A note on transition in discrete gauge groups in F-theory

Yusuke Kimura

1KEK Theory Center, Institute of Particle and Nuclear Studies, KEK, 1-1 Oho, Tsukuba, Ibaraki 305-0801, Japan
E-mail: kimurayu@post.kek.jp

Abstract

We observe a new puzzling physical phenomenon in F-theory on the multisection geometry, wherein a model without a gauge group transitions to another model with a discrete \(Z_n\) gauge group via Higgsing. This phenomenon may suggest an unknown aspect of F-theory compactification on multisection geometry lacking a global section. A possible interpretation of this puzzling physical phenomenon is proposed in this note. We also propose a possible interpretation of another unnatural physical phenomenon observed for F-theory on four-section geometry, wherein a discrete \(Z_2\) gauge group transitions to a discrete \(Z_4\) gauge group via Higgsing as described in the previous literature. These phenomena may suggest that a (discrete) gauge group is enhanced when a multisection splits into multisections of smaller degrees, and/or it splits into a global section and a multisection of smaller degree, in the moduli of multisection geometry.
1 Introduction

In F-theory formulation \cite{1, 2, 3}, theories are compactified on spaces that admit torus fibration. The modular parameter of an elliptic curve as a fiber of this fibration is identified with the axiodilaton in type IIB superstring; this enables axiodilaton to have $SL_2(\mathbb{Z})$ monodromy in F-theory formulation.

There are cases in which genus-one fibrations admit a global section, and those in which they do not have a global section. These two cases have different physical interpretations, as we now explain. Genus-one fibrations with a global section are frequently referred to as elliptic fibrations in the F-theory literature. Geometrically, that a genus-one fibration has a global section means that one can choose a point for each elliptic fiber over every point in the base space, and one can move this chosen point throughout the base space, yielding a smooth copy of the base space. Thus, to have a global section implies that one has a copy of the base space embedded in the total elliptic fibration.

F-theory models compactified on elliptic fibrations with a global section have been studied in the literature, e.g., in \cite{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31}. The number of $U(1)$ factors is identified with the rank of the group generated by the set of global sections \cite{3}, which is known as the “Mordell–Weil group,” and the rank of this group is positive when there are two or more independent sections. Therefore, the F-theory model has a $U(1)$ factor when it is compactified on an elliptic fibration with two (or more) independent sections.

F-theory models on genus-one fibrations lacking a global section have attracted interest recently, for reasons including the discrete gauge group \cite{47} that arises \cite{17} in this type of F-theory compactification. A genus-one fibration without a section still admits a multisection,
which “wraps around” over the base multiple times. Multisections of “degree” $n$, or simply “$n$-sections,” are those wrapped around over the base $n$ times and therefore they intersect with the fiber $n$ times. In F-theory on a genus-one fibration with a multisection, the degree of the multisection corresponds to the degree of a discrete gauge group \(^2\) forming in F-theory \([17]\); a discrete $\mathbb{Z}_n$ gauge group arises in F-theory on a genus-one fibration with an $n$-section.

Recent studies of F-theory compactifications on genus-one fibrations without a section can be found, e.g., in \([49, 47, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73]\). The aim of this note was to study the transitions in discrete gauge groups forming in F-theory on the moduli of multisection geometry and, as a result, we point out that we observed a physical phenomenon that might be interesting.

A transition from a discrete $\mathbb{Z}_2$ gauge group to a discrete $\mathbb{Z}_4$ gauge group was discussed in \([73]\), and it was pointed out there that this appears physically puzzling. The process in which a four-section splits into a pair of bisections, and this pair of bisections further splits into four sheets of global sections, was discussed in \([73]\). This process has a physical interpretation as $U(1)^3$ breaking down into a discrete $\mathbb{Z}_2$ gauge group, and this discrete $\mathbb{Z}_2$ gauge group further transitions to a discrete $\mathbb{Z}_4$ gauge group \([73]\) through a Higgsing process, along the lines of the argument as in \([17]\). The problem here is that, because this process is Higgsing, one would naturally expect a discrete $\mathbb{Z}_2$ gauge group to break down into a discrete gauge group of a smaller degree; however, the opposite to this expectation occurs here. A discrete $\mathbb{Z}_2$ gauge group appears to be “enhanced” to a discrete $\mathbb{Z}_4$ group, rather than breaking down into another discrete gauge group of smaller degree. This was the puzzle raised in \([73]\).

Our analysis in this study finds that this is not the only puzzling phenomenon that can be observed in F-theory models on the multisection geometry.

We point out a new physical phenomenon that appears unnatural in this note, by studying the structure of the multisection geometry. There are various ways a multisection splits into multisections of smaller degrees. By analyzing these structures, and by considering the physical interpretation of these structures, we observe the new puzzling physical phenomenon.

To be clear, the analysis of splitting of a multisection enables us to observe a transition process wherein an F-theory model that has neither a discrete gauge group nor a $U(1)$ appears to “break down” into another F-theory model with a discrete gauge group via Higgsing. Because an F-theory model transitions to another model with a discrete gauge group via Higgsing in this process, the original model is supposed to possess a larger gauge symmetry than the discrete gauge group that the model has after the transition. However, the original

---

\(^2\)The discrete part of the “Tate–Shafarevich group,” which is often denoted as $\text{III}$, of the Jacobian of the compactification space yields a discrete gauge group arising in F-theory compactification \([17]\). Given a Calabi–Yau genus-one fibration $Y$, the “Tate–Shafarevich group,” $\text{III}(J(Y))$, of the Jacobian $J(Y)$ can be considered. The discrete part of this Tate–Shafarevich group $\text{III}(J(Y))$ is then identified with the discrete gauge group arising in F-theory on the genus-one fibration $Y$ \([48]\).

\(^3\)F-theory on genus-one fibrations lacking a global section was investigated in \([74, 48]\).
F-theory model appears not to have any gauge group, simply because it does not have either a discrete gauge group or $U(1)$. This observation poses us a question: how should we interpret this transition process, and what is the gauge group that the original model possesses breaking down into a discrete gauge group of another model?

The argument that we use leading to this observation does not depend on the degree of a multisection; the observation generally holds for an F-theory model on any $n$-section geometry for $n \geq 3$. Given a generic F-theory model with a discrete $\mathbb{Z}_n$ gauge group for any $n \geq 3$, one can find some F-theory model without $U(1)$ or a discrete gauge group that undergoes transition to that model.

In addition, we propose a possible physical interpretation that can explain a puzzling phenomenon observed in [73] in which a discrete $\mathbb{Z}_2$ gauge group appears to be “enhanced” to a discrete $\mathbb{Z}_4$ gauge group via Higgsing. This proposed interpretation of a physically unnatural phenomenon hints at a physical interpretation of a puzzling phenomenon wherein an F-theory model without a discrete gauge group or $U(1)$ “breaks down” into another model with a discrete $\mathbb{Z}_n$ gauge group via Higgsing, which we have just mentioned. We give a description of this puzzling phenomenon in detail in section 2.1. We also discuss a possible interpretation of this puzzling physical phenomenon in section 3.2.

When we propose possible physical interpretations of the phenomena that we have discussed, we also find new open problems. We mention these problems at the end of this study.

At the level of geometry, the arguments we use in this note do not depend on the dimensionality of the space. However, when we consider four-dimensional (4D) F-theory models, the issue of flux [75, 76, 77, 78, 79] arises, and this can make the analysis complicated. To this end, we focus on six-dimensional (6D) F-theory models when we discuss possible physical interpretations of the puzzling phenomena in this study. [93, 94, 95] discussed structures of genus-one fibrations of 3-folds.

Recent studies on building F-theory models have focused on local F-theory models [96, 97, 98, 99]. To discuss the issues of gravity and the issues of early universe such as inflation, however, the global aspects of the geometry need to be studied. The geometries of compactification spaces are analyzed from the global perspective here.

