Distributed Endogenous Internal Model for Modal Consensus and Formation Control

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Abstract: In this paper, the problems of (modal) consensus and formation control are tackled for a group of heterogeneous agents described by linear dynamics and communicating over a network with fixed topology. The classic approach to these problems prescribes that each agent is provided with its own complete internal model of the desired dynamics (which can be viewed as an exosystem). In this paper, the novel concept of Distributed Endogenous Internal Model is introduced and discussed. Such internal model is characterized by two features: i) it is actually distributed over the network, i.e. no single agent is provided with a complete internal model; ii) it is endogenous, namely it is generated by exploiting the dynamics already available to the overall group of agents, through the local cooperation between each agent and its neighbors. As a consequence, each agent is capable of generating the desired steady-state distributed static control input by only exchanging information with its neighbors, without the need for additional dynamics anywhere in the network.

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1. INTRODUCTION

The problem of characterizing as well as enforcing meaningful collective behaviors of groups of - independent and possibly heterogeneous - agents has become increasingly crucial in recent years, as it naturally arises in several contexts and applications, encompassing social sciences, biology and engineering tasks, such as mobile robots, unmanned air vehicles (UAVs) or autonomous underwater vehicles, see e.g. Fax and Murray [2004], Olfati-Saber and Murray [2004], Jadamba et al. [2003], Moreau [2005], Lin et al. [2005]. It has been immediately recognized that the role played by the information exchange pattern and the communication topology among the components of the group is of paramount importance in the resulting collective coordination and consensus, Wieland et al. [2011].

It appears evident that the interest on such coordinated motions is intrinsically related to the ability of each agent of the group to achieve individual or common desired objectives, possibly despite the limited communication and available information. Since such objectives may be naturally formulated within the framework of output regulation problems, the above problem can be indeed interpreted in terms of a distributed output regulation task for each agent, Su and Huang [2012]. This intuition essentially motivated several works, see e.g. Sepulchre et al. [2008], Wieland et al. [2011], De Persis and Jayawardhana [2014], that relate the solvability of the consensus problem - or more in general of formation control tasks - to the presence of internal models, Francis and Wonham [1976], of the desired (common) behavior, which must be possessed by each agent of the group. Therefore, in recent years, a large number of practical solutions have been proposed along this direction, all characterized by the fact that the internal model is essentially embedded into each individual agent by means of additional dynamics, while the communication topology is fundamentally employed for the stabilization task. However, in modern control applications collective motions must be obtained for extremely large-scale (even huge, in some circumstances) networks of agents, e.g. power grids, where implementing a solution that avoids replicating internal models in each individual node of the network may be a critical advantage. Moreover, the interconnection among the nodes is typically not exploited to solve the coordination task, but rather the stabilization one, hence somewhat wasting the behaviors that can be already achieved by considering the dynamics of each agent, without the need for introducing additional dynamics anywhere in the network. This paper aims at addressing such issues by introducing internal models that are essentially distributed over the entire network and are not, in general, entirely possessed by any agent of the group. The main contribution of this paper consists in introducing the concept of distributed endogenous internal model and in employing such notion to tackle the problems of modal consensus and formation control. The rest of the paper is organized as follows. In Section 2, we formally introduce the modal consensus and the formation control problems, together with interesting insights and notable variations. The novel notion of distributed endogenous internal model is the topic of Section 3. The solutions to the two problems defined above by means of distributed internal models are dealt with in Sections 4 and 5, respectively. Finally, numerical simulations, involving some interesting scenarios from the literature, are discussed in Section 6, while conclusions are drawn in Section 7.
2. PRELIMINARIES AND PROBLEM DEFINITION

