The mKdV equation on a finite interval

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Abstract

We analyse an initial-boundary value problem for the mKdV equation on a finite interval by expressing the solution in terms of the solution of an associated matrix Riemann-Hilbert problem in the complex \( k \)-plane. This Riemann-Hilbert problem has explicit \((x,t)\)-dependence and it involves certain functions of \( k \) referred to as “spectral functions”. Some of these functions are defined in terms of the initial condition \( q(x,0) = q_0(x) \), while the remaining spectral functions are defined in terms of two sets of boundary values. We show that the spectral functions satisfy an algebraic “global relation” that characterize the boundary values in spectral terms.

1 Introduction

The general method for solving initial-boundary value problems for two-dimensional linear and integrable nonlinear PDEs announced in [2] and developed further in [2]–[4] is based on the simultaneous spectral analysis of the two eigenvalue equations of the associated Lax pair. It expresses the solution in terms of the solution of a matrix Riemann-Hilbert (RH) problem formulated in the complex plane of the spectral parameter. The spectral functions determining the RH problem are expressed in terms of the initial and boundary values of the solution. The fact that these values are in general related can be expressed in a simple way in terms of a global relation satisfied by the corresponding spectral functions.

The rigorous implementation of the method to the modified Korteweg–de Vries (mKdV) equation on the half-line is presented in [1]. In the present Note, this methodology is applied to the mKdV equation on a finite interval. The similar problem for the nonlinear Schrödinger equation is studied in [5].

The modified Korteweg–de Vries equation

\[
q_t - q_{xxx} + 6\lambda q^2 q_x = 0, \quad \lambda = \pm 1
\]

admits the Lax pair formulation

\[
\mu_x - ik\sigma_3 \mu = Q(x,t)\mu, \quad \mu_t + 4i k^3 \sigma_3 \mu = \tilde{Q}(x,t,k)\mu,
\]

where

\[
Q(x,t) = \frac{1}{2} \left( \begin{array}{cc}
q(x,t) & \lambda q(x,t)
\end{array} \right),
\tilde{Q}(x,t,k) = \frac{1}{2} \left( \begin{array}{cc}
q(x,t) & \lambda q(x,t)
\end{array} \right) e^{i k x}.
\]
where \( \sigma_3 = \text{diag}\{1, -1\} \), \( \bar{\sigma}_3 A := \sigma_3 A - A \sigma_3 \), \( e^{\bar{\sigma}_3 A} = e^{\sigma_3 A} A e^{-\sigma_3} \),
\[
Q(x,t) = \begin{pmatrix}
0 & q(x,t) \\
\lambda q(x,t) & 0
\end{pmatrix}, \quad \tilde{Q}(x,t,k) = -4k^2 Q - 2ik(Q^2 + Q_x)\sigma_3 - 2Q^3 + Q_{xx}.
\]

We study the initial-boundary value problem for the mKdV equation in the domain \( \{0 < x < L, 0 < t < T\} \), \( L < \infty, T \leq \infty \) using the following steps.

- Assuming that the solution \( q(x,t) \) of the mKdV equation exists, express it via the solution of a matrix Riemann-Hilbert problem. For this purpose:
  1. Define proper solutions of (2) sectionally analytic and bounded in \( k \in \mathbb{C} = \mathbb{C} \cup \{\infty\} \).
  2. Define spectral functions \( s(k) \), \( S(k) \), and \( S_1(k) \) such that:
    - They determine a Riemann-Hilbert problem.
    - \( s(k) \) is determined by the initial conditions \( q(x,0) = q_0(x), 0 < x < L \).
    - \( S(k) \) is determined by the boundary values \( q(0,t) = g_0(t), q_x(0,t) = g_1(t), q_{xx}(0,t) = g_2(t), 0 < t < T \).
    - \( S_1(k) \) is determined by the boundary values \( q(L,t) = f_0(t), q_x(L,t) = f_1(t), q_{xx}(L,t) = f_2(t), 0 < t < T \).
    - They satisfy an algebraic “global relation”, expressing the fact that \( q_0(x), \{g_j(t)\}_{j=0}^2, \{f_j(t)\}_{j=0}^2 \) being the initial and boundary conditions for the mKdV equation, cannot be chosen arbitrarily.

