GLOBAL STABILITY OF MINKOWSKI SPACE FOR THE EINSTEIN-MAXWELL-KLEIN-GORDON SYSTEM IN GENERALIZED WAVE COORDINATES

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Abstract. We prove global existence for Einstein’s equations with a charged scalar field for initial conditions sufficiently close to the Minkowski spacetime without matter. The proof relies on generalized wave coordinates adapted to the outgoing Schwarzschild light cones and the estimates for the massless Maxwell-Klein-Gordon system, on the background of metrics asymptotically approaching Schwarzschild at null infinity in such coordinates, by Kauffman [23]. The generalized wave coordinates are obtained from a change of variables, introduced in Lindblad [34], to asymptotically Schwarzschild coordinates at null infinity. This in particular gives a simplified proof in the vacuum case with precise asymptotics.

Contents

1. Introduction
2. Asymptotics of different types of terms and error terms
3. The geometric structure of Einstein’s equations in wave coordinates
4. Subtracting off terms that picks up the mass and charge contributions
5. The structure of the metric in asymptotically Schwarzschild coordinates
6. Einstein’s equations in asymptotically Schwarzschild coordinates
7. Vector fields applied to Einstein’s equations, commutators and higher order equations
8. The $L^2$ estimates for the wave equation
9. The Decay estimates for the wave equation
10. Energy bounds and decay estimates for Maxwell-Klein-Gordon
11. Precise statement of the theorem and the structure of the proof
12. The decay of the metric and the fields assuming weak energy bounds
13. The Energy bounds of the metric assuming the decay estimates
Appendix A. The Ricci curvature in terms of generalized wave coordinates
Appendix B. Einstein’s equations in generalized wave coordinates
Appendix C. Additional interior asymptotics
References

1. INTRODUCTION

Einstein’s equations state that the Ricci curvature of the space time metric satisfies

$$R_{\mu\nu} = \tilde{T}_{\mu\nu}, \quad \text{where} \quad \tilde{T}_{\mu\nu} = T_{\mu\nu} - g_{\mu\nu}g^{\alpha\beta}T_{\alpha\beta}/2,$$

and $T_{\mu\nu}$ is the energy momentum tensor of matter that satisfies the divergence free condition

$$\nabla^\nu T_{\mu\nu} = 0.$$

Einstein’s equations in harmonic or wave coordinates are a system of nonlinear wave equations

$$\Box^g g_{\mu\nu} = F_{\mu\nu}(g)|\partial g, \partial g| + \tilde{T}_{\mu\nu}, \quad \text{where} \quad \Box^g = g^{\alpha\beta}\partial_\alpha\partial_\beta,$$

is the reduced wave operator for a Lorentzian metric $g_{\alpha\beta}$, that satisfy the wave coordinate condition

$$g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta} = |g|^{-1/2}\partial_\alpha(|g|^{1/2}g^{\alpha\gamma}) = 0, \quad \text{where} \quad |g| = |\det (g)|,$$
that is preserved under the flow of (1.1). Here \( F(g)[\partial g, \partial g] \) is quadratic form in \( \partial g \) with coefficients depending on \( g \). Choquet-Bruhat \cite{5} proved local existence in these coordinates.

Christodoulou-Klainerman \cite{10} proved global existence for Einstein vacuum equations \( R_{\mu\nu} = \hat{T}_{\mu\nu} = 0 \) for asymptotically flat initial data:

\[
g_{ij}|_{t=0} = (1 + Mr^{-1}) \delta_{ij} + o(r^{-1-\gamma}), \quad \partial_t g_{ij}|_{t=0} = o(r^{-2-\gamma}), \quad r = |x|, \tag{1.3}
\]

with \( \gamma > 1/2 \), that are small perturbations of the Minkowski metric \( m_{\alpha\beta} \). Here \( M > 0 \), by the positive mass theorem. \cite{10} avoids using coordinates and instead uses equations for the full curvature tensor \( R_{\alpha\beta\gamma\delta} \) since it was believed the metric in harmonic coordinates would blow up for large times, because this is true for wave equations with quadratic nonlinearities without any extra structure.

However, Lindblad \cite{32} and Lindblad-Rodnianski \cite{35} identified a weak null structure in Einstein’s equations and formulated a weak null condition under which they expected that systems of nonlinear wave equations would have global solutions. Then in Lindblad-Rodnianski \cite{36, 37} they proved global existence of solutions to Einstein’s equations in wave coordinates for small asymptotically flat initial data with \( \gamma > 0 \) in the case of the energy momentum tensor \( T \) of a scalar field.

The proof in \cite{10} involved proving strong decay of various components of the curvature tensor that may not hold in the presence of matter, whereas the proof in \cite{36, 37} only relies on weaker decay for most components of the metric. Moreover the proof in \cite{10} uses null coordinate and vector fields adapted to the outgoing curved light cones or characteristic surfaces \( u = \text{const} \), where \( u \) solves the eikonal equation

\[
g^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0, \tag{1.4}
\]

whereas \cite{36, 37} only use vector fields and coordinates adapted to the background Minkowski space.

Due to the simpler and more perturbative approach in \cite{36, 37} it was followed by global existence for Einstein’s equations in wave coordinates coupled to the energy momentum tensor of Maxwell’s equations \cite{30}, of Vlasov matter \cite{39, 13} and of Klein-Gordon \cite{20, 29}.

Here we will prove global existence for Einstein’s equations coupled to Maxwell-Klein-Gordon. This however requires more precise asymptotics of the metric. There are strong estimates of Morawetz type for Maxwell-Klein-Gordon on Minkowski background see Lindblad-Sterbenz \cite{38}, but these estimates are not stable under perturbations as large as the difference between the metric and the Minkowski metric. However, they hold for the Schwarzschild metric.

In \cite{37} it was shown that solution with asymptotically flat data approaches the Schwarzschild metric with the same ADM mass, as \( r \to \infty \). Moreover, it was shown in Lindblad \cite{34} that the same is true for some critical components at null infinity and that the outgoing light cones for the metric converge to the outgoing light cones for the Schwarzschild metric, \( u^* = t - r^* = \text{const} \), where \( r^* = r + M \ln r \), i.e. there is a solution \( u \) to (1.4) such that \( u \sim u^* \) at null infinity. Moreover changing to the coordinates \( (\tilde{t}, \tilde{x}) \), with \( \tilde{t} = t \) and \( \tilde{x} = r^* x / r \), the precise asymptotics for the metric was given and a set of vector fields were constructed analogous to the commutation fields in Minkowski space, depending only on the ADM mass of initial data.

In these coordinates Kauffman \cite{23} proved that the same kind of Morawetz estimates as in \cite{38} hold for MKG on the background of a metric that is asymptotic to Schwarzschild at null infinity. The purpose of this paper is to couple MKG to Einstein using \cite{23} to estimate the MKG fields.

In order to deal with more difficult matter fields we prove global existence for Einstein’s equations directly in the new coordinates by composing the wave coordinates with the change of variables above. The change of variables can either be thought of as the new metric satisfying a generalized wave coordinate condition, cf. \cite{18, 17}, or the new metric satisfying the equation obtained by replacing derivatives with covariant derivatives with respect to the change of variables. This will introduce new error terms but they are fast decaying and easily estimated. However, the estimates of the quasilinear part simplifies substantially because the commutators of the vector fields with the wave operator in these coordinates decay much faster. Moreover, the estimates of the semilinear terms are also simplified substantially by using Lie derivatives, which as observed in \cite{34} preserve
the geometric weak null structure of the semilinear terms. As a byproduct of our proof we hence have an improved proof of the vacuum case as well, that we think will be useful to deal with more difficult matter fields. We will first formulate the result for a general matter field satisfying certain assumptions and then afterward we formulate the theorem in the coupled case specifically for MKG.

1.1. Einstein’s equations in generalized wave coordinates. Since $m$ is constant $h = g - m$ satisfies

$$
\Box^{g}h_{\mu\nu} = F_{\mu\nu}(g)[\partial h, \partial h] + \tilde{T}_{\mu\nu}, \quad \text{where} \quad F_{\mu\nu}(g)[\partial h, \partial h] = P(g)[\partial_{\mu}h, \partial_{\nu}h] + Q_{\mu\nu}(g)[\partial h, \partial h],
$$

(1.5)

where $Q$ is a sum of classical null forms with two metric contractions and $P$ has a weak null structure

$$
P(g)[k, p] = g^{\alpha\beta}k_{\alpha\beta}g^{\gamma\delta}p_{\gamma\delta}/4 - g^{\alpha\beta}g^{\gamma\delta}k_{\alpha\beta}p_{\gamma\delta}/2,
$$

(1.6)

see Section 3.1.4. Here $F(\cdots)[u_{1}, u_{2}]$ stands for functions that are separately linear in $u_{1}$ and $u_{2}$. With $H^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta} = -h^{\alpha\beta} + o^{\alpha\beta}(g)[h, h]$, where $h^{\alpha\beta} = m^{\alpha\delta}m^{\delta\gamma}h_{\gamma\delta}$, the wave coordinate condition become

$$
\partial_{\mu}(H^{\mu\nu} - m^{\mu\nu}m^{\alpha\beta}H^{\alpha\beta}/2) = W^{\nu}(g)[H, \partial H].
$$

(1.7)

1.1.1. The weak null structure, vector fields and Lie derivatives. To see the weak null structure we introduce a null frame $N$ of vectors tangential to the outgoing light cones plus a vector perpendicular to the cone:

$$
L = \partial_{t} - \partial_{r}, \quad L = \partial_{t} + \partial_{r}, \quad S_{1}, S_{2} \in S^{2}, \quad \langle S_{i}, S_{j} \rangle = \delta_{ij},
$$

It is well known that, for solutions of wave equations, derivatives tangential to the outgoing light cones $\mathcal{O} \in \mathcal{T} = \{L, S_{1}, S_{2}\}$ decay faster. Since $Q_{\mu\nu} = Q_{\mu\nu}(m)$ satisfy the classical null condition

$$
|Q_{\mu\nu}(\partial h, \partial k)| \lesssim |\partial h| |\partial k| + |\partial h| |\partial k|.
$$

(1.8)

Moreover, projecting derivatives onto the frame

$$
|\partial_{\mu}\phi - L_{\mu}\partial_{q}\phi| \lesssim |\partial h|, \quad \text{where} \quad \partial_{q} = (\partial_{t} - \partial_{h})/2, \quad L_{\mu} = m_{\mu\nu}L^{\nu},
$$

we see that the main term $P = P(m)$ has the following weak null structure:

$$
|P(\partial_{\mu}h, \partial_{\nu}k) - L_{\mu}L_{\nu}P(\partial_{q}h, \partial_{q}k)| \lesssim |\partial h| |\partial k| + |\partial h| |\partial k|.
$$

(1.9)

Furthermore, from the wave coordinate condition

$$
|L_{\nu}L_{\mu}\partial_{q}H^{\mu\nu}| \lesssim |\partial h| + |\partial h| |\partial h|,
$$

and expanding $P$ in a null frame using this it follows that

$$
|P(\partial_{q}h, \partial_{q}h)| \lesssim \sum_{T \in \mathcal{T}, U \in N} |\partial h_{TU}|^{2} + |\partial h| |\partial h| + |h| |\partial h|^{2}.
$$

(1.10)

The weak null structure is that the tangential components of the right hand side of (1.5) are decaying faster, due to (1.8)-(1.9), and the remaining component $F_{LL} = F_{\mu\nu}L^{\mu}L^{\nu}$ to highest order does not depend on that component of the solution $\partial_{q}h_{LL}$, due to (1.10). To highest order there is no mutual interaction between the components, similar to the structure of $\Box \phi = (\partial_{t}\psi)^{2}, \Box \psi = 0$.

In order to estimate the solution we also need higher order equations for vector fields applied to it. The vector fields $\Omega_{ab} = x_{a}\partial_{b} - x_{b}\partial_{a}$, where $x_{a} = m_{ab}x^{b}$ commute with $\Box$ and $S = x^{a}\partial_{a}$ satisfy $[\Box, S] = -2\Box$. Applying vector fields to $F_{\mu\nu}(m)[\partial h, \partial k]$ will produce lower order terms that no longer have the weak null structure, but, as observed in [34], if we instead apply Lie derivatives the weak null structure is preserved:

$$
\mathcal{L}_{Z}(F_{\mu\nu}(m)[\partial h, \partial k]) = F_{\mu\nu}(m)[\partial \hat{L}_{Z}h, k] + F_{\mu\nu}(m)[\partial h, \partial \hat{L}_{Z}k].
$$
1.1.2. Reduction to Schwarzschild at null infinity. We will make two reductions that will put us closer to Schwarzschild in the exterior and at null infinity. First we write $h_{\mu\nu} = h^0_{\mu\nu} + h^1_{\mu\nu}$ and $H^{\mu\nu} = H^0_{\mu\nu} + H^1_{\mu\nu}$, where

$$h^0_{\mu\nu} = \chi(\sqrt{r'}) M r^{-1} \delta_{\mu\nu}, \quad \text{and} \quad H^0_{\mu\nu} = -\chi(\sqrt{r'}) M r^{-1} \delta^{\mu\nu},$$

and $\chi$ is a cutoff function such that $\chi(s) = 1$ for $s \geq 3/4$ and $\chi(s) = 0$ for $s \leq 1/4$. Then we change variables

$$\bar{t} = t, \quad \bar{x} = r^s x/r, \quad \text{where} \quad r^s = r + \chi(\sqrt{r'}) M \ln r,$$

and with $x^0 = t$, $\bar{x}^0 = \bar{t}$, set

$$\bar{g}^{ab} = A^a_\\alpha A^b_\beta g_{\alpha\beta}, \quad \bar{g}_{ab} = A^a_\alpha A^b_\beta g_{\alpha\beta}, \quad \bar{\partial}_a = A^a_\alpha \partial_\alpha, \quad \text{where} \quad A^a_\alpha = \partial x^a/\partial x^\alpha, \quad A^a_\alpha = \partial x^a/\partial \bar{x}^a.$$

Then $\bar{h}_{ab} = h^0_{ab} = A^a_\\alpha A^b_\beta \bar{h}^{\alpha\beta}$ satisfies

$$\square \bar{h} = F_{cd}(\bar{g})[\bar{\partial} h^1, \bar{\partial} h^1] + \bar{T}_{cd} + R_{cd}^{\text{error}}, \quad \text{where} \quad \square \bar{g} = g^{ab} \bar{\partial}_a \bar{\partial}_b,$$

and $\bar{H}_{1}^{ab} = A_a A_b h_{\alpha\beta}$ satisfies

$$\bar{\partial}_c (\bar{H}_{1}^{cd} - m_{cd} m_{ab} \bar{H}^{ab}/2) = W^d(\bar{g})[\bar{H}_1, \bar{\partial} \bar{H}_1] + W^d_{\text{error}},$$

where the errors $R_{cd}^{\text{error}}$ and $W^d_{\text{error}}$ are decaying faster. What we have achieved with this is that initial data for critical components of $h^1$ and $H_1$ decay faster and that $\square \bar{g}$ more rapidly approaches a scalar multiple of the constant coefficient wave operator $\square = m_{ab} \bar{\partial}_a \bar{\partial}_b$ in the new coordinates. This will lead to that $h^1$ and $H_1$ are decaying faster in the exterior and to that the commutators between the wave operator with the vector field will decay faster. We will use the vector fields $\bar{\Omega}_{ab} = \bar{x}_a \bar{\partial}_b - \bar{x}_b \bar{\partial}_a$, that commute with $\square$ and $\bar{S} = \bar{x}^a \partial_a$ that satisfy $[\square, \bar{S}] = -2\bar{S}$.

1.1.3. Higher order energies. Let $\bar{Z}$ be any of the vector fields $\bar{\Omega}_{ab}, \bar{S}$, and $\bar{\partial}$, and for a multindex $I$ let $\bar{Z}^I$ respectively $\bar{L}^I_{\bar{Z}}$ denote any combination of $|I|$ vector fields respectively Lie derivatives. We will be using energies with an additional exterior weight

$$E_N(t) = \sum_{|I| \leq N} \int |\bar{\partial} \bar{L}^I_{\bar{Z}} h^1(t, x)|^2 w \, dx,$$

where $w(t, x) = \begin{cases} (1 + |r^s - t|)^{1+2\gamma}, & r^s > t, \\ \sim 1, & r^s \leq t, \end{cases}$

for some $0 < \gamma < 1$. These energies remains bounded for the linear homogeneous wave equations $\square \bar{g} \phi = 0$, and the only small growth is caused by that the inhomogeneous term $F_{\mu\nu}$ in (1.11) don’t satisfy the classical null condition. The extra weight in the exterior is meant to catch the extra decay of $h^1$ in the exterior, which in turn using the wave coordinate condition (1.12) gives additional decay for the critical components controlling the geometry of the light cones.

1.1.4. Conditions on matter. The energy momentum tensor has to satisfy some smallness conditions and be compatible with the weak null structure for the metric. In order for it to be compatible with the energy estimate for the metric we assume that for a sufficiently small $\varepsilon$, the following hold:

$$\| \bar{L}^I_{\bar{Z}} \bar{T}(t, \cdot) w^{1/2} \|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon^2 (1 + t)^{-1}, \quad \text{for} \quad |I| \leq N. \quad (1.13)$$

It is also natural to assume that with $q = r - t$ and $q_+ = q$ for $q > 0$ or $q_+ = 0$ for $q < 0$, we have

$$\| \bar{L}^I_{\bar{Z}} \bar{T}(t, \cdot) q_+^{\gamma} \|_{L^1} \lesssim \varepsilon^2, \quad |J| \leq N - 3. \quad (1.14)$$

We also assume that the following decay estimate holds for sufficiently small $\varepsilon$

$$|\bar{L}^K_{\bar{Z}} \bar{T}(t, x)| \lesssim \varepsilon^2 (t + r^s)^{-2} (t - r^s)^{-1} ((t - r^s)_+)^{-\gamma}, \quad |K| \leq N - 6. \quad (1.15)$$

In addition we must assume that $T$ is compatible with the weak null structure of Einstein’s equation; for some $s > 0$ and all $T \in \mathcal{T}, U \in \mathcal{N}$ we assume that

$$|\bar{L}^K_{\bar{Z}} \bar{T}(t, x)|_{TU} \lesssim \varepsilon^2 (t + r^s)^{-2-s} (t - r^s)^{-1+s} ((t - r^s)_+)^{-\gamma}, \quad |K| \leq N - 6. \quad (1.16)$$
1.1.5. *The existence theorem.* We are now ready to state our first result.

**Theorem 1.1.** Suppose that the energy momentum tensor \( \hat{T} \) satisfies the conditions (1.13), (1.14), (1.15) and (1.16) for some \( N \geq 11 \) and \( 0 < \gamma < 1 \). Then there is an \( \varepsilon_0 > 0 \) such that if \( \varepsilon < \varepsilon_0 \) and

\[
E_N(0) + M^2 \leq \varepsilon^2,
\]

then (1.5) has a global solution and there is a constant \( C \) such that for all \( t > 0 \)

\[
E_N(t) \leq C\varepsilon^2 (1 + t)^C\varepsilon.
\]

Moreover, with \( \varpi(q^*) = (q^*)^{1-\delta}(q_+^*)^\gamma \), for any \( \delta > 0 \), we have for \( |I| \leq N - 7 \)

\[
(1 + t + |x|) |(\partial \hat{T}^I Z^I)_{TV}(t, x) \varpi(q^*)| \lesssim \varepsilon,
\]

\[
(1 + t + |x|) |\partial \hat{T}^I Z^I(t, x) \varpi(q^*)| \lesssim \varepsilon \left(1 + \ln \frac{t^2 + r^*}{(t - r^*)^2}\right),
\]

and

\[
|\partial \hat{T}^I Z^I H_1|_{LT} + |\partial H^I H_1| \lesssim \varepsilon (1 + t + r)^{-2+2\delta}(q_+^*)^{-\gamma}(q^*)^{-\delta}.
\]

1.2. *Einstein with Maxwell-Klein-Gordon matter.* As an application of this general theorem we couple Einstein’s equations to the massless Maxwell-Klein-Gordon system, which has been studied extensively in the Minkowski spacetime and for which the associated trace-reversed energy momentum tensor \( \hat{T} \) satisfies the conditions (1.13), (1.14), (1.15) and (1.16) for small initial data. We first define the system and state basic stability results in perturbed spacetime, as proven in [23].

1.2.1. *The electromagnetic field and complex scalar field.* Given a real connection one-form \( A_\alpha \) and the consequent quantities \( F = dA, D_\alpha = \nabla_\alpha + iA_\alpha \), where \( \nabla_\alpha \) is covariant differentiation, \( \{\phi, F\} \) is a solution to the charge-scalar field system if

\[
\Box^C g \phi = D^\alpha D_\alpha \phi = 0, \tag{1.17}
\]

\[
\nabla^\beta F_{\alpha\beta} = \mathcal{I}(\phi D_\alpha \phi), \tag{1.18}
\]

\[
\nabla^\beta (\ast F)_{\alpha\beta} = 0,
\]

where \( \mathcal{I}, \mathcal{R} \) denote the imaginary and real parts of a quantity respectively. From a physical perspective it is useful to introduce the current

\[
J_\mu = \mathcal{I}(\phi D_\mu \phi).
\]

The energy momentum tensor \( T \) corresponding to a solution is given by

\[
T_{\alpha\beta}[\phi, F] = \mathcal{R} \left( D_\alpha \phi D_\beta \phi - g_{\alpha\beta} D_\gamma \phi D^\gamma \phi / 2 \right) + F_{\alpha\gamma} F_{\beta}^\gamma - \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta}. \tag{1.19}
\]

There is an ambiguity in the choice of \( A_\alpha \), called gauge freedom, which does not affect the system on a physical level, and which it is therefore not necessary to resolve. There is a choice of gauge, called Lorentz gauge, for which \( A_\alpha \) has a clear weak null structure and components decay like solutions of the corresponding asymptotic system, as proven by Candy-Kauffman-Lindblad [4] and He [16].

A characteristic difficulty for \( F \), which also has an analogue in the behavior of Einstein’s Vacuum Equations, is the decay of \( F \) in space:

\[
F_{\alpha\beta}(0, x) \sim C|\omega_1|x|^{-2} + o(|x|^{-2-\gamma}), \quad \partial \phi = o(|x|^{-2-\gamma}).
\]

The value of \( C \) scales with the charge \( q[F] \), which depends on \( \phi \), and is in general not 0 even if \( \phi \) is compactly supported. This decay rate causes issues when attempting to establish an energy estimate, as the (fractional) conformal Morawetz estimate we use requires a decay rate of \( O(r^{-2-\gamma}) \) for some \( \gamma > 0 \). In order to deal with this we construct a model field \( F^0 \) with the same charge as \( F \), and run analysis on the remainder quantity \( F^1 = F - F^0 \), which decays faster as \( r \rightarrow \infty \).
1.2.2. The existence theorem for the Einstein-Maxwell-Klein-Gordon system. For the fields \( \{\phi, F\} \), we define the weighted higher order energy

\[
Q_N(T) = \sup_{0 \leq t \leq T} \sum_{|I| \leq N} \int \left| D_I^L \tilde{D}\phi(t,x) \right|^2 + \left| \tilde{Z}^I F^1(t,x) \right|^2 \, w \, dx,
\]

where \( F^1 \) is the charge-free part of \( F \), as defined in (1.19) and \( q \) is the associated charge, which we will use later. Note the comparison to \( E_N(T) \).

**Theorem 1.2.** Einstein’s equations in harmonic coordinates (1.1) with energy momentum tensor (1.19) coupled with (1.17), (1.18) have global solutions for sufficiently small and smooth asymptotically flat initial data. Specifically, for \( \gamma > \frac{1}{2} \), there exists an \( \varepsilon_0 > 0 \) such that for all \( \varepsilon < \varepsilon_0 \), the coupled system has global solutions in time whenever

\[
E_N(0) + Q_N(0) + M^2 + q^2 \leq \varepsilon^2.
\]

Taking \( s < (1 + \gamma)/2 \), then, in addition to the bounds from Theorem 1.1, we have

\[
|D_I^L D_X^I \phi| + |\alpha[L_X^I F^1]| \leq C\varepsilon(t + r^*)^{-2}(t - r^*)^{1/2 - s}(r^* - t)^{s-1/2-\gamma},
\]

\[
|D^I X^I \phi| + |\beta[L_X^I F^1]| + |\sigma[L_X^I F^1]| \leq C\varepsilon(t + r^*)^{-1-s}(t - r^*)^{-1/2}(r^* - t)^{s-1/2-\gamma},
\]

\[
(t - r^*)^{-1}|D_X^I \phi| + |D_L^L D_X^I \phi| + |\alpha[L_X^I F^0]| \leq C\varepsilon(t + r^*)^{-1}(t - r^*)^{-1/2-s}(r^* - t)^{s-1/2-\gamma},
\]

for \( |I| \leq N - 4 \), and

\[
|\alpha[L_X^I F^0]| \leq C\varepsilon(t + r^*)^{-3}(t - r^*),
\]

\[
|\beta[L_X^I F^0]| + |\sigma[L_X^I F^0]| + |\alpha[L_X^I F^0]| \leq C\varepsilon(t + r^*)^{-2},
\]

for \( |I| \leq N \).

1.2.3. History of related results. An early global existence result for the Maxwell-Klein-Gordon system with small initial data in the Lorenz gauge was given by Choquet-Bruhat-Christodoulou [7]. However, this method required \( H^s \) bounds for components of the potential \( A \), which was incompatible with a nonzero charge, and furthermore, did not give any decay for solutions of the system. Eardley-Moncrief [11]-[12] showed global existence for general initial data bounded in \( H^s \) in a gauge adapted to the light cone. These existence results were later refined by Klainerman-Machedon [26], who improved the requirements on the the initial decay. However, these results did not give any insight on the rate of decay in time. Furthermore, the initial bounds were strongly dependent on the choice of gauge.

A study of asymptotic behavior of the system with small initial data was undertaken by Shu [41] in the massless case, and in the massive case by Psarrelli [40]. In each case, compact support of the initial data is assumed, modulo the presence of a charge. We restrict our discussion to the massless system, as the massive system is more closely modelled by the Klein-Gordon system, and so the treatment of asymptotics is substantially different. The approach in [41], though heuristic, provided a model for more complete global stability results and asymptotics for small initial data by Lindblad-Sterbenz [38] and Bieri-Miao-Shahshahani [3].

Asymptotics for large initial data were determined more recently by Yang [47] and Yang-Yu [48]. The first of these papers removed the assumption of smallness for the field \( F \), and the second removed it for \( \phi \) as well, though in each case the fields were assumed to be in certain weighted Sobolev spaces. Results for the massive case with small initial data (not necessarily compactly supported) were shown for Klainerman-Wang-Yang [28] and for large \( F \) by Fang-Wang-Yang [14].
2. Asymptotics of different types of terms and error terms

2.1. Asymptotics. In this section we will use asymptotics for solutions of wave equations to give a heuristic argument for why certain nonlinear effects are under control.

2.1.1. General nonlinear wave equations and the weak null condition. General wave equations with quadratic nonlinearities may blow up for large times even for small data as shown in John [21, 22] for □ ψ = ψ² or □ ψ = ψ△ ψ. The null condition is an algebraic condition on the structure of the nonlinear terms that guarantees small data global existence, e.g. □ ψ = ψ² − |∇x ψ|², see Christodoulou [9] and Klainerman [28]. The null condition is however not satisfied for Einstein’s equations in wave coordinates. Neither is it satisfied for the quasilinear equation □ ψ = cαβψ ∂α∂βψ, that resembles the quasilinear terms in Einstein’s equations, yet global existence was proven in Lindblad [32, 33] and Alinhac [1] for these equations. A simple semilinear system that violates the null condition yet trivially has global solutions is

□ ψ = (∂t ψ)², \quad \text{where} \quad □ ψ = 0. \quad (2.1)

The semilinear terms in Einstein’s equations resemble this system, see Section 3.1.4. Based on this Lindblad-Rodnianski [35] introduced the weak null condition and showed that it was satisfied for Einstein’s equations.

2.1.2. Asymptotics, decay along the light cone and total decay. A solution of a linear homogeneous wave equation □ ϕ = 0 with smooth initial data decaying like r⁻¹, decays like t⁻¹ and has a radiation field

ϕ(t, x) \sim \mathcal{F}(r - t, ω)/r, \quad \text{where} \quad |\mathcal{F}(q, ω)| + |q| |\mathcal{F}_q(q, ω)| \lesssim 1, \quad \langle q \rangle = 1 + |q|. \quad (2.2)

For spherically symmetric compactly supported data this is exact for large r. The same is true if

|\ □ \ ϕ | + r⁻² |△ ω ϕ | \lesssim r⁻¹ (t + r)⁻¹ - \varepsilon (t - r)⁻¹ + \varepsilon ((r - t)⁺)⁻¹, \quad \varepsilon > 0, \quad (2.3)

and data decays like r⁻¹. This is seen by expressing the wave operator in spherical coordinates:

□ ϕ = -r⁻¹(∂t + \partial_r)(∂t - \partial_r)(r ϕ) + r⁻² △ ω ϕ, \quad (2.4)

and integrating, in the t+r direction and in the t-r direction, to obtain a bound for rϕ and the asymptotics (2.2). The decay in (2.3) separates into decay along the light cone in the t+r direction and decay away from the light cone in the t-r direction, the sum of the two we call the total decay or homogeneity. The total decay of the solution ϕ is one and since differentiating twice will decrease the homogeneity by two this is consistent with a total decay of three for (2.3). Note also how the decay in t-r helped but only up to integrable decay. In the nonradial case when △ωϕ ≠ 0 the integration along characteristics can be replaced by the energy integral method in which case the bound for the tangential Laplacian is not needed.

2.1.3. Sources along light cones. However, general quadratic inhomogeneous terms as in (2.1) do not decay enough along the light cone for (2.3) to hold. In fact, by (2.2)

ϕ(t, x)² \sim \mathcal{F}_q(r - t, ω)²/r², \quad \text{where} \quad |\mathcal{F}_q(q, ω)| \lesssim \langle q \rangle⁻¹. \quad (2.5)

The asymptotics for the wave equation with such sources along light cones was studied in Lindblad [31]:

- □ ψ = n(r - t, ω)/r². \quad (2.6)

The solution is asymptotically given by a formula which leads to a log correction of the asymptotic behavior:

ψ(t, r, ω) \sim \int_{r-t}^{∞} \frac{1}{2r} \ln \left| \frac{t + q + r}{t + q - r} \right| n(q, ω) dq \sim \ln |r| \mathcal{F}_{log}(r - t, ω)/r + \mathcal{F}(r - t, ω)/r, \quad \text{when} \quad r \sim t, \quad (2.7)
In fact applying the wave operator (2.4) to the first expression in (2.7) shows that (2.6) holds up to a nonspherical error bounded by (2.8) which hence correspond to a solution with a radiation field without a logarithm, see Proposition 7.2 in [31].

Only certain combinations of quadratic terms that satisfy the classical null condition decay better, e.g.

$$\phi_t(t,x)^2 - |\nabla_x \phi|^2 \sim \mathcal{F}_q(r-t,\omega)^2/r^2 - \mathcal{F}_q(r-t,\omega)^2/r^2 \sim r^{-3}(t-r)^{-1}, \quad (2.8)$$

but semilinear terms in Einstein’s equations behave like (2.1) and hence will produce logarithms.

### 2.1.4. The asymptotics in the wave coordinate condition.

Assuming that we have asymptotics for $H^{\mu\nu}$ of the form $\mathcal{H}^{\mu\nu}(t-r,\omega)/r$ plus a similar term multiplied with a log, in order for this to be compatible with the wave coordinate condition (1.7) we must have the following relation:

$$L_\mu \partial_t (\mathcal{H}^{\mu\nu} - \frac{1}{2} m^{\mu\nu} m_{\alpha\beta} \mathcal{H}^{\alpha\beta}) = 0,$$

where $L_0 = -1$ and $L_i = \omega_i$, for $i = 1, 2, 3$. In particular $L_\mu L_0 \partial_\mu \mathcal{H}^{\mu\nu} = 0$. Since by the asymptotic flatness condition (1.3) $\mathcal{H}^{\mu\nu} L_\mu L_\nu = -2M$ and $\mathcal{H}^{\mu\nu} L_\mu L_\nu = 0$ when $t = 0$ it follows that

$$\mathcal{H}^{\mu\nu} L_\mu L_\nu = -2M. \quad (2.9)$$

### 2.1.5. The principal quasilinear part of the wave operator.

If we plug in the expansion into the wave operator we get that, up to lower order terms involving derivatives tangential to the outgoing light cone that decay better,

$$\Box^g - \Box = H^{\alpha\beta} \partial_\alpha \partial_\beta - 4^{-1} r^{-1} H^{\alpha\beta} L_\alpha L_\beta \partial_q^2 = 2^{-1} r^{-1} M \partial_q^2. \quad (2.10)$$

### 2.2. Inhomogeneous error terms.

By Section 2.1.3 general quadratic inhomogeneous terms, e.g. (2.5), just fail to decay enough whereas by Section 2.1.2 terms with any additional decay along the light cone, e.g. (2.8), will not qualitatively affect the asymptotic behaviour of the solution. Apart from possible logarithmic factors we expect the solution to decay like linear wave $\phi \sim t^{-1}$ and $\partial_\phi \sim t^{-1}(t-r)^{-1}$ and hence we expect the quadratic inhomogeneous terms to decay like $\sim t^{-2}(t-r)^{-2}$. In fact, a closer analysis shows that the components of the metric with additional logarithmic factors are not present in the quadratic semilinear nonlinearity for Einstein’s equations since the system satisfies the weak null condition. Anything with additional decay is to be considered a negligible error term. The first type of error term has additional decay everywhere $\sim t^{-2-a}(t-r)^{-2}$, $a > 0$ and the second type has additional decay along the light cone but the same total decay $\sim t^{-2-b}(t-r)^{-2+b}$, $b > 0$.

#### 2.2.1. Error terms with additional decay everywhere.

This is in particular true for cubic or higher order terms, which we will denote cubic error terms. Moreover changing coordinates, see Section 2.4.2 will produce lower order terms which additional decaying factors, which we will call covariant error terms, see Section 2.4.3 It is easy to estimate terms with additional decay everywhere.

#### 2.2.2. Error terms with additional decay along the light cone.

There are two types of inhomogeneous quadratic terms that, although they decay faster along the light cone, do not have faster total decay. The first are terms that satisfy the classical null condition e.g. (2.8). The second are terms produced when we subtract off a term to pick up the leading order spatial decay from the mass in initial data, see Section 2.3.1. Since these are proportional to the mass we will call these mass error terms. Although by Section 2.1.2 solutions to the wave equation with inhomogeneous terms with additional decay along the light cone will not distort the asymptotics along the light cones, we will also need additional exterior decay and for this we need the inhomogeneous terms to decay additionally in the exterior, see Section 2.3.
2.2.3. Commutator errors with no additional decay. The most difficult quadratic error terms show up when one commutes vector fields $Z$ with the reduced wave operator, in order to obtain wave equations also for product of vector fields applied to the solution $Z^i h_{\mu\nu}$. If we use the vector fields that commutes with the Minkowski wave operator, or satisfy $[Z, \Box] = c_Z \Box$, then the commutator with the reduced wave operator $\Box^g$ is roughly a product of vector fields applied to this wave operator plus terms of the form

$$ (Z^i H^{\alpha\beta}) \partial_\alpha \partial_\beta Z^K h_{\mu\nu}, \quad \text{for } |J| + |K| \leq |I|, \ |J| \geq 1. $$

(2.11)

Such terms are problematic but due to (2.9)-(2.10) the leading behavior is determined by a fixed tensor and as a result (2.11) are to leading order linear. (2.10) does however affect the asymptotic behaviour. To remedy this we introduce coordinates that asymptotically straighten out the light cones, see Section 2.4.

