A Kinematic Model for Understanding Rain Formation Efficiency of a Convective Cell

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Abstract A pure theoretical investigation of convective rain formation processes and formation efficiency (FE) is performed using a kinematic one-dimensional time-dependent model with warm rain microphysics. FE is defined as the ratio of total cloud-to-rainwater conversion to total condensation. FE is a component of precipitation efficiency, which is an important but poorly understood parameter in idealized climate models. This model represents a cloud by a cylindrical thermal bubble rising at constant velocity. The model focuses on the interaction between auto-conversion, collection, and lateral mixing about which no theory has been proposed. Taking the auto-conversion threshold into account, a criterion for rain formation and a semianalytical approximate solution of FE are found. The auto-conversion threshold limits the temporal and spatial extent of the "vigorou s rain formation region" where most of the rain is produced. The collection and auto-conversion compete with lateral mixing to determine the strength of rain formation within this region. The FE is predicted to be most sensitive to auto-conversion threshold, fractional entrainment rate, and initial bubble water vapor density.

Plain Language Summary A theory is proposed to understand how much rain a convective cloud can produce. The subsequent evaporation of raindrops after they fall out of the cloud is a separate problem and is not considered here. The rain formation depends on two competing factors: one is how fast it is for floating small droplets to grow into falling large droplets, and the other is how much water is thrust out of the cloud by turbulent flow before they are converted to rain. The cloud is modeled as a cylindrical hot air blob moving upward at a given uniform speed. The result shows that the cloud radius, initial water vapor content, and a cloud water conversion threshold governed by aerosol concentration are the most influential factors for the in-cloud rain formation process.

1. Introduction

The ensemble effect of moist convection plays an important role in modulating tropical water vapor distribution (e.g., Romps, 2014a; Sun & Lindzen, 1993). The cloud transports water from the boundary layer to the free troposphere. Some is recycled by precipitation locally, and the rest is detrained to the environment or left in the troposphere.

Precipitation efficiency (PE) was first introduced by Braham (1952) who defined it as the ratio of surface rain fall to the total vapor transported into the cloud. PE can be classified based on its denominator. As is summarized by Sui et al. (2005), the "large-scale PE" uses moisture convergence as the denominator, and "cloud-microphysical PE" uses condensation and deposition rates instead. PE has a wide application, including precipitation forecasting (Market et al., 2003), aerosol vertical transport (Bailey et al., 2015), tropical relative humidity (RH) estimation (Romps, 2014a), downdraft strength estimation in idealized models and cumulus parameterization schemes (e.g., Bechtold et al., 2001; Emanuel et al., 2014; Zhang & McFarlane, 1995). Romps (2016) concluded that PE lies between 0.2 and 0.5 for deep convection. Langhans et al. (2015) decomposed the conversion process into three kinds of efficiencies. In plain words, the condensation efficiency (CE) denotes the ratio of vapor involved in evaporation that condenses into non-falling cloud liquid water. Formation efficiency (FE) denotes the ratio of cloud water that is converted to rainwater. The rest of the water is detrained to the environment or just stays in the troposphere as nonprecipitating cloud wreck and eventually dissipates. Sedimentation efficiency (SE) denotes the ratio of rain that
survives subcloud evaporation and falls to the ground. The product of the three quantities is defined as drying ratio (DR), and the product of the latter two is defined as PE (essentially the cloud-microphysical PE): 
\[ \text{DR} = \text{CE} \times \text{FE} \times \text{SE}, \quad \text{and PE} = \text{FE} \times \text{SE}. \]

The diverse PE definitions prevent a clear comparison between papers. We try to summarize the previous observation and model results by separating their subjects into FE and SE. Factors that favor higher FE include the following:

a) Smaller aerosol particle concentration (more pristine environment), which shortens auto-conversion time scale (Suzuki et al., 2013).

b) Longer warm cloud depth. This increases the available time for warm cloud microphysics and weakens the role of ice particles, which are considered to be less efficient in collecting cloud water than rainwater (Market et al., 2003; McCaul et al., 2005).

c) Moderate updraft strength. Too weak an updraft does not powerfully hold raindrops and leaves too short a residence time (if updraft persists during the particle falling process) for raindrops to grow, and dynamically, it is even hard to maintain the storm itself against water loading (Market et al., 2003). Too strong an updraft transports too much water above frozen level and weakens the role of warm rain process (McCaul et al., 2005).

d) Longer updraft lifetime (relative to auto-conversion time scale), which provides longer time for cloud water conversion (Seifert & Stevens, 2010).

e) Lower cloud surface-to-volume ratio, which reduces the opportunity for entrainment and detrainment (Newton, 1966).

f) Higher free tropospheric RH, which reduces entrainment evaporation (Cohen & McCaul, 2007; Langhans et al., 2015; Newton, 1966).

g) For isolated convection, lower vertical wind shear (Fritsch & Chappell, 1980; Market et al., 2003; Schoenberg Ferrier et al., 1996). Strong shear tilts the cloud, shortening the path for in-cloud rain collection, and increases the cloud surface area, which favors entrainment and detrainment. However, vertical shear can also make convection organized and complicate the issue.

h) For idealized radiative convective equilibrium (RCE) simulation, more cloud condensate caused by higher SST leads to higher mean FE (Lutsko & Cronin, 2018).

Many other studies only mention precipitation rate rather than PE. For example, Wood et al. (2009) found that longer cloud liquid water path can increase marine stratocumulus cloud precipitation rate. Main factors that favor higher SE are higher subcloud RH and shorter subcloud layer depth (or in other words, lower lifted condensation level [LCL] height; e.g., Zhang & Fritsch, 1986; Market et al., 2003). Other factors influencing PE include environmental temperature (Cohen & McCaul, 2007; McCaul et al., 2005) and CAPE and CIN (Market et al., 2003), but we consider them to be indirect and act as the combination of more direct factors.

Theoretical modeling of CE, FE, and SE is difficult due to the interaction of factors shown above. CE is determined by the competition between condensation and vapor lateral mixing. Romps (2014a) solved a quantity equivalent to CE without considering supersaturation. FE is determined by the competition between cloud water detrainment, its conversion to rainwater, and the residue detained by auto-conversion threshold. Seifert and Stevens (2010) presented the first semiempirical FE expression of shallow cumulus cloud that considers the threshold behavior of auto-conversion, with fitting from data required. SE depends on in-cloud and subcloud rain evaporation (Langhans et al., 2015) and is coupled to downdraft. The evaporation amount in a downdraft depends on compressional warming, lateral mixing, and other microphysical parameters (e.g., Betts & Silva Dias, 1979; Srivastava, 1985). We are unaware of any theory for SE.

An isolated convection can serve as a simple start point. It can be viewed as a rising thermal bubble with a turbulent wake (Blyth et al., 1988). The wake is a mixture of the bubble and environmental air that extends from bubble rear to the LCL. The rain production depends on both the processes inside the bubble and those in the wake. In the bubble, active condensation, collision-coalescence, entrainment, and detrainment take place; in the wake, vigorous rain evaporation takes place (Langhans et al., 2015). The microphysics in the wake and subcloud layer depends on that in the bubble, so the problem is largely sequential, and the first step is to understand the bubble process.
As for the methodology, a hierarchy of idealized models can provide insights on rain formation process. The first is one-dimensional time-dependent cloud model (1DTD), which is a useful tool to study the lifecycle of isolated convection (e.g., Chen & Sun, 2002; Ferrier & Houze, 1989; Ogura & Takahashi, 1971). This model is a sister of ensemble plume model, which considers a bulk representation of multiple clouds and their interaction with the environment (e.g., Romps, 2010; Romps, 2014a; Yanai et al., 1973; Zhang & McFarlane, 1995). Furthermore, to isolate the rain formation problem from the complicated dynamics, one can prescribe updraft velocity to study a pure kinematic problem (Haiden, 1995; Haiden & Kahlig, 1988; Haiden & Kerschbaum, 1989; Kessler, 1969; Seifert & Stevens, 2010; Suzuki et al., 2013). This neglects the interaction of rain formation and updraft strength (e.g., water loading). In particular, Haiden and Kerschbaum (1989) studied the auto-conversion threshold behavior in windward orographic precipitation by sequentially solving the stage with and without auto-conversion. Haiden (1995) introduced the first integral method to treat the nonlinear collection in stratiform precipitation. However, a new mathematical method, which will be introduced in this paper, is needed if an auto-conversion threshold, collection, and lateral mixing coexist. An even simpler type is parcel model (e.g., Lee & Pruppacher, 1977), which views the updraft as a zero-dimension parcel and solves its thermodynamics, droplet growth, and mixing with the environment. Its drawback is not being able to depict nonlocal effects such as collection. We will not use parcel model in this work, but as will be shown later, the bubble top behavior does not involve collection and could be depicted as a parcel.

In this paper, we intend to construct a novel 1D kinematic model that considers supersaturation effect, auto-conversion (with threshold), collection, and lateral eddy mixing inside an ascending thermal bubble. It is simpler than most 1D model but is designed as a base camp for finding (semi)analytical solution. By assuming that the cloud processes below the thermal bubble do not influence the process inside the bubble, we only solve the rain condensation and formation processes (characterized by CE and FE) and leave rain evaporation process (SE) for future study. The CE is modified to include supersaturation effect. The FE theory starts from the 1DTD model and is the first one that incorporates lateral mixing, nonlinear auto-conversion, and collection together. Not only this model serves as a toy that people can play with to gain understanding, but also its (semi)analytical solution can be used as a module in idealized climate models (e.g., Romps, 2014a).

The paper is organized in the following way. The physical model framework is introduced in section 2. The condensation process is introduced in section 3. The rain formation process and its semianalytical solution are presented in section 4. The FE expression and the sensitivity are discussed in section 5. Section 6 summarizes the paper and discusses possible future work. The mathematical symbol system is summarized in Tables S1 and S2 in the supporting information.

2. Derivation of the Kinematic Model

As a start, the study object is a highly simplified convective cell that is strictly neither shallow nor deep convection. The bubble is assumed to ascend at uniform speed without stopping, so it does not resemble shallow convection where the equilibrium height is not significantly higher than the condensation height scale. As we will omit ice phase at current stage, it is also different from deep convection.

2.1. The Dynamical Setup

We present a few assumptions for the motion of the bubble, as is illustrated in Figure 1. The bubble is assumed to be a closed cylinder, with constant depth $d_b$, a constant radius $R$, and a constant vertical velocity $w_c$. It rises vertically in a shear-free quiescent environment from below the LCL to an infinitely high level.

The use of such cylinder geometry was inspired by Romps (2014b). The constant $w_c$ assumption is to facilitate the introduction of bubble-following coordinate. The $w_c$ value should represent the mean updraft speed in rain formation region of a real cloud. This assumption excludes the high-level dynamic detrainment (e.g., cloud anvil formation) in our model. The air compressibility in continuity equation is omitted, which causes error for deep convection. The decrease of air density with height can cause divergent flow and make the raindrop trajectory more complicated (Kessler, 1969). In this way, the bubble mass flux is constant with height, also an assumption used by Romps (2014b) for middle layer where entrainment and detrainment roughly balance (Romps, 2010).
depends on the buoyancy difference between the parcel and the environment, boundary-layer turbulence strength, drag, and entrainment dilution (Del Genio et al., 2007).

For shallow cumulus cloud, the bubble depth $d_b$ and radius $R$ scale as the boundary-layer thermal size and therefore the boundary-layer depth (Stull, 1985). For deep convection, the duration time and therefore updraft depth could be controlled by the emergence of downdraft (Markowski & Richardson, 2011). The radius $R$ of deep convective updraft generally increases with $w_c$ (Khairoutdinov et al., 2009).

### 2.2. The Thermodynamic Setup

We have six assumptions:

1. Ice phase is not included, so only phase change between vapor and liquid phase is considered. This makes the model only very qualitatively relevant to deep convection.
2. Supersaturation is considered by assigning a constant relaxation time scale $\tau_{sp}$.
3. The temperature and density difference between cloud and environment is small. This is a proper approximation for tropical maritime convection (Romps & Öktem, 2015) where cloud and environmental temperature difference is small. Such a quasi-neutral environment can be produced by vigorous moist convection and the following gravity wave adjustment (Emanuel, 1994). In fact, the evaporation due to dry air entrainment causes some cooling, which can excite more condensation, but we omit this complexity here.
4. An analogy of Boussinesq approximation is used. Air density of both the cloud and environment is regarded as constant in the transport process but is variable in calculating condensation.
5. Initially, all parcels in the bubble are assumed to be well mixed and have homogeneous property, so they have the same potential temperature and moisture content and therefore the same LCL. Following Romps (2014a), the saturated water vapor density $\rho_{vs}$ decreases exponentially with height with a height scale $H_c$. See Appendix A for detailed derivation.
6. The free tropospheric environment has constant RH.

