Steps towards the axiomatic foundations of the relativistic quantum field theory: Spin-statistics, commutation relations and CPT theorems

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A realistic physical axiomatic approach of the relativistic quantum field theory is presented. Following the action principle of Schwinger, a covariant and general formulation is obtained. The correspondence principle is not invoked and the commutation relations are not postulated but deduced. The most important theorems such as spin-statistics, and CPT are proved. The theory is constructed from the notion of basic field and system of basic fields. In comparison with others formulations, in our realistic approach fields are regarded as real things with symmetry properties. Finally, the general structure is contrasted with other formulations.

KEYWORDS: Schwinger principle, physical axiomatics, CPT theorem, spin-statistics theorem, quantized fields

1. INTRODUCTION

The most effective way of systematizing and elucidating a body of ideas, enhancing clarity and rigor, is by axiomatizing a theory. Although it has been proved to be very fruitful in mathematics, it has rarely been tried in physics. This is due, in part, to that a physical theory presents an additional difficulty, because it include a mathematical formalism but it is more than this. This something more is the physical meaning. And the way of attaching a physical meaning to a formalism has been a very elusive problem. In general, it proceeds informally by the use of analogies and heuristic clues. But many wrong interpretations originate in an informal analysis of the structure of the theory. Therefore, we think that the assignation of meaning must be formal in such a way that all the presuppositions and interpretation

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rules be explicit. Following this idea, our physical axiomatics consists not only of a logical organization of a theory (as a mathematical systems of axioms), but also of an adequate characterization of the physical meaning of the symbolism and an examination of the metatheoretical aspects. As a consequence, our axiomatic approach has a number of important advantages: (a) It avoid mistakes in the interpretation of the theory because the assignation of meaning is made by means of semantical axioms. (b) It avoid the intrusion of elements that are alien to physics (e.g. observers) because all the presuppositions of the theory are made explicit from the beginning. (c) The physical referent class of the theory (i.e. the set of physical objects it describes) can be identified from the primitive basis and the axiom systems. (d) It allows an formal analysis of the structure of the theory, identifying key concepts and hypothesis, and facilitating the control of the derivations of theorems. Armed with this approach, in this paper we present an axiomatic formulation of the relativistic quantum field theory (RQFT) and we analyze its general structure.

Before axiomatizing a field theory, it must be clearly stated if the concept of field is justified. This is one of the most disputed questions about foundations of any RQFT and, based on different conceptions, a number of formulations have been developed. The most of these, are oriented either to eliminate the notion of ‘field’ because fields are unobservables or to assign them only a computational role. Here, differing from the bulk of the formulations, we propose a framework in which ‘quantum fields’ are the prima matter of the theory. That is, we assume a realistic ontology in which fields are regarded as quantum entities (i.e. as real things) with a number of properties (association, transformation properties, mutual action or interaction, etc.) and we shall construct the theory from the notion of basic field and system of basic fields. From this point of view, the real existence of ‘fields’ is an important assumption about the basic structure of matter, in terms of which is intended to explain the observed properties of particles. Moreover, we base our approach on the action principle of Schwinger. This principle is stated as a variational equation for the action integral operator (the spacetime integral of the Lagrange function-operator) which is an infinitesimal alteration of the transformation function (the infinitesimal generator of unitary transformations). Thus, the action principle of Schwinger appear as a differential version of the integral formulation of quantum mechanics given by Feynman. As a result of this realistic axiomatic formulation, in this paper we deduce the most important theorems such as spin-statistics, CPT and commutation relations.

Our formulation has the following advantages when compared with others: (a) It provide a abstract (in the sense that it does not dependent of any particular representation of the field operators) and covariant formalism that does not use of the so called “correspondence principle”. We think that correspondence

\[3\text{The ontological concepts presupposed by any field theory were presented in the Ontological Background in ref. [10].}\]
principles must be avoided into the construction of physical theories because it is not a constitutive principle but an heuristic principle which is useful only as a conceptual test for compatibility of a theory with less refined theories. (b) The commutation relations between Bose and Fermi field operators are not postulated but deduced. (c) The spin-statistics relation is probed with great generality from the property of invariance under time reversal. This proof, differently from the original given by Pauli [13], follows from a direct argument either for integer spin or half-integer spin and in an independent way of the causality requirement, as is the case of the proof given by Weinberg [6, 7]. (d) The CPT theorem is proved from the general form postulated for the dynamical Lagrangian, and the mathematical properties of this formulation. Moreover, our proof follows avoiding to analyze the transformation properties for each kind of field, as has been proved by Lüders [14]. (e) In addition, other important results such as the field equations, the expression for the generators and the crossing symmetry theorem can also be deduced [15]. However, it should be stressed that several mathematical and physical problems will not be addressed in this paper. Among them, let us mention the mathematical structure of the distribution-valued field operators and Green functions, the role of the renormalization group and its physical interpretation, the structure of gauge symmetries and the associated Ward identities. Some of these problems will be dealt, hopefully, in forthcoming papers.

The article is organized as follows. In the second section, we present the physical axiomatics of RQFT. In the third section, we deduce the spin-statistics theorem. In the fourth section we obtain the commutation relations of field operators as a theorem. In the fifth, we deduce the fundamental theorem that must be satisfied by any RQFT: the CPT theorem. We give a simple example of this formulation in the section six. Finally, in the last section we compare our presentation with others, and we discuss our results.

2. PHYSICAL AXIOMATIC S OF RQFT

In this section we shall exhibit the axiomatic structure of RQFT. Firstly we shall list the set of ideas that the theory takes for granted. The formal background consists of all the logical and mathematical ideas it employs, and the material background consists of all the generic and specific physical theories it presupposes. As we shall see, RQFT presupposes no specific theory and for this reason it is called a fundamental theory.

2.1 FORMAL BACKGROUND

$P_1$ Bivalent logic.
2.2 MATERIAL BACKGROUND

P$_5$ Protophysics [19] (i.e. Physical probabilities, Chronology, Physical geometry,...).

2.3 PRIMITIVE BASIS

The conceptual space of the theory is generated by the basis B of primitive (or undefined) concepts, where:

\[ B = \langle M^4, \Sigma, \Sigma, \mathcal{H}, \mathcal{P}, \mathcal{A}, h, c, \mathcal{F}, \mathcal{P} \rangle. \]

The elements of this basis will be characterized both formally and semantically by the axiomatics of the theory and the derived theorems. According to their status in the theory, the axioms will be divided in three classes: mathematical [M], physical [P] and semantical [S].

2.4 AXIOMATIC BASIS

RQFT is a finite-axiomatizable theory, whose axiomatic basis is

\[ B_A(RQFT) = \bigwedge_{i=1}^{70} A_i \]

where the index i runs on the axioms.

2.5 DEFINITIONS

D$_1$ $K$ \(\overset{Df}{=}\) set of inertial reference systems.

D$_2$ eiv \(\overset{Df}{=}\) eigenvalues of \(\hat{A}\).

D$_3$ \(\langle \Phi_1 | \Phi_2 \rangle \) \(\overset{Df}{=}\) scalar product of vectors \(\Phi_1\) and \(\Phi_2\).

D$_4$ An interval \(d\tau^2 = dx^\mu dx_\mu\) between two points in spacetime will be called spacelike if \(d\tau^2 < 0\), timelike if \(d\tau^2 > 0\) and lightlike if \(d\tau^2 = 0\).

