SELF-DUAL VORTICES IN THE FRACTIONAL QUANTUM
HALL SYSTEM

XIN-HUI ZHANG, YI-SHI DUAN, YU-XIAO LIU, and LI ZHAO
Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, P. R. China

Abstract

Based on the $\phi$-mapping theory, we obtain an exact Bogomol’nyi self-dual equation with a
topological term, which is ignored in traditional self-dual equation, in the fractional quantum Hall
system. It is revealed that there exist self-dual vortices in the system. We investigate the inner
topological structure of the self-dual vortices and show that the topological charges of the vortices
are quantized by Hopf indices and Brouwer degrees. Furthermore, we study the branch processes
in detail. The vortices are found generating or annihilating at the limit points and encountering,
splitting or merging at the bifurcation points of the vector field $\vec{\phi}$.

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*Corresponding author; Electronic address: zhangxingh03@lzu.cn
I. INTRODUCTION

The fractional quantum Hall (FQH) effect is an example of the new physics that has emerged from the enormous progress made during the past few decades in material synthesis and device processing [1, 2]. Since its discovery [3], experiments on FQH systems have continued to reveal many new phenomena and surprises. Filling factors may take the values $\nu = 1/(2m \pm 1)$ or $\nu = m/(2mp \pm 1)$, in which $m$ and $p$ are integers. All these spark interest to theoretical work of the FQH system. In 1983 Laughlin proposed his celebrated wave functions as an explanation of the FQH effect for two-dimensional electron gas with filling factors $\nu = 1/(2m \pm 1)$ [4]. Shortly after that, hierarchical generalizations of these fully polarized states were also proposed for filling factors $m/(2mp \pm 1)$, with odd-denominator fractions [5, 6]. Subsequently, Jain pointed out that the FQH effect of electrons can be physically understood as a manifestation of the integer quantum Hall effect of composite fermionic objects consisting of electrons bound to an even number of flux quanta [7]. However, the observation of fractions such as $4/11$ and $5/13$ [8, 9] points to new physics beyond the integral quantum Hall effect of composite fermions. Recently, Ref. [10] presents extensive experimental evidence for the considerable strength of these interactions observing the appearance of FQH effect states at filling factors $\nu = 7/11, 4/13, 6/17$ and $15/17$, located between the minima of the primary FQH effect sequences.

Despite the success of the microscopic theories, it is important to develop an effective-field-theory model analogous to the Ginzburg-Landau theory of superconductivity. Zhang, Hansson and Kivelson constructed the bosonic Chern-Simon field theory (ZHK model) of the Laughlin states describing the FQH states as the Bose condensation of a (bosonic) field [11], from which all the essential features can be derived [12]. Girvin and MacDonald [13] and Read [14] also discovered a hidden order parameter in the FQH system. They proposed a field-theory model, containing a complex scalar field $\phi$ coupled to a vector field $(a_0, \vec{a})$ with a Chern-Simon action (or topological mass term). This model exhibits vortex solutions with finite energy and fractional charge which can be identified with Laughlin’s quasiparticles and quasiholes. By adding a natural magnetic term to the field-theory model of Girvin and MacDonald, Ref. [15] has shown that the FQH system admits stable topological as well as non-topological vortex solutions and the physical significance of the “topological” vortices corresponds to quasiparticles and quasiholes. These successes certainly make the
effective-field approach extremely appealing. Recently, the intrinsic spin Hall effect has been theoretically predicted for semiconductors with spin-orbit coupled band structures \[16, 17\]. It has been argued that the spin quantum Hall liquid is a novel state of matter with profound correlated properties described by a topological field theory \[18\].

We can see that the topological properties play an important role in the FQH system. The purpose of the paper is to study the FQH system from the point of view of topology. We introduce a topological method, the $\phi$-mapping topological current theory, which provides a powerful method in researching some topological properties. It has been effectively used to study topological characteristic of dislocations and disclinations continuum \[19\], the topological properties of magnetic monopoles in superconductors \[20\] and the knotted soliton in two-gap superconductors \[21\]. By making use of the $\phi$-mapping theory, the self-dual vortices are investigated in the frame of the ZHK model in the FQH system. The paper is arranged as follows. In Sec. \[\Pi\] by studying the Bogomol’nyi first-order self-dual equation, we point out that there exist self-dual vortices in the FQH system and research their inner topological structure. In Sec. \[\III\] we study the evolution of the self-dual vortices. The conclusion of this paper is given in Sec. \[\IV\].

