Cylindrically polarized Bessel–Gauss beams

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Abstract
We present a study of radially and azimuthally polarized Bessel–Gauss (BG) beams in both the paraxial and nonparaxial regime. We discuss the validity of the paraxial approximation and the form of the nonparaxial corrections for BG beams. We show that independently on the ratio between the Bessel aperture cone angle $\vartheta_0$ and the Gaussian beam divergence $\theta_0$, the nonparaxial corrections are always very small and therefore negligible. The explicit expressions for the nonparaxial vector electric field components are also reported.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Cylindrically polarized beams of light, i.e., optical beams whose polarization is non-uniformly distributed across the intensity pattern, have proven to be a very versatile tool, as their peculiar properties, such as the ability of producing a smaller focus [1] for example, demonstrated to be useful in various fields of research such as spectroscopy [2], microscopy [3], optical tweezing [4], material processing [5], propagation of linear and nonlinear waves in crystals [6–8], quantum information [9] and metrology [10]. This plethora of applications motivated the development of several different experimental techniques to generate such beams [11–15]. A detailed theoretical analysis of the properties of these beams and their application in the paraxial case can be found in [16].

Motivated by these many applications, different groups have then tried in the last years to provide a suitable extension of these beams to the nonparaxial case, by exploring the field of a strongly focused beam [17], by using complex dipole sources [18], introducing elegant Laguerre–Gauss beams in the nonparaxial regime [19], decentered Gaussian beams [20] and vector Bessel beams [21]. Recently we also contributed to this subject by proposing a direct and simple generalization of the formalism introduced by Holleczek et al [16], based on the use of Bessel beams to generate Hermite–Gaussian-like beams with zero total angular momentum [22].

Although Bessel beams are exact solutions of the Helmholtz equation, they are not physical states of the electromagnetic field, as they carry infinite energy [23]. Bessel–Gauss (BG) beams, on the other side, are also exact solutions of the Helmholtz equation, but with a finite energy spectrum [24–27], a feature that makes it possible to realize such beams experimentally [28, 29].

It is then the aim of this work to extend the results of [22] to the case of BG beams by deriving the expressions for the electric field of cylindrically polarized beams of light both in the paraxial and nonparaxial case. Since BG beams can be nowadays easily generated in an optical laboratory with the help of suitably programmed spatial light modulators [30, 31], we believe that the present work could serve as a toolbox for extending the framework of radially and azimuthally polarized states of light to the nonparaxial domain quite accurately and easily.

This work is organized as follows: in section 2 we briefly revise the paraxial and nonparaxial form of BG beams. These results are then used in section 3 to generate the cylindrically polarized vector fields in the paraxial regime, according to the method presented in [22]. In section 4, we briefly discuss the
various regimes of BG beams and how strong is the influence of nonparaxial correction in all these regimes. In section 5, the explicit expressions of the vector electric and magnetic fields of cylindrically polarized BG beams are given. Finally, conclusions are drawn in section 6.

2. Paraxial and nonparaxial BG beams

As it is well known, Bessel beams carry infinite energy, and therefore they do not represent physical solutions of the Helmholtz equation [23]. This peculiar characteristic is intimately related to the fact that the support of the angular spectrum of such beams is a ring of zero thickness of radius $K_0 = k_0 \sin \theta_0$ (being $\theta_0$ the characteristic cone angle of the Bessel beam) and represented by the Dirac delta $\delta(K - K_0)$, a highly singular function. A more realistic description of such beams is represented by BG beams, which can be thought as the equivalent of Bessel beams where the Dirac-delta ring in Fourier space is replaced with a circle of finite radial thickness associated to a Gaussian envelope, characterized by the waist $w_0$ [24]. Another possible interpretation of BG beams is that they generated by a superposition of tilted Gaussian beams whose axes of propagation are uniformly distributed on a surface of a cone of half aperture $\theta_0$ [32]. In contrast with pure Bessel beams, however, BG beams are not diffractionless anymore, even if they maintain their diffractionless character up to a maximal distance $D = w_0/\sin \theta_0$ [24], after which their Gaussian character dominates over the non-diffracting one given by the the Bessel part. BG beams are, however, still an exact solution of the Helmholtz equation, i.e.,

