The Limiting Distribution of the Number of Block Pairs in Type $B$ Set Partitions

David G. L. Wang
Beijing International Center for Mathematical Research
Peking University, Beijing 100871, P. R. China
wgl@math.pku.edu.cn

Abstract

It is a classical result of Harper that the limiting distribution of the number of blocks in partitions of the set \{1, 2, \ldots, n\} is normal. In this paper, using the saddle point method we prove the normality of the limiting distribution of the number of block pairs in set partitions of type $B_n$. Moreover, we obtain that the limiting distribution of the number of block pairs in $B_n$-partitions without zero-block is also normal.

1 Introduction

This paper is concerned with the limiting distribution of the number of block pairs of type $B_n$ set partitions. For ordinary set partitions, Harper [11] has established the normality of the limiting distribution of the number of blocks in partitions of the set \{1, 2, \ldots, n\}. For the asymptotic behavior concerning with the ordinary set partitions, see [6,9,10,14,17,18]. For the study on the limiting distribution of other combinatorial objects, see Flajolet and Sedgewick’s book [7] for instance.

The lattice of ordinary set partitions can be regarded as the intersection lattice for the hyperplane arrangement corresponding to the root system of type $A$, see Björner and Brenti [3] or Humphreys [13]. From this point of view, type $B$ set partitions are a generalization of ordinary partitions, see Reiner [19]. To be more precise, ordinary set partitions encode the intersections of hyperplanes in the hyperplane arrangement for the type $A$ root system, while the intersections of subsets of hyperplanes from the type $B$ hyperplane arrangement can be encoded by type $B$ set partitions, see Björner and Wachs [4]. A type $B_n$ set partition is a partition $\pi$ of the set

$$\{1, 2, \ldots, n, -1, -2, \ldots, -n\}$$

such that for any block $B$ of $\pi$, $-B$ is also a block of $\pi$, and there is at most one block, called zero-block, satisfying $B = -B$. We call $(B, -B)$ a block pair of $\pi$ if $B$ is not a zero-block.

Let $M_{n,k}$ be the number of $B_n$-partitions with $k$ block pairs. It is easy to deduce the recurrence relation

$$M_{n,k} = M_{n-1,k-1} + (2k + 1)M_{n-1,k}.$$ (1.1)
The main result of this paper is to derive the limiting distribution of the number of block pairs in \( B_n \)-partitions based on the above recurrence formula. Let \( \xi_n \) be the random variable of the number of block pairs in \( B_n \)-partitions. We shall prove that the limiting distribution of \( \xi_n \) is normal by using the saddle point method, which was introduced by Schrödinger [21], see also [2][5][8][16].

This paper is organized as follows. In Section 2, we present some facts about the saddle point of the generating function of the number of \( B_n \)-partitions. Section 3 is devoted to deduce the normality of the limiting distribution of \( \xi_n \). Using the same technique, we obtain the normality of the limiting distribution of the number of block-pairs in \( B_n \)-partitions without zero-block.

2 Preliminary lemmas

Let \( M_n \) be the number of \( B_n \)-partitions. In this section, we give some lemmas which will be used to derive an approximate formula for \( M_n \).

Let \( N_{n,k} \) be the number of \( B_n \)-partitions with \( k \) block pairs but with no zero-block. Denote by \( N_n \) the number of \( B_n \)-partitions without zero-block. It is easy to see that

\[
N_{n,k} = 2^{n-k}S(n,k).
\]

Since

\[
\sum_n S(n,k) \frac{x^n}{n!} = \frac{1}{k!} (e^x - 1)^k,
\]

see Stanley [22, page 34], we find that

\[
F_N(z) = \sum_{n \geq 0} N_n \frac{z^n}{n!} = \exp\left(\frac{e^{2z} - 1}{2}\right).
\]

It is also easy to see that

\[
M_n = \sum_{k} \binom{n}{k} N_k.
\]

It follows from (2.2) that

\[
F_M(z) = \sum_{n \geq 0} M_n \frac{z^n}{n!} = \exp\left(\frac{e^{2z} - 1}{2} + z\right).
\]

The saddle point of \( F_M(z) \) is defined to be the value \( z \) that minimizes \( z^{-n} F_M(z) \), i.e., the unique positive solution \( r_1 \) of the equation

\[
r_1 (e^{2r_1} + 1) = n.
\]
Similarly, the saddle point of $F_N(z)$ is the unique positive solution $r_0$ of the equation
\[ r_0 e^{2r_0} = n. \] (2.5)

For convenience, we consider the equation
\[ r \left( e^{2r} + c \right) = n. \] (2.6)

It reduces to (2.4) when $c = 1$, and to (2.5) when $c = 0$. It is easy to deduce the following approximation for the unique positive solution $r$ of (2.6).

