Quantum Supremacy for Simulating A Translation-Invariant Ising Spin Model

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We introduce an intermediate quantum computing model built from translation-invariant Ising-interacting spins. Despite being non-universal, the model cannot be classically efficiently simulated unless the polynomial hierarchy collapses. Equipped with the intrinsic single-instance-hardness property, a single fixed unitary evolution in our model is sufficient to produce classically intractable results, compared to several other models that rely on implementation of an ensemble of different unitaries (instances). We propose a feasible experimental scheme to implement our Hamiltonian model using cold atoms trapped in a square optical lattice. We formulate a procedure to certify the correct functioning of this quantum machine. The certification requires only a polynomial number of local measurements assuming measurement imperfections are sufficiently small.

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A universal quantum computer is believed to be able to solve certain tasks exponentially faster than the current computers [1, 2]. Over the past several decades, there has been tremendous progress in both theoretical and experimental developments of a quantum computer. In theory, pioneering quantum algorithms, including Shor’s factorization [3] and an algorithm for linear systems of equations [4], achieve exponential speedup compared with the best-known classical algorithms. However, formidable experimental challenges still lie ahead in building a universal quantum computer large enough to demonstrate quantum supremacy. This calls for simpler tasks to demonstrate exponential quantum speedup without the need for a universal machine.

Several intermediate computing models have been developed recently for this purpose. Examples include boson sampling [5], quantum circuits with commuting gates (IQP) [6, 7], sparse and “fault-tolerant” IQP [8, 9], the one-clean-qubit model [10, 11], evolution of two-qubit commuting Hamiltonians [12], quantum approximate optimization algorithm [13] and random or universal quantum circuit [14, 15]. These models fall into the category of sampling problems: the task of simulating the distribution sampled from the respective quantum system is believed to be classically intractable. In particular, if a classical computer can efficiently simulate the distribution to multiplicative errors, the polynomial hierarchy, a generalization of \( \text{P} \) and \( \text{NP} \) classes, will have to collapse to the third level [16, 17], which is believed to be unlikely in complexity theory. Several experiments (e.g. [20, 21]) have been reported for realization of boson sampling in small quantum systems using photons. However, the system size is still limited, which prohibits demonstration of quantum supremacy beyond classical tractability.

In this paper, we report three advancements towards demonstration of exponential quantum speedup in intermediate computing models. First, we formulate a new sampling model built from translation-invariant Ising-interacting spins, with strong connection to simulation of natural quantum many-body systems [22–25]. Our model only requires nearest-neighbor Ising-type interactions. The state preparation, the Hamiltonian and measurements are all constructed to be translation-invariant. Similar to Refs. [5, 7], we prove the distribution sampled from our model cannot be classically efficiently simulated based on complexity theory results under reasonable conjectures [6, 26–28]. An additional desirable feature of our model, which we call the ‘single-instance-hardness’ property, is that a single fixed circuit and measurement pattern are sufficient to produce a classically hard distribution once the system size is fixed. This differs from typical sampling problems, where an ensemble of instances (unitaries) with a large number of parameters is demanded for the hardness result to hold [5–15]. This feature offers a significant simplification for experiments since proof of quantum supremacy for this model requires implementation of only a single Hamiltonian and measurement pattern instead of a range of different realizations (typically an exponential number or even an infinite number). Ref. [5] also discussed the single-instance-hardness possibility in an abstract quantum circuit language, but no explicit circuit has been given thus far. Second, we propose a feasible experimental scheme to realize our model with cold atoms in optical lattices. The state preparation, engineering of time evolution and measurement techniques are achievable with the state-of-the-art technology. Unlike photonic systems, cold atomic systems are much easier to scale up and reach a system size intractable to classical machines. Finally, we devise a scheme to certify our proposed quantum machine based on extension of the techniques developed in Refs. [29, 30]. Certification of functionality is critically important for a sampling quantum machine as a correct sampling is hard to be verified. Our certification scheme only requires a polynomial number of local measurements, assuming the measurement imperfections are sufficiently small.
O consists of Ising interactions between any pairs of spins with a constant circuit depth. A general IQP [6, 7] model can be regarded as a special type of IQP on a tum system with \( q \) \( \{ +, - \} \) depth. Note that we are able to achieve a different complexity conjecture of average-case hardness. Ref. [8] proposed another type of IQP in constant circuit depth on the Raussendorf-Harrington-Goyal (RHG) lattice [33]. In their model, the classical hardness result is guaranteed with multiplicative errors under some local noise below a threshold. Their Hamiltonian is also translation-invariant but the measurements are not. Thus, this model and the general IQP do not have the single-instance-hardness property. The general interactions in IQP and the three-dimensional structure of the RHG lattice may be difficult to realize in experiments.

**Translation-invariant Ising model.**—Our main construction is based on measurement-based quantum computing models [34–36]. We first introduce a translation-invariant nonadaptive measurement-based quantum computation model with only one measurement basis required. With postselection, we show that it can simulate universal quantum computation. Next, we reinterpret the measurement-based model as a sampling model based on quantum simulation of two-dimensional (2D) spins with translation-invariant Ising interactions and local magnetic fields. It has been known that if a sampling model with postselection can simulate universal quantum computation, it will be hard to simulate classically with multiplicative error bounds unless the polynomial hierarchy collapses to the third level [6, 10, 12]. We therefore conclude that our quantum Ising model will be classically intractable if the polynomial hierarchy does not collapse [17].

Consider the brickwork state shown in Fig. 1(a), which has been used for universal blind quantum computation [37]. Each circle represents a qubit prepared in the state \( |+ \rangle = (|0 \rangle + |1 \rangle)/\sqrt{2} \). A line connecting two neighboring circles denotes a controlled-Z operation on the qubits. As illustrated in Fig. 1(b), a measurement on one qubit in \( X \) basis with measurement result \( s \) implements a gate \( HZ^\theta R_z(\theta) \), where \( H \) is the Hadamard gate and \( R_z(\theta) = e^{-i\theta Z/2} \) denotes a rotation on a single qubit. Ref. [37] proved that the model supports universal quantum computation given proper rotation angles \( \theta \) and measurement results \( s \) (see Supplemental Material [17] for details). An important attribute of this model is that the graph structure and measurement patterns are independent of the computation. We further improve the model by making the angles \( \theta \) translation-invariant. In terms of the sampling problem, this modification gives rise to the advantage of the single-instance-hardness property. It differs from other existing sampling problems, such as boson sampling, wherein an average over random quantum circuits is needed for the classical hardness result to hold.

