The s-cobordism theorem seen as a particular case of Latour’s theorem
Carlos Moraga Ferrandiz

To cite this version:
Carlos Moraga Ferrandiz. The s-cobordism theorem seen as a particular case of Latour’s theorem. 2014. hal-00915569v2

HAL Id: hal-00915569
https://hal.science/hal-00915569v2
Preprint submitted on 19 Jun 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
The \(s\)-cobordism theorem seen as a particular case of Latour’s theorem

C. Moraga Ferrándiz

Abstract

We show how Latour’s theorem (\cite{Lat94}) can be understood as a natural generalization of the \(s\)-cobordism theorem for cohomology classes \(u \in H^1(M; \mathbb{R})\). The \(s\)-cobordism theorem becomes a special degenerate case when \(u = 0\).

Keywords: Whitehead torsion, \(s\)-cobordism, Latour’s theorem, Morse-Novikov theory

MSC Classification: 57R80; 19J10

1 The \(s\)-cobordism theorem: the exact case

Two connected, closed and oriented manifolds \(N_0^n, N_1^n\) are cobordant if there exists a compact oriented manifold \(W^{n+1}\) such that \(\partial W^{n+1} = (N_0^n) \cup (N_1^n)\). Superscripts denote dimension while \((-N)\) represents the manifold \(N\) with reversed orientation. Such a triad \((W; N_0, N_1)\) is said to be an \(h\)-cobordism if both inclusions \(N_0 \xrightarrow{i_0} W \xleftarrow{i_1} N_1\) are homotopy equivalences.

Let \(\pi\) be the fundamental group of \(W\); we denote by \(\Lambda := \mathbb{Z}\left[\pi\right]\) its group ring. To each \(h\)-cobordism we can associate its torsion \(\tau(W, N_0)\) which lives in the Whitehead group \(\text{Wh}(\pi) := K_1(\Lambda)_{\pm}\) (see \cite{Coh73} for a definition).

The \(s\)-cobordism theorem, which can be found in \cite{Ker65}, states that \(\tau(W, N_0) = 0\) is a sufficient condition\(^1\) for \(W\) being diffeomorphic to \(N_0 \times [0, 1]\), provided \(n \geq 5\).

We can reformulate this theorem into a statement about non triviality of some functional space: consider \(\mathcal{F}\) the space of \(C^\infty\)-functions \(f : W \to [0, 1]\) such that \(f^{-1}(i) = N_i, i = 0, 1\) with the \(C^\infty\)-topology. Its subspace \(\mathcal{E}\) consisting of functions without critical points is non-empty if and only if \(W \cong N_0 \times [0, 1]\), as it suffices to pick some \(f \in \mathcal{E}\) and to integrate the vector field \(\nabla f / \|\nabla f\|\) relative to some Riemannian metric on \(W\) in order to find a diffeomorphism from \(N_0 \times [0, 1]\) to \(W\). We obtain so:

**Theorem 1 (Functional formulation of the \(s\)-cobordism theorem).** Let \(n \geq 5\),

\[\mathcal{E} \neq \emptyset \iff \tau(W, N_0) = 0\]

\(^1\)Trivially, the condition \(\tau(W, N_0) = 0\) is also necessary for \(W \cong N_0 \times [0, 1]\).
Remark 2. The relative homology $H_*(W, N_0)$ vanishes since $i_0$ is a homotopy equivalence. This is indeed a necessary condition for $E \neq \emptyset$, since the Morse complex $C_*(f)$ of a Morse function $f \in \mathcal{E}$ is zero in every degree, and the homology of $C_*(f)$ is isomorphic to $H_*(W, N_0)$ (see [Mil63]).

By Lefschetz duality, we deduce that $H^1(W, N_1) \approx H_n(W, N_0)$ vanishes. The same holds for $H^1(W, N_0) \approx H_n(W, N_1)$ by using the fact that $i_1$ is also a homotopy equivalence.

