Maximal freedom at minimum cost: linear large-scale structure in general modifications of gravity

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Abstract. We present a turnkey solution, ready for implementation in numerical codes, for the study of linear structure formation in general scalar-tensor models involving a single universally coupled scalar field. We show that the totality of cosmological information on the gravitational sector can be compressed — without any redundancy — into five independent and arbitrary functions of time only and one constant. These describe physical properties of the universe: the observable background expansion history, fractional matter density today, and four functions of time describing the properties of the dark energy. We show that two of those dark-energy property functions control the existence of anisotropic stress, the other two — dark-energy clustering, both of which are scale-dependent. All these properties can in principle be measured, but no information on the underlying theory of acceleration beyond this can be obtained. We present a translation between popular models of late-time acceleration (e.g. perfect fluids, $f(R)$, kinetic gravity braiding, galileons), as well as the effective field theory framework, and our formulation. In this way, implementing this formulation numerically would give a single tool which could consistently test the majority of models of late-time acceleration heretofore proposed.

Keywords: modified gravity, cosmological parameters from LSS, cosmological perturbation theory, dark energy theory

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1 Introduction

Even before the discovery of the acceleration of the expansion of the universe, dynamical mechanisms for generating such a phase in the evolution history were being proposed (e.g. quintessence [1, 2]). The actual discovery resulted in a very large volume of proposals for dynamical models of this late-time acceleration, both based on classical field theories (usually of a scalar field, but also modified theories of the gravitational metric, e.g. $f(R)$ gravity [3]) and on phenomenological approaches (e.g. see reviews [4, 5]). It was quickly understood that one could potentially get a handle on the actual mechanism behind the acceleration by comparing the behaviour of the background expansion history with that of the large-scale structure which evolves on this background [6].

It is really this comparison that is key to understanding the nature of dark energy. A detection of an expansion history incompatible with a $\Lambda$-cold-dark-matter (LCDM) universe does mean that an explanation alternative to the cosmological constant is required, but does not actually say much about the nature of the dynamical mechanism. Conversely, the LCDM framework describes the whole gravitational sector using one constant parameter ($\Omega_\Lambda$) measurable from the background expansion. The growth of structure is then completely fixed (the remaining theoretical uncertainties on non-linear scales notwithstanding) and a measurement of the properties of large-scale structure is necessary to ascertain that some
other mechanism with background expansion close to ΛCDM is not actually responsible. Just measuring the background is not enough.

Structure in the can be studied universe in two approaches. One can build model-independent null tests from observables which can be used to constrain properties of the models. This approach is well studied for the case of the evolution of the background [7–9]. In the case of perturbations, it was demonstrated that one can in principle reconstruct the gravitational potentials as a function of time and scale and therefore measure such properties as the anisotropic stress in a model-independent manner [10, 11]. For example, Euclid should be able to measure the slip parameter in such a model-independent manner to within a few percent, depending on the assumptions [12].

A much more usual approach is to take a particular dark energy model and fold in the evolution described by it into a full cosmological Boltzmann code, such as CAMB [13] or CLASS [14, 15] following this with a Monte Carlo exploration of the parameter space allowed by the totality of all the available observational data. The disadvantage of this method is that it is very computationally expensive since each model must be constrained separately, unless it is nested in some sort of larger model class. Typically one therefore reduces the models to a small number of parameters which capture the gross behaviour and tries to constrain the parameters’ allowed values. For example, it is well understood that, on the level of the background, the effective equation of state of the dark energy, which is in principle a free function of time, fully describes the evolution of the metric. It is frequently parameterised using the first two terms of a Taylor expansion [16, 17].

The description of perturbations is more complicated. Scalar perturbations dominate at linear scales and, in principle, all one needs to do is to relate the two scalar gravitational potentials to the matter distribution through some sort of parameterisation involving two functions of both scale and time [18–20]. The remaining question is how to do it in a way that models properties of the underlying dark-energy theory, rather than being an arbitrary parameterisation which could capture a deviation away from the concordance model but not necessarily the physics. To be specific, general relativity has a generally covariant structure with no scalar degrees of freedom of its own, which leads to a particular set of relations between the gravitational potentials, their derivatives, and the density and velocity perturbations of the matter. The idea is to find the appropriate ways of generalising such relationships in more complicated theories.

For example, one can appropriately take large-scale effects into consideration [21], but this is clearly not enough to fully specify the theory. This can be matched to small-scale behaviour using a phenomenological interpolation [22, 23]. Alternatively, requiring gauge invariance of the equations [24, 25] restricts the possible structures, but still leaves a very large number of free functions to be determined. One can then attempt to describe the dark energy through its field content and its effective fluid — or some sort of hopefully physical — properties [26–29]. Instead, one can assume that the dark-energy model preserves locality and is approximately quasi-static, which results in a particular behaviour of the perturbations with scale [10, 30–32].

An alternative approach is to consider an effective field theory (EFT) for perturbations on top of a Friedmann-Robertson-Walker (FRW) cosmological background. This method was introduced in [33] and has been developed in the context of general dark-energy cosmologies in [34–39] (see also the review [40]). The idea is that one writes down an action for perturbations containing all possible operators allowed by the symmetries of the FRW background and properties required of the underlying dark energy. These operators have arbitrary coeffi-
cients that are purely functions of time and represent the maximal freedom that is permitted in the equations of motion for perturbations. Not all these operators contribute in a distinct manner to the perturbations [38] but they do represent the maximal freedom that one has in deforming Einstein equations according to whichever DE properties are required. This approach to perturbations has been implemented numerically in ref. [41].

Our approach in this paper is similar in spirit to the EFT approach, although we restrict our attention to the Horndeski class of models [42]. The Horndeski models are the most general set of theories of a single extra scalar degree of freedom or, equivalently, which have equations of motion with at most second derivatives on any background [43]. The majority of universally coupled dark energy and modified gravity models belong to this class. We show that the evolution of linear perturbations can be completely described by specifying an arbitrary background evolution history, the constant $\Omega_{m0}$ — fractional matter density today — and four additional independent functions of time only which fully specify the effects of the dark-energy model. We provide the full set of equations for direct implementation in standard cosmological codes. Since most popular models of dynamical dark energy fall into the scope of this paper, they can be easily mapped onto our formulation. We present this mapping, demonstrating how a code modified for our formulation can immediately be used to model perturbations in these less general classes of dark-energy theories, allowing for a single code to be used to constrain many classes of models.

Our choice of these four functions is equivalent to a particular combination of the EFT operators of refs. [34, 35], but one which has the advantage of mapping directly onto physical effects. In addition, contrary to the EFT descriptions where some of the operators enter both the background and the perturbation properties, our formulation completely separates the two. We also explicitly demonstrate that the four two of these parameters control the existence of anisotropic stress while the other two control the clustering of the dark energy.

A non-detection of anisotropic stress or any deviation from the standard growth rate would constrain the parameters to lie close to their $\Lambda$CDM values of zero. Combined with an expansion history consistent with $\Lambda$CDM, such a result would imply that any dynamical dark-energy mechanism is responsible for no more than a small fraction of the acceleration, with the remainder being driven by the standard vacuum energy.

We begin in section 2 by describing the scope of our approach. In section 3, we define our formulation and present the full background and perturbation equations in Newtonian gauge in terms of our parameter functions. We discuss the requirements posed by stability considerations, connect our formulation to previous approaches and provide a mapping between popular models of dark energy onto our formulation. We discuss the effect of the dark energy on the physics of structure formation in section 4, discussing the validity of the quasi-static limit, constraints from Solar-System tests and the measurability of the dark energy parameters. We discuss our findings and conclude in section 5. In the appendix, we present the relevant equation in synchronous gauge, allowing for immediate inclusion in popular Boltzmann codes.

2 Scope: general scalar-tensor theories

In this paper, we discuss the properties of linear perturbations of scalar-tensor theories belonging to the Horndeski class of actions [42, 43] when evolving on a cosmological background. This action is the most general action for a single classical scalar field in the presence of gravity which does not result in any derivatives higher than second order in the equations of motion.
on any metric. We assume that the weak equivalence principle holds and therefore that all matter species external to the scalar-tensor system are coupled minimally and universally.

The combined action for gravity and the scalar is defined by

\[ S = \int d^4x \sqrt{-g} \left[ \sum_{i=2}^{5} \mathcal{L}_i + \mathcal{L}_m[g_{\mu\nu}] \right], \tag{2.1} \]

where \( g_{\mu\nu} \) is the metric to which the matter sector, described by \( \mathcal{L}_m \), is coupled.\(^1\) For the purpose of this paper, the matter sector should be thought of as describing all of dark matter, baryons, radiation and neutrinos; we will not differentiate between them here. The Lagrangians \( \mathcal{L}_i \) are usually written as

\[ \mathcal{L}_2 = K(\phi, X), \]

\[ \mathcal{L}_3 = -G_3(\phi, X) \Box \phi, \]

\[ \mathcal{L}_4 = G_4(\phi, X) R + G_{4X}(\phi, X) \left[ (\Box \phi)^2 - \phi_{;\mu\nu} \phi^{;\mu\nu} \right], \]

\[ \mathcal{L}_5 = G_5(\phi, X) G_{\mu\nu} \phi^{;\mu\nu} - \frac{1}{6} G_{5X}(\phi, X) \left[ (\Box \phi)^3 + 2 \phi_{;\mu} \phi_{;\nu} \phi_{;\alpha} - 3 \phi_{;\mu\nu} \phi^{;\mu\nu} \Box \phi \right]. \]

The couplings to gravity are completely fixed by the Lagrangians \( \mathcal{L}_{4-5} \). The four functions \( K(\phi, X) \) and \( G_i(\phi, X) \) are arbitrary functions of the scalar field \( \phi \) and its canonical kinetic term \( X = -\phi^{;\mu} \phi^{;\mu}/2 \). The subscript \( X \) represents a derivative w.r.t. \( X \) while \( \phi \) is a derivative w.r.t. the field \( \phi \). This class of models therefore comprises four functional degrees of freedom which will combine into the four degrees of freedom in our parameterisation.

This general class of actions includes essentially all universally coupled models of dark energy with one scalar degree of freedom: quintessence \([1, 2]\), Brans-Dicke models \([44]\), k-essence \([45, 46]\), kinetic gravity braiding \([47-49]\), galileons \([50, 51]\). Archetypal modified-gravity models such as \( f(R) \) \([3]\) and \( f(G) \) \([52]\) gravity are within our purview. Models such as Dvali-Gabadadze-Porrati (DGP) gravity \([53]\) or ghost-free massive gravity \([54-56]\) lie outside of our purview. However, their appropriately covariantised decoupling limits, Galileon Cosmology \([57]\) and proxy massive gravity \([58, 59]\), which should correctly capture the behaviour of the scalar degree of freedom, do belong to the Horndeski class and therefore are described by our formulation.

