Re-understanding the Electromagnetic 4-potential in the Orthogonal Frame Bundle

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There has been much debate about the separation of the spin and orbital angular momentum of electromagnetic field. The spin and orbital angular momentum cannot be made simultaneously gauge invariant and Lorentz covariant, and are not conserved separately. We find that the electromagnetic 4-potential depends on the local frame instead of the global one. The local transformation is the intrinsic degree of freedom of the electromagnetic field. Therefore, only considering the global Lorentz transformation and neglecting the local one may lead to the non-covariance of the electromagnetic 4-potential. By using the behavior of the electromagnetic 4-potential that satisfies the Coulomb gauge in the Lorentz coordinate transformation, we can construct the electromagnetic vector in the orthogonal frame bundle. The various physical quantities constructed by this vector satisfy the Lorentz covariance in the fiber bundle. This electromagnetic vector which is projected onto the space-time just is an electromagnetic 4-potential satisfied the Coulomb gauge, and the electromagnetic vector is gauge invariant.

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INTRODUCTION

There has been much debate about the separation of angular momentum of electromagnetic field into its spin and orbital parts[1, 2]. Neither of the spin and orbital angular momentum satisfies the gauge invariance[3], while the gauge invariance is an inevitable requirement of an observable physical quantity, and the spin and orbital angular momentum of the electromagnetic field can indeed be observed in some optical experiments[4, 5].

This problem can be solved by replacing electromagnetic potential with only its transverse part[6–8]. However, this definition broke the Lorentz covariance — Coulomb gauge is not Lorentz covariant. Lorentz covariance is a requirement of the principle of relativity, physical laws should not depend on reference frame.

Apart from the spin and orbital angular momentum can’t simultaneously be Lorentz covariant and gauge invariant, they are not conserved separately[9, 10]. K.Y. Bliokh et al. constructed a conserved spin and orbital angular momentum density[10], but this structure also depends on the Coulomb gauge, therefore they are not Lorentz covariant. Bliokh et al. consider that this phenomenon is consistent with the experimental operation because a local probe particle will always pick out a special laboratory reference frame. This explanation is not convincing. Any observable quantity must be observed and measured in a special laboratory reference frame, but the Lorentz violation of observation methods wouldn’t cause Lorentz violation of physical laws. Therefore, the mathematical form of the measurement result can’t depend on the reference frame, which is known as ‘observer Lorentz covariation’[11]. On the other hand, according to the gauge theory of gravitation[12], similar to $U(1)$ group, Lorentz covariation is in fact the gauge invariance of $SO(3,1)$ group, so it is an inevitable requirement of observable quantities.

However, Bliokh’s point of view gives us some inspiration. One is the specificity of the Coulomb gauge. The Coulomb gauge not only can be used to construct a conserved spin and orbital angular momentum, but also the canonical quantization procedure works very easily in this gauge[13]. The other is that the optical phenomenon is closely related to the reference frame. We know that physics laws are local[13, 14]. Then, what if an observable quantity of the electromagnetic field don’t depend on the global reference frame (coordinate system) but the local one (tetrad field)? In other words, the origin of non-covariance of electromagnetic 4-potential which satisfies Coulomb gauge, possibly is that we haven’t taken the transformation of tetrad field into account. In detail, when the coordinate system is transformed, the choice of tetrad has changed sneakily, although Coulomb gauge is broken, the transformation of tetrad will produce a phase to correct the deviation (see eq.(27)).

Anyway, this view has already fully implied that we must shift the perspective from space-time to Orthogonal Frame Bundle (see [15, 16]), the set of all local reference frames of a space-time point $q$ constitutes $q$’s fiber. As a result, each transformation $\Lambda$ of a ‘local reference frame’ becomes a intrinsic freedom degree of electromagnetic field. Physical quantity we observe is actually the projection to space-time of a high-dimensional electromagnetic vector. However, this electromagnetic vector has ‘Lorentz covariance’ in the high-dimensional ‘orthogonal frame bundle’. To trace back to the source, the essence of ‘Lorentz covariance’ means that the mathematical form of the physical quantity can’t be dependent on reference frames, so physical quantity should be a ‘geometric invariant’. Not all ‘geometric invariant’ must be vectors in space-time, we have narrowly replaced ‘Lorentz covariance’ with ‘Lorentz covariation of space-time vectors’. The electromagnetic vector in orthogonal frame bundle that we are going to construct is such an exam-
ple — it is indeed a geometric invariant but don’t have ‘Lorentz covariance’ in space-time where we can’t obtain all of its information.