Consider a multi-agent model consisting of $N$ heterogeneous linear systems described by equations

$$
\dot{x}_i = A_i x_i + B_i u_i, \quad (1a)
$$

$$
e_i = C_i x_i + Q_i w, \quad (1b)
$$

with $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, and $e_i(t) \in \mathbb{R}^{p_i}$, for $i = 1, \ldots, N$. For later use, define $\bar{n} = \sum_{i=1}^{N} n_i$, $A = \text{blkdiag}\{A_i\}$ and $B = \text{blkdiag}\{B_i\}$. The exogenous signal $w(t) \in \mathbb{R}^q$, which may be given several alternative interpretations as extensively discussed in the following, is generated by equations of the form

$$
\dot{w} = Sw, \quad (2)
$$

which is referred to as the exosystem, following the standard terminology used in output regulation theory. The communication topology is captured by a directed graph $\bar{G} = (\bar{V}, \bar{E})$, with vertex set $\bar{V} = \{v_1, \ldots, v_N\}$, each vertex associated to a system of the form of (1), and arc set $\bar{E} \subseteq \bar{V} \times \bar{V}$. The latter encodes the information exchange pattern, namely the $i$-th agent receives information from the $j$-th agent if and only if $\{v_j, v_i\} \in E$. The adjacency matrix $\bar{A}$ associated to the graph $\bar{G}$ is constructed by letting $a_{ij} = 1$ if and only if there is an arc between $v_j$ and $v_i$ and $a_{ij} = 0$ otherwise. Moreover, the notation $\bar{N}_i$ defines the set of neighbors from which the agent $i$ receives information. Consider in addition the matrix $L = \bar{A} \otimes I_{n_i}$. Finally, let $L_k$ denote the $k$-th block row of $L$, that is the $n_k \times \bar{n}$ matrix consisting of the rows from $(\sum_{j=1}^{k-1} n_j + 1)$ to $n_k$ of the matrix $L$. Despite the fact that the following definitions and derivations are carried out mainly in the presence of a time-invariant communication topology, it is not difficult to show that a time-varying topology are not a critical obstruction to the extension of such concepts.

Since the notion of steady-state response is instrumental for the following derivations, it is briefly recalled here. Towards this end, consider first the classical decomposition of the state space in terms of the so-called stable, center and unstable subspaces, denoted by $V^-, V^0$ and $V^+$, respectively. In particular, $V^-, V^0$ and $V^+$ are the subspaces generated by the generalized eigenvectors corresponding to eigenvalues of the dynamic matrix with negative, zero or positive real parts, respectively. It is well-known that $\mathbb{R}^n = V^- \oplus V^0 \oplus V^+$. Finally, let $\tilde{C}^- \triangleq \{\lambda \in \mathbb{C} : \text{re} [\lambda] < 0\}$ and let $\tilde{C}^- \triangleq \{\lambda \in \mathbb{C} : \text{re} [\lambda] \leq 0\}$.

**Definition 2.1.** Consider an autonomous linear system $\dot{x} = Ax$ and suppose that $\sigma(A) \subset \tilde{C}^-$. Then $x_{ss}(\cdot)$, the steady-state response from $x_0$, is defined as

$$
x_{ss}(t) = e^{At}P_{\mathbb{R}^0}(x_0), \quad (3)
$$

where $P_{\mathbb{R}^0}(x_0)$ denotes the projection of the initial condition $x_0$ on the subspace $V^0$.

It is worth noting that, differently from the classical notion of steady-state response of a forced dynamical system, the response $x_{ss}(t)$ in (3) depends on the initial condition $x_0$. Such definition is particularly useful in the following, where, as will be extensively discussed, the concept of distributed endogenous internal model allows the endogenous generation of desired signals. Two alternative formulations of the control objectives of interest, in terms of consensus or formation control, can be defined. In both cases, the resulting control law $u$ is said to be distributed if, for each agent $i \in \{1, \ldots, N\}$, the input $u_i$ depends only on information exchanged with agents belonging to the set of its neighbors $\bar{N}_i$.

**Definition 2.2.** (Modal Consensus). Consider the heterogeneous systems (1a), for $i = 1, \ldots, N$, and the exosystem (2). The Modal Consensus problem consists in determining distributed control inputs $u_i$, $i = 1, \ldots, N$ such that the steady-state responses $x_{ss,i}(\cdot)$ contain only modes of (2), and that for almost any initial condition $x(0) \in \mathbb{R}^n$ it holds that $x_{ss}(\cdot) \triangleq [x_{ss,1}(\cdot), \ldots, x_{ss,N}(\cdot)]'$ contains all the modes of (2). Moreover, Strong Modal Consensus is achieved if the steady-state response $x_{ss,i}(\cdot)$ of each agent $i = 1, \ldots, N$ contains all the modes of (2) for almost any initial condition $x(0) \in \mathbb{R}^n$.

