Relating Granger causality to directed information theory for networks of stochastic processes

Pierre-Olivier Amblard\textsuperscript{1,2} and Olivier J.J. Michel\textsuperscript{1}

\textsuperscript{1} GIPSAlab/CNRS UMR 5216/ BP46, 38402 Saint Martin d’Hères cedex, France
\textsuperscript{2} The University of Melbourne, Dept. of Math&Stat. Parkville, VIC, 3010, Australia

bidou.amblard@gipsa-lab.inpg.fr
olivier.michel@gipsa-lab.grenoble-inp.fr

Abstract

This paper addresses the problem of inferring circulation of information between multiple stochastic processes. We discuss two possible frameworks in which the problem can be studied: directed information theory and Granger causality. The main goal of the paper is to study the connection between these two frameworks. In the case of directed information theory, we stress the importance of Kramer’s causal conditioning. This type of conditioning is necessary not only in the definition of the directed information but also for handling causal side information. We also show how directed information decomposes into the sum of two measures, the first one related to Schreiber’s transfer entropy quantifies the dynamical aspects of causality, whereas the second one, termed instantaneous information exchange, quantifies the instantaneous aspect of causality. After having recalled the definition of Granger causality, we establish its connection with directed information theory. The connection is particularly studied in the Gaussian case, showing that Geweke’s measures of Granger causality correspond to the transfer entropy and the instantaneous information exchange. This allows to propose an information theoretic formulation of Granger causality.

keywords directed information, transfer entropy, Granger causality, graphical models

I. INTRODUCTION

The importance of the network paradigm for the analysis of complex systems, in fields ranging from biology and sociology to communication theory or computer science, gave rise recently to the emergence of new research interests referred to as network science or complex network \cite{9}, \cite{18}, \cite{50}. Characterizing the interactions between the nodes of such a network is a major issue for understanding its global behavior and identifying its topology. It is customary assumed that nodes may be observed via the recording of (possibly multivariate) time series at each of them, modeled as realizations of stochastic processes (see \cite{60}, \cite{61} for examples in biology, or \cite{43}, \cite{66}, \cite{12}, \cite{35} for applications in neurosciences). The assessment of an interaction between two nodes is then formulated as
a interaction detection/estimation problem between their associated time series. Determining the existence of edges between given nodes (or vertices) of a graph may be reformulated in a graphical modeling inference framework [74], [38], [15], [54], [39]. Describing connections in a graph requires to provide a definition for the interactions that will be carried by the edges connecting the nodes. Connectivity receives different interpretations in the neuroscience literature for instance, depending on whether it is ‘functional’, revealing some dependence, or ‘effective’ in the sense that it accounts for directivity [29], [66]. This differentiation in the terms describing connectivity raises the crucial issue of causality, that goes beyond the problem of simply detecting the existence or the strength of an edge linking two nodes.

Detecting whether a connection between two nodes can be given a direction or two can be addressed by identifying possible ‘master-slave’ relationships between nodes. Based on the measurements of two signals \( x_t \) and \( y_t \), the question is: ‘Does \( x_t \) influences \( y_t \) more than \( y_t \) influences \( x_t \)’. Addressing this problem requires the introduction of tools that account for asymmetries in the signals information exchanges.

Granger and others investigated this question using the concept of causality [24], [25], [20], [54] and emphasized that interaction between two processes is relative to the set of observed nodes. Actually, the possible interactions of the studied pair of nodes with other nodes from the network may profoundly alter the estimated type of connectivity. This leads to fundamental limitations of pairwise approaches for multiply connected network studies. Many authors addressed the topic of inferring causal relationship between interacting stochastic systems under the restriction of linear/Gaussian assumptions. In [20], [21] the author develops a general linear modeling approach in the time domain. A spectral domain definition of causal connectivity is proposed in [31], whose relationship with Granger causality is explored in [17]. However, all these techniques need to be extended or revisited to tackle nonlinearity and/or nonGaussianity.

Information-theoretic tools provide a means to go beyond Gaussianity. Mutual information characterizes the information exchanged between stochastic processes [13], [56]. It is however a symmetric measure and does not provide any insight on possible directionality. Many authors have managed to modify mutual information in order to obtain asymmetrical measures. These are for example Saito and Harashima’s transinformation [63], [32], [1], the coarse grained transinformation proposed by Palus et al. [51], [52], Schreiber’s transfer entropy [64], [30]. All these measures share common roots which are revealed using directed information theory.

A. Main contributions of the paper

This paper is an attempt to make sense of and to systematize the various definitions and measures of causal dependence that have been proposed to date. Actually, we claim that these measures can be reduced to directed information, with or without additional causal conditioning. Directed information introduced by Massey in 1990 [46] and based on the earlier results on Marko’s bidirectional information theory [45], is shown to be an adequate quantity to address the topic of causal conditioning within an information theoretic framework. Kramer, Tatikonda and others have used directed information to study communication problems in systems with feedback [34], [67], [71], [68]. Although their work aimed at developing new bounds on the capacity of channels with feedback and
optimizing directed information, most of their results allow better insight in causal connectivity problems for systems that may exhibit feedback.

Massey’s directed information will be extensively used to quantifying directed information flow between stochastic processes. We show how directed information, which is intimately linked to feedback, provides a nice answer to the question of characterizing directional influence between processes, in a fully general framework. A contribution of this paper is to describe the link between Granger causality and directed information theory, both in the bivariate and multivariate cases. It is shown that causal conditioning plays a key role as its main measure, directed information, can be used to assess causality, instantaneous coupling and feedback in graphs of stochastic processes. A main contribution is then a reformulation of Granger causality in terms of directed information theoretic concepts.

**B. Organization of the paper**

As outlined in the preceding sections, directed information plays a key role in defining information flows in networks [45], [63], [32], [46], [34], [67], [68], [60], [61], [65], [3], [5]. Section III gives a formal development of directed information following earlier works of Massey, Kramer and Tatikonda [46], [34], [67]. Feedback in the definition of directed information is revisited, together with its relation to Kramer’s causal conditioning [34]. This paper extends these latter ideas and shows that causally conditioned directed information is a means of measuring directed information in networks: it actually accounts for the existence of other nodes interacting with those studied. The link between directed information and transfer entropy [64] established in this section is a contribution of the paper. In section III we present Granger causality which relies on forward prediction. We particularly insist on the case of multivariate time series. Section IV is devoted to developing the connection between the present information theoretic framework and Granger causality. Although all results hold in a general framework explained in section IV-C, a particular attention is given to the Gaussian case. In this case directed information theory and Granger causality are shown to lead to equivalent tools to assess directional dependencies (see also [5]). This extends similar recent results independently obtained by Barnett et. al. [8] in the case of two interacting signals without instantaneous interaction. An enlightening illustration of the interactions between three time series is presented in section V for a particular Gaussian model.