2.3. Subtracting off a term that picks up the mass contribution and exterior decay. The Schwarzschild metric in harmonic coordinates is a solution of Einstein’s vacuum equations, (1.3) with $T = 0$ also satisfying the harmonic coordinate condition (1.7). Since this metric has an expansion in powers of $r^{-1}$ it follows that the first term in the expansion $Mr^{-1} \delta_{\mu\nu}$ satisfies (1.5) with $\hat{T} = 0$ up to terms of order $r^{-4}$ as well as (1.7) up to terms of order $r^{-3}$. In fact this is the contribution of the next term in the expansion. However, since we expect the solution to decay like $t^{-1}$ this approximation can only be valid for $r > t/2$, say. Therefore we will multiply with a cutoff function $\chi(t/r)$, where $\chi(s) = 1$ for $s \geq 3/4$ and $\chi(s) = 0$ for $s \leq 1/4$, and write $h_{\mu\nu} = h_{\mu\nu}^0 + h_{\mu\nu}^1$ and $H^{\mu\nu} = H_0^{\mu\nu} + H_1^{\mu\nu}$, where

$$ h_{\mu\nu}^0 = \chi(t/r) MR^{-1} \delta_{\mu\nu}, \quad \text{and} \quad H_0^{\mu\nu} = -\chi(t/r) MR^{-1} \delta^{\mu\nu}. $$

Since $H^{\alpha\beta} = (m+h)^{-1}\alpha_\beta - (m-1)\alpha_\beta = -h^{\alpha\beta} + O^{\alpha\beta}(g)[h, h]$, it follows that $H_1^{\alpha\beta} = -h^{1\alpha\beta} + O^{\alpha\beta}(g)[h, h]$.

2.3.1. Asymptotics and additional exterior decay. Solutions of linear homogeneous wave equations $\Box \varphi = 0$ with smooth initial data decaying like $r^{-1-\gamma}$, $\gamma > 0$, have radiation fields

$$ \varphi(t, x) \sim F(r - t, \omega)/r, \quad \text{where } |F(q, \omega)| + |q| |F_q(q, \omega)| \lesssim \langle q+ \rangle^{-\gamma}, $$

where $q_+ = \max\{q, 0\}$. The same is true if only

$$ |\Box \varphi| + r^{-2} \Delta \omega \varphi| \lesssim r^{-1} (t+r)^{-1-\epsilon} (t-r)^{-1+\epsilon} \langle (t-r)^+ \rangle^{-\gamma}, \quad \gamma > 0, \ \epsilon > 0. $$

(2.12)

as is seen by integrating along characteristics as in Section 2.1.2. When $\epsilon = 0$ we do get the logarithm but multiplied by the additional exterior decay $\langle (t-r)^+ \rangle^{-\gamma}$ as is seen in (2.7) with $|n(q, \omega)| \lesssim \langle q+ \rangle^{-1} (q_+)^{-\gamma}$.

2.3.2. Subtracting off a term picking up the mass in the wave equation. Since $r^{-1}$ is a fundamental solution of $\Delta$ one can also see directly that $\Box r^{-1} = 0$, for $r \neq 0$. The cutoff will introduce an error $\langle t+r \rangle^{-3}$ but only in the region $(t+1)/4 < r < 3(t+1)/4$. We have

$$ |\Box h_{\mu\nu}^0| \lesssim (t+r)^{-3}, \quad \text{when } (t+1)/4 < r < 3(t+1)/4, \quad \text{and } \Box h_{\mu\nu}^0 = 0, \quad \text{otherwise}. $$

(2.13)

As alluded to in Section 2.2.3 we want to subtract off $h_{\mu\nu}^0$ in order to pick up the leading behavior of $h_{\mu\nu}$ in the exterior region. Then $h_{\mu\nu}^0 = h_{\mu\nu} - h_{\mu\nu}^0$, will satisfy the same equation as $h_{\mu\nu}$, i.e. $\Box^g h_{\mu\nu}^0 = F_{\mu\nu} - \Box^g h_{\mu\nu}^0$, apart from an error which to leading order is of the form (2.13). The new error we introduced in this way as well as $F_{\mu\nu}$ have additional exterior decay as in (2.12) with any $\gamma \leq 1$. By the asymptotic flatness condition (1.3) data for $h_{\mu\nu}^1$ are decaying like $r^{-1-\gamma}$ for some $\gamma < 1$ so if we could replace $\Box^g$ with $\Box$ we could conclude from Section 2.3.1 that $h_{\mu\nu}^1$ will have additional exterior decay $\langle (t-r)^+ \rangle^{-\gamma}$. The above heuristic argument can be made into a proof in $L^2$ for $\Box^g$ by using energy inequalities with exterior weights.
2.3.3. Subtracting off a term picking up the mass in the wave coordinate condition. As pointed out above, \( M_r^{-1} \delta \mu^\nu \) has to be a solution to the wave coordinate condition (1.7) up to terms of order \( r^{-3} \). However since all terms in the left are order \( r^{-2} \) it has to be an exact solution of the left hand side. Multiplying with the cutoff function introduces an error as for the wave equation above

\[
|\partial_\mu (H_0^{\mu \nu} - \frac{1}{2} m^{\mu \nu} m_0^{\alpha \beta} H_0^{\alpha \beta})| \lesssim (t+r)^{-2}, \quad \text{when} \quad (t+1)/4 < r < 3(t+1)/4, \quad \text{and} \quad = 0, \quad \text{otherwise}.
\]

Subtracting this from (1.7) gives a similar equation for \( H_1^{\alpha \beta} = H^{\alpha \beta} - H_0^{\alpha \beta} \). This equation can then be integrated to show that the component (2.9) of \( H_1 \), that determines the bending of the light cones and the main component in the commutators, decays faster. The proof of this will however require that we first show estimates for all components with the additional exterior decay mentioned above.

2.4. Coordinates adapted to the outgoing characteristic surfaces of Schwarzschild. In order to unravel the effect of the quasilinear terms one can change to characteristic coordinates as in [10], but this is not explicit and loses regularity. Instead we use the asymptotic behavior of the metric to determine the characteristic surfaces asymptotically and use this to construct coordinates. Due to the wave coordinate condition (1.2) the outgoing light cones of a solution of the metric to determine the characteristic surfaces asymptotically and use this to construct coordinates. As pointed out in Section 2.3.2, the outgoing light cones of a solution with asymptotically flat data (1.3) approach those of the Schwarzschild metric with the same mass, which are described by the Regge-Wheeler coordinates.

2.4.1. The outgoing characteristic surfaces. The outgoing light cones or characteristic surfaces are level sets of the solution of the eikonal equation

\[
g^{\alpha \beta} \partial_\alpha u \partial_\beta u = 0. \quad (2.14)
\]

For the Schwarzschild metric there is a solution \( r^* = r + M \ln r \) and due to the wave coordinate condition there is a solution \( u \) of (2.14) such \( u \sim r^* \), as \( r > t/2 \to \infty \), see Lindblad [34]. In fact \( r^* \) is an approximate solution of (2.14) with \( g^{\alpha \beta} \) replaced by \( m_0^{\alpha \beta} = m^{\alpha \beta} + H_0^{\alpha \beta} \) up to terms of order \( (t+r)^{-2} \):

\[
m_0^{\alpha \beta} \partial_\alpha u^* \partial_\beta u^* = -(1 + M/r) + (1 - M/r)(dr/\rho r)^2 = O(M^2 r^{-2}) \quad \text{when} \quad r > t/2.
\]

2.4.2. Asymptotic Schwarzschild coordinates. We therefore make the change of variables

\[
\tilde{t} = t, \quad \tilde{x} = r^* x/r, \quad \text{where} \quad r^* = r + \tilde{\chi} \left( \frac{t}{r} \right) M \ln r,
\]

and \( \tilde{\chi} \) is as in Section 2.3.2. Let

\[
\tilde{g}^{ab} = A_a^a A_b^b g^{\alpha \beta}, \quad \text{where} \quad A_a^a = \frac{\partial \tilde{t}^a}{\partial x^a}, \quad A_a^b = \frac{\partial x^a}{\partial \tilde{x}^b}, \quad \tilde{\partial}_a = A_a^b \partial_b,
\]

where \( x^0 = t, \tilde{x}^0 = \tilde{t} \). Then if \( \tilde{m}^{ab} \) is the Minkowski metric in the \( \tilde{x} \) coordinates,

\[
\tilde{m}_0^{\alpha \beta} \partial_\alpha u^* \partial_\beta u^* = m_0^{\alpha \beta} \partial_\alpha u^* \partial_\beta u^*, \quad \text{and} \quad \tilde{m}^{ab} \partial_\alpha u^* \partial_\beta u^* = 0.
\]

We would like to deduce that as far as components determining the characteristic surfaces

\[
\tilde{m}_0^{ab} \sim (1 + M r^{-1}) \tilde{m}^{ab}.
\]

Since \( \tilde{g}^{ab} = \tilde{m}_0^{ab} + \tilde{H}^{ab} \) and we expect the critical components of \( H_1 \) to decay faster, we expect the reduced wave operator in the \((t, \tilde{x})\) coordinates:

\[
\Box = \Box \tilde{g} = \tilde{g}^{ab} \tilde{\partial}_a \tilde{\partial}_b,
\]

to asymptotically approach the constant coefficient Minkowski wave operator in these coordinates,

\[
\Box^* = \Box \tilde{m} = \tilde{m}^{ab} \tilde{\partial}_a \tilde{\partial}_b.
\]

One heuristic motivation is to look at the linearized Einstein’s Equations, as in Wald [13] Chapter 7.5. The \( h_{\mu \nu} \) terms are generated by a metric perturbation term like \( \frac{1}{r} \delta \), and the perturbation for a gauge transformation of the linearized system is generated by subtracting off the symmetric
part of $2\nabla_\alpha V_\beta$ for an arbitrary vector $V$ (where $\nabla$ is the Levi-Civita connection associated with the background (Minkowski) metric). Setting $V = \ln r \partial_r$ gives a perturbation with a nice null structure, specifically away from $r = 0$ we have
\[
\frac{1}{r} \delta_{\alpha\beta} - \partial_\alpha V_\beta - \partial_\beta V_\alpha = -\frac{1}{r} m_{\alpha\beta} - \frac{2\ln r - 2}{r}(\delta_{\alpha\beta} - \omega_\alpha \omega_\beta).
\]
In this gauge, $t - |x|, t + |x|$ are again optical functions.

2.4.3. Covariant formulation of Einstein’s equations. Let $\nabla_a$ be covariant differentiation with respect to the metric $\tilde{m}_{ab} = m_{\alpha\beta} A^\alpha A^\beta$ with Christoffel symbols $\tilde{\Gamma}_{ab}^c = \tilde{m}^{cd}(\partial_a \tilde{m}_{bd} + \partial_b \tilde{m}_{ad} - \partial_d \tilde{m}_{ab})/2 = O(M r^{-2} \ln r)$. Then $\nabla_a h$ is equal to $\tilde{\partial}_a h$ plus a correction of the form $\tilde{\Gamma} \cdot h$. Moreover Einstein’s equations can be written
\[
\tilde{g}^{ab} \nabla_a \nabla_b \tilde{h}_{cd} = \tilde{F}_{cd}(\tilde{g}) [\nabla \tilde{h}, \nabla \tilde{h}] + \tilde{T}_{cd},
\]
and the wave coordinate condition \((17)\) become
\[
\nabla_a (\tilde{H}^{ac} - \frac{1}{2} \tilde{m}^{ac} \tilde{m}_{bd} \tilde{H}^{bd}) = \tilde{W}^c(\tilde{g}) [\tilde{H}, \nabla \tilde{H}].
\]

2.4.4. Improved commutators in the new coordinates. If we use the vector fields $\tilde{Z}$ that commute with the Minkowski wave operator $\Box \tilde{m} = \tilde{m}^{ab} \partial_a \partial_b$, or satisfy $[\tilde{Z}, \Box \tilde{m}] = c_{\tilde{Z}} \partial^a \tilde{m}$ for some constant $c_{\tilde{Z}}$, then the commutator of $\tilde{Z}^I$ with the reduced wave operator $\Box \tilde{g} = \tilde{g}^{ab} \partial_a \partial_b$ is roughly a product of vector fields applied to this wave operator plus terms of the form
\[
(\tilde{Z}^J \tilde{g}^{ab}) \partial_b \tilde{\partial}_a \tilde{Z}^K h_{\mu \nu}, \quad \text{for } |J| + |K| \leq |I|, \quad |J| \geq 1.
\]
However in these coordinates
\[
\tilde{g}^{ab} = \tilde{m}^{ab} + \tilde{H}_I^{ab},
\]
Here the critical components of $\tilde{Z} \tilde{m}_I^{ab}, \tilde{Z} \tilde{H}_I^{ab}$ are under control as mentioned in Sections \ref{2.4.2} \ref{2.3.3}.

2.5. The energy momentum tensor of the charged scalar field.

2.5.1. How the metric enters into the energy momentum tensor. The energy momentum tensor in its simplest form can be written as
\[
T_{\alpha\beta}[F, \phi] = T^1_{\alpha\beta}(g)[F, F] + T^2_{\alpha\beta}(g)[D \phi, D \phi],
\]
where $T^1, T^2$ are quadratic in $F$ and $D \phi$ respectively. We wish to bound $L^\infty$ norms for low derivatives and $L^1$ and $L^2$ norms for higher derivatives. We can write
\[
T \sim (1 + O(|H| + |h|))(|F|^2 + |D \phi|^2)
\]
with derivatives of $T$ satisfying similar bounds with derivatives of $H, h, F, \phi$. We are most interested with quadratic terms, which appear in the Minkowski case, as well as cubic terms where a large number of derivatives fall on $H$, and for which a slightly different analysis is required. Other cubic terms can be handled similarly to the quadratic terms, with nicer decay, and higher order terms in general behave similarly. For the quadratic terms, the analysis closely mirrors that for Minkowski. On lower derivatives we have, for $T \in \mathcal{T}, U \in \mathcal{N}$, and $s > 1/2$
\[
|T[F, \phi](t, \cdot)| \lesssim \varepsilon (t + r^s)^{-2} (t - r^s)^{-2} ((t - r^s)_+)^{1-2s},
\]
\[
|T[F, \phi](t, \cdot)_{TU}| \lesssim \varepsilon (t + r^s)^{-2s} (t - r^s)^{-2s} ((t - r^s)_+)^{1-2s}.
\]
Similar bounds hold for $\hat{T}$, which are consistent with \((1.15)\) and \((1.16)\).

The metric perturbation induces additional considerations in the $L^2$ estimates. We have to consider the case where almost all derivatives fall on the metric $g$, for which we must instead use weighted energy estimates on the metric. The method of this is detailed in \cite{23}.
2.5.2. Subtracting off the charge contribution in the exterior. In the analysis of the Maxwell-Klein Gordon system, one issue that arises is the fact that, even for compactly supported initial data \( \Phi \), \( \mathbf{F} \), like \( \partial \theta \), will generally decay like \( r^{-2} \), except in the charge free case. This limits the weights we can use when attempting an energy estimate. We resolve this issue by subtracting off a fixed solution of Maxwell’s equations, \( \mathbf{F}^0 \), which picks up the asymptotic decay up to terms decaying like \( f(|x|^{-2-\gamma}) \). \( \mathbf{F}^0 \) satisfies Maxwell’s equations for a current \( J^0 \) which in the Minkowski spacetime is compactly supported in \( u \).

**Remark 2.1.** In the Minkowski space setting, we are subtracting off an exterior derivative of

\[
\overline{A} \sim \frac{q [\mathbf{F}]}{4\pi r} dt.
\]

As with \( h^0 \), these terms have additional decay along the light cone, but worse total decay at spatial infinity, see Section 2.2.

3. The geometric structure of Einstein’s equations in wave coordinates

3.1. The geometric structure. Generic wave equations with quadratic nonlinearities do not in general have global solutions for small data, but some extra cancellation such as the null condition or weak null condition is needed. In Section 2.1.1 we explained the need for the null condition, and what extra cancellation it achieves. For Einstein’s equations the weak null condition can only be seen in a null frame, which we introduce in Section 3.1.1. The geometric structure of Einstein’s equations in wave coordinates in a null frame really enters in three different places. The first cancellation originates from the wave coordinate condition in Section 3.1.2 and is then used to control the reduced wave operator in Section 3.1.4. The fact that the reduced wave operator by itself is under control is then used together with the null structure of the reduced system with inhomogeneous terms in Section 3.1.3.

3.1.1. The null frame, tangential derivatives and killing vector fields. By Huygen’s principle the solution of the constant coefficient wave equation emanating from an initial source at the origin propagates along the outgoing light cone \( t = r = |x| \). It is therefore natural to introduce a null frame \( N \) of vectors tangential to the outgoing light cones plus a vector perpendicular to the cone:

\[
L = \partial_t - \partial_r, \quad L = \partial t + \partial_r, \quad S_1, S_2 \in S^2, \quad \langle S_i, S_j \rangle = \delta_{ij},
\]

It is well known that, for solutions of wave equations, derivatives tangential to the outgoing light cones \( \mathcal{J} \in \mathcal{T} = \{L, S_1, S_2\} \) decay faster. In fact, derivatives of solutions to the homogeneous wave equation are also solutions since derivatives commute with the wave operator, and therefore decay as much. Moreover, for the generators of the Lorentz transformations and the scaling

\[
x^i \partial_{x^i} - x^i \partial_{x^i}, \quad x^i \partial_t + t \partial_{x^i}, \quad t \partial_t + x^i \partial_{x^i},
\]

the commutator with the wave operator is either 0 or a multiple of the wave operator. In any case, if \( \Box \phi = 0 \), then \( \Box Z \phi = 0 \) if \( Z \) is any of the vector fields (3.1), so \( Z \phi \) decays like a solution of the wave equation. Since the vector fields span the tangent space of the outgoing light cones

\[
|\overline{\partial} \phi| \lesssim (t+|x|)^{-1} \sum_{\mathbf{i}} |Z \phi|, \quad \text{and} \quad |\partial \phi| \lesssim (t-|x|)^{-1} \sum_{\mathbf{i}} |Z \phi|.
\]

Therefore, tangential derivatives decay better and neglecting tangential derivatives \( \overline{\partial} \phi \) of \( \phi \):

\[
|\partial_\mu \phi - L_\mu \partial_\phi| \lesssim |\overline{\partial} \phi|, \quad \text{where} \quad \partial_\phi = (\partial_r - \partial_t)/2, \quad L_\mu = m_{\mu \nu} L^\nu.
\]

In fact we have

\[
|\partial \phi|^2 = |\partial_1 \phi|^2 + |\partial_2 \phi|^2 + |\partial_3 \phi|^2 = |S_1 \phi|^2 + |S_2 \phi|^2, \quad \text{where} \quad \partial \phi = \partial \phi - \omega \partial_r, \quad \omega = x/r.
\]
3.1.2. The geometric structure of the wave coordinate condition. The wave coordinate condition can be written \( \partial_\alpha g^{\alpha \beta} - \frac{1}{2}g^{\alpha \beta}g_{\beta \gamma} \partial_\alpha g^{\beta \gamma} = 0 \) from which it follows that \( H^{\alpha \beta} = g^{\alpha \beta} - m^{\alpha \beta} \) satisfy
\[
\partial_\mu (H^{\mu \nu} - \frac{1}{2}m^{\mu \nu} m_{\alpha \beta} H^{\alpha \beta}) = W^{\nu}(g)[H, \partial H].
\] (3.4)
Expressing the divergence above in a null frame;
\[
\partial_\mu \tilde{H}^{\mu \nu} = L_\mu \partial_q \tilde{H}^{\mu \nu} - L_\mu \partial_s \tilde{H}^{\mu \nu} + S_1 \partial_s \tilde{H}^{\mu \nu} + S_2 \partial_s \tilde{H}^{\mu \nu},
\]
where \( \partial_q = (\partial_r - \partial_t)/2 \), \( \partial_s = (\partial_r + \partial_t)/2 \), and contracting with \( T_\mu \) in \( T \) respectively \( L_\nu \) it follows that
\[
|\partial_q H|_{LT} + |\partial_q \partial_k H| \lesssim |\overline{\partial} H| + |W|, \quad \text{where} \quad |W| \lesssim |H| |\partial H|.
\] (3.5)
Here \( H_{UV} = H^{\alpha \beta} U_\alpha V_\beta \), where \( U_\alpha = m_{\alpha \beta} U^\beta \) and
\[
|H|_{LT} = |H_{LL}| + |H_{LS_1}| + |H_{LS_2}| \quad \text{and} \quad \partial_k H = \delta^{AB} H_{AB}, \quad A, B \in S = \{S_1, S_2\}.
\]
3.1.3. The geometric null structure of the reduced wave operator. By [3.3] we have
\[
|H^{\alpha \beta} \partial_\alpha \partial_\beta \phi - H_{LL} \partial_\alpha^2 \phi| \lesssim |H| |\overline{\partial} \partial \phi|,
\] (3.6)
and by (3.2) the tangential derivatives in the right are better behaved. Here \( H_{LL} \) is controlled by the wave coordinate condition (7.12). In fact a more detailed analysis, see Section 4.1.1, shows that at null infinity \( H_{LL} \sim -2M/r \), consistent with (2.9). In Section 5 we will change coordinates to asymptotically remove \( -2M/r \) from \( H_{LL} \) in (3.6) so that the quasilinear terms will be lower order, while it will only introduce lower order corrections to the wave coordinate condition and the inhomogeneous terms.

3.1.4. The geometric structure of the inhomogeneous terms. Recall that the inhomogeneous term in Einstein’s vacuum equations has the form
\[
F_{\mu \nu}(g)[\partial h, \partial h] = P(g)[\partial_\mu h, \partial_\nu h] + Q_{\mu \nu}(g)[\partial h, \partial h],
\]
where \( Q \) is a combination of classical null forms and \( P \), given by (1.6), has a weak null structure that we will now describe. First, since \( Q_{\mu \nu} = Q_{\mu \nu}(m) \) satisfy the classical null condition
\[
|Q_{\mu \nu}(\partial h, \partial k)| \lesssim |\overline{\partial} h| |\partial k| + |\partial h| |\overline{\partial} k|.
\]
The main term \( P = P(m) \) can be further analyzed as follows. First we note that by (3.3)
\[
|P(\partial_\mu h, \partial_\nu k) - L_\mu L_\nu P(\partial_q h, \partial_q k)| \lesssim |\overline{\partial} h| |\partial k| + |\partial h| |\overline{\partial} k|.
\]
Expressing \( P(h, k) = P_N(h, k) \) in a null frame we have
\[
P_N(h, k) = -(h_{LL} k_{LL} + h_{LL} k_{LL})/8 - \delta^{CD} \delta^{C'D'} (2h_{CC'} k_{DD'} - h_{CD} k_{C'D'})/4 + \delta^{CD} (2h_{CL} k_{DL} + 2h_{CL} k_{DL} - h_{CD} k_{LL})/4.
\]
It follows that
\[
|P_N(h, k) - P_S(h, k)| \lesssim (|h|_{LT} + |\overline{\partial} h|)|k| + |h|(|k|_{LT} + |\overline{\partial} k|).
\] (3.7)
where
\[
P_S(D, E) = -\tilde{D}_{AB} \tilde{E}^{AB}/2, \quad A, B \in S, \quad \text{where} \quad \tilde{D}_{AB} = D_{AB} - \delta_{AB} \partial_\mu \partial_\nu /2.
\]
Hence
\[
|P(\partial_\mu h, \partial_\nu k) - L_\mu L_\nu P_S(\partial_q h, \partial_q k)| \lesssim (|\overline{\partial} h| + |\partial_\mu h| |\partial_\nu k| + |\partial_\mu h| |\overline{\partial} h| + |\partial h| ([|\overline{\partial} k| + |\partial_\mu k| |\partial_\nu h| + |\partial_\mu k| |\overline{\partial} k|)).
\]
Also using the wave coordinate condition (7.12) and that fact the \( H = -h + O(h^2) \) we get
\[
|P(\partial_\mu h, \partial_\nu h) - L_\mu L_\nu P_S(\partial_q h, \partial_q h)| \lesssim (\overline{\partial} h + |h| |\overline{\partial} h|) |\overline{\partial} h|.
\]
With respect to the null frame, the Einstein equations become
\[
(\Box g h)_{TU} \sim 0, \quad T \in T, U \in N \quad (\Box g h)_{LL} \sim 4P_S(\partial_q h, \partial_q h),
\]
since $T^\mu L_\mu = 0$ for $T \in \mathcal{T}$. Here, $P_S(\partial_q h, q_li)$ only depends on tangential components, for which, by the first equation, we have better control. Hence in a null frame as far as semilinear terms Einstein’s equations look like

$$\Box \phi = 0, \quad \Box \psi = (\partial_t \phi)^2.$$  

3.2. Lie derivatives and Commutators. In order to get estimates for higher derivatives we need to commute the system with vector fields $Z$. However, if one instead commutes with Lie derivatives along the vector fields $Z$ it turns that the geometric structure is preserved also for the lower order terms. Note that for a function $\phi$

$$\Box^g Z \phi = Z(\Box^g \phi) + 2g^{\alpha \beta} \partial_\alpha Z^\alpha \partial_\beta \phi \circ \Box^g (g^{\alpha \beta} \partial_\alpha \partial_\beta \phi) = Z(\Box^g \phi) - (L_Z g^{\alpha \beta}) \partial_\alpha \partial_\beta \phi,$$

where the fact that $\partial_\mu \partial_\nu Z^\lambda = 0$ for each $Z$ and $\mu, \nu, \lambda = 0, 1, 2, 3$ has been used. Here the Lie derivative applied to a $(r, s)$ tensor $K$ is defined by

$$L_Z K^{\alpha_1, \cdots, \alpha_r}_{\beta_1, \cdots, \beta_s} = Z K^{\alpha_1, \cdots, \alpha_r}_{\beta_1, \cdots, \beta_s} - \partial_\gamma Z^\alpha K^{\alpha \gamma, \cdots, \alpha_r}_{\beta_1, \cdots, \beta_s} - \cdots - \partial_\lambda Z^{\alpha_\lambda} K^{\alpha_1, \cdots, \alpha_r - \alpha_\lambda}_{\beta_1, \cdots, \beta_s} + \partial_\beta Z^\gamma K^{\alpha_1, \cdots, \alpha_r}_{\beta_\gamma, \beta_1, \cdots, \beta_s} + \cdots + \partial_\beta Z^\gamma K^{\alpha_1, \cdots, \alpha_r}_{\beta_1, \cdots, \beta_\gamma}.$$ 

More generally, for any vector field $Z = Z^\alpha \partial_\alpha$ with linear coefficients $Z^\alpha = c^{\alpha \beta} Z^\beta$ we have

$$L_Z \partial_\mu \cdots \partial_k K^{\alpha_1, \cdots, \alpha_r}_{\beta_1, \cdots, \beta_s} = \partial_\mu \cdots \partial_k L_Z K^{\alpha_1, \cdots, \alpha_r}_{\beta_1, \cdots, \beta_s}.$$  

(3.8)

The procedure simplifies further by commuting with a modified Lie derivative $\tilde{L}$, defined by

$$\tilde{L}_Z K^{\alpha_1, \cdots, \alpha_r}_{\beta_1, \cdots, \beta_s} = L_Z K^{\alpha_1, \cdots, \alpha_r}_{\beta_1, \cdots, \beta_s} + \frac{1}{4} (\partial_\gamma Z^\gamma) K^{\alpha_1, \cdots, \alpha_r}_{\beta_1, \cdots, \beta_s}.$$  

(3.9)

3.2.1. Commutators with the wave coordinate condition. The Lie derivative commutes with exterior differentiation which for any of the vector fields $Z$ leads to

$$\partial_\mu \tilde{L}_Z \hat{H}^{\mu \nu} = (\tilde{L}_Z + \frac{\partial_\mu Z^\gamma}{2}) \partial_\mu \hat{H}^{\mu \nu}.$$  

This applied to $\hat{H}_L^L$ in turn leads to higher order approximate wave coordinate conditions so the term $\tilde{L}_Z H_L^L$ is then controlled by using (3.12) and integrating in the $q = r - t$ direction to control $H_L^L$ itself.

3.2.2. Commutators with the reduced curved wave operator. The modified Lie derivative satisfies

$$\tilde{L}_Z m^{\alpha \beta} = 0,$$  

(3.10)

for each of the vector fields $Z$ and moreover

$$\Box^g \tilde{L}_Z \phi_{\mu \nu} = L_Z (\Box^g \phi_{\mu \nu}) - (\tilde{L}_Z g^{\alpha \beta}) \partial_\alpha \partial_\beta \phi_{\mu \nu},$$  

(3.11)

where in view of (3.10) $\tilde{L}_Z g^{\alpha \beta} = \tilde{L}_Z H^{\alpha \beta}$. By (3.6) the commutator can be controlled by the term $(\tilde{L}_Z H)_L L \partial_\gamma \phi_{\mu \nu}$, plus terms involving one tangential derivative

$$| (\tilde{L}_Z H^{\alpha \beta}) \partial_\alpha \partial_\beta \phi_{\mu \nu} | \lesssim |(\tilde{L}_Z H)_L L | | \partial \phi_{\mu \nu}| + |\tilde{L}_Z H| | \partial \phi_{\mu \nu} |.$$  

3.2.3. Commutators with the inhomogeneous terms. When applying the commutation formula (3.11) to the reduced Einstein equations (1.6) we have to estimate Lie derivatives of nonlinear terms: $\mathcal{L}_Z^f (F_{\mu \nu} (g) [\partial h, \partial k])$. Let $h_{\alpha \beta}$ and $k_{\alpha \beta}$ be $(0, 2)$ tensors and let $S_{\mu \nu} (g) [\partial h, \partial k]$ be a $(0, 2)$ tensor which is a quadratic form in the $(0, 3)$ tensors $\partial h$ and $\partial k$ with two contractions with the metric $g$ (in particular $P(g) [\partial h, \partial k]$ or $Q_{\mu \nu} (g) [\partial h, \partial k]$). Then $S_{\mu \nu} (g) [\partial h, \partial k] = S_{\mu \nu} (G, G) [\partial h, \partial k]$, i.e. it is bilinear in the inverse of the metric $G^{\alpha \beta} = g^{\alpha \beta}$. We have

$$\mathcal{L}_Z (S_{\mu \nu} (g) [\partial h, \partial k]) = S_{\mu \nu} (g) [\partial \mathcal{L}_Z h, k] + S_{\mu \nu} (g) [\partial h, \partial \mathcal{L}_Z k] + S_{\mu \nu} (\mathcal{L}_Z G, G) [\partial h, \partial k] + S_{\mu \nu} (G, \mathcal{L}_Z G) [\partial h, \partial k],$$  

(3.12)

so the Lie derivative preserves the desirable structure of the nonlinear terms $P(g) [\partial h, \partial k]$ and $Q_{\mu \nu} (g) [\partial h, \partial k]$. We remark that the last two terms in (3.12) are cubic and therefore much easier to control, see Section 2.2.1.
4. Subtracting off terms that picks up the mass and charge contributions

In order to bound the solution \( h \) to the wave equation in the weighted energy spaces we will be using, we need to subtract off an approximate solution to the homogenous wave equation \( h^0 \) which picks up the contribution from the initial mass in (3.3). Similarly one can subtract off the charge from the electromagnetic field. These will be explained in further detail when we get in to the specific norms that we will be using, e.g. in Section 12.1.2.

4.1. Subtracting off the mass.

4.1.1. Subtracting off a term that picks up the mass contribution from the wave coordinate condition. Let

\[
m_0^{\alpha\beta} = m^{\alpha\beta} + H_0^{\alpha\beta}, \quad \text{where} \quad H_0^{\alpha\beta} = -Mr^{-1}\tilde{\chi}(\frac{r}{1+t})\delta^{\alpha\beta},
\]

and \( \tilde{\chi}(s) = 1 \), when \( s > 3/4 \) and \( \tilde{\chi}(s) = 0 \) when \( s < 1/4 \). A calculation shows that the approximate wave coordinate condition (3.4) is approximately satisfied by \( H_0 \):

\[
W_{\text{mass}}^{\beta,0} = -\partial_\alpha(H_0^{\alpha\beta} - \frac{1}{2}m^{\alpha\beta}m_{\mu\nu}H_0^{\mu\nu}) = -M\delta^{\beta\alpha}\chi_0'(\frac{r}{1+t})r^{-2},
\]

where \( \chi_0'(s,) \) stands for functions supported when \(|s-1/2| < 1/4\). Hence by (3.4) \( H_1^{\alpha\beta} = H_0^{\alpha\beta} \) satisfy an approximate wave coordinate condition

\[
\partial_\alpha(H_1^{\alpha\beta} - \frac{1}{2}m^{\alpha\beta}m_{\mu\nu}H_1^{\mu\nu}) = W_0^\beta(g)[H,\partial H] + W_{\text{mass}}^{\beta,0}.
\]

Once we subtracted off \( H_0 \), the critical components of \( H_1 \) will have better decay as \( r \to \infty \) and by integrating the equation in the \( t-r \) direction using the null decomposition, everywhere in the exterior of the light cone. We can further write

\[
W_0^\beta(g)[H,\partial H] = W_0^\beta(g)[H_1,\partial H_1] + W_{\text{mass}}^{1,0} + W_{\text{mass}}^{2,0}[H_1] + W_{\text{mass}}^{3,0}[\partial H_1],
\]

where

\[
W_{\text{mass}}^{1,0} = W_0^\beta(g)[H_0,\partial H_0] = \chi^\beta_0(\frac{r}{1+t},\omega,g)Mr^{-3},
\]

\[
W_{\text{mass}}^{2,0}[H_1] = W_0^\beta(g)[H_1,\partial H_0] = \chi^\beta_0(\frac{r}{1+t},\omega,g)Mr^{-2}H_1^{\mu\nu},
\]

\[
W_{\text{mass}}^{3,0}[\partial H_1] = W_0^\beta(g)[H_0,\partial H_1] = \chi^\beta_0(\frac{r}{1+t},\omega,g)Mr^{-1}\partial_\alpha H_1^{\mu\nu},
\]

since

\[
\partial^\alpha H_0^{\mu\nu} = \chi^{\alpha\mu\nu}(\frac{r}{1+t},\omega)Mr^{-1-|\alpha|}.
\]

Here \( \chi^{\alpha\mu\nu}(s,\omega) \) and \( \chi^\beta_0(s,\omega,g) \) are bounded functions that vanish for \( s \leq 1/4 \).

4.1.2. The asymptotically Schwarzschild wave operator. We have

\[
\Box g = \Box m_0 + H_1^{\alpha\beta}\partial_\alpha\partial_\beta,
\]

where we expect the critical coefficient in front of the \( \partial_\alpha^2 \) term, i.e. \( H_1^{\alpha\beta}L_\alpha L_\beta \) to be small.

Expressing \( \Box m_0 = m_0^{\alpha\beta}\partial_\alpha\partial_\beta \) in spherical coordinates we get

\[
\Box m_0 \phi = (-\partial_t^2 + \Delta_x - \frac{M}{r^2}\tilde{\chi}(\frac{r}{1+t})(\partial_t^2 + \Delta_x)) \phi = \frac{1}{L}(-\partial_t^2 + \partial_r^2 - \frac{M}{r^2}\tilde{\chi}(\frac{r}{1+t})(\partial_t^2 + \partial_r^2))(r\phi) + (1 - \frac{M}{r^2}\tilde{\chi}(\frac{r}{1+t}))L_\phi.
\]
4.1.3. Subtracting off a term that picks up the mass contribution from the wave equation. Let
\[ h^0_{\alpha\beta} = Mr^{-1}\chi_{\mu}r^{\mu}\delta_{\alpha\beta}, \quad (4.5) \]
where \( \chi_{\mu} \) is as in (4.1). We have
\[ \partial^\mu h^0_{\mu\nu} = \chi^{\alpha}_{\mu}\left(\frac{r}{r+1}\right)^{-1}M r^{-1-|\alpha|}, \]
where \( \chi^{\alpha}_{\mu}(s,\omega) \) stands for bounded functions that vanish when \( s < 1/4 \). Using (4.4) we see that
\[ E^\text{mass}_{\mu\nu,0} = \Box h^0_{\mu\nu} = M\chi^\prime_{\mu\nu}\left(\frac{r}{r+1}\right)^{M/r}r^{-3}, \quad (4.7) \]
where \( \chi^\prime_{\mu\nu}(s,\cdot) \) stands for functions supported when \( |s-1/2| < 1/4 \), in the far interior away from the light cone. Moreover
\[ E^\text{mass}_{\mu\nu,1}[H_1] = H_1^{\alpha\beta}\partial_{\alpha}\partial_{\beta}h^0_{\mu\nu} = \chi^{\mu\nu\alpha\beta}_{\mu\nu}(\frac{r}{r+1},\omega)M r^{-3}H_1^{\alpha\beta}. \quad (4.8) \]
Hence
\[ E^\text{mass}_{\mu\nu} = \Box g h^0_{\mu\nu} = \Box h^0_{\mu\nu} + H_1^{\alpha\beta}\partial_{\alpha}\partial_{\beta}h^0_{\mu\nu} = E^\text{mass}_{\mu\nu,0} + E^\text{mass}_{\mu\nu,1}[H_1]. \quad (4.9) \]
The reason we need to subtract off \( h^0 \) is that \( h \) itself is not in the weighted energy space we need to use. The terms (4.7) decay enough to cover extra exterior weight in \( L^2 \), since they vanish in the exterior and along the light cone. The term (4.8) is also easy to control since it is linear in \( h \) and the weight can be absorbed in the norm of \( h^1 \). Since as we will see, in the weighted norms that we will be using
\[ \| \langle r^* - t \rangle^{-1}h^1 \| \lesssim \| \partial h^1 \|, \]
the term (4.9) still has an additional decaying factor of \( t^{-2} \).

