Bridgeland Stability of Line Bundles on Surfaces

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Abstract

We study the Bridgeland stability of line bundles on surfaces using Bridgeland stability conditions determined by divisors. We show that given a smooth projective surface $S$, a line bundle $L$ is always Bridgeland stable for those stability conditions if there are no curves $C \subseteq S$ of negative self-intersection. When a curve $C$ of negative self-intersection is present, $L$ is destabilized by $L(-C)$ for some stability conditions. We conjecture that line bundles of the form $L(-C)$ are the only objects that can destabilize $L$, and that torsion sheaves of the form $L(C)|_C$ are the only objects that can destabilize $L[1]$. We prove our conjecture in several cases, and in particular for Hirzebruch surfaces.

Contents

1 Introduction 2

2 Bridgeland Stability Conditions 4
  2.1 General Definition of Bridgeland Stability Conditions . 4
  2.2 Bridgeland Stability Conditions on a Surface . . . 5
  2.3 Slices of $\text{Stab}_{\text{div}}(S)$ . . . . . . . . . . . . . . . . . . . 6

3 Reduction to the case of $\mathcal{O}_S$ 7

4 Preliminaries on the Stability of $\mathcal{O}_S$ 9
  4.1 Subobjects of $\mathcal{O}_S$ and their walls . . . . . . . . . . . . . . . . 9
  4.2 Bridgeland Stability of $\mathcal{O}_S$ for $t >> 0$ . . . 13
  4.3 Bertram’s Lemma . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 14
1 Introduction

Let $S$ be a smooth projective surface. In this paper, we study Bridgeland stability for line bundles on $S$ using the geometric Bridgeland stability conditions introduced in [AB13] (see Section 2.2 for a precise definition). Bridgeland stability conditions can be seen as an extension of Mumford $\mu$-stability for sheaves to complexes of sheaves in the derived category, $D^b(\text{Coh} \ S)$. Line bundles are always Mumford slope-stable, as their only subobjects are ideal sheaves, but the situation is less constrained in the derived setting. For example, in the abelian subcategories we consider, a subobject of a line bundle is a sheaf, but may a priori have arbitrarily high rank. The quotient is a possibly two-term complex.

One might still expect line bundles to always be Bridgeland stable, and this is correct if $S$ has no curves $C$ of negative self-intersection (see the first part of Theorem 1.1 below). However, if there exists a curve $C$ on $S$ of negative self-intersection, then $L(-C)$ destabilizes $L$ for some Bridgeland stability conditions, and $L(C)|_C$ destabilizes $L[1]$. We make the following conjecture.

**Conjecture 1.** Given a surface $S$ and a stability condition $\sigma_{H,D}$ as in [AB13],

- the only objects that could destabilize a line bundle $L$ are line bundles of the form $L(-C)$ for a curve $C$ of negative self-intersection, and

- the only objects that could destabilize $L[1]$ are torsion sheaves of the form $L(C)|_C$ for a curve $C$ of negative self-intersection.
The goal of this paper is to prove the conjecture in several cases, and provide evidence for the conjecture for others. Specifically, we prove the following.

**Theorem 1.1.** The conjecture is true in the following cases:

- If $S$ does not have any curves of negative self-intersection.
- If the Picard rank of $S$ is 2, and there exists only one irreducible curve of negative self-intersection.

In particular, the conjecture is true for Hirzebruch surfaces. We cite Propositions 5.5 and 5.10 as further evidence for our conjecture in general. Proposition 5.5 establishes some structure of actually destabilizing subobjects for line bundles and their walls for surfaces of any Picard rank, while Proposition 5.10 proves a stronger version of the conjecture for a subset of stability conditions when $S$ has Picard rank 2 and two irreducible curves of negative self-intersection.

In [AB13] the stability of line bundles is proven for stability conditions $\sigma_{D,H}$ with $D = sH$, and is utilized in [ABCH13] and [BC] while classifying destabilizing walls for ideal sheaves of points on surfaces. When $D \neq sH$ the more algebraic proof of [AB13] using the Bogomolov inequality and Hodge Index Theorem fails and new techniques are required. We use Theorem 3.1 (Bertram’s Nested Wall Theorem) from [Mac] and Lemma 6.3 (which we refer to as Bertram’s Lemma) from [ABCH13] along with an analysis of the relative geometry of relevant walls in certain three-dimensional slices of the space of stability conditions. This technique lends itself well to induction, which is the primary method of proof used here.

We begin in Section 2 by introducing Bridgeland stability conditions, the stability conditions $\sigma_{D,H}$ of interest, as well as important slices of the space of stability conditions. In Section 3 we present an action by line bundles which allows us to consider only $\mathcal{O}_S$ in our questions of the stability of line bundles. In Section 4 we consider the basic structure of subobjects of $\mathcal{O}_S$ as well as present two already known results which will serve as important tools in the remainder. In Section 5 we prove our main results, but first consider the rank 1 subobjects of $\mathcal{O}_S$. The rank 1 subobjects form the base case for our main results, all of which use induction. The case of $\mathcal{O}_S[1]$ is then completed primarily using duality, which allows us to use our results for $\mathcal{O}_S$ except when $D.H = 0$. 

3
Acknowledgements

We would like to thank Arend Bayer, Cristian Martinez, and especially Aaron Bertram and Renzo Cavalieri for many useful discussions.

2 Bridgeland Stability Conditions

Bridgeland stability conditions (introduced in [Bri07]) give a notion of stability on the derived category of a variety. They generalize other classical notions of stability conditions, e.g., Mumford-slope stability. As with slope stability, we may deform our stability conditions (Bridgeland showed that the space of all stability conditions is a complex manifold) and the stability of objects can change. We first introduce these stability conditions in general, and then restrict our attention to surfaces in the next section. Our goal is to study Bridgeland stability of line bundles.

2.1 General Definition of Bridgeland Stability Conditions

Let $X$ be a smooth projective variety, $D(X) = D^b(\text{Coh}S)$ the bounded derived category of coherent sheaves on $X$, $K(X)$ its Grothendieck group, and $K_{\text{num}}(X)$ its quotient by the subgroup of classes $F$ such that $\chi(E, F) = 0$ for all $E \in D(X)$.

Definition. A full numerical stability condition on $X$ is a pair $\sigma = (Z, A)$ where

- $A$ is a heart of $D(X)$
- $Z : K_{\text{num}}(X) \rightarrow \mathbb{C}$ a group homomorphism called the central charge

satisfying properties 1, 2 and 3 below.

1 (Positivity) For all $0 \neq E \in A$, $Z(E) \in \{ re^{i\varphi} \mid r > 0, 0 < \varphi \leq 1 \}$.

To discuss stability for a given stability condition, we define for each $E \in D(X)$

$$\beta(E) = -\frac{\text{Re} Z(E)}{\text{Im} Z(E)} \in (-\infty, \infty]$$
For example, if \( Z(E) = -1 \) then \( \beta(E) = \infty \), and if \( Z(E) = \sqrt{-1} \) then \( \beta(E) = 0 \).

We say that \( E \in \mathcal{A} \) is \( \sigma \)-stable (resp. \( \sigma \)-semistable) if for all nontrivial \( F \hookrightarrow E \) in \( \mathcal{A} \) we have \( \beta(E) > \beta(F) \) (resp. \( \beta(E) \geq \beta(F) \)).

2 (Harder-Narasimhan Filtrations) For all \( E \in \mathcal{A} \) there exist objects \( E_1, \ldots, E_{n-1} \in \mathcal{A} \) such that

- \( 0 = E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_{n-1} \hookrightarrow E_n = E \) in \( \mathcal{A} \)
- \( E_{i+1}/E_i \) is \( \sigma \)-semistable for each \( i \)
- \( \beta(E_1/E_0) > \beta(E_2/E_1) > \cdots > \beta(E_n/E_{n-1}) \)

3 (Support Property) Choose a norm \( \| \cdot \| \) on \( K_{num}(X) \otimes \mathbb{R} \). There exist a \( C > 0 \) such that for all \( \sigma \)-semistable \( E \in D(X) \) we have \( C\|E\| \leq |Z(E)| \).

Remark 2.1. The support property guarantees us a nicely behaved wall and chamber structure for classes of objects - namely, the walls are locally finite, real codimension 1 submanifolds of the stability manifold and deleting the walls gives chambers where Bridgeland stability is constant (see [BM11, Proposition 3.3]). The support property is equivalent to Bridgeland’s notion of full, see [BM11, Proposition B.4].

We will say stability condition to mean full numerical stability condition.

2.2 Bridgeland Stability Conditions on a Surface

Let \( S \) be a smooth projective surface. The stability conditions that we are going to consider were defined in [ABI13]. They form a subset \( \text{Stab}_{\text{div}}(S) \) of stability conditions that depend on a choice of ample and general divisor (the “div” stands for “divisor”), and are well suited to computations. Let us recall their definition.

Let \( S \) be a smooth projective surface. Given two \( \mathbb{R} \)-divisors \( D, H \) with \( H \) ample, we define a stability condition \( \sigma_{D,H} = (Z_{D,H}, \mathcal{A}_{D,H}) \) on \( S \) as follows:

Consider the \( H \)-Mumford slope

\[
\mu_H(E) = \frac{c_1(E).H}{\text{rk}(E)H^2}.
\]
Let $\mathcal{A}_{D,H}$ be the tilt of the standard $t$-structure on $D(S)$ at $\mu_H(D) = \frac{D.H}{H^2}$ defined by $\mathcal{A}_{D,H} = \{ E \in D(S) \mid H^i(E) = 0 \text{ for } i \neq -1, 0, \text{ and } H^{-1}(E) \in \mathcal{F}_{D,H}, \ H^0(E) \in T_{D,H} \}$ where

- $T_{D,H} \subset \text{Coh}(S)$ is generated by torsion sheaves and $\mu_H$-stable sheaves $E$ with $\mu_H(E) > \frac{D.H}{H^2}$.
- $\mathcal{F}_{D,H} \subset \text{Coh}(S)$ is generated by $\mu_H$-stable sheaves $F$ with $\mu_H(F) \leq \frac{D.H}{H^2}$.

