Dualized gravity beyond linear approximation

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Received: 17 November 2021 / Accepted: 25 June 2022
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Abstract We will construct a loop space formalism for general relativity, and construct the Polyakov variables as connections for such a loop space. We will use these Polyakov variables to construct a dual theory of gravity beyond linear approximation. It will be demonstrated that this loop space duality reduces to the Hodge duality for linearized gravity. Furthermore, a loop space curvature will be constructed from this Polyakov variable. It will be shown that this loop space curvature vanishes in the absence of topological defects, and so it can be used to investigate gravitational monopoles. We will also construct the suitable monopole charge for such gravitational monopoles.

1 Introduction

The electromagnetic Hodge duality has been used to obtain interesting results [1,2], and so it is interesting to investigate such a Hodge duality for gravity. As the linearized gravity can be analyzed using gravitoelectromagnetism [3–6], the Hodge duality has been constructed for linearized gravity, and it has also been used to obtain a dual gravitational theory [7–10]. This Hodge duality for linearized gravity has been generalized to curved spacetime [11,12]. The dual gravity has been used in M-theory [13,14]. It has been proposed that the M-theory could have a (4, 0) phase in six dimensions [15]. The free (4, 0) six dimensional theory is obtained from five dimensional linearized supergravity theory. The reduction of this theory to four dimensions on a 2-torus has an $SL(2, \mathbb{Z})$ duality symmetry, which acts on the linearized gravitational sector of the theory. This interchanges the Bianchi identities and the linearized Einstein equations [13,14]. Thus, it produces a self-duality between strong and weak coupling regimes of the theory. It may be noted that the Hodge dual of linearized gravity has been studied in $E\,11$ generalized eleven dimensional geometry [16]. It has also been used to investigate $E\,7$ generalized eleven dimensions geometry [17]. Thus, important results in M-theory have been obtained using the gravitational version of Hodge duality for linearized gravity. The gravitons degree of freedom in five dimensions are similar to a specific field, and this field has been used to construct a dual theory of gravity [18]. The twisted duality condition relating the usual gravitational field to a tensor field has been constructed [19]. The action for the dual gravity has also been investigated [20]. It would be interesting to investigate generalization of this duality to full gravitational theory beyond its linearized limit. However, as it would contain non-linear terms, and it would resemble Yang–Mills theory. In fact, it is known that in the vierbein formalism, general relativity can be viewed as a gauge theory of Lorentz group, with spin connection as its gauge connection [21–26]. It may be noted that it has been proposed that we can solve problems with perturbative quantum gravity by using gauge connection as the dynamical variable [25]. The dimension reduction has been investigated using a dynamical gravitational Higgs mechanism with spin connection as the dynamical variable [26]. Thus, we can use the spin connection to construct such a dual model for gravity beyond its linearized approximation.

The problem with this approach is that it has not been possible to construct a generalization of Hodge duality to non-abelian gauge theories in spacetime. However, it is possible to construct such a duality in Yang–Mills theories using Polyakov variable [27–30]. It has also been demonstrated that this duality reduces to the electromagnetic Hodge duality for abelian gauge theories [27]. These Polyakov variables are constructed using Polyakov loops in loop space, which are holonomies of closed loops in spacetime [31,31–35]. These Polyakov loops in loop space resemble the Wilson
The Polyakov loops are gauge group-valued functions rather than simple numbers [31]. The connection in the Polyakov loop space is called Polyakov variable. As this Polyakov variable has been used to construct a non-abelian generalization of Hodge duality [27–30], and gravity can be constructed as a gauge theory of Lorentz group [21–26], we can construct Polyakov variable for gravity, and use it to construct a dual for gravity beyond linearized approximation. It may be noted that this duality has been used in Yang–Mills theories to obtain several interesting results. These results have been obtained using a non-abelian dual potential for Yang–Mills theories, which has been constructed using this duality [36,37]. In fact, this non-abelian dual potential has been used for construction of a Dualized Standard Model [36–40]. This Dualized Standard Model has been used for investigating the mass difference between various generations of fermions [41,42]. The Neutrino oscillations have also been studied using this dual potential [43]. The Dualized Standard Model has been used to investigate the off-diagonal elements of the CKM matrix [44]. As the construction of the ‘t Hooft’s order-disorder parameters require a non-abelian dual potential, the Polyakov loop space formalism can be used to construct ‘t Hooft’s order-disorder parameters [45–47].

The loop space curvature can be constructed using this Polyakov variable. This loop space curvature vanishes in absence of a monopole. Thus, it can be used to detect the existence of monopoles in spacetime. It has been possible to define a loop in this space of loops, and use it to obtain the charge for non-abelian monopoles [31,32]. It may be noted that such monopoles have been studied for various theories using loop space formalism. The superspace formalism has been used to supersymmetrize the Polyakov loop space, and this supersymmetric loop space has been used to analyze monopoles in supersymmetric gauge theories [48,49]. This is because it has been demonstrated that the supersymmetric loop space curvature would also vanish in absence of monopoles. Thus, it can be used to analyze topological defects in supersymmetric gauge theories. This supersymmetric loop space has also been used to construct a dual potential in supersymmetric gauge theories [28]. This has been generalized to a deformed supersymmetric gauge theory using deformed superspace [29]. Polyakov loops space has also been used for investigating the topological defects in fractional M2-branes [50]. It has been demonstrated that the Polyakov loops for M2-branes can be constructed using a supergauge potential, with a supergroup as its gauge group. The Polyakov loop space for M2-branes has been used to analyze topological defects of M2-branes. The topological defect has also been studied in a deformed gauge theory using Polyakov loops [51]. This is done by first deforming the gauge theory by minimal length in the background spacetime. This deformation of the gauge theory also deforms the Polyakov variables, which in turn deforms the loop space curvature. Thus, deformed Bianchi identity for such deformed gauge theories can be violated even if the original Bianchi identity is not violated. This can be viewed as the formation of topological defects from minimal length in the background geometry of spacetime. Thus, the geometry of spacetime can have important consequences for topological defects. So, it would be important to analyze Polyakov loops in general relativity.

