The Chiral de Rham Complex and
Positivity of the Equivariant Signature of the Loop Space

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Abstract

In this note we show that the positivity property of the equivariant signature of the loop space, first observed in [MS1] in the case of the even-dimensional projective spaces, is valid for Picard number 2 toric varieties. A new formula for the equivariant signature of the loop space in the case of a toric spin variety is derived.

0. Introduction

The equivariant signature of the loop space, an example of an elliptic type genus, is, in particular, a rule which assigns to a manifold \( X \) a power series in \( q \):

\[
X \mapsto \text{sign}(q, \mathcal{L}X) = b_0 + b_1 q + b_1 q^2 + \cdots.
\]

The intuition behind its definition (a well-known theorem in fact, see sect. 2) is that \( \text{sign}(q, \mathcal{L}X) \) is the equivariant index of a certain elliptic operator and therefore the coefficients \( b_j \) are integers, not necessarily positive since they are equal to the difference between the dimensions of two vector spaces. The cohomological computations of [MS1] fairly unexpectedly showed that in the case of the even-dimensional complex projective space \( \mathbb{P}^{2n} \) the series \( \text{sign}(q, \mathcal{L} \mathbb{P}^{2n}) \) has positive coefficients. Moreover, there is a graded vector space naturally associated to \( \mathbb{P}^{2n} \) so that \( b_j \) essentially equals the dimension of the \( j \)-th homogeneous component. Let us formulate this result more precisely.

Associated with any smooth manifold \( X \) there is a sheaf of vertex algebras \( \Omega^h_X \), the chiral de Rham complex [MSV]. It is graded:

\[
\Omega^h_X = \bigoplus_{j \geq 0} (\Omega^h_X)_j,
\]

and so are its cohomology groups. The following equality of formal power series was proven in [MS1] (cf. Theorem 4.1 below)

\[
\text{sign}(q, \mathcal{L} \mathbb{P}^{2n}) = 2 \sum_{j=0}^{\infty} \dim H^0(\mathbb{P}^{2n}, (\Omega^h_{\mathbb{P}^{2n}})_j)q^j - 1.
\]

One can say that the vector space \( H^0(\mathbb{P}^{2n}, \Omega^h_{\mathbb{P}^{2n}}) \), a vertex algebra in fact, provides a realization of \( \text{sign}(q, \mathcal{L} \mathbb{P}^{2n}) \).

The present note came out of the discussion of an earlier version of [MS1], and (0.1) appearing in the final version of [MS1] is a result of this discussion. We further use the Borisov-Libgober formula for the equivariant signature of the loop space in the case of a toric variety [BL] to extend the positivity result to the Picard number 2 toric varieties. (Recall that the projective spaces exhaust the class of Picard number 1 smooth toric varieties.) The very nature of this calculation does not
allow us to conclude whether there is a graded vector space such that it is naturally associated to the manifold in question and realizes this signature. The existence of such a vector space, be it a vertex algebra or not, remains an open question.

A similar computation combined with Witten’s rigidity theorem gives the following formula for the equivariant signature of the loop space in the case of a spin toric variety $X$ of complex dimension $d$ (cf. Theorem 6.2):

$$\text{sign}(q, LX) = \text{sign} X \frac{\epsilon^{-\frac{d}{2}}}{2^d},$$  \hspace{1cm} (0.2)$$

where $\text{sign} X$ is the signature of $X$ and $\epsilon$ a well-known modular form defined by (2.2) below.

The r.h.s. of this formula is not unfamiliar to vertex algebra specialists. Let $V$ be the vertex algebra of $2d$ free bosons coupled to $2d$ free fermions. $V$ is naturally graded:

$$V = \oplus_{j=0}^{\infty} V_j$$

so that

$$\sum_{j=0}^{\infty} \dim V_j q^j = \epsilon^{-\frac{d}{4}}.$$  \hspace{1cm} (0.3)$$

Having compared (0.1) to (0.2-0.3) one may perhaps be so bold as to conjecture that the vertex algebra $V$ is isomorphic to $H^0(X, \Omega^0_X)$. If correct this conjecture will be a natural extension of the realization result (0.1) to the spin-case.