Now define $Z_{D,H}$ by $Z_{D,H}(E) = -\int e^{-(D+H)i} \text{ch}(E)$. It is equal to

$$Z_{D,H}(E) = \left( -\text{ch}_2(E) + c_1(E).D - \frac{\text{rk}(E)}{2}(D^2 - H^2) \right) + i( c_1(E).H - \text{rk}(E)D.H )$$

By [AB13, Corollary 2.1] and [Tod13, Sections 3.6 & 3.7], $\sigma_{D,H}$ is a stability condition on $S$. Let $\text{Stab}_{\text{div}}(S)$ be the set of all such stability conditions. By the support property, $\text{Stab}_{\text{div}}(S) \cong (\text{Amp}(S) \oplus \text{Pic}(S))_R$ is a submanifold of the space of all stability conditions.

**Remark 2.2.** These are geometric stability conditions since for all $p \in S$, the skyscraper sheaf $\mathcal{C}_p \in \mathcal{A}_{D,H}$ is $\sigma_{D,H}$-stable with $Z_{D,H}(\mathcal{C}_p) = -1$ (the proof is the same as [ABCH13, Proposition 6.2.a]).

**Note.** When the $D$ and $H$ divisors have been fixed, we will often drop the $D,H$ subscript from $\sigma, Z, \mathcal{A}, T$, and $\mathcal{F}$.

### 2.3 Slices of $\text{Stab}_{\text{div}}(S)$

One of the features that makes $\text{Stab}_{\text{div}}(S)$ well suited to computations is its decomposition into well-behaved 3-spaces, each given by a choice of ample divisor and another divisor orthogonal to it. In these 3-spaces, walls of interest will be quadric surfaces and most of our work will begin by first choosing a particular 3-space to live in. Most of these concepts were introduced in [Mac].

Let $H$ be an ample divisor such that $H^2 = 1$. If $S$ has Picard rank 1, then the stability conditions in $\text{Stab}_{\text{div}}(S)$ are all of the form $\sigma_{sH,tH}$. It was
already proved in [AB13] that line bundles are always Bridgeland stable for these stability conditions.

Assume from now on that \( S \) has Picard rank greater than 1.

**Definition.** Choose a divisor \( G \) with \( G.H = 0 \) and \( G^2 = -1 \) (note that \( G^2 \leq 0 \) by the Hodge Index Theorem with \( G^2 = 0 \) iff \( G = 0 \)). Then, define \( S_{H,G} := \{ \sigma_{sH+uG,tH} \mid s, u, t \in \mathbb{R}, t > 0 \} \subset \text{Stab}_{\text{div}}(S) \).

From now on we assume that any divisors \( H, G \) are as above. We identify \( S_{H,G} \) with \( \{(s, u, t) \mid t > 0\} \) by \( (s, u, t) \leftrightarrow \sigma_{sH+uG,tH} \).

Each of the stability conditions \( \sigma_{D,H} \) defined in 2.2 can be seen as an element of a particular 3-dimensional slice. Indeed, we can just scale the \( H \) to ensure that \( H^2 = 1 \), and then choose \( G \) such that \( D = sH + uG \) so that \( \sigma_{D,H} \in S_{H,G} \). Thus these slices cover all of \( \text{Stab}_{\text{div}}(S) \).

**Note.** Though these spaces do not contain the plane \( t = 0 \), for convenience we will treat them as if they do. Also, we write \( (s, u) \) to mean \( (s, u, 0) \) and identify \( \sigma = \sigma_{sH+uG,tH} = (s, u, t) \).

Let \( E \in D(S) \) and set \( \text{ch}(E) = (\text{ch}_0(E), \text{ch}_1(E), \text{ch}_2(E)) = (r, c_1(E), c) \). We may write \( c_1(E) = d_h H + d_g G + \alpha \) where \( \alpha.H = \alpha.G = 0 \) and \( d_h, d_g \in \mathbb{R} \).

Specifically, we have \( d_h = c_1(E).H \) and \( d_g = -c_1(E).G \).

**Remark 2.3.** Given \( S_{H,G} \), the equality \( \mu_H(sH + uG) = \mu_H(E) \) is equivalent to \( s = \frac{\text{ch}_1(E).H}{rH^2} = \mu_H(E) \), and \( \mu_H(sH + uG) < \mu_H(E) \) iff \( s < \mu_H(E) \).

The vertical plane \( s = \mu_H(E) \) is very important, because, if \( E \) is a \( \mu_H \)-semistable sheaf, then \( s < \mu_H(E) \) iff \( E \in \mathcal{A}_{sH+uG,H} \), and \( s \geq \mu_H(E) \) iff \( E[1] \in \mathcal{A}_{sH+uG,H} \).

For each fixed value of \( u \), we denote by \( \Pi_u \) the vertical plane of stability conditions \( (s, u, t) \) of fixed \( u \)-value. It is parametrized by \( (s, t) \), and it will play a special role in our work (see Section 4.1 below for more details).

The central charge of a stability condition \( \sigma = \sigma_{sH+uG,tH} \) for an object \( E \) with \( \text{ch}(E) = (r, d_h H + d_g G + \alpha, c) \) is equal to

\[
Z(E) = \left( -c + sd_h - ud_g - \frac{r}{2}(s^2 - u^2 - t^2) \right) + i\left( td_h - rst \right).
\]

**3 Reduction to the case of \( \mathcal{O}_S \)**

The action of tensoring stability conditions by line bundles will allow us to restrict our attention from the stability of all line bundles to that of \( \mathcal{O}_S, \mathcal{O}_S[1] \).
Proving the action on $\text{Stab}(S)$ descends to $\text{Stab}_{\text{div}}(S)$ is straightforward but we provide it here for completeness.

**Lemma 3.1.** Let $S$ be a smooth projective surface with line bundle $\mathcal{O}_S(D')$. For any $\sigma_{D,H} \in \text{Stab}_{\text{div}}(S)$, we have $\mathcal{O}_S(D') \otimes \sigma = \sigma_{D+D',H}$.

**Proof.** For an autoequivalence $\Phi$ and stability condition $\sigma = (Z, A)$, let $\Phi \sigma = (\Phi Z, \Phi A)$. We first consider the central charge: We have $(\Phi Z)(E) = Z(\Phi^{-1}E)$. Thus, for $\Phi = \mathcal{O}_S(D') \otimes -$, we have

\[
(\Phi Z)(E) = Z_{D,H}(\Phi^{-1}E) = Z_{D,H}(\mathcal{O}_S(-D') \otimes E)
\]

\[
= - \int e^{-(D+ih)} \text{ch}(\mathcal{O}_S(-D') \otimes E)
\]

\[
= - \int e^{-(D+ih)} \text{ch}(\mathcal{O}_S(-D')).\text{ch}(E)
\]

\[
= - \int e^{-(D+ih)} \left(1[S] - D' + \frac{(-D')^2}{2}[pt]\right).\text{ch}(E)
\]

\[
= - \int e^{-(D+ih)} e^{-D'}.\text{ch}(E)
\]

\[
= - \int e^{-(D+D'+ih)} \text{ch}(E)
\]

\[
= Z_{D+D',H}(E)
\]

Next, the heart: we have $\Phi A = \Phi(A)$, i.e. the image of $A$ under the autoequivalence $\Phi$. Since

$E \in \text{Coh}(S)$ is $\mu_H$ – stable with $\mu_H(E) >$ (resp. $\geq$) $\mu_H(D)$ iff

$F := \mathcal{O}(D') \otimes E$ is $\mu_H$ – stable with $\mu_H(F) >$ (resp. $\geq$) $\mu_H(D + D')$

we have that

$\Phi A = \Phi(A_{D,H})$

\[
= \langle E \in \text{Coh}(S), \mu_H\text{-stable with } \mu_H(E) > \mu_H(D + D')\rangle;
\]

$F[1]$, where $F \in \text{Coh}(S)$ is $\mu_H$-stable with $\mu_H(E) \leq \mu_H(D + D')$

\[
= A_{D+D',H}
\]
We have shown $\mathcal{O}_S(D') \otimes \sigma_{D,H} = \sigma_{D+D',H}$. \hfill \Box

As an immediate corollary we have that $\mathcal{O}_S(D')$ is (semi)stable at $\sigma_{D,H}$ iff $\mathcal{O}_S$ is (semi)stable at $\sigma_{D-D',H}$ (and similarly for $\mathcal{O}_S[1]$).

4 Preliminaries on the Stability of $\mathcal{O}_S$

Walls are subsets of the stability manifold where the stability of objects can change. Our main interest lies in describing the chambers of stability for $\mathcal{O}_S$, which are bounded by walls corresponding to certain destabilizing objects. Let us start with a few definitions, and a description of the possible walls.

4.1 Subobjects of $\mathcal{O}_S$ and their walls

First of all, here is our generic definition of a wall.

**Definition.** Given two objects $E, B \in D(S)$, with $B$ Bridgeland-stable for at least one stability condition, we define the wall $W(E,B)$ as $\{ \sigma \in \text{Stab}(S) \mid (\text{Re } Z(E))(\text{Im } Z(B)) - (\text{Re } Z(B))(\text{Im } Z(E)) = 0 \}$. If at some $\sigma \in W(E,B)$ we have $E \subset B$ in $A$, we say that $W(E,B)$ is a **weakly destabilizing wall** for $B$. If at some $\sigma \in W(E,B)$ we have $E \subset B$ in $A$, and $B$ is Bridgeland $\sigma$-semistable, we say that $W(E,B)$ is an **actually destabilizing wall** for $B$.

Note that if $\text{Im } Z(E) \neq 0 \neq \text{Im } Z(B)$ then the defining condition is just $\beta(E) = \beta(B)$.

We are interested in the walls for $\mathcal{O}_S$ and $\mathcal{O}_S[1]$, and we start by studying the walls for $\mathcal{O}_S$. Note that, given a Bridgeland stability condition $\sigma$, we have $\mathcal{O}_S \in A$ iff $s < 0$, and $\mathcal{O}_S[1] \in A$ iff $s \geq 0$.

At each fixed value of $u$, Maciocia showed in [Mac, Section 2] that all walls for $\mathcal{O}_S$ in $\Pi_u$ are nested semicircles centered on the $s$-axis.

