MIN-MAX MINIMAL HYPERSURFACES WITH OBSTACLE

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Abstract. We study min-max theory for area functional among hypersurfaces constrained in a smooth manifold with boundary. A Schoen-Simon-type regularity result is proved for integral varifolds which satisfies a variational inequality and restricts to a stable minimal hypersurface in the interior. Based on this, we show that for any admissible family of sweepouts $\Pi$ in a compact manifold with boundary, there always exists a closed $C^{1,1}$ hypersurface with codimension $\geq 7$ singular set in the interior and having mean curvature pointing outward along boundary realizing the width $L(\Pi)$.

1. Introduction

Min-max theory for minimal hypersurfaces in a closed manifold is established by Almgren-Pitts in [4, 29] and by Schoen-Simon [30] for regularity in higher dimensions. In recent years, the abundance of closed minimal hypersurfaces has been well studied by [19, 22, 24, 26, 32].

We consider in this paper the following min-max problem in a manifold with boundary.

Question: Can we find a minimal hypersurface constrained in a compact smooth domain $M$ in an oriented Riemannian manifold $(\tilde{M}, g)$?

Clearly, the answer is no in general if one only focus on closed surfaces with vanishing mean curvature. Allowing an extension of subjects we would like to find, two distinct aspects of this problem are frequently studied.

• Free boundary problem A minimal hypersurface with free boundary in $M$ is a hypersurface with boundary $\partial M$, perpendicular to $\partial M$ and having vanishing mean curvature. The study of free boundary minimal surfaces has a long history, we refer the readers to [21] for a beautiful survey of the development.

• Obstacle problem Let $(\tilde{M}, g) = (\mathbb{R}^{n+1}, |dx|^2)$, $E_0 \subset \mathbb{R}^{n+1}$ be a bounded Caccioppoli set; Consider a minimizer $\hat{E}$ of the perimeter of Caccioppoli sets among

\[ \{ E \subset \mathbb{R}^{n+1} : E \Delta E_0 \subset M \} \]

The regularity of boundary of $\hat{E}$ is well studied in [5,23,27,33], and a non-parametric version is studied in [8,9,13,20] under weaker assumptions on $M$. We emphasis here that in general, the optimal regularity for $\partial \hat{E}$ is $C^{1,1}$ instead of $C^2$ where $\partial \hat{E}$ touch $\partial M$, and the mean curvature of $\partial \hat{E}$ does not necessarily vanish. We refer the readers to [16] for a brief introduction and generalizations when $M$ is not necessarily smooth. For later use in this paper, see subsection 2.4.

The min-max analogue of obstacle problem for area functional and other parametric elliptic integral is also studied recent year under further convexity assumption on $M$, and hence the stationarity of solutions valid by maximum principle. See [6,11,28,32].

The first goal of this paper is to prove the following,

Theorem 1.1. Let $m \geq 1$, $n \geq 2$, $Z_a(M)$ be the space of integral cycles supported in $M$, endowed with $F$-metric (see section 2.7), $\Phi_0 : I^m \to (Z_a(M), F)$ be a continuous map. Let $\Pi$ be the family of
sequences of $F$-continuous maps $S = \{\Phi_i : I^m \to Z_n(M)\}_{i \geq 1}$ which $C^0$-tends to $\Phi_0|_{\partial I^m}$ restricted on $\partial I^m$. Let

$$L(\Pi) := \inf_{S = \{\Phi_i\}_{i \geq 1} \in \Pi} \limsup_{i \to \infty} \sup_{x \in I^m} M(\Phi_i(x)) ; \quad L_{\Phi_0,\partial I^m} := \sup_{x \in \partial I^m} M(\Phi_0(x))$$

If $L(\Pi) > L_{\Phi_0,\partial I^m}$, then there exists a constrained embedded minimal hypersurface in $M$ with the same mass as $L(\Pi)$ (see definition 2.3). In particular, there’s an immersed $C^{1,1}$ hypersurface $\Sigma \hookrightarrow M$ such that

- $\mathcal{H}^n(\Sigma) = L(\Pi)$
- $\Sigma$ is minimal, locally stable and embedded with multiplicity and codim $\geq 7$ singularity in the interior of $M$;
- $\Sigma$ is multiple $C^{1,1}$ graphs near $\partial M$, and the mean curvature vector of $\Sigma$ on $\Sigma \cap \partial M$ points outward, i.e.
  $$\vec{H}_{\Sigma} \cdot \nu_M \leq 0$$

for inward normal field $\nu_M$ of $\partial M$

For the precise statement, see theorem 1.1 and the notations in section 2.

Our proof is based on the idea of min-max construction in [29] and the regularity result in [30]. Two major difficulties are the lack of Schoen-Simon-type regularity for stable varifolds near $\partial M$, and the invalidity of unique continuation result for minimal hypersurface with obstacle. To fix this, we propose a general structure theorem for a certain class of varifolds $V$ near $\partial M$, saying that locally $V$ is the sum of finitely many $C^1$-graphs over $\partial M$, see theorem 3.2. The improved regularity of min-max solution then follows from a standard replacement argument together with a uniform $C^{1,1}$ estimate for graphs satisfying a variational inequality. Based on this, the compactness is also proved for a family of constrained embedded stable minimal hypersurfaces, see corollary 3.3.

A natural question is when the hypersurface we construct above has vanishing mean curvature. By strong maximum principle [35], this is the case when $M$ is weakly mean convex. Without mean convexity, in section 5, we prove the following rigidity theorem,

**Theorem 1.2.** Let $\Sigma \subset N^{n+1}$ be a two-sided, non-degenerate smooth closed minimal hypersurface; $\epsilon \in (0,1)$. Then there’s $s_0 = s_0(\Sigma, N, \epsilon) > 0$ such that if $\delta \in (0, s_0)$, then any constrained embedded minimal hypersurface in the $\delta$-neighborhood $\text{Clos}(B_\delta(\Sigma))$ of $\Sigma$ with the mass between $\mathcal{H}^n(\Sigma)$ and $(2 - \epsilon)\mathcal{H}^n(\Sigma)$ is $\Sigma$ itself.

In view of the local min-max property by [34], we can see from theorem 1.1 and theorem 1.2 that a non-degenerate two-sided smooth minimal hypersurface could be reconstruct from min-max in a sufficiently small neighborhood. In an upcoming work, we generalize the approach above to construct minimal hypersurfaces lying nearby certain singular minimal hypersurface under any sufficiently small perturbation of metric.

**Organization of the paper.** In section 2 we introduce some notations and preliminaries for constrained stationary varifold, an object we shall deal with throughout this note. We also include a brief survey on obstacle problem in the minimizing setting in 2.4 and two $C^{1,1}$-regularity theorems we shall use later, the proof of which are in the appendix; section 3 is devoted to the proof of main structure theorem for constrained stationary varifolds that are stable in the interior. In section 4, we discuss the min-max constructions and prove a precise version of theorem 1.1. Section 5 discusses theorem 1.2.

**Acknowledgement.** I am grateful to my advisor Fernando Codá Marques for his constant support and guidance. I’m thankful to Chao Li and Xin Zhou for suggesting this problem and inspiring discussions and to Fanghua Lin for helpful explanations to his thesis [23] on obstacle problem. I would also thank Yangyang Li, Zhenhua Liu and Lu Wang for their interest in this work.
2. Preliminaries

2.1. Notations. Here are some notations we adopt throughout the article. In an $L$-dimensional Euclidean space $\mathbb{R}^L$,

- $\mathbb{B}_r^L(p)$ the open ball of radius $r$ centered at $p$
- $\mathbb{S}_r^{L-1}(p)$ the sphere of radius $r$ centered at $p$
- $A_{s,r}(p)$ the open annuli $\mathbb{B}_r(p) \setminus \overline{\mathbb{B}_s(p)}$ centered at $p$
- $\eta_{p,r}$ the map between $\mathbb{R}^L$, maps $x$ to $(x-p)/r$
- $G(L,k)$ the Grassmannian of $k$-dimensional unoriented linear subspace in $\mathbb{R}^L$
- $\mathcal{H}^k$ $k$-dimensional Hausdorff measure
- $\omega_k$ the volume of $k$-dimensional unit ball
- $\theta^k(x,\mu)$ the $k$-th density (if exists) of a Radon measure $\mu$ at $x$, i.e. $\lim_{r \to 0} \frac{1}{\omega_k r^k} \mu(B_r^L(x))$

Let $(M, g)$ be a compact oriented $n+1$-dimensional smooth manifold with boundary $\partial M \neq \emptyset$, $n \geq 2$; For sake of simplicity, we can always extend $(M, g)$ to a closed $n+1$-manifold $(\bar{M}, \bar{g})$ isometrically embedded in $\mathbb{R}^L$. We always work under the intrinsic topology of $\bar{M}$. Denote the intrinsic Riemannian metric by $\langle \cdot, \cdot \rangle$ and the Levi-Civita connection by $\nabla$. Also write

- $\text{Int}(A)$ the interior of a subset $A \subset \bar{M}$
- $\text{Clos}(A)$ the closure of a subset $A \subset \bar{M}$
- $\text{dist}_M$ the intrinsic distant function on $M$
- $\text{dist}_{\partial M}$ the intrinsic distant function on $\partial M$
- $B_r(p)$ the open geodesic ball in $\bar{M}$ of radius $r$ centered at $p$
- $B_r(E)$ the $r$ neighborhood of $E \subset \bar{M}$, $\bigcup_{p \in E} B_r(p)$
- $S_r(p)$ the geodesic shpere in $\bar{M}$ of radius $r$ centered at $p$
- $A_{s,r}(p)$ the open geodesic annuli in $\bar{M}$, $B_r(p) \setminus \overline{B_s(p)}$
- $B_r(p)$ the open geodesic ball in $\partial M$ of radius $r$ centered at $p$
- $\nu_{\partial M}$ the inward unit normal field of $\partial M$ with respect to $M$ extended to a neighborhood of $\partial M$ by $\nabla \nu_{\partial M} = 0$
- $\bar{H}_{\partial M} := -\text{div}_{\partial M}(\nu_{\partial M}) \cdot \nu_{\partial M} = -H_{\partial M} \cdot \nu_{\partial M}$, the mean curvature vector of $\partial M$
- $A_{\partial M} := \nabla \nu_{\partial M}$, the 2nd fundamental form of $\partial M$
- $T^*_p M := \{v \in T_p^*M : v \cdot \nu_{\partial M} \geq 0\}$, the inward pointed vectors in $T_p^*M$
- $\exp_M$ the exponential map of $\bar{M}$
- $\text{inj}(\partial M)$ the intrinsic injectivity radius of $\partial M$
- $\mathcal{X}_c(U)$ the space of compactly supported vector fields in a relative open subset $U \subset M$
- $\mathcal{X}_c^+(U) := \{X \in \mathcal{X}_c(U) : X_p \in T_p^*M, \forall p \in \partial M\}$
- $\mathcal{X}_c^{\text{tan}}(U) := \{X \in \mathcal{X}_c(U) : X_p \in T_p \partial M, \forall p \in \partial M\}$

Let $r_M$ be the relative injective radius of $\partial M$ in $\bar{M}$,

$$r_M := \inf \left\{ \text{inj}(\partial M), \sup \{r : (x,s) \mapsto \exp_M^s(s \cdot \nu_{\partial M}(x)) \text{ is a diffeomorphism on } \partial M \times (-r,r) \} \right\}$$
We shall work under Fermi coordinates in $\partial_M^s(\partial M)$,

$$\Phi : \partial M \times (-r_M, r_M) \to B_r^s(\partial M), \quad (x, s) \mapsto \exp^s_M(s \cdot \nu_M(x))$$

and shall simply write a point $p = \Phi(x, s) \in B_r^s(\partial M)$ as $(x, s)$. Under this notation, for $x \in \partial M$ and $r \in (0, r_M)$ write

$$C_r(x) \quad \text{the open cylinder in } \hat{M}, \{(y, s) : \text{dist}_M(y, x) < r, |s| < r\}$$

$$C_r^-(x) \quad \text{the open half cylinder in } M, \{(y, s) : \text{dist}_M(y, x) < r, 0 < s < r\}$$

$$\partial^+ C_r(x) \quad \text{the top boundary of } C_r(x), \{(y, r) : \text{dist}_M(y, x) < r\}$$

$$\partial^* C_r(x) \quad \text{the side boundary of } C_r(x), \{(y, s) : \text{dist}_M(y, x) = r, |s| < r\}$$

For $u \in C^1(U, (-r_M, r_M))$ for some domain $U \subset \partial M$, define

$$\text{graph}_M(u) := \{(x, u(x)) : x \in U\}$$

and orient it by choosing the normal field to $\text{graph}_M(u)$ having positive inner product with $\nu_M$.

We also recall some basic notions of currents and varifolds and refer readers to the standard references [11], [31] for details. The following are the spaces we shall work with,

$$\mathcal{I}_k(M) \quad \text{the space of } k\text{-dimensional integral currents in } \mathbb{R}^L \text{ with support in } M$$

$$\mathcal{Z}_k(M) := \{T \in \mathcal{I}_k(M) : \partial T = 0\}$$

$$\mathcal{V}_k(M) \quad \text{the closure of } \mathcal{Z}_k(M) \text{ under varifold weak topology}$$

$$\mathcal{I}_k^+(M) \quad \text{the space of integral } k\text{-varifolds in } M \text{ supported in } M$$

For an immersed $k$-dimensional submanifold $\Sigma$ in $M$ with finite volume, let $[\Sigma]$ be the associated integral $k\text{-varifold in } \mathcal{I}_k^+(M)$; If further $\Sigma$ is oriented, let $[\Sigma]$ be the associated integral $k\text{-current}$. For $T \in \mathcal{I}_k(M)$, let $[T]$ and $\|T\|$ be the associated integral varifold and Radon measure on $M$.

