THE HILBERT TRANSFORM DOES NOT MAP $L^1(Mw)$ TO $L^{1,\infty}(w)$

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Abstract. We disprove the following a priori estimate for the Hilbert transform $H$ and the Hardy Littlewood maximal operator $M$:

$$
\sup_{t>0} t w\{x \in \mathbb{R} : |Hf(x)| > t\} \leq C \int |f(x)| Mw(x) \, dx.
$$

This is a sequel to paper [5] by the first author, which shows the existence of a Haar multiplier operator for which the inequality holds.

1. Introduction and statement of main result

In [2], C. Fefferman and E. Stein observed the following a priori estimate for the Hardy Littlewood maximal operator $M$:

$$
\sup_{t>0} t w\{x \in \mathbb{R} : |Mf(x)| > t\} \leq C \int |f(x)| Mw(x) \, dx.
$$

Here the weight $w$ is a non-negative, locally integrable function, and $w(E)$ denotes the integral of the weight over the set $E$. We give a negative answer to the question whether such an inequality holds when the Hardy Littlewood maximal operator on the left hand side is replaced by the Hilbert transform. For a discussion of the history of this question we refer to [5].

Theorem 1.1. For each constant $C > 0$ there is a weight function $w$ on the real line and an integrable compactly supported function $f$ and a $t > 0$ such that

$$
t w\{x \in \mathbb{R} : |Hf(x)| > t\} \geq C \int |f(x)| Mw(x) \, dx.
$$

Similarly as in [5], we prove Theorem 1.1 as a consequence of the following:

Proposition 1.2. For each constant $C > 0$ there is an everywhere positive weight function $w$ on the real line and an integrable compactly supported function $f$ and a $t > 0$ such that

$$
t^2 w\{x \in \mathbb{R} : |Hf(x)| > t\} \geq C \int |f(x)|^2 \left( \frac{Mw(x)}{w(x)} \right)^2 w(x) \, dx.
$$

The reduction to Proposition 1.2 is taken from [1], we sketch the argument at the end of this paper. Following [5] further, we reduce Proposition 1.2 to the dual proposition:

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\end{itemize}
Proposition 1.3. For each constant $C$ there is a nontrivial weight $w$ on the real line such that

$$\|H(1_{[0,1]}w)\|_{L^2(w/(Mw)^2)} \geq C\|1_{[0,1]}\|_{L^2(w)}.$$  

Our construction of the weight $w$ is a somewhat simpler variant of the construction in [5]. It was discovered during a stimulating summer school on “Weighted estimates for singular integrals” at Lake Arrowhead, Oct 3-8. 2010.

2. Proof of Theorem 1.2

Recall that a triadic interval $I$ is of the form $[3^jn, 3^j(n+1))$ with integers $j, n$. Denote by $I^m$ the triadic interval of one third the length of $I$ which contains the center of $I$.

Fix an integer $k$ which will be chosen large enough depending on the constant $C$ in Proposition 1.3. Define $K_0$ to be $\{[0,1]\}$ and recursively for $i \geq 1$:

$$J_i := \{K^m : K \in K_{i-1}\},$$

$$K_i := \{K : K \text{ triadic, } |K| = 3^{-ik}, K \subset \bigcup_{J \in J_i} J\}.$$  

Proceeding recursively from the larger to the smaller intervals, we choose for each $J \in J := \bigcup_{i \geq 1} J_i$ a sign $\epsilon(J) \in \{-1, 1\}$. More precisely, $\epsilon(J)$ depends on the values $\epsilon(J')$ with $|J'| > |J|$. The exact choice will be specified below. Define for each $J \in J$ the interval $I(J)$ to be the triadic interval of length $31^{-k}|J|$ whose right endpoint equals the left endpoint of $J$ if $\epsilon(J) = 1$, and whose left endpoint equals the right endpoint of $J$ if $\epsilon(J) = -1$. Note that $I(J)$ has the same length as the intervals in $K_i$.

Next we define a sequence of absolutely continuous measures on $[0,1]$. We continue to use the same symbol for a measure and its Lebesgue density. Let $w_0$ be the uniform measure on $[0,1]^m \cup I([0,1])^m$ with total mass 1. Recursively we define the measure $w_i$ by the following properties: It coincides with $w_{i-1}$ on the complement of $\bigcup_{K \in K_i} K$. For $K \in K_i$ we have $w_i(K) = w_{i-1}(K)$ and the restriction of $w_i$ to $K$ is supported and uniformly distributed on $K^m \cup I(K^m)$.

