The Undecidability of Conditional Affine Information Inequalities and Conditional Independence Implication with a Binary Constraint

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Abstract

We establish the undecidability of conditional affine information inequalities, the undecidability of the conditional independence implication problem with a constraint that one random variable is binary, and the undecidability of the problem of deciding whether the intersection of the entropic region and a given affine subspace is empty. This is a step towards the conjecture on the undecidability of conditional independence implication. The undecidability is proved via a reduction from the periodic tiling problem (a variant of the domino problem). Hence, one can construct examples of the aforementioned problems that are independent of ZFC (assuming ZFC is consistent).

Index Terms

Information inequalities, entropic region, conditional independence implication, domino problem.

I. INTRODUCTION

The problem of characterizing the entropic region \( \Gamma_n^\ast \) and the almost-entropic region \( \overline{\Gamma_n^\ast} \) (the closure of \( \Gamma_n^\ast \)) is a fundamental problem in information theory \cite{1, 2, 3}. Its applications include network coding \cite{4, 5, 6, 7, 8}, secret sharing \cite{9, 10, 11, 12, 13, 14}, group theory \cite{15}, and automated proofs of capacity regions in multiuser coding settings \cite{16, 17, 18}.

Pippenger \cite{19} raised the question whether all inequalities among entropy and mutual information of random variables can be deduced from \( I(X;Y|Z) \geq 0 \) (i.e., Shannon-type inequalities). This was answered by Zhang and Yeung, who showed the first conditional non-Shannon-type inequality (i.e., showing that an inequality on entropy terms is implied by a collection of other such inequalities, but this implication cannot be proved using Shannon-type inequalities) in \cite{11}, and the first unconditional non-Shannon-type inequality in \cite{3}. More non-Shannon-type inequalities were discovered later in \cite{20, 21, 22, 23, 24}. In particular, Matuš \cite{22} showed that \( \Gamma_n^\ast \) is not polyhedral. Chan and Grant \cite{25} proved a non-linear information inequality, and conjectured that \( \Gamma_n^\ast \) is semialgebraic. It was shown by Gómez, Mejía and Montoya \cite{26} that \( \Gamma_n^\ast \) for \( n \geq 3 \) is not semialgebraic, though it is still unknown whether \( \overline{\Gamma_n^\ast} \) is semialgebraic in general.

To study unconditional linear information inequalities, it suffices to consider \( \overline{\Gamma_n^\ast} \), which is a convex cone. Nevertheless, if conditional information inequalities are of interest (e.g. for the conditional independence implication problem and secret sharing \cite{11}), we have to work with \( \Gamma_n^\ast \) which is non-convex. Kaced and Romashchenko \cite{27} showed the existence of conditional information inequalities that are not implied by unconditional linear information inequalities, and conditional information inequalities that are valid in \( \Gamma_n^\ast \) but not in \( \overline{\Gamma_n^\ast} \) (also see \cite{28}). This suggests that characterizing conditional information inequalities might be harder than characterizing unconditional ones (which is already extremely hard).

While there are algorithms that attempt to verify conditional information inequalities (e.g. \cite{16, 29, 30}), most of them only take Shannon-type inequalities into account, and hence fail to verify true non-Shannon-type inequalities. There are algorithms capable of verifying some non-Shannon-type inequalities \cite{23, 24, 13, 14, 18}, though there is no known algorithm that is capable of verifying every true conditional information inequality. It was unknown whether such algorithm can exist, that is, whether the problem of conditional information inequality is algorithmically decidable. See \cite{31, 26, 32, 33, 34, 35} for partial results on the decidability/undecidability of information inequalities and network coding.

A closely related problem is the conditional independence implication problem \cite{36, 37, 38, 39}, which is to decide whether a statement on the conditional independence among several random variables follows from a list of other such statements. Since conditional independence can be expressed as \( I(X;Y|Z) = 0 \), this problem can be reduced to conditional information inequalities. Pearl and Paz \cite{39} introduced a set of axioms (the semi-graphoid axioms) capable of solving a subset of conditional independence implication problems. This set of axioms was shown to be incomplete by Studený \cite{40}, who later showed in \cite{41} that there is no finite axiomization of probabilistic conditional independence in general.

Nevertheless, conditional independence implication among random variables with fixed cardinalities can be decided algorithmically, as shown by Niepert \cite{42}. Hannula et al. \cite{43} showed that if all random variables are binary, then the problem is
in EXPSPACE. Khamis, Kolaitis, Ngo and Suciu [35] proved that the general conditional independence implication problem is decidable in \( \Pi^1_1 \) in the arithmetical hierarchy. It is unknown whether the conditional independence implication problem is decidable in general. See [44], [45], [46], [47], [48], [49], [50], [51] for other partial results on the decidability/undecidability of conditional independence implication.

In this paper, which is an extended version of [52], we give a partial negative answer to the question on the algorithmic decidability of conditional information inequalities. We show that the problem of deciding whether the intersection of the entropic region and a given affine subspace is empty is undecidable. More precisely, we show the undecidability of the problem of deciding whether there exists \( v \in \Gamma^* \) such that \( A v = b \), where \( A \in \mathbb{Q}^{m \times (2^n-1)} \), \( b \in \mathbb{Q}^m \) are given rational matrix and vector, and \( n, m \geq 0 \) are given integers. We remark that affine or nonhomogeneous inequalities have appeared in existential results about random variables before (e.g. [53], [54]), so the study of affine constraints is unnatural.

We also prove the undecidability of the conditional independence implication problem with a binary constraint. Consider random variables \( X_1, \ldots, X_n, n \geq 2 \). This problem is undecidable: deciding whether \( I(X_{A_j}; X_{B_j} | X_{C_j}) = 0 \) for \( j = 1, \ldots, m \) and \( X_1 \) has cardinality at most 2 (i.e., \( X_1 \) is binary) implies that \( I(X_1; X_2) = 0 \), where \( A_j, B_j, C_j \subseteq \{1, \ldots, n\} \) are three given disjoint sets for each \( j = 1, \ldots, m \), and \( X_A := \{X_i\}_{i \in A} \) (see Corollary 5). To the best of the author’s knowledge, this is the first undecidability result about probabilistic conditional independence\(^1\) and hence is a step towards the conjecture on the undecidability of conditional independence implication. This undecidability result is perhaps surprising, considering that the conditional independence implication problem when the cardinalities of all random variables are bounded is decidable \( 2^{n} \) (see [42]), but it is counterintuitive that bounding the cardinality of only one random variable makes the problem undecidable (see Table 1).

Furthermore, we prove that the following problems are undecidable:

- (One linear equality and one affine equality) Deciding whether there exists \( v \in \Gamma^* \) such that \( a^TVv = 0 \) and \( v_{(1)} = 1 \), where \( a \in \mathbb{Q}^{2^n-1} \) and \( v_{(1)} \) denotes the entry of \( v \) that represents the entropy of the first random variable.

- (Conditional affine information inequality) Deciding the truth value of the conditional affine information inequality in the form
  \[
  v \in \Gamma^* \land a^Tv \leq 0 \land v_{(1)} \leq 1 \Rightarrow v_{(1)} = 0,
  \]
  where \( a \in \mathbb{Q}^{2^n-1} \) (i.e., deciding whether \( v \in \Gamma^* \), \( a^Tv \leq 0 \) and \( v_{(1)} \leq 1 \) implies \( v_{(1)} = 0 \)).

- (Boolean information constraint) Deciding the truth value of the Boolean information constraint (with strict affine inequality constraints) in the form \( \forall v \in \Gamma^* : a_1^Tv > b_1 \lor \cdots \lor a_Nv > b_N \).

We will show the undecidability of entropic region via a reduction from the periodic tiling problem. The domino problem introduced by Wang 62 concerns the problem of tiling the plane with unit square tiles. Given a set of tiles, where each tile has four colored edges, the problem is to decide whether it is possible to tile the plane using this set of tiles, such that touching edges of adjacent tiles have the same color. Each tile in the set can be used an unlimited number of times, though no rotation or reflection is allowed. It was proved by Berger 63 that deciding whether a finite set of tiles can tile the plane is undecidable by simulating a Turing machine using the tiles, and hence undecidability follows from the undecidability of the halting problem 63. Moreover, Gurevich and Koryakov 65 showed that the problem of deciding whether a finite set of tiles can tile the plane periodically (i.e., there exists a positive integer \( N \) such that the tiling remains the same after shifting upward by \( N \) or leftward by \( N \), or equivalently, it can tile a torus) is also undecidable. This is also proved by Mazoyer and Rapaport 66 for NW-deterministic tiles.

As a consequence, one can explicitly construct \( A \in \mathbb{Q}^{m \times (2^n-1)} \), \( b \in \mathbb{Q}^m \) such that the non-existence of \( v \in \Gamma^* \) with \( Av = b \) is unprovable in ZFC (assuming ZFC is consistent). Also, one can construct an example of a conditional independence implication problem with a binary constraint which is unprovable in ZFC. This is the fact that one can construct a Turing machine such that whether it halts is independent of ZFC, assuming ZFC is consistent (see e.g. 67, 68), and that the set of tiles in 63, 66 and the \( A, b \) in this paper (Theorem 1) are given constructively. Finding the smallest \( A \in \mathbb{Q}^{m \times (2^n-1)} \), \( b \in \mathbb{Q}^m \) such that \( \exists v \in \Gamma^* : Av = b \) is independent of ZFC is left for future studies.

We remark that the decidability of the following problems remain open (also see Table 1):

- Deciding the truth value of a conditional linear information inequality (our construction requires one affine inequality condition).

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1 Note that 52 only contains a brief version of the proof of the undecidability of the affine subspace intersection problem, and does not contain the proof for the conditional independence implication result.

