An existence result for a nonlinear transmission problems.

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August 25, 2014

Abstract

Let $\Omega^o$ and $\Omega^i$ be open bounded subsets of $\mathbb{R}^n$, $n \geq 2$, of class $C^{1,\alpha}$, and such that the closure of $\Omega^i$ is contained in $\Omega^o$. Let $f^o$ be a function in $C^{1,\alpha}(\partial \Omega^o)$ and let $F$ and $G$ be continuous functions from $\partial \Omega^i$ to $\mathbb{R}$. By exploiting an argument based on potential theory and on the Leray-Schauder principle we show that under suitable conditions on $F$ and $G$ there exists at least one pair of continuous functions $(u^o, u^i)$ such that

\[
\begin{cases}
\Delta u^o = 0 & \text{in } \Omega^o \setminus \text{cl} \Omega^i, \\
\Delta u^i = 0 & \text{in } \Omega^i, \\
u_{\Omega^o} \cdot \nabla u^o(x) - \nu_{\Omega^i} \cdot \nabla u^i(x) = G(x, u^i(x)) & \text{for all } x \in \partial \Omega^i,
\end{cases}
\]

where the last equality is attained in weak sense. In a simple example we show that such a pair of functions $(u^o, u^i)$ is in general not unique.

Keywords: nonlinear transmission problem; systems of nonlinear integral equations; fixed-point theorem; potential theory.

MSC2010: 35J65; 31B10; 45G15; 47H10.

1 Introduction

We study the existence of solutions of a boundary value problem with a nonlinear transmission conditions. We begin by introducing some notation. We fix once for all

a natural number $n \in \mathbb{N}$, $n \geq 2$, and a real number $\alpha \in ]0, 1[.$

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Here \( \mathbb{N} \) denotes the set of natural numbers including 0. Then we fix two sets \( \Omega^o \) and \( \Omega^i \) in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). The letter ‘o’ stands for ‘outer domain’ and the letter ‘i’ stands for ‘inner domain’. We assume that \( \Omega^o \) and \( \Omega^i \) satisfy the following condition:

\( \Omega^o \) and \( \Omega^i \) are open bounded subsets of \( \mathbb{R}^n \) of class \( C^{1,\alpha} \), \( \text{cl} \Omega^i \subseteq \Omega^o \), and the boundaries \( \partial \Omega^o \) and \( \partial \Omega^i \) are connected.

For the definition of functions and sets of the usual Schauder class \( C^{0,\alpha} \) and \( C^{1,\alpha} \), we refer for example to Gilbarg and Trudinger [13, § 6.2]. Here and in the sequel \( \text{cl} \Omega \) denotes the closure of \( \Omega \) for all \( \Omega \subseteq \mathbb{R}^n \). Then we consider the following nonlinear transmission boundary value problem for a pair of functions \((u^o, u^i)\) in \( C^{1,\alpha}(\text{cl} \Omega^o \setminus \Omega^i) \times C^{1,\alpha}(\text{cl} \Omega^i) \),

\[
\begin{align*}
\Delta u^o &= 0 & \text{in } \Omega^o \setminus \text{cl} \Omega^i, \\
\Delta u^i &= 0 & \text{in } \Omega^i, \\
 u^o(x) &= f^o(x) & \text{for all } x \in \partial \Omega^o, \\
 u^o(x) &= F(x, u^i(x)) & \text{for all } x \in \partial \Omega^i, \\
 \nu_{\Omega^i} \cdot \nabla u^o(x) - \nu_{\Omega^i} \cdot \nabla u^i(x) &= G(x, u^i(x)) & \text{for all } x \in \partial \Omega^i,
\end{align*}
\]

(1)

where \( f^o \) is a given function in \( C^{1,\alpha}(\partial \Omega^o) \), and \( F \) and \( G \) are given continuous functions from \( \partial \Omega^i \times \mathbb{R} \) to \( \mathbb{R} \).

The analysis of problems such as (1) is motivated by their applications in continuum mechanics. In particular, nonlinear transmission conditions of this kind arise in the study of composite structures glued together by thin adhesive layers which are thermally or mechanically very different from the components’ constituents. In modern material technology such composites are widely used (see, e.g., [23, 24, 29]), but the numerical treatment of the mathematical model by finite elements methods is still difficult, requires the introduction of highly inhomogeneous meshes, and often leads to poor accuracy and numerical instability. A convenient way to overcome this problem is to replace the thin layers by zero thickness interfaces between the composite’s components. Then one has to define on such interfaces suitable transmission conditions which incorporates the thermal and mechanical properties of the original layers. Such a procedure can be rigorously justified by an asymptotic method (see for example [25] and references therein) and leads to the introduction of boundary value problems with nonlinear transmission conditions.

We also observe that a large literature has been dedicated to the existence of solutions of nonlinear boundary value problems by means of variational techniques (see, e.g., Nečas [28] and Roubíček [30]). In particular, if it happens that problem (1) can be reformulated into an equation of the form \( -\text{div} A(x, U) \nabla U = 0 \), where \( A \) is a suitable Carathéodory function and the unknown function \( U \) belongs to
the Sobolev space $H^1(\Omega^o)$ and satisfies a Dirichlet condition on $\partial \Omega^o$, then the existence and uniqueness of a solution can be directly deduced by the results of Hlaváček, Křížek and Malý in [9]. This is for example the case when $G = 0$ and the function $F(x, t)$ of $(x, t) \in \partial \Omega^i \times \mathbb{R}$ is constant with respect to $x$, is differentiable with respect to $t$, and the partial differential $\partial_t F(x_0, \cdot)$ is Lipschitz continuous and satisfies the inequality $1/c < \partial_t F(x_0, t) < c$ for a constant $c > 0$ and for all $t \in \mathbb{R}$ (here $x_0$ is a fixed point of $\partial \Omega^i$).

In this paper instead we exploit a method based on potential theory to rewrite problem (1) into a suitable nonlinear system of integral equations which can be analysed by fixed-point theorems. Potential theoretic techniques have been largely exploited in literature to study existential and uniqueness problems for linear or semilinear partial differential equations with non linear boundary conditions. In particular, as far back as in 1921 Carleman [4] has considered the existence of harmonic functions $u$ in a domain $\Omega$ which satisfy a non-linear Robin condition $\nu_{\Omega} \cdot \nabla u = H(x, u)$ on the boundary $\partial \Omega$. Since then such a problem has received the attention of many authors, here we mention for example the contributions of Leray (cf., e.g., [14] and [20]), Nakamori and Suyama [27], Kühnelhöfer [15, 16], Cushing [6], and Efendiev, Schmitz, and Wendland [8]. For the case of domains with a small hole we also mention Lanza de Cristoforis [17] and [7], where nonlinear traction problems in elasticity are addressed. More recently an approach based on coupling of boundary integral and finite element methods has been developed in order to study exterior nonlinear boundary value problems with transmission conditions, we mention for example the papers of Berger [2], Berger, Warnecke, and Wendland [3], Costabel and Stephan [5], and Gatica and Hsiao [11, 12]. In particular, Barrenechea and Gatica considered in [1] the case when the jump of the normal derivative across the interface boundary depends nonlinearly on the Dirichlet data. Boundary integral methods have been applied also by Mityushev and Rogosin for the analysis of transmission problems in the two dimensional plane (cf. [26, Chap. 5]). Finally, we mention the nonlinear transmission problem in a domain with a small inclusion which shrinks to a point was investigated by Lanza de Cristoforis in [18] by an approach based on potential theory (see also Lanza de Cristoforis and Musolino [19] for a periodic analog).

We now describe the main result of this paper. To do so we introduce the following notation: if $H$ is a function from $\partial \Omega^i \times \mathbb{R}$, then we denote by $F_H$ the nonlinear non-autonomous composition operators which takes a function $f$ from $\partial \Omega^i$ to $\mathbb{R}$ to the function $F_H(f)$ defined by

$$F_H(f)(x) \equiv H(x, f(x)) \quad \forall x \in \partial \Omega^i.$$ 

Since $F$ and $G$ which define the nonlinear condition in (1) are assumed to be continuous from $\partial \Omega^i \times \mathbb{R}$, one easily verifies that $F_F$ and $F_G$ map $C^0(\partial \Omega^i)$ to
itself. Then we consider the following condition:

the composition operator \((I_{\Omega} + F)\) is invertible from \(C^0(\partial \Omega^i)\) to itself. \(\text{(2)}\)

Here \(I_{\Omega}\) denotes the identity operator from \(C^0(\partial \Omega^i)\) to itself. We observe that condition \((2)\) does not imply the invertibility of \(F(x, \cdot)\) for a fixed \(x \in \partial \Omega^i\). Indeed, if \(\text{id}_R\) is the identity function of \(\mathbb{R}\), then condition \((2)\) is equivalent to the condition that \((\text{id}_R + F(x, \cdot))\) is invertible in \(\mathbb{R}\) for all fixed \(x \in \partial \Omega^i\) and that \((\text{id}_R + F(x, \cdot))^{-1}(t)\) is a continuous function of \(x \in \partial \Omega^i\) for all fixed \(t \in \mathbb{R}\). Then we introduce a grow condition on \(F\) and \(G\): we assume that

there exist \(c_1, c_2 \in ]0, +\infty[\), \(\delta_1 \in ]1, +\infty[\), and \(\delta_2 \in [0, 1]\) such that

\[
|F(x, t)| \geq c_1 |t|^\delta_1 - (1/c_1) \quad \text{and} \quad |G(x, t)| \leq c_2 (1 + |F(x, t)|)^\delta_2 \quad \forall (x, t) \in \partial \Omega^i \times \mathbb{R}.
\]  \(\text{(3)}\)

We observe that the first condition \((3)\) is a super-linear grow condition for \(F\), while the second one is a sub-linear grow condition for \(G\) with respect to \(F\), which is a strictly weaker condition than the standard sub-linear condition \(|G(x, t)| \leq c_2 (1 + |t|)^\delta_2\).

