A Deterministic Parallel APSP Algorithm and its Applications

Adam Karczmarz$^1$ and Piotr Sankowski$^1$

$^1$Institute of Informatics, University of Warsaw, Poland

Abstract

In this paper we show a deterministic parallel all-pairs shortest paths algorithm for real-weighted directed graphs. The algorithm has $\tilde{O}(nm + (n/d)^3)$ work and $\tilde{O}(d)$ depth for any depth parameter $d \in [1, n]$. To the best of our knowledge, such a trade-off has only been previously described for the real-weighted single-source shortest paths problem using randomization [Bringmann et al., ICALP’17]. Moreover, our result improves upon the parallelism of the state-of-the-art randomized parallel algorithm for computing transitive closure, which has $\tilde{O}(nm + n^3/d^2)$ work and $\tilde{O}(d)$ depth [Ullman and Yannakakis, SIAM J. Comput. ’91].

Our APSP algorithm turns out to be a powerful tool for designing efficient planar graph algorithms in both parallel and sequential regimes. By suitably adjusting the depth parameter $d$ and applying known techniques, we obtain:

1. nearly work-efficient $\tilde{O}(n^{1/6})$-depth parallel algorithms for the real-weighted single-source shortest paths problem and finding a bipartite perfect matching in a planar graph,
2. an $\tilde{O}(n^{9/8})$-time sequential strongly polynomial algorithm for computing a minimum mean cycle or a minimum cost-to-time-ratio cycle of a planar graph,
3. a slightly faster algorithm for computing so-called external dense distance graphs of all pieces of a recursive decomposition of a planar graph.

One notable ingredient of our parallel APSP algorithm is a simple deterministic $\tilde{O}(nm)$-work $\tilde{O}(d)$-depth procedure for computing $O(n/d)$-size hitting sets of shortest $d$-hop paths between all pairs of vertices of a real-weighted digraph. Such hitting sets have also been called $d$-hub sets. Hub sets have previously proved especially useful in designing parallel or dynamic shortest paths algorithms and are typically obtained via random sampling. Our procedure implies, for example, an $O(nm)$-time deterministic algorithm for finding a shortest negative cycle of a real-weighted digraph. Such a near-optimal bound for this problem has been so far only achieved using a randomized algorithm [Orlin et al., Discret. Appl. Math. ’18].
1 Introduction

The all-pairs shortest paths problem (APSP) is one of the most fundamental graph problems. It has been studied in numerous variants, for many computational models and graph classes. In this paper we study the APSP problem on real-weighted, possibly sparse graphs in the parallel setting.

The efficiency of a parallel algorithm is usually characterized using the notions of work and depth (also called span or time). The work is the total number of primitive operations performed. The depth is the longest chain of sequential dependencies between these operations. A parallel algorithm of work $W(n)$ and depth $D(n)$ (where $n$ is the problem size) can be generally scheduled to run in $\tilde{O}(D(n))$ time using $\tilde{O}(W(n)/D(n))$ processors [10], where the $\tilde{O}(\cdot)$ notation suppresses $O(\text{polylog } n)$ factors. The quantity $W(n)/D(n)$ is often called the parallelism of a parallel algorithm. An algorithm is called nearly work-efficient if $W(n) = \tilde{O}(T(n))$, where $T(n)$ is the best known time bound needed to solve the problem using a sequential algorithm.

There exists a very simple folklore parallel algorithm for the APSP problem via repeatedly squaring the weighted adjacency matrix using min-plus product. It has $\tilde{O}(n^3)$ work and $O(\text{polylog } n)$ depth. This can be slightly tweaked to obtain polylogarithmic-factor improvement in work and depth [25]. However, these algorithms are nearly work-efficient only for dense graphs since the best known sequential algorithms run in $\tilde{O}(nm)$ time [30, 54]. Hence, we are missing APSP algorithms competitive with the state-of-the-art sequential algorithms even for moderately dense graphs.

Dealing with sparser graphs in the parallel setting turns out to be a much more challenging task even for an easier problem of computing the transitive closure. To the best of our knowledge, the state-of-the-art for parallel transitive closure for sparse graphs is the classical tradeoff of Ullman and Yannakakis [60]. They showed a Monte Carlo randomized parallel algorithm with $\tilde{O}(nm + n^3/d^3)$ work and $\tilde{O}(d)$ depth for any parameter $d \in [1, n]$. Hence, their algorithm is nearly work-efficient for $d = \tilde{\Omega}(n/\sqrt{m})$ and can achieve parallelism of order $\tilde{O}(m^{3/2})$ while being nearly work-efficient.

1.1 Our Results and Related Work

Our main result is a deterministic parallel all-pairs shortest paths algorithm for real-weighted directed graphs that improves upon the 30-year-old randomized transitive closure trade-off of [60].

**Theorem 1.** Let $G$ be a real-weighted digraph. For any $d \in [1, n]$, there exists a deterministic parallel algorithm computing all-pairs shortest paths in $G$ with $\tilde{O}(nm + (n/d)^3)$ work and $\tilde{O}(d)$ depth.

Observe that our algorithm is nearly work-efficient for $d = \Omega(n^{2/3}/m^{1/3})$, which is $\Omega(n^{1/3})$ for sparse graph. As a result, as long as the number of used processors is $p = \tilde{O}(n^{1/3}m^{4/3})$, we can compute all-pairs shortest paths in $\tilde{O}(nm/p)$ parallel time. To the best of our knowledge, such a tradeoff for real-weighted digraphs has only been achieved for single-source shortest paths (SSSP) and negative cycle detection problems [11]. Both of these results require randomization, whereas our algorithm is deterministic. Bringmann et al. [11] also show a deterministic variant of their algorithm with work $\tilde{O}(nmd + (n/d)^3)$ that is not nearly work-efficient unless the graph is dense.\footnote{Bringmann et al. [11, Theorem 19] mistakenly state the work of their algorithm to be $\tilde{O}(nm + (n/d)^3 + n^2d)$—even though in Section 4.1.2 they correctly bound the work in the first step of their algorithm to be $\Theta(nmd)$.}

In the aforementioned previous results [11, 60], randomization is used only for computing a small subset of $V$ that is, roughly speaking, guaranteed to contain some vertex of each “long” shortest path consisting of at least $h$ hops. Following [33], we call such set a $h$-hub set of $G$ (a
formal definition is deferred to Section 3\textsuperscript{2}. A classical argument of Ullman and Yannakakis \cite{ullman1981complexity} shows that a randomly sampled $\Theta((n/h) \cdot \log n)$-size subset of $V$ constitutes an $h$-hub set with high probability.

We show that for any $h \in [1, n]$, an $h$-hub set of size $\tilde{O}(n/h)$ can be computed deterministically using $\tilde{O}(nm)$ work and $\tilde{O}(h)$ depth. Our procedure works in presence of negative edge weights and even – to some extent – when negative cycles are allowed. The constructed hub set is guaranteed to hit all-pairs shortest $h$-hop paths (which are well-defined regardless of whether the APSP problem is feasible) unless the shortest (in terms of hops) negative cycle has at most $h$ edges. As a by-product, we also obtain the following result in the sequential regime.

**Theorem 2.** Let $G$ be a real-weighted directed graph. A shortest (in terms of hops) negative-weight cycle in $G$ can be found deterministically in $O(nm \log^2 n)$ time.

So far, an $\tilde{O}(nm)$ bound for the shortest negative cycle problem has only been obtained using a randomized algorithm by Orlin et al. \cite{orlin2010finding}\textsuperscript{3}. The best known deterministic algorithms \cite{zwick1999negative, zwick1999negative} require $\Omega(n^3)$ worst-case time. One can easily argue that our bound is the best possible, up to polylogarithmic factors, unless an unexpected algorithmic breakthrough is made, that would imply progress for other core problems as well. In particular, the shortest negative cycle captures the unweighted directed girth problem (for which the trivial $O(nm)$ bound stands for decades). Moreover, one should not hope for an $O(n^{3-\varepsilon})$ fast matrix multiplication-based algorithm since the shortest negative cycle problem also captures the negative triangle detection problem known to be subcubic-equivalent to the APSP problem \cite{young1970negative}.

We believe that our procedure for computing hub sets might be useful in obtaining other deterministic sequential and parallel algorithms. As a direct application of our APSP algorithm one can obtain, e.g., more efficient parallel algorithms for computing closeness centrality \cite{liben2015efficient}. Moreover, as shown below, we can use it to improve algorithms for several planar graph problems.

### 1.1.1 Applications for planar graphs

**Nearly work-efficient parallel algorithms.** Theorem 1 can be used to highly parallelize the framework used by Fakcharoenphol and Rao \cite{fakcharoenphol2004highly} to obtain nearly optimal algorithms for two fundamental planar graph problems.

**Theorem 3.** Let $G$ be a real-weighted planar digraph. There exists a deterministic parallel algorithm for negative cycle detection and single-source shortest paths in $G$ with $\tilde{O}(n)$ work and $\tilde{O}(n^{1/6})$ depth.

By a well-known duality-based reduction \cite{steinberg1988all}, this also implies feasible flow and bipartite perfect matching algorithms with the same bounds. It is worth noting that even though there exist $O(\text{polylog } n)$ depth algorithms (i.e., belonging to NC class) for finding perfect matchings in bipartite planar graphs, they are very far from being work-efficient \cite{zwick1999negative, steinberg1988all}. The same applies to the $\tilde{O}(\sqrt{n})$-depth algorithm implied by the interior-point method-based result for general graphs \cite{kolountzakis2004subcubic}. Since the $s, t$-max flow problem on planar graphs with capacities $[1, C]$ can be reduced to $O(\log C)$ feasible flow computations \cite{steinberg1988all}, this also yields a nearly work-efficient $\tilde{O}(n^{1/6})$-depth algorithm for maximum $s, t$-flow with polynomially bounded capacities. Similar bounds can be obtained for the

\textsuperscript{2}Zwick \cite{zwick1999negative} uses the name bridging set for an analogous concept. Some works also use the term hitting set, but hitting set is a more general notion, which in our paper is used in multiple different contexts.

\textsuperscript{3}The algorithm of \cite{orlin2010finding} runs in $O(nm \log n)$ expected time. However, if one aims at high-probability correctness, its running time is $O(nm \log^2 n)$ which matches our bound.
related replacement paths problem using the recent reduction [16] to the all-edge shortest cycles problem (see Section 4.2 and Appendix A).

To the best of our knowledge, the parallel complexity of the aforementioned problems on planar graphs has not achieved much attention in recent years. However, one can easily obtain near-optimal work and $\mathcal{O}(\sqrt{n})$-depth algorithm for these problems using the breakthrough framework of Fakcharoenphol and Rao [20]. In this framework, one repeatedly computes all-pairs shortest paths on certain dense distance graphs using multiple runs of a clever implementation of Dijkstra’s algorithm (so-called FR-Dijkstra). Unfortunately, it is not clear how to break the $\Omega(\sqrt{n})$ depth bound this way since Dijkstra’s algorithm is inherently sequential. Our improved depth bound is obtained by replacing the simple-minded Dijkstra-based APSP algorithm with that of Theorem 1. As we show, the Monge property of dense distance graphs that is crucial for the efficiency of FR-Dijkstra can also be employed in our algorithm to yield a significant parallel speed-up.

**Minimum mean and cost-to-time ratio cycle problems.** By plugging in our parallel negative cycle detection algorithm into Megiddo’s parametric search framework [48] we obtain improved strongly polynomial algorithms for the minimum mean cycle and minimum cost-to-time ratio cycle problems on planar graphs (for formal definitions of these problems, refer to Section 4.3).

**Theorem 4.** Let $G$ be a real-weighed planar graph. There exists an $\tilde{O}(n^{9/8})$-time strongly polynomial algorithm for computing a minimum ratio cycle (and thus also a minimum mean cycle) in $G$.

The minimum mean cycle and minimum cost-to-time ratio cycle problems are classical graph problems studied since the seventies. They are used to construct strongly polynomial algorithms for computing minimum-cost flows [59]. Moreover, via the cut-cycle duality in planar graphs, both problems have found practical applications in the area of image segmentation [29, 61–63].

It is known that both problems can be reduced to negative cycle detection via binary search [42]. However, this way we can obtain only weakly polynomial time algorithms with running times dependent on the magnitude of edge weights. For general graphs, the classical algorithm of Karp [34] solves the minimum mean cycle problem in $O(nm)$ time, matching the best known strongly polynomial negative cycle detection bound achieved by the Bellman-Ford algorithm. Karp’s algorithm (and other minimum mean cycle algorithms for general graphs [26, 65]) operates on limited-hop shortest paths. As a result it is not clear how to take advantage of planarity to speed it up\(^5\).

