Correlation functions in the Calogero–Sutherland model with open boundaries

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Calogero-Sutherland models of type $BC_N$ are known to be relevant to the physics of one-dimensional quantum impurity effects. Here we represent certain correlation functions of these models in terms of generalized hypergeometric functions. Their asymptotic behaviour supports the predictions of (boundary) conformal field theory for the orthogonality catastrophe and Friedel oscillations.
1 Introduction

One-dimensional models with inverse square interactions have attracted considerable interest in recent years. For the Calogero Sutherland (CS) models [1–3] describing particles moving on a continuous line the many-body ground state wave function is of Jastrow type and excitations can be written as a product of this pair product wave function and certain polynomials in the coordinates. Based on this observation the eigenvalues of the Hamiltonian can be found by means of an asymptotic Bethe Ansatz (ABA) solution [4]. Finite-size scaling analysis of the excitation spectrum and predictions of conformal field theory (CFT) have been used to study the critical behaviour of the CS model, leading to the identification of the universality class of the model with periodic boundary conditions as Luttinger liquid, i.e. a Gaussian model with central charge \( c = 1 \) [4, 5]. An interesting property of the CS models is that the compact form of the eigenstates allows for an explicit calculation of certain correlation functions [6–9], thus allowing to compare the asymptotic predictions of CFT to exact expressions derived in a microscopic model.

In addition to the models with periodic boundary conditions there exists a class of CS models lacking translational invariance, in particular the model of \( BC_N \)-type which is invariant under the action of the Weyl group of type \( B_N \) [3]. Again the ground state can be written in a compact form of Jastrow type, and the spectrum of excitations can be found [10, 11]. An analysis of the finite size corrections to the energies shows that the spectrum acquires contributions due to the ‘boundaries’ of the system and the low-energy critical behaviour is described by a \( c = 1 \) boundary CFT [12]. Again, the existence of a ‘simple’ expression for the ground state wave function opens the possibility to compare the predictions of boundary CFT for the asymptotic behaviour of correlation functions with exact results. These predictions are of great interest at present due to the possibility to extract observable properties of quantum impurity systems from finite size spectra (see e.g. [13–15]).

In the present paper we shall consider the \( BC_N \)-type CS model and compute matrix elements which allow to compare the exponents associated with the Anderson’s ‘orthogonality catastrophe’, i.e. the dependence of the overlap between ground states for different boundary conditions on the system size, and the asymptotic behaviour of (Friedel) density oscillations due to the existence of the boundary with the corresponding expressions for the Luttinger liquid [14–20].

First we will give a brief review of the properties of the CS model of \( BC_N \)-type as well as the predictions of boundary CFT relevant to the results of this paper. A form of the Hamiltonian
in a finite geometry especially convenient for our studies is [12]

\[ H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial q_j^2} + 2\lambda(\lambda - 1) \left( \frac{\pi}{2L} \right)^2 \sum_{j<k} \left\{ \frac{1}{\sin^2 \frac{\pi}{2L}(q_j - q_k)} + \frac{1}{\sin^2 \frac{\pi}{2L}(q_j + q_k)} \right\} 
+ \mu(\mu - 1) \left( \frac{\pi}{2L} \right)^2 \sum_{j=1}^{N} \frac{1}{\sin^2 \frac{\pi}{2L}q_j} + \nu(\nu - 1) \left( \frac{\pi}{2L} \right)^2 \sum_{j=1}^{N} \frac{1}{\cos^2 \frac{\pi}{2L}q_j} . \]  

(1)