**II. MEASURING DIRECTIONAL DEPENDENCE**

**A. Notations and basics**

Throughout the paper we consider discrete time, finite variance $E[|x|^2] < +\infty$ stochastic processes. Time samples are indexed by $\mathbb{Z}$; $x^n_k$ stands for the vector $(x_k, x_{k+1}, \ldots, x_n)$, whereas for $k = 1$, the index will be omitted for the sake of readability. Thus we identify the time series $\{x(k), k = 1, \ldots, n\}$ with the vector $x^n$. $E[x|.]$ will denote the expectation with respect to the probability measure describing $x$, whereas $E_p[.]$ will indicate that the expectation is taken with respect to the probability distribution $p$.

In all the paper, the random variables (vectors) considered are either purely discrete, or continuous with the added assumption that the probability measure is absolutely continuous with respect to the Lebesgue measure. Therefore,
all derivations hereafter are valid for either cases. Note however that existence of limits will be in general assumed when necessary and not proved.

Let $H(x^n) = -E_x[\log p(x^n)]$ be the entropy of a random vector $x^n$ whose density is $p$. Let the conditional entropy be defined as $H(x^n|y^n) = -E[\log p(x^n|y^n)]$. The mutual information $I(x^n; y^n)$ between vectors $x^n$ and $y^n$ is defined as [13]:

$$I(x^n; y^n) = H(y^n) - H(y^n|x^n)$$

$$= D_{KL} \left( p(x^n, y^n)||p(x^n)p(y^n) \right) \tag{1}$$

where $D_{KL}(p||q) = E_p[\log p(x)/q(x)]$ is the Kulback-Leibler divergence. It is 0 if and only if $p = q$ almost everywhere and is positive otherwise. The mutual information effectively measures independence since it is 0 if and only if $x^n$ and $y^n$ are independent random vectors. As $I(x^n; y^n) = I(y^n; x^n)$, mutual information cannot handle directional dependence.

Let $z^n$ be a third time series. It may be a multivariate process accounting for side information (all available observation but $x^n$ and $y^n$). To account for $z^n$, the conditional mutual information is introduced:

$$I(x^n; y^n|z^n) = E_z \left[ D_{KL} \left( p(x^n, y^n|z^n)||p(x^n|z^n)p(y^n|z^n) \right) \right] \tag{2}$$

$$= D_{KL} \left( p(x^n, y^n, z^n)||p(x^n|z^n)p(y^n|z^n)p(z^n) \right) \tag{3}$$

$I(x^n; y^n|z^n)$ is zero if and only if $x^n$ and $y^n$ are independent conditionally to $z^n$. Stated differently, conditional mutual information measures the divergence between the actual observations and those which would be observed under Markov assumption ($x \rightarrow z \rightarrow y$). Arrows may be misleading here, as by reversibility of Markov chains, the equality above holds also for ($y \rightarrow z \rightarrow x$). This again emphasizes the inability of mutual information to provide answers to the information flow directivity problem.

B. Directed information

Directed information was introduced by Massey [46], based on the previous concept of “bidirectional information” of Marko [45]. Bidirectional information focuses on the two nodes problem, but accounts for the respective roles of feedback and memory in the information flow.

1) Feedback and memory: Massey [46] noted that the joint probability distribution $p(x^n, y^n)$ can be written as a product of two terms:

$$\hat{p}(x^n|y^{n-1}) = \prod_{i=1}^{n} p(x_i|x^{i-1}, y^{i-1})$$

$$\hat{p}(y^n|x^n) = \prod_{i=1}^{n} p(y_i|x_i, y^{i-1})$$

$$p(x^n, y^n) = \hat{p}(x^n|y^{n-1})\hat{p}(y^n|x^n) \tag{4}$$

where for $i = 1$ the first terms are respectively $p(x_1)$ and $p(y_1|x_1)$. Assuming that $x$ is the input of a system that creates $y$, $\hat{p}(x^n|y^{n-1})$ can be viewed as a characterization of feedback in the system. Therefore the name feedback
factor: each of the factors controls the probability of the input $x$ at time $i$ conditionally to its past and to the past values of the output $y$. Likewise, the term $\overrightarrow{p}(y^n|x^n)$ will be referred to as the feedforward factor. The factorization leads to some remarks:

- In the absence of feedback in the link from $x$ to $y$, one has
  \[ p(x_i|x^{i-1}, y^{i-1}) = p(x_i|x^{i-1}) \quad \forall i \geq 2 \] (5)
  or equivalently
  \[ H(x_i|x^{i-1}, y^{i-1}) = H(x_i|x^{i-1}) \quad \forall i \geq 2 \] (6)

As a consequence:

- If the feedforward factor does not depend on the past, the link is memoryless:
  \[ p(y_i|x_i) = p(y_i|x^i, y^{i-1}) \quad \forall i \geq 1 \] (8)

- Let $D$ be the unit delay operator, such that $Dy_n = y_{n-1}$. We define $Dy^n = (0, y_1, y_2, \ldots, y_{n-1})$ for finite length sequences, in order to deal with edge effects while maintaining constant dimension for the studied time series. Then we have
  \[ \overrightarrow{p}(x^n|Dy^n) = \overrightarrow{p}(x^n|y^{n-1}) \] (9)

The feedback term in the link $x \to y$ is the feedforward term of the delayed sequence in the link $y \to x$.

2) Causal conditioning and directed information: In [34], Kramer introduced an original point of view, based upon the following remark. The conditional entropy is easily expanded (using Bayes rules) according to

\[ H(y^n|x^n) = \sum_{i=1}^{n} H(y_i|y^{i-1}, x^n) \] (10)

where each term in the sum is the conditional entropy of $y$ at time $i$ given its past and the whole observation of $x$: Causality (if any) in the dynamics $x \to y$ is thus not taken into account. Assuming that $x$ influences $y$ through some unknown process, Kramer proposed that the conditioning of $y$ at time $i$ should include $x$ from initial time up to time $i$ only. He named this causal conditioning, and defined causal conditional entropy as

\[ H(y^n|x^n) = \sum_{i=1}^{n} H(y_i|y^{i-1}, x^i) \] (11)

By plugging causal conditional entropy in the expression of mutual information in place of the conditional entropy, we obtain a definition of directed information:

\[ I(x^n \to y^n) = H(y^n) - H(y^n|x^n) = \sum_{i=1}^{n} I(x^i; y_i|y^{i-1}) \] (12)

