BPS Geodesics in $N = 2$ Supersymmetric Yang-Mills Theory

J. Schulze

and

N.P. Warner

*Physics Department, U.S.C.*
*University Park, Los Angeles, CA 90089*

We introduce some techniques for making a more global analysis of the existence of geodesics on a Seiberg-Witten Riemann surface with metric $ds^2 = |\lambda_{SW}|^2$. Because the existence of such geodesics implies the existence of BPS states in $N = 2$ supersymmetric Yang-Mills theory, one can use these methods to study the BPS spectrum in various phases of the Yang-Mills theory. By way of illustration, we show how, using our new methods, one can easily recover the known results for the $N = 2$ supersymmetric $SU(2)$ pure gauge theory, and we show in detail how it also works for the $N = 2$, $SU(2)$ theory coupled to a massive adjoint matter multiplet.
1. Introduction

One of the many remarkable features of string duality is that one has been able to use it to extract new statements about the strong coupling regime of supersymmetric field theories. In particular, one can re-derive the quantum effective actions of Seiberg and Witten \[1\] from the classical effective actions of type II string compactifications on K3-fibrations \[2,3\]. In the IIB theory, the Yang-Mills BPS states come from 3-branes, and when these are wrapped around 2-cycles in the fibration, the result can be reinterpreted as a six-dimensional self-dual string \[4,5,6\] compactified to four dimensions on the Seiberg-Witten Riemann surface, \( \Sigma \) \[2\]. The BPS states of Yang-Mills theory then become minimum energy winding configurations of the self-dual string on \( \Sigma \), where the (local) tension in the string is given by the Seiberg-Witten differential, \( \lambda_{\text{SW}} \).

In \[1\], the differential \( \lambda_{\text{SW}} \) was an object whose period integrals gave the central charges, and hence the masses of BPS states. Since one integrated it around cycles, one was only interested in its cohomology class – one was free to add the derivative of any meromorphic function. The stringy approach led to a sharper statement about the BPS states: it is only a specific local form of \( \lambda_{\text{SW}} \) that has the interpretation of a string tension on \( \Sigma \), and the existence of BPS states with given monopole and electric charges \( (\vec{g}, \vec{q}) \), is equivalent to the existence of a geodesic with these winding numbers on \( \Sigma \) with the metric

\[
    ds^2 = |\lambda_{\text{SW}}|^2.
\]

Thus a statement about the stability and existence of strong quantum BPS states is reduced to a classical computation.

There have been several attempts to use the foregoing as a tool to probe the BPS structure of the theory \[3,7,8,9\], but existence of geodesics can be subtle to establish, particularly if one proceeds numerically. One would also like to see precisely what happens as one crosses inside a curve of marginal stability, where some of the BPS geodesics must “cease to exist”. A proper understanding of this has so far proven elusive.

Our purpose in this paper is to introduce some analytical tools by which these issues can be addressed. The key idea is to look for “geodesic horizons”. These are maximal, closed geodesics that surround poles in \( \lambda_{\text{SW}} \), and have the property that once crossed by a BPS geodesic, they can never be recrossed (hence the name “horizon”) by a BPS geodesic of finite energy. We find that, at least in the \( SU(2) \) pure gauge theory, and in the \( SU(2) \) gauge theory with adjoint matter, these geodesic horizons confine the winding states on \( \Sigma \) in a manner that provides a simple geometric understanding of the BPS spectrum. Since
these geodesic horizons are straightforward to characterize and their behaviour at curves of marginal stability can be easily addressed, we anticipate that they will provide a valuable technique in using geodesic methods to analyze the BPS spectrum of more complex models than the ones discussed here.

We start in the next section by introducing geodesic horizons and deriving some of their properties. We then describe an illustrative, but unphysical "toy model". In section 3 we obtain some relatively simple analytic expressions for the indefinite integrals of the Seiberg-Witten differential in the softly broken $N = 4$, $SU(2)$ gauge theory, and in its $N = 2$ supersymmetric pure gauge limit. In section 4 we analyze the BPS spectra in various phases of these gauge theories by using geodesic horizons, and the "shadows" cast by such horizons.

2. BPS geodesics and horizons

There are a few implicit issues in the geodesic characterization of BPS states, and these need to be brought into the open. First, the geodesic characterization has only been carefully established for pure gauge and for the $N = 4$ supersymmetric model. While it is almost certainly true in greater generality, one still needs the stringy derivation in order to get the proper local form of $\lambda_{SW}$ so as to properly describe the local string tension. Such a stringy derivation has been obtained for the softly broken $N = 4$ supersymmetric, $SU(2)$ gauge theory [10]. More generally one appeals to the underlying integrable hierarchy to posit the proper local form of $\lambda_{SW}$. That is, one finds that the indefinite integral of the differential, $\lambda_{SW}$, selected by the string theory is the Hamilton–Jacobi function of the integrable hierarchy that underlies the construction of the effective action [11,12,13,14]. It is natural to assume that this remains true in general.

The second issue concerns what constitutes a BPS geodesic. The geometrical origin from 3-branes in ten dimensions implies that states with magnetic charge must begin and end at "branch points of the fibration" [3], whereas purely electric states merely wind with no specific base point. From the point of view of the Riemann surface, the branch point is a mere coordinate artefact – the invariant statement of this comes from the fact that $\lambda_{SW}$ has zeroes at the branch points of the fibration. We therefore take BPS geodesics to be those that begin and end at the zeroes of $\lambda_{SW}$.

Given a set of winding numbers $(\vec{g}, \vec{q})$ for a BPS state there is always a corresponding (minimum length) geodesic: one considers all curves with the same fixed end-points and the same winding numbers, and then minimizes the length within this homotopy class.
This geodesic is then either reducible or irreducible. A geodesic will be called reducible if it is the concatenation of two other BPS geodesic curves. It is thus reducible if it runs into a zero of $\lambda_{SW}$ at an intermediate point along its length. If the shortest curve in the proper homotopy class is reducible, then the corresponding BPS state is the sum of two others, and there is no fundamental Yang-Mills BPS state with these quantum numbers. The BPS spectrum is thus characterized by irreducible BPS geodesics, and transitions in the spectrum must correspond to irreducible geodesics becoming reducible.

