_{\gamma,B}$ on $\mathcal{M}_{1,\infty}$, similarly to \cite{20}. Observe that

$$\zeta_{\gamma,B_1+B_2}(A) = \zeta_{\gamma,B_1}(A) + \zeta_{\gamma,B_2}(A), \ B_1, B_2 \in \mathcal{N}, \ A \in \mathcal{M}_{1,\infty}.$$ 

**Lemma 9.** If $A \in \mathcal{M}_{1,\infty}$ and $B_n \to B$ in $\mathcal{N}$, then

$$\zeta_{\gamma,B_n}(A) \to \zeta_{\gamma,B}(A).$$

**Proof.** It is sufficient to prove the assertion for $A \in \mathcal{M}_{1,\infty}$. Since

$$|\tau(A^{1+s}B) - \tau(A^{1+s}B_n)| \leq \tau(A^{1+s})\|B - B_n\|,$$

we obtain

$$|\zeta_{\gamma,B}(A) - \zeta_{\gamma,B_n}(A)| \leq \zeta_{\gamma}(A)\|B - B_n\|.$$ 

\hfill \Box

The following lemma follows immediately from \cite{8} Lemma 3.3.

**Lemma 10.** Let $A, B \in B^+(H)$ and let $s > 0$. We have

i) $(B^{1/2}AB^{1/2})^{1+s} \leq B^{1/2}A^{1+s}B^{1/2}$ if $0 \leq B \leq 1$.

ii) $(B^{1/2}AB^{1/2})^{1+s} \geq B^{1/2}A^{1+s}B^{1/2}$ if $B \geq 1$.

The result below significantly strengthens \cite{5} Proposition 3.6 by removing all extra assumptions on the generalised limit $\gamma$.

**Proposition 11.** If $\gamma : L_\infty(0, \infty) \to \mathbb{R}$ is a generalised limit, then

$$\zeta_{\gamma,B}(A) = \zeta_{\gamma}(B^{1/2}AB^{1/2}), \ \forall A \in \mathcal{M}_{1,\infty}, \ B \in \mathcal{N}^+.$$ 

**Proof.** It is sufficient to prove the assertion for $A \in \mathcal{M}_{1,\infty}$. Suppose first that there are constants $0 < m \leq M < \infty$ such that $m \leq B \leq M$. Applying Lemma \cite{10} to the operators $A$ and $M^{-1}B$ (respectively, $m^{-1}B$), we have

$$m^sB^{1/2}A^{1+s}B^{1/2} \leq (B^{1/2}AB^{1/2})^{1+s} \leq M^sB^{1/2}A^{1+s}B^{1/2}.$$ 

Therefore,

$$\frac{1}{t}m^{1/t}\tau(A^{1+1/t}B) \leq \frac{1}{t}\tau((B^{1/2}AB^{1/2})^{1+1/t}) \leq \frac{1}{t}M^{1/t}\tau(A^{1+1/t}B).$$

Since $\gamma(|m^{1/t} - 1|) = 0$ and $\gamma(|M^{1/t} - 1|) = 0$, it follows from Remark \cite{6} that

$$\zeta_{\gamma,B}(A) = \zeta_{\gamma}(B^{1/2}AB^{1/2}).$$

For an arbitrary $B \in \mathcal{N}^+$, we set $B_n := Be^{B(1/n, \infty)} + 1/ne^{B[0, 1/n]}$, $n \geq 1$. From the first part of the proof, we have

$$\zeta_{\gamma,B_n}(A) = \zeta_{\gamma}(B_n^{1/2}AB_n^{1/2}).$$

Since $B_n^{1/2}AB_n^{1/2} \to B^{1/2}AB^{1/2}$ in $\mathcal{M}_{1,\infty}$, we have by Theorem \cite{8}

$$\zeta_{\gamma}(B_n^{1/2}AB_n^{1/2}) \to \zeta_{\gamma}(B^{1/2}AB^{1/2}).$$

On the other hand, by Lemma \cite{8} we have $\zeta_{\gamma,B}(A) \to \zeta_{\gamma,B}(A).$ \hfill \Box

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The following is our main result on the $\zeta$–function.

**Theorem 12.** If $\gamma : L_\infty(0, \infty) \to \mathbb{R}$ is a generalised limit, then

$$\zeta_{\gamma, B}(A) = \zeta_{\gamma}(AB), \quad \forall A \in \mathcal{M}_{1, \infty}, \ B \in \mathcal{N}.$$  

**Proof.** It is sufficient to prove the assertion for $B \in \mathcal{N}^+$. By Theorems 8 and 11 we know that $\zeta_{\gamma}$ is a Dixmier trace on $\mathcal{M}_{1, \infty}$. Hence, by (17), we have $\zeta_{\gamma}(B^{1/2}AB^{1/2}) = \zeta_{\gamma}(AB)$. The assertion follows now from Proposition 11. $\blacksquare$

Our remaining objective in this section is to provide strengthening of several formulae linking Dixmier traces and $\zeta$-functions from [3, 6].

**Lemma 13.** Let $A \in \mathcal{M}^{\uparrow}_{1, \infty}$. The mapping $s \to s^{-1}\zeta_{\gamma \circ \sigma_s}(A)$ is convex and, therefore, continuous.

**Proof.** For all $t, s > 0$, we have

$$s^{-1}\sigma_s\left(\frac{1}{t} (A^{1+1/t})\right) = \frac{1}{t} (A^{1+s/t}).$$

Therefore, for every $s > 0$

$$s^{-1}\zeta_{\gamma \circ \sigma_s} = \gamma\left(\frac{1}{t} \tau(A^{1+s/t})\right).$$

Let $\lambda_1 > 0$ and let $\lambda_1 + \lambda_2 = 1$. Since the mapping $t \to a^{1+t}$ is convex for every $a > 0$, it follows from the spectral theorem that the map $s \to A^s$ is also convex. Therefore, for all positive real numbers $s_1, s_2$ and $t$, we have

$$A^{1+(\lambda_1 s_1 + \lambda_2 s_2)/t} \leq \lambda_1 A^{1+s_1/t} + \lambda_2 A^{1+s_2/t}.$$ 

The assertion follows immediately. $\blacksquare$

Let $\gamma$ be a generalised limit on $L_\infty(0, \infty)$. Below, we will formally apply the notation $\zeta_{\gamma, B}(A)$ introduced in (3) to some unbounded positive operators $B$ on $H$.

**Lemma 14.** Let $A \in \mathcal{N}$ be a positive $\tau$–compact operator and let $B \geq 1$ be an unbounded operator commuting with $A$. If (the closure of) the product $AB \in \mathcal{M}_{1, \infty}$ and $AB^n \in \mathcal{N}$ for every $n \in \mathbb{N}$, then $\zeta_{\gamma}(AB) = \zeta_{\gamma, B}(A)$.

**Proof.** It follows from $AB = BA$ and $B \geq 1$ that $A^{1+s}B \leq (AB)^{1+s}$. The inequality $\zeta_{\gamma, B}(A) \leq \zeta_{\gamma}(AB)$ follows immediately.