For a relatively open subset $U \subset M$, let $\mathcal{F}_k(U), \mathcal{M}_U$ be the flat metric and mass norm on $\mathcal{I}_k(U)$, and $F$ be the metric compatible with weak convergence of varifolds in $\mathcal{M}$ on $\mathcal{I}_k(U)$. If $U = M$, we shall omit the subscript. The $F$-metric between integral currents is defined by $F(T, S) = F([T], [S]), S, T \in \mathcal{I}(M)$.

For a proper $C^1$ map $f$ between $\mathbb{R}^L$, $f_x$ be the push forward of varifold or current; For $E \subset M$ and $V \in \mathcal{V}_k(M)$, $V \lfloor E$ be the restriction of $V$ onto $E$.

For $X \in \mathcal{X}^+(M)$ and $V \in \mathcal{V}_k(M)$, let

$$\delta V(X) := \left. \frac{d}{dt} \right|_{t=0} M(e^{tX} \delta V) = \int \text{div}^S X(x) \, dV(x, S)$$

be the first variation of $V$, where $e^{tX}$ be the one-parameter family of diffeomorphism induced by $X$; $\text{div}^S X := \sum \langle \nabla_{e_i} X, e_i \rangle$, $\{e_1, ..., e_k\}$ be an orthonormal basis of $S$.

2.2. Constrained stationary varifolds. Throughout this subsection, let $M$ be as in subsection 2.1 $1 \leq k \leq n$, $U \subset M$ be a bounded relatively open subset. We note that the following definitions also generalize directly to properly embedded $M \subset \mathbb{R}^L$, not necessarily compact.

Definition 2.1. Call $V \in \mathcal{V}_k(M)$ stationary in $U$ if

$$\delta V(X) = 0 \quad \forall X \in \mathcal{X}_c(U)$$

Call $V \in \mathcal{V}_k(M)$ constrained stationary along $\partial M$ in $U$ if

$$\delta V(X) \geq 0 \quad \forall X \in \mathcal{X}_c^+(U)$$

Call $V$ stationary with free boundary along $\partial M$ in $U$ if

$$\delta V(X) = 0 \quad \forall X \in \mathcal{X}_c^{tan}(U)$$
We shall omit the description along $\partial M$ in both case if there’s no ambiguity. The same condition of a constrained stationary varifold is studied in [35], where it is called varifold minimizes area to first order. The notion of stationary varifold with free boundary was introduced in [17] for the study of free boundary minimal surfaces. Since $\mathcal{F}_c^{tan}(U) \subset \mathcal{F}_c^{\pm}(U)$ and $X \in \mathcal{F}_c^{tan}(U)$ iff $-X \in \mathcal{F}_c^{tan}(U)$ by definition, every constrained stationary varifold in $U$ is stationary with free boundary.

We list some basic results on constrained stationary varifolds, which will be used in the next few sections.

**Lemma 2.2** (Compactness). Let $(M_j, g_j) \subset \mathbb{R}^L$ be a family of manifold with boundary converging to $(M, g)$ in $C^0_{loc}$, $\mathcal{U} \subset \mathbb{R}^L$ be an open subset such that $U = \mathcal{U} \cap M$; Let $V_j \in \mathcal{V}_k(M_j)$ be constrained stationary in $\mathcal{U} \cap M$ and $V_j \rightharpoonup V \in \mathcal{V}_k(\mathbb{R}^L)$. Then, $V \in \mathcal{V}_k(M)$ is constrained stationary in $U$.

**Proof.** By the continuity of $\delta V(X)$ in $V$ w.r.t. varifold topology and in $X$ w.r.t. to $C^1$ topology. □

**Lemma 2.3** (Boundary Monotonicity). Let $\mathcal{K} \subset U \cap \partial M$ be a compact subset. Then there exists $R_0 = R_0(M, U, \mathcal{K}) > 0$, $\Lambda = \Lambda(M) > 0$ such that if $V \in \mathcal{V}_k(M)$ is a stationary varifold with free boundary along $\partial M$ in $U$, $p \in \mathcal{K}$, then the function

$$r \mapsto \frac{e^{\Lambda r}}{r^k} \|V((\mathcal{B}_r^L(p)))$$

is non-decreasing in $r \in (0, R_0)$. Moreover, when $(U, M, g) = (\mathbb{R}^n_{+1}, \mathbb{R}^{n+1}_{+1}, |dx|^2)$, if $\|V((B_r(p))/r^k$ is constant in $r \in (0, \infty)$, then $V$ is a cone, i.e.

$$(\eta_{p,r})_2V = V \quad \forall r \in (0, +\infty)$$

This is a corollary of [17] theorem 3.4, where $N = \partial M$ and an explicit monotonicity formula is proved.

**Corollary 2.4.** Suppose $V \in IV_k(M)$ is a stationary integral varifold with free boundary along $\partial M$ in $U$; $x \in \text{spt}(V) \cap U \cap \partial M$, then

$$\theta^k(x, \|V\|) \geq 2^{-k}$$

**Proof.** Since $V$ is integer multiplicity $k$-rectifiable, there exist (under Fermi coordinates) $p_j = (x_j, t_j) \to x$ such that $\theta^k(p_j, \|V\|) \geq 1$. Let $r_j := |x - x_j|$. Suppose WLOG that either $t_j \equiv 0$ or $t_j > 0$ for all $j \geq 1$. For every $0 < r << 1$, take $j > 1$ such that $r - r_j > 2t_j$. In the first case, by lemma 2.3

$$\frac{1}{\omega_k r^k} \|V((\mathcal{B}_r^L(x))) \geq \limsup_{j \to \infty} \frac{1}{\omega_k (r - r_j)^k} \|V((\mathcal{B}_{r-r_j}(x_j))) \geq 1$$

While in the second case,

$$\frac{1}{\omega_k r^k} \|V((\mathcal{B}_r^L(x))) \geq \limsup_{j \to \infty} \frac{1}{\omega_k (r - r_j)^k} \|V((\mathcal{B}_{r-r_j}^L(x_j)))$$

$$\geq \limsup_{j \to \infty} \frac{e^{\Lambda t_j}}{\omega_k (2t_j)^k} \|V((\mathcal{B}_{2t_j}^L(x_j)))$$

$$\geq \limsup_{j \to \infty} \frac{e^{\Lambda t_j}}{2^k \cdot \omega_k t_j^k} \|V((\mathcal{B}_r^L(p_j))) \geq 2^{-k}$$

where the second inequality follows from lemma 2.3 and the last inequality follows from interior monotonicity formula for stationary varifolds. Let $r \to 0$, the corollary is proved. □

**Lemma 2.5** (Strong maximum principle). Suppose $\mathcal{W} := U \cap \partial M$ is connected and mean convex, i.e. $\vec{H}_{\partial M} \cdot \nu_{\partial M} |_{\mathcal{W}} \geq 0$; $V \in TV_k(M)$ be a constrained stationary integral varifold along $\partial M$ in $U$. Suppose $\text{spt}(V) \cap \mathcal{W} \neq \emptyset$, then
But since the optimal regularity for constrained stationary varifolds. Suppose this subsection the following notion of constrained embedded hyper surfaces in order to describe \( M \).\n
\[ \text{Constrained embedded hypersurfaces.} \]

\[ \text{□} \]

independent of the converging sequences.\n
By lemma 2.2, 2.3, 2.6 and Allard compactness theorem \([31, \text{Chapter 8, 5.8}]\), up to a Proof. (1) is proved in \([35, \text{theorem 4}]\), so is (2) under the additional assumption that \( V \) is stationary. The only place stationarity is used is that \( \lambda \in \text{spt}(V) \cap W \) implies \( \theta^0(x, ||V||) \geq 1 \), which also holds for constrained stationary integral varifolds in mean convex domain by combining (1) and the proof of corollary 2.4. \( \Box \)

The following lemma guarantees that constrained stationary varifolds have locally bounded first variation. Recall \( r_{M} \) is defined in \([2.1]\).

**Lemma 2.6.** Let \( V \in \mathcal{V}_n(M) \) be constrained stationary in \( U \) with \( \|V\|(U) \leq \Lambda; U' \subset U \). Then \( \exists \ C = C(M,U,U') > 0 \) s.t.

\[ \delta V(X) \leq C\|V\|(U) \cdot \|X\|_{C^0} \quad \forall X \in \mathcal{X}_c(U') \]

Proof. We shall work under Fermi coordinate in \( B_{r_M}(\partial M) \), where \( \nu_{\partial M} \) is extended by \( \nabla_{\nu_{\partial M}}(\nu_{\partial M}) = 0 \). Let \( \eta \in \mathcal{C}^1(B_{r_M}(\partial M) \cap U), [0,1] \) and equal to 1 near \( \partial M \cap U' \). For every \( X \in \mathcal{X}_c(U') \) with \( \|X\|_{C^0} \leq 1 \), let \( X_0 := X - \eta \cdot (X,\nu_{\partial M})\nu_{\partial M} \). Notice that \( \pm X_0 \in \mathcal{X}_c(U) \) and \( 1 \pm (X,\nu_{\partial M}) \geq 0 \), hence

\[ \delta V(X) = \int \text{div}^S X_0 + \text{div}^S(\eta \cdot (X,\nu_{\partial M})\nu_{\partial M}) \, dV(x,S) \]

\[ = \int \text{div}^S(\eta \cdot (X,\nu_{\partial M}) - 1)\nu_{\partial M}) \, dV(x,S) + \int \text{div}^S(\eta \cdot \nu_{\partial M}) \, dV(x,S) \]

\[ \leq \int \text{div}^S(\eta \cdot \nu_{\partial M}) \, dV(x,S) \leq C(M,U,U')\|V\|(U). \]

where the first inequality follows from that \( \eta \cdot (1 - (X,\nu_{\partial M}))\nu_{\partial M} \in \mathcal{X}_c(U) \). Similarly,

\[ \delta V(X) = \int \text{div}^S(\eta \cdot (X,\nu_{\partial M}) + 1)\nu_{\partial M}) \, dV(x,S) - \int \text{div}^S(\eta \cdot \nu_{\partial M}) \, dV(x,S) \]

\[ \geq \int \text{div}^S(\eta \cdot \nu_{\partial M}) \, dV(x,S) \geq -C(M,U,U')\|V\|(U). \]

where \( C(M,U,U') := \|\eta \cdot \nu_{\partial M}\|_{C^1}. \)

\( \Box \)

**Corollary 2.7.** Let \( V \in \mathcal{V}_n(M) \) be a constrained stationary integral varifold in \( U \). Then \( \forall x \in \text{spt}(V) \cap U \cap \partial M, \) the tangent varifold of \( V \) at \( x \) exists and equals to \( m|T_x(\partial M)| \) for some \( m \in \mathbb{N} \).

Proof. By lemma 2.2, 2.3, 2.6 and Allard compactness theorem \([31, \text{Chapter 8, 5.8}]\), up to a subsequence of \( r_i \rightarrow 0, (\eta_{r_i},r_i)_{\mathcal{V}} \rightarrow C \in \mathcal{V}(T^*_x(\partial M)). \) Moreover, by the rigidity part of lemma 2.6, \( C \) is a constrained stationary cone along \( T^*_x(\partial M) \) centered at 0. Since \( 0 \in \text{spt}(C) \), by lemma 2.5, \( C = m|\partial M| + C_1 \) for some stationary varifold \( C_1 \) supported in \( T^*_x M \setminus T^*_x(\partial M) \) and \( m \in \mathbb{N} \). But since \( C_1 \) is also a cone, \( C_1 = 0 \) and \( C = m|\partial M| \) and hence is unique since \( m = \theta^0(x, \|V\|) \) is independent of the converging sequences. \( \Box \)

2.3. Constrained embedded hypersurfaces. A major problem in the regularity theory of constrained stationary varifolds is that one should not expect embeddedness near \( \partial M \). Consider for example, \( M = \mathbb{R}^{n+1} \setminus \mathbb{B}^{p+1}(0), V = [\mathbb{S}^p(0)]^{+1} = (x_{n+1} = 1) \in \mathcal{V}_n(M) \) is constrained stationary in \( M \) but is supported on an immersed submanifold with self-touching points. We introduce in this subsection the following notion of constrained embedded hypersurfaces in order to describe the optimal regularity for constrained stationary varifolds. Suppose \( U \subset M \) be a relatively open subset. Recall \( \text{Int}(U) = U \setminus \partial M. \)

**Definition 2.8.** Call \( V \in \mathcal{V}_n(U) \) a constrained embedded hypersurface with optimal regularity if \( \forall p \in U \), there’s a neighborhood \( U_p \) of \( p \) in \( U \), \( m \in \mathbb{N} \cup \{0\} \) and \( C^1,1 \) embedded hypersurfaces \( \Sigma_j \subset U_p \) without boundary, \( 1 \leq j \leq m \) such that
Obstacle problem. Problems in two different settings are well studied, mainly for hypersurfaces in the Euclidean space.