We are going to consider the s-cobordism theorem and the one from Latour as statements about the relative cohomologies $H^1(X; Y)$ and $H^1(X; Z)$ of a triad $(X; Y, Z)$. Since the only relative cohomology class of degree 1 to consider in the case of an h-cobordism is $u = 0$, we will talk about the exact case to refer to the context of this section.

2 The theorem of Latour

Consider now a closed manifold $M^{n+1}$. We ask $M$ to fiber over the circle $S^1$, which is equivalent by Tischler’s theorem [Tis70] to the existence of a non-singular closed 1-form on $M$.

We say that a cohomology class is non-singular if it is representable by a non-singular closed 1-form. It is clear that there is no chance for $u = 0 \in H^1(M; \mathbb{R})$ to be non-singular since $M$ is closed. Latour’s theorem characterizes degree one de Rham cohomology classes $0 \neq u$ that are non-singular. Within the context of this section, here is the statement:

Theorem 3 ([Lat94]). Let $n \geq 5$, and let $\Omega^n_{NS}$ denote the space of non-singular closed 1-forms representing $u$. We have:

$$\Omega^n_{NS} \neq \emptyset \iff \left\{ \begin{array}{l} H_*(M, -u) = 0, \\ \tau(-u) = 0, \\ u \text{ and } -u \text{ are stable.} \end{array} \right.$$  

Notice that a $\alpha \in \Omega^n_{NS}$ determines a whole ray $ru = [r\alpha], r \in \mathbb{R}^*$ of non-singular cohomology classes. These form a cone into $H^1(M; \mathbb{R})$. In particular $\Omega^n_{NS} \neq \emptyset \iff \Omega^\tau_{NS} \neq \emptyset$.

A degree one cohomology class can be seen as a morphism $u : \pi \to \mathbb{R}$ just by integrating representatives of loops in $M$. The Novikov ring associated to $u$, denoted by $\Lambda_u$, is a completion of the group ring $\Lambda$. Elements of $\Lambda_u$ are formal sums $\lambda := \sum n_i g_i, n_i \in \mathbb{Z}$ such that, for every fixed $C \in \mathbb{R}$, there are only finitely many terms $g_i$ verifying $u(g_i) < C$. The homology $H_*(M; -u)$ which appears in Latour’s theorem is the Novikov homology, which was first constructed in [Nov81]. The Novikov complex is the free finite $\Lambda_{-u}$-module $(\mathcal{N}^-_u := \Lambda_{-u} \otimes_{\Lambda} S_*(\tilde{M}), \partial_*)$, where $S_*(\tilde{M})$ denotes the simplicial/cellular chain complex of the universal cover of $M$ associated to a given triangulation/cell structure on $M$.

Remark 4. It is important to notice that Latour’s theorem, which is a property of the cohomology class $u$, is stated in terms related to $\Lambda_{-u}$-modules.

The second right-side condition of theorem 3 contains indeed the first: in order to define the torsion $\tau(-u)$, we need the Novikov complex to be acyclic. In this case, $\tau(-u)$ is defined as follows: by setting a base of $\mathcal{N}^-_u$, we obtain a contraction $\delta_s : \mathcal{N}^-_u \to \mathcal{N}^-_{u+1}$ as in [Mau67] §4.
The map $(\partial + \delta)_* : N_{ev}^u \to N_{od}^u$ is then an isomorphism and we can consider $S$, the class in $K_1(\Lambda_u)$ of its associated matrix in the fixed basis. This class may depend on the choice of the basis (compare to [Mil66, §7]); in order to remove this indeterminacy, Latour defined the Whitehead group associated to $-u$ as $Wh(-u) := K_1(\Lambda_u)$, where the class $[S]$ depends only on $-u$. Here, $T_{-u} := \pm \pi \cdot (1 + (u < 0)) \subset \Lambda^u$ is the subgroup of the so-called trivial units.

The torsion $\tau(-u)$ is defined by $[S] \in Wh(-u)$. An explanation about the stability condition of $\pm u$ is postponed to subsection 4.1.