Set $c_n := \|AB^{2n}\|$, $n \geq 1$ and observe that $BA^{1/2n} \leq c_n^{1/2n}$. Setting $B_n = Be^{A[0, c_n^{-1}]}$, we obtain

$$B_n A^{1/n} = BA^{1/2n} \cdot A^{1/2n} e^A[0, c_n^{-1}] \leq (c_n A)^{1/2n} e^A[0, c_n^{-1}] \leq 1. \tag{21}$$

It follows from (21) that $A^{1+1/t}B_n \geq (AB^n)^{1+n/t(n-1)}$. Thus,

$$\gamma(\frac{1}{t} \tau(A^{1+1/t}B_n)) \geq \gamma(\frac{1}{t} \tau((AB^n)^{1+n/t(n-1)})) = \frac{n-1}{n} \zeta_{\gamma \circ \sigma_{n/(n-1)}}(AB_n).$$
Since $A$ is $\tau$-compact, then $B - B_n$ is bounded operator with finite support. Due to the linearity with respect to $B$, we have

$$\zeta_{\gamma,B}(A) = \zeta_{\gamma,B_n}(A) \geq \frac{n-1}{n} \zeta_{\gamma \sigma_n/(n-1)}(AB_n) = \frac{n-1}{n} \zeta_{\gamma \sigma_n/(n-1)}(AB).$$

The assertion follows now from Lemma 13.

The following result is mainly known (see [5, 6]). Our proof is however much simpler than the arguments used there.

**Theorem 15.** If $\omega$ is a dilation invariant generalised limit such that the generalised limit $\omega \circ \log$ is still dilation invariant, then $\tau_\omega = \zeta_{\omega \circ \log}$.

**Proof.** It is sufficient to verify the equality $\tau_\omega = \zeta_{\omega \circ \log}$ on positive operators $A \in M_{1,\infty}^+$ such that $A \leq e^{-1}$. Define a continuously increasing function $\beta : (0, \infty) \to (0, \infty)$ by

$$\beta(u) := -\int_{ue^{-u}}^{\infty} \lambda dA(\lambda).$$

Let $h$ be as in Theorem 8 as applied to the above $\beta$. Define an operator $B \geq 1$ by the formula $A = Be^{-B}$ and set $C = e^{-B}$. We have

$$h(t) = \int_0^{\infty} e^{-u/t} d\beta(u) = -\int_{0}^{\infty} e^{-u(1+1/t)} u dA(ue^{-u}) \quad (\text{13}) \quad \tau(C^{1+1/t}B). \quad (22)$$

The conditions of Lemma 14 are valid for $B$ and $C$. Indeed, $B$ commutes with $C$, $BC = A \in M_{1,\infty}^+$ and $B^n e^{-B} \in \mathcal{N}$ for every $n \in \mathbb{N}$. By Lemma 14 we have

$$\zeta_{\omega \circ \log}(A) = \zeta_{\omega \circ \log,B}(C) = (\omega \circ \log)(h(t)/t).$$

By Theorem 2 we have

$$\tau_\omega(A) = \omega(-1/\log(t) \int_{0}^{\infty} \lambda dA(\lambda)) = (\omega \circ \log)(\beta(t)/t). \quad (23)$$

We can now conclude

$$\zeta_{\omega \circ \log}(A) \overset{\text{22}}{=} (\omega \circ \log)(h(t)/t) \overset{\text{Thm 8}}{=} (\omega \circ \log)(\beta(t)/t) \overset{\text{23}}{=} \tau_\omega(A).$$

The following corollary strengthens and extends the results of [8, Theorem 4.11] and [3, Theorem 3.8]. It follows immediately from Theorems 15 and 12.

**Corollary 16.** If $\omega$ is a dilation invariant generalised limit such that the generalised limit $\omega \circ \log$ is still dilation invariant, then

$$\tau_\omega(AB) = (\omega \circ \log)(1/t \tau(A^{1+1/t}B)), \quad \forall A \in M_{1,\infty}^+, \quad B \in \mathcal{N}.$$
5. The linearity criterion for functionals $\xi_\gamma$

In this section we focus on functionals $\xi_\gamma(\cdot)$ defined in (8). It was implicitly proved in [6, Theorem 5.2] that

$$M \left( t \to \frac{1}{t} \tau(\exp(-(tA)^{-q})) \right) \in L_\infty(0, \infty), \ \forall A \in \mathcal{M}_{1,\infty}^+$$

and therefore,

$$\xi_\gamma(A) := (\gamma \circ M) \left( t \to \frac{1}{t} \tau(\exp(-(tA)^{-q})) \right)$$

(24)

is finite for every $A \in \mathcal{M}_{1,\infty}^+$ and every generalised limit $\gamma$ on $L_\infty(0, \infty)$. We note, in passing that a stronger result than [6, Theorem 5.2] is established in Theorem 40 below. Let $A \in \mathcal{M}_{1,\infty}$. For a functional $\xi_\gamma$, we set

$$\xi_\gamma(A) := (\xi_\gamma(\Re(A)_+) - \xi_\gamma(\Re(A)_-)) + i(\xi_\gamma(\Im(A)_+) - \xi_\gamma(\Im(A)_-)).$$

(25)

It is probably a difficult task to describe the set of all generalised limits $\gamma$ for which (25) yields a linear functional $\xi_\gamma$. However, the class of linear functionals $\xi_\gamma$ is an easier object. Below in Proposition 18 we show that the sets of linear functionals $\{\xi_\gamma : \gamma \text{ is a generalised limit} \}$ and linear functionals $\{\xi_\omega : \omega \text{ is a dilation invariant generalised limit}\}$ coincide.

**Lemma 17.** For every locally integrable $z$ with $Mz \in L_\infty(0, \infty)$, we have

$$(M \circ \sigma_{s-1} - \sigma_{s-1} \circ M)(z) \in C_0^b(0, \infty), \ \forall s > 0.$$  

Here, $C_0^b(0, \infty)$ is the space of all bounded continuous functions tending to 0 at $\infty$.

**Proof.** Fix $s > 0$. The assertion follows by writing

$$(M \circ \sigma_{s-1} - \sigma_{s-1} \circ M)(z) = \frac{1}{\log(t)} \int_s^t z(u) \frac{du}{u} - \frac{1}{\log(st)} \int_1^s z(u) \frac{du}{u}$$

and noting that the assumption $Mz \in L_\infty(0, \infty)$ easily implies that

$$\frac{1}{\log(st)} \int_1^s z(u) \frac{du}{u} - \frac{1}{\log(t)} \int_1^s z(u) \frac{du}{u} \in C_0^b(0, \infty).$$

□

**Proposition 18.** Suppose that a generalised limit $\gamma$ on $L_\infty(0, \infty)$ is such that $\xi_\gamma$ is a linear functional on $\mathcal{M}_{1,\infty}$. Then, there exists a dilation invariant generalised limit $\omega$ on $L_\infty(0, \infty)$ such that $\xi_\gamma = \xi_\omega$.  

