Holographic entanglement negativity for disjoint subsystems in $\text{AdS}_{d+1}/\text{CFT}_d$

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Abstract

We propose a construction to compute the holographic entanglement negativity for bipartite mixed state configurations of two disjoint subsystems (in proximity) in higher dimensional conformal field theories ($\text{CFT}_d$) dual to bulk $\text{AdS}_{d+1}$ geometries. Our construction follows from the corresponding $\text{AdS}_3/\text{CFT}_2$ scenario and involves a specific algebraic sum of the areas of bulk co dimension two static minimal surfaces homologous to appropriate subsystems. Utilizing our construction we compute the holographic entanglement negativity for such bipartite mixed state configurations of two disjoint subsystems with long rectangular strip geometries in $\text{CFT}_d$s dual to bulk pure $\text{AdS}_{d+1}$ geometries and the $\text{AdS}_{d+1}$-Schwarzschild black holes.

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# Contents

1 Introduction 3

2 Entanglement negativity in $CFT_{1+1}$ 5  
  2.1 Entanglement negativity at large central charge 6  
  2.2 Holographic entanglement negativity for disjoint intervals in $CFT_{1+1}$ 7

3 Holographic entanglement negativity for $AdS_{d+1}/CFT_d$ 9

4 Holographic entanglement negativity for $AdS_{d+1}/CFT_d$ in vacuum 10

5 Holographic entanglement negativity for $AdS_{d+1}/CFT_d$ at finite temperature 12  
  5.1 Holographic entanglement negativity in the low temperature limit 13  
  5.2 Holographic entanglement negativity in the high temperature limit 15

6 Summary and Conclusions 16
1 Introduction

Quantum entanglement has emerged as a central issue in recent years in relation to a diverse range of phenomena from condensed matter physics to issues of quantum gravity. For bipartite pure states quantum entanglement is characterized through the entanglement entropy which is the von Neumann entropy of the reduced density matrix for the subsystem in question. This quantity is easily computable for quantum systems with finite number of degrees of freedom although for extended quantum systems like quantum field theories this is an extremely complex issue involving a density matrix with infinite number of eigenvalues. A formal definition of this quantity is possible in quantum field theories through a suitable replica technique but an explicit evaluation is in general computationally intractable. However for \((1+1)\)-dimensional conformal field theories \((CFT_{1+1})\) the entanglement entropy for bipartite states may be explicitly computed through the replica technique as described in \([1-4]\). For bipartite configurations involving two or more disjoint intervals it was observed that the entanglement entropy involved non universal functions that depend on the full operator content of the \(CFT_{1+1}\). Using monodromy techniques it could be shown \([5-7]\) that these non universal contributions were sub leading in the large central charge limit.

Although entanglement entropy is crucial for characterizing pure state entanglement it is invalid for mixed states as it receives contributions from irrelevant classical and quantum correlations for such states. This renders the characterization of mixed state entanglement to be a subtle and complex issue in quantum information theory requiring introduction of other suitable measures. Several such independent formal measures have been introduced in quantum information theory to characterize mixed state entanglement, most of which were computationally intractable. A computable measure termed as entanglement negativity (logarithmic negativity) for characterization of mixed state entanglement was introduced in a classic work by Vidal and Werner \([8]\). This was defined as the logarithm of the trace norm for the partially transposed density matrix for a bipartite system with respect to one of the subsystems and characterized an upper bound to the distillable entanglement of the mixed state. The entanglement negativity was found to be an entanglement monotone under local operations and classical communication (LOCC) however it was shown to be non convex by Plenio \([9]\). Despite the latter drawback entanglement negativity being an entanglement monotone served as an important computable measure for mixed state entanglement in quantum information theory. Interestingly as described in \([10-12]\) the entanglement negativity for bipartite pure and mixed states in \(CFT_{1+1}\) could be explicitly computed through a modification of the replica technique mentioned earlier. It was shown that the entanglement negativity for certain bipartite configurations in \(CFT_{1+1}\) also involves non universal functions, however a universal contribution could be extracted in the large central charge limit through the monodromy technique. Interestingly for the mixed state configuration of two disjoint intervals in proximity it could be numerically established that the entanglement negativity exhibits a phase transition \([13,14]\).

Following the computation of the entanglement entropy for bipartite states in \(CFT_{1+1}\) through the replica technique Ryu and Takayanagi (RT) \([15,16]\) proposed a holographic conjecture to compute the universal part of the entanglement entropy for bipartite states in generic \(d\)-dimensional \(CFT_d\) \((CFT_d)\) dual to bulk asymptotically \(AdS_{d+1}\) geometries in the context of the \(AdS_{d+1}/CFT_d\) correspondence. The holographic entanglement entropy was shown to be proportional to the area of a co-dimension two bulk \(AdS_{d+1}\) static minimal surface (RT surface) homologous to the subsystem. Their conjecture was utilized to compute the entanglement entropy for various bipartite states in holographic \(CFT_d\) \([17-24]\). Subsequently a covariant generalization of the RT conjecture was established by Hubeny, Rangamani and Takayanagi (HRT) in \([25]\). An explicit proof of the RT and the HRT conjectures for certain limited configurations was later established for the \(AdS_3/CFT_2\) scenario in \([26]\) and subsequently in \([27,28]\) for a generic \(AdS_{d+1}/CFT_d\) framework.

The developments described above regarding the holographic entanglement entropy naturally
led to the significant issue of a corresponding holographic description for the entanglement negativity of bipartite states in a dual CFT\(_d\) critical for mixed state entanglement characterization. In this context a holographic computation for the entanglement negativity of a bipartite pure state in a CFT\(_d\) was described in [29] in a generic AdS\(_{d+1}/\text{CFT}_d\) scenario. However the issue of a holographic prescription for generic bipartite mixed states in a dual CFT\(_d\) remained an outstanding issue. Interestingly in [30,31] a holographic entanglement negativity conjecture and its covariant generalization were proposed for the configuration of a single interval in a AdS\(_3/\text{CFT}_2\) framework. At finite temperature the above configuration described a mixed state in the corresponding dual CFT\(_{1+1}\) and the holographic entanglement negativity computed utilizing the conjecture reproduced the universal part of the corresponding replica technique results. A comprehensive large central charge analysis of the entanglement negativity utilizing the monodromy technique was developed in [32] for such bipartite mixed states in a CFT\(_{1+1}\) which provided a strong substantiation for the holographic construction. Subsequently a higher dimensional generalization of the holographic entanglement negativity conjecture was described in [33]. Applied to the configuration of single subsystems with long rectangular strip geometry in a CFT\(_d\) dual to a bulk pure AdS\(_{d+1}\) geometry and the AdS\(_{d+1}\)-Schwarzschild black hole reproduced certain universal features of entanglement negativity obtained for corresponding CFT\(_{1+1}\). Despite these significant progress a bulk proof for the holographic entanglement negativity conjecture similar to [26–28] remained an unresolved involved issue. Some recent interesting progress in this direction has been reported in [34–37] which reproduces some of the results described above in the context of the AdS\(_3/\text{CFT}_2\) framework, however the issue still needs further investigation and substantiation especially for the higher dimensional AdS\(_{d+1}/\text{CFT}_d\) scenario. We will elaborate further on this issue in the relevant section.

