Conservation of energy and momentum for an electromagnetic field propagating into a linear medium from the vacuum

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The form of the energy–momentum tensor when a quasimonochromatic field propagates into and through an antireflection-coated, sourceless, transparent, continuous, linear magneto-dielectric medium, initially at rest in the local frame, remains controversial. The Minkowski energy–momentum tensor is the main component of the electromagnetic conservation law. It has been known for over a century that the electromagnetic conservation law is unsound as evidenced by alternative energy–momentum tensors that have been proposed to ameliorate known physical deficiencies (violation of conservation of angular and linear momentum) and by the various material energy–momentum tensors and coupling forces that have been introduced to repair or complete the law. The extant resolution is to treat the continuum electromagnetic system as a subsystem and add a phenomenological material subsystem energy–momentum tensor. We show that the four-divergence of the total, electromagnetic plus material, energy–momentum tensor produces an energy continuity theorem in which the two non-zero terms depend on different powers of the refractive index $n$. Then the extant resolution of the Abraham–Minkowski controversy is self-inconsistent.

I. INTRODUCTION

A large region of otherwise empty space that completely contains, for times of interest between an initial time $t_i$ and a final time $t_f$ ($-\infty < t_i \leq t < t_f < \infty$), a finite quasimonochromatic field propagating toward and through a sourceless, transparent, continuous, linear magneto-dielectric medium that is initially at rest in the local frame is a thermodynamically closed system $\Sigma$, definitively, regardless of what subsystems that one chooses to identify [1]. In particular, any forces and whatever material motion that is imparted by the interaction with the field are part of the closed system along with the incident, refracted, transmitted, and reflected fields. While conservation principles can be unambiguously applied [2] to the thermodynamically closed system, just described, continuum electrodynamics has traditionally been formulated solely in terms of the macroscopic Maxwell field equations (see Eqs. (2.1)) and the constitutive relations, $D = E + \rho/\epsilon$ and $B = H + \mu H$.

The total energy–momentum tensor of the system, when the field is in the medium, has come to be viewed as being composed of an electromagnetic part and a material part [3, 4]. Presenting the emerging viewpoint in 1979, Brevik [5] commented that there exists “no unique prescription for the separation of this total energy–momentum tensor into a field part and a matter part.” In their 2007 review of the Abraham–Minkowski controversy, Pfeifer, Nieminen, Heckenberg, and Rubinsztein-Dunlop [6] present the extant viewpoint that “any electromagnetic energy–momentum tensor must always be accompanied by a counterpart material energy–momentum tensor, and that the division of the total energy–momentum tensor into these two components is entirely arbitrary.” If the separation is truly arbitrary, then the allowable electromagnetic and material components of the total momentum extend well beyond the two specific field and material pairs prescribed by Barnett [7] and Barnett and Loudon [8]. Arbitrary quantities cannot be verified experimentally and there are examples, in the experimental record [11, 12, 7], of experiments that prove the Minkowski electromagnetic momentum and experiments that prove the Abraham momentum although experiments are unable to discriminate between the two momentums [1, 2].

In this article, we derive the well-known electromagnetic conservation law [2, 3, 4, 6]

$$\partial_\beta T^{\alpha\beta}_{EM} = 0$$

(1.1)

as a formal (axiomatic) theorem of the Maxwell–Minkowski equations (macroscopic Maxwell equations) and constitutive relations for the electromagnetic field in a gradient-index antireflection-coated, sourceless, transparent, continuous, linear magneto-dielectric medium in the limit that the gradient Minkowski force $f_M = (\mathbf{E}^2 \nabla \varepsilon - \mathbf{H}^2 \nabla \mu)/2$ and the concomitant reflection can be neglected. (We use the conventions of summing over repeated indices on the same side of the equal sign, that Greek indices are elements of $\{0, 1, 2, 3\}$, and that Roman indices from the middle of the alphabet are in $\{1, 2, 3\}$.)

The electromagnetic Minkowski energy–momentum tensor [6, 9]

$$T_M^{\alpha\beta} = \begin{bmatrix}
\frac{1}{2} (\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) & (\mathbf{E} \times \mathbf{H})^1 & (\mathbf{E} \times \mathbf{H})^2 & (\mathbf{E} \times \mathbf{H})^3 \\
(\mathbf{D} \times \mathbf{B})^1 & W_{11} & W_{12} & W_{13} \\
(\mathbf{D} \times \mathbf{B})^2 & W_{21} & W_{22} & W_{23} \\
(\mathbf{D} \times \mathbf{B})^3 & W_{31} & W_{32} & W_{33}
\end{bmatrix},$$

(1.2a)

where

$$W^{ij} = -D^i E^j - B^i H^j + \frac{1}{2} (\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) \delta^{ij},$$

