Exploring the meson spectrum with twisted mass lattice QCD

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Numerical simulations with access to all possible meson quantum numbers, $J^{PC}$, are presented using two-flavor (up and down) quenched twisted mass lattice QCD with three different lattice spacings and four different quark masses. The connection between the quantum numbers $(P$ and $C)$ and the symmetries of the twisted mass action is discussed, as is the connection between $J$ and the lattice rotation group, for the 400 operators used in this study. Curve fitting of this large data set is accomplished by using an evolutionary fitting algorithm. Results are reported for conventional and exotic quantum numbers.

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\section{Introduction}

Although many mesons have been observed in nature, the spectrum of known mesons does not include all the states expected in Quantum Chromodynamics. Experimental searches and theoretical studies are continuing, but gaps in current knowledge leave room for new approaches. Lattice QCD is an established method for extracting numerical predictions directly from the underlying quantum field theory, but lattice explorations of the full spectrum of mesons still suffer from limitations. For reviews of both theory and experiment for light-quark mesons, see Refs. \textsuperscript{1,2,3}. A helpful framework for beginning the discussion of light-quark mesons is provided by the constituent quark model, where mesons are considered to be composed of a system of two bound quarks whose spins can couple to a singlet $(S = 0)$ or triplet $(S = 1)$ total spin which in turn can couple with relative angular momentum $L$ between the quarks to produce a total observed angular momentum $J$. Using spectroscopic notation for the states $2S+1L_J$, one obtains the familiar list of accessible $J^{PC}$, as shown in Table \textsuperscript{I}.

Since in QCD gluon fields are also present, it is possible for gluonic excitations in mesons to contribute non-trivially to the observed quantum numbers of a meson. Such hybrid mesons with the same $J^{PC}$ as conventional mesons are difficult to distinguish but, as shown in Table \textsuperscript{I}, there is a subclass of hybrid mesons, called exotic mesons, with quantum numbers unattainable in the quark model, eg. $0^{-}$, $0^{+-}$, $1^{-}$, $1^{--}$, $2^{+-}$, and $3^{++}$. These exotic mesons, for which there is yet no definitive experimental evidence, would offer a clear signature for excited gluon dynamics.

Twisted mass lattice QCD (tmLQCD) \textsuperscript{4,5} is used for the lattice simulations in this work. It is not clear \textit{a priori} whether tmLQCD is a very favorable action for lattice simulations of the full meson spectrum. The fact that tmLQCD does not respect parity $P$ may be seen as an obstacle, but we will show that it is not insurmountable. Meanwhile, tmLQCD has the advantages of offering a cost-effective approach to the chiral limit as well as a simple removal of the leading, i.e. $O(a)$, lattice spacing errors. In this work, it is shown that tmLQCD’s reduced symmetries can be understood and accounted for in practical simulations, thereby allowing the reader to make an informed decision as to whether this option is preferable for future studies of the meson spectrum. The issues specific to tmLQCD that affect our operators are discussed in Section IV.

Much of the discussion of meson operators is not specific to tmLQCD. See Section III for this tmLQCD-independent presentation. In particular, we choose meson correlators that can be built from quark and antiquark propagators originating at a single lattice site augmented by gauge field links defined on extended spatial paths.

All possible quantum numbers are obtained by using a variety of options for the paths of gauge fields connecting quark to anti-quark. (Gauge fields are subsequently smeared at both source and sink, but quarks are only smeared at the sink in this work.) With just one quark propagator inversion, these “excited glue” operators are minimally expensive, and the results of our simulations allow us to tabulate the relative strengths of the overlaps.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{S} & \textbf{2S+1L_J} & \textbf{J^{PC}} \\
\hline
$S=0$ (singlet) & $1^{S_0}$ & $0^+$ \\
$S=1$ (triplet) & $1^{P_1}$ & $1^-$ \\
& $1^{D_2}$ & $0^{++, 1^{++}, 2^{++}}$ \\
& $1^{F_3}$ & $2^{++, 3^{++}, 4^{++}}$ \\
\hline
\end{tabular}
\caption{States accessible in the constituent quark model for different total spin $S$ and orbital angular momentum $L$ labeled by spectroscopic notation $2S+1L_J$ with corresponding $J^{PC}$.}
\end{table}
that various mesons have with various operators. It is interesting to see the extent to which these “excited glue” operators couple to exotic mesons, and also to conventional quantum numbers.

Meson masses are extracted from lattice QCD simulations by fitting a linear combination of exponentials to correlators as a function of Euclidean time. It is a delicate business. For example, the number of exponentials that should be used for a certain fit depends on the particular channel being studied and also on the quality and quantity of data. Our study of the meson spectrum will include hundreds of correlators, some of which should be fit simultaneously since each meson will appear in multiple correlators. Because sink smearing is more easily varied than source smearing due to the expense of re-computing quark propagators, we want a fitting method that does not require computation of a complete correlator matrix, i.e. we want the freedom to consider more sink options than source options.

To address all of the delicate issues of fitting, the evolutionary algorithm introduced in Refs. [6, 7] is used. The workings of the algorithm are well-understood in terms of the basic principles underlying biological evolution, but the algorithm will identify the number of exponentials by fitting a linear combination of exponentials to correlators as a function of Euclidean time. It is a delicate business. For example, the number of exponentials that minimize the \( \chi^2/n_{dof} \) for a given fit. This black box method is also general enough to handle multi-correlator fits with no need of a complete correlator matrix. The data-fitting technique proposed in Refs. [6, 7] is independent of tmLQCD. This is its first application to such a large set of lattice QCD data, and the implementation is described in Section [V].

Section [V] explains the parameter choices used in our numerical simulations. The present study is exploratory, and we have therefore chosen to perform quenched simulations. Lessons learned in this study about extended meson operators, fitting algorithms, tmLQCD and the meson spectrum are applicable to future studies beyond the quenched approximation. Section [V] presents and discusses the results.

Section [VII] draws some conclusions about the meson spectrum. It also highlights properties of our chosen operators, comments on the appropriateness of the twisted mass action to this physics, and underscores the valuable qualities of the evolutionary fitting technique.

### II. LATTICE SYMMETRIZED MESON OPERATORS

Lighter quark mass calculations require improved statistics for hadron mass resolution. This requirement and the need to disentangle physical states with the same quantum numbers may be accomplished in part through the creation of more operators that represent the channel in question. The creation of more elaborate operators also allows for the study of hybrid and exotic mesons.

Operators with displaced quarks have been constructed using group theoretical techniques but their usage requires the calculation of quark propagators from multiple lattice sites. (See, for example, Refs. [8, 10, 11].) Consideration of the operators available through gluonic extension alone therefore is numerically expedient. Moreover, it allows investigation of the coupling of operators with “excited glue” to conventional and exotic states. This section provides a complete discussion of the operators (first introduced in Ref. [12]) used in our simulation.

#### A. Lattice symmetry group

While parity (\( P \)) and charge conjugation (\( C \)) may be conserved by lattice actions, the continuous rotational symmetry of nature is broken and one requires operators adapted to the symmetry group of the lattice. For mesons this is the octahedral group \( O \) with 24 elements. The group \( O \) has five conjugacy classes conventionally labeled \{ \( E, 3C_4, 8C_3, 6C_4, 6C_2 \) \} and therefore admits five (unitary) irreducible representations (irreps): two one-dimensional irreps \( A_1 \) and \( A_2 \), one two-dimensional irrep \( E \), and two three-dimensional irreps \( T_1 \) and \( T_2 \). The direct product of the parity, charge conjugation, and octahedral groups is denoted \( O^{PC} \).

For an operator adapted to the representation \( \Lambda^{PC} \), where \( \Lambda \in \{ A_1, A_2, E, T_1, T_2 \} \) is an irrep of \( O \) and \( P, C \in \{ +, - \} \), one identifies the possible physical states \( J^{PC} \) to which it corresponds using Table [II] which shows the number of copies \( n^J_\Lambda \) of irrep \( \Lambda \) contained in the reduction of the subduced representation of the continuum irrep \( J \) of \( SO(3) \) irrep \( J \). For example, an operator transforming as the irrep \( E^{++} \) could have spin content \( J^{PC} = 2^{++}, 4^{+-}, 5^{+-}, \ldots \). In theory one needs many operators of each \( \Lambda^{PC} \) to resolve the range of physical spins \( J^{PC} \) in the tower of states to which \( \Lambda^{PC} \) corresponds.

Table [II] is arrived at through a consideration of the character table of \( O \) shown in Table [III]. The trace of the rotation group matrix for spin \( J \) and rotation angle \( \theta(\xi) \)

### Table II: Number of copies \( n^J_\Lambda \) of the irrep \( \Lambda \) of \( O \) in the reduction of the subduced representation of the continuum irrep \( J \) of \( SO(3) \).

| \( J \) | \( A_1 \) | \( A_2 \) | \( E \) | \( T_1 \) | \( T_2 \) |
|-------|-------|-------|-------|-------|-------|
| 0     | 1     | 1     | 0     | 0     | 0     |
| 1     | 1     | 0     | 0     | 1     | 1     |
| 2     | 0     | 0     | 0     | 1     | 1     |
corresponding to class $\xi$ is given by

$$\chi^{(J)}(\xi) = \sin((J + 1/2)\theta(\xi))/\sin(\theta(\xi)/2).$$  \hspace{1cm} (1)

Those matrices in the continuous irrep $J$ which correspond to the subgroup $O$ of $SO(3)$ form a representation of $O$ which is, in general, now reducible. Equation (1) combined with Table III allows one to determine the subduction of continuum $J$ to discrete $\Lambda$ of $O$ via the decomposition formula

$$n_\Lambda^J = \frac{1}{g_o} \sum_\xi p_\xi \chi^{(\Lambda)}(\xi)^* \chi^{(J)}(\xi)$$  \hspace{1cm} (2)

for the number $n_\Lambda^J$ of copies of $\Lambda$ in the subduction of $J$. Here $p_\xi$ is the number of elements in class $\xi$ and $g_o$ is the order of the octahedral group.

### B. Operator building blocks

One may construct zero-momentum operators transforming as irreps of $O^{PC}$ from the space of operators spanned by

$$M_{j,k,a,b}(t) = \sum_x \bar{\psi}_a(x) U_{j,k}(x) \psi_b(x),$$  \hspace{1cm} (3)

where the gauge link part of our operators is defined via

$$U_{j,k}(x) \equiv U_j(x) U_k(x + \hat{j}) U_{-j}(x + \hat{j} + \hat{k}) U_{-k}(x + \hat{k}).$$  \hspace{1cm} (4)

Here $j,k = \pm 1, \pm 2, \pm 3; j \neq k$ and $a$ and $b$ are spinor indices for a total of 24 x 16 = 384 operators. The $\hat{j}$ denotes a four-vector of unit length along the spatial axis $j$.

See Figure 1 for a diagrammatic representation of the building block operator. Superpositions of such operators for meson spectrum analysis have been suggested and used previously in special cases [9, 14, 15, 16]. Evaluation of correlators constructed from operators of this form requires only calculation of propagators from a single source. Construction of operators transforming as irrep $O^{PC}$ is facilitated by observing that one can effectively consider the transformation of the spinor and link paths independently and then combine them via octahedral Clebsch-Gordan coefficients. Link smearing can be performed on operators at both the source and sink of the correlator, and quark smearing at one end to suppress high energy states with no change to the symmetry of the symmetrized operators.

### C. Spinor contribution to group structure

The contribution to the group structure due to spinor indices is determined by the 16 bilinears $\bar{\psi} F \psi$, where $F$ represents one of sixteen $4 \times 4$ matrices, \{ $I, \gamma_5, \gamma_4 \gamma_5, \gamma_i, \gamma_i \gamma_5, \sigma_{4i}, \epsilon_{ijk} \sigma_{jk}$ \}. The first four bilinears are scalars while the last four three-index objects are vectors under the rotation group in the Euclidean continuum.