II. SELF-DUAL VORTECES IN THE FQH SYSTEM

The FQH effect appears in a two-dimensional electron system in a strong magnetic field. It is known that in (2+1)-dimensional spacetime the kinetic action for a gauge field can be either the Maxwell term or the Chern-Simon term, or both. Ref. \[22\] has shown that there can be a Bogomol’nyi-type bound for the energy functional in a pure Abelian Chern-Simon theory. In this paper, we start with the Lagrangian of the ZHK model \[12\]

$$L_{ZHK} = -\frac{k}{2} \epsilon^{\mu\nu\rho} \partial_\mu a_\rho \partial_\nu a_\rho + i \phi^*(\partial_0 + i a_0) \phi$$

$$- \frac{1}{2m} |(\partial_i + i(a_i + A_i^{\text{ext}}))\phi|^2$$

$$- \frac{1}{2} \int d^2x' (|\phi(x)|^2 - n)V(x - x')(|\phi(x')|^2 - n),$$

which just contains the pure Abelian Chern-Simon term. Here $a_\mu$ is the statistical Chern-Simon gauge field, the external gauge field $A_i^{\text{ext}}$ describes the external magnetic field, the constant $n$ denotes a uniform condensate charge density. Since in a two-dimensional system,
a spinless electron may be represented as a hardcore boson carrying an odd integer of Dirac flux quanta, when the Chern-Simon coupling \( \kappa \) takes the values

\[
\kappa = \frac{1}{2\pi(2N - 1)} \quad (N \geq 1),
\]

one can regard this as the condensing of the fundamental fermion field into bosons by the attachment of an odd number of flux through the Chern-Simon coupling \([12]\). We consider a \( \delta \)-function contacted interaction with \( V(\vec{x} - \vec{x}') = \frac{1}{mk} \delta(\vec{x} - \vec{x}') \), then the potential can be written as \( V(\rho) = \frac{1}{2mk}(\rho - n)^2 \), in which \( \rho = \phi^*\phi \). The static energy functional for this model is

\[
\varepsilon_{ZHK} = \int d^2x \left[ \frac{1}{2m} |D_\pm \phi|^2 + \frac{1}{2m} (B^{ext} - \frac{1}{\kappa\rho})\rho \right] \quad + \frac{1}{2mk}(\rho - n)^2 \quad (3)
\]

Clearly, the minimum energy solutions correspond to the constant field solutions \( \phi = \sqrt{n} \), \( a_i = -A_i^{ext} \), \( a_0 = 0 \). In this case the Chern-Simon gauge field opposes and cancels the external field, thus the FQH effect can be viewed as the condensation of the hardcore bosons. Since the Chern-Simon constraint is \( b = -\frac{1}{\kappa\rho} \) \([23]\) (where \( b \) is the Chern-Simon gauge field tensor), we learn that the minimum energy solutions have density \( \rho = n = \kappa B^{ext} \). Here \( \kappa \) takes the values of Eq. (2). These are exactly the conditions for the uniform Laughlin states of filling fraction \( \nu = \frac{1}{2N-1} \), at which there exist \( (2N - 1) \) times as many vortices as there are electrons, each vortex representing a local charge deficit \( \frac{e}{2N-1} \). According to the identity \( |\tilde{D}\phi|^2 = |(D_1 \pm iD_2)\phi|^2 \equiv eB|\phi|^2 \pm \epsilon^{ij}\partial_i J_j \), in which \( J_j = \frac{1}{2i}[\phi^* D_j \phi - \phi(D_j \phi)^*] \), the static energy can be reexpressed as

\[
\varepsilon_{ZHK} = \int d^2x \left[ \frac{1}{2m} |D_\pm \phi|^2 \pm \frac{1}{2m}(B^{ext} - \frac{1}{\kappa\rho})\rho \right] \quad + \frac{1}{2mk}(\rho - n)^2 \quad (4)
\]

where the total magnetic \( B = B^{ext} + b \) and \( D_\pm \phi = (D_1 \pm D_2)\phi \). Then the energy is bounded below by a multiple of the total magnetic flux and obeys a Bogomol’nyi-type lower bound,
which is achieved by the field satisfying a set of first-order self-dual equations

\[ D_\pm \phi = 0, \quad B = B^{\text{ext}} - \frac{1}{\kappa} \rho. \]  