Therefore, given two objects $E_1$ and $E_2$, and a fixed value of $u$, we have that $W(E_1,\mathcal{O}_S) \cap \Pi_u$ and $W(E_2,\mathcal{O}_S) \cap \Pi_u$ are both semicircles, with one of them inside the other, unless they are equal.

**Definition.** We say that the wall $W(E_1,\mathcal{O}_S)$ is **inside** the wall $W(E_2,\mathcal{O}_S)$ at $u$ if the semicircle $W(E_1,\mathcal{O}_S) \cap \Pi_u$ is inside the semicircle $W(E_2,\mathcal{O}_S) \cap \Pi_u$ or equal to it. We will use the notation $W(E_1,\mathcal{O}_S) \cap \Pi_u \preceq W(E_2,\mathcal{O}_S) \cap \Pi_u$. 

9
Lemma 4.1. Let $\sigma \in \text{Stab}_{\text{div}}(S)$, and let $0 \to E \to O_S \to Q \to 0$ be a short exact sequence in $\mathcal{A}$. Then $E$ is a torsion-free sheaf, $H^0(Q)$ is a quotient of $O_S$ of rank 0, and the kernel of the map $O_S \to H^0(Q)$ is an ideal sheaf $I_Z(-C)$ for some effective curve $C$ and some zero-dimensional scheme $Z$.

Proof. The long exact sequence in cohomology associated to the short exact sequence shows that $E$ must be a sheaf, while $Q$ may have cohomologies in degrees $-1$ and $0$:

$$0 \to H^{-1}(Q) \to E \to O_S \to H^0(Q) \to 0.$$ 

If $H^0(Q)$ had rank 1, then it would have to be equal to $O_S$, and we would have that $H^{-1}(Q) = E = 0$, which is impossible. Therefore, $H^0(Q)$ is a quotient of $O_S$ of rank 0. Since $I_Z(-C)$ and $H^{-1}(Q)$ are both torsion-free sheaves, $E$ is also a torsion-free sheaf. $\square$

Here we study which forms the walls $W(E, O_S)$ can take. The intersection of a wall with the $t = 0$-plane is a conic through the origin, and we classify the wall based on invariants associated to $E$.

If $\text{ch}(E) = (r, d_h H + d_g G + \alpha_E, c)$, then we saw above that

$$Z(E) = \left( -c + sd_h - ud_g - \frac{r}{2}(s^2 - u^2 - t^2) \right) + i (td_h - rst),$$

and the equation of the wall $W(E, O_S)$ is

$$\frac{t}{2}(-d_h(s^2 + t^2 + u^2) + 2d_gsu + 2cs) = 0.$$ 

Since $t \neq 0$, this is equivalent to

$$-d_h(s^2 + t^2 + u^2) + 2d_gsu + 2cs = 0.$$ 

It is a quadric, and we will start by studying its intersection with the $t = 0$ plane:

$$-d_h(s^2 + u^2) + 2d_gsu + 2cs = 0.$$ 

We will abuse notation, and still refer to this equation as the wall $W(E, O_S)$. The determinant is equal to

$$\Delta = 4(d_g^2 - d_h^2).$$

If $\Delta = 0$, then the wall is a parabola. Since $s < 0$, the wall can only be a weakly destabilizing wall if $c > 0$, in which case the equation of the parabola is $-d_h(s \pm u)^2 + 2cs = 0$.

If $\Delta \neq 0$ and $c \neq 0$, straightforward calculations show the following:
• 0\(P\) and 2\(P\) are on the wall, where
  \[
P = -\frac{c}{d_g^2 - d_h^2} (d_h, d_g).
  \]
• The tangent line to the wall at 0\(P\) and 2\(P\) is vertical, i.e. \(s = \) constant.
• The tangent line to the wall is horizontal (i.e. \(u = \) constant) at the points where the conic intersects \(u = s\) and \(u = -s\). These are
  \[
  \left(\frac{c}{d_h - d_g}, \frac{c}{d_h - d_g}\right) \text{ and } \left(\frac{c}{d_h + d_g}, -\frac{c}{d_h + d_g}\right).
  \]
If \(\Delta < 0\), then the wall is a weakly destabilizing wall only if \(c > 0\), and it is an ellipse.
If \(\Delta > 0\), then there are three possibilities:
• If \(c = 0\), then the wall is a cone centered at \((0, 0)\).
• If \(c > 0\), then the wall is a hyperbola with center \(P\) to the right of \(s = 0\).
• If \(c < 0\), then the wall is a hyperbola with center \(P\) to the left of \(s = 0\).
  In this case, the asymptotes have slope
  \[
  d_g \pm \sqrt{d_g^2 - d_h^2} \over d_h.
  \]
Here is a summary of all possible weakly destabilizing walls (pictures drawn for \(d_h < 0\) and \(d_g > 0\)):

**Parabola.** When \(d_g^2 - d_h^2 = 0\) and \(c > 0\).

**Ellipse.** When \(d_g^2 - d_h^2 < 0\) and \(c > 0\).

**Cone.** When \(d_g^2 - d_h^2 > 0\) and \(c = 0\).

**Right Hyperbola.** When \(d_g^2 - d_h^2 > 0\) and \(c > 0\).

**Left Hyperbola.** When \(d_g^2 - d_h^2 > 0\) and \(c < 0\).
Lemma 4.2. Given two subobjects $E_1$ and $E_2$ of $\mathcal{O}_S$ in $A$, there exists at most one value of $u \neq 0$ such that

$$\mathcal{W}(E_1, \mathcal{O}_S) \cap \Pi_u = \mathcal{W}(E_2, \mathcal{O}_S) \cap \Pi_u,$$

unless the two walls coincide everywhere.

Proof. Suppose that $\mathcal{W}(E_1, \mathcal{O}_S)$ and $\mathcal{W}(E_2, \mathcal{O}_S)$ do not coincide everywhere. Looking at the intersection of the walls with the $t = 0$ plane, we see that they are two conics that intersect at $(0,0)$ with multiplicity two. Therefore, they can only intersect at at most two other points there. Since the walls $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ are nested semicircles centered on the $s$-axis, they intersect the $t = 0$ plane. Therefore, if $\mathcal{W}(E_1, \mathcal{O}_S) \cap \Pi_u = \mathcal{W}(E_2, \mathcal{O}_S) \cap \Pi_u$, the walls coincide at the two points where the semicircle intersects the $t = 0$ plane, and cannot intersect anywhere else there except at $(0,0)$. Therefore, the walls cannot coincide at any other value of $u \neq 0$. \qed
4.2 Bridgeland Stability of $\mathcal{O}_S$ for $t >> 0$

In this section, we prove that $\mathcal{O}_S$ is Bridgeland stable in $\mathcal{S}_{H,G}$ for $t >> 0$. This fact is already known. For example, it follows from the result in [Mac] that walls in each plane of the form $u = \text{constant}$ are disjoint circles that are bounded above. We give however a new proof that would easily generalize to the case of an object of the form $E$ or $E[1]$ with $E$ a $\mu$-stable sheaf.

Proposition 4.3. Let $H$ and $G$ be as above, and fix a divisor $sH + uG$ with $s < 0$. Then $\mathcal{O}_S$ is Bridgeland stable for the stability condition $\sigma_{sH+uG,tH}$ for $t >> 0$.

Proof. Let $\mathcal{A} = \mathcal{A}_{sH+uG,tH}$ (note that $\mathcal{A}_{sH+uG,tH}$ is independent of $t$), and let $E$ be subobject of $\mathcal{O}_S$ in $\mathcal{A}$. As we saw above, $E$ is torsion-free. Moreover, since $0 < \text{Im} Z(E) < \text{Im} Z(\mathcal{O}_S)$ for all $t$, we have that $\mu_H(E) < \mu_H(\mathcal{O}_S) = 0$. This implies that $\beta(E) < \beta(\mathcal{O}_S)$ for $t >> 0$. From our analysis of the possible walls above, we know that, for a fixed $(s,u)$, there is at most one value of $t$ such that $\beta(E) = \beta(\mathcal{O}_S)$. Moreover, if $\beta(E) < \beta(\mathcal{O}_S)$ at a given $\sigma_{s,u,t}$ then the same inequality holds for all $t > t_0$.

Hence the only way $\mathcal{O}_S$ could not be Bridgeland stable for large $t$ is if there are infinitely many Chern characters corresponding to subobjects $E$ of $\mathcal{O}_S$ with $\beta(E) \geq \beta(\mathcal{O}_S)$.

Suppose this is so and fix a value of $t$ (in particular, we can now assume that the subobjects in question are Bridgeland semistable). Now, for the fixed stability condition $\sigma_{sH+uG,tH}$, $\mathcal{O}_S$ has a HN-filtration of Bridgeland semistable objects $\mathcal{O}_S = \langle L_i \rangle_1^n$ with $L_1$ torsion-free and thus $\text{Im} Z(L_1) > 0$ and $\beta(L_1) < \infty$. We have that $E \in \mathcal{A}$ and $E \subset \mathcal{O}_S$ imply $\beta(L_1) \geq \beta(E)$ and since we also have $\text{Im} Z(E) < \text{Im} Z(\mathcal{O}_S)$ the images of these semistable subobjects are forced into a finite triangular region of the upper half-plane, specifically the region bounded by the ray through $Z(\mathcal{O}_S)$, the ray through $Z(L_1)$, and the horizontal line $\text{Im} z = \text{Im} Z(\mathcal{O}_S)$. But this is impossible by the support property. Thus $\mathcal{O}_S$ is Bridgeland stable for large $t$. □

Remark 4.4. Even when fixing $H$ and $G$, there is not necessarily a $t_0$ such that $\mathcal{O}_S$ is Bridgeland stable for all $(s,u,t)$ with $t > t_0$. As a matter of fact, due to the nature of the rank 1 walls, which are hyperboloids, whenever $\mathcal{O}_S$ is not Bridgeland stable somewhere in a space $\mathcal{S}_{H,G}$, there will be stability conditions for any $t$ for which $\mathcal{O}_S$ is not Bridgeland stable. However, for every fixed value of $u$, Maciocia proved in [Mac] that there exists a value $t_0(u)$ such...
that $\mathcal{O}_S$ is Bridgeland stable for $\sigma_{sH+uG,tH}$ for all $s < 0$ and $t > t_0(u)$. What happens is that the $t_0(u)$, if it is not equal to 0, goes to $\infty$ as $u$ goes to $\pm\infty$.