By exploiting an argument based on the invariance of the Leray-Schauder topological degree we show in Theorem 3.11 that, under conditions \((2)\) and \((3)\), there exists at least one pair of continuous functions \((\tilde{u}^0, \tilde{u}^i) \in C^0(\overline{\Omega^o} \setminus \Omega^i) \times C^0(\overline{\Omega^o})\) which satisfies the first four equations of \((1)\) and fulfils the fifth condition in a weak sense which will be clarified. However, we observe that the conditions \((2)\) and \((3)\) do not imply any uniqueness property for the solution \((\tilde{u}^0, \tilde{u}^i)\) (not even the local uniqueness).

This fact can be verified in a simple example: we take \(\Omega^o = R\overline{B}_n\), \(\Omega^i = r\overline{B}_n\), with \(r, R \in \mathbb{R}, r < R\), and with \(\overline{B}_n \equiv \{x \in \mathbb{R}^n : |x| < 1\}\), and we assume that \(f^o\) is constant and identically equal to a real number \(t^o \in \mathbb{R}\). Then we look for solutions of problem \((1)\) in the form

\[
u^o(x) = c^o + d^o \Gamma_n(|x|) \quad \text{for all } x \in cl\Omega^o \setminus \Omega^i, \quad \nu^i(x) = c^i \quad \text{for all } x \in cl\Omega^i \quad (4)\]

where \(c^o, d^o, c^i \in \mathbb{R}\) and

\[
\Gamma_n(t) \equiv \begin{cases} \frac{1}{2\pi} \log t & \text{if } n = 2, \\ \frac{t^{n-2}}{s_n(2-n)} & \text{if } n \geq 3, \end{cases} \quad \forall t \in ]0, +\infty[.
\]

Here \(s_n\) denotes the \((n-1)\)-dimensional measure of \(\partial \overline{B}_n\) (thus \(\Gamma_n(|x|) = S_n(x)\), where \(S_n\) denotes the standard fundamental solution of \(\Delta\), cf. equality \((7)\) below). Then the problem of finding a solution of \((1)\) is reduced to that of finding a real number \(c^i\) such that

\[
t^o + \frac{\Gamma_n(r) - \Gamma_n(R)}{\Gamma_n(r)} G(c^i) = F(c^i). \quad (5)\]
Once that we have \( c^i \) we can recover \( u^o \) and \( u^o \) by (4) with \( c^o = t^o - G(c^i) \Gamma_n(R)/\Gamma_n'(r) \) and \( d^o = G(c^i)/\Gamma_n'(r) \). Now let 

\[
F(t) \equiv t^3 - 2t^2 + t + 1 \quad \forall t \in \mathbb{R}
\]

and \( G \) be constant. One immediately verifies that \( F \) and \( G \) satisfies conditions (2) and (3). In addition, if we choose \( t^o, R, \) and \( r \) in such a way that the left hand side of (5) is equal to 1, then equation (5) has two solutions, \( c^i = 0 \) and \( c^i = 1 \), and thus the corresponding problem (1) has two different solutions (see Fig.1). If instead we take \( F(t) \equiv t^3 - 2t^2 + t + 1 \) for \( t < 0 \) and \( t > 1 \) and \( F(t) \equiv 1 \) for \( t \in [0, 1] \), then every \( c^i \) in \([0, 1]\) is a solution of (5) and the solutions of (1) are not locally unique in any reasonable norm.

Finally we observe that one of the main concerns of this paper is that of showing the existence of a solution of (1) under minimal and completely explicit conditions on \( F \) and \( G \). However, one may wish to have classical solutions in \( C^{1,\alpha}(\text{cl}\Omega^o \setminus \Omega^i) \times C^{1,\alpha}(\text{cl}\Omega^i) \) (or at least in \( H^1(\Omega^o \setminus \text{cl}\Omega^i) \times H^1(\Omega^i) \)) instead of the weak solutions in \( C^0(\text{cl}\Omega^o \setminus \Omega^i) \times C^0(\text{cl}\Omega^i) \) which are provided by Theorem 3.11. Thus, it is natural to ask what further conditions should one impose on \( F \) and \( G \) in order to obtain such a regularity. In Theorem 3.12 we show that, if

\[
(2 I_{\Omega^i} + \mathcal{F}_F)^{(-1)} \text{ and } \mathcal{F}_G \text{ map } C^{0,\alpha}(\partial\Omega^i) \text{ to itself,}
\]

then problem (1) has at least one weak solution in \( C^{0,\alpha}(\text{cl}\Omega^o \setminus \Omega^i) \times C^{0,\alpha}(\text{cl}\Omega^i) \). However, in order to obtain solutions in \( C^{1,\alpha}(\text{cl}\Omega^o \setminus \Omega^i) \times C^{1,\alpha}(\text{cl}\Omega^i) \) it does not suffice to increase the regularity of \( F \) and \( G \) and it seems that a completely different approach should be implemented.

The paper is organised as follows. Section 2 is a section of preliminaries were we introduce some classical notion of potential theory. Then in Section 3 we prove our main Theorems 3.11 and 3.12.

### 2 Classical notions of potential theory

We denote by \( S_n \) the function from \( \mathbb{R}^n \setminus \{0\} \) to \( \mathbb{R} \) defined by

\[
S_n(x) \equiv \begin{cases} \frac{1}{2\pi} \log |x| & \text{if } n = 2, \\ \frac{1}{s_n(2-n)} |x|^{2-n} & \text{if } n \geq 3, \end{cases} \quad \forall x \in \mathbb{R}^2 \setminus \{0\},
\]

Figure 1: the intersections of the blue graph with the red line correspond to solutions of (5) and in [0, 1] are not locally unique in any reasonable norm.
where $s_n$ denotes the $(n-1)$-dimensional measure of the unite sphere in $\mathbb{R}^n$. As is well known, $S_n$ is a fundamental solution for the Laplace operator in $\mathbb{R}^n$.

Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\phi \in L^2(\partial \Omega)$. Then $v_\Omega[\phi]$ denotes the single layer potential with density $\phi$. Namely,

$$v_\Omega[\phi](x) \equiv \int_{\partial \Omega} \phi(y) S_n(x - y) \, d\sigma_y \quad \forall x \in \mathbb{R}^n,$$

where $d\sigma$ denotes the area element on $\partial \Omega$. As is well known, $v_\Omega[\phi]$ is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$. If $\phi \in C^{0,\alpha}(\partial \Omega)$, then the restrictions $v_\Omega^+[\phi] \equiv v_\Omega[\phi]|_{\text{cl}\Omega}$ and $v_\Omega^-[\phi] \equiv v_\Omega[\phi]|_{\mathbb{R}^n\setminus \Omega}$ belong to $C^{1,\alpha}(\text{cl}\Omega)$ and to $C^{1,\alpha}_\text{loc}(\mathbb{R}^n \setminus \Omega)$, respectively. Here $C^{1,\alpha}_\text{loc}(\mathbb{R}^n \setminus \Omega)$ denotes the space of functions on $\mathbb{R}^n \setminus \Omega$ whose restrictions to $\text{cl}\Omega$ belong to $C^{1,\alpha}(\text{cl}\Omega)$ for all open bounded subsets $\Omega$ of $\mathbb{R}^n \setminus \Omega$.

If $\psi \in L^2(\partial \Omega)$, then $w_\Omega[\psi]$ denotes the double layer potential with density $\psi$. Namely,

$$w_\Omega[\psi](x) \equiv -\int_{\partial \Omega} \psi(y) \nu_\Omega(y) \cdot \nabla S_n(x - y) \, d\sigma_y \quad \forall x \in \mathbb{R}^n,$$

where $\nu_\Omega$ denotes the outer unit normal to $\partial \Omega$ and the symbol `$\cdot$' denotes the scalar product in $\mathbb{R}^n$. If $\psi \in C^{1,\alpha}(\partial \Omega)$, then the restriction $w_\Omega[\psi]|_{\partial \Omega}$ extends to a function $w_\Omega^+[\psi]$ of $C^{1,\alpha}(\text{cl}\Omega)$ and the restriction $w_\Omega[\psi]|_{\mathbb{R}^n\setminus \text{cl}\Omega}$ extends to a function $w_\Omega^-[\psi]$ of $C^{1,\alpha}_\text{loc}(\mathbb{R}^n \setminus \Omega)$.

Let

$$W_\Omega[\psi](x) \equiv -\int_{\partial \Omega} \psi(y) \nu_\Omega(y) \cdot \nabla S_n(x - y) \, d\sigma_y \quad \forall x \in \partial \Omega,$$

for all $\psi \in L^2(\partial \Omega)$, and

$$W_\Omega^+[\phi](x) \equiv \int_{\partial \Omega} \phi(y) \nu_\Omega(x) \cdot \nabla S_n(x - y) \, d\sigma_y \quad \forall x \in \partial \Omega,$$

for all $\phi \in L^2(\partial \Omega)$. As is well known $W_\Omega$ and $W_\Omega^+$ are compact operator from $L^2(\partial \Omega)$ to itself and are adjoint one to the other. We denote by $I_\Omega$ the identity map from $L^2(\partial \Omega)$ to itself. Then we have that

$$\pm \frac{1}{2} I_\Omega + W_\Omega \text{ and } \pm \frac{1}{2} I_\Omega + W_\Omega^+ \text{ are Fredholm operators from } L^2(\partial \Omega) \text{ to itself.}$$

In addition we have the following classical result of Schauder [31, 32]:

**Lemma 2.1.** Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\beta \in [0, 1]$. Then the map which takes $\psi$ to $W_\Omega[\psi]$ is continuous from $C^0(\partial \Omega)$ to $C^{0,\alpha}(\partial \Omega)$ and from $C^{1,\beta}(\partial \Omega)$ to $C^{1,\alpha}(\partial \Omega)$. The map which takes $\phi$ to $W_\Omega^+[\phi]$ is continuous from $C^{0,\beta}(\partial \Omega)$ to $C^{0,\alpha}(\partial \Omega)$.  

