Some resolving parameters with the minimum size for two specific graphs

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**Abstract**

A resolving set for a graph \( G \) is a set of vertices \( Q = \{q_1, ..., q_k\} \) such that, for all \( p \in V(G) \) the \( k \)-tuple \( (d(p, q_1), ..., d(p, q_k)) \) uniquely determines \( p \), where \( d(p, q_i) \) is considered as the minimum length of a shortest path from \( p \) to \( q_i \) in graph \( G \). In this paper, we consider the computational study of some resolving sets with the minimum size for the \( m \)-cylinder graph \( (C_n \Box P_k) \Box P_m \). The Boolean lattice \( BL_n \), \( n \geq 1 \), is the graph whose vertex set is the set of all subsets of \( [n] = \{1, 2, ..., n\} \), where two subsets \( X \) and \( Y \) are adjacent if their symmetric difference has precisely one element. In the graph \( BL_n \), the layer \( L_i \) is the family of \( i \)-subsets of \( [n] \). The subgraph \( BL_n(i, i+1) \) is the subgraph of \( BL_n \) induced by layers \( L_i \) and \( L_{i+1} \). Usually the graph \( BL_n(1, 2) \) is denoted by \( H(n) \). We study the minimum size of a resolving set, doubly resolving set and strong resolving set for the graph \( L(n) \), which is the line graph of \( H(n) \).

**Keywords:** resolving set; doubly resolving set; strong resolving set; line graph.

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1 **Introduction**

All graphs considered in this paper are assumed to be finite and connected. For notation and terminology not defined here, see [12]. A graphical representation of a vertex \( p \) of a connected graph \( G \) relative to an arranged subset \( Q = \{q_1, ..., q_k\} \) of vertices of \( G \) is defined as the \( k \)-tuple \( (d(p, q_1), ..., d(p, q_k)) \), and this \( k \)-tuple is denoted by \( r(p|Q) \), where \( d(p, q_i) \) is considered as the minimum length of a shortest path from \( p \) to \( q_i \) in the graph \( G \). If any vertices \( p \) and \( q \) that belong to \( V(G) \setminus Q \) have various representations with respect to the set \( Q \), then \( Q \) is called a resolving set for \( G \) [6]. Slater [28] considered the concept and notation of the metric dimension problem under the term locating
neighbors that $u$ is maximally distant from a vertex $v$. In the graph $G$ of $G$ is called strong resolving set of $G$ see [7] from a strong resolving set of vertices in a graph $G$ of graphs have been obtained, such as Cayley digraphs of split metacyclic groups [1], Mobius ladders [4], regular bipartite graphs [5], lexicographic product of graphs [15], certain families of Toeplitz graphs [22], layer sun graph and line graph of the layer sun graph [23], and Crystal Cubic Carbon $CCC(n)$ [31].

In 2007 Cáceres et al. [7] considered the concept and notation of a doubly resolving set of vertices of graph $G$ as follows. For any two vertices $u$ and $v$ in a graph $G$, we say that $u$ and $v$ are doubly resolved by $x, y \in V(G)$, if $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$, and we can see that a subset $Q = \{q_1, \ldots, q_t\}$ of vertices of a graph $G$ is a doubly resolving set of $G$, if for any vertices $x$ and $y$ in $G$ we have $r(x|Q) - r(y|Q) \neq \lambda l$, where $\lambda$ is an integer, and $I$ denotes the unit $l$-vector $(1, \ldots, 1)$, see [2]. The size of the smallest doubly resolving set of vertices in a graph $G$, is denoted by $\psi(G)$. For more results about doubly resolving set, see [9, 16, 30].

The notion of a strong metric dimension problem set of vertices of graph $G$ was introduced by Sebő and Tannier [29], indeed introduced a more restricted invariant than the metric dimension and this was further investigated by Oellermann and Peters-Fransen [26]. A vertex $u$ of a graph $G$ is called maximally distant from a vertex $v$ of $G$, if for every $w \in N_G(u)$, we have $d(v, w) \leq d(v, u)$, where $N_G(u)$ denote the set of neighbors that $u$ has in $G$. If $u$ is maximally distant from $v$ and $v$ is maximally distant from $u$, then $u$ and $v$ are said to be mutually maximally distant. A set $Q \subseteq V(G)$ is called strong resolving set of $G$, if for any various vertices $p$ and $q$ of $G$ there is a vertex of $Q$, say $r$ so that $p$ belongs to a shortest $q - r$ path or $q$ belongs to a shortest $p - r$ path. A strong metric basis of $G$, denoted by $sdim(G)$, is the size of the smallest strong resolving set of vertices in a graph $G$. For more results about this parameter, see [20, 21, 27].

The written papers about the resolving parameters for graphs, show that these parameters have a very important potential to solve a number of representative real life problems, which have been described in several works. For instance, they have been frequently used in graph theory, chemistry, coding theory, robotics and many other disciplines. Note that these problems are NP-hard, see [8, 10, 17, 19].

The Cartesian product of two graphs $G$ and $H$, is denoted by $G \square H$, is a graph with vertex set $V(G) \times V(H)$ and with edge set $E(G \times H)$ so that $(g_1, h_1)(g_2, h_2) \in E(G \square H)$, whenever $h_1 = h_2$ and $g_1, g_2 \in E(G)$, or $g_1 = g_2$ and $h_1, h_2 \in E(H)$. We denote by $C_n$ and $P_k$ the cycle on $n \geq 3$ and the path on $k \geq 3$ vertices, respectively. Also, we use $C_n \square P_k$ and $(C_n \square P_k) \square P_m$ to denote the cylinder graph and the $m$-cylinder graph, respectively.

The Boolean lattice $BL_n$, $n \geq 1$, is the graph whose vertex set is the set of all subsets of $[n] = \{1, 2, \ldots, n\}$, where two subsets $X$ and $Y$ are adjacent if their symmetric difference has precisely one element. In the graph $BL_n$, the layer $L_i$ is the family of $i$-subsets of $[n]$. The subgraph $BL_n(i, i + 1)$ is the subgraph of $BL_n$ induced by layers $L_i$ and $L_{i+1}$. Usually the graph $BL_n(1, 2)$ is denoted by $H(n)$. We study the minimum
size of a resolving set, doubly resolving set and strong resolving set for the graph $L(n)$, which is the line graph of $H(n)$ (see [24]).

In this paper, we consider the computational study of some resolving sets with the minimum size for the $m$-cylinder graph $(C_n \square P_k) \square P_m$ and the graph $L(n)$.