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4The semantical symbol \(\overset{Df}{=}\) will be used for ‘definition’, \(\overset{\Delta}{=}\) for ‘representation’, and \(\overset{d}{=}\) for the relation of ‘denotation’ (See ref. [19] for details.)
A continuous set of physically independent (i.e. spacelike) points forms a spacelike surface. Formally: $\forall k, s_\mu = \{x_\mu \in R^4/\frac{dx_\mu}{dx} < 0\}$.

### 2.6 AXIOMS

#### GROUP I: SPACETIME

A1 [M] $M^4 \equiv$ Minkowski four-dimensional space.

A2 [S] $M^4$ is physical spacetime.

A3 [S] $\forall p, (\exists x_\mu)_{R^4} (x_\mu = p$ and $x_\mu \triangleq$ coordinates).

A4 [S] $\forall k, (\forall x_\mu)_{R^4} (x_\mu \triangleq$ a spacetime point referred to the inertial reference system $k \in K$. The component $x_0 \triangleq$ an instant of time and the components $x_i \triangleq$ a space position).

#### GROUP II: F-SYSTEMS

A5 [M] $\Sigma, \overline{\Sigma}$: nonempty numerable sets.

A6 [S] $\forall \sigma, \sigma_i \triangleq$ a basic field).

A7 [S] $\forall \sigma, \sigma \in C(\sigma) = \{\sigma_1, ..., \sigma_N\} (\sigma \triangleq f$-system).

A8 [S] $\forall \sigma, \overline{\sigma}, \sigma \triangleq$ environment of some f-system). In particular, $(\overline{\sigma}_o \triangleq$ the empty environment).

A9 [S] $\forall \sigma, \overline{\sigma}, \sigma \in C(\sigma) \Rightarrow H_E \triangleq \otimes_{i=1}^N H_{E_i}$.

A10 [P] $\exists |0\rangle$ (normalized vacuum state).

#### GROUP III: STATES

A11 [M] $\forall \sigma, (\exists H_E = \langle \mathcal{L}, \mathcal{H}, \mathcal{L}' \rangle \equiv$ rigged Hilbert space).

A12 [P] There exists a one-to-one correspondence between states of $\sigma \in \Sigma$ and rays $\mathcal{R}_\sigma \subset \mathcal{H}$.

A13 [M] $\forall \sigma, C(\sigma) = \{\sigma_1, ..., \sigma_N\} \Rightarrow H_E = \otimes_{i=1}^N H_{E_i}$.

A14 [S] $\Phi(\sigma, k) \in \mathcal{R}_\sigma \triangleq$ basis vector that is the representative of the ray $\mathcal{R}_\sigma$ of the f-system $\sigma$ with respect to $k \in K$.

A15 [P] $\exists |0\rangle \in \mathcal{R}_\sigma$ (normalized vacuum state).

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5 $C(\sigma)$ denotes the composition of $\sigma$ (see Section 2 of ref.[10]).

6 We restrict here to closed systems and we consider that $\sigma$ denotes $\langle \sigma, \sigma_0 \rangle$.

7 In order to avoid unnecessary complexity in notation we are not going to explicit the dependence of the state on the system and on the reference system.
GROUP IV: OPERATORS AND PHYSICAL QUANTITIES

A16 [M] $\mathcal{P} \equiv$ nonempty family of applications over $\Sigma$.

A17 [M] $\mathcal{A} \equiv$ ring of operators over $\mathcal{H}_E$.

A18 [P] $(\forall P)_P (\exists \sigma)(P \in P(\sigma))$.

A19 [P] $(\forall P)_P (\exists \hat{A})_A (\hat{A} \triangleq P)$.

A20 [P] $(\forall \sigma)_\Sigma (\forall \hat{A})_A (\forall a) (e^{iv\hat{A}} = a \wedge \hat{A} \triangleq P \Rightarrow a$ is the sole value that $P$ takes on $\sigma)$.

A21 [M] $\bar{\hbar} = c = 1$.

A22 [M] (Linearity and Hermiticity) $(\forall \sigma)_\Sigma (\forall \hat{A})_A (\forall P)_P (\hat{A} \triangleq P \wedge \hat{A} \hat{U} = \hat{A} \hat{U}^{-1}) \Rightarrow \hat{U} \hat{A} \hat{U}^{-1} \triangleq P$.

A23 [P] (Probability Densities) $(\forall \sigma)_\Sigma (\forall \hat{A})_A (\forall P)_P (\forall |\Phi\rangle) (\forall \phi) (\hat{A} \triangleq P \Rightarrow |\hat{A}\phi\rangle \Rightarrow \text{probability density } \langle \Phi | a \rangle |a\phi\rangle$ corresponds to the property $P$ of the f-system $\sigma$).

A24 [P] (Unitary Operators) $(\forall \sigma)_\Sigma (\forall \hat{A})_A (\forall P)_P (\forall |\phi\rangle) (\forall U) (\hat{A} \triangleq P \Rightarrow \hat{U} \hat{A} \hat{U}^{-1} \triangleq P)$.

GROUP V: QUANTUM FIELDS AND FIELD OPERATORS

A25 [M] $\mathcal{F} \subset \mathcal{A} \equiv$ nonempty set of differentiable operators over $\mathcal{H}_E$.

A26 [M] $(\forall \sigma_i)_\Sigma (\exists \hat{X}_A)_P (\hat{X}_A \triangleq (\hat{X}_{A_1}, \ldots, \hat{X}_{A_k}, \ldots))$.

A27 [S] $(\forall \sigma_i)_\Sigma (\hat{X}_A \triangleq \text{field operator in the } A \text{ representation associated with the basic field } \sigma_i)$.

A28 [S] $(\forall \sigma_i)_\Sigma (\forall x)_M (\hat{X}_A(x) \triangleq \text{the amplitude of the basic field } \sigma_i \text{ at } x)$.

A29 [M] $(\forall \sigma)_\Sigma (\exists \chi)_P (\hat{\chi} \equiv (\hat{\chi}^1, \ldots, \hat{\chi}^l, \ldots))$.

A30 [S] $\hat{\chi} \triangleq \text{general field operator associated with the f-system } \sigma$.

A31 [M] $\hat{\chi}^l = \hat{\chi}^l$.

