On multidimensional cosmological solutions with scalar fields and 2-forms corresponding to rank-3 Lie algebras: acceleration and small variation of \( G \)

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Abstract

By means of a simple model we investigate the possibility of an accelerated expansion of a 3-dimensional subspace in the presence of the variation of the effective 4-dimensional constant obeying the experimental constraint. Multidimensional cosmological solutions with \( m \) 2-form fields and \( l \) scalar fields are presented. Solutions corresponding to rank-3 Lie algebras are singled out and discussed. Each of solutions contain two factor spaces: one-dimensional space \( M_1 \) and Ricci-flat space \( M_2 \). A 3-dimensional subspace of \( M_2 \) is interpreted as "our" space. We show that there exists a time interval where accelerated expansion of our 3D space is compatible with a small enough variation of the effective gravitational constant \( G(\tau) \). This interval contains \( \tau_0 \) which is the point of minimum of \( G(\tau) \) (here \( \tau \) is the synchronous time variable). Special solutions with three phantom scalar fields are analyzed. It is shown that in the vicinity of the point \( \tau_0 \) the time variation of \( G(\tau) \) decreases in the sequence of Lie algebras \( A_3, C_3 \) and \( B_3 \) when the solutions with asymptotically power-law behavior of scale-factors for \( \tau \to \infty \) are considered. Exact solutions with asymptotically exponential accelerated expansion of 3D space are also considered.
1 Introduction

One of the challenging problems of modern physics and cosmology is that of possible time-, location-, and scale-dependent variations of the fundamental physical constants, in particular, of Newton’s gravitational constant $G$. According to the observational data the variation of $G$ is admissible at the level of less $10^{-12}\text{yr}^{-1}$ and there exists a necessity in further theoretical and experimental developments of this problem. At present multidimensional cosmological models with diverse matter sources are widely used as a theoretical framework for describing possible time variations of fundamental physical constants, e.g. the gravitational constant $G$, see [1]-[10] and references therein.

This paper is devoted to the investigation of an accelerated expansion of our 3-dimensional space in presence of the variation of the effective 4-dimensional gravitational constant. Here we use the approach proposed in the papers [6]-[9]. In [8], multidimensional exact S-brane solutions with the intersection rules for branes corresponding to rank-2 Lie algebras were discussed. It was shown that there exists an interval of the synchronous time $\tau$ where the scale factor of our 3-dimensional space exhibits an accelerated expansion according to the observational data [11, 12] while the relative variation of the effective 4-dimensional gravitational constant is small enough with the Hubble parameter, see [13]-[16] and references therein.

In this paper, we study the variation of $G$ using a multidimensional gravitational model with an arbitrary number of dimensions, $m$ Abelian gauge 2-form fields and $l$ scalar fields. The 2-form fields contribute to 0-branes. Thus we have the multidimensional model with $m$ 0-branes. Our exact solutions are governed by polynomials $H_s$ corresponding to rank-3 Lie algebras. We aim to show that for this case there is an interval of the synchronous time $\tau$ where an accelerated expansion of “our” 3-dimensional space coexists with a small enough value of $G$, like in [8]. But we also intend to show that the accelerated expansion is possible in the presence of ”phantom” scalar fields in the model.

The paper is organized as follows. In Section 2, we consider the multidimensional gravitational model and exact solutions. We also single out solutions corresponding to rank-3 Lie algebras in Section 2. In Section 3, the solutions are examined for the presence of an accelerated expansion if there is a small variation of the gravitational constant. In Section 4, a special configuration with three phantom fields is considered and the expressions for the dimensionless variation of $G$ corresponding to the rank-3 Lie algebras are present. Section 5 is devoted to the investigation of the solutions with asymptotically exponential accelerated expansion of 3-dimensional space and Section 6 is a conclusion.