The geodesic equation for (1.1) has a trivial implicit solution

$$ w \equiv \int_{z_0}^{z} \lambda_{SW}(z) = \alpha \ t , \quad \text{(2.1)} $$

where $z_0$ is the starting point, $t$ is the parameter and $\alpha$ is a constant of integration. The solution is unique provided that the tangent vector is continuous, and the only way the tangent can fail to be continuous is if the curve runs into a zero of $\lambda_{SW}$ – i.e. if the geodesic is irreducible. Therefore the irreducible geodesics can be obtained by solving the initial value problem having used the winding numbers to fix the constant, $\alpha$. That is, if $\vec{a}$ and $\vec{a}_D$ are the periods of $\lambda_{SW}$, then the irreducible geodesic with charges $(\vec{g}, \vec{q})$, if it exists, must be obtained by taking $\alpha = \vec{q} \cdot \vec{a} + \vec{g} \cdot \vec{a}_D$ and letting $t$ run from 0 to 1. If this method produces a curve that does not terminate (at $t = 1$) at a zero of $\lambda_{SW}$, then the BPS geodesic with winding numbers $(\vec{g}, \vec{q})$ must be reducible.

The latter method is effective, and was used in [3,7], but it is somewhat implicit. To see transitions in the BPS spectrum it is easier to think in terms of minimizing the lengths of curves within a homotopy class, and finding that these curves move towards another zero of $\lambda_{SW}$ as one approaches a curve of marginal stability.

Finally, we note that since the geodesic equation is solved by (2.1), it follows that the length of a geodesic, $\Gamma$, is given by:

$$ L(\Gamma) = \sum_{\text{irr. segs. } \gamma} L(\gamma) = \sum_{\text{irr. segs. } \gamma} \left| \int_{\gamma} \lambda_{SW}(z) \right| . \quad \text{(2.2)} $$

2.1. Geodesic horizons

We wish to show how the existence of certain stable BPS states precludes the existence of others. To see this most clearly we will henceforth work in the covering space, $\tilde{\Sigma}$, of the Riemann surface, $\Sigma$, lifting the metric and curves in the obvious manner. In particular we will now take $\lambda_{SW}(z)$ and the integral (2.1) on $\tilde{\Sigma}$. We will also be interested in the
intersections of various geodesic curves, and by this we will mean intersections on \( \tilde{\Sigma} \). The idea is to see how BPS geodesics partition the covering space, and this is all a consequence of three simple facts: (i) geodesics are straight lines in the \( w \)-plane (where \( w \) is defined in (2.1)), (ii) geodesics representing BPS states of finite energy are finite line segments in the \( w \)-plane, and (iii) finite line segments (in the \( w \)-plane) that intersect more than once must coincide along a common segment. An irreducible BPS geodesic must consist of a single line segment in the \( w \)-plane, beginning and ending at zeroes of \( dw/dz = \lambda_{SW} \), but not encountering any other zeroes of \( \lambda_{SW} \) along its length. A reducible geodesic may still be a single line segment, but generically will be a collection of such segments, and might involve traversing the same segment twice.

One is thus tempted to conclude that a pair of irreducible geodesics can only intersect once, but there is a minor subtlety in (iii). Suppose two geodesics on \( \tilde{\Sigma} \) intersect at two points, \( P_1 \) and \( P_2 \), then either (a) the two geodesics coincide between \( P_1 \) and \( P_2 \) on \( \tilde{\Sigma} \), or (b) the geodesics lie on different branches of \( z(w) \), and these branches meet at \( P_1 \) and \( P_2 \). In the latter instance, \( P_1 \) and \( P_2 \) must either be poles or zeroes of \( \lambda_{SW} \). Thus if two irreducible geodesics intersect more than once then they are either identical or they have common endpoints.

We will say that a closed curve, \( \Gamma \), has the horizon property if any finite irreducible geodesic that crosses \( \Gamma \) can never recross it (or even return to it). We will call a closed curve a geodesic horizon if a) it is a closed geodesic that has the horizon property, and b) surrounds a region that contains no zeroes of \( \lambda_{SW} \). We will, however, allow geodesic horizons to pass through zeroes of \( \lambda_{SW} \), and we further define a geodesic horizon to be reducible if it passes through two or more distinct zeroes of \( \lambda_{SW} \). The irreducible components of a geodesic horizon are finite line segments in the \( w \)-plane. (In some circumstances there will be infinitely many concentric geodesic horizons, and so we will subsequently refine this definition to only mean the outermost, or maximal such horizon.)

The whole point of excluding zeroes from the interior is that it ensures that any irreducible geodesic that crosses a geodesic horizon can never meet a zero, and thus cannot represent a fundamental BPS state.

We now show that every pole, \( z_0 \), of \( \lambda_{SW} \) is surrounded by at least one geodesic horizon. Consider the family of simple, closed curves (in \( \tilde{\Sigma} \)) that satisfy the following:

(i) The only pole of \( \lambda_{SW} \) that they contain is \( z_0 \) itself.

(ii) They do not surround any zeroes of \( \lambda_{SW} \) (but can pass through such zeroes).
Within this homotopy class there is at least one curve of minimum length: A candidate is any geodesic polygon connecting zeroes of $\lambda_{SW}$, and surrounding the pole. One might find a shorter curve inside the polygon, but the curve of minimal length cannot be trivial since the length of the curves becomes infinite as they approach the pole. These closed curves of minimum length are geodesics, and the minimality of their length means that they must be simple (i.e. not self-crossing).

We now show that any such closed curve, $\Gamma$, has the horizon property. If $\Gamma$ is irreducible then the horizon property is almost obvious. The only finite irreducible geodesic (apart from $\Gamma$ itself) that can meet $\Gamma$ more than once is a geodesic that meets $\Gamma$ only at the (single) zero of $\lambda_{SW}$ that lies on $\Gamma$. Such a geodesic thus cannot be irreducible and cross $\Gamma$.