To fix the angle pattern, we use seven qubits to replace one white circle (Fig. 1(c)). The primary goal is to encode rotation angle values into measurement outcomes, so that measurement postselection effectively realizes all...
necessary rotation angles. The basic building block is
\[ HZ^s HR_z \left( -\frac{\theta}{2} \right) HZ^s HR_z \left( \frac{\theta}{2} \right) = R_z^s(\theta) \] (3)
which can be realized by measuring four connecting qubits in X basis with rotation angles \( \theta/2, 0, -\theta/2, 0 \) and postselecting the results to be 0, 0, 0, s. This equality furnishes a mechanism to conditionally perform the rotation \( R_z(\theta) \) based on the measurement result \( s \). Because of the Solovey-Kitaev theorem \([38]\), it is sufficient to implement \( HR_z(k\pi/4) \), \( k \in \{0, \cdots, 7\} \) for universal computation \([17]\). Writing \( k = s_1 s_2 s_3, s_i \in \{0, 1\} \) in binary form, we have
\[ Z^{s_3} HR_z \left( \frac{k\pi}{4} \right) Z^{s_3} = Z^{s_3} HR_z^s(\pi) R_z^s \left( \frac{\pi}{2} \right) R_z^s \left( \frac{\pi}{4} \right) Z^{s_3} \]
\[ = HR_z \left( -\frac{\pi}{8} \right) H Z^{s_3} HR_z \left( \frac{\pi}{4} \right) H Z^{s_2} \]
\[ = HR_z \left( -\frac{\pi}{4} \right) H Z^{s_2} H Z^{s_1} + \frac{s_3}{2} R_z \left( \frac{\pi}{8} \right) . \]
The extra term \( Z^{s_3} \) can be absorbed into the following gate and \( Z^{s_3} \) is left from the previous gate. Postselecting the measurement results as \( s_1 \oplus s_1', s_2, 0, s_2, 0, s_3, 0 \) with rotation angles \( \pi/8, 0, -\pi/4, 0, \pi/4, 0, -\pi/8 \), we can implement the gates \( H R_z(k\pi/4) \) with \( k = s_1 s_2 s_3 \).

We now recast the nonadaptive measurement-based computation model as a sampling problem. A distribution can be sampled by measuring each spin in Fig. 1 in X basis. The above procedure is only used to prove the universality of the nonadaptive measurement-based model with a fixed circuit under postselection. We remark that neither postselection nor adaptive measurements are required for sampling the distribution. The circuit can be implemented by a unitary time evolution under a local Hamiltonian
\[ \mathcal{H} = -\sum_{\langle i,j \rangle} J Z_i Z_j + \sum_i B_i Z_i \] (4)
starting from the initial state \( |\rangle \rangle \rangle \otimes |\rangle \rangle \rangle \), with \( m \times n \) being the number of spins. The second term imprints local rotation angles since \( e^{-iB_i Z_i} = R_z(\theta_i) \), where \( B_i = \theta_i/2 \) characterizes the local Zeeman field strength on spin \( i \). The evolution time and the reduced Planck constant \( \hbar \) are set to unity. The first term forms the controlled-Z operations with \( J = \pi/4 \), where \( \langle i,j \rangle \) represents nearest-neighbor pairs connected by a line in Fig. 1. This can be seen as
\[ C Z_{ij} = e^{i\pi/4} e^{-i\pi/4} Z_i Z_j = e^{i\pi/4} i (1 - Z_i) \otimes (1 - Z_j) \]
\[ = e^{i\pi/4} e^{-i\pi/4} Z_i Z_j e^{-i\pi/4} Z_i \otimes Z_j e^{i\pi/4} Z_i \otimes Z_j . \] (5)

The two local magnetic field terms in the equation above can be absorbed into rotation angles, without changing Fig. 1(c) (see Supplemental Material \([17]\) ). The distribution sampled from this fixed 2D Ising model cannot be simulated by a classical computer in polynomial time to multiplicative errors unless the polynomial hierarchy collapses.

**Implementation proposal with cold atoms.**—The Hamiltonian in Eq. (4) exhibits a few properties that make it amenable for experimental implementation. First of all, it only consists of commuting terms, so in experiment one can choose to break up the Hamiltonian and apply simpler terms in sequence. Second, the state preparation, the Hamiltonian and measurements are all translation-invariant. This may greatly simplify the implementation for setups that can engineer the required unit cell. Another merit of our model originates from the single-instance-hardness feature. It ensures the sampling distribution after a single fixed unitary operation is already hard to simulate classically.

Here, we put forward a feasible experimental scheme based on cold atoms in optical lattices. A major difficulty arises from the special geometry required in the brickwork state. We propose to circumvent this problem by starting from the 2D cluster state (square lattice geometry) and reducing it to the brickwork state. In theory, this can be achieved by the “break” and “bridge” operations with measurement postselection as shown in Fig. 2 (see Supplemental Material \([17]\) for more details). In experiment, postselection is again unnecessary with regard to sampling, but one incurs an additional cost of measuring in both X and Z basis (the measurement pattern is still translation-invariant though). As a by-product, this procedure offers a concrete single-instance-hardness protocol to produce classically non-simulatable distribution from the cluster state.

A complete experimental procedure is as follows. First, create a Mott-insulator state of cold atoms in 2D optical lattices with a central core of unit filling. One atom with two relevant atomic levels \( e.g., |F = 1, m_F = -1\rangle \) and \( |F = 2, m_F = -2\rangle \) hyperfine levels of \(^{87}\text{Rb}\) atoms) can be trapped in each site forming a square lattice of qubits. A 2D cluster state can be created in a single operational step by controlled collisional interaction \([39, 40]\). The basic idea involves entangling neighboring atoms by spin-dependent transport together with controlled on-site
collisions, which has been realized in experiment [40]. After generating the cluster state, one needs to impose the rotation angle pattern onto each qubit. This requires the ability to address individual atoms with diffraction-limited performance. Single-site addressing is currently one of the state-of-the-art quantum control techniques in cold atom experiments [41, 42]. In particular, by using a digital micro-mirror device, it is possible to engineer holographic beam shaping with arbitrary amplitude and phase control [42]. To imprint the individual phases, one can make use of spin-dependent AC Stark shifts [41] with beam amplitude patterns given by the rotation angles. The amplitude hologram controls the strength $B_1$ and realizes the second term in the Hamiltonian in Eq. (4).