Subsequent to the developments described in [30,31,33] in [38,39] a holographic entanglement negativity conjecture and its covariant extension was established for mixed state configurations of adjacent intervals in the context of the AdS\(_3/\text{CFT}_2\) correspondence. It was followed by a higher dimensional generalization of the above conjecture for such mixed states in CFT\(_d\)s dual to a bulk pure AdS\(_{d+1}\) geometry, AdS\(_{d+1}\)-Schwarzschild black holes and the AdS\(_{d+1}\)-Reissner Nordström black hole [40,41]. Recently the holographic entanglement negativity conjecture for the mixed state configuration of disjoint intervals in proximity in a CFT\(_{1+1}\) dual to static bulk AdS\(_3\) geometries was advanced in [42] and its covariant generalization in [43].

In the context of the progress described above in computing the holographic entanglement negativity for mixed states of disjoint intervals in a CFT\(_{1+1}\) in the AdS\(_3/\text{CFT}_2\) scenario [42,43] a higher dimensional generalization of the construction to a generic AdS\(_{d+1}/\text{CFT}_d\) framework assumes critical significance. These generalizations and their applications to specific higher dimensional examples of such mixed states in dual CFT\(_d\)s are expected to provide crucial insights into a generic higher dimensional holographic entanglement negativity conjecture and a possible approach towards an explicit proof. In this article we address this crucial issue and propose a holographic entanglement negativity construction for the mixed state of two disjoint subsystems in proximity for a AdS\(_{d+1}/\text{CFT}_d\) scenario. Following the AdS\(_3/\text{CFT}_2\) scenario discussed in [30–32,38,39,42,43] our holographic entanglement negativity construction involves a specific algebraic sum of the areas of co dimension two bulk static minimal surfaces homologous to specific subsystems. In this context utilizing our prescription we compute the holographic entanglement negativity for zero and finite temperature mixed states of two disjoint subsystems with infinite rectangular strip geometries in CFT\(_d\)s dual to bulk pure AdS\(_{d+1}\) and the AdS\(_{d+1}\)-Schwarzschild black holes respectively. The areas for the specific static minimal surfaces for the bulk AdS\(_{d+1}/\text{CFT}_d\)-Schwarzschild black hole geometry are obtained perturbatively for both low and high temperatures \(^1\). Interestingly for both the zero and finite temperature cases the entanglement negativity for the mixed state configuration of disjoint intervals in proximity are cut

\(^1\)Note that in [44] an exact evaluation of this has been provided through the Meijer G-functions however we adopt a perturbative approach which provides the necessary leading order behaviour for the area functional.
off independent as expected from the $AdS_3/CFT_2$ results. We observe that in the limit where the subsystems are considered to be adjacent, our results for the disjoint case exactly reproduce earlier results for adjacent subsystems in the context of an $AdS_{d+1}/CFT_d$ scenario described in [40]. We however mention here that in the light of recent developments in [35–37] our results may be modified by some overall constant numerical factor which arises from the bulk cosmic brane contribution due to the conical defect and is dependent on the bulk dimension and the shape of the entangling surface. This issue will be further elucidated in the relevant section.

This article is organized as follows, in section 2 we briefly review the computation of holographic entanglement negativity for bipartite mixed state configuration of two disjoint intervals in the context of the $AdS_3/CFT_2$ framework described in [42]. Subsequently in section 3 we propose a holographic entanglement negativity conjecture for mixed states of two disjoint subsystems in the $AdS_{d+1}/CFT_d$ scenario and describe its application to the zero temperature mixed state of two disjoint subsystems with long rectangular strip geometries in $CFT_d$s dual to a bulk pure $AdS_{d+1}$ geometry in section 4. Following this in section 5 we utilize our proposal to compute the holographic entanglement negativity for such mixed states at a finite temperature in a $CFT_d$ dual to a bulk $AdS_{d+1}$-Schwarzschild black hole. Finally we summarize our results in section 6 and present our conclusions.

## 2 Entanglement negativity in $CFT_{1+1}$

In this section we briefly review the computation of the entanglement negativity for bipartite states in a $CFT_{1+1}$ as described in [10–12] and subsequently describe the holographic entanglement negativity prescription for zero and finite temperature mixed states of two disjoint (spatial) intervals in proximity in $CFT_{1+1}$s dual to bulk pure $AdS_3$ and the BTZ black holes respectively. To this end we begin by considering a tripartition of a system in a pure state described by a $CFT_{1+1}$ into the spatial intervals $A_1$, $A_2$ and $B$ with $A = A_1 \cup A_2 = [u_1, v_1] \cup [u_2, v_2]$, and $B = A^c$ represents the rest of the system. The two intervals $A_1$ and $A_2$ are separated by the interval $A_s \subset B$ as shown in the Fig. 1.

![Figure 1: Schematic of two disjoint intervals $A_1$ and $A_2$ with rest of the system $B$ in a $CFT_{1+1}$](image)

The reduced density matrix for the subsystem $A$ which is in a mixed state is defined as $\rho_A = \text{Tr}_B \rho$ and $\rho_A^{T_2}$ is the partial transpose of the reduced density matrix with respect to the interval $A_2$. The entanglement negativity $\mathcal{E}$ for the subsystem $A$ described by the disjoint intervals $A_1$ and $A_2$ is then defined as the logarithm of the trace norm of the partially transposed reduced density matrix as

$$\mathcal{E} = \ln \text{Tr} |\rho_A^{T_2}|. \quad (2.1)$$

The entanglement negativity may now be obtained through a replica technique as discussed in [10, 11] to determine $\text{Tr}(\rho_A^{T_2})^{n_e}$ and the replica limit is given as the analytic continuation of even sequences of $n = n_e$ to $n_e \to 1$. This leads us to the following expression for the entanglement negativity of the subsystem $A$ as

$$\mathcal{E} = \lim_{n_e \to 1} \ln \text{Tr}(\rho_A^{T_2})^{n_e}. \quad (2.2)$$
For the configuration of the two disjoint intervals as shown in Fig. 1, the quantity $\text{Tr}(\rho_A^{T_x})_{\Gamma_v}$ in eq. (2.2) is expressed as a four point correlator of the twist fields on the complex plane $\mathbb{C}$ in the following way

$$\text{Tr}(\rho_A^{T_x})_{\Gamma_v} = \langle \mathcal{T}_{\Gamma_v}(u_1)\mathcal{T}_{\Gamma_v}(v_1)\mathcal{T}_{\Gamma_v}(u_2)\mathcal{T}_{\Gamma_v}(v_2) \rangle_{\mathbb{C}}. \quad (2.3)$$