2 The model

The model is governed by the action

$$S = \int d^D x \sqrt{|g|} \left\{ R[g] - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \frac{1}{2} \sum_{s=1}^{m} \exp[2\lambda_s(\varphi)](F^s)^2 \right\}$$

(2.1)

where $g = g_{MN}(x)dx^M \otimes dx^N$ is a D-dimensional metric of the pseudo-Euclidean signature $(-,+,\ldots,+)$, $F^s = dA^s = \frac{1}{2} F^s_{MN}dz^M \wedge dz^N$ is a 2-form of rank 2, $\varphi = (\varphi^\alpha) \in \mathbb{R}^l$ is a vector of scalar fields, $(h_{\alpha\beta})$ is a constant symmetric non-degenerate $l \times l$ matrix ($l \in \mathbb{N}$), $\lambda_s$ is a 1-form on $\mathbb{R}^l$: $\lambda_s(\varphi) = \lambda_{s\alpha} \varphi^\alpha$, with $s = 1,\ldots,m$ and $\alpha = 1,\ldots,l$. In (2.1) we denote $|g| = |\det(g_{MN})|$, $(F^s)^2 = F^s_{M_1M_2} F^s_{N_1N_2} g^{M_1N_1} g^{M_2N_2}$, $s = 1,\ldots,m$. 

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We consider the manifold
\[ M = (0, +\infty) \times M_1 \times M_2, \tag{2.2} \]
where \( M_1 \) is a one-dimensional manifold (say \( S^1 \) or \( \mathbb{R} \)) and \( M_2 \) is a \((D-2)\)-dimensional Ricci-flat manifold.

### 2.1 General solutions.

In what follows the subspace \( M_1 \) will support all forms \( A^s \). Let us consider a family of exact solutions to the field equations corresponding to the action (2.1) and depending on one variable \( \rho \). These solutions are defined on the manifold (2.2). The solutions read [25]
\[
g = \left( \prod_{s=1}^{m} H_s^{2h_s/(D-2)} \right) \left\{ w d\rho \otimes d\rho + \left( \prod_{s=1}^{m} H_s^{-2h_s} \right) \rho^2 d\phi \otimes d\phi + g^2 \right\}, \tag{2.3} \]
\[
\exp(\varphi^\alpha) = \prod_{s=1}^{m} H_s^{h_s \lambda_{s\alpha}}, \tag{2.4} \]
\[
F^s = -Q_s \left( \prod_{s'=1}^{m} H_{s'}^{-A_{ss'}} \right) \rho d\rho \wedge d\phi, \tag{2.5} \]
\[
s = 1, \ldots, m, \text{ where } w = \pm 1; \ g^1 = d\phi \otimes d\phi \text{ is a metric on } M_1 \text{ and } g^2 \text{ is a Ricci-flat metric on } M_2. \]

Functions \( H_s(z) > 0 \) with \( z = \rho^2 \), are defined on the interval \((0, +\infty)\) and obey the nonlinear differential equations
\[
\frac{d}{dz} \left( \frac{z d}{dz} H_s \right) = P_s \prod_{s'=1}^{m} H_{s'}^{-A_{ss'}}, \tag{2.6} \]
with the following boundary conditions imposed
\[
H_s(+0) = 1. \tag{2.7} \]

Parameters \( P_s \) are proportional to the charge density squared parameters in the following way:
\[
P_s = \frac{1}{4} K_s Q_s^2. \tag{2.8} \]

Parameters \( h_s \) satisfy the relations
\[
h_s = K_s^{-1}, \quad K_s = (U^s, U^s), \tag{2.9} \]
where \( K_s \neq 0 \) and the scalar products of the \( U^s \)-vectors belonging to \( \mathbb{R}^{n+l} \) are defined as follows
\[
(U^s, U^{s'}) = 1 + \frac{1}{2 - D} + \lambda_{ss'} h^{\alpha\beta}, \tag{2.10} \]
\[
s = 1, \ldots, m, \text{ with } (h^{\alpha\beta}) = (h_{\alpha\beta})^{-1} \text{ and } \lambda_s = h^{\alpha\beta} \lambda_{s\beta}. \text{ The } U^s \text{-vectors and the scalar products were specified in } [17, 18, 19]. \]

Here we put the matrix
\[
(A_{ss'}) = \left( 2(U^s, U^{s'})/(U^{s'}, U^{s'}) \right) \tag{2.11} \]
to be coinciding with the Cartan matrix for a simple Lie algebra $G$ of rank $m$. If we remember
the integrality condition for a root system and compare it with (2.11), we will notice a close
correspondence between $U^s$-vectors and roots.