Parity ($P$) and charge conjugation ($C$) of the bilinears are identified via

$$C \bar{\psi} C^\dagger = (\bar{\psi} C^\dagger)^T, \quad P \bar{\psi} P^\dagger = \bar{\gamma}_4 \psi,$$

$$C \bar{\psi} C^\dagger = - (\bar{\psi} C)^T, \quad P \bar{\psi} P^\dagger = \bar{\psi} \gamma_4, \hspace{1cm} (5)$$

where $C$ is the matrix implementing charge conjugation and we have suppressed the action on coordinates, which under parity sees $x = (x_1, x_2, x_3, x_4) \rightarrow x_P$, where

$$x_P = (-x_1, -x_2, -x_3, x_4).$$  \hspace{1cm} (6)

To classify the bilinears into irreps of $O$ one uses its character table found in Table III to project out the octahedral irreps from the representations generated by the bilinears. Since the bilinears are scalars and vectors in the continuum it follows that their respective spans are also invariant under $O$. By inspection it is straightforward to show that the character table for the scalar and vector bilinears is identical with that of $A_1$ and $T_1$ respectively. Since the multiplicity $n_\Lambda^J$ of irrep $\Lambda$ in any representation $J$ is given by Equation (2), it follows trivially that the scalar and vector bilinears form the basis of $A_1$ and $T_1$ irreps respectively.

Furthermore it may be verified that the $i^{th}$ vector bilin-
TABLE IV: Matrices $\Gamma$ for the generators $C_{4y}$ and $C_{4z}$ of the octahedral group for each irrep $\Lambda$.

| $\Lambda$ | $\Gamma^{(\Lambda)}(C_{4y})$ | $\Gamma^{(\Lambda)}(C_{4z})$ |
|-----------|-----------------------------|-----------------------------|
| $A_1$     | $[1]$                       | $[1]$                       |
| $A_2$     | $[-1]$                      | $[-1]$                      |
| $E$       | $\frac{1}{2} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ |
| $T_1$     | $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $T_2$     | $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ |

TABLE V: Octahedral symmetries of the spinor bilinears. The operators have been separated into quadrants according to their dependency on twist angle as discussed in Ref. [12].

| $F$ | $\Lambda^{PC}$ | $\lambda$ | $\alpha$ | $F$ | $\Lambda^{PC}$ | $\lambda$ | $\alpha$ |
|-----|----------------|-----------|---------|-----|----------------|-----------|---------|
| $I$  | $A_1^{++}$    | $1$       | $1$     | $T_1^{--}$ | $i$ | $1$ |
| $\gamma_5$ | $A_1^{+-}$    | $1$       | $1$     | $T_1^{++}$ | $i$ | $1$ |
| $\gamma_4$ | $A_1^{-+}$    | $1$       | $1$     | $T_1^{--}$ | $i$ | $2$ |
| $\gamma_4^{75}$ | $A_1^{--}$    | $1$       | $2$     | $T_1^{--}$ | $1$ | $1$ |

ear component transforms as the $i^{th}$ row for our choice of matrix representation $\Gamma^{(T)}(R)$ given in Table IV. Since the action on the spinor under rotation $R$ is $\psi \rightarrow S(R)\psi$ and $\psi \rightarrow S^\dagger(R)\psi$, this amounts to verifying that

$$S(R)F_iS^\dagger(R) = \sum_j \Gamma^{(T_1)}_j(R)F_j,$$

(7)

for the elements $R$ of $O$ and the four different vector bilinears $F$. Here $S(R)$ is the matrix implementing the rotation on the spinors. It is sufficient to verify Equation (7) for the two generators, namely the $\pi/2$ rotations about the $y$ and $z$ axes, $C_{4y}$ and $C_{4z}$, for which these matrices are given by

$$S(C_{4y}) = \frac{1}{\sqrt{2}} (1 + \gamma_1 \gamma_3),$$

(8)

$$S(C_{4z}) = \frac{1}{\sqrt{2}} (1 + \gamma_2 \gamma_1).$$

(9)

Hence the reduction of the spinor structure of our operators is given in Table V where now each bilinear may be classified uniquely by its irrep $\Lambda^{PC}$, row $\lambda$, and irrep multiplicity index $\alpha$ as $F^{\lambda}_{\alpha^{PC}}$.

FIG. 2: $PC$-symmetrized basis elements. We have suppressed the overall normalization factor of $1/2$.

TABLE VI: Character table showing $\chi^{(U^{PC})}(\xi)$ for representation $U^{PC}$ induced by $U^{PC}_{i,j}$. Here positive and negative are to be interpreted as $+1$ and $-1$ in the entries.

| $E$ | $3C_3^2$ | $8C_4$ | $6C_4$ | $6C_2$ | $0$ | $2P$ | $0$ | $(P + 1)C$ |
|-----|-----------|--------|--------|--------|-----|------|-----|--------|

D. Link contribution to group structure

As with the spinors, one can simplify the discussion of the rotational properties of the gauge links by considering first the parity and charge conjugation and only then the rotations in $O$. Parity of a link about the point $x$ is implemented via inversion about all three spatial axes as usual. Charge conjugation sees $U \rightarrow U^\dagger$, however to compensate for an overall transpose arising from the transformation of our quark bilinear, we formally transform $U \rightarrow U^\dagger$ to achieve the correct overall charge conjugation properties of our operators. The parity and charge conjugation contributions due to this link structure can then be taken into account by defining the $PC$-adapted superpositions,

$$U^{PC}_{i,j} = \frac{1}{2} (U_{i,k} + PU_{-j,-k} + CU_{k,j} + PCU_{-k,-j}),$$

(10)

where $k$ is defined via $\hat{k} = \hat{i} \times \hat{j}$. Diagrammatically, $U^{PC}_{i,j}$ is shown in Figure 2. For fixed $P$ and $C$ the space spanned by $U^{PC}_{i,j}$ is invariant under $O$ and will generate a representation $U^{PC}$ of the group. A basis for the six-dimensional space may be found by restricting $(i,j)$ to $\{(1,2), (1,3), (2,3), (2,1), (3,1), (3,2)\}$ and this will in turn induce the particular matrix representation $\Gamma^{(U^{PC})}_{(i,j)(m,n)}(R)$.

In order to determine how each of these four six-dimensional representations reduces, it is convenient to calculate their respective character tables. A straightforward consideration of the effect of a single element from each class of $O$ on the basis vectors allows one to find the character table for $U^{PC}$ shown in Table VI. Use of Equation (9) with $J$ as $U^{PC}$ reduces the latter to irreps $\Lambda^{PC}$ of $O^{PC}$ as shown in Table VII. So, for example, $U^{++} = A_1^{++} \oplus E^{++} \oplus T_2^{++}$.

To arrive at the explicit form of the lattice-symmetrized gauge links, note that, by Table VII each of our spaces reduces to no more than a single copy of each irrep. For such simply reducible representations the
TABLE VII: Reduction of $U^{PC}$ to irreps $\Lambda^{PC}$ of $O^{PC}$.

| $P$ | $C$ | $\Lambda^{PC}$ | $A_{1}^{PC}$ | $A_{2}^{PC}$ | $E^{PC}$ | $T_{1}^{PC}$ | $T_{2}^{PC}$ |
|-----|-----|----------------|--------------|--------------|--------|----------|----------|
| + + | + + | 1 | 0 | 1 | 0 | 1 | + + |
| + + | + + | 0 | 1 | 1 | 0 | + + | + + |
| + + | + + | 0 | 0 | 0 | 1 | + + | + ++ |
| + + | + + | 0 | 0 | 0 | 1 | + ++ | + ++ |

Reduction is straightforward\(^3\) and in our case one has the following formula for the symmetrized link fields:

$$U^{\Lambda^{PC}}_{\lambda}(x) = \frac{\sum_{(i,j)} \sum_{R \in O} \sum_{(m,n)} \Gamma_{\lambda_{\mu}}^{(U^{PC})} (R) \Gamma_{\lambda_{\mu}}^{(\Lambda)} (R)^{*} U^{PC}_{ij}(x)}{\sqrt{\sum_{(m,n)} \sum_{R \in O} \sum_{(m,n)} \Gamma_{\lambda_{\mu}}^{(U^{PC})} (R) \Gamma_{\lambda_{\mu}}^{(\Lambda)} (R)^{*}}}$$

where $(m,n)$ and $\mu$ are chosen so the denominator does not vanish and the accessible $\Lambda^{PC}$ are taken from Table \text{VII}. Here $\Gamma_{\lambda}^{(\Lambda)} (R)$ is the matrix representation for the irrep $\Lambda$ of dimension $d_{\Lambda}$ taken from Table \text{IV}. Lowercase Greek denotes row indices of the group $O$, and the sum is taken over the six pairs $(i,j)$ listed above.

The $U^{\Lambda^{PC}}_{\lambda}(x)$ transform as the row $\lambda$ of irrep $\Lambda^{PC}$ and may be uniquely identified by these latter parameters. (Since each $\Lambda^{PC}$ occurs only once in Table \text{VII} there is no need, unlike in the spinor case, to further identify a multiplicity for the irrep.) Using the explicit irrep generator matrices in Table \text{VII} to construct the full matrix representations for each irrep, one can evaluate Equation (11) to produce the final lattice-symmetrized gauge links. The results are tabulated in Table \text{VIII}. A diagram of a lattice-symmetrized gauge link structure is shown in Figure 3.

In Equation (10), all the link structures with $PC \neq ++$ are suppressed by powers of lattice spacing relative to $PC = +$. Further factors of lattice spacing can appear when Equation (10) is used to form a gauge structure like the one shown in Figure 3. Such overall factors of lattice spacing will not be of direct relevance to our study of the mass spectrum, but would be of greater interest for decay constants and matrix elements.

E. Total representation reduction and operator construction

Having classified the spinor and link components of our operators into irreps of $O^{PC}$, it remains to combine them and reduce all possible direct product representations.

\[^{3}\text{See Ref. [15] page 74] where the Clebsch-Gordan coefficients for a simply reducible direct product representation are given. In the case at hand one simply considers an arbitrary representation in place of the direct product to arrive at our result.\]
TABLE VIII: Shown are the lattice-symmetrized gauge links, $U_{ij}^{PC}(x)$, derived from Equation (11) in terms of the PC-adapted gauge field structures $U_{ij}^{PC}(x)$. Each subtable corresponds to a different PC combination and contains the lattice-symmetrized gauge link combinations along the top, denoted by $Λ^f$, and the PC-adapted structures along the left, denoted by $U(i,j)$. For irreps of dimension one the superscript denoting the row $λ$ of the lattice-symmetrized link structure is suppressed. The actual coefficient equals the sign of the entry times the square root of the absolute value of the entry in the table. Thus, for example, one has $U_{ij}^{PC} = (U_{ij}^{−} + U_{ij}^{+} + U_{ij}^{−} + U_{ij}^{+})/2$, which in turn can be expanded via Equation (10) to arrive at Figure 3.

| $PC$ | $A_1$ | $E$ | $E^2$ | $T_2$ | $T_2^*$ | $T_2^*$ |
|------|------|-----|-------|--------|---------|---------|
| $U(1,2)$ | 6/1 | 6/1 | 1/4 | −1/12 | 1/2/6 | 0 |
| $U(1,3)$ | 6/1 | 6/1 | 1/4 | −1/12 | 0 | 1/2 |
| $U(2,3)$ | 6/1 | 6/1 | −1/12 | 0 | 0 | 1/2 |
| $U(2,1)$ | 6/1 | 6/1 | −1/12 | 0 | 0 | −1/2 |
| $U(3,1)$ | 6/1 | 0/1 | 1/3 | 1/2 | 0 | 0 |
| $U(3,2)$ | 6/1 | 0/1 | −1/3 | 1/2 | 0 | 0 |

