On the Lagrangian capacity of convex or concave toric domains

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Abstract

We establish computational results concerning the Lagrangian capacity, originally defined by Cieliebak–Mohnke. More precisely, we show that the Lagrangian capacity of a 4-dimensional convex toric domain is equal to its diagonal. Working under the assumption that there is a suitable virtual perturbation scheme which defines the curve counts of linearized contact homology, we extend the previous result to any convex or concave toric domain. This result gives a positive answer to a conjecture of Cieliebak–Mohnke for the Lagrangian capacity of the ellipsoid.

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1 Introduction

1.1 Motivation

A symplectic capacity is a function $c$ that assigns to every symplectic manifold $(X, \omega)$ (in a restricted subclass) a number $c(X, \omega) \in [0, +\infty)$, satisfying

(Monotonicity) If there exists a symplectic embedding (possibly in a restricted subset of all symplectic embeddings) $(X, \omega_X) \rightarrow (Y, \omega_Y)$, then $c(X, \omega_X) \leq c(Y, \omega_Y)$;

(Conformality) If $\alpha > 0$ then $c(X, \alpha \omega_X) = \alpha c(X, \omega_X)$.

By the monotonicity property, symplectic capacities can provide obstructions to the existence of symplectic embeddings.

An example of a symplectic capacity is the Lagrangian capacity $c_L$, first defined in [CM18, Section 1.2]. It is defined as follows. If $(X, \omega)$ is a $2n$-dimensional symplectic manifold and $L \subset X$ is a Lagrangian submanifold, then the minimal symplectic area of $L$ is given by

$$A_{\min}(L) := \inf \{ \omega(\sigma) \mid \sigma \in \pi_2(X, L), \omega(\sigma) > 0 \}.$$ 

Then, the Lagrangian capacity of $(X, \omega)$ is given by

$$c_L(X, \omega) := \sup \{ A_{\min}(L) \mid L \subset X \text{ is an embedded Lagrangian torus} \}.$$ 

The main goal of this paper is to compute the Lagrangian capacity of (some) toric domains. A toric domain is a Liouville domain of the form $X_\Omega := \mu^{-1}(\Omega) \subset \mathbb{C}^n$, where $\Omega \subset \mathbb{R}_{\geq 0}^n$ and $\mu(z_1, \ldots, z_n) = \pi(|z_1|^2, \ldots, |z_n|^2)$.

Some examples of toric domains which are going to be relevant in this introduction are the ball $B(a)$, the cylinder $Z(a)$, the ellipsoid $E(a_1, \ldots, a_n)$ and the nondisjoint union of cylinders $N(a)$, which are given by

$$B(a) := \mu^{-1}(\Omega_B(a)), \quad \Omega_B(a) := \{ x \in \mathbb{R}_{\geq 0}^n \mid x_1 + \cdots + x_n \leq a \},$$
$$Z(a) := \mu^{-1}(\Omega_Z(a)), \quad \Omega_Z(a) := \{ x \in \mathbb{R}_{\geq 0}^n \mid x_1 \leq a \},$$
$$E(a_1, \ldots, a_n) := \mu^{-1}(\Omega_{E(a_1, \ldots, a_n)}), \quad \Omega_{E(a_1, \ldots, a_n)} := \{ x \in \mathbb{R}_{\geq 0}^n \mid x_1/a_1 + \cdots + x_n/a_n \leq 1 \},$$
$$N(a) := \mu^{-1}(\Omega_N(a)), \quad \Omega_N(a) := \{ x \in \mathbb{R}_{\geq 0}^n \mid \exists i = 1, \ldots, n: x_i \leq a \}.$$ 

The diagonal of a toric domain $X_\Omega$ is

$$\delta_\Omega := \max \{ a \mid (a, \ldots, a) \in \Omega \}.$$ 

1Unless otherwise stated, every symplectic manifold we will consider will be $2n$-dimensional.
It is easy to show (see Lemmas 3.6 and 3.7) that \( c_L(X_\Omega) \geq \delta_\Omega \) for any convex or concave toric domain \( X_\Omega \). Also, Cieliebak–Mohnke give the following results for the Lagrangian capacity of the ball and the cylinder.

**Proposition 3.8** ([CM18, Corollary 1.3]). The Lagrangian capacity of the ball is

\[
c_L(B(1)) = \frac{1}{n}.
\]

**Proposition 3.9** ([CM18, p. 215-216]). The Lagrangian capacity of the cylinder is

\[
c_L(Z(1)) = 1.
\]

In other words, if \( X_\Omega \) is the ball or the cylinder then \( c_L(X_\Omega) = \delta_\Omega \). This motivates the following conjecture by Cieliebak–Mohnke.

**Conjecture 3.10** ([CM18, Conjecture 1.5]). The Lagrangian capacity of the ellipsoid is

\[
c_L(E(a_1, \ldots, a_n)) = \left( \frac{1}{a_1} + \cdots + \frac{1}{a_n} \right)^{-1}.
\]

A more general form of the previous conjecture is the following.

**Conjecture 3.11.** If \( X_\Omega \) is a convex or concave toric domain then

\[
c_L(X_\Omega) = \delta_\Omega.
\]

So, more precisely, the goal of this paper is to prove Conjecture 3.11. We will offer two main results in this direction.

1. In Theorem 3.28, we prove that \( c_L(X_\Omega) = \delta_\Omega \) whenever \( X_\Omega \) is convex and 4-dimensional.

2. In Theorem 4.37, using techniques from contact homology we prove that \( c_L(X_\Omega) = \delta_\Omega \) for any convex or concave toric domain \( X_\Omega \). More specifically, in this case we are working under the assumption that there is a virtual perturbation scheme such that the linearized contact homology of a nondegenerate Liouville domain can be defined (see Section 4.1).

**Remark 1.1.** In [GPR], Jean Gutt, the Author and Vinicius Ramos explain that the proof of Theorem 4.37 that we will give carries over (with minor changes) to the case where the toric domain \( X_\Omega \) is not necessarily convex or concave, thus giving a formula for the Lagrangian capacity of a more general class of toric domains. More precisely, it is shown that \( c_L(X_\Omega) = \eta_\Omega \) for any toric domain \( X_\Omega \) with \((\eta_\Omega, \ldots, \eta_\Omega) \in \partial \Omega\), where \( \eta_\Omega := \sup \{ a \mid X_\Omega \subset N(a) \} \). Notice that if \( X_\Omega \) is convex or concave, then \( \eta_\Omega = \delta_\Omega \).

1.2 Main results

Notice that by the previous discussion, we only need to prove the hard inequality \( c_L(X_\Omega) \leq \delta_\Omega \). For this, we will need to use other symplectic capacities. The following is a list of the symplectic capacities we will use in this paper, in addition to the Lagrangian capacity:
(1) the Gutt–Hutchings capacities from [GH18], denoted by $c_{GH}^k$ (Definition 3.12);
(2) the $S^1$-equivariant symplectic homology capacities from [Iri21], denoted by $c_{S^1}^k$ (see Definition 3.17);
(3) the McDuff–Siegel capacities from [MS22], denoted by $\tilde{g}_k$ (see Definition 3.21);
(4) the higher symplectic capacities from [Sie20], denoted by $g_k$ (see Definition 4.8).

We now describe our results concerning the capacities mentioned so far. The key step in proving $c_L(X,\Omega) \leq \delta_\Omega$ is the following inequality between $c_L$ and $\tilde{g}_k$.

**Theorem 3.27.** If $(X,\lambda)$ is a Liouville domain then

$$c_L(X) \leq \inf_k \frac{\tilde{g}_k(X)}{k}.$$ 

Indeed, this result can be combined with the following results from [MS22] and [GH18].

**Proposition 3.24** ([MS22, Proposition 5.6.1]). If $X_\Omega$ is a 4-dimensional convex toric domain then

$$\tilde{g}_k(X_\Omega) = c_{GH}^k(X_\Omega).$$

**Lemma 3.16** ([GH18, Lemma 1.19]). $c_{GH}^k(N(a)) = a(k+n-1)$.

Combining the three previous results, we get the following particular case of Conjecture 3.11. Since the proof is short, we present it here as well.

**Theorem 3.28.** If $X_\Omega$ is a 4-dimensional convex toric domain then

$$c_L(X_\Omega) = \delta_\Omega.$$  

**Proof.** For every $k \in \mathbb{Z}_{\geq 1}$,

$$\begin{align*}
\delta_\Omega & \leq c_L(X_\Omega) \quad \text{[by Lemmas 3.6 and 3.7]} \\
& \leq \frac{\tilde{g}_k(X_\Omega)}{k} \quad \text{[by Theorem 3.27]} \\
& = \frac{c_{GH}^k(X_\Omega)}{k} \quad \text{[by Proposition 3.24]} \\
& \leq \frac{c_{GH}^k(N(\delta_\Omega))}{k} \quad [X_\Omega \text{ is convex, hence } X_\Omega \subset N(\delta_\Omega)] \\
& = \frac{\delta_\Omega(k+1)}{k} \quad \text{[by Lemma 3.16]}.
\end{align*}$$

The result follows by taking the infimum over $k$. \qed

Before we move on to the discussion about computations using linearized contact homology, we show one final result which uses only the properties of $S^1$-equivariant symplectic homology.

**Theorem 3.19.** If $(X,\lambda)$ is a Liouville domain, then
(1) $c_k^{\text{GH}}(X) \leq c_k^{S^1}(X)$;
(2) $c_k^{\text{GH}}(X) = c_k^{S^1}(X)$ provided that $X$ is star-shaped.

We now present another approach that can be used to compute $c_L$, using linearized contact homology. This has the disadvantage that at the time of writing, linearized contact homology has not yet been defined in the generality that we need (see Section 4.1 and more specifically Assumption 4.1). Using linearized contact homology, together with an augmentation map, one can define the higher symplectic capacities $g_k$. The key idea is that the capacities $g_k$ can be compared to $\tilde{g}_k$ and $c_k^{\text{GH}}$.

**Theorem 4.35** ([MS22, Section 3.4]). If $X$ is a Liouville domain then

$$\tilde{g}_k(X) \leq g_k(X).$$

**Theorem 4.36.** If $X$ is a Liouville domain such that $\pi_1(X) = 0$ and $2c_1(TX) = 0$ then

$$g_k(X) = c_k^{\text{GH}}(X).$$

These two results show that $\tilde{g}_k(X_{\Omega}) \leq c_k^{\text{GH}}(X_{\Omega})$ (under Assumption 4.1). Using the same proof as before, we conclude that $c_L(X_{\Omega}) = \delta_{\Omega}$.

**Theorem 4.37.** Under Assumption 4.1, if $X_{\Omega}$ is a convex or concave toric domain then

$$c_L(X_{\Omega}) = \delta_{\Omega}.$$

### 1.3 Outline

In Section 2, we review some basics about asymptotically cylindrical holomorphic curves in symplectizations. We start by reviewing the definitions of the various types of symplectic manifolds that we will work with, namely Liouville domains, star-shaped domains and toric domains. After, we consider asymptotically cylindrical holomorphic curves, as well as the moduli spaces that they form. We state the (virtual) dimension formula for these moduli spaces, as well as the SFT compactness theorem, which describes their compactifications. Finally, we give a list of properties of $S^1$-equivariant symplectic homology, which is required to define the Gutt–Hutchings capacities.

Section 3 is about symplectic capacities. The first three subsections are each devoted to defining and proving the properties of a specific capacity, namely the Lagrangian capacity $c_L$, the Gutt–Hutchings capacities $c_k^{\text{GH}}$ and the $S^1$-equivariant symplectic homology capacities $c_k^{S^1}$, and finally the McDuff–Siegel capacities $\tilde{g}_k$. In the subsection about the Lagrangian capacity, we also state the conjecture that we will try to solve in the remainder of the paper, i.e. $c_L(X_{\Omega}) = \delta_{\Omega}$ for a convex or concave toric domain $X_{\Omega}$. The final subsection is devoted to computations. We show that $c_L(X) \leq \inf_k \tilde{g}_k(X)/k$. We use this result to prove the conjecture in the case where $X_{\Omega}$ is 4-dimensional and convex.

Section 4 introduces the linearized contact homology of a nondegenerate Liouville domain. The idea is that using the linearized contact homology, one can define the higher symplectic capacities, which will allow us to prove $c_L(X_{\Omega}) = \delta_{\Omega}$ for any convex or concave toric domain $X_{\Omega}$ (but under the assumption that linearized contact homology and the
augmentation map are well-defined). We give a review of real linear Cauchy–Riemann operators on complex vector bundles, with a special emphasis on criteria for surjectivity in the case where the bundle has complex rank 1. We use this theory to prove that moduli spaces of curves in ellipsoids are transversely cut out and in particular that the augmentation map of an ellipsoid is an isomorphism. The final subsection is devoted to computations. We show that \( g_k(X) = c_k^{GH}(X) \), and use this result to prove our conjecture (again, under Assumption 4.1).

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## 2 Preliminaries on holomorphic curves

### 2.1 Liouville domains

A symplectic cobordism is a compact symplectic manifold \((X, \omega)\) with boundary \(\partial X\), together with a 1-form \(\lambda\) defined on an open neighbourhood of \(\partial X\), such that \(d\lambda = \omega\) and the restriction of \(\lambda\) to \(\partial X\) is a contact form. In this case, we let \(\partial^+ X\) (respectively \(\partial^- X\)) be the subset of \(\partial X\) where the orientation defined by \(\lambda|_{\partial X}\) as a contact form agrees with the boundary orientation (respectively negative boundary orientation). In the case where \(\lambda\) is defined on \(X\), we say that \((X, \lambda)\) is a Liouville cobordism. Finally, a Liouville domain is a Liouville cobordism \((X, \lambda)\) such that \(\partial^- X = \emptyset\).