$$\left(\nabla^2 + k_b^2\right)\psi_f(x, y, z) = 0,$$  \hspace{1cm} (1)

where $k_b = 2\pi/\lambda$ is the vacuum wave number. If we write the previous equation in cylindrical coordinates, BG solutions at $z = 0$ can be found according to Gori et al [32] to be as follows:

$$\psi_f(R, \phi, 0) = J_\ell(K_0 R) e^{-ik_0 z} e^{i\ell \phi},$$  \hspace{1cm} (2)

where $K_0 = k_0 \sin \theta_0$, $R = \sqrt{x^2 + y^2}$, $J_\ell(x)$ is the Bessel function of the first kind of order $\ell$ and $(R, \phi, z)$ are the usual cylindrical coordinates defined with respect to the main axis of propagation $\hat{z}$. The angular spectrum at $z = 0$ is then obtained by taking the 2D Fourier transform of equation (2), namely

$$\tilde{\psi}_f(K, \phi) = \frac{1}{2\pi} \int d^2 R \psi_f(R, \phi, 0) e^{-iK R}$$  \hspace{1cm} (3)

where $d^2 R = dx dy$, $K = \sqrt{K_x^2 + K_y^2}$, $K_x = K \cos \phi$, $K_y = K \sin \phi$, $R = x \hat{x} + y \hat{y}$ and $I_\ell(x)$ is the modified Bessel function of the first kind [33]. From the previous equation one can easily see that in the limit in which the waist $w_0 \rightarrow \infty$, equation (2) gives the traditional Bessel beam, as the Gaussian envelope goes to one. Correspondingly, the angular spectrum defined in equation (3) becomes

$$\lim_{w_0 \rightarrow \infty} \tilde{\psi}_f(K, \phi) = \lim_{w_0 \rightarrow \infty} \left[ \frac{w_0^2}{2\ell^2} \frac{i \ell \phi}{K_0} \left( \frac{K K_0}{2w_0^2} \right) e^{-\frac{x^2 + y^2}{4w_0^2}} \right]$$  \hspace{1cm} (4)

where in order to calculate the limit we used the following asymptotic expression of the modified Bessel function of the first kind in the vicinity of infinite [33]:

$$I_\ell(z) \approx \frac{e^z}{\sqrt{2\pi z^\ell}} \left[ \left(4\ell^2 - 1\right)\left(4\ell^2 - 3\right) + \ldots \right].$$  \hspace{1cm} (5)

Equation (4) is therefore the correct limit that leads to the angular spectrum of a Bessel beam.

To find the expression of the BG beam in the generic plane $z > 0$, we now propagate equation (3) according to the propagation rule of the angular spectrum [34], thus obtaining

$$\psi_f(R, z) = \frac{1}{2\pi} \int d^2 K \tilde{\psi}_f(K, \phi) e^{-i\mathbf{q} \cdot \mathbf{R}}$$  \hspace{1cm} (6)

where $d^2 K = dk_x dk_y$ and $N = (w_0^2/2) \exp\left[i\phi - K_0^2/(4w_0^2)\right]$. This expression is still exact but cannot be calculated analytically, due to the presence of the square root at the exponent of the last exponential function. However, in the paraxial limit one has that $K \ll K_0$ and the Taylor expansion of the square root around $K = 0$, i.e.,

$$\sqrt{1 - K^2/k_0^2} \approx 1 - \frac{1}{2} \left( \frac{K}{k_0} \right)^2 + \mathcal{O}\left( \frac{K}{k_0} \right)^4$$  \hspace{1cm} (7)

allows us to rewrite the angular spectrum propagator in the approximate form

$$e^{ik_0 z \sqrt{1 - K^2/k_0^2}} \approx e^{ik_0 z \exp\left(-iK^2/2k_0\right)},$$  \hspace{1cm} (8)

