Order parameter description of the Anderson-Mott transition

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Abstract

An order parameter description of the Anderson-Mott transition (AMT) is given. We first derive an order parameter field theory for the AMT, and then present a mean-field solution. It is shown that the mean-field critical exponents are exact above the upper critical dimension. Renormalization group methods are then used to show that a random-field like term is generated under renormalization. This leads to similarities between the AMT and random-field magnets, and to an upper critical dimension $d_c^+ = 6$ for the AMT. For $d < 6$, an $\epsilon = 6 - d$ expansion is used to calculate the critical exponents. To first order in $\epsilon$ they are found to coincide with the exponents for the random-field Ising model. We then discuss a general scaling theory for the AMT. Some well established scaling relations, such as Wegner’s scaling law, are found to be modified due to random-field effects. New experiments are proposed to test for random-field aspects of the AMT.

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I. INTRODUCTION

Metal-insulator transitions of purely electronic origin, i.e. those for which the structure of the ionic background does not play an essential role, are commonly divided into two categories. In one category the transition is triggered by electronic correlations, in the other it is driven by disorder. The first case is known as a Mott transition, and the second one as an Anderson transition. It is believed that for many real metal-insulator transitions both correlations and disorder are relevant, and this expectation is also borne out by model studies of disordered interacting electron systems. The resulting quantum phase transition, which carries aspects of both types of transitions, we call an Anderson-Mott transition (AMT).

Most of the theoretical work to date on the AMT has been based on a small disorder expansion near $d = 2$, which is believed to be the lower critical dimension for the AMT. A generalized matrix nonlinear $\sigma$-model (NL$\sigma$M) has proven very useful for this purpose. This model is a generalization of Wegner’s theory for the Anderson transition in noninteracting electronic systems. It has led to the identification of various universality classes for the AMT, as well as to an $\epsilon = d - 2$ expansion for the critical exponents. Complementary work has recently been done by adding disorder to models which display a Mott transition. In the limit of infinite dimensions the resulting models have been studied in some detail.

From a standard phase transition point of view these approaches to the AMT are quite unconventional. The standard procedure for solving phase transition problems is to (1) identify the relevant order parameter (OP), and possibly derive or postulate an effective field theory for the OP, (2) construct a Landau, or mean-field description of the phase transition, and (3) use renormalization group (RG) methods to identify the upper critical dimension, $d_+^c$, and to compute the critical exponents in an $\epsilon = d - 2$ expansion. RG methods can also be used to derive a general scaling description of the phase transition for $d < d_+^c$.

Early attempts to implement this standard program for the Anderson transition failed, because the most obvious simple OP, viz. the single-particle density of states (DOS) at the Fermi level, turned out to be uncritical. As a result, the Anderson transition can only be described by the NL$\sigma$M approach, or possibly in terms of a complicated functional OP. The situation is, however, fundamentally different at the AMT, where the DOS is generally believed to be critical, and to vanish at the transition. This raises the prospect of using the DOS for a conventional OP description of the AMT, making the AMT conceptually simpler than the Anderson transition.

In two previous short publications we have shown that these expectations are indeed justified, and have discussed such an OP approach to the AMT. The purpose of the present paper is to explain the technical details of the OP approach, and to present a number of additional results.

One of the most far-reaching implications of our approach is that the AMT is in some respects similar to magnetic transitions in random fields. We will present the technical reasons underlying this conclusion in Sec. below. Here we argue that a random-field structure of the theory could have been anticipated on general physical grounds. Let us consider a model of an interacting disordered electron gas. In terms of anticommuting Grassmann fields, $\bar{\psi}$ and $\psi$, the action can be written
\[ S = -\sum_{\sigma} \int dx \bar{\psi}_\sigma(x) \left[ \frac{\partial}{\partial \tau} - \frac{1}{2m} \nabla^2 - \mu + u(x) \right] \psi_\sigma(x) \]
\[ -\frac{\Gamma}{2} \sum_{\sigma_1\sigma_2} \int dx \bar{\psi}_{\sigma_1}(x) \bar{\psi}_{\sigma_2}(x) \psi_{\sigma_2}(x) \psi_{\sigma_1}(x) . \]  

Here \( x \equiv (x, \tau) \) with \( \tau \) denoting imaginary time, \( \int dx \equiv \int dx \int_{0}^{1/T} d\tau \), \( m \) is the electron mass, \( \mu \) is the chemical potential, \( \sigma \) is a spin label, and for simplicity we have assumed an instantaneous point-like electron-electron interaction with strength \( \Gamma \). \( u(x) \) is a random potential which represents the disorder. We assume \( u \) to be \( \delta \)-correlated, and to obey a Gaussian distribution with second moment
\[ \{u(x)u(y)\} = \frac{1}{2\pi N_F \tau_{el}} \delta(x - y) \]  

where the braces denote the disorder average, \( N_F \) is the bare DOS per spin at the Fermi energy, and \( \tau_{el} \) is the bare elastic mean-free time. For future reference we denote the disorder related part of the action by
\[ S_{\text{dis}} = -\sum_{\sigma} \int dx \ u(x) \bar{\psi}_\sigma(x) \psi_\sigma(x) = -\sum_{\sigma,n} \int dx \ u(x) \bar{\psi}_{\sigma,n}(x) \psi_{\sigma,n}(x) . \]

In the second equality in Eq. (1.2) a Matsubara frequency decomposition of \( \bar{\psi}(\tau) \) and \( \psi(\tau) \) has been used.

As mentioned above, the most obvious OP for the AMT is the single-particle DOS, \( N \), at the Fermi level. In terms of Grassmann variables this quantity is proportional to the zero-frequency limit of the expectation value of the composite fermionic variable \( \bar{\psi}\psi \):
\[ N = \text{Im} N(i\omega_n \to 0 + i0), \]
with
\[ N(i\omega_n) = \frac{-1}{2\pi N_F} \sum_{\sigma} \langle \bar{\psi}_{\sigma,n}(x) \psi_{\sigma,n}(x) \rangle , \]

where we have normalized the DOS by \( 2N_F \). Equations (1.2, 1.3) suggest that the OP for the AMT couples directly to the random potential \( u \), and that this random field (RF) term is structurally identical to the one that appears in magnetic RF problems. Notice that this structure appears in both interacting and noninteracting localization theories. However, in the interacting case there is an additional physical feature: The interaction will in general favor a local electron arrangement that is different from the one favored by the random potential. This kind of frustration is crucial for the generation of the typical RF effects which are known from RF magnets. One should therefore expect RF effects at the AMT, but not necessarily at the Anderson transition.

This conclusion has several important implications. For instance, one expects hyperscaling to be violated at the AMT due to a dangerous irrelevant variable, as it is in RF magnets. We will see below that this is indeed the case and that, as a result, Wegner’s scaling law relating the conductivity exponent, \( s \), to the correlation length exponent, \( \nu \), is modified.

The plan of this paper is as follows. In Sec. II we review the NL\( \sigma \)M approach to the AMT, and use it to derive an OP field theory for the transition. In Sec. III the OP field
theory is solved in the saddle point or mean-field approximation. We show that the mean-field solution is stable for \( d > d^+_c \), and that the mean-field exponents are exact in this regime. In Sec. \[ IV \] RG methods are used to prove that the theory has RF aspects, to establish that \( d^+_c = 6 \), and to set up an \( \epsilon = d - 6 \) expansion for the critical exponents. In Sec. \[ IV \] we discuss a general scaling description of the AMT. We conclude in Sec. \[ VI \] with a summary of the paper and a discussion of some open problems as well as of experimental implications of our results.

II. FORMALISM

In this section we start by briefly reviewing the field theoretic description of disordered interacting electron systems. We focus on the NLσM, which is a free field theory with nonlinear constraints relating massive and massless modes in the ordered phase, which in the case of the AMT is the metallic one. We then integrate out the massless excitations to obtain a field theory solely in terms of the massive modes, which are directly related to the OP for the transition, i.e. the DOS. This is different from the usual treatment of the NLσM, which integrates out the massive fluctuations to obtain a theory for the soft modes. Our procedure is closely analogous to the treatment of the \( O(n) \) symmetric NLσM in the limit of large \( n \).

A. The Model

We will start from the action given by Eq. (1.1a). The NLσM for the AMT can be derived from Eq. (1.1a) by assuming that all of the relevant physics can be expressed in terms of long-wavelength and low-frequency fluctuations of the number density, the spin density, and the single-particle spectral density. Technically this is achieved by making long-wavelength approximations, and by introducing classical composite variables that are related to the operators mentioned above. The quenched disorder is handled by means of the replica trick. The resulting action reads,

\[
S[\tilde{Q}] = -\frac{1}{2G} \int dx \, tr \left( \nabla \tilde{Q}(x) \right)^2 + 2H \int dx \, tr \left( \Omega \tilde{Q}(x) \right)
- \frac{\pi T}{4} \sum_{u,s,t} \int dx \, [\tilde{Q}(x) \gamma^{(u)} \tilde{Q}(x)] ,
\]

(2.1a)

where

\[
[\tilde{Q}(x) \gamma^{(s)} \tilde{Q}(x)] = K_s \sum_{n_1,n_2,n_3,n_4} \delta_{n_1+n_3,n_2+n_4} \sum_{\alpha} \sum_{r=0,3} (-)^r \times tr \left( (\tau_r \otimes s_0) \tilde{Q}_{n_1,n_2}^{\alpha\alpha} \right) tr \left( (\tau_r \otimes s_0) \tilde{Q}_{n_3,n_4}^{\alpha\alpha} \right) ,
\]

(2.1b)

and

\[
[\tilde{Q}(x) \gamma^{(t)} \tilde{Q}(x)] = -K_t \sum_{n_1,n_2,n_3,n_4} \delta_{n_1+n_3,n_2+n_4} \sum_{\alpha} \sum_{r=0,3} (-)^r \sum_{i=1}^{3} \times tr \left( (\tau_r \otimes s_i) \tilde{Q}_{n_1,n_2}^{\alpha\alpha} \right) tr \left( (\tau_r \otimes s_i) \tilde{Q}_{n_3,n_4}^{\alpha\alpha} \right) ,
\]

(2.1c)
Here \( \tilde{Q} \) is a classical field that is, roughly speaking, composed of two fermionic fields. It carries two Matsubara frequency labels, \( n \) and \( m \), and two replica labels, \( \alpha \) and \( \beta \). The matrix elements \( \tilde{Q}_{nm}^{\alpha\beta} \) are spin quaternions, with the quaternion degrees of freedom describing the particle-hole (\( \tilde{Q} \sim \bar{\psi}\psi \)) and particle-particle (\( \tilde{Q} \sim \bar{\psi}\bar{\psi} \)) channels, respectively. We will restrict ourselves to the particle-hole degrees of freedom. The matrix elements can then be expanded in a restricted spin-quaternion basis,

\[
\tilde{Q}_{nm}^{\alpha\beta} = \sum_{r=0,3} \sum_{i=0}^3 i^r \tilde{Q}_{nm}^{\alpha\beta} (\tau_r \otimes s_i),
\]

with \( \tau_{0,1,2,3} \) the quaternion basis, and \( s_{0,1,2,3} \) the spin basis (\( s_{1,2,3} \) are the Pauli matrices). The matrix \( \tilde{Q} \) is subject to the nonlinear constraints,

\[
\tilde{Q}^2 = 1,
\]

\[
tr \tilde{Q} = 0,
\]

\[
\tilde{Q}^\dagger = C^T \tilde{Q}^T C = \tilde{Q}.
\]

The last equation expresses the requirements of hermiticity and charge conjugation; the matrix \( C \) is block-diagonal with elements \( i \tau_1 \otimes s_2 \). Equation (2.3c) implies,

\[
i^r \tilde{Q}_{nm}^{\alpha\beta} = (-)^r S_i \tilde{Q}_{nm}^{\alpha\beta},
\]

with \( S_0 = 1 \) and \( S_{1,2,3} = -1 \).

