Extracting maximal entanglement from linear cluster states

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Quantum information processing architectures usually only allow for nearest-neighbour entanglement creation. In many cases, this prevents the direct generation of maximally entangled states, which are commonly used for many communication and computation tasks. Here, we show how to obtain maximally entangled GHZ states between vertices that are initially connected by a minimum number of connections, which specifically allows them to share linear cluster states. We prove that the largest GHZ state that can be obtained from a linear cluster state of \( n \) qubits, using local Clifford unitaries, local Pauli measurements and classical corrections, is of size \( \lfloor (n+3)/2 \rfloor \). We demonstrate exactly which qubit selection patterns are possible below this threshold and which are not, and implement the transformation on the IBMQ Montreal quantum device for linear cluster states of up to \( n = 19 \) qubits.

I. INTRODUCTION

Recent years have seen exciting developments in quantum computation and communication, both in theory and experiment. Building upon the year-long research on bipartite settings, focus has now also turned towards multipartite settings, where multiple vertices in a network share quantum resources between them. While the correlations of maximally entangled states, in particular Greenberger-Horne-Zeilinger (GHZ) states\textsuperscript{[1]} have naturally been the first to explore, other types of graph states have also been extensively examined\textsuperscript{[2–5]}. The states\textsuperscript{[1]} have naturally been the first to explore, other types of GHZ states\textsuperscript{(14,15)} showed how to extract three- and four-partite GHZ states from linear cluster states. Here, we focus on the transformations of linear cluster states\textsuperscript{[13]} to GHZ states, i.e. on transforming the graph states that are naturally distributable in linear quantum networks to maximally entangled states.

Such transformations require the removal of some of the qubits from the state by measuring them, such that only a selected subset of the qubits of the resource linear cluster state can in the end belong to the target GHZ state. We refer to these transformations as GHZ extractions. A previous study\textsuperscript{[14]} showed how to extract three- and four-partite GHZ states from linear cluster states. Here, we conclude this study by providing a complete characterisation of which GHZ extractions are possible and which are not. Very importantly, we provide a tight upper bound to the size of the largest GHZ state that can be extracted, equal to \( \lfloor (n+3)/2 \rfloor \); interestingly this is slightly higher than the one of \( n/2 \) conjectured in Ref.\textsuperscript{[13]}. In addition to our theoretical analysis, we perform experimental implementations of the GHZ extractions from linear cluster states with \( n \in \{5, 7, \ldots, 19\} \) qubits on the IBMQ Montreal device.

Our manuscript is organized as follows: The notation, technical terminology and main definitions are introduced in Section II. Section III contains the main theoretical results. In Section IV the experiments are introduced, discussed, and their results presented. Finally, Section V discusses the obtained results and the opportunities for future research. The technical details are diverted to topical appendices: Appendix A contains the proof of an important lemma stated in the theoretical section, Appendix B contains technical details regarding the post-processing steps during the extractions, and Appendix C contains technical details regarding the data analysis of the experimental section.

II. NOTATION AND TERMINOLOGY

In this work, two quantum graph states play a central role: we define linear cluster states \( |L\rangle \) and GHZ states \( \text{GHZ} \) as

\[
|L\rangle_{\{1,\ldots,n\}} := \frac{1}{2^n} \bigotimes_{i=1}^{n} (|0\rangle + |1\rangle \sigma_z^{i+1})
\]

\[
\text{GHZ}_{\{1,\ldots,m\}} := \frac{1}{\sqrt{2^m}} \left( \bigotimes_{i=1}^{m} |0\rangle + \bigotimes_{i=1}^{m} |1\rangle \right)
\]

and \( |L\rangle_V, |\text{GHZ}\rangle_V \) as the states corresponding to the vertex set \( V \). When context permits, with e.g. \( |L\rangle_n \) we denote the linear cluster state of size \( n \).