As we explain later, our approach is also closely related to the effective field theory method of describing linear perturbations on the Friedman cosmological metric. It was proven in ref. \([36]\) that the maximal theory for linear perturbation described by equations with no more than second derivatives is equivalent to that given by the Horndeski action. Although ref. \([36]\) notes that there is an extension allowed where the Einstein equations contain third derivatives, but the equation for the propagating scalar degree of freedom does not. Surprisingly, this structure seems to be repeated on non-linear level \([60]\).\(^2\)

Some models of dark energy break the equivalence principle between the baryonic and dark sectors (e.g. coupled quintessence \([62]\) or non-universal disformal couplings \([63, 64]\)) allowing, for example, for a much simpler evasion of Solar-System constraints. Our method could be extended to these models quite simply with the introduction of an additional parameter for the coupling to dark matter and the relative DM/baryon density. We must note

\(^1\)We use the \((- + + +)\) signature convention.

\(^2\)See also the possibly related ref. \([61]\).
that our framework does not in general cover such classes of models as: Lorentz-invariance-violating models (e.g. Hořava-Lifschitz models \[65, 66\]) or non-conservative fluids arising from high-temperature self interactions of a scalar (e.g. Dark Goo \[67\]).

3 Minimal description of dynamics

We assume that the universe is well described by small scalar perturbations on top of a FRW metric. We assume spatial flatness and use the Newtonian gauge with the notation of ref. \[68\] throughout the paper,\(^3\) i.e. the line element takes the form

\[ \text{ds}^2 = -(1 + 2\Psi)dt^2 + a^2(t)(1 - 2\Phi)d\mathbf{x}^2. \] (3.1)

We have chosen this gauge since the metric potentials \(\Phi\) and \(\Psi\) are in fact cosmological observables \[11\]. We present the main results in synchronous gauge in the appendix C.

The purpose of the presentation that follows in the rest of this section is dual. Firstly, it provides sufficient information to fully solve for the evolution of linear cosmological perturbations in any particular dark-energy model in the Horndeski class. Given an action for a model, all one is required to do is to compute the form of four functions \(\alpha_i\) defined in section 3.1 and solve the resulting equations of motion for the background and perturbations. On this level, the method gives a unified, “write once use many” approach for evolving linear large-scale structure in dark energy models. To obtain constraints on parameters of the Lagrangian, the algorithm would need to scan through potential initial conditions on the scalar field value, solving for the background evolution and then determining the values of the \(\alpha_i\) as a function of time for that model. This is what is being done today for example for covariant galileon models \[69\]; our formulation essentially provides a book-keeping device when seen from this perspective.

However, a potentially more fruitful approach would be one close to that taken by EFT methods: observations to date have determined that the background expansion history appears close to \(\Lambda\text{CDM}\). Since our description identifies the maximal number of functions independent of background expansion, the equations we provide can be used to constrain the forms of these property functions \(\alpha_i\) compatible with the observed quasi-\(\Lambda\text{CDM}\) background, without making any reference to a particular model or initial conditions for the scalar’s background value. In fact, the values of the functions \(\alpha_i\) are the maximum unambiguous information that can ever be extracted about such dark-energy models from the evolution of linear cosmological perturbations.

We now describe the physical meaning and origin of the four property functions \(\alpha_i\), before turning to the background equations and stability tests and finally the full perturbations equations involving all of these property functions.

3.1 Non-redundant set of linear-perturbation properties

It is well known that the evolution history of the cosmological background does not specify the dark-energy model (for example quintessence and \(f(R)\) gravity have different predictions for the growth of structure given the same expansion history). The main result of this paper is that we have formulated the possible linear perturbation theory for dark-energy models belonging to class (2.1) in terms of four functions of time, \(\alpha_i(t)\). Augmenting them with the \(^3\)Note that in this paper we used \(\Psi\) and \(\Phi\) instead of \(\psi\) and \(\phi\), being these potentials the correspondent gauge invariant variables in the Newtonian gauge.
background evolution history $H(t)$ and the matter density today $\tilde{\rho}_{m0}$ fully determines the evolution of large-scale structure. It is important to stress that for a general model all of these are completely independent. Equally importantly: this is the minimal set of functional parameters which is capable of describing all models within our scope.

When considering a particular model, with appropriate initial conditions for the background variables, the four functions $\alpha_i$ are determined by the Lagrangian and the value of the scalar field through the definitions (A.7)–(A.10). On one hand they can be considered a helpful compression of the perturbations equations. However, this is not the approach we advocate here. Given a fixed background and $\tilde{\rho}_{m0}$, any two trajectories in any two seemingly different models which have the same $\alpha_i$ exhibit absolutely no difference in their behaviour. This implies that no measurements of linear structure can ever say anything about the particular theory of dark energy beyond the measurement of $\rho_{m0}$ and $\alpha_i(t)$. This is the maximum extent to which dark-energy Horndeski Lagrangians can ever be reconstructed using linear-theory observations.

The above is strictly true only when the scalar rolls monotonically, i.e. $\dot{\phi} \neq 0$. If there are oscillations of the background values (e.g. see ref. [70]), then the equations need to be expanded and they do depend also on the value of $\dot{\phi}$. In such a case, the full model must be supplied and solved for.

Since the four DE property functions $\alpha_i$ are arbitrary and independent of the background, the matter sector and of each other, we argue that they should be thought of as being essentially the relevant independent physical properties of the dark energy and should be the ones targeted for measurement. We describe the physical meaning of these function here, while their analytic definition in terms of the Horndeski functions $K$ and $G_i$ can be found in appendix A,

- $\alpha_K$, kineticity. Kinetic energy of scalar perturbations arising directly from the action. It is the only contribution of perfect-fluid models but is not present at all in archetypal “modified gravity” models such as $f(R)$ and $f(G)$. Large values act to suppress the sound speed of scalar perturbations. Contribution from all of $K, G_3, G_4, G_5$.

- $\alpha_B$, braiding. Signifies braiding, or mixing of the kinetic terms of the scalar and metric. Contributes to kinetic energy of scalar perturbations indirectly, by backreacting through gravity. Second time derivatives of metric and scalar field to appear in equations of motion for both the scalar and the metric. Causes dark energy to cluster. Contributions from $G_3, G_4, G_5$.

- $\alpha_M$, Planck-mass run rate. Rate of evolution of the effective Planck mass. A constant redefinition of the Planck mass does not affect physics. Its time evolution in the Jordan frame of the observer creates anisotropic stress. Contributions from $G_4$ and $G_5$.

- $\alpha_T$, tensor speed excess. Deviation of the speed of gravitational waves from that of light. This violation of Lorentz-invariance for tensors also changes the response of the Newtonian potential $\Psi$ to matter sources even in the presence of no scalar perturbations, leading to anisotropic stress. Contributions from $G_4$ and $G_5$.

The existence of these four contributions can be roughly allocated to the four possible types of perturbation structures in the dark-energy EMT at linear order. Following ref. [26], we analyse the perturbations on “comoving” $\phi = \text{const.}$ (equal clock) hypersurfaces.
• The $\alpha_K$ contributions are essentially those of linear perturbations of the standard perfect-fluid EMT.

• Braiding $\alpha_B$ represents a space-like energy flux vector $q^i$ in this frame (a $T^{0i}$ contribution); essentially this is a generalisation of the imperfect contributions described for kinetic gravity braiding models in [49].

• The change in the speed of tensors $\alpha_T$ arises as a contribution in the DE EMT proportional to the curvature of the spatial comoving hypersurfaces, $(^3R_{\mu\nu})$, which contains the graviton gradient terms. The effect of this is to introduce an offset between the two metric potentials, i.e. anisotropic stress.

• Finally, $\alpha_M$ is a result of contributions in the EMT proportional to the Einstein tensor. On $\phi = \text{const.}$ hypersurfaces at linear order it has no impact. However, any variation of the Planck mass represents a conformal rescaling of the metric which splits the two gravitational potentials introducing anisotropic stress.

We have presented a list of $\alpha_i$’s for a selection of popular dark-energy model classes in table 1. Typically, any particular dark-energy model class will only turn on one of the $\alpha_i$’s (perfect fluids), or the values of multiple parameters will be related (e.g., $f(R)$). Thus measuring the appropriate values of the parameters allows for a determination of the class of models into which the dark energy falls. Standard parameterisations for particular models can easily be used by finding the appropriate conversion between them and the $\alpha_i$.

In principle, it might be possible to directly measure the parameters $\alpha_i$ as a function of redshift given good enough data (see the null tests of ref. [11] for a method), but most likely they will have to be inferred from the integrated total of cosmological structure-formation data. This means that a parameterisation is necessary, especially if this method is to be implemented in Boltzmann codes. For a general parameterisation we propose that

\[
\alpha_i = (1 - \tilde{\Omega}_m) \hat{\alpha}_i, \quad \hat{\alpha}_i = \text{const.},
\]

be used. This sort of parameterisation reflects the fact the values of $\alpha_i$ are driven by the same functions of the scalar field as the energy density and their derivatives. Thus naively one would expect them to be of similar size, i.e. $\hat{\alpha}_i \sim 1$. If any of the $\hat{\alpha}_i \gg 1$ (for example, in a perfect fluid model with $c_s^2 \ll 1 + w_X$), then this is a sign that the energy density and pressure both have some kind of cancellation which disappears when additional derivatives are taken to form the $\alpha_i$. Thus this kind of situation, which is in principle perfectly well allowed, can be considered evidence of tuning in the structure of the Lagrangian. Finally, the situation where all the $\hat{\alpha}_i \ll 1$ implies that the acceleration is mainly driven by a cosmological constant, with a marginal contribution from the dynamical dark-energy mechanism.

### 3.2 Cosmological background

The Friedmann equations can be obtained in the usual manner. The ambiguity is the fact that in Horndeski theories the Planck mass can be a function of time. In particular, the role of the Planck mass is played by

\[
M_P^2(\phi, X, H) \equiv 2 \left( G_4 - 2XG_{4X} + XG_{5\phi} - \dot{\phi} H XG_{5X} \right),
\]

which depends on time implicitly through the solution realised by the universe in $\phi$, $X$ and $H$. If the Planck mass $M_P^2$ is constant on the solution realised by the Universe then it can
Table 1. Parameter functions $\alpha_i$ for various classes of models determined from their actions using definitions (A.6)–(A.10). For example in quintessence models, background evolution determines absolutely everything about linear perturbations, while in k-essence we have an additional degree of freedom, the sound speed. Perfect-fluid models are described by just the kineticity $\alpha_K$. On the other hand, the archetypal “modified gravity” models such as $f(R)$ and $f(G)$ have $\alpha_K = 0$, but turn on other directions in the space of perturbations. Most simple models of dark energy are described by a single or possibly two functions forming a subspace in the full potential theory space spanned by the $\alpha_i$.