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As a consequence, the map which takes \( \psi \) to \( W_\Omega[\psi] \) is compact from \( C^{1,\alpha}(\partial\Omega) \) to itself and the map which takes \( \phi \) to \( W^*_\Omega[\phi] \) is compact from \( C^{0,\alpha}(\partial\Omega) \) to itself and one immediately deduces the validity of the following.

**Lemma 2.2.** The operators \( \pm \frac{1}{2} I_\Omega + W_\Omega \) are Fredholm of index 0 from \( C^0(\partial\Omega) \) to itself, from \( C^{0,\alpha}(\partial\Omega) \) to itself, and from \( C^{1,\alpha}(\partial\Omega) \) to itself. The operators \( \pm \frac{1}{2} I_\Omega + W^*_\Omega \) are Fredholm of index 0 from \( C^{0,\alpha}(\partial\Omega) \) to itself.

Moreover we have

\[
 w^\pm_\Omega(\psi)|_{\partial\Omega} = \pm \frac{1}{2} \psi + W_\Omega[\psi] \quad \text{and} \quad \nu_\Omega \cdot \nabla v^\pm_\Omega[\psi]|_{\partial\Omega} = \mp \frac{1}{2} \phi + W^*_\Omega[\psi] \quad (8)
\]

for all continuous function \( \psi \in C^0(\partial\Omega) \) (cf., e.g., Folland [10, Chap. 3]). If \( \psi \in C^{1,\alpha}(\partial\Omega) \) then we also have

\[
 \nu_\Omega \cdot \nabla w^\pm_\Omega[\psi]|_{\partial\Omega} = \nu_\Omega \cdot \nabla w^\pm_\Omega[\psi]|_{\partial\Omega}. \quad (9)
\]

We exploit the following notation: if \( \mathcal{X} \) is a subspace of \( L^1(\partial\Omega) \) then we denote by \( \mathcal{X}_0 \) the subspace of \( \mathcal{X} \) consisting of the functions which have 0 integral mean. Then we have the following classical lemma (cf., e.g., Folland [10, Chap. 3]).

**Lemma 2.3.** Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \) of class \( C^{1,\alpha} \). Let \( \Omega_1, \ldots, \Omega_N \) be the bounded connected components of \( \Omega \) and \( \Omega^-_1, \Omega^-_2, \ldots, \Omega^-_M \) be the connected components of \( \mathbb{R}^n \setminus \text{cl}\Omega \). Assume that \( \Omega^-_1, \ldots, \Omega^-_M \) are bounded and \( \Omega^-_0 \) is unbounded. Then the following statements hold.

(i) The map from \( \text{Ker}(\frac{1}{2} I_\Omega + W^*_\Omega) \) to \( \text{Ker}(\frac{1}{2} I_\Omega + W_\Omega) \) which takes \( \mu \) to \( v|_{\partial\Omega}, \mu|_{\partial\Omega} \) is bijective.

(ii) The map from \( \text{Ker}(\frac{1}{2} I_\Omega + W^*_\Omega)_0 \) to \( \text{Ker}(\frac{1}{2} I_\Omega + W_\Omega) \) which takes \( \mu \) to \( v|_{\partial\Omega}, \mu|_{\partial\Omega} \) is one to one. If \( n \geq 3 \), then the map from \( \text{Ker}(\frac{1}{2} I_\Omega + W^*_\Omega) \) to \( \text{Ker}(\frac{1}{2} I_\Omega + W_\Omega) \) which takes \( \mu \) to \( v|_{\partial\Omega}, \mu|_{\partial\Omega} \) is bijective.

(iii) \( \text{Ker}(\frac{1}{2} I_\Omega + W_\Omega) \) consists of the functions from \( \partial\Omega \) to \( \mathbb{R} \) which are constant on \( \partial\Omega^-_j \) for all \( j \in \{1, \ldots, M\} \) which are identically equal to 0 on \( \partial\Omega^-_0 \).

(iv) \( \text{Ker}(\frac{1}{2} I_\Omega + W_\Omega) \) consists of the functions from \( \partial\Omega \) to \( \mathbb{R} \) which are constant on \( \partial\Omega_j \) for all \( j \in \{1, \ldots, N\} \).

(v) If \( \phi \in \text{Ker}(\frac{1}{2} I_\Omega + W^*_\Omega) \) and \( \int_{\partial\Omega} \phi \psi \, d\sigma = 0 \) for all \( \psi \in \text{Ker}(\frac{1}{2} I_\Omega + W_\Omega) \), then \( \phi = 0 \).

(vi) If \( \phi \in \text{Ker}(\frac{1}{2} I_\Omega + W^*_\Omega) \) and \( \int_{\partial\Omega} \phi \psi \, d\sigma = 0 \) for all \( \psi \in \text{Ker}(\frac{1}{2} I_\Omega + W_\Omega) \), then \( \phi = 0 \).

Finally, we have the following technical Lemma.
Lemma 2.4. Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\psi \in L^2(\partial \Omega)$. Let $\beta \in [0, \alpha]$. If $(\frac{1}{2} I_\Omega + W^*_\Omega)\psi$ or $(-\frac{1}{2} I_\Omega + W^*_\Omega)\psi$ belongs to $C^{0,\beta}(\partial \Omega)$, then $\psi \in C^{0,\beta}(\partial \Omega)$.

Proof. If $(\frac{1}{2} I_\Omega + W^*_\Omega)\psi \in C^{0,\beta}(\partial \Omega)$, then a standard argument based on iterated kernels ensures that $\psi \in C^0(\partial \Omega)$. It follows that $W^*_\Omega\psi \in C^{0,\beta}(\partial \Omega)$ for all $\beta' \in [0, \alpha]$ (cf. Miranda [22, Chap. II, §14, IV]). Thus the lemma is proved for $\beta < \alpha$. If instead $\beta = \alpha$, then we observe that $W^*_\Omega\psi \in C^{0,\alpha}(\partial \Omega)$ by the membership of $\psi$ in $C^{0,\beta}(\partial \Omega)$ with $\beta' < \alpha$ (cf. Schauder [32, Hilfssatz XXII]). Then the validity of the Lemma follows.

3 Existence results

We prove in this section the existence of continuous solutions $(u^o, u^i)$ of problem (1) which satisfy the fifth equation of (1) in a weak sense which will be clarified.

As a first step we deduce in the following Lemma 3.1 a representation for a pair of harmonic functions in $C^{1,\alpha}(\text{cl}(\Omega^o \setminus \Omega^i)) \times C^{1,\alpha}(\text{cl}\Omega^i)$ in terms of a suitable combination of layer potential.

Lemma 3.1. The map from $C^{1,\alpha}(\partial \Omega^o) \times C^{1,\alpha}(\partial \Omega^i) \times C^{0,\alpha}(\partial \Omega^i)$ to the set of pairs of functions

$$
\left\{ (\phi^o, \phi^i) \in C^{1,\alpha}(\text{cl}(\Omega^o \setminus \Omega^i)) \times C^{1,\alpha}(\text{cl}\Omega^i) : 
\Delta \phi^o = 0 \text{ in } \Omega^o \setminus \text{cl}\Omega^i \text{ and } \Delta \phi^i = 0 \text{ in } \Omega^i \right\}
$$

which takes $(\mu^o, \mu, \eta)$ to the pair $(u^o[\mu^o, \mu, \eta], u^i[\mu^o, \mu, \eta])$ given by

$$
u^o[\mu^o, \mu, \eta] \equiv (w^+_\Omega[\mu^o] + w^-_\Omega[\mu] + v^-_\Omega[\eta]|_{\text{cl}(\Omega^o \setminus \Omega^i)}, \ u^i[\mu^o, \mu, \eta] \equiv w^-_\Omega[\mu],
$$

is bijective.

Proof. The map is well defined. Indeed $(u^o[\mu^o, \mu, \eta], u^i[\mu^o, \mu, \eta]) \in C^{1,\alpha}(\text{cl}(\Omega^o \setminus \Omega^i)) \times C^{1,\alpha}(\text{cl}\Omega^i)$ and $\Delta u^o[\mu^o, \mu, \eta] = 0$, $\Delta u^i[\mu^o, \mu, \eta] = 0$ for all $(\mu^o, \mu, \eta) \in C^{1,\alpha}(\partial \Omega^o) \times C^{1,\alpha}(\partial \Omega^i) \times C^{0,\alpha}(\partial \Omega^i)$. We now show that it is bijective. We take a pair of harmonic functions $(\phi^o, \phi^i)$ in $C^{1,\alpha}(\text{cl}(\Omega^o \setminus \Omega^i)) \times C^{1,\alpha}(\text{cl}\Omega^i)$ and we prove that there exists unique $(\mu^o, \mu, \eta) \in C^{1,\alpha}(\partial \Omega^o) \times C^{1,\alpha}(\partial \Omega^i) \times C^{0,\alpha}(\partial \Omega^i)$ such that $(u^o[\mu^o, \mu, \eta], u^i[\mu^o, \mu, \eta]) = (\phi^o, \phi^i)$. By the standard properties of the double layer potential there exists a unique $\mu \in C^{1,\alpha}(\partial \Omega^o)$ such that $u^i[\mu^o, \mu, \eta] = \phi^i$ (cf. [8]) and Lemma 2.3 (iii)). Then we have to show that there exists unique $(\mu^o, \eta) \in C^{1,\alpha}(\partial \Omega^o) \times C^{0,\alpha}(\partial \Omega^i)$ such that

