ABSTRACT. Let $f$ be a volume-preserving diffeomorphism of a closed $C^\infty$ two-dimensional Riemannian manifold $M$. In this paper, we prove the equivalence between the following conditions:
(a) $f$ belongs to the $C^1$-interior of the set of volume-preserving diffeomorphisms which satisfy the weak shadowing property.
(b) $f$ belongs to the $C^1$-interior of the set of volume-preserving diffeomorphisms which satisfy the limit weak shadowing property,
(c) $f$ is Anosov.

1. Introduction

Let $M$ be a closed $C^\infty$ $n$-dimensional Riemannian manifold, and let Diff($M$) be the space of diffeomorphisms of $M$ endowed with the $C^1$-topology. Denote by $d$ the distance on $M$ induced from a Riemannian metric $\| \cdot \|$ on the tangent bundle $TM$. Let $f : M \to M$ be a diffeomorphism, and let $\Lambda \subset M$ be a closed $f$-invariant set.

For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^b (-\infty \leq a < b \leq \infty)$ in $M$ is called a $\delta$-pseudo orbit of $f$ if $d(f(x_i), x_{i+1}) < \delta$ for all $a \leq i \leq b-1$. For given $x, y \in M$, we write $x \leadsto y$ if for any $\delta > 0$, there is a $\delta$-pseudo orbit $\{x_i\}_{i=a}^b (a < b)$ of $f$ such that $x_a = x$ and $x_b = y$. The set of points $\{x \in M : x \leadsto x\}$ is called the chain recurrent set of $f$ and is denoted by $\mathcal{C}R(f)$. If we denote the set of periodic points of $f$ by $P(f)$, then $P(f) \subset \Omega(f) \subset \mathcal{C}R(f)$. Here $\Omega(f)$ is the non-wandering set of $f$.

We say that $f$ has the shadowing property on $\Lambda$ if for any $\epsilon > 0$ there is $\delta > 0$ such that for any $\delta$-pseudo orbit $\{x_i\}_{i \in \mathbb{Z}} \subset \Lambda$ of $f$ there is $y \in M$ such that $d(f^i(y), x_i) < \epsilon$, for $i \in \mathbb{Z}$. Note that in this definition, the shadowing point $y \in M$ is not necessarily contained in

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We say that $f$ has the $C^1$-interior shadowing property if there is a $C^1$-neighborhood $U(f)$ of $f$ such that for any $g \in U(f)$, $g$ has the shadowing property.

The weak shadowing property was introduced in [11]. The weak shadowing property is investigated in [10, 11, 12, 13, 14]. Every diffeomorphism having the shadowing property has the weak shadowing property but the converse is not true. Indeed, an irrational rotation on the unit circle has the weak shadowing property but does not have the shadowing property.

Given $\epsilon > 0$, $\{x_i\}_{i \in \mathbb{Z}}$ is said to be weakly $\epsilon$-shadowed by $y \in M$ if $\{x_i\}_{i \in \mathbb{Z}} \subset B_\epsilon(O_f(y))$. Here $B_\epsilon(A) = \{y \in M : \text{there is } x \in A \text{ such that } d(x, y) < \epsilon\}$ is the $\epsilon$-neighborhood of a subset $A$ of $M$. We say that $f$ has the weak shadowing property if for every $\epsilon > 0$, there is $\delta > 0$ such that every $\delta$-pseudo orbit of $f$ can be weakly $\epsilon$-shadowed by some point. Note that $f$ has the weak shadowing property if and only if $f^n$ has the weak shadowing property for every $n \in \mathbb{Z}$. We say that $f$ has the $C^1$-interior weak shadowing property if there is a $C^1$-neighborhood $U(f)$ of $f$ such that for any $g \in U(f)$, $g$ has the weak shadowing property.