| Model Class                  | $\alpha_K$ | $\alpha_B$ | $\alpha_M$ | $\alpha_T$ |
|------------------------------|------------|------------|------------|------------|
| $\Lambda$CDM ($w_X \neq -1$) | 0          | 0          | 0          | 0          |
| cuscuton ($w_X \neq -1$)     | [71]       | 0          | 0          | 0          |
| quintessence                 | [1, 2]     | $(1 - \Omega_m)(1 + w_X)$ | 0          | 0          | 0          |
| k-essence/perfect fluid      | [45, 46]   | $(1 - \Omega_m)(1 + w_X)$ | 0          | 0          | 0          |
| kinetic gravity braiding     | [47–49]    | $\frac{m^2(n_m + \kappa_\phi)}{H^2 M^2_{\text{Pl}}}$ | $\frac{m_\kappa}{HM^2_{\text{Pl}}}$ | 0          | 0          |
| galileon cosmology           | [57]       | $-\frac{3}{2} \alpha_M^3 H^2 \epsilon^2 e^{2\phi/M}$ | $\frac{\alpha_K}{\text{Pl}} - \alpha_M$ | $\frac{-2\dot{\phi}}{HM_{\text{Pl}}}$ | 0          |
| BDK                         | [26]       | $\frac{\phi^2 K_{\phi\phi} e^{-\kappa}}{H^2 M^2}$ | $-\alpha_M$ | $\frac{\kappa}{\text{Pl}}$ | 0          |
| metric $f(R)$               | [3, 72]    | 0          | $-\alpha_M$ | $\frac{B\theta}{H^2}$ | 0          |
| MSG/Palatini $f(R)$         | [73, 74]   | $-\frac{3}{2} \alpha_M^2$ | $-\alpha_M$ | $\frac{2\dot{\phi}}{\text{Pl}}$ | 0          |
| $f$(Gauss-Bonnet)            | [52, 75, 76]| 0          | $\frac{-2H\dot{\xi}}{M^2 + H\dot{\xi}}$ | $\frac{\dot{H}\dot{\xi} + H\dot{\xi}}{H(M^2 + H\dot{\xi})}$ | $\frac{\dot{\xi} - H\dot{\xi}}{M^2 + H\dot{\xi}}$ |

$\alpha_M \equiv H^{-1} \frac{\text{d} \ln M^2}{\text{d} t}$.  

The Friedmann equations are

$$3H^2 = \tilde{\rho}_m + \tilde{\epsilon},$$

$$2\dot{H} + 3H^2 = -\tilde{p}_m - \tilde{\mathcal{P}},$$

where we have kept the left-hand side standard. $\tilde{\rho}_m$ and $\tilde{p}_m$ are the background energy density and pressure of all the matter components together, while $\tilde{\epsilon}$ and $\tilde{\mathcal{P}}$ are the background energy...
density and pressure of the dark energy.\footnote{Bear in mind that we have removed some terms that appear in the EMT by collecting them to form the Planck mass $M^2_*$, eq. (3.3).} We also sometimes use the matter equation of state parameter and the fractional matter density,

$$w_m \equiv \frac{\tilde{p}_m}{\tilde{\rho}_m}, \quad \tilde{\Omega}_m \equiv \frac{\tilde{\rho}_m}{3H^2}.$$  

(3.6)

We are using the tilde to indicate that all the quantities are divided by $M^2_*$, e.g.

$$\tilde{\rho}_m \equiv \frac{\rho_m}{M^2_*}.$$  

(3.7)

The full expressions for $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{P}}$ are given in the appendix A, eqs. (A.1) and (A.2). Here we will only note that the pressure term contains a dependence on $\ddot{\phi}$,

$$\tilde{\mathcal{P}} \equiv \tilde{P}(\phi, X, H) - \alpha_B \frac{H\ddot{\phi}}{\phi},$$  

(3.8)

where $\tilde{P}$ is the part of the pressure only depending on $\phi, X$ and $H$. The dependence on $\ddot{\phi}$ results from kinetic braiding, the mixing between the scalar and metric kinetic terms which was extensively described as an essential feature of kinetic gravity braiding models in refs. \cite{47, 49}. In general Horndeski models, the braiding $\alpha_B$ receives contributions from all of $G_{3-5}$ and is defined in the appendix in eq. (A.9).

As a result of absorbing a potentially time-evolving Planck mass in the definition of the tilded quantities such as (3.7), the energy density is not necessarily covariantly conserved,

$$\dot{\tilde{\rho}}_m + 3H(\tilde{\rho}_m + \tilde{p}_m) = -\alpha_M H \ddot{\rho}_m,$$

but rather is exchanged between the scalar and matter subsystems whenever the value of $M^2_*$ is changing. This presents a slight complication: one must either keep track of the value of $M^2_*$ but is then able to use the standard conservation laws for the untilded quantities. Alternatively, the densities of all matter species need to appropriately integrated whenever $\alpha_M \neq 0$ using eq. (3.9).

It is important to stress that, despite appearing in background equations, both $\alpha_B$ and $\alpha_M$ are only possible to determine through their influence on structure formation. Any expansion history can be realised for any choice of these two functions. However, the linear perturbation theory changes when these functions change and this physics cannot be replicated by changing the other parameters. They are thus linear-perturbation properties and their appearance in this section is stressed only to guide those working to constrain models with fully defined actions, rather than in the effective framework advocated here.

Given the non-conservation of the matter components whenever the Planck mass runs, there is an ambiguity as to the definition of the equation of state and its relation to evolution of energy density for the dark energy. We will choose to make an operational definition based on the Friedmann equations (3.5) and therefore on the dark-energy background configuration,

$$1 + w_X \equiv \frac{\tilde{\mathcal{E}} + \tilde{\mathcal{P}}}{\tilde{\mathcal{E}}} = -\frac{2H + \dot{\tilde{\rho}}_m + \tilde{p}_m}{3H^2 + \tilde{\rho}_m}.$$  

(3.10)
This choice does not in general fully determine the evolution of dark-energy energy density, but is a useful book-keeping device in the perturbation equations and stability analysis. We should note that the observation of the background expansion history only determines $H(z)/H_0$. The value of $\tilde{\rho}_{m0}$, the density of the matter sector today, is a free parameter that can only be determined through measurement of large-scale structure [10, 11]. Depending on different choices of $\tilde{\rho}_{m0}$, the history of $w_X$ will also differ given a background expansion history fixed by observations.

Finally, the scalar field value evolves according to the equation of motion which can be written succinctly as,

$$\dot{n} + 3Hn = \mathcal{P}_\phi,$$

where $n$, defined in the appendix in eq. (A.4), is a shift charge density which is covariantly conserved whenever the action exhibits a Noether symmetry with respect to constant shifts, $\phi \to \phi + \text{const}$. $\mathcal{P}_\phi$ is a partial derivative of the pressure with respect to $\phi$, defined by eq. (A.5). Whenever the shift symmetry is present, the models feature attractors where the charge density $n = 0$. These provide natural scaling solutions and post-inflationary conditions for the models [47, 77].

### 3.3 Background stability

One of the complications of the non-canonical Horndeski models is that backgrounds can become unstable to perturbations. The meaning of these instabilities is that the background solution found is no longer appropriate. This is a particular concern when no full action is given, but a more phenomenological approach we advocate here is taken. Thus one must ensure that instabilities are not present and discard a particular range of parameters if they are found.

Generally, the background can suffer either from ghost or gradient instabilities, or both. Gradient instabilities occur when the background evolves to a region where the speed of sound squared of the perturbations is negative, leading to an exponential destabilisation of the perturbations at small scales, with timescales of the order of the cutoff of the theory. Ghost instabilities occur when the sign of the kinetic term for of the background perturbations is wrong. Usually, they are discussed in the context of quantum stability since the vacuum can destabilise to produce ghost and normal modes, if ghosts are present, dynamical and coupled (see estimates of the rates of such destabilisation in [78]).

The second-order action for perturbations of the Horndeski action was first derived in ref. [79] where a 3+1 decomposition in the unitary gauge to obtain actions for the propagating modes: the tensors (gravitational waves) $h_{ij}$, the scalar $\zeta$ and the perfect-fluid matter sound waves, the last of which which we will not write out here,

$$S_2 = \int \! dt d^3x a^3 \left[ Q_S \left( \dot{\zeta}^2 - \frac{c_s^2}{a^2} \left( \partial_i \zeta \right)^2 \right) + Q_T \left( \dot{h}_{ij}^2 - \frac{c_s^2}{a^2} \left( \partial_i h_{ij} \right)^2 \right) \right],$$

with the stability of the background to the scalar modes requiring

$$Q_S = \frac{2M^2 D}{(2 - \alpha_B)^2} > 0, \quad D \equiv \alpha_K + \frac{3}{2} \alpha_B^2,$$

and

$$c_s^2 = -\frac{(2 - \alpha_B) \left[ H - \frac{1}{2} H^2 \alpha_B (1 + \alpha_T) - H^2 (\alpha_M - \alpha_T) \right] - H \dot{\alpha}_B + \tilde{\rho}_m + \tilde{p}_m}{H^2 D} > 0,$$
while the stability to the tensor modes requires
\[
Q_T = \frac{M_{\ast}^2}{8} > 0, 
\]
\[
c^2_T = 1 + \alpha_T > 0. 
\]

Taking all the conditions together also implies that the no-ghost condition for the scalar perturbations reduces to \( D > 0 \). We should note here that \( c_s \) is the speed of propagation of small scalar perturbations in the infinite frequency (eikonal) limit, \( k \to \infty \), and therefore a statement about the causal structure of the model. It is therefore not a function of scale but only of time. It is not in general equivalent to the frequently used function \( C^2 \) relating the dark-energy pressure and density perturbations, \( \delta P \equiv C^2 \delta \epsilon \), which usually depends on scale, but which is not necessarily related to stability (see e.g. the discussion in ref. [26]).

It is the expression for \( Q_T \) that validates why we chose to call \( M_{\ast}^2 \), eq. (3.3), the effective Planck mass: the unambiguous meaning of Planck mass is as the normalisation of the graviton kinetic term. The action for the tensor perturbations also clarifies the meaning of \( \alpha_T \): it is the deviation of the speed of tensors from the speed of light. The graviton gradient terms are contained in the Ricci scalar of the spatial hypersurface in unitary gauge, \( (3)R \).

When evaluating the suitability of backgrounds, one must ensure that all conditions (3.13) and (3.14) are satisfied at all times of interest. When \( \alpha_B \neq 0 \), the region of phase space where \( D = 0 \) represents a pressure singularity [47] and no trajectory ever evolves across it [80]. However, nothing in the background dynamics prevents a trajectory with \( c^2_s > 0 \) from crossing into an unstable region. It is also possible that theories which violate the null energy condition suffer from some sort of non-linear instability even when they do not exhibit the instabilities described above [81].

One may, in principle, also wish to constrain the viable models to such trajectories where the speeds of sound are subluminal. Subluminality is a necessary precondition for there to exist a standard Wilsonian ultraviolet completion of the action [82]. However, in general Horndeski models superluminalities always exist somewhere in the phase space, especially in the presence of matter external to the scalar [83]. There are not necessarily causal paradoxes in theories with superluminality [84, 85], but these theories must be ultraviolet completed in an alternative way, e.g. via classicalisation [86, 87].

