Lectures on Minimal Surfaces

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Abstract
Some elementary considerations are presented concerning Catenoids and their stability, separable minimal hypersurfaces, minimal surfaces obtainable by rotating shapes, minimal tori in $S^3$, and the minimality in $\mathbb{R}^{nk}$ of the ordered set of k orthogonal equal-length n-vectors.
I. Solitonic Catenoids

Let us start with the following simple question: what is the surface of least area connecting 2 circles of radius \( r \) lying (one above the other) in parallel planes a distance \( d \) apart?

As the answer (if it exists) should clearly be of the form

\[
\vec{x}(z, \varphi) = \begin{pmatrix}
f(z) \cos \varphi \\
f(z) \sin \varphi \\
z
\end{pmatrix} \quad \varphi \in [0, 2\pi]
\]  

(I.1)

one may determine the optimal surface by minimizing, among functions \( f \) taking the value \( r \) at the boundary (say \( z = \pm \frac{d}{2} \))

\[
A[\vec{x}] = A[f] = 2\pi \int_{-\frac{d}{2}}^{\frac{d}{2}} f \sqrt{1 + f'^2} \, dz,
\]

(I.2)

the area of an axially symmetric surface of the form (I.1). Varying \( f \) (\( f \to f + \varepsilon \), \( \varepsilon(\pm \frac{d}{2}) = 0 \)) gives

\[
A(f, f') := f \sqrt{1 + f'^2} \to f \sqrt{1 + f'^2} + \varepsilon \sqrt{1 + f'^2} + \frac{f \varepsilon' f'}{1 + f'^2} + \ldots, \text{resp.}
\]

\[
\sqrt{1 + f'^2} = \frac{\partial A}{\partial f} = \frac{d}{dz} \left( \frac{\partial A}{\partial f'} \right), \text{ i.e. } 1 + f'^2 = f f''
\]

hence

\[
f(z) = a \cosh \left( \frac{z}{a} + c \right).
\]

(I.3)

The boundary conditions at \( \pm \frac{d}{2} \) imply \( c = 0 \), and \( a \cosh \frac{d}{2a} = r \), which written as

\[
\frac{\cosh w}{w} = \rho; \quad w := \frac{d}{2a}, \quad \rho := \frac{2r}{d}
\]

(I.4)

is easily seen to generically, for \( \rho > \bar{\rho} \), have 2 solutions \( w_1 < w_2 \) (corresponding to an outer resp. inner 'Catenoid'), of closest approach to the \( z \)-axis \( a_1 > a_2 \), while no solution exists for \( \rho < \bar{\rho} \) (large distance, resp. small radius) and exactly one solution \( w_0 \), \( \cosh w_0 = \bar{\rho} \), at critical distance/radius.

Evaluating (I.2) for (I.3) gives

\[
A[f] = 4\pi a^2 \int_{0}^{\frac{d}{2a}} (\cosh r)^2 \, dr
\]

\[
= \pi a^2 \left( \frac{d}{a} + \sinh \frac{d}{a} \right)
\]

\[
= \frac{\pi d^2}{2} \left( \frac{1}{w} + \frac{\sinh 2w}{2w^2} \right) =: A(w)
\]

(I.5)

Naturally one would like to compare the two areas

\[
A_i := A(w_i) = \frac{\pi d^2}{2w_i} (1 + \rho \sinh w_i),
\]

(I.6)
as well as determine whether they are really local minima, or saddle points resp. maxima. Using (I.6), and then again (I.4), one finds that \( A_1 < A_2 \), as
\[
\frac{2w_1w_2}{\pi d^2} (A_2 - A_1) = (w_1 - w_2 + \cosh w_1 \sinh w_2 - \cosh w_2 \sinh w_1) = \sinh(w_2 - w_1) - (w_2 - w_1) > 0,
\]
while calculating the second variation of (I.2)
\[
(f \to f + \varepsilon, f = a \cosh \frac{z}{a}, \varepsilon(z) = \tilde{\varepsilon}(v := \frac{z}{a}))
\]
gives
\[
\delta^2 A = \pi \int_{-w}^{+w} \frac{\tilde{\varepsilon}(v)}{\cosh v} \left(-\partial_v^2 - \frac{2}{\cosh^2 v}\right) \frac{\tilde{\varepsilon}}{\cosh v} dv
\]
as
\[
A[f + \varepsilon] = 2\pi \int_{-\frac{w}{2}}^{+\frac{w}{2}} f \sqrt{1 + f'^2} \left(1 + \frac{\varepsilon}{f} + \ldots\right) \left(1 + \frac{\varepsilon f''}{1 + f'^2} + \frac{\varepsilon^2}{2(1 + f'^2)^2} + \ldots\right) dz
\]
\[
= A[f] + \delta A + \ldots
\]
\[
= \ldots + 2\pi \int_{-w}^{+w} \left(\frac{\varepsilon^2}{\cosh^2 v} + (\tilde{\varepsilon}^2) \cdot \frac{\sinh v}{\cosh v}\right) dv
\]
\[
= \ldots + \frac{2\pi}{2} \int_{-w}^{+w} \left\{-\tilde{\varepsilon} \tilde{\varepsilon}'' - \tilde{\varepsilon}^2 \left(\frac{\sinh v}{\cosh v}\right)' - \tilde{\varepsilon}^2 \left(\frac{\sinh v}{\cosh v}\right)\right\}
\]
\[
= \ldots + \frac{2\pi}{2} \left(\int_{-w}^{w} \frac{\tilde{\varepsilon}^2}{\cosh^2 v} dv - \int_{-w}^{w} \frac{\tilde{\varepsilon} \tilde{\varepsilon}''}{\cosh^2 v} dv - 3 \int_{-w}^{w} \frac{\tilde{\varepsilon}^2}{\cosh^4 v} dv\right),
\]
always using that the variation vanishes at the boundary, \( \tilde{\varepsilon}(\pm w) = 0 \). ( and leaving out terms of order 3 and higher)

The stability properties of the Catenoids therefore depend on the spectrum of the operator \( J_w \) given as
\[
J := -\partial_v^2 - \frac{2}{\cosh^2 v}
\]
acting on functions vanishing at the boundary of the interval \( I_w := [-w, +w] \).

Recalling that \( w = \frac{d}{2a} \) is determined by solving
\[
\cosh w = \frac{2r}{d} w =: \rho w,
\]
note that (for \( \rho = \tilde{\rho} \)) the straight line \( \rho \cdot w \) will be tangent to \( \cosh w \), hence \( \tilde{\rho} = \sinh w_0 \), and therefore
\[
w_0 \tanh w_0 = 1.
\]
This observation is important as it allows one to conclude that \( J^{(0)} := J_{w_0} \) is non-negative, as
\[
\psi_0(v) := 1 - v \tanh v,
\]
which can easily be seen to be annihilated by \( J^{(0)} \), being non-negative on \( I_{w_0} \) and vanishing on its boundary, must be the groundstate of \( J^{(0)} \), hence \( J^{(0)} \geq 0 \).

While this in particular means that surfaces corresponding to
\[
f_\gamma(z) = a_0 \cosh \frac{z}{a_0} \left(1 + \gamma \left(1 - \frac{z}{a_0 \tanh \frac{z}{a_0}}\right)\right)
\]
will have, up to ( and including ) second order in $\gamma \ll 1$, the same area than the critical Catenoid, the main virtue of knowing the lowest eigenvalue of $J^{(0)}$ to be zero is that it allows one to conclude that ( for $\rho > \bar{\rho}$ ) the outer Catenoid is stable while the inner one is unstable! For that one can either invoke the fact that the lowest eigenvalue of $J$ has to increase ( decrease ) when the length of the interval is decreased ($w_0 \searrow w_1$) resp increased ($w_0 \nearrow w_2$)-or explicitly argue as follows:

As is well known from integrable systems ( see e.g. chapter 15 of [1] ), while apparently less common knowledge in the context of minimal surfaces, $J$ can factorized, and be related to the free operator $\tilde{J} := -\partial^2$ :

$$J = -\partial^2 - \frac{2}{\cosh^2} = (-\partial + \tanh)(\partial + \tanh) - 1 =: L^\dagger L - 1 \quad \text{(I.14)}$$

$$LL^\dagger - 1 = (\partial + \tanh)(-\partial + \tanh) - 1 = -\partial^2 = \tilde{J}.$$ 

Hence ( forgetting for the moment the boundary condition, i.e. only on the level of solutions to differential equations ) solutions of $J\psi = E\psi$ can be constructed as

$$\psi_E = L^\dagger \phi_E, \quad -\partial^2 \phi_E = E\phi_E. \quad \text{(I.15)}$$

E.g. for $E = 0$ ($\phi_+^{(0)} = \text{const} \phi_-^{(0)} = (\text{const}) \cdot v$)

one obtains (for $\phi_-^{(0)} = -v$)

$$\psi_0^{(+)} = L^\dagger \phi_0^- = (-\partial + \tanh v)(-v) = 1 - v \tanh v, \quad \text{(I.16)}$$

explaining a way to derive the explicit form $\psi_0^{(+)}$. 

To explicitly construct the instability - mode of the inner Catenoid consider

$$\psi_k^{(+)}(v) = L^\dagger \left( \frac{-\sinh k}{k} \right) = \cosh kv - \tanh v \cdot \frac{\sinh k}{k}, \quad \text{(I.17)}$$

the normalization is taken to smoothly reduce to $\psi_0^{(+)}$ as $k \to 0$, $\psi_k(0) = 1$, and the superscript ( left out from now on ) indicating the parity of the function . While for generic $k$, $\psi_k$ will not vanish on the given boundary ($\pm v = w_2 = w_0 + \varepsilon, \varepsilon > 0$) but for some (‘minimal’) $k > 0$ one will have

$$\psi_k(\pm w_2) = \cosh kw_2 - \tanh w_2 \frac{\sinh kw_2}{k} = 0, \quad \text{(I.18)}$$

$\psi_k(v) > 0$ on $(-w_2, +w_2)$.

To conclude this, note that $\psi_k$, for fixed $k$ (and restricting to $v \geq 0$), is monotonically decreasing ( at least for $k$ not too large; for simplicity let us consider small $k > 0$, $\varepsilon = (w_2 - w_0) \ll 1$, close to the critical case ): 

$$\psi'_k = k \sinh kv - \frac{1}{\cosh^2 v} \frac{\sinh k}{k} - \tanh v \cosh k < 0. \quad \text{(I.19)}$$

For $k = 0$, $\psi_0$ vanishes at $w_0$ and then, in the (small) interval $[w_0, w_2]$ becomes negative. To conclude that ( for fixed $v$ ) $\psi_k$ is monotonically increasing with $k$ ( near zero ),

$$\frac{\partial \psi_k}{\partial k}(v) = v \sinh kv - v \tanh v \frac{\cosh k}{k} + \tanh v \frac{\sinh k}{k^2} > 0, \quad \text{(I.20)}$$

$^1$Many thanks to J.Choe for pointing out that, geometrically,$\psi_0^{(+)}$ is the projection of the position vector on to the surface normal.
one calculates the Taylor-expansion ( $v$ fixed )

$$\frac{\partial \psi_k}{\partial k}(v) = \frac{1}{k} \cdot 0 + \frac{k v^2}{3} \left( 3 - v \tanh v \right) + O(k^3).$$

(I.21)

Taylor expanding $\psi_k(\pm(w_0 + \varepsilon)) = 0$, cp. (I.18),

$$1 + \frac{k^2 (w_0 + \varepsilon)^2}{2} - \left( \tanh w_0 + \frac{\varepsilon}{\cosh^2 w_0} \right) \left( w_0 (1 + \frac{\varepsilon}{w_0}) + \frac{1}{6} k^2 (w_0 + \varepsilon)^3 \right) + O(k^4) \overset{!}{=} 0$$

(I.22)

yields

$$k^2 = \frac{3 \varepsilon}{w_0},$$

(I.23)

while Taylor expanding (I.17) gives

$$\psi_k(v) = (1 - v \tanh v) + \frac{k^2 v^2}{6} (3 - v \tanh v) + O(k^4).$$

(I.24)

As an interesting exercise one may compare/double check/ these results with 2 ( not-completely- ordinary. non-standard ) perturbation theory calculations:

Making in $J\psi_E = E\psi_E$ the Ansatz $\psi_E = \psi_0 + \delta E$ one gets, via (I.14),

$$(L^\dagger L - 1) \delta E = E(\psi_0 + \delta E) \approx E\psi_0$$

(I.25)

resp. ( acting with $L$ on both sides )

$$- \partial^2 (L \delta E) \approx EL(1 - v \tanh v) = -Ev,$$

(I.26)

i.e. ( integrating 2 times and approximating $\delta E$ by the exact solution of the approximated equation (I.25) )

$$L \delta E = E \left( \frac{v^3}{6} + \alpha + \beta v \right) \overset{!}{=} \frac{1}{\cosh}(\delta E \cosh)',$$

(I.27)

from which one deduces

$$\delta E = \frac{e}{\cosh v} - \frac{Ev^2}{6} (3 - v \tanh v) - (\beta + 1)(1 - v \tanh v)E$$

(I.28)

( setting $\alpha = 0$ for parity-reasons ).

