HALF-INTEGRABLE MODULES OVER ALGEBRAS OF TWISTED CHIRAL DIFFERENTIAL OPERATORS

D. CHEBOTAROV

Abstract. A module $M$ over a vertex algebra $V$ is half-integrable if $a_n$ act locally nilpotently on $M$ for all $a \in V$, $m \in M$, $n > 0$ We study half-integrable modules over sheaves of twisted chiral differential operators (TCDO) on a smooth variety $X$. We prove an equivalence between certain categories of half-integrable modules over TCDO and categories of (twisted) D-modules on $X$.

1. Introduction

Let $V$ be a graded vertex algebra and $M$ a $V$-module, i.e. we are given a map

$$V_\Delta \ni a \mapsto a^M(z) = \sum a_n z^{-n-\Delta} \in \text{End}_M[[z, z^{-1}]]$$

satisfying the usual axioms, cf. section 2.1 below.

We call $M$ half-integrable if for any $a \in V$ and $n > 0$ the action of $a_n$ on $M$ is locally nilpotent; that is, $M$ is integrable with respect to the positive part of the graded Borcherds Lie algebra of $V$.

Let $X$ be a smooth complex variety and $V$ a graded sheaf of vertex algebras on $X$ which is a vertex envelope of vertex algebroid $V_0 \oplus V_1$, see [GMS1], such that $V_0 = O_X$. The previous definition immediately carries over to this situation, and we will apply it to those $V$ that satisfy some extra conditions. In order to formulate these conditions, recall that $L := V_1/V_0(-1)\partial V_0$ is an $O_X$-Lie algebroid [GMS1]; in particular, there is an anchor, an $O_X$-Lie algebroid morphism: $L \to T_X$. We require:

- $L$ is a locally free $O_X$-module of finite rank;
- $L$ fits into an exact sequence

$$0 \to \mathfrak{h} \to L \to T_X \to 0$$

where $\mathfrak{h}$ is an abelian $O_X$-Lie algebra.

Such sheaves of vertex algebras $V$ form a natural class which includes sheaves of chiral twisted differential operators $\mathcal{D}_{X}^{ch,tw}$ defined in [AChM] and certain deformations of those.

Let $Z(V) \subset V$ be the subsheaf of $V$ with stalks $Z_{X,x} = Z(V_{X,x})$, $x \in X$, where $Z(V)$ denotes the center of $V$; we will call it the center of $V$.

Define $\mathcal{M}^{int_{+}}(V)$ to be the category of half-integrable $V$-modules that satisfy the following regularity condition: $z_n m = 0$ for $z \in Z(V)$, $n > 0$, $m \in \mathcal{M}$.

Then the following result holds. (Theorem 4.1).

**Theorem 1.** Let $V$ be as above and $\mathcal{M} \in \mathcal{M}^{int_{+}}(V)$. Then

1) The subsheaf

$$\text{Sing}\mathcal{M} = \{m \in \mathcal{M} : v_n m = 0 \text{~for~all~} v \in V, n > 0\}$$

generates $\mathcal{M}$;
2) \( \mathcal{M} \) possesses a filtration \( \mathcal{M} = \bigcup_{i \geq 0} \mathcal{M}_i \) such that \( \mathcal{V}_{1(n)} \mathcal{M}_j \subset \mathcal{M}_{i+j-n-1} \) and \( \mathcal{M}_0 = \text{Sing} \mathcal{M} \).

The result, as stated, is true in the analytic topology; when working algebraically one has to add one more requirement, cf. section 4.1.7.

Let \( X \) be connected and \( V \) as above. Suppose that \( h = \mathcal{O}_X \otimes_C \mathfrak{h}^V \) where \( \mathfrak{h}^V \) is a constant sheaf such that \([\mathcal{L}, \mathfrak{h}^V] = 0\). For \( \theta \in (\mathfrak{h}^V)^* \) consider the ideal \( I_\theta \) in \( U_{\mathcal{O}_X}(\mathcal{L}) \) generated by \( h - \theta(h), h \in \mathfrak{h}^V \).

Then \( U_{\mathcal{O}_X}(\mathcal{L})/I_\theta \) is an algebra of twisted differential operators (tdo) on \( X \). Let us denote it by \( \mathcal{D}_X^\theta \).

Let \( \mathcal{M}_\theta^{int+}(V) \) denote the full subcategory of modules \( \mathcal{M} \in \mathcal{M}_\theta^{int+}(V) \) such that for each \( h \in \mathfrak{h}^V , m \in \mathcal{M} \), \( h_0 m = \theta(h)m \). Being the Zhu algebra of \( V, U_{\mathcal{O}_X}(\mathcal{L}) \) acts on \( \mathcal{M}_0 = \text{Sing} \mathcal{M} \), and this action factors through \( U_{\mathcal{O}_X}(\mathcal{L})/I_\theta = \mathcal{D}_X^\theta \). This way we get a functor

\[
(1.1) \quad \text{Sing} : \mathcal{M}_\theta^{int+}(V) \to \mathcal{M}(\mathcal{D}_X^\theta)
\]

It is not an equivalence; however, it becomes one upon restriction to certain subcategories.

It is not hard to verify that the center \( Z(V) \) is a constant sheaf with values in an algebra of differential polynomials, generated by the subspace \( \mathfrak{z} := (\mathfrak{h}^V)^\perp = \{ h \in \mathfrak{h}^V : \langle h, \mathfrak{h}^V \rangle = 0 \} \), where \( \langle , \rangle : \mathfrak{h} \times \mathfrak{h} \to \mathcal{O}_X \) is induced by (1).

Fix a series \( \chi(z) \in \mathfrak{z}^*([z]), \chi(z) = \sum \chi_n z^{-n-1} \). We say that \( Z \) acts on \( \mathcal{M} \) via \( \chi(z) \) if for all \( h \in \mathfrak{z}, m \in \mathcal{M} \)

\[
h_n m = \chi_n(h)m, \quad n \in \mathbb{Z}
\]

Let \( \mathcal{M}_{\theta, \chi(z)}^{int+}(V) \) denote the full subcategory of \( \mathcal{M}_\theta^{int+}(V) \) consisting of modules with action of \( Z \) given by \( \chi(z) \). For this category to be nonzero, the obvious compatibility conditions \( \chi_0 = \theta|_\mathfrak{z} \) and \( \chi(z) \in \mathfrak{z}^*([z])z^{-1} \) have to be satisfied.

The following result holds. (Theorem 4.5 below).

**Theorem 2.** Let \( \chi(z) \in \mathfrak{z}^*([z]) \) be such that \( \chi_0 = \theta|_\mathfrak{z} \). Then the restriction of (1.1) to \( \mathcal{M}_{\theta, \chi(z)}^{int+}(V) \) is an equivalence.

As a particular case, this theorem can be applied when \( V \) is the sheaf of twisted chiral differential operators \( D_X^{ch, tw} \) introduced in [AChM]. In this case \( \mathfrak{h}^V \) and \( \mathfrak{z} \) are both equal to the trivial local system with fiber \( H^1(X, \Omega^1 \to \Omega^{2, ct})^* \) and the center \( Z(V) \) is the algebra of differential polynomials on the affine space \( H^1(X, \Omega^1 \to \Omega^{2, ct}) \). Theorem 2 becomes an equivalence between the category of half-integrable \( D_X^{ch, tw} \)-modules with central character \( \chi(z) \) and the category of modules over the tdo \( D_X^{cho} \).

Furthermore, when \( \mathfrak{h} = 0 \) this becomes an equivalence between the category of half-integrable modules over a CDO \( D_X^{ch} \) and the category of usual \( D_X \)-modules.

These results should be contrasted with the classic Kashiwara Lemma from the D-module theory ([Bo], [HTT]). Recall that if \( Y \subset X \) is a submanifold and \( J \) is the defining ideal, \( M \) is a \( D_X \)-module supported on \( Y \), and \( M_J \subset M \) the subsheaf annihilated by \( J \), then \( M_J \) is naturally a \( D_Y \)-module. Kashiwara’s Lemma asserts that \( M \to M_J \) sets up an equivalence of the category of \( D_X \)-modules supported on \( Y \) and the category of \( D_Y \)-modules.

