Explicit Calculations of Tensor Product Coefficients for $E_7$

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Abstract

We propose a new method to calculate coupling coefficients of $E_7$ tensor products. Our method is based on explicit use of $E_7$ characters in the definition of a tensor product.

When applying Weyl character formula for $E_7$, Lie algebra, one needs to make sums over 2903040 elements of $E_7$ Weyl group. To implement such enormous sums, we show we have a way which makes their calculations possible. This will be accomplished by decomposing an $E_7$ character into 72 participating $A_7$ characters.

Keywords: Coupling coefficients; Lie algebra; Irreducible representations; Subdominants; Tensor coupling coefficients

Introduction

Let $G = E_7$, $\Lambda$, $\Lambda'$ be two dominant weights of $G$, where $R(\Lambda)$ and $R(\Lambda')$ are corresponding irreducible representations. For general terms, we follow the book of Humphreys [1] as ever.

Tensor product of these two irreducible representations is defined by,

$$R(\Lambda) \otimes R(\Lambda') = R(\Lambda + \Lambda') + \sum_{\lambda \in \Lambda + \Lambda'} t(\lambda < \Lambda + \Lambda') \ (\lambda) \ (1.1)$$

where $S(\lambda + \lambda')$ is the set of $\Lambda + \Lambda'$ subdominants and $t(\lambda < \Lambda + \Lambda')$ s are tensor coupling coefficients. Though Steinberg formula is the best known way, a natural way to calculate tensor coupling coefficients is also to solve the equation

$$Ch(\Lambda) \otimes Ch(\Lambda') = Ch(\Lambda + \Lambda') + \sum_{\lambda \in \Lambda + \Lambda'} t(\lambda < \Lambda + \Lambda') \ (\lambda) \ (1.2)$$

for tensor coupling coefficients. $Ch(\lambda)$ here is the character of an irreducible representation $R(\lambda)$ which corresponds to a dominant weight $\lambda$ and it is defined by the famous Weyl Character formula:

$$Ch(\lambda') = \frac{A(\lambda')}{A(\rho)} \ (1.3)$$

where for a weight $\mu$ in general

$$A(\mu) = \sum_{\sigma \in G/\sigma} \epsilon(\sigma) e^{n(\mu)} \ (1.4)$$

$W(G)$ is the Weyl Group of $G$, and each and every element $\sigma$ is the so-called Weyl reflection while $\epsilon(\sigma)$ denotes its sign and $e^{n(\mu)}$’s here are known as formal exponentials. Throughout this work, we assume $\lambda' \otimes \lambda''$ denotes a strictly dominant weight defined for a dominant $\lambda'$ by

$$\lambda'{}^+ = \rho_{\lambda'} + \lambda' \ (1.5)$$

where $\rho_{\lambda'}$ is the Weyl vector of $G_{\lambda'}$.

The crucial fact here is that

$$||W(E_7)|| = 2903040 \ (1.6)$$

where $|S|$ denotes order of set $S$. It is easy to see then to implement the sum in (1.4) would not be realizable explicitly. We, instead, propose 72 specifically chosen Weyl reflections which give us $A_7$ dominant weights participating within the same $E_7$ Weyl orbit $W(\Lambda')$ for any $E_7$ dominant weight $\Lambda'$. As it is shown in the next section, this makes the evaluation of (1.4) realizable for $E_7$ but in terms of 72 $A_7$ characters and hence easily implementable.

A. Decomposition of $E_7$ Lie Algebra

For $i = 1, 2, \ldots, 7$, let $\lambda_i$’s and $\alpha_i$’s be respectively the fundamental dominant weights and simple roots of $A_7$ Lie algebra with the following Dynkin diagram (Figure 1).

![Figure 1: Dynkin diagram 1.](image1)

where $\rho_{\lambda_i} = \lambda_i + \ldots + \lambda_i$ is $A_7$ Weyl vector and $\Lambda$ is fundamental dominant weight of $E_7$ Lie algebra in according with the following Dynkin diagram, (Figure 2).