According to a conjecture suggested in [21, 22] solutions to eqs. (2.6), (2.7) (governed by the
Cartan matrix $(A_{ss'})$) are polynomials:

$$H_s(z) = 1 + \sum_{k=1}^{n_s} P_s^{(k)} z^k, \quad (2.12)$$

where

$$n_s = 2 \sum_{s'=1}^{n} A^{ss'} \quad (2.13)$$

and $(A^{ss'}) = (A_{ss'})^{-1}$. The integers $n_s$ are components of the twice dual Weyl vector in the basis
of simple co-roots [22, 23]. It should be also noted that the set of polynomials $H_s$ defines a special
solution to the open Toda chain equations corresponding to a simple Lie algebra $G$ [24].

These solutions are special cases of more general solutions from [21]. The solutions under
consideration may be verified just by the substitution into the equations of motion corresponding
to (2.1). It may be also obtained as a special case of the fluxbrane (for $w = +1, M_1 = S^1$) and
$S$-brane ($w = -1$) solutions from [21] and [22], respectively.

## 2.2 Solutions for Lie algebras of rank 3.

Let us single out a special class of exact solutions corresponding to rank-3 Lie algebras $A_3$, $B_3$
and $C_3$.

The Cartan matrix corresponding to Lie algebras rank 3 reads

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -k_1 \\ 0 & -k_2 & 2 \end{pmatrix}, \quad (2.14)$$

where here and in the sequel

$$k_1 = (1, 2, 1), \quad k_2 = (1, 1, 2), \quad (2.15)$$

for the Lie algebras $A_3$, $B_3$ and $C_3$, respectively.

Due to (2.11) and (2.14), we get

$$K_1 = K_2 = \frac{k_1}{k_2} K_3, \quad (2.16)$$

denoting $K_1 = K_2 = K$ and $K_3 = K'$, we get $K = \frac{k_1}{k_2} K'$.

Here we put $K < 0$ and use a special choice of $P_s$-parameters:

$$P_s = n_s P, \quad (2.17)$$

$P \neq 0$, where the integers $n_s$ are components of a twice dual Weyl vector in the basis of simple
co-roots:

$$(n_1, n_2, n_3) = (3, 4, 3), (6, 10, 6), (5, 8, 9) \quad (2.18)$$
for the Lie algebras $A_3$, $B_3$ and $C_3$ respectively. Then due to relations (2.8) and (2.17) we obtain

$$Q_s^2 = \frac{4n_s}{K_s} P. \quad (2.19)$$

The governing functions in this case have the following form

$$H_s = (1 + Pt^2)^{n_s} = X^{n_s}, \quad (2.20)$$

where

$$X = 1 + Pt^2 \quad (2.21)$$

and $t$ is time variable.

For general form of polynomials $H_s$ corresponding to the Lie algebras $A_3$, $B_3$, $C_3$ see Appendix A.

The solutions read

$$g = X^{2A} \left\{-dt \otimes dt + X^{-2B}l^2 d\phi \otimes d\phi + g^2 \right\}, \quad (2.22)$$

$$\exp(\varphi^a) = X^{B_1 \lambda_1^a + B_2 \lambda_2^a + B_3 \lambda_3^a}, \quad (2.23)$$

$$F^1 = -Q_1 X^{n_2 - 2n_1} dt \wedge d\phi, \quad (2.24)$$

$$F^2 = -Q_2 X^{n_1 - 2n_2 + k_1 n_3} dt \wedge d\phi, \quad (2.25)$$

$$F^3 = -Q_3 X^{k_2 n_2 - 2n_3} dt \wedge d\phi, \quad (2.26)$$

where

$$A = \frac{B}{D - 2}, \quad (2.27)$$

$$B_s = n_s K_s^{-1}, \quad B = \sum_{s=1}^{3} B_s, \quad (2.28)$$

$s = 1, 2, 3$.

The relations (2.24)-(2.26) mean that the only nonzero components of the electromagnetic field tensor are $F_{ty}^s = -F_{yt}^s = -Q_0 X^{-(n_1 A_1 + n_2 A_2 + n_3 A_3)} l_t$.