In the replica limit $n_v \to 1$ the four point function of the twist fields described above is expressed in [11] as

$$\lim_{n_v \to 1} \langle \mathcal{T}_{\Gamma_v}(u_1)\mathcal{T}_{\Gamma_v}(v_1)\mathcal{T}_{\Gamma_v}(u_2)\mathcal{T}_{\Gamma_v}(v_2) \rangle_{\mathbb{C}} = G(x). \quad (2.4)$$

The function $G(x)$ of the cross ratio $x = [(v_1 - u_1) (v_2 - u_2)] / [(u_2 - u_1) (v_2 - v_1)]$ in eq. (2.4) describes the non-universal part of the four point twist correlator which depends on the full operator content of the CFT$_{1+1}$. We briefly discuss and present a universal form for the above four point twist correlator in the large central charge limit obtained through the monodromy technique [5, 6, 13, 42].

### 2.1 Entanglement negativity at large central charge

The universal part of the four point function eq. (2.4) in the large central charge limit when the two disjoint intervals are in proximity (1/2 < $x$ < 1) may be expressed as [5, 13, 32]

$$\lim_{n_v \to 1} \langle \mathcal{T}_{\Gamma_v}(z_1)\mathcal{T}_{\Gamma_v}(z_2)\mathcal{T}_{\Gamma_v}(z_3)\mathcal{T}_{\Gamma_v}(z_4) \rangle_{\mathbb{C}} = (1 - x)^{2\hat{h}}, \quad (2.5)$$

where we have identified $u_1 \equiv z_1, v_1 \equiv z_2, u_2 \equiv z_3, v_2 \equiv z_4$ and the cross ratio $x$ is given by $(z_1 z_4)/(z_1 z_4)$, with $z_{ij} \equiv z_i - z_j$, and $\hat{h}$ is the conformal dimension of the operator with the dominant contribution in the corresponding conformal block expansion. The entanglement negativity for the bipartite mixed state configuration of disjoint intervals in proximity may then be obtained from eq. (2.2) & (2.5) as [42]

$$E = \frac{c}{4} \ln \left( \frac{|z_{13}| |z_{24}|}{|z_{14}| |z_{23}|} \right). \quad (2.6)$$

The entanglement negativity for the zero temperature mixed state of the two disjoint intervals in proximity is then determined from eq. (2.6) by substituting the lengths of the respective intervals as follows

$$E = \frac{c}{4} \ln \left( \frac{(l_1 + l_s) (l_2 + l_s)}{l_s (l_1 + l_2 + l_s)} \right). \quad (2.7)$$

Interestingly note however that the result described in the above equation is cut off independent in contrast to that for the mixed state of adjacent intervals [11].

The entanglement negativity for the corresponding mixed state of disjoint intervals at a finite temperature $T$ at large $c$ may be obtained as above through the conformal map $z \to w = (\beta / 2\pi) \ln z$ from the complex plane to the cylinder where the Euclidean time direction has now been compactified to a circle with circumference $\beta \equiv 1/T$ [12]. The four point function of the twist fields in eq. (2.3) transforms under the conformal map described above as follows

$$\langle \mathcal{T}_{\Gamma_v}(w_1)\mathcal{T}_{\Gamma_v}(w_2)\mathcal{T}_{\Gamma_v}(w_3)\mathcal{T}_{\Gamma_v}(w_4) \rangle_{\text{cyl}} = \prod_{i=1}^{4} \left[ \left( \frac{dw(z)}{dz} \right)^{-\Delta_i} \right]_{z=z_i} \times \langle \mathcal{T}_{\Gamma_v}(z_1)\mathcal{T}_{\Gamma_v}(z_2)\mathcal{T}_{\Gamma_v}(z_3)\mathcal{T}_{\Gamma_v}(z_4) \rangle_{\mathbb{C}}, \quad (2.8)$$

where $\Delta_i$ are the scaling dimensions of the twist fields at the locations $w = w_i$.

The entanglement negativity for the mixed state configuration of disjoint intervals in proximity at a finite temperature in the limit of large central charge $c$ is then obtained from eq. (2.5) upon utilizing the eqs. (2.2), (2.3) and (2.8), as follows
The above result is also cut off independent as earlier.

2.2 Holographic entanglement negativity for disjoint intervals in $\text{CFT}_{1+1}$

Following the extraction of the universal part of the entanglement negativity in the large central charge limit for mixed states of disjoint intervals in proximity we now outline the corresponding holographic construction as described in [42]. The two point function of the twist fields in a holographic $\text{CFT}_{1+1}$ can be expressed as

$$\langle T_{n_e}(z_i) T_{n_e}(z_j) \rangle_{\mathcal{C}} \sim |z_{ij}|^{-2 \Delta_T n_e}.$$  \hspace{1cm} (2.10)

From the $\text{AdS}_3/\text{CFT}_2$ lexicon, the two point function of the twist fields in eq. (2.10) (in the geodesic approximation) is described in terms of of the length $\mathcal{L}_{ij}$ of the bulk space like geodesic which is homologous to the interval in question as [16]

$$\langle T_{n_e}(z_i) T_{n_e}(z_j) \rangle_{\mathcal{C}} \sim \exp \left( - \frac{\Delta_T n_e \mathcal{L}_{ij}}{R} \right),$$ \hspace{1cm} (2.11)

where $R$ is the $\text{AdS}_3$ length scale.

Using eqs. (2.10) and (2.11), the four point twist correlator in eq. (2.5) may be written as

$$\lim_{n_e \to 1} \langle T_{n_e}(z_1) T_{n_e}(z_2) T_{n_e}(z_3) T_{n_e}(z_4) \rangle_{\mathcal{C}} = \exp \left[ \frac{c}{8R} \left( \mathcal{L}_{13} + \mathcal{L}_{24} - \mathcal{L}_{14} - \mathcal{L}_{23} \right) \right].$$  \hspace{1cm} (2.12)

We now utilize the eqs. (2.1), (2.2), (2.12) and Brown-Henneaux formula $c = \frac{3R}{2G_N^{(3)}}$ [45], to express the holographic entanglement negativity for the mixed state configuration of the two disjoint intervals in proximity as

$$\mathcal{E} = \frac{3}{16G_N^{(3)}} \left( \mathcal{L}_{A_1 \cup A_s} + \mathcal{L}_{A_2 \cup A_s} - \mathcal{L}_{A_1 \cup A_2 \cup A_s} - \mathcal{L}_{A_s} \right)$$

$$= \frac{3}{4} \left( S_{A_1 \cup A_s} + S_{A_2 \cup A_s} - S_{A_1 \cup A_2 \cup A_s} - S_{A_s} \right)$$

$$= \frac{3}{4} \left[ \mathcal{I} (A_1 \cup A_s, A_2) - \mathcal{I} (A_s, A_2) \right],$$ \hspace{1cm} (2.13)

where $\mathcal{I}(A_i, A_j)$ denotes the holographic mutual information between the subsystems $A_i$ and $A_j$ and given as $\mathcal{I}(A_i, A_j) = S_{A_i} + S_{A_j} - S_{A_i \cup A_j}$.
Figure 2: Schematic for the geodesics anchored on the intervals $A_1 \cup A_s$, $A_s \cup A_2$, $A_1 \cup A_s \cup A_2$ and $A_s$.

