$L^\infty$-ENERGY METHOD FOR A PARABOLIC SYSTEM WITH CONVECTION AND HYSTERESIS EFFECT

Dedicated to Professor Vladimir Georgiev on the occasion of his sixtieth birthday

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Abstract. The $L^\infty$-energy method is developed so as to handle nonlinear parabolic systems with convection and hysteresis effect. The system under consideration originates from a biological model where the hysteresis and convective effects are taken into account in the evolution of species. Some results for the existence of local and global solutions as well as the uniqueness of solution are presented.

1. Introduction. The present paper deals with the following system of nonlinear PDE's with convection and hysteresis effect:

$$
\frac{\partial \sigma}{\partial t} - \nabla \cdot (\nabla \sigma + \tilde{\lambda}(\sigma)) + \partial I^U(\sigma) \ni g(\sigma, U) \quad (x, t) \in Q_T = \Omega \times (0, T),
$$

$$
\frac{\partial u_i}{\partial t} - \nabla \cdot (\nabla u_i + \tilde{\mu}_i(u)) = h_i(\sigma, U) \quad (i = 1, \cdots, m) \quad (x, t) \in Q_T,
$$

$$
\frac{\partial \sigma}{\partial \nu} = 0, \quad \frac{\partial u_i}{\partial \nu} = 0 \quad (i = 1, \cdots, m) \quad (x, t) \in \Sigma_T = \partial \Omega \times (0, T),
$$

$$
\sigma(x, 0) = \sigma_0(x), \quad U(x, 0) = U_0(x) \quad x \in \Omega,
$$

where $U = (u_1, \cdots, u_m)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial \Omega$, $T > 0$, $N$ and $m$ are positive integers, $\nu$ is the unit outward normal vector on $\partial \Omega$ and $\sigma_0$, $U_0 = (u_{10}, \cdots, u_{m0})$ are given initial data and $\tilde{\lambda}, \tilde{\mu}_i$ and $\partial I^U(\cdot)$ of the indicator function $2000$ Mathematics Subject Classification. Primary: 35R70, 35K51; Secondary: 37N25, 47J40.

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\( I^U(\cdot) \) of the interval \([f_*(U), f^*(U)]\) which depends on the unknown function \( U \). The subdifferential \( \partial I^U(\sigma) \) is a set-valued mapping given by

\[
\partial I^U(\sigma) = \begin{cases} 
\emptyset & \text{if } \sigma > f^*(U) \text{ or } \sigma < f_*(U), \\
[0, +\infty) & \text{if } \sigma = f^*(U) > f_*(U), \\
\{0\} & \text{if } f_*(U) < \sigma < f^*(U), \\
(-\infty, 0] & \text{if } \sigma = f_*(U) < f^*(U), \\
\mathbb{R} & \text{if } \sigma = f^*(U) = f_*(U),
\end{cases}
\]

where \( f_*, f^* : \mathbb{R}^m \to \mathbb{R} \) are given functions such that \( f_* \leq f^* \) on \( \mathbb{R}^m \).

It is known that the hysteresis effect can be found in many phenomena in nature, for instances, phase transitions, plasticity, ferroelectricity, superconductivity, etc. In particular, there are indications for existence of hysteresis in various biological problems, see e.g., [10], [13]. Although population dynamics is an object of long-standing interest, the mathematical description of hysteresis effect in the processes in population dynamics has been considered only in a few papers. We refer the reader to the paper [7] which seems to be the first paper treating mathematically hysteresis phenomena in a biological problem, namely the authors treated bacterial growth in a petri dish modeled by hysteresis operator of relay type which describes the relation between the rate of the growth of the bacterial population and the \( pH \) level of the surrounding acid-buffer mix. The survey paper [14] treats applications of hysteresis in various natural phenomena. One of the chapters of [14] is devoted to the applications in biological problems and the authors of [14] also underscore the necessity of developing of new models for description of hysteresis in biological processes. It is known that some types of hysteresis operators can be represented by ordinary (or partial) differential inclusion containing the subdifferential of the indicator function of a closed set (whose length/shape could possibly depend on the unknown variables). This fact was pointed out by A. Visintin [24] and was used for analysis of many nonlinear phenomena, for example, the real-time control problems [8], solid-liquid phase transitions [6], [9], shape memory alloy problems [1], filtration problems [12] and this approach was used to some problems arising from population dynamics [2].

In this paper we shall apply \( L^\infty \)-energy method which was recently developed (see [22, 23, 18, 19, 20]) and found to be an effective tool applicable to various types of parabolic equations and systems including doubly nonlinear parabolic equations, porous medium equations, strongly nonlinear parabolic equations governed by the \( \infty \)-Laplacian, etc. The core of the \( L^\infty \)-energy method is the derivation of energy estimates in \( L^\infty \) even when any energy estimates could not be expected in \( L^p \) with \( 1 \leq p < \infty \). The aim of the present paper is to develop \( L^\infty \)-energy method so as to solve problems arising from population dynamics with hysteresis effect whose typical example is the system (1)-(4).

The existence of global solutions and the uniqueness of solution of (1)-(4) is already discussed in [16]. The advantage of our treatment based on \( L^\infty \)-energy method over [16] lies in the fact that it enables us to obtain easily the a priori bound of the \( L^\infty \)-norm of solutions, which leads to the existence of local and global solutions of (1)-(4) under much weaker assumptions on the space dimension \( N \), \( f_*, f^* \) and other given functions. Moreover it also makes it possible to set up a new strategy for the proof of the uniqueness of solution via the uniform \( L^\infty \)-estimate of solutions, which ameliorates that in [2] or [9].
2. Preliminary and main results. In this section we formulate our main results. To do this we first prepare some preliminary notes.

2.1. Preliminary notes. Let $H$ be the Hilbert space $L^2(\Omega)$ with the usual scalar product $(\cdot, \cdot)_H$ and norm $| \cdot |_H$ and denote $L^2(\Omega) := (L^2(\Omega))^m$. We designate by $V$ the Sobolev space $H^1(\Omega)$ equipped with the norm $|v|_V = (u, u)^{1/2}$, where $(u, v)_V = (u, v)_H + \langle a(u, v) \rangle$, $a(u, v) = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx$, $u, v \in V$. We also denote $H^1(\Omega) := (H^1(\Omega))^m$. Now we give the definition of solutions of the system (1)-(4).

Definition 2.1. A pair of functions $\{\sigma, U = (u_1, \cdots, u_m)\}$ is called a solution of the system (1)-(4) if the following hold.

(i) \( \sigma, u_i \in L^\infty(0, S; V \cap L^\infty(\Omega)) \cap L^2(0, S; H^2(\Omega)) \cap W^{1,2}(0, S; H^2(\Omega)) \) (\( i = 1, \cdots, m \)),

(ii) \( \frac{\partial \sigma}{\partial t} - \nabla \cdot (\nabla \sigma + \lambda_\sigma(\sigma)) + \partial I^U(\sigma) \ni g(\sigma, U) \) \( \in L^2(\Omega) \) a.e. $t \in (0, S)$,

(iii) \( \frac{\partial u_i}{\partial t} - \nabla_\sigma \cdot (\nabla u_i + \mu_i(u_i)) = h_i(\sigma, U) \) \( \in L^2(\Omega) \) a.e. $t \in (0, S)$, \( i = 1, \cdots, m \),

(iv) \( \frac{\partial \sigma}{\partial \nu} = 0, \frac{\partial u_i}{\partial \nu} = 0 \) \( \in L^2(\partial \Omega) \) a.e. $t \in (0, S)$ \( (i = 1, \cdots, m) \),

(v) \( \sigma(0) = \sigma_0, U(0) = U_0 = (u_{10}, \cdots, u_{m0}) \).

For simplicity, we denote by $\sigma'$ and $u_i'$ the time-derivatives $\frac{\partial \sigma}{\partial t}$ and $\frac{\partial u_i}{\partial t}$ of $\sigma$ and $u_i$ \( (i = 1, \cdots, m) \) respectively.

Note that the inclusion (ii) implies that

\( f_\sigma(U) \leq \sigma \leq f_\sigma^*(U) \) a.e. \( (x, t) \in Q_S \),

\( (\sigma'(t) - \nabla \cdot (\nabla \sigma(t) + \lambda_\sigma(\sigma(t)))) - g(\sigma(t), U(t)), \sigma(t) - z \) \( \in H^2(\Omega) \) a.e. $t \in (0, S)$, \( z \in L^2(\Omega) \) with $f_\sigma(U(t)) \leq z \leq f_\sigma^*(U(t))$ for a.e. \( (x, t) \in Q_S \).

We here introduce our basic assumptions:

(H1) \( \sigma_0, u_{0i} \in L^\infty(\Omega) \cap V \) \( (i = 1, \cdots, m) \), \( f_\sigma(U_0) \leq \sigma_0 \leq f_\sigma^*(U_0) \) a.e. in $\Omega$.

(H2) \( f_\sigma, f_\sigma^* \in C^2(\mathbb{R}^m; \mathbb{R}) \) and $f_\sigma \leq f_\sigma^*$ on $\mathbb{R}^m$.

(H3) \( \lambda_\sigma, \mu_i \) \( (i = 1, \cdots, m) \) are locally Lipschitz continuous functions from $\mathbb{R}$ into $\mathbb{R}^N$, and $g$ and $h_i \) \( (i = 1, \cdots, m) \) are locally Lipschitz continuous functions from $\mathbb{R} \times \mathbb{R}^m$ into $\mathbb{R}$.

(H4) There exists a positive constant $C$ such that

\[ g(\sigma, U) \sigma \leq C (|\sigma|^2 + |U|^2 + 1), \] (5)

\[ h_i(\sigma, U) u_i \leq C (|\sigma|^2 + |U|^2 + 1) \quad (i = 1, \cdots, m), \] (6)

\[ \max \{ |f_\sigma(U)|, |f_\sigma^*(U)| \} \leq C (|U| + 1), \] (7)

where $\sigma \in \mathbb{R}$, $U \in \mathbb{R}^m$ and $|U| = (\sum_{i=1}^m |u_i|^2)^{1/2}$.

