THEORY OF MULTIPOLE
SOLUTIONS TO THE SOURCELESS
GRAD-SHAFRANOV EQUATION
IN PLASMA PHYSICS

by

A. FERREIRA
Theory of multipole solutions to the sourceless Grad-Shafranov equation in plasma physics

A. Ferreira
R. Goiás, 1021, Jardim Santa Cruz
18700-140 Avaré, São Paulo, Brazil

Abstract

The rules to write out any one of the linearly independent functions belonging to the infinite set of those in polynomial form that satisfy the sourceless Grad-Shafranov equation as stated in the toroidal-polar coordinate system are established. It is found that a polynomial solution even in the poloidal angle is given by the product of an integral power of the radial coordinate variable by a complete polynomial of equal degree in this same variable with angular-dependent coefficient functions that are linear combinations of a finite number of Chebyshev polynomials in the cosine of the poloidal angle, the numerical coefficients of these being expressed in terms of the binomial numbers of Pascal’s arithmetic triangle. Tables of the ten polynomial solutions of the lowest degrees are provided in variables of the toroidal-polar and of the cylindrical coordinate systems.
I. INTRODUCTION

In the literature of Plasma Physics devoted to the equilibrium of the toroidal pinch, multipole solutions are generally understood as solutions to the Grad-Shafranov equation with no source terms as represented by the gradient of the plasma pressure and the gradient of half the squared toroidal field function (also called poloidal current function) in flux space, some examples of which have been known since the early days of thermonuclear fusion research [1], [2]. In the present paper we shall designate by multipole fields the magnetic fields in vacuum that are invariant under rotation about a fixed axis in space and whose field lines are entirely contained in the planes passing by the symmetry axis; the fluxes of such fields coincide then with the multipole solutions in the sense this expression is costumarily employed in Plasma Physics.  

Multipole solutions have found theoretical application in the solution of the free boundary problem of tokamak plasmas, and, by extension, in the calculation of external magnetic field configurations required by such confining devices and design of the system of coils intended to generate them [2]. As it is our aim to show in the article that follows this in the current issue of this journal [3], the utility of the multipole solutions goes beyond this scope, standing their use on the basis of a method of solution to the Grad-Shafranov equation for which the sources are constant in flux space. This multiplicity of utilizations then justifies the study of the solutions to the partial differential equation whose derivation we pass now to outline [4].

For frame of reference we choose the cylindrical coordinate system as illustrated

---

1 It should be noted that an azimuthal magnetic field that falls with the inverse of the distance away from the axis of rotational symmetry and that the presence of a conducting fluid with a uniform pressure distribution in space still do not provide the Grad-Shafranov equation with a source term. We shall exclude such magnetic field and material medium from our definition of a multipole field.
FIG. 1 The left-handed Cartesian coordinate system \((x, y, z)\), the left-handed cylindrical coordinate system \((R, \phi, z)\) and the right-handed toroidal-polar coordinate system \((r, \theta, \phi)\).

in Fig. 1, which is, among all rotational coordinate systems, the one with the simplest metrical properties\(^2\).

The multipole fields satisfy the equations:

\[
\nabla \cdot \vec{B} = 0 \quad (1.1)
\]

and

\[
\nabla \times \vec{B} = 0 \quad (1.2)
\]

\(^2\)The cyclic order of the unity vectors in the cylindrical system represented in Fig. 1, with the left hand rule being presumed, is \((\vec{e}_R, \vec{e}_\phi, \vec{k})\). Since this order is the same as that of the unity vectors of same names in the conventional right-handed cylindrical coordinate system, the results for all vector operations in both systems have equal forms. In particular, the terms in the expressions for the curl of a vector \((B_R, B_\phi, B_z)\) have the same respective signals in one and the other systems.
The solenoidal law, which in the coordinate system of our choice takes the form:

\[
\frac{1}{R} \frac{\partial}{\partial R} (RB_R) + \frac{\partial B_z}{\partial z} = 0 ,
\]

(1.3)
is automatically satisfied if the radial and the axial components of the magnetic field are written in terms of a scalar “stream function” \( \Psi \) as:

\[
B_R = -\frac{1}{R} \frac{\partial \Psi}{\partial z} ,
\]

(1.4)

\[
B_z = \frac{1}{R} \frac{\partial \Psi}{\partial R} .
\]

(1.5)

For a rotationally symmetric vector field with components lying on meridian planes the curl admits a nonnull projection only on the azimuthal \( \phi \)-direction, which, in the left-handed cylindrical coordinate system \((R, \phi, z)\) represented in Fig. 1, is expressed as:

\[
(\nabla \times \vec{B})_\phi = \frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R} .
\]

(1.6)

Replacing \( B_R \) and \( B_z \) as given by Eqs. (1.4) and (1.5) in Eq. (1.6), we can cast Ampère’s law into the form:

\[
R^2 \nabla . \left( \frac{1}{R^2} \nabla \Psi \right) = 0 ,
\]

(1.7)

which is, in vector notation, the equation governing the multipole fields. The stream function \( \Psi \) can be shown [4] to be physically the flux associated with the meridian (or poloidal) magnetic field divided by \( 2\pi \).

Equation (1.7), which we shall call the sourceless Grad-Shafranov equation or the multipole equation, is a partial differential equation of the elliptic type which, referring to the two coordinate systems most widely used in equilibrium studies, the cylindrical and the toroidal-polar (see Fig. 1), admits of infinitely many solutions in the form of polynomials in one coordinate variable with the meaning of length \((z\) or \(r\)), the coefficients of which are themselves polynomials, if the first of those two
systems is used, also in a linear coordinate variable \((R)\), or, if the second one is used, in the cosine of the angular coordinate \((\theta)\). With reference to an equatorial plane in space defined by the coordinate \(z = 0\), these solutions can be grouped in even ones and in odd ones, and, inside each parity group, linearly independent solutions can be singled out according to the degree of the polynomial they are in a linear coordinate variable (say). From a mathematical viewpoint it is these independent solutions that we shall recognize as the multipole solutions.

It is the object of the present paper to derive the infinite set of the even multipole solutions to the sourceless Grad-Shafranov equation according to the foregoing definition here given to them. Of the two previously cited coordinate systems, our choice will fall upon the toroidal-polar as the ground for the analytical work, since the developments in this system lead naturally to the representation of the angular dependence of the solutions in terms of orthogonal polynomials that have simple and well known properties.

We shall commence by assigning the multipole solutions the general form of polynomials in the radial coordinate variable with coefficients that are unknown functions of the cosine of the angular coordinate variable.\(^3\) From the construction of a few examples of the lowest degrees, the precise form of polynomial dependence the linearly independent solutions must bear on the radial coordinate variable is inferred. The ensuing problem to tackle is that of the dependence of the solutions on the angular variable, and accordingly it takes the guise, no longer of a partial differential equation in two variables, but that of ordinary differential equations in the angular one for the coefficient functions of the polynomials whose dependence on the radial coordinate variable has already been established. It is at this stage of the analysis that the representation of the angular dependences of the multipole

\(^3\)This is true for solutions of both parities; the odd ones are made so by effect of an overall multiplying sine factor.
solutions in terms of orthogonal polynomials appears forcefully as the analytical recourse to be adopted since they are advanced by the very form of the differential equations governing those dependences. Two families of orthogonal polynomials present themselves as natural candidates to play the role of basis for the (finite) expansion of the angular-dependent coefficients: that of the Chebyshev polynomials and that of the associated Legendre polynomials of order unity, of which we have given preference to the former, on account both of the simpler defining properties of its members and of the form we should lend to the solution as suggested by the isolated examples worked out.

The finite set of Chebyshev polynomials that are to enter a multipole solution of a given degree in the radial variable is found by a blend of general arguments and induction. The next task in the progression is then to determine the numerical coefficients by which they must be multiplied in a combination that satisfies the ordinary differential equations for the coefficient functions of the assumed polynomial solution.

At this level the problem will appear as transmuted from that of differential equations, partial and ordinary, into that of a restricted set of difference equations for the numerical coefficients of nonseparable solutions in two variables. Three are these difference equations, two of them ordinary and one partial, the solutions of the ordinary ones serving as boundary conditions to the partial one.

Besides the recurrent construction of the set of numerical coefficients entering the multipole solution of a particular degree they allow for, we show that these difference equations can be given solutions under the form of general expressions by which capability the step-by-step procedure of evaluation of the coefficients can be circumvented. Moreover, since the end values are ipso facto contained in the solution to the partial difference equation, this one comprises also the solutions to
the ordinary difference equations and a single expression which attends to all the coefficients belonging to any and all multipole solutions can be written.

Up to the present moment uses seem not to have been ever made of the odd multipole solutions, which however would find application in systems of confinement that would not exhibit up-down symmetry with respect to the equatorial plane of the torus. In the present paper we shall not pursue the full characterization of such odd solutions, limiting ourselves to showing how they arise along a process of systematic construction of polynomial solutions to the multipole equation.

II. THE RADIAL DEPENDENCE OF THE LINEARLY INDEPENDENT MULTIPOLe SOLUTIONS

As stated in Section I, we shall adopt the toroidal-polar coordinate system in which to express the multipole equation and to work out its solutions. This system, which is pictorially represented in Fig. 1, is formed by rotating the usual polar system of the plane about a fixed axis in space, which we identify with the $z$-axis. In the planar system, the polar axis is placed perpendicularly to the axis of rotation; the pole $A$ is located on the polar axis at a distance $R_A$ from that of rotational symmetry. As the plane of the system is revolved, the polar axis generates a plane, which we refer to as the equatorial plane, perpendicular to the $z$-axis. We shall call meridian plane any plane containing the $z$-axis. The coordinates of a point $P$ belonging to the meridian plane determined by the $z$-axis and a given position of the polar axis are then the radius $r$, which measures the distance from the pole $A$ to $P$, the polar angle $\theta$, the angle between the positive direction of the polar axis (the direction of growing distances from the $z$-axis) and the radius, and the azimuthal angle $\phi$, the angle of rotation given to a reference plane passing by the
axis of rotation to bring that plane from an arbitrarily chosen conventional position, to which we assign the azimuthal angle $\phi = 0$, to coinciding with the meridian plane containing the $z$-axis, the polar axis and the point $P$, such that $(r, \theta, \phi)$ form a positive set according to the right-hand rule. In this coordinate system Eq. (1.7) takes the form:

$$x^2(1 + x \mu) \frac{\partial^2 \psi}{\partial x^2} + (1 + x \mu)(1 - \mu^2) \frac{\partial^2 \psi}{\partial \mu^2} + x \frac{\partial \psi}{\partial x} - (x + \mu) \frac{\partial \psi}{\partial \mu} = 0 ,$$  

(2.1)

where $\mu \equiv \cos \theta$, 

$$x \equiv \frac{r}{R_A}$$  

(2.2)

is the radial coordinate normalized to $R_A$ and $\psi$ is the flux function $\Psi$ normalized to an arbitrary flux $\Psi_0$. From now on it is Eq. (2.1) that we shall refer to as the sourceless Grad-Shafranov equation or the multipole equation.

We shall recognize that Eq. (2.1) has infinitely many linearly independent solutions of polynomial form in the variable $x$ with coefficients that are polynomials in the variable $\mu$. Considered as functions of the angle $\theta$, these independent solutions can be grouped into two sets, of the even ones and of the odd ones; we shall designate the solutions belonging to the first set by the Greek letter $\varphi$ and those belonging to the second set by the Greek letter $\gamma$.

Since in Eq. (2.1) only derivatives of the unknown function appear, a constant is a solution, and we write:

$$\varphi^{(0)}(x, \mu) = 1 ,$$  

(2.3)

which we refer to as the even multipole solution of order zero.

A polynomial solution of the first degree in $x$ writes in general as:

$$\psi(x, \mu) = w_0(\mu) + w_1(\mu)x .$$  

(2.4)

To determine the forms of the functions $w_0(\mu)$ and $w_1(\mu)$ we substitute this expression in the multipole equation. Collecting terms of equal power in $x$ and
equating the resulting coefficient of each power to zero, we obtain from the zeroth power:
\[
(1 - \mu^2) \frac{d^2w_0}{d\mu^2} - \mu \frac{dw_0}{d\mu} = 0 .
\] (2.5)

The solution of this differential equation is:
\[
w_0(\mu) = a_0 + c_0 \arccos \mu ,
\] (2.6)

where \(a_0\) and \(c_0\) are two arbitrary constants. Since our interest is restricted to solutions that are periodic in the variable \(\theta = \arccos \mu\), we demand \(c_0\) to be zero, and \(w_0(\mu)\) thus reduces to:
\[
w_0(\mu) = a_0 .
\] (2.6a)

We next consider the equation resulting from putting the coefficient of the first power in \(x\) equal to zero. We have:
\[
(1 - \mu^2) \frac{d^2w_1}{d\mu^2} - \mu \frac{dw_1}{d\mu} + w_1 = -\mu (1 - \mu^2) \frac{d^2w_0}{d\mu^2} + \frac{dw_0}{d\mu} = 0 ,
\] (2.7)

the second equality coming from the use of equation (2.6a) for \(w_0(\mu)\) in the right hand side of the first equality. This equation can be recognized as an instance of Chebyshev’s differential equation [5], whose solution can be written as:
\[
w_1(\mu) = a_{11} T_1(\mu) + b_{10} \sqrt{1 - \mu^2} U_0(\mu) ,
\] (2.8)

where
\[
T_1(\mu) = \mu
\] (2.9)
is the Chebyshev polynomial type I of the first order,
\[
U_0(\mu) = 1
\] (2.10)
is the Chebyshev polynomial type II of the zeroth order, and $a_{11}$ and $b_{10}$ are arbitrary constants.

The terms now remaining in the equation obtained by substituting Eq. (2.4) in the multipole equation are all of the second power in $x$. At this last level in the succession of equations for the $\mu$-dependence of the assumed polynomial solution we meet a condition again regarding $w_1(\mu)$, to know:

$$
\mu (1 - \mu^2) \frac{d^2 w_1}{d\mu^2} - \frac{d w_1}{d\mu} = 0 .
$$

(2.11)

The solution of this differential equation can be easily obtained and is:

$$
w_1(\mu) = c_1 + d_1 \sqrt{1 - \mu^2} ,
$$

(2.12)

where $c_1$ and $d_1$ are arbitrary constants.

To make equal the two expressions we have for $w_1(\mu)$ we require that

$$
a_{11} = 0
$$

(2.13)

in Eq. (2.8) and that

$$
c_1 = 0 ,
$$

(2.14)

$$
d_1 = b_{10}
$$

(2.15)

in Eq. (2.12). Note that this choice for the free constants $a_{11}$ and $c_1$ suppresses the terms that are even in $\theta$ in each of the two distinct expressions for $w_1(\mu)$.

With $w_0(\mu)$ and $w_1(\mu)$ so determined, the expression that results for the solution of the form proposed in Eq. (2.4) is:

$$
\psi(x, \mu) = a_0 + b_{10} x \sqrt{1 - \mu^2} U_0(\mu) .
$$

(2.16)

The constant term $a_0$ is just a multiple of the multipole solution $\varphi^{(0)}(x, \mu)$, and the new, independent solution brought about by Eq. (2.16) is:

$$
\gamma^{(0)}(x, \mu) = x \sqrt{1 - \mu^2} U_0(\mu) ,
$$

(2.17)
where we have adopted, to designate it, the notation appropriate to an odd multipole solution in the angular variable, which it is. We shall call this the odd multipole solution of the zeroth order.

We next consider a polynomial solution of the second degree in $x$ for the multipole equation:

$$\psi(x, \mu) = w_0(\mu) + w_1(\mu)x + w_2(\mu)x^2 .$$  \hspace{1cm} (2.18)

The procedure of substitution in Eq. (2.1) and of equating the coefficients of the powers of $x$ to zero leads again, from the lowest power, to Eq. (2.5) for $w_0(\mu)$, which is solved by Eq. (2.6a) as before. From the equality of the term in $x$ to zero we obtain Eq. (2.7) for $w_1(\mu)$, the solution of which is given by Eq. (2.8). Since a condition like that of Eq. (2.11) is no longer found when we go to the next power of $x$, we keep both terms in Eq. (2.8) (that is to say, we take $a_{11} \neq 0$).

