Optimal control of algae growth by controlling $CO_2$ and nutrition flow using Pontryagin Maximum Principle

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Abstract. There are so many benefit of algae. One of them is using for renewable energy and sustainable in the future. The greater growth of algae will increasing biodiesel production and the increase of algae growth is influenced by glucose, nutrients and photosynthesis process. In this paper, the optimal control problem of the growth of algae is discussed. The objective function is to maximize the concentration of dry algae while the control is the flow of carbon dioxide and the nutrition. The solution is obtained by applying the Pontryagin Maximum Principle. and the result show that the concentration of algae increased more than 15%.

1. Introduction
Indonesia has abundant natural resources both in the oceans and inland. One of the natural resources in the ocean is algae. These natural resources have a lot of benefits, among others, as animal feed, etc.

With the increasing population growth, the need for energy is also increasing as well. Whereas the current reserves of energy with fossil fuels are running low due to less fossil fuel raw materials. Therefore, an alternative energy needed to replace fossil fuels is needed. In this research, it will be discussed about alga because require cheap cost used for the production of fuel / bio fuel, in addition, algae are also nature friendly because in the process of making not produce a lot of waste.

Many research has been made on optimal control and the growth or extraction of algae present in Indonesia. One of the research about optimal control was made by Mardlijah [1], about optimal control of tumor growth. the research about optimal control of algae was made by Nasria [2] and Hajar [3], and Nailul [4]. Nasria [2] research is about the optimal control of algae growth by controlling the nutrition flow only, Hajar [3] research is about the optimal control of algae growth by controlling the carbon dioxide flow only and Nailul [4] research is about optimal feeding strategy on microalgae growth.

So, in this research we combined both controlling carbon dioxide and nutrition flow to optimize the algae growth and we use the Pontragyn Maximum Principles method to solve this optimal control problem.
The paper is organized as follows: Section 2 describes the formulation of mathematical model. Section 3 describes the objective function of mathematical model. Section 4 describes derivation of optimal control. Section 5 deals with numerical simulation results. Conclusions are outlined in Section 6.

2. The mathematical model

For the mathematical model, notation, and terminology follows that of Thornton [5]. Variables and parameters of the model are described in Table 1. The following system of equations that captures the optimal control of algae growth is derived.

\[
\dot{A} = \alpha_A \left( \frac{\rho_{\text{max}} M}{M + M_{\text{turn}}} \right) S - (D_r + h_r)A \\
\dot{M} = -k_2 \alpha_A \left( \frac{\rho_{\text{max}} M}{M + M_{\text{turn}}} \right) S + I_m(t) \\
\dot{S} = \alpha_s C - k_3 \alpha_A \left( \frac{\rho_{\text{max}} M}{M + M_{\text{turn}}} \right) S - (D_r + h_r)S \\
\dot{C} = -k_1 \alpha_s C + I_c(t)
\]

with initial conditions \(A(0) = A_0, M(0) = M_0, S(0) = S_0\) and \(C(0) = C_0\).

| Parameter | Description |
|-----------|-------------|
| A         | Concentration of dry algae. |
| M         | Concentration of nutrients. |
| S         | Concentration of glucose |
| C         | Concentration of carbon dioxide |
| \(\rho_{\text{max}}\) | maximal nutrient concentration inside algae |
| \(M_{\text{turn}}\) | half-saturation constant for nutrient concentration inside algae |
| \(I_c(t)\) | inflow of carbon dioxide |
| \(I_m(t)\) | inflow of nutrients |
| \(D_r\) | algae death rate |
| \(h_r\) | algae harvest rate |
| \(\alpha_A\) | rate constant for biomass growth |
| \(\alpha_s\) | rate constant for photosynthesis |
| \(k_1\) | conversion rate of \(CO_2\) into \((CH_2O)_6\) |
| \(k_2\) | conversion rate of nutrients into dry algae |
| \(k_3\) | conversion rate of \((CH_2O)_6\) into dry algae |

3. Optimal control

The goal to be achieved in the optimal control problem in this research is to get the optimal the flow of carbon dioxide and the flow of nutrition by maximizing the concentration of dry algae. Matematically, this problem is to maximize the objective function below,

\[
J(u_1, u_2) = \int_{t_0}^{t_f} \left( A(t) - \frac{D}{2} I_c^2(t) - \frac{N}{2} I_m^2(t) \right) dt
\]

where \(I_c(t)\) dan \(I_m(t)\) is control variable. while \(D\) is weight factor in the flow of the carbon dioxide dan \(N\) is weight factor in the flow of the nutrition, \(t_0\) is start time, dan \(t_f\) is final time.
4. Pontryagin Maximum Principle

Based on Pontryagin Maximum Principle, solution of optimal control problem has two steps to solve specifically make the Hamiltonian function, maximize the hamiltonian function with solving the State, costate and boundary conditions equations.

