Extended BRST symmetries. Quantum approach

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Abstract

The aim of this lecture is to present in a comprehensible way what the BRST quantization means and how the "classical" master equation, action and BRST transformations have to be prolonged towards the same "quantum" items. The presentation will focus not only on the standard BRST symmetry, but on larger symmetries as $sp(2)$, both in the Lagrangean and in the Hamiltonian formalisms. How to find answers to these questions in more sophisticated cases will be illustrated by the example of a nonlinear system with open superalgebra.

1 Introduction

All fundamental interactions from nature are described by theories with internal symmetries or gauge theories. Such theories, as QED, QCD, electro-weak theory and gravity, ask for a special approach because of the unphysical degrees of freedom involved by the gauge invariance. For example, when quantizing such theories the direct computation of the path-integrals is meaningless since the integration over gauge directions in the measure would make the path-integral infinite-valued. The BRST technique [1] is one of the approaches which allows to overcome this difficulty. It assumes the replacement of the local (gauge) symmetry with a global (BRST) one. The BRST symmetry is expressed either as a differential operator $s$, or in a canonical form, by the antibracket $(\cdot, \cdot)$ in the Lagrangean (Batalin-Vilkovisky) case [2] and by the extended Poisson bracket $\{\cdot, \cdot\}$ in the Hamiltonian (Batalin-Fradkin-Vilkovisky) formulation [3]:

$$s\ast = (\ast, S) = \{\ast, \Omega\}.$$  \hspace{1cm} (1)

The BRST generator $S$ and the BRST charge $\Omega$ are both defined in extended spaces generated by the real and by the ghost-type variables. To perform path-integral calculations in this frame, it is necessary to remove the redundant gauge variables, that is to gauge-fix the action by choosing a gauge fermion $Y$. Elimination of gauge variables assures the BRST invariance of the action [4].
but not of the measure. A BRST transformation of the coordinates could generate non-trivial terms in the action. These terms can be exponentiated and adsorbed in the action (Fadeev-Popov trick). One obtain an extended action called "quantum" action.

In this paper we will explicitly show how the BRST formalism allows to construct well defined path-integrals and what is the concrete form of the quantum master equation in the special case of a physical system described by a nonlinear gauge algebra. The interest on such systems was initiated by the discovery of conformal field theories [5] which led to a new class of gauge theories with nonlinear gauge algebras, the so-called \( W_N \) algebras [6]. The BRST approach for non-linear superalgebras was developed in [7].

In the standard BRST approach the gauge fixing procedure asks for a "non-minimal sector" which supposes the introduction of new, supplementary variables. In the extended symmetries this non-minimal sector appears in a natural way. Now, the gauge fixing function can be constructed in a more simple way and the gauge fixing action is obtained easily. Singula difficulty which can be appear in the case of the extended BRST formalism is asigurarea of the invariance of the integrating measure, so that the average of the observables to be independent de the choice of the gauge. Ne propunem in this paper to show how can done this in the case of gauge theories with nonlinear gauge superalgebras.

2 Quantum Master Equations

2.1 The Standard BRST Approach

There are two main procedures in obtaining a quantum description of the gauge systems: the Dirac quantization method and the path integral method. In the Hamiltonian formalism the existence of the gauge symmetries is equivalent with the existence of some constraints imposed to the theory and this is why the Dirac quantization method, which is especially apropriate for the description of systems with constraints, is a very useful method in quantizing gauge theories. As we intend to built up the quantum equation for interacting fields, we will concentrate our approach on the path integrals' formalism which is more suitable for the quantum description of these systems. This approach to the quantization of systems with symmetries, both in the Lagrangean (BV) formalism and in the Hamiltonian (BFV) one, started from the Fadeev-Popov trick and ended with the development of the BRST technique. The gauge fixing procedure assures the BRST invariance of the gauge fixed action but not automatically of the generating functional \( Z_Y \) and of the expectation values \( \langle F \rangle_Y \). For a gauge theory with an action \( S[\phi, \phi^*] \) we have:

\[
Z_Y = \int d\phi^A \exp \left( \frac{i}{\hbar} S_Y[\phi^A, \phi^{A*} = \frac{\delta Y}{\delta \phi^A}] \right)
\]  

(2)

and

\[
\langle F \rangle_Y = \int d\phi^A F \exp \left( \frac{i}{\hbar} S_Y \right).
\]

(3)
The independence on the choice of $Y$ and the BRST invariance of (2) and (3), essential requirements for a consistent quantum description, asks more for the invariance of the measure $d\phi^A$ to the BRST transformation:

$$
\phi^A \rightarrow \phi'^A = \phi^A + (-1)^{\varepsilon_A} (\phi^A, S) \mu = \phi^A + (-1)^{\varepsilon_A} \frac{\delta S}{\delta \phi^A} \mu
$$

(4)

where $\mu$ is a constant, anti-commuting parameter. In the general case the transformation (4) leads to

$$
d\phi'^A = (1 - (-1)^{\varepsilon_A} \delta \frac{\delta S}{\delta \phi^A} \mu) d\phi^A = (1 - (\Delta S) \mu) d\phi^A
$$

(5)

and nontrivial terms in the measure could be generated. The invariance condition would impose:

$$
d\phi'^A = d\phi^A \Rightarrow \Delta S = 0; \Delta \equiv (-1)^{\varepsilon_A} \delta \frac{\delta S}{\delta \phi^A}.
$$

(6)

The condition (6) might not be fulfilled. Although, all the terms generated by $(\Delta S) \mu) d\phi^A$ can be exponentiated and absorbed as "quantum corrections" of the action $S$:

$$
W = S + \hbar W_1 + \hbar^2 W_2 + \cdots
$$

(7)

One obtains the "quantum action" $W$ which satisfies a Quantum Master Equation (QME), an extension of the classical master equation $(S, S) = 0$. It has the form:

$$
\frac{1}{2} (W, W) - i\hbar \Delta W = 0.
$$

(8)

The equation (8) can be written in the equivalent form

$$
\Delta e^{i\hbar W} = 0.
$$

(9)

It is important to note that $\Delta$ is not a true derivation, its action being given by the following rule:

$$
\Delta(\alpha \beta) = \alpha \Delta \beta - (-)^{\varepsilon_\alpha} (\Delta \alpha) \beta - i\hbar \Delta(\alpha, \beta).
$$

(10)

The classical BRST operator $s$ can be, it also, extended to a quantum one:

$$
s* \equiv (*, S) \rightarrow \sigma* \equiv (*, W) - i\hbar *
$$

(11)