In Eqs. (2.4), \( G = 2/\pi \sigma \), with \( \sigma \) the bare conductivity, is a measure of the disorder, and \( H = \pi N F / 2 \) is a frequency coupling parameter. \( K_s \) and \( K_t \) are bare interaction amplitudes in the spin singlet and spin triplet channels, respectively, and \( \Omega_{nm}^{\alpha\beta} = \delta_{nm} \delta_{\alpha\beta} (\tau_0 \otimes s_0) \omega_n \), with \( \omega_n = 2\pi T n \), is a bosonic frequency matrix. Notice that \( K_s < 0 \) for repulsive interactions.

The correlation functions of the \( \tilde{Q} \) determine the physical quantities. Correlations of \( \tilde{Q}_{nm} \) with \( nm < 0 \) determine the diffusive modes which describe charge, spin, and heat diffusion, while the DOS is determined by \( \langle \tilde{Q}_{nm}^{\alpha\alpha} \rangle \), cf. Eq. (3.3a) below. It is therefore convenient to separate \( \tilde{Q} \) into blocks,

\[
\tilde{Q}_{nm}^{\alpha\beta} = \Theta(nm) Q_{nm}^{\alpha\beta}(x) + \Theta(n)\Theta(-m) q_{nm}^{\alpha\beta}(x) + \Theta(-n)\Theta(m) (q^\dagger)_{nm}^{\alpha\beta}(x).
\]

Normally a NL\( \sigma \)M is treated by integrating out the massive modes, i.e. the \( Q_{nm} \) in this case, to obtain an effective theory for the massless modes, which here are the diffusion processes described by \( q \) and \( q^\dagger \). However, since our goal is to obtain a field theory for the OP for the AMT, \( Q_{nn} \), we will proceed differently, and instead integrate out the massless \( q \)-fields. To this end, we use a functional integral representation of the delta function,
\[
\prod_x \delta[\tilde{Q}^2(x) - 1] = \int D[\Lambda] \exp\left\{-\frac{1}{2G} \int dx \tr (\Lambda(x)[\tilde{Q}^2(x) - 1])\right\}, \tag{2.6}
\]

where \(\tr\) denotes a trace over all discrete indices, and the factor of \(1/2G\) has been inserted for convenience. Together with Eq. (2.1a) this allows us to write the action as,

\[
S_1[\tilde{Q}, \Lambda] = -\frac{1}{2G} \int dx \tr \left[\Lambda(x)[\tilde{Q}^2(x) - 1] + \left(\partial_x \tilde{Q}(x)\right)^2\right] + 2H \int dx \tr \left(\Omega \tilde{Q}(x)\right) - \frac{\pi T}{4} \sum_{u=s,t} [\tilde{Q}(x)\gamma^{(u)} \tilde{Q}(x)], \tag{2.7}
\]

We note that the constraint given by Eq. (2.3b) does not involve \(q\), so it can be imposed later. Furthermore, the contraints expressed by Eqs. (2.3c) or (2.4) do not couple independent blocks of \(\tilde{Q}\). Finally, by decomposing \(\Lambda\) into blocks like \(\tilde{Q}\) one sees that the elements \(\Lambda_{nm}\) with \(nm > 0\), together with the tracelessness condition, Eq. (2.3b), are sufficient for enforcing the constraint \(\tilde{Q}^2 = 1\). We can therefore restrict ourselves to \(\Lambda_{nm}\) with \(nm > 0\).

The partition function of the replicated theory (\(N\) replicas) is given by,

\[
Z^N = \int D[\tilde{Q}] \exp(S[\tilde{Q}]) = \int D[\tilde{Q}] D[\Lambda] \exp(S_1[\tilde{Q}, \Lambda]) = \int D[Q] D[\Lambda] \exp(S_2[Q, \Lambda]) \tag{2.8}
\]

In the last equality in Eq. (2.8) we have defined yet another action,

\[
S_2[Q, \Lambda] = \ln \int D[q, q'] \exp(S_1[\tilde{Q}, \Lambda]) \tag{2.9}
\]

by integrating out the massless \(q\)-fields. Since the action \(S_1\), Eq. (2.7), is quadratic in these fields, this can be done exactly. Formally one obtains,

\[
S_2[Q, \Lambda] = S_1[Q, \Lambda] - \frac{1}{\mathcal{Z}} T \tr \ln M[\Lambda] \tag{2.10a}
\]

Here \(T\tr\) denotes a trace over all degrees of freedom, and \(M\) is a matrix with elements,

\[
\frac{\delta_{\alpha_2 \alpha_4} \delta_{\mu_2 \mu_4}}{2G} \sum_{\tau_3, \tau_3} \tr \left(\tau_3 \tau_1 \tau_2^\dagger\right) \tr \left(s_{i_3} s_{i_1} s_{t_2}^\dagger i_{3 \tau_3} \Lambda_{\mu_3 \mu_1}^{\alpha_3 \alpha_1}(x)\right) + \frac{\delta_{\alpha_1 \alpha_3} \delta_{\mu_1 \mu_3}}{2G} \sum_{\tau_3, \tau_3} \tr \left(\tau_1 \tau_3 \tau_2^\dagger\right) \tr \left(s_{i_1} s_{i_3} s_{t_2}^\dagger i_{3 \tau_3} \Lambda_{\mu_2 \mu_4}^{\alpha_2 \alpha_4}(x)\right)
\]

\[
+ \frac{4}{\mathcal{Z}} G \delta_{\alpha_1 \alpha_3} \delta_{\alpha_2 \alpha_4} \delta_{\tau_1 \tau_2} \delta_{\nu_1 \tau_2} \left[\delta_{\mu_1 \mu_3} \delta_{\mu_2 \mu_4} \left(-\nabla^2 + \delta_{\alpha_1 \alpha_2} \delta_{\mu_1 \mu_4} \delta_{\mu_2 \mu_3} 2\pi TGK_{\nu_1}\right)\right], \tag{2.10b}
\]

with \(\nu_0 = s\) and \(\nu_{1,2,3} = t\). In giving Eq. (2.10a) we have omitted some terms which result from the interaction, i.e. the last term in Eq. (2.7), coupling \(Q\) and \(q\). These terms will turn out to be irrelevant (in the RG sense) because they are of higher order in the frequency.
B. The Order Parameter Field Theory

The field theory given by Eqs. (2.10) has been formulated in terms of the OP for the AMT and the auxiliary, or ghost, field \( \Lambda \). In order to obtain a theory solely in terms of \( Q \) we need to integrate out \( \Lambda \). We will do this perturbatively, and verify later that the neglected terms are either RG irrelevant, or provide analytic corrections to nonuniversal coefficients in the theory. Either way they do not modify the critical behavior. Physically this is to be expected, since it is easily established that the \( \Lambda \)-modes are not critical near the AMT.

To proceed we write the average of \( \Lambda \) as

\[
\langle i_r \Lambda_{\alpha\beta}^{nm}(x) \rangle = \delta_{r0} \delta_{i0} \delta_{nm} \ell_n ,
\]

with \( \ell_n \) to be determined later. We write

\[
i_r \Lambda_{\alpha\beta}^{nm}(x) = \langle i_r \Lambda_{\alpha\beta}^{nm}(x) \rangle + i_r \psi_{\alpha\beta}^{nm}(x) ,
\]

and expand in powers of \( \psi \) to determine a new OP action,

\[
S_3[Q] = \ln \int D[\psi] \exp(S_2[Q, \langle \Lambda \rangle + \psi]) .
\]

Integrating over \( \psi \) in Gaussian approximation, neglecting constant contributions to the action, and neglecting explicit interaction terms (see the end of this subsection for a justification), we obtain

\[
S_3[Q] = -\frac{1}{2G} \int dx \ tr \left[ (\nabla Q(x))^2 + \langle \Lambda \rangle Q^2(x) \right] + \frac{u}{2G^2} \int dx \ tr \left[ (1 - f)Q^2(x) \right] - \frac{u}{4G^2} \int dx \ tr \left[ Q^4(x) \right] - \frac{v}{4G^2} \int dx \left( tr_+ Q^2(x) \right) \left( tr_- Q^2(x) \right) ,
\]

where \( tr_\mp \) denotes 'half-traces' that sum over all replica labels but only over positive and negative frequencies, respectively: \( tr_+ = \sum_\alpha \sum_{n \geq 0} \), \( tr_- = \sum_\alpha \sum_{n < 0} \). \( f = f(\langle \Lambda \rangle) \) is a matrix with elements

\[
i_r f_{\alpha\beta}^{nm}(x) = \delta_{r0} \delta_{i0} \delta_{nm} f_n \text{ with,}
\]

\[
f_n = -\frac{G}{4} \int_p \sum_{m=-1}^{-\infty} \frac{2\pi T G}{(p^2 + \frac{1}{2} (\ell_n + \ell_m))^2} \sum_{i=0}^{3} K_{\nu_i} \left[ 1 + \sum_{n_1=0}^{n-m-1} \frac{2\pi T G K_{\nu_i}}{p^2 + \frac{1}{2} (\ell_{n_1} + \ell_{n_1-n+m})} \right]^{-1} ,
\]

for \( n \geq 0 \), where \( f_p = (2\pi)^{-d} \int dp \). For \( n \leq 0 \), the same expression holds with the sum over \( m \) from 0 to \( \infty \), and with \( n \) and \( m \) interchanged in the summand. The coefficients \( u \) and \( v \) in Eq. (2.13) are given by,

\[
u = -G/(df_n/d\ell_n)_{n=0} , \quad (2.15a)
\]

and

\[
u = -u^2 \int_p \frac{1}{p^2} . \quad (2.15b)
\]