4.1.4. Subtracting off a term that picks up the mass contribution from the inhomogeneous term. Let
\[ F^\text{mass}_{\mu\nu} = F^\text{mass}_{\mu\nu}(g)[\partial h, \partial h] - F^\text{mass}_{\mu\nu}(g)[\partial h^1, \partial h^1] = F^\text{mass}_{\mu\nu,0} + F^\text{mass}_{\mu\nu,1}[\partial h], \quad (4.10) \]
where
\[ F^\text{mass}_{\mu\nu,0} = F^\text{mass}_{\mu\nu}(g)[\partial h^0, \partial h^0] = \chi_{\mu\nu}\left(\frac{r}{r+1},\omega,\frac{M}{r},g\right)M^2 r^{-4}, \]
\[ F^\text{mass}_{\mu\nu,1}[\partial h] = 2F^\text{mass}_{\mu\nu}(g)[\partial h^0, \partial h^1] = M^2 r^{-2}\chi^{\alpha\beta\gamma}_{\mu\nu}\left(\frac{r}{r+1},\omega,\frac{M}{r},g\right)\partial_{\alpha}h^1_{\beta\gamma}. \quad (4.11) \]
Here the terms with linear and quadratic factors in \( \partial h^0 \) can be estimated without use of any special geometric structure only using the estimate (4.6). The reason we need to subtract off \( h^0 \) is that \( h \) itself is not in the weighted energy space we use. The terms (4.10) decay enough to cover the exterior weight. The term (4.11) that is linear in \( h^1 \) is even easy to control since the weight can be absorbed in the norm of \( h^1 \).

4.1.5. Subtracting off a term that picks up the mass contribution from Einstein’s equations and estimating the mass errors in exterior weighted norms. Modulo a mass error \( R^\text{mass}_{\mu\nu} = E^\text{mass}_{\mu\nu} + F^\text{mass}_{\mu\nu} \) we have
\[ \Box h^1_{\mu\nu} = F^\text{mass}_{\mu\nu}(g)[\partial h^1, \partial h^1] + R^\text{mass}_{\mu\nu} + \tilde{T}_{\mu\nu}. \quad (4.12) \]
We can write \( R^\text{mass}_{\mu\nu} = R^\text{mass}_{\mu\nu,0} + R^\text{mass}_{\mu\nu,1} \), where \( R^\text{mass}_{\mu\nu,0} = E^\text{mass}_{\mu\nu,0} + F^\text{mass}_{\mu\nu,0} \) and \( R^\text{mass}_{\mu\nu,1} = E^\text{mass}_{\mu\nu,1}[H_1] + F^\text{mass}_{\mu\nu,1}[\partial h] \).

We will be using a weighted \( L^p \) norms, for \( p = 2, \infty, 1 \),
\[ \| \phi(t, \cdot)w_{p,\gamma} \|_{L^p}, \quad \text{where} \quad w_{p,\gamma} = \langle (r^*-t) \rangle^{1-1/p+\gamma}, \quad 0 < \gamma < 1. \]
Using the weighted energy inequality (if \( p = 2 \)) we need to show that
\[ \int_0^\infty (1 + t)^{-2/p} \| R^\text{mass}_{\mu\nu,0}(t, \cdot)w_{p,\gamma} \|_{L^p} dt < CM, \]
which follows from (4.9) and (4.10). For \( p = 1, \infty \) this is used for the decay estimates.
4.2. Subtracting off the charge. In the Minkowski metric, given a field $F$ solving \((1.17)\), $F$ has a charge $q$ defined by the integral

$$q[F] = \int_{\mathbb{R}^3} J(-J^0(0,x)) \, dx,$$

where the quantity on the right can be defined in terms of initial data for $\phi$. This is invariant in time, since the current $J$ is by design divergence free. By an application of the divergence theorem in space,

$$\lim_{r \to \infty} \int_{S^2} r^2 \omega F_{0i}(0,r\omega) = q[F].$$

Consequently, unless $q=0$, $F$ can not decay uniformly faster than $r^{-2}$, so it is not bounded in our desired energy norms. We instead subtract off a well-defined $F^0$ with the same asymptotic decay as $F$, such that we can establish a meaningful energy estimate on the difference $F - F^0$. In the Minkowski spacetime, Lindblad and Sterbenz \cite{38} use the fact that $F = d(r^{-1} \, dt)$ is a solution of Maxwell’s equations away from the origin (with vanishing current) to construct a field $F^0$ which models the point charge in the exterior and is 0 inside the light cone. The associated current of $F^0$ is supported close to the light cone and decays rapidly in time. Kauffman \cite{23} constructed an analogous current for asymptotically Schwarzschild metrics which instead decays rapidly away from the light cone.

We can split the portion of the energy momentum tensor coming from $F$ into four parts, recalling the remainder $F^1 = F - F^0$. Using the notation

$$T[F,G]_{\alpha\beta} = F_{\alpha\gamma}G_{\gamma\beta} - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}G^{\gamma\delta},$$

we can write

$$T[F,F] = T[F^0,F^0] + 2T[F^0,F^1] + T[F^1,F^1],$$

which has the consequent bounds

$$|T[F^0,F^0]| \lesssim |q|^2 \bar{\chi}(r^*-t)(r^*+t)^{-4},$$

$$|T[F^0,F^1]| \lesssim |q| \bar{\chi}(r^*-t)(r^*+t)^{-2}|F^1|,$$

$$|T[F^1,F^1]|_{\|u\|T} \lesssim |F^1||F^1|, + |t||F^1|^2,$$

$$|T[F^1,F^1]| \lesssim |F^1|^2.$$

Here, we use the notation

$$|F^1| = |F^1|_{LS^1} + |F^1|_{LS^2} + |F^1|_{LS^3} + |F^1|_{S^1S_2} + |F^1|_{L^L};$$

that is, $|F^1|$ does not include the components with the worst decay. We note for now that the component decomposition for $|T[F^1,F^1]|$ is necessary to establish the bounds \((1.15)\) and \((1.16)\).

When we apply Propostion \ref{pro:5.2} we must treat $F^0$ and $F^1$ separately, using energy bounds when $F^1$ appears, and integrating decay bounds for $F^0$. Importantly, $F^1$ determines the asymptotic behavior of the metric $g$, as we have worse decay along the light cone.

5. The structure of the metric in asymptotically Schwarzschild coordinates

In Section \ref{sec:5.1} we will give an outline of the change to generalized wave coordinates, which we will use to asymptotically remove the first order correction $-2M/r$ from $H_{LL}$ in \((3.6)\), so that the remainder will show better decay along the light cone. We will see that this only introduces lower order corrections to the analysis of the wave coordinate condition and the inhomogeneous terms. The remainder of this section will be dedicated to proving (mostly technical) estimates on quantities which arise from this coordinate change. In Section \ref{sec:6} we will use these results to bound lower order corrections in Einstein’s equations which arise from the coordinate change.
5.1. Generalized Wave Coordinates asymptotically adapted to the outgoing light cones.

5.1.1. The leading behavior of the metric at null infinity and changes of coordinates that straighten out the characteristic surfaces at null infinity. As previously pointed out in Section 2.4.2, the metric to leading order approaches the Schwarzschild metric with the same mass, in the sense that the distance between light cones remains bounded. In order to study the precise asymptotic behavior, Lindblad [34] straightened out the light cones by a change of coordinates

\[ \tilde{t} = t, \quad \tilde{x} = r^* x / r, \quad \text{where} \quad r^* = r + \tilde{\chi}(\frac{r}{\sqrt{r^2 + M^2}}) M \ln r, \]

where \( \tilde{\chi} \) is as in (4.1). Therefore,

\[ \tilde{x}^a = x^a + M \omega^a \tilde{\chi} \ln r. \]

If

\[ A^a = \partial x^a / \partial x^\alpha = \delta^a_\alpha + M r^{-1} \ln r \chi^a_{\alpha}(\frac{r}{\sqrt{r^2 + M^2}}, \omega) + M r^{-1} \tilde{\chi}^a_{\alpha}(\frac{r}{\sqrt{r^2 + M^2}}, \omega), \tag{5.1} \]

then in the new coordinates \( m_0 \) satisfies

\[ \tilde{m}^{ab} = A^a A^b m_0^a \sim \tilde{m}^{ab}, \]

where \( \tilde{m}^{ab} \) is the inverse of the Minkowski metric in the new coordinates,

\[ \tilde{m} = -d \tilde{t}^2 + \sum_i d \tilde{x}_i^2. \]

In the new coordinates

\[ A^a A^b \partial_\alpha \partial_\beta = A^a A^b \partial_\alpha \partial_\beta \tilde{c} = \tilde{\partial}_a \tilde{\partial}_b - \tilde{\Gamma}^c_{ab} \partial_c, \tag{5.2} \]

where

\[ \tilde{\Gamma}^c_{ab} = -A^a A^b \partial_\alpha A^c_{\beta} = -A^a A^b \partial_\alpha \partial_\beta \tilde{x}^c = M r^{-2} \ln r \chi^c_{ab}(\frac{r}{\sqrt{r^2 + M^2}}, \omega, \frac{M}{r^2}, \frac{M \ln r}{r}), \tag{5.3} \]

are the Christoffel symbols for the metric \( \tilde{m}_{ab} = m_{\alpha \beta} A^a A^b \) in the new coordinates:

\[ \tilde{\Gamma}^c_{ab} = \tilde{m}^{cd} (\tilde{\partial}_a \tilde{m}_{bd} + \tilde{\partial}_b \tilde{m}_{ad} - \tilde{\partial}_d \tilde{m}_{ab}) / 2. \tag{5.4} \]

Here \( A^a_\alpha \) is the inverse of \( A^a_\alpha \). In fact \( \tilde{x}(x) \) is a wave map into \(( \mathbb{R}^{1+3}, \tilde{m} )\).

5.1.2. Change of coordinates and the generalized wave coordinate condition. If \( \tilde{g}^{ab} = A^a A^b g^{\alpha \beta} \) and \( \tilde{\Gamma}^c_{ab} \) are the Christoffel symbols of \( \tilde{g} \) then since the geometric wave operator is invariant

\[ \tilde{g}^{ab} \tilde{\partial}_a \tilde{\partial}_b - \tilde{g}^{ab} \tilde{\Gamma}^c_{ab} \partial_c = \tilde{g} = \tilde{g}^{\alpha \beta} \partial_\alpha \partial_\beta - g^{\alpha \beta} \Gamma^\delta_{\alpha \beta} \partial_\delta. \]

This combined with (5.2) gives, with \( \tilde{\Gamma}_c_{ab} \) given by (5.1),

\[ \tilde{g}^{ab} \tilde{\Gamma}_c_{ab} = \tilde{g}^{ab} \tilde{\Gamma}_c_{ab} + g^{\alpha \beta} \Gamma^\delta_{\alpha \beta} A^c_\delta. \tag{5.5} \]

If \( g \) is harmonic the last term vanishes and (5.5) is the generalized wave coordinate condition. Moreover

\[ \tilde{g}^{ab} \tilde{\Gamma}_c_{ab} = g^{\alpha \beta} \partial_\alpha \partial_\beta (\tilde{x}^c - x^c) = m^{\alpha \beta} \partial_\alpha \partial_\beta (\tilde{x}^c - x^c) + H^\alpha_1 \partial_\alpha \partial_\beta (\tilde{x}^c - x^c) \]

Furthermore,

\[ \tilde{g}^{ab} = \tilde{m}^{ab} + \tilde{H}^{ab}_1. \tag{5.6} \]

We shall see in Section 5.3.2 that

\[ \tilde{m}^{ab} \sim (1 + \frac{M}{r}) \tilde{m}^{ab}, \tag{5.7} \]

where \( \tilde{m}^{ab} \) is the Minkowski metric. In fact we shall see already a heuristic argument in Section 5.1.3 or for the linearized equations at the end of Section 2.4.2.
5.1.3. The asymptotically Schwarzschild wave operator in the new coordinates. By (5.2) applied to $m_0$, and recalling (4.4),

\[ \square \tilde{m}_0 = \square m_0 - \tilde{m}_0 \rho^\alpha \chi_{ab} \tilde{\partial}_c. \]

Let $\Phi(t, r^*, \omega) = r\phi(t, r\omega)$, where $r^* = r + M \tilde{\chi}(\frac{r}{r+1}) \ln r$. For $\frac{r}{r+1} > \frac{3}{4}$ we have $\tilde{\chi}(\frac{r}{r+1}) = 1$ and

\[ r \square m_0 \phi - r(1 - \frac{M}{r}) r^{-2} \Delta_{\omega} \phi = \left( - \partial_r^2 + \frac{2}{r} - \frac{M}{r} \right) \Phi = - (1 + \frac{M}{r}) \Phi_t + (1 - \frac{M}{r}) \left( \frac{dr}{d r} \right)^2 \Phi_{r^* r^*} + d_{\omega}^2 \Phi_{r^*}. \]

Hence in the new coordinates

\[ r \square m_0 \phi = (1 - \frac{M}{r}) (- \Phi_t + \Phi_{r^* r^*}) + r(1 - \frac{M}{r}) r^{-2} \Delta_{\omega} \Phi - (\frac{M}{r})^2 ((1 + \frac{M}{r}) \Phi_{r r^*} + (1 - \frac{M}{r}) \Phi_{r^*}), \quad \text{when} \quad \frac{r}{r+1} > \frac{3}{4}. \]

Hence $\tilde{m}_0$ approximately becomes $(1 + \frac{M}{r})$ times the Minkowski wave operator in the new coordinates:

\[ \tilde{\square} \phi = \frac{1}{r^2} \left( - \partial_t^2 + \partial_{r^*}^2 \right) (r^* \phi) + \frac{1}{r^4} \Delta_{\omega} \phi. \]

5.2. Estimates on error terms arising from the change in coordinates.

5.2.1. The modified fields and commutation properties. We recall that by $\tilde{Z}$ and $\tilde{X}$ we mean commutator fields in

\[ \tilde{\partial}_a, \quad \tilde{\Omega}_{ij} = \tilde{x}^i \tilde{\partial}_j - \tilde{x}^j \tilde{\partial}_i, \quad \tilde{\Omega}_{i0} = \tilde{x}^i \tilde{\partial}_0 + t \tilde{\partial}_i, \quad \tilde{S} = t \tilde{\partial}_t + \tilde{x}^i \tilde{\partial}_i, \]

where $a$ ranges from 0 to 3 and $i, j$ range from 1 to 3. These fields $\tilde{Z}$ commute nicely with each other:

\[ [\tilde{Z}_1, \tilde{Z}_2] = \sum_{|I| = 1} c_{12I} \tilde{Z}_I, \quad [\tilde{Z}_1, \tilde{\partial}_a] = \sum_{b=0}^3 c_{1a}^b \tilde{\partial}_b, \quad [\tilde{\partial}_a, \tilde{\partial}_b] = 0, \]

where all $c_{12I}, c_{1a}^b$ are $-1, 0, 1$.  

5.2.2. The change of coordinates. We now show estimates on error terms arising from the change in coordinates. The expansion (5.11) follows directly from the definition

\[ \partial_a \tilde{x}^0 = \delta_a^0, \]

\[ \partial_0 \tilde{x}^i = \delta_0^i - Mr^{-1} \ln r \omega^i \left( \frac{r}{r+1} \right)^2 \tilde{\chi}', \]

\[ \partial_j \tilde{x}^i = \delta_j^i + Mr^{-1} \ln r \left( (\delta_j^i - \omega_j \omega^i) \tilde{\chi}' + \omega_j \omega^i \left( \frac{r}{r+1} \right) \tilde{\chi}' \right) + Mr^{-1} \omega_j \omega^i \tilde{\chi}', \]

where $\tilde{\chi}' = \frac{1}{r+1} \chi$ are supported when their arguments are in $[1/4, 3/4]$ and $[1/4, \infty)$ respectively. We now consider functions $\psi$ which satisfy the following condition: for every multiindex $\alpha$ or $I$, there exist constants $C_\alpha, C_I$ such that the bounds

\[ |\partial^\alpha \psi| \leq C_\alpha (\ln r)^m / \rho^{n + |\alpha|}, \quad |Z^I \psi| \leq C_I (\ln r)^m / \rho^n, \]

hold everywhere in some region $R \subset \{ (t, x) : t \geq 0, |x| / \frac{r}{r+1} \geq \frac{1}{4} \}$. The second bound follows from the first by expanding each field $Z$. Additionally, if $\phi, \psi$ satisfy (5.13) for pairs $m_1, n_2$ and $m_2, n_2$ respectively, then $\phi \psi$ satisfies (5.13) for $m_1 + m_2, n_1 + n_2$.

Proposition 5.1. Take constants $K, M_0, M, N \geq 0$. Let $\psi$ be a finite sum of the form

\[ \psi = \sum_{k \leq K, m \leq M_0} (\ln r)^m (\frac{r}{t+1})^k \chi_{kmN}(\omega, \frac{r}{t+1}) + \sum_{k \leq K, m \leq M, n \geq N+1} (\ln r)^m (\frac{r}{t+1})^k \chi_{kmn}(\omega, \frac{r}{t+1}), \]

where $\chi_{knm}$ are smooth functions supported when $\frac{r}{t+1} \geq \frac{1}{4}$ and $\partial_y \chi(\omega, y)$ is supported in the region $y \geq \frac{3}{4}$. Then, $\psi$ satisfies the bounds (5.13) for $m = M_0, n = N$. 

Lemma 5.5. Let $\Lambda$ be the Christoffel symbols of $\tilde{m}_{ab} = m_{\alpha\beta} A^\alpha_a A^\beta_b$. Then
\[
|\mathcal{L}_X^{\tilde{\Gamma}}| \lesssim \frac{M(\ln (1 + r) + 1)}{(1 + r + t)^2}
\] (5.14)
Moreover if $\tilde{m}^{ab} = A^\alpha_a A^\beta_b m^{\alpha\beta}$ is the inverse of the Minkowski metric, written in the new coordinates then
\[
|\mathcal{L}_X^{\tilde{\Gamma}}(\tilde{m}^{ab} - \tilde{m}^{ab})| \lesssim \frac{M(\ln (1 + r) + 1)}{1 + r + t}
\] (5.15)
Proof. First, we consider (5.14), for which it suffices to replace the Lie derivatives with standard derivatives in each component. We write
\[
\tilde{\Gamma}^{ab}_c = \frac{1}{2} \tilde{m}^{ad}(\partial_b \tilde{m}_{dc} + \partial_d \tilde{m}_{bd} - \partial_d \tilde{m}_{bc})
\]
\[
= \frac{1}{2}(A^\alpha_a A^\beta_b m^{\alpha\beta})(A^\delta_b \partial_\beta (A^\delta_d A^\gamma_m m_{\delta\gamma}) + A^\gamma_c \partial_\gamma (A^\delta_b A^\delta_d m_{\delta\gamma}) - A^\delta_b \partial_\delta (A^\delta_b A^\gamma_m m_{\delta\gamma})).
\]
The result follows from Corollary 5.4. To prove (5.15), we use the decomposition
\[ \tilde{\partial}_i (\tilde{m}^{ab} - \hat{m}^{ab}) = \tilde{\partial}_i (m^{ab}) (A_\alpha^a (A_\beta^b - \delta_\beta^b) + (A_\alpha^a - \delta_\alpha^a) \delta_\beta^b), \]
and again apply Corollary 5.4. \hfill \square

5.3. The null structure of the metric in asymptotically Schwarzschild coordinates.

5.3.1. The null frame in asymptotically Schwarzschild coordinates. We take the null frame with respect to \( \tilde{m} \),
\[ \tilde{L} = \partial_{\tau^*} + \partial_{\nu^*} \quad \tilde{\nu} = \partial_{\tau^*} - \partial_{\nu^*} \quad \tilde{S}_i = \frac{r}{r^*} S_i. \]
Let derivatives tangential to the outgoing light cones in these coordinates be denoted by \( \tilde{\partial} \in \mathcal{F} = \{ \tilde{L}, \tilde{S}_1, \tilde{S}_2 \}. \)

All the estimates in Section 3.1.1 hold with \( \partial \) replaced by \( \tilde{\partial}, \tilde{\partial}_q \) by \( \tilde{\partial}_q \), the frame \( \{ L, S_1, S_2 \} \) replaced by \( \{ \tilde{L}, \tilde{S}_1, \tilde{S}_2 \} \) and the vector fields (3.1) replaced by (5.10).

We now estimate the difference between the fields \( \tilde{Z}^I \) and \( \tilde{X}^I \) applied to an arbitrary function:

**Lemma 5.6.** We have
\[ |\partial^a \tilde{Z}^I (\tilde{\partial}_{\mu} - \partial_{\mu}) \phi| \lesssim \frac{M \ln |1 + t + r|}{1 + t + r} \sum_{|\mu| \leq |l|, |\beta| \leq |\alpha|} |\partial^\beta \tilde{Z}^I \phi|. \] (5.16)
Moreover,
\[ (1 + \frac{M \ln |1 + r|}{1 + q^*})^{-1} \lesssim \frac{1 + |q^*|}{1 + |q|} \lesssim \left(1 + \frac{M \ln |1 + r|}{1 + |q|}\right), \] (5.17)
\[ C^{-1} \left(1 + \frac{M \ln |1 + r|}{1 + |q^*|}\right)^{-k} \sum_{|l| \leq k} |\tilde{Z}^I \phi| \lesssim \sum_{|l| \leq k} |\tilde{X}^I \phi| \leq C \left(1 + \frac{M \ln |1 + r|}{1 + |q|}\right)^k \sum_{|l| \leq k} |\tilde{Z}^I \phi|. \] (5.18)

**Proof.** We first prove (5.16). We write the left hand side as \( \partial^a \tilde{Z}^I ((A_\mu^a - \delta_\mu^a) \partial_\beta) \), from which we get
\[ |\partial^a \tilde{Z}^I (\tilde{\partial}_{\mu} - \partial_{\mu}) \phi| \lesssim \sum_{|l_1| + |l_2| \leq |l|} |\partial^{a_1} \tilde{Z}^I (A_\mu^a - \delta_\mu^a) ||\partial^{a_2} \tilde{Z}^I \partial_\beta \phi|. \]
The result follows from Proposition 5.1 and Corollary 5.3.

To prove the rightmost inequality of (5.17) we note that \( 1 + |q^*| \leq 1 + |q| + M \chi \ln r \) and divide by \( 1 + |q| \). Taking \( 1 + |q| \leq 1 + |q^*| + M \chi \ln r \) and dividing by \( 1 + |q^*| \) gives the left inequality.

We now look at the right hand inequality in (5.18). In the region \( \frac{1}{3} \leq \frac{r}{r^* + 1} \leq \frac{1}{4} \) this is trivial. For \( \frac{1}{4} \leq \frac{r}{r^* + 1} \leq \frac{3}{4} \), we don’t need to distinguish derivatives. and the inequality follows from expanding out and applying (5.2) We now consider the region \( \frac{3}{4} \leq \frac{r}{r^* + 1} \leq 1 \) (the region where it is greater than \( \frac{3}{4} \) this is trivial). Here, we must take the null decomposition into consideration. For any set of vector fields \( \tilde{X}^I \), we can commute fields and take the decompositions
\[ \tilde{X}^I = \sum_{j + k + |\alpha| + |\gamma| \leq k} \prod_{1 \leq i \leq 3} f_{j^i \gamma} (\omega, t^i \phi^*/r^i) (\omega^* \tilde{L})^j (u^* \tilde{L})^k (r^* \tilde{\phi}^i) \tilde{\partial}_\mu, \] which gives us
where $\gamma$ is a multiindex with components $\gamma_{ij}$, and $\partial^\alpha$ is a standard multiindex. We can replace $1 + u^*$ with $1 + u^*$ and $u^*$ with $1 + u^*$, which changes the values of $f$. We can write

$$(1 + u)L = \left(1 + \frac{M \ln r}{1 + u^*}\right) (1 + u^*) \tilde{L} - (1 + u) \left(\frac{M}{r} \partial_r\right),$$

$$(1 + u)L = \left(1 - \frac{M \ln r}{1 + u^*}\right) (1 + u^*) \tilde{L} + (1 + u) \left(\frac{M}{r} \partial_r\right),$$

$$(1 + u^*) \tilde{L} = \left(1 - \frac{M \ln r}{1 + u^*}\right) (1 + u)L + (1 + u^*) \left(\frac{M}{r} \partial_r\right),$$

$$(1 + u^*) \tilde{L} = \left(1 + \frac{M \ln r}{1 + u^*}\right) (1 + u)L - (1 + u^*) \left(\frac{M}{r} \partial_r\right).$$

Our result follows from iterating this and applying Proposition 5.2 when needed. The right hand side of the inequality, and the exterior case $r^* \geq t$, follow similarly. \qed

### 5.3.2. The asymptotically Schwarzschild metric in the new coordinates

Recall that

$$\tilde{m}_{0}^{ab} = (m^{\alpha \beta} - \frac{M \chi}{r} \delta^{\alpha \beta}) A^{a}_{\alpha} A^{b}_{\beta}$$

In asymptotically Schwarzschild coordinates, we expect $g_{\tilde{L} \tilde{L}}$ to decay faster than $r^{-1}$ along the light cone. We first show bounds on components $\tilde{m}_0$, but before that we must define certain quantities. Let $\tilde{\delta}^{ab}$ be the Euclidean metric on the spatial coordinates in this frame, with $\tilde{\delta}^{ab} = 1$ if $a = b > 0$ and 0 otherwise, and let $\tilde{\mathcal{S}}^{ab}$ be the tangential part, given by

$$\tilde{\mathcal{S}}^{00} = \tilde{\mathcal{S}}^{0i} = \tilde{\mathcal{S}}^{i0} = 0, \quad \tilde{\mathcal{S}}^{ij} = \delta^{ij} - \omega^i \omega^j$$

where as usual $i, j$ range from 1 to 3. In the inverse null decomposition neither $\tilde{m}$ nor $\tilde{\mathcal{S}}$ contains a term like $\tilde{L}^{a} \tilde{L}^{b}$. We now decompose $\tilde{m}_0$ as a weighted sum of these terms, plus a remainder which decays rapidly along the light cone in all components.

**Lemma 5.7.** We can decompose

$$\tilde{m}_{0}^{ab} = \kappa_0 \tilde{m}^{ab} + \kappa_1 \tilde{\mathcal{S}}^{ab} + \kappa_2 \omega^a \omega^b + \kappa_3 i_+^{ab},$$

where

$$\kappa_0 = 1 + \frac{M \chi}{r}, \quad \kappa_1 = (1 + \frac{M \chi}{r})^2 (1 - \frac{M \chi}{r}) - (1 + \frac{M \chi}{r}) = 2 \frac{M \ln r}{r} \chi(\frac{r}{t+1}, \frac{M \ln r}{r}, \frac{M}{r}) - 2 \chi \frac{M}{r},$$

and

$$\kappa_2 = \frac{(M \chi)^2}{r} (1 + \frac{M \chi}{r}), \quad i_+^{ab} = \tilde{\chi}'(\frac{r}{t+1}) \chi^{ab}(\frac{r}{t+1}, \frac{M \ln r}{r}, \frac{M}{r}, \omega), \quad \text{and} \quad \kappa_3 = \frac{M \ln r}{r},$$

for some smooth functions $\chi_1, \chi_2$. We have the following estimates on Lie derivatives of $\kappa_2 \tilde{\delta}^{ab}$ and $\kappa_3 i_+^{ab}$:

$$\sum_{a, b} \sum_{|I| \leq k} |(\mathcal{L}_{\tilde{\chi}}(\kappa_2 \tilde{\delta}))^{ab}| \lesssim \frac{M^2 (\ln(1 + r))^2}{(t + r^*)^2},$$

$$\sum_{a, b} \sum_{|I| \leq k} |(\mathcal{L}_{\tilde{\chi}}(\kappa_3 i_+))^{ab}| \lesssim \frac{M \ln (1 + r)}{(t + r^*)^2}.$$ 

The support of these are contained in the supports of $\tilde{\chi}$ and $\tilde{\chi}'$ respectively.

**Proof.** For the bounds on Lie derivatives of $\kappa_2 \tilde{\delta}^{ab}$ and $\kappa_3 i_+^{ab}$ it suffices to prove the bounds on derivatives of each component, for which we can use Proposition 5.1 and Corollary 5.4.

We expand $\tilde{m}_{0}^{ab}$ and use (5.12a, b, c) to get

$$\tilde{m}_{0}^{00} = - (1 - \frac{M \chi}{r}), \quad \tilde{m}_{0}^{0i} = - \frac{1}{r} (\frac{r}{t+1})^2 \left(1 - \frac{M \chi}{r}\right) \omega^i \chi' M \ln r.$$
The term $\tilde{m}_0^{ij}$ is equal to the corresponding component of $\kappa_0 \tilde{m}^{ab}$, and $\tilde{m}_0^{ij}$ appears in $\kappa_3 i_+$. The component bounds for this term follow directly from Proposition 5.1 and Corollary 5.4. For $\tilde{m}_0^{ij}$, the process is slightly longer. Writing $\tilde{m}_0^{ij} = m_0^{\alpha\beta} A_{i}^{\alpha} A_{j}^{\beta}$ we see that if $\alpha$ or $\beta$ are 0, then both must be 0 and we can absorb this term in $\kappa_3 i_+$ using Proposition 5.1 and Corollary 5.4. If neither is 0 we write

$$\tilde{m}_0^{ij} - m_0^{00} A_i^{0} A_j^{0} = \sum_{k,l=1}^{3} (1 - \frac{M \tilde{\chi}}{r}) \delta^{kl} A_k^{i} A_l^{j},$$

Equation (5.12c) is equivalent to

$$\partial_r \tilde{\chi} = \delta^i_j \left( 1 + \frac{M \tilde{\chi}}{r} \right) + \left( \frac{M \ln r - M \tilde{\chi}}{r} \right) \left( \delta^i_j - \omega_j \omega^j \right) + \frac{M \ln r}{r} \omega_j \omega^j \frac{r}{r+1} \tilde{\chi}'.$$

Terms containing $\chi'$ can be absorbed into $\kappa_3 i_+$ using Proposition 5.1 and Corollary 5.4. The remaining terms are

$$(1 - \frac{M \tilde{\chi}}{r}) \left( 1 + \frac{M \tilde{\chi}}{r} \right)^2 \delta^{ij} + \left( 1 - \frac{M \tilde{\chi}}{r} \right) \left( 2 \left( 1 + \frac{M \tilde{\chi}}{r} \right) \left( \frac{M \ln r - M \tilde{\chi}}{r} \right)^2 + \left( \frac{M \ln r - M \tilde{\chi}}{r} \right)^2 \right) \left( \delta^{ij} - \omega^i \omega^j \right). \tag{5.20}$$

We can expand this, using the identity

$$2 \left( 1 + \frac{M \tilde{\chi}}{r} \right) \left( \frac{M \ln r - M \tilde{\chi}}{r} \right)^2 + \left( \frac{M \ln r - M \tilde{\chi}}{r} \right)^2 = 1 + \frac{M \ln r}{r} - (1 + \frac{M \tilde{\chi}}{r})^2.$$

For $i,j > 0$ we now that $\delta^{ij} - \omega^i \omega^j$ so componentwise (5.20) is equal to

$$\kappa_0 \tilde{m}^{ij} + \kappa_1 \tilde{\mathcal{S}}^{ij} + \kappa_2 \omega^i \omega^j.$$

□

We now show that $\kappa_0 \tilde{m}^{ij}$, $\kappa_1 \tilde{\mathcal{S}}^{ij}$, $\kappa_2 \omega^i \omega^j$ behave nicely under repeated Lie differentiation. First, we consider $\tilde{m}$.

$$\mathcal{L}_{X}^{I} (\kappa_0 \tilde{m}^{ij})^{ab} = \kappa_0^{I} \tilde{m}^{ab},$$

for some bounded function $\kappa_0^{I}$. This follows from expanding and applying Proposition 5.1 and Corollary 5.4 whenever a derivative falls on $\kappa_0$, or noting that $\mathcal{L}_{X} \tilde{m}^{-1} = c X \tilde{m}^{-1}$ otherwise. Similar reasoning gives us the estimate

$$| \mathcal{L}_{X}^{I} (\kappa_0 \tilde{m})^{ab} | = \frac{1}{r} \chi_0 \left( \frac{r}{r+1}; \frac{M \ln r}{r} \right)$$

where $\chi_0$ is a bounded smooth function which is 0 when $\frac{r}{r+1} < \frac{1}{4}$. In order to deal with the term $\kappa_1 \tilde{\mathcal{S}}^{ij}$, we use the identity

$$\tilde{\mathcal{S}}^{ab} = r^{-2} \sum_{i<j} \tilde{\Omega}^{a}_{ij} \tilde{\Omega}^{b}_{ij}. $$

We now expand. If Lie derivatives fall on $r^{-2} \kappa_1$, we can again apply Proposition 5.1 and Corollary 5.4. If derivatives fall on $\tilde{\Omega}^{a}_{ij} \tilde{\Omega}^{b}_{ij}$, the commutator identities (5.11) and the Leibniz rule give us the following decomposition, which holds when $\frac{r}{r+1} > 1/4$: for $U, V \in \{ \tilde{L}, \tilde{L}_{X}, \tilde{S}_1, \tilde{S}_2 \}$, we have the pointwise estimates

$$\| \left( \mathcal{L}_{X}^{I} (\tilde{\Omega}^{a}_{ij} \tilde{\Omega}^{b}_{ij}) \right) U \bar{V} \| \lesssim (t + r^*)^2,$$

$$\| \left( \mathcal{L}_{X}^{I} (\tilde{\Omega}^{a}_{ij} \tilde{\Omega}^{b}_{ij}) \right) \bar{U} \bar{L} \| \lesssim (t + r^*) (t + r^*),$$

$$\| \left( \mathcal{L}_{X}^{I} (\tilde{\Omega}^{a}_{ij} \tilde{\Omega}^{b}_{ij}) \right) \bar{U} \bar{V} \| \lesssim (t - r^*)^2.$$ Combining this with the bounds $\tilde{X} \tilde{U} \lesssim (t + r^*)$, $\tilde{X} \tilde{L} \lesssim (t - r^*)$ gives the following:

Lemma 5.8. Given the inverse metric $\tilde{m}_{0}^{ab}$ under the change of coordinates, we can decompose

$$\left( \mathcal{L}_{X}^{I} \tilde{m} \right)^{ab} = \kappa_0^{I} \tilde{m}^{ab} + \tilde{\mathcal{S}}^{ab} + \left( \mathcal{L}_{X}^{I} R \right)^{ab}.$$
where

\[
\kappa_0^I \lesssim 1 \quad \text{ and } \quad |g^I_{UV}| \lesssim \frac{M \ln(1+(t+r^*))}{(t+r^*)} \chi_1 \\
|g^I_{LU}| \lesssim \frac{M(t+r^*) \ln(1+(t+r^*))}{(t+r^*)^2} \chi_1 \\
|\langle L^I_X R \rangle_{UV}| \lesssim \frac{M^2 \ln(1+r^*)^2}{(t+r^*)^2} \chi_1 + \frac{M \ln(1+r^*)}{(t+r^*)} \chi'_2,
\]

where \( \chi_1 \) and \( \chi'_2 \) are smooth functions functions bounded above by 1 with support respectively in the intervals \( \frac{-\pi}{2} \leq \tau \leq \frac{\pi}{2} \) and \( \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \) respectively.