Now the holographic entanglement negativity for the mixed state of disjoint intervals in the dual $CFT_{1+1}$ at zero temperature may be obtained using the holographic entanglement negativity conjecture described in eq. (2.13). The corresponding bulk configuration is described by a pure $AdS_3$ geometry which is expressed in the Poincaré coordinates as

$$ds^2 = \left(\frac{r^2}{R^2}\right) (-dt^2 + dx^2) + \left(\frac{r^2}{R^2}\right)^{-1} dr^2,$$  \hspace{1cm} (2.14)

where $R$ is the $AdS_3$ radius. The length $L_\gamma$ of the bulk space like geodesic homologous to an interval $\gamma$ (of length $l_\gamma$) in these coordinates is written as $[15, 16, 46, 47]$  

$$L_\gamma = 2R \ln \left(\frac{l_\gamma}{a}\right),$$  \hspace{1cm} (2.15)

where $a$ is the UV cut off for the $CFT_{1+1}$. Now it is possible to utilize the expression in eq. (2.15) to obtain the holographic entanglement negativity for the zero temperature mixed state in question from eq. (2.13) in the following form $[42]$  

$$E = \frac{3R}{8G^{(3)}_N} \ln \left[ \frac{(l_1 + l_s) (l_2 + l_s)}{l_s (l_1 + l_2 + l_s)} \right].$$  \hspace{1cm} (2.16)

On using the Brown-Henneaux formula $[45]$, the above result exactly matches with the $CFT_{1+1}$ replica technique results for the large central charge limit as given in eq. (2.7).

The corresponding holographic entanglement negativity for the finite temperature mixed state configuration of disjoint intervals (in proximity) in a $CFT_{1+1}$ may also be computed as above. For this case the dual bulk configuration is described by the Euclidean BTZ black string at a temperature $T$ $[15, 16, 46, 47]$ the metric for which is as follows

$$ds^2 = \frac{(r^2 - r_h^2)}{R^2} d\tau^2 + \frac{R^2}{(r^2 - r_h^2)} dr^2 + \frac{r^2}{R^2} d\phi^2.$$  \hspace{1cm} (2.17)

In the above equation $\tau$ represents the Euclidean time where the $\phi$ coordinate is uncompactified and $r = r_h$ denotes the event horizon. The corresponding length $L_\gamma$ of the bulk space like geodesic
anchored on an interval $\gamma$ (of length $l_\gamma$) for this geometry may be expressed as \cite{15,16,46,47}

$$L_\gamma = 2R \ln \left( \frac{\beta}{\pi a} \sinh \left( \frac{\pi l_\gamma \gamma}{\beta} \right) \right), \quad (2.18)$$

where $a$ is the UV cut off. Utilizing the above eqs. (2.18) and (2.13), the holographic entanglement negativity for the finite temperature mixed state of disjoint intervals in proximity may be computed as \cite{42}

$$\mathcal{E} = \frac{3R}{8G_N^{(3)}} \ln \left[ \frac{\sinh \frac{\pi l_s}{\beta} \sinh \frac{\pi l_{s+1}}{\beta}}{\sinh \frac{\pi l_s}{\beta} \sinh \frac{\pi l_{s+1}}{\beta}} \right]. \quad (2.19)$$

As earlier the holographic entanglement negativity in the above equation matches with the replica technique results in the large central charge limit described in eq. (2.9) on utilizing the Brown-Henneaux formula \cite{45}.

3 Holographic entanglement negativity for $AdS_{d+1}/CFT_d$

Following the computation of the holographic entanglement negativity reviewed in the last section for mixed states of disjoint intervals in the $AdS_3/CFT_2$ scenario \cite{42,43}, we proceed to propose a higher dimensional generalization in a generic $AdS_{d+1}/CFT_d$ framework. The construction for the $AdS_3/CFT_2$ scenario suggests that the corresponding higher dimensional construction would involve a similar algebraic sum of the areas of bulk co-dimension two static minimal surfaces\footnote{Note that these are geodesics homologous to appropriate intervals in the corresponding $AdS_3/CFT_2$ scenario.} homologous to the respective subsystems. The holographic entanglement negativity for such mixed state configurations in the $AdS_{d+1}/CFT_d$ scenario may then be expressed as follows

$$\mathcal{E} = \frac{3}{16G_N^{d+1}}(A_{1s} + A_{s2} - A_{12s} - A_s), \quad (3.1)$$

where $A_{ij}$ and $A_{ijk}$ are the areas of the co-dimension two static minimal surfaces anchored on the subsystems $A_i \cup A_j$ and $A_i \cup A_j \cup A_k$ respectively with $i = 1, 2, s$, as depicted in Fig. 3. Using the Ryu-Takayanagi prescription \cite{15}, the above expression for the holographic entanglement negativity may be expressed as follows

$$\mathcal{E} = \frac{3}{4}(S_{A_1 \cup A_s} + S_{A_s \cup A_2} - S_{A_1 \cup A_2 \cup A_s} - S_{A_s}) = \frac{3}{4}[\mathcal{I}(A_1 \cup A_s, A_2) - \mathcal{I}(A_s, A_2)], \quad (3.2)$$

where $\mathcal{I}(A_1 \cup A_s, A_2)$ and $\mathcal{I}(A_s, A_2)$ are the holographic mutual information between the respective subsystems. In the limit $A_s \to \emptyset$ where $\emptyset$ is the null set, we recover the holographic entanglement negativity for the mixed state of adjacent subsystems as described in \cite{40} \footnote{Note that the entanglement negativity and the mutual information are also related in the holographic limit for other mixed state configurations as reported in the literature.}. This serves as a strong indication for the overall consistency of our proposal for the holographic entanglement negativity for such mixed state configurations in the $AdS_{d+1}/CFT_d$ scenario.

