We associate a complete intersection singularity to a graded matrix factorization of size two of a polynomial in three variables. We show that we get an inverse to the reduction of singularities considered by C. T. C. Wall. We study this for the full strongly exceptional collections in the triangulated category of graded matrix factorizations constructed by H. Kajiura, K. Saito, and the second author.

0. Introduction

In his classification of singularities [16], C. T. C. Wall noticed that there is a relation between series of singularities given by a reduction. More precisely, he considers a linear reduction \( L \) of a complete intersection singularity in \( \mathbb{C}^4 \) to a hypersurface singularity in \( \mathbb{C}^3 \) and a reduction \( \Delta \) of a hypersurface singularity in \( \mathbb{C}^3 \) of corank 3 to one of corank 2. The aim of this paper is to show that as an inverse of such a reduction one can consider a graded matrix factorization of size two.

We consider weighted homogeneous deformations of an invertible polynomial. We use the approach of [1] to define graded matrix factorizations for such deformations. We classify weighted homogeneous deformations of invertible polynomials of three variables and we define Dolgachev numbers for them. We consider a graded matrix factorization of size two of such a deformation and associate a complete intersection singularity to it. We show that we get the original singularity back by \( L \)- or \( \Delta \)-reduction. Moreover we consider graded matrix factorizations of size two of invertible polynomials in two variables and show that they are related to a reduction between hypersurface singularities in \( \mathbb{C}^2 \) considered in [10].

In [12], H. Kajiura, K. Saito, and the second author constructed a full strongly exceptional collection in the triangulated category of graded matrix factorizations of a polynomial associated to a regular system of weights whose smallest exponents are equal to \(-1\). These polynomials define the 14 exceptional unimodal hypersurface
singularities, the 6 heads of the bimodal series of hypersurface singularities, and 2 further singularities. They are weighted homogeneous deformations of invertible polynomials of three variables. The exceptional collection is described by a certain quiver. The graded matrix factorizations of size two listed in the paper correspond to the outermost vertices of certain arms of the quiver. We show that they correspond to singularities where the corresponding length of the arm (which is one of the Dolgachev numbers) is increased by one. We thus obtain the picture of Fig. 2.

There is Arnold’s strange duality between the 14 exceptional unimodal hypersurface singularities and the extension [10] of it to the bimodal series and the complete intersection singularities which reduce to the 14 exceptional unimodal singularities and the 6 heads of the bimodal series. This can be considered as a special kind of mirror symmetry. As a corollary of our main result we get that the reduction is mirror dual to a special adjacency between these singularities.

1. Graded matrix factorizations for invertible polynomials

A polynomial \( f(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n] \) is called weighted homogeneous, if there are positive integers \( w_1, \ldots, w_n \) and \( d \) such that \( f(\lambda w_1 x_1, \ldots, \lambda w_n x_n) = \lambda^d f(x_1, \ldots, x_n) \) for \( \lambda \in \mathbb{C}^* \). We call \( (w_1, \ldots, w_n; d) \) a system of weights. If \( \gcd(w_1, \ldots, w_n, d) = 1 \), then the system of weights is called reduced.

**Definition 1.** A weighted homogeneous polynomial \( f(x_1, \ldots, x_n) \) is called invertible if the following conditions are satisfied

(i) it can be written

\[
f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ij}},
\]

where \( a_i \in \mathbb{C}^* \), \( E_{ij} \) are non-negative integers, and the \( n \times n \)-matrix \( E := (E_{ij}) \) is invertible over \( \mathbb{Q} \),

(ii) it has an isolated singularity at the origin.

Let \( f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ij}} \) be an invertible polynomial. Without loss of generality, we assume that \( \det E > 0 \). The canonical system of weights is the system of weights \( (w_1, \ldots, w_n; d) \) given by the unique solution of the equation

\[
E \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \det(E) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad d := \det(E).
\]

The canonical system of weights \( (w_1, \ldots, w_n; d) \) is in general non-reduced, define

\[
c_f := \gcd(w_1, \ldots, w_n, d).
\]
Definition 2. A **weighted homogeneous deformation of an invertible polynomial** is a polynomial of the form

\[ f(x_1, \ldots, x_n) = \sum_{i=1}^{N} a_i \prod_{j=1}^{n} x_j^{E_{ij}}, \]

where \( N \geq n \), \( a_i \in \mathbb{C}^* \), \( E_{ij} \) are non-negative integers, \( \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ij}} \) is an invertible polynomial with canonical weight system \((w_1, \ldots, w_n; d)\), and \( \prod_{j=1}^{n} x_j^{E_{ij}} \) are monomials of the same degree \( d \) for \( 1 \leq i \leq N \).

Note that the case \( N = n \) is permitted, i.e., the weighted homogeneous deformations of an invertible polynomial include the invertible polynomial itself.

Definition 3. Let \( f(x_1, \ldots, x_n) = \sum_{i=1}^{N} a_i \prod_{j=1}^{n} x_j^{E_{ij}} \) be a weighted homogeneous deformation of an invertible polynomial. Consider the free abelian group \( \oplus_{i=1}^{n} \mathbb{Z} \vec{x}_i \oplus \mathbb{Z} \vec{f} \) generated by the symbols \( \vec{x}_i \) for the variables \( x_i \) for \( i = 1, \ldots, n \) and the symbol \( \vec{f} \) for the polynomial \( f \). The **maximal grading** \( L_f \) of the invertible polynomial \( f \) is the abelian group defined by the quotient

\[ L_f := \bigoplus_{i=1}^{n} \mathbb{Z} \vec{x}_i \oplus \mathbb{Z} \vec{f} / I_f, \]

where \( I_f \) is the subgroup generated by the elements

\[ \vec{f} - \sum_{j=1}^{n} E_{ij} \vec{x}_j, \quad i = 1, \ldots, N. \]

Note that \( L_f \) is an abelian group of rank 1 which is not necessarily free. We have the degree map \( \deg : L_f \rightarrow \mathbb{Z} \) defined by \( \vec{x}_i \mapsto w_i/c_f, \vec{f} \mapsto d/c_f \), where \((w_1, \ldots, w_n; d)\) is the canonical system of weights of \( f \). This map is an isomorphism if and only if the canonical system of weights is reduced, see [8, Proposition 17].