This is a crude approximation in the tropics since RH is usually in C shape: small at middle level and large at lower and upper levels (Romps, 2014a). Here the detrained water is assumed not to influence environmental RH. Using the assumption of no air temperature and density difference between the cloud and environment, environmental water vapor density $\rho_{ve}$ is obtained:

$$\rho_{ve} = RH\rho_{vs}$$  \hspace{1cm} (1)

### 2.3. The Microphysics Parameterization

We use the classic Kessler’s bulk microphysics scheme (Kessler, 1969), which deals with warm rain and splits hydrometers into cloud water and rainwater. They are viewed as continuum: cloud water density $\rho_c$ (unit: $\text{kg m}^{-3}$), which follows air parcel, and rainwater density $\rho_r$ (unit: $\text{kg m}^{-3}$), which sediments. The conversion
process includes auto-conversion and rain collection. The terminal fall velocity of rainwater relative to the air flow is set as a constant $V_f$, as has been done in many idealized models (e.g., Emanuel, 1986; Hernandez-Duenas et al., 2013; Kessler, 1969). This precludes the stretching of the rain packet due to terminal fall speed difference between different sizes of raindrops (Kessler, 1969).

First, we introduce auto-conversion term. It is defined as the process of initial raindrop formation due to the collision-coalescence between small cloud droplets (Rogers & Yau 1989). Kessler (1969) observed that this process only starts when cloud water density reaches a threshold $\rho_{th}$ of around 1 g m$^{-3}$ and the auto-conversion rate (AUT, unit: kg m$^{-3}$ s$^{-1}$) can be approximated as a linear relaxation to the threshold with a time scale of $\tau_c$:

$$\text{AUT} \equiv \max \left( \frac{\rho_c - \rho_{th}}{\tau_c}, 0 \right)$$

Later works found that the threshold behavior lies more inherently in certain critical cloud droplet radius $r_c$, which is around 10 $\mu$m rather than density threshold (Liu et al., 2004). The two views can be linked. When cloud droplet number density is larger (e.g., polluted atmosphere), the total water amount needed by the droplets to reach $r_c$ is larger, so $\rho_{th}$ is larger (Phillips et al., 2002). Note that AUT can also be represented with other nonlinear function of $\rho_c$ (e.g., Berry, 1968) that is smoother than the “max” operator. The threshold makes FE < 1 possible even without lateral mixing, as is studied by Seifert and Stevens (2010).

The collection rate in Kessler scheme depends on both cloud and rainwater densities (Kessler, 1969):

$$\text{CLC} \equiv K' \rho_c \rho_f^2 \sim K \rho_c \rho_f$$

The coefficient $K'$ is proportional to collection efficiency and other parameters; see Table 4 of Kessler’s (1969) paper for detail. Haiden (1995) approximated the $\rho_c^{7/6}$ to be $\rho_c$ and introduced a modified collection coefficient $K$, which is estimated to be 6 m$^{-3}$ kg$^{-1}$ s$^{-1}$.

Grabowski (1998) showed that Kessler scheme can be extended to include ice phase by prescribing variable auto-conversion rate, collection (accretion) rate, and terminal fall velocity for different species, which are classified by temperature. This suggests that understanding the fundamental role of auto-conversion and collection is crucial for both warm and cold cloud precipitation formation, so our warm rain model may still have some implications for deep convection.

### 2.4. The Governing Equation

First, we introduce the time and vertical coordinate. The time origin $t = 0$ is the moment for bubble top to reach LCL. We will use two sets of vertical coordinates: the ground coordinate $z_g$ and bubble coordinate $z$. The former is static to the ground, with LCL as its origin. The latter ascends at uniform velocity $w_c$ with the bubble, with the bubble top as its origin, as is shown in Figure 1. Their transformation relationship is as follows:

$$z = z_g - w_c t$$

We start from the volume conservation law of water vapor density $\rho_v$, cloud liquid water density $\rho_c$, and rainwater density $\rho_r$ within a cylinder ascending at constant speed $w_c$. The bulk-plume assumption is used, which states that the cloud exchanges its horizontal mean quantity with the unsaturated environment (Romps, 2010), so the problem is reduced to 1D. The governing equations of the in-cloud horizontally averaged $\rho_v$, $\rho_c$, and $\rho_r$ are derived as follows:

$$\frac{\partial}{\partial t} \rho_v + w_c \frac{\partial}{\partial z} \rho_v = -f_c - \frac{\rho_v - \rho_w}{\tau_m}$$

$$\frac{\partial}{\partial t} \rho_c + w_c \frac{\partial}{\partial z} \rho_c = f_c - \left( \max \left( \frac{\rho_c - \rho_{th}}{\tau_c}, 0 \right) + K \rho_c \right) \frac{\rho_c}{\tau_m}$$

$$\frac{\partial}{\partial t} \rho_r + (w_c - V_f) \frac{\partial}{\partial z} \rho_r = \left( \max \left( \frac{\rho_r - \rho_{th}}{\tau_c}, 0 \right) + K \rho_r \right) - \frac{\rho_r}{\tau_{mr}}$$

Here $\tau_m$ is the vapor and cloud water lateral mixing time scale, and $\tau_{mr}$ is that for rainwater. $f_c$ is the conversion rate from vapor to cloud water with $\tau_{sp}$ as supersaturation relaxation time scale.
The conversion is basically diffusional growth process where supersaturated vapor condensates onto droplet to reduce supersaturation. The derivation of $\tau_{sp}$ is shown in Appendix A in the framework of Korolev and Mazin (2003). $\tau_{sp}$ is estimated to be at most tens of seconds for liquid clouds.

All species are susceptible to lateral turbulent mixing with the environment. The entrained environmental air is assumed to be distributed immediately and homogenously within the cloud at that height level. As the bubble is continuously ascending and producing condensation, the primary role of entrainment in this model is reducing supersaturation and slowing down condensation process, not including cloud droplet evaporation, which occurs in real cloud. Because the vapor and cloud water follow the air’s turbulent motion well, they are assumed to have the same lateral mixing time scale $\tau_m$. It is set to depend on fractional entrainment rate $\varepsilon$ and $w_c$ through eddy diffusivity parameterization of Asai and Kasahara (1967):

$$\tau_m \equiv \frac{1}{\varepsilon w_c}, \text{with } \varepsilon = \frac{2\pi}{R}$$

Here $\varepsilon$ is inversely proportional to bubble radius $R$ with $\alpha$ as a nondimensional mixing coefficient which is typically 0.1 for a plume (Turner, 1986). Essentially, we are transforming the fractional entrainment rate $\varepsilon$, which is based on length scale to time scale. Note that only two of $\tau_m$, $\varepsilon$, and $w_c$ are independent. To arrive at such a simple representation of lateral mixing, the following assumptions have been made:

1. Dynamic detrainment is not considered for simplicity. Turbulent entrainment and detrainment rates are assumed equal and constant with height.
2. Vertical eddy diffusion of all species by in-cloud turbulence is neglected for convenience.
3. Cloud top entrainment/detrainment is neglected. Their relative importance to lateral entrainment/ detrainment is still debated (de Rooy et al., 2013; Yeo & Romps, 2013).
4. The turbulent detrainment time scale of rainwater is set as $\tau_{mr}$. It is more complicated than water vapor and cloud water because raindrops do not closely follow the flow (e.g., Srivastava, 1985). We roughly estimate $\tau_{mr}$ to be of similar magnitude to $\tau_m$. In this case, we will use scale analysis to show that rainwater detrainment has important influence on bubble bottom precipitation but has little influence on FE.

As can be seen from equations (5) to (7), some of the water vapor carried by the bubble is lost to the environment through lateral mixing, and the rest becomes condensation, which drives cloud water evolution. Unlike water vapor, which can be solved independently, cloud and rainwater evolution is strongly coupled by the collection term. In the next section we will solve the water vapor evolution and condensation first.

### 3. The Condensation Process

This section deals with the conversion of bubble water vapor into cloud liquid water under the influence of lateral mixing and supersaturation. The solution for $\rho_v$ is derived in Appendix A. Using equations (8) and (A.14), the $f_c$ expression in bubble and ground coordinates is as follows:

$$f_c \equiv \frac{\rho_v - \rho_{ws}}{\tau_{sp}}$$

$$\text{(8)}$$

$$\text{for max}\{-d_b, -w_c t\} < z < 0 \text{(bubble coordinate)} \quad \text{and} \quad \text{for max}\{0, -d_b + w_c t\} < z < w_c t \text{(ground coordinate)}$$

$$\text{(10)}$$

where the constant $f_0$ is as follows:
\[
f_0 \equiv \frac{\rho_{w0}}{\rho_{w}} \frac{w_c}{H_s} \frac{1 - (1 - \text{RH}) \varepsilon H_s}{1 + \frac{\tau_m}{\tau_{sp}} \frac{1}{1 - \varepsilon}} \tag{11}
\]

Note that the condensation rate \( f_0 \) is only nonzero within the bubble and above LCL at the same time. Only the \( f_0 \geq 0 \) case where lateral mixing is not too strong to prohibit condensation is considered. The supersaturation effect serves as a “buffer zone” that makes condensation less concentrated. As each parcel has the same condensation experience (the bubble is well mixed), the CE is defined as the ratio of total condensation in unit volume of bubble air to the initial (or LCL) water vapor density \( \rho_{w0} \):

\[
\text{CE} \equiv \frac{\int_0^\infty f_c dv}{\rho_{w0}} = \frac{\tau_m}{\tau_m + \tau_{sp}} \left( 1 - \frac{(1 - \text{RH}) \varepsilon H_s}{\tau_m w_c} \right) = \frac{\tau_m}{\tau_m + \tau_{sp}} [1 - (1 - \text{RH}) \varepsilon H_s] \tag{12}
\]

Here we have used equations (9) and (10).

For thermodynamic equilibrium case where \( \tau_{sp} \to 0 \), CE degenerates to the result of Romps (2014a) in his equation (13) where higher RH, smaller \( \varepsilon \) and shorter \( H_s \) help increase CE. He was using a variable mass flux bulk-plume model with steady updraft, but the result is the same. In this case, CE is a bulk property that does not involve time scale: \( H_s \) can be interpreted as condensation scale height, and \( \varepsilon^{-1} \) can be interpreted as lateral mixing scale height.

Practically, as \( \tau_m \) is of hundreds of seconds, there is always \( \tau_m \gg \tau_{sp} \) for liquid clouds, so thermodynamic equilibrium assumption is quite accurate for CE in this model. Korolev and Mazin (2003) pointed out that the phase change relaxation is potentially important for mixed phase and ice cloud. Theoretically, the effects of supersaturation are not only to postpone condensation as is seen in equation (10) but also to reduce CE in the presence of lateral mixing. The latter is because longer \( \tau_{sp} \) extends the time for water to stay in vapor phase and leads to more loss to the environment through lateral mixing. The sensitivity tests of all the parameters are shown as the dashed black line in Figure 8 (plotted together with FE). Increasing \( \tau_{sp} \) from 0 to 500 s can roughly decrease CE by 0.2.

The total available condensation of the bubble CND (unit: kg m\(^{-2}\)) can be calculated from equation (12):

\[
\text{CND} \equiv \int_0^\infty \int_{-d}^{d} f_c dz dt = d_3 \rho_{w0} \text{CE} \tag{13}
\]

### 4. The Rain Formation Process

The nonlinearity of the rain formation problem lies in the auto-conversion \( \text{max}(\rho_c - \rho_{th})/\tau_c, 0) \), and the product \( \rho_c \rho_s \) in collection term. If auto-conversion threshold is omitted and further neglects cloud water detrainment, the nonlinear equation is analytically solvable by borrowing the first integral method introduced by Haiden (1995). In this section, we will first solve the model numerically and analyze the cloud and rainwater evolution. The result hints us to decompose the domain into four parts, which makes a semianalytical approximate solution for FE possible.

#### 4.1. Some Examples of Numerical Solution

We choose a typical tropical deep convection setup as a reference run whose parameters are shown in Table 1. The values are referred from other papers (not necessarily the exact ones) and common sense. The numerical solution uses finite-difference method, with second-order upwind advection scheme and third-order Runge-Kutta time-stepping scheme. In Figure 2, we plot the \( f_c, \rho_c, \rho_s \) of three runs: the reference run (\( \varepsilon = 0.33 \text{ km}^{-1}, \rho_{th} = 1 \text{ g m}^{-3} \)), the \( \rho_{th} = 2 \text{ g m}^{-3} \) run, and the comparative \( \varepsilon = 0.50 \text{ km}^{-1} \) run to study the effect of lateral mixing and auto-conversion threshold, which will be shown to be among the leading order influential factors of FE. All other parameters of the comparative runs are the same as the reference run. The \( \rho_c \) evolution at bubble top and \( \rho_s \) at LCL (if further multiplied by \( V_f \), it becomes precipitation flux) are additionally plotted in Figure 3 for clarification.