We also note that the charge density parameters $Q_s$ obey the following relations (see (2.16) and (2.19))

$$\frac{Q_1^2}{Q_2^2} = \frac{n_1 K_2}{n_2 K_1} = \left(\frac{3}{4}, \frac{3}{5}, \frac{5}{8}\right), \quad \frac{Q_2^2}{Q_3^2} = \frac{n_2 k_2}{n_3 k_1} = \left(\frac{4}{3}, \frac{5}{6}, \frac{16}{9}\right) \quad (2.29)$$

for the Lie algebras $A_3$, $B_3$ and $C_3$, respectively.

## 3 Solutions with acceleration

In what follows we use a "synchronous" time variable $\tau = \tau(t)$ :

$$\tau = \int_0^t d\bar{t}[X(\bar{t})]^A. \quad (3.1)$$
Recall that $P < 0$, $K < 0$ and hence $A < 0$. Consider two intervals of parameter $A$:

\begin{align*}
(i) & \quad A < -1, \\
(ii) & \quad -1 < A < 0.
\end{align*}

(A special case $A = -1$ will be investigated in other section.)

In both cases after integrating we get Gauss’s hypergeometric functions. If we look at the graphics of these functions (Fig. 1 and Fig. 2), we note that in case (i), the function $\tau = \tau(t)$ monotonically increases from 0 to $+\infty$, for $t \in (0, t_1)$, where $t_1 = |P|^{-1/2}$, while in case (ii) it is monotonically increases from 0 to a finite value $\tau_1 = \tau(t_1)$.

![Figure 1: The graph of the "synchronous" time variable $\tau$, for $A < -1$.](image1)

![Figure 2: The graph of the "synchronous" time variable $\tau$, for $-1 < A < 0$.](image2)

The solutions (2.22)-(2.26) are defined on the manifold

$$M = (0, t_1) \times M_1 \times M_2.$$  

(3.4)

Here we put $M_1 = S^1$ and

$$M_2 = R^3 \times (S^1)^{n-4},$$

(3.5)

where the subspace $R^3$ is our 3-dimensional space with the scale factor

$$a_3 = X^A.$$  

(3.6)

For the first branch (i) we get an asymptotical relation

$$a_3 \sim \text{const}\tau^\nu,$$

(3.7)

for $\tau \to +\infty$, where

$$\nu = \frac{A}{A + 1}$$

(3.8)

and, due to (3.2), $\nu > 1$. For the second branch (ii) we obtain

$$a_3 \sim \text{const}(\tau_1 - \tau)^\nu,$$

(3.9)

for $\tau \to \tau_1 - 0$, where $\nu < 0$, due to $-1 < A < 0$.  

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Thus, we get an asymptotic accelerated expansion of the 3D subspace $R^3$ in both cases (i) and (ii), and $a_3 \to +\infty$.

It may be verified that the accelerated expansion takes place for all $\tau > 0$, i.e.,

$$
\dot{a}_3 > 0, \quad \ddot{a}_3 > 0
$$

(3.10)

(see [8, 10]).

Figure 3 and Figure 4 show a behavior of the function $a_3(t)$ and its asymptotic forms.

![Figure 3: The scale factor $a_3(t)$ and its asymptotic form, for $A < -1$](image1)

![Figure 4: The scale factor $a_3(t)$ and its asymptotic form, for $-1 < A < 0$](image2)

Now we consider a variation of the effective constant $G$. According to (3.4)-(3.5) in our model we have two internal compact spaces $S^1$ and $(S^1)^{n-4}$ with scale factors

$$
a_1 = X^{A-B}t, \quad a_2 = X^{A},
$$

(3.11)

respectively. In Jordan’s frame the 4-dimensional gravitational "constant" is

$$
G = \text{const}a_1^{-d_1}a_2^{-d_2} = \text{const}X^{B-A(n-3)t^{-1}},
$$

(3.12)

where $d_1 = 1$ and $d_2 = n - 4$ are dimensions of factor spaces $S^1$ and $(S^1)^{n-4}$, respectively. Due to the formulas (2.27)-(2.28) for $A$ and $B$ parameters and remembering that $D = n + 1$ we rewrite the expression (3.12) as follows

$$
G = X^{2A}t^{-1}.
$$

(3.13)

The dimensionless variation of $G$ reads

$$
\delta = \dot{G}/(GH) = 2 + \frac{1 - |P|t^2}{2|P|t^2},
$$

(3.14)

where

$$
H = \frac{\dot{a}_3}{a_3}
$$

(3.15)

is the Hubble parameter of the 3D subspace $R^3$. Using the expression (3.14) we can find an extremum of the function $G(\tau)$ at the point $\tau_0$ corresponding to $t_0$ and

$$
t_0^2 = \frac{|P|^{-1}}{1 + 4|A|}.
$$

(3.16)
At this point the function $G(\tau)$ has a minimum and $\dot{G}$ vanishes thereafter.