RESTATE THE PROBLEMS

We assume that $\phi^\rho$ is a spin-1 vector field with mass, which is transformed as a vector representation of the Lorentz group\cite{17}.

$$U(\Lambda)\phi^\rho(x)U^{-1}(\Lambda) = \Lambda_\rho^\alpha \phi^\alpha (\Lambda x)$$

The canonical energy-momentum tensor $T^\mu_N$ of $\phi^\rho$ is usually defined as

$$T^\mu_N = \eta^{\mu\nu} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^\rho)} \phi^\rho$$

where $\mathcal{L}$ is the Lagrangian density. Using eq.(1), we can prove that,

$$U(\Lambda)T^\mu_N(x)U^{-1}(\Lambda) = \Lambda_\mu^\rho T^\rho_N(\Lambda x)$$

Therefore $T_N$ is a Lorentz tensor with Lorentz covariance. The Belinfante energy-momentum tensor was introduced by adding a intrinsic ‘spin’ term to the canonical energy-momentum tensor\cite{18},

$$T^\mu_B = T^\mu_N + \frac{1}{2} \partial_\rho (S^{\rho\mu} - S^{\mu\rho} - S^{\nu\rho})$$

where the spin term $\partial_\rho S^{\rho\mu}$ cancels out the antisymmetric part of $T_N$, leaving only the symmetrical part,

$$\frac{1}{2} \partial_\rho S^{\rho\mu} = -T^\mu_N$$

The famous physicist S. Weinberg gave the expression of spin $S^{\rho\nu}$ in the work \cite{13},

$$S^{\rho\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_\rho \phi^\mu)} \phi^\rho - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^\rho)} \phi^\rho$$

Obviously, the Lorentz covariance condition of spin is,

$$U(\Lambda)S^{\rho\nu}U^{-1}(\Lambda) = \Lambda_\rho^\alpha \Lambda_\nu^\rho \Lambda_\mu^\beta S^{\alpha\beta}(\Lambda x)$$

Problems arise in the construction of a massless vector field $A_\mu(x)$ modeled on a mass vector field $\phi^\rho$\cite{13}. Simply using the creation and annihilation operators $a^\dagger, a$ of photon, we can’t construct a 4-vector that satisfies (1), $A_\mu(x)$ can only have the following form,

$$A_\mu(x) = (2\pi)^{-\frac{3}{2}} \sum_{h=\pm 1} \int \frac{d^3p}{\sqrt{2p^0}} \left( e_\mu(p, h)e^{ip^\nu a(p, h)} + e_\mu^*(p, h)e^{-ip^\nu a^\dagger(p, h)} \right)$$

where $\mathbf{p}$ is 3-momentum, $p = (p^\rho, \mathbf{p})$ is a lightlike 4-momentum, and $h$ is the helicity of the photon. The coefficient $e_\mu(p, h)$ is polarization vector.

In this configuration, $A_\mu$ satisfies the Coulomb gauge (in vacuum).

$$A_0 = 0, \partial^\mu A_\mu = 0$$

Under a reference frame transformation, $A_\mu$ behaves as,

$$U(\Lambda)A_\mu(x)U^{-1}(\Lambda) = \Lambda_\rho^\sigma A_\sigma(\Lambda x) + \Lambda_\nu^\rho (\partial_\nu \Omega) (\Lambda x, \lambda)$$

