On twisted Verlinde formulae for modular categories

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Abstract

In this note, we describe two analogues of the Verlinde formula for modular categories in a twisted setting. The classical Verlinde formula for a modular category $\mathcal{C}$ describes the fusion coefficients of $\mathcal{C}$ in terms of the corresponding S-matrix $S(\mathcal{C})$. Now let us suppose that we also have an invertible $\mathcal{C}$-module category $\mathcal{M}$ equipped with a $\mathcal{C}$-module trace. This gives rise to a modular autoequivalence $F : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$. In this setting, we can define a crossed S-matrix $S(\mathcal{C}, \mathcal{M})$. As our first twisted analogue of the Verlinde formula, we will describe the fusion coefficients for $\mathcal{M}$ as a $\mathcal{C}$-module category in terms of the S-matrix $S(\mathcal{C})$ and the crossed S-matrix $S(\mathcal{C}, \mathcal{M})$. In this twisted setting, we can also define a twisted fusion $\mathbb{Q}$-algebra $K_{\text{Qab}}(\mathcal{C}, F)$. As another analogue of the Verlinde formula, we describe the fusion coefficients of the twisted fusion algebra in terms of the crossed S-matrix $S(\mathcal{C}, \mathcal{M})$.

1 Introduction

In this note we will describe two twisted analogues of the Verlinde formula for modular fusion categories, namely Theorems 1.1 and 1.4 below. We begin by describing the twisted setting in which we will work throughout this note.

1.1 Notation and conventions

Let $k$ be an algebraically closed field of characteristic zero. Throughout this note, $\mathcal{C}$ will denote a non-degenerate $k$-linear braided fusion category and $\mathcal{M}$ will denote a $\mathcal{C}$-module category which is invertible when considered as a $\mathcal{C}$-$\mathcal{C}$-bimodule category. We refer to [ENO1], [ENO2] for the theory of fusion categories and module categories over them. By [ENO2], the invertible $\mathcal{C}$-module category $\mathcal{M}$ gives rise to an associated braided monoidal autoequivalence $F : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ such that we have functorial crossed braiding isomorphisms

$$\beta_{\mathcal{C}, \mathcal{M}} : \mathcal{C} \otimes \mathcal{M} \xrightarrow{\sim} \mathcal{M} \otimes \mathcal{C},$$

$$\beta_{\mathcal{M}, \mathcal{C}} : \mathcal{M} \otimes \mathcal{C} \xrightarrow{\sim} F(\mathcal{C}) \otimes \mathcal{M}$$

for $C \in \mathcal{C}$, $M \in \mathcal{M}$ satisfying the suitable hexagonal identities. Moreover, as in [De1] §2.3, we can construct a braided $\mathbb{Z}/N\mathbb{Z}$-crossed category $\mathcal{D} = \bigoplus_{a \in \mathbb{Z}/N\mathbb{Z}} \mathcal{D}_a$ for some positive integer $N$ with trivial component $\mathcal{D}_0 \cong \mathcal{C}$ as a braided fusion category and $\mathcal{D}_1 \cong \mathcal{M}$ as a module category. We will assume the choice of such a $\mathcal{D}$ throughout the article.

For an abelian category $\mathcal{A}$, let $K(\mathcal{A})$ denote its Grothendieck group and let $\mathcal{O}_\mathcal{A}$ denote the set of its simple objects up to isomorphism. For an object $A \in \mathcal{A}$, we let $[A]$ denote its image in the Grothendieck group. In our setting, $K(\mathcal{C})$ is a commutative based ring (see [L]) with basis $\mathcal{O}_\mathcal{C}$ and $K(\mathcal{M})$ is a $K(\mathcal{C})$-module with $\mathbb{Z}$-basis $\mathcal{O}_\mathcal{M}$. For a commutative ring $R$, we let $K_R(\mathcal{A}) = K(\mathcal{A}) \otimes R$.

For most part of this note, we will assume that $\mathcal{C}$ is also equipped with a spherical structure making it a modular fusion category and that $\mathcal{M}$ is equipped with a compatible $\mathcal{C}$-module trace (see [S][De1] for details).
In this case, it follows that the braided autoequivalence $F$ is in fact a modular autoequivalence. This extra structure allows us to define categorical dimensions of objects of $\mathcal{C}$ and $\mathcal{M}$. We assume that the $\mathcal{C}$-module trace on $\mathcal{M}$ is normalized according to the conventions of [De1], so that $\dim \mathcal{C} = \sum_{M \in \mathcal{O}_\mathcal{M}} \dim^2 \mathcal{M}(M)$. With this convention it follows that the categorical dimensions of all objects of $\mathcal{C}$ and $\mathcal{M}$ are totally real cyclotomic integers in $k$. Also, the categorical dimension $\dim \mathcal{C} \in k$ is a totally positive cyclotomic integer. Note that in this setting it is also possible to construct our braided $\mathbb{Z}/\mathbb{N}\mathbb{Z}$-crossed $\mathcal{D}$ to have a compatible spherical structure (cf. [De1] §2.4).