This note is structured as follows. In section 2.1, we describe a physically puzzling phenomenon in F-theory on multisection geometry, in which an F-theory model without $U(1)$ or a discrete gauge group appears to “break down” into another F-theory model with a discrete gauge group via Higgsing. We also briefly review the physically unnatural phenomenon pointed out in [73] in section 2.2 wherein a discrete $\mathbb{Z}_2$ gauge group is “enhanced” to a discrete $\mathbb{Z}_4$ gauge group via Higgsing.

In section 3.1 we propose a possible physical interpretation that can explain the unnatural phenomenon discussed in [73]. Making use of this interpretation as a hint, we also propose a

---

See, e.g., [80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 57, 92] for recent progress on F-theory compactifications with four-form flux.
physical interpretation of the puzzling phenomenon that we describe in section 2.1 in section 3.2.

We discuss some questions that these interpretations raise and we also discuss open problems in section 4.

2 Transitions in discrete gauge groups in multisection geometry

2.1 Puzzling transitions of models with discrete gauge groups

We analyze the splitting process of multisections in the moduli of multisection geometry to observe a physical phenomenon that seems puzzling. As noted in the introduction, this is new and different from the unnatural physical phenomenon pointed out in [73].

We would like to demonstrate that when we study a certain splitting of an $n$-section, namely a multisection that intersects with a fiber $n$ times, the analysis leads to a physically puzzling phenomenon.

A generic member of the $n$-section geometry does not have a global section, and it only contains an $n$-section as a multisection of the smallest degree. However, when the coefficients of the equation of the genus-one fibration as parameters take special values, so certain polynomials, in the variables of the function field of the base space, plugged into the coordinate variables satisfy the equation of the genus-one fibration, this situation geometrically implies that the genus-one fibration admits a global section [58]. This is because this “solution” specifies a point in each fiber over the base space, yielding a copy of the base.

At this special point in the moduli of $n$-section geometry where the equation of the genus-one fibration admits a “solution” yielding a global section, an $n$-section splits into a global section and an $(n-1)$-section. For the case $n = 2$, a bisection simply splits into a pair of global sections. For $n \geq 3$, this process of the splitting of an $n$-section leads to an interpretation that appears physically puzzling, as we shall shortly show.

A discrete $\mathbb{Z}_n$ gauge group in F-theory forms on a genus-one fibration with an $n$-section [47] as mentioned in the introduction. When this $n$-section splits into a global section and an $(n-1)$-section in the moduli, the Mordell–Weil rank of the resulting elliptic fibration is zero. (This is because it only has one independent global section, i.e., the “zero-section.”) For this reason, F-theory on the resulting elliptic fibration has neither a discrete gauge group nor $U(1)$. When we view this process from a physical viewpoint, which reverses the geometric process of moving from a genus-one fibration with an $n$-section to an elliptic fibration with a global section and an $(n-1)$-section via the splitting of an $n$-section in the moduli, an F-theory model without $U(1)$ or a discrete gauge group transitions to another F-theory model with a discrete $\mathbb{Z}_n$ gauge group via Higgsing, along the lines of the argument as in [47].

This physical interpretation appears puzzling: the original F-theory model is supposed to possess some gauge group, which breaks down into a discrete $\mathbb{Z}_n$ gauge group via Higgsing. However, the original F-theory model has neither $U(1)$ nor a discrete gauge symmetry, and
we do not see any gauge group that can break down into a discrete $Z_n$ gauge group. Although this must be a Higgsing process, it appears a discrete gauge group “emerged from nothing.”

We propose a possible interpretation in section 3.2 that might explain this puzzling phenomenon. Before we discuss this interpretation, we review the unnatural physical phenomenon observed in [73] in section 2.2 because this is relevant to the contents in section 3. We also discuss possible explanations for the unnatural phenomenon observed in [73] in section 3.1. An interpretation of the unnatural phenomenon observed in [73] that we describe in section 3.1 gives a hint to interpret the puzzling phenomenon that we have just discussed.

The possible interpretations that we propose in section 3 also yield new questions and open problems. We mention this in section 4.

Before we move on to the next section, we would like to make a remark: if the degree $p$ of a discrete gauge group $Z_p$ is prime (forming in F-theory on a genus-one fibration with a $p$-section), when the $p$-section splits into multisections of smaller degrees (which are not necessarily global sections), say $m$- and $l$-sections, because $p$ is prime and $m + l = p$, $m$ and $l$ should be coprime. Thus, they in fact generate a global section: whenever the $p$-section of prime degree splits into multisections of smaller degrees, they generate a global section. (The specific case $p = 5$ is discussed in [73]. The statement is trivial for $p = 2, 3$.) For these situations (when $p \neq 2$), F-theory on the resulting elliptic fibration generically does not have a $U(1)$ factor because one expects only one independent section for the resulting elliptic fibration and, thus, the Mordell–Weil rank of the elliptic fibration is zero. The physical viewpoint of a $p$-section of prime degree splitting into multisections of smaller degrees yields a Higgsing process wherein an F-theory model without $U(1)$ or a discrete gauge group transitions to another F-theory model with a discrete $Z_p$ gauge group.

### 2.2 Review of transitions in discrete $Z_4$ gauge group

We review the physically unnatural process discussed in [73] in which an F-theory model with a discrete $Z_2$ gauge group transitions to another F-theory model with a discrete $Z_4$ gauge group.

A discrete $Z_4$ gauge group forms in F-theory on a genus-one fibration with a four-section. The genus-one curve obtained as a complete intersection of two quadric hypersurfaces in $\mathbb{P}^3$, fibered over any base space, yields a genus-one fibration with a four-section [55, 63, 73]. This genus-one fibration with a four-section is described by the following equation in a general form:

\[
\begin{align*}
  a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2 + 2a_5 x_1 x_2 + 2a_6 x_1 x_3 + & \\
  2a_7 x_1 x_4 + 2a_8 x_2 x_3 + 2a_9 x_2 x_4 + 2a_{10} x_3 x_4 = 0 & \\
  b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 + b_4 x_4^2 + 2b_5 x_1 x_2 + 2b_6 x_1 x_3 + & \\
  2b_7 x_1 x_4 + 2b_8 x_2 x_3 + 2b_9 x_2 x_4 + 2b_{10} x_3 x_4 = 0.
\end{align*}
\]

$[x_1 : x_2 : x_3 : x_4]$ denotes the coordinates of $\mathbb{P}^3$, and $a_i, b_j; i, j = 1, \ldots, 10$, are sections of line
 bundles of the base space\(^5\). The Jacobian fibration\(^6\) of this genus-one fibration (1) always exists, the types of the singular fibers and the discriminant locus of which are identical to the genus-one fibration. See, e.g., [49, 73] for the construction of the Jacobian fibration of the genus-one fibration (1).

For generic values of the parameters \(a_i, b_j\), the genus-one fibration (1) has a four-section to the fibration, but when the parameters take special values, the four-section splits into a pair of bisections as observed in [63, 73]. For example, four-section splits into a pair of bisections when the parameters take the following values\(^7\):

\[
\begin{align*}
a_3 &= 1, \quad a_8 = a_{10} = 0 \\
b_3 &= 0, b_4 = b_2, \quad b_8 = b_9 = b_{10} = 0.
\end{align*}
\]

(The parameters, \(a_1, a_2, a_4, a_5, a_6, a_7, a_9, b_1, b_2, b_5, b_6, b_7\), remain free.) When further conditions are imposed on the parameters\(^2\), bisections split into global sections\(^7\). For this special situation, two bisections further split into four sheets of global sections. This process, consisting of two steps, may be expressed concisely as follows:

\[
\text{four-section} \to \text{bisection} + \text{bisection} \to 4 \text{ sheets of global sections}. \tag{3}
\]

This process can be viewed from a physical viewpoint, which reverses the geometric order, \(U(1)^3\) gauge group in F-theory breaking down into a discrete \(\mathbb{Z}_2\) gauge group, and this discrete gauge group further transitions to a discrete \(\mathbb{Z}_4\) gauge group, via Higgsing, along the lines of the argument as in [47]. However, a discrete \(\mathbb{Z}_2\) gauge group transitions to a discrete \(\mathbb{Z}_4\) gauge group in this Higgsing process, and although it is natural to expect that a discrete \(\mathbb{Z}_2\) gauge group breaks down into another discrete gauge group of smaller degree, it rather appears “enhanced” to a discrete \(\mathbb{Z}_4\) gauge group, which seems physically unnatural. This is the unnatural phenomenon pointed out in [73].