For the reference run, the pattern and magnitude of \( \rho_c \) and \( \rho_s \) in Figure 2 qualitatively agree with the 1DTD model simulation of Ferrier and Houze (1989) shown in their Figure 10. \( \rho_c \) concentrates at the bubble’s upper
Table 1
The Parameters for the Reference Run and the Perturbation Range for the Sensitivity Tests

| Notation | Reference value | Perturbation range | Meaning |
|----------|-----------------|--------------------|---------|
| $\varepsilon$ | 0.33 km$^{-1}$ | 0.083 to 1 km$^{-1}$ab (range enlarged) | Fractional entrainment rate |
| $d_b$ | 2 km | 1 to 5 km$^2$ | Bubble depth |
| $w$ | 3 m s$^{-1}$ | 1 to 6 m s$^{-1}$bc,d | Updraft velocity |
| $\tau_c$ | $10^3$ s | 500 to $10^4$ s (range enlarged) | Auto-conversion time scale |
| $\tau_{sp}$ | 10 s | $10^{-10}$ to 500 s$^g$ (range enlarged) | Supersaturation relaxation time scale |
| $\tau_{mr}$ | $10^{10}$ s | 0.5 to 10$^{10}$ times of $r_m$ | Rainwater detrainment time scale |
| $K$ | $6 \times 10^3$ kg$^{-1}$ s$^{-1}$ | 3 to 8 m$^3$ kg$^{-1}$ s$^{-1}$ | Collection coefficient |
| $T_{LCL}/\rho_0$ | 293 K/17.1 g/3.30 km | 270 to 305 K$^h$/3.9 to 33.5 | LCL temperature/initial (LCL) vapor density/ the scale height of saturated vapor density |
| $\rho_{th}$ | 1 g m$^{-3}$ | 0.01 to 3 g m$^{-3}$e (range enlarged) | Auto-conversion cloud water density threshold |
| RH | 0.7 | 0.4 to 0.9$^{f}$ | Environmental relative humidity |

Note. The perturbation range is roughly chosen as the terrestrial climate regime, and some parameters are noted to be exaggerated for purely sensitivity study. The choice has referenced

- aFerrier and Houze (1989), bRomps (2010), cDel Genio et al. (2007), dRomps and Öktem (2015), eKessler (1969), fSuzuki et al. (2013), gKorolev and Mazin (2003), hHaiden (1995), iLawrence (2005), jParodi and Emanuel (2009), kRomps (2014a).

Part because the consumption of $\rho_c$ by collection is strong at both $\rho_c$ small at the bubble’s upper part, but it gradually grows large as rainwater falls. The auto-conversion region is enclosed by the $\rho_c = \rho_{th}$ contour line (the solid red line) in Figure 2. The span is controlled by bubble top $\rho_c$ evolution, because the $\rho_c$ there is the largest due to the absence of collection and it controls the collection process below by seeding rainwater. We call the bubble top auto-conversion start time as $t_1$ and the end time as $t_2$ when $\rho_c$ falls back to $\rho_{th}$ due to auto-conversion and detrainment. We get $t_1 \approx 117$ s (when bubble top is around 352 m above LCL); it ends around $t_2 \approx 3,208$ s (when bubble top is around 9,625 m above LCL). The $\rho_c$ at the lower part of the bubble is below $\rho_{th}$ because of the strong collection. Such distribution of $\rho_c > \rho_{th}$ region inspires us to decompose the time-space domain into four parts, as will be introduced in section 4.2.

Both larger $\rho_{th}$ and larger $\varepsilon$ are advertent to rain formation, but they act in different ways. As $\rho_{th}$ is raised from 1 to 2 g m$^{-3}$, the auto-conversion region shrinks smaller. Auto-conversion starts only a little later but stops much earlier, and it is more concentrated at bubble’s upper part. Figure 3 shows that the time evolution of bubble top $\rho_c$ has a larger peak value and overall magnitude than the reference run, mainly due to the weaker auto-conversion strength (the $\rho_c - \rho_{th}$ is smaller). The LCL rainwater starts a bit later, but it grows a bit faster and reaches a similar peak value due to the abundant un-auto-converted cloud water that favors collection, similar to the result of Kessler (1969). Then $\rho_c$ decays faster due to the earlier termination of auto-conversion.

As $\varepsilon$ is increased from 0.33 to 0.50 km$^{-1}$, $\rho_c$ is vertically more uniform within the bubble. This is because stronger detrainment reduces cloud water content in the bubble and therefore decelerates collection. The start time of bubble top’s auto-conversion does not change much because lateral mixing operates at longer time scale. However, the peak is much smaller, and the end time is much earlier because lateral mixing now has enough time to work. LCL precipitation has smaller magnitude and ends earlier.

In a word, the LCL precipitation start time for all three runs is similar. However, the smaller accumulated precipitation of larger $\rho_{th}$ case is due to the earlier auto-conversion end time, and that of larger $\varepsilon$ case is due to the generally weaker collection.

4.2. The Approximate Analytical Solution With Domain Decomposition

To study the rain formation efficiency, which is a bulky property that does not require exact time-dependent solution, we try to obtain an approximate solution of bubble bottom accumulated precipitation in the presence of the two nonlinear effects: auto-conversion threshold and collection.

The bubble part of the temporal-spatial domain is decomposed into four regions, as is shown in Figure 4. They are (in bubble coordinate) as follows: the preparation region (PR): $t < t_1 - z/V_T$; the dissipation region

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t_2 - z / V_T; the upper region (UR): t_1 - z / V_T \leq t \leq t_2 - z / V_T and -d_U \leq z \leq 0; and the lower region (LR): t_1 - z / V_T \leq t \leq t_2 - z / V_T and -d_L \leq z \leq -d_U. Such decomposition picks out a “vigorous rain formation region,” which consists of UR and LR. It is the region where rainwater is produced. Further, the auto-conversion region is assigned to be UR, and the rest as LR. All four regions have cloud water detrainment; only UR and LR have collection; only UR has auto-conversion. The t_1, t_2, and d_U that delineate the borders will be solved. This decomposition makes the auto-conversion a linear term inside UR.

Auto-conversion threshold acts to delay the initial rain formation and makes it finish earlier. The border of PR is the initial rain’s trajectory: It starts from bubble top at t = t_1 and falls at V_T speed within the bubble (in bubble coordinate). The border of DR is the terminal rain’s trajectory: It starts falling from bubble top at t = t_2. The t_1 and t_2 obey a transcendental equation, which could only be solved numerically, as is discussed in Appendix B. The height where bubble top reaches t_2 (point F in Figure 4) is z_g F = w_c t_2. This is the highest point of the rain formation process, so the dynamic detrainment over this height (if allowed) does not influence precipitation. Thus, our model is technically most suitable for shallow convection whose dynamic detrainment height (roughly the level of neutral buoyancy) is high enough to be over vigorous rain.

**Figure 2.** The time-height plots of the reference run (upper panel), the comparative run for \( \rho_{th} = 2 \text{ g m}^{-3} \) (middle panel), and the comparative run for \( \varepsilon = 0.50 \text{ km}^{-1} \) run (lower panel). The left column is condensation rate \( f_c \) (in unit of \( 10^{-2} \text{ g m}^{-3} \text{ s}^{-1} \)), the middle column is cloud water density \( \rho_c \) (in unit of \( \text{g m}^{-3} \)), and the right column is rainwater density \( \rho_r \) (in unit of \( \text{g m}^{-3} \)). In each plot the abscissa is time (hr), and the ordinate is distance above LCL (km). The red solid line in the \( \rho_c \) and \( \rho_r \) is \( \rho_{th} \) contour line. The dashed red line is the predicted auto-conversion region border as will be introduced in section 4.2: The one that spans from the lower left to the upper right is \( z = -d_U \) line (BE line in Figure 4), the one at the left is \( z_g = w_c t_1 + (w_c - V_T) (t - t_1) \) line (AC line in Figure 4), and the one at the right is \( z_g = w_c t_2 + (w_c - V_T) (t - t_2) \) line (DF line in Figure 4). The thin \( \rho_r \) anomaly line at bubble top in subplots (c), (f) and (i) is numerical artifacts.
region is de...A schematic diagram of the domain decomposition. The UR
Figure 4.
The time evolution of bubble top cloud water
2 run is in green.
The integration domain is shown in Figure 4 as ABEF and BCDE, respectively. Here
families. They try to reconstruct the contribution of the product form
The time evolution of bubble top cloud water
Now, we tackle with the collection term. As we are only interested in
in the bubble bottom total precipitation, we can design a linearized problem by construction that represents the bulk property well in sacrifice of the time-dependent information. We introduce two “mean values” of \( \rho_c \), which are a large constant \( \langle \rho_c \rangle_U \) in UR and a small constant \( \langle \rho_c \rangle_L \) in LR to replace the \( \rho_c \) in the collection term:

\[
\text{CLC} \approx \begin{cases} 
K(\rho_c)_U \rho_c, & -d_U \leq \rho_c \leq 0 \\
K(\rho_c)_L \rho_c, & -d_b \leq \rho_c \leq -d_U 
\end{cases}
\]  

(14)

To guarantee that the bubble bottom precipitation is correctly grasped by the linearized problem, the total collection of the linearized problem should approximate that of the original problem in UR and LR, respectively. The reason is that we only make approximation to the collection term and other terms are linear and identical to the original problem within each region. We assign \( \langle \rho_c \rangle_U \) and \( \langle \rho_c \rangle_L \) as certain temporal and spatial averages in UR and LR:

\[
\langle \rho_c \rangle_U = \frac{1}{\tau_U} \int_0^{t_U} \int_{-d_U}^{-z} \rho_c \, dz \, dt
\]

(15)

\[
\langle \rho_c \rangle_L = \frac{1}{\tau_L} \int_{-d_b}^{-d_U} \int_{-z_U}^{-z} \rho_c \, dz \, dt
\]

(16)

The integration domain is shown in Figure 4 as ABEF and BCDE, respectively. Here \( \tau_U \) and \( \tau_L \) are certain time scales. They try to reconstruct the contribution of the product form \( \rho_c \rho_r \) to the total collection that is lost in the linearized scheme. Based on the observation of the numerical solution in Figure 2, we set them as follows:

\[
\tau_U \equiv y_U \min \left\{ H_s / \omega_c + \left( \tau_m^{-1} + \tau_c^{-1} \right)^{-1}, \ t_2 - t_1 \right\}
\]

(17a)

\[
\tau_L \equiv y_L \min \left\{ H_s / \omega_c + \left( \tau_m^{-1} + \tau_c^{-1} \right)^{-1}, \ t_2 - t_1 \right\}
\]

(17b)

The first aspect is picking out the length of the time slot during which \( \rho_c \rho_r \) is not small, as is shown in the “min[ ]” operator. The second is to estimate the spatial-temporal overlapping of \( \rho_c \) and \( \rho_r \) as is represented by the fixed parameters \( y_U \) and \( y_L \).

Now, we introduce how the content of “\( \min[ ] \)” is determined. When \( \rho_{th} \) is large, \( \tau_U \) should depend on the start and end time of bubble top autoconversion \( t_1 \) and \( t_2 \). When \( \rho_{th} \) is small, \( t_2 \) could be very large and far beyond the rain formation time, so a new \( \tau_U \) estimation is demanded. In the latter case, the cloud water (and therefore collection process) is estimated to exist for a condensation time scale plus the decaying time scale determined by both autoconversion and lateral mixing: \( H_s / \omega_c + (\tau_m^{-1} + \tau_c^{-1})^{-1} \). As is shown in Figure 10, \( H_s / \omega_c + (\tau_m^{-1} + \tau_c^{-1})^{-1} \) is much smaller than \( t_2 - t_1 \) for most cases in the sensitivity test.

The \( y_U \) and \( y_L \) are used to represent the overlapping or the relative distribution of \( \rho_c \) and \( \rho_r \). For example, if the distribution of \( \rho_c \) and \( \rho_r \) in t-z diagram had little overlap, the product would be small, so \( \langle \rho_c \rangle_U \) should be small by setting a large \( y_U \), and vice versa. In the middle of the two extremes, if \( \rho_c \) were uniform whenever \( \rho_r \) is nonzero, \( \langle \rho_c \rangle_U \) would be exactly \( \rho_c \), and \( y_U \)
Figure 5. The dependence of $\eta_{max}$ (the maximum bubble top cloud water divided by $\rho_{max}$) on RH and $eH_s$. The increment for each line is 0.1. The white region is $\eta_{max} < 0$ ($f_0 < 0$) region, which is unphysical.

would equal to 1. In the bubble, $\rho_c$ is larger in UR and smaller in LR, while $\rho_c$ is smaller in UR and larger in LR. Thus, the spatial overlap is always small for both UR and LR. As for temporal overlap, in UR, both $\rho_c$ and $\rho_r$ are quite unsteady, and their distribution is similar, while $\rho_c$ in LR is quite steady (will be explained in section 4.2.3). Thus, the general overlap of UR is weaker than LR, and we empirically choose $\gamma_U = 3$ and $\gamma_L = 1.5$. Though the choice involves arbitrary factor, Figure 11 shows that the total collection of the semianalytical solution is very close to the numerical solution for most of the sensitivity tests, so $\gamma_U$ and $\gamma_L$ are universal.