A TCDO can be intuited about as an algebra of differential operators on the loop space \( LM \), see papers of Kapranov and Vasserot ([KV1, KV2]) for a rigorous treatment. The half-integrability condition can then be interpreted as the requirement that the module is supported on regular loops. Sing \( M \) then becomes an analogue of \( M_J \). Therefore, Theorem 2 appears to be a loop space (or vertex algebra) version of Kashiwara’s Lemma.
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2. Preliminaries.

We will recall the basic notions of vertex algebra following the exposition of [AChM]. All vector spaces will be over \( \mathbb{C} \). All vertex algebras considered in this article will be even \( \mathbb{Z}_{\geq 0} \)-graded vertex algebras.

2.1. Definitions. Let \( V \) be a vector space.

A field on \( V \) is a formal series
\[
a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1} \in (\text{End}V)[[z, z^{-1}]]
\]
such that for any \( v \in V \) one has \( a(n)v = 0 \) for sufficiently large \( n \).

Let \( \text{Fields}(V) \) denote the space of all fields on \( V \).

A vertex algebra is a vector space \( V \) with the following data:

- a linear map \( Y : V \to \text{Fields}(V) \), \( V \ni a \mapsto a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1} \)
- a vector \( |0\rangle \in V \), called vacuum vector
- a linear operator \( \partial : V \to V \), called translation operator

that satisfy the following axioms:

1. (Translation Covariance) \((\partial a)(z) = \partial_z a(z)\)
2. (Vacuum) \(|0\rangle(z) = \text{id};\ a(z)|0\rangle \in V[z] \text{ and } a(-1)|0\rangle = a\)
3. (Borcherds identity) \[
\sum_{j \geq 0} \binom{m}{j} (a_{n+j}b_{m+k-j}) = \sum_{j \geq 0} (-1)^j \binom{n}{j} \{a_{m+n-j}b_{k+j} - (-1)^n b_{n+k-j}a_{m+j} \}
\]

A vertex algebra \( V \) is graded if \( V = \bigoplus_{n \geq 0} V_n \) and for \( a \in V_i, b \in V_j \) we have
\[
a_{(k)}b \in V_{i+j-k-1}
\]
for all \( k \in \mathbb{Z} \). (We put \( V_i = 0 \) for \( i < 0 \).)

All vertex algebras in this article are graded vertex algebras.

We say that a vector \( v \in V_m \) has conformal weight \( m \) and write \( \Delta_v = m \).

If \( v \in V_m \) we denote \( v_k = v(k-m+1) \), this is the so-called conformal weight notation for operators. One has
\[
v_k V_m \subset V_{m-k}.
\]

A morphism of vertex algebras is a map \( f : V \to W \) that preserves vacuum and satisfies \( f(v_{(n)}v') = f(v)_{(n)}f(v') \).
A module over a vertex algebra $V$ is a vector space $M$ together with a map

$$Y^M : V \to \text{Fields}(M), \ a \to Y^M(a, z) = \sum_{n \in \mathbb{Z}} a^{M}_{(n)} z^{-n-1},$$

that satisfy the following axioms:

1. $|0\rangle^M_M = \text{id}_M$
2. (Borcherds identity)

$$\sum_{j \geq 0} \binom{n}{j} \{ a^M_{(n-j)} b^M_{(k+j)} - (-1)^n b^M_{(n-k-j)} a^M_{(m+j)} \}$$

A module $M$ over a graded vertex algebra $V$ is called graded if $M = \oplus_{n \geq 0} M_n$ with $v_k M_l \subset M_{l-k}$ (assuming $M_n = 0$ for negative $n$).

A morphism of modules over a vertex algebra $V$ is a map $f : M \to N$ that satisfies $f(v^M_{(n)} m) = v^N_{(n)} f(m)$ for $v \in V$, $m \in M$. $f$ is homogeneous if $f(M_k) \subset N_k$ for all $k$.

2.1.1. The Borcherds Lie algebra. To any vertex algebra $V$ one can associate a Lie algebra that acts on any $V$-module. It is the Borcherds Lie algebra of $V$ defined in the following way

$$\text{Lie}(V) = V \otimes \mathbb{C}[t, t^{-1}] / \langle (\partial a + (n + H)a) \otimes t^n \rangle$$

where $Ha = ka$ for $a \in V_k$. This is a Lie algebra with Lie bracket given by

$$[a \otimes t^n, b \otimes t^l] = \sum_{j \geq 0} \binom{n + \Delta a - 1}{j} (a_{(j)} b) \otimes t^{n+l}$$

for a homogeneous $a \in V_\Delta$ and $b \in V$, and extended linearly. Lie $(V)$ acts on any $V$-module $M$ by letting $a \otimes t^n$ act as $a_n$.

Lie $(V)$ has a natural grading, Lie $(V) = \oplus_{n \in \mathbb{Z}} \text{Lie}(V)_n$ with $\text{Lie}(V)_n$ equal to the image of $V \otimes t^n$.

2.2. Examples.

2.2.1. Commutative vertex algebras. A vertex algebra is said to be commutative if $a_{(n)} b = 0$ for $a, b$ in $V$ and $n \geq 0$. The structure of a commutative vertex algebras is equivalent to one of commutative associative algebra with a derivation.

If $W$ is a vector space we denote by $H_W$ the algebra of differential polynomials on $W$. As an associative algebra it is a polynomial algebra in variables $x_i, \partial x_i, \partial^{(2)} x_i, \ldots$ where $\{ x_i \}$ is a basis of $W^*$. A commutative vertex algebra structure on $H_W$ is uniquely determined by attaching the field $x(z) = e^{z \partial} x_i$ to $x \in W^*$.

$H_W$ is equipped with grading such that

$$\text{(H}_W)_0 = \mathbb{C}, \ (H_W)_1 = W^*.$$
2.2.2. Beta-gamma system. Define the Heisenberg Lie algebra to be the algebra with generators \( a^i_n, b^i_n, 1 \leq i \leq N \) and \( K \) that satisfy \([a^i_n, b^j_m] = \delta_{m,-n}\delta_{i,j}K, \) \([a^i_n, a^j_m] = 0, [b^i_n, b^j_m] = 0.\)

Its Fock representation \( M \) is defined to be the module induced from the one-dimensional representation \( C_1 \) of its subalgebra spanned by \( a^i_n, n \geq 0, b^i_m, m > 0 \) and \( K \) with \( K \) acting as identity and all the other generators acting as zero.

The beta-gamma system has \( M \) as an underlying vector space, the vertex algebra structure being determined by assigning the fields

\[
a^i(z) = \sum a^i_n z^{-n-1}, \quad b^i(z) = \sum b^i_n z^{-n}
\]
to \( a^i_{-1} \) and \( b^i_1 \) resp., where \( 1 \in \mathbb{C}. \)

This vertex algebra is given a grading so that the degree of operators \( a^i_n \) and \( b^i_m \) is \( n. \) In particular,

\[
M_0 = \mathbb{C}[b^1_0, ..., b^N_0], \quad M_1 = \bigoplus_{j=1}^N (b^1_0 M_0 \oplus a^1_{-1} M_0).
\]

2.3. Vertex algebroids.

2.3.1. Let \( V \) be a vertex algebra.

Define a 1-truncated vertex algebra to be a sextuple \( (V_0 \oplus V_1, [0], \partial, \{(-1), (0), (1)\}) \) where the operations \( \{(-1), (0), (1)\} \) satisfy all the axioms of a vertex algebra that make sense upon restricting to the subspace \( V_0 + V_1. \) (The precise definition can be found in [GMS1]). The category of 1-truncated vertex algebras will be denoted \( \mathcal{V}_{\text{tr}}. \)

The definition of vertex algebroid is a reformulation of that of a sheaf of 1-truncated vertex algebras.

