Stabilization in the Keller–Segel system
with signal-dependent sensitivity

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Abstract. This paper deals with the Keller–Segel system with signal-dependent sensitivity

\[ u_t = \Delta u - \chi \nabla \cdot (uS(v)\nabla v), \]
\[ v_t = \Delta v - v + u, \]

where \( \chi > 0 \) and \( S \) is a given function generalizing the sensitivity \( S(s) = \frac{1}{(a+s)^k} \), \( k > 1, a \geq 0 \), and shows exponential convergence of global classical solutions under an additional smallness condition condition for \( \chi > 0 \).

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1 Introduction

1.1 Long-term behaviour in chemotaxis models

Although mainly known for admitting solutions that blow up, the class of chemotaxis models also encompasses a large variety of systems whose solutions are global and bounded. In these situations the question of long-term behaviour of the solutions becomes significant. However, even in the most prototypical situation,

\[
\begin{aligned}
\frac{du}{dt} &= \Delta u - \chi \nabla \cdot (uS(v) \nabla v), \\
\frac{dv}{dt} &= \Delta v - v + u,
\end{aligned}
\]  

(1.1)

with \( S(v) \equiv 1 \) (and \( \chi > 0 \) so small that solutions in bounded two-dimensional domains are global), the answer to this question is not straightforward: Whereas each of the solutions converges to a stationary state, [3], the set of those steady state solutions is rather non-trivial, [1, 21, 20]. If \( S(v) \) takes a different form—prototypical choices being \( S(v) = \frac{1}{v} \) or \( S(v) = \frac{1}{(1+v)^k} \), see also [8, Sec. 2.2]—, the system (1.1) even loses the energy structure on which the proof of [3] is based.

In these situations, even global existence is only known under additional restrictions: If \( S(v) = \frac{1}{v} \), a smallness condition on \( \chi \) ([2, 24, 4, 12]), vastly different diffusion speeds ([5, 6]) or pursuance of a weaker solution concept ([24, 22, 13, 27]) have been needed for corresponding proofs.

Nevertheless, recently, posing even stricter smallness conditions on \( \chi \), Winkler and Yokota in [26] obtained global asymptotic stability of the homogeneous state \((\bar{u}_0, \bar{v}_0)\), thus highlighting the strong qualitative differences between the classical Keller–Segel model (where large perturbations of the stationary state usually lead to blow-up in finite time, see e.g. [25]) and chemotaxis systems with logarithmic sensitivity \((S(v) \nabla v = \nabla \log v)\).

For similar sensitivities, global existence has been assured in the radial setting and if the chemical diffuses fast ([5]), or, alternatively, whenever

\[
\chi < k(a + \eta)^{k-1} \sqrt{\frac{2}{n}},
\]

if \( S(v) \leq \frac{1}{(1+v)^k} \) with some \( k > 1, a \geq 0, [16] \), where \( \eta > 0 \) has the form discussed below.

It is the latter case that we want to examine in regards to its asymptotic behaviour in the present paper. This goal compels us to revisit the boundedness proof of [16], since now more quantitative information becomes necessary. Said bounds at hand, we can then begin adapting the reasoning of [26] about the large-time asymptotics.

In contrast to the situation there, not only smallness of the chemotaxis coefficient, but also, remarkably, largeness of the initial mass \( \int_{\Omega} u_0 \) lead to eventual equilibration.
1.2 Main results

In order to state the results, let us first introduce the precise setting: We investigate the chemotaxis system with signal-dependent sensitivity

\[
\begin{aligned}
&u_t = \Delta u - \chi \nabla \cdot (uS(v)\nabla v) \quad \text{in } \Omega \times (0, T), \\
v_t = \Delta v - v + u \quad \text{in } \Omega \times (0, T), \\
\partial_n u = \partial_n v = 0 \quad \text{in } \partial\Omega \times (0, T), \\
u(\cdot, 0) = u_0, \ v(\cdot, 0) = v_0 \quad \text{in } \Omega,
\end{aligned}
\]

(1.2)

where \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) is a bounded domain with smooth boundary \( \partial\Omega \), \( \partial_n \) denotes differentiation with respect to the outward normal of \( \partial\Omega \), \( \chi > 0 \) is a constant, \( S \) is a given function and \( u_0, v_0 \) are also given initial data satisfying

\[
0 \leq u_0 \in C^0(\overline{\Omega}) \setminus \{0\} \quad \text{and} \quad 0 \leq v_0 \in W^{1,\infty}(\Omega) \setminus \{0\}.
\]

(1.3)

The main result of this article will then be given by:

**Theorem 1.1.** For \( n \geq 2 \) let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundary. Let \( k > 1, M > 0 \) and \( v_* > 0, a \geq 0 \). Then there is \( \delta = \delta(a, k, M, v_*) > 0 \) such that for all functions

\[
S \in C^{1+\theta}((0, \infty)) \quad \text{for some } \theta \in (0, 1), \quad \text{with} \quad 0 \leq S(s) \leq \frac{1}{(a+s)^k},
\]

(1.4)

all initial data \( u_0, v_0 \) as in (1.3) and satisfying \( \int_{\Omega} u_0 = M \) and \( \min v_0 = v_* \) and for all \( \chi < \delta \), the problem (1.2) has a global classical solution

\[
\begin{aligned}
u & \in C^{2,1}(\overline{\Omega} \times (0, \infty)) \cap C^0(\overline{\Omega} \times [0, \infty)), \\
v & \in C^{2,1}(\overline{\Omega} \times (0, \infty)) \cap C^0(\overline{\Omega} \times [0, \infty)) \cap L^\infty((0, \infty); W^{1,\infty}(\Omega))
\end{aligned}
\]

(1.5)

and for this solution one can find \( \kappa > 0 \) and \( C > 0 \) such that

\[
\|u(\cdot, t) - \overline{u_0}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \overline{v_0}\|_{L^\infty(\Omega)} \leq Ce^{-\kappa t} \quad \text{for all } t > 0.
\]

(1.6)

**Remark 1.2.** In the case that \( a > 0, \delta \) can be chosen independently of \( M \) and \( v_* \).

The strategy for the proof of this result lies in identifying the functional

\[
\mathcal{F}(u, v)(t) := \int_{\Omega} (u(\cdot, t) - \overline{u_0})^2 + K \int_{\Omega} (v(\cdot, t) - \overline{v_0})^2
\]