(1.2b)

is formally derived as the main component of the electromagnetic conservation law, Eq. (1.1). The electromag-
nestic energy

\[ U_{em} = \int_{\Sigma} T_{00}^{\text{M}} \, dv = \frac{1}{2} \int_{\Sigma} (\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) \, dv \]  

(1.3)

and the electromagnetic Minkowski momentum

\[ G_M = \frac{1}{c} \int_{\Sigma} (T_{10}^{\text{M}}, T_{20}^{\text{M}}, T_{30}^{\text{M}}) \, dv = \int_{\Sigma} \frac{\mathbf{D} \times \mathbf{B}}{c} \, dv \]  

(1.4)

form a Lorentz four-vector \((U_{em}/c, -G_M)\) \([6, 8, 11, 14]\). The Minkowski energy–momentum tensor has a well-defined formal relationship to the electromagnetic conservation law and to its axioms, the Maxwell–Minkowski equations and constitutive relations.

If there were no problems with the electromagnetic conservation law then the Minkowski momentum and the Minkowski energy–momentum tensor, which are formally derived from the Maxwell–Minkowski equations and constitutive relations, would be settled physics. Instead, the Minkowski energy–momentum tensor remains enmeshed in an ancient controversy.

In 1909, Abraham \([12]\) noted that the Minkowski energy–momentum tensor, Eq. (1.2a), lacks transpose symmetry indicating violation of conservation of angular momentum. Abraham proposed a transpose symmetric energy–momentum tensor and an accompanying linear momentum \(G_A = G_M/n^2\) for the field in a simple linear dielectric. Transpose symmetry is typically considered to be a necessary characteristic of the total energy–momentum tensor and the Abraham tensor is extensively supported in the scientific literature \([1, 5, 13]\). Nowadays, the Abraham energy–momentum tensor is often treated as being incomplete because the Abraham linear momentum is not globally conserved. For now, we concentrate on the Minkowski energy–momentum tensor, which has its own historical imperative.

The lack of transpose symmetry in the Minkowski energy–momentum tensor has come to be deemed as acceptable based on the provenance of the electromagnetic conservation law, Eq. (1.1), as a theorem of the Maxwell–Minkowski equations and constitutive relations or based on appealing to the existence of transformations that diagonalize certain energy–momentum tensors \([6, 14]\). Alternatively, the Minkowski energy–momentum tensor is assumed to be incomplete, but fixable with the addition of a phenomenological material momentum energy–momentum tensor \([6]\). Both of these contradictory viewpoints are supported in the current scientific literature \([6, 11, 14]\).

The lack of transpose symmetry in the Minkowski energy–momentum tensor is not the only problem with the electromagnetic conservation law, Eq. (1.1). The finite quasimonochromatic field, initially in the free space portion of the thermodynamically closed system, propagates toward and then enters the sourceless, transparent, continuous, linear magneto-dielectric medium at normal incidence through a gradient-index antireflection coating. (The quasimonochromatic field has a constant amplitude throughout its duration except for a short smooth ramp up in amplitude at turn-on and a short smooth ramp down in amplitude at turn-off.) The field re-enters the vacuum through the gradient-index antireflection coating on the opposite side of the medium. The usual textbook \([3, 11, 12]\) constitutive relations for a quasimonochromatic field with center frequency \(\omega_p\) propagating through an anti-reflection coated, sourceless, transparent, continuous, linear magneto-dielectric medium, at rest in the local frame, (with dispersion treated in lowest-order \([18]\) are \(\mathbf{D} = \varepsilon(\omega_p, \mathbf{r})\mathbf{E}, \mathbf{B} = \mu(\omega_p, \mathbf{r})\mathbf{H}, \) and \(n = \sqrt{\varepsilon(\omega_p, \mathbf{r})}\mu(\omega_p, \mathbf{r})\).

In the Fresnel drag experiments \([19, 20]\), motion of the medium in the local frame is a specified condition of the system and the velocity in the local frame has an effect on the constitutive relations. Here, the medium is initially at rest in the local frame and unless the radiation pressure and duration are extraordinary, the velocity of the medium in the local frame will be quite small. Then the effect of the material velocity on the parameters, \(\varepsilon, \mu, n\), can be treated as negligible, as is done in the textbook derivation of Fresnel reflection \([9, 15, 17]\). Even more so for the current case because the gradient-index antireflection coating makes the acceleration and velocity of the material in the local frame negligible.

