On the automorphisms of quantum Weyl algebras

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Abstract

Motivated by Weyl algebra analogues of the Jacobian conjecture and the Tame Generators problem, we prove quantum versions of these problems for a family of analogues to the Weyl algebras. In particular, our results cover the Weyl-Hayashi algebras and tensor powers of a quantization of the first Weyl algebra which arises as a primitive factor algebra of $U_q^+(\mathfrak{so}_5)$.

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1 Introduction

For a field $k$, let $A := k\langle x_1, \ldots, x_n \rangle$ be the free associative algebra in $n$ variables and denote by $\text{Aut}A$ the automorphism group of $A$. An automorphism $\psi$ of $A$ is called elementary if it is of the form

$$\psi(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, \alpha x_i + F, x_{i+1}, \ldots, x_n),$$

where $\alpha \in k^* := k \setminus \{0\}$ and $F \in k\langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \rangle$. The subgroup of $\text{Aut}A$ generated by the elementary automorphisms is called the tame subgroup, and an element of this subgroup is called tame. An automorphism of $A$ not belonging to the tame subgroup is called wild. Study of the automorphisms of $A$, and of factor algebras of $A$, has been ubiquitous over the last hundred years (for a comprehensive overview see [13]). It was shown in [19] and [21] that the automorphisms groups of the polynomial ring and the free associative algebra in two variables are tame. Along with these results came the following natural problems:

(P1) Is every automorphism of free associative algebra in $n$ variables tame?

(P2) Is every automorphism of the commutative polynomial ring in $n$ variables tame?

Notably in [25], Nagata’s automorphism, proposed in [22], was shown to be wild yielding a negative answer to (P2) in three variables. In [29] the Anick automorphism of the free associative algebra in three variable was show to be wild also giving a negative answer to (P1). Both Nagata’s automorphism and the Anick automorphism are stably tame (see [26]), thus, lifting either automorphism to higher order spaces unfortunately does not produce further wild automorphisms. To the best of the authors knowledge the tame generators problems (P1) and (P2) remain unsolved for $n$ greater than 3 generators.

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In his foundational paper on the Weyl algebra \cite{12}, Dixmier showed that every automorphism of the first Weyl algebra is tame. Given that the $n^{th}$ Weyl algebra can be realised as a factor algebra of the free associative algebra in $2n$ variables, the natural Weyl algebra analogue of the tame generators problem follows:

(P3) Is every automorphism of the $n^{th}$ Weyl algebra tame?

Again, to the best of the authors' knowledge (P3) remains unsolved for $n$ greater than one. Given the existence of wild automorphism in the polynomial ring and free algebra cases, one might suspect that a similar result will follow for higher order Weyl algebras.

In \cite{20}, primitive factor algebras of Gelfand-Kirillov dimension 2 of the positive part of the quantized enveloping algebra $U_q(so_5)$ were classified. These can be thought of as quantum analogues of the first Weyl algebra. Among those are the algebras $A_{\alpha,q}$ with $\alpha \in k^*$, where $A_{\alpha,q}$ is the free associative algebra in three variables $e_1, e_2$ and $e_3$, subject to the commutation relations:

\begin{align*}
e_1 e_3 &= q^{-1} e_3 e_1, \\
e_2 e_3 &= q e_3 e_2 + \alpha, \\
e_2 e_1 &= q^{-1} e_1 e_2 - q^{-1} e_3, \\
e_3^2 + (q^2 - 1) e_3 e_1 e_2 + \alpha q(q + 1) e_1 &= 0.
\end{align*}

Setting $q = 1$ and $\alpha = 1$ we indeed get an algebra isomorphic to the first Weyl algebra. In \cite{20} these algebras are denoted $A_{0,0}$ and for simplicity we replace $q^2$ with $q$.

Let $\mathcal{H}^t_q$ denote the free associative algebra with generators $\Omega, \Omega^{-1}, \Psi$ and $\Psi^\dagger$ subject to the relations

\begin{align*}
\Omega \Omega^{-1} &= \Omega^{-1} \Omega = 1, \\
\Psi \Omega &= q \Omega \Psi, \quad \Psi^\dagger \Omega = q^{-1} \Omega \Psi^\dagger, \\
\Psi \Psi^\dagger &= \frac{q^t \Omega^t - q^{-t} \Omega^{-t}}{q^t - q^{-t}} \quad \text{and} \quad \Psi^\dagger \Psi = \frac{\Omega^t - \Omega^{-t}}{q^t - q^{-t}}. \quad (1)
\end{align*}

By setting $t = 1$ we retrieve the Weyl Hayashi algebra $\mathcal{H}^1_q$ studied in \cite{11} and \cite{10}. When $t = 2$ we get the original algebras introduced by Hayashi in \cite{14}. In this article we will consider the generalization $\mathcal{H}^t_q$ which covers both conventions. In \cite{14} Hayashi introduced $\mathcal{H}^2_q$ as $q$-analogue of the Weyl algebra to construct oscillator representations of quantum enveloping algebras. In \cite{16} it was shown that the algebras $\mathcal{H}^1_q$ arise as factor algebras of a $q$-analogue of the universal enveloping algebra of the Heisenberg Lie algebra. It was also shown in (\cite{20}, Section 3) that $\mathcal{H}^1_q$ appears as factor algebras of the positive part of the quantized enveloping algebra $U_q(so_5)$.

The tame generators problems, and in particular (P3), makes it natural to consider if the complexity of the automorphism group of quantum analogues of the $n^{th}$ Weyl algebra fundamentally changes as $n$ increases. In this article we arrive at analogues to the $n^{th}$ Weyl algebra by taking the tensor product (over the ground field) of $n$ copies of our first Weyl algebra analogues $A_{\alpha,q}$ and $\mathcal{H}^t_q$. By showing that these algebras are part of a family of generalized Weyl algebras that we call quantum Weyl analogue (qwa) algebras, we are able to define the notion of qua-tame (see Section 5 and specifically Definition 5.1). Using our definition we show that the automorphism groups of our analogues are well behaved as we increase the number of tensor copies. Precisely we prove the following quantum analogues to the tame generator problem:
Theorem 1.1. Every automorphism of $A_{\alpha,q} \otimes \ldots \otimes A_{\alpha,q}$ is $q$-wa-tame for $\alpha \in k^*$ and $q \in k^* \setminus \{z \mid z^2 \neq 1\}$.

Theorem 1.2. Every automorphism of $H_{q}^t \otimes \ldots \otimes H_{q}^t$ is $q$-wa-tame for $\alpha \in k^*$ and $q \in k^* \setminus \{z \mid z^{2t} \neq 1\}$.

In general computing the automorphism group of an algebra can be very difficult. Recently some progress has been made to produce a uniform approach to this problem for a large class of algebras (see [10]). In [11] the same authors use their approach to show that the automorphism group of tensor products of the so called $q$-quantum Weyl algebra is tame. Theorem 1.1 and Theorem 1.2 can be seen as a direct analogue to [11, Theorem 2].

Dixmier also made the now famous conjecture: Every endomorphism of the $n^{th}$ Weyl algebra is an automorphism. Tsuchimoto, [28], and Belov-Kanel and Kontsevich, [8], proved independently that the Dixmier Conjecture is stably equivalent to the Jacobian Conjecture of Keller [15]. It is natural to ask Dixmier’s question for related algebras (see [3, 23]), and especially generalizations and quantizations of the Weyl algebras (see [2, 9]). In [17], every endomorphism of $A_{\alpha,q}$ (and more generally simple quantum generalized Weyl algebras), when $q$ is not a root of unity, was shown to be an automorphism. In this article we show that every homomorphism between two of our analogues of $n^{th}$ Weyl algebra is invertible. Precisely, we prove the following theorems:

Theorem 1.3. If $q$ is a not a root of unity and $\alpha_i, \tilde{\alpha}_i \in k^*$ for $i \in \{1, \ldots, n\}$, then every homomorphism between $A_{\alpha_1,q} \otimes \ldots \otimes A_{\alpha_n,q}$ and $A_{\tilde{\alpha}_1,q} \otimes \ldots \otimes A_{\tilde{\alpha}_n,q}$ is invertible.

Theorem 1.4. If $q$ is a non root of unity, then every endomorphism of $H_{q}^t \otimes \ldots \otimes H_{q}^t$ is an automorphism.