The quantum version of the BRST invariance will be expressed by:

$$
\sigma F = 0 \iff \Delta (Fe^{i\hbar W}) = 0.
$$

(12)
2.2 The sp(2) extended BRST Quantization

Because of some difficulties the standard BRST approach met in the gauge fixing procedure, procedure which suposes the introduction of some supplementary variables from a non-minimal sector, extended BRST symmetries has been formulated. The main such extension is known as the sp(2) BRST symmetry and suposes the existence of two anticommuting differential operators, $s_1$ and $s_2$, which can be joined in a symplectic doublet $s_1, s_2$ with:

$$s = s_1 + s_2; \quad s^2 = 0.$$  \hfill (13)

In the Hamiltonian formalism [8], the sp(2) BRST symmetry is canonical generated by the BRST charges $\Omega_a$, $a = 1, 2$ with $\varepsilon(\Omega_a) = 1$. By defining an extended phase-space and a generalized Poisson bracket, the BRST charges will be given by:

$$s_a^\ast = \{\ast, \Omega_a\}; \quad a = 1, 2.$$  \hfill (14)

Their concrete form depends on the theory and it can be obtained by using the homological perturbations theory [4]. The extended Hamiltonian $H$ is the solution of the equations

$$\{H, \Omega^a\} = 0, \quad a = 1, 2$$

with boundary condition

$$H|_{Q=\lambda=0} = H_0(q, p).$$

It is obtained using the same homological perturbations theory [4]. The gauge fixing procedure leads to a gauge fixed Hamiltonian of the form

$$H_Y = H + \frac{1}{2} \varepsilon_{ab}\{\Omega^a, \{\Omega^b, Y\}\}$$

where $Y$ is the gauge fixing functional defined in terms of the real and ghost type coordinates.

Even if it is easier to develop the Hamiltonian formalism, it does not always lead to a covariant gauge fixing action. This is the way the passage to the Lagrangean formalism is indicated. It is in the Lagrangean frame where we should study the invariance of the generating functional $Z_Y$ and of the expectation values of the form (3) for observables.

In the Lagrangean case [9], two antibrackets are defined and the master equations take the form:

$$\frac{1}{2}(S, S)_a + V_a S = 0; \quad a = 1, 2 \quad \text{ (15)}$$

where $\{V_a, a = 1, 2\}$ represent two special "non-canonical" operators. There is a direct modality of obtaining the sp(2) BRST Lagrangean formalism, but it assumes the use of a very large spectrum of ghost generators. To avoid this unuseful extension, it is simpler at the classical level to construct the Lagrangean
formalism following its equivalence with the Hamiltonian one\cite{10}. We will present how the classical \(sp(2)\) theory can be extended at quantum level in the frame of path integrals formalism. We will not give here more details but we direct to\cite{11}, where quantum \(sp(2)\) master equations were obtained in the Hamiltonian and then in the Lagrangean schemes. In this quantum context, the invariance condition \((6)\) has as correspondent the equations:

\[
\Delta^a S = 0; \Delta^a \equiv (-1)^{\varepsilon_A} \frac{\partial^r}{\partial \phi^A} \frac{\partial^r}{\partial \phi_{Aa}} + V^a = (-1)^{\varepsilon_A} \frac{\partial^r}{\partial \phi^A} \frac{\partial^r}{\partial \phi_{Aa}} + \varepsilon^{ab} \phi^a_{Ab} \frac{\partial^r}{\partial \phi_A} \tag{16}
\]

and

\[
\Delta^a \Delta^b + \Delta^b \Delta^a = 0, \ a, b = 1, 2. \tag{17}
\]

If \(\Delta^a S \neq 0\), the operators \(\Delta^a, a = 1, 2\) determine the \(sp(2)\)- QME in the \(sp(2)\) symmetric formulation of gauge theories:

\[
\Delta^a e^{i W} = 0 \iff \frac{1}{2} (W, W)^a = i \hbar \Delta^a W, \ a = 1, 2 \tag{18}
\]

where \(W\) represents the "quantum action", the \(\hbar\)-order corrections being involved by integrating measure. In the next section we will come back to this problem, approaching it in more concrete and applied manner.

3 The Example of Nonlinear Superalgebras

Let us consider a system described in a phase-space \(M \equiv \{q^i, p_i, i = 1, \cdots, n\}\) with \(\varepsilon(q^i) = \varepsilon(p_i) = \varepsilon_i\) by a set of first class constraints \(\{G_\alpha, \alpha = 1, \cdots, m\}\) with \(\varepsilon(G_\alpha) = \varepsilon_\alpha\) and by the canonical action

\[
S_{can}[q, p, u] = \int dt \left[ \dot{q}^i p_i - H^{(0)}(q, p, u) \right], \ i = 1, \cdots, n \tag{19}
\]

\[
H^{(0)}(q, p, u) = H_0(q, p) + u^\alpha G_\alpha. \tag{20}
\]

The Lagrange multipliers \(\{u^\alpha, \alpha = 1, \cdots, m\}\) will play the key role in the establishment of the equivalence between the extended BRST Hamiltonian and Lagrangean formalisms. We suppose that the constraints satisfy involution relations of the form:

\[
\{G_\alpha, G_\beta\} = G_\gamma f^\gamma_{\alpha\beta} + G_\delta g^\delta_{\alpha\beta} \tag{21}
\]

The gauge algebra is given by the relations \((21)\) and by

\[
\{H_0, G_\alpha\} = G_\beta V^\beta_\alpha + G_\delta G_\beta U^\delta_\alpha. \tag{22}
\]

We consider the case when \(f^\gamma_{\alpha\beta}, g^\delta_{\alpha\beta}, V^\beta_\alpha, U^\delta_\alpha\) are true constants. We deal in this case with a quadratic non-linear Lie algebra, case which correspond to a constrained system with open algebra\cite{12}. An example of such system is the
Yang-Mills fields theory with the nilpotent BRST charge has a quadratic function of the Fadeev-Popov ghost fields.

We will develop the $sp(2)$ BRST Hamiltonian formalism for the previous nonlinear superalgebra. The extended phase space will be generated in this case by the real variables $\{q^i, p_i, i = 1, \ldots, n\}$ and by the ghost-type variables $\{Q^{\alpha a}, P_{\beta b}, \lambda^\alpha, \pi_\beta\}$. The last ones satisfy the relations:

$$\{Q^{\alpha a}, P_{\beta b}\} = \delta_\beta^\alpha \delta_b^a, \quad \{\lambda^\alpha, \pi_\beta\} = \delta_\beta^\alpha \delta_\beta^\alpha$$