Suppose that $\Gamma$ is reducible and that an irreducible geodesic, $\gamma$, meets $\Gamma$ at two points, $P_1$ and $P_2$. Further suppose the $\gamma$ actually crosses $\Gamma$ at one of these points, $P_1$. It follows that $P_1$ cannot be a zero of $\lambda_{SW}$, since this would violate the irreducibility of $\gamma$. Let $\Gamma_1$ and $\Gamma_2$ be segments of $\Gamma$ between $P_1$ and $P_2$, and let $\gamma_0$ be the segment of $\gamma$ between $P_1$ and $P_2$. Since $w(\gamma_0)$ is a single line segment between $w(P_1)$ and $w(P_2)$ in the $w$-plane, it follows that $L(\gamma_0) \leq L(\Gamma_i)$, $i = 1, 2$, with equality if and only if $w(\Gamma_i) = w(\gamma_0)$. Since $P_1$ is not a zero of $\lambda_{SW}$, the latter equality would imply that that $\Gamma_i = \gamma_0$. Now observe that both $\Gamma_1 \cup \gamma_0$ and $\Gamma_2 \cup \gamma_0$ are closed curves, neither contains a zero of $\lambda_{SW}$, and one of them contains the pole. Suppose that it is the former. Minimality of $\Gamma$ then requires $L(\Gamma_2) \leq L(\gamma_0)$, and hence there must be equality. From the comment above, we therefore conclude that $\gamma_0 = \Gamma_2$, and hence $\gamma$ cannot cross into the interior of $\Gamma$.

Thus any irreducible closed geodesic around a pole has the horizon property, but a reducible closed geodesic crucially needs to be the geodesic of globally minimal length in order to have the horizon property.

Consider now a closed geodesic $\Gamma$ of minimum length that surrounds several poles. The foregoing argument fails, but in an interesting way. If the geodesic $\gamma$ goes between the poles so that the total residue in both $\Gamma_1 \cup \gamma_0$ and $\Gamma_2 \cup \gamma_0$ is non-zero, then $\gamma$ can recross $\Gamma$. If however, one of the two residues is zero, then it follows from (2.2) that $L(\gamma_0) \leq L(\Gamma_i)$, and the argument still goes through. In terms of BPS states this means that a BPS geodesic can emerge from $\Gamma$, provided that the BPS state picks up a hypermultiplet charge from the interior that is different from the total hypermultiplet charge enclosed by $\Gamma$. We will refer to closed curves like $\Gamma$ as selective horizons.

1 This follows because $\Gamma$ must be simple.
We now need to refine the definition of irreducible geodesic horizons since there can be infinitely many of them (see the example below). If $\gamma_1$ and $\gamma_2$ are two irreducible geodesic horizons around the same pole, then one geodesic must lie inside the other. (They cannot cross since they are closed, and would thus have to cross twice.) To make the definition more useful, if there is more than one irreducible horizon around a pole then take the geodesic horizon to be the maximal one, that is, the one that is contained in no other such horizon. Again, we exclude the possibility of zeroes of $\lambda_{SW}$ from the interior of the region surrounded horizon.

By definition, a reducible horizon must encounter at least two zeroes of $\lambda_{SW}$ upon its length. It may thus be thought of as a sum of BPS states whose net electric and magnetic charges are zero. We now show that the (maximal) irreducible geodesic horizons must meet one zero of $\lambda_{SW}$.

First observe that an irreducible horizon can only surround a pole with a non-zero residue, and conversely, the horizon around a pole with no residue must necessarily be reducible. This is a trivial consequence of (2.2) and residue calculus. Let the residue of the pole be $m$, then closed irreducible geodesics around the pole are given by the straight lines between some point $w_0$ and the point $w_0 + 2\pi im$. Let $\gamma$ be the “outermost” such closed geodesic, and let $w_1$ be a point on it. Suppose that $\lambda_{SW}$ is non-zero everywhere on $\gamma$, then we can locally invert to get $z(w)$ around $\gamma$. Look at all straight lines, $S_{\epsilon}$, running from $w_1 + \epsilon$ to $w_1 + \epsilon + 2\pi im$, and consider preimages, $\gamma_{\epsilon}$, in the $z$-plane of $S_{\epsilon}$. Since $\gamma$ is maximal, then no matter how small one chooses $|\epsilon|$ there must always be some $\gamma_{\epsilon}$ that is an open curve. Let $z_1$ be the limit point where the $\gamma_{\epsilon}$’s first open out. By considering the integral along $\gamma_{\epsilon}$, and upon a small segment $\delta_z$ that closes $\gamma_{\epsilon}$ to a loop around the pole, one easily sees that $\lambda_{SW} = dw/dz$ must vanish at $z_1$.

To summarize, all geodesic horizons must be closed geodesics loops that run through at least one of the zeroes of $\lambda_{SW}$. They must consist of irreducible geodesic segments, which can be thought of as BPS states. Irreducible horizons can only surround poles with residues, and horizons around poles with no residue must necessarily be reducible. Most importantly, any irreducible geodesic that crosses a horizon can never recross the horizon, and so BPS geodesics, fundamental or composite, can only exist if they lie outside, or tangent to, such horizons. Thus there are “BPS horizon states” whose existence and behaviour determines the existence and behaviour of all the other BPS states.
2.2. A toy example

To illustrate these ideas we consider an unphysical example with many of the features of important physical examples.

Consider the conformal mapping:

\[ w = z + \frac{1}{2} \log \left( \frac{z-1}{z+1} \right) - \frac{i\pi}{2}. \]  

(2.3)

The corresponding differential is \( \lambda = \frac{z^2}{z^2-1} \) has a double zero at \( z = 0 \), and two simple poles with residues \( \pm 1/2 \). Since \( w \) has a triple zero at \( z = 0 \), straight lines in the \( w \)-plane through \( w = 0 \) turn a \( 60^\circ \) corner at \( z = 0 \). The right half-\( z \)-plane maps onto the combination of the right half-\( w \)-plane and the strip \( \{ w : \text{Re}(w) < 0, -\pi \leq \text{Im}(w) < 0 \} \) (see Fig. 1). The top and bottom of strip may be periodically identified, making the cylinder that can be associated with the conformal map \( z \rightarrow \log(z) \). The left half-\( z \)-plane maps in the reflected manner: to the left half-\( w \)-plane and the strip, or cylinder \( \{ w : \text{Re}(w) > 0, -\pi \leq \text{Im}(w) < 0 \} \). The two patches are glued together along two parts of the imaginary \( w \)-axis: \( \text{Im}(w) > 0 \) and \( \text{Im}(w) < -\pi \). The points \( w = 0 \) and \( w = -i\pi \) are to be identified. As a result, circling by \( 2\pi \) around \( z = 0 \) results in a circle of \( 6\pi \) in the \( w \)-plane.

