BOUNDEDNESS OF THE GAUSSIAN RIESZ POTENTIALS ON GAUSSIAN VARIABLE LEBESGUE SPACES

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Abstract. In this paper we prove the boundedness of the Gaussian Riesz potentials $I_\beta$, for $\beta \geq 1$ on $L^p(\gamma_d)$, the Gaussian variable Lebesgue spaces under a certain additional condition of regularity on $p(\cdot)$ following [5]. Additionally, this result trivially gives us an alternative proof of the boundedness of Gaussian Riesz potentials $I_\beta$ on Gaussian Lebesgue spaces $L^p(\gamma_d)$.

1. Introduction and Preliminaries

In the classical case, the Riesz potential of order $\beta > 0$ is defined as negative fractional powers of the negative Laplacian $-\Delta = -\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$,

(1.1) \((-\Delta)^{-\beta/2},\)

which means, using Fourier transform, that

(1.2) \(((-\Delta)^{-\beta/2} \hat{f})(\xi) = (2\pi|\xi|)^{-\beta} \hat{f}(\xi).\)

for more details; see [7], [8], [12].

Analogously, the Gaussian fractional integrals or Gaussian Riesz potentials can be also defined as negative fractional powers of the Ornstein-Uhlenbeck operator

(1.3) \((-L) = -\frac{1}{2}\Delta + \langle x, \nabla_x \rangle = -\sum_{i=1}^{d} \left[ \frac{1}{2} \frac{\partial^2}{\partial x_i^2} + x_i \frac{\partial}{\partial x_i} \right].\)

However, since the Ornstein-Uhlenbeck operator has eigenvalue 0, the negative powers are not defined on all of $L^2(\gamma_d)$, and therefore we need to be more careful with the definition. Let us consider

$$\Pi_0 f = f - \int_{\mathbb{R}^d} f(y) \gamma_d(dy),$$

for $f \in L^2(\gamma_d)$, the orthogonal projection on the orthogonal complement of the eigenspace corresponding to the eigenvalue 0.

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Definition 1.1. The Gaussian Fractional Integral or Gaussian Riesz potential of order \( \beta > 0 \), \( I_\beta \), is defined spectrally as,

\[
I_\beta = (-L)^{-\beta/2} \Pi_0,
\]

which means that for any multi-index \( \nu \), \(|\nu| > 0\) its action on the Hermite polynomial \( \vec{H}_\nu \) is given by

\[
I_\beta \vec{H}_\nu(x) = \frac{1}{|\nu|^{\beta/2}} \vec{H}_\nu(x),
\]

and for \( \nu = 0 = (0, ..., 0) \), \( I_\beta(\vec{H}_0) = 0 \).

By linearity, using the fact that the Hermite polynomials are an algebraic basis of \( \mathcal{P}(\mathbb{R}^d) \), \( I_\beta \) can be defined for any polynomial function \( f(x) = \sum_\nu \vec{f}_\nu(\nu) \vec{H}_\nu(x) \), where \( \vec{f}_\nu(\nu) = \frac{1}{|\nu|^{\beta/2}} \int_{\mathbb{R}^d} f(t) \vec{H}_\nu(t) dt \), as

\[
I_\beta f(x) = \sum_\nu \frac{\vec{f}_\nu(\nu)}{|\nu|^{\beta/2}} \vec{H}_\nu(x) = \sum_{k \geq 1} \frac{1}{k^{\beta/2}} J_k f(x),
\]

and similarly for \( f \in L^2(\gamma_d) \), as the Hermite polynomials are an orthogonal basis of \( L^2(\gamma_d) \).

It can be proved that the Gaussian Riesz potential \( I_\beta, \beta > 0 \), has the following integral representations, for \( f \) a polynomial function or \( f \in C_0^2(\mathbb{R}^d) \),

\[
I_\beta f(x) = \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} T_t(I - J_0) f(x) dt,
\]

with respect to the Ornstein-Uhlenbeck semigroup \( \{T_t\} \), and

\[
I_\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} P_t(I - J_0) f(x) dt,
\]

with respect to the Poisson-Hermite semigroup, \( \{P_t\} \).

Therefore, from (1.7) we have an explicit integral representation of \( I_\beta \) as

\[
I_\beta f(x) = \int_{\mathbb{R}^d} N_{\beta/2}(x, y) f(y) dy
\]

where the kernel \( N_{\beta/2} \) is defined as

\[
N_{\beta/2}(x, y) = \frac{1}{\pi^{d/2} \Gamma(\frac{\beta}{2})} \int_0^{+\infty} t^{\beta/2-1} \left( e^{-\frac{|y - x|^2}{2t} - \frac{t}{\beta}} - e^{-|y|^2} \right) dt
\]

For more details and background we refer to [14].
From (1.6) it is clear that the Gaussian Riesz potentials $I_{\beta}$ are the simplest Meyer’s multipliers (see for instance Theorem 6.2 of [14]), since in this case

$$m(k) = \frac{1}{k^\beta} = h(\frac{1}{k^\beta}),$$

with $h(x) = x$, the identity function and therefore their $L^p(\gamma_d)$-boundedness follows immediately.

In this paper we prove that Gaussian Riesz potentials $I_{\beta}$, for $\beta \geq 1$ are also bounded in $L^p(\gamma_d)$, the Gaussian variable Lebesgue spaces for certain exponent functions $p(\cdot)$ that will be determine later. For completeness, we will briefly review the notion of variable Lebesgue spaces.

Given $\mu$ a Borel measure, any $\mu$-measurable function $p(\cdot) : \mathbb{R}^d \to [1, \infty]$ is an exponent function; the set of all the exponent functions will be denoted by $\mathcal{P}(\mathbb{R}^d, \mu)$. For $E \subset \mathbb{R}^d$ we set

$$p_-(E) = \text{ess inf}_{x \in E} p(x) \quad \text{and} \quad p_+(E) = \text{ess sup}_{x \in E} p(x).$$

We use the abbreviations $p_+ = p_+(\mathbb{R}^d)$ and $p_- = p_-(\mathbb{R}^d)$.

**Definition 1.2.** Let $E \subset \mathbb{R}^d$. We say that $\alpha(\cdot) : E \to \mathbb{R}$ is locally log-Hölder continuous, and denote this by $\alpha(\cdot) \in LH_0(E)$, if there exists a constant $C_1 > 0$ such that

$$|\alpha(x) - \alpha(y)| \leq \frac{C_1}{\log(e + \frac{1}{|x-y|})}$$

for all $x, y \in E$. We say that $\alpha(\cdot)$ is log-Hölder continuous at infinity with base point at $x_0 \in \mathbb{R}^d$, and denote this by $\alpha(\cdot) \in LH_\infty(E)$, if there exist constants $\alpha_\infty \in \mathbb{R}$ and $C_2 > 0$ such that

$$|\alpha(x) - \alpha_\infty| \leq \frac{C_2}{\log(e + |x-x_0|)}$$

for all $x \in E$. We say that $\alpha(\cdot)$ is log-Hölder continuous, and denote this by $\alpha(\cdot) \in LH(E)$ if both conditions are satisfied. The maximum, $\max\{C_1, C_2\}$ is called the log-Hölder constant of $\alpha(\cdot)$.

**Definition 1.3.** We denote $p(\cdot) \in \mathcal{P}^{log}_d(\mathbb{R}^d)$, if $\frac{1}{p(\cdot)}$ is log-Hölder continuous and denote by $C_{log}(p)$ or $C_{log}$ the log-Hölder constant of $\frac{1}{p(\cdot)}$.

**Definition 1.4.** For a $\mu$-measurable function $f : \mathbb{R}^d \to \mathbb{R}$, we define the modular

$$\rho_{p(\cdot), \mu}(f) = \int_{\mathbb{R}^d \setminus \Omega_{\infty}} |f(x)|^{p(x)} \mu(dx) + ||f||_{L^{p(\cdot)}(\Omega_{\infty}, \mu)},$$

**Definition 1.5.** The variable exponent Lebesgue space on $\mathbb{R}^d$, $L^{p(\cdot)}(\mathbb{R}^d, \mu)$ consists on those $\mu$-measurable functions $f$ for which there exists $\lambda > 0$ such that
\[ \rho_{p(\cdot),\mu} \left( \frac{f}{\lambda} \right) < \infty, \text{ i.e.} \]

\[ L^{p(\cdot)}(\mathbb{R}^d, \mu) = \left\{ f : \mathbb{R}^d \to \mathbb{R} : f \text{ is } \mu\text{-measurable and } \rho_{p(\cdot),\mu} \left( \frac{f}{\lambda} \right) < \infty, \text{ for some } \lambda > 0 \right\} \]

and the norm

\[ \|f\|_{L^{p(\cdot)}(\mathbb{R}^d, \mu)} = \inf \left\{ \lambda > 0 : \rho_{p(\cdot),\mu}(f/\lambda) \leq 1 \right\}. \]

It is well known that, if \( p(\cdot) \in LH(\mathbb{R}^d) \) with \( 1 < p_- \leq p^+ < \infty \) the classical Hardy-Littlewood maximal function \( M \) is bounded on the variable Lebesgue space \( L^{p(\cdot)}(\mathbb{R}^d) \), see \([3]\). However, it is known that even though these are the sharpest possible point-wise conditions, they are not necessary. In \([6]\) a necessary and sufficient condition is given for the \( L^{p(\cdot)} \)-boundedness of \( M \), but it is not an easy to work condition. The class \( LH(\mathbb{R}^d) \) is also sufficient for the boundedness on \( L^{p(\cdot)} \)-spaces of classical singular integrals of Calderón-Zygmund type, see \([4, \text{ Theorem 5.39}]\).

If \( \mathcal{B} \) is a family of balls (or cubes) in \( \mathbb{R}^d \), we say that \( \mathcal{B} \) is \( N \)-finite if it has bounded overlappings for \( N \), i.e., \( \sum_{B \in \mathcal{B}} \chi_B(x) \leq N \) for all \( x \in \mathbb{R}^d \); in other words, there is at most \( N \) balls (resp. cubes) that intersect at the same time.

The following definition was introduced for the first time by Bereznoi in \([2]\), defined for a family of disjoint balls or cubes. In the context of variable spaces, it has been considered in \([6]\), allowing the family to have bounded overlappings.

**Definition 1.6.** Given an exponent \( p(\cdot) \in \mathcal{P}(\mathbb{R}^d) \), we will say that \( p(\cdot) \in \mathcal{G} \), if for every family of balls (or cubes) \( \mathcal{B} \) which is \( N \)-finite,

\[ \sum_{B \in \mathcal{B}} \|f \chi_B\|_{p(\cdot)} \|g \chi_B\|_{p'(\cdot)} \lesssim \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)} \]

for all functions \( f \in L^{p(\cdot)}(\mathbb{R}^d) \) and \( g \in L^{p'(\cdot)}(\mathbb{R}^d) \). The constant only depends on \( N \).