Let \( f(x_1, \ldots, x_n) \) be a weighted homogeneous deformation of an invertible polynomial. We recall the definition of an \( L_f \)-graded matrix factorization of \( f \), see [1] and the references therein.

Let \( S = \mathbb{C}[x_1, \ldots, x_n] \). This is naturally an \( L_f \)-graded algebra:

\[ S = \bigoplus_{\vec{l} \in L_f} S_{\vec{l}}. \]

For any \( L_f \)-graded \( S \)-module \( M \), let \( M(\vec{l}) \) denote the \( L_f \)-graded \( S \)-module, where the grading is shifted by \( \vec{l} \in L_f \):

\[ M(\vec{l}) = \bigoplus_{\vec{m} \in L_f} M(\vec{l} + \vec{m}). \]

The grading shift by \( \vec{l} \in L_f \) induces a functor \( (\vec{l}) \) on the category of finitely generated \( L_f \)-graded \( S \)-modules.
Definition 4. An $L_f$-graded matrix factorization of $f$ is
\[
\mathcal{F} := \left( \begin{array}{c}
F_0 & \xrightarrow{f_0} & F_1 \\
\xleftarrow{f_1} & & \end{array} \right),
\]
where $F_0$ and $F_1$ are $L_f$-graded free $S$-modules of finite rank and $f_0 : F_0 \to F_1$, $f_1 : F_1 \to F_0(\tilde{f})$ are morphisms such that $f_1 \circ f_0 = \tilde{f} \cdot \text{id}_{F_0}$ and $f_0(\tilde{f}) \circ f_1 = \tilde{f} \cdot \text{id}_{F_1}$. The rank of $F_1$ is equal to the rank of $F_0$ and it is called the size or the rank of the matrix factorization $\mathcal{F}$.

Let $f(x_1, \ldots, x_n)$ be a weighted homogeneous deformation of an invertible polynomial with reduced weight system $(w_1, \ldots, w_n; d)$. We define
\[
\varepsilon_f := w_1 + \cdots + w_n - d.
\]

Let $f(x, y, z)$ be a weighted homogeneous deformation of an invertible polynomial in three variables. We shall associate Dolgachev numbers to $f$.

Definition 5. The Dolgachev numbers of a weighted homogeneous deformation $f$ of an invertible polynomial is the $(m+2)$-tuple $(m \geq 1)$ of numbers $A = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m)$ defined by the embedding $R_f \hookrightarrow R_{A,\lambda}$ given by Table 1.

It follows from the embedding $R_f \hookrightarrow R_{A,\lambda}$ into the ring of a weighted projective line that $L_f \cong \mathbb{Z}$.

2. Graded matrix factorizations of size two

We shall now consider $L_f$-graded matrix factorizations of size two for the following two cases

(A) $f$ is a weighted homogeneous deformation of an invertible polynomial in three variables,
Table 1. The image of the generators in $R_{A,\lambda}$

| Type | $f(x, y, z)$ | $x$ | $y$ | $z$ |
|------|-------------|-----|-----|-----|
| I    | $-\prod_{i=3}^{m+2} (y^{p_2} - \lambda_i x^{p_1}) + z^{p_3}$ | $X_1$ | $X_2$ | $X_3 \cdots X_{m+2}$ |
| II   | $-x \prod_{i=3}^{m+2} (y^{p_2} - \lambda_i x^{p_1}) + z^{p_3}$ | $X_1^{p_3}$ | $X_2$ | $X_1 X_3 \cdots X_{m+2}$ |
| III  | $-xy \prod_{i=3}^{m+2} (y^{p_2} - \lambda_i x^{p_1}) + z^{p_3}$ | $X_1^{p_3}$ | $X_2^{p_3}$ | $X_1 X_2 X_3 \cdots X_{m+2}$ |
| IV   | $x^{q_1} y + y q_2 z - z^{q_3} x$ | $X_1^{p_3} X_2$ | $X_2^{p_1}$ | $X_1 X_3 \cdots X_{m+2}$ |

| Type | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ |
|------|-------------|-------------|-------------|
| I    | $\frac{p_1}{m}$ | $\frac{p_2}{m}$ | $p_3$ |
| II   | $\frac{p_3}{m} (p_1 - 1)$ | $\frac{p_2}{m}$ | $p_3$ |
| III  | $\frac{p_1}{m} (p_1 - 1)$ | $\frac{p_2}{m} (p_2 - 1)$ | $p_3$ |
| IV   | $\frac{p_3}{m} (p_1 - 1)$ | $\frac{p_1 p_2 - p_1 + 1}{m}$ | $p_3$ |
| V    | $q_2 q_3 - q_3 + 1$ | $q_3 q_1 - q_1 + 1$ | $q_1 q_2 - q_2 + 1$ |

(B) $f$ is an invertible polynomial in two variables.