### 3.4 Linear perturbations

The value of the scalar field value is not an observable. We can reparameterise the field by defining a new \( \tilde{\phi} \equiv \tilde{\phi}(\phi) \) and all observables must remain unchanged. This means that focussing on the scalar’s value introduces unobservable redundancy into the description. Instead, one can notice that the scalar-field gradient forms a natural four-velocity,
\[
\tilde{u}_\mu \equiv -\frac{\partial_\mu \phi}{\sqrt{2X}}, 
\]
which defines a comoving frame, provided that \( \partial_\mu \phi \) be timelike, \( X > 0 \). The meaning of \( \phi \) then is one of a clock and its gradient is the time direction for an observer at rest in this frame. The perturbation of the scalar, when appropriately normalised, can then be interpreted as a scalar potential for a peculiar velocity field,
\[
\tilde{v}_X \equiv -\frac{\delta \phi}{\phi}, 
\]
where the overdot denotes the derivative with respect to the coordinate time $t$. This velocity potential is then invariant under the reparameterisations of $\phi$ and allows us to write down a much simpler set of perturbations equations. Oscillating models, where the background field velocity $\dot{\phi}$ crosses zero, cannot be directly described using our formulation, since the singularities in (3.16) would have to be appropriately accounted for.

The main result of this paper is the non-redundant formulation of linear cosmological perturbation equations in a manner immediately implementable in codes for calculating linear large-scale structure such as CAMB [13] or CLASS [14, 88]. We present the Einstein equations below in the user-friendly Newtonian gauge and coordinate time, providing the conformal-time synchronous-gauge version in appendix C. We are not the first to derive these equations (see refs. [34, 35, 89] and section 3.5), but we argue in this paper that our choice of variables allows for the separation of different physical effects and a natural limit to concordance cosmology. This allows us to discuss the physics of general Horndeski theories with relative ease in section 4.

The effect of the presence of the Horndeski scalar is to introduce the dynamical velocity potential $v_X$, describing the perturbation of the scalar field through eq. (3.16). We keep these terms on the left-hand side of the perturbed Einstein equations to emphasise the gravity-like nature of this degree of freedom, resulting from the universality of its coupling. In addition, the scalar’s background configuration changes the coefficients of the gravitational potentials away from their standard values arising from the Einstein tensor. On the right-hand-side of the Einstein equations lie the standard contributions of all the matter sources in the cosmology: dark matter, baryons, photons and neutrinos.

The Hamiltonian constraint (Einstein (00) equation) takes the form:

$$3(2 - \alpha_B) H \dot{\Phi} + (6 - \alpha_K - 6 \alpha_B) H^2 \Psi + \frac{2 k^2 \Phi}{a^2}$$

(3.17)

$$- (\alpha_K + 3 \alpha_B) H^2 \dot{v}_X - \left( \alpha_B \frac{k^2}{a^2} - 3 \dot{H} \alpha_B + 3 \left(2 \dot{H} + \tilde{\rho}_m + \tilde{p}_m \right) \right) H v_X = -\tilde{\rho}_m \delta_m,$$

the momentum constraint (Einstein (0i) equation)

$$2 \dot{\Phi} + (2 - \alpha_B) H \Psi - \alpha_B H \dot{v}_X - \left(2 \dot{H} + \tilde{\rho}_m + \tilde{p}_m \right) v_X = -\left(\tilde{\rho}_m + \tilde{p}_m \right) v_m,$$

(3.18)

the anisotropy constraint (spatial traceless part of the Einstein equations)

$$\Psi - \left(1 + \alpha_T \right) \Phi - \left(\alpha_M - \alpha_T \right) H v_X = \tilde{p}_m \pi_m,$$

(3.19)

and the pressure equation (spatial trace part of the Einstein equations)

$$2 \ddot{\Phi} - \alpha_B H \ddot{v}_X + 2 \left(3 + \alpha_M \right) H \dot{\Phi} + (2 - \alpha_B) H \dot{\Psi}$$

$$+ \left[ H^2 (2 - \alpha_B) (3 + \alpha_M) - (\alpha_B H) + 4 \dot{H} - \left(2 \dot{H} + \tilde{\rho}_m + \tilde{p}_m \right) \right] \Psi$$

$$- \left[ 2 \dot{H} + \tilde{\rho}_m + \tilde{p}_m \right] + (\alpha_B H) + H^2 \alpha_B (3 + \alpha_M) \right) \dot{v}_X$$

$$- \left[ 2 \dot{H} + 2 \dot{H} \left(3 + \alpha_M \right) + \tilde{p}_m + \alpha_M H \tilde{p}_m \right] v_X = \delta p_m / M^2_*$$,

where we have kept the effective Planck mass on the right hand side explicitly to stress that the matter-pressure perturbation refers only to the matter sector and not any perturbations in the Planck mass, which are already included on the left-hand side.
Finally, the equation of motion for the scalar velocity potential $v_X$,

$$3H\alpha_B \ddot{\Phi} + H^2 \alpha_K \dddot{v}_X - 3 \left[ \left( 2\dot{H} + \dot{\rho}_m + \dot{\rho}_\Lambda \right) - H^2 \alpha_B \left( 3 + \alpha_M \right) - \left( \alpha_B H \right) \right] \ddot{\Phi}$$

$$+ \left( \alpha_K + 3\alpha_B \right) H^2 \dot{\Psi} - 2 \left( \alpha_M - \alpha_T \right) H \frac{k^2}{a^2} \Phi - \alpha_B H \frac{k^2}{a^2} \Psi$$

$$- \left[ 3 \left( 2\dot{H} + \dot{\rho}_m + \dot{\rho}_\Lambda \right) - H \left( 2\alpha_K + 9\alpha_B \right) \right] \dot{H} \dot{\Psi}$$

$$- H \left( \dot{\alpha}_K + 3\dot{\alpha}_B \right) - H^2 \left( 3 + \alpha_M \right) \left( \alpha_K + 3\alpha_B \right) \right] H \dot{\Psi}$$

$$+ \left[ 2\dot{H} \alpha_K + \dot{\alpha}_K H + H^2 \alpha_K \left( 3 + \alpha_M \right) \right] H \dot{v}_X + H^2 M^2 v_X$$

$$+ \left[ - \left( 2\dot{H} + \dot{\rho}_m + \dot{\rho}_\Lambda \right) + 2H^2 \left( \alpha_M - \alpha_T \right) + H^2 \alpha_B \left( 1 + \alpha_M \right) + \left( \alpha_B H \right) \right] \frac{k^2}{a^2} v_X = 0,$$

with

$$H^2 M^2 \equiv 3\dot{H} \left[ \dot{H} \left( 2 - \alpha_B \right) + \dot{\rho}_m + \dot{\rho}_\Lambda - H \dot{\alpha}_B \right] - 3H\alpha_B \left[ \dot{H} + \ddot{H} \left( 3 + \alpha_M \right) \right]. \quad (3.22)$$

The system is completed by the standard evolution for the perturbations of the combined matter sector — $\delta_m$, $v_m$, $\delta p_m$ and $\pi_m$ — obtained through the usual Boltzmann code.

The most important feature of this formulation of the equations is that the evolution equations depend only on the standard matter-sector perturbation quantities, the background expansion history $H(t)$ (and its derivatives $\dot{H}$ and $\ddot{H}$), the background energy density and pressure of the matter sector, and the values of the four dimensionless functions of time $\alpha_i(t)$ which fully define the properties of the dark-energy perturbations. All of these elements are in general completely independent of each other and can take arbitrary values. The values of the scalar field $\phi$ or the velocity $\dot{\phi}$ do not enter the any of the equations directly at all. This formulation is purely in terms of physical quantities.

The expansion history $H(t)/H_0$ has been quite precisely observed and is seen to match that of $\Lambda$CDM with parameter $\Omega_{m0}^{\Lambda\text{CDM}}$ quite closely (e.g. [90, 91]). One can vary this expansion history somewhat at the price of making it less compatible with distance measurements. Given that the matter sector is understood at linear order (DM is essentially non-interacting and follows geodesics, baryons and photons interact at high densities, neutrinos do not interact) the evolution of the background evolution of each of the components and perturbations is determined. However, it is frequently not appreciated that the energy density of matter today $\rho_{m0}$ is a free parameter that cannot be determined from the expansion history without a prior knowledge of the dark-energy evolution history [10]: one can keep $H(t)$ fixed to be e.g. exactly $\Lambda$CDM while altering the amount of dark matter and simultaneously appropriately changing the evolution of the equation of state. Essentially, $\Omega_{m0}$ can only be determined from the growth of perturbations even in simple dark-energy cases like quintessence.$^5$

Indeed, in our formulation, one can think of picking a value for $\Omega_{m0} \neq \Omega_{m0}^{\Lambda\text{CDM}}$ for a fixed $\Lambda$CDM expansion history as describing a dark-energy model which tracks the energy density of the matter sector at early times and therefore is a model of early dark energy (“EDE”) with an effective parameterisation similar to ref. [92]. Such tracker models are tightly constrained by cosmic-microwave-background data [93–95], but this is assuming that

$^5$Seen in this light, even cluster counts are a measure of the amplitude of the gravitational potential and therefore a non-linear measure of the perturbations and not of background evolution.
the dark energy does not cluster. A dark energy that tracks the background equation of state but clusters at all scales during recombination and afterwards would act exactly like dark matter and therefore is not constrained at all. Our formulation gives an easy way to investigate the viability of such models.

We note that in the limit \( \alpha_i \to 0 \) together with \( 2\dot{H} + \dot{\rho}_m \to 0 \), the equations (3.17)–(3.20) reduce to the standard \( \Lambda \)CDM Einstein equations, while the equation of motion for the scalar (3.21) identically vanishes, since the scalar degree of freedom is no longer present. This means that, for sufficiently small deviations from this limit, the growth of perturbations will be sufficiently close to that of \( \Lambda \)CDM and this parameterisation has the concordance model as its natural limit.

On the other hand, the limit \( \alpha_i \to 0 \) but with a non-\( \Lambda \)CDM expansion history is a little peculiar: eq. (3.21) reduces to a constraint. The constraint structure of general relativity is modified allowing for an arbitrary evolution of the background. In fact, this limit is equivalent to the *cuscuton* model of refs. \[71, 96\] which is a limit of the k-*essence* class of models where the sound speed is infinite. It should be noted that such a limit is the proper way of modelling perturbations within the context of a wCDM cosmology that preserves general covariance of the full action.

Finally, as is well known (e.g. \[72\]), exactly \( \Lambda \)CDM expansion history does not necessarily imply that perturbations grow as in \( \Lambda \)CDM. As discussed above, this is obviously true for models with EDE where \( \Omega_{m0} \neq \Omega_{m0}^{\Lambda \text{CDM}} \). However, even in the case \( 2\dot{H} + \dot{\rho}_m = 0 \) the Einstein equations differ by the arbitrary functions \( \alpha_i \), which alter the solutions on top of the concordance background and can change growth rates.

### 3.5 Connection to EFT

We have chosen the set of variables \( \alpha_i \) related to the form of the perturbations of the dark-energy EMT, since that is most simply related to physical properties of the perturbations. However, our approach is very much equivalent to the recent results obtained through effective-field-theory methods \[34–38\]. Since the Horndeski Lagrangian describes all theories with no more than second derivatives in their equations of motion on the FRW background, the two approaches differ by essentially a redefinition of variables for this subclass of models. The EFT framework does allow for an extension which involve third derivatives in the Einstein equations, which cancel once the constraints are solved \[36, 60\]. In principle an extra property function would have to be added to our formulation to describe such theories.