While the field is inside the medium, the electric field \(\mathbf{E}\) is reduced in amplitude \((\mu < \varepsilon\) at optical frequencies\) compared to the incident electric field \(\mathbf{E}_0\) in vacuum with \(\mathbf{E} = \sqrt{n/\varepsilon}\mathbf{E}_0\). Similarly, the magnetic field \(\mathbf{H}\) is enhanced in amplitude compared to the amplitude in the vacuum with \(\mathbf{H} = \sqrt{\varepsilon/n}\mathbf{H}_0\). The longitudinal width \(w = w_0/n\) of the field (in the direction of propagation) is reduced by a factor of \(n\) due to the reduced speed of light in a linear medium. Then the linear momentum

\[ G_M = \int_{\Sigma} \left( \varepsilon \sqrt{\frac{n}{\varepsilon}} \mathbf{E}_0 \times \mu \sqrt{\frac{\varepsilon}{n}} \mathbf{H}_0 \right) \, dv \]  

(1.5a)

\[ G_M \sim \frac{n|\mathbf{E}_0||\mathbf{H}_0|w_0A_0}{c} \]  

(1.5b)

that is obtained by evaluating the Minkowski linear momentum, Eq. (1.4), while the quasimonochromatic field, with cross-sectional area \(A_0\) inside the medium, is a nominal factor of \(n\) \([6, 11, 18, 21, 24]\) greater than the total linear momentum of the incident field. (The Abraham momentum \(G_A = G_M/n^2\) is a factor of \(n\) smaller than the incident linear momentum.)

The existence of a difference between the linear momentum of the incident field and the Minkowski linear momentum in the medium has been known for a long time and has been extensively documented in the scientific literature. In the old days, the difference in linear momentum was attributed to the action of a Minkowski pull-force by the field entering the medium \([11]\). More recently, it has been attributed to a material pseudomomentum \([22]\) and a canonical material momentum \([6, 8]\).

Nevertheless, it continues to be reported that the Minkowski linear momentum is provably conserved \([6]\)
electromagnetic energy, Eq. (1.3), is equal to the total
material is second-order in smallness and can be neglected
quadratic in the fields. The kinetic energy of the ma-
energy and linear momentum in a linear medium are both
linearly independent in the dynamical variable, the en-
In order to be simultaneously conserved, the total en-
mal theory is proven false then the axioms of the formal
The Minkowski energy–momentum tensor, where the latter is
conserved. We find that the total momentum formula [25]
We prove that the conservation law \( \partial_\beta T^\beta_{tot} = 0 \) that is
constructed from the total, electromagnetic plus mater-
Minkowski momentum is false because the four-
divergence of the total energy–momentum tensor pro-
duces an energy continuity equation in which the two
non-zero terms depend on different powers of the refrac-
tive index \( n \). The energy continuity equation \( \partial_\beta T^\beta_{tot} = 0 \)
is self-inconsistent and also violates Poynting’s theorem.
We conclude by reiterating that the electromagnetic
conservation law, Eq. (1.1), is a formal theorem of the
Maxwell–Minkowski equations and constitutive relations
for a quasimonochromatic field traversing a sourceless,
transparent, linear magneto-dielectric medium in the
limit that the Minkowski force \( f_M = (-E^2\nabla\varepsilon - H^2\nabla\mu)/2 \)
on the gradient-index antireflection coating can be ne-
eglected. We show that the incident linear momentum and
the globally conserved total linear momentum, Eq. (1.7),
contradict the Minkowski momentum, Eq. (1.4), and
thereby disproves the electromagnetic conservation law,
Eq. (1.1), in which the Minkowski momentum appears.
Because the electromagnetic conservation law is proven
to be false by global conservation, the axioms of the law,
the Maxwell–Minkowski equations and constitutive relations,
are proven to be false.

II. FORMAL THEORY OF THE MINKOWSKI
ENERGY–MOMENTUM TENSOR

The microscopic Maxwell equations for electromag-
netic fields in the vacuum of free space are fundamental.
When these laws are applied to the propagation of light in
a medium, one typically obtains the Maxwell–Minkowski equations [3, 13, 17]

\[
\frac{\nabla \times \mathbf{H}}{\partial (ct)} - \frac{\partial \mathbf{D}}{\partial (ct)} = \frac{\mathbf{J}_f}{c} \tag{2.1a}
\]

\[
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial (ct)} = 0 \tag{2.1b}
\]

\[
\nabla \cdot \mathbf{D} = \rho_f \tag{2.1c}
\]

\[
\nabla \cdot \mathbf{B} = 0 \tag{2.1d}
\]

for the macroscopic fields \( \mathbf{E}, \mathbf{B}, \mathbf{D}, \) and \( \mathbf{H} \). Here, \( \rho_f \) is the
free charge density and \( \mathbf{J}_f \) is the free current density.
Maxwell–Minkowski is the usual textbook representation of the macroscopic Maxwell field equations, although there are other representations [24, 28].