We propose a possible interpretation of this phenomenon in section 3.1. An interpretation that we propose in section 3.1 can be used as a hint to interpret the puzzling phenomenon that we previously discussed in section 2.1. Utilizing an interpretation that we give in section 3.1 as a hint, we propose a possible interpretation of the phenomenon discussed in section 2.1 in section 3.2.

Before we move on to the next section and propose a possible interpretation, we consider a minor variant of the splitting process of a four-section in [73] that we have reviewed. This

---

5 Certain conditions are imposed on the sections of line bundles, so when the Jacobian fibration is taken, the Weierstrass form of which is given by \(y^2 = x^3 + fx + g\), then the associated divisors \([f], [g]\), satisfy \([f] = -4K\), \([g] = -6K\), where \(K\) denotes the canonical divisor of the base, to ensure that the total space of fibration is Calabi–Yau, as described in [73].

6 Construction of the Jacobian fibration of an elliptic curve can be found in [100].

7 Bisection geometry can always be expressed as a double cover of a quartic polynomial [49, 47]. The coefficients of this quartic polynomial are sections of some line bundles over the base space, and when these coefficients assume special values, the bisection splits into a pair of global sections [47].
variant will be useful in the next section when we discuss a possible interpretation of the physical phenomenon pointed out in that we have just reviewed here.

After a four-section splits into two bisections, we can also consider a process wherein only one of the two bisections splits into a pair of global sections, instead of both bisections splitting into global sections. Concisely, we consider the following splitting process:

\[
\text{four-section} \rightarrow \text{bisection} + \text{bisection} \rightarrow 2 \text{ sheets of global sections} + \text{bisection}. \tag{4}
\]

For this case, the fibration on the right extreme in has two independent global sections and the rank of the Mordell–Weil group is one, thus F-theory on this fibration has a $U(1)$ gauge group. In this variant, from a physical viewpoint $U(1)$, instead of $U(1)^3$, breaks down into a discrete $Z_2$ gauge group, and a discrete $Z_2$ gauge group further transitions to a discrete $Z_4$ gauge group.

3 Physical interpretations of the puzzles

3.1 An interpretation of the transition from a discrete $Z_2$ gauge group to a discrete $Z_4$ gauge group

We propose a physical interpretation of the puzzle pointed out in , namely a physically unnatural phenomenon wherein a discrete $Z_2$ gauge group appears to be “enhanced” to a discrete $Z_4$ gauge group via Higgsing.

First, we consider in four-section geometry a specific form of complete intersection of two quadric hypersurfaces. The consideration of this example provides a motivation for our interpretation of the puzzle in , which we discuss shortly after we consider the example. We take the specific example as a starting point. We consider the complete intersection of the two quadric hypersurfaces given by the following equations:

\[
x_1^2 + x_2^2 + 2 f x_2 x_4 = 0 \tag{5}
\]

\[
x_2^2 + x_4^2 + 2 g x_1 x_3 = 0.
\]

As noted in section , $[x_1 : x_2 : x_3 : x_4]$ denotes the homogeneous coordinates of $\mathbb{P}^3$. Here $f, g$ are sections of line bundles over the base space. This complete intersection can be defined over any base space. The complete intersections of this form were considered in . The form corresponds to a special case of the more general form that we discussed previously, in which a four-section splits into a pair of bisections. Bisections are given by \{ $x_1 = 0$, $x_2 = i x_4$ \} and \{ $x_1 = 0$, $x_2 = -i x_4$ \}.

The Jacobian fibration of the complete intersection is given by \[59, 63\]

\[
\tau^2 = g^2 \lambda^4 - (f^2 g^2 + 1) \lambda^2 + f^2. \tag{6}
\]

(By a computation similar to those in \[59, 63\], we subtract the second equation times $\lambda$ as a variable from the first equation, then we arrange the coefficients of the resulting equation}
into a symmetric $4 \times 4$ matrix, as

\[
\begin{pmatrix}
1 & 0 & -\lambda g & 0 \\
0 & -\lambda & 0 & f \\
-\lambda g & 0 & 1 & 0 \\
0 & f & 0 & -\lambda
\end{pmatrix}.
\]

Taking the double cover of the determinant of this matrix, we arrive at the equation (6). We absorbed the minus sign on the right-hand side by replacing $\tau$ with $i \tau$.

This Jacobian fibration can be transformed into the (general) Weierstrass form as [63]

\[
y^2 = \frac{1}{4} x^3 - \frac{1}{2} (f^2 g^2 + 1) x^2 + \frac{1}{4} (f^2 g^2 - 1)^2 x.
\]  

(7)

The discriminant of the complete intersection (5) and the Jacobian fibration (6) is given by [63]

\[
\Delta = 16 f^2 g^2 (f^2 g^2 - 1)^4 = 16 f^2 g^2 (fg - 1)^4 (fg + 1)^4.
\]  

(8)

The 7-branes are wrapped on the components of the vanishing locus of the discriminant (8). The fibers lying over the seven-branes wrapped on the components $\{f = 0\}$ and $\{g = 0\}$ are type $I_2$, and the fibers lying over the seven-branes wrapped on the components $\{fg - 1 = 0\}$ and $\{fg + 1 = 0\}$ have type $I_4$. From the general Weierstrass form, it can be confirmed [63] that the type $I_4$ fibers over the seven-branes wrapped on the components $\{fg - 1 = 0\}$ and $\{fg + 1 = 0\}$ are split [101].

At the geometric level, the arguments of this example apply to both 4D and 6D F-theory compactifications. If we consider the physics of the transition of discrete gauge groups in 4D F-theory, however, the structure of the flux also needs to be studied. To this end, we only consider 6D F-theory models here as noted in the introduction.

As we explained previously, the complete intersection (5) is a bisection geometry. The condition where a bisection splits into a pair of global sections for bisection geometry is given in [47]. As described in [49, 47], a genus-one fibration with a bisection is given by the double cover of a quartic polynomial:

\[
\tau^2 = e_0 \lambda^4 + e_1 \lambda^3 + e_2 \lambda^2 + e_3 \lambda + e_4,
\]  

(9)

and when the parameter $e_4$ in the double cover (9) becomes a perfect square, bisection of the genus-one fibration splits into two global sections [47]. We find that the Jacobian (6) of the specific genus-one fibration we consider here has the term

\[
e_4 = f^2,
\]  

(10)

which is a perfect square. Owing to this, one may expect that the Jacobian fibration (6) should have two global sections, and thus it has Mordell–Weil rank one (or higher). However, when we study the structure of the global section more carefully, it turns out that this is not the case, and the Jacobian fibration (6) in fact has only one independent global section.
Let us discuss this point. It was described in [47] that when \( e_4 \) is a perfect square, \( e_4 = b^2/4 \), one can rewrite the equation (9) as

\[
(\tau + \frac{1}{2}b)(\tau - \frac{1}{2}b) = \lambda (e_0 \lambda^3 + e_1 \lambda^2 + e_2 \lambda + e_3),
\]  
(11)

and \( \lambda = \tau \pm \frac{1}{2}b = 0 \) yields two global sections. This holds for generic coefficients \( e_0, e_1, e_2, e_3 \), but when \( e_3 \) vanishes, \( e_3 = 0 \), these no longer yield two global sections. This is because, when \( e_3 = 0 \), equation (11) becomes

\[
(\tau + \frac{1}{2}b)(\tau - \frac{1}{2}b) = \lambda^2 (e_0 \lambda^2 + e_1 \lambda + e_2),
\]  
(12)

and the right-hand side of (12) has the factor \( \lambda^2 \), instead of just \( \lambda \) as in (11).