It can be seen that the transformation of reference frame will leads $A_\mu$ to produce a ‘gauge’ $\partial_\mu \Omega$ that depends on the reference frame. Where, $\Omega(x, \lambda)$ can be expressed as a linear combination of creation and annihilation operators. The literature \cite{13} didn’t give a explicit expression of $\Omega$. According to our calculations,

$$\Omega(x, \Lambda) = (2\pi)^{-\frac{3}{2}} \sum_{h=\pm 1} \int \frac{d^3p}{2\sqrt{p^0}} \left[ (\alpha(p, \Lambda) + i\hbar \beta(p, \Lambda)) e^{ip\cdot x} a(p, h) + (\alpha(p, \Lambda) - i\hbar \beta(p, \Lambda)) e^{-ip\cdot x} a^\dagger(p, h) \right]$$

where $\alpha, \beta$ are parameters that depend on the Lorentz transformation $\Lambda$ and the 4-momentum $p$. The little group representation of $\Lambda$ is expressed as $W = L^{-1}(\Lambda p)\Lambda L(p)$, $L(p)$ is the standard Lorentz transformation. The definitions of $\alpha, \beta$ are $\alpha = W^{0}_{1}$ and $\beta = W^{0}_{2}$.

From the above discussion, it can be seen that although both electromagnetic 4-potential $A_\mu$ and Lorentz vector $\phi^\rho$ own one index, their transformation properties are quite different. The gauge field theory \cite{19} tells us that $A_\mu$ is a gauge potential in the principal bundle whose structure group is $U(1)$, but $\phi^\rho$ is a vector (component) in the representation space. Therefore, if we directly apply the definitions of quantities induced by $\phi^\rho$ to the electromagnetic 4-potential $A_\mu$, it will inevitably lead to problems. But people are actually used to applying these definitions directly to electromagnetic fields. We believe that this is why the orbital and spin angular momentum of the electromagnetic field do not have Lorentz covariance and gauge invariance.

The Lagrangian and action of electromagnetic field in vacuum (no matter) can be expressed as,

$$S_\gamma = -\frac{1}{4} \int d^4x F_{\mu\nu}F^{\mu\nu}$$

where $F_{\mu\nu}(x) = \partial_\mu \omega_\nu(x) - \partial_\nu \omega_\mu(x)$ is the electromagnetic field tensor, $\omega_\mu(x) = A_\mu(x) + \partial_\mu \theta(x)$ is an electromagnetic 4-potential under any gauge. We temporally imitate eq.(2) to calculate the canonical energy-
momentum tensor of the electromagnetic field,
\[ T^\mu_\nu = \eta^{\mu\nu} \mathbf{L} - \frac{\partial \mathbf{L}}{\partial (\partial_\mu \phi^\rho)} \partial_\nu \phi^\rho \]
(13)
and imitate eq.(6) to calculate the spin term \( S^{\mu\nu} \) of the electromagnetic field,
\[ S^{\mu\nu} = \frac{\partial \mathbf{L}}{\partial (\partial_\mu \phi^\rho)} \phi^\rho - \frac{\partial \mathbf{L}}{\partial (\partial_\nu \phi^\rho)} \phi^\rho \]
(14)
and \( S^{\mu\nu} = F^{\mu\nu} \phi^\rho - F^{\nu\rho} \phi^\mu \)

According to eq.(13) and eq.(14), it is easy to calculate the 3-dimensional form of angular momentum density, which is a well-known content in textbooks[20], \( \mathbf{L} = E_j (r \times \nabla) \phi^j, \) \( \mathbf{S} = \mathbf{E} \times \mathbf{A}, \) where \( \phi^\mu = (\varphi, \mathbf{A}) \).

There isn’t local gauge invariance in eqs.(13),(14). Unlike the gauge transformation of a vector field \( \phi^\rho \), the gauge transformation of \( \phi^\mu(x) \) is not to multiply a local phase factor \( e^{i\chi(x)} \). \( \phi^\mu(x) \) is the gauge potential in U(1) algebra, i.e.[19],
\[ \phi^\mu(x) = \phi^\mu(x) + \partial_\mu \chi(x) \]
(15)
Substituting (15) into (13) and (14) will yield additional terms \( F^{\mu\nu} \phi^\rho \partial_\nu \chi \) and \( 2 F^{\mu\nu} \phi^\rho \partial_\nu \chi \). It can be seen that the canonical energy-momentum tensor and spin don’t satisfy the U(1) gauge invariance.