Now, we introduce the notion of the limit weak shadowing property which is introduced in [15]. We say that $f$ has the limit weak shadowing property if for every $\epsilon > 0$ there is $\delta > 0$ such that for any $\delta$-limit pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$, there exists $y \in M$ weakly $\epsilon$-shadowing $\{x_i\}_{i \in \mathbb{Z}}$, and, if in addition, $d(f(x_i), x_{i+1}) \to 0$ as $i \to \pm \infty$ then $d(O_f(y), x_i) \to 0$ as $i \to \pm \infty$. Clearly, the limit weak shadowing property is stronger than the weak shadowing property by definition. Note that $f$ has the limit weak shadowing property if and only if $f^n$ has the limit weak shadowing property for every $n \in \mathbb{Z}$. We say that $f$ has the $C^1$-interior limit weak shadowing property if there is a $C^1$-neighborhood $U(f)$ of $f$ such that for any $g \in U(f)$, $g$ has the limit weak shadowing property.

Note that if $f$ is topologically transitive then $f$ has the weak shadowing property and $f$ has the limit weak shadowing property.

The shadowing property usually plays an important role in the investigation of stability theory and ergodic theory. Sakai [13] showed that a diffeomorphism belonging to the $C^1$-interior of the set of all diffeomorphisms on a $C^\infty$ closed surface with weak shadowing property satisfies both Axiom A and the no-cycle condition. Thus we can restate the above facts as follows.

**Theorem 1.1.** Let $M$ be a closed two-dimensional manifold. A diffeomorphism $f$ belongs to the $C^1$-interior weak shadowing property if and only if $f$ satisfies both Axiom A and the no-cycle condition.

Hence the \( C^1 \)-interior weak shadowing property in the two-dimensional manifold is characterized as the \( \Omega \)-stability of the system by Theorem 2. The non-empty open set \( U \subset \text{Diff}(\mathbb{T}^3) \) such that every \( g \in U \) is topologically transitive but not Anosov. It is easy to see that every \( g \in U \) has the weak shadowing property but does not satisfy Axiom A and the no-cycle condition. In \cite{16}, the authors proved that an \( \Omega \)-stable diffeomorphism has the limit weak shadowing property. And Sakai \cite{15} showed that there is a diffeomorphism \( f \) on 2 torus belonging to the \( C^1 \)-interior of the set of diffeomorphisms possessing the limit weak shadowing property such that \( f \) does not satisfy the strong transversality condition. Thus we can restate the above facts as follows.

**Theorem 1.2.** Let \( \mathbb{T}^2 \) be the two dimensional torus. There is a diffeomorphism \( f \) belongs to the \( C^1 \)-interior limit weak shadowing property satisfying both Axiom A and the no-cycle condition, but not the strong transversality condition.

By the theorem, even though a diffeomorphism is contained in the \( C^1 \)-interior of the set of diffeomorphisms possessing the limit weak shadowing property, it does not necessarily satisfy the strong transversality condition.

A periodic point \( p \) of \( f \) is hyperbolic if \( Df^{\pi(p)} \) has eigenvalues with absolute values different of one, where \( \pi(p) \) is the period of \( p \). Denote by \( \mathcal{F}(M) \) the set of \( f \in \text{Diff}(M) \) such that there is a \( C^1 \)-neighborhood \( U(f) \) of \( f \) such that for any \( g \in U(f) \), every \( p \in P(g) \) is hyperbolic. It is proved that by Hayashi \cite{4} that \( f \in \mathcal{F}(M) \) if and only if \( f \) satisfies both Axiom A and the no-cycle condition.

Let \( \Lambda \) be a closed \( f \in \text{Diff}(M) \)-invariant set. We say that \( \Lambda \) is hyperbolic if the tangent bundle \( T_\Lambda M \) has a \( Df \)-invariant splitting \( E^s \oplus E^u \) and there exists constants \( C > 0 \) and \( 0 < \lambda < 1 \) such that

\[
\| D_xf^n |_{E^s} \| \leq C \lambda^n \quad \text{and} \quad \| D_xf^{-n} |_{E^u} \| \leq C \lambda^n
\]

for all \( x \in \Lambda \) and \( n \geq 0 \). If \( \Lambda = M \) then we say that \( f \) is an Anosov diffeomorphism.