The term in $x^2$ in the multipole equation gives rise to the equation:

$$(1 - \mu^2) \frac{d^2w_2}{d\mu^2} - \mu \frac{dw_2}{d\mu} + 4w_2 = -\mu(1 - \mu^2) \frac{d^2w_1}{d\mu^2} + \frac{dw_1}{d\mu} = a_{11} ,$$  \hspace{1cm} (2.19)

where the second equality follows from the use of Eq. (2.8) for $w_1(\mu)$ in the right hand side of the first equality. We are again in the presence of a Chebyshev equation, this time inhomogeneous, the complete solution of which is:

$$w_2(\mu) = a_{22}T_2(\mu) + b_{21}\sqrt{1 - \mu^2}U_1(\mu) + \frac{a_{11}}{4} ,$$  \hspace{1cm} (2.20)

where $T_2(\mu)$ and $U_1(\mu)$ are respectively the Chebyshev polynomial type I of the second order, defined by

$$T_2(\mu) = 2\mu^2 - 1 ,$$  \hspace{1cm} (2.21)

and the Chebyshev polynomial type II of the first order, defined by

$$U_1(\mu) = 2\mu ,$$  \hspace{1cm} (2.22)
and $a_{22}$ and $b_{21}$ are two arbitrary constants.

The constraint that results from equating the term in $x^3$ in the multipole equation to zero closes the set, and, the same as in the previous case of a polynomial solution of the first degree in $x$, takes on the form of a homogeneous differential equation for the $\mu$-dependent coefficient of the term of the highest power in $x$ in the proposed solution, in this way imposing a demand on $w_2(\mu)$ of its own. We have:

$$\mu(1 - \mu^2)\frac{d^2w_2}{d\mu^2} - \frac{dw_2}{d\mu} + 2\mu w_2 = 0.$$  \hspace{1cm} (2.23)

In general, for an assumed polynomial solution of any degree, by following the two opposite leads in the chain of equations for the coefficient functions, one corresponding to the equation associated with the lowest power, and the other corresponding to the equation associated with the highest power of $x$ in the multipole equation, we are able to generate two parallel sets of solutions for the $\mu$-dependent functions. All the consistency of the method of construction of polynomial solutions and in fact the very existence of such solutions hinge on the possibility of making equal these two sets by appropriately choosing the arbitrary constants appearing in both.

A possible manner of bringing $w_2(\mu)$, as given by Eq. (2.20), to satisfy Eq. (2.23) is simply by substituting the former into the latter and then choosing for the constants $a_{11}$, $a_{22}$ and $b_{21}$ the values that solve the ensuing system of algebraic equations. An alternative manner consists in comparing the solution of Eq. (2.23) with the expression for $w_2(\mu)$ stated in Eq. (2.20). Adopting this second one, we write down the solution of Eq. (2.23):

$$w_2(\mu) = c_2\mu^2 + d_2 \left[ \sqrt{1 - \mu^2} + \frac{\mu^2}{2} \ln \left( \frac{1 + \sqrt{1 - \mu^2}}{1 - \sqrt{1 - \mu^2}} \right) \right], \hspace{1cm} (2.24)$$

where $c_2$ and $d_2$ are constants, and then, by equating it to the solution for $w_2(\mu)$ as given by Eq. (2.20), we are led to determine the constants which appear in both as:

$$a_{22} = \frac{a_{11}}{4},$$  \hspace{1cm} (2.25)
\[ b_{21} = d_2 = 0, \]  
\[ c_2 = 2a_{22}. \]  

Note that these choices for \( b_{21} \) and \( d_2 \) have the effect of suppressing the odd functions in \( \theta \) that enter the expressions of the two solutions obtained for \( w_2(\mu) \).

With all these results at hand we are in position to state the polynomial solution of the second degree in \( x \) for the multipole equation as:

\[ \psi(x, \mu) = a_0 + b_{10}x \sqrt{1 - \mu^2} U_0(\mu) + a_{11} \left\{ T_1(\mu)x + \left[ \frac{1}{4}T_0(\mu) + \frac{1}{4}T_2(\mu) \right] x^2 \right\}. \]  \(2.28\)

Now, the first two terms on the right hand side are just a combination of the two previously found multipole solutions \( \varphi^{(0)}(x, \mu) \) and \( \gamma^{(0)}(x, \mu) \), and we recognize the third, new term, as a multiple of the even multipole solution of the first order, which we shall denote by \( \varphi^{(1)}(x, \mu) \):

\[ \varphi^{(1)}(x, \mu) = T_1(\mu)x + \left[ \frac{1}{4}T_0(\mu) + \frac{1}{4}T_2(\mu) \right] x^2. \]  \(2.29\)

If we next consider a polynomial solution of the third degree in \( x \), by following the same steps of the procedure we employed to derive the even and odd multipole solutions of lower degrees, we arrive at the odd multipole solution of the first order, which is:

\[ \gamma^{(1)}(x, \mu) = x \sqrt{1 - \mu^2} \left\{ U_1(\mu)x + \left[ \frac{1}{4}U_0(\mu) + \frac{1}{4}U_2(\mu) \right] x^2 \right\}, \]  \(2.30\)

where

\[ U_2(\mu) = 4\mu^2 - 1 \]  \(2.31\)

is the Chebyshev polynomial type II of the second order.

The examination of these and of higher order multipole solutions in this way obtained shows that they have definite parity, being either even or odd in the polar
angle $\theta$, the general form of the even ones of order $n$ being:

$$\varphi^{(n)}(x, \mu) = \sum_{k=n}^{2n} f^{(n)}_k(\mu) x^k ,$$

(2.32)

and that of the odd ones of order $n$ being:

$$\gamma^{(n)}(x, \mu) = x \sqrt{1 - \mu^2} \sum_{k=n}^{2n} g^{(n)}_k(\mu) x^k ,$$

(2.33)

where the angular functions $f^{(n)}_k(\mu)$ and $g^{(n)}_k(\mu)$ should be expected to express themselves naturally as combinations of Chebyshev polynomials type I and type II respectively. The problem then reduces to finding the coefficients of these combinations given the order $n$ of the multipole solution. In this paper we shall treat only the case of the solutions that are even with respect to the polar angle $\theta = \arccos \mu$.

III. THE DIFFERENTIAL EQUATIONS FOR THE ANGULAR DEPENDENCES OF THE MULTIPOLE SOLUTIONS; THE REPRESENTATION OF THE SOLUTIONS AS COMBINATIONS OF CHEBYSHEV POLYNOMIALS AND THE RECURRENCE RELATIONS FOR THEIR COEFFICIENTS

We start by introducing Eq. (2.32) in Eq. (2.1). Collecting terms of equal powers of $x$ and equating the resulting coefficient for each power to zero, we obtain three differential equations for the angular functions $f^{(n)}_k(\mu)$ according to the value of the exponent of $x$. From the term in $x^n$ we obtain an equation for $f^{(n)}_n(\mu)$, which is:

$$(1 - \mu^2) \frac{d^2 f^{(n)}_n}{d\mu^2} - \mu \frac{df^{(n)}_n}{d\mu} + n^2 f^{(n)}_n = 0 ,$$

(3.1)

and, from the term in $x^{2n+1}$, an equation for $f^{(n)}_{2n}(\mu)$, namely:

$$\mu(1 - \mu^2) \frac{d^2 f^{(n)}_{2n}}{d\mu^2} - \frac{df^{(n)}_{2n}}{d\mu} + 2n(2n - 1)\mu f^{(n)}_{2n} = 0 ,$$

(3.2)
both of which are homogeneous. The coefficient of $x^\ell$, for $\ell = n + 1, n + 2, \ldots, 2n$, gives rise to a differential relation involving two angular functions of subscripts differing by unity, which can be written as:

$$
(1 - \mu^2)\frac{d^2 f^{(n)}_{\ell}}{d\mu^2} - \mu \frac{df^{(n)}_{\ell}}{d\mu} + \ell^2 f_{\ell} = -\mu(1 - \mu^2)\frac{d^2 f^{(n)}_{\ell-1}}{d\mu^2} + \frac{df^{(n)}_{\ell-1}}{d\mu} - (\ell - 1)(\ell - 2)\mu f^{(n)}_{\ell-1}
$$

$(\ell = n + 1, n + 2, \ldots, 2n)$. \hfill (3.3)

We now proceed by considering each of these equations.

(1) The equation for $f^{(n)}_n(\mu)$

Equation (3.1) can be immediately recognized as being Chebyshev’s equation, whose even solution is the Chebyshev polynomial type I of order $n$, $T_n(\mu)$ \cite{5}. We write thus:

$$
f^{(n)}_n(\mu) = A^{(n)}_{n,n}T_n(\mu) \ (n = 0, 1, 2, 3, \ldots),
$$

(3.4)

where $A^{(n)}_{n,n}$ is a constant.

(2) The homogeneous equation for $f^{(n)}_{2n}(\mu)$

Although our ultimate aim is to express $f^{(n)}_{2n}(\mu)$ as a combination of Chebyshev polynomials, it is nonetheless of interest to consider also other representations which bear a more direct reference to the general pattern the differential equation we found to govern this function obeys. Indeed, the even solution in $\theta$ of Eq. (3.2) finds a concise representation as:

$$
f^{(n)}_{2n}(\mu) = \mu P^1_{2n-1}(\sqrt{1 - \mu^2}),
$$

(3.5)

where $P^1_{2n-1}$ is the associated Legendre function of degree $2n - 1$ and order 1 of the first kind \cite{5}. (The second independent solution to Eq. (3.2), which writes as:

$$
\mu Q^1_{2n-1}(\sqrt{1 - \mu^2}),
$$

14
being the associated Legendre function of degree $2n - 1$ and order 1 of the second kind, though it is regular in the whole interval of variation of $\mu$, $-1 \leq \mu \leq 1$, is however odd in the angle $\theta$ and must thence be rejected.) Another representation of $f_{2n}^{(n)} (\mu)$ that we shall find to be useful is obtained by noting that, upon the transformation:

$$\mu^2 = t ,$$

Eq. (3.2) becomes:

$$t(1 - t) \frac{d^2 f_{2n}^{(n)}}{dt^2} - \frac{t}{2} \frac{df_{2n}^{(n)}}{dt} + n \left( n - \frac{1}{2} \right) f_{2n}^{(n)} = 0 ,$$

which can be recognized as a form of the hypergeometric equation [5]. The solution of interest for $f_{2n}^{(n)} (\mu)$ can be written down immediately as:

$$f_{2n}^{(n)} (\mu) = \mu^2 \binom{-n + 1}{n + 1} 2F_1 \left( -n + 1, n + \frac{1}{2}; 2; \mu^2 \right) .$$

Since the first argument of the hypergeometric function $2F_1$ vanishes for $n = 1$ and is a negative integer for $n = 2, 3, 4, \ldots$, this solution is actually a polynomial of degree $n$ in $\mu^2$, for which the constant term is missing. (To obtain a convenient representation of the second solution to Eq. (3.2), the transformation of the independent variable $\mu^2 = 1 - v$ proves to be more advantageous than that of Eq. (3.6). By the sole consideration of the form into which the equation is converted when the free variable passes to be $v$, it becomes apparent that one of the independent solution for $f_{2n}^{(n)} (\mu)$ is [6]:

$$f_{2n}^{(n)} (\mu) = \sqrt{1 - \mu^2} \binom{1}{n, n \frac{3}{2}; 1 - \mu^2} .$$

Because of the sine term multiplying the hypergeometric function, this solution is odd in the variable $\theta$ and therefore to be abandoned.)

We now turn ourselves to our main objective with regard to the angular function $f_{2n}^{(n)} (\mu)$, which is to find for it a representation in terms of the Chebyshev polynomials. For this purpose we shall take as starting point the differential equation
for \( f_{2n}^{(n)}(\mu) \) itself rather than the explicit definitions we have found by solving it. Equation (3.2) can be concisely written as:

\[
\mathcal{H}_{2n}\{f_{2n}^{(n)}(\mu)\} = 0 ,
\]

(3.9)

where we have made use of the notation:

\[
\mathcal{H}_\ell \equiv \mu(1 - \mu^2) \frac{d^2}{d\mu^2} - \frac{d}{d\mu} + \ell(\ell - 1)\mu ,
\]

(3.10)

\( \ell \) being an integer. For the attainment of our aim we have to know which are the effects of the differential operator above introduced as it acts on the Chebyshev polynomials in general.

By having recourse to the frequently quoted relations for the Chebyshev polynomials [5], [7]:

\[
(1 - \mu^2) \frac{d^2 T_m}{d\mu^2} = \mu \frac{dT_m}{d\mu} - m^2 T_m(\mu) ,
\]

(3.11)

\[
(1 - \mu^2) \frac{dT_m}{d\mu} = \frac{m}{2} [T_{m-1}(\mu) - T_{m+1}(\mu)] ,
\]

(3.12)

\[
2\mu T_m(\mu) = T_{m+1}(\mu) + T_{m-1}(\mu) ,
\]

(3.13)

the following properties of the operator \( \mathcal{H}_\ell \) can be easily established:

\[
\mathcal{H}_\ell\{T_0(\mu)\} = \ell(\ell - 1)T_1(\mu) ,
\]

(3.14)

\[
\mathcal{H}_\ell\{T_m(\mu)\} = \frac{1}{2}(\ell+m)(\ell-m-1)T_{m-1}(\mu) + \frac{1}{2}(\ell-m)(\ell+m-1)T_{m+1}(\mu)
\]

(3.15)

\((m = 1, 2, 3, \ldots; \ell = 1, 2, \ldots, m)\).

Next, from the considerations on the form of \( f_{2n}^{(n)}(\mu) \) brought forth in the sequel of Eq. (3.8), it is apparent that this function can be suitably expressed as a finite combination of Chebyshev polynomials, restricted these to the ones of the orders smaller than and equal to \( 2n \), and even, to know:

\[
f_{2n}^{(n)}(\mu) = \sum_{\ell=0}^{n} A_{2n,2\ell}^{(n)} T_{2\ell}(\mu) ,
\]

(3.16)
where the $A_{2n,2\ell}^{(n)}$'s are constants.

If now we insert Eq. (3.16) in Eq. (3.9), by making use of the formulae stated in Eqs. (3.14) and (3.15), we are able to transform the differential equation for $f_{2n}^{(n)}(\mu)$ into a null identity for a combination of Chebyshev polynomials of odd orders, whose coefficients are combinations two by two respectively of the constants $A_{2n,2\ell}^{(n)}$'s. By appeal to the orthogonality property of the Chebyshev polynomials it follows that the coefficient of each polynomial in the identity must in its turn be null, and this gives a recursive pair of formulae for the constants. From the coefficient of $T_1(\mu)$ we get:

$$A_{2n,2}^{(n)} = -2\frac{n(2n-1)}{(n+1)(2n-3)}A_{2n,0}^{(n)}, \quad (3.17)$$

and from the coefficient of $T_{2\ell+1}(\mu)$ we get:

$$A_{2n,2\ell+2}^{(n)} = -\frac{(n-\ell)(2n+2\ell-1)}{(n+\ell+1)(2n-2\ell-3)}A_{2n,2\ell}^{(n)} \quad (\ell = 1, 2, \ldots, n-1). \quad (3.18)$$

Once $A_{2n,0}^{(n)}$ is specified, Eqs. (3.17) and (3.18) provide us with the means to evaluate the remaining $n$ coefficients $A_{2n,2\ell}^{(n)}$ that enter the making up of $f_{2n}^{(n)}(\mu)$ according to Eq. (3.16). We shall defer the accomplishment of this task to the next Section, leaving the matter as it is at this point, and proceed by considering the remaining differential relation for the angular dependences of the even multipole solutions we derived at the beginning of the present Section.

(3) The inhomogeneous equation for $f_{\ell}^{(n)}(\mu), \ell = n+1, n+2, \ldots, 2n$

For the treatment of Eq. (3.3) it is useful to introduce the differential operator:

$$\mathcal{T}_\ell \equiv (1 - \mu^2) \frac{d^2}{d\mu^2} - \mu \frac{d}{d\mu} + \ell^2, \quad (3.19)$$

which has the property:

$$\mathcal{T}_k \{T_m(\mu)\} = (k^2 - m^2)T_m(\mu) \quad (m = 0, 1, 2, \ldots), \quad (3.20)$$
as it can be easily proved with the help of Eq. (3.11). Recalling the definition of the operator $H_\ell$ from Eq. (3.10), we may restate Eq. (3.3) as:

$$
T_\ell \left\{ f^{(n)}_\ell (\mu) \right\} = -H_{\ell-1} \left\{ f^{(n)}_{\ell-1} (\mu) \right\} \quad (\ell = n + 1, n + 2, \ldots, 2n - 1, 2n).
$$

(3.21)

The first use we shall give to Eq. (3.21) is to find the general form the angular functions $f^{(n)}_\ell (\mu)$ must obey. For this purpose we shall view it as an inhomogeneous equation for the function $f^{(n)}_\ell (\mu)$, therefore assuming that the right hand side is known. We shall proceed by induction, starting by considering the equation for the index $\ell = n + 1$, which is:

$$
T_{n+1} \left\{ f^{(n)}_{n+1} (\mu) \right\} = -H_n \left\{ f^{(n)}_n (\mu) \right\} = nA_{n,n} T_{n-1} (\mu),
$$

(3.22)

the second equality coming from the property of the operator $H_n$ stated in Eq. (3.15) when the object function is $f^{(n)}_n (\mu)$ as given by Eq. (3.4).