In parallel to the pathology often encountered when considering algebras over nonzero characteristic fields, in quantum algebra, considering quantizations at roots of unity can be equally problematic. Given the current interest in reduction modulo $p$ techniques and results in the context of differential operators (see for instance [7, 18]), it is natural to extend the work in [17] to study the endomorphisms of quantum generalized Weyl algebras when $q$ is a root of unity. This case can be thought of as the quantum analogue of reduction modulo $p$ (see for instance [2]). Thus we extend the classification of endomorphisms used in the proof of Theorem 1.1 of [17], to include the case where $q$ is a root of unity other than $\pm 1$. We show that there exist non-invertible endomorphisms in this case (see Corollary 4.4).

2 Preliminaries

To prove our quantum analogues of the tame generators problem and the Dixmier conjecture, we will exploit that the algebras $A_{\alpha,q}$ and $H_{q}^t$ are isomorphic to generalized Weyl algebras of degree 1. Our strategy will then be to classify the homomorphisms between tensor products of these algebras.

Recall that for a $k$-algebra $R$, a ($k$-algebra) automorphism $\sigma$ of $R$, and a central element of $R$, say $a$, the generalized Weyl algebra $R(\sigma, a)$ of degree 1 is the algebra extension of $R$ by the two indeterminates $x$ and $y$ subject to the relations

\[ xy = \sigma(a), \quad yx = a, \quad xr = \sigma(r)x \quad \text{and} \quad yr = \sigma^{-1}(r)y \quad \text{for all} \ r \in R. \]
The isomorphisms and automorphisms of generalized Weyl algebras of degree 1 have been widely examined (see [6, 24, 27]). For $d \in \mathbb{N}^*$, $q \in \mathbb{k}^* \setminus \{z|z^d \neq 1\}$ and $\sigma \in \text{Aut}(\mathbb{k}[h^\pm])$ such that $\sigma(h) = qh$, we denote by $A(d, q)$ the generalized Weyl algebra $\mathbb{k}[h^\pm](\sigma, h^d - 1)$. Using Proposition 3.10 of [20] and Theorem A of [27] we have that $\mathcal{A}_{\alpha, q} \simeq A(1, q)$.

**Remark 2.1.** Given the isomorphism $\mathcal{A}_{\alpha, q} \simeq A(1, q)$ for all $\alpha \in \mathbb{k}^*$, Theorem 1.3 reduces to proving that every endomorphism of $A(1, q) \otimes \ldots \otimes A(1, q)$ is an automorphism when $q$ is a not a root of unity.

By the isomorphism which sends
\[
\Omega \mapsto h, \quad \Psi \mapsto x \quad \text{and} \quad \Psi^\dagger \mapsto y,
\]
we have that $\mathcal{H}_q \simeq A(2t, q)$. Since the algebras $\mathcal{A}_{\alpha, q}$ and $\mathcal{H}_q$ are analogues of the first Weyl algebra, we can produce analogues, and in the case of $\mathcal{A}_{\alpha, q}$ a quantization, of the $n$th Weyl algebra by taking a tensor product, over $\mathbb{k}$, of $n$ copies of the original algebra. Thus for $n, d \in \mathbb{N}^*$, $q := (q_1, \ldots , q_n) \in (\mathbb{k}^* \setminus \{z|z^d \neq 1\})^n$ and $\sigma_i \in \text{Aut}(\mathbb{k}[h_i^\pm])$ such that $\sigma_i(h_i) = q_i h_i$, we define the quantum Weyl analogue (qwa) algebras
\[
A(n, d, q) := \bigotimes_{i=1}^n \mathbb{k}[h_i^\pm](\sigma_i, h_i^d - 1).
\]

By extending the above isomorphisms we can realize the algebras
\[
A_{\alpha, q_1} \otimes \ldots \otimes A_{\alpha, q_n} \quad \text{and} \quad \mathcal{H}_{q_1, t} \otimes \ldots \otimes \mathcal{H}_{q_n, t}
\]
as members of the family of algebras $A(n, d, q)$.

Since the category of generalized Weyl algebras is closed under tensor product, $A(n, d, q)$ is a degree $n$ generalized Weyl algebra in the sense of [3]. For simplicity we fix the notation $N := \{1, \ldots , n\}$ and $a_d(h_i) = h_i^d - 1$. Precisely $A(n, d, q)$ is the $\mathbb{k}$-algebra generated by $x_i, y_i, h_i$ and $h_i^{-1}$ subject to the relations
\[
x_i h_i = q_i h_i x_i, \quad y_i h_i = q_i^{-1} h_i y_i, \quad x_i y_i = a_d(q_i h_i), \quad y_i x_i = a_d(h_i), \quad h_i^{\pm 1} h_i^{\mp 1} = 1
\]
and the commutation relations
\[
h_i h_j = h_j h_i, \quad h_i x_j = x_j h_i, \quad h_i y_j = y_j h_i,
\]
\[
x_i x_j = x_j x_i, \quad x_i y_j = y_j x_i \quad \text{and} \quad y_i y_j = y_j y_i
\]
for $i, j \in N$ and $i \neq j$.

The property that any degree $n$ generalized Weyl algebra is $\mathbb{Z}^n$-graded is integral to the proof of Theorems 1.1 and 1.3. Thus, we recall this grading from [4] applying it to $A(n, d, q)$. For a vector $k := (k_1, \ldots , k_n) \in \mathbb{Z}^n$ we set $w_k := w_{k_1}(1) \ldots w_{k_n}(n)$, where for $i \in N$ and $m \geq 0$ we have
\[
w_m(i) = x_i^m, \quad w_{-m}(i) = y_i^m \quad \text{and} \quad w_0(i) = 1.
\]
It follows from the relations of $A(n, d, q)$ that
\[
A(n, d, q) := \bigoplus_{k \in \mathbb{Z}^n} A_{(k)}
\]
is a $\mathbb{Z}^n$-graded algebra, where $A_{(\mathbb{k})} := \mathbb{k}[h_1^{\pm 1}, \ldots, h_n^{\pm 1}]$.

When classifying automorphisms or isomorphisms, it can often be illuminating to consider normal elements, since normality is preserved by invertible homomorphisms. Indeed, this approach was adopted in [24] to classify, up to isomorphism, quantum generalized Weyl algebras over a polynomial ring. For general homomorphisms, normality is not preserved. Instead, we exploit that any homomorphism maps invertible elements to invertible elements.

It is well known that any unit in $T$ is of the form $u^{\pm 1}$, with the relation

$$u_i v_i = q_i v_i u_i,$$

and define $T_q := \bigotimes_{i=1}^n T_i$. We now classify the units in $A(n, d, q)$.

**Lemma 2.2.** Any unit in $A(n, d, q)$ is of the form $\gamma h_1^{m_1} \cdots h_n^{m_n}$, where $\gamma \in \mathbb{k}^*$ and $m_1, \ldots, m_n \in \mathbb{Z}$.

**Proof.** Consider the two embeddings $\phi : A(n, d, q) \hookrightarrow T_q$ and $\phi' : A(n, d, q) \rightarrow T_q$ defined in the following ways:

$$\phi(x_i) = u_i, \quad \phi(y_i) = a_d(v_i)u_i^{-1} \quad \text{and} \quad \phi(h_i) = v_i$$

and

$$\phi'(x_i) = u_i^{-1}a_d(v_i), \quad \phi'(y_i) = u_i \quad \text{and} \quad \phi'(h_i) = v_i.$$ 

It is well known that any unit in $T_q$ is of the form $\gamma u_1^{r_1} \cdots u_n^{r_n} v_1^{m_1} \cdots v_n^{m_n}$, for $\gamma \in \mathbb{k}^*$ and $r_1, \ldots, r_n, m_1, \ldots, m_n \in \mathbb{Z}$. By the embedding $\phi$, it is necessary that a unit $u$ in $A(n, d, q)$ is of the form $u = \gamma x_1^{n_1} \cdots x_n^{n_d} h_1^{m_1} \cdots h_n^{m_n}$. Similarly by the embedding $\phi'$ we find that $u$ must be of the form $u = \gamma y_1^{n_1} \cdots y_n^{n_d} h_1^{m_1} \cdots h_n^{m_n}$. Comparing these expressions using the grading introduced in Equation (3) yields the desired result.

## 3 Classification of Homomorphisms

Before giving our classification of homomorphisms we introduce for simplicity, the following notation. For $0 < r \leq n$, let $R := \{1, \ldots, r\}$ and $\vec{\gamma} : = (\vec{\gamma}_1, \ldots, \vec{\gamma}_r) \in (\mathbb{k}^* \setminus \{z|z^d \neq 1\})^r$ and set $\vec{A}(r, d, \vec{\gamma}) := A(r, d, \vec{\gamma})$ distinguishing $\vec{A}(r, d, \vec{\gamma})$ from $A(n, d, q)$ by marking every generator and indeterminate of $\vec{A}(r, d, \vec{\gamma})$ with a tilde (for example $\tilde{h}_i$). In this section we classify the homomorphisms between $\vec{A}(r, d, \vec{\gamma})$ and $A(n, d, q)$.