![Figure 2: Dynkin diagram 2.](image2)

where $\rho_{\lambda_i} = \lambda_i + \ldots + \lambda_i$ is $A_7$ Weyl vector. We suggest following relations allows us to embed $A_7$ subalgebra into $E_7$ algebra:

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\[ \lambda_1 = \lambda_2 \]
\[ \lambda_2 = \lambda_1 + \lambda_3 \]
\[ \lambda_3 = 2 \lambda_3 \]
\[ \lambda_4 = 2 \lambda_3 + \lambda_4 \]
\[ \lambda_5 = \lambda_5 + \lambda_5 \]
\[ \lambda_6 = \lambda_4 \]
\[ \lambda_7 = \lambda_2 + \lambda_5 \quad (II.1) \]

This essentially means
\[ \frac{||W(E_7)||}{||W(A_7)||} = 72 \quad (II.2) \]

which tells us that there are at most 72 \( A_7 \) dominant weights inside a Weyl orbit \( W(\Lambda^+) \). Note here that it is exactly 72 when \( \Lambda^+ \) is a strictly dominant weight. From the now on, \( W(\mu) \) will always denotes the Weyl orbit of a weight \( \mu \).

As the main point of view of this work, we present in appendix, 72 Weyl reflections to give 72 \( A_7 \) dominant weights participating in the same \( E_7 \) Weyl orbit \( W(\Lambda^+) \) when they are exerted on the dominant weight \( \Lambda^+ \). To this end, the Weyl reflections with respect to simple roots \( \alpha_i \) will be called simple reflections \( \sigma_i \). We extend multiple products of simple reflections trivially by
\[ \sigma_{\frac{1}{2}}(\lambda) = \sigma_i(\sigma_{\frac{1}{2}}(\lambda)). \]

For \( s = 1, \ldots, 72 \), \( \Sigma(s) \)'s are 72 Weyl reflections mentioned above. As will also be seen by their definitions that,

1. \( e(\Sigma(s)) = +1 \quad s = 1, 2, \ldots, 36 \)
2. \( e(\Sigma(s)) = -1 \quad s = 37, 38, \ldots, 72 \)

Calculating Tensor Coupling Coefficients

Let us proceed in the instructive example
\[ R(\Lambda_1) \otimes R(\Lambda_3) = R(\Lambda_1 + \Lambda_3) + \sum_{j=1}^{36} m(j)(\theta_j) (III.1) \]
of (I.1). One can see that there are 39 sub-dominant weights \( \theta_j \) of \( \Lambda_1 + \Lambda_3 \):
\[ \theta_1 = \Lambda_1 + 2 \Lambda_3 \]
\[ \theta_2 = \Lambda_2 + \Lambda_3 + \Lambda_4 \]
\[ \theta_3 = \Lambda_1 + 2 \Lambda_2 \]
\[ \theta_4 = 2 \Lambda_1 + \Lambda_3 \]
\[ \theta_5 = \Lambda_1 + 2 \Lambda_2 \]
\[ \theta_6 = \Lambda_1 + \Lambda_3 + \Lambda_4 \]
\[ \theta_7 = \Lambda_1 + \Lambda_3 + 2 \Lambda_4 \]
\[ \theta_8 = \Lambda_1 + \Lambda_4 + \Lambda_5 \]
\[ \theta_9 = \Lambda_1 + \Lambda_5 + \Lambda_6 \]
\[ \theta_{10} = \Lambda_2 + \Lambda_3 + \Lambda_5 \]
\[ \theta_{11} = \Lambda_2 + \Lambda_3 + \Lambda_6 \]
\[ \theta_{12} = \Lambda_2 + \Lambda_3 + \Lambda_7 \]

To this end, we should care about specialization of formal exponentials [2]. Let us consider the so-called Fundamental Weights \( \mu_i \) which are defined for \( i = 1, \ldots, 8 \) as in the following [3]:
\[ a_i \equiv \mu_i - \mu_i, \quad (i = 1, \ldots, 7) \quad (III.2) \]
a\'s here are \( A_i \) simple roots mentioned above and the best way to calculate, and hence \( E_i \) characters is to use the specialization in terms of parameters \( u_i = e^{i \theta_i} \) which are subjects of the condition \( \mu_i + \mu_i + \cdots + \mu_i = 0 \) or
\[ u_i u_j u_k \cdots u_i = 1. \]