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1. Genera. In our brief review of the relevant genera we shall follow the book [HBJ].

Let $R$ be a commutative ring, $\Omega^*$ the cobordism ring. Recall that $\Omega^* \otimes \mathbb{Q} = \mathbb{Q}[P^2, P^4, ...]$, where $P^{2n}$ is the cobordism class of the complex projective space of dimension $2n$. According to Hirzebruch, a genus is a ring homomorphism

$$g : \Omega^* \otimes \mathbb{Q} \to R.$$  \hspace{1cm} (1.1)$$

An invertible even formal power series $Q(x) = a_0 + a_1 x^2 + \ldots$ with coefficients in $R$ defines a genus, to be denoted $g_Q$, as follows. Let $x_1, \ldots, x_n$ be formal variables and $p_i$ the $i$-th elementary symmetric function in $x_i^2$. Then

$$Q(x_1)Q(x_2) \ldots Q(x_n) = a_0^n (1 + K_1(p_1) + K_2(p_1, p_2) + \ldots)$$

where $K_i(p_1, \ldots, p_i)$ is a (uniquely determined) homogeneous polynomial of degree $2i$. For a manifold $X$ of dimension $4n$ define

$$\alpha_n(X) = a_0^n K_n(p_1, \ldots, p_n)[X]$$  \hspace{1cm} (1.2)$$
where $p_i(M)$, $i = 1, \ldots, n$, is the $i$-th the Pontryagin class of $X$ and $[X]$ is the fundamental class of $X$. In fact the rule $Q(x) \to g_Q$ sets up a 1-1 correspondence.

2. The elliptic genus and the equivariant signature of the loop space. The notion of an elliptic genus is due to Ochanine\cite{O}. A genus is called elliptic if it is associated to a series $Q(x)$ such that:

if $f(x) = \frac{x}{Q(x)}$, then

$$(f')^2 = 1 - 2\delta f^2 + \epsilon f^4,$$

(2.1)

for some parameters $\delta$ and $\epsilon$.

All solutions to (2.1) can be constructed as follows. Fix a lattice $L$ in $\mathbb{C}$ with generators $\omega_1, \omega_2$ and let $p_L(z)$ be the corresponding Weierstrass function. Then

$$f(x) = \frac{1}{\sqrt{p_L(x) - p_L(\omega_1/2)}}$$

is a solution to (2.1) with

$$\delta = -\frac{2}{3}p_L(\omega_1/2),$$

$$\epsilon = [p_L(\omega_1/2) - p_L(\omega_2/2)][p_L(\omega_1/2) - p_L((\omega_1 + \omega_2)/2)].$$

Therefore each elliptic genus equals (cf. (1.2))

$$g_x\sqrt{p_L(x) - p_L(\omega_1/2)}(\cdot),$$

for some $L$ and $\omega_1$, and $g_x\sqrt{p_L(x) - p_L(\omega_1/2)}(\cdot)$, as a function of $L$ and $\omega_1$, can be considered the universal elliptic genus.

Introducing the modular parameter $q = \exp(2\pi i \omega_2/\omega_1)$ one sees that all the defined expressions are naturally identified with functions of $q$. In particular, $\delta$ and $\epsilon$ become the standard generators of the ring $M_*(2)$ of modular forms on $\Gamma_0(2)$ so that $\delta \in M_2(2)$, $\epsilon \in M_4(2)$. Needed for (2.6) below is the following formula for $\epsilon$:

$$\epsilon = (2 \prod_{n=1}^{\infty} \frac{(1 + q^n)^2}{(1 - q^n)^2})^{-4},$$

(2.2)

Likewise, for any $X$, $g_x\sqrt{p_L(x) - p_L(\omega_1/2)}(X)$ is a function of $q$. We shall emphasize this by changing (and unburdening) the notation as follows:

$$och(q, X) = g_x\sqrt{p_L(x) - p_L(\omega_1/2)}(X).$$

In fact, for a manifold $X$ of real dimension $4k$, $och(q, X)$ is a weight $2k$ polynomial in $\delta$ counted with weight 2 and $\epsilon$ counted with weight 4. Therefore $och(q, X)$ is a weight $2k$ modular form on $\Gamma_0(2)$.