where the quadratic phase factor is the so-called Fresnel propagator and it is responsible for the paraxial propagation [34]. With this in mind, we can now calculate from equation (6) the form of the BG beam in the paraxial limit and retrieve the nonparaxial corrections as higher order correction to the paraxial limit. In order to do so, we first need to isolate the Fresnel term from the total propagator

$$e^{ik_0 z \sqrt{1 - K^2/k_0^2}} = e^{ik_0 z} \exp\left(-iK^2/2k_0\right) \left[ e^{ik_0 z \sqrt{1 - K^2/2k_0}} \right]$$  \hspace{1cm} (9)

and then perform a Taylor expansion of the nonparaxial part...
of the propagator (the one in square brackets in the previous equation), thus obtaining
\[ e^{i k_0 \sqrt{1 - k^2/k_0}} \approx 1 - i k_0 z \left( \frac{K}{k_0} \right)^4 
- \frac{i k_0 z}{16} \left( \frac{K}{k_0} \right)^6 + \ldots. \] (10)

By inserting this result into equation (6) we can then write the exact form of the BG beam in a series form as follows:
\[
\psi_\ell(R, z) \approx N \int_0^\infty dK K e^{-K^2 \left( \frac{1}{\rho^2} + \frac{i z}{2k_0} \right)} 
\times I_1 \left( -\frac{W_0^2 K K_0}{2 W_0^2} \right) I_\ell(KR) 
\times \left[ 1 - i k_0 z \left( \frac{K}{k_0} \right)^4 - \frac{i k_0 z}{16} \left( \frac{K}{k_0} \right)^6 \ldots \right],
\]
\[ = e^{i k_0 z} \left[ \psi_0^{(1)}(x, y, z) + \psi_\ell^{(1)}(x, y, z) + \psi_\ell^{(2)}(x, y, z) + \ldots \right]. \] (11)

This expression allows us to evaluate all the expansion terms, the lowest one being the paraxial approximation and the higher ones being the nonparaxial corrections.

The paraxial BG beam is then given by:
\[
\psi_\ell^{(0)}(R, \varphi, z) = N \int_0^\infty dK K e^{-K^2 \left( \frac{1}{\rho^2} + \frac{i z}{2k_0} \right)} 
\times I_1 \left( -\frac{W_0^2 K K_0}{2 W_0^2} \right) I_\ell(KR) 
\times \left[ 1 - i k_0 z \left( \frac{K}{k_0} \right)^4 - \frac{i k_0 z}{16} \left( \frac{K}{k_0} \right)^6 \ldots \right],
\]
\[ = e^{i k_0 z} \psi_0^{(0)}(x, y, z) + e^{i k_0 z} \psi_\ell^{(1)}(x, y, z) + e^{i k_0 z} \psi_\ell^{(2)}(x, y, z) + \ldots. \] (12)

where \( \rho = R/W_0 \), \( \theta = \sin \theta_0/\theta_0 \) and \( z = z_0 R \), with \( W_0 = \sqrt{2 z_0/\rho k_0} \) and \( \theta_0 = 2/(k_0 W_0) \) being the waist and the angular aperture of the beam, respectively. The manner we derived the expansion in equation (11) is different from the previously proposed one in [24], yet in full agreement with the latter. This is one of our main results.

According to equation (11), the first nonparaxial correction can be written in the following simple compact form:
\[
\psi_\ell^{(1)}(R, \varphi, z) = \frac{-i z N}{2 k_0} \int_0^\infty dK K^5 e^{-K^2 \left( \frac{1}{\rho^2} + \frac{i z}{2k_0} \right)} 
\times I_1 \left( -\frac{W_0^2 K K_0}{2} \right) I_\ell(KR) 
\times \left[ 1 - i k_0 z \left( \frac{K}{k_0} \right)^4 - \frac{i k_0 z}{16} \left( \frac{K}{k_0} \right)^6 \ldots \right].
\] (13)

