Abstract—Post-Quantum Cryptography (PQC) attempts to find cryptographic protocols resistant to attacks using Shor’s polynomial time algorithm for numerical field problems or Grover’s search algorithm. A mostly overlooked but valuable line of solutions is provided by non-commutative algebraic structures, specifically canonical protocols that rely on one-way trapdoor functions (OWTF). Here we develop an evolved algebraic framework which could be applied to different asymmetric protocols. The (nonic) trapdoor one-way function here selected is a fortified version of the Triple decomposition Problem (TDP) developed by Kurt. The original protocol relies on two linear and one quadratic algebraic public equation. As quadratic equations are much more difficult to cryptanalyze, an Algebraic Span Attack (ASA) developed by Boaz Tsaban, focus on the linear ones. This seems to break our previous work. As countermeasure, we present here an Extended TDP (cited as XTDP in this work). The main point is that the original public linear equations are transformed into quadratic ones and the same is accomplished for exchanged tokens between the entities. All details not presented here, could be found at the cited references.

Keywords—Post-Quantum Cryptography, Non-Commutative Cryptography, General Linear Group, Linear Algebra, Triple Decomposition Problem, OWTF, IND-CCA2.

1. INTRODUCTION.

Post-Quantum Cryptography (PQC) is a trend that has an official NIST status [1,2] and which aims to be resistant to quantum computers attacks like Shor [3] and Grover [4] algorithms. NIST initiated last year a process to solicit, evaluate, and standardize one or more quantum-resistant public-key cryptographic algorithms [1]. Particularly Shor algorithm provided a quantum way to break asymmetric protocols.

Security of a canonical non-commutative protocol always relies his security on a one-way trapdoor function (OWTF) [5]. For instance, in an algebraic context, the conjugacy search problem, decomposition problem, double coset problem, triple decomposition problem, factorization search problem, commutator based or simultaneous conjugacy search problem. All are hard problems assumed to belong to AWPP time-complexity (but out of BQP), which yield convenient computational security against current quantum attacks.

In a previous work [6], we developed a framework based on the GL(d, Z_p), working with d-dimensional non-singular matrices of elements in Z_p and presented a full solution of a KEM based on p=251 field and using Kurt’s TDP [5, 7]. More details could be consulted on that paper.

2. ALGEBRAIC SPAN ATTACK (ASA)

At time of publishing [6], we were not aware of the ASA attack developed by Tsaban and others [8]. It seems a fruitful way of cryptanalyzing non-commutative canonical protocols. As that paper declares, TDP is the only canonical protocol that is non-affected by earlier methods and that was also our own idea about it and the motivation to adopt it.

ASA concentrates on a weak point of TDP. This protocol manages public keys in format of two linear equations with two unknowns each and a quadratic one with three unknowns. Tsaban paper exposes the strong point of ASA: quadratic equations may be very difficult to solve, so he targets the linear equations with algebraic spans.

An obvious defense could be to extend TDP so that all public components are quadratic and that is the main point and purpose to present XTDP, an extension of the original framework. Further study will support the ASA resistance conjecture of XTDP or find an alternative way of cryptanalyzing it, perhaps by a new algebraic attack.

3. TDP REVISITED.

All details could be found at [5, 6, 7], but a short overview will help as an introduction to XTDP.

The public platform is a monoid G (or group) with two separate sets of five subsets each affected by invertibility and commutativity restrictions. In our framework, all subsets belongs to GL(d, Z_p). The two sets are defined as A and B and their subsets are:

\[ A : \{A_1, A_2, A_3, X_1, X_2\} \]
\[ B : \{B_1, B_2, B_3, Y_1, Y_2\} \]

Subjected to following restrictions:

\textbf{(invertibility)} \{X_1, X_2, Y_1, Y_2\} should be invertible. In our specific platform, all subsets are invertible.

\textbf{(commutativity)} \[ [A_3, Y_1]=\{A_3, Y_2\}=[B_3, X_1]=\{B_3, X_2\}=I \] in terms of group commutators.

Alice and Bob agree on using respectively A and B. All lowercase variables are random non-biased selections of uppercase sets (i.e., \(a_i \in A_i\)).
The protocol goes as follows:

1. Alice chooses \{a_1, a_2, a_3, x_0, x_1, x_2\} and computes:
   \[ u = a_1 x_0, \quad v = x_1 a_2 x_2, \quad w = x_2 a_3 \]
   The private key is \( (u, v, w) \) and the public key is \( (u, v, w) \).

2. Bob chooses \{b_1, b_2, b_3, y_0, y_1, y_2\} and computes:
   \[ p = b_1 y_0, \quad q = y_1 b_2 y_2, \quad r = y_2 b_3 \]
   The private key is \( (b_1, b_2, b_3) \) and the public key is \( (p, q, r) \).

3. Both exchange public keys and compute the common key:
   \[
   (\text{Alice}) \quad K_A = a_1 b_1 a_2 b_2 a_3 b_3 \\
   (\text{Bob}) \quad K_B = u b_1 v b_2 w b_3 = a_1 b_1 a_2 b_2 a_3 b_3
   \]

As could be verified, \{u, w, p, r\} are linear equations and \{v, q\} are quadratic. ASA avoid attacking the non-linear equations and target the linear ones.

4. XTDP.

This protocol transforms TDP into a stronger one changing all linear public equations into quadratic ones in a mostly symmetric way. This feature enhances true TDP solving of each public piece to cryptanalyze it. As a drawback, in case of volatile or session public keys, it needs a double pass of public information. Aside from this not too disturbing disadvantage, the XTDP based framework could be a fast and secure way to obtain a canonical non-commutative PQC solution.

The public platform is the general linear group GL(d, \( \mathbb{Z}_p \)) with two separate sets of seven disjoint subsets each affected by commutativity restrictions, as invertibility is a-priori assured. In our framework, the two sets are defined as A and B and their subsets are:

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A : \{ A_1, A_2, A_3, X_0, X_1, X_2, X_3 \} \\
B : \{ B_1, B_2, B_3, Y_0, Y_1, Y_2, Y_3 \}
\]

Subjected to following restrictions:

(commutativity) \[[A_i, Y_j]=[A_i, X_i]=[A_j, X_j]=[B_j, X_j]=[B_i, X_i]=[X_i, X_j]=1\] in terms of group commutators.

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