2. **Statement of the results**

A fundamental problem in differentiable dynamical systems is to understand how a robust dynamic property on the underlying manifold would influence the behavior of the tangent map on the tangent bundle. For instance, in \cite{6}, Mañé proved that any \( C^1 \) structurally stable diffeomorphism is an Axiom A diffeomorphism. And in \cite{9}, Palis extended this result to \( \Omega \)-stable diffeomorphisms.
Let $M$ be a compact $C^\infty$ $n$-dimensional Riemannian manifold endowed with a volume form $\omega$. Let $\mu$ denote the measure associated to $\omega$, that we call Lebesgue measure, and let $d$ denote the metric induced by the Riemannian structure. Denote by $\text{Diff}_\mu(M)$ the set of diffeomorphisms which preserves the Lebesgue measure $\mu$ endowed with the $C^1$-topology. In the volume preserving, the Axiom A condition is equivalent to the diffeomorphism be Anosov, since $\Omega(f) = M$ by Poincaré Recurrence Theorem. The purpose of this paper is to do this using the robust property.

We define the set $\mathcal{F}_\mu(M)$ as the set of diffeomorphisms $f \in \text{Diff}_\mu(M)$ which have a $C^1$-neighborhood $\mathcal{U}(f) \subset \text{Diff}_\mu(M)$ such that if for any $g \in \mathcal{U}(f)$, every periodic point of $g$ is hyperbolic. Note that $\mathcal{F}_\mu(M) \subset \mathcal{F}(M)$ (see [1, Corollary 1.2]).

Very recently, Arbieto and Catalan [1] proved that if a volume preserving diffeomorphism contained in $\mathcal{F}_\mu(M)$ then it is Anosov. From the above facts, we can restate as follows.

**Theorem 2.1.** Any diffeomorphism in $\mathcal{F}_\mu(M)$ is Anosov.

Very recently, Lee [8] showed that if a volume preserving diffeomorphisms on any dimensional manifold belongs to the $C^1$-interior expansive or $C^1$-interior shadowing property, then it is Anosov. As in the above theorems 1.1 and 1.2, we can’t extend on any dimensional manifold. Thus, we study the cases when a volume preserving diffeomorphism is in $C^1$-interior weak shadowing property or $C^1$-interior limit weak shadowing property on two-dimensional mainfold, then it is Anosov. Let $\text{intWS}_\mu(M)$ be denote the set of volume preserving diffeomorphisms in $\text{Diff}_\mu(M)$ satisfying the weak shadowing property, and let $\text{intLWS}_\mu(M)$ be denote the set of volume-preserving diffeomorphisms in $\text{Diff}_\mu(M)$ satisfying the limit weak shadowing property. Main thing to prove this paper is the following.

**Theorem 2.2.** Let $M$ be a two-dimensional manifold, and let $f \in \text{Diff}_\mu(M)$. We has that

$$\text{intWS}_\mu(M) = \text{intLWS}_\mu(M) = \text{AN}_\mu(M),$$

where $\text{AN}_\mu(M)$ is the set of Anosov volume preserving diffeomorphisms in $\text{Diff}_\mu(M)$.

3. **Proof of Theorem 2.2**

Let $M$ be a closed $C^\infty$ $n$-dimensional Riemannian manifold, and let $\text{Diff}(M)$ be the space of diffeomorphisms of $M$ endowed with the $C^1$-topology. Denote by $d$ the distance on $M$ induced from a Riemannian
metric \( \| \cdot \| \) on the tangent bundle \( TM \). Let \( f : M \to M \) be a diffeomorphism, and let \( \Lambda \subset M \) be a closed \( f \)-invariant set. From now, we study relation between a normally hyperbolic (see [3]) and the weak shadowing property as follows lemmas.

**Lemma 3.1.** Let \( f \in \text{Diff}(M) \), and let \( \Delta \subset \Lambda \) be a normally hyperbolic \( f \)-invariant submanifold of \( M \). Suppose that \( f \) has the weak shadowing property on \( \Lambda \). Then the shadowing point is in \( \Delta \).