It is also important to investigate Polyakov loop space formalism for general relativity as it can be used to analyze topological defects in general relativity. It may be noted that it is possible to construct gravitational monopole solutions in analogy with the usual monopole solutions. In fact, it has been demonstrated that dual supertranslation gauge symmetry can be used to construct a gravitational Wu–Yang monopole solution [52]. This is done using a specific metric describing the monopole solution, which is constructed from two overlapping patches on a sphere. This solution is separately regular on the two patches and differentiable in the region where they are overlapped. The gravitational monopole solutions have also been studied in K-theory [53]. This is done by reconstructing the moduli space for such solutions using the NHD construction. The properties of the Higgs field for a gravitational global monopoles have been investigated, and a bound for the maximum vacuum value of such a Higgs field has been obtained [54]. The spherical symmetry solutions for gravitating global monopoles have been constructed, and used to investigate models of topological inflation [55]. The gravitational monopoles have also been studied using the twistors for the moduli space of solutions. [56]. These results have been obtained by analyzing the gravitational analogue to Yang–Mills monopoles. However, as Polyakov loops can be used to properly analyze non-abelian monopoles [31,32], and general relativity can be viewed as a gauge theory of Lorentz group [21–26], it is important to construct gravitational Polyakov loop space.

2 Polyakov variable

We would like to point out that our investigation is classical, and is only done to investigate the existence of a dual symmetry for the theory. So, we treat classical general relativity as a non-abelian gauge theory, with spin connection as the gauge potential [21–26]. The loop space techniques developed for non-abelian gauge theory are then applied to general relativity [27–30]. It is important to study loop space formalism as it has been explicitly demonstrated that Hodge duality does not hold even for a $SU(2)$ gauge theory [57]. Hence, it is not possible to construct a Hodge dual $^*F$ for $F$, ...
such that $\ast F$ can be expressed in terms of a dual gauge potential $A$ as $\ast F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. However, the loop space formalism can be used to construct a generalized duality, which reduces to the Hodge duality in the abelian case [27–30]. It may be noted that such a generalized duality can only be expressed in loop space. However, it is important to address the question of existence of such a dual gauge potential, in 't Hooft order-disorder parameters [45–47] and Dualized Standard Model [36–40]. Thus, studies to prove the formal existence of such a dual potential is important for the construction of such models. In this paper, we hope to obtain a generalized duality for gravity, with the condition that it should reduce to the usual Hodge dual under suitable approximation (which in our case occurs when the gravity is suitably linearized). Our main concern in this paper are symmetry considerations to prove the existence of this dual symmetry. For the case of the strong-weak correspondence, it has been shown that Dirac quantization condition holds for symmetry. For the case of the strong-weak correspondence, it has been shown that Dirac quantization condition holds for

\[ C : \{ \xi^\mu(s) : s = 0 \to 2\pi, \xi^\mu(0) = \xi^\mu(2\pi) \}, \]

where $\xi^\mu(0) = \xi^\mu(2\pi)$ is the chosen (but arbitrary) base point [31–35]. Next we define the loop space variable

\[ \Phi[\xi] = P_s \exp i \oint_0^{2\pi} \omega^\mu(\xi(s)) \frac{d\xi^\mu}{ds}. \]

In this derivative, the limit is taken from the left. It may be noted as the loop variable $\Phi[\xi]$ only depends on $C$ and not the manner in which $C$ is parameterized, so labeling it with a fixed point is over complete. In fact, any other parameterization of $C$ will only change the variable in the integration and not the loop space variable $\Phi[\xi]$. We can obtain a expression relating $F_{\mu}^a[\xi,s]$ to the spacetime curvature $R_{\mu\nu}^{a\beta} \Sigma_{ab}$ by first defining a parallel transport from a point $\xi(s_1)$ to a point $\xi(s_2)$ as

\[ \Phi[\xi : s_1, s_2] = P_s \exp i \oint_{s_1}^{s_2} \omega^\mu(\xi(s)) \frac{d\xi^\mu}{ds}. \]

Thus, we can write the loop space connection as

\[ F_{\mu}[\xi,s] = F_{\mu}^{-1}[\xi,s] R_{\mu\nu}^{a\beta} \Sigma_{ab} \Phi[\xi,s]. \]

So, we parallel transport from a fixed point along a fixed path to another fixed point. After reaching that point, we took a detour then turned back along the same path till we reach the original point. The phase factor generated by going along the original path will be canceled by the phase factor generated
by coming back along it. However, there will be a contribution generated by the transport along the infinitesimal circuit along the final point. This contribution is proportional to the spacetime curvature at that point. This is similar to the procedure of defining the loop space curvature in Yang–Mills spacetime curvature at that point. This is similar to the projection generated by the transport along the infinitesimal circuit by coming back along it. However, there will be a contribution.

