Khovanov’s Heisenberg category, moments in free probability, and shifted symmetric functions

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Abstract

We establish an isomorphism between the center \( \text{End}_{\mathcal{H}}(1) \) of the Heisenberg category defined by Khovanov in [13] and the algebra \( \Lambda^* \) of shifted symmetric functions defined by Okounkov-Olshanski in [18]. We give a graphical description of the shifted power and Schur bases of \( \Lambda^* \) as elements of \( \text{End}_{\mathcal{H}}(1) \), and describe the curl generators of \( \text{End}_{\mathcal{H}}(1) \) in the language of shifted symmetric functions. This latter description makes use of the transition and co-transition measures of Kerov [10] and the noncommutative probability spaces of Biane [2].

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1 Introduction

In [13], Khovanov introduces a graphical calculus of oriented planar diagrams and uses it to define a linear monoidal category $\mathcal{H}'$, which he proposes as a categorification of the Heisenberg algebra. We denote by $\text{End}_{\mathcal{H}'}(\mathbb{1})$ the endormophism algebra of the monoidal unit in $\mathcal{H}'$. The commutative algebra $\text{End}_{\mathcal{H}'}(\mathbb{1})$ is, by definition, the algebra of closed oriented planar diagrams modulo the relations of the Khovanov graphical calculus. In his study of morphism spaces of $\mathcal{H}'$, Khovanov introduces two sets of generators for $\text{End}_{\mathcal{H}'}(\mathbb{1})$: the clockwise curls $t_c k \varepsilon$ and the counterclockwise curls $\tilde{t}_c k \varepsilon^2$. He then establishes algebra isomorphisms $\text{End}_{\mathcal{H}'}(\mathbb{1}) \cong \mathbb{C}[c_0, c_1, c_2, \ldots] \cong \mathbb{C}[\tilde{c}_2, \tilde{c}_3, \tilde{c}_4, \ldots]$, and describes a recursion for expressing the clockwise and counterclockwise curls in terms of each other. He then relates $\mathcal{H}'$ to representation theory by defining a sequence of monoidal functors $f_{H'}^n$ from $\mathcal{H}'$ to bimodule categories for symmetric groups. A consequence of the existence of these functors is the existence of surjective algebra homomorphisms,

$$f_{H'}^n : \text{End}_{\mathcal{H}'}(\mathbb{1}) \longrightarrow Z(\mathbb{C}[S_n]),$$

from $\text{End}_{\mathcal{H}'}(\mathbb{1})$ to the center of the group algebra of each symmetric group. Based in part on this, Khovanov suggests that there should be a close connection between $\text{End}_{\mathcal{H}'}(\mathbb{1})$ and the asymptotic representation theory of symmetric groups. Furthermore, one might hope that $\text{End}_{\mathcal{H}'}(\mathbb{1})$ in fact gives a diagrammatic description of some algebra of pre-existing combinatorial interest.

The main goal of the current paper is to make precise the connection between $\text{End}_{\mathcal{H}'}(\mathbb{1})$ and both the asymptotic representation theory of symmetric groups and algebraic combinatorics. We do this by establishing an isomorphism between

$$\varphi : \text{End}_{\mathcal{H}'}(\mathbb{1}) \longrightarrow \Lambda^*,$$

where $\Lambda^*$ is the shifted symmetric functions of Okounkov-Olshanski [18]. (See Theorem 5.3.) The algebra of shifted symmetric functions $\Lambda^*$ is a deformation of the algebra of symmetric functions. As is the case for $\text{End}_{\mathcal{H}'}(\mathbb{1})$, there are surjective algebra homomorphisms

$$f_{n}^{\Lambda^*} : \Lambda^* \longrightarrow Z(\mathbb{C}[S_n]),$$

to the center of the group algebra of each symmetric group. The isomorphism $\varphi : \text{End}_{\mathcal{H}'}(\mathbb{1}) \longrightarrow \Lambda^*$ is canonical, in that it intertwines the homomorphisms $f_{n}^{\Lambda'}$ and $f_{n}^{\Lambda^*}$.

The isomorphism $\varphi : \text{End}_{\mathcal{H}'}(\mathbb{1}) \longrightarrow \Lambda^*$ allows us to give a graphical description of several important bases of $\Lambda^*$. For example, the shifted power sum denoted $p_{\lambda}^\#$ in [18] appears in $\text{End}_{\mathcal{H}'}(\mathbb{1})$ as the closure of a permutation of cycle type $\lambda$. The shifted Schur function $s_{\lambda}^\#$ appears as the closure of a Young symmetrizer of type $\lambda$. (See Theorem 5.4).
In the other direction, it is also reasonable to ask for a description of the image of Khovanov’s curl generators $c_k$ and $\tilde{c}_k$ as elements of $\Lambda^*$. It turns out that the right language for such a description is that of noncommutative probability theory. In [10], Kerov introduces, for each partition $\lambda$, a pair of finitely supported probability measures on $\mathbb{R}$; these probability measures are known as the transition and co-transition measures, or sometimes as growth and decay. In work of Biane [2], these probability measures appear as the compactly-supported measures associated to self-adjoint operators on a noncommutative probability space, and as a result they are basic objects of interest at the intersection of representation theory and noncommutative probability theory. In particular, the moments and Boolean cumulants of the transition and co-transition measures may be regarded as elements of $\Lambda^*$. In Theorem 5.5, we show that the isomorphism $\varphi$ takes Khovanov’s curl generators $c_k$ and $\tilde{c}_k$ to scalar multiples of the $k$th moments of Kerov’s transition and co-transition measures. In fact, the close relationship between the transition and co-transition measures themselves yields two independent descriptions of the image of the curl generator $c_k$: it is equal to a scalar multiple of both the $k$th moment of the co-transition measure and the $(k + 2)$th Boolean cumulant of the transition measure. The observation that the Boolean cumulants of the transition measure are equal to the moments of the co-transition measure seems to be new, and is closely connected to the adjointness of induction and restriction functors between representation categories of symmetric groups. A dictionary between several of the bases of $\text{End}_{\mathcal{H}'}(1)$ and $\Lambda^*$ is given in Table 1 below.

The existence of a relationship between $\mathcal{H}'$ and free probability – and indeed, much of this paper – was anticipated by Khovanov in [13]. The relationship between generators of $\text{End}_{\mathcal{H}'}(1)$ and the noncommutative probability spaces of [2] may be seen as a further manifestation of the “planar structure” of free probability; the many connections between noncommutative probability and other mathematical subjects with planar structure are emphasized in the work of Guionnet, Jones and Shlyakhtenko [6].

In addition to the center of $\mathcal{H}'$, another algebra of interest in the study of $\mathcal{H}'$ is its trace (or zeroth Hochschild homology). The trace of $\mathcal{H}'$ is an infinite-dimensional noncommutative algebra, which may be defined diagrammatically as the algebra of diagrams on an annulus; the trace acts naturally on $\text{End}_{\mathcal{H}'}(1)$ by gluing annular diagrams around planar ones. In [4], the trace of $\mathcal{H}'$ is shown to be isomorphic to the $W_{1+\infty}$ algebra of conformal field theory. An action of $W_{1+\infty}$ on $\Lambda^*$ appears to be well known in the vertex algebra community, and such an action is constructed explicitly in the work of Lascoux-Thibon [14]. Thus the isomorphism $\varphi : \text{End}_{\mathcal{H}'}(1) \rightarrow \Lambda^*$ of Theorem 5.3, together with the main result of [4], gives a purely planar realization – via Khovanov’s graphical calculus – of Lascoux-Thibon’s construction.

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| $\Lambda^*$ | diagram in $\text{End}_{\mathcal{H}}(1)$ |
|------------|-----------------------------------|
| $p^\#_\lambda$ | ![Diagram](#) |
| $s^*_\lambda$ | ![Diagram](#) |
| $h^*_k$ | ![Diagram](#) |
| $e^*_k$ | ![Diagram](#) |
| $\hat{m}_k$ | ![Diagram](#) |
| $\hat{b}_{k+2} = p^\#_1 \hat{m}_k$ | ![Diagram](#) |

Table 1: A dictionary between $\Lambda^*$ and diagrams in $\text{End}_{\mathcal{H}}(1)$. 
2 The symmetric group and its normalized character theory

We begin by establishing notation related to partitions and Young diagrams. Let \( \mathcal{P}_n \) be the set of partitions of \( n \) and
\[
\mathcal{P} := \bigcup_{n \geq 0} \mathcal{P}_n.
\]
For this section let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathcal{P}_n \) and \( \mu = (\mu_1, \ldots, \mu_t) \in \mathcal{P}_k \) with \( n \geq k \). We assume that \( \lambda_1 \geq \cdots \geq \lambda_r > 0 \) and \( \mu_1 \geq \cdots \geq \mu_t > 0 \). When \( i > r \) (respectively \( i > t \)) we then understand \( \lambda_i = 0 \) (resp. \( \mu_i = 0 \)). We use the following notation throughout:

- \( n = \lambda_1 + \lambda_2 + \cdots + \lambda_r =: |\lambda| \).
- \( \lambda \cup \mu \) is the partition formed from the union of the parts of \( \lambda \) and \( \mu \).
- \( \mu \leq \lambda \) if \( \mu_i \leq \lambda_i \) for all \( i \geq 1 \). When this is the case, we write \( \lambda/\mu \) for the associated skew diagram.
- \( \phi_{k,n} : \mathcal{P}_k \hookrightarrow \mathcal{P}_n \) is the function defined by \( \phi_{k,n}(\mu) = \mu \cup 1^{n-k} \in \mathcal{P}_n \).