### 4.3 Bertram’s Lemma

The following lemma is a key tool in the proof of the main theorem. The lemma is essentially Lemma 6.3 in [ABCH13], adapted to our situation. It allows us to, in some situations, find walls higher than a given wall for $\mathcal{O}_S$ by removing a Mumford semistable factor from the subobject or quotient. In either case, rank strictly drops and this sets the stage for our induction proof characterizing the stability of $\mathcal{O}_S$.

Let $E$ be a subobject of $\mathcal{O}_S$ of rank $\geq 2$ in $\mathcal{A}$, and let $Q$ be the quotient.

Let $0 = E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_{n-1} \subseteq E_n = E$ be the Harder-Narasimhan filtration of $E$, and let $K_i = E_i/E_{i-1}$ (so that $K_1 = E_1$ and $K_n = E/E_{n-1}$). We have that $\mu_H(K_1) > \mu_H(K_2) > \cdots > \mu_H(K_n)$. Also, let $\overline{K} = K_n$.

Similarly, let $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{m-1} \subseteq F_m = H^{-1}(Q)$ be the Harder-Narasimhan filtration of $H^{-1}(Q)$, and let $J_i = F_i/F_{i-1}$ with $\overline{J} = J_1$.

We have that $E \subseteq \mathcal{O}_S \in \mathcal{A}$ iff $\mu_H(\overline{J}) \leq s < \mu_H(\overline{K})$.

At $s = \mu_H(\overline{K})$, we will consider the natural subsheaf $E_{n-1} \subseteq E$. At $s = \mu_H(\overline{J})$, we will consider the natural quotient sheaf $E \to E/\overline{J}$ (note that, as sheaves, $\overline{J} = J_1 = F_1 \subseteq H^{-1}(Q) \subseteq E$).

**Lemma 4.5 (Bertram’s Lemma).** Fix $H$ and $G$ as above, and let $E \subseteq \mathcal{O}_S$ in $\mathcal{A}$ for some $\sigma = \sigma_{sH+uG,tH} = (Z, \mathcal{A})$ such that $\sigma \in \mathcal{W}(E, \mathcal{O}_S)$.

1. If $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ intersects the line $s = \mu_H(\overline{K})$ for $t > 0$, then $\beta(E_{n-1}) > \beta(E)$ at $\sigma$, with $E_{n-1} \subseteq \mathcal{O}_S$ in $\mathcal{A}$ (in particular, $E$ is not $\mu_H$-semistable).

2. If $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ intersects the line $s = \mu_H(\overline{J})$ for $t > 0$, then $\beta(E/\overline{J}) > \beta(E)$ at $\sigma$, with $E/\overline{J} \subseteq \mathcal{O}_S$ in $\mathcal{A}$.

**Proof.** (1) Maciocia proved in [Mac] that, if $E$ is $\mu_H$-semistable, then $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ does not intersect the line $s = \mu_H(E)$. Therefore, if $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ intersects $s = \mu_H(\overline{K})$, $E$ cannot be $\mu_H$-semistable. Since $E \in \mathcal{A}_\sigma$, we have that $s(\sigma) < \mu_H(\overline{K})$. Because $\overline{K} \in \mathcal{A}$ iff $s < \mu_H(\overline{K})$, it follows that for all values of $s$ between $s(\sigma) \leq s < \mu_H(\overline{K})$ we have that $0 \to E_{n-1} \to E \to \overline{K} \to 0$ in $\mathcal{A}$. At $s = \mu_H(\overline{K})$, we have that $\text{Im } Z(\overline{K}) = 0$, and $\beta(\overline{K}) = -\infty$. Therefore, approaching $s = \mu_H(\overline{K})$ from $\sigma$ along $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$, we have that
\[ \beta(K) \to -\infty, \text{ and } \beta(E_{n-1}) > \beta(E) > \beta(K). \] Since the walls are nested in \( \Pi_u \), we must have \( \beta(E_{n-1}) > \beta(E) = \beta(\mathcal{O}_S) \) at \( \sigma \) as well.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6}
\caption{\( t = 0 \) plane}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7}
\caption{\( \Pi_u \) plane at \( u = u_0 \)}
\end{figure}

(2) Since \( E \subseteq \mathcal{O}_S \in \mathcal{A} \), we have that \( \mu_H(\mathcal{J}) \leq s(\sigma) < \mu_H(K) \). Therefore, for all values of \( s \) between \( \mu_H(\mathcal{J}) \leq s \leq s(\sigma) \), we have that \( E, \mathcal{J}[1] \in \mathcal{A} \), and there exists a short exact sequence \( 0 \to E \to E/\mathcal{J} \to \mathcal{J}[1] \to 0 \) in \( \mathcal{A} \).

At \( s = \mu_H(\mathcal{J}) \), we have that \( \text{Im } Z(\mathcal{J}[1]) = 0 \), and \( \beta(\mathcal{J}[1]) = \infty \). Therefore, approaching \( s = \mu_H(\mathcal{J}) \) from \( \sigma \) along \( \mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u \), we have that \( \beta(\mathcal{J}[1]) \to \infty \), and \( \beta(E) < \beta(E/\mathcal{J}) < \beta(\mathcal{J}[1]) \). Since the walls are nested in \( \Pi_u \), we must have \( \beta(E/\mathcal{J}) > \beta(E) = \beta(\mathcal{O}_S) \) at \( \sigma \) as well.

\begin{remark}
Since the surface \( S \) may have Picard rank larger than 1, we are forced to strengthen the hypotheses from those of [ABCH13], but the proof is exactly the same. We have restricted our attention to \( \mathcal{O}_S \), but the proof holds for any sheaf satisfying the Bogomolov inequality (even a torsion sheaf).
\end{remark}

\begin{remark}
Let us point out an important fact that will be needed later in the paper: When looking at \( E/\mathcal{J} \subseteq \mathcal{O}_S \) in \( \mathcal{A} \), if we call \( Q' \) the quotient of \( \mathcal{O}_S \) by \( E/\mathcal{J} \) in \( \mathcal{A} \), we have that \( H^{-1}(Q') = H^{-1}(Q)/\mathcal{J} \) and \( H^0(Q') = H^0(Q) \).
\end{remark}

\section{Bridgeland Stability of \( \mathcal{O}_S \)}

We prove Conjecture \[ \] for surfaces with (any Picard rank and) no curves of negative self-intersection as well as surfaces with Picard rank 2 and one irreducible curve of negative self-intersection. Proposition \[ \] serves as evidence for the conjecture on surfaces with Picard rank 2 and two irreducible curves of negative self-intersection.
5.1 Subobjects of $\mathcal{O}_S$ of rank 1

Understanding the rank 1 weakly destabilizing subobjects of $\mathcal{O}_S$ is crucial to our main results, all of which use induction. If a subobject $E \subset \mathcal{O}_S$ has rank 1, then Lemma 4.1 shows that $E$ must be equal to $\mathcal{I}_Z(-C)$ for some effective curve $C$ and some zero-dimensional scheme $Z$. We show here that for $\mathcal{I}_Z(-C)$ to weakly destabilize $\mathcal{O}_S$, we must have $C^2 < 0$.

**Proposition 5.1.** If $C^2 \geq 0$, then $\mathcal{I}_Z(-C)$ does not weakly destabilize $\mathcal{O}_S$ for any $\sigma$, i.e., there does not exist any $\sigma$ such that $\mathcal{I}_Z(-C) \subseteq \mathcal{O}_S$ in $\mathcal{A}$ and $\beta(\mathcal{I}_Z(-C)) = \beta(\mathcal{O}_S)$.

**Proof.** We have that $\text{ch}(\mathcal{I}_Z(-C)) = (1, -C, C^2/2 - l(Z))$, where $l(Z)$ is the length of $Z$. If $C = c_h H + c_g G + \alpha C$, with $\alpha C.H = \alpha C.G = 0$, then $C^2 = c_h^2 - c_g^2 + \alpha_C^2$, and $\alpha_C^2 \leq 0$ by the Hodge Index Theorem. Therefore, if $C^2 \geq 0$, then $c_h^2 - c_g^2 \geq 0$. The equation for the wall $W(\mathcal{I}_Z(-C), \mathcal{O}_S)$ simplifies to

$$c_h(s^2 + t^2 + u^2) - 2c_g su + (c_h^2 - c_g^2)s + 2\alpha_C^2 s - 2l(Z)s = 0.$$

If $c_h^2 - c_g^2 > 0$, then the wall is an ellipse going through $0P$ and $2P$ with

$$P = \frac{C^2/2 - l(Z)}{c_g^2 - c_h^2}(c_h, c_g).$$

The $s$-coordinate of $2P$ is

$$\frac{c_h^2 - c_g^2 + 2\alpha_C^2 - 2l(Z)}{c_g^2 - c_h^2}c_h \geq -c_h + \frac{2\alpha_C^2 - 2l(Z)}{c_g^2 - c_h^2}c_h \geq -c_h (< 0).$$

Therefore, the ellipse is contained in the region $s \geq -c_h$.

Since $\mathcal{I}_Z(-C) \in \mathcal{A}$, we have that $s < -c_h$, and therefore $\mathcal{I}_Z(-C)$ cannot weakly destabilize $\mathcal{O}_S$.

If $c_h^2 - c_g^2 = 0$, then $C^2 \geq 0$ implies that $C^2 = 0$. Therefore, $\text{ch}_2(\mathcal{I}_Z(-C)) = -l(Z) \leq 0$, and, as we saw in Section 4.1 where we listed the possible weakly destabilizing walls for $\mathcal{O}_S$, this wall cannot be a weakly destabilizing wall. □

**Remark 5.2.** Since $\mathcal{O}_S$ is Bridgeland stable for $t >> 0$ (by Proposition 4.3), we have from Proposition 5.1 that if $C^2 \geq 0$, then $\beta(\mathcal{I}_Z(-C)) < \beta(\mathcal{O}_S)$ whenever $\mathcal{I}_Z(-C) \subseteq \mathcal{O}_S$ in $\mathcal{A}$.
Remark 5.3. For $C$ a curve of negative self-intersection, we have two possibilities.

