Potential energy of mechanical system dynamics with non-holonomic constraints on the cylinder configuration space

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Abstract. The formulation of the dynamics of a mechanical system can be done by the method of the Port Controlled Hamiltonian System (PCHS), but this method still leaves a Lagrange multiplier. Furthermore, the dynamics can be formulated using another method which is more systematic, namely the Routhian Reduction method. The method illustrates a system that is subject to non-holonomic constraints and external force, so that the Lagrange multiplier can be removed from the equation. Before formulating the dynamics of a non-holonomic mechanical system, the researcher will analyze the potential energy that occurs in a system that moves in the cylinder configuration space. Potential energy is the main part that must be completed to formulate the motion system of an object, because Routhian reduction only reviews the kinetic energy and potential energy in a dynamic system. The dynamical system reviewed is an object that moves both translation and rotation with a non-holonomic constraint, namely the Tippe Top (TT). The author analyzes the potential energy of a mechanical system that moves in a cylinder configuration space with non-holonomic constraints. Method in this research is a mathematical theoretical study. This method can reduce the equation of TT’s motion with and without friction that moves on the surface of the cylinder clearly in the form of a set of differential equations. According the result of this riset, the potential energy for the TT with non-holonomic constraints that move on the surface in the tube can be determined by $U = m g (r(1 - \cos \eta) + (R - a \cos \theta) \cos \eta + a \sin \theta \cos \phi \sin \eta)$, transforming the TT’s Lagrangian that moves on a flat plane (Cartesian coordinates) to the tube coordinates, with reference to the height of the plane solved by coordinate transformation.

1. Introduction

A little toy called Tippe Top (TT) worked as a shortened ball with a peg as a handle. At the point when the toy turns quick enough on a level surface with the handle facing upward, the top will begin to flip around in a nutating way until it winds up turning on its handle. This illogical and captivating wonder is called reversal [1]. We study a model of the TT with the purpose of seeing how the elements of reversal pursues from the properties of the displaying conditions. Turns out, even the simplest model of a TT which displays the reversal wonder establish a non-integrable, non-linear dynamical arrangement of (at least) six degrees of freedom. This makes elements of the TT a provoking issue to dissect [2,3].

Because of the entrancing idea of the issue, examination of TT reversal has a rich history. During the 1950s a few papers [4,5] set up a working physical model for the Tippe Top, basically decreasing it to a rolling and skimming axisymmetric circle. There was additionally some discussion over what is the primary driver of the inversion phenomenon, when the focal point of mass moved along symmetry axis in the spherical part, or the coasting grating between the TT and the level supporting surface. Del
Campo [6] has indicated certainly that the gliding friction is the main instrument, inside the model, offering ascend to reversal.

This outcome was later certified by Cohen [7] in a paper where he exhibited the primary numerical recreations of TT reversal. Since the 1990s the center has moved to investigation of the numerical idea of the issue; examination of the integrable subcase of a moving TT [10, 11] and investigation of asymptotic conduct of the TT [2, 6, 9]. These works gave in addition to other things criteria to how the TT must be assembled and how quick it must be spun so as to empower reversal. The equations for purely rolling TT are known to be integrable since Chaplygin and Routh [12, 13]. When friction between TT and the supporting plane is presented the dynamical conditions gain two extra factors and become a non-integrable dynamical arrangement of 6 degrees of opportunity. A thorough examination of these conditions and of the elements of reversal stays an unexplored field. The vast majority of the work toward that path concentrated on numerical reenactments for different starting conditions and for different presumptions about the friction [14, 15]. This study surveys the current outcomes with respect to the asymptotics and elements of the model of the TT and gives investigation of the elements of reversal. We study specifically the partition condition for the rolling TT as a beginning device for comprehension reversing arrangements of the rolling and gliding Tippe Top. We show that for a specific scope of parameters this condition can be rearranged so exact appraisals of conduct of nutational arrangements can be found. In this preliminary research we survey the rudiments of rigid body movement, outline it for the infamous case of a Heavy Symmetric Top and condense the central matters in our investigation of elements of the TT. Lastly, to represent the intrinsic unpredictability of TT reversal, we numerically coordinate the conditions of movement for the rolling and gliding TT for a lot of physical parameters and initial conditions relating to an inverting TT. The subsequent diagrams of development of important functions and factors are talked at length.