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8Here the index $l$ runs over all the different representations. Usually the components $\hat{\chi}^l$ of each field operator $\hat{\chi}^l$ will be mutually dependent.
GROUP VI: POINCARE GROUP

A.32 [S] $P_d = \text{Full Poincaré group.}$

A.33 [S] $P_+ = \text{Proper Inhomogeneous Orthochronous Lorentz Group or Poincaré Group.}$

A.34 [S] $L_c = \text{Proper Homogeneous Orthochronous Lorentz Group or Lorentz Group.}$

A.35 [S] $(\forall g)P_+(g = (\Lambda, b) \overset{d}{=} \text{a generic element of the Poncaré Group, where } b \overset{d}{=} \text{a 4-dimensional translation and } \Lambda \overset{d}{=} \text{a Homogeneous Lorentz transformation } ).$

A.36 [M] The structure of Lie algebra of the Poincaré group is generated by the operators $\{\hat{H}, \hat{P_i}, \hat{K_i}, \hat{J_i}\} \subset A.$

A.37 [S] $\hat{H} \overset{d}{=} \text{the time translations generator.}$

A.38 [S] $(\forall \sigma)\Sigma(ei v \hat{H} \overset{d}{=} E \overset{\Lambda}{=} \text{the energy of } \sigma).$

A.39 [S] $\hat{P_i} \overset{d}{=} \text{the spatial translations generator.}$

A.40 [S] $(\forall \sigma)\Sigma(ei v \hat{P_i} \overset{d}{=} p_i \overset{\Lambda}{=} \text{the } i\text{-th component of linear momentum of } \sigma).$

A.41 [S] $\hat{K_i} \overset{d}{=} \text{the generator of the pure transformations of Lorentz.}$

A.42 [S] $\hat{J_i} \overset{d}{=} \text{the generator of spatial rotations.}$

A.43 [S] $(\forall \sigma)\Sigma(ei v \hat{J_i} \overset{d}{=} j_i \overset{\Lambda}{=} \text{the } i\text{-th component of angular momentum of } \sigma).$

A.44 [P] The vacuum state $|0\rangle$ is the state that is invariant under Poincaré transformations (up to a possible phase factor accounting for a constant energy).

GROUP VII: CONTINUOUS TRANSFORMATIONS: POINCARE

A.45 [M] Each A-component of the field operators associated to basic fields of spin $j$, denoted by $\hat{\chi}_AB^{j\lambda}$, transforms under an arbitrary Poincaré transformation as:

$$\hat{U}(\Lambda, b)\hat{\chi}_AB^{j\lambda}(x)\hat{U}^{-1}(\Lambda, b) = D^{(j)}_{AB}(\Lambda^{-1})^\lambda_{\lambda'}\hat{\chi}_{AB}^{j\lambda'}(\Lambda x + b),$$

where the matrix $(2j + 1)$-dimensional $D^{(j)}_{AB}(\Lambda)^\lambda_{\lambda'}$ is some irreducible representation $(A, B)$ of $\Lambda.$

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9 All elements of the Poincaré group, together with space inversion $I_s$, time reversal $I_t$ and their products is called the Full Poincaré group.

10 The representation labelled by $(A, B)$ has a dimensionality $(2A + 1)(2B + 1)$ where $A$ and $B$ assume values integer and/or half integer.
GROUP VIII: DISCRETE TRANSFORMATIONS: SPACE INVERSION

A₄₆ [S] (∀ Is)ₚₚ(Iₛ d = space inversion, represented in the Hilbert space by the unitary operator \( \hat{P} \triangleq \hat{U}(Iₛ) \).)

A₄₇ [M] The field operators associated to basic fields transform under spatial inversion as follows:

\[
\hat{P}\hat{\chi}^{\leftrightarrow AB}(x)\hat{P}^{-1} = D^{-1}(Iₛ)\hat{\chi}^{\leftrightarrow AB}(Iₛ x) = \nu_s\hat{\chi}^{\leftrightarrow AB}(Iₛ x)
\]

with \( \nu_s \) an arbitrary c-number of modulus one, i.e. a phase factor. Moreover,

\[
\hat{\chi}^{\leftrightarrow AB} = \hat{\chi}^{\leftrightarrow AB} \quad \text{with} \quad \hat{\chi}^{\leftrightarrow AB}(A, B) = \hat{\chi}^{\leftrightarrow BA}(B, A)
\]

and

\[
\hat{\chi}^{\leftrightarrow AB} = \hat{\chi}^{\leftrightarrow AA} \quad \text{with} \quad \hat{\chi}^{\leftrightarrow AA}(A, A) = 0
\]

GROUP IX: CHARGE OPERATOR

A₄₈ [M] (∀ \( \sigma_i \))(∃ \( \hat{Q} \))(\( \check{Q} \) has a discrete spectrum of real eigenvalues).

A₄₉ [M] (∀ \( \sigma_i \))(\( [\hat{Q}, \hat{H}] = [\hat{Q}, \hat{P}_1] = [\hat{Q}, \hat{K}_i] = [\hat{Q}, \hat{J}_i] = 0 \)).

A₅₀ [S] \( \hat{Q} \) d is the generator of gauge transformations of the first kind.

A₅₁ [S] (∀ \( \sigma_i \))(eiv\( \hat{Q} \) = q \( \hat{\gamma} \) the charge of the \( \sigma_i \)).

A₅₂ [P] The vacuum state \( |0\rangle \) is the state that is invariant under gauge transformations of the first kind.

GROUP X: DISCRETE TRANSFORMATIONS: CHARGE CONJUGATION

A₅₃ [S] \( I_c = \hat{C} \) charge conjugation, represented in the Hilbert space by the unitary operator \( \hat{C} = \hat{U}(I_c) \).

A₅₄ [M] The field operators associated to basic fields transform under charge conjugation as,

\[
\hat{C}\hat{\chi}^{\leftrightarrow AB_{12}}(x)\hat{C}^{-1} = D^{-1}(I_c)\hat{\chi}^{\leftrightarrow AB_{12}}(x) = \nu_c\hat{\chi}^{\leftrightarrow AB_{12}(\leftrightarrow -2)}(x)
\]

with \( \nu_c \) a phase factor. Moreover, for \( \hat{\leftrightarrow AB_1} \neq \hat{\leftrightarrow AB_2} \):

\[
\hat{\leftrightarrow AB_{12}} = \left( \begin{array}{c} \hat{\chi}^{\leftrightarrow AB_1} \\ \hat{\chi}^{\leftrightarrow AB_2} \end{array} \right) \quad \text{and} \quad \hat{\leftrightarrow AB_{12}(\leftrightarrow -2)} = \left( \begin{array}{c} \hat{\chi}^{\leftrightarrow AB_1} \\ -\hat{\chi}^{\leftrightarrow AB_2} \end{array} \right) \quad \text{with} \quad \hat{\leftrightarrow AB_{12}} = \hat{\leftrightarrow AB_1} \oplus \hat{\leftrightarrow AB_2}
\]

and

\[
\hat{\leftrightarrow AB_{12}} = \hat{\leftrightarrow AB} \quad \text{with} \quad \hat{\leftrightarrow AB_{12}} = \hat{\leftrightarrow AB} \quad \text{for} \quad \hat{\leftrightarrow AB_1} = \hat{\leftrightarrow AB_2}.
\]
GROUP XI: CONTINUOUS TRANSFORMATIONS: GAUGE

The field operators associated to basic fields transform under a gauge transformation of the first kind as:

\[ \hat{U}(\tau)\hat{\chi}^l(x)\hat{U}^{-1}(\tau) = e^{i\tau \xi} \hat{\chi}^l(x) , \]

where the unitary transformation \( \hat{U}(\tau) = e^{i\tau \hat{Q}} \) with \( \tau \) a constant and \( \xi \) an imaginary matrix.