3 Topological defects

The topological defects can be investigated using loop space formalism. We first observe that the $F_\mu[\xi]|_s$ represents the connection in this loop space. Now this connection $F_\mu[\xi]|_s$, can be used to define a functional covariant derivative as $\Delta_\mu(s) = \delta/\delta s^\mu(s) + i F_\mu[\xi]|_s$. This functional covariant derivative can be used to construct a loop space curvature term $G_{\mu v}[\xi(s_1, s_2)]$. It may be noted that the commutator of two such functional covariant derivatives can be expressed as $-iG_{\mu v}[\xi(s_1, s_2)]$. So, we can write the loop space curvature as

$$i[\Delta_\mu(s_1), \Delta_v(s_2)] = G_{\mu v}[\xi(s_1, s_2)] = \frac{\delta}{\delta s^\mu(s_2)} F_\mu[\xi]|_s - \frac{\delta}{\delta s^v(s_1)} F_\mu[\xi]|_s + i F_\mu[\xi]|_s F_\mu[\xi]|_s.]$$

(8)

Now topological defects occurring due to gravitational monopoles can be investigated using this loop space curvature. The gravitational Bianchi identity gets violated, if a gravitational monopole is present in the system. So, the violation of the gravitational Bianchi identity can be taken as an indication for the presence of gravitational monopoles. This violation can in turn be related to the loop space curvature. This can be done as the loop space curvature is formed from an infinitesimal circuit. So, we can start from one point in loop space move in a certain direction to another point in loop space. After reaching that point, we can travel in a different direction. We return by first moving in the first direction, followed by the second direction. Now by this motion in loop space, we can obtain the loop space curvature. Thus, to obtain the loop space curvature, we consider variations of the curve in two orthogonal directions $\lambda$ and $\kappa$. Now we can express three displaced curves as

$$\xi^\mu(s) = (\xi^\mu(s))_\lambda + \Delta^\mu\delta(s - s_1)$$

$$\xi^\mu(s) = (\xi^\mu(s))_\kappa + \Delta^\mu\delta(s - s_2)$$

$$\xi^\mu(s) = (\xi^\mu(s))_\kappa + \Delta^\mu\delta(s - s_2).$$

(9)

Here the variation vanishes if $\mu \neq \kappa$. Now it is possible to define the functional derivative of $F_\mu[\xi]|_s$ as

$$\frac{\delta}{\delta s^\kappa(s_2)} F_\mu[\xi]|_s = \lim_{\Delta \to 0} \lim_{\Delta' \to 0} \frac{D_{\kappa\lambda}}{\Delta \Delta'} [\Phi^{-1}[\xi]],$$

(10)

where $D_{\kappa\lambda}$ is defined such that the limit $\Delta' \to 0$ is taken before $\Delta \to 0$, and it also encloses a suitable volume [32]. We need to obtain the value of $\Phi^{-1}[\xi]$ to find an explicit expression for the loop space curvature. We can use parallel transport along these paths in loop space to obtain

$$\Phi[\xi_1] = \Phi[\xi] - i \int ds F[\xi : 2\pi, s] F[\xi(s)]\Phi[\xi : s, 0].$$

(11)

Here $F[\xi(s)]$ can be written as $F[\xi(s)] = R^{ab\mu
\nu} \Sigma_{\mu
\nu}(\xi(s)) (d\xi_\mu(s)/ds) \delta^\mu\delta(s - s_1)$. It is also possible to write $\Phi[\xi_2]$ as

$$\Phi[\xi_2] = \Phi[\xi] - i \int ds \Phi[\xi : 2\pi, s] F[\xi(s)]\Phi[\xi : s, 0].$$

(12)

Here $F[\xi(s)]$ can be expressed as $F[\xi(s)] = R^{ab\mu
\nu} \Sigma_{\mu
\nu}(\xi(s)) (d\xi_\mu(s)/ds) \delta^\mu\delta(s - s_2)$. Next consider $\Phi[\xi_3]$, and write

$$\Phi[\xi_3] = \Phi[\xi_1] - i \int ds \Phi[\xi : 2\pi, s] F[\xi(s)]\Phi[\xi : s, 0].$$

(13)

Here we can write $F[\xi(s)]$ as $F[\xi(s)] = R^{ab\mu
\nu} \Sigma_{\mu
\nu}(\xi(s)) (d\xi_\mu(s)/ds) \delta^\mu\delta(s - s_2)$. We can also obtain the value of $\Phi[\xi : 2\pi, s]$ and $\Phi[\xi : 1, 0]$. So, using all these expressions, we can write the functional derivatives of the loop space connection as

$$\frac{\delta}{\delta s^\mu(s_2)} F_\mu[\xi]|_s$$

$$= \Phi^{-1}[\xi : s_1, 0] R^{\mu ab} \Sigma_{ab}(\xi(s_2))$$

$$\times \left[ d\xi_\mu(s_1)/ds_1 \right] \Phi[\xi : s_1, 0] \delta(s_2 - s_1)$$

$$+ \Phi^{-1}[\xi : s_2, 0] R^{ab\mu
\nu} \Sigma_{\mu
\nu}(\xi(s_2)) \Phi[\xi : s_2, 0]$$

$$\times \left[ d\xi_\mu(s_2)/ds_2 \right] \delta(s_2 - s_1)$$

$$+ i [F_\mu[\xi]|_s, F_\mu[\xi]|_s] \theta(s_2 - s_1).$$

(14)
Thus, the loop space curvature is proportional to the gravitational Bianchi identity. The presence of monopoles will violate this gravitational Bianchi identity, [D_μ, R_{ντ}^{ab} Σ_{ab}] + [D_ν, R_{τμ}^{ab} Σ_{ab}] + [D_τ, R_{μν}^{ab} Σ_{ab}] ≠ 0. Thus, the loop space curvature will not vanish, when there is a gravitational monopole in spacetime. However, in absence of such a gravitational monopole, the gravitational Bianchi identity will not be violated, [D_μ, R_{ντ}^{ab} Σ_{ab}] + [D_ν, R_{τμ}^{ab} Σ_{ab}] + [D_τ, R_{μν}^{ab} Σ_{ab}] = 0. The loop space curvature will vanish, in absence of a gravitational monopole. So, the loop space curvature can be used to study gravitational monopoles.