**Theorem 1.1** (Teorema 7.3.22 of \([6]\)). If \( p(\cdot) \in LH(\mathbb{R}^d) \), then \( p(\cdot) \in \mathcal{G} \)

We will consider only variable Lebesgue spaces with respect to the Gaussian measure \( \gamma_d \), \( L^{p(\cdot)}(\mathbb{R}^d, \gamma_d) \). The next condition was introduced by E. Dalmasso and R. Scotto in \([5]\).

**Definition 1.7.** Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^d, \gamma_d) \), we say that \( p(\cdot) \in \mathcal{P}_\infty(\mathbb{R}^d) \) if there exist constants \( C_{\gamma_d} > 0 \) and \( p_\infty \geq 1 \) such that

\[ |p(x) - p_\infty| \leq \frac{C_{\gamma_d}}{|x|^2}, \]

for \( x \in \mathbb{R}^d \setminus \{(0,0,\ldots,0)\} \).

**Observation 1.1.** If \( p(\cdot) \in \mathcal{P}_\infty(\mathbb{R}^d) \), then \( p(\cdot) \in LH_\infty(\mathbb{R}^d) \)

**Lemma 1.1** (Lemma 2.5 of \([5]\)). If \( 1 < p_- \leq p_+ < \infty \), the following statements are equivalent:
with Observation 1.1 we need some technical results. Let \( p \in P^\infty_{\gamma_d}(\mathbb{R}^d) \)
and let \( C_1 \leq e^{-x^2(p'(x)/p_{\infty} - 1)} \leq C_2 \)
and \( C_2 \leq e^{-x^2(p'(x)/p_{\infty} - 1)} \leq C_2 \),
for all \( x \in \mathbb{R}^d \), where \( C_1 = e^{C_{\gamma d}/p_{\infty}} \) and \( C_2 = e^{C_{\gamma d}/p_{\infty}} \).

Definition 1.7 with Observation 1.1 and Lemma 1.1 end up strengthening the
regularity conditions on the exponent functions \( p(\cdot) \) to obtain the boundedness of the
Ornstein-Uhlenbeck semigroup \( \{T_t\} \), see [9]. As a consequence of Theorem 1.1, we have

**Corollary 1.1.** If \( p(\cdot) \in P^\infty_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d) \), then \( p(\cdot) \in G \).

As we have already mentioned, the main result in this paper is the proof that the Gaussian Riesz Potentials \( I_\beta \), for \( \beta \geq 1 \), are bounded on Gaussian variable Lebesgue spaces under the same condition of regularity on \( p(\cdot) \) considered by Dalmaso and Scotto [5].

**Theorem 1.2.** Let \( p(\cdot) \in P^\infty_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d) \) with \( 1 < p_- \leq p_+ < \infty \). Then for \( \beta \geq 1 \) there exists a constant \( C > 0 \), depending only on \( p, \beta \) and the dimension \( d \) such that
\[
\|I_\beta f\|_{p(\cdot),\gamma_d} \leq C\|f\|_{p(\cdot),\gamma_d},
\]
for any \( f \in L^{p(\cdot)}(\gamma_d) \).

Trivially, Theorem 1.2 gives us an alternative proof of the boundedness of the Gaussian Riesz Potentials \( I_\beta \), for \( \beta \geq 1 \) on Gaussian variable Lebesgue spaces \( L^{p(\cdot)}(\gamma_d) \), by simply taking the exponent function constant, but the constant \( C \) depends on \( \beta \) and the dimension, which is weaker than the estimate obtained using Meyer’s multiplier theorem mentioned above.

## 2. Proof of the Main Result.

In order to prove our main result, Theorem 1.2 we need some technical results.

**Lemma 2.1.** Let \( p(\cdot) : \mathbb{R}^d \to [0, \infty) \) be such that \( p(\cdot) \in LH_\infty(\mathbb{R}^d) \), \( 0 < p_\infty < \infty \),
and let \( R(x) = (e + |x|)^{-N} \), \( N > d/p_- \). Then there exists a constant \( C \) depending on \( d, N \) and the \( LH_\infty \) constant of \( p(\cdot) \) such that given any set \( E \) and any function \( F \) with \( 0 \leq F(y) \leq 1 \), for all \( y \in E \),
\[
\int_E F^{p(y)}(y)dy \leq C \int_E F^{p(\cdot)}(y)dy + \int_E R^{p_\infty}(y)dy,
\]
and
\[
\int_E F^{p(\cdot)}(y)dy \leq C \int_E F^{p(y)}(y)dy + \int_E R^{p_\infty}(y)dy.
\]

For the proof see Lemma 3.26 of [4].

**Lemma 2.2.** If \( \alpha > 0 \), there exists a constant \( C > 0 \) such that
\[
\int_0^1 (-\log(\sqrt{1-u}))^{\alpha - 1} du = CT(\alpha) \leq \infty
\]
Proof. Taking the change of variables \( t = -\log(\sqrt{1-u}) \) then \( u = 1 - e^{-2t} \) and \( du = 2e^{-2t}dt \). For \( \alpha > 0 \) we get

\[
\int_0^1 (-\log(\sqrt{1-u}))^{\alpha-1} du = 2 \int_0^{+\infty} t^{\alpha-1} e^{-2t} dt = C \Gamma(\alpha) < \infty
\]

\( \square \)

Lemma 2.3. For \( \beta > 0 \)

i)

\[
(2.4) \quad \int_{1/2}^1 \left(-\log(\sqrt{1-u})\right)^{\beta/2} \frac{1}{\sqrt{1-u}} du < \infty.
\]

ii)

\[
(2.5) \quad \int_{0}^{1/2} \left(-\log(\sqrt{1-u})\right)^{\beta/2} \frac{1}{u} du < \infty.
\]

Proof.

i) Using Hölder’s inequality with \( p = \frac{3}{2}, q = 3 \) and Lemma 2.2, with \( \alpha = \frac{3\beta}{2} + 1 \), we have that

\[
\int_{1/2}^1 \left(-\log(\sqrt{1-u})\right)^{\beta/2} \frac{1}{\sqrt{1-u}} du \leq \int_0^1 \left(-\log(\sqrt{1-u})\right)^{\beta/2} \frac{1}{\sqrt{1-u}} du \\
\leq \left( \int_0^1 \left(-\log(\sqrt{1-u})\right)^{3\beta/2} du \right)^{1/3} \left( \int_0^1 \frac{du}{(1-u)^{3/4}} \right)^{2/3} \\
= \left( \int_0^1 \left(-\log(\sqrt{1-u})\right)^{(3\beta+1)/2} du \right)^{1/3} \left( \int_0^1 \frac{du}{(1-u)^{3/4}} \right)^{2/3} < \infty
\]

ii) Let us rewrite the integral as

\[
\int_{0}^{1/2} \left(-\log(\sqrt{1-u})\right)^{\beta/2} \frac{1}{u} du = \int_{0}^{1/2} \left(-\log(\sqrt{1-u})\right)^{\beta/2} \left(-\log(\sqrt{1-u})\right)^{\beta-1} du,
\]

since \( \lim_{u \to 0} \left(-\log(\sqrt{1-u})\right) = 0 \) and \( \left(-\log(\sqrt{1-u})\right)/u \) is bounded in \((0, 1/2]\), then we have we have by (2.3)

\[
\int_{0}^{1/2} \left(-\log(\sqrt{1-u})\right)^{\beta/2} \frac{1}{u} du = \int_{0}^{1/2} \left(-\log(\sqrt{1-u})\right)^{\beta-1} \left(-\log(\sqrt{1-u})\right) du \\
\leq C \int_{0}^{1/2} \left(-\log(\sqrt{1-u})\right)^{\beta-1} du < \infty.
\]

\( \square \)

We are now ready to prove the main result, Theorem 1.2.
Proof. As usual, we split the operator $I_\beta$ in its local part and its global part

$$I_\beta f(x) = I_{\beta,L} f(x) + I_{\beta,G} f(x),$$

where

$$I_{\beta,L} f(x) = I_\beta(f \chi_{B_h})(x)$$

is the local part, and

$$I_{\beta,G} f(x) = I_\beta(f \chi_{B_c})(x)$$

is the global part, and for $x \in \mathbb{R}^d$ by taking $m(x) = 1 \wedge \frac{1}{|x|}$,

$B_\beta(x) := \{ y \in \mathbb{R}^d : |x - y| < dm(x) \}$ is an hiperbolic ball (admissible ball).

Let us take $\omega(s) = \frac{e^{-\frac{2|y-e^{-s}x|^2}{1-e^{-2s}}}}{(1-e^{-2s})^{\frac{d}{2}}}$, then, from (1.10)

$$N_{\beta/2}(x,y) = \frac{1}{\pi^{\frac{d}{2}} \Gamma(\frac{\beta}{2})} \int_0^{+\infty} t^{\frac{\beta}{2} - 1} (\omega(t) - \omega(+\infty)) \, dt$$

$$= \frac{1}{\pi^{\frac{d}{2}} \Gamma(\frac{\beta}{2})} \int_0^{+\infty} t^{\frac{\beta}{2} - 1} \left(- \int_t^{+\infty} \frac{\partial \omega(s)}{\partial s} \, ds \right) \, dt.$$

Thus, using Hardy’s inequality, see [12]

$$|N_{\beta/2}(x,y)| \leq \frac{1}{\pi^{\frac{d}{2}} \Gamma(\frac{\beta}{2})} \int_0^{+\infty} t^{\frac{\beta}{2} - 1} \int_t^{+\infty} \left| \frac{\partial \omega(s)}{\partial s} \right| \, ds \, dt \leq \frac{1}{\pi^{\frac{d}{2}} \Gamma(\frac{\beta}{2})} \frac{2}{\beta} \int_0^{+\infty} s^{\frac{\beta}{2}} \left| \frac{\partial \omega(s)}{\partial s} \right| \, ds.$$