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In writing Eq. (2.13) we have localized the coefficients \( u \) and \( v \) in space and imaginary time. Frequency and wavenumber dependent corrections to this result turn out to either renormalize nonuniversal coefficients, or to be RG irrelevant. Note that \( v \) is infinite for \( d \leq 4 \). \( u \) also contains singularities in \( d \leq 4 \). In the main text of this paper we restrict ourselves to the region \( d \approx d_c^+ = 6 \), and ignore these singularities. In Appendix A we show that, at the mean-field level, they are of no consequence for the critical behavior as long as \( d > 3 \). Beyond the mean-field level, these singularities are one of several severe difficulties one encounters. This point will be further discussed in Sec. VII. Finally, we note that our integrating out of \( \Lambda \) has obscured the condition
\[
\langle \psi \rangle = 0 ,
\]
which follows from Eq. (2.11b). By returning to the action \( S_2 \) one sees that Eq. (2.16a) implies,
\[
0 = 1 - f_n(\langle \Lambda \rangle) - \sum_{\beta, m, r, i} (-)^r S_i \langle i^{\alpha} Q_{\alpha \beta}^{\beta \alpha}(x) i^{\beta} Q_{\alpha \beta}^{\beta \alpha}(x) \rangle + \ldots ,
\]
with \( S_i \) given after Eq. (2.14). Corrections to Eq. (2.16b) come from higher orders in the \( \psi \)-expansion. We will show below that they do not change the critical behavior.

We conclude this section by mentioning two approximations that were used to derive Eqs. (2.13) and (2.16). First, the coefficients given in Eq. (2.13) follow by expanding \( S_2 \) in Eq. (2.12) to \( O(\psi^2) \), neglecting all terms of higher order in \( \psi \). Second, all explicit interaction terms, such as the last terms in Eq. (2.1a), or the terms mentioned after Eq. (2.10b), have been neglected in Eq. (2.13). In the next section we will use RG methods to argue that these approximations are justified, at least in the vicinity of \( d_c^+ \).

### III. MEAN-FIELD THEORY

In this section we solve the field theory derived in Sec. II in the mean-field or saddle point approximation. We will show that the mean-field theory describes a phase transition, and that it is an AMT. We then show that the saddle point solution is locally stable, and that the mean-field exponents are exact for \( d > d_c^+ \).

#### A. Mean-Field Solution

We can use either Eqs. (2.10) or (2.13) – (2.16) to construct a mean-field theory. We choose to use the latter. We look for saddle point (SP) solutions \( Q_{sp} \), \( \Lambda_{sp} \) that are spatially uniform and satisfy
\[
i_r^{\alpha \beta} (Q_{sp})_{nm}^{\alpha \beta} = \delta_{r0} \delta_{i0} \delta_{nm} \delta_{\alpha \beta} N_r^{(0)} ,
\]
\[
i_r^{\alpha \beta} (\Lambda_{sp})_{nm}^{\alpha \beta} = \delta_{r0} \delta_{i0} \delta_{nm} \delta_{\alpha \beta} \ell_r^{(0)} ,
\]
where the superscript \( (0) \) denotes the SP approximation. The replica, frequency, and spin-quaternion structures in Eqs. (3.1) are motivated by the fact that \( \langle i^{\alpha} Q_{nm}^{\beta \alpha} \rangle \) and \( \langle i^{\beta} \Lambda_{nm}^{\beta \alpha} \rangle \) have
these properties (cf. Eq. (2.11a)), and that in the mean-field approximation averages are
replaced by the corresponding SP values.
Equations (3.1) and (2.16b) yield
\[(N_n^{(0)})^2 = 1 - f_n(\langle \Lambda \rangle) \quad (3.2a)\]
Taking the extremum of the action \(S_3\), Eq. (2.13), with respect to \(Q\) or \(N_n^{(0)}\) gives
\[\ell_n^{(0)} = 2GH\omega_n/N_n^{(0)} \quad (3.2b)\]
where we have used Eq. (3.2a).
Let us discuss some aspects of these results. First, the DOS at a frequency or energy \(\Omega\),
measured from the Fermi surface, and normalized by \(2N_F\) (cf. Eq. (1.3)), is given by,
\[N(\Omega) = \text{Re} \langle 0_0 Q^\alpha_{n\alpha} (x) |_{\omega_n \rightarrow \Omega+i0} \equiv \text{Re} N_n |_{\omega_n \rightarrow \Omega+i0} \quad (3.3a)\]
In mean-field approximation this yields
\[N(\Omega) = N_n^{(0)} |_{\omega_n \rightarrow \Omega+i0} \quad (3.3b)\]
Clearly, the DOS decreases with increasing disorder \(G\). Furthermore, \(N(\Omega)\) is nonanalytic
at \(\Omega = 0\), see Appendix A. If we insert Eqs. (3.2) into Eq. (3.3), and iterate to first order
in \(G\), then we just recover the well-known 'Coulomb anomaly' of the DOS. With further
increasing disorder, \(N^{(0)}(\Omega = 0)\) eventually vanishes at a critical value \(G_c\) of \(G\). The mean-
field approximation thus describes a phase transition with a vanishing DOS at the Fermi
level, the hallmark of the AMT. (Transport properties we will discuss shortly). Note that in
the absence of interactions the quantity \(f\), Eq. (2.14), vanishes, and hence \(N^{(0)} \equiv 1\). This
reflects the fact that the DOS does not vanish at an Anderson transition.
Second, it is easy to determine the behavior of \(N^{(0)}\) in the vicinity of \(G_c\). This is true
even though Eqs. (3.2) and (2.14) represent an integral equation for \(N_n^{(0)}\). The point is that
\(\ell_m\) in Eq. (2.14) gets integrated over all frequencies. At nonzero \(m\) it is given by \(\omega_m\) times
a finite, nonuniversal prefactor which depends on \(N_n^{(0)}\), see Eq. (3.2b). This implies that we
can treat \(\ell_m\) as a finite, nonuniversal quantity. The distance from the critical point, \(t\), is
proportional to \(G - G_c\). In mean-field approximation \(t\) is given by,
\[t^{(0)} = 1 - f_n^{(0)} \quad (3.4)\]
Near the critical point we have
\[N_n^{(0)} = (t^{(0)})^{1/2} \quad (3.5)\]
which yields the mean-field value of the critical exponent \(\beta\),
\[\beta = 1/2 \quad (3.6)\]
Note that the sign of \(N_n^{(0)}\) in Eq. (3.5) is determined by the way the zero-frequency limit is
taken: \(N_n\) is an odd function of \(n\). In this respect the external Matsubara frequency acts like
a symmetry breaking external field at the mean-field AMT. The DOS, of course, is positive
definite since in order to obtain it from \(N_n\) the branch cut must always be approached from
above, Eqs. (3.3). To put it another way, $N_n$ is $i$ times a causal function that is an odd function of complex frequency (viz. the Green function). $N(\Omega)$ is the spectrum of that function, which is even in $\Omega$.

Third, it is not much more difficult to obtain all other critical exponents. Let us write $Q$ as its expectation value plus fluctuations, as we did with $\Lambda$ in Eq. (2.11b),

$$iQ^{\alpha\beta}_{nm} = \delta_{r0}\delta_{i0}\delta_{\alpha\beta}\delta_{nm}N_n + \sqrt{2G}i\phi^{\alpha\beta}_{nm}, \quad (3.7)$$

where the factor of $\sqrt{2G}$ has been inserted for later convenience. The action governing Gaussian fluctuations about the mean-field solution in the critical region then follows from Eq. (2.13) as,

$$S_3[\phi] = -\int d\mathbf{x} \, tr \left[ (\nabla\phi(\mathbf{x}))^2 + \ell^{(0)}\phi^2(\mathbf{x}) + \frac{2u}{G} \ell^{(0)}\phi^2(\mathbf{x}) \right] - \frac{v}{2G}\ell^{(0)} \int d\mathbf{x} \left( tr_+\phi(\mathbf{x})(tr_-\phi(\mathbf{x})) + O(\phi^3) \right) \quad (3.8)$$

All remaining critical exponents can now be read off Eq. (3.8): Comparing the first and the third terms on the r.h.s. yields the correlation length exponent $\nu = 1/2$. With $\ell^{(0)} \sim \omega/Q$, the first and the second term give the dynamical exponent $z = 3$. (This result one can also confirm by explicitly calculating $N(\Omega \to 0)$ at $G = G_c$ from the mean-field equation, see Appendix A). Finally, inspection shows that the $\phi-\phi$ correlation function near the transition in the limit of small frequencies and small wavenumbers has a standard Ornstein-Zernike form, which yields exponents $\gamma = 1$ and $\eta = 0$. We thus have standard mean-field values for all static exponents,

$$\beta = \nu = 1/2 \quad , \quad \gamma = 1 \quad , \quad \eta = 0 \quad , \quad \delta = 3 \quad , \quad (3.9a)$$

and for the dynamical exponent we have,

$$z = 3 \quad . \quad (3.9b)$$

Inspection of Eq. (3.8) further shows that the AMT saddle point is a local minimum and therefore stable.

Apart from the quantities whose critical properties are governed by the exponents given in Eqs. (3.9) we are also interested in the transport properties, and in some other thermodynamic quantities. Let us first consider the charge diffusion coefficient, $D_c$. It can be obtained from the NL$\sigma$M by a direct calculation of the particle-hole spin-singlet $q-q$ correlation function. To this end we consider the action $S_1$, Eq. (2.7). If we replace $Q$ and $\Lambda$ by their mean-field values, Eqs. (3.1), (3.2), we get an action that is quadratic in $q$. Inversion of the corresponding matrix yields the $q-q$ propagator in mean-field approximation. It has a diffusive structure with diffusion constant

$$D_c = \frac{N^{(0)}_{n=0}}{GH + GK_0N^{(0)}_{n=0}} \quad , \quad (3.10a)$$

that is, $D_c$ vanishes like the OP. The same argument applied to the particle-hole spin-triplet $q-q$ propagator and to a $q-q$ correlation that is off-diagonal in replica space, which contain the
spin and heat diffusion coefficients $D_s$ and $D_h$, respectively, establish that these transport coefficients also vanish like the OP,

\[ D_s = \frac{N_{n=0}(0)}{GH + GK_1N_{n=0}(0)} \]  \hspace{1cm} (3.10b)

\[ D_h = \frac{N_{n=0}}{GH} \]  \hspace{1cm} (3.10c)

Finally, we note that the SP approximation is not in any way related to a systematic small disorder ($G$) expansion. As a consequence, it is in general not possible to relate a $G$-expansion of the SP theory to previous work on the NLσM, or to many-body perturbation theory. An exception is the $G$-expansion of the OP itself, see the discussion after Eq. (3.3b), and Eq. (A1).