The $\tilde{O}(nm)$ bound has not been matched to date for the more general minimum cost-to-time ratio problem which seems to be the original inspiration for the invention of the parametric search technique [48] that later found other applications (e.g., [1]). This technique can be used to convert an efficient parallel negative cycle detection algorithm into a strongly polynomial minimum ratio cycle algorithm. The best known strongly polynomial $\tilde{O}(m^{3/4}n^{3/2})$ bound for the minimum ratio cycle problem is due to Bringmann et al. [11] and also follows by plugging their aforementioned parallel negative cycle detection algorithm into Megiddo’s framework.

It seems that no strongly polynomial algorithms for the minimum mean cycle problem to date have been designed specifically for planar graphs. However, by plugging in previously known parallel negative cycle detection algorithms [20, 44] into the parametric search framework, one would only obtain $\tilde{O}(n^{3/2})$-time strongly polynomial algorithms. Theorem 4 improves upon this significantly.

---

\(^4\) Also known as the minimum ratio cycle problem.

\(^5\) At-most-$h$-hop shortest paths connecting pairs of vertices of a single face of a plane graph do not seem to admit algorithmically useful properties, like the non-crossing property of usual shortest paths.
Computing external dense distance graphs. Finally, our parallel APSP algorithm can be used to improve the bound for computing so-called external dense distance graphs wrt. a planar graph’s recursive decomposition – an important black box with applications in computing maximum flows [46], minimum cuts [9], and constructing distance oracles [14,15,51].

Suppose a planar graph $G$ is recursively decomposed using small cycle separators [49] of size $O(\sqrt{n})$ until the obtained pieces have constant size. The decomposition procedure produces a binary tree $T(G)$ whose nodes correspond to subgraphs of $G$ (pieces). The boundary vertices $\partial H$ of a piece $H \in T(G)$ are vertices that $H$ shares with the remaining part $G - H$ of the entire graph $G$. We denote by $DDG_H$ the dense distance graph – a complete weighted graph on $\partial H$ whose edge weights represent distances between all pairs of vertices of $\partial H$ in $H$. Efficient construction of piecewise DDGs and FR-Dijkstra alone are enough to obtain e.g., nearly linear-space static and dynamic exact distance oracles with sublinear query time [20,31,37]. Originally, Fakcharoenphol and Rao [20] gave an $O(n \log^3 n)$ algorithm for computing each of the graphs $DDG_H$, $H \in T(G)$, inductively, based on the children graphs $DDG_{H_1}, DDG_{H_2}$. The key ingredient in their algorithm was the aforementioned fast implementation of Dijkstra’s algorithm on a dense distance graph.

Later, Klein [37] showed a more efficient, $O(n \log^2 n)$-time algorithm that computed every $DDG_H$ directly, by building the so-called multiple-source shortest paths (MSSP) data structure for each piece $H \in T(G)$ separately. This was possible since the total size of all pieces $H$ of the decomposition is $O(n \log n)$. However, some important applications [9,14,15,46,51] also require external graphs $DDG_{G-H}$ representing distances between $\partial H$ in the complement graph $G - H$ (for all $H \in T(G)$). In this case Klein’s approach fails since the total size of all possible graphs $G - H$ is $\Omega(n^2)$. Instead, one has to stick to the original inductive method of [20] and compute each $DDG_{G-H}$ based on $DDG_{G-H_1}$ and $DDG_{G-H_2}$ (where, again, $H$ is a parent of $H_1, H_2$ in $T(G)$). This takes $O(n \log^3 n)$ time through all $H$. We show that our parallel APSP algorithm can be used to obtain an algorithm whose running time almost matches the $O(n \log^2 n)$ MSSP-based bound.

**Lemma 5.** The external dense distance graphs $DDG_{G-H}$ for all $H \in T(G)$ can be computed in $O(n \log^2 n \cdot \log \log n \cdot \alpha(n))$ time.

### 1.2 Further Related Work

For general directed graphs, the parallel single-source shortest paths problem (SSSP) is much better understood. Ullman and Yannakakis [60] showed an $O(m \sqrt{n})$-work algorithm with $O(\sqrt{n})$ depth for unweighted digraphs. Besides the aforementioned result of Bringmann et al. [11] and $\tilde{O}(n)$-depth parallel implementations of Dijkstra’s algorithm [12], all known parallel exact SSSP algorithms work for weighted graphs with non-negative and integral weights bounded by $W$ and are weakly polynomial. Klein and Subramanian [41] generalized the bound of [60] to the weighted setting at the cost of $\tilde{O}(\log W)$ factor in work/depth bounds. In this setting, they also gave a nearly work-efficient polylog-depth algorithm for planar graphs [40]. Spencer [56] gave a trade-off algorithm with $\tilde{O}(n^3/d^2 + m) \log W$ work and $O(d)$ depth. Forster and Nanongkai [22] in turn have recently shown a trade-off algorithm with $\tilde{O}((md + mn/d + (n/d)^3) \log W)$ work and $\tilde{O}(d)$ depth.

Recently, following similar results for single-source reachability [21,45], Cao et al. [13] showed that $(1+\epsilon)$-approximate single-source shortest paths can be found using near-optimal work $\tilde{O}(m \log W)$ and $\tilde{O}(n^{1/2+o(1)})$ depth. Note that this SSSP algorithm, if run from every source, yields a nearly work-efficient (approximate) APSP algorithm, but with polynomially worse depth than ours.

Finally, parallel single-source shortest paths problems in undirected graphs have also received much attention, both in the exact [8,18,55] and the approximate [3,19,43] setting.
1.3 Technical Overview

Our parallel APSP algorithm is based on the techniques used in the state-of-the-art Monte Carlo randomized decremental all-pairs shortest paths algorithm for weighted digraphs, due to Bernstein [7]. Without loss of generality assume that our depth parameter is a power of two, i.e., \( d = 2^K \). Our algorithm makes use of a hierarchy of hub sets \( V = H_1, H_2, \ldots, H_d \subseteq V \) such that each \( H_{2^i} \) is a \( 2^i \)-hub set of \( G \) and has size \( O((n/2^i) \log n) \). Roughly speaking, this means that for all pairs \( u, v \in V \), some shortest \( u \rightarrow v \) path, if it consists of at least \( 2^i \) edges, contains a vertex of \( H_{2^i} \).

Let us start with describing a randomized version of our algorithm. A well-known fact attributed to Ullman and Yannakakis [60] states that picking each \( H_{2^i} \) to be a random \( \Theta((n/2^i) \cdot \log n) \)-subset of \( V \) guarantees that \( H_{2^i} \) forms a \( 2^i \)-hub-set of \( G \) with high probability.

Rather than using the inherently sequential Dijkstra’s algorithm, as sequential \( \tilde{O}(nm) \)-time APSP algorithms [30, 54] do, we rely exclusively on the Bellman-Ford algorithm. A variant of Bellman-Ford algorithm, given a source \( s \), maintains distance labels and performs a number of steps relaxing all edges in arbitrary order in \( O(m) \) time. After \( k \) steps, the distance label of \( v \in V \) is equal to the length \( \delta^k_G(s, v) \) of a shortest at-most-\( k \)-hop path from \( s \) to \( v \). In a single step, for each vertex we need to combine edge relaxations ending in this vertex, so a single step requires \( O(\text{polylog } n) \) depth. Consequently, performing \( k \) steps of Bellman-Ford requires \( O(mk) \) work but only \( \tilde{O}(k) \) depth.

The key idea behind Bernstein’s algorithm [7] is that computing shortest paths \( \delta_G(u, v) \) for \( (u, v) \in H_{2^i} \times V \) can be reduced to computing at-most-\( 2^{i+1} \)-hop shortest paths on a graph \( G_{s,t} \) obtained from \( G \) by augmenting it with \( |H_{2^{i+1}}| = O(n) \) auxiliary edges \( st \) for all \( t \in H_{2^{i+1}} \), such that the weight of \( st \) equals the distance \( \delta^i_G(s, t) \) from \( s \) to \( t \). This idea suggests an inductive algorithm that, given distances for all pairs in \((H_{2^{i+1}} \times V) \cup (V \times H_{2^{i+1}})\), where \( i < K \), lifts them to distances for all pairs in \((H_{2^i} \times V) \cup (V \times H_{2^i})\) using \( |H_{2^i}| \) parallel \( 2^{i+1} \)-step Bellman-Ford runs. These can be performed in \( O(|H_{2^i}| \cdot 2^{i+1} \cdot m) = O(nm \log n) \) work and \( \tilde{O}(2^i) \) depth. Eventually, since \( H_1 = V \), we obtain the all-pairs distances in \( G \). Summing through all inductive steps, the work is \( O(nm \log n \log d) \), whereas the depth is \( \tilde{O}(2^i + \ldots + 2^K) = \tilde{O}(2^K) = \tilde{O}(d) \).

The last thing missing in the above algorithm is the induction base, i.e., computing \( \delta_G(s, t) \) for all \( s, t \in H_d \) in only \( \tilde{O}(d) \) depth.

Computing hub sets deterministically. Now let us briefly describe how to derandomize the above parallel APSP algorithm by replacing sampling with a deterministic preprocessing step that computes all hub sets \( H_1, \ldots, H_d \) within \( \tilde{O}(nm) \) work and \( \tilde{O}(d) \) depth.

---

\( ^6 \)The all-pairs distances for \( H_d \) can be lifted to distances for all pairs in \((H_d \times V) \cup (V \times H_d)\) using \( \tilde{O}(nm) \) work and \( \tilde{O}(d) \) depth as in the inductive step by setting \( H_{2d} := H_d \).

\( ^7 \)The same strategy of switching to a dense-graph algorithm has also proved useful in obtaining a deterministic incremental algorithm for APSP in weighted directed graphs [33].
Previous results in the parallel and dynamic graph algorithms literature [11, 33, 35] that used deterministic $h$-hub sets typically obtained them by (1) computing (or maintaining – in the dynamic setting) at-most-$h$-hop shortest paths between all pairs of vertices $s, t \in V$, and (2) running a greedy $O(\log n)$-approximation algorithm for computing a $O((n/h) \log n)$-size hitting set of a family of $h$-element subsets of an $n$-element universe (see e.g., [35] for analysis). Unfortunately, the first step of this approach seems to require $\Omega(nmh)$ time.

To obtain an improved bound, we reuse the inductive approach. Specifically, we show that given a hub set $H_2$, one can obtain a $2^{i+1}$-hub set $H_{2^{i+1}}$ of size $O((n/2^i) \cdot \log n)$ by running the aforementioned greedy hitting set algorithm on shortest $2^i$-hop paths from $H_{2^i}$\(^8\). Constructing these paths using Bellman-Ford costs $O(|H_{2^i}| \cdot 2^i \cdot m) = O(nm \log n)$ time and requires $\tilde{O}(2^i)$ depth. Luckily, the deterministic greedy hitting set algorithm has its nearly work-efficient parallel version with $O(\text{polylog } n)$ depth [6]. Therefore, one can construct $H_d$ using $\tilde{O}(nm)$ work and $\tilde{O}(d)$ depth.

It is worth noting that the analysis of the above algorithm breaks if the shortest negative cycle in $G$ has at most $d$ edges. We stress that this is not a problem for the APSP application though, since any negative cycles in $G$ make the APSP problem infeasible. That being said, our deterministic hub set algorithm can be easily extended to correctly report the shortest negative cycle within the same bounds and thus match the randomized bound of [53] for the shortest negative cycle problem.

We also remark that even though the hub sets are very useful in designing dynamic graph algorithms, it is unlikely that our construction can help match the best known randomized bounds for dynamic problems (like [7]) using deterministic algorithms. The power of a randomly sampled hub set $H$ (and, possibly, the so-called oblivious adversary assumption) lies in the fact that $H$ retains its guarantees through all, in particular future, versions of a dynamic graph.