We will be more interested in a closely related genus, $sign(q, L X)$, proposed by Witten and called the formal equivariant signature of the loop space [HBJ]. By definition

$$sign(q, L X) = \cdot$$

(2.3a)
where
\[ Q(x) = x \frac{1 + e^{-x}}{1 - e^{-x}} \prod_{n=1}^{\infty} \frac{(1 + q^n e^{-x}) (1 + q^n e^x)}{(1 - q^n e^{-x}) (1 - q^n e^x)}. \] (2.3b)

This genus can be equivalently defined to be the signature of \( X \) twisted by the bundle
\[ W = \otimes_{n=1}^{\infty} S_{q^n} TX \otimes_{n=1}^{\infty} \Lambda_{q^n} TX, \] (2.4)
where we habitually set
\[ \Lambda_i E = \sum_{i=0}^{\infty} t^i \Lambda^i E, \] (2.5a)
\[ S_i E = \sum_{i=0}^{\infty} t^i S^i E. \] (2.5b)

Yet another possibility is to define \( \text{sign}(q, \mathcal{L}X) \) to be the index of the elliptic operator \( d + d^* \) acting on the global sections of the bundle (2.4).

We conclude this section by noting that \( \text{sign}(q, \mathcal{L}X) \) is related to the elliptic genus as follows:
\[ \text{sign}(q, \mathcal{L}X) = \text{och}(q, X) e^{-\frac{\epsilon}{2}}, \] (2.6)
where \( \epsilon \) is defined by (2.2) and \( \dim X = 4k \).

3. The chiral de Rham complex and the equivariant signature of the loop space. Defined in [MSV] for any smooth complex manifold \( X \) of complex dimension \( n \) there is a sheaf \( \Omega^ch_X(X) \) of vertex algebras over \( X \). Morally, \( \Omega^ch_X(X) \) is a semi-infinite de Rham complex on a “small” loop space with coefficients in distributions supported on the submanifold of analytically contractible loops. This vague assertion was made a theorem in [KV] after overcoming considerable technical difficulties.

\( \Omega^ch_X(X) \) is not a sheaf of \( \mathcal{O}_X \)-modules but it possesses a filtration such that the associated graded sheaf \( gr\Omega^ch_X \) is. In fact, \( gr\Omega^ch_X \) is associated to a holomorphic vector bundle and this vector bundle was explicitly described in [BL] as follows:
\[ gr\Omega^ch_X = \otimes_{n=1}^{\infty} \{ S_{q^n} TX \otimes S_{q^n} TX^* \otimes \Lambda_{q^n-1} T^* X \otimes \Lambda_{q^n-1} q^n TX \}, \] (3.1)
where we use the notation introduced in (2.5a,b). Thus \( gr\Omega^ch_X(X) \) is bi-graded so that the component of weight \((i, j)\) is the coefficient of \( q^i y^j \). In fact, the sheaf \( \Omega^ch_X \) is itself bi-graded :
\[ \Omega^ch_X = \bigoplus_{i \in \mathbb{Z}, j \geq 0} (\Omega^ch_X)^i_j, \] (3.2)
and this bi-grading descends to the graded object. In particular, \( \Omega^i_X \), i.e. the sheaf of holomorphic \( i \)-forms, canonically identifies with \((\Omega^ch_X)^i_0\):
\[ \Omega^i_X \xrightarrow{\sim} (\Omega^ch_X)^i_0, \] (3.3)
which partially justifies the name “a chiral de Rham complex”.