Now,

$$\frac{\partial \omega(s)}{\partial s} = \frac{(1-e^{-2s})^\frac{d}{2} e^{-\frac{1-e^{-2s}}{1-e^{-2s}}}}{(1-e^{-2s})^d} \left( \frac{-2(1-e^{-2s})(y-e^{-s}x) \cdot (e^{-s}x) + |y-e^{-s}x|^2 e^{-2s}}{(1-e^{-2s})^2} \right) \right. \left. - \frac{1}{(1-e^{-2s})^d} \left( \frac{d e^{-\frac{1-e^{-2s}}{1-e^{-2s}}}}{2} \right) \right)$$

$$= \omega(s) \left( \frac{-2(1-e^{-2s})(y-e^{-s}x) \cdot (e^{-s}x) + |y-e^{-s}x|^2 e^{-2s}}{(1-e^{-2s})^2} \right) \right. \left. - \frac{de^{-2s}}{(1-e^{-2s})} \right).$$
Then, taking $u = 1 - e^{-2s}$, $du = 2e^{-2s}ds$, i.e., $e^{-2s} = \sqrt{1 - u}$, we have

\[
|N_{\beta/2}(x, y)| \leq \int_0^1 \left( -\log(\sqrt{1 - u}) \right)^\beta e^{-\frac{|y - \sqrt{1 - u}x|^2}{u^2}} \\
\times \left( 2u|y - \sqrt{1 - u}x| \sqrt{1 - u} + |y - \sqrt{1 - u}x|^2(1 - u) + \frac{d(1 - u)}{u} \right) \frac{du}{2(1 - u)}
\]

\[
= \int_0^1 \left( -\log(\sqrt{1 - u}) \right)^\beta e^{-\frac{|y - \sqrt{1 - u}x|^2}{u^2}} \left( \frac{|y - \sqrt{1 - u}x||x|}{u \sqrt{1 - u}} + \frac{|y - \sqrt{1 - u}x|^2}{2u^2} + \frac{d}{2u} \right) du
\]

\[
= \int_0^1 \left( -\log(\sqrt{1 - u}) \right)^\beta e^{-\frac{|y - \sqrt{1 - u}x|^2}{u^2}} \frac{|y - \sqrt{1 - u}x||x|}{u \sqrt{1 - u}} du
\]

\[
+ \int_0^1 \left( -\log(\sqrt{1 - u}) \right)^\beta e^{-\frac{|y - \sqrt{1 - u}x|^2}{u^2}} \frac{|y - \sqrt{1 - u}x|^2}{2u} du
\]

\[
+ \int_0^1 \left( -\log(\sqrt{1 - u}) \right)^\beta e^{-\frac{|y - \sqrt{1 - u}x|^2}{u^2}} \frac{d}{2u} du
\]

\[
= I + II + III,
\]

where

\[
I = \int_0^1 \left( -\log(\sqrt{1 - u}) \right)^\beta e^{-\frac{|y - \sqrt{1 - u}x|^2}{u^2}} \frac{|y - \sqrt{1 - u}x||x|}{u \sqrt{1 - u}} du,
\]

\[
II = \int_0^1 \left( -\log(\sqrt{1 - u}) \right)^\beta e^{-\frac{|y - \sqrt{1 - u}x|^2}{u^2}} \frac{|y - \sqrt{1 - u}x|^2}{2u} du,
\]

and

\[
III = \int_0^1 \left( -\log(\sqrt{1 - u}) \right)^\beta e^{-\frac{|y - \sqrt{1 - u}x|^2}{u^2}} \frac{d}{2u} du.
\]

- Let us study the local part first. We need to bound each of the terms $I$, $II$ and $III$ in this part.

For $I$, since we are in the local part $|x - y| \leq \frac{d}{|x|}$, then we have $|x - y||x| \leq d$ therefore,

\[
|y - \sqrt{1 - u}x|^2 \geq (|y - x| - |x|(1 - \sqrt{1 - u}))^2
\]

\[
\geq |y - x|^2 - 2|x||y - x|\frac{u}{1 + \sqrt{1 - u}} \geq |y - x|^2 - 2d u.
\]

On the other hand, it is well known that, there exist $C > 0$ such that for any $x > 0$, $\alpha \geq 0$ and $c > 0$,

\[
x^\alpha e^{-cx^2} \leq C.
\]
Thus, using (2.9) and (2.10) twice, with $\alpha = 1$, $c = \frac{1}{2}$ and $\alpha = d - 1, c = \frac{1}{2}$, we get

\[
I = \int_0^1 \left( -\log(\sqrt{1} - u) \right)^{\frac{\beta}{2}} e^{\frac{|y - \sqrt{1 - ux}|^2}{2u}} \left( \int y - \sqrt{1 - ux} \right) \left( e^{\frac{|y - \sqrt{1 - ux}|^2}{2u}} \right) \frac{|x|}{\sqrt{u} \sqrt{1 - u}} du \\
\leq C|x| \int_0^1 \left( -\log(\sqrt{1} - u) \right)^{\frac{\beta}{2}} e^{\frac{|y - \sqrt{1 - ux}|^2}{2u}} \frac{du}{u \sqrt{1 - u}} \\
= C|x| \int_0^1 \left( -\log(\sqrt{1} - u) \right)^{\frac{\beta}{2}} e^{\frac{|y - \sqrt{1 - ux}|^2}{2u}} \frac{du}{u \sqrt{1 - u}} \\
\leq \frac{C|x|}{|x - y|^{d - 1}} \int_0^1 \left( -\log(\sqrt{1} - u) \right)^{\frac{\beta}{2}} \frac{du}{u \sqrt{1 - u}}.
\]

By Lemma 2.3 we get

\[
\int_0^1 \left( -\log(\sqrt{1} - u) \right)^{\frac{\beta}{2}} \frac{du}{u \sqrt{1 - u}} \\
= \int_0^{1/2} \left( -\log(\sqrt{1} - u) \right)^{\frac{\beta}{2}} \frac{du}{u \sqrt{1 - u}} + \int_{1/2}^1 \left( -\log(\sqrt{1} - u) \right)^{\frac{\beta}{2}} \frac{du}{u \sqrt{1 - u}} \\
\leq C \int_0^{1/2} \left( -\log(\sqrt{1} - u) \right)^{\frac{\beta}{2}} \frac{du}{u} + C \int_{1/2}^1 \left( -\log(\sqrt{1} - u) \right)^{\frac{\beta}{2}} \frac{du}{u \sqrt{1 - u}} < \infty,
\]

since $\frac{1}{u}$ is bounded on $[1/2, 1]$ and $\frac{1}{\sqrt{1 - u}}$ is bounded on $[0, 1/2]$. Thus, we have

\[
I \leq \frac{C|x|}{|x - y|^{d - 1}}.
\]

For $II$, we use again (2.9) and (2.10) with $\alpha = 2$ and $c = \frac{1}{2}$ we have

\[
II = \int_0^1 \left( -\log(\sqrt{1} - u) \right)^{\frac{\beta}{2}} e^{\frac{|y - \sqrt{1 - ux}|^2}{u^{d+1}}} \frac{|y - \sqrt{1 - ux}|^2}{2u} du \\
\leq C \int_0^1 \left( -\log(\sqrt{1} - u) \right)^{\frac{\beta}{2}} e^{\frac{|y - \sqrt{1 - ux}|^2}{u^{d+1}}} du \\
\leq C \int_0^1 \left( -\log(\sqrt{1} - u) \right)^{\frac{\beta}{2}} e^{\frac{|y - \sqrt{1 - ux}|^2}{u^{d+1}}} du \\
= C \int_0^{1/2} \left( -\log(\sqrt{1} - u) \right)^{\frac{\beta}{2} + 1} e^{\frac{|y - \sqrt{1 - ux}|^2}{u^{d+1}}} du + \int_{1/2}^1 \left( -\log(\sqrt{1} - u) \right)^{\frac{\beta}{2}} e^{\frac{|y - \sqrt{1 - ux}|^2}{u^{d+1}}} du.
\]
Since \( \lim_{u \to 0} \frac{- \log(\sqrt{1-u})}{u} = \frac{1}{2} \), this function is bounded on \([0, 1/2]\) and \( \frac{1}{u} \) is bounded on \([1/2, 1]\), thus we get

\[
\int \leq C \int_0^{1/2} \left( -\log(\sqrt{1-u}) \right)^{\frac{d}{2} - 1} e^{-\frac{\log(1-u)}{2u}} \, du + C \int_{1/2}^1 \left( -\log(\sqrt{1-u}) \right)^{\frac{d}{2}} e^{-\frac{\log(1-u)}{2u}} \, du
\]

\[
\leq C \int_0^1 \left( -\log(\sqrt{1-u}) \right)^{\frac{d}{2} - 1} e^{-\frac{\log(1-u)}{2u}} \, du + C \int_0^1 \left( -\log(\sqrt{1-u}) \right)^{\frac{d}{2}} e^{-\frac{\log(1-u)}{2u}} \, du
\]

\[
= CK_2(x-y) + CG_2(x-y),
\]

where

\[
K_2(x) := \int_0^1 \left( -\log(\sqrt{1-u}) \right)^{\frac{d}{2} - 1} e^{-\frac{\log(1-u)}{2u}} \, du,
\]

and

\[
G_2(x) := \int_0^1 \left( -\log(\sqrt{1-u}) \right)^{\frac{d}{2}} e^{-\frac{\log(1-u)}{2u}} \, du.
\]

Again by (2.10)

\[
G_2(x-y) = \int_0^1 \left( -\log(\sqrt{1-u}) \right)^{\frac{d}{2}} e^{-\frac{\log(1-u)}{2u}} \, du
\]

\[
\leq \frac{C}{|x-y|^{d-1}} \int_0^1 \left( -\log(\sqrt{1-u}) \right)^{\frac{d}{2}} \, du.
\]

Now, by Hölder’s inequality with \( p = \frac{3}{2} \), \( q = 3 \) we have

\[
\int_0^1 \left( -\log(\sqrt{1-u}) \right)^{\frac{d}{2}} \, du \leq \left( \int_0^1 \left( -\log(\sqrt{1-u}) \right)^{\frac{3d}{2}} \, du \right)^{1/3} \left( \int_0^1 \frac{du}{u^{3/4}} \right)^{2/3}.
\]

Thus, by Lemma (2.2) we obtain

\[
\int_0^1 \left( -\log(\sqrt{1-u}) \right)^{\frac{3d}{2}} \, du = \int_0^1 \left( -\log(\sqrt{1-u}) \right)^{\frac{3d}{2} + 1} \, du < \infty
\]

and trivially \( \int_0^1 \frac{du}{u^{3/4}} < \infty \).

Finally for \( III \), by analogous arguments as in \( II \), we get

\[
III = \int_0^1 \left( -\log(\sqrt{1-u}) \right)^{\frac{d}{2}} e^{-\frac{\log(1-u)}{2u}} \, du \leq CK_2(x-y) + CG_2(x-y).
\]
Therefore,

\[ |N_{\beta/2}(x, y)| \leq I + II + III \]

\[ \leq \frac{C|x|}{|x - y|^{d-1}} + C\mathcal{K}_2(x - y) + \frac{C}{|x - y|^{d-1}} \]

\[ \leq C\frac{|x| + 1}{|x - y|^{d-1}} + C\mathcal{K}_2(x - y) = C\mathcal{K}_3(x, y) + C\mathcal{K}_2(x - y), \]

where

\[ \mathcal{K}_3(x, y) := \frac{|x| + 1}{|x - y|^{d-1}}. \]

Thus, using the above estimates, we conclude that the local part \( I_{\beta, L} \) can be bounded as

\[ |I_{\beta, L} f(x)| = |\beta(f \chi_{
abla B_{\hat{r}}}(\cdot))(x)| = \left| \int_{B_{\hat{r}}(x)} N_{\beta/2}(x, y) f(y) \, dy \right| \]

\[ \leq \int_{B_{\hat{r}}(x)} \mathcal{K}_3(x, y)|f(y)| \, dy + \int_{B_{\hat{r}}(x)} \mathcal{K}_2(x - y)|f(y)| \, dy \]

\[ = IV + V. \]

Now, to bound IV and V, we need to take a countable family of admissible balls \( \mathcal{F} \) that satisfies the condition of Lemma 4.3 of [14]. In particular, \( \mathcal{F} \) verifies

i) For each \( B \in \mathcal{F} \) let \( \hat{B} = 2B \), then, the family of those balls \( \hat{\mathcal{F}} = \{B(0, 1), \{\hat{B}_{B \in \mathcal{F}}\} \} \) is a covering of \( \mathbb{R}^d \);

ii) \( \mathcal{F} \) has a bounded overlaps property;

iii) Every ball \( B \in \mathcal{F} \) is contained in an admissible ball, and therefore for any pair \( x, y \in B \), \( e^{-|x|^2} \sim e^{-|y|^2} \) with constants independent of \( B \);

iv) There exists a uniform positive constant \( C_d \) such that, if \( x \in B \in \mathcal{F} \) then \( B_{\hat{r}}(x) \subset C_d B := \hat{B} \). Moreover, the collection \( \hat{\mathcal{F}} = \{\hat{B}_{B \in \mathcal{F}}\} \) also satisfies the properties ii) and iii).