A vertex \( O_X \)-algebroid is a sheaf \( \mathcal{A} \) of \( \mathbb{C} \)-vector spaces equipped with \( \mathbb{C} \)-linear maps \( \pi : \mathcal{A} \to T_X \) and \( \partial : O_X \to \mathcal{A} \) satisfying \( \pi \circ \partial = 0 \) and with operations \( (1) : O_X \times \mathcal{A} \to \mathcal{A}, \) \( (0) : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \) and \( (1) : \mathcal{A} \times \mathcal{A} \to O_X \) satisfying axioms:

\begin{align*}
(2.6) \quad f(-1)(g(-1)v) - (fg)(-1)v &= \pi(v)(f)(-1)\partial(g) + \pi(v)(g)(-1)\partial(f) \\
(2.7) \quad x(0)(f(-1)y) &= \pi(x)(f)(-1)y + f(-1)(x(0)y) \\
(2.8) \quad x(0)y + y(0)x &= \partial(x(1)y) \\
(2.9) \quad \pi(f(-1)v) &= f\pi(v) \\
(2.10) \quad (f(-1)x)(1)y &= f(x(-1)y) - \pi(x)(\pi(y)(f)) \\
(2.11) \quad \pi(v)(x(1)y) &= (v(0)x)(1)y + x(1)(v(0)y) \\
(2.12) \quad \partial(fg) &= f(-1)\partial(g) + g(-1)\partial(f) \\
(2.13) \quad v(0)\partial(f) &= \partial(\pi(v)(f)) \\
(2.14) \quad v(1)\partial(f) &= \pi(v)(f)
\end{align*}

for \( v, x, y \in \mathcal{A}, f, g \in O_X. \) The map \( \pi \) is called the anchor of \( \mathcal{A}. \)

If \( \mathcal{V} = \bigoplus_{n \geq 0} \mathcal{V}_n \) is a (graded) sheaf of vertex algebras with \( \mathcal{V}_0 = O_X, \) then \( \mathcal{A} = \mathcal{V}_1 \) is a vertex algebroid with \( \partial \) equal to the translation operator and \( \pi \) sending \( x \in \mathcal{V}_1 \) to the derivation \( f \mapsto x(0)f. \)

2.3.2. Associated Lie algebroid. Recall that a Lie algebroid is a sheaf of \( O_X \)-modules \( L \) equipped with a Lie algebra bracket \( [\cdot, \cdot] \) and a morphism \( \pi : \mathcal{A} \to T_X \) of Lie algebra and \( O_X \)-modules called anchor that satisfies \([x, ay] = a[x, y] + \pi(x)(a)y, x, y \in \mathcal{A}, a \in O_X. \)
If $\mathcal{A}$ is a vertex algebroid, then the operation $(0)$ descends to that on $\mathcal{L}_\mathcal{A} = \mathcal{A}/\mathcal{O}_X(-1)\partial\mathcal{O}_X$ and makes it into a Lie algebroid, with the anchor induced by that of $\mathcal{A}$. $\mathcal{L}_\mathcal{A}$ is called the associated Lie algebroid of $\mathcal{A}$.

2.3.3. A vertex (resp., Lie) algebroid is transitive, if its anchor map $\pi$ is surjective.

Being a derivation, see (2.12), $\partial : \mathcal{O}_X \to \mathcal{A}$ lifts to $\Omega^1_X \to \mathcal{A}$. It follows from (2.14) that if $\mathcal{A}$ is transitive, then $\Omega^1_X \simeq \mathcal{O}_X(-1)\partial\mathcal{O}_X$ and $\mathcal{A}$ fits into an exact sequence
\[ 0 \to \Omega^1_X \to \mathcal{A} \to \mathcal{L} \to 0, \]
$\mathcal{L} = \mathcal{L}_\mathcal{A}$ being an extension
\[ 0 \to \mathfrak{h}(\mathcal{L}) \to \mathcal{L} \to \mathcal{T}_X \to 0 \]
where $\mathfrak{h}(\mathcal{L}) := \ker(\mathcal{L} \xrightarrow{\pi} \mathcal{T}_X)$ is an $\mathcal{O}_X$-Lie algebra.

Note that the pairing (1) on $\mathcal{A}$ induces a symmetric $\mathcal{L}_\mathcal{A}$-invariant $\mathcal{O}_X$-bilinear pairing on $\mathfrak{h}(\mathcal{L}_\mathcal{A})$ which will be denoted by $\langle \cdot, \cdot \rangle$.

We regard the pair $(\mathcal{L}_\mathcal{A}, \langle \cdot, \cdot \rangle)$ as ”classical data” underlying the vertex algebroid $\mathcal{A}$.

2.3.4. Truncation and vertex enveloping algebra functors. There is an obvious truncation functor $t : \text{Vert} \to \text{Vert}_{\leq 1}$ that assigns to every vertex algebra a 1-truncated vertex algebra. This functor admits a left adjoint $u : \text{Vert}_{\leq 1} \to \text{Vert}$ called a vertex enveloping algebra functor.

These functors have evident sheaf versions. In particular, one has the functor
\[ U : \text{VertAlg} \to \text{ShVert} \]
from the category of vertex algebroids to the category of sheaves of vertex algebras.

2.4. Chiral differential operators. A sheaf of vertex algebras $\mathcal{D}$ over $X$ is called a sheaf of chiral differential operators, CDO for short, if $\mathcal{D}$ is the vertex envelope of a vertex algebroid $\mathcal{A}$ that fits into an exact sequence of $\mathbb{C}$-vector spaces
\[ 0 \to \Omega^1_X \to \mathcal{A} \to \mathcal{T}_X \to 0. \]

A sheaf of chiral differential operators does not exist over any $X$, but it does exist locally on any smooth $X$.

To be more precise, a smooth affine variety $U = \text{Spec} A$ will be called suitable for chiralization if $\text{Der}(A)$ is a free $A$-module admitting an abelian frame $\{\tau_1, ..., \tau_n\}$. In this case there is a CDO over $U$, which is uniquely determined by the condition that $(\tau_i)_{(1)}(\tau_j) = (\tau_i)_{(0)}(\tau_j) = 0$. Denote this CDO by $\mathcal{D}^{ch}_{U,\tau}$.

**Theorem 2.1.** Let $U = \text{Spec} A$ be suitable for chiralization with a fixed abelian frame $\{\tau_i\} \subset \text{Der}A$.

(i) For each closed 3-form $\alpha \in \Omega^3_{A}^{\text{cl}}$ there is a CDO over $U$ that is uniquely determined by the conditions
\[ (\tau_i)_{(1)}\tau_j = 0, \ (\tau_i)_{(0)}\tau_j = \iota_{\tau_i,\tau_j}\alpha. \]
Denote this CDO by $\mathcal{D}^{ch}_{U,\tau}(\alpha)$.

(ii) Each CDO over $U$ is isomorphic to $\mathcal{D}^{ch}_{U,\tau}(\alpha)$ for some $\alpha$. 
(iii) \( \mathcal{D}_{U,\tau}(\alpha_1) \) and \( \mathcal{D}_{U,\tau}(\alpha_2) \) are isomorphic if and only if there is \( \beta \in \Omega^2_X \) such that \( d\beta = \alpha_1 - \alpha_2 \).

In this case the isomorphism is determined by the assignment \( \tau_i \mapsto \tau_i + \tau_\beta \).

If \( A = \mathbb{C}[x_1, \ldots, x_n] \), one can choose \( \partial/\partial x_j \), \( j = 1, \ldots, n \), for an abelian frame and check that the beta-gamma system \( M \) of sect. 2.2.2 is a unique up to isomorphism CDO over \( \mathbb{C}^n \). A passage from \( M \) to Theorem 2.1 is accomplished by the identifications \( b'_01 = x_j \), \( a'_01 = \partial/\partial x_j \).

3. Universal twisted CDO

In this section we recall the definition of the sheaf \( \mathcal{D}^{ch, tw}_X \) of twisted chiral differential operators (TCDO) \([\text{AChM}]\) corresponding to a given CDO \( \mathcal{D}^{ch} \) on a smooth projective variety \( X \).