(for some \( K > 0 \)) as eventual Lyapunov functional (cf. [26]). Apparent estimates for its dissipation rate show \( L^2 \)-convergence, which then by means of boundedness information for the solutions can be upgraded to \( L^\infty \)-convergence. One of the keys to construct an asymptotic estimate for
the Lyapunov functional is to obtain an asymptotic universal estimate for \( u_S(v) \). This estimate will be the objective of Section 2, where we will employ the function
\[
\varphi(s) := \exp \left\{ \frac{r}{(k - 1)(a + s)^{k-1}} \right\}, \quad s > 0,
\]
with some \( r > 0 \), which is similar to that used in [16], to establish the estimate as
\[
\limsup_{t \to \infty} \int_\Omega u^p \varphi(v) \leq C_1 \int_\Omega u_0
\]
with some \( C_1 > 0 \). Smoothing estimates for the heat semigroup will enable us to turn this into
\[
\limsup_{t \to \infty} \|u(\cdot, t) S(v(\cdot, t))\|_{L^\infty(\Omega)} \leq C_2
\]
with some \( C_2 > 0 \). If \( \chi C_2 \) is sufficiently small, this facilitates the Lyapunov type arguments alluded to above. (They will be given in Section 3.) It turns out that, due to \( k > 1 \) (cf. (2.12)), the “constants” \( C_1 \) and \( C_2 \) depend on \( M = \int_\Omega u_0 \) in such a way that actually large initial masses augment the chances for convergence:

**Theorem 1.3.** Let \( n \geq 2 \) and let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundary, let \( a \geq 0 \), \( k > 1 \), let \( v_* > 0 \) and \( \chi_0 \in (0, k(a + v_*)^{k-1} \sqrt{2/n}) \). Then there is \( M_0 > 0 \) such that for every \( M \geq M_0 \), for every \( \chi \in (0, \chi_0) \), every function \( S \) as in (1.4) and all initial data \( u_0, v_0 \) as in (1.3) that satisfy \( \int_\Omega u_0 = M \) and \( \min v_0 \geq v_* \), the problem (1.2) has a global classical solution converging as indicated in (1.6).

The proofs of both theorems (and of Remark 1.2) will be given at the end of Section 3.

**Notation.** While constants \( C_i \) are “local” to each proof, constants denoted by \( K_i \) are meant to be the same ones as introduced with the same name by a previous Lemma.

## 2 An asymptotic universal estimate for \( u_S(v) \)

Let us start with recalling some properties which have been established in previous studies and are fundamental when discussing results concerning the global existence of classical solutions in the setting of (1.2). For any \( T \in (0, \infty) \) the arguments employed in the proof of [6, Lemma 3.1] show that all classical solutions \((u, v)\) of (1.2) in \( \Omega \times (0, T) \) satisfy the following time-independent lower estimate for \( v \):
\[
v(x, t) \geq \eta \quad \text{for all } x \in \Omega \text{ and } t \in (0, T),
\]
where \( \eta \) is defined by
\[
\eta := 4 \left( 1 + \sqrt{1 + \frac{4v_*}{c_0 M}} \right)^{-2} v_*, \quad M = \int_\Omega u_0, \quad v_* = \min v_0,
\]
with \( c_0 > 0 \) being a lower bound for the fundamental solution of \( w_t = \Delta w - w \) with Neumann boundary condition. The formula in (2.1) is an explicit form of the expression given in [16, (1.5)]. In [16, Theorem 1.1], the above inequality was utilized to show existence of time-global classical solutions. We state it for reference in the next lemma.
Lemma 2.1. Assume that $S \in C^{1+\theta}([0,\infty))$, with some $\theta > 0$, satisfies (1.4) for some $a \geq 0$ and $k > 1$ and that $u_0, v_0$ fulfill (1.3). Let $\eta$ be given by (2.1) and suppose $\chi \in (0, k(a + \eta)k^{-1}\sqrt{2/\pi})$. Then the problem (1.2) possesses a unique global classical solution $(u, v)$ satisfying (1.5), $u > 0$ and $v > 0$ in $\Omega \times (0,\infty)$, and

$$u \in L^\infty(\Omega \times (0,\infty)) \quad \text{and} \quad v \in L^\infty(\Omega \times (0,\infty))$$

and moreover,

$$\int_{\Omega} u(\cdot, t) = \int_{\Omega} u_0 \quad \text{for all } t > 0.$$

In the case of $a = 0$, in order to control $S(v)$ for large time, we will need to consider asymptotic upper bounds on $\frac{1}{v^r}$ which do not depend on time. Hence, we will make use of the following lower estimate for $v$ which only depends on $\Omega$ and the mass of $u_0$, which was established in [26, Lemma 3.1].

Lemma 2.2. There exists $K_1 > 0$ such that whenever $(u, v)$ is a global classical solution of (1.2) for some $\chi > 0$ and some $(u_0, v_0)$ fulfilling (1.3), the inequality

$$\liminf_{t \to \infty} \inf_{x \in \Omega} v(x,t) \geq 2K_1 \int_{\Omega} u_0$$

holds.

In order to prepare a testing procedure suitable to our purpose, we will now consider a test function $\varphi \in C^2((0,\infty))$ which has a structural resemblance to the test function used in [16, Lemma 3.2]. To be precise, for $a \geq 0$ and $k > 1$ as in (1.4) and some $r > 0$ we define

$$\varphi(s) := \exp\left\{ \frac{r}{(k - 1)(a + s)^{k-1}} \right\} \quad \text{for } s > 0. \quad (2.2)$$

Obviously, from straightforward differentiation we find that

$$\varphi'(s) = -\frac{r}{(a + s)^k} \varphi(s) \quad \text{for all } s > 0, \quad (2.3)$$

which will be used in the next lemma to derive a differential inequality for functionals of the form $\int_{\Omega} u^p \varphi(v)$ with some $p > \frac{\alpha}{2}$.

Lemma 2.3. Let $r > 0$, $a \geq 0$, $k > 1$, $p > 1$, $\varepsilon \in (0,1)$, $\chi > 0$ and let $\varphi$ be the function defined by (2.2). Then for every global classical solution $(u, v)$ to (1.2), the inequality

$$\frac{d}{dt} \int_{\Omega} u^p \varphi(v) \leq -\varepsilon p(p - 1) \int_{\Omega} u^{p-2} \varphi(v) |\nabla u|^2 + \int_{\Omega} u^p H_{\varepsilon, p, r, \chi}(v) \varphi(v) |\nabla v|^2 + r \int_{\Omega} u^p \varphi(v) \frac{v - u}{(a + v)^k}$$