2 Results for the $m$-cylinder graph $(C_n \square P_k) \square P_m$

In this section, we consider a class of the form $(C_n \square P_k) \square P_m$, which we call it $m$-cylinder graph and we compute some resolving sets for the $m$-cylinder graph $(C_n \square P_k) \square P_m$. In fact, some resolving parameters of the cylinder graph $C_n \square P_k$ have been calculated. Detailed information and further references concerning the resolving parameters of the cylinder graph $C_n \square P_k$ can be found in [7, 18, 25, 27]. We compute some resolving parameters of the cylinder graph $C_n \square P_k$ with another approach. Also, we study some resolving sets for the $m$-cylinder graph $(C_n \square P_k) \square P_m$. For more results of families of graphs with constant metric, see [3, 14].

Observation 2.1 [7] For any connected graph $G$ and the path $P_n$ with $n$ vertices, $\beta(G \square P_n) \leq \beta(G) + 1$.

In the following, we construct a class of graphs so that this class is isomorphic to $(C_n \square P_k)$, and call this graph cylinder graph.

Definition 2.2 Let $n$ and $k$ be fixed positive integers, with $n, k \geq 3$ and $[n] = \{1, \ldots, n\}$. Suppose that $G$ is a graph with vertex set $\{x_1, \ldots, x_{nk}\}$ on the layers $V_1, V_2, \ldots, V_k$, where $V_p = \{x_{(p-1)n+1}, x_{(p-1)n+2}, \ldots, x_{(p-1)n+n}\}$ for $p \in \{1, \ldots, k\}$, and the edge set of graph $G$ is

$$E(G) = \{x_ix_j \mid x_i, x_j \in V_p, 1 \leq i < j \leq nk, j - i = 1 \text{ or } j - i = n - 1\} \cup \{x_ix_j \mid x_i \in V_q, x_j \in V_{q+1}, 1 \leq i < j \leq nk, 1 \leq q \leq k - 1, j - i = n\}.$$ 

We can see that this graph is isomorphic to the cylinder graph $C_n \square P_k$. So, we can assume throughout this article $V(C_n \square P_k) = \{x_1, \ldots, x_{nk}\}$, and we use $V_p$ to denote a layer of cylinder graph $C_n \square P_k$, where $V_p$, is defined already.

Now, we give a more elaborate description of cylinder graph $C_n \square P_k$, that are required to prove Theorems.

Remark 2.3 Let $e$ and $d$ be positive integers with $1 \leq e < d \leq nk$. We say that two vertices $x_e$ and $x_d$ in $C_n \square P_k$ are compatible, if $n|d - e$.

In the following, we construct a class of graphs so that this class is isomorphic to $(C_n \square P_k) \square P_m$, and call this graph $m$-cylinder graph.
Definition 2.4 Let $m \geq 2$ be an integer. Suppose $1 \leq i \leq m$ and consider $i$th copy of cylinder graph $C_n \square P_k$ with the vertex set $\{x_1^{(i)},...,x_{nk}^{(i)}\}$ on the layers $V_1^{(i)}, V_2^{(i)},...,V_k^{(i)}$, where it can be defined $V_p^{(i)}$ as similar $V_p$ on the vertex set $\{x_1^{(i)},...,x_{nk}^{(i)}\}$. Also, suppose that $K$ is a graph with vertex set $\{x_1^{(1)},...,x_{nk}^{(1)}\} \cup \cdots \cup \{x_1^{(m)},...,x_{nk}^{(m)}\}$ so that the vertex $x_i^{(r)}$ is adjacent to the vertex $x_i^{(r+1)}$ in $K$, for $1 \leq t \leq m$, and $1 \leq r \leq m-1$. We can see that the graph $K$ is isomorphic to $(C_n \square P_k) \square P_m$, and we call this graph $m$-cylinder graph.

Remark 2.5 Let $e$ and $d$ be positive integers with $1 \leq e < d \leq nk$. We say that two vertices $x_{e}^{(i)}$ and $x_{d}^{(i)}$ in $i$th copy of $C_n \square P_k$ are compatible, if $n|d - e$.

Theorem 2.6 If $n \geq 3$ is an odd integer, then the minimum size of a doubly resolving set of vertices for the cylinder graph $C_n \square P_k$ is 3.

Proof. We first consider the cylinder graph $C_n \square P_k$ with the vertex set $\{x_1,...,x_{nk}\}$ on the layers $V_1, V_2,...,V_k$. Based on [7], if $n$ is an odd integer, then the minimum size of a resolving set in $C_n \square P_k$ is 2. In particular, we can show that if $n$ is an odd integer, then all the elements of every minimum resolving set of vertices in $C_n \square P_k$ must lie in exactly one of the layers $V_1$ or $V_k$. Without lack of theory if we consider the layer $V_1$ of $C_n \square P_k$, then we can show that all the minimum resolving sets of vertices in the layer $V_1$ of $C_n \square P_k$ are the sets in the forms $M_i = \{x_i, x_{\lceil \frac{n}{2} \rceil +i-1}\}$, $1 \leq i \leq \lceil \frac{n}{2} \rceil$ and $N_j = \{x_j, x_{\lceil \frac{n}{2} \rceil +j}\}$, $1 \leq j \leq \lceil \frac{n}{2} \rceil$. On the other hand, the arranged subsets $M_i$ and $N_j$ of vertices in $C_n \square P_k$ cannot be doubly resolving sets for the cylinder graph $C_n \square P_k$, because for $1 \leq i \leq \lceil \frac{n}{2} \rceil$ and two compatible vertices $x_{i+n}$ and $x_{i+2n}$ with respect to $x_i$, we have $r(x_{i+n}|M_i) - r(x_{i+2n}|M_i) = -I$, where $I$ denotes the unit 2-vector $(1,1)$. It is also clear that for $1 \leq j \leq \lceil \frac{n}{2} \rceil$ the arranged subsets $N_j$ of vertices in $C_n \square P_k$ cannot be doubly resolving sets for the cylinder graph $C_n \square P_k$, and so the minimum size of a doubly resolving set in $C_n \square P_k$ is greater than 2. Now, we claim that the minimum size of a doubly resolving set of vertices in $C_n \square P_k$ is 3. So it is enough to find a doubly resolving set of vertices in $C_n \square P_k$ with the size 3. For each $1 \leq i \leq \lceil \frac{n}{2} \rceil$, if $A_i = M_i \cup x_c = \{x_i, x_{\lceil \frac{n}{2} \rceil +i-1}, x_c\}$ is an arranged subset of vertices in $C_n \square P_k$, where $x_c$ lies in the layer $V_k$ of $C_n \square P_k$ and $x_c$ is a compatible vertex with respect to $x_i$, then $A_i$ is a doubly resolving set of vertices in $C_n \square P_k$, because it is sufficient to show that for any compatible vertices $x_c$ and $x_d$ in $C_n \square P_k$, $r(x_c|A_i) - r(x_d|A_i) \neq \lambda I$. Suppose $x_c \in V_p$ and $x_d \in V_q$ are compatible vertices in $C_n \square P_k$, $1 \leq p < q \leq k$. So, $r(x_c|M_i) - r(x_d|M_i) = -\lambda I$, where $\lambda$ is a positive integer, and $I$ indicates the unit 2-vector $(1,1)$. We also see that for the compatible vertex $x_c$ with respect to $x_i$, $r(x_c|x_c) - r(x_d|x_c) = \lambda$. Therefore, $r(x_c|A_i) - r(x_d|A_i) \neq \lambda I$, where $I$ indicates the unit 3-vector $(1,1,1)$, and the claim is proved.