Furthermore, by investigating the above Bogomol’nyi self-dual equations, we will see there exist the self-dual vortices in the FQH system. It is known that the complex scalar field \( \phi = \phi^1 + i \phi^2 \) can be regard as the complex representation of a two-dimensional vector field \( \vec{\phi} = (\phi^1, \phi^2) \) over the base manifold, and it is actually a section of a complex line bundle on the base spacetime manifold. From the two-dimensional vector field \( \vec{\phi} = (\phi^1, \phi^2) \), we can define the unit vector field

\[ n^a = \frac{\dot{\phi}^a}{\| \phi \|}, \quad \| \phi \|^2 = \phi^* \phi. \]  

This is a reasonable representation, \( \phi^a \) is a two component vector field related to the order parameter field \( \phi \). Obviously, it can be looked upon as a smooth mapping between the two-dimensional space \( X \) (with the local coordinate \( x \)) and the two-dimensional Euclidean space \( \mathbb{R}^2 \), \( \vec{\phi}(x) \in \mathbb{R}^2 \) and \( n^a \) a section of the sphere bundle \( S(x) \). Clearly, the zero points of the \( \vec{\phi} \) field are just the singular points of the unit vector field. In the following, by virtue of the so-called \( \phi \)-mapping theory, we will point out that, when \( \phi^a \) field possesses several zero points, there exists a topological term, taking the form of the \( \delta \)-function. From the first case of self-dual equations \( D_\pm \phi = 0 \), by separating the real part from the imaginary, we obtain

\[ eA_\mu = \epsilon_{ab} n^a \partial_\mu n^b - (1/2) \epsilon^{\mu\nu} \partial_\nu \ln(\phi^* \phi). \]

From the second case, i.e. \( D_- \phi = 0 \), by repeating the same process, we get

\[ eA_\mu = \epsilon_{ab} n^a \partial_\mu n^b + (1/2) \epsilon^{\mu\nu} \partial_\nu \ln(\phi^* \phi). \]

Then we have \( eA_\mu = \epsilon_{ab} n^a \partial_\mu n^b \pm \frac{1}{2} \epsilon^{\mu\nu} \partial_\nu \ln(\phi^* \phi). \) In \( (2+1) \)-dimensional spacetime, the magnetic field is \( B = e^{ij} \partial_i A_j \). According to above equations, we have

\[ B = e^{\mu\nu} \epsilon_{ab} n^a \partial_\mu n^b + \nabla^2 \ln \| \phi \|^2. \]  

Noticing the relation \( \partial_\mu n^a = (\partial_\mu \phi^a)/\| \phi \| + \phi^a \partial_\mu (1/\| \phi \|) \) and the well-known Green function relation in \( \phi \)-space \( \partial_\mu \partial_a \ln \| \phi \| = 2\pi \delta^2(\vec{\phi}) (\partial_a = \partial/\partial \phi^a) \), one can prove that \( e^{\mu\nu} \epsilon_{ab} \partial_\mu n^a \partial_\nu n^b = 2\pi \delta^2(\vec{\phi}) D(\phi/x) \), where \( D(\phi/x) = (1/2) e^{\mu\nu} \epsilon_{mn} (\partial \phi^m/\partial x^\mu)(\partial \phi^n/\partial x^\nu) \) is the Jacobian. So the Bogomol’nyi self-dual equation [1] can be written as

\[ 2\pi \delta^2(\vec{\phi}) D(\phi/x) + \nabla^2 \ln \rho = B^{\text{ext}} - \frac{1}{\kappa} \rho, \]  

where \( \rho = \| \phi \|^2 \) is the field density. This equation is more exact than the usual self-dual equation in the ZHK model [23]

\[ \nabla \ln^2 \rho = B^{\text{ext}} - \frac{1}{\kappa} \rho, \]  

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in which the first term $2\pi \delta^2(\vec{\phi})D(\frac{\vec{\phi}}{x})$ on the LHS of Eq. (8) has been ignored all the time. From our previous work, obviously, the first term of Eq. (8) describes the topological properties of the self-dual FQH system. As for usual self-dual equation (9), it only describes the non-topological properties of the system.