(A) Let $S = \mathbb{C}[x, y, z]$ and let $f(x, y, z)$ be a weighted homogeneous deformation of an invertible polynomial of the form

$$f(x, y, z) = p_1(x, y, z) h_1(x, y, z) + p_2(x, y, z) h_2(x, y, z),$$

where $p_i \in S_{\tilde{p}_i}$ for some $\tilde{p}_i \in L_f, i = 1, 2,$ and $p_1, p_2$ forms a regular sequence in $S$. As in [5, Section 2] (cf. [1, Definition 2.11]), we consider the Koszul resolution of the $L_f$-graded $S$-module $M = S/(p_1, p_2)$

$$0 \rightarrow S(-\tilde{p}_1 - \tilde{p}_2) \rightarrow S(-\tilde{p}_1) \oplus S(-\tilde{p}_2) \rightarrow S \rightarrow S/(p_1, p_2) \rightarrow 0.$$

This yields the $L_f$-graded matrix factorization $Q := \begin{pmatrix} F_0 & \overset{f_0}{\rightarrow} & F_1 \end{pmatrix}$ of $f$ such that

$$F_0 = S(\tilde{0}) \oplus S(\tilde{f} - \tilde{p}_1 - \tilde{p}_2), \quad F_1 := S(\tilde{f} - \tilde{p}_1) \oplus S(\tilde{f} - \tilde{p}_2),$$

and $f_0$ and $f_1$ are given by the matrices

$$q_0 = \begin{pmatrix} h_1(x, y, z) - p_2(x, y, z) \\ h_2(x, y, z) - p_1(x, y, z) \end{pmatrix} \quad \text{and} \quad q_1 = \begin{pmatrix} p_1(x, y, z) & p_2(x, y, z) \\ -h_2(x, y, z) & h_1(x, y, z) \end{pmatrix}$$

respectively.

We associate to the matrix factorization a complete intersection singularity in $\mathbb{C}^4$. Denote the coordinates of $\mathbb{C}^4$ by $w, x, y, z$. 
**Definition 6.** The complete intersection singularity \((X_Q, 0)\) is the singularity defined by

\[
F_Q(w, x, y, z) = (F_{Q,1}(w, x, y, z), F_{Q,2}(w, x, y, z))
\]

\[
:= (-h_1(x, y, z) + w p_2(x, y, z), h_2(x, y, z) + w p_1(x, y, z)).
\]

We can get the function \(f\) back from \(F_Q\) by the linear reduction considered, in the case \(p_1(x, y, z) = y\) and \(p_2(x, y, z) = z\), by Wall [16, 7.9]. Namely, let \(L_w F_Q\) be the elimination of the variable \(w\) from the functions \(F_{Q,1}, F_{Q,2}\). This is given by the resultant \(R_w(F_{Q,1}, F_{Q,2})\) of the functions with respect to the variable \(w\). We have

\[
L_w F_Q = R_w(F_{Q,1}, F_{Q,2}) = \det q_1 = f.
\]

Now suppose that \(p_1(x, y, z) = y\), \(p_2(x, y, z) = z\), \(h_1(x, y, z) = z\) and \(h_1(x, y, z) = 2a(x, y)z + b(x, y)\). Then

\[
(F_{Q,1}(w, x, y, z), F_{Q,2}(w, x, y, z)) = (-h_1(x, y, z) + wz, z + wy)
\]

and the substitution \(z = -wy\) in \(F_{Q,1}\) gives

\[
F_Q(w, x, y) : = F_{Q,1}(w, x, y, -wy) = -w^2y - h_1(x, y, -wy)
\]

\[
= -w^2y + 2wy(a(x, y) - b(x, y)).
\]

Thus \((X_Q, 0)\) is in this case a hypersurface singularity defined by \(F_Q = 0\). The discriminant of the polynomial \(F_Q(w, x, y)\) in the variable \(w\) is

\[
\Delta_w F_Q(w, x, y) = y^2a(x, y)^2 - yb(x, y).
\]

This is the reduction \(\Delta_w F_Q\) of the polynomial \(F_Q\) considered by Wall in [16, 7.3]. The polynomial \(f\) is equal to

\[
f(x, y, z) = z^2 + 2ya(x, y)z + yb(x, y).
\]

Under the substitution \(\tilde{z} := z + ya(x, y)\) this becomes

\[
f(x, y, \tilde{z}) = \tilde{z}^2 - y^2a(x, y)^2 + yb(x, y) = \tilde{z}^2 - \Delta_w F_Q(w, x, y).
\]

**B.** Now let \(S = \mathbb{C}[x, y]\) and let \(f(x, y)\) be an invertible polynomial of the form \(f(x, y) = x^\alpha y^\beta + y^3\), where \(\alpha \geq 5\). We consider the Koszul resolution of the \(L_f\)-graded \(S\)-module \(M = S/(x^2, y^2)\)

\[
0 \rightarrow S(-2\bar{x} - 2\bar{y}) \rightarrow S(-2\bar{x}) \oplus S(-2\bar{y}) \rightarrow S \rightarrow S/(x^2, y^2) \rightarrow 0.
\]

