HOPF POWERS AND ORDERS FOR SOME BISMASH PRODUCTS

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INTRODUCTION

Let $H$ be a Hopf algebra over the field $k$. We study the $n$-th Hopf power map of $H$, a linear endomorphism $[n]: H \to H$. If $H = kG$ is a group algebra, then $[n]$ is the linear extension of the $n$-th power map on $G$. If $H$ is commutative, and thus represents an affine group scheme $G$, the $n$-th Hopf power map represents the endomorphism of the affine scheme $G$ given by taking the $n$-th power of all elements of the group $G(R)$ for each $k$-algebra $R$. In this sense the Hopf power map for commutative Hopf algebras is implicit in work of Gabriel [6]; explicitly it seems to appear first in a paper by Tate and Oort [16]. Kashina [9, 10] proposed studying the same map for general (non-commutative and non-cocommutative) Hopf algebras. If $H$ is commutative and represents the affine group scheme $G$, then the $n$-th Hopf power map for $H$ is trivial if and only if the $n$-th power map for each of the discrete groups $G(R)$ is trivial. Thus the smallest number such that $[n]$ is trivial can be viewed as a sort of joint exponent for all the groups $G(R)$, or the exponent of the algebraic group $G$. Kashina reproves a result of Gabriel [6, Prop. 8.5] using more Hopf-algebraic techniques: For an $n$-dimensional commutative Hopf algebra the $n$-th Hopf power map is trivial, i.e. the exponent of a finite group scheme over a field divides its order. On a more elementary level, the exponent of a finite group $G$ is also the least positive integer for which the $n$-th Hopf power map on the group algebra $kG$ is trivial.

For a general finite-dimensional Hopf algebra $H$, Kashina asks whether $[n]$ is trivial for some finite number $n$, preferably a divisor of $\dim H$. She proves several positive results, most notably that $[n]$ is trivial for the Drinfeld double of the group algebra of a group of exponent $n$; more generally, $[n]$ is trivial for $D(H)$ if it is for $H$ and $H^{op}$.

Etingof and Gelaki [5] mint the obvious term exponent of $H$ for the least $n$ such that $[n]$ is trivial. They point out that Kashina’s definition of $[n]$ is only suitable for involutive Hopf algebras, and define the right version for the general case. Among the various important results of [5] is that the exponent of a semisimple Hopf algebra $H$ over a field of characteristic 0 is indeed finite and divides $(\dim H)^3$, and the exponent is invariant under Drinfeld twists.

In the present paper we study the behavior of individual elements of $H$ under the Hopf power maps. For which $n$ are there nontrivial elements of $H$ whose $n$-th Hopf power is trivial? What are the possible Hopf orders of elements of $H$ (where the Hopf order of $h$ is the least positive integer $n$ such that the $n$-th power of $h$ is trivial)? Since the $n$-th Hopf power map for a group algebra $kG$ is the linearization...
of the $n$-th power map of $G$, these questions generalize (or linearize) questions about groups whose answers are well-known. Although we will see that the answers for group algebras deviate somewhat from the answers for groups, they are still quite easy and reasonable.

Before turning to more interesting examples, we propose a refinement of the above questions. The space $T_n(H)$ of all elements of $H$ whose $n$-th Hopf power is trivial is a linear subspace of $H$. The dimensions $t_n(H)$ of all these spaces $T_n(H)$, as well as the dimensions $t_{m,n}(H)$ of their pairwise intersections, are isomorphism invariants of $H$: in particular they tell us for which $n$ there exist nontrivial elements whose $n$-th power is trivial, and for which $n$ there exist elements of Hopf order $n$. If $H$ is a Kac Hopf algebra, the numbers $t_{m,n}$ have certain symmetries (apart from the obvious $t_{m,n} = t_{n,m}$).

We study the power maps, possible Hopf orders, and the dimensions $t_{m,n}$ for the (semisimple) bismash product Hopf algebras obtained from a factorizable group. Since explicit calculations for these examples are quite involved, we rely heavily on computer help; for most of our examples we use Maple to first compute the matrices of the Hopf power maps with respect to a suitable basis, and then to do computations with these matrices.

Here is a brief summary of some of our findings: A Hopf algebra $H$ can have elements whose Hopf order is a prime that does not divide the dimension of $H$. The numbers $t_{m,n}$ are not invariant under Drinfeld twists. While it is easy to check that the numbers $t_n$ are the same for $H$ and its dual $H^*$, the numbers $t_{m,n}(H)$ and $t_{m,n}(H^*)$ can be different. Also, the Hopf orders of elements that occur in $H$ can be different from the Hopf orders of elements that occur in $H^*$. (We should note at this point that the possible Hopf orders of elements of a group algebra and its dual are the same; computer experiments show that $t_{m,n}(QG) = t_{m,n}(Q^G)$ for all of the few groups that we have checked.)

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1. Definitions, and the (co)commutative case

Throughout the paper we work over a fixed base field $k$. Hopf algebras, vector spaces, and tensor products are understood to be over $k$. We use Sweedler notation in the form $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for comultiplication.

Definition 1.1. Let $H$ be a Hopf algebra. For $n \in \mathbb{Z}$, the $n$-th Hopf power map $[n] = [n]_H : H \to H$ is the $n$-th convolution power of the identity. Thus, writing $[n](h) := h^{[n]}$, we have $h^{[n]} = h_{(1)} h_{(2)} \cdots h_{(n)}$ for $n > 0$, $h^{[0]} = \varepsilon(h) 1$, and $h^{[n]} = S(h_{(1)}) S(h_{(2)}) \cdots S(h_{(-n)})$ for $n < 0$. The element $h^{[n]}$ is called the $n$-th Hopf power of $h$.

We will say that the $n$-th Hopf power of $h$ is trivial if $h^{[n]} = \varepsilon(h) 1$. The Hopf order of $h$ is the least positive integer $n$ such that the $n$-th Hopf power of $h$ is
trivial. The exponent of $H$ is the least positive integer $n$ such that the $n$-th Hopf power of every element of $H$ is trivial.

Etingof and Gelaki [5] have shown that the above definition of Hopf powers and the exponent is suitable only for involutive Hopf algebras. For the general case they give a better-behaved definition involving the square of the antipode. The most interesting class of Hopf algebras for the investigation of Hopf powers and the exponent still seems to be that of semisimple Hopf algebras over a field of characteristic zero. We will therefore stick to the above definition, even though we will not always make those assumptions on the Hopf algebras in consideration.

Most results will require the Hopf algebra $H$ to have finite exponent.

Remark 1.2. For all $m, n \in \mathbb{Z}$ we have $h^{[m+n]} = h^{(1)}[m] h^{(2)}[n]$, and $S(h^{[n]}) = S(h)^{[n]}$.

In particular, $h^{[n+ek]} = h^{[n]}$ for all $n, k \in \mathbb{Z}$, if $e$ denotes the exponent of $H$.

Definition 1.3. Let $H$ be a Hopf algebra. The $n$-th trivial power space of $H$ is

$$T_n := T_n(H) := \{ h \in H | h^{[n]} = \varepsilon(h)1 \} = \ker([n] - \eta \varepsilon).$$

The $n$-th trivial power dimension of $H$ is $t_n := t_n(H) := \dim T_n(H)$. We put $T_{m,n} := T_{m,n}(H) := T_m \cap T_n$, and $t_{m,n} := t_{m,n}(H) := \dim(T_{m,n})$.

Since $1^{[n]} = 1 = \varepsilon(1) \cdot 1$ for all $n$, all spaces $T_{m,n}$ and all dimensions $t_{m,n}$ are nonzero. Nontrivial elements with trivial $n$-th Hopf power exist whenever $t_n > 1$. Over an infinite field, the numbers $t_{m,n}$ also determine whether elements of a given Hopf order exist:

**Proposition 1.4.** Assume that $H$ is a finite dimensional Hopf algebra over an infinite base field $k$. Then the following are equivalent

1. $H$ contains an element of Hopf order $n$.
2. $T_n(H) \not\subset T_m(H)$ for all $m < n$.
3. $t_n(H) > t_{m,n}(H)$ for all $m < n$.

**Proof.** By definition there is an element of Hopf order $n$ if and only if $T_n(H)$ contains an element not in $\bigcup_{m<n} T_m(H)$, or if $T_n(H)$ properly contains $\bigcup_{m<n}(T_m(H) \cap T_n(H))$. But a vector space over $k$ cannot be written as the union of finitely many proper subspaces (see for example [7, Sec. 4.2, Ex. 21]), so this is equivalent to the condition that $T_n(H)$ properly contains each of the spaces $T_m(H) \cap T_n(H)$ for $m < n$. This is equivalent on one hand to the proper inequality of dimensions, and on the other hand to $T_n(H)$ not being contained in $T_m(H)$. \hfill \qed

**Corollary 1.5.** Let $H$ be a Hopf algebra with finite exponent $e$ over an infinite base field $k$. Then $H$ contains an element of Hopf order $e$.

**Proof.** By definition, $e$ is the least integer such that $H = T_e(H)$. If there is no element of Hopf order $e$, the preceding result shows that there is $m < e$ with $H = T_e(H) \subset T_m(H) \subset H$, a contradiction. \hfill \qed

If $H$ is semisimple over the complex numbers, then Kashina, Sommerhäuser, and Zhu in fact show that an integral in $H$ has Hopf order $e$, see the end of Section 3 in [11]. This is easy to check explicitly for a group algebra $kG$, and also for its dual (cf. Example 1.13 and Example 1.15 (3))

In the rest of this section we collect the facts on Hopf powers and the trivial power spaces for commutative or cocommutative Hopf algebras and compare to the more familiar notions for groups.
Definition 1.6. Let $H$ be a Hopf algebra. We will say that $H$ satisfies the power rule if

\[(h^m)^n = h^{mn}\]

holds for all $h \in H$, $m, n \in \mathbb{Z}$.

Lemma 1.7. Any cocommutative or commutative Hopf algebra satisfies the power rule.

This observation was made by Tate and Oort [16] for commutative Hopf algebras. A proof for positive exponents avoiding the language of group schemes is in Kashina’s paper [10]. Negative exponents are also easy to cover, more so perhaps after Remark 1.8 below.

Clearly the power rule also holds for the tensor product of two Hopf algebras that satisfy the power rule. It also carries over to dual Hopf algebras, quotient Hopf algebras, and Hopf subalgebras. However, we do not know of any more meaningful examples. We will still make some statements below for Hopf algebras satisfying the power rule, instead of just for cocommutative or commutative ones, simply because there is no additional difficulty. We also note that some proofs require only the power rule for positive exponents, but we have no example where this is satisfied, but the power rule for negative exponents is not.

Remark 1.8. (1) Since $h^{-1} = S(h)$, the power rule (1.1) specializes to $h^{-n} = S(h^n)$ for all $n \in \mathbb{Z}$ when we set $m = -1$. If we also specialize $n = -1$, we get $S^2(h) = h$, so a Hopf algebra satisfying the power rule is involutive. This should be read with a grain of salt, however: We have already remarked that the definition of Hopf powers that we use only fits with the definition of exponent given by Etingof and Gelaki if $H$ is involutive to begin with.

(2) Conversely, it is easy to check that the power rule for positive exponents together with the requirement that $h^{-n} = S(h^n)$ for all $n \in \mathbb{N}$ and $S^2 = \text{id}$ implies the power rule for general exponents.

(3) Let $H$ be a Hopf algebra satisfying the power rule. If the $n$-th Hopf power of $h \in H$ is trivial, and $m$ is a multiple of $n$, then the $m$-th Hopf power of $h$ is trivial.

