REMARKS ON THE LIECHTI-STRENNER’S EXAMPLES HAVING
SMALL DILATATIONS

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Abstract. We show that the Liechti-Strenner’s example for the closed non-orientable
surface of genus 2 in [LS18] is minimal among the monic polynomials with negative coef-
ficients of degree 2k - 1. We also show that the Liechti-Strenner’s example of orientation-
reversing homeomorphism for the closed orientable surface of genus 2k - 1 in [LS18] is
minimal among the monic polynomials with negative coefficients of degree 4k.

1. Introduction

Let \( \Sigma_g \) be a surface of finite type. A homeomorphism \( h \) of \( \Sigma_g \) is called pseudo-Anosov
if there is a pair of transversely measured foliations \( F^u \) and \( F^s \) in \( \Sigma \) and a real number
\( \lambda > 1 \) such that \( h(F^u) = \lambda F^u \) and \( h(F^s) = 1/\lambda F^s \) [Thu88, CB88]. The number \( \lambda \) is
called the dilatation of \( h \) and the logarithm of \( \lambda \) is called the topological entropy. The
set of dilatations of pseudo-Anosov homeomorphisms of the group of isotopy classes \( \Sigma_g \)
is discrete [AY81, Iva88]. In particular, there exists the minimal dilatation.

The dilatation of a pseudo-Anosov homeomorphism of \( \Sigma_g \) measures its dynamical com-
plexity. Furthermore, the collection of topological entropies has a geometric interpretation
as the collection of Teichmüller distances between Riemann surfaces of the same topolog-
ical type as \( \Sigma_g \) [Abi80]. In particular, the logarithm of the minimal dilatation of a genus
\( g \) surface gives the length of the systole for the genus \( g \) moduli space.

For an orientable surface \( S_g \), several results have been known on the bounds of the
minimal dilatation, \( \delta_g \), for all pseudo-Anosov homeomorphisms of \( S_g \). Penner gave an
upper and lower bounds for the dilatations of \( S_g \), and proved that as \( g \) tends to infinity,
the minimal dilatation tends to one (the logarithm of the minimal dilatation tends to
zero on the order of \( 1/g \) [Pen91]. The upper bound was improved by Bauer for closed
surfaces of genus \( g \geq 3 \) [Bau92].

However, the exact value of the minimal dilatation \( \delta_g \) of \( S_g \) has been found only when
the genus \( g \) is two [CH08].

More is known for the minimal dilatation of orientation-preserving pseudo-Anosov
homeomorphisms on \( S_g \) with orientable invariant foliations. Denote the minimal di-
latation of orientation-preserving pseudo-Anosov homeomorphisms on \( S_g \) with orientable
invariant foliations by \( \delta^+(S_g) \). The following Table 1 shows the known values.

The pseudo-Anosov homeomorphisms realizing \( \delta^+(S_g) \) in Table 1 were constructed by
Zhirov [Zhi95] for \( g = 2 \), Lanneau and Thiffeault [LT11] for \( g = 3 \) and 4, Leiniger [Lei04]
for \( G = 5 \), Kin and Takasawa [KT13] and Aaber and Dunfield [AD10] for \( g = 7 \), and

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stretch factors.
The minimal polynomial of $\delta^+(S_g)$ is given by:

| $g$ | $\delta^+(S_g) \approx$ | Minimal polynomial of $\delta^+(S_g)$ |
|-----|---------------------|----------------------------------|
| 1   | 2.61803             | $x^2 - 3x + 1$                   |
| 2   | 1.72208             | $x^4 - x^3 - x^2 - x + 1$        |
| 3   | 1.40127             | $x^6 - x^4 - x^3 - x^2 + 1$      |
| 4   | 1.28064             | $x^8 - x^5 - x^4 - x^3 + 1$      |
| 5   | 1.17628             | $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 = \frac{x^{12} - x^7 - x^6 - x^5 + 1}{x^2 - x + 1}$ |
| 6   | 1.15486             | $x^{14} + x^{13} - x^9 - x^8 - x^7 - x^6 - x^5 + x + 1$ |
| 7   | 1.12876             | $x^{16} - x^9 - x^8 - x^7 + 1$   |