Our resource state is the \( n \)-partite linear cluster state \( |L\rangle_{V_L} \). As a graph state it corresponds to a line graph on the vertices \( V_L := \{1, 2, \ldots, n\} \). Here, each vertex \( i \) corresponds to the \( i \)-th qubit of \( |L\rangle_{V_L} \) and the edges of the graph correspond to nearest-neighbour entangling controlled phase gates. This structure allows us to use the terms left and right neighbours.
of \( i \) to indicate any vertices \( h, j \) with \( h < i, i < j \), respectively; e.g. the direct left and right neighbours of \( i \) are \( i \pm 1 \).

Let \( V_G \subset V_L \) be a set of vertices for which we can extract a GHZ state from the linear cluster resource state. Performing Pauli measurements on the qubits corresponding to \( V_M := V_L \setminus V_G \), we obtain a post-measurement state which is locally equivalent to the \(|\text{GHZ}\rangle_{V_G}| \) state. By performing local operations based on the measurement outcomes, the state can then be locally transformed into this GHZ state.

This construction allows for \( V_G \) to inherit the neighbour structure from the linear network \( V_L \): For a vertex \( j \in V_G \), we use \( j_- \) and \( j_+ \) to indicate the left and right neighbour of \( j \) in \( V_G \), respectively. We refer to the smallest and largest element of \( V_G \) as the boundaries of the GHZ state. We finally define any selection of consecutive vertices \( i, i + 1, \ldots, i + k \in V_G \) as a \( k \)-island.

III. MAIN RESULTS

We now examine what are the different types of GHZ states one can obtain from a given linear cluster state. We first provide an upper bound for the size \(|V_G|\) of the extracted GHZ states, and we then show how to saturate it. In order to achieve this, we use Lemma 1 which provides an impossibility result for 2-islands (the proof can be found in App. A).

Lemma 1. No 2-island can have both a left and a right neighbour in \( V_G \). If two vertices \( i, i + 1 \) are in \( V_G \), then there is either no vertex to the left of \( i \) or no vertex to the right of \( i + 1 \).

Lemma 1 implies that all vertices \( i \) in the target GHZ state must be ‘isolated’ in the linear cluster state; \( i - 1 \) and \( i + 1 \) cannot be in \( V_G \) (with the exception of the boundaries). A corollary for 3-islands follows directly:

Corollary 1. If \( V_G \) contains a 3-island, then \(|V_G| = 3\).

Proof. Let \( i, i + 1, i + 2 \) be a 3-island in \( V_G \) and assume that \(|V_G| \geq 4\), i.e., that we have \( h < i \) or \( j > i + 2 \) in \( V_G \). This implies that either \( i, i + 1 \) form a 2-island with both left-neighbour \( h \) and right-neighbour \( i + 2 \) or \( i + 1, i + 2 \) form a 2-island with both left-neighbour \( i \) and right-neighbour \( j \). Both are in direct contradiction to Lemma 1.

By the same argument, \( k \)-islands with \( k \geq 4 \) are impossible. Ultimately, such \( k \)-islands would contain 3-islands in contradiction to Corollary 1. Figure illustrates examples.

This allows us to calculate the upper bound to \(|V_G|\).

Theorem 1. The size of a GHZ state extractable from an \( n \)-partite linear cluster state via local Clifford operations, local Pauli measurements, and local unitary corrections, is upper-bounded as \(|V_G| \leq \left\lfloor \frac{n+3}{2} \right\rfloor\).

Proof. As there are at most two 2-islands, for every other \( i \) in \( V_G \), both \( i \pm 1 \) were measured. Thus, to maximize \(|V_G|\), we may have \( 1, 2, n - 1, n \) in \( V_G \), and \( V_M \) containing every other vertex in between: For \( n \) odd, \( V_M = \{3, 5, \ldots, n - 2\} \); for \( n \) even \( V_M = \{3, 5, \ldots, n - 5, n - 3, n - 2\} \). In the even case, \( n - 2 \) must be measured due to Corollary 1. In both cases, \(|V_G| = n - |V_M|\) is upper bounded by \( \left\lfloor \frac{n+3}{2} \right\rfloor \).