On the basis of the Jacobi identities for the structure functions of the superalgebra, we can show that for any superalgebras $[17], [21], [22]$,

$$\Omega^\alpha = G_\alpha Q^{\alpha a} + \varepsilon^{ab} P_{ab} \lambda^\alpha + \frac{1}{2} (-)^\varepsilon \gamma^\lambda \left[ \mathcal{P} \gamma^\epsilon \left( f_{\alpha \beta}^\gamma + G_\delta g_{\alpha \beta}^\gamma \right) Q^\beta Q^{\alpha a} + \frac{1}{2} \varepsilon_{\gamma} \left( f_{\beta}^\gamma + G_\delta g_{\beta}^\gamma \right) \left( f_{\delta}^\gamma + G_\delta g_{\delta}^\gamma \right) \right]$$

where the Lagrange multipliers $u^{\alpha a}$, $\alpha = 1, \ldots, m$ are seen now as real fields. The action (24) is invariant at the gauge transformations

$$\delta q^i = a_i^\alpha (q, \dot{q})^{\alpha a}, \quad \delta u^{\alpha a} = \varepsilon^{\alpha a} - (V^\alpha + G_\delta U^\alpha) \varepsilon^\beta + (f_{\alpha \beta}^\gamma + G_\delta g_{\alpha \beta}^\gamma) u^\gamma \varepsilon^\beta$$

The Noether’s identities will have the form

$$\frac{\delta S_0}{\delta q^i} a_i^\alpha + \frac{\delta S_0}{\delta u^{\alpha a}} \left[ -(V^\alpha + G_\delta U^\alpha) \varepsilon^\beta + (f_{\alpha \beta}^\gamma + G_\delta g_{\alpha \beta}^\gamma) u^\gamma \varepsilon^\beta \right] \frac{d}{dt} \left( \delta u^{\alpha a} \right) = 0.$$  

Starting from (24) we will develop the $sp(2)$ BRST Lagrangean formalism [9]. The complete spectrum of the antifields is given in our case by

$$Q^{\alpha a} = \{Q_{aa}, \pi^{\alpha a}\} = \{q^a, Q^{\alpha ab}, \lambda^\alpha, \pi_\alpha, q^a, \pi^{\alpha a}\}, \quad a, b = 1, 2,$$

$$\overline{Q}_A \equiv \{\overline{Q}_A, \pi\} = \{\overline{q}, \overline{Q}_{aa}, \overline{\lambda}, \overline{\pi}\}, \quad a, b = 1, 2.$$  

It is well known that the Lagrangian dynamics is generated in a ”anticanonical” structure. The generator of the Lagrangian BRST symmetry is

$$S = S_0[q, u] + \cdots$$
Because $S_0$ is unique one we will consider that $S$ is unique and we will introduce two antibracket structures which have the same properties like in the standard theory:

$$(F, G)_\alpha = \frac{\delta^r F}{\delta Q^\alpha} \frac{\delta^l G}{\delta Q^*_\alpha} - \frac{\delta^r F}{\delta Q^*_\alpha} \frac{\delta^l G}{\delta Q^\alpha},$$

The functionals $F$ and $G$ are depend to $Q^\alpha$ and $Q^*_\alpha$. On the basis of the graduation properties and of the Grassmann parities [9] we define the pairs canonical conjugate in respect with these antibrackets

$$(Q^*_a, Q^b)_{\alpha} = -\delta^r_{\alpha} \delta_{ba}$$

where $Q^*_a$ are expressed by (26) and $Q^\alpha$ are the fields of the theory (real $q^i$, $u^\alpha$ and ghosts $Q^\alpha_a$, $\lambda^\alpha$)

$$Q^\alpha \equiv \{Q^\alpha, u^\alpha, \lambda^\alpha, Q^\beta_a, \alpha = 1, 2\}.$$ (29)

For the Lagrange multipliers we will have

$$\varepsilon(u^\alpha) = \varepsilon_\alpha, gh(u^\alpha) = 0.$$  

We observe that some antifields of the theory have a canonical conjugate in the antibracket structure (28) and other antifields not have canonical pair (27). So, the BRST differentials \{s^a, a = 1, 2\} will have the following decomposition

$$s^a = (s^a)^{can} + V^a, a = 1, 2$$  

where the non-canonical operators $V_a$ have the form [9]

$$V^a \equiv (-)\varepsilon(Q^\alpha) \varepsilon^a Q^*_b \frac{\delta^r}{\delta Q^b}.$$

The nilpotency condition for $s^a$ (30) leads to the master equations (15). For our irreducible theory, the proper solution of the master eqs (15) till terms linear in the antifields is

$$S = S_0 + \int dt \left( q^*_ia^i_a Q^{\alpha a} + u^*_a (V^\alpha + G_{\beta}^{\delta a} Q^{\beta a}) Q^{\beta a} +
\right.
\left.
+ u^*_a (f^{\alpha}_{\beta} + G_{\delta} g^{\delta a}_{\beta}) u^\gamma Q^{\beta a} (-)^{\varepsilon} + Q^{\alpha}_{a b} \left( \varepsilon^{a b} Q^{\gamma a} + \frac{1}{2} (-)^{\varepsilon} (f^{\alpha}_{\beta} + G_{\delta} g^{\delta a}_{\beta}) Q^{\gamma b} Q^{\beta a} \right) +
\right.
\left.
+ \lambda^\alpha_a \frac{1}{2} (-)^{\varepsilon + 1} (f^\alpha_{\beta} + G_{\delta} g^{\delta a}_{\beta}) \lambda^\beta Q^{\gamma a} + \right.
\left.
\bar{\lambda}_{a} \left( a^{i}_{a} \lambda^{i} + \frac{1}{2} a^{i}_{a} \frac{\delta a^{i}_{a}}{\delta q^{b}} Q^{\alpha e} Q^{\beta b} \varepsilon_{bc} \right) +
\right.
\left.
+ \bar{\lambda}_{a} \frac{1}{2} (-)^{\varepsilon} (f^\alpha_{\beta} + G_{\delta} g^{\delta a}_{\beta}) (f^\sigma_{\rho} + G_{\delta} g^{\delta a}_{\rho}) \varepsilon_{cd} Q^{\alpha c} Q^{\beta d} +
\right.
\left.
+ \bar{\lambda}_{a} \lambda^{a} + \bar{\lambda}_{a} \left( (V^\alpha_{\beta} + G_{\delta} U^{\delta a}_{\beta}) + (-)^{\varepsilon} (f^\alpha_{\beta} + G_{\delta} g^{\delta a}_{\beta}) u^\gamma \right) \lambda^\beta +
\right.$$  

$\text{7}$
\[ Q_{aa} \alpha_a (-)^{\varepsilon_a + 1} (f^\alpha_\sigma + G_\delta g^\delta_\rho \rho^\sigma)(\lambda^\beta \xi^\alpha_a - \frac{1}{6} (f^\alpha_\gamma_\rho + G_\delta g^\delta_\gamma_\rho) \xi^\alpha_a Q^{\rho \sigma} Q^{\beta \sigma}). \]  (32)