**Theorem 3.1.** 1. Let $\psi$ be a homomorphism from $\vec{A}(r, d, \vec{\gamma})$ to $A(n, d, q)$.

   (i) Then there exists a partial permutation $w : R \rightarrow N$, $(\tau_1, \ldots, \tau_r) \in \{0, 1\}^r$ and $(m_1, \ldots, m_r) \in (\mathbb{Z}^*)^r$ such that

   $$q^{-1}_{w(i)}^{\tau_i m_i} = \tilde{\gamma}_i$$

   (4)

   for $i \in R$.  


(ii) There exists a matrix \((t_{i,j}) \in \mathcal{M}_{r,n}(\mathbb{Z})\) such that
\[
q_{w(i)}^{t_{i,w(i)}(1-\tau_i)} q_{w(i)}^{-t_{i,w(i)} \tau_i} q_{w(l)}^{t_{i,w(l)} \tau_i} = 1
\]
for all \(i, l \in R\).

(iii) For \(i \in R\) there exist \(p_i(h_{w(i)}), p'_i(h_{w(i)}) \in \mathbb{k}[h_{w(i)}^{\pm 1}]\) such that
\[
p_i(h_{w(i)}) p'_i(h_{w(i)}) a_d(q_{w(i)}^{(1-\tau_i)} h_{w(i)}) = a_d(q_{w(i)}^{\tau_i} h_{w(i)}^{m_i}).
\]

(iv) The homomorphism \(\psi\) is defined on the generators of \(A(n,d,q)\) as follows:

- \(\psi(\tilde{h}_i) = \gamma_i h_{w(i)}^{m_i}\), where \(\gamma_i^{d_i} = \tilde{q}_i^{-\tau d_i}\).
- \(\psi(\tilde{x}_i) = p_i(h_{w(i)}) b_i h_{w(i)}^{t_{i,1}} \ldots h_{w(i)}^{t_{i,n}} x_{w(i)}^{(1-\tau_i)} y_{w(i)}^{\tau_i}\), where \(b_i \in \mathbb{k}^*\).
- \(\psi(\tilde{y}_i) = x_{w(i)}^{\tau_i} y_{w(i)}^{(1-\tau_i)} p'_i(h_{w(i)}) h_{w(i)}^{-t_{i,1}} \ldots h_{w(i)}^{-t_{i,n}}\).

2. Conversely, assume there exist a partial permutation \(w : R \to N\), \((\tau_1, \ldots, \tau_r) \in \{0,1\}^r\), \((m_1, \ldots, m_r) \in (\mathbb{Z}^*)^r\), \((b_1, \ldots, b_r), (\gamma_1, \ldots, \gamma_r) \in \mathbb{k}^*\), a matrix \((t_{i,j}) \in \mathcal{M}_{r,n}(\mathbb{Z})\) and \((p_i, p'_i), (\tilde{p}_i, \tilde{p}'_i) \in (\mathbb{k}[h_{w(i)}^{\pm 1}])^r\) such that Equations (4), (5) and (6) hold, and \(\gamma_i^{d_i} = \tilde{q}_i^{-\tau d_i}\) for \(i \in R\). Then, there exists a unique homomorphism \(\psi_{\tilde{A}}\) (where \(\tilde{A}\) encodes the information in the hypothesis) from \(\tilde{A}(r,d,\tilde{q})\) to \(A(n,d,q)\) defined on the generators of \(\tilde{A}(r,d,\tilde{q})\) as follows:

- \(\psi_{\tilde{A}}(\tilde{h}_i) = \gamma_i h_{w(i)}^{m_i}\).
- \(\psi_{\tilde{A}}(\tilde{x}_i) = p_i(h_{w(i)}) b_i h_{w(i)}^{t_{i,1}} \ldots h_{w(i)}^{t_{i,n}} x_{w(i)}^{(1-\tau_i)} y_{w(i)}^{\tau_i}\).
- \(\psi_{\tilde{A}}(\tilde{y}_i) = x_{w(i)}^{\tau_i} y_{w(i)}^{(1-\tau_i)} p'_i(h_{w(i)}) h_{w(i)}^{-t_{i,1}} \ldots h_{w(i)}^{-t_{i,n}}\).

Proof. We dedicate the rest of Section 3 to the proof of Theorem 3.1. For ease of understanding we break down our proof into four steps, giving summaries at the beginning and end of each step. Steps 1-3 combine to prove statement 1 of Theorem 3.1 and Step 4 proves statement 2.

3.1 Step 1

In Step 1 we will determine, for \(i \in R\), the action of a homomorphism \(\psi\) of \(\tilde{h}_i\). We will also show that \(\psi(\tilde{x}_i) \in A_{(\tilde{q})}\) and \(\psi(\tilde{y}_i) \in A_{(-\tilde{q})}\), where \(A_{(\tilde{q})}\) and \(A_{(-\tilde{q})}\) are as defined in Equation (3).

Let \(\psi\) be a homomorphism from \(\tilde{A}(r,d,\tilde{q})\) to \(A(n,d,q)\). Since units are preserved by homomorphisms, from Lemma 2.2 we deduce that, for all \(i \in R\),
\[
\psi(\tilde{h}_i) = \gamma_i h_{1}^{m_{i,1}} \ldots h_{n}^{m_{i,n}},
\]
where \(\gamma_i \in \mathbb{k}^*, m_{i,1}, \ldots, m_{i,n} \in \mathbb{Z}\).
We first prove that for all \( i \in R \) there exists \( l \in N \) such that \( m_{i,l} \neq 0 \). By contradiction assume \( m_{i,1} = \ldots = m_{i,n} = 0 \). Applying \( \psi \) to the relations \( \tilde{x}_i h_i = \tilde{q}_i h_i \tilde{x}_i \) and \( \tilde{y}_i h_i = \tilde{q}_i^{-1} h_i \tilde{y}_i \), we find that \( \psi(\tilde{x}_i) = 0 = \psi(\tilde{y}_i) \). Applying \( \psi \) to \( \tilde{y}_i \tilde{x}_i = a_d(\tilde{h}_i) \) and \( \tilde{x}_i \tilde{y}_i = a_d(\tilde{q}_i \tilde{h}_i) \) gives us that

\[
a_d(\gamma_i) = 0 = a_d(\tilde{q}_i \gamma_i)
\]

implying that \( \tilde{q}_i^d = 1 \) contradicting our assumption. Hence for all \( i \in R \) there exists \( l \in N \) such that \( m_{i,l} \neq 0 \).

Applying \( \psi \) to the relation \( \tilde{x}_i \tilde{y}_i = a_d(\tilde{q}_i \tilde{h}_i) \) we get

\[
\psi(\tilde{x}_i) \psi(\tilde{y}_i) = a_d(\tilde{q}_i \gamma_i h_1^{m_{i,1}} \ldots h_n^{m_{i,n}}).
\]

Using that \( A(n, d, q) \) is \( \mathbb{Z}^n \)-graded (see Equation (2)) we write

\[
\psi(\tilde{x}_i) = \sum_{k \in \mathbb{Z}^n} W_k \quad \text{and} \quad \psi(\tilde{y}_i) = \sum_{s \in \mathbb{Z}^n} W'_s
\]

where \( W_k, W'_s \in A(k) \) for all \( k \in \mathbb{Z}^n \) (and all but a finite number of them being equal to zero). Substituting these expressions into Equation (5) yields

\[
\left( \sum_{k \in \mathbb{Z}^n} W_k \right) \left( \sum_{s \in \mathbb{Z}^n} W'_s \right) = a_d(\tilde{q}_i \gamma_i h_1^{m_{i,1}} \ldots h_n^{m_{i,n}}).
\]