- Given \( s(k) \) and assuming that \( \{g_j(t)\}_{j=0}^2 \) and \( \{f_j(t)\}_{j=0}^2 \) are such that the associated \( S(k) \) and \( S_1(k) \) together with \( s(k) \) satisfy the global relation, prove that the solution of the Riemann-Hilbert problem constructed from \( s(k) \), \( S(k) \), and \( S_1(k) \) generates the solution of the initial-boundary value problem for the mKdV equation with initial data \( q(x,0) = q_0(x) \) and boundary values \( q(0,t) = g_0(t), q_x(0,t) = g_1(t), q_{xx}(0,t) = g_2(t), q(L,t) = f_0(t), q_x(L,t) = f_1(t), q_{xx}(L,t) = f_2(t) \).

2 Eigenfunctions and spectral functions

Assume that there exists a real-valued function \( q(x,t) \) with sufficient smoothness and decay satisfying (1) in \( \{0 < x < L, 0 < t < T\} \), \( T \leq \infty \). Define the eigenfunctions \( \mu_n(x,t,k) \), \( n = 1, 2, 3, 4 \) as matrix-valued solutions of the integral equations
\[
\mu_n(x,t,k) = I + \int_{(x_1,t_1)}^{(x,t)} e^{i(k(x-y) - 4k^3(t-\tau))\bar{\sigma}_3}(Q\mu_n dy + \tilde{Q}\mu_n d\tau),
\]
where \( (x_1,t_1) = (0,T), (x_2,t_2) = (0,0), (x_3,t_3) = (L,0), (x_4,t_4) = (L,T) \) and the paths of integration are chosen to be parallel to the \( x \) and \( t \) axes:
\[
\mu_1(x,t,k) = I + \int_0^x e^{ik(x-y)\bar{\sigma}_3}(Q\mu_1)(y,t,k) dy - e^{ikx\bar{\sigma}_3} \int_1^T e^{-4ik^3(t-\tau)\bar{\sigma}_3}(\tilde{Q}\mu_1)(0,\tau,k) d\tau,
\]
\[
\mu_4(x,t,k) = I - \int_x^L e^{ik(x-y)\bar{\sigma}_3}(Q\mu_4)(y,t,k) dy - e^{ik(x-L)\bar{\sigma}_3} \int_T e^{-4ik^3(t-\tau)\bar{\sigma}_3}(\tilde{Q}\mu_4)(L,\tau,k) d\tau.
\]
Equations for $\mu_2$ and $\mu_3$ are similar to those for $\mu_1$ and $\mu_4$, respectively, with the integral term $\int_0^t$ instead of $-\int_t^T$. The columns of $\mu_n = (\mu_n^{(1)} \mu_n^{(2)})$ are analytic and bounded in domains separated by the three lines $\{k \in \mathbb{C} \mid \text{Im } k^3 = 0\}$, see Figure 1:

$$\begin{array}{c}
\mu_1^{(1)}, \mu_3^{(2)} \text{ in IV } \cup \text{ VI; } \\
\mu_1^{(2)}, \mu_3^{(1)} \text{ in I } \cup \text{ III; } \\
\mu_2^{(1)}, \mu_4^{(2)} \text{ in V; } \\
\mu_2^{(2)}, \mu_4^{(1)} \text{ in II.}
\end{array}$$

