A Theorem of Equivalence between TransverseDiff Theories and Scalar-Tensor Gravity

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Abstract. Transverse Diffeomorphism (TDiff) theories are well-motivated theories of gravity from the quantum perspective, which are based upon a gauge symmetry principle. The main contribution of this work is to firmly establish a correspondence between TransverseDiff and the better-known scalar-tensor gravity — in its more general form —, a relation which is completely analogous to that between unimodular gravity and General Relativity. We then comment on observational aspects of TDiff. In connection with this proof, we derive a very general rule that determines under what conditions the procedure of fixing a gauge symmetry can be equivalently applied before the variational principle leading to the equations of motion, as opposed to the standard procedure, which takes place afterwards; this rule applies to gauge-fixing terms without derivatives.

1. Introduction

Symmetry principles have a great relevance in modern physics theories. Specifically, there is a widespread point of view on General Relativity, stemming from its possible quantum interpretation, that considers it as the theory of spacetime diffeomorphism (Diff) invariance for spin-two massless particles: one starts from a linearized gravity action for a symmetric rank-two tensor in Minkowski space and, under the guidance of the Diff symmetry principle, one is “naturally” led to the full Einstein theory in the non-linear regime [9, 5] (see however [17]). Under this view, the external symmetry group of (active) diffeomorphisms acts as a gauge group which proves essential for eliminating the unphysical ill-defined polarization modes (ghosts) of the massless graviton.

On the other hand, it was already pointed out that the necessary and sufficient symmetry group for a consistent description of the massless graviton is not the full group of diffeomorphisms (Diff(M)) but rather a maximal subgroup of it [23]. Transformations belonging to this subgroup have been dubbed transverse (TDiff(M)) [1, 2], since the parameter describing the gauge transformation at this linear level is transverse, \( \partial_\mu \xi^\mu = 0 \), such as in the transformation law for the linearized metric:

\[
h_{\mu\nu}(x) \rightarrow h_{\mu\nu}(x) + \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x), \quad \text{with} \quad \partial_\mu \xi^\mu(x) = 0.
\]

(1)

At the non-linear level, one can maintain the same restriction \( \partial_\mu \xi^\mu = 0 \) on the symmetry group so that, for example, the transformation of the metric reads:

\[
g_{\mu\nu}(x) \rightarrow g_{\mu\nu}(x) + \nabla_\mu \xi_\nu(x) + \nabla_\nu \xi_\mu(x), \quad \text{with} \quad \partial_\mu \xi^\mu(x) = 0.
\]

(2)
For a finite transformation, this corresponds to the subgroup of (finite) general coordinate transformations with Jacobian determinant equal to unity.

Following under this “quantum view”, consider that the true symmetry of nature were TDi, instead of Diff. Then, one is impelled to construct the most general action compatible with TDi symmetry. The result is that, since $g = -det(g_{\mu\nu})$ behaves as a scalar under this subgroup, one can add arbitrary functions of $g$ to the usual terms of the GR Lagrangian [2]. Therefore, the most general Lagrangian invariant under transverse diffeomorphisms and of second order in derivatives reads:

$$S_{TDiff} = -\frac{1}{2\kappa^2} \int d^4x \sqrt{g} \left[ f(\sqrt{g} - 1) R + 2f_\Lambda (\sqrt{g} - 1) \Lambda + \frac{1}{2} f_k (\sqrt{g} - 1) g^{\mu\nu} \partial_\mu \sqrt{g} \partial_\nu \sqrt{g} \right] + S_M$$

and the matter action may be taken to be of the form

$$S_M = \int d^4x \sqrt{g} L_{SM} [\psi_m, g_{\mu\nu}; \sqrt{g} - 1]$$

where we allow for an arbitrary extra-dependence on $g$ in the conventional Standard Model Lagrangian; also, we have expressed this dependence on $\sqrt{g} - 1$ instead of $g$ for later convenience. As it stands, the action given by 3, 4 is clearly non-covariant, but is formulated in a specific set of coordinates — up to TDiff transformations of coordinates. The quantum ultraviolet behavior of such a theory was discussed in [3]; its observational signatures were addressed in [4] — we will have some comments to add under the light shed by our findings.

This way of proceeding from the TDiff symmetry principle is not, however, the standard route that followed from the work of Bij, Dam&Ng [23]. Indeed, these same authors started to develop what is known as unimodular gravity (recent proposals include [8, 21]). We compare and contrast these two approaches in section 2.

