Nonlinear stability analysis of a collinear libration point in the planar circular restricted photogravitational three-body problem

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Abstract We investigate the problem on stability of collinear libration point in the photogravitational three-body problem. That is, we study the stability of stationary motion of an infinitesimal small body (particle) in gravitational field of two bodies (primaries) acting on the particle with radiation pressure repulsive forces. The primaries move in circular coplanar orbits about their barycenter and the particle lies in the plane of primaries motion. By using the method of normal forms and applying theorems of the Kolmogorov-Arnold-Moser theory we perform nonlinear stability study of collinear point, which located on the line connecting the primaries. Rigorous conclusions on stability in the sense of Lyapunov have been obtained both for non-resonant case and for cases of third and fourth order resonances. These results can be interesting for spaceflight and astronomical applications such as studies of dynamics of artificial satellites and observations of cosmic dust.

1. Introduction
The restricted photogravitational three-body problem is a natural generalization of the classical restricted three body problem. It deals with the motion of a small body (particle) in the Newtonian gravitational field of two bodies, called primaries. The particle is regarded as passive gravitating point, which does not affect the motion of the primaries. Besides of gravitational attraction the primaries act on the particle with radiation pressure repulsive forces. If the primaries move in circular coplanar orbits about their barycenter and the particle lies in the plane of primaries motion, then the problem is called the planar circular restricted photogravitational three-body problem.

The equations of particle motion admit three remarkable solutions (the so-called straight line solutions) describing stationary motions, in which the particle is located on the line connecting the primaries (see, for instance, [1]). The above stationary motions correspond to equilibrium positions in the frame, rotating with the primaries. These equilibrium positions are usually called Euler (or collinear) libration points. They exist both in photogravitational and in classical three-body problems. The equilibrium position located between the primaries is denoted by $L_1$.

The problem of existence and stability of collinear libration points in the planar circular restricted photogravitational three-body problem has been considered in many papers. It has been shown in [2,3]...
that the influence of radiation pressure repulsive forces can lead to linear stability of collinear libration point \( L_1 \), which is always unstable in the classical restricted circular three body problem. In [4,5] the existence and linear stability of collinear libration point \( L_1 \) both in the planar circular and elliptic restricted photogravitational three-body problem has been investigated for all values of parameters. The problem on stability study of \( L_1 \) in the sense of Lyapunov was considered in [6,7] for the cases of third and fourth order resonances. In particular, it has been established that the libration point \( L_1 \) is always stable in circular restricted photogravitational three-body problem if the fourth order resonance takes place. The nonlinear stability analyses of libration point \( L_1 \) has been performed in [8,9] for the partial case, when primaries have equal masses and act on the particle with equal radiation pressure repulsive forces. By using normal form method and the Kolmogorov-Arnold-Moser (KAM) theory the rigorous conclusions on stability in the sense of Lyapunov has been obtained both for non-resonant and resonant cases.

There are a lot of spaceflight applications of libration points. For instance, the point \( L_1 \) is suited for making observations of the Sun–Earth system, as well as libration points can be used for location of space-based observatories. That is why the problem of existence and stability of libration points for various generalizations of the restricted three-body problem attracts a lot of attention of researches.

In recent years, the stability of libration points in three-body problem has been considered in the presence of some perturbations. In [10-12] it was supposed that both primaries are oblate spheroids and the problem of existence and stability for the libration points was considered by taking into account combined effects of radiation forces and oblateness. In particular, the possible positions of \( L_1, L_2 \) and \( L_3 \) has been constructed in a form of series with respect to a small parameter and stability problem of the libration points in linear approximation has been considered. Similar investigation has been performed in [13-15] under the assumption that the primaries are triaxial rigid bodies.

It is worth noting that most of stability studies of \( L_1 \) have been performed either in linear approximation or for some special values of parameters. However, the stability study of the linearized system is not enough to establish the stability in the original non-linear system. In particular, in resonant cases unstable libration point can be linearly stable.