Theorem 2.7 If $n \geq 3$ is an odd integer, then the minimum size of a doubly resolving set of vertices for the $m$-cylinder graph $(C_n \square P_k) \square P_m$ is 4.
Proof. Suppose the \( m \)-cylinder graph \((C_n \square P_k) \square P_m\) is a graph with vertex set \( \{x_1^{(1)}, \ldots, x_{nk}^{(1)}\} \cup \ldots \cup \{x_1^{(m)}, \ldots, x_{nk}^{(m)}\} \) so that the vertex \( x_t^{(r)} \) is adjacent to \( x_{t+1}^{(r+1)} \) in \((C_n \square P_k) \square P_m\), for \( 1 \leq t \leq nk \), and \( 1 \leq r \leq m - 1 \). We also see that for any connected graph \( G \) and the path \( P_m \), \( \beta(G \square P_m) \leq \beta(G) + 1 \). So, by considering \( G = (C_n \square P_k) \) we have \( \beta(G \square P_m) \leq \beta(G) + 1 = 3 \). Moreover, it is not hard to see that for \( 1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil \), and \( 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor \), \( i \)th copies of the arranged sets \( A_i = \{x_i, x_{i+1}^{(r)}, x_c\} \) and \( B_j = \{x_j, x_{j+1}^{(r)}, x_c\} \), denoted by the sets \( A_i^{(1)} = \{x_i^{(1)}, x_{i+1}^{(1)}, x_c^{(1)}\} \) and \( B_j^{(1)} = \{x_j^{(1)}, x_{j+1}^{(1)}, x_c^{(1)}\} \), respectively, are resolving sets with the minimum size for the \( m \)-cylinder graph \((C_n \square P_k) \square P_m\). On the other hand for \( 1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil \) the arranged sets \( A_i^{(1)} = \{x_i^{(1)}, x_{i+1}^{(1)}, x_c^{(1)}\} \) of vertices \((C_n \square P_k) \square P_m\), cannot be doubly resolving sets for the \( m \)-cylinder graph \((C_n \square P_k) \square P_m\), because for each vertex \( x_t^{(m)} \) of \((C_n \square P_k) \square P_m\), we have
\[
r(x_t^{(m)}|A_i^{(1)}) = (d(x_t^{(1)}, x_{i+1}^{(1)}) + (m - 1), d(x_t^{(1)}, x_{i+1}^{(1)}) + (m - 1), d(x_t^{(1)}, x_c^{(1)}) + (m - 1)).
\]

With the same way for \( 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor \), by applying the same argument we can show that the arranged sets \( B_j^{(1)} = \{x_j^{(1)}, x_{j+1}^{(1)}, x_c^{(1)}\} \) of vertices \((C_n \square P_k) \square P_m\) cannot be doubly resolving sets for the \( m \)-cylinder graph \((C_n \square P_k) \square P_m\). So, the minimum size of a doubly resolving set of vertices in \((C_n \square P_k) \square P_m\) is greater than 3. Now, let \( D_i = A_i^{(1)} \cup x_c^{(m)} = \{x_i^{(1)}, x_{i+1}^{(1)}, x_c^{(1)}, x_c^{(m)}\} \) be an arranged subset of vertices of \((C_n \square P_k) \square P_m\), where \( x_c^{(m)} \) lies in the layer \( V_k^{(m)} \) of \((C_n \square P_k) \square P_m\). We show that the arranged subset \( D_i \) is a doubly resolving set of vertices in \((C_n \square P_k) \square P_m\). It is enough to show that for every two vertices \( x_t^{(r)} \) and \( x_s^{(s)} \), \( 1 \leq t \leq nk \), \( 1 \leq r < s \leq m \), \( r(x_t^{(r)}|D_i) - r(x_s^{(s)}|D_i) \neq -4I \), where \( I \) indicates the unit 4-vector \((1, \ldots, 1)\) and \( \lambda \) is a positive integer. For this purpose, let the distance between two vertices \( x_t^{(r)} \) and \( x_s^{(s)} \) in \((C_n \square P_k) \square P_m\) is \( \lambda \), then we can verify that, \( r(x_t^{(r)}|A_i^{(1)}) - r(x_s^{(s)}|A_i^{(1)}) = \lambda I \), where \( I \) indicates the unit 3-vector, and \( r(x_t^{(r)}|x_c^{(m)}) - r(x_s^{(s)}|x_c^{(m)}) = \lambda \). Thus the minimum size of a doubly resolving set of vertices in \((C_n \square P_k) \square P_m\) is 4.

Lemma 2.8 If \( n \geq 4 \) is an even integer, then the minimum size of a doubly resolving set of vertices for the cylinder graph \( C_n \square P_k \) is 4.

