Koszulity of finitely semi-graded algebras

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Abstract
In this paper we introduce the class of finitely semi-graded algebras which extends the connected graded algebras finitely generated in degree 1. The Koszul behavior of finitely semi-graded algebras is investigated by the distributivity of some associated lattice of ideals. The Hilbert series, the Poincaré series and the Yoneda algebra are defined for this class of algebras. Finitely semi-graded algebras include many important examples of non $N$-graded algebras finitely generated in degree one coming from mathematical physics, and for these concrete examples the Koszulity will be established, as well as, the explicit computation of its Hilbert and Poincaré series.

Key words and phrases. Graded algebras, Hilbert and Poincaré series, Yoneda algebra, distributive lattices, Koszul algebras, skew PBW extensions.

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1 Introduction
Finitely graded algebras over fields cover many important classes of non-commutative rings and algebras coming from mathematical physics; examples of these algebras are the multi-parameter quantum affine $n$-space, the Jordan plane, the Manin algebra $M_q(2)$, the multiplicative analogue of the Weyl algebra, among many others. There exists recent interest in developing the non-commutative projective algebraic geometry for finitely graded algebras (see for example [15], [16], [20], [21], [32]). However, for non $N$-graded algebras only few works in this direction have been realized ([12], [18]). Some examples of non $N$-graded algebras generated in degree one are the dispin algebra $U(osp(1, 2))$, the Woronowicz algebra $W_\nu(\mathfrak{su}(2, K))$, the quantum algebra $U'(\mathfrak{so}(3, K))$, the quantum symplectic space $O_q(\mathfrak{sp}(K^{2n}))$, some algebra of operators, among others. One of the most important algebraic properties studied in non-commutative algebraic geometry for graded algebras is the Koszulity. Koszul graded algebras were defined by Stewart B. Priddy in [26] and have many equivalent characterizations involving the Hilbert series, the Poincaré series, the Yoneda algebra and some associated lattices of vector spaces. In this paper we are interested in investigating the Koszul behavior for algebras over fields not being necessarily $N$-graded. For this purpose we will introduce in this work the finitely semi-graded algebras; these type of algebras extend finitely graded algebras over fields generated in degree one, and conform a particular subclass of finitely semi-graded rings defined in [18]. In order to study the Koszulity for finitely semi-graded algebras we will define its Hilbert series, the Poincaré series, the Yoneda algebra, and we will investigate some associated lattices of vector spaces similarly as this is done in the classical graded case.

For finitely semi-graded algebras we will prove the uniqueness of the Hilbert series (Corollary 3.5); in the proof we used a beautiful paper by Jason Bell and James J. Zhang ([7]), where this property was
induced $N$-filtration, so we will show that the Hilbert series of the algebra coincides with the Hilbert series of its associated graded algebra; the same will be proved for the Poincaré series. Theorem 3.9 shows that the Yoneda algebra of a finitely semi-graded algebra is isomorphic to the Yoneda algebra of its induced graded algebra. We will associate to a finitely semi-graded algebra a lattice of vector spaces defined with the ideal of relations of its presentation, and from a result that gives conditions over the distributiveness of this lattice (Theorem 4.4), we will define the semi-graded Koszul algebras, extending this way the classical notion of graded Koszul algebras. One important part of the present paper consists in giving many examples of finitely semi-graded algebras as well as examples of semi-graded Koszul algebras. Most of the examples that we will present arise in mathematical physics and can be interpreted as skew $PBW$ extensions. This class of non-commutative rings of polynomial type were introduced in [13], and they are a good global way of describing rings and algebras not being necessarily $\mathbb{N}$-graded. Thus, the general results that we will prove for finitely semi-graded algebras will be in particular applied to skew $PBW$ extensions; in Corollary 4.6 we explicitly computed the Hilbert series of skew $PBW$ extensions that are finitely semi-graded algebras over fields, covering this way many examples of algebras coming from quantum physics. A similar explicit computation was done in Corollary 4.10 and Example 4.11 for the $P$-Poincaré series. Finally, in Theorem 4.7 and Example 4.8 we present examples of non $\mathbb{N}$-graded algebras that have Koszul behavior, i.e., they are semi-graded Koszul.

The paper is organized in the following way: In the first section we review the basic facts on semi-graded rings and skew $PBW$ extensions that we need for the rest of the work. In the second section we introduce the semi-graded algebras and we present many examples of them. The list of examples include not only skew $PBW$ extensions that are algebras over fields, but also other non graded algebras that can not be described as skew extensions. The third section is dedicated to construct and prove the uniqueness of the Hilbert series, the $P$-Poincaré series and the Yoneda algebra of a finitely semi-graded algebra. In the last section we study the Koszul behavior of finitely semi-graded algebras and we will show that some non $\mathbb{N}$-graded algebras coming from quantum physics are semi-graded Koszul.

If not otherwise noted, all modules are left modules and $K$ will be an arbitrary field.

In order to appreciate better the results of the paper we recall first the definition of finitely graded algebras over fields and its Hilbert series (see [30]). Let $A$ be a $K$-algebra, $A$ is finitely graded if: (a) $A$ is $\mathbb{N}$-graded, i.e., $A$ has a graduation $A = \bigoplus_{n \geq 0} A_n$, $A_nA_m \subseteq A_{n+m}$ for every $n,m \geq 0$; (b) $A$ is connected, i.e., $A_0 = K$; (c) $A$ is finitely generated as $K$-algebra. Thus, $A$ is locally finite, i.e., $\dim_K A_n < \infty$ for every $n \geq 0$, and hence the Hilbert series of $A$ is defined by
\[ h_A(t) := \sum_{n=0}^{\infty} (\dim_K A_n) t^n. \]

### 1.1 Semi-graded rings and modules

In this starting subsection we recall the definition and some basic facts about semi-graded rings and modules, more details and the proofs omitted here can be found in [18].

**Definition 1.1.** Let $B$ be a ring. We say that $B$ is semi-graded (SG) if there exists a collection $\{B_n\}_{n \geq 0}$ of subgroups $B_n$ of the additive group $B^+$ such that the following conditions hold:

(i) $B = \bigoplus_{n \geq 0} B_n$.

(ii) For every $m,n \geq 0$, $B_mB_n \subseteq B_0 \oplus \cdots \oplus B_{m+n}$.

(iii) $1 \in B_0$.

The collection $\{B_n\}_{n \geq 0}$ is called a semi-gradation of $B$ and we say that the elements of $B_n$ are homogeneous of degree $n$. Let $B$ and $C$ be semi-graded rings and let $f : B \to C$ be a ring homomorphism, we say that $f$ is homogeneous if $f(B_n) \subseteq C_n$ for every $n \geq 0$. 


Definition 1.2. Let $B$ be a SG ring and let $M$ be a $B$-module. We say that $M$ is a $\mathbb{Z}$-semi-graded, or simply semi-graded, if there exists a collection $\{M_n\}_{n \in \mathbb{Z}}$ of subgroups $M_n$ of the additive group $M^+$ such that the following conditions hold:

(i) $M = \bigoplus_{n \in \mathbb{Z}} M_n$.

(ii) For every $m \geq 0$ and $n \in \mathbb{Z}$, $B_m M_n \subseteq \bigoplus_{k \leq m+n} M_k$.

The collection $\{M_n\}_{n \in \mathbb{Z}}$ is called a semi-graduation of $M$ and we say that the elements of $M_n$ are homogeneous of degree $n$. We say that $M$ is positively semi-graded, also called $\mathbb{N}$-semi-graded, if $M_n = 0$ for every $n < 0$. Let $f : M \rightarrow N$ be an homomorphism of $B$-modules, where $M$ and $N$ are semi-graded $B$-modules; we say that $f$ is homogeneous if $f(M_n) \subseteq N_n$ for every $n \in \mathbb{Z}$.

Let $B$ be a semi-graded ring and $M$ be a semi-graded $B$-module, let $N$ be a submodule of $M$ and $N_n := N \cap M_n$, $n \in \mathbb{Z}$; observe that the sum $\sum_n N_n$ is direct. This induces the following definition.

Definition 1.3. Let $B$ be a SG ring and $M$ be a semi-graded module over $B$. Let $N$ be a submodule of $M$, we say that $N$ is a semi-graded submodule of $M$ if $N = \bigoplus_{n \in \mathbb{Z}} N_n$.

We present next an important class of semi-graded rings that includes finitely graded algebras.

Definition 1.4. Let $B$ be a ring. We say that $B$ is finitely semi-graded ($FSG$) if $B$ satisfies the following conditions:

(i) $B$ is SG.

(ii) There exists finitely many elements $x_1, \ldots, x_n \in B$ such that the subring generated by $B_0$ and $x_1, \ldots, x_n$ coincides with $B$.

(iii) For every $n \geq 0$, $B_n$ is a free $B_0$-module of finite dimension.

Moreover, if $M$ is a $B$-module, we say that $M$ is finitely semi-graded if $M$ is semi-graded, finitely generated, and for every $n \in \mathbb{Z}$, $M_n$ is a free $B_0$-module of finite dimension.

From the definitions above we get the following elementary but key facts.

