Construction of a dual attractor for linear randomized systems of iterated functions

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Abstract. The Legendre transform is applied to the attractors of linear randomized systems of iterated functions $X_1$ and $X_2$, construction using the ordered (F1) and disordered (F2) algorithms. The original attractors $X_1$ and $X_2$ have identical similarity dimension $d_S(X_1) = d_S(X_2)$, but applying the Legendre transform to $X_1$ generates a set $X_1^*$ with a linearly ordered structure and similarity dimension $1 < d_S(X_1^*) < d_S(X_1)$, and applying the Legendre transform to $X_2$ generates a set $X_2^*$ with a disordered structure and similarity dimension $d_S(X_2) < d_S(X_2^*) < 2$.

1. Introduction
The problem of identifying similar mathematical structures has been known in mathematics at least since the time of Euler. The now solved Poincare problem is well known, which is connected with the identification of three-dimensional manifolds such as surfaces of a torus and a sphere. Such problems underlie methods for solving classification problems used in various applied fields: control of stock and financial markets [1], statistical data analysis and forecasting [2], analytical chemistry and chemometrics [3]. Usually these methods are based on the construction of some feature space and analysis of the topological characteristics of the sets formed in this space.

Another formulation of the identification problem is associated with attractors of the linear randomized systems of iterated functions, which can be constructed by different algorithms, but have identical topological characteristics. In this case, some transformation is required to highlight the latent features of the analyzed sets. From a geometric point of view, this transformation projects the analyzed sets from the feature space into the classification space, whose topological characteristics better correspond to the features of the problem [4].

2. Materials and methods
The object of our research is the discrete attractors of random iterated function systems (RIFS), which can be considered as an alternative to the classical continual models of Sierpinski fractal sets. Such sets have singular distribution functions; therefore, the analysis of their properties remains an important task at the present time.
2.1. F1 and F2 algorithms

The attractor of a RIFS on the plane can be defined by the relations [5]:

\[
\begin{align*}
    x_{1,n+1} &= \xi x_{1,n} + (1 - \xi) z_{1,j,n}; \\
    x_{2,n+1} &= \xi x_{2,n} + (1 - \xi) z_{2,j,n},
\end{align*}
\]  

(F1)

where \( n = 0, 1, \ldots, N - 1 \), which can be considered as a finite-difference scheme of the Langevin equation for a Wiener process with some parameter \( 0 < \xi < 1 \). Here, in contrast to the classical case, the random variable \( Z \) has a finite discrete distribution: \( \{ Z_i/p_i \}_{i=1}^{K} \), \( p_i > 0 \), \( \sum_{i=1}^{K} p_i = 1 \).

The construction of an attractor for this model can be performed by another procedure using the urn scheme [6] and the converging number series \( \mu \sum_{i=1}^{\infty} \xi^i = 1 \), where \( \mu \) is a normalization constant depending on the parameter \( \mu = \xi^{-1}(1 - \xi) \). At each step of the algorithm, the sums of subsets of the specified series are written in the form of rows of the matrix \( A_{n+1} \), and each element of \( a_{ij} \) is the sum of the members of the series, selected in accordance with the given distribution \( \{ p_i > 0, i = 1, 2, \ldots, K, \sum_{i=1}^{K} p_i = 1 \} \). The result of this algorithm can be represented as a matrix product [7]

\[ X = AZ. \]  

(F2)

Figure 1 shows a commutative diagram for constructing the attractors using the different algorithms. The path through the upper right corner of the diagram corresponds to the F1 algorithm, and the path through the lower left corner corresponds to the F2 algorithm.

2.2. Legendre transform

The Legendre transform is widely used in statistical physics and thermodynamics. In analytical mechanics, this transformation allows one to pass from the Lagrange function to the Hamilton function, which simplifies the solution of many variational problems.

To study the structure of fractal sets constructed using the above algorithms F1 and F2, we write a finite-difference analogue of the Legendre transform [8]. Let \( \{ x_{1,n}, x_{2,n} \}_{n=0}^{N-1} \) be a sample of points of the attractor on the plane. Then for \( x_{1,n+1} \neq x_{1,n} \) this transformation takes the form:

\[
\begin{align*}
    x_{1,n}^* &= x_{2,n+1} - x_{2,n}, \\
    x_{2,n}^* &= x_{2,n} - x_{1,n} x_{1,n}^*.
\end{align*}
\]  

(1)

This transformation puts each pair of points \((x_n, y_n)\) and \((x_{n+1}, y_{n+1})\) in correspondence with a point \((x_n^*, y_n^*)\), the coordinates of which are the parameters of the equation of the straight line passing through the points \((x_n, y_n)\) and \((x_{n+1}, y_{n+1})\).
3. Results and discussion

The attractors $X_1$ and $X_2$, obtained using the algorithms F1 and F2, not only have the same similarity dimension $d_S(X_1) = d_S(X_2)$ [9], but they are not visually distinguishable even with identical sample sizes. Let us compare the topological characteristics of the dual attractors $X_1^*$ and $X_2^*$ obtained from $X_1$ and $X_2$ using the Legendre transformation (1).