GROUP XII: DISCRETE TRANSFORMATIONS: TIME REVERSAL

\( (\forall I_t) \hat{P}F(I_t) = \text{time inversion, represented in the Hilbert space by the antiunitary operator } \hat{T} = \hat{\tilde{U}}(I_t).) \)

The field operators associated to basic fields transforms under time reversal as:

\[ \hat{T}\hat{\chi} \leftrightarrow \hat{\chi}^{AB_{12}}(I_t x) \hat{T}^{-1} = D^{-1}(I_t)\hat{\chi}^{*AB_{12}}(I_t x). \]

GROUP XIII: LAGRANGE OPERATOR

\( \hat{\mathcal{L}} \equiv \text{differentiable Hermitian scalar-operator on any region } O \subset M^4. \)

\( \hat{\mathcal{L}}^{\mu} \equiv \partial \hat{\mathcal{L}} / \partial(\partial^{\mu}\hat{\chi}) \)

\( \hat{\mathcal{L}}^x \equiv \hat{\mathcal{L}}^{\mu}(\partial^{\mu}\hat{\chi}). \)

\( \hat{\mathcal{L}}^{DF} \equiv \partial \hat{\mathcal{L}} / \partial(\partial^{\mu}\hat{\chi}) \)

The generalized momentum field operator \( \hat{\pi}^{\mu}_l \equiv \hat{\chi}^{\tau}(\hat{\mathcal{L}}^\mu)_l \)

where the \( \mathcal{L}^\mu \) are numerical matrices to be determined.

\( \hat{\mathcal{L}}^{DF} \equiv \sum_{n_1, n_2, \ldots, n_s} a \cdot \hat{\chi}^{\Pi_{k_1}^{n_1}(\hat{\mathcal{L}}^\mu)^{k_1}} \hat{\chi}^{\Pi_{k_2}^{n_2}(\hat{\mathcal{L}}^\mu)^{k_2}} \cdots \hat{\chi}^{\Pi_{k_s}^{n_s}(\hat{\mathcal{L}}^\mu)^{k_s}} \hat{\chi} \)

with the constraint that the sum \( k_1 + k_2 + \cdots + k_s \) is even, and where “\( a \)” represents a real constant that must be different for each member of the sum.

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We shall use the notation \( \partial^{\mu} = \partial / \partial(x^\mu). \)
A63 [P] (Invariance under transformations of the Poincaré group)

\[ \hat{U}(\Lambda, b)\hat{\mathcal{L}}[x]\hat{U}^{-1}(\Lambda, b) = \hat{\mathcal{L}}[\Lambda x + b]. \]

A64 [P] (Invariance under gauge transformations)

\[ \hat{U}(\tau)\hat{\mathcal{L}}[x]\hat{U}^{-1}(\tau) = \hat{\mathcal{L}}[x]. \]

A65 [P] (Kinematical invariance under space inversion)

\[ \hat{P}\hat{\mathcal{L}}_{Kin}[x]\hat{P}^{-1} = \hat{\mathcal{L}}_{Kin}[Ix]. \]

A66 [P] (Kinematical invariance under charge conjugation)

\[ \hat{C}\hat{\mathcal{L}}_{Kin}[x]\hat{C}^{-1} = \hat{\mathcal{L}}_{Kin}[Ix]. \]

A67 [P] (Kinematical invariance under time reversal)

\[ \hat{T}\hat{\mathcal{L}}_{Kin}[x]\hat{T}^{-1} = \hat{\mathcal{L}}^{*}_{Kin}[Ix]. \]

GROUP XV: STATIONARY ACTION PRINCIPLE

A68 [P] (\( \forall \sigma \))\((\forall \hat{\mathcal{L}})\). If \( \hat{W}_{12} \overset{Df}{=} \int_{s_2}^{s_1} dx \hat{\mathcal{L}}[x] \) then

\[ \delta \hat{W}_{12} = \hat{F}_1 - \hat{F}_2, \]

where \( \hat{F}_i \in \mathcal{A} \).

A69 [S] \( \hat{W}_{12} \overset{d}{=} \) Action integral operator.

A70 [S] (\( \forall \sigma \))\((\forall s_i)\)_\(M^i\)\((\forall \hat{F}_i)\)\(\mathcal{A} (\hat{F}_i \overset{\Delta}{=} \) the Hermitian generator of infinitesimal unitary transformation on the surface \( s_i \).

Remark 1 As we can see from A45, A47 and A54, the dimensionality of a given basic field will be determined by the transformation properties under the group \( \mathbf{L}_k \) enlarged by space inversion \( I_s \) (denoted by \( \mathbf{L}_C \) and called complete Lorentz group) and by \( \hat{Q} \) (denoted by \( \mathbf{L}_{CQ} \)). For instance, a scalar field transforms as the representation \((0, 0)\) with spin \( j = 0 \) and it will be the only A-component of a field operator representing a neutral basic field. The next irreducible representation
is either \((0, \frac{1}{2})\) or \((\frac{1}{2}, 0)\), both corresponding to a field with spin \(\frac{1}{2}\). But, in order to represent the basic field of the electron, we have to consider each of them as an A-component of the field operator i.e., the \((0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)\) representation of the Lorentz group enlarged by parity. This reflects the fact that the Dirac representation used to describe the electron is reducible. Therefore, an electro-positron basic field of spin \(\frac{1}{2}\) will be represented by an Hermitian field operator of 8 A-components (i.e. \(2 \cdot 2 \cdot (2A + 1)(2B + 1)\)). **Remark 2** Note that the requirement that the fields will be characterized by Hermitian operators is not restrictive because we work in a representation where other properties are not diagonalized.

**Remark 3** A general field \(\hat{\chi}\) representing a f-system can be a very complicated mathematical entity as:

\[
\hat{\chi} = (\hat{\chi}(0,0), \hat{\chi}(\frac{1}{4},\frac{1}{4}), \hat{\chi}(0,\frac{1}{4}), \hat{\chi}(\frac{1}{4},0), \hat{\chi}(1,1), \hat{\chi}(0,1), \hat{\chi}(1,0), \cdots)
\]

that can be written in a compact way as \(\hat{\chi} = (\hat{\phi}, \hat{\phi}_\mu, \hat{\nu}, \cdots)\). In this case, the general field transforms as a reducible representation where the matrix \(D(\Lambda^{-1})\) includes the representations given by the matrices \(1, \Lambda^\mu\nu, 2\)-spinor, etc. **Remark 4** Since the general field operator contains all the different representations, the \(\Lambda^{\mu}\nu\) states a very natural assumption: that the generalized canonical momentum field operators \(\hat{\pi}^\mu\) are included in the components of the general field \(\hat{\chi}\). Note that the sum on the \(r\)-index runs over all the components of the general field operator \(\hat{\chi}\). We use the following notation: when no confusion is possible we shall consider the expression without matricial indices, i.e., \(\hat{\pi}^\mu = \hat{\chi}^\mu\). **Remark 5** \(\Lambda^{\mu}\nu\) is not as restrictive as it looks since the fields \(\hat{\chi}\) include all the basic fields represented of the f-system. Moreover, the properties of the \((\hat{\mu}^\nu)_{\alpha\beta}\) matrices have not been specified yet. We will see that these properties have very important consequences.

### 3. SPIN-STATISTICS RELATION

In this section we shall obtain the spin-statistic theorem. We shall present the proofs of the theorems in an schematic way since our purpose is illustrating the role of the axioms.