From the equation 5 and the equations 1 to 4, the Hamiltonian function was given by:

\[
H (A, M, S, C, I_c, I_m, \lambda) = A(t) - \frac{D}{2} I_c^2 (t) - \frac{N}{2} I_m^2 (t) + \sum_{i=1}^{4} \lambda_i f_i
\]

\[
= A(t) - \frac{D}{2} I_c^2 (t) - \frac{N}{2} I_m^2 (t) + \\
\lambda_1 \left[ \alpha_A \left( \frac{\rho_{\text{max}}M}{M + M_{\text{turn}}} \right) S - (D_r + h_r)A \right] \\
\lambda_2 \left[ -k_2 \alpha_A \left( \frac{\rho_{\text{max}}M}{M + M_{\text{turn}}} \right) S + I_m (t) \right] \\
\lambda_3 \left[ \alpha_s C - k_3 \alpha_A \left( \frac{\rho_{\text{max}}M}{M + M_{\text{turn}}} \right) S - (D_r + h_r)S \right] \\
\lambda_4 \left[ -k_1 \alpha_s C + I_c (t) \right]
\]

(7)

So, the state Function are given by:

\[
\frac{\partial H}{\partial \lambda_1} = \alpha_A \left( \frac{\rho_{\text{max}}M}{M + M_{\text{turn}}} \right) S - (D_r + h_r)A
\]

\[
\frac{\partial H}{\partial \lambda_2} = -k_2 \alpha_A \left( \frac{\rho_{\text{max}}M}{M + M_{\text{turn}}} \right) S + I_m (t)
\]

\[
\frac{\partial H}{\partial \lambda_3} = \alpha_s C - k_3 \alpha_A \left( \frac{\rho_{\text{max}}M}{M + M_{\text{turn}}} \right) S - (D_r + h_r)S
\]

\[
\frac{\partial H}{\partial \lambda_4} = -k_1 \alpha_s C + I_c (t)
\]

and the costate Function are given by:

\[
\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial A} \\
= -1 + \lambda_1 (t) D_r + \lambda_1 (t) h_r
\]

\[
\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial M} \\
= \left[ \frac{\alpha_A \rho_{\text{max}}S (t)}{M (t) + M_{\text{turn}}} (-\lambda_1 (t) + \lambda_2 (t) k_2 + \lambda_3 (t) k_3) \\
+ \frac{\alpha_A \rho_{\text{max}}M (t) S (t)}{(M (t) + M_{\text{turn}})^2} (\lambda_1 (t) - \lambda_2 (t) k_2 - \lambda_3 (t) k_3) \right]
\]

\[
\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial S} \\
= \frac{\alpha_A \rho_{\text{max}}M (t)}{M (t) + M_{\text{turn}}} (-\lambda_1 (t) + \lambda_2 (t) k_2 + \lambda_3 (t) k_3) + \lambda_3 (t) (D_r + h_r)
\]

\[
\frac{d\lambda_4}{dt} = -\frac{\partial H}{\partial C} \\
= -\lambda_3 (t) \alpha_s + \lambda_4 (t) k_1 \alpha_s
\]
The control equation are given by:

\[ \frac{\partial H}{\partial I_c(t)} = 0 \]
\[ -DI_c(t) + \lambda_4(t) = 0 \]
\[ I_c(t) = \frac{\lambda_4}{D} \]

Because \( 0 \leq I_c(t) \leq 4 \), so that:

\[ I_c(t) = \begin{cases} 0 & I_c(t) \leq 0 \\ \frac{I_c(t)}{t} & 0 < I_c(t) < 4 \\ 4 & I_c(t) \geq 4 \end{cases} \]