6. Einstein’s equations in asymptotically Schwarzschild coordinates

6.1. Einstein’s equations in asymptotically Schwarzschild coordinates. Let \( \nabla_a \) be covariant differentiation with respect to the metric \( \tilde{m} \):

\[
\nabla_c \tilde{K}_{b_1 \ldots b_k} = \tilde{\partial}_c \tilde{K}_{b_1 \ldots b_k} + \tilde{\Gamma}^a_{dc} \tilde{K}_{b_1 \ldots b_k} \cdots + \tilde{\Gamma}^a_{dc} \tilde{K}_{b_1 \ldots b_k} - \tilde{\Gamma}^d_{b_1 c} \tilde{K}_{a_1 \ldots a_k} + \cdots - \tilde{\Gamma}^d_{b_1 c} \tilde{K}_{a_1 \ldots a_k}.
\]

Note that here \( \nabla \) is simply a tool to systematize the change of variables and does not depend on \( g \), but only on the Christoffel symbols \( \tilde{\Gamma} \) of \( \tilde{m} \), given in (6.4).

Let \( \tilde{H}^{cd} = A^c_b A^d_a H^{\delta} \). Since \( \partial_a H^{\alpha \beta} = m^{\alpha \alpha'} m^{\beta \beta'} \partial_a H_{\alpha' \beta'} = A^a_b m^{aa'} m^{bb'} \nabla_a H_{a' b'} \), it follows that

\[
A^a_b \partial_a H^{\alpha \beta} = \nabla_a \tilde{H}^{ac}.
\]

Let \( \tilde{h}_{cd} = A^c_b A^d_a \tilde{h}_{a \beta} \). Since \( A^a_b A^d_c \partial_a \partial_b \tilde{h}_{c \delta} = \nabla_c \nabla_h \tilde{h}_{cd} \) it follows that

\[
A^a_b A^d_c \partial_a \partial_b \tilde{h}_{c \delta} = \tilde{g}^{ab} \nabla_a \nabla_b \tilde{h}_{cd}.
\]

Since the covariant derivative satisfies \( A^a_b A^d_c \partial_a \partial_b \tilde{h}_{c \delta} = \nabla_c \nabla_h \tilde{h}_{cd} \) and since \( F \) consists of two contractions with the (inverse of the) metric it follows that

\[
A^a_b A^d_c F_{c \delta}(g) [\partial h, \partial k] = \tilde{F}_{cd}(g) [\nabla_h, \nabla_k],
\]

Lemma 6.1. We have

\[
\nabla_c \tilde{K}^{ac} = \partial_c \tilde{H}^{ac} + \tilde{W}_1^c [\nabla_c, \nabla_h^c],
\]

\[
\tilde{g}^{ab} \nabla_a \nabla_b \tilde{h}_{cd} = \tilde{g}^{ab} \nabla_a \nabla_b \tilde{h}_{cd} + E^1_{cd}(g) [\partial h, \partial k] + E^{2}_{cd}(g) [\partial \tilde{h}, \nabla_k],
\]

\[
\tilde{F}_{cd}(g) [\nabla_h, \nabla_k] = \tilde{F}_{cd}(g) [\partial h, \partial k] + \tilde{F}^{1}_{cd}(g) [\nabla_h, \tilde{h}, \nabla_k] + \tilde{F}^{2}_{cd}(g) [\tilde{h}, \partial h, \nabla_k] + \tilde{F}^{2}_{cd}(g) [\tilde{h}, \tilde{h}, \tilde{h}],
\]

Proof. (6.3) follows from \( \nabla_c \tilde{K}^{ac} = \partial_c \tilde{H}^{ac} + \tilde{W}_1^c [\nabla_c, \nabla_h^c] \) and (6.4) follows from

\[
\nabla_c \nabla_b \tilde{h}_{cd} = \partial_b \tilde{h}_{cd} - \tilde{\Gamma}^b_{bc} \tilde{h}_{cd} - \tilde{\Gamma}^b_{bd} \tilde{h}_{cd} - \tilde{\Gamma}^b_{cd} \tilde{h}_{bc} - \tilde{\Gamma}^b_{cd} \tilde{h}_{bcd} - \tilde{\Gamma}^b_{cd} \tilde{h}_{bc} - \tilde{\Gamma}^b_{cd} \tilde{h}_{bcd} - \tilde{\Gamma}^b_{cd} \tilde{h}_{c bd} - \tilde{\Gamma}^b_{cd} \tilde{h}_{c bd}.
\]

Finally, (6.5) follows from the definition of covariant derivative \( \nabla_c \tilde{h}_{cd} = \partial_b \tilde{h}_{cd} - \tilde{\Gamma}^b_{bc} \tilde{h}_{cd} - \tilde{\Gamma}^b_{bd} \tilde{h}_{cd} - \tilde{\Gamma}^b_{cd} \tilde{h}_{bc} - \tilde{\Gamma}^b_{cd} \tilde{h}_{bcd} - \tilde{\Gamma}^b_{cd} \tilde{h}_{c bd} - \tilde{\Gamma}^b_{cd} \tilde{h}_{c bd}.
\]

6.1.1. Einstein’s equations in the asymptotically Schwarzschild coordinates. By (6.1) and (6.2) Einstein’s equations (11.12) in the new coordinates become

\[
\tilde{g}^{ab} \nabla_a \nabla_b \tilde{h}_{cd} = \tilde{F}_{cd}(g) [\nabla_h, \nabla_k] + \tilde{\Gamma}_{cd}.
\]

Here by (4.9) respectively (4.10) we can write

\[
\tilde{g}^{ab} \nabla_a \nabla_b \tilde{h}_{cd} = \tilde{g}^{ab} \nabla_a \nabla_b \tilde{h}_{cd}^{1} + \tilde{E}_{cd,0} + \tilde{E}_{cd,1} [\tilde{H}_1],
\]

\[
\tilde{F}_{cd}(g) [\nabla_h, \nabla_k] = \tilde{F}_{cd}(g) [\nabla_h^{1}, \nabla_k] + \tilde{E}_{cd,0}^{mass}(g) + \tilde{E}_{cd,1}^{mass}(g) [\nabla_h^{1}],
\]
where
\[
\begin{align*}
\tilde{E}^{\text{mass}}_{cd,0} &= M \chi^{cd}_a \left( \frac{r}{1 + t} \right) M^{\ln r} M^{\ln r} r^{-2}, \\
\tilde{E}^{\text{mass}}_{cd,1} &= \chi^{cd} \left( \frac{r}{1 + t} \right) M^{\ln r} M^{\ln r} M r^{-3} \tilde{H}_{ab}^{1},
\end{align*}
\]
(6.6)
\[
\begin{align*}
\tilde{F}^{\text{mass}}_{cd,0} &= M^2 \chi^{cd}_a \left( \frac{r}{1 + t} \right) M^{\ln r} M^{\ln r} \omega \tilde{g}, \\
\tilde{F}^{\text{mass}}_{cd,1} &= \chi^{cd} \left( \frac{r}{1 + t} \right) M^{\ln r} M^{\ln r} \omega \tilde{g} M r^{-2} \nabla \tilde{h}_{ab}^{1}.
\end{align*}
\]
(6.7)

Here \(\chi^{cd}_a(s, \cdot)\) stands for functions with \(|s - 1/2| < 1/4\) in the support and \(\chi^{cd}_a(s, \cdot)\) and \(\chi^{cd}_a(s, \cdot)\) with \(|s - 1/2| < 1/4\) in the support. It follows that
\[
\tilde{g}^{ab} \nabla \tilde{g} \tilde{h}_{ab}^{1} + \tilde{E}^{\text{mass}}_{cd,0} + \tilde{E}^{\text{mass}}_{cd,1} \tilde{H}_{1} = \tilde{F}^{\text{mass}}_{cd}(\nabla \tilde{h}_{1}, \nabla \tilde{h}^{1}) + \tilde{F}^{\text{mass}}_{cd,0}(\tilde{g}) + \tilde{F}^{\text{mass}}_{cd,1}(\tilde{g}) \nabla \tilde{h}_{1}^{1} + \tilde{T}_{cd}
\]
(6.8)

The approximate wave coordinate condition \(\tilde{E}_{cd}^{\text{mass}}\) becomes
\[
\nabla_a \left( \tilde{H}_{1}^{ac} - \frac{1}{4} \tilde{h}^{ac} \tilde{m}_{bd} \tilde{H}_{1}^{bd} \right) = \tilde{W}^c(\tilde{g}) \tilde{H}_{1}, \nabla \tilde{H} = \tilde{W}^{c,0}_{\text{mass}},
\]
(6.9)

where
\[
\tilde{W}^c(\tilde{g}) \tilde{H}_{1}, \nabla \tilde{H} = \tilde{W}^c(\tilde{g}) \tilde{H}_{1}, \nabla \tilde{H} + \tilde{W}^{c,1}_{\text{mass}} + \tilde{W}^{c,2}_{\text{mass}} \tilde{H}_{1} + \tilde{W}^{c,3}_{\text{mass}} \nabla \tilde{H}_{1}.
\]

Here
\[
\tilde{W}^{c,0}_{\text{mass}} = -M \chi^{c,0}(\frac{r}{1 + t} M^{\ln r} M^{\ln r} \omega r^{-2}, \\
\tilde{W}^{c,1}_{\text{mass}} = \chi^{c,1}(\frac{r}{1 + t} M^{\ln r} M^{\ln r} M r^{-3}, \\
\tilde{W}^{c,2}_{\text{mass}} \tilde{H}_{1} = \chi^{c,2}(\frac{r}{1 + t} \omega, g) M r^{-2} \tilde{h}_{1}, \\
\tilde{W}^{c,3}_{\text{mass}} \nabla \tilde{H}_{1} = \chi^{c,3} \frac{r}{1 + t} \omega, g) M r^{-1} \nabla \tilde{h}_{1}^{1},
\]
with \(\chi^{c}(s, \cdot)\) denoting a function with \(|s - 1/2| < 1/4\) in the support.

6.2. The structure of the covariant error terms.

6.2.1. The covariant error terms. Lemma \(6.1\) says that we can replace the covariant derivatives by \(\tilde{g}\), up to terms involving at least an extra factor of \(\tilde{g}\), that by \(\tilde{g}^{ab} \tilde{g}^{cd} \tilde{h}_{cd}^{1} + \tilde{E}^{\text{mass}}_{cd} - \tilde{E}^{\text{cov}}_{cd} = \tilde{F}^{\text{cd}}(\tilde{g}) \tilde{H}_{1}, \tilde{H}_{1} + \tilde{F}^{\text{mass}}_{cd} + \tilde{F}^{\text{cov}}_{cd} + \tilde{T}_{cd},
\]
(6.10)

where with notation as in \(6.6\)-\(6.7\),
\[
\tilde{E}^{\text{mass}}_{cd} = \tilde{E}^{\text{mass}}_{cd,0} + \tilde{E}^{\text{mass}}_{cd,1} \tilde{H}_{1}, \\
\tilde{F}^{\text{mass}}_{cd} = \tilde{F}^{\text{mass}}_{cd,0}(\tilde{g}) + \tilde{F}^{\text{mass}}_{cd,1}(\tilde{g}) \nabla \tilde{h}_{1}^{1},
\]

and
\[
\tilde{E}^{\text{cov}}_{cd} = \tilde{E}^{\text{cov}}_{cd} \tilde{H}_{1} + \tilde{E}^{\text{cov}}_{cd} \tilde{H}_{1}, \\
\tilde{F}^{\text{cov}}_{cd} = \tilde{F}^{\text{cov}}_{cd} \tilde{H}_{1} + \tilde{F}^{\text{cov}}_{cd} \tilde{H}_{1},
\]
are the covariant error coming from replacing the covariant derivatives by partial derivatives in \(6.8\):
\[
\tilde{E}^{\text{mass}}_{cd,0} \tilde{H}_{1}^{1} = M r^{-3} \ln r \chi^{cd}_a(\frac{r}{1 + t} \omega, M^{\ln r} M^{\ln r} h)^{1} + M r^{-3} \chi^{cd}_a(\frac{r}{1 + t} \omega, M^{\ln r} M^{\ln r} h)^{1},
\]
(6.11)
\[
\tilde{E}^{\text{mass}}_{cd,1} \tilde{H}_{1}^{1} = M r^{-2} \ln r \chi^{cd}_a(\frac{r}{1 + t} \omega, M^{\ln r} M^{\ln r} h)^{1} + M r^{-2} \chi^{cd}_a(\frac{r}{1 + t} \omega, M^{\ln r} M^{\ln r} h)^{1},
\]
(6.12)

by \(6.6\), and by \(6.5\),
\[
\tilde{F}^{\text{mass}}_{cd,0} \tilde{H}_{1}^{1} = M r^{-3} \ln r \chi^{cd}_a(\frac{r}{1 + t} \omega, M^{\ln r} M^{\ln r} h)^{1} + M r^{-3} \chi^{cd}_a(\frac{r}{1 + t} \omega, M^{\ln r} M^{\ln r} h)^{1},
\]
(6.13)
\[
\tilde{F}^{\text{mass}}_{cd,1} \tilde{H}_{1}^{1} = M r^{-2} \ln r \chi^{cd}_a(\frac{r}{1 + t} \omega, M^{\ln r} M^{\ln r} h)^{1} + M r^{-2} \chi^{cd}_a(\frac{r}{1 + t} \omega, M^{\ln r} M^{\ln r} h)^{1},
\]
(6.14)

Moreover by \(6.9\),
\[
\tilde{D}_{a}(\tilde{H}_{1}^{ac} - \frac{1}{2} \tilde{m}^{ac} \tilde{m}_{bd} \tilde{H}_{1}^{bd}) = \tilde{W}^c(\tilde{g}) \tilde{H}_{1}, \tilde{H} + \tilde{W}^{c,0}_{\text{mass}} + \tilde{W}^{c,0}_{\text{cov}},
\]
(6.15)
where
\[ \tilde{W}^c(\tilde{g})[\tilde{H}, \tilde{\partial} \tilde{H}] = \tilde{W}^c(\tilde{g})[\tilde{H}_1, \tilde{\partial} \tilde{H}_1] + \tilde{W}^c_{\text{mass}} + \tilde{W}^c_{\text{mass}}[\tilde{H}_1] + \tilde{W}^c_{\text{mass}}[\tilde{\partial} \tilde{H}_1], \]
and
\[ \tilde{W}^c_{\text{cov}} = \tilde{W}^c_{\text{cov}}(h)[\tilde{H}_1] = M r^{-2} \ln r \chi^{c}_{ab}(\frac{r}{r+1}, \omega, \frac{M \ln r}{r}, h) H_1^{ab} + M r^{-2} \chi^{c}_{ab}(\frac{r}{r+1}, \omega, \frac{M \ln r}{r}, h) \tilde{H}_1^{ab}. \]

6.2.2. Alternative term picking up the mass. In view of (5.6)-(5.7) it may be natural to alternatively subtract of a term picking up the mass after having changed coordinates:
\[ \tilde{h}_{ab}^{0T} = M r^{-1} \chi^{c}(\frac{r}{r+1}) \delta_{ab}, \]
where \( \chi \) is as in (4.1). The difference
\[ \tilde{\partial}^a \tilde{h}_{ab}^{0T} - \tilde{\partial}^a \tilde{h}_{ab}^0 = O(M r^{-2-|a|} \ln r), \]
is of the same size as \( \hat{\Gamma} \) so one could include it in the covariant error terms described above.

6.3. The structure of the quadratic and higher order terms. Recall that the inhomogeneous term in Einstein’s vacuum equations has the form
\[ \tilde{F}_{ab}(\tilde{g})[\tilde{h}, \tilde{\partial} \tilde{h}] = P(\tilde{g})[\tilde{h} \tilde{\partial} h] + Q_{ab}(\tilde{g})[\tilde{h}, \tilde{\partial} \tilde{h}], \]
where \( Q \) is a combination of classical null forms and \( P \), given by (1.6), has a weak null structure. In view of (5.6)-(5.7) we expect that
\[ \tilde{F}_{ab}(\tilde{g})[\tilde{h}, \tilde{\partial} \tilde{h}] \sim \tilde{F}_{ab}(\tilde{m})[\tilde{h}, \tilde{\partial} \tilde{h}]. \]
Since \( \tilde{Q}_{ab} = Q_{ab}(\tilde{m}) \) satisfies the classical null condition with respect to the new coordinates,
\[ |\tilde{Q}_{ab}(\tilde{h}, \tilde{\partial} \tilde{k})| \lesssim |\tilde{\partial} \tilde{h}| |\tilde{\partial} \tilde{k}| + |\tilde{\partial} \tilde{h}| |\tilde{\partial} \tilde{k}|. \]
(6.16)
The main term \( \hat{P} = P(\tilde{m}) \) can be further analyzed as follows. First we note that by (3.3)
\[ |\hat{P}(\tilde{\partial} \tilde{h}, \tilde{\partial} \tilde{k}) - \tilde{L}_a \tilde{L}_b \hat{P}(\tilde{\partial} \tilde{h}, \tilde{\partial} \tilde{k})| \lesssim |\tilde{\partial} \tilde{h}| |\tilde{\partial} \tilde{k}| + |\tilde{\partial} \tilde{h}| |\tilde{\partial} \tilde{k}|. \]
(6.17)
where by (3.7)
\[ |\hat{P}(\tilde{\partial} \tilde{h}, \tilde{\partial} \tilde{k}) - P_S(\tilde{\partial} \tilde{h}, \tilde{\partial} \tilde{k})| \lesssim (|\tilde{\partial} \tilde{h}| |\tilde{\partial} \tilde{k}| + |\tilde{\partial} \tilde{h}| |\tilde{\partial} \tilde{k}| + |\tilde{\partial} \tilde{h}| |\tilde{\partial} \tilde{k}| + |\tilde{\partial} \tilde{h}| |\tilde{\partial} \tilde{k}|). \]
(6.18)
where
\[ P_S(D, E) = -\hat{D}_{AB} E^{AB}/2, \quad A, B \in S, \quad \text{where} \quad \hat{D}_{AB} = D_{AB} - \delta_{AB} \|D/2, \quad \|D = \delta^{AB} D_{AB}. \]
(6.19)

6.4. Commutators and Lie derivatives along the modified vector fields.

6.4.1. Lie derivatives along the modified vector fields. As for (5.8) we have
\[ \mathcal{L}_{\tilde{X}} \tilde{c}_1 \cdots \tilde{c}_k \tilde{K}_{b_1 \cdots b_s}^{a_1 \cdots a_r} = \tilde{c}_1 \cdots \tilde{c}_k \mathcal{L}_{\tilde{X}} \tilde{K}_{b_1 \cdots b_s}^{a_1 \cdots a_r}. \]
As in (5.9) we define the modified Lie derivative in the new coordinates by
\[ \hat{\mathcal{L}}_{\tilde{X}} \tilde{K}_{b_1 \cdots b_s}^{a_1 \cdots a_r} = \mathcal{L}_{\tilde{X}} \tilde{K}_{b_1 \cdots b_s}^{a_1 \cdots a_r} + \frac{r-s}{r-1} (\tilde{\partial}_e \tilde{X}^e) \tilde{K}_{b_1 \cdots b_s}^{a_1 \cdots a_r}. \]
Then if \( \hat{m}_{ab} = m_{ab} \) is the Minkowski metric in the \( \bar{x} \) coordinates then for the vector fields in (5.10) we have
\[ \hat{\mathcal{L}}_{\tilde{X}} \hat{m}_{ab} = 0. \]
(6.20)
6.4.2. The commutators of the equations with Lie derivatives along the modified vector fields. The Lie derivative commutes with exterior differentiation which for any of the vector fields \( \tilde X \) leads to
\[
\partial_c \tilde L_\tilde X \tilde H^{cd} = (\tilde L_{\tilde X} + \tilde \partial_c \tilde \partial_b) \tilde H^{cd}.
\]
(6.21)
Moreover,
\[
[\tilde L_{\tilde X}, \Box] \phi = -(\tilde L_{\tilde X} \tilde g^{ab}) \tilde \partial_a \tilde \partial_b \phi + \tilde g^{ab} (\tilde \partial_a \tilde \partial_b \tilde X^c) \tilde \partial_c \phi = -(\tilde L_{\tilde X} \tilde g^{ab}) \tilde \partial_a \partial_b \phi,
\]
since \( \tilde X^c \) are linear functions. Moreover
\[
\Box \tilde g \tilde L_{\tilde X} \phi_{cd} = \tilde L_{\tilde X} (\Box \tilde g_{\phi_{cd}}) - (\tilde L_{\tilde X} \tilde g^{ab}) \tilde \partial_a \partial_b \phi_{cd},
\]
(6.22)
where by (5.6)-(5.7) and (6.20)
\[
\tilde L_{\tilde X} \tilde g^{ab} \sim \tilde L_{\tilde X} \tilde H^{ab}.
\]
Finally if
\[
S_{ab}(\tilde g)[\tilde \partial h, \tilde \partial k] = S_{ab}(\tilde G)[\tilde \partial h, \tilde \partial k]
\]
is a quadratic form in the \((0,3)\) forms \( \partial h \) and \( \partial k \) with two contractions with the inverse of the metric \( \tilde G^{ab} = \tilde g^{ab} \), i.e. it is bilinear in \( \tilde G \), we have
\[
\tilde L_{\tilde X} (S_{ab}(\tilde g)[\tilde \partial h, \tilde \partial k]) = S_{ab}(\tilde g)[\tilde \partial \tilde L_{\tilde X} \tilde h, \tilde \partial \tilde k] + S_{ab}(\tilde g)[\tilde \partial \tilde L_{\tilde X} \tilde k, \tilde \partial \tilde h] + S_{ab}(\tilde g, \tilde X^c \tilde G)[\tilde \partial h, \tilde \partial k] + S_{ab}(\tilde G, \tilde L_{\tilde X} \tilde G)[\tilde \partial \tilde h, \tilde \partial \tilde k].
\]
(6.23)

6.4.3. The cubic error terms. We remark that the last term in (6.23) is cubic and therefore much easier to control, see Section 2.2.1 and Section 6.2.1.

7. Vector fields applied to Einstein’s equations, commutators and higher order equations

Modulo mass errors (that comes from replacing \( h \) by \( h^1 \)) and covariant errors (that comes from replacing \( \nabla_a \) by \( \tilde \partial_a \) and \( h^1 \) by \( h^1 \)) (6.10) the reduced wave operator \( \Box = \tilde g^{ab} \tilde \partial_a \tilde \partial_b \) applied to \( h^1 \) satisfy
\[
\Box \tilde h^{1}_{cd} = \tilde F_{cd}(\tilde g)[\tilde \partial h^1, \tilde h^1] + \tilde T_{cd} + \tilde R_{cd}^{\text{mass}} + \tilde R_{cd}^{\text{cov}},
\]
(7.1)
where \( \tilde R_{cd}^{\text{mass}} = \tilde F_{cd}^{\text{mass}} + \tilde F_{cd}^{\text{mass}} \) and \( \tilde R_{cd}^{\text{cov}} = \tilde F_{cd}^{\text{cov}} + \tilde F_{cd}^{\text{cov}} \) are given in Section 6.2. It was explained how to deal with the mass errors in Section 3 and covariant errors in Section 6 at lowest order. However, applying vector fields using the estimates in Section 3 for curved vector fields in terms of the non curved gives, apart from additional logarithmic factors, the same estimates for vector fields applied to these quantities. We can therefore neglect these error terms in the analysis that follows. Similarly by (6.15)
\[
\tilde \partial_a (\tilde H_{1ac}^{1} - \frac{1}{2} \tilde m_{ac} \tilde m_{bd} \tilde H_{1bd}^{1}) = \tilde W_c(\tilde g)[\tilde H_1, \tilde \partial \tilde H_1] + \tilde W_{cd}^{\text{mass}} + \tilde W_{cd}^{\text{cov}}.
\]
(7.2)

7.1. Higher order commutators with Einstein’s equations. We will now apply vector fields to Einstein’s equations (7.1) to get higher order equations. As we have seen the commutators improve if we use Lie derivatives and furthermore as will see below they also improve if we first multiply with \( \kappa = (1 + M \tilde \chi / r)^{-1} \), which just produces lower order terms of the same form. This gives a wave equation for the differentiated metric \( \tilde h_{11}^{1} = \tilde L_{\tilde X} \tilde h_{11}^{1} \):
\[
\Box \tilde h_{11}^{1} + \tilde R_{cd}^{\text{com}} I = \sum_{I' + I'' = I} \kappa I' \left( \tilde L_{\tilde X}^{I'} (\tilde F_{cd}(\tilde g)[\tilde \partial h^{1],1}, \tilde \partial \tilde h^{1j}] + \tilde R_{cd,0}^{I'} \tilde R_{cd}^{\text{mass}} I' + \tilde R_{cd,0}^{I'} \tilde R_{cd}^{\text{cov}} I' \right),
\]
(7.3)
when some of the vector fields in $\mathcal{L}^j_X$ fall on the metric $g$ in the nonlinearity $\tilde{F}_{cd}(g)[\partial h^1, \partial h^1]$. The remainder terms $\tilde{R}_{cd}^* I$ we will be estimated below. Moreover will show that we can write
$$\kappa^2 \tilde{F}_{cd}(g)[\partial h^1, \partial h^1] = L_c L_d \tilde{P}(\partial_t, \partial_t, \partial_t) + \kappa^2 \tilde{R}_{cd}^* I + \tilde{R}_{cd, 1}^*,$$
where $\tilde{P}$ is the constant coefficient quadratic form with the special geometric weak null structure mentioned before and $\tilde{R}_{cd}^* I$ consist of a quadratic form in $\partial h^1$, with at least one factor with only tangential derivatives, plus cubic terms $\tilde{R}_{cd, 1}^* I$. The cubic terms as well as the mass and covariant terms are lower order and easier to control.

7.1.1. The leading behavior of the inverse of the metric in the curved coordinates. We have
$$\tilde{g}^{ab} = \tilde{m}^{ab}_0 + \tilde{H}^{ab}_1.$$ 
We have seen in Section 5 that
$$\tilde{m}^{ab}_0 = \kappa_0 \tilde{m}^{ab}_0 + \kappa_1 \tilde{g}^{ab}_0 + \kappa_2 \tilde{\delta}^{ab} + \kappa_3 \tilde{t}^{ab},$$ 
where
$$\kappa_0 = 1 + \frac{M \tilde{X}}{r}, \quad \kappa_1 = (1 + \frac{M \ln r}{r})^2 (1 - \frac{M \tilde{X}}{r}) - (1 + \frac{M \tilde{X}}{r}) = 2 \tilde{X} \frac{M \ln r}{r} \chi_1 (\frac{r}{\pi + 1}, \frac{M \ln r}{r}, \frac{M}{r}) - 2 \tilde{X} \frac{M}{r},$$
and
$$\kappa_2 = \left( \frac{M \tilde{X}}{r} \right)^2, \quad i^{ab} = \tilde{X}' (\frac{r}{\pi + 1}) \chi^{ab} (\frac{r}{\pi + 1}, \frac{M \ln r}{r}, \frac{M}{r}, \omega), \quad \text{and} \quad \kappa_3 = \frac{M \ln r}{r},$$
for some smooth functions $\chi_1$ and $\chi^{ab}$. Here $\tilde{m}^{ab}_0$ is the Minkowski metric in the new coordinates, $\tilde{\delta}^{ab} = 1$, when $a, b = 1, 2, 3$ and $\tilde{\delta}^{00} = \tilde{\delta}^{ab} = 0$. For $r > 1/2$ (7.4) also follows from (5.8).

Let $\kappa = 1/\kappa_0$. Lemma (5.8) together with (3.2) gives
$$\left| \left( \mathcal{L}_X^j (\kappa \tilde{m}^{ab}_0 - \tilde{m}^{ab}_0) \right) \tilde{\partial}_a \tilde{\partial}_b \phi \right| \lesssim \frac{M \ln (1 + r)}{t + r} \sum_{|J| \leq 1} |\tilde{\partial} \tilde{Z}^J \phi|. $$
Moreover using (3.2) and (3.6) we get
$$\left| \left( \mathcal{L}_X^j (\kappa \tilde{H}^{ab}_1) \right) \tilde{\partial}_a \tilde{\partial}_b \phi \right| \lesssim \sum_{|J| \leq 1} \left( \left| \frac{\tilde{L}_X^j \tilde{H}^J_1}{t + r} \right| + \left| \frac{\tilde{L}_X^j \tilde{H}^J_1}{t - r} \right| \right) \sum_{|K| \leq 1} |\tilde{\partial} \tilde{Z}^K \phi|. $$
Hence
$$\left| \left( \mathcal{L}_X^j (\kappa \tilde{g}^{ab}_0 - \tilde{m}^{ab}_0) \right) \tilde{\partial}_a \tilde{\partial}_b \phi \right| \lesssim \sum_{|J| \leq 1} \left( \frac{M \ln (1 + r)}{t + r} \right)^2 + \left( \frac{M \ln (1 + r)}{t - r} \right)^2 \sum_{|K| \leq 1} |\tilde{\partial} \tilde{Z}^K \phi|. $$ (7.5)

7.1.2. Higher order commutators with the reduced wave operator. Since $\mathcal{L}_X^j \tilde{m}^{ab}_0 = 0$, it follows that with $\kappa = 1/\kappa_0$
$$\kappa \Box \mathcal{L}_X^j \phi_{cd} = \mathcal{L}_X^j (\kappa \Box \phi_{cd}) - (\mathcal{L}_X^j (\kappa \tilde{g}^{ab}_0 - \tilde{m}^{ab}_0)) \tilde{\partial}_a \tilde{\partial}_b \phi_{cd},$$ (7.6)
By repeated use of (6.22) (see also (7.6)) we have
$$\mathcal{L}_X^j (\kappa \tilde{h}^{ab}_1) = (\kappa \Box \mathcal{L}_X^j \tilde{h}^{ab}_1 + \kappa \tilde{R}_{cd, 1}^* I),$$ (7.7)
where
$$\tilde{R}_{cd, 1}^* I = \sum_{|J + K = I, |K| \leq |J|} \kappa^{-1} \left( \mathcal{L}_X^j (\kappa \tilde{g}^{ab}_0 - \tilde{m}^{ab}_0) \right) \tilde{\partial}_a \tilde{\partial}_b \mathcal{L}_X^j \tilde{h}^{ab}_1.$$
The commutator term above is the main problematic term to deal with. We will use the above expression for the curved wave operator with the energy estimate but for the decay estimates we will use the constant coefficient wave operator $\Box^* = \tilde{m}^{ab}_0 \tilde{\partial}_a \tilde{\partial}_b$. We have
$$\Box^* \mathcal{L}_X^j \tilde{h}^{ab}_1 = \mathcal{L}_X^j (\kappa \tilde{h}^{ab}_1) - \kappa \tilde{R}_{cd, 1}^* I^*,$$ (7.8)
where
$$\tilde{R}_{cd, 1}^* I = \sum_{|J + K = I, |K| \leq |J|} \kappa^{-1} \left( \mathcal{L}_X^j (\kappa \tilde{g}^{ab}_0 - \tilde{m}^{ab}_0) \right) \tilde{\partial}_a \tilde{\partial}_b \mathcal{L}_X^j \tilde{h}^{ab}_1.$$
In conclusion by (7.7) respectively (7.9) using (7.8) we have:

**Lemma 7.1.** We have

\[
\mathcal{L}_X^I (\kappa \Box \tilde{h}_c^1) = \kappa \Box \tilde{\mathcal{L}}_X^I \tilde{h}_c^1 + \kappa \tilde{R}_{cd}^{\text{com}, I},
\]

where

\[
\left| \tilde{R}_{cd}^{\text{com}, I} \right| \lesssim \sum_{|J|+|K| \leq |I|+1, |J| \leq |I|} \left( \frac{M \ln \langle t+r \rangle}{(t+r)^2} + \frac{|(\tilde{\mathcal{L}}_X^I \tilde{H}_1)|}{(t-r^s)} + \frac{|\tilde{\mathcal{L}}_X^I \tilde{H}_1|}{(t+r)} \right)|\partial \tilde{\mathcal{L}}_X^K \tilde{h}_1|.
\]

Moreover

\[
\Box^* \tilde{\mathcal{L}}_X^I \tilde{h}_c^1 = \mathcal{L}_X^I (\kappa \Box \tilde{h}_c^1) - \kappa \tilde{R}_{cd}^{\text{com}, *I},
\]

where

\[
\left| \tilde{R}_{cd}^{\text{com}, *I} \right| \lesssim \sum_{|J|+|K| \leq |I|+1, |J| \leq |I|} \left( \frac{M \ln \langle t+r \rangle}{(t+r)^2} + \frac{|(\tilde{\mathcal{L}}_X^I \tilde{H}_1)|}{(t-r^s)} + \frac{|\tilde{\mathcal{L}}_X^I \tilde{H}_1|}{(t+r)} \right)|\partial \tilde{\mathcal{L}}_X^K \tilde{h}_1|.
\]

### 7.1.3. Higher order commutators with the inhomogeneous term.

Finally we note that we have some additional structure of the nonlinear term:

\[
\tilde{F}_{ab}(\tilde{g})[\partial \tilde{h}, \partial \tilde{k}] = \tilde{F}_{ab}[\tilde{G}, \tilde{G}][\partial \tilde{h}, \partial \tilde{k}]
\]

is a quadratic form in the (0, 3) forms \( \partial \tilde{h} \) and \( \partial \tilde{k} \) with two contractions with \( \kappa \) times the inverse of the metric \( \tilde{G}^{ab} = \tilde{g}^{ab} \), i.e. it is bilinear in \( \tilde{G} \). Hence we have by (6.23):

\[
\mathcal{L}_X (\tilde{F}_{ab}(\tilde{g})[\partial \tilde{h}, \partial \tilde{k}]) = \tilde{F}_{ab}(\tilde{g})[\partial \tilde{\mathcal{L}}_X \tilde{h}, \partial \tilde{k}] + \tilde{F}_{ab}(\tilde{g})[\partial \tilde{h}, \partial \tilde{\mathcal{L}}_X \tilde{k}] + \tilde{F}_{ab}[\mathcal{L}_X \tilde{G}, \tilde{G}][\partial \tilde{h}, \partial \tilde{k}] + \tilde{F}_{ab}[\tilde{G}, \tilde{\mathcal{L}}_X ^G][\partial \tilde{h}, \partial \tilde{k}].
\]

Let \( \tilde{h}_c^1 = \tilde{L}_X^I \). By repeated use of (7.10) we get modulo cubic error terms

\[
\mathcal{L}_X^I (\tilde{\tilde{F}}_{cd}(\tilde{g})[\partial \tilde{h}_c^1, \partial \tilde{h}_c^1]) = \sum_{J+K = I} \tilde{F}_{cd}(\tilde{g})[\partial \tilde{h}_c^{1J}, \partial \tilde{h}_c^{1K}] + \tilde{R}_{cd, 0}^{\text{cube}, I}.
\]

Here with \( \tilde{G}^I = \tilde{L}_X \tilde{G}, \tilde{G}^{ab} = \tilde{g}^{ab} \),

\[
\tilde{R}_{cd, 0}^{\text{cube}, I} = \sum_{J+K+L+M = I, |L|+|M| \geq 1} \tilde{R}_{cd, 0}^{IJKLM}[\tilde{G}^L, \tilde{G}^M][\partial \tilde{h}_c^{1J}, \partial \tilde{h}_c^{1K}].
\]

Hence we have proven

**Lemma 7.2.** We have

\[
\mathcal{L}_X^I (\tilde{\tilde{F}}_{cd}(\tilde{g})[\partial \tilde{h}_c^1, \partial \tilde{h}_c^1]) = \sum_{J+K = I} \tilde{F}_{cd}(\tilde{g})[\partial \tilde{h}_c^{1J}, \partial \tilde{h}_c^{1K}] + \tilde{R}_{cd, 0}^{\text{cube}, I},
\]

where if \( |\tilde{\mathcal{L}}_X \tilde{H}_1| \lesssim 1 \), for \( |J| \leq N/2 \), then for \( |I| \leq N \)

\[
|\tilde{R}_{cd, 0}^{\text{cube}, I}| \lesssim \sum_{|J|+|K|+|L| \leq |I|, |L| \geq 1} \tilde{\mathcal{L}}_X^I \tilde{H}_1| \partial \tilde{h}_c^{1J} | |\partial \tilde{h}_c^{1K}|.
\]