An alternative way to linearize the collection term is to let $\rho_c$ be a constant and $\rho_c$ be a variable, as has been frequently used in idealized cloud-resolving models (e.g., Emanuel, 1986; Majda et al., 2010), cumulus parameterization (e.g., Zhang & McFarlane, 1995), and 1D cloud model (Ogura & Takahashi, 1971; Suzuki & Stephens, 2009). The advantage is being able to treat auto-conversion and collection in a unified way and introduce a bulk conversion time scale. However, such scheme neglects the physical nature of collection process—the positive feedback that more rainwater can collect more cloud water.

The procedure for the semianalytical solution is introduced in the following sections and is highlighted here as follows:

Step 1 (section 4.2.1): Judge whether rain can form by accurately solving the bubble top cloud water evolution. If rain can form, solve the bubble top auto-conversion start time $t_1$ and end time $t_2$ semianalytically.

Step 2 (section 4.2.2): Solve rain formation process in the UR analytically.

Step 3 (section 4.2.3): Solve rain formation process in the LR semianalytically.

In fact, we cannot get a one-line expression of FE for the semianalytical solution, just like finite-difference numerical solution. However, the main value lies in the understanding gained along the road—The good match with numerical solution validates the physical approximations we made. What is more, the semianalytical solution is computationally much cheaper than the finite-difference solution, which is a desirable property for climate modeling. An approximate analytical solution, which is a degeneration of the semianalytical procedure, is found for the special case $\rho_{th} = 0$ g m$^{-3}$ and will be discussed in section 5.

### 4.2.1. Rain Formation at Bubble Top

Let the bubble top $\rho_c$ be $\rho_{ct}$. As there is no rainwater at bubble top, the bubble top parcel evolves without the nonlocal collection effect and could be viewed as an independent parcel model. The governing equation for $\rho_{ct}$ in bubble coordinate is as follows:

$$\partial_t \rho_{ct} = f_c - \frac{\rho_{ct} - \rho_{th}}{\tau_c} - \max\left\{ \frac{\rho_{ct} - \rho_{th}}{\tau_c}, 0 \right\}.$$  (18)

If $\rho_{ct}$ cannot attain $\rho_{th}$, none elsewhere in the bubble can $\rho_c$ attain $\rho_{th}$. Thus, we first calculate the maximum attainable $\rho_{ct}$ without auto-conversion and compare it with $\rho_{th}$ to obtain a basic rain formation criterion.

Through the calculation in Appendix B, for infinitely fast supersaturation relaxation $\tau_{sp} \to 0$ case, the criterion is simplified to

Rain formation criterion: $\{\rho_{ct}\}_{max} = \eta_{max} \rho_{th} \geq \rho_{th},$  (19)

where $\eta_{max}$ is an efficiency that measures the maximum ratio of bubble vapor that can be converted to cloud water:

$$\eta_{max} \equiv \left[ 1 - (1 - \text{RH}) eH_s \right] \frac{[eH_s]^{\gamma_{Hs}} - [eH_s]^{\gamma_{Hs}}} {eH_s - 1}.$$  (20)

Whether $\rho_{ct}$ can reach $\rho_{th}$ depends on the vapor content itself and $\eta_{max}$. $\eta_{max}$ only depends on two non-dimensional parameters RH and $eH_s$, as is shown in Figure 5. The left middle bracket in equation (20) is
the CE for $\tau_m \to 0$ and represents the competition between condensation and vapor lateral mixing. The right middle bracket represents the competition between condensation and cloud water detrainment. Both competitions are characterized by $\varepsilon_{Hs}$. For weak lateral mixing situation (small $\varepsilon_{Hs}$), the dependence of $\eta_{max}$ on RH is weak because the parcel is not significantly influenced by the environment, and $\eta_{max}$ is anyhow close to 1. We conclude that larger $\varepsilon_{Hs}$, smaller RH, smaller $\rho_{vs}$, and larger $\rho_{th}$ make precipitation less likely. For the reference value $\varepsilon = 0.33 \text{ km}^{-1}$, $H_s = 3.30 \text{ km}$, and RH = 0.7, we have $\eta_{max} = 0.23$. To visualize the physics, we notice that $\rho_{vs}$ is directly related to LCL temperature (see Appendix A) and therefore decreases with increasing cloud bottom height and $\varepsilon$ is inversely proportional to cloud radius $R$. This matches our intuition that a narrow cloud with high bottom is less likely to precipitate. The cloud in Figure 6 is such an example. If rain formation criterion is satisfied, the solution to $\rho_{ct}$ is piecewise at the three intervals: $t < t_1$, $t_1 \leq t \leq t_2$, and $t > t_2$. They are documented in Appendix B.

4.2.2. Rain Formation in the UR

In UR region, autoconversion is active. We will not incorporate rainwater detrainment directly in the derivation but will use the solution to show that the addition of $\tau_{mr}$ only has tiny influence. From equations (5) to (7), we get the UR governing equation:

$$\partial_t \rho_c = f c - \frac{\rho_c - \rho_{th}}{\tau_c} + K(\rho_c U \rho_r) - \frac{\rho_c}{\tau_m} \quad (21a)$$

$$\partial_t \rho_r - V T \partial_z \rho_r = \left( \frac{\rho_c - \rho_{th}}{\tau_c} + K(\rho_c U \rho_r) \right) \quad (21b)$$

They can be transformed to

$$\partial_t \rho_{cU} = f_U - \frac{\rho_{cU}}{\tau_c} + \frac{\rho_r}{\tau_{rU}} \quad (22a)$$

$$\partial_t \rho_r - V T \partial_z \rho_r = \frac{\rho_{cU}}{\tau_c} + \frac{\rho_r}{\tau_{rU}} \quad (22b)$$

where

$$f_U \equiv f c - \frac{\rho_{th}}{\tau_m} \quad (23a)$$

$$\rho_{cU} \equiv \frac{\rho_c}{\tau_c} - \frac{\rho_{th}}{\tau_m} \quad (23b)$$

$$\tau_{rU} \equiv \frac{1}{K(\rho_c U) K((\rho_{cU} U + \rho_{th})} \quad (23c)$$

To solve precipitation strength at $z = -d U$, we implement the temporal integration operator $\int_{t_1}^{t_2} V_{U} \partial_z \rho_r dt$ on equations (22a) and (22b) to get:

$$\int_{t_1}^{t_2} \frac{\rho_{cU}}{\tau_c} + \left( \frac{1}{\tau_c} \right) \rho_{cU} \quad (24a)$$

$$- V T \frac{\rho_r}{\tau_{rU}} \quad (24b)$$

The integrations $\int_{t_1}^{t_2} \partial_z \rho_r dt$ and $\int_{t_1}^{t_2} V_{U} \partial_z \rho_r dt$ approximately vanish because $\rho_{cU}$ and $\rho_r$ are small at the edge of UR. The $\int_{t_1}^{t_2} \rho_{cU}$ is a function of $z$, but we approximate it to as the bubble top parcel’s accumulated condensation, which does not depend on $z$.

Figure 6. A photo of narrow nonprecipitating cumulus cloud taken by the authors near Duke University, NC, USA.
The tedious expression is put in Appendix D1.

Substituting equation (24a) into equation (24b) to eliminate \( \rho_c U \), we obtain a first-order ordinary differential equation (ODE) about \( \rho_r \):

\[
\frac{d\rho_r}{dz} + \frac{\sigma}{\tau_c V_T} \rho_r = -\frac{\int_U^{UR} \left( f_c(z-0) - \frac{\rho_{th}}{\tau_m} \right) dt}{V_T},
\]

where \( \sigma \) measures the relative strength of lateral mixing to auto-conversion:

\[
\sigma \equiv \frac{\tau_c}{\tau_c + \tau_m}.
\]

As bubble top should be free of rainwater physically, we use \( \rho_{th} U |_{z=0} = 0 \) as the boundary condition of equation (26) and solve it:

Figure 7. The same as the upper panel of Figure 2 but for \( \rho_{th} = 0.5 \text{ g m}^{-3} \).

Figure 8. The dependence of CE and FE on the 11 parameters for auto-conversion threshold \( \rho_{th} = 0.01 \text{ g m}^{-3} \) case (except for the subplot (d) of \( \rho_{th} \)). The dashed black line is CE, the solid black line is the FE from numerical solution, the solid red line is the FE from semianalytical solution, and the solid blue line is the FE\text{sim}, which is the simplified analytical solution (not shown in the subplot that changes \( \rho_{th} \)). The blue shadow denotes \( d_U = d_0 \) cases. The 11 subplots are arranged with descending order of FE sensitivity of \( \rho_{th} = 1 \text{ g m}^{-3} \) case, which is measured by the difference between the maximum and minimum FE of the numerical simulations for each parameter.
ρ_r \equiv f_U \frac{1 - \sigma}{\tau_r U} \left( e^{-\frac{\sigma}{\tau_r U V_T}} - 1 \right). \tag{28a}

\bar{\rho}_c U = \frac{\int_{-d_U}^{d_U} \rho_c U \, dz}{\tau_U} = \frac{\int_U^{\tau_U} - \bar{\rho}_c U \, \tau_U - \frac{1}{\tau_U}}{\tau_U \tau_m} \left[ 1 + \frac{(1 - \tau_U V_T)}{\tau_U V_T} \right]. \tag{28b}

In solving \bar{\rho}_c U we have used equations (24a) and (28a). Equations (28a) and (28b) show that there is more \rho_r in the lower part than the upper part and less \rho_c U in the lower part than the upper part, agreeing with the numerical solution in Figure 2. Thus, more cloud water is detrained in the bubble’s upper half. The vertical variation of \rho_c U is smaller when the collection length scale \tau_r U V_T is larger, corresponding to weaker collection.

The \rho_c U is calculated as the zero point of \bar{\rho}_c U in equation (28b):
\bar{\rho}_c U(z = -d_U) = 0 \Rightarrow d_U = \min \left\{ d_b, -\frac{\tau_r U V_T}{\tau} \frac{\ln(1 - \sigma)}{1 - \frac{1}{\tau_m}} \right\}. \tag{29}

The \rho_c U tends to be smaller than \rho_b when \tau_r U is small. In this case, both UR and LR exist, and \rho_c U is proportional to the collection length scale \tau_r U V_T. It corresponds to strong collection case as well as large \rho_b case. As \frac{1}{1 - \sigma} \ln(1 - \sigma) is a monotonically increasing function of \sigma that is always larger than 1, \rho_c U is increasingly larger than \rho_b with increasing \sigma (stronger lateral mixing). Thus, both stronger collection and weaker lateral mixing make \rho_c U smaller and \bar{\rho}_c U more concentrated at bubble’s upper part. For the sensitivity test of \rho_b = 1 g m^{-3} case (shown in Figure 9), d_U < \rho_b is always valid. For the \rho_b = 0.01 g m^{-3} case (shown in Figure 8), \rho_c U is more common. As such small \rho_b case is rare, the tedious solution of \bar{\rho}_c U and bubble bottom precipitation for \rho_c U = \rho_b case are derived in Appendix C. In the text below, we will only consider \rho_c U < \rho_b.

Substitute equation (28b) into equation (15), which is a closure that links \langle \rho_c U \rangle_U = \bar{\rho}_c U; we get its expression directly:
\langle \rho_c U \rangle_U d_{cU} = \frac{1}{d_U} \int_U^{\tau_U} \bar{\rho}_c U \, dz
= \frac{\int_U^{\tau_U} \rho_c U \, \tau_U - \frac{1}{\tau_U}}{\tau_U \tau_m} \left[ \frac{1}{\tau_U V_T} \ln(1 - \sigma) \right]. \tag{30}

The rainwater density at \rho_c U = -d_U is obtained by subtracting cloud water detrainment from total condensation:

Figure 9. The same as Figure 8 but for \rho_b = 1 g m^{-3} case. The simplified analytical solution \text{FE}_\text{sim} is not plotted because it does not apply to this case. In most cases we have \rho_b < \rho_b, so there is no blue shadow. The CE is the same as \rho_b = 0.01 g m^{-3} case and is plotted here only for reference.
It has used equations (24a), (29), and (30). Note that \( \tau_U \) is defined in equation (23c). The \( \mathcal{P}_r \) is very elegant—it is proportional to collection time scale \( \tau_U \), which carries most of the factors inside. When the collection effect is stronger, \( \tau_U \) and \( d_U \) are smaller, the accumulated rainwater before exhausting \( \rho_c \) turns out to be smaller (“drain the pond to catch the fish”). For the reference run, \( \tau_U \) is only 121 s. From Figure 2, we know the portion of rainwater formed at UR out of the total conversion is small. Its major role is seeding the LR and enhancing the collection there.

\[
\mathcal{P}_r \bigg|_{d_c=d_t<\tau_U} = \frac{\int_{U_t}^{U} \left( \frac{\rho_{\text{th}}}{U_{t}} \right) U_{t} \frac{\tau_{U}}{\tau_{m}} \, dU}{V_f} = \int_{U_t}^{U} \tau_{U} \tag{31}
\]

Figure 10. For \( \rho_{\text{th}} = 1 \text{ g m}^{-3} \) case, the 11 parameters sensitivity for bubble top auto-conversion time span \( t_2 - t_1 \) (solid black line), the \( H_{r_t}/w_t + (\tau_{m}^{-1} + \tau_{c}^{-1})^{-1} \) (solid red line), 10 times of the upper region collection time scale \( \tau_{UL} \) (dashed blue line), and 10 times of the lower region collection time scale \( \tau_{UL} \) (dashed green line). For each subplot the absissa is time (hr).