2.2. Main results. Our main results are stated as follows.

Theorem 2.2. Let (H1), (H2) and (H3) be satisfied. Then there exists a positive number $T_0$ depending on $|\sigma_0|_{L^\infty}$ and $\max_{1 \leq i \leq m} |u_{0i}|_{L^\infty}$ such that (1)-(4) admits a
solution \( (\sigma, U = (u_1, \cdots, u_m)) \) on \([0, T_0]\) satisfying
\[
\sigma, u_i \in C([0, T_0]; H^1(\Omega)) \cap L^\infty(0, T_0; L^\infty(\Omega)) \cap L^2(0, T_0; H^2(\Omega)) \\
\cap W^{1,2}(0, T_0; L^2(\Omega)) \quad (i = 1, \cdots, m).
\] (8)

**Theorem 2.3.** Let (H1), (H2), (H3) and (H4) be satisfied. Then (1)-(4) admits a global solution \( (\sigma, U = (u_1, \cdots, u_m)) \) on \([0, T]\) satisfying (8) with \( T_0 \) replaced by \( T \).

**Theorem 2.4.** Let \( N \leq 4 \) and assume that \( \bar{\lambda}, \bar{\mu}_i \in C^2(\mathbb{R}; \mathbb{R}^N) \) \((i = 1, \cdots, m)\), then the solution of (1)-(4) satisfying (8) is unique.

Since the system (1)-(4) has a biological interpretation, the non-negativity of the solution is an important information.

**Theorem 2.5.** Let (H3) be satisfied and assume the following conditions:
1. \( u_0 \geq 0 \) \((i = 1, \cdots, m)\), \( \sigma_0 \geq 0 \) a.e. \( \Omega \) and \( f^* \geq 0 \) on \( \mathbb{R}^m \),
2. \( h_i(\sigma, u_1, \cdots, u_{i-1}, 0, u_{i+1}, \cdots, u_m) = 0 \) \( \forall \sigma, u_j \in \mathbb{R} \) \((j = 1, \cdots, i-1, i+1, \cdots, m)\) and \( g(0, U) = 0 \) \( \forall U \in \mathbb{R}^m \).

Then the solution given in Theorem 2.2 or Theorem 2.3 is non-negative on \([0, T_0]\) or \([0, T]\).

**Remark 1.** The existence of global solutions is also discussed in [16], where it is assumed that \( \bar{\lambda}, \bar{\mu}_i \) \((i = 1, \cdots, m)\), \( g \) and \( h_i \) \((i = 1, \cdots, m)\) are Lipschitz continuous, \( 0 \leq f_s \leq f^* \leq 1 \) and the derivatives of \( f_s, f^* \) are bounded up to the second order. Our assumptions (5), (6), (7) are much weaker than those in [16].

3. **Proofs of main results.** We here give proofs of main results.

3.1. **Proof of Theorem 2.2.**

3.1.1. **Approximate equations.** To prove Theorem 2.2, we rely on "\(L^\infty\)-energy method" developed in [18, 19, 20]. In order to apply \(L^\infty\)-energy method, in most cases, we need to introduce approximate equations which admit solutions belonging to \(L^\infty(\Omega)\). For this purpose, there are several ways of approximation (see [18, 19, 20]). Here we apply a method to restrict the \(L^\infty\)-norm of solution by adding the subdifferential term of some indicator function to the original equation.

To this end, we introduce the restriction parameter \( M > 0 \) which will be fixed in the sequel and the indicator function of the closed convex set \( K_M := \{ u \in L^2(\Omega); |u(x)| \leq M \ \text{a.e.} \ x \in \Omega \} \) given by
\[
I_M(u) = \begin{cases} 
0 & \text{if } u \in K_M, \\
+\infty & \text{if } u \in L^2(\Omega) \setminus K_M.
\end{cases}
\]

Then \( I_M(\cdot) : H = L^2(\Omega) \to [0, +\infty] \) becomes a lower semi-continuous convex function and its subdifferential is given by (see [3, 4])
\[
D(\partial I_M) = K_M, \quad \partial I_M(u)(x) = \begin{cases} 
\emptyset & \text{if } M < |u(x)|, \\
\{0\} & \text{if } -M < u(x) < M, \\
[0, +\infty) & \text{if } u(x) = M, \\
(-\infty, 0] & \text{if } u(x) = -M.
\end{cases}
\]
Moreover we put
\[ \varphi_M(u) = \varphi(u) + I_M(u), \quad \varphi(u) = \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u(x)|^2 \, dx & \text{if } u \in H^1(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus H^1(\Omega). \end{cases} \]

Then \( \varphi_M(\cdot) : H = L^2(\Omega) \to [0, +\infty] \) becomes a lower semi-continuous convex function and its subdifferential is given by
\[
D(\partial \varphi_M) = D(\partial \varphi) \cap K_M, \quad \partial \varphi_M(u)(x) = -\Delta u(x) + \partial I_M(u)(x).
\]

To see this, it suffices to show that \( u \mapsto -\Delta u + \partial I_M(u) \) is maximal monotone, since it is clear that \( \partial \varphi_M(u) \supset -\Delta u + \partial I_M(u) \). Noting that the Yosida approximation \( (\partial I_M)_\delta(r) \) of \( \partial I_M(r) \) is Lipschitz continuous and monotone increasing in \( r \in \mathbb{R} \), we get
\[
(\partial \varphi(u), (\partial I_M)_\delta(u))_H = \int_{\Omega} (\partial I_M)_\delta(u(x)) \nabla u(x)^2 \geq 0 \quad \forall u \in D(\partial \varphi).
\]

Then Theorem 4.4 and Proposition 2.17 in [4] assure that \( u \mapsto -\Delta u + \partial I_M(u) \) is maximal monotone in \( L^2(\Omega) \).

We further introduce the Yosida approximation \( \partial I^\delta(\cdot) \) of the hysteresis operator \( \partial I^U(\cdot) \) for each \( \delta > 0 \) given below.

\[
\partial I^\delta(\sigma) = \frac{1}{\delta} [\sigma - f^+(U)]^+ - \frac{1}{\delta} [f_+(U) - \sigma]^+, \quad [r]^+ = \max(r, 0). \quad (11)
\]

Then our approximate equations are given by the following.

\[
\begin{align*}
\sigma' - \Delta \sigma + \partial I_M(\sigma) + \partial I^\delta(\sigma) &= g(\sigma, U) + \nabla \cdot \tilde{\lambda}(\sigma) \quad \text{in } Q_T, \quad (12) \\
u_i' - \Delta u_i + \partial I_M(u_i) &= h_i(\sigma, U) + \nabla \cdot \tilde{\mu}_i(u_i) \quad (i = 1, \ldots, m) \quad \text{in } Q_T, \quad (13) \\
\frac{\partial \sigma}{\partial \nu} &= 0, \quad \frac{\partial u_i}{\partial \nu} = 0 \quad (i = 1, \ldots, m) \quad \text{on } \Sigma_T, \quad (14) \\
\sigma(0, x) = \sigma_0(x), \quad u_i(0, x) = u_{i0}(x) \quad (i = 1, \ldots, m) \quad \text{in } \Omega. \quad (15)
\end{align*}
\]

We are going to show below that (12)-(15) admits a unique global solution for any \( \sigma_0, u_{i0} \in L^\infty(\Omega) \cap V \) by applying Corollary IV of [21] with the underlying Hilbert space \( \mathbb{H} := (L^2(\Omega))^{1+m} \), i.e., \( (\sigma, U) \in \mathbb{H} \) means \( \sigma \in L^2(\Omega), \quad U = (u_1, \ldots, u_m) \in L^2(\Omega) := (L^2(\Omega))^m \).

Indeed, we can rewrite the system (12)-(15) in the form of single evolution equation in \( \mathbb{H} \) as follows:

\[
W'(t) + \partial \phi_M(W(t)) + B(W(t)) \ni 0, \quad 0 < t < T, \quad (16)
\]

\[
W(0) = W_0, \quad (17)
\]

where \( W = \begin{pmatrix} \sigma \\ U \end{pmatrix}, \quad \phi_M(W) = \varphi_M(\sigma) + \sum_{i=1}^m \varphi_M(u_i), \quad B(W) = \begin{pmatrix} \partial I^\delta(\sigma) - g(\sigma, U) - \nabla \cdot \tilde{\lambda}(\sigma) \\ -h_1(\sigma, U) - \nabla \cdot \tilde{\mu}_1(u_1) \\ \ldots \ldots \quad (i = 1, \ldots, m) \\ -h_m(\sigma, U) - \nabla \cdot \tilde{\mu}_m(u_m) \end{pmatrix}. \]

We first note that \( \sigma, u_i \in D(\partial \varphi_M) \) implies that
\[
|\sigma|_{L^\infty} \leq M, \quad |u_i|_{L^\infty} \leq M \quad (i = 1, \ldots, m).
\]
Hence, by virtue of (H2), (H3) and (11), we find that
\[ \partial I^\|_k(\sigma), g(\sigma, U), h_i(\sigma, U) \text{ are all Lipschitz continuous from } H \text{ into } L^2(\Omega), \]
\[ |\partial I^\|_k(\sigma)|_H, |g(\sigma, U)|_H, |h_i(\sigma, U)|_H \leq C_M \quad \forall(\sigma, U) \in D(\partial \phi_M), \]
where \( C_M \) denotes the general constant depending on \( M \).
Furthermore (H3) implies
\[ |\nabla \cdot \tilde{X}(\sigma)|_H = |(\tilde{X})'(\sigma) \cdot \nabla \sigma|_H \leq C_M |\nabla \sigma|_H \quad \forall \sigma \in D(\partial \phi_M), \]
\[ |\nabla \cdot \tilde{\mu}_i(u_i)|_H = |(\tilde{\mu}_i)'(u_i) \cdot \nabla u_i|_H \leq C_M |\nabla u_i|_H \quad \forall u_i \in D(\partial \phi_M), \]
\[ (\tilde{X})'(\sigma) = (\partial \lambda^1/\partial x_1, \cdots, \partial \lambda^N/\partial x_N), \quad (\tilde{\mu}_i)'(u_i) = (\partial \mu_i^1/\partial x_1, \cdots, \partial \mu_i^N/\partial x_N). \]
Thus, by (18), (19) and (20), we obtain
\[ |B(W)|_H \leq C_M \left( |\nabla \sigma|_{L^2(\Omega)} + |\nabla U|_{L^2(\Omega)} + 1 \right) \]
\[ \leq C_M \left( \phi_M^{1/2}(W) + 1 \right) \quad \forall W \in D(\partial \phi_M), \]
which assures assumption (A.5) of [21].
On the other hand, by (21), we get
\[ (-\partial \phi_M(W) - B(W), W)_H + 2\phi_M(W) = (-B(W), W)_H \]
\[ \leq |B(W)|_H |W|_H \leq C_M \left( \phi_M^{1/2}(W) + 1 \right) |W|_H \]
\[ \leq \phi_M(W) + C_M \left( |W|_H^2 + 1 \right) \quad \forall W \in D(\partial \phi_M), \]
which assures assumption (A.6) of [21]. Thus, we can apply Theorem III and Corollary IV of [21] and conclude the existence of global solutions \((\sigma, U) \in L^\infty(\Omega) \times (L^\infty(\Omega))^m\) satisfying (8) with \( T_0 \) replaced by \( T \).