As he pointed out in his introduction, Latour’s strategy to prove theorem 3 is similar to that of the $s$-cobordism theorem; the goal of the present paper is to show that Latour’s theorem is indeed a natural generalization of $s$-cobordism theorem for relative cohomology classes.

3 A generalization framework

In Latour’s theorem, the notion of $u$-stability is related with unbounded primitives of $p^*(u)$ where $p : \hat{M} \to M$ is the abelian cover of $u$ having $\pi_1(\hat{M})$ equal to $\ker(u)$. If we try to extend this notion to a null class $u = 0$, the cover coincides with $\text{Id} : M \to M$ and we have no unbounded primitives of 0.

However, we only want to extend the notion of $u$-stability for null classes of the relative 1-cohomology of an $h$-cobordism. We replace so the notion of $h$-cobordism in the most trivially possible way in order to have unbounded primitives in the exact context when $u = 0 \in H^1(W, N_0) \cup H^1(W, N_1)$:

**Definition 5.** From any $h$-cobordism $(W; N_0, N_1)$, we construct the triad $(W_\pm; N_-, N_+)$ by setting:

- $N_- := N_0 \times (-\infty, 0], N_+ := N_1 \times [1, \infty)$ and
- $W_\pm := \bigcup_{\text{Id}_{N_0}} W \bigcup_{\text{Id}_{N_1}} N_+$.

We call $(W_\pm; N_-, N_+)$ the extended triad of $(W; N_0, N_1)$.

In particular the cohomologies of an $h$-cobordism and of its extended triad are the same and $W$ is trivial if and only if $W_\pm$ is diffeomorphic to $N_0 \times \mathbb{R}$. We can so state the $s$-cobordism theorem in terms of extended triads.

**Remark 6.** Of course, the extended triad is not strictly an $h$-cobordism since $W_\pm$ has no boundary, but the inclusion $i : (W; N_0, N_1) \hookrightarrow (W_\pm; N_-, N_+)$ is nevertheless a simple homotopy equivalence: any cell of, say $N_-$, is of the form $\Delta \times \mathbb{R}^-$ where $\Delta$ is a cell of $N_0$ and we have a natural collapse $c : N_- \to N_0$.

4 Comparison of the two theorems

Let us study how Latour’s conditions relative to $u \in H^1(M; \mathbb{R}) \setminus \{0\}$ of closed manifolds $M$ degenerate to the $s$-cobordism theorem condition for extended triads of $h$-cobordisms $(W; N_0, N_1)$.
as in section 3

Firstly, regard the closed manifold $M$ as the triad $(M; \emptyset_-, \emptyset_\pm)$ and the cohomology class as living in $u \in H^1(M; \mathbb{R}) = H^1(M, \emptyset_-; \mathbb{R})$. Latour’s conditions applied to $-u$ should be regarded as a statement about $-u \in H^1(W, \emptyset_\pm; \mathbb{R})$ since in this case, the associated Novikov complex is constructed using $\Lambda_\pm$-modules instead of $\Lambda_-\pm$-modules.

Secondly, consider the $h$-cobordism replaced by its extended triad $(W_\pm; N_-, N_\pm)$ as in definition 5. We distinguish the null-elements of the relative cohomologies by setting $H^1(W_\pm, N_-) = \{+0\}$ and $H^1(W_\pm, N_+) = \{-0\}$.

Now we study what happens to Latour’s conditions when they are interpreted relatively to the extended triad $(W_\pm; N_-, N_\pm)$ for $u = +0 \in H^1(W_\pm, N_-)$:

- The Novikov homology $H_*((W_\pm, N_-), -0)$ is computed from the complex $\mathcal{N}_{\pm}^{-0}$. This complex is $\Lambda_{-0} \otimes_{\Lambda} S_*(\tilde{W}_\pm, \tilde{N}_-)$ by definition, but the ring $\Lambda_{-0}$ trivially coincides with the group ring $\Lambda$. Hence the Novikov complex $\mathcal{N}_{\pm}^{-0}$ is nothing but $S_*(\tilde{W}_\pm, \tilde{N}_-)$. So $H_*((W_\pm, N_-), -0) = H_*\left(\tilde{W}_\pm, \tilde{N}_-\right)$ which is isomorphic to $H_*(\tilde{W}, \tilde{N}_0) = 0$ since both pairs are homotopy equivalent. The first condition of Latour is so trivially true for $h$-cobordisms as we have noticed on remark 2.