Calculating/ checking

$$(L^\dagger L - 1) \delta E = - \frac{e}{\cosh v} + \left( \partial^2 + \frac{2}{\cosh^2} \right) \frac{Ev^2}{6} (3 - v \tanh v)$$

$$= - \frac{e}{\cosh v} + E(1 - v \tanh v) \overset{!}{=} E\delta E_0$$

(I.29)

one finds $e = 0$, hence

$$\psi_E = (1 - E(\beta + 1))(1 - v \tanh v) - \frac{Ev^2}{6} (3 - v \tanh v);$$

(I.30)
while $\beta \neq -1$ just changes the normalization (hence now $\beta = -1$ for simplicity) $\psi^0(w_0 + \varepsilon)^{\dagger} = 0$ gives

\[
E(\varepsilon) = -\frac{(w_\varepsilon \tanh w_\varepsilon - 1) \cdot 6}{w_\varepsilon^2 (3 - w_\varepsilon \tanh w_\varepsilon)} \approx -\frac{3\varepsilon}{w_0},
\]

(I.31)

in accordance with (I.24), resp. (I.23).

The non-standard part of this derivation is the use of the $LL^\dagger$ structure that allows one to calculate the perturbed wavefunction without having to use all eigenstates of the unperturbed problem.

The subtlety, on the other hand, when wanting to use the standard first-order formula for the perturbed eigenvalue, is that $J_{w_\varepsilon}$ is a perturbation of $J_{w_0}$ only via the boundary condition $2$, to effectively have $J_{w_\varepsilon}$ in the standard form $J_{w_0} + \varepsilon J'$, note that $E_\varepsilon$ is the minimum of

\[
\frac{\int_{0}^{w_0} (\psi' \tanh^2 \psi - \frac{2\psi^2(1 + w_\varepsilon^2)}{\cosh^2 (\psi(1 + w_\varepsilon^2))}) d\hat{v}}{\int_{0}^{w_0} \psi^2 d\hat{v}}
\]

subject to $\psi(w_\varepsilon) = 0$.

Introducing

\[
\hat{v} := \frac{v}{\frac{w_\varepsilon}{w_0} + 1} \in [0, w_0], \quad \hat{\psi}(\hat{v}(v)) = \psi(v)
\]

(I.32) becomes equal to

\[
\frac{1}{(1 + \frac{w_\varepsilon}{w_0})^2} \int_{0}^{w_0} \hat{\psi}^2 d\hat{v}
\]

(I.34)

With $\cosh(\hat{v} + \frac{w_\varepsilon}{w_0}) \approx \cosh \hat{v} \left(1 + \frac{w_\varepsilon}{w_0} \tanh \hat{v} \right)$ one gets

\[
J' = -\frac{2}{\cosh^2 \hat{v}} \left(\frac{2}{w_0} - \frac{\hat{v} \tanh \hat{v}}{w_0} \right) = -\frac{4}{w_0} \hat{v} \tanh \hat{v} (1 - \hat{v} \tanh \hat{v}),
\]

so that the standard first-order formula for $E_\varepsilon$ gives

\[
E_\varepsilon = \varepsilon \frac{\int_{0}^{w_0} \hat{\psi}^2 \hat{J'}}{\int_{0}^{w_0} \hat{\psi}^2} \bigg|_{\hat{\psi}=(1-\hat{v} \tanh \hat{v})} = -\frac{4\varepsilon}{w_0} \frac{(J_0 - 3J_1 + 3J_2 - J_3)}{K_0 - 2K_1 + K_2}
\]

(I.35)

where

\[
J_n := \int_{0}^{w_0} v^n (\tanh v)^n \cosh^2 v \, dv, \quad K_n = \int_{0}^{w_0} v^n (\tanh v)^n dv.
\]

With the help of

\[
J_n = \frac{\tanh w_0}{n+1} - \frac{n}{n+1} K_{n+1} + \frac{n}{n+1} J_{n-1}
\]

(I.37)

(I.38)

\[\text{thanks to R. Hempel for pointing out to me the corresponding general treatment of boundary-perturbations given in [2].}\]
one finds
\[ J_0 = \frac{1}{w_0}, \quad J_1 = \frac{1}{w_0} - \frac{w_0}{z_0}, \quad J_2 = \frac{1}{w_0} - \frac{w_0}{3} - \frac{2}{3} K_1, \quad J_3 = \frac{1}{w_0} - \frac{w_0}{4} - \frac{1}{2} K_1 - \frac{3}{4} K_2 \] (I.39)
so that
\[ J_0 - 3J_1 + 3J_2 - J_3 = \frac{1}{4} w_0^3, \] (I.40)
and with \( K_0 = w_0, \quad K_2 = \frac{1}{3} w_0^3 - w_0 + 2 K_1 \) (note that the non-elementary \( K_1 \) cancels both in the numerator as well as the denominator) (I.36) becomes
\[ E_\varepsilon = \frac{-4 \varepsilon}{w_0} \cdot \frac{3}{4}, \] (I.41)
in agreement with (I.23) and (I.31).

Consider now fluctuations around the Catenoid \((\times \mathbb{R})\) as a minimal (hyper-)surface in \( \mathbb{R}^{3,1} \), i.e. a time-independent \((\dot{z} = 0)\) stationary point of the volume-functional (see e.g. [3])
\[
2\pi \int r dr dt \sqrt{1 - \dot{z}^2 + z'^2} = \int rdz dt |r'| \sqrt{1 - \frac{\dot{z}'^2}{r'^2} + \frac{1}{r'^2}} = \int rdtdz \sqrt{1 + r'^2 - \dot{r}'^2}, \] (I.42)
for (in general time-dependent) axially symmetric hypersurfaces in Minkowski-space,
\[
x^\mu = x^\mu(t, r, \varphi) = \begin{pmatrix} t \\ r \cos \varphi \\ r \sin \varphi \\ z(r, t) \end{pmatrix} = x^\mu(t, z, \varphi), \] (I.43)
with \((r, t) \rightarrow (z(r, t), t)\) implying
\[
\begin{pmatrix} z' \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \partial(z(t)) \\ \partial(r(t)) \end{pmatrix} = \begin{pmatrix} \partial(r(t)) \\ \partial(z(t)) \end{pmatrix}^{-1} = \begin{pmatrix} r' \ \\ 0 \ \\ 1 \end{pmatrix} \begin{pmatrix} 1 & -\dot{r} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\dot{r} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r' \\ 0 \ \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -\dot{r} \\ 0 & 1 \end{pmatrix}, \] (I.44)
and the variation of the rhs of (I.42) giving
\[
r''(1 - r'^2) - \dot{r}(1 + r'^2) + 2 \dot{r} r' \dot{r} = \frac{1}{r}(1 + r'^2 - \dot{r}'^2), \] (I.45)
the stationary Cateniod \( r(z, t) = r(z) = \cosh z \) indeed being a solution, as \( r'' = \frac{1}{r}(1 + r'^2) \). Linearization of (I.45) around \( \cosh z = r(z, t) - \varepsilon(z, t) \) gives
\[
\ddot{\varepsilon} + D\varepsilon = 0 \] (I.46)
with
\[
D = \frac{1}{\cosh^2 z} \left( \partial_z^2 - 2 \tanh z \partial_z + 1 \right). \] (I.47)
While \( D \) has two zero-modes,
\[ \varepsilon_+(z) = \cosh z - z \sinh z \]
\[ \varepsilon_-(z) = \sinh z \quad (I.48) \]

(corresponding to the 2 parameters in the time-independent solutions of \((I.45), \frac{1}{a} \cosh(az + b)\)), it is interesting to note that \(D\) has (exactly) one positive parity eigenfunction with energy \(\lesssim - \frac{8}{15}\) \(= \frac{\langle \psi, D\psi \rangle}{\langle \psi, \psi \rangle}\), where \(\psi = \frac{1}{\cosh z}\), while being positive on negative parity eigenfunctions, as

\[
D = \frac{1}{\cosh z} \left( \partial_z + \frac{1}{\sinh z \cosh z} \right) \left( -\partial_z + \frac{1}{\sinh z \cosh z} \right) \frac{1}{\cosh z} \\
= \left( \partial_z - \frac{1}{\cosh \frac{1}{s}} \right) \left( -\frac{1}{\cosh} \partial_z + \frac{1}{s} \right). 
\]

This having been noticed at least 10 years ago [J. Hoppe, unpublished note to G. Huisken], the question of non-linear stability was taken up, and answered, more recently. Let us mention a few facts/things related to the endeavour of trying to find a closed expression for the unstable mode of \((I.47)\), resp. (expressed in the coordinate \(y = \sinh z\), hence \(\partial_z = \cosh z \partial_y, \partial_z^2 = y \partial_y + (1 + y^2) \partial_y^2\), and compensating \(dz = \frac{1}{\sqrt{1+y^2}} dy\) by conjugation with \((1 + y^2)^{\frac{1}{2}}\),

\[
\tilde{D} = -(1 + y^2)^{-\frac{\gamma}{4}} \left( (1 + y^2) \partial_y^2 - y \partial_y + 1 \right) (1 + y^2)^{\frac{\gamma}{4}} \\
= -\partial_y^2 - \frac{1}{4(1 + y^2)} - \frac{5}{4} \frac{1}{(1 + y^2)^2} = -\partial_y^2 + \tilde{V}(y), 
\]

which also follows as

\[
(1 + y^2)^{-\frac{\gamma}{4}} \left( \left( -\partial_y + \frac{y}{1 + y^2} \right) \partial_y - \frac{1}{1 + y^2} \right) (1 + y^2)^{\frac{\gamma}{4}} \\
= \left( -\partial_y + \frac{1}{2} \frac{y}{1 + y^2} \right) \left( \partial_y + \frac{1}{2} \frac{y}{1 + y^2} \right) - \frac{1}{1 + y^2}, 
\]

noting (cp. [I.47], [I.45], [I.46])

\[
D = \frac{1}{\cosh z} \left( -\partial_z^2 - \frac{2}{\cosh^2 z} \right) \frac{1}{\cosh z} \\
= \frac{1}{\cosh z} \left[ (\partial_z + \tanh z) (\partial_z + \tanh z) - 1 \right] \frac{1}{\cosh}. 
\]

One reformulation of trying to solve

\[
\tilde{D}\tilde{\phi} = -\kappa^2 \tilde{\phi}, 
\]

thanks to J. Szefte\(\ldots\) for discussions and bringing [4] to my attention
\[ \int \tilde{\phi}^2 dy < \infty, \tilde{\phi}(y) \neq 0, \text{ arises from the factorization} \]
\[ \left( -\partial_y^2 + \tilde{V} + \kappa^2 \right)^{1} = \left( -\partial_y + U \right) \left( \partial_y + U \right) \geq 0 \] (I.54)
giving the Riccati-equation
\[ U' = U^2 - \tilde{V} - \kappa^2 = U^2 + \frac{B}{1 + y^2} + \frac{D}{(1 + y^2)^2} - \kappa^2 \] (I.55)
(with \( B = \frac{1}{4}, D = \frac{5}{4} \) in the case of interest), resp. (using that one can choose the eigenfunctions of (I.50) to be either odd, or - as in the case of the ground state - even, in both cases having \( U = -\frac{\tilde{\phi}'(y)}{\tilde{\phi}(y)} \) to be odd, and, with \( \tilde{\phi}(y) = \chi(x := y^2) \),
\[ W(x) := -2yU(y) = -\frac{\chi'(x)}{\chi(x)} \] (I.56)
having to satisfy
\[ 4x(W' + W^2) + 2W + \frac{B}{1 + x} + \frac{D}{1 + x} - \kappa^2 = 0. \] (I.57)
For \( \kappa = 0 \) (and \( B = \frac{1}{4}, D = \frac{5}{4} \)) a particular solution is
\[ W_0(x) = \frac{1}{4x} \left( \frac{2 + x}{1 + x} \right) = \frac{1}{4x} \left( 1 + \frac{1}{1 + x} \right), \] (I.58)
corresponding to the non-normalizable zero-mode \( \varepsilon_-(z) = \sinh z \), resp. \( \chi_0 \sim \frac{\sqrt{x}}{(x + 1)^{\frac{3}{4}}} \), resp. \( \frac{y}{(1 + y^2)^{\frac{3}{4}}} = \tilde{\phi}_0 \), and the Ansatz
\[ W = W_0 + \frac{1}{Y} \] (I.59)
then gives \( Y' - \frac{1}{2x} Y - 2W_0 Y = 1 \), resp.
\[ Y = \tilde{C} \frac{x^{\frac{3}{4}}}{\sqrt{x + 1}} + \frac{x^{\frac{3}{4}}}{\sqrt{x + 1}} \int \frac{\sqrt{x + 1}}{x^{\frac{3}{2}}} \] (I.60)
\[ = \tilde{C} \frac{x^{\frac{3}{4}}}{\sqrt{x + 1}} + \frac{x^{\frac{3}{4}}}{\sqrt{x + 1}} \left( 2 \ln \left( \sqrt{x} + \sqrt{1 + x} \right) - 2 \sqrt{1 + x} \right) \]
i.e.
\[ W(x) = \frac{1}{4x} \left( \frac{2 + x}{1 + x} \right) + \frac{1}{2 - \frac{3}{2} \sqrt{x + 1}} \left( \frac{C}{2} - \sqrt{1 + x} + \ln \left( \sqrt{x} + \sqrt{x + 1} \right) \right), \] (I.61)
which indeed agrees with the expression one gets from
\(-U(y) = \frac{\phi'_0}{\phi_0} = \frac{C}{2} \left( \frac{2+y^2}{(1+y^2)^2} \right) - \frac{A}{2} y \left[ \frac{y}{(1+y^2)^2} + \frac{2+y^2}{(1+y^2)^2} \ln \left( y + \sqrt{1+y^2} \right) \right] \]

\[ C = \frac{y}{(1+y^2)^{\frac{1}{2}}} + A \left[ (1+y^2)^{\frac{1}{2}} - \frac{y}{(1+y^2)^{\frac{1}{2}}} \ln \left( y + \sqrt{1+y^2} \right) \right] \]

\[ A \neq 0 \]

\begin{equation}
A \neq 0 \quad \frac{1}{2} y + \sqrt{1+y^2} - y \ln \left( y + \sqrt{1+y^2} \right)
\end{equation}