To exemplify (I.3) for \( E_7 \), we would like to give detailed calculation of \( Ch(\Lambda_1 + \Lambda_3) \). By applying 72 specifically chosen Weyl reflections on strictly dominant weight \( \rho_{\varepsilon_7} + \Lambda_1 + \Lambda_4 \), one can see we have the following decompositions:
\[ A(\rho_{\varepsilon_7} + \Lambda_1 + \Lambda_4) = \sum_{k=1}^{36} Ch(v_k) - \sum_{j=37}^{72} Ch(v_j) (III.3) \]

where
\(v_1 = 2 \lambda_1 + 4 \lambda_6 + 2 \lambda_9 + 12 \lambda_3 + 12 \lambda_6 + 2 \lambda_{12}
\)
\(v_2 = 3 \lambda_1 + 2 \lambda_6 + 4 \lambda_9 + 12 \lambda_3 + 12 \lambda_6 + 2 \lambda_{12}
\)
\(v_3 = 2 \lambda_1 + 2 \lambda_6 + 4 \lambda_9 + 12 \lambda_3 + 12 \lambda_6 + 2 \lambda_{12}
\)
\(v_4 = 4 \lambda_1 + 12 \lambda_3 + 2 \lambda_6 + 4 \lambda_9 + 12 \lambda_3 + 12 \lambda_6 + 2 \lambda_{12}
\)
\(v_5 = 2 \lambda_1 + 2 \lambda_6 + 4 \lambda_9 + 12 \lambda_3 + 12 \lambda_6 + 2 \lambda_{12}
\)
\(v_6 = 7 \lambda_1 + 2 \lambda_6 + 3 \lambda_9 + 12 \lambda_3 + 12 \lambda_6 + 2 \lambda_{12}
\)
\(v_7 = 7 \lambda_1 + 2 \lambda_6 + 3 \lambda_9 + 12 \lambda_3 + 12 \lambda_6 + 2 \lambda_{12}
\)
\(v_8 = 7 \lambda_1 + 2 \lambda_6 + 3 \lambda_9 + 12 \lambda_3 + 12 \lambda_6 + 2 \lambda_{12}
\)
\(v_9 = 7 \lambda_1 + 2 \lambda_6 + 3 \lambda_9 + 12 \lambda_3 + 12 \lambda_6 + 2 \lambda_{12}
\)

Note here that \(W(A_3)\) is the permutation group of 8 objects.

To display our result here, we use the following specialization of formal exponentials with only one free parameter \(x\):

\[u_1 = 1\]

\[u_2 = 2\]

\[u_3 = 3\]

\[u_4 = 4\]

\[u_5 = 5\]

\[u_6 = 6\]

\[u_7 = x\]

\[u_8 = 1/ (720 x)\] (III. 5)

In this specialization, one obtains the following one-parameter characters:

\[A(\rho_{u_n}) = -\frac{1}{2^2 \times 3 \times 5^2 \times 7^2} x \times \]

\[(-6 + x) \times (-5 + x) \times (-4 + x) \times (-3 + x) \times (-2 + x) \times \]

\[(-1 + x) \times (-1 + 720 x) \times (-1 + 1440 x) \times (-1 + 2160 x) \times \]

\[(-1 + 2880 x) \times (-1 + 3600 x) \times (-1 + 4320 x) \times (-1 + 720 x^2)\]
\( \Delta(r) = \frac{7}{2} n(n+1) + \frac{5}{2} n(n-1) + \frac{3}{2} n(n-2) + \frac{1}{2} n(n-3) \)
\( (r = x, y, z) \)

\( \chi(\Lambda) = \frac{1}{2} \times \begin{cases} 2 \cdot \chi(x)^2 & \text{if } \Lambda = \chi(x)^2 \\ 2 \cdot \chi(y)^2 & \text{if } \Lambda = \chi(y)^2 \\ 2 \cdot \chi(z)^2 & \text{if } \Lambda = \chi(z)^2 \end{cases} \)

\( \chi(\theta) = \frac{1}{2^n \cdot 3^m \cdot 5^p} \chi(x) \)