2 Some authors have considered the non-disjoint case 54, 55, whereas some authors require those sets to be disjoint 59, 56. We prove in Section VI-B that the two cases are actually Turing equivalent, and non-disjoint conditional independence relations can be expressed as disjoint conditional independence relations.

3 We remark that the embedded multivalued dependencies implication problem, a different but related problem, was shown to be undecidable in 57, 58. We also remark that the capacity of finite state machine channel is uncomputable 59 (also see 60, 61), though this is unrelated to the setting in this paper.

4 In 66, for any Turing machine, a set of NW-deterministic tiles is constructed such that the set of tiles admits a periodic tiling if and only if the Turing machine halts.
The decidability status of the conditional independence implication problem under various conditions on the set of random variables $A_j, B_j, C_j$ and cardinality bounds. "Cardinality bounds on some variables" means that we restrict $\text{card}(X_i) \leq k_i$ for given $k_1, \ldots, k_n \in \{1, 2, \ldots\} \cup \{\infty\}$. "Cardinality bounds on all variables" means $k_i$ is finite for all $i$.  

| Condition | No cardinality bounds | Cardinality bounds on some variables | Cardinality bounds on all variables |
|-----------|------------------------|--------------------------------------|----------------------------------|
| Disjoint $A_j, B_j, C_j = \emptyset$ | Decidable [44], [69] | Decidable [45] | Decidable [44], [42] |
| Disjoint $A_j, B_j, C_j$, $A_j \cup B_j \cup C_j = [n]$ (saturated CI) | Decidable [70], [45] | Decidable [45] | Decidable [45], [42] |
| General disjoint $A_j, B_j, C_j$ | Unknown | Undecidable (this paper) | Decidable [42] |

This paper is organized as follows. In Section II, we review the entropic region. In Section III, we introduce the concept of affine existential information predicate (an extension of existential information predicate [18]), the main tool in our proof. In Section IV, we prove the main result on the undecidability of the intersection of $\Gamma_n$ and an affine subspace. In Section V, we prove the undecidability of the satisfiability and implication problem of conditional independence with a binary constraint. In Section VI, we show the undecidability of related problems as corollaries of our main result.

**Notations**

Throughout this paper, entropy is in bits, and $\log$ is to the base 2. The binary entropy function (i.e., the entropy of Bern($h$)) is denoted as $H_b(h)$. Given a discrete random variable $X \in \mathcal{X}$, its cardinality (the size of its support) is denoted as $\text{card}(X) := |\{x \in \mathcal{X} : P(X = x) > 0\}|$. The set of rational real numbers is denoted as $\mathbb{Q}$. Given propositions $P, Q$, the logical conjunction (i.e., AND) is denoted as $P \land Q$.

Given vectors $v, w \in \mathbb{R}^k$, $v \succeq w$ means $v_i \geq w_i$ for any $i$. The sign function is denoted as $\text{sgn}(t) = 1\{t > 0\} - 1\{t < 0\}$. We write $[a..b] := \mathbb{Z} \cap [a, b]$, $[n] := \{1, \ldots, n\}$. We write $X^a_n := (X_a, X_{a+1}, \ldots, X_b)$, $X^n := X^1_n$. For finite set $S \subseteq \mathbb{N}$, write $X_S := (X_{a_1}, \ldots, X_{a_k})$, where $a_1, \ldots, a_k$ are the elements of $S$ in ascending order. We usually use $\Delta$ for matrix, $a, b, c, d$ for column vectors, $X, Y, Z, U, V, W$ for random variables, and $S$ for sets.

### II. Entropic Region

In this section, we briefly review the definition of the entropic region [11, 2, 6]. For a sequence of discrete random variables with finite entropies $X^n = (X_1, \ldots, X_n)$, its entropic vector [11] is defined as $h(X^n) = h \in \mathbb{R}^{2^n-1}$, where the entries of $h$ are indexed by nonempty subsets of $[n]$, and $h_S := H(X_S)$ (where $S \subseteq [n]$) is the joint entropy of $\{X_i\}_{i \in S}$. The entropic region [11] is the region of entropic vectors

$$\Gamma^*_n := \bigcup_{p \times n} \{h(X^n)\}$$

over all discrete joint distributions $p_{X^n}$. The problem of characterizing $\Gamma^*_n$ for $n \geq 4$ is open. A polyhedral outer bound $\Gamma_n$ characterized by Shannon-type inequalities is given in [3], which is the basis of the linear program for verifying linear information inequalities in [16, 2].

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6This follows from [45] Theorem 15 which uses only binary random variables to show the completeness of the semi-graphoid axioms, and that random variables with cardinality bound 1 can be ignored.

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III. AFFINE EXISTENTIAL INFORMATION PREDICATE

The concept of existential information predicate (EIP) was studied systematically in [18]. Before that, several existential statements on random variables and entropy terms have been studied, for example, the copy lemma [3], [24], strong functional representation lemma [71], [54], double Markov property [72], and the infinite divisibility of information [73].

Here we introduce an extension of EIP to allow affine inequalities, called affine existential information predicate (AEIP). An AEIP is a predicate on the random sequence $X^n$ in the form

$$\exists U^l: \mathbf{A}h(X^n, U^l) \succeq \mathbf{b},$$

where $\mathbf{A} \in \mathbb{R}^{m \times (2^n + l - 1)}$, $\mathbf{b} \in \mathbb{R}^m$, $m \geq 0$. The “$\exists U^l$” should be interpreted as “$\exists p_{U^l|X^n}$”, i.e., there exists random sequence $U^l$ dependent on $X^n$ such that $\mathbf{A}h(X^n, U^l) \succeq \mathbf{b}$. Denote the above predicate as $\text{AEIP}_{n, \mathbf{A}, \mathbf{b}}(X^n)$ (note that the value of $l$ can be deduced from $n$ and the width of $\mathbf{A}$).

While affine inequalities among entropy might seem unnatural since they are nonhomogeneous (considering most previous works on information inequalities are about linear inequalities, and the nonlinear inequality in [23] is homogeneous), it is not uncommon to have inequalities involving both entropy and constant terms in information theory, e.g., the expected length of Huffman code, and the Knuth-Yao scheme for random number generation [74]. Some specific examples of AEIPs have been studied previously. For example, the approximate infinite divisibility of information [53] states that the following AEIP holds for any random variable $X$ and any $l \geq 1$ (slightly relaxed since [53] requires $U_1, \ldots, U_l$ to be i.i.d.):

$$\exists U^l : U_1 \perp \cdots \perp U_l \wedge H(X|U^l) = 0$$

$$\wedge H(U_1) = \cdots = H(U_l) \leq \frac{1.59}{l} H(X) + 2.43.$$

Another example is the strong functional representation lemma [54], [75], which states that the following holds for any random variables $X, Y$:

$$\exists U : U \perp X \wedge H(Y|X, U) = 0$$

$$\wedge H(Y|U) \leq I(X; Y) + \log(I(X; Y) + 1) + 4.$$

While this is not itself an AEIP, it implies the following AEIP holds for any fixed $\gamma \geq 0$ and random variables $X, Y$:

$$\exists U : U \perp X \wedge H(Y|X, U) = 0$$

$$\wedge H(Y|U) \leq I(X; Y) + \frac{\log e}{\gamma + 1} (I(X; Y) - \gamma) + \log(\gamma + 1) + 4.$$

Also see [76] for some affine inequalities on the entropy of sums of random variables. These suggest that homogeneity is not a property we should expect from inequalities on entropy in general, and affine inequalities are not unnatural. We remark that AEIP can be handled by the Python Symbolic Information Theoretic Inequality Prover in [18].

Our goal is to show that the problem of deciding the truth value of $\text{AEIP}_{0, \mathbf{A}, \mathbf{b}}(\emptyset)$ for a given rational matrix $\mathbf{A}$ and rational vector $\mathbf{b}$ is undecidable, where $\emptyset$ means $n = 0$ ($X^n$ is empty).

We study some composition rules of AEIPs. For two AEIPs $\text{AEIP}_{n, \mathbf{A}, \mathbf{b}}(X^n)$, $\text{AEIP}_{n, \mathbf{C}, \mathbf{d}}(X^n)$, their conjunction (i.e., “and”) is

$$\exists U^l, V'^l : \mathbf{A}h(X^n, U^l) \succeq \mathbf{b}$$

$$\wedge \mathbf{C}h(X^n, V'^l) \succeq \mathbf{d},$$

which is clearly still an AEIP. Also, given an AEIP $\text{AEIP}_{n+l, \mathbf{A}, \mathbf{b}}(X^{n+l})$, a predicate on $X^n$ in the form:

$$\exists U^l : \text{AEIP}_{n+l, \mathbf{A}, \mathbf{b}}(X^n, U^l)$$

$$\wedge \mathbf{C}h(X^n, U^l) \succeq \mathbf{d}$$

is also an AEIP.

We will use these composition rules to construct a class of undecidable AEIPs in the next section. Note that while AEIP is quite general, we will only use AEIPs defined using conditional independence constraints (i.e., $I(X; Y|Z) = 0$) and cardinality constraints [2] throughout the proof. This will be important for results on conditional independence implication in Section V.

\(^7\)Source code of Python Symbolic Information Theoretic Inequality Prover (PSITIP) is available at [https://github.com/cheuktingli/psitip](https://github.com/cheuktingli/psitip)
IV. CONSTRUCTION OF TILING

In this section, we establish our main result, which is the undecidability of AEIPs with rational coefficients.