Example 2.1. If \( \mu = (3, 2, 1, 1, 1) \in \mathcal{P}_8 \) then \( \phi_{8,10}(\mu) = (3, 2, 1, 1, 1, 1, 1) \in \mathcal{P}_{10} \).

We freely identify \( \mu \in \mathcal{P} \) with its corresponding Young diagram, which we draw using Russian notation (see Example 2.2). If \( \square \) is a cell in the \( i \)th row and \( j \)th column of \( \mu \) then the content of \( \square \) is defined as
\[
\text{cont}(\square) := j - i.
\]
We say that a cell \( \square \notin \mu \) is \( i \)-addable with respect to \( \mu \) if it has content \( i \) and adding it to \( \mu \) gives a Young diagram. We say that a cell \( \square \in \mu \) is \( i \)-removable with respect to \( \mu \) if it has content \( i \) and removing it from \( \mu \) gives a Young diagram. We call two sequences \( a_1, \ldots, a_d \) and \( b_1, \ldots, b_{d-1} \) interlacing when
\[
a_1 < b_1 < a_2 < \cdots < a_{d-1} < b_{d-1} < a_d.
\]
The center of this pair of sequences is defined as the quantity
\[
(a_1 + \cdots + a_d) - (b_1 + \cdots + b_{d-1}).
\]
Each Young diagram \( \mu \) uniquely defines two integer valued interlacing sequences \( a_1, \ldots, a_d \) and \( b_1, \ldots, b_{d-1} \) where:

- \( a_1, \ldots, a_d \) is the ordered list of all \( a_j \) such that there exists an \( a_j \)-addable cell with respect to \( \mu \).
- \( b_1, \ldots, b_{d-1} \) is the ordered list of all \( b_j \) such that there exists a \( b_j \)-removable cell with respect to \( \mu \).

From this description it is clear that \( a_1, \ldots, a_d \) and \( b_1, \ldots, b_{d-1} \) are interlacing.
Example 2.2. Let $\mu = (4, 2, 1)$. Then $\mu$ yields the interlacing sequences

$$-3 < -1 < 1 < 4 \quad \text{and} \quad -2 < 0 < 3.$$ 

Proposition 2.3. \[\mathbb{1}\] If $a_1, \ldots, a_d$ and $b_1, \ldots, b_{d-1}$ are the pair of interlacing sequences associated to a Young diagram then their center is 0. Conversely, any pair of integer valued interlacing sequences with center 0 are associated to a Young diagram.

When $\mu \subseteq \lambda$ and $\lambda/\mu = \square$, then we write $\mu \nearrow \lambda$. In other words, $\mu \nearrow \lambda$ whenever we can obtain $\lambda$ from $\mu$ by adding a single cell. If $a_1, \ldots, a_d$ and $b_1, \ldots, b_{d-1}$ are the interlacing sequences associated to $\mu$, then we denote by $\mu^{(i)}$ the Young diagram that we get by adding a cell of content $a_i$, so that

$$\text{cont}(\mu^{(i)}/\mu) = a_i.$$ 

Similarly, we denote by $\mu_{(i)}$ the Young diagram that we get by removing a cell of content $b_i$ from $\mu$, so that

$$\text{cont}(\mu/\mu_{(i)}) = b_i.$$ 

Note that $\mu_{(i)} \nearrow \mu$, while $\mu \nearrow \mu^{(i)}$.

Example 2.4. If $\mu = (4, 2, 1)$ as in Example 2.2 we have

$$\begin{align*}
\mu^{(1)} &= (4, 2, 1, 1) \\
\mu^{(2)} &= (4, 2, 2) \\
\mu^{(3)} &= (4, 3, 1) \\
\mu^{(4)} &= (5, 2, 1)
\end{align*}$$

and

$$\begin{align*}
\mu_{(1)} &= (4, 2) \\
\mu_{(2)} &= (4, 1, 1) \\
\mu_{(3)} &= (3, 2, 1).
\end{align*}$$

Let $S_n$ be the symmetric group. $S_n$ is generated by Coxeter generators $s_1, \ldots, s_{n-1}$ where $s_i$ is the adjacent transposition $(i, i+1)$. We identify $\mathbb{C}[S_n] \cong \mathbb{C}$. If $g \in S_n$ has cycle type $\lambda \vdash n$, then we write $\text{sh}(g) := \lambda$. For $k \leq n$, there is an embedding $S_k \hookrightarrow S_n$ called the standard embedding which sends $S_k$ to the subgroup generated by $s_1, \ldots, s_{k-1}$, which stabilizes $\{k+1, \ldots, n\}$ pointwise.
We extend this embedding by linearity to get an embedding of group algebras which we denote by \( \iota_{k,n} : \mathbb{C}[S_k] \to \mathbb{C}[S_n] \). We write \( 1_k \) for the identity element in \( \mathbb{C}[S_k] \) so that \( \iota_{k,n}(1_k) = 1_n \). We write \( w_{0,n} \) for the longest element of \( S_n \).

For \( \lambda \vdash n \), let \( L^\lambda \) be the simple \( \mathbb{C}[S_n] \)-module corresponding to \( \lambda \), \( E_\lambda \) its associated Young idempotent, and \( \chi^\lambda : \mathbb{C}[S_n] \to \mathbb{C} \) its associated character. Abusing notation, we write \( \chi^\lambda(\mu) \) for \( \chi^\lambda(g) \) when \( \text{sh}(g) = \mu \) (this notation is well-defined since \( \chi^\lambda \) is a class function). The normalized character \( \tilde{\chi}^\lambda : \bigoplus_{k \leq n} \mathbb{C}[S_k] \to \mathbb{C} \) associated to \( \lambda \) is defined so that for \( x \in \mathbb{C}[S_k] \),

\[
\tilde{\chi}^\lambda(x) := \frac{\chi^\lambda(\iota_{k,n}(x))}{\dim L^\lambda} = \frac{\chi^\lambda(\iota_{k,n}(x))}{\chi^\lambda(1_n)}.
\]  

(2)

Let \( \mu = (\mu_1, \ldots, \mu_t) \vdash k \leq n \) and set \( \pi_\mu = 1_k \) if \( \mu = (1^k) \) and otherwise

\[
\pi_\mu = (s_{k-1} \ldots s_{k-\mu_t+1}) \ldots (s_{\mu_1+\mu_2-1} \ldots s_{\mu_1+1}) \ldots (s_{\mu_1-1} \ldots s_{2s_1})
\]

\[
= (k, k-1, \ldots, k - \mu_t + 1)(\mu_1 + \mu_2, \ldots, \mu_1 + 1) \ldots (\mu_1, \ldots, 2, 1) \in S_k.
\]

We define

\[
\sigma_{\mu,n} := w_{0,n}^{-1}(\iota_{k,n}(\pi_\mu))w_{0,n} \in S_n.
\]

Observe that \( \sigma_{\mu,n} \) has cycle type \( \phi_{k,n}(\mu) \) and fixes \( 1, 2, \ldots, n - k \) pointwise.

**Example 2.5.** Let \( \mu = (3, 2) \vdash 5 \), then

\[
\pi_\mu = (s_4)(s_2s_1) = (5, 4)(3, 2, 1)
\]

and we see that \( \text{sh}(\pi_\mu) = \mu \). For \( n = 8 \),

\[
\sigma_{\mu,8} = s_4s_6s_7 = (4, 5)(6, 7, 8),
\]

while for \( n = 10 \),

\[
\sigma_{\mu,10} = (6, 7)(8, 9, 10).
\]

The elements

\[
\{1_n, \sigma_{(2),n}, \sigma_{(3),n}, \ldots, \sigma_{(n),n}\} = \{1_n, s_{n-1}, s_{n-2} s_{n-1}, \ldots, s_1 s_2 \ldots s_{n-1}\}
\]

are the minimal length left coset representatives of \( S_{n-1} \) in \( S_n \). We extend this observation in the following lemma.