The usual solution is to replace \( \phi^\mu \) by its transverse part \( A_\mu[6-8] \), which always satisfies Coulomb gauge, so it is invariance under gauge transformation eq.(15),
\[ T^\mu_\nu = \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + F^{\mu\nu} \phi^\rho A_\rho \]
(16)
\[ S^{\mu\nu} = F^{\mu\nu} A_\nu - F^{\nu\rho} A^\rho \]
(17)
The 3-dimension form of orbital and spin angular momentum is corrected by \( \mathbf{L} = E_j (r \times \nabla) A_j, \mathbf{S} = \mathbf{E} \times \mathbf{A} \), where \( A_\mu = (0, \mathbf{A}^\bot) \) and \( \nabla \cdot \mathbf{A}^\bot = 0 \).

However, the frame transformation (10) of \( A_\mu \) cause a new problem immediately, which is \( T^\mu_\nu, S^{\mu\nu} \) no longer satisfy Lorentz covariance condition eqs.(3),(7), so do the orbital and spin angular momentum density induced by them. The source of the question is the dependency of Coulomb gauge on reference frame, if we make a boost transformation to the reference frame \( \tilde{x} = \Lambda x \), concomitantly, let \( A_\mu(x) \) transforms as the ‘classic way’ \( \tilde{A}_\mu(\tilde{x}) = \Lambda^\mu_\nu A_\nu(\Lambda^{-1} \tilde{x}), \) then \( \tilde{A}_\mu(\tilde{x}) \) no longer satisfies this gauge.

But the Belinfante energy momentum tensor that sums both the contributions of orbital and spin part
\[ T_B^{\mu\nu} = \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + F^{\mu\nu} F^{\nu_\rho} \]
(18)
is indices symmetrical, gauge invariant and Lorentz covariant. So it is a well defined observable quantity.

The origin of these difficulties, in the final analysis, is that \( A_\mu(x) \)’s transformation (10) is very different from vector \( \phi^\mu \), while we can’t simply assert that \( A_\mu(x) \) doesn’t have Lorentz covariance. After carefully considering, we can find that the essential requirement of Lorentz covariance is that the definition of a physical quantity must be independent of reference frames. That is to say, a physical quantity is a ‘geometric invariant’. The geometric invariants of different structures have different Lorentz transformation forms. For the most familiar example, the Christoffel symbol \( \Gamma^\rho_\mu \) is a projection to space-time of the torsion vanishing metric connection \( \omega \) in the orthogonal frame bundle. This projection mapping is related to reference frames, and the mathematical definition of \( \omega \) is independent of reference frames[15]. As a projection of a geometric invariant, the Christoffel symbol’s transformation is fully different from a tensor. We couldn’t simply treat the ‘Lorentz transformation’ narrowly as the ‘Lorentz transformation of tensors’.

If we can find a geometric invariant and whose projection to space-time just is the electromagnetic 4-potential, then we will prove that the electromagnetic 4-potential still is Lorentz covariant, and the problem will be solved.

THE ELECTROMAGNETIC BUNDLE VECTOR

Look at eq.(10), and we notice that for the unit Lorentz transformation \( \mathbb{1} \), the little group representation \( W(p, \mathbb{1}) = L^{-1}(\mathbb{1}p)\mathbb{1}L(p) = \mathbb{1} \), thus \( \alpha(p, \mathbb{1}) = \beta(p, \mathbb{1}) = 0 \), which leads to \( \Omega(x, \mathbb{1}) = 0 \). Rewrite eq.(10) as,
\[ U(\Lambda)(A_\mu(x) + \partial_\mu \Omega(x, \mathbb{1}))/U^{-1}(\Lambda) = \Lambda^\sigma_\rho (A_\rho(\Lambda x) + (\partial_\sigma \Omega)(\Lambda x, \Lambda)) \]
(19)
It can be seen that \( A_\rho + \partial_\rho \Omega \) behaves like a 4-vector under a reference frame transformation. We define \( \mathbb{A}_\rho(x, \Lambda) = A_\rho(x) + \partial_\rho \Omega(x, \Lambda) \)
(20)
then according to eq.(19), Lorentz transformation of \( \mathbb{A}_\rho \) and its derivative can be directly obtained,
\[ U(\Lambda)\mathbb{A}_\rho(x, \mathbb{1})U^{-1}(\Lambda) = \Lambda^\sigma_\rho \mathbb{A}_\sigma(\Lambda x, \Lambda) \]

