Abstract

Let \( a, b \in \mathbb{N} \) be relatively prime. We consider \( (a - 1)(b - 1)/2 \), which arises in the study of the \( pq \)-th cyclotomic polynomial, where \( p, q \) are distinct primes. We prove two possible representations of \( (a - 1)(b - 1)/2 \) as nonnegative, integral linear combinations of \( a \) and \( b \). Surprisingly, for each pair \( (a, b) \), only one of the two representations exists and the representation is also unique. We then investigate the representations of \( (F_n - 1)(F_{n+1} - 1)/2 \) and \( (F_n - 1)(F_{n+2} - 1)/2 \), where \( F_i \) is the \( i \)-th Fibonacci number, and observe several nice patterns.

1. Motivation and main results

The \( n \)-th cyclotomic polynomial is defined as

\[
\Phi_n(x) = \prod_{m=1,(m,n)=1}^{n} (x - e^{2\pi im/n}).
\]

Naturally, much work has been done on the values of the coefficients of \( \Phi_n(x) \). Numbers of the form \( (a - 1)(b - 1)/2 \) with \( (a, b) = 1 \) arise in the study of the midterm coefficient of the \( pq \)-th cyclotomic polynomial, where \( p, q \) are distinct primes. (Note that the degree of \( \Phi_{pq}(x) \) is \( \phi(pq) = (p - 1)(q - 1) \), where \( \phi \) is the Euler totient function, so its midterm coefficient has degree \( (p - 1)(q - 1)/2 \).) In particular, these polynomials have been fully characterized by the work of Beiter [Be], Carlitz [Ca] and Lam and Leung [LL]. These authors used different and very clever approaches.

In computing the midterm coefficient of \( \Phi_{pq}(x) \), Beiter sketched a proof that \( (p - 1)(q - 1)/2 \) can be uniquely written as \( \alpha q + \beta p + \delta \), where \( 0 \leq \alpha \leq p - 1, \beta \geq 0 \) and \( \delta \in \{0, 1\} \). In this article, we provide an alternate proof of the result applied to any relatively prime numbers.

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Theorem 1.1. Let $a, b \in \mathbb{N}$ be relatively prime. Consider two following equations.

\[
x a + y b = \frac{(a-1)(b-1)}{2}. \quad (1)
\]
\[
x a + y b + 1 = \frac{(a-1)(b-1)}{2}. \quad (2)
\]

Exactly one of the two equations has nonnegative integral solution(s) and the solution is unique.

Example 1.2. We observe that both representations of \((a-1)\frac{(b-1)}{2}\) can happen. If \(a = 3\) and \(b = 5\), we have \(1 \cdot 3 + 0 \cdot 5 + 1 = (3-1)(5-1)/2\). If \(a = 11\) and \(b = 31\), we have \(8 \cdot 11 + 2 \cdot 31 = (11-1)(31-1)/2\). Our theorem is also related to Problem E1637 [Mo], which states that for \(k \geq (a-1)(b-1)\), there exist nonnegative solution(s) to \(x a + y b = k\). Our theorem gives examples of \(k\) smaller than \((a-1)(b-1)\), which still make \(x a + y b = k\) have a unique nonnegative solution.

Corollary 1.3. Let \(p, q\) be distinct primes. Then \((p-1)(q-1)/2\) can be uniquely written as \(\alpha q + \beta p + \delta\) for some \(0 \leq \alpha \leq p-1, \beta \geq 0\) and \(\delta \in \{0, 1\}\).

This corollary is what Beiter used in computing the midterm coefficient of \(\Phi_{pq}(x)\). We now present another proof, which is shorter with the use of a strong theorem of Dresden that was not available when [Be] first appeared.

Alternate proof of Corollary 1.3. By [Dr, Theorem 1], the midterm coefficient of \(\Phi_{pq}(x)\) is odd. By [Be, Theorem 1], \((p-1)(q-1)/2 = \alpha q + \beta p + \delta\) in exactly one way.