So, the optimal control \( I_c(t) \) can be written as:

\[ I_c(t)^* = \min \left( 4, \max \left( 0, I_c(t) \right) \right) \]

or

\[ I_c(t)^* = \min \left( 4, \max \left( 0, \frac{\lambda_4}{D} \right) \right) \]

For getting the optimal system, then substitute \( I_c(t)^* \) dan \( I_m(t)^* \) to state equations dan costate equations.
equation, So it can be obtained as:

\[
\frac{\partial H}{\partial \lambda_1} = \alpha_A \left( \frac{\rho_{\text{max}} M}{M + M_{\text{turn}}} \right) S - (D_r + h_r)A
\]

\[
\frac{\partial H}{\partial \lambda_2} = -k_2 \alpha_A \left( \frac{\rho_{\text{max}} M}{M + M_{\text{turn}}} \right) S + \min \left( 4.2, \max \left( 0, \frac{\lambda_2}{D} \right) \right)
\]

\[
\frac{\partial H}{\partial \lambda_3} = \alpha_s C - k_2 \alpha_A \left( \frac{\rho_{\text{max}} M}{M + M_{\text{turn}}} \right) S - (D_r + h_r)S
\]

\[
\frac{\partial H}{\partial \lambda_4} = -k_1 \alpha_s C + \min \left( 4, \max \left( 0, \frac{\lambda_4}{D} \right) \right)
\]

\[
\frac{d\lambda_1}{dt} = -1 + \lambda_1 (t) D_r + \lambda_1 (t) h_r
\]

\[
\frac{d\lambda_2}{dt} = \left[ \frac{\alpha_A \rho_{\text{max}} S(t)}{M(t) + M_{\text{turn}}} \left( -\lambda_1 (t) + \lambda_2 (t) k_2 + \lambda_3 (t) k_3 \right) \\
+ \frac{\alpha_A \rho_{\text{max}} M(t) S(t)}{(M(t) + M_{\text{turn}})^2} (\lambda_1 (t) - \lambda_2 (t) k_2 - \lambda_3 (t) k_3) \right]
\]

\[
\frac{d\lambda_3}{dt} = \frac{\alpha_A \rho_{\text{max}} M(t)}{M(t) + M_{\text{turn}}} \left( -\lambda_1 (t) + \lambda_2 (t) k_2 + \lambda_3 (t) k_3 \right) + \lambda_3 (t) (D_r + h_r)
\]

\[
\frac{d\lambda_4}{dt} = -\lambda_3 (t) \alpha_s + \lambda_4 (t) k_1 \alpha_s
\]

5. Numerical simulation

This section discusses the numerical simulations of the optimality system and the corresponding results of varying the optimal controls \( I_c (t) \) and \( I_m (t) \) and some parameter choices and interpretations from various cases using baseline parameter values was given in the table 2 below, the weight factor \( D = 0.3 \) is carbon dioxide flow debit and \( N = 0.3 \) is nutrition flow debit with initial conditions \( A_0 = 3, M_0 = 0.4, S_0 = 10 \) and \( C_0 = 5 \)

| Parameter Values | Values |
|------------------|--------|
| \( \rho_{\text{max}} \) | 0.4 |
| \( M_{\text{turn}} \) | 4 |
| \( D_r \) | 0.46 |
| \( h_r \) | 0.4 |
| \( \alpha_A \) | 10.2 |
| \( \alpha_s \) | 67.6 |
| \( k_1 \) | 0.4 |
| \( k_2 \) | 0.05 |
| \( k_3 \) | 0.05 |
6. Conclusion
(i) From the simulation result is obtained the control variables are $I_m(t)^* = min\left(4.2, max\left(0, \frac{\lambda_2}{\lambda_1}\right)\right)$ and $I_c(t)^* = min\left(4, max\left(0, \frac{\lambda_2}{\lambda_1}\right)\right)$.

(ii) At the final time, $t_f = 10$, before the controller applied to the system, the concentration of dry algae is 34,146 gram. While after controlled applied, the concentration of dry algae is 39.2719 gram. So, the concentration of dry algae increase 15.010704 percent.

Acknowledgments
The research was funded by grant of Ministry of Research, Technology and Higher Education of the Republic of Indonesia (DIKTI) with number 010/SP2H/LT/DPRM/IV/2017.

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