Nilpotent Lie Groups
in Clifford Analysis and Mathematical Physics

Five Directions for Research

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Abstract

The aim of the paper is to popularise nilpotent Lie groups (notably the Heisenberg group and alike) in the context of Clifford analysis and related models of mathematical physics. It is argued that these groups are underinvestigated in comparison with other classical branches of analysis. We list five general directions which seem to be promising for further research.

1 Introduction

One simpleton can ask more questions than a hundred of wise men find answers.

The purpose of this short note is to advertise nilpotent Lie groups among researchers in Clifford analysis. It is the author’s feeling that this interesting subject is a fertile area still waiting for an appreciation.

The rôle of symmetries in mathematics and physics is widely acknowledged: the Erlangen program for geometries of F. Klein and the theory of relativity of A. Einstein are probably the most famous examples became common places already. The field of Clifford analysis is not an exception in this sense: the rôle of symmetries was appreciated from the very beginning, see e.g. [4, 30] (to mention only very few papers). Moreover reach groups
of symmetries are among corner stones which distinguish analytic function theories from the rest of real analysis [19, 21, 24]. This seems to be widely accepted today: in the proceedings of the recent conference in Ixtapa [31] seven out of total seventeen contributions explicitly investigate symmetries in Clifford analysis. But all of these paper concerned with the semisimple group of Möbius (conformal) transformations of Euclidean spaces. This group generalises the $SL_2(\mathbb{R})$—the group of overwhelming importance in mathematics in general [13, 23] and complex analysis in particular [20, 24].

On the other hand the Möbius group is not the only group which may be interesting in Clifford analysis. It was argued [11, 12] that the Heisenberg group is relevant in many diverse areas of mathematics and physics. The simplest confirmations are that for example

- Differentiation $\frac{\partial}{\partial x}$ and multiplication by $x$—two basic operation of not only analysis but also of umbral calculus in combinatorics [3, 29], for example—generate a representation of the Lie algebra of the Heisenberg group.

- Any quantum mechanical model gives a representation of the Heisenberg group.

Impressive continuation of the list can be found in [11, 12].

Clifford analysis is not only a subfield of analysis but also has reach and fertile connections with other branches: real harmonic analysis [26], several complex variables [10, 27], operator theory [14, 17], quantum theory [8], etc. Therefore it is naturally to ask about (cf. [11])

the rôle of the Heisenberg group in Clifford analysis.

Unfortunately it was not written enough on the subject. The early paper [10] just initiated the topic and the recent joint paper [5] only hinted about richness of a possible theory. Thus the whole field seems to be unexplored till now. In order to bring researchers’ attention to the above question we list in the next Section five (rather wide) directions for future advances.

### 2 Five Directions for Research

In the joint paper [5] with Jan Cnops we constructed two examples of spaces of monogenic functions generated by nilpotent Lie groups. The first example is based on the Heisenberg group and gives a monogenic space which is isometrically isomorphic to the classic Segal-Bargmann space $F_2(\mathbb{C}^n, e^{-|z|^2}dz)$
of holomorphic functions in $\mathbb{C}^n$ square integrable with respect to the Gaussian measure $e^{-|z|^2}dz$. The second example gives monogenic space of Segal-Bargmann type generated by a Heisenberg-like group with $n$ dimensional centre. The following propositions are motivated by these examples.

1. **Representation Theory**
   Representation theory of nilpotent Lie group by unitary operators in linear spaces over the field of complex numbers is completely described by the Kirillov theory of induced representations [15]. Particularly all irreducible unitary representations are induced by a character (one dimensional representation) of the centre of the group. Therefore the image of the centre is always one-dimensional. Representations in linear spaces with Clifford coefficients open a new possibility: unitary irreducible (in an appropriate sense) representations can have a multidimensional image of the groups centre [5]. Various aspects of such representations should be investigated.

2. **Harmonic Analysis**
   Clifford analysis technique is useful [26] for investigation of classic real harmonic analysis questions about singular integral operators [32]. Classical real analytical tools are closely related to the harmonic analysis on the Heisenberg and other nilpotent Lie groups [9, 12, 32]. A combination of both—the Heisenberg group and Clifford algebras—could combine the power of two approaches in a single device.

3. **Operator Theory**
   The Segal-Bargmann space $F_2$ and associated orthogonal projection $P : F_2 \to F_2$ produce an important class of Toeplitz operators $T_a = Pa(z, \bar{z})I$ [6] which is a base for the Berezin quantisation [2]. Connections between properties of an operator $P_a$ and its symbol $a(z, \bar{z})$ are the subject of important theory [7]. Moreover translations of Toeplitz operators to the Schrödinger representation gives interesting information on pseudodifferential operators and their symbolic calculus [12]. Monogenic space of the Segal-Bargmann type [5] could provide additional insights in these important relations.

4. **Functional Calculus and Spectrum**
   Functions of several operators can be defined by means of Weyl calculus [1], i.e. essentially using the Fourier transform and its connections with representations with the Heisenberg group [22]. Another opportunity of a functional calculus from Segal-Bargmann type spaces is also based on this group [22]. On the other hand a functional calculus can
be defined by the Cauchy formula for monogenic functions \[14, 17\]. Simultaneous usage of monogenic functions and nilpotent groups could give a fuller picture for functional calculus of operators and associated notions of joint spectrums.

5. Quantum Mechanics
Observables of coordinates and momentums satisfy to the Heisenberg commutation relations \([p, q] = i\hbar I\), thus generated algebra of observables representing the Heisenberg group. Even better: the representation theory of Heisenberg and other nilpotent Lie groups provides us with both non-relativistic classic and quantum description of the world and a correspondence between them \[18, 23, 28\]. On the other hand Clifford modules provide a natural description for spinor degrees of freedom of particles or fields. Therefore a mixture of these two objects provides a natural model for quantum particles with spin. Yet such models and their advantages should be worked out.

It should not be difficult to extend the list of open problems (see the epigraph).

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Any bibliography for a paper with a small size and a wide scope is necessarily grossly incomplete. I appreciate an understanding of readers who will not be disappointed if their relevant papers are not listed among (almost random) references as well as an excuse for the extensive self-citing.

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