We remark that the cubic errors \( \tilde{R}_{cd}^{\text{cube}, I} \) are much easier to control, see Section 7.2.4.1. The only issue may be when most derivatives fall on one factor of \( \tilde{G} \) and we have to estimate it in \( L^2 \), but we are going to handle worse terms of this form in the commutator errors.
7.1.4. The fine structure of the differentiated inhomogeneous term. Assuming the weak estimate $|\tilde{H}_1| \lesssim 1$ the difference

$$|\kappa^2 \tilde{F}_{ab}(\tilde{G}, \tilde{G})[\tilde{\partial} h, \tilde{\partial} k] - \tilde{F}_{ab}[\kappa \tilde{m}_0, \kappa \tilde{m}_0][\tilde{\partial} h, \tilde{\partial} k]| \lesssim |\tilde{H}_1| |\tilde{\partial} h| |\tilde{\partial} k|,$$

can be estimated by cubic terms and by (7.4)

$$|\tilde{F}_{ab}[\kappa \tilde{m}_0, \kappa \tilde{m}_0][\tilde{\partial} h, \tilde{\partial} k] - \tilde{F}_{ab}[\tilde{\partial} h, \tilde{\partial} k]| \lesssim \frac{M \ln (t+r)}{(t+r)} |\tilde{\partial} h| |\tilde{\partial} k|.$$

Moreover by, (6.16) and (6.17) with $\tilde{P}[\tilde{h}, \tilde{k}] = P(\tilde{m})[\tilde{h}, \tilde{k}]$ we have

$$|\tilde{F}_{ab}(\tilde{m})[\tilde{\partial} h, \tilde{\partial} k] - \tilde{L}_a \tilde{L}_b \tilde{P}(\tilde{\partial}_q \tilde{h}, \tilde{\partial}_q \tilde{k})| \lesssim |\tilde{\partial} h| |\tilde{\partial} k| + |\tilde{\partial} h| |\tilde{\partial} k|.$$

Hence we conclude that

$$|\kappa^2 \tilde{F}_{ab}(\tilde{G})[\tilde{\partial} h, \tilde{\partial} k] - \tilde{L}_a \tilde{L}_b \tilde{P}(\tilde{\partial}_q \tilde{h}, \tilde{\partial}_q \tilde{k})| \lesssim |\tilde{H}_1| |\tilde{\partial} h| |\tilde{\partial} k| + \left( |\tilde{H}_1| + \frac{M \ln (t+r)}{(t+r)} \right) |\tilde{\partial} h| |\tilde{\partial} k|.$$

Hence, also using (6.18) and (6.19) to estimate $\tilde{P}$ we have proven:

**Lemma 7.3.** Suppose that $|\tilde{L}_X^{\tilde{H}_1}| \lesssim 1$, for $|J| \leq N/2$. Then for $|I| \leq N$ with $\tilde{h}^{IJ} = \tilde{L}_X^{\tilde{H}_1}$ we have

$$\kappa^2 \tilde{F}_{cd}(\tilde{G})[\tilde{\partial} h^{IJ}, \tilde{\partial} h^{1K}] = \tilde{L}_a \tilde{L}_b \tilde{P}(\tilde{\partial}_q \tilde{h}^{IJ}, \tilde{\partial}_q \tilde{h}^{1K}) + \kappa^2 \tilde{R}_{cd}^{tan JK} + \kappa^2 \tilde{R}_{cd}^{cube JK},$$

where

$$|\tilde{R}_{cd}^{tan JK}| \lesssim |\tilde{\partial} h^{IJ}| |\tilde{\partial} h^{1K}| + |\tilde{\partial} h^{IJ}| |\tilde{\partial} h^{1K}|,$$

and with $\tilde{P}(\tilde{m})[\tilde{h}, \tilde{k}]$ we have

$$|\tilde{\partial}_q \tilde{h}^{IJ}| |\tilde{\partial}_q \tilde{h}^{1K}| \lesssim (|\tilde{\partial}_q \tilde{h}^{IJ}| \tilde{L}_T + |\tilde{\partial}_q \tilde{h}^{1K}| \tilde{\partial}_T + |\tilde{\partial}_q \tilde{h}^{1K}| \tilde{L}_T + |\tilde{\partial}_q \tilde{h}^{1K}| \tilde{T}_T + |\tilde{\partial}_q \tilde{h}^{1K}| \tilde{T}_T).$$

7.1.5. Higher order vector fields applied to the mass and covariant error terms in Einstein’s equations. We have $E_{cd}^{mass} = E_{cd,0}^{mass} + E_{cd,1}^{mass} [\tilde{H}_1]$ where it follows from (6.5) that

$$|L_X^{\tilde{E}_{cd,0}^{mass}}| \lesssim \frac{M}{(t+r)3} |\tilde{Z}^J \tilde{H}_1|,$$

Moreover $E_{cd}^{cov} = E_{cd,0}^{cov}[\tilde{H}_1] + E_{cd,1}^{cov}[\tilde{\partial} \tilde{H}_1]$ where it follows from (6.11) and (6.12) that

$$|L_X^{\tilde{E}_{cd,0}^{cov}}[\tilde{H}_1]| \lesssim \frac{M}{(t+r)^2} \sum_{|J| \leq |I|} |\tilde{Z}^J \tilde{H}_1|,$$

We have $\tilde{F}_{cd}^{mass} = F_{cd,0}^{mass} + F_{cd,1}^{mass} [\tilde{H}_1]$ where it follows from (6.7) that

$$|L_X^{\tilde{F}_{cd,0}^{mass}}| \lesssim \frac{M^2}{(t+r)^2} |\tilde{Z}^J \tilde{H}_1|,$$

Moreover $\tilde{F}_{cd}^{cov} = F_{cd,0}^{cov}[\tilde{H}_1] + F_{cd,1}^{cov}[\tilde{\partial} \tilde{H}_1]$ where it follows from (6.13) and (6.14) that

$$|L_X^{\tilde{F}_{cd,0}^{cov}}[\tilde{H}_1]| \lesssim \frac{M}{(t+r)^2} \sum_{|J| \leq |I|} |\tilde{Z}^J \tilde{H}_1|.$$
Lemma 7.4. Let \( \bar{R}^\text{mass}_{\cd} = \bar{E}^\text{mass}_{\cd} + \bar{F}^\text{mass}_{\cd} \) and \( \bar{R}^\text{cov}_{\cd} = \bar{E}^\text{cov}_{\cd} + \bar{F}^\text{cov}_{\cd} \) be as in (7.1) and Section 6.2 and let \( \bar{R}^\text{mass}_{\cd} l = \bar{L}_X \bar{R}^\text{mass}_{\cd} \) and \( \bar{R}^\text{cov}_{\cd} l = \bar{L}_X \bar{R}^\text{cov}_{\cd} \). We have

\[
|\bar{R}^\text{mass}_{\cd} l| + |\bar{R}^\text{cov}_{\cd} l| \lesssim \frac{\bar{M} |H| (r < 3t/4)}{(t+r)^3} + \frac{M^2}{(t+r)^3} + \frac{M (\ln (t+r)+1)}{(t+r)^3} \sum_{|J| \leq |I|} |\bar{Z}^J \bar{H}_1| + \frac{M (\ln (t+r)+1)}{(t+r)^2} \sum_{|J| \leq |I|} |\bar{\partial} \bar{Z}^J \bar{H}_1|.
\]

7.2. Higher order commutators with the wave coordinate condition. Let \( \bar{H}^{ac} = \bar{H}_1^{ac} - \frac{1}{2} \bar{m}^{ac} \bar{m}^{bd} \bar{H}_1^{bd} \). By repeated use of (6.21) we have

\[
\bar{\partial} \bar{L}^I_X \bar{H}_1^{cd} = \sum_{|J| \leq |I|} \bar{L}^J_X \bar{\partial} \bar{H}_1^{cd}.
\]

We claim that for any \( \bar{H}^{ab} \)

\[
|\partial q^* \bar{H}_1|_{\bar{L}^I_T} + |\partial q^* \bar{H}_1| \lesssim |\bar{\partial} \bar{H}_1| + |\text{div} \bar{F}_1|, \quad \text{if div} \bar{H}_1 = \bar{\partial}_a \bar{H}^{ab}, \quad \text{and} \quad \bar{H}^{ac} = \bar{H}^{ac} - \frac{1}{2} \bar{m}^{ac} \bar{m}^{bd} \bar{H}_1^{bd},
\]

(7.12)
Theorem 7.5. Let \( \bar{H}_1^I = \bar{L}^I_Z \bar{H}_1 \). We have

\[
|\partial q^* \bar{L}_Z \bar{H}_1|_{\bar{L}^I_T} + |\partial q^* \bar{L}_Z \bar{H}_1| \lesssim |\bar{\partial} \bar{L}_Z \bar{H}_1| + \sum_{|J| \leq |I|} |\bar{L}_Z \bar{H}_1| |\partial \bar{L}_Z \bar{H}_1| + \frac{M |\chi'(\frac{r}{t+r})|}{(t+r)^2} + \frac{M}{(t+r)^3} \sum_{|J| \leq |I|} |\bar{L}_Z \bar{H}_1| + \frac{M}{(t+r)^2} \sum_{|J| \leq |I|} |\bar{\partial} \bar{L}_Z \bar{H}_1|,
\]

(7.14)

where \( \chi'(s) \), is a function supported when \( 1/4 \leq s \leq 1/2 \).

8. The \( L^2 \) estimates for the wave equation

8.1. The energy estimate with asymptotically Schwarzschild coordinates and weights. In the energy estimate on Minkowski space, one can introduce a spacetime integral by multiplying by a weight \( w(q) \), with \( w' > 0 \), since \( \nabla w \) is a future directed null vector field. For the metric \( g \), the positive mass theorem implies \( \| \nabla w \|_g \sim 2Mr^{-1} \), so we instead adapt the weight to the approximate optical function \( q^* \):

\[
w = w(q^*) = \begin{cases} \frac{1}{2} \left( 1 + \frac{1}{2} q^* \right)^{1+2\gamma}, & q^* \geq 0, \\ \left( 1 - \frac{1}{2} q^* \right)^{-2\mu}, & q^* < 0. \end{cases}
\]

(8.1)

Then, \( \| \nabla w \|_g = o(r^{-1}) \) along the light cone. If \( \mu < 1/2 \), we have the basic inequality

\[(1 + 2\gamma)^{-1} (1 + |q^*|) \partial q^* w \leq w \leq 2(2\mu)^{-1} (1 + |q^*|)^{1+2\mu} \partial q^* w.
\]
The energy momentum tensor of the wave equation in the modified coordinates is

\[
\bar{T}_{ab}[\phi] = \bar{\partial}_a \phi \bar{\partial}_b \phi - \frac{1}{2} \bar{g}_{ab} \bar{g}^{cd} \bar{\partial}_c \phi \bar{\partial}_d \phi.
\]
This is consistent with the energy momentum tensor $T[\phi,0]$ for the Maxwell-Klein Gordon equations in modified coordinates. We can define the weighted scalar current

$$J^a_w[\phi] = -\bar{g}^{ab}\tilde{T}_b w[\phi] w = -\left(\bar{g}^{ab}\partial_b\phi \partial_t\phi - \delta^a_0 \bar{g}^{bc}\partial_b\phi \partial_c\phi \right)/2) w,$$

the weighted energy

$$E[\phi](T) = \sup_{t \in [0,T]} \int_{\Sigma_t} |\bar{\partial}\phi|^2 w \, d\bar{x},$$

and the weighted spacetime energy

$$S[\phi](T) = \int_0^T \int_{\Sigma_t} |\tilde{\partial}\phi|^2 w' \, d\bar{x} \, dt,$$

where $|\bar{\partial}\phi|^2 = |\bar{S}_1\phi|^2 + |\bar{S}_2\phi|^2 + |\bar{L}\phi|^2$ is the norm of the derivatives tangential to the outgoing curved light cones $r^* - t = q^*$, and $|\tilde{\partial}\phi|^2 = |\tilde{S}_1\phi|^2 + |\tilde{S}_2\phi|^2$ is the norm of the derivatives tangential to the sphere with constant $r^*$ and $t$.

**Theorem 8.1.** Take $\gamma, \mu > 0$. There exists an $\varepsilon_0 > 0$ (which depends on $\gamma, \mu$) such that, if $g$ satisfies the metric assumption

$$M \leq \varepsilon_0,$$

$$|\tilde{H}_1| + (t - r^*)|\tilde{\partial}\tilde{H}_1| + (t + r^*)|\bar{\partial}\tilde{H}_1| \leq \varepsilon_0(t - r^*)^{1/2 - \mu} + (t + r^*)^{-1/2 - \mu},$$

$$|\tilde{H}_1|_{\bar{L}L} + (t - r^*)|\tilde{\partial}\tilde{H}_1|_{\bar{L}L} + (t + r^*)|\bar{\partial}\tilde{H}_1|_{\bar{L}L} \leq \varepsilon_0(t - r^*)^{1/2 - 2\mu},$$

with

$$|\tilde{\partial}\tilde{H}_1|_{\bar{L}L} = \sum_{\tilde{u}\in\{\bar{L},\bar{L},\tilde{S}_1,\tilde{S}_2\}} \tilde{U}^a \tilde{U}^b \tilde{L}^c \tilde{L}^d \tilde{\partial}_a \tilde{H}_{bc},$$

$$|\bar{\partial}\tilde{H}_1|_{\bar{L}L}^2 = \sum_{\tilde{u}\in\{\bar{L},\tilde{S}_1,\tilde{S}_2\}} \tilde{U}^a \tilde{U}^b \tilde{L}^c \tilde{L}^d \tilde{\partial}_a \tilde{H}_{bc}^2$$

then we have

$$\sup_{t \in [0,T]} \int_{\Sigma_t} |\bar{\partial}\phi|^2 w \, d\bar{x} + \int_0^T \int_{\Sigma_t} |\tilde{\partial}\phi|^2 w' \, d\bar{x} \, dt \leq 8 \int_{\Sigma_0} |\tilde{\partial}\phi|^2 w \, d\bar{x} + 12 \int_0^T \int_{\Sigma_t} |\tilde{\partial}\phi||\tilde{\partial}\phi| w \, d\bar{x} \, dt.$$

**Remark 8.2.** If we had used pointwise bounds on components of $H$ with respect to the null frame in Minkowski space, we would have only been able to prove slowly growing energy, even in the case $\tilde{\partial}\phi = 0$ (cf. [37]).

Before proving this, we state some consequences of the metric assumptions.

**Proposition 8.3.** If we replace $H_1^{ab}$ with $(g - \tilde{m})^{ab}$, then for $\varepsilon_0 < 1$ the metric assumptions (8.2a) and (8.2b) hold up to a constant (replacing $\varepsilon_0$ with $C\varepsilon_0$ on the right hand side). We additionally have the following estimates:

$$|\tilde{\partial}_a \tilde{H}_1^{ab}\tilde{\partial}_b\phi| \leq C\varepsilon_0(t + r^*)^{-1/2 - \mu}|\tilde{\partial}\phi|^2 + (t - r^*)^{-1/2 - \mu}|\tilde{\partial}\phi||\tilde{\partial}\phi|,$$

$$|\tilde{\partial}_a \tilde{H}_1^{ab}\tilde{\partial}_b\phi| \leq C\varepsilon_0(t + r^*)^{-1/2 - 2\mu}|\tilde{\partial}\phi|^2 + (t - r^*)^{-1/2 - \mu}|\tilde{\partial}\phi||\tilde{\partial}\phi|.$$
and further expand $\tilde{m}^{ab}, \tilde{m}^{cd}$ in our null frame, which gives
\[ |\tilde{\partial}_a H_1^{ab} \tilde{\partial}_b \phi| \lesssim \varepsilon \langle t - r^* \rangle^{-2\mu} \langle t + r^* \rangle^{-1 - 2\mu} |\tilde{\partial} \phi| + |\tilde{\partial} H_1| (|\tilde{\partial} \phi| + (|\tilde{\partial} H_1| + |\tilde{\partial} H_1|_{L^1})) |\tilde{\partial} \phi| \]

Our estimate follows.

**Proof of Theorem 8.1** We apply the divergence theorem in the $(\tilde{t}, \tilde{x})$ coordinates to $\tilde{J}_w[\phi]$ on $[0, T] \times \mathbb{R}^3$.

\[
\int_{\Sigma_T} \tilde{J}_w^0[\phi] \, d\tilde{x} - \int_{\Sigma_0} \tilde{J}_w^0[\phi] \, d\tilde{x} = \int_0^T \int_{\Sigma_t} \tilde{\partial}_a (\tilde{J}_w^0[\phi]) \, d\tilde{x} \, d\tilde{t}. \tag{8.4}
\]

Defining
\[
E[\phi](t) = \int_{\Sigma_t} |\tilde{\partial} \phi|^2_w \, d\tilde{x},
\]

A pointwise calculation gives
\[
\int_{\Sigma_t} \left| \frac{1}{2} |\tilde{\partial} \phi|^2 - \tilde{T}_{00} \right| \, d\tilde{x} \lesssim \varepsilon_0 E[\phi](t).
\]

Additionally,
\[
|(-\tilde{g}^{0b} + \tilde{m}^{0b}) \tilde{T}[\phi]_{00} w| \lesssim \varepsilon_0 E[\phi](t).
\]

Therefore, for sufficiently small $\varepsilon_0$, the inequality
\[
\frac{1}{2} E[\phi](t) \leq \int_{\Sigma_t} \tilde{J}_w^0[\phi] \, d\tilde{x} \leq \frac{2}{3} E[\phi](t).
\]

holds for all $t$. Now we estimate the right hand side. We write

\[
-\tilde{\partial}_a (\tilde{g}^{ab} \tilde{T}[\phi]_{00} w) = (-\tilde{\partial}_a \tilde{g}^{ab}) \tilde{\partial}_b \phi \tilde{\partial}_c \phi w + \frac{1}{2} (\tilde{\partial}_a \tilde{g}^{bc}) \tilde{\partial}_b \phi \tilde{\partial}_c \phi w - \tilde{g}^{ab} \tilde{\partial}_a \tilde{\partial}_b \phi \tilde{\partial}_c \phi w - \tilde{g}^{ab} \tilde{T}_{00} \tilde{\partial}_a w
\]

In the far interior, \( r_{t+1} < \frac{3}{2} \), the bounds \([8,2]\) imply

\[
|\tilde{g} - \tilde{m}| \lesssim \varepsilon_0 \langle t \rangle^{-2\mu}, \quad |\tilde{\partial} \tilde{g}| \lesssim \varepsilon_0 \langle t \rangle^{-1 - 2\mu}, \quad |w| \lesssim 1, \quad |\tilde{\partial} w| \lesssim \langle t \rangle^{-1 - 2\mu},
\]

and consequently

\[
|(-\tilde{\partial}_a \tilde{g}^{ab}) \tilde{\partial}_b \phi \tilde{\partial}_c \phi w| + |(\tilde{\partial}_a \tilde{g}^{bc}) \tilde{\partial}_b \phi \tilde{\partial}_c \phi w| + |\tilde{g}^{ab} \tilde{T}_{00} \tilde{\partial}_a w + \frac{1}{2} \tilde{T}_{00} \tilde{\partial} w| \lesssim \varepsilon_0 \langle t \rangle^{-1 - 2\mu} |\tilde{\partial} \phi|^2_w,
\]

follows directly. Outside this region, we use the null decomposition. Lemma \([5,7]\) implies

\[
|\tilde{\partial} (\tilde{g}^{ab} - \tilde{H}_1^{ab})| \lesssim \frac{M \ln(1 + \langle t + r^* \rangle)}{(t + r^*)^2},
\]

and therefore

\[
|\tilde{\partial}_a (\tilde{g}^{ab} - \tilde{H}_1^{ab}) \tilde{\partial}_b \phi \tilde{\partial}_c \phi w| + |\tilde{\partial}_a (\tilde{g}^{bc} - \tilde{H}_1^{bc}) \tilde{\partial}_b \phi \tilde{\partial}_c \phi w| \lesssim \frac{M \ln(1 + \langle t + r^* \rangle)}{(t + r^*)^2} |\tilde{\partial} \phi|^2_w.
\]

The inequalities \([5,3]\) imply

\[
|\tilde{\partial}_a \tilde{H}_1^{ab} \tilde{\partial}_b \phi \tilde{\partial}_c \phi w| \lesssim \varepsilon_0 \langle t + r^* \rangle^{-1 - 2\mu} |\tilde{\partial} \phi|^2_w + \langle t - r^* \rangle^{-1 - 2\mu} |\tilde{\partial} \phi|^2_w,
\]

and therefore, we have the pointwise inequality

\[
|\tilde{\partial}_a \tilde{g}^{ab} \tilde{\partial}_b \phi \tilde{\partial}_c \phi w| + |\tilde{\partial}_a \tilde{g}^{bc} \tilde{\partial}_b \phi \tilde{\partial}_c \phi w| \lesssim \varepsilon_0 \langle t + r^* \rangle^{-1 - 2\mu} |\tilde{\partial} \phi|^2 + |\tilde{\partial} \phi|^2 w'.
\]
Since the integral of $\langle t \rangle^{-1-2\mu}$ is finite in time, this gives the bound
\[
\int_0^T \int_{\Sigma_t} \bar{\partial}_a g^{ab} \bar{\partial}_b \phi \tilde{\partial}_c \phi \, d\sigma^c \, d\tau \lesssim \varepsilon_0 (E[\phi](T) + S[\phi](T)).
\]

When $\tilde{\partial}$ falls on $w$ give the spacetime energy $S[\phi](T)$ plus error terms. We first bound
\[
|\tilde{g}^{ab} T_{0a} \tilde{\partial}_b w + \frac{1}{2} T_{0a} \bar{\omega}^{,a} w| \lesssim \varepsilon_0 \langle t-r^* \rangle^{1-2\mu} \langle t+r^* \rangle^{-1/2-\mu} |\tilde{\partial} \phi| |\tilde{\partial} \phi| + \langle t-r^* \rangle \langle t+r^* \rangle^{-1-2\mu} |\tilde{\partial} \phi|^2 w',
\]
which follows from the null decomposition and the first statement of Proposition 8.3. The inequality $\langle t-r^* \rangle w' \lesssim w$ and Hölder’s inequality allow us to bound the integral of the right hand side by a constant times $\varepsilon_0 (E[\phi](T) + S[\phi](T))$. Additionally,
\[
|\langle \tilde{T}_{L0} - \frac{1}{2} \tilde{\partial} \phi \tilde{\partial} \phi \rangle \rangle | \lesssim \varepsilon_0 \langle (t-r^*) (t+r^*)^{-1-2\mu} |\tilde{\partial} \phi|^2 + \varepsilon_0 (t-r^*)^{1-2\mu} (t+r^*)^{-1-2\mu} |\tilde{\partial} \phi| |\tilde{\partial} \phi| w'.
\]

We can bound this in the same way. Therefore,
\[
\int_0^T \int_{\Sigma_t} |\tilde{g}^{ab} T_{0a} \tilde{\partial}_b w + \frac{1}{4} |\tilde{\partial} \phi|^2 \bar{\omega}^{,a} w| \lesssim \varepsilon_0 (E[\phi](T) + S[\phi](T)).
\]

We recall that
\[
S[\phi](T) = - \int_0^T \int_{\Sigma_t} \frac{1}{4} |\tilde{\partial} \phi|^2 \bar{\omega}^{,a} w.
\]

Then,
\[
\int_0^T \int_{\Sigma_t} |\bar{\partial}_a (\tilde{J}_w^a[\phi]) - \bar{\partial} \phi \partial_t \phi - \frac{1}{4} |\tilde{\partial} \phi|^2 \bar{\omega}^{,a} w| \, d\sigma^c \, d\tau \lesssim \varepsilon_0 (E[\phi](T) + S[\phi](T)).
\]

To close the proof, we first rewrite (8.4) as
\[
\int_{\Sigma_T} \tilde{J}_w^0[\phi] \, d\sigma = \int_0^T \int_{\Sigma_t} (\bar{\partial}_a (\tilde{J}_w^a[\phi]) + \bar{\partial} \phi \partial_t \phi - \frac{1}{4} |\tilde{\partial} \phi|^2 \bar{\omega}^{,a} w) - \bar{\partial} \phi \partial_t \phi + \frac{1}{4} |\tilde{\partial} \phi|^2 \bar{\omega}^{,a} w \, d\sigma^c \, d\tau
\]

Therefore, there exists a $C$ depending on $\gamma, \mu$ such that
\[
\frac{1}{2} E[\phi](T) + \frac{1}{2} S[\phi](T) \leq \frac{1}{2} E[\phi](0) + C \varepsilon_0 (E[\phi](T) + S[\phi](T)) + \int_0^T \int_{\Sigma_t} |\bar{\partial}_t \phi| |\bar{\partial} \phi| w \, d\sigma \, d\tau
\]

We repeat this estimate with $T$ replaced with $T' \in [0, T]$ where $E[\phi]$ attains its maximum; i.e., $E[\phi](T') = E[\phi](T)$, and add the two estimates to get:
\[
\frac{1}{2} E[\phi](T) + \frac{1}{2} S[\phi](T) \leq \frac{1}{2} E[\phi](0) + 2C \varepsilon_0 (E[\phi](T) + S[\phi](T)) + 2 \int_0^T \int_{\Sigma_t} |\bar{\partial}_t \phi| |\bar{\partial} \phi| w \, d\sigma \, d\tau.
\]

Multiplying this by 6 and assuming $2C \varepsilon_0 < 1$ gives
\[
E[\phi](T) + S[\phi](T) \leq 8E[\phi](0) + 12 \int_0^T \int_{\Sigma_t} |\bar{\partial}_t \phi| |\bar{\partial} \phi| w \, d\sigma \, d\tau. \quad \square
\]

### 8.2. Poincaré lemmas with weights.

We restate results from [37], which will be particularly useful in bounding $L^2$ norms of Lie derivatives of $H$.

**Lemma 8.4.** Let $0 \leq a \leq 2$, $\mu > -1/2$, and $\gamma > 0$, and let $\phi$ be a differentiable function on $[0, \infty)$ such that $r^\gamma \phi$ vanishes at $\infty$. Then, for all $t \geq 0$, we have the inequality
\[
\int_0^\infty \frac{|\phi|^2 w_{\gamma, \mu}(r^* - t)}{(1 + |r^* - t|)^2 (1 + t + r^*)^a} r^* \, dr^* \leq C_{\gamma, \mu} \int_0^\infty \frac{|\partial_r \phi|^2 w_{\gamma, \mu}(r^* - t)}{(1 + t + r^*)^a} r^* \, dr^*,
\]

where
\[
w_{\gamma, \mu}(y) = \begin{cases} (1 + |y|)^{1+2\gamma}, & y \geq 0, \\ (1 + |y|)^{-2\mu}, & y < 0. \end{cases}
\]
Proof. Let \( f, g, \) and \( \phi \) be functions of \( r^* \). By a density argument, we can assume \( \phi \) is smooth and compactly supported in \([0, \infty)\). Then,
\[
\int_0^\infty (C f \partial_r \phi + g \phi)^2 \, dr^* - \int_0^\infty \partial_r (C f g \phi^2) \, dr^* \geq 0.
\]
as long as \( f g \phi^2 \) vanishes at 0 and \( \infty \). Therefore,
\[
\int_0^\infty C^2 |f \partial_r \phi|^2 \, dr^* \geq \int_0^\infty \left[ C \partial_r (fg) - g^2 \right] |\phi|^2 \, dr^*.
\]
Now set
\[
f(r^*) = r^{*1/2}(1 + t + r^*)^{-a/2},
g(r^*) = r^*(1 + |q^*|)^{-1} w_{\gamma,\mu}^{-1/2}(1 + t + r^*)^{-a/2}.
\]
Then,
\[
\partial_r (fg) = \left( \frac{2}{r^*} - \frac{a}{1 + t + r^*} - \frac{\text{sgn} q^*}{(1 + |q^*|)} + \frac{w_{\gamma,\mu}'(q^*)}{w_{\gamma,\mu}(q^*)} \right) f g.
\]
Since \( a < 2 < \frac{2}{r^*} - \frac{a}{1 + t + r^*} > 0 \), and
\[
- \frac{\text{sgn} q^*}{(1 + |q^*|)} + \frac{w_{\gamma,\mu}'(q^*)}{w_{\gamma,\mu}(q^*)} = \begin{cases} \frac{1+2\mu}{(1+|q^*|)}, & q^* < 0, \\ \frac{2\gamma}{(1+|q^*|)}, & q^* > 0. \end{cases}
\]
Setting \( C = 2 \max((2\gamma)^{-1}, (1 + 2\mu)^{-1}) \) gives our result. \( \square \)

This lemma has two immediate consequences, which will be useful in the fixed-time and spacetime energies.

**Corollary 8.5.** Recalling the definition \( (8.1) \), for fixed \( t > 0, -1 \leq b \leq 1, \) and \( \phi \in C^\infty_0(\mathbb{R}^3) \), the following estimate holds:
\[
\int_{\mathbb{R}^3} \frac{|\phi|^2}{(1 + |q^*|^2)(1 + t + r^*)^{1-b}} w \, d\bar{x} \leq C_{\gamma,\mu} \int_{\mathbb{R}^3} \frac{|\partial_r \phi|^2}{(1 + t + r^*)^{1-b}} w \, d\bar{x}. \tag{8.5}
\]
If additionally \( b < 2\gamma, \mu > 0, \) and \( q^*_- = \max(-q^*, 0) \), we additionally have the estimate
\[
\int_{\mathbb{R}^3} \frac{|\phi|^2(1 + |q^*|)^{-b}}{(1 + |q^*|^2)(1 + t + r^*)^{1-b}} \frac{w}{(1 + q^*_-)^{2\mu}} \, d\bar{x} \leq C_{b,\gamma,\mu} \int_{\mathbb{R}^3} |\partial_r \phi|^2 w' \, d\bar{x}. \tag{8.6}
\]

**Proof.** The inequality \( (8.5) \) follows directly from Lemma \( 8.4 \) along with the approximation \( w \leq w_{\gamma,\mu} + w_{\gamma,0} \leq 2w \), and \( (8.6) \) follows from Lemma \( 8.4 \) using \( w_{\gamma,\mu}^{-b/2,\mu+b/2} \) and the pointwise inequality
\[
\frac{(1 + |q^*|)^{-b}}{(1 + |t + r^*|^{1-b})} \frac{w}{(1 + q^*_-)^{2\mu}} \leq w'.
\]

**9. The Decay estimates for the wave equation**

We consider energy norms with the following weight function
\[
w_{p,\gamma}(t, x) = \langle q^* \rangle^{1-1/p-\gamma}, \quad 0 < \gamma < 1.
\]
In this section we will work with the flat wave operator in the curved coordinates
\[
\Box^* = \tilde{\nabla}^a \tilde{\nabla}_a \tilde{\nabla}_b = \tilde{\Box},
\]
taking advantage of already established formulas using the fundamental solution in flat coordinates.
9.1. Weighted Klainerman-Sobolev estimates. In this section we provide a straightforward generalization of the Klainerman-Sobolev inequalities, expressing pointwise decay in terms of the bounds on $L^2$ norms involving vector field $Z \in \mathcal{Z}$.

We have the following global Sobolev inequality, see [37]

**Proposition 9.1.** For any function $\phi \in C_0^\infty(\mathbb{R}^3)$ and an arbitrary $(t, x)$,

$$|\phi(t, x)|(1 + t + |q^*|)(1 + |q^*|)^{1/2}w_{2, \gamma}(t, x) \leq C\sum_{|l| \leq 3} |w_{2, \gamma}Z^l \phi(t, \cdot)|_{L^2}.$$

9.2. The weighted $L^1$-$L^\infty$ estimates. To get improve decay estimates in the interior we will use Hörmander’s $L^1$--$L^\infty$ estimates for the fundamental solution of $\Box$, see [19, 31]:

**Proposition 9.2.** Suppose that $w$ is a solution of $\Box^* u = F$ (i.e. the flat Minkowski wave operator) with vanishing data $u|_{t=0} = \partial_t u|_{t=0} = 0$. Then

$$|u(t, x)|(1 + t + |x|)w_{1, \gamma}(t, x) \leq C\sum_{|l| \leq 2} \int_0^t \int_{\mathbb{R}^3} \frac{|Z^l F(s, y)|}{1 + s + |y|} w_{1, \gamma}(s, y) dy \, ds.$$

In [31] the proof was without the weight, but by a domain of dependency argument the weight is larger in the support of the inhomogeneous term.

Also for the linear homogenous solution we have from [31]:

**Lemma 9.3.** If $v$ is the solution of $\Box^* v = 0$, with data $v|_{t=0} = v_0$ and $\partial_t v|_{t=0} = v_1$ then for any $\gamma > 0$;

$$(1 + t + r)v(t, x)w_{1, \gamma}(t, x) \leq C\sup_x ((1 + |x|)^{2+\gamma}|v_1(x)| + |\partial v_0(x)| + (1 + |x|)^{1+\gamma}|v_0(x)|).$$

**Proof.** The proof is an immediate consequence of Kirchoff’s formula

$$v(t, x) = t\int_{|\omega|=1} (v_1(x + t\omega) + \langle v_0(x + t\omega), \omega \rangle) \, dS(\omega) + \int_{|\omega|=1} v_0(x + t\omega) \, dS(\omega),$$

where $dS(\omega)$ is the normalized surface measure on $S^2$. Suppose that $x = re_1$, where $e_1 = (1, 0, 0)$. Then for $k = 1, 2$ we must estimate

$$\int_{1 + re_1 + t\omega}^{\tau} \frac{C d\omega}{1 + ((r-t\omega_1)^2 + t^2(1-\omega_2^2))^{(k+\gamma)/2}} \leq \int_0^2 \frac{C ds}{1 + ((r-t+s)^2 + t^2 s)^{(k+\gamma)/2}}.$$ 

If $k = 2$ we make the change of variables $t s = \tau$ to get an integral bounded by $C\tau^{-2} (r^* - r^*)^{-\gamma}$ and if $k = 1$, we make the change of variables $r s = \tau$ to get an integral bounded by $t^{-1} (r^* - r^*)^{-\gamma}$. This proves the result for $r < 2t$, say, but for $r > 2t$ it follows by inspection.

9.3. The weighted $L^\infty$-$L^\infty$ estimates. We will now derive sharp estimates for the first order derivatives, following [31] we have

**Lemma 9.4.** Let $D_t = \{(t, x); |t - |x|| \leq c_0 t\}$, for some constant $0 < c_0 < 1$ and let $\overline{w}(q^*)$ be any positive continuous function, where $q^* = r^* - t, r^* = r + M \ln r$. Suppose that $\Box^* \phi_{\mu\nu} = F_{\mu\nu}$. Let $U, V \in \{\widetilde{L}, \widetilde{L}, S_1, S_2\}$ and $\phi_{UV} = \phi_{\mu\nu} U^\mu V^\nu$. Then

$$(1 + t + |x|)|\partial_{\phi_{UV}}(t, x)\overline{w}(q^*)| \lesssim \sup_{|q^*|/4 \leq \tau \leq t} \sum_{|l| \leq 1} \|Z^l \phi(\tau, \cdot)\overline{w}\|_{L^\infty}$$

$$+ \int_{|q^*|/4}^t (1 + \tau)\|F_{UV}(\tau, \cdot)\overline{w}\|_{L^\infty(D_r)} + \sum_{|l| \leq 2} (1 + \tau)^{-1}\|Z^l \phi(\tau, \cdot)\overline{w}\|_{L^\infty(D_r)} \, d\tau.$$

**Proof.** Since $\Box \phi = -r^{-1}(\partial_t^2 - \partial_r^2)(r^2 \phi) + r^{-2} \Delta \phi$, where $\Delta \phi = \sum \Omega_{ij}^2$ and $|ZU| \leq C$, for $U \in \{A, B, L, \overline{L}\}$, it follows that

$$|\Box^*(\phi_{UV}) - U^\nu V^\mu \Box^* \phi_{\mu\nu}| \leq r^{-2} \sum_{|l| \leq 1} |Z^l \phi|.$$
We have
\[ \Box^* \phi = \frac{1}{r^*} 4 \partial_{s^*} \partial_{q^*} (r^* \phi) + \frac{1}{(r^*)^2} \Delta \omega \phi, \]
where \( \partial_{q^*} = (\partial_r - \partial_t)/2 \) and \( \partial_{s^*} = (\partial_r + \partial_t)/2 \). Hence
\[ |4 \partial_{s^*} \partial_{q^*} (r^* \phi) - r^* \Box^* \phi| \lesssim r^{-1} \sum_{|J| \leq 2} |Z^J \phi|, \]
so with \( s = t + r^* \)
\[ |\partial_{s^*} \partial_{q^*} (r^* \phi_{UV})| \lesssim r |(\Box^* \phi)_{UV}| + (t + r)^{-1} \sum_{|J| \leq 2} |Z^J \phi|, \quad |t - r^*| \leq c_0 t. \]
Integrating this along the flow lines of the vector field \( \partial_{s^*} \) from the boundary of \( D = \cup_{\tau \geq 0} D_\tau \) to any point inside \( D \), using that \( \bar{w} \) is essentially constant along the flow lines, gives that for any \( (t, x) \in D \)
\[ |\partial_q (r \phi_{UV} (t, x)) \bar{w}| \lesssim \sup_{0 \leq \tau \leq t} \sum_{|I| \leq 1} \| Z^I \phi (\tau, \cdot) \bar{w} \| 
+ \int_0^t \left( (1 + \tau) \| F_{UV} (\tau, \cdot) \bar{w} \|_{L^\infty (D_\tau)} + \sum_{|I| \leq 2} (1 + \tau)^{-1} \| Z^I \phi (\tau, \cdot) \bar{w} \|_{L^\infty (D_\tau)} \right) d\tau. \]
The lemma follows from also using (3.2)-(3.3) and that the estimate is trivially true when \( |r - t| \geq c_0 t. \)

10. Energy bounds and decay estimates for Maxwell-Klein-Gordon

We restate Theorem 1.1 from [23], along with results from Theorem 7.2 in that paper, which will form a portion of our bootstrap argument. This takes the form of a set of energy and decay estimates which will follow from the harmonic coordinate condition and the bootstrap assumption on the metric. In order to properly set up the argument, we must first define certain quantities arising from the MKG system which appear in decay rates and in the initial conditions. Next, we prove a result which expands on the identification \( h^1 \sim H_1 \), and which allows us to simplify the required metric bounds. Finally, we restate the main theorem and consequent decay estimates. In Section 12 we will confirm that the required bounds on the energy momentum tensor follow from the bootstrap assumption.