When \( e_3 \neq 0 \), the “horizontal” divisor, namely a global section, increases and the Mordell–Weil rank increases to one [47], but when \( e_3 = 0 \), the “vertical” divisor increases instead, and the fibration (12) acquires a type I_2 fiber.

A study of the discriminant of the genus-one fibration also suggests this. Let us provide a demonstration. The discriminant of the double cover of quartic polynomial with \( e_3 = 0 \), \( e_4 = b^2/4 \),

\[
\tau^2 = e_0 \lambda^4 + e_1 \lambda^3 + e_2 \lambda^2 + b^2/4
\]  
(13)

is given as

\[
\Delta \sim -\frac{27}{16} b^4 e_1^4 + 9 e_0 e_2 e_3^2 b^4 - e_2^2 e_1^2 b^2 + 4 e_0^2 b^6 - 8 e_0 e_2^2 b^4 + 4 e_0 e_3 b^2
\]  
(14)

The factor \( b^2 \) in the discriminant (14) represents a type I_2 fiber along the divisor \( \{ b = 0 \} \) in the base. In our example \( e_4 = f^2 \) and \( e_3 = 0 \), and we previously computed that the fibers lying, over the component \( \{ f = 0 \} \) have type I_2, agreeing with our analysis.

As noted in [47] by a symmetry argument applied to the double cover (9), a bisection splits into a pair of global sections also when \( e_0 \) is a perfect square (and \( e_1 \neq 0 \)). When \( e_1 = 0 \) for this situation, similar to what we have just discussed, the double cover instead acquires a type I_2 fiber. For our example, \( e_0 = g^2 \), and \( e_1 = 0 \). We computed previously that the fibers over the component \( \{ g = 0 \} \) have type I_2, agreeing with our analysis.

Owing to these arguments, we learn that the Jacobian fibration (6) has only one independent global section and the Mordell–Weil rank is zero [9]. If we consider adding variations to the coefficients \( a_1, \ldots, a_{10}, b_1, \ldots, b_{10} \) of the complete intersection (5) so the Jacobian has

---

8This situation is similar to the limit \( b \to 0 \) discussed in [47]. In the limit at which \( b \to 0 \), two global sections together form a type I_2 fiber (and these no longer yield two sections) [47]. For the limit \( b \to 0 \), the left-hand side of the double cover (11) becomes a square \( \tau^2 \), whereas in the case we considered, \( e_3 = 0 \), the right-hand side contains a square factor \( \lambda^2 \).

9We would like to remark that, when the specific case where

\[
f = t \tilde{f}, \quad g = t \tilde{g},
\]  

nonzero $e_3$ while keeping the term $e_4 = f^2$ in the Jacobian fibration $[6]$ fixed (or the Jacobian has nonzero $e_1$ while keeping the term $e_0 = g^2$ fixed) with the condition $[2]$ imposed on the coefficients $a_1, \ldots, a_{10}$, $b_1, \ldots, b_{10}$, we obtain a genus-one fibration with a bisection, the Jacobian fibration of which has two independent sections. Both $U(1)$ and a discrete $\mathbb{Z}_2$ gauge group form in F-theory on this deformed genus-one fibration. The transition from this deformed genus-one fibration to the complete intersection $[5]$ can be physically seen as the reverse of Higgsing in which an $SU(2)$ gauge group breaks down into $U(1)$ $[17]$. In this sense, the condition on the parameters yielding the complete intersection $[5]$ is more restrictive than the deformed genus-one fibration on which both $U(1)$ and a discrete $\mathbb{Z}_2$ gauge group form.

In bisection geometry contained in the four-section geometry, as the locus where a four-section splits into bisections, the complete intersection $[5]$ and the deformation of this that we have just described have enhanced gauge groups, $SU(2) \times \mathbb{Z}_2$ and $U(1) \times \mathbb{Z}_2$, respectively, unlike a generic member of bisection geometry on which only a discrete gauge group $\mathbb{Z}_2$ forms. If we consider transitions from a generic point in the four-section geometries to these points with enhanced gauge symmetries, the physical viewpoint yields the transitions of gauge groups from $SU(2) \times \mathbb{Z}_2$ or $U(1) \times \mathbb{Z}_2$ to a discrete $\mathbb{Z}_4$ gauge group, which gives a natural Higgsing process.

If the same argument applies to general situations, not limited to the specific example that we have just discussed, then the physically unnatural phenomenon pointed out in $[73]$ can admit a seemingly natural physical interpretation. We find a point in the moduli at which the Mordell–Weil rank of the Jacobian fibration of a genus-one fibration with a bisection increases to one. Namely, we try to find a genus-one fibration with a bisection lacking a global section, the Jacobian fibration of which has two global sections. F-theory compactification on this genus-one fibration has both a discrete $\mathbb{Z}_2$ gauge group and $U(1)$ $[10]$. In most situations in 6D F-theory, this $U(1)$ can be un-Higgsed to $SU(2)$ (or higher) without changing the structure of the base space $[17]$, and the specific example $[5]$ that we have just discussed precisely corresponds to one of such situations.

We choose to interpret that these “enhanced” models directly transition to an F-theory model with a discrete $\mathbb{Z}_4$ gauge group in the physical process; then $U(1) \times \mathbb{Z}_2$ (or $SU(2) \times \mathbb{Z}_2$) breaks down into a discrete $\mathbb{Z}_4$ gauge group and no apparent contradiction arises. We expect that, in the four-section geometry when a four-section splits into a pair of bisections, bisection geometries with these enhanced symmetries are likely to be chosen, rather than a generic model with a bisection.

There is no clear proof that this should be the correct interpretation, but this interpretation seems to resolve the unnatural transition of the gauge groups in the Higgsing process as as functions over specific base three-fold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is considered, the complete intersection $[5]$ yields a genus-one fibered Calabi–Yau four-fold analyzed in $[63]$. Similarly, when the specific case $f = g = t$ as functions over $\mathbb{P}^1$ as a base is considered, the complete intersection $[3]$ yields a genus-one fibered K3 surface analyzed in $[59]$. The Jacobian fibrations of these spaces have both Mordell–Weil rank zero $[59, 63]$, agreeing with our analysis here.

$^{10}$F-theory models on which both a discrete gauge group and $U(1)$ form were considered, e.g., in $[51, 102]$. 

10
considered in [73]. This interpretation yields a natural transition of the gauge groups in the process, and this fact provides a motivation for the interpretation.

It remains to confirm that the enhancement in gauge symmetry actually occurs generically in bisection geometries locus occurring in the four-section geometry, where a four-section splits into a pair of bisections.

As we noted at the end of section 2.2, we can consider the process in which a four-section splits into a pair of bisections, and one and only one of the two bisections further splits into a pair of global sections. The Mordell–Weil group of the resulting elliptic fibration has rank one when this occurs. One can confirm from a mathematical argument that a similar thing happens for the Jacobian fibrations: given a point in the moduli of bisection geometry, there is a deformation of the Jacobian fibration of the corresponding genus-one fibration to another Jacobian fibration with the Mordell–Weil rank one. The original genus-one fibration of this deformed Jacobian generically does not have a global section (and it has a bisection), thus both $U(1)$ and a discrete $\mathbb{Z}_2$ gauge group arise in F-theory on this genus-one fibration. We learn from this that for a generic genus-one fibration with a four-section, one can find some F-theory model with $U(1) \times \mathbb{Z}_2$ gauge group that undergoes transition to F-theory model on that genus-one fibration. This confirms that the enhancement in gauge symmetry indeed occurs generically.