On the basis of the graduation rules and Grassmann parities [8], [9] we can identify the following variables

\[ P_{aa} \equiv u^*_{aa}, \pi_\alpha \equiv \pi_\alpha. \]  (33)

These identifications will be very useful in the gauge fixing procedure [13]. On the basis of the identifications (33), we will have

\[ Y = f^\alpha(q^i) \pi_\alpha. \]  (34)

The following relations are valid

\[ q^i_\alpha = \frac{\delta Y}{\delta q^i} \left( \frac{1}{2} \varepsilon_{ab} V_b Y \right) = -\frac{\delta Y}{\delta q^i} f^\alpha(q^i) u^*_{aa}, \]  (35)

\[ \overline{q}_i = \frac{\delta Y}{\delta q^i} = -\frac{\delta Y}{\delta q^i} f^\alpha(q^i) \pi_\alpha \]  (36)

\[ u^\alpha = -\frac{\delta L}{\delta u^*_{aa}} \left( \frac{1}{2} \varepsilon_{ab} V_b Y \right) = f^\alpha(q^i) \]  (37)

the remaining antifields vanishing because of the choice (34) for \( Y \). The gauge fixed action will be

\[ S_1 Y = S_1 [Q^A, u^*_{aa}, \pi_\alpha, u^\alpha = f^\alpha(q^i), q^i_\alpha = -\frac{\delta Y}{\delta q^i} u^*_{aa}, \]  (38)

\[ \overline{q}_i = \frac{\delta Y}{\delta q^i} = -\frac{\delta Y}{\delta q^i} f^\alpha(q^i) \pi_\alpha]. \]

It leads to an effective action which is \( s_a \)-invariant and that can be further used in the path integral. This path integral can be written as

\[ Z^L_Y = \int DQ^A D u^*_{aa} D \pi_\alpha \exp(iS_1 Y). \]  (39)

We will introduce the condensed notations

\[ \phi^A \equiv \{ Q^A, u^*_{aa}, \pi_\alpha, u^\alpha = f^\alpha(q^i), q^i_\alpha = -\frac{\delta Y}{\delta q^i} u^*_{aa}, \]  (40)

\[ \pi_\alpha \} \}

and we will consider the following BRST transformations

\[ \phi^A \rightarrow \phi'^A = \phi^A - (s_a \phi^A) \mu_a (-1)^{f^A}. \]  (41)

where \( \mu_a \) are small fermionic constant parameters. The superjacobian of this transformations can be aproximate with supertrace (because it involves both fermionic and bosonic fields, the jacobian is a superdeterminant) and the new integrating measure will be

\[ D\phi'^A = \left( 1 - ((-1)^{f^A} \frac{\partial S}{\partial \phi^A} \frac{\partial S}{\partial \phi'^A} + V^a S) \mu_a \right) D\phi^A = (1 - (\Delta^a S) \mu_a) D\phi^A. \]  (42)

8
In the previous relation, the operators $\Delta^a, a = 1, 2$ have the form (16).

Exprimam $\Delta^a S$ using (29):

$$
\Delta^a S = \left( -\frac{\partial^r a^i_\beta}{\partial q^i} + \frac{1}{2} [(-1)^{\epsilon_\alpha (\epsilon_\beta + 1)} - (-)^{\epsilon_\alpha (\epsilon_\beta + 1)}] (f_\beta^\alpha + G_\delta^\delta_\beta_\alpha) \right) Q^{\delta a} +
$$

$$
\varepsilon^{ab} u_{ab}^* \left( -\delta^a \frac{d}{dt} - (V_\beta^a + G_\delta^\delta_\beta_\alpha + (-)^{\epsilon_\alpha} \frac{\partial^r f_\alpha^i(q)}{\partial q^i} a^i_\beta + (f_\beta^\alpha + G_\delta^\delta_\beta_\alpha) f_\gamma(q) \right) \lambda^\beta
$$

$$
-(-)^{\epsilon_\alpha} \frac{\partial^r f_\alpha^i(q)}{\partial q^i} \frac{1}{2} \partial^i_q \frac{\partial^i_\alpha}{\partial q^i} Q^{\beta b} \varepsilon_{bc} + \frac{1}{12} (-)^{\epsilon_\beta} (f_\beta^\alpha + G_\delta^\delta_\beta_\alpha) \varepsilon_{cd} Q^{\alpha a} Q^{\beta c} Q^{\beta d} - (-)^{\epsilon_\alpha} \varepsilon^{ab} u_{ab}^* \left( \frac{\delta^r}{\delta q^i} \left( \frac{1}{2} \varepsilon_{ab} V_\delta \delta^i \right) \right) +
$$

$$
+ \frac{1}{2} \partial^i_q \frac{\partial^i_\alpha}{\partial q^i} Q^{\beta c} \varepsilon_{bc} + \frac{1}{12} (-)^{\epsilon_\beta} (f_\beta^\alpha + G_\delta^\delta_\beta_\alpha) \varepsilon_{cd} Q^{\alpha a} Q^{\beta c} Q^{\beta d} - (-)^{\epsilon_\alpha} \varepsilon^{ab} u_{ab}^* \left( \frac{\delta^r}{\delta q^i} \left( \frac{1}{2} \varepsilon_{ab} V_\delta \delta^i \right) \right)
$$

The functions $G_\delta$ which appear in the previous relations are the constraints in which the real momenta was substitute (on the basis of their equations of motions) in function of $q, \dot{q}$. In the same way appear the functions $a^i_\alpha$ from

$$
\{q^i, G_\alpha\}.
$$

The antifields was eliminated by gauge fixing procedure (35) - (37) but considering the gauge fixing functional of the form $Y + \delta Y$. Then, $\Delta^a S = 0$ if

$$
\left( -\frac{\partial^r a^i_\beta}{\partial q^i} + \frac{1}{2} [(-1)^{\epsilon_\alpha (\epsilon_\beta + 1)} - (-)^{\epsilon_\alpha (\epsilon_\beta + 1)}] (f_\beta^\alpha + G_\delta^\delta_\beta_\alpha) \right) Q^{\delta a} = 0
$$

and if $\frac{\partial^r}{\partial q^i} \left( \frac{1}{2} \varepsilon_{ab} V_\delta \delta^i \right)$ and $\frac{\delta^r}{\delta q^i} \left( \frac{1}{2} \varepsilon_{ab} V_\delta \delta^i \right)$ anuleaza termenii din paranteza (cancel the terms) ce apar pe langa $\varepsilon^{ab} u_{ab}^*$. With other words, the measure remain invariant ($\Delta^a S = 0$) to the transformation (11) if $\delta Y$ induced by this transformation in the gauge fixing functional indeplineste cerintele de mai sus. If $\Delta^a S \neq 0$, the operators $\Delta^a, a = 1, 2$ determine the $sp(2)$- QME in the $sp(2)$ symmetric formulation of gauge theories, where the action $S_{Y + \delta Y}$ passes to a new "quantum action", $W$, containing the $h$—order corrections and satisfying the equations (18).