Consider the canonical symplectic potential of \(\mathbb{C}^n\), given by

\[
\lambda := \frac{1}{2} \sum_{j=1}^n (x^j dy^j - y^j dx^j).
\]

A star-shaped domain is a subset \(X \subset \mathbb{C}^n\) such that \((X, \lambda)\) is a Liouville domain. We will be interested in a further subclass of domains, namely toric domains. To define this notion, first consider the moment map \(\mu: \mathbb{C}^n \rightarrow \mathbb{R}^n_{\geq 0}\), which is given by

\[
\mu(z_1, \ldots, z_n) := \pi(|z_1|^2, \ldots, |z_n|^2),
\]

and define

\[
\Omega_X := \mu(X) \subset \mathbb{R}^n_{\geq 0}, \quad \text{for every } X \subset \mathbb{C}^n,
\]

\[
X_{\Omega} := \mu^{-1}(\Omega) \subset \mathbb{C}^n, \quad \text{for every } \Omega \subset \mathbb{R}^n_{\geq 0},
\]

\[
\delta_\Omega := \sup \{ \alpha \mid (\alpha, \ldots, \alpha) \in \Omega \}, \quad \text{for every } \Omega \subset \mathbb{R}^n_{\geq 0}.
\]

We call \(\delta_\Omega\) the diagonal of \(\Omega\). With this notation, a toric domain is a star-shaped domain \(X\) of the form \(X = X_{\Omega}\) for some \(\Omega \subset \mathbb{R}^n_{\geq 0}\). We say that a toric domain \(X_{\Omega}\) is convex if

\[
\hat{\Omega} := \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid (|x_1|, \ldots, |x_n|) \in \Omega \}
\]
is convex and that it is concave if $\mathbb{R}_{\geq 0}^n \setminus \Omega$ is convex. Some examples of toric domains are the ball $B(a)$, the cylinder $Z(a)$, the ellipsoid $E(a_1, \ldots, a_n)$, the cube $P(a)$ and the nondisjoint union of cylinders $N(a)\footnote{Strictly speaking, $Z(a)$, $N(a)$ are noncompact and $P(a)$, $N(a)$ have corners, so they do not fit into the definition of toric domain. We will mostly ignore this small discrepancy in nomenclature and refer to them as toric domains anyway.}$, which are given by

$$
B(a) := \mu^{-1}(\Omega_{B(a)}), \quad \Omega_{B(a)} := \{ x \in \mathbb{R}_{\geq 0}^n \mid x_1 + \cdots + x_n \leq a \},
$$

$$
Z(a) := \mu^{-1}(\Omega_{Z(a)}), \quad \Omega_{Z(a)} := \{ x \in \mathbb{R}_{\geq 0}^n \mid x_1 \leq a \},
$$

$$
E(a_1, \ldots, a_n) := \mu^{-1}(\Omega_{E(a_1, \ldots, a_n)}), \quad \Omega_{E(a_1, \ldots, a_n)} := \{ x \in \mathbb{R}_{\geq 0}^n \mid x_1/a_1 + \cdots + x_n/a_n \leq 1 \},
$$

$$
P(a) := \mu^{-1}(\Omega_{P(a)}), \quad \Omega_{P(a)} := \{ x \in \mathbb{R}_{\geq 0}^n \mid \forall i = 1, \ldots, n: x_i \leq a \},
$$

$$
N(a) := \mu^{-1}(\Omega_{N(a)}), \quad \Omega_{N(a)} := \{ x \in \mathbb{R}_{\geq 0}^n \mid \exists i = 1, \ldots, n: x_i \leq a \}.
$$

Any Liouville cobordism $(X, \lambda)$ has a Liouville vector field $Z$ which is defined by the equation $\lambda = \iota_Z d\lambda$. If $\varphi : (X, \lambda_X) \longrightarrow (Y, \lambda_Y)$ is an embedding between Liouville cobordisms, we say that $\varphi$ is

1. symplectic if $\varphi^* \lambda_Y - \lambda_X$ is closed;
2. generalized Liouville if $\varphi^* \lambda_Y - \lambda_X$ is closed and $(\varphi^* \lambda_Y - \lambda_X)|_{\partial X}$ is exact;
3. exact symplectic if $\varphi^* \lambda_Y - \lambda_X$ is exact;
4. Liouville if $\varphi^* \lambda_Y - \lambda_X = 0$.

The symplectization of a contact manifold $(M, \alpha)$ is the exact symplectic manifold $\mathbb{R} \times M$ whose symplectic potential is $e^r \alpha$, where $r$ denotes the coordinate on $\mathbb{R}$. If $(X, \omega, \lambda)$ is a symplectic cobordism, the completion of $X$ is given by gluing half-symplectizations at $\partial^+ X$, i.e.

$$
(\hat{X}, \hat{\lambda}) := (\mathbb{R}_{\geq 0} \times \partial^- X, e^r \lambda|_{\partial^- X}) \cup (X, \lambda) \cup (\mathbb{R}_{\geq 0} \times \partial^+ X, e^r \lambda|_{\partial^+ X}).
$$

where $r$ denotes the coordinate on $\mathbb{R}$. If $(X, \lambda_X)$ and $(Y, \lambda_Y)$ are Liouville cobordisms and $\varphi : (X, \lambda_X) \longrightarrow (Y, \lambda_Y)$ is a Liouville embedding such that $Z_X$ is $\varphi$-related to $Z_Y$, then one can define a Liouville embedding $\hat{\varphi} : (\hat{X}, \hat{\lambda}_X) \longrightarrow (\hat{Y}, \hat{\lambda}_Y)$. With these definitions, the operation of taking the completion is actually a functor.

### 2.2 Holomorphic curves

In this section, we review some basics about asymptotically cylindrical holomorphic curves. Standard references for this are [Hof93] and [EGH10]. Our presentation will be based on [Wen16] and [MS22, Section 2.1].

Let $(M, \alpha)$ be a contact manifold and consider its symplectization $(\mathbb{R} \times M, e^r \alpha)$. Recall that $M$ has a Reeb vector field $R$, given by

$$
\iota_R \alpha = 1, \quad \iota_R d\alpha = 0,
$$

and a contact distribution $\xi := \ker \alpha$. An almost complex structure $J$ on $\mathbb{R} \times M$ is cylindrical if $J(\partial_r) = R$, if $J(\xi) \subset \xi$, and if the almost complex structure $J : \xi \longrightarrow \xi$...
is compatible with $d\alpha$ and independent of $r$. Denote by $\mathcal{J}(M)$ the set of such $J$. If $(X,\omega,\lambda)$ is a symplectic cobordism and $J$ is an almost complex structure on $\hat{X}$, we say that $J$ is **cylindrical** if its restriction to each symplectization end is cylindrical. Denote by $\mathcal{J}(X)$ the set of such $J$. If $J\pm\in\mathcal{J}(\partial\pm X)$, denote
\[
\mathcal{J}^+(X) := \{ J \in \mathcal{J}(X) \mid J = J^+ \text{ on } \mathbb{R}_{\geq 0} \times \partial^+ X \},
\]
\[
\mathcal{J}^-(X) := \{ J \in \mathcal{J}(X) \mid J = J^- \text{ on } \mathbb{R}_{\leq 0} \times \partial^- X \},
\]
\[
\mathcal{J}_{J^+}^+(X) := \mathcal{J}^+(X) \cap \mathcal{J}^+(X).
\]

Let $(\Sigma,j)$ be a compact Riemann surface without boundary and $z^\pm := \{ z^\pm_1, \ldots, z^\pm_{p^\pm_0} \} \subset \Sigma$ be finite sets of positive and negative punctures, and denote $\hat{\Sigma} := \Sigma \setminus z^- \cup z^+$. An **asymptotically cylindrical holomorphic curve** is a holomorphic map $u: (\hat{\Sigma},j) \longrightarrow (\hat{X},J)$ such that $u$ is positively (respectively negatively) asymptotic to a Reeb orbit of $\partial^+ X$ (respectively $\partial^- X$) at every $z \in z^+$ (respectively $z \in z^-$). For more details see [Wen16]. We will denote by $\Gamma^\pm = (\gamma^\pm_1, \ldots, \gamma^\pm_{p^\pm_0})$ the tuples of Reeb orbits in $\partial^\pm X$ that $u$ is asymptotic to.

Define a piecewise smooth 2-form $\tilde{\omega} \in \Omega^2(\hat{X})$ by
\[
\tilde{\omega} := \begin{cases} 
    d\lambda|_{\partial^+ X} & \text{on } \mathbb{R}_{\geq 0} \times \partial^+ X, \\
    \omega & \text{on } X, \\
    d\lambda|_{\partial^- X} & \text{on } \mathbb{R}_{\leq 0} \times \partial^- X.
\end{cases}
\]

If $u$ is an asymptotically cylindrical holomorphic curve, its **energies** are given by
\[
E_{\tilde{\omega}}(u) := \int_{\hat{\Sigma}} u^* \tilde{\omega},
\]
\[
E_{\omega}(u) := \int_{\hat{\Sigma}} u^* \omega.
\]

In the case where $(X,\omega,\lambda)$ is a Liouville cobordism, Stokes’ theorem implies that
\[
0 \leq E_{\tilde{\omega}}(u) = A(\Gamma^+) - A(\Gamma^-),
\]
where
\[
A(\Gamma^+) := \sum_{i=1}^{p^+} A(\gamma^+_i), \quad A(\gamma^+_i) := \int_{\gamma^+_i} \lambda|_{\partial^+ X}.
\]

Here, $A(\gamma^+_i)$ is the **action** of the Reeb orbit $\gamma^+_i$. In particular, $u$ must have at least one positive puncture. Another useful result to rule out certain behaviours of holomorphic curves is the **maximum principle**, which works as follows. Suppose that the target of $u$ is a symplectization, i.e. $u = (a,f): \hat{\Sigma} \longrightarrow \mathbb{R} \times M$. The fact that $u$ is holomorphic with respect to a cylindrical almost complex structure implies that $\Delta a \geq 0$, where $\Delta$ denotes the Laplacian. By the maximum principle for elliptic partial differential operators, $a$ cannot have any local maxima. We finish this subsection with a result which we will need to prove Theorem 3.27.
Lemma 2.1. Assume that $\Sigma$ has no positive punctures. Let $(X, \omega, \lambda)$ be a symplectic cobordism, and $J \in J(X)$ be a cylindrical almost complex structure on $\hat{X}$. Assume that the canonical symplectic embedding

$$(\mathbb{R}_{\leq 0} \times \partial^{-} X, d(e^{-} \lambda|_{\partial^{-} X})) \mapsto (\hat{X}, \hat{\omega})$$

can be extended to a symplectic embedding

$$(\mathbb{R}_{\leq K} \times \partial^{-} X, d(e^{-} \lambda|_{\partial^{-} X})) \mapsto (\hat{X}, \hat{\omega})$$

for some $K > 0$. Let $u : \Sigma \to \hat{X}$ be a $J$-holomorphic curve which is negatively asymptotic to a tuple of Reeb orbits $\Gamma$ of $\partial^{-} X$. Consider the energies $E_{\hat{\omega}}(u)$ and $E_{\tilde{\omega}}(u)$. Then,

$$A(\Gamma) \leq \frac{1}{e^{K} - 1} E_{\hat{\omega}}(u),$$

$$E_{\tilde{\omega}}(u) \leq \frac{e^{K}}{e^{K} - 1} E_{\hat{\omega}}(u).$$

Proof. It is enough to show that

$$E_{\hat{\omega}}(u) - E_{\tilde{\omega}}(u) = A(\Gamma),$$

$$E_{\tilde{\omega}}(u) \geq e^{K} A(\Gamma),$$

since these equations imply Equations (5) and (6). Since $u$ has no positive punctures, the maximum principle implies that $u$ is contained in $\mathbb{R}_{\leq 0} \times \partial^{-} X \cup \bar{X}$. We prove Equation (7). For simplicity, denote $M = \partial^{-} X$ and $\alpha = \lambda|_{\partial^{-} X}$.

$$E_{\hat{\omega}}(u) - E_{\tilde{\omega}}(u) = \int_{\Sigma} u^* (\hat{\omega} - \tilde{\omega}) \quad \text{[by definition of } E_{\hat{\omega}} \text{ and } E_{\tilde{\omega}}\text{]}$$

$$= \int_{u^{-1}(\mathbb{R}_{\leq 0} \times M)} u^* d((e^{-} - 1)\alpha) \quad \text{[by definition of } \hat{\omega} \text{ and } \tilde{\omega}\text{]}$$

$$= A(\Gamma) \quad \text{[by Stokes’ theorem]}. $$

We prove Equation (8).

$$E_{\tilde{\omega}}(u) = \int_{\Sigma} u^* \tilde{\omega} \quad \text{[by definition of } E_{\tilde{\omega}}\text{]}$$

$$\geq \int_{u^{-1}(\mathbb{R}_{\leq K} \times M)} u^* d(e^{K} \alpha) \quad \text{[by definition of } \tilde{\omega} \text{ and } u^* \tilde{\omega} \geq 0\text{]}$$

$$= e^{K} \int_{u^{-1}(\{K\} \times M)} u^* \alpha \quad \text{[by Stokes’ theorem]}$$

$$= e^{K} \int_{u^{-1}(\mathbb{R}_{\leq K} \times M)} u^* d\alpha + e^{K} A(\Gamma) \quad \text{[by Stokes’ theorem]}$$

$$\geq e^{K} A(\Gamma) \quad \text{[since } J \text{ is cylindrical]}. $$

2.3 Moduli spaces

Let $\Gamma^{\pm} = (\gamma_{1}^{\pm}, \ldots, \gamma_{p}^{\pm})$ be a tuple of Reeb orbits in $\partial^{\pm} X$ and $J \in J(X)$ be a cylindrical almost complex structure on $\hat{X}$. Define a moduli space

$$\mathcal{M}_{X}^{\pm}(\Gamma^{+}, \Gamma^{-}) := \left\{ (\Sigma, u) \mid \Sigma \text{ is a connected closed Riemann surface of genus } 0 \text{ with punctures } z^{\pm} = \{z_{1}^{\pm}, \ldots, z_{p}^{\pm}\}, \\ u : \Sigma \to \hat{X} \text{ is holomorphic and asymptotic to } \Gamma^{\pm} \right\} / \sim, $$

9
where \((\Sigma_0, u_0) \sim (\Sigma_1, u_1)\) if and only if there exists a biholomorphism \(\phi: \Sigma_0 \to \Sigma_1\) such that \(u_1 \circ \phi = u_0\) and \(\phi(z_{0i}^\pm) = z_{1i}^\pm\) for every \(i = 1, \ldots, p^\pm\). If \(\Gamma^\pm = (\gamma_1^\pm, \ldots, \gamma_p^\pm)\) is a tuple of Reeb orbits on a contact manifold \(M\) and \(J \in \mathcal{J}(M)\), we define a moduli space \(\mathcal{M}_M^{J}(\Gamma^+, \Gamma^-)\) of holomorphic curves in \(\mathbb{R} \times M\) analogously. Since \(J\) is invariant with respect to translations in the \(\mathbb{R}\) direction, \(\mathcal{M}_M^{J}(\Gamma^+, \Gamma^-)\) admits an action of \(\mathbb{R}\) by composition on the target by a translation.