This yields the \(L_f\)-graded matrix factorization \(Q := \left( F_0 \xleftarrow{f_0} \xrightarrow{f_1} F_1 \right)\) of \(f\) such that

\[
F_0 = S(\bar{0}) \oplus S(\bar{f} - 2\bar{x} - 2\bar{y}), \quad F_1 := S(\bar{f} - 2\bar{x}) \oplus S(\bar{f} - 2\bar{y}),
\]

and \(f_0\) and \(f_1\) are given by the matrices

\[
q_0 = \begin{pmatrix} x^{\alpha-2}y^\beta \ y^2 \\ -y x^{\alpha-2}y^\beta \end{pmatrix} \quad \text{and} \quad q_1 = \begin{pmatrix} x^2 \ y^2 \\ -y x^{\alpha-2}y^\beta \end{pmatrix}
\]

respectively.
Definition 7. We define a singularity \((X_Q, 0)\) by
\[
F_Q(w, x, y) := (F_{Q,1}(w, x, y), F_{Q,2}(w, x, y)) = \left( x^{a-2}y^\beta + \frac{w}{x}y^2, y - \frac{w}{x}x^2 \right).
\]
If we substitute \(y = \frac{w}{x}x^2\) in \(F_{Q,1}\), we obtain the function
\[
F_Q(w, x) := F_{Q,1} \left( w, x, \frac{w}{x}x^2 \right) = w^3x + x^{a+\beta-2}w^\beta.
\]
Thus \((X_Q, 0)\) is in fact a hypersurface singularity defined by \(F_Q = 0\).

We get the function \(f\) back from \(F_Q\) by the reduction considered in [10, p. 19], namely \(F_Q\) is of the form \(F_Q(w, x) = w^3x + x^3A(w, x)\), where \(A(w, x) = x^{a+\beta-5}w^\beta\), and \(RF_Q(w, x) = w^3 + x^5A\left(\frac{w}{x}, x\right) = f(x, w)\).

3. Graded matrix factorizations of size two for regular systems of weights with \(\varepsilon = -1\)

We shall now consider the graded matrix factorizations of size two for the regular systems of weights with \(\varepsilon = -1\). They were determined in [12]. There are 22 such systems of weights. They define the Fuchsian hypersurface singularities of genus 0 (see, e.g., [7]). Let \(W = (a, b, c; h)\) be such a weight system. By definition, it is reduced. The number \(\varepsilon_W\) is defined by \(\varepsilon_W := a + b + c - h\). Let \(A_W = (\alpha_1, \ldots, \alpha_r)\) be the signature of the system of weights \(W\). Then \((\alpha_1, \ldots, \alpha_r)\) corresponds to the signature of the corresponding Fuchsian singularity. For \(r = 3\), there are 14 such weight systems which correspond to the 14 exceptional unimodal singularities. Here the polynomial \(f_W\) is an invertible polynomial and its canonical system of weights coincides with \((a, b, c; h)\). For \(r = 4\), there are 6 corresponding to the heads of the bimodal series. There are 2 more with \(r = 5\). The corresponding equations are weighted homogeneous deformations \(f_W\) of an invertible polynomial, see Table 2.

This again implies that \(L_{f_W} \cong \mathbb{Z}\). Thus all the regular systems of weights with \(\varepsilon_W = -1\) can be given by weighted homogeneous deformations \(f\) of invertible polynomials with \(\varepsilon_f = -1\) and \(L_f \cong \mathbb{Z}\) and the \(L_f\)-graded matrix factorizations of \(f\) correspond to the graded matrix factorizations considered in [12]. The Dolgachev numbers are equal to the signature of the system of weights.

It will turn out that the graded matrix factorizations of size 2 correspond to some of these singularities, but also to the Fuchsian complete intersection singularities of genus 0 (see [7]). There are 8+5+2 of them for certain signatures \((\alpha_1, \ldots, \alpha_r)\).

Let \(W\) be a regular system of weights with \(\varepsilon_W = -1\) and signature \(A_W = (\alpha_1, \ldots, \alpha_r)\). In [12], a full strongly exceptional collection in the triangulated category of graded matrix factorizations associated to \(W\) was constructed. The full strongly exceptional collection is described by the graph shown in Fig. 1.

It turns out that a graded matrix factorization of size 2 corresponds to the end point \(V_{i, \alpha_i}\) of some arm.

Theorem 1. Let \(f\) be a weighted homogeneous deformation of an invertible polynomial with \(\varepsilon_f = -1\) and \(L_f \cong \mathbb{Z}\). Let \(A = (\alpha_1, \ldots, \alpha_r)\) be the Dolgachev
Table 2. Type of $f_W$