Proof. We first consider the cylinder graph \( C_n \square P_k \) with the vertex set \( \{x_1, \ldots, x_{nk}\} \) on the layers \( V_1, V_2, \ldots, V_k \), is defined already. Based on [7], if \( n \) is an even integer, then the minimum size of a resolving set in \( C_n \square P_k \) is 3. In particular, using a similar argument in the proof of Theorem 2.6, we can prove that any subset of vertices in \( C_n \square P_k \) of size 3 cannot be a doubly resolving set for the cylinder graph \( C_n \square P_k \), and so the minimum size of a doubly resolving set of vertices for the cylinder graph \( C_n \square P_k \) is greater than 3. Now, we claim that if \( n \) is an even integer, then the minimum size of a doubly resolving set of vertices in \( C_n \square P_k \) is 4. For this purpose, let \( E_1 \) be an arranged subset of vertices of \( C_n \square P_k \) so that \( E_1 \) is a minimal resolving set for
Suppose the vertices in \( C_n \square P_k \) and all the elements of \( E_1 \) lie in exactly one of the layers \( V_1 \) or \( V_k \). Without loss of generality if we consider \( E_1 = \{x_1, x_2, x_{n+1}\} \), then we show that the arranged subset \( E_2 = E_1 \cup x_c = \{x_1, x_2, x_{n+1}, x_c\} \) of vertices in \( C_n \square P_k \), where \( x_c \) lies in the layer \( V_k \) of \( C_n \square P_k \) and \( x_c \) is a compatible vertex with respect to \( x_1 \), is one of the minimum doubly resolving sets for the cylinder graph \( C_n \square P_k \). It is enough to show that for any compatible vertices \( x_e \) and \( x_d \) in \( C_n \square P_k \), \( r(x_e|E_2) - r(x_d|E_2) \neq \lambda I \). Suppose \( x_e \in V_p \) and \( x_d \in V_q \) are compatible vertices in \( C_n \square P_k \), \( 1 \leq p < q \leq k \). Hence, \( r(x_e|E_1) - r(x_d|E_1) = \lambda \), where \( \lambda \) is a positive integer, and \( I \) indicates the unit 3-vector \((1,1,1)\). We also see that for the compatible vertex \( x_e \) with respect to \( x_1 \), \( r(x_e|x_c) - r(x_d|x_c) = \lambda \). So, \( r(x_e|E_2) - r(x_d|E_2) \neq \lambda I \), where \( I \) indicates the unit 4-vector \((1,...,1)\), and the claim is proved.

**Theorem 2.9** If \( n \geq 4 \) is an even integer, then the minimum size of a resolving set of vertices for the \( m \)-cylinder graph \( (C_n \square P_k) \square P_m \) is 4.

**Proof.** Suppose the \( m \)-cylinder graph \( (C_n \square P_k) \square P_m \) is a graph with vertex set \( \{x_{1}^{(1)},...,x_{n}^{(1)}\} \cup \cup \{x_{1}^{(m)},...,x_{n}^{(m)}\} \) so that the vertex \( x_t^{(r)} \) is adjacent to \( x_t^{(r+1)} \) in \( (C_n \square P_k) \square P_m \), for \( 1 \leq t \leq nk \) and \( 1 \leq r \leq m - 1 \). Therefore, none of the minimal resolving sets of \( C_n \square P_k \) cannot be a resolving set for the \( m \)-cylinder graph \( (C_n \square P_k) \square P_m \). So, the minimum size of a resolving set of vertices in \( (C_n \square P_k) \square P_m \) is greater than 3.

Now, we show that the minimum size of a resolving set of vertices in \( (C_n \square P_k) \square P_m \) is 4. Let \( x_1^{(1)} \) be a vertex in the layer \( V_1^{(1)} \) of \( (C_n \square P_k) \square P_m \) and \( x_{c}^{(1)} \) be a compatible vertex with respect to \( x_1^{(1)} \), where \( x_c^{(1)} \) lies in the layer \( V_k^{(1)} \) of \( (C_n \square P_k) \square P_m \). Based on Lemma 2.8, we know that \( 1^{th} \) copy of the arranged subset \( E_2 = \{x_1, x_2, x_{n+1}, x_c\} \), denoted by the set \( E_2^{(1)} = \{x_1^{(1)}, x_2^{(1)}, x_{n+1}^{(1)}, x_c^{(1)}\} \) is one of the minimum doubly resolving sets for the cylinder graph \( C_n \square P_k \). Besides, the vertex \( x_t^{(r)} \) is adjacent to the vertex \( x_t^{(r+1)} \) in \( (C_n \square P_k) \square P_m \), and hence the arranged set \( E_2^{(1)} \) is one of the resolving sets for the \( m \)-cylinder graph \( (C_n \square P_k) \square P_m \). Because for each vertex \( x_t^{(i)} \) of \( (C_n \square P_k) \square P_m \), we have
\[
r(x_t^{(i)}|E_2^{(1)}) = (d(x_t^{(i)}, x_1^{(1)}) + i - 1, d(x_t^{(i)}, x_2^{(1)}) + i - 1, d(x_t^{(i)}, x_{n+1}^{(1)}) + i - 1, d(x_t^{(i)}, x_c^{(1)}) + i - 1),
\]
so all the vertices \( \{x_1^{(1)},...,x_{n}^{(1)}\} \cup \cup \{x_1^{(m)},...,x_{n}^{(m)}\} \) of \( (C_n \square P_k) \square P_m \) have various representations with respect to the set \( E_2^{(1)} \). Thus the minimum size of a resolving set of vertices in \( (C_n \square P_k) \square P_m \) is 4.

**Theorem 2.10** If \( n \geq 4 \) is an even integer, then the minimum size of a doubly resolving set of vertices for the \( m \)-cylinder graph \( (C_n \square P_k) \square P_m \) is 5.

**Proof.** Suppose the \( m \)-cylinder graph \( (C_n \square P_k) \square P_m \) is a graph with vertex set \( \{x_1^{(1)},...,x_{n}^{(1)}\} \cup \cup \{x_1^{(m)},...,x_{n}^{(m)}\} \) so that the vertex \( x_t^{(r)} \) is adjacent to \( x_t^{(r+1)} \) in \( (C_n \square P_k) \square P_m \), for \( 1 \leq t \leq nk \) and \( 1 \leq r \leq m - 1 \). By the proof of Theorem 2.9, we know that the arranged set \( E_2^{(1)} = \{x_1^{(1)}, x_2^{(1)}, x_{n+1}^{(1)}, x_c^{(1)}\} \) of vertices of \( (C_n \square P_k) \square P_m \) is one of the resolving sets for the \( m \)-cylinder graph \( (C_n \square P_k) \square P_m \), so that the arranged
set $E_2^{(1)}$ cannot be a doubly resolving set for the $m$-cylinder graph $(C_n \square P_k) \square P_m$, and hence the minimum size of a doubly resolving set of vertices in $(C_n \square P_k) \square P_m$ is greater than 4. Now, let $E_3 = E_2^{(1)} \cup x_c^{(m)} = \{x_e^{(1)}, \frac{x_e^{(1)}}{2}, \frac{x_e^{(1)}}{2} + 1, x_c^{(1)}, x_c^{(m)}\}$ be an arranged subset of vertices of $(C_n \square P_k) \square P_m$, where $x_c^{(m)}$ lies in the layer $V_k^{(m)}$ of $(C_n \square P_k) \square P_m$. It is enough to show that for every two vertices $x_t^{(r)}$ and $x_t^{(s)}$, $1 \leq t \leq nk$, $1 \leq r < s \leq m$, $r(x_t^{(r)}|E_3) - r(x_t^{(s)}|E_3) \neq -\lambda I$, where $I$ indicates the unit 5-vector $(1, \ldots, 1)$ and $\lambda$ is a positive integer. For this purpose, let the distance between two the vertices $x_t^{(r)}$ and $x_t^{(s)}$ in $(C_n \square P_k) \square P_m$ is $\lambda$. Then we can verify that, $r(x_t^{(r)}|E_3^{(1)}) - r(x_t^{(s)}|E_3^{(1)}) = -\lambda I$, where $I$ indicates the unit 4-vector, and $r(x_t^{(r)}|x_c^{(m)}) - r(x_t^{(s)}|x_c^{(m)}) = \lambda$. Therefore, the arranged subset $E_3$ is one of the minimum doubly resolving sets of vertices for the $m$-cylinder graph $(C_n \square P_k) \square P_m$. Thus the minimum size of a doubly resolving set of vertices in $(C_n \square P_k) \square P_m$ is 5.