Equation (3.22) is an inhomogeneous Chebyshev differential equation, which admits as complementary solution of even parity in $\theta$ the Chebyshev polynomial type I of order $n + 1$, and whose particular solution should exhibit the same dependence on $\theta$ as the driving term, being therefore proportional to the Chebyshev polynomial of order $n - 1$. We write then for the complete solution:

$$
f^{(n)}_{n+1} (\mu) = A^{(n)}_{n+1,n+1} T_{n+1} (\mu) + A^{(n)}_{n+1,n-1} T_{n-1} (\mu),
$$

(3.23)

where $A^{(n)}_{n+1,n+1}$ and $A^{(n)}_{n+1,n-1}$ are constants.

We next consider Eq. (3.21) with the index $\ell$ taken to be equal to $n + 2$, namely:

$$
T_{n+2} \left\{ f^{(n)}_{n+2} (\mu) \right\} = -H_{n+1} \left\{ f^{(n)}_{n+1} (\mu) \right\} = \left[ (n+1)A^{(n)}_{n+1,n+1} - (2n - 1)A^{(n)}_{n+1,n-1} \right] T_n (\mu) - nA^{(n)}_{n+1,n-1} T_{n-2} (\mu),
$$

(3.24)

18
where we have used the form just derived for \( f_{n+1}^{(n)}(\mu) \) in the right hand side and again the property of Eq. (3.15) for the differential operator \( \mathcal{H}_{n+1} \). The even component of the complementary solution to Eq. (3.24) is a multiple of the Chebyshev polynomial type I of order \( n + 2 \), while the particular solution is a combination of the two Chebyshev polynomials that appear on the right hand side. The complete solution of even parity in the angle \( \theta \) then writes as:

\[
f^{(n)}_{n+2}(\mu) = A_{n+2,n+2}^{(n)} T_{n+2}(\mu) + A_{n+2,n}^{(n)} T_{n}(\mu) + A_{n+2,n-2}^{(n)} T_{n-2}(\mu),
\]

(3.25)

where the \( A \)’s are constants.

From a consideration of the form of \( f^{(n)}_n(\mu) \), given by Eq. (3.4), and of those of \( f_{n+1}^{(n)}(\mu) \) and \( f_{n+2}^{(n)}(\mu) \), given respectively by Eqs. (3.23) and (3.25), we infer that the form of the function \( f^{(n)}_{n+k}(\mu) \) in general must be:

\[
f^{(n)}_{n+k}(\mu) = \sum_{p=0}^{k} A_{n+k,n+k-2p}^{(n)} T_{n+k-2p}(\mu) \quad (k = 0, 1, 2, \ldots, n),
\]

(3.26)

where the \( A_{n+k,n+k-2p}^{(n)} \)’s are constants. Given that, according to Eq. (2.32), the even multipole solution of order \( n \), \( \varphi^{(n)}(x,\mu) \), depends on \( n + 1 \) angular functions \( f_i^{(n)}(\mu) \) (\( i = n, n+1, \ldots, 2n \)) and since by Eq. (3.26) each of these comprises \( i - n + 1 \) constants \( A_{i,j}^{(n)} \) (\( j = i, i+1, \ldots, 2n - i \)), the total number of constants needed to specify \( \varphi^{(n)}(x,\mu) \) completely equals \( (n+1)(n+2)/2 \), one of them remaining arbitrary, as it should be proper to the solution of a homogeneous equation of the second order with a definite parity; we shall take this constant, in terms of which all others will be expressed, as \( A_{2n,0}^{(n)} \). The problem of finding the multipole solution of order \( n \) to the sourceless Grad-Shafranov equation is then the problem of determining the set of these constants.

To deduce the recursion formulae connecting the coefficients that enter the representation of \( f^{(n)}_{n+k} \), we start by rewriting Eq. (3.21) as:

\[
\mathcal{T}_{n+k} \left\{ f^{(n)}_{n+k}(\mu) \right\} = -\mathcal{H}_{n+k-1} \left\{ f^{(n)}_{n+k-1}(\mu) \right\} \quad (k = 0, 1, 2, \ldots, n).
\]

(3.21′)
By inserting Eq. (3.26) in the left hand side of this equation and by using the property of Eq. (3.20) for the differential operator $T_{n+k}$, we obtain after a brief calculation:

$$T_{n+k} \left\{ f_{n+k}^{(n)}(\mu) \right\} = 4knA_{n+k,n-k}^{(n)}T_{n-k}(\mu) + 4 \sum_{p=1}^{k-1} p(n+k-p)A_{n+k,n+k-2p}^{(n)}T_{n+k-2p}(\mu)$$

$$(k = 1, 2, \ldots, n). \quad (3.27)$$

The reduction of the right hand side of Eq. (3.21') requires a somewhat lengthier manipulation than that of the left hand side, but it is otherwise straightforward. With the help of the formula stated in Eq. (3.15) we obtain:

$$-H_{n+k-1} \left\{ f_{n+k-1}^{(n)}(\mu) \right\} = -(2k-3)kA_{n+k-1,n-k+1}^{(n)}T_{n-k}(\mu)$$

$$- \sum_{p=1}^{k-1} \left[ p(2n+2k-2p-3)A_{n+k-1,n+k-2p+1}^{(n)} + (2p-3)(n+k-p)A_{n+k-1,n+k-2p+1}^{(n)} \right] T_{n+k-2p}(\mu)$$

$$(k = 1, 2, \ldots, n). \quad (3.28)$$

The process of equating the left and the right hand sides of Eq. (3.21'), as given by Eqs. (3.27) and (3.28) respectively, yields two recursion formulae for the coefficients $A_{i,j}^{(n)}$. The first one comes from the terms proportional to $T_{n-k}(\mu)$ on both sides of the equation and is:

$$A_{n+k-1,n-k+1}^{(n)} = \left( \frac{4k}{3-2k} \right) A_{n+k,n-k}^{(n)} \quad (k = n, n-1, n-2, \ldots, 2, 1). \quad (3.29)$$

(The order in which the values of $k$ are written above corresponds to the sequence in which the values of the coefficients are generated by systematic application of the formula, knowing the initial value $A_{2n,0}^{(n)}$.)

The second recursion formula stems from the equality of the coefficients multiplying $T_{n+k-2p}(\mu)$ on both sides of Eq. (3.21'), and is:

$$A_{n+k-1,n+k-2p+1}^{(n)} = -\frac{p}{2p-3} \left[ 4A_{n+k,n+k-2p+1}^{(n)} + \left( \frac{2n+2k-2p-3}{n+k-p} \right) A_{n+k-1,n+k-2p-1}^{(n)} \right]$$

$$(k = n, n-1, \ldots, 3, 2; \quad p = k - 1, k - 2, \ldots, 2, 1). \quad (3.30)$$
The set of difference equations constituted by Eq. (3.17) together with Eq. (3.18), Eq. (3.29) and Eq. (3.30) encompasses all the coefficients that are needed to specify the whole of the angular functions that enter the composition of a multipole solution of a given order and forms thus a complete formulation of the problem of finding the totality of even polynomial solutions to Eq. (2.1). In Section IV we shall show how the difference equations can be used to generate the coefficients by a step-by-step calculation procedure; next, partly in Section IV and partly in Section V, we shall derive a solution in closed form for each of the three difference equations, such that a coefficient knowingly belonging to the scope of one of them can be evaluated by the appropriate solving formula from the sole knowledge of its indices; finally, also in Section V, we shall show that the solutions found can be merged into a single formula, from which the whole of the coefficients pertaining to a multipole solution can be obtained in terms of $A_{2n,0}^{(n)}$, being enough for that to specify the order of the multipole.

IV. THE TRIANGLE OF COEFFICIENTS AND THE SOLUTIONS FOR THE PERIPHERAL COEFFICIENTS

An insight into the mathematical structure of the problem posed by the difference equations to which the original problem of the partial differential equation for the multipole fields was reduced in the last Section can be gained by displaying the Chebyshev coefficients $A_{i,j}^{(n)}$ for the solution $\varphi^{(n)}(x,\mu)$ of a given order $n$ in a Cartesian array of columns and rows in which the number attached to a column specifies the first suffix $i$ of the coefficients keeping position in that column and the number attached to a row specifies the second suffix $j$ of the coefficients belonging to that row. According to Eq. (3.26) the first suffix of the numerical coefficients
$A_{i,j}^{(n)}$ belonging to the set of those characterized by possessing a specified $i$ as the common first suffix is defined to be the subscript of the angular function $f_{i}^{(n)}(\mu)$ into whose composition they enter as factors multiplying Chebyshev polynomials, and, by Eq. (2.32), the subscripts of the angular functions that are summoned to participate in the combination that builds up the multipole solution $\varphi^{(n)}(x, \mu)$ rank from $n$ to $2n$. Thus the label of the columns in the array must run from $i = n$ to $i = 2n$. With regard to the second suffix of the coefficient $A_{i,j}^{(n)}$, which is equal to the order of the Chebyshev polynomial this coefficient multiplies in Eq. (3.26), it must be clear from this equation that its range of variation, considered in its wholeness the constellation of coefficients involved in the construction of the multipole solution $\varphi^{(n)}(x, \mu)$, is determined by the angular function $f_{2n}^{(n)}(\mu)$, the general expression of which is given by Eq. (3.16). From this we see that the label of the rows in the array must run from $j = 0$ to $j = 2n$.

Figure 2 exhibits such an array, in which $n$ is equal to 4 and the coefficient $A_{8,0}^{(n)}$ is taken to be unity. As this example illustrates, not all of the positions are to be filled, since, by virtue of the conventions we have adopted regarding notation, there are no coefficients associated with every pair of indices within the wideness of range of the matrix $(i, j)$. In general, the field of indices of the coefficients $A_{i,j}^{(n)}$ for a fixed $n$ is defined by:

$$i = 2n, 2n - 1, 2n - 2, \ldots, n + 1, n; \quad j = 2n - i, 2n - i + 2, \ldots, i - 2, i.$$  \hspace{1cm} (4.1)$$

The area enclosing the coefficients resembles that of a triangle, and for this reason we shall refer to this array as the triangle of coefficients. In due time we shall show that it bears a close kinship with the array of binomial coefficients known as Pascal’s triangle.

Each of the difference equations we have derived in Section III applies to the coefficients occupying a different region of the triangle according to a pattern of
correspondence which we pass to expound.

![Table of coefficients](image)

**FIG. 2** The triangle of the coefficients $A^{(n)}_{i,j}$ for $n = 4$. The coefficient $A^{(4)}_{8,0}$ is taken to be unity. The long arrows on the sides indicate the sequence in which the peripheral coefficients are generated by recursion starting with the coefficient $A^{(4)}_{8,0}$. The short interior arrows intend to signify that the element $A^{(4)}_{7,1}$ combines with the element $A^{(4)}_{8,2}$ to generate the coefficient $A^{(4)}_{7,3}$.

(1) The equation for the coefficients in the column on the right hand side of the triangle of coefficients

This is Eq. (3.18) together with Eq. (3.17), here reproduced as:

$$A^{(n)}_{2n,2\ell+2} = -(1 + \delta_{\ell,0}) \frac{(n-\ell)(2n + 2\ell - 1)}{(n+\ell+1)(2n - 2\ell - 3)} A^{(n)}_{2n,2\ell} \quad (\ell = 0, 1, 2, \ldots, n-1), \quad (4.2)$$

where we have made use of the Kronecker symbol $\delta_{\ell,0}$ in order to unify the expressions for $\ell = 0$ and for $\ell \neq 0$. We find it useful to define a new discrete variable $j$
relating to $\ell$ through:

$$j = 2\ell + 2,$$  

(4.3)

in terms of which Eq. (4.2) is restated as:

$$A_{2n,j}^{(n)} = -(1 + \delta_{j,2})\frac{(2n - j + 2)(2n + j - 3)}{(2n + j)(2n - j - 1)} A_{2n,j-2}^{(n)} \quad (j = 2, 4, \ldots, 2n - 2, 2n). \quad (4.4)$$

Note that, with this transformation, the free variable that appears in the recursion formula for the coefficients is now denoted by the same symbol that, as a suffix, indicates their positions in the column $i = 2n$. By choosing any value for $A_{2n,0}^{(n)}$ (unity, for example, as in the case illustrated by Fig. 2), all the coefficients belonging to the column on the right hand side of the triangle can be evaluated by setting $j = 2, 4, \ldots$, up to $2n$ in succession. The arrow on the side of the column $i = 8$ in Fig. 2 indicates the sequence in which the positions are filled according to this procedure.

A closed form solution to the homogeneous difference equation for $A_{2n,j}^{(n)}$ can be easily obtained. Writing down Eq. (4.4) for $j = 2, 4, \ldots$ up to a generic (even) value $j$ and then multiplying all the relations so obtained one by the other in succession, we arrive at:

$$A_{2n,j}^{(n)} = (-1)^{\frac{j}{2}} 2 \frac{(2n)(2n - 2) \cdots (2n - j + 4)(2n - j + 2)}{(2n + 2)(2n + 4) \cdots (2n + j - 2)(2n + j)} \times \frac{(2n - 1)(2n + 1) \cdots (2n + j - 5)(2n + j - 3)}{(2n - 3)(2n - 5) \cdots (2n - j + 1)(2n - j - 1)} A_{2n,0}^{(n)} \quad (j = 2, 4, \ldots, 2n). \quad (4.5)$$

This expression can be written in a more concise manner as:

$$A_{2n,j}^{(n)} = (-1)^{\frac{j}{2}} \left[ \frac{2n - 1}{(2n)_n} \right]^2 \frac{2}{(2n - 1 - j)(2n - 1 + j)} \left( \frac{2n + j}{n - \frac{j}{2}} \right) \left( \frac{2n - j}{n - \frac{j}{2}} \right) A_{2n,0}^{(n)} \quad (j = 2, 4, \ldots, 2n - 2, 2n), \quad (4.6)$$
where the symbol \( \binom{p}{q} \) stands for the binomial coefficient of \( p \) with respect to \( q \) as usually defined \[7\]. The constant \( A_{2n,0}^{(n)} \) remains arbitrary.

(2) The equation for the coefficients on the upper side of the triangle of coefficients

We now turn our attention to Eq. (3.29) in Section III. With the transformation of the independent variable:

\[
k = i - n + 1,
\]

(4.7)

this equation takes on the convenient form:

\[
A_{i,2n-i}^{(n)} = -4 \left( \frac{i - n + 1}{2i - 2n - 1} \right) A_{i+1,2n-i-1}^{(n)} \quad (i = 2n - 1, 2n - 2, \ldots, n),
\]

(4.8)

which can be recognized, the same as the previous equation for the coefficients on the right hand side of the triangle, as a first order, ordinary, homogeneous difference equation for \( A_{2n-j,j}^{(n)} \), whose solution depends on one arbitrary constant. It is seen that, as the independent variable is varied in accordance with the sequence of integers from \( i = 2n - 1 \) to \( i = n \), Eq. (4.8) provides us with a recursive scheme starting with \( A_{2n,0}^{(n)} \) to evaluate the coefficients whose positions are aligned along the top side of the triangle. In Fig. 2 the sequence in which the coefficients for the case \( n = 4 \) are generated is indicated by an arrow above the upper side of the triangle, the initiating coefficient being \( A_{8,0}^{(4)} = 1 \).