One can try to show that the moduli space \(\mathcal{M}_X^{J}(\Gamma^+, \Gamma^-)\) is transversely cut out by showing that the relevant linearized Cauchy–Riemann operator is surjective at every point of the moduli space. In this case, the moduli space is an orbifold whose dimension is given by the Fredholm index of the linearized Cauchy–Riemann operator. However, since the curves in \(\mathcal{M}_X^{J}(\Gamma^+, \Gamma^-)\) are not necessarily simple, this proof will in general not work, and we cannot say that the moduli space is an orbifold. However, the Fredholm theory part of the proof still works, which means that we still have a dimension formula.

In this case the expected dimension given by the Fredholm theory is usually called a virtual dimension. For the moduli space above, the virtual dimension at a point \(u\) is given by (see [BM04, Section 4])

\[
\text{virdim}_{u} \mathcal{M}_X^{J}(\Gamma^+, \Gamma^-) = (n - 3)(2 - p^+ - p^-) + c_1^\tau(u^*T\hat{X}) + \mu^{\text{CZ}}_\tau(\Gamma^+) - \mu^{\text{CZ}}(\Gamma^-),
\]

where \(\tau\) is a unitary trivialization of the contact distribution over each Reeb orbit, \(c_1^\tau\) is the first Chern class and \(\mu^{\text{CZ}}_\tau(\Gamma)\) is the sum of the Conley–Zehnder indices of the Reeb orbits in \(\Gamma\).

We now discuss curves satisfying a tangency constraint. Our presentation is based on [MS22, Section 2.2] and [CM18, Section 3]. Let \((X, \omega, \lambda)\) be a symplectic cobordism and \(x \in \text{int} X\). A symplectic divisor through \(x\) is a germ of a 2-codimensional symplectic submanifold \(D \subset X\) containing \(x\). A cylindrical almost complex structure \(J \in \mathcal{J}(X)\) is compatible with \(D\) if \(J\) is integrable near \(x\) and \(D\) is holomorphic with respect to \(J\). We denote by \(\mathcal{J}(X, D)\) the set of such almost complex structures. In this case, there are complex coordinates \((z_1, \ldots, z_n)\) near \(x\) such that \(D\) is given by \(h(z_1, \ldots, z_n) = 0\), where \(h(z_1, \ldots, z_n) = z_1\). Let \(u: \Sigma \to X\) be a \(J\)-holomorphic curve together with a marked point \(w \in \Sigma\). For \(k \geq 1\), we say that \(u\) has contact order \(k\) to \(D\) at \(x\) if \(u(w) = x\) and

\[
(h \circ u \circ \varphi)^{(1)}(0) = \cdots = (h \circ u \circ \varphi)^{(k-1)}(0) = 0,
\]

for some local biholomorphism \(\varphi: (\mathbb{C}, 0) \to (\Sigma, w)\). We point out that the condition of having “contact order \(k\)” as written above is equal to the condition of being “tangent of order \(k - 1\)” as defined in [CM18, Section 3]. Following [MS22], we will use the notation \(\langle \mathcal{T}^{(k)}x \rangle\) to denote moduli spaces of curves which have contact order \(k\), i.e. we will denote them by \(\mathcal{M}_X^{J}(\Gamma^+, \Gamma^-)\langle \mathcal{T}^{(k)}x \rangle\) and \(\mathcal{M}_M^{J}(\Gamma^+, \Gamma^-)\langle \mathcal{T}^{(k)}x \rangle\). The virtual dimension is given by (see [MS22, Equation (2.2.1)])

\[
\text{virdim}_{u} \mathcal{M}_X^{J}(\Gamma^+, \Gamma^-)\langle \mathcal{T}^{(k)}x \rangle = (n - 3)(2 - p^+ - p^-) + c_1^\tau(u^*T\hat{X}) + \mu^{\text{CZ}}_\tau(\Gamma^+) - \mu^{\text{CZ}}(\Gamma^-) - 2n - 2k + 4.
\]

We finish this subsection with two lemmas by Cieliebak–Mohnke which we will use in the proof of Theorem 3.27.
Lemma 2.2 ([CM18, Lemma 2.2]). Let $L$ be a compact $n$-dimensional manifold without boundary. Let $\text{Riem}(L)$ be the set of Riemannian metrics on $L$, equipped with the $C^2$-topology. If $g_0 \in \text{Riem}(L)$ is a Riemannian metric of nonpositive sectional curvature and $U \subset \text{Riem}(L)$ is an open neighbourhood of $g_0$, then for all $t_0 > 0$ there exists a Riemannian metric $g \in U$ on $L$ such that with respect to $g$, any closed geodesic $c$ in $L$ of length $\ell(c) \leq t_0$ is noncontractible, nondegenerate, and such that $0 \leq \mu_M(c) \leq n - 1$.

Lemma 2.3 ([CM18, Corollary 3.3]). Let $(L, g)$ be an $n$-dimensional Riemannian manifold with the property that for some $t_0 > 0$, all closed geodesics $\gamma$ of length $\ell(\gamma) \leq t_0$ are noncontractible and nondegenerate and have Morse index $\mu_M(\gamma) \leq n - 1$. Let $x \in T^*L$ and $D$ be a symplectic divisor through $x$. For generic $J$ every (not necessarily simple) punctured $J$-holomorphic sphere $\hat{C}$ in $T^*L$ which is asymptotic at the punctures to geodesics of length $\leq t_0$ and which has contact order $\hat{k}$ to $D$ at $x$ must have at least $\hat{k} + 1$ punctures.

### 2.4 SFT compactness

In this subsection we present the SFT compactness theorem, which describes the compactifications of the moduli spaces of the previous subsection. This theorem was first proven by Bourgeois–Eliashberg–Hofer–Wysocki–Zehnder [BEH+03]. Cieliebak–Mohrke [CM05] have given a proof of this theorem using different methods. Our presentation is based primarily on [CM18] and [MS22].

Let $(X, \omega, \lambda)$ be a symplectic cobordism and choose almost complex structures $J^{\pm} \in J(\partial^{\pm}X)$ and $J \in J^+_l(X)$. Let $\Gamma^{\pm} = (\gamma_1^{\pm}, \ldots, \gamma_p^{\pm})$ be a tuple of Reeb orbits in $\partial^{\pm}X$. For $1 \leq L \leq N$, let $\alpha^{\pm} = \lambda|_{\partial^{\pm}X}$ and define

\[
(X^{\nu}, \omega^{\nu}, \bar{\omega}^{\nu}, J^{\nu}) := \begin{cases} 
(\mathbb{R} \times \partial^-X, d(e^\nu \alpha^-), d\alpha^-, J^-) & \text{if } \nu = 1, \ldots, L - 1, \\
(\hat{X}, \hat{\omega}, \bar{\omega}, J) & \text{if } \nu = L, \\
(\mathbb{R} \times \partial^+X, d(e^\nu \alpha^+), d\alpha^+, J^+) & \text{if } \nu = L + 1, \ldots, N,
\end{cases}
\]

\[
(X^*, \omega^*, \bar{\omega}^*, J^*) := \prod_{\nu=1}^{N} (X^{\nu}, \omega^{\nu}, \bar{\omega}^{\nu}, J^{\nu}).
\]

The moduli space of holomorphic buildings, denoted $\mathcal{M}_X^J(\Gamma^+, \Gamma^-)$, is the set of tuples $F = (F^1, \ldots, F^N)$, where $F^{\nu} : \Sigma^\nu \to X^\nu$ is an asymptotically cylindrical nodal $J^{\nu}$-holomorphic curve in $X^\nu$ with sets of asymptotic Reeb orbits $\Gamma^{\nu}_\nu$. Here, each $F^{\nu}$ is possibly disconnected and if $X^\nu$ is a symplectization then $F^{\nu}$ is only defined up to translation in the $\mathbb{R}$ direction. We assume in addition that $F$ satisfies the following conditions.

1. The tuples of asymptotic Reeb orbits $\Gamma^{\pm}_{\nu}$ are such that

\[
\Gamma^- = \Gamma^-_1, \quad \Gamma^+_N = \Gamma^+_1, \quad \Gamma^+_\nu = \Gamma^-_{\nu+1} \quad \text{for every } \nu = 1, \ldots, N - 1.
\]

2. Define the graph of $F$ to be the graph whose vertices are the components of $F^1, \ldots, F^N$ and whose edges are determined by the asymptotic Reeb orbits. Then the graph of $F$ is a tree.
(3) The building $F$ has no symplectization levels consisting entirely of trivial cylinders, and any constant component of $F$ has negative Euler characteristic after removing all special points.

The **energy** of a holomorphic building $F = (F^1, \ldots, F^N)$ is $E_{\omega^*}(F) := \sum_{\nu=1}^N E_{\omega^\nu}(F^\nu)$. The moduli space $\mathcal{M}^i_X(\Gamma^+, \Gamma^-)$ admits a metrizable topology (see [BO16, Appendix B]). With this language, the SFT compactness theorem can be stated by saying that $\mathcal{M}^i_X(\Gamma^+, \Gamma^-)$ is compact.

We now consider the case where the almost complex structure on $\hat{X}$ is replaced by a family of almost complex structures obtained via **neck stretching**. Let $(X^\pm, \omega^\pm, \lambda^\pm)$ be symplectic cobordisms with common boundary

$$(M, \alpha) = (\partial^- X^+, \lambda^+|_{\partial^- X^+}) = (\partial^+ X^-, \lambda^-|_{\partial^+ X^-}).$$

Choose almost complex structures $J_M \in \mathcal{J}(M)$, $J_+ \in \mathcal{J}^{J_M}(X^+)$, and $J_- \in \mathcal{J}^{J_M}(X^-)$ and denote by $J_{\partial^\pm X^\pm} = J(\partial^\pm X^\pm)$ the induced cylindrical almost complex structure on $\mathbb{R} \times \partial^\pm X^\pm$. Let $(X, \omega, \lambda) := (X^-, \omega^-, \lambda^-) \oplus (X^+, \omega^+, \lambda^+)$ be the gluing of $X^-$ and $X^+$ along $M$. We wish to define a family of almost complex structures $(J_t)_{t \in \mathbb{R}_{\geq 0}} \subset \mathcal{J}(X)$. For every $t \geq 0$, let

$$X_t := X \cup_M [-t, 0] \times M \cup_M X^+.$$

There exists a canonical diffeomorphism $\phi_t : X \to X_t$. Define an almost complex structure $J_t$ on $X_t$ by

$$J_t := \begin{cases} J^\pm & \text{on } X^\pm, \\ J_M & \text{on } [-t, 0] \times M. \end{cases}$$

Denote also by $J_t$ the pullback of $J_t$ to $X$ along $\phi_t$, as well as the induced almost complex structure on the completion $\hat{X}$. Finally, consider the moduli space

$$\mathcal{M}^{(J_t)}_X(\Gamma^+, \Gamma^-) := \prod_{t \in \mathbb{R}_{\geq 0}} \mathcal{M}^{J_t}_X(\Gamma^+, \Gamma^-).$$

As before, we wish to define a suitable compactification for $\mathcal{M}^{(J_t)}_X(\Gamma^+, \Gamma^-)$. For $1 \leq L^- < L^+ \leq N$, let $\alpha^\pm := \lambda^\pm|_{\partial^\pm X^\pm}$ and define

$$(X^\nu, \omega^\nu, \tilde{\omega}^\nu, J^\nu) := \begin{cases} (\mathbb{R} \times \partial^- X^-, \dd(e^{\nu} \alpha^-), \dd \alpha^-, J_{\partial^- X^-}) & \text{if } \nu = 1, \ldots, L^- - 1, \\ (X^-, \omega^-, \tilde{\omega}^-, J^-) & \text{if } \nu = L^-, \\ (\mathbb{R} \times M, \dd(e^{\nu} \alpha), \dd \alpha, J_M) & \text{if } \nu = L^- + 1, \ldots, L^+ - 1, \\ (X^+, \omega^+, \tilde{\omega}^+, J^+) & \text{if } \nu = L^+, \\ (\mathbb{R} \times \partial^+ X^+, \dd(e^{\nu} \alpha^+), \dd \alpha^+, J_{\partial^+ X^+}) & \text{if } \nu = L^+ + 1, \ldots, N, \\ \prod_{\nu=1}^N (X^\nu, \omega^\nu, \tilde{\omega}^\nu, J^\nu). & \end{cases}$$

Define $\mathcal{M}^{(J_t)}_X(\Gamma^+, \Gamma^-)$ to be the set of tuples $F = (F^1, \ldots, F^N)$, where $F^\nu : \hat{\Sigma}^\nu \to X^\nu$ is an asymptotically cylindrical nodal $J^\nu$-holomorphic curve in $X^\nu$ with sets of asymptotic Reeb orbits $\Gamma^\nu_\pm$, such that $F$ satisfies conditions analogous to those of Items (1) to (3). Then, $\mathcal{M}^{(J_t)}_X(\Gamma^+, \Gamma^-)$ is compact.
Remark 2.4. The discussion above also applies to compactifications of moduli spaces of curves satisfying tangency constraints. The compactification $\mathcal{M}_X^j(\Gamma^+, \Gamma^-)\langle T^{(k)}x \rangle$ consists of buildings $F = (F^1, \ldots, F^N) \in \mathcal{M}_X(\Gamma^+, \Gamma^-)$ such that exactly one component $C$ of $F$ inherits the tangency constraint $\langle T^{(k)}x \rangle$, and which satisfy the following additional condition. Consider the graph obtained from the graph of $F$ by collapsing adjacent constant components to a point. Let $C_1, \ldots, C_p$ be the (necessarily nonconstant) components of $F$ which are adjacent to $C$ in the new graph. Then we require that there exist $k_1, \ldots, k_p \in \mathbb{Z}_{\geq 1}$ such that $k_1 + \cdots + k_p \geq k$ and $C_i$ satisfies the constraint $\langle T^{(k_i)}x \rangle$ for every $i = 1, \ldots, p$. This definition is natural to consider by [CM07, Lemma 7.2]. We can define $\mathcal{M}_X^j(\Gamma^+, \Gamma^-)\langle T^{(k)}x \rangle$ analogously.