**Remark 2.1.** Note that, in Definition 2.2, the need to neglect a set of initial conditions (having zero measure) is imposed by the fact that in a linear system as (1a) there necessarily exist such a set where some natural modes are not excited.

The statement of Definition 2.2 entails that the agents of the team should agree on the modal content (that is, the functions appearing) in their steady-state behavior; this is a weaker request with respect to the requirement in asymptotic tracking, where all agents are required to follow a specific trajectory (specified a priori by the exosystem’s initial condition), or the requirement in consensus, where all agents are required to follow a common trajectory (not specified a priori). In the setting of modal consensus, the exosystem (2) is interpreted as a tool by means of which the modal content at steady-state is specified, rather than $e.g.$ an actual reference generator. However, it is worth stressing that more demanding requirements can be achieved within the same framework introduced for modal consensus, essentially by considering suitable selections of the matrices $S$, $C_i$ and $Q_i$, for $i = 1, \ldots, N$, as discussed in details below. In output regulation problems, it is usual to reformulate a tracking objective as the task of pushing to zero the output error $e_i$, having the form in (1b). In the present context, in order to be able to control a formation of agents, it seems of interest to be able to have each agent follow a phase-shifted version of a common reference trajectory. Let, then, $w(t; w_0; t_0)$ denote the solution of (2) at time $t$ with initial condition $w(t_0) = w_0$ and an output matrix $C_w$ such that $C_w w$ can be seen as a reference signal, and consider the following definition.

**Definition 2.3.** (Formation Control). Consider the heterogeneous systems (1a) with output $y_i \triangleq C_i x_i$, for $i = 1, \ldots, N$, and the exosystem (2). Given $t_i \in \mathbb{R}_+$, $i = 1, \ldots, N$, the Formation Control problem consists in determining distributed control inputs $u_i$, $i = 1, \ldots, N$ such that

$$
\lim_{t \to \infty} y_i(t) - C_w w(t; w_0; t_i + \Delta) = 0, \quad (4)
$$

for some $\Delta \in \mathbb{R}$ and for all $w_0 \in \mathbb{R}^q$.

Note that (4) can be reformulated as the requirement of having $\lim_{t \to \infty} C_i(t) = 0$ with $e_i$ given by (1b), provided that $Q_i$ is chosen as

$$
Q_i = -C_w e^{S(t_i + \Delta)}. \quad (5)
$$
The reason for denoting the problem in Definition 2.3 as *Formation control* can be given as follows.

**Remark 2.2.** Interestingly, by completely or partially specifying the selections for the available parameters in the definition above of Formation Control, several notable problems can be derived. In fact, if it is imposed that all the phase-lags $t_i$ are identical, namely $t_i = t_j$, $i,j = 1,...,N$, then the requirement in (4), enforces *Synchronization*, for some $\Delta$, of the steady-state output responses of each agent. If, on the other hand, condition (4) is achieved for $\Delta = 0$, then (asymptotic) *Distributed Output Tracking* is obtained. Moreover, if the values of the constants $\Delta$ are arbitrary, then the requirement in (4), enforces an *arbitrary phase-locking problem*, whereas if the values are not given but a solution is obtained for some $t_i$, then a *phase-locking problem* is solved.

Based on the above remark, solving the formation control problem allows to steer an entire team of autonomous agents in coordinated or independent (individual) motions. The two problems stated above, together with additional alternative formulations along similar lines, can be approached and solved with the tool introduced in the following section, namely the notion of *distributed endogenous internal model*.

### 3. DISTRIBUTED ENDOGENOUS INTERNAL MODEL

The aim of this section consists in introducing the concept of distributed endogenous internal model. To begin with, consider the following standing assumption.