As mentioned in the introduction, a related holographic construction for the entanglement negativity of bipartite states in a $CFT_d$ has been proposed in \cite{35,37}. This alternative holographic proposal involves the bulk minimal entanglement wedge cross section for the mixed state configuration of subsystems in the dual $CFT_d$ with an overall constant numerical factor which arises from the cosmic brane for the bulk conical defect and is only dependent on the bulk dimension and possibly on the shape of the entangling surface. For the $AdS_3/CFT_2$ scenario their proposal correctly reproduces the corresponding $CFT_{1+1}$ replica technique results of \cite{11} for certain mixed state configurations. For the mixed state configuration of adjacent intervals their

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\[ \begin{align*}
L_\gamma &= 2R \ln \left( \frac{\beta}{\pi a} \sinh \left( \frac{\pi l_\gamma \gamma}{\beta} \right) \right), \\
\mathcal{E} &= \frac{3R}{8G_N^{(3)}} \ln \left[ \frac{\sinh \frac{\pi l_s}{\beta} \sinh \frac{\pi l_{s+1}}{\beta}}{\sinh \frac{\pi l_s}{\beta} \sinh \frac{\pi l_{s+1}}{\beta}} \right].
\end{align*} \]
results match those described in [38]. For the corresponding mixed state of disjoint intervals in a \( CFT_{1+1} \) their results match with those described in [42] in the strict proximity limit for which the cross ratio \( (x \to 1) \) apart from a constant numerical factor. However for the mixed state configuration of a single interval at a finite temperature their results reproduce the corresponding universal part of the replica technique results only in the limit of high and low temperatures. In particular the subtracted thermal entropy term in the universal part of the replica technique results described in [12] seems to be missing in their approach and this issue requires further investigation. The overall constant numerical factor for the holographic entanglement negativity however matches with other mixed state configurations as described in [30, 38]. Interestingly this demonstrates that the algebraic sum of geodesics involved in describing the holographic entanglement negativity in [30, 38] is proportional to the bulk minimal entanglement wedge cross section, which is an interesting geometrical issue and is expected to be valid for higher dimensions as well.

For higher dimensions the dimension dependent constant overall numerical factor for the holographic entanglement negativity in their approach is obtained from the entanglement negativity of the pure vacuum state in the \( CFT_d \) dual to the bulk pure \( AdS_{d+1} \) geometry [29] and is valid only for subsystems with spherical entangling surfaces. However the analysis in [33, 40, 41] and the present article involves configurations described by subsystems with long rectangular strip geometries and hence the corresponding results are not comparable. Given the relation between the geodesic combination and the minimal entanglement wedge cross section for the \( AdS_3/CFT_2 \) scenario mentioned above, it appears that even for higher dimensions the two distinct approaches are related. However the results for the holographic entanglement negativity in higher dimensions described in [33, 40, 41] may be possibly modified by some overall constant undetermined numerical factor which is dependent on the shape and geometry of the subsystems involved. Accordingly our higher dimensional results presented here may also be subject to a similar modification by a similar overall constant numerical factor.

4 Holographic entanglement negativity for \( AdS_{d+1}/CFT_d \) in vacuum

In this section we employ our holographic conjecture described in the last section to compute the entanglement negativity for the zero temperature mixed state of two disjoint subsystems (in proximity) in a dual \( CFT_d \). The subsystems are described by infinite \((d-1)\) dimensional spatial long rectangular strip geometries. In this case, the corresponding bulk geometry is the pure \( AdS_{d+1} \) space time whose metric in Poincaré coordinates is given as

\[
ds^2 = \frac{1}{z^2} \left( -dt^2 + dz^2 + \sum_{i=1}^{d-1} dx_i^2 \right).
\]

(4.1)

The corresponding rectangular strip geometries are specified by the subsystems \( A_1, A_2 \) and \( A_s \) as shown in Fig. 3 with

\[
x = x^1 = [-\frac{L}{2}, \frac{L}{2}], \quad x^i = [-\frac{L}{2}, \frac{L}{2}]; \quad i = 2, 3, ..., (d-1), \quad j = 1, 2, s.
\]

(4.2)

where \( L \) for the transverse coordinates are taken to be very large \( L \to \infty \).
We now briefly review the calculation for the area of bulk co-dimension two static minimal surface anchored on such infinite rectangular strip geometries described in [21]. The area functional in this case may be expressed as follows

\[ A = L^{d-2} \int_{\frac{l_1}{2}}^{\frac{l_2}{2}} dx \sqrt{1 + \left( \frac{dz}{dx} \right)^2} \cdot \frac{z}{z^{d-1}}. \]  \hspace{1cm} (4.3)

Upon extremization of the area functional above we arrive at the following differential equation

\[ \frac{dz}{dx} = \sqrt{\frac{z_{s}^{2(d-1)} - z^{2(d-1)}}{z^{2(d-1)}}}. \]  \hspace{1cm} (4.4)

Here, \( z = z_{s} \) is the turning point of the static minimal surface. Utilizing the eq. (4.4) and eq. (4.3), the area functional may be expressed as

\[ A = A_{div} + A_{finite}. \]  \hspace{1cm} (4.5)

where

\[ A_{div} = \frac{2}{d-2} \left( \frac{L}{a} \right)^{d-2}, \]  \hspace{1cm} (4.6)

\[ A_{finite} = \frac{2\sqrt{\pi} \Gamma \left( \frac{d+2}{2d-2} \right)}{\Gamma \left( \frac{1}{2d-2} \right)} \left( \frac{L}{z_{s}} \right)^{d-2} - S_{0} \left( \frac{L}{l_{s}} \right)^{d-2}, \]  \hspace{1cm} (4.7)

with \( a \) being the UV cut off and the constant \( S_{0} \) is given as

\[ S_{0} = -\frac{2^{d-1} \pi^{(d-1)/2}}{d-2} \left( \frac{\Gamma \left( \frac{d}{2d-2} \right)}{\Gamma \left( \frac{1}{2d-2} \right)} \right)^{d-1}. \]  \hspace{1cm} (4.8)

We now utilize eq. (3.1) to compute the holographic entanglement negativity for the zero temperature mixed state of the disjoint long rectangular strip geometries in proximity as

\[ E = \frac{3S_{0}}{16G_{N}^{(d+1)}} \left[ \left( \frac{L}{l_{1} + l_{s}} \right)^{d-2} + \left( \frac{L}{l_{2} + l_{s}} \right)^{d-2} - \left( \frac{L}{l_{1} + l_{2} + l_{s}} \right)^{d-2} - \left( \frac{L}{l_{s}} \right)^{d-2} \right]. \]  \hspace{1cm} (4.9)
Note that the above result is cut off independent as expected, unlike the case for the mixed state of adjacent subsystems as described in [40].