1. If $c^2_h - c^2_g < 0$ and $c^2_g - c^2_h \leq 0$ then $\mathcal{I}_Z(-C)$ does not weakly destabilize $\mathcal{O}_S$ in $\mathcal{S}_{H,G}$ (note that this is only possible for $S$ with Picard rank $\geq 3$).

2. If $c^2_h - c^2_g > 0$ then $\mathcal{W}(\mathcal{I}_Z(-C), \mathcal{O}_S)$ is a Left Hyperbola and is a weakly destabilizing wall for $\mathcal{O}_S$.

5.2 Surfaces with no curves of negative self-intersection

We characterize the stability of $\mathcal{O}_S$ when $S$ has no curves of negative self-intersection. This implies the first part of Theorem 1.1.

Theorem 5.4. If $S$ does not contain any curves of negative self-intersection, then $\mathcal{O}_S$ is always Bridgeland stable (whenever in $\mathcal{A}$).

Proof. Let $E \subseteq \mathcal{O}_S$ be a proper subobject in $\mathcal{A}$ for some $\sigma$. We prove that $\beta(E) < \beta(\mathcal{O}_S)$ for all $\sigma$ by induction on the rank of $E$.

If $E$ has rank 1, then the result follows from Proposition 5.1 and 4.3. Assume that $E$ has rank $r$, and that the result holds for any proper subobject of rank less than $r$. Choose a pair $H, G$ as above so that $\sigma \in \mathcal{S}_{H,G}$. From our study of the walls above, we know that the wall $\mathcal{W}(E, \mathcal{O}_S)$ is an Ellipse, a Cone, a Left or Right Hyperbola, or a Parabola. Using the same notation
as in Section 4.3, we know that $E \subseteq \mathcal{O}_S \in \mathcal{A}$ iff $\mu_H(J) \leq s < \mu_H(K)$. If we had that $\beta(E) \geq \beta(\mathcal{O}_S)$ for some $\sigma$ in that range, then $E$ would weakly destabilize $\mathcal{O}_S$. If the wall were an Ellipse, it would have to intersect $s = \mu_H(K)$, because these walls are connected and pass through $(0,0)$. If it were a Left Hyperbola, it would have to intersect $s = \mu_H(J)$. If it were a Right Hyperbola, a Parabola, or a Cone it would have to intersect $s = \mu_H(K)$ and/or $s = \mu_H(J)$. Regardless of the type of wall, we would have that, by Lemma 4.5, there would exist a proper subobject of $\mathcal{O}_S$ of higher $\beta$ and lower rank, contradicting our induction hypothesis.

5.3 Actual Walls are Left Hyperbolas

There are characteristics of actually destabilizing walls and subobjects which persist regardless of the Picard rank of $S$ or the composition of curves of negative self-intersection within $S$. We prove the following, and apply it in our study of surfaces with Picard rank 2 (Section 5.4).

**Proposition 5.5.** Let $E \subseteq \mathcal{O}_S$ in $\mathcal{A}$ with quotient $Q$. If the wall $W(E, \mathcal{O}_S)$ is a destabilizing wall, then it is a Left Hyperbola. Moreover, $C = c_1(H^0(Q))$ is a curve of negative self-intersection such that the wall $W(\mathcal{O}_S(-C), \mathcal{O}_S)$ is also a Left Hyperbola, and the wall $W(E, \mathcal{O}_S)$ is inside the wall $W(\mathcal{O}_S(-C), \mathcal{O}_S)$ for all $|u| >> 0$.

The proof follows mostly from two basic lemmas in which we prove the following two basic results:

- Every weakly destabilizing wall that is not a Left Hyperbola is inside a higher wall which is a Left Hyperbola.

- If $E \subseteq \mathcal{O}_S$ in $\mathcal{A}$ with quotient $Q$ has a weakly destabilizing wall that is a Left Hyperbola, then either $C = c_1(H^0(Q))$ is a curve of negative self-intersection or the wall is inside a higher wall that is not a Left Hyperbola.

Let $E \subseteq \mathcal{O}_S$ in $\mathcal{A}$, and let $Q$ be the quotient. We use the same notation as in Section 4.3 for the Harder-Narasimhan filtrations of $E$ and $H^{-1}(Q)$ with respect to the Mumford slope $\mu_H$.

**Lemma 5.6.** Let $E \subseteq \mathcal{O}_S$ in $\mathcal{A}$ such that the wall $W(E, \mathcal{O}_S)$ is a weakly destabilizing wall that is not a Left Hyperbola. Then, for some $E_i$ in the
Harder-Narasimhan filtration of $E$, the wall $\mathcal{W}(E_i, \mathcal{O}_S)$ is a Left Hyperbola such that the following is true: If there exists a stability condition $\sigma$ such that $E \subseteq \mathcal{O}_S$ in $\mathcal{A}_\sigma$, then $E_i \subset \mathcal{O}_S$ in $\mathcal{A}_\sigma$, and the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E_i, \mathcal{O}_S)$ at $u(\sigma)$. In particular, $E$ cannot actually destabilize $\mathcal{O}_S$.

Proof. We prove this by induction on the number of terms $n$ in the Harder-Narasimhan filtration of $E$.

If $n = 1$ (i.e. $E$ is $\mu_H$-semistable), then the wall $\mathcal{W}(E, \mathcal{O}_S)$ cannot be a weakly destabilizing wall without being a Left Hyperbola, because all other type of walls would intersect the line $s = \mu_H(E)$, contradicting Lemma [4.5].

Let now $n > 1$, and assume that the statement is true for all subobjects of $\mathcal{O}_S$ which have a weakly destabilizing wall that is not a Left Hyperbola, and have a Harder-Narasimhan filtration of length $< n$.

Since the wall $\mathcal{W}(E, \mathcal{O}_S)$ is not a Left Hyperbola, it will intersect the line $s = \mu_H(K)$. Consider a value of $u$ such that $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ intersects $s = \mu_H(K)$ at $t > 0$. By Lemma [4.5] we have that $\beta(E_{n-1}) > \beta(E)$ at the stability conditions $\sigma_{s,u}$ on $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ such that $s < \mu_H(K)$. Therefore, the wall $\mathcal{W}(E_{n-1}, \mathcal{O}_S)$ is also a weakly destabilizing wall.

If the wall $\mathcal{W}(E_{n-1}, \mathcal{O}_S)$ is not a Left Hyperbola, we can conclude by induction that there exists an $E_i$ such that the wall $\mathcal{W}(E_i, \mathcal{O}_S)$ is a Left Hyperbola and the wall $\mathcal{W}(E_{n-1}, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E_i, \mathcal{O}_S)$ for all $u(\sigma)$ for which $E_{n-1} \subseteq \mathcal{O}_S$ in $\mathcal{A}_\sigma$. If the wall $\mathcal{W}(E_{n-1}, \mathcal{O}_S)$ is a Left Hyperbola, let $i = n - 1$.

Let $\sigma$ be a stability condition such that $E \subseteq \mathcal{O}_S$ in $\mathcal{A}_\sigma$. We need to prove that the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E_i, \mathcal{O}_S)$ at $u(\sigma)$. First of all, notice that, if $E \subseteq \mathcal{O}_S$ in $\mathcal{A}_\sigma$, then $E_{n-1} \subseteq E \subseteq \mathcal{O}_S$ is $\mathcal{A}_\sigma$, because if $E \in \mathcal{A}_\sigma$, then $E_{n-1} \subseteq E \in \mathcal{A}_\sigma$.

Suppose now that $E$ weakly destabilizes $\mathcal{O}_S$ at $\sigma$ (i.e. $\beta_{s}(E) \geq \beta_{s}(\mathcal{O}_S)$), and suppose moreover that $u(\sigma) > 0$ (the proof for $u(\sigma) < 0$ is similar). Since the wall $\mathcal{W}(E, \mathcal{O}_S)$ is not a Left Hyperbola, its intersection with the planes $\Pi_u$ is non-empty for all $0 \leq u \leq u(\sigma)$. Since the wall $\mathcal{W}(E_i, \mathcal{O}_S)$ is a Left Hyperbola, on the other hand, its region in the $s < 0$ half-plane does not reach the $s$-axis. Therefore, for small positive values of $u$, $\mathcal{W}(E_i, \mathcal{O}_S) \cap \Pi_u$ is empty in the $s < 0$ region, and $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u$ is non-empty.

For every value of $u$ such that the wall $\mathcal{W}(E, \mathcal{O}_S)$ intersects $s = \mu_H(K)$ at $t > 0$, we know that

$$\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u \leq \mathcal{W}(E_{n-1}, \mathcal{O}_S) \cap \Pi_u \leq \mathcal{W}(E_i, \mathcal{O}_S) \cap \Pi_u$$
Lemma 4.5 and the induction hypothesis, respectively. The statement is still true at \( u_0 \) by continuity, where we denote by \( u_0 \) be the smallest value of \( u \) such that \( W(E, \mathcal{O}_S) \) intersects \( s = \mu_H(K) \) (the intersection will be at \( t = 0 \)).

Since the wall \( W(E_i, \mathcal{O}_S) \) is inside the wall \( W(E, \mathcal{O}_S) \) for small positive values of \( u \), while the wall \( W(E, \mathcal{O}_S) \) is inside the wall \( W(E_i, \mathcal{O}_S) \) at \( u_0 \), there must exist a value \( u_1 \) with \( 0 < u_1 < u_0 \) such that \( W(E, \mathcal{O}_S) \cap \Pi_{u_1} = W(E_i, \mathcal{O}_S) \cap \Pi_{u_1} \) (recall that, for each \( u \), the intersections of the walls with the plane \( \Pi_u \) are nested semicircles). Moreover,

\[
W(E, \mathcal{O}_S) \cap \Pi_u \leq W(E_i, \mathcal{O}_S) \cap \Pi_u
\]

for all \( u \geq u_1 \) by Lemma 4.2. Since \( E \in \mathcal{A}_\sigma \) only if \( s < \mu_H(J) \), we have that the part of the wall \( W(E, \mathcal{O}_S) \) where \( E \subseteq \mathcal{O}_S \) in \( \mathcal{A}_\sigma \) is contained in the region \( u \geq u_0 \), which is contained in \( u \geq u_1 \) where the wall \( W(E, \mathcal{O}_S) \) is inside the wall \( W(E_i, \mathcal{O}_S) \).