Remark 1.9. If $H$ is finite-dimensional and satisfies the power rule, hence is involutive, then it has finite exponent. This follows from more general and rather deeper results of Etingof and Gelaki. In the special case where $H$ is commutative (or cocommutative) it was shown by Gabriel [6]. If char $k = 0$, then by results of Larson and Radford [12] $H$ is semisimple and cosemisimple, and Etingof and Gelaki [5, Thm.4.3] show that $H$ has finite exponent, which in fact divides $\text{dim}(H)^3$. On the other hand, if $k$ has finite characteristic, then another result of Etingof and Gelaki [5, Cor.4.10] shows that $H$ has finite exponent (in the same sense as in our paper, since $H$ is involutive).

Proposition 1.10. Let $H$ be a Hopf algebra with finite exponent $e = \text{exp}(H)$ which satisfies the power rule. Then

(1) $t_n(H) = 1$ if and only if $\gcd(n, e) = 1$.

(2) There is $h \in H$ of Hopf order $n$ if and only if $n|e$.

(3) $T_{e-n}(H) = T_n(H)$ for $0 < n < e$.

(4) If $H$ is finite-dimensional, then $t_{e-n}(H) = t_n(H)$, and $t_{m,e-n}(H) = t_{m,n}(H) = t_{e-m,e-n}(H)$ for $0 < m, n < e$. 

Proof. Let $h \in H$. We show that if the $n$-th Hopf power of $h$ is trivial, then so is the $k$-th Hopf power, where $k = (e, n)$ is the greatest common divisor of $n$ and $e = \exp(H)$. Write $k = na + eb$ for some $a, b \in \mathbb{Z}$. Then $h^k = h^{na+eb} = h^{na} = \varepsilon(h)1$. This shows that $T_n(H)$ can only contain nontrivial elements if $(n, e) \neq 1$, and that the Hopf order of $h$ divides $e$.

Now let $n|e$. We first show that there is a nontrivial element whose $n$-th Hopf power is trivial. Assume otherwise, that is, assume that for every $h \in H$, $h^{[n]} = \varepsilon(h)1$ implies $h = \varepsilon(h)1$. Then $\varepsilon(h)1 = h^{[e]} = (h^{[e/n]})^{[n]}$ implies that $h^{[e/n]} = \varepsilon(h)1$ for all $h \in H$, contradicting the definition of the exponent. Now assume that there is no element of Hopf order $n$. This means that there is $m < n$ with $T_n(H) \subset T_m(H)$. But then $h^{[e/n]} \in T_n(H)$ for all $h \in H$, so $\varepsilon(h)1 = (h^{[e/n]})^{[m]} = h^{[em/n]}$ for all $h \in H$, contradicting once more the definition of the exponent. We have shown the other implication in (2). The missing implication in (1) follows.

For (3) let $h \in T_{e-n}$. Then $\varepsilon(h)1 = h^{[n]} = h^{[n-e]} = S(h^{[e-n]})$ and hence $h \in T_{e-n}$. Clearly (4) follows from (3).

For a more detailed comparison to the group case we make the following definition:

**Definition 1.11.** Let $G$ be a finite group. The $n$-th trivial power set of $G$ is

$$T'_n := T'_n(G) = \{g \in G|g^n = 1\}.$$ 

The $n$-th trivial power number of $G$ is $t'_n = |T'_n|$.

**Remark 1.12.** Let $G$ be a finite group with exponent $e = \exp(G)$, and $n \in \mathbb{N}$.

1. $T'_n = \{1\}$ if and only if $(n, e) = 1$.
2. $G$ contains an element of order $n$ if and only if $T'_n \nsubseteq \bigcup_{1 < m < n} T'_m$.
3. $T'_n = T'_{e-n}$, where $e = \exp(G)$. In particular $t'_n = t'_{e-n}$.
4. $T'_n \cap T'_m = T'_{\gcd(m, n)}$ for $m, n \in \mathbb{N}$. In particular, $g^m = 1$ if and only if the order of $g$ divides $m$.

The first property is exactly parallel to the first property in Proposition 1.10, though we will elaborate some on the difference between $T_n(kG)$ and $T'_n(G)$ below. The second statement in Proposition 1.10 is of course simply false for orders of group elements. The third property, which, again, parallels Proposition 1.10, perhaps deserves a short proof for comparison: If $g^n = 1$, then $g^{e-n} = g^e(g^n)^{-1} = 1$.

The fourth property is (standard and) easy to check just using the standard rules for taking powers in a group (write $\gcd(m, n) = rm + sn$, and conclude that for $g \in T'_n \cap T'_m$ we have $g^{\gcd(m,n)} = g^{rm+sn} = (g^m)^r (g^n)^s = 1$), but its obvious Hopf analog does not hold even for group algebras, as we will see below.

**Example 1.13.** Let $G$ be a finite group. Then $T'_n(G) \subset T_n(kG)$. Also, if $x, y \in G$ satisfy $x^n = y^n$, then $x - y \in T_n(kG)$. More generally, consider $h = \sum_{x \in G} \alpha_x x \in kG$. Then $h^{[n]} = \sum_{x \in G} \alpha_x x^n = \sum_{g \in G} \left(\sum_{x^n = g} \alpha_x\right) g$, and thus $h \in T_n$ if and only if the coefficients $\alpha_x \in k$ satisfy the equations $\sum_{x^n = g} \alpha_x = 0$ for all $g \in G \setminus \{1\}$ (or for all $g \in G \setminus \{1\}$ that are $n$-th powers to begin with). In particular, the space $T_n(kG)$ is spanned by $T'_n(G)$ along with the differences $x - y$ with $x^n = y^n$, or by these differences along with the neutral element 1. For any group $G$, the integral $\sum_{g \in G} g$ in the Hopf algebra $kG$ has Hopf order $\exp(kG)$.
Proof. We need only prove the last assertion.

Note that in a finite abelian group $A$, every element of $A$ is an $m$-th power if $m$ is prime to the order of $A$. Indeed, in the abelian case the map $A \ni a \mapsto a^m \in A$ is a group homomorphism, and if $g$ is in its Kernel, that is, $g^m = e$, then $m$ is a multiple of the order of $g$, which divides the order of $A$.

Now choose $g \in G$ of order $p^k$, where $k$ is maximal. Then $p_g$ has Hopf order $p$. In fact $g$ is not a $p$-th power by maximality of $k$. On the other hand, any $m < p$ is prime to $p$, and so $g$ is an $m$-th power even in the subgroup of order $p^k$ generated by $g$.

More generally, for any $1 \leq n \leq k$, the element $g^{p^{n-1}} \in G$ has order $p^{k-n+1}$, hence is not a $p^n$-th power in $G$ by maximality of $k$. Let $1 < m < p^n$. If $m$ is a power of $p$ then $g^{p^{n-1}}$ is obviously an $m$-th power of a power of $g$. Otherwise write $m = p^t \ell$ with $\ell$ prime to $p$ and $0 < t < n$. Then $g$ is an $\ell$-th power in the subgroup of order $p^k$ generated by $g$, say $g = h^\ell$, and hence $g^{p^{n-1}} = h^{\ell p^{n-1}} = (h^{p^{n-1-t}})^m$. \medskip

Before discussing a few examples, we will make some observations on general finite-dimensional Hopf algebras. From the descriptions of $T_n(k^G)$ and $T_n(kG)$ obtained in Example 1.13 and Example 1.15 (1), it is easy to check that these

Example 1.14. (1) In the symmetric group $S_4$, the elements $(1234)$ and $(1432)$, each of order 4, have the same square $(13)(24)$, so their difference has Hopf order 2. The same holds for $(1243) - (1342)$ and $(1423) - (1324)$. It is not hard to check that $\dim T_2(kS_4) = |T_2'(S_4)| + 3$.

(2) In the symmetric group $S_5$, consider the elements

\[
\begin{align*}
x &= (12)(345) \\
y &= (345) \\
a &= (12)(354) \\
b &= (354).
\end{align*}
\]

Since $x^2 = y^2 = (354)$, and $a^2 = b^2 = (354)$, the Hopf order of $h := x - y + b - a \in kS_5$ is 2. On the other hand $x^3 = a^3 = (12)$ and $y^3 = b^3 = \text{id}$, and thus also $h^{[3]} = 0 = \varepsilon(h)1$. In particular $t_{2,3}(kS_5)$

Example 1.15. Let $G$ be a finite group. Consider the Hopf algebra $H = k^G$ of $k$-valued functions on $G$, the dual of the group algebra $kG$. Let $p_g$ for $g \in G$ denote the elements of the canonical basis of $k^G$.

(1) Since $h_{(1)}(x)h_{(2)}(y) = h(xy)$ and $(hh')(x) = h(x)h'(x)$ for $h, h' \in H$ and $x, y \in G$, we have $h[j](x) = (h_{(1)}\cdots h_{(n)})(x) = h_{(1)}(x)\cdots h_{(n)}(x) = h(x^n)$. Since on the other hand $(\varepsilon(h)1)(x) = h(1)$, we see that the $n$-th Hopf power of $h \in k^G$ is trivial if and only if $h(x^n) = h(e)$ for all $x \in G$. In other words, $T_n(k^G)$ consists precisely of those functions that are constant on the set of all $n$-th powers in $G$.

(2) Let $1 \neq g \in G$ and $n \in \mathbb{N}$. Then $(p_g)^n = 0 = \varepsilon(p_g)1$ if and only if $g$ is not an $n$-th power in $G$. In particular, the Hopf order of $p_g$ is the least positive integer such that $g$ is not an $n$-th power in $G$.

(3) The $n$-th Hopf power of $p_1$ is trivial if and only if no element besides 1 is an $n$-th power in $G$, in other words, if and only if all $n$-th powers equal 1.

Thus the Hopf order of the integral $p_1$ of $k^G$ is the exponent of $G$.

(4) Let $p$ be a prime divisor of the order of $G$. Then there is $g \in G$ such that $p_g$ has Hopf order $p$. More generally, if $G$ contains an element of order $p^k$, then there is $g \in G$ such that $p_g$ has Hopf order $p^k$. 

\[\square\]
spaces have the same dimension. More generally, \( T_n(H) \) and \( T_n(H^*) \) have the same dimension for any finite-dimensional Hopf algebra \( H \), since these spaces are the kernels of an endomorphism of \( H \) and its dual map. This is (3) of the following Lemma, and also follows directly from (4), which is based on a closer analysis of the spaces \( T_n(H) \).

**Lemma 1.16.** Let \( H \) be a finite-dimensional Hopf algebra.

1. \( T_n(H) = k \cdot 1_H \oplus \text{Ker}(\lfloor n \rfloor) \).
2. \( t_n(H) = \dim(H) + 1 - \text{rank}(\lfloor n \rfloor) \).
3. \( t_n(H) = t_n(H^*) \).
4. If \( K \) is another finite-dimensional Hopf algebra, then

\[
t_n(H \otimes K) = t_n(H \otimes K^*)
= (t_n(H) - 1) \dim(K) + (t_n(K) - 1) \dim(H) + 1 - (t_n(H) - 1)(t_n(K) - 1)
\]

**Proof.** Since \( [n](1) = 1 \) and \( \varepsilon[n] = \varepsilon \), the \( n \)-th power map preserves the direct sum decomposition \( H = k \cdot 1_H \oplus \text{Ker}(\varepsilon) \), and it is the identity on the first summand. This implies (1) and (2). Since \( [n]_{H \otimes K} = [n]_H \otimes [n]_K \), and \( [n]_{K^*} = ([n]_K)^* \), the formula in (4) follows by substituting the rank formula \( \text{rank}(f \otimes g) = \text{rank}(f) \text{rank}(g) \) for homomorphisms \( f, g \).

**Remark 1.17.** In the examples we have computed, not only \( t_n(kG) = t_n(kG^*) \) and \( t_n(kG \otimes kF) = t_n(kG^* \otimes kF) \) for finite groups \( F, G \), which is proved in greater generality in the preceding Lemma, but also \( t_{m,n}(kG) = t_{m,n}(kG^*) \) and \( t_{m,n}(kG \otimes kF) = t_{m,n}(kG^* \otimes kF) \). We do not know if this holds for all finite groups \( F, G \).

