Chiral symmetry breaking generalizes in tensor theories

P. Diaz and J. A. Rosabal

Fields, Gravity and Strings @CTPU, Institute for Basic Science, 55, Expo-ro, Yuseong-gu, Daejeon, Korea, 34126.

(Dated: October 8, 2018)

In this letter we uncover a new facet of chiral symmetry and the implications of its breaking in some theories. By generalizing the concept of chiral symmetry, tensor theories naturally arise. This novel approach adds to the known uses of tensor theories (quantum gravity, holography, entanglement, etc.) a possible link to QCD phenomena.

I. INTRODUCTION

Chiral symmetry breaking has been very successful in describing the appearance and properties of some particles in the standard model [1,2]. It is known that if quarks were massless, QCD would be invariant under global transformations $SU_R(3) \times SU_L(3)$. This group acts on given bilinears of the fermion fields, schematically $\Phi^i_j(x) = M^i_\beta \bar{q}_\alpha(x) j^\beta_j(x)$, transforming in the fundamental-antifundamental representation of each $SU(3)$. Usually, $M = 1 + \gamma_5$, and the expectation value of $\Phi^i_j(x)$ is considered as the order parameter.

Quarks are not massless, so chiral symmetry is manifestly broken in Nature. The pattern we observe is

$$SU_R(3) \times SU_L(3) \to SU_R + SU_L(3),$$

where the subscript $R+L$ refers to the diagonal subgroup. This pattern of symmetry breaking can be extended to $N$ flavours, with remaining diagonal group $SU(N)$ [3].

It is natural to ask whether there exists a generalized notion of chiral group and chiral symmetry breaking and which objects would hold it. As a motivation to generalize the chiral symmetry group, we can use the concept of polyquark boundstates [4,5], which is an extension of the bilinears above-mentioned. We shall assume that there is a hypothetical polyquark state

$$\Phi_{j_1...j_d}(x) = M^{i_1...i_d}_{j_1...j_d} \bar{q}_{i_1}(x) \cdots q_{i_d}(x) j_{j_1}^1(x) \cdots j_{j_d}^d(x),$$

where tensor $M$ does not carry any symmetry and $i_k, j_k = 1, \ldots, N$, with $N > d$. As we see, the extended notion of chiral symmetry group is associated to tensors, a connection that we explore throughout this letter.

Tensor theories were popular in the nineties for their possible link to quantum gravity, after the success of matrix theories describing gravity in two dimensions [6]. Nowadays, there is a revival of the subject since they have been conjectured to be related to holography [21], in the context of SYK duality [22–23]. Besides, new applications of tensor theories in condensed matter physics are being developed, mainly since these theories are intimately related with the entanglement phenomenon. See, for instance, [24] and references therein.

Spontaneous symmetry breaking (SSB) is known to generate massless bosons (Goldstone modes). In this letter we show that generalized chiral symmetry breaking in tensor theories generates Goldstone bosons which get arranged into matrices. Thus, matrix effective theories emerge naturally in the context of tensor theories with SSB.

Random matrices models were first used in the realm of nuclear physics [25]. Since then, matrix models have been increasingly applied in many areas of mathematics and physics such as number theory, string theory, quantum gravity, holography, etc. Recently, matrix models have appeared to be intriguingly related to tensor theories [26–28]. However, the connection remains unclear so far.

In this letter we step forward in unravelling this relation by showing that SSB in tensor theories provides a link with matrix theories, full technical details can be found in [21]. Conceptually, this connection could clarify the relation between tensor theories and quantum gravity, and holography. Besides, as an application for QCD, if tensors were built on quarks as in (2), there is a chance for testing exotic bound states via the extended version of chiral symmetry breaking. These states would appear as extra octets, arranged into matrices, similar to the ordinary meson multiplet.