**Theorem 1.** The problem of deciding the truth value of \( \text{AEIP}_{A,b}(\emptyset) \) (i.e., \( \exists U^l : \text{Ah}(U^l) \supseteq b \)) given \( A \in \mathbb{Q}^{m \times (2^l - 1)} \), \( b \in \mathbb{Q}^m \), \( l, m \geq 0 \) is algorithmically undecidable. Therefore, the problem of deciding the truth value of \( \exists U^l : \text{Ah}(U^l) = b \) is also undecidable.

The proof of Theorem 1 is divided into several steps presented in the following subsections.

A. Construction of Tori/Grid

It was shown in [1] that if \( H(Y_1|Y_2, Y_3) = H(Y_2|Y_1, Y_3) = H(Y_3|Y_1, Y_2) = I(Y_1; Y_2) = I(Y_1; Y_3) = I(Y_2; Y_3) = 0 \), then \( Y_1 \) is uniformly distributed over its support (also true for \( Y_2, Y_3 \)), and they have the same cardinality. Write this as the AEIP:

\[
\begin{align*}
\text{TRIPLE}(Y_1, Y_2, Y_3) : & \quad H(Y_1|Y_2, Y_3) = H(Y_2|Y_1, Y_3) = H(Y_3|Y_1, Y_2) \\
& \quad = I(Y_1; Y_2) = I(Y_1; Y_3) = I(Y_2; Y_3) = 0.
\end{align*}
\]

(1)

Note that we use the notation \( \text{P}(X) : \) (formula with \( X \) as free variable) to define a predicate \( P \) instead of writing \( \text{P}(X) = \) (formula) to avoid having to write extra parentheses (e.g. we can write \( \text{P}(X) : H(X) = 0 \) instead of \( \text{P}(X) : (H(X) = 0) \)). Therefore, the predicate that \( X \) is uniformly distributed over its support can be expressed as the following AEIP:

\[
\begin{align*}
\text{UNIF}(X) : & \quad \exists U_1, U_2 : \text{TRIPLE}(X, U_1, U_2).
\end{align*}
\]

The predicate that \( X \) is uniform with cardinality \( k \geq 2 \) is given by the AEIP:

\[
\begin{align*}
\text{UNIF}_k(X) : & \quad \text{UNIF}(X) \land \alpha_k \leq H(X) \leq \alpha_{k+1},
\end{align*}
\]

where \( \alpha_k \) is a rational number such that \( \log(k - 1) < \alpha_k < \log k \).

Given this, one natural approach to prove the desired undecidability result is to show a reduction from the problem of deciding whether a Diophantine equation (a polynomial equation with integer coefficients and variables) has a solution, which is undecidable by Matiyasevich’s theorem [77]. We can represent the positive integer \( k \) by a uniform random variable with cardinality \( k \). It is possible to define equality, multiplication and comparison over positive integers using AEIPs (see Section V-A). Nevertheless, it is uncertain whether addition (or the successor function) over positive integers can be defined using AEIP (note that addition of entropy corresponds to multiplication of cardinality), and hence Matiyasevich’s theorem cannot be applied. We will instead show a reduction from the periodic tiling problem [65], another undecidable (and seemingly less related) problem.

Given two random variables \( X_1, X_2 \), the predicate that they are uniform with the same cardinality (pair \( X_1, X_2 \) is uniformly distributed over its support, and all vertices in their characteristic bipartite graph (i.e., a graph with edge \( (x_1, x_2) \) if and only if \( p_{X_1,X_2}(x_1, x_2) > 0 \)) have degree 2 (i.e., the bipartite graph consists of disjoint cycles) can be given by the AEIP:

\[
\begin{align*}
\text{CYCS}(X_1, X_2) : & \quad \exists U : \text{UNIF}(X_1) \land \text{UNIF}(X_2) \land \text{UNIF}(U) \\
& \quad \land I(X_1; U) = I(X_2; U) = 0 \\
& \quad \land H(X_1|X_2, U) = H(X_2|X_1, U) = 0 \\
& \quad \land H(U|X_1, X_2) = 0.
\end{align*}
\]

This can be proved by observing \( I(X_1; U) = H(X_2|X_1, U) = 0 \) implies that \( p_{X_2|X_1} = x_1 \) is degenerate or uniform over two values for any \( x_1 \), and the degenerate case is impossible since \( H(U|X_1, X_2) = 0 \). Hence all vertices have degree 2. For the other direction, if the bipartite graph consists of disjoint cycles, then we can color the edges in two colors so that no two edges sharing a vertex have the same color. The value of \( U \) corresponds to the color of the edge.

Tori can be formed by taking two independent copies of cycles. Define the AEIP

\[
\begin{align*}
\text{TORI}(X^2, Y^2) : & \quad \text{CYCS}(X^2) \land \text{CYCS}(Y^2) \\
& \quad \land I(X^2; Y^2) = 0.
\end{align*}
\]

Note that the characteristic bipartite graph between \( (X_1, Y_1) \) and \( (X_2, Y_2) \) is the disjoint union of a collection of tori (product of cycles). Each vertex of the tori is represented as a quadruple \( (x_1, x_2, y_1, y_2) \) in the support of \( (X_1, X_2, Y_1, Y_2) \). Refer to Figure 1. We will use the tori as the grid for the tiling problem.
B. Construction of Colors

We now describe a way to assign colors to the vertices of the tori. Intuitively, the color of a vertex can be represented as a random variable (which has values corresponding to colors). Nevertheless, this does not work since the values of a random variable cannot be identified using entropy, which is invariant under relabeling. Hence, we have to introduce a way to encode color information using conditional independence.

Consider the distribution $F \sim \text{Bern}(1/2)$, $G_1 | F \sim \text{Bern}((1 - F)/2)$, $G_2 | (F, G_1) \sim \text{Bern}(F/2)$ (i.e., the distribution $(F, G_1, G_2) \sim \text{Unif}(((0, 0, 0), (0, 1, 0), (1, 0, 0), (1, 0, 1)))$. We can check for this joint distribution (up to relabeling) by

$$\begin{align*}
\text{FLIP}(F, G_1, G_2) : \\
\exists U, Z_1, Z_2 : \text{UNIF}_4(U) \land \text{UNIF}_2(F) \\
\land H(F, G_1, G_2 | U) = I(G_1; G_2 | F) = 0 \\
\land \text{UNIF}_3(Z_1) \land I(Z_1; G_1) = H(U | G_1, Z_1) = 0 \\
\land \text{UNIF}_3(Z_2) \land I(Z_2; G_2) = H(U | G_2, Z_2) = 0.
\end{align*}$$

This is because $\text{UNIF}_4(U)$, $H(G_1 | U) = 0$, $\text{UNIF}_3(Z_1)$ and $I(Z_1; G_1) = H(U | G_1, Z_1) = 0$ implies $G_1$ either follows $\text{Unif}(\{4\})$ or $\text{Bern}(1/4)$ (up to relabeling). Same for $G_2$. Since $I(G_1; G_2 | F) = 0$, it is impossible to have $G_1 \sim \text{Unif}(\{4\})$ or $G_2 \sim \text{Unif}(\{4\})$. Hence we have the desired distribution. Refer to Figure 2 for an illustration.

We introduce a construction called switch: for $k \geq 4$,

$$\begin{align*}
\text{SW}(W^k, V^k, \bar{V}^k, F) : \\
\exists G : I(W^k; F, G) = 0 \\
\land \bigwedge_{i \in [k]} (\text{UNIF}_2(W_i) \\
\land H(V_i, \bar{V}_i | W_i, F) = I(V_i; \bar{V}_i | W_i) = 0 \\
\land \text{FLIP}(F, G, V_i) \land \text{FLIP}(F, G, \bar{V}_i)).
\end{align*}$$

We can regard $W^k \in \{0, 1\}^k$ as a sequence of Boolean-valued switches. Refer to Figures 2 and 3 for an illustration.

Since $\text{FLIP}(F, G, V_i)$, we assume $F \sim \text{Bern}(1/2)$, $G/F \sim \text{Bern}((1 - F)/2)$. Since $\text{UNIF}_2(W_i), \text{UNIF}_2(F)$ and $I(W_i; F) = 0$, we have $(W_i, F) \sim \text{Unif}(\{0, 1\})^2$. Since $\text{FLIP}(F, G, V_i)$, we can assume $V_i | F \sim \text{Bern}(F/2)$. Combining this with $H(V_i | W_i, F) = 0$, we either have $V_i = W_i F$ or $V_i = (1 - W_i) F$. Assume $V_i = (1 - W_i) F$ without loss of generality. Similarly, we either have $\bar{V}_i = W_i F$ or $\bar{V}_i = (1 - W_i) F$. The latter is impossible since $I(V_i; \bar{V}_i | W_i) = 0$. Hence we have $V_i = (1 - W_i) F$, $\bar{V}_i = W_i F$.