Lemma 5.7. Let \( E \subseteq \mathcal{O}_S \) in \( \mathcal{A} \) with quotient \( Q \) such that the wall \( W(E, \mathcal{O}_S) \) is a weakly destabilizing Left Hyperbola. Then:

- Either \( C = c_1(H^0(Q)) \) is a curve of negative self-intersection, and the wall \( W(\mathcal{O}_S(-C), \mathcal{O}_S) \) is a Left Hyperbola, or

- For some \( F_j \) in the Harder-Narasimhan filtration of \( H^{-1}(Q) \), the wall \( W(E/F_j, \mathcal{O}_S) \) is not a Left Hyperbola, and the following is true: If there exists a stability condition \( \sigma \) such that \( E \subseteq \mathcal{O}_S \) in \( \mathcal{A}_\sigma \), then \( E/F_j \subseteq \mathcal{O}_S \) in \( \mathcal{A}_\sigma \), and the wall \( W(E, \mathcal{O}_S) \) is inside the wall \( W(E/F_j, \mathcal{O}_S) \) at \( u(\sigma) \). In particular, \( E \) cannot actually destabilize \( \mathcal{O}_S \).

Proof. We prove this by induction on the number of terms \( m \) in the Harder-Narasimhan filtration of \( H^{-1}(Q) \) (including the case \( m = 0 \) corresponding to \( H^{-1}(Q) = 0 \)).

If \( m = 0 \) and \( H^{-1}(Q) = 0 \), then \( E = \mathcal{T}_Z(-C) \), and by Proposition 5.1, \( C = c_1(H^0(Q)) \) must be a curve of negative self-intersection for the wall \( W(E, \mathcal{O}_S) \) to be a weakly destabilizing wall.

Let now \( m > 0 \), and assume that the statement is true for all subobjects of \( \mathcal{O}_S \) which have a weakly destabilizing wall that is a Left Hyperbola, and a Harder-Narasimhan filtration for \( H^{-1}(\text{quotient}) \) of length \( < m \).

Since the wall \( W(E, \mathcal{O}_S) \) is a Left Hyperbola, it will intersect the line \( s = \mu_H(J) = \mu_H(F_i) \). Consider a value of \( u \) such that \( W(E, \mathcal{O}_S) \cap \Pi_u \)
intersects \( s = \mu_H(F_1) \) at \( t > 0 \). By Lemma 4.5, we have that \( \beta(E/F_1) > \beta(E) \) at those stability conditions. Therefore, the wall \( \mathcal{W}(E/F_1, \mathcal{O}_S) \) is also a weakly destabilizing wall.

Let \( Q_1 \) be the quotient for \( E/F_1 \subseteq \mathcal{O}_S \). By Remark 4.7, we have that \( H^0(Q_1) = H^0(Q) \) and \( H^{-1}(Q_1) = H^{-1}(Q)/F_1 \). In particular, the Harder-Narasimhan filtration of \( H^{-1}(Q_1) \) is simply \( F_j/F_1 \) (\( 1 \leq j \leq m \)), which has length \( m-1 \). Moreover, the quotients \( (E/F_1)/(F_j/F_1) \) are the quotients \( E/F_j \).

If the wall \( \mathcal{W}(E/F_1, \mathcal{O}_S) \) is a Left Hyperbola, we have by induction that at least one of the following two options is true:

1. \( C = c_1(H^0(Q_1)) \) is a curve of negative self-intersection, and the wall \( \mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S) \) is a Left Hyperbola. Since \( H^0(Q_1) = H^0(Q) \), we are done.

2. There exists an \( F_j/F_1 \) in the Harder-Narasimhan filtration of \( H^{-1}(Q_1) \) such that the wall \( \mathcal{W}(E/F_j, \mathcal{O}_S) \) is not a Left Hyperbola, and the wall \( \mathcal{W}(E/F_1, \mathcal{O}_S) \) is inside the wall \( \mathcal{W}(E/F_j, \mathcal{O}_S) \) for all \( u(\sigma) \) for which \( E/F_1 \subseteq \mathcal{O}_S \) in \( \mathcal{A}_\sigma \), with \( E/F_j \subseteq \mathcal{O}_S \) there.

If the wall \( \mathcal{W}(E/F_1, \mathcal{O}_S) \) is not a Left Hyperbola, let \( j = 1 \).

Let \( \sigma \) be a stability condition such that \( E \subseteq \mathcal{O}_S \) in \( \mathcal{A}_\sigma \). We now need to prove that the wall \( \mathcal{W}(E, \mathcal{O}_S) \) is inside the wall \( \mathcal{W}(E/F_j, \mathcal{O}_S) \) at \( u(\sigma) \).

First of all, notice that, if \( E \subseteq \mathcal{O}_S \) in \( \mathcal{A}_\sigma \), then \( E/F_1 \subseteq \mathcal{O}_S \subseteq \mathcal{A}_\sigma \).

As in the previous proof, assume that \( E \) weakly destabilizes \( \mathcal{O}_S \) at \( \sigma \), and that \( u(\sigma) > 0 \). We know that \( \mathcal{W}(E/F_j, \mathcal{O}_S) \cap \Pi_u \) is not empty for small positive values of \( u \), while \( \mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u \) is empty in the \( s < 0 \) region for \( u \) positive and sufficiently small.

Let \( u_0 \) be the largest value of \( u \) such that the wall \( \mathcal{W}(E, \mathcal{O}_S) \) intersects \( s = \mu_H(J) \) (the intersection will be at \( t = 0 \)). As in the previous proof we can use Lemma 4.5 the induction hypothesis, and continuity to show that

\[
\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{u_0} \preceq \mathcal{W}(E/F_1, \mathcal{O}_S) \cap \Pi_{u_0} \preceq \mathcal{W}(E/F_j, \mathcal{O}_S) \cap \Pi_{u_0}.
\]

Since \( \mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u \) is inside than \( \mathcal{W}(E/F_j, \mathcal{O}_S) \cap \Pi_u \) for small positive values of \( u \) and at \( u_0 \), it must be inside of it for all \( 0 < u \leq u_0 \). Otherwise, there would have to exist two values of \( u \) between 0 and \( u_0 \) where \( \mathcal{W}(E, \mathcal{O}_S) \cap \Pi_u = \mathcal{W}(E/F_j, \mathcal{O}_S) \cap \Pi_u \) which is not possible by Lemma 4.2.

Since \( E \subseteq \mathcal{O}_S \subseteq \mathcal{A}_\sigma \) only if \( s \geq \mu_H(J) \), we have that the part of the wall \( \mathcal{W}(E, \mathcal{O}_S) \) where \( E \subseteq \mathcal{O}_S \) in \( \mathcal{A}_\sigma \) is contained in the region \( u \leq u_0 \), where the wall \( \mathcal{W}(E, \mathcal{O}_S) \) is inside the wall \( \mathcal{W}(E/F_j, \mathcal{O}_S) \).

\( \Box \)
We can now prove Proposition 5.5. The only part of Proposition 5.5 that does not follow directly from the two lemmas is the part where we claim that the wall \( \mathcal{W}(E, \mathcal{O}_S) \) is inside the wall \( \mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S) \) for all \(|u| > 0\).

We will prove this by actually proving the following statement:

**Lemma 5.8.** Let \( E \subseteq \mathcal{O}_S \) in \( A \) be an actually destabilizing object with quotient \( Q \). For all \( F_j \) in the Harder-Narasimhan filtration of \( H^{-1}(Q) \), the wall \( \mathcal{W}(E/F_j, \mathcal{O}_S) \) is a Left Hyperbola, and the wall \( \mathcal{W}(E, \mathcal{O}_S) \) is inside the wall \( \mathcal{W}(E/F_j, \mathcal{O}_S) \) for all \(|u| > 0\).

The statement of the proposition follows from this lemma because, since \( F_m = H^{-1}(Q) \), \( E/F_m \) is a scheme \( Z \) for some zero-dimensional scheme \( Z \). Then, for \(|u| > 0\), the wall \( \mathcal{W}(E, \mathcal{O}_S) \) must be inside the wall \( \mathcal{W}(Z(-C), \mathcal{O}_S) \), which is inside the wall \( \mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S) \).

**Proof (of Lemma 5.8).** Assume that \( E \) actually destabilizes \( \mathcal{O}_S \) at a stability condition \( \sigma \), and that \( u(\sigma) > 0 \). We know that all of the walls \( \mathcal{W}(E/F_j, \mathcal{O}_S) \) must be Left Hyperbolas by the proof of the previous lemma. Indeed, if they were not Left Hyperbolas, then there would exist a \( j \) such that the wall \( \mathcal{W}(E, \mathcal{O}_S) \) would be inside the wall \( \mathcal{W}(E/F_j, \mathcal{O}_S) \) at \( u(\sigma) \) whenever \( E \subseteq \mathcal{O}_S \) in \( A \), making it impossible for the wall \( \mathcal{W}(E, \mathcal{O}_S) \) to be an actually destabilizing wall.

Let \( u_0 \) be the largest value of \( u \) such that \( \mathcal{W}(E, \mathcal{O}_S) \) intersects \( s = \mu_H(J) \) (the intersection will be at \( t = 0 \)). Then we know that the region where \( E \subseteq \mathcal{O}_S \) in \( A \) and \( \beta(E) \) is contained within \( s \geq \mu_H(F_1) \) and \( 0 < u < u_0 \). Moreover, as above, we have that \( \mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{u_0} \leq \mathcal{W}(E/F_1, \mathcal{O}_S) \cap \Pi_{u_0} \).