The function $G(\tau)$ monotonically decreases from $+\infty$ to $G_0 = G(\tau_0)$ for $\tau \in (0, \tau_0)$ and monotonically increases from $G_0$ to $+\infty$ for $\tau \in (\tau_0, \bar{\tau}_1)$. Here $\bar{\tau}_1 = +\infty$ for the case (i) and $\bar{\tau}_1 = \tau_1$ for the case (ii) (see Fig. 5 and Fig. 6).

![Figure 5](image1.png)  ![Figure 6](image2.png)

Figure 5: The plot of the gravitation constant $G(t)$, for $A < -1$

Figure 6: The plot of the gravitation constant $G(t)$, for $-1 < A < 0$

We should consider only solutions with the accelerated expansion of our space and small enough variations of the gravitational constant obeying the experimental constraint [13]-[16]

$$|\delta| < 0.001.$$  \hspace{1cm} (3.17)

Here, as in the model with two form fields and two scalar fields [8], $\tau$ is restricted to a certain range containing $\tau_0$. It follows from (3.14) that in the asymptotical regions (3.2) and (3.3) $\delta \to 2$, which is unacceptable due to the experimental bounds (3.17). This restriction is satisfied for a range containing the point $\tau_0$ where $\delta = 0$.

Calculating $\dot{G}$, in the linear approximation near $\tau_0$, we get the following approximate relation for the dimensionless parameter of relative variation of $G$:

$$\delta \approx (8 + 2|A|^{-1})H_0(\tau - \tau_0),$$  \hspace{1cm} (3.18)

where $H_0 = H(\tau_0)$. This relation gives approximate bounds for values of the time variable $\tau$ allowed by the restriction on $\dot{G}$. A derivation of this relation is given in Appendix B.

The solutions under consideration with $P < 0$, one-dimensional $M_1$ and 3D subspace $\mathbb{R}^3$ take place when the configuration of 2-form fields, the matrix $(h_{\alpha\beta})$ and the dilatonic coupling vectors $\lambda_a$, obey the relations (2.9), (2.10) and $K_s < 0$. This is possible when $(h_{\alpha\beta})$ is not positive-definite, otherwise all $K_s > 0$. Thus, there should be at least one scalar field with a negative kinetic term (i.e., a phantom scalar field).

4 Example: a model with three phantom fields

Let us consider the following example of the cosmological solutions: $l = 3$, $(h_{\alpha\beta}) = - (\delta_{\alpha\beta})$, $w = -1$, i.e. there are three phantom scalar fields.

Using (2.9), (2.10) and (2.16) we get that the coupling vectors obey the following relations:

$$\lambda^2 = \vec{\lambda}_1^2 = \vec{\lambda}_2^2 = 1 + \frac{1}{2-D} - K, \quad \vec{\lambda}_3^2 = 1 + \frac{1}{2-D} - K',$$  \hspace{1cm} (4.1)
where $K < 0$ and $K' = \frac{k_2}{k_1} K$, $k_1 = 1, 2, 1$, $k_2 = 1, 1, 2$ for algebras $A_3$, $B_3$ and $C_3$, respectively. It was verified (i.e. by the use of Mathematica) that the matrix of scalar products

\[
\begin{pmatrix}
M - K & M + \frac{1}{2} K & M \\
M + \frac{1}{2} K & M - K & M + \frac{k_2}{2} K \\
M & M + \frac{k_2}{2} K & M - \frac{k_1}{k_1} K
\end{pmatrix}
\]

with $M = 1 + \frac{1}{2 - D} > 0 \ (D \geq 5)$, is positive definite for all $K < 0$ and hence the set of vectors obeying (4.1), (4.2) does exist. Thereby \(\langle \vec{\lambda}_s \vec{\lambda}'_s \rangle\) is the Gramian matrix.