We will now list the numbers \( t_{i,j} \) for a few group algebras, and the trivial power numbers of the respective group for comparison. We begin with the symmetric group \( S_3 \) and the alternating group \( A_4 \), and discuss at the same time the format of tables we are also planning to use for more general examples of Hopf algebras. At the bottom of each of the tables (\( t_{i,j}(\mathbb{Q}G) \)) we have attached a list of the numbers \( t_i'(G) \). This of course only makes sense for group algebras. We have only printed the upper triangular part of the tables (\( t_{i,j}(H) \)); for any Hopf algebra \( H \) the obvious symmetry \( t_{m,n}(H) = t_{n,m}(H) \) makes the lower part redundant. Any rows or columns beyond the \( e \)-th, where \( e \) is the exponent (in this case, \( e = 6 \) for both groups) would be redundant since \( T_{n+e}(H) = T_n(H) \) by Remark 1.2. The \( e \)-th row and column are also redundant because \( T_e(H) = H \) and hence \( t_{e,n}(H) = t_n(H) = t_{n,n}(H) \); that is, the \( e \)-th column as well as the diagonal contains the numbers \( t_n(H) \). We have kept the first row and \((e - 1)\)-st column, which

\[
\begin{array}{cccccc}
 i,j (\mathbb{Q}S_3) & j = 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 4 & 1 & 4 & 1 & 1 \\
3 & 3 & 1 & 1 & 1 & 1 \\
4 & 3 & 1 & 1 & 1 & 1 \\
5 & 1 & 1 & 1 & 1 & 1
\end{array}
\quad
\begin{array}{cccccc}
 i,j (\mathbb{Q}A_4) & j = 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 4 & 1 & 4 & 1 & 1 \\
3 & 3 & 1 & 1 & 1 & 1 \\
4 & 3 & 1 & 1 & 1 & 1 \\
5 & 1 & 1 & 1 & 1 & 1
\end{array}
\]

**Table 1**
Table 2

| $t_{i,j}(\mathbb{Q}S_4)$ | $j = 1$ | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|---------------------------|---------|----|----|----|----|----|----|----|----|----|----|
| $i = 1$                   | 1 1 1 1 | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
| 2                         | 13 1 1 1 | 13 1 | 13 1 | 13 1 | 13 1 |
| 3                         | 9 1 1 1 9 1 1 9 1 1 |
| 4                         | 16 1 13 1 16 1 13 1 |
| 5                         | 1 1 1 1 1 1 1 1 1 1 |
| 6                         | 21 1 13 9 13 1 |
| 7                         | 1 1 1 1 1 |
| 8                         | 16 1 13 1 |
| 9                         | 9 1 1 |
| 10                        | 13 1 |
| 11                        | 1 |

will contain only 1’s for any finite-dimensional Hopf algebra: The first row because the first Hopf power is the identity, the last column because the $(e - 1)$-st Hopf power map is the antipode. For group algebras we have the additional symmetries in Proposition 1.10, (4) that would allow us to further reduce the table; one reason we didn’t do this is that only some of these symmetries will continue to hold in interesting more general examples. The other is that the table in the above form gives a convenient recipe to look up the numbers $n$ for which elements of Hopf order $n$ exist. We will return to this below.

Note that in the above examples $t'_i(G) = t_i(\mathbb{Q}G)$. We already know from Example 1.14 (1) that this will fail for $G = S_4$. Table 1 lists the numbers for this case. For each of the groups $G = S_3, A_4, S_4$ we see that $t_{i,j}(kG) = t_{\gcd(i,j)}(kG)$ for all $i, j$. This looks like a variant of the equation $|T'_i(G) \cap T'_j(G)| = t'_{\gcd(i,j)}(G)$ that follows from Remark 1.12 (4), and is the reason that we only print the numbers $t'_i(G)$ for a group, rather than a full square table of numbers. The observation on group algebras is misleading, however, as we have seen in Example 1.14 (2).

As we mentioned above, it is easy to read off from the tables for which numbers $n$ elements of Hopf order $n$ exist. We will spell out the recipe, which rephrases Proposition 1.4, for later reference, and we will illustrate it with the example $kS_4$, although we know the result here without looking from Proposition 1.10 (3).

Remark 1.18. For a finite dimensional Hopf algebra $H$ with finite exponent $e$ consider the upper triangular table $(t_{i,j}(H))$ for $1 \leq j \leq i < e$. The Hopf algebra $H$ contains a nontrivial element whose $n$-th power is trivial if and only if the $n$-th diagonal entry is greater than 1. It contains an element of Hopf order $n$ if and only if that diagonal entry is strictly larger than all the entries above it.

For example, we see that $\mathbb{Q}S_4$ contains nontrivial elements whose tenth Hopf power is trivial, since in Table 1 the tenth diagonal entry is 13. But there is no element of Hopf order 10, since the same number 13 appears several times above the diagonal in the tenth column. By contrast, there do exist elements of Hopf order six, since the sixth diagonal element 21 is strictly larger than all the numbers in the column above it.
2. Doubles and twists

Let $H$ be a bialgebra, and $\theta \in H \otimes H$ an invertible element. If $\theta$ satisfies the cocycle identity

$$ (\theta \otimes 1) \cdot (\Delta \otimes H)(\theta) = (1 \otimes \theta) \cdot (H \otimes \Delta(\theta)), $$

then $\Delta_\theta(h) = \theta \Delta(h) \theta^{-1}$ defines a new comultiplication making the algebra $H$ into a bialgebra $H_\theta$, called a Drinfeld twist [4] of $H$. If $(\varepsilon \otimes H)(\theta) = (H \otimes \varepsilon)(\theta) = 1$, the counit of $H_\theta$ is that of $H$. If $H$ is a Hopf algebra, so is $H_\theta$.

The dual construction was studied by Doi [2]: If a convolution invertible map $\sigma: H \otimes H \to k$ is a two-cocycle, that is, satisfies the identity

$$ \sigma(f(1) \otimes g(1))\sigma(f(2)g(2) \otimes h) = \sigma(g(1) \otimes h(1))\sigma(f \otimes g(2)h(2)) $$

for all $f, g, h \in H$, then

$$ g \cdot h := \sigma(g(1) \otimes h(1))g(2)h(2)\sigma^{-1}(g(3) \otimes h(3)) $$

defines a new multiplication making the coalgebra $H$ into a bialgebra $H^\sigma$, which we call a cocycle twist of $H$. If $\sigma(1 \otimes h) = \sigma(h \otimes 1) = \varepsilon(h)$ for all $h$, then $1_H$ is the unit of $H^\sigma$. If $H$ is a Hopf algebra, so is $H^\sigma$.

Etingof and Gelaki [5] show that their improved version of exponent of a finite-dimensional Hopf algebra is invariant under Drinfeld twists (and cocycle twists, since it is invariant under taking the dual). They also show that the exponent is invariant under the operation of taking the Drinfeld double; this is actually a special case of the result on cocycle twists: Doi and Takeuchi [3] have shown that the Drinfeld double can be constructed in a particularly smooth way as a cocycle twist. In this section we will start to investigate whether the trivial power dimensions $t_n(H)$, the property of a Hopf algebra to contain an element of Hopf order $n$, or the property to contain nontrivial elements whose $n$-th power is trivial, are invariant under Drinfeld twists, more particularly under those twists that construct Drinfeld doubles.

Let us recall first from [3] how certain cocycle twists including the double arise from skew pairings: A skew pairing between bialgebras $B$, $H$ is a map $\tau: B \otimes H \to k$ that satisfies

$$ \tau(bc \otimes h) = \tau(b \otimes h(1))\tau(c \otimes h(2)) $$

$$ \tau(b \otimes gh) = \tau(b(1) \otimes h)\tau(b(2) \otimes g) $$

$$ \tau(b \otimes 1) = \varepsilon(b) $$

$$ \tau(1 \otimes h) = \varepsilon(h). $$

If $\tau$ is convolution invertible, its inverse satisfies

$$ \tau^{-1}(bc \otimes h) = \tau(c \otimes h(1))\tau(b \otimes h(2)) $$

$$ \tau^{-1}(b \otimes gh) = \tau(b(1) \otimes g)\tau(b(2) \otimes h) $$

$$ \tau^{-1}(b \otimes 1) = \varepsilon(b) $$

$$ \tau^{-1}(1 \otimes h) = \varepsilon(h). $$

For an invertible skew pairing $\tau$, the map

$$ [\tau]: B \otimes H \otimes B \otimes H \ni b \otimes g \otimes c \otimes h \mapsto \varepsilon(b)\varepsilon(h)\tau(c \otimes g) \in k $$
is a two-cocycle on the tensor product bialgebra $B \otimes H$. We denote the cocycle twist by

$$B \rtimes \tau := (B \otimes H)^{[\tau]}$$

and note

$$(b \rtimes g)(c \rtimes h) = b\tau(c_{(1)} \otimes g_{(1)})c_{(2)} \rtimes g_{(2)}\tau^{-1}(c_{(3)} \otimes g_{(3)})h.$$ 

If $H$ is a finite-dimensional Hopf algebra, there is an obvious skew pairing $\tau : (H^*)^{cop} \otimes H \to k$. The Drinfeld double of $H$ can be obtained as

$$D(H) = (H^*)^{cop} \rtimes \tau.$$ 

Table 3 lists the results of our computations of the numbers $t_{m,n}(H)$, where $H$

| $t_{i,j}(D(QS_3))$ | $j = 1$ | $2$ | $3$ | $4$ | $5$ |
|---------------------|--------|-----|-----|-----|-----|
| $i = 1$             | 1      | 1   | 1   | 1   | 1   |
| $2$                 |        | 25  | 13  | 23  | 1   |
| $3$                 |        |     | 21  | 13  | 1   |
| $4$                 |        |     |     | 25  | 1   |
| $5$                 |        |     |     |     | 1   |

**Table 3**

is the Drinfeld double $D(QS_3)$. Comparing with Table 4, which does the same for

| $t_{i,j}(QS_3 \otimes QS_3)$ | $j = 1$ | $2$ | $3$ | $4$ | $5$ |
|-----------------------------|--------|-----|-----|-----|-----|
| $i = 1$                     | 1      | 1   | 1   | 1   | 1   |
| $2$                         |        | 28  | 13  | 28  | 1   |
| $3$                         |        |     | 21  | 13  | 1   |
| $4$                         |        |     |     | 28  | 1   |
| $5$                         |        |     |     |     | 1   |

**Table 4**

the tensor product $H = QS_3 \otimes QS_3$, we see that the numbers $t_n$ are not invariant under the specific cocycle twist that obtains the double from the tensor product.

Our calculations have shown that $t_{i,j}(H) = t_{i,j}(H^*)$ for $H = D(QS_3)$, so the analogous table for $D(QS_3)^*$ looks exactly like Table 3. In particular, the numbers $t_i(H)$ are not invariant under Drinfeld twists either.

Applying Remark 1.18 we see that $D(QS_3)$ has elements of Hopf order four. By Proposition 1.10 this is impossible for a Hopf algebra of exponent six satisfying the power rule — such as $QS_3 \otimes QS_3$, from which $D(QS_3)$ is obtained by a cocycle twist.

Let us also point out the inequality $t_{2,4}(D(QS_3)) = 23 < 25 = t_2(D(QS_3))$, which shows that $D(QS_3)$ contains elements whose second Hopf power is trivial, but whose fourth Hopf power is not.

Our next set of examples is based on the alternating group $A_4$. Table 5 lists the numbers $t_{i,j}(Q^{A4} \otimes QA_4)$.

