Abstract—In this paper, we discuss pinning consensus in networks of multiagents via impulsive controllers. In particular, we consider the case of using only one impulsive controller. We provide a sufficient condition to pin the network to a prescribed value. It is rigorously proven that in case the underlying graph of the network has spanning trees, the network can reach consensus on the prescribed value when the impulsive controller is imposed on the root with appropriate impulsive strength and impulse intervals. Interestingly, we find that the permissible range of the impulsive strength completely depends on the left eigenvector of the graph Laplacian corresponding to the zero eigenvalue and the pinning node we choose. The impulses can be very sparse, with the impulse intervals being lower bounded. Examples with numerical simulations are also provided to illustrate the theoretical results.

Index Terms—consensus, synchronization, multiagent systems, impulsive pinning control.

I. INTRODUCTION

Coordinated and cooperative control of teams of autonomous systems has received much attention in recent years. Significant research activity has been devoted to this area. In the cooperation, group of agents seek to reach agreement on a certain quantity of interest. This is the so-called consensus problem, which has a long history in computer science. Recently, consensus problem reappeared in the cooperative control of multi-agent systems and has gained renewed interests due to the broad applications of multi-agent systems. A great deal of papers have addressed this problem. For a review of this area, see the surveys [1], [2] and references therein.

The basic idea of consensus is that each agent updates its state based on the states of its neighbors and its own such that the states of all agents will converge to a common value. The interaction rule that specifies the information exchange between an agent and its neighbors is called the consensus algorithm.

The following is an example of continuous-time consensus algorithm:

\[
\dot{x}_i(t) = \sum_{j=1, j \neq i}^{n} a_{ij}[x_j(t) - x_i(t)], \quad i = 1, \ldots, n \tag{1}
\]

where \(x_i(t) \in \mathbb{R}\) is the state of agent \(i\) at time \(t\), \(a_{ij} \geq 0\) for \(i \neq j\) is the coupling strength from agent \(j\) to agent \(i\).

Let \(a_{ii} = -\sum_{j=1, j \neq i}^{n} a_{ij}\) for \(i = 1, 2, \ldots, n\), we can have

\[
\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}x_j(t), \quad i = 1, \ldots, n. \tag{2}
\]

A topic closely related to consensus is synchronization, which can be written as the following Linearly Coupled Ordinary Differential Equations (LCODEs):

\[
\frac{dx^i(t)}{dt} = f(x^i(t), t) + c \sum_{j=1}^{n} a_{ij}x^j(t), \quad i = 1, \ldots, n \tag{3}
\]

where \(x^i(t) \in \mathbb{R}^n\) is the state variable of the \(i\)th node at time \(t\), \(f: \mathbb{R}^n \times [0, +\infty) \rightarrow \mathbb{R}^n\) is a continuous map, \(A = [a_{ij}] \in \mathbb{R}^{n \times n}\) is the coupling matrix with zero-sum rows and \(a_{ij} \geq 0\), for \(i \neq j\), which is determined by the topological structure of the LCODEs.

There are lots of papers discussing synchronization in various circumstances. It is clear that the consensus is a special case of synchronization (\(f = 0, m = 1\)). Therefore, all the results concerning synchronization can apply to consensus.

It was shown in [3], [4] that under some assumptions, we have

\[
\lim_{t \to \infty} \|x^i(t) - \sum_{j=1}^{n} \xi_jx^j(t)\| = 0, \quad i = 1, \ldots, n, \tag{4}
\]

where \([\xi_1, \ldots, \xi_n]^T\) is the left eigenvector of \(A\) corresponding to the eigenvalue \(0\) satisfying \(\sum_{j=1}^{n} \xi_j = 1\).

Since in the consensus model,

\[
\sum_{j=1}^{n} \xi_jx_j(t) = \sum_{j=1}^{n} \xi_jx_j(0) \tag{5}
\]

for all \(t > 0\), we have

\[
\lim_{t \to \infty} |x^i(t) - \sum_{j=1}^{n} \xi_jx^j(0)| = 0, \quad i = 1, \ldots, n. \tag{6}
\]

It can be seen that the agreement value \(\sum_{j=1}^{n} \xi_jx^j(0)\) strongly depends on the initial value, which means that the
agreement value of the consensus algorithm is neutral stable (or semi-stable used in some papers). The concept of neutral stability is used in physics and other research fields. For example, the principal subspace extraction algorithms and principal component extraction algorithms discussed in [5]. A set of equilibrium points is called neutral stable for a system, if each equilibrium is Lyapunov stable, and every trajectory that starts in a neighborhood of an equilibrium converges to a possibly different equilibrium. Similarly, a set of manifolds is called neutral stable for a system, if every manifold is invariant, and when there is a small perturbation, the state will stay in another manifold and never return.