**Lemma 2.6.** For \( k < n \), the elements of the set

\[
\{\sigma_{(i_k),n} \sigma_{(i_{k-1}),n-1} \ldots \sigma_{(i_1),k+1} | 1 \leq i_j \leq j \}
\]

are the minimal length left coset representatives of \( S_k \) in \( S_n \). We denote this set by \( \mathcal{LC}_{k}^n \).
We note that $| \mathcal{LC}_k^n | = (n \downarrow n - k)$, where the *falling factorial power* is defined as

$$(x \downarrow k) = \begin{cases} x(x-1) \ldots (x-k+1), & \text{if } k = 1, 2, \ldots \\ 1, & \text{if } k = 0. \end{cases}$$

**Example 2.7.** We have

$$\mathcal{LC}_3^4 = \{ 14, s_3, s_2 s_3, s_1 s_2 s_3 \},$$

$$\mathcal{LC}_2^3 = \{ 13, s_2, s_1 s_2 \},$$

and

$$\mathcal{LC}_2^4 = \{ 14, s_3, s_2 s_3, s_1 s_2 s_3, \}
\{ s_2, s_3 s_2, s_2 s_3 s_2, s_1 s_2 s_3 s_2, \}
\{ s_1 s_2, s_3 s_2 s_2, s_2 s_3 s_1 s_2, s_1 s_2 s_3 s_1 s_2 \}. \tag{3}$$

### 2.1 The center of $\mathbb{C}[S_n]$  

For $\mu \vdash k \leq n$, set

$$C_{\mu,n} := \sum_{g \in S_n, \text{sh}(g) = \phi_{k,n}(\mu)} g.$$  

The elements $\{C_{\mu,n}\}_{\mu \vdash n}$ are a basis for the center of the symmetric group algebra, $Z(\mathbb{C}[S_n])$. We write $z_{\mu,n}$ for the size of the centralizer of an element in $S_n$ with cycle type $\phi_{k,n}(\mu)$. Note that when $\mu \vdash n$, then $z_{\mu,n} = z_{\mu}$.

**Definition 2.8.** For $\mu = (\mu_1, \ldots, \mu_t) \vdash k \leq n$, set

$$A_{\mu,n} := \sum_{g \in \mathcal{LC}_{n-k}^n} g \sigma_{\mu,n} g^{-1}. \tag{3}$$

We call $A_{\mu,n}$ the *normalized conjugacy class sum* associated to $\mu$ in $\mathbb{C}[S_n]$.

Alternatively, $A_{\mu,n}$ may be written as

$$A_{\mu,n} = \sum (i_1, \ldots, i_{\mu_1}) \ldots (i_{k-\mu+1}, \ldots, i_k) \tag{4}$$

where this sum is taken over all distinct $k$-tuples $(i_1, \ldots, i_k)$ of elements from $\{1, 2, \ldots, n\}$. From (4) an easy counting argument shows that

$$A_{\mu,n} = \frac{z_{\mu,n}}{(n-k)!} C_{\mu,n}. \tag{5}$$

It follows from (5) that $A_{\mu,n} \in Z(\mathbb{C}[S_n])$.  

8
Example 2.9. Let $k \leq n$. When $\mu = (k) \vdash k$, then $z_{(k),n} = k(n - k)!$ so that

$$A_{(k),n} = kC_{(k),n}.$$  

The elements $A_{\mu,n}$ are important in the study of the asymptotic character theory of symmetric groups \[12\]. They also appear in connection with the algebra of partial permutations \[8\]. If $\mu \vdash k \leq n$ and $\lambda \vdash n$ then

$$\hat{\chi}^\lambda(A_{\mu,n}) = (n \downarrow k) \frac{\chi^\lambda(\mu)}{\dim L^\lambda}. \quad (6)$$

The following is well-known.

**Proposition 2.10.** When restricted to $Z(C[S_n])$, the normalized character $\hat{\chi}^\lambda$ is an algebra homomorphism from $Z(C[S_n])$ to $\mathbb{C}$.

$Z(C[S_n])$ is also generated by symmetric polynomials in the Jucys-Murphy elements \{J_i\}$_{1 \leq i \leq n} \subseteq C[S_n]$, where

$$J_1 = 0, \quad \text{and} \quad J_k = (1,k) + (2,k) + \cdots + (k - 1,k), \quad 2 \leq k \leq n.$$  

We can also write

$$J_k = \sum_{i=1}^{k-1} s_i \cdots s_{k-1}s_{k-2} \cdots s_i. \quad (7)$$

2.2 The transition measure and co-transition measure

In this section we recall the notion of transition and co-transition measures, also known as growth and decay, respectively. Assume that $\lambda \vdash n$ and let $a_1, \ldots, a_d$ and $b_1, \ldots, b_{d-1}$ be the interlacing sequences associated to $\lambda$. Recall that $\lambda^{(1)}, \ldots, \lambda^{(d)}$ are the partitions of $n + 1$ such that $\text{cont}(\lambda^{(i)}/\lambda) = a_i$, while $\lambda_{(1)}, \ldots, \lambda_{(d-1)}$ are the partitions of $n - 1$ such that $\text{cont}(\lambda/\lambda_{(i)}) = b_i$.

For $1 \leq i \leq d$, the **transition probabilities** for $\lambda$ are defined as

$$\hat{q}_\lambda(\lambda^{(i)}) := \frac{\dim(L^{\lambda^{(i)}})}{(n + 1) \dim(L^\lambda)}.$$  

The **transition measure** $\widehat{\omega}_\lambda$ is then the probability measure on $\mathbb{R}$ defined by

$$\widehat{\omega}_\lambda := \sum_{i=1}^{d} \hat{q}_\lambda(\lambda^{(i)}) \delta_{a_i} \quad (8)$$

where $\delta_{a_i}$ is the Dirac delta measure with support on $a_i \in \mathbb{R}$. Dually, for $1 \leq i \leq d - 1$ the **co-transition probabilities** of $\lambda$ are

$$\check{q}_\lambda(\lambda_{(i)}) := \frac{\dim(L^{\lambda_{(i)}})}{\dim(L^\lambda)}.$$
and the co-transition measure $\tilde{\omega}_\lambda$ is

$$\tilde{\omega}_\lambda := \sum_{i=1}^{d-1} \tilde{q}_\lambda(\lambda^{(i)}) \delta_{b_i}. \quad (9)$$

These probability measures were first investigated by Kerov ([10], [11]). They are fundamental tools in the study of the asymptotic representation theory of symmetric groups and in the connection between asymptotic representation theory and free probability.

The $k$th moment associated to the transition measure $\omega_\lambda$ is given by

$$m_k(\lambda) = \sum_{i=1}^{d} a_i^k \tilde{q}_\lambda(\lambda^{(i)})$$

while the $k$th moment associated to the co-transition measure $\tilde{\omega}_\lambda$ is given by

$$\tilde{m}_k(\lambda) = \sum_{i=1}^{d-1} b_i^k \tilde{q}_\lambda(\lambda^{(i)}).$$

We write the moment generating series for the transition measure (resp. co-transition measure) as

$$\tilde{M}_\lambda(z) := \sum_{k=0}^{\infty} m_k(\lambda) z^{-k-1} \quad \text{and} \quad \tilde{M}_\lambda(z) := z - \sum_{k=0}^{\infty} |\lambda| \tilde{m}_k(\lambda) z^{-k-1}. \quad (10)$$

Note that we scale all coefficients of $\tilde{M}_\lambda(z)$ by $|\lambda|$ with the exception of the coefficient on $z$.

**Lemma 2.11.** For $\lambda \in \mathcal{P}$

$$\tilde{M}_\lambda(z) = (\tilde{M}_\lambda(z))^{-1}. \quad (10)$$

**Proof.** This follows directly from equation (2.3) and Lemma 5.1 in [11]. \qed

The boolean cumulants $\{\hat{b}_k(\lambda)\}_{k \geq 1}$ associated to $\tilde{\omega}_\lambda$ can be defined as the coefficients on the multiplicative inverse of $\tilde{M}_\lambda(z)$,

$$\hat{B}_\lambda(z) = z - \sum_{k=1}^{\infty} \hat{b}_{k+2}(\lambda) z^{-k-1} = (\tilde{M}_\lambda(z))^{-1}. \quad (11)$$

With Lemma 2.11 this definition immediately gives us the following fact.

**Proposition 2.12.** Let $\lambda \in \mathcal{P}$ and $k \geq 0$, then $\hat{b}_1(\lambda) = 0$ and

$$\hat{b}_{k+2}(\lambda) = |\lambda| \tilde{m}_k(\lambda). \quad (12)$$
Remark 2.13. The equality (11) can be rewritten as

$$\sum_{i=1}^{k} \hat{m}_{k-i}(\lambda) \hat{b}_{i}(\lambda) = \hat{m}_{k}(\lambda).$$

(13)

For general information about the relationship between moments, Boolean cumulants, and other families of cumulants see [1].

There is a more algebraic approach to the transition measure due to Biane [2]. Let

$$\text{pr}_{n-1} : \mathbb{C}[S_{n}] \to \mathbb{C}[S_{n-1}] \subset \mathbb{C}[S_{n}]$$

be the projection map defined on $S_{n}$ by

$$\text{pr}_{n-1}(g) = \begin{cases} g & \text{if } g(n) = n \\ 0 & \text{otherwise.} \end{cases}$$

In the context of probability theory, pr$_{n-1}$ is sometimes known as the conditional expectation.