**Lemma 2.1.** Let $c$ be a nonnegative integer. Let $r > 0$ be the unique positive solution of equation (2.6). Then we have
\[ r = \frac{\log n}{2} \left( 1 + O \left( \frac{\log \log n}{\log n} \right) \right), \]
\[ e^{2r} = \frac{2n}{\log n} \left( 1 + O \left( \frac{\log \log n}{\log n} \right) \right). \]

We will also need the following lemma.

**Lemma 2.2.** Let $h(x)$ be a continuous function defined on the closed interval $[a, b]$. Suppose that $h''(x)$ exists in the open interval $(a, b)$. Then for any $c \in (a, b)$, there exists $s \in (a, b)$ such that
\[ \frac{h(a)}{(a - b)(a - c)} + \frac{h(b)}{(b - a)(b - c)} + \frac{h(c)}{(c - a)(c - b)} = \frac{h''(s)}{2}. \] (2.7)

**Proof.** Let
\[ f_1(x) = (a - b)h(x) + (b - x)h(a) + (x - a)h(b), \]
\[ g_1(x) = (a - b)(b - x)(x - a). \]

Then the left hand side of (2.7) becomes $f_1(c)/g_1(c)$. Note that $f_1(a) = g_1(a) = 0$. By Cauchy’s mean value theorem, there exists $s_1 \in (a, c)$ such that
\[ \frac{f_1(c)}{g_1(c)} = \frac{f_1'(s_1)}{g_1'(s_1)} = \frac{f_2(a) - f_2(b)}{g_2(a) - g_2(b)}, \]
where $f_2(x) = h'(s_1)x - h(x)$ and $g_2(x) = x^2 - 2s_1x$. Again, by Cauchy’s mean value theorem, there exist $s_2 \in (a, b)$ and $s \in (a, b)$ such that
\[ \frac{f_1(c)}{g_1(c)} = \frac{f_2'(s_2)}{g_2'(s_2)} = \frac{h'(s_1) - h'(s_2)}{2s_1 - 2s_2} = \frac{h''(s)}{2}. \]

This completes the proof. □
Lemma 2.3. Let $c$ be a nonnegative integer, and $f(x) = x(e^{2x} + c)$. Suppose that $t_i$ ($i = 0, 1, 2$) is the unique positive number such that $f(t_i) = n + i$. Then we have

\begin{align*}
  t_1 - t_0 &= \frac{1}{2n} - \frac{1}{4nt_0} + O\left(\frac{1}{n \log^2 n}\right); \\
  t_2 - t_1 &= \frac{1}{2n} - \frac{1}{4nt_0} + O\left(\frac{1}{n \log^2 n}\right); \\
  2t_1 - t_0 - t_2 &= \frac{1}{n^2} + O\left(\frac{1}{n^2 \log n}\right), \\
  \frac{1}{t_0} + \frac{1}{t_2} - \frac{2}{t_1} &= O\left(\frac{1}{n^2 \log^2 n}\right).
\end{align*}

Proof. We first consider (2.8). By Cauchy’s mean value theorem, there exists $t$ such that $t_0 < t < t_1$ and

$$f(t_1) - f(t_0) = (t_1 - t_0)f'(t).$$

Since $f(t_1) - f(t_0) = 1$ and $f'(t) = (2t + 1)e^{2t} + c$, we have

$$\frac{1}{(2t_1 + 1)e^{2t_1} + c} \leq t_1 - t_0 \leq \frac{1}{(2t_0 + 1)e^{2t_0} + c}.$$ \hspace{1cm} (2.10)

It can be seen that both $\frac{1}{(2t_1 + 1)e^{2t_1} + c}$ and $\frac{1}{(2t_0 + 1)e^{2t_0} + c}$ have the same estimate

$$\frac{1}{2n} - \frac{1}{4nt_0} + O\left(\frac{1}{n \log^2 n}\right).$$

It follows that $t_1 - t_0$ also has the above estimate. Similarly, one can prove (2.9). The last two approximations are consequences of Lemma 2.2.