Here \( \lambda, \mu, \nu \) are positive coupling constants. The particles move on the interval \( 0 \leq q_j \leq L \). The two particle interaction terms consist of the usual inverse square interaction of particles moving on one half of the circle with circumference \( 2L \) plus a term from the interaction of the particle at \( q_j \) with the mirror image \( -q_k \equiv 2L - q_k \) of the particle at \( q_k \). The last two terms can be regarded as impurity potentials situated at the edges \( q = 0 \) (\( q = L \)) of the system with strength given by \( \mu (\nu) \). Eigenvalues and eigenstates of (1) have been determined explicitly \([10, 11]\), the many-particle ground state wavefunction is \([3]\)

\[ \Psi_0^{(\lambda)}(q_1, \ldots, q_N; \mu, \nu) = \prod_{j<k} \left| \sin \frac{\pi}{2L}(q_j - q_k) \sin \frac{\pi}{2L}(q_j + q_k) \right|^{\lambda} \prod_{\ell=1}^{N} \sin \frac{\pi}{2L}q_\ell^{\mu} \left| \cos \frac{\pi}{2L}q_\ell \right|^{\nu} \]  

(2)

As for the periodic CS models the spectrum can be reproduced exactly by means of the asymptotic Bethe Ansatz method \([10]\). Expanding the ground state energy in inverse powers of the system size the following finite size scaling form is found \([12]\]

\[ E_{N}^{(0)} = L\epsilon^{(0)} + 2f + \frac{2\pi v_F^2 \lambda}{L} (\Delta N_b)^2 - \frac{\pi v_F^2}{24L} \]  

\[ \Delta N_b = \frac{1}{2\lambda} (\mu + \nu - \lambda) \]  

(3)

where \( \epsilon^{(0)} \) and \( f \) are the bulk energy density and the boundary energy in the thermodynamic limit for fixed particle density \( n = N/L \), respectively. \( v_F = 2\pi\lambda n \) is the Fermi velocity of the elementary excitations. Similarly, the energy of an excited state with \( \Delta N \) additional particles and \( N_{ph} > 0 \) particle-hole excitations near the Fermi point is, to leading order in \( 1/L \),

\[ E - E_{N}^{(0)} - \mu c^{(0)} \Delta N \simeq \frac{2\pi v_F^2}{L} \left( \frac{\lambda}{4} (\Delta N + \Delta N_b)^2 - \frac{\lambda}{4} (\Delta N_b)^2 \right) + \frac{\pi v_F^2}{L} N_{ph} . \]  

(4)

Here we have absorbed a term \( \mu c^{(0)} \Delta N \) into the definition of the energy the system with \( \mu c^{(0)} = L\partial\epsilon^{(0)}/\partial N \) being the chemical potential.

The expressions (3), (4) should be compared with the corresponding prediction of CFT for models with free boundary conditions, namely

\[ E^{(x)} = L\epsilon^{(0)} + 2f - \frac{\pi v_F^2}{24L} x + \frac{\pi v_F^2}{L} x \]  

(5)
with the Virasoro central charge $c$ appearing in the universal amplitude of the $1/L$-term and
the critical exponent $x$ of the operator generating this state. It is well known that the long-
range nature of the interactions in CS models gives rise to non-universal $1/L$-contributions to
the ground state energy of these systems \cite{5,12}. The $\lambda$-dependence in the last term in (3)
is believed to be a consequence of this effect, thus yealding an incorrect value for the central
charge.

The other contribution of order $1/L$ to the ground state energy is a consequence of the scat-
tering due to the free boundary and the impurity potentials. If one applies boundary CFT to
obtain the surface critical exponents controlling the asymptotic behaviour of correlation func-
tions one has to distinguish operators connecting states corresponding to different boundary
conditions and operators inducing a change of particle number or particle-hole excitations in
the ground state corresponding to a given boundary condition \cite{14,15}. For the latter case the
phase shift $\Delta N_b$ should be absorbed into the the change of the number of particles
\begin{equation}
\hat{E}_N^{(0)} = E_N^{(0)} - \frac{2\pi v_F \lambda}{4L} (\Delta N_b)^2 , \quad \hat{\Delta}N = \Delta N + \Delta N_b
\end{equation}
to restore particle hole symmetry of the finite size spectrum (4). The resulting scaling dimension
of an operator $\phi$ corresponding to this situation is
\begin{equation}
x(\phi) = \frac{L}{\pi v_F} \left( E - \hat{E}_N^{(0)} \right)^2 = \frac{\lambda}{2} \left( \hat{\Delta}N \right)^2 + N_{ph}
\end{equation}
with integer $\hat{\Delta}N$.