For \( E \) to actually destabilize \( \mathcal{O}_S \) at \( \sigma \), we must have that \( \sigma(u) < u_0 \), and \( \mathcal{W}(E/F_1, \mathcal{O}_S) \cap \Pi_{u(\sigma)} \leq \mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{u(\sigma)} \). Using Lemma 4.2 as above, we see that since the wall \( \mathcal{W}(E/F_1, \mathcal{O}_S) \) is inside the wall \( \mathcal{W}(E, \mathcal{O}_S) \) at \( u(\sigma) \), and the wall \( \mathcal{W}(E, \mathcal{O}_S) \) is inside the wall \( \mathcal{W}(E/F_1, \mathcal{O}_S) \) at \( u_0 > u(\sigma) \), we know that the wall \( \mathcal{W}(E, \mathcal{O}_S) \) must stay inside for all \( u \geq u_0 \). Thus, if \( m = 1 \) we are done.

If \( m > 1 \), let \( u_1 > u_0 \) be the largest value of \( u \) such that \( \mathcal{W}(E/F_1, \mathcal{O}_S) \) intersects \( s = \mu_H(F_2/F_1) \). Then, \( \mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{u_1} \leq \mathcal{W}(E/F_1, \mathcal{O}_S) \cap \Pi_{u_1} \leq \mathcal{W}(E/F_2, \mathcal{O}_S) \cap \Pi_{u_1} \), as above.

Since the wall \( \mathcal{W}(E/F_2, \mathcal{O}_S) \) is inside the wall \( \mathcal{W}(E, \mathcal{O}_S) \) at \( u(\sigma) \), and the wall \( \mathcal{W}(E, \mathcal{O}_S) \) is inside the wall \( \mathcal{W}(E/F_1, \mathcal{O}_S) \) at \( u_1 > u_0 > u(\sigma) \), we know that the wall \( \mathcal{W}(E, \mathcal{O}_S) \) must stay inside for all \( u \geq u_1 \).
Continuing in a similar manner proves the statement for all $1 \leq j \leq m$. \hfill \qed

5.4 Surfaces of Picard Rank 2

We now restrict our attention to surfaces of Picard Rank 2, where we can describe the actually destabilizing walls for $\mathcal{O}_S$ more precisely. We pause to illuminate a fact which is helpful in this situation.

**Remark 5.9.** Let $S$ have Picard rank 2. Then for any line bundle $\mathcal{O}_S(D')$ and $\sigma_{D,H} \in \mathcal{S}_{H,G}$ we have $\mathcal{O}_S(D') \otimes \sigma_{D,H} = \sigma_{D+D',H} \in \mathcal{S}_{H,G}$.

Let $S$ be a surface of Picard Rank 2, and let $H$ and $G$ be as above. Moreover, let $C_1$ and $C_2$ be the generators of the cone of effective curves on $S$. Since $H$ is ample, we must have that $H = eC_1+fC_2$ for some $e, f > 0$. Then, $H.G = 0$ implies that $fC_2.G = -eC_1.G$. Therefore, $(C_1.G) \cdot (C_2.G) < 0$. Assume that $C_1.G > 0$ and $C_2.G < 0$.

We saw in Proposition 5.5 that every destabilizing wall is a Left Hyperbola. Moreover, if we let $\beta$ be as above. So we can assume that the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ at $u(\sigma)$. (A similar proof would work with $C_2$ in place of $C_1$ if $u(\sigma) \leq C_2.G$.)

If the rank of $E$ is 1, and $\beta(E) \geq \beta(\mathcal{O}_S)$ at $\sigma$, then $E = \mathcal{I}_Z(-C)$ for some curve $C$ of negative self-intersection and some zero-dimensional scheme $Z$. Since $\beta(\mathcal{O}_S(-C)) \geq \beta(\mathcal{I}_Z(-C)) \geq \beta(\mathcal{O}_S)$ at $\sigma$ with $u(\sigma) \geq C_1.G > 0$, we must have that $C = aC_1 + bC_2$ with $a > 0$. Therefore, $\mathcal{O}_S(-C) \subseteq \mathcal{O}_S(-C_1)$.

Since the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ for $u >> 0$ because of the slope of their asymptotes (see Section 4.1), and it is inside of it at $u = C_1.G$ because $\mathcal{O}_S(-C_1)$ is always Bridgeland-stable there, it must be inside of it for all $u \geq C_1.G$ by Lemma 4.2.

Assume now that $E$ has rank $r > 1$, and that the statement is true for all subobjects of $\mathcal{O}_S$ of rank $< r$.

By Lemma 5.6, we can assume that the wall $\mathcal{W}(E, \mathcal{O}_S)$ is a Left Hyperbola. Moreover, if we let $C = c_1(H^0(Q))$, where $Q$ is the quotient of $\mathcal{O}_S$ by $E$
in \(A\), we have that the wall \(W(\mathcal{O}_S(-C), \mathcal{O}_S)\) is also a Left Hyperbola, and that the wall \(W(E, \mathcal{O}_S)\) is inside the wall \(W(\mathcal{O}_S(-C), \mathcal{O}_S)\) for \(u >> 0\). By the rank 1 case, we know that the wall \(W(\mathcal{O}_S(-C), \mathcal{O}_S)\) is inside the wall \(W(\mathcal{O}_S(-C_1), \mathcal{O}_S)\) for all \(u \geq C_1.G\).

If the wall \(W(E, \mathcal{O}_S)\) is inside the wall \(W(\mathcal{O}_S(-C_1), \mathcal{O}_S)\) at \(u(\sigma)\), we are done. Supposing not, we have by Lemma 4.2 that the wall \(W(\mathcal{O}_S(-C_1), \mathcal{O}_S)\) is inside the wall \(W(E, \mathcal{O}_S)\) for all \(u \leq u(\sigma)\). Denote this statement by \((\star)\).

If \(W(E, \mathcal{O}_S) \cap \Pi_{u(\sigma)}\) intersects \(s = \mu_H(K)\) (using our usual notation for the Harder-Narasimhan filtration from Section 4.3), then it will be inside the wall \(W(E_{n-1}, \mathcal{O}_S) \cap \Pi_{u(\sigma)}\), which will be inside the wall \(W(\mathcal{O}_S(-C_1), \mathcal{O}_S) \cap \Pi_{u(\sigma)}\) by induction, contradicting \((\star)\). Thus \(W(E, \mathcal{O}_S) \cap \Pi_{u(\sigma)}\) does not intersect \(s = \mu_H(K)\), and since \(u(\sigma) \geq C_1.G\), \((\star)\) implies that the wall \(W(\mathcal{O}_S(-C_1), \mathcal{O}_S)\) is inside the wall \(W(E, \mathcal{O}_S)\) at \(u = C_1.G\). Since \(\mathcal{O}_S(-C_1)\) is Bridgeland stable at \(u = C_1.G\), this can only happen if \(E\) is not a subobject of \(\mathcal{O}_S(-C_1)\) in \(A\) at the points on the wall \(W(E, \mathcal{O}_S) \cap \Pi_{C_1.G}\). This means that the wall \(W(E, \mathcal{O}_S) \cap \Pi_u\) had to intersect \(s = \mu_H(K)\) for some \(C_1.G \leq u < u(\sigma)\).

Consider a value of \(u\) with \(C_1.G \leq u < u(\sigma)\) where the wall \(W(E, \mathcal{O}_S)\) intersects \(s = \mu_H(K)\). There, the wall \(W(E, \mathcal{O}_S)\) would have to be inside the wall \(W(E_{n-1}, \mathcal{O}_S)\) by Lemma 4.3 and the wall \(W(E_{n-1}, \mathcal{O}_S)\) would be inside \(W(\mathcal{O}_S(-C_1), \mathcal{O}_S)\) by induction hypothesis, contradicting \((\star)\). Thus the wall \(W(E, \mathcal{O}_S)\) is inside the wall \(W(\mathcal{O}_S(-C_1), \mathcal{O}_S)\) at \(u(\sigma)\). \(\square\)

We now prove the second part of our main Theorem 1.1 i.e., that Conjecture 1 is true for surfaces of Picard Rank 2 that only have one irreducible curve of negative self-intersection. We first give a lemma, then prove a result which is stronger than the conjecture in this situation.

**Lemma 5.11.** Let \(S\) be a surface of Picard Rank 2. Assume that the cone of effective curves is generated by \(C_1\) and \(C_2\), that \(C_1.G > 0\), and that \(C_1\) is the only irreducible curve in \(S\) of negative self-intersection. Let \(C\) be an effective curve. If \(C.G < 0\), then \(C^2 \geq 0\).

**Proof.** We saw above that we can write \(H = eC_1 + fC_2\) with \(e, f > 0\), and obtain \(fC_2.G = -eC_1.G\). Since \((eC_1 + fC_2)^2 = H^2 > 0\), we have that \(2efC_1.C_2 > -e^2C_1^2 - f^2C_2^2\).

Let \(C\) be an effective curve. We have that \(C = aC_1 + bC_2\) with \(a, b \geq 0\). Assume that \(C.G < 0\). Then, \(0 > fC.G = afC_1.G + bfC_2.G = (af - be)C_1.G\), and therefore, \(af - be < 0\).
Proposition 5.12. Let $S$ be a surface of Picard Rank 2. Assume that the cone of effective curves is generated by $C_1$ and $C_2$, that $C_1.G > 0$, and that $C_1$ is the only irreducible curve in $S$ of negative self-intersection. Then $\mathcal{O}_S$ is only destabilized by $\mathcal{O}_S(-C_1)$.

Proof. Note that, since $C_1$ is the only irreducible curve of negative self-intersection, then $\mathcal{O}_S(-C_2)$ does not weakly destabilize $\mathcal{O}_S$, and the wall $\mathcal{W}(\mathcal{O}_S(-C_2), \mathcal{O}_S)$ is empty in the region where $\mathcal{O}_S(-C_2) \in \mathcal{A}$.

We prove the following statement by induction on the rank of $E$: If $E \subseteq \mathcal{O}_S$ for some stability condition $\sigma$, and $\beta(E) \geq \beta(\mathcal{O}_S)$, then the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ at $u(\sigma)$.

If $u(\sigma) \geq C_1.G$, we already know this statement to be true by the proof of Proposition 5.10. We therefore only need to prove the statement in the case when $u(\sigma) < C_1.G$.