\((\text{for } A = 0, \quad -U = \frac{1}{2} \frac{2+y^2}{y(1+y^2)}, \quad \text{hence } W = \frac{U}{2y} = \frac{1}{2(1+y^2)}). \) To solve (I.57) for \(\kappa \neq 0, \) however, seems to be just as difficult as directly trying to find the groundstate of \(D, \) which, using that one knows explicitly (see e.g. [1]) the exact eigenfunctions of

\[ H = -\partial_z^2 - \frac{2}{\cosh^2 z}, \]

\begin{equation}
\psi(z) = \frac{1}{\sqrt{2 \cosh z}}, \quad \psi_k(z) = -(ik + \tanh z) e^{-ikz},
\end{equation}

satisfying

\begin{align}
H\psi(z) &= -\psi(z), \quad \int \psi(z)^2 = 1, \quad \int \psi(z)\psi_k = 0 \\
H\psi_k &= k^2 \psi_k, \quad \int \psi_k\psi_{k'} = (k^2 + 1)\delta(k-k')
\end{align}

\begin{equation}
\psi(z)\psi(z') + \int_{-\infty}^{+\infty} \frac{dk}{k^2 + 1} \psi_k(z)\psi_k(z') = \delta(z - z'),
\end{equation}

one could formulate as trying to find constants \(C_{-1}\) and \(C(k)\) satisfying

\[ -\frac{C_{-1}}{\sqrt{2} \cosh z} - \int_{-\infty}^{+\infty} C(k) \frac{ik + \tanh z}{\sqrt{k^2 + 1}} e^{-ikz} dk \]

\[ = -\kappa^2 \cosh^2 z \left( \frac{C_{-1}}{\sqrt{2} \cosh z} - \int_{-\infty}^{+\infty} C(k) \frac{ik + \tanh z}{\sqrt{k^2 + 1}} e^{-ikz} dk \right), \]

with the expression in brackets (on the rhs) being , when multiplied by \(\cosh z,\) square-integrable.
II. Separable Minimal Hypersurfaces and Rotating Shapes

For surfaces representable as graphs over (parts of) \( \mathbb{R}^2 \) the area is expressed as

\[
A[z] = \int \int \sqrt{1 + z_x^2 + z_y^2} \, dx dy, \tag{II.1}
\]

whose stationary points correspond to solutions \( z(x, y) \) of

\[
z_{xx}(1 + z_y^2) + z_{yy}(1 + z_x^2) = 2z_x z_y z_{xy}. \tag{II.2}
\]

Inserting the Ansatz \( z(x, y) = \zeta(f(x) + g(y)) \), and denoting the inverse of \(-\zeta\) by \( h \) one finds an equation for separable surfaces,

\[
\Sigma := \{ \vec{x} \in \mathbb{R}^3 | f(x) + g(y) + h(z) = 0 \}, \tag{II.3}
\]

to be 'minimal':

\[
f''(x) \left( g'^2(y) + h'^2(z) \right) + g''(y) \left( h'^2(z) + f'^2(x) \right) + h''(z) \left( f'^2(x) + g'^2(y) \right) = 0, \tag{II.4}
\]

to hold for all \((x, y, z) \in \Sigma\) (i.e. on \( \Sigma \)). While it is easy to verify the Catenoid

\[
x^2 + y^2 - (\cosh z)^2 = 0, \tag{II.5}
\]
as a solution (on \( \Sigma_5 \)) of (II.4), other elementary minimal surfaces (and in fact, after (II.5) historically the first known ones) are the helicoid,

\[
y \cos z = x \sin z, \tag{II.6}
\]
Scherk’s first,

\[
e^z \cos x = \cos y, \tag{II.7}
\]
and second,

\[
\sin z = \sinh x \sinh y, \tag{II.8}
\]
surface.

Exercise:

Show that (II.6)/(II.7)/(II.8) are of the form (II.3), \( \sum_{i=1}^3 f_i(x_i) = 0 \), with each of the functions satisfying

\[
f_i'^2 = a_i + b_i e^{\kappa f_i} + c_i e^{-\kappa f_i}, \tag{II.9}
\]
and derive the general conditions on the coefficients appearing in (II.9) to guarantee (II.3), i.e. \( \sum_{i \neq j} f_i'' f_j'^2 = 0 \), on (II.3).

Note that varying the 'area' (volume)

\[
A[u] := \int \delta(u(\vec{x})) |\nabla u| \, d^N x \tag{II.10}
\]
of a hypersurface described as a level set,

\[
\Sigma := \{ \vec{x} \in \mathbb{R}^N | u(\vec{x}) = 0 \}, \tag{II.11}
\]
yields the equation

\[(\nabla u)^2 \Delta u - \sum_{i,j=1}^{N} u_i u_j u_{ij} = 0, \tag{II.12}\]

to hold on \((\text{II.11})\). The separation Ansatz

\[u(\vec{x}) = \sum_{k} f_k(x_k),\]

then yields that

\[\sum_{i \neq j} f''_i f'^2_j = 0 \tag{II.13}\]

should hold on \(\sum f_i = 0\).

Existence and form of the solutions heavily depend on the dimension. While for \(N = 3\), \((\text{II.13})\) was completely solved already 130 years ago \([6]\), and the earliest attempt for \(N = 4\) seems to be in the Lorentzian context \([7]\), a complete classification for \(N \geq 4\) has been attacked (and more or less completed) recently together with J.Choe and V.Tkatjev.

For \(N = 3\), if none of the 3 functions is linear, solutions are of the form

\[f''_i = a_i + b_i e^{\sqrt{\mu} f_i} + c_i e^{-\sqrt{\mu} f_i}, \tag{II.14}\]

\(\mu \neq 0\), with the 9 constants linked by non-linear equations allowing for solutions in terms of 5 free constants; resp. (\(\mu = 0\))

\[f''_i = a_i + b_i f_i + c_i f^2_i, \tag{II.15}\]

with the coefficients satisfying another set of non-linear equations. Apart from the fully linear case

\[f_i = \alpha_i + \beta_i x_i \tag{II.16}\]

the, up to permutation, only other case (cp. \([N]\)) is

\[\begin{align*}
  f''_1 &= a_1 + b_1 e^{\lambda f_i} \\
  f''_2 &= a_2 + c_2 e^{-\lambda f_i} \\
  f_3(x_3) &= \alpha x_3 + \beta,
\end{align*} \tag{II.17}\]

including \((\text{II.7})\).

Other choices of sign combinations give \(u_+ = \pm \ln \cosh x\) and \(\pm \ln \sinh x := v_+\) as constituents (satisfying \(u_{+\pm}^2 = 1 - e^{\mp 2u_\pm}, u_{+\pm}'' = \mp (u_{+\pm}^2 - 1)\), resp. \(v_{+\pm}'' = \mp (v_{+\pm}^2 - 1)\), \(v_{+\pm}^2 = 1 + e^{\mp 2v_\pm}\), instead of \(w_\pm = \pm \ln \cos x\), which satisfies \(w_{\pm}^2 = -1 + e^{\mp w_\pm^2}, w_{\pm}'' = \mp (w_{\pm}^2 + 1)\)). While the solutions of \((\text{II.14})\) are in general elliptic functions, special cases
will yield trigonometric / hyperbolic expressions, such as \((\kappa = 4, \ b_1 = b_2 = c_3 = 0, \ a_1 = a_2 = -a_3 = b_3 = c_1 = c_2 = 1)\)

\[
\sin z = \sinh x \cdot \sinh y.
\] (II.19)

What about \(N \geq 4\)?

Nonlinear solutions of the form

\[
z(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n=N-1} z_i(x_i),
\] (II.20)

apparently do not exist:
while the resulting equation

\[
\sum_{i=1}^{n} z''_i \left(1 + \sum_{j \neq i} z^2_j\right) = 0
\] (II.21)

is ‘trivially’ solved for \(n = 2\), letting

\[
z''_1 = c(1 + z^2_1), \quad z''_2 = -c(1 + z^2_2),
\] (II.22)

\[
z''_i (z^2_j + c_{ij}) + z''_j (z^2_i + c_{ji}) = s_{ij},
\] (II.23)

does not (seem to) have any non-trivial solutions once \(n > 2\) (i.e. only the linear solution \(z_i(x_i) = a_i x_i + b_i\)).

For \(N = 4\), the Ansatz

\[
f^2_L = \alpha_L e^{+\kappa f_L} + \beta_L e^{-\kappa f_L},
\] (II.24)
yields solutions of (II.13), provided

\[
\alpha_L \alpha_K = \beta_L \beta_K,
\] (II.25)

\((L, K, L', K' \in \{1, 2, 3, 4\} \ all \ different!)\) e.g.

\[
\alpha_1 = \alpha_2 = 1 = \beta_3 = \beta_4
\] (II.26)

\[
\alpha_3 = \alpha_4 = -1 = \beta_1 = \beta_2,
\] yielding elliptic functions, resp.

\[
\Sigma_3 := \{ \vec{x} \in \mathbb{R}^4 \mid \varphi(x)\varphi(y) = \varphi(z)\varphi(v) \}
\] (II.27)

where \(\varphi\) is an elliptic Weierstrass-function, satisfying

\[
\varphi'^2 = 4\varphi(\varphi^2 - 1),
\] (II.28)

with \(\varphi\) real, taking its minimum, 1, at half-period (while diverging at 0, \(2w, \ldots\)).

Note that \(\alpha_3 = \alpha_4 = 0 = \beta_1 = \beta_2\) (and \(\kappa = 2, \alpha_1 = \alpha_2 = 1 = \beta_3 = \beta_4\)) i.e. \(f^2_{i=1,2} = e^{2f_i}, \ f^2_{i=3,4} = -e^{-2f_i}\) gives the known solution

\[
x_1 \cdot x_2 = x_3 \cdot x_4.
\] (II.29)
A slightly more elegant route to solving (II.13) is to note that
\[ f''_L(x_L) = \varepsilon_L Q_L(f_L(x_L)) = F_L(v_L) = f_L(x_L), \quad L = 1, 2, 3, 4, \quad \varepsilon_L \in \mathbb{R}, \]  
(II.30)
implying \(2f''_L f'_L = \varepsilon_L f'_L Q'_L\) will (when \(f'_L \neq 0\)) solve (II.13), resp.
\[ \sum_{L \neq K} \varepsilon_K \varepsilon_L Q_L(f'_L)Q'_K(f_K) = 0, \]  
(II.31)
provided
\[ Q_L(f_L)Q_K(f_K) + Q_K(f_K)Q'_L(f_L) = R_{LK}(f_L + f_K) \]  
(II.32)
with \(R_{LK} = R_{L'K'}\) (cp. (II.25)) any (!) odd function of its argument, resp. \(R_{LK} = (-)^{\sigma_{LK}} R_{L'K'}\) having parity \((-)^{\sigma_{LK}+1}\).
Examples are
\[ Q(f) = af(-x_1^2 + x_2^2 = x_3^2 + x_4^2 \text{ eg.)} \]
\[ Q(f) = \cosh f \text{ or } \sinh f \]
\[ Q(f) = \cos f \text{ or } \sin f, \]
(II.33)
the trigonometric \(Q\)’s giving elliptic solutions, like (II.27).

**Separable Minimal Hypersurfaces in \(\mathbb{R}^{N\geq 4}\):**

\[ \sum = \left\{ \overrightarrow{x} \in \mathbb{R}^N | u(\overrightarrow{x}) := \sum_{i=1}^N f_i(x_i) = 0 \right\} \]

\[(\nabla u)^2 \Delta u - u_i u_j u_{ij} = 0 \Rightarrow \sum f''_i f'_j = 0 \text{ on } u = 0.\]

Defining \(J_i(v_i = f_i(x_i)) := f''_i(x_i)\) (i.e. necessarily nonnegative) the basic equation to solve is
\[ J := \sum_{i \neq k} J_i J'_k \approx 0 \text{ (i.e. } = 0 \text{ on } \sum v_j = 0). \]  
(J)
Differentiating the basic equation \(J \approx 0\), using that \(F \approx 0\) implies \(\partial_{v_i} F - \partial_{u_i} F \approx 0\), and \(F \approx 0\) together with \(F\) not depending on one of the \(v_j\) implying \(F \equiv 0\), gives (applying \(\partial_k - \partial_i\) and denoting \(\Sigma_k := \sum_{i \neq k} J_i, \Sigma'_k := \sum_{i \neq k} J'_i\))
\[ J_{ki} := J_k \Sigma_k + J'_k \Sigma'_k - J'_i \Sigma_i - J_i \Sigma'_i \approx 0 \]  
(II.35)
as well as (applying \(\partial_{l} - \partial_{n}, ... \text{ all different}\))
\[ J_{ki,lm} := (J''_k - J''_i) (J'_l - J'_n) + (J''_i - J''_n) (J'_k - J'_l) \approx 0, \]  
(II.36)
and (applying again \(\partial_k - \partial_i\),)
\[ J_{(ki)^2,lm} = (J''_k + J''_i) (J'_l - J'_n) + (J''_l + J''_n) (J'_k - J'_i) \approx 0, \]  
(II.37)
hence, multiplying by \((J'_k - J'_i)\), and using (II.36),
\[ (J'_i - J'_n) \left\{ J''_k + J''_i (J'_k - J'_l) - (J''_l + J''_n) (J'_k - J'_l) \right\} \approx 0 \]  
(II.38)
For every $l, n$ with $(J'_l - J'_n) \neq 0$ one therefore has

$$(J''_k + J''_i)(J'_k - J'_i) - (J''_k + J''_i)(J'_k - J'_i) = 0 \quad (\text{II.39})$$

(for all distinct $(ki)$ different from $(ln)$).