It is straightforward to construct the electromagnetic continuity and the electromagnetic conservation law as formal identities of the Maxwell–Minkowski equations using the rules of algebra and calculus for scalars, vectors, matrices and tensors. We take the scalar product of Eq. (2.1a) with \( \mathbf{H} \), the scalar product of Eq. (2.1a) with \( \mathbf{E} \), subtract the results, and apply a common vector identity \( \nabla \cdot (\mathbf{X} \times \mathbf{Y}) = \mathbf{Y} \cdot (\nabla \times \mathbf{X}) - \mathbf{X} \cdot (\nabla \times \mathbf{Y}) \) to produce a continuity equation

\[
\frac{1}{c} \left( \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) + \nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\mathbf{J}_f}{c} \times \mathbf{E} \tag{2.2}
\]

that is a formal theorem, Poynting’s theorem, of the Maxwell–Minkowski equations. The quantity

\[
\mathbf{S} = c(\mathbf{E} \times \mathbf{H}) \tag{2.3}
\]

is identified as the Poynting energy flux vector.

Adding the vector product of \( \mathbf{B} \) with Eq. (2.1a), the vector product of \( \mathbf{D} \) with Eq. (2.1a), the product of Eq. (2.1a) with \( -\mathbf{H} \), and the product of Eq. (2.1a) with \( -\mathbf{E} \) produces the well-known continuity equation [3]

\[
\frac{\partial}{\partial t} \frac{\mathbf{D} \times \mathbf{B}}{c} + \mathbf{D} \times (\nabla \times \mathbf{E}) + \mathbf{B} \times (\nabla \times \mathbf{H})
- (\nabla \cdot \mathbf{D})\mathbf{E} - (\nabla \cdot \mathbf{B})\mathbf{H} = -\rho_f \mathbf{E} - \frac{1}{c} \mathbf{J}_f \times \mathbf{B} \tag{2.4}
\]

that is also a formal theorem of the Maxwell–Minkowski equations. The Minkowski momentum density

\[
\mathbf{g}_M = \frac{\mathbf{D} \times \mathbf{B}}{c} \tag{2.5}
\]

is identified in the first term of Eq. (2.4). The Minkowski momentum

\[
\mathbf{G}_M = \int_{\Sigma} \frac{\mathbf{D} \times \mathbf{B}}{c} \, dv \tag{2.6}
\]

is the Minkowski momentum density integrated over the volume of the system \( \Sigma \). The energy

\[
U = \frac{1}{2} \int_{\Sigma} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \, dv \tag{2.7}
\]

is the energy density

\[
u = (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})/2 \tag{2.8}
\]

integrated over the volume.

The rate at which the fields do work on a continuous distribution of charge and current

\[
w_{\text{mech}} = \mathbf{J}_f \cdot \mathbf{E} \tag{2.9}
\]

and the Lorentz force density law

\[
\frac{dp_{\text{mech}}}{dt} = \mathbf{f}_L = \rho_f \mathbf{E} + \frac{\mathbf{J}_f}{c} \times \mathbf{B} \tag{2.10}
\]

are physical interpretations of terms that are formally derived in the energy and momentum continuity theorems, Eqs. (2.2) and (2.4).

We derived the electromagnetic energy and momentum continuity equations as identities of the Maxwell–Minkowski equations, Eqs. (2.1). In contrast, derivations of the energy and momentum continuity equations in textbooks [3, 12, 17] typically start with postulating the work rate \( w_{\text{mech}} \), Eq. (2.9), and the Lorentz force density \( \mathbf{f}_L \), Eq. (2.10). Then the postulated sources, Eqs. (2.9) and (2.10), are expressed in terms of the fields by substitution of the Maxwell–Minkowski equations, Eqs. (2.1), to derive the energy and momentum continuity equations, Eqs. (2.2) and (2.4). Consequently, there is scientific inertia for the presence of the sources, \( \rho_f \) and \( \mathbf{J}_f \), and for treating continuum electrodynamics as an open system [3].

We seek to apply the axiomatic formal theory to a thermodynamically closed system that consists of a large finite volume \( \Sigma \) of otherwise-empty space that contains, as an initial condition at time \( t_i \), the finite quasimonochromatic electromagnetic field in the vacuum and a gradient-index antireflection-coated block of sourceless, transparent, linear magneto-dielectric material at rest in the local frame. In the absence of sources and sinks the total system is thermodynamically closed, definitively, regardless of what field and material subsystems that one might choose to identify. Because global conservation principles can be applied, without ambiguity, to thermodynamically closed systems, we specify that

\[
\rho_f = 0 \tag{2.11a}
\]

\[
\mathbf{J}_f = 0 \tag{2.11b}
\]

\[
[\varepsilon, \mu, n = \sqrt{\varepsilon \mu}] \in \mathbb{R} \geq 1 \, , \tag{2.11c}
\]

are additional axioms of the formal theory for our sourceless transparent system. Then the Maxwell–Ampère law, Eq. (2.1a), and the Gauss law, Eq. (2.1c), become homogeneous with a right-hand side of zero.