Figure 11. The 11 parameters sensitivity tests of the ratio of total auto-conversion to total condensation is shown as the blue lines. The ratio of total conversion to total condensation is shown as the red lines. Of them, the dashed lines denote numerical solution for \( \rho_{\text{th}} = 0.01 \text{ g m}^{-3} \) case, the solid lines denote the numerical solution for \( \rho_{\text{th}} = 1 \text{ g m}^{-3} \) case, and the dotted lines denote the semianalytical solution for \( \rho_{\text{th}} = 1 \text{ g m}^{-3} \) case.
Now we consider the role of rainwater detrainment. It will appear as $-\rho_r/\tau_{mr}$ at the right-hand side of equations (21b) and (22b). Compare it to the collection term $\rho_r/\tau_{mr}$, As $O(\tau_{mr})=O(\tau_m)\sim 10^3$ s, $\rho_r$ detrainment can be well neglected in the rainwater equation of UR unless lateral mixing is very strong.

There is one special regime to note. For roughly $\rho_{th} < 0.8$ g m$^{-3}$ regime (other parameters identical to the reference run), the bottom line of auto-conversion region becomes hard to grasp because $\rho_{th}$ is so small that the $\rho_c$ concentrates in a small portion of the auto-conversion region. Figure 7 shows an example for $\rho_{th} = 0.5$ g m$^{-3}$ where auto-conversion extends to the bubble bottom in numerical solution, but the semianalytical model predicts $d_{L2} \approx 0.5d_L$. As a result, a small change in $\rho_{th}$ can lead to a large shift of $d_{L2}$, that is why the bulky solution idea, which is designed only to qualitatively grasp the large $\rho_c$ region, fails to tell the correct lower boundary of auto-conversion region. However, even if the auto-conversion region is predicted to be too narrow by the semianalytical model, the FE error should be small. We use a scale analysis to prove this. Let the interior region $\rho_c$ scale be 1.5 g m$^{-3}$ and $K$ and $\tau_c$ be the reference value; the upper bound for the ratio of auto-conversion to collection in the lower part is as follows:

$$\frac{\text{AUT}}{\text{CLC}} = \frac{\tau_c^{-1}(\rho_r-\rho_{th})}{K\rho_r} < \frac{\tau_c^{-1}}{K\rho_r} = \frac{10^{-3}s^{-1}}{6 \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-1}\times0.0015 \text{ kg m}^{-3}} = \frac{1}{9}. \quad (32)$$

Thus, the missed lower part’s auto-conversion does not introduce significant error.

### 4.2.3. Rain Formation in the LR

In LR region, auto-conversion is inactive, so the problem is similar to the evolution of stratiform precipitation due to rainwater seeding from above (Haiden, 1995) but with the distinct lateral mixing. We consider rainwater detrainment in LR, which will be shown to be important for bubble bottom precipitation. The governing equation is as follows:

$$\frac{\partial}{\partial t} \rho_c = f_c = \frac{\rho_r}{\tau_{rl}} - \frac{\rho_c}{\tau_m}, \quad (33a)$$

$$\frac{\partial}{\partial z} \rho_c - V_T \frac{\partial}{\partial t} \rho_c = \frac{\rho_r}{\tau_{rl}} - \frac{\rho_c}{\tau_m}, \quad (33b)$$

where

$$\tau_{rl} = \frac{1}{K(\rho_c)_L} \quad \text{and} \quad \tau_m = \frac{1}{K(\rho_c)_m}. \quad (34)$$

As $\rho_r$ has grown large, the dominant physics is the fast collection of condensation by rainwater: $f_c \sim K\rho_r \rho_c$. No matter how strong the collection is, certain amount of $\rho_c$ is retained to guarantee that $K\rho_r \rho_c$ is nonzero, so $\rho_c$ is controlled by $\rho_r$ in LR. Such tiny but nonzero $\rho_c$ can be detrained, and the collection rate can be reduced.

Implement temporal integration operator $\int^{t_2}_{t_1} \frac{d\rho_c}{d\tau} \frac{1}{\tau} d\tau$ on equations (33a) and (33b), we get

$$\rho_c|_{t=t_2} - \rho_c|_{t=t_1} = \int_{t_1}^{t_2} \left[ \frac{\rho_r}{\tau_{rl}} - \frac{\rho_c}{\tau_m} \right] \frac{1}{\tau} dt, \quad (35a)$$

$$-V_T \frac{d\rho_c}{dz} = \frac{\rho_r}{\tau_{rl}} - \frac{\rho_c}{\tau_m}. \quad (35b)$$

The tendency term of $\rho_r$ equation vanishes because rainwater concentrates within LR but that of $\rho_c$ does not because the initial and terminal cloud water is not necessarily 0. The $\rho_c$ entering LR is nonzero when auto-conversion threshold is active, because the condensation has already produced some $\rho_c$, which accumulates without auto-conversion in PR region. It strengthens the collection in LR. The $\rho_c$ departing LR is always nonzero, and they serve as the “cloud wreck,” which is not washed out by precipitation.

Figure 2 tells that there is a fast rise of $\rho_c$ from 0 near LCL due to condensation and then a decrease due to the strong collection and may finally have a weak rise due to the residue condensation after most of the rainwater has fallen out. The condensation and collection decrease with time at similar rate, making $\rho_c$ roughly steady in LR.
above LCL. As a simplification, we assign \( \tilde{\rho}_{c(z=t_1-z)/V_z} \) based on the temporal average value \( \overline{\rho_c} / (t_1-t_1) \) and the percentage of the border line that lies within the condensation region \( \eta_t \):

\[
\tilde{\rho}_{c(z=t_1-z)/V_z} = \frac{\overline{\rho_c}}{t_2-t_1} \eta_t,
\]

where \( \eta_t \in [0,1] \) is a piecewise function of the model parameter rather than \( z \) and \( t \), as is introduced in Appendix D2. The terminal value \( \tilde{\rho}_{c(z=t_2-z)/V_z} \) is simply set as the temporal average value:

\[
\tilde{\rho}_{c(z=t_2-z)/V_z} = \frac{\overline{\rho_c}}{t_2-t_1}.
\]

The tendency term can be combined with the lateral mixing term by introducing a bulky damping term \( \bar{\tau}_m \), so equation (35a) can be rewritten as follows:

\[
0 = \varepsilon_c^{LR} - \frac{\overline{\rho_c}}{\bar{\tau}_m} - \frac{\overline{\rho_c}}{\tau_{m1}},
\]

where

\[
\varepsilon_c^{-1} = \varepsilon + (1-\eta_t)(t_2-t_1)^{-1}.
\]

The \( \varepsilon_c^{LR} \) is an approximation to \( \varepsilon_c \), and is independent of \( z \), as is shown in Appendix D3. Equations (35b) and (38) can be combined into a first-order ODE, but there are two candidates for the boundary condition: \( \overline{\rho_c} \big|_{z=-d_b} = \rho_{hi}(t_2-t_1) \) and \( \overline{\rho_c} \big|_{z=-d_U} \), which has been solved in equation (31). First, as we have discussed at the end of section 4.2.2, the \( \rho_c = \rho_{hi} \) contour line is not accurately captured by \( z = -d_b \) for small \( \rho_{hi} \), so the first candidate can cause large error in this case. Second, as we focus on rain formation, the consistency of rain-water is more important. Thus, we will use \( \overline{\rho_c} \big|_{z=-d_U} \) as the boundary condition.

Equation (35b) can be directly integrated as follows:

\[
\overline{\rho_c} = \overline{\rho_c} \big|_{z=-d_U} e^{-\frac{d_U}{\varepsilon_c}(\tau_{m1}^{LR} - \tau_{m1})},
\]

Substitute equation (40) into equation (38) and do vertical average from \( -d_b \) to \( -d_U \) and then use equation (34); we obtain a transcendental equation for \( \overline{\rho_c} \):

\[
\tau_{L1} \overline{\rho_c} \big|_{z=-d_b} + \frac{K(\overline{\rho_c} \big|_{z=-d_b})}{d_b-d_U} \left( \frac{Z_t}{\varepsilon_c} + \frac{d_U}{\varepsilon_c} \right) e^{-\frac{d_U}{\varepsilon_c}(\tau_{m1}^{LR} - \tau_{m1})} e^{\varepsilon_c} = \varepsilon_c^{LR} + \frac{\overline{\rho_c} \big|_{z=-d_U} + \frac{V_T}{d_b-d_U} \overline{K(\overline{\rho_c} \big|_{z=-d_U})} \left( \frac{(d_U-d_U)}{V_T} \right)}{d_b-d_U} \left( \frac{(d_U-d_U)}{V_T} \right) \left( \frac{(d_U-d_U)}{V_T} \right)^{-1} = \varepsilon_c^{LR}.
\]

The \( e^{(d_U-d_U)/(K(\overline{\rho_c} \big|_{z=-d_U})/V_T)} \) can be approximated as a Taylor series of \( (d_U-d_U)/(K(\overline{\rho_c} \big|_{z=-d_U})/V_T) \) and discarding terms higher than the cubic one will render sufficient accuracy. However, this leads to a cubic equation whose tedious solution is hard to interpret. Thus, we solve the full equation (41) numerically by choosing the most accurate value from a 10^4 long array of candidate \( \rho_{hi} \) within [0,5] g m^-3. Only one root seems to exist within physically meaningful range. \( \rho_{hi} \) is then used to calculate \( \tau_{L1} \) with equation (34). As will be shown in Figure 10, for \( \tau_{L1}^{LR} \) case, \( \tau_{L1} \) (for the reference run is 194 s) is clearly larger than \( \tau_{L1}^{LR} \), because LR has less cloud water than UR. \( \tau_{L1} \) is still larger than \( \tau_{L1}^{LR} \), but the scale is closer.

Though equation (41) is not analytically solvable, we can still gain some understanding from the equation itself, after doing Taylor expansion of the exponential term. Notice that both terms on the left-hand side are monotonically increasing with \( \rho_{hi} \). Thus, \( \rho_{hi} \) increases with larger LR condensation \( \varepsilon_c^{LR} \), shorter \( \tau_{L1} \), longer damping time scale \( \tau_{m1} \), and shorter rain residence time scale \( (d_b-d_U)/V_T \).
The bubble bottom precipitation is obtained by substituting \( \langle \rho \rangle_L \) into equation (40) and using equation (31):

\[
\text{PCP} = V_T \tau_f \int_{z_{-d_b}}^{z_{-d_b}} = V_T \tau_f \int_U U e^{\frac{\tau_f}{R_L^2} (\tau_{sp}^L - \tau_{sp}^L)}. \tag{42}
\]

Rainwater detrainment damps bottom precipitation not only through the direct effect shown as the \( e^{(d_b-d_c)/U} \) factor in equation (42) but also through damping collection shown as the decrease of \( \langle \rho \rangle_L \) (or increase of \( \tau_{sp} \); see equation (41)). The bubble total auto-conversion and collection can be calculated by adding total rainwater detrainment (using equation (40)) to PCP:

\[
\int_0^\infty \int_0^{d_b} (\text{AUT} + \text{CLC})dzdt = \text{PCP} + \int_0^{d_b} \int_U \tau_f \int_U U e^{\frac{\tau_f}{R_L^2} (\tau_{sp}^L - \tau_{sp}^L)} \left\{ 1 + \frac{\tau_{sp}^L}{\tau_{sp}^L - \tau_{sp}^L} \left[ \frac{d_b - d_{U}}{V_T} (\tau_{sp}^L - \tau_{sp}^L) - 1 \right] \right\}. \tag{43}
\]

The contribution from UR corresponds to the first term (“1”) in the large bracket, and that from LR is the second term. The total conversion only feels rainwater detrainment through its indirect effect on LR collection.

5. The Formation Efficiency

In this model, the total FE of a thermal bubble over its lifetime is defined as the ratio of total conversion (auto-conversion plus collection) in the bubble to the total condensation:

\[
\text{FE} = \frac{\int_0^\infty \int_0^{d_b} (\text{AUT} + \text{CLC})dzdt}{\text{CND}}. \tag{44}
\]

This is the FE definition that will be used in the numerical simulation. FE is calculated by substituting equations (13), (43) for \( d_U < d_b \) case, and (C.6) for \( d_U = d_b \) case into equation (44).