Noting that \( a_d(\tilde{q}_i \gamma_i h_1^{m_{i,1}} \ldots h_n^{m_{i,n}}) \in A(\emptyset) \) we find that \( \psi(\tilde{x}_i) = W_k \) and \( \psi(\tilde{y}_i) = W'_k \) for some \( k \in \mathbb{Z}^n \). Up to reordering the tensor product factors in \( A(n, d, q) \), it suffices to only consider the case where \( k = (k_1, \ldots, k_e, -k_{e+1}, \ldots, -k_n) \) with \( k_j \in \mathbb{Z}_{\geq 0} \) for \( j \in N \). First consider the case where \( k = \emptyset \). Thus \( \psi(\tilde{x}_i) = P_i(h_1, \ldots, h_n) \) for \( P_i(h_1, \ldots, h_n) \) a Laurent polynomial in the variables \( h_1, \ldots, h_n \). Applying \( \psi \) to the relation \( \tilde{x}_i \tilde{h}_i = \tilde{q}_i \tilde{h}_i \tilde{x}_i \) implies that \( P_i(h_1, \ldots, h_n) = 0 \) since \( \tilde{q}_i \neq 1 \). Now, applying \( \psi \) to the relation \( \tilde{y}_i \tilde{x}_i = a_d(\tilde{h}_i) \) gives us the contradiction \( \psi(a_d(\tilde{h}_i)) = 0 \). Thus there must be at least one nonzero entry in \( k \). We now have

\[
\psi(\tilde{x}_i) = P_i(h_1, \ldots, h_n)x_1^{k_1} \ldots x_e^{k_e} y_{e+1}^{k_{e+1}} \ldots y_{n,2}^{k_n},
\]

and

\[
\psi(\tilde{y}_i) = y_1^{k_1} \ldots y_e^{k_e} x_{e+1}^{k_{e+1}} \ldots x_{n,2}^{k_n} P'_i(h_1, \ldots, h_n)
\]

where \( P_i(h_1, \ldots, h_n) \) and \( P'_i(h_1, \ldots, h_n) \) are nonzero Laurent polynomials in the variables \( h_1, \ldots, h_n \). Thus we can rewrite Equation (5) as

\[
P_i(h_1, \ldots, h_n)x_1^{k_1} \ldots x_e^{k_e} y_{e+1}^{k_{e+1}} \ldots y_{n,2}^{k_n} P'_i(h_1, \ldots, h_n) = a_d(\tilde{q}_i \gamma_i h_1^{m_{i,1}} \ldots h_n^{m_{i,n}}).
\]

Standard manipulation (see [Equation 5]) of Equation (10) gives us that

\[
U_i(h_1, \ldots, h_n) \left( \prod_{s=1}^{e} \prod_{l=1}^{k_s} a_d(q_s^l h_s) \right) \left( \prod_{s=e+1}^{n} \prod_{l=0}^{k_{s-1}} a_d(q_s^{-l} h_s) \right) = a_d(\tilde{q}_i \gamma_i h_1^{m_{i,1}} \ldots h_n^{m_{i,n}})
\]
where \( U_i(h_1, \ldots, h_n) = P_i(h_1, \ldots, h_n)P'_i(h_1, \ldots, h_n) \). Using that \( a_d(X) = X^d - 1 \) we get

\[
U_i(h_1, \ldots, h_n) \left( \prod_{s=1}^{e} \prod_{l=1}^{k_s} \left( q_s^{ld} h_s^d - 1 \right) \right) \left( \prod_{s=e+1}^{n} \prod_{l=0}^{k_s-1} \left( q_s^{ld} h_s^d - 1 \right) \right) = a_d(\tilde{q}_i \gamma_i h_j^{m_i-1} \cdots h_n^{m_{i,n}}).
\] (11)

Pick \( j \in \mathbb{N} \) such that \( k_j \neq 0 \). Evaluating Equation (11) at \( h_j = q_j^{-1} \) if \( j \in \{1, \ldots, e\} \), or \( h_j = 1 \) if \( j \in \{e+1, \ldots, n\} \), implies that \( m_{i,s} = 0 \) for all \( s \in \mathbb{N} \setminus \{j\} \). We cannot repeat this process by evaluating at an alternate zero, since we have proved that at least one \( m_{i,t} \neq 0 \) with \( t \in \mathbb{N} \). Hence for each \( i \in \mathbb{N} \) there exists a unique \( j \in \mathbb{N} \) such that \( m_{i,j} \neq 0 \). Moreover \( \psi(h_i) = \gamma_i h_j^{m_{i,j}} \). We set \( w(i) := j \), but suppress this notation for simplicity until Step 3, where we show that the map \( w : \mathbb{N} \rightarrow \mathbb{N} \) is a partial permutation. Since the double subscript is now redundant we simplify our notation and set \( m_i := m_{i,j} \).

To summarize, in Step 1 we have shown that, for all \( i \in \mathbb{N} \),

\[
\psi(\tilde{h}_i) = \gamma_i h_j^{m_i}, \quad \psi(\tilde{x}_i) = P_i(h_1, \ldots, h_n)x_1^{k_1} \cdots x_e^{k_e} y_{e+1}^{k_{e+1}} \cdots y_n^{k_n}
\]

and \( \psi(\tilde{y}_i) = y_1^{k_1} \cdots y_e^{k_e} x_{e+1}^{k_{e+1}} \cdots x_n^{k_n}P'_i(h_1, \ldots, h_n) \),

where \( j = w(i) \in \mathbb{N}, \gamma_i \in \mathbb{K}, m_i \in \mathbb{Z} \) and \( P_i(h_1, \ldots, h_n), P'_i(h_1, \ldots, h_n) \in \mathbb{K}[h_1^{\pm 1}, \ldots, h_n^{\pm 1}] \) and \( (k_1, \ldots, k_n) \in (\mathbb{Z}_{\geq 0})^n \setminus \{(0, \ldots, 0)\} \).

3.2 Step 2

In Step 2 we will determine precisely, for \( i \in \mathbb{N} \), the action of a homomorphism on \( \tilde{x}_i \) and \( \tilde{y}_i \).

We also show that

\[
\gamma_i^d = \tilde{q}_i^{-\tau_i d}.
\]

Using the action of \( \psi \) on \( \tilde{h}_i \) we found in Step 1 we now rewrite Equation (11) as

\[
U_i(h_1, \ldots, h_n) \left( \prod_{s=1}^{e} \prod_{l=1}^{k_s} \left( q_s^{ld} h_s^d - 1 \right) \right) \left( \prod_{s=e+1}^{n} \prod_{l=0}^{k_s-1} \left( q_s^{ld} h_s^d - 1 \right) \right) = a_d(\tilde{q}_i \gamma_i h_j^{m_i}) = a_d(\tilde{q}_i \gamma_i h_j^{m_i})^d - 1.
\] (12)

Since the factors in the product of the left hand side of Equation (12) are not invertible (discounting \( U_i(h_1, \ldots, h_n) \)), comparing coefficients shows that \( k_j \) is the only nonzero entry in \( \mathbb{K} \). We can also conclude that \( U_i(h_1, \ldots, h_n) \) is a Laurent polynomial in \( h_j \) only and write \( U_i(h_1, \ldots, h_n) = U_i(h_j) \) to reflect this.

For simplicity we introduce notation to distinguish between the following two cases: Let \( \tau_i = 0 \) if \( \mathbb{K} = (0, \ldots, k_j, \ldots, 0) \) and \( \tau_i = 1 \) if \( \mathbb{K} = (0, \ldots, -k_j, \ldots, 0) \), for \( k_j > 0 \). Thus we can now write Equation (12) as

\[
U_i(h_j) \prod_{l=1-\tau_i}^{k_j-\tau_i} \left( q_j^{(l-1)\tau_i d h_j^{d}} - 1 \right) = \tilde{q}_i^{d} \gamma_i^{d} h_j^{m_{i,d}} - 1.
\] (13)
We will now prove by contradiction that $k_j = 1$. Assuming $k_j > 1$ we find that $q_j^{(-1)^{\tau_i}(1-\tau_i)}$ and $q_j^{(-1)^{\tau_i}(2-\tau_i)}$ are zeros of the left hand side of Equation (14), substituting these yields

$$
\left(q_j^{(-1)^{\tau_i}(1-\tau_i)}\right)^{m_i d} = \tilde{q}_i^{-d} = \left(q_j^{(-1)^{\tau_i}(2-\tau_i)}\right)^{m_i d},
$$

implying, by simple manipulation, that $q_j^{m_i d} = 1$. Applying $\psi$ to the relation $\tilde{x}_i \tilde{h}_i = \tilde{q}_i \tilde{h}_i \tilde{x}_i$ gives us

$$
P_i(h_1, \ldots, h_n) x_j^{k_j(1-\tau_i)} y_j^{k_j \tau_i} \gamma_i h_j^{m_i} = \tilde{q}_i \gamma_i h_j^{m_i} P_i(h_1, \ldots, h_n) x_j^{k_j(1-\tau_i)} y_j^{k_j \tau_i}.
$$

Simple manipulation indicates that

$$
q_j^{(-1)^{\tau_i} m_i k_j} = \tilde{q}_i.
$$

Equation (14) implies that $q_j^{(-1)^{\tau_i} m_i k_j} = \tilde{q}_i^d$, and by substituting $q_j^{m_i d} = 1$, we find that $\tilde{q}_i^d = 1$ which contradicts our assumptions and thus $k_j = 1$.