$$(w^+_\Omega[\mu^o] + v^-_\Omega[\eta]|_{\text{cl}(\Omega^o \setminus \Omega^i)} = w^-_\Omega[\mu]|_{\text{cl}(\Omega^o \setminus \Omega^i)} + \phi^o). \quad (10)$$
Let \( \psi^o \equiv w^{-}_\Omega[\mu]|_{\partial \Omega^o} + \phi^o|_{\partial \Omega^o} \) and \( \psi^i \equiv \nu_\Omega \cdot \nabla w^{-}_\Omega[\mu]|_{\partial \Omega^i} + \nu_\Omega \cdot \nabla \phi^o|_{\partial \Omega^i} \). Then \( \psi^o \in C^{1,\alpha}(\partial \Omega^o) \), \( \psi^i \in C^{0,\alpha}(\partial \Omega^o) \) and by the uniqueness of the classical solution of the Neumann-Dirichlet mixed boundary value problem, equation (10) is equivalent to

\[
\begin{align*}
\left( \frac{1}{2} I_{\Omega^o} + W_{\Omega^o} \right) \mu + v^{-}_\Omega[\eta]|_{\partial \Omega^o} &= \psi^o, \\
\left( \frac{1}{2} I_{\Omega^i} + W^*_{\Omega^i} \right) \eta + \nu_\Omega \cdot \nabla w^+_{\Omega^i}[\mu^o]|_{\partial \Omega^i} &= \psi^i
\end{align*}
\]

(see also (3)). By Lemmas 2.3 and 2.2 the operator which takes \((\mu, \eta)\) to \(((\frac{1}{2} I_{\Omega^o} + W_{\Omega^o})\mu, (\frac{1}{2} I_{\Omega^i} + W^*_{\Omega^i})\eta)\) is a linear isomorphism from \(C^{1,\alpha}(\partial \Omega^o) \times C^{0,\alpha}(\partial \Omega^i)\) to itself. Moreover, by the properties of the integral operators with real analytic kernels and no singularities, the operator which takes \((\mu, \eta)\) to \(((\frac{1}{2} I_{\Omega^o} + W_{\Omega^o})\mu + v^{-}_\Omega[\eta]|_{\partial \Omega^o}, (\frac{1}{2} I_{\Omega^i} + W^*_{\Omega^i})\eta + \nu_\Omega \cdot \nabla w^+_{\Omega^i}[\mu^o]|_{\partial \Omega^i})\) is compact from \(C^{1,\alpha}(\partial \Omega^o) \times C^{0,\alpha}(\partial \Omega^i)\) to itself. Hence, the operator which takes \((\mu, \eta)\) to \(((\frac{1}{2} I_{\Omega^o} + W_{\Omega^o})\mu + v^{-}_\Omega[\eta]|_{\partial \Omega^o}, (\frac{1}{2} I_{\Omega^i} + W^*_{\Omega^i})\eta + \nu_\Omega \cdot \nabla w^+_{\Omega^i}[\mu^o]|_{\partial \Omega^i})\) is a Fredholm operator of index 0 from \(C^{1,\alpha}(\partial \Omega^o) \times C^{0,\alpha}(\partial \Omega^i)\) to itself. Thus, to complete the proof it suffices to show that \((\psi^o, \psi^i)\) = (0, 0) in equation (11). If \(((\frac{1}{2} I_{\Omega^o} + W_{\Omega^o})\mu + v^{-}_\Omega[\eta]|_{\partial \Omega^o}, (\frac{1}{2} I_{\Omega^i} + W^*_{\Omega^i})\eta + \nu_\Omega \cdot \nabla w^+_{\Omega^i}[\mu^o]|_{\partial \Omega^i})\) = (0, 0), then by the jump properties (8) and by the uniqueness of the classical solution of the Neumann-Dirichlet mixed problem one deduces that \((w^+_{\Omega^i}[\mu^o] + v^{-}_\Omega[\eta])|_{\partial \Omega^i} = 0\). Hence \((w^+_{\Omega^i}[\mu^o] + v^{-}_\Omega[\eta])|_{\partial \Omega^i} = 0\) by the uniqueness of the classical solution of the Dirichlet problem in \(\Omega^i\) and by the continuity of \((w^+_{\Omega^i}[\mu^o] + v^{-}_\Omega[\eta])|_{\partial \Omega^i}\). Then \(\eta = \nu_\Omega \cdot \nabla w^+_{\Omega^i}[\mu^o] - \nu_\Omega \cdot \nabla v^{-}_\Omega[\eta]|_{\partial \Omega^i} = \nu_\Omega \cdot \nabla (w^+_{\Omega^i}[\mu^o] + v^{-}_\Omega[\eta])|_{\partial \Omega^i} - \nu_\Omega \cdot \nabla (w^+_{\Omega^i}[\mu^o] + v^{-}_\Omega[\eta])|_{\partial \Omega^i} = 0\). By (11) it follows that \(((\frac{1}{2} I_{\Omega^o} + W_{\Omega^o})\mu = 0\) and thus \(\mu = 0\) by Lemma 2.3 (iii).

In the following Lemma 3.2 we introduce an auxiliary integral operator which we denote by \(J\).

**Lemma 3.2.** Let

\[ J[\eta] \equiv \left( \frac{1}{2} I_{\Omega^i} + W^*_{\Omega^i} \right) \eta - \nu_\Omega \cdot \nabla w^+_{\Omega^i}\left[ \left( \frac{1}{2} I_{\Omega^o} + W_{\Omega^o} \right)^{-1} v^{-}_\Omega[\eta]|_{\partial \Omega^o} \right] \]

for all \(\eta \in L^2(\partial \Omega^i)\). Then the map which takes \(\eta\) to \(J[\eta]\) is an isomorphism from \(L^2(\partial \Omega^i)\) to itself, from \(C^0(\partial \Omega^i)\) to itself, and from \(C^{0,\alpha}(\partial \Omega^i)\) to itself.

**Proof.** By the properties of integral operators with real analytic kernels and no singularity, by the invertibility of \(\frac{1}{2} I_{\Omega^o} + W_{\Omega^o}\) in \(C^{1,\alpha}(\partial \Omega^o)\) (cf. Lemma 2.3 (iii) and Lemma 2.2), and by the continuity of the map \(w^+_{\Omega^i}[\cdot]\) from \(C^{1,\alpha}(\partial \Omega^p)\) to.
\( C^{1,\alpha}(\partial \Omega^o) \), one deduces that the operator which takes \( \eta \)

\[
\nu_{\Omega^i} \cdot \nabla w_{\Omega^i}^+ \left[ \left( \frac{1}{2} I_{\Omega^o} + W_{\Omega^o}\right)^{-1} v_{\Omega^i}[\eta]|_{\partial \Omega^i} \right] \mid_{\partial \Omega^i}
\]

is continuous from \( L^2(\partial \Omega^i) \) to \( C^{0,\alpha}(\partial \Omega^i) \). Then, by the compactness of \( W_{\Omega^o}^* \) in \( L^2(\partial \Omega^i) \) it follows that \( J \) is Fredholm operator of index 0 from \( L^2(\partial \Omega^i) \) to itself. Thus, to show that \( J \) is invertible from \( L^2(\partial \Omega^i) \) to itself suffices to prove that \( \eta = 0 \) if \( J[\eta] = 0 \). If \( \eta \in L^2(\partial \Omega^i) \) and \( J[\eta] = 0 \), then \( \left( \frac{1}{2} I_{\Omega^i} + W_{\Omega^o}^* \right) \eta \in C^{0,\alpha}(\partial \Omega^i) \). It follows that \( \eta \in C^{0,\alpha}(\partial \Omega^i) \) (cf. Lemma 2.4). Then, by setting \( \mu^o = \left( \frac{1}{2} I_{\Omega^o} + W_{\Omega^o}\right)^{-1} v_{\Omega^i}[\eta]|_{\partial \Omega^o} \) and by arguing as in the proof of Lemma 3.1 one verifies that \( \eta = 0 \). To prove that \( J \) is invertible from \( C^0(\partial \Omega^i) \) to itself we observe that \( J \) is continuous from \( C^0(\partial \Omega^i) \) to itself (because \( W_{\Omega^o}^* \) has a weak singularity). Moreover, if \( \eta \in L^2(\partial \Omega^i) \) and \( J[\eta] \in C^0(\partial \Omega^i) \) then \( \left( \frac{1}{2} I_{\Omega^i} + W_{\Omega^o}^* \right) \eta \in C^0(\partial \Omega^i) \), and thus Lemma 2.4 ensures that \( \eta \in C^0(\partial \Omega^i) \). Similarly, to prove that \( J \) is invertible from \( C^{0,\alpha}(\partial \Omega^i) \) to itself we observe that \( J \) is continuous from \( C^{0,\alpha}(\partial \Omega^i) \) to itself and that \( J[\eta] \in C^{0,\alpha}(\partial \Omega^i) \) implies \( \eta \in C^{0,\alpha}(\partial \Omega^i) \) for all \( \eta \in L^2(\partial \Omega^i) \).

Then we have the following Lemma 3.3 where we rewrite problem (1) into an equivalent system of boundary integral equations.