Now we compare the $A$ parameters corresponding to different Lie algebras $A_3$, $B_3$ and $C_3$, when the parameter $K$ is fixed. We get from the definition (2.27)

\[
A = \frac{1}{K(D - 2)} \left( n_1 + n_2 + n_3 \frac{k_1}{k_2} \right)
\]

or, in detail, (see (2.18))

\[
A_{(1)} = \frac{10}{K(D - 2)}, \quad A_{(2)} = \frac{28}{K(D - 2)}, \quad A_{(3)} = \frac{35}{2K(D - 2)}
\]

for Lie algebras $A_3$, $B_3$ and $C_3$, respectively. Hence,

\[
|A_{(1)}| < |A_{(3)}| < |A_{(2)}|.
\]

Due to the relation (3.18) for the dimensionless parameter of the relative variation of $G$ calculating in the leading approximation when $(\tau - \tau_0)$ is small, we get for approximate values of $\delta$: $\delta_{(1)}^{(op)} > \delta_{(3)}^{(op)} > \delta_{(2)}^{(op)}$ that means that the variation of $G$ (calculated near $\tau_0$) decreases in the sequence of Lie algebras $A_3$, $C_3$ and $B_3$, but the allowed interval $\Delta \tau = \tau - \tau_0 \ (obeying |\delta| < 0.001)$ increases in the sequence of Lie algebras $A_3$, $C_3$ and $B_3$. This effect could be strengthen (even drastically) when $|K_1|$ becomes larger. We note that for $|K| \to +\infty$ we get a strong coupling limit $\vec{\lambda}_a^2 \to +\infty$, $a = 1, 2, 3$, due to the relations (4.1).

5 Solutions with asymptotically exponential acceleration

We deal here with a special case of the solutions (2.22) - (2.24) when the parameter $A = -1$. As in the previous section we have three scalar “phantom” fields and the dimension is arbitrary.

In this case the relations for the vector couplings (4.1)-(4.2) remain unchanged. Due to $A = -1$ the synchronous time variable $\tau = \tau(t)$ is defined by the relation:

\[
\tau = \int_0^t dt [X(t)]^{-1} = \frac{\text{Arctanh}(\sqrt{|P|}t)}{\sqrt{|P|}}.
\]
The function $\tau = \tau(t)$ is monotonically increasing from 0 to $+\infty$ for $t \in (0, t_1)$, where $t_1 = |P|^{-1/2}$.

The solutions are defined on the manifold

$$M = (0, +\infty) \times M_1 \times M_2,$$

and $M_2$ is a $(D-2)$-dimensional Ricci-flat factor-space. Using the variable $\tau$ we can write the solutions in terms of the synchronous time variable:

$$g = -d\tau \otimes d\tau + \bar{X}^{2(D-3)}Y^2 dy \otimes dy + \bar{X}^{-2}g^2,$$

$$\varphi^a = -h \left( n_1 \lambda_{1\alpha} + n_2 \lambda_{2\alpha} + n_3 \lambda_{3\alpha} \frac{k_1}{k_2} \right) \ln \bar{X},$$

$$F^1 = -Q_1 \bar{X}^{n_2-2n_1+1} Y d\tau \wedge dy,$$

$$F^2 = -Q_2 \bar{X}^{n_1-2n_2+k_1n_3+1} Y d\tau \wedge dy,$$

$$F^3 = -Q_3 \bar{X}^{k_2n_2-2n_3+1} Y d\tau \wedge dy,$$

where

$$\bar{X} = \frac{1}{\cosh^2 \sqrt{|P|\tau}}, \quad Y = \frac{\tanh(\sqrt{|P|\tau})}{\sqrt{|P|}},$$

$$h = \frac{2 - D}{3 - D - \lambda^2(2 - D)}$$

the charge density parameters $Q_s$ satisfy (2.19), $s = 1, 2, 3$. Due to (4.5) $K(D-2) = (-10, -28, -35/2)$ for Lie algebras $A_3$, $B_3$ and $C_3$, respectively.

In the factor-space $M_2 = \mathbb{R}^3 \times (S^1)^{n-4}$ we single out ”our” three-dimensional space with the scale factor

$$a_3 = \bar{X}^{-1}.$$

and when $\tau \to +\infty$ we get the asymptotic formula

$$a_3 \sim \frac{1}{4} \exp(2\tau/t_1).$$

Hence our three-dimensional space expands exponentially and $a \to +\infty$ as $\tau \to +\infty$.