Note, since the derivation of Equation (14) did not rely of the assumption that $k_j > 1$, we have, by substituting $k_j = 1$,

$$
q_j^{(-1)^{\tau_i} m_i} = \tilde{q}_i.
$$

(15)

Substituting $k_j = 1$ into Equation (13) gives us

$$
U_i(h_j) \left(q_j^{(-1)^{\tau_i}(1-\tau_i)} h_j^d - 1\right) = \tilde{q}_i^d x_i^d h_j^{m_i d} - 1.
$$

(16)

Evaluating $h_j$ at $q_j^{(-1)^{\tau_i}(1-\tau_i)}$ in Equation (16) and using Equation (15) we can conclude that

$$
\gamma_i^d = \tilde{q}_i^{-\tau_i d}.
$$

(17)

Finally, since $U_i(h_j)$ is a Laurent polynomial in $h_j$ we have

$$
P_i(h_1, \ldots, h_n) = p_i(h_j) b_i h_1^{t_1,n} \ldots h_n^{t_n,n} \quad \text{and} \quad P'_i(h_1, \ldots, h_n) = p'_i(h_j) b_i^{-1} h_1^{-t_1,n} \ldots h_n^{-t_n,n}
$$

where $p_i(h_j) p'_i(h_j) = U_i(h_j)$, $b_i \in \mathbb{K}^*$ and $t_{i,1}, \ldots, t_{i,n} \in \mathbb{Z}$.

To summarize, in Step 2 we have shown that there exist $(\tau_1, \ldots, \tau_r) \in \{0, 1\}^r$, $(m_1, \ldots, m_r) \in (\mathbb{Z}^*)^r$, $(t_{i,1}, \ldots, t_{i,r}) \in \mathcal{M}_{r,n}(\mathbb{Z})$, $(b_1, \ldots, b_r)$, $(\gamma_1, \ldots, \gamma_r) \in (\mathbb{K}^*)^r$ and $(p_1, \ldots, p_r), (p'_1, \ldots, p'_r) \in (\mathbb{K}[h^{\pm 1}])^r$ such that, for all $i \in R$,

$$
\psi(\tilde{h}_i) = \gamma_i h_1^{m_i} \quad \text{and} \quad \psi(\tilde{x}_i) = p_i(h_j) b_i h_1^{t_1,n} \ldots h_n^{t_n,n} x_j^{(1-\tau_i)} y_j^{\tau_i}
$$

\[\text{and} \quad \psi(\tilde{y}_i) = x_j^{\tau_i} y_j^{(1-\tau_i)} p'_i(h_j) b_i^{-1} h_1^{-t_1,n} \ldots h_n^{-t_n,n},\]

and $\gamma_i^d = \tilde{q}_i^{-\tau_i d}$. 

\[9\]
In Step 3 we will show that the map \( w : R \rightarrow N \) from Step 1 is a partial permutation. We will also derive the necessary condition
\[
q_{w(i)}^\tau q_{w(i)}^\tau = 1,
\]
for \( i,l \in R \), which is required to ensure \( \psi \) is consistent on the commutation relations of \( \tilde{A}(r,d,\tilde{q}) \) (see Equation (22)).

For simplicity we state the action of \( \tilde{\psi} \) on \( \tilde{h}_i \) and \( \tilde{h}_c \) for \( i \neq e \in R \):
\[
\tilde{\psi} \left( \tilde{h}_i \right) = \gamma_i h_i^{\tau_1}, \quad \tilde{\psi} \left( \tilde{x}_i \right) = p_i(h_j) b_i h_i^{\tau_1} \ldots h_n^{\tau_n} x_j^{\tau_1} y_j^{\tau_1},
\]
and
\[
\tilde{\psi} \left( \tilde{h}_c \right) = \gamma_e h_c^{\tau_e}, \quad \tilde{\psi} \left( \tilde{x}_c \right) = p_e(h_k) b_e h_e^{\tau_1} \ldots h_n^{\tau_n} x_k^{\tau_1} y_k^{\tau_1},
\]
where for simplicity we set \( j := w(i) \) and \( k := w(e) \).

First we prove, by contradiction, that \( w \) is a partial permutation. Assume \( j = k \). Consider when \( \tau_i = \tau_e \) (due to the similarity in the calculation we leave the \( \tau_i \neq \tau_e \) to the reader (see Remark (22)). Applying \( \psi \) to the relation \( \tilde{x}_i \tilde{y}_e = \tilde{y}_e \tilde{x}_i \) yields
\[
p_i(h_j) b_i h_i^{\tau_1} \ldots h_n^{\tau_n} x_j^{\tau_1} y_j^{\tau_1} = \tilde{p}_i(h_j) b_e h_e^{\tau_1} \ldots h_n^{\tau_n} p_i(h_j) b_i h_i^{\tau_1} \ldots h_n^{\tau_n} x_j^{\tau_1} y_j^{\tau_1}.
\]
Rearrangement of Equation (18) gives us
\[
p_i(h_j) p'_e(h_j) a_d(q_j^{\tau_1} h_j) = P(h_j) a_d(q_j^{\tau_1} h_j).
\]
where \( P(h_j) \in \mathbb{K}[h_j^{\pm 1}] \). Evaluating Equation (19) at \( h_j = q_j^{\tau_1} \) yields
\[
p_i(q_j^{\tau_1}) p'_e(q_j^{\tau_1}) a_d(q_j^{1-2\tau_1}) = 0
\]
which implies, since \( a_d(q_j^{1-2\tau_1}) \neq 0 \), that \( p_i(q_j^{\tau_1}) = 0 \) or \( p'_e(q_j^{\tau_1}) = 0 \). Assuming \( p_i(q_j^{\tau_1}) = 0 \) and noting that \( p_i(h_j) p'_e(h_j) = U_i(h_j) \) we get that \( U_i(q_j^{\tau_1}) = 0 \). Thus by evaluating \( h_j = q_j^{\tau_1} \) in Equation (18) we get
\[
\tilde{d} \gamma_i d \left( q_j^{\tau_1} \right)^{m_d} - 1 = 0
\]
and by substituting for \( \gamma_i d \) using Equation (17) we get
\[
\tilde{d} \gamma_i d \left( q_j^{\tau_1} \right)^{m_d} = 1.
\]
By considering the cases where \( \tau_i = 0 \) and \( \tau_i = 1 \) separately and using Equation (15) we derive the contradiction \( \tilde{d} = 1 \). The case where \( p'_e(q_j^{\tau_1}) = 0 \) follows in exactly the same way.
**Remark 3.2.** The $\tau_i \neq \tau_e$ differs only insofar as we apply $\psi$ to the equation $\tilde{x}_i \tilde{x}_e = \tilde{x}_e \tilde{x}_i$ to derive a contradiction.

Since $j = w(i) \neq k = w(e)$ for all $i \neq e \in R$, the map $w : R \rightarrow N$ is a partial permutation and we have for all $i \in R$

$$\psi(\tilde{h}_i) = \gamma_i h_{w(i)}^{m_i}, \quad \psi(\tilde{x}_i) = P_i(h_{w(i)}^{t_{i,1}} \cdots h_{w(i)}^{t_{i,n}} x_{w(i)}^{(1-\tau_i)} y_{w(i)})$$

and

$$\psi(\tilde{y}_i) = x_{w(i)}^{r_{i}} y_{w(i)}^{t_{i}} P_i(h_{w(i)}) h_1^{t_{1,1}} \cdots h_n^{t_{n,1}}.$$

Finally applying $\psi$ to the relation $\tilde{x}_i \tilde{x}_i = \tilde{x}_i \tilde{x}_i$ (see the commutation relations (2)) yields the relation

$$q_{w(i)}^{t_{i,1}} q_{w(i)}^{t_{i,2}} q_{w(i)}^{t_{i,3}} = 1$$

as required.

*Thus we have completed the proof of part 1 of Theorem 3.4.*

### 3.4 Step 4

In Step 4 we will show that $\psi_\omega$ defines a homomorphism between $\tilde{A}(r, d, \tilde{q})$ and $A(n, d, \tilde{q})$.