1The term $0$ in $Dy^n = (0, y_1, y_2, \ldots, y_{n-1})$ indicates a wild card which plays no influence on conditioning, and makes sense as $y_0$ is not assumed observed.
Alternately, Tatikonda’s work [67] leads to express directed information as a Kullback-Leibler divergence:\footnote{The proofs rely on the use of the chain rule $I(X, Y; Z) = I(Y; Z|X) + I(X; Z)$ in the definition of the directed information.}

$$I(x^n \to y^n) = D_{KL}(p(x^n, y^n) || \hat{p}(x^n) | y^n_{n-1}) p(y^n) \bigg)$$

$$= E \left[ \log \frac{p(x^n, y^n)}{\hat{p}(x^n) | y^n_{n-1}) p(y^n) \bigg) \right]$$

$$= E \left[ \log \frac{\hat{p}(y^n | x^n)}{p(y^n) } \right]$$

(14)

The expression (14) highlights the importance of the feedback term when comparing mutual information with directed information: $p(x^n)$ in the expression of the mutual information is replaced by the feedback factor $\hat{p}(x^n | y^n_{n-1})$ in the definition directed information.

This result allows the derivation of many (in)equalities rapidly. First, as a divergence, the directed information is always positive. Then, since

$$I(x^n \to y^n) = E \left[ \log \left( \frac{p(x^n, y^n)}{\hat{p}(x^n | y^n_{n-1})} \times \frac{p(x^n)}{p(y^n)} \right) \right]$$

Using equations (9) and (14) we get

$$- E \left[ \log \frac{p(x^n)}{\hat{p}(x^n | y^n_{n-1})} \right] = I(Dy^n \to x^n)$$

(15)

Substituting this result into eq. (12) we obtain

$$I(x^n \to y^n) = I(x^n; y^n) + E \left[ \log \frac{p(x^n)}{\hat{p}(x^n | y^n_{n-1})} \right]$$

$$= I(x^n; y^n) - \sum_i I(x_i; y_i^{-1} | x_i^{-1})$$

$$= I(x^n; y^n) - I(Dy^n \to x^n)$$

(16)

Equation (17) is fundamental as it shows how mutual information splits into the sum of a feedforward information flow $I(x^n \to y^n)$ and a feedback information flow $I(Dy^n \to x^n)$. In this absence of feedback, $\hat{p}(x^n | y^n) = p(x^n)$ and $I(x^n; y^n) = I(x^n \to y^n)$. Equation (16) shows that the mutual information is always greater than the directed information, since $I(Dy^n \to x^n) = \sum_i I(x_i; y_i^{-1} | x_i^{-1}) \geq 0$. As a sum of positive terms, it is zero if and only if all the terms are zero:

$$I(x_i; y_i^{-1} | x_i^{-1}) = 0 \quad \forall i = 2, \ldots, n$$

or equivalently

$$H(x_i | x_i^{-1}, y_i^{-1}) = H(x_i | x_i^{-1}) \quad \forall i = 2, \ldots, n$$

(18)

This last equation states that without feedback, the past of $y$ does not influence the present of $x$ when conditioned on its own past. Alternately, one sees that if eq. (18) holds, then the sequence $y_i^{-1} \to x_i^{-1} \to x_i$ forms a Markov chain, for all $i$: again, the conditional probability of $x_i$ given its past does not depend on the past of $y$. Equalities
(18) can be considered as a definition of the absence of feedback from \( y \) to \( x \). All this findings are summarized in the following theorem:

**Theorem:** ([46] and [47]) The directed information is less than or equal to the mutual information, with equality if and only if there is no feedback.

This theorem implies that mutual information over-estimates the directed information between two processes in the presence of feedback. This was thoroughly studied in [34], [67], [71], [68], in a communication theoretic framework.

Summing the information flows in opposite directions gives:

\[
I(x^n \rightarrow y^n) + I(y^n \rightarrow x^n) = E \left[ \log \frac{p(x^n, y^n)}{\overline{p}(x^n \mid y^{n-1})p(y^n)} + \log \frac{p(x^n, y^n)}{\overline{p}(y^n \mid x^n)p(x^n)} \right]
\]

\[
= I(x^n; y^n) + E \left[ \log \frac{\overline{p}(y^n \mid x^n)}{\overline{p}(y^n \mid x^n)} \right]
\]

\[
= I(x^n; y^n) + I(x^n \rightarrow y^n \mid Dx^n)
\] (19)

where

\[
I(x^n \rightarrow y^n \mid Dx^n) = \sum_i I(x_i; y_i \mid y_i^{i-1}, x_i^{i-1})
\]

(20)

This proves \( I(x^n \rightarrow y^n) + I(y^n \rightarrow x^n) \) is symmetrical but is in general not equal to the mutual information, except if and only if \( I(x_i; y_i \mid y_i^{i-1}, x_i^{i-1}) = 0, \forall i = 1, \ldots, n \). Since the term in the sum is the mutual information between the present samples of the two processes conditioned on their joint past values, this measure is a measure of instantaneous dependence. The term \( I(x^n \rightarrow y^n \mid Dx^n) = I(y^n \rightarrow x^n \mid Dy^n) \) will thus be named the instantaneous information exchange between \( x \) and \( y \).

### C. Directed information rates

Entropy as well as mutual information are extensive quantities, increasing (in general) linearly with the length \( n \) of the recorded time series. Shannon’s information rate for stochastic processes compensates the linear growth by considering \( A_\infty(x) = \lim_{n \to +\infty} A_n(x)/n \) (if the limit exists), where \( A_n(x) \) denotes any information measure on the sample \( x \) of length \( n \).

For the important class of stationary processes (see e.g. [13]) the entropy rate turns out to be the limit of the conditional entropy:

\[
\lim_{n \to +\infty} \frac{1}{n} H(x^n) = \lim_{n \to +\infty} H(x_n \mid x^{n-1})
\] (21)

Kramer generalized this result for causal conditional entropies, thus defining the directed information rate for stationary processes as

\[
I_\infty(x \rightarrow y) = \lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} I(x_i; y_i \mid y_i^{i-1})
\]

\[
= \lim_{n \to +\infty} I(x^n; y^n \mid y^{n-1})
\] (22)
This result holds also for the instantaneous information exchange rate. Note that the proof of the result relies on the positivity of the entropy for discrete valued stochastic processes. For continuously valued processes, for which entropy can be negative, the proof is more involved and requires the methods developed in [56], [26], [27], see also [68].