holds on $(0,\infty)$, with

$$H_{\varepsilon, p, r, \chi}(s) := \frac{\chi p r S(s)}{(a + s)^k} - \frac{r^2}{(a + s)^{2k}} - \frac{kr}{(a + s)^{k+1}} + \left( \frac{2pr}{(a + s)^2} + \chi p(p - 1)S(s) \right)^2 \quad \text{for } s > 0. \quad (2.4)$$
Proof. From straightforward calculations, while relying on (2.3), we derive that
\[
\frac{d}{dt} \int_\Omega u^p \varphi(v) = p \int_\Omega u^{p-1} \varphi(v) (\Delta u - \chi \nabla \cdot (uS(v) \nabla v)) - r \int_\Omega u^p \frac{\varphi(v)}{(a + v)^k} (\Delta v - v + u) \tag{2.5}
\]
on \((0, \infty)\). Here, noting from integration by parts and (2.3) that
\[
p \int_\Omega u^{p-1} \varphi(v) \Delta u = -p(p - 1) \int_\Omega u^{p-2} \varphi(v) |\nabla u|^2 + pr \int_\Omega u^{p-1} \frac{\varphi(v)}{(a + v)^k} \nabla u \cdot \nabla v
\]
and
\[
-\chi p \int_\Omega u^{p-1} \varphi(v) \nabla \cdot (uS(v) \nabla v) = \chi p(p - 1) \int_\Omega u^{p-1} S(v) \varphi(v) \nabla u \cdot \nabla v
\]
\[-\chi pr \int_\Omega u^p S(v) \frac{\varphi(v)}{(a + v)^k} |\nabla v|^2
\]
as well as
\[
-r \int_\Omega u^p \frac{\varphi(v)}{(a + v)^k} \Delta v = rp \int_\Omega u^{p-1} \frac{\varphi(v)}{(a + v)^k} \nabla u \cdot \nabla v - r^2 \int_\Omega u^{p-1} \frac{\varphi(v)}{(a + v)^{2k}} |\nabla v|^2
\]
\[-kr \int_\Omega u^p \frac{\varphi(v)}{(a + v)^{k+1}} |\nabla v|^2
\]
hold on \((0, \infty)\), we obtain from (2.5) that
\[
\frac{d}{dt} \int_\Omega u^p \varphi(v)
\]
\[= -p(p - 1) \int_\Omega u^{p-2} \varphi(v) |\nabla u|^2 + \int_\Omega u^{p-1} \varphi(v) \left( \frac{2pr}{(a + v)^k} + \chi p(p - 1)S(v) \right) \nabla u \cdot \nabla v
\]
\[+ \int_\Omega u^p \left( -\chi pr S(v) \right) (a + v)^k
\]
\[-r^2 \frac{2pr}{(a + v)^{2k}} - kr \frac{2pr}{(a + v)^{k+1}} \right) \varphi(v) |\nabla v|^2 + r \int_\Omega u^p \varphi(v) \frac{v - u}{(a + v)^k}.
\]
Now we let \(\varepsilon \in (0, 1)\). Then from Young’s inequality we have
\[
u^{p-1} \varphi(v) \left( \frac{2pr}{(a + v)^k} + \chi p(p - 1)S(v) \right) \nabla u \cdot \nabla v \leq (1 - \varepsilon)p(p - 1)u^{p-2} \varphi(v) |\nabla u|^2
\]
\[+ \left( \frac{2pr}{(a + v)^k} + \chi p(p - 1)S(v) \right)^2 \frac{4(1 - \varepsilon)p(p - 1)}{u^p \varphi(v) |\nabla v|^2}
\]
and infer that
\[
\frac{d}{dt} \int_\Omega u^p \varphi(v) \leq -\varepsilon p(p - 1) \int_\Omega u^{p-2} \varphi(v) |\nabla u|^2 + \int_\Omega u^p H_{\varepsilon,p,r,\chi}(v) \varphi(v) |\nabla v|^2 + r \int_\Omega u^p \varphi(v) \frac{v - u}{(a + v)^k},
\]
where \(H_{\varepsilon,p,r,\chi}\) is the function defined by (2.4), which completes the proof. \(\square\)
Observing that the differential inequality only depends on \( \chi \) inside the function \( H_{\varepsilon,p,r,\chi} \), we can conclude that whenever the sign of \( H_{\varepsilon,p,r,\chi} \) is non-positive the chemotactic influence in this inequality is negligible. Our next aim is to verify that one can find a suitable combination of parameters \( \varepsilon,p,r \) and \( \chi_0 \) such that \( H_{\varepsilon,p,r,\chi} \) is bounded from above by zero independently of \( \chi \in (0, \chi_0) \).

**Lemma 2.4.** Let \( a \geq 0, k > 1 \) and \( \eta > 0 \). For all \( \chi_0 \in (0, k(a + \eta)^{k-1} \sqrt{\frac{2}{n}}) \) there exist \( p = p(\chi_0, a, k, \eta) > \frac{n}{2} \) and \( \varepsilon = \varepsilon(\chi_0, a, k, \eta) \in (0, 1) \) such that

\[
H_{\varepsilon,p,r,\chi}(s) \leq 0 \quad \text{for all } s \geq \eta
\]

holds for all \( \chi \in (0, \chi_0] \) with

\[
r := \frac{(p-1)\chi_0}{2} \sqrt{\frac{p}{1 + \varepsilon p - \varepsilon}}.
\]

**Proof.** Since \( \chi_0 < k(a + \eta)^{k-1} \sqrt{\frac{2}{n}} \), we can find \( p = p(\chi_0, a, k, \eta) \in (\frac{n}{2}, n) \) and \( \varepsilon = \varepsilon(\chi_0, a, k, \eta) \in (0, 1) \) such that

\[
\frac{\chi_0}{k(1-\varepsilon)} \sqrt{p(1 + \varepsilon p - \varepsilon)} + \varepsilon p \chi_0 \leq (a + \eta)^{k-1}.
\]

Now we let \( r \) be given as in (2.6). Then straightforward calculations using condition (1.4) and the fact \( \chi \leq \chi_0 \) imply that

\[
H_{\varepsilon,p,r,\chi}(s) = \frac{(1 + \varepsilon p - \varepsilon)}{(1-\varepsilon)(p-1)(a + s)^k} + \frac{\varepsilon p \chi r S(s)}{(1-\varepsilon)(a + s)^k} + \frac{p(p-1)\chi_0^2 S^2(s)}{4(1-\varepsilon)} \leq \frac{kr}{(a + s)^{k+1}}
\]

\[
\leq \frac{kr}{(a + s)^{2k}} \left( \frac{(1 + \varepsilon p - \varepsilon)}{k(1-\varepsilon)(p-1)} + \frac{\varepsilon p \chi_0}{k(1-\varepsilon)} + \frac{p(p-1)\chi_0^2}{4k(1-\varepsilon)r} - (a + s)^{k-1} \right).
\]

Here, noting from the definition of \( r \) and \( p, \varepsilon \) that

\[
\frac{(1 + \varepsilon p - \varepsilon)}{k(1-\varepsilon)(p-1)} + \frac{\varepsilon p \chi_0}{k(1-\varepsilon)} + \frac{p(p-1)\chi_0^2}{4k(1-\varepsilon)r} = \frac{\chi_0}{k(1-\varepsilon)} \sqrt{p(1 + \varepsilon p - \varepsilon)} + \varepsilon p \chi_0 \leq (a + \eta)^{k-1},
\]

we can verify that

\[
H_{\varepsilon,p,r,\chi}(s) \leq \frac{kr((a + \eta)^{k-1} - (a + s)^{k-1})}{(a + s)^{2k}} \leq 0 \quad \text{for all } s \geq \eta
\]

is valid. \( \square \)

Combining the Lemmata 2.3 and 2.4, we can now derive the following asymptotic \( L^p \)-estimate of the first solution component for a certain choice of \( p > \frac{n}{2} \).