**Assumption 3.1.** There exist matrices $\Pi_i \in \mathbb{R}^{n_i \times q}$ and $\Gamma_i \in \mathbb{R}^{m_i \times q}$ such that

$$\Pi_i S = A_i \Pi_i + B_i \Gamma_i,$$

$$0 = C_i \Pi_i + Q_i,$$

for $i = 1,...,N$. Let, in addition, $\Pi \in \mathbb{R}^{n \times q}$ be defined as

$$\Pi = [\Pi'_{1} \ldots \Pi'_{N}]'$$

**Remark 3.1.** As discussed in Galeani et al. [2015], Carnevale et al. [2013], in the case of redundant plants, namely having more inputs than outputs, the equations (6) admit an affine variety of solutions described as

$$\Pi_i(\theta) = \Pi_i^0 + \sum_{j=1}^{s} \theta_{ij} \Pi_{ij}^0 \quad \Gamma_i(\theta) = \Gamma_i^0 + \sum_{j=1}^{s} \theta_{ij} \Gamma_{ij}^0,$$

where $\Pi_i^0$, $\Gamma_i^0$ is any solution of (6) and $\Pi_{ij}^0$, $\Gamma_{ij}^0$, $j = 1, ..., s$, $s = (m_i - p_i)q$ are linearly independent solutions of the corresponding homogeneous equation, obtained from (6) for $Q_i = 0$. Such an abundance of solutions (and their parameterization provided in Galeani et al. [2015]) can be exploited as a useful degree of freedom in order to solve the forthcoming equations defining the distributed endogenous internal model.

In Wieland et al. [2011] it is shown that Assumption 3.1 is in fact a necessary condition for output synchronization, which is intimately related to the problems of modal consensus and formation control. It is then generally concluded in the literature that the required internal model for synchronization should be either contained in the agent dynamics or essentially introduced by the controller dynamics. In the following, we propose a different point of view on this task. The following statement provides the definition of an internal model that is distributed over the communication topology between the agents. To clarify the notation of the following definition, we let the group be the set of agents defined by the heterogeneous systems as in (1) with $i = 1,...,N$, while the extended group consists of the above agents together with a virtual agent described by (2), associated to the node 0.

**Definition 3.1.** *(Distributed Endogenous Internal Model).* Consider a multi-agent model as in (1) and a communication topology $\mathcal{G}$. The group possesses a *distributed endogenous internal model* if there exist $G_{i,j} \in \mathbb{R}^{m_i \times n_j}$, $i,j = 1,...,N$, such that

$$\Gamma_i = \sum_{j \in \mathcal{N}_i} G_{i,j} \Pi_j,$$

for all $i = 1,...,N$ such that $0 \not\in \mathcal{N}_i$.

The above definition entails that each agent is capable of generating the desired control action at steady-state to solve either modal consensus or formation control, i.e. $u_i(t) = \Gamma_i w(t)$, by simply interacting with the steady-state behaviors of its neighbors, i.e. $\bar{x}_{a,s}(t) = \Pi_j w(t)$, $j \in \mathcal{N}_i$, without the need for an explicit copy of the internal model for (2) locally defined at each agent.

**Proposition 3.1.** *(Invariance condition).* Consider a multi-agent model as in (1) and a communication topology $\mathcal{G}$. The group admits a *distributed endogenous internal model* if

$$\Gamma_i' \in \text{Im}\{(\text{blkdiag}\{L_i\} \cdot \Pi')\},$$

for all $i = 1,...,N$ such that $0 \not\in \mathcal{N}_i$.

The problems of Modal Consensus and Formation Control are tackled by exploiting the concept of distributed internal model in Sections 4 and 5, respectively.

### 4. MODAL CONSENSUS

The topic of this section consists in the solution of the Modal Consensus problem introduced in Definition 2.2. To this end, suppose that, without loss of generality, the matrix $\Pi$ introduced above is full column-rank, i.e. rank($\Pi$) = $q$. Then, let $\Pi^\perp$, with rank($\Pi^\perp$) = $\bar{n}$, $\bar{n} = \sum_{i=1}^{N} m_i$, be such that (\Pi^\perp)'\Pi = 0.