Proposition 2.14. For $\lambda \vdash n$,

$$\hat{m}_{k}(\lambda) = \tilde{\chi}^{\lambda}[\text{pr}_{n}(J_{n+1}^{k})]$$

(14)

and

$$\hat{b}_{k+2}(\lambda) = |\lambda| \hat{m}_{k}(\lambda) = \tilde{\chi}^{\lambda} \left( \sum_{i=1}^{n} s_{i} \ldots s_{n-1} J_{n+1}^{k} s_{n-1} \ldots s_{i} \right).$$

(15)

Proof. The statement of (14) appears in [3] Section 4. A detailed proof is given in Theorem 9.23 of [7]. To get (15) note that since characters are class functions,

$$\tilde{\chi}^{\lambda} \left( \sum_{i=1}^{n} s_{i} \ldots s_{n-1} J_{n+1}^{k} s_{n-1} \ldots s_{i} \right) = |\lambda| \tilde{\chi}^{\lambda}(J_{n}^{k}).$$

As $J_{n}$ eigenspaces, $L^{\lambda}$ decomposes as

$$L^{\lambda} \cong \bigoplus_{i=1}^{d-1} L^{\lambda(i)}$$

with $L^{\lambda(i)}$ corresponding to eigenvalue $b_{i}$ [20]. Hence,

$$|\lambda| \tilde{\chi}^{\lambda}(J_{n}^{k}) = |\lambda| \sum_{i=1}^{d-1} \frac{\dim(\lambda(i)) b_{i}^{k}}{\dim(\lambda)} = |\lambda| \hat{m}_{k}(\lambda) = \hat{b}_{k+2}(\lambda).$$

Proposition 2.14 is related to the fact that we are working in a noncommutative probability space (that is, a von Neumann algebra equipped with a normal faithful trace). In our case the algebra is $\text{End}(L^{\lambda}) \otimes M_{n+1}(\mathbb{C})$ and $\hat{\omega}_{\lambda}$ then arises from the distribution of a self-adjoint element in this algebra (see Proposition 3.3 in [2]).
3 Symmetric functions and shifted symmetric functions

In order to define the algebra of shifted symmetric functions, we first recall the classical symmetric functions. Let \( \Lambda_n \) be the algebra of symmetric polynomials over \( \mathbb{C} \) in \( x_1, \ldots, x_n \). This algebra is graded by polynomial degree. Recall that for \( n \geq 0 \) there is a homomorphism

\[
\Lambda_{n+1} \to \Lambda_n
\]  
(16)
given by setting \( x_{n+1} = 0 \) in \( \Lambda_{n+1} \). One can define the algebra of symmetric functions as the projective limit \( \Lambda = \lim_{\longrightarrow} \Lambda_n \) taken in the category of graded algebras. We recall three collections of algebraically independent generators of \( \Lambda \):

- elementary symmetric functions \( e_1, e_2, e_3, \ldots \),
- complete homogeneous symmetric functions \( h_1, h_2, h_3, \ldots \),
- power sum symmetric functions \( p_1, p_2, p_3, \ldots \).

For \( \{f_k\}_{k \geq 1} \) equal to any of these three sets of generators and \( \lambda = (\lambda_1, \ldots, \lambda_r) \) we write \( f_{\lambda} := f_{\lambda_1} \cdots f_{\lambda_r} \). We denote the basis of Schur functions by \( \{s_{\lambda}\}_{\lambda \in \mathcal{P}} \). We refer the reader to [16] and [19] for background on \( \Lambda \).

Let \( \Lambda^*_n \) be the algebra of polynomials over \( \mathbb{C} \) in \( x_1, \ldots, x_n \), which become symmetric in the new variables \( x_{i} = x_i - i \). This algebra is filtered by polynomial degree. In analogy to \( \Lambda_{n+1} \), setting \( x_{n+1} = 0 \) in \( \Lambda^*_n \) gives a homomorphism

\[
\Lambda^*_{n+1} \to \Lambda^*_n
\]
(17)
which respects the filtration. Using (17), set

\[
\Lambda^* := \lim_{\longrightarrow} \Lambda^*_n,
\]
where this limit is taken in the category of filtered algebras. \( \Lambda^* \) is called the algebra of shifted symmetric functions.

Because \( \Lambda^* \) is filtered, we can consider the associated graded algebra \( \operatorname{gr}(\Lambda^*) \).

**Proposition 3.1.** [18, Prop. 1.5] \( \operatorname{gr}(\Lambda^*) \) is canonically isomorphic to \( \Lambda \).

**Remark 3.2.** It is noted in Remark 1.7 of [18] that we may also view \( \Lambda^* \) as a deformation of \( \Lambda \). Let \( \Lambda^*_n(\theta) \) be the algebra of polynomials in \( x_1, \ldots, x_n \) which are symmetric in the new variables \( x_i = x_i + c - i\theta \) for \( 1 \leq i \leq n \) and where \( c \in \mathbb{C} \). Define \( \Lambda^*(\theta) = \lim_{\longrightarrow} \Lambda^*_n(\theta) \). Then \( \Lambda^*(0) = \Lambda \) and \( \Lambda^*(1) = \Lambda^* \). In fact for all \( \theta \neq 0 \), \( \Lambda^*(\theta) \cong \Lambda^* \).
3.1 Bases of $\Lambda^*$

In [18] Okounkov and Olshanski introduced a remarkable basis for $\Lambda^*$ called the shifted Schur functions. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ (note that here we allow components of a partition to be zero). The \textit{shifted Schur polynomial in $n$ variables, indexed by $\lambda$} is the ratio of two $n \times n$ determinants,

$$s^*_\lambda(x_1, \ldots, x_n) = \frac{\det[(x_i + n - i - i \downarrow \lambda_j + n - j)]}{\det[(x_i + n - i - j)]}, \quad (18)$$

where $1 \leq i, j \leq n$. This polynomial belongs to $\Lambda^*_n$. It is shown in [18] that

$$s^*_\lambda(x_1, \ldots, x_n, 0) = s^*_\lambda(x_1, \ldots, x_n). \quad (19)$$

This implies that for fixed $\lambda$, letting $n \to \infty$ gives a well-defined element $s^*_\lambda$ of $\Lambda^*$.

The elements $\{s^*_\lambda\}_{\lambda \in \mathcal{P}} \in \Lambda^*$ are called the \textit{shifted Schur functions} and form a basis for $\Lambda^*$. There is a linear map $\Lambda^* \to \text{gr}(\Lambda^*) \cong \Lambda$ which sends $f \in \Lambda^*$ to its top homogeneous component which is an element of $\Lambda$. Under this map

$$s^*_\lambda \mapsto s_\lambda$$

or alternatively,

$$s^*_\lambda = s_\lambda + \text{l.o.t.} \quad (20)$$

where l.o.t. means lower order terms in polynomial degree.

In analogy to the classical case, the \textit{elementary shifted functions} can be defined as $e^*_k := s^*_{(1^k)}$, while the \textit{complete shifted functions} can be defined as $h^*_k := s^*_{(k)}$. More explicitly:

$$e^*_k(x_1, x_2, \ldots) = \sum_{1 \leq i_1 < \cdots < i_k < \infty} (x_{i_1} + k - 1)(x_{i_2} + k - 2) \cdots x_{i_k}$$

and

$$h^*_k(x_1, x_2, \ldots) = \sum_{1 \leq i_1 < \cdots < i_k < \infty} (x_{i_1} - k + 1)(x_{i_2} - k + 2) \cdots x_{i_k}.$$

Let $F$ be the linear isomorphism $F : \Lambda \to \Lambda^*$ which sends $s_\lambda \mapsto s^*_\lambda$. Define the element $p^#_\lambda \in \Lambda^*$ to then be

$$p^#_\lambda := F(p_\lambda), \quad (21)$$

where $p_\lambda$ is the power sum symmetric function. The elements $p^#_\lambda$ are one of several shifted analogues of the power sums. For $\lambda \vdash n$, the transition coefficients between the power-sum and Schur bases are given by the character tables of the symmetric group (see [19]):

$$p_\lambda = \sum_{\mu \vdash n} \chi^\mu(\lambda)s_\mu.$$
It follows directly from definition (21) that
\[ p_\lambda^\# = \sum_{\mu \vdash n} \chi^\mu(\lambda) s_\mu^*. \]  
(22)

Note also that by (20) and (22),
\[ p_\lambda^\# = p_\lambda + \text{l.o.t.} \]  
(23)

Since the power symmetric functions \( p_1, p_2, \ldots \) are algebraically independent and generate \( \Lambda \), it follows from (23) that \( p_1^\#, p_2^\#, \ldots \) are algebraically independent and generate \( \Lambda^* \). Similarly, since \( \{p_\lambda\}_{\lambda \in \mathcal{P}} \) is a basis for \( \Lambda \), \( \{p_\lambda^\#\}_{\lambda \in \mathcal{P}} \) is a basis for \( \Lambda^* \). For more properties of the basis \( \{p_\lambda^\#\} \) see [9].