As we have explained above, our claim is that our variables are more concretely connected to physical properties of the perturbations and the energy-momentum tensor, whereas the EFT framework describes perturbations through coefficients of particular operators permitted by the symmetries of the Friedmann background. The benefit of our approach is that it cleanly separates the properties of the background expansion (\( \dot{H} \) and \( \Omega_{m0} \)) from those of the perturbations (\( \alpha_i \)), whereas the EFT approach mixes the two. Moreover, as noted by ref. \[38\], the EFT approach has some redundancy, which when removed reduces to the same number of degrees of freedom as we have in our minimal framework, five. We have provided a mapping between the EFT operators and our \( \alpha_i \) in table 2, as well as definitions employed by other authors in the past to connect all these approaches together.

### 4 Physics of Horndeski linear structure formation

We eliminate the scalar field \( v_X \) using the constraints (3.17)–(3.19) and the time derivative of eq. (3.19) to obtain a dynamical equation for the potential \( \Phi \) with a source driven by the
matter perturbations,
\[
\bar{\Phi} + \frac{\beta_1 \beta_2 + \beta_3 \alpha B^2 k^4}{\beta_1 + \alpha B^2 k^2} \Phi + \frac{\beta_1 \beta_4 + \beta_1 \beta_5 \frac{k^2}{\beta_1 + \alpha B^2 k^2} + \alpha B^2 k^2}{\beta_1 + \alpha B^2 k^2} \Phi = -\frac{1}{2} \dot{\rho}_m \frac{\beta_1 \beta_6 + \beta_7 \alpha B^2 k^2}{\beta_1 + \alpha B^2 k^2} \delta_m ,
\]
where the functions \( \beta_i \equiv \beta_i (t) \) are all functions of the property functions \( \alpha_i \) and the background \( H(t) \), defined in appendix B. We have neglected the matter velocity term since for the purposes of this section we will assume it that the matter is dust and therefore the velocities are irrelevant inside the cosmological horizon. We can obtain the Newtonian potential \( \Psi \) through a version of the constraint (3.19) with the scalar field eliminated,
\[
\alpha B^2 \frac{k^2}{a^2} \left[ \Psi - \Phi \left( 1 + \alpha_T + \frac{2(\alpha_M - \alpha_T)}{\alpha_B} \right) \right] + \beta_1 \left[ \Psi - \Phi \left( 1 + \alpha_T \right) \left( 1 - \frac{2DH^2(\alpha_M - \alpha_T)}{\beta_1} \right) \right] = (\alpha_M - \alpha_T) \left[ \alpha_B \dot{\rho}_m \delta_m - 2HD \dot{\Phi} \right] .
\]
Augmented with the standard evolution equations for dust perturbations
\[
\dot{\delta}_m - \frac{k^2}{a^2} v_m = 3 \dot{\Phi} , \quad \dot{v}_m = -\Psi ,
\]
eqs (4.1) and (4.2) form the complete dynamical system.

The equations presented in section 3.4 describe in full generality the evolution of first-order scalar perturbations in models within the scope defined in section 2, and they are in the natural form for implementation in numerical codes such as CAMB [13] and CLASS [14, 15]. However, for the purpose of interpreting the physics, it is helpful to eliminate the scalar field \( \nu_X \) and express all of the dynamics in terms of the variables familiar from the standard \( \Lambda \text{CDM} \) case.

We have written the equations (4.1) and (4.2) in a particular manner to make explicit the existence of a new transition scale in the behaviour of the dynamics, which we will call the braiding scale \( k_B \). Whenever there is any kinetic braiding, \( \alpha_B \neq 0 \), then the response of the gravitational potentials will transition between two separate regimes at the scale
\[
\frac{k_B^2}{a^2 H^2} = \frac{\beta_1}{\alpha B^2 H^2} = \frac{D}{\alpha B} \left[ (1 - \Omega_m) \left( 1 + w_X \right) + 2 \left( \alpha_M - \alpha_T \right) \right] + \frac{9}{2} \Omega_m .
\]
During matter domination, the braiding scale — if it at all exists, \( \alpha_B \neq 0 \) — typically lies at the cosmological horizon, but can lie inside when

\[
\frac{\alpha_K}{\alpha_B^2} \left( 1 - \tilde{\Omega}_m \right) (1 + w_X) \gg 1. \tag{4.5}
\]

The same braiding scale \( k_B \) appears in both the dynamical equation (4.1) and in the anisotropy constraint (4.2) and was discussed for the first time in ref. [26] in the context of a small subclass of the models being considered here.\(^6\) If \( k_B \) lies subhorizon, this is evidence of a hierarchy between the values of \( \alpha_K \) and \( \alpha_B \). In archetypal modified-gravity models such as \( f(R) \) and \( f(G) \), \( \alpha_K = 0 \) and therefore \( k_B \) always lies either close to or outside the cosmological horizon. The whole subhorizon regime always exhibits braiding in these models (in the linear regime).

When considering the observable impact, it is usual to compress any modifications from the concordance case into an effective Newton’s constant and the slip parameter. Since there are two potentials, two effective Newton’s constants can be defined,

\[
Y \equiv -\frac{2k^2\Psi}{a^2\bar{\rho}_m\delta_m}, \quad Z \equiv -\frac{2k^2\Phi}{a^2\bar{\rho}_m\delta_m}. \tag{4.6}
\]

One usually discusses \( Y \), since this is the term that enters directly in the equations for growth rate for matter perturbations (see e.g. [20]). However, it is actually \( \Phi \) that is the dynamical variable (related to the spatial curvature perturbation) and it turns out to generically have a simpler behaviour. Thus \( Z \) should be considered more amenable to parameterisation in the classes of models considered here. Since it is \( Z \) that appears in the Hamiltonian constraint, eq. (3.17), it is a deviation of \( Z \) from its subhorizon GR value of 1 that signifies that dark energy clusters.

An honestly model-independent observable that can be measured by comparing weak lensing and redshift-space distortions is the slip parameter describing the anisotropic stress,

\[
\bar{\eta} \equiv \frac{2\Psi}{\Psi + \Phi}. \tag{4.7}
\]

We have chosen a non-standard definition of this ratio to reflect the fact that projected measurement errors from Euclid are minimised for this particular combination and could be as low as a few percent under certain assumptions [12]. A parameterisation of one of the effective Newton’s constants (4.6) and the slip parameter (4.7) is sufficient to describe the dynamics of the matter sector and therefore to calculate the observables. These can be translated to observables in more frequent use by the community through

\[
Y = \frac{\bar{\eta}}{2-\bar{\eta}} Z, \quad \Sigma = \frac{2Z}{2-\bar{\eta}}. \tag{4.8}
\]

We devote the remainder of this section to describing the typical behaviour of these variables in the class of models within our scope to build an understanding of what can be expected in models which are fully consistent rather than purely phenomenological. First, one

\(^6\) It was named the imperfect scale in ref. [26], since the anisotropic stress vanished outside it. This was the result of a Brans-Dicke-type non-minimal coupling considered, but it is not the general behaviour, as shown in section 4.1.
can ignore all the scale dependence and seek the extreme quasi-static limit of the dynamics, $k \to \infty$, where

$$
Z_{QS} = 1 + \frac{\alpha_B^2 (1 + \alpha_T) + 2\alpha_B (\alpha_M - \alpha_T)}{2 D c_s^2},
$$

$$
\bar{\eta}_{QS} = 1 + \frac{2 (\beta_4 - c_s^2) (\alpha_M - \alpha_T) + \beta_7 \alpha_B \alpha_T}{2 (\beta_7 - c_s^2) (\alpha_M - \alpha_T) + \beta_7 (2\alpha_B + \alpha_B \alpha_T)}, \quad \alpha_B \neq 0.
$$

We note here that the dark energy clusters at small scales ($Z \neq 1$) only if there is braiding, $\alpha_B \neq 0$. Thus a detection of clustering of dark energy is unambiguous evidence of the presence of kinetic mixing of the scalar and graviton. Secondly, the limit for $\bar{\eta}$ taken in (4.9) does not exist when there is no braiding (see section 4.1 for an account of what happens when $\alpha_B = 0$).

In the following subsections, we discuss in more detail two limiting cases which put the above observation on firmer footing:

- No braiding, $\alpha_B = 0$; the braiding scale $k_B$ does not exist. The sound speed provides the only scale in the problem. Inside the Jeans scale, dark energy does not cluster, $Z = 1$. Anisotropic stress can be non-vanishing, with a slip parameter constant as a function of scale inside the Jeans length.

- Negligible standard kinetic term, $\alpha_K \ll \alpha_B^2$; the braiding scale $k_B$ is superhorizon. There is a single transition scale determined by the Compton mass of the scalar. DE clusters only inside the Compton scale. The slip parameter interpolates between two values with the transition also occurring at the Compton scale.

In principle, the fully general model will contain another scale (the braiding scale $k_B$) across which the behaviour transitions between these two behaviours (see the behaviour in ref. [26]). We defer a detailed description of the phenomenology to a numerical analysis, but stress that in general such properties as anisotropic stress and DE clustering are scale-independent only for very particular subclasses of dark-energy models.

4.1 Example: no braiding

Here we assume that there is no braiding: the graviton and the scalar do not mix kinetically at any scale, $\alpha_B = 0$. This set of models can be thought of as a generalisation of perfect-fluid models to include anisotropic stress. The new scale dependence in the coefficients of eqs. (4.1)–(4.2) disappears and we remain with

$$
\dot{\Phi} + H (4 + \alpha_M + 3\Upsilon) \dot{\Phi} + \left(\beta_4 + \frac{c_s^2 k^2}{a^2}\right) \Phi = -\frac{1}{2} c_s^2 \bar{\rho}_m \delta_m, \quad (4.10)
$$

for the dynamical equation. The variable $\Upsilon$ is defined in eq. (B.12); it can be thought of as a generalisation of the adiabatic sound speed for cases where the dark energy is not a perfect fluid. The structure of eq. (4.10) is fundamentally unchanged from the case of the perfect fluid ($\alpha_M = \alpha_T = 0$). In particular, inside the Jeans length the effective Newton’s constant

$$
Z \simeq 1 \quad \text{for} \quad \frac{k^2}{a^2} \gg \beta_4 / c_s^2, \quad (4.11)
$$
no matter what other modifications are present. Since \( \beta_4 \sim H^2 \), this range of validity of this quasi-static approximation is essentially determined by the sound speed of the dark energy.

Despite the fact that there is no modification in (4.11), and therefore — just as in the case of a perfect-fluid dark energy — it does not cluster, growth rates can be affected, since this sort of dark energy can carry anisotropic stress. The anisotropy constraint reduces to

\[
Hc_s^2\alpha_K \Psi + H \left[ c_s^2\alpha_K + 2(\alpha_M - \alpha_T) \right] (1 + \alpha_T) \Phi = -2(\alpha_M - \alpha_T) \dot{\Phi},
\]

i.e. it also loses any scale dependence of the coefficients as well as its dependence on \( \delta_m \).