TABLE IX: Reduction of product representations $Λ_f ⊗ Λ_u$ into irreps of $O$. Only irreps $A_1$ and $T_1$ are accessible for $Λ_f$. Reduction of other octahedral direct products may be inferred from Table [X].

| ⊗ | $A_1$ | $A_2$ | $E$ | $T_1$ | $T_2$ |
|----|------|------|-----|--------|---------|
| $A_1$ | 4/1 | 4/1 | 2/1 | 2/1 | 1/2 |
| $A_2$ | 4/1 | 4/1 | 2/1 | 2/1 | 1/2 |
| $E$ | 4/1 | 4/1 | 2/1 | 2/1 | 1/2 |
| $T_1$ | 12/1 | 12/1 | 10/1 | 10/1 | 8/1 |
| $T_2$ | 12/1 | 12/1 | 10/1 | 10/1 | 8/1 |

The lattice-symmetrized operators, $M_{ij}^{PC}(t)$, are finally

$$M_{ij}^{PC,Λ_f^aΠ_f^{C_j},Λ_u^bΠ_u} = \sum_{λ_f, λ_u} \left( \begin{array}{c} Λ_f \\ Λ_u \end{array} \right) \sum_R \left( \begin{array}{c} Λ_f \\ Λ_u \end{array} \right) F_{λ_f}^{\Lambda_f^aΠ_f^{C_j}U_{λ_u}^bΠ_u}(x) \psi(x) ,$$

where the allowed irreps for the spinor bilinear and link components are determined by Tables [V] and [VII]. These fix $P$ and $C$ for the operator while the irreps $λ$ of $O$ are those allowed by the C-G series in Table [IX]. Dirac indices on $P$ and color indices on $U$ and both indices on the spinors have been suppressed. We note that each operator is thus uniquely identified by its irrep $Λ$, its row $λ$, and the direct product from which it originates, $Λ_f^aΠ_f^{C_j} ⊗ Λ_u^bΠ_u$.

III. TWISTED MASS

It is a feature of twisted mass lattice QCD (tmLQCD) that, at maximal twist, $O(a)$ errors are absent from physical observables [4, 5]. Critical slow-down is softened in tmLQCD and thus it permits simulations with light quark masses. Unlike theories that require the tuning of separate parameters for the improvement of each individual operator, tmLQCD requires the tuning of only one, the standard mass parameter $m_0$. At maximal twist, information about the physical quark mass is given by the twisted mass parameter $μ$. However, the theory modifies the parity symmetry of QCD. The implication of this for meson correlators is examined in this section.
The tmLQCD fermion action is

\[ S_F = a^4 \sum_{q=u,d} \sum_{x,y} \bar{q}(x)S_q^{-1}(x,y)q(y), \]

where

\[ \sum_y S_q^{-1}(x,y)q(y) \]

\[ = \frac{1}{2a} \sum_\mu \gamma_\mu [U_\mu(x)q(x + a\hat{\mu}) - U_{\mu\dagger}(x - a\hat{\mu})q(x - a\hat{\mu})] \]

\[ - \frac{1}{2a} \sum_\mu [U_\mu(x)q(x + a\hat{\mu}) + U_{\mu\dagger}(x - a\hat{\mu})q(x - a\hat{\mu}) - 2q(x)] \]

\[ + [m_0 + i\gamma_5\mu_q]q(x) \]

\[ \text{(16)} \]
and

\[ \mu_u = -\mu_d \equiv \mu_0 \]  

(17)
is the twisted mass parameter. The tmLQCD action of Equations (15-16) is displayed in the “twisted basis”. Fermion propagators are computed in that basis and then converted to the “physical basis” via

\[ (S_u)_{\text{physical}} = e^{i\gamma_5 \pi/4} (S_u)_{\text{twisted}} e^{i\gamma_5 \pi/4}, \]  

(18)
and

\[ (S_d)_{\text{physical}} = e^{-i\gamma_5 \pi/4} (S_d)_{\text{twisted}} e^{-i\gamma_5 \pi/4}. \]  

(19)
Except for Equations (15-16) and (34), all discussions in this work are in the physical basis at maximal twist.

While parity \( P \) defined in standard fashion by Equation (5) is not a symmetry of the tmLQCD action, flavor-parity

\[ \tilde{P} \equiv F_2 P \]  

(20)
which combines the parity operation with a 180° rotation about the second isospin axis,

\[ F_2 u F_2^\dagger = -d, \quad F_2 d F_2^\dagger = u, \]
\[ F_2 \pi F_2^\dagger = -\pi, \quad F_2 \pi F_2^\dagger = \pi, \]  

(21)
is respected \( \mathbb{R} \). In our notation, \( F_2 \) is defined by

\[ F_2 \equiv e^{i\pi \tau_2} \]  

(22)
where \( \tau_2 \) is an isospin operator, defined in Ref. [10], whose action on quark fields can be inferred from the commutation relations in Equations (31) and (32) found therein.

Notice the relations involving reflections,

\[ S_u^{-1}(0, y; U) = \gamma_i \gamma_5 S_a^{-1}(0, y_i; U_i) \gamma_5 \gamma_i, \]  

(23)where the subscript \( i \) refers to the lattice coordinates and links after inversion in the \( i \)th direction. Applying this relation in all three spatial directions and a consideration of the inverse leads to a flavor-parity relation

\[ S_u(x, y; U) = \gamma_4 S_d(x; p, y; P_U) \gamma_4, \]  

(24)where \( x_p \) is defined by Equation (6) and \( U_p \) represents the links after inversion under spatial parity. The tmLQCD action also preserves charge conjugation,

\[ S_u(U) = C S_d^T (u^*) C^\dagger \]  

(25)
and has a Hermiticity relation,

\[ S_u(U) = \gamma_5 S_d^T (U) \gamma_5. \]  

(26)
In our simulations we consider charged-meson two-point correlators,

\[ C_{AB}(t) = \left\langle \sum_{\zeta_{\text{spins}}} \bar{u}(0) \gamma_4 (F^A U^A(0))^\dagger \gamma_4 d(0) d(x) F^B U^B(x) u(x) \right\rangle \]  

(27)and neutral-meson two-point correlators with disconnected contractions omitted,

\[ N_{AB}(t) = \left\langle \sum_{\zeta_{\text{spins}}} \bar{u}(0) \gamma_4 (F^A U^A(0))^\dagger \gamma_4 u(0) u(x) F^B U^B(x) u(x) \right\rangle \]  

(28)where each of \( U^A \) and \( U^B \) is one of the symmetrized link fields \( U^{PC}_\Lambda \) defined in Equation (11), and each of \( F^I \) and \( F^J \) is one of the 16 unique products of Dirac matrices \( F^{A_{PC}}_\Lambda \) defined in Table V. Choosing \( F^A \) to be Hermitian or anti-Hermitian and evaluating the (connected) contractions, a few lines of algebra leads to valuable conclusions:

- In the configuration average, \( C_{AA}(t) \) is real for tmLQCD fermions.
- In the configuration average, \( N_{AA}(t) \) is real for tmLQCD fermions.

Note that the imaginary part of \( N_{AA}(t) \) does not vanish if the flavor symmetrization, \( (u \leftrightarrow d) \), is omitted. In practice, correlators in the charged case are averaged as well but for improved statistics.

We are interested in correlators \( C'_{IJ}(t) \) and \( N'_{IJ}(t) \) between the fully-symmetrized source and sink meson operators \( I \) and \( J \) of the form \( M_{IJ}^{PC} \) given by Equation (14). These correlators involve C-G superpositions of \( C_{IJ}(t) \) and \( N_{AB}(t) \) respectively and the above results hold for \( C'_{IJ}(t) \) and \( N'_{IJ}(t) \) as well.

It is necessary to consider what effect the use of a twisted mass action will have on the group theory discussion from Section XI. The distinction between respecting \( P \) and not \( P \) is clarified by using the three symmetry relations, Equations (24-26), to evaluate two-point correlators of operators of given parity \( P \). Table XII shows the resulting orthogonalities among various creation/annihilation operators. Note that charged mesons, containing \( \bar{u}d \) or \( du \) creation operators, have different relations from neutral mesons, containing \( \bar{u}u \) or \( dd \) creation operators.

In Table XII the charged meson entries that vanish without a flavor interchange \( (u \leftrightarrow d) \) are direct consequences of \( PC \) symmetry. This is equivalent to the physical \( PG \) symmetry since \( G \equiv F_2 C \). These lines in the table prove that \( PG = -1 \) states (i.e. \( \Lambda^{+-} \) and \( \Lambda^{-+} \)) are orthogonal to \( PG = +1 \) states (i.e. \( \Lambda^{++} \) and \( \Lambda^{--} \)). For example, because the physical pion is an eigenstate
of $PG$ with eigenvalue $+1$, we know that no charged pion signal can appear within $A_1^{++}$ nor within $A_1^{--}$.

Unfortunately, the charged physical pion will in general couple to both the $A_1^{+-}$ and $A_1^{-+}$ channels. This is because it is an eigenstate of $P$ and $G$ separately, whereas tmLQCD does not respect these as good quantum numbers. Instead, tmLQCD respects $P$ and $C$, but the charged pion is not an eigenstate of those. Notice that the lines in Table XII that involve "$(u \rightarrow d)$" merely show that unphysical states of opposite flavor-parity are orthogonal, i.e.

$$\langle \gamma_4 I^\dagger \gamma_4 S_u J S_d \rangle - (u \leftrightarrow d) = 0$$

$\langle \gamma_4 I^\dagger \gamma_4 S_u J S_d \rangle + (u \leftrightarrow d) = 0$

Each factor in parentheses has definite flavor-parity, so finding that this matrix element vanishes (as was found in some entries in the table) means that the flavor-parities are orthogonal. Similarly, the charged physical mesons in $A_1^{++}$ and $A_1^{--}$ cannot be separated either.

In Table XII the neutral meson entries that vanish without using $(u \rightarrow d)$ are direct consequences of $C$ symmetry. These lines prove that $C = +1$ states (i.e. $A^{++}$ and $A^{--}$) are orthogonal to $C = -1$ states (i.e. $A^{+-}$ and $A^{-+}$). For example, because the physical neutral pion is an eigenstate of $C$ with eigenvalue $+1$, we know that no neutral pion signal can appear within $A_1^{++}$ nor within $A_1^{--}$.

In contrast to the case of the charged pion, the physical neutral pion is an eigenstate of $\bar{P}$, and since tmLQCD respects $P$ the pion will couple to $A_1^{++}$ and not to $A_1^{--}$. However, the $A_1^{++}$ channel will couple to the flavor-singlet pseudoscalar (i.e. the $SU(2)$ $\eta'$) and if we neglect disconnected diagrams, then the mass of the $SU(2)$ $\eta'$ is identical to the mass of the neutral pion. This degeneracy is not specific to the pion-$\eta'$ system; it occurs for any angular momentum irrep with any $C$. Therefore $\Lambda^{PC}$ contains the same spectrum of masses for both $P$ values in neutral channels.

As discussed in Ref. [12], all $\Lambda^{PC}$ combinations can be obtained from operators that are independent of twist angle. Since the present work is restricted to maximal twist, those operators are not emphasized. In either case, tmLQCD simulations will still contain mixing as discussed above: For charged mesons we cannot separate the $\Lambda^{++}$ and $\Lambda^{--}$ pair, nor can we separate the $\Lambda^{+-}$ and $\Lambda^{-+}$ pair. For neutral mesons we cannot separate the $\Lambda^{++}$ and $\Lambda^{--}$ pair, nor can we separate the $\Lambda^{+-}$ and $\Lambda^{-+}$ pair. A striking example is the appearance of a pion signal for the operator $\bar{u}(x)\gamma_4 d(x)$.