Remark 3.3. Let \( \lambda = (\lambda_1, \ldots, \lambda_r) \vdash n \). While it is true that in \( \Lambda \), \( p_{\lambda_1} \cdots p_{\lambda_r} = p_\lambda \), in general
\[ p_{\lambda_1}^\# \cdots p_{\lambda_r}^\# \neq p_\lambda^\#. \]

However, by (23)
\[ p_{\lambda_1}^\# \cdots p_{\lambda_r}^\# = p_\lambda^\# + \text{l.o.t.} \]

3.2 \( \Lambda^* \) as functions on \( \mathcal{P} \)

Let \( \text{Fun}(\mathcal{P}, \mathbb{C}) \) be the algebra of functions from \( \mathcal{P} \) to \( \mathbb{C} \) with pointwise multiplication. Viewing \( \mu = (\mu_1, \ldots, \mu_t) \vdash k \) as the sequence \( (\mu_1, \ldots, \mu_t, 0, 0, \ldots) \), we can evaluate \( f \in \Lambda^* \) on \( \mu \) by setting
\[ f(\mu) = f(\mu_1, \ldots, \mu_t, 0, 0, \ldots). \]  
(24)

Since \( (\mu_1, \ldots, \mu_t, 0, 0, \ldots) \) has only a finite number of nonzero values, it is clear that (24) is well-defined. In fact \( f \) is uniquely defined by its values on \( \mathcal{P} \). Thus \( \Lambda^* \) may be realized as a subalgebra of \( \text{Fun}(\mathcal{P}, \mathbb{C}) \). This fact is used repeatedly en route to establishing many of the fundamental results about shifted symmetric functions in [12] and [18].

For \( \lambda \vdash n \) and \( \alpha \) a cell in the Young diagram corresponding to \( \lambda \) let \( h(\alpha) \) be the hook length of \( \alpha \). Then set \( H(\lambda) \) as the product of all hooklengths in \( \lambda \),
\[ H(\lambda) := \prod_{\alpha \in \lambda} h(\alpha). \]

The following is known as the “Characterization Theorem” of [17].

Theorem 3.4. For \( \mu \vdash k \), \( s_\mu^* \) is the unique element of \( \Lambda^* \) such that \( \text{deg}(s_\mu^*) \leq k \) and
\[ s_\mu^*(\lambda) = \delta_{\mu\lambda} H(\mu) \]
for all \( \lambda \in \mathcal{P} \) such that \( |\lambda| \leq |\mu| \).
This theorem along with (22) then give the following proposition.

Proposition 3.5. [18] For \( \mu \vdash k \), \( \lambda \vdash n \),

\[
p^\#_\mu(\lambda) = \begin{cases} \frac{(n|k)}{\dim \mathbb{Z}^n} \chi^\lambda(\mu) & k \leq n \\ 0 & \text{otherwise.} \end{cases} \tag{25}
\]

Remark 3.6. We will later use the fact that \( p^\#_1 = \sum_{x_1 \leq \cdots \leq x_2} \lambda x_1 \cdots x_2 \), so that \( p^\#_1(\lambda) = |\lambda| \) for all \( \lambda \in \mathcal{P} \).

In Section 2.2 we introduced the moments \( \{\hat{m}_k(\lambda)\} \) (resp. \( \{\hat{m}_k(\lambda)\} \)) of the transition measure (resp. co-transition measure) associated to a partition \( \lambda \) and the corresponding Boolean cumulants \( \{\hat{b}_k(\lambda)\} \). We can interpret all of these as elements of \( \text{Fun}(\mathcal{P}, \mathbb{C}) \) via

\[
\lambda \xrightarrow{\hat{m}_k} \hat{m}_k(\lambda), \quad \lambda \xrightarrow{\hat{m}_k} \hat{m}_k(\lambda), \quad \text{and} \quad \lambda \xrightarrow{\hat{b}_k} \hat{b}_k(\lambda).
\]

We omit the partition argument from \( \hat{m}_k \), \( \hat{m}_k \), and \( \hat{b}_k \) in this context to emphasize that we are considering them as elements of \( \text{Fun}(\mathcal{P}, \mathbb{C}) \).

Proposition 3.7. [15] Theorem 6.4] As elements of \( \text{Fun}(\mathcal{P}, \mathbb{C}) \), \( \hat{m}_k \) and \( \hat{b}_k \) belong to \( \Lambda^* \).

Remark 3.8. In [15] Section 5, Lassalle shows that with the appropriate alphabet \( A_\lambda \) (which is specific to each partition \( \lambda \)),

\[
\hat{m}_k(\lambda) = h_k(A_\lambda) \quad \text{and} \quad \hat{b}_k(\lambda) = (-1)^{k-1} e_k(A_\lambda). \tag{26}
\]

4 The Heisenberg category \( \mathcal{H}' \)

In [13], Khovanov defined an additive \( \mathbb{C} \)-linear monoidal category \( \mathcal{H}' \) which we will call the Heisenberg category. The objects in \( \mathcal{H}' \) are generated by two objects \( Q_+ \) and \( Q_- \). Following the notation of [13], we denote \( Q_{\epsilon_1} \otimes \cdots \otimes Q_{\epsilon_m} \) by \( Q_\epsilon \) where \( \epsilon = \epsilon_1 \cdots \epsilon_m \) is a finite sequence of pluses and minuses. The unit object, \( 1 \), corresponds to the empty sequence \( Q_{\emptyset} \).

The collection of morphisms \( \text{Hom}_{\mathcal{H}'}(Q_\epsilon, Q_{\epsilon'}) \), for two sequences \( \epsilon \) and \( \epsilon' \) is the \( \mathbb{C} \)-vector space spanned by planar diagrams modulo some local relations. The diagrams are oriented compact 1-manifolds embedded in the strip \( \mathbb{R} \times [0,1] \), modulo rel boundary isotopies. The endpoints of the 1-manifolds are located at \( \{1, \ldots, m\} \times \{0\} \) and \( \{1, \ldots, n\} \times \{1\} \), where \( m \) and \( n \) are the lengths of \( \epsilon \) and \( \epsilon' \), respectively. Further, the orientation of the 1-manifold at the endpoints must match the signs in the sequences \( \epsilon \) and \( \epsilon' \). Triple intersections are not allowed.
Example 4.1. The diagram

\[
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (1,-1) -- (0,0);
\draw (0,-1) .. controls (0.5,-0.5) and (1.5,-0.5) .. (2,0);
\draw (0,1) .. controls (0.5,0.5) and (1.5,0.5) .. (2,0);
\end{tikzpicture}
\end{array}
\]

is a morphism from \( Q_{--} \) to \( Q_{++--} \).

The composition of two morphisms is achieved by stacking diagrams. The local relations for diagrams are:

\[
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (1,-1) -- (0,0);
\draw (0,-1) .. controls (0.5,-0.5) and (1.5,-0.5) .. (2,0);
\draw (0,1) .. controls (0.5,0.5) and (1.5,0.5) .. (2,0);
\end{tikzpicture}
\end{array} = \begin{array}{c}
\begin{tikzpicture}
\draw (0,-1) -- (1,0) -- (2,1) -- (1,0) -- (0,-1);
\end{tikzpicture}
\end{array}
\quad \quad \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (1,-1) -- (0,0);
\draw (0,-1) .. controls (0.5,-0.5) and (1.5,-0.5) .. (2,0);
\draw (0,1) .. controls (0.5,0.5) and (1.5,0.5) .. (2,0);
\end{tikzpicture}
\end{array} = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (1,-1) -- (0,0);
\draw (0,-1) .. controls (0.5,-0.5) and (1.5,-0.5) .. (2,0);
\draw (0,1) .. controls (0.5,0.5) and (1.5,0.5) .. (2,0);
\end{tikzpicture}
\end{array} = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) .. controls (0,1) and (0,-1) .. (0,0);
\end{tikzpicture}
\end{array}
\quad (27)
\end{array}
\]

\[
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (1,-1) -- (0,0);
\draw (0,-1) .. controls (0.5,-0.5) and (1.5,-0.5) .. (2,0);
\draw (0,1) .. controls (0.5,0.5) and (1.5,0.5) .. (2,0);
\end{tikzpicture}
\end{array} = 1 \quad \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (1,-1) -- (0,0);
\draw (0,-1) .. controls (0.5,-0.5) and (1.5,-0.5) .. (2,0);
\draw (0,1) .. controls (0.5,0.5) and (1.5,0.5) .. (2,0);
\end{tikzpicture}
\end{array} = 0 \quad (28)
\end{array}
\]

\[
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (1,-1) -- (0,0);
\draw (0,-1) .. controls (0.5,-0.5) and (1.5,-0.5) .. (2,0);
\draw (0,1) .. controls (0.5,0.5) and (1.5,0.5) .. (2,0);
\end{tikzpicture}
\end{array} = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (1,-1) -- (0,0);
\draw (0,-1) .. controls (0.5,-0.5) and (1.5,-0.5) .. (2,0);
\draw (0,1) .. controls (0.5,0.5) and (1.5,0.5) .. (2,0);
\end{tikzpicture}
\end{array} = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1,1) -- (2,0) -- (1,-1) -- (0,0);
\draw (0,-1) .. controls (0.5,-0.5) and (1.5,-0.5) .. (2,0);
\draw (0,1) .. controls (0.5,0.5) and (1.5,0.5) .. (2,0);
\end{tikzpicture}
\end{array} \quad (29)
\end{array}
\]

The relations (27) and (28) are motivated by the Heisenberg relation \( pq = qp + 1 \), where \( p \) and \( q \) are the two generators of the Heisenberg algebra, while the relations (29) are motivated by the symmetric group relations.