Rigid objects are particle systems that have a relative position between fixed particles, ie the distance between any two particles in the system is fixed [11,12]. An example of rigid body motion is the Tippe Top (TT) motion which is commonly called backing top. The formulation of the dynamics of the mechanical system can be done by the method of the Port Controlled Hamiltonia System (PCHS), but this method still leaves a Lagrange multiplier, so it cannot be solved until the equations of motion. Furthermore, the dynamics can be formulated using another method which is more systematic, namely the Routhian Reduction method. The method illustrates a system that is subject to non-holonomic constraints and external style, so that the Lagrange multiplier can be removed from the equation. Before formulating the dynamics of a non-holonomic mechanical system, the researcher will analyze the potential energy that occurs in a system that moves in the cylinder configuration space. Potential energy is the initial milestone in formulating the dynamics of a system, because Routhian reduction only reviews the kinetic energy and potential energy in a dynamic system. TT’s motion in various arenas is a daily example of a rigid body motion system with non-holonomic constraints, but with a study of mechanics that is not simple.

In the research conducted by Ciooci, et al [16] and Moffat [17, 18], TT’s equation was formulated for TT that moves in the flat plane using various methods such as the Euler equation and Maxwell-Bloch equation. The author is interested in analyzing the Dynamics of with and without friction in variable initial conditions based on Routhian Reduction. Before analyzing the motion of the TT without friction, the author will review the reverse TT in the flat plane with friction according to the research that Ariska has done before [19, 20]. Routhian reduction was chosen by the authors because this equation can formulate the dynamics of complex moving systems such as systems that move both translation and rotation [21]. In addition, Routhian reduction can also describe a dynamic system in the form of a system of differential equations [22].

The concept and use of robots has evolved, both for industry, health, research and household use. Research that develops a non-holonomic mechanical system like that of Murray et al. (1994) who developed a mathematical basis for understanding the manipulation of robots that behave unholonomically by formulating kinematics and controlling robotic manipulators [1]. Researchers are interested in analyzing the dynamics of rotational objects such as TT which have un holonomic dynamics through physics computation in order to get the equation of motion dynamics of TT for designing the arena of TT games in the curved plane.
The purpose of this study is to analyze the potential energy that occurs at TT that moves on the surface in a cylinder, reducing the potential energy equation of non-holonomic mechanical systems based on physics computing with the design of curved motion fields. This research is useful in designing the TT playing arena and increasing knowledge in understanding the unbalanced TT obstacles. This theory is useful in robotic technology and mechanical technology. Following is an illustration of the dynamics of the TT on the inner surface of the tube,

![Figure 1. TT played on the inner surface of the tube viewed from the front.](image)

Rotational dynamics are difficult to formulate with the Euler-Lagrange equation because rotational dynamics contain angular velocity which is generally not a direct derivative of time from general coordinates. This is because rotational generators are not commutative, so rotational dynamics are difficult if solved by the Euler-Lagrange equation [23]. This analysis is an attempt to understand the energy that works on the dynamics of TT, especially potential energy that contains a conservative force that is gravity, as well as its effect on TT dynamics when moving in a curved plane, especially cylinders.

2. Method
This research is a mathematical theoretical study. The study was conducted with a review of several libraries about the mechanical system in the case of TT that moves in the curved plane, namely the tube; the study uses the Euler equation in determining the system reference angle. The axis of rotation used with the 3D configuration space and the axis of rotation with two axes namely the Cartesian space \((x, y)\) as well as mathematical calculations using physics computing, especially based on Maple 18.

The dynamics of TT can be analyzed by assuming that the radius of the cylinder is much larger than the backing top \((r \gg R)\). Defined several axes in the dynamics of TT in the tube as follows:

1. Fixed axis toward space \((X, Y, Z)\).
2. Axis originating at the top point of contact \((X', Y', Z')\).
3. The axis which originates in the TT’s center of mass, and always remains against the TT \((1, 2, 3)\).
4. Axis originating at the center of the mass of the TT \((x, y, z)\).