Let \( \text{Sing}(V) := \{ p \in \text{spt}(V) \cap \text{Int}(U) : p \text{ is not a regular point of hypersurfaces above in } U_p \} \).

Here a hypersurface \( \Sigma \subset M \) is called \( C^{1,\alpha} \) if locally it’s the graph of some \( C^{1,\alpha} \) function over some \( \mathbb{R}^n \)-plane.

**Remark 2.9.** (1) When a constrained embedded \( V \) is constrained stationary along \( \partial M \), it’s clear that \( \text{sp}(V) \cap \text{Int}(U) \) is a minimal hypersurface in \( \text{Int}(U) \); Also for every \( x \in \partial M \cap U \), suppose for some neighborhood \( U_x \) of \( x \) that \( V \cup U_x = \sum_j |\text{graph}_{\partial M}(u_j)| \), since \( u_j \) are \( C^{1,\alpha} \) and by lemma \( \{ u_j = 0 \} \subset \{ y \in \partial M : H_{\partial M} \geq 0 \} \). Hence \( \Delta_{\partial M} u_j = 0 \) a.e. \( x \in \{ u_j = 0 \} \), \( |\text{graph}_{\partial M}(u_j)| \) is constrained stationary along \( \partial M \) and has \( L^\infty \)-mean curvature \( H_{\partial M} \cdot \chi_{\{u_j=0\}} \).

(2) Also note that a constrained embedded hypersurface \( V \) could be viewed as the associated varifold of some immersed hypersurface. More precisely, consider \( \Sigma \subset M \times \mathbb{R} \) defined by

\[
\Sigma := \{(p, j) : p \in \text{sp}(V) \setminus \text{Sing}(V), 1 \leq j \leq \theta^n(x, ||V||)\}
\]

and endow topology on \( \Sigma \) by specifying neighborhoods of points in \( \Sigma \):

- If \( (p, j) \in \text{Int}(M) \times \mathbb{N} \setminus \Sigma \), \( O \subset \Sigma \) is a neighborhood of \((p, j)\) iff \( \exists \ U \ni p \) an open subset of \( M \) such that \( U \times \{j\} \subset O \);

- If \( (p, j) \in \partial M \times \mathbb{N} \setminus \Sigma \), let \( U_p \) be a neighborhood of \( p \) in \( M \) such that \( V \cup U_p = \sum_{i=1}^m |\text{graph}_{\partial M}(u_i)| \), where \( 0 \leq u_1 \leq u_2 \leq \ldots \leq u_m \) are \( C^{1,\alpha} \) functions over \( U_p \cap \partial M \), if \( u_i(x) = u_j(x) \), then either they both vanish or \( u_i = u_j \) on the connected component of \( \{ u_i > 0 \} \) containing \( x \). By definition of \( \Sigma \), \( m_p \geq j \); for each \( x \in U_p \cap \partial M \), define \( l_j(x) := \inf \{ l \in \mathbb{N} : u_l(x) = u_j(x) \} \).

Now, \( O \subset \Sigma \) is a neighborhood of \((p, j)\) iff \( \exists \ U \subset \partial M \cap U \) an open neighborhood of \( p \) in \( \partial M \) such that

\[
\{ (x, u_j(x), j - l_j(x) + 1) : x \in U \} \subset O
\]

where recall that \((x, u_j(x))\) represent a point in \( \text{sp}(V) \) under Fermi coordinate near \( \partial M \).

We leave it to readers to verify that such \( \Sigma \to M \) by projection onto first invariant is a \( C^{1,1} \) immersed submanifold.

**Definition 2.10.** Let \( \{ V_j \} \), \( V \) be constrained embedded hypersurfaces with optimal regularity in \( U \).

Call \( V_j \to V \) \( C^1 \)-converges to \( V \) if \forall \( p \in \text{sp}(V) \setminus \text{Sing}(V) \), there’s some neighborhood \( U_p \cap \text{Sing}(V) = \emptyset \) of \( p \) such that

(1) If \( p \in \text{Int}(U) \), then \( U_p \cap \partial M = \emptyset \), \( V \cup U_p = m|\Sigma| \) for some \( m \in \mathbb{N} \), \( \Sigma \subset U_p \) properly embedded hypersurface and \( V \cup U_p = \sum_{j=1}^m |\text{graph}_{\partial M}(u_j)| \) for some \( u_j \) with \( |u_j|_{C^1(\Sigma)} \to 0 \) as \( j \to \infty \).

(2) If \( p \in \partial M \), then \( V \cup U_p = \sum_{j=1}^m |\text{graph}_{\partial M}(u_j)| \) for some \( m \in \mathbb{N} \), \( u_j \) \( C^{1,\alpha} \) \( \partial M \cap U_p \) and \( |u_j|_{C^1} \to 0 \).

2.4. **Obstacle problem.** Here we only focus on minimal hypersurfaces with smooth obstacle. Problems in two different settings are well studied, mainly for hypersurfaces in the Euclidean spaces.

**Non-parametric obstacle problem** Let \( \Omega \subset \mathbb{R}^n \) be a smooth bounded convex domain, \( \psi \in C^2(\bar{\Omega}) \), \( \varphi \in C^2(\partial \Omega) \). Consider the problem

(1) Minimize

\[
\mathcal{A}(u) := \int_{\Omega} \sqrt{1 + |\nabla u|^2} \, dx
\]

among all Lipschitz functions \( u \) such that \( u = \varphi \) on \( \partial \Omega \) and \( u \geq \psi \) in \( \Omega \).
The existence and uniqueness of solutions to problem (N) was studied by [20] and [13] under the weaker assumptions on \( \psi, \varphi \) and \( \Omega \). See also [10].

The regularity of minimizer \( \bar{u} \) in problem (N) was also studied in [20] and [13]. Assuming \( \psi \in H^{2,p} \), they are able to prove \( \bar{u} \in H^{2,p} \), \( \forall n < p < +\infty \). Using an approximating argument, [8] showed that \( \bar{u} \) is \( C^{1,1} \) provided \( \psi \in C^2(\Omega) \).

A linearization of problem (N) is studied in [9], where they considered the minimizer of energy functional \( E(u) := \int_\Omega |\nabla u|^2 \, dx \) instead of area functional \( \mathcal{A} \), and obtained \( C^{1,\alpha} \)-regularity for minimizers by a Harnack type inequality only assuming \( \psi \in C^{1,\alpha}(\Omega) \), \( 0 \leq \alpha \leq 1 \). Their approach was generalized in [23] to deal with the non-parametric obstacle problem for area functional.

The regularity result we shall use in the rest of this paper is

**Theorem 2.11 (Interior Regularity).** Let \( M \subset \mathbb{R}^l \) be as in subsection 2.1, \( \alpha \in (0,1) \). Then there exists \( r_0 = r_0(M) \in (0, r_M/2) \) such that, if \( r < r_0 \), \( y \in \partial M \), \( u \in C^1(\mathcal{B}_r(y), [0, r]) \cap C^1_{\text{loc}}(\mathcal{B}_r(y) \setminus \{y\}) \) with \( |\nabla u| \leq 1 \) and \( |\text{graph}_{\partial M}(u)| \) constrained stationary in \( C_r(y) \) along \( \partial M \), then \( u \in C^{1,1}_{\text{loc}}(\mathcal{B}_r(y)) \) and for every compact subset \( K \subset \mathcal{B}_r(y) \),

\[
|\nabla u(x) - \nabla u(x')| \leq C(M,K,r_0)|x-x'| \quad \text{for every} \ x,x' \in K.
\]

For later applications, we include the case with isolated singularity. The proof essentially follows [23]. For some technical reason, we have to assume a priori \( C^{1,\alpha} \) regularity instead of only \( C^1 \) regularity. But the point is we don’t need a priori a uniform \( C^{1,\alpha} \) bound. For sake of completeness, we include in the Appendix the proof of theorem 2.11 and the following theorem 2.12.

**Parametric obstacle problem** Let \( \Omega \subset \mathbb{R}^{n+1} \) be a bounded domain, \( E_0 \subset \mathbb{R}^{n+1} \) be a set of finite perimeter. Consider the following obstacle problem for minimizing boundary

(P) Minimize \( M(\partial[E]) \) among all Caccioppoli sets \( E \subset \mathbb{R}^{n+1} \) with \( \text{spt}(E) \subset \partial \Omega \).

Regularity of solution \( \bar{E} \) to Problem (P) was first studied by [27], in which he showed \( \partial \bar{E} \) is of class \( C^1 \) near \( \partial \Omega \) provided \( \partial \Omega \) is \( C^1 \) (Call a hypersurface of class \( C^{k,\alpha} \) if locally it’s the graph of some \( C^{k,\alpha} \) function ). Combined with the result in non-parametric problem, \( \partial \bar{E} \) is \( C^{1,\alpha} \) if \( \partial \Omega \) is \( C^{1,\alpha} \), \( 0 \leq \alpha \leq 1 \). Different approaches were exploited in [5] and [33] where \( C^1 \) regularity instead of only \( C^{1,\alpha} \) regularity (\( \alpha \in (0,1) \)) of \( \partial \bar{E} \) is proved assuming \( H^{2,p} \) or \( C^{1,\alpha} \) regularity of \( \partial \Omega \) correspondingly. We asserts that [33] proves a general regularity result on reduced boundary with good excess estimate (see also [27]), hence directly generalize to minimizing boundary with obstacle in a manifold. The following regularity for minimizing currents with obstacle in a compact manifold with boundary will be used in section 3.

**Theorem 2.12.** Let \( M \subset \mathbb{R}^l \) be as in subsection 2.1, \( U \subset M \) be a relative open subset with non-empty intersection with \( \partial M \). Suppose \( T \in Z_n(U) \) minimizes mass in \( \{ S \in Z_n(U) : \text{spt}(T-S) \subset U \} \). Then \( \forall y \in U \cap \partial M \), there’s a neighborhood \( U_y \) of \( y \) and \( C^{1,1} \) functions \( u_1 \geq u_2 \geq \ldots \geq u_q \geq 0 \), \( |\nabla u_1| \leq 1 \) such that

1. \( \pm T_y U_y = \sum_{j=1}^q [\text{graph}_{\partial M}(u_j)]_y U_y \).
2. For every \( x \in U_y \cap \partial M \), if \( u_i(x) = u_j(x) \), then either \( u_i(x) = u_j(x) = 0 \), or \( u_i \equiv u_j \) on the connected component of \( \{ u_i > 0 \} \) containing \( x \).

3. A structure theorem near the boundary

The goal of this section is to prove the following structure theorem of a class of constrained stationary varifold near \( \partial M \). Throughout the section, \( U \subset M \) be a relatively open subset with nonempty intersection with \( \partial M \), \( K \subset \partial M \cap U \) be a compact subset. Keep working under Fermi coordinates near \( \partial M \). Use \( x, y, z \) to denote points on \( \partial M \) and \( p = (x,t), q = (y,s) \) to denote points in \( \text{Int}(M) \). We shall not distinguish between \( x \) and \( (x,0) \).

**Definition 3.1.** Call \( V \in V_n(\text{Int}(U)) \) a stable minimal hypersurface with multiplicity, if there exists a countable family of disjoint properly embedded stable minimal hypersurfaces \( \{ \Sigma_j \}_{j \geq 1} \) with optimal regularity in \( \text{Int}(U) \) and \( m_j \in \mathbb{N} \) such that
• \( \bigcup_{j \geq 1} \Sigma_j \) has no accumulation point in Int\((U)\), i.e. \( \text{Clos}(\bigcup_{j \geq 1} \Sigma_j) \cap \text{Int}(U) = \bigcup_{j \geq 1} \Sigma_j \).

• \( V = \sum_{j} m_j |\Sigma_j| \)

Call points in \( \bigcup_{j \geq 1} \text{Reg}(\Sigma) \) regular points of \( V \) in \( \text{Int}(U) \).

Call \( V \in \mathcal{TV}(U) \) constrained stable varifold if \( V \) is constrained stationary in \( U \) and a stable minimal hypersurface with multiplicity in \( \text{Int}(U) \). Call it constrained embedded stable minimal hypersurface if in addition it’s a constrained embedded hypersurface.

**Theorem 3.2.** Suppose \( V \in \mathcal{TV}_0(M) \) be a constrained stable varifold in \( U \) along \( \partial M \) with \( \|V\|((U) \leq \Lambda. \)

Then \( \exists r_0 = r_0(M, U, \mathcal{K}, \Lambda) \in (0, \text{dist}(\mathcal{K}, U^0)/4), m_0 = m_0(M, U, \mathcal{K}, \Lambda) < +\infty \) such that, for every \( y \in \mathcal{K}, \)

\( V \circ C_{r_0}(y) = V_0 + V_+ \)

where \( V_+ \) is a stable minimal hypersurfaces with multiplicity in \( C_{r_0}(y) \) and \( \text{spt}(V_+) \cap B_{r_0/2}(\partial M) = \emptyset; \ V_0 = \sum_{j=1}^{m_0} |\text{graph}_{\partial M}(u_j)| \) for some \( 0 \leq m \leq m_0 \) and \( C^1 \) functions \( u_j : B_{r_0}(y) \to [0, 3r_0/4) \)

satisfying

1. \( \text{spt}(V_0) \cap \text{spt}(V_+) = \emptyset; \)
2. \( u_1 \geq u_2 \geq \ldots \geq u_m, |\nabla u_j| \leq 1/2; \)
3. If \( y' \in B_{r_0} \) such that \( u_i(y') = u_j(y'), \) then either \( u_i(y') = 0 \) or \( u_i \equiv u_j \) on the connected component of \( \{u_i > 0\} \) containing \( y'. \)

If further, \( u_j \in C^{1,1}(B_{r_0}(y)) \), then |\text{graph}_{\partial M}(u_j)| is a constrained stationary varifold in \( C_{r_0}(y) \).