Differentiating with respect to $v_k$ gives

$$J'''_k J'_k - J'''_i J'_i = J'''_k J'_i - J'''_i J'_k \quad (\text{II.40})$$

and then w.r.t. $v_i$, finally,

$$J'''_k J''_k - J'''_i J''_i = 0 \quad (\text{II.41})$$

i.e. (here derived for all $N > 3$ and all $i$ different from $l,n$ for which $J'_l \neq J'_n$; for $N=3$ see [3]), if $J_k$ is nonlinear,

$$J'''_i = cJ''_i; \quad (\text{II.42})$$

implying (and when inserted into (40), $e_i = e$)

$$J'''_i = cJ'_i + e_i \quad (\text{II.43})$$

$$J''_i = cJ'_i + e_i v_i + d_i$$

i.e. (if there are at least two nonlinear $J_j$’s)

$$J_i = \alpha_i e^{\sqrt{cv_i}} + \beta_i e^{-\sqrt{cv_i}} - \frac{d_i}{c} - \frac{e_i}{c} v_i \quad (\text{II.44})$$

for $c \neq 0$, and

$$J_i = \frac{e_i}{6} v_i^3 + \frac{d_i}{2} v_i^2 + b_i v_i + a_i \quad (\text{II.45})$$

for $c = 0$. Inserting (II.43) into (II.39) yields $e_i = e$ and (with some separation constant $d$)

$$(J''_i)^2 = c(J'_i)^2 + 2eJ'_i + d \quad (\text{II.46})$$

As the form (44/45) includes linear functions one can (if at least 2 of the $J_j$’s are nonlinear) simply insert it, for all $i$ and $k$, into (J) - finding that non-linearities actually are impossible if $N > 4$, and for $N=4$ (the linear parts having to vanish because of the single positive and negative exponentials necessitating opposite signs for the linear parts) the only possibility being

$$J_i(v_i) = \alpha_i e^{\sqrt{cv_i}} + \beta_i e^{-\sqrt{cv_i}} \quad (\text{II.47})$$

with $(iki’k’)$ all different

$$\alpha_i \alpha_k = \beta_i \beta_k' \quad (\text{II.48})$$

(which in particular implies that the products of the coefficients of the $v_i$-exponentials are independent of $i$, a condition that one also finds as a consequence of (39)).
Let us now discuss the special case that all $J_{\alpha=1} \neq 0$ are of the form $b_i v_i + a_i$, except $J_1(v_i)$. It is easy to see that all the $b_i$ must be the same ($J_1$ being non-linear) in which case

$$\sum_{\alpha \neq \beta} J'_\alpha J_\beta = \sum_{\alpha=2}^N b \sum_{\beta \neq \alpha} (bv_\beta + a_\beta) = (N - 2)b(b \sum v_\beta + \sum a_\beta) \approx (N - 2)b(-bv_1 + A)$$

(II.49)

$$J_1' \sum_\alpha J_\alpha + J_1 \sum J'_\alpha \approx J'_1(-bv_1 + a) + (N - 1)bJ_1$$

(II.50)

$$J(v := bv_1 - a) := J_1(v_1)$$

will have to satisfy

$$-J'v + (N - 1)J = (N - 2)v,$$

hence $J(v) = v + Cv^{N-1}$, i.e.

$$f_1^2(x_1) = (bf_1 - a) + C(bf_1 - a)^{N-1}.$$  

(II.51)

As $bf_\alpha + a_\alpha = J_\alpha = f_\alpha^2(x_\alpha)$ implies

$$f_\alpha = \frac{b}{4}(x_\alpha - t_\alpha)^2 - \frac{a_\alpha}{b}$$

(II.52)

the corresponding hypersurface $\Sigma := \{ \vec{x} \in \mathbb{R}^N, \sum f_\alpha + f_1 = 0 \}$, given by

$$\sum (x_\alpha - t_\alpha)^2 = \frac{4}{b}(-f_1 + \frac{a}{b}) = \frac{4}{b^2}(bf_1 - a) =: g(x_1) = r^2(x_1),$$

(II.53)

is also given by solving the ODE (cp.(51))

$$g^2 = 4g \left( -C \left( \frac{-b^2}{4}g \right)^{N-2} - 1 \right),$$

(II.54)

resp.

$$r^2 + 1 + C \left( \frac{-b^2 r^2}{4} \right)^{N-2} = 0.$$  

(II.55)

It is easy to check that one indeed gets the Weierstrass-function(s) $\wp(x_1)$ as solutions of (54)$_{N=4}$ and the catenoid(s) $r(z) = \frac{1}{e} \cosh(\varepsilon z + d)$ as solutions of (55)$_{N=3}$, $(\varepsilon := \frac{b}{2} \sqrt{+C})$.

Finally, if all $J_i$ are linear ($= b_i v_i + a_i$), one finds the condition $\sum_{i \neq j} b_j (b_i v_i + a_i) \approx 0$, i.e., with $B := \sum b_i$,

$$B b_i - b_i^2 = const$$

(II.56)

$$\sum a_i B = \sum a_i b_i.$$  

Solving the quadratic equation,

$$b_i = \frac{B}{2} \pm \sqrt{\frac{B^2}{4} - const.} = \frac{B}{2} (1 \pm \sqrt{1 - const}) = b_\pm,$$  

(II.57)
it follows that the $b_i$ can take only 2 different values, say $r < \frac{N}{2}$ times $\frac{B}{2}(1 + \sqrt{1-c})$ and $N - r$ times $\frac{B}{2}(1 - \sqrt{1-c})$, hence (summing II.57)

$$2 = \frac{2}{B} \sum b_i = r(1 + \sqrt{1-c}) + (N - r)(1 - \sqrt{1-c}) = N - (N - 2r)\sqrt{1-c},$$

which implies $\sqrt{1-c} = \frac{N - 2r}{N - 2r}$, i.e. after scaling (multiplying with $N - 2r$ and dividing by $|B|$): $r$ times $\pm(N - r - 1)$ and $(N - r)$ times $\mp(r - 1)$.

**Minimal Surfaces from Rotating Shapes**

What kind of $M$-dimensional objects can be moved such that a higher dimensional minimal surface (in some constant-curvature embedding space; $\mathbb{R}^N, \mathbb{R}^{1,N}, S^N, \ldots$) results? This question is more or less fully understood for the lowest dimension ($M = 1, \mathbb{R}^{N=3}$ being classic; this case can be reduced to $M = 0$, i.e. simple point motion generating the 1-dimensional object, e.g. as being the trace of a point on a circle rolling around another circle). On the other hand, as found more than 2 decades ago [7], the Ansatz

$$\left(x^\mu(t, \varphi)\right)_{\mu=0,1,2} = \left(t \quad R(wt) \quad \vec{u}(\varphi)\right)$$

for a minimal surface in $\mathbb{R}^{1,2}$, with

$$R(wt) \cdot \vec{u}(\varphi) = \begin{pmatrix} \cos(wt) & -\sin(wt) \\ \sin(wt) & \cos(wt) \end{pmatrix} \begin{pmatrix} u_1(\varphi) \\ u_2(\varphi) \end{pmatrix}$$

(describing the (constant angular velocity) rotation of a parametrized planar curve

$$\vec{u}(\varphi) = r(\varphi) \begin{pmatrix} \cos \theta(\varphi) \\ \sin \theta(\varphi) \end{pmatrix}$$

and leading to the equations

$$\partial_\alpha \left( \frac{1}{\sqrt{\vec{u}^2(1 - w^2r^2) + w^2(\vec{u} \times \vec{u})^2}} \begin{pmatrix} -\vec{u}^2 \\ w(\vec{u} \times \vec{u}) \\ w(\vec{u} \times \vec{u}) \end{pmatrix} \begin{pmatrix} \alpha \beta \\ \beta \alpha \end{pmatrix} \partial_\beta x^\mu \right)_{\mu=0,1,2} = 0,$$

due to (II.62)$_{\mu=0}$ implying

$$\partial_\varphi \begin{pmatrix} wr \sin \phi \\ \sqrt{1 - w^2r^2 \cos^2 \phi} \end{pmatrix} = 0$$

(II.63)
where \( \phi = \angle(\vec{u}, \vec{u}') \) is the angle between \( \vec{u} \) and \( \vec{u}' \), allows one to conclude that the shape of the curve \( \vec{u} \) is given by the simple equation

\[
w^2r^2(1 + \gamma \sin^2 \phi) = 1, \quad \gamma + 1 = \frac{1}{\gamma_0} (\text{if } w^2r^2 < 1)
\]

(II.64)

where \( \gamma \) is a constant of integration. This derivation of the shape is somewhat simpler than the standard one (calculating the mean-curvature from (II.59), giving

\[
\vec{u}'^2w^2(\vec{u}, A\vec{u}') = (w^2r^2 - 1)(\vec{u}', A\vec{u}''),
\]

(II.65)

with \( A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), resp. deriving a second-order differential equation for \( \theta \) as a function of \( r \), which (being of first order Bernoulli-type in \( g' = -\frac{\vec{g}}{w} \)) can be linearized, and twice integrated, the very last step being equivalent to solving (II.64) after using that

\[
\sin^2 \phi = \frac{r^2\theta'^2}{r^2 + r^2\theta'^2}(\varphi) = \frac{w^2r^2g'^2}{1 + w^2r^2g'^2}(r),
\]

(II.66)

i.e.

\[
g'^2(r) = \frac{1 - w}{w(w(1 + \gamma) - 1)} (w := w^2r^2).
\]

(II.67)

In order to compare with the corresponding Euclidean calculation (see e.g. [12]) one could rewrite (II.59) by substituting

\[
t = \frac{v}{w} + g(u(\varphi))
\]

(II.68)

and then notice that

\[
R(wt)\vec{u}(\varphi) = R(v)R(wg(u(\varphi)))) |\vec{u}| \begin{pmatrix} \cos\theta(\varphi) \\
\sin\theta(\varphi) \end{pmatrix} = R(v)u \begin{pmatrix} 1 \\
0 \end{pmatrix}
\]

(II.69)

if one defines \( g \) to undo the rotation \( R(\theta) \) by which \( \begin{pmatrix} \cos\theta \\
\sin\theta \end{pmatrix} \) results from \( \begin{pmatrix} 1 \\
0 \end{pmatrix} \) – and then choosing \( \varphi = r = |\vec{u}| = u \). The class of solutions (of (II.64), resp. (II.62), resp. (II.67)) considered in [7] were \( (n \neq k, nk > 0) \)

\[
\vec{u}(\varphi) = \frac{1}{2n} \begin{pmatrix} \cos n\varphi \\
\sin n\varphi \end{pmatrix} + \frac{1}{2k} \begin{pmatrix} \cos k\varphi \\
\sin k\varphi \end{pmatrix}
\]

(II.70)

for which, with
\[ c := \cos \left( \frac{n-k}{2} \varphi \right), s = \sin \left( \frac{n-k}{2} \varphi \right) \]

\[ w^2(\vec{u}^2) = 1 + \frac{4nk}{(n-k)^2}c^2 \]

\[ w^2 = \frac{4n^2k^2}{(n-k)^2}, \quad \vec{u}^2 = c^2, \quad \sin^2 \phi = \frac{c^2(1 + \frac{4nk}{(n-k)^2})}{1 + \frac{4nk}{(n-k)^2} + c^2}. \tag{II.71} \]

Note that they can be written (for later convenience) in the form

\[ 2\vec{u}_\pm(\varphi) = \left( \frac{1}{n} R^n(\varphi) \pm \frac{1}{k} R^k(\varphi) \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{II.72} \]

making the calculations leading to (II.71) very simple, just using

\[ R'(\varphi) = AR(\varphi) \]
\[ R^T = R^{-1} \]
\[ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -A^T = -A^{-1}; \]

\[ e.g. \ 2\vec{u}_\pm = A(R^n \pm R^k) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ 4\vec{u}^2 = (1, 0)(R^{-n} \pm R^{-k})(-A^2)(R^n \pm R^k) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]
\[ = (1, 0)(1 \pm 1 \pm R^{n-k} \pm R^{k-n}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]
\[ = 2(1 \pm \cos((n-k)\varphi)) = 4c^2, \tag{II.73} \]

resp. 4s^2. The above given frequency \( w = \frac{2nk}{k-n} \) is special to (II.70), alone for the following reason:

\[ \ddot{x}(t, \varphi) := R(\varphi t)\vec{u}(\varphi) \]
\[ = \left( \frac{1}{2n} R^n \left( \varphi + \frac{w}{n} t \right) + \frac{1}{2k} R^k \left( \varphi + \frac{w}{k} t \right) \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]
\[ = \left( \frac{1}{2n} R^n \left( \tilde{\varphi} + \tilde{t} \right) + \frac{1}{2k} R^k \left( \tilde{\varphi} - \tilde{t} \right) \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]
\[ =: \ddot{x}(\tilde{t} = t, \tilde{\varphi} := \varphi - \frac{n+k}{n-k} t) = \frac{1}{2} \vec{v}_+ (\tilde{\varphi} + \tilde{t}) - \frac{1}{2} \vec{v}_- (\tilde{\varphi} - \tilde{t}) \tag{II.74} \]
is of the same type as \( \vec{u} \) (only \( \varphi \to \tilde{\varphi} + \tilde{t}, \varphi \to \tilde{\varphi} - \tilde{t} \) by a reparametrization of \( \varphi \) alone (!), in the 2 terms) and \( \left( \frac{t}{\tilde{t}} \right) \) a difference of 2 Null-curves in \( \mathbb{R}^{1,2} \).