In the limit \( l_s \to a \) in the above eq. (4.9), where the subsystems are adjacent we arrive at the following expression for the holographic entanglement negativity

\[
E_{\text{Adjacent}} = \frac{3}{16G_N^{d+1}} \left[ \frac{2}{d-2} \left( \frac{L}{a} \right)^{d-2} + S_0 \left\{ \left( \frac{L}{l_1} \right)^{d-2} + \left( \frac{L}{l_2} \right)^{d-2} - \left( \frac{L}{l_1 + l_2} \right)^{d-2} \right\} \right].
\] (4.10)

Note that in arriving at the above expression we have added and subtracted the divergent part of the entanglement negativity from the right hand side of the eq. (4.9) and neglected sub leading terms in the limit \( l \gg a \). Interestingly the above result exactly matches with the corresponding holographic entanglement negativity for adjacent subsystems as described in [40]. This serves as a strong consistency check for our computations however as mentioned earlier our results for the holographic entanglement negativity in higher dimensions may be modified by an overall constant numerical factor which is dependent on the bulk dimension and possibly the subsystem geometry.

5 Holographic entanglement negativity for \( \text{AdS}_{d+1}/\text{CFT}_d \) at finite temperature

Having computed the holographic entanglement negativity for the zero temperature mixed state of two disjoint subsystems in proximity we turn our attention in this section to the corresponding finite temperature case where the dual bulk geometry is described by the \( \text{AdS}_{d+1} \)-Schwarzschild black hole. The metric for the \( \text{AdS}_{d+1} \)-Schwarzschild black hole with the \( \text{AdS} \) radius \( R = 1 \) is given as

\[
ds^2 = -r^2 \left( 1 - \frac{r^d}{r^d_h} \right) dt^2 + \frac{dr^2}{r^2 \left( 1 - \frac{r^d}{r^d_h} \right)} + r^2 d\vec{x}^2,
\] (5.1)

where \( \vec{x} \equiv (x, x^i) \) are the coordinates on the boundary and the horizon radius \( r_h \) is related to Hawking temperature as \( r_h = \frac{4 \pi T_d}{d} \). We begin by briefly reviewing the computation of the area of the bulk co-dimension two static minimal surface anchored on a subsystem of infinite rectangular strip geometry on the boundary in this case, as described in [21]. The area functional for a single long rectangular strip in this case may be expressed as

\[
A = L^{d-2} \int dr r^{d-2} \frac{1}{\sqrt{r^2 (1 - u^2)}} (1 - \frac{r^d}{r^d_c})^{-\frac{1}{2}},
\] (5.2)

Extremizing the above area functional we obtain

\[
\frac{l}{2} = \frac{1}{r_c} \int_0^1 \frac{u^{d-1} du}{\sqrt{(1 - u^{2d-2})}} (1 - \frac{r^d}{r^d_c} u^d)^{-\frac{1}{2}}, \quad u = \frac{r_c}{r},
\] (5.3)

where \( r_c \) is the turning point as earlier. Now eq. (5.2) may be re-expressed in terms of the variable \( u \) as

\[
A = 2L^{d-2} r_c^{-d-2} \int_0^1 \frac{du}{u^{d-1} \sqrt{(1 - u^{2d-2})}} (1 - \frac{r^d}{r^d_c} u^d)^{-\frac{1}{2}}.
\] (5.4)

As is evident the above integral diverges at the lower limit and as earlier we express the area functional in the following form

\[
A = A_{\text{div}} + A_{\text{finite}}.
\] (5.5)
where $\mathcal{A}_{\text{div}}$ is the temperature independent divergent part and $\mathcal{A}_{\text{finite}}$ is the temperature dependent finite part. The divergent and the finite parts may be written as

$$\mathcal{A}_{\text{div}} = \frac{2}{d-2} \left( \frac{L}{a} \right)^{d-2}$$

where

$$\mathcal{A}_{\text{finite}} = 2L^{d-2} r_c^{d-2} \left[ \frac{\sqrt{\pi \Gamma} \left( - \frac{d-2}{2(d-1)} \right)}{2(d-1) \Gamma \left( \frac{1}{2(d-1)} \right)} + \sum_{n=1}^{\infty} \left( \frac{1}{2(d-1)} \right)^{n+1} \frac{1}{\Gamma(1+n) \Gamma \left( \frac{dn+1}{2(d-1)} \right)} \right] \frac{r_h}{r_c^{nd}}$$

where the series $\mathcal{A}_{\text{finite}}$ always converges for $r_c > r_h$. Finally utilizing our conjecture described in eq. (3.1) the holographic entanglement negativity for the mixed state configuration of two disjoint subsystems of long rectangular strip geometries in question may be expressed as

$$\mathcal{E} = \frac{3L^{d-2}}{16G_N} \left[ r_{c,1,s}^{d-2} \left\{ \frac{\sqrt{\pi \Gamma} \left( - \frac{d-2}{2(d-1)} \right)}{(d-1) \Gamma \left( \frac{1}{2(d-1)} \right)} + \sum_{n=1}^{\infty} \left( \frac{1}{(d-1)} \right)^{n+1} \frac{1}{\Gamma(1+n) \Gamma \left( \frac{dn+1}{2(d-1)} \right)} \right] \frac{r_h}{r_{c,1,s}^{nd}} \right]$$

$$+ r_{c,2,s}^{d-2} \left\{ \frac{\sqrt{\pi \Gamma} \left( - \frac{d-2}{2(d-1)} \right)}{(d-1) \Gamma \left( \frac{1}{2(d-1)} \right)} + \sum_{n=1}^{\infty} \left( \frac{1}{(d-1)} \right)^{n+1} \frac{1}{\Gamma(1+n) \Gamma \left( \frac{dn+1}{2(d-1)} \right)} \right] \frac{r_h}{r_{c,2,s}^{nd}} \right]$$

$$- r_{c,1,2,s}^{d-2} \left\{ \frac{\sqrt{\pi \Gamma} \left( - \frac{d-2}{2(d-1)} \right)}{(d-1) \Gamma \left( \frac{1}{2(d-1)} \right)} + \sum_{n=1}^{\infty} \left( \frac{1}{(d-1)} \right)^{n+1} \frac{1}{\Gamma(1+n) \Gamma \left( \frac{dn+1}{2(d-1)} \right)} \right] \frac{r_h}{r_{c,1,2,s}^{nd}} \right]$$

$$- r_{c,1,s}^{d-2} \left\{ \frac{\sqrt{\pi \Gamma} \left( - \frac{d-2}{2(d-1)} \right)}{(d-1) \Gamma \left( \frac{1}{2(d-1)} \right)} + \sum_{n=1}^{\infty} \left( \frac{1}{(d-1)} \right)^{n+1} \frac{1}{\Gamma(1+n) \Gamma \left( \frac{dn+1}{2(d-1)} \right)} \right] \frac{r_h}{r_{c,1,s}^{nd}} \right] \right] \right]$$

5.1 Holographic entanglement negativity in the low temperature limit

The low temperature limit for the integral in eq. (5.3) is described by the regime $Tl \ll 1$ or $(r_h l \ll 1)$ as the turning point for the static minimal surface remains far away from the horizon at $r_h$. The quantity $r_c$ describing the turning point for the static minimal surface in the bulk described in eq. (5.3) may now be evaluated perturbatively as a series expansion in $r_h l$. The finite part of the area in this case may be expressed as

$$\mathcal{A}_{\text{finite}} = S_0 \left( \frac{L}{a} \right)^{d-2} \left[ 1 + S_1 (r_h l)^d + O((r_h l)^{2d}) \right]$$