**Theorem 2.11** If $n \geq 3$ is an integer, then the minimum size of a strong resolving set of vertices for the $m$-cylinder graph $(C_n \square P_k) \square P_m$ is $2n$.

**Proof.** Suppose the $m$-cylinder graph $(C_n \square P_k) \square P_m$ is a graph with vertex set $\{x_1^{(1)}, \ldots, x_{nk}^{(1)}\} \cup \ldots \cup \{x_1^{(m)}, \ldots, x_{nk}^{(m)}\}$ so that the vertex $x_t^{(r)}$ is adjacent to $x_t^{(r+1)}$ in $(C_n \square P_k) \square P_m$, for $1 \leq t \leq nk$, and $1 \leq r \leq m - 1$. We know that each vertex of the layer $V_1^{(1)}$ is maximally distant from a vertex of the layer $V_k^{(m)}$ and each vertex of the layer $V_k^{(m)}$ is maximally distant from a vertex of the layer $V_1^{(1)}$. In particular, each vertex of the layer $V_1^{(1)}$ is maximally distant from a vertex of the layer $V_k^{(m)}$ and each vertex of the layer $V_k^{(1)}$ is maximally distant from a vertex of the layer $V_1^{(m)}$, and so the minimum size of a strong resolving set of vertices for the $m$-cylinder graph $(C_n \square P_k) \square P_m$ is greater than or equal to $2n$. Because it is well known that for every pair of mutually maximally distant vertices $u$ and $v$ of a connected graph $G$ and for every strong metric basis $S$ of $G$, it follows that $u \in S$ or $v \in S$. Suppose the set $\{x_1^{(1)}, \ldots, x_{nk}^{(1)}\}$ is an arranged subset of vertices in the layer $V_1^{(1)}$ of $(C_n \square P_k) \square P_m$ and suppose that the set $\{x_1^{(m)}, \ldots, x_{nk}^{(m)}\}$ is an arranged subset of vertices in the layer $V_1^{(m)}$ of $(C_n \square P_k) \square P_m$. Now, let $T = \{x_1^{(1)}, \ldots, x_{nk}^{(1)}\} \cup \{x_1^{(m)}, \ldots, x_{nk}^{(m)}\}$ be an arranged subset of vertices of $(C_n \square P_k) \square P_m$. In the following cases we show that the arranged set $T$, is one of the minimum strong resolving sets of vertices for the $m$-cylinder graph $(C_n \square P_k) \square P_m$. For this purpose let $x_e^{(i)}$ and $x_d^{(j)}$ be two various vertices of $(C_n \square P_k) \square P_m$, $1 \leq i, j \leq m$, $1 \leq e, d \leq nk$ and $1 \leq r \leq n$.

Case 1. If $i = j$, then $x_e^{(i)}$ and $x_d^{(i)}$ lie in $i^{th}$ copy of $(C_n \square P_k)$ with vertex set $\{x_e^{(i)}, \ldots, x_e^{(i)}\}$ so that $i^{th}$ copy of $(C_n \square P_k)$ is a subgraph of $(C_n \square P_k) \square P_m$. Since $i = j$, then we can assume that $e < d$, because $x_e^{(i)}$ and $x_d^{(i)}$ are various vertices.

Case 1.1. If both vertices $x_e^{(i)}$ and $x_d^{(i)}$ are compatible in $i^{th}$ copy of $(C_n \square P_k)$ relative to $x_r^{(i)} \in V_t^{(i)}$, then there is the vertex $x_r^{(i)} \in V_t^{(i)} \subset T$ so that $x_e^{(i)}$ belongs to a shortest
Case 1.2. Suppose both vertices $x^{(i)}_e$ and $x^{(i)}_d$ are not compatible in $i^{th}$ copy of $(C_n \square P_k)$, and lie in various layers or lie in the same layer in $i^{th}$ copy of $(C_n \square P_k)$, also let $x^{(i)}_r \in V^{(i)}_1$, be a compatible vertex relative to $x^{(i)}_e$. Hence there is the vertex $x^{(m)}_r \in V^{(m)}_1 \subset T$ so that $x^{(i)}_e$ belongs to a shortest $x^{(m)}_r - x^{(i)}_d$ path, say as $x^{(m)}_r, ..., x^{(i)}_r, ..., x^{(i)}_e, ..., x^{(i)}_d$.

Case 2. If $i \neq j$, then $x^{(i)}_e$ lies in $i^{th}$ copy of $(C_n \square P_k)$ with vertex set $\{x^{(i)}_1, ..., x^{(i)}_{n_k}\}$, and $x^{(j)}_d$ lies in $j^{th}$ copy of $(C_n \square P_k)$ with vertex set $\{x^{(j)}_1, ..., x^{(j)}_{n_k}\}$. In this case we can assume that $i < j$.

Case 2.1. If $e = d$ and $x^{(j)}_r \in V^{(j)}_1$ is a compatible vertex relative to $x^{(j)}_d$, then there is the vertex $x^{(m)}_r \in V^{(m)}_1 \subset T$ so that $x^{(j)}_d$ belongs to a shortest $x^{(m)}_r - x^{(i)}_e$ path, say as $x^{(m)}_r, ..., x^{(j)}_r, ..., x^{(i)}_e, ..., x^{(i)}_d$.

Case 2.2. If $e < d$, also $x^{(i)}_e$ and $x^{(j)}_d$ lie in various layers of $(C_n \square P_k) \square P_m$ or $x^{(i)}_e$ and $x^{(j)}_d$ lie in the same layer of $(C_n \square P_k) \square P_m$ and $x^{(i)}_r \in V^{(i)}_1$ is a compatible vertex relative to $x^{(i)}_e$, then there is the vertex $x^{(1)}_r \in V^{(1)}_1 \subset T$ so that $x^{(i)}_e$ belongs to a shortest $x^{(1)}_r - x^{(j)}_d$ path, say as $x^{(1)}_r, ..., x^{(i)}_r, ..., x^{(i)}_e, ..., x^{(i)}_d$.