**Proposition 4.1.** *(Modal Consensus).* Given $N$ agents as in (1), together with the exosystem (2), and a communication topology $\mathcal{G}$, suppose that Assumption 3.1 holds disregarding (6b) and the group admits a distributed internal model. Then, the modal consensus problem is solved by $u_i = \sum_{j \in \mathcal{N}_i} (G_{i,j} + K_{i,j})x_j$, provided i) $\sigma((\Pi^\perp)'\Pi\Pi^\perp + (\Pi^\perp)'BK\Pi^\perp) \subset \mathbb{C}^-$; ii) $\Pi$ is observable.

for some block matrix $K \in \mathbb{R}^{\bar{n} \times \bar{n}}$, $\bar{n} = \sum_{i=1}^{N} m_i$, such that $K_{i,j} = 0$ if $j \not\in \mathcal{N}_i$.

If the conditions above hold with (ii) replaced by

iii) the pairs $(\Pi, S)$, $i = 1,...,N$, are observable, then the *strong modal consensus* problem is solved.
Interestingly, the Modal Consensus problem is solved by
static decentralized control laws, provided the group of
agents admits a distributed endogenous internal model;
in particular, no additional dynamics is needed in order
to ensure the presence of a suitable internal model. Moreover,
the external attractivity of the consensus subspace, namely
the subspace on which each agent behaves according to the
dynamics of $S$, is recast into a stabilization problem for
a reduced-order system, with the structure of the control
gain provided by the communication topology. This aspect
is further discussed in the following statement from the
computational point of view.

**Proposition 4.2.** Given $N$ agents as in (1), together with
the exosystem (2), and a communication topology $\mathcal{G}$,
that is, a communication topology $\mathcal{G}$, such that
Assumption 3.1 holds disregarding (6$b$) and the group
admits a distributed internal model. Then, the \textit{(strong) modal consensus} problem is solved by $u_i = \sum_{j \in N_i} (G_{i,j} + K_{i,j})x_j$, provided
i) there exists $P_e \in \mathbb{R}^{(n-q)\times(n-q)}$, $P_e = P_e^T > 0$ such that:
\begin{equation}
He(P_e((\Pi^+)^T\Pi^+ + (\Pi^+)^T K\Pi K^T)) < 0.
\end{equation}

Condition $i'$) of Proposition 4.2, compared to item $i)$ of
Proposition 4.1, more clearly highlights an interesting
aspect of the previous construction: increasing the number of
modes to be reproduced by the group of agent - i.e.
the complexity of the signal to be agreed on - actually
decreases the complexity of the (external) stabilization
task, by reducing the dimension of the Lyapunov matrix
$P_e$. This is due to the fact that a larger portion of the
components of the composite system are employed by
the network itself to create the distributed copy of the
exosystem.

**Remark 4.1.** It is worth interpreting the classical \textit{Laplacian control law} in the consensus problem in terms of the
definition of modal consensus and distributed endogenous
internal model given above. Towards this end, in the case
of single integrators, the consensus (1a) are described by
$A_i = 0$ and $B_i = 1$, together with $C_i = -Q_i$, $i = 1, \ldots, N$.
The \textit{mode} on which all the agents must agree is defined by a constant function of time, obtained
by letting $S = 0$ in (2). It can be shown that equations
(6) are solved by $\Pi_i = 1$ and $\Gamma_i = 0$ for each agent,
which clearly satisfy the requirements for a distributed
endogenous internal model - with all the $G_{i,j,s}$ equal to zero
- as in Definition 3.1. This in turn yields a combined matrix
$\Pi$ defined as $\Pi = [1, 1, \ldots, 1]^T$. Finally, letting $K = L$, it
follows that condition $i)$ of Proposition 5.1 holds, since
the eigenvalues of the matrix $BL$ belong to $\mathbb{C}^-$ apart
from a single eigenvalue at zero, which is however the one
associated to the consensus subspace defined by $\Pi$, while
item $ii)$ holds by definition of Laplacian matrix.

5. FORMATION CONTROL
The objective of this section is to discuss the solution to
the Formation Control problem as in Definition 2.3.

\textit{Proposition 5.1. (Formation Control).} Given $N$ agents as
in (1), together with the exosystem (2), and a
communication topology $\mathcal{G}$, suppose that Assumption 3.1 holds
with $Q_i = -C_iw^{-S(\pi + \Delta)}$, for some $\Delta \in \mathbb{R}$, and that
the \textit{extended group} admits a distributed internal model.
Then, the formation control problem is solved by $u_i = \sum_{j \in N_i} (G_{i,j} + K_{i,j})x_j$, provided
i) $\sigma((\Pi^+)^T\Pi^+ + (\Pi^+)^T K\Pi K^T) \subset \mathbb{C}^-$;
ii) $K\Pi = 0$.

