FAILURE OF STRONG APPROXIMATION ON AN AFFINE CONE

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Abstract. We use the Brauer–Manin obstruction to strong approximation on a punctured affine cone to explain why some mod p solutions to a homogeneous Diophantine equation of degree 2 cannot be lifted to coprime integer solutions.

1. Introduction

Let \( Y \subset \mathbb{P}^3_\mathbb{Q} \) be the quadric surface defined by the equation

\[
X_0^2 + 47X_1^2 = 103X_2^2 + (17 \times 47 \times 103)X_3^2.
\]

One can easily check that \( Y \) is everywhere locally soluble, and so has rational points. Being a quadric surface, \( Y \) satisfies weak approximation. In particular, if we fix a prime \( p \), then any smooth point on the reduction of \( Y \) at \( p \) lifts to a rational point of \( Y \). Given that a point on the reduction of \( Y \) is given by \((\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \mathbb{F}_p^4 \) satisfying (1), and a point of \( Y(\mathbb{Q}) \) can be given by coprime integers \((x_0, x_1, x_2, x_3) \in \mathbb{Z}^4 \) satisfying (1), one might be tempted to think that every \( \mathbb{F}_p \)-solution \((\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \) can be lifted to a coprime integer solution \((x_0, x_1, x_2, x_3) \).

However, at the end of the article [2], it was remarked that \( Y \) has the following interesting feature: if \((\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \) is a solution to (1) over \( \mathbb{F}_p \), then at most half of the non-zero scalar multiples of \((\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \mathbb{F}_p^4 \) can be lifted to coprime 4-tuples \((x_0, x_1, x_2, x_3) \in \mathbb{Z}^4 \) defining a point of \( Y \). That observation was a byproduct of the calculation of the Brauer–Manin obstruction to rational points on a diagonal quartic surface related to \( Y \). In this note we will interpret the observation as a failure of strong approximation on the punctured affine cone over \( Y \), and will show that this failure is itself due to a Brauer–Manin obstruction.

The same phenomenon has been observed by Lindqvist [7] in the case of the quadric surface \( X_0^2 - pqX_1^2 - X_2X_3 \), for \( p, q \) odd primes congruent to 1 modulo 8. We expect that example also to be explained by a Brauer–Manin obstruction.

Following Colliot-Thélène and Xu [3], for a variety \( X \) over \( \mathbb{Q} \), we define \( X(\mathbb{A}_\mathbb{Q}) \) to be the set of adelic points of \( X \), that is, the restricted product of \( X(\mathbb{Q}_v) \) for all places \( v \), with respect to the subsets \( X(\mathbb{Z}_v) \). (One needs to choose a model of \( X \) to make sense of the notation \( X(\mathbb{Z}_v) \), but since any two models agree outside a finite set of primes the resulting definition of \( X(\mathbb{A}_\mathbb{Q}^\infty) \) does not depend on the choice of model.) Similarly, define \( X(\mathbb{A}_\mathbb{Q}^\infty) \) to be the set of adelic points of \( X \) away from \( \infty \), that is, the restricted product of \( X(\mathbb{Q}_v) \) for \( v \neq \infty \) with respect to the subsets \( X(\mathbb{Z}_v) \). Assuming that \( X \) has points over every completion of \( \mathbb{Q} \), we say that \( X \) satisfies strong approximation away from \( \infty \) if the image of the diagonal map \( X(\mathbb{Q}) \to X(\mathbb{A}_\mathbb{Q}^\infty) \) is dense.

If a variety \( X \) does not satisfy strong approximation, this can sometimes be explained by a Brauer–Manin obstruction. Define

\[
X(\mathbb{A}_\mathbb{Q})^{Br} = \{ (P_v) \in X(\mathbb{A}_\mathbb{Q}) \mid \sum_v \text{inv}_v A(P_v) = 0 \text{ for all } A \in Br X \},
\]
and define \( X(A_q^\infty)^{Br} \) to be the image of \( X(A_q)^{Br} \) under the natural projection map \( X(A_q) \to X(A_q^\infty) \). Then \( X(A_q^\infty)^{Br} \) is a closed subset of \( X(A_q^\infty) \) that contains the image of \( X(Q) \). If \( X(A_q^\infty)^{Br} \neq X(A_q^\infty) \), we say that there is a Brauer–Manin obstruction to strong approximation away from \( \infty \) on \( X \).