Comparison to the transitive closure algorithm of [60]. Ullman and Yannakakis [60] use a single randomly sampled $d$-hub set $H_d$ of size $O((n/d) \log n)$. In similar way they compute reachability between the nodes of $H_d$ in $\tilde{O}(nm + (n/d)^3)$ work and $\tilde{O}(d)$ depth – they first apply limited-depth BFS from all $H_d$, and then use repeated squaring. Adding the $\tilde{O}((n/d)^2)$ “shortcuts” between $H_d$ to $G$ reduces $G$’s diameter to $\tilde{O}(d)$. Finally, a limited-depth BFS is run from each source in parallel. Since the augmented graph has $\tilde{O}(m + (n/d)^2)$ edges, this takes $\tilde{O}(nm + n^3/d^2)$ work and $\tilde{O}(d)$ depth. The efficiency of this approach crucially relies on the fact that BFS has near-linear work and, as a result, does not generalize to real-weighted graphs which require using Bellman-Ford.

Planar graph applications. All of the numerous consequences of our parallel APSP algorithm for planar graphs essentially follow by showing improved parallel and sequential bounds for the following problem, which we call dense distance graph APSP (DDG APSP).

Let $H_1, H_2 \in \mathcal{T}(G)$ be two pieces of a recursive decomposition of $G$ and let $b = |\partial H_1| + |\partial H_2|$. We would like to compute all-pairs shortest paths in the graph $\text{DDG}_{H_1} \cup \text{DDG}_{H_2}$.

A parallel algorithm solving the above problem using $T(b, d) = \Omega(b^2)$ work and $\tilde{O}(d)$ depth, plugged in the framework of Fakcharoenphol and Rao [20], implies:

- An $\tilde{O}((n + T(\sqrt{n}, d))$-work and $\tilde{O}(d)$-depth algorithm for negative-cycle detection on real-weighted planar graphs. Via known reductions [16, 50], the same bound can be achieved for the feasible flow, bipartite perfect matching, and replacement paths problems.

\(^8\)A somewhat similar trick has been used in [33] for improving the state-of-the-art randomized partially dynamic APSP algorithms [5, 7] from Monte Carlo to Las Vegas.
• \( \widetilde{O}(nd + T(\sqrt{n}, d)) \)-time sequential strongly polynomial algorithms for minimum mean and minimum cost-to-time ratio cycle problems.

Moreover, a sequential algorithm solving the DDG APSP problem in \( S(b) = \Omega(b^2) \) time implies an \( O(S(\sqrt{n}) \log n) \) algorithm for computing all external dense distance graphs \( DDG_{G-H} \) for \( H \in \mathcal{T}(G) \).

Fakcharoenphol and Rao’s [20] algorithm for solving DDP APSP uses Johnson’s [30] approach: first a feasible price function is computed using Bellman-Ford algorithm to reduce the task to the non-negatively weighted case. Subsequently, Dijkstra’s algorithm is run from each of \( S(b) \) sources. Fakcharoenphol and Rao gave a very efficient Dijkstra implementation running in \( O(b \log^2 b) \) time on \( DDG_{H_1} \cup DDG_{H_2} \). They also showed that a single step of Bellman-Ford can be performed on \( DDG_{H_1} \cup DDG_{H_2} \) in \( O(b \log b) \) time. Klein et al. [39] later noticed that a Bellman-Ford step can be performed in \( \widetilde{O}(b \cdot \alpha(b)) \). By these bounds, the dense distance graph APSP can be solved sequentially in \( O(b^2 \log b) \) time and using a parallel algorithm with \( \widetilde{O}(b^2) \) work and \( \widetilde{O}(b) \) depth.

We show that the dense distance graph APSP problem can be solved using a parallel algorithm with work \( \widetilde{O}(b^2 + (b/d)^3) \) and depth \( \widetilde{O}(d) \). This essentially follows by using a more efficient parallel implementation of a Bellman-Ford step in the algorithm of Theorem 1. As observed in [20], a Bellman-Ford step on a dense distance graph can be rephrased as computing column minima of a certain matrix with Monge property (a Monge matrix, in short). Since column minima of a Monge matrix can be computed using a polylogarithmic time parallel algorithm [2], a single Bellman-Ford step can be implemented using only \( \widetilde{O}(b) \) work and \( O(\polylog n) \) depth, even though \( DDG_{H_1} \cup DDG_{H_2} \) has \( \Omega(b^2) \) edges.

Moreover, we show a faster sequential algorithm for DDG APSP with \( O(b^2 \log b \cdot \log \log b \cdot \alpha(b)) \) running time. First, we observe that our parallel APSP algorithm (given a feasible price function) has a sequential implementation with \( O\left(n \cdot B(n, m) \cdot \log n \cdot \log d + \left(\frac{b}{d} \log n\right) \cdot D(n, m)\right) \) time, where \( B(n, m) \) denotes the cost of a single Bellman-Ford step, and \( D(n, m) \) denotes the cost of a single Dijkstra step (on a graph with \( n \) vertices and \( m \) edges). Second, we leverage the asymmetry between the \( O(b \cdot \alpha(b)) \) cost of a Bellman-Ford step and the \( O(b \log^2 b) \) cost of running FR-Dijkstra on \( DDG_{H_1} \cup DDG_{H_2} \). To obtain our improved bound, it is enough to choose \( d = \log^2 b \).

2 Preliminaries

In this paper we deal with real-weighted directed graphs. We write \( V(G) \) and \( E(G) \) to denote the sets of vertices and edges of \( G \), respectively. A graph \( H \) is a subgraph of \( G \), which we denote by \( H \subseteq G \), if and only if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). We write \( uv \in E(G) \) when referring to edges of \( G \) and use \( w_G(uv) \) to denote the weight of \( uv \). If \( uv \notin E \), we assume \( w_G(uv) = \infty \).

A sequence of edges \( P = e_1 \ldots e_k \), where \( k \geq 1 \) and \( e_i = u_iv_i \in E(G) \), is called an \( s \rightarrow t \) path in \( G \) if \( s = u_1, v_k = t \) and \( v_{i-1} = u_i \) for each \( i = 2, \ldots, k \). For brevity we sometimes also express \( P \) as a sequence of \( k + 1 \) vertices \( u_1u_2 \ldots u_kv_k \) or as a subgraph of \( G \) with vertices \( \{u_1, \ldots, u_k, v_k\} \) and edges \( \{e_1, \ldots, e_k\} \). The hop-length \( |P| \) is equal to the number of edges in \( P \). We also say that \( P \) is a \( k \)-hop path. The length of the path \( \ell(P) \) is defined as \( \ell(P) = \sum_{i=1}^{k} w_G(e_i) \). For convenience, we sometimes consider a single edge \( uv \) as a path of hop-length 1. If \( P_1 \) is a \( u \rightarrow v \) path and \( P_2 \) is a \( v \rightarrow w \) path, we denote by \( P_1P_2 \) (or simply \( P_1P_2 \)) a path obtained by concatenating \( P_1 \) with \( P_2 \).

We define \( \delta^k_G(u, v) \) to be the length of the shortest path from \( u \) to \( v \) among paths of at most \( k \) edges. Formally, \( \delta^k_G(u, v) = \min\{\ell(P) : u \rightarrow v = P \subseteq G \text{ and } |P| \leq k\} \). The distance \( \delta_G(u, v) \) between the vertices \( u, v \in V(G) \) is the length of the shortest \( u \rightarrow v \) path in \( G \), or \( \infty \), if no \( u \rightarrow v \) path exists in \( G \). In other words, \( \delta_G(u, v) = \min_{k \geq 0} \delta^k_G(u, v) \). Note that the distance is well-defined only if \( G \) contains no negative cycles. It is well known that \( G \) has no negative cycles
if and only if there exists a feasible price function \( p : v \rightarrow \mathbb{R} \) satisfying \( w_G(e) + p(u) - p(v) \geq 0 \) for all \( uv = e \in E(G) \). We define minimal paths as follows.

**Definition 6.** We call a \( u \rightarrow v \) path \( P \subseteq G \) minimal if \( \ell(P) = \delta^{|P|}_G(u, v) < \delta^{|P|-1}_G(u, v) \).

**Observation 7.** All subpaths of a minimal path are also minimal.

If the graph \( G \) that we refer to is clear from the context, we sometimes omit the subscript \( G \) and write \( w(uv), \delta(u, v), \delta^k(u, v) \) etc. instead of \( w_G(uv), \delta_G(u, v), \delta^k_G(u, v) \), etc., respectively.

**Parallel model.** Formally, we use the work-depth model as used in recent literature on parallel reachability and shortest paths, e.g., [21, 43]. An algorithm in this PRAM model differs from a sequential algorithm only by the inclusion of the parallel foreach loops. The work of an algorithm is the total number of operations performed if all parallel foreach loops were executed sequentially. The depth of an algorithm is the total number of operations performed by sequential steps, plus the sum, over all parallel foreach loops, of the maximum (sequential) iteration cost of a loop. In order to not focus on low-level details, we specify neither what is the depth overhead of a sequential algorithm only by the inclusion of the parallel foreach loops, nor what is the precise shared memory access model (e.g., EREW, CREW). Instead, we state all our parallel work and depth bounds using \( O(\cdot) \) notation that suppresses polylogarithmic factors. This is justified by the existence of general reductions between these different PRAM variants that yield only polylogarithmic multiplicative overhead in work and depth [28].

**Bellman-Ford algorithm.** The Bellman-Ford algorithm is a classical algorithm for computing shortest paths from a single source \( s \) or detecting a negative cycle if one exists. It maintains a distance label vector \( d : V \rightarrow \mathbb{R} \), where initially \( d(s) = 0 \) and \( d(v) = \infty \) for all \( v \in V \setminus \{s\} \), and proceeds in steps. Classically, a Bellman-Ford step consists of performing edge relaxations, i.e., substitutions \( d(v) := \min(d(v), d(u) + w(uv)) \) for all edges \( e = uv \), in any order. It is well-known that: (1) if \( d(v) \neq \infty \) then \( d(v) \) is a length of some \( s \rightarrow v \) path in \( G \), (2) after \( k \) Bellman-Ford steps we have \( d(v) \leq \delta^k_G(s, v) \) for all \( v \in V \), (3) if \( d(u) + w(uv) < d(v) \) for some \( uv \in E \) after \( n - 1 \) steps, then \( G \) has a negative cycle, (4) if \( G \) has no negative cycle and \( s \) can reach every other vertex, then the obtained distance label vector constitutes a feasible price function of \( G \). A single Bellman-Ford step clearly takes \( O(m) \) time, so the Bellman-Ford algorithm runs in \( O(mk) \) time if \( k \) steps are performed. However, in general, the results of individual relaxations in a single step may depend on the results of relaxations that happened earlier in that step.

In this paper we actually use a variant of the Bellman-Ford algorithm that might converge to the answer slower, but is easier to reason about. Namely, in a single step, for all \( v \) at once, we replace \( d(v) \) with \( \min(d(v), \min_{uv \in E}(d(u) + w(uv))) \). Equivalently, at the beginning of a step we could store a copy of vector \( d \) as \( d' \), and then again relax the edges in any order, where the relaxation is now a substitution \( d'(v) := \min(d'(v), d'(u) + w(uv)) \). It is easy to prove that in this variant, after \( k \) steps we have precisely \( d(v) = \delta^k_G(v) \) for all \( v \in V \). The properties (3) and (4) of the classical Bellman-Ford algorithm hold for this variant as well. Moreover, the result of each relaxation now only depends on the distance labels in the previous step. Consequently, for this variant, a single Bellman-Ford step can be clearly performed in parallel using \( O(m) \) work and \( O(\text{polylog } n) \) depth.

**Hitting sets.** Let \( F \) be a collection of subsets of some universe \( U \). Then \( X \subseteq U \) is called a hitting set of \( F \) if \( X \cap S \neq \emptyset \) for all \( S \in F \).

**Lemma 8 ([6, 35]).** Let \( \Pi \) be a collection of \( k \) simple \( h \)-hop paths of \( G \) (i.e., \( k \) \((h+1)\)-element subsets of \( V(G) \)). A hitting set \( \Pi \) of size \( O((n/h) \cdot \log k) \) can be computed in a deterministic way:
• sequentially using a greedy algorithm in $O(kh + n)$ time,
• using a parallel algorithm with $O(\text{polylog } n)$ depth and $\tilde{O}(kh + n)$ work.\(^9\)

3 A Parallel APSP Algorithm

In this section we describe the main result of this paper. Our algorithm will use the concept of an $h$-hub set, as defined below.

**Definition 9.** We call a set $H_h \subseteq V$ a $h$-hub set if for all $u, v \in V$ such that $\delta_G^h(u, v) < \delta_G^{h-1}(u, v)$ there exists a minimal path $P = u \rightarrow v$ in $G$ such that $|P| = h$ and $P$ goes through a vertex of $H_h$.