4 Generalized dual transform

We will use the loop space variable to construct duality for general relativity. To analyze this duality, we define a new loop space variable E_μ[ξ] as

\[ E_μ[ξ] = \Phi[ξ : s, 0]F_μ[ξ|s]\Phi^{-1}[ξ : s, 0] \]

It may be noted here the variable E_μ[ξ|s] only dependent on the segment of the loop ξ between two points s − ϵ/2 to \( s + ϵ/2 \) (here the δ function has been replaced by a function with a finite width ϵ). Thus, we can now define loop derivative for E_μ[ξ|s] using the standard procedure, and take the limit ϵ → 0 at the end of our calculations. To construct a second loop derivative at the same point s, we again consider the δ function as a function with a finite width ϵ’, and define the second derivative on this segment of such a loop. Then after taking these two loop derivatives, we first by take the limit ϵ’ → 0, and then we take the limit ϵ → 0. It is possible to construct a loop space formalism for general relativity using E_μ[ξ|s].

Now using a suitable Heaviside θ-function, we can write the functional derivative of E_μ[ξ|s] as

\[ \frac{\delta}{\delta ξ^ν(s_1)} E_μ[ξ|s] = \Phi[ξ : s, 0][\frac{\delta}{\delta ξ^ν(s_1)} F_μ[ξ|s]] + iθ(s - s_1)[F_μ[ξ|s'], F_μ[ξ|s]]\Phi^{-1}[ξ : s, 0] \]

This loop space connection can be used to construct the loop space curvature as

\[ G_μν[ξ : s_1, s_2] = \Phi^{-1}[ξ : s, 0][\frac{\delta}{\delta ξ^ν(s_1)} E_μ[ξ|s_2]] - \delta E_ν[ξ|s_1]|Φ[ξ : s, 0] \]

Now we observe that G_μν[ξ : s_1, s_2] vanishes, in the absence of a monopole. So, this is because in absence of a monopole, the functional derivatives of E_μ[ξ|s] satisfy

\[ \frac{\delta}{\delta ξ^ν(s_1)} E_μ[ξ|s_2] - \delta E_ν[ξ|s_1] = 0. \]

So, the loop space description of general relativity can be constructed using E_μ[ξ|s] is equivalent to its loop space description constructed using F_μ[ξ|s].

Now we can express a dual potential to a spin connection of the general relativity, and thus construct a dual gravitational theory. This can be done by introducing a set of dual set of variables \( \tilde{E}_μ[η|t] \) which would be dual to the original loop variable E_μ[ξ|s]. Here η is another parametrized loop with parameter t, which is different from the original parametrized loop ξ with the parameter s. We can express this dual set of variable \( \tilde{E}_μ[η|t] \) as

\[ \mu^{-1}(η(t)) \tilde{E}_μ[η|t]μ(η(t)) = \frac{2}{N} \epsilon_{μνστ} \]

\[ \times \frac{dη^ν}{dt} \int \delta ξ ds E^σ[ξ|s] \frac{dξ^ν}{ds} \left( \frac{dξ^ν}{ds} \frac{dξ^σ}{ds} \right)^{-1} \times \delta(ξ(s) - η(t)) \]

This integral is an functional integral in the loop space, and so we will obtain an infinite constant after performing functional integral [27–30]. We have introduced a normalization constant \( \tilde{N} \) to remove this infinite constant. Now in analogy with Yang–Mills theory, we can express this normalization constant as [30]

\[ \tilde{N} = \int_0^{2π} ds \int \Pi_π dδξ(π) \]

Here \( μ(χ) \) is a transformation matrix which transforms the general relativity to its dual theory. We need to use a formal regularization procedure to properly define this duality transformation [27–30]. The loop derivative operates on a segment of the loop. We view E_μ[ξ|s] as a segmental quantity, with width \( ϵ = s_+ - s_- \) at ξ^σ. After the integration has been performed, we can set \( ϵ → 0 \). So, \( dξ^σ/ds \) can be viewed as \( (ξ^σ(s_+) - ξ^σ(s_-))/ϵ \), and when \( ϵ → 0 \), it becomes a tangent to the loop ξ^σ at s. We can also regard \( E_μ[η|t] \) as a segmental quantity with width \( ϵ' = t_+ - t_- \), and so \( dη^σ/dt \) becomes the tangent to the loop η^σ at t, when \( ϵ' → 0 \). Here the limit \( ϵ' → 0 \) is again taken after the integration has been performed. We can let \( ϵ' < ϵ \), and view \( δ(ξ(s) - η(t)) \) as the
segmental quantity from \( s = t_- \) to \( s = t_+ \). This way we can perform the loop space integral.