In this section, we will do sensitivity tests on 11 parameters: \( \varepsilon, \rho_{\rho_{\text{th}}, \rho_{\text{th}}}, d_b, V_T, \text{RH}, \tau_{sp}, K, w_c, \) and \( \tau_{sp} \) and pick out the most sensitive ones. The dependence of FE on each parameter for both numerical simulation and semianalytical solution is shown in Figure 8 for \( \rho_{\text{th}} = 0.01 \text{ g m}^{-3} \) and Figure 9 for \( \rho_{\text{th}} = 1 \text{ g m}^{-3} \) case. For each subplot, only one parameter is changing. The perturbation range for that parameter and the reference values for other parameters are shown in Table 1. The range is roughly the possible value in terrestrial climate, except that of \( \varepsilon, \rho_{\rho_{\text{th}}, \rho_{\text{th}}}, \tau_{sp}, \) and \( \tau_{sp} \) are enlarged for theoretical interests. For reference run, \( \tau_{sp} \) is set to infinity due to its uncertainty. In general, the semianalytical solution matches well with the numerical solution: The magnitude is similar, and all the trends are qualitatively correct. The trends also agree with the previous papers’ results summarized in section 1. Note that the performance for \( \tau_{sp} \) is least accurate.

Figure 10 shows that \( t_2 - t_1 \) is generally larger than \( H_i/w_c + (\tau_{sp}^L - \tau_{sp}^L) \) for \( \rho_{\text{th}} = 1 \text{ g m}^{-3} \), so the \( \tau_{sp} \) and \( \tau_{sp} \) shown in equations (17a) and (17b) will mostly depend on the latter. Figure 11 shows the contribution of AUT and CLC to FE for both the numerical and the semianalytical solution, with \( \text{FE}_{\text{lsu}, \text{AUT}} \) denoted as the AUT contribution for FE in semianalytical solution and \( \text{FE}_{\text{lsu}, \text{CLC}} \) for that of CLC. Using equations (15), (24a), and (44), we get

\[
\text{FE}_{\text{lsu}, \text{AUT}} = \frac{\int_0^{d_b} \int_U U / \tau_{sp} dz}{\text{CND}} = \frac{U d_U \langle \rho_{\rho_{\text{th}}} \rangle_U / \tau_{sp}}{\text{CND}}, \quad \text{FE}_{\text{lsu}, \text{CLC}} = \text{FE} - \text{FE}_{\text{lsu}, \text{AUT}}. \tag{45}
\]

The \( \rho_{\rho_{\text{th}}} \) (or equivalently \( T_{\text{LCL}} \)), fractional entrainment rate \( \varepsilon \), and threshold \( \rho_{\text{th}} \) are the leading order factors; one of the reasons is that they are involved in the rain formation criterion discussed in section 4.2.1. \( \rho_{\text{th}} \) is the physically most important one: It is the only parameter that can make zero precipitation possible.

For understanding other sensitivities that do not critically depend on auto-conversion threshold, we present an analytical FE for a special regime, which satisfies \( \rho_{\text{th}} = 0 \) and \( d_U < d_b \). The latter is valid when collection is not too weak and lateral mixing is not too strong. The cloud water detrainment in LR is further neglected. Rainwater detrainment, which is later on shown to have little influence on FE, is completely neglected.
Then we get the simplified case formation efficiency FE$_{sim}$ after using equations (13), (29), (30), (31), and (43) and by approximating LR condensation (directly converted to rainwater) as CND($d_b-d_U$)/$d_b$:

$$ FE_{sim} = \frac{V_T \gamma_U \left( 1 + \frac{H_s}{\nu_c} \tau_m \right)}{K_d \rho_{v0} \gamma_{v0} CE} \frac{1 + \tau_U \left( \tau_c^{\prime} + \tau_m^{\prime} \right) \ln(1 - \sigma)}{1 + \frac{H_s}{\nu_c} \tau_c^{\prime} \tau_m^{\prime} \ln(1 - \sigma)} $$

(46)

In the last equality we have used equation (17a), and $\tau_U = \gamma_U \left[ H_s/\nu_c + \left( \tau_m^{\prime} + \tau_c^{\prime} \right)^{-1} \right]$ is used because $t_2 - t_1 \to \infty$ for $\rho_{v0} = 0$ gm$^{-3}$ case. The FE$_{sim}$ is plotted as the solid blue line in Figure 8. Note that FE$_{sim}$ does not apply to the blue shadow region in Figure 8, which denotes $d_U = d_b$ region. Within $d_U < d_b$ region, FE$_{sim}$ generally overestimates FE due to the neglect of cloud water detrainment in LR, but it basically grasps the trend. FE$_{sim}$ increases with condensation amount $\rho_{v0} d_b$, manifesting the nonlinearity due to collection effect.

The FE$_{sim}$, which depends on nine parameters, can be compactly expressed with five nondimensional parameters, utilizing characteristic length scale $\epsilon$,$^{-1}$, velocity scale $\nu_c$, and water density scale $V_T/(K_d)$:

$$ FE_{sim} = 1 - \frac{\gamma_U \left( 1 + \frac{\bar{\nu}_p}{\nu_c} \right)}{\bar{\rho}_{v0} \left[ 1 - \left( 1 - RH \right) H_s \right]} $$(47)

where

$$ \bar{\rho}_{v0} = \frac{K_d}{V_T}, \bar{H}_s = \epsilon H_s, \bar{\nu}_p = \epsilon \nu_p, \bar{\tau}_e = \epsilon \tau_e, \text{ and RH}. $$

$\rho_{v0}$ is rescaled with precipitation-related parameters: Even if $\rho_{v0}$ is small, its role can be compensated by larger $K$ and longer rain residence time $d_b/V_T$. This confirms that one of the nonlinear factors that make FE depend on $\rho_{v0}$ is collection process (the other is autoconversion threshold, which is not included in FE$_{sim}$). The rescaling of $\bar{H}_s$, $\bar{\nu}_p$, and $\bar{\tau}_e$ all involve $\epsilon$, showing the almost ubiquitous competition with mixing in rain formation process. It is obvious that higher FE$_{sim}$ is favored by larger $\bar{\rho}_{v0}$ due to the nonlinearity of collection term, smaller $\bar{\nu}_p$ and higher RH due to the raise of CE. The dependence on $\bar{H}_s$ and $\bar{\tau}_e$ involves multiple parameters. FE$_{sim}$ decreases with increasing $\bar{H}_s$ due to the cooperation of smaller CE (denominator) and less concentrated condensation and therefore collection (numerator).

Doing Taylor expansion on $\ln(1 + \bar{\tau}_e)$ in the numerator shows that FE$_{sim}$ always grows with increasing $\bar{\tau}_e^{-1}$, due to the direct effect of stronger auto-conversion and the subsequent collection, as well as the indirect effect of more concentrated collection. Note that for infinitely fast auto-conversion ($\bar{\tau}_e \to 0$), FE should converge to 1 but FE$_{sim}$ and even the semianalytical solution do not. This is because $\tau_U$ is based on inaccurate intuitive estimation and for $\bar{\tau}_e \to 0$, it leads to a finite UR cloud water amount $\langle \rho_{d0} \rangle_{d_U < d_b}$ in equation (30), which should be 0.

The dependence mechanism of each parameter is discussed in more detail below.

### 5.1. Initial Vapor Content $\rho_{v0}$ and Saturated Vapor Scale Height $H_s$

As is shown in Appendix A, knowing any one of $\rho_{v0}$, $H_s$, and LCL temperature $T_{LCL}$ leads to the other two. Larger $T_{LCL}$ corresponds to larger $\rho_{v0}$ and $H_s$. For tropical marine boundary layer, which is close to saturation, a higher SST corresponds to moister mixed layer air and therefore larger $\rho_{v0}$ and larger $H_s$.

From Figure 9, we see that as $\rho_{v0}$ increases, FE grows steeply from close to 0. Such sharp transition is due to both the nonlinearity in auto-conversion threshold and collection. Larger $\rho_{v0}$ extends the auto-conversion available time $t_2 - t_1$, so the “vigorous rain formation region” UR and LR are enlarged, and the forcing $\tau^U_{LR}$ and $\tau^L_{LR}$ are much larger. Such increase is further amplified by the nonlinear collection effect.
To quantify the effect of growing $H_s$ at the same time, we conduct an additional test where $\rho_{\text{aut}}$ is fixed and $H_s$ increases from 2.6 to 3.6 km (wider than the interval of the normal sensitivity test). FE decreases by 0.1, in the opposite direction to increasing $\rho_{\text{aut}}$, but is much less significant. The mechanism of the dependence of $H_s$ has been discussed in its nondimensional form $\bar{H}_s$.

Our model agrees with the simulation of Lutsko and Cronin (2018) where higher SST leads to higher FE, and our theory supports and expands their inference that auto-conversion threshold plays an important role.

5.2. Fractional Entrainment Rate $\varepsilon$

Larger $\varepsilon$ physically corresponds to smaller cloud radius $R$ and indicates stronger lateral mixing. The FE decreases with increasing $\varepsilon$ almost linearly, agreeing with the qualitative results of Newton (1966) and Cohen and McCaul (2007). As $(1 - \text{FE})$ is proportional to the total detrainment $\int_0^\infty \int_{-\bar{d}_b}^{\bar{d}_b} \rho_c \, dz \, dt$, the quasi-linear dependence of FE on $\varepsilon$ implies that the total integration $\int_0^\infty \int_{-\bar{d}_b}^{\bar{d}_b} \rho_c \, dz \, dt$ is almost invariant with $\varepsilon$. This is verified by the weak dependence of total AUT ($\rho_c \varepsilon^{-1}$) on $\varepsilon$ shown in Figure 11, for $\rho_{\text{th}} = 0.01 \text{ g m}^{-3}$ (except for very small FE) case. It is because the stronger $\rho_c$ detrainment reduces the strength of auto-conversion but enlarges UR depth (see equations 28b and 29) by suppressing collection. For $\rho_{\text{th}}=1 \text{ g m}^{-3}$ case, the threshold makes the total $\rho_c$ integration not directly related to total AUT, and both AUT and CLC decrease with increasing $\varepsilon$.

5.3. Auto-Conversion Threshold $\rho_{\text{th}}$ and Time Scale $\tau_c$

Both $\rho_{\text{th}}$ and $\tau_c$ are auto-conversion parameters that depend on aerosol concentration. Figure 9 shows that as $\rho_{\text{th}}$ increases from 0 to 3 g m$^{-3}$, FE decreases from around 0.75 to 0.40 almost linearly. For $\rho_{\text{th}}=1 \text{ g m}^{-3}$, FE drops increasingly slower as $\tau_c$ increases from 0 to 10$^4$ s and remains around 0.45 for $\tau_c = 10^4$ s. The decaying slope of the sensitivity with increasing $\tau_c$ is predicted to be too steep in our semianalytical and simplified analytical model, likely due to the error in linearizing the collection term. Both $\rho_{\text{th}}$ and $\tau_c$ influence AUT significantly, but they have little influence on CLC except for very large $\rho_{\text{th}}$ where AUT approaches 0. Our explanation for the weak sensitivity on CLC is as follows: When auto-conversion is weaker (larger $\rho_{\text{th}}$ or $\tau_c$), the initial rain is expected to be weaker, but there is more collectable cloud water in turn to make it up. Thus, the two AUT parameters mainly influence AUT process itself.

The specific ways that $\rho_{\text{th}}$ and $\tau_c$ influence FE are also different. $\rho_{\text{th}}$ can change both the size of auto-conversion region (characterized by $t_2 - t_1$ and $d_U$) and the auto-conversion rate there. On the other hand, $\tau_c$ has little influence on $t_2 - t_1$ and moderate influence on $d_U$ (around 20% $d_U$ difference between $\tau_c = 500$ and 2,000 s with other parameters taking reference values), so it basically only changes the auto-conversion rate and is less decisive.

5.4. Rain Mean Terminal Fall Velocity $V_T$ and Collection Coefficient $K$

Both parameters are associated with rainwater. FE increases with $K$ and decreases with $V_T$. This is well captured by the semianalytical model and roughly acceptable by the special case FE$_{\text{lim}}$ where $V_T$ and $K$ always appear together as $V_T/K$. Physically, larger $K$ directly leads to more collection; smaller $V_T$ increases the duration time of rainwater in the bubble and indirectly increases collection. As is shown in Figure 11, for both $V_T$ and $K$, weaker collection increases auto-conversion because $\rho_c$ is more abundant, but they are too small to compensate for the decrease of collection. In the limit of $V_T \rightarrow \infty$, equation (7) shows that to make the rain advection term finite, there must be $\delta p / \delta z \rightarrow 0$. As $\rho_c$ is 0 at bubble top, this means that $\rho_c$ is tiny everywhere, and the precipitation is due solely to auto-conversion.

However, $V_T$ and $K$ in Kessler scheme are not completely independent. For example, as both utilize single raindrop's terminal fall velocity, which uses high Reynolds number drag law in their derivation, both are proportional to the square root of gravity (Kessler, 1969). Thus, in a planet of higher gravity, the rain not only falls faster but also collects faster, and $V_T/K$ does not change.