### D. Transfer entropy and instantaneous information exchange

Introduced by Schreiber in [64], [30], transfer entropy evaluates the deviation of the observed data from a model assuming the following joint Markov property

\[
p(y_n | y_{n-k+1}^{n-1}, x_{n-l+1}^{n-1}) = p(y_n | y_{n-k+1}^{n-1})
\]  

(23)

This leads to the following definition

\[
T(x_{n-l+1}^{n-1} \rightarrow y_{n-k+1}^{n}) = E \left[ \log \frac{p(y_n | y_{n-k+1}^{n-1}, x_{n-l+1}^{n-1})}{p(y_n | y_{n-k+1}^{n-1})} \right]
\]  

(24)

Then \(T(x_{n-l+1}^{n-1} \rightarrow y_{n-k+1}^{n}) = 0\) iff eq. (23) is satisfied. Although in the original definition the past of \(x\) in the conditioning may begin at a different time \(m \neq n\), for practical reasons \(m = n\) is considered. Actually, no a priori information is available about possible delays, and setting \(m = n\) allows to compare the transfer entropy with the directed information.

By expressing the transfer entropy as a difference of conditional entropies, we get

\[
T(x_{n-l+1}^{n-1} \rightarrow y_{n-k+1}^{n}) = H(y_n | y_{n-k+1}^{n-1}) - H(y_n | y_{n-k+1}^{n-1}, x_{n-l+1}^{n-1})
\]

\[
= I(x_{n-l+1}^{n-1}; y_{n-k+1}^{n})
\]

(25)

For \(l = n = k\), the identity \(I(x, y; z|w) = I(x; z|w) + I(y; z|x, w)\) leads to

\[
I(x_n; y_n | y_{n-1}^{n-1}) = I(x_{n-1}; y_{n-1}^{n-1}) + I(x_n; y_n | x_{n-1}^{n-1}, y_{n-1}^{n-1})
\]

\[
= T(x_{n-1}^{n-1} \rightarrow y_n^{n-1}) + I(x_n; y_n | x_{n-1}^{n-1}, y_{n-1}^{n-1})
\]

(26)

For stationary processes, letting \(n \rightarrow \infty\) and provided the limits exist, we obtain for the rates

\[
I_\infty(x \rightarrow y) = T_\infty(x \rightarrow y) + I_\infty(x \rightarrow y || Dx)
\]

(27)

Transfer entropy is the part of the directed information that measures the causal influence of the past of \(x\) onto the present of \(y\). However it does not take into account the possible instantaneous dependence of one time series on another, which is handled by directed information.

Moreover, only \(I(x_{i-1}^{i}; y_i | y_{i-1}^{i-1})\) is considered in \(T\), instead of its sum over \(i\) in the directed information. Thus stationarity is implicitly assumed and the transfer entropy has the same meaning as a rate. Summing over \(n\) in eq. (26), the following decomposition of the directed information is obtained

\[
I(x_n \rightarrow y_n) = I(Dx_n \rightarrow y_n) + I(x_n \rightarrow y_n || Dx_n)
\]

(28)

Eq. (28) establishes that the influence of one process on another may be decomposed into two terms accounting for the past and for instantaneous contributions respectively.
E. Accounting for side information

The preceding definitions all aim at proposing definitions of information exchange between \( x \) and \( y \); the possible information gained from possible connections with the rest of the network is not taken into account. The other possibly observed time series are hereafter referred to as side information. The available side information at time \( n \) is noted \( z^n \). Then, two conditional quantities are introduced: conditional directed information and causally conditioned directed information.

\[
I(x^n \rightarrow y^n | z^n) = H(y^n | z^n) - H(y^n | x^n, z^n) 
\]

(29)

\[
I(x^n \rightarrow y^n | z^n) = H(y^n | z^n) - H(y^n | x^n, z^n) 
\]

(30)

where

\[
H(y^n | x^n | z^n) = H(y^n, x^n | z^n) - H(x^n | z^n) 
\]

(31)

\[
H(y^n | x^n | z^n) = \sum_{i=1}^{n} H(y_i | y^{i-1}, x^i, z^n) 
\]

(32)

In these equations, following [34], conditioning goes from left to right: the first conditioning type met is the one applied.

Note that for usual conditioning, variables do not need to be synchronous with the others and can have any dimension. The synchronicity constraint appear in the new definitions above.

For conditional directed information, a conservation law similar to eq. (19) holds:

\[
I(x^n \rightarrow y^n | z^n) + I(Dy^n \rightarrow x^n | z^n) = I(x^n; y^n | z^n) 
\]

(33)

Furthermore, conditional mutual and directed information are equal if and only if

\[
H(x_i | x^{i-1}, y^{i-1}, z^n) = H(x_i | x^{i-1}, z^n) \quad \forall i = 1, \ldots, n
\]

(34)

which means that given the whole observation of the side information, there is no feedback from \( y \) to \( x \). Otherwise stated, if there is feedback from \( y \) to \( x \) and if \( I(Dy^n \rightarrow x^n | z^n) = 0 \), the feedback from \( y \) to \( x \) goes through \( z \).

Finally, let us mention that conditioning with respect to some stationary time series \( z \) similarly leads to define the causal directed information rate as

\[
I_\infty(x \rightarrow y | z) = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^{n} I(x^i; y_i | y^{i-1}, z^i) 
\]

(35)

\[
= \lim_{n \rightarrow +\infty} I(x^n; y_n | y^{n-1}, z^n) 
\]

(36)

This concludes the presentation of directed information. We have put emphasis on the importance of Kramer’s causal conditioning, both for the definition of directed information and for taking into account side information. We have also proven that Schreiber’s transfer entropy is that part of directed information dedicated to the strict sense causal information flow (not accounting for simultaneous coupling). Next section revisits Granger causality as another means for assessing influences between time series.
III. GRANGER CAUSALITY BETWEEN MULTIPLE TIME SERIES

A. Granger’s definition of causality

Although no universally well accepted definition of causality exists, Granger approach of causality is often preferred for two major reasons. The first reason is to be found in the apparent simplicity of the definitions and axioms proposed in the early papers, that suggests the following probabilistic approach for causality: $y_n$ is said to cause $x_n$ if

$$\text{Prob} \left( x_n \in A | \Omega_{n-1} \right) \neq \text{Prob} \left( x_n | \Omega_{n-1} \setminus y^{n-1} \right)$$

for any subset $A$. $\Omega_n$ was called by Granger “all the information available in the universe” at time $n$, whereas $\Omega_n \setminus y^n$ stands for all information except $y^n$. In practice, $\Omega_n \setminus (x^n, y^n)$ is the side information $z^n$.

The second reason is that, in his 1980 paper [24], Granger introduced a set of operational definitions, thus allowing to derive practical testing procedures. These procedures require the introduction of models for testing causality, although Granger’s approach and definitions are fully general; furthermore, Granger’s approach raises the important issues below:

1) Full causality is expressed in terms of probability and leads to relationships between probability density functions. Restricting causality to relations defined on mean quantities is less stringent and allows more practical approaches.