An interesting analytic question is whether one can directly conclude that \( u_j \in C^{1,1}. \)

In view of theorem 2.11 a direct corollary of theorem 3.2 is the compactness of constrained embedded stable minimal hypersurfaces. The proof is left to readers.

**Corollary 3.3.** Let \( \{V_j\}_{j \geq 1} \) be a sequence of constrained embedded stable minimal hypersurfaces in \( U \) along \( \partial M \), with \( \|V_j\|((U) \leq \Lambda. \)

Then up to a subsequence, there exists a constrained embedded stable minimal hypersurface \( V_f \in \mathcal{TV}_0(U) \) such that \( V_f \) converges to \( V \) in varifold sense and in \( C^1 \) sense (see definition 2.10).

The rest of this section is devoted to the proof of theorem 3.2. We start with a tilt estimate of constrained stable varifolds near \( \partial M \).

**Lemma 3.4.** Let \( V \) be in theorem 3.2. There exists a continuous function \( \rho : (0, 1/2) \to (0, \text{dist}(\mathcal{K}, U^0)/2) \) depending only on \( M, U, \mathcal{K}, \Lambda \) such that, for every \( \epsilon \in (0, 1/2), \) if \( p = (y, t) \in \mathcal{K} \times (0, \rho(y)) \cap \text{spt}(V), \) then \( V \) is regular near \( p \) and \( 1 - |\rho_V \cdot v_{\partial M}|(p)^2 < \epsilon, \) where \( \rho_V \) denotes the unit normal of \( V. \)

**Proof.** We shall argue by contradiction. If the statement is not true, then there’s some \( \epsilon > 0, \ V_i \in \mathcal{TV}_0(M) \) constrained stationary in \( U \), stable minimal hypersurfaces in \( \text{Int}(U) \) with \( \|V_i\|((U) \leq \Lambda \) and \( p_i = (y_i, t_i) \in \text{spt}(V) \cap \text{Int}(U) \) with \( t_i \to 0, y_i \in \mathcal{K} \) such that one of the following holds,

a. \( p_i \) is a singular point of \( V_i \circ \text{Int}(U) \)

b. \( p_i \) is a regular point but \( 1 - |\rho_V \cdot v_{\partial M}|(p_i)^2 \geq \epsilon \)

Suppose \( y_i \to y_0 \in \mathcal{K}. \) By lemma 2.6 2.3 2.2 and Allard compactness 3.1, up to a subsequence,

\( (\eta_{y_i, t_i})_t V_i \overset{\ast}{\to} P \)

for some \( P \in \mathcal{TV}_0(T_{y_0}^+ M) \) constrained stationary along \( T_{y_0} \partial M. \) Thus by lemma 2.20, \( P = k|T_{y_0} \partial M| + P_1 \) for some \( k \in \{0\} \cup \mathbb{N} \) and stationary integral varifold \( P_1 \) in \( T_{y_0} M \) with \( \text{spt}(P_1) \subset \text{Int}(T_{y_0} M). \)

On the other hand, since \( V_i \circ \text{Int}(U) \) are stable minimal hypersurfaces, by Schoen-Simon compactness 3.2, \( P_1 \) is a stable minimal hypersurface in \( T_{y_0} M. \) Since \( \text{spt}(P_1) \subset \text{Int}(T_{y_0}^+ M), \) by strong maximum principle 18, \( P_1 = \sum_j |L_j| \) for some hyperplane \( L_j \subset T_{y_0} M \) parallel to \( T_{y_0} \partial M. \)
Now that, consider (up to a subsequence) \( \eta_{b_0,t}(p_1) \to p_0 \in \text{Int}(T^+_{2\rho_0} M) \). Since \( p_0 \) is a smooth point of \( P_t \) with tangent plane parallel to \( T^0_{2\rho_0} \partial M \), by Schoen-Simon’s \( \epsilon \)-regularity theorem 39, theorem 1, neither of (a) (b) is true. \( \square \)

From now on, let \( \rho \) be the function in lemma 3.4 with respect to some \( \mathcal{K}' \subset U \cap \partial M \) such that \( \mathcal{K} \subset \text{Int}(\mathcal{K}') \). We shall also assume, WLOG, that \( \rho < \min\{1, \text{dist}(\mathcal{K}, (\mathcal{K}')^c)\} \). The following is a direct corollary of lemma 3.4 and the proof is left to readers.

**Corollary 3.5.** For every \( k \in \mathbb{N} \) and \( p = (y,t) \in \mathcal{K} \times (0, \rho(1/(10k^2))/2) \cap \text{spt}(V) \), the connected component of \( \text{spt}(V) \cap C_{2\rho_0}(y) \) containing \( p \) is the graph of some function \( u \in C^\infty(B_{2\rho_0}(y)) \) with

\[
0 < |u| < 2t, \quad |\nabla u| < 1/(2k)
\]

Now we proceed the proof of Theorem 3.2. For \( y \in \mathcal{K} \), let \( r_0 = \rho(1/4000)/2 \). Suppose \( V \in C^\infty_{3\rho_0}(y) = \bigcup_{j \geq 1} m_j \Sigma_j \) where \( \Sigma_j \) are connected, disjoint stable minimal hypersurfaces. Let \( J := \{ j \geq 1 : \Sigma_j \cap B_{r_0/2}(\partial M) = \emptyset \} \); \( V_+ := \sum_{j \in J} m_j \Sigma_j \) and \( V_0 = V \setminus V_+ \). By corollary 3.4, \( \text{spt}(V_0) \subset B_{3\rho_0/4}(\partial M) \), \( \text{spt}(V_0) \cap \text{spt}(V_+) = \emptyset \) and \( V_0 \) is also constrained stationary in \( C_{3\rho_0}(y) \) among \( \partial M \cap M \).

Define \( m : B_{r_0}(y) \to \mathbb{Z} \cup \{+\infty\} \) by,

\[
m(x) = \begin{cases} 0 & \text{if } V_0 = 0 \\ \sum_{p=(x,t) \in \text{spt}(V_0)} \theta^n(p, ||V_0||) & \text{if } V_0 \neq 0 \end{cases}
\]

Note that by corollary 2.21 and lemma 3.4, \( \theta^n(p, ||V||) \in \mathbb{Z} \) for \( p \in \text{spt}(V_0) \cap C_{r_0}(y) \), hence \( m(x) \in \mathbb{Z} \cup \{+\infty\} \). We shall show that \( m \) is constant on \( B_{r_0}(y_0) \), bounded uniformly from above.

**Lemma 3.6.** \( m(x) \leq m_0 \) for some \( m_0 = m_0(M, U, \mathcal{K}, \Lambda, r_0) \) and every \( x \in B_{r_0}(y) \).

**Proof.** For every \( x \in B_{r_0}(y) \) fixed, let \( \{ t_0 > 0 : (x,t) \in \text{spt}(V_0) \} = \{ t_1 > t_2 > \ldots > 0 \} \), \( t_1 < 3r_0/4 \); and let \( m_i := \theta^n((x,t_i), ||V_0||) \in \mathbb{N} \). By corollary 3.5, for each \( i \geq 1 \), the connected component of \( \text{spt}(V_0 \cap C_{2t_i}(x)) \) containing \( (x,t_i) \), denoted by \( S_i \), is a smooth minimal graph over \( B_{2t_i}(x) \), and \( \{S_i\}_{i \geq 1} \) are disjoint. Moreover

\[
||S_i||(||B_{2t_i}(x)||) \geq e^{-\Lambda_1 t_i} \cdot \omega_n t_i^n
\]

for some \( \Lambda_1 = \Lambda(M) \geq 0 \) by interior monotonicity of area.

Also, by lemma 2.3, suppose WLOG \( \Lambda_1 \geq 0 \) such that \( t \mapsto e^{\Lambda_1 t} ||V_0||(||B_t(x)||)/t^n \) is non-decreasing on \( (0,2r_0) \).

Therefore,

\[
\frac{e^{\Lambda_1 2r_0}}{\omega_n(2r_0)^n} ||V_0||(B_{2r_0}(x)) \geq \frac{e^{\Lambda_1 (2t_1)}}{\omega_n(2t_1)^n} ||V_0||(B_{2t_1}(x))
\]

\[
= \frac{e^{\Lambda_1 (2t_1)}}{\omega_n(2t_1)^n} \cdot n_1 ||S_1||(B_{2t_1}(x)) + \frac{e^{\Lambda_1 (2t_1)}}{\omega_n(2t_1)^n} ||V_1||(B_{2t_1}(x))
\]

\[
\geq n_1 \cdot 2^{-n} + \frac{e^{\Lambda_1 (2t_2)}}{\omega_n(2t_2)^n} ||V_1||(B_{2t_2}(x))
\]

\[
= n_1 \cdot 2^{-n} + \frac{e^{\Lambda_1 (2t_2)}}{\omega_n(2t_2)^n} \cdot n_2 ||S_2||(B_{2t_2}(x)) + \frac{e^{\Lambda_1 (2t_2)}}{\omega_n(2t_2)^n} ||V_2||(B_{2t_2}(x))
\]

\[
\geq (n_1 + n_2) \cdot 2^{-n} + \frac{e^{\Lambda_1 (2t_3)}}{\omega_n(2t_3)^n} ||V_2||(B_{2t_3}(x))
\]

\[
\cdots
\]

\[
\geq \sum_{i \geq 1} n_i \cdot 2^{-n} + \theta^n(x, ||V_0||)
\]

where \( \tilde{V}_i \) is a constrained stationary integral varifold in \( C_{2t_i}(x) \) defined inductively by

\[
\tilde{V}_0 := V_0; \quad \tilde{V}_i := (\tilde{V}_{i-1} - n_i ||S_i||) \cup C_{2t_i}(x)
\]
Hence, the lemma follows from \( \|V_0\|\(U\) \leq \|V\|\(U\) \leq \Lambda \). \( \square \)

Let \( \bar{m} := \sup_{x \in B_{\rho_0}(y)} m(x) \). WLOG \( \bar{m} > 0 \). We are going to define inductively the functions \( u_i \) in theorem 3.2. Let

\[
\begin{align*}
\bar{V}_i &:= V_{\rho_0}(y); \\
\text{For } 1 \leq i \leq \bar{m}, &\quad u_i(x) := \sup\{ t : (x, t) \in \text{spt}(\bar{V}_{i-1}) \} , \quad x \in B_{\rho_0}(y); \\
\text{For } 1 \leq i \leq \bar{m}, &\quad \bar{V}_i := \bar{V}_{i-1} - \{ \text{graph}_{\partial M}(u_i) \}; \\
\text{For } 1 \leq i \leq \bar{m}, &\quad m_i(x) := \begin{cases} 0, & \text{if } \bar{V}_i = 0 \\ \sum_{p = (x, t) \in \text{spt}(\bar{V}_i)} \theta^n(p, \|\bar{V}_i\|), & \text{if } \bar{V}_i \neq 0 \end{cases} \\
\end{align*}
\]

where we set \( \sup \emptyset = -\infty \). The following lemma guarantees they are well-defined. (We set \( m_0(x) := m(x) \).)

**Lemma 3.7.** For \( u_i, \bar{V}_i \) defined above, \( 1 \leq i \leq \bar{m} \),

1. \( u_i \) are nonnegative \( C^1 \) functions on \( B_{\rho_0}(y) \), and is \( C^\infty \) in \( \{ x : u_i(x) > 0 \} \).
2. \( \bar{V}_i \) are integral varifolds in \( C_{\rho_0}(y) \) such that \( \bar{V}_i \cap C^+_{\rho_0}(y) \) are stable minimal hypersurfaces with multiplicity supported in \( \text{spt}(\bar{V}_0) \).
3. \( \forall x \in \text{spt}(\bar{V}_i) \cap B_{\rho_0}(y) \), the tangent varifold of \( \bar{V}_i \) at \( x \) is \( k|T_x \partial M| \) for some \( k \in \mathbb{N} \).
4. \( m_i(x) = m_{i-1}(x) - 1, \forall x \in B_{\rho_0}(y) \).

**Remark 3.8.** By (4) of lemma 3.7 and that \( m_i \geq 0 \), \( m_{i-1} = 0 \) and \( V_0 = \sum_{i=1}^{\bar{m}} \{ \text{graph}_{\partial M}(u_i) \} \).

Together with (1), this proves the first part of theorem 3.8.