A closed form solution for Eq. (4.8) can be derived by the usual procedure of writing down the equations for \( i = 2n - 1, 2n - 2, \ldots, \) down to a generic \( i \) and then multiplying all of them together. The result is:

\[
A_{i,2n-i}^{(n)} = (-4)^{2n-i} \frac{n(n-1)(n-2) \cdots (i-n+2)(i-n+1)}{(2n-3)(2n-5) \cdots (2i-2n+1)(2i-2n-1)} A_{2n,0}^{(n)},
\]

(4.9)

\( (i = 2n - 1, 2n - 2, \ldots, n). \)
With the help of the symbol for the binomial coefficients this expression can be recast as

\[ A_{i,2n-i}^{(n)} = (-1)^i 2^{3(2n-i)} \frac{2n-1}{2i-2n-1} \binom{2i-2n}{i-n} A_{2n,0}^{(n)} \]

\[ (i = 2n-1, 2n-2, \ldots, n+1, n). \]  

(4.10)

(3) The equation for the internal coefficients

We shall call internal coefficients all those that belong neither to the column on the right hand side nor to the upper side of the triangle of coefficients, including thus under this denomination also the coefficients that fill in the positions along the down side. For all of them the governing equation is Eq. (3.30). If we introduce the transformation defined by

\[
\begin{align*}
  k &= i - n + 1, \\
  p &= 1 + \frac{1}{2} (i - j),
\end{align*}
\]

(4.11)

it can be brought to the form:

\[ A_{i,j}^{(n)} = \left( \frac{i - j + 2}{i - j - 1} \right) \left( \frac{i + j - 3}{i + j} \right) A_{i,j-2}^{(n)} + 2A_{i+1,j-1}^{(n)} \]

\[ (i = 2n-1, 2n-2, \ldots, n+2, n+1; j = 2n-i+2, 2n-i+4, \ldots, i-2, i). \]  

(4.12)

The use of Eq. (4.12) as a recursion formula to evaluate the internal coefficients and complete the filling in of the positions still vacant in the triangle requires that the positions in the column on the right hand side and those along the upper skew side be already filled up. For illustration of this requirement and of the computational scheme brought forth by the above mentioned difference equation, the arrows in Fig. 2 directed from the element \( A_{7,1}^{(4)} \) to the element \( A_{8,2}^{(4)} \) and from this one to the element \( A_{7,3}^{(4)} \) intend to indicate that the last-named follows up an operation performed on the two first.
Distinctly from the two other previously considered difference equations, Eq. (4.12) is a partial difference equation and one of the second order, since it relates the unknown function $A^{(n)}_{i,j}$ in three neighboring positions not aligned on the plane $(i,j)$. Two of the positions, however, belong to the same column and this makes it possible to treat the equation as an ordinary difference equation of the first order in the variable designating the row for the coefficients belonging to this common column, the free variable designative of the column itself being seen as a parameter. This approach requires that the third coefficient present in the equation, whose place in the $(i,j)$ diagram falls on a neighboring column, be assumed to be known, and leads to a recursion relation between columns rather than a relation between single elements, as in Eq. (4.12).

We rewrite Eq. (4.12) under the form of an inhomogeneous equation as:

$$A^{(n)}_{i,j} - h(i, j - 2)A^{(n)}_{i,j-2} = g(i + 1, j - 1)A^{(n)}_{i+1,j-1},$$

(4.12')

where we have introduced the functions:

$$h(i, j) = -\frac{(i - j)(i + j - 1)}{(i - j - 3)(i + j + 2)}$$

(4.13)

and

$$g(i, j) = -2\left(\frac{i - j}{i - j - 3}\right),$$

(4.14)

and where the right hand side is taken as the forcing term, supposed to be known. The resolution of this difference equation requires, as a first step, that it be multiplied by a “summing factor” $[9]$, the analog of an integrating factor for a first order differential equation, the effect of which is to convert the left hand side into an exact “discrete differential”. To find out which is this factor we multiply Eq. (4.12') by the reciprocal of a function $W^{(n)}(i, j)$, unknown as yet, and obtain:

$$\frac{A^{(n)}_{i,j}}{W^{(n)}(i, j)} - \frac{h(i, j - 2)A^{(n)}_{i,j-2}}{W^{(n)}(i, j)} = \frac{g(i + 1, j - 1)A^{(n)}_{i+1,j-1}}{W^{(n)}(i, j)}.$$  

(4.15)
To make the left hand side assume the desired form, we must choose \( W(n)(i, j) \) in such way that the equality:

\[
W(n)(i, j) = h(i, j - 2)W(n)(i, j - 2) \quad (i = 2n - 1, 2n - 2, \ldots, n + 1;
\]

\[
j = 2n - i + 2, 2n - i + 4, \ldots, i)
\]

be true, in which Eq. (4.15) can be written as:

\[
a^{(n)}_{i, j} - a^{(n)}_{i, j - 2} = g(i + 1, j - 1) W(n)(i + 1, j - 1) a^{(n)}_{i+1, j-1},
\]

where we have employed the notation:

\[
a^{(n)}_{i, j} \equiv \frac{A^{(n)}_{i, j}}{W(n)(i, j)}.
\]

The condition imposed upon \( W(n)(i, j) \), which translates by Eq. (4.16), can be viewed as a recursion formula for the dependence of \( W(n)(i, j) \) on the variable \( j \). No demand is made on its end value, which can be chosen \emph{ad libitum}, and regarding the convenience coming from simplicity, we take

\[
W(n)(i, 2n - i) = 1.
\]

The solution to Eq. (4.16) then flows from the method generally applicable to first order, homogeneous difference equations, and is:

\[
W(n)(i, j) = h(i, 2n - i)h(i, 2n - i + 2) \cdots h(i, j - 2).
\]

By use of Eq. (4.13) for \( h(i, j) \), and making appeal to the symbol of binomial coefficients to give a concise representation to the product on the right hand side, this expression can be brought to the form:

\[
W(n)(i, j) = (-1)^{\frac{i+j}{2}} \frac{2n - 1}{(2n)^{\frac{i+j}{2}}} \frac{2i - 2n - 1}{(i - j - 1)(i + j - 1)} \frac{\binom{i+j}{i} \binom{i-j}{i-j}}{\binom{2n-2}{i-1-n}},
\]

\[
(i = 2n - 1, 2n - 2, \ldots, n + 1; j = 2n - i + 2, 2n - i + 4, \ldots, i).
\]

28
With the help of this formula, the factor that appears on the right hand side of Eq. (4.17) can be readily shown to be:

\[
\frac{W^{(n)}(i+1, j-1)}{W^{(n)}(i, j)} = -\frac{4}{g(i+1, j-1)} \left( \frac{i-n+1}{2i-2n-1} \right),
\]

(4.22)

where we have made use of the notation for the function \(g(i, j)\) introduced by Eq. (4.14).

We thus have for the weighted coefficients \(a^{(n)}_{i,j}\) the equation:

\[
a^{(n)}_{i,j} - a^{(n)}_{i,j-2} = -4 \left( \frac{i-n+1}{2i-2n-1} \right) a^{(n)}_{i+1,j-1}
\]

\[(i = 2n - 1, 2n - 2, \ldots, n + 1; j = 2n - i + 2, 2n - i + 4, \ldots, i). \quad (4.23)
\]

Considered as an ordinary difference equation for the elements aligned along the column labelled by \(j\) in the triangle of coefficients, the end condition for Eq. (4.23) is provided by Eq. (4.8), here rewritten as:

\[
a^{(n)}_{i,2n-i} = -4 \left( \frac{i-n+1}{2i-2n-1} \right) a^{(n)}_{i+1,2n-i-1} \quad (i = 2n - 1, 2n - 2, \ldots, n), \quad (4.8')
\]

where we have employed the equality between the weighted coefficients and the coefficients themselves at the positions on the top of the columns:

\[
a^{(n)}_{k,2n-k} = A^{(n)}_{k,2n-k}, \quad (4.24)
\]

which is itself a consequence of the adoption of Eq. (4.19) as end condition for \(W^{(n)}(i, j)\).

By writing Eq. (4.23) for a fixed \(i\) and \(j = 2n - i + 2, 2n - i + 4, \ldots\) up to a generic \(j\) in succession, and then adding all the equations so obtained, we arrive at the solution for the weighted coefficients \(a^{(n)}_{i,j}\) belonging to the \(i\)-column in terms of a summation carried on coefficients belonging to the neighboring \(i+1\)-column:

\[
a^{(n)}_{i,j} = -4 \left( \frac{i-n+1}{2i-2n-1} \right)^{\frac{i}{2i-2n-1}} \sum_{\ell=0}^{n} a^{(n)}_{i+1,2n-i-1+2\ell}
\]

\[(i = 2n - 1, 2n - 2, \ldots, n + 1; j = 2n - i, 2n - i + 2, \ldots, i - 2, i). \quad (4.25)
\]
Regarding future use, it is also of interest to point out here that, from Eqs. (4.18), (4.6) and (4.21), the weighted coefficients for the column on the right hand side of the triangle take on the values:

\[ a^{(n)}_{2n,0} = A^{(n)}_{2n,0}, \]  
\[ a^{(n)}_{2n,j} = 2A^{(n)}_{2n,0} \quad (j = 2, 4, \ldots, 2n). \]  

(4.26a)  
(4.26b)

We conclude this Section by illustrating the use of the several recursion formulae for the coefficients we have derived by means of a numerical application.

**Example.** Evaluation of the Chebyshev coefficients for the multipole solution of order \( n = 4 \).

(a) We start by the coefficients belonging to the column on the right hand side of the triangle of coefficients. The recursion formula to be employed in this case is that of Eq. (4.4), which, for \( n = 4 \), is written as:

\[ A^{(4)}_{8,j} = -(1 + \delta_j,2) \frac{(10 - j)(5 + j)}{(8 + j)(7 - j)} A^{(4)}_{8,j-2} \quad (j = 2, 4, 6, 8). \]  

(4.27)

Assuming that \( A^{(4)}_{8,0} = 1 \), we obtain in succession:

\[ A^{(4)}_{8,2} = -\frac{56}{25}, \quad A^{(4)}_{8,4} = \frac{84}{25}, \quad A^{(4)}_{8,6} = -\frac{264}{25}, \quad A^{(4)}_{8,8} = -\frac{429}{25}. \]  

(4.28)

(b) We now evaluate the coefficients for the column adjacent to that on the right hand side of the triangle, using Eq. (4.23) as a recurrence formula between contiguous rows. For \( n = 4 \) and \( i = 7 \), this is:

\[ a^{(4)}_{7,j} = a^{(4)}_{7,j-2} - \frac{16}{5} a^{(4)}_{8,j-1} \quad (j = 3, 5, 7). \]  

(4.29)

The weighted coefficient on the top of the column must be evaluated from Eq. (4.25), which furnishes:

\[ a^{(4)}_{7,1} = -\frac{16}{5} a^{(4)}_{8,0}. \]  

(4.30)
The other ones that are required in precedence to the use of Eq. (4.29) are those making up the column corresponding to \( i = 8 \) and are promptly given by Eqs. (4.26a) and (4.26b). We have:

\[
a_{8,0}^{(4)} = 1, \quad a_{8,2}^{(4)} = a_{8,4} = a_{8,6} = a_{8,8}^{(4)} = 2.
\] (4.31)

We then obtain from Eqs. (4.30), (4.31) and from the repeated use of Eq. (4.29):

\[
a_{7,1}^{(4)} = -\frac{16}{5}, \quad a_{7,3}^{(4)} = -\frac{48}{5}, \quad a_{7,5}^{(4)} = -16, \quad a_{7,7}^{(4)} = -\frac{112}{5}.
\] (4.32)

The bridge between the weighted coefficients \( a_{7,j}^{(4)} \) and the coefficients \( A_{7,j}^{(4)} \) is the function \( W^{(4)}(7,j) \), the knowledge of which requires the knowledge of the function \( h(7,j) \). By resorting to Eq. (4.13) we obtain for the values of \( j \) of interest:

\[
h(7,1) = -\frac{7}{5}, \quad h(7,3) = -3, \quad h(7,5) = \frac{11}{7}.
\] (4.33)

With this, the values of \( W^{(4)}(7,j) \) can be computed recursively from Eq. (4.16), which, for \( i = 7 \), becomes:

\[
W^{(4)}(7,j) = h(7,j - 2)W^{(4)}(7,j - 2).
\] (4.34)

Starting with:

\[
W^{(4)}(7,1) = 1,
\] (4.35)

in obedience to the end condition stated in Eq. (4.19), now obligatory as a matter of consistency with the values ascribed to \( a_{8,j}^{(4)} \) in Eq. (4.31), by putting \( j = 3, 5 \) and \( 7 \) in Eq. (4.34) we obtain one after the other:

\[
W^{(4)}(7,3) = -\frac{7}{5}, \quad W^{(4)}(7,5) = \frac{21}{5}, \quad W^{(4)}(7,7) = \frac{33}{5}.
\] (4.36)

The values of the Chebyshev coefficients \( A_{7,j}^{(4)} \), according to Eq. (4.18), are given by the product of the weighted coefficients \( a_{7,j}^{(4)} \) and the values of the function \( W^{(4)}(7,j) \), and this leads to:

\[
A_{7,1}^{(4)} = -\frac{16}{5}, \quad A_{7,3}^{(4)} = \frac{336}{25}, \quad A_{7,5}^{(4)} = -\frac{336}{5}, \quad A_{7,7}^{(4)} = -\frac{3696}{25}.
\] (4.37)
(c) We now complete the filling in of the vacancies in the triangle of coefficients using, for the purpose of illustration, the two other recursion formulae we have derived, namely, the one given by Eq. (4.8), which applies to the coefficients high up on the columns, and that of Eq. (4.12'), which generates the internal coefficients. For $n = 4$ the first of these is:

$$A_{i,8-i}^{(4)} = -4 \left( \frac{i - 3}{2i - 9} \right) A_{i+1,7-i}^{(4)} \quad (i = 7, 6, 5, 4),$$  \hspace{1cm} (4.38)

and gives, in addition to the value of $A_{7,1}^{(4)}$ already found, the following ones:

$$A_{6,2}^{(4)} = \frac{64}{5}, \quad A_{5,3}^{(4)} = -\frac{512}{5}, \quad A_{4,4}^{(4)} = -\frac{2048}{5}.$$  \hspace{1cm} (4.39)

For the column $i = 6$, Eq. (4.12') becomes for $j = 4$ and $j = 6$, respectively:

$$A_{6,4}^{(4)} = h(6, 2)A_{6,2}^{(4)} + g(7, 3)A_{7,3}^{(4)},$$  \hspace{1cm} (4.40)

$$A_{6,6}^{(4)} = h(6, 4)A_{6,4}^{(4)} + g(7, 5)A_{7,5}^{(4)}.$$  \hspace{1cm} (4.41)

Referring to Eqs. (4.13) and (4.14) we determine:

$$\begin{aligned}
    h(6, 2) &= -\frac{14}{5}, \quad h(6, 4) = \frac{3}{2} ; \\
    g(7, 3) &= -8, \quad g(7, 5) = 4,
\end{aligned}$$  \hspace{1cm} (4.42)

from which and from the values already known for the coefficients on the right hand side of Eqs. (4.40) and (4.41) we obtain:

$$A_{6,4}^{(4)} = -\frac{3584}{25}, \quad A_{6,6}^{(4)} = -\frac{12096}{25}.$$  \hspace{1cm} (4.43)

Finally, putting $i = 5, \quad j = 5$ in Eq. (4.12') we have the relation for the last coefficient still missing in the triangle:

$$A_{5,5}^{(4)} = h(5, 3)A_{5,3}^{(4)} + g(6, 4)A_{6,4}^{(4)},$$  \hspace{1cm} (4.44)

which, with

$$h(5, 3) = \frac{7}{5} \quad \text{and} \quad g(6, 4) = 4,$$  \hspace{1cm} (4.45)
yields:

\[ A_{5,5}^{(4)} = -\frac{3584}{5}. \]  

(4.46)

Having found all the Chebyshev coefficients, we can now establish the multipole solution of the fourth order, whose form, according to Eq. (2.32), is

\[
\varphi^{(4)}(x, \mu) = f_4^{(4)}(\mu)x^4 + f_5^{(4)}(\mu)x^5 + f_6^{(4)}(\mu)x^6 + f_7^{(4)}(\mu)x^7 + f_8^{(4)}(\mu)x^8. 
\]  

(4.47)

Considering that the Chebyshev polynomial \( T_n(\mu = \cos \theta) \) \( (n = 0, 1, 2, \ldots) \) is identical with the trigonometric function \( \cos n\theta \), and recalling the general expression for \( f_n^{(n)}(\mu) \) as given by Eq. (3.26), the angular functions that enter the constitution of \( \varphi^{(4)}(x, \mu) \) of the present case can be written as:

\[
f_4^{(4)}(\mu) = A_{4,4}^{(4)}T_4(\mu) = -\frac{2048}{5}\cos 4\theta, \]