Since $I(W^k; F) = 0$, the conditional distribution of $F$ given $W^k = w^k$ is $\text{Bern}(1/2)$. Using $V_i = (1 - W_i) F$, $\bar{V}_i = W_i F$, for $S, \bar{S} \subseteq [k]$,

$$H(F | V_S, \bar{V}_S, W^k = w^k) = \text{sat}(w^k, S, \bar{S}),$$

Figure 1. The tori constructed. The labels on the x-axis give the support of $(X_1, X_2)$ (consisting of two cycles), whereas the labels on the y-axis give the support of $(Y_1, Y_2)$ (consisting of one cycle). Their product is a collection of two tori.
Figure 2. Left: An illustration for FLIP$(F, G, V)$, showing the 4 possible combinations of $(F, G, V)$ which are equally likely. Middle: An illustration for $F, G, V_i, W_i$ satisfying $SW(W^k, V^k, V^k, F)$. The x-axis is $W_i$, which is independent of $F$ (the y-axis). Since FLIP$(F, G, V_i)$, we can assume $G|F \sim Bern((1 - F)/2)$. Since $I(V_i; F, G) = 0$, we have $G|\{(W_i, F) = (0, 0)\} \sim Bern(1/2)$ and $G|\{(W_i, F) = (1, 0)\} \sim Bern(1/2)$ (represented by dividing the cells $(W_i, F) = (0, 0)$ and $(1, 0)$ into halves in the figure). Since FLIP$(F, G, V_i)$ and $H(V_i|W_i, F) = 0$, we can assume $V_i = 1$ if and only if $(W_i, F) = (0, 1)$ (another choice is $(W_i, F) = (1, 1)$). Right: An illustration for $F, G, V_i, W_i, W_4$ in $SW(W^k, V^k, V^k, F)$. We cannot have $V_i = 1$ if and only if $(W_i, F) = (0, 1)$ since $I(V_i; \{V_i|W_i\}) = 0$. Hence $V_i = 1$ if and only if $(W_i, F) = (1, 1)$. The purpose of $G$ is to force $V_i$ and $V_i$ to be 1 for the same value of $F$.

\[
\begin{array}{ccc}
F = 1 & G = 0 & V = 1 \\
F = 0 & G = 1 & V = 0 \\
W_i = 0 & V_i = 1 \\
W_i = 1 & V_i = 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
F = 1 & G = 0 & V = 0 \\
F = 0 & G = 1 & V = 1 \\
W_i = 0 & V_i = 0 \\
W_i = 1 & V_i = 0 \\
\end{array}
\]

Figure 3. The table of the values of $V^k, \bar{V}^k$ (which are functions of $W^k, F$) for different values of $W^k, F$ when $k = 4$. The x-axis denotes different values of $W^4$ (there are 6 possible combinations according to the constraints in COL, which are referred as colors), whereas the y-axis denotes the two values of $F \in \{0, 1\}$. If $V_i = 1$ for a combination of $W^k$ and $F$, the corresponding cell in the table is marked “$V_i$” in blue (similar for $V_i$ in red). This table can be obtained from the relation $V_i = (1 - W_i)F, V_i = W_iF$. where

\[
\text{sat}(w^k, S, \bar{S}) := \left(\prod_{i \in S} w_i \right) \left(\prod_{i \in \bar{S}} (1 - w_i)\right) \in \{0, 1\},
\]

which is 1 if and only if $w_i = 1$ for all $i \in S$, and $w_i = 0$ for all $i \in \bar{S}$. We have

\[
H(F|V_S, \bar{V}_S, W^k) = E[\text{sat}(W^k, S, \bar{S})].
\]

In particular, for $w^k \in \{0, 1\}^k,$

\[
H(F|V_{i\{w_i=1\}}, \bar{V}_{i\{w_i=0\}}, W^k) = P(W^k = w^k).
\]

There are a total of $2^k$ choices of $W^k$. We reduce the number of choices by restricting that for any $w^k$ with $p_{W^k}(w^k) > 0$, there exists $j \in [k - 1]$ such that either $w_i = 1\{i = j\}$ for $i \in [k]$ (in this case, call the color of $w^k$ to be $j$, a positive color), or $w_i = 1\{i \neq j\}$ for $i \in [k]$ (in this case, call the color of $w^k$ to be $-j$, a negative color). Let the set of such $w^k$ be $T_k \subseteq \{0, 1\}^k$. There are $|T_k| = 2(k - 1)$ choices of colors. Let $\text{col}: T_k \rightarrow [-k - 1, \ldots, -1] \cup [1, k - 1]$ such that $\text{col}(w^k)$ is the color of $w^k$, and write $\text{sgn}(\text{col}(w^k)) := \text{sgn}(\text{col}(w^k))$ for the sign of the color. Note that all $w^k$ of positive color has $w_k = 0$, and all $w^k$ of negative color has $w_k = 1$. Define the coloring AEIP as

\[
\text{COL}(W^k, V^k, \bar{V}^k, F) := SW(W^k, V^k, \bar{V}^k, F) \wedge \bigwedge_{w^k \in \{0, 1\}^k \setminus T_k} \left( H(F|V_{i\{w_i=1\}}, \bar{V}_{i\{w_i=0\}}, W^k) = 0 \right).
\]

Refer to Figure 3 for an illustration. Note that the “$\wedge$” is a conjunction of finitely many choices of $w^k$ (that does not depend on the random variables), and hence the resultant predicate is still an AEIP. The last condition ensures that $P(W^k = w^k) = 0$ for $w^k \notin T_k$, and $W^k \in T_k$ with probability 1.
C. Color Constraints on Edges

We now consider applying the colors to vertices. Let $X$ be a random variable representing a uniformly chosen random vertex. We assign a color to each vertex (i.e., the color $W^k$ is a function of $X$), as given by the following AEIP:

$$\text{COLD}(X, W^k, V^k, \tilde{V}^k, F) :$$
$$\text{COL}(W^k, V^k, \tilde{V}^k, F)$$
$$\land H(W^k|X) = I(V^k, \tilde{V}^k, F; X|W^k) = 0.$$  

Let $W^k = w^k(X)$ be a function of $X$. Let $E$ be a random variable where $H(E|X) = 0$ and $p_{X|E=e}$ is uniform over $l$ values for any $e$. Regard $E$ as groups of vertices of size $l$. Letting $p_{X|E=e} = \text{Unif}\{x(e,1), \ldots, x(e,l)\}$, we have the conditional distribution

$$(F, V_S, \tilde{V}_S)|\{E = e\}$$
$$\sim l^{-1} \sum_{i=1}^l p_F|V_S, \tilde{V}_S|W^k=w^k(x(e,i)).$$

which is due to the conditional independence $I(V^k, \tilde{V}^k, F; X|W^k) = 0$.

We can enforce certain rules on the colors of vertices in a group. For example, to enforce that for each group, all of the vertices must have positive color (recall that $w^k$ has positive color if and only if $w_k = 0$), we can use the condition

$$H(F|V_k), \tilde{V}_0, E) = 0.$$  

This is because $V_k = (1 - W_k)F$, so if $F$ is a function of $V_k$ and $E$, we must have $W_k = 0$ with probability 1. Note that we will not actually enforce this.

Consider the case $l = 2$, and we want to enforce that each group contains two vertices of the same signs of colors. If all of the vertices have negative color, since $V_k = (1 - W_k)F = 0$, there exists $U \sim \text{Bern}(1/2)$ (independent of $E$, $V_k$) such that $H(F|V_k, E, U) = 0$ (simply take $U = F$). If exactly one of the two vertices has negative color, since $V_k = (1 - W_k)F$, we have $V_k = 0$ with probability $1/2$, and $V_k = F$ with probability $1/2$. We have $H(F|V_k, E = e) \sim \text{Bern}(1/3)$, and hence there does not exist $U \sim \text{Bern}(1/2)$ (independent of $E$, $V_k$) such that $H(F|V_k, E, U) = 0$. Hence the condition that each group contains two vertices of the same signs of colors can be checked using the AEIP $\text{SAT_{#1/2},(k)} , \emptyset(E, W^k, V^k, \tilde{V}^k, F)$, where

$$\text{SAT}_{#1/2, S, \tilde{S}}(E, W^k, V^k, \tilde{V}^k, F) :$$
$$\exists U : \text{UNIF}_2(U) \land I(U; E, V_S, \tilde{V}_{\tilde{S}}) = 0$$
$$\land H(F|V_S, \tilde{V}_{\tilde{S}}, E, U) = 0. \quad (4)$$

This means that for each group, the number of vertices $w^k$ with $\text{sat}(w^k, S, \tilde{S}) = 1$ cannot be exactly one (out of $l = 2$ vertices). Refer to Figure 4 for an illustration.

Now consider $l = 2$ and we want to enforce that the number of vertices $w^k$ with $\text{sat}(w^k, S, \tilde{S}) = 1$ in each group is at most one. Using a similar argument, we have

$$\text{SAT}_{\leq 1/2, S, \tilde{S}}(E, W^k, V^k, \tilde{V}^k, F) :$$
$$\exists U : \text{UNIF}_2(U) \land I(U; E, V_S, \tilde{V}_{\tilde{S}}) = 0$$
$$\land H(F|V_S, \tilde{V}_{\tilde{S}}, E, U) = 0. \quad (5)$$

Note that a function of a random variable following $\text{Unif}([3])$ can be $\text{Bern}(1/3)$, but cannot be $\text{Bern}(1/2)$.

Now consider $l = 4$ and we want to enforce that the number of vertices $w^k$ with $\text{sat}(w^k, S, \tilde{S}) = 1$ in each group is at most three. Note that $F = 1$ if $V_S \neq 0$ or $\tilde{V}_{\tilde{S}} \neq 0$, and $F|\{V_S = 0, \tilde{V}_{\tilde{S}} = 0, E = e\} \sim \text{Bern}(a_e/(4 + a_e))$, where $a_e$ is the number of vertices $w^k$ with $\text{sat}(w^k, S, \tilde{S}) = 1$ in the group $E = e$. Using a similar argument, we have

$$\text{SAT}_{\leq 3/4, S, \tilde{S}}(E, W^k, V^k, \tilde{V}^k, F) :$$
$$\exists U : \text{UNIF}_{103}(U) \land I(U; E, V_S, \tilde{V}_{\tilde{S}}) = 0$$
$$\land H(F|V_S, \tilde{V}_{\tilde{S}}, E, U) = 0. \quad (6)$$

Note that a function of a random variable following $\text{Unif}([103])$ can be $\text{Bern}(1/5)$, $\text{Bern}(1/3)$ or $\text{Bern}(3/7)$, but cannot be $\text{Bern}(1/2)$.  