An interesting alternative formulation of the formation
control problem, namely the autonomous formation con-
rol with initial deployment, is presented and tackled in
the following section.

5.1 Autonomous Formation Control problem with Initial
Deployment
In this section, the output synchronization problem is
approached and solved in an unconventional fashion: it
is assumed that the agents can be initially deployed by a
\textit{centralized authority}, by exploiting the knowledge of
the initial condition $w(0)$ of the exosystem (2), and then
the desired errors with respect to the state of a reference
generator are asymptotically driven to zero without any
additional communication with external exosystems.

\textit{Definition 5.1.} Consider the heterogeneous systems (1a-
1b), for $i = 1, \ldots, N$, and the exosystem (2). Given $t_i \in \mathbb{R}_+$, $i = 1, \ldots, N$ and an initial condition $w_0 \in \mathbb{R}^n$
the \textit{Autonomous Formation Control problem with Initial
Deployment} consists in determining distributed control
inputs $u_i$, $i = 1, \ldots, N$ and an initial condition $x(0) \in \mathbb{R}^n$ such that (4) holds, for some $\Delta \in \mathbb{R}$, without any
knowledge of $w(t)$ for all $t > 0$.

Note that, despite the fact that each agent may know in
advance, via a centralized authority, the initial condition
of the exosystem $w_0$, the requirement of static distributed
control inputs $u_i$ entails that the agents do not possess in
general a copy of the dynamics of the signal generator,
namely individual internal models.

\textit{Remark 5.1.} The interest of the problem may be motivated
by those practical applications in which the actual refer-
ence generator, or leader, may not be able, for several
reasons, to remain in contact with at least one agent of the
team during the formation control or the coordinated
motion. This is for instance the case in marine and underwater
applications as well as in search and rescue operations, in
which the agents could be only initially deployed in a \textit{safe
region}, i.e. the sea surface or the border of the inaccessible
region, respectively, and the task must then be performed
autonomously.

Towards the solution to the autonomous formation control
problem with initial deployment, suppose that the team of
agents possesses a distributed endogenous internal model,
according to Definition 3.1, and consider first the change
of coordinates for the composite system (1) with $u_i = \sum_{j \in N_i} (G_{i,j} + K_{i,j})x_j + v_i$, $i = 1, \ldots, N$, defined by $z = T x$, with
$T^{-1} = [[(\Pi^+)^T, \Pi]]$. In the transformed coordinates, the
composite system becomes, by the definition of $\Pi$ and of distributed endogenous internal model,
\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} = 
\begin{bmatrix}
\bar{A}_{11} & 0 \\
\bar{A}_{21} & S
\end{bmatrix} \begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2
\end{bmatrix} + 
\begin{bmatrix}
\bar{B}_1 \\
\bar{B}_2
\end{bmatrix} v.
\]

(12)

It is interesting to point out that, in the $z$-coordinates, the components of the state $z_2$ behave essentially as (endogenous) signal generator, which is in principle capable of generating signals identical to those generated by the classical exosystem (2), provided it is suitably initialized, as the classical exosystem, and somehow decoupled by the influence of the component $z_1$, differently from what happens with (2). These aspects are the topic of the discussion in the remaining of this section. To provide a concise statement of the following result, let $C = [C_1^T, C_2^T]^T = CT^{-1}$.

**Proposition 5.2.** Given $N$ agents as in (1), together with the exosystem (2), and a communication topology $\mathcal{G}$, suppose that Assumption 3.1 holds with $Q_i = -C_we^{-S(t+\Delta)}$, for some $\Delta \in \mathbb{R}$, and that the group admits a distributed internal model. Suppose that items $i)$-$ii)$ of Proposition 5.1 hold.