3 The limiting distribution

Recall that $\xi_n$ is the random variable of the number of block pairs in a $B_n$-partition. Denote by $E(\xi_n)$ the expectation of $\xi_n$, and $V(\xi_n)$ the variance of $\xi_n$. Below is the main result of this paper.

Theorem 3.1. The limiting distribution of the random variable $\xi_n$ is normal. In other words, the random variable

$$\frac{\xi_n - E(\xi_n)}{\sqrt{V(\xi_n)}}$$

has an asymptotically standard normal distribution as $n$ tends to infinity.
There are various sufficient conditions on a random variable which ensures a normal limiting distribution, see Sachkov [20]. Let $\eta_n$ be a random variable of certain statistic of some combinatorial objects on a set $A_n$. Let $a_n(k)$ be the number of elements of $A_n$ with the statistic equal to $k$. Consider the polynomial

$$P_n(x) = \sum_k a_n(k)x^k.$$ 

The following criterion was used by Harper [11], see also Bender [1].

**Proposition 3.2.** The limiting distribution of $\eta_n$ is normal, if the $P_n(x)$ distinct real roots and the variance of $a_n(k)$ tends to infinity as $n \to \infty$.

We shall prove Theorem 3.1 with the aid of Proposition 3.2. Recall that $M_{n,k}$ is the number of $B_n$-partitions with $k$ block pairs. Consider the polynomial

$$M_n(x) = \sum_k M_{n,k}x^k.$$ (3.1)

For example,

$$M_1(x) = 1 + x,$$

$$M_2(x) = 1 + 4x + x^2,$$

$$M_3(x) = 1 + 13x + 9x^2 + x^3.$$ 

**Theorem 3.3.** For any $n \geq 1$, the polynomial $M_n(x)$ has $n$ distinct real roots.

**Proof.** The proof is similar to the proof of Harper for ordinary partitions. We prove it by induction on $n$. It is clear that the theorem holds for $n = 1, 2$. We assume that it holds for all $n \leq m - 1$, where $m \geq 3$. Let

$$G_n(x) = \sqrt{x} e^{\frac{x}{2}} M_n(x).$$ (3.2)

Differentiating $M_n(x)$ with respect to $x$ and using the recurrence (1.1), we obtain that

$$M_n(x) = (1 + x)M_{n-1}(x) + 2xM'_{n-1}(x).$$ (3.3)

Multiplying both sides of (3.3) by $\sqrt{x} e^{\frac{x}{2}}$ yields

$$G_n(x) = 2xG'_{n-1}(x).$$ (3.4)

By the induction hypothesis, we may assume that $M_{m-1}(x)$ has roots $x_1, x_2, \ldots, x_{m-1}$ where $x_1 < x_2 < \cdots < x_{m-1} < 0$. Observe that

$$\lim_{x \to -\infty} G_n(x) = 0.$$
From (3.2) it can be seen that $G_{m-1}(x)$ has $m+1$ roots
\[-\infty, x_1, x_2, \ldots, x_{m-1}, 0.\]

By Rolle’s theorem, in each of the $m$ open intervals
\[(-\infty, x_1), (x_1, x_2), \ldots, (x_{m-1}, 0),\]
there exists a point $y$ such that $G'_{m-1}(y) = 0$. Suppose that
\[G'_{m-1}(y_1) = G'_{m-1}(y_2) = \cdots = G'_{m-1}(y_m) = 0,\]
where $y_1 < y_2 < \cdots < y_m < 0$. By (3.4), the function $G_m(x)$ has $m+2$ roots
\[-\infty, y_1, y_2, \ldots, y_m, 0.\]

Because of (3.2), we see that $y_1, y_2, \ldots, y_m$ are $m$ distinct negative roots of $M_m(x)$. This completes the proof.

It should be mentioned that Theorem 3.3 can also be deduced from the criteria of Liu and Wang [15]. The following theorem gives an estimate of $M_n$.

**Theorem 3.4.** We have
\[M_n = \frac{1}{\sqrt{2\pi r_1 + 1}} \exp\left(2nr_1 - n + \frac{n}{2r_1} + 2r_1 - 1\right)\left[1 + O\left(\frac{\log^{7/2} n}{\sqrt{n}}\right)\right],\] (3.5)
where $r_1$ is the unique positive solution of the equation $r_1(e^{2r_1} + 1) = n$.