The aim of this paper is to obtain rigorous conclusions on the stability of \( L_1 \) in the sense of Lyapunov for almost all parameters values including the cases of resonances. In the next sections we perform nonlinear stability analysis by using normal form method and KAM theory.

2. Problem statement and equations of motion

Let us introduce a rotating coordinate system \( Oxyz \), such that its origin \( O \) is the mass center of primaries \( P_1 \) and \( P_2 \) (see figure 1). The axis \( OX \) passes through the primaries, axis \( OZ \) is perpendicular to the plane of bodies motion and the axis \( Oy \) complements the coordinate system to right hand and orthogonal one.

![Figure 1. Coordinate system.](image)

Now we pass to dimensionless coordinates \( \xi, \eta \)

\[
x = r\xi, \quad y = r\eta, \quad r = \text{const},
\]

(1)

where \( x, y \) are coordinates of the particle \( P \) in system \( Oxyz \) and \( r \) is the distance between primaries.

The equations of motions of the particle \( P \) can be written in the form of canonical system
with the following Hamiltonian
\[ H = \frac{1}{2}(p_\xi^2 + p_\eta^2) + p_\xi \eta - \frac{Q_1(1 - \mu)}{r_1} - \frac{Q_2 \mu}{r_2}, \]
where \( r_1, r_2 \) are distances between the particle and primaries
\[ r_1 = \sqrt{(\xi + \mu)^2 + \eta^2}, \quad r_2 = \sqrt{(\xi + \mu - 1)^2 + \eta^2}, \quad \mu = \frac{m_2}{m_1 + m_2}. \]

By \( Q_1 \) and \( Q_2 \) are denoted so-called coefficients of mass reduction.

In what follows we assume that the values of gravitational forces are grater then the values of radiation pressure forces. It means, \( Q_1 \) and \( Q_2 \) belong to the interval \([0; 1]\). If \( Q_i = 1 \) \((i = 1, 2)\) then the radiation pressure forces are equal to zero.

The equations of motion admit the stationary solution
\[ \xi = \xi_*, \quad \eta = 0, \quad p_\xi = 0, \quad p_\eta = \xi_*, \] (3)
which corresponds to the collinear point \( L_1 \) (see figure 2). Here \( \xi_* \) is a real root of the equation
\[ \xi_* - \frac{Q_1(1 - \mu)}{(\xi_* + \mu)(\xi_* + \mu - 1)} - \frac{Q_2 \mu}{(\xi_* + \mu - 1)(\xi_* + \mu - 1)} = 0, \]
such that \( \xi_* \in (-\mu; 1 - \mu). \)

Figure 2. Collinear point \( L_1. \)

Let us now introduce the perturbations \( q_i, p_i \) \((i = 1, 2)\)
\[ \xi = \xi_* + q_1, \quad \eta = q_2, \quad p_\xi = p_1, \quad p_\eta = \xi_* + p_1, \]
and expand the Hamiltonian in the power series with respect to \( q_i, p_i. \)
\[ H = \frac{1}{2}(p_\xi^2 + p_\eta^2) + p_\xi q_2 - p_\eta q_1 - a q_1^2 + \frac{1}{2} a q_2^2 + b q_1^2 - \frac{3}{2} b q_1 q_2 - c q_1^3 + 3 c q_1^2 q_2 - \frac{3}{8} c q_2^4 + \cdots, \] (4)
where the constant coefficient is omitted and other coefficients reads
\[ a = \frac{Q_1(1 - \mu)}{|\xi_* + \mu|^3} + \frac{Q_2 \mu}{|\xi_* + \mu - 1|^3}, \]
\[ b = \frac{Q_1(1 - \mu)}{|\xi_* + \mu|(\xi_* + \mu)^3} + \frac{Q_2 \mu}{|\xi_* + \mu - 1|(\xi_* + \mu - 1)^3}, \]
\[ c = \frac{Q_1(1 - \mu)}{|\xi_* + \mu|^5} + \frac{Q_2 \mu}{|\xi_* + \mu - 1|^5}. \]

In next sections we perform rigorous stability analysis of the trivial solution of the system with the Hamiltonian (4) and obtain conclusions on its stability in the sense of Lyapunov.