\[ U(\Lambda)\partial_\sigma \mathbb{A}_\rho(x, \mathbb{1})U^{-1}(\Lambda) = \Lambda^\nu_\sigma \Lambda^\rho_\mu (\partial_\nu \mathbb{A}_\mu)(\Lambda x, \Lambda) \]
(21)
Although \( \mathbb{A}_\rho \) is similar to a 4-vector, we noticed that \( \mathbb{A}_\rho \) depends on both \( x \) and \( \Lambda \). This shows that \( \mathbb{A}_\rho \) is not a vector field in space-time. If we take \( (x^\rho, \Lambda^\nu_\rho) \) as a coordinate, we immediately recall that these will constitute a coordinate domain of the ‘orthogonal frame bundle’. For a point \( (q, e_0, e_1, e_2, e_3) \) in the orthogonal frame bundle, \( q \) is a point in space-time whose coordinate is \( x^\rho \), then any orthonormal base \( (e_0, e_1, e_2, e_3) \) of the tangent space of \( q \) can be expressed as \( e_\nu = \Lambda^\nu_\rho e_\rho, \) so \( \Lambda^\nu_\nu \) is taken as the coordinate of this base. Therefore \( \mathbb{A}_\rho \)
is a vector field in the orthogonal frame bundle. $\mathcal{A}_\rho$ is a component of the vector field and $\mathcal{A} = \mathcal{A}_\rho$ $dx^\rho$ just is a vector field.

One point need to be explained is that $dx^\rho$ is not a vector in space-time, but in orthogonal frame bundle. Both $dx^\rho$ and $dA^\rho_\sigma$ constitute the vector base of cotangent space of $(q, e_0, e_1, e_2, e_3)$. We rewrite eq. (20) as a coordinate independent form, and $(q, e_0, e_1, e_2, e_3)$ can be simplified as $(q, e)$.

$$\mathcal{A}(q, e) = \mathcal{A}_\rho(q, e) \, dx^\rho + A^\nu_\rho(q, e) \, d\Lambda^\nu, \quad \mathcal{A}_\rho(q, e) = A_\rho(x) + \partial_\rho \Omega(x, \Lambda), \quad A^\nu_\rho(q, e) = 0$$  \hspace{1cm} (22)

It is necessary to discuss the performance of the $\mathcal{A}$’s projection to space-time. Only this projection can be directly observed. If we choose a fixed base $e(q)$ at each space-time point $q$, then $e(q)$ is called a tetrad.

Applying the pullback mapping $e^*$ induced by the tetrad $e(q)$ to $\mathcal{A}$, we can get a vector field $(e^*\mathcal{A})$ in space-time. Using eq. (22), we can get the component form of $(e^*\mathcal{A})$, while setting $e_\nu(q) = \Lambda^\nu_\nu(q) \partial_\mu$ where $\Lambda^\nu_\nu(q)$ is related to $q$,

$$(e^*\mathcal{A})_\mu(q) = A_\rho(q, e(q)) + A^\nu_\rho(q, e(q)) \frac{\partial \Lambda^\nu_\nu(q)}{\partial x^\rho}$$  \hspace{1cm} (23)

it is to say,

$$(e^*\mathcal{A})(q) = A_\rho(q, e(q)) \, dx^\rho = (A_\rho(x) + \partial_\rho \Omega(x, \Lambda)) \, dx^\rho$$  \hspace{1cm} (24)

where $dx^\rho$ becomes a vector in space-time again.

Obviously, pullback $e^*$ is a projection. The tetrad $e(q)$ represents the (local) laboratory reference frame that we are observing. We regard $e(q)$ as a Lorentz gauge.