3.1. Twisted differential operators. A sheaf of twisted differential operators (TDO) is a sheaf of filtered \( \mathcal{O}_X \)-algebras such that the corresponding graded sheaf is (the push-forward of) \( \mathcal{O}_{T^*X} \), see \([\text{BB2}]\). The set of isomorphism classes of such sheaves is in 1-1 correspondence with \( H^1(X, \Omega^{1,2}_{X}) \).

Denote by \( \mathcal{D}^\lambda_X \) a TDO that corresponds to \( \lambda \in H^1(X, \Omega^{1,2}_{X}) \). If \( \dim H^1(X, \Omega^{1,2}_{X}) < \infty \), then it is easy to construct a universal TDO, that is to say, a family of sheaves with base \( H^1(X, \Omega^{1,2}_{X}) \) so that the sheaf that corresponds to a point \( \lambda \in H^1(X, \Omega^{1,2}_{X}) \) is isomorphic to \( \mathcal{D}^\lambda_X \). The construction is as follows.

Let \( X \) be a smooth projective variety. Then \( \dim H^1(X, \Omega^{1,2}_{X}) < \infty \) and there exists an affine cover \( \mathcal{U} \) so that \( H^1(\mathcal{U}, \Omega^{1,2}_{X}) = H^1(X, \Omega^{1,2}_{X}) \).

Let \( \Lambda = \hat{H}^1(\mathcal{U}, \Omega^{1,2}_{X}) \). We fix a lifting \( \hat{H}^1(\mathcal{U}, \Omega^{1,2}_{X}) \rightarrow \hat{Z}^1(\mathcal{U}, \Omega^{1,2}_{X}) \) and identify the former with the subspace of the latter defined by this lifting. Thus, each \( \lambda \in \Lambda \) is a pair of cochains \( \lambda = ((\lambda^1_{ij}), (\lambda^2_{ij})) \) with \( \lambda^1_{ij} \in \Omega^1(U_i \cap U_j) \), \( \lambda^2_{ij} \in \Omega^{2, cl}(U_i) \), satisfying \( d_{DR} \lambda^1_{ij} = d_C \lambda^2_{ij} \) and \( d_C \lambda^1_{ij} = 0 \).

For \( \lambda = (\lambda^1_{ij}, \lambda^2_{ij}) \in \Lambda \) denote \( \mathcal{D}^\lambda \) the corresponding sheaf of twisted differential operators. One can consider \( \mathcal{D}^\lambda \) as an enveloping algebra of the Lie algebroid \( \mathcal{T}^\lambda = \mathcal{D}^\lambda_1 \) \([\text{BB2}]\). As an \( \mathcal{O}_X \)-module, \( \mathcal{T}^\lambda \) is an extension

\[
0 \longrightarrow \mathcal{O}_X 1 \longrightarrow \mathcal{T}^\lambda \longrightarrow \mathcal{T}_X \longrightarrow 0
\]

given by \((\lambda^1_{ij})\). The Lie algebra structure on \( \mathcal{T}^\lambda_\mathcal{U}_i \) is given by \([\xi, \eta]_{\mathcal{T}^\lambda_\mathcal{U}_i} = [\xi, \eta] + i \xi \eta \lambda^2_{ij} \) and \([1, \mathcal{T}^\lambda_\mathcal{U}_i] = 0 \).

Let \( \{\lambda^1_i\} \) and \( \{\lambda^2_i\} \) are dual bases of \( \Lambda^* = H^1(X, \Omega^{1,2}_{X}) \) and \( \Lambda \) respectively. Denote by \( k \) the dimension of \( \Lambda \).

Define \( \mathcal{T}^{tw} \) to be an abelian extension

\[
0 \longrightarrow \mathcal{O}_X \otimes \Lambda^* \longrightarrow \mathcal{T}^{tw}_X \overset{\pi}{\longrightarrow} \mathcal{T}_X \longrightarrow 0
\]

such that \([\Lambda^*, \mathcal{T}^{tw}] = 0 \) and there exist sections \( \nabla_i : \mathcal{T}^{tw}_\mathcal{U}_i \rightarrow \mathcal{T}^{tw}_\mathcal{U}_i \) of \( \pi \) satisfying

\[
\nabla_j(\xi) - \nabla_i(\xi) = \sum_r i \xi \lambda^{(1)}_{ij}(U_{ij}) \lambda^*_r
\]

(3.1)

\[
[\nabla_i(\xi), \nabla_i(\eta)] - \nabla_i([\xi, \eta]) = \sum_r i \xi \eta \lambda^{(2)}_{ij}(U_{ij}) \lambda^*_r
\]

(3.2)

It is clear that the pair \((\mathcal{T}^{tw}, \mathcal{O}_X \otimes \Lambda^* \rightarrow \mathcal{T}^{tw})\) is independent of the choices made.

We call the universal enveloping algebra \( \mathcal{D}^{tw} = U(\mathcal{T}^{tw}) \) the universal sheaf of twisted differential operators.
3.2. A universal twisted CDO. Let $ch_2(X) = 0$ It is proved in [GMS1] that $X$ carries at least one CDO. To each CDO $D_X^{ch}$ one can attach a universal twisted CDO, $D_X^{ch,tw}$, a sheaf of vertex algebras whose ”underlying” Lie algebroid is $D_X^{tw}$. Let us place ourselves in the situation of the previous section, where we had a fixed affine cover $U = \{ U_i \}$ of a projective algebraic manifold $X$, dual bases $\{ \lambda_i \} \in H^1(X, \Omega_X^{1,2>})$, $\{ \lambda_i^* \} \in H^1(X, \Omega_X^{1,2>})^*$, and a lifting $H^1(X, \Omega_X^{1,2>}) \rightarrow Z^1(\Omega, \Omega_X^{1,2>})$.

We can assume that $U_i$ are suitable for chiralization. Let us fix, for each $i$, an abelian basis $\tau_{(i)}^1, \tau_{(i)}^2, ...$ of $\Gamma(U_i, T_X)$. Then the CDO $D^{ch}$ is given by a collection of 3-forms $\alpha^{(i)} \in \Gamma(U_i, \Omega_X^{0,3})$ (cf. sect. 2.4, Theorem 2.1) and transition maps $g_{ij} : D^{ch}_{U_i}|_{U_i \cap U_j} \rightarrow D^{ch}_{U_j}|_{U_i \cap U_j}$. Let us as well fix splittings $T_{U_i} \rightarrow D^{ch}_{U_i}$ and view $g_{ij}$ as maps $g_{ij} : (T_{U_j} \oplus \Omega_{U_j})|_{U_i \cap U_j} \rightarrow (T_{U_i} \oplus \Omega_{U_i})|_{U_i \cap U_j}$,

The universal sheaf of twisted chiral differential operators $D^{ch,tw}_X$ (TCDO for short) corresponding to $D^{ch}_X$ is a vertex envelope of the $O_X$-vertex algebroid $A^{tw}$ determined by the following:

- the associated Lie algebroid of $A^{tw}$ is $T^{tw}_X$;
- there are embeddings $T^{tw}_{U_i} \rightarrow A_{U_i}$, such that

$$\tau^{(i)}_{(0)} \tau^{(i)}_{m} = \sum_{i} t^{(i)}_{ji} t^{(i)}_{jm} \alpha^{(i)} + \sum_{k} t^{(i)}_{ji} t^{(i)}_{km} \lambda^{(2)}_k(U_i) \lambda^*_k$$

$$\lambda^{(1)}_k x = 0, \ x \in A^{tw}$$

- the transition function from $U_j$ to $U_i$ is given by

$$(3.3) \quad g^{tw}_{ij}(\xi) = g_{ij}(\xi) - \sum_{k} t^{(i)}_{ji} \lambda^{(1)}_k(U_i \cap U_j) \lambda^*_k$$

See [AChM] for a detailed construction.

3.3. Locally trivial twisted CDO. Observe that there is an embedding

$$(3.4) \quad H^1(X, \Omega_X^{1,2}) \hookrightarrow H^1(X, \Omega_X^{1,2>})$$

The space $H^1(X, \Omega_X^{1,2})$ classifies locally trivial twisted differential operators, those that are locally isomorphic to $D_X$. Thus for each $\lambda \in H^1(X, \Omega_X^{1,2})$, there is a unique up to isomorphism TDO $\alpha^\lambda_X$ such that for each sufficiently small open $U \subset X$, $\alpha^\lambda_X|_U$ is isomorphic to $D_U$. Let us see what this means at the level of the universal TDO.