3. Linear stability analysis
The stability study we start from the analysis of linear system with the following Hamiltonian
The stability of the linear system can be established by analyzing the roots of its characteristic equation. The characteristic equation of the linear system reads

\[ \lambda^4 - a\lambda^2 + 2a\lambda^2 - 2a^2 + a + 1 = 0. \] (9)

For \( a \notin \left[ \frac{8}{9}; 1 \right] \), the equation (9) has roots with nonzero real parts. It means that the trivial solution is unstable. If \( a \in \left( \frac{8}{9}; 1 \right) \), then the equation (9) has two pairs of pure imaginary simple roots: \( \pm i\omega_1, \pm i\omega_2 \), where \( \omega_1 = \left( 1 - \frac{1}{2}a + \frac{1}{2}\sqrt{9a^2 - 8a} \right)^{1/2} \), \( \omega_2 = \left( 1 - \frac{1}{2}a - \frac{1}{2}\sqrt{9a^2 - 8a} \right)^{1/2} \). In this case the trivial solution is stable in linear approximation. The condition \( a \in \left( \frac{8}{9}; 1 \right) \) specifies a domain in the three-dimensional space of parameters \( \mu, Q_1, Q_2 \). In this domain the trivial solution of the linear system is stable. On the figure 3 the cross section of parameter space at \( \mu = 0.4 \) is shown. The area of stability in the linear approximation is indicated by gray color.

The stability in the linear approximation does not mean the stability of the complete system with the Hamiltonian (4). It means that to obtain conclusions on stability of \( L_1 \) in the sense of Lyapunov, it is necessary to perform a nonlinear stability analysis taking into account terms of degree higher than two in the Hamiltonian expansion (4).

\[ H_2 = \frac{1}{2} \left( p_1^2 + p_2^2 \right) + p_1 q_2 - p_2 q_1 - a q_1^2 + \frac{1}{2} a q_2^2. \] (8)

The stability analysis in non-resonant case

In what follows we suppose that \( a \in \left( \frac{8}{9}; 1 \right) \), i.e. \( L_1 \) is stable in the linear approximation. Rigorous conclusions on stability for \( a \in \left( \frac{8}{9}; 1 \right) \) can be obtained by using methods of KAM theory. In particular, the theorem on stability for autonomous Hamiltonian systems with two degrees of freedom [16-19] can be applied. To this end, we normalize the Hamiltonian (4), i.e. we construct canonical changes of variables, which bring the Hamiltonian into the most convenient for stability study normal form.
first stage of the normalization, we perform the following linear autonomous canonical change of variables

\[ q_1 = a_{11}x_1 + a_{12}x_2, \quad q_1 = b_{11}y_1 + b_{12}y_2, \]
\[ p_1 = b_{21}y_1 + b_{22}y_2, \quad p_2 = a_{21}x_1 + a_{22}x_2, \]

where

\[ a_{11} = \frac{\kappa_1(\omega_1^2 - a + 1)}{\omega_1}, \quad a_{21} = \frac{\kappa_1(-\omega_1^2 - a + 1)}{\omega_1}, \]
\[ b_{11} = 2\delta_1\kappa_1, \quad b_{21} = \delta_1\kappa_1(\omega_1^2 - a - 1), \]
\[ \kappa_1 = \left(\frac{\omega_1}{\omega_1^4 - 2\omega_1a^2 + a^2 + 2\omega_1^2 + 2a - 3}\right)^{1/2}. \]

The above change of variables normalizes the quadratic part (8) of the Hamiltonian (4). In the variables \( x_i, y_i \) (\( i = 1, 2 \)) the quadratic part takes the following normal form

\[ H_2 = \frac{1}{2}\omega_1(x_1^2 + y_1^2) - \frac{1}{2}\omega_2(x_2^2 + y_2^2). \]

The change of variables (10) can be constructed, for instance, by the method described in [16,20].