In order to investigate the topological properties of the FQH system in more detail, let's introduce a topological current in terms of the topological term

$$J^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \epsilon_{ab} n^a \partial_\nu n^b = \delta^2(\vec{\phi}) D^\mu(\frac{\vec{\phi}}{x}),$$

(10)

where $D^\mu(\phi/x) = (1/2) \epsilon^{\mu\nu\lambda} \epsilon_{mn} (\partial_\nu \phi^m/\partial x^\nu)(\partial_\lambda \phi^n/\partial x^\lambda)$ is the Jacobian vector. From Eq. (10), we can see that the topological current does not vanish only at the zero points of the $\vec{\phi}$ field. So it is necessary to study the zero points of the $\vec{\phi}$ field to determine the nonzero solution of the topological current. The implicit function theory [24] shows that under the regular condition $D^\mu(\phi/x) \neq 0$, the general solutions of

$$\phi^1(x^1, x^2, t) = 0, \quad \phi^2(x^1, x^2, t) = 0,$$

(11)

can be expressed as $x^a = x_k^a(t)$ ($k = 1, 2, \cdots, N$), which represent the world lines of $N$ moving isolated singular points. These singular solutions are just the self-dual vortices in the FQH system. In $\delta$-function theory [25], one can prove that

$$\delta^2(\vec{\phi}) = \sum_{k=1}^N \frac{\beta_k}{|D(\frac{\vec{\phi}}{x})|_{\vec{x}_k}} \delta^2(\vec{x} - \vec{x}_k).$$

(12)

Here the positive integer $\beta_k$ is the Hopf index of the $\phi$-mapping, which means that when $\vec{x}$ covers the neighborhood of the zero point $\vec{x}_k$ once, the vector field $\vec{\phi}$ covers the corresponding region in $\phi$ space $\beta_k$ times. With the definition of vector Jacobian, we can obtain the general velocity of the $k$-th vortices

$$v_k^\mu = \frac{dx_k^\mu}{dt} = \left. \frac{D^\mu(\phi/x)}{D(\phi/x)} \right|_{\vec{x}_k}, \quad v_0 = 1.$$

(13)

Then the topological current $J^\mu$ can be written as the form of the current and the density of the system of $N$ classical point particles with topological charge $W_t = \beta_t \eta_t$ moving in the (2+1)-dimensional spacetime

$$\vec{J} = \sum_{k=1}^N \beta_k \eta_k \vec{v}_k \delta^2(\vec{x} - \vec{x}_k),$$

$$\rho = J^0 = \sum_{k=1}^N \beta_k \eta_k \delta^2(\vec{x} - \vec{x}_k),$$

(14)
where $\eta_k = \text{sgn}(D(\phi/x)|_{x_k}) = \pm 1$ is the Brouwer degree of the $\phi$-mapping. It is clear to see that Eq. (14) shows the movement of the self-dual vortices in spacetime. So, the total charge of the FQH system can be written as

$$Q = \int \rho(x)d^2x = \sum_{k=1}^{N} \beta_k \eta_k.$$  \hspace{1cm} (15)

It is obvious to learn that there exist $N$ isolated vortices, of which the $k$-th vortex possesses charge $\beta_k \eta_k$. And $\eta_k = +1$ corresponds to the vortex, while $\eta_k = -1$ corresponds to the antivortex.