Finally, spin measurements can be performed on each site, with single-site-resolved imaging techniques [43, 44]. Because some spins have to be measured in $Z$ basis, they should be rotated by individual addressing techniques before all atoms can be measured in $X$ basis.

Simulation and certification with variation distance errors. — So far, we have shown that our Ising spin model is classically intractable with multiplicative error bounds. Similar to what have been attained in boson sampling [5] and IQP [7], we can also prove classical hardness to variation distance error bounds if we assume the “worst-case” hardness result can be extended to “average-case”. More specifically, let us define the partition function of

$$H_x = H + \frac{\pi}{2} \sum_i x_i Z_i,$$  

for $x_i \in \{0, 1\}$

(6)

to be $Z_x = \text{tr}(e^{-\beta H_x})$, setting the imaginary temperature unit as $\beta \equiv 1/k_B T = i$. In Supplemental Material [17], we prove that approximating $|Z_x|^2/2^{mn}$ by $|\bar{Z}_x|^2/2^{mn}$ to a mixture of multiplicative and additive errors such that

$$|\frac{|Z_x|^2}{2^{mn}} - \frac{|\bar{Z}_x|^2}{2^{mn}}| \leq \frac{1}{\text{poly}(n)} \left| \frac{|Z_x|^2}{2^{mn}} \right| + \frac{\epsilon}{\delta}(1 + o(1))$$  

(7)

with $\epsilon/\delta < 1/2$ is #P-hard in the worst-case. Our classical intractability result requires lifting the #P-hardness of the estimation from the worst-case to the average-case: picking any $1 - \delta$ fraction of instances $x$, it is still #P-hard. This conjecture is similar to the one used in Ref. [7] except that they reduced the mixture of errors to simply multiplicative errors. All the known classically intractable quantum sampling models with variation distance errors require a similar average-case complexity conjecture.

Thus, with reasonable assumptions, our Ising spin model is also classically intractable with variation distance bounds. Using techniques similar to those in Refs. [29, 30], we can in addition certify the correct functioning of a quantum device, with only a polynomial number of local measurements. Suppose $\{q'_x\}$ is the distribution sampled from our quantum device with the final state $\rho'$ (state before measurement); the ideal ones are denoted as $\{q_x\}$ and $\rho$. The total variation distance between distributions $\{q_x\}$ and $\{q'_x\}$ can be bounded by [1]:

$$\sum_x |q_x - q'_x| \leq D(\rho, \rho'),$$  

(8)

where $D(\rho, \rho') = \text{tr}((\rho - \rho')/2)$ is the trace distance between states $\rho$ and $\rho'$. Hence, if we can bound the trace distance $D(\rho, \rho') < \epsilon$, we can also bound the total variation distance. Note, however, this does not allow us to estimate $q_x$ in experiment: statistical errors always kick in to thwart any polynomial-time efforts to estimate the distribution due to the exponential suppression of some $q_x$. We bypass statistical errors by assuming the correctness of quantum mechanics. To sample from $\{q'_x\}$ in experiment though, measurement imperfections may cause deviations in variation distance. However, if measurement imperfections on each spin are local and bounded by $O(\epsilon/(mn))$ [17], we can still correctly certify the quantum device. Below, we show how to bound $D(\rho, \rho')$ by a polynomial number of local measurements.

As a graph state, the brickwork state in Fig. 1(a)(c), is the unique ground state of the 4-local Hamiltonian

$$H_{\text{brickwork}} = \sum_i \frac{1 - X_i \prod_j \text{neighbor of } i Z_j}{2},$$  

(9)

Each qubit $i$ is connected to at most three neighboring ones, and the energy gap from the ground state is 1. The ideal state $\rho$ is the brickwork state acted by some single qubit rotations $R_z(\theta_i)$. It is therefore the unique ground state of the Hamiltonian

$$H'_{\text{brickwork}} = \prod_i R_z(\theta_i) H_{\text{brickwork}} \prod_i R_z^*(\theta_i)$$

$$= \sum_i \frac{1 - R_z(\theta_i) X_i R_z^*(\theta_i) \prod_j \text{neighbor of } i Z_j}{2}.$$

This Hamiltonian is still 4-local, with ground state energy gap 1. Using the weak-membership quantum state certification protocol in Ref. [29], one can measure each local term of $H'_{\text{brickwork}}$ by a polynomial number of times to obtain a good estimation of $\langle H'_{\text{brickwork}} \rangle$ averaged over $\rho'$. The estimation will be efficient due to Hoeffding’s bound and the finite norm of each local term. Since the ground state energy gap is constant, $\langle H'_{\text{brickwork}} \rangle > 0$ implies a finite component of excited states is present in $\rho'$. Conversely, a small $\langle H'_{\text{brickwork}} \rangle$ will be able to bound $D(\rho, \rho')$. More quantitatively, we show in Supplemental Material [17] that with confidence level $1 - 2^{-O(r)}$, using $O(m^2 n^2 r/\epsilon^4)$ measurements on each local term is sufficient to certify $\sum_x |q_x - q'_x| \leq \epsilon$, provided the measurement imperfections on each spin are bounded by $O(\epsilon/(mn))$. Similar hardness and certification results hold if we start from the cluster state as in our experimental proposal [17]. In that case, 5-local measurements are needed.
The IQP certification protocol developed in Ref. [29] requires a much stronger quantum simulator than the IQP simulator itself since they need to generate all the history states [45]. In contrast, our certification protocol only requires preparing the state $\rho'$ itself. This is relevant in light of demonstrating quantum supremacy [46] using practical quantum many-body systems, instead of resorting to a universal quantum simulation device.

Discussion.—In summary, we have introduced a translation-invariant Ising spin model and shown that it is classically intractable unless the polynomial hierarchy collapses. Because our average-case conjecture bypasses the anticoncentration property used in Refs. [5, 7, 9], the classical simulability result under constant-strength local noise [9] may not apply to our model. Whether our model is robust to noise requires further analysis. There is also a natural connection between our model and sampling models of random quantum circuits such as the one in Ref. [14]: measurement on qubits in the first $n−1$ columns in our model corresponds to choosing one instance of a random circuit due to the relation between our model and measurement-based quantum computing. With the advantageous single-instance-hardness property, the amenability to experimental implementation and certification of the quantum machine, we develop a full picture of using our model to demonstrate quantum supremacy. This may shed light on the likely exponential gap in computational power between a classical and a quantum machine.