4 Conclusions

The problem of the quantum extension of the BRST procedure has been investigated, both in the BFV and in the BV formalisms. The main questions arising in our study were: What is the form of the quantum master equation? How is looking and what is the meaning of the quantum action? How is it possible to extend the standard BRST quantization procedure in order to obtain $sp(2)$ or $sp(3)$ quantum master equations?
We tried to offer comprehensible answers to all these questions and to illustrate them by considering the case of nonlinear gauge theories which mix bosonic and fermionic fields and whose gauge superalgebra is defined by nonlinear relations of the form (21), (22). For such a special system considered in [14] we implemented the Hamiltonian $sp(2)$ formalism and we pointed out the condition for which the path integrals are BRST invariant and how the gauge fixing function has to be chosen in order to assure the invariance when it is not intrinsic for the model.

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Description of Nonlinear Phenomena in the Atmospheric Dynamics through Linear Wave type Equations

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Abstract

The paper tackles with a procedure which allow to extend some linear, wave type models currently used in describing phenomena appearing in atmosphere to the study of nonlinear models. More concretely, we present a practical way to generate the largest class of \((1 + 1)\)-dimensional second order partial differential equations (pdes) of a given form which could be reduced to an imposed ordinary wave type equation. This class generalize the ordinary differential equation describing the equatorial trapped waves generated in a continuously stratified ocean and will be obtained following the Lie symmetry and similarity reduction procedures. Moreover, some concrete nonlinear second order differential equations will be proposed as possible candidates for replacing more complicated, nonintegrable systems, as the Rossby type equation.

Keywords: Nonlinear dynamical systems, Lie symmetries, Similarity reduction procedure, Rossby type symmetries.

1 Introduction

A rich variety of complex phenomena occurring in many physical fields, including the atmospheric dynamics, are described by linear differential equations which allow a simple handling of the constraints which appear in the system’s evolution. Although, the linearized models often do not adequately describe the dynamics of the processes as a whole and it is very simple to shift the system to a region in which the linear behavior is no longer valid. This is why, in order to capture the real behavior, to accurately estimate and control the complex systems in all their regimes, nonlinear models must be defined. In this case, the linear differential equations could appear as approximations to the nonlinear systems, valid under restricted conditions.

The price to be paid when nonlinearity is taken into consideration appears in the investigation of the exact solutions of the attached equations. There are not standard methods of solving nonlinear differential equations, they are usually depend on the form of the equations and on their particular symmetries. Many interesting nonlinear models have been proposed over the last years [1] and a lot of methods have been developed in order to find solutions of equations describing these nonlinear phenomena. Some of the most important methods [7] are the inverse scattering method [8], the Darboux and Bäcklund transformations [9], the Hirota bilinear method [10], the Lie symmetry analysis [11, 12, 13], etc. By applying these methods, many types of specific solutions have been obtained. For example, solitary waves or solitons, which have no analogue for linear partial differential equations, are very important for the nonlinear dynamical systems.

In this paper we shall concentrate our attention to the Lie group method. It is well-known that this method is a powerful and direct approach to construct many types of exact solutions of nonlinear differential equations, such as soliton solutions, power series solutions, fundamental solutions [14, 15], and so on. The existence of the operators associated with the Lie group of infinitesimal transformations allows the reduction of equations to simpler ones. The similarity reduction method for example is an important way of transforming a \((1 + 1)\)–dimensional pde into an ordinary differential one. We shall concretely consider the inverse symmetry problem [16] and
we shall generate the largest class of second order \((1 + 1)\)–dimensional pdes which generalize the ordinary, wave type, differential equation describing the equatorial trapped waves generated in a continuously stratified ocean. Practically, four types of waves appears in this case and have to be found among the solutions of the equation: Kelvin waves, Rossby waves, inertia-gravity waves and mixed Rossby-gravity waves. In the Boussinesq approximation and on an equatorial \(\beta\)-plane, the equation which describe the \(m\)-th oscillation mode of the wave’s vertical velocity \(\phi_m(z)\) on the direction \(z\) has the form \([23]\):

\[
\frac{\partial^2 \phi_m(z)}{\partial z^2} + \frac{N_m^2(z)}{C_m^2} \phi_m(z) = 0
\] (1)

One consider for the velocity the boundary conditions:

\[
\phi_m(z = -H) = 0 \text{ (ocean floor)}
\]
\[
\phi_m(z = 0) = 0 \text{ (ocean surface)}
\]

In the equation (1), \(C_m\) is a constant and \(N_m(z)\) represents the "buoyancy" frequency. Measurements made during El Niño events by 2 Japanese stations in the equatorial Pacific \([23]\) show that the buoyancy frequency \(N_m(z)\) has strong variations with the water depth close to the surface and practically vanishes for higher depths. We notice that in the first case (at the surface, \(z \in [0, 300] \text{ m}\)) one can approximate \(N_m(z)\) with an averaged value around \(N(z) = 2 \cdot 10^{-4} \text{ m/s}^{-1}\) So, the equation (1) can be linearized in one of the following forms:

\[
\ddot{\phi}(z) = 0; z \geq 300
\] (2)

\[
\ddot{\phi}(z) + k^2 \phi(z) = 0; z \in [0, 300]; k \equiv \frac{N}{C} = \text{const.}
\] (3)

In this paper, we shall consider the two previous wave type equations and we shall see how they can be extended towards \((1 + 1)\)–second order differential equations with the same group of symmetry as the initial ordinary wave type equations have.

The outline of this paper is as follow: after this introductory notes, in Section 2, we shall obtain the general determining system for a chosen class of \((1+1)\)–dimensional models. The system will be generated by using the Lie symmetry approach and by asking for an imposed form of the similarity reduction equation. More exactly, we shall generate a class of \((1 + 1)\)–second order differential equations which by similarity reduction come to the wave forms \((2)\) and, respectively, \((3)\). The general results of the second section will be particularized in Section 3, when concrete examples of two dimensional equations with similar solutions as the ordinary wave equations \((2)\) and \((3)\) will be generated. Moreover, we shall compute the form of the second order partial differential equation which admit an imposed form of symmetry, specific for the two dimensional Rossby type equation. So we shall be able to replace the study of this last strongly nonintegrable equation with a simpler class of equations observing similar symmetries. Some concluding remarks will end the paper.