| $W$         | $f_W$                                             | Type   | $p_1$, $p_2$, $p_3$ | $A = A_W$ |
|-------------|---------------------------------------------------|--------|---------------------|-----------|
| 6,14,21:42  | $-(y^3 - x^7) + z^2$                              | I, $m = 1$ | 3,7,2               | 3,7,2     |
| 4,10,15:30  | $-x(y^5 - x^2) + z^2$                             | II$_1$, $m = 1$ | 3,5,2               | 4,5,2     |
| 3,8,12:24   | $-x(y^4 - x) + z^3$                               | II$_1$, $m = 1$ | 2,4,3               | 3,4,3     |
| 6,8,15:30   | $-x(y^3 - x^4) + z^2$                             | II$_1$, $m = 1$ | 5,3,2               | 8,3,2     |
| 4,6,11:22   | $-xy(y^2 - x^3) + z^2$                            | III, $m = 1$ | 4,3,2               | 6,4,2     |
| 3,5:9:18    | $-x(y^3 - x) + yz^3$                              | IV, $m = 1$ | 2,3,3               | 3,5,3     |
| 4,5:10:20   | $-x(y^2 - x) + z^5$                               | II$_1$, $m = 1$ | 2,2,5               | 5,2,5     |
| 3,4:8:16    | $-x(y^2 - x) + yz^4$                              | IV, $m = 1$ | 2,2,4               | 4,3,4     |
| 6,8,9:24    | $-x(y^2 - x^3) + z^3$                             | II$_1$, $m = 1$ | 4,2,3               | 9,2,3     |
| 4,6:7:18    | $-x(y^3 - x^2) + yz^2$                            | IV, $m = 1$ | 3,3,2               | 4,7,2     |
| 3,5:6:15    | $-xy(y - x^2) + z^3$                              | III, $m = 1$ | 3,2,3               | 6,3,3     |
| 4,5:6:16    | $-x(y^2 - x^3) + yz^2$                            | IV, $m = 1$ | 4,2,2               | 6,5,2     |
| 3,4:5:13    | $-z(xz - y^2) + yx^3$                             | V       | 3,2,2               | 3,4,5     |
| 3,4:4:12    | $-xy(y - x) + z^4$                                | III, $m = 1$ | 2,2,4               | 4,4,4     |
| 2,6:9:18    | $-x(y^3 - x)(y^3 - \lambda_4 x) + z^2$           | II$_1$, $m = 2$ | 3,6,2               | 2,3,2,2   |
| 2,4:7:14    | $-xy(y - x^2)(y - \lambda_4 x^2) + z^2$          | III, $m = 2$ | 5,3,2               | 4,2,2,2   |
| 2,4:5:12    | $-x(y^2 - x)(y^2 - \lambda_4 x) + yz^2$          | IV, $m = 2$ | 3,4,2               | 2,5,2,2   |
| 2,3:6:12    | $-(z^3 - x)(z^3 - \lambda_4 x) + xy^2$           | II$_2$, $m = 2$ | 2,2,6               | 3,3,2,2   |
| 2,3:4:10    | $-(z^2 - x)(z^2 - \lambda_4 x) + xy^3$           | IV, $m = 2$ | 2,3,4               | 2,2,2,3   |
| 2,3:3:9     | $-x(y - x^3)(y - \lambda_4 x^2) + yz^2$          | IV, $m = 2$ | 2,5,2               | 4,3,2,2   |
| 2,2:5:10    | $-xy(y - x)(y - \lambda_4 x)(y - \lambda_5 x) + z^2$ | III, $m = 3$ | 3,2,3               | 3,2,3,3   |
| 2,2:3:8     | $-x(y - x)(y - \lambda_4 x)(y - \lambda_5 x) + yz^2$ | IV, $m = 3$ | 4,4,2               | 2,2,2,2   |

Fig. 1. The graph $T_{\alpha_1,\ldots,\alpha_r}$.
numbers of $f$. If the $L_f$-graded matrix factorization $V_{i,\alpha_i}$ is of rank 2, then the corresponding singularity $F_{V_{i,\alpha_i}}$ has the signature $(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r)$ with $\tilde{\alpha}_i = \alpha_i + 1$ and $\tilde{\alpha}_j = \alpha_j$ for $j \neq i$.

**Proof.** We consider the matrix factorizations for the cases of Table 1 and examine conditions under which there exist maps of the local ring of the corresponding complete intersection singularity to the ring of a weighted projective line of type $(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r)$. We indicate the matrix factorization, the mapping, the index $i$ where we have an increase of $\alpha_i$ by one, and the condition under which the mapping gives an embedding.

(I) $m \geq 2$, $m|p_1$, $m|p_2$. We consider the matrix factorization

$$q_0 = \left( \prod_{i=4}^{m+2} \left( y_{\frac{p_2}{m}} - \lambda_i x_{\frac{p_1}{m}} \right) \frac{-z}{z^{p_3-1}} \right) \left( -y_{\frac{p_2}{m}} - x_{\frac{p_1}{m}} \right).$$

Mapping:

$$x = X_1 X_3^{\frac{p_2}{m}}, \quad y = X_2 X_3^{\frac{p_1}{m}}, \quad z = X_3^{(m-1)\frac{p_1 p_2}{m^2}} X_4 \cdots X_{m+2},$$

Increase of $\alpha_3$ under the condition:

$$(m - 1)\frac{p_1 p_2}{m^2} (p_3 - 1) = \frac{p_1 p_2}{m^2} + p_3 + 1.$$  \hspace{1cm} (1)

(II) $m \geq 1$, $m|(p_1 - 1)$, $m|p_2$. We have two essentially different matrix factorizations:

(a) $$q_0 = \left( \prod_{i=3}^{m+2} \left( y_{\frac{p_2}{m}} - \lambda_i x_{\frac{p_1}{m}} \right) \frac{-z}{z^{p_3-1}} \right).$$

Mapping:

$$x = X_1^{p_1+\frac{m+2}{p_1-1}}, \quad y = X_1 X_2, \quad z = X_1^{p_2} X_3 \cdots X_{m+2},$$

Increase of $\alpha_1$ under the condition:

$$p_2(p_3 - 1)(p_1 - 1) = p_3(p_1 - 1) + m + p_2.$$  \hspace{1cm} (2)

(b) $$q_0 = \left( x \prod_{i=4}^{m+2} \left( y_{\frac{p_2}{m}} - \lambda_i x_{\frac{p_1-1}{m}} \right) \frac{-z}{z^{p_3-1}} \right) \left( -y_{\frac{p_2}{m}} - x_{\frac{p_1-1}{m}} \right).$$

Mapping:

$$x = X_1^{p_3} X_3^{\frac{p_2}{m}}, \quad y = X_2 X_3^{\frac{p_1-1}{m}}, \quad z = X_1 X_3^{\frac{p_2}{m}+(m-1)\frac{(p_1-1)p_2}{m^2}} X_4 \cdots X_{m+2},$$
\[ w = X_1^{p_3-1} X_4^{p_3-1} \cdots X_{m+2}^{p_3-1} \]