On another note, there is a significant lesson in the work by Kretschmann [12], when objecting to Einstein’s views: every spacetime theory admits a generally covariant representation (see also [15, 16]). For TDiff, a representation of this kind can be found, e.g., in [18]. It introduces an absolute prior-geometric object [22], namely a background density $\tilde{g}$, so that all occurrences of $g$ in the scalar Lagrangian appear in the covariant form $g/\tilde{g}$ (then again, when the coordinates employed are such that $\tilde{g}$ equals the unit scalar density we recover straightforwardly our action 3). We go one step further and introduce the notion of physically-equivalent theories. We find that, under certain simple boundary conditions, TDiff theory resembles a general scalar-tensor theory, to the same extent that General Relativity constrained to the harmonic gauge — or any other gauge — corresponds in its physical consequences to General Relativity in its covariant standard form (we are just “fixing the gauge”):

$$\delta \left( \int d^4x \sqrt{g} R \right) = 0 \quad \iff \quad \left[ \delta \left( \int d^4x \sqrt{\tilde{g}} R \right) = 0 \right]_{\partial_\mu (\sqrt{\tilde{g}} g^{\mu\nu}) = 0}$$

By general scalar-tensor theory we mean that the gravitational scalar is arbitrarily present all through the Lagrangian, i.e., not the standard presentation of scalar-tensor theory [11, 10, 6], which employs the principle of universal coupling in the matter Lagrangian or the equivalent concept of a metric theory of gravity (see for example [22, 24]), and which was developed to rescue the (Weak) Equivalence Principle (see however [7] regarding its convenience). The idea of this correspondence between TDiff and general scalar-tensor gravity has already been touched upon in previous studies [3, 4], and has even made use of, but now our aim is to definitively establish it; we address it in section 3. Finally, we conclude in section 4.
2. TDiff Gravity vs. Unimodular theories

The TransverseDiff symmetry at the linear level can be implemented in two different fashions in the Lagrangian, which give rise to two very different theories at the non-linear level.

The first route was explored in the aforementioned seminal paper [23]. One should note that the transversality condition in 1 is directly affecting the transformation rule of the trace of \( h_{\mu\nu} \): \( h_{\mu\nu} \to h_{\mu\nu}^{0} + \partial_{\mu} \xi^{\nu} \), such that this quantity is an invariant. Therefore, one can decide to restrict the value of the trace in the theory to have a fixed value, typically \( h_{\mu\nu}^{0} = 0 \). In the full non-linear regime, this naturally translates into the condition \( g = -det(g_{\mu\nu}) = 1 \), which explains the name unimodular gravity, but one could also fix the determinant of the metric to any arbitrary function.

Therefore, unimodular gravity is based upon a reduction of the functional space on which the Einstein-Hilbert action is defined:

\[
S_{EH}[g_{\mu\nu}] = -\frac{1}{2\kappa^{2}} \int d^{4}x \sqrt{g} R[g_{\mu\nu}] \rightarrow S_{UG}[\hat{g}_{\mu\nu}] = -\frac{1}{2\kappa^{2}} \int d^{4}x \epsilon_{0} R[\hat{g}_{\mu\nu}]
\]

where \( \epsilon_{0} \) is some fixed scalar density, usually taken unity. The symmetries of this action are the “volume preserving diffeomorphisms (VPD)”, which respect \( g(x) = g(x) \) (and at the infinitesimal level \( \delta g = \nabla_{\mu} \xi^{\mu} = 0 \)). Note that these are not the same as 2 unless \( \epsilon_{0}(x) = \text{const.} \). We have 9 e.o.m.’s \( \delta S/\delta \hat{g}_{\mu\nu} = 0 \) (of which only 6 are independent, in virtue of the VPD symmetry) for 9 functions \( g_{\mu\nu} \). This theory resembles General Relativity with a gauge-fixing constraint on the metric. In fact, the only difference at the classical level is an arbitrary integration constant that appears, and which plays the role of a cosmological constant.

On the other hand, the approach that we are interested in, named “TDiff gravity” after the symmetry, is based upon application of a symmetry principle on the full space of metrics \( g_{\mu\nu} \) (see 3 and 2). There are 10 e.o.m.’s: \( \delta S/\delta \hat{g}_{\mu\nu} = 0 \) (only 7 independent in virtue of TDi-based Bianchi identities), for 10 functions \( g_{\mu\nu} \). This theory is connected to scalar-tensor theory with a gauge-fixing constraint, as we will see.