Proposition 1.5. Let $B = \bigoplus_{n \geq 0} B_n$ be a SG ring. Then,

(i) $B_0$ is a subring of $B$. Moreover, for any $n \geq 0$, $B_0 \oplus \cdots \oplus B_n$ is a $B_0-B_0$-bimodule, as well as $B$.

(ii) $B$ has a standard $\mathbb{N}$-filtration given by

$$F_n(B) := B_0 \oplus \cdots \oplus B_n.$$  \hspace{1cm} (1.1)

(iii) The associated graded ring $Gr(B)$ satisfies

$$Gr(B)_n \cong B_n, \text{ for every } n \geq 0 \text{ (isomorphism of abelian groups)}.$$  

(iv) Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a semi-graded $B$-module and $N$ a submodule of $M$. The following conditions are equivalent:

(a) $N$ is semi-graded.

(b) For every $z \in N$, the homogeneous components of $z$ are in $N$.

(c) $M/N$ is semi-graded with semi-gradation given by

$$(M/N)_n := (M_n + N)/N, \text{ } n \in \mathbb{Z}.$$  

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Remark 1.6. (i) According to (iv)-(b) in the previous proposition, if \( N \) is a semi-graded submodule of \( M \), then \( N \) can be generated by homogeneous elements; however, if \( N \) is a submodule of \( M \) generated by homogeneous elements, then we can not asserts that \( N \) is semi-graded.

(ii) Let \( B \) be a SG ring, as we saw in (ii) of the previous proposition, \( B \) is \( \mathbb{N} \)-filtered. Conversely, if we assume that \( B \) is a \( \mathbb{N} \)-filtered ring with filtration \( \{ F_n(B) \}_{n \geq 0} \) such that for any \( n \geq 0 \), \( F_n(B)/F_{n-1}(B) \) is \( F_0(B) \)-projective, then it is easy to prove that \( B \) is SG with semi-graduation \( \{ B_n \}_{n \geq 0} \) given by \( B_0 := F_0(B) \) and \( B_n \) is such that \( F_{n-1}(B) \oplus B_n = F_n(B), \ n \geq 1 \).

(iii) If \( B \) is a FSG ring, then for every \( n \geq 0 \), \( Gr(B)_n \cong B_n \) as \( B_0 \)-modules.

(iv) Observe if \( B \) is FSG ring, then \( B_0B_p = B_p \) for every \( p \geq 0 \), and if \( M \) is finitely semi-graded, then \( B_0M_n = M_n \) for all \( n \in \mathbb{Z} \).

We conclude this subsection recalling one of the invariants that we will study later for finitely semi-graded algebras. In \([13]\) the authors introduced the notion of generalized Hilbert series for finitely semi-graded rings.

**Definition 1.7.** Let \( B = \bigoplus_{n \geq 0} B_n \) be a FSG ring. The generalized Hilbert series of \( B \) is defined by

\[
Gh_B(t) := \sum_{n=0}^{\infty} (\dim B_n) t^n.
\]

**Remark 1.8.** (i) Note that if \( K \) is a field and \( B \) is a finitely graded \( K \)-algebra, then the generalized Hilbert series coincides with the usual Hilbert series, i.e., \( Gh_B(t) = h_B(t) \).

(ii) Observe that if a FSG ring \( B \) has another semi-graduation \( B = \bigoplus_{n \geq 0} C_n \), then its generalized Hilbert series can change, i.e., the notion of generalized Hilbert series depends on the semi-graduation, in particular on \( B_0 \). For example, consider the usual real polynomial ring in two variables \( B := \mathbb{R}[x, y] \), then \( Gh_B(t) = \frac{1}{1-t^2} \), but if we view this ring as \( B = \langle \mathbb{R}[x] \rangle[y] \) then \( C_0 = \mathbb{R}[x] \), its generalized Hilbert series is \( \frac{1}{1-t} \). However, in Section 3 we will introduce the semi-graded algebras over fields and for them we will discuss the uniqueness of the Hilbert series based in a recent paper by Bell and Zhang \((7)\).

### 1.2 Skew PBW extensions

As was pointed out above, finitely graded algebras over fields are examples of FSG rings. In order to present many other examples of FSG rings not being necessarily graded algebras, we recall in this subsection the notion of skew PBW extension defined firstly in \([13]\).

**Definition 1.9** \(([13])\). Let \( R \) and \( A \) be rings. We say that \( A \) is a skew PBW extension of \( R \) (also called a \( \sigma \)-PBW extension of \( R \)) if the following conditions hold:

(i) \( R \subseteq A \).

(ii) There exist finitely many elements \( x_1, \ldots, x_n \in A \) such that \( A \) is a left \( R \)-free module with basis

\[
\text{Mon}(A) := \{ x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \}, \text{ with } \mathbb{N} := \{0, 1, 2, \ldots \}.
\]

The set \( \text{Mon}(A) \) is called the set of standard monomials of \( A \).

(iii) For every \( 1 \leq i \leq n \) and \( r \in R - \{0\} \) there exists \( c_{i,r} \in R - \{0\} \) such that

\[
x_i r - c_{i,r} x_i \in R.
\]  

(iv) For every \( 1 \leq i, j \leq n \) there exists \( c_{i,j} \in R - \{0\} \) such that

\[
x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n.
\]

Under these conditions we will write \( A := \sigma(R)[x_1, \ldots, x_n] \).
Example 1.10. Many important algebras and rings coming from mathematical physics are particular examples of skew PBW extensions: Habitual ring of polynomials in several variables, Weyl algebras, enveloping algebras of finite dimensional Lie algebras, algebra of q-differential operators, many important types of Ore algebras, algebras of diffusion type, additive and multiplicative analogues of the Weyl algebra, dispin algebra \( \mathcal{U}(osp(1, 2)) \), quantum algebra \( \mathcal{U}(so(3, K)) \), Woronowicz algebra \( \mathcal{W}_c(\mathfrak{sl}(2, K)) \), Manin algebra \( \mathcal{O}_q(M_2(K)) \), coordinate algebra of the quantum group \( SL_q(2) \), q-Heisenberg algebra \( H_n(q) \), Hayashi algebra \( W_q(J) \), differential operators on a quantum space \( D_q(S_q) \), Witten’s deformation of \( \mathcal{U}(\mathfrak{sl}(2, K)) \), multiparameter Weyl algebra \( A_n^{Q, T}(K) \), quantum symplectic space \( \mathcal{O}_q(K^{2n}) \), some quadratic algebras in 3 variables, some 3-dimensional skew polynomial algebras, particular types of Sklyanin algebras, homogenized enveloping algebra \( A(G) \), Sridharan enveloping algebra of 3-dimensional Lie algebra \( G \), among many others. For a precise definition of any of these rings and algebras see [17], [27], [31], [32], [33].

Associated to a skew PBW extension \( A = \sigma(R)(x_1, \ldots, x_n) \), there are \( n \) injective endomorphisms \( \sigma_1, \ldots, \sigma_n \) of \( R \) and \( \sigma_i \)-derivations, as the following proposition shows.

Proposition 1.11 ([13]). Let \( A \) be a skew PBW extension of \( R \). Then, for every \( 1 \leq i \leq n \), there exist an injective ring endomorphism \( \sigma_i : R \to R \) and a \( \sigma_i \)-derivation \( \delta_i : R \to R \) such that

\[
x_i r = \sigma_i(r)x_i + \delta_i(r),
\]

for each \( r \in R \).

A particular case of skew PBW extension is when all derivations \( \delta_i \) are zero. Another interesting case is when all \( \sigma_i \) are bijective and the constants \( c_{ij} \) are invertible. We recall the following definition.

Definition 1.12 ([13], [31], [32], [33]). Let \( A \) be a skew PBW extension.

(a) \( A \) is quasi-commutative if the conditions (iii) and (iv) in Definition 1.9 are replaced by

(iii’) For every \( 1 \leq i \leq n \) and \( r \in R - \{0\} \) there exists \( c_{i,r} \in R - \{0\} \) such that

\[
x_i r = c_{i,r}x_i.
\]

(iv’) For every \( 1 \leq i, j \leq n \) there exists \( c_{i,j} \in R - \{0\} \) such that

\[
x_j x_i = c_{i,j} x_i x_j.
\]

(b) \( A \) is bijective if \( \sigma_i \) is bijective for every \( 1 \leq i \leq n \) and \( c_{i,j} \) is invertible for any \( 1 \leq i, j \leq n \).

(c) \( A \) is constant if the condition (ii) in Definition 1.9 is replaced by: For every \( 1 \leq i \leq n \) and \( r \in R \),

\[
x_i r = r x_i.
\]

(d) \( A \) is pre-commutative if the condition (iv) in Definition 1.9 is replaced by: For any \( 1 \leq i, j \leq n \) there exists \( c_{i,j} \in R \setminus \{0\} \) such that

\[
x_j x_i - c_{i,j} x_i x_j \in Rx_1 + \cdots + Rx_n.
\]

(e) \( A \) is called semi-commutative if \( A \) is quasi-commutative and constant.