Our next set of examples is based on the alternating group $A_4$. Table 5 lists the numbers $t_{i,j}(Q^{A4} \otimes QA_4)$. As in the example of $S_5$, the numbers for the cocycle twist $D(QA_4)$, listed in Table 6, are different. From Table 7 we see that $t_{i,j}(H) \neq t_{i,j}(H^*)$ for $H = D(QA_4)$. Observe, however, that the Hopf orders of elements that occur
are still the same for $H$ and $H^*$ here: In each case all Hopf orders in $\{1, \ldots, 4\}$ are possible.

Each of the tables we have seen so far is symmetric with respect to the anti-diagonal; this means we have $t_{e-m,e-n} = t_{n,m}$, or equivalently $t_{e-m,e-n} = t_{m,n}$ for all $m,n$. For Hopf algebras that satisfy the power rule this follows from the more general rule $t_{m,n} = t_{m,e-n}$; both were observed in Proposition 1.10 (4). The more general symmetry $t_{m,n} = t_{m,e-n}$ does not hold for doubles of group algebras: For example $t_{2,2}(D(QA_4)) \neq t_{2,4}(D(QA_4))$ although $4 = e - 2$. We will show, however, that the observed mirror symmetry about the anti-diagonal does hold for a large class of Hopf algebras. Of course once we prove that, part of the tables is actually redundant: We could have left out the part below, or the part above the anti-diagonal. We will make use of this for bigger tables below, but whenever possible we have left the redundant data in the picture to facilitate the procedure of looking up the numbers $n$ for which elements of Hopf order $n$ exist.

**Definition 2.1.** Let $k$ be a field with involution $k \ni x \mapsto \overline{x} \in k$. A $*$-Hopf algebra over $k$ is a Hopf algebra $H$ with a semilinear involution $* : H \to H$ on the algebra $H$ such that $\Delta : H \to H \otimes H$ and $\varepsilon : H \to k$ are homomorphisms of $*$-algebras. Semisimple $*$-Hopf algebras over the complex numbers are called Kac Hopf algebras.
Proposition 2.2. Let $H$ be a finite-dimensional Hopf algebra with finite exponent $e$, and $1 \leq n < e$.

1. $T_n(H) = T_{e-n}(H^{op})$.
2. $\exp(H^{op}) = \exp(H)$.
3. If $H$ is a $*$-Hopf algebra, then $T_n(H)$ and $T_{e-n}(H)$ correspond to each other under the involution of $H$.
4. If $H$ is a $*$-Hopf algebra, or isomorphic to its opposite Hopf algebra, then $t_n(H)$ and $t_{m,n}(H)$ for $1 \leq m, n < e$.

Proof. Let $0 \leq n \leq e$, and $h \in T_n(H)$. Then $\varepsilon(h)1 = h^{[n]} = h^{[n-e]} = S(h_{(1)}) \cdot \cdots \cdot S(h_{(e-n)})$. Applying $S^{-1}$ yields $\varepsilon(h)1 = h_{(e-n)} \cdots \cdot h_{(1)}$, and thus $T_n(H) \subset T_{e-n}(H^{op})$. In the special case $n = 0$ this means $\exp(H^{op}) \leq e = \exp(H)$. By symmetry this proves (1) and (2).

Now if we have a Hopf algebra isomorphism $f : H \to H^{op}$, then (1) implies $f(T_n(H)) = T_{e-n}(H)$, and hence the claims in (4). If (3) and the claims in (4) for $*$-Hopf algebras are not strictly a special case of this, it is only for a technical reason: The involution of $H$ is not linear. Still, if we apply the involution to $\varepsilon(h)1 = h_{(e-n)} \cdots \cdot h_{(1)}$ for $h \in T_n(H) = T_{e-n}(H^{op})$, we get $\varepsilon(h^*)1 = \varepsilon(h)1 = (h_{(e-n)} \cdots \cdot h_{(1)})^* \cdots \cdot (h_{(e-n)})^* = (h^*)_{(1)} \cdots \cdot (h^*)_{(e-n)}$, so again $(T_n(H))^* \subset T_{e-n}(H)$, and equality follows by symmetry. □

The equality in (2) was obtained for the improved version of exponent by Etingof and Gelaki [5, Cor.2.6], but with a less elementary proof. It is also proved in a different way by Kashina, Sommerh"{a}user, and Zhu [11].

Since the Drinfeld doubles of finite groups over $\mathbb{C}$ are Kac algebras, Proposition 2.2 (3) explains the mirror symmetry we observed in the tables above.

Our next example is the Drinfeld double of the symmetric group $S_4$. As in the previous examples, we can spot many Hopf orders from Table 8 which would be impossible for Hopf algebras that satisfy the power rule. However, the primes 5 and 7 are not possible Hopf orders; there do not even exist nontrivial elements whose fifth or seventh Hopf powers are trivial. Thus, for the group $G = S_4$, we see that nontrivial elements of $D(QG)$ whose $n$-th Hopf power is trivial exist only for those

| $t_{i,j}(D(QS_4))$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---------------------|---|---|---|---|---|---|---|---|---|----|----|
| 1                   | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1  |
| 2                   | 415 | 217 | 395 | 1 | 401 | 1 | 401 | 217 | 386 | 1  |
| 3                   | 313 | 241 | 1  | 305 | 1  | 241 | 304 | 217 | 1  | 1  |
| 4                   | 484 | 1   | 449 | 1  | 443 | 241 | 401 | 1  |    |    |
| 5                   | 1   | 1   | 1   | 1 | 1 | 1 | 1 | 1 |    |    |    |
| 6                   | 535 | 1   | 449 | 305 | 401 | 1 |    |    |    |    |    |
| 7                   | 1   | 1   | 1   | 1 | 1 | 1 | 1 | 1 |    |    |    |
| 8                   | 484 | 241 | 395 | 1  |    |    |    |    |    |    |    |
| 9                   | 313 | 217 | 1   |    |    |    |    |    |    |    |    |
| 10                  | 415 | 1   |    |    |    |    |    |    |    |    |    |
| 11                  | 1   |    |    |    |    |    |    |    |    |    |    |
n where the tensor product $\mathbb{Q}^G \otimes \mathbb{Q}G$, or, for that matter, the group algebra $\mathbb{Q}G$, already contains such elements.

This limited version of an “invariance” of the behavior of trivial power dimensions under special cocycle twists is proved for arbitrary groups in the following Proposition. It rules out, in particular, the existence of elements of a double $D(kG)$ whose Hopf order is a prime that does not divide the dimension.

**Proposition 2.3.** Let $\tau: B \otimes H \to k$ be an invertible skew pairing of bialgebras.

If $n \in \mathbb{N}$ satisfies $t_n(H) = t_n(B) = 1$, and $H$ is cocommutative, then also $t_n(B \bowtie \tau H) = 1$.

In particular, if $G$ is a finite group, and $n$ is prime to the order of $G$, then the only elements in the double $D(kG)$ that have trivial $n$-th Hopf power are the scalar multiples of the identity.

**Proof.** We first essentially follow Kashina [10, Sec. 3] to find a formula for the $n$-th power map on $B \bowtie H$. We rewrite multiplication in $B \bowtie H$ in the form

$$ (b \bowtie g)(c \bowtie h) = b\tau(c_{(1)} \otimes g_{(1)})c_{(2)} \bowtie g_{(2)}\tau^{-1}(c_{(3)} \otimes g_{(3)})h $$

Using the actions

$$ H \otimes B \ni h \otimes b \mapsto h \mapsto b = \tau(b_{(1)} \otimes h)b_{(2)} \in B $$

$$ H \otimes B \ni h \otimes b \mapsto h \mapsto b = \tau^{-1}(b \otimes h_{(2)})h_{(1)} \in H. $$

It is easy to check that $\mapsto$ makes $B$ an $H$-module algebra, satisfying in addition $\Delta(h \mapsto b) = h \mapsto b_{(1)} \otimes b_{(2)}$, and that $\mapsto$ makes $H$ a $B$-module algebra, satisfying in addition $\Delta(h \mapsto b) = h_{(1)} \otimes h_{(2)} \mapsto b$. We consider the bijection

$$ F: B \otimes H \to B \bowtie H $$

$$ b \otimes h \mapsto \tau(b_{(1)} \otimes h_{(1)})b_{(2)} \bowtie h_{(2)}\tau^{-1}(b_{(3)} \otimes h_{(3)}) $$

$$ = h_{(1)} \mapsto b_{(2)} \bowtie h_{(2)} \mapsto b_{(2)} $$

$$ = (1 \bowtie h)(b \bowtie 1) $$

and will prove, by induction on $n$,

$$ [n]F(b \otimes h) = h_{(1)} \mapsto b_{(1)}^{[n]} \bowtie h_{(2)}^{[n]} \mapsto b_{(2)}. $$

The case $n = 1$ is just the definition of multiplication in $B \bowtie H$. Assuming the formula for $n$, we first get

$$ (b \bowtie h)^{[n+1]} = (b_{(1)} \bowtie 1)((1 \bowtie h_{(1)})(b_{(2)} \bowtie 1))^{[n]}(1 \bowtie h_{(2)}) $$

$$ = b_{(1)}(h_{(1)} \mapsto b_{(2)}^{[n]} \bowtie (h_{(2)}^{[n]} \mapsto b_{(3)})) $$

In this expression we substitute $h_{(1)} \mapsto b_{(1)} \bowtie h_{(2)} \mapsto b_{(2)}$ for $b \bowtie h$, using

$$ \Delta(h_{(1)} \mapsto b_{(1)}) \otimes \Delta(h_{(2)} \mapsto b_{(2)}) = h_{(1)} \mapsto b_{(1)} \otimes b_{(2)} \otimes h_{(2)} \otimes h_{(3)} \mapsto b_{(3)}, $$

and find

$$ [n+1]F(b \otimes h) = (h_{(1)} \mapsto b_{(1)} \bowtie h_{(2)} \mapsto b_{(2)})^{[n+1]} $$

$$ = (h_{(1)} \mapsto b_{(1)})(h_{(2)} \mapsto b_{(2)}^{[n]} \bowtie (h_{(3)}^{[n]} \mapsto b_{(3)})(h_{(4)} \mapsto b_{(4)}) $$

$$ = h_{(1)} \mapsto b_{(1)}b_{(2)}^{[n]} \bowtie h_{(2)}^{[n]}h_{(3)} \mapsto b_{(3)} $$

$$ = h_{(1)} \mapsto b_{(1)}^{[n+1]} \bowtie h_{(2)}^{[n+1]} \mapsto b_{(2)} $$
Now consider the bijections
\[ P: B \otimes H \ni b \otimes h \mapsto h(1) \rightarrow b \otimes h(2) \in B \otimes H \]
\[ Q: B \otimes H \ni b \otimes h \mapsto b(1) \otimes h(1)\tau^{-1}(b(2) \otimes h(2)) \in B \otimes H. \]
It is easy to check that
\[ Q^n(b \otimes h) = b(1) \otimes h(1)\tau^{-1}(b(2) \otimes h(2)^[n]). \]
If \( H \) is cocommutative, then \([n]_H\) is a coalgebra map, and we finally have
\[ [n]_{B \bowtie H}F(b \otimes h) = h(1) \rightarrow b(1)^-[n]\tau^{-1}(b(2) \otimes h(3)^[n]) \bowtie h(2)^[n] \]
\[ = (B \otimes [n]_H)P([n]_B \otimes H)Q^n(b \otimes h). \]
In particular, if \( t_n(H) = t_n(B) = 1 \), so that \([n]_B\) and \([n]_H\) are bijective, then \([n]_{B \bowtie H}\) is bijective, and so \( t_n(B \bowtie H) = 1. \)  

To close the section, note that according to Table 9, the dual \( D(\mathbb{Q}S_4)^* \) of the double of \( S_4 \) does not contain an element of Hopf order 10, whereas \( D(\mathbb{Q}S_4) \) does.