In terms of Chech cocycles the image of embedding (3.4) is described by those $(\lambda^{(1)}, \lambda^{(2)})$, see section 3.1, where $\lambda^{(2)} = 0$, and this forces $\lambda^{(1)}$ to be closed. Picking a collection of such cocycles that represent a basis of $H^1(X, \Omega_X^{1,2})$ we can repeat the constructions of sections 3.1 and 3.2 to obtain a sheaf $\alpha^{ch,tw}_X$. It is defined by gluing pieces isomorphic (as vertex algebras) to $D^{ch}_{U_i} \otimes H_X$ with transition functions as in (3.3); here $H_X$ is the vertex algebra of differential polynomials on $H^1(X, \Omega_X^{1,2})$. We will call the sheaf $\alpha^{ch,tw}_X$ the universal locally trivial sheaf of twisted chiral differential operators.

3.3.1. Example. Let us construct a sheaf of TCDO on $X = \mathbb{P}^1$. We have $\mathbb{P}^1 = \mathbb{C}_0 \cup \mathbb{C}_\infty$, a cover $\mathbb{g} = \{ \mathbb{C}_0, \mathbb{C}_\infty \}$, where $\mathbb{C}_0$ is $\mathbb{C}$ with coordinate $x$, $\mathbb{C}_\infty$ is $\mathbb{C}$ with coordinate $y$, with the transition function $x \rightarrow 1/y$ over $\mathbb{C}^* = \mathbb{C}_0 \cap \mathbb{C}_\infty$.

Defined over $\mathbb{C}_0$ and $\mathbb{C}_\infty$ are the standard CDOs, $D^{ch}_{C_0}$ and $D^{ch}_{C_\infty}$. The spaces of global sections of these sheaves are polynomials in $\partial^m(x)$, $\partial^m(\partial_x)$ (or $\partial^n(y)$, $\partial^n(\partial_y)$ in the latter case), where $\partial$ is the translation operator, so that, cf. sect. 2.4,

$$(\partial_x)^{(0)} x = (\partial_y)^{(0)} y = 1.$$
There is a unique up to isomorphism CDO on \( \mathbb{P}^1 \), \( \mathcal{D}^{ch}_{\mathbb{P}^1} \), it is defined by gluing \( \mathcal{D}^{ch}_{\mathcal{C}^0} \) and \( \mathcal{D}^{ch}_{\mathcal{C}_\infty} \) over \( \mathbb{C}^* \) as follows [MSV]:

(3.5) \[ x \mapsto 1/y, \ \partial_x \mapsto (-\partial_y)(-1)(y^2) - 2\partial(x). \]

The twisted version is as follows.

Since \( \dim \mathbb{P}^1 = 1 \),

\[ H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1} \to \Omega^2_{\mathbb{P}^1}, cl) = H^1(\mathbb{P}^1, \Omega^{1, cl}_{\mathbb{P}^1}) \]

so all twisted CDO on \( \mathbb{P}^1 \) are locally trivial. Furthermore, \( H^1(\mathbb{P}^1, \Omega^{1, cl}_{\mathbb{P}^1}) = \mathbb{C} \) and is spanned by the cocycle \( \mathbb{C} \mathcal{O} \cap \mathcal{C}_\infty \mapsto dx/x \). We have \( H_{\mathbb{P}^1} = \mathbb{C}[\lambda^*, \partial(\lambda^*), ....] \). Let \( \mathcal{D}^{ch, tw}_{\mathcal{C}^0} = \mathcal{D}^{ch}_{\mathcal{C}^0} \otimes H_{\mathbb{P}^1}, \mathcal{D}^{ch, tw}_{\mathcal{C}_\infty} = \mathcal{D}^{ch}_{\mathcal{C}_\infty} \otimes H_{\mathbb{P}^1} \) and define \( \mathcal{D}^{ch, tw} \) by gluing \( \mathcal{D}^{ch, tw}_{\mathcal{C}^0} \) onto \( \mathcal{D}^{ch, tw}_{\mathcal{C}_\infty} \) via

(3.6) \[ \lambda^* \mapsto \lambda^*, \ x \mapsto 1/y, \ \partial_x \mapsto -(\partial_y)(-1)y^2 - 2\partial(y) + y(-1)\lambda^*. \]

All of the above is easily verified by direct computations, cf. [MSV].

4. MAIN RESULT

In this section we prove the existence of a filtration for modules \( \mathcal{M} \) that satisfy certain integrability condition and use it to prove a generalization of Theorem 5.2. in [AChM].

We work in a setup that is slightly more general than that of TCDO with an intention to apply the results to certain deformations of TCDO defined in [Ch].

Let \( \mathcal{V} \) be a sheaf of vertex algebras. We will call a sheaf of vector spaces \( \mathcal{M} \) a \( \mathcal{V} \)-module if for each open \( U \subset X \) \( \mathcal{M}(U) \) is a \( \mathcal{V}(U) \)-module and the restriction morphisms \( \mathcal{M}(U) \to \mathcal{M}(U') \), \( U' \subset U \), are \( \mathcal{V}(U) \)-module morphisms, with the \( \mathcal{V}(U) \)-module structure on \( \mathcal{M}(U') \) given by pull-back.

Recall that to each graded vertex algebra \( V \) one can associate \( \text{Lie } V \), the Borcherds Lie algebra of \( V \) (cf. section 2.1.1).

We say that a \( \mathcal{V} \)-module \( \mathcal{M} \) is half-integrable iff each point \( x \in X \) has a neighbourhood \( U \) with \( \mathcal{M}(U) \) a \( \text{Lie } (\mathcal{V}(U))_+ \)-integrable module, that is to say, \( \text{Lie } (\mathcal{V})_+ \) acts on \( M \) by locally nilpotent operators.

We call the center of \( \mathcal{V} \) the subsheaf \( \mathcal{Z}(\mathcal{V}) \subset \mathcal{V} \) with stalks \( \mathcal{Z}_{X,x} = \mathcal{Z}(\mathcal{V}_{X,x}), x \in X \) where \( \mathcal{Z}(\mathcal{V}) \) denotes the center of \( \mathcal{V} \).

Define the category \( \mathcal{M}^{int+}(\mathcal{V}) \) as the full subcategory of the category of \( \mathcal{V} \)-modules \( \mathcal{M} \) such that

1. \( \mathcal{M} \) is half-integrable
2. \( h_n m = 0 \), for all \( h \in \mathcal{Z}(\mathcal{V}), m \in \mathcal{M} \).

Our goal in this section is the proof of the Theorem below. We work in the analytic topology. In section 4.1.7 we present a version of this result which works in Zariski topology.

**Theorem 4.1.** Suppose \( \mathcal{V} \) is a vertex envelope of a transitive vertex algebroid \( \mathcal{A} \) whose associated Lie algebroid \( \mathcal{L} = \mathcal{A}/\mathcal{O}^1_X \) is locally free of finite rank and fits into an exact sequence

(4.1) \[ 0 \to \mathfrak{h} \to \mathcal{L} \to \mathcal{T}_X \to 0 \]

in which \( \mathfrak{h} \) is an abelian \( \mathcal{O}_X \)-Lie algebra. Let \( \mathcal{M} \in \mathcal{M}^{int+}(\mathcal{V}) \). Then

1. \( \mathcal{M} \) is generated by the subsheaf

\[ \text{Sing } \mathcal{M} = \{ m \in \mathcal{M} : v_n m = 0 \text{ for all } v \in \mathcal{V}, n > 0 \} \]

2. There is a filtration \( \{ \mathcal{M}_i \}_{i \geq 0} \) on \( \mathcal{M} \) with \( \mathcal{M}_0 = \text{Sing } \mathcal{M} \), compatible with the grading of \( \mathcal{V} \).
4.1. Proof.