The only scale dependence that can appear in the slip parameter is through \( \dot{\Phi} \), which cannot be neglected here, especially if \( c_s^2 \ll 1 \).

### 4.2 Example: negligible standard kinetic term

In this section, we assume that the kinetic term of the scalar mode is mainly produced through mixing with the graviton rather than directly, \( \alpha_B^2 \gg \alpha_K \). For the sake of simplicity of presentation, we also choose \( \alpha_T = 0 \). These kind of models can be considered as a generalization of the \( f(R) \) theories, since the braiding \( \alpha_B \) and the Planck-mass run rate \( \alpha_M \) are uncorrelated here (see table 1). More generally, we are essentially describing the behaviour of other modified gravity models such as \( f(G) \) and models with large braiding such as Imperfect Dark Energy of ref. [47].

The positivity of the sound speed (3.13) provides an upper limit to the magnitude of the braiding \( \alpha_B \). We will take it to be

\[
\alpha_B \lesssim \tilde{\Omega}_X.
\]

Under this assumption, the braiding scale (4.4) always lies superhorizon,

\[
\frac{k_B^2}{a^2} \equiv \frac{\beta_1}{\alpha_B^2} \lesssim \mathcal{O}(H^2),
\]

and therefore inside the observable domain the dark energy scalar is always kinetically mixed with the graviton. In this subhorizon regime, the evolution equation (4.1) reduces to

\[
\ddot{\Phi} + (3 + \alpha_M) H \dot{\Phi} + \left( \frac{\beta_1 \beta_5}{\alpha_B^2} + c_s^2 \frac{k^2}{a^2} \right) \Phi \simeq -\frac{1}{2} \dot{\rho}_m \left( \frac{\beta_1 \beta_6}{\alpha_B^2 k^2} + \beta_7 \right) \delta_m.
\]

Yet again, a scale is present in this equation

\[
\frac{k_C^2}{a^2} \equiv \frac{\beta_1 \beta_5}{\alpha_B^2 c_s^2},
\]

which in the context of \( f(R) \) gravity is called the Compton mass scale [97]. On either side of this scale, in the quasi-static limit, we have

\[
Z = \frac{\beta_6}{\beta_5} \quad aH \ll k \ll k_C,
\]

\[
Z = \frac{\beta_7}{c_s^2} = 1 + \frac{2}{3} \frac{\alpha_M}{\alpha_B c_s^2} \quad k \gg k_C.
\]

\(^7\)This is an example of the failure of the quasi-static approximation, see section 4.3.
The requirement that $Z$ remain of the order of 1 in order to not catastrophically affect structure formation ensures that the coefficients of $\Phi$ and $\delta_m$ in eq. (4.16) both transition in their behaviour around the same scale. Thus, despite the rather peculiar $k$-dependence, the behaviour of the braided models is to interpolate between two different values of the effective Newton’s constant $Z$ across the Compton scale. It is typical for $\beta_5 \gg 1$ before dark-energy domination, ensuring that most of the scales relevant to structure formation observations are super-Compton. We also note that $Z$ is naturally close to 1 when the parameters $\alpha_i$ are not tuned to create large hierarchies.

The only scale present in the coefficients of the anisotropy equation (4.2) is the scale (4.15) which is superhorizon at all times. Therefore the only scale dependence in the slip parameter is a result of the scale dependence in $Z$, thus we obtain

$$\bar{\eta} \equiv 1 - \frac{\alpha_M}{\alpha_B} (\alpha_M + 2\Upsilon) \quad aH \ll k \ll k_C,$$

$$\bar{\eta} \equiv 1 + \frac{\alpha_M}{\alpha_B} \left(1 + 2\frac{\alpha_M}{\alpha_B}\right) \quad k \gg k_C,$$

where $\Upsilon$ is defined in eq. (B.12) and is a variable that reduces to the adiabatic sound speed in the case of a perfect fluid. Thus in general the anisotropic stress can be non-vanishing both inside and outside the Compton scale.

A natural limit of this class of models is $f(R)$ gravity for which $\alpha_M = -\alpha_B$. It can easily be checked that at super-Compton scales both $Z$ and $\bar{\eta}$ are very close to their $\Lambda$CDM values of 1 (with corrections of order $\alpha_M$), while inside the Compton scale we have the standard result $Z = 2/3$ and $\bar{\eta} = 4/3$.

4.3 Comment on quasi-static limit

Typically when discussing modified-gravity models, the quasi-static approximation (QS) is used to obtain the effective Newton’s constant $Z$ and the slip parameter $\bar{\eta}$. This involves neglecting all terms time derivatives in eq. (3.21), turning a dynamical equation into a constraint between the value of the velocity potential $v_X$ and the gravitational potential. This result is then used in the Hamiltonian constraint (3.17) and the anisotropy constraint (3.19) under a similar approximation to obtain $Z$ and $\bar{\eta}$. Does this procedure give the same result as the quasi-static limit of eq. (4.1) which was obtained without such approximations?

It turns out that the two results are identical in the $k \to \infty$ limit (4.9). However the term subleading in $k^2$ are affected. The reason for this is that both equation (3.18) and (3.20) do not have any scale dependence. As a result, eliminating the time derivatives of $v_X$ in eq. (3.21) does not affect any of the $k^2$ terms. However the other terms are changed, leading to non-negligible differences at larger scales. One must therefore be careful about how the quasi-static limit is taken when calculating e.g. the Compton mass, as in section 4.2.

A separate issue is whether the QS limit is at all a good approximation to the full dynamics at small-enough scales. In reality, taking the QS approximation is equivalent to turning a full degree of freedom into a constraint. In principle the dynamics of the late universe are described by the coupled system of equations (4.1) and conservation for the matter EMT (4.3). One should investigate the normal modes of this coupled system and ask what their behaviour is. It may well be that an instability exists which the quasi-static limit would hide.
In particular, the QS approximation removes from consideration the oscillating modes which solve the homogeneous version of eq. (4.1). Whether those decay or grow compared to the solutions obtained in the QS limit depends, among others, on the friction term in (4.1), i.e. on the values of $\beta_2$ and $\beta_3$. If all the $\alpha_i \lesssim 1 - \Omega_m$, then $\beta_2 \sim 4H + O(\alpha_i H)$ and $\beta_3 \sim 3H + O(\alpha_i H)$ and therefore there is no large difference between the friction terms on any scales. Inside the braiding scale, the homogeneous mode may be marginally more unstable, but probably this is not something particularly dangerous for most models. The details do however depend on the behaviour of the mass term and should be studied more extensively.

4.4 Constraints beyond large-scale structure

In addition to the effect on cosmological large-scale structure driven by the modifications in section 3.4, the realised dynamical dark energy mechanism must satisfy additional constraints (see e.g. refs. [98, 99] and references therein):

1. Bing-Bang Nucleosynthesis constrains the expansion rate to be within approximately 10% of the standard one imputed using local measurements of Newton’s constant. This is a restriction on the evolution history of the Planck mass $M_*$.

2. Shapiro-time-delay tests in the Solar System constrain the parameterised post-Newtonian parameter $|\gamma - 1| < 10^{-5}$. This is a restriction on the anisotropic stress present around the solar solution, i.e. the slip parameter $\bar{\eta}$.

3. Binary-pulsar orbits decay in a manner consistent with general relativity. A new gravitationally coupled scalar is a new channel for radiation and therefore accelerated orbit evolution [100].

4. If gravity is slower than the speed of light, Čerenkov radiation into gravitons would be produced by particles moving sufficiently quickly, such as cosmic rays.

One must be careful in how one interprets these constraints in the context of dynamical dark energy. The linear perturbation equations (3.17)–(3.21) must remain valid whenever the background has all the symmetries of an FRW universe, i.e. well inside any localised distribution of matter which is sufficiently homogeneous and isotropic. However, it is not necessarily true that the local values of the parameters $\alpha_i$ are the same as those for the cosmology at large scales. As a result of screening around localised objects, a transition region can appear where eqs. (3.17)–(3.21) are no longer valid and the solution interpolates between the cosmological and the local values of the $\alpha_i$ (see e.g. [101, 102] and references therein for a discussion of screening mechanisms).

The constraints (1) and (2) above should therefore be seen as constraints on the local or Solar System values of the $\alpha_i$ and not on their cosmological values. In particular, constraint (1) in our context should be interpreted as

$$\left| \Omega_m(t_{\text{BBN}})Y_{\text{SS}} \frac{M_*^2(t_{\text{BBN}})}{M_{\text{SS}}^2} - 1 \right| \lesssim 10\% .$$

(4.20)

No measurement of the pure Planck mass exists, but rather the force felt by masses in the Solar-System is a sum of both the gravitational and the scalar force and is described by the local values of the effective Newton’s constant $Y_{\text{SS}}$ and the the Solar-System value of $M_{\ast, \text{SS}}$.
which may not be the same as the one at large distances today. On the other hand, the constraints on the expansion rate during BBN are sensitive to both the value of the Planck mass and on the presence of any early dark energy at that time.

Similarly, constraint (2) is sensitive to the Solar System value of the slip parameter, i.e.

\[ |\bar{\eta}_{SS} - 1| = \frac{\bar{\eta}_{SS} - 1}{2} \lesssim 10^{-5}, \]

which is related to the cosmological one through the screening mechanism, just as in ref. [103]. The knowledge of the screening mechanism is required to answer whether screening can happen in the Solar System, but this requires the study of higher-order perturbation theory and cannot be answered within the context of the linear one. Only in models where no screening occurs, the Solar-System values are the same as the cosmological ones and the constraints above apply directly.

Binary-pulsar orbits in principle constrain our models at various locations in the Galaxy. However, since neutron stars are much more compact than the Sun, the level of screening is likely to be much stronger than in the Solar System and therefore the constraints they provide are related to even denser environments than those of the Solar System.

Finally, constraint (4) is relevant both to the local tests but also the wider cosmological scales. The requirement that no gravitational Čerenkov radiation be emitted is a lower bound on the speed of gravity

\[ \alpha_T \gtrsim -10^{-15}, \]

but no such upper limit exists on cosmological scales [104–106].

5 Discussion and conclusions

In this paper, we have presented a turnkey solution for the study of linear cosmological perturbations in general dark-energy and modified-gravity models described by the Horndeski Lagrangian, with a formulation focussing on non-redundant variables which lead to physical effects. Our approach is essentially equivalent to the recent work within the framework of effective field theory, amounting to a redefinition of variables in this context. However, we make explicit the variables which independently affect the behaviour of perturbations and which are constrainable by observations. The dark-energy/gravity sector of an arbitrary general Horndeski model can be described by five functions of time and one constant:

1. A completely arbitrary dimensionless background expansion history \( H(t)/H_0 \), which is all that observations of distances, using supernovae, BAOs can map out (in principle, spatial curvature \( \Omega_k \) can also be allowed as an extra constant parameter and unambiguously measured using longitudinal BAO [10]).