(4.48a)

\[
f_5^{(4)}(\mu) = A_{5,5}^{(4)}T_5(\mu) + A_{5,3}^{(4)}T_3(\mu) = -\frac{512}{5}\cos 3\theta - \frac{3584}{5}\cos 5\theta, \]

(4.48b)

\[
f_6^{(4)}(\mu) = A_{6,2}^{(4)}T_2(\mu) + A_{6,4}^{(4)}T_4(\mu) + A_{6,6}^{(4)}T_6(\mu) = \frac{64}{5}\cos 2\theta - \frac{3584}{25}\cos 4\theta - \frac{12096}{25}\cos 6\theta, \]

(4.48c)

\[
f_7^{(4)}(\mu) = A_{7,1}^{(4)}T_1(\mu) + A_{7,3}^{(4)}T_3(\mu) + A_{7,5}^{(4)}T_5(\mu) + A_{7,7}^{(4)}T_7(\mu) = -\frac{16}{5}\cos \theta + \frac{336}{25}\cos 3\theta - \frac{336}{5}\cos 5\theta - \frac{3696}{25}\cos 7\theta, \]

(4.48d)

\[
f_8^{(4)}(\mu) = A_{8,0}^{(4)}T_0(\mu) + A_{8,2}^{(4)}T_2(\mu) + A_{8,4}^{(4)}T_4(\mu) + A_{8,6}^{(4)}T_6(\mu) + A_{8,8}^{(4)}T_8(\mu) = 1 - \frac{56}{25}\cos 2\theta + \frac{84}{25}\cos 4\theta - \frac{264}{25}\cos 6\theta - \frac{429}{25}\cos 8\theta. \]

(4.48e)

Alternatively, using [7], [10]:

\[
T_3(\mu) = -3\mu + 4\mu^3, \]

(4.49)

\[
T_4(\mu) = 1 - 8\mu^2 + 8\mu^4, \]

(4.50)
\[ T_5(\mu) = 5\mu - 20\mu^3 + \mu^5, \quad (4.51) \]
\[ T_6(\mu) = -1 + 18\mu^2 - 48\mu^4 + 32\mu^6, \quad (4.52) \]
\[ T_7(\mu) = -7\mu + 56\mu^3 - 112\mu^5 + 64\mu^7, \quad (4.53) \]
\[ T_8(\mu) = 1 - 32\mu^2 + 160\mu^4 - 256\mu^6 + 128\mu^8, \quad (4.54) \]

and \( T_1(\mu) \) and \( T_2(\mu) \) as given by Eqs. (2.9) and (2.21), these same angular functions can be stated in the form:

\[ f_4(4)(\mu) = \frac{2048}{5}(-1 + 8\mu^2 - 8\mu^4), \quad (4.55a) \]
\[ f_5(4)(\mu) = \frac{4096}{5}(-4\mu + 17\mu^3 - 14\mu^5), \quad (4.55b) \]
\[ f_6(4)(\mu) = \frac{1024}{25}(8 - 184\mu^2 + 539\mu^4 - 378\mu^6), \quad (4.55c) \]
\[ f_7(4)(\mu) = \frac{1024}{25}(16\mu - 168\mu^3 + 378\mu^5 - 231\mu^7), \quad (4.55d) \]
\[ f_8(4)(\mu) = \frac{128}{25}(64\mu^2 - 432\mu^4 + 792\mu^6 - 429\mu^8). \quad (4.55e) \]

Combining the angular functions \( f_4(4)(\mu), f_5(4)(\mu), \ldots, f_8(4)(\mu) \) listed above according to Eq. (4.47), the multipole solution of order \( n = 4 \) is obtained. In Appendix A and in Appendix B the expressions for \( \varphi_4(x, \theta) \) and for \( \varphi_4(x, \mu) \), as combinations of harmonics of the angle \( \theta \) and as polynomials in \( \mu \) respectively, are written in full after being divided by a normalization factor equal to 512.

V. SOLUTION TO THE DIFFERENCE EQUATION FOR THE INTERNAL COEFFICIENTS. UNIFIED EXPRESSION FOR THE CHEBYSHEV COEFFICIENTS OF THE MULTIPOLe SOLUTIONS

In the previous Section the problem of the determination of the Chebyshev coefficients \( A_{ij}^{(n)} \) for the multipole solution of order \( n \) to the sourceless Grad-Shafranov
equation was reduced to that of a partial difference equation of the second order subjected to boundary conditions on two frontiers of the domain of the two variables $i$ and $j$ in which the problem is stated, one corresponding to the “line” $i + j = 2n$ and the other to the “line” $i = 2n$. The boundary conditions take the form of values imposed on the dependent variable which are themselves fixed by two ordinary difference equations, for which solutions were obtained in terms of a single coefficient which remains arbitrary. In the present Section we establish the solution to the partial difference equation apt to the inner region and show that it makes possible to express the coefficients $A_{i,j}^{(n)}$ by a single formula which encompasses all domains pertaining to the statement of the problem, the border sites and the inside alike.

We commence by considering Eq. (4.23) for the weighted coefficients $a_{i,j}^{(n)}$, in which only one of the multiplying coefficients to the unknown function in the three terms of which it is constituted is not a constant. The fact that this coefficient depends on only one of the two independent variables opens the way to converting the equation into one of constant coefficients. Indeed, if we write the dependent variable as:

$$a_{i,j}^{(n)} = p^{(n)}(i)b_{i,j}^{(n)}, \quad (5.1)$$

Eq. (4.23) becomes:

$$p^{(n)}(i)\left[b_{i,j}^{(n)} - b_{i,j-2}^{(n)}\right] = -4 \left(\frac{i - n + 1}{2i - 2n - 1}\right)p^{(n)}(i + 1)b_{i+1,j-1}^{(n)}$$

$$(i = 2n - 1, 2n - 2, \ldots, n + 1; \quad j = 2n - i + 2, 2n - i + 4, \ldots, i). \quad (5.2)$$

If now we put:

$$p^{(n)}(i) = -4 \left(\frac{i - n + 1}{2i - 2n - 1}\right)p^{(n)}(i + 1), \quad (5.3)$$

then the equation governing the transformed function $b_{i,j}^{(n)}$ is:

$$b_{i,j}^{(n)} - b_{i,j-2}^{(n)} - b_{i+1,j-1}^{(n)} = 0$$

$$(i = 2n - 1, 2n - 2, \ldots, n + 1; \quad j = 2n - i + 2, 2n - i + 4, \ldots, i). \quad (5.4)$$
which bears the feature of being of constant coefficients.

We first take care of Eq. (5.3), which can be easily solved. Writing it for

\[ i = 2n - 1, 2n - 2, \ldots, \text{down to a generic } i \geq n + 1, \]

and then multiplying the equations so obtained one by the other all together, we reach:

\[
p^{(n)}(i) = (-1)^{2n-i} 2^{3(2n-i)} \left( \frac{2n - 1}{2i - 2n - 1} \right) \binom{2n-2}{2} \binom{2n}{i} \binom{2i}{2n} p^{(n)}(2n)
\]

\[ (i = 2n - 1, 2n - 2, \ldots, n + 1). \quad (5.5) \]

We are free to choose the end value for \( p^{(n)}(i) \) and in the benefit of simplicity we put \( p^{(n)}(2n) = 1 \).

The attack to Eq. (5.4) has to be preceded by the specification of the boundary conditions to be applied to the function \( b^{(n)}_{i,j} \). For the column making the right hand side frontier of the domain of the free variables, \( \text{id est} \), for \( i = 2n \), from Eq. (4.26b) and Eq. (5.1) we have:

\[
b^{(n)}_{2n,j} = 2A^{(n)}_{2n,0} \quad (j = 0, 1, \ldots, 2n).
\]

\[ (5.6) \]

For the positions high up in the columns, having recourse to the connection between \( a^{(n)}_{2n-j,j} \) and \( b^{(n)}_{2n-j,j} \) coming from Eq. (5.1), then evaluating \( p^{(n)}(2n - j) \) according to Eq. (5.5), and finally recalling the frontier values for the weighted coefficients \( a^{(n)}_{i,j} \) and for the coefficient \( A^{(n)}_{i,j} \) as they come stated in Eqs. (4.24) and (4.10) respectively, we are able to establish that:

\[
b^{(n)}_{2n-j,j} = A^{(n)}_{2n,0} \quad (j = 0, 1, \ldots, n).
\]

\[ (5.7) \]

Having determined all the boundary conditions to be satisfied by the solution of Eq. (5.4) we turn our attention to the equation itself. A more familiar form can be given to it by introducing new independent variables \( k \) and \( \ell \) related to the ones we
have been using until now through the equations:

\[
\begin{aligned}
    k &= n - \frac{i - j}{2}, \\
    \ell &= \frac{i + j}{2} - n.
\end{aligned}
\]  

(5.8)

The ranges of variation of \( k \) and \( \ell \) are respectively:

\[
\begin{aligned}
    k &= 0, 1, 2, \ldots, n - 1, n; \\
    \ell &= 0, 1, 2, \ldots, k - 1, k .
\end{aligned}
\]  

(5.9)

Displayed in a Cartesian arrangement, the domain of the pair of variables \((k, \ell)\) still shows to be a triangular one but at variance with that of the variables \((i, j)\), it contains no positions associated with holes, since the allowed values for one and the other variables obey a complete sequence of integers (starting with zero). Note that the inverse transformation to the one defined by Eq. (5.8) is:

\[
\begin{aligned}
    i &= 2n - k + \ell , \\
    j &= k + \ell .
\end{aligned}
\]  

(5.10)

Following the transformation of free variables \((i, j) \rightarrow (k, \ell)\), the dependent variable transforms as \(b_{i,j}^{(n)} \rightarrow F_{k,\ell}^{(n)}\), and, in place of Eq. (5.4) we have:

\[
F_{k,\ell}^{(n)} = F_{k-1,\ell-1}^{(n)} + F_{k-1,\ell}^{(n)}
\]

\(k = 2, 3, \ldots, n - 1, n; \ \ell = 1, 2, \ldots, k - 1). \tag{5.11}

After consultation to Eqs. (5.6) and (5.7) on the boundary conditions for \(b_{i,j}^{(n)}\), we find that they translate for \(F_{k,\ell}^{(n)}\) as:

\[
F_{k,k}^{(n)} = 2A_{2n,0}^{(n)} \quad (k = 1, 2, \ldots, n) \tag{5.12a}
\]

and

\[
F_{k,0}^{(n)} = A_{2n,0}^{(n)} \quad (k = 0, 1, 2, \ldots, n). \tag{5.12b}
\]
The equation we have been able to derive for $F_{k,\ell}^{(n)}$ could be called Stifel’s equation, since it bears the precise form of the formula associated with the name of Stifel that relates three contiguous binomial coefficients and that provides the basis for the step-by-step procedure of construction of Tartaglia’s triangle (also known as Pascal’s arithmetic triangle) for the coefficients of the binomial expansion. The solution of Eq. (5.11), however, is not a multiple of the binomial coefficient $\binom{k}{\ell}$ because of the factor 2 multiplying the arbitrary constant $A_{2n,0}^{(n)}$ that appears in Eq. (5.12a) for the values of $F_{k,\ell}^{(n)}$ along the boundary $\ell = k$. Perhaps the easiest way to establish the solution to the problem we have in hand is by inspecting a version of Pascal’s triangle that associates two different constant values with the two lines that delimit the available area for its expansion respectively, distinctly from the classical version in which just a single value is assumed for both.

Consider a quantity $G_{k,\ell}$ ($k = 2, 3, \ldots; \ell = 1, 2, \ldots, k - 1$) that obeys Stifel’s relation the same as $F_{k,\ell}^{(n)}$ in Eq. (5.11), and that takes on the value $G_{k,\ell} = 1 + c$, $c$ being an arbitrary constant, at the side boundary $\ell = k$ ($k \geq 1$) of the domain of the free variables, while keeping the reference value $G_{k,\ell} = 1$ along the upper boundary ($\ell = 0, k = 0, 1, 2, \ldots$). A limited extension of the arithmetic triangle for this case is represented in Fig. 3.

Each position in the table is filled by summing the constituents in two neighboring positions, both of which in the column immediately to the left side of that position, one in the upper row, and the other, below the latter, in the same row. The value of the entry in a position $(k, \ell)$ that is obtained by following this rule is made up of two parcels which can be traced back each to two analogous sets of starting values respectively, but imposed on displaced boundaries. The first parcel is the same that we would have with $c = 0$ and there is no need to say more about it than that it is the binomial coefficient $\binom{k}{\ell}$ of the classical Pascal’s triangle. The
second parcel, which contains the constant $c$ as a factor, is obtained by filling all positions in the row below the uppermost one having $k \geq 1$ with the constant value $c$ and the ones along the side boundary having $\ell \geq 2$ also with $c$, and applying the step-by-step building procedure of the table thereafter. This means that the number multiplying $c$ in the position $(k, \ell)$ is still a binomial coefficient, but shifted with respect to the one unrelated to $c$ by one column and one row, namely, it is the coefficient $\binom{k-1}{\ell-1}$. The addition of the two parcels gives then the tabulated quantity in Fig. 3 as:

$$G_{k,\ell} = \binom{k}{\ell} + c \binom{k-1}{\ell-1}, \quad (5.13)$$

or, which is the same, as:

$$G_{k,\ell} = \left(\frac{k + c\ell}{k}\right) \binom{k}{\ell}. \quad (5.14)$$

\[
\begin{array}{cccccccccc}
\hline
\ell & k \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
1 & 1+c & 2+c & 3+c & 4+c & 5+c & 6+c & 7+c & 8+c & 9+c & \cdots \\
2 & 1+c & 3+2c & 6+3c & 10+4c & 15+5c & 21+6c & 28+7c & \cdots \\
3 & 1+c & 4+3c & 10+6c & 20+10c & 35+15c & 56+21c & \cdots \\
4 & 1+c & 5+4c & 15+10c & 35+20c & 70+35c & 126+56c & \cdots \\
5 & 1+c & 6+5c & 21+15c & 56+35c & 126+70c & \cdots \\
6 & 1+c & 7+6c & 28+21c & 84+56c & \cdots \\
7 & 1+c & 8+7c & 36+28c & \cdots \\
8 & 1+c & 9+8c & \cdots \\
9 & 1+c & \cdots \\
\vdots & & & & & & & & & & \cdots \\
\hline
\end{array}
\]

**FIG. 3** Modified Pascal’s triangle with boundary values $G_{k,0} = 1$ ($k = 0, 1, \ldots$) and $G_{\ell,\ell} = 1 + c$ ($\ell = 1, 2, \ldots$).
To apply this result to the problem for $F_{n}^{(n)}$, defined by Eq. (5.11) together with Eqs. (5.12a) and (5.12b), all we have to do is to multiply it by $A_{2n,0}^{(n)}$ and to put $c = 1$. We get in this way:

$$F_{n}^{(n)} = \binom{k + \ell}{k} \binom{k}{\ell} A_{2n,0}^{(n)}$$

$$(k = 2, 3, \ldots, n - 1, n; \ell = 1, 2, \ldots, k - 1). \quad (5.15)$$

We now go through the way back from $F_{k,\ell}^{(n)}$ to $A_{i,j}^{(n)}$. Replacing the variables $k$ and $\ell$ in Eq. (5.15) by $i$ and $j$ respectively according to Eq. (5.8), we transform $F_{k,\ell}^{(n)}$ on the left hand side to $b_{i,j}^{(n)}$. The last-mentioned function, in the following of Eqs. (5.1) and (4.18), connects to the Chebyshev coefficient $A_{i,j}^{(n)}$ by way of the relation:

$$A_{i,j}^{(n)} = W^{(n)}(i, j)p^{(n)}(i)b_{i,j}^{(n)} \quad . \quad (5.16)$$

Using Eq. (4.21) for $W^{(n)}(i, j)$ and Eq. (5.5) for $p^{(n)}(i)$, the resulting expression that we obtain for $A_{i,j}^{(n)}$ can be recast as:

$$A_{i,j}^{(n)} = (-1)^{n - \frac{i+j}{2}} \left[ \frac{2n - 1}{\binom{2n}{n}} \right]^{2} 2^{3(2n-i)}$$

$$\times \frac{2j}{(i - j - 1)(i + j - 1)(2n - i + j)} \binom{i + j}{\frac{i+j}{2}} \binom{i - j}{\frac{i-j}{2}} \binom{n - \frac{i+j}{2}}{\frac{i+j}{2} - n} A_{2n,0}^{(n)} \quad . \quad (5.17)$$

The range of validity of this formula is that which suffices to cover all positions in the triangle of coefficients except the one at the vertex on the top of the right hand side, corresponding to $i = 2n$, $j = 0$, which we know to be occupied by the arbitrary coefficient $A_{2n,0}^{(n)}$, and can be stated as:

$$i = 2n, 2n - 1, 2n - 2, \ldots, n + 1, n; \quad j = 2n - i \neq 0, 2n - i + 2, 2n - i + 4, \ldots, i - 2, i. \quad \{5.18\}$$

This completes the solution to the problem. In Appendices A and B we provide the reader with tables of the multipole solutions of the orders $n = 0$ to $n = 9$ in
the toroidal-polar coordinate system, in which the overall multiplying constant for 
$n > 0$ was taken to be:

$$A^{(n)}_{2n,0} = \frac{1}{2^{2n+1}},$$

(5.19)
a choice intended to bring the numbers in general to more manageable dimensions 
for high values of $n$ than those of the ones afforded by the value $A^{(n)}_{2n,0} = 1$ adopted 
in the main text for illustrative purposes, while still keeping them of order unity for 
low values of $n$.