In [5], the manifold discussed is neutral stable, and if the algorithm is restricted to the manifold, the Stiefel manifold is stable. Instead, in [6], the equilibrium is neutral stable. A set of equilibrium points is called neutral stable for a system, if each equilibrium is Lyapunov stable, and every point is neutral stable. However, in some cases, it is desired that all states converge to a prescribed value, say, some \( s \in \mathbb{R} \). For example, in a military system, if one wants to use a missile network to attack some object of the enemy, then it is required that all the missiles from different military bases should finally hit the same point (see [19]). Generally, for this purpose, one can make every state \( x_i(t) \) converge to \( s \) by imposing a negative feedback term \(-[x_i(t) - s]\) to agent \( i \). However, due to the interaction of the network, it is not necessary to impose controllers on all the nodes. This is the basic idea of the pinning control technique, which is an effective class of control schemes. Generally, in a pinning control scheme, we only need to impose controllers on a small fraction of the nodes. This is a big advantage because in large complex networks, it is usually difficult if not impossible to add controllers to all the nodes. Recently, pinning strategies have been used in the control of dynamical networks. For example, decentralized adaptive pinning strategies have been proposed in [26, 27] for controlled synchronization of complex networks. And pinning consensus algorithms have been proposed in [10, 20].

Most works on pinning control consider pinning a fraction of the nodes. However, there are a few works that consider pinning only one node. In [7], it was proved that if \( \epsilon > 0 \), the following coupled dynamical network with a single controller

\[
\begin{align*}
\frac{dx^1(t)}{dt} &= f(x^1(t), t) + \epsilon \sum_{j=1}^{n} a_{1j}x^j(t) - c[x^1(t) - s(t)], \\
\frac{dx^2(t)}{dt} &= f(x^2(t), t) + c \sum_{j=1}^{n} a_{2j}x^j(t), \quad i = 2, \ldots, n
\end{align*}
\]

(7)

can pin the complex dynamical network \( x(s) \) to \( s(t) \), if \( c \) is chosen suitably. Therefore, the following coupled network with a single controller

\[
\begin{align*}
\dot{x}_1(t) &= \sum_{j=1}^{n} a_{1j}x_j(t) - \epsilon[x_1(t) - s], \\
\dot{x}_i(t) &= \sum_{j=1}^{n} a_{ij}x_j(t), \quad i = 2, \ldots, n
\end{align*}
\]

(8)
can make every state \( x_i(t) \) converge to \( s \).

It is worth noticing that the above mentioned works all consider continuous time feedback controllers and the disadvantage of such controllers lies in that the controller must be imposed at every time \( t \). So it is not applicable to systems which can not endure continuous disturbances. One can ask if we can pin the network only at a very sparse time sequence to make every state \( x_i(t) \) converge to \( s \) for the consensus algorithm [3].

Actually, to avoid such disadvantages, some discontinuous control schemes, such as act-and-wait concept control [11, 12], intermittent control [13, 14] and impulsive technique [9, 15–19] have already been developed and used in the control of dynamical systems. Particularly, in recent years, impulsive technique has been successfully used in many areas such as neural networks [2], control of spacecraft [16], secure communications [17] and so on.

Compared to continuous-time controllers, impulsive controllers have some obvious advantages. First, we only need to impose controllers at a very sparse sequence of time points. Besides, it is typically simpler and easier to implement. Recently, impulsive control techniques have been used in the controlled synchronization and consensus of complex networks. For example, in [24], an impulsive distributed control scheme was proposed to synchronize dynamical complex networks with both system delay and multiple coupling delays. In [23], impulsive control technique has been used in projective synchronization of drive-response networks of coupled chaotic systems. In [25], the authors used impulsive control technique to synchronize stochastic discrete-time networks. In [19], the authors proposed an impulsive hybrid control scheme for the consensus of a network with nonidentical nodes. Yet in these works, the controllers are imposed on all the nodes of the networks. To take advantage of both the impulsive and pinning control techniques, impulsive pinning technique has been proposed which combines these two control techniques as a whole. That is, the impulsive controllers are imposed only on a small fraction of the nodes. For example, in [21, 22], impulsive pinning control technique is used to stabilize and synchronize complex networks of dynamical systems. In this paper, we will introduce this technique into the pinning consensus algorithm. We show if the underlying graph has spanning trees, then a single impulsive controller imposed on one root is able to drive the network to reach consensus on a given value when the impulsive strength is in a permissible range and the impulse is sparse enough.

The rest of the paper is organized as follows. In Section II, some mathematical preliminaries are presented; In Section III, the sufficient conditions for pinning consensus via one impulsive controller on strongly connected graphs are proposed and proved; The results are extended to graphs with spanning trees in Section IV. Examples with numerical simulations are provided in Section V to illustrate the theoretical results; And the paper is concluded in Section VI.

II. Mathematical Preliminaries

In this section, we present some notations, definitions and lemmas concerning matrix and graph theory that will be used later.
First, we introduce following definitions and notations from [4].