B. The Mean-Field Transition as a Renormalization Group Fixed Point

Here we discuss the mean-field solution presented in the last subsection in terms of a RG fixed point (FP). We will see that the mean-field critical behavior is exact for \( d > d_c^+ \). We will also use the RG to derive scaling arguments which allow us to determine the critical behavior of thermodynamic susceptibilities, such as the specific heat coefficient and the density susceptibility $\partial n/\partial \mu$, and of the electrical conductivity.

Consider the action given by Eq. (2.13), or Eq. (3.8). Our parameter space is spanned by $\mu = \{c, t, H, K_{s,t}, u_1, u_2\}$. Here $c$, whose bare value is $1/2G$, is the coefficient of the gradient squared term in Eq. (2.13), and $t$ is a measure of the distance from criticality. Its bare value is $t(0)u/2G^2$ with $t(0)$ given by Eq. (3.4). $u_1 = u/4G^2$ and $u_2 = v/4G^2$ are the quartic coupling constants, $H$ is the frequency coupling, and $K_{s,t}$ are the interaction coupling constants. Since, in anticipation of the RG arguments below, we have dropped the explicit interaction terms from Eq. (2.13), this action contains $K_{s,t}$ only implicitly via the matrix f. The explicit interaction dependence is easily restored by adding to Eq. (2.13) the last term in Eq. (2.1a) with $\tilde{Q}$ replaced by $Q$. The contributions omitted from Eqs. (2.10) yield a term of the same structure. Structurally, Eq. (2.13) is just a matrix version of standard $\phi^4$-theory. Consequently, in looking for a mean-field FP, we follow Ref. [8] and [18]. We define the scale dimension of a length to be $-1$, and require the exponent $\eta$, i.e. the anomalous dimension of the $Q$-field, to be zero. The exact scale dimension of $Q$ is then $[Q] = (d-2)/2$. Power counting then yields the scale dimensions of the various coupling constants,

\[ [c] = 0 \]  \hspace{1cm} [t] = 2 \]  \hspace{1cm} [u_1] = [u_2] = 4 - d \]  \hspace{1cm} [HT] = (d+2)/2 \]  \hspace{1cm} [K_{s,t}] = 2

(3.11a)

Sometimes the dimension of temperature, $T$, or frequency, $\Omega$, is chosen to be $[T] = [\Omega] = d$, which results in $[H] = 1 - d/2$ and $[K_{s,t}] = 2 - d$. This choice lumps the anomalous part of the dynamical scaling exponent into the scaling behavior of $H$, and is convenient in considerations near the lower critical dimension $d_c^+ = 2$. Near the upper critical dimension, $d_c^+$, the scaling dimension of $H$ or its equivalent is usually chosen to be zero. Here we adopt this choice. Then
Here $\tilde{z}$ is a 'naive' dynamical scaling exponent. As we will see below, the effective dynamical exponent $z$ given in Eq. (3.9b) results from $\tilde{z}$ if the existence of dangerous irrelevant variables is taken into account. Equations (3.11a),(3.11d) give

$$[T] = [\Omega] = \tilde{z} = d/2 + 1 .$$  \hspace{1cm} (3.11b)

With either choice for $[T]$, $[K_{s,t}] < 0$ for $d > 2$, that is, the electron-electron interaction is irrelevant. This justifies, a posteriori, our omitting the interaction terms in Eq. (2.13). The same reasoning implies that the quartic coupling constants $u_{1,2}$ are irrelevant for $d > 4$. Terms generated by expanding the action $S_2$ in Eq. (2.12) to higher than second order in $\psi$ turn out to be even more irrelevant than $u_1$ and $u_2$, as we will see below. We thus have a Gaussian FP $\mu^* = (c,0,0,\ldots)$ which superficially appears to be stable for $d > 4$. The relevant parameters are $t$ and $T$ or $\Omega$, and the correlation length exponent is $\nu = 1/[t] = 1/2$. $c$ is marginal, and all other coupling constants are irrelevant. This Gaussian FP obviously corresponds to the saddle point solution discussed in the previous subsection.

The zero-loop RG analysis presented above would imply that the mean-field critical behavior is exact for $d > 4$, unless the RG, at some stage in the renormalization process, generated new terms in the action which are relevant for $d > 4$. From general power counting it follows that the only such terms one has to worry about are local terms of $O(Q^2)$ in Eq. (2.13). Such terms would be relevant and grow like $b^2$. We will see in the next section that this indeed happens, because random-field like terms of this type are generated by the renormalization process. The net result will be that the upper critical dimension is $d^c_\ast = 6$ rather than $d^c_\ast = 4$, but the mean-field critical behavior is still exact for $d > d^c_\ast = 6$. Also, important aspects of the scaling arguments that follow from the above zero-loop RG considerations will remain valid. Before we turn to a more sophisticated RG approach, we therefore discuss scaling.

The RG arguments given above imply that the OP satisfies a homogeneity relation,

$$N(t, \Omega, u_1, u_2, \ldots) = b^{1-d/2} N(tb^{1/\nu}, \Omega b^{\tilde{z}}, u_1b^{4-d}, u_2b^{4-d}, \ldots) \hspace{1cm} (3.12a)$$

with $\nu = 1/2$ and $\tilde{z} = d/2 + 1$. The exponents $\beta$ and $z$ follow from Eq. (3.12a) by means of standard arguments. The crucial point is that $u \sim u_1 \sim u_2$ is a dangerous irrelevant variable (DIV): Solving the saddle point equations explicitly yields $N(t, \Omega = 0, u \to 0) \sim u^{-1/2}$, and $N(t = 0, \Omega, u \to 0) \sim u^{-1/3}$. Taking this into account, Eq. (3.12a) yields $\beta = 1/2$ and $z = 3$ in agreement with Eqs. (3.9). In order to drop the $u_i$ from Eq. (3.12a) we have to change the scale dimension of $N$, $(d-2)/2$, and the exponent $\tilde{z}$, to their effective values 1 and $z$, respectively,

$$N(t, \Omega) = b^{-1} N(tb^{1/\nu}, \Omega b^z) \hspace{1cm} (3.12b)$$

Let us also determine the singular or critical parts $\chi_{\text{sing}}$ of the thermodynamic susceptibilities $\partial n/\partial \mu$, $\gamma$, and $\chi_s$, where $\gamma_V = \lim_{T \to 0} C_V/T$ is the specific heat coefficient, and $\chi_s$ is the spin susceptibility. To be specific, let us consider $\gamma_V$. The singular part of the free energy, $f_{\text{sing}}$, satisfies the relation

$$f_{\text{sing}}(t, T, u, \ldots) = b^{-(d+z)} f_{\text{sing}}(tb^{1/\nu}, Tb^\tilde{z}, ub^{4-d}, \ldots) \hspace{1cm} (3.13a)$$
In the critical region, $f_{\text{sing}} \sim uQ^4 \sim 1/u$, so that $u$ is a DIV for $f_{\text{sing}}$ as well. This implies,

$$f_{\text{sing}}(t, T) = b^{-(4+z)} f_{\text{sing}}(tb^{1/\nu}, Tb^z).$$

(3.13b)

Differentiating twice with respect to $T$ yields a homogeneity relation and the critical behavior for $\gamma_{V,\text{sing}}$.

More generally, we note that all of these thermodynamic susceptibilities scale like an inverse volume times a time, so their naive scale dimension is $d - \tilde{z}$. The DIV changes this to $4 - z = 1$, so that the result for $\gamma_{V,\text{sing}}$ actually holds for all of the $\chi_{\text{sing}}$, viz.,

$$\chi_{\text{sing}}(t, T) = b^{-(4-z)} \chi_{\text{sing}}(tb^{1/\nu}, Tb^z) = b^{-1} \chi_{\text{sing}}(tb^2, Tb^3),$$

(3.14a)

or,

$$\chi_{\text{sing}}(t, T = 0) \sim t^{1/2}, \quad \chi_{\text{sing}}(t = 0, T) \sim T^{1/3},$$

(3.14b)

where we have specialized to the mean-field exponents.

We next consider transport properties. The charge, spin, or heat diffusion coefficients, which we denote collectively by $D$, all scale like a length squared times a frequency, so that

$$D(t, \Omega) = b^{2-z} D(tb^{1/\nu}, \Omega b^z) = b^{-1} D(tb^2, \Omega b^3),$$

(3.15)

so the diffusion coefficients scale like the OP, Eq. (3.12b), in agreement with the explicit result, Eqs. (3.10). We are also interested in the scaling behavior of the electrical conductivity, $\sigma = D_c \partial n / \partial \mu$. The behavior of $\sigma$ depends on whether or not $\partial n / \partial \mu$ has an analytic background contribution in addition to the critical contribution given by Eqs. (3.14). In general it is hard to see why $\partial n / \partial \mu$, or any of the thermodynamic susceptibilities, should not have an analytic background contribution, although Ref. [13] has given some arguments why the background may be missing in the particular model under consideration. If $\partial n / \partial \mu$ has indeed no background term, then

$$\sigma(t, \Omega) = b^{-2} \sigma(tb^2, \Omega b^3).$$

(3.16a)

The conductivity exponent $s$, defined by $\sigma(t, \Omega = 0) \sim t^s$, is

$$s = 2\nu = 1.$$

(3.16b)

If $\partial n / \partial \mu$ does have an analytic background, then $\sigma$ will scale like the diffusion coefficient, so that

$$s = \nu = 1/2.$$

(3.16c)

We conclude this section by considering the terms of higher than quadratic order in $\psi$ that were neglected in deriving Eq. (2.13). Expanding the $Tr \ln$-term in the action $S_2$, Eq. (2.10a), in powers of $\psi$, and localizing each term both in real space and in imaginary time, leads to all possible powers of $\psi$. Schematically we denote these terms by

$$S_\psi = \sum_{n=1}^{\infty} w_n \int dx \; \psi^n(x).$$

(3.17)
where the $w_n$ are in general matrices.