### III. THE EVOLUTION OF THE SELF-DUAL VORTICES

In the above section, we have studied the topological properties of the self-dual vortices in the case that the vector order parameter $\vec{\phi}$ only consists of regular points, i.e., $D^\mu(\phi/x) \neq 0$ is hold true. However, when the regular condition fails, branch processes will occur. Usually there are two kinds of branch points, namely the limit points and the bifurcation points. In this section, we will study the evolution of the self-dual vortices in the FQH system. From Eq. (13), we can learn that the velocity of the $k$-th zero point is determined by

$$\frac{dx^i}{dt} = \frac{D^i(\phi/x)}{D^0(\phi/x)}|_{x=x_k}, \hspace{1cm} (i = 1, 2),$$  \hspace{1cm} (16)

where $D^0(\phi/x) = D(\phi/x)$ is the usual Jacobian. It is obvious that when $D^0(\phi/x) = 0$, at the very point $(t^*, \vec{x}^*)$ the velocity is not unique in the neighborhood of $(t^*, \vec{x}^*)$. If the Jacobian $D^1(\phi/x)|_{(t^*, \vec{x}^*)} \neq 0$, we can use the Jacobian $D^1(\phi/x)$ instead of $D^0(\phi/x)$ for the purpose of using the implicit function theorem \[24\]. Then we have a unique solution of Eqs. (11) in the neighborhood of the very point $(t^*, \vec{x}^*)$

$$t = t(x^1), \hspace{0.5cm} x^2 = x^2(x^1).$$  \hspace{1cm} (17)

We call the critical points $(t^*, \vec{x}^*)$ the limit points. In the present case, we know that

$$\frac{dx^1}{dt} = \frac{D^1(\phi/x)}{D^0(\phi/x)}|_{(t^*, \vec{x}^*)} = \infty, \hspace{0.5cm} \frac{dt}{dx^1}|_{(t^*, \vec{x}^*)} = 0.$$  \hspace{1cm} (18)

Then, the Taylor expansion of $t = t(x^1)$ at the limit point $(t^*, \vec{x}^*)$ is

$$t - t^* = \frac{1}{2} \left. \frac{d^2t}{(dx^1)^2} \right|_{(t^*, \vec{x}^*)} (x^1 - x^{1*})^2.$$  \hspace{1cm} (19)
which is a parabola in $x^1 - t$ plane. From Eq. (19) we can obtain two solutions $x^1_1$ and $x^1_2$, which give two branch solutions (world lines of vortices). If $\frac{d^2 x^1}{(dx^1)^2} > 0$, we have the branch solutions for $t > t^*$; otherwise, we have the branch solutions for $t < t^*$ [see FIG. 1]. These two cases are related to the generation and annihilation of the self-dual vortices.

![Graph](image_url)

FIG. 1: (a) A pair of vortices with opposite charges generate at the limit point, i.e., the origin of vortices. (b) A pair of vortices with opposite charges annihilate at the limit point.

Since the topological current is identically conserved, the topological charges of these two generated or annihilated vortices must be opposite at the limit point, i.e.,

$$\beta_k \eta_k = -\beta_k \eta_k,$$

which shows that $\beta_k = \beta_k$ and $\eta_k = -\eta_k$. One can see that the Brouwer degree $\eta_k$ is indefinite at the limit points, i.e. it can change discontinuously at limit points along the world lines of self-dual vortices (from $\pm 1$ to $\mp 1$).

In the following, the more complicated case will be discussed. We have the restrictions of Eq. (11) at the bifurcation point $(t^*, \vec{x}^*)$,

$$D^1 \left( \frac{\phi}{\vec{x}} \right) \bigg|_{\vec{x}^*} = 0, \quad D^2 \left( \frac{\phi}{\vec{x}} \right) \bigg|_{\vec{x}^*} = 0.$$

Without loss of generality, we discuss only the branch of the velocity component $(dx^1/dt)$ at $(t^*, \vec{x}^*)$. It is known that the Taylor expansion of the solutions of Eq. (11) in the neighborhood of $(t^*, \vec{x}^*)$ can generally be expressed as

$$A(x^1 - x^{1*})^2 + 2B(x^1 - x^{1*})(t - t^*) + C(t - t^*)^2 + \cdots = 0,$$
where \( A, B \) and \( C \) are three constants. Then the above Taylor expansion can lead to

\[
A \left( \frac{dx^1}{dt} \right)^2 + 2B \left( \frac{dx^1}{dt} \right) + C = 0,
\]

and

\[
C \left( \frac{dt}{dx^1} \right)^2 + 2B \frac{dt}{dx^1} + A = 0.
\]

The solutions of Eqs. (23) and (24) give different motion directions of the zero point at the bifurcation point. There are four possible cases, which will show the physical meanings of the bifurcation points.