**Lemma 2.5.** Let \( M_0 \geq 0, a \geq 0, k > 1 \) and suppose that \( S \) satisfies (1.4). Then for all \( \chi_0 \in (0, k(a + K_1 M_0)^{k-1} \sqrt{\frac{2}{n}}) \) there exist \( p = p(\chi_0, a, k, M_0) > \frac{n}{2} \) and \( K_2 = K_2(\chi_0, a, k, M_0) \) such that whenever \((u, v)\) is a global classical solution of (1.2) with \( \chi \leq \chi_0 \) and \((u_0, v_0)\) fulfilling (1.3) as well as \( \int_{\Omega} u_0 = M \geq M_0 \), we have

\[
\limsup_{t \to \infty} \|u(\cdot, t)\|_{L^p(\Omega)} \leq K_2 \int_{\Omega} u_0.
\]
Proof. We first note that, aided by Lemma 2.2, we can find \( t_0 > 0 \) such that
\[
v(x, t) \geq K_1 M \geq K_1 M_0 \quad \text{for all } x \in \Omega \text{ and all } t > t_0.
\]
Then combining Lemma 2.3 and Lemma 2.4 and choosing \( p = p(\chi_0, a, k, K_1 M_0) \) and \( \varepsilon = \varepsilon(\chi_0, a, k, K_1 M_0) \) as in the latter, we derive that with \( \varphi \) as in (2.2)
\[
\frac{d}{dt} \int_{\Omega} u^p \varphi(v) \leq -\varepsilon(p-1) \int_{\Omega} u^{p-2} \varphi(v) |\nabla u|^2 + r \int_{\Omega} u^p \varphi(v) \frac{v - u}{(a + v)^k}
\]
holds for all \( t > t_0 \). From positivity of \( u \) and the definition of \( r > 0 \) in (2.6) we obtain that
\[
\frac{d}{dt} \int_{\Omega} u^p \varphi(v) \leq -\varepsilon(p-1) \int_{\Omega} u^{p-2} \varphi(v) |\nabla u|^2 + C_1 \int_{\Omega} u^p \varphi(v)
\]
for all \( t > t_1 \) with \( C_1 := \frac{(p-1)\chi_0}{2} \sqrt{\frac{p}{1+\varepsilon p - \varepsilon}} \frac{m}{m_*} \), where \( m_* := \max\{\frac{a}{k-1}, K_1 M_0\} \). Here we note that
\[
C_\varphi := \exp\left\{ -\frac{(p-1)\chi_0}{2(k-1)(a+K_1 M_0)^{k-1}} \sqrt{\frac{p}{1+\varepsilon p - \varepsilon}} \right\} \leq \varphi(s) \leq 1 \quad \text{for all } s \geq K_1 M_0. \quad (2.8)
\]
Thanks to the upper estimate in (2.8), the Gagliardo–Nirenberg inequality and the mass conservation law entail that on \((t_0, \infty)\)
\[
\int_{\Omega} u^p \varphi(v) \leq \int_{\Omega} u^p = \|u^p\|_{L^2(\Omega)}^2 \leq C_{GN} \left( \|\nabla u_p\|_{L^2(\Omega)}^2 + \|u_p\|_{L^p(\Omega)}^2 \right)^{\frac{b}{2}} \|u_p\|_{L^p(\Omega)}^{2(1-b)}
\]
\[
= C_{GN} \left( \int_{\Omega} |\nabla u_p|^2 + M^p \right)^{\frac{b}{2}} M^{p(1-b)}
\]
holds with \( b := \frac{(p-1)\chi_0}{(p-1)n+2} \in (0, 1) \) and some \( C_{GN} > 0 \), which means that
\[
\int_{\Omega} u^{p-2} \varphi(v) |\nabla u|^2 \geq \frac{4C_\varphi}{p^2} \int_{\Omega} |\nabla u_p|^2 \geq \frac{4C_\varphi}{p^2} \left( C_{GN} M^{p(1-b)} \right)^{-\frac{1}{b}} \left( \int_{\Omega} u^p \varphi(v) \right)^{\frac{1}{b}} - M^p
\]
for all \( t > t_0 \). Therefore we have from Young’s inequality that
\[
\frac{d}{dt} \int_{\Omega} u^p \varphi(v) \leq -C_2 (C_{GN} M^{p(1-b)})^{-\frac{1}{b}} \left( \int_{\Omega} u^p \varphi(v) \right)^{\frac{1}{b}} + C_2 M^p + C_1 \int_{\Omega} u^p \varphi(v)
\]
\[
\leq -\frac{C_2 (C_{GN} M^{p(1-b)})^{-\frac{1}{b}}}{2} \left( \int_{\Omega} u^p \varphi(v) \right)^{\frac{1}{b}} + C_3 M^p
\]
for all \( t > t_0 \), where \( C_2 := \frac{4\varepsilon(p-1)C_{\varphi}}{p} \) and \( C_3 := C_2 + (1-b)(C_{GN})^{1-p}(\frac{2b}{C_2})^{\frac{b}{p}}(\Omega) \), which with (2.8) implies that
\[
\limsup_{t \to \infty} \|u(\cdot, t)\|_{L^p(\Omega)} \leq \frac{1}{C_{\varphi}^p} \left( \limsup_{t \to \infty} \int_{\Omega} u^p \varphi(v) \right)^{\frac{1}{p}} \leq \frac{1}{C_{\varphi}^p} \left( \frac{2C_3 M^p}{C_2} \left( C_{GN} M^{p(1-b)} \right)^{\frac{1}{b}} \right)^{\frac{1}{p}} \leq \left( \frac{C_{GN}}{C_{\varphi}} \right)^{\frac{1}{p}} \left( \frac{2C_3}{C_2} \right)^{\frac{b}{p}} M
\]
because $\frac{1}{b} > 1$. Thus, (2.7) holds with $K_2 := (\frac{C_{\text{max}}}{C_\omega})^{\frac{1}{b}} (\frac{2C_\sigma}{C_2})^{\frac{b}{p}}$.

Still striving for an asymptotic $L^\infty$-estimate for $uS(v)$, we nevertheless need to obtain additional regularity information on $\nabla v$, since when estimating $u$, we lack control on the crucial term $uS(v)\nabla v$ with our current knowledge. In particular, an $L^p$-estimate for some $q_0 > n$ would suffice for our purpose. Fortunately, the regularity of $\nabla v$ is directly linked to the $L^p$-regularity of $u$, as illustrated by the following result (cf. [26, Lemma 3.2]).

**Lemma 2.6.** Let $\mu \geq 1$ and $\lambda \geq 1$ be such that $\lambda < \frac{np}{(n-p)q}$. Then there is $K_3 = K_3(\mu, \lambda) > 0$ such that whenever $(u, v)$ is a global classical solution of (1.2), for any $S, u_0, v_0$ as in (1.4) and (1.3), respectively, then the inequality

$$\limsup_{t \to \infty} \| v(\cdot, t) \|_{W^{1, \lambda}(\Omega)} \leq K_3 \limsup_{t \to \infty} \| u(\cdot, t) \|_{L^p(\Omega)}$$

holds.