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Proof. Fix \( s > 0 \) and observe that
\[
\left( t \mapsto \frac{1}{t} \tau(\exp(-(tA)^{-q})) \right) = s\sigma_{s^{-1}} \left( t \mapsto \frac{1}{t} \tau(\exp(-(tA)^{-q})) \right).
\]
Therefore,
\[
\xi_\gamma(sA) = s(\gamma \circ M \circ \sigma_{s^{-1}})\left( \frac{1}{t} \tau(\exp(-(tA)^{-q})) \right).
\]
By the assumption, we have \( \xi_\gamma(sA) = s\xi_\gamma(A) \) and appealing to Lemma 17, we obtain
\[
\xi_\gamma(A) = (\gamma \circ \sigma_{s^{-1}} \circ M)\left( \frac{1}{t} \tau(\exp(-(tA)^{-q})) \right), \quad \forall s > 0.
\]
(27)

Let \( E \) be the linear span of the functions
\[
t \mapsto M\left( \frac{1}{t} \tau(\exp(-(tA)^{-q})) \right), \quad A \in \mathcal{M}_{1,\infty}^+
\]
and let \( F := E + C_0^b(0, \infty) \). We claim that the space \( F \) is dilation invariant. Indeed, it follows from Lemma 17 and (26) that every function
\[
\sigma_{s^{-1}} \left( t \mapsto M\left( \frac{1}{t} \tau(\exp(-(tA)^{-q})) \right) \right)
\]
belongs to the set
\[
s^{-1} \left( t \mapsto M\left( \frac{1}{t} \tau(\exp(-(tsA)^{-q})) \right) \right) + C_0^b(0, \infty).
\]
It follows from (29) that \( \gamma \circ \sigma_{s^{-1}} = \gamma \) on \( F \). By the invariant form of the Hahn-Banach theorem (see [13, p. 157]) applied to the group of dilations \( \{\sigma_s\}_{s>0} \), we see that \( \gamma|_F \) can be extended to a dilation invariant generalised limit \( \omega \) on \( L_\infty(0, \infty) \).

The following lemma can be found in [24]. We present a shorter proof for convenience of the reader.

Lemma 19. If \( \omega \) is a dilation invariant generalised limit on \( L_\infty(0, \infty) \), then
\[
\xi_\omega(A) = \Gamma(1 + \frac{1}{q})\left( \omega \circ M \right)\left( \frac{1}{t} d_A\left( \frac{1}{u} \right) \right), \quad \forall A \in \mathcal{M}_{1,\infty}^+.
\]
(28)

Proof. It follows from (13) that
\[
\tau(\exp(-(tA)^{-q})) = \int_0^\infty e^{-(u/t)^q} d_A\left( \frac{1}{u} \right).
\]
(29)

Setting \( \beta(u) = d_A(1/u) \), multiplying both sides of (29) by \( 1/t \) and applying Theorem 5 to \( \omega \circ M \) (which is dilation invariant, see [8]), we obtain (28).
Lemma 20. Let $A \in \mathcal{M}_{1,\infty}$ and let $\omega$ be a dilation invariant generalised limit on $L_\infty(0, \infty)$. We have

$$\xi_\omega(A) = \Gamma(1 + \frac{1}{q})\omega\left(\frac{1}{\log(1+t)} \tau((A - \frac{1}{t})e^{A}(\frac{1}{t}, \infty))\right).$$

(30)

Proof. In view of Lemma 19, it is sufficient to show that right hand sides of (28) and (30) coincide. This easily follows from the following computation, where we use integration by parts

$$M\left(\frac{1}{t}d_A(\frac{1}{t})\right) = \frac{1}{\log(t)} \int_1^t d_A(\frac{1}{s}) \frac{ds}{s^2} = \frac{1}{\log(t)} \int_{1/t}^1 d_A(u)du =$$

$$= \frac{1}{\log(t)} ud_A(u)|_{1/t} - \frac{1}{\log(t)} \int_{1/t}^1 udd_A(u) = \frac{1}{\log(t)} \tau((A - \frac{1}{t})e^{A}(\frac{1}{t}, \infty)) + o(1).$$

Lemma 21. Let $\omega$ be a dilation invariant generalised limit on $L_\infty(0, \infty)$ and let $A, B \in \mathcal{M}_{1,\infty}$ be such that $B \ll A$. We have $\xi_\omega(B) \leq \xi_\omega(A)$.

Proof. The assertion follows from Lemma 20 and Theorem 4.

The following is the main result of this section.

Theorem 22. For any dilation invariant generalised limit $\omega$ on $L_\infty(0, \infty)$, the functional $\xi_\omega$ given by (25) is linear and fully symmetric on $\mathcal{M}_{1,\infty}$.

Proof. The assertion follows from Lemma 21 provided we have shown that

$$\xi_\omega(A + B) = \xi_\omega(A) + \xi_\omega(B), \ \forall A, B \in \mathcal{M}_{1,\infty}.$$  

(31)

To this end, we observe first that since $\omega$ and $\omega \circ M$ are dilation invariant, it follows from Lemma 21 and (15) that

$$\xi_\omega(A + B) = \xi_\omega(\mu(A) + \mu(B)), \ \forall A, B \in \mathcal{M}_{1,\infty}.$$

Now, let $C$ and $D$ be disjoint copies of $A$ and $B$ (see Section 2). Thus, we have

$$\xi_\omega(C + D) = \xi_\omega(\mu(C) + \mu(D)) = \xi_\omega(\mu(A) + \mu(B)) = \xi_\omega(A + B).$$

However, the equality

$$\xi_\omega(C + D) = \xi_\omega(C) + \xi_\omega(D)$$

for positive operators $C$ and $D$ such that $CD = 0$ follows immediately from the definition (24). Since the equalities $\xi_\omega(A) = \xi_\omega(C), \xi_\omega(B) = \xi_\omega(D)$ are obvious, we arrive at (31). 

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6. Every fully symmetric functional has form $\xi_\omega$

It follows from Theorem 22 and Theorem 1 that the functional $\xi_\omega$ is a fully symmetric functional on $\mathcal{M}_{1,\infty}$ whenever $\omega$ is a dilation invariant generalised limit $\omega$ on $L_\infty(0,\infty)$. In this section, we show the converse.

Define a (non-linear) operator $T : \mathcal{M}_{1,\infty}^{+} \to L_\infty(0,\infty)$ by the formula

\[(TA)(t) = \frac{1}{\log(1+t)} \tau((A - \frac{1}{t})e^A(\frac{1}{t},\infty)), \quad t > 0. \quad (32)\]

We need some properties of the operator $T$. Firstly, we show that it is additive on certain pairs of $A, B \in \mathcal{M}_{1,\infty}^{+}$.

**Lemma 23.** Let $A, B \in \mathcal{M}_{1,\infty}^{+}$ be such that $AB = BA = 0$. It follows that $T(A + B) = TA + TB$.

**Proof.** It follows immediately from the assumption that

\[(A + B - \frac{1}{t})e^{A+B}(\frac{1}{t},\infty) = (A - \frac{1}{t})e^A(\frac{1}{t},\infty) + (B - \frac{1}{t})e^B(\frac{1}{t},\infty). \]

Next, we explain the connection of the operator $T$ with fully symmetric functionals on $\mathcal{M}_{1,\infty}$.

**Lemma 24.** Let the operators $A, B \in \mathcal{M}_{1,\infty}^{+}$ be such that $TB \preceq TA$. For every fully symmetric functional $\varphi$ on $\mathcal{M}_{1,\infty}$, we have $\varphi(B) \leq \varphi(A)$.

**Proof.** It follows immediately from the definition (32) that

\[\tau((B - \frac{1}{t})e^B(\frac{1}{t},\infty)) \leq \tau((A - \frac{1}{t})e^A(\frac{1}{t},\infty)), \quad \forall t > 0. \]

Applying Theorem 1 we obtain $B \preceq A$ and so $\varphi(B) \leq \varphi(A)$.