The \( w \)-plane thus looks like the \( z \)-plane but with two semi-infinite cylinders sewn into it: one on each side of the line \( \text{Re}(w) = 0, -\pi < \text{Im}(w) < 0 \). Straight lines parallel to the imaginary \( w \)-axis, and lying in the strips are closed geodesic loops around the simple poles in the \( z \)-plane. The geodesic horizons are both mapped to the straight line in the \( w \)-plane across the necks of the “cylinders,” i.e. running along \( \text{Re}(w) = 0, -\pi \leq \text{Im}(w) \leq 0 \) (see Fig. 1.). Clearly, any straight line in the \( w \)-plane that crosses one of these horizons will spiral down the cylinder, never to return.
Fig. 1: The shaded region shows the section of the \( w \)-plane that maps onto \( \text{Re}(z) > 0 \). The second diagram shows the \( z \)-plane, and the curves correspond to straight lines parallel to the imaginary \( w \) axis. Note the two lobes that make up the geodesic horizons and the closed orbits inside these horizons.

Finally, there is a useful physical model of the BPS geodesics that is valid for any meromorphic map \( w(z) \). The real and imaginary parts of \( w \) are harmonic functions of \( z \), and so straight lines in the \( w \)-plane can be thought of as equipotentials of some two-dimensional field in the \( z \)-plane. In models with logarithmic singularities, this perspective is perhaps most useful when we look at lines that select the real part of the logarithm. The singularities can then be thought of as charges. In the example above, straight lines parallel to the imaginary \( w \)-axis can be thought of as equipotentials of a uniform electric field in the \( x \)-direction that has been perturbed by equal and opposite charges at \( z = 1 \) and \( z = -1 \) (see Fig. 1).

3. \( N = 2, SU(2) \) Yang-Mills theory with adjoint matter

3.1. The curve and differential

In the formulation of [14,15,16], the Riemann surface and differential for \( N = 2, SU(2) \) Yang-Mills theory with adjoint matter can be defined as follows. One starts with the Weierstraß torus,

\[
\tilde{y}^2 = 4 (\tilde{x} - e_1)(\tilde{x} - e_2)(\tilde{x} - e_3),
\]

(3.1)

and constructs a genus two double cover via

\[
\tilde{t}^2 - u = m^2 \tilde{x}.
\]

(3.2)
The Weierstraß torus can be uniformized in the familiar manner by taking \( \tilde{x} = \varphi(\tilde{\xi}) \), \( \tilde{y} = \varphi'(\tilde{\xi}) \). To avoid confusion later, we will denote the modular parameter of this torus by \( \tilde{\tau} \). The differential is

\[
\lambda_{SW} = \tilde{t} \, d\tilde{\xi} = \tilde{t} \, \frac{d\tilde{x}}{\tilde{y}} .
\]

(3.3)

The genus two curve defined by (3.2), and differential (3.3) have an involution symmetry \( \tilde{y} \to -\tilde{y}, \tilde{t} \to -\tilde{t} \), and the torus of the \( SU(2) \) effective action is obtained by dividing out this symmetry. That is, one takes \( \tilde{z} = \tilde{t}\tilde{y}/m \), and replaces \( \tilde{t}^2 \) using (3.2), to obtain

\[
\tilde{z}^2 = (\tilde{x} + u/m^2)(\tilde{x} - e_1)(\tilde{x} - e_2)(\tilde{x} - e_3) .
\]

(3.4)

One can map this to the cubic form of \( \mathbb{I} \) by using a fractional linear transformation \( \mathbb{I} \), however we will map it to different cubic form that is better adapted to the study of the \( m \to \infty \) limit. To this end, introduce

\[
x = m^2(\tilde{x} + b)/(\tilde{x} + u/m^2) \quad \text{and} \quad y = (x - m^2)^2 \tilde{z}/m
\]

where \( b = \frac{1}{3e_1}(e_1^2 + 2e_2e_3) \). After some judicious rescaling of \( x \) and \( y \) the curve and differential reduce to the form:

\[
y^2 = (x - \mu^2)(x^2 - \Lambda^4) , \quad \lambda_{SW} = c_0 \frac{(x - \mu^2)}{(x - m^2)} \frac{dx}{y} ,
\]

(3.5)

where

\[
\Lambda^2 = \frac{m^2}{3e_1} (e_2 - e_3) , \quad \mu^2 = m^2 \frac{u - m^2b}{u + m^2e_1} , \quad c_0 = m \sqrt{\frac{m^4 - \Lambda^4}{m^2 - \mu^2}} .
\]

(3.6)

The whole point of this formulation is that the curve is exactly that of the pure gauge theory. Indeed to get this limit one takes \( m \to \infty \) along with \( \tilde{\tau} \to i\infty \). One then has \( (e_2 - e_3)/e_1 \to 24 e^{2\pi i \tilde{\tau}} \), and one takes the double limit so that \( \Lambda \) remains finite. One must also make the infinite additive renormalization: \( u \to u + m^2b \), in order that the parameter \( \mu \) remain finite. We now take \( \Lambda, \mu \) and \( m \) as the fundamental parameters of the theory, and we will let \( \tau \) denote the Teichmüller parameter of the torus (3.3). One can easily verify that \( \lambda_{SW} \) does indeed have the property that if one differentiates it with respect to \( \mu \) one gets the holomorphic differential.

