Fitting algorithm of sine wave with partial period waveforms and non-uniform sampling based on least-square method

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Abstract. A novel four-parameter sine wave curve-fitting method for partial period and non-uniform sampling is presented. Based on the three-parameter sine wave curve-fitting method, the principle of this method is to turn the optimization of four parameters (amplitude, frequency, phase and offset) into the optimization of only one parameter (frequency). Compared to other curve-fitting methods, the characteristics of this method is that it does not need initial parameter estimation, and can be used to fitting partial period sinusoid, even with only 1/5 signal periods or less. It is shown that this method has wide convergence interval and excellent robustness. The validity and feasibility of this method have been proved by both the simulations and experiment. The four-parameter sine wave curve-fitting method can be applied to the parameter estimation of sinusoidal wave with non-uniform sampling series of partial period, especially it is very useful in the control of ultra lower frequency sine wave parameters.

Keywords: non-uniform sampling, partial period, sine wave, curve-fitting, parameter estimation, calibration

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1. Introduction

Sinusoid can be characterized by four parameters, the amplitude, frequency, phase and DC bias. The four-parameter curve-fitting of sinusoid is a basic algorithm in engineering measurements, and has been studied widely and deeply by many researchers [1-20]. Generally, initial parameter estimation is required in most curve-fitting methods. If the initial parameter is far enough from the target value, the final fitting results may not be always convergence to the target value.

In some applications, such as calibrations of ultra-low frequency vibration and ultra-low frequency sinusoid generator, the frequency may be very low, 10⁻⁴Hz for example, that it takes a very long time to sample a whole signal period. So a method to estimate all the four sinusoidal parameters from partial signal period is required in the control, adjustment, test and calibration of sine wave parameters. Because the sampling series of signal is sometimes less than one period, and maybe it is only a little
section of a period, it is difficult to estimate the initial parameters that many sinusoidal curve-fitting methods can’t be used. Thus, it is an important and valuable work to estimate sinusoidal parameters from a partial period of signal.

In this paper, a partial period sine wave curve-fitting method with non-uniform sampling is presented. It can also be used to the uniform and random sampling sinusoidal series with both amplitude and time information known.

2. Three-parameter sinusoidal curve-fitting method

2.1. Principle of three-parameter sinusoidal fitting method [10]

Suppose an ideal sinusoidal wave can be expressed as

\[
y(t) = E_1 \cos(2\pi ft) + E_2 \sin(2\pi ft) + Q = E \cos(2\pi ft + \Phi) + Q
\]

where, \(y_1, y_2, ..., y_n\) are the sampling data at time points \(t_1, t_2, ..., t_n\). The three-parameter sinusoidal curve-fitting process is, with known frequency \(f\), to search for the \(A_1, B_1\) and \(C\) that can minimize the \(\varepsilon\) in the following equation.

\[
\varepsilon = \sum_{i=1}^{n} \left[ y_i - A_1 \cos(2\pi f t_i) - B_1 \sin(2\pi f t_i) - C \right]^2
\]

Then a fitting equation is obtained, as shown below.

\[
y'(i) = A_1 \cos(2\pi f t_i) + B_1 \sin(2\pi f t_i) + C = A \cos(2\pi f t_i + \theta) + C
\]

\[
A = \sqrt{A_1^2 + B_1^2}
\]

\[
\theta = \begin{cases} 
\arctan\left(\frac{-B_1}{A_1}\right); & A_1 \geq 0 \\
\arctan\left(\frac{-B_1}{A_1}\right) + \pi; & A_1 < 0
\end{cases}
\]

The residual error of fitting is

\[
\rho = \sqrt{\frac{\varepsilon}{n}}
\]

\[
\varepsilon = \sum_{i=1}^{n} (y_i - y'(i))^2
\]

Since this is a closed arithmetic, the curve-fitting results can always be attained.

2.2. Discussion

The preceding three-parameter curve-fitting process is on the hypothesis that the frequency \(f\) is known. Especially, suppose \(t_{\text{min}} = \min \{t_i\} = t_1\) and \(t_{\text{max}} = \max \{t_i\} = t_n (i = 1, ..., n)\), then the average sampling rate is \(v = (n-1)/(t_{\text{max}} - t_{\text{min}})\), the average sampling period is \(\Delta t = 1/v\), and the digital angular frequency is \(\omega = 2\pi f / v\).