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\[
ch(\theta_l) = \frac{7}{2^3 \times 3^2 \times 5^2 \times x} (\text{for } x = 70782069928080 + 2^3 \times 3 \times 7 \times 19 \times 101 \times 2437 \times 7883 \times 45307 \times x + 3^3 \times 839 \times 2591 \times 5665016976003 \times x^2 + 5^2 \times 127 \times 74352660418728544311 \times x^3 + 2^3 \times 5 \times 83 \times 263 \times 6791 \times 13901 \times 522761 \times 27245893 \times x^4 + 2^2 \times 1439 \times 6833 \times 57110377 \times 29440768222739 \times x^5 + 2^2 \times 3 \times 7 \times 2833 \times 5453185631 \times 1412153017963 \times x^6 + 2^2 \times 3 \times 5 \times 7 \times 37577587 \times 1652664511 \times 109347145543 \times x^7 + 2^2 \times 3 \times 5 \times 7 \times 2923 \times 5453185631 \times 1412153017963 \times x^8 + 2^2 \times 3 \times 5^4 \times 1439 \times 6833 \times 57110377 \times 29440768222739 \times x^9 + 2^4 \times 3^2 \times 5 \times 83 \times 263 \times 6791 \times 13901 \times 522761 \times 27245893 \times x^{10} + 2^4 \times 3^5 \times 5 \times 7 \times 127 \times 74352660418728544311 \times x^{11} + 2^3 \times 5^2 \times 839 \times 2591 \times 5665016976003 \times x^{12} + 2^3 \times 5^2 \times 5 \times 7 \times 19 \times 101 \times 2437 \times 7883 \times 45307 \times x^{13} + 2^3 \times 3^5 \times 5^2 \times 19 \times 101 \times 2437 \times 7883 \times x^{14})
\]

\[
ch(\theta_l) = \frac{1}{2^3 \times 3 \times 5^2 \times x} (\text{for } x = 5 \times 11 \times 1730263 \times 475374719 + 3 \times 5 \times 7 \times 111 \times 4933 \times 15316029186509 \times x + 5 \times 7 \times 13 \times 17283128778277827303 \times x^3 + 3 \times 7^2 \times 205820307755599239132761 \times x + 2^2 \times 3^3 \times 41 \times 707660359 \times 113581475950821 \times x + 2^3 \times 3^4 \times 41 \times 707660359 \times 113581475950821 \times x^2 + 2^3 \times 3^5 \times 41 \times 1252496251 \times 205416650707448 \times x + 2^3 \times 3^6 \times 41 \times 169031 \times 2819023 \times 22718868350084081 \times x + 2^3 \times 3^7 \times 41 \times 1252496251 \times 205416650707448 \times x^2 + 2^3 \times 3^8 \times 41 \times 707660359 \times 113581475950821 \times x^3 + 2^2 \times 3^2 \times 5^2 \times 7 \times 111 \times 4933 \times 15316029186509 \times x^2 + 2^3 \times 3^3 \times 5^2 \times 11 \times 1730263 \times 475374719 \times x^3)
\]