**Definition 1:** Suppose $A = [a_{ij}]_{j=1}^{n} \in \mathbb{R}^{n \times n}$. If
1) $a_{ij} \geq 0, \ i \neq j, \ a_{ii} = -\sum_{j=1, j \neq i}^{n} a_{ij}, \ i = 1, \ldots, n$;
2) real parts of eigenvalues of $A$ are all negative except an eigenvalue 0 with multiplicity 1,
then we say $A \in A_1$.

**Definition 2:** Suppose $A = [a_{ij}]_{j=1}^{n} \in \mathbb{R}^{n \times n}$. If
1) $a_{ij} \geq 0, \ i \neq j, \ a_{ii} = -\sum_{j=1, j \neq i}^{n} a_{ij}, \ i = 1, \ldots, n$;
2) $A$ is irreducible.

Then we say $A \in A_2$.

It is clear that $A_2 \subseteq A_1$.

By Gersgorin theorem and Perron Frobenius theorem, we have the following result.

**Lemma 1:** If $A \in A_1$, then the following items are valid:
1) If $\lambda$ is an eigenvalue of $A$ and $\lambda \neq 0$, then $Re(\lambda) < 0$;
2) $A$ has an eigenvalue 0 with multiplicity 1 and the right eigenvector $[1, 1, \ldots, 1]^T$;
3) Suppose $\xi = [\xi_1, \xi_2, \ldots, \xi_n]^T \in \mathbb{R}^n$ (without loss of generality, assume $\sum_{i=1}^{n} \xi_i = 1$) is the left eigenvector of $A$ corresponding to eigenvalue 0. Then, $\xi_i \geq 0$ holds for all $i = 1, \ldots, n$; more precisely,
4) $A \in A_2$ if and only if $\xi_i > 0$ holds for all $i = 1, \ldots, n$;
5) $A$ is reducible if and only if for some $i$, $\xi_i = 0$. In such case, by suitable rearrangement, assume that $\xi^T = [\xi_1, \xi_2, \ldots, \xi_p]^T \in \mathbb{R}^p$, with all $\xi_i > 0$, $i = 1, \ldots, p$ and $\xi^T = [\xi_{p+1}, \xi_{p+2}, \ldots, \xi_n]^T \in \mathbb{R}^{n-p}$ with all $\xi_j = 0$, $p+1 \leq j \leq n$. Then, $A$ can be rewritten as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where $A_{11} \in \mathbb{R}^{p \times p}$ is irreducible and $A_{22} = 0$.

**Remark 1:** By Lemma 1, for $A \in A_2$, let $\Xi = \text{diag}[\xi_1, \xi_2, \ldots, \xi_n]$ be the diagonal matrix generated by the left eigenvector of $A$ corresponding to the eigenvalue 0. Then $A = A^T \Xi \Xi A$ is symmetric. Therefore, its eigenvalues are real and satisfy 0 = $\lambda_1 > \lambda_2 > \lambda_3 \geq \cdots \geq \lambda_n$.

A **weighted directed graph** of order $n$ is denoted by a triple $\{V, E, A\}$, where $V = \{v_1, \ldots, v_n\}$ is the vertex set, $E \subseteq V \times V$ is the edge set, $e_{ij} = (v_i, v_j) \in E$ if and only if there is an edge from vertex $v_i$ to $v_j$, and $A = [a_{ij}], i, j = 1, \ldots, n$, is the weight matrix which is a nonnegative matrix such that for $i, j \in \{1, \ldots, n\}, a_{ij} > 0$ if and only if $i \neq j$ and $e_{ij} \in E$. For a weighted directed graph $\mathcal{G}$ of order $n$, the graph Laplacian $L(\mathcal{G}) = [l_{ij}]_{j=1}^{n}$ can be defined from the weight matrix $A$ in the following way:

$$l_{ij} = \begin{cases} -a_{ij} & i \neq j \\ \sum_{k=1, k \neq i}^{n} a_{ik} & j = i. \end{cases}$$

A **(directed) path** of length $l$ from vertex $v_i$ to $v_j$ is a sequence of $l + 1$ distinct vertices $v_{r_1}, \ldots, v_{r_{l+1}}$ with $v_{r_1} = v_i$ and $v_{r_{l+1}} = v_j$ such that $(v_{r_k}, v_{r_{k+1}}) \in E(\mathcal{G})$ for $k = 1, \ldots, l$.

A graph $\mathcal{G}$ is strongly connected if for any two vertices $v$ and $w$ of $\mathcal{G}$, there is a directed path from $v$ to $w$. A graph $\mathcal{G}$ contains a spanning (directed) tree if there exists a vertex $v_i$ such that for all other vertices $v_j$ there’s a directed path from $v_i$ to $v_j$, and $v_i$ is called the root.

**Remark 2:** From graph theory, a graph is strongly connected if and only if its graph Laplacian $L$ satisfies $-L \in A_2$.