To obtain the conformal dimension of boundary condition changing operators finite size
energies corresponding to states subject to different boundary conditions have to be compared
\cite{14,15,17}. Consequently, only one of the two phase shifts $\Delta N_b, \Delta N_b'$ arising in these expressions
can be absorbed into a shift of the particle number as in (3). For the operator $\psi_{bb'}$ connecting
the gound states corresponding to different boundary conditions this leads to an operator
dimension
\begin{equation}
x(\psi_{bb'}) = \frac{\lambda}{2} (\Delta N_b - \Delta N_{b'})^2 .
\end{equation}
This exponent determines the orthogonality exponent $\langle 0_b|0_{b'} \rangle \propto L^{-x}$ and the related X-ray
edge singularity arising from a sudden change of the boundary potential \cite{14,15,17,21}.

The asymptotic behaviour of correlation functions in the bulk is still determined by the con-
formal dimension of the corresponding operator. In fact, an $n$-point function of the semiinfinite
system is subject to different boundary conditions but obeys the same differential equation as
the $2n$-point function including the mirror positions in the system without boundary \cite{22}.
Hence, the critical exponent for the asymptotic behaviour of the single particle density $\langle \rho(q) \rangle$
due to the presence of the boundary can be obtained from the density density correlation function for the corresponding system without a boundary, namely the periodic CS model. The most dominant term is due to backscattering processes with momentum \( \pm 2k_F \equiv 2\pi n \) and decays asymptotically as [4]:

\[
\langle \rho(q) \rangle - n \sim \frac{\cos(2 k_F q)}{q^{1/\lambda}}.
\]  

(9)

2 Overlap integrals

To compute the overlap integral between ground states of (1) corresponding to different values of the boundary field strengths \( \mu, \nu \) we will make use their representation in terms of so called Selberg correlation integrals (see [3] and references therein)

\[
S_{n,m}(\lambda_1, \lambda_2, \lambda; x_1, \ldots, x_m) = \left( \prod_{\ell=1}^{n} \int_0^1 dt_\ell \prod_{p=1}^m (t_\ell - x_p) \right) D_{\lambda_1,\lambda_2,\lambda}(t_1, \ldots, t_n) \tag{10}
\]

with

\[
D_{\lambda_1,\lambda_2,\lambda}(t_1, \ldots, t_n) = \prod_{\ell=1}^n t_\ell^{\lambda_1-1}(1-t_\ell)^{\lambda_2-1} \prod_{1 \leq j<k \leq n} |t_k-t_j|^\lambda. \tag{11}
\]

For \( m = 0 \) the integrals can be evaluated with result expressed in terms of Gamma-functions

\[
S_{n,0}(\lambda_1, \lambda_2, \lambda) = \prod_{j=1}^n \frac{\Gamma(1 + \lambda_j/2)\Gamma(\lambda_1 + \lambda(j - 1)/2)\Gamma(\lambda_2 + \lambda(j - 1)/2)}{\Gamma(1 + \lambda/2)\Gamma(\lambda_1 + \lambda_2 + \lambda(n + j - 2)/2)} . \tag{12}
\]

We first note that the normalization integral of (2)

\[
N_{\lambda,\mu,\nu} = \left( \prod_{\ell=1}^N \int_0^L dq_\ell \right) |\Psi_0^{(\lambda)}(q_1, \ldots, q_N; \mu, \nu)|^2 \tag{13}
\]

is of the form (10): Substituting \( t_\ell = \frac{1}{2} \left( 1 - \cos \frac{\pi}{L} q_\ell \right) \) we obtain

\[
N_{\lambda,\mu,\nu} = \left( \prod_{\ell=1}^N \frac{L}{2\pi} \int_0^1 dt_\ell t_\ell^{\mu-\frac{1}{2}}(1-t_\ell)^{\nu-\frac{1}{2}} \right) \prod_{1 \leq j<k \leq n} |t_k-t_j|^{2\lambda} = \left( \frac{L}{2\pi} \right)^N S_{N,0} \left( \mu + \frac{1}{2}, \nu + \frac{1}{2}, 2\lambda \right) . \tag{14}
\]