We now return to the variety \( Y \) defined above. Let \( X \subset A_4^4 \) be the punctured affine cone over \( Y \): that is, \( X \) is the complement of the point \( O = (0, 0, 0, 0) \) in the affine variety defined by the equation (1). There is a natural morphism \( \pi : X \to Y \) given by restricting the usual quotient map \( A^4 \setminus \{O\} \to P^3 \), so that \( X \) is realised as a \( G_m \)-torsor over \( Y \). To talk about integral points, we must choose a model: let \( X \subset A_3^3 \) be the complement of the section \( (0, 0, 0, 0) \) in the scheme defined by the equation (1) over \( \mathbb{Z} \). If we let \( f \in \mathbb{Z}[X_0, X_1, X_2, X_3] \) be the polynomial defining \( Y \), then the integral points of \( X \) are given by

\[
X(\mathbb{Z}) = \{(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4 \mid x_0, x_1, x_2, x_3 \text{ coprime}, f(x_0, x_1, x_2, x_3) = 0\}.
\]

**Theorem 1.1.** The group \( Br X/Br \mathbb{Q} \) has order 2; a generator is given by the quaternion algebra \((17, g)\), where \( g \in \mathbb{Z}[X_0, X_1, X_2, X_3] \) is a homogeneous linear form defining the tangent hyperplane to \( X \) at a rational point \( P \in X(Q) \). There is a Brauer–Manin obstruction to strong approximation on \( X \) away from \( \infty \). Specifically, for any smooth point \( \bar{Q} \in X(\mathbb{F}_{17}) \), at most half of the scalar multiples of \( \bar{Q} \) lift to integer points of \( X \).

It is interesting to compare this result with the “easy fibration method” of [3, Proposition 3.1]. We have a fibration \( \pi : X \to Y \), and the base \( Y \) satisfies strong approximation. However, the fibres are isomorphic to \( G_m \), which drastically fails to satisfy strong approximation, so we cannot use that method to conclude anything about strong approximation on \( X \).

## 2. Quadric surfaces

In this section we gather some basic facts about quadric surfaces. Any non-singular quadric surface \( Y \subset P^3 \) over a field \( k \) of characteristic different from 2 may be defined by an equation of the form \( \mathbf{x}^T \mathbf{M} \mathbf{x} = 0 \), where \( \mathbf{M} \) is an invertible \( 4 \times 4 \) matrix with entries in \( k \). We define \( \Delta_Y \in k^\times/(k^\times)^2 \) to be the class of the determinant of \( \mathbf{M} \), which is easily seen to be invariant under linear changes of coordinates. If \( k \) is an algebraic closure of \( k \) and \( \bar{Y} \) is the base change of \( Y \) to \( k \), then \( Pic \bar{Y} \) is isomorphic to \( \mathbb{Z}^2 \), generated by the classes of the two families of lines on \( \bar{Y} \) [4, Example II.6.6.1].

**Lemma 2.1.** Let \( k \) be a field of characteristic not equal to 2, and let \( Y \subset P^3_k \) be a non-singular quadric surface. Then the two families of lines on \( Y \) are defined over the field \( k(\sqrt{\Delta_Y}) \), and are conjugate to each other.