Figure 1: Domains of boundedness of eigenfunctions

Thus, in each domain I, . . . , VI, one has a bounded $2 \times 2$ matrix-valued eigenfunction, consisting of the appropriate vectors $\mu_n^{(l)}$. The eigenfunctions $\mu_j$ are related by

$$\begin{align*}
\mu_3(x, t, k) &= \mu_2(x, t, k)e^{i(kx-4k^3t)\hat{\sigma}_3} s(k), \\
\mu_1(x, t, k) &= \mu_2(x, t, k)e^{i(kx-4k^3t)\hat{\sigma}_3} S(k), \\
\mu_4(x, t, k) &= \mu_3(x, t, k)e^{i(kx-4k^3t)\hat{\sigma}_3} e^{-ikL\hat{\sigma}_3} S_1(k),
\end{align*}$$

where the *spectral (matrix-valued) functions* are defined as follows:

$$\begin{align*}
s(k) &= \left( \begin{array}{c} a(k) \\ \lambda b(k) \end{array} \right) := \mu_3(0, 0, k), \\
S(k) &= \left( \begin{array}{c} A(k) \\ \lambda B(k) \end{array} \right) := \mu_1(0, 0, k), \\
S_1(k) &= \left( \begin{array}{c} A_1(k) \\ \lambda B_1(k) \end{array} \right) := \mu_4(L, 0, k).
\end{align*}$$

The direct and inverse spectral maps

$$\begin{align*}
\{g_0(x)\} &\leftrightarrow \{a(k), b(k)\}, \\
\{g_0(t), g_1(t), g_2(t)\} &\leftrightarrow \{A(k), B(k)\}, \\
\{f_0(t), f_1(t), f_2(t)\} &\leftrightarrow \{A_1(k), B_1(k)\}
\end{align*}$$

are well-defined [1]. They correspond to the separate spectral maps for the $x$-problem ($t = 0$) and $t$-problems ($x = 0$ and $x = L$) from the Lax pair [2].
3 Global relation

Evaluating equations (1) and (3) at \( x = 0, t = T \) and writing \( \mu_3(0, 0, k), \mu_2(0, T, k), \) and \( \mu_4(L, 0, k) \) in terms of \( s(k), S(k), \) and \( S_1(k) \), respectively, we obtain

\[
S^{-1}(k)s(k) \left[ e^{-ikL\delta_3}S_1(k) \right] = I - e^{4ikT\delta_3} \int_0^L e^{-iky\delta_3}(Q\mu_4)(y, T, k)dy. \tag{7}
\]

- For \( T < \infty \), the \((1, 2)\) coefficient of (7) is \((k \in \mathbb{C})\)

\[
e^{-2ikL} \left( a(k)A(k) - \lambda b(k)B(k) \right) B_1(k) - (a(k)B(k) - b(k)A(k)) A_1(k) = e^{8ikT}c(k), \tag{8}
\]

where \( c(k) = \int_0^L e^{-2iky}(Q\mu_4)_{12}(y, T, k)dy \) is an entire function which is \( O \left( (1 + e^{-2ikL})/k \right) \) as \( k \to \infty \).

- For \( T = \infty \), the \((1, 2)\) coefficient of (7) becomes

\[
e^{-2ikL} \left( a(k)A(k) - \lambda b(k)B(k) \right) B_1(k) - (a(k)B(k) - b(k)A(k)) A_1(k) = 0, \tag{9}
\]

which is valid for \( k \in I \cup III \cup V \).

Equation (8) for \( T < \infty \), or (9) for \( T = \infty \), is an algebraic relation between the spectral functions. We call it “global relation”, because it express, in spectral terms, the relations between the initial and boundary values of a solution of the mKdV equation. The global relation can be used to characterize the unknown boundary values in a well-posed boundary value problems, say, \( g_2(t), f_1(t), \) and \( f_2(t) \) in terms of the boundary conditions \( \{g_0(x), g_0(t), g_1(t), f_0(t)\} \).