Consider the Euler character \( \text{Eu}(\Omega^ch_X)(q, y) \), which by definition is a formal Laurent power series in \( q, y \) such that the coefficient of \( q^i \) is the Euler characteristic
\[ \text{sign}(q, \mathcal{L}X) \text{och}(q, X) e^{-\frac{\epsilon}{2}}, \] (2.6)
where \( \epsilon \) is defined by (2.2) and \( \dim X = 4k \).
of the component \((\Omega^\text{ch}_X)^i_j\). Define \(\text{Eu}(\Omega^\text{ch}_X)(q)\) to be \(\text{Eu}(\Omega^\text{ch}_X)(q, 1)\) Applications of \(\Omega^\text{ch}_X\) to elliptic genera are based on the following observation due to [BL]:

\[
\text{Eu}(\Omega^\text{ch}_X)(q) = \text{sign}(q, \mathcal{L}X).
\] (3.4)

Let us prove (3.4) for the sake of completeness. Since the Euler characteristic does not change under the passage to the graded object, we can write

\[
\text{Eu}(\Omega^\text{ch}_X)(q) = \text{Eu}(\text{gr} \Omega^\text{ch}_X)(q) = \int_X \text{ch}((\text{gr} \Omega^\text{ch}_X)tdX),
\] (3.5)

where the 2nd equality follows from the Riemann-Roch Theorem, and \(tdX\) is the Todd genus of \(X\). We now compute \(\text{ch}((\text{gr} \Omega^\text{ch}_X)\) by using (3.1) and the multiplicativity of \(\text{ch}\) to the effect that

\[
\text{ch}((\text{gr} \Omega^\text{ch}_X) = \prod_{n=1}^{\infty} \{\text{ch}(S_{q^n}TX)\text{ch}(S_{q^n}TX^*)\text{ch}(\Lambda_{q^{n-1}}T^*X)\text{ch}(\Lambda_{q^n}TX)\}. \quad (3.6)
\]

As is well known for a \(k\) dimensional manifold \(X\)

\[
\text{ch}\Lambda_{q^{n-1}}TX = \prod_{i=1}^{k} (1 + q^{n-1}e^{x_i}), \quad \text{ch}\Lambda_{q^{n-1}}TX^* = \prod_{i=1}^{k} (1 + q^{n-1}e^{-x_i}),
\]

\[
\text{ch}S_{q^n}TX = \prod_{i=1}^{k} \frac{1}{1 - q^n e^{x_i}}, \quad \text{ch}S_{q^n}TX^* = \prod_{i=1}^{k} \frac{1}{1 - q^n e^{-x_i}},
\]

\[
\text{td}X = \prod_{i=1}^{k} \frac{x_i}{1 - e^{-x_i}}.
\]

Plugging these in the right hand side of (3.6) we readily see that the integrand in the right hand side of (3.5) is \(Q(x_1)Q(x_2) \cdots Q(x_k)\), where \(Q(x)\) is the series (2.3b). Therefore (3.4) is identical to the definition (1.2) with \(Q(x)\) given by (2.3b). \(\square\)

Note that \(\text{Eu}(\Omega^\text{ch}_X)(0)\) is equal to the signature of the manifold \(X\).

4. Positivity of the equivariant signature of the loop space.

Introduce the formal character

\[
\text{ch}H^i(X, \Omega^\text{ch}_X) = \sum_{j=0}^{\infty} \dim H^i(X, (\Omega^\text{ch}_X)_j)q^j,
\]

where we ignore the upper-index grading, cf. (3.2). (It follows easily from (3.1) that the dimensions entering the formal character are all finite.)

**Theorem 4.1.** The following character formula is valid

\[
2\text{ch}H^0(\mathbb{P}^{2n}, \Omega^\text{ch}_{\mathbb{P}^{2n}}) - 1 = \text{sign}(q, \mathcal{L}\mathbb{P}^{2n}).
\]

**Corollary 4.2.** The series \(\text{sign}(q, \mathcal{L}\mathbb{P}^{2n})\) has positive coefficients and the coefficients against positive powers of \(q\) are even.
Corollary 4.2 is an immediate consequence of Theorem 4.1 whereas Theorem 4.1 is a simple consequence of certain properties of the chiral de Rham complex over projective spaces discovered in [MS1]. We shall list these properties here and reproduce from [MS1] a simple computation leading to Theorem 4.1.