**Proof.** We may assume that the matrix \( \mathbf{M} \) defining \( Y \) is diagonal, with entries \( p, q, r, s \). Following [4, Section IV.3.2], we explicitly compute an open subvariety of the Fano scheme of lines on \( Y \) by calculating the conditions for the line through \( (1 : 0 : a : b) \) and \( (0 : 1 : c : d) \) to lie in \( Y \). The resulting affine piece of the Fano scheme is given by

\[
\{p + ra^2 + sb^2 = 0, rac + sbd = 0, q + rc^2 + sd^2 = 0\} \subset A_k^4 = Spec k[a, b, c, d].
\]

This is easily verified to consist of two geometric components, each a plane conic, one contained in the plane \( qra = -\sqrt{\Delta_Y}d, qsb = \sqrt{\Delta_Y}c \) and the other in the conjugate plane. \( \square \)
Lemma 2.2. Let $Y$ be a non-singular quadric surface over the finite field $\mathbb{F}_q$, with $q$ odd. Then

$$\#Y(\mathbb{F}_q) = \begin{cases} q^2 + 2q + 1 & \text{if } \Delta_Y \in (\mathbb{F}_q^\times)^2; \\ q^2 + 1 & \text{otherwise}. \end{cases}$$

Proof. This can be computed directly, but we recall how to obtain it from the Lefschetz trace formula for étale cohomology. Let $\ell$ be a prime not equal to $p$. Let $\bar{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$, let $\bar{Y}$ be the base change of $Y$ to $\bar{\mathbb{F}}_q$, and let $F: Y \to \bar{Y}$ be the Frobenius morphism. The Lefschetz trace formula states that $\#Y(\mathbb{F}_q)$ can be calculated as

$$\#Y(\mathbb{F}_q) = \sum_{i=0}^{4} (-1)^i \text{Tr}(F^i|H^i(\bar{Y}, \mathbb{Q}_\ell)).$$

Because $Y$ is smooth and projective, there are isomorphisms of Galois modules $H^0(Y, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$ and $H^1(Y, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-2)$ (see [S] VI.11.1). We have $\bar{Y} \cong \mathbb{P}^1 \times \mathbb{P}^1$. The standard calculation of the cohomology groups of projective space [5, VI.5.6], and the Künneth formula [8, Corollary VI.8.13], give $H^i(Y, \mathbb{Q}_\ell) = 0$ for $i$ odd, and show that $H^2(Y, \mathbb{Q}_\ell)$ has dimension 2. This reduces the formula to

$$\#Y(\mathbb{F}_q) = q^2 + 1 + \text{Tr}(F^2|H^2(\bar{Y}, \mathbb{Q}_\ell)).$$

Moreover, the cycle class map (arising from the Kummer sequence) gives a Galois-equivariant injective homomorphism

$$\text{Pic } \bar{Y} \otimes \mathbb{Z} \mathbb{Q}_\ell \to H^2(\bar{Y}, \mathbb{Q}_\ell(1)),$$

which by counting dimensions must be an isomorphism. If $\Delta_Y$ is a square in $\mathbb{F}_q$, then the Galois action is trivial and we obtain (after twisting) $\text{Tr}(F^2|H^2(\bar{Y}, \mathbb{Q}_\ell)) = 2q$. If $\Delta_Y$ is not a square in $\mathbb{F}_q$, then $F^2$ acts on $\text{Pic } \bar{Y} \cong \mathbb{Z}^2$ by switching the two factors, so with trace zero. In either case we obtain the claimed number of points. (Note that, in the first case, $Y$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, so we should not be surprised that it has $(q + 1)^2$ points.) \hfill \Box

3. Proof of the theorem

Firstly, we calculate the Brauer group of $X$; it is convenient to do so in more generality.

Lemma 3.1. Let $k$ be a field of characteristic zero, let $Y \subset \mathbb{P}^3_k$ be a smooth quadric surface, and let $X \subset \mathbb{A}^3_k$ be the punctured affine cone over $Y$. If $\Delta_Y \in (k^\times)^2$, then we have $\text{Br } X = \text{Br } k$. Otherwise, suppose that $X$ has a $k$-rational point $P$, and let $g$ be a homogeneous linear form defining the tangent hyperplane to $X$ at $P$. Then $\text{Br } X/\text{Br } k$ has order 2, and is generated by the class of the quaternion algebra $(\Delta_Y, g)$. This class does not depend on the choice of $P$.