Let us examine the Gaussian $\psi$-$\psi$ correlation function in momentum space. For $d > 4$ it goes to a constant at small momenta, for the same reason for which $v$ in Eq. (2.15b) is finite for $d > 4$. If we follow Ref. [18] in defining the scale dimension of a field via the behavior of its Gaussian propagator, then we obtain,

$$[\psi] = d/2 \quad .$$

(3.18)

We note that other choices for the scale dimension of $\psi$, e.g. $[\psi] = d - 2$ (motivated by $[Q] = (d - 2)/2$ and the fact that $\Lambda$ is conjugate to $Q^2$), are also possible, but the final result is independent of this choice.

We next observe that for fixed $d$, the $w_n$ are divergent if $n$ is large enough. Scaling wavenumbers with the correlation length $\xi$, we have $w_n \sim \xi^{2n-d}$ for $n \geq d/2$. Therefore, for $n \geq d/2$, $w_{n+1}/w_n \sim \xi^2$. Since each successive term in Eq. (3.17) carries one more power of $\psi$, we obtain for the scale dimensions of the $w_n$,

$$[w_n] = -\frac{d}{2}(n - 2) \quad \text{for} \quad n \leq d/2 \quad ,$$

(3.19a)

$$[w_{n+1}] = [w_n] - d/2 \quad \text{for} \quad n < d/2 \quad ,$$

(3.19b)

and

$$[w_{n+1}] = [w_n] - \frac{d - 4}{2} \quad \text{for} \quad n \geq d/2 \quad .$$

(3.19c)

The important conclusions from these power counting arguments are that all of the $w_n$ with $n > 2$ are RG irrelevant for $d > 4$, that for fixed $d$ they become more irrelevant as $n$ increases, and that they all become marginal in $d = 4$. Anticipating that we will conclude in the next subsection that $d_c^+ = 6$, this implies that we were justified in neglecting higher than second order terms in $\psi$, provided we work either in $d \geq d_c^+$, or perturbatively (in the sense of an $\epsilon$-expansion) in $d < \sim d_c^+$.

We conclude this section with three final comments. (1) It is obvious that the nonlocal gradient corrections to Eq. (3.17) are even more irrelevant than the leading terms studied. (2) If one chooses $[\psi] = d - 2$ instead of Eq. (3.18), then the r.h.s. of Eq. (3.19a) gets replaced by $-(n - 2)(d - 2) - (d - 4)$, and $d/2$ and $(d - 4)/2$ in Eqs. (3.19b, 3.19c) get replaced by $d - 2$ and $d - 4$, respectively. Obviously, all conclusions drawn above remain valid. (3) The power counting arguments employed so far do not distinguish between terms of different symmetry at a given order in powers of the field. The terms neglected in the $\psi$-expansion could lead to terms with different symmetries than the ones present in our $Q$-action, Eq. (2.13), and, because of complications to be discussed in the next subsection, such terms could in principle modify the universality classes for the AMT, even though they are irrelevant by power counting. We will show in Sec. [VI] and in Appendix [B] that all possible terms of this kind are generated by the RG anyway, starting with Eq. (2.13), and that they do not modify the critical behavior near $d = 6$. 

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IV. RENORMALIZATION, AND RANDOM FIELD ASPECTS

In this section we consider the renormalization of the field theory derived in Sec. II. First we show that a Wilson-type RG procedure generates additional terms in the action. The most interesting ones are of random-field (RF) type. Structurally they are identical to the terms that would have resulted if the original, unreplicated action had had terms like,

\[ S_{\text{RF}} = \sum_{n,\sigma} \int d\mathbf{x} \ h_n(x) \bar{\psi}_{\sigma,n}(x)\psi_{\sigma,n}(x), \]

with \( h_n \) a Gaussian random field with a second moment given by,

\[ \{ h_n(x) h_m(x) \} = \Theta(nm) \frac{\Delta^2}{4G} \delta(x - y). \]

Here \( \Delta/4G \) is the strength of the random field, with the factor \( 1/4G \) inserted for convenience. Standard arguments imply that such an RF term yields \( d + c = 6 \) instead of \( d + c = 4 \).

We then use a \( 6 - \epsilon \) expansion to describe the AMT in \( d = 6 - \epsilon \) dimensions. A stable RG fixed point is found, and the critical exponents are calculated to \( O(\epsilon) \).

A. Perturbation Theory, and the Renormalization Group

Let us consider Eq. (2.13) at criticality, where \( \langle \Lambda \rangle \) and \( \langle Q \rangle \) vanish at zero frequency. Consider the quartic term,

\[ S_4 = -\frac{u}{4G^2} \int d\mathbf{x} \ tr \ Q^4(x), \]

and the momentum-shell decomposition

\[ Q(x) = Q^< (x) + Q^> (x). \]

Here \( Q^> \) has only Fourier components between \( b^{-1} \) and unity, with \( b \) the RG length scale factor, and \( Q^< \) has only Fourier components between zero and \( b^{-1} \), where we have set an ultraviolet momentum cutoff equal to unity. Inserting Eq. (4.3) into Eq. (4.2), examining terms \( \sim (Q^<)^2 \), and averaging over the \( Q^> \) degrees of freedom, yields to \( O(u) \) a term \( S_{2,2} \),

\[ S_{2,2} = \frac{u}{8G} \int_{k> \bar{k}} \frac{1}{k^2} \int d\mathbf{x} \sum_{i=\pm} (tr_i Q^< (x))^2. \]

Here \( f_{k,> \bar{k}} \) denotes an integral over the region \( b^{-1} \leq k \leq 1 \). Examining the structure of \( S_{2,2} \) we see that it is indeed a RF like term, generated by renormalization at one-loop order.

Similarly, if we consider terms of \( O(u^2) \), we find that renormalization generates new types of quartic terms such as \( (tr_+ Q^2)(tr_- Q^2) \) and \( (tr_+ Q^2)^2 \). Physically, these new contributions reflect the fact that in general all of the terms in the Landau theory of the AMT are random. For example, a random quadratic term leads to \( (tr_+ Q^2)^2 \). All of these quartic terms are RG irrelevant for \( d > 4 \). However, one of the main technical points about RF problems is that the coupling constant, \( \Delta \), of a term like \( S_{2,2} \), Eq. (4.4), scales like \( b^2 \) near the Gaussian FP.
Since the quartic coupling constants, $u_i$, scale like $b^{4-d}$ near this FP, products $\Delta u_i$ of a RF coupling constant and a quartic coupling constant scale like $b^{6-d}$, i.e. the product is marginal in $d = 6$. This leads to $d_c^+ = 6$. In Appendix B we show that, at least to one-loop order, of all possible quartic terms only the one that appears in the original action, i.e. $u \, tr \, Q^4$, couples to the RF term. This simplifies the problem substantially. It means that a term with the structure of $S_{2,2}$, Eq. (1.4), is the only one that has to be added to Eq. (2.13) in order to perform a one-loop RG analysis. Expanding $Q$ about its average value, Eq. (3.7), we can write the effective action as,

$$S[\phi] = - \int d\mathbf{x} \, tr \left[ \phi(\mathbf{x})(-\partial^2 + \langle \Lambda \rangle) \phi(\mathbf{x}) + \frac{u}{G} \left( \langle \langle Q \rangle \phi(\mathbf{x}) \rangle^2 + \langle Q \rangle^2 \phi^2(\mathbf{x}) \right) \right]$$

$$+ \frac{\Delta}{2} \int d\mathbf{x} \sum_{i=0}^{\infty} \left( tr_i \phi(\mathbf{x}) \right)^2 - u \int d\mathbf{x} \, tr \, \phi^4(\mathbf{x})$$

$$- \frac{2u}{\sqrt{2G}} \int d\mathbf{x} \, tr \left[ \langle Q \rangle \phi^3(\mathbf{x}) + \phi(\mathbf{x}) \langle Q \rangle \phi^2(\mathbf{x}) \right]$$

$$- \frac{u}{G} \int d\mathbf{x} \, tr \left[ A \left( \phi^2(\mathbf{x}) + \frac{2}{\sqrt{2G}} \langle Q \rangle \phi(\mathbf{x}) \right) \right] - \frac{2}{\sqrt{2G}} \int d\mathbf{x} \, tr \left[ B \phi(\mathbf{x}) \right] \ , \quad (4.5a)$$

with $A$ and $B$ given by

$$A(\langle Q \rangle, \langle \Lambda \rangle) = \langle Q \rangle^2 - 1 + f(\langle \Lambda \rangle) \ , \quad B(\langle Q \rangle, \langle \Lambda \rangle) = \langle \Lambda \rangle \langle Q \rangle - 2GH\Omega \ , \quad (4.5b)$$

Note that $\langle Q \rangle$ and $\langle \Lambda \rangle$ are determined by the conditions $\langle \phi \rangle = \langle \psi \rangle = 0$. At zero-loop order, these conditions yield $A(\langle Q \rangle, \langle \Lambda \rangle) = B(\langle Q \rangle, \langle \Lambda \rangle) = 0$. This is just the zero-loop equation of state discussed in Sec. [11].

To perform a loop expansion we need the Gaussian (G) propagator for the field theory defined by Eqs. (1.3). In the replica limit we find,

$$\left< \prod_{\nu} \phi_{n_{\nu} \nu}^{\alpha_{1} \alpha_{2}}(\mathbf{k}) \prod_{\nu} \phi_{n_{\nu} \nu}^{\alpha_{3} \alpha_{4}}(\mathbf{p}) \right>^{(G)} = (2\pi)^d \delta(\mathbf{k} + \mathbf{p}) \frac{\delta_{ij}}{16(k^2 + m_{12}^2)}$$

$$\times \left[ \delta_{i3} \delta_{24} + (-)^r \delta_{i4} \delta_{23} + \frac{4\Delta \Theta(n_{1}n_{3})}{k^2 + m_{12}} \delta_{r0} \delta_{i0} \delta_{12} \delta_{34} \right] \ , \quad (4.6a)$$

with $1 \equiv (n_{1}, \alpha_{1})$, etc., and,

$$m_{12} = (\ell_{n_{1}} + \ell_{n_{2}})/2 + u(N_{n_{1}} + N_{n_{2}})^2 \ . \quad (4.6b)$$

Note that the last term in Eq. (4.6a) diverges like $k^{-4}$ at criticality. In propagator language, this is what changes the upper critical dimension from $d^+_c = 4$ to $d^+_c = 6$. Alternatively, one can leave the RF term out of the propagator and treat it as an interaction vertex. Also note that this anomalously divergent critical fluctuation is only present in the spin-singlet channel, and only for correlations that are diagonal in frequency and replica space. This is in accord with our discussion in Sec. [11].