**Proof.** Let $r = r_1$. Applying Cauchy’s formula and the generating function (2.3), we have
\[\frac{M_n}{n!} = \frac{1}{2\pi i} \int_{|z|=r} \frac{F(z)}{z^{n+1}} dz = \frac{1}{2\pi r^n \sqrt{e}} \int_{|\theta|\leq\pi} e^A d\theta,\] (3.6)
where
\[A = \frac{1}{2} e^{2r \cos \theta} + r e^{i\theta} - n\theta i.\] (3.7)

We divide the integral in (3.6) into two parts as
\[\int_{|\theta|\leq\pi} e^A d\theta = \int_{|\theta|\leq\theta_0} e^A d\theta + \int_{\theta_0 \leq |\theta| \leq \pi} e^A d\theta,\] (3.8)
where
\[\theta_0 = \sqrt{\frac{2\log n}{n}}.\]

Let $\Re(A)$ denote the real part of $A$, and $\Im(A)$ the imaginary part. It follows from (3.7) that
\[\Re(A) = \frac{1}{2} e^{2r \cos \theta} \cos(2r \sin \theta) + r \cos \theta,\] (3.9)
\[\Im(A) = \frac{1}{2} e^{2r \cos \theta} \sin(2r \sin \theta) + r \sin \theta - n\theta.\]
For the part $\int_{|\theta| \leq \theta_0} e^A d\theta$, we have

$$\Re(A) = \frac{e^{2r} + 2r}{2} - \frac{n(2r + 1)}{2} \theta^2 + O\left(n r^2 \theta_0^4\right), \quad (3.10)$$

$$\Im(A) = O\left(n r^2 \theta_0^3\right).$$

Substituting them into $e^A = e^{\Re(A) + i \Im(A)}$, we get

$$\int_{|\theta| \leq \theta_0} e^A d\theta = \exp\left(\frac{e^{2r} + 2r}{2}\right) \int_{|\theta| \leq \theta_0} e^{-m\theta^2} d\theta \left(1 + O\left(n r^2 \theta_0^3\right)\right), \quad (3.11)$$

where $m = (2r + 1)n/2$. Note that

$$\int_{-\infty}^{\infty} x e^{-t^2} dt = o\left(e^{-x^2}\right), \quad \text{as} \quad x \to \infty.$$  

The integral in (3.11) can be estimated as follows

$$\int_{|\theta| \leq \theta_0} e^{-m\theta^2} d\theta = \frac{1}{\sqrt{m}} \left(\sqrt{\pi} - 2 \int_{\sqrt{m/n}}^{\infty} e^{-t^2} dt\right) = \frac{\sqrt{\pi}}{m} \left(1 + o\left(e^{-r}\right)\right). \quad (3.12)$$

By (3.11) and (3.12), we find

$$\int_{|\theta| \leq \theta_0} e^A d\theta = \exp\left(\frac{e^{2r} + 2r}{2}\right) \sqrt{\frac{2\pi}{(2r + 1)n}} \left(1 + O\left(n r^2 \theta_0^3\right)\right). \quad (3.13)$$

Now we estimate the integration $\int_{|\theta| \leq \pi} e^A d\theta$. By (3.9), we have

$$\int_{\theta_0 \leq |\theta| \leq \pi} e^A d\theta \leq 2\pi \max_{\theta_0 \leq \theta \leq \pi} e^{\Re(A)} \leq 2\pi \exp\left(\frac{1}{2}e^{2r \cos \theta_0} + r\right).$$

Since

$$2r \cos \theta_0 = 2r - r \theta_0^2 + O\left(r \theta_0^4\right),$$

we get

$$\int_{\theta_0 \leq |\theta| \leq \pi} e^A d\theta = O\left(\exp\left(\frac{e^{2r}}{2} - \frac{n \theta_0^2}{2} + r\right)\right).$$

It is easy to check that

$$\lim_{n \to \infty} \frac{\exp\left(\frac{e^{2r}}{2} - \frac{n \theta_0^2}{2} + r\right)}{\exp\left(\frac{e^{2r} + 2r}{2}\right) \sqrt{\frac{2\pi}{(2r + 1)n}}} n r^2 \theta_0^3 = 0. \quad (3.14)$$