Remark 1.13. Later below we need the following classification given in [31], [32] and [33] of skew PBW extensions of Example 1.10. The extensions are classified as constant (C), bijective (B), pre-commutative (P), quasi-commutative (QC) and semi-commutative (SC); in the tables the symbols \( \ast \) and \( \checkmark \) denote negation and affirmation, respectively.
If $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ is a skew PBW extension of the ring $R$, then, as was observed in Proposition \ref{prop:skew_extension}, $A$ induces injective endomorphisms $\sigma_k : R \to R$ and $\sigma_k$-derivations $\delta_k : R \to R$, $1 \leq k \leq n$. From the Definition \ref{def:skew_pbw_extension}, there exists a unique finite set of constants $c_{ij}, d_{ij}, \alpha_{ij}^{(k)} \in R$, $c_{ij} \neq 0$, such that

$$x_j x_i = c_{ij} x_i x_j + \alpha_{ij}^{(1)} x_1 + \cdots + \alpha_{ij}^{(n)} x_n + d_{ij}, \text{ for every } 1 \leq i < j \leq n. \quad (1.8)$$
Definition 1.14. Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a skew PBW extension. $R$, $n$, $\sigma_k, \delta_k, c_{ij}, d_{ij}, a_{ij}^{(k)}$, with $1 \leq i < j \leq n, 1 \leq k \leq n$, defined as before, are called the parameters of $A$.

Some notation will be useful in what follows.

Definition 1.15. Let $A$ be a skew PBW extension of $R$.

(i) For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

(ii) For $X = x^\alpha \in \text{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$.

(iii) Let $0 \neq f \in A$, $t(f)$ is the finite set of terms that conform $f$, i.e., if $f = c_1 X_1 + \cdots + c_t X_t$, with $X_i \in \text{Mon}(A)$ and $c_i \in R - \{0\}$, then $t(f) := \{c_1 X_1, \ldots, c_t X_t\}$.

(iv) Let $f$ be as in (iii), then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$.

Skew PBW extensions have been enough investigated, many ring and homological properties of them have been studied, as well as their Gröbner theory ([1], [2], [3], [4], [13], [14], [17], [18], [19], [27], [28], [29], [34]). We conclude this introductory section with some known results about skew PBW extensions and semi-graded rings that will use in the present paper.

Theorem 1.16 ([17]). Let $A$ be an arbitrary skew PBW extension of the ring $R$. Then, $A$ is a $\mathbb{N}$-filtered ring with filtration given by

$$F_m := \begin{cases} R, & \text{if } m = 0 \\ \{f \in A | \deg(f) \leq m\}, & \text{if } m \geq 1, \end{cases}$$

and the graded ring $\text{Gr}(A)$ is a quasi-commutative skew PBW extension of $R$. If the parameters that define $A$ are as in Definition 1.14, then the parameters that define $\text{Gr}(A)$ are $R, n, \sigma_k, c_{ij}$, with $1 \leq i < j \leq n, 1 \leq k \leq n$. Moreover, if $A$ is bijective, then $\text{Gr}(A)$ is bijective.

Proposition 1.17 ([15]). (i) Any $\mathbb{N}$-graded ring is SG.

(ii) Let $K$ be a field. Any finitely graded $K$-algebra is a FSG ring.

(iii) Any skew PBW extension is a FSG ring.

For skew PBW extensions the generalized Hilbert series has been computed explicitly.

Theorem 1.18 ([15]). Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be an arbitrary skew PBW extension. Then,

$$Gh_A(t) = \frac{1}{(1 - t)^n}.$$ (1.9)

Remark 1.19. (i) Note that the class of SG rings includes properly the class of $\mathbb{N}$-graded rings: In fact, the enveloping algebra of any finite-dimensional Lie algebra proves this statement. This example proves also that the class of FSG rings includes properly the class of finitely graded algebras.

(ii) The class of FSG rings includes properly the class of skew PBW extensions: For this consider the Artin-Schelter regular algebra of global dimension 3 defined by the following relations:

$$yx = xy + z^2, \quad zy = yz + x^2, \quad zx = xz + y^2.$$ 

Observe that this algebra is a particular case of a Sklyanin algebra which in general are defined by the following relations:

$$axy + bxy + cz^2 = 0, \quad azy + byz + cx^2 = 0, \quad axz + bzx + cy^2 = 0, \quad a, b, c \in K.$$ 

7
2 Finitely semi-graded algebras

In the present section we define the finitely semi-graded algebras. In all of examples that we will study, in particular, the semi-graded Koszul algebras that we will introduce later, they are additionally finitely presented. Let us recall first this notion. Let $B$ be a finitely generated $K$-algebra, so there exist finitely many elements $g_1, \ldots, g_n \in B$ that generate $B$ as $K$-algebra and we have the $K$-algebra homomorphism $f : K\{x_1, \ldots, x_n\} \to B$, with $f(x_i) := g_i$, $1 \leq i \leq n$; let $I := \ker(f)$, then we get a presentation of $B$:

$$B \cong K\{x_1, \ldots, x_n\}/I. \quad \text{(2.1)}$$

Recall that $B$ is said to be finitely presented if $I$ is finitely generated.

2.1 Definition

In the previous section we defined the finitely semi-graded rings and we observed that they generalize finitely graded algebras over fields and skew PBW extensions. In this section we will be concentrated in some particular class of this type of rings which satisfy some other extra natural conditions.

Definition 2.1. Let $B$ be a $K$-algebra. We say that $B$ is finitely semi-graded (FSG) if the following conditions hold:

(i) $B$ is a FSG ring with semi-gradation $B = \bigoplus_{p \geq 0} B_p$.

(ii) For every $p, q \geq 1$, $B_pB_q \subseteq B_1 \oplus \cdots \oplus B_{p+q}$.

(iii) $B$ is connected, i.e., $B_0 = K$.

(iv) $B$ is generated in degree 1.

Remark 2.2. Let $B$ be a FSG $K$-algebra;

(i) Since $B$ is locally finite and $B$ is finitely generated in degree 1, then any $K$-basis of $B_1$ generates $B$ as $K$-algebra.

(ii) The canonical projection $\varepsilon : B \to K$ is a homomorphism of $K$-algebras, called the augmentation map, with $\ker(\varepsilon) = \bigoplus_{n \geq 1} B_n$. Therefore, the class of FSG algebras is contained in the class of augmented algebras, i.e., algebras with augmentation (see [24]), however, as we will see, a semi-gradation is a nice tool for defining some invariants useful for the study of the algebra. $B_{\geq 1} := \bigoplus_{n \geq 1} B_n$ is called the augmentation ideal. Thus, $K$ becomes into a left and right $B$-module with products given by $b \cdot \lambda := b_0\lambda$, $\lambda \cdot b := \lambda b_0$, with $b \in B$, $\lambda \in K$ and $b_0$ is the homogeneous component of $b$ of degree zero.

(iii) It is well known that $B$ is finitely graded if and only if the ideal $I$ in (2.1) is homogeneous (20). In general, finitely semi-graded algebras do not need to be finitely presented. Any finitely graded algebra generated in degree 1 is FSG, but $B := K\{x, y\}/(xy - x)$ with semi-gradation $B_n := K \langle y^kx^{n-k} | 0 \leq k \leq n \rangle$, $n \geq 0$, is a FSG algebra and it is not finitely graded generated in degree 1. Thus, the class of FSG algebras includes properly all finitely graded algebras generated in degree 1.

(iv) Any FSG algebra is $\mathbb{N}$-filtered (see Proposition 1.5), but note that the Weyl algebra $A_1(K) = K\{t, x\}/(xt - tx - 1)$ is $\mathbb{N}$-filtered but not FSG, i.e., the class of FSG algebras do not coincide with the class of $\mathbb{N}$-filtered algebras.

Proposition 2.3. Let $B$ be a FSG algebra over $K$. Then $B_{\geq 1}$ is the unique two-sided maximal ideal of $B$ semi-graded as left ideal.

Proof. From Remark 2.2 we have that $B_{\geq 1}$ is a two-sided maximal ideal of $B$, and of course, semi-graded as left ideal. Let $I$ be another two-sided maximal ideal of $B$ semi-graded as left ideal; since $I$ is proper, $I \cap B_0 = I \cap K = 0$; let $x \in I$, then $x = x_0 + x_1 + \cdots + x_n$, with $x_i \in B_i$, $1 \leq i \leq n$, but since $I$ is semi-graded, $x_i \in I$ for every $i$, so $x_0 = 0$, and hence, $x \in B_{\geq 1}$. Thus, $I \subseteq B_{\geq 1}$, and hence, $I = B_{\geq 1}$. \qed
2.2 Examples of $FSG$ algebras

In this subsection we present a wide list of $FSG$ algebras, many of them, within the class of skew $PBW$ extensions. For the explicit set of generators and relations for these algebras see [17], [27], [31], [32], [33].

**Example 2.4** (Skew $PBW$ extensions that are $FSG$ algebras). Note that a skew $PBW$ extension of the field $K$ is a $FSG$ algebra if and only if it is constant and pre-commutative. Thus, we have:

(i) By the classification presented in the tables of Remark 1.13 the following skew $PBW$ extensions of the field $K$ are $FSG$ algebras: The classical polynomial algebra; the particular Sklyanin algebra; the universal enveloping algebra of a Lie algebra; the quantum algebra $U'(so(3, K))$; the dispin algebra; the Woronowicz algebra; the quantum Heisenberg algebra; nine types 3-dimensional skew polynomial algebras; six types of Sridharan enveloping algebra of 3-dimensional Lie algebras.