Here $S_0$ is the same constant as given in eq. (4.8) and $S_1$ is another constant given by

$$S_1 = \frac{\Gamma \left( \frac{1}{2(d-1)} \right)^{d+1}}{2^{d+1} \pi^{d/2} \Gamma \left( \frac{d}{2(d-1)} \right)^d} \left( \frac{\Gamma \left( \frac{1}{d-1} \right)}{\Gamma(1 - \frac{d-2}{2(d-1)})} \frac{2^{d-2}}{\sqrt{\pi(d+1)}} \right)$$

The holographic entanglement negativity at low temperatures for the mixed state of two disjoint subsystems with long rectangular strip geometries as described in Fig. 4 may now be
perturbatively computed utilizing our conjecture eq. (3.1) and the eqs. (5.5), (5.9), (5.10) as follows

\[ E = \frac{3}{16G_N^{(d+1)}} \left[ S_0 \left\{ \left( \frac{L}{l_1 + l_s} \right)^{d-2} + \left( \frac{L}{l_2 + l_s} \right)^{d-2} - \left( \frac{L}{l_1 + l_2 + l_s} \right)^{d-2} \right\} \right. 
+ S_0 S_1 L^{d-2} \left( \frac{4\pi T}{d} \right)^d \left\{ (l_1 + l_s)^2 + (l_2 + l_s)^2 - (l_1 + l_2 + l_s)^2 - l_s^2 \right\} 
+ S_0 \left\{ \left( \frac{L}{l_1 + l_s} \right)^{d-2} O(T(l_1 + l_s))^{2d} + \left( \frac{L}{l_2 + l_s} \right)^{d-2} O(T(l_2 + l_s))^{2d} 
- \left( \frac{L}{l_1 + l_2 + l_s} \right)^{d-2} O(T(l_1 + l_2 + l_s))^{2d} \right\} - S_0 \left( \frac{L}{l_s} \right)^{d-2} O(Tl_s)^{2d} \right] . \] (5.11)

Note that this result is also cut off independent in contrast with the case for the mixed state configuration of adjacent intervals in [40]. The first term on the right hand side of the above equation arises from the contribution of the \( AdS_{d+1} \) vacuum described in eq. (4.9) and is temperature independent. The remaining terms are the finite temperature corrections to the holographic entanglement negativity at low temperatures which is similar to the case of the mixed state of adjacent intervals as reported in [40].

Using similar arguments as described in section 4 we may obtain the corresponding holographic entanglement negativity for the mixed state of two adjacent subsystems at low temperature through the adjacent limit \( l_s \to a \) in eq. (5.11) as

\[ E_{Adjacent} = \frac{3}{16G_N^{(d+1)}} \left[ \frac{2}{d-2} \left( \frac{L}{a} \right)^{d-2} + S_0 \left\{ \left( \frac{L}{l_1} \right)^{d-2} + \left( \frac{L}{l_2} \right)^{d-2} - \left( \frac{L}{l_1 + l_2} \right)^{d-2} \right\} \right. 
- k l_1 l_2 L^{d-2} T^d + S_0 \left\{ \left( \frac{L}{l_1} \right) O(Tl_1)^{2d} + \left( \frac{L}{l_2} \right) O(Tl_2)^{2d} \right\} \right] , \] (5.12)

where \( k = 2(4\pi)^d S_0 S_1 \) is a constant. As earlier for the zero temperature mixed state dual to the bulk pure \( AdS_{d+1} \) geometry, the above result reproduces the corresponding holographic
entanglement negativity for adjacent intervals [40] at low temperatures further validating our construction.

5.2 Holographic entanglement negativity in the high temperature limit

We now proceed to calculate the entanglement negativity for the mixed state of disjoint subsystems as depicted in Fig. 5 at high temperature limit $T_l \gg 1 (r_h l \gg 1)$. In the above limit the co dimension two bulk static minimal surface homologous to a subsystem in the dual CFT$_d$ approaches the black hole horizon, hence the turning point radius $r_c$ is large and $r_c \approx r_h$.

As described in [21] the integral in eq. (5.3) can be expanded in terms of $\epsilon = (\frac{r_c}{r_h} - 1)$ and solving it up to the leading order leads to the following expression

$$\epsilon = C_1 \exp(-\sqrt{\frac{d(d-1)}{2}} l r_h).$$ \hspace{1cm} (5.13)

Here $C_1$ is a constant which is given by

$$C_1 = \frac{1}{d} \exp \left[ \sqrt{\frac{d(d-1)}{2}} \left\{ \frac{2\sqrt{\pi} \Gamma \left( \frac{d}{2(d-1)} \right)}{\Gamma \left( \frac{1}{2(d-1)} \right)} \right. \right. \right.$$

$$+ \left. \left. 2 \sum_{n=1}^{\infty} \left( \frac{1}{1 + n d} \Gamma \left( n + \frac{1}{2} \right) \Gamma \left( \frac{d(n+1)}{2(d-1)} \right) \Gamma \left( \frac{d n + 1}{2} \right) \right) - \frac{1}{\sqrt{2d(d-1)}} \right) \right].$$ \hspace{1cm} (5.14)

The area of a bulk static minimal surface homologous to the subsystem in question at a high temperature may then be obtained in terms of $r_h = \frac{4\pi T_d}{d}$ by writing eq. (5.7) as an expansion of $\epsilon$ and using eqs. (5.13), (5.6) and (5.5) as follows

$$A = \frac{2}{d-2} \left( \frac{L}{a} \right)^{d-2} + \left( \frac{4\pi}{d} \right)^{d-1} \left[ V T^{d-1} + \frac{C_2 d}{8\pi A'} T^{d-2} \right.$$

$$- \frac{C_1}{8\pi} \sqrt{2d(d-1)} A' T^{d-2} \exp \left\{ -\sqrt{d-1)/2d} 4\pi T_l \right\} + \ldots \right].$$ \hspace{1cm} (5.15)
where \( V = l L^{d-2} \) is the volume of the subsystem and \( A' = 2L^{d-2} \) is the area of a single long (large \( L \)) rectangular strip. Here \( C_2 \) is another constant which is given as

\[
C_2 = 2 \left[ -\frac{\sqrt{\pi}(d-1)\Gamma\left(\frac{d}{2(d-1)}\right)}{(d-2)\Gamma\left(\frac{1}{2}\right)} + \sum_{n=1}^{\infty} \frac{1}{1 + nd(d(n-1) + 2)} \frac{d-1}{\Gamma(n+1/2)\Gamma(n+1)} \frac{\Gamma\left(\frac{d(n+1)}{2d-2}\right)}{\Gamma(n)} \right]. \quad (5.16)
\]