It is straightforward to derive the homogeneous energy and momentum continuity equations

\[
\frac{1}{c} \left( \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) + \nabla \cdot (\mathbf{E} \times \mathbf{H}) = 0 \tag{2.12a}
\]

\[
\frac{\partial}{\partial t} \frac{\mathbf{D} \times \mathbf{B}}{c} + \mathbf{D} \times (\nabla \times \mathbf{E}) + \mathbf{B} \times (\nabla \times \mathbf{H})
- (\nabla \cdot \mathbf{D})\mathbf{E} - (\nabla \cdot \mathbf{B})\mathbf{H} = 0 \tag{2.12b}
\]
as theorems of the homogeneous Maxwell–Minkowski equations
\[ \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial (ct)} = 0 \quad (2.13a) \]
\[ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial (ct)} = 0 \quad (2.13b) \]
\[ \nabla \cdot \mathbf{D} = 0 \quad (2.13c) \]
\[ \nabla \cdot \mathbf{B} = 0 \quad (2.13d) \]
using the same procedure that was used to derive Eqs. (2.2) and (2.4).

The usual textbook derivation of the continuity laws by postulating Eqs. (2.9) and (2.10) is not applicable to a neutral linear medium in which \( \rho_f \) and \( \mathbf{J}_f \), and the associated work rate \( w_{\text{mech}} \) and Lorentz force density \( \mathbf{f}_L \), do not exist. Deriving the energy and momentum continuity theorems by formally combining the Maxwell equations as axioms in the manner described is arguably more fundamental than the usual textbook procedure because the formal axiomatic procedure works for a medium without, as well as with, sources \( \rho_f \) and \( \mathbf{J}_f \).

We define a simple linear medium as a sourceless, transparent, isotropic, homogeneous, continuous linear magneto-dielectric medium. Here, \( \varepsilon \) is a continuum abstraction of the electric permittivity, \( \mu \) is a continuum abstraction of the magnetic permeability, and \( n \) is the macroscopic refractive index. Dispersion is treated in lowest-order [18] such that \( \varepsilon, \mu, \) and \( n \) depend on the center frequency \( \omega_p \) of the incident quasimonochromatic field. We do not include additional orders of dispersion [2] because that is an exercise in complexity for a second-order consequence. Unless the radiation is of extraordinary intensity and duration, the velocity of the antireflection-coated material that is initially at rest in the Laboratory Frame of Reference will be minimal and neglecting the effects of the material motion on the permittivity, permeability, and refractive index is an “extremely accurate approximation indeed” [1].

Treating the continuum electrodynamics in lowest order, the permittivity, permeability, and refractive index are constant in time and a simple continuous function of location in space. The usual constitutive relations [18], in lowest order,
\[ \mathbf{D} = \varepsilon(\omega_p, \mathbf{r}) \mathbf{E} \quad (2.14a) \]
\[ \mathbf{B} = \mu(\omega_p, \mathbf{r}) \mathbf{H} \quad (2.14b) \]
\[ n = \sqrt{\varepsilon(\omega_p, \mathbf{r})/\mu(\omega_p, \mathbf{r})}, \quad (2.14c) \]
where the permittivity, permeability, and refractive index are evaluated at the center frequency \( \omega_p \) of the quasimonochromatic field, are additional axioms for the theoretical model of our system. The material parameters are piecewise homogeneous (homogeneous material and homogeneous vacuum) except for a short smooth transition modeled as a gradient-index antireflection coating.

Although it is always possible to start with a fully microscopic model of the field and material, any microstructure of the field and medium has been eliminated in the continuum limit. The continuous material and continuous field cannot be reliably un-averaged or re-quantized at the microscopic scale once the continuum limit is invoked.