1.2 Twisted Verlinde formulae

We now state the two twisted analogues of the Verlinde formula. The first version below, describes the fusion coefficients for the $\mathcal{C}$-module category $\mathcal{M}$. Let us assume now that $\mathcal{C}$ is equipped with a normalized $\mathcal{C}$-module trace as in [De1]. Let $\mathcal{O}_{\mathcal{C},\mathcal{M}}$ denote the $\mathcal{C}$-module category defined in [De1]. In this setting, we have:

**Theorem 1.1.** Let $C \in \mathcal{O}_{\mathcal{C},\mathcal{M}}$, $M \in \mathcal{O}_{\mathcal{M}}$. Then the multiplicity of $N$ in $C \otimes M$ is given by

$$a_{C,M}^N = \frac{1}{\dim \mathcal{C}} \sum_{D \in \mathcal{O}_{\mathcal{C},\mathcal{M}}} \frac{S(C,D,C) \cdot S(C,M,D,M) \cdot S(M,N)}{\dim \mathcal{D}}.$$  (4)

In particular, the two expressions on the right hand side are equal to a non-negative integer.

This is the main result of this note. It will be proved in [De1] using results from [De1, De2]. We will also prove a non-spherical version (Theorem 1.1) of this result.

**Remark 1.2.** Recall that the crossed S-matrix $S(C,M)$ is only well-defined up to rescaling by roots of unity (cf. [De1] Rem. 2.2). However it is clear that this does not affect the Verlinde formula once any such scaling factors cancel out on the right hand side.

We now describe another twisted analogue of the Verlinde formula that appeared in [De1]. This formula describes the fusion coefficients of a twisted version of the fusion ring. Let us recall the construction of this twisted fusion ring from [De1]. The autoequivalence $F : \mathcal{C} \to \mathcal{C}$ induces a braided monoidal action of $\mathcal{C}$ on $\mathcal{C}$. Consider the Grothendieck ring $K(\mathcal{C}^\otimes)$ of the $\mathbb{Z}$-equivariantization of $\mathcal{C}$. Now consider the trivial modular category $\text{Vec}$ of finite dimensional $k$-vector spaces and equip it with the trivial action of $\mathcal{C}$. We have a ring homomorphism $K(\text{Vec}^\otimes) \to K(\mathcal{C}^\otimes)$. We also have a ring homomorphism $K(\text{Vec}^\otimes) \to k$ that takes the class of the object $(V,\psi : V \cong V) \in \text{Vec}^\otimes$ to $\text{tr}(\psi) \in k$. Define the twisted fusion $k$-algebra $K_k(\mathcal{C},F)$ to be $K(\mathcal{C}^\otimes) \otimes_{K(\text{Vec}^\otimes)} k$.

\[\text{Note that the autoequivalence } F : \mathcal{C} \to \mathcal{C} \text{ induces a permutation } F : \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}. \text{ It can be shown (cf. } \text{De1}) \text{ that the number of fixed points } |\mathcal{O}_{\mathcal{C}}^F| \text{ equals } |\mathcal{O}_{\mathcal{C}}| \text{ although in general there is no canonical bijection between the two sets.}\]
Here is another construction of this algebra. We refer to [De1] §2.3 for the details. Recall that for a sufficiently large $N$, we have constructed the braided $\mathbb{Z}/N\mathbb{Z}$-crossed category $\mathcal{D}$. In particular, we have a braided action of the finite group $\mathbb{Z}/N\mathbb{Z}$ on $\mathcal{C}$ and we can form the braided fusion category $\mathcal{C}^{\mathbb{Z}/N\mathbb{Z}}$. Let $\omega$ be a primitive $N$-th root of unity and consider the $\mathbb{Z}[\omega]$-algebra $K(\mathcal{C}^{\mathbb{Z}/N\mathbb{Z}}) \otimes \mathbb{Z}[\omega]$. Consider the involution of the ring $\mathbb{Z}[\omega]$ mapping $\omega$ to $\omega^{-1}$. The rigid duality on $\mathcal{C}^{\mathbb{Z}/N\mathbb{Z}}$ provides $K(\mathcal{C}^{\mathbb{Z}/N\mathbb{Z}}) \otimes \mathbb{Z}[\omega]$ with the structure of a Frobenius $\mathbb{Z}[\omega]$-algebra. Define $K_{\mathbb{Z}[\omega]}(\mathcal{C}, F)$ to be the quotient of $K(\mathcal{C}^{\mathbb{Z}/N\mathbb{Z}}) \otimes \mathbb{Z}[\omega]$ by the ideal generated by the element $[(1, \omega)] - \omega \cdot [1, \text{id}_1]$. The twisted fusion $k$-algebra $K_k(\mathcal{C}, F)$ is then defined to be the extension by scalars to $k$. In fact, it will often be more helpful for us to consider the twisted fusion $\mathbb{Q}^{ab}$-algebra $K_{\mathbb{Q}^{ab}}(\mathcal{C}, F) := K_{\mathbb{Z}[\omega]}(\mathcal{C}, F) \otimes_{\mathbb{Z}[\omega]} \mathbb{Q}^{ab}$, where $\mathbb{Q}^{ab}$ is the cyclotomic subfield of $k$ obtained by attaching all roots of unity to $\mathbb{Q}$.

Remark 1.3. From now on, for each $C \in \mathcal{O}_\mathcal{C}$, we fix $\psi : F(C) \cong C$ above denotes a (choice of a) $\mathbb{Z}/N\mathbb{Z}$-equivariance structure on $C$. This is only well-defined up to scaling by an $N$-th root of unity. Hence, this basis is well-defined only up to scaling by $N$-th roots of unity.