Now, let us consider the variation of $G$. Using the mechanism examined earlier we get the approximate relation for dimensionless variation of $G$

$$\delta \approx 10H_0(\tau - \tau_0).$$

6 Conclusions

We have considered a family of exact cosmological solutions in the multidimensional model with scalar and Abelian gauge fields. We have singled out the solutions corresponding to rank-3 Lie algebras.

Here, as for electric S-brane solutions [8], we have found that there exists a time interval where accelerated expansion of our 3-dimensional space is compatible with a small enough value of $\dot{G}/G$ obeying the experimental bounds. This interval contains a point of minimum of the function $G(\tau)$.
denoted as $\tau_0$. It was shown there should be at least one scalar field with negative kinetic term to ensure an accelerated expansion of 3D space.

We have analyzed the special solutions with three phantom scalar fields for the Lie algebras $A_3, B_3, C_3$. In the vicinity of the point $\tau_0$ the time variation of $G(\tau)$ (calculated in the linear approximation) decreases in the sequence of Lie algebras $A_3, C_3$ and $B_3$. A generalization of this result to the case of $m$-forms will be a subject of a separate publication.

Among these solutions an example of solutions with exponential dependence of the scale factor $a_3$ (w.r.t. synchronous time variable) was presented.

### Appendix

#### A General form of polynomials

**$A_3$-case.** The polynomials for the $A_3$-case read as follows

\[
H_1 = 1 + P_1 z + \frac{1}{4} P_1 P_2 z^2 + \frac{1}{36} P_1 P_2 P_3 z^3, \\
H_2 = 1 + P_2 z + \left( \frac{1}{4} P_1 P_2 + \frac{1}{4} P_2 P_3 \right) z^2 + \frac{1}{9} P_1 P_2 P_3 z^3 + \frac{1}{144} P_1 P_2^2 P_3 z^4, \\
H_3 = 1 + P_3 z + \frac{1}{4} P_2 P_3 z^2 + \frac{1}{36} P_1 P_2 P_3 z^3. 
\]

**$B_3$-polynomials.**

For the Lie algebra $B_3$ we get the following polynomials

\[
H_1 = 1 + P_1 z + \frac{1}{4} P_1 P_2 z^2 + \frac{1}{18} P_1 P_2 P_3 z^3 + \frac{1}{144} P_1 P_2 P_3^2 z^4 + \frac{1}{3600} P_1 P_2^2 P_3^2 z^5 + \frac{1}{129600} P_1^2 P_2^2 P_3^2 z^6, \\
H_2 = 1 + P_2 z + \left( \frac{1}{4} P_1 P_2 + \frac{1}{2} P_2 P_3 \right) z^2 + \left( \frac{1}{9} P_1 P_2 P_3 + \frac{2}{9} P_1 P_2 P_3 \right) z^3 + \left( \frac{1}{144} P_1 P_2^2 P_3 \right) z^4 + \frac{1}{72} P_1 P_2 P_3 \right) z^5 + \frac{7}{600} P_1 P_2^2 P_3^2 z^6 + \left( \frac{1}{16} P_1 P_2 P_3 \right) z^7 + \left( \frac{1}{1600} P_1 P_2^3 P_3^3 \right) z^8 + \frac{1}{2592} P_1 P_2^3 P_3^3 z^9 + \frac{1}{4665600} P_1^2 P_2^3 P_3^3 z^{10} + \frac{1}{518400} P_1^3 P_2^3 P_3^3 z^{11}, \\
H_3 = 1 + P_3 z + \frac{1}{4} P_2 P_3 z^2 + \left( \frac{1}{36} P_1 P_2 P_3 + \frac{1}{36} P_2 P_3^2 \right) z^3 + \frac{1}{144} P_1 P_2 P_3^2 z^4 + \frac{1}{3600} P_1 P_2^2 P_3^2 z^5 + \frac{1}{129600} P_1^2 P_2^2 P_3^2 z^6. 
\]