The basic transformation of the three coordinate systems is

\[
\begin{bmatrix}
\hat{e}_x \\
\hat{e}_y \\
\hat{e}_z
\end{bmatrix}
= \begin{bmatrix}
\cos \phi & \sin \phi & 0 \\
-sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\hat{e}'_x \\
\hat{e}'_y \\
\hat{e}'_z
\end{bmatrix}
\]

With the transformation back as follows

\[
\begin{bmatrix}
\hat{e}'_x \\
\hat{e}'_y \\
\hat{e}'_z
\end{bmatrix}
= \begin{bmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\hat{e}_x \\
\hat{e}_y \\
\hat{e}_z
\end{bmatrix}
\]

The next transformation is

\[
\begin{bmatrix}
\hat{e}_x \\
\hat{e}_y \\
\hat{e}_z
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \eta & \sin \eta \\
0 & -\sin \eta & \cos \eta
\end{bmatrix}
\begin{bmatrix}
\hat{e}'_x \\
\hat{e}'_y \\
\hat{e}'_z
\end{bmatrix}
\]
with
\[
\begin{bmatrix}
\dot{e}_x \\
\dot{e}_y \\
\dot{e}_z
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \eta & -\sin \eta \\
0 & \sin \eta & \cos \eta
\end{bmatrix}
\begin{bmatrix}
\dot{e}'_x \\
\dot{e}'_y \\
\dot{e}'_z
\end{bmatrix}
\]

So, it is obtained the following transformation,
\[
\begin{bmatrix}
\dot{e}_x \\
\dot{e}_y \\
\dot{e}_z
\end{bmatrix} =
\begin{bmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos \eta \sin \phi \dot{e}_x + \cos \eta \cos \phi \dot{e}_y - \sin \eta \dot{e}_z \\
\sin \eta \sin \phi \dot{e}_x + \sin \eta \cos \phi \dot{e}_y + \cos \eta \dot{e}_z
\end{bmatrix}
\]

3. Result And Discussion
Based on the transformation of coordinates from cartesian coordinates into tube coordinates, the potential energy of the system can be determined. By using the potential energy equation, analysis is required to obtain a dynamic and complex system height value because the TT moves translation as well as moving rotation in the curved configuration space. The system scheme can be seen as follows,

![Figure 2](image1)

**Figure 2.** TT played on the inner surface of the tube viewed from the side.

In Figure (2) TT is played on the inner surface of the tube which is seen from the side, whereas if the TT that moves in the surface in the tube viewed from the front can be observed through the following picture,

![Figure 3](image2)

**Figure 3.** TT played on the inner surface of the tube viewed from the front.

Based on Figure 2 and 3 it can be seen that determining the height value of a system that moves in the curved configuration space is the hardest part, of course in finding the value of the distance of the TT’s radius to the center of the curvature of the system configuration space. Based on Figure (4), it can be seen a schematic analysis of determining the height of the system in determining the potential energy equation.
Figure 4. TT scheme on the inner surface of the tube.

Figure 4 can be clarified with the schematic below.

Figure 5. TT scheme on the inner surface of the tube.

TT always comes in contact with the surface of the tube. The contact point between the TT and the floor is called P. The position of P relative to the observer on the floor is

\[
\mathbf{r}_p = -\sin\theta \, \mathbf{e}_y - (R - a\cos\theta) \mathbf{e}_z
\]

The position vector \( \mathbf{r}_p \) with respect to the observer on the floor is

\[
\mathbf{r}_{p(o)} = X_p \mathbf{e}_X + r \sin\eta \, \mathbf{e}_Y + r(1 - \cos\eta) \mathbf{e}_Z
\]

The position vector of the center of mass of TT with respect to the observer is

\[
\mathbf{r}_{CM} = \mathbf{r}_{p(o)} + \mathbf{r}_p
\]

\[
\mathbf{r}_{CM} = X_p \mathbf{e}_X + r \sin\eta \, \mathbf{e}_Y + r(1 - \cos\eta) \mathbf{e}_Z - \sin\eta \, \mathbf{e}_Y + \sin\eta \, \mathbf{e}_Z
\]

Then

\[
\dot{\mathbf{r}}_{CM} = \dot{X} \mathbf{e}_X + \left[ r \dot{\eta} \cos\eta + a\dot{\theta} \cos\theta \cos\phi \cos\eta - a\sin\theta \sin\phi \cos\eta - a\phi \sin\theta \cos\phi \sin\eta - (R - a\cos\theta) \dot{\eta} \cos\eta
\]

\[
- a\theta \sin\theta \sin\eta \right] \mathbf{e}_Y + \left[ r \dot{\eta} \sin\eta + a\dot{\cos}\theta \cos\phi \sin\eta - a\phi \sin\theta \sin\phi \sin\eta + a\phi \dot{\sin}\theta \cos\eta \right] \mathbf{e}_Z
\]

To simplify the above equation, it is assumed that \( a, R \ll r \) and \( \dot{\eta} \ll \), so the term that survives in the equation above is