Now the overlap between two states (4) for different sets of boundary fields \( (\mu, \nu) \) and \( (\mu', \nu') \) can be expressed as the ratio of Selberg correlation integrals

\[
|\langle \mu, \nu | \mu', \nu' \rangle|^2 = \frac{(S_{N,0}((\mu + \mu' + 1)/2, (\nu + \nu' + 1)/2, 2\lambda))^2}{S_{N,0}(\mu + 1/2, \nu + 1/2, 2\lambda)S_{N,0}(\mu' + 1/2, \nu' + 1/2, 2\lambda)} . \tag{15}
\]
Using (12) we obtain

\[ \ln |\langle \mu, \nu | \mu', \nu' \rangle| = \frac{1}{2} \sum_{j=0}^{N-1} \left( f(\mu, \mu', \frac{1}{2} + \lambda j) + f(\nu, \nu', \frac{1}{2} + \lambda j) \right. \\
- \left. f(\mu + \nu, \mu' + \nu', 1 + \lambda (N + j - 1)) \right) \]  

with

\[ f(\mu, \mu', x) = \ln \frac{\Gamma^2 \left( \frac{1}{2} (\mu + \mu') + x \right)}{\Gamma (\mu + x) \Gamma (\mu' + x)} \approx \left( \mu - \mu' \right)^2 \left( -\frac{1}{4x} + \frac{1}{8x^2} (\mu + \mu' - 1) + \cdots \right). \]  

For large system size \( L \) and fixed density \( \rho = N/L \) the overlap integral (16) is determined by the logarithmic divergence of the first two contributions to the sum at \( j = N - 1 \) giving

\[ \ln |\langle \mu, \nu | \mu', \nu' \rangle| \propto \frac{1}{8\lambda} (\mu - \mu')^2 + (\nu - \nu')^2 \ln L \]  

which is in perfect agreement with (3) and the prediction (8) of boundary CFT (note that the operator connecting \(|\mu, \nu \rangle\) and \(|\mu', \nu' \rangle\) in this situation is a product of two boundary changing operators \( \psi_{\mu \mu'} \) and \( \psi_{\nu \nu'} \) acting at the boundary at \( q = 0 \) and \( q = L \), respectively [17]).

### 3 Friedel oscillations

We consider now a spatial dependence for single particle density

\[ \langle \rho(q) \rangle = \frac{1}{N_{\lambda, \mu, \nu}} \left( \prod_{\ell=2}^{N} \int_{0}^{L} dq_{\ell} \right) |\Psi_{0}^{(\lambda)}(q, q_{2}, \ldots, q_{N}; \mu, \nu)|^2. \]  

For integer values \( \lambda \) the function \( \langle \rho(q) \rangle \) can be expressed in terms of the Selberg integrals (10)

\[ \langle \rho(q) \rangle = \frac{2\pi}{L} x^{\mu} (1-x)^{\nu} \frac{S_{N-1,2\lambda} (\mu + \frac{1}{2}, \nu + \frac{1}{2}, 2\lambda; x_{1} = \ldots = x_{m})}{S_{N,0} (\mu + \frac{1}{2}, \nu + \frac{1}{2}, 2\lambda)}, \]  

where \( m = 2\lambda \) and \( x_{1} = \ldots = x_{m} = x = \sin^{2}(\pi q/2L) \). Eq. (20) can be rewritten as

\[ \langle \rho(q) \rangle = \frac{2\pi}{L} x^{\mu} (1-x)^{\nu} \frac{S_{N-1,0} (\mu + \frac{1}{2} + 2\lambda, \nu + \frac{1}{2}, 2\lambda)}{S_{N,0} (\mu + \frac{1}{2}, \nu + \frac{1}{2}, 2\lambda)} \times \, _2F_1^{(\lambda)} (-N + 1, \frac{1}{\lambda}(\mu + \nu + m) + N - 2, \frac{1}{\lambda}(\mu - \frac{1}{2} + m), x_{1} = \ldots = x_{m}), \]  