### D. Construction of Colored Tori

For \( k \geq 1 \), the **colored tori** is defined as

\[
\begin{align*}
C_{\text{TORI}}(X^2, Y^2, W^k, V^k, \bar{V}^k, F) : \\
\text{TORI}(X^2, Y^2) \\
\wedge \text{COLD}((X^2, Y^2), W^k, V^k, \bar{V}^k, F).
\end{align*}
\]

Write \( R \) for the support of \( p_{X^2, Y^2} \). Since \( W_i \) is a function of \( X^2, Y^2 \), we denote that function as \( w_i(x_1, x_2, y_1, y_2) = w_i(x^2, y^2) \), and let \( w^k(x^2, y^2) = \{ w_i(x^2, y^2) \}_{i \in [k]} \) for \( (x^2, y^2) \in \mathbb{R} \). Each vertex of the tori is assigned one of \( 2(k-1) \) colors.

We remark that since (3) enforces \( \text{UNIF}_2(W_i) \), the number of vertices with \( w_i(x^2, y^2) = 0 \) must be equal to the number of vertices with \( w_i(x^2, y^2) = 1 \). This is satisfied if for each \( j \in [k-1] \), the number of vertices with color \( j \) is equal to the number of vertices with color \( -j \). This will be addressed in Section IV-E.

Conditioned on \( (X_1, X_2, Y_1) = (x_1, x_2, y_1) \), the distribution of \( (X^2, Y^2) \) is uniform over two values. The tuple \( (x_1, x_2, y_1) \) corresponds to an edge in the tori, which we call a **vertical edge** (since the two vertices have different vertical coordinates \( y_2 \)). We can check whether the two vertices have the same signs of colors using (4):

\[
\text{SAT}_{\neq 1/2, \{k\}, \emptyset}((X_1, X_2, Y_1), W^k, V^k, \bar{V}^k, F).
\]

Similar for conditioning on \( (X_1, X_2, Y_2) \) (vertical edge), \( (X_1, Y_1, Y_2) \) (horizontal edge) or \( (X_2, Y_1, Y_2) \) (horizontal edge). Hence we can use AEIPs to force the signs of colors to be constant within the same torus, and we call it the sign of the torus. Note that different tori within the collection of tori may have different signs.

We now require that \( k - 1 \geq 8 \) is a multiple of 4. Divide the colors into 4 groups according to the remainder modulo 4 of their absolute values (i.e., color \( j \) is in group \( [j] \mod 4 \), where we assume \( a \mod 4 \in [4] \) instead of \([0..3]\)). We enforce that each horizontal edge either connect a group 1 vertex and a group 2 vertex (the group of a vertex is the group of its color), or connect a group 3 vertex and a group 4 vertex. Also, each vertical edge either connect a group 1 vertex and a group 4 vertex, or connect a group 2 vertex and a group 3 vertex. Note that \( \text{sat}(w^k, [0..3]) = \text{sat}(\bar{w}^k, [0..3]) = 1 \) if and only if

![Figure 4. An illustration of SAT\(_{\neq 1/2, \{k\}, \emptyset}\) (second row) and SAT\(_{\leq 1/2, \{k\}, \emptyset}\) (third row) for \( k = 4 \). Refer to Figure 3 for the meaning of the axes. For the first case (first column) where the two sequences \( W^k \) on the two vertices are both positive, we have \( F = V_4 \), and hence there exists \( U \sim \text{Bern}(1/2) \) (independent of \( V_4 \)) where \( F \) is a function of \((V_4, U)\), and SAT\(_{\neq 1/2, \{k\}, \emptyset}\) is true (the hatch patterns in the figure represent different values of \( U \)). For the second case (second column) where the two sequences \( W^k \) have different signs, we have \( F|\{V_4 = 0\} \sim \text{Bern}(1/3) \) and \( F = 1 \) when \( V_4 = 1 \), and hence there does not exist such \( U \sim \text{Bern}(1/2) \), and SAT\(_{\neq 1/2, \{k\}, \emptyset}\) is false. For the third case (third column) where the two sequences \( W^k \) are both negative, we have \( V_4 = 0 \) independent of \( F \sim \text{Bern}(1/2) \), and hence there exists such \( U \sim \text{Bern}(1/2) \) (take \( U = F \)), and SAT\(_{\neq 1/2, \{k\}, \emptyset}\) is true. The cases for SAT\(_{\leq 1/2, \{k\}, \emptyset}\) are similar.](image)
where $a$ faces in the N, E, S, W directions respectively. The grid is rotated 45° of the torus (see Figure 5). Note that the N, E, S, W directions are rotated face 22 face of the edges.

We use type conditioned on $(X_1, Y_1) = (x_1, y_1)$, the distribution of $(X^2, Y^2)$ is uniform over 4 values. We call these 4 vertices a type 11 face. Due to the aforementioned constraint on the groups, each face has vertices of all 4 groups. Regarding the group 1, 2, 3, 4 vertices of a type 11 face as the north (N), east (E), south (S) and west (W) directions respectively, we can fix the orientation of the torus (see Figure 5). Note that the N, E, S, W directions are rotated 45° compared to the horizontal/vertical directions of the edges.

Conditioned on $(X_2, Y_2) = (x_2, y_2)$, the distribution of $(X^2, Y^2)$ is uniform over 4 values. We call these 4 vertices a type 22 face. Note that the group 1, 2, 3, 4 vertices of a type 22 face are pointing to the S, W, N and E directions respectively.

We use type 11 and type 22 faces as our grid (ignore type 12 and type 21 faces). Each type 11 face has 4 neighboring type 22 faces in the N, E, S, W directions respectively. The grid is rotated 45° compared to the horizontal/vertical directions of the edges.

E. Reduction from the Periodic Tiling Problem

We now show a reduction from the periodic tiling problem. A tiles can be represented as an integer sequence of length 4, with entries representing the colors of the N,E,S,W edges respectively. Assume there are $(k-1)/4$ different colors among the edges of tiles (tile colors), and let the set of tiles be $C \subseteq \left[\left((k-1)/4\right)\right]$. Assume $k \geq 8$ without loss of generality. Recall that the number of colors of vertices of the tori is $2(k-1)$ (vertex colors). Each tile color $j \in \left[\left((k-1)/4\right)\right]$ corresponds to 8 vertex colors: $4j - 3, 4j - 2, 4j - 1, 4j, -(4j - 3), -(4j - 2), -(4j - 1), -4j$, one per combination of sign and group.

To ensure that each type 11 face has colors belonging to one of the tiles in $C$, the set of absolute values of colors of the 4 vertices must be in the form

$$\{4c_1 - 3, 4c_2 - 2, 4c_3 - 1, 4c_4\}$$

for some $c \in C$ (e.g. $4c_1 - 3$ is a positive group 1 color, which appears in the north vertex of a type 11 face in a positive torus, so this means the north vertex has a vertex color that corresponds to the tile color $c_1$). Rotations and reflections are disallowed since the vertex colors contain group (and hence orientation) information. Write $C_{11} := \{\{4c_1 - 3, 4c_2 - 2, 4c_3 - 1, 4c_4\} : c \in C\}$.

Similarly, to enforce that each type 22 face has colors belonging to $C$, the set of absolute values of colors must be in the form

$$\{4c_1 - 1, 4c_2, 4c_3 - 3, 4c_4 - 2\}$$
for some \( c \in \mathcal{C} \). Define \( \mathcal{C}_{22} \) similarly. These constraints can be enforced in a similar manner as \([\text{7}]\) using \([\text{6}]\). The constraint that touching edges of adjacent tiles match is automatically enforced since two adjacent faces share a vertex. Therefore, the final AEIP is given by

\[
\text{TTORI}_C : \quad \exists X^2, Y^2, W^k, V^k, \bar{V}^k, F : \\
\text{OTORI}(X^2, Y^2, W^k, V^k, \bar{V}^k, F) \\
\land \quad \bigwedge_{j_1, \ldots, j_4 \in \Z, \not\equiv 0} \left( \left( \text{SAT} \leq 3/4, 0, [k] \right) \land \left( (X_1, Y_1), W^k, V^k, \bar{V}^k, F \right) \land \left( (X_2, Y_2), W^k, V^k, \bar{V}^k, F \right) \right) \\
\land \quad \bigwedge_{j_1, \ldots, j_4 \in \Z, \not\equiv 0} \left( \left( \text{SAT} \leq 3/4, 0, [k] \right) \land \left( (X_1, Y_1), W^k, V^k, \bar{V}^k, F \right) \land \left( (X_2, Y_2), W^k, V^k, \bar{V}^k, F \right) \right).
\]