If further, \( u_j \in C^{1,1}(B_{\rho_0}(y)) \), then (denote \( \{ \text{graph}_{\partial M}(u_j) \} \) by \( \Gamma \) for simplicity) the mean curvature of \( \Gamma \) is in \( L^\infty(\text{spt}(\bar{V}_i)) \) satisfying

\( \bar{H}_\Gamma = \bar{H}_{BM} \cdot H^n - \text{a.e. on } \{ u_i = 0 \} \)

Combined with lemma 3.8, we see \( \Gamma = \text{graph}_{\partial M}(u_i) \), hence \( \Gamma \) is constrained stationary in \( C_{\rho_0}(y) \) along \( \partial M \). This proves the second part of theorem 3.8.

**Proof of lemma 3.7:** It’s sufficient to prove (1)-(4) inductively in \( i \) assuming they hold for \( 1 \leq l < i \).

1. \( u_i \) is clearly upper semi-continuous by definition; While by corollary 3.6, together with the assumption that \( \text{spt}(\bar{V}_{i-1}) \subset \text{spt}(V_0) \) and \( \bar{V}_{i-1} \cap C^+_{\rho_0}(y) \) is a stable minimal hypersurface with multiplicity, \( u_i \) is lower semi-continuous, and then smooth, in open subset \( \{ u_i > 0 \} \subset B_{\rho_0}(y) \).

Note also that by assumption (4) for \( 1 \leq i \leq \bar{m} \), \( \text{spt}(\bar{V}_{i-1}) \neq \emptyset \). Hence \( \{ u_i \geq 0 \} \neq \emptyset \). To prove the continuity of \( u_i \), it suffices to verify that \( u_i \geq 0 \). If not, then by upper semi-continuity of \( u_i \), there exists an open ball \( B \subset \{ u_i < 0 \} \) and some point \( z \in \partial B \cap \{ u \geq 0 \} \). Moreover, by the continuity of \( u_i \) on \( \{ u_i > 0 \} \), \( u_i(z) = 0 \). By definition, \( z \in \text{spt}(\bar{V}_i) \), then by the assumption (3) for \( i-1 \),

\( (\eta_{\lambda,\lambda} y) \bar{V}_i \rightharpoonup k|T_z \partial M| \quad \text{as } \lambda \to 0 \)

for some \( k \geq 1 \). On the other hand, by definition, \( B \times (-r_0, r_0) \cap \text{spt}(\bar{V}_i) = \emptyset \), hence the weak limit above should be supported in a half space. Here comes a contradiction.

That \( u_i \) is differentiable and \( C^1 \) on \( B_{\rho_0}(y) \) follows from corollary 3.6 and lemma 3.4.

2. Integrality of \( \bar{V}_i \) follows from assumption (3) for \( i-1 \), that \( u_i \in C^1(B_{\rho_0}(y)) \) from (1) and that \( \text{graph}_{\partial M}(u_i) \subset \text{spt}(\bar{V}_{i-1}) \) by definition. Hence \( \text{spt}(\bar{V}_i) \subset \text{spt}(\bar{V}_{i-1}) \subset \ldots \subset \text{spt}(V_0) \). Since \( V_{\rho_0}(y) \) is smooth minimal hypersurface with multiplicity and \( \{ u_i \}_{1 \leq i \leq \bar{m}} \) are \( C^1 \), \( \bar{V}_i \cap C^+_{\rho_0}(y) \) is also a smooth minimal hypersurface with multiplicity.
for some $k \in \mathbb{Z}$. Thus, to prove (3) for $i$, it suffices to verify that $\theta^n(x, V_i) > 0$. The proof is similar to the one of corollary 2.4. The only difference is that it’s unclear whether $V_i$ is constrained stationary in $C_0(y)$ or not, so lemma 2.3 does not apply directly. In what follows, we shall establish a quasi-boundary monotonicity for $V_i$.

Claim: $\forall \epsilon > 0$, there’s an $r_1 \in (0, r_0)$ such that, for every $0 < \sigma < \eta < r_1$ and every $x \in B_{r_0}(y)$ such that $B_{2\rho}^r(x) \subset C_0(y)$,

\[
(3.2) \quad \frac{e^{\Lambda n}}{\omega_{n}\eta^{n}}\|\nabla\partial M v\|((B_{r}^r(x))) \geq \frac{e^{\Lambda \sigma}}{\omega_{n}\sigma^{n}}\|\nabla\partial M v\|((B_{r}^{r}(x))) - 2\epsilon
\]

Notice that by taking $\epsilon$ sufficiently small in (3.2), the proof of corollary 2.4 generalized direct to prove the density lower bound at every point in $spt(V_i)$.

Proof of the claim: First notice that $\exists r_2 > 0$ and $k > 4$ such that, for every $0 < r < r_2$, every $B_r(z) \subset B_{r_0}(y)$ and every $v \in C^1(B_r(z))$ satisfying

\[
\frac{1}{r}|v| + |\nabla\partial M v| \leq \frac{1}{k} \quad \text{on } B_r(z),
\]

we have

\[
(3.3) \quad \frac{e^{\Lambda \sigma}}{\omega_{n}\sigma^{n}}\|\nabla\partial M v\|((B_{r}^{r}(z)) - 1) \leq \epsilon
\]

where $\Lambda$ is in lemma 2.3.

Now take $r_1 := \min\{r_2, \rho(1/(40k^2)), 1/k\}/10$, where $\rho$ is specified in corollary 3.5. For every $0 < r < r_1$ and every $B_r(x) \subset B_{r_0}(y)$, by our choice of $r_1$ and corollary 3.5, $\forall 1 \leq l \leq i$, either

\[
(3.4) \quad \frac{1}{r}|u_l| < \frac{1}{2k}, \quad |\nabla\partial M u_l| \leq \frac{1}{2k} \quad \text{on } B_{2r}(x)
\]

or

\[
(3.5) \quad u_l > 0 \quad \text{on } B_{2r}(x)
\]

Note that if $u_l$ satisfies (3.5) on $B_{2r}(x)$, then it satisfies (3.5) on every $B_l(x) \subset B_{2r}(x)$. Also note that once $u_l > 0$ on $B_{2r}(x)$, $|\nabla\partial M u_l|$ is a stationary varifold in $B_{2r}(x) \times (-r_0, r_0)$ and hence $V_0 - |\nabla\partial M u_l|$ is constrained stationary in $B_{2r}(x) \times (-r_0, r_0)$.

Now that $\forall 0 < \sigma < \eta < r_1$, $\forall B_{2\rho}^r(x) \subset B_{r_0}(y)$, let

\[
I_1 := \{l : 1 \leq l \leq i, u_l \text{ satisfies } (3.4) \text{ on } B_l(x) \text{ for every } t \in [\sigma, \eta]\}
\]

\[
I_2 := \{l : 1 \leq l \leq i, u_l \text{ satisfies } (3.5) \text{ on } B_l(x) \text{ for every } t \in [\sigma, \eta]\}
\]

\[
I_3 := \{1, 2, \ldots, l\} \setminus (I_1 \cup I_2)
\]

And for $l \in I_3$, let $t_l := \inf\{t \in [\sigma, \eta] : (3.5) \text{ fails on } B_{2r}(x)\}$. After relabeling the index, WLOG $I_3 = \{1, 2, \ldots, q\}$ and $t_1 \leq t_2 \leq \ldots t_q$, $q \in \mathbb{N} \cup \{0\}$. Denote for simplicity,

\[
a(x, t; V) := \frac{e^{\lambda t}}{\omega_{n}\lambda^{n}}\|V\|((B_{r}^{r}(x))) ; \quad a_l(x, t) := a(x, t; |\nabla\partial M u_l|)\]
and \( \Gamma_l := \text{graph}_M(w_l) \). Then

\[
a(x, \eta; \mathring{V}_i) = a(x, \eta; V_0 - \sum_{l \in I_2} \Gamma_l - \sum_{l=1}^{q} a_l(x, \eta) - \sum_{k \in I_1} a_k(x, \eta) \)
\[
\geq a(x, t_q; V_0 - \sum_{l \in I_2} \Gamma_l - \sum_{l=1}^{q} (a_l(x, t_q) + 2\epsilon) - \sum_{k \in I_1} (a_k(x, \sigma) + 2\epsilon) \)
\[
\geq a(x, t_q; V_0 - \sum_{l \in I_2} \Gamma_l - \sum_{l=1}^{q} (a_l(x, t_q) - \sum_{k \in I_1} a_k(x, \sigma) - 2\bar{m}\epsilon) \)
\[
\geq a(x, t_{q-1}; V_0 - \sum_{l \in I_2} \Gamma_l - \sum_{l=1}^{q} a_l(x, t_{q-1}) - a_k(x, \sigma) - 2\bar{m}\epsilon \)
\[
\geq a(x, t_{q-1}; V_0 - \sum_{l \in I_2} \Gamma_l - \sum_{l=1}^{q} a_l(x, t_{q-1}) - a_k(x, \sigma) - 4\bar{m}\epsilon \)
\[
\ldots \ldots \ldots
\]
\[
\geq a(x, t_1; V_0 - \sum_{l \in I_2} \Gamma_l - \sum_{l=1}^{q} \Gamma_l) - \sum_{k \in I_1} a_k(x, \sigma) - 2q\bar{m}\epsilon \)
\[
\geq a(x, \sigma; V_0 - \sum_{l \in I_2 \cup I_1} \Gamma_l) - \sum_{k \in I_1} a_k(x, \sigma) - 2q\bar{m}\epsilon \)
\[
\geq a(x, \sigma; \mathring{V}_i) - 2q\bar{m}^2\epsilon
\]

where the first inequality follows from \( \text{(3.3)} \), lemma \( \text{(2.3)} \) and that \( V_0 - \sum_{l \in I_2} \Gamma_l \) is constrained stationary in \( B_{\eta}(x) \times (-r_0, r_0) \); the third inequality follows form \( \text{(3.3)} \), lemma \( \text{(2.3)} \) and that \( V_0 - \sum_{l \in I_2} \Gamma_l - \Gamma_q \) is stationary in \( B_{2\eta}(x) \times (-r_0, r_0) \) and so on. This completes the proof of claim.

(4) follows immediately from (3) and corollary \( \text{(3.5)} \). \( \Box \)

Remark 3.9. From the proof we can also see that \( r_0 = r_0(M, U, K, \Lambda) \) can be chose to continuously depend on \( (M, \partial M) \) in \( C^2 \) topology.

4. Min-max with Obstacle

4.1. Almost minimizer and regularity. The notion of almost minimizing varifolds is introduced in \( \text{(2.9)} \) to prove the regularity of stationary varifolds obtained from min-max construction. We employ the similar notions here. The only difference is that everything is supported in an ambient manifold with boundary, making the regularity result more subtle near the boundary. \( M \) will be as in subsection \( \text{(2.1)} \) \( U \subset M \) be a relative open subset.

Definition 4.1. Call \( \{T^{(l)}\}_{l=0}^{q} \subset I_n(U) \) a \( \delta \)-sequence in \( U \) start at \( T^{(0)} \) if \( \forall 1 \leq l \leq q, \)

- \( \text{spt}(T^{(l)} - T^{(0)}) \subset U \)
- \( M_U(T^{(l)}) \leq M_U(T^{(0)}) + \delta \)
- \( M_U(T^{(l)} - T^{(l-1)}) \leq \delta \)

Call \( V \in \mathcal{V}_n(M) \) almost minimizing in \( U \), if there exist \( \epsilon_i \to 0, \delta_i \to 0, T_i \in I_n(M) \) with \( |T_i| \to V \) such that for every \( i \) and every \( \delta_i \)-sequence \( \{T^{(l)}\}_{l=0}^{q} \) in \( U \) start at \( T_i \), we have

\[
M_U(T^{(l)}) \geq M_U(T_i) - \epsilon_i
\]
Note that by definition, if $V$ is not almost minimizing in $U$, then $\exists \, \epsilon > 0$ such that $\forall \, T \in I_\epsilon(M)$ with $F([T], V) < \epsilon$ and $\forall \delta > 0$, there exists some $\delta$-sequence $\{T^{(\delta)}\}_{\delta=0}^\infty$ start at $T$ with $M_U(T^{(\delta)}) \leq M_U(T) - \epsilon$

**Theorem 4.2.** Let $V \in \mathcal{V}_\epsilon(M)$ be almost minimizing in $U$. Then $V$ is a constrained embedded stable minimal hypersurface in $U$ with optimal regularity.

**Proof.** First see that by [29], $V \cap \text{Int}(U)$ is a stable minimal hypersurface with optimal regularity; moreover, a similar argument in [29, 3.3] yields that $V$ is constrained stationary in $U$. Now we focus on the behavior of $V$ near $\partial M$. The proof of integrality of $V$ is almost the same as in [29, Chapter 3], with aid of the lemma in subsection 2.2. We briefly go through the proof.

- **Replacement.** $\forall \, U' \subset U$ be a subdomain. Since $V$ is almost minimizing in $U$, let $\epsilon_i \to 0$, $\delta_i \to 0$ and $|T_i| \to V$ be as in the definition. For each $i$, let $T^*_i$ be a mass minimizer among $\mathcal{E}(T_i, Clos(U')) := \{T^{(\delta)} \in \mathcal{E} : \exists \, \delta > 0 \text{ such that } spt(T^{(\delta)} - T_i) \subset Clos(U')\}$

The existence of such $T^*_i \in \mathcal{E}(T_i, Clos(U'))$ follows from Federer-Fleming compactness [12] and discretization-interpolation theorem [25, 13.1, 14.1].