Next, given a pair of consecutive Fibonacci numbers \((F_n, F_{n+1})\), we investigate the nonnegative, integral solutions to

\[
F_n x + F_{n+1} y = (F_n - 1)(F_{n+1} - 1)/2 \quad (3)
\]
\[
1 + F_n x + F_{n+1} y = (F_n - 1)(F_{n+1} - 1)/2 \quad (4)
\]

Due to Theorem 1.1 and \((F_n, F_{n+1}) = 1\), we know that exactly one of these equations has a unique nonnegative, integral solution. By convention, we index the Fibonacci sequence as follows

\[
F_1 = 1, \ F_2 = 1, \ F_3 = 2, \ F_4 = 3, \ F_5 = 5, \ldots
\]

The following table provides the first several cases.

\footnote{due to the Cassini’s identity: \(F_{n+1}^2 - F_n F_{n+2} = (-1)^n\).}
We observe two patterns here. First, Equation (3) and Equation (4) are used alternatively with period 3. Second, the first and the third row of each period give $x = y$. The two patterns are summarized by the following theorem.

**Theorem 1.4.** For $k \geq 1$, the following formulas are correct.

$$\frac{1}{2}(F_{6k} - 1)F_{6k} + \frac{1}{2}(F_{6k} - 1)F_{6k+1} = \frac{(F_{6k} - 1)(F_{6k+1} - 1)}{2}.$$ (5)

$$\frac{1}{2}(F_{6k+1} - 1)F_{6k+1} + \frac{1}{2}(F_{6k+1} - 1)F_{6k+2} = \frac{(F_{6k+1} - 1)(F_{6k+2} - 1)}{2}.$$ (6)

$$\frac{1}{2}(F_{6k+1} - 1)F_{6k+2} + \frac{1}{2}(F_{6k+1} - 1)F_{6k+3} = \frac{(F_{6k+2} - 1)(F_{6k+3} - 1)}{2}.$$ (7)

$$1 + \frac{1}{2}(F_{6k+2} - 1)F_{6k+2} + \frac{1}{2}(F_{6k+2} - 1)F_{6k+4} = \frac{(F_{6k+3} - 1)(F_{6k+4} - 1)}{2}.$$ (8)

$$1 + \frac{1}{2}(F_{6k+4} - 1)F_{6k+4} + \frac{1}{2}(F_{6k+4} - 1)F_{6k+5} = \frac{(F_{6k+5} - 1)(F_{6k+6} - 1)}{2}.$$ (9)

$$1 + \frac{1}{2}(F_{6k+4} - 1)F_{6k+5} + \frac{1}{2}(F_{6k+4} - 1)F_{6k+6} = \frac{(F_{6k+5} - 1)(F_{6k+6} - 1)}{2}.$$ (10)

**Remark 1.5.** If $n$ is a multiple of 3, then $F_n$ is even [SE]. Let $k \in \mathbb{N}$. Because $(F_{3k}, F_{3k+1}) = (F_{3k+2}, F_{3k+3}) = 1$, $F_{3k+1}$ and $F_{3k+2}$ are odd. Hence, $F_n$ is even if and only if $n$ is a multiple of 3. Therefore, if $n \neq 0 \mod 3$, $\frac{1}{2}(F_n - 1)$ is a nonnegative integer.

Similarly, given $(F_n, F_{n+2})$, we can consider two following equations

$$F_n x + F_{n+2} y = (F_n - 1)(F_{n+2} - 1)/2$$ (11)

$$1 + F_n x + F_{n+2} y = (F_n - 1)(F_{n+2} - 1)/2$$ (12)

Due to Theorem 1.4 and $(F_n, F_{n+2}) = 1$, we know that exactly one of these equations has a unique nonnegative, integral solution. The following table provides the first several cases.