Namely, the remainder of $\left|\int_{\theta_0 \leq |\theta| \leq \pi} e^A d\theta\right|$ is smaller than the remainder of $\left|\int_{|\theta| \leq \theta_0} e^A d\theta\right|$. By (3.13), we have

$$\int_{|\theta| \leq \pi} e^A d\theta = \exp\left(\frac{e^{2r} + 2r}{2}\right) \sqrt{\frac{2\pi}{(2r + 1)n}} \left(1 + O\left(n r^2 \theta_0^3\right)\right).$$
Hence by (3.6) and Stirling’s formula
\[
  n! = \frac{\sqrt{2\pi n} n^n}{e^n} \left( 1 + O\left( n^{-1} \right) \right),
\]
we have
\[
  M_n = \frac{1}{\sqrt{2r+1}} \left( \frac{n}{r} \right)^n \exp\left( \frac{n}{2r} - n + r - 1 \right) \left[ 1 + O\left( \frac{\log \frac{7}{2} n}{\sqrt{n}} \right) \right]. \tag{3.15}
\]
By Equation (2.4) and Lemma 2.1 we find
\[
  \left( \frac{n}{r} \right)^n = e^{2nr + r} \left( 1 + O\left( \frac{\log^2 n}{n} \right) \right).
\]
Together with (3.15), we arrive at (3.5). This completes the proof. \( \Box \)

As will be seen in the next theorem, the remainder \( O\left( \frac{\log^2 n}{\sqrt{n}} \right) \) plays an essential role in estimating the variance \( V(\xi_n) \).

**Theorem 3.5.** We have
\[
  E(\xi_n) = \frac{M_{n+1}}{2M_n} - 1 \sim \frac{n}{\log n}, \tag{3.16}
\]
\[
  V(\xi_n) = \frac{M_{n+2}}{4M_n} - \frac{M_{n+1}^2}{4M_n^2} - \frac{1}{2} \sim \frac{n}{\log^2 n}. \tag{3.17}
\]

**Proof.** It can be easily checked that the expectation and the variance of \( \xi_n \) can be expressed by
\[
  E(\xi_n) = \frac{M'_n(1)}{M_n},
\]
\[
  V(\xi_n) = E(\xi_n) - E(\xi_n)^2 + \frac{M''_n(1)}{M_n}.
\]
Thus we can deduce the exact formulas in (3.16) and (3.17). In view of Theorem 3.4, Lemma 2.1 and Lemma 2.3 we find
\[
  \frac{M_{n+1}}{2M_n} - 1 \sim \frac{n}{\log n}.
\]

We now proceed to derive the approximation in (3.17). Suppose that
\[
  t_i(e^{2t_i} + 1) = n + i,
\]
for \( i = 0, 1, 2 \). By Theorem 3.4 we have
\[
  \frac{M_{n+2}}{M_n} - \frac{M_{n+1}^2}{M_n^2} = \left( \sqrt{\frac{2t_0 + 1}{2t_2 + 1}} e^{A} - \frac{2t_0 + 1}{2t_1 + 1} e^{B} \right) \left( 1 + O\left( \frac{\log^{7/2} n}{\sqrt{n}} \right) \right). \tag{3.18}
\]
where
\[ A = 4t_2 + \left( 2nt_2 - 2nt_0 - 2 + \frac{1}{t_2} \right) - \left( \frac{n}{2t_0} - \frac{n}{2t_2} \right) + (2t_2 - 2t_0), \]
\[ B = 4t_1 + \left( 4nt_1 - 4nt_0 - 2 + \frac{1}{t_1} \right) - \left( \frac{n}{t_0} - \frac{n}{t_1} \right) + (4t_1 - 4t_0). \]

By Lemma 2.1, both \( \sqrt{\frac{2t_0 + 1}{2t_1 + 1}} \) and \( \frac{2t_0 + 1}{2t_1 + 1} \) can be estimated by \( 1 + O\left( \frac{1}{n \log n} \right) \). Because of the estimates in Lemma 2.3, (3.18) simplifies to
\[ \frac{M_{n+2}}{M_n} - \frac{M_{n+1}^2}{M_n^2} = \left( e^A - e^B \right) \left( 1 + O\left( \frac{\log^{7/2} n}{\sqrt{n}} \right) \right). \] (3.19)