Remark 2.5. We point out that in [MS22, Definition 2.2.1], the compactification of Remark 2.4 is denoted by $\mathcal{M}_X^j(\Gamma^+, \Gamma^-)\langle T^{(k)}x \rangle$, while the notation $\mathcal{M}_X(\Gamma^+, \Gamma^-)\langle T^{(k)}x \rangle$ is used to denote the moduli space of buildings $F = (F^1, \ldots, F^N) \in \mathcal{M}_X(\Gamma^+, \Gamma^-)$ such that exactly one component $C$ of $F$ inherits the tangency constraint $\langle T^{(k)}x \rangle$, but which do not necessarily satisfy the additional condition of Remark 2.4.

The following lemma will be useful to us in proving Theorem 3.27.

Lemma 2.6 ([CM18, Lemma 2.8]). The homology class $A := [F] \in H_2(X; \mathbb{Z})$ of a nonconstant broken holomorphic curve $F: (\Sigma^*, j) \longrightarrow (X^*, J^*)$ satisfies $\omega(A) > 0$.

2.5 $S^1$-equivariant symplectic homology

If $(X, \lambda)$ is a nondegenerate Liouville domain, one can define its $S^1$-equivariant symplectic homology, denoted $SH^{S^1}(X, \lambda)$. The presentation we will give will be based on [GH18]. Other references discussing $S^1$-equivariant symplectic homology are [Gut14, Gut17, BO13, BO10, BO16, Sei08]. The $S^1$-equivariant symplectic homology is a $\mathbb{Q}$-module which has the following structural properties.

1. **Action filtration.** For every $a, b \in \mathbb{R}$ we have $\mathbb{Q}$-modules and maps

   $\iota^a: SH^{S^1, a}(X, \lambda) \longrightarrow SH^{S^1}(X, \lambda),$

   $\iota^{b,a}: SH^{S^1, a}(X, \lambda) \longrightarrow SH^{S^1, b}(X, \lambda),$

   which compose in a functorial way. In particular, we can define the $S^1$-equivariant symplectic homology associated to intervals $(a, b] \subset \mathbb{R}$ and $(a, +\infty) \subset \mathbb{R}$ by taking the quotient:

   $\quad SH^{S^1, (a, b]}(X, \lambda) := SH^{S^1, b}(X, \lambda)/\iota^{b,a}(SH^{S^1, a}(X, \lambda)),$

   $\quad SH^{S^1, (a, +\infty)}(X, \lambda) := SH^{S^1}(X, \lambda)/\iota^a(SH^{S^1, a}(X, \lambda)).$

   The positive $S^1$-equivariant symplectic homology is given by $SH^{S^1, +}(X, \lambda) = SH^{S^1, (\varepsilon, +\infty)}(X, \lambda)$, where $\varepsilon$ is half of the minimal action of a Reeb orbit in $\partial X$.

2. **U map.** There is a map $U: SH^{S^1}(X, \lambda) \longrightarrow SH^{S^1}(X, \lambda)$ which respects the action filtration, i.e. there exist maps $U^a: SH^{S^1, a}(X, \lambda) \longrightarrow SH^{S^1, a}(X, \lambda)$ such that $\iota^a \circ U^a = U \circ \iota^a$ and $\iota^{b,a} \circ U^a = U^b \circ \iota^{b,a}$.
(3) $\delta$ map. There is a map $\delta : SH^{S^1}(X, \lambda) \to H_\bullet(\mathbb{B}S^1; \mathbb{Q}) \otimes H_\bullet(X, \partial X; \mathbb{Q})$, which is of the form $\delta := \alpha \circ \delta_0$. Here, $\alpha : SH^{S^1, e}(X) \to H_\bullet(\mathbb{B}S^1; \mathbb{Q}) \otimes H_\bullet(X, \partial X; \mathbb{Q})$ is an isomorphism and $\delta_0$ is the continuation map of the long exact homology sequence

$$\cdots \to SH^{S^1}(X) \to SH^{S^1, +}(X) \xrightarrow{\delta_0} SH^{S^1, e}(X) \to \cdots$$

(4) Viterbo transfer map. If $\varphi : (X, \lambda_X) \to (Y, \lambda_Y)$ is a generalized Liouville embedding with $\varphi(X) \subset \text{int}(Y)$, one can define a map $\varphi_! : SH^{S^1}(Y) \to SH^{S^1}(X)$. This map has the following properties. First, $\varphi_!$ commutes with the action filtration, in the sense that for each $a \in \mathbb{R}$ there exists $\varphi^a_! : SH^{S^1, a}(Y) \to SH^{S^1, a}(X)$ such that $\iota^a_X \circ \varphi^a_! = \varphi_! \circ \iota^a_Y$ and $\iota^{b,a}_X \circ \varphi^a_! = \varphi^b_! \circ \iota^{b,a}_Y$. Second, $\varphi_!$ commutes with the $U$ maps, i.e. $\varphi^a_! \circ U^b_Y = U^b_X \circ \varphi^a_!$. Finally, $\varphi_!$ commutes with the $\delta$ map, i.e. $\delta_X \circ \varphi_! = (1 \otimes \rho) \circ \delta_Y$, where $\rho : H_\bullet(Y, \partial Y; \mathbb{Q}) \to H_\bullet(X, \partial X; \mathbb{Q})$ is the composition

$$H_\bullet(Y, \partial Y; \mathbb{Q}) \xrightarrow{\rho} H_\bullet(Y, Y \setminus \varphi(\text{int} X); \mathbb{Q}) \xleftarrow{\rho} H_\bullet(X, \partial X; \mathbb{Q})$$

(5) Grading. In the case where $\pi_1(X) = 0$ and $c_1(TX)|_{\pi_2(X)} = 0$, the $S^1$-equivariant symplectic homology admits an integer grading. With respect to this grading, the maps $\iota^a$, $\iota^{b,a}$ and $\varphi_!$ are of degree 0 and the $U$ map is of degree $-2$.

(6) Star-shaped domains. Suppose that $(X, \lambda)$ is a star-shaped domain. Then,

$$SH^{S^1}_\bullet(X, \lambda) \cong \begin{cases} \mathbb{Q} & \text{if } \bullet \in n - 1 + 2\mathbb{Z}_{\geq 1}, \\ 0 & \text{otherwise} \end{cases}$$

and $\delta : SH^{S^1}_{n-1+2k}(X, \lambda) \to H_{2k-2}(\mathbb{B}S^1; \mathbb{Q}) \otimes H_{2n}(X, \partial X; \mathbb{Q})$ is an isomorphism.

3 Computation using only classical transversality

3.1 Lagrangian capacity

Here, we define the Lagrangian capacity (Definition 3.3) and state its properties (Proposition 3.4). One of the main goals of this paper is to study whether the Lagrangian capacity can be computed in some cases, for example for toric domains. In the end of the section, we state some easy inequalities concerning the Lagrangian capacity (Lemmas 3.6 and 3.7), known computations (Propositions 3.8 and 3.9) and finally the main conjecture of this paper (Conjecture 3.11), which is inspired by all the previous results. The Lagrangian capacity is defined in terms of the minimal area of Lagrangian submanifolds, which we now define.

Definition 3.1. Let $(X, \omega)$ be a symplectic manifold. If $L$ is a Lagrangian submanifold of $X$, then we define the minimal symplectic area of $L$, denoted $A_{\min}(L)$, by

$$A_{\min}(L) := \inf\{\omega(\sigma) \mid \sigma \in \pi_2(X, L), \omega(\sigma) > 0\}.$$
Lemma 3.2. Let \((X, \lambda)\) be an exact symplectic manifold and \(L \subset X\) be a Lagrangian submanifold. If \(\pi_1(X) = 0\), then
\[
A_{\min}(L) = \inf \{\lambda(\rho) \mid \rho \in \pi_1(L), \lambda(\rho) > 0\}.
\]

Proof. The diagram
\[
\begin{array}{ccc}
\pi_2(X, L) & \xrightarrow{\partial} & \pi_1(L) \\
\omega & \downarrow & \lambda \\
& \downarrow \iota & \pi_1(X) \\
& \iota \circ \lambda & \iota
\end{array}
\]
commutes, where \(\partial([\sigma]) = [\sigma|_{S^1}]\), and the top row is exact. \(\Box\)

Definition 3.3 ([CM18, Section 1.2]). Let \((X, \omega)\) be a symplectic manifold. We define the Lagrangian capacity of \((X, \omega)\), denoted \(c_L(X, \omega)\), by
\[
c_L(X, \omega) := \sup \{A_{\min}(L) \mid L \subset X \text{ is an embedded Lagrangian torus}\}.
\]

Proposition 3.4 ([CM18, Section 1.2]). The Lagrangian capacity \(c_L\) satisfies:

- **(Monotonicity)** If \((X, \omega_X) \longrightarrow (Y, \omega_Y)\) is a symplectic embedding with \(\pi_2(Y, \iota(X)) = 0\), then \(c_L(X, \omega_X) \leq c_L(Y, \omega_Y)\).

- **(Conformality)** If \(\alpha \neq 0\), then \(c_L(X, \alpha \omega) = |\alpha| c_L(X, \omega)\).

We now wish to show that if \(X_{\Omega}\) is a convex or concave toric domain, then \(c_L(X_{\Omega}) \geq \delta_{\Omega} := \sup \{a \mid (a, \ldots, a) \in \Omega\}\). For this, we consider the following symplectic capacity.

Definition 3.5 ([GH18, Definition 1.17]). If \((X, \omega)\) is a symplectic manifold, its cube capacity is given by
\[
c_P(X, \omega) := \sup \{a \mid \text{there exists a symplectic embedding } P(a) \longrightarrow X\}.
\]

Lemma 3.6. If \(X\) is a star-shaped domain, then \(c_L(X) \geq c_P(X)\).

Proof. Let \(\iota: P(a) \longrightarrow X\) be a symplectic embedding, for some \(a > 0\). We want to show that \(c_L(X) \geq a\). Define \(T = \mu^{-1}(a, \ldots, a) \subset \partial P(a)\) and \(L = \iota(T)\). Then,
\[
c_L(X) \geq A_{\min}(L) \quad \text{[by definition of } c_L]\nn = A_{\min}(T) \quad \text{[since } X\text{ is star-shaped]}\nn = a \quad \text{[by Lemma 3.2]}.
\]

\(\Box\)

Lemma 3.7. If \(X_{\Omega}\) is a convex or concave toric domain, then \(c_P(X_{\Omega}) \geq \delta_{\Omega}\).

Proof. Since \(X_{\Omega}\) is a convex or concave toric domain, we have that \(P(\delta_{\Omega}) \subset X_{\Omega}\). The result follows by definition of \(c_P\). \(\Box\)
Actually, Gutt–Hutchings show that \( c_P(X_\Omega) = \delta_\Omega \) for any convex or concave toric domain \( X_\Omega \) ([GH18, Theorem 1.18]). However, for our purposes we will only need the inequality in Lemma 3.7. We now consider the results by Cieliebak–Mohnke for the Lagrangian capacity of the ball and the cylinder.

**Proposition 3.8** ([CM18, Corollary 1.3]). The Lagrangian capacity of the ball is

\[
c_L(B(1)) = \frac{1}{n}.
\]

**Proposition 3.9** ([CM18, p. 215-216]). The Lagrangian capacity of the cylinder is

\[
c_L(Z(1)) = 1.
\]

By Lemmas 3.6 and 3.7, if \( X_\Omega \) is a convex or concave toric domain then \( c_L(X_\Omega) \geq \delta_\Omega \). But as we have seen in Propositions 3.8 and 3.9, if \( X_\Omega \) is the ball or the cylinder then \( c_L(X_\Omega) = \delta_\Omega \). This motivates Conjecture 3.10 below for the Lagrangian capacity of an ellipsoid, and more generally Conjecture 3.11 below for the Lagrangian capacity of any convex or concave toric domain.

**Conjecture 3.10** ([CM18, Conjecture 1.5]). The Lagrangian capacity of the ellipsoid is

\[
c_L(E(a_1, \ldots, a_n)) = \left( \frac{1}{a_1} + \cdots + \frac{1}{a_n} \right)^{-1}.
\]

**Conjecture 3.11.** If \( X_\Omega \) is a convex or concave toric domain then

\[
c_L(X_\Omega) = \delta_\Omega.
\]

In Theorems 3.28 and 4.37 we present our results concerning Conjecture 3.11.

### 3.2 Gutt–Hutchings capacities

In this subsection we will define the Gutt–Hutchings capacities (Definition 3.12) and the \( S^1 \)-equivariant symplectic homology capacities (Definition 3.17), and list their properties (Theorems 3.14 and 3.18 respectively). We will also compare the two capacities (Theorem 3.19). The definition of these capacities relies on \( S^1 \)-equivariant symplectic homology. In the commutative diagram below, we display the modules and maps which will play a role in this subsection, for a nondegenerate Liouville domain \( X \).

\[
\begin{array}{cccc}
SH^{S^1,(\epsilon,a)}(X) & \overset{\delta_0}{\longrightarrow} & SH^{S^1,\varepsilon}(X) & \overset{\iota_0}{\longrightarrow} & SH^{S^1,a}(X) \\
\downarrow{\alpha} & & \downarrow{\iota} & & \downarrow{\alpha} \\
SH^{S^1,\varepsilon}(X) & \overset{\delta_0}{\longrightarrow} & H_*(BS^1; \mathbb{Q}) \otimes H_*(X, \partial X; \mathbb{Q})
\end{array}
\]

We point out that every vertex in the above diagram has a \( U \) map and every map in the diagram commutes with this \( U \) map. Specifically, all the \( S^1 \)-equivariant symplectic
homologies have the $U$ map, and $H_\bullet(BS^1; \mathbb{Q}) \otimes H_\bullet(X, \partial X; \mathbb{Q}) \cong \mathbb{Q}[u] \otimes H_\bullet(X, \partial X; \mathbb{Q})$ has the map $U := u^{-1} \otimes \text{id}$. We will also make use of a version of Diagram (10) in the case where $X$ is star-shaped, namely Diagram (11) below. In this case, the modules in the diagram admit gradings and every map is considered to be a map in a specific degree. By [GH18, Proposition 3.1], $\delta$ and $\delta_0$ are isomorphisms.