**T\(_1\) (\(\hat{F}(\delta\chi)\) Generating Operator)** The generating operator \(\hat{F}(\delta\chi)\) is given by:

\[
\hat{F}(\delta\chi) = \int ds\hat{\pi}_f\delta\hat{\chi}^f
\]

**Proof:** By steps: (1) The operator \(\hat{F} = \int_s ds\mu(\hat{\pi}_{\mu}^\mu\delta\hat{\chi}^f + \hat{\mathcal{L}}\delta\hat{\chi}^f)\) is obtained from \(\delta\mathcal{W}_{12} = \int_{s_2}^{s_1} \delta(dx)\hat{\mathcal{L}} + \int_{s_2}^{s_1} d\delta\hat{\mathcal{L}}\) and considering the expressions \(\delta(dx) = dx\hat{\delta}_\mu dx^\mu\) and \(\delta\hat{\mathcal{L}} = \hat{\delta}\hat{\mathcal{L}} + \frac{d\delta\hat{\mathcal{L}}}{dx}\delta\hat{\chi}^f\) with \(\hat{\delta}\hat{\mathcal{L}}[x] = \frac{d\hat{\delta}\hat{\mathcal{L}}}{d\chi^f}\delta\hat{\chi}^f + \hat{\pi}_{\mu}^\mu\partial_{\mu}\delta\hat{\chi}^f\) where \(\delta\hat{\chi}^f(x)\) is the total variation \(\delta\hat{\chi}^f(x) = \hat{\delta}\hat{\chi}^f(x) - \hat{\chi}^f(x)\). (2) Replacing \(\hat{\delta}\hat{\chi}^f = \delta\hat{\chi}^f - \partial_{\nu}\hat{\chi}^f\delta\chi^\nu\) into the expression of \(\hat{F}\) we obtain \(\hat{F}(\delta\chi) = \int_s ds\mu\hat{\pi}_{\mu}^\mu\delta\hat{\chi}^f\) and \(\hat{F}(\delta\pi) = -\int_s ds\nu\hat{T}_{\nu}^\mu\delta\chi^\nu\) with \(\hat{T}_{\nu}^\mu = \hat{\pi}_{\nu}^\mu\partial_{\nu}\hat{\chi}^\mu - \hat{\mathcal{L}}\delta\chi^\nu\). (3) Taking \(ds_{\mu} = n_{\mu}ds\) and \(\hat{\pi}_{\mu}^\mu = n_{\mu}\hat{\pi}_{\mu}^\mu\) (with \(n_{\mu}\) a unit timelike vector and \(ds\) the numerical measure of the surface element) we have \(\hat{F}(\delta\chi) = \int ds\hat{\pi}_f\delta\hat{\chi}^f\).

**T\(_2\) (\(\hat{F}(\delta\pi)\) Generating Operator)** The generating operator \(\hat{F}(\delta\pi)\) is given by:

\[
\hat{F}' = \hat{F}(\delta\pi) = -\int ds\hat{\pi}_f\delta\hat{\chi}^f.
\]
Proof: (1) Consider the Lagrangian $\mathcal{L}' = \mathcal{L} - \partial_{\mu}f^\mu$ (which is equivalent to $\mathcal{L}$), from $\mathcal{A}_{68}$ we have $\delta \hat{W}_{12}^\prime = \hat{F}_{1}^\prime - \hat{F}_{2}^\prime$ with $\hat{F}_{i}^\prime = \hat{F}_{i} - \delta \hat{W}_{i}$ and where $\hat{W}_{i}^\prime \equiv \int_{s_{i}} ds_{\mu} \hat{f}_{\mu}$. (2) Taking $\hat{f}_{\mu} = \hat{\pi}_{\mu} \hat{\chi}_{\mu}$ into the expression of the generator $\hat{F}_{1}^\prime$ and using $\mathcal{T}_{1}, \mathcal{T}_{2}, \mathcal{A}_{31}, \mathcal{A}_{61}$ and the hermiticity condition of $\hat{F}$ stated in $\mathcal{A}_{105}$. 

$\mathcal{T}_{3}$ (Symmetric Infinitesimal Generator) If $\hat{F}_{\mathrm{sym}}^\prime \equiv \frac{1}{2} [\hat{F}(\delta \chi) + \hat{F}(\delta \pi)]$ then

$$\hat{F}_{\mathrm{sym}}^\prime = \frac{1}{2} \int ds [\hat{\chi}^{r}(U_{\mu}^{\mu})_{rl} \delta \hat{\chi}^{l} - \delta \hat{\chi}^{r}(U_{\mu}^{\mu})_{rl} \hat{\chi}^{l}] \quad \text{where} \quad U_{\mu}^{\mu} = -U_{\mu}^{\mu}. $$

Proof: Using $\mathcal{T}_{1}, \mathcal{T}_{2}, \mathcal{A}_{31}, \mathcal{A}_{61}$ and the hermiticity condition of $\hat{F}$ stated in $\mathcal{A}_{105}$. 

$\mathcal{T}_{4}$ (Equivalent Generating Operators) (a) The generating operators $\hat{F}(\delta \chi), \hat{F}(\delta \pi)$ and $\hat{F}_{\mathrm{sym}}$ are equivalents. (b) $\hat{\pi}_{\mu} \delta \hat{\chi}^{l} = -\delta \hat{\chi}_{\mu} \hat{\chi}^{l}$. 

Proof: They are obtained from equivalents Lagrangian (i.e. differing each other by a divergence). Then, use the expressions of $\hat{F}(\delta \chi)$ and $\hat{F}(\delta \pi)$. 

$\mathcal{T}_{5}$ The matrices $U_{\mu}^{\mu}$ can be decomposed in a symmetric and an antisymmetric parts, that is, $U_{\mu}^{\mu} = U_{S}^{\mu} + U_{A}^{\mu}$ where each part satisfy:

$$U_{S}^{\mu T} = U_{S}^{\mu}, \quad U_{A}^{\mu T} = -U_{A}^{\mu}. $$

Proof: From matrix algebra. 

$\mathcal{T}_{6}$ The symmetric matrices $U_{S}^{\mu}$ are imaginary and the antisymmetric matrices $U_{A}^{\mu}$ are real, i.e.:

$$(U_{S}^{\mu})^{*} = -U_{S}^{\mu}, \quad (U_{A}^{\mu})^{*} = U_{A}^{\mu}. $$

Proof: From $\mathcal{T}_{4}$ and $\mathcal{T}_{5}$. 

$\mathcal{T}_{7}$ There are two different classes of field operators that satisfy the following commutation relations according to the different properties of the matrices $U_{\mu}^{\mu}$:

$$[U_{S}^{\mu}(x), \delta \hat{\chi}(x')]_{+} = 0, \quad [U_{A}^{\mu}(x), \delta \hat{\chi}(x')]_{-} = 0. $$

Proof:

1. Let $u_{\mu}$ be symmetric, then $\hat{\chi}U_{S}^{\mu} = U_{S}^{\mu} \hat{\chi}$. Thus we have from $\mathcal{T}_{13}$

$$(U_{S}^{\mu})^{*} = -U_{S}^{\mu} \Rightarrow [U_{S}^{\mu}(x), \delta \hat{\chi}(x')]_{+} = 0. $$

2. Let $u_{\mu}$ be antisymmetric, then $\hat{\chi}U_{A}^{\mu} = -U_{A}^{\mu} \hat{\chi}$. So, again from $\mathcal{T}_{13}$

$$(-U_{A}^{\mu})^{*} = U_{A}^{\mu} \Rightarrow [U_{A}^{\mu}(x), \delta \hat{\chi}]_{-} = 0. $$

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These commutation relations have been obtained for only one point \( x \). The expressions for arbitrary different points \( x \) and \( x' \) are obtained by the compatibility requirement for operators located at distinct points of a spacelike surface.

\[ D_6 \] (Bose and Fermi Field Operators)

(a) Fermi Operators: The field operators that satisfy the first group of commutation relations of \( T_7 \) will be called *Fermi field operators* and will be denoted by \( \hat{\psi} \).