At the second stage of normalization we normalize the cubic and quartic parts of the Hamiltonian. It can be done by means of a canonical near-identity change of variables \( x_i, y_i \to u_i, v_i \). The generating function for such a change has the form

\[ S = x_i v_i + S^{(3)}(x_i, v_i) \quad (i = 1, 2), \]

where \( S^{(3)}(x_i, v_i) \) is a convergent power series, which starts from terms of order three. The Deprit-Hori method [16,21] can be used to calculate the coefficients of the generating function (12) and coefficients of normal form of the Hamiltonian up terms of fourth order.

We note that in non-resonant case, the Hamiltonian normal form can differ from ones in resonant cases. That is why, cases of resonances should be considered separately from the general non-resonant case. In the general non-resonant case the following inequality \( n_1 \omega_1 + n_2 \omega_2 \neq 0 \) holds for integer \( n_1 \) and \( n_2 \) satisfying the non-resonant condition \( |n_1| + |n_2| \leq 4 \).

It is easy to see that \( \omega_1 > \omega_2 > 0 \) at \( a \in \left(\frac{8}{7}; 1\right) \). Thus, if \( \omega_1 \neq 2\omega_2 \) and \( \omega_1 \neq 3\omega_2 \), then the above non-resonant condition is fulfilled. By using the explicit formulas for \( \omega_1, \omega_2 \) given in section 3, it is easy to show that \( \omega_1 = 2\omega_2 \) at \( a = \frac{41+\sqrt{145}}{108} \) and \( \omega_1 = 3\omega_2 \) at \( a = \frac{68+6\sqrt{5}}{209} \).

Let us assume that \( a \neq \frac{41+\sqrt{145}}{108} \) and \( a \neq \frac{68+6\sqrt{5}}{209} \). Then the general non-resonant case takes place. In this case the Hamiltonian normal form reads

\[ H = \omega_1 r_1 - \omega_2 r_2 + c_{20}r_1^2 + c_{11}r_1r_2 + c_{02}r_2^2 + O((r_1 + r_2)^{5/2}), \]

where the canonical variables \( r_i, \varphi_i \) are introduced by formulæ \( u_i = \sqrt{2r_i} \sin \varphi_i, v_i = \sqrt{2r_i} \cos \varphi_i \).

If \( a = c_{20}a^2 + c_{11}a + c_{02}a^2 \neq 0 \), then it follows from the Arnold-Moser theorem [16-19] the stability in the sense of Lyapunov for the trivial solution of the system with Hamiltonian (13).

By using the Deprit-Hori method we have calculated the coefficients \( c_{20}, c_{11}, c_{02} \) and obtained \( \Delta \) in the following explicit form

\[ \Delta(a, b, c) = \frac{-3}{16a(9a - 8)(54a^2 - 41a - 9)(a - 1)(2a + 1)^2} \delta, \]

where

\[ \delta = 11664a^7c - 7776a^6b^2 - 14688a^6c + 6336a^5b^2 - 8856a^5c + 12110a^4b^2 + 17196a^4c - 19009a^3b^2 - 6483a^3c + 16439a^2b^2 - 3189a^2c - 5256ab^2 + 3384ac - 2520b^2 + 648c. \]
Thus, for values of parameters satisfying the inequalities

\[ a \neq \frac{41+5\sqrt{145}}{108}, \quad a \neq \frac{68+60\sqrt{5}}{209} \quad \text{and} \quad \Delta \neq 0 \]

the libration point \( L_1 \) is stable in the sense of Lyapunov.