The definition of the Euler characteristic and (3.5) give

\[
\text{sign}(q, \mathcal{L}_\mathbb{P}^{2n}) = \sum_{i=0}^{2n} (-1)^i \text{ch}H^i(\mathbb{P}^{2n}, \Omega^ch_\mathbb{P}^{2n}). \tag{4.1}
\]

We now make use of the following

**Theorem [MS1].** The natural embedding of sheaves \( \Omega^*_\mathbb{P}^{n} \hookrightarrow \Omega^{ch}_\mathbb{P}^{n} \) due to (3.3) provides an isomorphism

\[
H^i(\mathbb{P}^{n}, \Omega^*_{\mathbb{P}^{n}}) \sim H^i(\mathbb{P}^{n}, \Omega^{ch}_{\mathbb{P}^{n}}).
\]

for \( 0 < i < n \).

This assertion reduces (4.1) to

\[
\text{sign}(q, \mathcal{L}_\mathbb{P}^{2n}) = \text{ch}H^0(\mathbb{P}^{2n}, \Omega^ch_{\mathbb{P}^{2n}}) + \text{ch}H^{2n}(\mathbb{P}^{2n}, \Omega^ch_{\mathbb{P}^{2n}}) - 1. \tag{4.2}
\]

To conclude it remains to notice that due to the chiral Poincaré duality [MS2]

\[
\text{ch}H^0(\mathbb{P}^{2n}, \Omega^ch_{\mathbb{P}^{2n}}) = \text{ch}H^{2n}(\mathbb{P}^{2n}, \Omega^ch_{\mathbb{P}^{2n}}).
\]

□

**Remark** The vector space \( H^0(\mathbb{P}^{n}, \Omega^{ch}_{\mathbb{P}^{n}}) \) is a vertex algebra. Thanks to Theorem 4.1, the known [HBJ] modular properties of the equivariant signature of the loop space say that \( \text{ch}H^0(\mathbb{P}^{n}, \Omega^{ch}_{\mathbb{P}^{n}}) \) is a modular function when \( n = 0 \text{ mod } 4 \). The modular properties of characters have been the hallmark of vertex algebra theory, see for example [Z].

5. Extending the positivity result

5.1 It seems natural to ask in what generality the positivity result (Corollary 4.2) holds true. Since according to [BR] every cobordism class contains a non-singular toric variety, not every non-singular toric variety \( X \) has positive \( \text{sign}(q, \mathcal{L}X) \). We will show nevertheless that apart from projective spaces there is a class of toric varieties with this property. To do so, we will have to rely on the calculations from [BL] instead of sheaves of vertex algebras.

When talking about toric varieties we shall keep to the following notation. Let \( e_i, 1 \leq i \leq d \), be the standard basis of \( \mathbb{Z}^d \); thus the \( j \)-th component of \( e_i \) is \( \delta_{ij} \). Define the inner product

\[\mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}, \ x, y \mapsto x \cdot y\]

by the requirement that

\[\delta \cdot \delta = \delta \]
This identifies \( \mathbb{Z}^d \) and hence \( \mathbb{R}^d \) with their duals.

By \( \Sigma \) we denote a complete, regular fan in \( \mathbb{R}^d \). This means, in particular, that each cone \( C^* \in \Sigma \) is spanned by part of a basis of \( \mathbb{Z}^d \), and we denote this spanning set by \( |C^*| \).

Associated to \( \Sigma \) there is a smooth compact toric variety of complex dimension \( d \) to be denoted \( X_\Sigma \) or simply \( X \) if no confusion is likely to arise.

The following formula holds true [BL]

\[
\text{sign}(q, \mathcal{L}X) = \sum_{m \in \mathbb{Z}^d} \sum_{C^* \in \Sigma} (-1)^{\text{codim} C^*} \left( \prod_{n \in |C^*|} \frac{1}{1 + q^{m \cdot n}} \right) \varepsilon^{-d/4}. \tag{5.1}
\]

Note that to make sense out of this expression one has to expand each factor

\[
\frac{1}{1 + q^{m \cdot n}}
\]

at \( q = 0 \) and then convince oneself that the sum of thus arising power series with respect to \( m \in \mathbb{Z}^d \) makes sense as a formal power series.

**Remark** The formula in Theorem 5.5 in [BL] where we borrowed (5.1) contains an extra factor \((-1)^{d/2}\). We drop it so as to conform to the standard notation.