It suffices to show that $\psi_\omega$ is consistent on the defining relations of $\tilde{A}(r, d, \tilde{q})$. For simplicity we set $\psi_\omega := \psi$. Thus

$$\psi(\tilde{x}_i) \psi(\tilde{h}_i) = P_i(h_{w(i)}^{t_{i,1}} \cdots h_{w(i)}^{t_{i,n}} x_{w(i)}^{(1-\tau_i)} y_{w(i)}^{m_i})$$

and

$$\psi(\tilde{y}_i) \psi(\tilde{h}_i) = P_i(q_{w(i)}^{r_{i}} q_{w(i)}^{t_{i}} h_{w(i)}^{m_i}).$$

Next consider

$$\psi(\tilde{y}_i) \psi(\tilde{x}_i) = P_i(h_{w(i)}^{t_{i,1}} y_{w(i)}^{m_i})$$

$$= P_i(q_{w(i)}^{r_{i}} h_{w(i)}^{m_i}) a_d(q_{w(i)}^{t_{i}} h_{w(i)}^{m_i}).$$

By hypothesis we have the equality

$$P_i(h_{w(i)}^{t_{i,1}}) a_d(q_{w(i)}^{t_{i}} h_{w(i)}^{m_i}) = a_d(q_{w(i)}^{t_{i}} h_{w(i)}^{m_i}).$$

Substituting $q_{w(i)}^{r_{i}} h_{w(i)}^{m_i}$ into Equation (22) gives us

$$P_i(q_{w(i)}^{r_{i}} h_{w(i)}^{m_i}) a_d(q_{w(i)}^{t_{i}} h_{w(i)}^{m_i}) = a_d(q_{w(i)}^{t_{i}} h_{w(i)}^{m_i}).$$

which in combination with Equation (21) yields the desired result that

$$\psi(\tilde{y}_i) \psi(\tilde{x}_i) = \psi(a_d(h_i)).$$
Similarly consider
\[
\psi(\tilde{x}_i) \psi(\tilde{y}_i) = p_i(h_{w(i)}) x_{w(i)}^{(1-\tau_i)} y_{w(i)}^{(1-\tau_i)} p_i'(h_{w(i)}) \\
= p_i(h_{w(i)}) p_i'(h_{w(i)}) x_{w(i)}^{(1-\tau_i)} y_{w(i)}^{(1-\tau_i)} \\
= p_i(h_{w(i)}) p_i'(h_{w(i)}) a_d(q_{w(i)}^{(1-\tau_i)} h_{w(i)})
\]
which by the hypothesis stated in Equation (22) gives
\[
\psi(\tilde{x}_i) \psi(\tilde{y}_i) = a_d(\tilde{q}_i \gamma_i h_{w(i)}^{m_i}) = \psi(a(\tilde{q}_i \tilde{h}_i)).
\]

Since the images of \(\tilde{h}_i\) and \(\tilde{h}_l\) commute (see Equation (2)) it is clear that \(\psi\) is consistent on the relation \(\tilde{h}_i \tilde{h}_l = \tilde{h}_l \tilde{h}_i\). For the same reason, \(\psi\) is consistent on the relations \(\tilde{h}_i \tilde{x}_l = \tilde{x}_l \tilde{h}_i\) and \(\tilde{h}_i \tilde{y}_l = \tilde{y}_l \tilde{h}_i\).

Finally for \(i \neq l\), consider
\[
\psi(\tilde{x}_i) \psi(\tilde{x}_l) = p_i(h_{w(i)}) b_i h_{1,i}^{l_i} \ldots h_{n,i}^{l_i,n} x_{w(i)}^{(1-\tau_i)} y_{w(i)}^{(1-\tau_i)} p_l(h_{w(l)}) b_l h_{1,l}^{l_l} \ldots h_{n,l}^{l_l,n} x_{w(l)}^{(1-\tau_l)} y_{w(l)}^{(1-\tau_l)}
\]
which after rearrangement and application of the hypothesis
\[
\frac{t_{i,w(i)}(1-\tau_i)}{q_{w(i)}} \frac{1}{q_{w(i)}} = 1
\]
gives us \(\psi(\tilde{x}_i) \psi(\tilde{x}_l) = \psi(\tilde{x}_l) \psi(\tilde{x}_i)\). Similarly \(\psi(\tilde{y}_i) \psi(\tilde{y}_l) = \psi(\tilde{y}_l) \psi(\tilde{y}_i)\) and \(\psi(\tilde{x}_i) \psi(\tilde{y}_l) = \psi(\tilde{y}_l) \psi(\tilde{x}_i)\). By universal property the algebra \(\tilde{A}(r, d, q)\), the map \(\psi\) defines an homomorphism from \(\tilde{A}(r, d, q)\) to \(A(n, d, q)\).

Thus we have completed the proof of part 2 of Theorem 1.1.

4 A quantum Dixmier analogue

Before proving Theorem 1.3 and Theorem 1.4 we give the general form of an endomorphism of \(A(n, d, q)\) subject to our required technical assumptions. Recall that \(\sigma_i\) is the automorphism of \(k[h_{i,1}^T, \ldots, h_{i,n}^T]\) defined by \(\sigma_i(h_{i,j}) = qh_{i,j}\) and \(\sigma_i(h_{j}) = h_{j}\) for \(j \neq i\).

**Corollary 4.1.** Let \(q = (q, \ldots, q)\) for \(q \in k^*\) a non root of unity. Then every endomorphism of \(A(n, d, q)\) is of the form:
\[
\psi(h_i) = \gamma_i h_{w(i)}^{(1-\tau_i)} \quad \psi(x_i) = e_i x_{w(i)}^{(1-\tau_i)} y_{w(i)}^{(1-\tau_i)} \quad \psi(y_i) = x_{w(i)}^{(1-\tau_i)} y_{w(i)}^{(1-\tau_i)}
\]
where \(w\) is a permutation of \(N\), \((\tau_1, \ldots, \tau_n) \in \{0, 1\}^n\), \((\gamma_1, \ldots, \gamma_n) \in (k^*)^n\) such that \(\gamma_i^d = q^{-\tau_d}\), and \(e_i, e_i'\) are units of \(A(n, d, q)\), such that \(e_i e_i' = (-1)^{\tau_i} h_{w(i)}^{dr}\) and
\[
e_i \sigma_i^{-1-2\tau_i}(e_i) = e_i \sigma_i^{-1-2\tau_i}(e_i),
\]
for all \(i \neq l \in N\).

**Remark 4.2.** By a simple calculation, we can see that when \(e_i := p_i(h_{w(i)}) b_i h_{1,i}^{l_i} \ldots h_{n,i}^{l_i,n}\) and \(e_i' := p_i'(h_{w(i)}) b_i h_{1,i}^{-l_i} \ldots h_{n,i}^{-l_i,n}\) (as in the statement of Theorem 1.1), Equation (23) is equivalent to Equation (22).
Proof. Let \( \psi \) be an endomorphism of \( A(n,d,q) \). By Theorem 3.1 the endomorphism \( \psi \) acts on the generators of \( A(n,d,q) \) as follows:

\[
\psi(h_i) = \gamma_i h_{w(i)}^{m_i}, \quad \psi(x_i) = p_i(h_{w(i)}) b_i h_1^{t_i} \ldots h_n^{t_i,n} x_{w(i)}^{(1 - \tau_i) \tau_i} \quad \text{and} \quad \psi(y_i) = x_{w(i)}^{\tau_i} y_{w(i)}^{(1 - \tau_i) \tau_i} b_i^{1 - t_i} h_1^{t_i} \ldots h_n^{t_i,n},
\]

where the parameters \( w, \gamma_i, m_i, t_i, \tau_i, p_i(h_{w(i)}), b_i \) are as stated in the statement of Theorem 3.1 and thus satisfy Equations (4), (5), and (6). By Equation (4), and since \( q \) is not a root of unity, we have that \( m_i = (-1)^{\tau_i} \) for all \( i \in N \). By substituting for \( m_i \) in Equation (4) and comparing coefficients of \( h_{w(i)} \), we find that \( p_i(h_{w(i)}) \) and \( b_i(h_{w(i)}) \) are monomials in \( h_{w(i)} \) such that \( p_i(h_{w(i)}) b_i(h_{w(i)}) = (-h_{w(i)}^{-d})^{\tau_i} \). We set \( e_i := p_i(h_{w(i)}) b_i h_1^{t_i} \ldots h_n^{t_i,n} \) and \( e_i' := b_i(h_{w(i)}) b_i^{-1} h_1^{t_i} \ldots h_n^{t_i,n} \) and it is clear \( e_i e_i' = (-h_{w(i)}^{-d})^{\tau_i} \). For simplicity we state an updated form of an endomorphism of \( A(n,d,q) \):

\[
\psi(h_i) = \gamma_i h_{w(i)}^{(-1)^{\tau_i}}, \quad \psi(x_i) = e_i x_{w(i)}^{(1 - \tau_i) \tau_i} y_{w(i)}^{\tau_i} \quad \text{and} \quad \psi(y_i) = x_{w(i)}^{\tau_i} y_{w(i)}^{(1 - \tau_i) \tau_i} e_i'.
\]

Finally, Equation (5) is equivalent to the relation \( e_i e_i'^{1 - 2 \tau_i}(c_i) = e_i e_i'^{1 - 2 \tau_i}(c_i) \), for all \( i \neq l \in N \). This is easily seen by applying \( \psi \) to the relation \( x_i x_l = x_l x_i \).