**Lemma 3.3.** Let condition (2) hold. Let \( (\mu^o, \mu, \eta) \in C^{1,\alpha}(\partial \Omega^o) \times C^{1,\alpha}(\partial \Omega^i) \times C^{0,\alpha}(\partial \Omega^i) \). Then \( (u^o|\mu^o, \mu, \eta), u^i|\mu^o, \mu, \eta) \) is a solution of (1) if and only if

\[
\mu^o = \left( \frac{1}{2} I_{\Omega^o} + W_{\Omega^o}\right)^{-1} \left( f^o - w_{\Omega^o}^-[\mu]|_{\partial \Omega^o} - v_{\Omega^i}[\eta]|_{\partial \Omega^i} \right),
\]

\[
\mu = \left( \frac{1}{2} I_{\Omega^i} + W_{\Omega^o}\right)^{-1} \left[ \left( 2 I_{\Omega^i} + \mathcal{F}_F\right)^{-1} \left( w_{\Omega^i}^+[\mu^o]|_{\partial \Omega^i} + v_{\Omega^i}[\eta]|_{\partial \Omega^i} + 2W_{\Omega^i} \mu \right) \right],
\]

\[
\eta = J^{-1} \left[ \mathcal{F}_G \circ \left( 2 I_{\Omega^i} + \mathcal{F}_F\right)^{-1} \left( w_{\Omega^i}^+[\mu^o]|_{\partial \Omega^i} + v_{\Omega^i}[\eta]|_{\partial \Omega^i} + 2W_{\Omega^i} \mu \right) \right. 
\]

\[
- \nu_{\Omega^i} \cdot \nabla w_{\Omega^i}^+ \left[ \left( \frac{1}{2} I_{\Omega^o} + W_{\Omega^o}\right)^{-1} \left( f^o - w_{\Omega^o}^-[\mu]|_{\partial \Omega^o} \right) \right].
\]

(12)

**Proof.** Note that \( \nu_{\Omega^i} \cdot \nabla w_{\Omega^i}^-[\mu](x) - \nu_{\Omega^i} \cdot \nabla w_{\Omega^i}^+[\mu](x) = 0 \) by the membership of \( \mu \) in \( C^{1,\alpha}(\partial \Omega^i) \) (cf. (9)). Then the validity of the statement is a consequence of Lemma 3.1 of the jump properties of single and double layer potentials (cf. (8)), of the invertibility of \( \left( \frac{1}{2} I_{\Omega^o} + W_{\Omega^o}\right) \) in \( C^{1,\alpha}(\partial \Omega^o) \) and of \( \left( \frac{1}{2} I_{\Omega^i} + W_{\Omega^o}\right) \) and \( J \) in \( L^2(\partial \Omega^i) \), and of condition (2).

We prove the existence of a solution \((\tilde{\mu}^o, \tilde{\mu}, \tilde{\eta})\) in \( C^{1,\alpha}(\partial \Omega^i) \times C^0(\partial \Omega^i) \times C^0(\partial \Omega^i) \) of the system of equations in (12) by using the Leray-Schauder principle, which follows by the invariance of the Leray-Schauder degree (see e.g. Gilbarg and Trudinger [13, Theorem 11.3]).
Theorem 3.4. Let \( X \) be a Banach space. Let \( T \) be a continuous (nonlinear) operator from \( X \) to itself which maps bounded sets to sets with a compact closure. Suppose that there exists a constant \( M \in \mathbb{R}^+ \) such that \( \|x\|_X \leq M \) for all \( (x, t) \in X \times [0, 1] \) satisfying \( x = tT(x) \). Then \( T \) has a fixed point \( x \in X \) such that \( \|x\|_X \leq M \).

In order to apply this principle, we introduce in the following Lemma an elementary consequence of conditions (2) and (3).

Lemma 3.5. Let conditions (2) and (3) hold. Then there exist \( C_1, C_2, C_3, C_4 \in \mathbb{R}^+ \) such that

\[
\|(2\Omega + F_F)^{(-1)}f\|_{C^0(\partial \Omega')} \leq C_1(C_2 + \|f\|_{C^0(\partial \Omega')}^{1/\delta_1}) \tag{13}
\]

and

\[
\|F_G \circ (2\Omega + F_F)^{(-1)}f\|_{C^0(\partial \Omega')} \leq C_3(C_4 + \|f\|_{C^0(\partial \Omega')}^{\delta_2}). \tag{14}
\]

Proof. To prove (13) we observe that the first inequality in (3) implies that there exist \( c_1^*, c_2^* \in \mathbb{R}^+ \) such that \( |2t + F(x, t)| \geq c_1^*|t|^{\delta_1} - c_2^* \) for all \( (x, t) \in \partial \Omega' \times \mathbb{R} \). Thus we have \( \|(2\Omega + F_F)g\|_{C^0(\partial \Omega')} \geq c_1^*\|g\|_{C^0(\partial \Omega')}^{\delta_1} - c_2^* \) for all \( g \in C^0(\partial \Omega') \) and the validity of (13) follows by taking \( g = (2\Omega + F_F)^{(-1)}f \). To prove (14) we observe that the second inequality in (3) implies that there exist \( c_3^*, c_4^* \in \mathbb{R}^+ \) such that \( |G(x, t)| \leq c_3^*(c_4^* + |2t + F(x, t)|)^{\delta_2} \) for all \( (x, t) \in \partial \Omega' \times \mathbb{R} \). Then we have \( \|F_G g\|_{C^0(\partial \Omega')} \leq c_3^*(c_4^* + \|(2\Omega + F_F)g\|_{C^0(\partial \Omega')}^{\delta_2}) \) for all \( g \in C^0(\partial \Omega') \) and the validity of (14) follows by condition (2) and by taking \( g = (2\Omega + F_F)^{(-1)}f \). \( \square \)

Then we have the following.

Proposition 3.6. Let conditions (2) and (3) hold. Then the nonlinear system (12) has at least one solution \((\tilde{\mu}, \tilde{\eta})\) in \( C^{1, \alpha}(\partial \Omega') \times C^{0}(\partial \Omega') \times C^{0}(\partial \Omega') \).

Proof. We plan to apply Theorem 3.4 with \( \mathcal{X} = C^{1, \alpha}(\partial \Omega') \times C^{0}(\partial \Omega') \times C^{0}(\partial \Omega') \) and \( T \equiv (T^0, T_1, T_2) \) given by

\[
T^0(\tilde{\mu}, \tilde{\eta}) \equiv \left( \frac{1}{2} I_{\Omega'} + W_{\Omega'} \right)^{(-1)}(f^0 - w_{\Omega'}[\tilde{\mu}]_{\partial \Omega'} - v_{\Omega'}[\tilde{\eta}]_{\partial \Omega'})
\]

\[
T_1(\tilde{\mu}, \tilde{\eta}) \equiv \left( \frac{1}{2} I_{\Omega'} + W_{\Omega'} \right)^{(-1)} \left[ (2\Omega + F_F)^{(-1)} \left( w_{\Omega'}^+[\tilde{\mu}]_{\partial \Omega'} + v_{\Omega'}[\tilde{\eta}]_{\partial \Omega'} + 2W_{\Omega'}\tilde{\mu} \right) \right],
\]

\[
T_2(\tilde{\mu}, \tilde{\eta}) \equiv J^{(-1)} \left[ F_G \circ (2\Omega + F_F)^{(-1)} \left( w_{\Omega'}^+[\tilde{\mu}]_{\partial \Omega'} + v_{\Omega'}[\tilde{\eta}]_{\partial \Omega'} + 2W_{\Omega'}\tilde{\mu} \right) \right.
\]

\[
- \nu_{\Omega'} \cdot \nabla w_{\Omega'}^+ \left( \frac{1}{2} I_{\Omega'} + W_{\Omega'} \right)^{(-1)}(f^0 - w_{\Omega'}^-[\tilde{\mu}]_{\partial \Omega'})], \tag{15}
\]
for all $(\tilde{\mu}, \tilde{\mu}, \tilde{\eta}) \in C^{1,\alpha}(\partial \Omega^0) \times C^0(\partial \Omega^0) \times C^0(\partial \Omega^0)$. We first verify that $T$ is continuous from $C^{1,\alpha}(\partial \Omega^0) \times C^0(\partial \Omega^0) \times C^0(\partial \Omega^0)$ to itself and maps bounded sets to sets with compact closure. To do so, we consider separately $T_0$, $T_1$ and $T_2$. By Lemmas 2.2 and 2.3 one deduces that $\frac{1}{2} I_{\Omega^0} + W_{\Omega^0}$ is an isomorphism from $C^{1,\alpha}(\partial \Omega^0)$ to itself. In particular, $(\frac{1}{2} I_{\Omega^0} + W_{\Omega^0})^{-1}$ is continuous from $C^{1,\alpha}(\partial \Omega^0)$ to itself. Moreover, by the properties of integral operators with real analytic kernel and no singularities $w_{\Omega^0}^{-1} \cdot |_{\partial \Omega^0}$ and $v_{\Omega^0}^{-1} \cdot |_{\partial \Omega^0}$ are compact from $C^0(\partial \Omega^0)$ to $C^{1,\alpha}(\partial \Omega^0)$. It follows that $T_0$ is continuous from $C^{1,\alpha}(\partial \Omega^0) \times C^0(\partial \Omega^0) \times C^0(\partial \Omega^0)$ to $C^{1,\alpha}(\partial \Omega^0)$ and maps bounded sets to sets with compact closure. We now consider $T_1$. By arguing as above for $(\frac{1}{2} I_{\Omega^0} + W_{\Omega^0})^{-1}$ one verifies that $(\frac{1}{2} I_{\Omega^0} + W_{\Omega^0})^{-1}$ is continuous from $C^0(\partial \Omega^0)$ to itself. By assumption (2) also the map $(2 I_{\Omega^0} + F_{\Omega^0})^{-1}$ is continuous from $C^0(\partial \Omega^0)$ to itself. Then, by the properties of integral operators with real analytic kernel and no singularities $w_{\Omega^0} \cdot |_{\partial \Omega^0}$ is compact from $C^{1,\alpha}(\partial \Omega^0)$ to $C^0(\partial \Omega^0)$ and, by Lemma 2.1 $v_{\Omega^0} \cdot |_{\partial \Omega^0}$ and $W_{\Omega^0}$ are compact from $C^0(\partial \Omega^0)$ to itself. If follows that $T_1$ is continuous from $C^{1,\alpha}(\partial \Omega^0) \times C^0(\partial \Omega^0) \times C^0(\partial \Omega^0)$ to $C^0(\partial \Omega^0)$ and maps bounded sets to sets with compact closure. Finally we consider $T_2$. By Lemma 3.2 (ii) the operator $J^{-1}$ is continuous from $C^0(\partial \Omega^0)$ to itself. By the continuity of $G$ and by condition (2), the map $F_G \circ (2 I_{\Omega^0} + F_{\Omega^0})^{-1}$ is continuous from $C^0(\partial \Omega^0)$ to itself. By Lemma 2.1 $v_{\Omega^0} \cdot |_{\partial \Omega^0}$ and $W_{\Omega^0}$ are compact from $C^0(\partial \Omega^0)$ to itself. By the properties of integral operators with real analytic kernel and no singularities and by the continuity of $(\frac{1}{2} I_{\Omega^0} + W_{\Omega^0})^{-1}$ from $C^{1,\alpha}(\partial \Omega^0)$ to itself, the maps $w_{\Omega^0}^{-1} \cdot |_{\partial \Omega^0}$ is compact from $C^{1,\alpha}(\partial \Omega^0)$ to $C^0(\partial \Omega^0)$, and the map $v_{\Omega^0} \cdot \nabla w_{\Omega^0}^{-1} \cdot \left((\frac{1}{2} I_{\Omega^0} + W_{\Omega^0})^{-1} w_{\Omega^0}^{-1} \cdot |_{\partial \Omega^0}\right)$ is compact from $C^0(\partial \Omega^0)$ to itself. Accordingly $T_2$ is continuous from $C^{1,\alpha}(\partial \Omega^0) \times C^0(\partial \Omega^0) \times C^0(\partial \Omega^0)$ to $C^0(\partial \Omega^0)$ and maps bounded sets to sets with compact closure.