1 In the case of \( n \) being even, there is more than one such pattern. While we have chosen here to measure the two consecutive vertices, \( n - 3 \) and \( n - 2 \),
For example, the largest GHZ state that can be extracted from the 7-qubit linear cluster state shown in Figure 1 is the GHZ state where \( V_G = \{1, 2, 4, 6, 7\} \) and \( V_M = \{3, 5\} \). Figure 1 further shows all possible and impossible selections of \( V_G \) to extract the GHZ state.

We now show that there is a set of measurements that saturates the bound of Theorem 1 by explicitly giving such a measurement pattern. For \( n \leq 5 \) this pattern was shown in Ref. [14]. For the general case, let us consider a case distinction with respect to the parity of \( n \):

For odd \( n \), we can choose \( V_M = \{2i + 1\}_{i=0}^{n-3} \) and every corresponding qubit to be measured in the \( \sigma_z \)-basis; we refer to this measurement pattern as the maximal pattern.

The linear cluster state is a stabilizer state, i.e. it is an element of the shared +1 eigenspace of the operators \( \{l_i = \sigma_z^{-1} \sigma_x^{i+1} \}_{i=1}^n \), where \( \sigma_z^0 \) and \( \sigma_z^{n+1} \) are set equal to the identity. This set of operators forms the set of canonical generators of an Abelian subgroup of the \( n \)-qubit Pauli group known as the stabilizer of the linear cluster states. For an overview of the stabilizer formalism and stabilizer measurements in particular see [15], [16].

Consider the generator transformation

\[
l_2 \rightarrow \tilde{l}_2 = l_2 l_{4} \ldots l_{n-4} l_{n-2} := \sigma_x^1 \prod_{2 \in V_L} \sigma_x^{2i},
\]

which ensures that \( \tilde{l}_2 \) and all odd-indexed generators commute with all measurement operators \( \{\sigma_x^{j0}\}_{j \in V_M} \). The post-measurement state is determined by replacing the other \( |V_M| \) generators \( \{l_{2i}\}_{i=2}^{n-2} \) with the measurement operators – together with a multiplicative phase depending on the respective measurement outcome. Then (after removing the support on the measured qubits and applying a Hadamard transformation to 1 and \( n \)) the post-measurement state on \( V_G \) is characterized by the generators \( \sigma_x^0 \) and \( \{m_{j0} \sigma_x^{j0} \sigma_x^{j0+1}\}_{j0 \in V_G \setminus \{n\}} \), where the \( m_{j0} = \pm 1 \) are phases due to the measurement outcomes. These phases can be accounted for by applying \( \sigma_x \) operations to the selection of the nodes, recovering the generators of the \( \text{GHZ}_{V_G} \) state. The number of measurements implies \( |V_G| = n - |\{3, 5, \ldots, n-2\}| = \frac{n+3}{2} \) which saturates the bound for odd \( n \).

For even \( n \), it suffices to observe that a \( \sigma_x \)-measurement on the qubit corresponding to \( n \) yields a linear cluster state \( |L\rangle_{\{1,2,\ldots,n-1\}} \) up to a randomized \( \sigma_x^{-1} \)-correction depending on the measurement outcome. In analogy to the odd parity case we then obtain \( V_M = \{3, 5, \ldots, n-3, n\} \) such that \( |V_G| = n - |V_M| = \frac{2n+2}{2} = \frac{n+2}{2} \) for even \( n \).

Note that the even-case analysis above also applies for measuring an ‘internal’ node in the \( \sigma_x \)-basis, rather than the first or last; this does introduce a Clifford rotation on the two neighbours of the node which needs to be accounted for [17]. The resulting state is then also LOCC equivalent to an \( (n-1) \)-partite linear cluster state on the remaining nodes, from which in turn a \( \text{GHZ}_{n+2} \) state can be extracted through the maximal pattern. This approach can be extended to more measurements, where additional ‘inside’ nodes are measured in the \( \sigma_y \)-basis, and ‘outside’ nodes are measured in the \( \sigma_z \)-basis. It is straightforward to see that any choice \( V_G \) allowed by Lemma 1 can be seen as arising from such a setting.