### 3. Bismash products

We will now compute the numbers \( t_{m,n}(H) \) for some bismash products obtained from a matched pair of groups, obtained in turn from a factorizable group. We will recall these notions and relevant facts, using Masuoka’s survey [15] as a general reference.

A group \( L \) is called factorizable into subgroups \( F, G \subset L \) if \( FG = L \) and \( F \cap G = \{1\} \), that is, if every \( \ell \in L \) can be written uniquely as a product \( \ell = ax \) with \( a \in F \) and \( x \in G \).

A matched pair of groups \((F, G, \triangleright, \triangleleft)\) is a pair of groups \( F, G \) with group actions
\[ G \triangleleft F \quad \overset{\triangleright}{\rightarrow} F \]
of each group on the underlying set of the other, such that
\[ x \triangleright ab = (x \triangleright a)((x \triangleleft a) \triangleright b) \]
\[ xy \triangleleft a = (x \triangleleft (y \triangleright a))(y \triangleleft a) \]
hold for all \( a, b \in F, x, y \in G \).

| \( t_{i,j}(D(\mathbb{Q}S_4)^*) \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 415 | 205 | 383 | 1 | 415 | 1 | 415 | 205 | 359 | 1 |
| 3 | 313 | 241 | 1 | 301 | 1 | 241 | 307 | 205 | 1 |
| 4 | 484 | 1 | 463 | 1 | 452 | 241 | 415 | 1 |
| 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 | 535 | 1 | 463 | 301 | 415 | 1 |
| 7 | 1 | 1 | 1 | 1 |
| 8 | 484 | 241 | 383 | 1 |
| 9 | 313 | 205 | 1 |
| 10 | 415 | 1 |
| 11 | 1 |

**Table 9**


A factorizable group gives rise to a matched pair of groups, if we define the mutual actions by the formula \( xa = (x \triangleright a)(x \triangleleft a) \) for \( a \in F, x \in G \), that is, let \( x \triangleright a \in F \) and \( x \triangleleft a \in G \) be the — by hypothesis unique — elements of \( F \) and \( G \) whose product is \( xa \).

Conversely, given a matched pair \((F, G, \triangleright, \triangleleft)\), we can define a group \( F \bowtie G \) with underlying set \( F \times G \) and multiplication
\[
(a, x)(b, y) = (a(x \triangleright b), (x \triangleleft b)y)
\]
for \( a, b \in F, x, y \in G \). This group \( L = F \bowtie G \) is factorizable into the subgroups \( F' = F \times \{1\} \cong F \) and \( G' = \{1\} \times G \cong G' \).

Let \((F, G, \triangleright, \triangleleft)\) be a matched pair, with \( G \) finite. Then one can define a Hopf algebra structure on \( H = k^G \otimes kF =: k^G \# kF \), called a bismash product of \( k^G \) and \( kF \) by
\[
(p_x \# a)(p_y \# b) = \delta_{x\triangleleft a, y} p_x \# ab
\]
\[
\Delta(p_x \# a) = \sum_{y \in G} p_{xy^{-1}} \# y \triangleright a \otimes p_y \# a
\]
\[
\varepsilon(p_x \# a) = \delta_{1,x}
\]
for \( a, b \in F, x, y \in G \); the unit element is \( 1 \# 1 = \sum_{x \in G} p_x \# 1 \). We can view \( k^G \cong k^G \# \mathbb{1} \) as a Hopf subalgebra, and \( kF \cong \mathbb{1} \# kF \subset k^G \# kF \) as a subalgebra of \( k^G \# kF \).

A special case of the bismash product construction is the Drinfeld double of the group algebra of a finite group \( G \). The relevant matched pair is \((G, G, \triangleright, \triangleleft)\) with trivial action \( \triangleright \), and adjoint action \( x \triangleleft a = a^{-1}xa \). The actions can be viewed as coming from the factorization of the group \( L = G \times G \) into two subgroups isomorphic to \( G \), the first being the diagonal \( \{(g, g) | g \in G\} \), the second \( \{1\} \times G \). The bismash product \( k^G \# kG \) is isomorphic to the Drinfeld double \( D(kG) \).

We return to the general case of a matched pair \((F, G, \triangleright, \triangleleft)\). The bismash product \( k^G \# kF \) is the neutral element in the Opext group of Hopf algebra extensions
\[
(3.1)
\]
The general middle term of such a short exact sequence is a bicrossproduct, which has its multiplication and comultiplication deformed in addition by two 2-cocycles. If \( k = \mathbb{C} \), then any middle term of an extension \( (3.1) \) is a Kac algebra by a result of Masuoka [14, Rem.2.4]. For the bismash product itself, the *-structure can be found in Kac’ paper [8].

Without giving details on the isomorphism, we note that the dual of a bismash product can be viewed as a bismash product itself:
\[
(k^G \# kF)^* \cong k^F \# kG.
\]

**Proposition 3.1.** Let \((F, G, \triangleright, \triangleleft)\) be a matched pair of finite groups.

Then \( \exp(k^G \# kF) = \exp(F \bowtie G) \).

**Proof.** By a result of Beggs, Gould, and Majid [1], the Drinfeld double \( D(k^G \# kF) \) is a Drinfeld twist of the Drinfeld double \( D(k[G \bowtie F]) \). By results of Etingof and Gelaki [5] we have already cited, the exponent is invariant under such twists, and also invariant under taking the Drinfeld double. \( \square \)

We shall now discuss the results of our computations of the trivial power dimensions \( t_{m,n}(Q^G \# QF) \) for some matched pairs of groups. The computations were
done with the help of Maple; part of the code will be presented in Section 4. In fact we have already presented results for special bismash products, namely the Drinfeld doubles of some group algebras, in the preceding section.

The next matched pair we consider is obtained from the symmetric group $S_n$:

**Lemma 3.2.** $S_n$ is factorizable into subgroups $F = S_{n-1} \subset S_n$ (the copy of $S_{n-1}$ in $S_n$ that fixes $n$), and $G \cong C_n$, the cyclic group of order $n$ generated by the $n$-cycle $\tau = (123 \ldots n)$. Explicitly, for $\sigma \in S_n$ we have $\sigma = \sigma_1 \sigma_2$ with $\sigma_2 = \tau^k \in G$, where $k = n - \sigma^{-1}(n)$, and $\sigma_1 = \sigma \sigma_2^{-1} \in F$.

**Proof.** It is easy to see that we have a factorizable group as stated, since the two subgroups have trivial intersection, and their orders multiply to give the order of $S_n$. If we want to factor $\sigma \in S_n$ into a product $\sigma_1 \sigma_2$ with $\sigma_2 = \tau^k \in G$ and $\sigma_1 \in S_{n-1}$, we have to find an exponent $k$ such that $\sigma_1 = \sigma \tau^{-k}$ does not move $n$, that is, we have to make sure that $\tau^{-k}(n) = \sigma^{-1}(n)$, that is $n = \tau^k \sigma^{-1}(n)$, or simply $k = n - \sigma^{-1}(n)$.\hfill \Box

The matched pair arising from the factorizable group $S_n$ as in the Lemma was studied in particular by Masuoka [13, Expl.1.3], who shows that the Opext group for this matched pair is trivial; in other words, the only middle term of an extension (3.1) giving rise to this matched pair is the bismash product.

**Remark 3.3.** The same formula as in the preceding Lemma also computes the factorization of an element $\sigma \in A_n$ for odd $n$ in the factorizable group $A_n \cong A_{n-1} \bowtie C_n$. The bismash product $kC_n \# kA_{n-1}$ is naturally a Hopf subalgebra of $kC_n \# kS_{n-1}$.

Tables 10 and 11 list the dimensions $t_{i,j}(Q^{C_4} \# QS_3)$ and its dual $Q^{S_3} \# QC_4$, respectively. Note that we know the exponent in each case to be that of $S_4$, namely 12. Using Remark 1.18 we find:

**Corollary 3.4.** The 24-dimensional Hopf algebra $H = Q^{C_4} \# QS_3$, of exponent 12, contains elements of Hopf order $n$ for each $1 \leq n \leq 10$ except $n = 7$ and $n = 10$. In particular, it contains elements whose Hopf order is a prime that does not divide the dimension (or the exponent) of $H$. 

| $t_{i,j}(Q^{C_4} \# QS_3)$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---------------------------|---|---|---|---|---|---|---|---|---|----|----|
| $i = 1$                   | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1   | 1   |
| $2$                       | 12 | 8 | 5 | 12 | 3 | 12 | 5 | 8 | 1   |     |    |
| $3$                       | 9  | 3 | 6 | 5  | 1 | 5  | 5 | 1 | 1   |     |    |
| $4$                       | 15 | 12| 5 | 11 | 1 | 12 | 1 |   |     |     |    |
| $5$                       | 7  | 5 | 3 | 5  | 3 | 1  |   |   |     |     |    |
| $6$                       | 18 | 7 | 12| 6  | 12| 1  |   |   |     |     |    |
| $7$                       | 7  | 3 | 3 | 5  | 1 |     |   |   |     |     |    |
| $8$                       | 15 | 5 | 8 | 1  |   |     |   |   |     |     |    |
| $9$                       | 9  | 1 | 1 |     |   |     |   |   |     |     |    |
| $10$                      |    |   | 12|     |   |     |   |   |     |     |    |
| $11$                      |    |   | 1 |     |   |     |   |   |     |     |    |

**Table 10**
The dual Hopf algebra $H^* = \mathbb{Q}S_3 \# \mathbb{Q}C_4$ contains elements of Hopf order $n$ for each $1 \leq n \leq 10$ except $n = 7$ and $n = 9$. In particular, the Hopf orders of elements that occur in $H$ and its dual are different.

Remark 3.5. While Tables 10 and 11 give us the full information on which Hopf orders of elements are possible, they do not tell us which individual elements have those orders. By a separate calculation one can verify that

$$p_{(1 3)(2 4)}#(2 3) - p_{(1 4 3 2)}#(1 2) + p_{(1 3)(2 4)}#(1 3 2) \in \mathbb{Q}C_4 \# \mathbb{Q}S_3 =: H$$

has Hopf order five.

It is clear that all elements of the Hopf subalgebra $\mathbb{Q}C_4 \subset H$ have Hopf orders 1, 2, or 4. One can verify that the elements of the subalgebra $\mathbb{Q}S_3 \subset H$ have Hopf orders 1, 2, 6 or 12. While the last number may be surprising, these possible orders are at least not prime to the exponent of $H$, or of $\mathbb{Q}S_3$. Even if elements of the form $1#a$ with $a \in S_3$ are, in this sense, well-behaved under the Hopf power maps, their behavior does not seem to be easy to understand and predict. For example, $1#(13)$ has Hopf order 2, while $1#(12)$ has Hopf order 12. Note also that 3 is not among the Hopf orders of elements in $\mathbb{Q}S_3 \subset H$.

One can also verify that all elements of the standard basis $p_a \otimes a$ have Hopf orders 1, 2, 3, 4, or 12. Again, this does not mean that the orders of such elements would be easy to understand. For example, both $p_{(1 2 3 4)} \in \mathbb{Q}C_4$ and $(1 2) \in \mathbb{Q}S_3$ have Hopf order 2, but $p_{(1 2 3 4)}#(1 2) \in H$ has Hopf order 3. While $p_{(1 3)(2 4)} \in \mathbb{Q}C_4$ has Hopf order 4, the element $p_{(1 3)(2 4)}#(1 2) \in H$ has Hopf order 2. And $p_{(1 2 3 4)}#(1 2 3)$ has Hopf order 2.