10.1. Setting up the initial value problem. We recall from Section 2.5.2 that in order to best capture the decay of \( F \) we decompose it into \( F^0 + F^1 \) where \( F^0 \) is a fixed field depending on the charge \( q \) which picks up the symmetric part of the decay in the exterior, and \( F^1 \) is the remainder. More precisely, we define
\[ q(t) = \int_{\Sigma_t} -\sqrt{|g|} J^0 dx, \]
where the volume form is taken with respect to \( m \). Since \( J \) is divergence free with respect to \( \nabla \), as long as \( \phi \) has good decay in space, we can take \( q \) to be constant in time. Then, for a smooth increasing function \( \chi_F (x) \) which is identically \( 0 \) for \( x \leq 0 \) and \( 1 \) for \( x \geq 1 \), we define the 2-forms \( F^0, F^1 \) by
\[ F^0_{0i} = \frac{\omega_i q \chi_F (r^* - t - 2) \partial_t (r^*)}{4\pi} \frac{1}{r^{s^2}}, \quad F^0_{ij} = F^0_{00} = 0, \quad F^1 = F - F^0. \]
Although the \( F^0, F^1 \) decomposition is useful in our calculations, when writing the initial conditions there is another useful decomposition. We may decompose \( F \) into its electric and magnetic field components, respectively defined by
\[ E_i = F_{0i}, \quad B_i = \varepsilon_{ijk} F^{jk}. \]
This decomposition is of course not preserved after Lie differentiation with respect to the Lorentz boosts. However, it has the nice property that the poor decay of \( F \) in space can be localized to
the curl-free part of $E$. If we define $E^{df}$ to be the divergence-free part of $E$ (in the Helmholtz decomposition), then $E^{df}_i \sim F_{0i}^1$.

For a $(0,k)$ tensor $T$ and a complex function $\phi$, we define the initial data norms
\[
\|T\|_{H^{N,s_0}}^2 = \sum_{|\alpha| \leq N} \sum_{\alpha \in \{1,2,3\}} \int_{\mathbb{R}^3} (1 + r^2)^{s_0 + |\alpha|} |\tilde{\partial}^\alpha T(\tilde{\partial}_{a_1}, \ldots, \tilde{\partial}_{a_k})|^2 \, dx,
\]
\[
\|\phi\|_{H^{N,s_0}}^2 = \sum_{|\alpha| \leq N} \int_{\mathbb{R}^3} (1 + r^2)^{s_0 + |\alpha|} |\tilde{\partial}^\alpha \phi|^2 \, dx,
\]
where $\tilde{\partial}$ are spatial derivatives in modified coordinates, and $\alpha$ is a multiindex.

**10.2. The metric bounds.** When dealing with derivatives of the energy momentum tensor, we must deal with terms containing both the metric and the inverse metric, so it is useful to establish similar energy and decay estimates for $h^1$ and $H_1$.

**Proposition 10.1.** Let $g_{\alpha \beta} = m_{\alpha \beta} + h_{\alpha \beta}^0 + h_{\alpha \beta}^1$, and let $g^{\alpha \beta} = m^{\alpha \beta} + H_0^{\alpha \beta} + H_1^{\alpha \beta}$, as in Section 2.3. Suppose further that $|Z^I(h_{\alpha \beta}^0)| + |Z^I(h_{\alpha \beta}^1)| \leq \varepsilon (t + r)^{-1+\delta}$ for some collection of vector fields $Z$, a multiindex $I$ with $|I| \leq N$, and a small constant $\delta \in (0,1/2)$. Then, for vector fields $X,Y$ with bounded components, and for sufficiently small $\varepsilon$,
\[
|X^\alpha Y^\beta Z^I(h_{\alpha \beta})| \leq |X^\alpha Y^\beta Z^I(h_{\alpha \beta}^1)| + C\varepsilon^2 (t + r)^{-2+2\delta}.
\]

**Proof.** We first apply the fields $Z$ to the identity
\[
g^{\alpha \beta} = g^{\alpha \gamma} g_{\gamma \delta} g^{\delta \beta},
\]
which, via induction, gives the preliminary estimate
\[
|Z^I(h_{\alpha \beta})| \leq 2\varepsilon (t + r)^{-1+\delta},
\]
for sufficiently small $\varepsilon$. We can now apply $Z^I$ to the identity
\[
g_{\alpha \beta} = g_{\alpha \gamma} g^{\gamma \delta} g_{\delta \beta},
\]
which gives
\[
|Z^I(H_{\alpha \beta}) + Z^I(h_{\alpha \beta})| \leq C\varepsilon^2 (t + r)^{-2+2\delta}.
\]
Taking the identity $H_{\alpha \beta} = -h_{\alpha \beta}^0$ and contracting with the fields $X,Y$ gives our result. \qed

A similar energy result holds, which follows from a pointwise calculation, again taking derivatives of the identity $g_{\alpha \beta} = g_{\alpha \gamma} g^{\gamma \delta} g_{\delta \beta}$, then taking an $L^\infty$ bound on $H$ or $h$ when applicable:

**Proposition 10.2.** Let $\sigma$ be a positive function, and take $N$ such that $|Z^I(h_{\alpha \beta}^0)| + |Z^I(h_{\alpha \beta}^1)| \leq \varepsilon (t + r)^{-1+\delta'}$ for $|I| \leq N/2$. Then, for a given region $\Omega$ with volume element $dV$,
\[
\int_{\Omega} |Z^I(H_{\alpha \beta})|^2 \sigma dV \leq \int_{\Omega} |Z^I(h_{\alpha \beta}^1)|^2 \sigma dV + C\varepsilon \sum_{|J| \leq |I|-1} \int_{\Omega} (t + r)^{-2+2\delta} (|Z^J h|^2 + |Z^J H|^2) \sigma dV.
\]

**10.3. Energy bounds and decay estimates for Maxwell-Klein-Gordon on a fixed background.** We can now state the relevant result from [23], using the initial norms and decompositions defined in Section 10.1.

**Theorem 10.3.** Let $(M,g)$ be a asymptotically flat Lorentzian manifold which admits a global system of coordinates $(t,x_1,x_2,x_3)$ (not necessarily satisfying Einstein’s equations), which are harmonic with respect to $g$ and satisfy the initial splitting condition $g_{0i} = 0$. Define the consequent quantities $M, \chi, h^0, h^1, H_0, H_1$ as in Section 2.3. Additionally, take real constants $s, s_0, \gamma, \mu, \delta$ satisfying $\mu < 1/2, \gamma > 1/2, \frac{1}{2} < s < 1 < s_0 < \frac{3}{2}$, and $4\delta < \min(1-2\mu, s-\frac{1}{2}, 1-s, \gamma-\frac{1}{2}, 1-\gamma, 1+\gamma-2s)$,
and take an integer $N \geq 11$. For $0 < T \leq \infty$, define the intervals $I_T = [0, T]$. There exists a constant $\varepsilon_g > 0$ depending only on $(s, s_0, \gamma, \mu, \delta, N)$ such that, for all $T$, if $M, h^1$ satisfies the decay estimates

\begin{align}
M &< \varepsilon_g, \quad \text{(10.1a)} \\
\sup_{t \in I_T} \| \mathcal{L}_X^I h^1 \|_t &< \varepsilon_g \langle t + r \rangle^{-1+\delta}, \quad \text{(10.1b)} \\
\sup_{t \in I_T} \| \mathcal{L}_X^I h^1 \|_t &< \varepsilon_g \left( \frac{t}{t+r} \right)^\gamma \langle t + r \rangle^{-1+\delta}, \quad \text{(10.1c)}
\end{align}

for $|I| \leq N - 6, t \in I_T$, and the energy estimates

\begin{align}
\left\| \left| \mathcal{L}_X^I h^1 \right| w^1_\gamma \right\|_{L^2(\mathbb{R}^3)} + \left\| \langle r^*-t \rangle^{-1} \left| \mathcal{L}_X^I h^1 \right| w^1_\gamma \right\|_{L^2(\mathbb{R}^3)} &\leq \varepsilon_g (1+t)^{\delta/2}, \quad \text{(10.2a)} \\
\left\| \left( \langle \mathcal{L}_X^I h^1 \|_\mathcal{L} \right) \left| w^1_\gamma \right| \right\|_{L^2(I_T \times \mathbb{R}^3)} &\leq \varepsilon_g (1+t)^{\delta/2}, \quad \text{(10.2b)} \\
\left\| \langle r^*-t \rangle^{-1} \langle r^*+t \rangle^{-1+s} \left| \mathcal{L}_X^I h^1 \right| \mathcal{L}(w^1_\gamma) \right\|_{L^2(\mathbb{R}^3)} &\leq \varepsilon_g, \quad \text{(10.2c)}
\end{align}

for $|I| \leq N$ and $t \in I_T$, and

\[
w_\gamma = \begin{cases} 
1 + \frac{1 + t - t^*}{2} & r^* \leq t, \\
1 + \frac{1 + t - t^*}{1+2\gamma} & r^* \geq t,
\end{cases}
\]

then the system \((1.17)\) is well-posed for small initial data. Specifically, there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and for all one-forms $E_0, B_0$ and scalar functions $\phi_0, \dot{\phi}_0$ on $\mathbb{R}^3$ satisfying

\[
\|E_0\|_{H^{k,s_0}} + \|B_0\|_{H^{k,s_0}} + \|D\phi_0\|_{H^{k,s_0}} + \|\dot{\phi}_0\|_{H^{k,s_0}} < \varepsilon_0,
\]

\[
(10.3)
\]

for a given $A$, there exists a solution to the system \((1.17)\) on $(t, x) \in I_T \times \mathbb{R}^3$, with

\[
\begin{align*}
F_0(0, x) &= (E_0)_i(x), \quad \epsilon_{ijk} F^{jk}(0, x) = (B_0)_i, \\
\phi(0, x) &= \phi_0(x), \quad \dot{\phi}(0, x) = \dot{\phi}_0.
\end{align*}
\]

All constants, including $\varepsilon_0$ and $\varepsilon_g$, can be taken independently of $T$. We also have decay estimates, which we adapt here from Theorems 5.6 and 6.6 of \cite{23}, as well as energy estimates, which we adapt from Theorems 3.1 and 4.2. For $|I| \leq N - 4$, we have

\begin{align}
|D_L^I D_X^I \phi| + |\alpha| [L_X^I F^1] | &\leq C \varepsilon \langle t + r^* \rangle^{-2} \langle (r^* - t)^s \rangle^{s-s_{-s_0}}, \quad \text{(10.4a)} \\
|D_L^I D_X^I \phi| + |\rho| [L_X^I F^1] | + |\sigma| [L_X^I F^1] | &\leq C \varepsilon \langle t + r^* \rangle^{-1-s} \langle (r^* - t)^s \rangle^{s-s_{-s_0}}, \quad \text{(10.4b)} \\
\langle r^* - t \rangle^{-1} |D_L^I D_X^I \phi| + |D_L^I D_X^I \phi| + |\alpha| [L_X^I F^1] | &\leq C \varepsilon \langle t + r^* \rangle^{-1-s} \langle (r^* - t)^s \rangle^{s-s_{-s_0}}. \quad \text{(10.4c)}
\end{align}

By the estimate (2.49) in \cite{23}, $F^0$ is supported in $r^* \geq t + 2$ and satisfies the following bounds for $|I| \leq N$:

\begin{align}
|\alpha| [L_X^I F^0] | &\leq C \varepsilon \langle t + r^* \rangle^{-3} \langle (r^* - t)^s \rangle, \quad \text{(10.5a)} \\
|\rho| [L_X^I F^0] | + |\sigma| [L_X^I F^0] | + |\alpha| [L_X^I F^0] | &\leq C \varepsilon \langle t + r^* \rangle^{-2}. \quad \text{(10.5b)}
\end{align}
Furthermore, the following energy bounds hold for $|I| \leq N$ and $t \in I_T$:
\[
\int_{\Sigma_t} (t+r^*)^{2s} (|[\mathcal{L}_X^I F]|^2 + |\sigma [\mathcal{L}_X^I F]|^2 + |\alpha [\mathcal{L}_X^I F]|^2 + (t-r^*)^{2s} |\alpha [\mathcal{L}_X^I F]|^2) \langle (t-r^*)_- \rangle^{2s-2} \leq C\varepsilon^2,
\]

\[
\int_{\Sigma_t} (t+r^*)^{2s} (|D_L D_X^I F|^2 + |\partial D_X^I F|^2 + |\phi|^2) \langle (t-r^*)_- \rangle^{2s-2} \leq C\varepsilon^2.
\]

Remark 10.4. The version of this theorem which appears in [23] requires the estimates (10.1) and (10.2) for both $h$ and $H$; however, we can reduce this using Propositions 10.1 and 10.2. Additionally, it follows from direct computation and Corollary 5.3 that the estimates (10.1, 10.2), are equivalent to the corresponding estimates with $h, H$ replaced by $\tilde{h}, \tilde{H}$, as Lie derivatives are unaffected by the change in coordinates.

Remark 10.5. In the application to the full system, we will require $s_0 = \gamma + \frac{1}{2}$ to close the argument. However, afterwards, we may repeat this estimate with higher $s_0$ to show better decay estimates for $\{\phi, F\}$ in the far exterior. This is a consequence of the fact that terms that are quadratic in $\phi, F$ appear by themselves in the right hand side of the equation for $h^1$, but in the MKG system $h^1$ only appears coupled with $\phi$ and $F$ terms.

Since $[D_a, D_b] \phi = iF_{ab} \phi$, we must deal with commutator terms which arise even in the Minkowski background.

Corollary 10.6. For $\tilde{U} \in \{\tilde{L}, \tilde{L}, \tilde{S}_1, \tilde{S}_2\}$, the decay bounds for $\phi$ in the estimates (10.3) hold up to a constant if $D_{\tilde{U}} D_X^I \phi$ is replaced with $\tilde{U}^a \tilde{X}^b (\mathcal{L}_X^I F)_{ab} D_X^I \phi$, with $|I_1| + |I_2| + 1 \leq |I|$. Additionally,

\[
\sup_{t \in I_T} \int_{\Sigma_t} (t-r^*)^{2s} |\tilde{U}^a \tilde{X}^b (\mathcal{L}_X^I F)_{ab} D_X^I \phi|^2 \langle (t-r^*)_- \rangle^{2s-2} \leq C\varepsilon^2.
\]

Proof. We first take the following bounds for a 2-form $G$, which follow from the null decompositions of the fields $\tilde{X}$:

\[
|G(\tilde{L}, \tilde{X})| \lesssim (t+r^*)|\alpha[G]| + (t-r^*)|\rho[G]| + |G|,
\]

\[
|G(\tilde{L}, \tilde{X})| \lesssim (t+r^*)|\sigma[G]|,
\]

\[
|G(\tilde{S}_i, \tilde{X})| \lesssim (t+r^*) (|\alpha[G]| + |\sigma[G]|) + (t-r^*)|\sigma[G]|.
\]

The decay bound then follows from (10.4) and (10.5). For the energy bound, we decompose $F = F^0 + F^1$. For the terms containing $F^0$, our result follows from (10.5), and (10.6). For terms containing derivatives of $F^1$ and $\phi$, for $|I| \geq 6$, we can apply (10.4) to $F^1$ or $\phi$, and bound the remainder with (10.6).

10.4. Energy and Decay bounds for the energy momentum tensor. We can use the decay estimates stated in the previous section to prove the sharp decay of the energy momentum tensor.

Corollary 10.7. If $H$ satisfies the estimates in (8.2) and $F, \phi$ satisfy (10.4), (10.5), then the following estimates hold for $|I| \leq k - 6$, $\mathcal{T} = \{\tilde{L}, \tilde{S}_1, \tilde{S}_2\}, \mathcal{U} = \{\tilde{L}, \tilde{L}, \tilde{S}_1, \tilde{S}_2\}$:

\[
|\mathcal{L}_X^I \mathcal{T}|_{\mathcal{U} \mathcal{V}} \lesssim \varepsilon^2 (t+r^*)^{-2} (t-r^*)^{-2} \langle (t-r^*)_+ \rangle^{1-2s},
\]

\[
|\mathcal{L}_X^I \mathcal{T}|_{\mathcal{T} \mathcal{U}} \lesssim \varepsilon^2 (t+r^*)^{-2-s} (t-r^*)^{-2+s} \langle (t-r^*)_+ \rangle^{1-2s}.
\]

Remark 10.8. The sharp decay rate in the exterior comes from $F^0$, which cannot be expected to decay faster than $r^{-2}$. 

Proof. We expand
\[
\hat{T}[\phi, F]_{ab} = \Re(D_a \phi D_b \phi) + g^{cd} F_{ac} F_{bd} - \tilde{g}_{ab} g^{ce} g^{df} F_{cd} F_{df}/4.
\]
We decompose \(\hat{T}[\phi, F] = \hat{T}_{\hat{m}}[\phi, F] + \hat{T}_{\hat{h}}[\phi, F]\) such that
\[
\begin{align*}
\hat{T}_{\hat{m}}[\phi, F]_{ab} &= \Re(D_a \phi D_b \phi) + \hat{m}^{cd} F_{ac} F_{bd} - \tilde{m}_{ab} \tilde{m}^{ce} \tilde{m}^{df} F_{cd} F_{df}/4, \\
\hat{T}_{\hat{h}}[\phi, F]_{ab} &= (g^{cd} - \hat{m}^{cd}) F_{ac} F_{bd} - (\tilde{g}_{ab} g^{ce} g^{df} - \tilde{m}_{ab} \tilde{m}^{ce} \tilde{m}^{df}) F_{cd} F_{df}/4.
\end{align*}
\]
As with the decomposition \(F\) in the Einstein portion of the system, we note that \(\hat{T}_{\hat{m}}[\phi, F]_{ab}\) behaves nicely with respect to the null decomposition, and \(\hat{T}_{\hat{h}}[\phi, F]_{ab}\) consists of cubic terms. Additionally, since \([D_a, D_b] \phi = i F_{ab} \phi\),
\[
L_{\hat{X}}(\Re(D_a \phi D_b \psi)) = \Re(D_a D_b \phi D_b \psi) + D_a \phi D_b D_b \phi - i \hat{X}^c F_{ac} \phi D_b \phi - i \hat{X}^c F_{bc} \phi D_a \phi,
\]
and
\[
L_{\hat{X}}(\Im(\phi D_b \psi)) = \Im(D_b \phi D_b \psi + \phi D_b D_b \psi - i \hat{X}^c F_{bc} \phi D_a \phi).
\]
Then,
\[
\sum_{|I| \leq N-6} \|L_{\hat{X}}^I T_{\hat{h}}[\phi, F]_{ab}\| \lesssim c_g (t + r^*)^{-1/2} \sum_{|I| \leq N-6} \|L_{\hat{X}}^I F\|^2,
\]
using (10.1), along with the estimate \(-1 + \delta < -s\). The estimate on \(T_{\hat{m}}[\phi, F]_{ab}\) follows directly from the null decomposition, as well as Corollary 10.6 for the commutator terms.

For higher derivatives of \(\hat{T}\) we can also establish energy bounds which will be useful in propagating the bootstrap assumption.

Corollary 10.9. For sufficiently small \(c_g\), and \(\gamma = s_0 - \frac{1}{2}\), the following inequality holds:
\[
\|L_{\hat{X}}^I \hat{T}[\phi, F](t, \cdot) w^{1/2}\|_{L^2(\mathbb{R}^3)} \lesssim \frac{\varepsilon^2}{1 + t}.
\]
Proof. We again split \(\hat{T} = \hat{T}_{\hat{m}} + \hat{T}_{\hat{h}}\). An \(L^2 - L^\infty\) estimate along with Corollary 10.6 gives
\[
\|L_{\hat{X}}^I \hat{T}_{\hat{m}}[\phi, F](t, \cdot) w^{1/2}\|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon (t)^{-1} \|\hat{T} - r^{*})^{-1} (|D\phi| + |F|^1 + \|i \hat{X} F||\phi)| w^{1/2}\|_{L^2(\mathbb{R}^3)} + \varepsilon^2 (t + r)^{-4} w^{1/2}\|_{L^2(\mathbb{R}^3)}.
\]
The second term on the right comes from the \(|F|^2\) terms appearing in \(T_{\hat{m}}\). The final bound follows from Corollary 10.6 for the first term and direct integration for the second, using the inequality \(1 + 2\gamma < 3\). In the case of \(T_{\hat{h}}\), methods used for \(T_{\hat{m}}\) as well as Lemma 5.8 and Corollary 10.6 can be used to bound all terms except those where the most derivatives fall on \(h^1\) and \(H_1\). We can therefore reduce this to
\[
\|L_{\hat{X}}^I T_{\hat{h}}[\phi, F](t, \cdot) w^{1/2}\|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon^2 \|t + r\|^{-2}(|L_{\hat{X}}^I h_1|^1 + |L_{\hat{X}}^I H_1|^1) w^{1/2}\|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon^2 \varepsilon g(t)^{-1}.
\]
This follows from our bootstrap assumption.

11. Precise statement of the theorem and the structure of the proof

In this section we give the precise statement of the theorem. We also start the proof that will be given over the next few sections as well as concluding the proof using the results from the next few sections.
11.1. The energies and statement of the theorem. For a given $0 < \gamma < 1$, $0 < \mu < 1 - \gamma$, we define the energy at time $t$,\[ E_N(T) = \sup_{0 \leq t \leq T} \sum_{|I| \leq N} |\vec{\partial} \tilde{Z}^I h^1(t,x)|^2 w \, dx, \] where $w(t,x) = \left\{ \begin{array}{ll} (1 + |r^*-t|)^{1+2\gamma} & r^* > t, \\ 1 & r^* \leq t. \end{array} \right.$ and space time norm\[ S_N(T) = \sum_{|I| \leq N} \int_0^T \int |\vec{\partial} \tilde{Z}^I h^1(t,x)|^2 w' \, dx \, dt, \] where $|\vec{\partial} h|^2 = |\vec{\partial} h|^2 + \tilde{|L} h|^2$ is the norm of the derivatives tangential to the outgoing curved light cones $r^*-t = q^*$, where $|\vec{\partial} h|^2 = \sum_{i=1,2} |S_i h|^2$ is the norm of the derivatives tangential to the sphere. Moreover let\[ Q_N(T) = \sup_{0 \leq t \leq T} \sum_{|I| \leq N} \int |D_Z^I \tilde{D} \phi(t,x)|^2 + |\tilde{Z}^I F^1(t,x)|^2 w \, dx, \] where $D_Z^I \psi = \tilde{D}^I \tilde{D} a$, and $\tilde{D} a = (\tilde{\partial}_a + i \tilde{A}_a) \psi$.\[ \textbf{Theorem 11.1.} Suppose that } 1/2 < \gamma < 1, \text{ and } N \geq 11. \text{ There are constants } C_N, C'_N < \infty, \mu > 0, \text{ and } \varepsilon > 0 \text{ such that if for some } \varepsilon \leq \varepsilon_N \[ E_N(0) + Q_N(0) + M^2 + q^2 \leq \varepsilon^2, \] then Einstein-MKG have a global solution satisfying\[ E_N(t) + S_N(t) \leq C_N \varepsilon^2 (1+t)^C h^\varepsilon, \text{ and } Q_N(t) \leq C_N \varepsilon^2. \] (11.2) Additionally, the decay estimates in Section 12 with $\delta$ replaced by $C'_N \varepsilon$ hold, as do (10.4) and (10.5) for $s > 1/2$ satisfying $2s < 1 + \gamma$. Moreover, in the case case $Q_N(0) = 0$, i.e. the vacuum case, then the same result holds if $0 < \gamma < 1$.\[ \textbf{11.2. The bootstrap energy assumptions and the structure of the proof.} \text{ Given } 1/2 < \gamma < 1 \text{ (or } 0 < \gamma < 1 \text{ in the vacuum case) pick a } \delta \text{ so small that } 1/2 + \delta < \gamma < 1 - \delta \text{ (or } 8\delta < \gamma < 1 - 8\delta \text{ in the vacuum case). We will start by making the bootstrap assumptions that for some } C_b < \infty \[ E_N(t) + S_N(t) \leq C_b \varepsilon^2 (1+t)^{\delta}, \text{ for } 0 \leq t \leq T, \] (11.3) and\[ Q_N(t) \leq C_b \varepsilon^2 (1+t)^{\delta}, \text{ for } 0 \leq t \leq T. \] (11.4) All energies are continuous, so it suffices to show that for some $C_b$ the estimates (11.3), (11.4) imply the same estimates with $C_b$ replaced by $C_b/2$.\[ \textbf{11.2.1. Getting back the bootstrap for the fields.} \text{ In the first part of Section 12 we will show that the energy bounds required for Theorem 10.3 follow from the bootstrap assumption (11.3) combined with the initial assumption (11.1) for sufficiently small $\varepsilon$ depending on } C_b. \text{ Specifically, we select } \varepsilon_b' = \varepsilon_b'(C_b) > 0 \text{ small enough that } C_b \varepsilon_b'^2 < \varepsilon_g^2, \text{ where } \varepsilon_g \text{ is as in Theorem 10.3 and } C \text{ refers to the constants which we use to retrieve the energy bounds (10.1), (10.2) from (11.3). Then there exist quantities } C_Q, \varepsilon_Q \text{ such that, if } \varepsilon < \varepsilon_Q \text{ and } \varepsilon < \varepsilon_b', \[ Q_N(t) \leq C_Q \varepsilon^2. \] (11.5) In other words, we recover the bootstrap (11.4) for $C_b \geq C_Q$. It is important to note that, provided $C_b \varepsilon_b'^2 < \varepsilon_g^2$, we may select $C_Q$ and $\varepsilon_Q$ independent of $C_b, \varepsilon_b'$ such that (11.5) holds. This bound on $Q_N$, combined with decay estimates for $F$ and $\phi$, will give an $L^2$ bound for the energy momentum tensor for the Maxwell Klein Gordon system.
11.2.2. Getting back the metric bootstrap. We will prove in Section 13, see (13.1), that the bootstrap assumptions imply that for some constant $C_N$ (with $C_N \geq C_Q$), that is independent of the constants $C_b$, $T$, $\delta$, and constants $C'_b = C'_b(C_b) < \infty$ and $\varepsilon_b = \varepsilon_b(C_b) > 0$, that depend continuously on $C_b$ but are independent of $T$, $\delta$, we have

$$E_N(t) + S_N(t) \leq C_N \varepsilon^2(1 + t)^C_b \varepsilon,$$

for $0 \leq t \leq T$, if $\varepsilon \leq \varepsilon_b$.

However, we can pick $C'_b$ and $\varepsilon_b$ just depending on $N$ as well. In fact, this is achieved by picking $C_b = 2C_N$ assuming that $C_N \geq 1$ as we may, and the further picking $0 < \varepsilon_N \leq \min(\varepsilon_Q, \varepsilon_b(2C_N), \varepsilon'_b(2C_N))$ so small that $\varepsilon NC'_b(2C_N) \leq \delta$. Once we found the universal constant $C_N$, through the estimates, we can now go back and make the more intelligent bootstrap assumption that (11.3) hold with $C_b = 2C_N$. It follows that (11.4) holds also. Therefore, we may extend the bootstrap argument to all time, which implies that (11.2) holds globally.

11.2.3. The structure of the proof. We will appeal to the results stated in Section 10 in a nested way in the proof. First we will show that the apriori bounds stated above imply weak decay estimates for the metric, and also $L^2$ estimates for the components controlled by the wave coordinate condition. This in turn is sufficient to satisfy the assumptions on the metric needed to prove the results in Section 10. Then using the decay estimates for the fields from Section 10 we will be able to prove the sharp decay estimates for the metric which are needed to prove the sharper $L^2$ estimates for the metric which gives back the apriori bounded for the metric. Moreover the sharper $L^2$ bounds for the field also follows from Section 10.

12. The decay of the metric and the fields assuming weak energy bounds

In this section we will prove the decay estimates of the metric and fields assuming the apriori energy bounds in the previous section.

12.1. The weak decay of the metric.

12.1.1. The weak decay of the metric and the fields using Klainerman-Sobolev. It follows from the weighted Klainerman-Sobolev and the assumed bounds that

$$|\partial \tilde{Z} J h^1(t, x)| + |\tilde{Z}^J F^1| \leq C \varepsilon(1 + t)^{\delta/2}(1 + t + |q^*|)^{-1}(1 + |q^*|)^{-1/2}(q^1)^{-1/2-\gamma}, |J| \leq N - 3.$$

Let us for simplicity now assume $\gamma > 1/2$. Integrating this from initial data in the $t - r$ direction also gives

$$|\tilde{Z}^J h^1(t, x)| \leq C \varepsilon(1 + t)^{\delta/2}(1 + t + |q^*|)^{-1}(1 + |q^*|)^{1/2}(q^1)^{1/2-\gamma}, |J| \leq N - 3.$$ (12.1)

12.1.2. The $L^1$ bound of the inhomogeneous terms. We will now show that

$$\|\tilde{Z}^J F(g)[\partial h, \partial \phi](t, \cdot)\|_{L^1(w_1, \gamma)} + \|\tilde{Z}^J T(t, \cdot)\|_{L^1(w_1, \gamma)} \lesssim \varepsilon^2(1 + t)^\delta, |J| \leq N - 3.$$ (12.2)

The first part of (12.2) is easy so let us first do the second part. Again the part for $F^1$ follows directly from the $L^2$ bounds although there are some issues with subtracting of the charge in order to get the bound for $F$. These are similar to subtracting of the mass for $h$. The main difficulty is therefore the estimate for $\phi$.

We have

$$Q_{\alpha\beta}[D\phi] = \Re \left(D_\alpha \phi \cdot \overline{D_\beta \phi} - g_{\alpha\beta} g^{\mu\nu} D_\mu \phi \cdot \overline{D_\nu \phi}/2\right),$$

where $D_\alpha \phi = \bar{\partial}_\alpha \phi + i A_\alpha \phi$ and

$$\tilde{Z} \Re \left(D_\alpha \phi \cdot \overline{D_\beta \psi}\right) = \Re \left(D_{\bar{\alpha}} D_\alpha \phi \cdot \overline{D_\beta \psi}\right) + \Re \left(D_{\alpha} \phi \cdot \overline{D_\beta \psi}\right).$$

In view of (12.1) it follows that for $|J| \leq N - 3$

$$|\tilde{Z}^J Q_{\alpha\beta}[D\phi]| \lesssim \sum_{|K| \leq |J|} |D_K \tilde{Z} \tilde{D} \phi|^2.$$
We have
\[ Q_{\alpha\beta}[F] = g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu} - \frac{1}{4} g_{\alpha\beta} g^{\mu\nu} F_{\mu\gamma} F_{\nu\delta}, \]
and hence in view of (12.1) it follows that for \(|J| \leq N - 3\)
\[ |\tilde{Z}^J Q_{\alpha\beta}[F]| \leq \sum_{|K| \leq |J|} |\tilde{Z}^K F|^2 \leq \sum_{|K| \leq |J|} |\tilde{Z}^K F|^2 + \sum_{|K| \leq |J|} |\tilde{Z}^K F^0|^2. \]
We also have
\[ |\tilde{Z}^J F(g)| \partial h, \partial h| \leq \sum_{|K| \leq |J|} |\tilde{Z}^K \partial h|^2 \leq \sum_{|K| \leq |J|} |\tilde{Z}^K \partial h|^2 + \sum_{|K| \leq |J|} |\tilde{Z}^K \partial h^0|^2. \]
We have
\[ \sum_{|K| \leq |J|} |\tilde{Z}^K F^0|^2 + \sum_{|K| \leq |J|} |\tilde{Z}^K \partial h|^2 \lesssim \frac{\varepsilon^2}{(1 + t + r)^4}. \]
It follows that
\[ \sum_{|K| \leq |J|} \int |\tilde{Z}^K F^0|^2 + |\tilde{Z}^K \partial h|^2 \, w_{1,\gamma} \, dx \lesssim \frac{\varepsilon^2}{(1 + t)^{1-\gamma}}, \quad \gamma < 1. \]
This together with the bootstrap assumptions (11.3)-(11.4) proves (12.2), since \(w_{1,\gamma} \lesssim w\).

12.1.3. The improved weak decay of the metric and the fields using Hörmander’s \(L^1 - L^\infty\) estimate. We will now show that

**Lemma 12.1.**
\[ |\tilde{Z}^J h^1(t, x)| \leq C \varepsilon (1 + t)^\delta (1 + t + |q^*|)^{-1} (q^*_+)^{-\gamma}, \quad |J| \leq N - 5, \hspace{1cm} (12.3) \]
and
\[ (1 + t + |q^*|) |\partial \tilde{Z}^J h^1(t, x)| + (|q^*|) |\tilde{Z}^J h^1(t, x)| \leq C \varepsilon (1 + t)^\delta (1 + t + |q^*|)^{-1} (q^*_+)^{-\gamma}, \quad |J| \leq N - 6. \hspace{1cm} (12.4) \]

**Proof.** (12.4) follows directly from (12.3). To prove (12.3) we will apply Proposition 9.2 and Lemma 9.3 to \(\Box^* \tilde{L}_X^I \tilde{h}^1\). Applying \(\tilde{L}_X^I\) to (7.1) using Lemma 7.1 and Lemma 7.4
\[ |\Box^* \tilde{L}_X^I \tilde{h}^1| \lesssim |\Box^* \tilde{L}_X^I \tilde{h}^1| - \frac{M}{(t + r)^3} + \frac{M^2}{(t + r)^4} + M \int |\tilde{Z}^J \tilde{H}_1| + \frac{M}{(t + r)^3} \sum_{|J| \leq |I|} |\tilde{Z}^J \tilde{H}_1|. \]
We have already estimated \(|\tilde{L}_X^I \tilde{F}(g)| \partial \tilde{h}, \tilde{h}| + |\tilde{L}_X^I \tilde{T}|\) in the weighted \(L^1\) norm, and using Cauchy Schwarz the other terms are bounded by
\[ \frac{M}{(t + r)^3} H(r < t/2) + \frac{M^2(1 + \ln (t + r))^2}{(t + r)^4} + \sum_{|K| \leq |J|} |\tilde{Z}^K \tilde{h}^1|^2 + \sum_{|J| \leq |I|} \left( \frac{M}{(t + r)^3} \right)^2. \]
We already dealt with terms similar to the first two terms and the first sum in Section 12.1.2. For the first term one uses that it vanishes when \(r > t\) so that it can absorb any exterior weight. It therefore only remains to take the weighted \(L^1\) norm of the last sum. However, this follows from that \(H_1 = -h^1 + O(h^2)\) and Corollary 8.5 since \(w_{1,\gamma} \lesssim w\). In conclusion we have
\[ \int |\Box^* \tilde{L}_X^I \tilde{h}^1| \, w_{1,\gamma} \, dx \lesssim \varepsilon (1 + t)^\delta. \]
This together with Proposition 9.2 and Lemma 9.3 gives (12.3). \(\square\)
12.2. The strong decay estimates from the wave coordinate condition. By (12.14) we have
\[
|\partial_q \tilde{\mathcal{L}}^L Z H_1|_{LT} + |\partial_q t \tilde{\mathcal{L}}^L Z H_1| \lesssim \varepsilon (1 + t + r)^{-2+2\delta} \langle q_+^* \rangle^{-\gamma} \langle q_-^* \rangle^{-\delta},
\]
(12.6)
\[
\tilde{\mathcal{L}}^L Z H_1|_{LT} + \# \tilde{\mathcal{L}}^L Z H_1 \lesssim \varepsilon (1 + t + r)^{-1-\gamma+\delta} + \varepsilon (1 + t)^{-2+2\delta} \langle q_-^* \rangle^{1-\delta} \lesssim \varepsilon (1 + t + r)^{-1-\gamma+\delta} \langle q_-^* \rangle^{\gamma}.
\]
(12.7)

Proposition 12.2. For \(|I| \leq N - 6\) we have
\[
|\partial_q \tilde{\mathcal{L}}^L Z H_1|_{LT} + |\partial_q t \tilde{\mathcal{L}}^L Z H_1| \lesssim \varepsilon (1 + t + r)^{-2+2\delta} \langle q_+^* \rangle^{-\gamma} \langle q_-^* \rangle^{-\delta},
\]
and
\[
(1 + t + r)^3 \sum_{|J| \leq |I|} \left| \tilde{\mathcal{L}}^L Z h_1 \right| + \frac{M \varepsilon(1 + t + r)}{(1 + t + r)^{2+2\delta} \langle q_+^* \rangle^{\gamma}}.
\]

from which the first part follows. The second follows from integrating the first in the \(q^*\) direction from initial data where it is bounded by \(\varepsilon (1 + r)^{-1-\gamma+\delta}\), because of Sobolev’s lemma and the fact the energy is bounded initially.