We need to identify suitable quantities in general relativity, which will appear as monopoles in the dual theory, defined using \( \tilde{E}_\mu [\eta | t] \). Furthermore, we would also require that monopoles in the original theory to be equivalent to such quantities in the dual theory. It is known that in Yang–Mills theories, the action of gauge covariant derivative on the curvature tensor of general relativity, \( \mathcal{Y}_\mu = D^\nu R^{ab}_{\mu \nu} \Sigma_{ab} \). It may be noted here that the covariant derivative is again defined with the spin connection as the gauge potential. Now using the Polyakov loop space formalism, we can observe that this quantity can be related to the non-vanishing functional divergence, \( [\delta / \delta \xi_\mu(s)] F_\mu[\xi | s] \). However, as the loop space can also be constructed using \( E_\mu[\xi | s] \), we can also relate it to non-vanishing functional divergence of \( E_\mu[\xi | s] \) as

\[
\frac{\delta}{\delta \xi_\mu(s)} E_\mu[\xi | s] = \Phi[\xi : s, 0] \frac{\delta}{\delta \xi_\mu(s)} F_\mu[\xi | s] \Phi^{-1}[\xi : s, 0]. \tag{22}
\]

In presence of a gravitational monopole, in general relativity the loop space curvature \( G_{\mu \nu}[\xi : s_1, s_2] \), does not vanish. This loop space curvature can now be expressed in terms of a functional curl of \( E_\mu[\xi | s] \) as

\[
G_{\mu \nu}[\xi : s_1, s_2] = \Phi^{-1}[\xi : s, 0] \left\{ \frac{\delta}{\delta \xi_\nu(s_1)} E_\mu[\xi | s_2] \right\} \Phi[\xi : s, 0] \tag{23}
\]

We have observed that the curl of \( E_\mu[\xi | s] \) can be used to investigate the presence of a monopole in general relativity. So, the functional curl of \( \tilde{E}_\mu[\eta | t] \) can also be used to investigate the presence of a gravitational monopole in the dual gravitational theory. To show that \( \mathcal{Y}_\mu = D^\nu R^{ab}_{\mu \nu} \Sigma_{ab} \) in general relativity is a monopole in the dual gravity, we have to demonstrate that the non-vanishing functional divergence of \( E_\mu[\xi | s_2] \) will produce to a non-vanishing functional curl of the dual variable \( \tilde{E}_\mu[\eta | t] \).

Now the functional divergence of the dual loop space variable \( \tilde{E}_\mu[\eta | t] \) can be expressed using the transformation matrix \( \mu^{-1}(\eta(t)) \). Thus, using a suitable transformation, we can relate it to the functional divergence of the loop space variable in general relativity as

\[
e^{\lambda_{\mu \alpha \beta}} \frac{\delta}{\delta \lambda(t)} \left\{ \mu^{-1}(\eta(t)) \tilde{E}_\mu[\eta | t] \mu(\eta(t)) \right\} = -2 N \epsilon^{\lambda_{\mu \alpha \beta}} \epsilon_{\mu \nu \rho \sigma} \frac{d \eta^\nu}{d \tau} \int \delta \xi ds \times \left\{ \frac{\delta}{\delta \xi(\xi)} E^\rho[\xi | s] \right\} \\
\times \left( \frac{d \xi^\alpha}{ds} \frac{d \xi^\beta}{ds} \right)^{-1} \delta(\xi(s) - \eta(t)) \tag{24}
\]

Here in the delta function \( \delta(\xi(s) - \eta(t)) \), \( \eta(t) \) can be viewed as a little segment, such that it coincides with \( \xi(s) \) for \( s = t_- \to t_+ \). Now using these segmental loop space quantities, we can express the functional divergence of the dual loop space variable \( \tilde{E}_\mu[\eta | t] \) as

\[
e^{\lambda_{\mu \alpha \beta}} \frac{\delta}{\delta \lambda(t)} \left\{ \mu^{-1}(\eta(t)) \tilde{E}_\mu[\eta | t] \mu(\eta(t)) \right\} = -2 N \int \delta \xi ds \left\{ \frac{d \eta^\theta}{d \tau} \times \frac{d \eta^\alpha}{d \tau} \frac{d \eta^\beta}{d \tau} \right\} \times \left( \frac{d \xi^\gamma}{ds} \frac{d \xi^\delta}{ds} \right)^{-1} \delta(\xi(s) - \eta(t)) \tag{25}
\]

multiplying this by \( \epsilon_{\mu \alpha \beta} / 2 \). So, dual loop space variable, we can write

\[
\mu^{-1}(\eta(t)) \left\{ \frac{\delta}{\delta \eta(\xi)} \tilde{E}_\mu[\eta | t] \right\} = - \frac{1}{N} \int \delta \xi ds \epsilon_{\mu \alpha \beta} \left\{ \left( \frac{d \eta^\theta}{d \tau} \frac{d \xi^\alpha}{d \tau} - \frac{d \eta^\theta}{d \tau} \frac{d \xi^\beta}{d \tau} \right) \frac{\delta}{\delta \eta(s) \delta \xi(s)} \right\} E^\rho[\xi | s] \\
\times \left( \frac{d \xi^\gamma}{ds} \frac{d \xi^\delta}{ds} \right)^{-1} \delta(\xi(s) - \eta(t)) \tag{26}
\]