Since the algebras \( \mathcal{A}_{\alpha,q} \otimes \ldots \otimes \mathcal{A}_{\alpha,q} \) and \( \mathcal{H}_q \otimes \ldots \otimes \mathcal{H}_q \) are isomorphic to \( A(n,1,q) \) and \( A(n,2t,q) \) respectively (see Section 2), Theorems 1.3 and 1.4 are specializations of the following corollary to Theorem 3.1 (see Remark 2.1 as to why this is sufficient).

**Corollary 4.3.** Let \( q = (q, \ldots, q) \) for \( q \in k^* \) a non root of unity. Then, every endomorphism of \( A(n,d,q) \) is an automorphism.

**Proof.** Let \( \psi \) be defined as in the statement of Corollary 4.1. We will construct a candidate inverse of \( \psi \), say \( \phi \), and show that \( \phi \) is a endomorphism of \( A(n,d,q) \). It is clear that \( \phi(h_{w(i)}) = \gamma_i^{(-1)^{\tau_i}} h_i^{(-1)^{\tau_i}} \) is a well defined automorphism when restricted to \( k[h_1^{\pm 1}, \ldots, h_n^{\pm 1}] \). Thus we propose the following candidate inverse of \( \psi \):

\[
\phi(h_{w(i)}) = \gamma_i^{(-1)^{\tau_i}} h_i^{(-1)^{\tau_i}}, \quad \phi(x_{w(i)}) = e_i(x_i^{(1 - \tau_i) \tau_i} y_i^{\tau_i}) x_i^{1 - \tau_i} y_i^{\tau_i}, \quad \text{and} \quad \phi(y_{w(i)}) = x_i^{\tau_i} y_i^{1 - \tau_i} (e_i')^{1 - \tau_i}. \]

We will now show that \( \phi \) is a well defined endomorphism of \( A(n,d,q) \) by checking the conditions of Corollary 4.1. Since \( \gamma_i^{d} = q^{-\tau_i d} \), a brief computation shows that \( \gamma_i^{(-1)^{\tau_i}} d = q^{-\tau_i d} \). Next we show that

\[
\left( \phi(e_i^{(1 - \tau_i) \tau_i})^{-1} \phi(e_i^{(1 - \tau_i) \tau_i}) \right) \left( \sigma_i^{-1} \phi(e_i^{(-\tau_i)}) \phi(e_i')^{-(1 - \tau_i)} \right) = (-h_{w(i)}^{-d})^{\tau_i}. \tag{24}
\]

By rearranging the left hand side of Equation (24) we get \( \phi(e_i e_i')^{(1 - \tau_i) \tau_i} \), which by the substitution \( e_i e_i' = (-h_{w(i)}^{-d})^{\tau_i} \) gives

\[
\phi(-h_{w(i)}^{-d})^{-(1 - \tau_i) \tau_i} \sigma_i^{-1} \phi(-h_{w(i)}^{-d})^{-(1 - \tau_i) \tau_i}. \tag{25}
\]
It is easy to see that when $\tau_i = 0$, Equation (26) is equal to 1 as required. Thus we set $\tau_i = 1$ in Equation (26) and find

$$\sigma_i^{-1}(\phi(-h_w^{-d})^{-1}) = -\sigma_i^{-1}(\gamma_i h_i^{-1})^d = -q^{-d}h_i^d = -h_i^d$$

as required (note the last step follows by the substitution $\gamma_i^d = q^{-d}$).

Next we show that

$$\phi(x_{w(i)})\phi(x_{l(i)}) = \phi(x_{w(l)})\phi(x_{w(i)}).$$

At this point we return to the notation of Theorem 3.1 and precisely express the units $e_i$ and $e_i'$.

We set

$$p_{l}(h_{w(i)}) = (-h_{w(i)})^\tau_l,$$

and $p_{l}(h_{w(i)}), p'_{l}(h_{w(i)})$ are monomials in $h_{w(i)}$, and

$$q^{l_i}(1-\tau_l)q^{-l_i}q^{-l_i}(1-\tau_l)q^{l_i} = 1,$$

which is an equivalent condition to Equation (27). For simplicity, we make the following observation regarding the way $\phi(x_{w(i)})$ and $\phi(x_{l(i)})$ commute. To show that Equation (27) holds, it is clear that we need only consider the coefficients that appear as the $h_l$ component of $\phi(x_{w(i)})$ passes the $x_l, y_l$ terms in $\phi(x_{l(i)})$ and as the $h_i$ component of $\phi(x_{l(i)})$ passes $x_l, y_l$ terms in $\phi(x_{w(i)})$. We reflect this observation in our notation by representing all of the unnecessary information by ellipses. We highlight that $(-1)^{\tau_l} = (1 - 2\tau_l)$. Thus

$$\phi(x_{w(i)})\phi(x_{l(i)}) = \left(\cdots h_i^{(\tau_l-1)(-1)^{\tau_l}t_{i,w(i)}\cdots h_i^{(\tau_l-1)(-1)^{\tau_l}t_{i,w(i)}\cdots x_l^{1-\tau_l}y_l^{\tau_l}}\right) \left(\cdots h_l^{(\tau_l-1)(-1)^{\tau_l}t_{i,w(i)}\cdots h_l^{(\tau_l-1)(-1)^{\tau_l}t_{i,w(i)}\cdots x_l^{1-\tau_l}y_l^{\tau_l}}\right)$$

$$= \left(q^{(2\tau_l-1)(2\tau_l-1)(-1)^{\tau_l}t_{i,w(i)}\cdots q^{(1-2\tau_l)(2\tau_l-1)(-1)^{\tau_l}t_{i,w(i)}}\phi(x_{l(i)})\phi(x_{w(i)})\right),$$

$$= \left(q^{(-1)^{\tau_l}t_{i,w(i)}(1-\tau_l)+(-1)^{\tau_l}t_{l,w(i)}(2\tau_l-1)(2\tau_l-1)}\phi(x_{w(i)})\phi(x_{l(i)}))\right)$$

which in combination with Equation (27) gives the desired result. Note that it is easier to apply Equation (27) if we consider the choices of $\tau_l$ and $\tau_i$ separately. We leave to the reader the calculations to show that $\phi$ is consistent on the remaining relations (see Equation (2)). These follow in a similar way. Thus we have shown that $\phi$ conforms to the necessary conditions from Theorem 3.1 to be an endomorphism of $A(n, d, q)$. By direct computation we can see $\psi\phi = \phi\psi = id$.

We will now offer a counter example to show that our quantum Dixmier analogue is false when $q$ is a root of unity.

**Corollary 4.4.** There exist non-invertible endomorphisms of $A(n, d, q)$ when (at least) one coordinate $q := q_i$ of $q$ is a root of unity.
Proof. Let $q$ be a $t^{th}$ root of unity. It is enough to find an example of a non-invertible endomorphism of $k[h^{\pm 1}](\sigma, h^d - 1)$ where $\sigma(h) = qh$. Define the polynomial $U(h) = \sum_{l=0}^{t} u_l h^d$, where $u_i = q^d$ for $0 \leq i \leq t$ so that

$$U(h)((qh)^{d} - 1) = (qh)^{d(t+1)} - 1.$$  

Then it follows from Theorem 3.1 that we define an endomorphism $\psi$ of $k[h^{\pm 1}](\sigma, h^d - 1)$ by setting $\psi(h) = h^{l+1}$, $\psi(x) = U(h)x$ and $\psi(y) = y$. Since by assumption $t > 1$ we can see that $\psi$ is not invertible by considering the action on $h$. By taking a tensor product with $n-1$ copies of the identity, we can lift $\psi$ to a non-invertible endomorphism of $A(n, d, q)$.