4 The Riemann-Hilbert problem

Define a sectionally holomorphic, matrix-valued function \( M(x, t, k) \):

\[
M = \begin{cases}
\frac{\mu_3^{(1)}}{a(k)} \frac{\mu_2^{(2)}}{d_1(k)}, & k \in I \cup III, \ |k| > R \\
\frac{\mu_3^{(1)}}{a(k)} \frac{\mu_2^{(2)}}{d_1(k)}, & k \in II, \ |k| > R \\
\frac{\mu_3^{(1)}}{a(k)} \frac{\mu_2^{(2)}}{d_1(k)}, & k \in IV \cup VI, \ |k| > R \\
\frac{\mu_3^{(1)}}{a(k)} \frac{\mu_2^{(2)}}{d_1(k)}, & k \in V, \ |k| > R \\
\mu_2, & |k| < R,
\end{cases} \tag{10}
\]

where \( d(k) = a(k)A(k) - \lambda b(k)B(k), \ d_1(k) = a(k)A_1(k) + \lambda e^{-2ikL}b(k)B_1(k), \) and \( R \) is large enough so that all possible zeros of \( a(k), d(k), \) and \( d_1(k) \) in \( \text{Im} \ k \leq 0 \) are in the disk \( |k| < R \).

Denote by \( \Sigma \) the contour \( \{k \mid \text{Im} \ k^3 = 0\} \cup \{k \mid |k| = R\} \) (Figure 2).
Then the limit values $M_\pm(x, t, k)$ (as $k$ approaches $\Sigma$ from $\Omega_\pm$) of $M(x, t, k)$ are related on $\Sigma$ by a jump matrix:

$$M_-(x, t, k) = M_+(x, t, k) e^{(ikx - 4i k^3 t) \sigma_3} J_0(k) e^{-(ikx - 4i k^3 t) \sigma_3}, \quad k \in \Sigma,$$

where

$$J_0(k) = \begin{cases} 
\begin{pmatrix} 1 & -\lambda \Gamma(k) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda \Gamma_1(k) & 1 \end{pmatrix}, & \text{arg } k = \frac{\pi}{3}, \frac{2\pi}{3}; \quad |k| > R \\
\begin{pmatrix} 1 & -\lambda \Gamma(k) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - \lambda |\gamma(k)|^2 & \gamma(k) \\ -\lambda \bar{\gamma}(k) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \Gamma(k) & 1 \end{pmatrix}, & \text{arg } k = 0, \pi; \quad |k| > R \\
\begin{pmatrix} A(k) \bar{d}(k) & -B(k) \bar{d}(k) \\ -\lambda b(k) \bar{a}(k) & \lambda \Gamma_2(k) \bar{a}(k) \end{pmatrix}, & \text{arg } k \in \left(0, \frac{\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \pi\right); \quad |k| = R \\
\begin{pmatrix} 0 & 1 \\ \lambda \Gamma_2(k) & \bar{a}(k) \end{pmatrix}, & \text{arg } k \in \left(\frac{\pi}{3}, \frac{2\pi}{3}\right); \quad |k| = R 
\end{cases}$$

for $k \in \Sigma$, $\text{Im} \ k \geq 0$,

$J_0(k) = \text{diag}\{-1, \lambda\} J_0^*(\bar{k}) \text{diag}\{-1, \lambda\}$ for $k \in \Sigma$, $\text{Im} \ k < 0$,

$J_0(k) = I$ for $k \in \Sigma$, $|k| < R$.

Here

$$\gamma(k) = \frac{b(k)}{a(k)}$$

$$\Gamma(k) = \lambda \frac{B(k)/A(k)}{a(k) \left(a(k) - \lambda b(k) (B(k)/A(k))\right)}$$

$$\Gamma_1(k) = \frac{e^{-2ikL} a(k) (B_1(k)/A_1(k))}{a(k) + \lambda e^{-2ikL} b(k) (B_1(k)/A_1(k))}$$

$$\Gamma_2(k) = a(k) \frac{e^{-2ikL} a(k) (B_1(k)/A_1(k)) + b(k)}{a(k) + \lambda e^{-2ikL} b(k) (B_1(k)/A_1(k)).}$$
Therefore, the jump data in (11) are determined by \( a(k) \) and \( b(k) \) for \( k \in \mathbb{C}, |k| \geq R \) and by \( B(k)/A(k) \) and \( B_1(k)/A_1(k) \) for \( k \in \Pi \cup \Pi \cup \mathbb{V}, |k| \geq R \).