Moreover, $T^*_i$ is locally constrained minimizing in $U'$ and satisfies the mass bound $||T^*_i||(U') - \epsilon_i \leq ||T^*_i||(U') \leq ||T_i||(U')$

Hence, by theorem 2.12 $|T^*_i| |U' \text{ is a constrained embedded stable minimal hypersurface in } U'$.

Then by corollary 2.6 when $i \to \infty$, $|T^*_i| \to V^* \in \mathcal{V}_\epsilon(U)$ satisfying

- (a) $V^*$ is almost minimizing in $U$ and $V^* \cap U'$ is a constrained embedded stable minimal hypersurface;
- (b) $V^* \cap \text{Clos}(U')^c = V \cap \text{Clos}(U')^c$;
- (c) $\|V^*\|(U) = \|V\|(U)$

Those $V^* \in \mathcal{V}_\epsilon(U)$ satisfying (a)-(c) above will be called replacements of $V$ in $U'$.

- **Rectifiability.** Thanks to Allard rectifiability theorem [21] Chapter 8, 5.5 and lemma 2.6 it suffices to show that $\theta^n(x, ||V||) > 0 \forall \, x \in \partial M \cap U$. $\forall \, x \in \text{spt}(V) \cap U \cap \partial M$, let $r_j \in \{0, \text{dist}_M(x, M \setminus U)/2\}, r_j \to 0$; $V^*_j$ be a replacement of $V$ in $A_{r_j/2}(x)$. Then by lemma 2.3 $\|V^*_j\|((A_{r_j/4, r_j/4}) \cap \partial M) > 0$ for $j > 1$. By the same trick as in corollary 2.4 $\|V^*_j\|((A_{r_j/4, r_j/4}) \cap \partial M) > 0$. Hence, $\theta^n(x, ||V||) \geq c(n, M) > 0$. Now, $\theta^n(x, ||V||) \geq c(n, M) > 0$.

- **Integrality.** For $V^*_j$ constructed above, by lemma 2.2 and 2.3 up to a subsequence, $(\gamma_n, r_j)_j \rightarrow P, (n, r_j)_j \rightarrow Q \rightarrow P, (n, r_j)_j \rightarrow Q \rightarrow P, Q \in \mathcal{V}_\epsilon(\mathbb{R}^{n+1})$ are constrained stationary along $\mathbb{R}^n \times \{0\}$. (We identify $(T^*_j, M, T^*_j \partial M)$ with $(\mathbb{R}^{n+1}, \mathbb{R}^n \times \{0\})$ for simplicity.) Moreover, by definition of replacements, $Q_{A_{1/2}(0)}$ is constrained embedded stable minimal hypersurface, hence stable minimal hypersurface by lemma 2.5. And $P = Q$ outside $\text{Clos}(A_{1/2}(0))$. By lemma 2.5, $P, Q$ are cones and hence $P = Q$ is a stable minimal cone with optimal regularity supported in $\mathbb{R}^{n+1}$. By strong maximum principle, $P = Q = m|\mathbb{R}^n \times \{0\}|$ for some $m \in \mathbb{N}$. Therefore $\theta^n(x, ||V||) = m \in \mathbb{N}$. This proves $V \in \mathcal{T}_\epsilon(U)$.

Now by theorem 3.2 $V$ is locally multiple $C^1$ graphs near $\partial M$. To get $C^{1,1}$ regularity of the graphical function, we use again the replacements. In what follows, we keep working under Fermi coordinates.

Let $x \in \partial M \cap U$, $r_0 = r_0(M, U, \{x\}, ||V||(U)) > 0$ be in theorem 3.2 $U_x \subset C_{r_0}(x)$ be a neighborhood of $x$ such that $V \cap U_x = \sum_{i=1}^{m} [\text{graph}_{\partial M}(u_i)]$, where $u_1 \geq u_2 \geq \ldots \geq u_m \in C^1(U_x \cap \partial M, \mathbb{R}^n)$, $|\nabla u_i| \leq 1/2$ and $\{u_i = 0\} \neq \emptyset$. For any $p = (y, t) \in \text{graph}_{\partial M}(u_i) \cap \text{Int}(M)$, let $V^*$ be a replacement of $V$ in $U_x \setminus \text{Clos}(B_{1/2}(p))$. By theorem 3.2 $V^* \cap U_x = [\text{graph}_{\partial M}(u^*)] + V^*$ for
some $\tilde{V}^* \in TV_n(U_x)$ and $u^* = u_t$ on the connected component of $\{u_t > 0\}$ containing $y$ by unique continuation; Since $V^*$ is a constrained embedded stable minimal hypersurface in $U_x$, by theorem 2.11 $u^* \in C^{1,1}(\partial M \cap U)$ and for every $K \subset \partial M \cap U$ compact, 
\[
\|u^*\|_{C^{1,1}(K)} \leq C(M, r_0, K)
\]
By applying the replacement on every connected component of $spt(V) \cap Int(U_x)$, we see $u_t \in C^{1,1}(\partial M \cap U)$. By the arbitrariness of $i$ and $x$ and second part of theorem 3.2 $V$ is a constrained embedded stable minimal hypersurface in $U$.

Theorem 4.3. Let $(M, g)$ be a compact $n+1$ manifold with boundary, $\Phi_0 : X \to (Z_n(M), F)$ be a continuous map (i.e. continuous in $F$-metric). Let $\Pi$ be the space of sequence of continuous maps $\{\Phi_i : X \to (Z_n(M), F)\}_{i \geq 1}$ satisfying 
\[
\sup\{F(\Phi_i(x), \Phi_0(x)) : x \in Z\} \to 0 \text{ as } i \to \infty
\]
And define the width 
\[
L(S) := \limsup_{i \to \infty} \sup_{x \in X} M(\Phi_i(x))
\]
\[
L(\Pi) := \inf_{S \in \Pi} L(S)
\]
\[
L_{\Phi_0, Z} := \sup_{x \in Z} M(\Phi_0(x))
\]

The goal of this section is to prove

Theorem 4.3. Let $(M, g)$ be a compact $n+1$ manifold with boundary, $\Phi_0 : X \to (Z_n(M), F)$ be a continuous map as above. If $L(\Pi) > L_{\Phi_0, Z}$, then there's a constrained embedded minimal hypersurface $V \in TV_n(M)$ with optimal regularity such that $\|V\|(M) = L(\Pi)$.

With aid of theorem 4.2 the proof is almost the same as in [29] Chapter 4. We briefly go through the major part of the argument.

- Minimizing sequence and pull tight. By a diagonalization process, there exists some $S \in \Pi$ such that $L(S) = L(\Pi)$. Call such $S$ a minimising sequence.

  For a minimizing sequence $S = \{\Phi_i\}_{i \in \Pi}$, denote 
  \[
  C(S) := \{V \in TV_n(M) : V_n = \lim_{j \to \infty} |\Phi_{i_j}(x_j)|, \|V\|(M) = L(S)\}
  \]
  called the critical set of $S$. Following [29] 4.3 and [37], we can construct $H : [0, 1] \times Z_n(M) \cap \{\|T\|(M) \leq 2L(\Pi)\} \to Z_n(M) \cap \{\|T\|(M) \leq 2L(\Pi)\}$ continuous in $F$-metric such that 
  (i) $H(0, -) = id$; 
  (ii) $H(t, T) = T$, $\forall t \in [0, 1]$, $\forall T \in V^\infty := \{\text{constrained stationary varifolds in } M\} \cup \{\Phi(x) : x \in Z\}$; 
  (iii) There's an $L : [0, +\infty) \to [0, +\infty)$ continuous, $L(0) = 0$, $L(\tau) > 0$ if $\tau > 0$ such that 
  \[
  \|H(1, T)\|(M) \leq \|T\|(M) - L(F(T, V^\infty))
  \]
  (iv) $\forall \epsilon > 0, \exists \delta > 0$ such that if $F(T, |\Phi_0(x)|) \leq \delta$ for some $x \in Z$, then $F(H(t, T), |\Phi_0(x)|) \leq \epsilon$ for all $t \in [0, 1]$.

Hence, if $S \in \Pi$ is a minimizing sequence, then $S' := \{H(1, \Phi_i)\}_{i > 1} \in \Pi$ is also a minimizing sequence such that $0 \neq C(S') \subset C(S)$ consists of constrained stationary varifolds in $M$. Call such $S'$ a pulled tight sequence.
• Discretization and interpolation. Given a pulled-tight sequence $S = \{\Phi_i\}_{i \geq 1} \in \Pi$, by the interpolation theorem [25, 13.1] and diagonalization process, there exist $k_i \to \infty$, $\delta_i \to 0$ and $\varphi_i : X(k_i) \to Z_n(M)$ such that

(i) $\sup \{F(\varphi_i(x), \Phi_i(x)) : x \in X(k_i)\} \leq \delta_i$;

(ii) $\sup \{F(\varphi_i(x), \varphi_i(y)) : x, y \in \alpha \cap X\}$ for some $\alpha \in I(m, k_i m) \leq \delta_i$;

(iii) $f(\varphi_i) \to 0$ as $i \to \infty$;

(iv) $L(\{\varphi_i\}, i) := \limsup_{i \to \infty} \sup_{x \in X(k_i)} \|\varphi_i(x)\|(M) = L(\Pi)$.

In particular,

$C(\{\varphi_i\}_{i \geq 1}) := \{V : V = \lim_{j \to \infty} |\varphi_i(x_j)|\}$ for some $i_j \to \infty$ and $x_j \in X(k_i)\}; \|V\|(M) = L(\Pi) \subset C(S)$

Conversely, given a sequence of $k_i \to \infty$, $\delta_i \to 0$ and $\varphi_i : X(k_i) \to Z_n(M)$ satisfying (ii), (iii) above and $\varphi_i(x) \in \Pi$. The interpolation theorem in [3, section 6]), (see also [25, 14.1]), there exists $S = \{\Phi_i\}_{i \geq 1} \in \Pi$ such that

$L(S) = L(\{\varphi_i\}_{i \geq 1})$

• Pitts’ deformation. Let $\{\varphi_i : X(k_i) \to Z_n(M)\}_{i \geq 1}$ be satisfying (i), (ii), (iii) above and such that

$L(\{\varphi_i\}_{i \geq 1}) > L_{\Phi_0, \mathbb{Z}}$, $C(\{\varphi_i\}_{i \geq 1}) \subset \{\text{constrained stationary varifolds in } M\}$

Assuming $V \in C(\{\varphi_i\}_{i \geq 1})$, $\exists x \in M$ such that $\forall r > 0, \exists s \in (0, r)$ with $V$ not almost minimizing in $\text{A}_{s, r}(x)$, [29] introduced a deformation process to construct $\{\varphi_i' : X(k_i') \to Z_n(M)\}$ satisfying (i), (ii), (iii) and

$L(\{\varphi_i'\}_{i \geq 1}) < L(\{\varphi_i\}_{i \geq 1})$

Applying this to the interpolation theorem above, one see that given a pulled-tight minimizing sequence $S \in \Pi$, $\exists V \in C(S)$ such that $\forall x \in M, \exists r_x > 0$ and $V$ being almost minimizing $\text{A}_{s, r}(x)$, $\forall s \in (0, r)$. By theorem 4.11 and 2.11, $V$ is a constrained embedded minimal hypersurface. This complete the proof of theorem 4.13.

Remark 4.4. By Pitts’ combinatorial argument [29, 4.9, 4.10], there’s a constant $N(m) \in \mathbb{N}$ such that in theorem 4.5, we can choose $V$ such that if $p \in M$ and $\{A_j := \text{A}_{r_j, s_j}(p)\}_{j=1}^{\infty}$, then $V$ is almost minimizing in at least one of $A_j \cap M$. Hence in particular $V$ is a constrained embedded stable minimal hypersurface in that $A_j \cap M$.

5. A local rigidity for constrained embedded minimal hypersurfaces

We close this article by pointing out the following local rigidity theorem for constrained embedded minimal hypersurface. Let $(N, g)$ be a closed $n + 1$ manifold; $(\Sigma, \nu) \subset N$ be a connected, two-sided, closed smooth minimal hypersurface; $r_{\Sigma}$ be the injective radius of $\Sigma$ in $N$, i.e.

$r_{\Sigma} := \sup \{r : \Phi : \Sigma \times (-r, r) \to N, (x, t) \mapsto \exp^N_{x}(t\nu(x)) \text{ is a diffeomorphism onto its image}\}$

Work under Fermi coordinates in $\Phi(\Sigma \times (-\Sigma, r_{\Sigma}))$. For $K > 1$ and $w \in C^\infty(\Sigma, (0, r_{\Sigma}))$, call the domain

$\mathcal{N}_w := \{(x, t) \in N : x \in \Sigma, -w^- \leq t \leq w^+\}$

a $K$-uniform neighborhood of $\Sigma$, if

(1) $\sup_{x \in \Sigma, t \in \pm w^+} w^+(x) \leq K \inf_{x \in \Sigma, t \in \pm w^-} w^-(x)$;

(2) The mean curvature $H_{\pm}$ of $\text{graph}_{\Sigma}(\pm w^\pm)$ satisfies

$|H_{\pm}| \leq K \sup_{\Sigma} w^\pm$.
Note that for a smooth minimal hypersurface as above, the δ-neighborhood $\mathcal{N} := \{ p \in N : \text{dist}(p, \Sigma) \leq \delta \}$ and the neighborhoods $\mathcal{N}_{w_1} := \{ (x, t) : x \in \Sigma, |t| \leq \delta w_1(x) \}$ given by first eigenfunction $w_1$ of Jacobi operator of $\Sigma$ are both $K$-uniform neighborhood of $\Sigma$ for every sufficiently small $\delta$ and $K$ independent of $\delta$.