Now let $t \in [0, 1]$ and assume that $(\tilde{\mu}, \tilde{\mu}, \tilde{\eta}) = tT(\tilde{\mu}, \tilde{\mu}, \tilde{\eta})$. We show that there exists a constant $M \in [0, +\infty]$ (which does not depend on $t$) such that

$$\| (\tilde{\mu}, \tilde{\mu}, \tilde{\eta}) \|_{C^{1,\alpha}(\partial \Omega^0) \times C^0(\partial \Omega^0) \times C^0(\partial \Omega^0)} \leq M.$$  

If $(\tilde{\mu}, \tilde{\mu}, \tilde{\eta}) = tT(\tilde{\mu}, \tilde{\mu}, \tilde{\eta})$, then

$$\| \tilde{\mu} \|_{C^{1,\alpha}(\partial \Omega^0)} \leq \| T_0(\tilde{\mu}, \tilde{\mu}, \tilde{\eta}) \|_{C^{1,\alpha}(\partial \Omega^0)},$$

$$\| \tilde{\mu} \|_{C^0(\partial \Omega^0)} \leq \| T_1(\tilde{\mu}, \tilde{\mu}, \tilde{\eta}) \|_{C^0(\partial \Omega^0)},$$

$$\| \tilde{\eta} \|_{C^0(\partial \Omega^0)} \leq \| T_2(\tilde{\mu}, \tilde{\mu}, \tilde{\eta}) \|_{C^0(\partial \Omega^0)}.$$  

Then, there exists a constant $m_1 \in [0, +\infty]$ which depends only on the norm of the bounded linear operator $(\frac{1}{2} I_{\Omega^0} + W_{\Omega^0})^{-1}$ from $C^{1,\alpha}(\partial \Omega^0)$ to itself, on $\| f \|_{C^{1,\alpha}(\partial \Omega^0)}$, and on the norm of the linear bounded operators $w_{\Omega^0}^{-1} \cdot |_{\partial \Omega^0}$ and $v_{\Omega^0}^{-1} \cdot |_{\partial \Omega^0}$ from $C^0(\partial \Omega^0)$ to $C^{1,\alpha}(\partial \Omega^0)$, such that

$$\| \tilde{\mu} \|_{C^{1,\alpha}(\partial \Omega^0)} \leq m_1 (1 + \| \tilde{\mu} \|_{C^0(\partial \Omega^0)} + \| \tilde{\eta} \|_{C^0(\partial \Omega^0)}).$$  

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By the second inequality of (16), we deduce that there exist real constants $m_2, m_3 \in ]0, +\infty[\) which depend on the norm of the linear bounded operator $(\frac{1}{2}I_{\Omega^0} + W_{\Omega^0})^{(-1)}$ from $C^0(\partial \Omega^0)$ to itself, on the constants $C_1$ and $C_2$ of Lemma 3.5 on the norm of the linear bounded operator $w_{\Omega^0}^+([\partial \Omega^0)$ from $C^{1, \alpha}(\partial \Omega^0)$ to $C^0(\partial \Omega^0)$, and on the norm of the linear bounded operators $v_{\Omega^0}[\partial \Omega^0$ and $W_{\Omega^0}$ from $C^0(\partial \Omega^0)$ to itself such that

$$\|\tilde{\mu}\|_{C^0(\partial \Omega^0)} \leq m_2(m_3 + \|\tilde{\mu}^o\|_{C^{1, \alpha}(\partial \Omega^0)} + \|\tilde{\mu}\|_{C^0(\partial \Omega^0)} + \|\tilde{\eta}\|_{C^0(\partial \Omega^0)})^{1/\delta_1}.$$  (18)

By the third inequality of (16) we deduce that there exist real constants $m_4, m_5, m_6 \in ]0, +\infty[\) which depend on the norm of the linear bounded operator $J^{(-1)}$ from $C^0(\partial \Omega^0)$ to itself, on the constants $C_3$ and $C_4$ of Lemma 3.5 on the norm of the linear bounded operator $w_{\Omega^0}^+([\partial \Omega^0)$ from $C^{1, \alpha}(\partial \Omega^0)$ to $C^0(\partial \Omega^0)$, on the norm of the linear bounded operators $v_{\Omega^0}[\partial \Omega^0$ and $W_{\Omega^0}$ from $C^0(\partial \Omega^0)$ to itself, on the norm of $\nu_{\Omega^0} \cdot \nabla w_{\Omega^0}^{+([\frac{1}{2}I_{\Omega^0} + W_{\Omega^0})^{(-1)}f^o}\Omega^0$ in $C^0(\partial \Omega^0)$, and on the norm of the bounded linear operator $\nu_{\Omega^0} \cdot \nabla w_{\Omega^0}^{+([\frac{1}{2}I_{\Omega^0} + W_{\Omega^0})^{(-1)}w_{\Omega^0}^-([\partial \Omega^0]$ from $C^0(\partial \Omega^0)$ to itself, such that

$$\|\tilde{\eta}\|_{C^0(\partial \Omega^0)} \leq m_4\left[(m_5 + \|\tilde{\mu}^o\|_{C^{1, \alpha}(\partial \Omega^0)} + \|\tilde{\mu}\|_{C^0(\partial \Omega^0)} + \|\tilde{\eta}\|_{C^0(\partial \Omega^0)})^{\delta_2} + 1 + m_6\|\tilde{\mu}\|_{C^0(\partial \Omega^0)}\right].$$  (19)

Then, by inequalities (17), (18), and (19) one deduces that there exists real constants $M_1, M_2, M_3 \in [0, +\infty[\), which depend on $m_1, \ldots, m_6$, such that

$$\|\tilde{\mu}^o\|_{C^{1, \alpha}(\partial \Omega^0)} + \|\tilde{\mu}\|_{C^0(\partial \Omega^0)} + \|\tilde{\eta}\|_{C^0(\partial \Omega^0)} \leq M_1 + M_2(M_3 + \|\tilde{\mu}^o\|_{C^{1, \alpha}(\partial \Omega^0)} + \|\tilde{\mu}\|_{C^0(\partial \Omega^0)} + \|\tilde{\eta}\|_{C^0(\partial \Omega^0)})^{\delta_*}$$

with $\delta_* \equiv \inf\{1/\delta_1, \delta_2\} \in [0, 1]$. Then a straightforward calculation shows that

$$\|\tilde{\mu}^o\|_{C^{1, \alpha}(\partial \Omega^0)} + \|\tilde{\mu}\|_{C^0(\partial \Omega^0)} + \|\tilde{\eta}\|_{C^0(\partial \Omega^0)} \leq \sup\left\{M_3, (M_1M_3^{-\delta_*} + M_2^{\delta_*})^{1/(1-\delta_*)}\right\}.$$

Now the validity of the statement follows by Theorem 3.4

With a further regularity request on $F$ and $G$ we can find a solution of (12) in $C^{1, \alpha}(\partial \Omega^0) \times C^{0, \alpha}(\partial \Omega^0) \times C^{0, \alpha}(\partial \Omega^0)$.

**Proposition 3.7.** Let conditions (3), (4), and (6) hold. Then the nonlinear system (12) has at least one solution $(\tilde{\mu}^o, \tilde{\mu}, \tilde{\eta})$ in $C^{1, \alpha}(\partial \Omega^0) \times C^{0, \alpha}(\partial \Omega^0) \times C^{0, \alpha}(\partial \Omega^0)$.

