Anyonic excitations of hardcore anyons

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Strongly interacting many-body systems consisting of fermions or bosons can host exotic quasi-particles with anyonic statistics. Here, we demonstrate that many-body systems of anyons can also form anyonic quasi-particles. The charge and statistics of the emergent anyons can be different from those of the original anyons.

I. INTRODUCTION

The contemporary literature provides many examples of two-dimensional systems consisting of many interacting bosons or fermions that have quasi-particle excitations with unusual properties. These unusual properties are, for example, that the quasi-particle excitations carry a fractional charge, the existence of nontrivial ground state degeneracies on topologically nontrivial surfaces such as a torus and that the mutual statistics are neither bosonic nor fermionic, but of exotic type 5, 6.

Much work has already been done to investigate and classify the different types of anyons that can appear in a lattice in the complex plane. 5, 6

We use $n_j$ to denote the number of particles on site $j$. The considered states take the form

$$\psi_q = \sum_{n_1, n_2, \ldots, n_N} \psi_q |n_1, n_2, \ldots, n_N\rangle,$$  

where

$$\psi_q = C^{-1} \delta_n \prod_{i<j} (z_i - z_j)^{q n_i} \prod_{i \neq j} (z_i - z_j)^{-n_i}.$$  

Here, $C$ is a real normalization constant and $\delta_n$ is one if there are $\sum_i n_i = N/q$ particles in the system and zero otherwise. For integer $q$, the states (2) are the normal Laughlin states except that both the particles and the background charge are restricted to be on the specified lattice sites. 5, 6

We can also express the states in the alternative form

$$\psi_q \propto \delta_n \prod_{i<j} (Z_i - Z_j)^q \prod_{i,j \mid Z_i \not= Z_j} (Z_i - Z_j)^{-1},$$  

where $Z_j \in \{z_1, z_2, \ldots, z_N\}$ is the position of the $j$th particle. From this expression, we observe that the wave-function acquires a phase factor $e^{2\pi i q}$ if one of the particles is moved counter clockwise around one of the other particles. The state hence describes fermions if $q$ is odd, hardcore bosons if $q$ is even, and hardcore anyons if $q$ is non-integer.

III. ANYONS AND BRAIDING

For the normal Laughlin states with integer $q$, one can add $Q$ anyons of charge $p_j/q$, with $p_j \in \mathbb{N}$, at the positions $w_j$ in the complex plane by introducing an extra factor in the wavefunction and reducing the number of particles accordingly:

$$\psi_{q, \vec{w}} = C_{\vec{w}}^{-1} \delta_n \prod_{i,j} (w_i - z_j)^{p_i n_j} \prod_{i<j} (z_i - z_j)^{q n_i} \times \prod_{i \neq j} (z_i - z_j)^{-n_i}.$$  

Here, $\vec{w} = (w_1, w_2, \ldots, w_Q)$, $C_{\vec{w}}$ is a real normalization constant, and $\delta_n = 1$ for $\sum_{j=1}^Q p_j = (N - \sum_{j=1}^Q p_j)/q$.
and $\delta_n = 0$ otherwise. We now investigate what happens if we do the same for the states with non-integer $q$.

We would like to determine the result of braiding the coordinate $w_k$ around the coordinate $w_j$. The Berry phase $\theta$ acquired by the wave function when $w_k$ is moved along the closed path $c$ is

$$\theta_k = i \int_c \frac{\partial \langle \psi_{q,w} \rangle}{\partial w_k} dw_k + \text{c.c.}$$

$$= \frac{p_k}{2} \int_c \sum_i \frac{\langle \psi_{q,w} | n_i | \psi_{q,w} \rangle}{w_k - z_i} dw_k + \text{c.c.}, \quad (5)$$

where c.c. is the complex conjugate of the first term. We are interested in $\Delta \theta_k = \theta_{k,\text{in}} - \theta_{k,\text{out}}$, where $\theta_{k,\text{in}}$ ($\theta_{k,\text{out}}$) is the Berry phase when $w_j$ is well inside (outside) the closed path $c$. We have

$$\Delta \theta_k = i \frac{p_k}{2} \int_c \sum_i \frac{\langle n_i \rangle_{\text{in}} - \langle n_i \rangle_{\text{out}}}{w_k - z_i} dw_k + \text{c.c.} \quad (6)$$

In the normal Laughlin state at $q = 3$, the anyons are screened, and hence the density of particles is only modified locally if anyons are present. If screening also applies to the state $| n \rangle$ for $q$ non-integer, we have that $\langle n_i \rangle_{\text{in}} - \langle n_i \rangle_{\text{out}}$ is nonzero only close to the two possible positions of $w_j$ and is independent of $w_k$. We can then take $\langle n_i \rangle_{\text{in}} - \langle n_i \rangle_{\text{out}}$ outside the integral, which leads to

$$\Delta \theta_k = -2\pi p_k \sum_{i \text{ inside } c} (\langle n_i \rangle_{\text{in}} - \langle n_i \rangle_{\text{out}}). \quad (7)$$

The sum is precisely minus the charge of the anyon at position $w_j$, and it follows that

$$\Delta \theta_k = 2\pi p_k p_j / q. \quad (8)$$

This is the same result as the Berry phase for the Laughlin states, except that $q$ is now a non-integer. We hence conclude that the state $| n \rangle$ hosts anyons with charges $p_j / q$ with $p_j \in \mathbb{N}$ provided the state is screening, i.e. provided $\langle n_i \rangle$ for the state with anyons only differ from $\langle n_i \rangle$ for the state without anyons in the vicinity of the anyons. In Fig. 1 we show numerical results for $q = 3/2$ and $q = 5/2$, and we indeed observe screening.