The loop space functional derivatives vanish for local quantities. Thus, the action of the loop space functional derivatives on the transformation matrix \( \mu^{-1}(\eta(t)) \) and \( \mu(\eta(t)) \) will vanish. The functional divergence of the loop space variable \( E^\rho[\xi | s] \) can be related to the functional curl of the dual loop space variable \( \tilde{E}_\mu[\eta | t] \). Thus, the quantity \( \mathcal{Y}_\mu = D^\nu R^{ab}_{\mu \nu} \Sigma_{ab} \) in general relativity appears as monopole in the dual gravitational theory. Furthermore, it can be argued that the vanishing of the functional divergence of the loop space variable \( E^\rho[\xi | s] \) can be related to the vanishing of functional curl of the dual loop space variable \( \tilde{E}_\mu[\eta | t] \). So, if the term \( \mathcal{Y}_\mu = D^\nu R^{ab}_{\mu \nu} \Sigma_{ab} \) vanishes in general relativity, then the dual gravitational theory will not contain any monopoles.
We will demonstrate that this duality is invertible. This can be done by first observing that

\[
\frac{2}{N} \epsilon^{a\beta\mu\lambda} \times \frac{d\xi_\beta}{du} \int \delta\eta dt \mu^{-1}(\eta(t)) E_\mu[\eta|t][\mu(\eta(t))] \frac{d\eta_\mu}{dt} \\
\times \left( \frac{d\eta_\mu}{ds} \frac{d\eta_\sigma}{ds} \right)^{-1} \delta(\eta(t) - \xi(u)) \\
= -\frac{4}{N^2} \epsilon^{a\beta\mu\lambda} \epsilon_{\mu\nu\sigma\rho} \frac{d\xi_\beta}{u} \int \delta\eta dt \\
\times \frac{d\eta_\mu}{dt} \frac{d\eta_\nu}{dt} \left( \frac{d\eta_\sigma}{dt} \right)^{-1} \delta(\eta(t) - \xi(u)) \\
\times \int \delta\xi ds E^\nu[\xi|s] \frac{d\xi_\rho}{ds} \left( \frac{d\xi_\gamma}{ds} \frac{d\xi_\lambda}{ds} \right)^{-1} \delta(\xi(s) - \eta(t)).
\]

(27)

Now we integrate over \(d\eta/dt\), and obtain a factor \(N\delta^\nu_\lambda/4\).

So, we can express the right-hand side of this equation as

\[
\frac{2}{N} \epsilon^{a\beta\mu\lambda} \times \frac{d\xi_\beta}{du} \int \delta\xi ds E^\rho[\xi|s] \frac{d\xi_\rho}{ds} \\
\times \left( \frac{d\xi_\gamma}{ds} \frac{d\xi_\lambda}{ds} \right)^{-1} \delta(\xi(s) - \eta(t)).
\]

(28)

Now we observe that this is anti-symmetric in the indices \(\rho\) and \(\sigma\). As it is possible for \(d\xi/du\) and \(d\xi/ds\) to be parallel, can equate them using \(\delta(\xi(s) - \eta(u))\). Thus, using \(E^\mu[\xi|u]\), this expression can be written as

\[
\mu(\xi(u)) E_\alpha[\xi(u)] \mu^{-1}(\xi(u)) = \frac{2}{N} \epsilon^{a\beta\mu\lambda} \times \frac{d\xi_\beta}{du} \int \delta\eta dt E_\mu[\eta|t] \\
\times \frac{d\eta_\mu}{dt} \left( \frac{d\eta_\sigma}{dt} \right)^{-1} \delta(\eta(t) - \xi(u)).
\]

(29)

So, it is possible to start from dual loop space variable and obtain the loop space variable of general relativity. Thus, this duality transformation satisfies all the properties of a duality transformation.

Now we will demonstrate that this duality reduces to the usual Hodge duality for linearized gravity. However, as we have analyzed the theory using spin connection as a dynamical variable, we will also define the linearized gravity with the spin connection as the dynamical variable. The curvature tensor can now be expressed in terms of the the spin connection as [21–24,26]

\[
R^{ab}_{\mu\nu} = R^{ab}_{\mu\nu(lin)} + \omega^{\alpha}_{\mu} \omega^{b}_{\nu} - \omega^{\alpha}_{\nu} \omega^{b}_{\mu},
\]

(30)

where \(R^{ab}_{\mu\nu(lin)} = \partial_{\mu} \omega^{ab}_{\nu} - \partial_{\nu} \omega^{ab}_{\mu}\) is the linearized curvature tensor. It may be noted that this model of linearized gravity has been linearized with respect to the spin connection, and so it is different from the linearized gravity, which is linearized with respect to the metric [3–6]. However, as both these theories can be used to construct the same gauge invariant observables, both of these approach to linearized gravity are equally valid. The Hodge duality for gravity linearized with respect to the metric has produced interesting results [7–10]. Here we will investigate the Hodge duality for linearized gravity, which has been linearized with respect to the spin connection. This can be seen by using \(E_\mu[\eta|t]\) to write

\[
\mu^{-1}(x) \tilde{R}^{ab}_{\mu\nu(lin)} \Sigma_{ab}(x)(x) \mu(x) \\
= -\frac{2}{N} \epsilon_{\mu\nu\sigma\rho} \int \delta\xi ds E^\rho[\xi|s] \\
\times \left( \frac{d\xi_\sigma}{ds} \frac{d\xi_\rho}{ds} \right)^{-1} \delta(x - \xi(s)).
\]

(31)

Here we will perform the integral first, and then take the width of the segment in \(E_\mu[\xi|s]\) to zero. Now for the linearized gravity with a the linearized curvature \(R^{ab}_{\mu\nu(lin)}\) (linearized with respect to the spin connection), we can write

\[
\tilde{R}^{ab}_{\mu\nu(lin)} \Sigma_{ab}(x)(x) \\
= -\frac{2}{N} \epsilon_{\mu\nu\sigma\rho} \int \delta\xi ds R^{ab}_{\mu\nu(lin)} \Sigma_{ab}(\xi(s)) \frac{d\xi_\sigma}{ds} \frac{d\xi_\rho}{ds} \\
\times \left( \frac{d\xi_\sigma}{ds} \frac{d\xi_\rho}{ds} \right)^{-1} \delta(x - \xi(s)) \\
= -\frac{1}{2} \epsilon_{\mu\nu\sigma\rho} R^{\sigma\rho ab}_{\mu\nu(lin)} \Sigma_{ab}(x).
\]

(32)

The expression on the right-hand side is exactly the Hodge dual for the tensor of linearized curvature \(R^{ab}_{\mu\nu(lin)}\Sigma_{ab}(x)\). However, with non-linear terms, we are not able to express this duality as Hodge dual. Thus, we have shown that the Hodge duality would hold for linearized gravity, linearized with respect to spin connection.