The explicit expression of equation (13) evaluated for arbitrary \( \ell \) is quite cumbersome and, for sake of clarity, it will not be reported here. However, in the present work we are interested in the circumstances \( \ell = \pm 1 \) solely and in these cases the formulas are much simpler:
\[
\psi_\ell^{(1)}(R, \varphi, z) \big|_{\ell = \pm 1} = \frac{e^{i k_0 z}}{1 + i \zeta} \left( \frac{1 + i \zeta}{\rho^2 + i \zeta \Theta^2} \right) \times \left[ 1 \pm \frac{1}{2} \left( \rho^2 + i \zeta \Theta^2 \right) \right] \times J_\ell \left( \frac{2 \rho \Theta}{1 + i \zeta} \right). \]

3. Cylindrically polarized paraxial BG beams

Now that we have correctly calculated the exact form of a paraxial BG beam and its nonparaxial corrections at all orders (each of them can be simply evaluated analytically thanks to the Gaussian form of the integrals), we can now build the Hermite–Gaussian-like BG beams, by combining the paraxial solutions with \( \ell = 1 \) and \( \ell = -1 \) as follows:
\[
\phi_{10}(R, \varphi, z) = \frac{1}{\sqrt{2} \sqrt{2} \sqrt{2}} \left[ \psi_0^{(0)}(R, \varphi, z) + \psi_{-1}^{(0)}(R, \varphi, z) \right], \quad (15a)
\]
\[
\phi_{01}(R, \varphi, z) = \frac{-i}{\sqrt{2} \sqrt{2} \sqrt{2}} \left[ \psi_0^{(0)}(R, \varphi, z) - \psi_{-1}^{(0)}(R, \varphi, z) \right]. \quad (15b)
\]

where \( \psi_{0}^{(0)}(R, \varphi, z) \) and \( \psi_{1}^{(0)}(R, \varphi, z) \) are defined by the equation (12) for \( \ell = \pm 1 \) respectively. A sketch of the function \( \phi_{10}(R, \varphi, z) \) in \( z = 0 \) and its comparison with the Hermite–Gaussian beam \( \text{HG}_{10}(x, y) \) is reported in figure 1. As can be noted, the two functions have the same Cartesian symmetry. Moreover, figure 1 also shows that unlike the case of real Bessel beams [22] (figure 1(c)), BG beams do not present any ring outside the paraxial region. This is a consequence of the fact that their angular spectrum is tailored with a Gaussian function, instead of being a simple Dirac delta function. By analogy with [16], we can then build a four dimensional space spanned by the basis formed by the Cartesian product of \{\( \mathbf{u}_0, \mathbf{u}_1 \}\} mode basis defined above and the polarization vectors \( \{\mathbf{x}, \mathbf{y}\} \), namely
\[
\{ \phi_{10}, \phi_{01} \} \otimes \{ \mathbf{x}, \mathbf{y} \} = \{ \phi_{10} \mathbf{x}, \phi_{01} \mathbf{x}, \phi_{01} \mathbf{y}, \phi_{10} \mathbf{y} \}. \quad (16)
\]
Radially \( (\mathbf{u}_R) \) and azimuthally \( (\mathbf{u}_A) \) polarized beams can be then easily obtained as linear combinations of these four modes as follows:
\[ \phi_{10}(R, \varphi, 0) = \pm \phi_{10}(R, \varphi, 0) \]
\[ \hat{u}_R(x, y, 0) \]
\[ \hat{u}_R(x, y, 0) \]

where the \( \pm \) sign refers to co-rotating and counter-rotating modes respectively [16]. The polarization patterns of these paraxial modes are shown in figures 2 and 3.