Our second analogue of the Verlinde formula computes the fusion coefficients for the algebra $K_{\mathbb{Q}^{ab}}(\mathcal{C}, F)$ in this special basis in terms of the crossed S-matrix $S(\mathcal{C}, \mathcal{M})$. For $C, C', D \in \mathcal{O}_\mathcal{C}$, let $a_{C, C'}^D \in \mathbb{Q}^{ab}$ be such that

$$[C, \psi_C] \cdot [C', \psi_{C'}] = \sum_{D \in \mathcal{O}_\mathcal{C}} a_{C, C'}^D \cdot [D, \psi_D] \text{ in } K_{\mathbb{Q}^{ab}}(\mathcal{C}, F).$$

From the second construction of the algebra $K_{\mathbb{Q}^{ab}}(\mathcal{C}, F)$ it is clear that these fusion coefficients $a_{C, C'}^D$, in fact lie in $\mathbb{Z}[\omega]$. Recall that $S(\mathcal{C}, \mathcal{M})$ is a $\mathcal{O}_\mathcal{C} \times \mathcal{O}_\mathcal{M}$-matrix. We now state the second twisted analogue of the Verlinde formula:

**Theorem 1.4.** ([De1], Thm. 2.12.) The fusion coefficients for the twisted fusion algebra $K_{\mathbb{Q}^{ab}}(\mathcal{C}, F)$ are given by

$$a_{C, C'}^D = \frac{1}{\dim \mathcal{C}} \sum_{M \in \mathcal{M}} \frac{S(\mathcal{C}, \mathcal{M})_{C, M} \cdot S(\mathcal{C}, \mathcal{M})_{C', M} \cdot S(\mathcal{C}, \mathcal{M})_{D, M}}{\dim \mathcal{M}}$$

for all $C, C', D \in \mathcal{O}_\mathcal{C}$. In particular, the expression of the right hand side above lies in $\mathbb{Z}[\omega]$.

We will prove a non-spherical analogue of this result in §.2.

2. **Characters of Grothendieck algebras**

In this section we gather general facts about characters of the various Grothendieck algebras and their relationship with S-matrices.

2.1. **Crossed S-matrices**

Let us begin by recalling the definition of a crossed S-matrix. We refer to [De1] for details. We work in the setting where $\mathcal{C}$ is a modular category and $\mathcal{M}$ has a normalized $\mathcal{C}$-module trace. In particular, this means that for any endomorphism $\gamma : M \to M$ in $\mathcal{M}$, the trace $\text{tr}_{\mathcal{M}}(\gamma)$ is defined. Let $C \in \mathcal{O}_\mathcal{C}$ and let $M \in \mathcal{O}_\mathcal{M}$.
Recall that for such a $C$, we have fixed a choice of a $\mathbb{Z}/N\mathbb{Z}$-equivariance structure $\psi_C : F(C) \xrightarrow{\simeq} C$. Consider the composition
\[
\gamma_{C,\psi_C,M} : C \otimes M \xrightarrow{\beta_{C,M}} M \otimes C \xrightarrow{\beta_{M,C}} F(C) \otimes M \xrightarrow{\psi_C \otimes \text{id}_M} C \otimes M \text{ in } \mathcal{M}.
\] (7)
The crossed S-matrix $S(\mathcal{C},\mathcal{M})$ is the $\mathcal{O}_\mathcal{M}^C \times \mathcal{O}_\mathcal{M}$-matrix whose $(C,M)$-th entry is
\[
S(C,M) := \text{tr}_\mathcal{M}(\gamma_{C,\psi_C,M}),
\] (8)
where the trace is computed using the trace in the $\mathcal{C}$-module category $\mathcal{M}$. In the special case when $\mathcal{M} = \mathcal{C}$, we recover the S-matrix $S(C)$ of $\mathcal{C}$.

We now state some properties of the crossed S-matrices. Recall that we have defined the (commutative) Frobenius $\mathbb{Q}^{ab}$-algebra (see [A] [De1] for details, see also [K2] below) $K_{Qab}(\mathcal{C},F)$ with our choice of basis \{$(C,\psi)[C \in \mathcal{O}_C^F]$\}. On the other hand consider the commutative semisimple $\mathbb{Q}^{ab}$-algebra $\text{Fun}_{Qab}(\mathcal{O}_\mathcal{M})$ of $\mathbb{Q}^{ab}$-valued functions on the set $\mathcal{O}_\mathcal{M}$ with pointwise multiplication. We have:

**Theorem 2.1.** (cf. [De1], Thm. 2.9 and 2.12) (i) For every $C \in \mathcal{O}_C^F$, $M \in \mathcal{O}_\mathcal{M}$, $S(\mathcal{C},\mathcal{M})_{C,M}$ and $S(\mathcal{C},\mathcal{M})_{C,M}$ are cyclotomic integers in $k$.
(ii) The Fourier transform $\Phi : K_{Qab}(\mathcal{C},F) \rightarrow \text{Fun}_{Qab}(\mathcal{O}_\mathcal{M})$, defined on our special basis by
\[
\Phi([C,\psi_C]) : \mathcal{O}_\mathcal{M} \ni M \mapsto S(\mathcal{C},\mathcal{M})_{C,M} \in \mathbb{Q}^{ab}
\] (9)
is an isomorphism of $\mathbb{Q}^{ab}$-algebras. In other words, for every $M \in \mathcal{O}_\mathcal{M}$, $[C,\psi_C] \mapsto S(\mathcal{C},\mathcal{M})_{C,M}$ defines a character $\varphi_M : K_{Qab}(\mathcal{C},F) \rightarrow \mathbb{Q}^{ab}$ and this defines an identification $\mathcal{O}_\mathcal{M} \cong \text{Irrep}(K_{Qab}(\mathcal{C},F))$.
(iii) The matrix $\frac{1}{\sqrt{\dim \mathcal{C}}} \cdot S(\mathcal{C},\mathcal{M})$ is unitary:
\[
S(\mathcal{C},\mathcal{M}) \cdot S(\mathcal{C},\mathcal{M})^T = \dim \mathcal{C} \cdot I = S(\mathcal{C},\mathcal{M})^T \cdot S(\mathcal{C},\mathcal{M}).
\] (10)