### III. Pinning Consensus on Strongly Connected Graphs

Consider the following consensus algorithm with a single impulsive controller:

$$\begin{cases} \dot{x}(t) = -Lx(t), & t \neq t_k, \\
\Delta x_k(x_k) = b_k[s - x_r(t_k)], & k = 0, 1, 2, \ldots \quad (9) \\
\Delta t_k(t_k) = 0, & i \neq r. \end{cases}$$

where $L = [l_{ij}]$ is the graph Laplacian of the underlying graph, $b_k$ is the strength of the impulse at time $t_k$, and $0 = t_0 < t_1 < t_2 < \cdots$.

Without loss of generality, in the following, we always assume $s = 0$ (by letting $y(t) = x(t) - s$ and consider the new system of $y$) and $r = 1$ (by suitable rearrangement when necessary). In this case, we need to do is to prove

$$\lim_{t \to \infty} x_i(t) = 0, \quad i = 1, \ldots, n \quad (10)$$

for the following system

$$\begin{cases} \dot{x}_i(t) = -\sum_{j=1}^{n} x_j(t), & i = 1, \ldots, n, \ t \neq t_k, \\
x_i(t_k^+) = (1 - b_k)x_i(t_k^-), \\
x_i(t_k^-) = x_i(t_k^-), & i = 2, 3, \ldots, n. \end{cases} \quad (11)$$

Given $x(t) = [x_1(t), \ldots, x_n(t)]^T$, denote

$$\bar{x}(t) = \sum_{i=1}^{n} \xi_i x_i(t), \quad (12)$$

where $[\xi_1, \ldots, \xi_n]^T$ is the left eigenvector of $L$ corresponding to the eigenvalue 0 satisfying $\sum_{i=1}^{n} \xi_i = 1$, and $\Delta t_k = t_{k+1} - t_k, k = 0, 1, 2, 3, \ldots$.

We also define the following Lyapunov function

$$V(x(t)) = \sum_{i=1}^{n} [x_i(t) - \bar{x}(t)]^2. \quad (13)$$

**Remark 3:** Quantity $\bar{x}(t)$ and function $V(x(t))$ were introduced in [4] to discuss synchronization. $\bar{x}(t) = [\bar{x}_1(t), \ldots, \bar{x}_n(t)]^T$ is the non-orthogonal projection of $[x_1(t), \ldots, x_n(t)]^T$ on the synchronization manifold $S = \{[x_1, \ldots, x_n]^T \in \mathbb{R}^m : x_i = x_j, \ i, j = 1, \ldots, n\}$, where $x_i = [x_{i1}, \ldots, x_{in}]^T \in \mathbb{R}^m, i = 1, \ldots, n$, and $x_{i1}$ represents the transpose of $x_i$. $V(t)$ is some distance from $[x_1(t), \ldots, x_n(t)]^T$ to the synchronization manifold $S$. And synchronization is equivalent to the distance goes to zero when time $t$ goes to infinity, i.e.,

$$\lim_{t \to \infty} V(t) = 0. \quad (14)$$

With the two functions $\bar{x}(t)$ and $V(t)$, we will prove the system with one impulsive controller (11) can reach consensus on 0 by proving

$$\lim_{t \to \infty} V(t) = 0 \quad (15)$$
simultaneously.

The following theorem is the main result of this paper.

**Theorem 1:** Suppose $-L \in A_2$, or equivalently, the underlying graph is strongly connected, and there exist $0 < \eta_1 \leq \eta_2 < 1/\xi_1$ such that $b_k \in [\eta_1, \eta_2]$ for each $k$. If $\bar{x}(0) \neq 0$, then there is a constant $T > 0$ such that (11) will reach consensus on $\bar{x}$, when $\Delta t_k \geq T$ for each $k$.

**Remark 4:** It is interesting to note that the permissible range of the impulsive strength is dependent on $\xi_1$ and decreasing with $\xi_1$. Indeed, in a strongly connected graph, $\xi_1 < 1$, we can always choose $\eta_2 > 1$. Actually, in a network of $n$ nodes, $\min_i \xi_i \leq 1/n$. So, by properly choosing the pinning node, we can always let $\eta_2 > n$ except for the cases $\xi_i = 1/n$ for each $i$, in which $\eta_2 < n$ but can be arbitrarily close to $n$.

The proof of Theorem 1 is divided into several steps. First, we prove

**Lemma 2:** If $-L \in A_2$, then

$$V(t_{k+1}^-) \leq V(t_k^-) e^{\max_i \{\xi_i\} \Delta t_k},$$

(17)

where $\lambda_2 > 0$ is the smallest positive eigenvalue of the symmetric matrix $\Xi L + L^T \Xi$.