*Proof.* Suppose that \( f \) has the weak shadowing property on \( \Lambda \). For any \( \epsilon > 0 \), let \( B_\epsilon(\Delta) \) be the \( \epsilon \)-neighborhood of \( \Delta \). Since \( \Delta \) is a normally hyperbolic, we can choose \( k > 0 \) and \( \epsilon_1 > 0 \) such that for any \( x \in B_{\epsilon_1}(\Delta) \setminus \Delta \), \( d(f^k(x), \Delta) > \epsilon_1 \). Let \( 0 < \delta < \epsilon_1 \) be the number of the weak shadowing property of \( f|_\Lambda \) for \( \epsilon_1 \). Since \( f \) has the weak shadowing property on \( \Lambda \), \( f \) must have the weak shadowing property on \( \Delta \). Thus for any \( \delta \)-pseudo orbit \( \{x_i\}_{i \in \mathbb{Z}} \subset \Delta \) of \( f \), we can find a point \( y \in B_{\epsilon_1}(\Delta) \) such that \( \{x_i\}_{i \in \mathbb{Z}} \subset B_{\epsilon_1}(O_f(y)) \). But, if \( y \in B_{\epsilon_1}(\Delta) \setminus \Delta \) then from the above facts, we can choose \( k > 0 \) such that \( d(x_k, O_f(y)) > \epsilon_1 \). This is a contradiction. Thus if \( f \) has the weak shadowing property on \( \Lambda \), then the shadowing point is in \( \Delta \).

\( \square \)

**Lemma 3.2.** Let \( f \in \text{Diff}(M) \), and let \( \Delta \subset \Lambda \) be a normally hyperbolic \( f \)-invariant submanifold of \( M \). Suppose that \( f \) has the weak shadowing property on \( \Lambda \). Then if \( \Delta \) is an arc or a disk, then \( f|_\Delta \) is not the identity map.

*Proof.* Let \( \Delta \subset \Lambda \) be a normally hyperbolic for \( f \), and let \( \Delta \) be an arc. Suppose that \( f \) has the weak shadowing property on \( \Lambda \). We will use the method of proof by contradiction. Assume that \( f|_\Delta \) is an identity map. Let \( l = \mathrm{diam}(\Delta) \). Take \( \epsilon = l/4 \). Let \( 0 < \delta < \epsilon \) be the number of the weak shadowing property of \( f \). Then we construct \( \delta \)-pseudo orbit \( \xi \) of \( f \) as follows; For fix \( k > 0 \), choose distinct points \( x_1, x_2, \ldots, x_k \in \Delta \) such that

(a) \( d(x_i, x_{i+1}) < \delta \) for \( i = 1, \ldots, k-1 \),

(b) \( x_1 = x \) and \( d(x, x_k) > 2\epsilon \).

Define \( \xi = \{y_i\}_{i \in \mathbb{Z}} \) by \( y_{k+i} = x_i \) for \( i \in \mathbb{Z} \) and \( j = 0, 1, \ldots, k-1 \). Since \( f \) has the weak shadowing property on \( \Lambda \), by Lemma 3.1 we can find a point \( z \in \Delta \) such that \( \{x_i\}_{i \in \mathbb{Z}} \subset B_\epsilon(O_f(z)) \). Since \( f|_\Delta \) is the identity map, we can find \( l > 0 \) such that \( d(y_i, O_f(z)) > \epsilon \). This is a contradiction.

\( \square \)

Let \( M \) be a compact \( C^\infty \) \( n \)-dimensional Riemannian manifold endowed with a volume form \( \omega \), and let \( f \in \text{Diff}_u(M) \). To prove the
results, we will use the following is the well-known Franks’ lemma for the conservative case, stated and proved in [2 Proposition 7.4].

**Lemma 3.3.** Let \( f \in \text{Diff}^1_\mu(M) \), and \( U \) be a \( C^1 \)-neighborhood of \( f \) in \( \text{Diff}^1_\mu(M) \). Then there exist a \( C^1 \)-neighborhood \( U_0 \subset U \) of \( f \) and \( \epsilon > 0 \) such that if \( g \in U_0 \), any finite \( f \)-invariant set \( E = \{x_1, \ldots, x_m\} \), any neighborhood \( U \) of \( E \) and any volume-preserving linear maps \( L_j : T_{x_j}M \to T_{g(x_j)}M \) with \( \|L_j - D_{x_j}g\| \leq \epsilon \) for all \( j = 1, \ldots, m \), there is a conservative diffeomorphism \( g_1 \in U \) coinciding with \( f \) on \( E \) and out of \( U \), and \( D_{x_j}g_1 = L_j \) for all \( j = 1, \ldots, m \).