IV. CURVE FITTING WITH EVOLUTIONARY ALGORITHMS

There are several motivations for our use of evolutionary fitting. In general, the sheer quantity of fitting to be done requires an automated black box method, an objective being pursued by others as well [13]. An important part of this goal is that the method not depend on any subjective parameters such as timestep fit ranges, a fixed number of states, or initial parameter values. This is not simply to speed up the fitting process but for re-
producibility as well. While evolutionary fitting requires 
mutation and breeding steps to be identified as well as 
probabilities to be fixed in the algorithm, the goal of 
minimizing the $\chi^2/\text{n.dof}$ is objective. Any two fit 
algorithms, evolutionary or not, can have their results on the 
same data readily compared by this statistic and either 
have the fits confirmed to be equivalent or the better fit 
selected.\footnote{With such comparisons, fitting algorithms themselves may 
thereby evolve.}

Other reasons for using this fitting method are specific 
to our problem. We require a method that extracts a set of 
common states from multiple correlators in the same 
channel. This is often accomplished using the variational 
fitting method \cite{19, 20, 21}. Due to the large number 
of operators present, however, we run only the dia-
ogonal correlators of the correlator matrix and not the full 
correlator matrix generated by putting all possible com-
binations of operators in a given channel at source and 
sink. The latter would have been required to use the 
variational method. We also smear our source and sink 
operators differently, which results in a correlator matrix 
which is not Hermitian, which is also required for that 
method. Evolutionary fitting in principle allows us to 
identify continuum angular momentum states ($J$) across 
lattice symmetry channels which have only octahedral 
symmetry. Finally, a method that is able to identify po-
tentially weak contamination signals occurring due to the 
use of twisted mass is also of value.

We have detailed the algorithm itself in Refs. \cite{6} and 
\cite{7}. Here a summary is presented to provide con-
text for evaluation of the method.

The idea behind an evolutionary algorithm is to con-
sider candidate solutions to a problem as individual or-
organisms in a population. This population is allowed to 
mutate and breed to produce successive generations. The 
types of operations involved in mutating an individual or 
breeding a pair of individuals will be specific to the 
problem at hand. These must be sufficient for the organ-
isms to have a good probability of being able to explore 
the solution space. Coupled with a function to measure 
the fitness of an organism which in turn influences which 
organisms will be allowed to populate the next genera-
tion, evolution is allowed to take its course with succes-
sive generations approaching a better solution. Different 
individuals in the population will have desirable charac-
teristics which will be disseminated over time with high 
probability to the rest of the population. In a sense, the 
population is able to explore the solution space to the 
problem in parallel, making it an effective technique in 
general for finding good, and sometimes unexpected, so-
lutions to complex problems.

In the context of lattice QCD, the organism is a fit 
function that is a linear superposition of an \textit{a priori} un-
known number of exponential functions whose exponents 
determine the spectrum of masses in the correlator. The 
simplest case involves a fit to a single correlator and we 
have detailed this elsewhere \cite{6, 7}. In the current situ-
ation we have multiple correlators for a single channel 
sharing the same energy states $\{E_m : m = 1, \ldots, m_{\text{max}}\}$. 
The total fit function to all the data is then a set of fit 
functions, one per correlator, each of the form:

$$G^{(i)}(t) = \sum_{n=0}^{n_{\text{max}}^{(i)}} Z_n^{(i)} \left( e^{-E_n^{(i)} t} + e^{-E_n^{(i)} (T-t)} \right). \quad (30)$$

Here $i$ is the correlator index, $n_{\text{max}}^{(i)}$ is the (variable) num-er of states found in that correlator, $Z_n$ is the coefficient 
for the energy state $E_n$, and $T$ is the temporal extent of 
the lattice. The form of the fit function reflects the fact 
that we are only fitting diagonal correlators in the cor-
relator matrix (essentially the same operator at source and 
sink though we do allow for different smearings at 
source and sink), and that we are using periodic bound-
dary conditions. The desired best fit minimizes $\chi^2/\text{n.dof}$ 
of the total fit function $G$. Assuming the datasets are 
not correlated in any way, $\chi^2(G)$ is simply the sum of 
$\chi^2(G^{(i)})$, the correlated $\chi^2$ on the $i$th correlator \cite{7}. 
The number of degrees of freedom is

$$n_{\text{dof}}(G) = n_{\text{data}} - m_{\text{max}} - \sum_{i=1}^{i_{\text{max}}} n_{\text{max}}^{(i)}, \quad (31)$$

where $n_{\text{data}}$ is the number of timesteps fit in each corre-
lator times the number of correlators ($i_{\text{max}}$). The fitness of the organism, $f(G)$, which we desire to maximize is 
therefore defined to be $-\chi^2(G)/n_{\text{dof}}(G)$. One notes that 
the degrees of freedom fluctuate with the number of pa-
rameters (states and coefficients) in a particular organ-
ism. The complication of searching such a discontinuous 
function space as well as the independence from initial 
conditions (the initial population is chosen at random) 
are some of the principal advantages of the evolutionary 
fitting method.

The information required to construct a given fit or-
ganism is coded in its genotype. The subfit for dataset 
number $i$ is represented by a list of $n_{\text{max}}^{(i)}$ coefficients 
$\{Z_n^{(i)}, I_n^{(i)}\}$ where $I \in \{1, \ldots, m_{\text{max}}\}$ is an integer index 
indicating to which of the $m_{\text{max}}$ energy states $E_m$ the 
coefficient is associated. In summary, for a fit of $i_{\text{max}}$ 
correlators the complete genotype is of the form:

\textbf{Fit Genotype} 
$$= (\text{Dataset coefficients, Mass list})$$
$$= ((\text{Dataset 1 coeffs, \ldots, Dataset } i_{\text{max}} \text{ coeffs}),$$
$$\text{Mass list}) \quad (32)$$
with

\[
\text{Dataset } i \text{ coeffs } = (Z^{(i)}_1, I^{(i)}_1, \ldots, Z^{(i)}_{\text{max}}, I^{(i)}_{\text{max}})
\]

\[
\text{Mass list } = (E_1, \ldots, E_{\text{max}}).
\]

With the above genotype defined, operations for mutation and breeding become apparent. The genotype is a hierarchy of lists so we coded procedures that proceeded recursively through the structures present. Mutations of lists involve mutating the elements in the list. For a list of elements of the same type, adding or removing a random element, and if order is meaningful to the list, reordering it, are other possible mutations. Breeding (or crossover) of two lists will produce two new lists containing parts of each, and potentially of different lengths if the lists contain elements of different types. Ultimately one has to mutate numbers, either reals or integers, and this can be accomplished by adding a Gaussian random variable or flipping bits in a binary representation respectively. (Our integer index has to be interpreted modulo the number of energy states \(m_{\text{max}}\) so that it always maps to an individual state.) Breeding of numbers can be done by randomly interpolating the real numbers or exchanging subsets of the bits of the integers.

It is of value to introduce special mutations to the full genotype. One mutation does a fixed number of steps of a Newtonian optimization of the fit function of a single organism. Here we used the Levenberg-Marquardt [23, 24] method with the current functional form defined by the organism’s genotype and the values of its individual parameters as the initial conditions. Also we introduced a reduction mutation which would convert genotypes that represented the same function into a common form so that the fitting algorithm would converge to a single representation of the solution. Consult Refs. [6, 7] for further detail of these steps.

V. SIMULATION DETAILS

The quenched configurations used for this work were previously discussed in Ref. [25], and the details are reproduced in Table XIII of this work for completeness. In addition to tmLQCD fermions, a Wilson fermion was considered for purposes of comparison and those parameters are also listed in Table XIII. Note that at each of the three \(\beta\) values, the four quark masses are comparable by their approximate ratios with the physical strange quark mass: \(m_s, m_s/2, m_s/3,\) and \(m_s/6\). The Wilson quark at \(\beta = 6.0\) was chosen close to \(m_s/2\).

The tuning to maximal twist was performed by varying \(m_0\) for each \(\mu_0\) until \(\omega\) became \(\pi/2\) as defined by [26, 27].

\[
\tan \omega = i \frac{\sum \langle V_4(\vec{x},t)P(0) \rangle}{\sum \langle A_4(\vec{x},t)P(0) \rangle},
\]

where \(P, V_4, A_4\) are the local bilinears for charged mesons with pseudoscalar, vector, and axial vector quantum numbers respectively.

The diagonal correlators of the 384 extended and 16 local operators from Section II were calculated for each lattice spacing using degenerate quarks at our four quark masses in both the charged and neutral channels for twisted mass and at the single quark mass for Wilson. Neutral twisted mass channels did not include the so-called disconnected contributions, i.e. contractions between a quark and anti-quark within the source (or sink) operator.

Smearing was used to reduce contributions from excited states to correlators at the shortest time extension. Gaussian quark smearing was performed at the sink and stout link smearing at both source and sink as described in Ref. [29]. Quark smearing parameters were \(\alpha = 0.15\) and \(\beta = 64\) for all lattice spacings. Stout link smearing used \(\rho = 0.15\) for \(\beta = 5.85\) and \(\rho = 0.2\) for \(\beta = 6.0\) and \(\beta = 6.2\) and \(\rho = 16\) for all three lattice spacings. For comparison, correlators of unsmeared operators were also computed.

Before fitting, diagonal correlators corresponding to operators that differed only in their row \(\lambda\) were averaged since these must be the same statistically by symmetry. Specifically, the diagonal correlators from operators in Equation (33) with the same \((\Lambda^P, \Lambda^P, \Lambda^I, \Lambda^I, C_{\mu}, \Lambda^I)\) were averaged, leaving the number of correlators to be fit for each channel given by Table XI for a fixed quark mass, fixed type (twisted mass charged, twisted mass neutral, and Wilson), fixed lattice spacing, and fixed smearing.

For each such set of correlators, evolutionary fits were done to the data at all timesteps using an overall population of 480 organisms. Two fits were done for each dataset to test fit consistency. The first was stopped at exactly 600 generations. The second run, the results of which were used, performed at least 600 generations but was allowed to continue up to 1200 generations, stopping in between only if no improvement in the best genotype of a given generation was seen for 200 generations. The genotype was allowed to contain up to 8 masses, and the fit for each correlator could have up to 8 coefficients pointing to elements of the mass list. All coefficients and masses were restricted to be positive. Once the evolutionary algorithm had obtained a best fit function for the dataset, bootstrap errors [30] were generated by fitting this function to bootstrap configurations.

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5. A modern list-based language, Python [24], was convenient for this implementation and fast enough for our purposes.

6. See Ref. [29] for explicit definitions of these parameters.

7. The population was distributed over 4 islands with parameters \(N_{\text{elite}} = 5, N_{\text{diversity}} = 5, N_{\text{mutant}} = 20\). See Refs. [3, 4].