The proof of the uniqueness of solution of system (12)-(15) is standard, since \( \tilde{X}, \tilde{\mu}_i, g, h_i(i = 1, \cdots, m) \) can be regarded as globally Lipschitz continuous functions by virtue of the boundedness of solutions in \( L^\infty(\Omega) \).

### 3.1.2. A priori estimates
Now let \((\sigma, U)\) be the unique solution of the approximate system (12)-(15). We are going to establish some a priori estimates for \((\sigma, U)\) which are independent of \( M \) and \( \delta \).

#### \( L^\infty \)-estimate of \( U \):
First, multiply the \( i \)-th equation of (13) by \(|u_i|^{r-2}u_i\) with \( r \geq 2 \) and integrate over \( \Omega \). Noting that \((\partial I^\|_k(u_i), |u_i|^{r-2}u_i)|_H \geq 0\), we get
\[ \frac{1}{r} \frac{d}{dt}|u_i(t)|_{L^r}^r + (r-1) \int_{\Omega} |u_i|^{r-2} |\nabla u_i|^2 dx = I_1 + I_2, \]
\[ I_1 = \int_{\Omega} (\tilde{\mu}_i)'(u_i) \cdot \nabla u_i |u_i|^{r-2} u_i dx \]
\[ \leq \frac{(r-1)}{2} \int_{\Omega} |u_i|^{r-2} |\nabla u_i|^2 dx + \frac{1}{2(r-1)} |(\tilde{\mu}_i)'(u_i)|_{L^\infty}^2 |u_i|_{L^r}^r, \]
\[ I_2 = \int_{\Omega} h_i(\sigma, U) |u_i|^{r-2} u_i dx \leq |h_i(\sigma, U)|_{L^\infty} |u_i|_{L^r}^{r-1}. \]
Here dividing both sides of (24) by \(|u_i|_{L^r}^{r-1}\), we obtain by (25) and (26)
\[ \frac{d}{dt}|u_i(t)|_{L^r} \leq |h_i(\sigma, U)|_{L^\infty} + \frac{1}{2(r-1)} |(\tilde{\mu}_i)'(u_i)|_{L^\infty}^2 |u_i|_{L^r}. \]
Integrating (27) with respect to $t$ over $(0,t)$ and letting $r \to \infty$, we can see by (H3) that there exists a monotone increasing function $\ell_1(\cdot)$ such that

$$|U(t)|_{L^\infty} \leq |U_0|_{L^\infty} + \int_0^t \ell_1(|U(s)|_{L^\infty} + |\sigma(s)|_{L^\infty}) \, ds.$$  \hspace{1cm} (28)

Here $|U(t)|_{L^\infty} = \max_{1 \leq i \leq m} |u_i(t)|_{L^\infty}$ and we used the fact that $\lim_{r \to \infty} |v|_{L^r} = |v|_{L^\infty}$ for all $v \in L^\infty(\Omega)$ (see Lemma 2.1 of [20]).

**Remark 2.** Apparently the argument for deriving (27)-(28) from (24)-(26) would not be rigorous, since the meaning of this procedure becomes obscure when the divisor $|u_i|_{L^{r-1}}$ attains zero. However the following proposition with

Proposition 1. Let $y(\cdot)$ be a nonnegative absolutely continuous function on $[0,S]$ satisfying

$$\frac{1}{r} \frac{d}{ds} y(s)^r \leq \Phi(s,y(s)) y(s)^{r-k} \quad \text{a.e. } s \in (0,S),$$  \hspace{1cm} (29)

where $\Phi(s,y(s))$ is a nonnegative measurable function such that $\Phi(s,y(s)) \in L^1(0,S)$, $r$ is a positive integer and $k$ is an integer such that $1 \leq k < r$. Then we have

$$\frac{1}{k} y(t)^k \leq \frac{1}{k} y(0)^k + \int_0^t \Phi(s,y(s)) \, ds \quad \forall t \in [0,S].$$  \hspace{1cm} (30)

**Proof.** Let $y(s) > 0$ for all $s \in [0,S]$, then dividing (29) by $y(s)^{r-k} > 0$, we get

$$\frac{1}{k} \frac{d}{ds} y(s)^k \leq \Phi(s,y(s)) \quad \text{a.e. } s \in (0,S).$$  \hspace{1cm} (31)

Then integrating this over $(0,t)$, we get (30). Suppose that $Z(y) := \{ s \in [0,S] ; y(s) = 0 \} \neq \emptyset$ and put $t_m = \min Z(y)$. For all $t \in Z(y)$, (30) is obvious, since the right-hand side of (30) is nonnegative. Let $t \in [0,S] \setminus Z(y)$ and $t_m < t$, then by the continuity of $y(t)$, there exists $t_1 \in [t_m,t)$ such that

$$y(t_1) = 0 \quad \text{and} \quad y(s) > 0 \quad \forall s \in (t_1,t).$$

Hence, for $s \in (t_1,t)$, dividing (29) by $y(s)^{r-k} > 0$, we get

$$\frac{1}{k} \frac{d}{ds} y(s)^k \leq \Phi(s,y(s)) \quad \text{a.e. } s \in (t_1,t).$$

Then integrating this over $(t_1,t)$, we get

$$\frac{1}{k} y(t)^k \leq \frac{1}{k} y(t_1)^k + \int_{t_1}^t \Phi(s,y(s)) \, ds \leq \frac{1}{k} y(0)^k + \int_0^t \Phi(s,y(s)) \, ds.$$  \hspace{1cm} (31)

As for the case where $t \in [0,S] \setminus Z(y)$ and $t_m > t$, we see that $y(s) > 0$ for all $s \in (0,t)$. Then repeating the same procedure as for (31) with $(0,S)$ replaced by $(0,t)$, we obtain (30).

**$L^\infty$-estimate of $\sigma$:** In order to derive the $L^\infty$-estimate for $\sigma$ from above, we define

$$u_*(t) = \sup \{ |f_*(U(x,s))| ; x \in \Omega, 0 \leq s \leq t \}.  \hspace{1cm} (32)$$
Multiply (12) by \(|\sigma(s) - u_*(t)| - 2[\sigma(s) - u_*(t)]^+ (s \in (0, t))\) with \(r \geq 2\) and integrate over \(\Omega\). Noting that \(\sigma(s) > u_*(t) \geq 0\) holds where the integrant does not vanish, which implies \((\partial I_J(\sigma(s)), |\sigma(s) - u_*(t)| - 2[\sigma(s) - u_*(t)]^+)_{H} \geq 0\), we obtain

\[
\frac{1}{r} \frac{d}{ds} |(\sigma(s) - u_*(t))|^r_{L^r} + (r - 1) \int_{\Omega} |(\sigma(s) - u_*(t))|^r_{L^r} |\nabla \sigma(s)|^2 dx
+ J_1 \leq J_2 + J_3,
\]

(33)

\[J_1 = \int_{\Omega} \partial I_J^U(\sigma(s)) |\sigma(s) - u_*(t)| - 2[\sigma(s) - u_*(t)]^+ dx = \frac{1}{\delta} \int_{\Omega} |(\sigma(s) - f^*(U(s)))^+| |\sigma(s) - u_*(t)| - 2[\sigma(s) - u_*(t)]^+ dx
- \frac{1}{\delta} \int_{\Omega(\sigma(s) \geq u_*(t))} |f_*(U(s)) - \sigma(s)|^+ |\sigma(s) - u_*(t)| - 2[\sigma(s) - u_*(t)]^+ dx \geq 0,
\]

(34)

\[J_2 = \int_{\Omega} \int_{\Omega} |(\bar{\lambda}_I(\sigma(s)) |\nabla \sigma(s)| |(\sigma(s) - u_*(t))|^r - 1 dx
\leq \frac{(r - 1)}{2} \int_{\Omega} |(\sigma(s) - u_*(t))|^r_{L^r} |\nabla \sigma(s)|^2 dx
+ \frac{1}{2(r - 1)} |(\bar{\lambda}_I(\sigma(s))|^2_{L^2} |(\sigma(s) - u_*(t))|^r_{L^r},
\]

\[J_3 = \int_{\Omega} g(\sigma, U) |(\sigma(s) - u_*(t))|^r_{L^r} dx \leq |g(\sigma, U)| |(\sigma(s) - u_*(t))|^r_{L^r}.
\]

Here dividing both sides of (33) by \(|(\sigma(s) - u_*(t))|^r_{L^r} - 1\) (see Remark 2), integrating with respect to \(s\) over \((0, t)\) and letting \(r \to \infty\), we can see as in (28) that there exists a monotone increasing function \(\ell_2(\cdot)\) such that

\[|(\sigma(t) - u_*(t))|^r_{L^r} \leq |(\sigma_0 - u_*(t))|^r_{L^r} + \int_0^t \ell_2((U(s)|_{L^r} + |\sigma(s)|_{L^r}) ds.
\]

(35)

In order to get the estimate for \(\sigma\) from below for each \(t\), we set \(u^*(t)\) by

\[u^*(t) = \inf \{ -|f^*(U(x, s))| : x \in \Omega, 0 \leq s \leq t \}.
\]

(36)

Multiplying (12) by \(-|u^*(t) - \sigma(s)| - 2[u^*(t) - \sigma(s)]^+ (s \in (0, t))\) and integrating over \(\Omega\), by the argument similar to that for (35), we find that there exists a monotone increasing function \(\ell_3(\cdot)\) such that

\[|u^*(t) - \sigma(t)|_{L^\infty} \leq |u_*(t) - \sigma_0|_{L^\infty} + \int_0^t \ell_3((U(s)|_{L^\infty} + |\sigma(s)|_{L^\infty}) ds.
\]

(37)