\[
\left( \frac{t}{\tilde{t}} \right) = \frac{1}{2} \left( \varphi + t = \varphi_+ \right) - \frac{1}{2} \left( \varphi - t = \varphi_- \right),
\]

(II.75)

While a lot is known about Null curves in relation with minimal surfaces in \( \mathbb{R}^{1,2} \), and their Weierstraß representations, the crucial question is whether (and if yes, how) any of these structures can be also used for \( M > 1 \). (The \( \varphi_\pm \) decomposition certainly is special to \( M + 1 = 2 \)). The most appealing seems to be that the technical simplifications following from (II.72) are matched by the important geometric property of the Epicycloids (having \( |n - k| \) cusps) as being obtained by rolling circles (with one point marked) around circles. Consider (see e.g. ["Epicycloids", Wikipedia]) rolling a circle of radius \( a \) around a circle of radius \( b \) (centered at \( \vec{0} \)); then

\[
\vec{x}(\psi) = R(\psi) \underbrace{R_{\vec{e}=\vec{a}+\vec{b}}(\vec{a} \psi) \vec{b}}_{\vec{e}+R(\vec{b} \psi)\vec{e}+\vec{c}}.
\]

\[
= R(\psi) \begin{pmatrix} a + b - R(\psi) \left( \frac{b}{a} \psi \right) a \\ R((1+\frac{b}{a})\psi) \end{pmatrix}
\]

\[
= (a + b) \begin{pmatrix} \cos \psi \\ \sin \psi \end{pmatrix} - a \begin{pmatrix} \cos(1 + \frac{b}{a})\psi \\ \sin(1 + \frac{b}{a})\psi \end{pmatrix}
\]

\[
\psi = \chi a \begin{pmatrix} \cos(\chi a) \\ \sin(\chi a) \end{pmatrix} - a \begin{pmatrix} \cos(\chi(a+b)) \\ \sin(\chi(a+b)) \end{pmatrix}
\]

(II.76)

For \( a = n \) and \( a + b = k \) this is proportional to

\[
\vec{u}_- = \frac{1}{2n} \begin{pmatrix} c_n \\ s_n \end{pmatrix} - \frac{1}{2k} \begin{pmatrix} c_k \\ s_k \end{pmatrix}
\]

(II.77)

(i.e. not (II.70), but with a relative sign); choosing \( \chi = \varphi + \frac{n \pi}{n-k} \) (however) one obtains

\[
\vec{u}_- \left( \varphi + \frac{n \pi}{n-k} \right) = \left( \frac{1}{2n} R^n(\varphi) R \left( \frac{n \pi}{n-k} \right) - \frac{1}{2k} R^k(\varphi) R \left( \frac{k \pi}{n-k} \right) \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

\[
= R \left( \frac{n \pi}{n-k} \right) \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \left( \frac{1}{2n} R^n(\varphi) + \frac{1}{2k} R^k(\varphi) \right) \right\} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

\[
= R \left( \frac{n \pi}{n-k} \right) \vec{u}_+(\varphi).
\]

(II.78)
The shape equation (II.64), \( w^2 r^2 (1 + \gamma \sin^2 \phi) = 1 \), for the curve \( \vec{u}(\varphi) = \left( \begin{array}{c} u_1(\varphi) \\ u_2(\varphi) \end{array} \right) = r(\varphi) \left( \begin{array}{c} \cos \theta(\varphi) \\ \sin \theta(\varphi) \end{array} \right) \), \( \phi := \angle(\vec{u}, \vec{u}') \), can for \( r' \neq 0 \neq r \) be written as

\[
\frac{1}{w^2 r^2} - 1 = \gamma \sin^2 \phi = \frac{\gamma r^2 \theta'^2(\varphi)}{r^2 + r^2 \theta'^2(\varphi)} = \gamma \frac{r^2 \hat{\theta}'^2(r)}{1 + r^2 \theta'^2(r)},
\]

when taking \( r \) to parametrized the curve, i.e. using \( \theta(\varphi) = \hat{\theta}(r(\varphi)) \Rightarrow \theta' = \frac{d\hat{\theta}}{dr} \cdot \dot{r} \), implying

\[
r^2 \hat{\theta}'^2 = \frac{w - 1}{1 - \delta w},
\]

with \( \delta := 1 + \gamma = \frac{1}{\delta_0} \) and \( w := w^2 r^2 \), i.e.

\[
\pm \int d\hat{\theta} = \int \sqrt{\frac{w - 1}{1 - \delta w}} \frac{dr}{r} = \frac{1}{2 \sqrt{\delta}} \int \frac{dw}{w} \sqrt{\frac{w - 1}{\gamma_0 - w}} =: \gamma_0 J.
\]

Calculating \( J \) with the substitution

\[
v := \sqrt{\frac{w - 1}{\gamma_0 - w}} = \frac{\sin \psi}{\cos \psi} = \tan \psi, \ \psi \in \left( 0, \frac{\pi}{2} \right),
\]

with

\[
w = \gamma_0 \sin^2 \psi + \cos^2 \psi,
\]

one obtains

\[
\pm (\hat{\theta}(r) - \theta_0) = \gamma_0 \arctan v - \arctan(\gamma_0 v),
\]

resp.

\[
\pm \tan(\hat{\theta} - \theta_0) = \frac{\tan(\gamma_0 \psi) - \gamma_0 \tan \psi}{1 + \gamma_0 \tan \psi + \tan(\gamma_0 \psi)} = \frac{(a - b) \sin_0 \cos - (a + b) \sin \cos_0}{(a - b) \cos_0 \cos + (a + b) \sin \sin_0}, \ \gamma_0 = \frac{a + b}{a - b}, \ b > 0,
\]

with \( \sin := \sin \left( a - \frac{b}{2} \varphi \right), \ \cos := \cos \left( a - \frac{b}{2} \varphi \right) \), resp. \( \sin_0 := \sin \left( a + \frac{b}{2} \varphi \right), \ \cos_0 := \cos \left( a + \frac{b}{2} \varphi \right) \),

\[
= \frac{b \sin a \varphi - a \sin b \varphi}{b \cos a \varphi - a \cos b \varphi},
\]

which is! the tangent of \( \theta \) of

\[
\lambda \vec{u}(\varphi) = \frac{1}{2a} \left( \begin{array}{c} \cos a \varphi \\ \sin a \varphi \end{array} \right) - \frac{1}{2b} \left( \begin{array}{c} \cos b \varphi \\ \sin b \varphi \end{array} \right),
\]

cp (II.76).
III. Minimal Tori in $S^3$ and Stiefel manifolds

One way of formulating the problem of finding minimal (hyper) surfaces in spheres (rather than in $\mathbb{R}^N$) is to subject the usual parametric area functional

$$A[\vec{x}] = \int \sqrt{g} \, d^M \varphi$$

$$g = \text{det}(\partial_a \vec{x} \partial_b \vec{x})_{a,b=1,\ldots,M}$$

(III.1)

to the constraint $\vec{x}^2 (\varphi) = 1$, i.e. to consider

$$S[\vec{x}] := A[\vec{x}] + \frac{1}{2} \int d^M \varphi \; \lambda(\vec{x}^2 - 1),$$

(III.2)

whose stationary points $\vec{x} (\varphi)$ are then easily seen to satisfy

$$\Delta \vec{x} := \frac{1}{\sqrt{g}} \partial_a \sqrt{g} g^{ab} \partial_b \vec{x} = \lambda \; \vec{x} = -M \; \vec{x},$$

(III.3)

where the last equality in (III.3) easily follows by multiplying the inner part with $\vec{x}$ (and noting that $\vec{x}^2 (\varphi) = 1$ implies $\vec{x} \partial_a \vec{x} = 0$ and, differentiating again, $\partial_b \vec{x} \partial_a \vec{x} + \vec{x} (\partial^2_{ab} \vec{x}) = 0$, hence $\vec{x} \Delta \vec{x} = g^{ab} \vec{x} \partial^2_{ab} \vec{x} = -M$).

The celebrated clifford torus CT being

$$\vec{x}^T (\varphi^1, \varphi^2)^M=2 = \frac{1}{\sqrt{2}} (\cos \varphi^1, \sin \varphi^1, \cos \varphi^2, \sin \varphi^2)$$

(III.4)

let us try (as in [8]) to find solutions as graphs over CT, i.e. of the form

$$\vec{x} (\varphi^1, \varphi^2) = \begin{pmatrix} \cos \theta \cos \varphi^1 \\ \cos \theta \sin \varphi^1 \\ \sin \theta \cos \varphi^2 \\ \sin \theta \sin \varphi^2 \end{pmatrix} =: \begin{pmatrix} c \cos_1 \\ c \sin_1 \\ s \cos_2 \\ s \sin_2 \end{pmatrix}.$$ 

(III.5)

$$\partial_1 \vec{x} = c \begin{pmatrix} -s_1 \\ c_1 \\ 0 \\ 0 \end{pmatrix} + \theta_1 \begin{pmatrix} -c_1 \\ 0 \\ s_1 \\ 0 \end{pmatrix} = c \; \vec{e}_1 + \theta_1 \; \vec{e}$$

(III.6)

$$\partial_2 \vec{x} = s \begin{pmatrix} 0 \\ 0 \\ -s_2 \\ c_2 \end{pmatrix} + \theta_2 \; \vec{e}, \; \theta_a := \frac{\partial \theta}{\partial \varphi^a}$$
imply
\[
\mathcal{g}_{ab} = \begin{pmatrix}
c^2 + \theta_1^2 & \theta_1 \theta_2 \\
\theta_1 \theta_2 & s^2 + \theta_2^2
\end{pmatrix},
\]
and the Ansatz \( \vec{m} = \alpha_1 \vec{e}_1 + \alpha_2 \vec{e}_2 + \alpha \vec{e} \), automatically orthogonal to \( \vec{x} \), and easily giving
\[
\vec{m} \parallel \vec{e} = -\theta_1 \vec{e}_1 - \theta_2 \vec{e}_2,
\]
when requiring orthogonality with (III.6), implies that
\[
\mathcal{g}_{ab} \mathcal{h}_{ab} = 0 \quad (\text{which is the non-trivial content of (III.3), as its components in the direction(s) of } \partial_1 \vec{x}, \partial_2 \vec{x} \text{ and } \vec{x} \text{ are trivially satisfied})
\]
corresponds to
\[
\begin{pmatrix}
\mathcal{g} \mathcal{h} \\
\mathcal{g} \mathcal{h}_1 + (s^2 - c^2) \mathcal{h}_2 - 2 \theta_1 \theta_2 (sc \theta_1 \theta_2 + s^2 - c^2)
\end{pmatrix}
\]
so that \( \mathcal{g}_{ab} \mathcal{h}_{ab} = 0 \) (which is the non-trivial content of (III.3), as its components in the direction(s) of \( \partial_1 \vec{x}, \partial_2 \vec{x} \) and \( \vec{x} \) are trivially satisfied) corresponds to
\[
\begin{pmatrix}
\mathcal{g} \mathcal{h} \\
\mathcal{g} \mathcal{h}_1 + (s^2 - c^2) \mathcal{h}_2 - 2 \theta_1 \theta_2 (sc \theta_1 \theta_2 + s^2 - c^2)
\end{pmatrix}
\]
also following directly by varying (cp. (III.7), (III.2), (III.5))
\[
S[\theta] := \int \sqrt{c^2 s^2 + (k^2 s^2 + l^2 c^2) \theta^2} \sqrt{c^2 s^2 + (k^2 s^2 + l^2 c^2) \theta^2} d\varphi^1 d\varphi^2.
\]
For \( \theta = \theta(t := k \varphi^1 + l \varphi^2) \), (III.10) reduces to the (highly non-linear) ODE
\[
sc \{ sc \theta (k^2 s^2 + l^2 c^2) + \theta^2 \left[(l^2 - k^2)s^2 c^2 + 2s^4 k^2 - 2c^2 l^2\right] + s^2 c^2 (s^2 - c^2)\} = 0
\]
corresponding to stationary points of (cp. (III.11), the overall sign put in for later convenience)
\[
\int \left(L := -\sqrt{c^2 s^2 + (k^2 s^2 + l^2 c^2) \theta^2}\right) dt.
\]
Switching now to physical terminology, interpreting \( \theta \) as the t ime-dependent position of a ‘particle’ moving in some ‘potential’ one identifies a conserved quantity, resp. integration of (III.12) via the ‘Hamiltonian’
\[
K := \dot{\theta} \frac{\partial L}{\partial \dot{\theta}} = \ldots = \frac{s^2 c^2}{\sqrt{c^2 s^2 + (k^2 s^2 + l^2 c^2) \theta^2}} = \text{const} =: E,
\]
i.e.
\[
\dot{\theta}^2 + \frac{c^2 s^2}{k^2 s^2 + l^2 c^2} \left(1 - \frac{c^2 s^2}{E^2}\right) = 0,
\]
with the second term an ‘effective potential’ (of the mass \( \frac{1}{2} \) particle) for a ‘zero-energy’ solution (with respect to which the positive integration constant \( E \) should perhaps
better be denoted by $\kappa^2$, to reflect the somewhat dangerous double interpretation of 'Energy') as the rhs (III.15) and, expressing (III.14) in terms of $\pi := \frac{\partial L}{\partial \theta}$,

$$ |\sin \theta \cos \theta| \sqrt{1 - \frac{\pi^2}{k^2 s^2 + l^2 c^2}}. \quad (\text{III.16}) $$