**IV. STATES ON THE TORUS**

In this section, we specialize to the case $q$ half integer for simplicity. From (8), we observe that anyons for which the value of $p_j$ differ by a multiple of $q$ have the same braiding properties. Since $q$ is half integer, and $p_j$ is integer, two possible values of $p_j$ cannot differ by an odd number of $q$. The anyons found in the previous section hence include $2q$ different types of anyons. These give rise to $2q$ degenerate states on the torus, which we will now find.

We define a torus from two vectors $\omega_1$ and $\omega_2$ in the complex plane. Without loss of generality, we shall assume that $\omega_1$ is a real, positive number and that $\omega_2$ has a positive imaginary part. We identify points in the complex plane that differ by $n \omega_1 + m \omega_2$ for some choice of integers $n$ and $m$, which gives the periodicity. The modular parameter is defined as $\tau = \omega_2 / \omega_1$. We shall also find it convenient to use the scaled coordinates $\xi_j = z_j / \omega_1$ in place of $z_j$.

We obtain the states on the torus by utilizing that the squared norm of the state $| n \rangle$ can be expressed as a conformal field theory correlator. Specifically,

$$\langle \cdot | e^{i(qn_1 - 1)/2} \xi \rangle = \ldots : e^{i(qN - 1)/2} \xi \rangle \langle \ldots \rangle_{\text{plane}} \propto |\psi_q|^2 \quad (9)$$

where $\ldots$ means normal ordering, $\phi(z,j)$ is the field of a massless, free boson, $\bar{z}$ is the complex conjugate of $z$, and $\langle \ldots \rangle$ is the vacuum expectation value in the conformal field theory. The states on the torus are then...
FIG. 2. Overlap \( \langle \psi_l | \psi_b \rangle \) of the states on an \( L \times L \) square lattice for \( q = 3/2 \) computed with Monte Carlo simulations, where \( |\psi_i\rangle \equiv |\psi_{l/q+a,b}\rangle \) with \( a = b = 0 \) (upper left), \( a = 1/2, b = 0 \) (upper right), \( a = 0, b = 1/2 \) (lower left), and \( a = b = 1/2 \) (lower right). The overlaps in the figure are scaled such that \( \sum_{i=1}^{L^2} \langle \psi_i | \psi_i \rangle = 1 \). It is seen that the states \( |\psi_i\rangle \) become orthogonal for large \( L \). The insets show the same data on a semi-log scale, and the lines are linear fits.

obtained by evaluating the conformal field theory correlator on the torus rather than in the plane. On the torus the correlator decomposes as

\[
\langle \psi^\prime | e^{i(qn_1-1)\phi(z_i,\bar{z}_i)/\sqrt{q}} \cdots e^{i(qn_N-1)\phi(z_N,\bar{z}_N)/\sqrt{q}} | \psi \rangle_{\text{torus}} \propto \sum_{l=0}^{2q-1} \sum_{a \in \{0,1/2\}} \sum_{b \in \{0,1/2\}} |\psi_{l/q+a,b}\rangle^2,
\]

and each of the \( |\psi_{l/q+a,b}\rangle \) are considered to be a state on the torus.

Following a derivation similar to the one in Ref. 3 for integer \( q \), we find explicitly

\[
|\psi_{l/q+a,b}\rangle = C_{l,a,b}^{-1} \delta_{n} \prod_{i<j} \theta^{1/2} \left[ \frac{1}{2} \right] (\xi_i - \xi_j, \tau)^{q n_i n_j} \times \theta^{(l/q + a)/b} \left[ \frac{1}{2} \right] (\xi_i - \xi_j, \tau)^{-n_i}
\]

where \( \xi \equiv \sum_{i=1}^{N} \zeta_i (q n_i - 1) \) and

\[
\theta^{[a/b]}(\xi, \tau) \equiv \sum_{n \in \mathbb{Z}} e^{i \pi \tau (n+a)^2 + 2i \pi (n+a)(\xi+b)}
\]

is the theta function. The possible values of \( a \) and \( b \) correspond to different boundary conditions and should hence be ignored when counting the number of states on the torus. Provided the states are linearly independent, we hence have \( 2q \) states on the torus.

To test whether the states are indeed independent, we have done a Metropolis Monte Carlo simulation to determine the overlap between the states for the case of \( q = 3/2 \). Figure 2 shows the overlap as a function of the system size for \( L \times L \) lattices. The overlap is seen to decay rapidly, indicating that the states are orthogonal in the thermodynamic limit.
V. CONCLUSION

We have found that systems of anyons can support the formation of anyonic quasiparticles, and that the charge and braiding properties of the emergent anyons can differ from the properties of the original anyons. We have also shown that Laughlin states with noninteger $q$ provide models of anyons, where these phenomena occur. Braiding one of the original anyons around another one gives a phase factor of $e^{i2\pi q}$ on the wavefunction, while braiding one of the emergent anyons of charge $p_k/q$ around another one of charge $p_j/q$ with $p_k$ and $p_j$ positive integers gives a phase factor of $e^{i2\pi p_k p_j/q}$ on the wavefunction if there is screening in the system. We have shown numerically that screening occurs for $q = 3/2$ and $q = 5/2$. Given the plasma analogy for the Laughlin wavefunctions, we expect screening to occur for all $q$ below a certain value (which is about 70 for continuous systems$^{10}$).

The work motivates several further investigations. In particular, it would be interesting to see which types of anyons can be hosted by systems of various types of non-Abelian anyons. It would also be interesting to look for possible physical implementations and to investigate which implications the possibility to form anyons of anyons have on which types of quantum gates can be done within a given system.

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