8. Bootstrap fits were done using the Levenberg-Marquardt
TABLE XIII: The parameters used for simulations in this work. Lattice spacings are taken from Ref. [26] using \( r_0 = 0.5 \) fm. Each \((am_0,a_{\mu 0})\) pair is the result of tuning to maximal twist as discussed in Ref. [23], except the Wilson case of course.

| \( \beta \) | \( a \) [fm] | \#sites | \#configurations | \( am_0 \)    | \( a_{\mu 0} \) | twist angle (degrees)  |
|------------|-----------|---------|------------------|-------------|-------------|----------------------|
| 5.85       | 0.123     | \( 20^4 \times 40 \) | 600 | -0.8965 | 0.0376 | 90.0±0.3     |
|            |           |         |                  | -0.9071 | 0.0188 | 90.2±0.6     |
|            |           |         |                  | -0.9110 | 0.01252 | 90.6±0.8    |
|            |           |         |                  | -0.9150 | 0.00627 | 90.6±1.6    |
| 6.0        | 0.093     | \( 20^4 \times 48 \) | 600 | -0.8110 | 0.030 | 90.4±0.4     |
|            |           |         |                  | -0.8170 | 0.015 | 91.0±0.7     |
|            |           |         |                  | -0.8195 | 0.010 | 92.5±1.0     |
|            |           |         |                  | -0.8210 | 0.005 | 95.5±2.1     |
| -0.7835    | 0.0       |         |                  |            |            | (Wilson)          |
| 6.2        | 0.068     | \( 28^3 \times 56 \) | 200 | -0.7337 | 0.021649 | 89.1±0.8     |
|            |           |         |                  | -0.7367 | 0.010825 | 87.3±1.8     |
|            |           |         |                  | -0.7378 | 0.007216 | 86.3±2.8     |
|            |           |         |                  | -0.7389 | 0.003608 | 86.4±4.5     |

VI. RESULTS

A. Observations on curve fitting

To assess the ability of the fitting algorithm to obtain accurately good fits to our data, we present in Figure 4 a histogram of the \( \chi^2/n_{dof} \) for all 320 twisted mass fits at each lattice spacing.\(^9\) While the plot clearly shows that all the fits are reasonable, falling largely in the range 0.9 – 1.4, the fits for the finest lattice spacing, \( \beta = 6.2 \), while having the same shape of distribution, have their mean shifted upward from that of the coarser spacings by about 0.3. At the finer lattice spacing more states are resolved so one possibility would be that the fitting algorithm has not had sufficient time to find the true minima. However, in comparing these fits that were allowed to go up to 1200 generations with those required to stop at 600 generations, no marked improvement in the histogram is seen. A good fit for all our data was achieved at 600 generations so this is not the source of the discrepancy.

The actual source of this discrepancy can be traced to lower statistics, as our \( \beta = 6.2 \) data involved only 200 configurations compared to 600 at the coarser lattice spacings. In Ref. [10] we fit subsets of the configurations of our \( \beta = 6.0 \) data to see what the effect of poor statistics would be on the result. The subfit on only 200 configurations shown in that paper clearly demonstrates an increase in the \( \chi^2/n_{dof} \) comparable to what we find here with our \( \beta = 6.2 \) data. Since our poorer \( \beta = 6.2 \) fits can be explained by fewer statistics, we conclude that where good fits to the data exist they are found by the fitting algorithm. It is notable that the shape of this histogram can be used to assess the minimum number of configurations required for a proper simulation. Presumably, once a sufficient number of configurations have been used that states are reliably identified, one will get a good distribution of \( \chi^2/n_{dof} \) centered on one.\(^{10}\) A greater number

\( \chi^2/n_{dof} \) functional form and the initial parameters taken from the best fit found by the evolutionary algorithm on the actual data.

\(^9\) These fits were for each of the 20 \( \Lambda^{PC} \) channels at four degenerate quark masses with correlators both neutral and charged, smeared and unsmeared. Later we also did fits to the combined \( E \) and \( T_2 \) channels as well as for the \( A_2 \) channel combining smeared and unsmeared correlators. There, fit quality statistics were in line with these, with \( \beta = 6.2 \) once again systematically higher.

\(^{10}\) Obviously the number of correlators fit in each channel varied.
of configurations would then serve to lower the error on the states found and narrow the overall distribution.\footnote{One may wonder why our smeared and unsmeared correlators had comparable histograms if the smeared data had to resolve a fewer number of states. Why this should be the case is likely due to the asymmetrical smearing that is done to our source and sink operators. While it is true that we are removing excited states by smearing, in our case we are producing noisier correlators with the result that our statistics remain wanting for our smeared correlators at our finest lattice spacing. Our baryon study with the same smearing showed a similar result.}

Since the errors of the parameters were obtained by bootstrapping the functional form found by the evolutionary algorithm, it was also possible to calculate the variation in the $\chi^2/n_{dof}$ for each fit. Figure 4 includes histograms of the standard deviation $\sigma_{\chi^2/n_{dof}}$ of our fits. For the coarser lattice spacings with 600 configurations one finds that there is a relatively narrow distribution of the $\chi^2/n_{dof}$ across bootstrap configurations. This gives some confidence that the fitting algorithm has found a stable functional form for the fit. For $\beta = 6.2$, however, the variation is found to be wider. That this is due to the fewer configurations available at this spacing can also be confirmed by consulting Ref. \cite{6} which shows that our subfit to 200 configurations displayed a similar trend. Similar findings are obtained by plotting the standard quality of fit $Q$, as discussed in Ref. \cite{31}.

Because our evolutionary algorithm fitting function was designed to fit correlators that are sums of decaying exponentials only, this causes a systematic error in the $A^{++}_1$ channel where, due to a quenching artifact, this is not the functional form of the correlator. In quenched lattice QCD one has a ghost contribution in the scalar correlator due to the $\eta^\prime\pi$ intermediate state being light and of negative norm in this approximation.\footnote{In parentheses after each number the octahedral irrep of the multiplicity $\alpha$ from Table V.} In our data this effect is most pronounced at the lightest quark mass in the $\beta = 6.2$ charged channel. See Figure 5 where our five correlators in this channel are plotted. Because we only have good $C$ and product $PG$ in our neutral and charged channels respectively, this artifact will appear in other $A_1$ channels.\footnote{Because the coefficients in our fit functions are constrained to be positive, the effect of negative ghost contributions is primarily to produce a poor fit. Several of the higher outliers of $\chi^2/n_{dof}$ in Figure 4 are readily traced to $A_1$ channels containing this ghost contribution.}

In order to evaluate the consistency of the fitting algorithm we did at least two runs in each channel and found no significant variation in the results. Where difference occurred it would tend to be a single state with large error found in a single correlator. Given a finite number of generations in a run this is to be expected.

The fitting algorithm appears to have resolved states predictably. Unsmearred channels found greater numbers of states than did smeared channels. In some channels no state was found, which affirms that the method does not claim signals in everything. Occasionally a clearly spurious state with large error would appear in a single correlator which is also expected from the statistical variation of the data itself. Finer lattice spacings in the unsmeared case found more states since a greater number of excitations could be resolved. Occasionally the $\beta = 6.2$ lattice spacing appeared to resolve an intermediate state not seen in the other spacings. While the bifurcation of states could certainly occur at finer spacing, the fewer configurations at $\beta = 6.2$ do not allow us to make a strong conclusion from this observation.

Occasionally states of low energy would appear in some channels with large error below the obvious ground state. Such errant signals are a consequence of fitting at Euclidean times far from the source, where correlators are noisy. They may be an artifact of time step correlation not being entirely removed through the use of the correlated $\chi^2$. We regard the presence of these noisy spurious states as a reminder that we are using a true black event and thus smearing allowed us to fit many correlators in

\[ \chi^2/n_{dof} \]

\[ Q \]

\[ C \]

\[ PG \]

\[ A^{++}_1 \]

\[ \eta^\prime\pi \]

\[ \alpha \]

\[ \beta \]

\[ T^{+}_{\pm-} \]

\[ T^{++}_1 \]

\[ T^{0}_1 \]

\[ T^{--}_1 \]
the same channel quickly.

As part of the analysis we tried to fit all smeared correlators corresponding to the \( J = 4 \) channel which includes all four octahedral irreps \((A_1, E, T_1, \text{and} T_2)\) to which its subduced representation reduces for a given \( PC \). Such fits involved fitting on the order of forty correlators simultaneously and, although taking proportionately longer, were, in the end, tractable by the fitting algorithm.

B. Observations on twisted mass

Figures 6 and 7 show fits with both charged and neutral operators in the \( A_1 \) and \( T_1 \) channels, which couple to spin zero and one respectively in the continuum. Results at four different degenerate quark masses are shown. The sizes of the points are scaled to give a qualitative impression of the significance of the state in the datasets via the factor:

\[
\frac{2}{\pi} \arctan \left( \frac{1}{N} \sum_{i=1}^{N} \frac{Z^{(i)}}{\epsilon^{(i)}} \right) . \tag{35}
\]

Here \( N \) is the number of correlators involved in the fit, \( Z^{(i)} \) is the coefficient on the \( i \)th correlator for the given energy state (taken to be zero if there was no coefficient found on that dataset) and \( \epsilon^{(i)} \) is the bootstrap error of the given coefficient. The arctan function is used to ensure the scale factor ranges from 0 to 1. The utility of the scaling is that it facilitates the identification of the same state across lattice spacings and quark masses for extrapolation purposes. The scale factor is not numerically involved in any extrapolation however.

In Figures 6 and 7, one is able to see easily the effect of the twisted mass channel contamination discussed in Section III. In Figure 6 the ground state pseudoscalar ("pion") is clearly visible across quark masses where it should be in the \( A_1^{++} \) channel for both the charged and neutral correlators. For the charged operators, channels are expected to contaminate others with the same \( \rho \) and pion contamination. The charged pion shows up clearly as contamination in the \( A_1^{++} \) channel as predicted. That it is contamination is clear not only because there is no corresponding signal in the neutral channel but also the larger error and smaller scales are both indicative of a weaker signal than the authentic one in the \( A_1^{++} \) channel. Comparison of the \( A_1^{++} \) channel with the \( A_1^{--} \) channel similarly shows contamination of the authentic scalar \( A_1^{++} \) ground state signal in its twinned channel.

Turning to the neutral case, charge conjugation is respected and one sees the pion now contaminating the \( A_1^{++} \) channel. States in the latter channel in turn are contaminating the \( A_1^{--} \) channel where one clearly sees weak contamination states interspersed with authentic states. It is a testament to the fitting algorithm that it is able to distinguish these errant states.

The strength of the ground state vector ("\( \rho \) meson") allows us to confirm similarly the contamination relations for those channels not evident in the scalar case. The \( \rho \) is clearly identified in both charged and neutral \( T_1^{--} \) channels in Figure 7. Consideration of the contamination in the charged channel suggests the particle should appear in the \( T_1^{++} \) channel, which it clearly does. In the neutral case the \( \rho \) contamination is appearing in the \( T_1^{--} \) channel as expected. It is worth noting that contamination states are not appearing in unexpected channels. The strength of the pion and the \( \rho \) as signals, even in contamination, would make them easily discernible.

When contamination is weak or when the actual states in the channel being contaminated are themselves weak one may see distortions in the actual state if the two states are nearby or cannot be distinguished. Higher statistics are required to disentangle correctly the greater number of states that will appear in a given channel. The value of being able to compare neutral and charged channels which get contaminated differently is readily apparent.

Overall, contamination appears to decrease with finer lattice spacing. At lighter quark mass the contamination also lessens but so does the actual signal in the original channel. In our simulation, smearing reduced the contamination strength and in our smeared fits it was only possible to identify reliably ground state \( \rho \) and pion contamination. The decrease in contamination with lattice spacing and the effect of smearing can be seen in the fit plots in Section VI C. The fits in Figures 6 and 7 belonging to data both unsmeared and at our coarsest lattice spacing, are for illustration of the contamination effect. The smeared data we actually fit for our results appear to have limited contamination identified by the fitting algorithm.

C. Meson spectroscopy

The data from the smeared \( T_1^{--} \) and \( T_1^{++} \) channels are provided in Figures 8 and 9 respectively. The sizes of all data points are scaled by Eq. (35) so tiny points are often devoid of any physics content. Similar plots for all other \( \Lambda^{PC} \) appear in Ref. [51].