It is convenient to denote a right curl by a dot on a strand, and a sequence of \( d \) right curls by a dot with a \( d \) next to it:
A right curl can be moved across intersection points, according to the following “dot-sliding relations” [13]:

\[
\begin{align*}
\begin{array}{c}
\includegraphics{dot_sliding_1} \\
\Rightarrow \\
\includegraphics{dot_sliding_2}
\end{array}
\end{align*}
\]

This observation easily generalizes to

\[
\begin{align*}
\begin{array}{c}
\includegraphics{generalized_dot_sliding_1} \\
\Rightarrow \\
\includegraphics{generalized_dot_sliding_2}
\end{array}
\end{align*}
\]

Another consequence of relations (27)-(29) are the “bubble moves” [13]:

\[
\begin{align*}
\begin{array}{c}
\includegraphics{bubble_move_1} \\
\Rightarrow \\
\includegraphics{bubble_move_2}
\end{array}
\end{align*}
\]

Note that relations (29) imply that there is a homomorphism \( T_n : \mathbb{C}[S_n] \to \text{End}_{\mathcal{H}'}(Q_+^n) \) which sends
The endomorphism algebra \( \text{End}_{H'}(\mathbb{1}) \)

Let \( \text{End}_{H'}(\mathbb{1}) \) denote the center of \( H' \), that is, the algebra of endomorphisms of the monoidal unit object \( \mathbb{1} \). Diagrammatically, the algebra \( \text{End}_{H'}(\mathbb{1}) \) is the commutative \( \mathbb{C} \)-algebra spanned by all closed diagrams, with multiplication given by juxtaposition of diagrams. The algebra structure of \( \text{End}_{H'}(\mathbb{1}) \) was determined by Khovanov in [13]. Let \( \mathbb{C}[c_0, c_1, c_2, \ldots] \) be the polynomial algebra in countably many indeterminants \( \{c_i\}_{i \geq 0} \).

**Theorem 4.2.** [13] Prop. 3] The map \( \psi_0 : \mathbb{C}[c_0, c_1, \ldots] \to \text{End}_{H'}(\mathbb{1}) \) which sends

\[
\begin{array}{c}
c_k \\
\psi_0 \\
k
\end{array}
\]

is an algebra isomorphism.

Henceforth we will freely identify \( c_k \) with its image in \( \text{End}_{H'}(\mathbb{1}) \). Another natural set of diagrams to consider are the counterclockwise-oriented circles with \( k \) right-twist curls on them. Set

\[
\hat{c}_k :=
\]

Diagrammatically, for \( x \in \mathbb{C}[S_n] \) we set

\[
T_n(x) =:
\]

The appearance of the group algebra \( \mathbb{C}[S_n] \) as endomorphisms in \( \mathcal{H}' \) is responsible for the connection between \( \mathcal{H}' \) and the representation theory of symmetric groups.
It follows from the relations in (28) that \( \tilde{c}_0 = 1 \) and \( \tilde{c}_1 = 0 \).

**Lemma 4.3.** [13, Prop. 2] For \( k > 0 \),

\[
\tilde{c}_{k+1} = \sum_{i=0}^{k-1} \tilde{c}_i c_{k-1-i}.
\]

The final class of elements in \( \text{End}_{\mathcal{H}'}(\mathbb{1}) \) we consider are those arising from the closure of permutations (that is, closures of morphisms in the image of \( \mathcal{T}_n \)). We define

\[
\tilde{c}_k \sim \text{\# strands}
\]

For \( \lambda = (\lambda_1, \ldots, \lambda_r) \vdash n \), let

\[
\lambda := \lambda_1 \ldots
\]

then we define

\[
\alpha_\lambda := \lambda
\]

with \( \alpha_k := \alpha_{(k)} \).

Lemma 4.4 below shows that we could replace the permutation in (36) by the image under \( \mathcal{T}_n \) of any \( g \in S_n \) such that \( \text{sh}(g) = \lambda \). We choose (36) because it will be convenient for later calculations.
Lemma 4.4. Suppose that \( g_1, g_2 \in S_n \) are conjugate, so that \( \text{sh}(g_1) = \text{sh}(g_2) \). Then

\[
\begin{array}{c}
g_1 \\
\end{array}
\quad = \quad
\begin{array}{c}
g_2 \\
\end{array}
\]

Proof. This is an easy diagrammatic argument which uses the fact that \( g_1 = hg_2h^{-1} \) for some \( h \in S_n \). Replacing \( g_1 \) by \( hg_2h^{-1} \), we slide \( h \) around the diagram to cancel it with \( h^{-1} \).

4.2 Diagrams as bimodule homomorphisms

In order to establish an isomorphism between \( \text{End}_{H'}(\mathbb{1}) \) and \( \Lambda^* \), we will make use of some representations of the monoidal category \( H' \) constructed in \[13\].

To describe these representations, we start by setting some notation for \( p \in C \), \( q \in S \), \( k \in S \), \( \text{Ind}^p_n : S_k \to S_n \) and \( \text{Res}^p_n : S_n \to S_k \). Suppose that \( k_1, k_2 \leq n \). We write:

- \((n)\) for \( \mathbb{C}[S_n] \) considered as a \( (\mathbb{C}[S_n], \mathbb{C}[S_n])\)-bimodule.
- \((n)_{k_2}\) for \( \mathbb{C}[S_n] \) considered as a \( (\mathbb{C}[S_n], \mathbb{C}[S_{k_2}])\)-bimodule.
- \(k_1(n)\) for \( \mathbb{C}[S_n] \) considered as a \( (\mathbb{C}[S_{k_1}], \mathbb{C}[S_n])\)-bimodule.
- \(k_1(n)_{k_2}\) for \( \mathbb{C}[S_n] \) considered as a \( (\mathbb{C}[S_{k_1}], \mathbb{C}[S_{k_2}])\)-bimodule.

Let \( S' \) be the category whose objects are compositions of induction and restriction functors of symmetric groups. We write

\[
\text{Ind}_{p_n}^{n+1} := \text{Ind}_{S_n}^{S_{n+1}} \quad \text{and} \quad \text{Res}_{p_n}^{n+1} := \text{Res}_{S_n}^{S_{n+1}}.
\]

Since induction from \( S_n \) to \( S_{n+1} \) is given by tensoring on the left by \( (n + 1)_n \) and restriction from \( S_{n+1} \) to \( S_n \) is given by tensoring on the left by \( n(n + 1) \), the objects in \( S' \) can be reinterpreted as \( (\mathbb{C}[S_{k_1}], \mathbb{C}[S_{k_2}])\)-bimodules for \( k_1, k_2 \geq 0 \).

Example 4.5. One object in \( S' \) is the composition

\[
\text{Res}_{S_3}^4 \circ \text{Ind}_{S_3}^3 \circ \text{Ind}_{S_3}^4 \circ \text{Res}_{S_3}^4.
\]

In the language of bimodules, this is the \( (\mathbb{C}[S_4], \mathbb{C}[S_4])\)-bimodule

\[
4(5)_{(4)}(4)_{(4)}(4).
\]
The morphisms in $S'$ are certain natural transformations of these compositions (or, equivalently, certain bimodule homomorphisms). Like $H'$, morphisms in $S'$ can be presented diagrammatically as oriented compact 1-manifolds embedded in $\mathbb{R} \times [0,1]$. Unlike $H'$, in $S'$ we label the regions of the strip $\mathbb{R} \times [0,1]$ by non-negative integers, so that if there is an upwards oriented line separating two regions and the right region is labeled by $n$, then the left region must be labeled by $n+1$. The diagram

$$
n + 1 \quad \uparrow \quad n
$$

denotes the identity endomorphism of the induction functor $\text{Ind}^{n+1}_n$ or alternatively the identity endomorphism of the bimodule $(n+1)_n$.

If there is a downward oriented line separating two regions and the right is labeled by $n + 1$ then the left must be labeled by $n$. The diagram

$$
n \quad \downarrow \quad n + 1
$$

denotes the identity endomorphism of the restriction functor $\text{Res}^{n+1}_n$ or alternatively the identity endomorphism of the bimodule $n(n+1)$.

The bimodule maps associated to the four U-turns are:

$$
n \quad \xrightarrow{n + 1} \quad (n + 1)_n(n + 1) \rightarrow (n + 1), \quad g \otimes h \mapsto gh, \quad g, h \in S_{n+1}, \quad (38)
$$

$$
n + 1 \quad \xrightarrow{n} \quad (n) \rightarrow n(n + 1)_n, \quad g \mapsto g, \quad g \in S_n, \quad (39)
$$

$$
n \quad \xrightarrow{n + 1} \quad n(n + 1)_n \rightarrow (n), \quad g \mapsto \text{pr}_n(g) = \begin{cases} g & g(n + 1) = n + 1 \\ 0 & \text{otherwise} \end{cases}, \quad (40)
$$

$$
n \quad \xrightarrow{n + 1} \quad n + 1 \rightarrow (n + 1)_n(n + 1), \quad (41)
$$

where the last map is determined by the condition that

$$
1_{n+1} \mapsto \sum_{i=1}^{n+1} s_i s_{i+1} \cdots s_n \otimes s_n \cdots s_{i+1} s_i = \sum_{g \in \mathcal{C}_{n+1}^n} g \otimes g^{-1}.
$$
Finally, the upward crossing is the bimodule map

\[
\begin{array}{c}
\begin{array}{c}
\uparrow
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\downarrow
\end{array}
\end{array}
\]

\[n, \quad (n + 2)n \rightarrow (n + 2)n, \quad g \rightarrow gs_{n+1}, \quad g \in S_{n+2}.
\] (42)

Any diagram that has a region labeled with a negative number is set to 0. It is shown in [13] that all diagrams are compatible with isotopy.