Finally, we show that \( \text{TTORI}_C \) is true if and only if the tiles in \( \mathcal{C} \) can tile the plane periodically. For the “only if” direction, assume \( \text{TTORI}_C \) is true. Consider any one of the torus, and assume the set of values of \( (X^2, Y^2) \) over that torus is

\[
\left( \bigcup_{i \in \Z} \{ (i, i), (i, i + 1) \} \right) \times \left( \bigcup_{j \in \Z} \{ (j, j), (j, j + 1) \} \right)
\]

without loss of generality, where \( \Z = \Z/a\Z \) is the cyclic group of order \( a \). For simplicity, we relabel the elements of the above set by the mapping \( (i_1, i_2, j_1, j_2) \mapsto (i_1 + i_2, j_1 + j_2) \in \Z_{2a} \times \Z_{2b} \), so we have a coloring of the torus \( \Z_{2a} \times \Z_{2b} \). Repeat this coloring \( l/a \) times horizontally and \( l/b \) times vertically, where \( l = \text{lcm}(a, b) \), to obtain a coloring of the torus \( \Z_{2l} \times \Z_{2l} \). We call this set the colored torus. Note that each type 11 face has a set of four vertices in the form \( \{2i, 2i + 1\} \times \{2j, 2j + 1\} \), and each type 22 face has a set of four vertices in the form \( \{2i + 1, 2i + 2\} \times \{2j + 1, 2j + 2\} \). Therefore each type 11 or 22 face (we call this an even face) has a set of four vertices in the form \( \{i, i + 1\} \times \{j, j + 1\} \) where \( i + j \) is even. We call this the even face at position \( (i, j) \in \text{even}(\Z_{2l}^2) = \{ (i', j') \in \Z_{2l}^2 : i' + j' \text{ even} \} \). The even face at \( (i, j) \) has neighbors (i.e., the other even faces that shares one vertex with this even face) \( (i \pm 1, j \pm 1) \). Define the mapping \( f : \Z_{2l}^2 \to \text{even}(\Z_{2l}^2) \) from the tiling torus \( \Z_{2l}^2 \) to the even faces of the colored torus, by \( f(u, v) := (u + v, u - v) \). Note that the even face \( f(u, v) \) has neighbors \( f(u + 1, v), f(u - 1, v), f(u, v + 1), f(u, v - 1) \). We now define a tiling over the tiling torus \( \Z_{2l}^2 \), where each point \( (u, v) \in \Z_{2l}^2 \) corresponds to a square tile, and the right edge of this tile has color which equals to the color of the common vertex between even face \( f(u, v) \) and \( f(u + 1, v) \) in the colored torus (this vertex is \( (u + v + 1, u - v + 1) \)), and similar for the other edges. Hence we have a tiling of the plane with period \( 2l \). Refer to the top figure in Figure [7].

![Figure 5. The oriented tori. The label on each vertex is its group. Solid grey squares are type 11 faces, whereas hatched grey squares are type 22 faces.](image)
For the “if” direction, assume the tiles in \( C \) can tile the plane periodically, and let the period be \( a \) (i.e., it tiles \( \mathbb{Z}_a^2 \)). Using a similar argument as the “only if” part, we have a coloring over \( \mathbb{Z}_a^2 \). To ensure that for each \( j \in [k] \), the number of vertices with color \( j \) is equal to the number of vertices with color \(-j\), so \( \text{UNIF}_2(W_j) \) in (3) is satisfied, we use two copies of \( \mathbb{Z}_a^2 \), one with positive colors and the other with negative colors. The collection of tori in \((X^2, Y^2)\) consists of these two tori. Refer to the bottom figure in Figure 7.

For the second part of Theorem 1, the undecidability of the truth value of \( \exists U^i : \text{Ah}(U^i) \succeq b \) follows from a reduction from \( \exists U^i : \text{Ah}(U^i) \succeq b \) if and only if \( \exists U^i, V^m : \text{Ah}(U^i) - [H(V_1), \ldots, H(V_m)]^T = b \). This completes the proof of Theorem 1.

V. Undecidability of Conditional Independence Implication with a Binary Constraint

In this section, we show that the above proof also establish the following result on the undecidability of the existence of random variables satisfying some conditional independence relations and the constraint that the first random variable is a binary uniform random variable.

In the study of conditional independence implication, some authors allow the three sets \( A_j, B_j, C_j \) of random variables (in the conditional independence relation \( I(X_{A_j}; X_{B_j} | X_{C_j}) = 0 \)) to be non-disjoint \([36, 55, 42] \), whereas some authors require those sets to be disjoint \([39, 56] \) (call \( I(X_{A_j}; X_{B_j} | X_{C_j}) = 0 \) a disjoint conditional independence relation in this case). The non-disjoint case allows functional dependency in the form \( H(X_{A_j} | X_{C_j}) = 0 \) (by letting \( B_j = A_j \)), whereas the disjoint case disallows this. We will argue in Section V.B that the two cases are Turing equivalent by showing a reduction from the non-disjoint case to the disjoint case. This appears to be a new result, considering \([56] \) (remark after Lemma 6) explicitly disallow non-disjoint sets of random variables on the ground that functional dependency is fundamentally different from conditional independence relations (our result shows that this is not true, as one can express functional dependency using conditional independence).

We first prove the non-disjoint case, and then prove the disjoint case.

A. Non-disjoint Case

We first prove the following result that allows non-disjoint \( A_j, B_j, C_j \).

**Theorem 2.** The problem of deciding whether there exists random variables \( X^n \) such that \( I(X_{A_j}; X_{B_j} | X_{C_j}) = 0 \) for all \( j \in [m] \) and \( X_1 \sim \text{Bern}(1/2) \), where \( A_j, B_j, C_j \subseteq [n] \), \( n \) and \( m \) are given, is undecidable.
Proof: Note that throughout the proof of Theorem 1, we have only used conditional independence constraints in the form $I(X; Y|Z) = 0$ (note that $H(X|Z) = I(X; X|Z)$; this also includes UNIF$(X)$, and UNIF$_a(Y)$ for $a \in \{2, 3, 4, 105\}$. In this section, we will use only conditional independence constraints and UNIF$_2$, and show that these constraints suffice.

Using (1), we can define the following AEIP that checks whether $Y,Z$ are both uniform and have the same cardinality:

$$\exists U^3: \text{TRIPLE}(Y,U_1,U_2) \land \text{TRIPLE}(Z,U_3,U_2).$$

This is because if $Y,Z$ are both uniform and $\text{card}(Y) = \text{card}(Z)$, we can let $U_1$ independent of $(Y,Z)$ and has the same distribution, and there exists $U_2,U_3$ such that TRIPLE$(Y,U_1,U_2) \land \text{TRIPLE}(X_1,U_1,U_3)$ holds.

Then define the following AEIP that checks whether the random variables $Y_1,\ldots,Y_l,G$ are uniform, and $\prod_i \text{card}(Y_i) = \text{card}(G)$:

$$\exists Z^l,U: \bigwedge_i (\text{UNIF}=(Y_i; Z_i) \land I(Z_i; Z_i^{-1}) = 0)$$

$$\land \text{UNIF}=(G; U) \land H(U|Z^l) = H(Z^l|U) = 0.$$

The following AEIP checks whether $Y,G$ are uniform and $(\text{card}(Y))^k = \text{card}(G)$:

$$\exists \prod_{k}(Y; G) : \text{PROD}((Y,\ldots,Y); G).$$
The following AEIP checks whether $Y, G$ are uniform, $b := \text{card}(Y)$ is a divisor of $a := \text{card}(G)$, and $b \geq \sqrt{a}$:

\[
\text{GESQRT}(Y; G) :
\exists Z, W, U, V : \text{UNIF}_a(Y; Z)
\land \text{UNIF}(W) \land I(W; Z) = 0
\land \text{UNIF}_a(G; U) \land H(U|Z, W) = H(Z, W|U) = 0
\land \text{UNIF}(Z; V) \land H(U|Z, V) = 0.
\]

To check that this AEIP implies $b \geq \sqrt{a}$, note that $\text{card}(V) = b$, and hence $\text{card}((Y, V)) \leq b^2$. Since $H(U|Y, V) = 0$, we have $a \leq b^2$. To check that $b \geq \sqrt{a}$ implies this AEIP, assume $U = (U_1, U_2) \sim \text{Unif}([0, a/b])$. Take $Z = U_1, V = U_1 + U_2 \mod a$ (assume mod takes values in $[a]$ instead of $[0..a-1]$). We have $Z, V \sim \text{Unif}([b])$ and $H(U|Z, V) = 0$, and the AEIP holds.

Using this, we can check whether $Y, Z$ are uniform and $\text{card}(Y) \leq \text{card}(Z)$:

\[
\text{LE}(Y; Z) :
\exists U : \text{PROD}(U; Y, Z) \land \text{GESQRT}(Z; U).
\]

Let $p_k, q_k$ be positive integers such that $\log(k - 1) < p_k/q_k < \log k$ for $k \geq 2$. Then $k$ is the only positive integer $i$ satisfying $2^{p_k} \leq i^{p_k}$ and $i^{p_{k+1}} \leq 2^{p_{k+1}}$. Hence we can state $\text{UNIF}_k$ as

\[
\text{UNIF}_k(Y) :
\exists U, V^2, W^2 : \text{UNIF}_2(U) \land \text{UNIF}(Y)
\land \text{POW}_{p_k}(U; V_1) \land \text{POW}_{q_k}(Y; W_1) \land \text{LE}(V_1; W_1)
\land \text{POW}_{p_{k+1}}(U; V_2) \land \text{POW}_{q_{k+1}}(Y; W_2) \land \text{LE}(V_2; W_2).
\]

Therefore, the undecidable AEIP in Theorem 1 can be stated using only conditional independence constraints and $\text{UNIF}_2$. To complete the proof, note that if $X_1 \sim \text{Bern}(1/2)$, then we can check $\text{UNIF}_2(Y)$ by $\text{UNIF}_a(Y; X_1)$, and hence one uniform bit suffices. 

As a result, we have the undecidability of the following implication problem.

**Corollary 3.** Fix any integer $r \geq 2$. The problem of deciding whether the following implication holds: $I(X_{A_j}; X_{B_j}|X_{C_j}) = 0$ for all $j \in [m]$ and $\text{card}(X_1) \leq r$ implies $H(X_1) = 0$, where $A_j, B_j, C_j \subseteq [n], n$ and $m$ are given, is undecidable.