- Since the set of trivial units $T_{-0} = \pm \pi$, the group $\text{Wh}(-0)$ defined by Latour reduces to the usual Whitehead group $\text{Wh}(\pi)$. The torsion $\tau(-0)$ is $\tau(W_\pm, N_-)$, since $\mathcal{N}_{\pm}^{-0} = S_*(\tilde{W}_\pm, \tilde{N}_-)$. But the latter torsion coincides with the Milnor torsion $\tau(W, N_0)$ since the pairs $(W_\pm, N_-)$ and $(W, N_0)$ are simply homotopy equivalent by remark 6. The condition $\tau(-0) = 0$ of Latour is so the equivalent condition of theorem 4 for an $h$-cobordism to be trivial.

**Remark 7.** The corresponding statements about $u = -0 \in H^1(W_\pm, N_\pm)$ yield the vanishing of the relative homology $H_*\left(\tilde{W}, \tilde{N}_1\right)$ and associated torsion $\tau(W, N_1)$, which is an equivalent formulation of the $s$-cobordism theorem.

Note that the previous observations do not need the notion of extended triad and can be applied to the $h$-cobordism $(W; N_0, N_1)$ directly. We have established so far that the first two conditions of Latour’s theorem reduce to theorem 1 when applied to an $h$-cobordism or to its extended triad. We need so to prove that the third condition relative to stability holds trivially when reducing to $u = \pm 0$. This will be proved below in proposition 9 where the convenience of the concept of extended triad will become more apparent.

### 4.1 The stability condition

To prove his theorem, Latour showed that every Morse closed 1-form $\alpha$ representing $u$ gives raise to a complex $C_*(\alpha)$ of $\Lambda_-\pm$-modules which is simply equivalent to the Novikov complex $\mathcal{N}_{\pm}^{-u}$. The two first conditions that we have analyzed allow one to proceed as in the $s$-cobordism theorem in order to recurrently eliminate zeros of index/coindex $i$ by eventually adding zeroes of index/coindex $i + 2$, apart from the case $i = 2$ which is special. Adding $\pm u$-stability, Latour
obtained a sufficient condition to handle with this special case (compare with [Dam00]). Since critical points of index/coindex 2 do not represent a natural obstruction in the exact case, ±-stability should hold trivially. Let us recall what u-stability means, as in [Lat94]:

Consider $p : \hat{M} \to M$ the covering whose fundamental group is $\ker u$. Its transformation group is $\frac{\pi_1(M)}{\ker u} \approx \mathbb{Z}^{\text{irr}(u)}$. Since the class $p^*(u)$ vanishes, any closed 1-form $\alpha$ representing $u$ admits a primitive: a function $f : \hat{M} \to \mathbb{R}$ verifying $df = p^*(\alpha)$ and $f(g \cdot x) = u(g) + f(x)$ for every pair $(g, x)$ in $\mathbb{Z}^{\text{irr}(u)} \times \hat{M}$. It is easy to see that for every $t \in \mathbb{R}$, $\hat{f}^{-1}((t, \infty))$ has only one connected component where $\hat{f}$ is not bounded; denote it by $\hat{M}_t$. The inclusions $(\hat{M}_s \hookrightarrow \hat{M}_t)_{s \geq t}$ induce a projective system $P(u) := \left(\pi_1(\hat{M}_t)\right)_{t \in \mathbb{R}}$.

Latour showed that this system does not depend on the choice of $\hat{f}$ but only on $u$, up to projective isomorphism (see [Lat94, Lemme 5.7]). The $u$-stability is a condition about $P(u)$.