The differential, \( \lambda_{SW} \), has two simple poles with residues \( \pm m \), and a double zero at \( x = \mu^2 \). In the pure gauge limit the two poles coalesce into a double pole. In the massless, \( N = 4 \) limit \( (m^2 \to \mu^2) \) these two poles move onto the double zero and all three annihilate each other. The finite mass theory thus looks like a lattice repetition of the toy example above. Before proceeding to a more detailed analysis of the horizons we wish to give some rather useful explicit formula for \( w \) as a function of \( z \).
3.2. Integrating $\lambda_{SW}$

One can evaluate the indefinite integral in (2.1), and to do this we go to the isogenous double cover and uniformize it. That is, set $x = \mu^2 + (\Lambda^2 + \mu^2) t^2$, and then take $t = \text{cn}(\xi, k)/\text{sn}(\xi, k)$, where $\text{sn}$ and $\text{cn}$ are Jacobi elliptic functions, and $\xi$ is the flat coordinate of the torus. One then finds that

$$\lambda_{SW} = -c \frac{(1 - \text{sn}^2(\xi, k))}{1 - k^2 \text{sn}^2(\alpha, k) \text{sn}^2(\xi, k)} d\xi,$$

where

$$k^2 \equiv \frac{\theta_2(0|\tau)^4}{\theta_3(0|\tau)^4} = \frac{2\Lambda^2}{\Lambda^2 + \mu^2}, \quad \text{sn}^2(\alpha, k) = \frac{\Lambda^2 + m^2}{2\Lambda^2},$$

$$c = 2m \sqrt{\frac{m^4 - \Lambda^4}{(m^2 - \mu^2)(\Lambda^2 + \mu^2)}} = 2mk^2 \frac{\text{sn}(\alpha, k) \text{cn}(\alpha, k)}{\text{dn}(\alpha, k)}.$$

For later convenience, we note that in the pure gauge limit, the corresponding result is:

$$\lambda_{SW} = 2\sqrt{\Lambda^2 + \mu^2} \frac{\text{cn}^2(\xi, k)}{\text{sn}^2(\xi, k)} d\xi.$$

The indefinite integrals of these differentials are elliptic integrals of the third and second kinds respectively. With a little massaging the integral of (3.7) can be reduced to

$$w = w_0 + (2m \ Z(\alpha) - c) \ \xi + m \ \log \left( \frac{\Theta(\xi - \alpha)}{\Theta(\xi + \alpha)} \right),$$

where $w_0$ is a constant of integration and, following the notation of [17],

$$\Theta(\xi) \equiv \theta_4 \left( \frac{\xi}{\theta_3(0|\tau)^2} \right) \tau; \quad Z(\xi) = \frac{\Theta'(\xi)}{\Theta(\xi)}.$$

For the pure gauge theory one gets

$$w = w_0 - \frac{2}{K} \sqrt{\Lambda^2 + \mu^2} \left( E \ \xi + \frac{\pi}{2} \frac{\theta_1'(|\tau|/2K)}{\theta_3(0|\tau) \theta_1(\tau)} \right),$$

where, following the usual convention $E$ and $K$ are elliptic periods. Explicitly, one has:

$$K \equiv \frac{\pi}{2} \theta_3(0|\tau)^2, \quad E \equiv \frac{1}{3} \left( 2 - k^2 - \theta_4''(0|\tau)/(\theta_1'(0|\tau) \ \theta_3(0|\tau)^4) \right) K.$$

Expressions for the integral of $\lambda_{SW}$ in the pure gauge theory were also given in [18,19].
From (3.10) it is trivial to read off the periods $a$ and $a_D$. One gets

$$a = 2(2m \text{Z}(\alpha) - c) K + 2\pi i n_1 m,$$

$$a_D = 2(2m \text{Z}(\alpha) - c) i K' + 2\pi i m(\alpha/K + n_2),$$

(3.14)

where $K' = -i\tau K$, and $n_1$ and $n_2$ are the winding numbers around the simple poles at $\xi = iK' \pm \alpha$. Similarly, for (3.12) one gets

$$a = 4 \sqrt{\Lambda^2 + \mu^2} E,$$

$$a_D = 2 \sqrt{\Lambda^2 + \mu^2} \left(2\tau E - \frac{\pi}{K}\right).$$

(3.15)

These expressions enable one to analytically solve for the geodesics without numerical integration. In particular, one can find all the geodesics through some point, $\xi_0$, by plotting all the curves on which $\arg(w - w(\xi_0))$ is constant. This approach was used to create Figs. 2 and 3.

Fig. 2: The lines shown are geodesics drawn in the $z$-plane. The horizontal and vertical axes are marked off in units of $K$ and $K'$ (the half-periods), and we have taken $\tau = i$, or $K = K'$. There are zeroes of $\lambda_{SW}$ at the points $(2p + 1, 2q)$, and double poles at the points $(2p, 2q)$, $p, q \in \mathbb{Z}$. The horizons around the poles at $(2n, 0)$ are evident, and are comprised of geodesics corresponding to $W$-bosons.

4. Stability of BPS states in $SU(2)$ Yang-Mills theory

We now wish to show how to obtain the BPS spectrum of $N = 2$, $SU(2)$ Yang-Mills theory, and of the same model coupled to an adjoint hypermultiplet, by finding the
geodesic horizons. To do this one simply needs to look for all geodesics that start and finish at zeroes and surround poles of $\lambda_{SW}$. We start by showing how the geodesic method easily replicates the known results for the pure gauge theory \[1,20,18\].

4.1. The pure gauge theory

The Seiberg-Witten differential has a double pole (with vanishing residue) and a double zero. In the uniformization used in the last section, they are located at $\xi = K$ and $\xi = 0 \pmod{2K}$ and $2iK'$ respectively. The geodesic horizon is thus reducible, its irreducible parts must correspond to a collection of BPS states. For large $\mu$, the $W$-boson is the BPS state of lowest mass, and so the geodesic horizon for large $\mu$ must be the pair of geodesics corresponding to a $W^+$ and $W^-$ connecting two zeroes above and below the double pole. This fact is born out by the manifest horizons in Fig. 2. These horizons mean that any BPS state can only cross the $W$-boson trajectory at the zero of $\lambda_{SW}$. It follows immediately that any BPS state of magnetic charge larger than one must be reducible to BPS states of magnetic charge one. Thus the existence of the stable $W$-boson implies that only the $W$-boson itself, and states with magnetic charge one can be stable.\[2\]