Let $w$ be the weak limit of the sequence $w_i$ and note that $w$ is supported on $\bigcup_{J \in J} I(J)$. For $K \in K_i$, $J \in J_i$, $x \in I(J)$, and any triadic interval $K'$ with $|K'| \geq |K|$ we have

$$w(x) = \frac{w(I(J))}{|I(J)|} = \frac{w(K)}{|K|} \geq \frac{w(K')}{|K'|}.$$  

We claim that for $J \in J$ and $x \in I(J)^m$ we have

$$Mw(x) \leq 7w(x).$$  

To see this, let $I$ be a (not necessarily triadic) interval containing $x$. If $I$ is contained in $I(J)$, then by the first identity of (2) the average of $w$ over $I$ equals $w(x)$. If $I$ is not
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contained in $I(J)$, then $|I| \geq |I(J)|/3$. Let $K'$ be the collection of triadic intervals of length $|I(J)|$ which intersect $I$ and note that

$$\sum_{K' \in K'} |K'| \leq |I| + 2|I(J)| \leq 7|I|$$

because at most two intervals in $K'$ are not entirely covered by $I$. With (2) we conclude that the average of $w$ over $I$ is no more than $7w(x)$, which completes the proof of (3).

**Lemma 2.1.** For $K \in K_i$, $J = K^m$, $x \in I(J)^m$, and $k > 3000$ we have

$$|Hw(x)| \geq \frac{k}{3}w(x) \ .$$

This Lemma proves Proposition 1.3, because with (3) and since $w$ is constant on every $I(J)$ we have

$$49\|Hw\|^2_{L^2(w/(Mw)^2)} \geq \left(\frac{k^2}{9}\right) \sum_{J \in J} \int_{I(J)^m} w(y) \, dy \geq \left(\frac{k^2}{27}\right) \|1_{[0,1]}\|^2_{L^2(w)} \ .$$

**Proof of Lemma 2.1:** We split the principal value integral for $Hw(x)$ into six summands:

(4) \hspace{1cm} p.v. \int_{I(J)} \frac{w(y)}{y-x} \, dy

(5) \hspace{1cm} + \int_{J} \frac{w(y)}{y-x} \, dy

(6) \hspace{1cm} + \int_{K^c} \frac{w(y)}{y-x} - \frac{w(y)}{y-c(J)} \, dy

(7) \hspace{1cm} + \int_{(\cup_{K_i} K')} \frac{w(y)}{y-x} \, dy

(8) \hspace{1cm} + \sum_{K' \in K \setminus \{K\}} \int_{K'} \frac{w(y)}{y-c(J)} - \frac{w(y)}{c(K') - c(J)} \, dy

(9) \hspace{1cm} + \sum_{K' \in K \setminus \{K\}} \int_{K'} \frac{w(y)}{c(K') - c(J)} \, dy \ .

The terms (7) and (9) remain unchanged if we replace $w$ by $w_i$ and hence depend only on the choices of $\epsilon(J')$ with $|J'| > |J|$. The integrand of (5) is positive or negative depending on $\epsilon(J)$. Specify the choice of $\epsilon(J)$ so that the sign of (5) equals the sign of (7)+(9). If the latter is zero, we may arbitrarily set $\epsilon(J) = 1$. We estimate

$$|(5)| \geq \sum_{K' \in K_{i+1}, K' \subset J} \int_{K'} \frac{w(y)}{|y-x|} \, dy \geq \sum_{K' \in K_{i+1}, K' \subset J} \sup_{y \in K'} |y-x| \frac{w(K')}{|y-x|}$$
\[ \geq \sum_{n=1}^{3^k} \frac{1}{n+1} \frac{w(I(J))}{|I(J)|} \geq \frac{(k/2)w(x)}{2} \]