(b) Bose Operators: The field operators that satisfy the second group of the commutation relations of \( T_7 \) will be called *Bose field operators* and will be denoted by \( \hat{\phi} \).

**Remark** \( D_6 \): Conventionally assigns a name to field operators associated to symmetric and antisymmetric matrices \( U^\mu \) that satisfy the anticommutation or commutation relations respectively. Thus, it is clear that we have not obtained the spin-statistic relation, because we have not specified any spin value for each different class of field operators.

\[ T_9 \] (Lagrange Operator) The Lagrange operator can be expressed in the general form:

\[
\hat{L} = \hat{L}_{\text{Kin}} + \hat{L}_{\text{Dyn}}
\]

where

\[
\hat{L}_{\text{Kin}} = \frac{1}{2} \left[ \dot{\chi}^r (U^\mu)_{rl} \partial_\mu \dot{\chi}^l - \partial_\mu \dot{\chi}^r (U^\mu)_{rl} \dot{\chi}^l \right], \quad \hat{L}_{\text{Dyn}} = -\hat{T}_\mu^\mu.
\]

**Proof:** From the expression of \( \hat{T}_\mu^\nu = \dot{\pi}^\mu_\nu \partial_\nu \dot{\chi}^l - \dot{\chi}^r (U^\mu)_{rl} \dot{\chi}^l \) (obtained from \( A_{58} \)) with \( \mu = \nu \), and using \( A_{51}, A_{58} \) and \( T_3 \). But, in order to have an Hermitian operator (\( A_{58} \)) we must add a term, i.e.,

\[
\hat{L} = \dot{\chi}^r (U^\mu)_{rl} \partial_\mu \dot{\chi}^l - \hat{T}_\mu^\mu - \frac{1}{2} \partial_\mu [\dot{\chi}^r (U^\mu)_{rl} \dot{\chi}^l].
\]

\[ T_9 \] The field operator associated to an \( f \)-system transforms under temporal inversion as a reducible representation, i.e.,

\[
\hat{T} \hat{\chi}(x) \hat{T}^{-1} = D^{-1}(I_t) \hat{\chi}^\ast(I_t x)
\]

with

\[
D(I_t) = \begin{pmatrix}
D_1 & 0 & \cdots & 0 \\
0 & D_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \cdots & D_j
\end{pmatrix}
\]

where \( D_j(I_t) \) is a matrix transformation of the \( j \)-th basic field.

**Proof:** From \( A_{58} \) and \( A_{58} \).
The matrices \( U^\mu \) transform under time reversal as:

\[
D(I_t) U^\mu D^{-1}(I_t) = (-1)^{\delta_{\mu 0}} (\pm) U^\mu
\]

where the matrices \( D(I_t) \) are imaginary for Fermi field operators and real for Bose field operators, with \((+)\) for antisymmetrical and \((-)\) for symmetrical \( U^\mu \) matrices.

**Proof:** We split the general field operator \( \hat{\chi} \) in terms of Bose and Fermi field operators, i.e. \( \hat{\chi} = (\hat{\phi}, \hat{\psi}) \). According to \( T \) the kinematical Lagrange operator expressed in this way (it is sufficient to consider the asymmetrical version) is,

\[
\hat{L}_{Kin} = \hat{\chi} U^\mu \partial_\mu \hat{\chi} = \hat{\phi} U^\mu \partial_\mu \hat{\phi} + \hat{\psi} U^\mu \partial_\mu \hat{\psi},
\]

applying the time reversal operator \( \hat{T} \) we have:

\[
\hat{T} \hat{L}_{Kin} \hat{T}^{-1} = \hat{\phi}^* \hat{T} U^\mu A \hat{T}^{-1} \partial_\mu \hat{\phi}^* + \hat{\psi}^* \hat{T} U^\mu S \hat{T}^{-1} \partial_\mu \hat{\psi}^*,
\]

where

\[
\partial_\mu = (-1)^{\delta_{\mu 0}} \partial_\mu.
\]

Comparing with (see \( A \)),

\[
\hat{L}_{Kin}^* = \hat{\phi}^* U^\mu A^* \partial_\mu \hat{\phi}^* + \hat{\psi}^* U^\mu S^* \partial_\mu \hat{\psi}^*,
\]

we finally obtain, from \( T \):

\[
U^\mu A \hat{T} = (-1)^{\delta_{\mu 0}} U^\mu A,
\]

\[
U^\mu S \hat{T} = (-1)^{\delta_{\mu 0}} (-) U^\mu S.
\]

Independent of the last theorem, we can prove a property of the \( D(I_t) \) matrix:

**T** The matricial representation \( D(I_t) \) is imaginary only for fields of half integer spin and real for fields of integer spin.

**Proof:** We must consider complex Lorentz transformations acting upon an A-component of \( \hat{\chi} \). Thus, we define \( J_4 \) and \( J_0 \) with \( J_0^* = -J_0 \) so that \( J_4^* = J_4 \). The \( \hat{T} \) operator of \( \hat{\chi} \) can be written as a rotation of \( \hat{P}_i \) where \( \hat{P}_i \) is the \( x_i \) space-inversion operator,

\[
\hat{T} = e^{-i J_4^* \hat{P}_i} e^{+i J_4 \hat{P}_i}
\]

using that \( \hat{P}_i \hat{J}_0 \hat{P}_i = -\hat{J}_0 \) we have,

\[
\hat{T} = e^{-i \hat{J}_4 \hat{P}_i} \hat{P}_i,
\]

complex conjugating and replacing,

\[
\hat{T}^* = e^{+i \hat{J}_4 \hat{P}_i} \hat{P}_i = e^{+2i \hat{J}_4 \hat{P}_i} \hat{T}.
\]

Since in the \((A, B)\) representation of Lorentz group, the operator \( i \hat{J}_0 \) is given, in our new notation, by \( \hat{J}_4 = \hat{A} - \hat{B} \). Denoting the eigenvalues of \( \hat{J}_4 \) by \( j \), we have for the matrix representation of \( \hat{T} \):

\[
D(I_t)^* = e^{+2i \pi j} D(I_t)
\]
with $D(I_t)^j = D(I_t)$ for $j$ integer and $D(I_t)^j = -D(I_t)$ for $j$ half integer. The theorem follows since the matrix $D(I_t)$ is a reducible representation as given by $T_{10}$.

We will close this section with a fundamental result of any relativistic quantum theory of fields:

$\mathbf{T_{12}}$ (Spin-Statistics Relation) Fields with half integer spin are represented by Fermi field operators, and fields with integer spin are represented by Bose field operators.

**Proof:** Comparing $T_{11}$ with $T_{10}$ noting that they are obtained independently.

**Remark 1** The derivation of the spin-statistics theorem presented here follows the proof presented by Schwinger in ref. [11, 12]. Modifications of his original derivation based on different assumptions were presented by himself on several occasions during the rest of his life (see for example [20]). A nice and comprehensive account of this theorem can be found in ref. [21].