The autonomous formation control problem with initial deployment is solved by $u_i = \sum_{j \in \mathcal{N}_i} (G_{ij} + K_{ij})x_j$ and $x(0) = T^{-1}z^*$, with $z^*_2(0) = -C_jQ_j(0) - \bar{X}(z_1(0))$ for any $z_1(0) \in \mathbb{R}^{n-q}$, with $X \in \mathbb{R}^{T \times (n_u - q)}$ such that
\[
X(\bar{A}_{11} + \bar{B}_1 K) - SX + \bar{A}_{21} = 0.
\]

(13) \hfill \Box

**Remark 5.2.** It is worth stressing the fact that, since the matrices $(\bar{A}_{11} + \bar{B}_1 K)$ and $S$ are spectrally disjoint, i.e. $\Lambda(\bar{A}_{11} + \bar{B}_1 K) \cap \Lambda(S) = \emptyset$, the Sylvester equation (13) admits a (unique) solution $X$ for any $\bar{A}_{21}$, hence it does not hinder the feasibility of the autonomous formation control problem with initial deployment. \hfill \Box

**Remark 5.3.** The solution above imposes restrictions on the selection only of the initial condition for $z_2(0)$, while $z_1(0)$ could be arbitrary and, typically, $z_1(0)$, namely the off-the-manifold dynamics, is much larger than $z_2(0)$. \hfill \Box

6. STEERED PARTICLES IN THE PLANE

In this section we consider the model of $N$ identical particles evolving in the plane at constant (unit) speed, described by the equations (see Sepulchre et al. [2008] for more detailed discussions on such models)
\[
\begin{align}
\dot{r}_k &= e^{i\theta_k}, \\
\dot{\theta}_k &= u_k,
\end{align}
\]

(14a)

(14b)

for $k = 1, \ldots, N$, with $r_k(t) = x_k(t) + iy_k(t) \in \mathbb{C}$, in complex notation, denoting the position of the particle in the plane, while $\theta_k(t) \in \mathbb{R}$ denotes its orientation. It is assumed that only relative information is exchanged among the agents, namely the steering control input $u_k$ depends only on the differences $\theta_i - \theta_j, i, j = 1, \ldots, N$, within an all-to-all communication topology. The notation $\vartheta = [\theta_1, \ldots, \theta_N]^T$ compactly describes all the orientations of the particles. The control objective consists in inducing collective and organized behaviors of the group of particles, by acting merely on their relative orientations (phase stabilization). Towards this end, the following definitions are in order.

**Definition 6.1.** Consider the group of particles (14). Then, the particles are said to be phase-locked if
\[
\vartheta_{ss,i}(t) - \vartheta_{ss,j}(t) = 0,
\]

for all $i, j = 1, \ldots, N$ and for all $t \geq 0$, whereas the particles are said to be exactly synchronized if
\[
\vartheta_{ss,i}(t) - \vartheta_{ss,j}(t) = 0,
\]

for all $i, j = 1, \ldots, N$ and for all $t \geq 0$. \hfill \Box

The above task may be naturally formulated within the framework of the Formation Control introduced in Definition 2.3, by considering the virtual exosystem (2) with the matrices $S$ and $C_w$ described by
\[
S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_w = \begin{bmatrix} 1 & 0 \end{bmatrix},
\]

respectively. The desired phase stabilization may be then achieved by letting, as proposed in Proposition 5.1, the matrices $Q_i$ defined as $Q_i = -C_we^{-S(t+\Delta)}$, with $t_i = (2\pi/N)i$, thus nominally obtaining the desired spacing among the phases of the group of particles. The solutions to the Francis equations (6) are then easily obtained as
\[
\Pi_i = Q_i = \begin{bmatrix} -1, 2m/N \end{bmatrix},
\]

(18a)

\[
\Gamma_i = \Pi_i S = \begin{bmatrix} -1 \end{bmatrix}.
\]

(18b)

Such solutions induce the change of coordinates defined as $T^{-1} = ([\Pi_i]^T)^T \Pi_i$. Moreover, the distributed endogenous internal model is straightforwardly implemented by defining the $i$-th row of the matrix $G$ as
\[
G_i = [0, \ldots, 0, g_i - g_i, 0, \ldots, 0]
\]