That (III.15) is a consequence of (III.12) (the reverse is trivially verified) can of course also be checked directly (without referring to physics terminology): (III.12) is, after dividing by $s^2 c^2 =: r(\theta)$, and multiplying by $2 \theta$, of the form $(L = -\sqrt{r + f^2} \quad f(\theta) := k^2 s^2 + l^2 c^2)$

$$ 2 f \ddot{\theta} + f' \dot{\theta}^3 = \dot{r}' + 2 f' r' \dot{\theta}^3, $$

$$ \dot{\theta} \left( r' + 2(f \dot{\theta}^2) \frac{r'}{r} \right) = \left( f \dot{\theta}^2 \right)' = \frac{d}{dt} (G(\theta(t))), \quad (\text{III.17}) $$

giving the first order ODE

$$ G' = r' + 2G' \frac{r'}{r}, \text{ resp} $$

$$ r^2 \left( \frac{G}{r^2} \right)' = r' $$

whose solution is $\frac{G}{r} = -\frac{1}{r} + \text{const.}$, i.e.

$$ Cr^2 - r = G(\theta(t)) = f \dot{\theta}^2 \quad \text{(III.18)} $$

with $C = \frac{1}{E^2} > 0$, as the right hand side is manifestly non-negative.

While the case $k = 0$ (resp. $l = 0$), solvable with the help of elliptic integrals, is well discussed in the differential geometry community (cp. [9]) the case $k = l \neq 0$ can be solved in terms of elementary functions as fallows: with

$$ a := \frac{1}{2E}, \quad (\text{III.15}) \text{ reads} $$

$$ \dot{\alpha}^2 + \sin^2 \alpha \left( 1 - a^2 \sin^2 \alpha \right) = 0. \quad (\text{III.19}) $$

Assuming $a = \cosh \gamma \geq 1, \quad (E \leq \frac{1}{2})$ in order to have turning-points, i.e. $\dot{\alpha} = 0$ for some $t$) the particle ($\alpha$) oscillates between $\alpha_- = \arcsin 2E < \frac{\pi}{2}$, and $\alpha_+ = \pi - \arcsin 2E$, while direct integration of (III.19), yields

$$ \varphi - \varphi_0 = \pm \int \frac{d\alpha}{\sin \alpha \sqrt{a^2 \sin^2 \alpha - 1}} = \mp \arctan \frac{\cos \alpha}{\sqrt{a^2 \sin^2 \alpha - 1}} \quad (\text{III.20}) $$

$$ \sin \alpha(\varphi) = \frac{1}{\sqrt{\cos^2(\varphi - \varphi_0) + \cosh^2 \gamma \sin^2(\varphi - \varphi_0)}} $$

$$ \cos \alpha(\varphi) = \frac{-\sin \gamma \sin(\varphi - \varphi_0)}{\sqrt{\cos^2(\varphi - \varphi_0) + \cosh^2 \gamma \sin^2(\varphi - \varphi_0)}} \quad (\text{III.21}) $$
i.e. a one parameter class ($e := \sinh \gamma$) of ‘minimal’ (extremal) tori in $S^3$, 

$$\vec{x}_e(\varphi^1, \varphi^2) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 - \frac{e \sin(\varphi - \varphi_0)}{\sqrt{1 + e^2 \sin^2}}} \\ \frac{\sin \varphi^1}{\cos \varphi^2} \\ \frac{\cos \varphi^1}{\sin \varphi^2} \end{pmatrix}.$$ (III.22)

While it is easy to see that they are without intersections (i.e. embedded) the consequence, namely that they must be congruent to (III.4), is stunning.

As (III.20) appears identically in the equation for geodesics on $S^2$ (i.e. great circles), and the Hopf map (s.b.) applied to (III.22) gives

$$\begin{pmatrix} \cos 2\theta \\ \sin 2\theta \cos(\varphi^2 - \varphi^1) \\ \sin 2\theta \sin(\varphi^2 - \varphi^1) \end{pmatrix}$$ (III.23)

( and the signs of $k = \pm l$ were never used above, so that one could as well have defined $t$ to be $\varphi^2 - \varphi^1$(rather than $\varphi^2 + \varphi^1$ for which one would need a ‘conjugate’ Hopf map) one may view the solutions (III.22) as a one parameter clars of inverse images of great circles on $S^2$.

The problem however rests in the vast freedom in the construction, which becomes apparent when trying to fix the details, using quaternions:

$$q = (q_0, q_1, q_2, q_3) \xrightarrow{\hat{\cdot}} \tilde{q} = \begin{pmatrix} q_0 - iq_3 \\ -iq_1 + q_2 \\ q_0 + iq_3 \end{pmatrix} \cong q_0 + iq_1 + j q_2 + k q_3,$$ (III.24)

using either matrix multiplications for $\tilde{q}$, or

$$(t, \vec{x}) \cdot (s, \vec{y}) = (ts - \vec{x} \cdot \vec{y}, t \vec{y} + s \vec{x} + \vec{x} \times \vec{y}),$$ (III.25)

resp. $i^2 = j^2 = k^2 = -1, ij = k, ...$ .

Defining an anti automorphism via

$$\tilde{q} := q_0 - iq_1 + j q_2 + h q_3$$ (III.26)

the Hopf-map from $S^3$ to $S^2$ can be given as

$$\pi(q) := \tilde{q}q,$$ (III.27)

nicely fitting with the action of $S^3$ onto itself by right-multiplication ($q \rightarrow q r$), and

$$\pi(q r) = \tilde{r} \pi(q) r$$ (III.28)
defining an action of $S^4$ on $S^2$ ($q_1 \equiv 0$). In coordinates one finds
\[
\begin{pmatrix}
t'

y'
z'
\end{pmatrix}
= \begin{pmatrix}
t^2 + x^2 - y^2 - z^2

2(tz - xy)

2(ty + xz)
\end{pmatrix}
\]
for $\pi(q)$ and for (III.28)
\[
\begin{pmatrix}
t''
y''
z''
\end{pmatrix}
= \begin{pmatrix}
1 - 2(r_2^2 + r_3^2)

2(r_1 r_3 - r_0 r_2)

-2(r_0 r_3 + r_1 r_2)

2(r_0 r_2 + r_1 r_3)

1 - 2(r_1^2 + r_2^2)

2(r_0 r_1 - r_2 r_3)

2(r_0 r_3 - r_1 r_2)

1 - 2(r_1^2 + r_3^2)
\end{pmatrix}
\begin{pmatrix}
t'
y'
z'
\end{pmatrix}
\]
i.e. written as an ordinary SO(III.3) transformation on $S^2$. As $\pi(e^{i\varphi}q) = \pi(q)$, and for the great circle
\[
\begin{pmatrix}
\cos 2\theta \\
\sin 2\theta \cos \beta \\
\sin 2\theta \sin \beta
\end{pmatrix}
= \begin{pmatrix}
\cos \beta & 0 & \sin \beta \\
0 & 1 & 0 \\
-\sin \beta & 0 & \cos \beta
\end{pmatrix}
\]
with the constant angle $\beta$ related to the parameter $e$ in (III.22), while a rotation
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \tilde{\gamma} & -\sin \tilde{\gamma} \\
0 & \sin \tilde{\gamma} & \cos \tilde{\gamma}
\end{pmatrix}
\]
with possibly non-trivial $\tilde{\gamma}(\varphi^1, \varphi^2)$, geometrically leaves the equation invariant, one finds 3 different transformations to possibly act on the ordinary Clifford-torus,
\[
\tilde{x}^T \sim (\tilde{\varphi}^1, \tilde{\varphi}^2) = \frac{1}{\sqrt{2}} (\cos \tilde{\varphi}^1, \sin \tilde{\varphi}^1, \cos \tilde{\varphi}^2, \sin \tilde{\varphi}^2),
\]
(III.32)

namely:
cos $\rho$ + $i$ sin $\rho$ from the left (as that does not change $\pi(q)$), $(-\sin \tilde{\gamma} + i \cos \tilde{\gamma})$ from the right-leaving the equation invariant, while transforming $(c_1 s_1, c_2, s_2)^T$ to
\[
\begin{pmatrix}
-\sin \tilde{\gamma} & 0 & 0 & \cos \tilde{\gamma} \\
0 & -\sin \tilde{\gamma} & \cos \tilde{\gamma} & 0 \\
0 & -\cos \tilde{\gamma} & -\sin \tilde{\gamma} & 0 \\
\cos \tilde{\gamma} & 0 & 0 & -\sin \tilde{\gamma}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
s_1 \\
c_2 \\
s_2
\end{pmatrix}
\]
(III.33)
as well as $\cos \frac{\beta}{2} - \sin \frac{\beta}{2} \cdot k$ from the right (corresponding to (III.31). Leaving out (III.33) (i.e. choosing $-\sin \tilde{\gamma} = 1$) for the moment, one would find
\[
\cos \rho \begin{pmatrix}
c_1 c + s_2 s \\
c_1 s_2 - c_2 s \\
c_2 + s_1 \\
c_2 - s_1
\end{pmatrix}
+ \sin \rho \begin{pmatrix}
sc_2 - cs_1 \\
cc_1 + ss_2 \\
cc_2 + ss_2 \\
cc_1 + ss_2
\end{pmatrix}
= \sqrt{2} \begin{pmatrix}
\cos \theta \\
\sin \theta \\
c_1 \\
sc_2
\end{pmatrix}
\]
(III.34)
which is ‘ almost ’ (but not quite) solvable, (note that even if it was, the question why $\rho(\varphi^1, \varphi^2)$ does not destroy minimality would still have to be answered).
Note that the right-action $q \to qr$ may also be written as

$$q \to \begin{pmatrix} r_0 & -\vec{r}^T \\ \vec{r} & (r_0 \mathbf{1} - \vec{r} \times) \end{pmatrix} \begin{pmatrix} q_0 \\ \vec{q} \end{pmatrix}$$  \tag{III.35}$$

Instead of trying to match the freedom in the quaternion description of the Hopf-map relation between geoderics on $S^2$ and minimal surfaces in $S^3$ let us note that in [ACH13] an explicit reparametrisation,

$$\tilde{\varphi}^1 = \varphi^1 + \int^{\tilde{r}} u = \varphi^1 + f_1(\varphi), \quad \tilde{\varphi}^2 = \varphi^2 + \int^{\tilde{r}} v = \varphi^2 + f_2(\varphi),$$  \tag{III.36}$$

was constructed that transforms the metric induced from (III.32), $\tilde{g}_{ab} = \frac{1}{2} \delta_{ab}$, into the metric induced from (III.22), (cp. (III.7), (III.22), (III.15).

$c = \cos \theta(\varphi), s = \sin \theta(\varphi); \frac{1}{2E} = \sqrt{1 + \epsilon^2}$

$$g_{ab} = \begin{pmatrix} c^2 + \frac{\vartheta^2}{\tilde{\vartheta}} & \frac{\vartheta}{\tilde{\vartheta}} \\ \frac{\vartheta}{\tilde{\vartheta}} & s^2 + \frac{\vartheta^2}{\tilde{\vartheta}} \end{pmatrix} = \begin{pmatrix} c^4 + s^4c^4 \epsilon^2 & s^2c^2 \left( \frac{3s^2c^2}{\epsilon^2} - 1 \right) \\ s^2c^2 \left( \frac{3s^2c^2}{\epsilon^2} - 1 \right) & s^4 + \frac{s^4c^4}{\epsilon^2} \end{pmatrix}$$

$$= \begin{pmatrix} g + c^4 & g - s^2c^2 \\ g - s^2c^2 & g + s^4 \end{pmatrix} = J^T(\tilde{g}_{ab})J = \frac{1}{2} J^T J$$

$$= \frac{1}{2} \begin{pmatrix} 1 + 2u + u^2 + v^2 & u + v + u^2 + v^2 \\ u + v + u^2 + v^2 & 1 + 2v + u^2 + v^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (1 + u)^2 + v^2 & u(1 + u) + v(1 + v) \\ u(1 + u) + v(1 + v) & (1 + v)^2 + u^2 \end{pmatrix},$$  \tag{III.37}$$

$$J := \left( \frac{\partial \tilde{\varphi}^a}{\partial \varphi^b} \right) = \begin{pmatrix} 1 + u & u \\ v & 1 + v \end{pmatrix},$$  \tag{III.38}$$

from which

$$u = -s^2 + \frac{s^2c^2}{E} = \sqrt{g} - s^2, \quad v = -c^2 + \frac{s^2c^2}{E} = \sqrt{g} - c^2, \tag{III.39}$$

follows. Note that $u - v = c^2 - s^2$, and

$$\sqrt{g} = \sqrt{s^2c^2 + \vartheta^2} = \frac{s^2c^2}{E}. \tag{III.40}$$

Let us now show that also the second fundamental forms are transformed into each other.
under the transformation (III.36), with \( u \) and \( v \) given by (III.39), resp.

\[
\begin{align*}
  u &= \frac{1}{2} \left( 1 + \frac{e \sin \sqrt{1 + e^2 \sin^2}}{\sqrt{1 + e^2 \sin^2}} \right) \sqrt{1 + e^2 \left( 1 - \frac{e \sin \sqrt{1 + e^2 \sin^2}}{\sqrt{1 + e^2 \sin^2}} \right)} - 1 \\
  v &= \frac{1}{2} \left( 1 - \frac{e \sin \sqrt{1 + e^2 \sin^2}}{\sqrt{1 + e^2 \sin^2}} \right) \sqrt{1 + e^2 \left( 1 + \frac{e \sin \sqrt{1 + e^2 \sin^2}}{\sqrt{1 + e^2 \sin^2}} \right)} - 1 \\
&= \frac{1}{2} \left[ -1 - \frac{e \sin \sqrt{1 + e^2 \sin^2}}{\sqrt{1 + e^2 \sin^2}} + \frac{1 + e^2}{1 + e^2 \sin^2} \right]
\end{align*}
\]

\( \sin = \sin(\varphi - \varphi^c) \).