**Remark 3.4.** Let \( f \in \text{Diff}^1_\mu(M) \). From the Moser’s Theorem (see [7]), there is a smooth conservative change of coordinates \( \varphi_x : U(x) \to T_xM \) such that \( \varphi_x(0) = 0 \), where \( U(x) \) is a small neighborhood of \( x \in M \).

**Proposition 3.5.** Let \( M \) be a closed \( C^\infty \) two-dimensional manifold. If \( f \in \text{int} WS_\mu(M) \), then every periodic point of \( f \) is hyperbolic.

**Proof.** Take \( f \in \text{int} WS_\mu(M) \), and \( U(f) \) a \( C^1 \)-neighborhood of \( f \in \text{int} E_\mu(M) \). Let \( \epsilon > 0 \) and \( V(f) \subset U_0(f) \) corresponding number and \( C^1 \)-neighborhood given by Lemma 3.3. We will derive a contradiction, we may assume that there exists a nonhyperbolic periodic point \( p \in P(g) \) for some \( g \in V(f) \). To simplify the notation in the proof, we may assume that \( g(p) = p \). Then there is at least one eigenvalue \( \lambda \) of \( D_p g \) such that \( |\lambda| = 1 \).

By making use of the Lemma 3.3, we linearize \( g \) at \( p \) with respect to Moser’s Theorem; that is, by choosing \( \alpha > 0 \) sufficiently small we construct \( g_1 \) \( C^1 \)-nearby \( g \) such that

\[
g_1(x) = \begin{cases} 
\varphi_p^{-1} \circ D_p g \circ \varphi_p(x) & \text{if } x \in B_\alpha(p), \\
g(x) & \text{if } x \notin B_{4\alpha}(p).
\end{cases}
\]

Then \( g_1(p) = g(p) = p \).

First, we may assume that \( \lambda \in \mathbb{R} \) with \( \lambda = 1 \). Let \( v \) be the associated non-zero eigenvector such that \( \|v\| = \alpha/4 \). Then we can get a small arc \( I_v = \{tv : -1 \leq t \leq 1\} \subset \varphi_p(B_\alpha(p)) \). Take \( \epsilon_1 = \alpha/8 \). Let \( 0 < \delta < \epsilon \) be the number of the weak shadowing property of \( g_1 \). Then by our construction of \( g_1 \), \( \varphi_p^{-1}(I_v) \subset B_\alpha(p) \). Then, it is clear that \( \varphi_p^{-1}(I_v) \) is a normally hyperbolic for \( g_1 \). Put \( J_p = \varphi_p(I_v) \). For the above \( \delta > 0 \), we construct \( \delta \)-pseudo orbit \( \xi = \{x_i\}_{i \in \mathbb{Z}} \subset J_p \) as follows: For fix \( k \in \mathbb{Z} \), choose distinct points \( x_0 = p, x_1, x_2, \ldots, x_k \) in \( J_p \) such that

\begin{enumerate}
\item \( d(x_i, x_{i+1}) < \delta \) for \( i = 0, 1, \ldots, k-1 \),
\item \( d(x_{-i-1}, x_{-i}) < \delta \) for \( i = 0, 1, \ldots, k-1 \),
\item \( x_0 = x \) and \( d(x_{-k}, x_k) > 2\epsilon_1 \).
\end{enumerate}
Then assume that \( \alpha > 0 \) and \( \xi \). Now we define \( g \in \mathbb{R} \) and properties. Thus we can find a point \( y \in M \) such that \( \{x_i\}_{i \in \mathbb{Z}} \subset B_{\epsilon_1}(O_{g_i}(y)) \). For any \( v \in I_v, \varphi^{-1}_p(v) \in J_p \subset B_{\alpha}(p) \) and
\[
g_1(\varphi^{-1}_p(v)) = \varphi^{-1}_p \circ D_p g \circ \varphi_p(\varphi^{-1}_p(v)).
\]
Then \( g_1(\varphi^{-1}_p(v)) = \varphi^{-1}_p(v) \). Thus \( g'_1(J_p) = J_p \) for some \( l > 0 \).

Since \( g_1 \) has the weak shadowing property, by Lemma 3.1, the point \( y \in J_p \). But, by Lemma 3.2, the identity map does not have the weak shadowing property. Thus \( g_1|J_p \) does not have the weak shadowing property.