The following randomized construction of $h$-hub sets is attributed to Ullman and Yannakakis [60].

**Fact 10 ([60]).** For any $h \in [1, n]$, a random $\Theta(\frac{n}{h} \log n)$-element subset of $V$ constitutes an $h$-hub set of $G$ with high probability\(^10\).

First, we will show how having hub sets for various values $h$ leads to a randomized algorithm, whereas the deterministic construction of hubs will follow.

3.1 The Algorithm

In the remaining part of this section we assume that $G$ has no negative cycles. We will deal with negative cycle detection later on.

Let $d \in [1, n]$ be a parameter that controls the depth of our algorithm. By possibly decreasing the demanded depth by just a constant factor, we can assume, without loss of generality, that $d = 2^K$, where $K$ is an integer. We first show how to compute APSP given hub sets $V = H_1, H_2, \ldots, H_{2^d}$, where each $H_i$ has size $O((n/2^d) \cdot \log n)$. By Fact 10, all these hub sets can be obtained with $\tilde{O}(n)$ work and $O(\text{polylog } n)$ depth using sampling.

The first step of our algorithm is to compute shortest $\leq (d + 1)$-hop distances $\delta_{d+1}^G(s, t)$ for all $s, t \in H_d$. This can be done by running $(d + 1)$ steps of the Bellman-Ford algorithm from all $s \in H_d$ in parallel using $\tilde{O}(|H_d| \cdot d \cdot m) = \tilde{O}(nm)$ work and $\tilde{O}(d)$ depth, or in $O(nm \log n)$ time sequentially.

Let $G_d$ be defined as a complete graph on $H_d$ with weights given by $w_{G_d}(uv) = \delta_{d+1}^G(u, v)$ for all $u, v \in H_d$. The second step is computing the actual shortest paths $\delta_G(s, t)$ for all $s, t \in H_d$ by running the APSP algorithm based on repeated-squaring (of the weighted adjacency matrix using min-plus product) on the graph $G_d$.

**Lemma 11.** Let $s, t \in H_d$. Then $\delta_{G_d}(s, t) = \delta_G(s, t)$.

**Proof.** Since the edge lengths in $G_d$ encode path lengths in $G$, we clearly have $\delta_{G_d}(s, t) \geq \delta_G(s, t)$. Let $h$ be the hop-length of a minimal shortest $s \rightarrow t$ path in $G$, i.e., minimum $h$ such that $\delta_G^h(s, t) = \delta_G(s, t)$. We prove $\delta_{G_d}(s, t) \leq \delta_G(s, t)$ by induction on $h$.

If $h \leq d + 1$, then $\delta_G(s, t) = \delta_{d+1}^G(s, t) = w_{G_d}(st) \geq \delta_{G_d}(s, t)$. Otherwise, suppose $h > d + 1$ and consider some minimal shortest $s \rightarrow t$ path $P$ in $G$. Let us write $P = P'QP''$, where $|P'|, |P''| \geq 1$, $|Q| = d$, and $Q$ is a $p \rightarrow q$ path, $p, q \in V$. Since $P$ is minimal, so is $Q$ (by Observation 7) and, as a result, we have $\delta_{d}^G(p, q) > \delta_{d}^G(p, q)$. Therefore, by the definition of a $d$-hub set, there exists a

\(^9\)The algorithm of Berger et al. [6] actually produces a hitting set a constant-factor larger than the greedy algorithm.

\(^10\)Here we abuse the notation slightly. Formally, by increasing the constant $c \geq 1$ hidden in the $\Theta$ notation, one can achieve probability $1 - 1/n^c$. 

path \( Q' = p \rightarrow q \) in \( G \) with length \( \delta^*_G(p, q) = \ell(Q) \), \( |Q'| = |Q| \), and going through a vertex \( z \in H_d \). Consequently, \( P'Q'P'' \) is a minimal shortest \( s \rightarrow t \) path with \( z \in H_d \) as an intermediate vertex. By Observation 7, this implies \( \delta^h_{-1}(s, z) = \delta_G(s, z) \) and \( \delta^h_{-1}(z, t) = \delta_G(z, t) \). By the inductive assumption we conclude \( \delta_{G_d}(s, z) = \delta_G(s, z) \) and \( \delta_{G_d}(z, t) = \delta_G(z, t) \). Finally, by \( z \in V(G_d) \) and triangle inequality we obtain \( \delta_{G_d}(s, t) \leq \delta_{G_d}(s, z) + \delta_{G_d}(z, t) = \delta_G(s, z) + \delta_G(z, t) = \delta_G(s, t) \). \( \square \)

The repeated squaring APSP algorithm has \( \tilde{O}(|H_d|^3) = \tilde{O}((n/d)^3) \) work and \( O(\text{polylog } n) \) depth (one can also think of the min-plus product as \( 2 \times |n| \) Bellman-Ford steps that can be performed in parallel, and hence a single product requires \( O(\text{polylog } n) \) depth). If one wanted to implement this step sequentially, the Floyd-Warshall APSP algorithm would yield \( O((n/d)^3 \log^3 n) \) time.

Finally, the last step is to inductively compute, for each \( k = K, \ldots, \) down to 0, the distances \( \delta_G(s, t) \) for all pairs \((s, t) \in (H_{2k} \times V) \cup (V \times H_{2k}) \). Recall that we have set \( H_0 = V \), so after completing the step for \( k = 0 \), this will give the all-pairs distance matrix as desired.

For convenience, let us set \( H_{2k+1} := H_{2k} \). Let us focus on computing \( \delta_{G}(s, t) \) for \( s \in H_{2k} \) and all \( t \in V \) assuming that the steps for larger \( k \) have already been completed and so the distances \( \delta_{G}(u, v) \) for \((u, v) \in (H_{2k+1} \times V) \cup (V \times H_{2k+1}) \) are already known. Actually, for the inductive step we only require these distances for the paths from \((H_{2k+1} \times H_{2k}) \cup (H_{2k} \times H_{2k+1}) \), which implies that in the first step (for \( k = K \)) they can be retrieved from the distance matrix of \( G_d \). Computing \( \delta_{G}(s, t) \) for \((s, t) \in V \times H_{2k} \) is symmetric. Let \( G_{k,s} \) be \( G \) with an auxiliary edge \( e_v = sv \) of weight \( w_{G_{k,s}}(e_v) = \delta_G(s, v) \) added for all \( v \in H_{2k+1} \). Observe that all the auxiliary edges’ weights have been already computed in the previous step. We compute the desired distances \( \delta_{G}(s, t) \) by running \( 2^{k+1} + 1 \) steps of the Bellman-Ford algorithm from each \( s \in H_d \) (in parallel). This costs \( \tilde{O}(|H_{2k}| \cdot m \cdot 2^k) = \tilde{O}(mn) \) work and \( \tilde{O}(2^k) \) depth in parallel, or \( O(nm \log n) \) time sequentially.

Note that through all \( k \), the total work is \( \tilde{O}(nmn) \), whereas the depth is \( \tilde{O}(d) \). The sequential time cost of the final phase is \( O(nm \log n \log d) \). The correctness of the above algorithm follows from the lemma below.

**Lemma 12.** For any \((s, t) \in H_{2k} \times V\), \( \delta_{G_{k,s}}^{2^{k+1}+1}(s, t) = \delta_G(s, t) \).

**Proof.** Let \( b := 2^{k+1} + 1 \). Since \( G \subseteq G_{k,s} \), and the auxiliary edges encode some path lengths in \( G \), we clearly have \( \delta_{G_{k,s}}^{b}(s, t) \geq \delta_{G_{k,s}}(s, t) = \delta_G(s, t) \).

Let us now prove \( \delta_{G_{k,s}}^{b}(s, t) \leq \delta_G(s, t) \). Let \( P \) be a minimal shortest \( s \rightarrow t \) path in \( G \). If \( |P| \leq b \), then our claim holds by \( G \subseteq G_{k,s} \). Suppose \(|P| > b \). Let us write \( P = QR \), where \(|R| = 2^{k+1} \) and \( R = u \rightarrow t \). Observe that the path \( R \) is shortest and minimal. Hence, by the definition of \( H_{2k+1} \), there exists a minimal shortest path \( R' = u \rightarrow t \) going through a vertex \( z \in H_{2k+1} \) and \(|R'| = 2^{k+1} \). Therefore, \( QR' \) is also a minimal shortest \( s \rightarrow t \) path. Let us express \( QR' \) as \( P_1P_2 \), where \( P_2 \) is a minimal shortest \( z \rightarrow t \) path. Note that we have \(|P_2| \leq 2^{k+1} \). Moreover, \( \ell(P_1) = \delta_G(s, z) = w_{G_{k,s}}(e_z) \). Hence, the \( s \rightarrow t \) path \( e_zP_2 \) of length \( \delta_G(s, t) \) is contained in \( G_{k,s} \) and its hop-length does not exceed \( 2^{k+1} + 1 = b \). Its existence implies \( \delta_{G_{k,s}}^{b}(s, t) \leq \delta_G(s, t) \). \( \square \)

Hence, we obtain the following lemma.

**Lemma 13.** Let \( d = 2^K \leq n \) and assume that \( G \) does not contain a negative cycle. Given a collection of sets \( H_{2k} \), where \( k = 0, \ldots, K \), such that \( H_{2k} \) is a \( 2^K \)-hub set of \( G \) of size \( O((n/2^K) \log n) \), all-pairs shortest paths can be computed in parallel using \( \tilde{O}(nm) \) work and \( \tilde{O}(d) \) depth, or sequentially in \( O(nm \log n \log d) \) time.
3.2 Deterministic Construction of Hubs

In this section we show how to construct hub sets in a deterministic way. We start with a few technical lemmas.

**Lemma 14.** Let $H_h$ be a $h$-hub set of $G$. Let $\Pi$ be a collection of minimal $h$-hop paths $P_{st} = s \rightarrow t$, one path for each pair $(s, t) \in H_h \times V$ for which such a minimal path exists.

Let $B$ be a hitting set of $\Pi$. Then $B$ is a $2h$-hub set of $G$.

**Proof.** We need to prove that, for each $u, v \in V$ such that $\delta_G^h(u, v) < \delta_G^{2h-1}(u, v)$, there exists a minimal $2h$-hop $u \rightarrow v$ path in $G$ going through a vertex of $B$. To this end, let $P$ be some minimal $u \rightarrow v$ path such that $|P| = 2h$. Split $P$ evenly into two paths $P_1P_2$ of hop-length $h$. Note that, by Observation 7, every subpath of $P$, in particular $P_1 = u \rightarrow w$, is minimal. Since $H_h$ is a hub set, there is a vertex $z \in H_h$ on some minimal path $Q = u \rightarrow w$ such that $|Q| = h$. Let $P' = QP_2$, and let us express $P' = RST$ so that $S = z \rightarrow y$ is a path satisfying $|S| = h$. Note that by minimality of $P'$, $S$ is also minimal. Therefore, $\delta_G^h(z, y) < \delta_G^{h-1}(z, y)$, and thus we have a minimal $z \rightarrow y$ path $P_{zy}$ of hop-length $h$ in $\Pi$. Finally, observe that $RP_{zy}T$ is a minimal $u \rightarrow v$ path of hop-length $2h$ and goes through a vertex of $B$ by the definition of a hitting set of $\Pi$. $\square$

**Lemma 15.** Suppose that $G$ has no negative cycles with at most $h$ edges. Then every minimal $h$-hop path in $G$ is simple.

**Proof.** A non-simple path contains a cycle. If a minimal $h$-hop path $P$ contained a non-negative cycle, there would exist a path of hop-length $< h$ and length no more than $\ell(P)$, thus contradicting minimality of $P$. If $P$ contained a negative cycle, the cycle would have at most $h$ edges. $\square$

**Lemma 16.** Suppose that $G$ has no negative cycles with at most $h$ edges. Let $H_h$ be a $h$-hub set of $G$. Then in $O(|H_h|hm)$ time we can:

- construct a $2h$-hub set $H_{2h}$ of $G$ such that $|H_{2h}| = O((n/h) \cdot \log n)$,
- find a shortest negative cycle in $G$ with no more than $2h$ edges, if one exists.