Numerical computations of equilibria are usually performed in the cylindrical 
system $(R, z, \phi)$ (see Fig. 1), the coordinate $\phi$ being ignorable, and it is useful to 
have the multipole solutions also expressed in this coordinate system. By defining 
the normalized cylindrical coordinates as:

$$\begin{align*}
\rho & \equiv \frac{R}{R_A} \\
Z & \equiv \frac{z}{R_A},
\end{align*}$$

(5.20)
transformation from the toroidal-polar system can be achieved by means of the 
formulae:

$$\begin{align*}
x &= \sqrt{(\rho - 1)^2 + Z^2}, \\
\mu &= \frac{\rho - 1}{\sqrt{(\rho - 1)^2 + Z^2}}.
\end{align*}$$

(5.21)
The ensuing expressions for the multipole solutions contain only even powers of 
$\rho$ and $Z$, and we thus find it convenient to introduce the variables:

$$\begin{align*}
\xi &= \rho^2 & \text{and} \\
\nu &= Z^2,
\end{align*}$$

(5.22a,b)
in terms of which results are presented in Appendix C. The multipole equation they 
satisfy then writes as:

$$2\xi \frac{\partial^2 \psi}{\partial \xi^2} + 2\nu \frac{\partial^2 \psi}{\partial \nu^2} + \frac{\partial \psi}{\partial \nu} = 0.$$
V.1. Field lines and flux surfaces

The discussion on the geometrical properties of the lines of force of the magnetic fields associated with the multipole solutions is more adequately conducted if we refer to the variables $\rho$ and $Z$ of the cylindrical system rather than to the variables $x$ and $\mu$ of the toroidal-polar one, both because of the greater simplicity the expressions assume in the former system and because of the symmetry the flux functions possess with respect to the equatorial plane of the magnetic configurations.

Magnetic field lines lying on the flux surface associated with the multipole solution of order $n$ are described by the equation:

$$\varphi^{(n)}(\rho, Z) = C,$$

where $C$ is a constant. Each value of $C$ specifies a field line on a meridian plane and the flux surface where it lies is generated by revolving it about the axis of rotational symmetry of the configuration (the $z$-axis). Since the quantity that ultimately bears a physical meaning is the magnetic field rather than the flux function, any constant can be added to the latter with no physical consequence whatsoever and no absolute meaning can be attached to the constant $C$ in Eq. (5.24). For the present discussion, however, it is natural to associate the null value of the flux function with the surface containing the “stagnation axis” (in the cases it does occur), which one appears as a circumference of a circle of radius $R_A$ and centre at $z = 0$ lying on the equatorial plane.

As general properties of the lines of force of the multipole fields we may say that, except for the ones of the two lowest orders, they comprise a variable number of (real) branches on each side of the equator line (on a plane $\Phi = \text{constant}$) dependent on the value of $C$ in Eq. (5.24), the least of these being one and the maximal equaling the order of the multipole; that they do not close upon themselves but extend to
infinity, as it should be expected on physical grounds, and thus that they do not
encircle a point of null field which would be identified with a magnetic axis; and that
for \( C = 0 \) in Eq. (5.24) they become separatrices, meaning this that they converge
to or diverge from a stagnation point, which for all of them is located at \( Z = 0, \rho = 1 \).

The multipole solution of zero order, \( \varphi^{(0)}(\rho, Z) = 1 \), corresponds of course to a
null magnetic field. The multipole solution of order \( n = 1 \) is given by
\[
\varphi^{(1)}(\rho, Z) = \frac{1}{4}(\rho^2 - 1),
\]
and the field lines associated with it are described by
\[
\rho = \text{constant},
\]
which means that they are straight lines parallel to the axis of rotational symmetry
of the magnetic configuration. This is the only case in which the magnetic field,
being uniform, vanishes at no point in space; for all other multipole solutions, both
the radial and the axial components of the magnetic field vanish at the point \( \rho = 1, Z = 0 \),
thus justifying the designation of “stagnation point” given to it.

Field lines associated with the multipole solution of order \( n = 2 \):
\[
\varphi^{(2)}(\rho, Z) = \rho^2 Z^2 - \frac{1}{4} \rho^4 + \frac{1}{2} \rho^2 - \frac{1}{4}
\]
representative of the various geometrical patterns that can be distinguished are
depicted in Fig. 4. The field line corresponding to the value \( C = 0 \) for the constant
in Eq. (5.24) touches the equator line \( Z = 0 \) at the radial coordinate \( \rho = 1 \) and
divides the upper and the lower half-planes into three regions each. Looking at
the upper half-plane, the region comprised between its two branches is the domain
of the lines associated with positive values of \( C \). The field lines corresponding to
negative values of \( C \) with \( |C| < 1/4 \) are composed of two branches apart, one of
Distinctive patterns assumed by the level curves of the multipole solution of order $n = 2$: $\varphi^{(2)}(\rho, Z) = C$, according to the values taken by $C$. The curve drawn in thick line corresponds to $C = 0$ and can be identified with the (trace of the) separatrix of the multipole field (on the plane $\phi = \text{constant}$). The curve lying in the domain of the plane $(\rho, Z)$ external to the region delimited by the branches of the separatrix and showing two symmetrical branches with respect to the $\rho$-axis corresponds to $C = 1$. Inside the domain delimited by the branches of the separatrix, the curve closest to its borders corresponds to $C = -1/8$, and has two separate branches, one lying on the left of the stagnation point $\rho = 1, Z = 0$, and the other on the right of it. Next to the latter it is shown the right branch of the curve for the critical value $C = -1/4$, whose left branch coincides with the $Z$-axis. Finally within the domain enclosed by the branches of the separatrix situated on the right of the stagnation point and the farthest away from that point it is seen the curve for $C = -1/2$, which has only this branch as real.
which is immersed in the region extending from the left branch of the separatrix \( C = 0 \) to the axis \( \rho = 0 \), while the other one belongs to the region on the right hand side of the right branch of the separatrix. The first of the two branches coincides with the axis \( \rho = 0 \) as the absolute value of \( C \) is increased to \( 1/4 \) and disappears (the function describing it becoming complex) upon further increase of \( |C| \); the one branch that remains for negative values of \( C \) with absolute value greater than \( 1/4 \) keeps still within the limits of the same region containing the branches for smaller values of \( |C| \), on the right hand side of the right branch of the separatrix, and is further removed to the right as \( |C| \) is increased.

For the multipole solution of order \( n = 3 \), which in cylindrical coordinates is expressed as:

\[
\varphi^{(3)}(\rho, Z) = \rho^2 Z^4 - \frac{3}{2} \rho^2 (\rho^2 - 1) Z^2 + \frac{1}{8} (\rho^2 - 1)^3,
\]

Fig. 5 shows the several zones into which the meridian plane is divided by the branches of the separatrix and by the two branches of the field line corresponding to the critical value \( C = -1/8 \) of the constant in Eq. (5.24). Starting from the \( Z \)-axis in the upper half-plane and moving clockwise towards the \( \rho \)-axis we traverse in succession:

(a) the region on the left of the stagnation point \( \rho = 1, Z = 0 \) comprised between the \( Z \)-axis and the first branch of the separatrix to be encountered, which contains one branch of the field lines corresponding to \( C < 0, |C| < 1/8 \);

(b) the region delimited by the first and the second branches of the separatrix, which lodges one branch of the field lines with \( C > 0 \);

(c) the region on the right of the stagnation point comprehended between the second branch of the separatrix, the second branch of the critical field line.
(the first one being represented by the $Z$-axis) and the third branch of the separatrix, which is the domain of the second branch of the curves having $C < 0$, $|C| < 1/8$;

FIG. 5  Particular level curves for the multipole solution of order $n = 3$: $\varphi^{(3)}(\rho, Z) = C$ on the meridian plane $\phi = \text{constant}$. With reference to the half-plane above the trace of the equator plane $Z = 0$ one sees in the figure: (a) the three branches of the separatrix, defined by $C = 0$, which are drawn in thick line and have as common point that located at $\rho = 1$, $Z = 0$; (b) the branches of the critical field line, for which $C = -1/8$, one of them coinciding with the $Z$-axis and the other internal to the region delimited by the branches of the separatrix on the right of the stagnation point.

(d) the region enclosed by the second branch of the critical field line, where the only real branch of the field lines that have $C < 0$, $|C| > 1/8$ is immersed;
(e) the region upbounded by the third branch of the separatrix and, within the realm of the upper half-plane, down bounded by the $\rho$-axis, which contains the second branch of the field lines associated with $C > 0$.

In general, a multipole solution of order $n$ gives rise to $n$ branches of the separatrix on each side of the equator line, which divide the upper and the lower half-planes into $n+1$ zones each; the $Z$-axis coincides with one branch of the field line associated with the (negative) value $C_0$ of the constant $C$ obtained by putting $\rho = 0$ in the expression of the multipole solution; the zone limited on the left by the axis $\rho = 0$ contains one branch of field lines defined by negative values of the constant $C$ in Eq. (5.24) whose absolute values are smaller than the absolute value of $C_0$; going through the upper half plane clockwise we traverse zones enclosing branches of lines of force associated with negative values of $C$ that alternate with zones that are the terrain of branches associated with positive values of $C$.

VI. SUMMARY

The general form of a multipole solution to the sourceless Grad-Shafranov equation that is even in regard to the half-spaces above and below the equatorial plane in the toroidal-polar coordinate system is stated in Eq. (2.32) as a polynomial in the radial coordinate normalized to $R_A$, the distance from the axis of rotational symmetry to the pole of the coordinate system. The numerical value of the exponent of the lowest power in this polynomial defines the order by which a particular multipole solution is identified. The angular-dependent coefficient functions of the powers of the radial coordinate variable are represented in Eq. (3.26) as combinations of Chebyshev polynomials of the first kind having the cosine of the poloidal angle as argument. The numerical coefficients of the Chebyshev polynomials in the
combinations can be conveniently displayed in a triangular array, an example of which is given in Fig. 2 for the multipole solution of order \( n = 4 \), and admit of being calculated by means of two alternative sets of laws of succession, both of which require, as starting value, that the coefficient \( A^{(n)}_{2n,0} \), located at the upper vertex on the right hand side of the triangle, be specified arbitrarily.

The first set comprehends Eq. (4.2), Eq. (4.8) and Eq. (4.12), which give shape to the rules for the sequential generation of the coefficients pertaining to the sites forming the column on the right hand side of the array, to those disposed along its upper side and to the remaining ones covering the two dimensional domain on the left of the first and below the second of these two sides, respectively.

The second set addresses to the weighted coefficients rather than to the coefficients themselves and consists of Eq. (4.8'), Eq. (4.26a) in conjunction with Eq. (4.26b), and Eq. (4.23) or equivalently Eq. (4.25), which parallel the equations of the first set in scope and have the same domains of application as these. The connection between the weighted coefficients and the coefficients proper is given by Eq. (4.18); its use requires the knowledge of the reciprocal of the function \( W^{(n)}(i,j) \), which one can be evaluated with the help of Eq. (4.16) taking unity as the initial value of the sequence corresponding to a fixed \( i \), as specified by Eq. (4.19), and recalling that the function \( h(i,j) \) is defined by Eq. (4.13).

Besides being calculable by these two recursive schemes, the coefficients can be obtained from a single expression, given by Eq. (5.17), which encompasses the solutions to the complete set of difference equations for the Chebyshev coefficients and whose range of validity reaches every site in the triangle of coefficients.

Except for the multipole solution of order \( n = 1 \), which gives a magnetic field constant and parallel to the axis of rotational symmetry, the multipole fields in general vanish at the radial coordinate \( r = 0 \).
Tables of the even multipole solutions of order \( n = 0 \) to \( n = 9 \) in variables of the toroidal-polar coordinate system are provided in Appendices A and B, and in variables of the cylindrical coordinate system in Appendix C.

**APPENDIX A: EXPRESSIONS FOR THE EVEN MULTPOLE SOLUTIONS OF ORDERS \( n = 0 \) TO \( n = 9 \) IN THE TOROIDAL-POLAR COORDINATE SYSTEM IN TERMS OF THE HARMONICS OF THE POLAR ANGLE \( \theta \)**

The variables are: \( x = r/R_A \), where \( r \) is the radial coordinate on the meridian plane and \( R_A \) is the distance from the axis of rotational symmetry to the pole of the coordinate system measured on the meridian plane; and \( \theta \), the polar angle on the meridian plane.

The multipole solutions, when their angular dependences are expressed in terms of the harmonics of the polar angle \( \theta \), can be stated in the following general form:

\[
\varphi^{(n)}(x, \theta) = \sum_{j=0}^{2n} M_{jn}(x) \cos j\theta \quad (n = 1, 2, 3, \ldots),
\]

where the coefficients \( M_{jn}(x) \) are given by:

(a) for \( j = 0, 1, 2, \ldots, n, \)

\[
M_{jn}(x) = \sum_{l=0}^{L_1} A_{2n-j+2l,j}^{(n)} x^{2n-j+2l}
\]

with

\[
L_1 = \begin{cases} 
\frac{j}{2} & \text{for } j = 0 \text{ or even} \\
\frac{j-1}{2} & \text{for } j \text{ odd};
\end{cases}
\]
(b) for $j = n + 1, n + 2, \ldots, 2n$,

$$M_{jn}(x) = \sum_{l=0}^{L_2} A_{2j+2l,j}^{(n)} x^{j+2l}$$  \hfill (A.4)

with

$$L_2 = \begin{cases} \frac{2n - j}{2} & \text{for } j \text{ even} \\ \frac{2n - 1 - j}{2} & \text{for } j \text{ odd}. \end{cases} \hfill (A.5)$$

The Chebychev coefficients $A^{(n)}_{k,j}$ are evaluated by means of the formula in Eq. (5.17) or by the recursive rules stated in Section IV.