Proof. Let $\bar{k}$ be an algebraic closure of $k$, and let $\bar{X}$ and $\bar{Y}$ denote the base changes to $\bar{k}$ of $X$ and $Y$, respectively. By [E] Theorem 2.2, we have $\text{Br } (\bar{X}) \cong \text{Br } (\bar{Y})$; but $\bar{Y}$ is a rational variety, so its Brauer group is trivial. So it remains to compute the algebraic Brauer group of $X$.

We claim that there are no non-constant invertible regular functions on $X$. Indeed, let $C \subset \mathbb{A}^3_k$ be the (non-punctured) affine cone over $Y$. Because $C$ is Cohen–Macaulay and $(0, 0, 0, 0)$ is of codimension $\geq 2$ in $C$, we have

$$k[X] = k[C] = k[X_0, X_1, X_2, X_3]/(f)$$

where $f$ is the homogeneous polynomial defining $Y$. This is a graded ring and its invertible elements must all have degree 0, so are constant.
The Hochschild–Serre spectral sequence gives an injection \( \text{Br} X/\text{Br} k \to H^1(k, \text{Pic} \tilde{X}) \). (Here we use \( k[X]^\times = k^\times \) and \( \text{Br} \tilde{X} = 0 \).) By [6, Exercise II.6.3], there is an exact sequence

\[
0 \to \mathbb{Z} \to \text{Pic} \tilde{Y} \xrightarrow{\pi^*} \text{Pic} \tilde{X} \to 0,
\]

where \( \pi : X \to Y \) is the natural projection and the first map sends 1 to the class of a hyperplane section of \( \tilde{Y} \). Using Lemma 2.1 shows that \( \text{Pic} \tilde{X} \) is isomorphic to \( \mathbb{Z} \), with \( G = \text{Gal}(k(\sqrt{\Delta_Y})/k) \) acting by \(-1\). The inflation-restriction sequence shows \( H^1(k, \text{Pic} \tilde{X}) \cong H^1(G, \text{Pic} \tilde{X}) \). If \( \Delta_Y \) is a square, then this group is trivial, and we conclude that \( \text{Br} X/\text{Br} k \) is also trivial. Otherwise \( G = \{1, \sigma\} \) has order 2, and we have

\[
H^1(G, \text{Pic} \tilde{X}) \cong \hat{H}^1(G, \text{Pic} \tilde{X}) = \frac{\ker(1 + \sigma)}{\ker(1 - \sigma)} = \mathbb{Z}^{2\sigma}.
\]

To conclude, it is sufficient to show that the algebra \((\Delta_Y, g)\) is non-trivial in \( \text{Br} X/\text{Br} k \). Because the polynomial \( g \) also defines the tangent plane to \( Y \) at \( \pi(P) \), the divisor \((g)\) is equal to \( \sigma^* (L + L') \), where \( L \) is a line passing through \( \pi(P) \) and \( L' \) is its conjugate. By [1, Proposition 4.17], this shows that \((\Delta_Y, g)\) is a non-trivial element of order 2 in \( \text{Br} X/\text{Br} k \). (The reference works with a smooth projective variety, but the proof generalises easily to any smooth \( X \) with \( k[X]^\times = k^\times \).)

We now return to the specific case where \( X \) is the punctured affine cone over the quadric surface defined by the equation (1). We will need to be more careful about constant algebras than we have been up to this point. Recall that \( X(\mathbb{Z}) \) consists of points \( P = (x_0, x_1, x_2, x_3) \) where \( x_0, x_1, x_2, x_3 \) are coprime integers satisfying the equation (1). Given such a \( P \), we define the linear form

\[
\ell_P = x_0 X_0 + 47 x_1 X_1 - 103 x_2 X_2 - (17 \times 47 \times 103)x_3 X_3 \in \mathbb{Z}[X_0, X_1, X_2, X_3]
\]

and the quaternion algebra \( A_P = (17, \ell_P) \in \text{Br} X \). Note that the linear form \( \ell_P \) does indeed define the tangent plane to \( X \) at \( P \), so Lemma 3.1 shows that \( A_P \) represents the unique non-trivial class in \( \text{Br} X/\text{Br} \mathbb{Q} \). We will now evaluate the Brauer–Manin obstruction associated to \( A_P \).