3. Correspondence between TDiff gravity and a general scalar-tensor theory.

**Theorem.** Of correspondence between scalar-tensor and TDiff gravity.

Let us have a TDiff theory of the general form 34, abbreviated as

\[
S_{TDiff} = \int d^{4}x \sqrt{g(x)} \mathcal{L}(g_{\mu\nu}(x), \sqrt{g(x)} - 1; \psi(x)),
\]

formulated in an open patch of coordinates \( B \), with boundary \( \partial B \), and where the \( \psi(x) \) stands for a generic matter field. Consider now the corresponding scalar-tensor theory with action

\[
S_{ST} = \int d^{4}x \sqrt{g(x)} \mathcal{L}(g_{\mu\nu}(x), \phi(x); \psi(x)),
\]

where the explicit dependency on the determinant of the metric \( \sqrt{g} - 1 \) has been replaced by the new scalar field \( \phi \). Then, the set of (classical) solutions in both theories that fulfill that \( \mathcal{L} \to 0 \) at the boundary \( \partial B \) is the same, provided that we consider “physically equivalent” all solutions in a gauge orbit; specifically, we take the partial gauge fixing \( \sqrt{g} - 1 = \phi \) on the scalar-tensor solutions:

\[
\{(g_{\mu\nu}, \psi)|\delta S_{TDiff}[g_{\mu\nu}, \psi] = 0, \mathcal{L}|_{\partial B} = 0 \} \equiv \{(g_{\mu\nu}, \phi, \psi)|\delta S_{ST}[g_{\mu\nu}, \phi; \psi] = 0, \mathcal{L}|_{\partial B} = 0 \} |_{\phi = \sqrt{g} - 1}
\]

See [14] for the proof.
Let us make some comments following the theorem. First, a totally analogous theorem exists which connects GR and unimodular gravity. Indeed, the action for unimodular gravity can be written with the help of a Lagrange multiplier as

$$S_{UG} = \int d^4x \left[ \sqrt{g(x)} R(x) + \lambda(x)(\sqrt{g(x)} - 1) \right]$$

and, following a similar reasoning, one can establish the equivalence of the solutions of both theories when the Lagrangian goes to zero at the boundary. This result is already well-known, since the integration constant playing the role of a cosmological constant that appears in unimodular gravity — with respect to GR — is non-zero at the boundary.

Actually, a far more general conclusion can be drawn from our line of reasoning: for any Lagrangian theory with a globally-defined gauge symmetry, it is equivalent to impose a gauge-fixing without derivatives before or after the calculation of variations, provided that we only consider (classical) solutions that make the Lagrangian vanish at the boundary of spacetime. This adds up to the result in [19, 20] (Hamiltonian and Lagrangian treatment, respectively), that “the effect of plugging the gauge fixing constraint into the Lagrangian can be compensated by adding to the equations of motion for the reduced [restricted] theory some constraints that have disappeared as such along the process”.

Second, this theorem somehow contradicts common beliefs of other authors, as in [18].

Third, we have expressed the explicit dependence on $g$ of the TDi theory by $\sqrt{g} - 1$ throughout the paper, so that it vanishes in Minkowski space with (canonical) cartesian coordinates. Whether the requirement $\frac{\partial L}{\partial B} \to 0$ is “natural” or too restrictive is left to the reader.

4. Conclusions

Let us summarize our results. We started by imposing a TDiff gauge symmetry group to the gravitational Lagrangian based on a symmetric rank-two tensor $g_{\mu\nu}$, as opposed to the standard Diff symmetry that leads us to General Relativity. Since the symmetry group is smaller, we thus get to a more general Lagrangian. Then, the theorem stated in section 3 tells us that, modulo some boundary conditions, one can view this theory just as a general scalar-tensor theory that has been presented in a particular gauge, namely $\phi = \sqrt{g} - 1$, but whose inherent symmetry group is again Diff (it is “physically-equivalent”, in the sense of the introduction). At the end of the day, we have “gained” a scalar degree of freedom that was not postulated, but appears automatically due to the lack of symmetry. In a way, the TDiff group can be thus considered to give rise to general-scalar tensor theory. This present knowledge builds up from previous works [3, 4]. In addition, some collateral consequences of the theorem with a wider scope of application are presented at the end of section 3: these confront the question of “whether it makes a difference to impose the gauge-fixing before or after the calculus of variations”.

Regarding the observational signatures of TDiff theories at the classical level, one can therefore rely on this equivalence to take advantage from the abundant literature that studies the bounds on the scalar-tensor theory. The same metric / universal coupling requirement that is usually imposed for scalar-tensor theory — which was mentioned in the introduction — can be applied directly in the TDiff Lagrangian by considering only matter Lagrangians without explicit dependence on $g$.

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