Given \( B \in \mathcal{F}, \) if \( x \in B \) then \( B_{\hat{r}}(x) \subset \hat{B} \), we get,

\[ IV = (1 + |x|) \sum_{k=0}^\infty \int_{2^{-(k+1)}C_d x (x) \leq |x| < 2^{-k}C_d x (x)} \frac{|f(y)| \chi_{\hat{B}}}{|x - y|^{d-1}} \, dy \]

\[ \leq C_d 2^d M(f \chi_{\hat{B}})(x)(1 + |x|) m(x) \sum_{k=0}^\infty 2^{-(k+1)} \leq C M(f \chi_{\hat{B}})(x) \chi_B(x), \]

where \( M(g) \) is the classical Hardy-Littlewood maximal function of the function \( g \).
On the other hand, let us consider the function \( \varphi(y) = \frac{1}{\pi y^2} e^{-\frac{1}{2} |y|^2} \), then
\[
\int_{\mathbb{R}^d} \varphi(y) dy = 1.
\]
It is well known that \( \varphi \) is a non-increasing radial function, and given \( t > 0 \), we rescale this function as \( \varphi(\sqrt{t} y) = \frac{1}{\sqrt{t}} \varphi(y) \), and, since \( 0 \leq \varphi \in L^1(\mathbb{R}^d) \), \( \{ \varphi(\sqrt{t}) \}_{t>0} \) is the classical (Gauss-Weierstrass) approximation of the identity in \( \mathbb{R}^d \). Then, since
\[
\int_0^1 \left( -\log(\sqrt{1-t}) \right)^{\frac{d}{2}} dt < \infty,
\]
we get
\[
V = \int_{B_h(x)} K_2(x-y) |f(y)| dy = \int_{B_h(x)} \left( \int_0^1 \varphi(\sqrt{t} (x-y)) \left( -\log(\sqrt{1-t}) \right)^{\frac{d}{2}} dt \right) |f(y)| dy
\]
\[
\leq \int_{B_h(x)} \left( \sup_{t>0} \varphi(\sqrt{t} (x-y)) \right) \left( \int_0^1 \left( -\log(\sqrt{1-t}) \right)^{\frac{d}{2}} dt \right) |f(y)| dy
\]
\[
\leq C \int_{B_h(x)} \left( \sup_{t>0} \varphi(\sqrt{t} (x-y)) \right) |f(y)| dy.
\]
Again, using the family \( \mathcal{F} \), if \( x \in B \) then \( B_h(x) \subset \hat{B} \). By a similar argument as before and as in as result in Stein’s book [12, Chapter II §4, Theorem 4],
\[
V = \int_{B_h(x)} K_2(x-y) |f(y)| dy \leq C \int_{\mathbb{R}^d} \left( \sup_{t>0} \varphi(\sqrt{t} (x-y)) \right) |f(y)| \chi_{\hat{B}}(y) dy
\]
\[
\leq \sum_{B \in \mathcal{F}} \left( \int_{t>0} \varphi(\sqrt{t} |f(x)|) \right) \chi_B(x) \leq \sum_{B \in \mathcal{F}} M(f \chi_B)(x) \chi_B(x),
\]
which yields, \( |I_{\beta,L} f(x)| \leq \sum_{B \in \mathcal{F}} M(f \chi_B)(x) \chi_B(x) \).

Then, for \( f \in L^{\mu_1}(\mathbb{R}^d, \gamma_d) \) we will use the characterization of the norm by duality, we get
\[
(2.11) \quad \left\| I_{\beta,L} f \right\|_{p(\cdot), \gamma_d} \leq 2 \sup_{\|g\|_{p(\cdot), \gamma_d} \leq 1} \int_{\mathbb{R}^d} |I_{\beta,L} f(x)| |g(x)| \gamma_d(dx).
\]

Using the estimates above, we get
\[
\int_{\mathbb{R}^d} |I_{\beta,L} f(x)| |g(x)| \gamma_d(dx) \leq \sum_{B \in \mathcal{F}} \int_B M(f \chi_{B(\cdot)})(x) |g(x)| e^{-|x|^2} dx
\]
\[
\approx \sum_{B \in \mathcal{F}} e^{-|c_B|^2} \int_B M(f \chi_{B(\cdot)})(x) |g(x)| dx,
\]
where \( c_B \) is the center of \( B \) and \( \hat{B} \) and we have used property iii) above, i.e. that over each ball of the family \( \mathcal{F} \), the values of \( \gamma_d \) are all equivalent.
Applying Hölder’s inequality for \( p(\cdot) \) and \( p'(\cdot) \) with respect of the Lebesgue measure and the boundedness of \( \mathcal{M} \) on \( L^{p(\cdot)}(\mathbb{R}^d) \), we get

\[
\int_{\mathbb{R}^d} |I_{\beta,L,f}(x)| |g(x)| \gamma_d(dx) \leq \sum_{B \in \mathcal{F}} e^{-|x|^2} \left\| \mathcal{M}(fX_{B^c}) \right\|_{p(\cdot)} \left\| gX_B \right\|_{p'(\cdot)}
\leq \sum_{B \in \mathcal{F}} e^{-|x|^2} \left\| fX_B \right\|_{p(\cdot)} \left\| gX_B \right\|_{p'(\cdot)}
\leq \sum_{B \in \mathcal{F}} e^{-|x|^2} \frac{1}{p_\infty} \left\| fX_B \right\|_{p(\cdot)} e^{-|x|^2/p_\infty} \left\| gX_B \right\|_{p'(\cdot)}.
\]

(2.12)

Since \( p \in P_{\gamma_d}^\infty(\mathbb{R}^d) \) and \( p_- > 1 \) then \( p' \in P_{\gamma_d}^\infty(\mathbb{R}^d) \). Thus, from Lemma 1.4, for every \( x \in \mathbb{R}^d \)

\[
e^{-|x|^2(p(x)/p_\infty - 1)} \leq C_1 \text{ and } e^{-|x|^2(p'(x)/p_\infty - 1)} \leq C_2.
\]

Moreover, since the values of the Gaussian measure \( \gamma_d \) are all equivalent on any ball \( \hat{B} \in \hat{\mathcal{F}} \), we have

\[
\int_\hat{B} \left( \frac{|f(y)|}{\left\| fX_B \right\|_{p(\cdot),\gamma_d}} \right)^{p(y)} dy \leq \int_\hat{B} \left( \frac{|f(y)|}{\left\| fX_B \right\|_{p(\cdot),\gamma_d}} \right)^{p(y)} e^{-|y|^2(p(x)/p_\infty - 1)} \gamma_d(dy) \leq 1,
\]

which yields

\[
e^{-|x|^2/p_\infty} \left\| fX_B \right\|_{p(\cdot)} \leq \left\| fX_B \right\|_{p(\cdot),\gamma_d}.
\]

Similarly, by the second inequality of (2.13) we also get

\[
e^{-|x|^2/p_\infty} \left\| gX_B \right\|_{p'(\cdot)} \leq \left\| gX_B \right\|_{p'(\cdot),\gamma_d}.
\]

Replacing both estimates in (2.12) we obtain

\[
\int_{\mathbb{R}^d} |I_{\beta,L,f}(x)| |g(x)| \gamma_d(dx) \leq \sum_{B \in \mathcal{F}} \left\| fX_B \right\|_{p(\cdot),\gamma_d} \left\| gX_B \right\|_{p'(\cdot),\gamma_d}
= \sum_{B \in \mathcal{F}} \left\| fX_B e^{-|x|^2/p(\cdot)} \right\|_{p(\cdot)} \left\| gX_B e^{-|x|^2/p'(\cdot)} \right\|_{p'(\cdot)}.
\]

Since the family of balls \( \hat{\mathcal{F}} \) has bounded overlaps, from Corollary 1.1 applied to \( f e^{-|x|^2/p(\cdot)} \in L^{p(\cdot)}(\mathbb{R}^d) \) and \( g e^{-|x|^2/p'(\cdot)} \in L^{p'(\cdot)}(\mathbb{R}^d) \), it follows that

\[
\int_{\mathbb{R}^d} |I_{\beta,L,f}(x)| |g(x)| \gamma_d(dx) \leq \left\| f \right\|_{p(\cdot),\gamma_d} \left\| g \right\|_{p'(\cdot),\gamma_d}.
\]

Taking supremum over all functions \( g \) with \( \left\| g \right\|_{p'(\cdot),\gamma_d} \leq 1 \), from (2.11) we get finally

\[
\left\| I_{\beta,L,f} \right\|_{p(\cdot),\gamma_d} \leq C \left\| f \right\|_{p(\cdot),\gamma_d}.
\]
Therefore, the local part $I_{\beta,L}$ is bounded in $L^p_{\text{loc}}(\gamma_d)$.

- Now, let us study the global part. Again, since
  \[|N_{\beta/2}(x,y)| \leq I + II + III,\]
  we need to estimate each term in this part. As usual for the global part, the arguments are completely different but are based on the following technical result, obtained by S. Pérez [10, Lemma 3.1], see also [14, §4.5]. To simplify the notation, in what follows we denote
  \[a = a(x,y) := |x|^2 + |y|^2,\]
  \[b = b(x,y) := 2 \langle x, y \rangle,\]
  \[u(t) = u(t; x, y) := \frac{|y - \sqrt{1-t}x|^2}{t} = \frac{a}{t} - \frac{\sqrt{1-t}}{t} b - |x|^2,\]
  \[t_0 = 2 \frac{\sqrt{a^2 - b^2}}{a + \sqrt{a^2 - b^2}}.\]
  then
  \[u(t_0) = \frac{\sqrt{a^2 - b^2}}{2} + \frac{a}{2} - |x|^2 = \frac{|y|^2 - |x|^2}{2} + \frac{\sqrt{a^2 - b^2}}{2}\]
  and
  \[t_0 \sim \frac{\sqrt{a^2 - b^2}}{a} \sim \frac{\sqrt{a - b}}{\sqrt{a + b}} = \frac{|x - y|}{|x + y|}.\]
  It is well known that $t_0 < 1$, the minimum of $u(t)$ is attained at $t_0$ and
  \[\frac{1}{t_0^{d/2}} \leq |x + y|^d.\]
  For details and other properties of these terms, see [10], [11] or [14].