The remaining terms are small error terms, we estimate with \( \delta = |I(J)m| : \)

\[ |(4)| = \left| \int_{I(J) \setminus [x-\delta,x+\delta]} \frac{w(y)}{y-x} dy \right| \leq 3w(x) , \]

\[ |(6)| \leq 4 \sum_{|K'|=|K|, K' \neq K} \int_{K'} \frac{|x-c(J)|}{|y-c(J)|} w(y) dy \]

\[ \leq 8 \sum_{|K'|=|K|, K' \neq K} \frac{|x-c(J)|}{|c(K')-c(J)|} w(K') \]

\[ \leq 16 \sum_{n=1}^{\infty} \frac{1}{(n-3/4)^2} \frac{w(I(J))}{|I(J)|} \leq 200w(x) , \]

\[ |(8)| \leq 4 \sum_{K' \in K, K' \neq K} \int_{K'} \frac{|y-c(K')|}{|c(K')-c(J)|} w(y) dy \]

and the last expression is dominated by the same final bound as (6). Putting all estimates together, we have

\[ |(4) + (5) + (6) + (7) + (8) + (9)| \]

\[ \geq |(5) + (7) + (9)| - |(4)| - |(6)| - |(8)| \]

\[ \geq |(5)| - |(4)| - |(6)| - |(8)| \]

\[ \geq (k/2 - 403)w(x) . \]

This completes the proof of Lemma 2.1 and thus Theorem 1.3.

3. Remarks

3.1. More general kernels. The construction can be generalized to apply to more general kernels, including those with even symmetry, such as for example \( \text{Re}(|x|^{-1+\alpha}) \) with \( \alpha \neq 0 \). Choose \( J \) to be the union of \( 3^k - 1 \) not necessarily adjacent but appropriately chosen intervals of length \( 3^{-k}|K| \) contained in \( K \), and \( I(J) \) an appropriate further interval of this length well inside \( K \), so that the kernel of the Calderon Zygmund operator for \( x \in I(J)m \) has sufficient positive or negative bias on \( J \).

3.2. Weights in Theorem 1.1. We specify weights satisfying Theorem (1.1). Fix a constant \( C \) as in Proposition (1.3) and consider \( k \) and the weight \( w \) constructed above. We slightly change \( w \) to make it positive by adding \( ec^{-x^2} \) for sufficiently small \( c \) so as to not change the conclusion of Proposition (1.3). We may normalize the measure to be probability measure and call the remaining measure \( w \) again. The conclusion of Proposition 1.3 can be written:

\[ (\int (Hw(x))^2 \frac{w(x)}{(Mw(x))^2} dx)^{1/2} \geq C . \]
Multiplying both sides of (10) by the left hand side of (10), setting \( f = (Hw)w/(Mw)^2 \) and using essential self-duality of \( H \) we obtain
\[
\int w(x)Hf(x) \, dx \geq C\left( \int f(x)^2 \frac{(Mw(x))^2}{(w(x))^2} w(x) \, dx \right)^{1/2}.
\]

Letting \( f^* \) be the non-increasing rearrangement of \( Hf \) on \([0,1]\), we may estimate the left hand side of (11)
\[
\int_0^1 f^*(y) \, dy \leq 2 \sup_{y \in [0,1]} y^{1/2} f^*(y) = 2 \sup_{t > 0} w(\{x : |Hf(x)| \geq t\})^{1/2} t.
\]

Hence Proposition 1.2 holds for the constant \( C/2 \) with the weight \( w \) and some existentially chosen \( t \). Now let \( E \) be the set on the left hand side of Proposition 1.2 for the given \( w, f, \) and appropriate \( t \), then we have
\[
M(w1_E)(x) = \sup_{x \in I} \int_I w \frac{1_{1_E}}{1_{1_w}} \leq Mw(x)Mw1_E(x),
\]
where \( M_w \) denotes the Hardy Littlewood maximal function with respect to the weight \( w \). With Hölder’s inequality we obtain
\[
\int |f(x)|M(w1_E)(x) \, dx \leq \left( \int |f(x)|^2 \frac{Mw(x)}{w(x)} \, dx \right)^{1/2} \|Mw1_E\|_{L^2(w)}.
\]

With the Hardy Littlewood maximal theorem with respect to the weight \( w \) we can estimate \( \|Mw1_E\|_{L^2(w)} \) by \( w(E)^{1/2} \). This shows that Theorem 1.1 holds for the weight \( w1_E \).

3.3. \( A_1 \) weights. It remains open to date whether the a priori inequality
\[
(t \ w \{x \in \mathbb{R} : |Hf(x)| > t\} \leq C\|w\|_{A_1} \int |f(x)|w(x) \, dx
\]
holds, where the \( A_1 \) constant is defined as \( \|w\|_{A_1} := \|Mw/w\|_{\infty} \). Our construction in this paper does not seem to address this question. The recent preprint [4] has announced that the analogue of (12) for Haar multipliers is false. In [3], a version of (12) has been proved with an additional logarithmic factor in the \( A_1 \) constant of the weight.

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