2 Determining equations for the Lie symmetry group

Let us consider the class of general dynamical systems described in a \((1 + 1)\)-dimensional space \((x, t)\) by a second order partial differential equation of the form:

\[
u_t = A(x, t)\nu_{2x} + B(x, t)\nu_x + C(x, t)\nu \Rightarrow \Omega(x, t, u, \nu_x, \nu_t, \nu_{2x}) = 0
\] (4)

Our aim is to select from the general dynamical systems described by \((4)\) the class of differential equations which admit a similarity reduction to wave type equations of the form \((2)\) and \((3)\). As feed-back, the solution of the wave equations will be used in order to obtain a solution for \((4)\). The
procedure that will be followed firstly implies to obtain the system of determining equations for the Lie symmetry group of (4). Then, an additional system of partial differential equations will be generated by imposing that (4) possess a reduced similarity equation of the wave type. Finally, this last system and the Lie determining equations will be solved and coefficient functions $A(x,t)$, $B(x,t)$, $C(x,t)$ will be obtained.

In this section we shall apply the Lie symmetry approach for the equation (4). Let us consider a one-parameter Lie group of infinitesimal transformations:

$$\bar{x} = x + \varepsilon \xi(t,x,u), \quad \bar{t} = t + \varepsilon \varphi(t,x,u), \quad \bar{u} = u + \varepsilon \eta(t,x,u)$$

with a small parameter $\varepsilon \ll 1$. The Lie symmetry operator associated with the above group of transformations can be written as follows:

$$U(x,t,u) = \varphi(t,x,u) \frac{\partial}{\partial t} + \xi(x,t,u) \frac{\partial}{\partial x} + \eta(x,t,u) \frac{\partial}{\partial u}$$

The second order equation $\Omega(x,t,u,u_x,u_t,u_{2x}) = 0$ of the form (4) is invariant under the action of the operator (5) if and only if the following condition [11] is verified:

$$U^{(2)}(\Omega) |_{\Omega=0} = 0$$

where $U^{(2)}$ is the second extension of the generator (6). A concrete computation shows that the coefficient functions from (11) and (6), $A(x,t), B(x,t), C(x,t), \varphi(x,t,u), \xi(x,t,u), \eta(x,t,u)$, must satisfy the equation:

$$(\varphi A_t + \xi A_x)u_{2x} + (\varphi B_t + \xi B_x)u_x + \varphi C_t u + \xi C_x u + C \eta + B \eta^x - \eta^t + A \eta^{2x} = 0$$

Coefficient functions $\eta^x$, $\eta^t$, $\eta^{2x}$ appear in the process of extension of $U$ towards $U^{(2)}$ and their general expressions are given in (11). Using these expressions in (8) and asking for the vanishing of the coefficients of each monomial in the derivatives of $u(t,x)$, we obtain the following differential system:

$$\varphi_x = 0; \quad \varphi_u = 0; \quad \xi_u = 0; \quad \eta_{2u} = 0; \quad \varphi A_t + \xi A_x + A \varphi_t - 2A \xi_x = 0;$$
$$- \varphi B_t - \xi B_x + B \xi_x - \xi_t - B \varphi_t - 2A \eta_{xu} + A \xi_{2x} = 0$$
$$- \varphi C_t u - \xi C_x u - C \eta - B \eta_{xu} + \eta_t + C \eta_u - \varphi C u - A \eta_{2x} = 0$$

The first four equations of the system (9) lead, for coefficient functions $\varphi(t,u), \xi(x,t,u), \eta(x,t,u)$, to the following reduced dependences:

$$\varphi = \varphi(t), \quad \xi = \xi(x,t), \quad \eta = M(x,t)u$$

Consequently, the remaining equations of (9) become:

$$\varphi A_t + \xi A_x + A \varphi_t - 2A \xi_x = 0;$$
$$- \varphi B_t - \xi B_x + B \xi_x - \xi_t - B \varphi_t - 2AM_x + A \xi_{2x} = 0$$
$$- \varphi C_t - \xi C_x - BM_x + M_t - \varphi C - AM_{2x} = 0$$

with 6 unknown functions: $A(x,t), B(x,t), C(x,t)$ provided by the evolutionary equation (11) and $\varphi(t), \xi(x,t), \eta(x,t,u)$ introduced by the symmetry group of transformations (5) and described by the relations (10).
3 Similarity reduction procedure

Let us consider now the similarity reduction procedure. In this section, some particular choices for the system \((\text{11})\) will be considered. We shall find equations describing concrete dynamical systems which admit reduction through the similarity procedure to ordinary wave type equations of the form \((\text{2})\) and \((\text{3})\). For the moment, we restrict the forms \((\text{10})\) of the infinitesimals \(\varphi(t), \xi(x,t), \eta(x,t,u)\) to the following separable expressions:

\[
\varphi = \varphi(t), \quad \xi = \xi(x,t) = \xi_1(x)\xi_2(t), \quad \eta = M(x,t)u = M_1(x)M_2(t)u
\]  

(12)

The Lie operator \((\text{6})\) becomes:

\[
U(x,t,u) = \varphi(t)\frac{\partial}{\partial t} + \xi_1(x)\xi_2(t)\frac{\partial}{\partial x} + M_1(x)M_2(t)u\frac{\partial}{\partial u}
\]  

(13)

The general expressions of the invariants could be found if we should consider the characteristic equations associated with the new generator \((\text{13})\). These equations are:

\[
\frac{dt}{\varphi(t)} = \frac{dx}{\xi_1(x)\xi_2(t)} = \frac{du}{M_1(x)M_2(t)u}
\]  

(14)

By integrating the previous equations, two invariants are obtained with the following expressions:

\[
I_1 = \exp\left(\int \frac{1}{\xi_1(x)} dx - \int \frac{\xi_2(t)}{\varphi(t)} dt\right), \quad I_2 = u\exp\left(-\int \frac{M_2(t)}{\xi_2(t)} \int \frac{M_1(x)}{\xi_1(x)} dx\right)
\]  

(15)

In the similarity reduction procedure two similarity variables have to be considered:

\[
I_1 = z, \quad I_2 = \phi(z)
\]  

(16)

The invariants \((\text{16})\) allow us, by an appropriate change of coordinates \(\{u, x, t\} \rightarrow \{z, \phi(z)\}\), to reduce the initial \((1 + 1)\) dimensional equation \((\text{11})\) to an ordinary differential equation of the form:

\[
\Omega'[z, \phi(z), \dot{\phi}(z), ...] = 0
\]  

(17)

3.1 Homogeneous wave type equation

Our aim is now to select from the general dynamical systems described by \((\text{11})\) the class of differential equations for which the equation \((\text{17})\) can be reduced at a wave type equation of the form \((\text{2})\):

\[
\frac{d^2\phi(z)}{dz^2} = 0 \Leftrightarrow \phi(z) = az + b
\]  

(18)

where \(a\) and \(b\) are arbitrary constants.