As we remarked previously, in 6D F-theory $U(1)$ arising in F-theory on the genus-one fibration with a bisection can be un-Higgsed to $SU(2)$ in most situations as discussed in [47].

It might be a possible interpretation that when a four-section splits into a pair of bisections, F-theory on the corresponding bisection geometry “avoids” a generic point, and the points in the bisection geometry on which a discrete gauge symmetry is enhanced (such as $U(1) \times \mathbb{Z}_2$ or $SU(2) \times \mathbb{Z}_2$) are likely to be chosen.

Based on the argument we have just made, next we discuss a possible physical interpretation of the phenomenon we discussed in section 2.1.

Before we move on to the next section, we would like to make a remark: a mathematical analysis can find an indication that the Mordell–Weil rank of the Jacobian fibrations tends to increase to one along the bisection geometries locus in the four-section geometry, where a four-section splits into a pair of bisections. This analysis can support our proposal here. Some aspects of this will be discussed in [103].

---

\[11\] We would like to provide a sketch of a proof of this when bisection locus in the four-section geometry is described by the complete intersection [1] with certain condition such as [2] imposed: one can consider the associated double cover of a quartic polynomial of the complete intersection [1]., and the Jacobian of the associated double cover yields the Jacobian of the genus-one fibration with a bisection described as complete intersection [1]. [39] [73]. The associated double cover generally has the form [9], and when either $e_0$ or $e_4$ is a perfect square, the associated double cover generically admits two global sections [47]. For these situations, the associated double cover yields the Jacobian fibration of the genus-one fibration with a bisection, and the associated double cover has the Mordell–Weil rank one. Therefore, one only needs to consider a particular deformation so either $e_0$ or $e_4$ becomes a perfect square, which yields the Jacobian fibration with Mordell–Weil rank one.
3.2 Proposal of a solution to transition from a model without $U(1)$ or discrete gauge group to a model with a discrete $\mathbb{Z}_n$ gauge group

By using a physical interpretation of the unnatural phenomenon pointed out in [73] that we proposed in section 3.1, we discuss a possible interpretation of the puzzle we considered in section 2.1. As observed in section 2.1 in the multisection geometry, an F-theory model without a gauge group transitions to another model with a discrete $\mathbb{Z}_n$ gauge group via Higgsing. What appears puzzling here is that, although there is supposed to be some gauge group that is breaking down into the discrete $\mathbb{Z}_n$ gauge group via Higgsing, the discrete $\mathbb{Z}_n$ gauge group rather appears to arise “from nothing.” If there appears to be no gauge group that breaks down into the discrete $\mathbb{Z}_n$ gauge group, how should we interpret this process?

A natural physical interpretation might be that, similar to what we proposed in section 3.1, the model corresponding to the point in the moduli of $n$-section geometry at which an $n$-section splits into a global section and an $(n-1)$-section deforms to another model with an enhanced gauge symmetry, such as a non-Abelian gauge group or $U(1)$. If so, this allows us to view the process as a non-Abelian gauge group or $U(1)$ breaking down into a discrete $\mathbb{Z}_n$ gauge group, which is natural.

Given a general multisection (of degree greater than two), the splitting processes of the multisection into a global section and a multisection as an intermediate step, before it finally splits into multiple sheets of global sections, are prevalent in the multisection geometry. Therefore, the puzzling process of an F-theory model without having a gauge group undergoing transition to another model with a discrete $\mathbb{Z}_n$ gauge group via a Higgsing process inevitably appears in the moduli. Some physical interpretation, such as the one we proposed, seems to be required as a “way out” to avoid this puzzling transition from no gauge symmetry to a discrete $\mathbb{Z}_n$ gauge symmetry via Higgsing.

The physical interpretation that we proposed is possible when a model with a single global section (with an $(n-1)$-section) in the moduli of an $n$-section admits a deformation to another model with two global sections, or a model with a non-Abelian gauge group such as $SU(2)$. This deformation is, in fact, possible for generic 6D F-theory models. Let us provide a sketch of a proof of this.

Similar to the process in which an $n$-section splits into a global section and an $(n-1)$-section, one can consider the iteration where the resulting $(n-1)$-section splits into a global section and an $(n-2)$-section. Putting the two steps together, one can view it as an $n$-section splitting into two global sections and an $(n-2)$-section. The $U(1)$ gauge group arises in F-theory on the resulting elliptic fibration with two global sections. As discussed in [47], in 6D F-theory this $U(1)$ can be further un-Higgsed to $SU(2)$ in most situations.

Briefly, given a locus in the $n$-section moduli at which an $n$-section splits into a global section and an $(n-1)$-section, a point with two global sections or a point where $U(1)$ is further enhanced to $SU(2)$ can be found. We consider a transition from a generic point in the moduli

\[12\text{As we stated in the introduction and section 2.1 the puzzling physical phenomenon that we discussed in section 2.1 occurs when } n \geq 3. \text{ We only consider the situations } n \geq 3 \text{ here.}\]
with an $n$-section to such points with enhanced gauge symmetries. The physical viewpoint of this reverses the order, and $U(1)$ or $SU(2)$ breaks down into a discrete $\mathbb{Z}_n$, yielding a natural Higgsing process of gauge groups. We take this as a physical interpretation of the puzzling phenomenon we observed in section 2.1. This at least provides one natural interpretation of the phenomenon.

4 Open problems and some remarks

In this note, we have pointed out a new puzzle, which is different from the unnatural phenomenon observed in [73], by analyzing the splitting process of a multisection into a global section and a multisection of smaller degree; in other words, we observed a phenomenon in which an F-theory model with no $U(1)$ and no discrete gauge group transitions to another F-theory model with a discrete $\mathbb{Z}_n$ gauge group via Higgsing. Because the original model in this transition does not have a gauge symmetry, a discrete gauge group appears to arise “from nothing” in the Higgsing process, and this appears physically puzzling.

We have proposed a possible interpretation that when an $n$-section splits into a global section and a multisection of smaller degree, or splits into multisections of smaller degrees, a discrete gauge group in the model corresponding to the point of splitting in the $n$-section geometry tends to become enhanced to a larger symmetry, and we have argued that this might explain the phenomena observed in this study, and that observed in [73]. If this is true, a gauge group breaks down into a smaller discrete gauge group in the corresponding process, and this yields a natural physical interpretation. There are various ways in which an $n$-section splits into multisections of smaller degrees. Does our interpretation suggest that when multisection splits, a specific way of splitting of an $n$-section in the $n$-section geometry is chosen so the corresponding F-theory model has an enhanced gauge symmetry? It might be interesting to study whether there is any reason that other ways of splitting of multisections are ruled out owing to some physical mechanism, or whether there is a reason that the ways of splitting in which the corresponding model has an enhanced gauge symmetry are favored over other possibilities. This will be a likely direction of future study.

The puzzling physical phenomena we observed in this study and observed in [73], at the level of geometry, do not depend on the dimensions. However, we only considered 6D F-theory when we proposed physical interpretations of these phenomena in this note. This is owing to the issue of flux for 4D F-theory: the superpotential generated by flux may alter the arguments that worked without the insertion of a flux, as noted in [47]. Meanwhile, the effect of inserting a flux may explain our proposal that models with enhanced gauge symmetries are chosen in the multisection geometry when a multisection splits into multisections of smaller degrees. It might be interesting to consider this possibility, and this is also a likely direction of future study.