II. EXTENDED CHIRAL GROUP AND SSB

In this section, we define the extended chiral symmetry breaking patterns. For the sake of generality we consider unitary groups instead of $SU(N)$. We propose the extension of the chiral symmetry $U_R(N) \times U_L(N)$ into

$$G_{d\bar{d}}(N) = \prod_{k=1}^d [U_k(N) \times U_k(\bar{N})],$$

with $d \geq 1$, where $d = 1$ reproduces the ordinary chiral symmetry. The object which naturally transforms under (3) is a tensor field,

$$\Phi_{j_1...j_d}(x) = (g_1)_{m_1}^{i_1} \cdots (g_d)_{m_d}^{i_d} (g_1)_{j_1}^{m_1} \cdots (g_d)_{j_d}^{m_d} \Phi_{i_1...i_d}(x),$$

where $g_k \in U_k(N)$.

Unlike the ordinary chiral symmetry breaking, the generalized notion contains a richer pattern structure. For instance, for $d = 2$ (this could be a tetraquark state as
The tensor \( v^{i_1 \ldots i_d} \) is an invariant of \( G_d(N) \), which determines the SSB pattern. The \( \epsilon \)-term technique is intimately related to the Ward-Takahashi identities which are used to identify the Goldstone bosons of the symmetry breaking induced by \( v \), \([31]\).

### III. SSB Patterns

In this section we explore the structure of the \( \epsilon \)-term, specified by the tensor \( v \), that drives the different SSB patterns \([8]\). In ordinary chiral symmetry breaking, the \( v \) tensor which drives the breaking into the diagonal group is \( v^i_j = \epsilon v^i_j \), with \( v \in \mathbb{C} \), and the order parameter \( \langle \Phi(x) \rangle \rangle = a \delta^i_1 \). For inducing the SSB patterns \([8]\), the natural extension is constructed using Kronecker deltas. Thus, in general, \( v \) can be

\[
v^{i_1 \ldots i_d} = \sum_{\sigma \in S_d} v_\sigma \delta^{i_1 \ldots i_d}_{j_1 \ldots j_{\sigma(d)}},
\]

where \( \delta^{i_1 \ldots i_d}_{j_1 \ldots j_{\sigma(d)}} = \delta^{i_1}_{j_1} \cdots \delta^{i_d}_{j_{\sigma(d)}} \). We have denoted \( S_d \) the group of permutations which has \( d! \) elements. However, as we will argue only two terms in \([13]\) are needed in order to induce \textit{any} SSB pattern of the type \([8]\).

In order to explore the relation between \( v \) and the SSB patterns, we first point out that the role of each monomial \( \delta \) in \([13]\) is to link indices in pairs, up and downstairs. Accordingly, each monomial produces the SSB pattern

\[
\delta^{i_1 \ldots i_d}_{j_{\sigma(1)} \ldots j_{\sigma(d)}} \longrightarrow \prod_{k=1}^d \text{Diag}[U_k(N) \times U_{\sigma(k)}(N)],
\]

which is \( G_d(N) \) with the notation introduced in \([8]\). For \( d = 1 \) the ordinary chiral symmetry breaking is recovered. For \( d = 2 \), we have the patterns \([3] \) or \([4] \), associated to the two permutations of \( S_2 \).

The sum of two or more monomials results in the intersection of the groups induced by each monomial in \([15]\). For example, for \( d = 2 \), the linear combination

\[
v^{11}_{12} = v^{11}_{12} + v^{22}_{22} \]

corresponds to the intersection of the groups \([4] \) and \([6] \), resulting in the pattern \([7] \).

Because of the increasing complexity of \([8]\) with \( d \), one might think that for \( d > 2 \), more monomial contributions would be needed in order to induce a given SSB pattern. Surprisingly, only two monomials are enough.

To prove this statement we develop a diagrammatic correspondence. In accordance with \([14]\), we will graphically represent the effect of the monomial in the SSB as

\[
\delta^{i_1 \ldots i_d}_{j_{\sigma(1)} \ldots j_{\sigma(d)}} \longrightarrow \begin{vmatrix} g_1 & \cdots & g_d \\ \vdots & \ddots & \vdots \\ g_{\sigma(1)} & \cdots & g_{\sigma(d)} \end{vmatrix},
\]

where \( g_k \) denotes a generic element of \( U_k(N) \).
Following the examples of $d = 2$, the two monomials are mapped as
\[ \delta^{ij}_{j'i'} \rightarrow g_1 \quad g_2 \quad g_1 \quad g_2 \quad \delta^{ij}_{j'i'} \rightarrow g_1 \quad g_2 \quad g_2 \quad g_1. \] (16)