In Figure 8, the extrapolated ground state vector meson (the \( \rho \)) is in agreement with its physical value, and a single excited state is resolved. The neutral \( \beta = 5.85 \) excited state, while present at every quark mass, appears to be interfering with the ground state. The resulting poor chiral extrapolations of the neutral \( \beta = 5.85 \) data will be excluded from the continuum extrapolation of both the ground and excited states. At the heaviest quark mass the ground state for the neutral \( \beta = 6.0 \) fit is bifurcated (the points are indistinguishable on this plot) presumably because statistical variation made this a slightly more probable fit. Due to limited statistics in the \( \beta = 6.2 \) channel (recall Section VI A), there may be increased systematic errors that arise during the fitting at \( \beta = 6.2 \). One might therefore prefer to omit \( \beta = 6.2 \) data from continuum extrapolations.
FIG. 6: The results of the fits to the four $A_1$ channels at $\beta = 5.85$ for the unsmeared correlators. For each of the charged (C) and neutral (N) channels, there are four fits for the quark masses going from the heaviest (mass #1) to the lightest (mass #4). Points are scaled to the $\rho$ mass found in the smeared charged mass #2 channel. Point sizes are scaled to reflect their significance as given by Equation (35).

FIG. 7: The results of the fits to the four $T_1$ channels at $\beta = 5.85$ for the unsmeared correlators. Notation is identical with Figure 6.

FIG. 8: Fits to smeared $T_1^{-+}$ correlators are shown. Experimental measurements [33] are plotted for comparison.
FIG. 9: Fits to smeared $T_{-}^{++}$ correlators are shown. A clear signal appears between 2 and 3 GeV which has been extrapolated. The lower points seem to suggest that the fitting algorithm is detecting some other state, likely channel contamination due to tmLQCD. This is an exotic channel. Experimental claims [33] are plotted for comparison.
two β continuum extrapolations. Extrapolations that omit the extrapolations, are given in Table XIV along with con-
pared directly to Ref. [9], and good agreement is found = 6 β.
In Figure 10, results in the exotic $T_1^{-+}$ channel are plotted. A clear signal is evident between 2 and 3 GeV, while tiny points also appear in the small mass range. These tiny points may be attributed in part to contamination with the other $T_1$ channels due to tmLQCD’s symmetry structure, and in part to statistical variation within the fitting procedure. At our largest quark mass, which is approximately the strange quark mass, our results are consistent with the lattice study of Ref. [9] which used displaced quarks to generate angular momenta in the operators for quenched simulations at $\beta = 6.0$.

The results of all our simulations, after linear chiral extrapolations, are given in Table XIV along with continuum extrapolations. Extrapolations that omit the $\beta = 6.2$ data, i.e. simple averages of data at the other two $\beta$ values, are provided in Table XIV for all $\Lambda^{PC}$. The predictions for charged meson masses are also shown graphically in Figure 10, together with the experimental mass spectrum.

Note that our results from the heaviest quark mass at $\beta = 6.0$ (provided explicitly in Ref. [31]) can be compared directly to Ref. [9], and good agreement is found for the available channels: $a_0, \pi, \rho, b_1, a_1$ and $a_2$. The exotic states which we identify in the $0^{++}, 1^{-+}$, and $2^{++}$ channels, for the strange quark mass at $\beta = 6.0$, are also found to be comparable to values which they ascribe to those same $J^{PC}$.

Our result for the $1^{-+}$ from Table XV agrees well with the general consensus of quenched [16, 34, 35, 36] and dynamical [37, 38] lattice studies that the lightest exotic meson is $1^{-+}$ with a mass near 1.9 GeV. We also note that the flux-tube model predicts exotic hybrids $0^{++}$, $1^{-+}$, and $2^{++}$ near 1.9 GeV in the light isovector channel.

Although results of our simulations with Wilson fermions have not been reported here, they did provide independent confirmation of which signals were actually contamination that arose due to the twisted mass term.

In all but the $A_2$ channels, the smeared operators produced better fits than unsmeared operators. For $A_2^{PC}$, it was found that just fitting the smeared channel tended to produce failures in the bootstrap fits which are used to generate the errors of the parameters. The reason for this failure is poor statistics due to a combination of the relatively few correlators in each $A_2$ channel (no row averaging occurs in a one-dimensional channel) and the weakness of the expected states in the channel since $A_2$ couples only to continuum $J \geq 3$ as shown in Table II. As our motivation for fitting smeared states is to reduce the influence of high energy states and since the $A_2$ channels have very few low lying states even in the unsmeared case, we refit all $A_2$ correlators, both smeared and not, simultaneously. This gives sufficient statistics to make our bootstrap fits converge properly.

Visual inspection of the fits to the $E$ and $T_2$ channels independently, in each $PC$ combination, revealed that results were remarkably similar for any state found. This is to be expected since, by Table II the lowest angular momentum to which they both couple is $J = 2$. With the assumption that the few states found across these channels are of this angular momentum, we do combined fits of all correlators of $E$ and $T_2$ together for each $PC$. In these combined fits the algorithm finds the states to be statistically the same (i.e. it does not bifurcate any of the states), thus allowing us to conclude that our results are commensurate with ascribing a value $J = 2$ to the common state. Furthermore it allows us to use all the available data to extract its value. We thus tabulate results only for the combined $(E, T_2)$ fits, and not them separately. Our lattice spacing is fine enough that given our statistics it is unlikely that any splitting between $E$ and $T_2$ states for a common $J = 2$ state would be detectable, and it is not. The remaining states in our fits are assigned their most probable continuum $J$ values based on Table II; namely $J = 0, 1, 3$ for irreps $A_1$, $T_1$, and $A_2$ respectively.

D. Observations on operators

To give insight into the nature of the operators which contribute to a particular state we have produced Table XVII in Appendix A. There one finds the operators

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12 Technically $J = 4$ is also a logical possibility since its subduced representation reduces to both irreps $E$ and $T_2$ as well.
13 Such splitting can be observed across continuum subduction channels on coarser lattices. See Ref. [39].
14 While $J = 3$ could also appear in $T_1$ and $T_2$ channels as well, the likely weakness of such a signal made a combined fitting of these channels inappropriate in comparison to the $J = 2$ case, given the presence of lower $J$ in $T_1$ and $T_2$ channels and the limitations of our statistics.
TABLE XIV: Extrapolations of each resolved channel labeled by its irrep ($A^{PC}$) and its type, charged (C) or neutral (N), are given. States designated by an asterisk indicate a higher energy state in the same channel. The inferred continuum quantum number $J$ of the state is also given. Chiral extrapolations (linear) for each $\beta$ are shown along with their $\chi^2/\nu_{dof}$ in subsequent columns. The continuum extrapolation of these results is shown in the last column. We have omitted the pion ($A_1^{++}$) from the table. All energies are in GeV.

| Channel | Irrep(s) | $\beta = 5.85$ | $\beta = 6.0$ | $\beta = 6.2$ | Final |
|---------|---------|---------------|---------------|---------------|-------|
|         | J/C/N   | $\chi^2/\nu$ | $\chi^2/\nu$ | $\chi^2/\nu$ | $\chi^2/\nu$ |
| $A_1^{++}$ | 0 | C | 1.35(3) | 1.579 | 1.36(3) | 0.131 | 1.40(6) | 0.015 | 1.40(6) | 0.231 |
|         |       | N | 1.01(4) | 2.566 | 1.21(4) | 0.198 | 1.20(8) | 0.012 | 1.39(8) | 1.683 |
| $A_1^{+-}$ | 0 | C | 3.3(3) | 0.015 | 4.2(4) | 1.576 | 5.1(10) | 0.023 | 5.6(8) | 0.053 |
|         |       | N | 3.5(4) | 0.688 | 2.5(3) | 0.014 | 3.1(6) | 0.194 | 2.1(7) | 2.209 |
| $A_1^{--}$ | 0 | C | 0.31(17) | 2.815 | 1.96(18) | 0.048 | 2.08(10) | 0.303 | 2.2(3)$^a$ | ...
|         |       | N | 1.47(16) | 0.027 | 1.69(10) | 0.268 | 1.96(13) | 1.548 | 2.14(19) | 0.309 |
| $T_1^{++}$ | 1 | C | 0.98(5) | 0.371 | 1.25(5) | 0.990 | 1.6(2) | 0.376 | 1.65(12) | 0.461 |
|         |       | N | 1.33(2) | 0.101 | 1.428(17) | 0.003 | 1.52(10) | 0.010 | 1.57(5) | 0.061 |
| $T_1^{++}$ | 1 | C | 2.4(3) | 0.843 | 2.3(3) | 0.213 | 3.3(6) | 0.315 | 2.9(6) | 1.688 |
|         |       | N | 3.0(7) | 0.109 | 2.4(3) | 0.298 | 5.8(13) | 0.459 | 2.9(10) | 7.283 |
| $T_1^{-+}$ | 1 | C | 1.40(3) | 1.507 | 1.48(3) | 1.018 | 1.74(8) | 0.961 | 1.69(7) | 4.136 |
|         |       | N | 1.14(6) | 0.715 | 1.62(6) | 0.018 | 1.57(7) | 0.199 | 1.90(10) | 6.728 |
| $T_1^{+-}$ | 1 | C | 2.6(4) | 1.095 | 2.7(3) | 0.535 | 2.8(6) | 0.111 | 2.9(7) | 0.207 |
|         |       | N | 3.1(4) | 0.283 | 2.8(4) | 0.021 | 3.2(4) | 1.833 | 3.1(6) | 0.591 |
| $T_1^{-}$ | 1 | C | 1.80(18) | 0.774 | 2.15(18) | 0.008 | 2.4(4) | 0.258 | 2.7(4) | 0.022 |
|         |       | N | 1.7(2) | 0.566 | 2.3(2) | 0.027 | 2.9(5) | 0.000 | 3.3(5) | 0.157 |
| $T_1^{--}$ | 1 | C | 0.767(19) | 0.141 | 0.764(16) | 0.036 | 0.79(3) | 0.066 | 0.78(3) | 0.573 |
|         |       | N | 0.797(6) | 0.508 | 0.812(12) | 0.515 | 0.80(2) | 0.122 | 0.79(5)$^a$ | ...
| $T_1^{--}$ | 1 | C | 1.18(12) | 0.006 | 1.50(7) | 0.030 | 1.82(18) | 0.021 | 2.00(19) | 0.253 |
|         |       | N | 0.84(2) | 3.926 | 1.63(10) | 0.199 | 1.88(16) | 0.154 | 2.2(4)$^a$ | ...
| $(E, T_2)^{++}$ | 2 | C | 0.4(2) | 0.340 | 0.75(19) | 0.290 | 1.0(4) | 0.244 | 1.3(4) | 0.003 |
|         |       | N | 0.6(4) | 0.049 | 0.52(13) | 0.179 | 0.5(3) | 0.051 | 0.4(4) | 0.002 |
| $(E, T_2)^{++}$ | 2 | C | 3.7(7) | 0.196 | 2.7(4) | 0.063 | 2.9(9) | 0.127 | 1.9(10) | 0.468 |
|         |       | N | 2.4(3) | 2.355 | 3.0(4) | 0.103 | 3.2(8) | 0.037 | 3.7(8) | 0.023 |
| $(E, T_2)^{+-}$ | 2 | C | 2.8(6) | 0.057 | 2.7(4) | 0.065 | 2.9(8) | 0.018 | 2.8(9) | 0.066 |
|         |       | N | 3.2(5) | 0.005 | 2.9(4) | 0.031 | 3.1(6) | 0.026 | 2.9(8) | 0.172 |
| $(E, T_2)^{-+}$ | 2 | C | 2.3(2) | 0.334 | 2.9(4) | 0.025 | 3.3(9) | 0.121 | 3.7(7) | 0.001 |
|         |       | N | 2.4(2) | 4.133 | 2.8(4) | 0.001 | 3.0(7) | 0.344 | 3.2(7) | 0.001 |
| $A_2^{++}$ | 3 | C | 0.5(7) | ... | 1.8(5) | 0.002 | 3.5(8) | 0.052 | 4.3(10) | 0.573 |
|         |       | N | 0.7(4) | ... | 10.8(10) | 0.005 | 3.2(8) | 0.017 | 4.4(11) | 0.912 |
| $A_2^{+}$ | 3 | C | 1.3(4) | 0.230 | 1.4(7) | 0.672 | 1.1(6) | 0.072 | 1.1(8) | 0.113 |
|         |       | N | 1.5(5) | 0.376 | 1.2(4) | 0.088 | 1.2(4) | 0.664 | 1.0(6) | 0.009 |

$^a$Continuum extrapolation excluded $\beta = 5.85$ data point.

which are found to contribute to each extrapolated state, written in terms of their local quark and extended gauge field content from which they are constructed. The nature of our operators is such that only one irrep is found in any such product as shown in Table[XI] so this labeling uniquely identifies the operator.