**Proof:** Let $\hat{p}_3, \hat{q}_3, \hat{p}_4, \hat{q}_4$ be integers such that $\hat{p}_3/\hat{q}_3 < \log 3 < \hat{p}_4/\hat{q}_4$, and there is no positive integer $k$ such that $\hat{p}_3/\hat{q}_3 < (\log k)/(\log 3) < \hat{p}_4/\hat{q}_4$. We define the following AEIP that checks whether $Y$ is uniform and $\text{card}(Y) \leq 2$, given that $\text{card}(Y) \leq 3$:

\[
\text{UNIF}_{\leq 2\leq 3}(Y) :
\exists U, V, W : \text{UNIF}(Y) \land \text{UNIF}(U)
\land \text{POW}_{\hat{p}_3}(Y; V_1) \land \text{POW}_{\hat{q}_3}(U; W_1) \land \text{LE}(V_1; W_1)
\land \text{POW}_{\hat{p}_4}(Y; V_2) \land \text{POW}_{\hat{q}_4}(U; W_2) \land \text{LE}(V_2; W_2).
\]

This is similar to [8] except that $Y, U$ are swapped, and we have $\text{UNIF}(Y)$ instead of $\text{UNIF}_2(Y)$. Hence, if $\text{card}(Y) = 2$, then the AEIP is satisfied by $\text{card}(U) = 3$. If $\text{card}(Y) = 3$, then since $\hat{p}_3/\hat{q}_3 < (\log k)/(\log 3) < \hat{p}_4/\hat{q}_4$ has no solution, the AEIP is not satisfied.

We show the corollary by a reduction from the decision problem in Theorem 2

\[
\neg \left( \exists X^n : X_1 \sim \text{Bern}(1/2) \land \forall j \in [m] : I(X_{A_j}; X_{B_j}|X_{C_j}) = 0 \right)
\]

\[
\iff \forall X^n : \left( \left( X_1 \sim \text{Bern}(1/2) \land \bigwedge_{j \in [m]} (I(X_{A_j}; X_{B_j}|X_{C_j}) = 0) \right) \rightarrow H(X_1) = 0 \right)
\]

\[
\iff \forall X^n, Y : \left( X_1 \sim \text{Bern}(1/2) \land \text{POW}_{\log r}(Y; X_1) \land \text{card}(Y) \leq r \land \text{UNIF}_{\leq 2\leq 3}(X_1) \land \bigwedge_{j \in [m]} (I(X_{A_j}; X_{B_j}|X_{C_j}) = 0) \rightarrow H(Y) = 0 \right)
\]

\[
\iff \forall X^n, Y : \left( (\text{POW}_{\log r}(Y; X_1) \land \text{card}(Y) \leq r \land \text{UNIF}_{\leq 2\leq 3}(X_1)
\right)
\[ \bigwedge_{j \in [m]} (I(X_{A_j}; X_{B_j}|X_{C_j}) = 0) \rightarrow H(Y) = 0 \],

which is a statement in the form given in the corollary, where (a) is because \( P \rightarrow Q \equiv \neg P \lor Q \equiv \neg P \) if \( P \rightarrow \neg Q \) holds, and (b) is because \( \text{card}(Y) \leq r \) and \( \text{UNIF}_{\leq 3}(X_1) \) are implied by \( X_1 \sim \text{Bern}(1/2) \) and are redundant. To show (c), since \( (P \land R) \rightarrow Q \equiv \neg P \lor Q \equiv \neg P \) if \( P \rightarrow (Q \lor R) \) holds, it suffices to show that \( \text{POW}_{\log r}(X_1; Y) \), \( \text{card}(Y) \leq r \) and \( \text{UNIF}_{\leq 3}(X_1) \) implies \( H(Y) = 0 \) or \( \text{card}(X_1) = 2 \). If \( \text{POW}_{\log r}(X_1; Y) \), \( \text{card}(Y) \leq r \) and \( \text{UNIF}_{\leq 3}(X_1) \), we have \( (\text{card}(X_1))^{\log r} = \text{card}(Y) \leq r \), and \( \text{card}(X_1) \leq r^{1/\log r} \). It is straightforward to check that \( r^{1/\log r} < 4 \) if \( r \geq 2 \), and hence \( \text{card}(X_1) \in \{1, 2, 3\} \). Since \( \text{UNIF}_{\leq 3}(X_1) \) holds, we have \( \text{card}(X_1) \in \{1, 2, 3\} \), and the result follows. \( \blacksquare \)

Note that Corollary 3 implies that there does not exist a program that, when given a list of conditional independence constraints, outputs the minimum possible \( \text{card}(X_1) \) other than 1 (i.e., the second smallest possible \( \text{card}(X_1) \); in case \( \text{card}(X_1) \) can only be 1, the program should output 1).

**B. Disjoint Case**

We show that the disjunct case of the conditional independence implication problem is equivalent to the non-disjoint case.

**Theorem 4.** For any \( n, m \) and (not necessarily disjoint) \( A_j, B_j, C_j \subseteq [n] \) for \( j \in [0..m] \), there exists \( \tilde{n}, \tilde{m} \) and disjoint \( \tilde{A}_j, \tilde{B}_j, \tilde{C}_j \subseteq [\tilde{n}] \) for \( j \in [0..\tilde{m}] \) such that the implication \( \bigwedge_{j \in [m]} (I(X_{\tilde{A}_j}; X_{\tilde{B}_j}|X_{\tilde{C}_j}) = 0) \Rightarrow (I(X_{\tilde{A}_1}; X_{\tilde{B}_1}|X_{\tilde{C}_1}) = 0) \) holds if and only if the implication \( \bigwedge_{j \in [\tilde{m}]} (I(Y_{\tilde{A}_j}; Y_{\tilde{B}_j}|Y_{\tilde{C}_j}) = 0) \Rightarrow (I(Y_{\tilde{A}_1}; Y_{\tilde{B}_1}|Y_{\tilde{C}_1}) = 0) \) holds. Moreover, the sets \( \tilde{A}_j, \tilde{B}_j, \tilde{C}_j \) can be computed in polynomial time (with respect to the input size \( mn \)).

**Proof:** Two random variables \( Y, Z \) are called **perfectly resolvable** if there exists \( X \) such that \( I(Y; Z|X) = H(X|Y) = H(X|Z) = 0 \) [78]. Define

\[
\text{RESC}(Y, Z, X) : I(Y; Z|X) = H(X|Y) = H(X|Z) = 0,
\]

\[
\text{RES}(Y, Z) : \exists X : \text{RESC}(Y, Z, X).
\]

Note that if \( Y, Z \) are perfectly resolvable, then \( X \) is uniquely determined from \( Y, Z \) since it is the Gács-Körner common part [79] of \( Y \) and \( Z \).

For each \( X_i \), let \( Y_i, Z_i \) be random variables such that they are perfectly resolvable and \( \text{RESC}(Y_i, Z_i, X_i) \) holds. To prove Theorem 4, we will express (not necessarily disjoint) conditional independence relations \( I(X_{A_i}; X_{B_i}|X_{C_i}) = 0 \) using disjoint conditional independence relations on \( \{Y_i, Z_i\} \).

First, we redefine \( \text{RES}(Y, Z) \) using only disjoint conditional independence relations. Let

\[
\text{RES3}(Y^3) : I(Y_1; Y_2|Y_3) = I(Y_2; Y_3|Y_1) = I(Y_3; Y_1|Y_2) = 0.
\]

We will show that \( \text{RES}(Y, Z) \) can be equivalently stated as

\[ \exists U : \text{RES3}(Y, Z, U). \]

The direction \( \text{RES}(Y, Z) \Rightarrow \exists U : \text{RES3}(Y, Z, U) \) follows from taking \( U = X \). To prove the other direction, assume \( \text{RES3}(Y, Z, U) \). Invoke the double Markov property [72]:

\[
I(U; Y|Z) = I(U; Z|Y) = 0
\]

\[ \Rightarrow \exists V : H(V|Y) = H(V|Z) = I(Y; Z; U|V) = 0. \]

This together with \( I(Y; Z|U) = 0 \) implies \( I(Y; Z|U) = 0 \). Hence \( \text{RES}(Y, Z) \) follows from taking \( X = V \). Note that this also implies \( H(V|U) = 0 \), and hence \( V \) is the Gács-Körner common part of \( X \) and \( Y \), that of \( X \) and \( U \), and also that of \( Y \) and \( U \). Write

\[
\text{EQ}(F, G) : H(F|G) = H(G|F) = 0
\]

for the condition that \( F, G \) are informationally equivalent. We then express \( \text{EQ}(X_i, X_j) \) for distinct \( i, j \) using disjoint conditional independence relations. We will prove that \( \text{EQ}(X_1, X_2) \) if and only if

\[
\text{EQRES}(Y_1, Z_1, Y_2, Z_2) : \\
\exists U^2 : \text{RES3}(Y_1, Z_1, U_1), \land \text{RES3}(Z_1, U_1, U_2) \\
\land \text{RES3}(U_1, U_2, Y_2) \land \text{RES3}(U_2, Y_2, Z_2).
\]

Recall that \( \text{RES3}(Y_i, Z_i, X_i) \) holds. For the “only if” part, take \( U_1 = U_2 = X_1 \). For the “if” part, we have shown that \( \text{RES3}(Y_1, Z_1, U_1) \) implies that the common part of \( Y_1 \) and \( Z_1 \) is the same as that of \( Z_1 \) and \( U_1 \), and hence the common part
of $Z_1$ and $U_1$ is also $X_1$. Repeating this argument, we deduce that the common part of $Y_2$ and $Z_2$ is also $X_1$. The result follows.

We will create 3 copies of $X^n$. Let $Y^{3n}, Z^{3n}$ satisfy $EQRES(Y_i, Z_i, Y_{i+n}, Z_{i+n})$ for $i \in [2n]$. This ensures that if $RESC(Y_i, Z_i, X_i)$, then $EQ(X_i, X_{i+n})$ for $i \in [2n]$ (i.e., $X_i, X_{i+n}, X_{i+2n}$ are informationally equivalent).