![Diagram](image)

**Definition 3.12** ([GH18, Definition 4.1]). If $k \in \mathbb{Z}_{\geq 1}$ and $(X, \lambda)$ is a nondegenerate Liouville domain, the Gutt–Hutchings capacities of $X$, denoted $c_k^{GH}(X)$, are defined as follows. Consider the map

$$
\delta \circ U^{k-1} \circ \iota^a : SH_{{\mathcal{S}},(\varepsilon,a)}(X) \longrightarrow H_\bullet(BS^1; \mathbb{Q}) \otimes H_\bullet(X, \partial X; \mathbb{Q})
$$

from Diagram (10). Then, we define

$$
c_k^{GH}(X) := \inf\{ a > 0 \mid \text{im}(\delta \circ U^{k-1} \circ \iota^a) \cap [pt] \otimes [X] \}
$$

**Remark 13.** In this paper, we consider symplectic capacities $c_k^{GH}$ (see Definition 3.12), $c_k^S$ (see Definition 3.17), $\tilde{g}_k$ (see Definition 3.21) and $g_k$ (see Definition 4.8). All these capacities are defined for nondegenerate Liouville domains, but their definition can be extended to Liouville domains which are not necessarily nondegenerate as in [GH18, Section 4.2]. In addition, if we wish to prove inequalities involving the capacities above, it will be enough to prove these inequalities for Liouville domains which are nondegenerate.

**Theorem 3.14** ([GH18, Theorem 1.24]). The functions $c_k^{GH}$ of Liouville domains satisfy the following axioms, for all equidimensional Liouville domains $(X, \lambda_X)$ and $(Y, \lambda_Y)$:

- **(Monotonicity)** If $X \longrightarrow Y$ is a generalized Liouville embedding then $c_k^{GH}(X) \leq c_k^{GH}(Y)$.

- **(Conformality)** If $\alpha > 0$ then $c_k^{GH}(X, \alpha \lambda_X) = \alpha c_k^{GH}(X, \lambda_X)$.

- **(Nondecreasing)** $c_1^{GH}(X) \leq c_2^{GH}(X) \leq \cdots \leq +\infty$.

- **(Reeb orbits)** If $c_k^{GH}(X) < +\infty$, then $c_k^{GH}(X) = \mathcal{A}(\gamma)$ for some Reeb orbit $\gamma$ which is contractible in $X$.

The following lemma provides an alternative definition of $c_k^{GH}$, in the spirit of [FHW94].

**Lemma 3.15.** Let $(X, \lambda)$ be a nondegenerate Liouville domain such that $\pi_1(X) = 0$ and $c_1(TX)|_{\pi_1(X)} = 0$. Let $E \subset \mathbb{C}^n$ be a nondegenerate star-shaped domain and suppose that $\phi : E \longrightarrow X$ is a symplectic embedding. Consider the map

$$
SH_{n-1+2k}^{S^1,(\varepsilon,a)}(X) \xrightarrow{\iota^a} SH_{n-1+2k}^{S^1,+}(X) \xrightarrow{\phi^!} SH_{n-1+2k}^{S^1,+}(E)
$$

Then, $c_k^{GH}(X) = \inf\{a > 0 \mid \phi^! \circ \iota^a \text{ is nonzero}\}$. 

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Proof. For every \( a \in \mathbb{R} \) consider the following commutative diagram:

\[
\begin{align*}
SH_{n-1+2k}^{S^1,(e,a)}(X) \xrightarrow{\phi^*} SH_{n-1+2k}^{S^1+}(X) \xrightarrow{\phi \rho^{-1}} SH_{n+1}^{S^1}(X) \xrightarrow{\delta_E} H_0(BS^1) \otimes H_{2n}(X,\partial X) \\
SH_{n-1+2k}^{S^1,(e,a)}(E) \xrightarrow{\phi^*} SH_{n-1+2k}^{S^1+}(E) \xrightarrow{\phi \rho^{-1}} SH_{n+1}^{S^1}(E) \xrightarrow{\delta_E} H_0(BS^1) \otimes H_{2n}(E,\partial E)
\end{align*}
\]

By [GH18, Proposition 3.1] and since \( E \) is star-shaped, the maps \( U_E \) and \( \delta_E \) are isomorphisms. Since \( \rho([X]) = [E] \), the map \( \rho \) is an isomorphism. By definition, \( c_{k}^{GH} \) is the infimum over \( a \) such that the top arrow is surjective. This condition is equivalent to \( \phi_1 \circ \iota_X^k \) being nonzero.

The following computation will be useful to us in the proofs of Theorems 3.28 and 4.37.

**Lemma 3.16** ([GH18, Lemma 1.19]). \( c_{k}^{GH}(N(a)) = a(k + n - 1) \).

We now consider other capacities which can be defined using \( S^1 \)-equivariant symplectic homology.

**Definition 3.17** ([Iri21, Section 2.5]). If \( k \in \mathbb{Z}_{\geq 1} \) and \((X, \lambda)\) is a nondegenerate Liouville domain, the \( S^1 \)-equivariant symplectic homology capacities of \( X \), denoted \( c_k^{S^1}(X) \), are defined as follows. Consider the map

\[
\iota^a \phi: H_*(BS^1; \mathbb{Q}) \otimes H_*(X, \partial X; \mathbb{Q}) \rightarrow SH^{S^1,a}(X)
\]

from Diagram (10). Then, we define

\[
c_k^{S^1}(X) := \inf\{ a > 0 \mid \iota^a \phi^{-1}([CP^{k-1}] \otimes [X]) = 0 \}.
\]

We now state the properties that the capacities \( c_k^{S^1} \) satisfy. For the sake of completeness, we include proofs as well.

**Theorem 3.18.** The functions \( c_k^{S^1} \) of Liouville domains satisfy the following axioms, for all Liouville domains \((X, \lambda_X)\) and \((Y, \lambda_Y)\) of the same dimension:

- **(Monotonicity)** If \( X \rightarrow Y \) is a generalized Liouville embedding then \( c_k^{S^1}(X) \leq c_k^{S^1}(Y) \).

- **(Conformality)** If \( \mu > 0 \) then \( c_k^{S^1}(X, \mu \lambda_X) = \mu c_k^{S^1}(X, \lambda_X) \).

- **(Nondecreasing)** \( c_1^{S^1}(X) \leq c_2^{S^1}(X) \leq \cdots \leq +\infty \).

**Proof.** We prove monotonicity. Consider the following commutative diagram:

\[
\begin{align*}
H_*(BS^1; \mathbb{Q}) \otimes H_*(Y, \partial Y; \mathbb{Q}) \xrightarrow{\iota_Y^a \phi} SH^{S^1,a}(Y) \xrightarrow{\phi^*} SH^{S^1,a}(X) \\
H_*(BS^1; \mathbb{Q}) \otimes H_*(X, \partial X; \mathbb{Q}) \xrightarrow{\iota_X^a \phi} SH^{S^1,a}(X)
\end{align*}
\]

If \( \iota_Y^a \phi \circ \alpha_Y^{-1}([CP^{k-1}] \otimes [Y]) = 0 \), then

\[
\iota_X^a \phi \circ \alpha_X^{-1}([CP^{k-1}] \otimes [X]) = 0.
\]
= \iota_X^{a,\varepsilon} \circ \alpha_X^{-1} \circ (\text{id} \otimes \rho)([\mathbb{CP}^{k-1}] \otimes [Y]) \quad \text{[since } \rho([Y]) = [X]\text{]} \\
= \phi_t \circ \iota_{\bar{Y}}^{c,\varepsilon} \circ \alpha_{\bar{Y}}^{-1}( [\mathbb{CP}^{k-1}] \otimes [Y]) \quad \text{[by Diagram (12)]} \\
= 0 \quad \text{[by assumption].}

To prove conformality, choose \( \varepsilon > 0 \) such that \( \varepsilon, \mu \varepsilon < \min \text{Spec}(\partial X, \lambda |_{\partial X}) \). Since the diagram

\[
\begin{array}{ccc}
H_\bullet(BS^1; \mathbb{Q}) \otimes H_\bullet(X, \partial X; \mathbb{Q}) & \xrightarrow{\alpha_{\lambda}} & SH^{S^1, \varepsilon}(X, \lambda) \\
& & \xrightarrow{\iota_X^{a,\varepsilon}} \xrightarrow{\iota_X^{a,\mu}} SH^{S^1, \alpha}(X, \mu \lambda)
\end{array}
\]

commutes (by \([GH18, \text{Proposition 3.1}]\), the result follows.

To prove the nondecreasing property, note that if \( \iota^{a,\varepsilon} \circ \alpha^{-1}( [\mathbb{CP}^k] \otimes [X]) = 0 \), then

\[
\iota^{a,\varepsilon} \circ \alpha^{-1}( [\mathbb{CP}^{k-1}] \otimes [X]) = 0 \quad \text{[by definition of kernel]}
\]

\[
\Rightarrow \alpha^{-1}( [\mathbb{CP}^{k-1}] \otimes [X]) \in \ker (\iota^{a,\varepsilon}) \quad \text{[by definition of kernel]}
\]

\[
\Rightarrow (\iota^{a,\varepsilon} \circ \alpha^{-1}( [\mathbb{CP}^{k-1}] \otimes [X])) \in \text{im } \delta_0 \quad \text{[since the top row of (10) is exact]}
\]

\[
\Rightarrow [\mathbb{CP}^{k-1}] \otimes [X] \in \text{im } (\alpha \circ \delta_0) \quad \text{[by definition of image]}
\]

\[
\Rightarrow [\mathbb{CP}^{k-1}] \otimes [X] \in \text{im } (\delta \circ \iota^{a,\varepsilon}) \quad \text{[since Diagram (10) commutes]}
\]

\[
\Rightarrow [\text{pt}] \otimes [X] \in \text{im } (U^{k-1} \circ \delta \circ \iota^a) \quad \text{[since } U^{k-1}( [\mathbb{CP}^{k-1}] \otimes [X]) = [\text{pt}] \otimes [X]\text{]}
\]

\[
\Rightarrow [\text{pt}] \otimes [X] \in \text{im } (\delta \circ U^{k-1} \circ \iota^a) \quad \text{[since } \delta \text{ and } U \text{ commute, by assumption].}
\]

\[\text{Theorem 3.19. If } (X, \lambda) \text{ is a Liouville domain, then}
\]

(1) \( c_k^{GH}(X) \leq c_k^{S^1}(X) \);

(2) \( c_k^{GH}(X) = c_k^{S^1}(X) \) provided that \( X \) is star-shaped.

\[\text{Proof. By Remark 3.13, we may assume that } X \text{ is nondegenerate. Since}
\]

\[
\iota^{a,\varepsilon} \circ \alpha^{-1}( [\mathbb{CP}^{k-1}] \otimes [X]) = 0
\]

\[
\quad \Leftrightarrow \alpha^{-1}( [\mathbb{CP}^{k-1}] \otimes [X]) \in \ker (\iota^{a,\varepsilon}) \quad \text{[by definition of kernel]}
\]

\[
\quad \Leftrightarrow (\iota^{a,\varepsilon} \circ \alpha^{-1}( [\mathbb{CP}^{k-1}] \otimes [X])) \in \text{im } \delta_0 \quad \text{[since the top row of (10) is exact]}
\]

\[
\quad \Leftrightarrow [\mathbb{CP}^{k-1}] \otimes [X] \in \text{im } (\alpha \circ \delta_0) \quad \text{[by definition of image]}
\]

\[
\quad \Leftrightarrow [\mathbb{CP}^{k-1}] \otimes [X] \in \text{im } (\delta \circ \iota^{a,\varepsilon}) \quad \text{[since Diagram (10) commutes]}
\]

\[
\quad \Rightarrow [\text{pt}] \otimes [X] \in \text{im } (U^{k-1} \circ \delta \circ \iota^a) \quad \text{[since } U^{k-1}( [\mathbb{CP}^{k-1}] \otimes [X]) = [\text{pt}] \otimes [X]\text{]}
\]

\[
\quad \Rightarrow [\text{pt}] \otimes [X] \in \text{im } (\delta \circ U^{k-1} \circ \iota^a) \quad \text{[since } \delta \text{ and } U \text{ commute, by assumption].}
\]

we have that \( c_k^{GH}(X) \leq c_k^{S^1}(X) \). If \( X \) is a star-shaped domain, we can view the maps of the computation above as being the maps in Diagram (11), i.e. they are defined in a specific degree. In this case, \( U^{k-1} : H_{2k-2}(BS^1) \otimes H_{2n}(X, \partial X) \to H_0(BS^1) \otimes H_{2n}(X, \partial X) \) is an isomorphism, and therefore the implication in the previous computation is actually an equivalence. \[\square\]

\[\text{Remark 3.20. The capacities } c_k^{GH} \text{ and } c_k^{S^1} \text{ are defined in terms of a certain homology class being in the kernel or in the image of a map with domain or target the } S^1\text{-equivariant symplectic homology. Other authors have constructed capacities in an analogous manner, for example Viterbo [Vit92, Definition 2.1] and [Vit99, Section 5.3], Schwarz [Sch00, Definition 2.6] and Ginzburg–Shon [GS18, Section 3.1].}\]
3.3 McDuff–Siegel capacities

We now define the McDuff–Siegel capacities. These will assist us in our goal of proving Conjecture 3.11 (at least in particular cases) because they can be compared with the Lagrangian capacity (Theorem 3.27) and with the Gutt–Hutchings capacities (Proposition 3.24).