**Definition 8.** A cohomology class $u \in H^1(M; \mathbb{R})$ is stable if there exists an increasing sequence $(t_n)_{n \in \mathbb{N}} \to \infty$ where the restrictions to the images of $P(u)$ are isomorphisms. More precisely, if we set $I_n := \text{Im}\left((\pi_1)_*(t_n^{t_{n+1}})\right)$ and $j_n := (\pi_1)_*(t_{n+1}^t)|_{I_{n+1}}$, then $j_n : I_{n+1} \to I_n$ are isomorphisms for every $n \in \mathbb{N}$.

The next proposition shows how $u$-stability reduces to a condition which holds trivially for extended triads of h-cobordisms.

**Proposition 9.** The extended triad $(W_\pm; N_-, N_+)$ of any h-cobordism is $\pm$-stable.

**Proof.** Let us deal with $(-0)$-stability. Here $\{-0\} = H^1(W_\pm, N_+)$. In this situation $\ker(-0)$ is identified with the whole $\pi_1(W_\pm, N_+)$, and the covering pair $(W_\pm, N_+)$ to consider coincides with the pair $(W_\pm, N_+)$ itself. By relative de Rham theory (see [BTS2, Ch.1, §6] for example), the class $-0$ is represented by the pair $(df, f|_{N+})$ with $f : W_\pm \to \mathbb{R}$. We are free to choose $f$ verifying $f(x, t) = t$ for every $(x, t) \in N_+ \cup N_-$; since $W \hookrightarrow W_\pm$ is compact, there exists some $1 \leq t_0 \in \mathbb{R}$ such that for every $t \geq t_0$, the unique unbounded component $W_t$ of $f^{-1}([t, \infty))$ equals $N_1 \times [t, \infty)$. The projective system $\pi_1(W_t)$ is constantly $\pi_1(N_1)$ with inclusions inducing the identity if $t \geq t_0$. By choosing any increasing sequence $(t_n)$ starting at $t_0$, stability for the class $-0$ holds.

**Acknowledgements:** This paper was conceived and written at Fall 2013, during a JSPS Short-term post-doctoral fellowship. The author would like to express his gratitude to the Graduate School of Mathematics of Tokyo for their warm welcome and for having provided an excellent research environment.
References

[BT82] R. Bott and L.W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[Coh73] M.M. Cohen. *A course in simple-homotopy theory*. Springer-Verlag, New York, 1973. Graduate Texts in Mathematics, Vol. 10.

[Dam00] M. Damian. Formes fermées non singulières et propriétés de finitude des groupes. *Ann. Sci. École Norm. Sup. (4)*, 33(3):301–320, 2000.

[Ker65] M.A. Kervaire. Le théorème de Barden-Mazur-Stallings. *Comment. Math. Helv.*, 40:31–42, 1965.

[Lat94] F. Latour. Existence de 1-formes fermées non singulières dans une classe de cohomologie de de Rham. *Inst. Hautes Études Sci. Publ. Math.*, (80):135–194 (1995), 1994.

[Mau67] S. Maumary. Type simple d’homotopie (Théorie algébrique) *in Torsion et type simple d’homotopie*. Exposés faits au séminaire de Topologie de l’Université de Lausanne. LNM 48. Springer-Verlag, Berlin, 1967.

[Mil63] J. Milnor. *Morse theory*. Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963.

[Mil66] J. Milnor. Whitehead torsion. *Bull. Amer. Math. Soc.*, 72:358–426, 1966.

[Nov81] S.P. Novikov. Multivalued functions and functionals. An analogue of the Morse theory. *Dokl. Akad. Nauk SSSR*, 260(1):31–35, 1981.

[Tis70] D. Tischler. On fibering certain foliated manifolds over $S^1$. *Topology*, 9:153–154, 1970.

Graduate School of Mathematical Sciences, the University of Tokyo
3-8-1 Komaba Meguro-ku, Tokyo 153-8914, Japan.

Telephone: (+81) (0)3-5465-8292
E-mail address: carlos@ms.u-tokyo.ac.jp