Finally, if \( \lambda \in \mathbb{C} \), then to avoid the notational complexity, we may assume that \( g(p) = p \). As in the first case, by Lemma 3.3 there are \( \alpha > 0 \) and \( g_1 \in V(f) \) such that \( g_1(p) = g(p) = p \) and
\[
g_1(x) = \begin{cases} \varphi^{-1}_p \circ D_p g \circ \varphi_p(x) & \text{if } x \in B_{\alpha}(p), \\ g(x) & \text{if } x \notin B_{4\alpha}(p). \end{cases}
\]

With a \( C^1 \)-small modification of the map \( D_p g \), we may suppose that there is \( l > 0 \) (the minimum number) such that \( D_p g^l(v) = v \) for any \( v \in \varphi_p(B_{\alpha}(p)) \subset T_p M \). Then, we can go on with the previous argument in order to reach the same contradiction. Thus, every periodic point of \( f \in int WS_\mu(M) \) is hyperbolic.

For any \( \epsilon > 0 \), let \( B_{\epsilon}(\Delta) \) be the \( \epsilon \)-neighborhood of \( \Delta \).

**Lemma 3.6.** Let \( f \in \text{Diff}(M) \), and let \( \Delta \subset \Lambda \) be a normally hyperbolic \( f \)-invariant submanifold of \( M \). Suppose that \( f \) has the limit weak shadowing property on \( \Lambda \). Then the shadowing point is in \( \Delta \).

**Proof.** Suppose that \( f \) has the limit weak shadowing property on \( \Lambda \). Since \( \Delta \) is a normally hyperbolic, we can choose \( k > 0 \) and \( \epsilon_1 > 0 \) such that for any \( x \in B_{\epsilon_1}(\Delta) \setminus \Delta, d(f^k(x), \Delta) > \epsilon_1 \). Let \( 0 < \delta < \epsilon_1 \) be the number of the limit weak shadowing property of \( f|\Lambda \) for \( \epsilon_1 \). Since \( f \) has the limit weak shadowing property on \( \Lambda \), \( f \) must have the limit weak shadowing property on \( \Delta \). Thus for any \( \delta \)-limit pseudo orbit \( \{x_i\}_{i \in \mathbb{Z}} \subset \Delta \) of \( f \), we can find a point \( y \in B_{\epsilon_1}(\Delta) \) such that \( \{x_i\}_{i \in \mathbb{Z}} \subset B_{\epsilon_1}(O_f(y)) \) then and \( d(x_i, O_f(y)) \to \infty \) as \( i \to \pm \infty \). But, if \( y \in B_{\epsilon_1}(\Delta) \setminus \Delta \) then from the above facts, we can choose \( k > 0 \) such that \( d(\Delta, O_f(y)) > \epsilon_1 \). This is a contradiction. Thus if \( f \) has the limit weak shadowing property on \( \Lambda \), then the shadowing point is in \( \Delta \). \( \square \)
Lemma 3.7. Let $f \in \text{Diff}(M)$, and let $\Delta \subset \Lambda$ be a normally hyperbolic $f$-invariant submanifold of $M$. Suppose that $f$ has the limit weak shadowing property on $\Lambda$. Then if $\Delta$ is an arc or a disk, then $f|_{\Delta} : \Delta \to \Delta$ is not the identity map.

Proof. Let $\Delta \subset \Lambda$ be a normally hyperbolic for $f$, and let $\Delta$ be an arc. Suppose that $f$ has the limit weak shadowing property on $\Lambda$. We will use the method of proof by contradiction. Assume that $f|_{\Delta}$ is an identity map. Let $l = \text{diam}(\Delta)$. Take $\epsilon = l/4$. Let $0 < \delta < \epsilon$ be the number of the weak shadowing property of $f$. Then we construct $\delta$-limit pseudo orbit $\xi$ of $f$ as follows: For fix $k > 0$, choose distinct points $x_0, x_1, x_2, \ldots, x_k \in \Delta$ such that

(a) $d(x_i, x_{i+1}) < \delta$ for $i = 0, \ldots, k - 1,$
(b) $d(x_{-i-1}, x_{-i}) < \delta$ for $i = 0, \ldots, k - 1,$
(c) $x_{-k-j} = x_{-k}$ for $j \geq 0$, and $x_{k+j} = x_k$ for $j \geq 0$.