After copying, we can express the (not necessarily disjoint) conditional independence relations $I(X_A; X_B | X_C) = 0$ as $I(X_A; X_{B+n} | X_{C+2n}) = 0$, where we write $B + n = \{b + n : b \in B\}$, ensuring the three sets $A, B + n, C + 2n$ are disjoint.

To complete the proof, we express the conditional independence relation $I(X_A; X_B | X_C) = 0$ ($A, B, C \subseteq [n]$) using disjoint conditional independence relations on $Y^{3n}, Z^{3n}$. Without loss of generality, we can assume

$$I(Y_i; Y_{3n-i} | i, Z_{3n-i} | i) = I(Z_i; Y_{3n-i} | i, Z_{3n-i} | i) = 0$$

for $i \in [3n]$. It is always possible to find such $Y^{3n}, Z^{3n}$ given $X^n$ since we can take $Y_i = Z_i = X_i$. Given this assumption, for $A, B, C \subseteq [n]$, $I(X_A; X_B | X_C) = 0$ holds if and only if

$$I(Y_A; Y_{B+n} | Y_{C+2n}) = 0.$$ 

To show this, first we have $I(X_A; X_B | X_C) = 0 \iff I(X_A; X_{B+n} | X_{C+2n}) = 0$. Let $a \in A$. Since

$$I(Y_a; Y_{(B+n)\cup(C+2n)}, Z_{(B+n)\cup(C+2n)} | a) = 0,$$

and $H(X_a | Z_a) = I(Y_a; Z_a | a) = 0$ (due to $RESC(Y_a, Z_a, X_a)$), we have

$$I(Y_a; Y_{(B+n)\cup(C+2n)}, Z_{(B+n)\cup(C+2n)} | a) = 0.$$ 

This implies that $I(X_A; X_{B+n} | X_{C+2n}) = 0 \iff I(X_A \setminus a; X_{B+n} | X_{C+2n}) = 0$. Repeating this argument (note that this argument is also valid for $X_{C+2n}$), we have $I(X_A; X_{B+n} | X_{C+2n}) = 0 \iff I(Y_A; Y_{B+n} | Y_{C+2n}) = 0$. In sum, the implication

$$\bigwedge_{i \in [m]} (I(X_{A_i}; X_{B_j} | X_{C_j}) = 0 \implies I(X_{A_i}; X_{B_0+n} | X_{C_0+2n}) = 0)$$

holds if and only if the following implication holds:

$$\bigwedge_{i \in [2n]} EQRES(Y_i, Z_i, Y_{i+n}, Z_{i+n})$$

and these disjoint conditional independence relations hold if and only if $I(X_{A_i}; X_{B_j} | X_{C_j}) = 0$ for all $j \in [m]$ and $card(X_i) \leq r$ implies $I(Y_i; Z_i) = 0$ holds if and only if the implication

$$I(X_{A_j}; X_{B_j} | X_{C_j}) = 0$$

for all $j \in [m] and card(X_i) \leq r$ implies $H(X_i) = 0$ holds.

We then prove the disjoint case of Corollary 5 by a reduction from the non-disjoint case.

**Corollary 5.** Fix any integer $r \geq 2$. The problem of deciding whether the following implication holds: $I(X_{A_j}; X_{B_j} | X_{C_j}) = 0$ for all $j \in [m]$ and $card(X_i) \leq r$ implies $I(X_1; X_2) = 0$, where the three sets $A_j, B_j, C_j \subseteq [n]$ are disjoint for any $j$, and $n \geq 2$ and $m$ are given, is undecidable.

**Proof:** We show a reduction from Corollary 5. Let $A_j, B_j, C_j \subseteq [n]$ be not necessarily disjoint sets. By the construction in Theorem 4, letting $Y_i, Z_i$ be random variables such that $RESC(Y_i, Z_i, X_i)$ holds, we can find disjoint conditional independence relations on $Y^n, Z^n, U^n$ (where $U^n$ are some extra auxiliary random variables) such that there exists $Y^n, Z^n, U^n$ such that $RESC(Y_i, Z_i, X_i)$ and these disjoint conditional independence relations hold if and only if $I(X_{A_j}; X_{B_j} | X_{C_j}) = 0$ for all $j$. The cardinality constraint $card(X_i) \leq r$ can be enforced by $card(Y_i) \leq r$ (since $X_i$ is a function of $Y_i$). The consequent $H(X_i) = 0$ can be expressed as $I(Y_i; Z_i) = 0$. Therefore, the implication “$RESC(Y_i, Z_i)$ for all $i \in [n]$, the aforementioned disjoint conditional independence relations hold, and $card(X_i) \leq r$ implies $I(Y_i; Z_i) = 0$ holds if and only if the implication

$I(X_{A_j}; X_{B_j} | X_{C_j}) = 0$ for all $j \in [m]$ and card$(X_i) \leq r$ implies $H(X_i) = 0$ holds.”
VI. UNDECIDABILITY OF RELATED PROBLEMS

We establish several undecidability results as corollaries of Theorem 1 and Theorem 2.

Corollary 6. The problem of deciding the truth value of \( v \in \Gamma_n^* \land a^T v \leq 0 \land v_{(1)} \leq 1 \Rightarrow v_{(1)} = 0 \), where \( a \in \mathbb{Q}^{2^n-1} \) and \( v_{(1)} \) denotes the entry of \( v \) that represents the entropy of the first random variable, is undecidable.

Proof: We show a reduction from the decision problem in Theorem 2 to deciding whether there exists random variables \( X^n \) such that \( I(X_{A_j}; X_{B_j}|X_{C_j}) = 0 \) for all \( j \in [m] \) and \( X_1 \sim \text{Bern}(1/2) \). Since UNIF(\( X_1 \)) can be expressed as conditional independence constraints, assume it is expressed as

\[
\exists X''_{n+1}: I(X_{A_{m+1}}; X_{B_{m+1}}|X_{C_{m+1}}) = \cdots = I(X_{A_{m'}}; X_{B_{m'}}|X_{C_{m'}}) = 0.
\]

Since \( \text{UNIF}(X_1) \land 0 < H(X_1) \leq 1 \) is equivalent to \( X_1 \sim \text{Bern}(1/2) \), we have

\[
\exists X^n: X_1 \sim \text{Bern}(1/2) \land \bigwedge_{j \in [m]} (I(X_{A_j}; X_{B_j}|X_{C_j}) = 0)
\]

\[
\Leftrightarrow \exists X''^n: 0 < H(X_1) \leq 1 \land \bigwedge_{j \in [m']} (I(X_{A_j}; X_{B_j}|X_{C_j}) = 0)
\]

\[
\Leftrightarrow \exists X^n': 0 < H(X_1) \leq 1 \land \sum_{j=1}^{m'} I(X_{A_j}; X_{B_j}|X_{C_j}) \leq 0
\]

\[
\Leftrightarrow \neg (\forall X^n: H(X_1) \leq 1 \land \sum_{j=1}^{m'} I(X_{A_j}; X_{B_j}|X_{C_j}) \leq 0 \Rightarrow H(X_1) = 0),
\]

which is the negation (the “\( \neg \)" sign means negation) of a statement in the form given in Corollary 6.

Corollary 7. The problem of deciding the truth value of \( \exists v \in \Gamma_n^*: a^Tv = 0 \land v_{(1)} = 1 \), where \( a \in \mathbb{Q}^{2^n-1} \) and \( v_{(1)} \) denotes the entry of \( v \) that represents the entropy of the first random variable, is undecidable.

Proof: We show a reduction from the decision problem in Theorem 2 to deciding whether there exists random variables \( X^n \) such that \( I(X_{A_j}; X_{B_j}|X_{C_j}) = 0 \) for all \( j \in [m] \) and \( X_1 \sim \text{Bern}(1/2) \). Since UNIF(\( X_1 \)) can be expressed as conditional independence constraints, assume it is expressed as

\[
\exists X''_{n+1}: I(X_{A_{m+1}}; X_{B_{m+1}}|X_{C_{m+1}}) = \cdots = I(X_{A_{m'}}; X_{B_{m'}}|X_{C_{m'}}) = 0.
\]

Since \( \text{UNIF}(X_1) \land 0 < H(X_1) \leq 1 \) is equivalent to \( X_1 \sim \text{Bern}(1/2) \), we have

\[
\exists X^n: X_1 \sim \text{Bern}(1/2) \land \bigwedge_{j \in [m]} (I(X_{A_j}; X_{B_j}|X_{C_j}) = 0)
\]

\[
\Leftrightarrow \exists X''^n: 0 < H(X_1) \leq 1 \land \bigwedge_{j \in [m']} (I(X_{A_j}; X_{B_j}|X_{C_j}) = 0)
\]

\[
\Leftrightarrow \exists X^n': 0 < H(X_1) \leq 1 \land \sum_{j=1}^{m'} I(X_{A_j}; X_{B_j}|X_{C_j}) \leq 0
\]

\[
\Leftrightarrow \neg (\forall X^n: H(X_1) \leq 1 \land \sum_{j=1}^{m'} I(X_{A_j}; X_{B_j}|X_{C_j}) \leq 0 \Rightarrow H(X_1) = 0),
\]

which is in the form given in Corollary 7.

Corollary 8. The problem of deciding the truth value of \( \forall v \in \Gamma_n^*: a_i^Tv > b_1 \lor \cdots \lor a_i^Tv > b_N \), where \( a_i \in \mathbb{Q}^{2^n-1} \) and \( b_i \in \mathbb{Q} \), is undecidable.

Proof: This can be seen by negating \( \exists U^1: A_h(U^1) \leq b \).
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