where \( _2F_1^{(\lambda)} \) is a generalized hypergeometric function of \( m \) variables (see [3]). For finite systems this expression for the single particle density can be evaluated by using the fact that \( _2F_1^{(\lambda)} \) for equal arguments can be written in tems of Jack symmetric polynomials [23]. In Fig. 1 we have plotted (21) in this representation for a system of \( N = 10 \) particles.
In the thermodynamic limit we derive the asymptotic behavior of \( \langle \rho(q) \rangle \) for \( 1 \ll q \ll N \) using the integral representation [7]

\[
\int_{-\infty}^{\infty} \frac{\cosh \left( \frac{q}{2} \sqrt{t} \right)}{\cosh \left( \frac{x_1}{2} \sqrt{t} \right)} \frac{dt}{(1 + xt)^{x_1 + \lambda_1 + \lambda_2 + \lambda(m - 1)}} = \frac{1}{(x_1 + \lambda_1 + \lambda_2 + \lambda(m - 1))^{1/2}} \int_{-\infty}^{\infty} \frac{dt}{(1 + xt)^{x_1 + \lambda_1 + \lambda_2 + \lambda(m - 1)}}.
\]

(22)

Omitting the \( x \)-independent factor we find from (21)

\[
\langle \rho(q) \rangle \propto x^{\mu}(1 - x)^{\nu} \prod_{\ell=1}^{m} \int_{0}^{\infty} dt_\ell \exp \left\{ \frac{x_1 t_\ell^2}{(1 + t_\ell)^{2 + (\nu - 1)/\lambda}} \right\} \prod_{j<k} |t_j - t_k|^{2/\lambda},
\]

where \( \bar{n} = N - 1 \). Near the boundary \( q \ll 1 \) we obtain

\[
\langle \rho(q) \rangle \propto q^{2 \mu}(1 - x)^{\nu} \prod_{\ell=1}^{m} \int_{0}^{\infty} dt_\ell \exp \left( g^2 t_\ell - \frac{1}{t_\ell} \right) \prod_{j<k} |t_j - t_k|^{2/\lambda}.
\]

(23)

In order to calculate this divergent integral we analytically continue it to imaginary \( g \) and obtain after rescaling \( t \rightarrow t/\bar{n} \)

\[
\langle \rho(q) \rangle \propto q^{2 \mu} \prod_{\ell=1}^{m} \int_{0}^{\infty} dt_\ell \exp \left( g^2 t_\ell - \frac{1}{t_\ell} \right) \prod_{j<k} |t_j - t_k|^{2/\lambda}.
\]

(24)

For \( 1 \ll q \ll N \) we have \( x = \sin^2(\pi q/2L) \rightarrow g^2/\bar{n}^2, g = k_F q/2 \). After rescaling \( t_\ell \rightarrow \bar{n} t_\ell \) and using the identity \( \lim_{n \to \infty} (1 + y/n)^n = \exp y \) we obtain from (23)

\[
\langle \rho(q) \rangle \propto q^{2 \mu} \prod_{\ell=1}^{m} \int_{0}^{\infty} dt_\ell \exp \left( g^2 t_\ell - \frac{1}{t_\ell} \right) \prod_{j<k} |t_j - t_k|^{2/\lambda}.
\]

(25)

In the general case the main oscillating term is obtained by integration over an region where only two variables are near extremum points. Then the integration over these two variables gives rise to a factor

\[
\cos \frac{2k_F q}{q^{1/\lambda}}.
\]

(26)