2) Instantaneous dependence may be added to the causal relationships, e.g. by adding $y_n$ to the set of observations available at time $n - 1$. This leads to a weak concept as it is no longer possible to discriminate between instantaneous causation of $x$ by $y$, of $y$ by $x$ or of feedback, at least without imposing extra structures to the data models.

3) It assumed that $\Omega_n$ is separable: $\Omega_n \setminus y^n$ must be defined. This point is crucial for practical issues: the causal relationship between $x^n$ and $y^n$ (if any) is intrinsically related to the set of available knowledge at time $n$. Adding a new subset of observations, e.g. a new time series, may lead to different conclusions when testing causal dependencies between $x$ and $y$.

4) If $y_n$ is found to cause $x_n$ with respect to some observation set, this does not preclude the possibility that $x_n$ causes $y_n$ if there exists some feedback between the two series.

Item 2 above motivated Geweke’s approaches [20], [21], discussed below. Item 3 and 4 highlights the importance of conditioning the information measures to the set of available observations (related to nodes that may be connected to either $x$ or $y$), in order to identify causal information flows between any pair of nodes in a multi-connected network. As a central purpose of this paper is to relate Granger causality and directed information in presence of side information, the practical point of view suggested by Geweke is adopted. It consists in introducing a linear model for the observations.
B. Geweke’s approach

Geweke proposed measures of (causal) linear dependencies and feedback between two multivariate Gaussian time series \( x \) and \( y \). A third time series \( z \) is introduced as side information. This series allows to account for the influence of other nodes interacting with either \( x \) or \( y \), as this may happen in networks where many different time series or multivariate random processes are recorded. The following parametric model is assumed,

\[
\begin{align*}
\begin{cases}
x_n &= \sum_{s=1}^{\infty} A_{i,s} x_{n-s} + \sum_{s=0}^{\infty} B_{i,s} y_{n-s} + \sum_{s=0}^{b} \alpha_{i,s} z_s + u_{i,t} \\
y_n &= \sum_{s=0}^{\infty} C_{i,s} x_{n-s} + \sum_{s=1}^{\infty} D_{i,s} y_{n-s} + \sum_{s=0}^{b} \beta_{i,s} z_s + v_{i,t}
\end{cases}
\end{align*}
\] (38)

Accounting for all \( z \) corresponds to \( b = \infty \) in eq. (38), whereas causally conditioning on \( z \) is obtained by setting \( b = n - 1 \). Furthermore, it is assumed that all the processes studied are jointly Gaussian. Thus the analysis can be restricted to second order statistics only.

Under the assumption that the coefficients \( \alpha_{i,s} \) and \( \beta_{i,s} \) are set to zero (leading back to original Geweke’s model), we easily see that eq. (38) can actually handle three different dependence models indexed by \( i = \{1, 2, 3\} \), as defined below :

- \( i = 1 \): no coupling exists, \( B_{1,s} = 0, C_{1,s} = 0, \forall s \) and the prediction residuals \( u_{1,t} \) and \( v_{1,t} \) are white and independent random processes.
- \( i = 2 \), both series are only dynamically coupled : \( B_{1,0} = 0, C_{1,0} = 0 \) and the prediction residues are white random processes. Linear prediction properties allow to show that the cross correlation function of \( u_{2,t} \) and \( v_{2,t} \) is different from zero for the null delay only : \( \Gamma_{2,uv}(t) = \sigma^2 \delta(t) \).
- \( i = 3 \); the time series are coupled : \( B_{3,s} \neq 0, C_{3,s} \neq 0, \forall s \) and the residues \( u_{3,t} \) and \( v_{3,t} \) are white, but are no longer independent.

Note that models 2 and 3 differ only by the presence (model 3) or absence (model 2) of instantaneous coupling. It can be shown that these models are ‘equivalent’ if \( \sigma^2 \neq 0 \), thus allowing to compute an invertible linear mapping that transforms model 2 into a model of type 3. This confirms that model 3 leads to some weak concept, as already quoted previously. The same analysis and conclusions hold when the coefficients \( \alpha_{i,s} \) and \( \beta_{i,s} \) are restored.

C. Measures of dependence and feedback.

Geweke [20], [21] introduced dependence measures constructed on the covariances of the residues \( u_{i,t} \) and \( v_{i,t} \) in (38). We briefly recall these measures. Let

\[
\varepsilon^2_{\infty}(x_n|x^{n-1}, y^{l}, z^{b}) = \lim_{n \to +\infty} \varepsilon^2(x_n|x^{n-1}, y^{l}, z^{b})
\] (39)

for \( l = n \) or \( n - 1 \) according to the considered model.

\( \varepsilon^2_{\infty}(x_n|x^{n-1}, y^{l}, z^{b}) \) is the asymptotic variance\(^3\) of the prediction residue when predicting \( x_n \) from the observation

\(^3\)The presence of \( n \) in the notation \( \varepsilon_{\infty}^2(\cdot) \) is an abuse of notation, but is adopted to keep track of the variables involved in this one-step forward prediction.
of \( x^{n-1} \), \( y^l \) and \( z^b \). For multivariate processes, \( \varepsilon^2() \) is given by the determinant \( \det \Gamma_i,() \) of the covariance matrix of the residues.

Depending on the value of \( b \) in (38), and following Geweke, the following measures are proposed for \( b = \infty \):

\[
F_{y \rightarrow x | z} = \log \frac{\varepsilon_\infty(y^n|x^{n-1}, y^{n-1}, z^{\infty})}{\varepsilon_\infty(x^n|x^{n-1}, y^{n-1}, z^{\infty})}
\]

\[
F_{x \rightarrow y | z} = \log \frac{\varepsilon_\infty(y_n|x^{n-1}, y^{n-1}, z^{\infty})}{\varepsilon_\infty(x_n|x^{n-1}, y^{n-1}, z^{\infty})}
\]

\[
F_{x,y} | z = \log \frac{\varepsilon_\infty(x_n|x^{n-1}, y^{n-1}, z^{\infty})}{\varepsilon_\infty(x_n|x^{n-1}, y^{n}, z^{\infty})}
\]

(40)

and for causal conditioning, \( b = n - 1 \):

\[
F_{y \rightarrow x || z} = \log \frac{\varepsilon_\infty(x_n|x^{n-1}, z^{n-1})}{\varepsilon_\infty(x_n|x^{n-1}, y^{n-1}, z^{n-1})}
\]

\[
F_{x \rightarrow y || z} = \log \frac{\varepsilon_\infty(y_n|x^{n-1}, z^{n-1})}{\varepsilon_\infty(y_n|x^{n-1}, y^{n-1}, z^{n-1})}
\]

\[
F_{x,y} || z = \log \frac{\varepsilon_\infty(x_n|x^{n-1}, y^{n-1}, z^{n-1})}{\varepsilon_\infty(x_n|x^{n-1}, y^{n}, z^{n-1})}
\]

(41)

Note that these measures are greater or equal to zero.