![FIG. 2: (a) One vortex splits into two vortices. (b) Two vortices merge into one. (c) Two vortices tangentially intersect.](image)

Case 1 \((A \neq 0)\). For \( \Delta = 4(B^2 - AC) = 0 \), from Eq. (23), we get only one motion direction of the zero point at the bifurcation point: \( (dx^1/dt)_{1,2} = -B/A \), which includes three sub-cases: (a) one vortex splits into two vortices; (b) two vortices merge into one; (c) two vortices tangentially intersect at the bifurcation point [see FIG. 2].

Case 2 \((A \neq 0)\). For \( \Delta = 4(B^2 - AC) > 0 \), from Eq. (23), we get two different motion directions of the zero point: \( (dx^1/dt) = (-B \pm \sqrt{B^2 - AC})/A \), which is shown in FIG. 2.

This is the intersection of two vortices, which means that the two vortices meet and then depart at the bifurcation point.

Case 3 \((A = 0, C \neq 0)\). For \( \Delta = 4(B^2 - AC) > 0 \), from Eq. (24) we have

\[
\frac{dt}{dx^1} \bigg|_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{C} = 0, \quad - \frac{2B}{C}.
\]
FIG. 3: Two vortices meet and then depart at the bifurcation point.

There are two important cases: (a) one world line resolves into three world lines, i.e., one vortex splits into three vortices at the bifurcation point. (b) Three world lines merge into one world line, i.e., three vortices merge into one vortex at the bifurcation point [see FIG. 4].

FIG. 4: (a) Three vortices merge into one at the bifurcation point. (b) One vortex splits into three vortices at the bifurcation point.

Case 4 (A=C=0). Eqs. (23) and (24) give respectively

$$\frac{dx_1}{dt} = 0, \quad \frac{dt}{dx_1} = 0.$$  

(26)

This case is obvious, see FIG. 5

These cases reveal the evolution of the self-dual vortices in the FQH system. Beside two vortices encounter and then depart at the bifurcation point along different branch curves, it
FIG. 5: (a) Three vortices merge into one at the bifurcation point. (b) One vortex splits into three vortices at the bifurcation point.

also includes splitting and merging of vortices. When a multicharged vortex moves through the bifurcation point, it may split into several vortices along different branch curves. On the contrary, several vortices can merge into one vortex at the bifurcation point.

The identical conversation of the topological charges shows the sum of the topological charges of these final vortices must be equal to that of the original vortices at the bifurcation point, i.e.,

$$\sum_i \beta_k \eta_{k_i} = \sum_f \beta_{k_f} \eta_{k_f}.$$  \hspace{1cm} (27)

Furthermore, from the above discussion, we can see that the generation, annihilation, and bifurcation of vortices are not gradually changed, but suddenly changed at the critical points.

### IV. CONCLUSION

In conclusion, using the $\phi$-mapping topological current theory, the topological properties of self-dual vortices in the FQH system are studied. In Sec. II, we obtain an exact Bogomol’nyi self-dual equation with topological term, which is ignored in traditional self-dual equation. It is revealed that there are self-dual vortices in the FQH system and the topological charges are determined by Hopf indices and Brouwer degrees. In Sec. III, we point out that the self-dual vortices generate or annihilate at the limit points and encounter, split or merge at the bifurcation. It is shown that the topological charges of the vortices are
preserved in the branch processes during the evolution of these self-dual vortices.

At last, it should be pointed out that in the present paper we treat the vortices as mathematical lines, i.e., the width of a vortex line is zero. This description is obtained in the approximation that the curvature radius of a vortex line is much larger than the width of the vortex line [20].

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