Let us fix $x \in \mathbb{R}^d$ and consider $E_x = \{y \in \mathbb{R}^d : b > 0\}$.

- Case $b \leq 0$: First, for $0 < \epsilon < 1$, using inequality (2.10) we have
  \[I = \int_0^1 (-\log(\sqrt{1-t}))^{\frac{d}{2}} e^{-u(t)} \frac{|y - \sqrt{1-t}x|}{t^{d/4}} \frac{|x|}{t \sqrt{1-t}} dt\]
  \[= |x| \int_0^1 (-\log(\sqrt{1-t}))^{\frac{d}{2}} e^{-u(t) - (1-\epsilon)u(t)} \frac{|y - \sqrt{1-t}x|}{t^{d/4}} \frac{dt}{\sqrt{t \sqrt{1-t}}}\]
  \[\leq C\epsilon |x| \int_0^1 (-\log(\sqrt{1-t}))^{\frac{d}{2}} e^{-(1-\epsilon)u(t)} \frac{dt}{t^{d/4}} \frac{1}{\sqrt{t \sqrt{1-t}}}.\]
Since \( \left( -\log(\sqrt{1-t}) \right)^{\frac{g}{\beta}} \) is continuous on \([0, 1/2]\) and therefore bounded, we get

\[
\int_{0}^{1/2} \left( -\log(\sqrt{1-t}) \right)^{\frac{g}{\beta}} \frac{e^{-(1-\epsilon)|a(t)|}}{t^{d+1}} \frac{dt}{\sqrt{1-t}} \leq C_{\beta} \int_{0}^{1/2} \frac{e^{-(1-\epsilon)|a(t)|}}{t^{d+1}} \frac{dt}{\sqrt{1-t}},
\]

and since \(0 < t < 1/2\), \( \frac{1}{\sqrt{t}} < \frac{1}{\epsilon} \) and then, by (2.14), we get

\[
\int_{0}^{1/2} e^{-(1-\epsilon)|a(t)|} dt \leq e^{(1-\epsilon)|a|^{2}} \int_{0}^{1/2} e^{-(1-\epsilon)\frac{2}{\beta}} \frac{dt}{t^{d+1}} \leq C_{\beta} e^{(1-\epsilon)|a|^{2}}.
\]

Analogously,

\[
\int_{1/2}^{1} \left( -\log(\sqrt{1-t}) \right)^{\frac{g}{\beta}} \frac{e^{-(1-\epsilon)|a(t)|}}{t^{d+1}} \frac{dt}{\sqrt{1-t}} \leq e^{(1-\epsilon)|a|^{2}} \int_{1/2}^{1} \left( -\log(\sqrt{1-t}) \right)^{\frac{g}{\beta}} \frac{e^{-(1-\epsilon)\frac{2}{\beta}}}{t^{d+1}} \frac{dt}{\sqrt{1-t}} \leq C_{d} e^{(1-\epsilon)|a|^{2}} \int_{1/2}^{1} \left( -\log(\sqrt{1-t}) \right)^{\frac{g}{\beta}} \frac{e^{-(1-\epsilon)\frac{2}{\beta}}}{\sqrt{1-t}} dt,
\]

since \( \frac{1}{t^{d+1}} \) is bounded on \([1/2, 1]\), also as \(1/2 < t < 1\) we have \(-\frac{2}{\beta} < -a\) and then by Lemma 2.3

\[
\int_{1/2}^{1} \left( -\log(\sqrt{1-t}) \right)^{\frac{g}{\beta}} \frac{e^{-(1-\epsilon)\frac{2}{\beta}}}{\sqrt{1-t}} dt \leq e^{-(1-\epsilon)a} \int_{1/2}^{1} \left( -\log(\sqrt{1-t}) \right)^{\frac{g}{\beta}} \frac{dt}{\sqrt{1-t}} = C_{\beta} e^{-(1-\epsilon)a}.
\]

Thus,

\[
\int_{1/2}^{1} \left( -\log(\sqrt{1-t}) \right)^{\frac{g}{\beta}} \frac{e^{-(1-\epsilon)|a(t)|}}{t^{d+1}} \frac{dt}{\sqrt{1-t}} \leq C_{d} e^{(1-\epsilon)|a|^{2}} e^{-(1-\epsilon)a} = C e^{-(1-\epsilon)|a|^{2}}.
\]

Therefore

\[
I \leq C_{\epsilon} |a| e^{-(1-\epsilon)|a|^{2}}.
\]
Now, using again inequality (2.10), we get

$$II = \int_0^1 (\log \sqrt{1 - t})^\frac{q}{2} e^{-u(t)} \frac{e^{-u(t)} |y - \sqrt{1 - tx}|^2}{2t^2} dt$$

$$= \frac{1}{2} \int_0^1 (\log \sqrt{1 - t})^\frac{q}{2} e^{-u(t) - (1 - \epsilon)u(t)} \frac{1}{t^{\frac{q}{2} + 1}} dt$$

$$\leq C\epsilon \int_0^1 (\log \sqrt{1 - t})^\frac{q}{2} e^{-e^{-1}\epsilon u(t)} dt$$

$$\leq C\epsilon e^{(1 - \epsilon)\frac{a}{2}} \int_0^1 (\log \sqrt{1 - t})^\frac{q}{2} e^{-e^{-1}\epsilon a^2} dt.$$  

Now,

$$\int_0^{1/2} (\log \sqrt{1 - t})^\frac{q}{2} e^{-e^{-1}\epsilon a^2} dt \leq C \int_0^{1/2} e^{-e^{-1}\epsilon a^2} dt \leq C \int_0^1 e^{-e^{-1}\epsilon a^2} dt.$$  

Therefore, by taking the change of variables $s = a(\frac{1}{t} - 1)$, we get

$$\int_0^1 e^{-e^{-1}\epsilon a^2} dt = \int_0^{\infty} e^{-e^{-1}\epsilon (s + a)} \left( \frac{s + a}{a} \right)^{d-1} \frac{a}{(s + a)^2} ds$$

$$= \frac{e^{-e^{-1}\epsilon a^2}}{a^2} \int_0^{\infty} e^{-e^{-1}\epsilon s} (s + a)^{d-1} ds$$

$$\leq C\epsilon e^{-e^{-1}\epsilon a^2} \int_0^{\infty} e^{-e^{-1}\epsilon s} (s^{d-1} + a^{d-1}) ds$$

$$\leq C\epsilon e^{-e^{-1}\epsilon a^2} \left( \Gamma \left( \frac{d}{2} \right) + a^{d-1} \right)$$

$$= C\epsilon e^{-e^{-1}\epsilon a^2} \left( \frac{\Gamma \left( \frac{d}{2} \right)}{a^{d-1}} \right)$$

(2.14)

since $a \geq \frac{d}{4}$. Analogously, by Lemma 2.2

$$\int_{1/2}^1 (\log \sqrt{1 - t})^\frac{q}{2} e^{-e^{-1}\epsilon a^2} dt \leq C \int_{1/2}^1 (\log \sqrt{1 - t})^\frac{q}{2} e^{-e^{-1}\epsilon a^2} dt$$

$$\leq C \int_{1/2}^1 (\log \sqrt{1 - t})^\frac{q}{2} e^{-e^{-1}\epsilon a^2} dt$$

$$\leq C\epsilon e^{-e^{-1}\epsilon a^2} \int_0^1 (\log \sqrt{1 - t})^\frac{q}{2} dt$$

$$= C\epsilon e^{-e^{-1}\epsilon a^2}.$$  

Then,

$$II \leq C\epsilon e^{(1 - \epsilon)\frac{a^2}{4}} e^{-e^{-1}\epsilon a^2} = C\epsilon e^{(1 - \epsilon)\frac{a^2}{4}} e^{-e^{-1}\epsilon (|y|^2 + |x|^2)} = C\epsilon e^{(1 - \epsilon)\frac{a^2}{4}}.$$
Finally,

\[ III = \int_0^1 \left( -\log(\sqrt{1-t}) \right)^2 e^{-\frac{d(t)}{t^2}} \frac{dt}{t^{d+1}} \leq d \int_0^1 \left( -\log(\sqrt{1-t}) \right)^{\frac{d}{2}} e^{-\frac{d}{t^{d+1}}} \frac{dt}{t^{d+1}} \]

\[ = Ce^{d\|x\|^2} \int_0^1 \left( -\log(\sqrt{1-t}) \right)^{\frac{d}{2}} e^{-\frac{d}{t^{d+1}}} dt. \]

Now,

\[ \int_0^{1/2} \left( -\log(\sqrt{1-t}) \right)^{\frac{d}{2}} e^{-\frac{d}{t^{d+1}}} dt \leq C \int_0^{1/2} e^{-\frac{d}{t^{d+1}}} dt \leq C \int_0^1 e^{-\frac{d}{t^{d+1}}} dt \]

since \( -\log(\sqrt{1-t}) \) is bounded in \([0, 1/2]\), by taking

\[ s = a\left(\frac{1}{t} - 1\right), \quad ds = -\frac{a}{t^2} dt, \]

we get

\[ \int_0^1 e^{-\frac{d}{t^{d+1}}} dt = \int_0^{+\infty} e^{-(s+a)\frac{d}{a}} \left( \frac{s}{s+a} \right)^{\frac{d}{a}+1} \left( \frac{a}{s+a} \right)^{\frac{d}{a}} ds \]

\[ = \frac{e^{-|x|^2}}{a^{\frac{d}{a}}} \int_0^{+\infty} e^{-s} (s+a)^{\frac{d}{a}-1} ds. \]

On the other hand,

\[ (s+a)^{\frac{d}{a}-1} = \left( \frac{s}{s+a} + \frac{a}{s+a} \right) \leq C \left( \frac{s^{\frac{d}{a}} + a^{\frac{d}{a}}}{s+a} \right) \leq C \left( \frac{s^{\frac{d}{a}}}{s} + \frac{a^{\frac{d}{a}}}{a} \right) = C \left( s^{\frac{d}{a}-1} + a^{\frac{d}{a}-1} \right). \]