5 Monopole charge

Now as we have analyzed general relativity as a gauge theory, with spin connection as the gauge potential, it was also possible to analyze the topological defects in this gauge theory. This was done using the loop space formalism. The loop space formalism has been used to obtain non-abelian monopole charge. This can be done by constructing a loop in the loop space. As this can be done for any non-abelian gauge theory, it can also be done for the gauge theory with Lorentz group as the gauge group. So, we will obtain gravitational monopoles by constructing a loop in the loop space of spin connection. A loop in the loop space can be constructed using the loop space connection \(F^\mu[\xi|s]\). A loop in the loop space can be defined using \(\Sigma\), where \(\Sigma\) is defined as [32]

\[
\Sigma : \{\xi^\mu(t : s) : s = 0 \rightarrow 2\pi, \ t = 0 \rightarrow 2\pi\}.
\]

(33)
with $\xi^\mu(t : 0)$ and $\xi^\mu(0 : s)$ are given by

$$\begin{align*}
\xi^\mu(t : 0) &= \xi^\mu(t : 2\pi), \quad t = 0 \rightarrow 2\pi, \\
\xi^\mu(0 : s) &= \xi^\mu(2\pi : s), \quad s = 0 \rightarrow 2\pi. \\tag{34}
\end{align*}$$

Now it is possible to represent closed loop $C(t)$ at every point $t$ on $\xi^\mu(t : s)$ as

$$C(t) : [\xi^\mu(t : s), s = 0 \rightarrow 2\pi]. \\tag{35}$$

Now as $t$ varies, $C(t)$ traces out a closed loop. This closed loop shrinks to a point at $t = 0$ and $t = 2\pi$. Here the loops $C(0)$ and $C(2\pi)$ shrink to a point. Now we can construct a loop in this loop space, using $F_\mu[\xi[t, s]]$ as the gauge connection as

$$\Theta(\Sigma) = P_t \exp i \int_0^{2\pi} dt \int_0^{2\pi} F_\mu[\xi[t, s]] \times \frac{\partial \xi^\mu[\xi[t, s]]}{\partial t}. \\tag{36}$$

A parameterized surface in the spacetime is used to define this loop in the loop space. If we denote by $\Omega X$ loop space, then this (parametrized) surface can be thought of as an element of the double loop space $\Omega^2 X$. This surface encloses a volume, and the gravitational monopole can be present inside such a volume. So, this loop of loop space, can be used to measure a gravitational monopole.

It may be noted that $F_\mu[\xi[t, s]]$ is a connection in the loop space, and it is used to construct the holonomy of the loop in loop space. Thus, $F_\mu[\xi[t, s]]$ in the double loop space is analogous to $\omega^a_{\mu b} \Sigma_{ab}$ in the loop space. So, $G_\mu[\xi(s_1, s_2)]$ in the double loop space is analogous to $R^a_{\mu \nu \lambda \kappa} \Sigma_{ab}$ in the loop space. However, $G_\mu[\xi(s_1, s_2)]$ can be constructed using the logarithmic derivative of $\Theta(\Sigma)$. Thus, it can also be viewed as the connection in the loop space. So, the loop space curvature can be viewed as a connection in the loop space. Now $\Theta(\Sigma)$ measures the gravitational monopole charge. So, for the gravitational monopole charge $\xi$ enclosed by the surface $\Sigma$, we can write $\Theta(\Sigma) = \xi$ [32]. Now we can write the Euclidean version of $SO(3, 1)$ as $SO(4)$, and we can construct $\xi$ for it. It may be noted that for three-dimensional gravity, we can write the Euclidean version of $SO(2, 1)$ as $SO(3)$. Since monopole charges are given by elements of the fundamental groups, and fundamental groups of $SO(3)$ and $SO(4)$ are both isomorphic to $\mathbb{Z}_2$, monopoles for the two cases are represented by the same topological obstruction. Now for $SO(3)$ the monopole charge is $I$ when the monopole is not present, and $-I$ when it is present. Now for $s_1 \neq s_2$, $G_\mu[\xi(s_1, s_2)]$ does not enclose any volume. So, we can write $\Theta(\Sigma) = I$, where $I$ is the group identity. Thus, for $s_1 \neq s_2$ logarithmic derivative of $\Theta$ vanishes, and so there is no monopole charge. This also occurs when $s_1 = s_2$, but the monopole world-line $Y^\mu(\tau)$ does not intersect with $\xi(s)$. So, even in that case, we can write $\Theta(\Sigma) = I$. However, when $s_1 = s_2$, and $\xi(s)$ intersect the monopole world-line $Y^\mu(\tau)$, we can write (with exp $i \pi \kappa = \xi$) [32],

$$G_\mu[\xi(s_1, s_2)] = \frac{-\pi}{g} \int d\tau \kappa[\xi[\tau]] \times \frac{d\xi^\mu(\tau)}{ds} \frac{dY^\mu(\tau)}{d\tau} \delta(\xi(s) - Y(\tau))\delta(s_1 - s_2). \\tag{37}$$

So, the monopole charge for general relativity can be explicitly constructed using this loop of loop space. It may be noted that even for non-abelian gauge theory, such non-abelian monopole can only be investigated using loop space formalism [27–30]. As the general relativity can be studied as a gauge theory of Lorentz group, such a monopole charge in the dual theory can also be studied only using the double loop space. In fact, the construction of such a dual theory is only possible in loop space formalism. Thus, to construct a dual gravitational theory beyond linear approximation, we have to analyze general relativity using the loop space formalism.