The previous solution, written in terms of the initial variable \((x, t)\), leads to the following form of the solution \(u(x,t)\) of \((\text{11})\):

\[
u(x,t) = \left[a \exp\left(\int \frac{1}{\xi_1(x)} dx - \int \frac{\xi_2(t)}{\varphi(t)} dt\right) + b\right] \exp\left(-\int \frac{M_2(t)}{\xi_2(t)} \int \frac{M_1(x)}{\xi_1(x)} dx\right)
\]  

(19)

For convenience reasons, we shall impose the following relations to be valid:

\[
\frac{\xi_2(t)}{\varphi(t)} = q, \quad \frac{M_2(t)}{\xi_2(t)} = v, \quad \int \frac{1}{\xi_1(x)} dx \equiv P(x), \quad \int \frac{M_1(x)}{\xi_1(x)} dx \equiv R(x)
\]  

(20)

with \(q, v\) arbitrary constants.
In terms of notations (20), the infinitesimals (12) and the solution (19) become:

\[
\varphi = \varphi(t), \quad \xi = A \frac{\varphi(t)}{P(x)}, \quad \eta = qv \varphi(t) \frac{\dot{R}(x)}{P(x)} u
\]  
(21)

\[
u(x, t) = [a \exp(P(x) - qt) + b] \exp(\nu R(x))
\]  
(22)

The solution (22) must verify the equation (4) which describes the analyzed model. This condition generates a differential system of the form:

\[
0 = q + 2vA(x, t) \dot{R}(x) \dot{P}(x) + v^2A(x, t)[\dot{R}(x)]^2 + A(x, t) \dot{P}(x) + A(x, t)[\dot{P}(x)]^2 + vA(x, t) \ddot{R}(x) + B(x, t) P(x) + vB(x, t) \dot{R}(x) + C(x, t)
\]

\[
0 = v^2A(x, t)[\dot{R}(x)]^2 + vA(x, t) \ddot{R}(x) + vB(x, t) \dot{R}(x) + C(x, t)
\]  
(23)

For an unitary analysis, it is necessary to describe the differential system (14), obtained in the previous subsection, in terms of the functions \(P(x)\) and \(R(x)\) introduced by (20). Using the expressions (21) we obtain the following differential system:

\[
\varphi A_x P_x^2 + q \varphi A_x P_x + \varphi_t A^2 + 2q \varphi A P_{2x} = 0
\]

\[
\varphi B_x P_x^2 + q \varphi B_x P_x + \varphi B P_{x2} P_x^2 + q \varphi_t P_x^3 + \varphi_t B P_x^3 + 2vq \varphi A R_{x2} x^2 - 2vq \varphi A R_{x2} P_{x2} + q \varphi A P_{3x} P_x^2 - 2q \varphi A P_{2x} = 0
\]  
(24)

\[
\varphi C_t P_x^4 + q \varphi C_x P_x^2 + q \varphi B R_{x2} P_x^3 - q \varphi B R_{x2} P_{x2} P_x^2 + \varphi_t C P_x^4 + q \varphi A R_{3x} P_x^3 - 2q \varphi A R_{x2} P_{x2} P_x^2 + q \varphi A R_{2x} P_{x2} P_x^2 - q \varphi R_{x2} P_x^3 = 0
\]

**Conclusion:** Our problem is to find the class of \((1 + 1)\) evolutionary equations of type (4) which could be reduced by the similarity approach to an ordinary wave type equation. Solving this problem is equivalent with searching the solutions of the system described by equations (23) and (24).

**Remark 1:** The system (23, 24) can be solved following two paths: (i) by choosing a concrete dynamical system, that is to say concrete expressions for the functions \(A(x, t), B(x, t), C(x, t)\) and trying to find out if this equation admits or not solution of the type (22). Now the unknown functions of the system are \(\varphi(t), P(x), R(x)\) defined by (20); (ii) by considering \(A(x, t), B(x, t), C(x, t)\) as unknown functions and by choosing \(\varphi(t), P(x), R(x)\). This is the way we shall follow in the next section.

**Remark 2:** In the case (ii) the general solutions obtained by computational way can be expressed as:

\[
A(x, t) = F \left( \frac{qt - P(x)}{q} \right) \exp[G(x)]
\]

\[
B(x, t) = \left[ -2F \left( \frac{qt - P(x)}{q} \right) \right] \left( \frac{1}{2} [\dot{P}(x)]^2 + v \dot{R}(x) \dot{P}(x) + \frac{1}{2} \ddot{P}(x) \right) \exp[-G(x)] - q
\]

\[
C(x, t) = v \left[ F \left( \frac{qt - P(x)}{q} \right) \right] \left[ \dot{R}(x) \dot{P}(x) + \dot{P}(x)(-\dot{R}(x) + \dot{R}(x)(v \dot{R}(x) + \ddot{P}(x))) \right] \exp[-G(x)] + q \dot{R}(x)
\]  
(25)

where

\[
G(x) = -\int^x 2(D(2))(P)(a)\varphi \left( \frac{P(a) + qt - P(x)}{q} \right) q + [D(P)(a)] \varphi \left( \frac{P(a) + qt - P(x)}{q} \right) q
\]

\[
\int D(P)(a) \varphi \left( \frac{P(a) + qt - P(x)}{q} \right) q
\]  
(26)

These solutions are valid for arbitrary constants \(q, v\) and for an arbitrary function \(F \left( \frac{qt - P(x)}{q} \right) \).
3.2 Harmonic oscillators

Let us consider now that the similarity reduction equation is an ordinary oscilator type equation of the form (3):

\[
\frac{d^2 \phi(z)}{dz^2} + k^2 \phi(z) = 0
\]  

(27)

It has the solution:

\[
\phi(z) = a \sin(kz) + b \cos(kz)
\]  

(28)

Here \(k, a\) and \(b\) are arbitrary constants.

To write down the previous solution in terms of the initial variable \(u(x,t)\) means that the solution of (4) should have the form:

\[
u(x,t) = \exp \left( \int M_1(x) \frac{\xi_1(x)}{\xi_1(t)} dx - \int \frac{\xi_2(t)}{\varphi(t)} dt \right) \sin \left( \sqrt{k} \exp \left( \int M_1(x) \frac{\xi_1(x)}{\xi_1(t)} dx - \int \frac{\xi_2(t)}{\varphi(t)} dt \right) \right) + (29)
\]

For convenience reasons, we shall impose again the following relations to be valid:

\[
\frac{\xi_2(t)}{\varphi(t)} = q, \quad \frac{M_2(t)}{\xi_2(t)} = v, \quad \int M_1(x) \frac{\xi_1(x)}{\xi_1(t)} dx = P(x), \quad \int M_1(x) \frac{\xi_1(x)}{\xi_1(t)} dx = R(x)
\]  

(30)

with \(q, v\) arbitrary constants.