5.2 Now we extend the result of Theorem 4.1 to a larger class of toric variates. Recall that the Picard number of a toric variety is the difference between the dimension of the variety and the number of one dimensional cones. Each smooth Picard number one toric variety is a projective space. The Picard number two toric varieties, which can be viewed as generalized Hirzebruch’s surfaces, were classified in [Kl]. Let us formulate this result.

We give ourselves a triple of integers \( d, s, r \) such that \( 1 < d \), \( 1 < s < d + 1 \), \( r = d - s + 1 \), and an increasing sequence of non-negative integers \( a_1, \ldots, a_r \). Define the following vectors in \( \mathbb{Z}^d \):

\[
\begin{align*}
&u_i = e_i, 0 < i < r + 1; \\
&u_{r+1} = -\sum_{i=1}^{r} u_i; \\
&v_j = e_{r+j}, 0 < j < s; \\
&v_s = \sum_{i=1}^{r} a_i e_i - \sum_{j=1}^{s-1} v_j;
\end{align*}
\]

We set \( U = \{u_1, \ldots, u_{r+1}\} \), \( V = \{v_1, \ldots, v_s\} \) and let \( C^*_{ij} \subset \mathbb{R}^d \), \( 0 < i < r + 2 \), \( 0 < j < s + 1 \), be the cone spanned by \( U \cup V \setminus \{u_i, v_j\} \). One checks that there is a uniquely determined regular complete fan such that \( \{C^*_{ij}, 0 < i < r + 2, 0 < j < s + 1\} \) is the set of \( d \)-dimensional cones. We denote this fan by \( \sum_d(a_1, \ldots, a_r) \) and the corresponding toric variety by \( X_\sum(a_1, \ldots, a_r) \).
**Theorem 5.1** [Kl] Every compact smooth toric variety of complex dimension $d$ with $d+2$ generators is isomorphic to precisely one of the varieties $X_d(a_1, \ldots a_r)$.

**Theorem 5.2**

1. For $d$ even

$$\text{sign}(q, \mathbb{L}^d) = \sum_{m \in \mathbb{Z}^d} \frac{2}{(1 + q^{-m_1-\ldots-m_d}) \prod_{i=1}^{d} (1 + q^{m_i})} \epsilon^{-d/4}, \quad (5.2)$$

where $m = (m_1 \ldots m_d)$. The series in the RHS has positive coefficients.

2. For a smooth toric variety $X$ of even complex dimension $d$ with Picard number 2, $\text{sign}(q, \mathcal{L}X)$ is 0 if $s$ is even. Otherwise

$$\text{sign}(q, \mathcal{L}X) = \sum_{m \in \mathbb{Z}^d} \frac{2 + 2q^{m \cdot (v_1 + \ldots v_s)}}{\prod_{i=1}^{r+1} (1 + q^{m \cdot u_i}) \prod_{j=1}^{s} (1 + q^{m \cdot v_j})} \epsilon^{-d/4} \quad (5.3)$$

The series in the RHS has positive coefficients.

**Remark.** Statement 1. of Theorem 5.2 is an alternative to the chiral de Rham complex approach of sect. 4. But note that formula (5.2) is of a different nature than that in Theorem 4.1, and the comparison of (5.2) and Theorem 4.1 may give rise to non-trivial combinatorial identities.

**Proof.** Both of the statements follow from (5.1) and the explicit description of the fans of the toric varieties in question.

1. Let $k_1 = e_1, \ldots, k_d = e_d, k_{d+1} = -e_1 \ldots -e_d$. The fan defining $\mathbb{P}^d$ consists of the cones spanned by all proper subsets of the set $\{k_1, \ldots, k_{d+1}\}$.

Now observe that (5.1) rewrites as follows:

$$\text{sign}(q, \mathcal{L}X) = \epsilon^{-d/4} \sum_{m \in M} \sum_{C^* \in \Sigma} (-1)^{\text{codim}C^*} \left( \prod_{k \in |C^*|} \frac{1}{1 + q^k} \right), \quad (5.4)$$

where $q^k$ is an element of the group ring of $\mathbb{Z}^d$ and $< m, q^k > = q^{m \cdot k}$.