| $n$ | $F_n$ | $F_{n+1}$ | Which equation | $x$ | $y$ |
|-----|-------|-----------|----------------|-----|-----|
| 3   | 2     | 3         | (4)            | 0   | 0   |
| 4   | 3     | 5         | (4)            | 1   | 0   |
| 5   | 5     | 8         | (4)            | 1   | 1   |
| 6   | 8     | 13        | (3)            | 2   | 2   |
| 7   | 13    | 21        | (3)            | 6   | 2   |
| 8   | 21    | 34        | (3)            | 6   | 6   |
| 9   | 34    | 55        | (4)            | 10  | 10  |
| 10  | 55    | 89        | (4)            | 27  | 10  |
| 11  | 89    | 144       | (4)            | 27  | 27  |
| 12  | 144   | 233       | (3)            | 44  | 44  |
| 13  | 233   | 377       | (3)            | 116 | 44  |
| 14  | 377   | 610       | (3)            | 116 | 116 |
Again, Equation (11) and Equation (12) appear alternately with period 3. The following theorem is compatible to Theorem 1.4 and the proof is similar, so we omit it.

**Theorem 1.6.** For \( k \geq 0 \), the following formulas are correct.

\[
\frac{F_{6k+2} - 1}{2} F_{6k+1} + \frac{F_{6k-1} - 1}{2} F_{6k+3} = \frac{(F_{6k+1} - 1)(F_{6k+3} - 1)}{2} \tag{13}
\]

\[
\frac{F_{6k+2} - 1}{2} F_{6k+2} + \frac{F_{6k+1} - 1}{2} F_{6k+4} = \frac{(F_{6k+2} - 1)(F_{6k+4} - 1)}{2} \tag{14}
\]

\[
\frac{F_{6k+4} - 1}{2} F_{6k+3} + \frac{F_{6k+1} - 1}{2} F_{6k+5} = \frac{(F_{6k+3} - 1)(F_{6k+5} - 1)}{2} \tag{15}
\]

\[
1 + \frac{F_{6k+5} - 1}{2} F_{6k+4} + \frac{F_{6k+2} - 1}{2} F_{6k+6} = \frac{(F_{6k+4} - 1)(F_{6k+6} - 1)}{2} \tag{16}
\]

\[
1 + \frac{F_{6k+5} - 1}{2} F_{6k+5} + \frac{F_{6k+4} - 1}{2} F_{6k+7} = \frac{(F_{6k+5} - 1)(F_{6k+7} - 1)}{2} \tag{17}
\]

\[
1 + \frac{F_{6k+1} - 1}{2} F_{6k} + \frac{F_{6k-2} - 1}{2} F_{6k+2} = \frac{(F_{6k} - 1)(F_{6k+2} - 1)}{2}. \tag{18}
\]

2. Proofs

We first prove the following lemma.

**Lemma 2.1.** For any integers \( n, x, y, a, b \) with \( a, b \) positive and \((a, b) = 1\), we consider the equation \( xa + yb = n \). If there is a solution \( r, s \) with \( r < b \) and \( s < 0 \), then there are no solutions with \( x, y \) nonnegative.

**Proof.** All solutions are of the form \((x, y) = (r + tb, s - ta)\) for some \( t \in \mathbb{Z} \). To get \( y \geq 0 \), we must have \( t < 0 \), but then \( x < 0 \). \( \square \)
Proof of Theorem 1.1. Let \( k = (a - 1)(b - 1)/2 \).

Let \( 0 \leq r_1 \leq b - 1 \) be chosen such that \( ar_1 \equiv k \mod b \) and \( s_1 := (k - ar_1)/b \).

Let \( 0 \leq r_2 \leq b - 1 \) be chosen such that \( ar_2 \equiv (k - 1) \mod b \) and \( s_2 := (k - 1 - ar_2)/b \).

Observe that
\[
a(r_1 + r_2) = ar_1 + ar_2 \equiv 2k - 1 = a(b - 1) - b \equiv a(b - 1) \mod b.
\]
So, \( b \mid (r_1 + r_2 - (b - 1)) \) and so, \( r_1 + r_2 = b - 1 \). We compute
\[
s_1 + s_2 = k - ar_1 + k - ar_2 - 1 \quad \text{divides} \quad b \quad \text{and} \quad s_2 = \frac{k - ar_2}{b}.
\]

Hence, exactly one of \( s_1, s_2 \) is nonnegative. By definition, \( r_1 a + s_1 b = k \) and \( r_2 a + s_2 b + 1 = k \). Therefore, we have shown that either Equation \((1)\) or Equation \((2)\) has a nonnegative solution \((x, y)\), while the other equation has no nonnegative solutions due to Lemma 2.1.