**Lemma 3.2.** Fix \( P \in X(\mathbb{Z}) \). Then, for any place \( v \) of \( \mathbb{Q} \) for which 17 is a square in \( \mathbb{Q}_v \), we have \( \text{inv}_v A_P(Q) = 0 \) for all \( Q \in X(\mathbb{Q}_v) \).

**Proof.** The homomorphism \( \text{Br} X \to \text{Br} \mathbb{Q}_v \) given by evaluation at \( Q \) factors through \( \text{Br}(X \times_{\mathbb{Q}} \mathbb{Q}_v) \), but the image of \( A_P \) in this group is zero.

Note that Lemma 3.2 applies in particular to \( v = \infty, v = 2, v = 47 \) and \( v = 103 \).

For the following lemma, let \( \mathcal{Y} \) be the model for \( Y \) over \( \mathbb{Z} \) defined by the equation (1), and extend \( \pi \) to the natural projection \( X \to \mathcal{Y} \).

**Lemma 3.3.** Fix \( P \in X(\mathbb{Z}) \). Let \( p \neq 17 \) be a prime such that 17 is not a square in \( \mathbb{Q}_p \), and let \( Q \in X(\mathbb{Z}_p) \) be such that \( \pi(Q) \neq \pi(P) \) (mod \( p \)). Then \( \text{inv}_p A_P(Q) = 0 \).

**Proof.** If \( \ell_P(Q) \) is not divisible by \( p \), then \( \ell_P(Q) \) is a norm from the unramified extension \( \mathbb{Q}_p(\sqrt{17})/\mathbb{Q}_p \) and therefore we have \( \text{inv}_p A_P(Q) = 0 \).

Now suppose that \( \ell_P(Q) \) is divisible by \( p \). Denote by \( \tilde{Y} \) the base change of \( \mathcal{Y} \) to \( \mathbb{F}_p \). Let \( \tilde{P}, \tilde{Q} \in \tilde{Y}(\mathbb{F}_p) \) be the reductions modulo \( p \) of \( \pi(P), \pi(Q) \) respectively. The variety \( \tilde{Y} \) is a smooth quadric over \( \mathbb{F}_p \), and the tangent space \( T_p \tilde{Y} \) is cut out by the reduction modulo \( p \) of the linear form \( \ell_P \). By Lemma 2.1, the scheme \( \tilde{Y} \cap \{\ell_P = 0\} \) consists of two lines that are conjugate over \( \mathbb{F}_p(\sqrt{17}) \). Therefore the only point of \( \tilde{Y}(\mathbb{F}_p) \) at which \( \ell_P \) vanishes is \( \tilde{P} \). It follows that \( \ell_P(Q) \) can only be divisible by \( p \) if \( \tilde{Q} \) coincides with \( \tilde{P} \).

**Lemma 3.4.** Let \( P, P' \in X(\mathbb{Z}) \) be two points. Then \( A_P \) and \( A_{P'} \) lie in the same class in \( \text{Br} X \).
Lemma shows that \( \tilde{\text{Fix}} \) the result. so it does not matter which we use to evaluate the invariant. Lemma 3.3 then gives Lemma 2.2 shows that \( \tilde{\text{Fix}} \) the result. For \( \alpha \) for which 17 is a square in \( Q_p \), Lemma 3.2 shows \( \text{inv} \) the algebras \( A_p \) and \( A_p' \) lie in the same class in \( Br \). By Lemma 3.1, we already know that the difference \( \text{inv} = 0 \) is zero if and only if \( \tilde{\text{Fix}} \) the result. Lemma 3.3 shows \( \text{inv} \) the algebras \( A_p \) and \( A_p' \) lie in the same class in \( Br \), and these do not lift to points of \( X \). More specifically, for any smooth point \( Q \in X(F_17) \), half of the scalar multiples of \( Q \) lie in the image of \( U \), showing that they do not lift to integer points of \( X \).

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