2. The value of fractional matter density \( \Omega_{m0} \). This is a constant not determined by measurements of the background expansion history since one can always change the amount of dark matter and replace it with an appropriately evolving dark energy. More appropriately, one should think of \( \Omega_{m0} \) as being a parameter which can only be measured by observing the evolution of cosmological perturbations, whether in the cosmic microwave background or at late times.

3. Four arbitrary dimensionless property functions \( \alpha_i(t) \) defining the effect of the dark energy model on the evolution of linear perturbations and independent from the above and each other in the most general case.
In addition, to convert these predictions to physical units, a value of $H_0$, the Hubble parameter today, is necessary. Local measurements of supernovae are affected by cosmic variance and therefore not quite true measurements of the averaged $H_0$, especially in the sense independent of the dark-energy model \cite{107, 108}. Alternatively, measurements of age differences of red galaxies can be used, as proposed in refs. \cite{109, 110} (see also refs. \cite{111–113}).

All the subclasses of dark-energy models in our scope are described as lying in a subspace of the four dimensions spanned by $\alpha_i$. For example, quintessence models offer no freedom whatsoever once (1) and (2) have been fixed; each of perfect fluid (k-essence), $f(R)$ and $f(G)$ models are completely specified by (1) and (2) and then a one-dimensional subspace relating the functions $\alpha_i$ to each other. We have specified the restrictions on the full freedom that various popular subclasses of dark energy models imply in table 1.

Even though there exists a Horndeski model which would describe any background for any choice of the functions $\alpha_i$, not all such configurations are stable. For any particular choice of background expansion history and $\Omega_m$, only certain ranges of $\alpha_i$ are permitted, as implied by the inequalities given in section 3.3. The background is unstable to scalar modes whenever there are ghosts, $\alpha_K < -\frac{3}{2} \alpha_B^2$, or the sound speed squared $c_s^2$ is smaller than zero. In particular, this means that any perfect-fluid model ($\alpha_M = \alpha_B = \alpha_T = 0$) is unstable whenever the DE has $w_X < -1$. In addition, one must pay attention to the tensor modes: whenever the Planck mass $M^2_\ast$ is evolving, it cannot be allowed to become negative in the past. The stability conditions specified here when tested for over the whole relevant evolution history should be considered a prerequisite for the acceptance of a particular linear solution, even if linear codes would seemingly allow an instability within the region of interest.

The evolution of perturbations is completely described by the set of modified Einstein equations (3.17)–(3.21) (or, alternatively, their synchronous-gauge versions in appendix C). All the effects of dynamical dark energy are included as modifications to the left-hand side of the equations, with the contributions from the matter EMT on the right-hand side standard in all but one way: if the Planck mass is evolving ($\alpha_M \neq 0$), the energy density of none of the matter species is conserved, but must be tracked according to eq. (3.9).

In order to access the physics of these models, we have presented the full evolution equations for the gravitational potentials with the unobservable scalar degree of freedom eliminated in section 4. We have demonstrated the following:

- The effect of dark energy on the effective Newton’s constant $Z$ is simpler than on $Y$ and parameterisations should in general focus on $Z$ and $\bar{\eta}$.

- There is a braiding scale $k_B$ driven by the competition between the kineticity $\alpha_K$ and kinetic braiding $\alpha_B$, eq. (4.4). This scale may be subhorizon and separates two behaviours in $Z$:
  
  - No braiding, $\alpha_B = 0$: perfect-fluid-like behaviour for $Z$. $Z = 1$ inside Jeans length, and usual gravitational instability outside it.
  
  - Negligible kineticity, $\alpha_K$ small: dark energy clusters, $Z \neq 1$. Transition between two regimes at scales related to Compton mass of the scalar.

- Anisotropic stress, $\bar{\eta} \neq 1$, can only be present at linear order whenever the tensor modes are non-minimally coupled, i.e. either $\alpha_M \neq 0$ or $\alpha_T \neq 0$. Its presence is completely independent of values of $\alpha_K$ and $\alpha_B$, but its value depends on all property functions.

If no deviation from the $\Lambda$CDM expansion history is detected and the growth of perturbations is consistent with $\alpha_i = 0$ up to some level of precision, then one should think of this result
as a statement that dynamical dark energy is responsible for at most a small part of the acceleration mechanism. This sort of argument was put forward in the case of models with chameleon screening in ref. [114]. The remainder of the acceleration can then be considered to be driven by a pure non-dynamical cosmological constant. For the ΛCDM expansion history, there essentially always exists a family of models which interpolate between the cosmological constant as the only source of acceleration and models with \( \alpha_i \approx \mathcal{O} \left( 1 - \tilde{\Omega}_m \right) \) which are fully dynamical.

The formulation we have proposed is limited in scope to universally coupled dark-energy models. However, it can be extended to also cover non-universal couplings with the addition of new parameters (some relative density of the non-universally coupled species and the rate of evolution of the coupling). In principle, only the right-hand side of the Einstein equations and the (non)-conservation equation for the differently coupled species need be modified. Adding additional scalar degrees of freedom to the dark energy is a much more complicated proposition. No full description of such a multi-scalar model exists (the multi-field Horndeski action conjectured in [115] does not seem to contain all possible terms [116]), although one can make progress through EFT methods [39]. The number of free \( \alpha_i \)-like parameters is likely to increase tremendously in addition to these models’ having new unobservable \( \psi_X \)-like degrees of freedom. Therefore the predictivity and the measurability of such models is likely to be much poorer.

The most important conclusion of this paper is that for the gravity-like models within our scope, there are no properties beyond those specified above which would affect any observations. Measuring the background \( H(t), \tilde{\Omega}_m \) and the four functions \( \alpha_i \) is the maximum that can be done using linear structure formation. If there is no evidence that the \( \alpha_i \) are different from zero up to some precision then dynamical dark energy does not contribute by more than this precision to the acceleration mechanism. The question of whether a more fundamental mechanism than \( \Lambda \) is present becomes moot, since it would be irrelevant to the dynamics. In a sense, measuring the \( \alpha_i \) with the best precision is the goal of large-scale-structure measurements within the context of dark-energy cosmology.

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A Definitions of evolution variables

The energy density and pressure of a dark energy described by a Horndeski Lagrangian (2.1) are

\[
M^4_\gamma \dot{\tilde{\mathcal{E}}} = -K + 2X (K_X - G_{3\phi}) + 6\dot{\phi} H (XG_{3X} - G_{4\phi} - 2XG_{4\phi X}) \\
+ 12H^2 X (G_{4X} + 2XG_{4XX} - G_{5\phi} - XG_{5\phi X}) + 4\dot{\phi} H^3 X (G_{5X} + XG_{5XX}),
\]

\[
M^2_\gamma \dot{\tilde{P}} = -K - 2X (G_{3\phi} - 2G_{4\phi\phi}) + 4\dot{\phi} H (G_{4\phi} - 2XG_{4\phi X} + XG_{5\phi\phi}) \\
- M^2_\gamma \alpha_B H \frac{\ddot{\phi}}{\phi} - 4H^2 X^2 G_{5\phi X} + 2\dot{\phi} H^3 XG_{5X}.
\]
where we have already absorbed the contribution to the Planck mass $M_\ast$. The equation of motion for the background values of the scalar field (3.11) is an equation for the evolution of the shift charge,

$$\dot{n} + 3Hn = \mathcal{P}_\phi$$  \hspace{1cm} (A.3)

with the charge density

$$n \equiv \dot{\phi} (K_X - 2G_{3\phi}) + 6HX (G_{3X} - 2G_{4\phi X}) + 6H^2 \dot{\phi} (G_{4X} + 2XG_{4XX} - G_{5\phi} - XG_{5\phi X}) +$$

$$+ 2H^3 X (3G_{5X} + 2XG_{5XX}) ,$$  \hspace{1cm} (A.4)

and the non-conservation driven by a violation of the shift symmetry through the term

$$\mathcal{P}_\phi \equiv K_\phi - 2XG_{3\phi \phi} + 2\ddot{\phi} \left( XG_{3\phi X} + 3H\dot{\phi} G_{4\phi X} \right) + 6\dot{H}G_{4\phi} +$$

$$+ 6H^2 (2G_{4\phi} + 2XG_{4\phi X} - XG_{5\phi X}) + 2H^3 \dot{\phi} XG_{5\phi X} .$$  \hspace{1cm} (A.5)

The evolution of perturbations on a particular background is determined by four independent and dimensionless functions of time, $\alpha_i$, which can be determined for any particular Lagrangian by

$$M^2_i \equiv 2 \left( G_4 - 2XG_{4X} + XG_{5\phi} - \dot{\phi} H XG_{5X} \right) ,$$  \hspace{1cm} (A.6)

$$HM^2_\alpha \equiv \frac{d}{dt} M^2_\alpha ,$$  \hspace{1cm} (A.7)

$$H^2 M^2_\alpha \equiv 2X (K_X + 2XK_{XX} - 2G_{3\phi} - 2XG_{3\phi X}) +$$

$$+ 12\dot{\phi} XH (G_{3X} + XG_{3XX} - 3G_{4\phi X} - 2XG_{4\phi X X}) +$$

$$+ 12XH^2 (G_{4X} + 8XG_{4XX} + 4X^2 G_{4XXX}) -$$

$$- 12XH^2 (G_{5\phi} + 5XG_{5\phi X} + 2X^2 G_{5\phi XX}) +$$

$$+ 4\dot{H}XH^3 (3G_{5X} + 7XG_{5XX} + 2X^2 G_{5XXX}) ,$$  \hspace{1cm} (A.8)

$$HM^2_\alpha \equiv 2\dot{\phi} (XG_{3X} - G_{4\phi} - 2XG_{4\phi X}) +$$

$$+ 8XH (G_{4X} + 2XG_{4XX} - G_{5\phi} - XG_{5\phi X}) +$$

$$+ 2\ddot{\phi} XH^2 (3G_{5X} + 2XG_{5XX}) ,$$  \hspace{1cm} (A.9)

$$M^2_\alpha \equiv 2X \left( 2G_{4X} - 2G_{5\phi} \right) \left( \ddot{\phi} - \dot{\phi} H \right) G_{5X} ,$$  \hspace{1cm} (A.10)

### B Scale dependence in dynamics

Since the value of the scalar perturbation $\nu_X$ is not an observable, it is helpful to eliminate it by solving all the constraints in the Einstein equations to produce an evolution equation for the Newtonian potential $\Phi$, eq. (4.1),

$$\ddot{\Phi} + \frac{\beta_1 \beta_2 + \beta_3 \alpha_B^2 \frac{k^2}{a^2}}{\beta_1 + \alpha_B^2 \frac{k^2}{a^2}} \ddot{\phi} + \frac{\beta_1 \beta_4 + \beta_1 \beta_5 \frac{k^2}{a^2} + \epsilon_s \alpha_B^2 \frac{k^4}{a^4}}{\beta_1 + \alpha_B^2 \frac{k^2}{a^2}} \dot{\Phi} =$$

$$- \frac{1}{2} \rho_m \left[ \frac{\beta_1 \beta_6 + \beta_7 \alpha_B^2 \frac{k^2}{a^2} \delta_m + (1 + w_m) \frac{\beta_1 \beta_8 + \beta_9 \alpha_B^2 \frac{k^2}{a^2} v_m}{\beta_1 + \alpha_B^2 \frac{k^2}{a^2}} v_m \right] ,$$  \hspace{1cm} (B.1)
and an anisotropy constraint that relates the two potentials \( \Phi \) and \( \Psi \), eq. (4.2),

\[
\alpha_B^2 \frac{k^2}{a^2} \left[ \Psi - \Phi \left( 1 + \alpha_T + \frac{2 (\alpha_M - \alpha_T)}{\alpha_B} \right) + \beta_1 \left[ \Psi - \Phi \left( 1 + \alpha_T \right) \left( 1 - \frac{2DH^2 (\alpha_M - \alpha_T)}{\beta_1} \right) \right] \right] = \\
(\alpha_M - \alpha_T) \left[ \alpha_B \rho_m \delta_m - 2HD\Phi - H (3\alpha_B + \alpha_K) (\bar{\rho}_m + \bar{\rho}_m) v_m \right].
\]

Both equations are repeated here for convenience without neglecting the matter velocity \( v_m \).