(ii) Many skew $PBW$ extensions in the first table of Remark 1.13 are marked as non constant, however, reconsidering the ring of coefficients, some of them can be also viewed as skew $PBW$ extensions of the base field $K$; this way, they are $FSG$ algebras over $K$: The algebra of shift operators; the algebra of discrete linear systems; the multiplicative analogue of the Weyl algebra; the algebra of linear partial shift operators; the algebra of linear partial q-dilation operators.

(iii) In the class of skew quantum polynomials (see [17]) the multi-parameter quantum affine $n$-space is another example of skew $PBW$ extension of the field $K$ that is a $FSG$ (actually finitely graded) algebra. In particular, this is the case for the quantum plane.

(iv) The following skew $PBW$ extensions of the field $K$ are $FSG$ but not finitely graded: The universal enveloping algebra of a Lie algebra; the quantum algebra $U'(so(3, K))$; the dispin algebra; the Woronowicz algebra; the quantum Heisenberg algebra; eight of the nine types 3-dimensional skew polynomial algebras; five of the six types of Sridharan enveloping algebra of 3-dimensional Lie algebras.

**Example 2.5** ($FSG$ algebras that are not skew $PBW$ extensions of $K$). The following algebras are $FSG$ but not skew $PBW$ extensions of the base field $K$ (however, in every example below the algebra is a skew $PBW$ extension of some other subring):

(i) The Jordan plane $A$ is the free $K$-algebra generated by $x, y$ with relation $yx = xy + x^2$, so $A = K\{x, y\}/(yx - xy - x^2)$. $A$ is not a skew $PBW$ extension of $K$, but of course, it is a $FSG$ algebra over $K$; actually, it is a finitely graded algebra over $K$ (observe that $A$ can be viewed as a skew $PBW$ extension of $K[x]$, i.e., $A = \sigma(K[x])(y)$).

(ii) The $K$-algebra in Example 1.18 of [30] is not a skew $PBW$ extension of $K$:

$$A = K\{x, y, z\}/(z^2 - xy - yx, zx - xz, zy - yz).$$

However, $A$ is a $FSG$ algebra, actually, it is a finitely graded algebra over $K$ (note that $A$ can be viewed as a skew $PBW$ extension of $K[z]$: $A = \sigma(K[z])\{x, y\}$).

(iii) The following examples are similar to the previous: The homogenized enveloping algebra $A(G)$; algebras of diffusion type; the Manin algebra, or more generally, the algebra $O_q(M_n(K))$ of quantum matrices; the complex algebra $V_q(\mathfrak{s}l_3(\mathbb{C}))$; the algebra $U$: the Witten’s deformation of $U(\mathfrak{sl}(2, K))$; the quantum symplectic space $O_q(\mathfrak{sp}(K^{2n}))$; some quadratic algebras in 3 variables.

**Example 2.6** ($FSG$ algebras that are not skew $PBW$ extensions). The following $FSG$ algebras are not skew $PBW$ extensions:

(i) Consider the Sklyanin algebra with $c \neq 0$ (see Remark 1.19), then $S$ is not a skew PBW extension, but clearly it is a $FSG$ algebra over $K$.

(ii) The finitely graded $K$-algebra in Example 1.17 of [30]:

$$B = K\{x, y\}/(yx^2 - x^2y, y^2x - xy^2).$$

(iii) Any monomial quadratic algebra

$$B = K\{x_1, \ldots, x_n\}/\langle x_i x_j, (i, j) \in S \rangle,$$
with $S$ any finite set of pairs of indices $(\ref{proof})$.

(iv) $B = K\{x, y, u\}/(yu, xu, uw)$ $(\ref{proof})$.

(v) $B = K\{x, y\}/(x^2, y^2, x, y)$ $(\ref{proof})$.

(vi) $B = K\{x, y\}/(x^2 - xy, yx, y^3)$ $(\ref{proof})$.

(vii) $B = K\{w, x, y, z\}/(z^2y^3, y^3, x^2w, x^2w, zy^3x)$ $(\ref{proof})$.

(viii) $B = K\{x, y, z\}/(x^4, yx^3, x^3z)$ $(\ref{proof})$.

(ix) $B = K\{x, y, z\}/(xz - x, xz - yz, xz - yx^3, y^3z, x^2z, y^4)$ $(\ref{proof})$.

(x) $B = K\{x, y, z, w\}/(y^2z, x^2, x^2, gw^2, y^2w^2, xg - gx, yg - gy, wg - gw, zg - gz)$ $(\ref{proof})$.

(xi) $B = K\{x, y\}/(x^2y - y^2, x, y^3 - y^3, x, y^3)$ $(\ref{proof})$.

(xii) $B = K\{x, y\}/(xy, x, y^3)$ $(\ref{proof})$.

3 Some invariants associated to $FSG$ algebras

Now we will study some invariants associated to finitely semi-graded algebras: The Hilbert series, the Yoneda algebra and the Poincaré series. The topics that we will consider here for $FSG$ algebras extend some well known results on finitely graded algebras.

3.1 The Hilbert series

In Definition $(\ref{definition})$ we presented the notion of generalized Hilbert series of a $FSG$ ring. We will prove next that if $B$ is a $FSG$ algebra over a field $K$, then $Gr_B(B)$ is well-defined, i.e., it does not depend on the semi-gradation (compare with Remark $(\ref{remark})$). This theorem was proved recently by Jason Bell and James J. Zhang in $(\ref{reference})$ for connected graded algebras finitely generated in degree 1, we will apply the Bell-Zhang result to our semi-graded algebras.

Theorem 3.1 $(\ref{reference})$. Let $A$ and $B$ be connected graded algebras finitely generated in degree 1. Then, $A \cong B$ as $K$-algebras if and only if $A \cong B$ as graded algebras.

Corollary 3.2 $(\ref{reference})$. Let $A$ be a connected graded algebra finitely generated in degree 1. If $A$ has two gradations $A = \bigoplus_{n \geq 0} A_n = \bigoplus_{n \geq 0} B_n$, then there exists an algebra automorphism $\phi : A \to A$ such that $\phi(A_n) = B_n$ for every $n \geq 0$. In particular, $\dim_K A_n = \dim_K B_n$ for every for every $n \geq 0$, and the Hilbert series of $A$ is well-defined. Moreover, if $\text{Aut}(A) = \text{Aut}_{Gr}(A)$, then $A_n = B_n$ for every $n \geq 0$.

We will prove that the generalized Hilbert series of $FSG$ algebras is well-defined.

Proposition 3.3. If $B$ is a $FSG$ algebra, then $Gr(B)$ is a connected graded algebra finitely generated in degree 1.

Proof. This is a direct consequence of part (iii) of Proposition $(\ref{proof})$.

Theorem 3.4. Let $B$ and $C$ be $FSG$ algebras over the field $K$. If $\phi : B \to C$ is a homogeneous isomorphism of $K$-algebras, then $Gr(B) \cong Gr(C)$ as graded algebras.

Proof. From the previous proposition we know that $Gr(B)$ and $Gr(C)$ are connected graded algebras finitely generated in degree 1; according to Theorem 3.1 we only have to show that $Gr(B)$ and $Gr(C)$ are isomorphic as $K$-algebras. For every $n \geq 0$ we have the homomorphism of $K$-vector spaces $\phi_n : Gr(B)_n \to Gr(C)_n$, $b_n \mapsto c_n$, with $\phi(b_n) := c_n$ (observe that $Gr(B)_n \cong B_n$ and $Gr(C)_n \cong C_n$ as $K$-vector spaces); from this we obtain a homomorphism of $K$-vector spaces $\phi : Gr(B) \to Gr(C)$ such that $\phi \circ \mu_n = \mu_n$, for every $n \geq 0$, where $\mu_n : Gr(B)_n \to Gr(B)$ is the canonical injection. Considering $\varphi := \phi^{-1}$ we get a homomorphism of $K$-vector spaces $\varphi : Gr(C) \to Gr(B)$ such that $\varphi \circ \nu_n = \nu_n$, for every $n \geq 0$, where $\nu_n : Gr(C)_n \to Gr(C)$ is the canonical injection. But observe that $\phi \circ \varphi = i_{Gr(C)}$ and $\varphi \circ \phi = i_{Gr(B)}$. In fact, $\varphi\phi(b_n) = \varphi\phi(\mu_n(b_n)) = \varphi\mu_n(b_n) = \varphi(\nu_n(c_n)) = \varphi\nu_n(c_n) = \varphi_n(c_n) = \phi^{-1}(c_n) = b_n$.

In a similar way we can prove the first identity. It is obvious that $\phi$ is multiplicative.
Corollary 3.5. Let $B$ be a $FSG$ algebra. If $B$ has two semi-graduations $A = \bigoplus_{n \geq 0} B_n = \bigoplus_{n \geq 0} C_n$, then $\dim_K B_n = \dim_K C_n$ for every for every $n \geq 0$, and the generalized Hilbert series of $B$ is well-defined. Moreover, $Gh_B(t) = h_{G\phi(B)}(t)$.