4.1.1. The center. First of all, let us describe the center of any such \( V \). For that, let us look more closely at (4.22). Since \( \mathfrak{h} \) is abelian, it possesses the structure of a \( \mathcal{D}_X \)-module. It is locally free as an \( \mathcal{O}_X \)-module and therefore, is of the form \( \mathfrak{h} = \mathcal{O}_X \otimes \mathbb{C} \mathfrak{h}^V \) where \( \mathfrak{h}^V \) denotes the subsheaf of horizontal sections of \( \mathfrak{h} \).

The pairing \((1)\) induces an \( \mathcal{O}_X \)-bilinear \( \mathcal{L} \)-invariant symmetric pairing \( \langle , \rangle : \mathfrak{h} \times \mathfrak{h} \to \mathcal{O}_X \) which restricts to \( \langle , \rangle : \mathfrak{h}^V \times \mathfrak{h}^V \to \mathbb{C}_X \).

Let \( \mathfrak{z} = \{ h \in \mathfrak{h}^V : \langle h, \mathfrak{h}^V \rangle = 0 \} \). Thus \( \mathfrak{z} \) is the kernel of \( \langle , \rangle \) restricted to \( \mathfrak{h}^V \).

**Lemma 4.2.** There exists a lifting \( s : \mathfrak{z} \to \mathcal{A} \), i.e. \( \pi s = \text{id}_\mathfrak{z} \), such that \( s(h), h \in \mathfrak{z} \) generate a subalgebra in \( \Gamma(X, \mathcal{V}) \) that is central in every \( \Gamma(U, \mathcal{V}) \), \( U \subset X \). Moreover, such a lifting is unique.

**Proof.** Let \( s' : \mathfrak{z} \to \mathcal{A} \) be any (local) lifting. Fix a basis \( \mathfrak{z}_i \) in \( \mathcal{T}(U) \) and the dual basis \( \{ \omega_i \} \) in \( \Omega^1(U) \). Fix arbitrary lifts \( \tilde{\mathfrak{z}}_i \) of \( \mathfrak{z}_i \) in \( \mathcal{A} \). For \( s'(h) \) to be central, \( s'(h)(1) \tilde{\mathfrak{z}}_i \) must be zero for all \( i \). This may fail, and so we are forced to change \( s' \). It is clear that

\[
s(h) = s'(h) - (\tilde{\mathfrak{z}}_i(1)s'(h))\omega_i, \ h \in \mathfrak{z}
\]

satisfies the desired condition \( s(h)(1) \tilde{\mathfrak{z}}_i = 0 \) for all \( h \in \mathfrak{z} \) and all basis elements \( \mathfrak{z}_i \); furthermore, the latter condition determines \( s(h) \) uniquely. Therefore, \( s \) is unique and extends globally.

It remains to show that \( \tilde{\mathfrak{z}}_i(0)s(h) = 0 \) for all \( i \)

Now, \( [\mathfrak{z}_i, h] = 0 \) in \( \mathcal{L} \) (since \( \mathfrak{z} \subset \mathfrak{h}^V \)). Therefore \( \tilde{\mathfrak{z}}_i(0)s(h) \) is in \( \Omega^1 \). However, for any \( 1 \leq j \leq \text{dim} \ X \)

\[
(\tilde{\mathfrak{z}}_i(0)s(h))(1)\tilde{\mathfrak{z}}_j = \tilde{\mathfrak{z}}_i(0)(s(h)(1)\tilde{\mathfrak{z}}_j) - s(h)(1)(\tilde{\mathfrak{z}}_i(0)\tilde{\mathfrak{z}}_j) = 0
\]

Thus \( \tilde{\mathfrak{z}}_i(0)s(h) \) must be zero. \( \square \)

Let \( U \subset X \) be suitable for chiralization (cf. 2.4) Choose an abelian basis \( \{ \mathfrak{z}_i \} \) of \( \mathcal{T}(U) \), the dual basis \( \{ \omega_i \} \) in \( \Omega^1(U) \) and a basis \( \{ h_k \} \) of \( \mathfrak{h}^V(U) \).

Choose any lifting \( \mathcal{L} \to \mathcal{A} \) extending that of Lemma 4.2 and identify \( \mathfrak{z}_i \) and \( h_k \) with the corresponding lifts.

Let \( V \) be the vertex enveloping algebra of \( \mathcal{A}(U) \). \( V \) is generated by \( \mathcal{O}_U \), vector fields \( \{ \mathfrak{z}_i \} \) and elements \( h_k \).

The following relations hold in \( V \).

\[
\begin{align*}
(4.2) \quad [\omega_{i,n}, \omega_{j,l}] &= 0 \\
(4.3) \quad [\mathfrak{z}_i, \omega_{j,l}] &= n\delta_{i,j}\delta_{n+l,0} \text{id} \\
(4.4) \quad [\mathfrak{z}_i, \mathfrak{z}_j] &= \alpha_{n+l} + \sum_k (f_k(-1)\lambda_k)_{n+l} \\
(4.5) \quad [h_{k,n}, h_{l,m}] &= n(h_k, h_l)\delta_{n+m,0} \text{id} \\
(4.6) \quad [\mathfrak{z}_i, h_{r,m}] &= \beta_{n+m}
\end{align*}
\]

where \( \alpha \in \Omega^1(U) \), \( f_k \in \mathcal{O}(U) \) and \( \beta \in \Omega^1(U) \) may depend on \( \mathfrak{z}_i \), \( \mathfrak{z}_j \) and \( h_r \) resp.

Using (4.3 - 4.6), it is easy to show that the subalgebra generated by \( \Gamma(U, s(\mathfrak{z})) \), see Lemma 4.2, is all of \( Z(V) \).
4.1.2. Let us now describe the strategy of proving Theorem 4.1, (1). The statement is local and we continue working on a suitable for chiralization subset $U$.

Let $M = \mathcal{M}(U)$.

Denote $\mathrm{Sing} M = \{m \in M : v_n m = 0 \text{ for all } v \in V, n > 0\}$

Introduce the filtration:

\begin{equation}
0 \subset M^{\Omega,h} \subset M^{\Omega} \subset M
\end{equation}

where

\begin{align*}
M^{\Omega} &= \{m \in M : \omega_{i,n} m = 0 \text{ for all } n > 0, i\} \\
M^{\Omega,h} &= \{m \in M^{\Omega} : h_{r,n} m = 0 \text{ for all } n > 0, r\}
\end{align*}

Let $[\mathrm{Sing} M]$ be the submodule of $M$ generated by $\mathrm{Sing} M$.

We show, step by step, that each of the terms in the filtration (4.7) is generated by $\mathrm{Sing} M$.

4.1.3. Step 1. We show that $[\mathrm{Sing} M]$ contains the subspace $M^{\Omega,h}$.

Let us introduce a filtration on $M^{\Omega,h}$ indexed by functions $d: \{1, 2, \ldots, \dim X\} \times \mathbb{Z}^+ \to \mathbb{Z}^+$ vanishing at all but finitely many pairs $(i, n)$.

Call any such function a \textit{degree vector}.

For a degree vector $d$ define

\begin{equation}
M^{\Omega,h}(d) = \bigcap_{i,n > 0} \ker \tau^{d(i,n)+1}_{i,n}.
\end{equation}

It is clear from definitions that $M^{\Omega,h} = \bigcup M^{\Omega,h}(d)$ and $M^{\Omega,h}(0) = [\mathrm{Sing} M]$.

Let us show that each $M^{\Omega,h}(d)$ is a subspace of $[\mathrm{Sing} M]$ using the induction on the \textit{length} $|d| = \sum_{i,n} d(i,n)$ of $d$. The base of induction is established in the line above.

Let $d \neq 0$ be a degree vector. Suppose $M^{\Omega,h}(d') \subset [\mathrm{Sing} M]$ for all $d'$ of smaller length.

Fix $(i, n)$ such that $k := d(i,n) > 0$ and let $d'$ be equal to $d$ everywhere except at $(i, n)$ where $d'(i,n) = d(i,n) - 1$. By induction assumption, $M^{\Omega,h}(d') \subset [\mathrm{Sing} M]$.