Finally, note that while Lemma 1 does allow 2-islands on the boundaries of the extracted GHZ states, they do not necessarily have to be contained in them. For example, \( \text{GHZ}_{\{1,3,5,7\}} \) can be extracted from \( |L\rangle_{\{1,\ldots,7\}} \) as shown in Figure 1. Rigorously stated, this pattern does not arise from any of the maximal patterns defined above, but can instead be considered as a maximal pattern \( \text{GHZ}_{\{0,1,3,5,7,8\}} \) extracted from \( |L\rangle_{\{0,\ldots,8\}} \). Here, the additional qubits corresponding to 0 and 8 are just “virtual” and not really there; they simply help visualize all possible patterns: \( \text{GHZ}_{\{1,3,5,7\}} \) can be extracted from the “virtual” state \( |GHZ}_{\{0,1,3,5,7,8\}} \) by measuring qubits 0 and 8 in the \( \sigma_z \)-basis. The measurements on the other qubits are unaffected by this; the physical measurements of 2, 4, 6 to obtain \( \text{GHZ}_{\{1,3,5,7\}} \) from \( |L\rangle_{\{1,\ldots,7\}} \) are exactly the same as the ones that would be required to obtain \( \text{GHZ}_{\{0,1,3,5,7,8\}} \) from the “virtual” \( |L\rangle_{\{0,\ldots,8\}} \). In this sense, all possible selections of \( V_G \) can be seen as subsets of the maximal measurement patterns defined above.

These measurement patterns result in states LOCC equivalent to GHZ states; for explicit calculations of the necessary corrections to obtain the GHZ states themselves we refer to the supplementary material in [17].

IV. EXPERIMENTAL IMPLEMENTATION

We used the IBMQ Montreal device to experimentally demonstrate our protocol for the maximal extraction of GHZ states from resource linear cluster states. For odd \( n \in \{5,7,\ldots,19\} \) we prepared the state

\[
|\psi\rangle_n = \bigotimes_{i \text{ odd}} H_i |L\rangle_{\{1,\ldots,n\}},
\]

i.e. the linear cluster state with every odd qubit rotated to the \( \sigma_x \)-basis. We then extract GHZ states for \( m \in \{\frac{n+1}{2}\} = \{4,5,\ldots,11\} \) using the maximal pattern described in the previous section.

To benchmark the results, we compute an estimate for the lower bound of the fidelity for both the linear cluster states and the GHZ states extracted from them. For the linear cluster states we use methods adapted from [13] using insights originally presented in [19]; two measurement settings suffice to estimate the lower bound – one in which all qubits are measured in the \( \sigma_x \)-basis, and one in which all qubits are measured in the \( \sigma_y \)-basis. For the GHZ states we derive a similar technique. Again, two measurement settings suffice – one where all the qubits of the GHZ state are measured in the \( \sigma_z \)-basis, and one where all the qubits of the GHZ are measured in the \( \sigma_x \)-basis. Both these measurement settings are performed in

other possibilities would have been to measure consecutive vertices further to the left and measure only the even vertices to the right. Another option would have been to measure not two consecutive vertices, but a vertex of one of the 2 islands, i.e. either 1,2,n − 1 or n. It is important to note that all resulting sets \( V_M \) have the same size.
v.

Figure 2: The circuit on the left and on the right are equal; the circuit on the left is the standard preparation of the $|L\rangle_n$ state. The generalization to $|L\rangle_n$ for higher (odd) $n$ follows naturally. The circuit on the right is the compiled version for the IBMQ Montreal chip. By not implementing the single-qubit gates in the grey box, the circuit depth is reduced by 25%. This results in a rotated linear cluster state $|\psi\rangle$ as defined in Eq. (1). This state carries the same entanglement properties as the $|L\rangle_n$ state, and most notably can still be used to extract $|GHZ\rangle$ states by adapting the measurement bases.