**Proof.** To find $H_{2h}$, we first compute shortest $\leq h$-hop paths from all $s \in H_h$ to all $v \in V$ using $h$ steps of Bellman-Ford algorithm in $O(|H_h|hm)$ time. Note that indeed minimal $h$-hop paths can be inferred from the Bellman-Ford execution by storing (1) the distance labels $d_i$ from all the steps, and (2) the predecessor vectors $p_i$ such that

$$d_{i+1}(v) = \min_{uv \in E} \{d_i(u) + w(uv)\} = d_i(p_i(v)) + w(p_i(v)v).$$

Clearly, an $s \rightarrow v$ minimal path of hop-length $h$ exists if $d_h(v) < d_{h-1}(v)$ and it can be easily reconstructed from the predecessor vectors. Since no negative cycle in $G$ has $\leq h$ edges, by Lemma 15, the computed minimal $h$-hop paths are all simple. As a result, by Lemma 8, we can compute a hitting set $B$ of size $O((n/h)\log n)$ of these paths in $O(|H_h| \cdot n) \cdot h$ time. By Lemma 14, $B$ constitutes a $2h$-hub set of $G$.

Suppose a shortest negative cycle $C = v_1 \ldots v_k$ with $|C| = k \in (h, 2h]$ exists in $G$. Since $C$ has a minimal number of hops (in particular, it is simple), the subpath $P = v_1 \rightarrow v_{h+1}$ of $C$ satisfies $\delta_G^h(v_1, v_{h+1}) \leq \ell(P) < \delta_G^{h-1}(v_1, v_{h+1})$. As a result, and by the definition of $H_h$, there exists a $h$-hop minimal path $P' = v_1 \rightarrow v_{h+1}$ such that $z \in H_h \cap V(P')$ and $\ell(P') \leq \ell(P)$. Hence, if we replace the prefix $P$ of $C$ with $P'$, we obtain another shortest negative cycle $C'$ with $|C'| \in (h, 2h]$ and going through $z$. It follows that $\delta_G^{2h}(z, z) < 0$. By performing $2h$ Bellman-Ford steps from all $z \in H_h$
independently we can thus find the smallest \( k \leq 2h \) such that \( \delta^k(z, z) < 0 \) for some \( z \in H_h \), if one exists. This \( k \) clearly equals \(|C|\). Since we have to check all \( z \in H_h \), this takes \( O(|H_h| \delta m) \) time through all \( z \).

We are now ready to describe the preprocessing step of our APSP algorithm that computes the hub sets \( H_1, \ldots, H_{d=2K} \) and possibly detects the shortest negative cycle. Let us first discuss a sequential algorithm. We proceed inductively, using Lemma 16.

We set \( H_{20} = V \). For each \( k = 0, \ldots, K - 1 \) we proceed as follows. We maintain an invariant that \( G \) contains no negative cycles of hop-length at most \( 2^k \) and \( |H_k| = O((n/2^k)^\log n) \). Hence, the invariant is true initially for \( k = 0 \) since, clearly, there are no single-edge negative cycles. By our invariant, we can apply Lemma 16 for \( h = 2^k \) and thus in \( O(nm \log n) \) time either detect a shortest negative cycle with no more than \( 2h \) edges (and declare the APSP problem infeasible) or construct a \( 2^{k+1} \)-hub set \( H_{2k+1} \) of size \( O((n/2^{k+1})\log n) \) and guarantee the invariant for \( k + 1 \).

The above sequential algorithm trivially implies a parallel algorithm: all the above \( O(\log d) \) steps of our computation can be implemented using a number of parallel invocations of an \( O(d) \)-step Bellman-Ford algorithm and an \( O(\text{polylog} \ n) \)-depth parallel computation (as given in Lemma 8) of a small hitting set of a collection of simple paths with a total hop-length of \( O(n^2 \log n) \) (the collection in step \( k \) has \( O((n^2/2^k) \cdot \log n) \) paths of length \( \Theta(2^k) \)). We thus obtain the following theorem.

**Theorem 17.** Let \( d = 2^K \leq n \). Then, a shortest negative cycle of \( G \), provided it has hop-length at most \( d \), can be computed deterministically:

- in \( O(nm \log n \log d) \) time using a sequential algorithm,
- using a parallel algorithm with \( \tilde{O}(nm) \) work and \( \tilde{O}(d) \) depth.

If no negative cycle of hop-length at most \( d \) exists in \( G \), then the algorithm can produce, within the same time bounds, a collection of sets \( H_{2k} \), where \( k = 0, \ldots, K \), such that \( H_{2k} \) is a \( 2^k \)-hub set of \( G \) of size \( O((n/2^k) \cdot \log n) \).

Together with Lemma 13 this finishes the proof of our main theorem.

**Theorem 1.** Let \( G \) be a real-weighted digraph. For any \( d \in [1, n] \), there exists a deterministic parallel algorithm computing all-pairs shortest paths in \( G \) with \( \tilde{O}(nm + (n/d)^3) \) work and \( \tilde{O}(d) \) depth.

As a corollary, we also obtain the following theorem.

**Theorem 2.** Let \( G \) be a real-weighted directed graph. A shortest (in terms of hops) negative-weight cycle in \( G \) can be found deterministically in \( O(nm \log^2 n) \) time.

## 4 Planar Graph Algorithms

In this section we present our improved planar graph algorithms. We start from describing the framework used by Fakcharoenphol and Rao [20] to solve the negative cycle detection problem on planar graphs in near-linear time. This framework forms the basis for all our parallel and sequential algorithms. We remark that the negative cycle detection algorithm of [20] has been subsequently improved by Klein et al. [39], whereas the currently best known bound \( O(n \log^2 n / \log \log n) \) is due to Mozes and Wulff-Nilsen [52]. These algorithms take a slightly simpler recursive approach, but the fundamental difference compared to [20] is using Klein’s MSSP algorithm [37] on each piece \( H \).
with reduced costs for computing $DDG_H$ in $O(|V(H)| + |\partial H|^2 \log n)$ time. This approach allows to avoid using the costly FR-Dijkstra (Lemma 19) and thus saves at least a $O(\log n)$ factor in the running time. However, it seems that Klein’s MSSP algorithm is inherently sequential and this is why in the following we will stick to the original approach of [20].

Transferring the approach of [20] to the parallel setting directly leads to $\tilde{O}(\sqrt{n})$-depth work-efficient parallel algorithms for single source shortest paths and negative cycle detection. We first show how to improve the depth to $\tilde{O}(n^{1/6})$ using our parallel APSP algorithm. Using this, we later show improved algorithms for minimum cost-to-time ratio cycle problem and external dense distance graphs computation.

4.1 The Framework of Fakcharoenphol and Rao [20]

Let $H$ be a weighted plane digraph with a distinguished set $\partial H$ of boundary vertices that necessarily lie on a $O(1)$ faces of $H$. We denote by $DDG_H$ (a dense distance graph) the complete weighted graph on $\partial H$ whose edge weights represent distances between all pairs of vertices of $\partial H$ in $H$. Fakcharoenphol and Rao [20] developed the concept of a dense distance graph along with efficient algorithms for constructing and processing DDGs as a way to obtain their breakthrough $O(n \log^3 n)$-time algorithm for negative cycle detection in a real-weighted planar digraph. We now review a variant of this algorithm using slightly more modern terminology.

After augmenting $G$ to be connected and triangulated, $G$ is recursively decomposed using small cycle separators [49] of size $O(\sqrt{n})$ until the obtained pieces have constant size. The decomposition procedure produces in $O(n \log n)$ time a binary tree $T(G)$ whose nodes correspond to subgraphs of $G$ (pieces), with the root being all of $G$ and the leaves being pieces of constant size. We identify each piece $H$ with the node representing it in $T(G)$. We can thus abuse notation and write $H \in T(G)$. The boundary vertices $\partial H$ of a piece $H$ are vertices that $H$ shares with some other piece $Q \in T(G)$ that is not $H$’s ancestor. For convenience we extend the boundary set $\partial L$ of a leaf piece $L$ to its entire vertex set $V(L)$. It is known that (see e.g., [9, 38]) one can additionally guarantee that for each piece $H \in T(G)$, (1) $H$ is connected, (2) $\partial H$ lies on some $O(1)$ faces of $H$, and (3) $|\partial H| = O(\sqrt{n})$. Moreover, one can assume that $\sum_{H \in T(G)} |V(H)| = O(n \log n)$ and $\sum_{H \in T(G)} |\partial H|^2 = O(n \log n)$.

Given the decomposition, the algorithm processes the pieces $H \in T(G)$ bottom-up. Clearly, if any piece contains a negative cycle, the whole $G$ does so as well. On the other hand, if $H$ contains no negative cycle, the dense distance graph $DDG_H$ on $\partial H$ is well-defined. Therefore, the computation for a piece $H$ either detects a negative cycle in $H$ or produces $DDG_H$ otherwise. Let $H_1, H_2$ be the children of the node $H$ in $T(G)$ and suppose neither of them contains a negative cycle. Let $H' = DDG_{H_1} \cup DDG_{H_2}$. It can be easily shown that (1) $H'$ contains a negative cycle if and only if $H'$ contains a negative cycle, (2) for any $u, v \in \partial H$, if $H$ contains no negative cycle, then $\delta_H(u, v) = \delta_{H'}(u, v)$. Consequently, in the algorithm of [20], for each piece $H$ we first run Bellman-Ford-based SSSP algorithm on $H'$ which either detects a negative cycle or produces a feasible price function $p$ on $H'$. The second step is to compute all-pairs shortest paths on $H'$ by running $|\partial H_1 \cup \partial H_2|$ Dijkstra-based single-source computations with edge costs in $H'$ reduced with $p$. The graph $DDG_H$ can be easily obtained from the computed distance matrix since $\partial H \subseteq \partial H_1 \cup \partial H_2$.

Since each $DDG_H$ has $|\partial H|^2$ edges, using Bellman-Ford and Dijkstra naively would lead to $\tilde{O} \left( \sum_{H \in T(G)} |\partial H|^3 \right) = \tilde{O}(n^{3/2})$ running time. The main contribution of [20] lies in showing that the special structure of a dense distance graph (that is, the distance matrix behind $DDG_H$ consists of two so-called staircase Monge matrices) can be leveraged to speed up naive implementations of these algorithms. The original implementations of [20] have been slightly improved, and currently
we have the following bounds.

**Lemma 18** ([20, 39]). A single step of Bellman-Ford algorithm can be simulated on $DDG_{H_1} \cup DDG_{H_2}$ in $O((|\partial H_1| + |\partial H_2|)\alpha(n))$ time, where $\alpha(n)$ is the inverse Ackermann function.

The above lemma is a simple application of the algorithm of Klawe and Kleitman [36] for computing row minima of a staircase $m \times m$ Monge matrix in $O(m\alpha(m))$ time.

**Lemma 19** (FR-Dijkstra [20, 23]). Given a feasible price function $p$ on $DDG_{H_1} \cup DDG_{H_2}$, one can simulate Dijkstra’s algorithm on $DDG_{H_1} \cup DDG_{H_2}$ in $O((|\partial H_1| + |\partial H_2|)\log^2 n \log \log n)$ time.

The above lemmas imply that the running time of Fakcharoenphol and Rao’s algorithm is $O\left(\sum_{H \in \mathcal{T}(G)} |\partial H| \log^2 n/\log^2 \log n\right) = O(n \log^3 n/\log^2 \log n)$.

**A parallel implementation.** We note that the above algorithm can be easily turned into a parallel algorithm with $\tilde{O}(n)$ work and $\tilde{O}(\sqrt{n})$ depth, as explained below.

First, computing the decomposition within $\tilde{O}(n)$ work and $O(\text{polylog } n)$ depth follows from the fact that $O(\sqrt{n})$-size simple cycle separators can be computed within these bounds [40, 49]. Next, it is known that row minima of a staircase Monge matrix can be computed using near-linear work and $O(\text{polylog } n)$ depth [2]. This implies the following.

**Lemma 20.** A single step of Bellman-Ford algorithm can be simulated on $DDG_{H_1} \cup DDG_{H_2}$ using a parallel algorithm with $\tilde{O}(\sqrt{n})$ work and $\tilde{O}(\sqrt{n})$ depth.

Consequently, the Bellman-Ford algorithm can be simulated on $DDG_{H_1} \cup DDG_{H_2}$ using $\tilde{O}((|\partial H_1| + |\partial H_2|)^2)$ work and $\tilde{O}(|\partial H_1| + |\partial H_2|) = \tilde{O}(\sqrt{n})$ depth.

Finally, since each of $|\partial H|$ Dijkstra’s algorithm runs used to compute $DDG_H$ based on $DDG_{H_1}$ and $DDG_{H_2}$ are independent, they can be performed in parallel to give $\tilde{O}(\sqrt{n})$ depth bound on this step. As the depth of $\mathcal{T}(G)$ is $O(\log n)$, the total work is $\tilde{O}(n)$, whereas the total depth is $\tilde{O}(\sqrt{n})$.