If a set of coordinates $x(q)$ is selected, then a natural selection of a tetrad is $e(q) = (\partial_0, \partial_1, \partial_2, \partial_3)$. This also is the default choice of optical experiments, so that the influence of the local reference frame (tetrad) is ignored. The coordinate of $(q, e(q))$ are $(x^\sigma, \delta^\nu_\mu)$. Due to $\partial_\rho \Omega(x^\sigma, \delta^\nu_\mu) = 0$, $(e^*\mathcal{A})_\rho(q) = A_\rho(x)$. If we choose another coordinate $\tilde{x} = \Lambda x$, the tetrad field will also be reselected as $\tilde{e}(q) = (\partial_0, \partial_1, \partial_2, \partial_3)$. Naturally, $\mathcal{A}$’s projection under $\tilde{e}(q)$ is $$(e^*\mathcal{A})(q) = \tilde{A}_\rho(\tilde{x}, e(q)) \, d\tilde{x}^\rho = \tilde{A}_\rho(\tilde{x}) \, d\tilde{x}^\rho$$  \hspace{1cm} (25)

From the other perspective to calculate $\tilde{e}^*\mathcal{A}$. Notice that $\tilde{e}(q) = e(q) \Lambda^{-1}$. Substituting $x = \Lambda^{-1} \tilde{x}$ and $\tilde{e}(q)$ into eq. (24), we can get,

$$(e^*\mathcal{A})(q) = A_\rho(q, \tilde{e}(q)) \, d\tilde{x}^\rho = (A_\rho(x) + \partial_\rho \Omega(x, \Lambda^{-1})) \, d\tilde{x}^\rho = (A_\sigma(\Lambda^{-1} \tilde{x}) + (\partial_\sigma \Omega)(\Lambda^{-1} \tilde{x}, \Lambda^{-1})) \, A^\rho_\sigma \, d\tilde{x}^\rho$$  \hspace{1cm} (26)

Contrast eq. (25) and eq. (26), there must be

$$\tilde{A}_\rho(\tilde{x}) = \Lambda^\rho_\sigma A_\sigma(\Lambda^{-1} \tilde{x}) + \Lambda^\rho_\sigma (\partial_\sigma \Omega)(\Lambda^{-1} \tilde{x}, \Lambda^{-1})$$  \hspace{1cm} (27)

where both $A_\rho(x)$ and $\tilde{A}_\rho(\tilde{x})$ satisfy the Coulomb gauge (9), so they are not Lorentz vectors in space-time. This is expected, due to the electromagnetic 4-potential we have observed is just a projection of the vector in orthogonal frame bundle, thus, it is one-side and naturally can’t satisfy the Lorentz covariance. The vector $\mathcal{A}(q, e)$ is the complete picture of the electromagnetic field. The eq. (27) just proves that the definition of $\mathcal{A}$ is independent of reference frames. No matter in which reference frame $x$, the $A_\rho(x)$ in the frame that satisfies the Coulomb gauge is finally promoted to be the same vector $\mathcal{A}$. So $\mathcal{A}$ is the geometric invariant that we are looking for.

Since $A_\rho(x)$ in eq. (20) must satisfy the Coulomb gauge (9), $\mathcal{A}$ doesn’t change when $\mathcal{A}_\rho(x)$ makes a gauge transformation $\mathcal{A}_\rho(x) + \partial_\rho e(x)$. So $\mathcal{A}$ has obvious $U(1)$ gauge invariance.

It is natural to replace $\mathcal{A}_\rho, A_\rho$ with $\mathcal{A}_\rho, A_\rho$ in each physical quantity definition. Since $F_{\mu\nu} = 2\partial_\rho A_\nu = 2\partial_\nu A_\rho$, it can be seen that the form of the electromagnetic field tensor and the Lagrangian will not change. However, this substitution has implied the promotion of scalar, vector, and tensor from the space-time to the orthogonal frame bundle.