By Cauchy’s mean value theorem, there exists a constant \( C \) such that \( B < C < A \) and
\[ e^A - e^B = (A - B)e^C. \] (3.20)

On one hand, Lemma 2.3 yields
\[ A - B = \left( 4t_2 - 4t_1 + \frac{1}{t_2} - \frac{1}{t_1} \right) - (2n + 2)(2t_1 - t_0 - t_2) + \frac{n}{2} \left( \frac{1}{t_0} + \frac{1}{t_2} - \frac{2}{t_1} \right) \]
\[ = \frac{1}{n} \left( 1 + O\left( \frac{1}{n^2 \log n} \right) \right). \] (3.21)

On the other hand, by Lemma 2.1 we find that
\[ e^C = \frac{4n^2}{\log^2 n} \left( 1 + O\left( \frac{\log \log n}{\log n} \right) \right). \] (3.22)

Substituting (3.22) and (3.21) into (3.20), we deduce that
\[ e^A - e^B = \frac{4n^2}{\log^2 n} \left( 1 + O\left( \frac{\log \log n}{\log n} \right) \right). \] (3.23)

Substituting (3.23) into (3.19), we obtain the approximation of \( V(\xi_n) \). This completes the proof. \qed

By (3.17), we see that \( V(\xi_n) \) tends to infinity as \( n \to \infty \). Hence Theorem 3.1 follows from Theorem 3.3 and Proposition 3.2.

For \( B_n \)-partitions without zero-block, we have an analogous limiting distribution. Using the saddle point method as in the proof of Theorem 3.4, we obtain the following estimates of \( N_n \).

**Theorem 3.6.** We have
\[ N_n = \frac{1}{\sqrt{(2r_0 + 1)}} \exp\left( 2nr_0 - n + \frac{n}{2r_0} - \frac{1}{2} \right) \left( 1 + O\left( \frac{\log^{7/2} n}{\sqrt{n}} \right) \right) \quad (3.24) \]
\[ \sim \frac{1}{\sqrt{\log n}} \exp\left( 2nr_0 - n + \frac{n}{2r_0} - \frac{1}{2} \right), \quad (3.25) \]
where \( r_0 \) is the unique positive solution of the equation \( r_0 e^{2r_0} = n \).
We remark that the approximation (3.25) can also be proved by Hayman’s theorem [12].

**Corollary 3.7.** We have

\[
\frac{N_n}{M_n} \sim \sqrt{\frac{\log n}{2n}}. \tag{3.26}
\]

**Proof.** Let \( r_0 e^{2r_0} = n \) and \( r_1 (e^{2r_1} + 1) = n \). By Theorem 3.4 and Lemma 2.1, we obtain that

\[
M_n \sim \frac{2n}{\log^{3/2} n} \exp\left(2nr_1 - n + \frac{n}{2r_1} - 1\right).
\]

Using (3.25), we get

\[
\frac{N_n}{M_n} \sim \frac{\log n}{2n} \exp\left(2n(r_0 - r_1) - n\frac{r_0 - r_1}{2r_0 r_1} + 1\right). \tag{3.27}
\]

By Cauchy’s mean value theorem, we have

\[
n(r_0 - r_1) = \frac{r_0}{2} - \frac{1}{4} + O\left(\frac{1}{\log n}\right). \tag{3.28}
\]

Thus (3.26) follows from (3.27) and (3.28). This completes the proof.

Recall that \( N_{n, k} \) is the number of \( B_n \)-partitions without zero-block having \( k \) block pairs. It can be verified that for any \( n \geq 1 \), the polynomial

\[
N_n(x) = \sum_k N_{n, k} x^k
\]

has \( n \) distinct real roots. Let \( \xi'_n \) be the random variable of the number of block pairs in \( B_n \)-partitions without zero-block. Using the same argument as that for \( \xi_n \), we find

\[
E(\xi'_n) = \frac{N_{n+1}}{2N_n} - 1 \sim \frac{n}{\log n},
\]

\[
V(\xi'_n) = \frac{N_{n+2}}{4N_n} - \frac{N_{n+1}^2}{4N_n^2} - \frac{1}{2} \sim \frac{n}{\log^2 n}.
\]

Hence \( V(\xi'_n) \) tends to infinity as \( n \) does. By Proposition 3.2, we are led to the following assertion.

**Theorem 3.8.** The limiting distribution of the random variable \( \xi'_n \) is normal.

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