Remarks:

- \( F_{x,y|z} \) and \( F_{x,y||z} \) can be shown to symmetric with respect to \( x \) and \( y \) [20], [21]. This is not the case for the other measures: if strictly positive, they indicate a direction in the coupling relation.

- Causally conditional on \( z \), \( F_{x,y||z} \) measures the linear feedback from \( x \) to \( y \) and \( F_{x,y||z} \) measures the instantaneous linear feedback, as introduced by Geweke.

- In [62], Rissanen and Wax introduce measures which are no longer constructed from the variances of the prediction residues but rather from a quantity of information (measured in bits) that is required for performing linear prediction. One cannot afford to deal with infinite order in the regression models, and these approaches are equivalent to Geweke’s. In [62], the information contained in the model order selection is taken into account. We will not develop this aspect in this paper.

IV. DIRECTED INFORMATION AND GRANGER CAUSALITY

We begin by studying the linear Gaussian case, and close the section by a more general discussion.

A. Gaussian linear models

Although it is not fully general, the Gaussian case allows to develop interesting insights into directed information. Furthermore, it provides a bridge between directed information theory and causal inference in networks, as partly described in an earlier work [5], [8]. The calculations below are conducted without taking observations others than \( x \) and \( y \), as it is straightforward to generalize in the presence of side information.
Let $H(y^k) = 1/2 \log(2\pi e)^k |\det \Gamma_{y^k}|$ be the entropy of the $k$ dimensional Gaussian random vector $y^k$ of covariance matrix $\Gamma_{y^k}$. Using block matrices properties, we have

$$\det \Gamma_{y^k} = \varepsilon^2(y_k|y^{k-1}) \det \Gamma_{y^{k-1}}$$

(42)

where $\varepsilon^2(y_k|y^{k-1})$ is the linear prediction error of $y$ at time $k$ given its past [11]. Then, the entropy increase is

$$H(y^k) - H(y^{k-1}) = \frac{1}{2} \log \left| \frac{\det \Gamma_{y^k}}{\det \Gamma_{y^{k-1}}} \right|$$

(43)

$$= \frac{1}{2} \log \varepsilon^2(y_k|y^{k-1})$$

(44)

Let $\varepsilon^2(y_k|y^{k-1}, x^k)$ be the power of the linear estimation error of $y_k$ given its past and the observation of $x$ up to time $k$. Since $I(x^k; y_k|y^{k-1}) = H(y^k) - H(y^{k-1}) - H(x^k; y^k) + H(x^k; y^{k-1})$, the conditional mutual information and the directed mutual information respectively write

$$I(x^k; y_k|y^{k-1}) = \frac{1}{2} \log \frac{\varepsilon^2(y_k|y^{k-1}, x^k)}{\varepsilon^2(y_k|y^{k-1}, x^k)}$$

(45)

$$I(x^n \to y^n) = \frac{1}{2} \sum_{i=1}^n \log \frac{\varepsilon^2(y_i|y^{i-1}, x^i)}{\varepsilon^2(y_i|y^{i-1}, x^i)}$$

(46)

If furthermore the vectors considered above are built from jointly stationary Gaussian processes, letting $n \to \infty$ in eq. (45) gives the directed information rates:

$$I_\infty(x \to y) = \frac{1}{2} \log \frac{\varepsilon^2_\infty(y_k|y^{k-1})}{\varepsilon^2_\infty(y_k|y^{k-1}, x^k)}$$

(47)

where $\varepsilon^2_\infty(y_k|y^{k-1})$ is the asymptotic power of the one step linear prediction error. By reformulating eq. (47) as

$$\varepsilon^2_\infty(y_k|y^{k-1}, x^k) = e^{-2I_\infty(x \to y)} \varepsilon^2_\infty(y_k|y^{k-1})$$

(48)

shows that the directed information rate measures the advantage of including the process $x$ into the prediction of process $y$.

If side information is available as a time series $z$, and if $x$, $y$ and $z$ are jointly stationary, the same arguments as above lead to

$$\varepsilon^2_\infty(y_k|y^{k-1}, x^k, z^{k-1}) = e^{-2I_\infty(x \to y||Dz)} \varepsilon^2_\infty(y_k|y^{k-1}, z^{k-1})$$

(49)

where recall that $Dz$ stands for the delayed time series. This equation highlights that causal conditional directed information has the same meaning as directed information, provided that we are measuring the information gained by considering $x$ in the prediction of $y$ given its past and the past of $z$.

\[ \text{Note that if } y \text{ is a stationary stochastic process, the limit of the entropy difference in eq. (44) is nothing but the entropy rate. Thus, taking the limit of eq. (44) exhibits the well known relation between entropy rate and asymptotic one step linear prediction [13].} \]
B. Relations between Granger and Massey’s approaches

To relate previous results to Granger causality, the contribution of the past values must be separated from those related to instantaneous coupling in the directed information expressions. A natural framework is provided by transfer entropy.

From eq. (28), under the assumption that the studied processes are jointly Gaussian, arguments similar to those used in the previous paragraph lead to

\[
I(Dx^n \to y^n) = \frac{1}{2} \sum_{i=1}^{n} \log \frac{\varepsilon^2(y_i|y^{i-1}, x^{i-1})}{\varepsilon^2(y_i|y^{i-1}, x^{i-1})} \quad (50)
\]

\[
I(x^n \to y^n || Dx^n) = \frac{1}{2} \sum_{i=1}^{n} \log \frac{\varepsilon^2(y_i|y^{i-1}, x^{i-1})}{\varepsilon^2(y_i|y^{i-1}, x^{i-1})} \quad (51)
\]

Likewise, causally conditioned directed information decomposes as

\[
I(x^n \to y^n || Dz^n) = I(Dx^n \to y^n || Dz^n) + I(x^n \to y^n || Dx^n, Dz^n) \quad (52)
\]

Expressing conditional information as a function of prediction error variance we get

\[
I(x^n \to y^n || Dz^n) = \frac{1}{2} \sum_{i=1}^{n} \log \frac{\varepsilon^2(y_i|y^{i-1}, z^{i-1})}{\varepsilon^2(y_i|y^{i-1}, x^{i-1})} + \frac{1}{2} \sum_{i=1}^{n} \log \frac{\varepsilon^2(y_i|y^{i-1}, x^{i-1}, z^{i-1})}{\varepsilon^2(y_i|y^{i-1}, x^{i-1})}
\]

The first term accounts for the influence of the past of \( x \) onto \( y \), whereas the second term evaluates the instantaneous influence of \( x \) on \( y \), provided \( z \) is (causally) observed.