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where \( Q \) is often used in complexity theory. \( \Sigma \), which means the number of witnesses. This class is defined as \( \Sigma_p \): nondeterministic polynomial. A language \( L \) is in \( \Sigma_p \) if for some \( i \geq 1 \), \( \forall \), \( \exists \), or \( \exists \) depending on whether \( i \) is even or odd, respectively. And

\[
x \in L \Leftrightarrow \exists u \in \{0,1\}^{|x|} \text{ s.t. } M(x,u) = 1.
\]

Polynomial hierarchy is in some sense a generalization of \( \Sigma_p \).

**Definition 2** (\( \Sigma_p, \text{PH} \): polynomial hierarchy). For \( i \geq 1 \), a language \( L \) is in \( \Sigma_i^p \) if there exists a polynomial \( q \) and a polynomial time classical Turing Machine \( M \) such that for every \( x \in \{0,1\}^* \)

\[
x \in L \Leftrightarrow \exists u_1 \in \{0,1\}^{q(|x|)} \forall u_2 \in \{0,1\}^{q(|x|)} \cdots Q_i u_i \in \{0,1\}^{q(|x|)} \text{ s.t. } M(x,u_1, u_2, \cdots, u_i) = 1,
\]

where \( Q_i \) denotes \( \forall \) or \( \exists \) depending on whether \( i \) is even or odd, respectively. And

\[
\text{PH} = \bigcup_i \Sigma_i^p.
\]

Note that \( \text{NP} = \Sigma_1^p \) and one can generalize \( i \) to 0 such that \( \text{P} = \Sigma_0^p \). Clearly, \( \Sigma_i^p \subseteq \Sigma_{i+1}^p \subseteq \text{PH} \). Most computer scientists believe \( \text{P} \neq \text{NP} \). A generalization of this conjecture is that for every \( i \), \( \Sigma_i^p \) is strictly contained in \( \Sigma_{i+1}^p \), which means \( \Sigma_i^p \neq \Sigma_{i+1}^p \). It can also be stated as “the polynomial hierarchy does not collapse”. This conjecture is often used in complexity theory.

There is another way to generalize the class \( \text{NP} \). According to the above definition, it only requires knowing whether there exists at least one witness such that the Turing machine accepts. Counting problems need to compute the number of witnesses. This class is defined as
Definition 3 (♯P). A function $f$ is in ♯P if there exists a polynomial $q$ and a polynomial time classical Turing machine $M$ such that for every $x \in \{0,1\}^*$

$$f(x) = \#\{y \in \{0,1\}^{q(|x|)} : M(x,y) = 1\}.$$ 

The following two complexity classes are directly related to sampling problems. One complexity class is postBQP defined in Ref. [28]. This complexity class characterizes the computational power of a universal quantum computer given the ability to do postselection. The other is a classical analog, postBPP, defined in Ref. [27].

Definition 4 (postBQP, postBPP). A language $L$ is in postBQP/postBPP if there exists a uniform (which means can be generated by a classical polynomial Turing Machine) family of polynomial size quantum/classical circuits $Q_n/C_n$ such that for every $x \in \{0,1\}^*$, after applying $Q_n/C_n$ to the state

- the probability measuring registers $P/\tilde{P}$ (called postselection registers) in the state $|0 \cdots 0\rangle/0 \cdots 0$ is nonzero;
- if $x \in L$, then conditioned on measuring $P/\tilde{P}$ on state $|0 \cdots 0\rangle/0 \cdots 0$, the probability measuring the output register on state $|1\rangle/1$ is at least $a$ (completeness error);
- if $x \notin L$, then conditioned on measuring $P/\tilde{P}$ on state $|0 \cdots 0\rangle/0 \cdots 0$, the probability measuring the output register on state $|1\rangle/1$ is at most $b$ (soundness error).

where $a - b > 1/poly(n)$.

Some relations between these classes are included in the following theorem.

Theorem 1. The first is Toda’s theorem [16], the second is proved in Ref. [28], and the third is proved in Ref. [27]:

$$\text{PH} \subseteq P^{\#P}$$

$$P^{\#P} = P^{\text{postBQP}}$$

$$\text{postBPP} \subseteq \Sigma_3^p.$$ 

In order to simulate postBQP by postselection, we need to define an output register $O/\tilde{O}$ which gives the result of the decision problem, and a postselection register $P/\tilde{P}$ of which the result is postselected to be some string of $\{0,1\}$. The key point is that we can change the definition slightly without changing the classes postBQP and postBPP: replacing the result of the register $P/\tilde{P}$ by

$$|0 \cdots 0\rangle/0 \cdots 0 \rightarrow |s_1 \cdots s_{m \times n-1}\rangle/s_1 \cdots s_{m \times n-1}.$$ (10)

This is crucial to our result.

Suppose the result in the output register is $x$. Classical simulability with multiplicative error implies

$$\frac{1}{c} q_{x \cdot 1 \cdots s_{m \times n-1}} \leq p_{x \cdot 1 \cdots s_{m \times n-1}} \leq cq_{x \cdot 1 \cdots s_{m \times n-1}}$$ (11)

where the probability $\{q\}$ is sampled by our model, denoted as Ising and $\{p\}$ is sampled by a classical polynomial probabilistic Turing machine, shorted as BPP; the first digit $x$ is in the register $O/\tilde{O}$ and other digits $s_1 \cdots s_{m \times n-1}$ are in the register $P/\tilde{P}$; This is equivalent to the definition of Eq. (2) in the main text if we choose $\gamma = \min(1-1/c, c-1)$.