**Lemma 25.** Let $A, B \in \mathcal{M}_{1,\infty}^{+}$. For every fully symmetric functional $\varphi$ on $\mathcal{M}_{1,\infty}$, we have

\[\varphi(B) - \varphi(A) \leq \|\varphi\|_{\mathcal{M}_{1,\infty}} \limsup_{t \to \infty}(TB - TA)(t).\]

**Proof.** Without loss of generality, $\|\varphi\|_{\mathcal{M}_{1,\infty}} = 1$. Denote the right hand side by $c$ and suppose that $c \geq 0$ (the case when $c < 0$ is treated similarly). Fix $\varepsilon > 0$. We have $(TB - TA)(t) \leq c + \varepsilon$ for all sufficiently large $t$. Let $C$ be an operator with $\mu(t, C) = (c + 2\varepsilon)/(1 + t)$. We have $TB \leq TA + TC$ for all sufficiently large $t$. Let $A_1$ and $C_1$ be disjoint copies of $A$ and $C$, respectively. It follows from Lemma 23 that $TB(t) \leq T(A_1 + C_1)(t)$ for all sufficiently large $t$. Choose $0 < \delta$ small enough to guarantee $TB_2(t) \leq T(A_1 + C_1)(t)$ for all $t > 0$, where $B_2 := \min\{B, \delta\}$. By Corollary 24 we have $\varphi(B_2) \leq \varphi(A_1) + \varphi(C_1)$, or equivalently $\varphi(B) \leq \varphi(A) + c + 2\varepsilon$. Since $\varepsilon$ is arbitrarily small, we are done.
Lemma 26. Let \( A_1, \ldots, A_n \in M_{1,\infty}^+ \) and let \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \) for some \( n \geq 1 \). For every fully symmetric functional \( \varphi \) on \( M_{1,\infty} \) we have

\[
\sum_{k=1}^{n} \lambda_k \varphi(A_k) \leq \limsup_{t \to \infty} \sum_{k=1}^{n} \lambda_k (T A_k)(t). \tag{33}
\]

Proof. Both sides of the inequality (33) depend continuously on the \( \lambda_k \)'s. Without loss of generality, we may assume that all \( \lambda_k \in \mathbb{Q} \). Multiplying both sides by the common denominator, we may assume that all \( \lambda_k \in \mathbb{Z} \).

Writing \( \lambda_k A_k = |\lambda_k| \sum_{k=1}^{n} \text{sgn}(\lambda_k) A_k \) we see that it is sufficient to prove (33) only for the case when \( \lambda_k = \pm 1 \) for every \( k \).

Let \( \{B_k\} \) be a disjoint copy sequence of \( \{A_k\} \). Both sides of the inequality (33) do not change if we replace \( A_k \) with \( B_k \). Without loss of generality, the operators \( A_k A_j = 0 \), \( k \neq j \). By Lemma 25 we have

\[
\sum_{k=1}^{n} \lambda_k \varphi(A_k) = \varphi(\sum_{\lambda_k=1} \lambda_k A_k) - \varphi(\sum_{\lambda_k=-1} \lambda_k A_k) \leq \limsup_{t \to \infty} T(\sum_{\lambda_k=1} \lambda_k A_k) - T(\sum_{\lambda_k=-1} \lambda_k A_k)(t).
\]

Since \( A_k A_j = 0 \) for all \( k \neq j \), we have by Lemma 23 that

\[
T(\sum_{\lambda_k=1} \lambda_k A_k) - T(\sum_{\lambda_k=-1} \lambda_k A_k) = \sum_{k=1}^{n} \lambda_k T A_k
\]

and the assertion follows. \( \square \)

Lemma 27. Let \( E \) be the linear span of \( T M_{1,\infty}^+ \) and \( C_0^0(0, \infty) \). For every \( s > 0 \) we have \( \sigma_s E = E \).

Proof. It follows from the definition (32) that for every \( s > 0 \), we have

\[
\sigma_s T A \in sT(s^{-1} A) + C_0^0(0, \infty), \quad \forall A \in M_{1,\infty}^+.
\]

Since \( A_k A_j = 0 \) for all \( k \neq j \), we have by Lemma 23 that

\[
T(\sum_{\lambda_k=1} \lambda_k A_k) - T(\sum_{\lambda_k=-1} \lambda_k A_k) = \sum_{k=1}^{n} \lambda_k T A_k
\]

and the assertion follows. \( \square \)

Definition 28. For every \( z \in E \) such that

\[
z \in \sum_{k=1}^{n} \lambda_k T A_k + C_0^\infty(0, \infty)
\]
we set

$$\rho(z) = \sum_{k=1}^{n} \lambda_k \varphi(A_k).$$

That $\rho$ is well-defined is proved below.

**Lemma 29.** The linear functional $\rho : E \to \mathbb{R}$ is well-defined. For every $z \in E$, we have

$$\rho(z) \leq \limsup_{t \to \infty} z(t).$$

**Proof.** Let $z \in E$ be such that

$$z \in \sum_{k=1}^{n} \lambda_k T A_k + C^b_0(0, \infty), \quad z \in \sum_{k=1}^{m} \mu_k T B_k + C^b_0(0, \infty).$$

We have

$$\sum_{k=1}^{n} \lambda_k T A_k - \sum_{k=1}^{m} \mu_k T B_k \in C^b_0(0, \infty).$$

It follows from Lemma 26 that

$$\sum_{k=1}^{n} \lambda_k \varphi(A_k) = \sum_{k=1}^{m} \mu_k \varphi(B_k),$$

so that $\rho$ is well-defined.

The second assertion directly follows from Lemma 29.

**Lemma 30.** Let $\varphi$ be a normalised fully symmetric functional on $M_{1,\infty}$. There exists a dilation invariant generalised limit $\omega$ on $L^{\infty}(0, \infty)$ such that $\varphi(A) = \omega(T A)$ for every $A \in M^+_1$.

**Proof.** For every $A \in M^+_1$, we have

$$\rho(s \sigma_s T A) = \rho(s \varphi(s^{-1} A)) = \varphi(s^{-1} A) = \rho(T A).$$

Therefore, $\rho$ is $\sigma_s$–invariant on $E$. It follows from Lemma 29 that

$$\rho(z) \leq \limsup_{t \to \infty} z(t), \quad z \in E.$$

By the invariant form of the Hahn-Banach theorem (see [13, p. 157]) applied to the group of dilations $\{\sigma_s\}_{s>0}$, we can extend $\rho$ to a dilation invariant generalised limit on $L^{\infty}(0, \infty)$.

The following assertion is the main result of this section. It permits representation of a fully symmetric functional $\varphi$ via heat kernel formulae.

**Theorem 31.** Let $\varphi$ be a fully symmetric functional on $M_{1,\infty}$. There exists dilation invariant generalised limit $\omega$ on $L^{\infty}(0, \infty)$ such that $\varphi = const \cdot \xi_\omega$. 18
Proof. It follows from Lemma \[30\] that there exists a dilation invariant generalised limit \( \omega \) such that
\[
\varphi(A) = \omega \left( \frac{1}{\log(1 + t)} \tau((A - \frac{1}{t})e^{A}(\frac{1}{t}, \infty)) \right).
\]
The assertion follows now from Lemma \[20\].

7. A counterexample

It is known (see \[24, Theorem 33\] and the more general result in Corollary \[51\] below) that the equality
\[
\xi_\omega(A) = \Gamma(1 + \frac{1}{q})\tau_\omega(A), \quad A \in \mathcal{M}_1^{+}\infty
\]
holds for every \( M\)–invariant generalised limit \( \omega \) on \( L_\infty(0, \infty) \) (see also earlier results with more restrictive assumptions on \( \omega \) in \[3, Theorem 4.1\] and \[6, Theorem 5.2\]). In view of Theorem \[31\] and Theorem \[1\] it is quite natural to ask whether the equality above holds for every dilation invariant generalised limit \( \omega \). In this section we prove that this is not the case.