At the curve of marginal stability one knows that the $W^{\pm}$-bosons become unstable to decay into states of monopole and electric charges $\pm(1,0)$ and $\pm(1,-1)$. This is very clearly seen in even the most rudimentary numerical calculation of the geodesic horizons (see, for example, Fig. 3). As $\tau$ approaches the curve of marginal stability, the $W^{\pm}$-boson geodesic horizons deform to the zeroes of $\lambda_{SW}$ located at $\xi = K \pm 2iK' \pmod{2K}$ and $2K'$. At and beyond the curve of marginal stability, the geodesic horizon around the pole is the quadrilateral with edges $(\pm1,0)$ and $(\mp1,\pm1)$. These quadrilateral horizons in every fundamental region of the torus completely block passage of BPS states in every direction: To get a winding number of more than $\pm1$ in either the $(1,0)$ or $(-1,1)$ directions, a BPS state can only pass through the zeroes at the quadrilateral corners, and thus must be decomposable. Thus the fact that the $W$ boson is unstable to a monopole-dyon pair inside the curve of marginal stability implies that the monopole and dyon are the only stable BPS states.

\[2\] The instability of states of charge $(n,0)$ and $(0,n)$ is obvious in the geodesic approach for the reasons outlined in \[3\].
Fig. 3: These two diagrams show several geodesics and a single geodesic horizon around a double pole at (0, 2). The first diagram has $\tau = 0.3 + 1.0i$, and the second has $\tau = 0.3 + 0.654i$. The curve of marginal stability passes through the point $\tau = 0.3 + 0.651i$. As one approaches the curve of marginal stability, the geodesic horizon deforms to the zeroes at (1, 2) and (3, −2). On and inside the curve of marginal stability the geodesic horizon becomes the quadrilateral with vertices (1, 0), (1, 2), (3, 0) and (3, −2).

One can, in fact, infer the entire structure of the BPS spectrum without doing any computations or simulations. From the diagrams in [3] one can immediately see that there is a curve (homotopic to a circle) in the parameter space on which the $W$-boson is marginally stable to a monopole and a dyon. Thus the geodesic horizon must deform and touch another zero of $\lambda_{SW}$ as described above. The fact that the quadrilateral horizon persists inside the curve of marginal stability is then inferred from the monodromies of the theory. Specifically, one considers what happens as $\tau$ of the torus is decreased from $i\infty$ to near the real axis. When $Im(\tau)$ is decreased far enough, one must get to a modular inversion of the strong coupling region again. This means that the geodesic horizons must skip from surrounding $a$-cycles of the torus to surrounding some combination of $a$ and $b$-cycles. Such horizon jumps can only occur at curves of marginal stability, and the only way that it can happen so that the monodromies of the theory is respected is if the quadrilateral forms and persists inside the curve of marginal stability, and then it detaches to leave a horizon along the $a$ or $b$ cycle depending on where one crosses the curve marginal stability considered as a function of $\tau$.

The beauty of the foregoing argument is that it only cares about the modular properties of the theory, and how the zeroes and poles (the divisor) of $\lambda_{SW}$ moves as a function of the parameters.
4.2. The softly broken $N = 4$ theory

This differential $\lambda_{SW}$ now has three independent periods, and as we will see, the BPS spectrum is richer, but computable.

Consider first the unbroken $N = 4$ theory. As can be seen from (3.5), the differential $\lambda_{SW}$ collapses to the holomorphic differential of the torus:

$$y^2 = x(x^2 - \Lambda^4), \quad \lambda_{SW} = c_0 \frac{dx}{y} = c_0 d\xi.$$

The geodesics are thus straight lines on the torus itself, and so the spectrum consists of any dyon $(g, q)$ with $g$ and $q$ relatively prime integers [21,22].

Turning on the mass, $m$, of the hypermultiplet causes the differential, $\lambda_{SW}$, to develop a double zero and two simple poles with residues $\pm m$. The geodesic horizons near the zeroes and poles look like those of the toy model – two lobes around the poles with the lobes touching at the double zero. The conformal mapping, $w(z)$, given by (3.10) is thus relatively easy to describe: The $w$-plane is broken into fundamental regions of periods $a$ and $a_D$. Inside each such region there is a cut parallel to the $Im(w)$-axis, and of length $2\pi m$. As in the toy example, the cut in the $w$-plane is doubly covered by the two geodesic horizons in the $z$-plane, and to the left and right of the $w$-plane cuts are semi-infinite cylinders. If one does not cross these cuts the mapping $z(w)$ is single valued. Any straight line through a cut must pass into a cylinder, and so viable BPS geodesics are straight lines that avoid all the cuts.

The cuts or geodesic horizons affect BPS geodesics in two different ways. First, they confine the geodesics to either terminate on a zero, or “squeeze through” the “passes” between two such horizons. Secondly, they mean that BPS states can only pass around the pole on one side and not on the other: the horizon thus imposes restrictions on the hypermultiplet charges that a BPS geodesics can pick-up. To be more specific, given a BPS geodesic, we can always compare its hypermultiplet charge to that of a fixed reference state such as a composite of monopoles and $W$-bosons (or $(1,1)$-dyons). The BPS geodesic must either wind around both members of a pair of simple poles, thus collecting a net zero contribution to its hypermultiplet charge, or it must terminate at the zero to which the poles’ geodesic horizons connect. From the perspective of the $z$-plane there are apparently sixteen ways in which one can connect two zeroes, but only four of them will be straight lines in the $w$-plane (see, for example, Fig. 4). One can thus easily see that BPS states with fixed magnetic and electric charges can come in “multiplets”, with at most most four members, and with with hypermultiplet charges of $0, \pm m$. For example, Fig. 4 shows the
possible BPS states with electric charge +1 and magnetic charge 0. These four states have a very simple perturbative interpretation. We are breaking $N = 4$, $SU(2)$ Yang-Mills to an $N = 2$, $U(1)$ gauge theory, and so the charge +1 BPS states should not only include the $W^+, N = 2$ vector multiplet, but also should contain a charge +1, complex hypermultiplet, which contains states with opposite hypermultiplet charge. The four contours represent two copies of the $W^+$ boson and the pair of charged hypermultiplet states. Since the $N = 4$ theory is $SL(2, \mathbb{Z})$ invariant, we can work perturbatively around any winding number state, and reach the same conclusion. Thus we learn that even at strong coupling, and for any value of the mass $m$, there are no new BPS states other than the ones that one would expect from perturbation theory around all the phases of the softly broken $N = 4$ theory. Our purpose now is to show the extent to which the $N = 4$ BPS states destabilize as the mass is increased.