Namely, using (III.39), (III.12) and (III.15) \((k = l = 1 \text{ for simplicity})\) one finds

\[
\begin{align*}
  h_{ab} &= \frac{s c^2}{E} \left( \begin{array}{cc} 2c^2 & c^2 - s^2 \\ c^2 - s^2 & -2s^2 \end{array} \right) = \sqrt{g} \left( \begin{array}{cc} 2c^2 & c^2 - s^2 \\ c^2 - s^2 & -2s^2 \end{array} \right) \\
  &= \frac{1}{2} \left[ -1 - \frac{e \sin \sqrt{1 + e^2 \sin^2}}{\sqrt{1 + e^2 \sin^2}} + \frac{1 + e^2}{1 + e^2 \sin^2} \right]
\end{align*}
\]

for the minimal surfaces (III.22). Note that due to \(-4s^2c^2 - (c^2 - s^2)^2 = -1, h = -g \) (hence \( K = -1, R = 0 \)) is manifest, just as \( g^{ab}h_{ba} = 0 \).

Transforming on the other hand the second fundamental form of (III.32)

\[
\tilde{h}_{ab} = \frac{1}{2} \left( \begin{array}{cc} -1 & 0 \\ 0 & +1 \end{array} \right)
\]

choosing the orientation via \( \vec{n} := \frac{1}{\sqrt{2}} \left( \begin{array}{c} -\tilde{c}_1 \\ -\tilde{s}_1 \\ +\tilde{c}_2 \\ +\tilde{s}_2 \end{array} \right) \) one finds

\[
\begin{align*}
  h_{ab} : &= -\frac{1}{2} J^T \left( \begin{array}{cc} -1 & 0 \\ 0 & +1 \end{array} \right) J \\
  &= -\frac{1}{2} \left( \begin{array}{cc} v^2 - (u + 1)^2 & v(v + 1) - u(u + 1) \\ v(v + 1) - u(u + 1) & (v + 1)^2 - u^2 \end{array} \right) \\
  &= +\frac{1}{2} \left( \begin{array}{cc} (u + 1)^2 - v^2 & u(u + 1) - v(v + 1) \\ u(u + 1) - v(v + 1) & u^2 - (v + 1)^2 \end{array} \right) \\
  &= \frac{1}{2} \left( \begin{array}{cc} (\sqrt{g} + c^2)^2 - (\sqrt{g} - c^2)^2 & (\sqrt{g} - s^2)(\sqrt{g} + c^2) - (\sqrt{g} - c^2)(\sqrt{g} + s^2) \\ (\sqrt{g} - s^2)(\sqrt{g} + c^2) - (\sqrt{g} - c^2)(\sqrt{g} + s^2) & (\sqrt{g} - s^2)^2 - (\sqrt{g} + s^2)^2 \end{array} \right) \\
  &= \sqrt{g} \left( \begin{array}{cc} 2c^2 & (c^2 - s^2) \\ (c^2 - s^2) & -2s^2 \end{array} \right)
\end{align*}
\]

(III.44)
With both first and second fundamental forms coinciding, insertion of (III.36) into (III.32), yielding

\[
\hat{x} = \cos \varphi_1 \frac{1}{\sqrt{2}} \begin{pmatrix}
\cos f_1 \\
\sin f_1 \\
\cos(f_2 + \varphi) \\
\sin(f_2 + \varphi)
\end{pmatrix} + \sin \varphi_1 \frac{1}{\sqrt{2}} \begin{pmatrix}
-sin f_1 \\
\cos f_1 \\
\sin(f_2 + \varphi) \\
-cos(f_2 + \varphi)
\end{pmatrix} \tag{III.45}
\]

\[
= \cos \varphi_1 \hat{e}_1 (\varphi) + \sin \varphi_1 \hat{e}_2 (\varphi),
\]

should differ from (III.22), resp. \((\varphi^2 = \varphi - \varphi^1)\)

\[
\hat{x} = \cos \varphi_1 \begin{pmatrix}
\cos \theta(\varphi) \\
0 \\
\sin \theta \cos \varphi \\
\sin \theta \sin \varphi
\end{pmatrix} + \sin \varphi_1 \begin{pmatrix}
0 \\
\cos \theta(\varphi) \\
\sin \theta \sin \varphi \\
-\sin \theta \cos \varphi
\end{pmatrix} \tag{III.46}
\]

\[
= \cos \varphi_1 \hat{e}_1 (\varphi) + \sin \varphi_1 \hat{e}_2 (\varphi),
\]

only by a fixed (\(\varphi\)-independent) orthogonal transformation \(S\), i.e. should hold that

\[
\hat{e}_i (\varphi) = S \hat{e}_i (\varphi). \tag{III.47}
\]

While at first glance hard to believe (as e.g. implying that the 4 components of \(\hat{e}_1 (\varphi)\), resp. \(\hat{e}_2 (\varphi)\), must be linearly dependent \(^4\) i.e. define a \(\varphi\)-independent hyperplane -one of the components of \(\hat{e}_1 (\varphi)\), resp. \(\hat{e}_2 (\varphi)\), being zero) help comes from the (proof of) the fundamental theorem for surfaces, which (given equality of first and second fundamental forms) constructs \(\hat{x}\), resp. \(\hat{x}\), from given initial conditions. Choosing for simplicity (and without lose of generality) \(\varphi^0 = 0\) and \(\varphi^1 = 0 = \varphi^2\) as well as the integration constants defining \(f_1\) and \(f_2\) such that \(f_1(0,0) = 0 = f_2(0,0)\), and noting that (cp. (III.22)) \(\theta_0 = \theta(0,0) = \frac{\pi}{4}\) as well as \(\theta_0 \equiv \theta (0,0) = \frac{\pi}{2}\), whereas \(f_1(0,0) = u_0 = u(0,0) = \frac{1}{2} (\sqrt{1 + e^2} - 1) = v(0,0) = v_0 = f_2(0,0)\), one finds

\[
\hat{x}_0 := \hat{x} (0,0) = \hat{x} (0,0) = x_0 = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 \\
0 \\
0 \\
1
\end{pmatrix} =: n_0 = \hat{n}_0
\]

\[
\hat{x}_+ := (\partial_1 + \partial_2) \hat{x} (0,0) = \frac{1}{\sqrt{2}} \begin{pmatrix}
-f \varepsilon \\
1 \\
e \varepsilon \\
1
\end{pmatrix} =: \sqrt{1 + e^2} \frac{1}{\sqrt{2}} \begin{pmatrix}
-sin \varepsilon \\
cos \varepsilon \\
\sin \varepsilon \\
cos \varepsilon
\end{pmatrix} =: \sqrt{1 + e^2} \hat{n}_+
\]

\(^4\) I am grateful to J.Arnold, M.Bordemann and B.Durhuus for helpful discussions when trying to resolve this puzzle.
\[ \vec{x}_+ := \left( (\partial_1 + \partial_2) \vec{x}_+ \right) (0, 0) = \sqrt{1 + e^2 \frac{1}{\sqrt{2}}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} =: \sqrt{1 + e^2} \vec{n}_+ \]

\[ \vec{x}_- := \left( (\partial_1 - \partial_2) \vec{x}_- \right) (0, 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} =: \vec{n}_- \]

\[ \vec{x}_- := \left( (\partial_1 - \partial_2) \vec{x}_- \right) (0, 0) =: \vec{n}_- = \vec{n}_- \]

Hence one is looking for an orthogonal transformation \((R)\) leaving \(\vec{n}_0\) and \(\vec{n}_-\) fixed, transforming \(\vec{n}_+\) into \(\vec{n}_+\), i.e.

\[ R \vec{n}_+ = \vec{n}_+ = \cos \varepsilon \vec{n}_+ + \sin \varepsilon \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} =: \vec{n}_+ \]

hence (choosing \(R\) to be a \(\varepsilon\)-rotation with \((\vec{n}_+, \vec{n}_-)\)-plane)

\[ R \vec{n}_- = -\sin \varepsilon \vec{n}_- + \cos \varepsilon \vec{n}_- \]

Instead of directly verifying \((III.47) (s=R)\) it is instructive to consider the action of \(R\) on \((III.32)\), resp.

\[ \vec{x}_+ (\tilde{\varphi}^1, \tilde{\varphi}^2) = \frac{1}{2} (\cos \tilde{\varphi}^1 + \cos \tilde{\varphi}^2) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} - \frac{1}{2} (\cos \tilde{\varphi}^1 - \cos \tilde{\varphi}^2) \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} 
+ \frac{1}{2} (\sin \tilde{\varphi}^1 + \sin \tilde{\varphi}^2) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} + \frac{1}{2} (\sin \tilde{\varphi}^1 - \sin \tilde{\varphi}^2) \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} \]

\[ = \frac{1}{2} \left\{ (\tilde{c}_1 + \tilde{c}_2) \vec{n}_0 - (\tilde{c}_1 - \tilde{c}_2) \vec{n}_+ + (\tilde{s}_1 + \tilde{s}_2) \vec{n}_+ + (\tilde{s}_1 - \tilde{s}_2) \vec{n}_- \right\} \]

i.e.

\[ R \vec{x}_+ = \frac{1}{2} \left\{ (\tilde{c}_1 + \tilde{c}_2) \vec{n}_0 - (\tilde{c}_1 - \tilde{c}_2) \begin{pmatrix} \cos \varepsilon \vec{n}_- - \sin \varepsilon \vec{n}_+ \end{pmatrix} 
+ (\tilde{s}_1 + \tilde{s}_2) \begin{pmatrix} \cos \varepsilon \vec{n}_+ + \sin \varepsilon \vec{n}_0 \end{pmatrix} + (\tilde{s}_1 - \tilde{s}_2) \vec{n}_- \right\} 
+ (\tilde{s}_1 - \tilde{s}_2) \left\{ \begin{pmatrix} \cos \varepsilon \vec{n}_+ + \sin \varepsilon \vec{n}_0 \end{pmatrix} \right\} \]

\[ = \frac{1}{2} \left\{ (\tilde{c}_1 + \tilde{c}_2) \vec{n}_0 - (\tilde{c}_1 - \tilde{c}_2) \begin{pmatrix} \cos \varepsilon \vec{n}_- - \sin \varepsilon \vec{n}_+ \end{pmatrix} 
+ (\tilde{s}_1 + \tilde{s}_2) \begin{pmatrix} \cos \varepsilon \vec{n}_+ + \sin \varepsilon \vec{n}_0 \end{pmatrix} + (\tilde{s}_1 - \tilde{s}_2) \vec{n}_- \right\} 
+ (\tilde{s}_1 - \tilde{s}_2) \begin{pmatrix} \cos \varepsilon \vec{n}_+ + \sin \varepsilon \vec{n}_0 \end{pmatrix} \]

\[ = \frac{1}{2} \left\{ (\tilde{c}_1 + \tilde{c}_2) \vec{n}_0 - (\tilde{c}_1 - \tilde{c}_2) \begin{pmatrix} \cos \varepsilon \vec{n}_0 \end{pmatrix} + (\tilde{s}_1 + \tilde{s}_2) \begin{pmatrix} \cos \varepsilon \vec{n}_0 \end{pmatrix} + (\tilde{s}_1 - \tilde{s}_2) \vec{n}_- \right\} 
+ (\tilde{s}_1 - \tilde{s}_2) \begin{pmatrix} \cos \varepsilon \vec{n}_0 \end{pmatrix} \]
which is supposed to equal (III.22),