In terms of notations (30), the infinitesimals (12) and the solution (29) become:

\[
\varphi = \varphi(t), \quad \xi = q \varphi(t) \frac{\varphi(t)}{P(x)}, \quad \eta = q v \frac{\varphi(t) R(x)}{P(x)}
\]  

(31)

\[
u(x,t) = [a \sin(k \exp (P(x) - qt)) + b \cos(k \exp (P(x) - qt))] \exp (v R(x))
\]  

(32)

The solution (32) must verify the equation (4) which describes the analyzed model. This condition generates the vanishing of the coefficient function \(A(x,t)\) and two other differential equations of the form:

\[
A(x,t) = 0
\]  

\[
q + B(x,t) P_t(x) = 0
\]  

\[
v B(x,t) \dot{R}(x) + C(x,t) = 0
\]  

(33)

For an unitary analysis, it is again necessary to describe the general differential system (11) obtained in the previous section, in terms of the functions \(P(x)\) and \(R(x)\) introduced by (30). Taking into account the equations (31), we obtain the following differential system:

\[
0 = \varphi B_t P^4_x + q \varphi B_x P^3_x + q \varphi B P^2_{2x} P^2_x + q \varphi^2 P^3_x + \varphi B P^4_x
\]

\[
0 = \varphi C_t P^4_x + q \varphi C_x P^3_x + q \varphi B R_{2x} P^3_x - q \varphi B R_x P^2_{2x} P^2_x +
\]

\[
+ \varphi C P^4_x - q \varphi R_x P^3_x
\]  

(34)

The system (33)-(34) can be solved following two paths: \(i\) by choosing a concrete dynamical system, that is to say concrete expressions for the functions \(B(x,t), C(x,t)\) and trying to find out
if this equation admits or not solution of the type \( \Box[32] \). Now the unknown functions of the system are \( \varphi(t), P(x), R(x) \) defined by \( \Box[30] \); (ii) by considering \( B(x,t), C(x,t) \) as unknown functions and by choosing \( \varphi(t), P(x), R(x) \).

This second case is the way we are interested in to follow and, in this case, the general solutions obtained by computational way can be expressed as:

\[
B(x,t) = \frac{-q}{P(x)}, \quad C(x,t) = \frac{vq\dot{R}(x)}{P(x)}
\]  
(35)

or in terms of the coefficient functions \( \varphi(t), \xi(x,t), M(x,t) \) which appear in the general Lie symmetry operator \( \Box[??] \), in the equivalent forms:

\[
B(x,t) = \frac{-\xi(x,t)}{\varphi(t)}, \quad C(x,t) = \frac{M(x,t)}{\varphi(t)}
\]

### 3.3 Rossby type symmetries

The equation for coupled gravity, inertial and Rossby waves in a rotating, stratified atmosphere using the \( \beta \)-plane approximation (which simplifies the spherical geometry whilst retaining the essential dynamics) and the Boussinesq approximation which filters out higher frequency acoustic waves can be written in \((2 + 1)\)-dimensions in the form \( \Box[24] \):

\[
\frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u_g = -\beta \frac{\partial u_g}{\partial x}
\]  
(36)

As we mentioned, this equation describes the coupling between the inertial, the gravity and the Rossby waves, but also the shallow water in an ocean of depth \( H \). It was proven \( \Box[25] \) that reducing the model to \((1 + 1)\)-dimensions, \((x,t)\), it admits some very simple Lie symmetries of the form:

\[
\varphi(t) = ct + c_1, \quad \xi(x,t) = cx + f(t), \quad \eta(u) = -3cu
\]  
(37)

with \( f(t) \) arbitrary function and \( c_0, c \) arbitrary constants. Despite this fact, it is still difficult to find explicite solutions for the equation \( \Box[36] \). This is why, we shall consider another approach: we shall impose the Lie symmetries \( \Box[37] \) to our general equation \( \Box[11] \) and we shall try to find the class of equations which observe them. This means that we have in fact to impose the Rossby symmetries \( \Box[37] \) to the system \( \Box[11] \). It will take the form:

\[
\begin{align*}
(ct + c_1)A_t + (cx + c_2)A_x + 3cA &= 0 \\
-(ct + c_1)B_t - (cx + c_2)A_x - 2cB &= 0 \\
-(ct + c_1)C_t - (cx + c_2)C_x - cC &= 0
\end{align*}
\]

with the unknown functions \( A(x,t), B(x,t), C(x,t) \).

This system admits the solutions:

\[
\begin{align*}
A(x,t) &= \frac{F(x(\varphi(t) - c_2t))}{(ct + c_1)^3} = \frac{F(x\varphi(t) - c_2t)}{[\varphi(t)]^3} \\
B(x,t) &= \frac{G(x(\varphi(t) - c_2t))}{(ct + c_1)^2} = \frac{G(x\varphi(t) - c_2t)}{[\varphi(t)]^2} \\
C(x,t) &= \frac{H(x(\varphi(t) - c_2t))}{ct + c_1} = \frac{H(x\varphi(t) - c_2t)}{\varphi(t)}
\end{align*}
\]

with \( F, G, H \) arbitrary functions of their arguments. These expressions give us equations of the form \( \Box[11] \) which are equivalent from the point of view of their symmetries with the Rossby equation.
4 Conclusions

The problem of finding exact solutions for nonlinear differential equations plays an important role in the study of nonlinear dynamics. There are many ways of tackling with it. One of them is based on the Lie symmetry method. This method supposes to find the symmetries of the system and, on this basis, to try to determine the general or some particular solutions of the equations. There is a direct approach in which the symmetries of a given equation are obtained, but also an inverse problem has been formulated [10]. A step forward for this latter approach is represented by the use of similarity reduction, a procedure which allows the reduction of the number of degrees of freedom and, by that, simplifies the problem of solving the equation. This paper used this approach and determined a class of (1 + 1) dimensional second order differential equations which can be reduced to ordinary wave-type equations with simple solutions. Using the Lie symmetry and the similarity reduction procedures, some particular cases of the equation (11) arise as good candidates of equations which could be used as generalization of the linear wave type equations describing complex atmospheric phenomena. Moreover, following our method, we were able to write down the solutions of these equations, solutions which otherwise could be derived by computational methods, but in a very complicated form. Another interesting results of our paper consisted in the fact that a complicated, nonintegrable equation, the Rossby equation, could be replaced by another, simpler equation, which have similar symmetries. The paper is important both by these results, but also as a methodological approach in finding exact solutions through similarity reduction procedure. We have shown how, starting from a particular form of solution for the reduced equation we could recover the solution of a most complicated problem, defined in a space with more than one dimensions. We tackled out a particular case, looking only for linear solutions of the reduced equation and considering that the coefficient functions appearing in the symmetry operators are separable. The problem can be extended for other cases, too.

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