**Proof.** Let $T$ be as in (15). By Proposition 3.5 there exists $(\tilde{\mu}^o, \tilde{\mu}, \tilde{\eta})$ in $C^{1, \alpha}(\partial \Omega^0) \times C^0(\partial \Omega^0) \times C^0(\partial \Omega^0)$ such that $(\tilde{\mu}^o, \tilde{\mu}, \tilde{\eta}) = T(\tilde{\mu}^o, \tilde{\mu}, \tilde{\eta})$. Then, by the mapping properties of integral operators with real analytic kernels and no singularities...
we have that \( w_{1,\mu}^-[\tilde{\mu}]_{|\partial\Omega^o} \) and \( v_{1,\Omega}^-[\tilde{\eta}]_{|\partial\Omega^o} \) belong to \( C^{1,\alpha}(\partial\Omega^o) \), that \( w_{1,\mu}^+[\tilde{\nu}]_{|\partial\Omega^o} \) belongs to \( C^{1,\alpha}(\partial\Omega^o) \), and that \( v_{1,\Omega}^+\nabla w_{1,\Omega}^+[\tilde{\psi}]_{|\partial\Omega^o} \) belongs to \( C^{0,\alpha}(\partial\Omega^o) \) for all \( \tilde{\psi} \in C^0(\partial\Omega^o) \). By a classical result in potential theory (cf. e.g., Miranda \cite{Miranda}, Chap. II, §14, III]) we have that \( v_{1,\Omega}^-[\tilde{\eta}]_{|\partial\Omega^o} \in C^{0,\alpha}(\partial\Omega^i) \) and by Lemma 2.1 we have that \( W_{1,\Omega}^-\mu \in C^{0,\alpha}(\partial\Omega^i) \). Then, by the invertibility of \( \frac{1}{2}I_{\Omega^o} + W_{1,\Omega}^- \) in \( C^{1,\alpha}(\partial\Omega^o) \) and of \( \frac{1}{2}I_{\Omega^o} + W_{1,\Omega}^+ \) in \( C^{1,\alpha}(\partial\Omega^i) \) (cf. Lemma 2.1 and 2.3 (iii)), by the invertibility of \( J \) in \( C^{0,\alpha} \) (cf. Lemma 3.2 (iii)), and by assumption \( (\tilde{T}) \) it follows that \( T(\tilde{\mu}, \tilde{\mu}, \tilde{\eta}) \in C^{1,\alpha}(\partial\Omega^o) \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i) \). Thus \((\tilde{\mu}^o, \tilde{\mu}, \tilde{\eta}) \in C^{1,\alpha}(\partial\Omega^o) \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i) \) and our proof is completed.

In the following Theorem 3.11 we show that under conditions (2) and (3) there exists a solution \((\tilde{w}^o, \tilde{w}^i) \in C^0(cl\Omega^o \setminus \Omega^i) \times C^0(cl\Omega^i) \) such that \( \Delta \tilde{w}^o = 0 \) in \( \Omega^o \setminus \Omega^i \) and \( \Delta \tilde{w}^i = 0 \) in \( \Omega^i \). Then \([\nu_{1,\Omega} \cdot \nabla \tilde{w}^o \cdot \tilde{w}^i]_{|\partial\Omega^o} \) denotes the distribution on \( \Omega^o \) defined by

\[
\langle [\nu_{1,\Omega} \cdot \nabla (\tilde{w}^o - \tilde{w}^i)]_{|\partial\Omega^o}, \phi \rangle := \int_{\partial\Omega^o} (\tilde{w}^o_{|\partial\Omega^o} - \tilde{w}^i_{|\partial\Omega^o}) (\nu_{1,\Omega} \cdot \nabla \phi_{|\partial\Omega^o}) \, d\sigma + \int_{\Omega^o \setminus \Omega^i} \tilde{w}^o \Delta \phi \, dx + \int_{\Omega^i} \tilde{w}^i \Delta \phi \, dx
\]

for all test functions \( \phi \in C_c^\infty(\Omega^o) \).

One immediately verifies that the map which takes \((\tilde{w}^o, \tilde{w}^i) \) to \([\nu_{1,\Omega} \cdot \nabla (\tilde{w}^o - \tilde{w}^i)]_{|\partial\Omega^o} \) is continuous. Namely we have the following.

**Lemma 3.9.** Let \((\tilde{w}^o, \tilde{w}^i) \) be a pair of functions of \( C^0(cl\Omega^o \setminus \Omega^i) \times C^0(cl\Omega^i) \) such that \( \Delta \tilde{w}^o = 0 \) in \( \Omega^o \setminus cl\Omega^i \) and \( \Delta \tilde{w}^i = 0 \) in \( \Omega^i \). Let \( \{(\tilde{w}^o_j, \tilde{w}^i_j)\}_{j \in \mathbb{N}} \) be a sequence in \( C^0(cl\Omega^o \setminus \Omega^i) \times C^0(cl\Omega^i) \) such that \( \Delta \tilde{w}^o_j = 0 \) in \( \Omega^o \setminus cl\Omega^i \) and \( \Delta \tilde{w}^i_j = 0 \) in \( \Omega^i \) for all \( j \in \mathbb{N} \). If \( \lim_{j \to +\infty} \tilde{w}^o_j = \tilde{w}^o \) in \( C^0(cl\Omega^o \setminus \Omega^i) \) and \( \lim_{j \to +\infty} \tilde{w}^i_j = \tilde{w}^i \) in \( C^0(cl\Omega^i) \) then

\[
\lim_{j \to +\infty} \langle [\nu_{1,\Omega} \cdot \nabla (\tilde{w}^o_j - \tilde{w}^i_j)]_{|\partial\Omega^o}, \phi \rangle = \langle [\nu_{1,\Omega} \cdot \nabla (\tilde{w}^o - \tilde{w}^i)]_{|\partial\Omega^o}, \phi \rangle \quad \forall \phi \in C_c^\infty(\Omega^o) .
\]

Moreover, if \((w^o, w^i) \) belongs to \( C^1(cl\Omega^o \setminus \Omega^i) \times C^1(cl\Omega^i) \), and \( \Delta w^o = 0 \) in \( \Omega^o \setminus cl\Omega^i \), \( \Delta w^i = 0 \) in \( \Omega^i \), then \([\nu_{1,\Omega} \cdot \nabla (w^o - w^i)]_{|\partial\Omega^o} \) coincides with \((\nu_{1,\Omega} \cdot \nabla w^o - \nu_{1,\Omega} \cdot \nabla w^i)_{|\partial\Omega^o} \). Namely we have

\[
\langle [\nu_{1,\Omega} \cdot \nabla (w^o - w^i)]_{|\partial\Omega^o}, \phi \rangle = \int_{\partial\Omega^o} (\nu_{1,\Omega} \cdot \nabla w^o(x) - \nu_{1,\Omega} \cdot \nabla w^i(x)) \phi(x) \, d\sigma_x
\]
for all $\phi \in C_c^\infty(\Omega^o)$ and for all pair of functions $(w^o, \tilde{w}^i) \in C^1(\text{cl}\Omega^o \setminus \Omega^i) \times C^1(\text{cl}\Omega^i)$ such that $\Delta w^o = 0$ in $\Omega^o \setminus \text{cl}\Omega^i$ and $\Delta \tilde{w}^i = 0$ in $\Omega^i$. Then we can prove that $[\nu_{\partial\Omega} \cdot \nabla (\tilde{w}^o - \tilde{w}^i)]_w$ is supported on $\partial\Omega^i$.

**Lemma 3.10.** For all $(\tilde{w}^o, \tilde{w}^i) \in C^0(\text{cl}\Omega^o \setminus \Omega^i) \times C^0(\text{cl}\Omega^i)$ such that $\Delta \tilde{w}^o = 0$ in $\Omega^o \setminus \text{cl}\Omega^i$ and $\Delta \tilde{w}^i = 0$ in $\Omega^i$ the support of $[\nu_{\partial\Omega} \cdot \nabla (\tilde{w}^o - \tilde{w}^i)]_w$ is contained in $\partial\Omega^i$.

**Proof.** Let $\phi_0 \in C_c^\infty(\Omega^o)$ be such that $\phi_0|_{\partial\Omega^o} = 0$. By a classical argument one can prove that there exists a sequence $\{(w^o_j, w^i_j)\}_{j \in \mathbb{N}}$ of harmonic functions in $C^{1,\alpha}(\text{cl}(\Omega^o \setminus \Omega^i)) \times C^{1,\alpha}(\text{cl}\Omega^i)$ such that $\lim_{j \to +\infty} w^o_j = \tilde{w}^o$ in $C^0(\text{cl}\Omega^o \setminus \Omega^i)$ and $\lim_{j \to +\infty} w^i_j = \tilde{w}^i$ in $C^0(\text{cl}\Omega^i)$. Then we have

$$\langle [\nu_{\partial\Omega} \cdot \nabla (w^o_j - w^i_j)]_w, \phi_0 \rangle = \int_{\partial\Omega^i} (\nu_{\partial\Omega} \cdot \nabla w^o_j(x) - \nu_{\partial\Omega} \cdot \nabla w^i_j(x)) \phi_0(x) \, d\sigma_x = 0$$

for all $j \in \mathbb{N}$. Moreover $\lim_{j \to +\infty} \langle [\nu_{\partial\Omega} \cdot \nabla (w^o_j - w^i_j)]_w, \phi_0 \rangle = \langle [\nu_{\partial\Omega} \cdot \nabla (w^o - w^i)]_w, \phi_0 \rangle$ by Lemma 3.9 and thus $\langle [\nu_{\partial\Omega} \cdot \nabla (\tilde{w}^o - \tilde{w}^i)]_w, \phi_0 \rangle = 0$. □

We are now ready to prove the main result of this section.