To quantify the significance of an operator’s diagonal correlator to a given channel two additional pieces of information are given in Table[XVI]. For each such operator the number of times it was found contributing to the state out of the twelve possible lattice spacing and quark mass combinations that were fit is given. In some cases a small contribution may not be statistically significant and as such the fitting algorithm may not to give a coefficient for it. Secondly, the table provides the largest magnitude of the coefficient $Z$ found for that state, among those fits that had a coefficient. These latter values are scaled by their error so as to give a measure of their significance. The table is organized to give the most significant operators contributing to a channel first in the list.

Several observations may be made from Table[XVI]. For one, operators with a non-scalar gauge structure still contribute to ground states. For the $\rho$ meson ($T_1^{-+}$ ground state) the greatest contributions come from those operators which have a quark structure of $T_1^{-+}$, either a local operator or with an extended $A_1^{++}$ gauge field. However, as can be seen in the table, one also has a signif-
TABLE XV: Weighted average of the chirally extrapolated $\beta = 5.85$ and $\beta = 6.0$ lattice spacing results of each resolved channel labeled by its irrep $(\Lambda^{PC})$ and its type (charged or neutral) are given. States designated by an asterisk indicate a higher energy state in the same channel. The inferred continuum quantum number $J$ of the state is also given. We have omitted the pion ($\pi^{\pm}$) from the table. All energies are in GeV.

| Channel $A_{1}^{++}$ | Irrep(s) $J$ | Charged $\pm$ | Neutral $\mp$ |
|----------------------|-------------|---------------|---------------|
| $A_{1}^{++}$         | 0           | 1.35(19)      | 1.12(3)       |
| $A_{1}^{+-}$         | 0           | 3.6(2)        | 2.9(3)        |
| $A_{1}^{-+}$         | 0           | 1.96(18)$^a$  | 1.63(8)       |
| $T_{1}^{++}$         | 1           | 1.12(3)       | 1.39(14)      |
| $T_{1}^{+-}$         | 1           | 2.4(2)        | 2.5(2)        |
| $T_{1}^{-+}$         | 1           | 1.44(2)       | 1.37(4)       |
| $T_{1}^{--}$         | 1           | 2.7(2)        | 2.9(3)        |
| $T_{1}^{1+}$         | 1           | 1.98(13)      | 1.98(16)      |
| $T_{1}^{0-}$         | 1           | 0.765(12)     | 0.812(12)$^a$ |
| $(E, T_{2})^{++}$    | 2           | 1.42(6)       | 1.63(10)$^a$  |
| $(E, T_{2})^{+-}$    | 2           | 0.61(15)      | 0.53(13)      |
| $(E, T_{2})^{-+}$    | 2           | 2.9(3)        | 2.6(3)        |
| $(E, T_{2})^{--}$    | 2           | 2.8(3)        | 3.0(3)        |
| $A_{2}^{++}$         | 3           | 1.3(4)        | 0.7(4)        |
| $A_{2}^{0-}$         | 3           | 1.3(4)        | 1.3(3)        |

$^a$Excluded $\beta = 5.85$ data point.

significant overlap with states of the form $A_{1}^{++} \otimes T_{1}^{--}$ in which the gauge field is actually providing the vector nature of the state.\(^{15}\) The excited state $T_{1}^{--}$ shows, on the other hand, a much greater relative contribution from the vector gauge field operators, but still dominant contributions from the vector quark operators. For the ground states $T_{1}^{++}$ and $T_{1}^{--}$ the operators with quark vector structure dominate, however for their excited $T_{1}^{++}$ and $T_{1}^{--}$ states it is seen that a greater contribution comes from the operators with vector gauge field structure. This observation is an example of how lattice studies of the spectrum can inform discussion of the distinction between hybrid vector mesons and conventional vector mesons which is made in models like the flux-tube model.\(^{11}\)

Table XVI also grants further insight into the geometry of related states. In the constituent quark model, for instance, the $a_{0}, a_{1},$ and $b_{1}$ states belong to a common $P$-wave and hence are considered to share common angular momentum.\(^{8}\) If one looks at the main contribution to these states in their corresponding octahedral channels, $A_{1}^{++}, T_{1}^{++},$ and $T_{1}^{--}$ respectively, it is observed that after the local operators the main contributor in each of these channels is of the form $X^{PC} \otimes T_{1}^{+-}$, where $X^{PC}$ is whatever local spin structure is required to produce the correct channel. This suggests that just as common spatial angular momenta ($L = 1$) are traditionally considered to unite these channels, so too does similar gauge field structure.

For those states we identified as $J = 2$ with our common fits to $E$ and $T_{2}$ channels, Table XVII shows further affirmation of this identification. In the continuum, five $J = 2$ operators can be formed with the coupling of a $J = 1$ quark and $J = 1$ gauge field structure. On the lattice these five states still appear in a coupling of $T_{1}$ quark and $T_{1}$ gauge field, but now in the subduction of $J = 2$ as a two-dimensional $E$ irrep and a three-dimensional $T_{2}$ irrep. As such one expects these $E$ and $T_{2}$ operators produced via $T_{1} \otimes T_{1}$ corresponding to an actual $J = 2$ state to have similar properties.\(^{16}\) A consideration of our eight $(E, T_{2})$ common fits shows that the contributions of such $E$ operators (denoted by $T_{1} \otimes T_{1}$) have comparable counts and significance to their related $T_{2}$ operators (denoted by $T_{1} \otimes T_{1}$) in all cases. Only in the comparably weak $2^{+-}$ ground state fit, and there only in the neutral channel, is the pattern less obvious. For all these states the importance of the gluonic degrees of freedom is readily apparent, in contradistinction to the limitations imposed by a simple quark model for two local quarks.

VII. CONCLUSION

Simulations with twisted mass lattice QCD (tmLQCD) have been used to explore the spectrum of mesons having all possible values of angular momentum, spin, and parity. Numerical results are consistent with those obtained by other authors using other lattice actions. As seen in Figure 10, we are still far from seeing the number of mesons claimed experimentally, and uncertainties are still large, but a viable methodology has been put into place. In particular, questions related to the appropriateness of tmLQCD for this physics can now be answered, the usefulness of operators containing local quark and anti-quark fields can be evaluated, and the valuable qualities of an evolutionary fitting method can be confirmed.

Because tmLQCD breaks the parity symmetry of the strong interactions, it is not clear a priori whether it can be used to determine the spectrum of hadrons having definite quantum numbers $J^{PC}$, which appear as $\Lambda^{PC}$ on a discrete lattice. In the case of both isovector and isoscalar mesons, we have identified orthogonality relations among certain operators which are valid in tmLQCD for any chosen twist angle. We have also emphasized that some states cannot be separated, such as $\Lambda^{++}, \Lambda^{--}$ in the

\(^{15}\) For the ground state pion ($\pi^{\pm}$), not shown in Table XVI it is to be remarked that there is similarly a clear ground state signal from the $T_{1}^{--} \otimes T_{1}^{--}$ operators.

\(^{16}\) However, since one is row averaging over three operators in the $T_{2}$ case compared to two in the $E$ case the correspondence is not expected to be identical due to differing statistics.
isovector case and $\Lambda^{++, \Lambda^{-}}$ in the isoscalar case. Nevertheless, simultaneous fits to multiple correlators allow the complete spectrum to be obtained (as proposed in Ref. [3]), and comparison of the isovector spectrum with its isoscalar counterpart serves to clarify the PC quantum numbers of the states. During this work, simulations with the Wilson action were performed also in order to confirm these claims.

Mesons that require only one quark propagator, i.e. operators with a common local source for mass-degenerate quark and anti-quark, minimize computational expense. Connecting the quark and anti-quark to spatially extended gauge fields permitted all possible meson quantum numbers, but how well do these operators overlap with the physical states of QCD? Our numerical explorations produced signals for both conventional and exotic mesons. The strength of each operator’s overlap was tabulated, so that future studies can make informed choices of operators for the various meson channels. Gauge field smearing was included at source and sink, while quark field smearing was performed only at the sink.

The use of an evolutionary fitting algorithm avoided the danger of human bias during data analysis. The algorithm was able to fit multiple correlators simultaneously, with some states shared (or not) across data sets. The algorithm itself determined how many states were present based on statistical significance, and all time steps beyond the source were included in all fits. Survival of the fittest was defined to mean minimizing $\chi^2/n_{\text{dof}}$. The algorithm even served to alert us to a case of insufficient statistics in one ensemble of lattices; this information appeared as larger $\chi^2/n_{\text{dof}}$ values output from the fit, which could only be reduced toward unity by increased statistics.

Based on this work, we conclude that a detailed study of the full meson spectrum using tmLQCD is feasible. However, to obtain precise results, substantially more configurations would be required than we have used, and additional classes of operators should be considered.

Quark smearing at the source, to match the sink, should improve the signal-to-noise ratio. With more precise meson mass determinations at a few $\beta$ values, one could see directly whether the continuum extrapolation of tmLQCD offers a significant advantage over other lattice actions, commensurate with the drawback of tmLQCD’s lack of parity conservation. It would also be interesting to consider the viability of using tmLQCD for a thorough study of the baryon spectrum.

### APPENDIX A: OPERATOR CONTRIBUTION

The operators contributing to each extrapolated state are listed in Table XVII. Operators are identified by the direct product of quark local irrep and extended gauge field irrep from which they are projected, $\Lambda^{\alpha \mu \nu \sigma} P_\mu C_\nu \otimes \Lambda^{\mu \nu} C$. Operators without an extended gauge structure given are local quark operators. Quark irreps from the second multiplicity have a superscript 2. In the case of the $(E,T_2)$ fits we designate $T_2$ operators with an $\otimes$ and $E$ operators with $\otimes$. In the $A_2$ fits where both smeared and unsmeared correlators were fit the unsmeared correlators are distinguished similarly with $\otimes$. After each operator the number of lattice spacing-quark mass channels in which a coefficient was identified for the state is given (maximum 12) followed by the maximum value of the coefficient $Z$ divided by its error over those channels.