In the case of uniform sampling, equation (1) can be expressed as

\[
y(i) = E_1 \cos(\omega \cdot i) + E_2 \sin(\omega \cdot i) + Q = E \cos(\omega \cdot i + \Phi) + Q
\]

The curve-fitting equation is

\[
y'(i) = A_1 \cos(\omega \cdot i) + B_1 \sin(\omega \cdot i) + C = A \cos(\omega \cdot i + \theta) + C
\]

Suppose the digital angular frequency is \(\omega\) instead of \(\omega\), by using a sine wave series of 1/3 periods, we can get the relationship between the normalized error \(\rho/E\) and the frequency ratio \(w/\omega\) as shown in figure 1, using the three-parameter curve-fitting method. Figure 2 shows the details of the beginning part where \(\rho/E \leq 0.05\) of figure 1. Here, \(\omega = 2\pi \times 10000\), \(E = 4\), \(Q = 0\), and \(\Phi\) are 0°, 35°, 70°, 105°, 140°, 175°, 210°, 245°, 280°, 315° and 350°, respectively. The three-parameter curve-fit is executed by replacing \(\omega\) with \(w\).
It is shown in figure 1 and figure 2 that, for the curve-fitting of partial period sine wave with known frequency $\omega$, the extremum of $\rho/E$ of three-parameter curve-fitting exists and it is alone in the interval of $[0, 2\omega]$. The extremum is at the point where the frequency is $\omega$. It is also shown that the waveform of $\rho/E$ is not affected by variations of amplitude and phase, but the amplitude of $\rho/E$ is affected by the variation of initial phase. When the fitting frequency $w$ is bigger than $2\omega$, the values of $\rho/E$ are bigger than that when $w$ is smaller than $2\omega$. This can be used as the criterion to determine the upper boundary of convergence, and the lower boundary can be a small frequency that is close enough to zero. The point where $\rho/E$ has the minimum value can be obtained by searching $\omega$ in the interval of $[0, 2\omega]$. The same conclusion can also be reached by using other parts of sinusoidal period waveform to make the fitting.

3. Four-parameter sine wave curve-fitting algorithm with partial period and non-uniform sampling

3.1. Principle and process
By rebuilding the preceding curve-fitting method, we can obtain a convergent four-parameter sinusoidal curve-fitting method, which can be used to partial period sinusoidal waveforms with non-uniform sampling. Suppose the average sampling rate is $v$, the true frequency to be estimated is $f_0$, the length of sampling series is $p$ periods of sinusoidal waveform ($0<p<1$), and the time interval of $p$ signal periods is $\tau$. Then $f_0<1/\tau$. Select a factor $q$ that is small enough ($q=10^{-3}$ for example) to make $f_0>q/\tau$. To each frequency $f$ in the interval $[q/\tau, 2/\tau]$, we know that the extremum of square sum of residuals $df$ is in existence and alone. So, the four-dimension search for amplitude, frequency, phase and offset can be transformed into one-dimension search for frequency and $df$. In the interval of $[q/\tau, 2/\tau]$, the convergence of the four-parameter sinusoidal curve-fitting process, which is realized by using the three-parameter curve-fitting method, can be assured. The procedures of the four-parameter fitting are as following.

(1) Suppose the condition for iteration to stop is $h_c$.
(2) Obtain the time interval $\tau$ of sampling the partial signal period from the sampling series $y_1$, $y_2$, ..., $y_p$ at known time points $t_1$, $t_2$, ..., $t_p$ using points counting method, $\tau = t_p - t_1$. Calculate the average sampling rate, $v=(n-1)/(t_p-t_1)$. Then select a factor $q$ that is small enough to make $f_0>q/\tau$. Determine the range in which the target frequency $f_0$ exists, $[q/\tau, 2/\tau]$.
(3) Determine the left boundary frequency, $f_L=q/\tau$, and the right boundary frequency, $f_R=2/\tau$, of the iteration.
(4) Let the median frequency be $f_{Md}=f_L+0.618\times(f_R-f_L)$ and $f_r=f_R-0.618\times(f_R-f_L)$.
(5) Execute the three-parameter curve fit at $f_L$, $f_R$, $f_M$ and $f_T$, respectively, to get $A_L$, $\theta_L$, $C_L$, $\rho_L$, $A_R$, $\theta_R$, $C_R$, $\rho_R$, $A_{Md}$, $\theta_{Md}$, $C_{Md}$, $\rho_{Md}$, $A_T$, $\theta_T$, $C_T$ and $\rho_T$.
(6) If $\rho_{Md}-\rho_L$, then $f_0=[f_M,f_L]$. Let $\rho=\rho_{Md}, f_0=f_M, f_r=f_M, f_r=f_M+0.618\times(f_M-f_L)$.
If $\rho_{Md}=\rho_T$, then $f_0=[f_M,f_T]$. Let $\rho=\rho_T, f_0=f_M, f_r=f_M, f_r=f_M+0.618\times(f_M-f_L)$. 

Figure 1. Variation of normalized error $\rho/E$ with frequency ratio $w/\omega$ in curve-fitting of $1/3$ period. 
Figure 2. Details of Figure 1 in the range of $\rho/E \leq 0.05$. 