Here, instead of (34), we used the following relation:

\[J^*_1 = -\int_{\Omega} \partial I_J^U(\sigma(s)) |(u^*(t) - \sigma(s)) - 2[u^*(t) - \sigma(s)]^+ dx
= \frac{1}{\delta} \int_{\Omega} f^*(U(s)) \geq u^*(t) \geq \sigma(s) \int_{\Omega} |(\sigma(s) - f^*(U(s)))^+ |(u^*(t) - \sigma(s)) - 2[u^*(t) - \sigma(s)]^+ dx
+ \frac{1}{\delta} \int_{\Omega} |f_*(U(s)) - \sigma(s)|^+ |(\sigma(s) - u_*(t))|^r_{L^r} |\sigma(s) - u_*(t)| - 2[\sigma(s) - u_*(t)]^+ dx \geq 0.
\]

Here noting that there exists a monotone increasing function \(\ell_4(\cdot)\) such that \(2(|u_*(t) + |u^*(t)|) \leq \ell_4((sup_{0 \leq s \leq t} |U(s)|_{L^\infty})\) in view of (32) and (36), we find that there exists
a monotone increasing function $\ell_5(\cdot)$ such that
\[
|\sigma(t)|_{L^\infty} \leq |\sigma_0|_{L^\infty} + \ell_4\left( \sup_{0 \leq s \leq t} |U(s)|_{L^\infty} \right) + \int_0^t \ell_5(\sigma(s))_{L^\infty} \, ds. \tag{38}
\]
Here put
\[
X(t) = \sup_{0 \leq s \leq t} |\sigma(s)|_{L^\infty}, \quad Y(t) = \sup_{0 \leq s \leq t} |U(s)|_{L^\infty},
\]
then (28) and (38) give
\[
Y(t) \leq |U_0|_{L^\infty} + \int_0^t \ell_1(X(s) + Y(s)) \, ds, \tag{39}
\]
\[
X(t) \leq |\sigma_0|_{L^\infty} + \ell_4(Y(t)) + \int_0^t \ell_5(X(s) + Y(s)) \, ds. \tag{40}
\]
Therefore $Z(t) := X(t) + Y(t)$ satisfies
\[
Z(t) \leq Z(0) + \ell_4\left( Z(0) + \int_0^t \ell_1(Z(s)) \, ds \right) + \int_0^t \ell_5(Z(s)) \, ds, \tag{41}
\]
where $\ell_6(r) = \ell_1(r) + \ell_5(r)$.

Here we prepare the following lemma, which is a generalization of Lemma 2.2 of [20], which corresponds to the case where $m_1(\cdot) \equiv 0$ in (42) below.

**Lemma 3.1.** Let $y(t)$ be a measurable function on $[0,T]$ and suppose that there exist $y_0$, $y_1 \in \mathbb{R}$ and monotone non-decreasing functions $m_1(\cdot), m_2(\cdot), m_3(\cdot) : \mathbb{R} \to [0,\infty)$ such that
\[
y(t) \leq y_0 + m_1\left( y_1 + \int_0^t m_2(y(s)) \, ds \right) + \int_0^t m_3(y(s)) \, ds \quad \text{a.e.} \ t \in (0, T). \tag{42}
\]
Then there exists a number $T_0$ depending on $y_0$, $y_1$, $m_1(\cdot)$, $m_2(\cdot)$ and $m_3(\cdot)$ such that
\[
y(t) \leq y_0 + m_1(y_1 + 1) + 3 \quad \text{a.e.} \ t \in (0, T_0). \tag{43}
\]

**Proof.** Put $z(t) = y_0 + m_1\left( y_1 + \int_0^t m_2(y(s)) \, ds \right) + \int_0^t m_3(y(s)) \, ds$, then $z(t)$ is a monotone non-decreasing function defined for all $t \in [0, T]$ and $y(t) \leq z(t)$ a.e. $t \in (0, T)$. So $z(t)$ satisfies
\[
z(t) \leq y_0 + m_1\left( y_1 + \int_0^t m_2(z(s)) \, ds \right) + \int_0^t m_3(z(s)) \, ds \quad \forall t \in (0, T). \tag{44}
\]
We here claim that there exists a number $T_0$ depending on $y_0$, $y_1$, $m_1(\cdot)$, $m_2(\cdot)$ and $m_3(\cdot)$ such that
\[
z(t) \leq L := y_0 + m_1(y_1 + 1) + 3 \quad \forall t \in [0, T_0]. \tag{45}
\]
We put
\[
t_1 := \sup \left\{ t \in [0, T) ; z(s) \leq L \quad \forall s \in [0, t] \right\}. \tag{46}
\]
Since the set of discontinuous points of a monotone non-decreasing function is at most countable, there exist $\rho_1 \in (0, 1)$ and $\rho_2 \in (0, \rho_1)$ such that
\[
m_1(r) \text{ is continuous at } r = y_1 + \rho_1, \tag{47}
\]
\[
m_1(r) \text{ is continuous at } r = y_1 + \int_0^{t_1} m_2(z(s)) \, ds + \rho_2. \tag{48}
\]
Hence, by (47), we can take $T_0 \in (0, T]$ such that
\[
m_1(y_1 + \rho_1 + m_2(L) T_0) \leq m_1(y_1 + \rho_1) + 1, \quad m_3(L) T_0 \leq 1. \tag{49}
\]
We note that (46) gives
\[ z(t) \leq L \quad \forall t \in (0, t_1) \quad \text{and} \quad z(t) > L \quad \forall t \in (t_1, T]. \tag{50} \]
Suppose that (45) does not hold, then we get \( t_1 < T_0 \). Take \( t = t_0^1 := t_1 + 1/n \) in (44) and let \( n \to \infty \). Then from (48), (50) and (49), we derive
\[
L \leq y_0 + m_1(y_1 + \rho_2 + \int_0^{t_1} m_2(z(s)) \, ds) + \int_0^{t_1} m_3(z(s)) \, ds
\leq y_0 + m_1(y_1 + \rho_2 + m_2(L) T_0) + m_3(L) T_0
\leq y_0 + m_1(y_1 + \rho_1 + m_2(L) T_0) + m_3(L) T_0
\leq y_0 + m_1(y_1 + \rho_1) + 2 < L = y_0 + m_1(y_1 + 1) + 3,
\]
which leads to a contradiction. Thus we verify (45), which implies (43).

Then applying Lemma 3.1 with \( y_0 = y_1 = Z(0) \), \( m_1(\cdot) = \ell_4(\cdot) \), \( m_2(\cdot) = \ell_1(\cdot) \), \( m_3(\cdot) = \ell_0(\cdot) \), we can deduce that there exists \( T_0 \in (0, T] \) depending on \( Z(0) \), \( \ell_1(\cdot) \), \( \ell_4(\cdot) \) and \( \ell_0(\cdot) \) such that
\[ Z(t) \leq Z(0) + \ell_4(Z(0) + 1) + 3 \quad \text{a.e.} \ t \in (0, T_0). \]
Hence taking \( M \) large enough such that \( M = Z(0) + \ell_4(Z(0) + 1) + 4 \), we conclude
\[ |\sigma(t)|_{L^\infty} + |U(t)|_{L^\infty} \leq M - 1 \quad \text{a.e.} \ t \in (0, T_0). \tag{51} \]
Consequently (51) yields
\[ \partial I_M(\sigma(t)) = \partial I_M(u_i(t)) = \{0\} \quad (i = 1, \ldots, m) \quad \text{a.e.} \ t \in (0, T_0). \tag{52} \]

A Priori Estimates in \( L^2(\Omega) \):

Furthermore, by (H3) and (51), we can regard \( g, \bar{X}, h_i, \bar{\mu}_i \) as global Lipschitz functions, by which we can establish further a priori estimates in \( L^2(\Omega) \) (cf. [2, 16]).

For simplicity of the notation we denote in the sequel by \( C(\alpha, \beta, \cdots) \) various positive constants whose value can vary from line to line (and possibly depend on the parameters \( \alpha, \beta, \cdots \)). Multiplying (13) by \( u_i'(t) \), we get by (51) and (20)
\[
|u_i'(t)|^2_H + \frac{1}{2} \frac{d}{dt} |\nabla u_i(t)|^2_H \leq C^1_M |u_i'(t)|_H + C_M |\nabla u_i(t)|_H |u_i'(t)|_H,
\]
where \( C^1_M = \sup \{|h_i(\sigma, U)|_{|\Omega|^{1/2}}; |\sigma| + |U| \leq M\} \). Then Gronwall’s inequality gives
\[
\sup_{0 \leq t \leq T_0} |\nabla u_i(t)|^2_H + \int_0^{T_0} |u_i'(t)|^2_H \, dt \leq C(M, |\nabla u_{i0}|_H). \tag{53} \]
We next multiply (13) by \( -\Delta u_i \), then by the same reasoning as for (53), we get
\[
\sup_{0 \leq t \leq T_0} |\nabla u_i(t)|^2_H + \int_0^{T_0} |\Delta u_i(t)|^2_H \, dt \leq C(M, |\nabla u_{i0}|_H). \tag{54} \]
Thus we obtain
\[
\sup_{0 \leq t \leq T_0} \|U(t)\|^2_{L^2} + \|U'(t)^2\|_{L^2(0, T_0; L^2(\Omega))} + |\Delta U|^2_{L^2(0, T_0; L^2(\Omega))} \leq C(M, |\nabla U_0|_{L^2}). \tag{55} \]

Here in order to derive a priori estimates for \( \sigma \) in \( L^2(\Omega) \), we prepare the following lemma.
Lemma 3.2. Let \( \{ \sigma, U \} \) be a solution of (12)-(15) on \([0, T_0]\) satisfying (8). Then the function
\[
(I_U^U(\sigma))(t) = \frac{1}{\delta^2} [\sigma(t) - f^*(U(t))]^+ + \frac{1}{2\delta} [f_*(U(t)) - \sigma(t)]^+ \]
is absolutely continuous on \([0, T_0]\) and it holds that
\[
\frac{d}{dt} I_U^U(\sigma) \leq (\partial I_U^U(\sigma), \sigma')_H + C_M |\partial I_U^U(\sigma)|_H \left( \sum_{i=1}^m |u_i'|_H \right) \quad a.e. \ t \in (0, T_0).
\]