### 4.2 A Faster Nearly Work-Efficient Parallel SSSP Algorithm

In this section we apply our parallel APSP algorithm to obtain a polynomially smaller bound on the depth required to detect a negative cycle and to compute single-source shortest paths in a real-weighted planar digraph.

First, observe that having an APSP algorithm that can handle negative cycle detection on the fly allows us to replace the two-phase Bellman-Ford-Dijkstra approach in the framework of Fakcharoenphol and Rao. Indeed, then for each piece $H$ all we do is computing $DDG_H$ by running the all-pairs shortest paths algorithm directly on $DDG_{H_1} \cup DDG_{H_2}$, without using price functions that reduce the problem to the non-negatively weighted case.

Recall that the algorithm of Theorem 1 is a certain combination of a number of limited-hop Bellman-Ford invocations on various graphs $G'$ obtained from $G$ by adding $O(n)$ auxiliary edges, and a single run of a repeated squaring algorithm. In fact, if we were able to execute a single Bellman-Ford step on such a graph $G'$ using $O(t(n, m))$ work and $O(\text{polylog } n)$ depth, the algorithm of Theorem 1 would have $\tilde{O}(n^2 + n \cdot t(n, m) + (n/d)^3)$ work and $\tilde{O}(d)$ depth for any $d$. We use this fact to prove the following lemma.

**Lemma 21.** Let $H_1, H_2$ be the children of a node $H \in \mathcal{T}(G)$. Let $b = |\partial H_1| + |\partial H_2|$. We can compute all-pairs shortest paths in $DDG_{H_1} \cup DDG_{H_2}$ (and thus obtain $DDG_H$), using a parallel
algorithm with $\tilde{O}(b^2)$ work and $\tilde{O}(b^{1/3})$ depth. If the problem is infeasible, the algorithm detects a negative cycle within the same bounds.

**Proof.** Let $X$ be the graph $DDG_{H_1} \cup DDG_{H_2}$ with some $O(b)$ auxiliary edges added. Every Bellman-Ford step in the algorithm of Theorem 1 run on $DDG_{H_1} \cup DDG_{H_2}$ is performed on a graph of this form. Hence, it is enough to show how to perform a Bellman-Ford step on $X$, which can have $\Omega(b^2)$ edges. Since in a Bellman-Ford step the order of edge relaxations is arbitrary, we can first relax all edges of $DDG_{H_1} \cup DDG_{H_2}$ and then all the auxiliary edges. Relaxing the former takes $O(b)$ work and $O(p\text{olylog } n)$ depth by Lemma 20. The latter can be relaxed within the same bounds naively. Hence, the APSP algorithm of Theorem 1 can be implemented so that it has $\tilde{O}(b^2 + (b/d)^3)$ work and $\tilde{O}(d)$ depth for any $d$. By choosing $d = b^{1/3}$ we obtain the desired bounds. \hfill \square

**Theorem 3.** Let $G$ be a real-weighted planar digraph. There exists a deterministic parallel algorithm for negative cycle detection and single-source shortest paths in $G$ with $\tilde{O}(n)$ work and $\tilde{O}(n^{1/6})$ depth.

**Proof.** To obtain a parallel negative cycle detection algorithm, we simply replace the Bellman-Ford and Dijkstra steps in the computation for each piece $H \in T(G)$ in the algorithm of Fakcharoenphol and Rao [20] by a single computation of $DDG_H$ from $DDG_{H_1}$ and $DDG_{H_2}$ as in Lemma 21. The work remains $\tilde{O}\left(\sum_{H \in T(G)} |\partial H|^2\right) = \tilde{O}(n)$, whereas the depth is

$$\tilde{O}\left(\max_{H \in T(G)} |\partial H|^{1/3}\right) = \tilde{O}\left((\sqrt{n})^{1/3}\right) = \tilde{O}(n^{1/6}).$$

In order to solve the single-source shortest paths problem one can use a trick first described by Cohen [17] (and also used in [32]). Denote by $H^*$ a complete graph on $\partial H_1 \cup \partial H_2$ whose edge weights represent distances in $DDG_{H_1} \cup DDG_{H_2}$. Note that $H^*$ is precisely the graph that we compute using our new APSP algorithm to obtain $DDG_H \subseteq H^*$. Now, consider the graph

$$G^* = \left(\bigcup_{\text{non-leaf } H \in T(G)} H^*\right) \cup \left(\bigcup_{\text{leaf } L \in T(G)} L\right).$$

Since $G \subseteq G^*$ and the edges in $G^*$ correspond to paths in $G$, it is clear that $\delta_G(s, t) = \delta_{G^*}(s, t)$ for all $s, t \in V(G)$. However, as proven in [17,32], the graph $G^*$, although no longer planar, has hop-diameter $O(\log n)$. In other words, we in fact have $\delta_{G^*}^{O(\log n)}(s, t) = \delta_G(s, t)$ for all $s, t \in V$. As a result, shortest paths in $G$ from a single source $s$ can be found by running $O(\log n)$ simple-minded Bellman-Ford steps from $s$ on $G^*$. Since $|E(G^*)| = \tilde{O}\left(n + \sum_{H \in T(G)} |\partial H|^2\right) = \tilde{O}(n)$, this takes $\tilde{O}(n)$ work and $O(p\text{olylog } n)$ depth. \hfill \square

**Corollary 22.** The following problems on planar directed graphs have nearly work-efficient algorithms with $\tilde{O}(n^{1/6})$ depth:

1. computing a feasible flow\footnote{That is, given a vertex demand function $b : V \to \mathbb{R}$, a flow $f$ such that the excess $e_f(v)$ of each vertex $v$ is $b(v)$.} for real-weighted capacities,

2. computing a bipartite perfect matching,

3. computing a maximum $s, t$-flow for polynomially bounded integral edge capacities,
(4) finding a shortest cycle going through each $e \in E(G)$,

(5) finding $s, t$-replacement paths.

Proof. For the feasible flow problem, Miller and Naor [50] gave a duality-based nearly work-efficient, $O(polylog n)$-depth reduction to the real-weighted single-source shortest paths problem.

Bipartite perfect matching is directly reducible to the feasible flow problem by setting the vertex demands on one side of the graph to $-1$ and on the other side to 1.

The maximum $s, t$-flow problem can be reduced at the cost of $O(\log nC)$ multiplicative overhead to the feasible flow problem via binary search over the flow value, where the edge capacities are from $\mathbb{Z} \cap [0, C]$. This implies an $O(n)$ work and $\tilde{O}(n^{1/6})$ depth algorithm for that problem if $C = poly n$.

In Appendix A we describe how external dense distance graphs, defined and discussed later in this Section, can be used to compute the shortest cycles through all edges within desired bounds.

Finally, Chechik and Nechushtan [16] have recently shown that one can reduce the $s, t$-replacement paths problem (using a planarity-preserving near-linear time reduction) to computing the shortest cycles through all edges of a fixed shortest path of a graph. All the reduction does is basically computing a shortest $s \rightarrow t$ path $P$ in $G$ and reversing it. Therefore, by Theorem 3, the reduction can be performed in a nearly work-efficient manner using $\tilde{O}(n^{1/6})$ depth. The final step is to compute shortest cycles through all edges of the reversed path $P$, which can be done by item (4). \hfill \square

### 4.3 Minimum Cost-To-Time Ratio Cycle

In this section we assume that each edge $e$ of a planar digraph $G$ is assigned, besides a real weight $w(e)$, a time parameter $t(e) \in \mathbb{R}_{\geq 0}$. Our goal is to compute a directed cycle $C \subseteq G$ minimizing the value $\lambda^* = \frac{\sum_{e \in C} w(e)}{\sum_{e \in C} t(e)}$. Such a cycle is called a minimum cost-to-time ratio cycle, or minimum ratio cycle, in short. In the special case when $t(e) = 1$ for all $e \in E$, $C$ is called a minimum mean cycle.

**Parametric search.** It is well-known that one can reduce the minimum ratio cycle problem to the negative cycle detection problem using binary search, as follows. The binary search algorithm maintains an interval $[\lambda_1, \lambda_2]$ such that $\lambda^* \in [\lambda_1, \lambda_2]$. Given some $\lambda \in [\lambda_1, \lambda_2]$, we wish to decide whether $\lambda^* < \lambda$ or $\lambda \leq \lambda^*$. Note that $\lambda^* < \lambda$ if and only if there exists a cycle $C$ such that

$$\sum_{e \in C} w(e) - \lambda \cdot t(e) < 0.$$ 

This condition can be clearly checked by running a negative cycle detection algorithm on the graph $G_\lambda$ obtained from $G$ by changing the edge weight function to $w_\lambda(e) := w(e) - \lambda t(e)$. By picking $\lambda = (\lambda_1 + \lambda_2)/2$, this also allows us to shrink the interval $[\lambda_1, \lambda_2]$ by half using a single negative cycle detection step. If all the edge weights and times are integers whose absolute values are bounded by $W$, the algorithm stops in $O(\log (nW))$ steps. Since negative cycle detection can be solved in $O(n)$ time for planar graphs, this leads to a weakly polynomial $\tilde{O}(n \log W)$-time algorithm for the minimum ratio cycle problem.

Megiddo’s parametric search technique can be used to convert a strongly polynomial parallel negative cycle detection algorithm into a strongly polynomial time minimum ratio cycle algorithm. All known strongly polynomial algorithms for this problem use variants of this technique.

Suppose we have two strongly polynomial negative cycle detection algorithms: (1) a parallel one $\mathcal{P}$ with work $W(n, m)$ and depth $D(n, m)$, and (2) a sequential one $\mathcal{S}$ with running time $T(n, m)$. Additionally, suppose the parallel algorithm operates on edge weights/times (and all the stored values dependent on the weights) only by either additions/subtractions and comparisons.
The idea is, conceptually, to simulate (sequentially) the parallel algorithm $\mathcal{P}$ “generically” on all the possible graphs $G_\lambda$ with $\lambda \in [\lambda_1, \lambda_2]$ (where $[\lambda_1, \lambda_2]$ shrinks in time). This, in particular, means that the edge weights of $G_\lambda$ and all the stored values (e.g., the distance labels in Bellman-Ford algorithm) are linear functions of the form $-\lambda \cdot a + b$. Adding and subtracting linear functions can be done straightforwardly. Moreover, adding or subtracting such linear functions clearly leads to functions of the same form. However, the result of a comparison of two values parameterized by $\lambda$ generally depends on $\lambda$, and different results of such a comparison might cause different flow of $\mathcal{P}$ in the future – we would need to handle both branches of the algorithm’s flow if we wanted to detect a negative cycle in all $G_\lambda$. However, this is not our goal: we only care about locating $\lambda^*$. So, instead, whenever $\mathcal{P}$ performs a comparison $-\lambda a + b < -\lambda c + d$, we compute the breakpoint $x = (d - b)/(c - a)$ (for $c \neq a$) and use the sequential negative cycle detection algorithm on $G_x$ to test whether $\lambda^* < x$ or $\lambda^* \geq x$. This allows us to decide which branch would be chosen for $\lambda = \lambda^*$ and discard the other branch (i.e., shrink $[\lambda_1, \lambda_2]$ to either $[\lambda_1, x]$ or $[x, \lambda_2]$). Note that this already implies a strongly polynomial bound of $O(W(n, m) \cdot T(n, m))$ since $\mathcal{P}$ performs $O(W(n, m))$ comparisons.

However, one can use the parallelism of $\mathcal{P}$ to do better. In each of $O(D(n, m))$ parallel steps $s$, $\mathcal{P}$ performs some number $p_s$ of comparisons, whose results depend on where $\lambda^*$ lies relatively to some breakpoints $\lambda_1 = x_0 \leq x_1 < \ldots < x_{p_s} \leq x_{p_s+1} = \lambda_2$, where the sum of $p_s$ over all parallel steps $s$ is $O(W(n, m))$. Note that the breakpoints can be sorted in $O(p_s)$ time. Afterwards, we can find such $x_i$ that $\lambda^* \in [x_i, x_{i+1}]$ via binary search using $O(\log (W(n, m)))$ sequential negative cycle detection runs. Observe that this allows us to choose the correct branch for all the comparisons in parallel step $s$ at once in just $O(T(n, m))$ time. Consequently, we obtain a (sequential) strongly polynomial algorithm with $O(W(n, m) + D(n, m) \cdot T(n, m))$ running time.