| Channel | Irrep(s) | C/N | Contributing Operators |
|---------|----------|-----|------------------------|
| $A_1^{++}$ | $C_{A_1^{++}}(12, 9.52)$, $C_{A_1^{--}}(12, 9.46)$, $C_{T_1^{--}}(7.3, 9.7)$, $T_1^{--} \otimes T_1^{-}(2, 2.49)$ | | |
| $A_1^{--}$ | $C_{A_1^{--}}(12, 9.48)$, $C_{A_1^{--}}(12, 9.48)$, $C_{T_1^{--}}(9.434)$, $T_1^{--} \otimes T_1^{-}(3.92)$ | | |
| $A_1^{-+}$ | $C_{A_1^{-+}}(12, 9.25)$, $C_{A_1^{-+}}(12, 9.25)$, $C_{T_1^{-+}}(10.74)$, $T_1^{-+} \otimes T_1^{-}(4.60)$, $T_1^{-} \otimes T_1^{-}(4, 4.11)$ | | |
| $A_1^{-+}$ | $C_{A_1^{-+}}(12, 9.52)$, $C_{A_1^{-+}}(12, 9.52)$, $C_{T_1^{-+}}(11.49)$, $T_1^{-+} \otimes T_1^{-}(5.31)$, $T_1^{-} \otimes T_1^{-}(2, 1.30)$ | | |
| $T_1^{++}$ | $C_{T_1^{++}}(8, 29.5)$, $C_{T_1^{++}}(8, 28.9)$, $C_{T_1^{++}}(8, 11.3)$, $T_1^{--} \otimes T_1^{-}(8, 3.57)$, $T_1^{--} \otimes T_1^{-}(9.42)$, $T_1^{--} \otimes T_1^{-}(4, 3.11)$, $T_1^{--} \otimes T_1^{-}(4, 0.92)$, $A_2^{++} \otimes A_1^{++}(3, 4.36)$ | | |
| $T_1^{-+}$ | $C_{T_1^{-+}}(12, 26.9)$, $C_{T_1^{-+}}(12, 26.9)$, $A_2^{--} \otimes A_2^{--}(10, 5.53)$, $T_2^{--} \otimes T_1^{-}(8, 3.26)$, $A_1^{--}(7, 5.58)$, $T_1^{--} \otimes T_1^{-}(6.438)$, $T_1^{--} \otimes T_1^{-}(5.508)$, $T_2^{--} \otimes T_1^{-}(4.492)$ | | |
| $T_1^{++}$ | $C_{T_1^{++}}(11, 36.8)$, $C_{T_1^{++}}(11, 36.7)$, $T_1^{--} \otimes T_1^{-}(7.247)$, $T_1^{--} \otimes T_1^{-}(7.200)$, $T_2^{--} \otimes T_1^{-}(3.214)$, $T_2^{--} \otimes T_1^{-}(3.212)$, $A_1^{--} \otimes T_1^{-}(3.132)$, $T_2^{--} \otimes T_1^{-}(2.145)$, $T_1^{--} \otimes T_1^{-}(2.139)$, $T_1^{--} \otimes T_1^{-}(2.136)$, $T_2^{--} \otimes T_1^{-}(1.424)$ | | |
| $T_1^{-+}$ | $C_{T_1^{-+}}(12, 61.8)$, $C_{T_1^{-+}}(12, 50.0)$, $T_1^{++} \otimes T_1^{-}(8, 5.43)$, $T_1^{++} \otimes T_1^{-}(7.354)$, $T_2^{--} \otimes T_1^{-}(7.169)$, $A_2^{--} \otimes T_1^{-}(6.197)$, $T_1^{--} \otimes T_2^{--}(5.416)$, $T_1^{--} \otimes T_1^{-}(5.298)$, $T_2^{--} \otimes T_1^{-}(5.080)$, $T_2^{--} \otimes T_1^{-}(3.777)$, $T_1^{--} \otimes T_1^{-}(1.457)$, $A_2^{--} \otimes T_1^{-}(1.260)$, $T_2^{--} \otimes T_1^{-}(1.240)$, $T_1^{--} \otimes T_1^{-}(1.233)$ | | |
| $T_1^{++}$ | $C_{T_1^{++}}(12, 65.6)$, $C_{T_1^{++}}(11, 6.70)$, $T_1^{++} \otimes T_1^{-}(8, 9.12)$, $T_1^{++} \otimes T_1^{-}(7.9.46)$, $T_1^{--} \otimes T_1^{-}(6.5.24)$, $T_1^{++} \otimes T_1^{-}(6.2.76)$, $T_2^{--} \otimes T_2^{--}(6, 2.55)$, $T_1^{++} \otimes T_1^{-}(6, 1.96)$, $A_2^{--} \otimes T_1^{-}(5.4.11)$, $T_1^{++} \otimes T_1^{-}(5.2.46)$, $T_2^{--} \otimes T_1^{-}(5.1.68)$, $T_1^{--} \otimes T_1^{-}(3.4.45)$ | | |
| Channel | Irrep(s) | C/N | Contributing Operators |
|---------|---------|-----|------------------------|
| $T_{1^{-}}$ | $A_{1}^{-} \otimes T_{1}^{-}$ | (12.602), $T_{1}^{-} \otimes T_{1}^{-}$ | (11.680), $T_{1}^{-} \otimes A_{1}^{++}$ | (9.379), $T_{1}^{-} \otimes A_{1}^{+}$ | (9.535), $A_{1}^{+} \otimes T_{1}^{-}$ | (8.2.24), $T_{2}^{-} \otimes T_{2}^{-}$ | (7.1.31), $T_{2}^{-} \otimes T_{2}^{-}$ | (5.2.71), $T_{1}^{-} \otimes E^{++}$ | (5.2.67), $T_{1}^{-} \otimes T_{2}^{-}$ | (5.1.86), $T_{1}^{-} \otimes T_{1}^{-}$ | (4.2.47), $T_{1}^{-} \otimes E^{++}$ | (4.1.05) |
| $T_{2}^{-}$ | $C_{1} \otimes A_{1}^{+}$ | (12.38.6), $T_{1}^{-} \otimes T_{1}^{-}$ | (12.35.3), $T_{1}^{-} \otimes T_{1}^{-}$ | (8.6.19), $A_{1}^{+} \otimes T_{1}^{-}$ | (7.2.78), $A_{1}^{+} \otimes T_{1}^{-}$ | (5.2.56), $T_{1}^{-} \otimes E^{++}$ | (5.2.41), $T_{1}^{-} \otimes T_{1}^{-}$ | (4.8.33), $T_{2}^{-} \otimes T_{1}^{-}$ | (4.1.59), $A_{1}^{+} \otimes T_{1}^{-}$ | (4.1.41), $T_{1}^{-} \otimes T_{2}^{-}$ | (2.1.64), $T_{1}^{-} \otimes T_{1}^{-}$ | (2.0.98) |
| $T_{3}^{-}$ | $A_{1}^{+} \otimes A_{1}^{+}$ | (12.29.9), $T_{1}^{-} \otimes T_{1}^{-}$ | (12.29.6), $T_{1}^{-} \otimes T_{1}^{-}$ | (8.4.67), $A_{1}^{+} \otimes T_{1}^{-}$ | (8.3.92), $T_{1}^{-} \otimes A_{1}^{+}$ | (3.3.44), $A_{1}^{+} \otimes T_{1}^{-}$ | (3.2.34), $T_{1}^{-} \otimes E^{++}$ | (3.2.30), $T_{1}^{-} \otimes T_{1}^{-}$ | (3.1.16), $A_{1}^{+} \otimes T_{1}^{-}$ | (3.1.16), $T_{1}^{-} \otimes T_{2}^{-}$ | (2.1.16), $A_{1}^{+} \otimes T_{2}^{-}$ | (2.0.97), $T_{1}^{-} \otimes T_{1}^{-}$ | (1.9.06), $T_{1}^{-} \otimes T_{2}^{-}$ | (1.0.67) |

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[1] S. Godfrey and J. Napolitano, Rev. Mod. Phys. 71, 1411 (1999), hep-ph/9811410.
[2] F. E. Close, Int. J. Mod. Phys. A17, 3239 (2002), hep-ph/0110081.
[3] E. Klempt and A. Zaitsev, Phys. Rept. 454, 1 (2007), arXiv:0708.4016 [hep-ph].
[4] R. Frezzotti and G. C. Rossi, JHEP 08, 007 (2004), hep-lat/0306014.
[5] R. Frezzotti, P. A. Grassi, S. Sint, and P. Weisz (Alpha), JHEP 08, 058 (2001), hep-lat/0101001.
[6] G. M. von Hippel, R. Lewis, and R. G. Petry, Comput. Phys. Commun. 178, 713 (2008), arXiv:0707.2788 [hep-lat].
[7] G. M. von Hippel, R. Lewis, and R. G. Petry, PoS LAT2007, 043 (2007), arXiv:0710.0014 [hep-lat].
[8] P. Boucaud et al. (ETM) (2008), arXiv:0803.0224 [hep-lat].
[9] P. Lacock, C. Michael, P. Boyle, and P. Rowland (UKQCD), Phys. Rev. D54, 6997 (1996), hep-lat/9605025.
[10] S. Basak et al., Phys. Rev. D72, 094506 (2005), hep-lat/0506029.
[11] S. Basak et al. (Lattice Hadron Physics (LHPC)), Phys. Rev. D72, 074501 (2005), hep-lat/0508018.
[12] D. Harnett, R. Lewis, and R. G. Petry, PoS LAT2006, 194 (2006), hep-lat/0609071.
[13] R. C. Johnson, Phys. Lett. B114, 147 (1982).
[14] J. E. Mandula, Phys. Lett. B135, 155 (1984).
[15] I. H. Jorysz and C. Michael, Nucl. Phys. B302, 448 (1988).
[16] C. W. Bernard et al. (MILC), Phys. Rev. D56, 7039 (1997), hep-lat/9707008.
[17] J. F. Cornwell, Group theory in physics: An introduction (Academic Press, San Diego, 1997).
[18] H.-W. Lin and S. D. Cohen (2007), arXiv:0709.1902 [hep-lat].
[19] C. Michael, Nucl. Phys. B259, 58 (1985).
[20] T. Burch, C. Gattringer, L. Y. Glozman, C. Hagen,
and C. B. Lang, Phys. Rev. D73, 017502 (2006), hep-lat/0511054.

[21] M. Luscher and U. Wolff, Nucl. Phys. B339, 222 (1990).

[22] Available from http://www.python.org/.

[23] K. Levenberg, Quart. Appl. Math. 2, 164 (1944).

[24] D. Marquardt, SIAM J. Appl. math. 11, 431 (1963).

[25] A. M. Abdel-Rehim, R. Lewis, R. M. Woloshyn, and J. M. S. Wu, Phys. Rev. D74, 014507 (2006), hep-lat/0601036.

[26] K. Jansen, A. Shindler, C. Urbach, and I. Wetzorke (XLF), Phys. Lett. B586, 432 (2004), hep-lat/0312013.

[27] A. M. Abdel-Rehim, R. Lewis, and R. M. Woloshyn, Phys. Rev. D71, 094505 (2005), hep-lat/0503007.

[28] F. Farchioni et al., Nucl. Phys. Proc. Suppl. 140, 240 (2005), hep-lat/0409098.

[29] A. M. Abdel-Rehim, R. Lewis, R. G. Petry, and R. M. Woloshyn, PoS LAT2006, 164 (2006), hep-lat/0610004.

[30] B. Efron, The Jackknife, the Bootstrap and Other Re-sampling Plans (SIAM, Philadelphia, 1982).

[31] R. G. Petry, University of Regina Ph.D. thesis (2008).

[32] W. A. Bardeen, A. Duncan, E. Eichten, N. Isgur, and H. Thacker, Phys. Rev. D65, 014509 (2001), hep-lat/0106008.

[33] W. M. Yao et al. (Particle Data Group), J. Phys. G33, 1 (2006).

[34] Z.-H. Mei and X.-Q. Luo, Int. J. Mod. Phys. A18, 5713 (2003), hep-lat/0206012.

[35] J. N. Hedditch et al., Phys. Rev. D72, 114507 (2005), hep-lat/0509106.

[36] M. S. Cook and H. R. Fiebig, Phys. Rev. D74, 094501 (2006), hep-lat/0609010.

[37] P. Lacock and K. Schilling (TXL), Nucl. Phys. Proc. Suppl. 73, 261 (1999), hep-lat/9809022.

[38] C. Bernard et al., Phys. Rev. D68, 074505 (2003), hep-lat/0301024.

[39] N. Isgur, R. Kokoski, and J. E. Paton, Phys. Rev. Lett. 54, 869 (1985).

[40] C. J. Morningstar and M. Peardon, Phys. Rev. D 56, 4043 (1997).

[41] F. E. Close (1995), hep-ph/9511442.