With postselection, we can define postIsing. The output probability is

$$R(x) = \frac{q_{x \cdot 1 \cdots s_{m \times n-1}}}{q_{0 \cdot 1 \cdots s_{m \times n-1}} + q_{1 \cdot 1 \cdots s_{m \times n-1}}}.$$ 

The output probability of the corresponding postBPP is

$$\tilde{R}(x) = \frac{p_{x \cdot 1 \cdots s_{m \times n-1}}}{p_{0 \cdot 1 \cdots s_{m \times n-1}} + p_{1 \cdot 1 \cdots s_{m \times n-1}}}.$$ 

According to the definition of multiplicative error Eq. (11), we have

$$\frac{1}{c^3} R(x) \leq \tilde{R}(x) \leq c^3 R(x),$$
With this inequality and if \( c < \sqrt{2} \) (so \( \gamma < 1/2 \)),
\[
|\tilde{R}(0) - \tilde{R}(1)| > 0 \quad \text{(not scaling with the problem size)} \Rightarrow |R(0) - R(1)| > 0.
\]
This condition means that if there is a gap between completeness and soundness error in Ising, there will also be a gap for the BPP simulator:
\[
\text{postIsing} \subseteq \text{postBPP}. \tag{12}
\]
If we can further prove
\[
\text{postBQP} \subseteq \text{postIsing}, \tag{13}
\]
which means Ising with postselection can simulate universal quantum computer. Combined with theorem 1, we have
\[
\text{PH} \subseteq \text{P}^{\#P} = \text{P}^{\text{postBQP}} = \text{P}^{\text{postIsing}} \subseteq \text{P}^{\text{postBPP}} \subseteq \Sigma^p_3, \tag{14}
\]
which means the polynomial hierarchy collapses to the third level. This contradicts with the generalization of the \( \text{P} \neq \text{NP} \) conjecture
\[
\Sigma^p_3 \subset \text{PH}. \tag{15}
\]
Here, we adopt the same idea of proof as in Ref. [6].

**Universal Quantum Computation with the Brickwork State**

Ref. [37] has given the proof of universality. For completeness, we briefly review the result. Fig. 3(a) shows how to choose different angles to get any single qubit gates and the CNOT gate. They are known to be universal. Fig. 3(b) shows how to combine two qubit gates together to implement universal quantum computation.

**MAGNETIC FIELD IN THE ISING SPIN MODEL**

In this section, we show that the extra local magnetic fields can be absorbed into the magnetic fields of Fig. 1(c) of the main text of our paper. We have three separate cases:

- For those spins that only couple with one other spin, there is an extra magnetic field \( R_z(\pi/2) \). This spin must be on the left or the right boundary of the brickwork state. We can regard it as an ordinary unitary operation acting on the input. It can be eliminated by acting \( R_z(-\pi/2) \) on the remaining quantum circuits.

- For those spins that couple with two other spins, there is an extra magnetic field \( R_z(\pi) \). These spins will be acted on by an extra \( Z \) gate. It can be eliminated by flipping the measurement result.

- For those spins that couple with three other spins, there is an extra magnetic field \( R_z(3\pi/2) \). These spins must have a vertical coupling; according to Fig. 1(c) of the main text, we can make the rotation angle \( \theta \) on those spins to be \( \pi/8 + 3\pi/2 = \pi/8 - \pi/2 \mod 2\pi \). It can be eliminated by flipping the measurement result from \( s_2 \) to \( s_2 \oplus 1 \) and from \( s_3 \) to \( s_3 \oplus s_2 \).

**break and bridge operations**

In the main text of our paper, we introduced the “break” and “bridge” operations. Here, we include more details of how to reduce a cluster state to a brickwork state by those operations. For the three qubit cluster state in Fig. 4(a), the red circle is rotated by \( R_z(\pi/2) \). The operations acting on qubits 1 and 2 controlled by qubit 0 can be written as
\[
e^{-i\pi/4} \frac{1}{\sqrt{2}} |0\rangle_0 \otimes I_1 \otimes I_2 + i|1\rangle_0 \otimes Z_1 \otimes Z_2. \tag{16}
\]
FIG. 3. Implementing universal quantum computation with the brickwork state. These figures are similar to the ones in Ref. [37].

with an extra global phase. Therefore, by postselecting qubit 0 being $|0\rangle$ by measuring $Z$, we have the operation $I_1 \otimes I_2$ on qubits 1 and 2, implementing the break operation. By postselecting qubit 0 being $|+\rangle$ by measuring $X$, we have

$$e^{-i\pi/4}\sqrt{2} (I_1 \otimes I_2 + iZ_1 \otimes Z_2) = e^{-i\pi/4}e^{i\pi/4}Z_1 \otimes Z_2.$$  (17)

This is the same as the time evolution of the Ising interaction in the Hamiltonian (Eq. 4 and Eq. 5 of the main text), implementing the bridge operation.

Fig. 4(b) demonstrates how to convert the cluster state to other graph states such as the brickwork state by the break and bridge operations.

Simulation with variation Distance Errors

This is the most technical part of the computational complex theory in this paper, so we divide it into three parts.

A $\#P$-hard problem in worst-case

First of all, we introduce a problem that is $\#P$-hard in worst-case. Later, we will find that our classically-intractable result for simulating our Ising spin model depends on a conjecture that lifts this problem from worst-case hardness to average-case hardness.

Suppose the probability of measuring result $x = x_1 \cdots x_i \cdots x_{m \times n}, x_i \in \{0, 1\}$ from the quantum sampler is $q_x$ with

$$q_x = \left| \bigotimes_{i=1}^{m \times n} (|+\rangle_x e^{-iHt}|+\rangle)^{\otimes m \times n} \right|^2 = \frac{|\langle 0|C_x|0\rangle|^2}{2^{mn-m}}$$  (18)
where $C_x$ is a polynomial size quantum circuit which can be implemented by choosing proper measurement results $x$ and $1/2^{mn-m}$ comes from equal probability for measurement in measurement-based quantum computing. We will show that approximating $q_x$ by $\tilde{q}_x$ to the following error

$$|\tilde{q}_x - q_x| \leq \frac{q_x}{\text{poly}(n)} + \frac{c}{2^{mn}}$$

is $\#P$-hard, where $c$ can be any constant $0 \leq c < 1/2$.

Suppose $f(z)$ is some boolean function which can be computed efficiently by a classical computer. Define

$$\text{gap}(f) \equiv |\{z : f(z) = 0\}| - |\{z : f(z) = 1\}| = \sum_z (-1)^{f(z)}$$

and $\tilde{\text{gap}}(f)^2 \equiv 2^{mn} \tilde{q}_x$. Consider the polynomial size quantum circuit $C_x$ doing the following operation on $|0\rangle^\otimes m\ (m = 2r)$

\[
\text{Hadamard gate: } |0\rangle^\otimes r |0\rangle^\otimes r \implies |0\rangle^\otimes m - r \sum_z |z\rangle \frac{\sqrt{2^r}}{\sqrt{2^r}}
\]

computing $f(z)$:

\[
\implies |0\rangle^\otimes r - 1 \sum_z |f(z)\rangle |z\rangle \frac{\sqrt{2^r}}{\sqrt{2^r}}
\]

applying $Z$ and uncomputing:

\[
\implies |0\rangle^\otimes r \sum_z (-1)^{f(z)} |z\rangle \frac{\sqrt{2^r}}{\sqrt{2^r}}
\]

\[
\text{Hadamard gate: } \implies |0\rangle^\otimes m \sum_z (-1)^{f(z)} \frac{\sqrt{2^r}}{2^r} + |\text{other terms}\rangle,
\]

which means

$$q_x = \frac{|\langle 0|C_x|0\rangle|^2}{2^{mn-m}} = \frac{\text{gap}(f)^2}{2^{mn}}.$$  