Lemma 32. Let \( \omega \) be a dilation invariant generalised limit on \( L_\infty(0, \infty) \). For every \( s > 1 \), we have
\[
\omega \left( \sum_k \chi_{[e^{e^k}, se^{e^k}]} \right) = 0. \quad (35)
\]
\[
\omega \left( \sum_k \chi_{[e^{e^k+k}, e^{e^k+k}]} \right) = 0. \quad (36)
\]

Proof. Denote the left hand side of \((35)\) by \( f(s) \). Due to the dilation invariance of \( \omega \), we have
\[
f(s) = \omega \left( \sum_k \chi_{[e^{e^k}, se^{e^k}]} \right) = f(st) - f(t), \quad s, t > 1.
\]
Since \( f \) is monotone and bounded, we have \( f = 0 \).

Denote the left hand side of \((36)\) by \( g(s) \). Due to the dilation invariance of \( \omega \), we have
\[
g(s) = \omega \left( \sum_k \chi_{[e^{e^k+k} / st, e^{e^k+k} / t]} \right) = g(st) - g(t), \quad s, t > 1.
\]
Since \( g \) is monotone and bounded, we have \( g = 0 \).

Lemma 33. Let \( \omega \) be a dilation invariant generalised limit on \( L_\infty(0, \infty) \). We have
\[
i) \quad \omega \left( \sum_k \frac{t}{\log(t)} e^{-e^k} \chi_{[e^{e^{k+k-1}}, e^{e^{k+k-1}}]}(t) \right) = 0.
\]
\[ \omega \left( \sum_k \frac{1}{t \log(t)} e^{k+e^k} \chi_{[e^k, e^{k+1}]}(t) \right) = 0. \]

**Proof.** We only prove the first assertion. Proof of the second one is similar.

Fix \( s > 1 \). We have

\[
\frac{t}{\log(t)} e^{-e^k} \leq \frac{2}{s} + 2 e^{-e^k/2}, \quad \forall t \leq e^{k+e^k}/s, \quad \forall k \geq 1
\]

and, therefore,

\[
\sum_k \frac{t}{\log(t)} e^{-e^k} \chi_{[e^{k-1}+e^k-1, e^{k+e^k}]}(t) \leq \frac{2}{s} + \sum_k \chi_{[e^{k+e^k}/s, e^{k+e^k}]}(t) + 2 \sum_k e^{-e^k/2} \chi_{[e^{k-1}+e^k-1, e^{k+e^k}]}(t).
\]

Clearly,

\[ \omega \left( \sum_k e^{-e^k/2} \chi_{[e^{k-1}+e^k-1, e^{k+e^k}]}(t) \right) = 0. \]

It follows from the Lemma 32 that

\[ \omega \left( \sum_k \frac{t}{\log(t)} e^{-e^k} \chi_{[e^{k-1}+e^k-1, e^{k+e^k}]}(t) \right) \leq \frac{2}{s}. \]

Since \( s \) is arbitrarily large, we have

\[ \omega \left( \sum_k \frac{t}{\log(t)} e^{-e^k} \chi_{[e^{k-1}+e^k-1, e^{k+e^k}]}(t) \right) = 0. \]

\[ \square \]

**Lemma 34.** There exists a dilation invariant generalised limit \( \omega \) on \( L_\infty(0, \infty) \) such that

\[ \omega \left( \sum_k \chi_{[e^k, e^{k+e^k}]} \right) = 1, \quad \omega \left( \sum_k \chi_{[e^{k+e^k}, e^{k+1}]} \right) = 0. \]

**Proof.** Define a positive, homogeneous functional \( \pi \) on \( L_\infty(0, \infty) \) by the formula

\[ \pi(x) = \limsup_{N \to \infty} \frac{1}{\log(\log(N))} \int_N^{N \log(N)} x(s) \frac{ds}{s}. \]

It is verified in \[24, Lemma 4\] that every \( \omega \in L_\infty(0, \infty)^* \) satisfying \( \omega \leq \pi \) is dilation invariant. Observing that

\[ \pi \left( \sum_k \chi_{[e^k, e^{k+e^k}]} \right) = 1, \]

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let us select $\omega \in L_\infty(0, \infty)^*$ satisfying $\omega \leq \pi$ and such that
$$\omega(\sum_k \chi_{[e^{e_k+e_k}, e^{e_k}+1])} = 1.$$ Therefore,
$$\omega(\sum_k \chi_{[e^{e_k+e_k}, e^{e_k}+1])} = 1 - \omega(\sum_k \chi_{[e^{e_k}, e^{e_k}+1)}) = 0.
$$

Define a function $x$ by the formula
$$x = \sup_{k \in \mathbb{N}} e^{-e_k} \chi_{[0, e^{e_k+1})}.$$ (37)

Fix $k \geq 1$. For every $t \in [e^{k-1+e_k-1}, e^{e_k}]$, we have
$$\frac{1}{\log(1+t)} \int_0^t x(s)ds \leq e^{1-k} \int_0^{e^{e_k}} x(s)ds \leq e^{1-k} \sum_{n=1}^k e^{-e_n} \cdot e^{n+e_n} \leq \frac{e^2}{e-1},$$
which guarantees $x \in M_{1,\infty}$.

**Lemma 35.** Let $x$ be as in (37) and let $\omega$ be as in Lemma 34. We have $\tau_\omega(x) = (e - 1)^{-1}$.

**Proof.** Fix $t \in [e^{k-1+e_k-1}, e^{e_k}]$. We have
$$\int_0^t x(u)du = \frac{e^k}{e-1} + te^{-e_k} + O(1).$$

It follows that
$$\tau_\omega(x) = (e - 1)^{-1} \omega(\sum_k \frac{e^k}{\log(t)} \chi_{[e^{k-1+e_k-1}, e^{e_k}+1)}(t)) +$$
$$+ \omega(\sum_k \frac{t}{\log(t)} e^{-e_k} \chi_{[e^{k-1+e_k-1}, e^{e_k}+1]}(t)).$$

By Lemma 33, the second generalised limit above vanishes. We claim that the first generalised limit above is 1. Indeed,
$$\sum_k \frac{e^k}{\log(t)} \chi_{[e^{k-1+e_k-1}, e^{e_k}+1)}(t) \geq (1 + o(1)) \sum_k \chi_{[e^{e_k}, e^{e_k}+1)}(t)$$
and
$$\sum_k \frac{e^k}{\log(t)} \chi_{[e^{k-1+e_k-1}, e^{e_k}+1]}(t) \leq \sum_k \chi_{[e^{e_k}, e^{e_k}+1]}(t) + e \sum_k \chi_{[e^{k-1+e_k-1}, e^{e_k}]}.$$

The claim follows from Lemma 34. \qed
Lemma 36. Let $x$ be as in (37) and let $\omega$ be as in Lemma 34. We have

$$\xi_\omega(x) = \frac{e}{e - 1} \Gamma\left(1 + \frac{1}{q}\right).$$

Proof. Fix $t \in [e^{ek}, e^{ek+1})$. We have

$$\int_{x > 1/t} (x(u) - \frac{1}{t}) du = \frac{e^{k+1}}{e - 1} - \frac{1}{t} e^{k+e^k} + O(1).$$

This estimate and Lemma 20 yield

$$\frac{1}{\Gamma(1 + 1/q)} \xi_\omega(x) = \frac{e}{e - 1} \omega\left(\sum_k \frac{e^k}{\log(t)} \chi_{[e^{ek}, e^{ek+1}]}(t)\right) -$$

$$- \omega\left(\sum_k \frac{1}{t \log(t)} e^{k+e^k} \chi_{[e^{ek}, e^{ek+1}]}(t)\right).$$

It follows from Lemma 33 that the second generalised limit is 0. We claim that the first generalised limit is 1. Indeed,

$$\sum_k \frac{e^k}{\log(t)} \chi_{[e^{ek}, e^{ek+1}]}(t) \geq (1 + o(1)) \sum_k \chi_{[e^{ek}, e^{ek+1}]}$$

and

$$\sum_k \frac{e^k}{\log(t)} \chi_{[e^{ek}, e^{ek+1}]}(t) \leq 1.$$ 

The claim follows from Lemma 34.