In a more indirect way, the example $\mathbb{Q}C_4 \# \mathbb{Q}S_3$ also helps to answer a question raised in Section 2 about the behavior of Hopf powers under Drinfeld twists:

Example 3.6. Consider the factorizable group $S_4$, and the bismash product $\mathbb{Q}C_4 \# \mathbb{Q}S_3$. From Proposition 2.3 we know that $t_n(D(kS_4)) \neq 1$ if and only if $n$ is not prime to 4!. But up to a Drinfeld twist, $D(kS_4)$ is isomorphic to $D(\mathbb{Q}C_4 \# \mathbb{Q}S_3)$ by [1]. On the other hand, we have seen that $\mathbb{Q}C_4 \# \mathbb{Q}S_3$, hence also its double, contains elements of Hopf order 5. This shows that the property of having nontrivial elements with trivial $n$-th Hopf power is not invariant under twisting.

| $t_{i,j}(\mathbb{Q}S_3 \# \mathbb{Q}C_4)$ | $j = 1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|----------------------------------------|--------|---|---|---|---|---|---|---|---|----|----|
| $i = 1$                                | 1      | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1   | 1   |
| 2                                     |        | 12| 1 | 6 | 5 | 10| 3 | 10| 5 | 7   | 1   |
| 3                                     |        |   | 9 | 5 | 3 | 9 | 5 | 1 | 5 | 5   | 1   |
| 4                                     |        |   | 15| 3 | 10| 5 | 11| 1 | 10| 1   |     |
| 5                                     |        |   |   | 7 | 7 | 5 | 5 | 3 | 1   |     |
| 6                                     |        |   |   | 18| 7 | 10| 9 | 10| 1   |     |
| 7                                     |        |   |   | 7 | 3 | 3 | 5 | 1   |     |
| 8                                     |        |   | 15| 5 | 6 | 1   |     |
| 9                                     |        |   | 9 | 1 | 1 |     |     |
| 10                                    |        | 12| 1 |     |     |     |
| 11                                    |        | 1 |     |     |     |     |

Table 11

The dual Hopf algebra $H^* = \mathbb{Q}S_3 \# \mathbb{Q}C_4$ contains elements of Hopf order $n$ for each $1 \leq n \leq 10$ except $n = 7$ and $n = 9$. In particular, the Hopf orders of elements that occur in $H$ and its dual are different.
As we already saw for the double of $kA_4$, we have $t_{i,j}(H) \neq t_{i,j}(H^*)$ in general. However, in the last example as well as for the doubles studied in Section 2, we see that the columns with prime index are the same. Note, though, that this only holds for the printed parts above the diagonal (otherwise the printed parts of the prime rows would have to agree as well). Still, $t_{pi}(H) = t_{pi}(H^*)$ for $p$ prime and $i < p$ implies that those prime numbers $p$ such that there is an element of Hopf order $p$ are the same for $H$ and $H^*$ (namely, 2, 3, and 5). We will have to consider a larger example to see that the last observation fails to be true in general, and that we may have $t_{pq}(H) \neq t_{pq}(H^*)$ even for two primes $p, q$. Tables 12 and 13 are for the bismash product $H = \mathbb{Q}C_5 \# \mathbb{Q}A_4$ and its dual, associated to the factorizable group $A_5 \cong A_4 \triangleright C_5$. The exponent of $A_5$ is $e = 30$, and thus it is preferable to use the symmetry Proposition 2.2 to cut away redundant parts of the table and save space.

We have printed the matrix $(t_{i,j}(H))$ for column indices up to half the exponent, i.e. we have printed the first 15 columns only. Since $t_{i,j}(H) = t_{e-i,e-j}(H)$, the columns

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| 1   | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1   | 1   | 1   | 1   | 1   | 1   |
| 2   | 1 | 32| 13| 18| 17| 18| 9 | 20| 1  | 18  | 11  | 26  | 11  | 12  | 21  |
| 3   | 1 | 13| 37| 15| 17| 17| 13| 17| 14 | 17  | 21  | 17  | 1   | 29  |    |
| 4   | 1 | 18| 15| 34| 7 | 20| 13| 22| 13  | 26  | 13  | 22  | 11  | 16  | 19  |
| 5   | 1 | 17| 17| 7 | 25| 13| 9 | 17 | 1   | 13  | 17  | 9   | 1   | 25  |    |
| 6   | 1 | 18| 17| 20| 13| 36| 1  | 18 | 5   | 14  | 13  | 25  | 1   | 14  | 17  |
| 7   | 1 | 9 | 13| 13| 9 | 1 | 21 | 9 | 9   | 21  | 7   | 17  | 13  | 1   | 17  |
| 8   | 1 | 20| 17| 22| 17| 18| 9 | 32 | 1   | 22  | 9   | 18  | 9   | 12  | 17  |
| 9   | 1 | 14| 13| 1 | 5 | 9 | 1 | 21 | 3   | 13  | 7   | 13  | 9   | 1   | 17  |
| 10  | 1 | 18| 17| 26| 13| 14| 21| 22| 13   | 38  | 7   | 26  | 17  | 14  | 25  |
| 11  | 1 | 11| 13| 13| 13| 13| 7 | 9 | 7   | 7   | 19  | 15  | 7   | 1   | 19  |
| 12  | 1 | 26| 21| 22| 17| 25| 17| 18| 13  | 26 | 15  | 44  | 13  | 10  | 29  |
| 13  | 1 | 11| 17| 11| 9 | 1 | 13| 9 | 9   | 17  | 7   | 13  | 21  | 1   | 21  |
| 14  | 1 | 12 | 1 | 14 | 1 | 12 | 1 | 14 | 1   | 10  | 1   | 16  | 1   |    |    |
| 15  | 1 | 21| 29| 19| 25| 17| 17| 17 | 17  | 25  | 19  | 29  | 21  | 1   | 41  |
| 16  | 1 | 12 | 1 | 16 | 1 | 12 | 1 | 14 | 1   | 10  | 1   | 16  | 1   |    |    |
| 17  | 1 | 21| 13| 7 | 17 | 9 | 9 | 13  | 1   | 9   | 11  | 17  | 11  | 1   | 21  |
| 18  | 1 | 18| 29| 24| 17 | 25| 13 | 26| 13  | 26  | 13  | 30  | 17  | 10  | 20  |
| 19  | 1 | 7 | 15| 19 | 7 | 9 | 7 | 7   | 13  | 13  | 13  | 11  | 1   | 19  |    |
| 20  | 1 | 26| 21| 20| 25 | 26| 9 | 26 | 1   | 26  | 13  | 26  | 9   | 14  | 25  |
| 21  | 1 | 9 | 14| 7 | 13 | 17 | 1 | 9 | 7   | 1   | 13  | 13  | 1   | 1   | 17  |
| 22  | 1 | 20| 13| 20 | 9 | 14 | 17 | 16 | 9   | 26  | 7   | 26  | 13  | 12  | 17  |
| 23  | 1 | 13| 21| 7 | 17 | 13 | 9 | 17 | 1   | 9   | 13  | 9   | 1   | 17  |    |
| 24  | 1 | 10| 13| 26 | 1 | 22 | 13 | 14 | 17  | 26  | 7   | 25  | 9   | 14  | 17  |
| 25  | 1 | 9 | 17| 13 | 13 | 1 | 17 | 9 | 13  | 25  | 7   | 17  | 17  | 1   | 25  |
| 26  | 1 | 22| 17| 28 | 13 | 26 | 7 | 20 | 7   | 20  | 19  | 24  | 7   | 16  | 19  |
| 27  | 1 | 17| 23| 17 | 17 | 13 | 21 | 13 | 14  | 21  | 15  | 29  | 13  | 1   | 29  |
| 28  | 1 | 18| 17| 22 | 9 | 10 | 13 | 20 | 9   | 26  | 7   | 18  | 21  | 12  | 21  |
| 29  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |

Table 12
we cut off appear upside down in reverse order in the printed part of the table, that is, the 16-th column is the 14-th column upside down, etc. We have boldfaced the diagonal elements to make them easier to spot. Now we have the following recipe to decide if there is an element of Hopf order \( n \) in \( H \). If \( n \leq e/2 \), check if the \( n \)-th diagonal element is strictly larger than all the numbers in the column above it; there exists an element of Hopf order \( n \) if and only if this is the case. If \( n > e/2 \), check if the \( (e-n) \)-th diagonal element is strictly larger than all the numbers in the column below it; there exists an element of Hopf order \( n \) if and only if this is the case. For example, \( H = \mathbb{Q}C_5 \# \mathbb{Q}A_4 \) does not contain an element of Hopf order 17; since 17 > 15, and 30 − 17 = 13, this is checked by finding the 13-th diagonal element, which is 21, in the thirteenth column two places below the diagonal. Observe that the dual Hopf algebra \( H^* = \mathbb{Q}A_4 \# \mathbb{Q}C_5 \) does contain an element of Hopf order 17, since 21 does not appear below the diagonal in the 13-th column of the table for \( H^* \). Thus we have an example where \( H \) does not contain an element of a certain prime

\[
\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 32 & 13 & 22 & 17 & 20 & 9 & 20 & 1 & 24 & 9 & 28 & 9 & 12 & 17 \\
3 & 1 & 13 & 37 & 17 & 17 & 17 & 13 & 17 & 21 & 15 & 25 & 21 & 1 & 33 \\
4 & 1 & 22 & 17 & 34 & 7 & 22 & 11 & 18 & 13 & 28 & 13 & 26 & 13 & 16 & 19 \\
5 & 1 & 17 & 17 & 7 & 25 & 13 & 9 & 17 & 1 & 9 & 13 & 17 & 9 & 1 & 21 \\
6 & 1 & 20 & 17 & 22 & 13 & 36 & 1 & 20 & 9 & 16 & 13 & 28 & 9 & 16 & 21 \\
7 & 1 & 9 & 13 & 11 & 9 & 1 & 21 & 11 & 9 & 17 & 7 & 17 & 13 & 1 & 21 \\
8 & 1 & 20 & 17 & 18 & 17 & 20 & 11 & 32 & 1 & 20 & 11 & 20 & 9 & 12 & 21 \\
9 & 1 & 1 & 17 & 13 & 1 & 9 & 9 & 1 & 21 & 13 & 7 & 17 & 9 & 1 & 21 \\
10 & 1 & 24 & 21 & 28 & 9 & 16 & 17 & 20 & 13 & 38 & 7 & 28 & 21 & 16 & 21 \\
11 & 1 & 9 & 15 & 13 & 13 & 13 & 7 & 11 & 7 & 7 & 19 & 13 & 7 & 1 & 19 \\
12 & 1 & 28 & 25 & 26 & 17 & 28 & 17 & 20 & 17 & 28 & 13 & 44 & 13 & 12 & 33 \\
13 & 1 & 9 & 21 & 13 & 9 & 1 & 13 & 9 & 9 & 21 & 7 & 13 & 21 & 1 & 17 \\
14 & 1 & 12 & 1 & 16 & 1 & 16 & 1 & 12 & 1 & 16 & 1 & 12 & 1 & 16 & 1 \\
15 & 1 & 17 & 33 & 19 & 21 & 21 & 21 & 21 & 21 & 19 & 33 & 17 & 1 & 41 \\
16 & 1 & 12 & 1 & 16 & 1 & 16 & 1 & 12 & 1 & 16 & 1 & 12 & 1 & 16 & 1 \\
17 & 1 & 17 & 13 & 7 & 17 & 13 & 9 & 13 & 1 & 9 & 9 & 17 & 9 & 1 & 17 \\
18 & 1 & 20 & 33 & 24 & 17 & 28 & 13 & 28 & 17 & 28 & 15 & 32 & 17 & 12 & 33 \\
19 & 1 & 7 & 13 & 19 & 7 & 7 & 11 & 7 & 13 & 13 & 15 & 9 & 1 & 19 \\
20 & 1 & 28 & 17 & 22 & 21 & 28 & 13 & 28 & 13 & 28 & 13 & 28 & 9 & 16 & 21 \\
21 & 1 & 9 & 17 & 7 & 13 & 21 & 1 & 9 & 9 & 1 & 13 & 17 & 1 & 1 & 21 \\
22 & 1 & 20 & 13 & 22 & 9 & 12 & 21 & 18 & 9 & 28 & 7 & 28 & 13 & 12 & 21 \\
23 & 1 & 13 & 17 & 7 & 17 & 9 & 11 & 21 & 1 & 9 & 11 & 13 & 9 & 1 & 21 \\
24 & 1 & 16 & 21 & 28 & 1 & 24 & 9 & 12 & 21 & 28 & 7 & 28 & 13 & 16 & 21 \\
25 & 1 & 9 & 17 & 13 & 11 & 1 & 17 & 9 & 13 & 21 & 7 & 17 & 17 & 1 & 21 \\
26 & 1 & 20 & 15 & 28 & 13 & 28 & 7 & 22 & 7 & 22 & 19 & 24 & 7 & 16 & 19 \\
27 & 1 & 17 & 25 & 15 & 17 & 21 & 17 & 13 & 17 & 17 & 13 & 33 & 13 & 1 & 33 \\
28 & 1 & 16 & 17 & 20 & 9 & 16 & 13 & 20 & 9 & 28 & 7 & 20 & 17 & 12 & 17 \\
29 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Table 13
The total number of possible orders is the same, namely 22, for both algebraic closures. We have already seen one example (the double of \( t \)), while 17 and 25 occur only in \( Q(\mathbb{C}^3) \). The same phenomenon occurs for the prime 13, with 4, 5, 13, 21, 25, 29, 37, 43, 47, 53, 59 occurring only in \( Q(\mathbb{C}^3) \). In Tables 14 and 15, while 45, 53, 59, 61, 67, 73 occur in neither.