Case 2.3. If $e > d$, also $x^{(i)}_e$ and $x^{(j)}_d$ lie in various layers of $(C_n \square P_k) \square P_m$ or $x^{(i)}_e$ and $x^{(j)}_d$ lie in the same layer of $(C_n \square P_k) \square P_m$ and $x^{(j)}_r \in V^{(j)}_1$ is a compatible vertex relative to $x^{(j)}_d$, then there is the vertex $x^{(m)}_r \in V^{(m)}_1 \subset T$ so that $x^{(j)}_d$ belongs to a shortest $x^{(m)}_r - x^{(i)}_e$ path, say as $x^{(m)}_r, ..., x^{(j)}_r, ..., x^{(j)}_d, ..., x^{(i)}_e$.

3 Results for the graph $L(n)$

Let $n$ be a fixed positive integer, with $n \geq 5$ and $[n] = \{1, ..., n\}$ and let $i, j \in [n], i \neq j, 1 \leq i \leq n-1, 2 \leq j \leq n$. Suppose that $G$ is a graph with vertex set $W_1 \cup ... \cup W_n$, where $W_r = \{v_r, v_i v_j\}$, for $1 \leq r \leq n$. We say that two various vertices $\{v_r, v_i v_j\}$ and $\{v_k, v_p v_q\}$ are adjacent in $G$ if and only if $v_r = v_k$ or $v_i v_j = v_p v_q$, that is $i = p$ and $j = q$. It is not hard to see that this family of graphs is isomorphic to the the graph $L(n)$, is defined in [24]. Based on [24] we can see that $L(n)$ is a connected vertex transitive graph of valency $n - 1$, with diameter 3, and the order $n(n - 1)$. It is easy to see that every $W_r$ is a clique of size $n - 1$ in the graph $L(n)$. We also undertake the necessary task of introducing some of the basic notation for this class of graphs. We say that two cliques $W_r$ and $W_k$ are adjacent in $L(n)$, if there is a vertex in clique $W_r$ so that this vertex is adjacent to exactly one vertex of clique $W_k$, $r, k \in [n], r \neq k$. Also, for any clique $W_r$ in $G = L(n)$ we use $N(W_r) = \bigcup_{w \in W_r} N_G(w)$ to denote the vertices in the all cliques $W_k$, say $w_k, 1 \leq k \leq n$ and $k \neq r$ so that $w_k$ is adjacent to
Theorem 3.2 If \( L \) set in graph \( C \) let \( v \) that \( d \) Suppose that \( \) Proof. 

\[ \{ v \} \to \text{the vertex } L \text{ of vertices of } W \text{ adjacent to a vertex of each clique } W \] 

resolving sets with the minimum size for the graph \( L(n) \).

Lemma 3.1 Let \( n \) be a fixed positive integer, with \( n \geq 5 \). If \( 1 \leq r \leq n \), then each subset of \( N(W_r) \) of size at least \( n - 2 \) can be a resolving set for \( L(n) \).

Proof. Suppose that \( V(L(n)) = W_1 \cup \ldots \cup W_n \), where the set \( W_r = \{ v_r, v_i \} \) denote a clique of size \( n - 1 \) in the graph \( L(n) \). We know that \( N(W_r) \) indicate the vertices in the all cliques \( W_k \), say \( w_k \), \( 1 \leq k \leq n \) and \( k \neq r \) so that \( w_k \) is adjacent to one vertex of the clique \( W_r \), also we can see that the cardinality of \( N(W_r) \) is \( n - 1 \). Since \( L(n) \) is a vertex transitive graph, then without loss of generality we may consider the clique \( W_1 \). So \( N(W_1) = \{ y_2, \ldots, y_{n-1}, y_n \} \), where \( y_k = \{ v_{k1}, v_{k2} \} \in W_k \).

If we consider \( C_1 = N(W_1) - \{ y_{n-1}, y_n \} = \{ y_2, \ldots, y_{n-2} \} \), then there are exactly two vertices \( \{ v_1, v_{1n-1} \}, \{ v_1, v_{1n-1} \} \in W_1 \) so that \( r(\{ v_1, v_{1n-1} \}|C_1) = r(\{ v_1, v_{1n} \}|C_1) = (2, \ldots, 2) \), and so we can verify that the set \( C_1 \) cannot be a resolving set for \( L(n) \), and so any subset of \( N(W_r) \) of size \( n - 3 \) cannot be a resolving set for \( L(n) \). Now, we take \( C_2 = N(W_1) - y_n = \{ y_2, \ldots, y_{n-1} \} \) and we show that all the vertices in \( V(L(n)) \setminus C_2 \) have different representations relative to \( C_2 \). If we consider the vertex \( \{ v_1, v_{1n-1} \} \in W_1 \), then the vertex \( \{ v_1, v_{1n-1} \} \) is adjacent to the vertex \( y_{n-1} \in W_{n-1} \), and hence \( r(\{ v_1, v_{1n-1} \}|C_2) = r(\{ v_1, v_{1n} \}|C_2) \). Also, every vertex \( w \) in the clique \( W_1 \) is adjacent to exactly a vertex of each clique \( W_j \), \( 2 \leq j \leq n \). So, all the vertices \( w \in W_1 \) have various metric representations relative to the subset \( C_2 \). In particular, for every vertex \( w \in W_r, 2 \leq r \leq n - 1 \) so that \( w \notin N(W_1) \) and each \( y_s \in C_2, 2 \leq s \leq n - 1 \), if \( w, y_s \) lie in a clique \( W_s \), \( 2 \leq s \leq n - 1 \), then we have \( d(w, y_s) = 1 \); otherwise \( d(w, y_s) \geq 2 \).

Moreover, all the vertices in the clique \( W_n \) have various metric representations relative to the subset \( C_2 \), because for every vertex \( w \) in the clique \( W_n \) such that \( w \) is not equal to the vertex \( \{ v_n, v_{1n} \} \) in the clique \( W_n \), there is exactly one element \( y_s \in C_2 \) such that \( d(w, y_s) = 2 \); otherwise \( d(w, y_s) > 2, 2 \leq s \leq n - 1 \). Finally, for the vertex \( y_n = \{ v_n, v_{1n} \} \) in the clique \( W_n \) and every element \( y_s \in C_2 \) we have \( d(w, y_s) = 3 \).

Thus the arranged subset \( C_2 = \{ y_2, \ldots, y_{n-1} \} \) of vertices in \( L(n) \) is a resolving set for \( L(n) \) of size \( n - 2 \), and so each subset of \( N(W_r) \) of size \( n - 2 \) is a resolving set for \( L(n) \).