\section{A quantum tame generators problem}

For the entirety of this section let $q = (q, \ldots, q)$ for $q \in k^* \setminus \{z | z^d \neq 1\}$. Also, recall from Section 2 that $A(1, d, q) \simeq k[h^{\pm 1}](\sigma, h^d - 1)$ where $\sigma(h) = qh$. Since $A(n, d, q)$ has a nontrivial group of units (See Lemma 2.2) we can find automorphisms of $A(n, d, q)$ which are not tame. For example consider the automorphism of $A(n, d, q)$ defined in the following way

$$h_i \mapsto h_i, \ x_i \mapsto h_ix_i, \text{ and } y_i \mapsto y_i h_i^{-1}. \quad (29)$$

Since we are interested in determining whether the complexity of the automorphisms of $A(n, d, q)$ fundamentally change as $n$ increases, we will take inspiration from the traditional definition of tame to define an $A(n, d, q)$ specific quantum analogue which we will denote qwa-tame. Before stating the definition of qwa-tame, we will highlight three natural families of automorphisms each of which is inspired by a family of tame automorphisms. The first two families arise from the fact that both the polynomials in $n$ variables and the algebra $A(n, d, q)$ can be constructed as $n$ tensor copies of $k[x]$ and $k[h^{\pm 1}](\sigma, h^d - 1)$ respectively.

By this construction we can pick $g \in \text{Aut}(k[h^{\pm 1}](\sigma, h^d - 1))$ and lift to an automorphism $\phi_g := g \otimes 1 \otimes \ldots \otimes 1$ of $A(n, d, q)$. For our second family, we associate to each permutation $\varpi$ of $N$ a (unique) automorphism $\chi_{\varpi}$ of $A(n, d, q)$ defined as follows:

$$\chi_{\varpi}(h_i) = h_{\varpi(i)}, \ \chi_{\varpi}(x_i) = x_{\varpi(i)} \text{ and } \chi_{\varpi}(y_i) = y_{\varpi(i)}.$$

Finally we introduce a family to include automorphisms arising from non-trivial group of units of $A(n, d, q)$ (for instance see Equation (29)) thus generalizing the scalar automorphisms to the following family. Recall that $\sigma_i \in \text{Aut}(k[h_1^{\pm 1}, \ldots, h_n^{\pm 1}])$ such that $\sigma_i(h_i) = qh_i$, and $\sigma_i(h_j) = h_j$ for $j \neq i$. For a vector of units in $A(n, d, q)$, say $\varpi := (u_1, \ldots, u_n)$, such that $u_i \sigma_i(u_i) = u_i \sigma_i(u_i)$ for $l \neq i$ (note this encodes Equation (33)), there exists a (unique) automorphism $\xi_{\varpi}$ of $A(n, d, q)$ defined as follows:

$$\xi_{\varpi}(h_i) = h_i, \ \xi_{\varpi}(x_i) = u_i x_i \text{ and } \xi_{\varpi}(y_i) = y_i u_i^{-1}.$$

\textbf{Definition 5.1.} Let $\psi$ be an automorphism of $A(n, d, q)$, we say that $\psi$ is qwa-tame if $\psi$ is in the subgroup generated by the families of automorphisms $\phi_g, \chi_{\varpi}$ and $\xi_{\varpi}$.
To enable us to practically apply Definition 5.1 we recall from [17] and [27] the classification of automorphisms of $k[h^\pm |\sigma, h^d - 1]$.

**Proposition 5.2.** Let $\psi$ be an automorphism of $k[h^\pm |\sigma, h^d - 1]$. Then $\psi$ is defined on the generators of $k[h^\pm |\sigma, h^d - 1]$ in the following way:

$$
\psi(h) = \gamma h(\gamma^{-1})^\tau, \quad \psi(x) = ux(1-\tau)y^\tau \quad \text{and} \quad \psi(y) = y(1-\tau)x^\tau u',
$$

where $\tau \in \{0, 1\}$, $\gamma = (q^{-d})^\tau$ and $u, u' \in k[h^{\pm 1}]$ such that $uu' = (-h^{-d})^\tau$.

Since the algebras $A_{\alpha,q} \otimes \ldots \otimes A_{\alpha,q}$ and $A_{q} \otimes \ldots \otimes A_{q}$ are isomorphic to $A(n, 1, q)$ and $A(n, 2, q)$ respectively (see Section 2), Theorems 1.1 and 1.2 are specializations of the following corollary to Theorem 3.1.

**Corollary 5.3.** Every automorphism of $A(n, d, q)$ is qwa-tame.

**Proof.** Let $\psi$ be an automorphism of $A(n, d, q)$. By Corollary 4.1 we have that $\psi$ acts on the generators of $A(n, d, q)$ as follows:

$$
\psi(h_i) = \gamma_i h_i(\gamma_i^{-1})\tau_i, \quad \psi(x_i) = e_i x_i(1-\tau_i)y_i^\tau \quad \text{and} \quad \psi(y_i) = x_i^\tau y_i(1-\tau_i)e_i',
$$

where the parameters $\gamma_i, \tau_i, e_i$ and $e_i'$ are as in the statement of Corollary 4.1. By applying qwa-tame automorphisms (see Definition 5.1), we will reduce $\psi$ to an obvious qwa-tame automorphism. Applying the automorphism $\chi_{w^{-1}}$ gives us

$$
\chi_{w^{-1}}\psi(h_i) = \gamma_i h_i(\gamma_i^{-1})\tau_i, \quad \chi_{w^{-1}}\psi(x_i) = x_i(1-\tau_i)y_i^\tau \quad \text{and} \quad \chi_{w^{-1}}\psi(y_i) = x_i^\tau y_i(1-\tau_i)e_i'.
$$

Next we fix the notation $\phi_i^{(j)} := \phi(i,j)\phi_j$, where $g_j$ is the automorphism of $k[h^\pm |\sigma, h^d - 1]$ defined by

$$
g_j(h_j) = \gamma_j h_j(\gamma_j^{-1})\tau_j, \quad g_j(x_j) = p_{j\tau_j} y_j^\tau x_j(1-\tau_j) \quad \text{and} \quad g_j(y_j) = x_j^\tau y_j(1-\tau_j)(p_j')^\tau_j
$$

with $p_j, p_j' \in k[h^\pm]$ such that $p_j p_j' = (-h^{-d})^\tau_j$. Moreover, we let $G := \phi_1^{(1,1)} \phi_2^{(1,2)} \ldots \phi_n^{(1,n)}$. Thus $G$ is a qwa-tame automorphism of $A(n, d, q)$ and we have:

$$
G(h_i) = \gamma_i h_i(\gamma_i^{-1})\tau_i, \quad G(x_i) = p_{i\tau_i} y_i^\tau x_i(1-\tau_i) \quad \text{and} \quad G(y_i) = x_i^\tau y_i(1-\tau_i)(p_i')^\tau_i
$$

for all $i \in N$. We can easily check that the action of $G\chi_{w^{-1}}\psi$ on the generators of $A(n, d, q)$ is given by:

$$
G\chi_{w^{-1}}\psi(h_i) = h_i, \quad G\chi_{w^{-1}}\psi(x_i) = G(\chi_{w^{-1}}(e_i))(p_i')^\tau_i x_i, \quad \text{and} \quad G\chi_{w^{-1}}\psi(y_i) = y_i G(\chi_{w^{-1}}(e_i'))(p_i')^\tau_i.
$$

Since $G\chi_{w^{-1}}\psi$ is an automorphism of $A(n, d, q)$, the units $G(\chi_{w^{-1}}(e_i))(p_i')^\tau_i$ and $G(\chi_{w^{-1}}(e_i'))(p_i')^\tau_i$ must decompose in the following way:

$$
G(\chi_{w^{-1}}(e_i))(p_i')^\tau_i = U_i \quad \text{and} \quad G(\chi_{w^{-1}}(e_i'))(p_i')^\tau_i = U_i^{-1}
$$

where $U_i$ is a unit of $A(n, d, q)$ such that $U_i \sigma_i(U_i) = U_i \sigma_i(U_i)$ for $l \neq i$. Applying the qwa-tame automorphism $\xi_{U_i}$, where $\xi_{U_i} := (U_i^{-1}, \ldots, U_n^{-1})$, yields

$$
\xi_{U_i}G\chi_{w^{-1}}\psi(h_i) = h_i, \quad \xi_{U_i}G\chi_{w^{-1}}\psi(x_i) = x_i, \quad \text{and} \quad G\chi_{w^{-1}}\psi(y_i) = y_i.
$$

Thus $\psi = \chi_{w^{-1}}G^{-1}\xi_{U_i}^{-1}$ is qwa-tame. □
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