The following theorem delivers the promised counterexample.

Theorem 37. There exists $A \in M_{1,\infty}$ and dilation invariant generalised limit $\omega$ on $L_\infty(0, \infty)$ such that

$$\Gamma(1 + \frac{1}{q}) \tau_\omega(A) < \xi_\omega(A).$$

Proof. For brevity, we assume that the von Neumann algebra $\mathcal{N}$ is of type $II$ (the argument can be easily adjusted when $\mathcal{N}$ is of type $I$). Let $x$ be as in (37) and let $A \in M_{1,\infty}^+$ be such that $x = \mu(A)$. The assertion follows from Lemmas 33 and 36.

8. Correctness of the definition for generalised heat kernel formulae

Let $\omega$ be a dilation invariant generalised limit on $L_\infty(0, \infty)$ and let $B \in \mathcal{N}$. Following [1], we consider the functionals on $M_{1,\infty}^+$ defined by the formula

$$\xi_{\omega,B,f}(A) = (\omega \circ M)(t \rightarrow \frac{1}{t} \tau(f(tA)B)).$$ (38)
The main result of this section, Theorem 40, shows that the function
\[
M \left( t \to \frac{1}{t} \tau(f(tA)B) \right)
\]
is bounded, and so the formula (38) is well-defined.

**Lemma 38.** Let \( A \in M_{1,\infty}^+ \). We have \( \tau(A^2 e^A[0,1/t]) = O(t^{-1} \log(t)) \) as \( t \to \infty \).

**Proof.** Let \( c := \|A\|_{1,\infty} \). We have \( \mu(s, A) \prec c(1 + s)^{-1} \). Fix \( t > 0 \). Define a decreasing function \( x_t \in M_{1,\infty}(0, \infty) \) by setting
\[
x_t(s) = \begin{cases} \frac{\log(1 + ct \log(t))}{t \log(t)}, & 0 \leq s \leq ct \log(t) \\ \frac{c}{1 + s}, & s > ct \log(t). \end{cases}
\]
Define a decreasing function \( y_t \in M_{1,\infty}(0, \infty) \) by setting
\[
y_t(s) = \mu(A) \chi_{\{\mu(A) \leq 1/t\}}(s) + \frac{1}{t} \chi_{\{\mu(A) \geq 1/t\}}(s), \quad s > 0.
\]
We claim that \( y_t \prec x_t \). Indeed, \( y_t(s) \leq 1/t \leq x_t(s) \) for \( s \leq ct \log(t) \) and
\[
\int_0^s y_t(u) du \leq c \int_0^s \frac{du}{1 + u} = \int_0^s x_t(u) du
\]
for \( s > ct \log(t) \).

It follows that
\[
\tau(A^2 e^A[0, 1/t]) \leq \int_0^\infty y_t^2(s) ds \leq \int_0^\infty x_t^2(s) ds.
\]
We have
\[
\int_0^\infty x_t^2(s) ds = \frac{c \log^2(1 + ct \log(t))}{t \log(t)} + \int_{ct \log(t)}^\infty \frac{c^2}{(1 + s)^2} ds \leq 5c \frac{\log(t)}{t}.
\]
\( \square \)

**Lemma 39.** Let \( f(t) = t^2 \chi_{[0,1]}(t) \) and let \( A \in M_{1,\infty}^+ \). We have
\[
t \to M\left( \frac{1}{t} \tau(f(tA)) \right) \in L_{\infty}(0, \infty).
\]

**Proof.** For fixed \( t > 0 \), we have
\[
M\left( \frac{1}{t} \tau(f(tA)) \right) = \frac{1}{\log(t)} \int_1^t \tau(A^2 e^A[0, 1/s]) ds = \frac{1}{\log(t)} \tau(A^2 \int_1^t e^A[0, 1/s] ds).
\]
Integrating by parts, we obtain
\[
\int_1^t e^A[0, 1/s] ds = se^A[0, 1/s] |_1^t - \int_1^t sde^A[0, 1/s] = se^A[0, 1/s] |_1^t + \int_{1/t}^1 u^{-1} ds e^{A[1/t, u]}
\]
\[ = O(1) + A^{-1} e^{A[\frac{1}{t}, \infty]} + t e^{A[0, \frac{1}{t}]} \]

Therefore,

\[ M(\frac{1}{t} \tau(f(tA))) = \frac{1}{\log(t)} \tau(A^{1/2} \infty) + \frac{t}{\log(t)} \tau(A^2 e^{A[0, \frac{1}{t}]} + O(\frac{1}{\log(t)})) \]

It follows from the definitions of \( \| \cdot \|_{1, \infty} \) and \( d_A(\cdot) \) that for every \( A \in M_{1,\infty} \) and every \( t > 0 \), we have

\[ d_A(\frac{1}{t}) \leq \max\{1, \|A\|_{1,\infty}\} \log(1 + t) \]

Clearly,

\[ \frac{1}{\log(t)} \tau(A^{1/2} \infty) = \frac{1}{\log(t)} \int_0^{d_A(1/t)} \mu(s, A) ds \leq \frac{\log(d_A(1/t))}{\log(t)} \|A\|_{1,\infty} \in L_\infty \]

The assertion follows now from the Lemma [38].

**Theorem 40.** Let a bounded function \( f \in C^2[0, \infty) \) be such that \( f(0) = f'(0) = 0 \). Let \( A \in M_{1,\infty}^+ \) and let \( B \in N \). We have

\[ M \left( t \to \frac{1}{t} \tau(f(tA)B) \right) \in L_\infty(0, \infty) \]

**Proof.** Due to the well known inequality \( \tau(CB) \leq \tau(\|C\|B) \), it suffices to prove the theorem only when \( B = 1 \). In this case, for the function \( f(t) := t^2 \chi_{[0,1]}(t) \), the assertion follows from Lemma [38]. If \( f(t) := \chi_{(1,\infty)}(t) \) then it holds trivially. Thus, it holds for the function \( f(t) := \min\{1, t^2\} \). Finally, observe that the assumptions on \( f \) guarantee that there exists a constant \( c > 0 \) such that \( |f(t)| \leq c \min\{1, t^2\} \).

Since the function \( t \to \exp(-t^{-q}) \) satisfies the assumptions of Theorem 40, we obtain the following corollary, which was implicitly proved in [6, Theorem 5.2].