Proof. We consider the identical isomorphism $i_B : B \rightarrow B$, by Theorem 3.4 there exists an isomorphism of graded algebras $\phi : Gr_1(B) \rightarrow Gr_2(B)$, where $Gr_1(B)$ is the graded algebra associated to the semi-graduation $\{B_n\}_{n \geq 0}$ and $Gr_2(B)$ is the graded algebra associated to $\{C_n\}_{n \geq 0}$; from the proof of Corollary 3.4 we know that $\dim_K (Gr_1(B)_n) = \dim_K (Gr_2(B)_n)$ for every $n \geq 0$, but from the part (iii) of Proposition 1.5 $Gr_1(B)_n \cong B_n$ and $Gr_2(B)_n \cong C_n$, moreover, these isomorphisms are $K$-linear, so $\dim_K B_n = \dim_K C_n$ for every for every $n \geq 0$. □

Corollary 3.6. Each of the algebras presented in Examples 2.4, 2.5 and 2.6 have generalized Hilbert series well-defined. In addition, let $A = \sigma(K)(x_1, \ldots, x_n)$ be a skew PBW extension of the field $K$; if $A$ is a $FSG$ algebra, then the generalized Hilbert series is well-defined and given by

$$Gh_A(t) = \frac{1}{(1-t)^n}.$$ 

Proof. Direct consequence of the previous corollary and Theorem 1.18 □

Example 3.7. In this example we show that the condition (iv) in Definition 2.1 is necessary in order to the generalized Hilbert series of $FSG$ algebras be well-defined. Let $\mathcal{L}$ be the 3-dimensional (Heisenberg) Lie algebra that has $K$-basis $\{x, y, z\}$ with Lie bracket

$$[x, y] = z, \quad [x, z] = 0, \quad [y, z] = 0.$$ 

The universal enveloping algebra $\mathcal{U}(\mathcal{L})$ is connected graded with $\deg x = \deg y = 1$, $\deg z = 2$. With this grading, the homogeneous component of degree 1 of $\mathcal{U}(\mathcal{L})$ is $Kx + Ky$. Thus, $\mathcal{U}(\mathcal{L})$ is not generated in degree 1, i.e., with this grading, $\mathcal{U}(\mathcal{L})$ can not be viewed as $FSG$ algebra. In this case the generalized Hilbert series is

$$\frac{1}{(1-t)^3(1-t^2)}.$$ 

On the other hand, $\mathcal{U}(\mathcal{L})$ is $FSG$ by setting $\deg x = \deg y = \deg z = 1$. According to Corollary 3.6 in this case the generalized Hilbert series is

$$\frac{1}{(1-t)^n}.$$ 

3.2 The Yoneda algebra

The collection $SGR - B$ of semi-graded modules over $B$ is an abelian category, where the morphisms are the homogeneous $B$-homomorphisms; $K$ is an object of this category with the trivial semi-graduation given by $K_0 := K$ and $K_n := 0$ for $n \neq 0$. We can associate to $B$ the $Yoneda$ algebra defined by

$$E(B) := \bigoplus_{i \geq 0} Ext^i_B(K, K);$$ 

(3.1)

recall that in any abelian category the $Ext^i_B(K, K)$ groups can be computed either by projective resolutions of $K$ or by extensions of $K$. Here we will take in account both equivalent interpretations; the first one will be used in the proof of Theorem 3.3. For the second interpretation (see [35]), the groups $Ext^i_B(K, K)$ are defined by equivalence classes of exact sequences of finite length with semi-graded $B$-modules and homogeneous $B$-homomorphisms from $K$ to $K$:

$$\xi : 0 \rightarrow K \rightarrow X_1 \rightarrow \cdots \rightarrow X_1 \rightarrow K \rightarrow 0;$$
the addition in $\text{Ext}_B^K(K, K)$ is the Baer sum (see [35], Section 3.4):

$$\xi : 0 \to K \to X_i \to \cdots \to X_1 \to K \to 0,$$

$$\chi : 0 \to K \to X'_i \to \cdots \to X'_1 \to K \to 0,$$

$$[\xi] \oplus [\chi] : 0 \to K \to Y_i \to X_{i-1} \oplus X'_{i-1} \to \cdots \to X_2 \oplus X'_2 \to Y_1 \to K \to 0,$$

where $Y_i$ is the pullback of homomorphisms $X_1 \to K$ and $X'_1 \to K$, and $Y_1$ is the pushout of $K \to X_i$ and $K \to X'_i$. The zero element of $\text{Ext}_B^K(K, K)$ is the class of any split sequence $\xi$.

The product in $E(B)$ is given by concatenation of sequences:

$$\text{Ext}_B^K(K, K) \times \text{Ext}_B^K(K, K) \to \text{Ext}_B^{i+j}(K, K)$$

$$(\langle \xi \rangle, \langle \chi \rangle) \mapsto \langle \xi \rangle \langle \chi \rangle := \langle \chi \xi \rangle,$$

where

$$\xi : 0 \to K \to X_i \to \cdots \to X_1 \to K \to 0,$$

$$\chi : 0 \to K \to X'_i \to \cdots \to X'_1 \to K \to 0,$$

$$\chi \xi : 0 \to K \to X'_j \to \cdots \to X'_1 \to X_1 \to \cdots \to X_1 \to K \to 0.$$

Note that the unit of $E(B)$ is the equivalence class of $0 \to K \xrightarrow{i_K} K \to 0$.

Thus, $E(B) = \bigoplus_{n \geq 0} E^n(B)$ is a connected $\mathbb{N}$-graded algebra, where $E^n(B) := \text{Ext}_B^n(K, K)$ is a $K$-vector space. Observe that definition [3.1] extends the usual notion of Yoneda algebra of graded algebras.

### 3.3 The Poincaré series

Another invariant that we want to consider is the Poincaré series; let $B$ be a FSG algebra; as we observed above, $E(B)$ is connected and graded; if $E(B)$ is finitely generated, then $E(B)$ is locally finite, and hence, the Poincaré series of $B$ is defined as the Hilbert series of $E(B)$, i.e.,

$$P_B(t) := \sum_{n=0}^{\infty} (\dim_K E^n_B(K, K)) t^n. \quad (3.2)$$

By Corollary 3.2, $P_B(t)$ is well-defined if $E(B)$ is generated in degree 1. In our next theorem we will show that $E(B)$ is isomorphic to $E(Gr(B))$ and $P_B(t)$ can be also defined by the $\text{Tor}$ vector spaces. We will prove this using a known result for finitely graded algebras.

**Lemma 3.8** ([5], [11]). Let $B$ be a finitely graded algebra. Then for every $n \geq 0$, it has the following isomorphism of $K$-vector spaces:

$$\text{Tor}_n^B(K, K) \cong \text{Ext}_B^n(K, K).$$

**Theorem 3.9.** Let $B$ be a FSG algebra. Then,

(i) For every $n \geq 0$,

$$\text{Tor}_n^B(K, K) \cong \text{Ext}_B^n(K, K).$$

(ii) $E(B) \cong E(Gr(B))$.

**Proof.** (i) We will proof that for every $n \geq 0$,

$$\text{Tor}_n^B(K, K) \cong \text{Tor}_n^{Gr(B)}(K, K) \text{ and } \text{Ext}_B^n(K, K) \cong \text{Ext}_B^n(K, K) \quad (3.3)$$

as $K$-vector spaces. If so, then the corollary follows from Proposition [3.3] and Lemma 3.8.

It is clear that
For $n \geq 1$, consider a $Gr(B)$-free homogeneous resolution of $K$

$$\cdots \rightarrow Gr(B)(X_{n-1}) \xrightarrow{\alpha_n} \cdots \xrightarrow{\alpha_2} Gr(B)(X_0) \xrightarrow{\alpha_1} Gr(B)(X_n) \xrightarrow{\alpha_0} K \rightarrow 0,$$

and the $B$-free homogeneous resolution of $K$

$$\cdots \rightarrow B(X_{n}) \xrightarrow{\beta_n} B(X_{n-1}) \xrightarrow{\beta_{n-1}} \cdots \xrightarrow{\beta_2} B(X_0) \xrightarrow{\beta_1} B(0) \rightarrow 0,$$

where $Gr(B(X_n)) = Gr(B)(X_n)$, $Gr(\beta_n) = \alpha_n$ for every $n$ (see [22], Theorem 4.4; note that the filtration of $B(X_n)$ is exhaustive and discrete). Applying $K \otimes_{Gr(B)} -$, $K \otimes_{B} -$, $Hom_{Gr(B)}(-, K)$ and $Hom_{B}(-, K)$ we get the complexes of $K$-vector spaces

$$\cdots \rightarrow K \otimes_{Gr(B)} Gr(B)(X_{n}) \xrightarrow{i_K \otimes \alpha_n} K \otimes_{Gr(B)} Gr(B)(X_{n-1}) \xrightarrow{i_K \otimes \alpha_{n-1}} \cdots \xrightarrow{i_K \otimes \alpha_1} K \otimes_{Gr(B)} Gr(B)(X_0) \xrightarrow{i_K \otimes 0} K \rightarrow 0,$$