Table 1. The known values of $\delta^+(S_g)$

The minimal polynomial of $\delta^+(N_g)$ is given by:

| $g$ | $\delta^+(N_g) \approx$ | Minimal polynomial of $\delta^+(N_g)$ |
|-----|---------------------|----------------------------------|
| 4   | 1.83929             | $x^3 - x^2 - x - 1$               |
| 5   | 1.51288             | $x^4 - x^3 - x^2 + x - 1$         |
| 6   | 1.42911             | $x^5 - x^3 - x^2 - 1$             |
| 7   | 1.42198             | $x^6 - x^5 - x^4 - x^3 + x - 1$   |
| 8   | 1.28845             | $x^7 - x^4 - x^3 - 1$             |
| 10  | 1.21728             | $x^{11} - x^6 - x^5 - 1$          |
| 12  | 1.17429             | $x^{13} - x^7 - x^6 - 1$          |
| 14  | 1.14551             | $x^{15} - x^8 - x^7 - 1$          |
| 16  | 1.12488             | $x^{16} - x^9 - x^8 - x^7 + 1$    |
| 18  | 1.10938             | $x^{17} - x^9 - x^8 - 1$          |
| 20  | 1.09730             | $x^{19} - x^{10} - x^9 - 1$       |

Table 2. The known values of $\delta^+(N_g)$

Hironaka [Hir10] for $g = 8$. Hironaka [Hir10] then showed that all of the examples above except the $g = 7$ example arise from the fibration of the link complement of $6_2^2$. From genus 6 to genus 8, each example identified the lower bound calculated by Lanneau and Thiffeault [LT11] as the minimal dilatation.

Recently, Liechti and Strenner [LS18] determined the minimal dilatation of pseudo-Anosov homeomorphisms with orientable invariant foliations on the closed nonorientable surfaces of genus 4, 5, 6, 7, 8, 10, 12, 14, 16, 18 and 20 and the minimal dilatation of orientation-reversing pseudo-Anosov homeomorphisms with orientable invariant foliations on the closed orientable surfaces of genus 1, 3, 5, 7, 9, and 11. Denote by $N_g$ the closed nonorientable surface of genus $g$ and by $\delta^+(N_g)$ the minimal dilatation among pseudo-Anosov homeomorphisms of $N_g$ with an orientable invariant foliation. Denote the minimal dilatation among orientation-reversing pseudo-Anosov homeomorphisms on $S_g$ with orientable invariant foliations by $\delta_{\text{rev}}^+(S_g)$. The values worked by Liechti and Strenner [LS18] are in the following Table 2 and Table 3.

The main purpose of the paper is to show that the Liechti-Strenner’s example for the closed non-orientable surface of genus $2k$ in [LS18] is minimal among the monic polynomials with negative coefficients of degree $2k-1$ and show that the Liechti-Strenner’s example of orientation-reversing homeomorphism for the closed orientable surface of genus
Remarks on the Liechti-Strenner’s examples having small dilatations

Table 3. The known values of $\delta_{\text{rev}}^+(S_g)$

| $g$ | $\delta_{\text{rev}}^+(S_g)$ | Minimal polynomial of $\delta_{\text{rev}}^+(S_g)$ |
|-----|-------------------------------|-------------------------------------------------|
| 1   | 1.61803                       | $x^2 - x - 1$                                   |
| 3   | 1.25207                       | $x^3 - x^2 - x^3 - 1$                           |
| 5   | 1.15973                       | $x^{12} - x^{11} - x^5 - 1$                     |
| 7   | 1.11707                       | $x^{16} - x^{15} - x^7 - 1$                     |
| 9   | 1.09244                       | $x^{20} - x^{19} - x^9 - 1$                     |
| 11  | 1.07638                       | $x^{24} - x^{23} - x^{11} - 1$                  |

$2k - 1$ in [LS18] is minimal among the monic polynomials with negative coefficients of degree $4k$.

**Theorem 1.1.** Denote by $N_{2k}$ the closed nonorientable surface of genus $2k$. For all $k \geq 2$,

$$x^{2k-1} - x^k - x^{k-1} - 1$$

gives the minimal dilatation among the polynomials

$$x^{2k-1} - a_{2k-2}x^{2k-2} \cdots - a_1 x - 1$$

where whose associated pseudo-Anosov homeomorphisms on $N_{2k}$ have orientable invariant foliations and where $a_i \geq 0$ for $1 \leq i \leq 2k - 2$.