5.5. Bubble Thickness $d_b$

FE increases with $d_b$. As in most cases cloud water is concentrated at the upper part of the bubble, an extension of the bubble depth generally does not influence the auto-conversion process. It only extends the LR and increases the total collection there.
5.6. Tropospheric Relative Humidity RH and Phase Change Relaxation Time Scale \( \tau_{sp} \)
RH and \( \tau_{sp} \) indirectly influence FE through CE. As both the auto-conversion threshold and collection make the total conversion super-linear to the condensation forcing, FE increases with RH and decreases with \( \tau_{sp} \).

5.7. Updraft Velocity \( w_c \)
The FE decreases weakly with increasing \( w_c \) as a result of multiple competitive effects. On one hand, larger \( w_c \) makes condensation more impulsive and makes cloud water produced in a shorter time. On the other hand, the lateral mixing is stronger as is shown in the decrease of \( \tau_{sp} = (\rho_{th})^{-1} \). As for the rain formation criterion shown in equation (19), the competition breaks even, and the criterion is independent of \( w_c \). As for the rain formation process, larger \( w_c \) increases the peak magnitude of \( \rho_{wc} \), which helps it climb over the auto-conversion threshold and increases collection (through decreasing \( \tau_{fl} \)) at the same time, but cloud water detrainment is also stronger. In general, lateral mixing is more dominant in our sensitivity tests, so FE decreases with \( w_c \). Note that the semianalytical FE does not do well for \( w_c < 2 \text{ m s}^{-1} \) regime of \( \rho_{th} = 0.01 \text{ g m}^{-3} \) case.

Now we combine dynamical and kinematic factors to see how CE and FE depend on convective strength for a liquid cloud. A stronger convection tends to have larger CAPE, which could be due to higher initial vapor density \( \rho_{v0} \) and leads to larger \( w_c \), larger radius and therefore smaller \( \epsilon \) (Khairoutdinov et al., 2009), and larger \( d_b \) if the bubble is supposed to have unit aspect ratio. Thus, CE will increase with convective strength. All these factors except the less influential kinematic effect of \( w_c \) support higher FE for stronger convection.

5.8. Rain Detrainment Time Scale \( \tau_{mr} \)
\( \tau_{mr} \) can significantly influence the bubble bottom precipitation rate but much less for FE. The reason is that FE only feels rainwater detrainment through its indirect effect on collection. Figures 8 and 9 show that FE drops less than 0.05 as \( \tau_{mr} \) decreases from \( \infty \) to \( \tau_{mr}/2 \) for both \( \rho_{th} = 0.01 \text{ g m}^{-3} \) and \( \rho_{th} = 1 \text{ g m}^{-3} \) cases, as is captured by our semianalytical FE. In another perspective, in the lower part of the bubble where rainwater accumulates most efficiently through collection, collection rate is mainly constrained by condensation rate, so some rainwater loss does not make a difference.

6. Conclusions
The bulky property of rain formation process can be depicted by CE, which denotes the conversion from vapor to cloud water, the FE, which denotes the conversion from cloud water to rainwater, and SE, which denotes the ratio of rainwater that can reach the surface (Langhans et al., 2015). All three quantities are important in idealized climate models. For cumulus cloud, an analogy of CE has been derived by Romps (2014a) without allowing supersaturation. A semiempirical theory of FE without considering lateral mixing has been proposed by Seifert and Stevens (2010). Thus, a systematic theory of FE and SE is particularly desirable. One difficulty is understanding the detail of each microphysical process. The other is understanding their interactions based on simplified microphysical parameterization. This paper pursues the latter and provides a theoretical investigation of CE and FE and leaves SE for future work.

We constructed a very idealized one-dimensional kinematic model of uniformly ascending cylindrical bubble with auto-conversion, collection, and lateral mixing parameterizations adapted from classic schemes (e.g., Asai & Kasahara, 1967; Kessler, 1969). As the important ice phase is sacrificed in this preliminary investigation, the results are quantitatively more relevant to warm rain-dominated regime. As the dynamic detrainment is not considered, it is also different from shallow cumulus cloud whose height is constrained by stratification.

A modified CE that considers supersaturation is analytical solved. This effect acts as a \( \tau_{mr}/(\tau_{mr}+\tau_{sp}) \) damping factor multiplied on the original formulae of Romps (2014a). Physically, longer supersaturation relaxation time scale \( \tau_{sp} \) gives lateral mixing of vapor more opportunity and decreases CE. In practice, \( \tau_{sp} \) is at most tens of seconds for liquid cloud, so supersaturation has little influence on CE and FE within this model.
The FE depends on complicated interactions between auto-conversion, collection, and lateral mixing. The pure cloud water’s auto-conversion to rainwater can only start if the cloud water reaches a threshold. Neglecting supersaturation, we used it to establish a criterion for whether precipitation can occur. It depends on initial vapor content \( \rho_{v0} \), auto-conversion threshold \( \rho_{th} \), environmental RH, and the condensation scale height rescaled by fractional entrainment rate \( \epsilon \).

If rain can form, we can obtain a semianalytical solution of FE, which depends on 11 (kinematically) independent parameters and a simplified fully analytical FE. The latter is only for zero auto-conversion threshold case and could be compactly expressed with five nondimensional parameters. The idea is to identify a “vigorous rain formation region” whose temporal and spatial extent is constrained by bubble top auto-conversion threshold. We further decompose this region into the UR where cloud water is abundant and auto-conversion is playing an important role and the LR where there is generally no auto-conversion and collection is dominant. The collection term is linearized by solving for a collection time scale, which depends on certain average cloud water density in each region. The average thickness of UR is analytically found to be proportional to the collection length scale there.

In the sensitivity tests of FE for the 11 parameters, the semianalytical solution and the simplified analytical solution (only for zero auto-conversion threshold) agree qualitatively well with the numerical solution. The trends also agree with previous papers’ results summarized in section 1. Physically, the FE is determined by the competition between lateral mixing and all the conversion processes, which is additionally limited by the auto-conversion threshold. The most sensitive parameters include auto-conversion threshold \( \rho_{th} \), bubble initial vapor density \( \rho_{v0} \), and fractional entrainment rate \( \epsilon \). In real world, they are related to aerosol concentration, SST, and cloud radius. When \( \rho_{v0} \) is just large enough to produce rain by overcoming auto-conversion threshold, FE grows steeply. FE increases quasi-linearly with decreasing fractional entrainment rate \( \epsilon \). FE is weakly sensitive to changing updraft speed \( w_c \) as a kinematic parameter alone due to the near offset between the change in collection and lateral mixing. In real world, stronger convection tends to have larger \( \rho_{v0} \), larger \( w_c \), and smaller \( \epsilon \) at the same time, so FE is expected to be larger.

The model is far from complete. First, the semianalytical solution still fails to capture the dependence of FE on \( \tau_c \) accurately, probably due to the error in linearizing the collection term. Second, we need to validate this idealized setup with large eddy simulation or observation. Third, we need to dig out how these parameters depend on more inherent microphysical parameters such as CCN density, as well as boundary layer and free troposphere property. Possible specific extensions may involve ice phase, a concentrated dynamic detrainment at a height below the auto-conversion’s terminal height, and developing an SE model coupled with a downdraft plume.

**Appendix A: The Simplified Thermodynamics**

First, we derive the saturated vapor density \( \rho_{vs} \) largely following the simplification used by Romps (2014a). The saturated vapor pressure \( e_s \) is solved from Clausius-Clapeyron equation with the assumption that latent heat \( L_v \) (using \( 2.4 \times 10^6 \text{ J kg}^{-1} \)) is independent of temperature \( T \):

\[
e_s = e_{s0} e^{-\frac{R_v T}{L_v}},
\]

where \( R_v \) (using \( 461.5 \text{ J kg}^{-1} \text{ K}^{-1} \)) is water vapor gas constant and \( e_{s0} \) is a constant determined at LCL temperature \( T_{LCL} \), with relatively more accurate Clausius-Clapeyron equation (Emanuel, 1994). The \( \rho_{vs} \) is obtained with equation of state:

\[
\rho_{vs} = \frac{e_s}{R_v T} = \frac{e_{s0}}{R_v T} e^{-\frac{\epsilon}{\epsilon}}
\]

Let the model atmosphere has constant temperature lapse rate \( \Gamma \) (using a value close to moist adiabat: \( 5.0 \times 10^{-3} \text{ K m}^{-1} \)), and let the temperature in ground coordinate (with LCL as the origin \( z_g = 0 \) ) be

\[
T = T_{LCL} - \Gamma z_g
\]

We then conduct a series of approximation to obtain the \( \rho_{vs} \) that decays exponentially with height:
\[ \rho_{\text{vs}} = \frac{\epsilon_{\text{vs}}}{R_v(T_{\text{LCL}} - \Gamma_\theta)} e^{\frac{-z}{2}} \frac{1}{e^\frac{-z}{2}} \approx \frac{\epsilon_{\text{vs}}}{R_v T_{\text{LCL}}} e^{\frac{-z}{2}} \frac{1}{e^\frac{-z}{2}} = \rho_{\text{vs}} e^{-\frac{z}{2}}, \quad (A.4) \]

where \( \rho_{\text{vs}} \) is the \( \rho_{\text{vs}} \) at LCL and \( H_s \) is its scale height:

\[ \rho_{\text{vs}} \equiv \frac{\rho_{\text{vs}}}{R_v T_{\text{LCL}}} e^{\frac{-z}{2}}. \quad (A.5) \]

\[ H_s \equiv \frac{R_v T^2_{\text{LCL}}}{\Gamma_\theta}. \quad (A.6) \]

Note that either of them has a one-to-one relationship with \( T_{\text{LCL}} \). Second, we solve supersaturation relaxation time scale \( \tau_{\text{sp}} \) in the framework of Korolev and Mazin (2003). The diffusional growth rate of cloud density \( \rho_c \) (denoted as \( \frac{d\rho_c}{dt}|_{\text{diff}} \)) is related to single droplet (with radius \( r_w \)) growth rate through

\[ \frac{d\rho_c}{dt}|_{\text{diff}} = \rho_w N_w \int_0^\infty f_w(r_w) \frac{r_w^2}{\pi} \frac{dr_w}{dt} dr_w. \quad (A.7) \]

Here \( f_w(r_w) \) is drop size spectrum, \( N_w \) is droplet number concentration, and \( \rho_w \) is liquid water density. The growth rate of the single droplet radius \( r_w \) obeys the diffusional growth equation:

\[ \frac{dr_w}{dt}|_{\text{diff}} = \frac{A_w}{r_w} \left( \frac{\rho_w - \rho_{\text{vs}}}{\rho_{\text{vs}}} \right), \quad \text{with} \ A_w = \left( \frac{\rho_w T^2_{\text{LCL}}}{D_T R_v T_{\text{LCL}}} + \frac{\rho_w R_v T_{\text{LCL}}}{\epsilon v D_T} \right)^{-1}. \quad (A.8) \]

Here \( D_T \) is air heat conductivity, and \( D_v \) is vapor diffusion coefficient. Substituting equation (A.8) into equation (A.7), we get

\[ \frac{d\rho_c}{dt} \mid_{\text{diff}} = \frac{\rho_w - \rho_{\text{vs}}}{\tau_{\text{sp}}}, \quad (A.9) \]

where \( \tau_{\text{sp}} \) is

\[ \tau_{\text{sp}} \equiv \frac{\rho_{\text{vs}}}{4\pi \rho_w A_w N_w r_w}, \quad \text{with} \quad r_w = \int_0^\infty f_w(r_w) r_w \frac{dr_w}{dt}. \quad (A.10) \]

Now we estimate the magnitude of \( \tau_{\text{sp}} \). Korolev and Mazin (2003) estimated the available range of \( N_w r_w^2 \) for liquid cloud to be within \([400,10^4]\) m\(^{-2}\). Using \( T = 293 \) K, \( \rho_{\text{vs}} = 17.1 \) g m\(^{-3}\) (regarded as a constant), \( \rho_w = 10^3 \) kg m\(^{-3}\), and use the Table 7.1 of Rogers and Yau (1989): \( D_T = 2.55 \times 10^{-2} \) J m\(^{-1}\) s\(^{-1}\) K\(^{-1}\) and \( D_v = 2.52 \times 10^{-5} \) m\(^2\) s\(^{-1}\), we estimate the range of \( \tau_{\text{sp}} \) to be within \([1.1,27.3]\) s.