Increase of \( \alpha_3 \) under the condition:

\[
\left( \frac{p_2}{m} + (m-1) \frac{(p_1-1)p_2}{m^2} \right) (p_3 - 1) = \frac{(p_1-1)p_2}{m^2} + p_3 + 1. \tag{3}
\]

\[(\text{II}_2) m \geq 2, m | p_1, m | p_3. \] We consider the matrix factorization

\[ q_0 = \left( \prod_{i=4}^{m+2} \left( z \frac{p_3}{m} - \lambda_i x \frac{p_1}{m} \right) \begin{pmatrix} -y \\ xyp_2 \end{pmatrix} - \left( z \frac{p_3}{m} - x \frac{p_1}{m} \right) \right). \]

Mapping:

\[ x = X_1^{p_3} X_3 \frac{p_1}{m}, \quad y = X_3^{(m-1) \frac{p_1 p_3}{m^2}} X_4 \cdots X_{m+2}, \quad z = X_1 X_2 X_3 \frac{p_1}{m}, \quad w = X_1^{(m-1) \frac{p_3}{m}} X_4^{p_2-1} \cdots X_{m+2}^{p_2-1} \]

Increase of \( \alpha_3 \) under the condition:

\[
\frac{p_3}{m} + (m-1) \frac{p_1 p_3}{m^2} (p_2 - 1) = (m-1) \frac{p_1 p_3}{m^2} + p_2 + 1. \tag{4}
\]

One can show that the other possible matrix factorization

\[ q_0 = \left( \prod_{i=4}^{m+2} \left( z \frac{p_3}{m} - \lambda_i x \frac{p_1}{m} \right) \begin{pmatrix} -x \\ yp_2 \end{pmatrix} - \left( z \frac{p_3}{m} - x \frac{p_1}{m} \right) \right) \]

has no solution.

\[(\text{III}) m \geq 1, m | (p_1 - 1), m | (p_2 - 1). \] We have the essentially different matrix factorizations:

(a)

\[ q_0 = \left( y \prod_{i=3}^{m+2} \left( y \frac{p_2-1}{m} - \lambda_i x \frac{p_1-1}{m} \right) - z \begin{pmatrix} -x \\ z \end{pmatrix} \right). \]

Mapping:

\[ x = X_1^{p_3 + \frac{(m-1)+p_2}{p_1-1}}, \quad y = X_1 X_3^{p_3}, \quad z = X_1^{p_2} X_2 X_3 \cdots X_{m+2}, \quad w = X_2^{p_3-1} X_3^{p_3-1} \cdots X_{m+2}^{p_3-1} \]

Increase of \( \alpha_1 \) under the condition:

\[
p_2(p_3 - 1)(p_1 - 1) = p_3(p_1 - 1) + (m-1) + p_2. \tag{5}
\]

(b)

\[ q_0 = \left( xy \prod_{i=4}^{m+2} \left( y \frac{p_2-1}{m} - \lambda_i x \frac{p_1-1}{m} \right) z \frac{p_3-1}{m} - (y \frac{p_2-1}{m} - x \frac{p_1-1}{m}) \right). \]
Graded matrix factorizations of size two and reduction

Mapping:

\[ x = X_1^{\frac{p_1-1}{m}} X_3^{\frac{p_2-1}{m}} , \quad y = X_2^{p_3} X_3^{\frac{p_1-1}{m}} , \quad z = X_1 X_2 X_3^{\frac{p_1-1}{m} + \frac{p_2-1}{m} + (m-1) \frac{(p_1-1)(p_2-1)}{m^2}} \]

\[ X_4 \cdots X_{m+2}, \quad w = X_1^{p_3-1} X_2^{p_3-1} X_4^{p_3-1} \cdots X_{m+2}^{p_3-1} \]

Increase of \( \alpha_3 \) under the condition:

\[
\left( \frac{p_1-1}{m} + \frac{p_2-1}{m} + (m-1) \frac{(p_1-1)(p_2-1)}{m^2} \right) (p_3-1) = \left( \frac{(p_1-1)(p_2-1)}{m^2} \right) + p_3 + 1. \tag{6}
\]

(IV) \( m \geq 1, m | (p_1-1), m | p_2 \). We have three essentially different factorizations.

(a)

\[ q_0 = \left( \prod_{i=3}^{m+2} \left( \frac{y^{\frac{p_2}{m}} - \lambda_i x^{\frac{p_1-1}{m}}}{y z^{p_3-1}} \right) \right). \]

Mapping:

\[ x = X_1^{p_3 + \frac{m+p_2}{p_1-1}} X_2, \quad y = X_1 X_2^{p_1}, \quad z = X_1 X_2 X_3 \cdots X_{m+2}, \]

\[ w = X_2^{p_1-1} X_3^{p_3-1} \cdots X_{m+2}^{p_3-1} \]

Increase of \( \alpha_1 \) under the condition:

\[ p_2(p_3-1)(p_1-1) = p_3(p_1-1) + (m + 1 - p_1) + p_2. \tag{7} \]

(b)

\[ q_0 = \left( \prod_{i=3}^{m+2} \left( \frac{y^{\frac{p_2}{m}} - \lambda_i x^{\frac{p_1-1}{m}}}{y z^{p_3-1}} \right) \right). \]

Mapping:

\[ x = X_1^{p_3} X_2^{p_3}, \quad y = X_2^{p_1 + \frac{(p_1-1)(p_3-1)+m}{p_2}}, \quad z = X_1 X_2 X_3 \cdots X_{m+2}, \]

\[ w = X_3^{p_3} \cdots X_{m+2}^{p_3} \]

Increase of \( \alpha_2 \) under the condition:

\[ p_2(p_1(p_3-1) - p_3) = (p_1-1)(p_3-1) + m. \tag{8} \]

This equation is equivalent to Eq. (7).