The multipole solutions referred to in the title of this Appendix are listed below.

$$\varphi^{(0)}(x, \theta) = 1,$$ \hfill (A.6)

$$\varphi^{(1)}(x, \theta) = \frac{1}{8} x^2 \cos(2\theta) + \frac{1}{2} \cos(\theta) x + \frac{x^2}{8},$$ \hfill (A.7)

$$\varphi^{(2)}(x, \theta) = -\frac{5}{32} \cos(4\theta) x^4 - \frac{3}{4} \cos(3\theta) x^3 + \left(-\frac{1}{8} x^4 - x^2\right) \cos(2\theta)$$

$$-\frac{1}{4} \cos(\theta) x^3 + \frac{x^4}{32},$$ \hfill (A.8)

$$\varphi^{(3)}(x, \theta) = \frac{21}{256} \cos(6\theta) x^6 + \frac{35}{64} \cos(5\theta) x^5 + \left(-\frac{7}{128} x^6 + \frac{5}{4} x^4\right) \cos(4\theta)$$

$$+\left(\frac{15}{64} x^5 + x^3\right) \cos(3\theta) + \left(-\frac{5}{256} x^6 + \frac{1}{4} x^4\right) \cos(2\theta)$$

$$-\frac{1}{32} \cos(\theta) x^5 + \frac{x^6}{128},$$ \hfill (A.9)

$$\varphi^{(4)}(x, \theta) = -\frac{429}{12800} \cos(8\theta) x^8 - \frac{231}{800} \cos(7\theta) x^7 + \left(-\frac{33}{1600} x^8 - \frac{189}{200} x^6\right) \cos(6\theta)$$

$$+\left(-\frac{21}{160} x^7 - \frac{7}{5} x^5\right) \cos(5\theta) + \left(\frac{21}{3200} x^8 - \frac{7}{25} x^6 - \frac{4}{5} x^4\right) \cos(4\theta)$$

$$+\left(\frac{21}{800} x^7 - \frac{1}{5} x^5\right) \cos(3\theta) + \left(-\frac{7}{160} x^8 + \frac{1}{40} x^6\right) \cos(2\theta)$$
$$\frac{1}{160} \cos(\theta)x^7 + \frac{x^8}{512} , \quad (A.10)$$

$$\varphi(5)(x, \theta) = \frac{2431}{200704} \cos(10\theta)x^{10} + \frac{6435}{50176} \cos(9\theta)x^9$$
$$+ (\frac{100352}{429} \frac{x^{10} + \frac{429}{784} x^8}{297} \cos(8\theta) + (\frac{429}{7168} x^9 + \frac{33}{28} x^7) \cos(7\theta)$$
$$+ (-\frac{200704}{165} x^{10} + \frac{15}{56} x^8 + \frac{9}{7} x^6) \cos(6\theta)$$
$$+ (\frac{12544}{33} x^9 + \frac{15}{112} x^7 + \frac{4}{7} x^5) \cos(5\theta)$$
$$+ (\frac{25088}{129024} x^{10} - \frac{3}{224} x^8 + \frac{1}{7} x^6) \cos(4\theta) + (\frac{9}{1792} x^9 - \frac{1}{56} x^7) \cos(3\theta)$$
$$+ (-\frac{14336}{14336} x^{10} + \frac{1}{224} x^8) \cos(2\theta) - \frac{5}{3584} \cos(\theta)x^9 + \frac{x^{10}}{2048} , \quad (A.11)$$

$$\varphi(6)(x, \theta) = -\frac{4199}{1032192} \cos(12\theta)x^{12} - \frac{46189}{903168} \cos(11\theta)x^{11}$$
$$+ (\frac{1806336}{2431} x^{12} - \frac{60775}{225792} x^{10}) \cos(10\theta)$$
$$+ (\frac{100352}{4704} x^{11} - \frac{4704}{3575} x^9) \cos(9\theta)$$
$$+ (\frac{3612672}{715} x^{12} - \frac{715}{7056} x^{10} - \frac{715}{588} x^8) \cos(8\theta)$$
$$+ (\frac{129024}{672} x^{11} - \frac{143}{22} x^9 - \frac{22}{21} x^7) \cos(7\theta)$$
$$+ (\frac{715}{1806336} x^{12} + \frac{429}{25088} x^{10} - \frac{11}{49} x^8 - \frac{8}{21} x^6) \cos(6\theta)$$
$$+ (\frac{301056}{715} x^{11} + \frac{2352}{21} x^9 - \frac{2}{21} x^7) \cos(5\theta)$$
$$+ (\frac{2408448}{55} x^{12} - \frac{11}{2352} x^{10} + \frac{1}{84} x^8) \cos(4\theta)$$
$$+ (\frac{50176}{336} x^{11} - \frac{1}{336} x^9) \cos(3\theta) + (-\frac{11}{43008} x^{12} + \frac{5}{5376} x^{10}) \cos(2\theta)$$
$$- \frac{1}{3072} \cos(\theta)x^{11} + \frac{x^{12}}{8192} , \quad (A.12)$$

$$\varphi(7)(x, \theta) = \frac{185725}{142737408} \cos(14\theta)x^{14} + \frac{676039}{35684352} \cos(13\theta)x^{13}$$
$$+ (\frac{52003}{71368704} x^{14} + \frac{29393}{247808} x^{12}) \cos(12\theta)$$
$$+ (\frac{29393}{29393} x^{13} + \frac{20995}{50688} x^{11}) \cos(11\theta)$$

51
\[
\varphi^{(8)}(x, \theta) = -\frac{1077205}{2680291328} x^{16} + \frac{557175}{83759104} x^{15} - \frac{87751543215}{7233685376} x^{14} - \frac{36554001}{63366784} x^{13} - \frac{8075}{56628} x^{12} - \frac{119}{1287} x^{11} - \frac{8075}{56628} x^{10} - \frac{119}{1287} x^9\cos(10 \theta)
\]

\[
+(-\frac{595}{52003} x^{16} + \frac{1507663872}{2261} x^{15} + \frac{56525}{14496768} x^{14} - \frac{1190}{1287} x^{13} - \frac{119}{1287} x^{12} - \frac{119}{1287} x^{11} - \frac{119}{1287} x^{10} - \frac{119}{1287} x^9)\cos(9 \theta)
\]

\[
+(-\frac{119}{595} x^{16} + \frac{1507663872}{2261} x^{15} + \frac{56525}{14496768} x^{14} - \frac{1190}{1287} x^{13} - \frac{119}{1287} x^{12} - \frac{119}{1287} x^{11} - \frac{119}{1287} x^{10} - \frac{119}{1287} x^9)\cos(8 \theta)
\]

\[
+(-\frac{119}{595} x^{16} + \frac{1507663872}{2261} x^{15} + \frac{56525}{14496768} x^{14} - \frac{1190}{1287} x^{13} - \frac{119}{1287} x^{12} - \frac{119}{1287} x^{11} - \frac{119}{1287} x^{10} - \frac{119}{1287} x^9)\cos(7 \theta)
\]

\[
+(-\frac{119}{595} x^{16} + \frac{1507663872}{2261} x^{15} + \frac{56525}{14496768} x^{14} - \frac{1190}{1287} x^{13} - \frac{119}{1287} x^{12} - \frac{119}{1287} x^{11} - \frac{119}{1287} x^{10} - \frac{119}{1287} x^9)\cos(6 \theta)
\]

\[
+(-\frac{119}{595} x^{16} + \frac{1507663872}{2261} x^{15} + \frac{56525}{14496768} x^{14} - \frac{1190}{1287} x^{13} - \frac{119}{1287} x^{12} - \frac{119}{1287} x^{11} - \frac{119}{1287} x^{10} - \frac{119}{1287} x^9)\cos(5 \theta)
\]

\[
+(-\frac{119}{595} x^{16} + \frac{1507663872}{2261} x^{15} + \frac{56525}{14496768} x^{14} - \frac{1190}{1287} x^{13} - \frac{119}{1287} x^{12} - \frac{119}{1287} x^{11} - \frac{119}{1287} x^{10} - \frac{119}{1287} x^9)\cos(4 \theta)
\]

\[
+(-\frac{119}{595} x^{16} + \frac{1507663872}{2261} x^{15} + \frac{56525}{14496768} x^{14} - \frac{1190}{1287} x^{13} - \frac{119}{1287} x^{12} - \frac{119}{1287} x^{11} - \frac{119}{1287} x^{10} - \frac{119}{1287} x^9)\cos(3 \theta)
\]

\[
+(-\frac{119}{595} x^{16} + \frac{1507663872}{2261} x^{15} + \frac{56525}{14496768} x^{14} - \frac{1190}{1287} x^{13} - \frac{119}{1287} x^{12} - \frac{119}{1287} x^{11} - \frac{119}{1287} x^{10} - \frac{119}{1287} x^9)\cos(2 \theta)
\]

\[
+\left(-\frac{119}{595} x^{16} + \frac{1507663872}{2261} x^{15} + \frac{56525}{14496768} x^{14} - \frac{1190}{1287} x^{13} - \frac{119}{1287} x^{12} - \frac{119}{1287} x^{11} - \frac{119}{1287} x^{10} - \frac{119}{1287} x^9\right)\cos(\theta)\right)
\]

\[
\]
\[ \varphi^{(9)}(x, \theta) = \frac{1178589}{9746513920} \cos(18\theta)x^{18} + \frac{60108039}{26802913280} \cos(17\theta)x^{17} + \frac{53605826560}{5816907} x^{18} + \frac{104698880}{26898343} x^{16} \cos(16\theta) + \frac{5360582656}{1938969} x^{17} + \frac{5234944}{1938969} x^{15} \cos(15\theta) + \frac{364021}{107211653120} x^{18} + \frac{418795520}{66861} x^{16} + \frac{364021}{52003} x^{14} \cos(14\theta) + \frac{257720320}{66861} x^{17} + \frac{1610752}{66861} x^{15} + \frac{629200}{1092003} x^{14} \cos(13\theta) + \frac{6700728320}{7429} x^{17} - \frac{41879552}{15827} x^{16} + \frac{13087360}{257720320} x^{14} \cos(12\theta) + \frac{60915712}{37145} x^{17} - \frac{9518080}{114400} x^{15} + \frac{2584}{3575} x^{11} \cos(11\theta) + \frac{5360582656}{2261} x^{18} + \frac{83759104}{201344} x^{16} - \frac{2261}{15730} x^{12} + \frac{272}{715} x^{10} \cos(10\theta) + \frac{468027}{6700728320} x^{17} + \frac{6783}{4026880} x^{15} - \frac{8721}{629200} x^{13} + \frac{306}{3575} x^{11} + \frac{64}{715} x^{9} \cos(9\theta) + \frac{364021}{6700728320} x^{18} - \frac{2261}{8053760} x^{16} + \frac{323}{125840} x^{14} - \frac{110789}{34} x^{12} + \frac{16}{715} x^{10} \cos(8\theta) + \frac{257720320}{47481} x^{17} - \frac{2261}{4026880} x^{15} + \frac{119}{57200} x^{13} - \frac{2}{715} x^{11} \cos(7\theta) + \frac{10308812800}{6783} x^{18} + \frac{20349}{161075200} x^{16} - \frac{51}{91520} x^{14} + \frac{1}{1430} x^{12} \cos(6\theta) + \frac{257720320}{969} x^{17} + \frac{119}{357} x^{15} - \frac{1}{4576} x^{13} \cos(5\theta) + \frac{234291200}{153} x^{17} - \frac{732160}{21} x^{16} + \frac{91520}{91520} x^{14} \cos(4\theta) + \frac{10649600}{51} x^{17} - \frac{732160}{51} x^{15} \cos(3\theta) + \frac{13107200}{266240} x^{18} + \frac{3}{x^{16}} \cos(2\theta) - \frac{3}{655360} \cos(\theta)x^{17} + \frac{524288}{x^{18}} \]
APPENDIX B: EXPRESSIONS FOR THE EVEN MULTIPOLE SOLUTIONS OF ORDERS \( n = 0 \) TO \( n = 9 \) IN THE TOROIDAL-POLAR COORDINATE SYSTEM AS BIVARIATE POLYNOMIALS IN THE NORMALIZED RADIAL VARIABLE AND IN THE COSINE OF THE POLOIDAL ANGLE

The variables are: \( x \), as defined in Appendix A, and \( \mu \equiv \cos \theta \).

\[
\phi^{(0)}(x, \mu) = 1 ,
\]

(B.0)

\[
\phi^{(1)}(x, \mu) = \frac{1}{2} \mu x + \frac{1}{4} \mu^2 x^2 ,
\]

(B.1)

\[
\phi^{(2)}(x, \mu) = (-\frac{5}{4} \mu^4 + \mu^2) x^4 + (-3 \mu^3 + 2 \mu) x^3 + (-2 \mu^2 + 1) x^2 ,
\]

(B.2)

\[
\phi^{(3)}(x, \mu) = (\frac{21}{8} \mu^6 - \frac{7}{2} \mu^4 + \mu^2) x^6 + (\frac{35}{4} \mu^5 - 10 \mu^3 + 2 \mu) x^5 + (10 \mu^4 - \frac{19}{2} \mu^2 + 1) x^4 + (4 \mu^3 - 3 \mu) x^3 ,
\]

(B.3)

\[
\phi^{(4)}(x, \mu) = (-\frac{429}{100} \mu^8 + \frac{198}{25} \mu^6 - \frac{108}{25} \mu^4 + \frac{16}{25} \mu^2) x^8 + (-\frac{462}{25} \mu^7 + \frac{756}{25} \mu^5 - \frac{336}{25} \mu^3 + \frac{32}{25} \mu) x^7 + (-\frac{756}{25} \mu^6 + \frac{1078}{25} \mu^4 - \frac{368}{25} \mu^2 + \frac{16}{25}) x^6 + (-\frac{112}{5} \mu^5 + \frac{136}{5} \mu^3 - \frac{32}{5} \mu) x^5 + (-\frac{32}{5} \mu^4 + \frac{32}{5} \mu^2 - \frac{4}{5}) x^4 ,
\]

(B.4)

\[
\phi^{(5)}(x, \mu) = (\frac{2431}{392} \mu^{10} - \frac{715}{49} \mu^8 + \frac{572}{49} \mu^6 - \frac{176}{49} \mu^4 + \frac{16}{49} \mu^2) x^{10} + (\frac{6435}{196} \mu^9 - \frac{3432}{49} \mu^7 + \frac{2376}{49} \mu^5 - \frac{576}{49} \mu^3 + \frac{32}{49} \mu) x^9 + (\frac{3432}{49} \mu^8 - \frac{6567}{49} \mu^6 + \frac{3834}{49} \mu^4 - \frac{680}{49} \mu^2 + \frac{16}{49}) x^8 + (\frac{528}{7} \mu^7 - \frac{894}{7} \mu^5 + \frac{424}{7} \mu^3 - \frac{48}{7} \mu) x^7 + (\frac{288}{7} \mu^6 - \frac{424}{7} \mu^4 + 22 \mu^2 - \frac{8}{7}) x^6
\]

54
\[
+ \left( \frac{64}{7} \mu^5 - \frac{80}{7} \mu^3 + \frac{20}{7} \mu \right) x^5 , \tag{B.5}
\]

\[
\varphi(6)(x, \mu) = \left( -\frac{4199}{504} \mu^{12} + \frac{20995}{882} \mu^{10} - \frac{11050}{441} \mu^8 + \frac{5200}{441} \mu^6 - \frac{1040}{441} \mu^4 + \frac{64}{441} \mu^2 \right) x^{12} \\
- \left( \frac{46189}{882} \mu^{11} + \frac{60775}{441} \mu^9 - \frac{57200}{441} \mu^7 + \frac{22880}{441} \mu^5 - \frac{3520}{441} \mu^3 + \frac{128}{441} \mu \right) x^{11} \\
- \left( \frac{147}{147} \mu^{10} + \frac{292435}{147} \mu^8 - \frac{121264}{147} \mu^6 + \frac{39952}{147} \mu^4 - \frac{4352}{147} \mu^2 + \frac{64}{147} \right) x^{10} \\
- \left( \frac{28600}{147} \mu^9 + \frac{62348}{147} \mu^7 - \frac{44704}{147} \mu^5 + \frac{11584}{147} \mu^3 - \frac{256}{147} \mu \right) x^9 \\
- \left( \frac{147}{147} \mu^8 + \frac{44704}{147} \mu^6 - \frac{27002}{147} \mu^4 + \frac{1704}{147} \mu^2 - \frac{48}{147} \right) x^8 \\
- \left( \frac{1408}{21} \mu^7 + \frac{2432}{21} \mu^5 - \frac{1192}{21} \mu^3 + \frac{48}{7} \mu \right) x^7 + \left( -\frac{256}{21} \mu^6 + \frac{128}{7} \mu^4 - \frac{48}{21} \mu^2 + \frac{8}{21} \right) x^6 , \tag{B.6}
\]

\[
\varphi(7)(x, \mu) = \left( -\frac{185725}{17424} \mu^{14} + \frac{52003}{1452} \mu^{12} + \frac{11305}{242} \mu^{10} - \frac{32300}{242} \mu^8 - \frac{3400}{126} \mu^6 - \frac{160}{126} \mu^4 + \frac{64}{126} \mu^2 \right) x^{14} \\
+ \left( \frac{676039}{126} \mu^{13} - \frac{29393}{126} \mu^{11} + \frac{104975}{126} \mu^9 - \frac{176800}{126} \mu^7 - \frac{5200}{126} \mu^5 + \frac{1664}{126} \mu^3 + \frac{128}{126} \mu \right) x^{13} \\
+ \left( \frac{121}{121} \mu^{12} - \frac{4356}{121} \mu^{10} + \frac{2178}{121} \mu^8 - \frac{1089}{121} \mu^6 + \frac{86320}{121} \mu^4 - \frac{1089}{121} \mu^2 + \frac{1089}{121} \mu \right) x^{12} \\
+ \left( \frac{44200}{99} \mu^{11} - \frac{107900}{99} \mu^9 + \frac{91429}{99} \mu^7 - \frac{31192}{99} \mu^5 + \frac{3632}{99} \mu^3 - \frac{64}{99} \mu \right) x^{11} \\
+ \left( \frac{99}{99} \mu^{10} - \frac{198}{99} \mu^8 + \frac{99}{99} \mu^6 - \frac{31192}{99} \mu^4 + \frac{3632}{99} \mu^2 - \frac{64}{99} \right) x^{10} \\
+ \left( \frac{3120}{33} \mu^9 - \frac{20696}{33} \mu^7 + \frac{15154}{33} \mu^5 - \frac{4064}{33} \mu^3 + \frac{96}{33} \mu \right) x^9 \\
+ \left( \frac{3328}{33} \mu^8 - \frac{6592}{33} \mu^6 + \frac{4064}{33} \mu^4 - \frac{796}{33} \mu^2 + \frac{8}{33} \right) x^8 \\
+ \left( \frac{512}{33} \mu^7 - \frac{896}{33} \mu^5 + \frac{448}{33} \mu^3 - \frac{56}{33} \mu \right) x^7 , \tag{B.7}
\]