In light of this result and Lemma 2.5 we can now draw on quite standard smoothing properties of the Neumann heat-semigroup to derive an asymptotic $L^\infty$-estimate for $u$.

**Lemma 2.7.** Let $\sigma \in (0, \lambda_1)$, where $\lambda_1 > 0$ denotes the first nonzero eigenvalue of the Neumann Laplacian in $\Omega$. Let $a \geq 0$, $k > 1$ and $M_0 \geq 0$. For all $\chi_0 \in (0, k(a + K_1M_0)^{k^{-1}} \sqrt{\frac{2}{n}})$ there are $\theta = \theta(\chi_0, a, k, M_0) > n$ and $\alpha = \alpha(\chi_0, a, k, M_0) < 1$ as well as $K_4 > 0$ and $K_5 > 0$ such that the following holds: If $(u, v)$ is a global classical solution of (1.2) with $\chi \leq \chi_0$ and $(u_0, v_0)$ satisfying (1.3) and $\int_\Omega u_0 = M \geq M_0$, then

$$\limsup_{t \to \infty} \| A^\alpha u(\cdot, t) \|_{L^p(\Omega)} \leq K_4 \int_\Omega u_0,$$

where $A$ denotes the sectorial realization of $-\Delta + \sigma$ in $L^\theta(\Omega)$ under homogeneous Neumann boundary conditions; moreover,

$$\limsup_{t \to \infty} \| u(\cdot, t) \|_{L^\infty(\Omega)} \leq \frac{K_5}{2} \int_\Omega u_0.$$

**Proof.** From Lemmata 2.2, 2.5 and 2.6 we can find $p > \frac{n}{2}$, $q \in (n, \frac{np}{(n-p)q})$ and $t_0 > 0$ such that

$$v(x, t) \geq K_1 \int_\Omega u_0 \quad \text{for all } x \in \Omega \text{ and all } t > t_0 \quad (2.9)$$

and such that

$$\| u(\cdot, t) \|_{L^p(\Omega)} \leq 2K_2 \int_\Omega u_0, \quad \| v(\cdot, t) \|_{W^{1, q}(\Omega)} \leq 2K_2K_3 \int_\Omega u_0 \quad \text{for all } t > t_0 \quad (2.10)$$

hold. Now for $\theta \in (n, q)$ we let $\alpha \in \left( \frac{n}{2} \min \left\{ 1 - \frac{n}{2} \left( \frac{1}{p} \left( \frac{1}{p} - \frac{1}{q} \right) \right), \frac{1}{2} \right\} \right)$ and put $\beta := \alpha + \frac{4}{2} \left( \frac{1}{p} - \frac{1}{q} \right) < 1$, $\gamma := \frac{1}{2} + \alpha < 1$, and moreover, for $T > t_0$ we define

$$Z(T) := \sup_{t \in (t_0, T]} \left( 1 + (t - t_0)^{-\beta} \right)^{-1} \| A^\alpha u(\cdot, t) \|_{L^p(\Omega)}. $$

9
Aided by the variation-of-constants representation of $u$, we have that, due to (2.9) and (2.10),

$$
\|A^\alpha u(\cdot, t)\|_{L^p(\Omega)} \leq \left\| A^\alpha e^{(t-t_0)\Delta} u(\cdot, t_0) \right\|_{L^p(\Omega)} + \chi \int_{t_0}^t \left\| A^\alpha e^{(t-s)\Delta} \nabla \cdot (uS(v)\nabla v)(\cdot, s) \right\|_{L^p(\Omega)} ds
$$

holds for all $t > t_0$. Here we note from the continuous embedding $D(A^\alpha) \hookrightarrow L^\infty(\Omega)$ (see [7, Theorem 1.6.1])

$$
\|\varphi\|_{L^\infty(\Omega)} \leq C_E \|A^\alpha \varphi\|_{L^p(\Omega)} \quad \text{for all } \varphi \in D(A^\alpha)
$$

(2.11)

with some $C_E > 0$ that for all $s > t_0$

$$
\left\| (uS(v)\nabla v)(\cdot, s) \right\|_{L^p(\Omega)} \leq \frac{1}{(a + K_1 \int_\Omega u_0)^k} \|u(\cdot, s)\|_{L^1(\Omega)}^{1-c} \|u(\cdot, s)\|_{L^\infty(\Omega)} \|\nabla v(\cdot, s)\|_{L^p(\Omega)}
$$

$$
\leq \frac{2C_E K_2 \int_\Omega u_0}{(a + K_1 \int_\Omega u_0)^k} \left( \int_\Omega u_0 \right)^{1-c} Z^c(T) \left( 1 + (s - t_0)^{-\beta} \right)^c
$$

$$
\leq \frac{2C_E K_2 K_3}{(a + K_1 M)^k} \left( M^{2-c} \right) Z^c(T) \left( 1 + (s - t_0)^{-\beta} \right)^c
$$

with $c := 1 - \frac{\alpha - \beta}{2p} \in (0, 1)$.

Noting from our choice of $\sigma$ and the known smoothing properties of the Neumann heat semigroup (see [7, Theorem 1.4.3] and [23, Lemma 1.3 (iv)]) that

$$
\left\| A^\alpha e^{(t-t_0)\Delta} u(\cdot, t_0) \right\|_{L^p(\Omega)} \leq C_{S_1} \left( 1 + (t - t_0)^{-\beta} \right) \|u(\cdot, t_0)\|_{L^p(\Omega)}
$$

$$
\leq 2K_2 C_{S_1} \left( 1 + (t - t_0)^{-\beta} \right) \int_\Omega u_0
$$

and

$$
\chi \left\| A^\alpha e^{(t-s)\Delta} \nabla \cdot (uS(v)\nabla v)(\cdot, s) \right\|_{L^p(\Omega)}
$$

$$
\leq C_{S_2} \chi_0(t - s)^{-\gamma} e^{-\lambda(t-s)} \left\| (uS(v)\nabla v)(\cdot, s) \right\|_{L^p(\Omega)}
$$

$$
\leq C_1 \frac{M^{2-c}}{(a + K_1 M)^k} Z^c(T)(t - s)^{-\gamma} e^{-\lambda(t-s)} \left( 1 + (t - t_0)^{-\beta} \right)^c
$$

with some $C_{S_1}, C_{S_2}, \lambda > 0$ and $C_1 := 2C_E C_{S_2} K_2 K_3 \chi_0$, we have from the inequality

$$
\int_{t_0}^t (t - s)^{-\gamma} e^{-\lambda(t-s)} \left( 1 + (s - t_0)^{-\beta} \right)^c ds \leq L \left( 1 + (t - t_0)^{-\beta} \right) \quad \text{for all } t > t_0
$$