The electromagnetic continuity equations, Eqs. (2.12), can be written row-wise as a single differential equation [3, 32] using the constitutive relations, Eqs. (2.14). We can write Eq. (2.12b) in component form as
\[ \frac{\partial (\mathbf{D} \times \mathbf{B})}{\partial (ct)} + \sum_j \frac{\partial}{\partial x_j} W^{ij} = -\frac{\varepsilon E^2}{2} \nabla \varepsilon - \frac{\mu H^2}{2} \nabla \mu. \quad (2.15) \]

Then Eqs. (2.12a) and (2.1b) can be written as a single equation [2],
\[ \partial_\beta T^{ab}_M = f^a_M, \quad (2.16) \]
where
\[ T^{ab}_M = \begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix} \quad (2.17a) \]
\[ W^{ij} = -D^i E^j - B^i H^j + \frac{1}{2}(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) \delta^{ij} \quad (2.17b) \]
\[ f^a_M = \left( 0, -\frac{\varepsilon E^2}{2} \nabla \varepsilon - \frac{\mu H^2}{2} \nabla \mu \right) \quad (2.17c) \]
\[ \partial_\beta \left( \frac{\partial}{\partial (ct)} \right) = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right). \quad (2.17d) \]

Here, Eq. (2.10) is explicitly proved to be a formal (axiomatic) theorem of the sourceless electromagnetic continuity equations, Eqs. (2.12), and the constitutive relations, Eqs. (2.14). Moreover, Eq. (2.16) has been derived as a formal theorem of the Maxwell–Minkowski equations, Eqs. (2.1), constitutive relations, Eqs. (2.14), and sources (2.11), by way of the electromagnetic continuity equations, Eqs. (2.12). The Minkowski energy–momentum tensor, Eq. (2.17a), has been formally derived for a neutral simple linear medium from the Maxwell–Minkowski equations and constitutive relations as the main component of the theorem, Eq. (2.16).

In the limit that the gradient-index antireflection coating is sufficiently smooth that we can neglect reflection, then the Minkowski force \( f^a_M \) becomes negligible such that Eq. (2.16) becomes
\[ \partial_\beta T^{ab}_M = 0, \quad (2.18) \]
which is widely known as the electromagnetic conservation law \[9\]. The vanishing right-hand side of Eq. \[2.18\] proves that there is no macroscopic force from the macroscopic electromagnetic system acting on the medium. The macroscopic medium remains kinematically stationary in the local frame of reference. Absent reflection, which is the case here, the electromagnetic energy and the Minkowski momentum constitute a Lorentz four-vector \((U_{em}/c, -G_{M})\) \[2, 3, 6, 11\], and the electromagnetic subsystem is thermodynamically closed.

Except, the Minkowski momentum, Eq. \[1.4\], that is obtained from the Minkowski energy–momentum tensor in the electromagnetic conservation law for the finite quasimonochromatic field in a linear medium is a factor of \(n\) greater than the momentum of the same field that is incident from the vacuum, Eq. \[1.5\]. Then Minkowski linear momentum is not globally conserved, by a non-negligible factor of \(n\) \[6, 11, 18, 21–24\], and the electromagnetic energy and the Minkowski momentum do not constitute a Lorentz four-vector.

### III. ENERGY AND MOMENTUM CONSERVATION

In the continuum limit, the fields are all macroscopic fields. The total energy and total momentum are both quadratic in the macroscopic fields and both must have the same dependence on the macroscopic parameters, \(\varepsilon\), \(\mu\), and \(n\) in order to be simultaneously conserved.

The electromagnetic fields can be written in terms of the vector potential \(A\) as

\[
E = -\nabla \phi - \frac{\partial A}{\partial (ct)}
\]

\[
B = \nabla \times A.
\]

In the absence of sources, \(\rho_f\) and \(J_f\), we can use the Coulomb gauge in which \(\phi = 0\). We then apply the envelope function \[18\]

\[
A(r, t) = \frac{1}{2} \left( \tilde{A}(r, t)e^{-i(\omega t - k_0 r)} + c.c. \right)
\]

(3.1)

to solve

\[
\frac{1}{2c} \int_{\Sigma} (D \cdot E + B \cdot H) \, dv = \int_{\Sigma} \frac{\zeta |E \times H|}{c} \, dv
\]

(3.2)

for the unknown factor \(\zeta\).

We obtain the nominal factor

\[
\zeta = n
\]

(3.3)

for fields with slowly varying envelopes, i.e., quasimonochromatic fields, propagating through an antireflection coated, sourceless, transparent, continuous, linear magneto-dielectric medium that is initially at rest in the local frame.

As discussed above Eq. \[1.5\], the amplitudes of the electric and magnetic fields in the medium are different from their amplitudes in the vacuum, but the changes offset such that the product \(EH\) remains constant. Using the solution \(\zeta = n\) of Eq. \[3.1\], the momentum

\[
G_{tot} = \int_{\Sigma} \frac{nE \times H}{c} \, dv
\]

(3.4)

is the globally conserved momentum counterpart of the globally conserved energy for quasimonochromatic fields.