The definition of canonical energy-momentum tensor $T_N$ is modified to,

$$T^\mu_\nu(q, e) = -\frac{1}{4} q^{\rho\sigma} F_{\rho\sigma}(q) F^{\nu\sigma}(q) + F^{\mu\sigma}(q) \partial_\nu A_\rho(q, e)$$  \hspace{1cm} (28)

Then we can prove that $T_N$ is Lorentz covariant,

$$U(A)T^\mu_\nu(x, A)U^{-1}(A) = \Lambda^\rho_\mu \Lambda^\nu_\sigma T^\rho_\sigma (\Lambda x, \Lambda)$$  \hspace{1cm} (29)

Due to the $U(1)$ gauge invariance of $\mathcal{A}$, the gauge invariance of $T_N$ is self-evident.

Similarly, the spin $S^{\rho\nu}$ is modified to,

$$S^{\rho\nu} = F^{\rho\nu} A^\sigma - F^{\rho\sigma} A^\nu$$  \hspace{1cm} (30)

The Belinfante energy-momentum tensor $T_B$ remains unchanged under $\mathcal{A} \rightarrow \tilde{\mathcal{A}}$.

In the laboratory reference frame, we select the tetrad $e(q) = (\partial_0, \partial_1, \partial_2, \partial_3)$ and the projection of the canonical energy-momentum tensor $T_N$ is

$$(e^*T_N)^\mu_\nu(x) = T^\mu_\nu(x) = -\frac{1}{4} q^{\rho\sigma} F_{\rho\sigma} F^{\nu\sigma} + F^{\mu\sigma} \partial_\nu A_\rho$$  \hspace{1cm} (31)

Naturally, the projection of spin is

$$(e^*S)^{\rho\nu} = F^{\rho\nu} A^\sigma - F^{\rho\sigma} A^\nu$$  \hspace{1cm} (32)

Similarly, we can give the projection of angular momentum density. These results above are consistent with eqs. (16), (17). We can see, not that eqs. (16), (17) don’t have the Lorentz covariance, but that $T_N$ and $S$ which
only reflect part of information, is not complete physical picture, while the \( T\mathcal{N} \) and \( S \) just is.

Since the projection of \( \mathcal{A} \) maintains the Coulomb gauge in any reference frame after being promoted to a bundle tensor, the conserved orbital and spin angular momentum constructed by Bliokh in \([10]\) will also be meaningful in any reference frame.

**SUMMARY**

The problem about orbital and spin angular momentum of electromagnetic field can’t simultaneously be gauge invariant and Lorentz covariant becomes clear through in the framework of quantum field theory. With a Lorentz transformation, the electromagnetic 4-potential that satisfies the Coulomb gauge will produce a phase \( \Omega \) that depends on this transformation and breaks the Lorentz covariant form of electromagnetic 4-potential. To promote the electromagnetic 4-potential as a vector in the orthogonal frame bundle can possess Lorentz covariance, which suggests that the non-covariance is due to our negligence of the transformation of the local reference frame. That’s to say, the electromagnetic object at a place \((x^\mu, \partial_\nu)\) should be covariant with one at \((\Lambda^\mu_\nu x^\nu, \Lambda^\nu_\mu \partial_\mu)\), but we are accustomed to comparing the electromagnetic objects at \((x^\mu, \partial_\nu)\) and \((\Lambda^\mu_\nu x^\nu, \partial_\nu)\). The local reference frame also plays a role as an internal degree of freedom.

As we know, the orthogonal frame bundle are closely related to the gravitational effects [21], and the Belinfante energy-momentum tensor of electromagnetic field can indeed make space-time to generate curvature [22], although the magnitude is too small to be observed. On the one hand, strictly speaking, in curved space-time, the local frame \( e(q) = (\partial_0, \partial_1, \partial_2, \partial_3) \) is not orthonormal, hence is not a tetrad, thus we can’t make such a choice. On the other hand, electromagnetic 4-potential as a gauge potential in the principal bundle whose structure group is \( U(1) \), meanwhile is a projection of a vector in the orthogonal frame bundle. We can see electromagnetic 4-potential is a link connecting the two bundles. A natural question is, what kind of relationship has been established by it between electromagnetic and gravitational interactions? We will explore this further.

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