Therefore,

\[ \int_0^1 e^{-\frac{d}{t^{d+1}}} dt \leq Ce^{-|x|^2} \left( \int_0^{+\infty} e^{-s} s^{\frac{d}{a}-1} ds + \int_0^{+\infty} e^{-s} a^{\frac{d}{a}-1} ds \right) \]

\[ = Ce^{-|x|^2} \left( \frac{\Gamma\left(\frac{d}{2}\right)}{a^{\frac{d}{a}}} + a^{\frac{d}{a}-1} \right) = Ce^{-|x|^2} \left( \frac{\Gamma\left(\frac{d}{2}\right)}{a^{\frac{d}{a}}} + \frac{1}{a} \right). \]

Thus,

\[ \int_0^1 e^{-\frac{d}{t^{d+1}}} dt \leq Ce^{-|x|^2}, \quad \text{as } a \geq \frac{d}{2}. \]

For \( \frac{1}{2} < t < 1, \quad -a > -\frac{d}{2} > -2a. \) Hence, by Lemma 2.2

\[ \int_{1/2}^1 \left( -\log(\sqrt{1-t}) \right)^{\frac{d}{2}} e^{-\frac{d}{t^{d+1}}} dt \leq C \int_{1/2}^1 \left( -\log(\sqrt{1-t}) \right)^{\frac{d}{2}} e^{-a} dt \]

\[ \leq Ce^{-a} \int_0^1 \left( -\log(\sqrt{1-t}) \right)^{\frac{d}{2}} dt \]

\[ = Ce^{-|x|^2}. \]
Thus, \[ III \leq Ce^{|x|^2} e^{-|y|^2} = Ce^{-|y|^2} \leq Ce^{-(1-\epsilon)|y|^2}. \]

In other words, \[ I \leq Ce^{|x|e^{-(1-\epsilon)|y|^2}} \]

and
\[ II, III \leq Ce^{-\epsilon^2} \leq Ce^{-(1-\epsilon)|y|^2}. \]

Thus, for \( b \leq 0 \)
\[ |N_{\beta/2}(x, y)| \leq Ce(|x| + 1)e^{-(1-\epsilon)|y|^2}. \]

Next, we take \( 0 < \epsilon < 1/p' \) and \( \bar{\epsilon} = 1/p' - \epsilon = 1 - \epsilon - 1/p' \). Then, \( \bar{\epsilon} > 0 \) and \( 1 - \epsilon = \bar{\epsilon} + 1/p' \). Therefore, for \( f \in L^{p_i}(\mathbb{R}^d, \gamma_d) \) with \( \|f\|_{p_i, \gamma_d} = 1 \), using H"older’s inequality

\[
\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |N_{\beta/2}(x, y)||f(y)|dy \right)^{p(x)} \gamma_d(dx) \leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} (|x| + 1)e^{-(1-\epsilon)|y|^2}|f(y)|dy \right)^{p(x)} \gamma_d(dx)
\]

\[
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} e^{-(\bar{\epsilon}+1/p')|y|^2}|f(y)|dy \right)^{p(x)} (|x| + 1)^{p(x)} \gamma_d(dx)
\]

\[
\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |f(y)|^{p} e^{-\epsilon|y|^2} dy \right)^{p(x)/p} \left( \int_{\mathbb{R}^d} e^{-\beta|y|^2} dy \right)^{p(x)/p'} (|x| + 1)^{p(x)} \gamma_d(dx)
\]

\[
\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |f(y)|^{p} e^{-\epsilon|y|^2} dy \right)^{\frac{p(x)}{p}} (|x| + 1)^{p(x)} \gamma_d(dx)
\]

and, since \( \rho_{p_i, \gamma_d}(f) \leq 1 \),

\[
\int_{\mathbb{R}^d} |f(y)|^{p} e^{-\epsilon|y|^2} dy \leq \int_{|f| \geq 1} |f(y)|^{p} \gamma_d(dy) + \int_{|f| < 1} |f(y)|^{p} \gamma_d(dy)
\]

\[
\leq \int_{|f| \geq 1} |f(y)|^{p} \gamma_d(dy) + 1 \leq \rho_{p_i, \gamma_d}(f) + 1 \leq 2.
\]

Thus,

\[
\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |f(y)|^{p} dy \right)^{p(x)} \gamma_d(dx) \leq 2^{\nu_2} \int_{\mathbb{R}^d} (|x| + 1)^{p(x)} \gamma_d(dx) = C_{d, p}.
\]

Hence,

\[
\|I_0(f \chi_{E_i})\|_{p_i, \gamma_d} \leq C_{d, p}.
\]
Case $b > 0$:

In this case, $I$ is a very problematic term so we will discuss it at the end.

Again, by inequality (2.10)

$$II = \frac{1}{2} \int_0^1 (-\log(\sqrt{1-t}))^{\beta/2} \frac{e^{-\eta(t)}}{t^{d/2+1}} \left| y - \sqrt{1-t} x \right|^2 dt$$

$$= \frac{1}{2} \int_0^1 (-\log(\sqrt{1-t}))^{\beta/2} \frac{e^{-(1-\epsilon)\eta(t)}}{t^{d/2+1}} \frac{e^{-\epsilon t}}{t} \left| y - \sqrt{1-t} x \right|^2 dt$$

$$\leq C_\epsilon \int_0^1 (-\log(\sqrt{1-t}))^{\beta/2} \frac{e^{-(1-\epsilon)\eta(t)}}{t^{d/2+1}} dt.$$  

By using inequality (4.44) of [14], see also [10], we get

$$e^{-\eta(t)} \leq \frac{2^d e^{-\eta(t_0)}}{t_0^{d/2}}. \quad (2.15)$$

Thus,

$$\frac{e^{-(1-\epsilon)\eta(t)}}{t^{d/2+1}} \leq \left( \frac{e^{-\eta(t_0)}}{t_0^{d/2}} \right)^{1-\epsilon} \frac{1}{t^{d(1-\epsilon)/2}} \frac{1}{t^{\epsilon d/2}}. \quad (2.16)$$

Now, splitting the above integral into two the integrals on $[0, 1/2]$ and $[1/2, 1]$; we have for the first integral using (2.16),

$$\int_0^{1/2} (-\log(\sqrt{1-t}))^{\beta/2} \frac{e^{-(1-\epsilon)\eta(t)}}{t^{d/2+1}} dt \leq e^{-(1-\epsilon)\eta(t_0)} \int_0^{1/2} \frac{(-\log(\sqrt{1-t}))^{\beta/2}}{t^{d(1-\epsilon)/2}} dt.$$

Set $r = \min(\frac{\beta}{2}, \frac{1}{2})$, then $0 < r < 1$ and by taking $\epsilon > 0$ such that $\frac{d}{2} < \frac{\beta}{2} - r$ we get

$$\lim_{t \to 0^+} \frac{(-\log(\sqrt{1-t}))^{\beta/2}}{t^{d/2+r}} = \lim_{t \to 0^+} \left( \frac{(-\log(\sqrt{1-t}))}{t} \right)^{\beta/2} t^{\beta/2-(cd/2+r)} = 0,$$

thus, $\frac{(-\log(\sqrt{1-t}))^{\beta/2}}{t^{d/2+r}}$ is bounded on $(0, 1/2]$, and hence

$$\int_0^{1/2} \frac{(-\log(\sqrt{1-t}))^{\beta/2}}{t^{\epsilon d/2+1}} dt = \int_0^{1/2} \frac{(-\log(\sqrt{1-t}))^{\beta/2}}{t^{\epsilon d/2+r}} \frac{dt}{t^{1-r}} \leq C_\epsilon \int_0^{1/2} \frac{dt}{t^{1-r}} = C_\epsilon \beta.$$
Then,

\[
(2.17) \quad \int_0^{1/2} \left( -\log(\sqrt{1-t}) \right)^{\beta/2} \frac{e^{-(1-\epsilon)u(t)}}{t^{d/2+1}} \, dt \leq C_e \epsilon \beta \frac{e^{-(1-\epsilon)u(t_0)}}{t_0^{d(1-\epsilon)/2}}.
\]

For the integral on \([1/2, 1]\) we have, using again (2.16) and Lemma 2.2

\[
\int_{1/2}^{1} \left( -\log(\sqrt{1-t}) \right)^{\beta/2} \frac{e^{-(1-\epsilon)u(t)}}{t^{d/2+1}} \, dt = \int_{1/2}^{1} \left( -\log(\sqrt{1-t}) \right)^{\beta/2} \left( \frac{e^{-u(t)}}{t^{d/2}} \right)^{(1-\epsilon)} \frac{dt}{t^{d(1-\epsilon)/2}} 
\]

\[
\leq C_e \frac{e^{-(1-\epsilon)u(t_0)}}{t_0^{d(1-\epsilon)/2}} \int_{1/2}^{1} \left( -\log(\sqrt{1-t}) \right)^{\beta/2} \, dt 
\]

\[
\leq C_e \frac{e^{-(1-\epsilon)u(t_0)}}{t_0^{d(1-\epsilon)/2}} \int_{0}^{1} \left( -\log(\sqrt{1-t}) \right)^{\beta/2} \, dt 
\]

\[
= C_e \epsilon \beta \frac{e^{-(1-\epsilon)u(t_0)}}{t_0^{d(1-\epsilon)/2}}.
\]

Therefore, since \(t_0 < 1\) and \(\frac{\beta}{2}(1-\epsilon) < \frac{\beta}{4}\), we get

\[
II \leq C_e \epsilon \beta \frac{e^{-(1-\epsilon)u(t_0)}}{t_0^{d(1-\epsilon)/2}} \leq C_e \frac{e^{-(1-\epsilon)u(t_0)}}{t_0^{d(1-\epsilon)/2}}.
\]

Now, using (2.15) and (2.5) we get

\[
III = \frac{1}{4} \int_{0}^{1} \left( -\log(\sqrt{1-t}) \right)^{\beta/2} \frac{e^{-u(t)}}{t^{d/2}} \frac{dt}{t} \leq \frac{e^{-u(t_0)}}{t_0^{d/2}} \int_{0}^{1} \left( -\log(\sqrt{1-t}) \right)^{\beta/2} \frac{dt}{t} 
\]

\[
= C_e \frac{e^{-u(t_0)}}{t_0^{d/2}} \leq C_e \frac{e^{-(1-\epsilon)u(t_0)}}{t_0^{d(1-\epsilon)/2}}.
\]

Now, to estimate \(I\), we use again inequality (2.10)

\[
I = \int_{0}^{1} \left( -\log(\sqrt{1-t}) \right)^{\beta/2} \frac{e^{-u(t)}}{t^{d/2}} \left| \frac{y - \sqrt{1-t}}{\sqrt{t}} \right| \frac{dx}{\sqrt{1-t}}.
\]

\[
\leq C_e |x| \int_{0}^{1} \left( -\log(\sqrt{1-t}) \right)^{\beta/2} \frac{e^{-u(t)}}{t^{d/2}} \frac{dt}{\sqrt{1-t}}.
\]