Thus, Eq. (19) implies

$$|\tilde{\text{gap}}(f)^2 - \text{gap}(f)^2| \leq \frac{\text{gap}(f)^2}{\text{poly}(n)} + c.$$
This condition implies $\hat{\text{gap}}(f)^2$ can estimate $\text{gap}(f)^2$ to multiplicative errors since $c < 1/2$:

$$|\text{gap}(f)^2 - \hat{\text{gap}}(f)^2| \leq (c + o(1)) \cdot \text{gap}(f)^2.$$  

(24)

This is because $\text{gap}(f)^2$ is an integer: if $\text{gap}(f)^2 = 0$, then $\hat{\text{gap}}(f)^2 < 1/2$ such that we can infer $\text{gap}(f)^2 = 0$, which means $|\text{gap}(f)^2 - \hat{\text{gap}}(f)^2| = 0$; if $\text{gap}(f)^2 \geq 1$, then $c \leq c \cdot \text{gap}(f)^2$. Ref. [7] proved that approximating $\text{gap}(f)^2$ to multiplicative errors is $\#P$-hard (actually, they proved that if $f$ is some special boolean function, it is $\text{GapP}$-complete, but this implies the result we need). This proves the worst-case hardness result.

Define the partition function with imaginary temperature $\beta \equiv 1/k_B T = i$ as

$$Z_x = \text{tre}^{-i(H + \sum x_i \pi/2Z_i)} = \sum_{z \in \{+1, -1\}^n} e^{i(\sum_{i,j} x_i \pi/2z_j + \sum_i B'_i z_i)}$$

(25)

where $B'_i$ depends on $x_i$. Then,

$$q_x = \left| \prod_i (+x_i e^{-iHt}|+) \otimes m \times n \right|^2 = \left| (|+) \otimes m \times n e^{-i(H + \sum x_i \pi/2Z_i)t} |+\right\rangle \otimes m \times n \right|^2 = \frac{|Z_x|^2}{2^{2mn}},$$

(26) where $|+\rangle = Z^x |+\rangle$ are the bases of $X$. Restating the above conclusion in terms of the partition function, we get

**Theorem 2.** Approximating the partition function to the following error

$$\left| \frac{|Z_x|^2}{2^{2mn}} - \frac{|Z_\beta|^2}{2^{2mn}} \right| \leq \frac{1}{\text{poly}(n)} \frac{|Z_x|^2}{2^{2mn}} + c$$

(29)

is $\#P$-hard in the worst-case, if $0 \leq c < 1/2$. (Notice that the range of $|Z_x|^2/2^{mn}$ is from 0 to $2^{2mn}$ instead of from 0 to 1.)

*Classically-intractable for simulation with variation distance error*

The main ingredient is Stockmeyer’s theorem [18] (see Ref. [5] or Ref. [7] for the statement here):

**Theorem 3.** There exists an $\text{FBPP}^{\text{NP}}$ algorithm which can approximate

$$P = \Pr_x [f(z) = 1] = \frac{1}{2^r} \sum_{z \in \{0, 1\}^r} f(z)$$

(30)

by $\tilde{P}$, for any boolean function $f : \{0, 1\}^r \rightarrow \{0, 1\}$, to multiplicative error $|\tilde{P} - P| \leq P/\text{poly}(n)$ if $f(z)$ can be computed efficiently given $z$.

The probability of any distribution that can be classically efficiently sampled is such kind of $P$: the distribution is produced by tossing the coin and regarding $z$ as the sequence of coin-tossing results, the probability of a specific event is the union of some $z$ such that $f(z) = 1$. Hence the above theorem states that any probability in a distribution sampled by a polynomial classical algorithm can be approximated to multiplicative errors in $\text{FBPP}^{\text{NP}}$, which is contained in the third level of the polynomial hierarchy [5, 7, 18]. The probability in the distribution sampled by a quantum algorithm is not $P$ since it involves sums of negative numbers. It can be proved that if $f : \{0, 1\}^r \rightarrow \{-1, 1\}$, it will still be $\#P$-hard to approximate the sum to multiplicative errors.

Assume there is a classical sampler that can sample from the distribution $\{p_x\}$. According to Stockmeyer’s theorem, $\tilde{p_x}$ can be computed in the third level of the polynomial hierarchy such that $|\tilde{p_x} - p_x| \leq p_x/\text{poly}(n)$. If the distribution $\{p_x\}$ can approximate $\{q_x\}$ to variation distance, i.e., $\sum_x |p_x - q_x| \leq \epsilon$. Then $E_x [\epsilon] \leq \epsilon/2^{2mn}$. Using Markov inequality

$$\Pr_x [p_x - q_x \geq \epsilon/2^{2mn}] \leq \delta,$$

(31)
we get

\[ |\tilde{p}_x - q_x| \leq |\tilde{p}_x - p_x| + |p_x - q_x| \]

Stockmeyer’s theorem:

\[
\begin{align*}
&\leq \frac{p_x}{\text{poly}(n)} + |p_x - q_x| \\
&\leq q_x + \frac{|p_x - q_x|}{\text{poly}(n)} + |p_x - q_x| \\
&= \frac{q_x}{\text{poly}(n)} + \left(1 + \frac{1}{\text{poly}(n)}\right)|p_x - q_x| \\
&\text{with } \geq 1 - \delta \text{ fraction of } x
\end{align*}
\]

classically simunable assumption & Markov inequality:

\[
\frac{q_x}{\text{poly}(n)} + \frac{\epsilon(1 + o(1))}{2^{mn}\delta}.
\]

We have shown that approximating \( q_x \) to a mixture of multiplicative and additive errors in Eq. (32) is \( \#P \)-hard in the worst-case if \( \epsilon/\delta < 1/2 \). Lifting this worst-case hardness result to average-case result, we will get the desired result: If for any \( 1 - \delta \) fraction of instances \( x \), approximating \( q_x \) to the mixture of the multiplicative and additive errors in Eq. (32) is still \( \#P \)-hard; then if we assume there is a classical sampler that can simulate the distribution of our Ising spin model to variation distance errors, there will exist a \( \text{BPP}^{\text{NP}} \) algorithm that can solve \( \#P \)-hard problems, implying the collapse of the polynomial hierarchy.