The above result says that the crossed S-matrix is essentially the character table of the twisted fusion algebra $K_{Qab}(\mathcal{C},F)$ in the basis $\{[C,\psi_C]\}_{C \in \mathcal{O}_C^F}$. Next, we relate the crossed S-matrix to a certain twisted character table.

### 2.2 Twisted characters and crossed S-matrices

We have constructed an auxiliary braided $\mathbb{Z}/N\mathbb{Z}$-crossed category $\mathcal{D}$. Consider the $\mathbb{Z}/N\mathbb{Z}$-graded based $\mathbb{Q}^{ab}$-algebra $K_{Qab}(\mathcal{D}) = \bigoplus_{a \in \mathbb{Z}/N\mathbb{Z}} K_{Qab}(\mathcal{D}_a)$ with basis $\{[D]\}_{D \in \mathcal{O}_\mathcal{D}}$. It is known (see [ENO1], Cor. 8.53) that all the irreducible representations of $K_{Qab}(\mathcal{D})$ and $K_{Qab}(\mathcal{C},F)$ are in fact defined over $\mathbb{Q}^{ab}$ (see [De1]).

Now $K_{Qab}(\mathcal{D})$ is a $\mathbb{Z}/N\mathbb{Z}$-graded Frobenius $\mathbb{Q}^{ab}$-algebra and hence we have the notion of twisted characters and their orthogonality relations (cf. [A]) which we recall now. By [Da], the $\mathbb{Z}/N\mathbb{Z}$-graded algebra $K_{Qab}(\mathcal{D})$ induces a partial $\mathbb{Z}/N\mathbb{Z}$-action on the set of irreducible representations of the identity component $K_{Qab}(\mathcal{C})$. On the other hand, the braided monoidal $\mathbb{Z}/N\mathbb{Z}$ action on $\mathcal{C}$ induces an action of $\mathbb{Z}/N\mathbb{Z}$ on the algebra $K_{Qab}(\mathcal{C})$ and hence also on $\text{Irrep}(K_{Qab}(\mathcal{C}))$. However, this action does not agree with the partial action obtained above. Using the argument from [De2] Lem. 3.5 we can describe the partial action:

**Lemma 2.2.** For $a \in \mathbb{Z}/N\mathbb{Z}$, $\rho \in \text{Irrep}(K_{Qab}(\mathcal{C}))$, let $(a)\rho \in \text{Irrep}(K_{Qab}(\mathcal{C})) \cup \{0\}$ denote the partial action. For $a \in \mathbb{Z}/N\mathbb{Z}$, we have the algebra automorphism $F^a : K_{Qab}(\mathcal{C}) \rightarrow K_{Qab}(\mathcal{C})$. Then we have
\[
(a)\rho = \begin{cases} 
\rho & \text{if } \rho \circ F^a = \rho, \\
0 & \text{else}.
\end{cases}
\] (11)

In particular, the fixed points of the partial action and the action match, so the notation $\text{Irrep}(K_{Qab}(\mathcal{C}))^{\mathbb{Z}/N\mathbb{Z}}$ is unambiguous.
By the results of [Da, De2], each $\rho \in \text{Irrep}(K_{Qab}(\mathcal{E}))^{Z/NZ}$ can be extended to a 1-dimensional character $\bar{\rho} : K_{Qab}(\mathcal{O}) \to Q_{ab}$. There are in fact $N$ such extensions and these extensions differ on $K_{Qab}(\mathcal{M} = \mathcal{O}_1)$ up to scaling by the $N$-th roots of unity. For $\rho \in \text{Irrep}(K_{Qab}(\mathcal{E}))^{Z/NZ}$, its twisted character is the linear functional $\bar{\chi}_\rho := \overline{\rho|_{K_{Qab}(\mathcal{O})}} : K_{Qab}(\mathcal{M}) \to Q_{ab}$. As we have noted, the twisted character $\bar{\chi}_\rho$ is well-defined up to scaling by $N$-th roots of unity.

Since $K_{Qab}(\mathcal{O})$ is a Frobenius $Q_{ab}$-$*$-algebra, we will identify (as in [De2]) the linear dual $K_{Qab}(\mathcal{M})^{*}$ with $K_{Qab}(\mathcal{M}^{-1} = \mathcal{O}^{-1})$. The basis $\{(D)\}_{D \in O_{ab}}$ of $K_{Qab}(\mathcal{O})$ is orthonormal with respect to the standard positive definite Hermitian form denoted by $\langle \cdot, \cdot \rangle$ (cf. [De2]). Hence, under the identification $K_{Qab}(\mathcal{M})^{*} = K_{Qab}(\mathcal{M}^{-1})$, a functional $\varphi \in K_{Qab}(\mathcal{M})^{*}$ corresponds to the element $\sum_{M \in O_{ab}} \varphi([M])[M^{*}] \in K_{Qab}(\mathcal{M}^{-1})$.