$$\cdots \rightarrow K \otimes_B B(X_{n}) \xrightarrow{i_K \otimes \beta_n} K \otimes_B B(X_{n-1}) \xrightarrow{i_K \otimes \beta_{n-1}} \cdots \xrightarrow{i_K \otimes \beta_2} K \otimes_B B(0) \xrightarrow{i_K \otimes 0} K \rightarrow 0,$$

since for every $n \geq 0$ we have the isomorphism of $K$-vector spaces $B_p \cong Gr(B)_p$ (Proposition 4.5), then for every $n \geq 0$ we obtain an isomorphism of $K$-vector spaces $B^{(X_n)} \cong Gr(B)(X_n)$, and from this we get:

$$Tor^B_n(K, K) = \ker(i_K \otimes \alpha_n)/\text{Im}(i_K \otimes \alpha_{n+1}) \cong \ker(i_K \otimes \beta_n)/\text{Im}(i_K \otimes \beta_{n+1}) = Tor^B_n(K, K),$$

$$Ext^B_n(K, K) = \ker(\alpha_{n+1})/\text{Im}(\alpha_n) \cong \ker(\beta_{n+1})/\text{Im}(\beta_n) = Ext^B_n(K, K).$$

In fact, we have the following commutative diagrams of $K$-vector spaces

$$\begin{array}{ccc}
B(X_{n+1}) & \xrightarrow{\beta_{n+1}} & B(X_n) \\
\downarrow \theta_{n+1} & & \downarrow \theta_n \\
Gr(B)(X_{n+1}) & \xrightarrow{\alpha_{n+1}} & Gr(B)(X_n)
\end{array}$$

$$\begin{array}{ccc}
K \otimes_B B(X_{n+1}) & \xrightarrow{i_K \otimes \beta_{n+1}} & K \otimes_B B(X_n) \\
\downarrow f_{n+1} & & \downarrow f_n \\
K \otimes_{Gr(B)} Gr(B)(X_{n+1}) & \xrightarrow{i_K \otimes 0} & K \otimes_{Gr(B)} Gr(B)(X_n)
\end{array}$$

$$\begin{array}{ccc}
K \otimes_{Gr(B)} Gr(B)(X_{n+1}) & \xrightarrow{i_K \otimes 0} & K \otimes_{Gr(B)} Gr(B)(X_n) \\
\downarrow g_n & & \downarrow g_{n+1} \\
Hom_B(B(X_{n+1}), K) & \xrightarrow{\beta_{n+1}} & Hom_B(B(X_n), K)
\end{array}$$

$$\begin{array}{ccc}
\downarrow g_n & & \downarrow g_{n+1} \\
Hom_{Gr(B)}(Gr(B)(X_{n+1}), K) & \xrightarrow{\alpha_{n+1}} & Hom_{Gr(B)}(Gr(B)(X_n), K)
\end{array}$$

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where $f_n$ is the isomorphism of $K$-vector spaces induced by the functions $f_n'(k, b) := k \otimes \theta_n(b)$, $f_n''(k, c) := k \otimes \theta_n^{-1}(c)$, with $k \in K$, $b \in B^{(X_n)}$, $c \in \text{Gr}_{n}^{(B)}(X_n)$, and $g_n$ is the isomorphism of $K$-vector spaces defined by $g_n(h) := h \theta_n^{-1}$, where $h \in \text{Hom}_{B}(B^{(X_n)}, K)$ (observe that $h \theta_n^{-1}$ is a $Gr(B)$-homomorphism since $h$ is homogeneous).

(ii) We will show that $E(B) \cong E(Gr(B))$ as graded $K$-algebras: From [3.3], for every $i \geq 0$ we have the isomorphism of $K$-vector spaces $E(B)_i = Ext^i_B(K, K) \cong Ext^i_{Gr(B)}(K, K) = E(Gr(B))_i$, then taking $\gamma := \bigoplus_{i \geq 0} \gamma_i$ we get the isomorphism of $K$-vector spaces $E(B) \cong E(Gr(B))$. It is clear that the product is well defined and $\alpha(1) = 1$. It only remains to show that $\gamma$ is multiplicative. For this it is enough to prove that $\gamma(z_i z_j) = \gamma(z_i) \gamma(z_j)$, for every $z_i \in E(B)_i$, $z_j \in E(B)_j$ and $i, j \geq 0$. Recall that the product in $E(B)$ is defined in the following way (see [10] or also [3.3]): As in (i), consider the resolution \{ $B^{(X_i)} \xrightarrow{\beta_i} B^{(X_{i-1})}$ \}_{i \geq 0} of $K$ that defines the groups $Ext^i_B(K, K)$, with $B^{(X_{i+1})} := K$; moreover, let $\mathcal{F} \in Ext^i_B(K, K) = \ker \beta_{i+1}/\text{Im} \beta_i$ with $f \in \ker \beta_{i+1} \subseteq \text{Hom}_B(B^{(X_i)}, K)$ and $\mathcal{G} \in Ext^i_B(K, K) = \ker \beta_{i+1}/\text{Im} \beta_i$ with $g \in \ker \beta_{i+1} \subseteq \text{Hom}_B(B^{(X_i)}, K)$, then the product is defined by

$$Ext^i_B(K, K) \times Ext^i_B(K, K) \to Ext^{i+j}_B(K, K)$$

$$(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} g' \mathcal{G},$$

where $g' : B^{(X_{i+j})} \to B^{(X_i)}$ is defined inductively by the following commutative diagrams:

Note that this product is well defined and $fg' \in \ker \beta_{i+j+1}$. Applying $\gamma_{i+j}$ we get the claimed.

**Corollary 3.10.** Let $B$ be a FSG algebra such that $E(B)$ is finitely generated in degree 1, then $P_{B}(t)$ is well-defined and it is also given by

$$P_{B}(t) = \sum_{n=0}^{\infty} (\dim K \text{Tor}^B_n(K, K)) t^n. \quad (3.4)$$

Moreover, $P_{B}(t) = P_{Gr(B)}(t)$.

**Proof.** This follows from [3.2] and [3.3].

### 4 Koszulity

Koszul algebras were defined by Stewart B. Priddy in [20]. Later in 2001, Roland Berger in [8] introduces a generalization of Koszul algebras which are called *generalized Koszul algebras* or *N-Koszul algebras*. The 2-Koszul algebras of Roland Berger are the Koszul algebras of Priddy (for the definition of Koszul algebras adopted in this paper see Remark [4.4]). N-Koszul algebras are finitely graded where all generators of the ideal $I$ of relations are homogeneous and have the same degree $N \geq 2$. In 2008 Thomas Cassidy and Brad Shelton ([11]) generalize the N-Koszul algebras introducing the $K_2$ algebras; these type of algebras accept that the generators of $I$ have different degrees, but again all generators are homogeneous since the $K_2$ algebras are graded. Later, Phan in [24] extended this notion to $K_m$ algebras for any $m \geq 1$.

In this section we study the semi-graded version of Koszulity, and for this purpose we will follow the lattice interpretation of this notion (see [5], [6], [8], [11], [25]).
4.1 Semi-graded Koszul algebras

Recall that a lattice is a collection $L$ endowed with two idempotent commutative and associative binary operations $\land, \lor : L \times L \to L$ satisfying the following absorption identities: $a \land (a \lor b) = a$, $(a \land b) \lor b = b$. A sublattice of a lattice $L$ is a non empty subset of $L$ closed under $\land$ and $\lor$. A lattice is called distributive if it satisfies the following distributivity identity: $a \land (b \lor c) = (a \land b) \lor (a \land c)$. If $X \subseteq L$, the sublattice generated by $X$, denoted $[X]$, consists of all elements of $L$ that can be obtained from the elements of $X$ by the operations $\land$ and $\lor$. We will say that $X$ is distributive if $[X]$ is a distributive lattice. The (direct) product of the family of lattices $\{L_\omega\}_{\omega \in \Omega}$ is defined as follow:

$$\prod_{\Omega} L_\omega := \prod_{\Omega} L_\omega, \land, \lor,$$

which is the cartesian product with $\land$ and $\lor$ operating component-wise. A semidirect product of the family $\{L_\omega\}_{\omega \in \Omega}$ is a sublattice $L$ of $\prod_{\Omega} L_\omega$ such that for every $\omega_0 \in \Omega$, the composition $L \hookrightarrow \prod_{\Omega} L_\omega \to L_{\omega_0}$ is surjective.

**Proposition 4.1** ([5]). If $L$ is a semidirect product of the family $\{L_\omega\}_{\omega \in \Omega}$, then $L$ is distributive if and only if for all $\omega \in \Omega$, $L_\omega$ is distributive.

Let $K$ be a field and $V$ be a $K$-vector space, the set $L(V)$ of all its linear subspaces is a lattice with respect to the operations of sum and intersection.