Proof. The assertion follows from the following direct calculation.
\[
\frac{d}{dt} I_U^U(\sigma) \sigma' - \sum_{i=1}^m \frac{\partial f_*(U)}{\partial u_i}(U) u_i'(t) \bigg|_H \frac{1}{\delta} \bigg( [f_*(U) - \sigma]^+ \bigg) \bigg( \frac{\partial f_*(U)}{\partial u_i}(U) u_i'(t) - \sigma' \bigg) \bigg|_H \leq (\partial I_U^U(\sigma), \sigma')_H + C_M \frac{1}{\delta} [\sigma - f^*(U)]^+ + \frac{1}{\delta} [f_*(U) - \sigma]^+ \bigg( \sum_{i=1}^m |u_i'|_H \bigg)
\]
\[
= (\partial I_U^U(\sigma), \sigma')_H + C_M |\partial I_U^U(\sigma)|_H \left( \sum_{i=1}^m |u_i'|_H \right).
\]

We first multiply (12) by \( \sigma' \), then by Lemma 3.2 and (19), we get
\[
\frac{1}{\delta} [\sigma - f^*(U)]^+ + \frac{1}{\delta} [f_*(U) - \sigma]^+ - \frac{1}{2} \Delta \sigma_H \leq C_M |\partial I_U^U(\sigma)|_H \left( \sum_{i=1}^m |u_i'|_H \right) + |g(\sigma, U)|_H |\sigma'|_H + C_M |\nabla \sigma|_H |\sigma'|_H
\]
\[
\leq \frac{1}{4} |\partial I_U^U(\sigma)|_H^2 + \frac{1}{2} |\sigma'|_H^2 + C_M \left( \sum_{i=1}^m |u_i'|_H \right)^2 + C_M \left( |\nabla \sigma|_H^2 + 1 \right).
\]

Here we note that
\[
(\partial I_U^U(\sigma), -\Delta \sigma)_H = \frac{1}{\delta} [\sigma - f^*(U)]^+ - \frac{1}{\delta} [f_*(U) - \sigma]^+ - \Delta \sigma
\]
\[
= \frac{1}{\delta} \nabla [\sigma - f^*(U)]^+ - \frac{1}{\delta} \nabla [f_*(U) - \sigma]^+, \nabla \sigma)_H
\]
\[
= \frac{1}{\delta} \nabla [\sigma - f^*(U)]^+ \bigg( \sigma - f^*(U) \bigg)_H + \frac{1}{\delta} \nabla [\sigma - f^*(U)]^+, \nabla f^*(U) \bigg)_H
\]
\[
= \frac{1}{\delta} \nabla [\sigma - f^*(U)]^+ \bigg( \sigma - f^*(U) \bigg)_H + \frac{1}{\delta} \nabla [\sigma - f^*(U)]^+, -\Delta f^*(U) \bigg)_H
\]
\[
+ \frac{1}{\delta} \nabla [f_*(U) - \sigma]^+ \bigg( \sigma - f^*(U) \bigg)_H + \frac{1}{\delta} \nabla [f_*(U) - \sigma]^+, \Delta f_*(U) \bigg)_H
\]
Applying Young’s inequality, we obtain

\[ \geq -\frac{1}{8\sigma^2} \left\{ \left[ |\sigma - f^*(U)|^2 \right]_H + \left[ |f_*(U) - \sigma| \right]^2_H \right\} \]

\[ - 2 |\Delta f^*(U)|^2_H - 2 |\Delta f_*(U)|^2_H \]

\[ \geq -\frac{1}{8} \left| \partial I^U_\delta(\sigma) \right|^2_H - 2 |\Delta f^*(U)|^2_H - 2 |\Delta f_*(U)|^2_H. \quad (58) \]

We also have, by (H3)

\[
(g(\sigma, U), -\Delta \sigma)_H \leq \int_\Omega \left\{ \frac{\partial g(\sigma, U)}{\partial \sigma} \left| \nabla \sigma \right|^2 + \sum_{i=1}^m \frac{\partial g(\sigma, U)}{\partial u_i} \nabla u_i \cdot \nabla \sigma \right\} \, dx
\]

\[ \leq CM \left( |\nabla \sigma|_H^2 + |\nabla U|^2 \right), \quad (59) \]

where \( C_M = C^u_M = \sup \{ \left| \frac{\partial g}{\partial \sigma}(\sigma, U) \right|_{L\infty} + \sum_{i=1}^m \left| \frac{\partial g}{\partial u_i}(\sigma, U) \right|_{L\infty} : |\sigma|_{L\infty}, |U|_{L\infty} \leq M \} \).

Multiplying (12) by \(-\Delta \sigma\) and using (19), (58) and (59), we get

\[
\frac{1}{2} \frac{d}{dt} |\nabla \sigma|_H^2 + \frac{1}{2} |\Delta \sigma|_H^2 \leq -\frac{1}{8} \left| \partial I^U_\delta(\sigma) \right|^2_H + 2 |\Delta f^*(U)|^2_H + 2 |\Delta f_*(U)|^2_H
\]

\[ + C_M \left( |\nabla \sigma|_H^2 + |\nabla U|^2 \right) + \frac{C^2_M}{2} |\nabla \sigma|_H^2. \quad (60) \]

Next multiplying (12) by \( \partial I^U_\delta(\sigma) \) and using Lemma 3.2, (19) and (58), we have

\[
\frac{d}{dt} I^U_\delta(\sigma) + |\partial I^U_\delta(\sigma)|^2_H
\]

\[ \leq C_M \left| \partial I^U_\delta(\sigma) \right|_H \left( \sum_{i=1}^m |u'_i|_H \right) + \frac{1}{8} \left| \partial I^U_\delta(\sigma) \right|^2_H + 2 |\Delta f^*(U)|^2_H + 2 |\Delta f_*(U)|^2_H
\]

\[ + |g(\sigma, U)|_H |\partial I^U_\delta(\sigma)|_H + C_M |\nabla \sigma|_H |\partial I^U_\delta(\sigma)|_H. \]

Applying Young’s inequality, we obtain

\[
\frac{d}{dt} I^U_\delta(\sigma) + \frac{1}{2} |\partial I^U_\delta(\sigma)|^2_H \leq 2m C^2_M |U'|^2_{L^2} + 2 |\Delta f^*(U)|^2_H + 2 |\Delta f_*(U)|^2_H
\]

\[ + 2 C^2_M \left( \left| \nabla \sigma \right|^2_H + 1 \right). \quad (61) \]

We here note that

\[
|\Delta f_*(U)|^2_H + |\Delta f^*(U)|^2_H \leq C_M \left( \sum_{i=1}^m |\nabla u_i|^4_{L^4(\Omega)} + \sum_{i=1}^m |\Delta u_i|_H^2 \right),
\]

\[
|\nabla u_i|^4_{L^4} = \sum_{j,k=1}^N \int_\Omega (\partial_j u_i)^2 (\partial_k u_i)^2 \, dx = \sum_{j,k=1}^N \int_\Omega (\partial_k (\partial_j u_i)^2 (\partial_k u_i) \, u_i \, dx
\]

\[ \leq |u_i|_{L^\infty} \sum_{j,k=1}^N \int_\Omega ((\partial_j u_i)^2 |\partial_k u_i|^2 + 2 |\partial_j u_i \partial_k u_i | \partial \partial_j u_i) \, dx
\]

\[ \leq C_M \left| \nabla u_i \right|^{\frac{4}{3}}_{L^4} \left( |\Delta u_i|_H + |u_i|_H \right)
\]

\[ \leq \frac{1}{2} |\nabla u_i|^4_{L^4} + C_M \left( \left| \nabla u_i \right|^2_{L^2} + 1 \right). \]

Hence we obtain

\[
|\Delta f_*(U)|^2_H + |\Delta f^*(U)|^2_H \leq C_M \left( 1 + |\Delta U|^2_{L^2} \right). \quad (62) \]

Thus, in view of (60), (61) and (62), we get

\[
\frac{d}{dt} \left\{ I^U_\delta(\sigma) + \frac{1}{2} |\nabla \sigma|^2_H \right\} + \frac{1}{2} |\Delta \sigma|^2_H + \frac{3}{8} |\partial I^U_\delta(\sigma)|^2_H
\]
\[ \leq C_M (1 + \|\nabla \sigma \|_H^2 + \|\nabla U \|_{L^2}^2 + \|U' \|_{L^2}^2 + \|\Delta U \|_{L^2}^2). \]  

(63)

Now adding (57) and (63), we obtain
\[
\frac{d}{dt} \left\{ 2 I_{\theta}^U (\sigma) + \|\nabla \sigma \|_H^2 \right\} + \frac{1}{2} \|\Delta \sigma \|_H^2 + \frac{1}{2} \|\sigma' \|_H^2 + \frac{1}{8} \|\partial I_{\theta}^U (\sigma) \|_H^2 
\leq C_M (1 + \|\nabla \sigma \|_H^2 + \|\nabla U \|_{L^2}^2 + \|U' \|_{L^2}^2 + \|\Delta U \|_{L^2}^2). \]

(64)

Consequently, from (55), we derive
\[
\sup_{0 \leq t \leq T_0} \|\nabla \sigma \|_H^2 + \|\sigma' \|_{L^2(0, T_0; H)}^2 + \|\Delta \sigma \|_{L^2(0, T_0; H)}^2 
+ \|\partial I_{\theta}^U (\sigma) \|_{L^2(0, T_0; H)}^2 \leq C(M, \|\nabla U_0 \|_{L^2}, \|\nabla \sigma_0 \|_H). \]

(65)