The twisted characters $\{\bar{\chi}_\rho \rho \in \text{Irrep}(K_{Qab}(\mathcal{E}))^{Z/NZ}\}$ form an orthogonal basis of $K_{Qab}(\mathcal{M})^{*} = K_{Qab}(\mathcal{M}^{-1})$ with respect to the standard positive definite Hermitian form (see [A, De2] for details). For each $\rho \in \text{Irrep}(K_{Qab}(\mathcal{E}))^{Z/NZ}$, let $\bar{\alpha}_\rho \in K_{Qab}(\mathcal{M}^{-1})$ be the element corresponding to the twisted character $\bar{\chi}_\rho \in K_{Qab}(\mathcal{M})^{*}$, namely

$$\bar{\alpha}_\rho = \sum_{M \in O_{ab}} \bar{\chi}_\rho([M])[M^{*}] \in K_{Qab}(\mathcal{M}^{-1}).$$

Similarly, for each $\rho \in \text{Irrep}(K_{Qab}(\mathcal{E}))$, we have the element $\alpha_\rho = \sum_{C \in O_{ab}} \rho([C])[C^{*}] \in K_{Qab}(\mathcal{E})$ corresponding to $\rho \in K_{Qab}(\mathcal{E})^{*}$. Note that $K_{Qab}(\mathcal{M}^{-1})$ is a $K_{Qab}(\mathcal{E})$-module. In this notation, we have the following:

**Theorem 2.3.** (cf. [L1, O1, De2] §2.7.) (i) Let $\rho, \rho' \in \text{Irrep}(K_{Qab}(\mathcal{E}))$, then $\rho'(\alpha_\rho) = 0$ if $\rho' \neq \rho$ and $f_{\rho'} := \rho(\alpha_\rho) \in Q_{ab}$ is a totally positive cyclotomic integer known as the formal codegree of $\rho$. In other words, $\bar{\alpha}_\rho \in K_{Qab}(\mathcal{E})$ is the minimal idempotent corresponding to $\rho \in \text{Irrep}(K_{Qab}(\mathcal{E}))$.

(ii) For $\rho, \rho' \in \text{Irrep}(K_{Qab}(\mathcal{E}))$, we have orthogonality of characters, $\langle \alpha_{\rho}, \alpha_{\rho'} \rangle = \delta_{\rho_{\rho'}} \cdot f_{\rho'}$.

(iii) Let $\rho \in \text{Irrep}(K_{Qab}(\mathcal{E}))^{Z/NZ}$ and $\bar{\alpha}_\rho \in K_{Qab}(\mathcal{M}^{-1})$ corresponding to the twisted character $\bar{\chi}_\rho \in K_{Qab}(\mathcal{M})^{*}$. Then for any $\rho' \in \text{Irrep}(K_{Qab}(\mathcal{E}))$, we have

$$\alpha_{\rho'} \cdot \bar{\alpha}_\rho = \begin{cases} 0 & \text{if } \rho' \neq \rho, \\ f_{\rho'} \cdot \bar{\alpha}_\rho & \text{if } \rho' = \rho. \end{cases}$$

We have a direct sum decomposition

$$K_{Qab}(\mathcal{M}^{-1}) = \bigoplus_{\rho \in \text{Irrep}(K_{Qab}(\mathcal{E}))^{Z/NZ}} Q_{ab} \cdot \bar{\alpha}_\rho \text{ as a } K_{Qab}(\mathcal{E})\text{-module.}$$

(iv) For $\rho, \rho' \in \text{Irrep}(K_{Qab}(\mathcal{E}))^{Z/NZ}$, we have orthogonality of twisted characters, $\langle \bar{\alpha}_\rho, \bar{\alpha}_{\rho'} \rangle = \delta_{\rho_{\rho'}} \cdot f_{\rho'}$.

Now assume that $\mathcal{E}$ is modular, $\mathcal{M}$ has a $\mathcal{E}$-module trace and that $\mathcal{O}$ is chosen to have a compatible spherical structure. In this case, we can identify $O_{ab} \cong \text{Irrep}(K_{Qab}(\mathcal{E}))$ (cf. [DNO1], also Thm. 2.1 above) and the action of $Z/NZ$ on both sides matches. For $C \in O_{ab}$, the map $\varphi_{C} : K_{Qab}(\mathcal{E}) \to Q_{ab}$, $[D] \mapsto S(\varphi_{C}, D)_{\dim \mathcal{E}}$ defines the corresponding irreducible representation. For each $C \in O_{ab} \subseteq \text{Irrep}(K_{Qab}(\mathcal{E}))^{Z/NZ}$, we have chosen an equivariantization $(C, \psi_{C}) \in Z/NZ \subseteq G^{Z/NZ}$. As observed in [De2] §4.4, this (choice of equivariantization) determines an extension of $\varphi_{C}$ to a character $\tilde{\varphi}_{C} = \varphi_{C, \psi_{C}} : K_{Qab}(\mathcal{O}) \to Q_{ab}$. As in loc. cit., we obtain:

**Theorem 2.4.** (i) For each $C \in O_{ab} \cong \text{Irrep}(K_{Qab}(\mathcal{E}))^{Z/NZ}$, the twisted character $\tilde{\chi}_{C} = \bar{\chi}_{\tilde{\varphi}_{C}} : K_{Qab}(\mathcal{M}) \to Q_{ab}$ (determined by our fixed choice of $\psi_{C} : F(C) \to \mathcal{E}$) is given by

$$\tilde{\chi}_{C}([M]) = \frac{S(\varphi_{C}, \mathcal{M})_{C,M}}{\dim \mathcal{E}} \text{ for each } M \in O_{ab}.$$

Hence in the spherical setting, we have $\tilde{\alpha}_{C} = \sum_{M \in O_{ab}} \frac{S(\varphi_{C}, \mathcal{M})_{C,M}}{\dim \mathcal{E}} \cdot [M^{*}] \in K_{Qab}(M^{-1})$.

(ii) For any $C \in O_{ab} \cong \text{Irrep}(K_{Qab}(\mathcal{E}))$, the formal codegree $f_{C} = f_{\varphi_{C}}$ of the representation $\varphi_{C}$ equals the totally positive cyclotomic integer $\frac{\dim \mathcal{E}}{\dim \mathcal{E}}$. 


Proof. The proof of (i) follows from the argument in [De2, §4.3]. To prove (ii), take $\mathcal{M} = \mathcal{E}$. Then by Theorems 2.1(iii), 2.3(ii) and statement (i) above, we obtain

$$f_C = \langle \alpha_C, \alpha_C \rangle = \sum_{C' \in \mathcal{O}_\mathcal{E}} \frac{S(\langle C, C' \rangle \mathcal{E}), \mathcal{E}}{\dim \mathcal{E}} = \frac{\dim \mathcal{E}}{\dim \mathcal{E}}.$$

The above result says that in the spherical case, the crossed S-matrix is essentially the twisted character table of $K_{Q_{ab}}(\mathcal{E})$.

3 Proof of the main results

We can now complete the proof of the main Theorem 1.1. Theorem 1.4 is already proved in [De1]. Here, we will state and prove versions of both the twisted Verlinde formulae without assuming the existence of spherical structures.

3.1 Fusion coefficients for the module category

Let us first prove a version of Theorem 1.1 without spherical structures. In this case the S-matrix and crossed S-matrix are not available. We will instead work with the character table and the twisted character table of $K_{Q_{ab}}(\mathcal{E})$.

Let us now consider the two character table matrices. Let $\tilde{\chi}$ be the Irrep$(\mathcal{E})$ matrix with $(\rho, M)$-th entry defined as $\tilde{\chi}_{\rho,M} = \tilde{\chi}_{\rho}([M])$. We will now describe the product $[C^*] \cdot [M^*] \in K_{Q_{ab}}(\mathcal{M}^{-1})$ for $C \in \mathcal{O}_\mathcal{E}, M \in \mathcal{O}_\mathcal{M}$.

By Theorem 2.3(ii) and (iv), we have $\tilde{\chi} \quad \tilde{\chi}^T = \text{Codeg}$ and $\tilde{\chi} \cdot \tilde{\chi}^T = \tilde{\text{Codeg}}$, where $\text{Codeg}$ (resp. $\tilde{\text{Codeg}}$) is the Irrep$(K_{Q_{ab}}(\mathcal{E})) \times \text{Irrep}(K_{Q_{ab}}(\mathcal{E}))$ (resp. Irrep$(K_{Q_{ab}}(\mathcal{E}))^Z/NZ \times \text{Irrep}(K_{Q_{ab}}(\mathcal{E}))^Z/NZ$) diagonal matrix with $(\rho, \rho)$-th entry $f_{\rho}$, the formal codegree of $\rho$.

By our definitions above (see also [12]) we have

$$\begin{pmatrix} \alpha_{\rho} \\ \vdots \\ \alpha_{\rho} \end{pmatrix} = \tilde{\chi} \cdot \begin{pmatrix} C^* \\ \vdots \\ C^* \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{\alpha}_{\rho} \\ \vdots \\ \tilde{\alpha}_{\rho} \end{pmatrix} = \tilde{\chi} \cdot \begin{pmatrix} [M^*] \\ \vdots \\ [M^*] \end{pmatrix}.$$ 

Hence $\begin{pmatrix} \alpha_{\rho} \\ \vdots \\ \alpha_{\rho} \end{pmatrix} = \begin{pmatrix} \alpha_{\rho} \\ \vdots \\ \alpha_{\rho} \end{pmatrix} = \begin{pmatrix} \alpha_{\rho} \\ \vdots \\ \alpha_{\rho} \end{pmatrix}$

We can now prove the following non-spherical analogue of Theorem 1.1

**Theorem 1.1’.** For $C \in \mathcal{O}_\mathcal{E}$ and $M, N \in \mathcal{O}_\mathcal{M}$, we have

$$a^N_{C, M} = a^N_{C^*, M^*} = \sum_{\rho \in \text{Irrep}(K_{Q_{ab}}(\mathcal{E}))^Z/NZ} \frac{\rho([C]) \cdot \tilde{\chi}_\rho([M]) \cdot \tilde{\chi}_\rho([N])}{f_\rho} = \sum_{\rho \in \text{Irrep}(K_{Q_{ab}}(\mathcal{E}))^Z/NZ} \frac{\rho([C]) \cdot \tilde{\chi}_\rho([M]) \cdot \tilde{\chi}_\rho([N])}{f_\rho}.$$ 