3.1. Legendre transform for Sierpinski sets

As an example, Figures 2 show the Legendre transform (1) for Sierpinski triangular sets, constructed using the algorithms F1 and F2.

The Sierpinski triangular set $X_{1,3}$ constructed by the F1 algorithm at $\mu = 1$ with $d_S(X_{1,3}) = \frac{\ln 3}{\ln 2} \approx 1.585$, and the result of applying the transformation (1) to it, is shown on the left in Figure 2. On the right in Figure 2 we can see the Sierpinski triangular set $X_{2,3}$ constructed using the F2 algorithm and the result of applying the same transformation (1) to it. Note that set $X_{2,3}$ is topologically identical to set $X_{1,3}$, in particular, $d_S(X_{1,3}) = d_S(X_{2,3})$.

For ease of comparison, the original and their dual attractors are shown in the same coordinate system by red and blue points. To generate all attractors, the authors used the “RIFS” package published in 2012 for the free system of statistical analysis and modeling R under the license GNU GPL-3 [10].

As another example, Figures 4 and 3 show the Legendre transform (1) for Sierpinski hexagonal and square sets, constructed using the algorithms F1 and F2 at $\mu = 2$ with $d_S(X_{1,6}) = \frac{\ln 6}{\ln 3} \approx 1.631$ and $d_S(X_{1,4}) = \frac{\ln 8}{\ln 3} \approx 1.893$.

It can be shown that the similarity dimensions in the F1 cases satisfy inequalities $1 < d_S(X_1^*) < d_S(X_1)$, and in the F2 cases they satisfy inequalities $d_S(X_2) < d_S(X_2^*) < 2$. Taking into account equality $d_S(X_1) = d_S(X_2)$, the system of inequalities can be written in the form

$$1 < d_S(X_1^*) < d_S(X_1) = d_S(X_2) < d_S(X_2^*) < 2.$$  \hspace{1cm} (2)

Comparing the results presented in Figures 2–4, we note that the Legendre transform reduces the influence of topological characteristics, but increases the influence of the regularity of the algorithm used to construct attractors.
3.2. Legendre transform properties

If we introduce into consideration the operator $\mathcal{L}$, denoting the Legendre transform, then we can write that

$$X^* = \mathcal{L}X,$$

(3)

where $X^*$ is the Legendre transform of the RIFS attractor $X$. Then by analogy with the continuous Legendre transformation we will call the set $X^*$ dual to the set $X$.

Considering the properties of the finite-difference analogue of the Legendre transform, we note that (1) in our case is not involutive [8]. Indeed, when we reapply the Legendre operator (1), we get

$$\mathcal{L}^1(X) = X^*, \quad \mathcal{L}^2(X) = \mathcal{L}(X^*) = -X,$$

$$\mathcal{L}^3(X) = \mathcal{L}(-X) = -X^*, \quad \mathcal{L}^4(X) = \mathcal{L}(-X^*) = X.$$
As a result, the transform (1) can be represented by a fourth-order cyclic group with basic elements \( \{ I, L, L^2, L^3 \} \).

**Table 1.** Cayley table for Legendre transform.

| *   | I   | L   | L^2 | L^3 |
|-----|-----|-----|-----|-----|
| I   | I   | L   | L^2 | L^3 |
| L   | L   | I   | L^2 | L^3 |
| L^2 | L^2 | L^3 | I   | L   |
| L^3 | L^3 | I   | L   | L^2 |

It can be noted that the identity transformation \( I \) plays the role of the identity element of the multiplicative group, and the group as a whole is isomorphic to the group of basic elements of a complex number \( \{1, i, -1, -i\} \).

4. **Conclusion**

In this work, we applied a finite-difference analogue of the Legendre transform to identify the algorithms used to construct the RIFS attractors \( X_1 \) or \( X_2 \). The cases considered above allow us to rank the similarity dimensions for the original \( X \) and dual \( X^* \) attractors of the RIFS in the form (2). As a further study of the Legendre transformation properties, we propose to consider the invariance of dual fractal sets under homeomorphic transformations [11].

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