If \( \Delta = 0 \), then we have the case of degeneration, when the problem of stability cannot be solved by taking into account terms of order four in the Hamiltonian expansion (4). To study the stability in the case of degeneration it is necessary to take into account terms of order six or higher. The equality \( \Delta = 0 \) defines a two-dimensional surface of degeneration in the three-dimensional space of the parameters. For \( Q_i \in [0, 0.3] \) \((i = 1, 2)\) the curve of degeneration obtained by the cross section of the above surface with plane \( \mu = 0.4 \) is indicated in figure 4.

![Figure 4. The curve of degeneration (for \( \mu = 0.4 \)).](image1)

![Figure 5. Resonant curves (for \( \mu = 0.4 \)).](image2)

Let us now recall that the restricted photogravitational three-body problem is a good mathematical model for study of dynamics of small celestial bodies such as cosmic microparticle. That is why one can expect the existence of cloud-like clusters of microparticle (the so-called cosmic dust) near the stable libration point \( L_1 \) of a binary star system.

### 5. Stability analysis in the case of resonances

In resonant cases the Hamiltonian normal form includes an additional (resonant) term. In this section we consider the third and fourth order resonances. It is worth noting that in three-dimensional space of parameters \( \mu, Q_1, Q_2 \), the third and fourth order resonance take place on two-dimensional resonant surfaces given by the equation \( a = \frac{41+5\sqrt{145}}{108} \) and \( a = \frac{68+60\sqrt{5}}{209} \) respectively, where \( a \) is determined by formula (5). These resonant surfaces lie within the domain of stability in the linear approximation. For \( Q_i \in [0, 0.3] \) \((i = 1, 2)\) the resonant curves obtained by the cross section of the resonant surfaces with the plane \( \mu = 0.4 \) are indicated in figure 5.

At first, we consider the third order resonance \( \omega_1 = 2\omega_2 \), i.e. we assume that \( a = \frac{41+5\sqrt{145}}{108} \). In this case by a suitable change of canonical variables, the Hamilton function can be reduced to the following normal form [16]

\[
H = 2\omega_2 r_1 - \omega_2 r_2 - 2\sqrt{2}k_{12}r_2\sqrt{r_1}\sin (\varphi_1 + 2\varphi_2) + O((r_1 + r_2)^2). \tag{15}
\]

The coefficient \( k_{12} \) obtained by using the Depri-Hori method reads

\[
k_{12} = \frac{81\sqrt{6}(61\sqrt{145} - 731)(105 - 3\sqrt{145})^{1/4}}{2(8360 - 640\sqrt{145})(475 - 29\sqrt{145})^{1/2} b}. \tag{16}
\]
If \( k_{12} \neq 0 \), then it follows from Markov's theorem [16] the instability of the collinear libration point. The special case \( k_{12} = 0 \) can appear only if \( b = 0 \). In this case the Hamiltonian normal form does not include resonant part up to terms of order four. In other words, at \( k_{12} = 0 \) the normalized Hamiltonian has form (13) and the above-mentioned Arnold-Moser theorem can be applied to solve the stability question. That is, the stability takes place if \( \Delta \neq 0 \). By substituting \( a = \frac{41 + 5\sqrt[3]{45}}{108} \) and \( b = 0 \) in (14), we have

\[
\Delta \left( \frac{41 + 5\sqrt[3]{45}}{108}, 0, c \right) = -\frac{3}{640} (10207 + 617\sqrt{145})c.
\]

It is easy to see from (7) that \( c > 0 \) for any values of parameters. Hence, in the special case \( a = \frac{41 + 5\sqrt[3]{45}}{108}, b = 0 \) the libration point \( L_1 \) is stable in the sense of Lyapunov. Let us also note that the special case takes place on the curve, which lies on the resonance surface in three-dimensional space of parameters, \( Q_1, Q_2 \). This curve is given by the equations \( a = \frac{41 + 5\sqrt[3]{45}}{108}, b = 0 \), where \( a \) and \( b \) are determined by formulas (5) and (6).