Theorem 3.2 If \( n \geq 5 \) is a fixed positive integer, then the minimum size of a resolving set in graph \( L(n) \) is \( n - 2 \).

Proof. Suppose that \( V(L(n)) = W_1 \cup \ldots \cup W_n \), where the set \( W_r = \{ v_r, v_i \} \) indicate a clique of size \( n - 1 \) in the graph \( L(n) \) for \( 1 \leq r \leq n \). Let \( D_1 = \{ W_{1r}, W_{2r}, \ldots, W_{nr} \} \) be a subset of vertices of \( L(n) \) of size \( (n - 1)(n - 3) \), consisting of some of the cliques of \( L(n) \) and let \( D_2 = \{ W_{n-2r}, W_{n-1r}, W_n \} \) be a subset of vertices of \( L(n) \), consisting of exactly three cliques of \( L(n) \). Now, let \( D_3 \) be a subset of \( D_2 \), consisting of exactly one clique of \( D_2 \), say \( W_n \), and let \( D_3 = \{ W_n \} \).

Thus there are exactly two distinct vertices in \( D_3 = \{ W_n \} \) say \( x \) and \( y \) so that \( x \) is adjacent to a vertex of \( W_{n-1} \) and \( y \) is adjacent to a vertex of \( W_{n-2} \), and hence the metric representations of two vertices \( x \) and \( y \) are identical relative to \( D_1 \). Therefore,
the set \( D_1 = \{W_1, W_2, \ldots, W_{n-3}\} \) cannot be a resolving set for the graph \( L(n) \). and Theorem 3.4, we know that the subset \( \{w_1, w_2, \ldots, w_{n-3}\} \) of vertices of \( L(n) \) cannot be a resolving set for graph \( L(n) \). So the cardinality of a minimum resolving set for graph \( L(n) \) must be greater than or equal to \( n - 2 \). In particular, based on the Lemma 3.1, the arranged subset \( C_2 = N(W_1) - y_n = \{y_2, \ldots, y_{n-1}\} \) of vertices in \( L(n) \) is a resolving set for \( L(n) \) of size \( n - 2 \), and so the minimum size of a resolving set in graph \( L(n) \) is \( n - 2 \).

**Lemma 3.3** Consider the graph \( L(n) \) with vertex set \( W_1 \cup \ldots \cup W_n \) for \( n \geq 5 \). Any subset of \( N(W_r) \) of size \( n - 2 \) cannot be a doubly resolving set for \( L(n) \).

**Proof.** Since \( L(n) \) is a vertex transitive graph, then without loss of generality we may consider the clique \( W_1 \). So if we take \( C_2 = N(W_1) - y_n = \{y_2, \ldots, y_{n-1}\} \), where for \( 2 \leq k \leq n \) we have \( y_k = \{v_k, v_1v_k\} \in V_k \), then by Lemma 3.1 and Theorem 3.2, the subset \( C_2 = N(W_1) - y_n = \{y_2, \ldots, y_{n-1}\} \) of vertices in \( L(n) \) is a minimum resolving set for \( L(n) \) of size \( n - 2 \). Hence by considering the vertices \( u = \{v_1, v_1v_n\} \in W_1 \) and \( y_n = \{v_n, v_1v_n\} \in W_n \), we see that \( d(u, v) - d(u, s) = d(y_n, r) - d(y_n, s) \) for elements \( r, s \in C_2 \), because for each element \( z \in C_2 \) we have \( d(u, z) = 2 \) and \( d(y_n, z) = 3 \). Thus the subset \( C_2 = N(W_1) - y_n = \{y_2, \ldots, y_{n-1}\} \) of vertices in \( L(n) \) cannot be a doubly resolving set for \( L(n) \), and so any subset \( N(W_r) \) of graph \( L(n) \) of size \( n - 2 \) cannot be a doubly resolving set for \( L(n) \).

**Theorem 3.4** If \( n \geq 5 \) is a fixed positive integer, then the minimum size of a doubly resolving set in graph \( L(n) \) is \( n - 1 \).

**Proof.** Suppose that \( V(L(n)) = W_1 \cup \ldots \cup W_n \), where \( W_r = \{v_r, v_1v_j\} \mid v_r = v_1 \) or \( v_r = v_j \}. By Lemma 3.1, and Theorem 3.2, the subset \( C_2 = N(W_1) - y_n = \{y_2, \ldots, y_{n-1}\} \) of vertices in \( L(n) \) is a minimum resolving set for \( L(n) \) of size \( n - 2 \), where \( y_k = \{v_k, v_1v_k\} \in W_k \) for \( 2 \leq k \leq n \). Also, from Lemma 3.3 we know that the subset \( C_2 \) is not a doubly resolving set for \( L(n) \), and hence the minimum size of a doubly resolving set in \( L(n) \) is greater than or equal to \( n - 1 \). Now, if we take \( C_3 = N(W_1) = \{y_2, \ldots, y_{n-1}, y_n\} \), where \( y_k = \{v_k, v_1v_k\} \in W_k \), then based on Lemma 3.1, we know that the subset \( C_3 = N(W_1) = \{y_2, \ldots, y_{n-1}, y_n\} \) of vertices in \( L(n) \) is a resolving set for \( L(n) \) of size \( n - 1 \). We show that \( C_3 \) is a doubly resolving set for \( L(n) \).

It will be enough to show that for any two vertices \( u \) and \( v \) in \( L(n) \), there exist elements \( x \) and \( y \) from \( C_3 \) so that \( d(u, x) - d(u, y) \neq d(v, x) - d(v, y) \). Consider two vertices \( u \) and \( v \) in \( L(n) \). Then the result can be deduced from the following cases:

Case 1. Suppose, both vertices \( u \) and \( v \) lie in the clique \( W_1 \). Hence, there exists an element \( x \in C_3 \) so that \( x \in W_r \) and \( x \) is adjacent to \( u \), also, there exists an element \( y \in C_3 \) so that \( y \in W_k \) and \( y \) is adjacent to \( v \) for some \( r, k \in [n] - 1 \), \( r \neq k \); and hence \( -1 = 1 - 2 = d(u, x) - d(u, y) \neq d(v, x) - d(v, y) = 2 - 1 = 1 \).

Case 2. Suppose, both vertices \( u \) and \( v \) lie in the clique \( W_r \), \( r \in [n] - 1 \), so that \( u, v \notin C_3 \). Hence, there exists an element \( x \in C_3 \) so that \( x \in W_r \) and \( d(u, x) = d(v, x) = 1 \),
also there exists an element \( y \in C_3 \) so that \( y \in W_k, r \neq k \), and \( d(u, y) = 2 \), \( d(v, y) = 3 \) or \( d(u, y) = 3, d(v, y) = 2 \). Thus \( d(u, x) - d(u, y) \neq d(v, x) - d(v, y) \).