The situation in which a four-section splits into a pair of bisections generalizes to those of multisections of higher degrees. For example, when a multisection has degree a multiple of two, say $2n$, we expect that there is a situation in which the 2n-section splits into a
pair of \(n\)-sections. A multisection of degree \(3n\) splitting into a triplet of \(n\)-sections yields another example. From physical viewpoint, an F-theory model with a discrete \(\mathbb{Z}_n\) gauge group transitions to another model with a discrete \(\mathbb{Z}_{2n}\) gauge group, and an F-theory model with a discrete \(\mathbb{Z}_n\) gauge group undergoes transition to another model with a discrete \(\mathbb{Z}_{3n}\) gauge group, via Higgsing, respectively. These processes also appear puzzling. Does our physical interpretation proposed in section 3.1 also apply to these splitting processes of multisections of higher degrees? Studying these is also a likely target of future study.

Acknowledgments

We would like to thank Shun’ya Mizoguchi and Shigeru Mukai for discussions.

References

[1] C. Vafa, “Evidence for F-theory”, *Nucl. Phys. B* **469** (1996) 403 [arXiv:hep-th/9602022].
[2] D. R. Morrison and C. Vafa, “Compactifications of F-theory on Calabi-Yau threefolds. 1”, *Nucl. Phys. B* **473** (1996) 74 [arXiv:hep-th/9602114].
[3] D. R. Morrison and C. Vafa, “Compactifications of F-theory on Calabi-Yau threefolds. 2”, *Nucl. Phys. B* **476** (1996) 437 [arXiv:hep-th/9603161].
[4] D. R. Morrison and D. S. Park, “F-Theory and the Mordell-Weil Group of Elliptically-Fibered Calabi-Yau Threefolds”, *JHEP* **10** (2012) 128 [arXiv:1208.2695 [hep-th]].
[5] C. Mayrhofer, E. Palti and T. Weigand, “U(1) symmetries in F-theory GUTs with multiple sections”, *JHEP* **03** (2013) 098 [arXiv:1211.6742 [hep-th]].
[6] V. Braun, T. W. Grimm and J. Keitel, “New Global F-theory GUTs with U(1) symmetries”, *JHEP* **09** (2013) 154 [arXiv:1302.1854 [hep-th]].
[7] J. Borchmann, C. Mayrhofer, E. Palti and T. Weigand, “Elliptic fibrations for \(SU(5) \times U(1) \times U(1)\) F-theory vacua”, *Phys. Rev. D* **88** (2013) no.4 046005 [arXiv:1303.5054 [hep-th]].
[8] M. Cvetič, D. Klevers and H. Piragua, “F-Theory Compactifications with Multiple U(1)-Factors: Constructing Elliptic Fibrations with Rational Sections”, *JHEP* **06** (2013) 067 [arXiv:1303.6970 [hep-th]].
[9] V. Braun, T. W. Grimm and J. Keitel, “Geometric Engineering in Toric F-Theory and GUTs with U(1) Gauge Factors,” *JHEP* **12** (2013) 069 [arXiv:1306.0577 [hep-th]].
[10] M. Cvetič, A. Grassi, D. Klevers and H. Piragua, “Chiral Four-Dimensional F-Theory Compactifications With \(SU(5)\) and Multiple U(1)-Factors”, *JHEP* **04** (2014) 010 [arXiv:1306.3987 [hep-th]].
[11] M. Cvetič, D. Klevers and H. Piragua, “F-Theory Compactifications with Multiple U(1)-Factors: Addendum”, *JHEP* **12** (2013) 056 [arXiv:1307.6425 [hep-th]].
[12] M. Cvetič, D. Klevers, H. Piragua and P. Song, “Elliptic fibrations with rank three Mordell-Weil group: F-theory with U(1) x U(1) x U(1) gauge symmetry,” JHEP 1403 (2014) 021 [arXiv:1310.0463 [hep-th]].

[13] I. Antoniadis and G. K. Leontaris, “F-GUTs with Mordell-Weil U(1)’s,” Phys. Lett. B735 (2014) 226–230 [arXiv:1404.6720 [hep-th]].

[14] M. Esole, M. J. Kang and S.-T. Yau, “A New Model for Elliptic Fibrations with a Rank One Mordell-Weil Group: I. Singular Fibers and Semi-Stable Degenerations”, arXiv:1410.0003 [hep-th].

[15] C. Lawrie, S. Schäfer-Nameki and J.-M. Wong, “F-theory and All Things Rational: Surveying U(1) Symmetries with Rational Sections”, JHEP 09 (2015) 144 [arXiv:1504.05593 [hep-th]].

[16] M. Cvetič, D. Klevers, H. Piragua and W. Taylor, “General U(1) x U(1) F-theory compactifications and beyond: geometry of unHiggsings and novel matter structure,” JHEP 1511 (2015) 204 [arXiv:1507.05954 [hep-th]].

[17] M. Cvetič, A. Grassi, D. Klevers, M. Poretschkin and P. Song, “Origin of Abelian Gauge Symmetries in Heterotic/F-theory Duality,” JHEP 1604 (2016) 041 [arXiv:1511.08208 [hep-th]].

[18] D. R. Morrison and D. S. Park, “Tall sections from non-minimal transformations”, JHEP 10 (2016) 033 [arXiv:1606.07444 [hep-th]].

[19] D. R. Morrison, D. S. Park and W. Taylor, “Non-Higgsable abelian gauge symmetry and F-theory on fiber products of rational elliptic surfaces”, Adv. Theor. Math. Phys. 22 (2018) 177–245 [arXiv:1610.06929 [hep-th]].

[20] M. Bies, C. Mayrhofer and T. Weigand, “Gauge Backgrounds and Zero-Mode Counting in F-Theory”, JHEP 11 (2017) 081 [arXiv:1706.04616 [hep-th]].

[21] M. Cvetič and L. Lin, “The Global Gauge Group Structure of F-theory Compactification with U(1)s”, JHEP 01 (2018) 157 [arXiv:1706.08521 [hep-th]].

[22] M. Bies, C. Mayrhofer and T. Weigand, “Algebraic Cycles and Local Anomalies in F-Theory”, JHEP 11 (2017) 100 [arXiv:1706.08528 [hep-th]].

[23] Y. Kimura and S. Mizoguchi, “Enhancements in F-theory models on moduli spaces of K3 surfaces with ADE rank 17”, PTEP 2018 no. 4 (2018) 043B05 [arXiv:1712.08539 [hep-th]].

[24] Y. Kimura, “F-theory models on K3 surfaces with various Mordell-Weil ranks - constructions that use quadratic base change of rational elliptic surfaces”, JHEP 05 (2018) 048 [arXiv:1802.05195 [hep-th]].

[25] S.-J. Lee, D. Regalado and T. Weigand, “6d SCFTs and U(1) Flavour Symmetries”, JHEP 11 (2018) 147 [arXiv:1803.07998 [hep-th]].

[26] S. Mizoguchi and T. Tani, “Non-Cartan Mordell-Weil lattices of rational elliptic surfaces and heterotic/F-theory compactifications”, JHEP 03 (2019) 121 [arXiv:1808.08001 [hep-th]].
[27] F. M. Cianci, D. K. Mayorga Pena and R. Valandro, “High U(1) charges in type IIB models and their F-theory lift”, *JHEP* **04** (2019) 012 [arXiv:1811.11777 [hep-th]].

[28] W. Taylor and A. P. Turner, “Generic matter representations in 6D supergravity theories”, *JHEP* **05** (2019) 081 [arXiv:1901.02012 [hep-th]].

[29] Y. Kimura, “F-theory models with 3 to 8 U(1) factors on K3 surfaces” [arXiv:1903.03608 [hep-th]].

[30] M. Esole and P. Jefferson, “The Geometry of SO(3), SO(5), and SO(6) models” [arXiv:1905.12620 [hep-th]].

[31] S.-J. Lee and T. Weigand, “Swampland Bounds on the Abelian Gauge Sector”, *Phys. Rev.* **D100** (2019) no.2 026015 [arXiv:1905.13213 [hep-th]].