4. Nonparaxial corrections

A BG beam is characterized by two competing parameters: the Bessel cone angle \( \theta_0 \) and the width \( w_0 \) of the Gaussian beam composing the spectrum or, alternatively, its angular spread \( \theta_0 = 2/(k_0w_0) \). Depending on the relative weight of these two parameters, according to [32] we can define three different regimes that are schematically represented in figure 4. The first of these regimes corresponds to have \( \theta_0/\theta_{01} < 1 \) with \( \theta_{01} \ll 1 \). In this case, as it is reported in detail in [32] for the fundamental BG beam, we expect that the central region of the beam (whose radius is approximately \( \xi_m/K_0 \), being \( \xi_m \) the first zero of the function \( J_m(\xi) \)) closely resembles the central part of a Gaussian beam, as the beam waist \( w_0 \) of the component Gaussian beams is less than the central radius of the BG beam. This corresponds to the most paraxial situation. We therefore expect that in this case (figure 4(a)) the contribution of the nonparaxial corrections would be negligible. To show this, in figure 4(d) we report a section (along the plane \( y = 0 \)) of the scalar first order correction \( \psi_{10}^{(1)}(x, z) \). As can be seen, the intensity of the first nonparaxial order of equation (11) is of the order of \( 10^{-6} \) and it can be therefore neglected.
The second regime that we can analyze is given by $\theta_0/\theta_0 > 100$ (figure 4(c)). In this regime, the Gaussian beam components are well separated and the spot size of each single component diffracts during the propagation along $z$. However, up to a distance $D$ defined as the distance from $z = 0$ at which a Gaussian beam component has receded from the $z$-axis by a quantity $w_0$ [32], the beam remains diffractionless. Although in this case one could intuitively say that the contributions of higher order nonparaxial terms in equation (11) are higher than the previous case, figure 4(f) shows that also in this case
the nonparaxial corrections are negligible with respect to the paraxial part of the beam, having an intensity $10^6$ times smaller that their paraxial counterpart.

For the sake of completeness, we present also the intermediate case $\theta_0/\theta_0 \approx 1$, where the component Gaussian beams overlap strongly during propagation (figure 4(b)). Also in this case, however, as it appears clear from figure 4(e), the effects of the nonparaxial corrections to equation (11) are negligible.

5. Electric and magnetic fields

The modes obtained from equation (17) and depicted in figures 2 and 3 are strictly paraxial. As already explained in [22], however, since $\hat{u}_{R,A}$ are paraxial modes, they are not exact solutions of the Helmholtz equation (1). In order to fix this problem, in principle, all the nonparaxial corrections to equation (13) must be taken into account and once the non-paraxial part of the beam, having an intensity $10^6$ times smaller that their paraxial counterpart, this problem, in principle, all the nonparaxial corrections to equation (11) into the definition of the Hermite–Gauss-like beams given by equation (15) instead of $u_{R,A}[0]$, they can be used as Hertz vectors to determine the correct form of the electric and magnetic fields, according to the following equations [22]:

\[
E(r, t) = \nabla \times \left[ \nabla \times \mathbf{H}(r, t) \right], \\
B(r, t) = \frac{1}{c^2} \frac{\partial}{\partial t} \left[ \nabla \times \mathbf{H}(r, t) \right],
\]

(18)

where $\mathbf{H}(r, t) = \hat{\mathbf{u}}_{R,A} \exp(-i\omega t)$ depending on which kind of polarization one wants to attribute to the fields. However, as we discussed in the previous section, the nonparaxial corrections are always very small and they can be neglected irrespectively on the relative weight between the two characterizing parameters of a BG beam, namely $\theta_0$ and $\theta_0$, provided that both angles are paraxial. It is therefore sufficient to use the paraxial modes $\hat{u}_{R,A}$ given by equation (17) as Hertz potentials to generate the nonparaxial electric and magnetic fields.