with some $L = L(\beta, \gamma, \lambda, c) > 0$, obtained in [26, Lemma 3.5], that

$$
\left( 1 + (t - t_0)^{-\beta} \right)^{-1} \|A^\alpha u(\cdot, t)\|_{L^p(\Omega)} \leq 2K_2 C_{S_1} \int_\Omega u_0 + C_1 L \frac{M^{2-c}}{(a + K_1 M)^k} Z^c(T).
$$
This together with Young’s inequality
\[ C_1L \left( \int_{\Omega} u_0 \right)^{1-c} Z^c(T) \leq (1 - c)(2c)^{\frac{c}{1-c}} \left( C_1L \frac{M^{2-c}}{(a + K_1M)^k} \right)^{\frac{1}{1-c}} + \frac{1}{2} Z(T) \]
enables us to see that
\[ Z(T) \leq C_2 \left( 1 + \left( \frac{M}{(a + K_1M)^k} \right)^{\frac{1}{1-c}} \right) \int_{\Omega} u_0 \quad \text{for all } T > t_0 \]
with \( C_2 := 2 \max \{2K_2C_{S_1}, (1 - c)(2c)^{\frac{c}{1-c}}(C_1L)^{\frac{1}{1-c}} \} \). Therefore we attain that
\[ \|A^\alpha u(\cdot, t)\|_{L^p(\Omega)} \leq \left( 1 + (t - t_0)^{-\beta} \right) Z(T) \]
\[ \leq C_3 \left( 1 + \left( \frac{M}{(a + K_1M)^k} \right)^{\frac{1}{1-c}} \right) \int_{\Omega} u_0 \quad \text{for all } t > t_0 + 1 \]
with \( C_3 := 2C_2 \). Here, we finally set \( K_4 := C_3 \left( \frac{m_*}{(a + K_1M)^k} \right)^{\frac{1}{1-c}} \), where \( m_* = \max \{ \frac{a}{K_1(k-1)}, M_0 \} \).

To verify the second assertion, we make use of the first part of the lemma, to find that there exists some \( t_1 > 0 \) such that
\[ \|A^\alpha u(\cdot, t)\|_{L^p(\Omega)} \leq 2K_4 \int_{\Omega} u_0 \]
is valid for all \( t > t_1 \). Then, we employ (2.11) to find that
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq 2C_EK_4 \int_{\Omega} u_0 \quad \text{for all } t > t_1 \]
which, by choice of \( K_5 := 4C_EK_4 \), completes the proof. \( \square \)

We can now establish an asymptotic universal bound on \( uS(v) \), which will be a key point in the proof of Theorem 1.1.

**Lemma 2.8.** Let \( a \geq 0, k > 1 \) and \( M_0 \geq 0 \). For all \( \chi \in (0, k(a + K_1M_0)^{k-1} \sqrt{\frac{2}{n}}) \) and with \( K_5 = K_5(\chi, a, k, M_0) \) as in the previous lemma, the following holds: Whenever \((u, v)\) is a global classical solution of (1.2) with \( \chi \leq \chi_0 \) and \((u_0, v_0)\) satisfying (1.3) and \( \int_{\Omega} u_0 = M \geq M_0 \), we have
\[ \limsup_{t \to \infty} \|u(\cdot, t)S(v(\cdot, t))\|_{L^\infty(\Omega)} \leq K_5 \frac{M}{(a + K_1M)^k}. \]  
(2.12)

**Proof.** Thanks to Lemmata 2.2 and 2.7, we can find \( t_0 > 0 \) such that
\[ v(x, t) \geq K_1 \int_{\Omega} u_0 \quad \text{for all } x \in \Omega, \; t > t_0 \]
and such that
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K_5 \int_\Omega u_0 \quad \text{for all } t > t_0. \]
Hence, we immediately obtain that for all \( t > t_0 \),
\[ \|u(\cdot, t)S(v(\cdot, t))\|_{L^\infty(\Omega)} \leq \frac{K_5 \int_\Omega u_0}{(a + K_1 \int_\Omega u_0)^k}. \]

3 An asymptotic estimate for the Lyapunov functional

As the existence part of the main theorem is covered by the previous section, we will now turn
our attention to verifying the desired convergence result. Inspired by the approach undertaken
in [26], we will consider the functional
\[ F(u, v)(t) := \int_\Omega (u(\cdot, t) - \bar{\Omega})^2 + K \int_\Omega (v(\cdot, t) - \bar{\Omega})^2 \]  
(3.1)
with some \( K > 0 \), which, at least for later times, acts as a Lyapunov function al to the system
under consideration. We start by establishing a differential inequality for the first solution
component.

**Lemma 3.1.** Let \( a \geq 0, k > 1, M_0 \geq 0 \) and \( \chi_0 \in (0, k(a + K_1 M_0)^{k-1} \sqrt{2/\pi}) \). Then every global
classical solution \((u, v)\) of (1.2) with some \( S \) as in (1.4), \( \chi \leq \chi_0 \) and \((u_0, v_0)\) fulfilling (1.3) and
\( \int_\Omega u_0 = M \geq M_0 \) satisfies
\[ \frac{d}{dt} \int_\Omega (u - \bar{\Omega})^2 + \int_\Omega |\nabla u|^2 \leq 4K_5^2 \frac{M^2}{(a + K_1 M)^{2k}} \chi^2 \int_\Omega |\nabla v|^2 \quad \text{on } (t^*, \infty) \]
for some \( t^* > 0 \), with \( K_5 = K_5(\chi_0, a, k, M_0) \) provided by Lemma 2.7.

**Proof.** From Lemma 2.8 there is \( t_0 > 0 \) such that
\[ u(\cdot, t)S(v(\cdot, t)) \leq 2K_5 \frac{M}{(a + K_1 M)^{k}} \quad \text{for all } t > t_0. \]
Then testing the first equation of (1.2) by \( \frac{1}{2}(u - u_0) \) and using integration by parts show
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega (u - u_0)^2 = - \int_\Omega |\nabla u|^2 + \chi \int_\Omega uS(v) \nabla u \cdot \nabla v \quad \text{for all } t > t_0. \]  
(3.2)
Here we use Young’s inequality to see that
\[ \chi \int_\Omega uS(v) \nabla u \cdot \nabla v \leq \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{\chi^2}{2} \int_\Omega |uS(v)|^2 |\nabla v|^2 \]
\[ \leq \frac{1}{2} \int_\Omega |\nabla u|^2 + 2\chi^2 K_5^2 \frac{M^2}{(a + K_1 M)^{2k}} \int_\Omega |\nabla v|^2 \]  
(3.3)
ia valid for all \( t > t_0 \). Therefore a combination of (3.2) and (3.3) directly implies this lemma. \( \square \)
In the next lemma we will investigate the time-evolution of the second part of the Lyapunov-functional.