Then $\xi = \{ \ldots, x_{-k}, x_{-k}, x_{-k+1}, \ldots, x_{-1}, x, x_1, \ldots, x_k, x_k, \ldots, \}$ is a $\delta$-limit pseudo orbit of $f$. Clearly, $\xi \subset \Delta$. Since $f$ has the weak shadowing property on $\Lambda$, by Lemma 3.6, we can find a point $z \in \Delta$ such that $(x_i)_{i \in \mathbb{Z}} \subset B_r(\mathcal{O}_f(z))$, and $d(x_i, \mathcal{O}_f(z)) \to \infty$, as $i \to \pm \infty$. Since $f|_{\Delta}$ is the indentity map, we can find $l > 0$ such that $d(y_l, \mathcal{O}_f(z)) > \epsilon$. This is a contradiction. \hfill \Box

Proposition 3.8. Let $M$ be a closed $C^\infty$ two-dimensional manifold. If $f \in \text{int} \text{LWS}_\alpha(M)$ then every periodic point of $f$ is hyperbolic.

Proof. Take $f \in \text{int} \text{LWS}_\alpha(M)$, and $\mathcal{U}(f)$ a $C^1$-neighborhood of $f \in \text{int} \text{LWS}_\alpha(M)$. Let $\epsilon > 0$ and $\mathcal{V}(f) \subset \mathcal{U}_0(f)$ corresponding number and $C^1$-neighborhood given by Lemma 3.3. To derive a contradiction, we may assume that there exists a nonhyperbolic periodic point $p \in P(g)$ for some $g \in \mathcal{V}(f)$. To simplify the notation in the proof, we may assume that $g(p) = p$. Then as in the proof of Proposition 3.3, we can take $\alpha > 0$ sufficiently small, and a smooth map $\varphi_p : B_\alpha(p) \to T_p M$. Form the above construction, we can make an arc $\mathcal{J}_p \subset B_\alpha(p)$ and for $g_1 \in \mathcal{V}(f)$, $\mathcal{J}_p$ is a $g_1$-invariant normally hyperbolic. Take $\epsilon_1 = (\text{length} \mathcal{J}_p)/4$, let $0 < \delta < \epsilon_1$ be the number of the limit weak shadowing property for $g_1$. Form now, we construct $\delta$-limit pseudo orbit of $g_1$ as follows; For fix $k > 0$, choose distinct points $x_0 = 0, x_1, \ldots, x_k \in \mathcal{J}_p$ such that

(a) $d(x_i, x_{i+1}) < \delta$ for $i = 0, \ldots, k - 1,$
(b) $d(x_{-k-i}, x_{-i}) < \delta$ for $i = 0, \ldots, k - 1,$
(c) $x_0 = x$ and $d(x_{-k}, x_k) > 2\epsilon_1$,
(d) $x_{-k-j} = x_{-k}$ and $x_{k+j} = x_k$ for $j \geq 0$. 


Then \( \xi = \{ \ldots, x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 = p, x_1, \ldots, x_k, x_{k+1}, \ldots \} \) is a \( \delta \)-limit pseudo orbit of \( g_1 \) and \( \xi \subset J_p \). Since \( J_p \) is a normally hyperbolic for \( g_1 \), by Lemma 3.6 the shadowing point \( y \in J_p \). Since \( g_1|_{J_p} \) is the identity map, By Lemma 3.7 \( g_1 \) does not have the limit weak shadowing property on \( J_p \). This is a contradiction.

Finally, if \( \lambda \in \mathbb{C} \), then as in the proof of Proposition 3.2 for \( g_1 \in \mathcal{V}(f) \), we can take \( l > 0 \) such that \( D_p g_1^l(v) = v \) for any \( v \in \varphi_p(B_\alpha(p)) \subset T_p M \). Then from the previous argument in order to reach the same contradiction. Thus, every periodic point of \( f \in \text{intLWS}_\mu(M) \) is hyperbolic.

\[ \square \]

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