6 Conclusion

In this paper, we have used the spin connection to construct Polyakov loop space formalism for general relativity. This was done as general relativity can be viewed as a gauge theory of Lorentz group. This Polyakov loop was then used to obtain Polyakov variable as the connection in the loop space. Furthermore, we also have constructed the loop space curvature using Polyakov variable. It was demonstrated that this loop space curvature is proportional to the gravitational Bianchi identity in spacetime. Thus, using the analogy with Yang–Mills monopoles, it was argued that this loop space curvature could be used to analyze the gravitational monopoles. We also argued that the loop space can be constructed using a new set of loop variables. These new loop variables were used to construct a dual gravitational theory beyond its linear approximation. It was observed that a certain quantity in the original theory produced a monopole in the dual theory. We also demonstrated that such a quantity in the dual theory produced the monopole in the original theory. We explicitly showed that the loop space duality reduces to Hodge duality for linearized gravity. It may be noted that we constructed the loop space with spin connection as the dynamical variable, and so, we linearized the gravity with respect to the spin connection. However, both the spin connection and metric can be viewed as dynamical variables, as they both can be used to construct gauge invariant observables. The advantage of using spin connection was that the loop space formalism could be constructed with the spin connection in analogy with the Yang–Mills theory, with spin connection as the non-abelian gauge potential. We have also analyzed the loop in...
the space of loops. We used this double loop space formalism to construct a gravitational monopole charge.

It may be noted that it would be interesting to construct the loop space using metric as a dynamic variable. Then it would be possible to show that this loop space reduces to the usual Hodge duality for linearized gravity [7–10]. This can be done by first defining the loop space variable in such a way, that the loop space connection can be related to the curvature tensor in spacetime. This loop space connection can be related to a new set of loop variables. These new loop variables can then be used to construct a dual gravitational theory using metric as a dynamic variable. It would also be interesting to investigate the relation between the loop space constructed with spin connection as the dynamic variable, and the loop space constructed with metric as the dynamical variable. Furthermore, as the Hodge duality for linearized gravity has been generalized to curved spacetime [11,12], it would be interesting to investigate such limits of loop space formalism of gravity, with metric as a dynamic variable. It is known that the Hodge dual geometry has interesting applications in M-theory [13,14], so it would be interesting to generalize it to the loop space duality in M-theory. It may be noted that it has been proposed that a six dimensional phases of M-theory can be obtained from five dimensional linearized supergravity theory [15]. In fact, the Bianchi identities have been exchanged with a suitable quantity defined in linearized gravity for such a system [13,14]. As the interchanging of the gravitational Bianchi identities with a quantity defined in the original gravitational theory can be studied using loop space formalism, it would be interesting to investigate such phases in M-theory using loop space formalism. The loop space formalism can also be used to analyze dual generalized geometries. This is because Hodge duality has been used to construct the dual geometry of an $E_{11}$ generalized geometry [16] and an $E_{7}$ generalized geometry [17]. So, it would be possible to use the loop space to construct dual generalized geometries. This loop space formalism can also be used to investigate topological defects in generalized geometries. As these generalized geometries are important in M-theory, and so the loop space formalism can be used to investigate interesting structures in M-theory. Here we will first require the spin connection for the generalized geometry, and then we can use that spin connection to construct a loop space formalism for it. We can also then use the generalized spin connection to construct loop space duality in generalized geometry. It would be interesting to investigate the generalization of this duality from Hodge duality for generalized geometries. We expect that at a linearized level, this duality will reduce to Hodge duality for generalized geometry [16,17].

It would be interesting to analyze a deformation of this theory, by analyzing a deformation of general relativity. In fact, the deformation of a non-abelian gauge theory by a minimal length in the background geometry of spacetime has been investigated using the loop space formalism [51]. It was demonstrated that even if the original gauge theory did not have a topological defect, it was possible for the deformed gauge theory to have a topological defect due to a minimal length in spacetime. The deformation of general relativity from such a minimal length has been investigated using spin connections [21]. So, it would be interesting to construct the loop space variable for this deformed gravitational theory. This deformed loop space variable could then be used to obtain a deformed loop space curvature. It would be interesting to analyze the effect of such deformation on gravitational monopoles. It is expected that the deformation of the general relativity by a minimal length will also deform the gravitational Bianchi identities. Thus, even if the original Bianchi identity is satisfied, it could be violated due to such a minimal length. So, it would be possible for the gravitational loop space curvature not to vanish, even in absence of a gravitational monopole. This could mean that a minimal length can produce a gravitational monopole. This has already been demonstrated for Yang–Mills theories [51]. So, it would be instructive to see if the same phenomenon occurs in general relativity.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: This is a theoretical paper, and contains no experimental data.]

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Funded by SCOAP3. SCOAP3 supports the goals of the International Year of Basic Sciences for Sustainable Development.

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