We have already seen one example (the double of \( S_4 \)) in Section 2 where a Hopf algebra and its dual admit different total numbers of Hopf orders. Another example is the bismark product of the factorizable group \( S_5 \) in Tables 14 and 15. While 45 of the numbers in \( \{2, \ldots, 58\} \) are Hopf orders of elements of \( Q(\mathbb{C}^3) \), only 31 are Hopf orders of elements of \( Q(\mathbb{C}^3) \). While in this last example the percentage of possible Hopf orders that actually occur is rather small compared to the other examples in this section (let alone the doubles of small groups in Tables 3, 6, and 7, where all conceivable orders occurred), it is still much larger than in group algebras.

| t | t_{ij} = t_{ij}(Q(\mathbb{C}^3) \# Q(\mathbb{S}_4)) |
|---|---|
| 2  | 1  |
| 3  | 1  |
| 4  | 1  |
| 5  | 1  |
| 6  | 1  |
| 7  | 1  |
| 8  | 1  |
| 9  | 1  |
| 10 | 1  |
| 11 | 1  |
| 12 | 1  |
| 13 | 1  |
| 14 | 1  |
| 15 | 1  |
| 16 | 1  |
| 17 | 1  |
| 18 | 1  |
| 19 | 1  |
| 20 | 1  |

Hopf order, while \( H^* \) does. The same phenomenon occurs for the prime 13, with the roles of \( H \) and \( H^* \) reversed. Note also that \( t_{3,13}(H) = 17 \neq 21 = t_{3,13}(H^*) \). Of course, one can make a complete list of the orders of elements of \( H \) and \( H^* \). In addition to 1 and the exponent, which are always possible orders, the candidates are the numbers in \( \{2, \ldots, e - 2\} \). For \( H = Q(\mathbb{C}^3) \# Q(\mathbb{A}_4) \), the orders 13 and 21 occur only in \( H \), while 17 and 25 occur only in \( H^* \), and 14, 16, 19, 23 occur in neither. The total number of possible orders is the same, namely 22, for both \( H \) and \( H^* \).
Although the picture might change when larger dimensions are considered, it seems quite unlikely that further computations will lead us to examples where a prime divides the exponent of a bismash product Hopf algebra \( H \), but \( p \) is not the Hopf order of an element of \( H \). Note that if \( H = kG \# kF \), then \( p \), a divisor of the order of \( F \cong G \), has to divide the order of either \( F \) or \( G \); then \( p \) occurs as the Hopf order of an element of one of the Hopf algebras \( kF \) and \( kG \). These are Hopf subalgebras in \( H^* \) and \( H \), respectively, so that \( p \) is at least the Hopf order of an element of \( F \# H \). Since the trivial power dimensions are invariant under taking the dual, this also implies that nontrivial elements with trivial \( p \)-th Hopf power exist both in \( H \) and \( H^* \).

According to Tables 12, 13, 14 and 15 the relevant primes 2, 3, 5, 7 occur as Hopf orders both in \( H \) and \( H^* \) for \( H = \mathbb{Q}

\text{Table 15.} \quad t_{ij} = t_{ij}(Q^S_i \# Q^S_j)

| \( i \) | \( j \) | \( t_{ij} \) |
|-------|-------|-------------|
| 1     | 2     | 1           |
| 1     | 3     | 1           |
| 1     | 4     | 1           |
| 1     | 5     | 1           |
| 1     | 6     | 1           |
| 1     | 7     | 1           |
| 1     | 8     | 1           |
| 1     | 9     | 1           |
| 1     | 10    | 1           |
| 1     | 11    | 1           |
| 1     | 12    | 1           |
| 1     | 13    | 1           |
| 1     | 14    | 1           |
| 1     | 15    | 1           |
| 1     | 16    | 1           |
| 1     | 17    | 1           |
| 1     | 18    | 1           |
| 1     | 19    | 1           |
| 1     | 20    | 1           |
| 1     | 21    | 1           |
| 1     | 22    | 1           |
| 1     | 23    | 1           |
| 1     | 24    | 1           |
| 1     | 25    | 1           |
| 1     | 26    | 1           |
| 1     | 27    | 1           |
| 1     | 28    | 1           |
| 1     | 29    | 1           |
| 1     | 30    | 1           |
| 2     | 3     | 1           |
| 2     | 4     | 1           |
| 2     | 5     | 1           |
| 2     | 6     | 1           |
| 2     | 7     | 1           |
| 2     | 8     | 1           |
| 2     | 9     | 1           |
| 2     | 10    | 1           |
| 2     | 11    | 1           |
| 2     | 12    | 1           |
| 2     | 13    | 1           |
| 2     | 14    | 1           |
| 2     | 15    | 1           |
| 2     | 16    | 1           |
| 2     | 17    | 1           |
| 2     | 18    | 1           |
| 2     | 19    | 1           |
| 2     | 20    | 1           |
| 2     | 21    | 1           |
| 2     | 22    | 1           |
| 2     | 23    | 1           |
| 2     | 24    | 1           |
| 2     | 25    | 1           |
| 2     | 26    | 1           |
| 2     | 27    | 1           |
| 2     | 28    | 1           |
| 2     | 29    | 1           |
| 2     | 30    | 1           |
| 3     | 3     | 1           |
| 3     | 4     | 1           |
| 3     | 5     | 1           |
| 3     | 6     | 1           |
| 3     | 7     | 1           |
| 3     | 8     | 1           |
| 3     | 9     | 1           |
| 3     | 10    | 1           |
| 3     | 11    | 1           |
| 3     | 12    | 1           |
| 3     | 13    | 1           |
| 3     | 14    | 1           |
| 3     | 15    | 1           |
| 3     | 16    | 1           |
| 3     | 17    | 1           |
| 3     | 18    | 1           |
| 3     | 19    | 1           |
| 3     | 20    | 1           |
| 3     | 21    | 1           |
| 3     | 22    | 1           |
| 3     | 23    | 1           |
| 3     | 24    | 1           |
| 3     | 25    | 1           |
| 3     | 26    | 1           |
| 3     | 27    | 1           |
| 3     | 28    | 1           |
| 3     | 29    | 1           |
| 3     | 30    | 1           |
| 4     | 4     | 1           |
| 4     | 5     | 1           |
| 4     | 6     | 1           |
| 4     | 7     | 1           |
| 4     | 8     | 1           |
| 4     | 9     | 1           |
| 4     | 10    | 1           |
| 4     | 11    | 1           |
| 4     | 12    | 1           |
| 4     | 13    | 1           |
| 4     | 14    | 1           |
| 4     | 15    | 1           |
| 4     | 16    | 1           |
| 4     | 17    | 1           |
| 4     | 18    | 1           |
| 4     | 19    | 1           |
| 4     | 20    | 1           |
| 4     | 21    | 1           |
| 4     | 22    | 1           |
| 4     | 23    | 1           |
| 4     | 24    | 1           |
| 4     | 25    | 1           |
| 4     | 26    | 1           |
| 4     | 27    | 1           |
| 4     | 28    | 1           |
| 4     | 29    | 1           |
| 4     | 30    | 1           |

\( t_{ij} := t_{ij}(Q^S_i \# Q^S_j) \)
\[ Q_1#Q_3 \subset Q^{C_5}#Q_4 \] are 1, 2, 4, 12, and 30. In particular, the subalgebra \( Q A_4 \subset Q^{C_5}#Q_4 \) also contains no elements of Hopf order 3, since we can naturally view \( Q A_4 \subset Q^{C_5}#Q_4 \) as a Hopf subalgebra. The element 
\[ p_{(1\ 2\ 3\ 4\ 5)}#(1\ 2\ 4) \in Q^{C_5}#Q A_4 \subset Q^{C_5}#Q S_4 \]
has Hopf order 3.

4. Maple code

We will now present the Maple programming used to compute the numerical results that we have discussed in the preceding sections.

For the moment, we assume that we have already implemented in Maple a matched pair \((F, G, \triangleright, \triangleleft)\) of finite groups, including explicit bijections \(E_M: M \rightarrow \{1, \ldots, |M|\}\) for \(M = F, G\). We will talk later about how to actually provide the following items:

- \text{NumfromF}: A maple procedure implementing \(E_F\).
- \text{FfromNum}: A maple procedure implementing \(E_F^{-1}\).
- \text{ordF}: The order of \(F\).
- \text{multF}: A maple procedure that computes the product of two elements of \(F\).
- \text{oneF}: The neutral element of \(F\).
- \text{invF}: A maple procedure that computes the inverse of an element in \(F\).
- \text{NumfromG}: The same as \text{NumfromF} for the group \(G\). We also have \text{GfromNum}, \text{ordG}, \text{multG}, \text{invG}, \text{oneG}.
- \text{hit}: The procedure call \text{hit}(x, a) should compute \(x \triangleright a\).
- \text{hitby}: The procedure call \text{hitby}(x, a) should compute \(x \triangleleft a\).
- \text{expL}: The exponent of the group \(L = F \triangleright \triangleleft G\).