**B. The \( \epsilon \)-Expansion**

We now follow the standard Wilson-type RG approach to construct differential recursion relations for the parameters in Eqs. (1.3). Let us first consider the critical theory. Then
we can put $\langle Q \rangle = \langle \Lambda \rangle = 0$, and consider only the renormalizations of $\Delta$ and $u$. The gradient-squared term is not renormalized at one-loop order, so the field renormalization factor, $\zeta = b^{-(d-2+\eta)/2}$, is

$$\zeta = b^{-(d-2)/2+O(\epsilon^2)} \quad ,$$

(4.7)
or, equivalently, the critical exponent $\eta$ is,

$$\eta = O(\epsilon^2) \quad .$$

(4.8)

To one-loop order we find the following recursion relations for $u$ and $\Delta$,

$$u' = \zeta^4 b^d \left[ 1 - \frac{9}{2} u \Delta \int_{k,>} \frac{1}{k^6} \right] \quad ,$$

(4.9a)

$$\Delta' = \zeta^2 b^d \Delta \quad .$$

(4.9b)

Taking a differential RG approach, we write

$$\int_{k,>} \frac{1}{k^6} = S_6 \ln b + O(\epsilon) \quad ,$$

(4.10)

where $S_d = \bar{S}_d/(2\pi)^d$ with $\bar{S}_d$ the surface of the $(d-1)$-sphere. Defining

$$g = u \Delta \quad ,$$

(4.11a)

we obtain the flow equation,

$$\frac{dg}{d \ln b} = \epsilon g - \frac{9}{2} S_6 g^2 + O(g^3) \quad .$$

(4.11b)

We see that for $\epsilon > 0$, or $d < 6$, there is, besides the unstable Gaussian FP, a new nontrivial stable FP. The FP value of $g$ is,

$$g^* = \frac{2\epsilon}{9 S_6} + O(\epsilon^2) \quad .$$

(4.11c)

An important feature of Eq. (4.9a) is that the coefficient does not depend on whether or not the spin-triplet degrees of freedom ($i = 1, 2, 3$) are present. This is different from the situation near $d = 2$, where the presence or absence of the triplet channel changes the universality class of the AMT. Structurally, the insensitivity to the triplet channel in the present case is due to the fact that the spin-singlet propagator is more strongly infrared divergent than any other term because it couples to the random field. Consequently, the leading critical singularities are controlled by the spin-singlet degrees of freedom.

Next we determine the correlation length exponent, $\nu$. Consider the disordered or insulator phase, where $N_{n=0} = 0$ and $\ell_{n=0} \equiv \ell \neq 0$. A straightforward renormalization of $\langle \Lambda \rangle$, or $\ell$, in Eq. (1.5a), yields,

$$\frac{d\ell}{d \ln b} = 2\ell - g S_6 \ell + O(g^2) \quad .$$

(4.12)
Similarly, a one-loop renormalization of \( A \) and \( B \) in Eq. (4.5a) gives the equation of state to that order,

\[
N^2 = 1 - f(\ell) - \frac{Gg}{4u} \sum_p \frac{1}{(p^2 + \ell)^2}, \quad (4.13a)
\]

\[
\ell N = 2GH\Omega - \frac{gN}{2} \sum_p \frac{1}{(p^2 + \ell)^2}, \quad (4.13b)
\]

where both \( N \) and \( \ell \) should be considered to be functions of \( \Omega \). Using Eq. (4.11c) in Eq. (4.12) we find,

\[
\ell(b) \sim b^{2[1-\epsilon/9+O(\epsilon^2)]}. \quad (4.14)
\]

In order to determine \( \nu \), we finally need the relation between \( \ell \) and the distance from the critical point, \( t \). To this end, we expand the r.h.s. of Eq. (4.13a) for small \( \ell \). The \( \ell \)-independent contribution is \( t \). At linear order in \( d = 6 \) one finds a term proportional to \( \ell \), and one proportional to \( \ell \ln \ell \). The prefactors of these terms are related by Eq. (2.15a). Replacing \( g \) by \( g^* \), we can exponentiate and obtain,

\[
\ell \sim t^{1+\epsilon/18+O(\epsilon^2)}. \quad (4.15)
\]

Combining Eqs. (4.14) and (4.15) gives,

\[
t(b) \sim b^{2-\epsilon/3}, \quad (4.16)
\]

which identifies the exponent \( \nu \) as \( 1/\nu = 2 - \epsilon/3 + O(\epsilon^2) \), or

\[
\nu = \frac{1}{2} + \frac{\epsilon}{12}. \quad (4.17)
\]

Equations (4.8, 4.17) give \( \eta \) and \( \nu \) to first order in \( \epsilon \). Standard scaling arguments yield all other static exponents. We find,

\[
\gamma = 1 + \frac{\epsilon}{6} + O(\epsilon^2),
\]

\[
\beta = \frac{1}{2} - \frac{\epsilon}{6} + O(\epsilon^2),
\]

\[
\delta = 3 + \epsilon + O(\epsilon^2). \quad (4.18)
\]

In order to obtain these results we have used the fact that \( u \) is dangerously irrelevant even for \( d < 6 \), as it is in magnetic RF problems. The arguments are the same as those given in Sec. (11) for \( d > 6 \), except that Eqs. (4.9a) and (4.11) imply that \( u \) scales like

\[
u(b) = u(b = 0) b^{-2}, \quad (4.19)
\]

instead of \( u(b) \sim b^{4-d} \). The net result is that in all \( d \)-dependent scaling laws \( d \) is replaced by \( d-2 \). We note that the exponent values given by Eqs. (4.8, 4.17) and (4.18) are identical with the corresponding ones for the RF Ising model. This may not be too surprising. Since the
random potential couples only to the ‘longitudinal’ field components, $\partial_t Q_{nm}$, the problem is structurally very similar to an anisotropic RF magnetic model studied by Aharony, which also yielded RF Ising exponents.

Finally, we need to determine the dynamical exponent $z$. For this purpose we consider Eq. (4.13b), which relates $\ell$, $N$, and $\Omega$. Expanding the r.h.s. for small $\ell$, going to criticality, and exponentiating yields,

$$N \ell^{1+\epsilon/9+O(\epsilon^2)} \sim \Omega .$$

Using Eq. (4.15) with $t$ replaced by $N^{1/\beta}$ yields,

$$z = 3 - \frac{\epsilon}{2} + O(\epsilon^2) .$$

Note that $z = \delta\beta/\nu = y_h$, with $y_h$ the exponent of the field that is conjugate to the OP. This was to be expected, since the RG did not couple different frequencies, so that, effectively, $\Omega$ in Eq. (2.1a) literally acts as the field conjugate to the OP. The technical reason for this simplification is that the electron-electron interaction terms turned out to be irrelevant. The physical reason is that the static RF fluctuations dominate over the quantum fluctuations which couple different frequencies together.

V. SCALING DESCRIPTION OF THE ANDERSON-MOTT TRANSITION

In Sec. III we gave scaling considerations that applied to the regime $d > d_c^+$, where mean-field theory gives the correct critical behavior. Here we generalize this scaling description in the light of the RG analysis in Sec. IV.

We first ask which of the general concepts discussed in Sec. III will survive. One important general feature is the presence of a dangerous irrelevant variable (DIV), $u$. Suppose that $u$ is characterized by an exponent $\theta$, $u(b) \sim b^{-\theta}$. One-loop perturbation theory in Sec. IV gave $\theta = 2 + O(\epsilon)$. Although there is reason to believe that $\theta = 2$ to all orders in the $6 - \epsilon$ expansion, this is probably misleading, see the discussion in Sec. VI, and we keep $\theta$ general. This adds a third independent exponent to the usual two independent static exponents. In addition, there is the dynamical critical exponent $\tilde{z}$, cf. Eq. (3.11b). One effect of the DIV is to effectively change $\tilde{z}$ to $z$. In Sec. IV we found $z$ to be not independent, but rather to be equal to the exponent $y_h$ of the field conjugate to the OP. We saw that this was due to the RF fluctuations being stronger than the quantum fluctuations, and the resulting lack of frequency mixing. The dominance of RF fluctuations seems to be a general feature of RF problems. However, it is possible that in low enough dimensions the explicit interaction terms could become relevant, which would lead to frequency mixing, and to an independent exponent $z$. For simplicity, we ignore this possibility here, and thus write our first scaling law as,

$$z = y_h = \delta\beta/\nu .$$

The OP obeys a scaling or homogeneity relation,

$$N(t, \Omega, u, \ldots) = b^{(2-d-n)/2} N(tb^{1/\nu}, \Omega b^{\tilde{z}}, ub^{-\theta}, \ldots) ,$$

$$19$$
which upon elimination of the \( u \) turns into
\[
N(t, \Omega) = b^{2+\theta-d-\eta/2} \ N(tb^{1/\nu}, \Omega b^z) .
\] (5.2b)
This relates the OP exponent \( \beta \) to the three independent exponents \( \nu, \eta, \) and \( \theta \) through the scaling law,
\[
\beta = \frac{\nu}{2} (d - \theta - 2 + \eta) .
\] (5.3a)
The remaining static exponents are given by the usual scaling laws, with \( d \to d - \theta \) due to the violation of hyperscaling by the \( \text{DIV} \),
\[
\delta = (d - \theta + 2 - \eta) \nu/2\beta ,
\gamma = \nu(2 - \eta) .
\] (5.3b)

Now we consider the thermodynamic susceptibilities \( \partial n/\partial \mu \), \( \gamma_{\nu} \), and \( \chi_{s} \). From the general argument given in connection with Eqs. (3.14) we expect all of them to share the same critical behavior. Denoting their singular parts collectively by \( \chi_{\text{sing}} \) again, the generalization of Eq. (3.14a) reads,
\[
\chi_{\text{sing}}(t, T) = b^{-d+\theta+z} \chi_{\text{sing}}(tb^{1/\nu}, Tb^z) .
\] (5.4)
This links the static critical behavior of the thermodynamic susceptibilities, characterized by an exponent \( \kappa \), \( \chi_{\text{sing}}(t) \sim t^\kappa \), to that of the OP,
\[
\kappa = \beta ,
\] (5.5)
where we have used the scaling laws, Eqs. (5.1, 5.3b). We see that as a consequence of the dominant RF fluctuations all of the thermodynamic susceptibilities scale like the OP. This is what we had found in mean-field theory, Eq. (3.14b), and we now see that this is generally valid.