**Remark 2** The explicit use of the so called “local causality” requirement, expressed as the commutation relation between field operators, is not necessary in this approach as it is the case for the proof of free fields (see ref. [13, 14]) or for interacting fields (see ref. [22, 23]). Thus, it can seem that this assumption is not needed in our proof. However, as we will see, “local causality” is a consequence derivable from the expression of the generating operators.

## 4. COMMUTATION RELATIONS

In this section the commutation relations of field operators will be deduced from a very general property of the generators. First we state a theorem concerning this general property,

$\mathbf{T_{13}}$ Let $\hat{F}$ be an infinitesimal Hermitian generator. The unitary operator defined by $\hat{U} \equiv 1 + i\hat{F}$ acts on an arbitrary operator $(\hat{A})_\mathcal{A}$ as:

$$\delta \hat{A} = \frac{1}{i}[\hat{A}, \hat{F}].$$

**Proof:** Evaluate $\hat{A}' = \hat{U}\hat{A}\hat{U}^{-1}$.

$\mathbf{T_{14}}$ The commutation relations of the field operators with the generators are given by:

$$[\hat{U}^\mu \chi(x), \hat{F}(\delta \chi)]_\pm = i\hat{U}^\mu \delta \chi.$$

**Proof:** Using $T_{13}$

$\mathbf{T_{15}}$ (“Equal-time” Commutation Relations) The covariant generalization of the equal-time commutation relations of the field operators are given by:

$$[\hat{U}^\mu \chi(x), \chi(x')\hat{U}^\mu]_\pm = i\hat{U}^\mu \delta_\mu(x - x').$$
where $\delta_s(x - x')$ is defined by $f(x) = \int_s ds' \delta_s(x - x')f(x')$. The commutators ($-$) are used for Bose field operators and anticommutators ($+$) for Fermi field operators.

**Proof:** From $\text{T}_{14}$ and using the expression of $\hat{F}(\delta\chi)$ (or its symmetric version).

$\text{T}_{16}$ (Bose and Fermi “Equal-time” Commutation Relations) In terms of Bose and Fermi field operators, we have:

\[
\left[\mathcal{U}_A^\mu \hat{\phi}(x), \hat{\phi}(x')\mathcal{U}_A^\mu\right]_-= i\mathcal{U}_A^\mu \delta_s(x - x'),
\]

\[
\left[\mathcal{U}_B^\nu \hat{\psi}(x), \hat{\psi}(x')\mathcal{U}_B^\nu\right]_+= i\mathcal{U}_B^\nu \delta_s(x - x'),
\]

\[
\left[\mathcal{U}_B^\nu \hat{\psi}(x), \hat{\phi}(x')\mathcal{U}_A^\mu\right]_{\pm} = 0.
\]

**Proof:** From $\text{T}_{15}$ and $\text{D}_6$.

**Remark 1** $\text{T}_{14}$ can equally be proved using the symmetric expression $\hat{F}^{sym}$ given in $\text{T}_3$.

**Remark 2** The commutation relations are a covariant generalization of the commutation relations for equal-time. To see this, note that for $n_\mu = (1, 0, 0, 0)$ we have $ds = d^3x$ and $\delta_s(x - x')$ become $\delta^3(x - x')$. **Remark 3** Since the above fields are characterized by Hermitian operators their commutators does not vanish. However, it is clear that in terms of the non-hermitian operators like $\hat{\chi}^+$ and $\hat{\chi}^-$, they vanish.

### 5. CPT THEOREM

In this section we give another fundamental theorem that must be satisfied by any relativistic quantum theory of fields. In order to deduce it, we must obtain first some preliminary results,

$\text{T}_{17}$ A matrix representation of $\hat{P}$ is given by,

\[
D^{-1}(I_s) = \nu_s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for} \quad A \neq B
\]

\[
D^{-1}(I_s) = \nu_s \quad \text{for} \quad A = B.
\]

**Proof:** From $\text{A}_{17}$.

$\text{T}_{18}$ The matrix representation of $\hat{P}$ acting upon a field operator $\hat{\chi} \mapsto \hat{\chi}_{AB_{12}}^{+}$ to $\hat{\chi}_{AB_1}^{+} \neq \hat{\chi}_{AB_2}^{+}$ is given by:

\[
D^{-1}(I_s) = \begin{pmatrix} D^{-1}_1(I_s) & 0 \\ 0 & D^{-1}_2(I_s) \end{pmatrix}.
\]

**Proof:** From $\text{T}_{16}$ and $\text{A}_{17}$. 

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A field operator $\hat{\chi}$ associated to an f-system transforms under spatial inversion as a reducible representation, i.e.,

$$\hat{P}\hat{\chi}(x)\hat{P}^{-1} = D^{-1}(I_s)\hat{\chi}(I_s x)$$

with

$$D(I_s) = \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & D_j \end{pmatrix}$$

where $D_j(I_s) = d$ is a matrix transformation for the $j$-th basic field.

**Proof:** From $A_{29}$, $A_{47}$, and $T_{18}$.

$T_{20}$ A field operator $\hat{\chi}$ associated to an f-system transforms under charge conjugation as a reducible representation, i.e.,

$$\hat{C}\hat{\chi}(x)\hat{C}^{-1} = D^{-1}(I_c)\hat{\chi}(I_c x)$$

with

$$D(I_c) = \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & D_j \end{pmatrix}$$

where $D_j(I_c) = d$ is a matrix transformation of the $j$-th basic field.

**Proof:** From $A_{29}$ and $A_{54}$.

$T_{21}$ (P Transformation) The matrices $\mathcal{U}^\mu$ transform under space inversion as:

$$D(I_s)\mathcal{U}^\mu D^{-1}(I_s) = -(-1)^{\delta_{\omega\nu}}\mathcal{U}^\mu.$$

**Proof:** From $A_{18}$, $T_{19}$, and $T_{18}$.

$T_{22}$ (PT Transformation) The $\mathcal{U}^\mu$ matrices transform under the combined $PT$ transformation as:

$$D(I_s)D(I_t)\mathcal{U}^\mu D^{-1}(I_t)D^{-1}(I_s) = \mp\mathcal{U}^\mu$$

where $(-)$ and $(+)$ corresponds to the antisymmetric and symmetric matrices respectively.

**Proof:** From $T_{10}$ and $T_{21}$.
quantum field theory. Note that thanks to our abstract formulation, the proof follows without to state the transformation properties for each particular representation of the field operators.

6. EXAMPLE: SPIN 1 AND SPIN $\frac{1}{2}$ FIELDS IN INTERACTION

In this section we present an example, that follows from our general formalism in the case of a system of interacting fields of spin 1 and spin $\frac{1}{2}$. In this particular framework, the Lagrange operator and the
expression for the generators will be obtained. First, a useful definition:

**D** (Fundamental and Non-fundamental Field Operators) Let the timelike vector \( n_\mu = (1, 0, 0, 0) \) be such that \( ds_\mu \equiv dx^3 \). The field operators \( \hat{\pi}^0_l \) and \( \hat{\chi}^l \) that appear in the expressions of the generators will be called *fundamental field operators* and they are the independent variables which obey the equations of motion. The remainder ones will be called *non-fundamental field operators* and they are the dependent variables which obey the constraint equations.