In the general case the main oscillating term is obtained by integration over an region where only two variables are near extremum points. Then the integration over these two variables gives rise to a factor

\[
\frac{\cos 2k_F q}{q^{1/\lambda + 1}}
\]

while the other integrals of type \( \prod \int e^{-gt} f(t) \) contribute a factor of order \( (\frac{1}{g})^{m-2} \). Substituting to Eq. (26) we reproduce the result (27) for any value of the coupling constant \( \lambda \). Again, this result is in complete agreement with the boundary CFT prediction (3).
4 Free fermionic case

Note that in the free fermionic case $\lambda = 1$ the Friedel oscillations can be investigated in more detail due to a possibility to express the Selberg integral $S_{n,2}$ in terms of Appell’s hypergeometric function $F_4$:

$$S_{n,2}(\lambda_1, \lambda_2, \lambda; x_1, x_2) = (-1)^n S_{n,0}(\lambda_1 + 1, \lambda_2 + 1, \lambda) F_4(a, b, c - 1, c_2; (1 - x_1)(1 - x_2)),$$  \hspace{1cm} (28)

where

$$a = -n, \quad b = \frac{2}{\lambda}(\lambda_1 + \lambda_2 + 1) + n - 1, \quad c_1 = \frac{2\lambda_1}{\lambda}, \quad c_2 = \frac{2\lambda_2}{\lambda}.$$  \hspace{1cm} (29)

Substituting (28) to eq.(20) we find

$$\langle \rho(q) \rangle \propto x^\mu(1 - x)^\nu F_4(-\bar{n}, \mu + \nu + 1 + \bar{n}, \mu + \frac{1}{2}, \nu + \frac{1}{2}, x^2, (1 - x)^2),$$  \hspace{1cm} (30)

where $x = \sin^2(k_F q/2N)$ as before. The Appell function $F_4(\alpha, \beta, \gamma_1, \gamma_2, x, y)$ satisfies the following system of equations

$$x(1 - x)Z''_{xx} - y^2 Z''_{yy} - 2xy Z''_{xy} + [\gamma_1 - (\alpha + \beta + 1)x]Z'_x - (\alpha + \beta + 1)y Z'_y - \alpha \beta Z = 0,$$

$$y(1 - y)Z''_{yy} - x^2 Z''_{xx} - 2xy Z''_{xy} + [\gamma_2 - (\alpha + \beta + 1)y]Z'_y - (\alpha + \beta + 1)x Z'_x - \alpha \beta Z = 0.$$  \hspace{1cm} (31)

In the limit $q \ll N$ we obtain after simple manipulations

$$\langle \rho(q) \rangle \propto Z(u, v)|_{u=2q^2,v=0},$$  \hspace{1cm} (32)

where $Z(u, v)$ is the solution of equations

$$Z''_{uv} = \frac{\mu(\mu - 1)Z}{(u + v)^2},$$

$$2uZ''_{uu} - 2vZ''_{vv} - 2(u - v)Z''_{uv} + 3Z'_v + k_F^2 Z = 0.$$  \hspace{1cm} (33)

For $u \gg 1$ in the first order variables $u$ and $v$ are separated each from other. We have as a result

$$\langle \rho(q) \rangle \propto n + \frac{1}{q^{1/2}} Z_\delta(2k_F q),$$  \hspace{1cm} (34)

where $Z_\delta$ is the Bessel function and

$$\delta = \sqrt{1/4 + 8\mu(\mu - 1)}.$$  

The asymptotic behavior of (34) is in agreement with general result (27).
5 Summary and conclusion

We have studied the asymptotic behaviour of the overlap integral between ground states of the \( BC_N \) type Calogero Sutherland model corresponding to different strengths of the boundary fields and the Friedel oscillations of the single particle density due to the presence of a boundary. Our results are in agreement with those obtained by applying the predictions of boundary conformal field theory to the finite size spectra of these systems. In particular, it has been established that non universal exponents showing a continuous dependence on the values of the boundary fields can arise in the correlation functions of boundary changing operators.

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Figure 1: Single particle density oscillations for a system of $N = 10$ particles with $\lambda = 2$, $\mu = 3$ and $\nu = 1$. 

![Graph showing single particle density oscillations](image-url)