Restating the above conclusion in terms of the partition function, we get

**Theorem 4.** If approximating the partition function to the following error

\[
\sqrt{\frac{|Z_x|^2}{2^{mn}}} - \frac{|Z_x|^2}{2^{mn}} \leq \frac{1}{\text{poly}(n)} \frac{|Z_x|^2}{2^{mn}} + \frac{\epsilon}{\delta}
\]

is also \( \#P \)-hard for any \( 1 - \delta \) fraction of instances \( x \), then simulating the distribution sampled by our Ising spin model to the variation distance \( \epsilon \) is classically intractable, otherwise the polynomial hierarchy will collapse.

**Intuition of our average-case hardness conjecture**

Substitute \( q_x \) in Eq. (32) by Eq. (18)

\[
\left|\langle 0|C_x|0\rangle^2 - \langle 0|C_x|0\rangle^2\right| \leq \frac{|\langle 0|C_x|0\rangle^2|}{\text{poly}(n)} + \frac{\epsilon(1 + o(1))}{2^{mn}\delta}
\]

(34)

where \( |\langle 0|C_x|0\rangle^2| \) is an estimation of \( |\langle 0|C_x|0\rangle^2| \) and \( m \) is the width of the circuit \( C_x \). The circuit \( C_x \) is formed by random 2-qubit gates layer by layer (\( n \) layers) similar to Fig. 3(b). Except some single qubit gates on the boundary, each 2-qubit gate has the form shown in Fig. 5, where the angles \( \alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta' \) are chosen from \( \{0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4\} \) randomly and independently. This can be verified directly by choosing random measurement results on blue circles in Fig. 1(c) of the main text. If either \( \delta \) or \( \delta' \) is different from 0 or \( \pi \), this 2-qubit gate will produce entanglement on some product states. In our opinion, with high probability, this kind of circuits will likely produce highly entangled states. Therefore, we conjecture that calculating the amplitudes of the circuit to the error in Eq. (34) is \( \#P \)-hard in the average-case.

There is a natural connection between our model and sampling models of random quantum circuits like the one in Ref. [14]. In Ref. [14], the quantum circuit is basically \( \sqrt{n} \) layers of single qubit gates (chosen from \( \{X^{1/2}, Y^{1/2}, R_z(\pi/4)\} \) randomly) and control-Z gates applied to \( \sqrt{n} \times \sqrt{n} \) input qubits on square lattice. The intuition of classical hardness of this sampling problem is from the relation between quantum chaos and random quantum circuits. The distribution produced by their sampling model is expected to satisfy the Porter-Thomas distribution [19] with a sufficient circuit depth. This is supported by numerical simulations in Ref. [14]. Then there is a large fraction of \( |\langle 0|U|z\rangle|^2 \geq 1/2^m \) where \( U \) is a random circuit, which implies that approximating output probabilities to multiplicative errors is \( \#P \)-hard in average-case and the noncollapse of the polynomial hierarchy is sufficient to prove the classical hardness result. Although the ensembles used in our model and the one in Ref. [14] are different, we think there is no fundamental difference since they both try to produce sufficiently random quantum circuits. Besides, it is expected that the distribution of our model approaches the Porter-Thomas distribution if \( n \sim m \) because the
FIG. 5. Random 2-qubit gate in \( C_\alpha \). \( \alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta' \) are chosen from \( \{0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4\} \) randomly and independently.

“input” in our model is on a linear array (the depth is expected to grow as \( n^{1/D} \) for a \( D \) dimensional qubit lattice. See corresponding discussions in Ref. [14]). Therefore, we should be able to convert Eq. (33) and Eq. (34) with multiplicative errors in our conjecture to be similar to the one in Ref. [7].

Certification to variation Distance Errors

With the reasonable assumption that the errors of \( X \) measurements are local and small (scales as \( O(1/mn) \)), we can certify whether the distribution sampled by a quantum sampler in the laboratory satisfies the variation distance bound. First, we give the condition that the measurement errors should satisfy; then we reduce the certification can certify whether the distribution sampled by a quantum sampler in the laboratory satisfies the variation distance bound. Therefore, we should be able to convert Eq. (33) and Eq. (34) with multiplicative errors in our conjecture to be similar to the one in Ref. [7].

\[
\sum_{x} |q_x - q'_x| \leq D(\rho, \rho').
\]

(35)

So if the measurements are perfect, bounding the trace distance will imply that variation distance is bounded.

Let us consider measurement imperfections. Denote the ideal measurement as a quantum operator \( \mathcal{E} \) and the imperfect one as \( \mathcal{E}' \). If the measurement errors are small and local, \( \mathcal{E}' \) can be approximated as

\[
\mathcal{E}' \approx \mathcal{E} \circ (\mathcal{I} + \varepsilon \sum_{i} \omega_i)
\]

(36)

where \( \mathcal{I} \) is the identity quantum operation, \( \omega_i \) is some local operation around spin \( i \), and \( \varepsilon \) is some small number. Bounding the variation distance can be reduced by

\[
\sum_{x} |q_x - q'_x| = D(\mathcal{E}(\rho), \mathcal{E}'(\rho')) \leq D(\mathcal{E}(\rho), \mathcal{E}(\rho')) + D(\mathcal{E}(\rho'), \mathcal{E}'(\rho')) \leq D(\rho, \rho') + D(\mathcal{E}(\rho'), \mathcal{E}'(\rho')).
\]

(37)

The term \( D(\rho, \rho') \) characterizes the error produced in the process of preparing the final state (time evolution and initial state preparation errors). The term \( D(\mathcal{E}(\rho'), \mathcal{E}'(\rho')) \) characterizes the error due to imperfect measurements.