**Proof:** Denote $\delta x(t) = [x_1(t) - \bar{x}(t), \cdots, x_n(t) - \bar{x}(t)]^T$.

Then

$$\dot{V}(t) = -2 \sum_{i=1}^n \xi_i [x_i(t) - \bar{x}(t)] [\sum_{j=1}^n l_{ij} x_j(t)]$$

$$= -2 \sum_{i=1}^n \sum_{j=1}^n \xi_i l_{ij} [x_i(t) - \bar{x}(t)] [x_j(t) - \bar{x}(t)]$$

$$= -\delta x(t)^T [\Xi L + L^T \Xi] \delta x(t)$$

$$\leq -\lambda_2 ||\delta x(t)||^2$$

$$\leq -\lambda_2 \max_i \{\xi_i\} V(t).$$

This implies (17).

**Remark 5:** By routine approach, it is desired to prove $V(t_{k+1}^-) \leq CV(t_k^-)$ for some constant $C$. Unfortunately, it is difficult to prove it directly. Instead, we prove following Lemma.

**Lemma 3:** Let $\epsilon$, $\eta_1$, $\eta_2$ be constants satisfying $0 < \eta_1 \leq \eta_2 < 1/\xi_1$, $0 < \epsilon < \min \{\eta_1, 1/\eta_2 - \xi_1\}$, the impulsive strength $b_k \in [\eta_1, \eta_2]$ for each $k$, $x(t)$ is a solution of the system (11). If

$$\Delta t_k \geq \frac{\max_i \{\xi_i\}}{\lambda_2} \ln \left( \frac{\xi_1 \sqrt{V(t_k^-)}}{\epsilon^2 x^2(t_k^-)} \right), \quad k = 0, 1, 2, \cdots ,$$

(18)

then, we have

$$|\bar{x}(t_{k+1})| \leq [1 - \eta_1 (\xi_1 - \epsilon)] |\bar{x}(t_k^-)|$$

and

$$\frac{V(t_{k+1}^-)}{x^2(t_{k+1}^-)} \leq \frac{[2 + 2\eta_1^2(1 - \xi_1)] \epsilon^2/\xi_1 + 4\eta_1^2 \xi_1(1 - \xi_1)}{[1 - \eta_2 (\xi_1 + \epsilon)]^2}$$

(19)

for $k = 0, 1, 2, \cdots$.

**Proof:** First, by (17), we have

$$V(t_{k+1}^-) \leq V(t_k^-) e^{\max_i \{\xi_i\} \Delta t_k},$$

(21)

which implies

$$|\bar{x}(t_{k+1})| \leq \sqrt{V(t_{k+1}^-)/\xi_1} \leq \frac{\epsilon}{\xi_1} |\bar{x}(t_{k+1})|. \tag{22}$$

By (11), we have

$$\bar{x}(t_{k+1}^-) = \bar{x}(t_{k+1}^-) - b_{k+1} \xi_1 x_1(t_{k+1}^-)$$

$$= (1 - b_{k+1} \xi_1) \bar{x}(t_{k+1}^-) + b_{k+1} \xi_1 (\bar{x}(t_{k+1}^-) - x_1(t_{k+1}^-)). \tag{23}$$

Thus, for $k = 0, 1, 2, \cdots$,

$$\begin{cases}
|\bar{x}(t_{k+1}^-)| \geq |1 - b_{k+1} (\xi_1 + \epsilon)| |\bar{x}(t_{k+1}^-)| \\
|\bar{x}(t_{k+1}^-)| \leq |1 - b_{k+1} (\xi_1 - \epsilon)| |\bar{x}(t_{k+1}^-)|.
\end{cases} \tag{24}$$

which implies

$$\begin{cases}
|\bar{x}(t_{k+1}^-)| \geq |1 - \eta_2 (\xi_1 + \epsilon)| |\bar{x}(t_{k+1}^-)| \\
|\bar{x}(t_{k+1}^-)| \leq |1 - \eta_1 (\xi_1 - \epsilon)| |\bar{x}(t_{k+1}^-)|.
\end{cases} \tag{25}$$

Noting $\bar{x}(t_{k+1}^-) = \bar{x}(t_{k+1})$, we have

$$|\bar{x}(t_{k+1}^-)| \leq |1 - \eta_1 (\xi_1 - \epsilon)| |\bar{x}(t_{k+1}^-)|,$$

(26)

which is just the inequality (19).

On the other hand, noting the fact that $\bar{x}(t_{k+1}^-) = \bar{x}(t_{k+1}^-)$ and (19), we have

$$\frac{V(t_{k+1}^-)}{x^2(t_{k+1}^-)} \leq \frac{V(t_{k+1}^-) e^{\max_i \{\xi_i\} \Delta t_k}}{x^2(t_k^-)} = \frac{\epsilon^2}{\xi_1}. \tag{27}$$