**$C_3$-polynomials.**

The $C_3$-polynomials have the following form
\[ H_1 = 1 + P_1 z + \frac{1}{4} P_1 P_2 z^2 + \frac{1}{36} P_1 P_2 P_3 z^3 + \frac{1}{576} P_1 P_2^2 P_3 z^4 + \frac{1}{14400} P_1^2 P_2^2 P_3 z^5, \quad (A.7) \]

\[ H_2 = 1 + P_2 z + \left( \frac{1}{4} P_1 P_2 + \frac{1}{4} P_2 P_3 \right) z^2 + \left( \frac{1}{36} P_1^2 P_2 P_3 + \frac{1}{9} P_1 P_2 P_3 \right) z^3 + \frac{7}{288} P_1 P_2^2 P_3 z^4 \quad (A.8) \]

\[ + \left( \frac{1}{576} P_1 P_2^3 P_3 + \frac{1}{900} P_1^2 P_2^2 P_3 \right) z^5 + \left( \frac{1}{6400} P_1^2 P_3^2 P_3 + \frac{1}{20736} P_1 P_2^3 P_3 \right) z^6 \]
\[ + \frac{1}{129600} P_1^2 P_2^3 P_3 z^7 + \frac{1}{8294400} P_1^2 P_4^2 P_3 z^8, \]

\[ H_3 = 1 + P_3 z + \frac{1}{2} P_2 P_3 z^2 + \left( \frac{1}{18} P_1 P_2 P_3 + \frac{1}{9} P_2^2 P_3 \right) z^3 + \left( \frac{1}{144} P_2^2 P_3^2 + \frac{1}{32} P_1 P_2^2 P_3 \right) z^4 \quad (A.9) \]
\[ + \left( \frac{1}{288} P_1 P_2^2 P_3^2 + \frac{1}{400} P_1^2 P_2^2 P_3 \right) z^5 + \left( \frac{1}{2025} P_1^2 P_3^2 P_3 + \frac{1}{10368} P_1 P_2^3 P_3 \right) z^6 \]
\[ + \frac{1}{28800} P_1^2 P_2^3 P_3^2 z^7 + \frac{1}{921600} P_1^2 P_4^2 P_3 z^8 + \frac{1}{74649600} P_1^2 P_4^2 P_3 z^9. \]

\section{B The dimensionless variation of $G$}

Here we give a derivation of the relation (3.18) for an approximate value of the dimensionless parameter of relative variation of $G$.

We start with the relation (3.14) written in the following form

\[ \delta = \frac{\dot{G}}{(GH)} = \frac{t^2 - t_0^2}{|AP||t_0^2|}, \quad (B.10) \]

where $H = \dot{a}_3/a_3$ is the Hubble parameter and $t_0$ is defined in (3.16). (We recall that here and in what follows $\dot{f} = df/d\tau$.) In the vicinity of the point $t_0$ we get in linear approximation

\[ \delta \approx \frac{\Delta t}{|AP||t_0^2|}. \quad (B.11) \]

Using the synchronous time variable $\tau = \tau(t)$ we get

\[ \delta \approx \left( \frac{dt}{d\tau} \right)_0 \frac{\Delta \tau}{|AP||t_0^2|} = \frac{\Delta \tau}{X_0 A|AP||t_0^2}, \quad (B.12) \]

$\left( d\tau/dt = X^A \right)$. Here the subscript "0" refers to $t_0$. From (3.6) we can find $\dot{a}_3$

\[ \dot{a}_3 = \frac{dt}{d\tau} \frac{da_3}{dt} = \frac{2|A||P|t}{X}, \quad (B.13) \]

and thereby for the Hubble parameter we get

\[ H_0 = \left( \frac{\dot{a}_3}{a_3} \right)_0 = \frac{2|A||P|t_0}{X_0 A+1}. \quad (B.14) \]

Then, it follows from (B.12) and (B.14) that

\[ \delta \approx \frac{X_0}{2A^2 P_2 t_0^2} H_0 \Delta \tau. \quad (B.15) \]

Since (see (2.21) and (3.16))

\[ X_0 = \frac{4|A|}{1 + 4|A|}, \quad (B.16) \]

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the pre-factor in (B.15) reads (see (3.16)):

$$\Pi = \frac{X_0}{2A^2P^2t^4_0} = 8 + 2|A|^{-1}$$  \hspace{1cm} (B.17)

and we are led to the relation

$$\delta \approx \Pi H_0(\tau - \tau_0),$$  \hspace{1cm} (B.18)

coinciding with (3.18).

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