**Lemma 3.2.** Every global classical solution \((u, v)\) of (1.2), for any choice of initial data permitted by (1.3) and \(S\) as in (1.4), satisfies

\[
\frac{d}{dt} \int_{\Omega} (v - \overline{u_0})^2 + 2 \int_{\Omega} |\nabla v|^2 + \int_{\Omega} (v - \overline{u_0})^2 \leq \int_{\Omega} (u - \overline{u_0})^2 \quad \text{for all } t > 0.
\]

**Proof.** Testing the second equation of (1.2) by \(v - \overline{u_0}\), we have from the Young’s inequality that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (v - \overline{u_0})^2 = - \int_{\Omega} |\nabla v|^2 - \int_{\Omega} (v - \overline{u_0})^2 + \int_{\Omega} (u - \overline{u_0})(v - \overline{u_0}) \leq - \int_{\Omega} |\nabla v|^2 - \frac{1}{2} \int_{\Omega} (v - \overline{u_0})^2 + \frac{1}{2} \int_{\Omega} (u - \overline{u_0})^2
\]

holds for all \(t > 0\).

Combining the previous two lemmata, for suitable choice of \(K > 0\), we can make use of (3.1) to obtain convergence of solutions towards the spatial mean of \(u_0\) in \(L^2(\Omega)\) with an exponential rate.

**Lemma 3.3.** Let \(a \geq 0, k > 1, M_0 \geq 0\) and \(\chi_0 \in (0, k(a + K_1 M_0)^{k-1} \sqrt{2/\pi})\). Then there exists \(\delta = \delta(\chi_0, a, k, M_0) > 0\) such that, if \(\chi \leq \chi_0\) and \(M \geq M_0\) satisfy

\[
\frac{M}{(a + K_1 M)^k} \chi < \delta,
\]

then for every global solution \((u, v)\) of (1.2) with \(S, u_0, v_0\) as in (1.4) and (1.3) and with \(\int_{\Omega} u_0 = M\) there are \(K_6, \ell > 0\) and \(t^* > 0\) such that

\[
\|u(\cdot, t) - \overline{u_0}\|_{L^2(\Omega)} + \|v(\cdot, t) - \overline{u_0}\|_{L^2(\Omega)} \leq K_6 e^{-\ell t} \quad \text{for all } t > t^*.
\]

**Proof.** We first note from Poincaré’s inequality that there is \(C_P > 0\) such that

\[
\int_{\Omega} (\varphi - \overline{\varphi})^2 \leq C_P \int_{\Omega} |\nabla \varphi|^2 \quad \text{for all } \varphi \in W^{1,2}(\Omega).
\]

We put

\[
\delta := \frac{1}{K_5} \sqrt{\frac{1}{2C_P}}
\]

and assume that \(\frac{M}{(a + K_1 M)^k} \chi < \delta\). Then we can choose \(K > 0\) such that

\[
K \in \left[ 2K_5^2 \left( \frac{M^2}{(a + K_1 M)^{2k}} \chi^2 \frac{1}{C_P} \right) \right].
\]
Combination of Lemmata 3.1 and 3.2 entails that
\[
\frac{d}{dt} \left( \int_{\Omega} (u - \overline{u})^2 + K \int_{\Omega} (v - \overline{u})^2 \right) + \frac{1}{C_p} \int_{\Omega} (u - \overline{u})^2 + 2K \int_{\Omega} \| \nabla v \|^2 + K \int_{\Omega} (v - \overline{u})^2 \\
\leq 4K^2 \frac{M^2}{(a + K_1 M)^2 \chi^2} \int_{\Omega} \| \nabla v \|^2 + K \int_{\Omega} (u - \overline{u})^2
\]
holds for all \( t > t_0 \), where we utilized that
\[
\int_{\Omega} (u - \overline{u})^2 = \int_{\Omega} (u - \overline{w})^2 \leq C_p \int_{\Omega} \| \nabla u \|^2 \quad \text{for all } t > 0.
\]
Then, aided by the definition of \( K > 0 \), we have that
\[
\frac{d}{dt} \left( \int_{\Omega} (u - \overline{u})^2 + K \int_{\Omega} (v - \overline{u})^2 \right) + C_1 \left( \int_{\Omega} (u - \overline{u})^2 + K \int_{\Omega} (v - \overline{u})^2 \right) \leq 0
\]
with \( C_1 := \min\{1, \frac{1}{C_p} - K\} \). This means that (3.4) holds with some \( K_0 > 0 \) and some \( \ell > 0 \). \( \Box \)

In fact, drawing on the bounds provided by Lemmata 2.6 and 2.7, we can refine the exponential convergence in \( L^2(\Omega) \) to an exponential convergence in \( L^\infty(\Omega) \).

**Lemma 3.4.** Under the assumptions of Lemma 3.3, there exist \( K_7 > 0, \kappa > 0 \) and \( t_* > 0 \) such that
\[
\| u - \overline{u_0} \|_{L^\infty(\Omega)} + \| v - \overline{u_0} \|_{L^\infty(\Omega)} \leq K_7 e^{-\kappa t} \quad \text{for all } t > t_*.
\] (3.5)

**Proof.** The proof is based on the arguments in [26, Proof of Theorem 1.1]. Let \( p \in \left( \frac{n}{q}, n \right) \) be as in Lemma 2.5 and let \( q > n \) be such that \( q < \frac{np}{n-p} \). In light of Lemmata 2.5 and 2.6, we can find \( t_0 > 0 \) such that
\[
\| v(\cdot, t) \|_{W^{1,q}(\Omega)} \leq 2K_2K_3 \int_{\Omega} u_0 \quad \text{for all } t > t_0.
\]
Then the Gagliardo–Nirenberg inequality enables us to see that
\[
\| v(\cdot, t) - \overline{u_0} \|_{L^\infty(\Omega)} \leq C_1 \| v(\cdot, t) - \overline{u_0} \|_{W^{1,q}(\Omega)}^{d_1} \| v(\cdot, t) - \overline{u_0} \|_{L^2(\Omega)}^{1-d_1} \leq C_1 \left( 2K_2K_3 + \| \Omega \|^{\frac{1}{q} \overline{u_0}} \right)^{d_1} \| v(\cdot, t) - \overline{u_0} \|_{L^2(\Omega)}^{1-d_1}
\]
with some \( C_1 > 0 \) and \( d_1 := \frac{nq}{2q+nq-2n} \in (0, 1) \), which with Lemma 3.3 shows that there is \( t_1 > t_0 \) such that
\[
\| v(\cdot, t) - \overline{u_0} \|_{L^\infty(\Omega)} \leq C_2 e^{-(1-d_1)\ell t} \quad \text{for all } t > t_1
\]
with \( C_2 := C_1K_6(2K_2K_3 + \| \Omega \|^{\frac{1}{q} \overline{u_0}})^{d_1} \). On the other hand, we next verify that
\[
\| u(\cdot, t) - \overline{u_0} \|_{L^\infty(\Omega)} \leq Ce^{-\kappa t} \quad \text{for all } t > T.
\] (3.6)
with some \( C, k, T > 0 \). Let \( \theta > n \) and let \( \alpha \in \left( \frac{\theta}{2}, 1 \right) \). Letting \( A \) again denote the sectorial realization of \( -\Delta + \sigma \) in \( L^\theta(\Omega) \) under homogeneous Neumann boundary conditions, from Lemma 2.7 we can find \( t_2 > t_1 \) such that

\[
\| A^\alpha u(\cdot, t) \|_{L^\theta(\Omega)} \leq 2K_4 \int_\Omega u_0 \quad \text{for all } t > t_2.
\]  