Figure 3: A lower bound of the fidelity of the rotated linear cluster and $GHZ$ states prepared on the IBMQ Montreal device. We prepared rotated (see Eq. (1)) linear cluster states $|\psi\rangle_n$ for $n \in \{5, 7, \ldots, 19\}$ and extracted $|GHZ\rangle_m$ states for $m \in \{4, 5, \ldots, 11\}$ using the maximal pattern introduced in Sec. III. For states with a higher number of qubits, the lower bound on the linear cluster state is increasingly worse due to a technical aspect of the estimation method (see App. C for details). The results are ordered such that each linear cluster state $|L\rangle_n$ is paired with the $|GHZ\rangle_m$ state extracted from it.

parallel to the $\sigma_z$-measurements of the qubits not included in the $GHZ$ state that are required for the extraction. All measurements are repeated 32 000 times to calculate estimates for the expectation values.

Figure 3 shows the lower bounds on the fidelity for all linear cluster states that we generated with the IBMQ Montreal device—as well as for the GHZ states we extracted from them. Note that our estimation method imposes a relative penalty for linear cluster state fidelity estimations compared to GHZ state fidelity estimations. However, this does not mean that the fidelity of the GHZ states is truly higher than that of the linear cluster states from which they were extracted: It simply means that we have used a method of bounding the fidelity from below, which works comparatively better for GHZ states than it does for linear cluster states. For the details of the estimation method we refer to App. C.

V. DISCUSSION

In this letter, we considered how to establish maximally entangled states between nodes that are share entanglement only with only a small number of nearest neighbours. In particular, given a linear cluster state shared between the nodes, we showed what are the possible GHZ states that can be obtained, the later being an indispensable resource in many quantum communication protocols including secret sharing [20], electronic voting [21] and anonymous conference key agreement [22].

Our results demonstrate that this process is possible but costly, since almost half of the linear cluster state qubits must be measured to obtain a GHZ state on the remaining qubits. Very importantly, we showed that there is in fact a tight upper bound to the size of the GHZ state we can obtain, higher than the one previously conjectured [13], thus solving a longstanding open problem. We finally gave an exhaustive characterization of all possible GHZ states that can be extracted and provided a constructive method to obtain them, including the calculations for the necessary local rotations on the remaining qubits.

Our theoretical results are complemented by an implementation on IBM’s superconducting quantum hardware, where this near-neighbour architecture is inherent. With fidelities of 80% and higher for up to nine qubits, the results show that the generation of multi-partite entangled states is possible. It is also evident from the experimental results that our method of extracting GHZ states does not compromise the fidelity of the target states compared to the resource states, since only local operations are required. Since the generation of linear cluster states can be, depending on the specific setting, experimentally more feasible, our approach shows a potentially more robust method of generating GHZ states.

Finally, extending our methods to other simple graph state resources does not seem trivial and requires further research. We note, however, that deriving an analogous characterization for ring graph states, i.e. graph states in which the leftmost and rightmost qubits of an otherwise linear cluster state are also connected, is straightforward using our methods. In this case, only a single 2-island is possible, so the upper bound for $|V_G|$ becomes $\left\lfloor \frac{n+1}{2} \right\rfloor$.

VI. ACKNOWLEDGEMENTS

A.P., J.d.J. and F.H. acknowledge support from the German Research Foundation (DFG, Emmy Noether Grant No. 418294583) and from the European Union via the Quantum Internet Alliance project. Upon completion of this work, we became aware of work similarly motivated [23, 24].

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and $i + 1$ coincides exactly with (the generators of) the GHZ state - up to local Clifford rotations. We will now show that, from a reversible transformation of the set $\{i_0\}$, it is impossible to obtain such a set of generators when $j, i + 1, k \in V_G$. This directly implies that a measurement pattern such that the GHZ state can be obtained is not possible.