Finally, we consider the transport coefficients. The homogeneity relation for the diffusion coefficients, Eq. (3.15), does not contain \( d \) explicitly, and therefore is generally valid. If we denote the static exponent for the diffusion coefficients by \( s_D \), \( D(t) \sim t^{s_D} \), we find the following scaling law,
\[
s_D = z - 2 = \beta - \nu\eta .
\] (5.6)
The mean-field result that the \( D \) scale like the OP is therefore valid only to the extent that \( \eta = 0 \). The behavior of the electrical conductivity \( \sigma \), which is related to the charge diffusion coefficient by means of an Einstein relation, \( \sigma = D_c \partial n/\partial \mu \), depends again on whether or not \( \partial n/\partial \mu \) has an analytic background contribution. Generally one would expect such a background to be present. The conductivity then scales like the charge diffusion coefficient,
\[
\sigma(t, \Omega) = b^{2-z} \sigma(tb^{1/\nu}, Tb^z) ,
\] (5.7a)
and the static conductivity exponent \( s \) (cf. Eqs. (3.16)), is given by
\[ s = \frac{\nu}{2}(d-2-\theta-\eta) \quad . \]  

(5.7b)

In models or physical situations where \( \partial n/\partial \mu \) has no analytic background one gets instead,

\[ \sigma(t, \Omega) = b^{2-d+\theta} \sigma(t b^{1/\nu}, \Omega b^z) \quad , \]  

(5.7c)

which leads to

\[ s = \nu(d-2-\theta) \quad . \]  

(5.7d)

In either case, Wegner’s scaling law \( s = \nu(d-2) \) \cite{[17]} which previously had been found to hold for the AMT\cite{[18]} as well as the Anderson transition\cite{[26]}, is violated, unless Eq. (5.7b) holds, and \( \theta = 2-d-\eta \). This has profound consequences which we will discuss in the next section.

**VI. DISCUSSION**

Our results suggest that a crucial physical aspect has been left out of all previous analyses of the AMT in general, and the NL\( \sigma \)M commonly used to describe it in particular. This aspect is the random-field (RF) nature of the transition, the presence of which has been made plausible in the Introduction, and which has been confirmed by the explicit RG calculation in Sec. [IV]. In our formulation of the problem the RF aspects follow naturally from the fact that the random potential couples to the OP for the AMT, cf. Eq. (1.2). In this sense an OP description of the AMT is necessary in order for the RF features to emerge in a straightforward way, and the absence of an OP theory has been the reason why these features have not been noted earlier.

The picture of the AMT that emerges from our OP theory with RF aspects differs crucially in many respects, both physical and technical, from the one that had previously been obtained by working in the vicinity of \( d = 2 \). Most importantly, the AMT according to the present picture is driven by the behavior of the OP, rather than by the soft modes as in the \( 2+\epsilon \) expansion. Indeed, we integrated out the soft modes at an early stage in Sec. [II], and due to the high dimensionality we are working in they never influence the leading nonanalytic behavior. Working in a high dimensionality (i.e., in the vicinity of \( d = 6 \)) was in turn forced upon us by the RF aspects of the theory: It is the dominance of the RF fluctuations over the quantum fluctuations, which drive the transition in a \( 2+\epsilon \) description, that shifts the upper critical dimension from \( d^+_c = 4 \) to \( d^+_c = 6 \). This is in exact analogy to the case of RF magnets, where the RF fluctuations dominate over the thermal fluctuations, which is why the RF transition is sometimes referred to as a zero-temperature fixed point [27].

As a consequence of the irrelevance of the soft modes we find only one universality class, irrespective of whether or not the spin-triplet channel is present. In contrast, the rich variety of universality classes near \( d = 2 \) resulted from the influence of the diffusive modes in that channel, the number of which can be controlled by adding magnetic impurities, a magnetic field, or spin-orbit scattering to the model [5,6]. For dimensionalities \( d < 4 \) one expects the soft modes to play an important role again. In mean-field theory this is readily seen explicitly, see Appendix [A]. However, beyond mean-field theory the behavior of the model, and even the nature of the transition, are unclear in that regime, as they are in the magnetic models, see below.
Another important difference between the present picture and all previous work on the AMT is that the electron-electron interaction turned out to be irrelevant, in the RG sense, for all \( d > 2 \). This is another manifestation of the subordinate role that quantum fluctuations play in our theory. This leads to an enormous technical simplification over the work near \( d = 2 \), since it eliminates the frequency coupling that made the latter extremely cumbersome. A physical implication of this simplification is that we find orthodox dynamical scaling, while near \( d = 2 \) there are several critical time scales, and corresponding dynamical critical exponents, and one has only what is known as 'weak dynamical scaling'.

Although it is irrelevant for the critical behavior, the electron-electron interaction is of course necessary for the AMT to exist, since for noninteracting electrons one has an Anderson transition with an uncritical DOS. This point is correctly reflected by the theory, as can be seen from Eq. (2.14) and has been mentioned after Eqs. (3.3). The electron-electron interaction thus plays a rather trivial, although crucial, role in the theory: Though irrelevant for the critical behavior, it ensures that the phase transition under consideration is accessible. Indeed, for \( K_{s,t} \to 0 \) the critical behavior increases without bound, \( G_c \to \infty \). This suggests a number of distinct phase transition scenarios. The simplest one is that for sufficiently small interaction constants, or large \( G_c \), the AMT discussed here gets preempted by some other phase transition, such as a pure Anderson transition. This scenario is particularly likely if \( K_t = 0 \), and if the electron-electron interaction is short-ranged, since in this case \( K_s \) is irrelevant near the Anderson transition FP, at least near \( d = 2 \). A different possibility is that in the above picture the Anderson transition is replaced by an AMT that is related to the one studied near \( d = 2 \) for the case when either both \( K_s \) and \( K_t \) are nonzero, or the electron-electron interaction is of long range. This picture leaves room for (1) the Anderson transition, (2) the type of AMT discussed in this paper, and (3) the type of AMT discussed before. Which transition is actually realized will depend on the relative strengths of the disorder and the electron-electron interaction in the various channels.

All thermodynamic susceptibilities, including \( \partial n / \partial \mu \), are found to be critical in the present theory. This is again in contrast to the results in \( d = 2 + \epsilon \), where \( \gamma_V \) and \( \chi_s \) may or may not be critical, depending on the universality class, but \( \partial n / \partial \mu \) is never critical. The reason is that the critical behavior in \( d = 2 + \epsilon \) derives from logarithmic singularities in \( d = 2 \), and \( \partial n / \partial \mu \), as a frequency integral over a quantity (the DOS) that is only logarithmically singular itself, cannot have any logarithmic corrections to any order in perturbation theory. We note, however, that on general physical grounds one would actually expect a nonanalyticity in \( \partial n / \partial \mu \), given a nonanalytic DOS. The frequency integration that takes one from the DOS to \( \partial n / \partial \mu \) may weaken the nonanalyticity, but it cannot completely remove it. In that respect the result in \( d = 2 + \epsilon \) is hard to understand, and the present one is physically more plausible. It also bears some resemblance to the case of a Mott transition in a Hubbard model, where \( \partial n / \partial \mu \) is also critical. We emphasize, however, that for the AMT a critical \( \partial n / \partial \mu \) does not necessarily mean that \( \partial n / \partial \mu \) vanishes at the transition, i.e. that the system becomes incompressible. As we have mentioned in Secs. 11 and 12, one expects in general a nonvanishing analytic background contribution to all thermodynamic quantities except for the OP. In special models, or for special parameter values, this background contribution could be absent for some or all of the thermodynamic susceptibilities, but one would expect a manifest physical reason for this, like, e.g., a symmetry. Ref. 13 argued that the present model might be such a special case. This suggestion was based on an explicit
representation of the susceptibilities in terms of the model parameters $H, K_s, $ and $K_t$ and the scaling to zero under RG iterations of the latter, but no more general physical argument has been given. Even if the suggestion was correct for this particular model, however, one would expect more general models to contain noncritical background contributions. For instance, in Ref. it has been argued that local moment effects, which are absent from our model, lead to a spin susceptibility that diverges as $T \to 0$ both in the metallic and in the insulating phases, but shows no critical behavior at the MIT. Such noncritical contributions can enter additively to the critical ones given by Eqs. (5.4). Of course this would not change the critical behavior of anything except possibly the conductivity: Depending on whether or not there is a noncritical background contribution to $\partial n/\partial \mu$, the critical exponent $s$ for the conductivity is given by either Eq. (5.7b) or by Eq. (5.7d). Alternatively, noncritical processes not included in our model might lead to bare interaction constants in the action, Eq. (2.7), that diverge at $T = 0$. Such terms at most would cause some of the soft modes in our model to be absent. Because of the irrelevance of the soft modes mentioned above this would not modify the critical behavior for $d > 4$.

The remarkable analogy which we have found to exist between the AMT and RF magnetic transitions implies that the AMT will also inherit the complications that are known to exist for RF magnets, most of which are not quite understood. For instance, it was mentioned in Sec. V that the exponent $\theta$ is probably equal to $2$ to all orders in perturbation theory, as it is in RF magnets. However, it is also known that this result is misleading, and that nonperturbative effects are likely to play an important role in general RF problems. This is known as the ’dimensional reduction problem’. A possible physical consequence of nonperturbative effects is the appearance of non-power law dynamical critical behavior, and, more generally, features of the RF phase transition problem that resemble those of a glass transition. If this is true for the magnetic RF problem, then one should expect the same for the AMT. A related question is what happens for $d < 4$. Fisher has shown that an infinite number of RF-type operators all become marginal in $d = 4$. Together with the fact that the soft modes will become important for $d < 4$ this means that dealing with the problem in $d < 4$ will require techniques substantially different from the ones employed above.

There are also some problems that are germane to the AMT that have not been addressed yet. In this work we have considered only the particle-hole degrees of freedom, the particle-particle or Cooper channel has been omitted. In $d = 2 + \epsilon$ the Cooper channel is known to lead to substantial technical problems, which have not been resolved entirely. It remains to be seen whether the Cooper channel is amenable to an easier treatment within the OP formulation of the AMT. Also, we have restricted ourselves to a short-ranged model interaction. Our main motivation for this restriction is that a Coulomb interaction raises questions concerning the treatment of screening and the plasmon mode which are most naturally addressed within the context of the general field theory given by Eqs. (1.1), rather than the NL$\sigma$M employed in this paper. Studying the underlying field theory is also attractive since the main motivation for using the NL$\sigma$M, viz. its construction as an effective model for the soft modes in the problem, disappears with the realization that the soft modes are irrelevant, at least for $d > 4$. We leave these problems for future investigations.