Furthermore, by the assumption $\sum_{i=1}^n \xi_i = 1$ and inequality $(a + b)^2 \leq 2(a^2 + b^2)$, we have

$$V(t_{k+1}^-) = \xi_1 [x_1(t_{k+1}^-) - \bar{x}(t_{k+1}^-)]^2$$

$$+ \sum_{i=2}^n \xi_i [x_i(t_{k+1}^-) - \bar{x}(t_{k+1}^-)]^2$$

$$= \xi_1 [x_1(t_{k+1}^-) - \bar{x}(t_{k+1}^-)] - b_{k+1} (1 - \xi_1) x_1(t_{k+1}^-)]^2$$

$$+ \sum_{i=2}^n \xi_i [x_i(t_{k+1}^-) - \bar{x}(t_{k+1}^-)] + b_{k+1} \xi_1 x_1(t_{k+1}^-)]^2$$

$$\leq 2 \left\{ \sum_{i=1}^n \xi_i [x_1(t_{k+1}^-) - \bar{x}(t_{k+1}^-)] \right\}$$

$$+ b_{k+1}^2 \xi_1 (1 - \xi_1) x_1^2(t_{k+1}) + b_{k+1}^2 \xi_1^2 \sum_{i=2}^n \xi_i^2 x_i^2(t_{k+1})$$

$$= 2V(t_{k+1}^-) + 2b_{k+1}^2 \xi_1 (1 - \xi_1) x_1^2(t_{k+1})$$

$$\leq 2V(t_{k+1}^-) + 2b_{k+1}^2 \xi_1 (1 - \xi_1) x_1^2(t_{k+1}). \tag{28}$$
By (25) and (26), we have

\[
\frac{V(t_{k+1})}{x^2(t_{k+1})} \leq 2V(t_{k+1}) + 2\eta_2^2\xi_1(1-\xi_1)x^2(t_{k+1}) \leq 2V(t_{k+1}) \leq \frac{2\eta_2^2\xi_1(1-\xi_1)}{[1-\eta_2(\xi_1 + \epsilon)]^2x^2(t_{k+1})} + \frac{4\eta_2^2\xi_1(1-\xi_1)}{[1-\eta_2(\xi_1 + \epsilon)]^2x^2(t_{k+1})} + \frac{4\eta_2^2\xi_1(1-\xi_1)}{[1-\eta_2(\xi_1 + \epsilon)]^2x^2(t_{k+1})} + \frac{4\eta_2^2\xi_1(1-\xi_1)}{[1-\eta_2(\xi_1 + \epsilon)]^2x^2(t_{k+1})}
\]

Thus, we have

\[
\lim_{t \to \infty} x_i(t) = \lim_{t \to \infty} x_i(t) - \bar{x}(t) + \lim_{t \to \infty} \bar{x}(t) = 0.
\]

The proof is completed.

Remark 6: In [21], Zhou et.al discussed pinning complex delayed dynamical networks by a single impulsive controller. In that paper, the authors proposed a novel model. However, the coupling matrix \( A \) is assumed to be a symmetric irreducible matrix and orthogonal eigen-decomposition is used and plays a key role. Therefore, the approach can not apply to our case.

Remark 7: In [22], Lu et.al, discussed synchronization control for nonlinear stochastic dynamical networks by impulsive pinning strategy. In that strategy, at each impulse time point \( t_k \), the authors select several nodes with largest errors, and adding controllers to those nodes. Therefore, one needs to observe all states \( x_i(t_k) \) at each \( t_k \). In our strategy, we only need to know the state \( x_1(t_k) \) and one controller is enough.

IV. PINNING CONSENSUS ON GRAPHS WITH SPANNING TREES

In this section, we will generalize the results obtained in previous section to graphs with spanning trees. In such case, by suitable arrangement, we can assume that \( L \) has the following block form:

\[
L = \begin{bmatrix}
L_{11} & 0 & \cdots & 0 \\
L_{21} & L_{22} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
L_{m1} & \cdots & \cdots & L_{mm}
\end{bmatrix}
\]

where \(-L_{ii} \in \mathbb{R}^{p_i \times p_i}\) is irreducible, and \([L_{i1}, \ldots, L_{i(i-1)}] \neq 0\) for \( i = 2, \ldots, m \). Let \([\xi_1, \ldots, \xi_m]^T\) be the normalized left eigenvector of \( L \) corresponding to the eigenvalue 0. From Lemma 1, \( \xi_i > 0 \) for \( i = 1, \ldots, n \), and \( \xi_i = 0 \) for \( i = p_1 + 1, \ldots, n \). Thus \( \bar{x}(t) = \sum_{i=1}^{p_1} \xi_i x_i(t) \).

We will prove

Theorem 2: Suppose the underlying graph is of the form (32), and there exist \( 0 < \eta_1 \leq \eta_2 < 1/\xi_1 \) such that \( \eta_1 \leq b_k \leq \eta_2 \) for each \( k \). If \( \bar{x}(0) \neq 0 \), then the consensus algorithm (11) can reach consensus on a given value \( s \) when \( \Delta t_k \geq T \) for a large enough \( T \).