Let us now assume that \( a = \frac{6a + 60\sqrt[3]{5}}{209} \), i.e. the fourth order resonance \( \omega_1 = 3\omega_2 \) takes place. At fourth order resonance the Hamiltonian normal form reads [16]

\[
H = 3\omega_2r_1 - \omega_2r_2 + c_20r_2^2 + c_11r_1r_2 + c_02r_2^2 + Br_2\sqrt{r_1r_2}\sin(\varphi_1 + 3\varphi_2) + O((r_1 + r_2)^{3/2}).
\]

In accordance with Markov's theorem [16] the stability condition for the system with Hamiltonian (17) reads: \(|3\sqrt{3}B| < |C| \), where \( C = c_{20} + 3c_{11} + 9c_{02} \). The coefficients of the Hamiltonian normal form (17) obtained by using the Depri-Hori method have the following form

\[
C = \frac{1045(6\sqrt{5} - 35)^2\gamma}{896(2322610\sqrt{5} - 5216425)} \cdot \frac{1}{4}, B = \frac{(7315 - 1254\sqrt{5})^2(151550\sqrt{5} - 184645)}{645945891800000} \cdot \beta,
\]

\[
\gamma = |(2024261214700322525980719774\sqrt{5} - 4526385684665734120415502645)b^2 + (107923587936003422551455375 - 48264895611203532412075050\sqrt{5})c|,
\]

\[
\beta = (2245 + 982\sqrt{5})^{1/2} |(11390400\sqrt{5} + 19052675)c - 46359753b^2|.
\]

The numerical calculations have shown that the above stability condition is always fulfilled. It yields the stability of \( L_1 \) in the sense of Lyapunov.

6. Conclusion

Let's summarize the conclusions on the stability study. If \( a \in \left( \frac{9}{\sqrt{5}}; 1 \right) \), then the collinear libration point \( L_1 \) is stable in the linear approximation, otherwise it is unstable in the sense of Lyapunov. The stability of \( L_1 \) in the sense of Lyapunov takes place if \( a \in \left( \frac{9}{\sqrt{5}}, \frac{41 + 5\sqrt[3]{45}}{108} \right) \cup \left( \frac{41 + 5\sqrt[3]{45}}{108}, 1 \right) \) and \( \Delta \neq 0 \). At \( a = \frac{41 + 5\sqrt[3]{45}}{108} \) and \( b \neq 0 \) the collinear libration point \( L_1 \) is unstable in the sense of Lyapunov, in spite of its stability in the linear approximation. The above instability is caused by the third order resonance. However, for \( a = \frac{41 + 5\sqrt[3]{45}}{108} \) and \( b = 0 \) the \( L_1 \) is stable in the sense of Lyapunov in the presence of the resonance mentioned. The conclusions on stability obtained in this paper agrees with the results [7, 9]. In the case of a third-order resonance, the above conclusions refine the results of [6], where the stability at \( b = 0 \) has not been established.
If $\Delta = 0$, then the case of degeneration takes place, when the problem of stability cannot be solved by taking into account terms of order four in the Hamiltonian. To study the stability in this case it is necessary to take into account terms of order six or higher. On the boundaries of the stability domain, i.e. at $a = 1$ and $a = \frac{8}{9}$ the resonances of first ($\omega_2 = 0$) and second ($\omega_1 = \omega_2$) order take place, respectively. In these cases an additional non-linear study is necessary to obtain rigorous conclusions on stability of the collinear libration point $L_1$.

The results obtained in this paper can be used in the satellite dynamics and astronomy. In particular, conclusions on stability are important for planning of space missions in the neighborhood of $L_1$ as well as they should be taken into account at the stage of selection of binary star systems, which are suitable for observation of cosmic dust near the libration point $L_1$.