5
\[
\dot{r}_{CM} \approx \dot{\hat{x}} + r \dot{\eta} \cos \eta \dot{\hat{y}} + (r \dot{\eta} \sin \eta + a \dot{\theta} \sin \theta \cos \eta) \dot{\hat{z}}
\]

So that,
\[
|\dot{r}_{CM}|^2 = \dot{x}^2 + (r \dot{\eta} \cos \eta)^2 + (r \dot{\eta} \sin \eta + a \dot{\theta} \sin \theta \cos \eta)^2
\]

with the height of the back gasing mass center measured from the floor
\[
h = r_{CM} \cdot \hat{z} = r(1 - \cos \eta) + \sin \theta \cos \phi \sin \eta + (R - a \cos \theta) \cos \eta
\]

TT angular velocity is
\[
\omega = \dot{\phi} \hat{z} + \dot{\psi} (\cos \theta \hat{e}_x - \sin \theta \hat{e}_y) + \dot{\theta} \hat{e}_z
\]

\[
= (\dot{\phi} + \dot{\psi} \cos \theta) \hat{e}_z - \dot{\psi} \sin \theta \hat{e}_x + \dot{\theta} \hat{e}_x
\]

The linear velocity of the contact point P is relative to the center of mass
\[
v_p(\text{rel}) = \vec{\alpha} \times \vec{r}_{rel}
\]

\[
= (a \phi + R \dot{\psi}) \sin \theta \hat{e}_x + \dot{\theta} (R - a \cos \theta) \hat{e}_y - a \dot{\theta} \sin \theta \hat{e}_z
\]

The frictional force between TT with the surface of the tube is
\[
F_f = -\mu F_N |v_p|
\]

\[
|F_N| \approx mg \cos \eta + m \ddot{z} + mr\dot{\eta}^2
\]

Based on Figure 5, the schematic in determining the potential energy of the TT moving on the inner surface of the tube is
\[
U = mgh
\]

\[
= mg(r(1 - \cos \eta) + (R - a \cos \theta) \cos \eta + \sin \theta \cos \phi \sin \eta)
\]

in such a way that, Lagrangian TT that moves in the inner surface of the tube can be obtained by transforming TT moves on a flat plane (Cartesian coordinates) to the coordinates of the tube,
\[
T = \frac{1}{2} m (\dot{x}^2 + r^2 \dot{\eta}^2 + (a \dot{\theta} \sin \theta \cos \eta)^2 + 2 r \dot{\theta} \dot{\eta} \sin \theta \sin \eta \cos \eta) + \frac{1}{2} l \dot{\theta}^2 + \frac{1}{2} l \sin^2 \theta \dot{\phi}^2 + \frac{1}{2} l \dot{\phi}^2 + \frac{1}{2} l^2 (\dot{\psi} + \dot{\phi} \cos \theta)^2
\]

The reverse TT played on the inner surface of the tube is
\[
L = \frac{1}{2} m (\dot{x}^2 + r^2 \dot{\eta}^2 + (a \dot{\theta} \sin \theta \cos \eta)^2 + 2 r \dot{\theta} \dot{\eta} \sin \theta \sin \eta \cos \eta) + \frac{1}{2} l \dot{\theta}^2 + \frac{1}{2} l \sin^2 \theta \dot{\phi}^2 + \frac{1}{2} l \dot{\phi}^2 + \frac{1}{2} l^2 (\dot{\psi} + \dot{\phi} \cos \theta)^2
\]

\[
- mg(r(1 - \cos \eta) + (R - a \cos \theta) \cos \eta + \sin \theta \cos \phi \sin \eta)
\]

4. Conclusion

The potential energy for the TT with non-holonomic constraints that move on the surface in the tube can be determined by transforming the TT’s Lagrangian that moves on a flat plane (Cartesian coordinates) to the tube coordinates, with reference to the height of the plane solved by coordinate transformation. The TT potential energy equation that moves on the surface in the tube is,
\[
U = mg(r(1 - \cos \eta) + (R - a \cos \theta) \cos \eta + \sin \theta \cos \phi \sin \eta)
\]

This potential energy equation is the beginning of the formulation of TT dynamics that move on the surface in the tube. TT’s potential energy that moves on the surface in the tube is the formulation of the mechanical system dynamics can be done by the method of the Port Controlled Hamiltonian System.
(PCHS), but this method still leaves a Lagrange multiplier. Furthermore, dynamics can be formulated using another method which is more systematic, namely the Routhian Reduction method. The method describes a system that is subject to non-holonomic constraints and external force, so that the Lagrange multiplier can be removed from the equation.

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