Diagrams representing the intersection of two groups $G_d(N)$ and $G_d'(N)$ will be called “intersection diagrams.” They are built by concatenation of two diagrams of the type (19), associated with the permutations $\sigma$ and $\sigma'$. Identical elements upstairs are joint, and the same with the barred elements downstairs. This is exemplified in Fig. 1 which represents the intersection of the two monomials (19). The diagram of Fig. 1 contains only one loop, with closed loops. We would like to emphasize that such procedure always fully connect the plain diagram. This happens because $\sum_{\alpha=1}^{\omega} n_\alpha = d$, and the groups $H_\alpha$ do not intersect each other, as stated in (9).

In order to complete the diagram we assign a subscript to each element according to the given SSB pattern. For this, we pick a loop, associated to some $H_\alpha$, and write the labels of the different unitary groups that $H_\alpha$ contains. For example, for $n_\alpha = 2$, we generically have
\[ H_\alpha = U_a \times U_b \times U_c \times U_d. \] (18)

Then the labels corresponding to $H_\alpha$ may be chosen as in Fig. 2.

A complete diagram is obtained by applying the same prescription on each $H_\alpha$. Finally, the two monomials which drive the desired SSB pattern can be read off from the full diagram. Now, since the SSB pattern is arbitrary, then any SSB pattern can be induced by only two monomials. This concludes the proof.

As a remark, this result suggests that a candidate for the order parameter built from the tensor field, no matter the theory, can always be taken as
\[ \langle \Phi_{j_1 \ldots j_d}(x) \rangle = a \delta^{i_1 \ldots i_d}_{j_1 \ldots j_d} + b \delta^{i_1 \ldots i_d}_{j_1 \ldots j_d}. \] (19)

**IV. GOLDSTONE BOSONS = MATRIX FIELDS**

In this section, we identify the massless modes that come along with different SSB patterns and we show that they get arranged into matrix fields. Assuming, as usual, that there are no other massless modes (33), the matrix fields are the relevant content of the effective theory. In order to make the link between both theories more precise, we show explicitly how Goldstone bosons are derivated from a tensor field.

As anticipated, we make use of the powerful $\epsilon$-term technique, which we now outline, and can be found in detail in (31). This technique takes advantage of the invariance of the functional integral (11) under the transformations (31). After performing (31), (11) reads as a function of $g_k = e^{i\theta_k T_a}$ and $g_k = e^{i\theta_k T_b}$ of $U_k(N)$ and $U_k(N)$, respectively. This implies that the derivatives of (11) with respect to the parameters $\theta_k$ and $\theta_k$ vanish.
Further functional derivatives with respect to the sources $J$ and $\bar{J}$ lead to a tower of identities among the Green functions. These are precisely the Ward-Takahashi identities. The first functional derivatives allow to identify the Goldstone bosons [32].

Proceeding in this way we find the Goldstone bosons $B^k_{a_j}(x) = (B^k_{a_j})^i_j(x)(T^i_a)^j$. By inspection, one can check that the number of degrees of freedom of the effective theory matches (11). Furthermore, they get arranged into matrices $(B^k_{a_j})^i_j(x)$, which in terms of the tensor field are given by

\[
(B^k_{a_j})^i_j(x) = \Phi^{i_1 \ldots i_d}_{j_1 \ldots j_d} \Phi^{j_1 \ldots j_d}_{i_1 \ldots i_d}(x) - \Phi^{j_1 \ldots j_d}_{i_1 \ldots i_d} \Phi^{i_1 \ldots i_d}_{j_1 \ldots j_d}(x),
\]

\[
(B^k_{a_j})^i_j(x) = \bar{\Phi}^{i_1 \ldots i_d}_{j_1 \ldots j_d} \bar{\Phi}^{j_1 \ldots j_d}_{i_1 \ldots i_d}(x) - \bar{\Phi}^{j_1 \ldots j_d}_{i_1 \ldots i_d} \bar{\Phi}^{i_1 \ldots i_d}_{j_1 \ldots j_d}(x),
\]

where $i, j$ in the RHS of (20) are in the slot $k, \bar{k} = 1, \ldots, d$.