(16)

**Proof.** For $C \in \mathcal{O}_\mathcal{E}, M \in \mathcal{O}_\mathcal{M}$, let us first compute $[C^*] \cdot [M^*] \in K_{Q_{ab}}(\mathcal{M}^{-1})$. By the previous equations

$$[C^*] \cdot [M^*] = \left( \sum_{\rho \in \text{Irrep}(K_{Q_{ab}}(\mathcal{E}))} \frac{\rho([C]) \cdot \alpha_{\rho}}{f_\rho} \right) \cdot \left( \sum_{\rho \in \text{Irrep}(K_{Q_{ab}}(\mathcal{E}))^Z/NZ} \frac{\tilde{\chi}_\rho([M]) \cdot \tilde{\alpha}_{\rho}}{f_\rho} \right).$$

(17)
\[ \rho([C]) \cdot \chi_{\rho}([M]) \cdot \overline{\alpha}_{\rho} \quad \cdots \text{by Thm. 2.3(iii)} \]  

\[ = \sum_{\rho \in \text{Irrep}(K_{Qa_{\mathbb{Q}}(\mathcal{O}_{\mathcal{E}})} \mathbb{Z}/N\mathbb{Z})} \frac{\rho([C]) \cdot \chi_{\rho}([M])}{f_{\rho}} \cdot \overline{\chi}_{\rho}([N]) \cdot [N^*] \]  

Hence for each \( N \in \mathcal{O}_{\mathcal{M}} \), the multiplicity of \([N^*]\) in \([C^*] \cdot [M^*]\) is given by

\[ a_{C^*,M^*}^N = \sum_{\rho \in \text{Irrep}(K_{Qa_{\mathbb{Q}}(\mathcal{O}_{\mathcal{E}})} \mathbb{Z}/N\mathbb{Z})} \frac{\rho([C]) \cdot \chi_{\rho}([M]) \cdot \overline{\chi}_{\rho}([N])}{f_{\rho}}. \]

Now it is clear that we have \((C \otimes M)^* \cong C^* \otimes M^*\) as objects of \( \mathcal{M}^{-1} \). Hence we have \( a_{C,M}^N = a_{C^*,M^*}^N \).

Moreover, since the multiplicities are integers, we can take the ‘complex conjugation’ in the summation and (noting that \( f_{\rho} \) is a totally positive cyclotomic integer) the theorem follows.

**Proof of Theorem 1.1.** Now we assume that we are in the spherical setting. In this setting we have identifications \( \mathcal{O}_{\mathcal{E}} \cong \text{Irrep}(K_{Qa_{\mathbb{Q}}(\mathcal{E})}) \) and \( \mathcal{O}_{D \mathcal{E}} \cong \text{Irrep}(K_{Qa_{\mathbb{Q}}(\mathcal{E})}) \mathbb{Z}/N\mathbb{Z} \). Now suppose that \( D \in \mathcal{O}_{\mathcal{E}} \) corresponds to \( \rho \in \text{Irrep}(K_{Qa_{\mathbb{Q}}(\mathcal{E})}) \mathbb{Z}/N\mathbb{Z} \), then we have

\[ \rho([C]) = \frac{S(\mathcal{E}, D, C)}{\text{dim}_{\mathcal{E}} D} \cdot \overline{\chi}_{\rho}([M]) = \frac{S(\mathcal{E}, D, M)}{\text{dim}_{\mathcal{E}} D} \cdot \overline{\chi}_{\rho}([N]) = \frac{S(\mathcal{E}, \mathcal{M}, D, N)}{\text{dim}_{\mathcal{E}} D} \text{ and } f_{\rho} = \frac{\text{dim}_{\mathcal{E}} D}{\text{dim}_{\mathcal{D}} D}. \]

The first equality above follows from either Theorem 2.4(i) or 2.4(ii) applied to the case \( \mathcal{M} = \mathcal{E} \). The next three equalities follow from Theorem 2.4(i) and (ii). This combined with Theorem 1.4 completes the proof.

### 3.2 Fusion coefficients for the twisted fusion algebra

Theorem 1.4 is proved in [De1]. Here we will only derive a non-spherical version of this Verlinde formula. The role of the S-matrix will be played by the character table of the twisted fusion algebra \( K_{Qa_{\mathbb{Q}}(\mathcal{E}, F)} \) in the basis \( \{[C, \psi_C] : C \in \mathcal{O}_{\mathcal{E}} \} \).