Let $m \in M^{\Omega,h}(d)$. Since $m \in \ker \tau^{k+1}_{i,n}$, one has

\begin{equation}
0 = \omega_{i,-n} \tau^{k+1}_{i,n} m = \tau^k_{i,n} (-n km + \omega_{i,-n} \tau_{i,n} m)
\end{equation}

Introduce the elements

\begin{align*}
m' &= \tau_{i,n} m, \\
m'' &= n km - \omega_{i,-n} m'.
\end{align*}

Then

\begin{equation}
m = \frac{1}{nk} (m'' + \omega_{i,-n} m')
\end{equation}

Hence, to show $m \in [\mathrm{Sing} M]$ it suffices to show that $m'$, $m''$ lie in $[\mathrm{Sing} M]$.

Let us show $m' \in M^{\Omega,h}(d')$.

One has $\tau^k_{i,n} m' = \tau^{k+1}_{i,n} m = 0$. To show $\tau^{d(j,l)+1}_{j,l} \tau_{i,n} m = 0$ it suffices to show

\begin{equation}
[\tau_{j,l}, \tau_{i,n}] \tau^{q}_{j,l} m = 0, \quad q \geq 0.
\end{equation}

But that follows from the fact that $\tau_{j,l} M^{\Omega,h} \subset M^{\Omega,h}$, which is a consequence of (4.3), (4.6).
Finally, to see that $m''$ is in $F^{d'}$, we need to check that $\tau^{d'(j,l)+1} m' = 0$ for all $(j,l)$. For $(j,l) \neq (i,n)$ this follows immediately from (4.3) and the fact that $m' \in F^{d'}$. The case $(j,l) = (i,n)$ is clear due to (4.9).

**Remark 4.3.** Note that (4.9) is a particular case of the following observation used in Steps 2 and 3 as well and originating in Kashiwara’s lemma.

Let $A$ and $B$ be linear operators on a space $V$ such that $[A,B]$ commutes with $A$.

Suppose $A^{n+1} m = 0$ for $m \in V$, $n \geq 0$. Then

$$A^n \cdot (n[B,A] + BA)m = 0$$

Indeed, $0 = BA^{n+1}m = [B,A]Am + A^n BAM = A^n(n[B,A] + BA)m \quad \square$\

Steps 2 and 3 are proved in essentially the same way. The reader may safely skip the rest of the proof. However, for the sake of completeness we will keep the same level of detail.

4.1.4. **Step 2.** We show $M^\Omega \subset [\text{Sing} M]$.

In the case $\langle \cdot, \cdot \rangle = 0$ the assumptions of the theorem imply $h_{r,n} = 0$ on $M$ so that $M^\Omega = M^\Omega_{\nondeg}$ and there’s nothing to prove.

Let us prove the claim in the case $\mathfrak{z}$ is a proper subset of $h^\vee_{\nondeg}$.

Let $\{b_r\}$ be an orthonormal basis for some complement $h^\vee_{\nondeg}$ to $\mathfrak{z}$ in $h^\vee$. In particular

$$[b_{k,n}, b_{l,m}] = n\delta_{k,l} \delta_{n+m,0} \text{id}$$

Introduce the filtration

$$M^\Omega(d) = \left\{ m \in M^\Omega : (b_r)^{d(r,n)+1} m = 0, \forall n > 0 \forall r \right\}$$

where $d : \{1, \ldots, \dim h^\vee_{\nondeg}\} \times \mathbb{Z}_+ \to \mathbb{Z}_+$ is a degree vector.

Then, clearly, $M^\Omega(0) = M^\Omega_{\nondeg}$ and $M^\Omega = \bigcup_d M^\Omega(d)$.

Suppose that $d \neq 0$ is a degree vector and suppose $M^\Omega(d')$ is a subspace of $[\text{Sing} M]$ for all $d'$ of smaller length. Let us show $M^\Omega(d) \subset [\text{Sing} M]$.

Fix some $i,n$ such that $d(i,n) > 0$ and define $d'$ by $d'(j,l) = d(j,l) - \delta_{ij}\delta_{nl}$. Let $k = d(i,n)$.

Let $m \in M^\Omega(d)$.

We have (cf. (4.12))

$$0 = (b_{i,n})^k (-nkm + b_{i,-n}b_{i,n}m)$$

Introduce the elements

$$m' = b_{i,n}m,$$

$$m'' = -nkm + b_{i,-n}m'$$

Since $\omega$’s commute with $b_r$’s, these elements are in $M^\Omega$ whenever $m$ is. We wish to show that they are in fact in $M^\Omega(d')$ and therefore, in $[\text{Sing} M]$. This would imply that

$$m = \frac{1}{nk}(-m'' + b_{i,-n}m')$$

is in $[\text{Sing} M]$ as well.
It is clear from (4.13) that \( m' \in M^\Omega(d') \).

Let us show that \( m'' \in M^\Omega(d') \). We need to show \( m'' \in \ker(b_{s, m}d's, m+1) \) for all \( s \) and all \( m > 0 \); but this follows from (4.13) in case \( (s, m) \neq (i, n) \) and from (4.14) in case \( (s, m) = (i, n) \). Thus \( m'' \in M^\Omega(d') \).

4.1.5. Step 3. We complete the proof by showing \( M \subset [\text{Sing } M] \).

Introduce a filtration on \( M \) indexed by degree vectors \( d : \{1, 2, \ldots, \dim X\} \times \mathbb{Z}_+ \rightarrow \mathbb{Z}_+ \) where for each \( d \) we define

\[
M(d) = \bigcap_{i, n > 0} \ker(\omega_{i,n}^{d(i,n)+1}).
\]

Clearly, \( M = \bigcup M(d) \) and \( M(0) = M^\Omega \).

In order to show that \( M(0) \subset [\text{Sing } M] \) implies \( M(d) \subset [\text{Sing } M] \) for all \( d \) we will use induction on the length \( |d| = \sum_{i,n} d(i,n) \). Thus, we suppose that \( d \neq 0 \) is a degree vector and \( M(d') \) is a subspace of \( [\text{Sing } M] \) for all \( d' \) of smaller length. Let us show \( M(d) \subset [\text{Sing } M] \)

Fix some \( i, n \) such that \( d(i,n) > 0 \) and define \( d' \) by \( d'(j,l) = d(j,l) - \delta_{ij}\delta_{nl} \). Let \( k = d(i,n) \).

Let \( m \in M(d) \).

Since \( m \in \ker \omega_{i,n}^{k+1} \), one has (cf. (4.12))

\[
0 = \tau_{i,-n}\omega_{i,n}^{k+1}m = \omega_{i,n}^{k}(-nk\cdot m + \tau_{i,-n}\omega_{i,n}m)
\]

It follows from (4.2-4.4) and (4.18) that the elements

\[
\begin{align*}
m' &= \omega_{i,n}m, \\
m'' &= nk\cdot m - \tau_{i,-n}m'
\end{align*}
\]

belong to \( M(d') \subset [\text{Sing } M] \). Therefore,

\[
m = \frac{1}{nk}(m'' + \tau_{i,-n}m')
\]

is also in \( [\text{Sing } M] \). \( \Box \)

4.1.6. Proof of (2). Let us work locally as in the proof of part (1) and keep all the notation used there. Thus we have \( U \subset X, M = \mathcal{M}(U), V \) is the vertex envelope of \( \mathcal{A}(U) \).

Define \( \text{Lie}(V)_+ \) to be the subalgebra of the Borcherds Lie algebra \( \text{Lie}(V) \) spanned by \( v \otimes t^n \) with positive \( n \); and let \( \text{Lie}(V)_{\leq 0} \) be the image of \( V \otimes \mathbb{C}[t^{-1}] \) in \( \text{Lie}(V) \). Then \( \text{Lie}(V) = \text{Lie}(V)_+ \oplus \text{Lie}(V)_{\leq 0} \)

The enveloping algebras \( U(\text{Lie}(V)), U(\text{Lie}(V)_+) \) have a natural grading defined by \( \deg v_{n_1} \ldots v_{n_k} = n_1 + \cdots + n_k \).