(c)

\[ q_0 = \left( \prod_{i=4}^{m+2} \left( \frac{y^{\frac{p_2}{m}} - \lambda_i x^{\frac{p_1-1}{m}}}{y z^{p_3-1}} \right) \right). \]
Mapping:
\[ x = X_1^{p_3} X_2 X_3^{\frac{p_2}{m}}, \quad y = X_2^{p_1} X_3^{\frac{p_1-1}{m}}, \quad z = X_1 X_3^{\frac{p_2}{m} + (m-1) \frac{(p_1-1)p_2}{m^2}} X_4 \cdots X_{m+2}, \]
\[ w = X_1^{p_3-1} X_2^{1+(m-1) \frac{p_1-1}{m}} X_4^{p_3-1} \cdots X_{m+2}^{p_3-1}. \]

Increase of \( \alpha_3 \) under the condition:
\[ \left( \frac{p_2}{m} + (m-1) \frac{(p_1-1)p_2}{m^2} \right) (p_3 - 1) + \frac{p_1 - 1}{m} = \frac{(p_1 - 1)p_2}{m^2} + p_3 + 1. \]
(9)

Similarly as in (II2), one can show that the other matrix factorization
\[ q_0 = \left( x \prod_{i=4}^{m+2} \left( y \frac{p_2}{m} - \lambda_i x \frac{p_1-1}{m} \right) \quad -y \quad z^{p_3} \right) \]
has no solution.

(d) We also have to consider a special matrix factorization in the case \( m = 1 \).
We can write the equation as
\[ f(x, y, z) = -y(xy^{p_2-1} - z^{p_3}) + x^{p_1}. \]
Then we have the matrix factorization
\[ q_0 = \left( x \frac{p_2}{m}^{p_3-1} - z^{p_3} - x \right) \]
Mapping:
\[ x = X_2^{p_3} X_3, \quad y = x_2^{p_1+\frac{3}{p_2-1}}, \quad z = X_1 X_2 X_3, \quad w = X_3^{p_1(p_1-1)} \]
Increase of \( \alpha_2 \) under the condition:
\[ p_3(p_1 - 1)(p_2 - 1) = p_1(p_2 - 1) + 2. \]
(10)
This equation is equivalent to Eq. (8) with \( m = 1 \).
(V) The equation is
\[ f(x, y, z) = -z(xz^{p_3-1} - y^{p_2}) + yx^{p_1}. \]
Then we have two matrix factorizations:
(a)
\[ q_0 = \left( xz^{p_3-1} - y^{p_2} - x \right) \]
Mapping:
\[ x = X_2^{p_2} X_3^{p_3}, \quad y = X_1^{p_3} X_2 X_3, \quad z = X_1 X_2^{p_1p_2-p_2+1}, \quad w = X_1^{p_3-1} X_3^{p_1p_2-p_2+1} \]
Increase of \( \alpha_2 \) under the condition:
\[ p_2(p_1 - 1)(p_3 - 1) = (p_1 - 1)(p_3 - 1) + 2. \]
(11)
Remark 2. The reduction is a relation between the singularities. Not all possible extensions of the signatures are realised. For example, the following pairs

\[
q_0 = \begin{pmatrix}
x z^{p_3 - 1} - y^{p_2} - y
x^{p_1}
- z
\end{pmatrix}
\]

Mapping:

\[
x = X_1 X_2 X_3^{p_2}, \quad y = X_1^{p_1} X_3^{p_1 - p_1 + 1} X_3, \quad z = X_1^{p_1} X_2^{p_1}, \quad w = X_2^{p_1} X_3^{p_1 + 2}
\]

Increase of \(\alpha_1\) under the condition (11).

In the cases I, II(b), III(b), and IV(c), one can exchange the factor with \(\lambda_3 = 1\) in the lower right corner of the matrix factorization by any of the factors corresponding to \(\lambda_i, i = 4, \ldots, m + 2\), and interchange \(X_3\) and \(X_i\). This leads to an increase of \(\alpha_i\) instead of \(\alpha_3\). In Case III(a), one can interchange the variables \(x\) and \(y\) to get an increase of \(\alpha_2\).

The solutions to the Eqs. (1)–(11) are precisely the entries of Table 2 and two extra ones. Equation (1) has the solution I, \(m = 2\), \((p_1, p_2, p_3) = (4, 4, 3), A = (2, 2, 3, 3)\). This corresponds to the minimally elliptic singularity \(V'_1\) of [6] with weight system \(W = (3, 3, 4; 12)\) and \(\varepsilon_W = -2\). Equation (9) has the additional solution IV, \(m = 1\), \((p_1, p_2, p_3) = (3, 3, 4), A = (8, 7, 4)\). This corresponds to a singularity with the weight system \(W = (12, 8, 7; 36)\) with \(\varepsilon_W = -9\).

Remark 1. The proof shows that the statement of Theorem 1 only holds in this very special situation. Namely, it shows that, except for two exceptional cases, the weighted homogeneous deformations \(f\) of an invertible polynomial with \(\varepsilon_f = -1\) are the only weighted homogeneous deformations of an invertible polynomial with the property that there exists an \(L_{f^{-}}\)-graded matrix factorization \(Q\) of size 2 such that the corresponding singularity \(F_Q\) has the signature \((\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r)\) with \(\tilde{\alpha}_i = \alpha_i + 1\) and \(\tilde{\alpha}_j = \alpha_j\) for \(j \neq i\) where \(\alpha = (\alpha_1, \ldots, \alpha_r)\) are the Dolgachev numbers of \(f\).