3.1.3. Convergence. Thus (55) and (65) assure the uniform estimates for solutions \((\sigma, U)\) of (12)-(15) denoted by \((\sigma_\delta, U_\delta)\) with respect to the parameter \(\delta\). More precisely, \(\{\sigma_\delta\}_{\delta > 0}, \{u_{i, \delta}\}_{\delta > 0} \) (\(i = 1, \cdots, m\)) are bounded in \(W^{1,2}(0, T_0; L^2(\Omega)) \cap L^\infty(0, T_0; V) \cap L^2(0, T_0; H^2(\Omega)), \{\partial I_{\theta}^U (\sigma_\delta)\}_{\delta > 0}\) is bounded in \(L^2(0, T_0; L^2(\Omega))\), and \(\{U_{\delta}(\sigma_\delta)\}_{\delta > 0}\) is bounded in \(L^\infty(0, T)\). Therefore, by Ascoli’s Theorem, we can extract a subsequence \(\delta_n \searrow 0\) such that
\[
\sigma_{\delta_n} \rightarrow \sigma \quad \text{strongly in } C([0, T_0]; L^2(\Omega)) \text{ and } \text{a.e. } (x, t) \in Q_{T_0},
\]
weakly star in \(L^\infty(0, T_0; L^\infty(\Omega) \cap V), \Delta \sigma_{\delta_n} \rightarrow \Delta \sigma \quad \text{weakly in } L^2(0, T_0; L^2(\Omega)), \sigma'_{\delta_n} \rightarrow \sigma' \quad \text{weakly in } L^2(0, T_0; L^2(\Omega)), u_{i, \delta_n} \rightarrow u_i \quad \text{strongly in } C([0, T_0]; L^2(\Omega)) \text{ and } \text{a.e. } (x, t) \in Q_{T_0},
\]
weakly star in \(L^\infty(0, T_0; L^\infty(\Omega) \cap V), \Delta u_{i, \delta_n} \rightarrow \Delta u_i \quad \text{weakly in } L^2(0, T_0; L^2(\Omega)), u_{i, \delta_n} \rightarrow u_i \quad \text{weakly in } L^2(0, T_0; L^2(\Omega)), g(\sigma_{\delta_n}, U_{\delta_n}) \rightarrow g(\sigma, U) \quad \text{strongly in } L^2(0, T_0; L^2(\Omega)), h_i(\sigma_{\delta_n}, U_{\delta_n}) \rightarrow h_i(\sigma, U) \quad \text{strongly in } L^2(0, T_0; L^2(\Omega)), \nabla \cdot \tilde{\lambda}(\sigma_{\delta_n}) \rightarrow \nabla \cdot \tilde{\lambda}(\sigma) \quad \text{weakly in } L^2(0, T_0; L^2(\Omega)), \nabla \cdot \tilde{\mu}(u_{i, \delta_n}) \rightarrow \nabla \cdot \tilde{\mu}(u_i) \quad \text{weakly in } L^2(0, T_0; L^2(\Omega)), \partial I_{\delta_n}^{U_{\delta_n}}(\sigma_{\delta_n}) \rightarrow g \quad \text{weakly in } L^2(0, T_0; L^2(\Omega)).
\]
We also note that the point-wise convergence of \(\sigma_{\delta_n}\) and \(U_{\delta_n}\) gives \(f_*(U) \leq \sigma \leq f_*(U)\) a.e. in \(Q_{T_0}\), which assures
\[
\sigma(t) \in D(I_{U(t)}), \text{ i.e., } I_{U(t)}^{U(t)}(\sigma(t)) = 0 \quad \text{for a.e. } t \in (0, T_0). \]

(66)

For any \(w(t) \in D(I_{U(t)})\), we put
\[ w_n(t) = \max \{ f_*(U_{\delta_n}(t)), \min \{ f_*(U_{\delta_n}(t)), w(t) \} \}. \]

Then it is easy to see that \(w_n(t) \in D(I_{U_{\delta_n}(t)})\) and
\[ w_n(t) \rightarrow w(t) \quad \text{ a.e. } (x, t) \in Q_{T_0} \quad \text{ and strongly in } L^2(Q_{T_0}), \]
where we used Lebesgue’s dominant convergence theorem. Then, since
\[ \partial I_{\delta_n}^{U_{\delta_n}}(\sigma_{\delta_n}) \in \partial I_{U_{\delta_n}}(J_n \sigma_{\delta_n}), \quad J_n = (I + \delta_n \partial I_{U_{\delta_n}})^{-1}, \]
we see that the definition of subdifferential operator gives
\[ 0 = I_{U,\sigma_n}(u_n(t)) - I_{U,\sigma_n}(J_n \sigma_{\delta_n}(t)) \]
\[ \geq \langle \partial I_{U,\sigma_n}^{\delta_n}(\sigma_{\delta_n}(t)), w_n(t) - J_n \sigma_{\delta_n}(t) \rangle_H. \]  
(68)

Here recalling that \(|J_n \sigma_{\delta_n} - \sigma_{\delta_n}|_H = \delta_n \| \partial I_{U,\sigma_n}^{\delta_n}(\sigma) \|_H\), we get
\[ J_n \sigma_{\delta_n} \to \sigma \quad \text{strongly in } L^2(Q_{T_0}). \]  
(69)

Let \(n \to \infty\) in (68). Then since \(I_{U,\sigma_n}(\cdot)\) converges to \(I_u(\cdot)\) in the sense of Mosco (see [17]), (67) and (69) yield
\[ I_u(w(t)) - I_u(\sigma(t)) = 0 \geq \langle g(t), w(t) - \sigma(t) \rangle_H \quad \forall w \in D(I_{U}(t)), \]
which implies that \(g(t) \in \partial I_{U}(\sigma(t))\) for a.e. \(t \in (0, T_0)\). Thus we see that \((\sigma, U)\) gives a solution of (1)-(4).

3.2. Proof of Theorem 2.3: Global solution. In this subsection, we give a proof for Theorem 2.3. The strategy of the proof is the same as in the previous section. So in order to derive the global solution, it suffices to establish a priori estimates global in \(t \in [0, T]\) under (H4).

Plugging (6) of (H4) in \(I_2\) given by (26), we get
\[ I_2 = \int_\Omega h_2(\sigma, U) u_i |u_i|^\gamma - 2 dx \leq C \int_\Omega \left( |\sigma|^2 + |U|^2 + 1 \right) |u_i|^\gamma - 2 dx \]
\[ \leq C \left( |\sigma|_{L^r}^2 + |U|_{L^r}^2 + 1 \right) |u_i|^\gamma - 2. \]  
(70)

Then dividing both sides of (24) by \(|u_i|^\gamma - 2\) (see Remark 2 and apply Proposition 1 with \(k = 2\)), we obtain by (25) and (70)
\[ \frac{1}{2} \frac{d}{dt} |u_i(t)|_{L^r}^2 \leq C \left( |\sigma|_{L^r}^2 + |U|_{L^r}^2 + 1 \right) + \frac{1}{2(r - 1)} |(\tilde{\mu}_i)'(u_i)|_{L^\infty}^2 |u_i|_{L^r}^2. \]
Integrating this with respect to \(t\) over \((0, t)\) and letting \(r \to \infty\), we obtain
\[ |U(t)|_{L^\infty}^2 \leq |U_0|_{L^\infty}^2 + \int_0^t C \left( |U(s)|_{L^\infty}^2 + |\sigma(s)|_{L^\infty}^2 + 1 \right) ds. \]  
(71)

Now we look at \(J_3\) in (33). Making use of (5) and recalling that \(u_*(t) \geq 0\), we now get
\[ J_3 = \int_\Omega g(\sigma, U) |(\sigma - u_*(t))|^\gamma - 1 dx \]
\[ \leq \int_\Omega g(\sigma, U) |(\sigma - u_*(t))|^\gamma - 1 dx \]
\[ \leq \int_\Omega g(\sigma, U) |(\sigma - u_*(t))|^{\gamma - 2} dx \]
\[ \leq C \int_\Omega \left( |\sigma|^2 + |U(s)|^2 + 1 \right) |(\sigma - u_*(t))|^\gamma - 2 dx \]
\[ \leq C \left( |\sigma|_{L^r}^2 + |U(s)|_{L^r}^2 + 1 \right) |(\sigma - u_*(t))|^\gamma - 2. \]  
(72)

Then we obtain
\[ |(\sigma - u_*(t))|^\gamma_{L^\infty} \leq |\sigma_0 - u_*(t)|_{L^\infty}^\gamma + \int_0^t C \left( |U(s)|_{L^\infty}^2 + |\sigma(s)|_{L^\infty}^2 + 1 \right) ds. \]  
(73)
We here note that (7) gives \(|u_*(t)| + |u^*(t)| \) ≤ \(C \sup_{0 \leq s \leq t} |U(s)|_{L^\infty}\). Thus in parallel with (38), we now have

\[
|\sigma(t)|_{L^\infty}^2 \leq C_4(|\sigma_0|_{L^\infty}^2 + \sup_{0 \leq s \leq t} |U(s)|_{L^\infty}^2) + \int_0^t C_5 \left( |U(s)|_{L^\infty}^2 + |\sigma(s)|_{L^\infty}^2 + 1 \right) ds \tag{74}
\]

for some constants \(C_4\) and \(C_5\). Here put

\[
\hat{X}(t) = \sup_{0 \leq s \leq t} |\sigma(s)|_{L^\infty}^2, \quad \hat{Y}(t) = \sup_{0 \leq s \leq t} |U(s)|_{L^\infty}^2,
\]

then (71) and (74) give

\[
\hat{Y}(t) \leq |U_0|_{L^\infty}^2 + \int_0^t C \left( \hat{X}(s) + \hat{Y}(s) + 1 \right) ds, \tag{75}
\]

\[
\hat{X}(t) \leq C_4(|\sigma_0|_{L^\infty}^2 + \hat{Y}(t)) + \int_0^t C_5(\hat{X}(s) + \hat{Y}(s) + 1) ds. \tag{76}
\]

Therefore (75) \(\times (C_4 + 1)\) + (76) yields

\[
\hat{X}(t) + \hat{Y}(t) \leq (C_4 + 1)(|U_0|_{L^\infty}^2 + |\sigma_0|_{L^\infty}^2) + (C(C_4 + 1) + C_5) \int_0^t (\hat{X}(s) + \hat{Y}(s)) ds.
\]

Thus applying Gronwall’s inequality, we can derive the a priori bound for \(\sup_{0 \leq t \leq T} |\sigma(t)|_{L^\infty} + |U(t)|_{L^\infty}\). To complete the proof of Theorem 2.3, it suffices to repeat the same procedures as in the previous section.