**Theorem 5.1.** Let $\Sigma \subset N$ be a non-degenerate closed minimal hypersurface as above; $K > 1$; $\epsilon \in (0, 1)$. Then $\exists s_0 = s_0(K, \epsilon, \Sigma, N) > 0$ such that if $\mathcal{N} = \mathcal{N}_{w_1}$ is a $K$-uniform neighborhood of $\Sigma$ and $\sup_{\Sigma} |w_\pm| \leq s_0$, and $V$ is a constrained embedded minimal hypersurface in $\mathcal{N}$ with

$$\mathcal{H}^n(\Sigma) \leq \|V\|(N) \leq (2 - \epsilon)\mathcal{H}^n(\Sigma)$$

then $V = |\Sigma|$.

We emphasize that having the same mass is crucial for such a theorem to be true. In fact, for an unstable minimal hypersurface $\Sigma$, let $w^+ = w^- = \delta w_1$, where $w_1$ is the first eigenfunction of Jacobi operator, $\delta \ll 1$, then each component of $\partial N_{w_1}$ is constrained embedded minimal in $N_{w_1}$.

**Proof.** We prove by contradiction. First recall some basic facts on minimal surface equation on $\Sigma$.

Let $F = F(x, z, p)$, $x \in \Sigma$, $z \in (-r_\Sigma, r_\Sigma)$, $p \in T_x\Sigma$ be the area integrand for graph over $\Sigma$, i.e.

$$\mathcal{H}^n(\text{graph}_\Sigma(\phi)) = \int_{\Sigma} F(x, \phi(x), \nabla \phi(x)) \, dx \quad \forall \phi \in C^1(\Sigma, (-r_\Sigma, r_\Sigma))$$

Here are some basic propositions we shall use later,

(a) $F(x, 0, 0) = 1$; $F_z(x, 0, 0) := \partial_z F(x, 0, 0) = 0$, $F_p := \partial_p F(x, 0, 0) = 0$;

$$F_{pp}(x, 0, 0)^2 = \delta^{ij}, \quad F_{zp}(x, 0, 0) = 0, \quad F_{zz}(x, 0, 0) = -|A_\Sigma|^2 - \text{Ric}_N(\nu_\Sigma, \nu_\Sigma).$$

(b) $\|F\|_{C^2} \leq C(\Sigma, N)$.

(c) If further $\phi \in C^1 \cap W^{2,2}(\Sigma)$, then

$$\mathcal{M} \phi := -\text{div} F_p(x, \phi, \nabla \phi) + F_z(x, \phi, \nabla \phi) = \frac{1}{\sqrt{1 + |\nabla \phi|^2}} H$$

where $H$ is the mean curvature of $\text{graph}_\Sigma(\phi)$ with normal field having positive inner product with $\nu$.

(d) $L^2 \phi := \int_{\Sigma} \mathcal{M}(s \phi) = -(\Delta + |A_\Sigma|^2 + \text{Ric}_N(\nu, \nu)) \phi$ be the Jacobi operator of $\Sigma$.

Suppose otherwise, and there are $K$-uniform neighborhood $\mathcal{N}_j = \mathcal{N}_{w_1}$ of $\Sigma$ with $\sup |w_\pm| \to 0$ and $|\Sigma| = |T_\Sigma(\mathcal{N}_j)|$ constrained embedded minimal hypersurfaces in $\mathcal{N}_j$ with $\|\Gamma_j\|(N) \in [\|\Sigma\|(N), (2 - \epsilon)\|\Sigma\|(N)]$. The strategy is to show these $\{\Gamma_j\}$ induces a nontrivial Jacobi field on $\Sigma$, which contradicts to the non-degeneracy assumption.

Recall by Remark 2.3 (1), $\Gamma_j$ has mean curvature uniformly bounded by the mean curvature of $\partial \mathcal{N}_j$, hence tends to 0 in $L^\infty$. By Allard compactness theorem [1][34], up to a subsequence, $\Gamma_j \to \Gamma_\infty$ for some stationary integral varifold $\Gamma_\infty \in \mathcal{IV}_0(N)$ supported in $\Sigma$ and having mass between $\|\Sigma\|(N)$ and $(2 - \epsilon)\|\Sigma\|(N)$. Hence $\Gamma_\infty = |\Sigma|$, and by Allard regularity [1][34], for $j >> 1$, there exist $u_j \in C^{1,\alpha} \cap W^{2,2}(\Sigma)$ such that $\Gamma_j = |\text{graph}\Sigma(u_j)|$, $u_j \to 0$ in $C^\alpha$, not identically 0 and

$$\mathcal{M} u_j = 0 \quad \text{on } \{-w_j^- < u_j < w_j^+\}$$

$$\|u_j\|_{L^\infty} \leq \sup_{i \in \pm} K w_j^i(x) \quad \text{on } \Sigma$$

where the last two inequality follows from Remark 2.3 (1). Note that $\{u_j = \pm w_j^+\} \neq \emptyset$, otherwise by standard elliptic estimate, $u_j/\|u_j\|_{L^2}$ will $C^\infty$-converge to some nontrivial Jacobi field on $\Sigma$. Thus,

$$\inf w_j^\pm \leq \|u_j\|_{C^\alpha(\Sigma)} \leq \sup w_j^\pm$$
Also by standard elliptic estimate,
\[ \|u_j\|_{C^{1,\alpha}} + \|u_j\|_{W^{2,2}} \leq C(\Sigma, N)(\|u_j\|_{C^0} + \|\mathcal{A}u_j\|_{C^0}) \leq C(\Sigma, N, K)\|u_j\|_{C^0} \]
Hence, let \( c_j := \|u_j\|_{C^0} \), up to a subsequence, \( u_j := u_j/c_j \rightarrow \hat{u}_\infty \in C^{1,\alpha} \cap W^{2,2}(\Sigma) \) in \( C^1 \). Moreover, \( \|\hat{u}_\infty\|_{C^0} = 1 \) and by (5.1), (5.2) and definition of \( K\)-uniform domain,
\[ L_{\Sigma} \hat{u}_\infty = 0 \quad \text{on} \{\|\hat{u}_\infty\| < 1/K\} \]  
(5.3)
Now we make use of the constraint on mass to show that \( \int_{\Sigma} \hat{u}_\infty \cdot L_{\Sigma} \hat{u}_\infty \geq 0 \). Combining this with (5.3), we see that \( \hat{u}_\infty \) is a nontrivial Jacobi field on \( \Sigma \), which contradicts to the non-degeneracy. Observe that
\[ 0 \leq \|\Gamma_j\|((N) - \|\Sigma\|((N)) = \int_{\Sigma} F(x, u_j, \nabla u_j) - 1 \, dx \]
(5.4)
\[ = \int_{\Sigma} \left( \int_0^1 F_p(x, su_j, s\nabla u_j) \, ds \right) \cdot \nabla u_j + \left( \int_0^1 F_z(x, su_j, s\nabla u_j) \right) \cdot u_j \, dx \]
\[ = \frac{1}{2} \int_{\Sigma} F_p(x, u_j, \nabla u_j) \cdot \nabla u_j + F_z(x, u_j, \nabla u_j) u_j \, dx + \mathcal{R} \]
where by (a),(b) and standard elliptic estimate, \( |\mathcal{R}| \leq C(\Sigma, N, K) c_j^3 \). Here we use the fact that \[ \int_0^1 f(s) \, ds - \frac{1}{2}(f(0) + f(1)) = \frac{1}{2} \int_0^1 f''(s)(s^2 - s) \, ds \]
Multiply (5.4) by \( c_j^{-2} \) and let \( j \rightarrow \infty \) one get,
\[ \int_{\Sigma} \hat{u}_\infty \cdot L_{\Sigma} \hat{u}_\infty = \int_{\Sigma} |\nabla \hat{u}_\infty|^2 - \|A_{\Sigma}\|^2 + Ric_N(\nu, \nu))\hat{u}_\infty^2 \, dx \geq 0 \]
which completes the proof. \( \square \)

**Remark 5.2.** An interesting question is when the constrained embedded minimal hypersurface obtained by min-max construction in theorem 4.3 is a minimal hypersurface (i.e. has vanishing mean curvature). When \( \partial M \) is minimal, this is clearly true by strong maximum principle; In general, is this true when one assume further that the width is invariant under local perturbation of \( \partial M \)? Is this true under assumption above for generically perturbed boundary?

**Appendix A. Improved regularity for obstacle problems**

The goal of this section is to prove theorem 2.11 and 2.12. First recall that under local coordinate chart \((\mathcal{U}, x')\) of \( \partial M \), if \( u \in C^4(\mathcal{U}, (-r_M/2, r_M/2)) \), then \( \mathcal{H}^n(\text{graph}_M(u)) = \oint_{\mathcal{U}} F(x, u, du) \), where \( 0 < F(x, z, p) \in C^\infty(\mathcal{U} \times (-r_M, r_M) \times \mathbb{R}^n) \) satisfies
\[ \partial_{pp}^2 F(x, z, p)^{ij} \xi_i \xi_j \geq \lambda(x, z, p) |\xi|^2 > 0 \quad \forall \xi \in \mathbb{R}^n \]  
(A.1)
\[ \|F\|_{C^4(\mathcal{U} \times [-r_M/2, r_M/2]) \times \mathbb{R}^n} \leq C(M, g, R) \]  
(A.2)
If \( u \geq 0 \) and \( \text{graph}_M(u) \) is constrained stationary in \( M \), then
\[ 0 \leq \frac{d}{dt} \bigg|_{t=0} \int_{\mathcal{U}} F(x, u + t\phi, du + t\phi) \, dx \]  
(A.3)
\[ \leq \int_{\mathcal{U}} \partial_{p} F(x, u, du) \cdot d\phi + \partial_t F(x, u, du) \phi \, dx \]
For every \( \phi \in C^2_c(\mathcal{U}) \) such that for \( 0 < t << 1, u + t\phi \geq 0 \).

We shall deal with more general quasi-linear 2nd order elliptic operator of divergence form. For simplicity, write \( B_r(x) := B^0_r(x) \subset \mathbb{R}^n \), \( B_r := B_r(0) \) and \( \Omega_r := B_r \times [-1, 1] \times B_1 \). Let \( \lambda, \Lambda > 0 \),
\( \alpha \in (0, 1) \) be constant, \( A = A(x, z, p) \in C^3(B_2 \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n) \), \( B = B(x, z, p) \in C^2(B_2 \times \mathbb{R} \times \mathbb{R}^n) \) satisfying
\[
(\text{A.4}) \quad \partial_p A^j |_{B_2 \times [-1,1] \times B_1} \cdot \xi_i \xi_j \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n
\]
Let \( \mathcal{M} u := -\text{div}(A(x, u, \nabla u)) + B(x, u, \nabla u) \). Consider the variational inequality for a function \( u \in C^1(B_2, \mathbb{R}^n) \)
\[
(\text{A.5}) \quad \int_{B_2} A(x, u, \nabla u) \cdot \nabla \phi + B(x, u, \nabla u) \phi \, dx \geq 0 \quad \forall \phi \in C^1(B_2, \mathbb{R}^n)
\]

Clearly, \((\text{A.3}) \) implies \((\text{A.5}) \), where \( A = \partial_u F \) and \( B = \partial_x F \).

**Theorem A.1.** \( \exists \delta_0 = \delta_0(\lambda, \Lambda, n, \alpha) \in (0, 1) \) such that if \( A, B \) satisfies \((\text{A.4}) \), \( 0 \leq u \in C^{1,\alpha}_{\text{loc}}(B_2 \setminus \{0\}) \cap C^1(B_2) \) satisfies \((\text{A.3}) \) and that \( \|u\|_{C^1(B_2)} \leq \delta_0 \), then \( u \in C^{1,\alpha}_{\text{loc}}(B_2) \) and
\[
\|u\|_{C^{1,\alpha}(B_2)} \leq C(\lambda, \Lambda, n, \alpha)
\]

In view of the proof of lemma 3.4, theorem 2.11 follows from theorem A.1.

The major effort is made to prove the following lemma,

**Lemma A.2.** There exists \( \delta_0 = \delta_0(\lambda, \Lambda, n, \alpha) \in (0, 1) \), \( \eta_0 = \eta_0(\lambda, \Lambda, n, \alpha) \in (0, \delta_0) \) s.t. if \( u \in \text{image} \) \((\text{A.7}) \) and further satisfying
\[
(\text{A.6}) \quad \|u\|_{C^{1,\alpha}(B_2)} \leq 2\delta_0
\]
and \( \eta < \eta_0 \) such that the elliptic coefficients satisfies
\[
(\text{A.7}) \quad \|\partial_x A\|_{C^2(B_2)} + \|\partial_x A\|_{C^2(B_2)} + \|B\|_{C^2(B_2)} \leq \eta
\]
Then \( \|u\|_{C^{1,\alpha}(B_1)} \leq C(\lambda, \Lambda, n, \alpha, \eta)\|u\|_{C^{1,\alpha}(B_2)} + \eta \).