Planar graphs. Note that all our algorithms (and therefore also the algorithm of Lemma 20) indeed operate on edge weights only by performing additions and comparisons. Hence, by plugging the algorithm of Theorem 3 as both the parallel and sequential algorithm into the parametric search framework, we already obtain an $\tilde{O}(n^{7/6})$-time strongly polynomial minimum ratio cycle algorithm (then, $T(n, m), W(n, m) \in O(n)$ and $D(n, m) = \tilde{O}(n^{1/6})$).

However, we can do better by slightly decreasing the depth of the parallel algorithm of Theorem 3 at the cost of increasing its work (we stress that we still stick to using a sequential algorithm with $T(n, m) = \tilde{O}(n)$). Recall from the proof of Lemma 21 that we could actually achieve depth $\tilde{O}(d)$ within work bounded by

$$\tilde{O}\left(n + \sum_{H \in \mathcal{T}(G)} |\partial H|^3/d^3\right) = \tilde{O}\left(n + \sqrt{\frac{n}{d^3}} \sum_{H \in \mathcal{T}(G)} |\partial H|^2\right) = \tilde{O}(n + n^{3/2}/d^3).$$

As a result, we can obtain a minimum ratio cycle algorithm that runs in time $\tilde{O}(dn + n^{3/2}/d^3)$ for any $d$. We balance these terms by choosing $d = n^{1/8}$ and obtain the following theorem.

**Theorem 4.** Let $G$ be a real-weighed planar graph. There exists an $\tilde{O}(n^{9/8})$-time strongly polynomial algorithm for computing a minimum ratio cycle (and thus also a minimum mean cycle) in $G$.

### 4.4 Faster Computation of External DDGs

Piecewise dense distance graphs $DDG_H$ for $H \in \mathcal{T}(G)$ have numerous other applications in sequential planar graph algorithms beyond negative cycle detection. Typically, though, they are computed
using the aforementioned Klein’s MSSP data structure [37] in \( O((|H| + |\partial H|^2) \log n) \) time rather than inductively in \( O\left(\left(\frac{|\partial H_1|^2 + |\partial H_2|^2}{\log^2 n} \log \log n\right)\right) \) time using FR-Dijkstra. However, in some situations, only the latter inductive method can be applied. One such application is computing so-called external dense distance graphs, i.e., the graphs \( DDG_{G-H} \) for all \( H \in T(G) \), where \( G - H \) is the graph obtained from \( G \) by removing the vertices \( V(H) \setminus \partial H \). We set \( \partial(G - H) := \partial H \) since indeed \( \partial H \) contains all vertices that \( G - H \) shares with \( H \). One can also argue that if the faces of \( H \) containing \( \partial H \) are simple and disjoint, \( \partial H \) can be assumed to lie on \( O(1) \) faces of \( G - H \) as well.

It is well-known (and also not difficult to prove, see, e.g., [51]) that if \( H_2 \) is a sibling of node \( H_1 \), whereas \( H \) is \( H_1 \)'s parent in \( T(G) \), then \( DDG_{G-H} \) can be obtained by computing all-pairs shortest paths on \( DDG_{G-H} \cup DDG_{H_2} \). Using FR-Dijkstra (Lemma 19), this takes \( O\left(\left(\frac{|\partial H_1|^2 + |\partial H_2|^2}{\log^2 n} \log \log n\right)\right) \) time and thus \( O\left(\left(\frac{n \log^3 n}{\log \log n}\right)\right) \) time through all pieces \( H \in T(G) \). On the other hand, using MSSP to compute all the external dense distance graphs is very inefficient, since clearly \( \sum_{H \in T(G)}|V(G - H)| = \Omega(n^2) \). In fact, the inductive FR-Dijkstra-based approach has been so far the only known way to compute external dense distance graphs efficiently. External DDGs alone can be used to compute, for example, shortest cycles through all edges of the graph (see Appendix A). They have also proved useful in obtaining very efficient algorithms for maximum flow and minimum cut related problems [9, 46]. Moreover, the computation of external dense distance graphs is the bottleneck of construction algorithms for, e.g., so-called cycle-MSSP [51] data structure, or distance oracles supporting failed vertices [15].

The below lemma shows that by combining the sequential version of our APSP algorithm with FR-Dijkstra we can obtain faster – by a factor of almost \( \Theta(\log n) \) – algorithm for inductively computing dense distance graphs. For simplicity, we focus on computing \( DDG_H \) from \( DDG_{H_1} \cup DDG_{H_2} \), as the algorithm for computing \( DDG_{G-H} \) (as argued above) is identical.

**Lemma 23.** Let \( H_1, H_2 \) be the children of node \( H \in T(G) \). Let \( b = |\partial H_1| + |\partial H_2| \). We can compute all-pairs shortest paths in \( DDG_{H_1} \cup DDG_{H_2} \) in \( O(b^2 \log n \cdot \log \log n \cdot \alpha(n)) \) sequential time. If the problem is infeasible, the algorithm finds a negative cycle within the same bounds.

**Proof.** By Lemma 18, in \( O(b^2 \cdot \alpha(n)) \) time we can either find a negative cycle in \( DDG_{H_1} \cup DDG_{H_2} \) or compute a feasible price function \( p \) on that graph. The price function allows us to make the weights in each \( DDG_{H_i} \) non-negative. Now we use the sequential version of the algorithm of Theorem 1 with two changes. First, we use a combination of Lemma 18 and \( O(b) \) naive relaxations for each Bellman-Ford step performed as we did in the proof of Lemma 21. Even more importantly, we replace the \( O((n/d)^3 \log^3 n) \) time Floyd-Warshall-based APSP computation of shortest paths between all \( s, t \in H_d \) with \( |H_d| \) invocations of FR-Dijkstra, as given in Lemma 19. As a result, the running time of the APSP algorithm becomes

\[
O\left(b \cdot (b \cdot \alpha(n)) \cdot \log n \log d + \left(\frac{b}{d} \log n\right) \cdot b \log^2 n\right) = O\left(b^2 \log n \cdot \left(\alpha(n) \log d + \frac{\log^2 n}{d}\right)\right).
\]

We obtain the desired bound by choosing \( d = \log^2 n \).

**Lemma 23** easily implies the following lemma.

**Lemma 5.** The external dense distance graphs \( DDG_{G-H} \) for all \( H \in T(G) \) can be computed in \( O(n \log^2 n \cdot \log \log n \cdot \alpha(n)) \) time.

---

\(^{12}\) Dealing with non-simple or non-disjoint faces is merely a tedious technical difficulty (see e.g., [14, 27, 31]) – they can be avoided by suitably extending the input graph along with its decomposition.
References

[1] Pankaj K. Agarwal, Micha Sharir, and Sivan Toledo. Applications of parametric searching in geometric optimization. *J. Algorithms*, 17(3):292–318, 1994. doi:10.1006/jagm.1994.1038.

[2] Alok Aggarwal, Dina Kravets, James K. Park, and Sandeep Sen. Parallel searching in generalized monge arrays. *Algorithmica*, 19(3):291–317, 1997. doi:10.1007/PL00009175.

[3] Alexandr Andoni, Clifford Stein, and Peilin Zhong. Parallel approximate undirected shortest paths via low hop emulators. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing, STOC 2020, Chicago, IL, USA, June 22-26, 2020*, pages 322–335, 2020. doi:10.1145/3357713.3384321.

[4] D. A. Bader and K. Madduri. Parallel algorithms for evaluating centrality indices in real-world networks. In *2006 International Conference on Parallel Processing (ICPP’06)*, pages 539–550, 2006.

[5] Surender Baswana, Ramesh Hariharan, and Sandeep Sen. Improved decremental algorithms for maintaining transitive closure and all-pairs shortest paths. *J. Algorithms*, 62(2):74–92, 2007. doi:10.1016/j.jalgor.2004.08.004.

[6] Bonnie Berger, John Rompel, and Peter W. Shor. Efficient NC algorithms for set cover with applications to learning and geometry. *J. Comput. Syst. Sci.*, 49(3):454–477, 1994. doi:10.1016/S0022-0000(05)80068-6.

[7] Aaron Bernstein. Maintaining shortest paths under deletions in weighted directed graphs. *SIAM J. Comput.*, 45(2):548–574, 2016. doi:10.1137/130938670.

[8] Guy E. Blelloch, Yan Gu, Yihan Sun, and Kanat Tangwongsan. Parallel shortest paths using radius stepping. In *Proceedings of the 28th ACM Symposium on Parallelism in Algorithms and Architectures, SPAA 2016, Asilomar State Beach/Pacific Grove, CA, USA, July 11-13, 2016*, pages 443–454, 2016. doi:10.1145/2935764.2935765.

[9] Glencora Borradaile, Piotr Sankowski, and Christian Wulff-Nilsen. Min st-cut oracle for planar graphs with near-linear preprocessing time. *ACM Trans. Algorithms*, 11(3):16:1–16:29, 2015. doi:10.1145/2684068.

[10] Richard P. Brent. The parallel evaluation of general arithmetic expressions. *J. ACM*, 21(2):201–206, 1974. doi:10.1145/321812.321815.

[11] Karl Bringmann, Thomas Dueholm Hansen, and Sebastian Krinninger. Improved algorithms for computing the cycle of minimum cost-to-time ratio in directed graphs. In *44th International Colloquium on Automata, Languages, and Programming, ICALP 2017, July 10-14, 2017, Warsaw, Poland*, pages 124:1–124:16, 2017. doi:10.4230/LIPIcs.ICALP.2017.124.

[12] Gerth Stolting Brodal, Jesper Larsson Träff, and Christos D. Zaroliagis. A parallel priority queue with constant time operations. *J. Parallel Distributed Comput.*, 49(1):4–21, 1998. doi:10.1006/jpdc.1998.1425.

[13] Nairen Cao, Jeremy T. Fineman, and Katina Russell. Efficient construction of directed hopsets and parallel approximate shortest paths. In Konstantin Makarychev, Yury Makarychev, Madhu Tulsiani, Gautam Kamath, and Julia Chuzhoy, editors, *Proceedings of the 52nd Annual*
[14] Panagiotis Charalampopoulos, Pawel Gawrychowski, Shay Mozes, and Oren Weimann. Almost optimal distance oracles for planar graphs. In Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, STOC 2019, pages 138–151, 2019. doi:10.1145/3313276.3316316.

[15] Panagiotis Charalampopoulos, Shay Mozes, and Benjamin Tebeka. Exact distance oracles for planar graphs with failing vertices. In Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2019, pages 2110–2123, 2019. doi:10.1137/1.9781611975482.127.

[16] Shiri Chechik and Moran Nechushtan. Simplifying and unifying replacement paths algorithms in weighted directed graphs. In 47th International Colloquium on Automata, Languages, and Programming, ICALP 2020, July 8-11, 2020, Saarbrücken, Germany (Virtual Conference), pages 29:1–29:12, 2020. doi:10.4230/LIPIcs.ICALP.2020.29.

[17] Edith Cohen. Efficient parallel shortest-paths in digraphs with a separator decomposition. J. Algorithms, 21(2):331–357, 1996. doi:10.1006/jagm.1996.0048.

[18] Edith Cohen. Using selective path-doubling for parallel shortest-path computations. J. Algorithms, 22(1):30–56, 1997. doi:10.1006/jagm.1996.0813.

[19] Edith Cohen. Polylog-time and near-linear work approximation scheme for undirected shortest paths. J. ACM, 47(1):132–166, 2000. doi:10.1145/331605.331610.

[20] Jittat Fakcharoenphol and Satish Rao. Planar graphs, negative weight edges, shortest paths, and near linear time. J. Comput. Syst. Sci., 72(5):868–889, 2006. doi:10.1016/j.jcss.2005.05.007.

[21] Jeremy T. Fineman. Nearly work-efficient parallel algorithm for digraph reachability. In Ilias Diakonikolas, David Kempe, and Monika Henzinger, editors, Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2018, Los Angeles, CA, USA, June 25-29, 2018, pages 457–470. ACM, 2018. doi:10.1145/3188745.3188926.

[22] Sebastian Forster and Danupon Nanongkai. A faster distributed single-source shortest paths algorithm. In Mikkel Thorup, editor, 59th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2018, Paris, France, October 7-9, 2018, pages 686–697. IEEE Computer Society, 2018. doi:10.1109/FOCS.2018.00071.