![Diagram](image)

**Fig. 4**: The irreducible geodesics with $(g, q) = (0, \pm 1)$. The first figure shows them in the $z$-plane and the second shows them in the $w$-plane. The shaded regions in the $z$-plane are the interiors of the horizons, and in the shaded lines in the $w$-plane are branch cuts.

If one approaches the pure gauge limit, then the simple poles in the $z$-plane approach one another at a point a half-period away from the zeroes. The geodesic horizons of the two approaching simple poles flatten at the ends, but cannot touch since they are irreducible geodesics. Thus until the two simple poles actually coincide, there is always a “hypermultiplet pass” between these horizons. The distance through this path gets infinitely long as $m \to \infty$. Put another way, the two $W$-boson geodesics, labeled $A$ and $B$ in Fig. 4, form a selective horizon in the sense of section 2 – one can recross it by weaving in between the simple poles that it encloses.

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3 The geodesic horizon itself can be thought of as the neutral hypermultiplet state.
The corresponding picture in the $w$-plane is as follows: The $w$-plane has cuts that periodically repeat with periods $a$ and $a_D$, and the length of the cuts is proportional to $m$. Outside the curve of marginal stability of the $W$-boson, the cuts fall into well-defined rows that do not interleave, creating an open channel between one row of cuts and the next. The edges of these channels are defined by the periodic repetition of the $W$-boson geodesic (see Fig. 5). As $m$ gets large the only fundamental BPS states that have small mass correspond to those irreducible geodesics that remain within the small channel between the tops of one row of cuts and the bottoms of the immediately neighbouring row of cuts. These states have charges $(g, q)$ equal to $(0, n)$ or $(\pm 1, n)$, and become the stable states of the pure gauge limit. A chasm of width $m$ opens out under the other states. However the identification in the $z$-plane of the endpoints of the cuts provides bridges of length zero across the chasm. The cost of using these bridges is the loss of irreducibility for the BPS state. Inside the curve of marginal stability of the $W$-boson, the rows of cuts interleave.

We can refine the $w$-plane picture to give much more specific information about individual states. Choose some cut $C_0$ as a base “point”. There are then stable BPS states associated with another cut, $C_1$, if and only if one can draw straight lines between the end(s) of $C_1$ and the end(s) of $C_0$ in such a manner that these straight lines do not meet any other cuts in between. Suppose that there is indeed a full complement of four BPS states associated with $C_1$. Draw the straight lines on the $w$-plane associated with these states and extend these lines beyond $C_1$ (see Fig. 5). Think of these extended lines as the edges of a shadow cast by $C_1$. That is, think of $C_0$ as an extended light source, and $C_1$ as a casting a shadow. Note that the light source and the shadow casting object have the same size. The lines that we have drawn define the edges of the umbra and penumbra of the shadow. If a point $w$ lies in the umbra of the shadow of $C_1$ then no straight line can connect it to $C_0$ without passing through $C_1$. If $w$ lies in the penumbra of the shadow, then a straight line (that does not cross $C_1$) can connect $w$ to either the top or bottom of $C_0$ but not to both. Therefore the extent to which there are stable BPS states of charge $(g, q)$ can be determined by the extent to which the cut, $C$, located at $ga_D + qa$ relative to $C_0$, lies in the shadows of the cuts between $C$ and $C_1$. We can, of course, transfer these cuts and shadows to the $z$-plane and think of shadows as being cast by geodesic horizons.

Thus, given the mass, $m$ and the Higgs vev, one can use the formulae of section 3 to compute $a$ and $a_D$, and it then becomes an elementary geometric exercise to determine if there is a stable BPS states with a given set of charges.
Fig. 5: The \( w \)-plane of the model with adjoint matter. The heavy vertical lines are branch cuts whose length is proportional to the mass \( m \). The horizontal dotted lines consist of \( W \)-boson geodesics, and define the sides of a channel between rows of cuts. The shaded regions show the umbra and penumbra of a shadow cast by the cut \( C_1 \).

The idea of shadows can be thought of as a natural geometric extension of the fact that stable BPS states in the \( N = 4 \) theory must have \((g, q)\) relatively prime: here the shadows are the shadows cast by the points \((g/h, q/h)\) where \( h \) is the highest common divisor of \( g \) and \( q \). As one turns on the mass, the shadows get fatter. There are some obvious general statements that can be made. First, if \( g \) and \( q \) are not relatively prime then the entire corresponding cut lies in the full umbra of the cut at \((g/h, q/h)\), and so all BPS states with magnetic and electric charges \((g, q)\) are unstable - they cannot be stabilized by picking up hypermultiplet charge.

More generally, as \( g \) and \( q \) (now taken to be relatively prime) get larger and larger, there are potentially more shadows that can fall on them, and so as \( m \) increases, the states with larger \( g \) and \( q \) will generically destabilize first. Since the width of an umbra does not change as one moves along it, but the width of the penumbra increases with distance, one will also generically find that some components of a geodesic multiplet will destabilize long before the other components of the same multiplet destabilize. Indeed it is rather easy to convince oneself that for fixed \( m \), there are always suitably large values of \( g \) and \( q \) beyond which the corresponding cuts are always in some penumbra\(^4\). However, the same is not true of an umbra since the opening angle of the umbra is zero. Indeed, if one is outside the curve of marginal stability of the \( W \)-boson, one can check that there is always one state with charges \((g, q) = (-n, n+1)\) for any \( n \): the top of \( C_0 \) can be always be connected via

\(^4\) The set of interiors of all penumbras form an open cover of any closed, finite angular interval not including the two directions in which the cut \( C_0 \) points. There is thus always a finite subcover of such an interval.
a straight line to the bottom of the cut corresponding to charges \((-n, n + 1\)). One can also easily see that in this region of moduli space there is a full geodesic multiplet of states with charges \((-n, 1\)), but these have mass larger than \(m\) and hence decouple in the pure gauge limit.