From these data we proceed to compute \(\text{ordL} := \text{ordF} \cdot \text{ordG}\) and a procedure to enumerate the elements of the standard basis of \(H = kG#kF\). The element \(p_x#a\) will be represented in Maple as a two-element list \([x, a]\). The following procedures enumerate the elements in the standard basis in inverse lexicographic order:

\begin{verbatim}
> NumfromB := proc(h)
> option remember;
> (NumfromF(h[2])-1)*ordG+NumfromG(h[1]);
> end proc;
>
> BfromNum := proc(i)
> option remember;
> [ GfromNum(irem(i-1,ordG)+1),
>   FfromNum(iquo(i-1,ordG)+1) ];
> end proc;

Based on these enumeration procedures, we will represent a general element of \(H\) as a column vector of length \(\text{ordL}\). To handle vectors and matrices, we use Maple’s \text{LinearAlgebra} package. The result of multiplying two basis elements will either be zero or a basis element. We represent the result as a general element, that is, a column vector:

\begin{verbatim}
> multBB := proc(h,j)
> x := h[1]; a := h[2]; y := j[1]; b := j[2];
> if hitby(x,a)=y
> then UnitVector(NumfromB([x,multF(a,b)]),ordL);
> end if;
>
> end proc;
\end{verbatim}
Besides multiplying two basis elements, we also need to be able to multiply a general element of $H$ and an element of the standard basis. This is of course easily reduced to the multiplication of basis elements.

```maple
> multHB:=proc(V,h);
> R:=ZeroVector(ordL); # a register to sum into
> for i to ordL do
> if not V[i]=0 then
> B:=V[i]*multBB(BfromNum(i),h); # the i-th summand
> end if;
> end do;
> R:=R+B; # is added to the register
> end proc;

Note that $\Delta(p_x \# a)$ is a sum of simple tensors that happen to be tensor products of two elements of the standard basis. Thus, we can represent the result of comultiplication on a basis element as a list of two-element lists of basis elements. This is computed by the following procedure:

```maple
> comult:=proc(h);
> x:=h[1]; a:=h[2]; # so h=[x,a]
> [seq([[multG(x,invG(GfromNum(i)))],
> hit(GfromNum(i),a)],
> i=1..ordG)];
> end proc;
```

The map $[n]: H \rightarrow H$ is naturally represented in Maple by its matrix with respect to the standard basis, in the convention that the matrix for $f: H \rightarrow H$ with respect to a basis $h_i$ is the matrix $m_{ij}$ with $f(h_j) = \sum m_{ij} h_i$, that is, the $i$-th column vector of the matrix is the coordinate vector of the image under $f$ of the $i$-th basis vector. The map $[1]$ is the identity, represented by the unit matrix. To compute $[n]$ recursively, we use the formula $h^{[n+1]} = h_1^{[n]} h_2$. We assume we are given the matrix $A$ representing the $n$-th power map, and we wish to compute the matrix representing the $(n+1)$-st power map. Recall that $h_1 \otimes h_2$, for a basis element $h$, is represented as a list of two-element lists. For each element in that list, which represents a simple tensor $f \otimes g$, we compute the image of $f$ under the $n$-th power map, simply by looking up the relevant column in the matrix $A$. Then we multiply the element represented by that column and the basis element $g$, using the procedure `multHB` given above. This we do for each simple tensor $f \otimes g$ in the list which is the output of the comultiplication procedure applied to $h$, and sum up the results. This gives one column of the desired matrix for the next power map.

```maple
> NextPowerMatrix:=proc(A);
> # The parameter A is assumed to be the matrix representing
> # the n-th Hopf power endomorphism of H. The result of
> # NextPowerMatrix should be the matrix representing the
> # n+1-st Hopf power endomorphism.
```

```maple
> NextPowerMatrix:=proc(A);
> # The parameter A is assumed to be the matrix representing
> # the n-th Hopf power endomorphism of H. The result of
> # NextPowerMatrix should be the matrix representing the
> # n+1-st Hopf power endomorphism.
```
R:=Matrix(ordL,ordL); # a register to compute

# the resulting matrix in

for i to ordL do # run through the basis of H

T:=comult(BfromNum(i)); # comultiply basis element

LengthofT:=ordG; # to get a list of this length

C:=ZeroVector(ordL); # a register to compute

# the i-th column of the result

for j to LengthofT do # go through the summands

D:=A[1..ordL,NumfromB(T[j][1])];

# look up the column in the previous matrix

# corresponding to the first tensor factor in

# the summand under consideration. So D represents

# the previous Hopf power of that tensor factor.

C:=C+multHB(D,T[j][2]); # multiply this with the second

# tensor factor, and add to

# the register.

end do;

R[i..ordL,i]:=C; # store this in the i-th column.

end do;

end proc;

To compute all the matrices, say A_1, ..., A_{euler} for the i-th power maps, we invoke

A[1]:=IdentityMatrix(ordL);

for l from 2 to expL do

A[l]:=NextPowerMatrix(A[l-1])

end do;

The matrix for the map \( \eta \epsilon \): \( H \to H \), which maps \( p \# x \) to \( \sum p_y \# 1 \) if \( x = 1 \), and to zero otherwise, is computed by the following procedure:

\[
\eta \epsilon := \text{Matrix}(\text{ordL}, \text{ordL}, (i,j) \rightarrow \begin{cases} 1 & \text{if } \text{BfromNum}(j)[1]=\text{oneG} \text{ and } \text{BfromNum}(i)[2]=\text{oneF} \\ 0 & \text{else} \end{cases})
\]

We can then compute bases for the spaces \( T_n \) by

for i to expL-1 do

T[i]:=NullSpace(A[i]-etaepsilon)

end do;

and the dimensions \( t_{i,j}(H) \) as the number of elements in a basis for the intersection of two such spaces by

\[
t[i,j]:=\text{nops(IntersectionBasis([T[i],T[j]]))};
\]

If we want to know the Hopf order of a specific element of \( H \) (represented by a vector of length the order of \( L \)), we can compute it, given the matrices for the power maps as above, by

\[
\text{HopfOrder}:=\text{proc}(h)
\]

# computes the Hopf order of an element

i:=1; # the least possible Hopf order is 1

while not Equal(A[i].h,etaepsilon.h) do

end while;
i:=i+1 # while the i-th power is not trivial, add one
end do;
i; # so this is the least i for which the i-th power is trivial
end proc;

In Section 3 we gave an example of an element of Hopf order 5 in $H = kC_4 \# kS_3$. This was found more or less by trial and error using the procedure HopfOrder. The elements of the basis $T[5]$ computed by Maple for the fifth trivial power space were found to have Hopf orders 2 and 3. The sum of a basis element of Hopf order 2 and one of Hopf order 3 had Hopf order 5.

To deal with $H^*$, it is sufficient to observe that the matrices for the $n$-th power map and $\eta \epsilon$ for $H^*$ can be found by transposing the matrices computed for $H$.

Now we should go about providing the numbers and procedures necessary to deal with the groups $F, G$, which we had assumed to be given above.

First, we implement the factorizable group $S_n \cong S_{n-1} \bowtie C_n$. We wrote down the factorization of $\sigma \in S_n$ as a product $\sigma = \sigma_1 \sigma_2$ for unique $\sigma_1 \in S_{n-1}$ and $\sigma_2 = (1 \ldots n)^k \in C_n$ in Lemma 3.2. The three procedures $S_nfactor2exp, S_nfactor2$, and $S_nfactor1$ below compute $k, \sigma_2,$ and $\sigma_1$, respectively, given $\sigma$ and the rank $n$ of the symmetric group as their two arguments. They are based on Maple’s group package. As auxiliary procedures we provide $applyinvperm$, which applies the inverse of its first argument, a permutation, to its second, a number, and $ncyclepower$ which provides the powers of the standard $n$-cycle, given $n$ as its first, and the exponent as its second argument. We provide and use a procedure multperms for multiplying permutations in the order used in this paper (acting on the left of elements), reversing the convention in Maple’s group package.

> Snfactor2exp:=proc(sigma,n)
>     option remember;
>     n-applyinvperm(sigma,n);
> end proc;
>
> Snfactor2:=proc(sigma,n)
>     option remember;
>     ncyclepower(n,Snfactor2exp(sigma,n));
> end proc;
>
> Snfactor1:=proc(sigma,n)
>     option remember;
>     multperms(sigma,invperm(Snfactor2(sigma,n)));
> end proc;
>
> applyinvperm:=proc(sigma,i);
>     # applies the inverse of permutation sigma to element i
>     j:=i; # the result will be j
>     for cyc in sigma do # for each cycle (a list)
>         if member(i,cyc,'k') # look if it contains i,
>             # and remember at which position
>             then if k=1 # if in the first position
>                 then j:=cyc[nops(cyc)] # the inverse cycle
>                 # maps i to the last element
> else j:=cyc[k-1] # otherwise to the predecessor
> end if
> end if
> end do;
> j;
> end proc;
>
> ncyclepower:=proc(n,e);
> convert([seq(i+ e mod n +1,i=0..n-1)],’disjcyc’);
> end proc;
>
> multperms:=proc(x,y);
> mulperms(y,x);
> end proc;

Now we fix a Maple variable \( N \) for the rank of a symmetric group (in practice, \( N = 4,5 \) were practicable and interesting), and set up a matched pair \((F, G, \triangleleft, \triangleright)\) with \( F \cong S_{N-1} \) and \( G \cong C_N \). Recall that the operations of the two groups on each other are obtained from a factorization in \( S_N \) of the product \( xa \) of \( x \in G \) and \( a \in F \).

> hit:=proc(x,a)
> Snfactor1(multperms(x,a),N);
> end proc;
>
> hitby:=proc(x,a)
> Snfactor2(multperms(x,a),N);
> end proc;

We also need all the other procedures for handling \( F \) and \( G \) that we assumed to exist above:

> FList:=elements(permgroup(N-1,\{ncycle(N-1),[[1,2]]\}));
> # To set up a bijection between elements of F and
> # numbers, we get a list of all elements of F.
> # Conversion then consists in looking up elements
> # in the list.
>
> ordF:=(N-1)!
> NumfromF:=proc(a)
> member(a,FList,’i’);i;
> end proc;
>
> FfromNum:=proc(i)
> FList[i];
> end proc;
>
> multF:=proc(a,b)
> # Multiplication in F is that of permutations.
> multperms(a,b);
> end proc;
> invF:=proc(a)
>  # Inverse in F is that for permutations.
>  invperm(a);
> end proc;
>
> oneF:=[];
>
> ordG:=N;
>
> NumfromG:=proc(x)
>  # The number we assign to a power of the N-cycle is
>  # the exponent, also the distance it moves 1.
>  applyperm(x,1);
> end proc;
>
> GfromNum:=proc(n)
>  ncyclepower(N,n-1);
> end proc;
>
> multG:=proc(s,t)
>  mulperms(t,s);
> end proc;
>
> invG:=proc(s)
>  invperm(s);
> end proc;
>
> oneG:=[];
>
> Since we are only dealing with a few ranks \(N\), it is much easier to simply enter the exponent \(\expL\) by hand than to devise a Maple procedure to compute it, so \(\expL=12\) for \(N=4\), and \(\expL=60\) for \(N=5\).

The same \hit\ and \hitby\ procedures that stem from the factorization of \(S_N\) can equally well be used for the factorization of \(A_N\), for odd \(N\) (in practice, \(N=5\)) into subgroups isomorphic to \(A_{N-1}\) and \(C_N\), respectively. The group \(G\) is the same as above, and the procedures for \(F\) only change where we set up a list of elements of \(F\) to enumerate the elements. For \(N=5\) we use

\[\text{Flist:=elements(permgroup(4,[[[1, 2, 3]], [[2, 3, 4]]])};\]
\[\text{expL:=30;}\]
to produce a list of all the elements of \(A_4\).

Next, let us discuss how to do our computations for the Drinfeld double of a group \(G\): We set up numbers and procedures \text{NumfromG, GfromNum, ordG, multG, invG}\ as before, and use the same procedures again for the group \(F=G\). The exponent of the group \(L \cong G \times G\) is the same as the exponent of \(G\). The action \(\triangleright\) should be trivial, and the action \(<\) should be the adjoint action of \(G\) on itself from the right:

\[\text{hit:=proc(x,a) a}; \text{end proc;}\]
>
hitby:=proc(x,a);
    option remember;
    multG(invG(a),multG(x,a));
end proc;

The computations for the Hopf algebra $k^G \otimes kG$, from which the double $D(kG)$ is obtained by a dual Drinfeld twist, can be done by defining both actions to be trivial. Computations for a group algebra or its dual can be done by defining one of the two groups $F, G$ as well as both actions to be trivial. Of course this is not particularly efficient, and was really done by different Maple procedures. Also, the computations for $k^G \otimes kG$ can be done more efficiently by computing the power matrices for $k^G$ first, and then taking their Kronecker products with their respective transposes, to obtain the power matrices for $k^G \otimes kG$. We have omitted listing these additional procedures to keep the paper to a reasonable length.

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