Third, we solve water vapor and condensation from equation (5) with \( \rho_{\text{vs}} \) in equation (A.4). Equation (5) is rearranged to let supersaturation vapor density \( \rho_s \equiv \rho_v - \rho_{\text{vs}} \) be the prognostic variable. Combining equations (1), (5), and (A.4), we get

\[ \partial_t \rho_s + w_c \partial_z \rho_s = -\rho_s \left( \frac{1}{\tau_{\text{sp}}} + \frac{1}{\tau_m} \right) - \left( \frac{1}{\tau_m} \right) \left( \frac{1}{\tau_m} \right) \left( \rho_s \partial_z w_c \right) = -\rho_s \left( \frac{1}{\tau_{\text{sp}}} + \frac{1}{\tau_m} \right) - \left( \frac{1}{\tau_m} \right) \left( \rho_s \partial_z w_c \right). \quad (A.11) \]

The bubble is assumed to be well mixed, with homogeneous potential temperature and water vapor density. Thus, the parcels have the common LCL and the same condensation experience. We will solve the \( \rho_c \) for the parcel at bubble top \( z = 0 \). Substituting equation (A.4) into equation (A.11), the temporal evolution of the parcel is governed by
The solution to \( \rho _{\text{ct}} \) without auto-conversion is as follows:

\[
\rho _{\text{ct}} = f_0 \left\{ \frac{e^{\frac{\rho _{\text{ct}}}{\kappa _c}} - e^{\frac{\rho _{\text{ct}}}{\kappa _m}}}{-w_c/H_{S} + 1/\tau _m} + \tau _p \left[e^{-\left(\frac{\rho _{\text{ct}}}{\kappa _c}\right)} - e^{-\left(\frac{\rho _{\text{ct}}}{\kappa _m}\right)}\right] \right\}. \tag{B.1}
\]

Here \( f_0 \) is defined in equation (11). As is shown in Figure 3, \( \rho _{\text{ct}} \) generally increases steeply with time first and then slowly decays. A rain formation criterion is obtained by comparing \( \rho _{\text{th}} \) with the maximum cloud water density the bubble top parcel can attain \( \rho _{\text{ct,max}} \). When supersaturation relaxation is infinitely fast \( \tau _p \rightarrow 0 \), \( \rho _{\text{ct,max}} \) has an analytical expression:

Rain formation criterion:

\[
\{\rho _{\text{ct}}\}_{\text{max}} = [1 - (1 - \text{RH})e H_s] \left[\frac{e [e H_s]^{\frac{1}{\tau_m}} - [e H_s]^{\frac{1}{\tau_c}}}{e H_s - 1}\right] \rho _{\text{sat}} \geq \rho _{\text{th}}, \tag{B.2}
\]

\[
t_{\text{max}} = -\ln(e H_s) \frac{e H_s}{1 - e H_s} \tau _m. \tag{B.3}
\]

The maximum \( \rho _{\text{ct}} \) is attained at time \( t_{\text{max}} \).

If rain formation criterion is satisfied, the solution to \( \rho _{\text{ct}} \) at the three intervals: \( t < t_1, t_1 \leq t \leq t_2, \) and \( t > t_2, \) is solved below. The governing equation and initial condition for the three intervals are as follows:

\[
\frac{d \rho _{\text{ct}}}{dt} = f e^{-\frac{\rho _{\text{ct}}}{\tau _c}}, \rho _{\text{ct}|t=0} = 0, \tag{B.4a}
\]

\[
\frac{d \rho _{\text{ct}}}{dt} = f e^{-\frac{\rho _{\text{ct}}}{\tau _m}} - \frac{\rho _{\text{ct}} - \rho _{\text{th}}}{\tau _c}, \rho _{\text{ct}|t=t_1} = \rho _{\text{th}}, \tag{B.4b}
\]

\[
\frac{d \rho _{\text{ct}}}{dt} = f e^{-\frac{\rho _{\text{ct}}}{\tau _m}}, \rho _{\text{ct}|t=t_2} = \rho _{\text{th}}. \tag{B.4c}
\]

The solutions are as follows:

\[
\rho _{\text{ct}} = f_0 \left\{ \frac{e^{\frac{\rho _{\text{ct}}}{\kappa _c}} - e^{\frac{\rho _{\text{ct}}}{\kappa _m}}}{-w_c/H_{S} + 1/\tau _m} + \tau _p \left[e^{-\left(\frac{\rho _{\text{ct}}}{\kappa _c}\right)} - e^{-\left(\frac{\rho _{\text{ct}}}{\kappa _m}\right)}\right] \right\}, 0 < t < t_1, \tag{B.5a}
\]

\[
\rho _{\text{ct}} = \rho _{\text{th}} + e^{-\left(\frac{\rho _{\text{ct}}}{\kappa _c}\right)} f_0 \left[\frac{e^{-\left(\frac{\rho _{\text{ct}}}{\kappa _c}\right)} - e^{-\left(\frac{\rho _{\text{ct}}}{\kappa _m}\right)}}{-w_c/H_{S} + 1/\tau _m + 1/\tau _c} + \frac{1}{\tau _m + 1/\tau _c} \right] \tag{B.5b}
\]

\[
-\frac{\rho _{\text{th}} e^{-\left(\frac{\rho _{\text{ct}}}{\kappa _c}\right)} - e^{-\left(\frac{\rho _{\text{ct}}}{\kappa _m}\right)}}{\tau _m + 1/\tau _c} \right\}, t_1 \leq t < t_2.
\]
\[
\rho_c = \rho_{bh} + e^{-\frac{\rho_c}{\rho_{th}}} \left[ f_0 \left( e^{-\frac{\rho_c}{\rho_{th}}} - e^{-\frac{\rho_{th}}{\rho_{th}}} \right) \right. + \int_{t_1}^{t_2} \left( e^{-\frac{\rho_c}{\rho_{th}}} - e^{-\frac{\rho_{th}}{\rho_{th}}} \right) \rho_{th} \left( e^{\frac{\rho_c}{\rho_{th}}} - e^{\frac{\rho_{th}}{\rho_{th}}} \right) dt \right].
\]

Here \( f_0 \) is defined in equation 11. The \( t_1 \) and \( t_2 \) can only be numerically solved. This is the only step that inherently prevents the semianalytical model from being fully analytical (the other less unsurmountable one is equation (41) in solving LR cloud water). First, we calculate the value of \( \rho_c \) in equation (B.5a) for an array of increasing \( t \) from \( t = 0 \). When \( \rho_c \) crosses \( \rho_{th} \), the \( t \) is defined as \( t_1 \). Second, we calculate equation (B.5b) for an array of increasing \( t \) from \( t = t_1 \) and obtain \( t_2 \) when \( \rho_c \) drops back to \( \rho_{th} \).

**Appendix C: The Bubble Bottom Precipitation Without the LR**

When collection is relatively weak, \( \rho_c \) is not that concentrated at the upper part of the bubble, and auto-conversion can occur throughout the bubble depth. Thus, UR extends down to the bubble bottom, and the LR is squeezed out. The occurrence of this regime is determined by equation (29). The mean cloud water \( \langle \rho_c \rangle_{U, d_i = d_b} \) and bubble bottom precipitation are derived below.

Using the definition of \( \langle \rho_c \rangle_U \) in equation (15), we get

\[
\langle \rho_c \rangle_U = \frac{1}{d_i} \int_{U_0}^{U} \left( \rho_c - \rho_{th} \right) dz = \frac{f_{UR}}{\tau_U} \left( C_1 + C_2 \right).
\]

This is a transcendental equation that is hard to solve directly. Instead, we seek for an approximate solution by doing Taylor expansion on the exponential term, and truncation to the cubic term yields enough accuracy:

\[
e^{-\Omega_{th}(\rho_c)} \approx 1 + \frac{\sigma_d K (\rho_c)}{V_T} + \frac{1}{2} \left( \frac{\sigma_d K (\rho_c)}{V_T} \right)^2 + \frac{1}{6} \left( \frac{\sigma_d K (\rho_c)}{V_T} \right)^3.
\]

Substituting equation (C.2) into equation (C.1), we get a quadratic equation about \( \langle \rho_c \rangle_U \):

\[
a \langle \rho_c \rangle_U^2 + b \langle \rho_c \rangle_U - 1 = 0,
\]

where

\[
a = \frac{11 - \sigma}{6} \left( \frac{\sigma_d K}{V_T} \right)^2, \quad b = \frac{\tau_U (C_1 + C_2)}{f_{UR} V_T} + \frac{1}{2} \left( \frac{\sigma_d K (\rho_c)}{V_T} \right).\]

It has one and only one positive root, so the approximate solution to \( \langle \rho_c \rangle_U \) without LR is as follows:

\[
\langle \rho_c \rangle_U d_i = d_b = \frac{-b + \sqrt{b^2 + 4a}}{2a}.
\]

We then use \( \langle \rho_c \rangle_U \) to express bubble bottom precipitation, which equals to the total auto-conversion and collection:

\[
\int_0^\infty \int_{d_b} \left( \text{AUT + CLC} \right) dt = \text{PCP} = \frac{V_T f_{UR}}{\tau_m} = \int_{d_i = d_b}^{\infty} \langle \rho_c \rangle_U d_i = \frac{\langle \rho_c \rangle_U}{\tau_m} d_b.
\]

**Appendix D: Some Tedium Expressions Used in the Semianalytical Solution**

**D1. The \( f_{UR} \)**

\( f_{UR} \) is the approximated time-integrated forcing \( (f_c - \rho_{th}/\tau_m) \) in UR for unit volume of air. Use equation (10), and notice that it is more convenient to be treated as a vertical integration; we get
\[\int_{t_1}^{t_2} \left( f_{c|z|0} \frac{\rho_{th}}{\tau_m} \right) dt = \frac{1}{w_c} \int_{z_g}^{z_g'} \left( f - \frac{\rho_{th}}{\tau_m} \right) dz_g \]

\[= \int_{0}^{t_2-t_1} \tau_{\eta} \left( e^{-\frac{z_g'}{\tau_{\eta}}} - e^{-\frac{z_g}{\tau_{\eta}}} \right) + \frac{w_c}{\tau_{\eta}} \left[ e^{-\frac{z_g'}{\tau_{\eta}}} - e^{-\frac{z_g}{\tau_{\eta}}} \right] \left( f \tau_{\eta} + \frac{1}{\tau_{\eta}} \right) \left( \int_{z_g}^{z_g'} \frac{\rho_{th}}{\tau_m} dt \right) \]

Here, \( z_{gA} \) and \( z_{gF} \) are the ground coordinate position of the bubble top parcel when it reaches \( t_1 \) and \( t_2 \), respectively (points A and F in Figure 4):

\[z_{gA} = w_c t_1 \quad \text{and} \quad z_{gF} = w_c t_2.\]  

(D.1)

**D2. The \( \rho_{ct|t-t_1-z/V_T} \)**

\( \rho_{ct|t-t_1-z/V_T} \) is the \( \rho_{ct} \) for a parcel that is entering LR. If the border line \( t = t_1 - z/V_T \) is completely below LCL (e.g., small \( \rho_{th} \)), there is no cloud water entering LR; if the line is completely above LCL, the \( \rho_{ct} \) has grown to a quasi-steady value, which is close to the mean value \( \frac{\rho_{ct}}{t_2-t_1} \). The transition between the two extremes is treated as a linear linkage between them. We set the \( \rho_{ct|t-t_1-z/V_T} \) as a number, which is proportional to the length of the LR border line (line BC in Figure 4) that is above LCL. Let the ground coordinate height of the corner point (the point C in Figure 4 where \( z = -d_b \) crosses \( t = t_1 - z/V_T \)) be \( z_{gC} \) and that for the corner (the point B in Figure 4 where \( z = -d_c \) crosses \( t = t_1 - z/V_T \)) be \( z_{gB} \):

\[z_{gC} = w_c t_1 + \frac{(w_c - V_T) d_b}{V_T} \quad \text{and} \quad z_{gB} = w_c t_1 + \frac{(w_c - V_T) d_U}{V_T}.\]  

We have

\[\rho_{ct|t-t_1-z/V_T} = \frac{\rho_{ct}}{t_2-t_1} \eta_t;\]  

where \( \eta_t \) is a piecewise function:

\[\eta_t \equiv \begin{cases} 0, & z_{gB} < 0 \\ \frac{z_{gB} - z_{gC}}{z_{gB} - z_{gC}}, & z_{gC} \leq 0 \leq z_{gB} \\ 1, & z_{gC} > 0 \end{cases}.\]  

(D.5)

**D3. The \( f_{cLR} \)**

\( f_{cLR} \) is a constant-value approximation to the \( f_c \) in LR. It is approximated as the accumulated condensation experienced by a parcel that locates at the middle height \( z = -\frac{(d_c + d_b)}{2} \) in bubble coordinate, starting from the time it reaches LCL till it leaves LR. This is a good approximation for most of our sensitivity tests. It may produce large error when the auto-conversion starts very late due to very large \( \rho_{th} \). In ground coordinate, the height at which it starts condensation is denoted as \( z_{g\text{ent}} \), and that for the parcel to leave LR is \( z_{g\text{lea}} \):

\[z_{g\text{ent}} = 0 \quad \text{and} \quad z_{g\text{lea}} = \max \left\{ 0, w_c t_2 + \frac{(w_c - V_T) (d_U + d_b)}{2V_T} \right\}.\]  

Here \( z_{g\text{lea}} \) is constrained to be positive to avoid negative \( f_{cLR} \). The accumulated condensation is calculated using \( f_c \) in equation (10):
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**Erratum**

The originally published version of this article featured typesetting errors in some math on pages 4404, 4406, 4408, 4410, 4411, 4419, and 4420, as well as in equations (14), (21a), (21b), (23c), (30), (41), and (A.13). These errors have been corrected and this may be considered the official version of record.