Again, splitting the above integral into two the integrals on \([0, 1/2]\) and \([1/2, 1]\). For the second integral, using (2.15), that \(t \geq 1/2\) and Lemma 2.3, we get

\[
\int_{1/2}^{1} \left( -\log(\sqrt{1-t}) \right)^{\beta/2} \frac{e^{-u(t)}}{t^{d/2}} \frac{dt}{\sqrt{1-t}} \leq \frac{e^{-(1-\epsilon)u(t_0)}}{t_0^{d(1-\epsilon)/2}} \int_{1/2}^{1} \left( -\log(\sqrt{1-t}) \right)^{\beta/2} \frac{dt}{\sqrt{1-t}} 
\]

\[
= C_e \frac{e^{-(1-\epsilon)u(t_0)}}{t_0^{d(1-\epsilon)/2}}.
\]

Next, for the integral on \([0, 1/2]\) we need to consider two cases:
* Case $\beta > 0$ and $d = 1$: by Lemma 2.2 and the fact that $\frac{\log(\sqrt{1-t})}{t}$ is bounded on $(0, 1/2]$,

$$
\int_0^{1/2} \left( -\log(\sqrt{1-t}) \right)^{\frac{\beta}{2}} \frac{e^{-(1-\epsilon)u(t)}}{t^{d/2} \sqrt{1-t}} dt = \int_0^{1/2} \left( -\log(\sqrt{1-t}) \right)^{\frac{\beta}{2}} \frac{e^{-(1-\epsilon)u(t)}}{t^{d/2} \sqrt{1-t}} \frac{dt}{\sqrt{1-t}}
$$

$$
\leq C_\beta \int_0^{1/2} \left( -\log(\sqrt{1-t}) \right)^{\frac{\beta}{2}-1} e^{-(1-\epsilon)u(t)} \frac{dt}{\sqrt{1-t}}
$$

$$
\leq C_\beta e^{-(1-\epsilon)u(0)} \int_0^{1/2} \left( -\log(\sqrt{1-t}) \right)^{\frac{\beta}{2}-1} dt
$$

* Case $\beta \geq 1$ and $d \geq 2$: by taking $\epsilon < \frac{2}{\beta}$

$$
\int_0^{1/2} \left( -\log(\sqrt{1-t}) \right)^{\frac{\beta}{2}} \frac{e^{-(1-\epsilon)u(t)}}{t^{d/2} \sqrt{1-t}} dt \leq C \int_0^{1/2} \left( -\log(\sqrt{1-t}) \right)^{\frac{\beta}{2}-(1-\epsilon)u(t)} \frac{dt}{t^{d/2} \sqrt{1-t}}
$$

$$
= C \int_0^{1/2} \left( -\log(\sqrt{1-t}) \right)^{\frac{\beta}{2}} \frac{e^{-(1-\epsilon)u(t)}}{t^{d/2} \sqrt{1-t}} \frac{dt}{t^{d/2} \sqrt{1-t}}
$$

$$
= C \int_0^{1/2} \left( -\log(\sqrt{1-t}) \right)^{\frac{\beta}{2}} \frac{e^{-(1-\epsilon)u(t)}}{t^{d/2} \sqrt{1-t}} \frac{dt}{t^{d/2} \sqrt{1-t}}
$$

$$
\leq C_\beta e^{-(1-\epsilon)u(0)} \int_0^{1/2} \left( -\log(\sqrt{1-t}) \right)^{\frac{\beta}{2}} \frac{e^{\epsilon - \frac{\beta}{2}}u(t)}{t^{d/2} \sqrt{1-t}} \frac{dt}{t^{d/2} \sqrt{1-t}}
$$

Since $(-\log(\sqrt{1-t}))^{\frac{\beta}{2}}$ is continuous on $[0, 1/2]$ for $\beta \geq 1$ and proceeding in analogous way as in Lemma 4.36 of [14], we get

$$
\int_0^{1/2} \left( -\log(\sqrt{1-t}) \right)^{\frac{\beta}{2}} \frac{e^{-(1-\epsilon)u(t)}}{t^{d/2} \sqrt{1-t}} dt \leq C_\beta e^{-(1-\epsilon)u(0)} \int_0^{1/2} \frac{e^{(\epsilon - \frac{\beta}{2})u(t)} dt}{t^{d/2} \sqrt{1-t}}
$$

$$
\leq C_\beta e^{-(1-\epsilon)u(0)} e^{(\epsilon - \frac{\beta}{2})u(0)}
$$

$$
= C_\beta \frac{e^{-(1-\epsilon)u(0)}}{t_0^{d/2-1}} = C_\beta \frac{e^{-(1-\epsilon)u(0)}}{t_0^{d/2-1}}.
$$

Thus,

$$
I \leq C_\epsilon \beta \left[ 1 + \frac{1}{t_0^{d/2}} \right] e^{-(1-\epsilon)u(0)}.
$$
Finally, since $\frac{1}{r^2} \leq |x+y|^d$, $t_0 < 1$, and $|x| \leq |x+y|$ as $b > 0$, we have

* For $|x| < 1$,

$$
I \leq C_{\epsilon, \beta}|x| \left(1 + \frac{1}{t_0^2}\right) e^{-(1-\epsilon)u(t_0)} \leq C_{\epsilon, \beta} \frac{1}{t_0^2} e^{-(1-\epsilon)u(t_0)}
$$

$$
\leq C_{\epsilon, \beta}|x+y|^d e^{-(1-\epsilon)u(t_0)}.
$$

* For $|x| \geq 1$, since $b > 0$, $|x| \leq |x+y|

$$
I \leq C_{\epsilon, \beta} |x| e^{-(1-\epsilon)u(t_0)} + C_{\epsilon, \beta} |x| t_0 \frac{e^{-(1-\epsilon)u(t_0)}}{t_0^2}
$$

$$
\leq C_{\epsilon, \beta}|x+y|^d e^{-(1-\epsilon)u(t_0)} + C_{\epsilon, \beta} t_0 |x||x+y|^d e^{-(1-\epsilon)u(t_0)}.
$$

Given that $t_0 \leq C \frac{|x-y|}{|x+y|}$ and the fact that $|x| \leq |x+y|$ we get, for $|x-y| < 1$,

$$
|x| t_0 \leq C \frac{|x||x-y|}{|x+y|} \leq C.
$$

Thus, for $|x-y| < 1$,

$$
I \leq C_{\epsilon, \beta}|x+y|^d e^{-(1-\epsilon)u(t_0)},
$$

and for $|x-y| \geq 1$,

$$
I \leq C_{\epsilon, \beta}|x+y|^{d+1} e^{-(1-\epsilon)u(t_0)}.
$$

Hence, we conclude that

* $|N_{\beta/2}(x,y)| \leq C_{\epsilon, \beta}|x+y|^d e^{-(1-\epsilon)u(t_0)}$, for either $|x| \leq 1$, or for $|x| \geq 1$ with $|x-y| < 1$.

* $|N_{\beta/2}(x,y)| \leq C_{\epsilon, \beta}|x+y|^{d+1} e^{-(1-\epsilon)u(t_0)}$, for $|x| \geq 1$ with $|x-y| \geq 1$.

Now, as $b > 0$, for $f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$ with $\|f\|_{p(\cdot), \gamma_d} = 1$, we have that

$$
\int_{\mathbb{R}^d} \int_{B_{\gamma}(x) \cap E_x} |N_{\beta/2}(x,y)| f(y)dy \gamma_d(dx) = \int_{|x| < 1} \int_{B_{\gamma}(x) \cap E_x} |N_{\beta/2}(x,y)| f(y)dy \gamma_d(dx)
$$

$$
+ \int_{|x| \geq 1} \int_{B_{\gamma}(x) \cap E_x, |x-y| < 1} |N_{\beta/2}(x,y)| f(y)dy + \int_{B_{\gamma}(x) \cap E_x, |x-y| \geq 1} |N_{\beta/2}(x,y)| f(y)dy \gamma_d(dx)
$$
\[
\int_{|x|\leq 1} \left( \int_{B(x, \epsilon) \cap B_{x}} |N_{\beta/2}(x, y)| |f(y)|\,dy \right)^{p(x)} \gamma_{d}(dx) \\
+ C \int_{|x|> 1} \left( \int_{B(x, \epsilon) \cap B_{x}, |x-y|< 1} |N_{\beta/2}(x, y)| |f(y)|\,dy \right)^{p(x)} \gamma_{d}(dx) \\
\leq C_{\epsilon, \beta} \int_{\mathbb{R}^{d}} \left( \int_{\mathbb{R}^{d}} \int_{B_{x} \cap B_{x}, |x-y|\geq 1} |x+y|^{d+1} e^{-(1-\epsilon)u(y)} |f(y)|\,dy \right)^{p(x)} \gamma_{d}(dx) \\
+ C_{\epsilon, \beta} \int_{\mathbb{R}^{d}} \left( \int_{\mathbb{R}^{d}} \int_{B_{x} \cap B_{x}, |x-y|\geq 1} |x+y|^{d+1} e^{-(1-\epsilon)u(y)} |f(y)|\,dy \right)^{p(x)} \gamma_{d}(dx) \\
= C_{\epsilon, \beta} \int_{\mathbb{R}^{d}} \left( \int_{\mathbb{R}^{d}} \int_{B_{x} \cap B_{x}} |x+y|^{d} e^{-(1-\epsilon)u(y)} \frac{dy}{|x-y|^{2}} \frac{dy}{|x-y|^{2}} |f(y)|\,dy \right)^{p(x)} \gamma_{d}(dx) \\
+ C_{\epsilon, \beta} \int_{\mathbb{R}^{d}} \left( \int_{\mathbb{R}^{d}} \int_{B_{x} \cap B_{x}, |x-y|\geq 1} |x+y|^{d+1} e^{-(1-\epsilon)u(y)} \frac{dy}{|x-y|^{2}} \frac{dy}{|x-y|^{2}} |f(y)|\,dy \right)^{p(x)} \gamma_{d}(dx) \\
\text{Since } p(\cdot) \in \mathcal{P}_{\gamma, \delta}^{\infty}(\mathbb{R}^{d}), \text{ we obtain that } e^{\frac{dy}{|x-y|^{2}} \frac{dy}{|x-y|^{2}} |f(y)|\,dy} \\
\text{and by the Cauchy-Schwartz inequality we have, } \int_{\mathbb{R}^{d}} |x+y|^{d} e^{-(1-\epsilon)u(y)} \frac{dy}{|x-y|^{2}} \frac{dy}{|x-y|^{2}} |f(y)|\,dy \\
\leq \int_{\mathbb{R}^{d}} P(x, y) |f(y)|e^{-\frac{dy}{|x-y|^{2}} \frac{dy}{|x-y|^{2}} |f(y)|\,dy}, \\
\text{and} \\
\int_{\mathbb{R}^{d}} |x+y|^{d+1} e^{-(1-\epsilon)u(y)} \frac{dy}{|x-y|^{2}} \frac{dy}{|x-y|^{2}} |f(y)|\,dy \\
\leq \int_{\mathbb{R}^{d}} Q(x, y) |f(y)|e^{-\frac{dy}{|x-y|^{2}} \frac{dy}{|x-y|^{2}} |f(y)|\,dy}, \\
\text{where} \\
P(x, y) = |x+y|^{d} e^{-\alpha_{\infty} |x+y|^{2}}, \quad Q(x, y) = |x+y|^{d+1} e^{-\alpha_{\infty} |x+y|^{2}} \\
\text{and} \\
\alpha_{\infty} = \left( \frac{1-\epsilon}{2} - \frac{1}{p_{\infty}} - \frac{1-\epsilon}{2} \right). \]