Here we report the explicit expression of the components (in normalized cylindrical coordinates $\{\rho, \phi, \zeta\}$) of the electric field as deriving from equation (18), that read explicitly as follows:

\[
E_{R+}(r, t) = \frac{1}{(\zeta - i)^2} \left\{ i \left[ 2\rho^2 \left( \zeta^3 + 3\Theta^2 - \rho^2 + 1 \right) \right. \\
+ \Theta^2 \left( 2i\zeta + \Theta^2 + 2 \right) + \rho^4 \right\} I(2\theta) \left\{ \frac{2\theta \rho}{-i + \zeta} \right\} \\
- 2i\theta \left( 2i\zeta + \Theta^2 - 2\rho^2 + 3 \right) \\
\times I(0) \left( \frac{2\theta \rho}{-i + \zeta} \right) e^{(\rho, \theta, \zeta, t)},
\]

(19a)

\[
E_{R-}(r, t) = \frac{1}{(\zeta - i)^2} \left\{ i \left[ 2\rho^2 \left( \zeta^3 + 3\Theta^2 - \rho^2 + 1 \right) \right. \\
+ \Theta^2 \left( 2i\zeta + \Theta^2 + 2 \right) + \rho^4 \right\} I(2\theta) \left\{ \frac{2\theta \rho}{-i + \zeta} \right\} \\
- 2i\theta \left( 2i\zeta + \Theta^2 - 2\rho^2 + 3 \right) \\
\times I(0) \left( \frac{2\theta \rho}{-i + \zeta} \right) e^{(\rho, \theta, \zeta, t)},
\]

(19b)

\[
E_{K+}(r, t) = 0,
\]

(19c)

\[
E_{K-}(r, t) = 0,
\]

(21a)

\[
E_{K+}(r, t) = \frac{\cos (2\phi)}{(\zeta - i)^2} \left\{ i \left[ 2\rho^2 \left( \zeta^3 + 3\Theta^2 - \rho^2 + 1 \right) \right. \\
+ \Theta^2 \left( 2i\zeta + \Theta^2 + 2 \right) + \rho^4 \right\} I(2\theta) \left\{ \frac{2\theta \rho}{-i + \zeta} \right\} \\
- 2i\theta \left( 2i\zeta + \Theta^2 - 2\rho^2 + 3 \right) \\
\times I(0) \left( \frac{2\theta \rho}{-i + \zeta} \right) e^{(\rho, \theta, \zeta, t)},
\]

(20a)

\[
E_{K-}(r, t) = \frac{\sin (2\phi)}{(\zeta - i)^2} \left\{ i \left[ 2\rho^2 \left( \zeta^3 + 3\Theta^2 - \rho^2 + 1 \right) \right. \\
+ \Theta^2 \left( 2i\zeta + \Theta^2 + 2 \right) + \rho^4 \right\} I(2\theta) \left\{ \frac{2\theta \rho}{-i + \zeta} \right\} \\
- 2i\theta \left( 2i\zeta + \Theta^2 - 2\rho^2 + 3 \right) \\
\times I(0) \left( \frac{2\theta \rho}{-i + \zeta} \right) e^{(\rho, \theta, \zeta, t)},
\]

(20b)

\[
E_{K+}(r, t) = \frac{-\cos (2\phi)}{(\zeta - i)^2} \left\{ i \left[ 2\rho^2 \left( \zeta^3 + 3\Theta^2 - \rho^2 + 1 \right) \right. \\
+ \Theta^2 \left( 2i\zeta + \Theta^2 + 2 \right) + \rho^4 \right\} I(2\theta) \left\{ \frac{2\theta \rho}{-i + \zeta} \right\} \\
- 2i\theta \left( 2i\zeta + \Theta^2 - 2\rho^2 + 3 \right) \\
\times I(0) \left( \frac{2\theta \rho}{-i + \zeta} \right) e^{(\rho, \theta, \zeta, t)},
\]

(20c)

for the co-rotating radially polarized electric field

for the counter-rotating radially polarized electric field
\[
E_{\vec{k}}^c(r, t) = \frac{i}{(\zeta - i)^3} \left[ \Theta^2 (2\zeta(3\zeta^2 - \rho^2) - 12\zeta \rho^2 + 6) \\
+ \rho^2 (2\zeta(2\zeta^2 - 5\zeta) + \rho^2 - 6) + \Theta^4 \right] \\
\times I_z \left( \frac{2\Theta \rho}{-i + \zeta} \right) \\
- 2\Theta \rho \left[ \zeta(-4\zeta^2 + 11i) + 2\Theta^2 - 2\rho^2 + 7 \right] \\
\times I_0 \left( \frac{2\Theta \rho}{-i + \zeta} \right) e^{i(\rho, \Theta, \zeta, t)},
\]
(21b)