Finally, let us discuss experimental implications of our results. A result of great importance in this respect is the violation of Wegner scaling expressed by Eqs. (7.7). This removes
the requirement \( s \geq 2(d-2)/d \) that followed from Wegner scaling (\( s = \nu(d-2) \)) together with the lower bound \( \nu \geq 2/d \). This requirement had led to severe problems with the interpretation of a certain class of experiments on doped semiconductors, most notably Si:P, which observed \( s \approx 1/2 \) in \( d = 3 \). While a dangerous irrelevant variable had been envisaged for some time as a possible way out of this problem, previous attempts to argue for the existence of one had been found to be erroneous.

Another question concerns possible experiments to test whether or not the AMT has indeed a RF character. In this context an interesting point is that RF problems are known to contain an anomalously divergent correlation function. The point is that in the presence of a random field the quantity \( \{ \langle O(x) \rangle \langle O(y) \rangle \} \), with \( O \) the OP, \( \langle \ldots \rangle \) the thermodynamic average, and \( \{ \ldots \} \) the disorder average, is nonzero even if \( \{ \langle O(x) \rangle \} \) vanishes. In the present case the anomalous correlation takes the form

\[
C(x, y) = \{N(x)N(y)\} - N^2,
\]

with \( N(x) \) the unaveraged local DOS at the Fermi level. At the critical point the Fourier transform of \( C \) behaves like

\[
C(k \to 0) \sim k^{-2+\eta-\theta}.
\]

The same correlation function, but without the anomalously strong divergence due to the RF physics, has been considered in connection with mesoscopic systems and it is measurable, at least in principle. An experimental observation of the strong divergence predicted by Eq. \( (6.1b) \) would be a strong indication of the presence of RF features at the AMT. It would also be interesting to experimentally look for indications of glassy behavior in the form of the abovementioned anomalously slow relaxation processes that one expects in a RF system.

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**APPENDIX A: THE MEAN-FIELD SOLUTION FOR \( D < 6 \)**

There is no reason to believe that mean-field theory is a particularly good approximation in the physical dimension \( d = 3 \), or anywhere else below \( d = 6 \). Here we determine the mean-field results for \( d < 6 \) anyway. We do so partly for completeness, and partly to illustrate certain interesting aspects and problems which at the mean-field level are easily soluble because mean-field theory lacks the complications of the RF aspects of the transition, and which may be of relevance for a more complete theory in \( d = 3 \). Throughout this appendix we suppress the superscript \( (0) \) that denoted the mean-field approximation in Sec. 11.

Before we turn to the critical behavior we consider the frequency dependence of the OP in the metallic phase. For small frequencies we find,

\[
N(\Omega \to 0) = N(\Omega = 0) + \text{const} \times \begin{cases} 
|\Omega| & \text{for } d > 4 \\
|\Omega|^{(d-2)/2} & \text{for } 2 < d \leq 4
\end{cases}.
\]

\[
(A1)
\]
This low-frequency nonanalyticity, or long-time tail, is the so-called Coulomb anomaly first found by Altshuler and Aronov in perturbation theory. Our mean-field theory correctly reproduces the perturbative result. In Sec. IV we have seen that Ω is the field conjugate to the OP \( N \). This allows for an interesting interpretation of the Coulomb anomaly: It is the AMT analogue of the fact that in an \( O(n) \) ferromagnet with \( n \geq 2 \) the magnetic susceptibility diverges everywhere in the ordered phase. This is, of course, a consequence of the Goldstone modes which become increasingly important for \( d < 4 \).

Now we show that the critical behavior is given by Eqs. (3.9) for all \( d > 3 \). We start with the equation of state, Eqs. (3.2). It is obvious that Eq. (3.5) holds independently of the dimensionality, so we have \( \beta = 1/2 \) for all \( d \). If we expand the equation of state for small frequencies at criticality we obtain for \( d > 3 \),

\[
N(\Omega \to 0) = |\Omega|^{1/3}. \tag{A2}
\]

This determines the exponents \( \delta = \beta/\nu z = 3 \) for \( d > 3 \). For \( d = 3 \) there is a logarithmic correction to the \( \Omega^{1/3} \), and for \( d < 3 \) the exponent changes. The fact that the exponent is \( d \)-independent for \( d > 3 \), rather than for \( d > 4 \) is a manifestation of the quantum nature of the mean-field AMT: A quantum phase transition in \( d \) dimensions is expected to be similar to the corresponding classical transition in \( d + 1 \) dimensions. Note that this is not the case for the RF phase transition discussed in the main text because near \( d = 6 \) we found that the interaction terms, which would lead to frequency mixing, are RG irrelevant.

In order to determine the remaining exponents we turn to Eq. (3.8). Obviously \( \eta = 0 \) for all \( d \). In order to determine \( \nu \) we must deal with the divergencies that occur in the coefficients \( u \) and \( v \) in \( d \leq 4 \). To protect the divergency, we keep the momentum dependence of \( u \) and \( v \), which we then scale with the correlation length, \( \xi \). The integral for \( u \) is dominated by the small frequency region, and we find,

\[
u \sim \begin{cases} 
\text{const} & \text{for } d \geq 4 \\
\text{const} + t^{1/2} \xi^{4-d} & \text{for } 3 < d < 4
\end{cases} \tag{A3}
\]

The scaling of the correlation length is now obtained by equating the first and the third term in Eq. (3.8). We obtain \( \xi \sim t^{1/2} \), and hence \( \nu = 1/2 \) for all \( d > 3 \). Again there are logarithmic corrections to scaling in \( d = 3 \). Equation (A3) immediately gives \( \gamma = 2\nu = 1 \), and from \( z = y_h = \delta \beta/\nu \) we obtain \( z = y_h = 3 \). The divergence of \( v \) is of no consequence: Since \( u \) is effectively constant for \( d > 3 \), \( v \) scales like \( \xi^{4-d} \sim t^{-(4-d)/2} \), so the product \( vt \) in Eq. (3.8) scales to zero. At the mean-field level, Eqs. (3.3) thus hold for \( d > 3 \).

**APPENDIX B: TERMS GENERATED BY THE RENORMALIZATION GROUP**

In this appendix we show that the new quartic terms generated by the RG, which were not in the original action, do not modify the RF fixed point discussed in Sec. IV, at least near \( d = 6 \) to one-loop order. The argument is that these new terms do not couple to either of the coupling constants \( g \) or \( \ell \) whose flow equations are given by Eqs. (4.11B) and (4.12), respectively.

If we repeat the steps below Eq. (4.2) to \( O(u^2) \), then the RF term in the two-point propagator, Eq. (4.6a), leads to the following new quartic terms in the action
\begin{align}
S_{4,1} &= v_1 \int dx \, \text{tr} \left( Q^2(x) \right) \text{tr} \left( Q^2(x) \right), \quad \text{(B1a)} \\
S_{4,2} &= v_2 \int dx \, \left[ \text{tr} \left( Q^2(x) \right) \right]^2, \quad \text{(B1b)} \\
S_{4,3} &= v_3 \int dx \, \text{tr} \left( Q^2(x) \right) \left[ \text{tr} \left( Q(x) \right) \right]^2, \quad \text{(B1c)} \\
S_{4,4} &= v_4 \int dx \, \left[ \text{tr} \left( Q(x) \right) \right]^4, \quad \text{(B1d)}
\end{align}

and additional terms given by \( \text{tr} \) in Eqs. (B1) replaced by \( \text{tr} \). Note that these quartic terms are distinguished from \( S_4 \), Eq. (4.2), by the appearance of additional traces. Ultimately these additional traces generate extra replica sums that cause these terms not to couple into Eqs. (4.11b) and (4.12).

First we argue that these terms cannot contribute to Eq. (4.12). We use Eqs. (4.3) and (B1), and examine the \( (Q < )^2 \)-terms. The RF singularity is necessary to obtain logarithmically singular contributions in \( d = 6 \), so it is sufficient to use the last term in Eq. (4.6a) for the internal propagator, \( \langle Q^2 Q^2 \rangle \). This either leads to terms that vanish in the replica limit, or to terms proportional to \( [\text{tr} Q]^2 \). The latter terms are nonsingular renormalizations of \( \Delta \), and can be neglected near \( d = 6 \).

Next we consider the renormalizations of \( O(uv) \) and \( O(v^2) \) of \( S_4 \) or \( u \), Eq. (4.2), where \( v \) can stand for any of the coupling constants in Eqs. (B1). Let us consider the term of \( O(v^2) \) specifically, the arguments for all other terms follow the same lines. In schematic notation, the replica structure of this term is,

\[ S^2_{4,2} \sim Q^{\alpha_1 \alpha_2} Q^{\alpha_2 \alpha_1} Q^{\alpha_3 \alpha_4} Q^{\alpha_4 \alpha_3} Q^{\beta_1 \beta_2} Q^{\beta_2 \beta_1} Q^{\beta_3 \beta_4} Q^{\beta_4 \beta_3}. \]  

We now use Eq. (4.3), examine the \( (Q < )^4 \)-term, and require that one of the internal propagators be proportional to the last term in Eq. (4.6a) in order to obtain a logarithmically singular term in \( d = 6 \). The resulting replica structure is either \( (\text{tr} Q^2)^2 \), or \( (\text{tr} Q)^2 \text{tr} Q^2 \). We conclude that this term does not contribute to the renormalization of \( u \).

Similar considerations lead to the general conclusion that none of the terms given by Eqs. (B1) renormalize \( S_4 \), Eq. (4.2).
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27 We expect the same conclusion to apply to the AMT.
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29 See, e.g., D. S. Fisher, Phys. Rev. B 31, 7233 (1985), and references therein.
Questions concerning the scaling of $\sigma$ and the background contribution to $\partial n/\partial \mu$ could be answered by a direct calculation of $\sigma$ as a correlation function. Naively, a source term for $\sigma$ will be $Q^2$, since $\sigma$ is related to a two-particle Green function. The mean-field scaling of $Q$ then suggests $[\sigma] = d - 2$. Taking into account the DIV, this is consistent with Eqs. (3.16b) and (5.7d). More work is needed to confirm or refute this conjecture.