\[
\begin{pmatrix}
1 + c & -s & 1 - c & -s \\
\frac{1}{\sqrt{2}} \begin{pmatrix}
\tilde{c}_1 \\
\tilde{s}_1 \\
\tilde{c}_2 \\
\tilde{s}_2
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
1 + c & s & 1 - c & s \\
-1 - c & s & 1 + c & s \\
-1 - c & s & 1 + c & s \\
-1 & 1 + c & s & 1 + c
\end{pmatrix}
\begin{pmatrix}
\cos \theta c_1 \\
\cos \theta s_1 \\
\sin \theta c_2 \\
\sin \theta s_2
\end{pmatrix}
= \frac{1}{\sqrt{2}}
\begin{pmatrix}
\cos \theta c_1 \\
\cos \theta s_1 \\
\sin \theta (c_1 \cos \varphi + s_1 \sin \varphi) \\
\sin \theta (c_1 \sin \varphi - s_1 \cos \varphi)
\end{pmatrix},
\]  
(III.53)

the orthogonal 4 × 4 matrix on the left \((c = \cos \varepsilon = \frac{1}{\sqrt{1+\varepsilon^2}}, \ s = \sin \varepsilon = \frac{\varepsilon}{\sqrt{1+\varepsilon^2}})\) being the looked for (cp. (III.47)) matrix \(S(= R)\). Simple inversion

\[
\begin{pmatrix}
\cos \theta c_1 \\
\cos \theta s_1 \\
\sin \theta (c_1 \cos \varphi + s_1 \sin \varphi) \\
\sin \theta (c_1 \sin \varphi - s_1 \cos \varphi)
\end{pmatrix}
\]
(III.54)

then, writing the l.h.s. as

\[
\begin{pmatrix}
\cos f_1 & -\sin f_1 \\
\sin f_1 & \cos f_1 \\
0 & 0 \\
0 & \begin{pmatrix}
\cos(f_2 + \varphi) & +\sin(f_2 + \varphi) \\
\sin(f_2 + \varphi) & -\cos(f_2 + \varphi)
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
s_1 \\
c_1 \\
s_1
\end{pmatrix}
\]
(III.55)

turns out to be consistently equivalent to

\[
\sqrt{2} \cos f_1 = (1 + c) \cos \theta + (1 - c) \sin \theta \cos \varphi + s \sin \theta \sin \varphi
\]

\[
\sqrt{2} \sin f_1 = -s \cos \theta + s \sin \theta \cos \varphi - (1 - c) \sin \theta \sin \varphi
\]

\[
\sqrt{2} \cos(f_2 + \varphi) = (1 - c) \cos \theta + \sin \theta [(1 + c) \cos \varphi - s \sin \varphi]
\]

\[
\sqrt{2} \sin(f_2 + \varphi) = -s \cos \theta + \sin \theta [(1 + c) \sin \varphi + s \cos \varphi].
\]

As the sum of the first 2 squares (crucially using \(e \tan 2\theta \sin \varphi = -1\)) gives indeed 2, one could in principle forget eqs (III.37)-(III.44) and simply define \(f_1\) and \(f_2\) (cp. (III.36)) by (III.56).

The hyperplane property for \(\tilde{c}_i(\varphi)\) finally follows, as (III.56) implies

\[
s \cos(f_2 + \varphi) + (1 - c) \sin(f_2 + \varphi) = \sqrt{2} s \sin \theta \cos \varphi
\]

\[
= s \cos f_1 + (1 + c) \sin f_1
\]
Pedestrian level-set proof that the Stiefel-manifolds $\sum_{n,k}$ are minimal in the corresponding sphere $S^{nk-1}(\sqrt{k})$, resp. the cone $\sum_{n,k}$ minimal in $\mathbb{R}^{nk}$: consider $k$ orthogonal $n-$vectors $\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(n)} \in \mathbb{R}^n$, of equal length, i.e. the constraints $(a,b = 1 \cdots k)$

\[ u^{(a \neq b)} = u^{[ab]} := \vec{x}^{(a)} \cdot \vec{x}^{(b)} \overset{!}{=} 0 \quad (\# = \frac{k(k - 1)}{2}) \]  

\[ v^{(a)} := \frac{1}{2} \left( (\vec{x}^{(a)})^2 - (\vec{x}^{(a+1)})^2 \right) \overset{!}{=} 0 \quad (a = 1 \cdots k - 1) \]  

defining $\sum_{n,k} \subset \mathbb{R}^{nk} (\text{dim} = nk - (k - 1)(1 + \frac{k}{2}) = nk - \frac{k(k+1)}{2} + 1)$.

\[ \text{Area}(\sum_{n,k}) = \int d^{nk}x \prod_{a=1}^{k-1} \delta(v^{(a)}(x)) \prod_{a < b} \delta(u^{[ab]}(x)) \sqrt{\det M}, \]  

where $M$ is the matrix formed by the scalar products of the gradients of the $K = \frac{1}{2}(k - 1)(k + 2) = \frac{k(k+1)}{2} - 1$ constraints $W^{(A)}_{A=1 \cdots K}$ (i.e. $M_{AB} := \nabla W^{(A)} \cdot \nabla W^{(B)}$). The minimality condition then reads (see e.g. [11])

\[ \delta \left( \prod_B W^{(B)} \right) \cdot Tr(P \cdot \partial^2 W^{(A)}) \overset{!}{=} 0, \quad A = 1 \cdots K \]  

where $P$ is the projector onto the subspace orthogonal to all the $\nabla W^{(A)}$ (and projects onto the tangent-space of $\sum_{n,k}$ when $\chi := \prod_A W^{(A)} = 0$),

\[ P_{ij} = \delta_{ij} - \partial_i W^{(A)} (M^{-1})^{AB} \partial_j W^{(B)}. \]  

For $\sum_{n,k}$, resp. (III.57)/(III.58) one can calculate (III.61), resp. $\hat{P} := P|_{\chi=0}$ explicitly, and prove that (III.60) is satisfied.

Simple(st) non-trivial example: $n = 3, k = 2$

\[ u^{[12]}(\vec{y}) = u := x_1 x_4 + x_2 x_5 + x_3 x_6 (\overset{!}{=} \vec{x} \cdot \vec{y}) = 0 \]

\[ v^{(1)}(\vec{y}) = v := \frac{1}{2} \left( x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 - x_6^2 \right) = \frac{1}{2} (x^2 - y^2)^2 \overset{!}{=} 0 \]

(2 orthogonal vectors of equal length)

\[ ^3\text{an elegant, earlier, proof [10] has been communicated to me by J.Choe.} \]
\[ \nabla u = \begin{pmatrix} \vec{y} \\ \vec{x} \end{pmatrix}, \quad \nabla v = \begin{pmatrix} \vec{x} \\ -\vec{y} \end{pmatrix}, \quad (\nabla^2 u)^2 = (\nabla^2 v)^2 = \frac{r^2}{x^2 + y^2} = 2s^2, \quad \nabla u \cdot \nabla v = 0 \]

\[ M = r^2 \mathbb{1}, \quad M^{-1} = \frac{1}{r^2} \mathbb{1} \]

\[ P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{r^2} \begin{pmatrix} y_1^2 & y_\alpha y_\beta \\ y_2^2 & x_1 y_1 \\ y_3^2 & x_2 y_2 \\ x_1^2 & x_3 y_3 \end{pmatrix} \]

\[ \partial^2 u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \partial^2 v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \delta(v) \text{Tr}(P \partial^2 u) = 0, \quad \delta(u) \text{Tr}(P \partial^2 v) = 0 \]
even without \( \delta(u) \cdot \delta(v) \)

\[ K = \frac{2 \cdot 3}{2} - 1 = 2, \quad \text{dim} \hat{\Sigma}_{3,2} = 6 - 2 = 4, \quad \text{dim} \Sigma_{3,2} = 3. \]

Another example: \( n = 3, k = 3 \)

\[ u_3 = u^{[12]} = \vec{x} \cdot \vec{y}, \quad u_1 = u^{[23]} = \vec{y} \cdot \vec{z}, \quad u_2 = u^{[13]} = \vec{x} \cdot \vec{z} \]

\[ v^{(1)} = \frac{1}{2} (\vec{x}^2 - \vec{y}^2), \quad v^{(2)} = \frac{1}{2} (\vec{y}^2 - \vec{z}^2) \]

\[ \nabla u^{[12]} = \begin{pmatrix} \vec{y} \\ \vec{x} \end{pmatrix}, \quad \nabla u^{[23]} = \begin{pmatrix} 0 \\ \vec{z} \end{pmatrix}, \quad \nabla u^{[13]} = \begin{pmatrix} \vec{z} \\ 0 \end{pmatrix} \]

\[ \nabla v^{(1)} = \begin{pmatrix} \vec{x} \\ -\vec{y} \end{pmatrix}, \quad \nabla v^{(2)} = \begin{pmatrix} 0 \\ \vec{y} \end{pmatrix} \]

\[ K = 5, \quad \text{dim} \sum_{3,3} = 4 \subset \mathbb{R}^9, \quad \text{dim} \Sigma_{3,3} = 3 \subset S^8 \]
on \hat{\Sigma}_{3,3}, \text{ with } s^2 = \frac{r^2}{y}, \quad (\vec{x}^2 = \vec{y}^2 = \vec{z}^2), \text{ and } \hat{M} := M|_{\chi=0} \]

\(^6\)special thanks to G.Linardopoulos and T.Turgut for discussions going back to 2016/17
\[ \hat{M} = s^2 \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & -1 \\ -1 & 2 & 0 \end{pmatrix}, \quad \hat{M}^{-1} = \frac{1}{s^2} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \hat{M}^{-1} \]

\[ \partial^2 u_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \partial^2 u_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \]

\[ \partial^2 v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \partial^2 v_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

\[ \hat{P} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2s^2} \left[ \begin{pmatrix} y_\alpha y_\beta & y_\alpha x_\beta & 0 \\ x_\alpha y_\beta & x_\alpha x_\beta & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & z_\alpha z_\beta & z_\alpha y_\beta \\ 0 & y_\alpha z_\beta & y_\alpha y_\beta \end{pmatrix} \right] 
+ \begin{pmatrix} z_\alpha z_\beta & 0 & z_\alpha x_\beta \\ 0 & 0 & 0 \\ x_\alpha z_\beta & x_\alpha x_\beta \end{pmatrix} \right] - \frac{1}{3s^2} \left[ \begin{pmatrix} 0 & x_\alpha y_\beta & -x_\alpha z_\beta \\ 0 & -y_\alpha y_\beta & y_\alpha z_\beta \\ 0 & 0 & 0 \end{pmatrix} \right] - \frac{1}{3s^2} \begin{pmatrix} 0 & 0 & 0 \\ y_\alpha x_\beta & -y_\alpha y_\beta & 0 \\ -z_\alpha x_\beta & z_\alpha y_\beta & 0 \end{pmatrix} 
- \frac{2}{3s^2} \begin{pmatrix} x_\alpha x_\beta & -x_\alpha y_\beta & 0 \\ -y_\alpha x_\beta & y_\alpha y_\beta & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{2}{3s^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -y_\alpha y_\beta & -y_\alpha z_\beta \\ 0 & z_\alpha z_\beta & z_\alpha y_\beta \end{pmatrix} \]

\[ \langle k = 4 : \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \rangle = \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 4 \end{pmatrix} \]

To verify (III.60) for the general case (any \( k \leq n \in \mathbb{N} \)) is completely straightforward, except for the fact that \( \hat{\nabla} v^{(a)} \cdot \hat{\nabla} v^{(a+1)} \neq 0 \).

\[ \hat{M} = s^2 \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \]

however has the explicit inverse

\[ \hat{M}^{-1} = \frac{1}{ks^2} \left( \begin{pmatrix} k & 1 \\ 0 & Q \end{pmatrix} \right) = \hat{M}^{-1} \]

with the \((k - 1) \times (k - 1)\) matrix \( Q \) having the matrix elements

\[ Q_{\alpha \beta} = \min(a', b') \cdot k - a'b'. \]

(III.62)
With

\[ \partial_c u^{(a<b)} = \delta_{ca} x_{bi} + \delta_{cb} x_{ai} \]
\[ \partial_c v^{(a')} = \delta_{ca'} x_{a'i} - \delta_{ca'+1} x_{a'+1,i} \]

the projector \( P \) becomes:

\[
\hat{P}_{ci,dj} = \delta_{cd} \delta_{ij} - \frac{1}{2s^2} \sum (\delta_{ca} x_{bi} + \delta_{cb} x_{ai})(\delta_{da} x_{bj} + \delta_{db} x_{aj}) \\
- \frac{1}{ks^2} \sum_{a,b=1}^{k-1} (\delta_{ca} x_{ai} - \delta_{c,a+1} x_{a+1,i})(k \cdot \min(a, b) - a \cdot b)(\delta_{db} x_{bj} - \delta_{d,b+1} x_{b+1,j}).
\]

(III.63)

When acting on

\[
(\partial^2 u^{(a<b)})_{ci,dj} = \delta_{ij}(\delta_{ca} \delta_{db} + \delta_{cb} \delta_{da})
\]
\[ (\partial^2 v^{(e)})_{ci,dj} = \delta_{ij}(\delta_{ce} \delta_{de} + \delta_{c,e+1} \delta_{d,e+1}) \] (III.64)

and then taking the trace, the only non-trivial part is the action of the last term in (III.63) on (III.64), which is proportional to something that on the constraint manifold vanishes (here calculated/displayed for non-boundary e),

\[
\frac{1}{ks^2} x_e^2 \left[ (ek - e^2) + (ke - 1) - (e - 1)^2 \right] - \frac{1}{ks^2} x_{e+1}^2 \left[ (ke + 1) - (e + 1)^2 \right] \approx 0.
\]

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