**Proposition 4.2** ([25]). Let $V$ be a vector space and $X_1, \ldots, X_n \subseteq V$ be a finite collection of subspaces of $V$. The following conditions are equivalent:

(i) The collection $X_1, \ldots, X_n$ is distributive.

(ii) There exists a basis $B := \{\omega_i\}_{i \in C}$ of $V$ such that each of the subspaces $X_i$ is the linear span of a set of vectors $\omega_i$.

(iii) There exists a basis $B$ of $V$ such that $B \cap X_i$ is a basis of $X_i$, for every $1 \leq i \leq n$.

With the previous elementary facts about lattices, we have the following notions associated to any FSG algebra presented as in [21] (compare with [5]).

**Definition 4.3.** Let $B = K\{x_1, \ldots, x_n\}/I$ be a FSG algebra. The lattice associated to $B$ is the sublattice $L(B)$ of subspaces of the free algebra $F := K\{x_1, \ldots, x_n\}$ generated by $\{F_{\geq 1}, \mathbb{P}F_{\geq 1}\{s, g, h \geq 0\}$. For any integer $j \geq 2$, the $j$-th lattice associated to $B$ is defined by

$$L_j(B) := \left\{ \{F_sI_gF_h|s, h \geq 0, g \geq 2, s + g + h = j\} \right\} \subset \left\{ \text{subspaces of } F_j; \cap, + \right\},$$

where $F_sI_gF_h$ is the subspace of $F_j$ consisting of finite sums of elements of the form $abc$, with $a \in F_s$, $b \in I_g$, $c \in F_h$, and

$$I_g := \{a_g \in F_g|a_g \text{ is the } g \text{-th component of some element in } I\}.$$

For any two-sided ideal $H$ of $F$, the $K$-subspace $H_g$ is defined similarly. From now on in this section we will denote $F := K\{x_1, \ldots, x_n\}$.

**Theorem 4.4.** Let $B = K\{x_1, \ldots, x_n\}/I$ be a FSG algebra with $I = \langle b_1, \ldots, b_m \rangle$ such that $b_i \in F_{\geq 1}$ for $1 \leq i \leq m$. Then $L(B)$ is a semidirect product of the family of lattices

$$\{L_j(B) \cup \{0, F_j\}_{j \geq 2}\} \cup \{\{0, K\}, \{0, F_i\}\}.$$

In particular, $L(B)$ is distributive if and only if for all $j \geq 2$, $L_j(B)$ is distributive.
Proof. The proof of Lemma 2.4 in [5] can be easily adapted.

Step 1. For any $j \geq 2$ and any $X \in L_j(B)$ we have $0 \subseteq X \subseteq F_j$. So $L_j(B) \cup \{0, F_j\}$ is in fact a lattice.

Step 2. If $s \geq 0$, $g \geq 1$, $h \geq 0$ and $j \geq 2 + s + h$, then

$$(F^s_{\geq 1} I^g F^h_{\geq 1})_j = F_s(I^g)_j - F^{s+h} F^g F^{h+1}.$$

We only have to prove that $(F^s_{\geq 1} I^g F^h_{\geq 1})_j \subseteq F_s(I^g)_j - F^{s+h} F^g F^{h+1}$ since the other containment is trivial. Recall that one element of $(F^s_{\geq 1} I^g F^h_{\geq 1})_j$ is the $j$-th component of some element of $F^s_{\geq 1} I^g F^h_{\geq 1}$; let $z_j \in (F^s_{\geq 1} I^g F^h_{\geq 1})_j$, then there exists $y \in F^s_{\geq 1} I^g F^h_{\geq 1}$ such that $z_j$ is the $j$-th component of $y$; the element $y$ is a finite sum of elements of the form $abc$, with $a \in F^s_{\geq 1} = F_{s+1}$, $b \in I^g$ and $c \in F^h_{\geq 1} = F_{h+1}$, so the $j$-th component of $y$ is a sum of the $j$-th components of elements of the form $a_k b a_t$, with $k \geq s$, $b \in I^g$ and $t \geq h$, but since $F_k = F_{s+1} F_{k-s}$ for $k \geq s$ and $F_{s+1} = F_{s+1} F_{h+1}$ for $t \geq h$, then the $j$-th component of $a_k b a_t$ is the $j$-th component $a_s(a_{k-s} b a_{t-h}) a_h$, i.e., it is an element of $F_s(I^g)_j - F^{s+h} F^g F^{h+1}$.

Step 3. For $g \geq 1$ and $j \geq 2$,

$$(I^g)_j = \sum F_{k_0} t_i, F_{k_1} t_i, \ldots, F_{k_{g-1}} t_{i_g},$$

where the sum is taken over all relevant $k_0, \ldots, k_g, l_1, \ldots, l_g$ such that $\sum m k_m + \sum n l_n = j$. Indeed, if $p \in I^g$, then $p$ is a finite sum of elements of the form $a^{(0)} p_1 a^{(1)} p_2 \cdots a^{(g-1)} p_g a^{(g)}$, with $a^{(r)} \in F$, $p_i \in \{b_1, \ldots, b_m\}$, $0 \leq r \leq g$, $1 \leq i \leq g$.

Step 4. For any $g \geq 2$ and any $2g + 1$ non-negative integers $k_0, \ldots, k_g, l_1, \ldots, l_g$ we have

$$F_{k_0} t_{i_1}, F_{k_1} t_{i_2}, \ldots, F_{k_{g-1}} t_{i_g}, F_{k_g} = \bigcap_{a=1}^g F_{k_0 + l_1 + \cdots + k_{a-1} + l_a} t_{i_a} F_{k_0 + \cdots + k_g}.$$

In fact, let $q = a_0 p_1 a_1 \cdots p_g a_g \in F_{k_0} t_{i_1}, F_{k_1} t_{i_2}, \ldots, F_{k_{g-1}} t_{i_g}, F_{k_g}$, with $a_r \in F_{k_r}$, $p_i \in F_{i_i}$, $0 \leq r \leq g$, $1 \leq i \leq g$, then $q \in F_{k_0 + l_1 + \cdots + k_{a-1} + l_a} t_{i_a} F_{k_0 + \cdots + k_g}$ for every $1 \leq a \leq g$; the converse follows from the fact that for any $a \in F - \{0\}$ homogeneous with $a = b c = d e$, then $b, c, d, e$ are homogeneous; in addition, if $b \in F_k$, $d \in F_l$ with $t \geq s$, then there is $f$ such that $a = b f e$, $d = b f$ and $c = f e$.

Step 5. For any $s \geq 0$, $g \geq 1$, $h \geq 0$ and $j \geq 1 + s + h$ we have $(F^s_{\geq 1} I^g F^h_{\geq 1})_j = 0$ since $b_i \in F_{s+1}$ for $1 \leq i \leq m$; likewise, for $j < g$, $(I^g)_j = 0$.

From these steps, $L(B)$ is a sublattice of the product of the given family, i.e.,

$$L(B) \rightarrow \{0, K\} \times \{0, F_1\} \times \bigcap_{j \geq 2} \{L_j(B) \cup \{0, F_j\}\}.$$

Finally, fix $j \geq 2$, then $L(B) \rightarrow L_j(B) \cup \{0, F_j\}$ is a lattice surjective map since: (a) $(I^g)_j = 0$ if $j < g$; (b) $(F^s_{\geq 1})_j = F_j$ if $j \geq s$; (c) if $s, h \geq 0$, $g \geq 2$ and $s + g + h = j$, then $F_s I^g F^h = (F^s_{\geq 1} I^g F^h_{\geq 1})_j$. The cases $j = 0, 1$ can be proved by the same method. Thus, $L(B)$ is a semidirect product of the given family. □

**Definition 4.5.** Let $B = K[\{x_1, \ldots, x_n\}] / I$ be a FSG algebra. We say that $B$ is semi-graded Koszul, denoted $SK$, if $B$ satisfies the following conditions:

(i) $B$ is finitely presented with $I = \langle b_1, \ldots, b_m \rangle$ and $b_i \in F_{\geq 1}$ for $1 \leq i \leq m$.

(ii) $L(B)$ is distributive.

**Remark 4.6.** (i) In the present paper we adopt the following definition of Koszul algebras (see [5], [6], [8], [11], [25]). Let $B$ be a $K$-algebra; it is said that $B$ is Koszul if $B$ satisfies the following conditions: (a) $B$ is $\mathbb{N}$-graded, connected, finitely generated in degree one; (b) $B$ is quadratic, i.e., the ideal $I$ in (2.1) is finitely generated by homogeneous elements of degree 2; (c) $L(B)$ is distributive.

(ii) From (i) it is clear that any Koszul algebra is $SK$. Many examples of skew PBW extensions are actually Koszul algebras. In [32] and [33] was proved that the following skew PBW extensions are Koszul.
algebras: The classical polynomial algebra; the particular Sklyanin algebra; the multiplicative analogue of
the Weyl algebra; the algebra of linear partial $q$-dilation operators; the multi-parameter quantum affine $n$-
space, in particular, the quantum plane; the 3-dimensional skew polynomial algebra with $\{\alpha, \beta, \gamma\} \neq 3$;
the Sridharan enveloping algebra of 3-dimensional Lie algebra with $[x, y] = [y, z] = [z, x] = 0$; The Jordan
plane; algebras of diffusion type; $A(G)$; the algebra $U$; the Manin algebra, or more generally, the algebra
$O_q(M_n(K))$ of quantum matrices; some quadratic algebras in 3 variables.