In this section, we provide the equations of motion for linear perturbations in synchronous and where the sound speed of the scalar mode is

\[
c^2_s = \frac{3 - 2\alpha_B}{2} (\bar{\rho}_m + \bar{\rho}_m) - D \left[ \Phi + \rho_m \right],
\]

(4.3)

\[
\rho_m - \beta_1 \frac{2H (2 + \alpha_M)}{3H \gamma},
\]

(4.4)

\[
\beta_3 = H (3 + \alpha_M) + \frac{\alpha_K}{D} \left( \frac{\dot{\alpha}_K}{\alpha_K} - 2 \frac{\dot{\alpha}_B}{\alpha_B} \right),
\]

(4.5)

\[
\beta_4 = (1 + \alpha_T) \left[ 2\dot{H} + H^2 (3 + 3\gamma + \alpha_M) \right] + \dot{\alpha}_T H,
\]

(4.6)

\[
\beta_5 = c_s^2 + \frac{\alpha_B \beta_4}{H D}, \quad \frac{H \alpha_B^2}{\beta_1} (1 + \alpha_T) (\beta_3 - \beta_2) + \frac{\alpha_B^2 \beta_4}{\beta_1},
\]

(4.7)

\[
\frac{\beta_6}{H D} - \frac{\alpha_B (\beta_3 - \beta_2)}{H D},
\]

(4.8)

\[
\beta_7 = c_s^2 - w_m + \frac{\alpha_B^2}{2D} (1 + \alpha_T + 3w_m) + 2 (\alpha_M - \alpha_T) \alpha_B,
\]

(4.9)

\[
\beta_8 = \beta_9 + \frac{(\beta_2 - \beta_3) (\alpha_K + 3\alpha_B)}{D},
\]

(4.10)

\[
\beta_9 = \beta_3 - H (4 + 3c_s^2 + \alpha_M + \alpha_T),
\]

(4.11)

where we have defined

\[
\delta_h \equiv \dot{H} - H^2 (\alpha_M - \alpha_T),
\]

(4.12)

\[
D \equiv \alpha_K + \frac{3}{2} \alpha_B^2,
\]

(4.13)

\[
3\beta_1 H \gamma \equiv 2D \left[ \delta_h + (3 + \alpha_M) H \delta_h \right] - H (\bar{\rho}_m + \bar{\rho}_m) (3\alpha_B + \alpha_K) (\alpha_M - \alpha_T)
\]

\[
+ \alpha_K (\bar{\rho}_m + \alpha_M H \bar{\rho}_m) + \frac{\alpha_K \alpha_B^2}{2D} (\bar{\rho}_m + \bar{\rho}_m) \left( \frac{\dot{\alpha}_K}{\alpha_K} - 2 \frac{\dot{\alpha}_B}{\alpha_B} \right),
\]

and where the sound speed of the scalar mode is

\[
c_a^2 = - \frac{\left( 2 - \alpha_B \right) \left[ \delta_h - \frac{1}{2} H^2 \alpha_B (1 + \alpha_T) \right] - H \dot{\alpha}_B + \bar{\rho}_m + \bar{\rho}_m}{H^2 D}.
\]

C Perturbation equations in synchronous gauge

In this section, we provide the equations of motion for linear perturbations in synchronous gauge and using conformal time. This is to allow for direct implementation of our formulation in codes as CAMB [13] or CLASS [14, 15]. For the metric potentials in synchronous gauge,
we continue to use the notation of ref. [68]. Note that with respect to the main text, we have made the redefinitions
\[ v_X \rightarrow aV_X , \]
\[ \mathcal{H} \equiv aH = \frac{\dot{a}}{a} , \]
for notational simplicity. The equations in this section are equivalent to those derived in ref. [41] in the effective field theory formalism. As we discuss in section 3.5, the advantage of our formulation is the fact that it is non-redundant, splits background and perturbation contributions and is more directly related to physical effects, as shown in section 4.

The Einstein time-time equation is
\[ 2k^2 \eta = -\frac{\rho_m\delta m a^2}{M_*^2} + \frac{\mathcal{H}}{2} (2 - \alpha_B) \, h' + \mathcal{H}^2 (\alpha_K + 3\alpha_B) \, V_X' \quad \text{(C.1)} \]
\[ + \left[ \alpha_B k^2 - 3 (\mathcal{H}^2 - \mathcal{H}') (2 - \alpha_B) + \mathcal{H}^2 (\alpha_K + 3\alpha_B) + 3a^2 \frac{\rho_m + p_m}{M_*^2} \right] \mathcal{H} V_X . \]

The Einstein time-space equation is
\[ 2\eta' = \frac{a^2}{M_*^2} (\rho_m + p_m) v_m + \alpha_B \mathcal{H} V_X' - \left[ 2 (\mathcal{H}^2 - \mathcal{H}') - \mathcal{H}^2 \mathcal{H}_B - a^2 \frac{\rho_m + p_m}{M_*^2} \right] V_X . \quad \text{(C.2)} \]

The Einstein space-space traceless equation is
\[ 3\eta'' + \frac{h''}{2} + \mathcal{H} (2 + \alpha_M) \left( 3\eta' + \frac{h'}{2} \right) - k^2 (1 + \alpha_T) \eta = \mathcal{H} k^2 (\alpha_M - \alpha_T) \, V_X - \frac{a^2 \mathcal{P} \Pi_m}{M_*^2} . \quad \text{(C.3)} \]

The Einstein space-space trace equation is
\[ h'' = -\frac{3a^2}{M_*^2} \delta \rho_m - \mathcal{H} (2 + \alpha_M) \, h' + 2k^2 (1 + \alpha_T) \eta - 3\mathcal{H} \alpha_B V_X'' + 2\mathcal{H} k^2 (\alpha_M - \alpha_T) \, V_X \quad \text{(C.4)} \]
\[ - 3 \left[ 2 (\mathcal{H}^2 - \mathcal{H}') + \mathcal{H} \mathcal{H}_B' + \alpha_B \mathcal{H}' + \mathcal{H}^2 \alpha_B (3 + \alpha_M) \right] V_X' \]
\[ - 3 \left[ 2 h'' + \mathcal{H}^2 \mathcal{H}_B' - \mathcal{H}^3 (2 + \alpha_M) (2 - \alpha_B) + 2 \mathcal{H} \mathcal{H}' (\alpha_M + \alpha_B) \right] V_X \]
\[ - 3a^2 \frac{\rho_m + p_m}{M_*^2} [V_X' + \mathcal{H} V_X] - \frac{3a^2 \rho_m}{M_*^2} V_X . \]

The scalar-field equation of motion is
\[ \mathcal{H}^2 \left( \alpha_K + \frac{3}{2} \alpha_B^2 \right) V_X'' + AV_X' + (3B + Ck^2) \, V_X - \frac{C}{2} h' + \frac{3\alpha_B}{2M_*^2} a^2 \delta \rho_m \]
\[ + [2 (\alpha_M - \alpha_T) + \alpha_B (1 + \alpha_T)] \frac{\rho_m \mathcal{H} \delta m a^2}{2M_*^2} = 0 , \quad \text{(C.5)} \]
where
\[ A \equiv \frac{1}{2} \mathcal{H}^3 (2 - \alpha_B) [\alpha_K (1 + \alpha_T) - 3\alpha_B (1 - \alpha_T + \alpha_M)] + \mathcal{H} \left( \alpha_K + \frac{3}{2} \alpha_B^2 \right) (\mathcal{H}^2 + \mathcal{H}') \]
\[ + \mathcal{H} (\alpha_K \mathcal{H}' + \mathcal{H} \alpha_K') + 3\mathcal{H} \alpha_B \left( \mathcal{H}' + \frac{1}{2} \mathcal{H} \alpha_B' \right) + \frac{3 (\rho_m + p_m) \mathcal{H} \alpha_B a^2}{2M_*^2} , \]

\(^8\)Note that CAMB redefines time derivatives of \( \eta \) and \( h \) using the scheme: \( h' \rightarrow 2kZ \) and \( \eta' \rightarrow k/3 (\sigma_+ - \bar{z}) \).
\[ B \equiv a^2 \rho_m + \frac{p_m}{M_*^2} \left[ \mathcal{H}' + \frac{1}{2} \mathcal{H}^2 \alpha_T (2 - \alpha_B) - \mathcal{H}^2 (1 + \alpha_m) \right] + \frac{\mathcal{H} \alpha_B a^2 p'_m}{2 M_*^2} \\
+ \mathcal{H}^2 (2 - \alpha_B) \left[ \frac{1}{6} \alpha_K \mathcal{H}^2 (1 + \alpha_T) - \mathcal{H}^2 (1 - \alpha_B) \alpha_T - \mathcal{H}' \alpha_m \right] \\
+ \mathcal{H}^2 (2 - \alpha_B)^2 \left( \mathcal{H}^2 \alpha_M + \mathcal{H}' \alpha_T \right) + (2 - \alpha_B) (\mathcal{H}^2 - \mathcal{H}')^2 \\
+ \mathcal{H}^3 \left( \frac{1}{3} \alpha_K' + \alpha_B \right) + \left( \frac{1}{2} \mathcal{H}^2 \alpha_B - \mathcal{H}' \right) \mathcal{H} \alpha_B' + \left( \alpha_K + \frac{3}{2} \alpha_B \right) \mathcal{H}^2 \mathcal{H}', \\
C \equiv \mathcal{H} \alpha_B' + (2 - \alpha_B) \left[ \mathcal{H}^2 \alpha_M - \mathcal{H}' + \frac{1}{2} \mathcal{H}^2 (2 - 2 \alpha_T + \alpha_B + \alpha_B \alpha_T) \right] - a^2 \frac{\rho_m + p_m}{M_*^2}. \]

Note that compared to the Newtonian-gauge equation (3.21), we have diagonalised the kinetic term for the scalar by eliminating the highest derivatives of the metric $\mathcal{h}''$ and $\eta''$ as well as $k^2 \eta$ using eqs. (C.1) and (C.4). This should simplify the implementation in the Boltzmann codes.

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