We still have that \( K_{Qa_{\mathbb{Q}}(\mathcal{E}, F)} \) is a commutative Frobenius \( \mathbb{Q}^a_{\mathbb{Q}} \)-algebra. Let us recall what this means. We refer to [De1] for details. Firstly, we have a non-degenerate linear functional \( \lambda : K_{Qa_{\mathbb{Q}}(\mathcal{E}, F)} \rightarrow \mathbb{Q}^a_{\mathbb{Q}} \) defined by \( \lambda([1, \text{id}_1]) = 1 \) and \( \lambda([C, \psi_C]) = 0 \) for \( 1 \not\cong C \in \mathcal{O}_{\mathcal{E}} \) providing \( K_{Qa_{\mathbb{Q}}(\mathcal{E}, F)} \) with the structure of a Frobenius algebra. In particular we have an identification \( K_{Qa_{\mathbb{Q}}(\mathcal{E}, F)} \cong K_{Qa_{\mathbb{Q}}(\mathcal{E}, F)} \) as \( K_{Qa_{\mathbb{Q}}(\mathcal{E}, F)} \)-modules. The two bases \( \{[C, \psi_C] : C \in \mathcal{O}_{\mathcal{E}} \} \) and \( \{[C^*, \psi_C^{-1}] : C \in \mathcal{O}_{\mathcal{E}} \} \) of the Frobenius algebra \( K_{Qa_{\mathbb{Q}}(\mathcal{E}, F)} \) are dual in the usual sense. Moreover, there is a \( \mathbb{Q}^a_{\mathbb{Q}} \)-semilinear anti-involution \( (\cdot)^* : K_{Qa_{\mathbb{Q}}(\mathcal{E}, F)} \rightarrow K_{Qa_{\mathbb{Q}}(\mathcal{E}, F)} \) mapping a basis element \([C, \psi_C] \mapsto [C^*, \psi_C^{-1}]\) and extended semilinearly. Then the Hermitian form \( \langle \cdot, \cdot \rangle \) on \( K_{Qa_{\mathbb{Q}}(\mathcal{E}, F)} \) defined by \( \langle a, b \rangle = \lambda(ab^*) \) is (totally) positive definite and our special basis \( \{[C, \psi_C] : C \in \mathcal{O}_{\mathcal{E}} \} \) is orthonormal.

We reiterate that by [ENO1], all irreducible characters of \( K_{Qa_{\mathbb{Q}}(\mathcal{E}, F)} \) are defined over \( \mathbb{Q}^a_{\mathbb{Q}} \). For a character \( \varphi : K_{Qa_{\mathbb{Q}}(\mathcal{E}, F)} \rightarrow \mathbb{Q}^a_{\mathbb{Q}} \), we have the corresponding element \( \alpha_{\varphi} = \sum_{C \in \mathcal{O}_{\mathcal{E}}} \varphi([C^*, \psi_C^{-1}])[C, \psi_C] \) in \( K_{Qa_{\mathbb{Q}}(\mathcal{E}, F)} \). From generalities about Frobenius \( * \)-algebras and the definition of \( K_{Qa_{\mathbb{Q}}(\mathcal{E}, F)} \), it follows that for \( \varphi' \in \text{Irrep}(K_{Qa_{\mathbb{Q}}(\mathcal{E}, F)}) \), \( \varphi'(\alpha_{\varphi}) = 0 \) if \( \varphi' \neq \varphi \) and \( f_{\varphi} := \varphi(\alpha_{\varphi}) \).
is a totally positive cyclotomic integer known as the formal codegree of \( \varphi \). We have the orthogonality relations \( (\alpha_\varphi, \alpha_{\varphi'}) = \delta_{\varphi\varphi'} \cdot f_\varphi \). The element \( \frac{\alpha_\varphi}{f_\varphi} \in K_{Qab}(\mathcal{C}, F) \) is the minimal idempotent corresponding to the irreducible representation \( \varphi \). It follows that (see [A, Lem. 2.5]) \( \alpha^{\ast}_\varphi = \alpha_\varphi \) and hence \( \varphi([C^{\ast}, \psi_C^{-1}]) = \varphi([C, \psi_C]) \).

Let \( \text{Ch}_F \) be the \( \text{Irrep}(K_{Qab}(\mathcal{C}, F)) \times O^F_C \) matrix with \( (\varphi, C) \)-th entry \( \varphi([C, \psi_C]) \). We have \( \text{Ch}_F \cdot \overline{\text{Ch}_F^T} = \text{Codeg}_F \), where \( \text{Codeg}_F \) is the \( \text{Irrep}(K_{Qab}(\mathcal{C}, F)) \times \text{Irrep}(K_{Qab}(\mathcal{C}, F)) \) matrix whose \( (\varphi, \varphi) \)-th entry is the formal codegree \( f_\varphi \). By our definitions \[
\begin{pmatrix}
\alpha_\varphi \\
\frac{f_\varphi}{\overline{\text{Ch}_F^T}} \\
\vdots
\end{pmatrix} = \text{Ch}_F \cdot \begin{pmatrix}
[C^{\ast}, \psi_C^{-1}] \\
[C, \psi_C] \\
\vdots
\end{pmatrix}
\]
Hence proceeding as in the proof of Theorem 1.1 we obtain (compare with [1.4]):

**Theorem 1.4'.** For \( C, C', D \in O^F_C \) we have

\[
a_{C,C'}^D = \sum_{\varphi \in \text{Irrep}(K_{Qab}(\mathcal{C}, F))} \varphi([C, \psi_C]) \cdot \varphi([C', \psi_{C'}]) \cdot \varphi([D, \psi_D]) \cdot f_\varphi .
\]

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**Remark 3.1.** Theorem 1.4 (in the spherical setting) follows from this using Theorem 2.1(ii) and its consequence that for \( M \in O_{\mathcal{C}, \mathcal{F}} \), the formal codegree of the associated representation \( \varphi_M \) is given by \( f_{\varphi_M} = \frac{\dim \mathcal{C}}{\dim \mathcal{F}} \cdot M \).

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