**Definition 3.21** ([MS22, Definition 3.3.1]). Let \((X, \lambda)\) be a nondegenerate Liouville domain. For \(k \in \mathbb{Z}_{\geq 1}\), we define the **McDuff–Siegel capacities** of \(X\), denoted \(\tilde{g}_k(X)\), as follows. Choose \(x \in \text{int} X\) and \(D\) a symplectic divisor at \(x\). Then,

\[
\tilde{g}_k(X) := \sup_{J \in \mathcal{J}(X,D)} \inf_{\gamma} \mathcal{A}(\gamma),
\]

where the infimum is over Reeb orbits \(\gamma\) such that \(\mathcal{M}_X^J(\gamma)\langle T^{(k)}x \rangle \neq \emptyset\).

**Remark 3.22.** Actually, the McDuff–Siegel capacities (given as in [MS22, Definition 3.3.1]) are a family of symplectic capacities \(\tilde{g}_k \leq \tilde{g}_1\) indexed by \(\ell,k \in \mathbb{Z}_{\geq 1}\). The capacity \(\tilde{g}_k\) from Definition 3.21 is the capacity \(\tilde{g}_\ell_k\) from [MS22, Definition 3.3.1]. We point out that in [MS22], the notation \(\tilde{g}_k\) is used for the case \(\ell = \infty\), while we use this notation for the case \(\ell = 1\). A similar discussion holds for the higher symplectic capacities \(\tilde{g}_k\) of Definition 4.8.

**Theorem 3.23** ([MS22, Theorem 3.3.2]). The functions \(\tilde{g}_k\) are independent of the choices of \(x\) and \(D\) and satisfy the following properties, for all nondegenerate Liouville domains \((X, \lambda_X)\) and \((Y, \lambda_Y)\) of the same dimension:

- **(Monotonicity)** If \(X \to Y\) is a generalized Liouville embedding then \(\tilde{g}_k(X) \leq \tilde{g}_k(Y)\).
- **(Conformality)** If \(\alpha > 0\) then \(\tilde{g}_k(X, \alpha \lambda_X) = \alpha \tilde{g}_k(X, \lambda_X)\).
- **(Nondecreasing)** \(\tilde{g}_1(X) \leq \tilde{g}_2(X) \leq \cdots \leq +\infty\).

**Proposition 3.24** ([MS22, Proposition 5.6.1]). If \(X_\Omega\) is a 4-dimensional convex toric domain then

\[
\tilde{g}_k(X_\Omega) = c_{k}^{\text{GH}}(X_\Omega).
\]

Finally, we state two stabilization results which we will use in Section 4.6. The fact that will be relevant to us is Lemma 3.26 (1), which we will use to argue that the moduli space of curves in an ellipsoid satisfying a point constraint is independent of the dimension of the ellipsoid.

**Lemma 3.25** ([MS22, Lemma 3.6.2]). Let \((X, \lambda)\) be a Liouville domain. For any \(c, \varepsilon \in \mathbb{R}_{>0}\), there is a subdomain with smooth boundary \(\tilde{X} \subset X \times B^2(c)\) such that:

1. The Liouville vector field \(Z_\tilde{X} = Z_X + Z_{B^2(c)}\) is outwardly transverse along \(\partial \tilde{X}\).
2. \(X \times \{0\} \subset \tilde{X}\) and the Reeb vector field of \(\partial \tilde{X}\) is tangent to \(\partial X \times \{0\}\).
3. Any Reeb orbit of the contact form \((\lambda + \lambda_0)|_{\partial \tilde{X}}\) (where \(\lambda_0 = 1/2(x dy - y dx)\)) with action less than \(c - \varepsilon\) is entirely contained in \(\partial X \times \{0\}\) and has normal Conley–Zehnder index equal to 1.
Lemma 3.26 ([MS22, Lemma 3.6.3]). Let \( X \) be a Liouville domain, and let \( \tilde{X} \) be a smoothing of \( X \times B^2(c) \) as in Lemma 3.25.

(1) Let \( J \in \mathcal{J}(\tilde{X}) \) be a cylindrical almost complex structure on the completion of \( \tilde{X} \) for which \( \tilde{X} \times \{0\} \) is \( J \)-holomorphic. Let \( C \) be an asymptotically cylindrical \( J \)-holomorphic curve in \( \tilde{X} \), all of whose asymptotic Reeb orbits are nondegenerate and lie in \( \partial X \times \{0\} \) with normal Conley–Zehnder index 1. Then \( C \) is either disjoint from the slice \( \tilde{X} \times \{0\} \) or entirely contained in it.

(2) Let \( J \in \mathcal{J}(\partial \tilde{X}) \) be a cylindrical almost complex structure on the symplectization of \( \partial \tilde{X} \) for which \( \mathbb{R} \times \partial X \times \{0\} \) is \( J \)-holomorphic. Let \( C \) be an asymptotically cylindrical \( J \)-holomorphic curve in \( \mathbb{R} \times \partial X \), all of whose asymptotic Reeb orbits are nondegenerate and lie in \( \partial X \times \{0\} \) with normal Conley–Zehnder index 1. Then \( C \) is either disjoint from the slice \( \mathbb{R} \times \partial X \times \{0\} \) or entirely contained in it. Moreover, only the latter is possible if \( C \) has at least one negative puncture.

3.4 Computations without contact homology

We now state and prove one of our main theorems, which is going to be a key step in proving that \( c_L(X_{\Omega}) = \delta_\Omega \). The proof uses techniques similar to those used in the proof of [CM18, Theorem 1.1].

Theorem 3.27. If \((X, \lambda)\) is a Liouville domain then

\[
c_L(X) \leq \inf_k \frac{\tilde{g}_k(X)}{k}.
\]

Proof. By Remark 3.13, we may assume that \( X \) is nondegenerate. Let \( k \in \mathbb{Z}_{\geq 1} \) and \( L \subset \text{int} X \) be an embedded Lagrangian torus. We wish to show that for every \( \varepsilon > 0 \) there exists \( \sigma \in \pi_2(X, L) \) such that \( 0 < \omega(\sigma) \leq \tilde{g}_k(X)/k + \varepsilon \). Define

\[
a := \tilde{g}_k(X), \\
K_1 := \ln(2), \\
K_2 := \ln(1 + a/\varepsilon k), \\
K := \max\{K_1, K_2\}, \\
\delta := e^{-K}, \\
\ell_0 := a/\delta.
\]

By Lemma 2.2 and the Lagrangian neighbourhood theorem, there exists a Riemannian metric \( g \) on \( L \) and a symplectic embedding \( \phi : D^* L \to X \) such that \( \phi(D^* L) \subset \text{int} X \), \( \phi|_L = \text{id}_L \) and such that if \( \gamma \) is a closed geodesic in \( L \) with length \( \ell(\gamma) \leq \ell_0 \) then \( \gamma \) is noncontractible, nondegenerate and satisfies \( 0 \leq \mu_M(\gamma) \leq n - 1 \).

Let \( D^*_\delta L \) be the codisk bundle of radius \( \delta \). Notice that \( \delta \) has been chosen in such a way that the symplectic embedding \( \phi : D^*_\delta L \to X \) can be seen as an embedding like that of Lemma 2.1. Define symplectic cobordisms

\[
(X^+, \omega^+) := (X \setminus \phi(D^*_\delta L), \omega),
\]

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For exactly one and let \( D \). Here, it is implicit that we are considering the restriction of the form \( \lambda_{T^*L} \) on \( T^*L \) to \( D_\delta^*L \) or \( S_\delta^*L \). Then, \( (X, \omega) = (X^-, \omega^-) \oplus (X^+, \omega^+) \). Recall that there are piecewise smooth 2-forms \( \tilde{\omega} \in \Omega^2(\tilde{X}) \) and \( \tilde{\omega}^\pm \in \Omega^2(\tilde{X}^\pm) \) given as in Equation (1). Choose \( x \in \text{int} \phi(D_\delta^*L) \) and let \( D \subset \phi(D_\delta^*L) \) be a symplectic divisor through \( x \). Choose also generic almost complex structures

\[
J_M \in \mathcal{J}(M), \\
J^+ \in \mathcal{J}_M(X^+), \\
J^- \in \mathcal{J}^{J_M}(X^-, D),
\]

and denote by \( J_{\partial X} \in \mathcal{J}(\partial X) \) the “restriction” of \( J^+ \) to \( \mathbb{R} \times \partial X \). Let \( (J_t)_t \subset \mathcal{J}(X, D) \) be the corresponding neck stretching family of almost complex structures. Since \( a = \tilde{g}_k(X) \), for every \( t \) there exists a Reeb orbit \( \gamma_t \) in \( \partial X = \partial^+X^+ \) and a \( J_t \)-holomorphic curve \( u_t \in \mathcal{M}^{J_t}_X(\gamma_t) \langle T^{(k)}x \rangle \) such that \( A(\gamma_t) \leq a \). Since \( \partial X \) has nondegenerate Reeb orbits, there are only finitely many Reeb orbits in \( \partial X \) with action less than \( a \). Therefore, possibly after going to a subsequence, we may assume that \( \gamma_t := \gamma_0 \) is independent of \( t \).

The curves \( u_t \) satisfy the energy bound \( E_\omega(u_t) \leq a \). By the SFT compactness theorem, the sequence \((u_t)_t\) converges to a holomorphic building

\[
F = (F^1, \ldots, F^{L_0-1}, F^{L_0}, F^{L_0+1}, \ldots, F^N) \in \mathcal{M}^{(J_t)_t}_X(\gamma_0) \langle T^{(k)}x \rangle,
\]

where

\[
(X^\nu, \omega^\nu, \tilde{\omega}^\nu, J^\nu) := \begin{cases} 
(T^*L, d\lambda_{T^*L}, \tilde{\omega}^-, J^-) & \text{if } \nu = 1, \\
(\mathbb{R} \times M, d(e^\nu \alpha), \alpha, J_M) & \text{if } \nu = 2, \ldots, L_0 - 1, \\
(\tilde{X} \setminus L, \tilde{\omega}, \tilde{\omega}^+, J^+) & \text{if } \nu = L_0, \\
(\mathbb{R} \times \partial X, d(e^\nu \lambda|_{\partial X}), d\lambda|_{\partial X}, J_{\partial X}) & \text{if } \nu = L_0 + 1, \ldots, N,
\end{cases}
\]

and \( F^\nu \) is a \( J^\nu \)-holomorphic curve in \( X^\nu \) with asymptotic Reeb orbits \( \Gamma^\nu_\pm \) (see Fig. 1). The holomorphic building \( F \) satisfies the energy bound

\[
E_\omega^\nu(F) := \sum_{\nu=1}^N E_{\omega^\nu}(F^\nu) \leq a. \tag{13}
\]

Moreover, by Lemma 2.6, \( F \) has no nodes. Let \( C \) be the component of \( F \) in \( X^- \) which carries the tangency constraint \( \langle T^{(k)}x \rangle \). Then, \( C \) is positively asymptotic to Reeb orbits \( (\gamma_1, \ldots, \gamma_p) \) of \( M \). For \( \mu = 1, \ldots, p \), let \( C_\mu \) be the subtree emanating from \( C \) at \( \gamma_\mu \). For exactly one \( \mu = 1, \ldots, p \), the top level of the subtree \( C_\mu \) is positively asymptotic to \( \gamma_0 \), and we may assume without loss of generality that this is true for \( \mu = 1 \). By the
maximum principle, \( C_\mu \) has a component in \( X^{L_0} = \hat{X} \setminus L \) for every \( \mu = 2, \ldots, p \). Also by the maximum principle, there do not exist components of \( C_\mu \) in \( X^{L_0} = \hat{X} \setminus L \) which intersect \( \mathbb{R}_{\geq 0} \times \partial X \) or components of \( C_\mu \) in the top symplectization layers \( X^{L_0+1}, \ldots, X^N \), for every \( \mu = 2, \ldots, p \).

We claim that if \( \gamma \) is a Reeb orbit in \( M \) which is an asymptote of \( F^\nu \) for some \( \nu = 2, \ldots, L_0 - 1 \), then \( \mathcal{A}(\gamma) \leq a \). To see this, notice that

\[
a \geq E_{2\cdot}(F) \quad [\text{by Equation (13)}] \\
\geq E_{2\cdot N}(F^N) \quad [\text{by monotonicity of } E] \\
\geq (e^K - 1) \mathcal{A}(\Gamma_N^-) \quad [\text{by Lemma 2.1}] \\
\geq \mathcal{A}(\Gamma_N^-) \quad [\text{since } K \geq K_1] \\
\geq \mathcal{A}(\Gamma_\nu^-) \quad [\text{by Eq. (4)}]
\]

for every \( \nu = 2, \ldots, L_0 - 1 \). Every such \( \gamma \) has a corresponding geodesic in \( L \) (which by abuse of notation we denote also by \( \gamma \)) such that \( \ell(\gamma) = \mathcal{A}(\gamma)/\delta \leq a/\delta = \ell_0 \). Hence, by our choice of Riemannian metric, the geodesic \( \gamma \) is noncontractible, nondegenerate and such that \( \mu_{\text{ad}}(\gamma) \leq n - 1 \). Therefore, the Reeb orbit \( \gamma \) is noncontractible, nondegenerate and such that \( \mu_{\text{CZ}}(\gamma) \leq n - 1 \).

We claim that if \( D \) is a component of \( C_\mu \) for some \( \mu = 2, \ldots, p \) and \( D \) is a plane, then \( D \) is in \( X^{L_0} = \hat{X} \setminus L \). Assume by contradiction otherwise. Notice that since \( D \) is a plane, \( D \) is asymptotic to a unique Reeb orbit \( \gamma \) in \( M = S^*_3 L \) with corresponding noncontractible geodesic \( \gamma \) in \( L \). We will derive a contradiction by defining a filling disk for \( \gamma \). If \( D \) is in a symplectization layer \( \mathbb{R} \times S^*_3 L \), then the map \( \pi \circ D \), where \( \pi : \mathbb{R} \times S^*_3 L \rightarrow L \) is the projection, is a filling disk for the geodesic \( \gamma \). If \( D \) is in the bottom level, i.e. \( X^1 = T^* L \),

Figure 1: The holomorphic building \( F = (F^1, \ldots, F^N) \) in the case \( L_0 = N = p = 3 \)
then the map $\pi \circ D$, where $\pi : T^*L \to L$ is the projection, is also a filling disk. This proves the claim.