3.3. **Proof of Theorem 2.4: Uniqueness.** In this subsection, we give a proof for Theorem 2.4. Let \(\{\sigma_1, U_1\}\) and \(\{\sigma_2, U_2\}\) be two solutions of the system (1)-(4) on \([0, S]\) in the sense of Definition 2.1 satisfying (8) with \(T_0\) replaced by \(S\). We denote \(\sigma = \sigma_1 - \sigma_2, \quad U = U_1 - U_2\). For simplicity of the notation in the sequel we denote by \(C\) various positive constants. We define the auxiliary function \(L(x, t)\) by

\[
L(x, t) := \max \{ |f_s(U_1(x, t)) - f_s(U_2(x, t))|, |f^*(U_1(x, t)) - f^*(U_2(x, t))| \}
\]

and put

\[
\tilde{\sigma}_1(x, t) = \sigma_1(x, t) - [\sigma(x, t) - L(x, t)]^+, \quad \tilde{\sigma}_2(x, t) = \sigma_2(x, t) + [\sigma(x, t) - L(x, t)]^+. \tag{77}
\]

We here claim that

\[
f_s(U_1(x, t)) \leq \tilde{\sigma}_1(x, t) \leq f^*(U_1(x, t)) \quad \text{a.e. in } Q_S \quad (i = 1, 2). \tag{79}
\]

In fact, for \(i = 1\) it is clear that \(\tilde{\sigma}_1(x, t) \leq \sigma_1(x, t) \leq f^*(U_1(x, t)) \) a.e. in \(Q_S\). The estimate from below of (79) is obvious for the case \([\sigma(x, t) - L(x, t)]^+ = 0\). As for the case where \([\sigma(x, t) - L(x, t)]^+ > 0\), we get

\[
\tilde{\sigma}_1(x, t) = \sigma_1(x, t) - \sigma(x, t) + L(x, t)
\]

\[
\geq \sigma_2(x, t) + f_s(U_1(x, t)) - f_s(U_2(x, t)) \geq f_s(U_1(x, t)) \quad \text{a.e. in } Q_S.
\]

Thus (79) is verified for \(i = 1\) and the verification for \(i = 2\) can be done analogously.

We get two inequalities by putting \(\sigma = \sigma_1, \quad z = \tilde{\sigma}_1\) and \(\sigma = \sigma_2, \quad z = \tilde{\sigma}_2\) in (ii)b of Definition 2.1. Taking the difference of these two inequalities, we have

\[
(\sigma'(t), [\sigma(t) - L(t)]^+)_H + a(\sigma(t), [\sigma(t) - L(t)]^+)
\leq (g(\sigma_1(t), U_1(t)) - g(\sigma_2(t), U_2(t)), [\sigma(t) - L(t)]^+)_H
\]

\[
+ \left( (\tilde{\lambda})'(\sigma_1(t)) \cdot \nabla \sigma_1(t) - (\tilde{\lambda})'(\sigma_2(t)) \cdot \nabla \sigma_2(t), [\sigma(t) - L(t)]^+ \right)_H.
\]
Hence we obtain
\[
\frac{1}{2} \frac{d}{dt} \left| \sigma(t) - L(t) \right|_H^+ + |\nabla \sigma(t) - L(t)|_H^+ + |L'(t)|_H^+ + a(L(t), \sigma(t) - L(t))^+ \\
\leq (g(\sigma_1(t), U_1(t)) - g(\sigma_2(t), U_2(t)), |\sigma(t) - L(t)|_H^+) \\
+ \left( (\vec{\lambda})'(\sigma_1(t)) \cdot \nabla \sigma_1(t) - (\vec{\lambda})'(\sigma_2(t)) \cdot \nabla \sigma_2(t), [\sigma(t) - L(t)]^+ \right)_H. \tag{80}
\]

Denote
\[
I_1(t) = (L'(t), [\sigma(t) - L(t)]^+)_H \quad \text{and} \quad I_2(t) = a(L(t), [\sigma(t) - L(t)]^+).
\]

In order to estimate terms \(I_1\) and \(I_2\), we need estimates for the derivatives of \(L\).

For simplicity of the notation, put \(A_\ast = |f_\ast(U_1(x, t)) - f_\ast(U_2(x, t))|\) and \(A^* = |f^*(U_1(x, t)) - f^*(U_2(x, t))|\). Then we have
\[
|L'(x, t)| = \begin{cases} 
|f_\ast(U_1) U'_1(x, t) - f_\ast(U_2) U'_2(x, t)| & \text{if } A_\ast \geq A^* \\
|f^*(U_1) U_1'(x, t) - f^*(U_2) U_2'(x, t)| & \text{if } A_\ast \leq A^*
\end{cases}
\]
\[
\leq \begin{cases} 
|f_\ast(U_1) U'(x, t) + (f_\ast(U_1) - f_\ast(U_2)) U'_2(x, t)| & \text{if } A_\ast \geq A^* \\
|f^*(U_1) U'(x, t) + (f^*(U_1) - f^*(U_2)) U'_2(x, t)| & \text{if } A_\ast \leq A^*
\end{cases}
\]
\[
\leq C |U'(x, t)| + C |U(x, t)| |U'_2(x, t)|. \tag{81}
\]

Analogously we have
\[
|\nabla L(x, t)| \leq C |\nabla U(x, t)| + C |U(x, t)| |\nabla U'_2(x, t)|. \tag{82}
\]

Therefore, since \(H^1 \subset L^4\) for \(N \leq 4\), we get
\[
|I_1(t)| \leq C \int_\Omega |U'(t)||\sigma(t) - L(t)|^+_H d x \\
+ C \int_\Omega |U(t)||U'_2(t)||\sigma(t) - L(t)|^+_H d x \\
\leq \varepsilon |U'(t)|_{L^2}^2 + C_\varepsilon |\sigma(t) - L(t)|^+_H d x \\
+ C |U(t)||U'_2(t)||\sigma(t) - L(t)|^+_H d x \\
\leq \varepsilon |U'(t)|_{L^2}^2 + C_\varepsilon |\sigma(t) - L(t)|^+_H d x + \varepsilon |\sigma(t) - L(t)|^+_H d x \\
+ C_\varepsilon |U(t)||U'_2(t)||\nabla U'_2(t)||_{H^1}^2 \quad \text{a.e. } t \in (0, S). \tag{83}
\]

Analogously we have
\[
|I_2(t)| \leq C \int_\Omega |\nabla U(t)||\nabla [\sigma(t) - L(t)]^+_H d x \\
+ C \int_\Omega |U(t)||\nabla U_2(t)||\nabla [\sigma(t) - L(t)]^+_H d x \\
\leq \varepsilon |\nabla [\sigma(t) - L(t)]^+_H|_{L^2}^2 + C_\varepsilon |\nabla U(t)||_{L^2}^2 \\
+ C_\varepsilon |U(t)||_{H^1}|\nabla U_2(t)||_{H^1}^2 \quad \text{a.e. } t \in (0, S). \tag{84}
\]

As for the terms on the right hand side of (80), we get by (H3) and assumption \(\vec{\lambda} \in C^2(\mathbb{R}; \mathbb{R}^N)\)
\[
| (g(\sigma_1(t), U_1(t)) - g(\sigma_2(t), U_2(t)), [\sigma(t) - L(t)]^+)_H | \\
\leq C \int_\Omega (|\sigma(t, x)| + |U(t, x)|)|\sigma(t, x) - L(t, x)|^+_H d x
\]
\[ \begin{align*}
\leq C\left(|\sigma(t)|_{L^2}^2 + |U(t)|_{L^2}^2 + |\sigma(t) - L(t)|_{L^2}^2 \right) \quad \text{a.e. } t \in (0, S),
\end{align*} \tag{85} \]
\[ \left| \left( \tilde{\lambda}'(\sigma_1(t)) \cdot \nabla \sigma_1(t) - (\tilde{\lambda}')(\sigma_2(t)) \cdot \nabla \sigma_2(t), [\sigma(t) - L(t)]^+ \right)_H \right| \]
\[ \leq \int_{\Omega} \left| (\tilde{\lambda}')(\sigma_1(t)) \cdot \nabla \sigma_1(t) + (\tilde{\lambda}')(\sigma_1(t) - (\tilde{\lambda})'(\sigma_2(t))) \cdot \nabla \sigma_2(t) \right| |\sigma(t) - L(t, x)|^+ dx \]
\[ \leq C \int_{\Omega} \left( |\nabla \sigma(t)| + |\sigma(t)||\nabla \sigma_2(t)| \right) |\sigma(t) - L(t, x)|^+ dx. \quad \text{ (86)} \]

We here put
\[ I_3(t) = C \int_{\Omega} |\nabla \sigma(t)| |\sigma(t) - L(t)|^+ dx, \]
\[ I_4(t) = C \int_{\Omega} |\sigma(t)||\nabla \sigma_2(t)| |\sigma(t) - L(t)|^+ dx. \]

Then in view of (82), we have
\[ I_3(t) \leq C \int_{\Omega} |\nabla \sigma(t) - L(t)|^+ |\sigma(t) - L(t)|^+ dx \]
\[ + C \int_{\Omega} |\nabla L(t)||\sigma(t) - L(t)|^+ dx \]
\[ \leq \varepsilon |\nabla \sigma(t) - L(t)|^2_{L^2} + C \varepsilon |\sigma(t) - L(t)|^2_{L^2} + C |\nabla L(t)|^2_{L^2} \]
\[ \leq \varepsilon |\nabla \sigma(t) - L(t)|^2_{L^2} + C \varepsilon |\sigma(t) - L(t)|^2_{L^2} \]
\[ + C |\nabla U(t)|^2_{H^1} + C |U(t)|^2_{H^1} |\nabla \sigma_2(t)|^2_{H^1}. \quad \text{ (87)} \]

\[ I_4(t) \leq C |\sigma(t)|_{L^2} |\nabla \sigma_2(t)|_{H^1} |\sigma(t) - L(t)|^+_{H^1} \]
\[ \leq \varepsilon |\sigma(t) - L(t)|^2_{H^1} + C \varepsilon |\sigma(t)|^2_{L^2} |\nabla \sigma_2(t)|^2_{H^1}. \quad \text{ (88)} \]

Then plugging (87) and (88) in (86), we get
\[ \left| \left( \tilde{\lambda}'(\sigma_1(t)) \cdot \nabla \sigma_1(t) - (\tilde{\lambda}')(\sigma_2(t)) \cdot \nabla \sigma_2(t), [\sigma(t) - L(t)]^+ \right)_H \right| \]
\[ \leq \varepsilon |\nabla \sigma(t) - L(t)|^2_{L^2} + C \varepsilon |\sigma(t) - L(t)|^2_{L^2} + C |\nabla L(t)|^2_{L^2} \]
\[ + |U(t)|^2_{H^1} + |U_2(t)|^2_{H^1} + C \varepsilon |\sigma(t)|^2_{L^2} |\nabla \sigma_2(t)|^2_{H^1}. \quad \text{ a.e. } t \in (0, S). \quad \text{ (89)} \]