We divide the certification of the variation distance error into two parts:

\[
\begin{align*}
D(\rho, \rho') & \leq \varepsilon_d \\
D(\mathcal{E}(\rho'), \mathcal{E}'(\rho')) & \leq \varepsilon_m \\
\varepsilon_d + \varepsilon_m & \leq \varepsilon.
\end{align*}
\]

(38)

The error due to imperfect measurements is

\[
\|\mathcal{E}'(\sigma) - \mathcal{E}(\sigma)\| \approx \|\varepsilon \mathcal{E} \circ \left( \sum_{i} \omega_i \right)(\sigma)\| \leq mn \varepsilon
\]

(39)

where \( \sigma \) is some arbitrary density matrix. So as long as the measurement error on every spin can be made smaller than \( \varepsilon = \varepsilon_m/(mn) \), it can be guaranteed that the total measurement error is bounded by \( \varepsilon_m \).
The remaining is to certify whether \( D(\rho, \rho') \leq \epsilon_d \). We reduce the problem to certifying whether the state produced in the laboratory is close to the ideal state, which is made to be the ground state of a given local gapped Hamiltonian. The method in Ref. [29] can achieve this task. Recall a lemma in Ref. [29]:

**Lemma 1.** Suppose \( \rho \) is the ground state of \( H = \sum_{\lambda} h_{\lambda} \) where \( h_{\lambda} \) is a local Hermitian operator, the ground state is unique and the ground state energy is 0. To estimate \( \text{tr}(h_{\lambda} \rho') \) where \( \rho' \) is the state produced in the laboratory, \( M \) measurements on \( \rho' \) in the basis of \( h_{\lambda} \) are needed. By summing over all the estimations of \( h_{\lambda} \), we can get an estimation of \( \text{tr}(H \rho') \). By this estimation, we can estimate \( F(\rho, \rho') = \text{tr}(\rho \rho') \) by \( F^* \) where

\[
\text{Pr}[|F^* - F| \leq \epsilon'] \geq 1 - \alpha.
\] (40)

If we choose \( M \) as

\[
M \geq \frac{Jm^2n^2}{2\Delta^2 \epsilon'^2} \ln \left( - \frac{mn + 1}{\ln(1 - \alpha)} \right) \approx \frac{Jm^2n^2}{2\Delta^2 \epsilon'^2} \left( \ln mn + \ln \frac{1}{\alpha} \right) \text{ for } m, n \text{ large and } \alpha \text{ small}
\]

(41)

where \( \Delta \) is the energy gap and \( J = \max_{\lambda} \|h_{\lambda}\| \).

Because \( D(\rho, \rho') \leq \sqrt{1 - F^2(\rho, \rho')} \), \( F(\rho, \rho') \geq \sqrt{1 - \epsilon_d^2} \) implies \( D(\rho, \rho') \leq \epsilon_d \). So we require

\[
F^* \geq \sqrt{1 - \epsilon_d^2} + \epsilon'.
\] (42)

In our problem, the Hamiltonian is

\[
H'_{\text{brickwork}} = \frac{1}{2} \sum_i \left( I - R_2(\theta_i)X_i R_2^*(\theta_i) \prod_{j \in \text{neighbor of } i} Z_j \right)
\]

(43)

on the brickwork lattice as shown in Fig. 1 of the main text, and \( J = 1, \Delta = 1 \).

If we choose \( \epsilon_d = O(\epsilon), \epsilon_m = O(\epsilon) \) and \( \epsilon' = O(\epsilon^2) \), then we need to measure each local term in the Hamiltonian \( M = O(m^2n^2r/\epsilon^4) \) times to get a confidence level of \( 1 - 2^{-O(r)} \). The certification protocol is therefore efficient.

**Hardness of Classically Simulating the Square Lattice Model to variation Distance Errors**

When doing break and bridge operations, we need to measure \( Z \) being \( |0\rangle \) and \( X \) being \( |+\rangle \) on the red circles in Fig. 4, but the results \( |1\rangle \) and \( |-\rangle \) are also present as we sample. According to Eq. (16), we can conclude

- Measuring \( Z \) on qubit 0, the probabilities of getting \( |0\rangle \) and \( |1\rangle \) are both 1/2. When the result is \( |1\rangle \), the operation is \( iZ_1 \otimes Z_2 \), so the effect is just flipping the measurement result on the blue circles in Fig. 4.

- Measuring \( X \) on qubit 0, the probability of getting \( |+\rangle \) and \( |-\rangle \) are also 1/2 each. When the result is \( |-\rangle \), the operation is

\[
\frac{e^{-i\pi/4}}{\sqrt{2}} (I_1 \otimes I_2 - iZ_1 \otimes Z_2) = e^{-i\pi/4} e^{-i\pi/4} \otimes Z_2.
\]

(44)

Since

\[
e^{-i\pi/4} Z_1 \otimes Z_2 = -i e^{i\pi/4} Z_1 \otimes Z_2 Z_1 \otimes Z_2,
\]

(45)

the effect is also flipping the measurement result on the blue circles.

Denote the measurement result on blue circles as \( x' \) and result on red circles as \( y \) (for the bridge operation, denote \( |+\rangle \) as 0) and \( q_x \) is the probability of measuring \( x \) on the brickwork model. Because the effect of \( y \) may be just flipping some bit of \( x \), given \( y \), we can infer \( x \) and \( x' \) from each other. Besides, \( q_y = \sum_x q_x' y = 1/2^r \) where \( r \) is the number of red circles (actually, \( r = 3mn - 2m - 2n + 1 \)) and \( q_x' |y = q_x \), so

\[
\sum_y q_x' y = \sum_y q_x' y q_y = \sum_y \frac{1}{2^r} q_x = q_x.
\]

(46)
Suppose there exists a quantum sampler that can generate a distribution \( \{ p_{x',y} \} \) to approximate the distribution of square lattice model to variation distance errors:

\[
\sum_{x',y} |p_{x',y} - q_{x',y}| \leq \epsilon.
\] (47)

We can then define a new classical sampler to simulate the distribution of the brickwork model: suppose the outcome is \( x', y \) and define the result to be \( x \) (\( x', y \) can determine a unique \( x \)), so the probability of getting \( x \) is \( p_x = \sum_y p_{x',y} \), implying

\[
\sum_{x',y} |p_{x',y} - q_{x',y}| = \sum_{x,y} |p_{x',y} - q_{x',y}|
\geq \sum_x \left( \sum_y p_{x',y} - \sum_y q_{x',y} \right)
= \sum_x |p_x - q_x|.
\] (48)

The first equality is because given \( y, x \) and \( x' \) can determine each other. The last equality is due to the definition of \( p_x \) and Eq. (46). This implies that there exists a classical sampler to simulate the brickwork model. So the hardness result of the square lattice model is based on the same conjectures (polynomial hierarchy does not collapse and Theorem 4).