(3.7)

Now we fix \( \alpha_0 \in \left( \frac{n}{2}, \alpha \right) \). Then the embedding \( D(A^{\alpha_0}) \hookrightarrow L^\infty(\Omega) \) enables us to find a constant \( C_3 > 0 \) such that

\[
\| \varphi \|_{L^\infty(\Omega)} \leq C_3 \| A^{\alpha_0} \varphi \|_{L^\theta(\Omega)} \quad \text{for all } \varphi \in D(A^{\alpha_0})
\]  

(3.8)

holds. Now noticing from a standard interpolation inequality

\[
\| A^{\alpha_0} \varphi \|_{L^\theta(\Omega)} \leq C_4 \| A^\alpha \varphi \|_{L^\theta(\Omega)}^{d_2} \| \varphi \|_{L^\infty(\Omega)}^{1-d_2}
\]

with some \( C_4 > 0 \) and \( d_2 := \frac{\alpha_0}{\alpha} \in (0, 1) \), and combination with Hölder’s inequality and (3.8)

\[
\| \varphi \|_{L^\theta(\Omega)} \leq \| \varphi \|_{L^\infty(\Omega)} \| \varphi \|_{L^2(\Omega)}^{\frac{2}{\theta}} \leq C_3 \| A^{\alpha_0} \varphi \|_{L^\theta(\Omega)}^{2(1-d_2)} \| \varphi \|_{L^2(\Omega)}^{2(1-d_2)}
\]

for all \( \varphi \in D(A^{\alpha_0}) \), we find that

\[
\| A^{\alpha_0} \varphi \|_{L^\theta(\Omega)} \leq C_6 \| A^{\alpha_0} \varphi \|_{L^\theta(\Omega)}^{d_2 \theta} \| \varphi \|_{L^\theta(\Omega)}^{2(1-d_2)}
\]  

for all \( \varphi \in D(A^\alpha) \),

with \( C_5 := \left( C_3 \frac{d_2 \theta}{\theta} \right)^{\frac{d_2 \theta}{\theta} \frac{2}{\theta}} \). Hence, we establish from (3.8) that

\[
\| \varphi \|_{L^\infty(\Omega)} \leq C_3 C_5 \| A^{\alpha_0} \varphi \|_{L^\theta(\Omega)}^{d_2 \theta} \| \varphi \|_{L^\infty(\Omega)}^{d_2 \theta}
\]  

for all \( \varphi \in D(A^\alpha) \).

Applying this to \( \varphi := u(\cdot, t) - \overline{u}_0 \), we can attain from (3.7) and Lemma 3.3 that there is \( t_3 > t_2 \) such that

\[
\| u(\cdot, t) - \overline{u}_0 \|_{L^\infty(\Omega)} \leq C_3 C_5 \left( \| A^\alpha (u(\cdot, t) - \overline{u}_0) \|_{L^\theta(\Omega)} \right)^{\frac{d_2 \theta}{2\theta + 2(1-d_2)}} \| u(\cdot, t) - \overline{u}_0 \|_{L^\infty(\Omega)}^{\frac{2(1-d_2)}{2\theta + 2(1-d_2)}}
\]

\[
\leq C_3 C_5 \left( 2K_4 \int_\Omega u_0 + \| A^{\alpha_0} \overline{u}_0 \|_{L^\theta(\Omega)} \right)^{\frac{d_2 \theta}{2\theta + 2(1-d_2)}} K_6^{\frac{2(1-d_2)}{2\theta + 2(1-d_2)}} e^{-\frac{2(1-d_2)}{2\theta + 2(1-d_2)}}
\]

for all \( t > t_3 \), which concludes the proof.

Finally, collecting three of the previous results we can establish Theorem 1.1.

**Proof of Theorem 1.1 and Remark 1.2.** Given \( M, v_*, a, k \) as in the theorem, we let \( \eta \) be as in (2.1), \( M_0 := M \), choose \( \chi_0 \in (0, k(a + \eta)^{k-1} \sqrt{\frac{2}{n}}) \) and pick \( \delta_1 := \delta(\chi_0, a, k, M_0) \) from Lemma 3.3. We define

\[
\delta := \min \left\{ \chi_0, k(a + K_1 M_0)^{k-1} \sqrt{\frac{2}{n}}, \frac{(a + K_1 M)^k}{M} \delta_1 \right\}.
\]

(3.9)
Then Lemma 2.1 is applicable and guarantees global existence of the solution, and Lemma 3.4 ensures the convergence statement and estimate (1.6) on \((t_* , \infty)\). Due to continuity of the solutions, upon proper choice of the constants, the estimate holds on \((0 , \infty)\) as claimed in (1.6).

If \(a > 0\), we could instead choose \(\eta = 0\) and \(M_0 := 0\) and in (3.9) replace \(\frac{(a + K_1 M)^k}{M}\) by the positive number \(\inf \{\frac{(a + K_1 \mu)^k}{\mu} \mid \mu > 0\}\) (and the interval \((0, \delta)\) would still be nonempty), thus removing any dependence of \(\delta\) on \(M\) and \(v_*\).

At this point, also the proof of the second theorem follows easily:

**Proof of Theorem 1.3.** We let \(M_1\) be so large that \(\eta' := 4 \left(1 + \sqrt{1 + \frac{3v_c}{c_0 M_1}}\right)^{-2} v_*\) (cf. (2.1)) satisfies

\[
k(a + \eta')^{k-1} \sqrt{\frac{2}{n}} > \chi_0
\]

and that \(k(a + K_1 M_1)^{k-1} \sqrt{\frac{2}{n}} > \chi_0\). Again taking \(\delta_1 := \delta(\chi_0, a, k, M_1)\) from Lemma 3.3, by choosing

\[
M_0 \geq M_1 \text{ such that } \frac{(a + K_1 M)^k}{M} > \frac{\chi_0}{\delta_1} \text{ for all } M \geq M_0,
\]

we can apparently ensure that for \(M \geq M_0\) and \(\chi \in (0, \chi_0)\) we have that

\[
\chi < \delta := \min \left\{\chi_0, k(a + K_1 M_0)^{k-1} \sqrt{\frac{2}{n}}, \frac{(a + K_1 M)^k}{M} \delta_1\right\}
\]

and hence may apply Lemma 2.1 and Lemma 3.4 as in the proof of Theorem 1.1. \(\square\)

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