Proof: Let \( x(t) = [X_1^T(t), \ldots, X_m^T(t)]^T \) with \( X_i(t) = \sum_{j=m_i+1}^{m_{i+1}} x_j(t) \), where \( m_1 = 0 \) and \( m_{i+1} = m_i + p_i \). Since \( \bar{x}(0) = \sum_{i=1}^{p_1} \xi_i x_i(0) \neq 0 \), by applying Theorem 1 to the subsystem of \( X_i(t) \), we can find \( T > 0 \) such that if \( \Delta t_k \geq T \) for each \( k \), then

\[
\lim_{t \to \infty} x_i(t) = 0, \quad i = 1, \ldots, p_1.
\]

Consider the subsystem of \( X_{p_1+1}(t) \), we have:

\[
\dot{X}_{p_1+1}(t) = -L_{21}X_{p_1+1}(t) - L_{22}X_{p_2}(t) - L_{23}X_{p_3}(t) - \cdots - L_{2m}X_{p_m}(t) - \cdots - L_{2(m-1)}X_{p_{m-1}}(t) - L_{2m}X_{p_m}(t) - L_{2m+1}X_{p_{m+1}}(t) - \cdots - L_{2m+n}X_{p_{m+n}}(t) \]

(33)
Denote $Y_2(t) = -L_2X_1(t)$. Then (33) can be rewritten as:
\[ \dot{X}_2(t) = -L_{22}X_2(t) + Y_2(t) \] (34)
Thus,
\[ X_2(t) = e^{-L_{22}t}X_2(0) + \int_0^t e^{-L_{22}(t-s)}Y_2(s)ds. \] (35)
Since the $L_{22} \neq 0$, at least one row sum of $L_{22}$ is negative, which implies that $L_{22}$ is a non-singular M-matrix and its eigenvalues $\mu_1, \cdots, \mu_{p_2}$ can be arranged as $0 < Re(\mu_1) \leq \cdots \leq Re(\mu_{p_2})$. Then,
\[ \|e^{-L_{22}t}\| \leq K e^{-Re(\mu_1)t} \]
for some constant $K > 0$. And
\[
\|X_2(t)\| \leq K\|X_2(0)\|e^{-Re(\mu_1)t} + K \int_0^t e^{-Re(\mu_1)(t-s)}\|Y_2(s)\|ds
\]
It is obvious that
\[
\lim_{t \to \infty} K\|X_2(0)\|e^{-Re(\mu_1)t} = 0
\]
To show
\[
\lim_{t \to \infty} \|X_2(t)\| = 0,
\]
we only need to estimate the second term on the right-hand side of (35).
Since $\lim_{t \to \infty} \|Y_2(t)\| = 0$, for any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $\|Y_2(t)\| \leq \epsilon$ for each $t \geq t_\epsilon$. Furthermore, $Y_2(t)$ is uniformly bounded. Let $Y_2 > 0$ be an upper bound of $Y_2(t)$. Then for $t > t_\epsilon + \frac{1}{Re(\mu_1)} \ln \frac{Y_2}{\epsilon}$,
\[
\int_0^t e^{-Re(\mu_1)(t-s)}\|Y_2(s)\|ds = \int_0^t e^{-Re(\mu_1)(t-s)}\|Y_2(s)\|ds + \int_{t_\epsilon}^t e^{-Re(\mu_1)(t-s)}\|Y_2(s)\|ds 
\]
\[
\leq Y_2 \int_0^t e^{-Re(\mu_1)(t-s)}ds + \int_{t_\epsilon}^t e^{-Re(\mu_1)(t-s)}ds 
\]
\[
= Y_2 \frac{1}{Re(\mu_1)}(1 - e^{-Re(\mu_1)t}) 
\]
\[
+ \frac{\epsilon}{Re(\mu_1)}(1 - e^{-Re(\mu_1)t_\epsilon}) 
\]
\[
\leq \frac{\epsilon}{Re(\mu_1)}(1 - e^{-Re(\mu_1)t_\epsilon}) + \frac{\epsilon}{Re(\mu_1)} 
\]
Because $\epsilon$ is arbitrary, we have
\[
\lim_{t \to \infty} \int_0^t e^{-Re(\mu_1)(t-s)}\|Y_2(s)\|ds = 0
\]
Thus,
\[
\lim_{t \to \infty} \|X_2(t)\| = 0.
\]
For $i = 3, \cdots, n$, we have
\[ \dot{X}_i(t) = -L_iX_i(t) - Y_i(t), \]
where $Y_i(t) = \sum_{j=1}^{i-1} L_{ij}X_j(t)$.
By induction, if we already have
\[
\lim_{t \to \infty} \|X_j(t)\| = 0
\]
for $j = 1, \cdots, i - 1$, then we have
\[
\lim_{t \to \infty} \|X_i(t)\| = 0. \] (36)
By a similar analysis as above, we can show that
\[
\lim_{t \to \infty} \|X_i(t)\| = 0.
\]
Similarly, we can have a corollary from Theorem 4 when the impulse strength is constant.
**Corollary 3:** Suppose the underlying graph is of the form (32), and $b_k = b \in (0, 1/\xi_1)$ for each $k$. If $\dot{x}(0) \neq 0$, then the consensus algorithm (11) can reach consensus on a given value $s$ when $\Delta t_k > T$ for a large enough $T$.