The expression

$$\sum_{C^* \in \Sigma} (-1)^{\text{codim}C^*} \left( \prod_{k \in |C^*|} \frac{1}{1 + q^k} \right), \quad (5.5)$$

(appearing in the r.h.s. of (5.4)) in the case of the projective space simplifies as follows:

$$\sum_B (-1)^{d - \# B} \prod_{k \in B} \frac{1}{1 + q^k}, \quad (5.6)$$

where the summation is performed over all proper subsets $B \subseteq \{k_1, \ldots, k_{d+1}\}$ and $\# B$ is the number of elements in $B$. Converting (5.6) to the common denominator we obtain

$$\sum_B (-1)^{d - \# B} \prod_{k \in B} (1 + q^k), \quad (5.7)$$
where $B$ denotes the complement of $B$. The multiple use of the binomial identity

$$
\sum_{i=0}^{n} (-1)^i C^i_n = 0 \text{ if } n > 0 \quad (5.8)
$$

allows us to collect the like terms in the numerator of (5.7). The result is

$$
1 + (-1)^d q^{k_1 + \cdots + k_n + 1} \prod_{i=1}^{d+1} (1 + q^{k_i}) \quad (5.9)
$$

Since $k_1 + k_2 + \cdots + k_{d+1} = 0$, it is zero if $d$ is odd (as it should) and

$$
\frac{2}{\prod_{i=1}^{d+1} (1 + q^{k_i})} \quad (5.10)
$$

otherwise. Plugging (5.10) in the r.h.s. of (5.4) and performing pairing with $m \in \mathbb{Z}^d$ we obtain (5.2), as desired. It is clear that every term in the denominator of (5.2) cancels against the appropriate term in $\epsilon^{-d/4}$. Indeed, for a fixed $m \in \mathbb{Z}^d$ the denominator of (5.2) contains at most $d + 1$ factors $(1 + q^n)$ for each $n \in \mathbb{Z}$ whereas $\epsilon^{-d/4}$, see (2.2), contains $2d$ such factors in the numerator. Therefore all the coefficients in the series $\text{sign}(q, \mathcal{L}^d)$ are positive.

2. Proof of the second statement is similar. Theorem 5.1 combined with (5.1) gives the following analogue of (5.4):

$$
\text{sign}(q, \mathcal{L}X) = \epsilon^{-d/4} \sum_{m \in M} < m, \sum_{I,J} \prod_{i \in I} (1 + q^{u_i}) \prod_{j \in J} (1 + q^{v_j}) >, \quad (5.11)
$$

where $I$ and $J$ are proper subsets of $\{1, \ldots, r + 1\}$ and $\{1, \ldots, s\}$ respectively and the vectors $u_i, v_j$ are those defined in the beginning of 5.2.

Converting the sum $\sum_{I,J}$ in (5.11) to the common denominator we obtain

$$
\sum_{I,J} \frac{(-1)^{d-\#I-\#J}}{\prod_{i=1}^{r+1} (1 + q^{u_i}) \prod_{j=1}^{s} (1 + q^{v_j})} \quad (5.12)
$$

Now observe that the numerator of (5.12) is the product of two factors analogous to the numerator of (5.7) – one is the numerator of (5.7) with $B$ replaced with $\{u_i, i \in I\}$, another is also with $B$ replaced with $\{v_j, j \in J\}$. Therefore identity (5.8), which allowed us to pass from (5.7) to (5.9), allows us to analogously rewrite (5.12) as follows:

$$
\frac{(1 + (-1)^r q^{u_1 + u_2 + \cdots + u_{r+1}})(1 + (-1)^s q^{v_1 + v_2 + \cdots + v_s})}{\prod_{i=1}^{r+1} (1 + q^{u_i}) \prod_{j=1}^{s} (1 + q^{v_j})} \quad (5.13)
$$