So, summarizing our previous results, we know that for every $\mu = 2, \ldots, p$ there is a holomorphic plane $D_\mu$ in $X^{L_0} \setminus (\mathbb{R}_{\geq 0} \times \partial X) = X \setminus L$. For each plane $D_\mu$ there is a corresponding disk in $X$ with boundary on $L$, which we denote also by $D_\mu$. It is enough to show that $E_\omega(D_{\mu_0}) \leq a/k + \varepsilon$ for some $\mu_0 = 2, \ldots, p$. By Lemma 2.3, $p \geq k + 1 \geq 2$.

By definition of average, there exists $\mu_0 = 2, \ldots, p$ such that

$$E_\omega(D_{\mu_0}) \leq \frac{1}{p-1} \sum_{\mu=2}^{p} E_\omega(D_\mu) \quad \text{[by definition of average]}$$

$$= \frac{E_\omega(D_2 \cup \cdots \cup D_p)}{p-1} \quad \text{[since energy is additive]}$$

$$\leq \frac{e^K}{e^K-1} \frac{a}{p-1} \quad \text{[by Lemma 2.1]}$$

$$\leq \frac{e^K}{e^K-1} \frac{a}{p} \quad \text{[by Equation (13)]}$$

$$\leq \frac{a}{k} \quad \text{[since $p \geq k + 1$]}$$

$$\leq \frac{a}{k} + \varepsilon \quad \text{[since $K \geq K_2$].}$$

**Theorem 3.28.** If $X_\Omega$ is a 4-dimensional convex toric domain then

$$c_L(X_\Omega) = \delta_\Omega.$$

**Proof.** For every $k \in \mathbb{Z}_{\geq 1}$,

$$\delta_\Omega \leq c_P(X_\Omega) \quad \text{[by Lemma 3.7]}$$

$$\leq c_L(X_\Omega) \quad \text{[by Lemma 3.6]}$$

$$\leq \tilde{g}_k(X_\Omega) \quad \text{[by Theorem 3.27]}$$

$$= \frac{c_k^{GH}(X_\Omega)}{k} \quad \text{[by Proposition 3.24]}$$

$$\leq \frac{c_k^{GH}(N(\delta_\Omega))}{k} \quad \text{[by Lemma 3.16].}$$

The result follows by taking the infimum over $k$. 

The result follows by taking the infimum over $k$. 

**4 Computations using the higher symplectic capacities**

**4.1 Assumptions on virtual perturbation scheme**

In this subsection, we wish to use techniques from contact homology to prove Conjecture 3.11. Consider the proof of Theorem 3.28: to prove the inequality $c_L(X_\Omega) \leq \delta_\Omega$. 


we needed to use the fact that \( \tilde{g}_k(X_\Omega) \leq c_{k}^{\mathrm{GH}}(X_\Omega) \) (which is true if \( X_\Omega \) is convex and 4-dimensional). Our approach here will be to consider the capacities \( g_k \) from [Sie20], which satisfy \( \tilde{g}_k(X) \leq g_k(X) = c_{k}^{\mathrm{GH}}(X) \). As we will see, \( g_k(X) \) is defined using the linearized contact homology of \( X \), where \( X \) is any nondegenerate Liouville domain.

Very briefly, the linearized contact homology chain complex, denoted \( CC(X) \), is generated by the good Reeb orbits of \( \partial X \), and therefore maps whose domain is \( CC(X) \) should count holomorphic curves which are asymptotic to Reeb orbits. The “naive” way to define such counts of holomorphic curves would be to show that they are the elements of a moduli space which is a compact, 0-dimensional orbifold. However, there is the possibility that a curve is multiply covered. This means that in general it is no longer possible to show that the moduli spaces are transversely cut out, and therefore we do not have access to counts of moduli spaces of holomorphic curves (or at least not in the usual sense of the notion of signed count). In the case where the Liouville domain is 4-dimensional, there exists the possibility of using automatic transversality techniques to show that the moduli spaces are regular. This is the approach taken by Wendl [Wen10]. Nelson [Nel15], Hutchings–Nelson [HN16] and Bao–Honda [BH18] use automatic transversality to define cylindrical contact homology.

In order to define contact homology in more general contexts, one needs to use a suitable notion of virtual count, which is obtained through a virtual perturbation scheme. This was done by Pardon [Par16, Par19] to define contact homology in greater generality. The theory of polyfolds by Hofer–Wysocki–Zehnder [HWZ21] can also be used to define virtual moduli counts. Alternative approaches using Kuranishi structures (see [FOOO10a, FOOO10b]) have been given by Ishikawa [Ish18] and Bao–Honda [BH21].

Unfortunately, linearized contact homology is not yet defined in the generality we need. Indeed, in order to prove Conjecture 3.11, we need to use the capacities \( g_k \). These are defined using the linearized contact homology and an augmentation map which counts curves satisfying a tangency constraint. As far as we know, the current work on defining virtual moduli counts does not yet deal with moduli spaces of curves satisfying tangency constraints.

So, during this section, we will work under assumption that it is possible to define a virtual perturbation scheme which makes the invariants and maps described above well-defined (this is expected to be the case).

**Assumption 4.1.** We assume the existence of a virtual perturbation scheme which to every compactified moduli space \( \overline{M} \) of asymptotically cylindrical holomorphic curves (in a symplectization or in a Liouville cobordism, possibly satisfying a tangency constraint) assigns a virtual count \( \#^{\text{vir}} \overline{M} \). We will assume in addition that the virtual perturbation scheme has the following properties.

1. If \( \#^{\text{vir}} \overline{M} \neq 0 \) then \( \text{virdim} \overline{M} = 0 \);
2. If \( \overline{M} \) is transversely cut out then \( \#^{\text{vir}} \overline{M} = \# \overline{M} \). In particular, if \( \overline{M} \) is empty then \( \#^{\text{vir}} \overline{M} = 0 \);
3. The virtual count of the boundary of a moduli space (defined as a sum of virtual counts of the moduli spaces that constitute the codimension one boundary strata)
is zero. In particular, the expected algebraic identities (\(\partial^2 = 0\) for differentials, \(\varepsilon \circ \partial = 0\) for augmentations) hold, as well as independence of auxiliary choices of almost complex structure and symplectic divisor.

### 4.2 Linearized contact homology

In this subsection, we define the linearized contact homology of a nondegenerate Liouville domain \(X\). This is the homology of a chain complex \(CC(X)\), which is described in Definition 4.3. We also define an augmentation map (Definition 4.7), which is necessary to define the capacities \(g_k\).

**Definition 4.2.** Let \((M, \alpha)\) be a contact manifold and \(\gamma\) be a Reeb orbit in \(M\). We say that \(\gamma\) is **bad** if \(\mu_{\text{CZ}}(\gamma) - \mu_{\text{CZ}}(\gamma_0)\) is odd, where \(\gamma_0\) is the simple Reeb orbit that corresponds to \(\gamma\). We say that \(\gamma\) is **good** if it is not bad.

Since the parity of the Conley–Zehnder index of a Reeb orbit is independent of the choice of trivialization, the definition above is well posed.

**Definition 4.3.** If \((X, \lambda)\) is a nondegenerate Liouville domain, the **linearized contact homology chain complex** of \(X\), denoted \(CC(X)\), is a chain complex given as follows. First, let \(CC(X)\) be the vector space over \(\mathbb{Q}\) generated by the set of good Reeb orbits of \((\partial X, \lambda|_{\partial X})\). The differential of \(CC(X)\), denoted \(\partial\), is given as follows. Choose \(J \in \mathcal{J}(X)\). If \(\gamma\) is a good Reeb orbit of \(\partial X\), we define

\[
\partial \gamma = \sum_{\eta} \langle \partial \gamma, \eta \rangle \eta,
\]

where \(\langle \partial \gamma, \eta \rangle\) is the virtual count (with combinatorial weights) of holomorphic curves in \(\mathbb{R} \times \partial X\) with one positive asymptote \(\gamma\), one negative asymptote \(\eta\), and \(k \geq 0\) extra negative asymptotes \(\alpha_1, \ldots, \alpha_k\) (called **anchors**), each weighted by the count of holomorphic planes in \(\hat{X}\) asymptotic to \(\alpha_j\) (see Fig. 2).

![Figure 2: A holomorphic curve with anchors contributing to the coefficient \(\langle \partial \gamma, \eta \rangle\)](image)
By assumption on the virtual perturbation scheme, \( \partial \circ \partial = 0 \) and \( CC(X) \) is independent (up to chain homotopy equivalence) of the choice of almost complex structure \( J \).

**Remark 4.4.** In general, the Conley–Zehnder index of a Reeb orbit is well-defined as an element of \( \mathbb{Z}_2 \). Therefore, the complex \( CC(X) \) has a \( \mathbb{Z}_2 \)-grading given by \( \deg(\gamma) := \mu_{CZ}(\gamma) \), and with respect to this definition the differential \( \partial \) has degree \(-1\). If \( \pi_1(X) = 0 \) and \( 2c_1(TX) = 0 \), then the Conley–Zehnder index of Reeb orbit is well-defined as an element of \( \mathbb{Z} \), which means that \( CC(X) \) is \( \mathbb{Z} \)-graded.

**Definition 4.5.** For every \( a \in \mathbb{R} \), we denote by \( CC^a(X) \) the submodule of \( CC(X) \) generated by the good Reeb orbits \( \gamma \) with action \( A(\gamma) \leq a \). We call this filtration the **action filtration** of \( CC(X) \).

In the next lemma, we check that this filtration is compatible with the differential.

**Lemma 4.6.** \( \partial(CC^a(X)) \subset CC^a(X) \).

**Proof.** Let \( \gamma, \eta \) be good Reeb orbits such that

\[
\begin{align*}
A(\gamma) & \leq a, \\
\langle \partial \gamma, \eta \rangle & \neq 0.
\end{align*}
\]

We wish to show that \( A(\eta) \leq a \). Since \( \langle \partial \gamma, \eta \rangle \neq 0 \) and by assumption on the virtual perturbation scheme, there exists a tuple of Reeb orbits \( \Gamma = (\eta, \alpha_1, \ldots, \alpha_p) \) and a (nontrivial) punctured \( J \)-holomorphic sphere in \( \mathbb{R} \times \partial X \) with positive asymptote \( \gamma \) and negative asymptotes \( \Gamma \). Then,

\[
\begin{align*}
A(\eta) & \leq A(\Gamma) \quad [\text{since } \eta \in \Gamma] \\
& \leq A(\gamma) \quad [\text{by Equation (4)}] \\
& \leq a \quad [\text{by assumption on } \gamma].
\end{align*}
\]

**Definition 4.7.** Consider the complex \( (CC(X), \partial) \). For each \( k \in \mathbb{Z}_{\geq 1} \), we define an augmentation \( \epsilon_k : CC(X) \to \mathbb{Q} \) as follows. Choose \( x \in \text{int} \, X \), a symplectic divisor \( D \) at \( x \), and an almost complex structure \( J \in \mathcal{J}(X, D) \). Then, for every good Reeb orbit \( \gamma \) define \( \epsilon_k(\gamma) \) to be the virtual count of \( J \)-holomorphic planes in \( \hat{X} \) which are positively asymptotic to \( \gamma \) and have contact order \( k \) to \( D \) at \( x \) (see Fig. 3).

![Figure 3: A holomorphic curve contributing to the count \( \epsilon_k(\gamma) \)](image)
By Equation (9), $\epsilon_k$ is a map $\epsilon_k : CC_{n-1+2k}(X) \to \mathbb{Q}$. By assumption on the virtual perturbation scheme, $\epsilon_k$ is an augmentation, i.e. $\epsilon_k \circ \partial = 0$. Therefore, there is a corresponding map $\epsilon_k : CH_{n-1+2k}(X) \to \mathbb{Q}$ on homology. In addition, $\epsilon_k$ is independent (up to chain homotopy) of the choices of $x, D, J$.

### 4.3 Higher symplectic capacities

**Definition 4.8** ([Sie20, Section 6.1]). Let $k, \ell \in \mathbb{Z}_{\geq 1}$ and $(X, \lambda)$ be a nondegenerate Liouville domain. The higher symplectic capacities of $X$ are given by

$$g_k(X) := \inf \{ a > 0 \mid \epsilon_k : CH^a(X) \to \mathbb{Q} \text{ is nonzero} \}.$$

The capacities $g_k$ will be useful to us because they have similarities with the McDuff–Siegel capacities $\tilde{g}_k$, but also with the Gutt–Hutchings capacities $c_{GH}^k$. More specifically:

1. Both $g_k$ and $\tilde{g}_k$ are related to the energy of holomorphic planes in $X$ which are asymptotic to a Reeb orbit and satisfy a tangency constraint. In Theorem 4.35, we will actually see that $\tilde{g}_k(X) \leq g_k(X)$. The capacities $g_k$ can be thought of as the SFT counterparts of $\tilde{g}_k$, or alternatively the capacities $\tilde{g}_k$ can be thought of as the counterparts of $g_k$ whose definition does not require the holomorphic curves to be regular.

2. Both $g_k$ and $c_{GH}^k$ are defined in terms of a map on homology being nonzero. In the case of $g_k$, we consider the linearized contact homology, and in the case of $c_{GH}^k$ the invariant in question is $S^1$-equivariant symplectic homology. Taking into consideration the Bourgeois–Oancea isomorphism (see [BO16]) between linearized contact homology and positive $S^1$-equivariant symplectic homology, one can think of $g_k$ and $c_{GH}^k$ as restatements of one another under this isomorphism. This is the idea behind the proof of Theorem 4.36, where we show that $g_k(X) = c_{GH}^k(X)$.

**Remark 4.9.** In the case where $X$ is only an exact symplectic manifold instead of a Liouville domain, we do not have access to an action filtration on $CC(X)$. However, it is possible to define linearized contact homology with coefficients in a Novikov ring $\Lambda_{\geq 0}$, in which case a coefficient in $\Lambda_{\geq 0}$ encodes the energy of a holomorphic curve. This is the approach taken in [Sie20] to define the capacities $g_k$. It is not obvious that the definition of $g_k$ we give and the one in [Sie20] are equivalent. However, Definition 4.8 seems to be the natural analogue when we have access to an action filtration, and in addition the definition we provide will be enough for our purposes.