\[
ch(\theta_r) = \frac{1}{2^3 \times 3 \times 5^2 \times x} (\text{for } x = 2^3 \times 3 \times 5 \times 7 \times 6311 \times 700299 + 3 \times 7^2 \times 45307 \times 6311 \times 700299 \times x + 3 \times 5^2 \times 7 \times 17 \times 191 \times 30444055140860819 \times x^2 + 2^3 \times 13 \times 8540249266566531445399 \times x^3 + 2^2 \times 3^2 \times 5 \times 7 \times 137 \times 1619903 \times 40109150434499081 \times x^4 + 2^4 \times 3^3 \times 5 \times 7 \times 13 \times 17283128778277827303 \times x^5 + 2^4 \times 3^4 \times 5 \times 7 \times 11 \times 4933 \times 15316029186509 \times x^6 + 2^4 \times 3^5 \times 5 \times 7 \times 19 \times 101 \times 2437 \times 7883 \times 45307 \times x^7 + 2^4 \times 3^6 \times 5 \times 7 \times 19 \times 101 \times 2437 \times 7883 \times x^8)
\]
\[ ch(\theta_s) = \frac{1}{2^s \times 3^s \times 5^s \times x^7} \]
\[ ch(\theta_s) = \frac{7}{2^s \times 3^s \times 5^s \times x^7} \]
\[
\begin{align*}
\text{ch}(\theta_3) &= \frac{7}{2^n \cdot 3^n \cdot 5^n \cdot x^n} (2^s \cdot 5^s \cdot 11^s \cdot 163450973 + 7^{10} \cdot 11^{18} \cdot 163450973 + 5^{10} \cdot 7^{17} \cdot 2^{11} \cdot 8^{20} \cdot 6^{22} + 2^{12} \cdot 7^{14} \cdot 3^{20} \cdot 11^{23} + 2^{13} \cdot 5^{17} \cdot 3^{21} + 2^{14} \cdot 5^{17} \cdot 7^{21} + 2^{15} \cdot 5^{17} \cdot 11^{21} + 2^{16} \cdot 5^{17} \cdot 17^{21}) \\
\text{ch}(\theta_4) &= \frac{1}{2^n \cdot 3^n \cdot 5^n \cdot x^n} (2^s \cdot 3^s \cdot 5^s \cdot 7^s \cdot 11^s \cdot 14^s \cdot 5^s + 3^s \cdot 5^s \cdot 7^s \cdot 11^s \cdot 16^s \cdot 17^s + 5^s \cdot 7^s \cdot 13^s \cdot 17^s + 7^s \cdot 11^s \cdot 13^s \cdot 17^s + 11^s \cdot 13^s \cdot 17^s + 17^s \cdot 13^s \cdot 17^s + 13^s \cdot 17^s) \\
\end{align*}
\]
\( ch(\theta_2) = \frac{7}{2 \times 3 \times 5 \times x} \)

\[
3 \times 7 \times 41 \times 13449 + 5 \times 25841 \times 172357 \times x + \\
2^8 \times 3 \times 7 \times 17 \times 151 \times 229 \times 76637 \times x^2 + \\
2^3 \times 3^3 \times 7 \times 41 \times 8271569177 \times x^3 + \\
2^3 \times 3^3 \times 5 \times 7 \times 17 \times 151 \times 229 \times 76837 \times x^4 + \\
2^3 \times 3^3 \times 5^3 \times 25841 \times 172357 \times x^5 + \\
2^7 \times 3^3 \times 5^3 \times 7 \times 41 \times 13469 \times x^6.
\]

\( ch(\theta_3) = \frac{1}{2 \times 3 \times 5 \times x} \)

\[
3 \times 7 \times 37 \times 87365639293 + \\
3^2 \times 7 \times 17 \times 25158739 \times 163675601 \times x + \\
2^7 \times 3^7 \times 293207 \times 1532112833 \times x^2 + \\
2^3 \times 3^3 \times 5 \times 7 \times 19 \times 4966016564031243349 \times x^3 + \\
2^7 \times 3^7 \times 5 \times 7 \times 39411116 \times 1532112833 \times x^4 + \\
2^3 \times 3^3 \times 5 \times 7 \times 71 \times 15185739 \times 163675601 \times x^5 + \\
2^3 \times 3^3 \times 5 \times 7 \times 37 \times 87365639293 \times x^6.
\]

\( ch(\theta_4) = \frac{1}{2 \times 3 \times 5 \times x} \)

\[
2^8 \times 3 \times 3 \times 5 \times 7 \times 25841 \times 45307 \times x + \\
5 \times 7 \times 25841 \times 115259 \times 66653 \times x^2 + \\
3 \times 7 \times 55717 \times 2804293 \times 133115567 \times x^3 + \\
2 \times 3^3 \times 8 \times 86399 \times 95972373277265 \times x^4 + \\
2^3 \times 3^3 \times 5 \times 7 \times 5 \times 71 \times 19 \times 372825737 \times 955373927 \times x^5 + \\
2^3 \times 3^3 \times 5 \times 7 \times 71 \times 38241 \times 162705053 \times 3733221 \times x^6 + \\
2^3 \times 3 \times 5 \times 7 \times 39411116 \times 1532112833 \times x^7 + \\
2^3 \times 3 \times 5 \times 7 \times 42 \times 7 \times 13 \times 705053 \times 3733221 \times x^8 + \\
2^3 \times 3 \times 5 \times 7 \times 42 \times 7 \times 13 \times 705053 \times 3733221 \times x^9 + \\
2^3 \times 3 \times 5 \times 7 \times 25841 \times 45307 \times x^{10}.
\]