Since by definition (see the beginning of 5.1)

$$
\sum_{i=0}^{r+1} u_i = 0,
$$

we have

$$
\sum_{i=0}^{s} v_i = 0 \quad (5.14)
$$

Therefore, setting

$$
\tau = \frac{\sum_{i=1}^{r+1} u_i}{2} = 0 \quad (5.15)
$$

we observe that (5.12) is just (5.9) with $\tau$ replaced by $\tau$. Hence, after cancellation of the appropriate terms in the numerator of (5.15) the result is

$$
\frac{2}{\prod_{i=1}^{d+1} (1 + q^{k_i})} \quad (5.16)
$$

as desired.
(5.13) vanishes if $r$ is odd and equals
\[
\frac{2(1 + q^{v_1+v_2+\cdots+v_s})}{\prod_{i=1}^{r+1}(1 + q^{u_i}) \prod_{j=1}^{s}(1 + q^{v_j})}
\]  
(5.14)
otherwise. (Note that $(-1)^{s-1}$ has disappeared because, $d$ being even, the relation $d = r + s - 1$ forces $r$ and $s$ to have different parity.)

Plugging (5.14) in the r.h.s. of (5.11) and performing pairing with $m \in \mathbb{Z}^d$ we obtain desired formula (5.3).

To show that series (5.3) has positive coefficients we observe that for a fixed $m \in \mathbb{Z}^d$ the denominator of (5.3) contains at most $r + s + 1$, that is, $d + 2$ factors $(1 + q^n)$ for each $n \in \mathbb{Z}$ whereas $\epsilon^{-d/4}$, see (2.2), contains $2d$ such factors in the numerator. Therefore, having carried out the cancellations we make (5.3) into a sum of power series with positive coefficients. □

6. Toric spin varieties

Recall Witten’s rigidity theorem [W] proved in [BT]. Let a torus $T^n$ act on a manifold $X$. This action lifts to an action on the holomorphic bundle $gr\Omega^c_X$, see (3.1) Therefore each cohomology group $H^i(X, gr\Omega^c_X)$ becomes a $T^n$-module – a direct sum of the torus characters $t \mapsto t^m$, $t \in T^n$, $m \in \mathbb{Z}^n$, in fact. Formula (3.4) then implies that $\text{sign}(q, LX)$ is a formal sum of the torus characters and one can think of $\text{sign}(q, LX)$ as a function of $t \in T^n$ with values in $\mathbb{C}[[q]]$.

**Theorem 6.1** [BT] If $X$ is a spin manifold equipped with an action of a torus $T^n$, then $\text{sign}(q, LX)$ is a constant function of $t \in T^n$.

**Theorem 6.2** For a toric spin variety $X$ of complex dimension $d$,
\[
\text{sign}(q, LX) = \text{sign}X \epsilon^{-\frac{d}{2d}}.
\]

**Proof.** Any toric variety $X$ carries the natural action of a torus; hence $\text{sign}(q, LX)$ is a formal sum of the torus characters. Formula (5.1) sharpens accordingly [BL]:
\[
\text{sign}(q, LX) = \sum_{m \in M} t^m \sum_{C^* \in \Sigma} (-1)^{\text{codim}C^*} \left( \prod_{i=1, \ldots, \text{dim}C^*}(1 + q^{m \cdot n_i}) \right) \epsilon^{-d/4}. \quad (6.1)
\]

Theorem 6.1 implies that only $t^0$ may appear in the r.h.s. of (6.1) with non-zero coefficient. Therefore
\[
\text{sign}(q, LX) = C \epsilon^{-d/4} \quad (6.2)
\]
for some constant $C$. To compute $C$ recall that
\[
\text{sign}(0, LX) = \text{sign}X.
\]
Therefore, having specialized (6.2) to $q = 0$ we obtain
\[
\text{sign}X = C2d.
\]
and Theorem 6.2 follows. □

Remarks.
1) According to [HS] $\text{sign}(q, \mathcal{L}X) = \text{sign}X$ if $X$ is a homogeneous space and a spin manifold at the same time. Therefore Theorem 6.2 is an extension of this result to spin toric varieties.

2) Theorem 6.2 suggests that perhaps the vertex algebra of $2d$ bosons coupled to $2d$ fermions provides a natural realization of $\text{sign}(q, \mathcal{L}X)$ in the case of a toric spin manifold as discussed in greater detail in the introduction.

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