**Theorem 1.2.** Denote by $S_{2k-1}$ the closed orientable surface of genus $2k - 1$. For all $k \geq 2$,

$$x^{4k} - x^{2k+1} - x^{2k-1} - 1$$

gives the minimal dilatation among the polynomials

$$x^{4k} - a_{4k-1}x^{4k-1} \cdots - a_1 x - 1$$

where whose associated pseudo-Anosov homeomorphisms on $S_{2k-1}$ are orientation reversing with orientable invariant foliations and where $a_i \geq 0$ for $1 \leq i \leq 4k - 1$.

## 2. Perron-Frobenius matrix

**Definition.** Let $M = (m_{ij})$ and $N = (n_{ij})$ be two nonnegative $d \times d$ matrices. We say $M > N$ if $m_{ij} \geq n_{ij}$ for all $i, j \in \{1, 2, \ldots, d\}$ and strict inequality holds for at least one entry.

**Lemma 2.1** (Perron-Frobenius). Let $T$ and $L$ be two Perron-Frobenius matrices such that $T > L$, then the spectral radius $\lambda_T$ of $T$ is strictly bigger than the spectral radius $\lambda_L$ of $L$.

**Proof.** Let $t > 0$ be a left eigenvector of $T$ corresponding to $\lambda_T$ and let $l > 0$ be a right eigenvector of $L$ corresponding to $\lambda_L$. Then

$$Tl > Ll = \lambda_L l$$

and

$$\lambda_T tl = tTl > \lambda_L tl.$$

Therefore $\lambda_T > \lambda_L$. \qed
Proposition 2.2. The transpose of the Frobenius companion matrix of 
\[ x^{2k-1} - a_{2k-2}x^{2k-2} \cdots - a_1 x - 1 \]
is
\[
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & a_1 & a_2 & \cdots & a_{2k-2}
\end{bmatrix},
\]
and the transpose of the Frobenius companion matrix of 
\[ x^{4k} - a_{4k-1}x^{4k-1} \cdots - a_1 x - 1 \]
is
\[
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & a_1 & a_2 & \cdots & a_{4k}
\end{bmatrix},
\]

Proposition 2.3. [LS18, Proposition 4.1] Let \( \psi : N_g \to N_g \) be a pseudo-Anosov map with a transversely orientable invariant foliation on the closed non-orientable surface \( N_g \) of genus \( g \). Then its stretch factor \( \lambda \) is a root of a (not necessarily irreducible) polynomial \( p(x) \in \mathbb{Z}[x] \) with the following properties:

1. \( \deg(p) = g - 1 \)
2. \( p(x) \) is monic and its constant coefficient is \( \pm 1 \)
3. The absolute values of the roots of \( p(x) \) other than \( \lambda \) lie in the open interval \( (\lambda^{-1}, \lambda) \). In particular, \( p(x) \) is not reciprocal or anti-reciprocal.
4. \( p(x) \) is reciprocal mod 2.

Proposition 2.4. [LS18, Proposition 4.3] Let \( \psi : S_g \to S_g \) be an orientation-reversing pseudo-Anosov map with transversely orientable invariant foliations. Then its stretch factor \( \lambda \) is a root of a (not necessarily irreducible) polynomial \( p(x) \in \mathbb{Z}[x] \) with the following properties:

1. \( \deg(p) = 2g \)
2. \( p(x) \) is monic and its constant coefficient is \( (-1)^g \)
3. \( p(x) = (-1)^g x^{2g}p(-x^{-1}) \)
4. The absolute values of the roots of \( p(x) \) other than \( \lambda \) and \(-\lambda^{-1}\) lie in the open interval \( (\lambda^{-1}, \lambda) \).

Lemma 2.5. Let \( A \) be the adjacent matrix for a graph \( G \) with \( n \) vertices. Then \( A \) is primitive if and only if \( G \) is strongly connected and the gcd of the lengths of the loops in \( G(A) \) is one.

Proof. See [DM67, Chapter 6, Section 1, Remarks, 6, 8].

Lemma 2.6. (1) If \( x^{2k-1} - a_{2k-2}x^{2k-2} \cdots - a_1 x - 1 \) gives the dilatation on \( N_{2k} \), then \( a_i \equiv a_{2k-1-i} \mod 2 \) and at least for one \( a_i \) with \( \gcd(2k-1, 2k-1-i) = 1 \), \( a_i \neq 0 \) for \( 1 \leq i \leq k-1 \).
Lemma 2.7. Let $k \geq 2$.