**Remark 4.6.** Closed diagrams in \( S' \) with outside region labeled by \( n \) correspond to \( (\mathbb{C}[S_n], \mathbb{C}[S_n]) \)-bimodule endomorphisms of \( (n) \). The algebra of such bimodule endomorphisms is isomorphic to \( Z(\mathbb{C}[S_n]) \) via the map which sends \( f \in \text{End}(\mathbb{C}[S_n], \mathbb{C}[S_n]) \) to \( f(1_n) \). Thus closed diagrams in \( S' \) may be regarded as elements of the center of the group algebra.

Khovanov shows that the diagrams in \( S' \) satisfy the defining relations for morphisms in \( \mathcal{H}' \). As a result, given an endomorphism of \( \mathcal{H}' \), after labeling the far right region by a non-negative integer, one obtains a well-defined bimodule homomorphism in \( S' \). An additional relation that can be calculated directly from the definitions of oriented cups and caps is the following:

\[
\begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}
\]

\[n + 1 = n + 1.
\] (43)

In other words, the endomorphism \( c_0 \in \text{End}_{\mathcal{H}'}(\mathbb{1}) \) becomes the scalar \( n + 1 \) in \( Z(\mathbb{C}[S_{n+1}]) \).

\( S' \) is the direct sum of categories

\[
S' = \bigoplus_{k=0}^{\infty} S'_k,
\]

where \( S'_k \) contains all objects such that induction or restriction starts at \( k \) (i.e. the rightmost region of the diagram is labeled by \( k \)). There are functors \( f^\mathcal{H}_k : \mathcal{H}' \rightarrow S'_k \) such that the object \( c_0 \) takes to a composition of induction and restriction functors with + sent to \( \text{Ind}^{i+1}_i \) and - sent to \( \text{Res}^{i-1}_i \) where \( i \) in each case is determined by the requirement that induction/restriction begin from \( S_k \). \( f^\mathcal{H}_k \) takes a diagram from \( \mathcal{H}' \) to \( S'_k \) by labeling regions so that the rightmost region is labeled with a \( k \) and then interpreting the diagram as an element of \( S'_k \).

**Example 4.7.** \( f^H_5 : \mathcal{H}' \rightarrow S'_5 \) takes

\[
\begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}
\]

\[\begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}
\]

\[\begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}
\]

\[\begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}
\]

\[\begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}
\]

\[\begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}
\]

\[\begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\circ
\end{array}
\end{array}
\]

In the remainder of this section we calculate the image of a number of important diagrams in \( \mathcal{H}' \) under the functors \( f^\mathcal{H}_k \).
Lemma 4.8. [13 Section 4] The diagram

![Diagram](image)

is the endomorphism of \((n)_{n-1}\) which is right multiplication by \(J_n\).

Proof. The right twist curl can be written as the composition of a cup, a crossing, and a cap.

![Composition Diagram](image)

Applying the endomorphism to \(1_n\) gives

\[
1_n \mapsto \sum_{i=1}^{n-1} s_i \cdots s_{n-2} \otimes s_{n-2} \cdots s_i \mapsto \sum_{i=1}^{n-1} s_i \cdots s_{n-2}s_{n-1} \otimes s_{n-2} \cdots s_i \\
\mapsto \sum_{i=1}^{n-1} s_i \cdots s_{n-2}s_{n-1}s_{n-2} \cdots s_i = J_n
\]

where the equality holds by (7).

Lemma 4.9. Let \(k \leq n\):

1. The diagram

![Diagram](image)

corresponds to the bimodule homomorphism \((n) \to (n)_{n-k}(n)\) which sends

\[
1_n \mapsto \sum_{g \in \mathcal{L}^n_{n-k}} g \otimes g^{-1}.
\]
The isomorphism \( \varphi : \text{End}_{H'}(\mathbb{1}) \rightarrow \Lambda^* \)

2. Let \( \mu \vdash k \) and \( x_1, x_2 \in (n) \). The diagram

\[
\begin{array}{c}
\begin{array}{c}
\mu \\
\vdots \\
\vdots \\
n-k \\
n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\mu \\
\vdots \\
\vdots \\
n-k \\
n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\mu \\
\vdots \\
\vdots \\
n-k \\
n
\end{array}
\end{array}
\end{array}
\]

corresponds to the bimodule homomorphism \((n)_{n-k}(n) \rightarrow (n)_{n-k}(n)\) which sends

\[x_1 \otimes x_2 \mapsto x_1 \sigma_{\mu,n} \otimes x_2.\]

**Proof.** These follow from direct calculation using the definitions of cups, caps, and crossings. \(\square\)

**Lemma 4.10.** As elements of \(Z(\mathbb{C}[S_n])\):

1. \( f_n^{H'}(c_k) = \sum_{i=1}^{n} s_i \cdots s_{n-1} j_n^k s_{n-1} \cdots s_i, \)

2. \( f_n^{H'}(\tilde{c}_k) = \text{pr}_n(j_n^k). \)

3. \( f_n^{H'}(\alpha_\mu) = \begin{cases} A_{\mu,n} & \text{if } |\mu| \leq n \\ 0 & \text{otherwise}. \end{cases} \)

**Proof.** \((1)-(2)\) are found in [13] Section 4 and can be computed from the definitions of cups and caps and Lemma 4.8. \((3)\) can be computed by composing the maps in Lemma 4.9 with a sequence of \(|\mu|\) nested clockwise oriented caps from \(35\). When \( |\mu| > n \) then \( f_n^{H'}(\alpha_\mu) \) will have its inner region labeled by \( n - |\mu| < 0 \) and will therefore be 0. \(\square\)

5  **The isomorphism** \( \varphi : \text{End}_{H'}(\mathbb{1}) \rightarrow \Lambda^* \)

In this section we establish the algebra isomorphism \( \text{End}_{H'}(\mathbb{1}) \cong \Lambda^* \). The proof is somewhat analogous to Ivanov and Kerov’s proof of a related isomorphism connecting shifted symmetric functions to the representation theory of symmetric groups (see Theorem 9.1 in [8]).

In [13] Section 4, Khovanov defines a grading on \( \text{End}_{H'}(\mathbb{1}) \) by setting

\[ \text{deg}(c_0) := 0, \quad \text{deg}(c_k) = k + 1, \quad \text{for } k \geq 1. \]

\[(44)\]

We will consider the increasing filtration induced by this grading. A relationship between the elements \( \{c_k\}_{k \geq 0} \) and \( \{\alpha_k\}_{k \geq 1} \) is then given in terms of this filtration as follows.

**Proposition 5.1.** For any \( k \geq 1 \),

\[ \alpha_k = c_{k-1} + \text{l.o.t.} \]
The isomorphism $\varphi : \text{End}_{\mathcal{H}'}(\mathbb{I}) \to \Lambda^*$

Proof. This follows from repeated application of the dot sliding moves (30)-(31) and bubble sliding move (32). Notice that with each application of these moves, we get a single term from the same filtered part plus additional terms of lower degree. \qed

Since the elements $c_0, c_1, \ldots$ are algebraically independent generators of $\text{End}_{\mathcal{H}'}(\mathbb{I})$, we immediately obtain the following.

Corollary 5.2. The elements $\alpha_1, \alpha_2, \ldots$ are algebraically independent generators of $\text{End}_{\mathcal{H}'}(\mathbb{I})$.

For any $\lambda \vdash n$, composing $f_n^{\mathcal{H}'}$ with the normalized character $\tilde{\chi}^\lambda$ gives a map

$$(\tilde{\chi}^\lambda \circ f_n^{\mathcal{H}'}): \text{End}_{\mathcal{H}'}(\mathbb{I}) \to \mathbb{C}$$

and allows us to define a homomorphism $\varphi : \text{End}_{\mathcal{H}'}(\mathbb{I}) \to \text{Fun}(\mathcal{P}, \mathbb{C})$. Specifically, for $x \in \text{End}_{\mathcal{H}'}(\mathbb{I})$, we write

$$[\varphi(x)](\lambda) := (\tilde{\chi}^\lambda \circ f_n^{\mathcal{H}'})(x).$$

Combining Lemma 4.10.3 with (6) implies that for $\mu \vdash k$

$$[\varphi(\alpha_\mu)](\lambda) = \begin{cases} \frac{\binom{n}{k}}{\text{dim} L^\lambda} \chi^\lambda(\mu) & \text{if } k \leq n \\ 0 & \text{otherwise}. \end{cases}$$ (45)