**Theorem 3.11.** Assume that $F$ and $G$ satisfy (2) and (3). Then there exists $(\tilde{u}^o, \tilde{u}^i) \in C^0(\text{cl}\Omega^o \setminus \Omega^i) \times C^0(\text{cl}\Omega^i)$ such that

$$\begin{cases}
\Delta \tilde{u}^o = 0 & \text{in } \Omega^o \setminus \text{cl}\Omega^i, \\
\Delta \tilde{u}^i = 0 & \text{in } \Omega^i, \\
\tilde{u}^o(x) = f(x) & \text{for all } x \in \partial\Omega^o, \\
\tilde{u}^i(x) = F(x, \tilde{u}^i(x)) & \text{for all } x \in \partial\Omega^i, \\
\langle [\nu_{\partial\Omega} \cdot \nabla (\tilde{u}^o - \tilde{u}^i)]_w, \phi \rangle = \int_{\partial\Omega^i} G(x, \tilde{u}^i(x)) \phi(x) \, d\sigma_x & \text{for all } \phi \in C_c^\infty(\Omega^o). 
\end{cases}$$

(20)

**Proof.** Let $(\tilde{\mu}^o, \tilde{\mu}^i) \in C^{1,\alpha}(\partial\Omega^o) \times C^0(\partial\Omega^i) \times C^0(\partial\Omega^i)$ be as in Proposition 3.6 and define

$$\tilde{u}^o \equiv (w^o_{\partial\Omega}[\tilde{\mu}^o] + w^i_{\partial\Omega}[\tilde{\mu}^i] + v^i_{\partial\Omega}[\tilde{\eta}]) |_{\text{cl}\Omega^o \setminus \Omega^i}, \quad \tilde{u}^i \equiv w^i_{\partial\Omega}[\tilde{\mu}^i].$$

Then the pair $(\tilde{u}^o, \tilde{u}^i)$ belongs to $C^0(\text{cl}\Omega^o \setminus \Omega^i) \times C^0(\text{cl}\Omega^i)$ (cf. Folland [? Chap. 3]) and satisfies the first four conditions of (20) (see also (3)). We now prove that $(\tilde{u}^o, \tilde{u}^i)$ satisfies also the fifth condition of (20).

By a standard argument one proves that there exists a sequence $\{v^i_j\}_{j \in \mathbb{N}}$ of harmonic functions in $C^{1,\alpha}(\text{cl}\Omega^i)$ such that

$$\lim_{j \to +\infty} v^i_j = \tilde{u}^i \quad \text{in } C^0(\text{cl}\Omega^i).$$

(21)
By Lemmas \ref{lem2.2} and \ref{lem2.3} we have that \( \frac{1}{2} I_{\Omega^i} + W_{\Omega^i} \) is an isomorphism from \( C^0(\partial\Omega^i) \) to itself and from \( C^{1,\alpha}(\partial\Omega^i) \) to itself. Then, by (8) one verifies that there exists \( \mu_j \in C^{1,\alpha}(\partial\Omega^i) \) such that \( v_j^i = w_{\Omega^i}^+ [\mu_j] \) for all \( j \in \mathbb{N} \). Moreover, by the continuity of \( \left( \frac{1}{2} I_{\Omega^i} + W_{\Omega^i} \right)^{(-1)} \) from \( C^0(\partial\Omega^i) \) to itself, we have

\[
\lim_{j \to +\infty} \mu_j = \lim_{j \to +\infty} \left( \frac{1}{2} I_{\Omega^i} + W_{\Omega^i} \right)^{(-1)} v_j^i|_{\partial\Omega^i} = \left( \frac{1}{2} I_{\Omega^i} + W_{\Omega^i} \right)^{(-1)} \tilde{u}^i|_{\partial\Omega^i} = \tilde{\mu} \quad \text{in} \quad C^0(\partial\Omega^i). \tag{22}
\]

Then we set

\[
\mu^\circ_j \equiv \left( \frac{1}{2} I_{\Omega^o} + W_{\Omega^o} \right)^{(-1)} (f^o - w_{\Omega^o}^- [\mu_j]|_{\partial\Omega^o} - v_{\Omega^o} [\tilde{\eta}]|_{\partial\Omega^o}) \quad \forall j \in \mathbb{N}.
\]

By Lemmas \ref{lem2.2} and \ref{lem2.3} we have that \( \frac{1}{2} I_{\Omega^o} + W_{\Omega^o} \) is an isomorphism from \( C^{1,\alpha}(\partial\Omega^o) \) to itself and from \( C^0(\partial\Omega^o) \) to itself. In particular, \( \left( \frac{1}{2} I_{\Omega^o} + W_{\Omega^o} \right)^{(-1)} \) is continuous from \( C^0(\partial\Omega^o) \) to itself and maps \( C^{1,\alpha}(\partial\Omega^o) \) to itself. Moreover, by the properties of integral operators with real analytic kernel and no singularities \( w_{\Omega} \) is continuous from \( C^0(\partial\Omega^o) \) to \( C^{1,\alpha}(\partial\Omega^o) \) and \( f^o - v_{\Omega^o} [\tilde{\eta}]|_{\partial\Omega^o} \) belongs to \( C^{1,\alpha}(\partial\Omega^o) \). It follows that \( \mu^\circ_j \in C^{1,\alpha}(\partial\Omega^o) \) for all \( j \in \mathbb{N} \) and that

\[
\lim_{j \to +\infty} \mu^\circ_j = \lim_{j \to +\infty} \left( \frac{1}{2} I_{\Omega^o} + W_{\Omega^o} \right)^{(-1)} (f^o - w_{\Omega^o}^- [\mu_j]|_{\partial\Omega^o} - v_{\Omega^o} [\tilde{\eta}]|_{\partial\Omega^o}) = \mu^\circ \quad \text{in} \quad C^0(\partial\Omega^o). \tag{23}
\]

Now let

\[
v^\circ_j \equiv (w_{\Omega^o}^+[\mu^\circ_j] + w_{\Omega^o}^- [\mu_j] + v_{\Omega^o}^- [\tilde{\eta}])|_{\text{cl}\Omega^o \setminus \Omega^i} \quad \forall j \in \mathbb{N}.
\]

By classical potential theory the maps \( w_{\Omega^o}^+ [\cdot]|_{\text{cl}\Omega^o \setminus \Omega^i} \) and \( w_{\Omega^o}^- [\cdot]|_{\text{cl}\Omega^o \setminus \Omega^i} \) are continuous from \( C^{0,\alpha}(\partial\Omega^o) \) to \( C^{0,\alpha}(\text{cl}\Omega^o \setminus \Omega^i) \) and from \( C^{0,\alpha}(\partial\Omega^o) \) to \( C^{0,\alpha}(\text{cl}\Omega^o \setminus \Omega^i) \), respectively. Then, by (22) and (23) it follows that

\[
\lim_{j \to +\infty} v^\circ_j = \tilde{u}^o \quad \text{in} \quad C^0(\text{cl}\Omega^o \setminus \Omega^i). \tag{24}
\]

In addition, by the jump formulas (8) and (9) one verifies that the pair \((v^\circ_j, v^i_j)\) satisfies the equality

\[
\nu_{\Omega^i} \cdot \nabla v^\circ_j (x) - \nu_{\Omega^o} \cdot \nabla v^i_j (x) = G(x, \tilde{u}^i (x)) \quad \forall x \in \partial\Omega^i
\]

for all \( j \in \mathbb{N} \). Hence, by Lemma 3.9 and the by the limit relations in (21) and in (24) we deduce that \((\tilde{u}^o, \tilde{u}^i)\) satisfies the fifth condition in problem (20). Now the theorem is proved. \( \square \)

If in addition \( F \) and \( G \) satisfy assumption (6), then the pair \((\tilde{u}^o, \tilde{u}^i)\) belongs to \( C^{0,\alpha}(\text{cl}\Omega^o \setminus \Omega^i) \times C^{0,\alpha}(\text{cl}\Omega^i). \)
Theorem 3.12. Assume that $F$ and $G$ satisfy [2], [3], and [6]. Then there exists $(\tilde{u}^o, \tilde{u}^i) \in C^{0,\alpha}(\partial\Omega^o \setminus \Omega^i) \times C^{0,\alpha}(\partial\Omega^i)$ such that

\[
\begin{align*}
\Delta \tilde{u}^o &= 0 & \text{in } \Omega^o \setminus \partial \Omega^i, \\
\Delta \tilde{u}^i &= 0 & \text{in } \Omega^o, \\
\tilde{u}^o(x) &= f(x) & \text{for all } x \in \partial \Omega^o, \\
\tilde{u}^o(x) - \tilde{u}^i(x) &= F(x, \tilde{u}^i(x)) & \text{for all } x \in \partial \Omega^i, \\
\langle [\nu_{\Omega^i} \cdot \nabla (\tilde{u}^o - \tilde{u}^i)]_w, \phi \rangle &= \int_{\partial \Omega^i} G(x, \tilde{u}^i(x)) \phi(x) \, d\sigma_x & \text{for all } \phi \in C^\infty_c(\partial \Omega^o).
\end{align*}
\]

Proof. If $(\tilde{\mu}^o, \tilde{\mu}^i, \tilde{\eta}) \in C^{1,\alpha}(\partial \Omega^o) \times C^{0,\alpha}(\partial \Omega^i) \times C^{0,\alpha}(\partial \Omega^i)$ is as in Proposition 3.12 and

\[
\tilde{u}^o \equiv (w_{\Omega^o}^{\tilde{\mu}^o} + w_{\Omega^i}^{\tilde{\mu}^o} + v_{\Omega^i}^{\tilde{\eta}})_{\partial \Omega^o \setminus \Omega^i}, \quad \tilde{u}^i \equiv w_{\Omega^i}^{\tilde{\mu}^i},
\]

then the pair $(\tilde{u}^o, \tilde{u}^i)$ belongs to $C^{0,\alpha}(\partial \Omega^o \setminus \Omega^i) \times C^{0,\alpha}(\partial \Omega^i)$ (cf. Miranda [21]) and we can prove that it satisfies the conditions of [20] by arguing as in the proof of Theorem 3.11.

\[\square\]

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