**Theorem 1.** Let \( q_0(x) \in S(\mathbb{R}^+) \). Suppose that the sets of functions \( \{q_j(t)\}_{j=0}^2 \) and \( \{f_j(t)\}_{j=0}^2 \) are such that the associated spectral functions \( s(k) \), \( S(k) \), and \( S_1(k) \) satisfy the global relation (8) for \( T < \infty \), or (4) for \( T = \infty \), where \( c(k) \) is an entire function such that \( c(k) = O\left((1 + e^{-2kL}/k)\right) \) as \( |k| \rightarrow \infty \).

Let \( M(x, t, k) \) be a solution of the following \( 2 \times 2 \) matrix RH problem:

- \( M \) is sectionally holomorphic in \( k \in \mathbb{C} \setminus \Sigma \).
- At \( k \in \Sigma \), \( M \) satisfies the jump conditions (11), where \( J_0 \) is defined in terms of the spectral functions \( a, b, A, B, A_1 \), and \( B_1 \) by eqs. (12), (13).
- \( M(x, t, k) = I + O\left(1/k\right) \) as \( k \rightarrow \infty \).

Then:

(i) \( M(x, t, k) \) exists and is unique;

(ii) \( q(x, t) := -2i \lim_{k \rightarrow \infty} (kM(x, t, k))_{12} \) satisfies the mKdV equation (11);

(iii) \( q(x, t) \) satisfies the initial condition \( q(x, 0) = q_0(x) \) and boundary conditions \( q(0, t) = g_0(t), q_x(0, t) = g_1(t), q_{xx}(0, t) = g_2(t), \) and \( q(L, t) = f_0(t), q_x(L, t) = f_1(t), q_{xx}(L, t) = f_2(t) \).

**Sketch of proof.** The unique solvability of the RH problem is a consequence of a “vanishing lemma” for the associated RH problem with vanishing condition at infinity \( M = O(1/k) \), \( k \rightarrow \infty \).

The proof that the function \( q(x, t) \) thus constructed solves the mKdV equation is straightforward and follows the proof in the case of the whole line problem.

The proof that \( q \) satisfies the initial condition \( q(x, 0) = q_0(x) \) follows from the fact that it is possible to map the RH problem for \( M(x, 0, k) \) to that for a sectionally holomorphic function \( M(x)(x, k) \) corresponding to the spectral problem for the \( x \)-part of the Lax pair (2): \( M(x)(x, k) = M(x, 0, k)P(x)(x, k) \) where \( P(x) \) is sectionally holomorphic and \( P(x) = I + P_{\text{off}} \), with \( P_{\text{off}}(x, k) \) off-diagonal and exponentially decaying as \( k \rightarrow \infty \) for \( \text{Im} k \neq 0 \).

The proof that \( q \) satisfies the boundary conditions is, in turn, based on the consideration of the maps \( M(0, t, k) \rightarrow M^{(t)}(t, k) \) and \( M(L, t, k) \rightarrow M_1^{(t)}(t, k) \), where \( M^{(t)}(t, k) \) and \( M_1^{(t)}(t, k) \) correspond to the spectral problems for the \( t \)-equation in the Lax pair (2) at \( x = 0 \) and \( x = L: M^{(t)}(t, k) = M(0, t, k)P^{(t)}(t, k), M_1^{(t)}(t, k) = M(L, t, k)P_1^{(t)}(t, k) \).

In this case, it is the global relation (8), or (4), that guarantees that \( P^{(t)} = P_{\text{diag}}^{(t)} + P_{\text{off}}^{(t)} \), where \( P_{\text{diag}}^{(t)} \) is diagonal, \( P_{\text{diag}}^{(t)} = I + O\left(1/k\right) \), and \( P_{\text{off}}^{(t)}(t, k) \) is off-diagonal and exponentially decaying as \( k \rightarrow \infty \), and similarly for \( P_1^{(t)}(t, k) \).
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