The momentum density of the field inside the medium

\[
g_{tot} = \int_{\Sigma} \frac{nE \times H}{c} \, dv
\]

(3.5)

is a factor of \(n\) greater than the momentum density of the incident field in the vacuum \[6, 11, 18, 21–24\]. Integrating the enhanced momentum density, Eq. \[3.5\], over the narrower pulse when the field is inside the medium proves that the momentum, Eq. \[3.4\], is globally conserved.

Global conservation of the momentum, Eq. \[3.4\], in the absence of any significant reflection, was previously demonstrated by the current author using a finite-difference time-domain numerical solution of the wave equation with numerical integration of the electromagnetic momentum, Eq. \[3.4\], of the propagated field at various points in time \[21\].

Global conservation of momentum for an antireflection coated linear medium is consistent with global conservation of energy. The electromagnetic energy density in the linear medium, Eq. \[2.3\], is well-known to be a factor of \(n\) greater than the energy density in the vacuum. Integrating the enhanced energy density over the narrower pulse demonstrates global conservation of the electromagnetic energy, Eq. \[2.7\]. The enhanced energy density is due to the narrower pulse, not to a material energy density. Similarly, the enhanced momentum density, Eq. \[3.5\], is due to the narrower pulse, not to a material-motion momentum density. Experiments, which are typically performed with cw fields, and the definition of photons in the medium must take the reduced volume of the field in the medium into account.

The difference between global conservation of energy and global conservation of momentum is that reflection changes the sign of momentum, but not energy. If part of the field is reflected, we must impute a momentum to the material that is twice the magnitude of the momentum of the reflected field and in the direction of propagation. Then,

\[
G_{\text{forward}} = \int_{\Sigma^+} \frac{nE \times H}{c} \, dv
\]

(3.6a)

\[
G_{\text{backward}} = -\int_{\Sigma^-} \frac{E \times H}{c} \, dv
\]

(3.6b)

\[
G_{\text{mat}} = 2 \int_{\Sigma^-} \frac{E \times H}{c} \, dv
\]

(3.6c)
where \( \Sigma^- \) is the vacuum side of the plane of incidence \((n = 1)\) and \( \Sigma^+ \) is the space on the material side of the plane of incidence \((n)\). If any of the field has exited the medium, then there is additional reflection and additional material momentum. As long as the refracted field remains in the medium, the total, field plus material, momentum is

\[
G_{\text{tot}} = G_{\text{forward}} + G_{\text{backward}} + G_{\text{mat}}.
\]  

(3.7)

If reflection is negligible, as is the main case here, there is no need to assign a kinematic momentum to the material and the kinetic energy is nil. The electromagnetic energy and electromagnetic momentum are simultaneously conserved and the electromagnetic system is thermodynamically closed such that

\[
G_{\text{tot}} = G_{\text{em}} = \int_{\Sigma} \frac{nE \times H}{c} dv.
\]  

(3.8)

IV. MATERIAL MOMENTUM APPROACH

In this section, we review the basic details of the way that the total energy–momentum tensor is presented in the scientific literature. The extant procedure is to add a phenomenological material momentum \( G_{\text{mat}} \) to the Minkowski momentum and write the tautology [7, 30]

\[
G_{\text{tot}} = G_M + G_{\text{mat}}.
\]  

(4.1)

such that the whole is the sum of the parts. This is part of a larger view of the total system as consisting of an electromagnetic subsystem and a material subsystem. The conservation law for the total system, Ref. [6] Eq. (10),

\[
\partial_\beta T^{\alpha \beta}_{\text{tot}} = \partial_\beta \left( T^{\alpha \beta}_M + T^{\alpha \beta}_{\text{mat}} \right) = 0,
\]  

(4.2)

is the tautological sum of the conservation law for the electromagnetic conservation law for the electromagnetic subsystem and the conservation law for the material motion subsystem. In this scenario, the two subsystems are assumed to be coupled by equal and opposite forces, \( \pm F_M \), in terms of the gradients of the permittivity and permeability.

The usual model for the material is the flow of particles of dust in the continuum limit. In their comprehensive review article, Pfeifer, Nieminen, Heckenberg, and Rubinsztein-Dunlop [6] present the total, electromagnetic plus dust, energy–momentum tensor

\[
T^{\alpha \beta}_{\text{tot}} = \left[ \frac{1}{2} (D \cdot E + B \cdot H) + \rho_0 c^2 \left( E \times H \right) + \rho_0 c^2 \right] W_{\text{tot}},
\]  

(4.3)

where \( \rho_0 \) is the density of the dust and \( W_{\text{tot}}^{ij} = W^{ij} + \rho_0 \rho v_i v_j \).