Case 3. Suppose that \( u \) and \( v \) are two distinct vertices in \( L(n) \) so that \( u \in W_1 \) and \( v \in W_r, r \in [n] - 1 \). Hence \( d(u, v) = t \), for \( 1 \leq t \leq 3 \).

Case 3.1. If \( t = 1 \), then \( v \in C_3 \). So if we consider \( x = v \) and \( v \neq y \in C_3 \), then we have \( d(u, x) - d(u, y) \neq d(v, x) - d(v, y) \).

Case 3.2. If \( t = 2 \), then in this case may be \( v \in C_3 \) or \( v \notin C_3 \). If \( v \in C_3 \), then there exists an element \( x \in C_3 \) so that \( x \in W_k, k \in [n] - 1, r \neq k \) and \( d(u, x) = 1, d(v, x) = 3 \). So if we consider \( v = y \), then we have \( -1 = 1 - 2 = d(u, x) - d(u, y) \neq d(v, x) - d(v, y) = 3 - 0 = 3 \). If \( v \notin C_3 \), then there exists an element \( x \in W_r \) so that \( x \in C_3 \) and \( d(u, x) = d(v, x) = 1 \), also there exists an element \( y \in C_3 \) so that \( y \in W_k, k \in [n] - \{1, r\} \), and \( d(u, y) = 2, d(v, y) = 3 \) or \( d(u, y) = 3, d(v, y) = 2 \), and hence we have \( d(u, x) - d(u, y) \neq d(v, x) - d(v, y) \).

Case 3.3. If \( t = 3 \), then there exists an element \( x \in W_r \) so that \( x \in C_3 \) and \( d(u, x) = 2, d(v, x) = 1 \), also there exists an element \( y \in C_3 \) so that \( y \in W_k, k \in [n] - \{1, r\} \), and \( d(u, y) = 1, d(v, y) = 3 \), and hence we have \( d(u, x) - d(u, y) \neq d(v, x) - d(v, y) \).

Case 4. Suppose that \( u \) and \( v \) are two distinct vertices in \( L(n) \) so that \( u \in W_r \) and \( v \in W_k, r, k \in [n] - 1, r \neq k \). If both two vertices \( u \) and \( v \) lie in \( C_3 \) or exactly one of them vertices lies in \( C_3 \), then there is nothing to prove. Now suppose that both two vertices \( u, v \notin C_3 \). Hence there exist elements \( x \in C_3 \) and \( y \in C_3 \) so that \( x \in W_r \) and \( y \in W_k \), and hence we have \( d(u, x) - d(u, y) \neq d(v, x) - d(v, y) \).

**Proposition 3.5** If \( n \geq 5 \) is a fixed positive integer, then for \( 1 \leq r \leq n \), any set \( N(W_r) \) of size \( n-1 \) cannot be a strong resolving set for \( L(n) \).

**Proof.** Suppose that \( V(L(n)) = W_1 \cup ... \cup W_n \), where the set \( W_r = \{v_r, v_i v_j \mid v_r = v_i \text{ or } v_r = v_j\} \) indicate a clique of size \( n - 1 \) in the graph \( L(n) \) for \( 1 \leq r \leq n \). Without loss of generality, if we consider the clique \( W_1 \) and we take \( C_3 = N(W_1) = \{y_2, ..., y_{n-1}, y_n\} \), where for \( 2 \leq k \leq n \) we have \( y_k = \{v_k, v_1 v_k\} \in W_k \), then By Lemma 3.1, we know that for the clique \( W_1 \) in \( L(n) \), the subset \( C_3 = N(W_1) = \{y_2, ..., y_{n-1}, y_n\} \) of vertices in \( L(n) \) is a resolving set for \( L(n) \) of size \( n-1 \). By considering various vertices \( w_1 \in W_r \) and \( w_2 \in W_k, 1 < r, k < n, r \neq k \), so that \( d(w_1, w_2) = 3 \) and \( w_1, w_2 \notin C_3 \), there is not a \( y_r \in C_3 \) so that \( w_1 \) belongs to a shortest \( w_2 - y_r \) path or \( w_2 \) belongs to a shortest \( w_1 - y_r \) path. Thus the set \( C_3 = N(W_1) = \{y_2, ..., y_{n-1}, y_n\} \) cannot be a strong resolving set for \( L(n) \), and so any set \( N(W_r) \) of graph \( L(n) \) of size \( n-1 \) cannot be a strong resolving set for \( L(n) \).

**Theorem 3.6** If \( n \geq 5 \) is a fixed positive integer, then the minimum size of a strong resolving set in graph \( L(n) \) is \( n(n-2) \).
Proof. Suppose that $V(L(n)) = W_1 \cup \ldots \cup W_n$, where the set $W_r = \{v_r, v_i, v_j \mid v_r = v_i \text{ or } v_r = v_j\}$ indicate a clique of size $n - 1$ in graph $L(n)$ for $1 \leq r \leq n$. Without loss of generality if we consider the vertex $\{v_1, v_1, v_2\}$ in the clique $W_1$, then there are exactly $(n - 2)$ vertices in any cliques $W_3, W_4, \ldots, W_n$, so that the distance between the vertex $\{v_1, v_1, v_2\} \in W_1$ and these vertices in any cliques $W_3, W_4, \ldots, W_n$ is 3, and hence these vertices must lie in every minimal strong resolving set of $L(n)$. Note that the cardinality of these vertices is $(n - 2)(n - 2)$. On the other hand if we take $C_3 = N(W_1) = \{y_2, \ldots, y_{n-1}, y_n\}$, where for $2 \leq k \leq n$ we have $y_k = \{v_k, v_1, v_k\} \in W_k$, then the distance between two distinct vertices of $N(W_1)$ is 3, and so $n - 2$ vertices of $N(W_1)$ must lie in every minimal strong resolving set of $L(n)$, we may consider these vertices are $y_3, \ldots, y_{n-1}, y_n$. If we now consider the cliques $W_1$ and $W_2$, then there are exactly $(n - 2)$ vertices in the clique $W_1$, so that the distance these vertices from $(n - 2)$ vertices in the clique $W_2$ is 3, and hence we may assume that $(n - 2)$ vertices of the clique $W_2$ so that the distance between these vertices from $(n - 2)$ vertices of $W_1$ is 3, must lie in every minimal strong resolving set of $L(n)$. Thus the minimum size of a strong resolving set in the graph $L(n)$ is $n(n - 2)$.

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