(A set of) generators for the GHZ state are, $\{\sigma_x^{i_0}\} \cup \{\sigma_x^{i_0} \sigma^z_{i_0}\}_{i_0 \in V_G}$, where it is implied that $\sigma^{z}_{i_0} = 1$ when $i_0 \notin V_G$. Focusing on $i$ and $i + 1$, the only generators with non-trivial support are $\{\alpha, \beta, \gamma, \delta\} = \{\sigma_0^{i_0}, \sigma_0^{i_0} \sigma_{i_0+1}, \sigma_{i_0+1} \sigma_0^{i_0}, \sigma_0^{i_0+1}\}$, where $a_1, a_{i+1}, b_1, b_{i+1} \in \{x, y, z\}$ reflect the fact that the state is locally equivalent to the GHZ state. This implies that $a_i \neq b_1$ and $a_{i+1} \neq b_{i+1}$.

Similarly, only the generators $l_{i-1}, l_i, l_{i+1}$ and $l_{i+2}$ of the linear cluster state (i.e., those with support on $i$ or $i + 1$) can have a non-trivial contribution to the generator transformation on the vertices in question. Therefore, w.l.o.g., we can focus on just these four generators and restrict our attention to vertices $i$ and $i + 1$. If we show that there is no reversible transformation of $\{l_k\}_{k=0,1,2}$ to obtain $\{\alpha, \beta, \gamma, \delta\}$ when only considering these nodes, the lemma follows. We show there is no such transformation by exhaustive contradiction.

There are three different ways of creating generator $\alpha$: i) $\alpha \propto l_{i-1} = \sigma_0^{i_0}$, ii) $\alpha \propto l_{i} \circ l_{i+2} = \sigma_y^{i_0}$, iii) $\alpha \propto l_{i-1} \circ l_{i} \circ l_{i+2} = \sigma_0^{i_0}$, where $\alpha \propto l_{i-1}$ should be read as `$l_{i-1}$ takes the role of $\alpha$' and $\circ$ denotes the (qubit-wise) product (e.g., $l_i \circ l_{i+1} = \sigma_z^{i_0} \sigma_y^{i_0} = x\sigma_y^{i_0}$, where the last equality is up to an irrelevant global phase). Similarly, there are three different ways of creating generator $\beta$: i) $\beta \propto l_{i+2} = \sigma_z^{i_0} + z^{i+2}$, ii) $\beta \propto l_{i-1} \circ l_{i+1} = \sigma_z^{i_0} + z^{i+2}$, iii) $\beta \propto l_{i-1} \circ l_{i+1} = \sigma_z^{i_0} + z^{i+2}$. Picking e.g. i) and ii) one sees that $\beta$ is fixed at $\sigma_z^{i_0} + z^{i+2}$. But this is $l_{i-1} \circ l_{i+2} \propto \alpha \circ \gamma$, which would not be a reversible transformation of the generators $l_{i-1}, l_{i}, l_{i+1}$ and $l_{i+2}$. Any other pair from $\{i_0\}, \{i, ii, iii\}$ and $\{iii, jj, jjj\}$ would also necessitate such a non-reversible transformation.

In essence, when viewing the generators as vectors over $\mathbb{F}_2^n$ through the binary representation [25], the argument follows from the observation that (the vector associated with) $\beta$ lies in the subspace spanned by (the vectors associated with) $\alpha$ and $\gamma$. As such there can never be a reversible (i.e. basis-preserving) operation on the vectors associated with $l_{i-1}, l_{i}, l_{i+1}$ and $l_{i+2}$ that obtains $\alpha, \beta$ and $\gamma$.

### Appendix B: Local-Clifford corrections to obtain GHZ states.

We provide a jupyter notebook for determining the required correction operations under [17].