[23] Pawel Gawrychowski and Adam Karczmarz. Improved bounds for shortest paths in dense distance graphs. In 45th International Colloquium on Automata, Languages, and Programming, ICALP 2018, pages 61:1–61:15, 2018. doi:10.4230/LIPIcs.ICALP.2018.61.

[24] Andrew V. Goldberg, Serge A. Plotkin, David B. Shmoys, and Éva Tardos. Using interior-point methods for fast parallel algorithms for bipartite matching and related problems. SIAM J. Comput., 21(1):140–150, February 1992. doi:10.1137/0221011.

[25] Y. Han, V. Pan, and John Reif. Efficient parallel algorithms for computing all pair shortest paths in directed graphs. In Proceedings of the Fourth Annual ACM Symposium on Parallel Algorithms and Architectures, SPAA ’92, page 353–362, New York, NY, USA, 1992. Association for Computing Machinery. doi:10.1145/140901.141913.
[26] Mark Hartmann and James B. Orlin. Finding minimum cost to time ratio cycles with small integral transit times. *Networks*, 23(6):567–574, 1993. doi:10.1002/net.3230230607.

[27] Giuseppe F. Italiano, Adam Karczmarz, Jakub Łącki, and Piotr Sankowski. Decremental single-source reachability in planar digraphs. In *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017*, pages 1108–1121. ACM, 2017. doi:10.1145/3055399.3055480.

[28] Joseph JáJá. *An Introduction to Parallel Algorithms*. Addison-Wesley, 1992.

[29] Ian Jermyn and Hiroshi Ishikawa. Globally optimal regions and boundaries as minimum ratio weight cycles. *IEEE Trans. Pattern Anal. Mach. Intell.*, 23(10):1075–1088, 2001. doi:10.1109/34.954599.

[30] Donald B. Johnson. Efficient algorithms for shortest paths in sparse networks. *J. ACM*, 24(1):1–13, 1977. doi:10.1145/321992.321993.

[31] Haim Kaplan, Shay Mozes, Yahav Nussbaum, and Micha Sharir. Submatrix maximum queries in monge matrices and partial monge matrices, and their applications. *ACM Trans. Algorithms*, 13(2):26:1–26:42, 2017. doi:10.1145/3039873.

[32] Adam Karczmarz. Decremental transitive closure and shortest paths for planar digraphs and beyond. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018*, pages 73–92. SIAM, 2018. doi:10.1137/1.9781611975031.5.

[33] Adam Karczmarz and Jakub Łącki. Reliable hubs for partially-dynamic all-pairs shortest paths in directed graphs. In *27th Annual European Symposium on Algorithms, ESA 2019*, pages 65:1–65:15, 2019. doi:10.4230/LIPIcs.ESA.2019.65.

[34] Richard M. Karp. A characterization of the minimum cycle mean in a digraph. *Discret. Math.*, 23(3):309–311, 1978. doi:10.1016/0012-365X(78)90011-0.

[35] Valerie King. Fully dynamic algorithms for maintaining all-pairs shortest paths and transitive closure in digraphs. In *40th Annual Symposium on Foundations of Computer Science, FOCS ’99, 17-18 October, 1999, New York, NY, USA*, pages 81–91, 1999. doi:10.1109/SFFCS.1999.814580.

[36] Maria M. Klawe and Daniel J. Kleitman. An almost linear time algorithm for generalized matrix searching. *SIAM J. Discret. Math.*, 3(1):81–97, 1990. doi:10.1137/0403009.

[37] Philip N. Klein. Multiple-source shortest paths in planar graphs. In *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2005*, pages 146–155, 2005.

[38] Philip N. Klein, Shay Mozes, and Christian Sommer. Structured recursive separator decompositions for planar graphs in linear time. In *Symposium on Theory of Computing Conference, STOC’13*, pages 505–514, 2013. doi:10.1145/2488608.2488672.

[39] Philip N. Klein, Shay Mozes, and Oren Weimann. Shortest paths in directed planar graphs with negative lengths: A linear-space $O(n\log^2 n)$-time algorithm. *ACM Trans. Algorithms*, 6(2):30:1–30:18, 2010. doi:10.1145/1721837.1721846.
Philip N. Klein and Sairam Subramanian. A linear-processor polylog-time algorithm for shortest paths in planar graphs. In 34th Annual Symposium on Foundations of Computer Science, Palo Alto, California, USA, 3-5 November 1993, pages 259–270, 1993. doi:10.1109/SFCS.1993.366861.

Philip N. Klein and Sairam Subramanian. A randomized parallel algorithm for single-source shortest paths. J. Algorithms, 25(2):205–220, 1997. doi:10.1006/jagm.1997.0888.

Eugene L. Lawler. Optimal cycles in doubly weighted linear graphs. In Theory of Graphs: International Symposium, pages 209–213, 1966.

Jason Li. Faster parallel algorithm for approximate shortest path. In Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing, STOC 2020, Chicago, IL, USA, June 22-26, 2020, pages 308–321, 2020. doi:10.1145/3357713.3384268.

Richard J. Lipton and Robert Endre Tarjan. Applications of a planar separator theorem. SIAM J. Comput., 9(3):615–627, 1980. doi:10.1137/0209046.

Yang P. Liu, Arun Jambulapati, and Aaron Sidford. Parallel reachability in almost linear work and square root depth. In David Zuckerman, editor, 60th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2019, Baltimore, Maryland, USA, November 20-22, 2019, pages 1664–1686. IEEE Computer Society, 2019. doi:10.1109/FOCS.2019.00098.

Jakub Łącki and Yahav Neussbaum and Piotr Sankowski and Christian Wulff-Nilsen. Single source - all sinks max flows in planar digraphs. In 53rd Annual IEEE Symposium on Foundations of Computer Science, FOCS 2012, New Brunswick, NJ, USA, October 20-23, 2012, pages 599–608, 2012. doi:10.1109/FOCS.2012.66.

Meena Mahajan and Kasturi R. Varadarajan. A new nc-algorithm for finding a perfect matching in bipartite planar and small genus graphs (extended abstract). In Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing, STOC ’00, page 351–357, New York, NY, USA, 2000. Association for Computing Machinery. doi:10.1145/335305.335346.

Nimrod Megiddo. Applying parallel computation algorithms in the design of serial algorithms. J. ACM, 30(4):852–865, 1983. doi:10.1145/2157.322410.

Gary L. Miller. Finding small simple cycle separators for 2-connected planar graphs. J. Comput. Syst. Sci., 32(3):265–279, 1986. doi:10.1016/0022-0000(86)90030-9.

Gary L. Miller and Joseph Naor. Flow in planar graphs with multiple sources and sinks. SIAM J. Comput., 24(5):1002–1017, 1995. doi:10.1137/S009753978962997.

Shay Mozes and Christian Sommer. Exact distance oracles for planar graphs. In Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2012, pages 209–222, 2012. doi:10.1137/1.9781611973099.19.

Shay Mozes and Christian Wulff-Nilsen. Shortest paths in planar graphs with real lengths in $O(n \log^2 n / \log \log n)$ time. In Algorithms - ESA 2010, 18th Annual European Symposium. Proceedings, Part II, pages 206–217, 2010. doi:10.1007/978-3-642-15781-3_18.

James B. Orlin, K. Subramaniam, and Piotr J. Wojciechowski. Randomized algorithms for finding the shortest negative cost cycle in networks. Discret. Appl. Math., 236:387–394, 2018. doi:10.1016/j.dam.2017.10.011.
[54] Seth Pettie. A new approach to all-pairs shortest paths on real-weighted graphs. *Theor. Comput. Sci.*, 312(1):47–74, 2004. doi:10.1016/S0304-3975(03)00402-X.

[55] Hanmao Shi and Thomas H. Spencer. Time-work tradeoffs of the single-source shortest paths problem. *J. Algorithms*, 30(1):19–32, 1999. doi:10.1006/jagm.1998.0968.

[56] Thomas H. Spencer. Time-work tradeoffs for parallel algorithms. *J. ACM*, 44(5):742–778, 1997. doi:10.1145/265910.265923.

[57] K. Subramani. Optimal length resolution refutations of difference constraint systems. *J. Autom. Reasoning*, 43(2):121–137, 2009. doi:10.1007/s10817-009-9139-4.

[58] K. Subramani, Matthew D. Williamson, and Xiaofeng Gu. Improved algorithms for optimal length resolution refutation in difference constraint systems. *Formal Asp. Comput.*, 25(2):319–341, 2013. doi:10.1007/s00165-011-0186-3.

[59] Éva Tardos. A strongly polynomial minimum cost circulation algorithm. *Combinatorica*, 5(3):247–256, 1985. doi:10.1007/BF02579369.

[60] Jeffrey D. Ullman and Mihalis Yannakakis. High-probability parallel transitive-closure algorithms. *SIAM J. Comput.*, 20(1):100–125, 1991. doi:10.1137/0220006.

[61] Olga Veksler. Stereo correspondence with compact windows via minimum ratio cycle. *IEEE Trans. Pattern Anal. Mach. Intell.*, 24(12):1654–1660, 2002. doi:10.1109/TPAMI.2002.1114859.

[62] Song Wang and Jeffrey Mark Siskind. Image segmentation with minimum mean cut. In Proceedings of the Eighth International Conference On Computer Vision (ICCV-01), Vancouver, British Columbia, Canada, July 7-14, 2001 - Volume 1, pages 517–524, 2001. doi:10.1109/ICCV.2001.10090.

[63] Song Wang and Jeffrey Mark Siskind. Image segmentation with ratio cut. *IEEE Trans. Pattern Anal. Mach. Intell.*, 25(6):675–690, 2003. doi:10.1109/TPAMI.2003.1201819.

[64] Virginia Vassilevska Williams and R. Ryan Williams. Subcubic equivalences between path, matrix, and triangle problems. *J. ACM*, 65(5):27:1–27:38, 2018. doi:10.1145/3186893.

[65] Neal E. Young, Robert Endre Tarjan, and James B. Orlin. Faster parametric shortest path and minimum-balance algorithms. *Networks*, 21(2):205–221, 1991. doi:10.1002/net.3230210206.

[66] Uri Zwick. All pairs shortest paths using bridging sets and rectangular matrix multiplication. *J. ACM*, 49(3):289–317, 2002. doi:10.1145/567112.567114.
A Computing All-Edges Shortest Cycles

The following corollary follows easily by Lemma 21 and extending the algorithm behind Theorem 3 to also compute external DDGs.

**Corollary 24.** All external dense distance graphs $DDG_{G-H}$ can be computed in parallel using $\tilde{O}(n)$ work and $\tilde{O}(n^{1/6})$ depth.

This allows us to prove the next lemma.

**Lemma 25.** Let $G$ be a real-weighted planar digraph with no negative cycles. Then, one can compute for each $e \in E(G)$ the shortest cycle going through $e$:

- sequentially in $O(n \log^2 n \cdot \log \log n \cdot \alpha(n))$ time,
- in parallel using $\tilde{O}(n)$ work and $\tilde{O}(n^{1/6})$ depth.

**Proof.** Let $T(G)$ be the recursive decomposition of $G$. By Lemma 5, we can compute the external distance graphs $DDG_{G-H}$ for all $H \in T(G)$ in $O(n \log^2 n \cdot \log \log n \cdot \alpha(n))$ time. By Corollary 24, the same task can be completed within $\tilde{O}(n)$ work and $\tilde{O}(n^{1/6})$ depth.

Let $L_e$ be any leaf node $L_e \in T(G)$ containing $e = uv$. The length of the shortest cycle going through $e$ is clearly $\delta_G(v, u) + w(e)$. So we need to compute $\delta_G(v, u)$. Since $\partial L_e = V(L_e)$, $u, v \in \partial L_e$.

Consider the graph $G_e = DDG_{G-L_e} \cup L_e$. We show that $\delta_G(v, u) = \delta_{G_e}(v, u)$. Indeed, consider a shortest $v \rightarrow u$ path $P = P_1 \ldots P_k$ in $G$ such that each $P_i$ is either a maximal subpath entirely contained in $G - L_e$, or a single edge in $L_e$. Consider the graph $G_e = DDG_{G-L_e} \cup L_e$. We show that $\delta_G(v, u) = \delta_{G_e}(v, u)$.

Since each $G_e$ has $O(1)$ size, computing shortest paths in all $G_e$ takes linear extra time. As the computations for each $G_e$ are independent, they can be parallelized within $\tilde{O}(n)$ work and $O(\text{polylog } n)$ depth. \qed