(19)

with $g_i$ and $-g_i$ being the $i$-th and the $(i+1)$-th (possibly modul $N$) element of $G_{ii,j}, i, j = 1, \ldots, N$ respectively, and $g_{i} = -1/(t_{i} - t_{i-1})$ (together with $K$ such that items $i)$ and $ii)$ of Proposition 5.1 hold and $\sum_{j=1}^{N} K_{ij} = 0, i = 1, \ldots, N$, to satisfy the relative information constraint. Note that the resulting cumulative control law $u = (G + K)\vartheta$ is linear and time-invariant. To provide a concise statement, let the vector $\vartheta_{0}$ denote the initial orientations of the particles and $[\vartheta_{2,1}^*, \vartheta_{2,2}^*]^T = \vartheta_{2,1}^* = 2\pi/3 + \chi_{1,0}; \vartheta_{2,2}^* \in \mathbb{R}^{N-2}$, with $[\vartheta_{1,0}; \vartheta_{2,2}^*]^T = T\vartheta_{0}$, and $X \in \mathbb{R}^{2 \times (N-2)}$ defined as in (13). Finally, let $\vartheta_{0}^* = T^{-1}[0, \vartheta_{2,2}^*]^T$.

**Proposition 6.1.** Consider the group of particles (14) exchanging only relative information within an all-to-all communication network. Consider the cumulative control action $u = (G + K)\vartheta$. Then,

i) If $\vartheta_{2,2} \neq 0$, the particles are phase-locked and move in circular orbits of radius $\omega_{\vartheta_2} = |\vartheta_{2,2}^*|^{-1}$ such that
\[
\vartheta_{ss,i}(t) - \vartheta_{ss,j}(t) = \frac{2\pi}{N}|\vartheta_{2,2}^*|,
\]

(20)

for all $t \geq 0$ and $i = 1, \ldots, N - 1$. Moreover, the center of mass of the particles moves in circular orbit of radius
\[
\vartheta_c = \omega_{\vartheta_2}\left[\alpha_1 \cos(\alpha_2) \alpha_2 \sin(\alpha_2) \right],
\]

(21)

with $\alpha_1 = \frac{1}{N} \sum_{j=1}^{N} \cos(\theta_{j,0}), \alpha_2 = \frac{1}{N} \sum_{j=1}^{N} \sin(\theta_{j,0})$;

ii) If $\vartheta_{2,2} = 0$, the particles are exactly synchronized and move, together with the center of mass, in straight lines in the direction of $\vartheta_{2,1}^*$. \hfill \Box

**Remark 6.1.** For arbitrary $\vartheta_0$, the LTI control law $u = (G + K)\vartheta$ solves a modal consensus problem with respect to the modes of a double integrator: the particles agree on
the (common) radius of the circular orbits, obtained as a weighted sum of the initial orientations. ▲

Remark 6.2. A specific collective behavior of the particles in (14) is enforced if one is allowed to select $\bar{\theta}_0$. Since the crucial value is the scalar $z_2$, one could achieve a desired motion by considering arbitrary orientations for $N - 1$ agents and by selecting the initial orientation of a single agent, i.e., such that $z_2 = \omega_{eq,d}$, with $\omega_{eq,d}$ denoting the desired radius. If one is interested in straight motions, this may be achieved by considering arbitrary orientations for $N - 2$ agents and by selecting the initial orientations of two agents, i.e., such that $z_2 = 0$ and $z_1$ represents the desired common orientation. ▲

In the following simulations we let $N = 6$. In the first scenario, depicted in Figure 1, the initial orientations of the particles are such that $z_2 = 1$, hence they agree on moving in circular orbits of radius equal to one. The orientations are phase-locked with a relative phase-lag of $\pi/3$. The virtual particle (black line) displayed in Figure 1 describes the evolution of the center of mass of the particles, which converges to a constant value after

Finally, in the last scenario displayed in Figure 2, the initial orientations are such that $z_2 = 0$, hence the phases are exactly synchronized on $z_1$ and the particles move on the plane in the same direction.

7. CONCLUSIONS

The concept of distributed endogenous internal model has been introduced and employed to solve the problems of modal consensus and formation control. More precisely, such internal models are characterized by the property of being actually distributed over the entire network and not, in general, entirely possessed by any agent of the group. This is achieved by exploiting the dynamics already available in the group of agents, without the need for introducing additional dynamics anywhere on the network.

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