Finally, letting \( n \to \infty \), the following relations between directed information measures and generalized Geweke’s indices are obtained:

\[
I_{\infty}(Dx \to y) = \frac{1}{2} \log \frac{\varepsilon^2_{\infty}(y_i|y^{i-1}, x^{i-1})}{\varepsilon^2(y_i|y^{i-1}, x^{i-1})} = F_{x \to y}
\]

\[
I_{\infty}(x \to y || Dx) = \frac{1}{2} \log \frac{\varepsilon^2_{\infty}(y_i|y^{i-1}, x^{i-1})}{\varepsilon^2_{\infty}(y_i|y^{i-1}, x^{i-1}, z^{i-1})} = F_{x,y}
\]

\[
I_{\infty}(Dx \to y || z) = \frac{1}{2} \log \frac{\varepsilon^2_{\infty}(y_i|y^{i-1}, x^{i-1}, z^{i-1})}{\varepsilon^2(y_i|y^{i-1}, x^{i-1}, z^{i-1})} = F_{x \to y || z}
\]

\[
I_{\infty}(x \to y || Dx, z) = \frac{1}{2} \log \frac{\varepsilon^2_{\infty}(y_i|y^{i-1}, x^{i-1}, z^{i-1})}{\varepsilon^2_{\infty}(y_i|y^{i-1}, x^{i-1}, z^{i-1})} = F_{x, y || z}
\]

This proves that for Gaussian processes, directed information rates (causal conditional or not) and Geweke’s indices are in perfect match.

C. Directed Information as a generalized Granger’s approach

These results are obtained under Gaussian assumptions and are closely related to linear prediction theory. However, the equivalence between Granger’s approach and directed information can hold in a more general framework by proposing the following information theoretic based definitions of causal dependence:

1) \( x_t \) is not a cause of \( y_t \) with respect to \( z_t \) if and only if \( I_{\infty}(Dx \to y || Dz) = 0 \)

2) \( x_t \) is not instantaneously causal to \( y_t \) with respect to \( z_t \) if and only if \( I_{\infty}(x \to y || Dx, Dz) = 0 \)
These directed information based definitions generalize Granger’s approach. Furthermore, these new definitions allow to infer graphical models for multivariate time series: This builds a strong connection between the present framework and Granger causality graphs developed by Eichler and Dalhaus [14]. This connection is further explored in [7] and in the recent work [58].

V. APPLICATION TO MULTIVARIATE GAUSSIAN PROCESSES

To illustrate the preceding results, we study the information flow between components of a multivariate Gaussian process. To stress the importance of causal conditioning and of availability of side information, we separate the bivariate analysis from the multivariate analysis. Furthermore, we particularly concentrate on a first order autoregressive model.

Let $X_n = CX_{n-1} + W_n$ be a multidimensional stationary, zero-mean, Gaussian process. $W_n$ is a Gaussian white multidimensional noise with correlation matrix $\Gamma_w$ (not necessarily diagonal). The off-diagonal terms in matrix $C$ describe the interactions between the components of $X$. $c_{ij}$ denotes the coupling coefficient from component $i$ to component $j$. The correlation matrix of $X$ is a solution of the equation

$$\Gamma_X = CTX C^t + \Gamma_w \quad (53)$$

Main directed information measures are firstly evaluated on a bivariate process. Then side information is assumed to be observed, and the same information measures are reconsidered for different coupling models.

A. Bivariate AR(1) model

Let $[v_n, w_n]^t = W_n$ and $\sigma_v, \sigma_w$ be their standard deviations and $\gamma_{vw}$ their correlation coefficient. Let $X_n = [x_n, y_n]^t$. $\Gamma_X$ is computed by solving eq. (53) as a function of the coupling coefficients between $x_n$ and $y_n$. The initial condition $(x_1, y_1)$ is assumed to follow the same distribution as $(x_n, y_n)$ to ensure the absence of transients.

Under these assumptions, some computations lead to express the mutual and directed information as

$$I(x^n; y^n) = \frac{n-1}{2} \log \left( \frac{(c_{yx}^2 \sigma_y^2 + \sigma_y^2)(c_{xy}^2 \sigma_x^2 + \sigma_x^2)}{c_{yx}^2 \sigma_y^2 \sigma_x^2 + \gamma_{xy}^2 \sigma_x^2} \right) + I(x_1; y_1) \quad (54)$$

$$I(x^n \rightarrow y^n) = \frac{n-1}{2} \log \left( \frac{c_{yx}^2 \sigma_y^2 + \sigma_y^2}{c_{yx}^2 \sigma_y^2 \sigma_w^2 - \gamma_{xy}^2 \sigma_w^2} \right) + I(x_1; y_1) + \frac{n-1}{2} \log (\sigma_y^2) \quad (55)$$

$$I(y^n \rightarrow x^n) = \frac{n-1}{2} \log \left( \frac{c_{xy}^2 \sigma_x^2 + \sigma_x^2}{c_{xy}^2 \sigma_x^2 \sigma_w^2 - \gamma_{xy}^2 \sigma_w^2} \right) + I(x_1; y_1) + \frac{n-1}{2} \log (\sigma_w^2) \quad (56)$$

where $I(x_1; y_1) = -1/2 \log (1 - \gamma_{xy}^2 / (\sigma_x^2 \sigma_y^2))$ and $\gamma_{xy}$ stands for the correlation between $x$ and $y$.

Equations (54,55,56) raise some comments:

1) The directed information is clearly asymmetric.
2) On one hand, the left hand side of the conservation equation (19) is given by summing equations (55) and (56). On the other hand summing the mutual information (54) and
\[ I(x^n \rightarrow y^n\|Dx^n) = I(x^n; y^n) + \sum_{i\geq 2} I(x_i, y_i|y^{i-1}, x^{i-1}) \]
(57)
\[ = n - \frac{1}{2} \log \left( \frac{\sigma_w^2 \sigma_v^2}{\sigma_w^2 \sigma_v^2 - \gamma_{vw}^2} \right) + I(x_1; y_1) \]
(58)
gives as expected the right hand side of the conservation equation (19). We recover the fact that for independent noise components ($\gamma_{vw} = 0$) the sum of the directed information flowing in opposite directions is equal to the mutual information. This is however not the case in general.