It is easy to see that \( \alpha_{\infty} > 0 \) if \( \epsilon < \frac{1}{p'_{\infty}} \).
Therefore, in order to make sense of all the estimates above we need to take
\[ 0 < \epsilon < \min \left\{ \frac{1}{d}, \frac{1}{p_\infty}, \frac{2}{d \left( \frac{\beta}{2} - r \right)} \right\}. \]

Observe that \( P(x, y) \) is the same kernel considered in the proof of Theorem 3.5, page 416 of [5], so we can conclude that
\[
\int_{\mathbb{R}^d} \left( \int_{B_r(x) \cap E_x} P(x, y)|f(y)|e^{-|y|^2/p(y)} dy \right)^{p(x)} dx \leq C.
\]

On the other hand, it can be proved that \( Q(x, y) \) is integrable on each variable and the value of each integral is independent of \( x \) and \( y \). Now, we use an analogous argument as in [5] for \( Q(x, y) \). Taking,
\[
J = \int_{\mathbb{R}^d} Q(x, y)|f(y)|e^{-|y|^2/p(y)} dy,
\]
and using Hölder’s inequality, we obtain
\[
J \leq \| Q(x, \cdot) \|_{p'(\cdot)} \| f e^{-|y|^2/p(\cdot)} \|_{p(\cdot)} \leq \| Q(x, \cdot) \|_{p'(\cdot)}.
\]

and,
\[
\int_{\mathbb{R}^d} Q(x, y)^{p'(y)} dy = \int_{\mathbb{R}^d} |x + y|^{(d+1)p'(y)} e^{-\alpha_\infty|x+y|p(y)} dy
\]
\[
\leq \int_{|x+y|<1} |x + y|^{(d+1)p'(y)} e^{-\alpha_\infty|x+y|} dy + \int_{|x+y|\geq1} |x + y|^{(d+1)p'_\infty} e^{-\alpha_\infty|x+y|} dy
\]
\[
\leq \int_{\mathbb{R}^d} \left( |z|^{(d+1)p'_\infty} + |z|^{(d+1)p'_\infty} \right) e^{-\alpha_\infty|z|} dz = C_{p,d}.
\]

Thus, \( J \leq \| Q(x, \cdot) \|_{p'(\cdot)} \leq C_{p,d} \), and therefore
\[
\frac{1}{C_{p,d}} \int_{\mathbb{R}^d} Q(x, y)|f(y)|e^{-|y|^2/p(y)} dy \leq 1.
\]

We set \( g(y) = |f(y)|e^{-|y|^2/p(y)} = g_1(y) + g_2(y) \), where \( g_1 = g\chi_{\{g \geq 1\}} \) and \( g_2 = g\chi_{\{g < 1\}} \), then
\[
\int_{\mathbb{R}^d} \rho(x) dx = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} Q(x, y)|f(y)|e^{-|y|^2/p(y)} dy \right)^{p(x)} dx
\]
\[
\leq \int_{\mathbb{R}^d} \left( \frac{1}{C_{p,d}} \int_{\mathbb{R}^d} Q(x, y)g(y)dy \right)^{p(x)} dx
\]
\[
\leq \int_{\mathbb{R}^d} \left( \frac{1}{C_{p,d}} \int_{\mathbb{R}^d} Q(x, y)g_1(y)dy \right)^{p(x)} dx + \int_{\mathbb{R}^d} \left( \frac{1}{C_{p,d}} \int_{\mathbb{R}^d} Q(x, y)g_2(y)dy \right)^{p(x)} dx.
\]
By Hölder’s inequality and Fubini’s theorem

\[
\int_{\mathbb{R}^d} \left( \frac{1}{C_{p,d}} \int_{\mathbb{R}^d} Q(x,y)g_1(y)dy \right)^{p(x)} dx \leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} Q(x,y)g_1(y)dy \right)^{p(x)} dx
\]

\[
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} Q^{\frac{1}{p-1}}(x,y)Q^{\frac{1}{p-1}}(x,y)g_1(y)dy \right)^{p(x)} dx
\]

\[
\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} Q(x,y)dy \right)^{p(x)/p} \left( \int_{\mathbb{R}^d} Q(x,y)g_1^p(y)dy \right) dx
\]

\[
= C_p \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} Q(x,y)g_1^p(y)dy dx
\]

\[
= C_p \int_{\mathbb{R}^d} g_1^p(y) \left( \int_{\mathbb{R}^d} Q(x,y)dy \right) dy = C_p \int_{\mathbb{R}^d} g_1^p(y) dy.
\]

Therefore,

\[
\int_{\mathbb{R}^d} \left( \frac{1}{C_{p,d}} \int_{\mathbb{R}^d} Q(x,y)g_1(y)dy \right)^{p(x)} dx \leq \int_{\mathbb{R}^d} g_1^p(y)dy \leq \int_{\mathbb{R}^d} |f(y)|^p e^{-|y|^2} dy \leq \rho_p(\gamma_d(f)) \leq 1.
\]

On the other hand, applying the inequality (2.1) in Lemma 2.1, since

\[ G(x) := \frac{1}{C_{p,d}} \int_{\mathbb{R}^d} Q(x,y)g_2(y)dy \leq 1, \]

we obtain that

\[
\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \frac{1}{C_{p,d}} Q(x,y)g_2(y)dy \right)^{p(x)} dx = \int_{\mathbb{R}^d} (G(x))^{p(x)} dx
\]

\[
\leq \int_{\mathbb{R}^d} (G(x))^{p_\infty} dx + \int_{\mathbb{R}^d} \frac{dx}{(e + |x|)^{dp_\infty}}
\]

\[
\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} Q(x,y)g_2(y)dy \right)^{p_\infty} + C_{d,p}.
\]

Finally, to estimate the integral

\[
\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} Q(x,y)g_2(y)dy \right)^{p_\infty} dx,
\]

we proceed in an analogous way, applying Hölder’s inequality to the exponent \( p_\infty \), Fubini’s theorem and inequality (2.2) in Lemma 2.1, we get

\[
\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} Q(x,y)g_2(y)dy \right)^{p_\infty} dx \leq \int_{\mathbb{R}^d} g_2^{p_\infty}(y)dy
\]

\[
\leq \int_{\mathbb{R}^d} g_2^{p_\infty}(y)dy + \int_{\mathbb{R}^d} \frac{dy}{(e + |y|)^{dp_\infty}}
\]

\[
\leq \int_{\mathbb{R}^d} |f(y)|^{p_\infty} e^{-|y|^2} dy + C_{d,p} \leq 1 + C_{d,p}.
\]
Thus,
\[
\int_{\mathbb{R}^d} \left( \int_{B^c(x) \cap E} |N_{\beta/2}(x, y)| \left| f(y) \right| dy \right)^{p(x)} \gamma_d(dx) \leq C_{d,p}.
\]

With this, we obtain that \( \|I_{\beta}(f \chi_{B^c(x) \cap E})\|_{p(\cdot),\gamma_d} \leq C_{d,p}. \)
We conclude that
\[
\|I_{\beta,G}(f)\|_{p(\cdot),\gamma_d} = \|I_{\beta}(f \chi_{B^c(x)})\|_{p(\cdot),\gamma_d} \leq C,
\]
and by homogeneity of the norm,
\[
\|I_{\beta,G}(f)\|_{p(\cdot),\gamma_d} \leq C\|f\|_{p(\cdot),\gamma_d}, \text{ for all function } f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d).
\]
Therefore, the global part \( I_{\beta,G} \) is bounded in \( L^{p(\cdot)}(\gamma_d) \) and the proof is complete.

\[\square\]

REFERENCES

[1] Adamowicz, T. Harjulehto, P. and Hästö, P. Maximal Operator in Variable Exponent Lebesgue Spaces on Unbounded Quasimetric Measure Space. Math. Scand. 116 (2015), no.1, pp 5-22.
[2] Berezhnoi E.I. Two-weighted estimations for the Hardy-Littlewood maximal function in ideal Banach spaces. Proc Amer Math Soc. 1999;127(1):79–87; Disponible en: http://dx.doi.org/10.1090/S0002-9939-99-04998-9
[3] Cruz-Uribe, D. & Fiorenza, A., Neugebauer, C. J. The maximal function on variable \( L^p \) spaces. Ann Acad Sci Fenn Math. 2003 28(1) 223–238.
[4] Cruz-Uribe, D. & Fiorenza, A. Variable Lebesgue Spaces Foundations and Harmonic Analysis, Applied and Numerical Harmonic Analysis Birkhäuser-Springer, Basel, (2013)
[5] Dalmasso, E. & Scotto, R. (2017) Riesz transforms on variable Lebesgue spaces with Gaussian measure, Integral Transforms and Special Functions, 28:5, 403-420, DOI: 10.1080/10652469.2017.1296835
[6] Diening, L., Harjulehto, P., Hästö, P. H. and Růžička, M. Lebesgue and Sobolev spaces with variable exponents. Lecture Notes in Mathematics, 2017. Springer, Heidelberg, 2011.
[7] Duoandikoetxea, J. Fourier Analysis. Graduated Studies in Mathematics, Volume 29, AMS R.I (2001).
[8] Grafakos, L. Classical Fourier Analysis GTM 249-50. 2nd. edition. Springer-Verlag (2008).
[9] Moreno, J., Pineda, E., & Urbina, W. Boundedness of the maximal function of the Ornstein-Uhlenbeck semigroup on variable Lebesgue spaces with respect to the Gaussian measure and consequences. Revista Colombiana de Matemáticas. Vol. 55 Núm. 1, 21–41 (2021).
[10] S. Pérez The local part and the strong type for operators related to the Gauss measure. J. Geom. Anal. 11 (2001), no. 3, 491–507. MR1857854 (2002h:42027)
[11] Pérez, S. Estimaciones puntuales y en normas para operadores relacionados con el semigrupo de Ornstein-Uhlenbeck, Memorias para optar al título de Doctora, Departamento de Matemáticas, Universidad Autónoma de Madrid.
[12] Stein E. Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press. Princeton, New Jersey, (1970).
[13] Urbina, W. Singular Integrals with respect to the Gaussian measure. Scuola Normale Superiore di Pisa. Classe di Science. Serie IV Vol XVII, 4 (1990) 531–567. MR10993708 (92d:42010)
[14] Urbina W. *Gaussian Harmonic Analysis*, Springer Monographs in Math. Springer Verlag, Switzerland AG (2019).

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