**Corollary 41.** For every \( q > 0 \) and every \( A \in M_{1,\infty}^+ \), we have

\[ M \left( t \to \frac{1}{t} \tau(\exp(-tA)^{-q})) \right) \in L_\infty(0, \infty) \]

**9. Reduction theorem for generalised heat kernel formulae**

The results of this section extend and generalise those of [3, Theorem 4.1] and [1, Theorem 5.2]. We also give an answer to the question asked in [1, page 52]. We explicitly prove that the functional \( \xi_{\omega,B,f} \) (extended to \( M_{1,\infty} \) as in [25]) is linear on \( M_{1,\infty} \).
**Lemma 42.** Let \( f \in C^2[0, \infty) \) be such that \( f(0) = f'(0) = 0 \). Let \( A \in \mathcal{M}^{+}_{1, \infty} \) and let \( B \in \mathcal{N} \). For every dilation invariant generalised limit \( \omega \) on \( L_{\infty}(0, \infty) \), we have

\[
\lim_{\varepsilon \to 0} (\omega \circ M)(\frac{1}{t} \tau(f(tAe^{A}[0, \varepsilon])B)) = 0.
\]

**Proof.** Since \( |f(t)| \leq \text{const} \cdot t^2 \) for \( t \in [0, 1] \), it is sufficient to prove the assertion for \( f(t) = t^2 \). As in the proof of Theorem 40, it is sufficient to assume that \( B = 1 \).

By Theorem 40 for every \( \varepsilon > 0 \) we have

\[
M \left( t \mapsto \frac{1}{t} \tau((tAe^{A}[0, \frac{\varepsilon}{t}])^2) \right) \in L_{\infty}(0, \infty).
\]

Since \( \omega \) is dilation invariant, we conclude

\[
(\omega \circ M)(\frac{1}{t} \tau((tAe^{A}[0, \frac{\varepsilon}{t}])^2)) = \varepsilon (\omega \circ M)(\frac{1}{t} \tau((tAe^{A}[0, \frac{1}{t}])^2)).
\]

The assertion follows immediately. \( \square \)

**Lemma 43.** Let \( f \in L_{\infty}(0, \infty) \) be such that \( f(0) = 0 \). Let \( A \in \mathcal{M}^{+}_{1, \infty} \) and let \( B \in \mathcal{N} \). For every dilation invariant generalised limit \( \omega \) on \( L_{\infty}(0, \infty) \), we have

\[
\lim_{\varepsilon \to 0} (\omega \circ M)(\frac{1}{t} \tau(f(tAe^{A}(\frac{1}{\varepsilon}, \infty))B)) = 0.
\]

**Proof.** As before, we may assume that \( B = 1 \). It is clear that

\[
f(tAe^{A}(\frac{1}{\varepsilon}, \infty)) \leq \|f\|e^{A}(\frac{1}{\varepsilon}, \infty).
\]

Since \( \omega \circ M \) is dilation invariant, we obtain

\[
(\omega \circ M)(\frac{1}{t} \tau(e^{A}(\frac{1}{\varepsilon}, \infty))) = \varepsilon (\omega \circ M)(\frac{1}{t} d_A(\frac{1}{t})).
\]

The assertion follows immediately. \( \square \)

**Lemma 44.** Let \( f : \mathbb{R}_+ \to \mathbb{R} \) be monotone on \([a, b]\) and such that \( f(0) = 0 \). Let \( A \in \mathcal{M}^{+}_{1, \infty} \) and let \( B \in \mathcal{N} \). For every dilation invariant generalised limit \( \omega \) on \( L_{\infty}(0, \infty) \) we have

\[
(\omega \circ M)(\frac{1}{t} \tau(f(tAe^{A}[a, \frac{b}{t}])B)) = \left( \int_{a}^{b} f(s) \frac{ds}{s^2} \right) \cdot (\omega \circ M)(\frac{1}{t} \tau(e^{A}(\frac{1}{t}, \infty))B)).
\]

**Proof.** Without loss of generality, we may assume that \( f \) is increasing on \([a, b]\) and that \( B \geq 0 \).

Let \( a = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n = b \). For every given \( t > 0 \), we have

\[
e^{A}(\frac{a}{t}, \frac{b}{t}) = \sum_{k=0}^{n-1} e^{A}(\frac{a_k}{t}, \frac{a_{k+1}}{t}).
\]

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Since $f$ is increasing on $[a, b]$ and $f(0) = 0$, we have
\[
    f(a_k) e^{A \frac{a_k}{t}} \leq f(\tau(e^{A \frac{a_k}{t}})) \leq f(a_{k+1}) e^{A \frac{a_{k+1}}{t}}.
\]
Therefore,
\[
    (\omega \circ M)(\frac{1}{t} \tau(f(\tau(e^{A \frac{a}{t}})))) \leq \sum_{k=0}^{n-1} f(a_{k+1})(\omega \circ M)(\frac{1}{t} \tau(e^{A \frac{a_k}{t}}))).
\]
and
\[
    (\omega \circ M)(\frac{1}{t} \tau(f(\tau(e^{A \frac{a}{t}})))) \geq \sum_{k=0}^{n-1} f(a_k)(\omega \circ M)(\frac{1}{t} \tau(e^{A \frac{a_{k+1}}{t}}))).
\]
We have
\[
    e^{A \frac{a_k}{t}} = e^{A \frac{a_k}{t}}(\frac{1}{t}, \infty) - e^{A \frac{a_{k+1}}{t}}(\frac{1}{t}, \infty).
\]
For all $c > 0$, we have
\[
    (\omega \circ M)(\frac{1}{t} \tau(e^{A \frac{c}{t}})) = c^{-1}(\omega \circ M)(\frac{1}{t} \tau(e^{A \frac{1}{t}})).
\]
Therefore,
\[
    (\omega \circ M)(\frac{1}{t} \tau(f(\tau(e^{A \frac{a_k}{t}})))) = \left(\frac{1}{t} - \frac{1}{a_k}\right)(\omega \circ M)(\frac{1}{t} \tau(e^{A \frac{1}{t}})).
\]
Hence,
\[
    (\omega \circ M)(\frac{1}{t} \tau(f(\tau(e^{A \frac{a_k}{t}})))) \leq \left(\frac{1}{t} - \frac{1}{a_k}\right)(\omega \circ M)(\frac{1}{t} \tau(e^{A \frac{1}{t}})).
\]
Both coefficients in the latter formula tend to $\int_a^b f(s) ds$. \hfill \Box

**Lemma 45.** Let a bounded function $f \in C^2[0, \infty)$ be such that $f(0) = f'(0) = 0$. Let $A \in M_1^\infty$ and let $B \in \mathcal{N}$. For every dilation invariant generalised limit $\omega$ on $L^\infty(0, \infty)$ we have
\[
    \xi_{\omega, B, f}(A) = \left(\int_0^\infty f(s) \frac{ds}{s^2}(\omega \circ M)(\frac{1}{t} \tau(e^{A \frac{1}{t}}, \infty))\right).
\]

**Proof.** Let $f$ satisfy the assumptions above. Observe that the assertion of Lemma 44 holds for the function $f|_{[a, b]}$, where $0 < a < b < \infty$. Indeed, every such function is a function of bounded variation and therefore may be written as a difference of two monotone functions. Now the assertion follows from Lemmas 42, 43, 44 by setting $a := \varepsilon$ and $b := \varepsilon^{-1}$ and letting $\varepsilon \to 0$. \hfill \Box
**Corollary 46.** Let a bounded function \( f \in C^2[0, \infty) \) be such that \( f(0) = f'(0) = 0 \). Let \( A \in M_{1,\infty}^+ \) and let \( B \in \mathcal{N}^+ \). For every dilation invariant generalised limit \( \omega \) on \( L_\infty(0, \infty) \) we have

\[
\xi_{\omega, B, f}(A) = \left( \int_0^\infty f(s) \frac{ds}{s^2} \right) \omega \left( \frac{1}{\log(1 + t)} \tau((A - \frac{1}{t})e^{A}(\frac{1}{t}, \infty)B) \right).
\]