Note that (20) defines $2dN \times N$-matrices, which exceeds (10). However, as explicitly shown in (31), the fields in (20) are not linearly independent. The number of linearly independent modes match exactly (11). The matrices in (20) would be the generalization of the meson octet in the Standard Model if the tensor field were built from quarks as in (2).

V. CONCLUSION AND OUTLOOK

In this letter, we have generalized the concept of chiral group $U_R(N) \times U_L(N)$ to $G_{\text{ad}}(N)$ in (3), and chiral symmetry breaking in (5). In this extended setup, the SSB patterns lead to diagonal subgroups of $G_{\text{ad}}(N)$. These SSB patterns are induced by $G_{\text{ad}}(N)$-invariant tensors $v$, whose general form is shown in (13). In order to elucidate the intricate relation between the monomial constituents of $v$ and the SSB patterns, we develop a diagrammatic correspondence. Besides, the correspondence provides a straightforward way of visualizing the SSB patterns. Surprisingly, from diagram inspection, we conclude that only two (complex) parameters are needed to induce any SSB, and suggests an extremely simple form for the order parameters [19]. Another central result of this work is the identification of the Goldstone bosons, arranged as matrix fields. They are explicitly written in terms of the original tensor modes and $v$ in (20).

We would like to highlight the possible role of the extended chiral symmetry in Nature, as it appears in different branches in physics. For instance, at high energies, tensor theories are relevant in quantum gravity and recently, in holography via SYK. Although the relation remains still unclear. We hope that the link to matrix models that we propose could conceptually clarify this matter. Nevertheless, the experimental verification of the predictions coming from tensor theories are out of reach with the current particle accelerators, whereas their effective low energy matrix theories, proposed in this letter, could in principle be observed.

A more promising scenario for testing the generalized chiral symmetry breaking is QCD. According to what we have exposed, the existence of exotic bound states in QCD could be checked. For instance, if a tetraquark state exists as in (2), at low energies it should be seen as

- Either, two matrix fields $(B^1)^j_j(x)$ and $(B^2)^j_j(x)$, transforming in the adjoint of two different diagonal $SU(3)$ groups. This would correspond to the analog of the SSB patterns [5] or [6].

- Or, three matrix fields $(B^{1,2,3})^j_j(x)$ transforming in the adjoint of the remaining diagonal group $SU(3)$, associated to the analog to the SSB pattern [4].

Since quarks are massive, the ordinary chiral symmetry breaking is not spontaneously but manifestly broken in Nature. A similar scenario is expected for the generalized chiral symmetry. As the known mesons in QCD, the matrix fields mentioned above will not be massless. However, they could be light, what would make them testable.

ACKNOWLEDGEMENTS

We are grateful to Robert de Mello Koch and Junchen Rong for fruitful discussions.