Example 1. We consider the case \(W = (4, 10, 15; 30), A_W = (2, 4, 5), f_W = x^5 y + y^3 + z^2, f_W' = x^5 y + y^3\). This is the singularity \(E_{13}\) in Arnold’s notation. We consider the graded matrix factorization of \(f_W'\) given by

\[
q_0 = \begin{pmatrix}
x^3 y - y^2
y
x^2
\end{pmatrix}
\]

Then \(F_Q (w, x) = w^3 x + wx^4\). This is the singularity \(Z_{12}\) with signature \((2, 4, 6)\).
Table 3. Dolgachev numbers, Kodaira types, and grades

| Type of Dolgachev numbers | Kodaira type | $\ell$ | $D = -Z^2$ |
|---------------------------|-------------|-------|-----------|
| $2, 3, c; c \geq 7$      | II$(k)$     | 1     | $c - 6$   |
| $2, b, c; b, c \geq 4$  | III$(k_1, k_2)$ | 2 | $b + c - 8$ |
| $a, b, c; a, b, c \geq 3$ | IV$(k_1, k_2, k_3)$ | 3 | $a + b + c - 9$ |
| $a, b, c, d; a, b, c, d \geq 2$ | $I^*_0(k_1, k_2, k_3, k_4)$ | 4 | $a + b + c + d - 8$ |

The isolated hypersurface singularities defined by regular systems of weights with $\varepsilon = -1$ and $r \leq 4$ and the isolated complete intersection singularities which reduce to them are Kodaira singularities [10] or minimally elliptic singularities [13]. The minimal resolution is of Kodaira type II, III, IV or $I^*_0$ (see [10]) and it carries a fundamental cycle $Z$ (see [3, p. 132]). The self-intersection number of the fundamental cycle $D := -Z^2$ is called the grade of the singularity [10]. Let $(X, 0)$ be any of the singularities of Table 2 with Dolgachev numbers $\alpha_1, \ldots, \alpha_r$ and $r \leq 4$ or of the isolated complete intersection singularities which reduce to them. According to [10] or [13], there is the correspondence between certain types of Dolgachev numbers, Kodaira types, the number $\ell$ of parameters $k_i$ of the Kodaira type, and the grades indicated in Table 3 (see also [15, p. 515]).

By [16, Corollary 7.9.2], the Kodaira type and hence the number $\ell$ of a singularity and its reduction are the same.

The possible reductions between these singularities are visualized in Fig. 2. Here a usual arrow refers to a reduction corresponding to Theorem 1, a dashed arrow to the special reduction $R$ described in Sect. 2(B). The Dolgachev numbers of the regular systems of weights for $\varepsilon = -1$ are denoted by usual letters, those of the remaining singularities are denoted by scriptstyle letters. We also indicate the names of Arnold for the hypersurface singularities [2] and of Wall [16] (see also [7]) for the complete intersection singularities.

On the other hand, each of these singularities has another triple or quadruple of numbers, the Gabrielov numbers, see [8] for the definition in the case of the hypersurface singularities and [9] in the case of the complete intersection singularities. They also have types like the Dolgachev numbers, but for them the columns corresponding to $\ell$ and $D$ are interchanged, see Table 4. Note that the grade $D$ corresponds to the type of Gabrielov numbers, see [13, Theorem 3.13]. More precisely, according to [13, Theorem 3.13], for $1 \leq D \leq 3$, the singularity is a hypersurface singularity and, for $D = 4$, it is a complete intersection singularity. For $D = 1$, the singularity is given by an equation of the form $z^2 + y^3 + g(x, y, z) = 0$, where the polynomial $g(x, y, z)$ has only terms of degree $\geq 4$, see also [13, Table 1]. For $D = 2$, the singularity is given by an equation of the form $z^2 + h(x, y) = 0$, where the polynomial $h(x, y)$ has only terms of degree $\geq 4$, see also [13, Table 2]. Finally, for $D = 3$, the equation is $f(x, y, z) = 0$, where the polynomial $f(x, y, z)$ has only terms of degree $\geq 3$.

The hypersurface singularities with $r = 3$ are the 14 exceptional unimodal singularities. There is Arnold’s strange duality [2] between these singularities which interchanges Dolgachev and Gabrielov numbers.
There is another partial order between singularities, namely the adjacency. We consider a special type of adjacency, namely the $s$-adjacency of singularities as defined by H. Laufer in [14, Definition 4.12]. The $s$-adjacency corresponds to a deformation with constant grade. By [14, Theorem 4.13], the $s$-adjacencies between the 14 exceptional unimodal singularities are given by Fig. 3, where the numbers correspond to the Gabrielov numbers. This is the pyramid of adjacencies given in [2, p. 255].
Fig. 3. The pyramid of s-adjacencies between the 14 exceptional unimodal singularities

Remark 3. Note that the s-adjacency is a special case of adjacency. For example, the general adjacency between the 14 exceptional unimodal singularities corresponds to the natural partial ordering between the Gabrielov numbers, see [4].

A comparison of Fig. 2 restricted to the 14 exceptional unimodal singularities, Fig. 3, and Arnold’s strange duality yields the following corollary of Theorem 1.

Corollary 1. Denote Arnold’s strange duality by $X \leftrightarrow X^*$. If a singularity $Y$ is a reduction of $X$, then the dual singularity $X^*$ is s-adjacent to the dual singularity $Y^*$.

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Declarations

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