3) The information rates are obtained by letting $n \rightarrow \infty$ in eq. (55), (56):
\[ I_{\infty}(x \rightarrow y) = \frac{1}{2} \log \left( \frac{\sigma_w^2 \sigma_v^2}{\sigma_w^2 \sigma_v^2 - \gamma_{vw}^2} \right) \]
(59)
\[ I_{\infty}(y \rightarrow x) = \frac{1}{2} \log \left( \frac{\sigma_w^2 \sigma_v^2}{\sigma_w^2 \sigma_v^2 - \gamma_{vw}^2} \right) \]
(60)
This shows that if e.g. $c_{yx} = 0$, we observe that a coupling is equal to zero in one direction, the directed information rate from $y$ to $x$ satisfies
\[ \frac{1}{2} \log \left( \frac{\sigma_w^2 \sigma_v^2}{\sigma_w^2 \sigma_v^2 - \gamma_{vw}^2} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} I(x^n \rightarrow y^n\|Dx^n) \]
(61)
The right hand side of the equality may be interpreted as a lower bound for the directed informations rates. In particular, when $\Gamma_w$ is diagonal, this bound is zero.

This corresponds to the decomposition (28) for the rates, $I_{\infty}(x \rightarrow y) = I_{\infty}(Dx \rightarrow y) + I_{\infty}(x \rightarrow y\|Dx)$. The first term $I_{\infty}(Dx \rightarrow y) = \frac{1}{2} \log \left( 1 + \frac{\sigma_x^2 \sigma_v^2}{\sigma_w^2} \right)$ is Schreiber’s transfer entropy, or the directed information from the past of $x$ to $y$ (this term is equal to the first index of Geweke in this case). The second term $I_{\infty}(x \rightarrow y\|Dx)$ corresponds to the second of Geweke’s indices and measures the instantaneous coupling between the time series.

4) Directed information increases with the coupling strength, as expected for a measure of information flow.

B. Multivariate AR(1) model

Let $X_n = [z_n, x_n, y_n]^T$, $W_n = [u_n, c_n, w_n]^T$ be a three dimensional Gaussian stationary zero mean process satisfying the AR(1) equation, satisfying the same set of hypothesis and notations as above. We study two cases described in figure (1), where the arrows indicate the coupling direction.

The distributions of the variables $y_n|y_{n-1}$ and $y_n|y_{n-1}, x_{n-1}$ required in the calculation of e.g. $I_{\infty}(x \rightarrow y)$ are difficult to obtain explicitly. Actually, even if $X$ is a Markov process, the components are not. However since we deal with and AR(1) process, $p(y_n|X_{n-1}) = p(y_n|X_{n-1})$ and $p(x_n, y_n|X_{n-1}) = p(x_n, y_n|X_{n-1})$. The goal is to evaluate $I_{\infty}(y \rightarrow x||Dz)$. As $I(y^n; x_n|x_{n-1}, z_{n-1}) = I(y_{n-1}; x_n|x_{n-1}, z_{n-1}) + I(y_n; x_n|X_{n-1})$, one has
\[ I_{\infty}(y \rightarrow x||Dz) = \lim_{n \rightarrow \infty} I(y^n; x_n|x_{n-1}, z_{n-1}) \]
\[ = \lim_{n \rightarrow \infty} I(y_{n-1}; x_n|x_{n-1}, z_{n-1}) - 1/2 \log(1 - \gamma_{vw}^2/(\sigma_w^2 \sigma_v^2)) \]
(62)
where $\gamma_{vw}$ is the correlation coefficient between $v_n$ and $w_n$.

In case B (see figure 1), there is feedback from $y$ to $x$. Since conditioning is over the past of $x$ and $z$ and since there is no feedback from $z$ to $y$, $x_n|(x, z)^{n-1}$ is normally distributed with variance $c_{yx}^2 \sigma_y^2 + \sigma_v^2$. Thus, we obtain for this case

$$I_{B,\infty}(y \rightarrow x||Dz) = \frac{1}{2} \log(1 + \frac{c_{yx}^2 \sigma_y^2}{\sigma_v^2}) - \frac{1}{2} \log(1 + \frac{\gamma_{vw}^2}{\sigma_v^2 \sigma_w^2})$$

Setting $c_{yx} = 0$ we get for case A,

$$I_{A,\infty}(y \rightarrow x||Dz) = -(1/2) \log(1 - \frac{\gamma_{vw}^2}{\sigma_v^2 \sigma_w^2})$$

which is the instantaneous exchange rate between $x$ and $y$. If the noise components $v$ and $w$ are independent, the causal conditional directed information is zero.

The preceding illustration highlights the ability of causal conditioning to deal with different feedback scenarios in multiply connected stochastic networks. Figure 2 illustrates the inference result and the difference obtained if the third time series is not taken into account.

VI. Conclusion

In this paper, we have revisited the directed information theoretic concept introduced by Massey, Marko and Kramer. A special attention has been paid to the key role played by causal conditioning. This turns out be a central issue for characterizing information flows in the case where side information may be available. We propose a unified framework to enable a comparative study of mutual information, conditional mutual information with directed information in the context of networks of stochastic processes. Schreiber’s transfer entropy, a widely used concept in physics and neuroscience, is also shown to be easily interpreted with directed information tools.

The second section describes and discusses Granger causality and its practical issues. Geweke’s work serves as a reference in our discussion, and allows to provide a means to establish that Granger causality and directed information lead to equivalent measures in the Gaussian linear case. Based upon the previous analysis, a possible extension of Granger causality definition is proposed. The extended definitions rely upon information theoretic criterion rather than probabilities, and allow to recover Granger’s formulation in the linear Gaussian case. This new extended formulation of Granger causality is of some practical importance for estimation issues. Actually, some recent works presented some advances in this direction; in [59], directed information estimators are derived from spike trains models; in [72], Kraskov and Leonenko entropy estimators are used for estimating entropy transfer. In [7], the authors recourse to directed information in a graphical modeling context; Their equivalence with generative graphs for analyzing complex systems is studied in [58]. The main contribution of the present paper is to provide a unified view that allow to recast causality and directed information within a unique framework.

Estimation issues were not mentioned in this study, as it may deserve a full paper per-se, and are deferred to a future work.
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Fig. 1. Networks of three Gaussian processes studied in the paper. An arrow represents a coupling coefficient not equal to zero from the past of one signal to the other. In frame A, there is no direct feedback between any of the signals. However, a feedback from $y$ to $x$ exists through $z$. In frame B, there is also a direct feedback from $y$ to $x$. The arrows coming from the outside of the network represent the inputs, i.e. the dynamical noise $W$ in the AR model.

Fig. 2. Networks of three Gaussian processes studied in the paper. The left plot corresponds to the correct model and to the inferred network when causal conditional directed information is used. The network on the right is obtained if the analysis is only pairwise, when directed information is used between two signals without causal conditioning over the remaining signals.