\[
\varphi(8)(x, \mu) = \left( \frac{1077205}{184041} \mu^{16} + \frac{104060}{184041} \mu^{14} - \frac{14560840}{184041} \mu^{12} + \frac{1648672}{184041} \mu^{10} - \frac{5064640}{184041} \mu^8 \\
+ \frac{1157632}{184041} \mu^6 - \frac{121856}{184041} \mu^4 + \frac{20449}{184041} \mu^2 x^{16} + \left( -\frac{20449}{184041} \mu^{15} + \frac{184041}{184041} \mu^{13} \\
- \frac{20449}{184041} \mu^{11} + \frac{61347}{184041} \mu^9 - \frac{28940800}{184041} \mu^7 + \frac{609280}{184041} \mu^5 - \frac{143360}{184041} \mu^3 \\
+ \frac{212663200}{184041} \mu^15 + \left( -\frac{184041}{184041} \mu^{14} + \frac{184041}{184041} \mu^{12} - \frac{20449}{184041} \mu^{10} \\
+ \frac{184041}{184041} \mu^8 - \frac{69632000}{184041} \mu^6 + \frac{1169920}{184041} \mu^4 - \frac{561152}{184041} \mu^2 + \frac{4096}{184041} \right) \right) x^{14}.
\]
\[\begin{align*}
&- \frac{11648672}{14157} \mu^{13} + \frac{36863344}{14157} \mu^{11} - \frac{44548160}{14157} \mu^9 + \frac{25568000}{14157} \mu^7 - \frac{7014400}{14157} \mu^5 \\
&+ \frac{800768}{14157} \mu^3 - \frac{8192}{14157} \mu x^{13} + \left( \frac{5064640}{14157} \mu^{12} + \frac{44548160}{14157} \mu^{10} - \frac{48665560}{14157} \mu^8 \right) x^{12} \\
&+ \frac{24267200}{14157} \mu^6 - \frac{5384960}{14157} \mu^4 + \frac{424960}{14157} \mu^2 - \frac{5120}{14157} \mu x^{12} \\
&- \frac{1157632}{14157} \mu^{11} + \frac{3124736}{14157} \mu^9 - \frac{3050176}{14157} \mu^7 + \frac{1291648}{14157} \mu^5 - \frac{220160}{14157} \mu^3 + \frac{10240}{14157} \mu x^{11} \\
&- \frac{1287}{14157} \mu^{10} + \frac{1502720}{14157} \mu^8 - \frac{1291648}{14157} \mu^6 + \frac{34624}{14157} \mu^4 - \frac{54272}{14157} \mu^2 + \frac{1024}{14157} \mu x^{10} \\
&+ \frac{20480}{14157} \mu^9 - \frac{137216}{14157} \mu^7 - \frac{101888}{14157} \mu^5 - \frac{27904}{14157} \mu^3 - \frac{2048}{14157} \mu x^9 \\
&+ \frac{8192}{429} \mu^8 + \frac{16384}{429} \mu^6 - \frac{10240}{429} \mu^4 + \frac{2048}{429} \mu^2 - \frac{64}{429} \mu x^8,
\end{align*}\]  
(B.8)

\[\begin{align*}
\phi^{(9)}(x, \mu) &= \left( \frac{1178589}{74360} \mu^{18} - \frac{7071534}{102245} \mu^{16} + \frac{12774384}{102245} \mu^{14} - \frac{12333888}{102245} \mu^{12} + \frac{1370432}{20449} \mu^{10} \\
&- \frac{10963456}{102245} \mu^8 + \frac{1906688}{155648} \mu^6 - \frac{511225}{102245} \mu^4 + \frac{511225}{102245} \mu^2 \right) x^{18} + \left( \frac{408980}{511225} \mu^7 \right) x^{17} \\
&+ \frac{102245}{511225} \mu^{15} + \frac{102245}{511225} \mu^{13} - \frac{93189376}{511225} \mu^{11} + \frac{46594688}{511225} \mu^9 - \frac{64827392}{511225} \mu^7 \\
&+ \frac{9261056}{511225} \mu^{16} + \frac{8192}{429} \mu^{17} \mu^7 - \frac{102245}{8192} \mu x^{17} + \left( \frac{62047008}{102245} \mu^6 - \frac{18587358}{7865} \mu^5 \right) x^{16} \\
&+ \frac{380037924}{102245} \mu^{12} + \frac{30868908}{102245} \mu^{10} + \frac{137251744}{102245} \mu^8 - \frac{162647296}{102245} \mu^6 \\
&+ \frac{18339328}{102245} \mu^{11} + \frac{739328}{4096} \mu^9 \mu^6 - \frac{29953728}{4096} \mu^5 - \frac{109622324}{102245} \mu^{13} x^{15} \\
&+ \frac{511225}{511225} \mu^{14} + \frac{511225}{511225} \mu^{12} + \frac{511225}{511225} \mu^{10} \right) x^{16} + \left( \frac{20449}{39325} \mu^{15} - \frac{20449}{39325} \mu^{13} \right) x^{14} \\
&+ \frac{93189376}{39325} \mu^{13} - \frac{297924368}{39325} \mu^{11} + \frac{36312176}{39325} \mu^9 - \frac{16269952}{39325} \mu^7 + \frac{11860864}{39325} \mu^5 \\
&+ \frac{280576}{39325} \mu^3 + \frac{9216}{39325} \mu x^{13} + \left( \frac{64827392}{39325} \mu^{12} - \frac{191588096}{39325} \mu^{10} + \frac{16269952}{39325} \mu^8 \\
&- \frac{1573}{107020512} \mu^6 + \frac{970688}{79360} \mu^4 - \frac{1024}{1573} \mu^2 + \frac{1573}{1024} \mu x^{12} \\
&- \frac{39325}{107020512} \mu^{11} + \frac{1573}{1573} \mu^9 + \frac{709648}{3575} \mu^7 - \frac{3050176}{3575} \mu^5 - \frac{21248}{143} \mu^3 - \frac{1024}{143} \mu x^{11} \\
&+ \left( \frac{139264}{715} \mu^{10} - \frac{26624}{715} \mu^8 - \frac{300544}{715} \mu^6 + \frac{21248}{715} \mu^4 + \frac{13088}{715} \mu^2 - \frac{256}{715} \mu x^9 \right) + \left( \frac{16384}{715} \mu^9 - \frac{36864}{715} \mu^7 - \frac{27648}{715} \mu^5 - \frac{1536}{143} \mu^3 + \frac{576}{715} \mu x^9 \right).
\end{align*}\]  
(B.9)
APPENDIX C: EXPRESSIONS FOR THE EVEN MULTIPLE SOLUTIONS OF ORDERS \( n = 0 \) TO \( n = 9 \) IN THE CYLINDRICAL COORDINATE SYSTEM

For simplicity of notation the variables are chosen to be:

\[
\xi \equiv \left( \frac{R}{R_A} \right)^2, \quad \nu \equiv \left( \frac{z}{R_A} \right)^2,
\]

where \( R \) and \( z \) are the cylindrical coordinates of a point and \( R_A \) is the radial coordinate defining the circle on the equatorial plane where the gradients of the multipole solutions are designated to be zero.

\[
\varphi^{(0)}(\xi, \nu) = 1, \quad (C.0)
\]

\[
\varphi^{(1)}(\xi, \nu) = \frac{1}{4}(\xi - 1), \quad (C.1)
\]

\[
\varphi^{(2)}(\xi, \nu) = -\frac{(\xi - 1)^2}{4} + \nu(\xi - 1) + \nu, \quad (C.2)
\]

\[
\varphi^{(3)}(\xi, \nu) = \frac{(\xi - 1)^3}{8} - \frac{3\nu(\xi - 1)^2}{2} + (\nu^2 - \frac{3}{2}\nu)(\xi - 1) + \nu^2, \quad (C.3)
\]

\[
\varphi^{(4)}(\xi, \nu) = -\frac{(\xi - 1)^4}{20} + \frac{6\nu(\xi - 1)^3}{5} + (-\frac{12}{5}\nu^2 + \frac{6}{5}\nu)(\xi - 1)^2
\]
\[
+ (\frac{16}{25}\nu^3 - \frac{16}{5}\nu^2)(\xi - 1) - \frac{4\nu^2}{5} + \frac{16\nu^3}{25}, \quad (C.4)
\]

\[
\varphi^{(5)}(\xi, \nu) = \frac{(\xi - 1)^5}{56} - \frac{5\nu(\xi - 1)^4}{7} + (\frac{20}{7}\nu^2 - \frac{5}{7}\nu)(\xi - 1)^3
\]
\[
+ (-\frac{16}{7}\nu^3 + \frac{30}{7}\nu^2)(\xi - 1)^2
\]
\[
+ (\frac{16}{49}\nu^4 - \frac{24}{7}\nu^3 + \frac{10}{7}\nu^2)(\xi - 1) + \frac{16\nu^4}{49} - \frac{8\nu^3}{7}, \quad (C.5)
\]
\[ \varphi^{(6)}(\xi, \nu) = \frac{(-\xi - 1)^6}{168} + \frac{5\nu(\xi - 1)^5}{14} + (-\frac{50}{21}\nu^2 + \frac{5}{14}\nu)(\xi - 1)^4 + \left(\frac{80}{21}\nu^3 - \frac{80}{21}\nu^2\right)(\xi - 1)^3 + \left(\frac{80}{49}\nu^4 + \frac{48}{7}\nu^3 - \frac{10}{7}\nu^2\right)(\xi - 1)^2 \\
+ \left(\frac{64}{441}\nu^5 - \frac{128}{49}\nu^4 + \frac{24}{7}\nu^3\right)(\xi - 1) - \frac{48\nu^4}{49} + \frac{8\nu^3}{21} + \frac{64\nu^5}{441}, \quad (C.6) \]

\[ \varphi^{(7)}(\xi, \nu) = \frac{\xi - 1)^7 - 7\nu(\xi - 1)^6}{528} + \frac{35\nu^2 - 7\nu}{44}(\xi - 1)^5 \\
+ \left(-\frac{140}{33}\nu^3 + \frac{175}{66}\nu^2\right)(\xi - 1)^4 + \left(\frac{40}{11}\nu^4 - \frac{280}{33}\nu^3 + \frac{35}{33}\nu^2\right)(\xi - 1)^3 \\
+ \left(-\frac{32}{33}\nu^5 + \frac{80}{11}\nu^4 - \frac{56}{11}\nu^3\right)(\xi - 1)^2 \\
+ \left(\frac{64}{1089}\nu^6 - \frac{160}{99}\nu^5 + \frac{48}{11}\nu^4 - \frac{28}{33}\nu^3\right)(\xi - 1) \\
+ \frac{8\nu^4}{11} + \frac{64\nu^6}{1089} - \frac{64\nu^5}{99}, \quad (C.7) \]

\[ \varphi^{(8)}(\xi, \nu) = \frac{(-\xi - 1)^8}{1716} + \frac{28\nu(\xi - 1)^7}{429} + \left(-\frac{392}{429}\nu^2 + \frac{28}{429}\nu\right)(\xi - 1)^6 \\
+ \left(-\frac{224}{143}\nu^3 + \frac{5120}{429}\nu^2\right)(\xi - 1)^5 + \left(-\frac{2240}{429}\nu^4 + \frac{1120}{143}\nu^3 - \frac{280}{429}\nu^2\right)(\xi - 1)^4 \\
+ \left(-\frac{3584}{1287}\nu^5 + \frac{1120}{429}\nu^4 - \frac{28}{33}\nu^3\right)(\xi - 1)^3 \\
+ \left(-\frac{14157}{4096}\nu^6 + \frac{2560}{143}\nu^5 - \frac{1280}{429}\nu^4 + \frac{448}{143}\nu^3\right)(\xi - 1)^2 \\
+ \left(\frac{184041}{4096}\nu^7 - \frac{5120}{4719}\nu^6 + \frac{1024}{1287}\nu^5 - \frac{64}{429}\nu^4\right)(\xi - 1) \\
+ \frac{4096\nu^7}{184041} - \frac{5120\nu^6}{14157} + \frac{1024\nu^5}{1287} - \frac{64\nu^4}{429}, \quad (C.8) \]

\[ \varphi^{(9)}(\xi, \nu) = \frac{(-\xi - 1)^9}{5720} - \frac{18\nu(\xi - 1)^8}{715} + \left(\frac{336}{715}\nu^2 - \frac{18}{715}\nu\right)(\xi - 1)^7 \\
+ \left(-\frac{9408}{3575}\nu^3 + \frac{588}{715}\nu^2\right)(\xi - 1)^6 \\
+ \left(\frac{4032}{3575}\nu^4 - \frac{21168}{715}\nu^3 + \frac{252}{715}\nu^2\right)(\xi - 1)^5 \\
+ \left(-\frac{3854}{715}\nu^5 + \frac{2016}{143}\nu^4 - \frac{3024}{715}\nu^3\right)(\xi - 1)^4 \\
+ \left(\frac{14336}{7865}\nu^6 - \frac{1792}{143}\nu^5 + \frac{1728}{143}\nu^4 - \frac{672}{715}\nu^3\right)(\xi - 1)^3 \\
+ \frac{14336\nu^6}{7865} - \frac{1792\nu^5}{143} + \frac{1728\nu^4}{143} - \frac{672\nu^3}{715}, \quad (C.9) \]
\[(C.9)\]
\[
+\left(\frac{-24576}{102245}\nu^7 + \frac{32256}{7865}\nu^6 - \frac{1536}{143}\nu^5 + \frac{576}{143}\nu^4\right)(\xi - 1)^2
\]
\[
+\left(\frac{4096}{511225}\nu^8 - \frac{43008}{102245}\nu^7 + \frac{4608}{1573}\nu^6 - \frac{512}{143}\nu^5 + \frac{288}{715}\nu^4\right)(\xi - 1)
\]
\[
-\frac{18432\nu^7}{102245} + \frac{1024\nu^6}{1573} - \frac{256\nu^5}{715} + \frac{4096\nu^4}{511225}.
\]

DEDICATION

This paper is dedicated to the memory of Professor Gumercindo Lima who, as professor of the author in high school, taught him the relations in combinatorial analysis that are applied in the theory here presented.

[1] L. E. Zakharov and V. D. Shafranov, Sov. Phys. Tech. Phys. 18, 151 (1973).
[2] T. Takeda and S. Tokuda, J. Comput. Phys. 93, 1 (1991).
[3] A. Ferreira, Companion arXiv paper
[4] G. Bateman, MHD Instabilities (The MIT Press, Cambridge, Massachusetts, and London, England, 1978), pages 66 to 68.
[5] G. Arfken, Mathematical Methods for Physicists (Academic Press, New York, San Francisco, London, Second Edition, 1970).
[6] E. T. Copson, An Introduction to the Theory of Functions of a Complex Variable (Oxford at the Clarendon Press), Section 10.33.
[7] M. Abramowitz and I. Segun, Eds. Handbook of Mathematical Functions (Dover Publications, Inc., New York, 1968).
[8] M. R. Spiegel, Calculus of Finite Differences and Difference Equations, Schaum’s Outline Series, McGraw-Hill Book Company, New York, 1971, Exercise 6.70.
[9] C. M. Bender and S. A. Orszag, Advanced Mathematical Methods for Scientists and Engineers (McGraw-Hill Book Company, Singapore, 1987), Chapter 2, Example 4.
[10] Maple V Release 5 and Maple 7. Waterloo Maple Inc., Waterloo, Ontario.