Theorem 5.3. The map $\varphi$ induces an algebra isomorphism $\text{End}_{\mathcal{H}'}(\mathbb{I}) \to \Lambda^* \subseteq \text{Fun}(\mathcal{P}, \mathbb{C})$ with

$$\alpha_\mu \; \mapsto \; p^n_\mu.$$ 

Proof. Let $\lambda \vdash n$, $\varphi$ is an algebra homomorphism because $f_n^{\mathcal{H}'}$ is a homomorphism from $\text{End}_{\mathcal{H}'}(\mathbb{I})$ to $Z(\mathbb{C}[S_n])$ and $\tilde{\chi}^\lambda$ is a homomorphism when restricted to $Z(\mathbb{C}[S_n])$. By Proposition 5.5 and (45), $\alpha_\mu$ maps to $p^n_\mu$. Since the $\{p^n_\mu\}_{k \geq 1}$ (respectively $\{\alpha_\mu\}_{\mu \vdash k}$) are algebraically independent generators of $\Lambda^*$ (resp. $\text{End}_{\mathcal{H}'}(\mathbb{I})$), $\varphi$ must be an isomorphism. \qed

Note that Theorem 5.3 along with Lemma 4.4 imply that when $\mu \vdash n$,

$$C_{\mu,n} \; = \; \frac{n!}{z_{\mu,n}} \; \lambda \; \overset{\varphi}{\longrightarrow} \; \frac{n!}{z_{\mu,n}} p^n_\mu.$$ (46)

For $\lambda \vdash n$ recall that $E_\lambda$ is the Young idempotent associated to $\lambda$. 25
The isomorphism $\varphi : \text{End}_{\mathcal{H}'}(\mathbb{1}) \rightarrow \Lambda^*$

**Theorem 5.4.** The isomorphism $\varphi$ sends

$$\frac{1}{\dim L^\lambda} E_\lambda \xrightarrow{\varphi} s^*_\lambda.$$

**Proof.** Recall that

$$\left( \frac{1}{\dim L^\lambda} \right) E_\lambda = \sum_{\mu \vdash n} \frac{\chi^\lambda(\mu)}{n!} C_{\mu,n},$$

while

$$s^*_\lambda = \sum_{\mu \vdash n} \frac{\chi^\lambda(\mu)}{\zeta_{\mu,n}} p^\#_\mu.$$

The result then follows from (46).

The previous theorems gave graphical realizations of some important bases of $\Lambda^*$. Now we go the other way, and describe Khovanov’s curl generators $\tilde{c}_k$ and $c_k$ as elements of $\Lambda^*$. It is this description that makes an explicit connection between $\mathcal{H}'$ and the transition and co-transition measures of Kerov.

**Theorem 5.5.** The isomorphism $\varphi$ sends:

1. $\tilde{c}_k \mapsto \tilde{m}_k \in \Lambda^*$,
2. $c_k \mapsto p^\#_1 \tilde{m}_k = \tilde{b}_{k+2} \in \Lambda^*$.

**Proof.** Let $\lambda \vdash n$, then from Lemma 4.10 and Proposition 2.14 we have

$$[\varphi(\tilde{c}_k)](\lambda) = \tilde{\chi}^\lambda(\text{pr}_n(J_n^k)) = \tilde{m}_k(\lambda)$$

and

$$[\varphi(c_k)](\lambda) = \tilde{\chi}^\lambda \left( \sum_{i=1}^n s_i \cdots s_{n-1} J_n^k s_{n-1} \cdots s_i \right) = p^\#_1(\lambda)\tilde{m}_k(\lambda) = \tilde{b}_{k+2}(\lambda).$$

\[\Box\]
Remark 5.6. In [5], Farahat and Higman used the inductive structure of symmetric groups to construct a \( \mathbb{C} \)-algebra known as the Farahat-Higman algebra \( \mathcal{K}_\mathbb{C} \) (see also Example 24, Section I.7, [16]). It follows from, for example [8], that there is an algebra isomorphism \( \mathcal{K}_\mathbb{C} \cong \Lambda^* \), and the functors \( f^\mathcal{H}_n \) can also be used to give a direct isomorphism between \( \text{End}_{\mathcal{H}'}(\mathbb{I}) \) and \( \mathcal{K}_\mathbb{C} \). So in principle all of the appearances of shifted symmetric functions in the previous sections could be rephrased in the language of the Farahat-Higman algebra.

Remark 5.7. Theorem 5.5 and Remark 3.8 together imply that the recursive relationships for \( t_{p^m k u} \) and \( t_{p^b k u} \) in Remark 2.13 and \( t_{p^c k u} \) and \( \tilde{t}_{p^c k u} \) in Lemma 4.3 are both consequences of the well-known relationship between the elementary and homogeneous symmetric functions:

\[
\sum_{i=0}^{k} (-1)^i e_i h_{n-i} = 0.
\]

Example 5.8. In \( \Lambda^* \) we have \( p_{(2)}^# p_{(2)}^# = p_{(2,2)}^# + 4p_{(3)}^# + 2p_{(1,1)}^# \). In \( \text{End}_{\mathcal{H}'}(\mathbb{I}) \) the local relations can be used to compute the corresponding equation:

\[
\begin{array}{c}
\text{Diagram 1} \\
= 4 + \text{Diagram 2} + 2 \text{Diagram 3}
\end{array}
\]

5.1 Involutions on \( \text{End}_{\mathcal{H}'}(\mathbb{I}) \)

In [13], Khovanov introduced three involutive autoequivalences on \( \mathcal{H}' \). Only one of these, which we denote as \( \xi \), acts non-trivially on \( \text{End}_{\mathcal{H}'}(\mathbb{I}) \) where it gives an involutive algebra automorphism. For \( D \in \text{Hom}_{\mathcal{H}'}(Q_{\ell_1}, Q_{\ell_2}) \), we have

\[
\xi(D) := (-1)^{c(D)} D
\]

where \( c(D) \) is the total number of dots and crossings in the diagram. Thus, in \( \text{End}_{\mathcal{H}'}(\mathbb{I}) \):

\[
\begin{align*}
\xi(c_k) & \rightarrow (-1)^k c_k, \\
\xi(\tilde{c}_k) & \rightarrow (-1)^k \tilde{c}_k, \\
\xi(\alpha_k) & \rightarrow (-1)^{k-1} \alpha_k.
\end{align*}
\]

In Section 4 of [18], Okounkov and Olshanski identified an involutive algebra automorphism \( I : \Lambda^* \rightarrow \Lambda^* \) which acts on \( f \in \Lambda^* \) such that for \( \lambda \in \mathcal{P} \),

\[
[I(f)](\lambda) = f(\lambda'),
\]

27
where $\lambda'$ is the conjugate partition to $\lambda$. In particular

\begin{align}
I(s_{\lambda}^*) &= s_{\lambda'}^*, \\
I(c_k^*) &= h_k^*, \\
I(p_k^\#) &= (-1)^{k-1} p_k^\#.
\end{align}

**Proposition 5.9.** The involution $\xi$ on $\text{End}_{\mathcal{H}'}(\mathds{1})$ coincides with the involution $I$ on $\Lambda^*$.

**Proof.** This follows from the fact that $\{\alpha_k\}_{k \geq 1}$, (respectively $\{p_k^\#\}_{k \geq 1}$) generate $\text{End}_{\mathcal{H}'}(\mathds{1})$ (resp. $\Lambda^*$), $\varphi(\alpha_k) = p_k^\#$, and a comparison of (49) and (52). \qed

### 5.2 A graphical construction of the action of $W_{1+\infty}$ on $\Lambda^*$

In [4], the trace $\text{Tr}(\mathcal{H}')$ (or zeroth Hochschild homology) of $\mathcal{H}'$ is shown to be isomorphic as an algebra to a quotient of the W-algebra $W_{1+\infty}$. Like the center $\text{End}_{\mathcal{H}'}(\mathds{1})$, which is the algebra of closed planar diagrams, the trace $\text{Tr}(\mathcal{H}')$ has a purely graphical description, as the space of annular diagrams modulo Khovanov’s local diagrammatic relations. More precisely, the underlying vector space of $\text{Tr}(\mathcal{H}')$ is isomorphic to the span of annular diagrams, where an annular diagram $\tilde{f}$ is by definition a diagram obtained by taking an endomorphism $f \in \text{End}_{\mathcal{H}'}(X)$ for some object $X \in \mathcal{H}'$, and closing it up to the right in an annulus. The multiplication in $\text{Tr}(\mathcal{H}')$ is given by gluing annuli around one another:

![Graphical Construction](image)

The action of $\text{Tr}(\mathcal{H}')$ on $\text{End}_{\mathcal{H}'}(\mathds{1})$ then acquires a graphical description: given an annular diagram $\tilde{f} \in \text{Tr}(\mathcal{H}')$ and a closed planar diagram $f \in \text{End}_{\mathcal{H}'}(\mathds{1})$, the closed planar diagram $\tilde{f}g \in \text{End}_{\mathcal{H}'}(\mathds{1})$ is given by inserting a planar neighborhood of the closed diagram $g$ into the middle of the annulus:
Thus, via the isomorphisms

$$\text{End}_\mathcal{H}(\mathbb{1}) \cong \Lambda^*, \quad \text{Tr}(\mathcal{H}) \cong W_{1+\infty}$$

of Theorem 5.3 and [4], respectively, we obtain a purely graphical construction of the action of $W_{1+\infty}$ on $\Lambda^*$. Such an action was first considered by Lascoux-Thibon in [14].

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