[1] Y. Nambu, “Axial vector current conservation in weak interactions,” Phys. Rev. Lett. 4, 380 (1960).
[2] M. Gell-Mann, “The Eightfold Way: A Theory of strong interaction symmetry,” California Institute of Technology Report CTSL-20, (1961).
[3] S. R. Coleman and E. Witten, “Chiral Symmetry Breakdown in Large N Chromodynamics,” Phys. Rev. Lett. 45, 100 (1980).
[4] S. L. Olsen, “A New Hadron Spectroscopy,” Front. Phys. (Beijing) 10, no. 2, 121 (2015).
[5] A. Ali, J. S. Lange and S. Stone, “Exotics: Heavy Pentaquarks and Tetraquarks,” Prog. Part. Nucl. Phys. 97, 123 (2017).
[6] P. Di Francesco, P. H. Ginsparg and J. Zinn-Justin, “2-D Gravity and random matrices,” Phys. Rept. 254, 1 (1995).
[7] R. Gurau, “Colored Group Field Theory,” Commun. Math. Phys. 304, 69 (2011).
[8] R. Gurau, “The 1/N expansion of colored tensor models,” Annales Henri Poincare 12, 829 (2011).
[9] R. Gurau and V. Rivasseau, “The 1/N expansion of colored tensor models in arbitrary dimension,” EPL 95, no. 5, 50004 (2011).
[10] R. Gurau, “The complete 1/N expansion of colored tensor models in arbitrary dimension,” Annales Henri Poincare 13, 399 (2012).
[11] R. Gurau and J. P. Ryan, “Colored Tensor Models - a review,” SIGMA 8, 020 (2012).
[12] I. R. Klebanov and G. Tarnopolsky, “On Large N Limit of Symmetric Traceless Tensor Models,” JHEP 1710, 037 (2017).
[13] S. Giombi, I. R. Klebanov and G. Tarnopolsky, “Bosonic tensor models at large N and small ε,” Phys. Rev. D 96, no. 10, 106014 (2017).
[14] K. Bulycheva, I. R. Klebanov, A. Milekhin and G. Tarnopolsky, “Spectra of Operators in Large N Tensor Models,” Phys. Rev. D 97, no. 2, 026016 (2018).
[15] S. Giombi, I. R. Klebanov, F. Popov, S. Prakash and G. Tarnopolsky, “Prismatic Large N Models for Bosonic Tensors,” arXiv:1808.09434 [hep-th].
[16] I. R. Klebanov, F. Popov and G. Tarnopolsky, “TASI Lectures on Large N Tensor Models,” arXiv:1808.09434 [hep-th].
[17] F. Ferrari, V. Rivasseau and G. Valette, “A New Large N Expansion for General Matrix-Tensor Models,” arXiv:1709.07366 [hep-th].
[18] J. Ben Geloun and S. Ramgoolam, “Counting Tensor Model Observables and Branched Covers of the 2-Sphere,” arXiv:1507.04909 [hep-th].
[19] J. Ben Geloun and S. Ramgoolam, “Tensor Models, Kronecker coefficients and Permutation Centralizer Algebras,” JHEP 1711, 092 (2017).
[20] H. Itoyama, A. Mironov and A. Morozov, “Rainbow tensor model with enhanced symmetry and extreme melonic dominance,” Phys. Lett. B 771, 180 (2017).
[21] E. Witten, “An SYK-Like Model Without Disorder,” arXiv:1610.09758 [hep-th].
[22] T. Azeyanagi, F. Ferrari and F. I. Schaposnik Massolo, “Phase Diagram of Planar Matrix Quantum Mechanics, Tensor, and Sachdev-Ye-Kitaev Models,” Phys. Rev. Lett. 120, no. 6, 061602 (2018).
[23] J. Yoon, “SYK Models and SYK-like Tensor Models with Global Symmetry,” JHEP 1710, 183 (2017).
[24] A. Jevicki, K. Suzuki and J. Yoon, “Bi-Local Holography in the SYK Model,” JHEP 1607, 007 (2016).
[25] S. R. Das, A. Ghosh, A. Jevicki and K. Suzuki, “Space-Time in the SYK Model,” JHEP 1807, 184 (2018).
[26] R. Vasseur, A. C. Potter, Y. Z. You and A. W. W. Ludwig, “Entanglement Transitions from Holographic Random Tensor Networks,” arXiv:1807.07082 [cond-mat.stat-mech].
[27] E. P. Wigner, “Characteristic Vectors of Bordered Matrices With Infinite Dimensions,” Annals of Mathematics 62, No. 3 (1955), pp. 548-564.
[28] P. Diaz, “Tensor and Matrix models: a one-night stand or a lifetime romance?,,” JHEP 1806, 140 (2018).
[29] V. Bonzom and F. Combes, “Tensor models from the viewpoint of matrix models: the case of loop models on random surfaces,” Ann. Inst. H. Poincare Comb. Phys. Interact. 2, no. 2, 1 (2015), arXiv:1304.4152 [hep-th].
[30] C. Krishnan, K. V. P. Kumar and S. Sanyal, “Random Matrices and Holographic Tensor Models,” JHEP 1706, 036 (2017), arXiv:1703.08155 [hep-th].