Thus plugging (83), (84), (85) and (86) in (80), we obtain
\[ \frac{1}{2} \frac{d}{dt} \left| [\sigma(t) - L(t)]^+ \right|_{L^2}^2 + |\nabla \sigma(t) - L(t)|^2_{L^2} \]
\[ \leq \varepsilon |U'(t)|^2_{L^2} + C \varepsilon |\sigma(t) - L(t)|^2_{L^2} + C \varepsilon |\nabla \sigma(t) - L(t)|^2_{L^2} \]
\[ + C \left( |\nabla U(t)|^2_{L^2} + |U(t)|^2_{L^2} \right) \left( 1 + |U_2(t)|^2_{H^1} + |\nabla U_2(t)|^2_{H^1} \right) \]
\[ + C \left( |\sigma(t)|^2_{L^2} + |U_2(t)|^2_{L^2} \right) + C \varepsilon |\sigma(t)|^2_{L^2} |\nabla \sigma_2(t)|^2_{H^1} \quad \text{ a.e. } t \in (0, S). \quad \text{ (90)} \]

Here, instead of (77) and (78), we put
\[ \hat{\sigma}_1(x, t) = \sigma_1(x, t) + [-\sigma(x, t) - L(x, t)]^+, \quad \text{ (91)} \]
\[ \hat{\sigma}_2(x, t) = \sigma_2(x, t) - [-\sigma(x, t) - L(x, t)]^+. \quad \text{ (92)} \]

Then we can show
\[ f_i(U_i(x, t)) \leq \hat{\sigma}_i(x, t) \leq f^*(U_i(x, t)) \quad \text{ a.e. in } Q_S \quad (i = 1, 2). \quad \text{ (93)} \]
and can get two inequalities by putting \( \sigma = \sigma_1, \ z = \tilde{\sigma}_1 \) and \( \sigma = \sigma_2, \ z = \tilde{\sigma}_2 \) in (ii b) of Definition 2.1. Taking the difference of these two inequalities, we now get

\[
(-\sigma' (t), [-\sigma(t) - L(t)]^+)_H + a(-\sigma(t), [-\sigma(t) - L(t)]^+)
\]

\[
\leq -(g(\sigma_1(t), U_1(t)) - g(\sigma_2(t), U_2(t)), [-\sigma(t) - L(t)]^+)_H
\]

\[
-(\tilde{\lambda})(\sigma_1(t)) \cdot \nabla \sigma_1(t) - (\tilde{\lambda})(\sigma_2(t)) \cdot \nabla \sigma_2(t), [-\sigma(t) - L(t)]^+)_H.
\]

Thus analogously we can obtain

\[
\frac{1}{2} \frac{d}{dt} \left[ -\sigma(t) - L(t) \right]_2^2 + \left| \nabla [-\sigma(t) - L(t)]^+ \right|_2^2
\]

\[
\leq \varepsilon \left| U'(t) \right|_2^2 + C \varepsilon \left| [-\sigma(t) - L(t)]^+ \right|_2^2 + \varepsilon \left| \nabla [-\sigma(t) - L(t)]^+ \right|_2^2
\]

\[
+ C \left( \left| \nabla U(t) \right|_2^2 + \left| U(t) \right|_2^2 \right) \left( 1 + \left| U_2(t) \right|_2^2 + \left| \nabla U(t) \right|_2^2 \right)
\]

\[
+ C \left( \left| \sigma(t) \right|_2^2 + \left| U(t) \right|_2^2 \right) \left| \nabla \sigma_2(t) \right|_2^2, \ \text{a.e.} \ t \in (0, S). \quad (94)
\]

Now multiplying the difference of the respective parts of (2) by \( \tilde{u}_i = u_{i,1} - u_{i,2} \) and recalling that \( \tilde{\mu}_i \in C^2(\mathbb{R}; \mathbb{R}^N) \), we have

\[
\frac{1}{2} \frac{d}{dt} \left| \tilde{u}_i(t) \right|_2^2 + \left| \nabla \tilde{u}_i(t) \right|_2^2
\]

\[
= \left( (\tilde{\mu}_i)( \nabla u_{i,1} - (\tilde{\mu}_i)'(u_{i,2}) \nabla u_{i,2}, \tilde{u}_i) \right)_H + (h_i(\sigma_1, U_1) - h_i(\sigma_2, U_2), \tilde{u}_i)_H
\]

\[
\leq C \left( \left| \nabla \tilde{u}_i \right|_2^2 + \left| u_{i,1} \right|_2^2 \left| \nabla u_{i,1} \right|_2^2 \right) + C \left( \| \sigma \|_H + \| U \|_2^2 \right) \left| \tilde{u}_i \right|_H.
\]

Thus we obtain

\[
\frac{d}{dt} \left| U(t) \right|_2^2 + \left| \nabla U(t) \right|_2^2
\]

\[
\leq C \left( \left| U(t) \right|_2^2 + \left| \sigma(t) \right|_2^2 + \left| U(t) \right|_2^2 \left( 1 + \left| \nabla U_2(t) \right|_2^2 \right) \right), \ \text{a.e. in} \ (0, S). \quad (95)
\]

Analogously, multiplying the difference of the respective parts of (2) by \( \tilde{u}_i' \), we obtain

\[
\left| U'(t) \right|_2^2 + \frac{d}{dt} \left| \nabla U(t) \right|_2^2
\]

\[
\leq C \left( \left| U(t) \right|_2^2 + \left| \sigma(t) \right|_2^2 + \left| U(t) \right|_2^2 \left( 1 + \left| \nabla U_2(t) \right|_2^2 \right) \right), \ \text{a.e.} \ t \in (0, S). \quad (96)
\]

Now we claim that the following estimate holds

\[
|\sigma(x, t)| \leq L(x, t) + |\sigma(x, t) - L(x, t)|^+ + [-\sigma(x, t) - L(x, t)]^+ \text{ in } Q_S. \quad (97)
\]

To show this, it suffices to check for the following three cases:

1. Let \( \sigma(x, t) \leq -L(x, t) < 0 \). Then the l.h.s. = \( -\sigma \) and the r.h.s. = \( -\sigma - L = -\sigma \).
2. Let \( -L(x, t) \leq \sigma(x, t) \leq L(x, t) \). Then \( |\sigma - L|^+ = [-\sigma - L]^+ = 0 \) and the l.h.s. = \( |\sigma| \leq L = \text{r.h.s.} \).
3. Let \( 0 < L(x, t) \leq \sigma(x, t) \). Then \( [-\sigma - L]^+ = 0 \) and the l.h.s. = \( \sigma \) and the r.h.s. = \( L + \sigma - L = \sigma \).

Hence we get

\[
|\sigma(t)|_2^2 \leq C \left( |L(t)|_2^2 + ||\sigma(t) - L(t)||_2^2 + ||[-\sigma(t) - L(t)]^+||_2^2 \right).
\]

Since \( |L(t)|_2^2 \leq C \left| U(t) \right|_2^2 \), we then have

\[
|\sigma(t)|_2^2 \leq C \left( \left| U(t) \right|_2^2 + ||\sigma(t) - L(t)||_2^2 + ||[-\sigma(t) - L(t)]^+||_2^2 \right). \quad (98)
\]
Now making a linear combination of estimates (90), (94), (95) and (96) with suitably chosen small \( \varepsilon > 0 \), we obtain

\[
\frac{d}{dt} \left\{ ||\sigma(t) - L(t)||^2_{L^2} + ||-\sigma(t) - L(t)||^2_{L^2} + |U(t)|^2_{L^2} + |\nabla U(t)|^2_{L^2} \right\}
\leq C \left\{ ||\sigma(t) - L(t)||^2_{L^2} + ||-\sigma(t) - L(t)||^2_{L^2} + |\sigma(t)|^2_{L^2} + (|U(t)|^2_{L^2} + |\nabla U(t)|^2_{L^2}) \left( 1 + |U'_2(t)|^2_{L^2} + |\nabla U_2(t)|^2_{L^2} \right) + |\sigma(t)|^2_{L^2} + |\nabla \sigma_2(t)|^2_{H^1} \right\}. \tag{99}
\]

Put

\[ I(t) := ||\sigma(t) - L(t)||^2_{L^2} + ||-\sigma(t) - L(t)||^2_{L^2} + |U(t)|^2_{L^2} + |\nabla U(t)|^2_{L^2}. \]

Then, from (98) and (99), we deduce

\[
\frac{d}{dt} I(t) \leq a(t) I(t) \quad \text{a.e. } t \in (0, S),
\]

where

\[ a(t) = C \left( 1 + |U'_2(t)|^2_{L^2} + |\nabla U_2(t)|^2_{L^2} + |\nabla \sigma_2(t)|^2_{H^1} \right) \in L^1(0, S). \]

Thus, by Gronwall’s inequality and the fact \( I(0) = 0 \), we conclude that \( I(t) = 0 \) for \( t \in [0, S] \) which proves the uniqueness of solution. \( \square \)

3.4. Proof of Theorem 2.5: Positivity. In this section, we discuss the non-negativity of solutions. Put

\[ \sigma^-(t) := \max(-\sigma(t), 0) \quad \text{and} \quad u_i^-(t) = \max(-u_i(t), 0). \]

Multiplying (2) by \( u_i^-(t) \), we get by (9) and (10)

\[
\frac{1}{2} \frac{d}{dt} |u_i^-(t)|^2_{L^2} + |\nabla u_i^-(t)|^2_{L^2} \leq \int_{\Omega} (\bar{u}_i(t) - u_i(t)) \cdot u_i^-(t) \, dx - \int_{\Omega} (h_i(\sigma, U) - h_i(\sigma, U_i^0)) \, u_i^-(t) \, dx \leq \frac{1}{2} |\nabla u_i^-(t)|^2_{L^2} + C |u_i^-(t)|^2_{L^2},
\]

where \( U_i^0 = (u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_m) \). Since \( u_i^-(0) = 0 \), Gronwall’s inequality assures that \( u_i^-(t) \equiv 0 \), i.e., \( u(x, t) \geq 0 \) for all \( t \).

Multiply (1) by \( \sigma^-(t) \). Noting that \( f^* \geq 0 \) implies \( \langle \partial I^U(\sigma), \sigma^- \rangle \leq 0 \), we get

\[
\frac{1}{2} \frac{d}{dt} |\sigma^-(t)|^2_{L^2} + |\nabla \sigma^-(t)|^2_{L^2} \leq \int_{\Omega} (\bar{\lambda}^*(\sigma(t)) \cdot \nabla \sigma^-)(t) \, dx - \int_{\Omega} (g(\sigma, U) - g(0, U)) \, \sigma^-(t) \, dx \leq \frac{1}{2} |\nabla \sigma^-(t)|^2_{L^2} + C |\sigma^-(t)|^2_{L^2}.
\]

Then as above, we can conclude that \( \sigma(x, t) \geq 0 \) for all \( t \). \( \square \)

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