Now, we consider the general field:

\[
\hat{x} = (\hat{x}(\frac{1}{2}, \frac{1}{2}), \hat{x}(0,1) \oplus (1,0), \hat{x}(1,\frac{1}{2}) \oplus (\frac{1}{2},0), \hat{x}(\frac{1}{2},0)) \equiv (\hat{A}_\nu, \hat{F}_{\mu\nu}, \hat{\psi}_1, \hat{\psi}_2) \overset{d}{=} (\hat{\phi}, \hat{\psi})
\]

where \( \hat{\phi} \) and \( \hat{\psi} \) have 10 and 8 A-components respectively with,

\[
\dot{\hat{\phi}} \overset{d}{=} (\dot{\hat{A}}_\nu, \dot{\hat{F}}_{\mu\nu}) \quad \text{and} \quad \dot{\hat{\psi}} \overset{d}{=} (\dot{\hat{\psi}}_1, \dot{\hat{\psi}}_2),
\]

describing basic fields of 1 and \( \frac{1}{2} \) spin. With the matrices,

\[
U^\mu_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad U^\mu_S = i\alpha^\mu 1 = \begin{pmatrix} i\alpha^\mu & 0 \\ 0 & i\alpha^\mu \end{pmatrix},
\]

the kinematical Lagrangian can be written as (using \( T^S \)):

\[
\hat{L}_{Kin} = \hat{L}^{1}_{Kin} + \hat{L}^{\frac{1}{2}}_{Kin},
\]

with:

\[
\hat{L}^{1}_{Kin} = -\hat{F}^{\mu\nu} \partial_\mu \hat{A}_\nu + \hat{A}_\nu \partial_\mu \hat{F}^{\mu\nu} \quad \text{and} \quad \hat{L}^{\frac{1}{2}}_{Kin} = i\hat{\psi} \alpha^\mu 1 \partial_\mu \hat{\psi}.
\]

The dynamical Lagrangian assume the following form (using \( A^{62} \)):

\[
\hat{L}_{Dyn} = \hat{L}^{1}_{Dyn} + \hat{L}^{\frac{1}{2}}_{Dyn} + \hat{L}^{int}_{Dyn},
\]

where the interaction terms of the fields with themselves (i.e. “autointeraction”) are given by:

\[
\hat{L}^{1}_{Dyn} = \hat{\phi} U^\mu_A U^\nu_A \hat{\phi} = -a \hat{A}_\mu \hat{A}^\mu - b \hat{F}_{\mu\nu} \hat{F}^{\mu\nu},
\]

\[
\hat{L}^{\frac{1}{2}}_{Dyn} = \hat{\psi} U^\mu_S U^\nu_S \hat{\psi} = -c \hat{\psi} \alpha^\mu 1 \alpha^\mu \hat{\psi}.
\]
The interaction Lagrangian (between different fields) must be conjectured, but taking into account the transformation properties of the Lagrange operator, we propose:

\[ \hat{L}_{\text{int}}^{\text{Dyn}} = \hat{A}_\mu \mathcal{U}_A^\mu \psi \mathcal{U}_A^\alpha \hat{\psi} \mathcal{U}_A^\mu \hat{\psi} \] with \( \mathcal{U}_A^\mu = \xi \).

Evaluating the generators we have:

\[ \hat{F}(\delta A_\nu, \delta F^{\mu\nu}, \delta \psi) = -\int ds_0 (\hat{F}_0^{0\nu} \delta \hat{A}_\nu - i\alpha^0 \hat{\psi} \delta \hat{\psi}). \]

It follows that the fundamental fields are (using \( \mathbf{D}_8 \)) \( (\hat{A}_k, \hat{F}_0^k, \hat{\psi}) \) and the non-fundamental fields are \( (\hat{A}_0, \hat{F}^{kl}) \). Obviously, we do not pretend to obtain electrodynamics, since we have only demanded gauge invariance of the first kind.

### 7. CONCLUSIONS

We have presented a physical axiomatization of the RQFT which has a number of important advantages:

1. The requirement of kinematical invariance under time reversal imposes a restriction upon the field operators: the spin-statistics relation. Pauli’s proof [13] of spin-statistics proceeds differently for integer and half-integer spin. The proofs given by Lüders and Zumino [22] and, similarly, by Burgoyne [23] for interacting fields, follow an indirect argument: the wrong relation between spin and statistics cannot be postulated in a relativistic quantum field theory without to come in contradiction. On the other hand, Weinberg [6] shows that “causality” is satisfied only with the correct statistics and with crossing symmetry. In all these cases the theorem is proved invoking the “causality” requirement. Here we follow a direct argument for both integer and half-integer spin and, in contrast with the other formulations, without the explicit use of the “causality” requirement.

2. The commutation relations of the field operators are not postulated as usual: they are derived from the stationary action principle, in particular from the generators associated with a given spacelike surface. However, the condition of physical independence of different points of a spacelike surface, is implicitly expressed in the structure of the generating operators because these are constructed from field operators attached on a spacelike surface.

3. The CPT theorem is proved using the mathematical advantages of this formulation: the different transformation properties of the \( \mathcal{U}^\mu \) matrices and the general form postulated of the dynamical Lagrangian. As it is well known, the proof given by Lüders [14] of CPT theorem uses the spin-statistics relation. In our case this assumption is stated in the different transformation properties.
We must point out that the quantum theory of fields is a very general framework that, by adding special postulates and subsidiary assumptions, reduces to particular theories as the known case of electrodynamics. In the last section we provide an example of this reduction for an interacting f-system.

The proof of the spin-statistics relation is presented here into the relativistic framework. However, a recent paper by Peshkin [24] has tried to prove the spin-statistics theorem from rotational properties of the non-relativistic wave function for the case of spinless particles opening thus a vivid debate on the subject.

However, that proof is based on the assumption that exchange of identical particles can be represented as a physical transportation. This is a misunderstanding, as has been pointed out in several references [25, 26], that the use of a formal semantic would have helped to avoid. Indeed, as Sudharshan [26] has stated that this kind of problems “would not arise if we were to exercise perfect semantic precision”.

Shaji and Sudharsan [27] have developed another proof, much more clear and physically well motivated. It hinges, however, on the possibility of representing non-relativistic fields as hermitean operators. In our previous axiomatization, we concluded that Galilean invariance forbids such a representation. Indeed, no Galilean field operator of non-null mass can be hermitian (see [25, 10]) because Galilean group admits only projective representations. The proof may be valid for a “gas in a box” and similar systems, which are models of macroscopic bodies, since Galilean invariance is broken for those systems. However, this issue merits a separate paper for its analysis.

Last but not least, in this approach fields are considered as things with properties which are represented by operators that satisfy certain symmetry transformations. This means that, in our conception, fields are more fundamental than both: particles and symmetries, since symmetries are symmetries of properties of things and without things, there are no symmetries. Indeed, fields are unobservable but they should not be regarded as auxiliary devices with no physical meaning since the concept of quantum fields provides a mechanism of interaction explicitly expressed by the form of the Lagrangian operator. On the contrary, this mechanism is lost if fields and their mathematical referents are regarded only as auxiliary computational devices. In sum, we think that the concept of field must be regarded as a deep basic hypothesis in term of which we try to explain the behavior of matter.

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