The next theorem gives a wide list of $SK$ algebras within the class of skew PBW extensions. If at
least one of the constants $a_{ij}^{(k_1, j)}$ is non zero, then the algebra is not Koszul but it is $SK$.

**Theorem 4.7.** If $A$ is a skew PBW extension of a field $K$ with presentation $A = K\{x_1, \ldots, x_n\}/I$, where

$I = \langle x_j x_i - c_{ij} x_i x_j - a_{ij}^{(k_1, j)} x_{k_1, i} | c_{ij}, a_{ij}^{(k_1, j)} \in K, c_{ij} \neq 0, 1 \leq j < i \leq n \rangle$,

then $A$ is $SK$.

**Proof.** Note that $A$ is a $FSG$ algebra. Let $F := K\{x_1, \ldots, x_n\}$, $N := \{x_1, \ldots, x_n\}$, and $J := \{k_{i,j} \in \{1, \ldots, n\} | a_{k_{i,j}} \neq 0, 1 \leq i < j \leq n\}$. We are going to show that $L_m(A)$ is distributive lattice for $m \geq 2$.

If $|J| = n$, we define

$$B_m := \left( \bigcup_{r=1}^{m} D_r^{(m)} \right),$$

where

$$D_r^{(m)} := \{a_1 \cdots a_{r-1} a_{r+1} \cdots a_m | a_t \in N, t = 1, \ldots, r - 1, r + 1, \ldots, n; 1 \leq i \leq n\};$$

$B_m$ is a basis of $F_m$. Now, consider $F_s I_g F_h \leq F_m$ with $s, h \geq 0, g \geq 2$ and $s + g + h = m$. Since $F_s I_g F_h$
is generated by $D_{s+1}^{(m)}, \ldots, D_{s+g}^{(m)}$, then $F_s I_g F_h \cap B_m = \bigcup_{r=s+1}^{s+g} D_r^{(m)}$, which is a basis of $F_s I_g F_h$.

If $|J| = n - 1$, define

$$B_m := \left( \bigcup_{r=1}^{m} D_r^{(m)} \right) \cup \{x_i^{m}\},$$

where $l \notin J$, and

$$D_r^{(m)} := \{a_1 \cdots a_{r-1} a_{r+1} \cdots a_m | a_t \in N, t = 1, \ldots, r - 1, r + 1, \ldots, n; i \in J\};$$

again $B_m$ is a basis of $F_m$. As before, consider $F_s I_g F_h \leq F_m$ with $s, h \geq 0, g \geq 2$ and $s + g + h = m$; since
$F_s I_g F_h$ is generated by $D_{s+1}^{(m)}, \ldots, D_{s+g}^{(m)}$, then $F_s I_g F_h \cap B_m = \bigcup_{r=s+1}^{s+g} D_r^{(m)}$, which is a basis of $F_s I_g F_h$.

If $|J| \leq n - 2$, we define

$$B_m := \left( \bigcup_{r=1}^{m-1} B_r^{(m)} \right) \cup \left( \bigcup_{r=1}^{m-1} C_r^{(m)} \right) \cup \left( \bigcup_{r=1}^{m} D_r^{(m)} \right) \cup E,$$

where

$$B_r^{(m)} := \{a_1 \cdots a_{r-1} x_j a_{r+2} \cdots a_m | a_t \in N; t = 1, 2, \ldots, r - 1, r + 2, \ldots, m; i, j \notin J; i < j\},$$

$$C_r^{(m)} := \{a_1 \cdots a_{r-1} x_j a_{r+2} \cdots a_m | a_t \in N; t = 1, 2, \ldots, r - 1, r + 2, \ldots, m; i \notin J; i < j\},$$

$$D_r^{(m)} := \{a_1 \cdots a_{r-1} a_{r+2} \cdots a_m | a_t \in N; t = 1, 2, \ldots, r - 1, r + 2, \ldots, n; i \notin J\},$$

$$E := \{x_i^{m} | i \notin J\}.$$
$B_m$ is a basis of $A_m$; consider $F_sI_yF_h \leq F_m$ with $s, h \geq 0$, $g \geq 2$ and $s + g + h = m$; since $F_sI_yF_h$ is generated by $C_{s+1}^{(m)}, \ldots, C_{s+g-2}^{(m)}, D_{s+1}^{(m)}, \ldots, D_{s+g}^{(m)}$, then $F_sI_yF_h \cap B_m = \bigcup_{r=s+1}^{s+g} C_r^{(m)} \cup \bigcup_{r=s+1}^{s+g} D_r$, which is a basis of $F_sI_yF_h$.

**Example 4.8.** (i) The following algebras satisfy the conditions of the previous theorem, and hence, they are $SK$ (but not Koszul): The dispin algebra $U(osp(1, 2))$; the $q$-Heisenberg algebra; the quantum algebra $U'(sl(3, K))$; the Woronowicz algebra $W_n(sl(2, K))$; the algebra $S_h$ of shift operators; the algebra $D$ for multidimensional discrete linear systems; the algebra of linear partial shift operators.

(ii) The following algebras do not satisfy the conditions of the previous theorem, but by direct computation we proved that the lattice $L(B)$ is distributive, so they are $SK$ (but not Koszul): The algebra $V_q(sl_3(\mathbb{C}))$; the Witten’s deformation of $U(sl(2, K))$; the quantum symplectic space $O_q(sp(K^{2n}))$.

### 4.2 Poincaré series of skew PBW extensions

Now we compute the Poincaré series of skew PBW extensions of $K$ that are FSG algebras.

**Theorem 4.9.** Let $A = \sigma(K)\langle x_1, \ldots, x_n \rangle$ be a skew PBW extension of the field $K$ that is a Koszul algebra, then the Poincaré series of $A$ is well-defined and given by $P_A(t) = (1 + t)^n$.

**Proof.** Since $A$ is Koszul, then $h_A(t)P_A(-t) = 1$ and $E(A)$ is Koszul, whence $E(A)$ is finitely generated in degree 1 (see [8], [11], or [25]); therefore the theorem follows from Corollaries 3.9 and 3.10.

**Corollary 4.10.** Let $A = \sigma(K)\langle x_1, \ldots, x_n \rangle$ be a skew PBW extension of the field $K$ such that $A$ is a FSG algebra. Then $E(A)$ is Koszul and the Poincaré series of $A$ is well-defined and given by $P_A(t) = (1 + t)^n$.

**Proof.** As we observed in Example 2.4, $A$ is constant and pre-commutative, so $Gr(A)$ is a multi-parameter quantum affine $n$-space, whence $Gr(A)$ is Koszul. From Theorem 3.9, $E(A)$ is Koszul, so $E(A)$ is finitely generated in degree one, hence the corollary follows from the previous theorem and Corollary 3.10.

**Example 4.11.** From Remark 4.6, Example 4.8 and Corollary 4.10, we present next the Poincaré series of some skew PBW extensions of the base field $K$ that are $SK$ algebras and for which the Yoneda algebra is Koszul:

| $SK$ algebra                                                                 | $P_A(t)$         |
|------------------------------------------------------------------------------|------------------|
| Classical polynomial algebra $K[x_1, \ldots, x_n]$                           | $(1 + t)^n$      |
| Some universal enveloping algebras of a Lie algebras $U(\mathfrak{g})$       | $(1 + t)^n$      |
| Some Sridharan enveloping algebras of 3-dimensional Lie algebras              | $(1 + t)^n$      |
| Particular Sylvain algebra $S_h$                                              | $(1 + t)^n$      |
| Algebra of shift operators $S_h$                                              | $(1 + t)^n$      |
| Algebra of discrete linear systems $K[t_1, \ldots, t_n][x_1; \sigma_1] \cdots [x_n; \sigma_n]$ | $(1 + t)^{n+m}$ |
| Linear partial shift operators $K[t_1, \ldots, t_n][E_1, \ldots, E_m]$      | $(1 + t)^{n+m}$ |
| L. Partial $q$-dilation operators $K[t_1, \ldots, t_n][H_1^{(q)}, \ldots, H_m^{(q)}]$ | $(1 + t)^{n+m}$ |
| Multiplicative analogue of the Weyl algebra $O_n(\lambda_{11})$              | $(1 + t)^n$      |
| Quantum algebra $U'(sl(3, K))$                                                | $(1 + t)^n$      |
| Some 3-dimensional skew polynomial algebras                                  | $(1 + t)^n$      |
| Dispin algebra $U(osp(1, 2))$                                                | $(1 + t)^n$      |
| Woronowicz algebra $W_n(sl(2, K))$                                           | $(1 + t)^n$      |
| $q$-Heisenberg algebra $H_n(q)$                                              | $(1 + t)^n$      |
| Multi-parameter quantum affine $n$-space                                       | $(1 + t)^n$      |

Table 1: Poincaré series of some skew PBW extensions of $K$ that are $SK$ algebras.

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