\[
E_{\vec{k}}^r(r, t) = 0,
\]
(21c)

for the counter-rotating azimuthally polarized electric field

\[
E_{\vec{k}}^r(r, t) = \frac{\sin(2\phi)}{(\zeta - i)^3 \rho^2} \left[ 2\rho^4 (-i\zeta - 3\Theta^2 - 1) \\
+ \rho^2 (2(1 + i\zeta)\Theta^2 \\
- 4i(\zeta - i)^3 + \Theta^4) + 4(\zeta - i)^4 + \rho^6 \right] \\
\times I_0 \left( \frac{2\Theta \rho}{-i + \zeta} \right) - 2i\Theta \rho \left[ 3i\zeta + 2\Theta^2 + 3 \right] \\
+ 2(1 - i\zeta)^3 - 2\rho^4 \\
\times I_0 \left( \frac{2\Theta \rho}{-i + \zeta} \right) e^{i(\rho, \Theta, \zeta, t)},
\]
(22a)

\[
E_{\vec{k}}^l(r, t) = \frac{\cos(2\phi)}{(\zeta - i)^2 \rho^2} \\
\times \left[ -i \left[ 2\rho^2 (-2\zeta^2 + 5i\zeta + 3\Theta^2 + 3) \\
+ \rho^2 (2(-3 + \zeta(2\zeta^2 - 5\zeta))\Theta^2 \\
- 4i(\zeta - i)^3 - \Theta^4) + 4(\zeta - i)^4 - \rho^6 \right] I_z \left( \frac{2\Theta \rho}{-i + \zeta} \right) \\
- 2\Theta \rho \left[ \zeta(-4\zeta^2 + 11i) + 2\Theta^2 + 7 \right] \\
- 2i(\zeta - i)^3 - 2\rho^4 \right] I_0 \left( \frac{2\Theta \rho}{-i + \zeta} \right) e^{i(\rho, \Theta, \zeta, t)},
\]
(22b)

\[
E_{\vec{k}}^s(r, t) = \frac{2\sin(2\phi)}{(\zeta - i)^2 \rho^2} \left[ \rho (3i\zeta + 3\Theta^2 - \rho^2 + 3) \\
\times I_z \left( \frac{2\Theta \rho}{-i + \zeta} \right) \\
- i\Theta (\rho^2 - 3\rho^2) I_z \left( \frac{2\Theta \rho}{-i + \zeta} \right) \right] e^{i(\rho, \Theta, \zeta, t)},
\]
(22c)

for the counter-rotating azimuthally polarized electric field. In all these expressions

\[
\chi(\rho, \Theta, \zeta, t) = \frac{-\zeta \Theta^2 + i\rho^2}{\zeta - i} - i\omega t,
\]
(23)

\[\mathbf{r} = (\rho e^\theta + \phi \hat{\phi} + \zeta \hat{z})\] and \(I_z(x)\) are the modified Bessel functions of the first kind, which are related with the usual Bessel functions \(J_i(x)\) by the relations \(I_i(x) = (-i)^i J_i(ix)\) [33]. The calculation of the explicit expression of the components of the magnetic field is left to the reader.

6. Conclusions

In this work we have theoretically investigated the cylindrically polarized modes associated to BG beams. We have derived the paraxial form of a BG beam in a plane \(\zeta \neq 0\) by propagating the angular spectrum and we have expressed the full nonparaxial BG field as a paraxial contribution \(\psi_{\ell 0}^{(0)}(R, \theta, z)\) plus a series of nonparaxial corrections and we have analyzed their role in three different regimes defined by the ratio \(\theta_{00}/\theta_0\). We have shown that independently on the considered regime (corresponding to how much nonparaxial the BG beam is), the nonparaxial corrections are always very small and therefore their contribution can be neglected.

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