[32] T. Kobayashi, H. P. Nilles, F. Ploger, S. Raby and M. Ratz, “Stringy origin of non-Abelian discrete flavor symmetries,” *Nucl.Phys.* **B768** (2007) 135 [arXiv: hep-ph/0611020].

[33] H. Abe, K.-S. Choi, T. Kobayashi and H. Ohki, “Non-Abelian Discrete Flavor Symmetries from Magnetized/Intersecting Brane Models,” *Nucl.Phys.* **B820** (2009) 317 [arXiv:0904.2631 [hep-ph]].

[34] T. Banks and N. Seiberg, “Symmetries and Strings in Field Theory and Gravity”, *Phys. Rev.* **D83** (2011) 084019 [arXiv:1011.5120 [hep-th]].

[35] S. Hellerman and E. Sharpe, “Sums over topological sectors and quantization of Fayet-Iliopoulos parameters”, *Adv.Theor.Math.Phys.* **15** (2011) 1141–1199 [arXiv:1012.5999 [hep-th]].

[36] P. G. Camara, L. E. Ibanez and F. Marchesano, “RR photons”, *JHEP* **09** (2011) 110 [arXiv:1106.0060 [hep-th]].

[37] M. Berasaluce-Gonzalez, L. E. Ibanez, P. Soler and A. M. Uranga, “Discrete gauge symmetries in D-brane models”, *JHEP* **12** (2011) 113 [arXiv:1106.4169 [hep-th]].

[38] L. E. Ibanez, A. N. Schellekens and A. M. Uranga, “Discrete Gauge Symmetries in Discrete MSSM-like Orientifolds”, *Nucl.Phys.* **B865** (2012) 509–540 [arXiv:1205.5364 [hep-th]].

[39] M. Berasaluce-Gonzalez, P. G. Camara, F. Marchesano, D. Regalado and A. M. Uranga, “Non-Abelian discrete gauge symmetries in 4d string models”, *JHEP* **09** (2012) 059 [arXiv:1206.2383 [hep-th]].

[40] M. Berasaluce-Gonzalez, P. G. Camara, F. Marchesano and A. M. Uranga, “Zp charged branes in flux compactifications”, *JHEP* **04** (2013) 138 [arXiv:1211.5317 [hep-th]].

[41] F. Marchesano, D. Regalado and L. Vazquez-Mercado, “Discrete flavor symmetries in D-brane models”, *JHEP* **09** (2013) 028 [arXiv:1306.1284 [hep-th]].

[42] G. Honecker and W. Staessens, “To Tilt or Not To Tilt: Discrete Gauge Symmetries in Global Intersecting D-Brane Models”, *JHEP* **10** (2013) 146 [arXiv:1303.4415 [hep-th]].
[43] M. Berasaluce-Gonzalez, G. Ramirez and A. M. Uranga, “Antisymmetric tensor $Z_p$ gauge symmetries in field theory and string theory”, *JHEP* **01** (2014) 059 [arXiv:1310.5582 [hep-th]].

[44] A. Karozas, S. F. King, G. K. Leontaris and A. Meadowcroft, “Discrete Family Symmetry from F-Theory GUTs”, *JHEP* **09** (2014) 107 [arXiv:1406.6290 [hep-ph]].

[45] G. Honecker and W. Staessens, “Discrete Abelian gauge symmetries and axions”, *J.Phys.Conf.Ser.* **631** (2015) no.1 [arXiv:1502.00985 [hep-th]].

[46] T. W. Grimm, T. G. Pugh and D. Regalado, “Non-Abelian discrete gauge symmetries in F-theory”, *JHEP* **02** (2016) 066 [arXiv:1504.06272 [hep-th]].

[47] D. R. Morrison and W. Taylor, “Sections, multisections, and $U(1)$ fields in F-theory”, *J. Singularities* **15** (2016) 126–149 [arXiv:1404.1527 [hep-th]].

[48] J. de Boer, R. Dijkgraaf, K. Hori, A. Keurentjes, J. Morgan, D. R. Morrison and S. Sethi, “Triples, fluxes, and strings”, *Adv. Theor. Math. Phys.* **4** (2002) 995–1186 [arXiv: hep-th/0103170].

[49] V. Braun and D. R. Morrison, “F-theory on Genus-One Fibrations”, *JHEP* **08** (2014) 132 [arXiv:1401.7844 [hep-th]].

[50] L. B. Anderson, I. Garcia-Etxebarria, T. W. Grimm and J. Keitel, “Physics of F-theory compactifications without section”, *JHEP* **12** (2014) 156 [arXiv:1406.5180 [hep-th]].

[51] D. Klevers, D. K. Mayorga Pena, P. K. Oehlmann, H. Piragua and J. Reuter, “F-Theory on all Toric Hypersurface Fibrations and its Higgs Branches”, *JHEP* **01** (2015) 142 [arXiv:1408.4808 [hep-th]].

[52] I. Garcia-Etxebarria, T. W. Grimm and J. Keitel, “Yukawas and discrete symmetries in F-theory compactifications without section”, *JHEP* **11** (2014) 125 [arXiv:1408.6448 [hep-th]].

[53] C. Mayrhofer, E. Palti, O. Till and T. Weigand, “Discrete Gauge Symmetries by Higgsing in four-dimensional F-Theory Compactifications”, *JHEP* **12** (2014) 068 [arXiv:1408.6831 [hep-th]].

[54] C. Mayrhofer, E. Palti, O. Till and T. Weigand, “On Discrete Symmetries and Torsion Homology in F-Theory”, *JHEP* **06** (2015) 029 [arXiv:1410.7814 [hep-th]].

[55] V. Braun, T. W. Grimm and J. Keitel, “Complete Intersection Fibers in F-Theory”, *JHEP* **03** (2015) 125 [arXiv:1411.2615 [hep-th]].

[56] M. Cvetič, R. Donagi, D. Klevers, H. Piragua and M. Poretschkin, “F-theory vacua with $\mathbb{Z}_3$ gauge symmetry”, *Nucl. Phys.* **B898** (2015) 736–750 [arXiv:1502.06953 [hep-th]].

[57] L. Lin, C. Mayrhofer, O. Till and T. Weigand, “Fluxes in F-theory Compactifications on Genus-One Fibrations”, *JHEP* **01** (2016) 098 [arXiv:1508.00162 [hep-th]].

[58] Y. Kimura, “Gauge Groups and Matter Fields on Some Models of F-theory without Section”, *JHEP* **03** (2016) 042 [arXiv:1511.06912 [hep-th]].
Y. Kimura, “Gauge symmetries and matter fields in F-theory models without section-compactifications on double cover and Fermat quartic K3 constructions times K3”, Adv. Theor. Math. Phys. 21 (2017) no.8, 2087–2114 [arXiv:1603.03212 [hep-th]].

P.-K. Oehlmann, J. Reuter and T. Schimannek, “Mordell-Weil Torsion in the Mirror of Multi-Sections”, JHEP 12 (2016) 031 [arXiv:1604.00011 [hep-th]].

Y. Kimura, “Gauge groups and matter spectra in F-theory compactifications on genus-one fibered Calabi-Yau 4-folds without section - Hypersurface and double cover constructions”, Adv. Theor. Math. Phys. 22 (2018) no.6, 1489–1533 [arXiv:1607.02978 [hep-th]].

M. Cvetič, A. Grassi and M. Poretschkin, “Discrete Symmetries in Heterotic/F-theory Duality and Mirror Symmetry”, JHEP 06 (2017) 156 [arXiv:1607.03176 [hep-th]].

Y. Kimura, “Discrete Gauge Groups in F-theory Models on Genus-One Fibered Calabi-Yau 4-folds without Section”, JHEP 04 (2017) 168 [arXiv:1608.07219 [hep-th]].

Y. Kimura, “K3 surfaces without section as double covers of Halphen surfaces, and F-theory compactifications”, PTEP 2018 (2018) 043B06 [arXiv:1801.06525 [hep-th]].