(7) Judge whether $|\Delta(p_{0.3})/\rho_{i0}|<\varepsilon_{c}$ is true. If it is true, then stop the iteration. If $\rho_{i}=\rho_{f}$, get the curve-fitting parameters $A=A_{c}, f=f_{c}, \theta=\theta_{c}$ and $C=C_{c}$. If $\rho_{i}=\rho_{m}$, get the curve-fitting parameters $A=A_{m}, f=f_{m}, \theta=\theta_{m}$ and $C=C_{m}$. The fitting process ends.

(8) If $|\Delta(p_{0.3})/\rho_{i0}|<\varepsilon_{c}$ is not true, repeat (5)–(7).

### 3.2. Convergence

The four-parameter sinusoidal curve-fitting is an iteration process, so it also needs to be convergent. As denoted in figure 1, for the sampling series of $p$ ($0<p<1$) periods, the convergence interval is (0, 2$f_{0}$], where, $f_{0}$ is the true value of signal frequency.

### 4. Simulations

Assuming that the measurement range is -5V~+5V, the amplitude of sine wave signal is 4V, the DC bias is 0V, the initial phase is 100°, the frequency is 6Hz, the uniform sampling rate is 8kSa/s, the sampling interval is 0.125ms, the sampling number is 400, and the sampling series covers about 0.3 waveform periods, we can get the four-parameter sine wave curve-fitting results, as shown in table 1, by using the method discussed above. The fitting curve is shown in figure 3, along with the uniform sampling curve. Figure 4 shows the difference between the sampling series and its curve-fitting.

#### Table 1. Results of partial period sine wave curve-fitting

| Parameter | Nominal value | Figure 3 Uniform sampling ($p=0.3$) | Figure 5 Non-uniform sampling ($p=0.4$) | Figure 7 Non-uniform sampling ($p=1.2$) |
|-----------|---------------|-------------------------------------|-------------------------------------|-------------------------------------|
| $A$       | 4 V           | 4.003 V                             | 4.000000 V                         | 4.000000 V                         |
| $\Delta A$| 0 V           | 2.6 mV                              | 0.24 $\mu$V                        | 0 V                                |
| $\Delta A/A$| 0            | 6.5x10^{-4}                        | 6x10^{-8}                          | 0                                  |
| $f$       | 6 Hz          | 5.996 Hz                            | 6.000001 Hz                        | 6.000000 Hz                        |
| $\Delta f$| 0 Hz          | -4 mHz                              | 1 $\mu$Hz                          | 0 Hz                               |
| $\Delta f/f$| 0          | -6.7x10^{-4}                        | 1.7x10^{-7}                        | 0                                  |
| $\theta$  | 100°          | 100.289°                            | 99.999981°                         | 100.0000038°                       |
| $\Delta \theta$| 0            | 0.289°                              | -0.00002°                          | 0.0000038°                         |
| $\Delta \theta/\theta$| 0          | 2.9x10^{-3}                        | -2x10^{-7}                         | 4x10^{-6}                          |
| $C$       | 0 V           | -2.082 mV                           | -0.476 $\mu$V                      | 42.7 nV                            |
| $\Delta C$| 0 V           | -2.082 mV                           | -0.476 $\mu$V                      | 42.7 nV                            |
| $\Delta C/A$| 0          | -5.2x10^{-4}                        | 1.2x10^{-7}                        | 1.1x10^{-6}                        |
| $\rho$    | 0 V           | 95.7 $\mu$V                         | 1.61 $\mu$V                        | 2.46 $\mu$V                        |

**Figure 3.** Uniform sampling points $y_{i}$ and curve-fitting points $\hat{y}_{i}$ ($p=0.3$)

**Figure 4.** Difference between uniform sampling series and its curve-fitting $y_{i}$, $\hat{y}_{i}$
Keeping the parameters of sine wave model fixed and executing the random rate sampling, we can get the random sampling series and their fitting-curves by using the method in this paper, which are shown in figure 5 to figure 8. Figure 5 shows the non-uniform sampling points \( y_i \) and their curve-fitting points \( \hat{y}(i) \) in about 0.4 periods. The sampling interval is \((RND-0.4)\times20\tau_0\), RDN is a random number with uniform distribution in \([0,1]\). It can be found that, since the sampling intervals are non-uniform and vary randomly, the form of sine wave curve drawn in the mode of uniform sampling in figure 5 varies evidently.

Figure 6 is the difference between the sampling series and its curve-fitting. It can be seen that the sampling data are fitted well. Figure 7 shows the non-uniform sampling points \( y_i \) and their curve-fitting points \( \hat{y}(i) \) in about 1.2 periods. The sampling interval is \((RND-0.3)\times20\tau_0\). Figure 8 is the difference between the sampling series and its curve-fitting. It can be found that the data are also fitted well. The curve-fitting parameters of non-uniform sampling are also listed in table 1. The data in table 1 indicates that the curve-fitting error of non-uniform sampling is smaller than that of uniform sampling. The difference is distinct enough. The reason for this needs to be further studied. Maybe it is because the residual error is more like the white noise in non-uniform series curve-fitting, as shown in figure 6 and figure 8, but it is like system error in uniform sampling series curve-fitting, as shown in figure 4.