**Proof of theorem A.7** Observe that if \( x_0 \in B_1, r \in (0, 1), \) let \( \tilde{u}(y) := u(x_0 + ry)/r \), then
\[
\mathcal{M} u(x) = \frac{1}{r} \left[-\text{div}(\tilde{A}(y, \tilde{u}(y), \nabla \tilde{u}(y)) + \tilde{B}(y, \tilde{u}(y), \nabla \tilde{u}(y)))\right]_{y=(x-x_0)/r}
\]
where
\[
\tilde{A}(y, z, p) := A(x_0 + ry, rz, p)
\]
\[
\tilde{B}(y, z, p) := rB(x_0 + ry, rz, p)
\]
Hence, by lemma A.2 for \( u \) satisfying the assumptions in theorem A.1 \( u \in C^{1,\alpha}_{\text{loc}}(B_2 \setminus \{0\}) \). Thus, for a.e. \( x \in B_2 \setminus \{0\} \), \( \mathcal{M} u(x) = B(x, 0, 0) \). Together with \((\text{A.5}) \) and that \( \|u\|_{C^1(B_2)} \leq 1 \), by a cutting off argument, \( \mathcal{M} u(x) = B(x, 0, 0) \cdot \chi_{[0,\infty)} \) in the weak sense on \( B_2 \). Hence by standard elliptic estimate [13] theorem 3.1, \( \exists \beta = \beta(\lambda, \Lambda, n) \in (0, 1) \) such that \( \|u\|_{C^{1,\beta}(B_3/2)} \leq C(\lambda, \Lambda, n) \). Then repeat the rescaling argument above, by lemma A.2, theorem A.1 is proved.

Now we start to prove lemma A.2.

**Lemma A.3.** \( \exists \eta_1 = \eta_1(\lambda, \Lambda, n, \alpha) \in (0, 1) \) such that if \( \eta_1 \leq \eta_1 \) and \( A, B \) satisfies \((\text{A.4}) \) and \((\text{A.7}) \),
\[
\|\phi\|_{C^{1,\alpha}(\partial B_1)} \leq \eta_1
\]
Then the equation
\[
(\text{A.8}) \quad \begin{cases} \mathcal{M} u = 0 & \text{in } B_1 \\ u = \phi & \text{on } \partial B_1 \end{cases}
\]
has a unique solution \( u \in C^{1,\alpha}(\text{Clos}(B_1)) \) satisfying
\[
(\text{A.9}) \quad \|u\|_{C^{1,\alpha}} \leq C(\lambda, \Lambda, n, \alpha)(\|\phi\|_{C^{1,\alpha}} + \eta)
\]
The uniqueness follows directly from comparison principle, see [15] Chapter 10; We assert that the existence result does not follow directly from the classical Schauder estimate, since the latter requires boundary value to be $C^{2,\alpha}$. Instead, one need to work in the space $H_a^{(b)}$ introduced [14], where $a = 2 + \alpha$, $b = 1 + \alpha$. By [14] theorem 5.1 and a fixed point argument, there exists $u \in H_a^{(b)}(B_1)$ satisfying (A.8) with uniformly bounded $H_a^{(b)}$-norm; and (A.3) follows from [14] lemma 2.1.

Proof of lemma A.2. Let $\delta_0, \eta_0 \in (0, \eta_1)$ TBD; $u$ be in lemma A.2

**Step 1** $\forall x_0 \in B_{3/2} \cap \{u = 0\}$, $\forall r \in (0, 1/2)$, let $v = v_{x_0, r} \in C^{1,\alpha}(B_r(x_0))$ be the solution of

\[
\begin{aligned}
J v &= 0 \quad \text{in } B_r(x_0) \\
v &= u \quad \text{on } \partial B_r(x_0)
\end{aligned}
\]

given by lemma A.3. Also, let $\bar{v}$ be the solution of

\[
\begin{aligned}
J \bar{v} &= 0 \quad \text{in } B_r(x_0) \\
\bar{v} &= 0 \quad \text{on } \partial B_r(x_0)
\end{aligned}
\]

Also note that $\|u\|_{C^1(B_r)}$, $\|v\|_{C^1(B_r)}$, $\bar{v}\|_{C^1(B_r)} \leq C(\lambda, \Lambda, n, \alpha)(\eta_0 + \delta_0)$ and $J u \geq 0$ in the weak sense. Hence, by taking $\eta_0, \delta_0 << 1$ and using the maximum principle [15] theorem 10.10,

(A.10) $0 \leq u(x) - v(x) \leq C(\lambda, \Lambda, n, \alpha) \sup_{\partial(\lambda u > 0)} u - v \leq C(\lambda, \Lambda, n, \alpha)\|v\|_{C^0(B_r(x_0))}$

for every $x \in B_r(x_0)$. And $\bar{v} \leq v$ on $B_r(x_0)$. Applying the classical $C^0$ estimate on $\bar{v}(y) := \bar{v}(x_0 + ry)/r$, one get

(A.11) $\|\bar{v}\|_{C^0(B_r(x_0))} \leq C(\lambda, \Lambda, n, \alpha) r^2 \|B(\cdot, 0, 0)\|_{C^0(B_r)}$

Since $\bar{v} - \bar{v} \geq 0$ on $B_r(x_0)$, $J \bar{v} = \bar{v} = 0$ and $\bar{v}(x_0) \leq u(x_0) = 0$, by (A.4) and Harnack inequality,

(A.12) $\|v - \bar{v}\|_{C^0(B_{r/2}(x_0))} \leq C(\lambda, \Lambda, n, \alpha)(v(x_0) - \bar{v}(x_0)) \leq C(\lambda, \Lambda, n, \alpha)|\bar{v}(x_0)|$

Combining (A.10), (A.11) and (A.12) one derive

(A.13) $0 \leq u \leq C(\lambda, \Lambda, n, \alpha)\|B(\cdot, 0, 0)\|_{C^0(B_{r/2}(x_0))}^2 \quad \text{on } B_{r/2}(x_0)$

**Step 2** $\forall x_0 \in \{u = 0\}$, $\nabla u(x_0) = 0$; $\forall x_0 \in \{u > 0\} \cap B_{3/2}$, if $l := \text{dist}(x_0, \{u = 0\}) < 1/8$, let $y_0 \in \{u = 0\}$ such that $|x_0 - y_0| = l$. Then by (A.13),

$\|u\|_{C^0(B_l(x_0))} \leq \|u\|_{C^0(B_{l/2}(y_0))} \leq C(\lambda, \Lambda, n, \alpha) l^2$

Hence by interior gradient estimate,

(A.14) $\frac{1}{l} \|\nabla u\|_{C^0(B_{l/2}(y_0))} + \|\nabla^2 u\|_{C^0(B_{l/2}(y_0))} \leq C(\lambda, \Lambda, n, \alpha)$

Now that, $\forall x_1, x_2 \in B_1$, $\rho := |x_1 - x_2| < 1/20$, $l_1 := \text{dist}(x_1, \{u = 0\})$, $l_1 \leq l_2$.

(i) If $l_2 \geq 1/8$, then directly by gradient estimate

$|\nabla u(x_1) - \nabla u(x_2)| \leq \rho \sup_{B_{\rho}(x_2)} \|\nabla^2 u\| \leq C(\lambda, \Lambda, n, \alpha)|x_1 - x_2|$

(ii) If $2\rho \leq l_2 < 1/8$, then by (A.14),

$|\nabla u(x_1) - \nabla u(x_2)| \leq \rho \sup_{B_{\rho/2}(x_2)} \|\nabla^2 u\| \leq C(\lambda, \Lambda, n, \alpha)|x_1 - x_2|$  

(iii) If $l_2 \leq 2\rho$, then also by (A.14),

$|\nabla u(x_1) - \nabla u(x_2)| \leq \|\nabla u(x_1)\| + \|\nabla u(x_2)\| \leq C(\lambda, \Lambda, n, \alpha)(l_1 + l_2) \leq C(\lambda, \Lambda, n, \alpha)|x_1 - x_2|$

This proves the $C^{1,1}$ bound of $u$. \qed
The regularity results in literature focus on minimizing boundaries; for general closed integral current, the strategy is to first decompose them into sum of boundary of Caccioppoli sets. One subtlety is that a priori the graphs may have incompatible orientations. We rule out this by choosing $U_0$ small enough.

Recall that $M^{n+1} \subset \mathbb{R}^L$ is a compact submanifold with boundary, extended to a closed manifold $\bar{M}^{n+1} \subset \mathbb{R}^L$; $\bar{U} \subset M$ is relatively open and $U := \bar{U} \cap M$. Suppose $U \cap \partial M \neq \emptyset$. $T \in Z_n(U)$ minimize mass among $\{ S \in Z_n(U) : \text{spt}(T - S) \subset U \}$. Call such $T$ constrained minimizing in $U$.

WLOG $U, \bar{U}, V := \bar{U} \setminus U$ are contractible and $M_U(T) > 0$. By decomposition theorem [31, Chap 6, 3.14], there’s a decreasing sequence of $\mathcal{H}^{n+1}$-measurable subsets $\{ U_j \subset U \}_{j \in \mathbb{Z}}$ such that

$$T = \sum_{j \in \mathbb{Z}} \partial[U_j]; \quad ||T|| = \sum_{j \in \mathbb{Z}} ||\partial[U_j]||$$

In particular, $M_W(T) = \sum_{j \in \mathbb{Z}} M_W(\partial[U_j])$ for all $W \subset \bar{U}$, $\text{spt}(\partial[U_j]) \subset U$ and $T_j := \partial[U_j]$ are also constrained minimizers in $U$ (along $\partial M$). Thus for each $j$, either $\text{spt}[U_j]$ or $\text{spt}([\bar{U} - U_j])$ contains $V$. WLOG, $\text{spt}(\partial[U_j]) \cap \partial M \neq \emptyset$ and $\text{spt}[U_0] \supset V$. Hence $\forall j \leq 0, \text{spt}[U_j] \supset V$; and $\forall j > 0$, either $\text{spt}[U_j] \supset V$ or $\text{spt}[U_j] \subset \text{spt}[U_0] \setminus V$.

Claim: $\forall K \subset U \cap \partial M$ compact, $\exists r_1 = r_1(M, U, K, \delta) > 0$ such that if $T_0 = \partial[U_0], T_1 = \partial[U_1]$ are constrained minimizers in $U$ and $U_0 \subset V, U_1 \subset U_0 \setminus V; y \in K \cap \text{spt}(T_0)$. Then $U_1 \cap B_{r_1}(y) = \emptyset$.

Clearly, with this claim, for each $j \in \mathbb{Z}, T_j \cap B_{r_1}(y) = \partial[U_j] \cap B_{r_1}(y)$ is constrained minimizing in $B_{r_1}(y)$, where $U_j := U_j \cap B_{r_1}(y)$ is either empty or containing $V$. Hence, by [27] and [33], $T_j \cap B_{r_1}(y)$ is $C^{1,\alpha}$ for some $\alpha \in (0, 1)$. Then by [21] $T$ is $C^{1,1}$ multiple graphs restricting to a smaller subset $U_0$. This proves (1) of theorem 2.1 and (2) follows from unique minimals of hypersurfaces.

Proof of the claim: By [27], for every $\delta \in (0, 1), \exists r_2 = r_2(M, U, K, \delta) > 0$ and $u_0 \in C^2(U \cap \partial M)$ such that

$$(A.15) \quad T_{i,j} := \partial[U_i^{(j)}] \text{ being constrained minimizing boundary in } U_i, \text{ where } i = 0, 1, j \geq 1, U_0^{(j)} \subset V, U_1^{(j)} \subset U_0^{(j)} \setminus V \text{ and } U_0^{(j)} \cap B_{1/j}(y_j) \neq \emptyset. \text{ Then } \text{spt}(T_1^{(j)}) \cap B_{1/j}(y_j) \neq \emptyset.$$

Assuming the claim fails, then $\exists y_j \in K, T_i^{(j)} = \partial[U_i^{(j)}]$ being constrained minimizing boundary in $U_i$, where $i = 0, 1, j \geq 1, U_0^{(j)} \subset V, U_1^{(j)} \subset U_0^{(j)} \setminus V$ and $U_1^{(j)} \cap B_{1/j}(y_j) \neq \emptyset$. Then $\text{spt}(T_0^{(j)}) \cap B_{1/j}(y_j) \neq \emptyset$.

Let $j \to \infty, y_j \to y_\infty \in K$, and consider $(y_{0,j}, 1/j) T_1^{(j)} \to T_1^{(\infty)}$ being constrained minimizing in $T_1^{(\infty)} \subset M \cong \mathbb{R}^{n+1}$. The convergent is both in current and in varifold sense. Therefore, by (A.15) and that $U_0^{(j)} \subset U_0^{(\infty)} \setminus V, T_1^{(\infty)} = \partial[U_0^{(\infty)}]$ for some $\text{spt}(U_1^{(\infty)}) \subset T_{y_\infty} \partial M$, which is $n + 1$ negligible. In particular, $T_1^{(\infty)} = 0$; On the other hand, by $\text{spt}(T_0^{(j)}) \cap B_{1/j}(y_j) \neq \emptyset$, lemma 2.3 and corollary 2.4 $T_1^{(\infty)} \neq 0$ and we get a contradiction.

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