The holographic entanglement negativity at high temperatures for the mixed state of disjoint subsystems of long rectangular strip geometries may now be obtained from our conjecture by utilizing eq. (5.15) and (3.1) in the following form

\[
E = \frac{3}{16G_N^{d+1}} \left( \frac{4\pi}{d} \right) \left( \frac{A}{a^{d-2}} \right) \left[ \frac{C_1}{4\pi} \sqrt{2d(d-1)AT^{d-2}} - \exp \left\{ -\sqrt{\frac{d-1}{2d}} 4\pi T(l_1 + l_s) \right\} - \exp \left\{ -\sqrt{\frac{d-1}{2d}} 4\pi T(l_2 + l_s) \right\} + \exp \left\{ -\sqrt{\frac{d-1}{2d}} 4\pi Tl_s \right\} + \ldots \right]. \quad (5.17)
\]

Here \( A = L^{d-2} \) is the area of the entangling surface between the two adjacent long rectangular strips (in proximity) in the dual \( CFT_d \) and the ellipses represent the higher order correction terms. As earlier this result is also cut off independent. Note that in the high temperature limit the volume dependent thermal terms in eq. (5.15) cancel between the two disjoint subsystems leading to an expression for the holographic entanglement negativity that is proportional to the transverse area of the subsystems. This is expected from a quantum information perspective as the entanglement negativity characterizes an upper bound to the distillable entanglement for the mixed state under consideration and should not involve volume dependent thermal contributions. Note that the subtraction for the thermal part in this case is more subtle than for the mixed state configuration of a single interval at a finite temperature described in [33].

Following a similar procedure as described for the low temperature regime it is possible to obtain the holographic entanglement negativity in the limit when the two subsystems are adjacent with \( l_s \to a \) as

\[
E_{Adjacent} = \frac{3}{16G_N^{d+1}} \left( \frac{A}{a^{d-2}} \right) \left[ \frac{C_1}{4\pi} \sqrt{2d(d-1)AT^{d-2}} \right] + \frac{3}{16G_N^{d+1}} \left( \frac{4\pi}{d} \right) \left( \frac{A}{a^{d-2}} \right) \left[ \frac{C_2d}{4\pi} AT^{d-2} \right] - \frac{C_1}{4\pi} \sqrt{2d(d-1)AT^{d-2}} \left\{ \exp \left( -\sqrt{(d-1)/2d} 4\pi Tl_1 \right) + \exp \left( -\sqrt{(d-1)/2d} 4\pi Tl_2 \right) \right\} + \ldots \]. \quad (5.18)
\]

Once again this matches exactly with the adjacent interval results described in [40] and is a consistency check for our construction subject to the modification by an overall constant numerical factor as mentioned earlier.

6 Summary and Conclusions

To summarize, we have proposed a holographic entanglement negativity construction for bipartite mixed states of disjoint subsystems in \( CFT_d \)s dual to bulk \( AdS_{d+1} \) configurations. Our
construction arising from the corresponding $AdS_3/CFT_2$ results for such mixed state configurations described in [42], involves a specific algebraic sum of the areas of bulk co dimension two static minimal surfaces homologous to appropriate subsystems on the boundary. As a substantiation and consistency check we have utilized our proposal to compute the holographic entanglement negativity for specific examples of such mixed state configurations described by subsystems with long $(d - 1)$ dimensional spatial rectangular strip geometries in the dual $CFT_d$. In this context utilizing our construction, we have computed the holographic entanglement negativity for both zero and finite temperature mixed states described by such disjoint subsystems of long rectangular strip geometries dual to bulk pure $AdS_{d+1}$ and the $AdS_{d+1}$-Schwarzschild black hole geometries respectively. In the latter case the area integrals for the co dimension two Ryu-Takayanagi surfaces were computed perturbatively for both low and high temperature regimes. It is observed that at low temperatures the dominant contribution to the holographic entanglement negativity arises from the pure $AdS_{d+1}$ vacuum with sub leading finite temperature corrections. In the high temperature limit we observe a subtle cancellation between the volume dependent thermal contributions which renders the holographic entanglement negativity to be proportional to the transverse area of the subsystems which conforms to quantum information theory expectations. Furthermore the results are cut off independent as expected from the results for the corresponding $AdS_3/CFT_2$ scenario. Interestingly we have exactly reproduced the corresponding results for mixed states of adjacent subsystems in a dual $CFT_d$ through the adjacent limit described earlier in the literature from our results. This provides a further consistency check for our construction.

As mentioned earlier an alternative holographic entanglement negativity conjecture has been proposed and explored in [35–37] which describes the holographic entanglement negativity as being proportional to the minimal entanglement wedge cross section where the proportionality constant is a dimension dependent pre factor arising from the cosmic brane for the bulk conical defect geometry. Although interesting this proposal requires further investigation as it reproduces the universal part of the replica technique results for the dual $CFT_{1+1}$ for the mixed state configuration of a single interval at a finite temperature only in the limit of high and low temperatures in the $AdS_3/CFT_2$ scenario. Specifically the subtracted thermal entropy term in the universal part of the replica technique results described in the literature seems to be missing. However their proposal correctly reproduces the corresponding replica technique results for other mixed state configurations which indicates that the mismatch described above needs to be investigated and resolved. Interestingly from a comparison of the results from the two distinct approaches it appears that the geodesic combinations involved in the description of the holographic entanglement negativity in the former approach is proportional to the bulk minimal entanglement wedge cross section for the latter. This suggests that the two approaches are closely related however in higher dimensions the results obtained from the former approach may be modified by an undetermined overall constant numerical factor which is dependent on the bulk dimensions and possibly the subsystem geometry. Accordingly our results presented here are also subject to such a possible modification by an overall constant numerical factor. However the issue is still far from being completely resolved especially in the context of a generic higher dimensional $AdS_{d+1}/CFT_d$ scenario and requires further investigation. This is expected to lead to interesting geometrical insights for the bulk configuration and provide a clear and unambiguous holographic proposal for the entanglement negativity in the $AdS_{d+1}/CFT_d$ context and a possible bulk proof for the conjecture which remains a non trivial open issue. It would be very interesting to compute the holographic entanglement negativity for subsystems described by more general geometries other than the long rectangular strip geometries considered here. However this is a difficult exercise as the structure of bulk RT surfaces for more general subsystem geometries in the dual $CFT_d$ are not clear. It is naturally expected that a clear holographic conjecture for the entanglement negativity in a generic $AdS_{d+1}/CFT_d$ scenario and its proof will lead to significant insights into other phenomena which involve mixed state entanglement such as topological phases, quantum
phase transitions, quantum quenches, strongly coupled condensed matter systems apart from issues of quantum gravity. We expect to report further significant insights into these fascinating issues in the near future.

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