12.3. The additional \(L^2\) bounds from the wave coordinate condition. Using (12.3)-(12.4) in (12.5) and the fact that \(H_1 = -h^1 + O(h^2)\) we have
\[
|\partial_q \tilde{\mathcal{L}}^L Z H_1|_{LT} + |\partial_q t \tilde{\mathcal{L}}^L Z H_1| \lesssim \varepsilon (1 + t + r)^{-2+2\delta} \langle q_+^* \rangle^{-\gamma} \langle q_-^* \rangle^{-\delta},
\]
(12.8)

The following \(L^2\) will be used both in the energy estimate for the fields and for the metric:

Lemma 12.3. If \(|I| \geq 6\) then
\[
\int_0^T \int_0^\infty \left( |\partial_q \tilde{\mathcal{L}}^L Z H_1|^2_{LT} + |\partial_q t \tilde{\mathcal{L}}^L Z H_1|^2 \right) w' dx dt \\
\lesssim \int_0^T \int_0^\infty \left| \tilde{\mathcal{L}} Z h^1(t, x) \right|^2 w' dx dt + \left( M^2 + \varepsilon^2 \right) \int_0^T \frac{dt}{(1 + t)^{2-2\delta}} \int_0^\infty \left| \tilde{\mathcal{L}} Z h^1(t, x) \right|^2 w dx.
\]

Proof. Using the first part of Corollary 8.3 and the fact that \(w' \sim w(q^*)^{-1}(q_-^*)^{-2\mu}\) in (12.8) gives the result.

The following \(L^2\) bounds are needed estimates for the fields.
Lemma 12.4. If $0 \leq b \leq \gamma$ and $|I| \geq 6$ then
\[
\int_0^T \left( \frac{\langle \tilde{L}_Z H_1 \rangle_{L^2}}{(q)^2} + \frac{|\nabla \tilde{L}_Z H_1|^2}{(q)^2} \right) \left( \frac{\langle q^s \rangle}{t+r^s} \right)^{1-b} \, w \, dx \, dt \\
\lesssim \int_0^T \left\| \nabla Z_h^1(t,x) \right\|^2 w \, dx \, dt + (M^2 + \varepsilon^2) \left( 1 + \sum_{|J| \leq |I|} \int_0^T \frac{dt}{(1 + t)^{2-2\delta}} \int \left| \nabla \tilde{Z}_h^1(t,x) \right|^2 dx \right).
\]

Proof. Using the second part of Corollary 8.5 and the fact that $w' \sim w(q^s)^{-1}(q^s)^{-2\mu}$ and the previous lemma gives the result. 

Hence we have the additional apriori assumption
\[
\int_0^T \int_{I_t} R_{\text{tan}}^2 \left( \frac{(1 + t)^{1+2\nu}}{\varepsilon} \right) w \, dx \, dt \lesssim \sum_{|J| \leq |I|} C \varepsilon \int_0^T \int_{I_t} \left| \nabla h^1 \right|^2 w \, dx \, dt \lesssim C \varepsilon S_N(T).
\]

12.4. The sharp decay estimates for the fields. We can now appeal to the results of Section 10 to further bound the metric estimates in the bootstrap assumption. We first show that the bootstrap assumption implies the required bound on metric components.

Proposition 12.5. For sufficiently small $\varepsilon$, the bounds (10.1) and (10.2) follow from (11.3).

Proof. The bound (11.3) immediately implies
\[
\sup_{t \in [0,T]} \left\| \nabla L^I_Z h^1 w^{1/2} \right\|_{L^2(\mathbb{R}^3)} + \left\| \nabla L^I_Z h^1 w^{1/2} \right\|_{L^2([0,T] \times \mathbb{R}^3)} \leq C \varepsilon C_b^{-1/2}(1 + T)^{\delta/2},
\]
The estimate
\[
\left\| (t-r^s)^{-1}s (t+r^s)^{s-1} \left| L^I_X h^1 \right| w^{1/2} \right\|_{L^2(\mathbb{R}^3)} \leq C \varepsilon C_b^{-1/2}(1 + T)^{\delta/2},
\]
follows directly from Corollary 8.5 in spherical coordinates. Likewise, (12.9) gives
\[
\left\| \nabla L^I_X h^1 \left| \nabla \phi \right| w^{1/2} \right\|_{L^2([0,T] \times \mathbb{R}^3)} \leq C \varepsilon (1 + T')^{\delta/2},
\]

noting that the difference consists of similar components containing lower order derivatives. To prove (10.2a) we convert to spherical coordinates and appeal to Corollary 8.5 more directly, specifically the estimate (8.6). For any $s \in (\frac{1}{2} + 4\delta, 1 - 4\delta)$ we take $b = 2s - 1 + 2\delta$, so $b \in (10\delta, 1 - 6\delta)$.
\[
\left\| (t-r^s)^{-1-s} (t+r^s)^{s-1} \left| L^I_X h^1 \right| \left| \nabla \phi \right| \right\|_{L^2([0,T] \times \mathbb{R}^3)} \lesssim T^{\delta/2} \left\| (t-r^s)^{-1-s} (t+r^s)^{s-1} \right\|_{L^2([0,T] \times \mathbb{R}^3)} \lesssim T^{\delta/2} \left\| (t-r)^{-\frac{3}{2} - \frac{b}{2} + \delta} (t+r)^{\frac{b-1}{2}} \right\|_{L^2([0,T] \times \mathbb{R}^3)}.
\]
Since $w \geq 1$ and $\mu < 1 - \gamma < \frac{1}{2} - 4\delta$, it follows that
\[
(t-r^s)^{-\frac{1}{2} + \delta} \lesssim \frac{w}{(1 + q^s)^\mu},
\]
so we may apply (8.6) and (12.9), then sum over dyadic regions in time $[2^k, 2^{k+1}]$. 

It remains to show that the initial conditions (10.3) follow from the initial assumption (11.1) for sufficiently small $\varepsilon$.

Proposition 12.6. There exists a fixed constant $C$ such that
\[
\| E_0 \|_{H^{N,1/2+\gamma}}^2 + \| B_0 \|_{H^{N,1/2+\gamma}}^2 + \| D\phi_0 \|_{H^{N,1/2+\gamma}}^2 + \| \dot{\phi}_0 \|_{H^{N,1/2+\gamma}}^2 < C(1 + C_b \varepsilon^2) Q_N(0).
\]
Lemma 12.10. The quantities \( \| D \phi_0 \|_{H^{N,1/2+\gamma}}^2 + \| \dot{\phi}_0 \|_{H^{N,1/2+\gamma}}^2 \) follow immediately from the definitions. For the estimates on the fields, we first use the standard frame in modified coordinates. This allows us to convert between Lie derivatives \( \mathcal{L}_Z \) and standard derivatives \( \bar{Z} \) without modifying the estimates, as the difference consists of lower order derivatives of the same weight. The estimate on \( \| B_0 \|_{H^{N,1/2+\gamma}}^2 \) therefore follows from the definition as well as (10.1) in the case where most derivatives fall on \( F^1 \), or a Klainerman-Sobolev estimate similar to the worst components of \( F^1 \) in (10.4), combined with (10.2), in the case where most derivatives fall on \( \varepsilon_{ijj}k \). The bound on \( \| E_0^d \|_{H^{N,1/2+\gamma}}^2 \) follows from an elliptic estimate as well as a Hardy inequality on the current, and follows from Lemma 10.1 in [38] and Hölder’s Inequality.

\[ Q_N(I) \leq C \varepsilon^2, \]

and the bounds (10.4) hold for \( s_0 = 1/2 + \gamma, s = 1/2 + \gamma/2 - 3\delta \).

Proof. We first note that \( 1 - s = (1 - \gamma)/2 + 3\delta \geq \delta, s - 1/2 \geq 1/4 \geq \delta, 1 - 2\mu \geq (2\gamma - 1)/2 \) and \( s = 1/2 + 3\delta, s_0 \geq 1 \). Therefore, the restrictions on \( s, s_0 \) and \( \delta \) in Theorem 10.3 are satisfied. This follows from Equations (10.6a) and (10.6b), along with the identity (10.8) and Corollary 10.6, which handle the commutator terms.

12.5. The energy and decay estimates for the energy momentum tensor. Corollaries 10.7 and 10.9 follow directly from Theorem 12.7.

Proposition 12.8. Given constants \( \gamma, \delta, \mu, N \) satisfying \( N \geq 11, 1/2 < \gamma < 1, 0 < \mu < 1 - \gamma, \) and \( 0 < \delta\delta < \min(1 - 2\mu, \gamma - 1/2, 1 - \gamma) \), as well as the initial bounds (11.1), and the bootstrap assumption (11.3), then for sufficiently small \( \varepsilon \), the following bounds hold for \( |I| \leq N - 6 \) and \( s = (1 + \gamma)/2-3\delta \):

\[ |(\mathcal{L}_X^I \bar{T})[F, \phi](t, x)| \leq \varepsilon^2 (t + r)^{-2} (t - r)^{-2/3} |I|^{-1/2}, \]

\[ |(\mathcal{L}_X^I \bar{T})[F, \phi](t, x)|_{\mathcal{U}T} \leq \varepsilon^2 (t + r)^{-2} (t - r)^{-2/3} |I|^{-1/2}. \]

We state a revised form of Corollary 10.9.

Lemma 12.9. Given constants \( \gamma, \delta, \mu, N \) satisfying \( N \geq 11, 1/2 < \gamma < 1, 0 < \mu < 1 - \gamma, \) and \( 0 < \delta\delta < \min(1 - 2\mu, \gamma - 1/2, 1 - \gamma) \), as well as the initial bounds (11.1), and the bootstrap assumption (11.3), then for sufficiently small \( \varepsilon \), the following bound holds:

\[ \int_{\Sigma_t} |(\mathcal{L}_X^I \bar{T})[F, \phi](t, x)|^2 dx \leq \frac{C \varepsilon^2}{1 + t}. \]

12.6. The sharp decay estimates for the metric. From Lemma 9.4, the decay estimates we have proven so far and to estimate the commutator \( \Box^s \mathcal{L}_X^I h_{cd} - \mathcal{L}_X^I (\kappa \Box h_{cd}) \) we get

Lemma 12.10. Let \( D_t = \{(t, x); |t - x| < c_0 t\} \), for some \( 0 < c_0 < 1 \). With \( \overline{w}(q^*) = \langle q^* \rangle^{-1} \langle q^* \rangle^{1/2} \) we have

\[ (1 + t + |x|) |(\partial \mathcal{L}_Z^I h_{1}^{1})_{UV}(t, x) \overline{w}(q^*)| \leq \varepsilon + \int_{|q^*|/4}^t (1 + \tau) \| (\mathcal{L}_X^I (\kappa \Box h_{1}^{1}))_{UV}(\tau, \cdot) \overline{w}\|_{L^\infty(D_\tau)} d\tau, \]

\[ |I| \leq N - 7. \]

Proof. By Lemma 9.4

\[ (1 + t + |x|) |(\partial \mathcal{L}_Z^I h_{1}^{1})_{UV}(t, x) \overline{w}(q^*)| \leq \sup_{|q^*|/4 \leq \tau \leq t} \sum_{|I| \leq |I| + 1} \| \mathcal{L}_Z^I h_{1}^{1}(\tau, \cdot) \overline{w}\|_{L^\infty} \]

\[ + \int_{|q^*|/4}^t (1 + \tau) \| (\Box^s \mathcal{L}_Z^I h_{1}^{1})_{UV}(\tau, \cdot) \overline{w}\|_{L^\infty(D_\tau)} + \sum_{|I| \leq |I| + 1} (1 + \tau)^{-1} \| \mathcal{L}_Z^I h_{1}^{1}(\tau, \cdot) \overline{w}\|_{L^\infty(D_\tau)} d\tau. \]
Using the estimate (12.3):
\[ |\mathcal{L}_Z^t h^I(t, x)| \lesssim \varepsilon (1 + t)^\delta (1 + t + |q^*|)^{-1} \langle q^*_+ \rangle^{-\gamma}, \quad |J| \leq N - 5 \]

it follows that
\[
(1 + t + |x|) |(\partial \mathcal{L}_Z^t h^I)_{UV}(t, x) \bar{w}(q^*)| \lesssim \varepsilon + \int_{|q^*|/4}^{t} (1 + \tau) \| (\kappa \tilde{\mathcal{L}}_X^t \tilde{h}^I)_{UV}(\tau, \cdot) \bar{w} \|_{L^\infty(D, \tau)} d\tau, \quad |I| \leq N - 7.
\]

By Lemma 7.1
\[
|\Box^s \mathcal{L}_X^t \tilde{h}_{cd}^I - \mathcal{L}_X^t \tilde{h}_{cd}^I(\kappa \tilde{\mathcal{L}}_X^t \tilde{h}_{cd}^I)| \lesssim \sum_{|J| + |K| \leq |I| + 1, |J| \leq |I|} \left( M \ln (t + r) \frac{(\kappa \tilde{\mathcal{L}}_X^t \tilde{h}_1)_{\bar{w}}}{(t + r)^2} + \frac{\| \tilde{\mathcal{L}}_X^t \tilde{h}_1 \|_{L^\infty}(t + r)}{(t + r)^2} \right) \| \partial \mathcal{L}_Z^t \tilde{h}^I \|_{L^\infty(D, \tau)}.
\]

Using the estimates (12.3), (12.4) and (12.7) we obtain
\[
|\Box^s \mathcal{L}_X^t \tilde{h}_{cd}^I - \mathcal{L}_X^t \tilde{h}_{cd}^I(\kappa \tilde{\mathcal{L}}_X^t \tilde{h}_{cd}^I)| \lesssim \left( \frac{(\kappa \tilde{\mathcal{L}}_X^t \tilde{h}_1)_{\bar{w}}}{(t + r)^2} + \frac{(\kappa \tilde{\mathcal{L}}_X^t \tilde{h}_1)_{\bar{w}}}{(t + r)^2} \right) \| \partial \mathcal{L}_Z^t \tilde{h}^I \|_{L^\infty(D, \tau)}.
\]

Hence
\[
\int_{|q^*|/4}^{t} (1 + \tau) |(\partial \mathcal{L}_Z^t h^I)_{UV}(t, x) \bar{w}(q^*)| \lesssim \varepsilon + \int_{|q^*|/4}^{t} (1 + \tau) \| (\mathcal{L}_Z^t (\kappa^2 (\tilde{F} (\tilde{\mathcal{L}}_X^t \tilde{h}^I) + \tilde{T})) )_{UV}(\tau, \cdot) \bar{w} \|_{L^\infty(D, \tau)} d\tau.
\]

The lemma follows from this and (12.11).

Also using Lemma 7.3 we get

**Lemma 12.11.** Let \( D_t = \{(t, x); |t - x| \leq c_0 t\} \), for some constant \( 0 < c_0 < 1 \), and \( \bar{w}(q^*) = \langle q^* \rangle^{1-\delta} \langle q^*_+ \rangle^\gamma \). Then for \( |I| \leq N - 7 \)
\[
(1 + t + |x|) |(\partial \mathcal{L}_Z^t h^I)_{UV}(t, x) \bar{w}(q^*)| \lesssim \varepsilon + \int_{|q^*|/4}^{t} (1 + \tau) \| (\mathcal{L}_Z^t (\kappa^2 (\tilde{F} (\tilde{\mathcal{L}}_X^t \tilde{h}^I) + \tilde{T})) )_{UV}(\tau, \cdot) \bar{w} \|_{L^\infty(D, \tau)} d\tau.
\]

**Proof.** Using (7.3) and Lemma 12.10 it suffices to show
\[
\int_{|q^*|/4}^{t} (1 + \tau) \| (|\tilde{R}_{cd}^{\text{mass}} I| + |\tilde{R}_{cd}^{\text{cov}} I| + |\tilde{R}_{cd0}^{\text{cube}} I|) (\tau, \cdot) \bar{w} \|_{L^\infty(D, \tau)} d\tau \lesssim \varepsilon.
\]

This follows from (12.3), (12.4), and Lemmas 7.2 and 7.3 which imply the pointwise bound
\[
|\tilde{R}_{cd}^{\text{mass}} I| + |\tilde{R}_{cd}^{\text{cov}} I| + |\tilde{R}_{cd0}^{\text{cube}} I| \lesssim \varepsilon ((t + r^*)^{-3+3\delta} \langle q^*_+ \rangle^{-1} + (t + r^*)^{-3} \langle q^*_+ \rangle^{-1}).
\]

The result follows from direct integration. \( \square \)

**Proposition 12.12.** Let \( D_t = \{(t, x); |t - x| \leq c_0 t\} \) for some constant \( 0 < c_0 < 1 \) and \( \bar{w}(q^*) = \langle q^* \rangle^{1-\delta} \langle q^*_+ \rangle^\gamma \). Let \( |I| \leq N - 7 \). Then if \( T \in \{L, S_1, S_2\} \) we have
\[
(1 + t + |x|) |(\partial \mathcal{L}_Z^t h^I)_{UV}(t, x) \bar{w}(q^*)| \lesssim \varepsilon + \sum_{|J| \leq |I|} \| (\mathcal{L}_Z^t \tilde{T})_{UV}(\tau, \cdot) \bar{w} \|_{L^\infty(D, \tau)} d\tau. \quad (12.12)
\]

On the other hand
\[
(1 + t + |x|) |(\partial \mathcal{L}_Z^t h^I)_{UV}(t, x) \bar{w}(q^*)| \lesssim \varepsilon + \sum_{|J| \leq |I|} \| (\mathcal{L}_Z^t \tilde{T})(\tau, \cdot) \bar{w} \|_{L^\infty(D, \tau)} d\tau
\]
\[
+ \sum_{J + K = I} \int_{|q^*|/4}^{t} (1 + \tau) \| (\tilde{h}^I)_{\bar{T} \bar{T}} (\tau, \cdot) \bar{w} \|_{L^\infty(D, \tau)} d\tau. \quad (12.13)
\]
Proof. By Lemma \[7.2\] Lemma \[7.3\] and \[12.3\]-\[12.4\] we have
\[
\left| \mathcal{L}_X^I(\kappa^2 \mathcal{F}_{ab}(\mathbf{g}) |\overline{\partial h^1}, \overline{\partial h^1}|) - \sum_{J+K=I} \overline{\mathcal{L}_a \mathcal{L}_b \mathcal{P}(\partial_q \overline{\partial h^1}_I, \partial_q \overline{\partial h^1}_K)} \right|
\lesssim \sum_{J+K=I} |\overline{\partial h^1}_I| |\overline{\partial h^1}_K| + \frac{M \ln (t+r)}{(t+r)} \sum_{J+K=I} |\overline{\partial h^1}_I| |\overline{\partial h^1}_K| + \sum_{|J|+|K|+|L| \leq |I|} \mathcal{L}_X^L \mathcal{H}_1 |\overline{\partial h^1}_I| |\overline{\partial h^1}_K|
\lesssim \varepsilon^2 (1 + t + |q^*|)^{-3+3\delta} \langle q^* \rangle^{-1} \langle q^*_+ \rangle^{-\gamma} + \varepsilon^2 (1 + t + |q^*|)^{-3+3\delta} \langle q^* \rangle^{-2} \langle q^*_+ \rangle^{-\gamma}.
\]
It follows that
\[
\int_{|q^*|/4}^t (1 + \tau) \left\| \left( \mathcal{L}_X^I(\kappa^2 \mathcal{F}_{ab}(\mathbf{g}) |\overline{\partial h^1}, \overline{\partial h^1}|) - \sum_{J+K=I} \overline{\mathcal{L}_a \mathcal{L}_b \mathcal{P}(\partial_q \overline{\partial h^1}_I, \partial_q \overline{\partial h^1}_K)} \right)(\tau, \cdot) \overline{\mathbf{w}} \right\|_{L^\infty(D_\tau)} d\tau \lesssim \varepsilon.
\]
The first estimate follows directly from this using Lemma \[12.11\]. To prove the second estimate we use Lemma \[7.3\] and \[12.6\] and \[12.3\]:
\[
\left| \mathcal{P}(\partial_q \overline{\partial h^1}_I, \partial_q \overline{\partial h^1}_K) \right|
\lesssim \left( |\partial_q \overline{\partial h^1}_I|_{L^\infty} + |\partial_q \overline{\partial h^1}_I| \right) |\overline{\partial h^1}_K| + |\overline{\partial h^1}_I| \left( |\partial_q \overline{\partial h^1}_K|_{L^\infty} + |\partial_q \overline{\partial h^1}_K| \right) + |\overline{\partial h^1}_I|_{L^\infty} |\overline{\partial h^1}_K|_{L^\infty}
\lesssim \varepsilon^2 (1 + t + |q^*|)^{-3+3\delta} \langle q^* \rangle^{-1} \langle q^*_+ \rangle^{-\gamma} + |\overline{\partial h^1}_I|_{L^\infty} |\overline{\partial h^1}_K|_{L^\infty},
\]
from which the second estimate follows. \(\square\)

**Proposition 12.13.** For \(|I| \leq N - 7\) we have
\[
(1 + t + |x|) \left\| \left( \mathcal{L}_Z^I \overline{\partial h^1}_I \right)_T(t, x) \overline{\mathbf{w}}(q^*) \right\| \lesssim \varepsilon.
\]
Moreover,
\[
(1 + t + |x|) \left\| \left( \mathcal{L}_Z^I \overline{\partial h^1}_I \right)_U(t, x) \overline{\mathbf{w}}(q^*) \right\| \lesssim \varepsilon \left( 1 + \ln \frac{t+r^*}{(t-r^*)} \right).
\]

**Proof.** By Proposition \[12.8\] we have
\[
\int_{|q^*|/4}^t \left( \mathcal{L}_Z^I \overline{\mathbf{w}} \right)_T \left( \tau, \cdot \right) d\tau \lesssim \varepsilon \int_{|q^*|/4}^t \left( \mathcal{L}_Z^I \overline{\mathbf{w}} \right)_T \left( \tau, \cdot \right) d\tau \lesssim \varepsilon \int_{|q^*|/4}^t \left( \mathcal{L}_Z^I \overline{\mathbf{w}} \right)_T \left( \tau, \cdot \right) d\tau \lesssim \varepsilon \left( \frac{t}{(t-r^*)} \right).
\]
which together with \[12.12\] implies \[12.14\], and
\[
\int_{|q^*|/4}^t \left( \mathcal{L}_Z^I \overline{\mathbf{w}} \right)_U \left( \tau, \cdot \right) d\tau \lesssim \varepsilon \int_{|q^*|/4}^t \left( \mathcal{L}_Z^I \overline{\mathbf{w}} \right)_U \left( \tau, \cdot \right) d\tau \lesssim \varepsilon \int_{|q^*|/4}^t \left( \mathcal{L}_Z^I \overline{\mathbf{w}} \right)_U \left( \tau, \cdot \right) d\tau \lesssim \varepsilon \ln \left( \frac{t}{(t-r^*)} \right).
\]
Moreover using \[12.13\] we get
\[
\int_{|q^*|/4}^t \left( \mathcal{L}_Z^I \overline{\partial h^1}_I \right)_T \left( \tau, \cdot \right) d\tau \lesssim \varepsilon \int_{|q^*|/4}^t \left( \mathcal{L}_Z^I \overline{\partial h^1}_I \right)_T \left( \tau, \cdot \right) d\tau \lesssim \varepsilon \int_{|q^*|/4}^t \left( \mathcal{L}_Z^I \overline{\partial h^1}_I \right)_T \left( \tau, \cdot \right) d\tau \lesssim \varepsilon \ln \left( \frac{t}{(t-r^*)} \right).
\]

\[12.15\] follows from using the last two estimates in \[12.13\]. \(\square\)

### 13. The Energy Bounds of the Metric Assuming the Decay Estimates

In the previous section we got improved decay estimates for certain components of the metric. With the energy estimate we can not differentiate between different components at the highest order. However in the proof we need to use that we have better decay estimates for tangential
components. In particular, the assumptions of the basic energy estimate in Theorem 8.1 hold. We will now apply this estimate to the differentiated equations (7.3):

$$\nabla^{11}_{cd} + \tilde{R}_{cd} \text{com} I = \sum_{I', I'' = I}^{I'} \kappa_{I''} \left( \sum_{J + I = I'} \tilde{F}_{cd}(\tilde{g}) \left[ \overline{\partial h}^{11}_{IJ}, \overline{\partial h}^{11}_{IK} \right] + \tilde{R}_{cd,0}^{\text{cub}} I' + \tilde{T}_{cd} I' + \tilde{R}_{cd}^{\text{mass}} I' + \tilde{R}_{cd}^{\text{cov}} I' \right),$$

where

$$\kappa^{2} \tilde{F}_{cd}(\tilde{g}) \left[ \overline{\partial h}^{11}_{IJ}, \overline{\partial h}^{11}_{IK} \right] = \tilde{L}_{c} \tilde{L}_{d} \tilde{P} \left( \partial_{t} \tilde{h}^{11}_{IJ}, \partial_{t} \tilde{h}^{11}_{IK} \right) + \kappa^{2} \tilde{R}_{cd}^{\text{tan} JK} + \kappa^{2} \tilde{R}_{cd}^{\text{cub} JK}.$$

We will now use the decay estimates for lower derivatives we obtained in the previous section in the estimates in Section 7 for the various remainder terms $\tilde{R}^{\bullet}$. We first note that since $H_1 = -h^1 + O(h^2)$ using the decay estimates in the previous section we see that higher norms of $h^1$ and $H_1$ are equivalent, with constants depending on the lower norms, and below we will use this frequently without mentioning it.

13.0.1. The main quadratic semilinear error terms. By Lemma 7.3 and (12.4)

$$|\tilde{R}_{cd}^{\text{tan} JK}| \lesssim |\overline{\partial h}^{11}_{IJ}||\overline{\partial h}^{11}_{IK}| \lesssim \frac{\varepsilon \langle q_+^* \rangle^{-\gamma}}{(t + r)^{1-\delta} \langle q^* \rangle} \sum_{|J| \leq |I|} |\overline{\partial h}^{11}_{IJ}| + \frac{\varepsilon \langle q_+^* \rangle^{-\gamma}}{(t + r)^{2-\delta}} \sum_{|J| \leq |I|} |\overline{\partial h}^{11}_{IK}|.$$

Moreover, for any $\delta' > \delta$

$$|\tilde{P} \left( \partial_{t} \tilde{h}^{11}_{IJ}, \partial_{t} \tilde{h}^{11}_{IK} \right)| \lesssim \left( |\partial_{t} \tilde{h}^{11}_{IJ}||\tilde{L}_{\tilde{T}}| + |\partial_{t} \tilde{h}^{11}_{JK}| |\tilde{L}_{T}| \right) + |\overline{\partial h}^{11}_{IJ}| \left( |\partial_{t} \tilde{h}^{11}_{JK}| |\tilde{L}_{T}| + |\partial_{t} \tilde{h}^{11}_{JK}| |\tilde{L}_{T}| \right)

\lesssim \frac{\varepsilon \langle q_+^* \rangle^{-\gamma}}{(t + r)^{1-\delta} \langle q^* \rangle} \left( |\partial_{t} \tilde{h}^{11}_{IJ}| |\tilde{L}_{T}| + |\partial_{t} \tilde{h}^{11}_{JK}| |\tilde{L}_{T}| \right) + \frac{\varepsilon \langle q_+^* \rangle^{-\gamma}}{(t + r)^{1-\delta} \langle q^* \rangle} \sum_{|J| \leq |I|} |\overline{\partial h}^{11}_{IJ}|.$$

13.0.2. The quasilinear commutator error terms. By Lemma 7.1 and the estimates in the previous section and using that $M \lesssim \varepsilon$ we have

$$|\tilde{R}_{cd}^{\text{com} I}| \lesssim \frac{\varepsilon \langle q_+^* \rangle^{-\gamma}}{(t + r)^{1-\delta}} \sum_{|J| \leq |I|} \left( \frac{|\overline{\partial h}^{11}_{IJ}| |\tilde{L}_{T}|}{(t + r)^{1-\delta}} + \frac{|\tilde{H}^{11}_{J}|}{(t + r)^{1-\delta}} \right) + \frac{\varepsilon}{(t + r)^{1-\gamma\delta}} \sum_{|J| \leq |I|} |\overline{\partial h}^{11}_{IJ}|.$$

The commutator terms with the Minkowski vector fields would have been highest order error terms but because we use modified coordinates and vector fields these are lower order.

13.0.3. The lower order error terms. There is much more room in the mass, covariant and cubic error terms. By Lemma 7.4

$$|\tilde{R}_{cd}^{\text{mass} I}| + |\tilde{R}_{cd}^{\text{cov} I}| \lesssim \frac{M \langle \ln \langle t + r \rangle + 1 \rangle}{(t + r)^{3}} \sum_{|J| \leq |I|} |\tilde{H}^{11}_{J}| + \frac{M \langle \ln \langle t + r \rangle + 1 \rangle}{(t + r)^{2}} \sum_{|J| \leq |I|} |\overline{\partial h}^{11}_{IJ}|.$$

By Lemma 7.3 and the estimates in the previous section where

$$|\tilde{R}_{cd,1}^{\text{cub} JK}| \lesssim \left( \frac{M \langle \ln \langle t + r \rangle \rangle}{(t + r)^{3}} \right) |\overline{\partial h}^{11}_{IJ}||\overline{\partial h}^{11}_{IK}| \lesssim \frac{\varepsilon \langle q_+^* \rangle^{-\gamma}}{(t + r)^{2-\delta} \langle q^* \rangle} \sum_{|J| \leq |I|} |\overline{\partial h}^{11}_{IJ}|$$

and by Lemma 7.2

$$|\tilde{R}_{cd,0}^{\text{cub} I}| \lesssim \sum_{|J| \leq |I|, |L| \leq |J|, |I| \geq 1} |\tilde{H}^{11}_{I}| |\overline{\partial h}^{11}_{IJ}||\overline{\partial h}^{11}_{IK}| \lesssim \frac{\varepsilon \langle q_+^* \rangle^{-\gamma}}{(t + r)^{2-\delta} \langle q^* \rangle} \sum_{|J| \leq |I|-1} |\overline{\partial h}^{11}_{IJ}| + \frac{\varepsilon \langle q_+^* \rangle^{-2\gamma}}{(t + r)^{2-\delta} \langle q^* \rangle^2} \sum_{|J| \leq |I|} |\tilde{H}^{11}_{I}|.$$
13.1. The energy estimates. With the weighted energies,

\[ E_N(T) = \sup_{0 \leq t \leq T} \sum_{|J| \leq N} \int |\bar{\partial} Z h^1(t, x)|^2 w \, dx, \]

where \( w(t, x) = \begin{cases} (1 + |r^* - t|)^{1+2\gamma}, & r^* > t, \\ 1 + (1 + |r^* - t|)^{-2\mu}, & r^* \leq t. \end{cases} \)

and

\[ S_N(T) = \sum_{|J| \leq N} \int_0^T \int |\bar{\partial} Z h^1(t, x)|^2 w' \, dx \, dt, \]

where \( w'(t, x) = \begin{cases} (1 + 2\gamma)(1 + |r^* - t|)^{2\gamma}, & r^* > t, \\ 2\mu(1 + |r^* - t|)^{-1-2\mu}, & r^* \leq t. \end{cases} \)

we have by Theorem 8.1

\[ E_N(T) + S_N(T) \leq 8E_N(T)(0) + 16 \sum_{|J| \leq N} \int_0^T \left( \int_{\Sigma_t} |\bar{\partial} h^{1J}|^2 w \, dx \right)^{1/2} E_N(t)^{1/2} \, dt. \]

Summing up we can identify essentially five types of error terms that we will estimate

\[ |\bar{\partial} h^{1J}| \leq R_{lead} + R_{low} + R_{tan} + R_{coord} + R_{mass} + |\hat{T}'|. \]

Here the leading order terms are bounded by

\[ R_{lead} = \frac{C\varepsilon}{1 + t} \sum_{|J| \leq |I|} |\bar{\partial} h^{1J}|, \]

which can be estimated by \( E_N(t) \) but are the worst that will lead to exponential growth like \( t^{C\varepsilon} \).

The lower terms \( R_{low} = R_{low,1} + R_{low,0} \) are for some fixed \( 0 < \gamma' < \gamma \)

\[ R_{low,1} = \frac{C\varepsilon}{(1 + t)^{1+\gamma'}} \sum_{|J| \leq |I|} |\bar{\partial} h^{1J}|, \quad \text{and} \quad R_{low,0} = \frac{C\varepsilon}{(1 + t)^{1+\gamma'}} \sum_{|J| \leq |I|} |h^{1J}| \]

where in the norms with the weights we are considering the second sum is bounded by the first using Hardy’s inequality, Corollary 8.5. The tangential terms can be bounded by

\[ R_{tan} = \frac{C\varepsilon(q^*_{+})^{-\gamma}}{(1 + t)^{1/2+\nu(q^*)}} \sum_{|J| \leq |I|} |\bar{\partial} h^{1J}|, \]

for any fixed \( \nu \) such that \( 0 < \nu < 1/2 \), which can be estimated in terms of \( S_N \). The wave coordinate terms are bounded by \( R_{coord} = R_{coord,1} + R_{coord,0} \)

\[ R_{coord,1} = \frac{C\varepsilon(q^*_{+})^{-\gamma}}{(1 + t)^{1/2+\nu(q^*)}} \sum_{|J| \leq |I|} \left( |\partial q^* \bar{\partial} h^{1J}|_{\bar{L}T} + |\partial q^* \bar{\partial} h^{1J}| \right), \]

and

\[ R_{coord,0} = \frac{C\varepsilon(q^*_{+})^{-\gamma}}{(1 + t)^{1/2+\nu(q^*)}} \sum_{|J| \leq |I|} \left( \frac{|\hat{L}^2 H_{1L}|_{\bar{L}T}}{q^*_{+}} + \frac{|\gamma_{\hat{L}^2 H_{1}}|}{q^*_{+}} \right), \]

can be estimated in terms of the tangential terms using the wave coordinate condition as in Lemma 12.4 and Hardy’s inequality. For this we also need to pick \( \nu \) and \( \nu' > 0 \) so that \( 1 - 2(\nu - \nu') < \gamma \).

Finally, the mass terms are bounded by

\[ R_{mass} = \frac{CM}{(t+r)^3(q^*_{+})}, \]

which can be bounded directly in terms of \( M \) that we assume is small.