If the rank of $E$ is 1, and $\beta(E) \geq \beta(\mathcal{O}_S)$ at $\sigma$, then $E = \mathcal{I}_Z(-C)$ for some curve $C$ of negative self-intersection and some zero-dimensional scheme $Z$. Suppose, by contradiction, that the wall $\mathcal{W}(E, \mathcal{O}_S)$ were not inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ at $u(\sigma)$. Choose a $\sigma' \in \mathcal{W}(E, \mathcal{O}) \cap \Pi_{u(\sigma)}$. Then at $\sigma'$ we have $\beta(\mathcal{O}_S(-C)) \geq \beta(L_Z(-C)) > \beta(\mathcal{O}_S(-C_1))$. Therefore, at $\mathcal{O}_S(C_1) \otimes \sigma'$ we have $\beta(\mathcal{O}_S(-C + C_1)) > \beta(\mathcal{O}_S)$. This means that $\mathcal{O}_S(-C + C_1)$ weakly destabilizes $\mathcal{O}_S$ at $\mathcal{O}_S(C_1) \otimes \sigma'$. By Proposition 5.1, we must have that $(C - C_1)^2 < 0$, and by Lemma 5.11 $(C - C_1).G > 0$. This implies that the wall $\mathcal{W}(\mathcal{O}_S(-C + C_1), \mathcal{O}_S)$ is a Left Hyperbola that could only weakly destabilize $\mathcal{O}_S$ in the region $u > 0$, but we have $u(\mathcal{O}_S(C_1) \otimes \sigma') = u(\sigma) - C_1.G < 0$.

Assume now that $E$ has rank $r > 1$, and that the statement is true for all subobjects of $\mathcal{O}_S$ of rank $r < r$.

By Lemma 5.6, we can assume that the wall $\mathcal{W}(E, \mathcal{O}_S)$ is a Left Hyperbola. Moreover, if we let $C = c_1(H^0(Q))$, we have that the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ is also a Left Hyperbola, and that the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ for $u >> 0$. By the rank 1 case, we know that the wall $\mathcal{W}(\mathcal{O}_S(-C), \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ for all $u$.

If the wall $\mathcal{W}(E, \mathcal{O}_S)$ is inside the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ at $u(\sigma)$, we are done. Supposing not, we have by Lemma 1.2 that the wall $\mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S)$ is inside the wall $\mathcal{W}(E, \mathcal{O}_S)$ for all $u \leq u(\sigma)$. Denote this statement by $(*).

If $\mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{u(\sigma)}$ intersects $s = \mu_H(K)$ (using our usual notation for the Harder-Narasimhan filtration from Section 4.3), then it will be inside the
wall \( \mathcal{W}(E_{n-1}, \mathcal{O}_S) \cap \Pi_{u(\sigma)} \), which will be inside the wall \( \mathcal{W}(\mathcal{O}_S(-C_1), \mathcal{O}_S) \cap \Pi_{u(\sigma)} \) by induction, contradicting (\( \ast \)).

Thus \( \mathcal{W}(E, \mathcal{O}_S) \cap \Pi_{u(\sigma)} \) does not intersect \( s = \mu_H(K) \). Choose a \( \sigma' \in \mathcal{W}(E, \mathcal{O}_S)\Pi_{u(\sigma)} \). Then \( \beta(E) > \beta(\mathcal{O}_S(-C_1)) \) at \( \sigma' \) implies \( \beta(E(C_1)) > \beta(\mathcal{O}_S) \) at \( \mathcal{O}_S(C_1) \otimes \sigma' \). Therefore, \( E(C_1) \) weakly destabilizes \( \mathcal{O}_S \) at \( \mathcal{O}_S(C_1) \otimes \sigma' \).

Note that \( u(\mathcal{O}_S(C_1) \otimes \sigma') = u(\sigma) - C_1.G < 0 \). Consider the highest semi-circular wall at \( u(\mathcal{O}_S(C_1) \otimes \sigma') \), corresponding to an object \( E' \subseteq \mathcal{O}_S \). Since \( E' \) actually destabilizes \( \mathcal{O}_S \), we have that the wall \( \mathcal{W}(E', \mathcal{O}_S) \) must be a Left Hyperbola. However, the proof of Proposition [5.10] shows that the wall \( \mathcal{W}(E', \mathcal{O}_S) \) would have to be inside the wall \( \mathcal{W}(\mathcal{O}_S(-C_2), \mathcal{O}_S) \) for all \( u \leq C_2.G \). But this cannot be true, because the wall \( \mathcal{W}(E, \mathcal{O}_S) \) is a Left Hyperbola, and the wall \( \mathcal{W}(\mathcal{O}_S(-C_2), \mathcal{O}_S) \) is empty. \( \square \)

6 Bridgeland Stability of \( \mathcal{O}_S[1] \)

We now move on to studying the stability of \( \mathcal{O}_S[1] \). This can be done via duality, except for the stability conditions \( \sigma_{D, H} \) with \( D = uG \), i.e. \( s = 0 \).

Note that, given a Bridgeland stability condition \( \sigma \), \( \mathcal{O}_S[1] \in \mathcal{A} \) iff \( s \geq 0 \).

6.1 Subobjects of \( \mathcal{O}_S[1] \)

Let \( \sigma \) be a Bridgeland stability condition, and let \( E \) be a proper subobject of \( \mathcal{O}_S[1] \) in \( \mathcal{A} \). We have a short exact sequence \( 0 \rightarrow E \rightarrow \mathcal{O}_S[1] \rightarrow Q' \rightarrow 0 \) in \( \mathcal{A} \) for some \( Q' \in \mathcal{A} \). The long exact sequence in cohomology is

\[
0 \rightarrow H^{-1}(E) \rightarrow \mathcal{O}_S \rightarrow H^{-1}(Q) \rightarrow H^0(E) \rightarrow 0,
\]

and therefore \( Q' = H^{-1}(Q)[1] \) is the shift of a sheaf. We will denote \( H^{-1}(Q) \) by \( Q \). Also, since \( H^{-1}(Q) \in \mathcal{F} \) is torsion-free, we have that either \( H^{-1}(E) = \mathcal{O}_S \) or \( H^{-1}(E) = 0 \). However, if \( H^{-1}(E) = \mathcal{O}_S \), then \( H^{-1}(Q) = H^0(E) \), and this is not possible, since the first sheaf is in \( \mathcal{F} \), and the second one is in \( \mathcal{T} \). Therefore, \( H^{-1}(E) = 0 \), and \( E = H^0(E) \) is a sheaf.

To summarize, if \( E \subseteq \mathcal{O}_S[1] \) is a proper subobject in \( \mathcal{A} \), then \( E \) is a sheaf in \( \mathcal{T} \), and the quotient is of the form \( Q[1] \) for some sheaf \( Q \in \mathcal{F} \). We have a short exact sequence of sheaves

\[
0 \rightarrow \mathcal{O}_S \rightarrow Q \rightarrow E \rightarrow 0.
\]
6.2 The $s = 0$ case

Let us start by proving that $\mathcal{O}_S[1]$ is Bridgeland stable when $s = 0$.

**Lemma 6.1.** If $s = 0$, $\mathcal{O}[1]$ has no proper subobjects in $\mathcal{A}$, and is therefore Bridgeland stable.

**Proof.** Let $E \subseteq \mathcal{O}_S[1]$ be a proper subobject of $\mathcal{O}_S[1]$ in $\mathcal{A}$ with quotient $Q[1]$ as above. Since $s = 0$, we have that $\text{Im} Z(\mathcal{O}_S[1]) = 0$. Since $\text{Im} Z(\mathcal{O}_S[1]) = \text{Im} Z(E) + \text{Im} Z(Q[1])$, and they all have non-negative imaginary parts, we must have that $\text{Im} Z(E) = \text{Im} Z(Q[1]) = 0$. Therefore, all three objects have maximal phase. The only objects in $\mathcal{T}$ of maximal phase are torsion sheaves supported in dimension 0. We therefore have the short exact sequence of sheaves $0 \to \mathcal{O}_S \to Q \to E \to 0$, with $E$ a torsion sheaf supported in dimension 0. But this cannot happen unless the sequence splits, in which case $Q$ would have torsion, which is impossible. Therefore, if $s = 0$, $\mathcal{O}_S[1]$ cannot have proper subobjects, and is Bridgeland stable. \qed

We can therefore assume that $s > 0$.

6.3 Duality

The following duality result allows us to apply results on the stability of $\mathcal{O}_S$ to that of $\mathcal{O}_S[1]$. It follows as in [Mar] Lemma 3.2 with a slightly different choice of functor. Specifically, we consider the functor $E \mapsto E^\vee := R\text{Hom}(E, \mathcal{O}_S)[1]$. Note that $\mathcal{O}_S^\vee = \mathcal{O}_S[1]$ and vice versa.

**Lemma 6.2.** Let $D$ be a divisor with $D.H < 0$. Then $\mathcal{O}_S$ is $\sigma_{D,H}$-(semi)stable if and only if $\mathcal{O}_S[1]$ is $\sigma_{-D,H}$-(semi)stable.

**Proof.** This follows from [Mar] Lemma 3.2(d)]. \qed

Note that if $\mathcal{O}_S(-C) \subseteq \mathcal{O}_S$ destabilizes $\mathcal{O}_S$ at $\sigma_{D,H}$, then applying $(\_)^\vee$ shows that the quotient $\mathcal{O}_S[1] \to \mathcal{O}_S(C)[1]$ destabilizes $\mathcal{O}_S[1]$ at $\sigma_{-D,H}$. The kernel of the map $\mathcal{O}_S[1] \to \mathcal{O}_S(C)[1]$ in $\mathcal{A}$ is $\mathcal{O}_S(C)|_C$. Thus Theorem 5.4, Proposition 5.12, and Lemma 6.1 yield the following result.

**Proposition 6.3.**

1. Let $S$ be a surface of any Picard rank such that there are no curves $C \subset S$ with $C^2 < 0$. Then $\mathcal{O}_S[1]$ is $\sigma$-stable whenever in $\mathcal{A}$.  

27
2. Let $S$ be a surface of Picard Rank 2. Assume that the cone of effective curves is generated by $C_1$ and $C_2$, that $C_1 G > 0$, and that $C_1$ is the only irreducible curve in $S$ of negative self-intersection. Then $\mathcal{O}_S[1]$ is only destabilized by $\mathcal{O}_S(C_1)|_{C_1}$.

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