**Proof.** It follows from the definition of Cesaro operator \( M \) that

\[
M \left( t \to \frac{1}{t} \tau(e^{A}(\frac{1}{t}, \infty)B) \right) = \frac{1}{\log(t)} \int_1^t \tau(e^{A}(\frac{1}{s}, \infty)B) ds.
\]

Integrating by parts, we obtain

\[
\frac{1}{\log(t)} \int_1^t \tau(e^{A}(\frac{1}{s}, \infty)B) \frac{ds}{s^2} = \frac{1}{\log(t)} \int_{1/t}^1 \tau(e^{A}(u, \infty)B) du =
\]

\[
= \frac{1}{\log(t)} \cdot u \tau(e^{A}(u, \infty)B) \big|_{1/t}^1 - \frac{1}{\log(t)} \int_{1/t}^1 u d\tau(e^{A}(u, \infty)B) =
\]

\[
= \frac{-1}{t \log(t)} \cdot \tau(e^{A}(\frac{1}{t}, \infty)B) + \frac{-1}{t \log(t)} \tau \left( \int_{1/t}^\infty u e^{A}(u, \infty)B \right) + o(1).
\]

Evidently,

\[
-\tau \left( \int_{1/t}^\infty u e^{A}(u, \infty)B \right) = \tau(A e^{A}(\frac{1}{t}, \infty)B).
\]

Therefore,

\[
M \left( t \to \frac{1}{t} \tau(e^{A}(\frac{1}{t}, \infty)B) \right) = \frac{1}{\log(t)} \tau((A - \frac{1}{t})e^{A}(\frac{1}{t}, \infty)B) + o(1).
\]

The assertion follows now from Lemma 45. \( \square \)

The first assertion in lemma below can be found in \([3, \text{Theorem 11}]\). For the second assertion we refer to \([2, \text{Theorem 3.5}]\).

**Lemma 47.** Let \( A, B \in B^+(H) \) and let \( f \) be convex continuous function such that \( f(0) = 0 \). We have

i) \( \tau(B^{1/2}f(A)B^{1/2}) \geq \tau(f(B^{1/2}AB^{1/2})) \) if \( B \leq 1 \).

ii) \( \tau(B^{1/2}f(A)B^{1/2}) \leq \tau(f(B^{1/2}AB^{1/2})) \) if \( B \geq 1 \).

We show in the following lemma that \( \xi_{\omega, B, f} \) depends continuously on \( B \).

**Lemma 48.** If \( A \in M_{1,\infty}^+ \) and let \( B_n, B \in \mathcal{N}, n \geq 1 \), then

\[
\| \xi_{\omega, B_n}(A) - \xi_{\omega, B}(A) \| \leq \xi_{\omega}(A) \cdot \| B_n - B \|. \]
Proof. The assertion follows from the inequality

\[ |\tau(f(tA)B_n) - \tau(f(tA)B)| \leq \tau(f(tA)) \cdot \|B_n - B\|. \]

The following theorem extends the results of [5, 6] and gives an affirmative answer to the question stated in [1]. It also shows that the functionals \( \xi_{\omega,B,f}(\cdot) \) are linear functionals on \( \mathcal{M}_{1,\infty} \) for a wide class of functions \( f \).

**Theorem 49.** Let a bounded function \( f \in C^2(0,\infty) \) be such that \( f(0) = f'(0) = 0 \). Let \( A \in \mathcal{M}_{1,\infty} \) and let \( B \in \mathcal{N} \). For every dilation invariant generalised limit \( \omega \) on \( L_\infty(0,\infty) \) we have

\[ \xi_{\omega,B,f}(A) = \frac{1}{\Gamma(1 + 1/q)} \left( \int_0^\infty f(s) \frac{ds}{s^2} \right) \xi_{\omega}(AB). \quad (39) \]

**Proof.** It follows from Theorem 22 that \( \xi_{\omega} \) is linear and fully symmetric. By Theorem 1 and (17), we have \( \xi_{\omega}(B_{1/2}AB_{1/2}) = \xi_{\omega}(AB) \).

Recall that function \( u \to (u - 1/t)_+ \) is convex. It follows from Lemma 47 that

i) \( \tau((A - \frac{1}{t})_+ B) \geq \tau((B^{1/2} AB^{1/2} - \frac{1}{t})_+ ) \) if \( B \leq 1 \).

ii) \( \tau((A - \frac{1}{t})_+ B) \leq \tau((B^{1/2} AB^{1/2} - \frac{1}{t})_+ ) \) if \( B \geq 1 \).

It follows from Corollary 46 that for \( 0 \leq B \leq 1 \) we have

\[ \xi_{\omega,B,f}(A) \geq \frac{1}{\Gamma(1 + 1/q)} \left( \int_0^\infty f(s) \frac{ds}{s^2} \right) \xi_{\omega}(B^{1/2} AB^{1/2}). \quad (40) \]

Since both sides are homogeneous, the inequality (40) is valid for every \( B \).

It follows from (40) that for \( B \geq 1 \) we have

\[ \xi_{\omega,B,f}(A) \leq \frac{1}{\Gamma(1 + 1/q)} \left( \int_0^\infty f(s) \frac{ds}{s^2} \right) \xi_{\omega}(B^{1/2} AB^{1/2}). \quad (41) \]

Since both sides are homogeneous, the inequality (41) is valid if \( B \) is bounded from below by a strictly positive constant.

Thus, we have the equality (39) valid for every \( B \) bounded from below by a strictly positive constant. Set \( B_n = Bc^B(1/n,\infty) + 1/n e^B[0,1/n] \). It follows that equality (39) holds with \( B \) replaced with \( B_n \) throughout. By Lemma 48 we have \( \xi_{\omega,B_n,f}(A) \to \xi_{\omega,B,f}(A) \). Since \( AB_n \to AB \) in \( \mathcal{M}_{1,\infty} \) and since \( \xi_{\omega} \) is bounded on \( \mathcal{M}_{1,\infty} \), we have \( \xi_{\omega}(AB_n) \to \xi_{\omega}(AB) \). The assertion follows immediately.

The following corollary treats the case of classical heat kernel formulae. We use the notation

\[ \xi_{\omega,B}(A) = (\omega \circ M)(\frac{1}{t} \tau(\exp(-(tA)^{-q})B)). \]
Corollary 50. Let $A \in M_{1,\infty}^+$ and let $B \in \mathcal{N}$. For every dilation invariant generalised limit $\omega$ on $L_\infty(0, \infty)$ we have $\xi_{\omega,B}(A) = \xi_\omega(AB)$.

Proof. Use $f(t) = \exp(-t^{-q})$ in Theorem 49 and observe that
\[
\int_0^\infty f(s) \frac{ds}{s^2} = \Gamma(1 + \frac{1}{q}).
\]

The following assertion extends [24, Theorem 33].

Corollary 51. Let $A \in M_{1,\infty}^+$ and let $B \in \mathcal{N}$. For every dilation invariant generalised limit $\omega$ on $L_\infty(0, \infty)$ such that $\omega = \omega \circ M$, we have
\[
\xi_{\omega,B}(A) = \Gamma(1 + \frac{1}{q})\tau_\omega(AB).
\]

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