**Figure 5.** Non-uniform sampling points \( y_i \) and curve-fitting points \( \hat{y}(i) \) (\( p \approx 0.4 \))

**Figure 6.** Difference between non-uniform sampling series and its fitting curve \( y_i \) - \( \hat{y}(i) \)

**Figure 7.** Non-uniform sampling points \( y_i \) and curve-fitting points \( \hat{y}(i) \) (\( p \approx 1.2 \))

**Figure 8.** Difference between non-uniform sampling series and its fitting curve \( y_i \) - \( \hat{y}(i) \)

5. **Experiment and result**

By using an ultra-low frequency vibration standard as inspiriting source, a sinusoidal vibration signal is generated. The displacement amplitude and frequency of the sinusoid are 36.32 mm and 0.040000
Hz, respectively. The signal is measured using a displacement transducer of model ASQ-1CA. The data are collected using a NI PXI-6281 data acquisition system, the A/D of which is 18 Bits, the measurement range is ±2.5V and the sampling rate is 200 Samples/s. The sampling series contains 5000 data, which distribute in approximate 1 period of sinusoidal waveform.

The four-parameter sinusoidal curve-fitting are carried out using the method discussed above and the following results are obtained. The amplitude is 1771.02mV, frequency is 0.04056Hz, phase is 230.098°, offset is 120.53mV, the root mean square of residual error ρ is 30.157mV and the total waveform distortion is 2.4%. Both the fitting curve and sampling data are shown in figure 9, and their difference is shown in figure 10.

By cutting out two parts of sampling series in figure 9, the uniform sampling series is turned into a non-uniform sampling series, as shown in figure 11. The results of four-parameter sinusoidal curve-fitting in this case are: amplitude is 1746.93mV, frequency is 0.04042Hz, phase is 231.354°, offset is 123.94mV, root mean square of residual error ρ is 34.154mV and the total waveform distortion is 2.8%. Both the fitting curve and the sampling data are shown in figure 11, and their difference is shown in figure 12.

6. Discussion

Through the simulation and experiment above, we can see that the four-parameter sinusoidal curve-fitting method can be used to partial period sine wave fitting with non-uniform sampling. The amplitude, frequency, phase and DC offset of a sinusoidal wave can be obtained using this method. Although the original purpose of this method is to solve the partial period sine wave fitting problem, in
fact, it can be used to any sinusoidal sampling series less than 2 periods directly. A distinguished feature of this method is that it fits for any sine wave sampling series, whether it is uniform sampling or not. Especially, it is very useful in the case that some abnormal parts are removed from a uniform sampling series. A uniform sampling is often turned into a non-uniform sampling for some reasons, such as the period is too long or the experiment condition is varied during the data acquisition. In random sampling, there are even some cases in which the time sequence reversal may occur. The method discussed above is valid in all these cases. It has shown excellent properties of robustness and convergence, and it can be used to solve various engineering problems above.

It is shown from the simulation that, compared with the uniform sampling, the sinusoidal curve-fitting error of non-uniform sampling is much smaller. In the simulation of non-uniform sampling above, the amplitude difference is $6 \times 10^{-8}$, frequency difference is $1.7 \times 10^{-7}$, phase difference is $-0.00002^\circ$ and the DC offset difference is $1.2 \times 10^{-7}$ (refer to amplitude). The simulation results are proved by the experiment. The simulation experiment also shows that, compared with the multi-period series curve-fitting, the error of partial period curve-fitting is a little larger, especially when the DC bias is large. The result of parameter estimation is affected by many factors, such as information losing, waveform distortion, sampling jitter, series number and period width, etc. Normally, correct results can be obtained with a sampling length over 1/5 sine wave period by use of the partial period curve-fitting method presented above.

7. Conclusions
A four-parameter curve-fitting method for partial period sinusoidal waveform with non-uniform sampling is presented in this paper. The process of four-parameter curve-fitting is introduced and the convergence interval is given. The characteristics of this method is that it does not need initial parameter estimation, and can be used to fitting partial period sinusoid. Knowing the sampling series and sampling moment is enough for this method. Simulations and experiment are carried out to prove the validity of this method, and the curve-fitting results in random sampling and non-random sampling are compared. It can be concluded that the four-parameter sine wave curve-fitting method has excellent properties of robustness and convergence. It can be used to partial period sinusoidal curve-fitting with non-uniform sampling. The method has a special value to ultra-low frequency parameter estimation.

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