It remains to prove that Equation \((1)\) has at most one nonnegative solution. (Similar claim and proof hold for Equation \((2)\).) Let \((x_1, y_1)\) and \((x_2, y_2)\) be two nonnegative solutions of Equation \((1)\). Clearly, \( 0 \leq x_1, x_2 \leq (b - 1) \), so \( |x_1 - x_2| \leq b - 1 \). Because \( x_1 a + y_1 b = x_2 a + y_2 b \), \( (x_1 - x_2) a = -(y_1 - y_2) b \). Since \( (a, b) = 1, b \) divides \( x_1 - x_2 \), which, along with \( |x_1 - x_2| \leq b - 1 \), implies that \( x_1 = x_2 \). It follows that \( y_1 = y_2 \). Hence, Equation \((1)\) has at most one nonnegative solution. \(\square\)

Proof of Theorem 1.4. We prove Formulas \((5), (6)\) and \((8)\). Proofs for the remaining formulas are similar. We use Identity \((d7) \ [LL2]\) repeatedly
\[
F_n F_{n+3} = F_{n+1} F_{n+2} + (-1)^{n-1}.
\](19)

Write
\[
\frac{1}{2}(F_{6k} - 1)F_{6k} + \frac{1}{2}(F_{6k-1} - 1)F_{6k+1}
= \frac{1}{2}(F_{6k} - 1)(F_{6k} + F_{6k+1})
= \frac{1}{2}F_{6k}F_{6k+1} - \frac{1}{2}(F_{6k} + F_{6k+1})
= \frac{F_{6k}F_{6k+1} + 1}{2} - \frac{1}{2}(F_{6k} + F_{6k+1}) \quad \text{due to} \quad (19)
= \frac{(F_{6k} - 1)(F_{6k+1} - 1)}{2}.
\]

This proves Formula \((5)\).
Next, we have
\[
\frac{1}{2}(F_{6k+1} - 1)F_{6k+1} + \frac{1}{2}(F_{6k-1} - 1)F_{6k+2}
\]
\[
= \frac{1}{2}F_{6k+1}^2 - \frac{1}{2}F_{6k+1} + \frac{1}{2}F_{6k-1}F_{6k+2} - \frac{1}{2}F_{6k+2}
\]
\[
= \frac{1}{2}F_{6k+1}^2 - \frac{1}{2}F_{6k+1} + \frac{1}{2}(F_{6k}F_{6k+1} + 1) - \frac{1}{2}F_{6k+2} \quad \text{due to (14)}
\]
\[
= \frac{1}{2}F_{6k+1}^2 - \frac{1}{2}F_{6k+1} + \frac{1}{2}((F_{6k+2} - F_{6k+1})F_{6k+1} + 1) - \frac{1}{2}F_{6k+2}
\]
\[
= \frac{1}{2}F_{6k+1}F_{6k+2} - \frac{1}{2}(F_{6k+1} + F_{6k+2}) + \frac{1}{2} = \left(\frac{F_{6k+1} - 1)(F_{6k+2} - 1)}{2}\right)
\]

This proves Formula (14).

Finally, we prove Formula (15). The left side is
\[
1 + \frac{1}{2}(F_{6k+2} - 1)F_{6k+3} + \frac{1}{2}(F_{6k+2} - 1)F_{6k+4}
\]
\[
= 1 + \frac{1}{2}F_{6k+2}F_{6k+5} - \frac{1}{2}(F_{6k+3} + F_{6k+4})
\]
\[
= 1 + \frac{1}{2}(F_{6k+3}F_{6k+4} - 1) - \frac{1}{2}(F_{6k+3} + F_{6k+4})
\]
\[
= \frac{(F_{6k+3} - 1)(F_{6k+4} - 1)}{2},
\]

which is the right side.

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