Define the following subspaces of \( M \):

\[
M_n = \{ m : U(\text{Lie}(V)_+)k\cdot m = 0, \text{ for all } k > n \}
\]

It is clear that \( M_n \subset M_{n+1}, M_{-1} = \{0\} \) and \( M_0 = \text{Sing } M \).

The decomposition \( U(\text{Lie}(V)) \cong U(\text{Lie}(V)_{\leq 0}) \otimes U(\text{Lie}(V)_+) \) implies that \( M_n = \tilde{M}_n \) where

\[
\tilde{M}_n = \{ m \in M : U(\text{Lie}(V))k\cdot m = 0 \text{ for all } k > n \}
\]
Note that $\tilde{M}_n$ is compatible with the grading of $V$, i.e. $v_k\tilde{M}_n \subset \tilde{M}_{n-k}$. Hence, the submodule $[\text{Sing} M]$ generated by $\text{Sing} M = M_0$ is a subset of the union of $\tilde{M}_n$. But $[\text{Sing} M] = M$ by part (1) and thus, $M_n$ is an (exhaustive) filtration.

Since $V$ is a graded sheaf, the formula (4.21) makes sense globally, giving the desired filtration $M_n$. □

4.1.7. The algebraic case. Theorem 4.1 remains true in Zariski topology, as long as one imposes one additional restriction. Since algebraic $D$-modules may have non-algebraic solutions, $h^\nabla$ may fail to have “right size”. Thus, one has to demand that $h$ be equal to $O_X \otimes h^\nabla$. To be more precise, one has:

**Theorem 4.4.** Suppose $V$ is a vertex envelope of a transitive vertex algebroid $A$ whose associated Lie algebroid $L = A/\Omega$ is locally free of finite rank and fits into an exact sequence

$$0 \rightarrow h \rightarrow L \rightarrow T_X \rightarrow 0$$

in which $h = O_X \otimes h^\nabla$ for a locally constant sheaf $h^\nabla$ such that $[L, h^\nabla] = 0$.

Let $M \in M^{\text{int}+}(V)$. Then:

1. $M$ is generated by the subsheaf $\text{Sing} M = \{m \in M : v_nm = 0 \text{ for all } v \in V, n > 0\}$

2. There is a filtration $\{M_i\}_{i \geq 0}$ on $M$ with $M_0 = \text{Sing} M$, compatible with the grading of $V$.

The proof is identical to that of Theorem 4.1.

4.2. Equivalence. Now we prove a generalization of Theorem 5.2 in [AChM] in the setup of the previous section. This is a result establishing the equivalence between certain subcategories of $M^{\text{int}+}(V)$ and categories of modules over sheaves of $tdo$, based on a version of Zhu correspondence. The result is true in both analytic and Zariski topology.

Let $V$ be as in Theorem 4.1 and assume further that $h = O_X \otimes h^\nabla$ with $h^\nabla$ a constant sheaf. By abuse of language we denote its fiber by $h^\nabla$ and write $(h^\nabla)^*$ and $z^*$ for vector space duals of $h^\nabla$ and $z := \{h \in h^\nabla : \langle h, s \rangle = 0, s \in h^\nabla\}$

Let us fix \( \theta \in (h^\nabla)^* \), \( \chi = \sum_{n \in \mathbb{Z}} \chi_n z^{-n-1} \in z^*((z)) \).

Define the category $M_{\theta, \chi(z)}^{\text{int}+}(V)$ to be the full subcategory of $M^{\text{int}+}(V)$ consisting of modules $M$ satisfying:

1. For all $h \in h^\nabla$, $m \in M$

   $$h_0 m = \theta(h) \cdot m,$$

2. For all $c \in z$, $n \in \mathbb{Z}$

   $$c_n m = \chi_n(c) \cdot m,$$

For $M_{\theta, \chi(z)}^{\text{int}+}(V)$ to be nonzero, an evident compatibility condition has to be satisfied:

$$\theta|_z = \chi_0$$
Furthermore, the half-integrability condition dictates

\[(4.26) \quad \chi_n = 0, \quad n > 0\]

for any nonzero module in \(\mathcal{M}_{\theta,\chi(z)}^{int+}(\mathcal{V})\). In other words, \(\chi(z)\) has regular singularity at 0.

Let \(\mathcal{M} \in \mathcal{M}_{\theta,\chi(z)}^{int+}(\mathcal{V})\) be arbitrary. By Theorem 4.1, \(\mathcal{M}\) is a \(\mathbb{Z}_{\geq 0}\)-filtered module over a sheaf of (graded) vertex algebras \(\mathcal{V}\). Thus, \(\mathcal{M}_0 = \text{Sing}\, \mathcal{M}\) is equipped with an action of the Zhu algebra of \(\mathcal{V}\), or, rather, a sheafified version of it. Furthermore, the action of \(\text{Zhu}(\mathcal{V})\) on \(\mathcal{M}_0 = \text{Sing}\, \mathcal{M}\) factors through \(\text{Zhu}(\mathcal{V})/I_\theta\) where \(I_\theta\) is the ideal generated by \(h - \theta(h)\), \(h \in \mathfrak{h}^\mathcal{V}\). As \(\text{Zhu}(\mathcal{V}) = U(\mathcal{L})\) ([AChM], Theorem 3.1), the quotient algebra \(\text{Zhu}(\mathcal{V})/I_\theta\) is a TDO, to be denoted \(D_X^\mathcal{V}\).

Thus we have a functor

\[(4.27) \quad \text{Sing} : \mathcal{M}_{\theta,\chi(z)}^{int+}(\mathcal{V}) \to \mathcal{M}(D_X^\mathcal{V})\]

**Theorem 4.5.** Assume that the conditions \((4.25, 4.26)\) are satisfied. Then the functor \((4.27)\) is an equivalence of categories.

This theorem is a generalization of Theorem 5.2. ([AChM] and has an almost identical proof, after some modifications; therefore, we only list the differences; an interested reader may easily supply all the details following the proof in [AChM]. Recall that the strategy of proof was to construct a left adjoint \(\text{Zhu}^*(\mathcal{V})\) to \((4.27)\) and to prove that it is a quasi-inverse. Our proof proceeds in the same way except for the following:

- in the construction of the left adjoint functor one has to change \(H^1(X, \Omega_X^{1,2})\) to \(\mathfrak{z}\)
- the quasi-inverse property (Lemma 5.3, op.cit.) is changed as follows: one adds to the polynomial algebra \(P\) indeterminates \(L_{-n}\) corresponding to \(l_r\), where \(\{l_r\}\) is an orthogonal basis of a complement to \(\mathfrak{z}\) in \(\mathfrak{h}^\mathcal{V}\). The proof of this new statement is identical to that of Lemma 5.3.

In the rest, one proceeds literally in the same manner as in op.cit. □

As an example, consider the case of TCDO, \(\mathcal{V} = D_X^{ch,tw}\). In this case \(\mathcal{L} = D^{tw}, \mathfrak{h}^\mathcal{V} = H^1(X, \Omega_X^{1,2})^*\) and \(\langle , \rangle\) is trivial, so that \(\mathfrak{z} = H^1(X, \Omega_X^{1,2})^*\) and \((4.25)\) translates into the condition \(\theta = \chi_0 \in H^1(X, \Omega_X^{1,2})\). Thus, we can simplify the notation and write \(\mathcal{M}_{\chi(z)}^{int+}(D_X^{ch,tw})\) instead of \(\mathcal{M}_{\theta,\chi(z)}^{int+}(D_X^{ch,tw})\).

Theorem 4.5 specializes to the following variant of [AChM], Theorem 5.2.

**Theorem 4.6.** Suppose that \(\chi(z) \in H^1(X, \Omega_X^{1,2})((z))\) has regular singularity. Then the functor

\[
\text{Sing} : \mathcal{M}_{\theta,\chi(z)}^{int+}(D_X^{ch,tw}) \to D_X^\mathcal{V} - \text{mod}
\]

is an equivalence.

When \(\mathcal{V}\) is a CDO, i.e. \(\mathfrak{h} = 0\), one obtains the equivalence between half-integrable modules over a CDO \(\mathcal{V}\) and the category of \(D\)-modules over \(X\).

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