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References

[1] Moulton F R 1960 An Introduction to Celestial Mechanics (New York: Macmillan Company) p 455

[2] Kunitsyn A L and Tureshbaev A T 1983 The collinear libration points in the photogravitational three-body problem. Sov. Astron. Lett. 9 228

[3] Kunitsyn A L and Tureshbaev A T 1985 On the collinear libration points in the photogravitational three-body problem. Celestial Mech. 35 105

[4] Tkhai V N and Zimovshikov A S 2009 The possible existence of cloud-like clusters of microparticles at the libration points of a binary star. Astronomy Reports 53 552 https://doi.org/10.1134/S1063772909060079

[5] Zimovshikov A S and Tkhai V N 2010 Stability diagrams for a heterogeneous ensemble of particles at the collinear libration points of the photogravitational three-body problem. J. Appl. Math. Mech. 74 158 https://doi.org/10.1016/j.jappmathmech.2010.05.005

[6] Tkhai N V 2011 On stability of the collinear libration points under internal third-order resonance. Automat. Rem. Contr. 72 1906 https://doi.org/10.1134/S0005117911090128

[7] Tkhai N V 2012 Stability of the collinear libration points of the photogravitational three-body problem with an internal fourth order resonance. J. Appl. Math. Mech. 76 441 https://doi.org/10.1016/j.jappmathmech.2012.09.011

[8] Bardin B S and Avdushkin A N 2018 Stability analysis of an equilibrium position in the photogravitational Sitnikov problem. AIP Conf. Proc. 1959, 040002 https://doi.org/10.1063/1.5034605

[9] Bardin B S and Avdushkin A N 2020 Stability of the collinear point $L_1$ in the planar restricted photogravitational three-body problem in the case of equal masses of primaries. IOP Conf. Ser.: Mater. Sci. Eng. 927 012015 https://doi.org/10.1088/1757-899X/927/1/012015

[10] Abouelmagd E I and El-Shaboury S M 2012 Periodic orbits under combined effects of oblateness and radiation in the restricted problem of three bodies. Astrophys. Space Sci. 341(2) 331 https://doi.org/10.1007/s10509-012-1093-7

[11] Abouelmagd E I 2013 The effect of photogravitational force and oblateness in the perturbed restricted three-body problem. Astrophys. Space Sci. 346(1) 51 https://doi.org/10.1007/s10509-013-1439-9

[12] Abouelmagd E I and Guirao J L G 2016 On the perturbed restricted three-body problem. Applied Mathematics and Nonlinear Sciences 1(1) 123 https://doi.org/10.21042/AMNS.2016.1.00010

[13] Alzahrai F, Abouelmagd E I, Guirao J L G and Hobiny A 2017 On the libration collinear points in the restricted three-body problem. Open Phys. 15(1) 58 https://doi.org/10.1515/phys-2017-0007

[14] Alamri S Z, Abd El-Bar S E and Seadawy A R 2018 The collinear equilibrium points in the
restricted three body problem with triaxial primaries. *Open Phys.* **16**(1) 525
https://doi.org/10.1515/phys-2018-0069

[15] Selim H H, Guirao J L G and Abouelmagd E I 2019 Libration points in the restricted three-body problem: Euler angles, existence and stability. *Discrete Cont. Dyn.-S* **12**(4&5) 703
doi: 10.3934/dcdss.2019044

[16] Markeev A P 1978 *Libration Points in Celestial Mechanics and Space Dynamics* (in Russian) (Moscow: Nauka) p 312

[17] Arnold V I 1961 The stability of the equilibrium position of a Hamiltonian system of ordinary differential equations in the general elliptic case. *Soviet Math. Dokl.* 2 247

[18] Arnold V I 1963 Small denominators and problems of stability of motion in classical and celestial mechanics. *Russ. Math. Surv.* **18** 85

[19] Siegel C L and Moser J K 1971 *Lectures on Celestial Mechanics* (New York: Springer) p 290

[20] Markeev A P 2009 *Linear Hamiltonian Systems and Certain Stability Problems of Satellite’s Motion about Its Center of Mass* (in Russian) (Moscow, Izhevsk: Regular and Chaotic Dynamics, Institute of Computer Research) p 396

[21] Giacaglia G E O 1972 *Perturbation Method in Non-Linear Systems* (New York: Springer) p 369