V. NUMERICAL SIMULATIONS

In this section we will provide two simple examples to illustrate the theoretical results. The first example considers a strongly connected graphs. And the second one concerns a graph that is not strongly connected but has a spanning tree.

**A. Example 1**

In the first example, we consider a directed circular network. (Fig. 1 shows an example of a circular network with 10 nodes.)

It is obvious that this network is strongly connected. If we assign each edge with weight 1, then the graph Laplacian is
\[ L = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\
-1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & -1 & 1 \\
\end{bmatrix}
\]

Then we have $\xi_i = 0.01$ for each $i$, and $\lambda_2 = 3.9465 \times 10^{-5}$.

Randomly choose an initial value $x(0)$ whose $\bar{x}(0) = 0.4886$. The objective is to drive the network to reach a consensus on value 0. After calculation, we have
\[
V(0) = 0.01 \sum_{i=1}^{100} [x_i(0) - \bar{x}(0)]^2 = 0.5935,
\]
\[
V(0)/\bar{x}^2(0) = 2.4856.
\]
Let $b_k = 11$ for each $k$, then we can set $\eta_1 = \eta_2 = 11$. Choose $\epsilon = 0.00999$. Then,
\[
C = \frac{2 + 4n_2^2(1 - \xi_1)}{1 - \eta_2(\xi_1 + \epsilon)} = 15.7641.
\]
Then we get the lower bound for the duration between each successive impulse is
\[
T = \frac{\max_i \{\xi_i\}}{\lambda_2} \ln \frac{C\xi_1}{\epsilon^2} = 1.8662 \times 10^3.
\]

In the simulation, we set \(\Delta t_k = 1867\) for each \(k\). The simulation result is presented in Figs.4, Fig.2 shows the trajectories of the network, and Fig.3 shows the variations of the trajectories with respect to time \(t\) which is defined as
\[
\text{var}(t) = \sum_{i=1}^{n} |x_i(t)|.
\]

It can be seen that the network will asymptotically reach a consensus on value 0.

B. Example 2

In this example, we consider a network that is not strongly connected but has a spanning tree. We start from a circular network with 10 nodes (shown in Fig.1) and construct a larger network by randomly adding new nodes to the network. At each step, randomly choose a node \(i\) from the existing network, then a new node \(j\) is added to the network such that there is a directed edge from \(i\) to \(j\). Continuing this procedure until the network has 100 nodes, we obtain a graph that has spanning trees but is not strongly connected. If we assign each edge with weight 1, then in the graph Laplacian \((32)\),
\[
L_{11} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}
\]

Thus \(\xi_1 = 0.1\) for \(1 \leq i \leq 10\), \(\xi_i = 0\) for \(11 \leq i \leq 100\), and \(\lambda_2 = 0.3820\). Randomly choose the initial value \(x(0)\) where \(x(0) = 0.3909\). The objective is to drive the network to reach consensus on the value 0. After calculation, we have:
\[
V(0) = \sum_{i=1}^{10} \xi_i |x_i(0) - \bar{x}(0)|^2 = 0.6369.
\]
\[
V(0)/\bar{x}^2(0) = 4.1677.
\]

Let \(b_k = 5\) for each \(k\), then we can set \(\eta_1 = \eta_2 = 5\). Choose \(\epsilon = 0.09\). Then we have
\[
C = \frac{[2 + 4\eta_2^2(1 - \xi_1)]\epsilon^2/\xi_1 + 4\eta_2^2\xi_1(1 - \xi_1)}{[1 - \eta_2(\xi_1 + \epsilon)]^2} = 7.3280.
\]

Thus the lower bound for the intervals between each successive impulse is
\[
T = \frac{\max_i \{\xi_i\}}{\lambda_2} \ln \frac{C\xi_1}{\epsilon^2} = 14.8720.
\]

In the simulation, we choose \(\Delta t_k = 15\). The simulation result is presented in Figs.4, 5. It can be seen that the network will asymptotically reach a consensus on 0.
Fig. 4. Pinning consensus to 0 on the graph that has spanning trees.

VI. CONCLUSIONS

In this paper, we investigate pinning consensus in networks of multiagents via a single impulsive controller. First, we prove a sufficient condition for a network with a strongly connected underlying graph to reach consensus on a given value. Then we extend the result to networks with a spanning tree. Interestingly, we find the permissible range of the impulsive strength is determined by the left eigenvector of the graph Laplacian corresponding to the zero eigenvalue and the pinning node we choose. Besides, a sparse enough impulsive pinning on one node can always drive the network to reach consensus on a prescribed value. Examples with numerical simulations are also provided to illustrate the theoretical results. The pinning synchronization in complex networks via a single impulsive controller is an interesting issue, which will be worked out soon.

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