There are also always some states of charge \((\pm 1, n)\), and when \(n\) small enough compared to \(|a_D/m|\) there will always be a full multiplet of geodesics\(^5\). As \(n\) increases, some of the geodesics are excluded by shadows. For large \(n\) (or large \(m\)) there is only one geodesic left in the multiplet, and it is one of the geodesics that was previously identified as representing an \(N = 2\) hypermultiplet. This identification is nicely consistent with the pure gauge limit in which these states become the dyons.

4.3. Curves of marginal stability

We wish to conclude this section by making one or two comments related to curves of marginal stability for the massive model. Our intent is not to make an exhaustive study here, but draw attention to some interesting features. To get such curves, one first imagines fixing one of the two scale independent complex moduli, \(\alpha\) and \(\tau\), and then varying the other. One then chooses a particular physical state, and then one seeks curves (in the space of the variable modulus) on which that state becomes unstable. There are, of course, infinitely many possibilities for the massive model, but they are easily characterized: One simply draws the state of interest on the \(w\)-plane (a straight line on Fig. 5), and varies the parameters until that state touches the top or bottom of some intervening cut, at which point the physical state of interest becomes a composite (reducible) state.

As mentioned above, the \(W\)-boson’s curve of marginal stability is heralded by the interleaving of the rows of cuts in Fig. 5. It is also interesting to note that if one crosses the curve of marginal stability of the \(W\)-boson then even though it decays into a monopole and a dyon, the other (hypermultiplet) geodesics with the same magnetic and electric charge as the \(W^\pm\) remain stable until the interleaving of cuts disturbs them, too.

Finally we turn to the curves of marginal stability for the geodesic horizons (neutral hypermultiplets) themselves. This is relevant to understanding what happens to the horizons and BPS states as one smoothly changes the position of the poles of \(\lambda_{SW}\). That is, one smoothly changes \(\alpha\) in (3.10) by any period \(2K\) or \(2iK'\) of the fundamental region in the \(z\)-plane. This must deform the horizons in the direction in which \(\alpha\) moves, but, if

\[^5\] In defining \(a_D\) we allowed arbitrary additions of \(2\pi im\). This is indexed by \(n_2\) in (3.14). Here we choose the value of \(n_2\) that gives the smallest value to \(|a_D|\).
one changes $\alpha$ by a period, the horizons must return to their original configuration. This means that the horizon must shift its attachment from one zero of $\lambda_{SW}$ in $\tilde{\Sigma}$ to another zero. It does this by momentarily becoming reducible and connecting to two zeroes of $\lambda_{SW}$ that are separated by a period. There are thus lines of marginal stability for the horizons. If one changes $\alpha$ by $2K$ then the line of marginal stability is a subset of the curve $\text{Im}(a/(2\pi im)) = 0$, and similarly if one changes $\alpha$ by $2K'$ then the line of marginal stability is contained in the curve $\text{Im}(a_D/(2\pi im)) = 0$. On either side of these lines of marginal stability the geodesic horizon detaches from one zero or the other and becomes irreducible again.

It is amusing to note that if one is on such curves of marginal stability, then the geodesic horizons generate a wall much like that of the pure gauge theory, and if the wall is in the direction of the $W$-bosons then all the irreducible BPS states with $|g| > 1$ cease to be stable. Unlike the pure gauge theory, this instability of BPS states is ephemeral – it only happens on the curve of marginal stability, and not in some region to one side of it. If one is at the intersection of the curves of marginal stability in both the $K$ and $K'$ directions (which also means that one is also on the curve of marginal stability of the $W$-boson), then the geodesic horizon attaches to three zeroes of $\lambda_{SW}$, and one momentarily gets something just like the “quadrilateral horizon” of the pure gauge theory, with similar consequences for the BPS spectrum.

5. Conclusions

We have shown that the BPS geodesics can be used with relative ease to give a simple geometric understanding of the stability of BPS states in $N = 2$, $SU(2)$ pure gauge theory and in the theory with adjoint matter. The key to the analysis is to find geodesics that partition the covering space of the Seiberg-Witten Riemann surface. This led us to introduce geodesic horizons and the shadows cast by such horizons. It seems very likely that this approach will find application in other models.

Since the geodesic method was first introduced in [3], it was evident that the poles of $\lambda_{SW}$ played a crucial role in destabilizing BPS geodesics. As can be see from this work, the zeroes of $\lambda_{SW}$ are just as important, and the degree of the poles is crucial to the horizon structure. Thus we believe that the general structure of the BPS spectrum is ultimately determined by the full divisor class of $\lambda_{SW}$. For example, if one takes the pure gauge

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6 There are other states that destabilize on other parts of the set $\text{Im}(a/(2\pi im)) = 0$. 

theory and adds matter in the fundamental representation, then the divisor class of $\lambda_{SW}$ changes, even if the matter is massless. It is this fact that must be the crucial ingredient in understanding the appearance of new stable BPS states as the number of flavours is increased.

Since we have exhibited the implicit general solution to the geodesic problem, we cannot help but note its remarkable form. First, the expression in terms of theta functions leads to natural conjectures for generalizations to higher genus surfaces. (Indeed, if true, this is how the divisor class of $\lambda_{SW}$ would mathematically determine the geodesic problem.) On a far more speculative level, there might be some interesting topological interpretations of the theta functions. Recall that the BPS geodesics represent the physical states of a topologically twisted self-dual string [3]. The characterization of these stringy states as “straight lines in the $w$-plane” is also all too reminiscent of the characterization of solitons in massive $N = 2$ supersymmetric field theories in two dimensions [23]. It is thus tempting to try to interpret $w$ as some kind of effective superpotential for the massive string. If $w$ is linear in $\xi$ then it is a priori a trivial superpotential. However, the topological charge of a soliton is given by $\Delta w$, and so a linear superpotential is appropriate to a free theory with winding states. While it may be a coincidence, it would be interesting to try to interpret the logarithms of the theta functions as some kind of effective action (or superpotential) induced by integrating out bosonic and fermionic degrees of freedom that have been twisted by some parameter $\alpha$. These degrees of freedom could conceivably be found in the other modes of the 3-brane from which the self-dual string descends.

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