\( ch(\theta_5) = \frac{1}{2 \times 3 \times 5 \times x} \)

\[
2^8 \times 3 \times 3 \times 5 \times 7 \times 25841 \times 45307 \times x + \\
5 \times 7 \times 25841 \times 115259 \times 66653 \times x^2 + \\
3 \times 7 \times 55717 \times 2804293 \times 133115567 \times x^3 + \\
2 \times 3^3 \times 8 \times 86399 \times 95972373277265 \times x^4 + \\
2^3 \times 3 \times 5 \times 7 \times 5 \times 71 \times 19 \times 372825737 \times 955373927 \times x^5 + \\
2^3 \times 3 \times 5 \times 7 \times 71 \times 38241 \times 162705053 \times 3733221 \times x^6 + \\
2^3 \times 3 \times 5 \times 7 \times 42 \times 7 \times 13 \times 705053 \times 3733221 \times x^7 + \\
2^3 \times 3 \times 5 \times 7 \times 42 \times 7 \times 13 \times 705053 \times 3733221 \times x^8 + \\
2^3 \times 3 \times 5 \times 7 \times 25841 \times 45307 \times x^{10}.
\]

\( ch(\theta_6) = \frac{7}{2 \times 3 \times 5 \times x} \)

\[
2^3 \times 3 \times 5 \times 7 \times 25841 \times 45307 \times x + \\
2^3 \times 3 \times 5 \times 7 \times 17 \times 45307 \times x + \\
3 \times 7 \times 17 \times 115259 \times 66653 \times x^2 + \\
5 \times 2512868354279147 \times x^3 + \\
2^3 \times 3 \times 5 \times 7 \times 1061 \times 218249 \times 31482709 \times x^4 + \\
2^3 \times 3 \times 5 \times 7 \times 2069 \times 634759 \times 1884431 \times x^5 + \\
2^3 \times 3 \times 5 \times 7 \times 1061 \times 218249 \times 31482709 \times x^6 + \\
2^3 \times 3 \times 5 \times 7 \times 2512868354279147 \times x^7 + \\
2^3 \times 3 \times 5 \times 7 \times 17 \times 115259 \times 66653 \times x^8 + \\
2^3 \times 3 \times 5 \times 7 \times 17 \times 45307 \times x^9 + \\
2^3 \times 3 \times 5 \times 7 \times 17 \times x^{10}.
\]

\( ch(\theta_7) = \frac{1}{2 \times 3 \times 5 \times x} \)

\[
2^3 \times 5 \times 25841 \times 45307 \times x + \\
3 \times 7 \times 19 \times 83 \times 2459 \times 45307 \times x^2 + \\
2^3 \times 3 \times 7 \times 43 \times 53841866849 \times x^3 + \\
2^3 \times 3 \times 5 \times 7 \times 109 \times 401 \times 1543 \times 1789 \times x^4 + \\
2^3 \times 3 \times 5 \times 7 \times 43 \times 53841866849 \times x^5 + \\
2^3 \times 3 \times 5 \times 7 \times 19 \times 83 \times 2459 \times 45307 \times x^6 + \\
2^3 \times 3 \times 5 \times 7 \times 25841 \times 45307 \times x^7 + \\
2^3 \times 3 \times 5 \times 7 \times 25841 \times x^8.
\]
Now, one can see that the characters above fulfill the following equation:

$$\chi(\lambda) + \chi(\Lambda) = \chi(\mu_1 + \mu_2).$$

One should note however that, the 1-parameter specialization (III.5) above is not enough to find all the tensor coupling coefficients completely so we saw that at least 3-parameters specializations will be sufficient, which we used the following one.

$$u_1 = 1$$
$$u_2 = 2$$
$$u_3 = 3$$
$$u_4 = 4$$
$$u_5 = x$$
$$u_6 = y$$
$$u_7 = z$$
$$u_8 = 1/(24 \times y \times z)$$

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