L. B. Anderson, A. Grassi, J. Gray and P.-K. Oehlmann, “F-theory on Quotient Threefolds with (2,0) Discrete Superconformal Matter”, JHEP 06 (2018) 098 [arXiv:1801.08658 [hep-th]].

Y. Kimura, “$SU(n) \times \mathbb{Z}_2$ in F-theory on K3 surfaces without section as double covers of Halphen surfaces” [arXiv:1806.01727 [hep-th]].

T. Weigand, “F-theory”, PoS TASI2017 (2018) 016 [arXiv:1806.01854 [hep-th]].

M. Cvetič, L. Lin, M. Liu and P.-K. Oehlmann, “An F-theory Realization of the Chiral MSSM with $\mathbb{Z}_2$-Parity”, JHEP 09 (2018) 089 [arXiv:1807.01320 [hep-th]].

M. Cvetič and L. Lin, “TASI Lectures on Abelian and Discrete Symmetries in F-theory”, PoS TASI2017 (2018) 020 [arXiv:1809.00012 [hep-th]].

Y.-C. Huang and W. Taylor, “On the prevalence of elliptic and genus one fibrations among toric hypersurface Calabi-Yau threefolds”, JHEP 03 (2019) 014 [arXiv:1809.05160 [hep-th]].

Y. Kimura, “Nongeometric heterotic strings and dual F-theory with enhanced gauge groups”, JHEP 02 (2019) 036 [arXiv:1810.07657 [hep-th]].

Y. Kimura, “Unbroken $E_7 \times E_7$ nongeometric heterotic strings, stable degenerations and enhanced gauge groups in F-theory duals” [arXiv:1902.00944 [hep-th]].

Y. Kimura, “Discrete gauge groups in certain F-theory models in six dimensions”, JHEP 07 (2019) 027 [arXiv:1905.03775 [hep-th]].

P. Berglund, J. Ellis, A. E. Faraggi, D. V. Nanopoulos and Z. Qiu, “Elevating the free fermion $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold model to a compactification of F-theory”, Int. Jour. of Mod. Phys. A 15 (2000) 1345–1362 [arXiv:hep-th/9812141].

K. Becker and M. Becker, “M theory on eight manifolds”, Nucl. Phys. B477 (1996) 155–167 [arXiv: hep-th/9605053].
[76] S. Sethi, C. Vafa and E. Witten, “Constraints on low dimensional string compactifications”, Nucl. Phys. B 480 (1996) 213–224. [arXiv: hep-th/9606122].

[77] E. Witten, “On flux quantization in M theory and the effective action”, J. Geom. Phys. 22 (1997) 1–13 [arXiv: hep-th/9609122].

[78] S. Gukov, C. Vafa and E. Witten, “CFT’s from Calabi-Yau four folds”, Nucl. Phys. B584 (2000) 69–108 [arXiv: hep-th/9906070].

[79] K. Dasgupta, G. Rajesh and S. Sethi, “M theory, orientifolds and G-flux”, JHEP 08 (1999) 023 [arXiv: hep-th/9908088].

[80] J. Marsano, N. Saulina and S. Schäfer-Nameki, “ A Note on G-Fluxes for F-theory Model Building”, JHEP 11 (2010) 088 [arXiv:1006.0483 [hep-th]].

[81] A. Collinucci and R. Savelli, “On Flux Quantization in F-Theory”, JHEP 02 (2012) 015 [arXiv:1011.6388 [hep-th]].

[82] J. Marsano, N. Saulina and S. Schäfer-Nameki, “G-flux, M5 instantons, and U(1) symmetries in F-theory”, Phys. Rev. D87 (2013) 066007 [arXiv:1107.1718 [hep-th]].

[83] A. P. Braun, A. Collinucci and R. Valandro, “G-flux in F-theory and algebraic cycles”, Nucl. Phys. B 856 (2012) 129 [arXiv:1107.5337 [hep-th]].

[84] J. Marsano and S. Schäfer-Nameki, “Yukawas, G-flux, and Spectral Covers from Resolved Calabi-Yau’s”, JHEP 11 (2011) 098 [arXiv:1108.1794 [hep-th]].

[85] S. Krause, C. Mayrhofer and T. Weigand, “$G_4$ flux, chiral matter and singularity resolution in F-theory compactifications”, Nucl. Phys. B 858 (2012) 1–47 [arXiv:1109.3454 [hep-th]].

[86] T. W. Grimm and H. Hayashi, “F-theory fluxes, Chirality and Chern-Simons theories”, JHEP 03 (2012) 027 [arXiv:1111.1232 [hep-th]].

[87] S. Krause, C. Mayrhofer and T. Weigand, “Gauge Fluxes in F-theory and Type IIB Orientifolds”, JHEP 08 (2012) 119 [arXiv:1202.3138 [hep-th]].

[88] K. Intriligator, H. Jockers, P. Mayr, D. R. Morrison and M. R. Plesser, “Conifold Transitions in M-theory on Calabi-Yau Fourfolds with Background Fluxes”, Adv. Theor. Math. Phys. 17 (2013) 601 [arXiv:1203.6662 [hep-th]].

[89] M. Kuntzler and S. Schäfer-Nameki, “ G-flux and Spectral Divisors”, JHEP 11 (2012) 025 [arXiv:1205.5688 [hep-th]].

[90] M. Cvetič, T. W. Grimm and D. Klevers, “Anomaly Cancellation And Abelian Gauge Symmetries In F-theory”, JHEP 02 (2013) 101 [arXiv:1210.6034 [hep-th]].

[91] A. P. Braun, A. Collinucci and R. Valandro, “Hypercharge flux in F-theory and the stable Sen limit”, JHEP 07 (2014) 121 [arXiv:1402.4096 [hep-th]].

[92] S. Schäfer-Nameki and T. Weigand, “F-theory and 2d (0,2) Theories”, JHEP 05 (2016) 059 [arXiv:1601.02015 [hep-th]].
[93] N. Nakayama, “On Weierstrass Models”, *Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata*, (1988), 405–431.

[94] I. Dolgachev and M. Gross, “Elliptic Three-folds I: Ogg-Shafarevich Theory”, *Journal of Algebraic Geometry* 3, (1994), 39–80.

[95] M. Gross, “Elliptic Three-folds II: Multiple Fibres”, *Trans. Amer. Math. Soc.* 349, (1997), 3409–3468.

[96] R. Donagi and M. Wijnholt, “Model Building with F-Theory”, *Adv. Theor. Math. Phys.* 15 (2011) no.5, 1237–1317 [arXiv:0802.2969 [hep-th]].

[97] C. Beasley, J. J. Heckman and C. Vafa, “GUTs and Exceptional Branes in F-theory -I”, *JHEP* 01 (2009) 058 [arXiv:0802.3391 [hep-th]].

[98] C. Beasley, J. J. Heckman and C. Vafa, “GUTs and Exceptional Branes in F-theory -II: Experimental Predictions”, *JHEP* 01 (2009) 059 [arXiv:0806.0102 [hep-th]].

[99] R. Donagi and M. Wijnholt, “Breaking GUT Groups in F-Theory”, *Adv. Theor. Math. Phys.* 15 (2011) 1523–1603 [arXiv:0808.2223 [hep-th]].

[100] J. W. S. Cassels, *Lectures on Elliptic Curves*, London Math. Society Student Texts 24, Cambridge University Press (1991).

[101] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov and C. Vafa, “Geometric singularities and enhanced gauge symmetries”, *Nucl. Phys.* B 481 (1996) 215 [arXiv:hep-th/9605200].

[102] T. W. Grimm, A. Kapfer and D. Klevers, “The Arithmetic of Elliptic Fibrations in Gauge Theories on a Circle”, *JHEP* 06 (2016) 112 [arXiv:1510.04281 [hep-th]].

[103] Y. Kimura, to appear.