We apply the four-divergence operator to the total energy–momentum tensor, as shown in Eq. 4.2. The first row, \( \partial_\beta T^{\alpha \beta}_{\text{tot}} \), is the energy continuity equation

\[
\frac{1}{c} \left( E \cdot \frac{\partial D}{\partial t} + H \cdot \frac{\partial B}{\partial t} \right) + \nabla \cdot (E \times H) + \nabla \cdot \rho_0 c^2 v = 0.
\]  

(4.4)

Pfeifer, Nieminen, Heckenberg, and Rubinsztein-Dunlop, Ref. [6] Eq. (44), use global conservation of total momentum to show that

\[
\rho_0 v = (n - 1) \frac{E \times H}{c}.
\]  

(4.5)

Substituting Eq. (4.5) into Eqs. (4.3) and (4.4) produces

\[
T^{\alpha \beta}_{\text{tot}} = \left[ \frac{\tau (D \cdot E + B \cdot H) + \rho_0 c^2 (E \times H)}{n(E \times H)} \right] W_{\text{tot}},
\]  

(4.6)

and

\[
\frac{1}{c} \left( E \cdot \frac{\partial D}{\partial t} + H \cdot \frac{\partial B}{\partial t} \right) + \nabla \cdot (E \times H) = 0.
\]  

(4.7)

The total, field plus matter, momentum \( G_{\text{tot}} = \int_{\Sigma} c^{-1} (T^{10}_{\text{tot}}, T^{20}_{\text{tot}}, T^{30}_{\text{tot}}) dv, \) Eq. (3.3), is globally conserved. Note that the energy continuity equation, Eq. (4.4), obviously violates Poynting’s theorem. The next step makes it clear how much of a problem that is because the two nonzero terms in Eq. (4.7) depend on different powers of \( n \). Equation (4.7) is manifestly false.

Partanen, Häyrynen, Oksanen, and Tulikki [31] propose that photons couple to the material system to form mass-polariton quasiparticles. Assuming that the Minkowski momentum is conserved, they write the total, field plus matter, energy–momentum tensor as

\[
T^{\alpha \beta}_{\text{MP}} = \left[ \frac{n^2}{c} (D \cdot E + B \cdot H) n^2 (E \times H) \right] W_{\text{tot}}.
\]  

(4.8)

In this case, the energy conservation law \( \partial_\beta T^{\alpha \beta}_{\text{MP}} = 0 \) is self-consistent and is apparently consistent with Poynting’s theorem in a linear medium because the common factor \( n^2 \) can be factored out of the energy continuity equation \( \partial_\beta T^{\alpha \beta}_{\text{MP}} = 0 \):

\[
\frac{n^2}{c} \left( E \cdot \frac{\partial D}{\partial t} + H \cdot \frac{\partial B}{\partial t} \right) + \nabla \cdot n^2 (E \times H) = 0.
\]  

(4.9)

However, the mass-polariton energy–momentum tensor, Eq. (4.8), is false because the total, field plus matter, momentum \( G_{\text{MP}} = \int_{\Sigma} c^{-1} (T^{10}_{\text{MP}}, T^{20}_{\text{MP}}, T^{30}_{\text{MP}}) dv \) is equal to the Minkowski momentum and is not globally conserved, Eq. (3.3). The total energy \( U_{\text{tot}} = \int_{\Sigma} T^{10}_{\text{MP}} dv \) is not globally conserved either.

The electromagnetic plus dust tensor treatment of the total energy–momentum tensor and the mass polariton model are illustrative of the many theories that are based on adding a material subsystem energy–momentum tensor to the electromagnetic subsystem energy–momentum tensor (or adding a material momentum to the linear momentum). However, the two subsystem energy–momentum tensors do not necessarily add in the fashion...
of Eq. (12) if the dynamical laws of each subsystem are different. This can be illustrated by considering two non-interacting systems, one electromagnetic and one kinematic. The two conservation laws can be combined in accordance with Eq. (12) with zero external forces on the two systems, but the four-divergence of the sum energy–momentum tensor mixes the closed systems together.

V. CONCLUSION

The Maxwell–Minkowski equations are not fundamental laws of physics, they are macroscopic equations that are phenomenologically extrapolated from a perturbation of the fundamental microscopic Maxwell equations. The non-negligible factor of \( n \) difference between the Minkowski linear momentum and the globally conserved total momentum in this simplest-case thermodynamically closed system proves that there is an essential contradiction between global conservation of the total momentum and the electromagnetic conservation law that is derived as an identity of the Maxwell–Minkowski equations and constitutive relations in a closed system that consists of a quasimonochromatic field, initially in vacuum, incident on a gradient-index antireflection-coated, sourceless, simple linear magneto-dielectric material, initially at rest in the vacuum local frame. The disproof of the electromagnetic conservation law by global conservation of energy and momentum in a thermodynamically closed system proves that the axioms, the Maxwell–Minkowski equations and constitutive relations, are formally false.

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