13.1.1. The leading and lower order terms. We have

\[ \int_0^T \left( \int_{\Sigma_t} R_{low}^2 w \, dx \right)^{1/2} E_N(t)^{1/2} \, dt \leq \int_0^T \frac{C\varepsilon}{1 + t} E_N(t)^{1/2} \, dt. \]

and using Corollary 8.5 to estimate \( R_{low,0} \) in terms of \( R_{low,1} \) we have

\[ \int_0^T \left( \int_{\Sigma_t} R_{low}^2 w \, dx \right)^{1/2} E_N(t)^{1/2} \, dt \leq \int_0^T \frac{C\varepsilon}{(1 + t)^{1+\gamma'}} E_N(t)^{1/2} \, dt. \]
13.1.2. The tangential and coordinate terms. By Cauchy-Schwarz in time
\[ \int_0^T \left( \int_{\Sigma_t} R_{\text{tan}}^2 w \, dx \right)^{1/2} E_N(t)^{1/2} \, dt \leq \left( \int_0^T \int_{\Sigma_t} R_{\text{tan}}^2 \frac{(1+t)^{1+2\nu}}{\varepsilon} w \, dx \, dt \right)^{1/2} \left( \int_0^T \varepsilon E_N(t) \frac{(1+t)^{1+2\nu}}{1+2\nu} \, dt \right)^{1/2}. \]
Here
\[ \int_0^T \int_{\Sigma_t} R_{\text{tan}}^2 \frac{(1+t)^{1+2\nu}}{\varepsilon} w \, dx \, dt \lesssim \sum_{|J| \leq |I|} C \varepsilon \int_0^T \int_{\Sigma_t} \frac{\left| \partial h^J \right|^2}{\langle q^* \rangle^2} w \, dx \, dt \lesssim C \varepsilon S_N(T). \]
Hence, using that \( 2ab \leq a^2 + b^2 \),
\[ \int_0^T \left( \int_{\Sigma_t} R_{\text{tan}}^2 w \, dx \right)^{1/2} E_N(t)^{1/2} \, dt \leq C' \varepsilon S_N(T) + C' \int_0^T \frac{\varepsilon}{(1+t)^{1+2\nu}} E_N(t) \, dt. \]
Since
\[ \int_0^T \int_{\Sigma_t} R_{\text{coord},1}^2 \frac{(1+t)^{1+2\nu'}}{\varepsilon} w \, dx \, dt \leq \sum_{|J| \leq |I|} C \varepsilon \int_0^T \int_{\Sigma_t} \frac{\left| \partial h^J \right|^2}{\langle q^* \rangle^2} w \, dx \, dt, \]
by Lemma [12.3] \( R_{\text{coord},1} \) satisfy the same bound as \( R_{\text{tan}} \). By Lemma [12.4] we have
\[ \int_0^T \int_{\Sigma_t} R_{\text{coord},0}^2 \frac{(1+t)^{1+2\nu'}}{\varepsilon} w \, dx \, dt \lesssim C \varepsilon S_N(T) + \int_0^T \frac{\varepsilon}{(1+t)^{1+2\nu'}} E_N(t) \, dt, \]
since we picked \( \nu \) and \( \nu' \) so that in Lemma [12.4] \( b = 1 - 2(\nu - \nu') < \gamma \). Hence, we also have
\[ \int_0^T \left( \int_{\Sigma_t} R_{\text{coord}}^2 w \, dx \right)^{1/2} E_N(t)^{1/2} \, dt \leq C' \varepsilon S_N(T) + C' \int_0^T \frac{\varepsilon}{(1+t)^{1+2\nu'}} E_N(t) \, dt. \]

13.1.3. The mass terms. We have if \( \gamma < 1 \)
\[ \int_{\Sigma_t} R_{\text{mass}}^2 w \, dx = C_N M^2 \int_0^\infty \frac{\langle q^*_+ \rangle^{1+2\gamma} t^2 \, dr}{(t+r)^6 \langle q^*_+ \rangle^2} \leq \frac{C_N M^2}{(t+1)^{4-2\gamma}}. \]
Hence
\[ \int_0^T \left( \int_{\Sigma_t} R_{\text{mass}}^2 w \, dx \right)^{1/2} E_N(t)^{1/2} \, dt \leq \int_0^T \frac{C_N M}{(1+t)^{2-\gamma}} E_N(t)^{1/2} \, dt. \]

13.1.4. The fields terms. By Proposition [12.9]
\[ \int_0^T \left( \int_{\Sigma_t} |\tilde{T}|^2 w \, dx \right)^{1/2} E_N(t)^{1/2} \, dt \leq \int_0^T \frac{C_N \varepsilon^2}{1+t} E_N(t)^{1/2} \, dt. \]

13.1.5. The final energy estimate. Also using that \( E_N(0) \leq \varepsilon^2 \) we obtain
\[ E_N(T) + S_N(T) \leq 8 \varepsilon^2 + C \varepsilon S_N(T) + \int_0^T \frac{C \varepsilon}{1+t} E_N(t) \, dt + \int_0^T \frac{C N \varepsilon^2}{1+t} E_N(t)^{1/2} \, dt + \int_0^T \frac{C_N M}{(1+t)^{2-\gamma}} E_N(t)^{1/2} \, dt \]
Next we pick \( \varepsilon \) so small that \( C \varepsilon < 1/2 \) so the term with \( S_N(T) \) in the right can be absorbed in the left. Let \( G(T)^2 \) denote the right hand side after this term has been removed. Then
\[ 2G(T)G'(t) \leq \frac{C \varepsilon}{1+t} G(t)^2 + \frac{C_N \varepsilon^2}{1+t} G(t) + \frac{C_N M}{(1+t)^{2-\gamma}} G(t) \]
Dividing by \( G(t) \) and multiplying by the integrating factor \( (1+t)^{-C \varepsilon} \) gives
\[ 2 \frac{d}{dt} \left( G(t)(1+t)^{-C \varepsilon} \right) \leq \frac{C_N \varepsilon^2}{(1+t)^{1+C \varepsilon}} + \frac{C_N M}{(1+t)^{2-\gamma+C \varepsilon}} \]
and integrating and dividing by the same integrating factor gives since also \( M \leq \varepsilon \)
\[ G(t) \leq G(0)(1+t)^{C \varepsilon} + C_N \varepsilon(1+t)^{C \varepsilon}, \]
Hence, for some other constant $C_N$, that is independent of $C_b$ and $T$ in the bootstrap assumptions (11.3), and constants $C'_b < \infty$ and $\varepsilon_b > 0$, that may depend depends on $C_b$ but not on $T$, we have

$$E_N(t) \leq C_N\varepsilon^2 (1 + t)C'_b\varepsilon, \quad \text{for} \quad 0 \leq t \leq T, \quad \text{and} \quad \varepsilon \leq \varepsilon_b \quad (13.1)$$

If we choose $C_b = 2C_N$ and $\varepsilon$ so that $C'_b\varepsilon < \delta$ we get back a better estimate than we started with.

**Appendix A. The Ricci curvature in terms of generalized wave coordinates**

Here we derive the expression for Einstein’s equations for the metric $g$ in terms of a quantity that is assumed to be under control by a generalized wave coordinate condition. In this paper we will only use these equations in the case this quantity vanishes. However, in Appendix B we show that the equations we use can alternatively be derived as expressing the metric in generalized wave coordinates.

We consider the generalized harmonic coordinate condition

$$\Gamma^\alpha = \Gamma^\beta_\gamma g^{\beta\gamma}, \quad (A.1a)$$

where $\Gamma$ is a known vector. Using this notation, the harmonic coordinate condition is equivalent to the condition $\Gamma^\alpha = 0$. We can rewrite (A.1a) as

$$g^{\alpha\beta} \partial_\alpha g_{\beta\gamma} = \Gamma_\gamma + \Gamma^\delta_\gamma. \quad (A.1b)$$

The Ricci curvature tensor

$$R_{\alpha\beta} = \partial_\gamma \Gamma^\gamma_\alpha - \partial_\beta \Gamma^\gamma_\alpha + \Gamma^\gamma_\rho \Gamma^\rho_\alpha - \Gamma^\gamma_\alpha \Gamma^\rho_\rho, \quad (A.2)$$

satisfies the following identity

**Lemma A.1.**

$$R_{\alpha\beta} = -\frac{1}{2} \Box g_{\alpha\beta} + \frac{1}{2} \Gamma^\delta_\beta \partial_\delta g_{\alpha\beta} + \frac{1}{2} (\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha) - \frac{1}{2} \Gamma_\alpha \Gamma_\beta + \frac{1}{2} \Gamma^\gamma_\alpha \Gamma^\rho_\beta - \frac{1}{4} g^{\gamma\delta} g^{\rho\sigma} \partial_\alpha g_{\rho\sigma} \partial_\beta g_{\gamma\delta} + Q_{\alpha\beta},$$

where $Q_{\alpha\beta}$ is a linear combination of classical null forms.

**Proof.** For the proof we expand each of the four terms in (A.2). First,

$$\partial_\gamma \Gamma^\gamma_\alpha = \frac{1}{2} \left( \partial_\gamma g^{\gamma\delta} \right) \left( \partial_\alpha g_{\gamma\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta} \right) + \frac{1}{2} g^{\alpha\delta} \left( \partial_\alpha \partial_\gamma g_{\beta\delta} + \partial_\beta \partial_\gamma g_{\alpha\delta} - \partial_\delta \partial_\beta g_{\alpha\beta} \right)$$

$$= \frac{1}{2} \left( \partial_\gamma g^{\gamma\delta} \right) \left( \partial_\alpha g_{\gamma\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta} \right) + \frac{1}{2} \left( \partial_\alpha \left( g^{\gamma\delta} \partial_\gamma g_{\beta\delta} \right) + \partial_\beta \left( g^{\gamma\delta} \partial_\gamma g_{\alpha\delta} \right) - g^{\gamma\delta} \partial_\gamma \partial_\beta g_{\alpha\delta} \right)$$

$$- \frac{1}{2} \partial_\alpha g^{\gamma\delta} \partial_\gamma g_{\beta\delta} - \frac{1}{2} \partial_\beta g^{\gamma\delta} \partial_\gamma g_{\alpha\delta}.$$ 

It follows from (A.1b) that

$$-\frac{1}{2} \partial_\gamma g^{\gamma\delta} = \frac{1}{2} \Gamma^\delta_\beta + \frac{1}{4} g^{\rho\sigma} g^{\gamma\delta} \partial_\gamma g_{\rho\sigma},$$

and therefore,

$$-\frac{1}{2} \partial_\gamma g^{\gamma\delta} \partial_\delta g_{\alpha\beta} = \frac{1}{2} \Gamma^\delta_\beta \partial_\delta g_{\alpha\beta} + \frac{1}{4} g^{\rho\sigma} g^{\gamma\delta} \partial_\gamma g_{\rho\sigma} \partial_\delta g_{\alpha\beta}.$$ 

If we define

$$Q_{1\alpha\beta} = \frac{1}{2} \left( \partial_\gamma g^{\gamma\delta} \partial_\alpha g_{\beta\gamma} - \partial_\alpha g^{\gamma\delta} \partial_\gamma g_{\beta\alpha} \right) + \frac{1}{2} \left( \partial_\gamma g^{\gamma\delta} \partial_\beta g_{\alpha\delta} - \partial_\beta g^{\gamma\delta} \partial_\gamma g_{\alpha\delta} \right) + \frac{1}{4} g^{\rho\sigma} g^{\gamma\delta} \partial_\gamma g_{\rho\sigma} \partial_\delta g_{\alpha\beta},$$

and apply (A.1b) to rewrite

$$\frac{1}{2} \partial_\alpha \left( g^{\gamma\delta} \partial_\gamma g_{\beta\delta} \right) = \frac{1}{2} \partial_\alpha \Gamma_\beta + \frac{1}{4} \partial_\alpha g^{\gamma\delta} \partial_\beta g_{\gamma\delta} + \frac{1}{4} g^{\gamma\delta} \partial_\alpha \partial_\beta g_{\gamma\delta},$$

$$\frac{1}{2} \partial_\beta \left( g^{\gamma\delta} \partial_\gamma g_{\alpha\delta} \right) = \frac{1}{2} \partial_\beta \Gamma_\alpha + \frac{1}{4} \partial_\beta g^{\gamma\delta} \partial_\alpha g_{\gamma\delta} + \frac{1}{4} g^{\gamma\delta} \partial_\alpha \partial_\beta g_{\gamma\delta},$$

$$\frac{1}{2} \partial_\gamma g^{\gamma\delta} \partial_\delta g_{\alpha\beta} = \frac{1}{2} \partial_\gamma \Gamma_\beta + \frac{1}{4} \partial_\gamma g^{\rho\sigma} g^{\gamma\delta} \partial_\rho g_{\sigma\delta} + \frac{1}{4} g^{\gamma\delta} \partial_\gamma \partial_\rho g_{\sigma\delta}.$$
we have the following identity:
\[
\partial_\gamma \Gamma_{\alpha\beta}^\gamma = \frac{1}{2} \Gamma^\delta \partial_\delta g_{\alpha\beta} + \frac{1}{2} (\partial_\alpha \Gamma_{\beta} - \partial_\beta \Gamma_{\alpha}) + \frac{1}{4} (\partial_\alpha g^\gamma \partial_{\beta\gamma} + \partial_\beta g^\gamma \partial_{\alpha\gamma}) + \frac{1}{2} g^\gamma \partial_\alpha \partial_\beta g_{\gamma\delta} - \frac{1}{2} \Box g_{\alpha\beta} + Q_{\alpha\beta}^\delta.
\]

Now we recall the identity
\[
\Gamma_{\alpha\gamma}^\gamma = \frac{1}{2} g^\gamma \partial_\alpha g_{\gamma\delta}.
\]

It follows that
\[
-\partial_\beta \Gamma_{\alpha\gamma}^\gamma = -\frac{1}{2} \partial_\beta g^\gamma \partial_\alpha g_{\gamma\delta} - \frac{1}{2} g^\gamma \partial_\alpha \partial_\beta g_{\gamma\delta}.
\]

We can define
\[
Q_{\alpha\beta}^2 = \frac{1}{4} (\partial_\alpha g^\gamma \partial_{\beta\gamma} g_{\delta\delta} - \partial_\beta g^\gamma \partial_{\alpha\gamma} g_{\delta\delta})
\]
in order to rewrite
\[
-\partial_\beta \Gamma_{\alpha\gamma}^\gamma = -\frac{1}{4} (\partial_\alpha g^\gamma \partial_{\beta\gamma} g_{\delta\delta} + \partial_\beta g^\gamma \partial_{\alpha\gamma} g_{\delta\delta}) - \frac{1}{2} g^\gamma \partial_\alpha \partial_\beta g_{\gamma\delta} + Q_{\alpha\beta}^2.
\]

Next, we expand
\[
\Gamma_{\alpha\rho}^\gamma \Gamma_{\alpha\beta}^\rho = \frac{1}{4} g^{\rho\sigma} g^\gamma \partial_\rho g_{\beta\delta} (\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta}).
\]

We define
\[
Q_{\alpha\beta}^2 = \frac{1}{4} g^{\rho\sigma} g^\gamma \partial_\rho g_{\beta\delta} (\partial_\alpha g_{\beta\sigma} - \partial_\alpha g_{\beta\sigma} - \partial_\alpha g_{\beta\sigma} - \partial_\alpha g_{\beta\sigma}) + (\partial_\rho g_{\beta\delta} \partial_\beta g_{\alpha\sigma} - \partial_\beta g_{\beta\delta} \partial_\alpha g_{\alpha\sigma} - \partial_\beta g_{\beta\delta} \partial_\alpha g_{\alpha\sigma} - \partial_\beta g_{\beta\delta} \partial_\alpha g_{\alpha\sigma})
\]

and rewrite
\[
\frac{1}{4} g^\gamma \partial_\alpha g_{\beta\delta} g^{\rho\sigma} \partial_\rho g_{\beta\sigma} = \frac{1}{4} g^\gamma \partial_\alpha g_{\beta\delta} \left( \Gamma_{\beta} + \frac{1}{2} g^{\rho\sigma} \partial_\beta g_{\rho\sigma} \right)
\]
\[
\frac{1}{4} g^\gamma \partial_\beta g_{\beta\delta} g^{\rho\sigma} \partial_\rho g_{\alpha\sigma} = \frac{1}{4} g^\gamma \partial_\beta g_{\beta\delta} \left( \Gamma_{\alpha} + \frac{1}{2} g^{\rho\sigma} \partial_\alpha g_{\rho\sigma} \right).
\]

We therefore have
\[
\Gamma_{\alpha\rho}^\gamma \Gamma_{\alpha\beta}^\rho = Q_{\alpha\beta}^2 + \frac{1}{2} (\Gamma_{\alpha\gamma}^\gamma \Gamma_{\beta\gamma} + \Gamma_{\beta\gamma}^\gamma \Gamma_{\alpha\gamma}) + \Gamma_{\alpha\gamma}^\gamma \Gamma_{\beta\rho}^\rho
\]

Now we take the final term,
\[
-\Gamma_{\alpha\rho}^\gamma \Gamma_{\beta\gamma}^\rho = -\frac{1}{4} g^\gamma \partial_\alpha g_{\beta\delta} (\partial_\gamma g_{\beta\sigma} + \partial_\gamma g_{\beta\sigma} - \partial_\gamma g_{\beta\sigma} - \partial_\gamma g_{\beta\sigma}).
\]

We expand it and deal with it term-by-term. We first take
\[
-\frac{1}{4} g^\gamma \partial_\alpha g_{\beta\delta} (\partial_\gamma g_{\beta\sigma} - \partial_\gamma g_{\beta\sigma}) = -\frac{1}{4} g^\gamma \partial_\alpha g_{\beta\delta} (\partial_\gamma g_{\beta\sigma} - \partial_\gamma g_{\beta\sigma})
\]
\[
- (\partial_\alpha g_{\beta\delta} \partial_\gamma g_{\beta\sigma} - \partial_\alpha g_{\beta\delta} \partial_\gamma g_{\beta\sigma}) = -\frac{1}{4} g^\gamma \partial_\alpha g_{\beta\delta} (\partial_\gamma g_{\beta\sigma} - \partial_\gamma g_{\beta\sigma}) (\partial_\alpha g_{\beta\delta} \partial_\gamma g_{\beta\sigma} - \partial_\alpha g_{\beta\delta} \partial_\gamma g_{\beta\sigma}).
\]

The two terms that cancel out are identical, which follows straightforwardly from renaming contracted indices and noting symmetry of \( g \). Similarly,
\[
-\frac{1}{4} g^\gamma \partial_\alpha g_{\beta\delta} \partial_\gamma g_{\beta\sigma} (\partial_\rho g_{\alpha\delta} - \partial_\delta g_{\alpha\rho}) = -\frac{1}{4} g^\gamma \partial_\alpha g_{\beta\delta} (\partial_\gamma g_{\beta\sigma} - \partial_\gamma g_{\beta\sigma}) (\partial_\rho g_{\alpha\delta} - \partial_\delta g_{\alpha\rho}) (\partial_\gamma g_{\beta\sigma} - \partial_\gamma g_{\beta\sigma}).
\]

Next, we take
\[
-\frac{1}{4} g^\gamma \partial_\alpha g_{\beta\delta} \partial_\gamma g_{\beta\sigma} + \partial_\delta g_{\alpha\rho} \partial_\gamma g_{\beta\sigma}).
\]
By renaming indices we have
\[-\frac{1}{4}g^{\gamma\delta}g^{\rho\sigma}(\partial_{\rho}g_{\alpha\delta}\partial_{\gamma}g_{\beta\sigma} + \partial_{\delta}g_{\alpha\rho}\partial_{\sigma}g_{\beta\gamma}) = -\frac{1}{2}g^{\gamma\delta}g^{\rho\sigma}(\partial_{\rho}g_{\alpha\delta}\partial_{\gamma}g_{\beta\sigma} + \partial_{\delta}g_{\alpha\rho}\partial_{\sigma}g_{\beta\gamma})]\n\quad= -\frac{1}{2}g^{\gamma\delta}g^{\rho\sigma}((\partial_{\rho}g_{\alpha\delta}\partial_{\gamma}g_{\beta\sigma} - \partial_{\gamma}g_{\alpha\delta}\partial_{\rho}g_{\beta\sigma}) + \partial_{\gamma}g_{\alpha\delta}\partial_{\rho}g_{\beta\sigma}) - \frac{1}{2}(\Gamma_{\alpha} + \Gamma_{\gamma})(\Gamma_{\beta} + \Gamma_{\delta}).\]

Defining
\[Q_{\alpha\beta}^4 = -\frac{1}{4}g^{\gamma\delta}g^{\rho\sigma}((\partial_{\alpha}g_{\delta\rho}\partial_{\beta}g_{\gamma\sigma} - \partial_{\beta}g_{\delta\rho}\partial_{\alpha}g_{\gamma\sigma}) - (\partial_{\alpha}g_{\delta\rho}\partial_{\sigma}g_{\beta\gamma} - \partial_{\sigma}g_{\delta\rho}\partial_{\alpha}g_{\beta\gamma})) - \frac{1}{4}g^{\gamma\delta}g^{\rho\sigma}((\partial_{\beta}g_{\sigma\rho}\partial_{\alpha}g_{\gamma\delta} - \partial_{\alpha}g_{\sigma\rho}\partial_{\beta}g_{\gamma\delta}) - (\partial_{\beta}g_{\sigma\rho}\partial_{\delta}g_{\alpha\gamma} - \partial_{\delta}g_{\sigma\rho}\partial_{\beta}g_{\alpha\gamma})) + \frac{1}{4}g^{\gamma\delta}g^{\rho\sigma}(\partial_{\rho}g_{\alpha\delta}\partial_{\gamma}g_{\beta\sigma} + \partial_{\gamma}g_{\alpha\delta}\partial_{\rho}g_{\beta\sigma} - \frac{1}{4}g^{\gamma\delta}g^{\rho\sigma}(\partial_{\rho}g_{\alpha\delta}\partial_{\gamma}g_{\beta\sigma} - \partial_{\gamma}g_{\alpha\delta}\partial_{\rho}g_{\beta\sigma}),\]

we can expand
\[-\Gamma_{\alpha\rho} \Gamma_{\beta\gamma}^\rho = -\frac{1}{4}g^{\gamma\delta}g^{\rho\sigma}\partial_{\alpha}g_{\delta\rho}\partial_{\beta}g_{\gamma\sigma} - \frac{1}{2}(\Gamma_{\alpha} + \Gamma_{\gamma})(\Gamma_{\beta} + \Gamma_{\delta}) + Q_{\alpha\beta}^4.\]

We can combine everything to get
\[R_{\alpha\beta} = \frac{1}{2}\Box g_{\alpha\beta} + \frac{1}{2}(\partial_{\alpha}\Gamma_{\beta} + \partial_{\beta}\Gamma_{\alpha}) - \frac{1}{2}\Gamma_{\alpha}\Gamma_{\beta} + \frac{1}{2}\Gamma_{\alpha}\Gamma_{\beta}^\rho - \frac{1}{4}g^{\gamma\delta}g^{\rho\sigma}\partial_{\alpha}g_{\delta\rho}\partial_{\beta}g_{\gamma\sigma} + Q_{\alpha\beta}^4 + Q_{\alpha\beta}^2 + Q_{\alpha\beta}^3 + Q_{\alpha\beta}^4.\]

A.0.1. The expression for Ricci curvature in terms of the wave operator. For a general metric \(g\) twice the Ricci curvature can by Lemma A.1 be written
\[2R_{\mu\nu} = -\Box g_{\mu\nu} + \partial_{\mu}\Gamma_{\nu} + \partial_{\nu}\Gamma_{\mu} + F_{\mu\nu}(g)[\partial g_{\mu\nu}] - \Gamma_{\nu}H_{\mu} + \Gamma_{\mu}H_{\nu},\]

where \(\Gamma_{\mu} = g_{\mu\beta}g^{\alpha\beta}\Gamma_{\alpha\beta}^\gamma, (A.3)\) and \(F_{\mu\nu}\) as in \((4.5)\). The Einstein vacuum equations in harmonic coordinates are \(R_{\mu\nu} = 0\) and \(\Gamma_{\mu} = 0\). The Minkowski metric and the Schwarzschild expressed in harmonic coordinates satisfy these. Since \(m_0\) in \((4.3)\) is the leading term in the expansion of the Schwarzschild metric it is therefore not surprising that it approximately satisfies these. By a similar calculation as above using \((4.2)\) we have
\[\Gamma_{\mu}^0 = m_{0}^{\alpha\beta}\Gamma_{\alpha\beta}^0 = m_{0}^{\alpha\beta} - \frac{1}{2}m_{0}^{\alpha\beta}m_{0}^{-1}\partial_{\alpha}m_{0}^{-\mu} = \partial_{\alpha}H_{\mu}^0 - \frac{1}{2}m_{0}^{-\mu}m_{0}^{\alpha\beta}\partial_{\alpha}H_{\mu}^{0\beta} + W^{\mu}(H_0)[H_0, \partial H_0] = \chi'(\frac{r}{1+\chi})M\delta^{(\beta)}r^{-2} + \chi''(\frac{r}{1+\chi}, \omega, \frac{M}{r})M^2r^{-3}.\]

Also using \((4.7)\) it follows that
\[2R_{\mu\nu}^0 = -\Box m_{0\mu\nu} + \partial_{\mu}\Gamma_{\nu}^0 + \partial_{\nu}\Gamma_{\mu}^0 + \Gamma_{\beta}^\delta\partial_{\beta}m_{0\mu\nu} + F_{\mu\nu}(m_0)[\partial m_0, \partial m_0] - \Gamma_{\beta}^\mu r_0^\nu\]
\[= \chi'_{2\nu}(\frac{r}{1+\chi}, \omega, \frac{M}{r})M^{-3} + \chi'_{2\nu}(\frac{r}{1+\chi}, \omega, \frac{M}{r})M^2r^{-4}.\]
B.0.1. The reduced Einstein’s equations in generalized wave coordinates. By (A.3) Einstein’s equations $\tilde{R}_{ab} = 0$ in generalized wave coordinates are

$$\tilde{\Box} \tilde{g}_{ab} - \tilde{\partial}_a \tilde{W}_b(\tilde{g}) - \tilde{\partial}_b \tilde{W}_a(\tilde{g}) - \tilde{W}^c(\tilde{g}) \tilde{\partial}_c \tilde{g}_{ab} = \tilde{F}_{ab}(\tilde{g})[\tilde{\partial} \tilde{g}, \tilde{\partial} \tilde{g}] - \tilde{W}_a(\tilde{g}) \tilde{W}_b(\tilde{g}) + \tilde{T}_{ab}, \quad (B.1)$$

where $\tilde{\Box} \tilde{g} = \tilde{g}^{ab} \tilde{\partial}_a \tilde{\partial}_b$, $\tilde{W}_c = \tilde{g}^{cd} \tilde{W}_d$ and $\tilde{W}^c$ are some given functions of the coordinate $\tilde{x}$ and the metric $\tilde{g}$ not depending on its derivative. One can show that if $\tilde{g}$ satisfies the reduced equations (B.1) and $\tilde{\Gamma}^c_{ab}$ are its Christoffel symbols, then the generalized wave coordinate condition

$$\tilde{g}^{ab} \tilde{\Gamma}^c_{ab} = \tilde{W}^c(\tilde{g})$$

holds if it holds initially. In particular if $g$ is the metric expressed in wave coordinates $x$, and we choose new coordinates $\tilde{x} = \tilde{x}(x)$ which are given fixed functions of the original coordinates then by (5.5) - (5.4)

$$\tilde{W}^c(\tilde{g}) = \tilde{g}^{ab} \tilde{\Gamma}^c_{ab} = \tilde{W}^c(\tilde{g})[\tilde{\Gamma}], \quad (B.2)$$

where $\tilde{\Gamma}$ given by (5.4) are the Christoffel symbols of the fixed metric $\tilde{m}$ and the last equality indicates that it is linear in $\tilde{\Gamma}$ which by itself is $O(M(t + r)^{-2} \ln(t + r))$. By (B.1) and (B.2)

$$\tilde{\Box} \tilde{g}_{ab} - \tilde{\partial}_a \tilde{W}_b(\tilde{g})[\tilde{\Gamma}] - \tilde{\partial}_b \tilde{W}_a(\tilde{g})[\tilde{\Gamma}] - \tilde{W}^c(\tilde{g})[\tilde{\Gamma}] \tilde{\partial}_c \tilde{g}_{ab} = \tilde{F}_{ab}(\tilde{g})[\tilde{\partial} \tilde{g}, \tilde{\partial} \tilde{g}] - \tilde{W}_a(\tilde{g})[\tilde{\Gamma}, \tilde{\Gamma}] + \tilde{T}_{ab},$$

where

$$\tilde{W}_c(\tilde{g})[\tilde{\Gamma}] = \tilde{g}^{cd} \tilde{W}_d(\tilde{g})[\tilde{\Gamma}], \quad \text{and} \quad \tilde{W}_{ab}(\tilde{g})[\tilde{\Gamma}, \tilde{\Gamma}] = \tilde{W}_a(\tilde{g})[\tilde{\Gamma}] \tilde{W}_b(\tilde{g})[\tilde{\Gamma}]$$

Now let $h_{\alpha \beta} = g_{\alpha \beta} - m_{\alpha \beta}$ and $H_{\alpha \beta} = g^{\alpha \beta} - m^{\alpha \beta} = -h_{\alpha \beta} + O(h^2)$

$$\tilde{h}_{ab} = \tilde{g}_{ab} - \tilde{m}_{ab} = A_a^\alpha A_b^\beta h_{\alpha \beta} \quad \text{and} \quad \tilde{H}^{ab} = \tilde{g}^{ab} - \tilde{m}^{ab} = A_a^\alpha A_b^\beta H_{\alpha \beta}.$$\n
Since in particular the Minkowski metric is a solution of the vacuum equations we have get by (B.1) and (B.2)

$$\tilde{\Box} \tilde{m}_{ab} - \tilde{\partial}_a \tilde{W}_b(\tilde{m})[\tilde{\Gamma}] - \tilde{\partial}_b \tilde{W}_a(\tilde{m})[\tilde{\Gamma}] - \tilde{W}^c(\tilde{m})[\tilde{\Gamma}] \tilde{\partial}_c \tilde{m}_{ab} = \tilde{F}_{ab}(\tilde{m})[\tilde{\partial} \tilde{m}, \tilde{\partial} \tilde{m}] - \tilde{W}_a(\tilde{m})[\tilde{\Gamma}, \tilde{\Gamma}] + \tilde{T}_{ab},$$

Subtracting the two equations we get and equation for the difference

$$\tilde{\Box} \tilde{h}_{ab} - \tilde{H}^{cd} \tilde{\partial}_c \tilde{\partial}_d \tilde{m}_{ab} - \tilde{\partial}_a \tilde{W}_b(h, \tilde{m})[\tilde{\Gamma}] - \tilde{\partial}_b \tilde{W}_a(h, \tilde{m})[\tilde{\Gamma}] - \tilde{W}^c(h, \tilde{m})[\tilde{\Gamma}] \tilde{\partial}_c \tilde{h}_{ab} - \tilde{W}^c_2(h, \tilde{\Gamma}) \tilde{\partial}_c \tilde{m}_{ab} = \tilde{F}_{ab}(\tilde{g})[\tilde{\partial} \tilde{g}, \tilde{\partial} \tilde{g}] - \tilde{F}_{ab}(\tilde{m})[\tilde{\partial} \tilde{m}, \tilde{\partial} \tilde{m}] - \tilde{W}_a(\tilde{m})[\tilde{\Gamma}, \tilde{\Gamma}] + \tilde{T}_{ab},$$

where

$$\tilde{W}_c(h, \tilde{m})[\tilde{\Gamma}] = \tilde{W}_c(\tilde{g})[\tilde{\Gamma}] - \tilde{W}_c(\tilde{m})[\tilde{\Gamma}], \quad \tilde{W}_a(h, \tilde{m})[\tilde{\Gamma}, \tilde{\Gamma}] = \tilde{W}_a(\tilde{g})[\tilde{\Gamma}, \tilde{\Gamma}] - \tilde{W}_a(\tilde{m})[\tilde{\Gamma}, \tilde{\Gamma}]$$

$$\tilde{W}^c(\tilde{g})[\tilde{\Gamma}] \tilde{\partial}_c \tilde{h}_{ab} + \tilde{W}^c_2(\tilde{H}, \tilde{\Gamma}) \tilde{\partial}_c \tilde{m}_{ab} = \tilde{W}^c(\tilde{g})[\tilde{\Gamma}] \tilde{\partial}_c \tilde{g}_{ab} - \tilde{W}^c(\tilde{m})[\tilde{\Gamma}] \tilde{\partial}_c \tilde{m}_{ab}.$$\n
Here $W(\cdots)[u_1, \ldots, u_k]$ stands for functions that are separately linear in each of the arguments $u_1, \ldots, u_k$.

B.0.2. The generalized wave coordinate condition. We have

$$\tilde{\partial}_a g^{ac} - \frac{1}{2} \tilde{g}^{ac} g_{bd} \tilde{\partial}_a \tilde{g}^{bd} = \tilde{W}^c(\tilde{g})[\tilde{\Gamma}]$$

and in particular

$$\tilde{\partial}_a \tilde{m}^{ac} - \frac{1}{2} \tilde{m}^{ac} \tilde{m}_{bd} \tilde{\partial}_a \tilde{m}^{bd} = \tilde{W}^c(\tilde{m})[\tilde{\Gamma}].$$

It follows that $\tilde{H}^{ab} = \tilde{g}^{ab} - \tilde{m}^{ab}$ satisfy

$$\tilde{\partial}_a (\tilde{H}^{ac} - \frac{1}{2} \tilde{m}^{ac} \tilde{m}_{bd} \tilde{H}^{bd}) = \tilde{W}^c_1(\tilde{m}, \tilde{H})[\tilde{\Gamma}, \tilde{\partial} \tilde{m}] + \tilde{W}^c_2(\tilde{m}, \tilde{H})[\tilde{\Gamma}, \tilde{\partial} \tilde{H}].$$
C.0.1. Subtracting off a better approximation from the homogeneous wave equation picking up the mass in the exterior. We can get an improved approximation to the solution \( L \) of the wave equation in the interior of the light cone by instead of having a cutoff which is a function of \( r/t \) have a cutoff which is a function of \( r^* - t \). This however only works in the wave equation but not in the solution of the wave coordinate condition \( H \). This is not needed for the energy estimate and existence but only for more precise asymptotics in the interior, see [34] for the detailed proof. Here we just want to show how the error terms are under control because of the wave coordinate condition. One can use this and the other results in this paper to give a more direct proof of the asymptotics in [34].

Let \( \square^* = \tilde{m}^{ab} \partial_a \tilde{\partial}_b \) and

\[
\tilde{h}^{1e}_{ab} = \tilde{h}_{ab} - \tilde{h}^0_{ab}, \quad \text{where} \quad \tilde{h}^0_{ab} = \frac{M}{r^*} \tilde{\chi}(r^* - t) \delta_{ab}
\]

We have

\[
\square^* \tilde{h}^0_{cd} = 0
\]

Moreover, with \( L^a_b = \tilde{a}_a(r^* - t), \omega_0 = 0, \omega_i = x_i/|x|, \) for \( i \geq 1 \) and \( S_{0\beta} = S_{\alpha 0} = 0 \) and \( S_{ij} = \delta_{ij} - \omega_i \omega_j \)

\[
\tilde{\partial_a} \tilde{\partial_a} \tilde{h}^0_{cd} = (L^a_b \tilde{\chi}'(r^* - t) - \frac{S_{ab}}{r} \tilde{\chi}(r^* - t)) M \delta_{cd}.
\]

Hence with \( R = (0, \omega) \)

\[
\tilde{H}^{ab}_1 \tilde{\partial_a} \tilde{\partial_a} \tilde{h}^0_{cd} = \left( \tilde{H}_1 L^\ast L^* \tilde{\chi}(r^* - t) + (\frac{\tilde{\omega} \tilde{H}_1 - 2 \tilde{H}_1 L^\ast R}{r^2}) \tilde{\chi}(r^* - t) + (\frac{\tilde{\omega} \tilde{H}_1}{r^2}) \tilde{\chi}(r^* - t) \right) M \delta_{cd}.
\]

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