### Appendix C: Estimation of lower bound for fidelity in experimental implementation.

We here provide details for the method of estimation of the lower bound of the fidelity of both the linear cluster state and GHZ state, that has been used in the experimental implementation. The method is presented in and adapted from [18].
using insights originally presented in [19]. The state that is prepared is
\[ |\psi\rangle_n = \bigotimes_{i \text{ odd}} H^i |L\rangle_n, \]
which is a linear cluster state rotated by Clifford operations and thus a stabilizer state. Note that the generators \( G^L \) for the stabilizer of \( |\psi\rangle \) can be grouped into ‘odd’ generators \( G^L_o = \{ \sigma^{-i-1}_z \sigma^i_x \sigma^i_y \}_{i \text{ odd}} \) and ‘even’ generators \( G^L_e = \{ \sigma^{-i-1}_z \sigma^i_x \sigma^i_y \}_{i \text{ even}} \), where again \( \sigma^0_z = \sigma^{n+1}_z = 1 \). The fidelity of the prepared state \( \rho \) with the rotated linear cluster state is \( F(\rho, |\psi\rangle_n) = tr[\rho |\psi\rangle\langle \psi|_n] \).

Writing \( G_{o(e)} = \Pi_{g \in G^L_{o(e)}} \frac{1+g}{2} \) and using \( |\psi\rangle\langle \psi|_n = \Pi_{g \in G} \frac{1+g}{2} = G_o G_e \), we can write
\[
F(\rho, |\psi\rangle_n) = tr[G_o G_e \rho] = tr[G_o \rho] + tr[G_e \rho] - tr[\rho] + tr[K \rho],
\]
where \( K = (I - G_o)(I - G_e) \). \( K \) is positive semidefinite and thus we can discard the last term to obtain a lower bound for the fidelity:
\[
F(\rho, |\psi\rangle_n) \geq E[G_o] + E[G_e] - 1,
\]
where \( E[G_{o(e)}] = \frac{1}{|S_{o(e)}|} \sum_{\sigma \in S_{o(e)}} tr[\rho \sigma] \) with \( S_{o(e)} = \{ G^L_{o(e)} \} \) the subgroup generated by the ‘odd’ (‘even’) generators of \( |\psi\rangle_n \). Notably, all terms \( tr[\rho \sigma] \) comprise of only \( \sigma_z \)-basis (\( \sigma \in S_e \)) or \( \sigma_z \)-basis (\( \sigma \in S_o \)) measurements. This means that just two measurement settings suffice to estimate the lower bound: measuring all vertices in the \( \sigma_z \)-basis, and measuring all vertices in the \( \sigma_x \)-basis. By repeating these measurements 32000 times and obtaining the outcome statistics, we estimate all terms \( tr[\rho \sigma] \) by selecting the outcomes associated with the \( +1 \) and \( -1 \) eigenspaces of all different observables.

For the GHZ state we use a similar method, where we now group the generators \( G^G \) of the GHZ state into \( G^G_o = \{ \sigma^j_z \}_{j \in V_o} \) and \( G^G_e = \{ \sigma^j_x \}_{j \in V_e} \), which again allows for an estimate of the lower bound with just two measurement settings. A caveat is that now there is only one ‘odd’ generator and thus \( E[G^G_o] = \frac{1}{2} tr[\rho] + \frac{1}{2} tr[\rho \sigma^j_z] \).

\[ \text{Figure 4: Lower bound on the fidelity using an adapted estimate method. In comparison with Figure3 positive terms that favour the GHZ states are dropped, which renders a lower but more equal estimate on the fidelities for all states.} \]

By definition \( tr[\rho I] = 1 \) and therefore the expectation value is more skewed towards 1 than for the linear cluster state estimation. In other words it gives a higher bound on the fidelity when compared to the linear cluster state, since \( G^L_o = O(2^n) \) and as such the identity does not have such a strong impact on the estimate, especially for larger linear cluster states. To give another comparison between the two states, Figure4 contains the same results as Figure3 from the main text, but with the identity-term omitted. This gives a lower but more equal estimate for both classes of states.