(1) \[ x^{2k-1} - x^k - x^{k-1} - 1 \]
gives the minimum spectral radius among the polynomials $x^{2k-1} - a_{2k-2}x^{2k-2} - \cdots - a_1x - 1$ with $a_i = a_{2k-i-1} = 1$ for only one $i$ ($1 \leq i \leq k - 1$) and $a_j = 0$ for all $j$ with $j \neq i$ and $j \neq 2k - 1 - i$.

(2) \[ x^{4k} - x^{2k+1} - x^{2k-1} - 1 \]
gives the minimum spectral radius among the polynomials $x^{4k} - a_{4k-1}x^{4k-1} - \cdots - a_1x - 1$ with $a_i = a_{4k-i} = 1$ for only one $i$ ($1 \leq i \leq 2k - 1$) and $a_j = 0$ for all $j$ with $j \neq i$ and $j \neq 4k - i$.

Proof. (1) Let $g(x) = x^{2k-1} - x^{2k-1-i} - x^i - 1$. Then $g(1) = -2 < 0$ and $g(2) = 2^{2k-1} - 2^{2k-1-i} - 2^i - 1 = (2^{2k-1-i} - 1)(2^i - 1) - 2 > 0$. $g'(x) = (2k - 1)x^{2k-2} - (2k - 1 - i)x^{2k-2-i} - ix^{i-1}$. $g'(1) = (2k - 1) - (2k - 1 - i) - i = 0$ and $g'(x) = x^{2k-2}((2k - 1) - (2k - 1 - i)x^{-i} - ix^{-2k+i+1}) \geq 0$ for $x \geq 1$. Hence the largest root of $g(x)$ lies between 1 and 2 and it is the only root bigger than 1. We can regard $g$ as a two variable polynomial $G(x, i)$.

\[ \frac{\partial G}{\partial i} = x^i \ln x \left(x^{2k-2i-1} - 1\right) \geq 0 \]

if $x \geq 1$. Hence

\[ x^{2k-1} - x^k - x^{k-1} - 1 \]
gives the minimum spectral radius.

(2) Let $g(x)$ be $g(x) = x^{4k} - x^{4k-i} - x^i - 1$. Then $g(1) = -2 < 0$ and $g(2) = 2^{4k} - 2^{4k-i} - 2^i - 1 = (2^{4k-i} - 1)(2^i - 1) - 2 > 0$. $g'(x) = (4k)x^{4k-1} - (4k - i)x^{4k-i-1} - ix^{i-1}$. $g'(1) = 4k - (4k - i) - i = 0$ and $g'(x) = x^{4k-1}((4k) - (4k - i - 1)x^{-i} - ix^{-4k+i}) \geq 0$ for $x \geq 1$. Hence the largest root of $g(x)$ lies between 1 and 2 and it is the only root bigger than 1. We can regard $g$ as a two variable polynomial $G(x, i)$.

\[ \frac{\partial G}{\partial i} = x^i \ln x \left(x^{4k-2i} - 1\right) \geq 0 \]

if $x \geq 1$. Hence

\[ x^{4k} - x^{2k+1} - x^{2k-1} - 1 \]
gives the minimum spectral radius.
3. Proof of Theorem 1.1 and Theorem 1.2

3.1. **Proof of Theorem 1.1.** Suppose $\lambda$ is the minimal dilatation of a pseudo-Anosov homeomorphism with a transversely orientable invariant foliation on the closed non-orientable surface $N_{2k}$ of genus $2k$ among the dilatations whose associated monic polynomials have negative coefficients and degree $2k - 1$. Since such polynomials can be considered as the characteristic polynomial of nonnegative Frobenius companion matrix which is also primitive,

$$x^{2k-1} - x^k - x^{k-1} - 1$$

gives the minimal dilation by Lemma 2.6 (1), Lemma 2.7 (1) and Lemma 2.1.

3.2. **Proof of Theorem 1.2.** Suppose $\lambda$ is the minimal dilatation of a pseudo-Anosov homeomorphism which is orientation reversing with orientable invariant foliations on the closed orientable surface $S_{2k-1}$ of genus $2k - 1$ among the dilatations whose associated monic polynomials have negative coefficients and degree $4k$. Since such polynomials can be considered as the characteristic polynomial of nonnegative Frobenius companion matrix which is also primitive,

$$x^{4k} - x^{2k+1} - x^{2k-1} - 1$$

gives the minimal dilation by Lemma 2.6 (2), Lemma 2.7 (2) and Lemma 2.1.

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