**Theorem 4.10.** The functions $g_k$ satisfy the following properties, for all nondegenerate Liouville domains $(X, \lambda_X)$ and $(Y, \lambda_Y)$ of the same dimension:

- (Monotonicity) If $X \to Y$ is an exact symplectic embedding then $g_k(X) \leq g_k(Y)$.

- (Conformality) If $\mu > 0$ then $g_k(X, \mu \lambda_X) = \mu g_k(X, \lambda_X)$.

- (Reeb orbits) If $\pi_1(X) = 0$, $2c_1(TX) = 0$ and $g_k(X) < +\infty$, then there exists a Reeb orbit $\gamma$ such that $g_k(X) = A(\gamma)$ and $\mu_{CZ}(\gamma) = n - 1 + 2k$. 

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Proof. We prove monotonicity. If \((X, \lambda^X) \rightarrow (Y, \lambda^Y)\) is an exact symplectic embedding, then it is possible to define a Viterbo transfer map \(CH(Y) \rightarrow CH(X)\). This map respects the action filtration as well as the augmentation maps, i.e. the diagram

\[
\begin{array}{ccc}
CH^a(Y) & \longrightarrow & CH(Y) \\
\downarrow & & \downarrow \\
CH^a(X) & \longrightarrow & CH(X)
\end{array}
\]

\[
\begin{array}{ccc}
\epsilon_k^Y & & \epsilon_k^X \\
\end{array}
\]

commutes. The result then follows by definition of \(\tilde{g}_k\).

We prove conformality. If \(\gamma\) is a Reeb orbit of \((\partial X, \lambda_{|\partial X})\) of action \(A_{\lambda}(\gamma)\) then \(\gamma\) is a Reeb orbit of \((\partial X, \mu\lambda_{|\partial X})\) of action \(A_{\mu\lambda}(\gamma) = \mu A_{\lambda}(\gamma)\). Therefore, there is a diagram

\[
\begin{array}{ccc}
CH^a(X, \lambda) & \longrightarrow & CH(X, \lambda) \\
\downarrow & & \downarrow \\
CH^{ma}(X, \mu\lambda) & \longrightarrow & CH(X, \mu\lambda)
\end{array}
\]

\[
\begin{array}{ccc}
\epsilon_k^a & & \epsilon_k^m \\
\end{array}
\]

Again, the result follows by definition of \(g_k\).

We prove the Reeb orbits property. Choose a point \(x \in \text{int} \ X\), a symplectic divisor \(D\) through \(x\) and an almost complex structure \(J \in J(X, D)\). Consider the complex \(CC(X)\), computed with respect to \(J\). By assumption and definition of \(g_k\),

\[
+\infty > g_k(X) = \inf \{a > 0 \mid \epsilon_k : CH^a(X) \rightarrow \mathbb{Q} \text{ is nonzero}\}
\]

\[
= \inf \{A(\beta) \mid \beta \in CC(X) \text{ is such that } \epsilon_k(\beta) \neq 0 \text{ and } \partial \beta = 0\},
\]

where \(\beta = \sum_{i=1}^{m} a_i \gamma_i\) is a linear combination of Reeb orbits and \(A(\beta) := \max_{i=1,\ldots,m} \gamma_i\).

Since the action spectrum of \((\partial X, \lambda_{|\partial X})\) is a discrete subset of \(\mathbb{R}\), we conclude that in the above expression the infimum is a minimum. More precisely, there exists \(\beta = \sum_{i=1}^{m} a_i \gamma_i \in CC_{n-1+2k}(X)\) such that \(\epsilon_k(\beta) \neq 0\) and \(g_k(X) = A(\beta)\). One of the orbits in this linear combination is such that \(A_{\gamma_i} = A(\beta) = g_k(X)\).

Remark 4.11. In [GH18, Theorem 1.6] (respectively [GH18, Theorem 1.14]) Gutt–Hutchings give formulas for \(c_{GH}^k\) of a convex (respectively concave) toric domain. However, the given proofs only depend on specific properties of the Gutt–Hutchings capacity and not on the definition of the capacity itself. These properties are monotonicity, conformality, a “Reeb orbits” property similar to the one of Theorem 4.10, and finally that the capacity be finite on star-shaped domains. If we showed that \(g_k\) is finite on star-shaped domains, we would conclude that \(g_k = c_{GH}^k\) on convex or concave toric domains, because in this case both capacities would be given by the formulas in the previously mentioned theorems. Showing that \(g_k\) is finite boils down to showing that the augmentation map is nonzero, which we will do in Section 4.6. However, in Theorem 4.36 we will use this information in combination with the Bourgeois–Oancea isomorphism to conclude that \(g_k(X) = c_{GH}^k(X)\) for any nondegenerate Liouville domain \(X\). Therefore, the proof suggested above will not be necessary, although it is a proof of \(g_k(X) = c_{GH}^k(X)\) alternative to that of Theorem 4.36 when \(X\) is a convex or concave toric domain.
4.4 Cauchy–Riemann operators on bundles

In order to show that \( g_k(X) = c_k^{GH}(X) \), we will need to show that the augmentation map of a small ellipsoid in \( X \) is nonzero (see the proof of Theorem 4.36). Recall that the augmentation map counts holomorphic curves satisfying a tangency constraint. In Section 4.6, we will explicitly compute how many such holomorphic curves there are. However, a count obtained by explicit methods will not necessarily agree with the virtual count that appears in the definition of the augmentation map. By assumption on the virtual perturbation scheme, it does agree if the relevant moduli space is transversely cut out.

Therefore, in this subsection and the next we will describe the framework that allows us to show that this moduli space is transversely cut out. This subsection deals with the theory of real linear Cauchy–Riemann operators on line bundles, and our main reference is [Wen10]. The outline is as follows. First, we review the basic definitions about real linear Cauchy–Riemann operators. By the Riemann–Roch theorem (Theorem 4.12), these operators are Fredholm and their index can be computed from a number of topological quantities associated to them. We will make special use of a criterion by Wendl (Proposition 4.13) which guarantees that a real linear Cauchy–Riemann operator defined on a complex line bundle is surjective. For our purposes, we will also need an adaptation of this result to the case where the operator is accompanied by an evaluation map, which we state in Lemma 4.16. We now state the assumptions for the rest of this subsection.

Let \( (\Sigma, j) \) be a compact Riemann surface without boundary, of genus \( g \), with sets of positive and negative punctures \( z^\pm = \{ z_1^\pm, \ldots, z_p^\pm \} \). Denote \( z = z^+ \cup z^- \) and \( \hat{\Sigma} = \Sigma \setminus z \).

Choose cylindrical coordinates \((s, t)\) near each puncture \( z \in z \) and denote \( U_z \subset \hat{\Sigma} \) the domain of the coordinates \((s, t)\).

We assume that we are given an asymptotically Hermitian vector bundle

\[
(E, J) \longrightarrow \hat{\Sigma}, \quad (E_z, J_z, \omega_z) \longrightarrow S^1, \quad \text{for each } z \in z
\]

over \( \hat{\Sigma} \) (see [Wen16, p. 68]). If \( \tau = (\tau_z)_{z \in z} \) is an asymptotic trivialization of \( E \) (i.e. each \( \tau_z \) is a unitary trivialization of \( (E_z, J_z, \omega_z) \), see [Wen16, p. 68]), then one can define Sobolev spaces of sections of \( E \), denoted by \( W^{k,p}(E) \), and weighted Sobolev spaces of sections of \( E \), denoted by \( W^{k,p,\delta}(E) \). Let \( D \) be a real linear Cauchy–Riemann operator on \( E \) (see [MS12, Definition C.1.5]), together with corresponding asymptotic operators \( (A_z)_{z \in z} \) (see [Wen16, Definition 3.25]). Some topological quantities which are going to be relevant to us are:

1. The **Euler characteristic** of \( \hat{\Sigma} \), which is given by \( \chi(\hat{\Sigma}) = 2 - 2g - \# z \);
2. The **relative first Chern number** of \( E \) (with respect to the trivialization \( \tau \)), which is an integer denoted by \( c_1^\tau(E) \in \mathbb{Z} \) (see [Wen16, Definition 5.1]);
3. The **Conley–Zehnder** index (with respect to the trivialization \( \tau \)) of an asymptotic operator \( A_z \), which is an integer denoted by \( \mu_{CZ}(A_z) \) (see [Wen16, Definitions 3.30 and 3.31]).

Using these quantities, we can state the following version of the Riemann–Roch theorem.
Theorem 4.12 (Riemann–Roch, [Wen16, Theorem 5.4]). The operator $D$ is Fredholm and its (real) Fredholm index is given by

$$\text{ind } D = n\chi(\tilde{\Sigma}) + 2c_1^r(E) + \sum_{z \in \mathbb{Z}^+} \mu_{C_\tau}(A_z) - \sum_{z \in \mathbb{Z}^-} \mu_{C_\tau}(A_z).$$

For the rest of this subsection, we restrict ourselves to the case where $n = \text{rank}_E = 1$. Our goal is to state a criterion that guarantees surjectivity of $D$. This criterion depends on other topological quantities whose definition we now recall (see [Wen10, Section 2.2]).

For every $\lambda$ in the spectrum of $A_z$, let $w^\tau(\lambda)$ be the winding number of any nontrivial section in the $\lambda$-eigenspace of $A_z$ (computed with respect to the trivialization $\tau$). Define the **winding numbers**

$$\alpha^+_{\tau}(A_z) := \max \{ w^\tau(\lambda) | \lambda < 0 \text{ is in the spectrum of } A_z \},$$

$$\alpha^-_{\tau}(A_z) := \min \{ w^\tau(\lambda) | \lambda > 0 \text{ is in the spectrum of } A_z \}.$$

The **parity** (the reason for this name is Equation (14) below) and associated sets of even and odd punctures are given by

$$p(A_z) := \alpha^+_{\tau}(A_z) - \alpha^-_{\tau}(A_z) \in \{0, 1\},$$

$$z_0 := \{ z \in \mathbb{Z} | p(A_z) = 0 \},$$

$$z_1 := \{ z \in \mathbb{Z} | p(A_z) = 1 \}.$$

Finally, the **adjusted first Chern number** is given by

$$c_1(E, A_z) = c_1^r(E) + \sum_{z \in \mathbb{Z}^+} \alpha^-(A_z) - \sum_{z \in \mathbb{Z}^-} \alpha^+(A_z).$$

These quantities satisfy the following equations.

$$\mu_{C_\tau}(A_z) = 2\alpha^-_{\tau}(A_z) + p(A_z) = 2\alpha^-_{\tau}(A_z) - p(A_z), \quad (14)$$

$$2c_1(E, A_z) = \text{ind } D - 2 - 2g + \#z_0. \quad (15)$$

Proposition 4.13 ([Wen10, Proposition 2.2]).

1. If $\text{ind } D \leq 0$ and $c_1(E, A_z) < 0$ then $D$ is injective.
2. If $\text{ind } D \geq 0$ and $c_1(E, A_z) < \text{ind } D$ then $D$ is surjective.

We will apply the proposition above to moduli spaces of punctured spheres which have no even punctures. The following corollary is just a restatement of the previous proposition in this simpler case.

Corollary 4.14. Assume that $g = 0$ and $\#z_0 = 0$. Then,

1. If $\text{ind } D \leq 0$ then $D$ is injective.
2. If $\text{ind } D \geq 0$ then $D$ is surjective.

Proof. By Proposition 4.13 and Equation (15).
We now wish to deal with the case where $D$ is taken together with an evaluation map (see Lemma 4.16 below). The tools we need to prove this result are explained in the following remark.

**Remark 4.15** ([Wen10, p. 362-363]). Suppose that $\ker D \neq \{0\}$. If $\xi \in \ker D \setminus \{0\}$, it is possible to show that $\xi$ has only a finite number of zeros, all of positive order, i.e. if $w$ is a zero of $\xi$ then $\text{ord}(\xi; w) > 0$. For every $z \in Z$, there is an asymptotic winding number $\text{wind}_z^\tau(\xi) \in \mathbb{Z}$, which has the properties

\[
\begin{align*}
z \in Z^+ & \implies \text{wind}_z^\tau(\xi) \leq \alpha_+^\tau(A_z), \\
z \in Z^- & \implies \text{wind}_z^\tau(\xi) \geq \alpha_+^\tau(A_z).
\end{align*}
\]

Define the asymptotic vanishing of $\xi$, denoted $Z_\infty(\xi)$, and the count of zeros, denoted $Z(\xi)$, by

\[
\begin{align*}
Z_\infty(\xi) & := \sum_{z \in Z^+} \left( \alpha_+^\tau(A_z) - \text{wind}_z^\tau(\xi) \right) + \sum_{z \in Z^-} \left( \text{wind}_z^\tau(\xi) - \alpha_+^\tau(A_z) \right) \in \mathbb{Z}_{\geq 0}, \\
Z(\xi) & := \sum_{w \in \xi^{-1}(0)} \text{ord}(\xi; w) \in \mathbb{Z}_{\geq 0}.
\end{align*}
\]

In this case, we have the formula (see [Wen10, Equation 2.7])

\[
c_1(E, A_z) = Z(\xi) + Z_\infty(\xi). \quad (16)
\]

**Lemma 4.16.** Let $w \in \hat{\Sigma}$ be a point and $E: W^{1,p}(\hat{\Sigma}, E) \rightarrow E_w$ be the evaluation map at $w$, i.e. $E(\xi) = \xi_w$. Assume that $g = 0$ and $\#Z_0 = 0$. If $\text{ind } D = 2$ then $D \oplus E: W^{1,p}(\hat{\Sigma}, E) \rightarrow L^p(\hat{\Sigma}, \text{Hom}^{0,1}(\hat{T}\Sigma, E)) \oplus E_w$ is surjective.

**Proof.** It is enough to show that the maps

\[
D: W^{1,p}(\hat{\Sigma}, E) \rightarrow L^p(\hat{\Sigma}, \text{Hom}^{0,1}(\hat{T}\Sigma, E)), \\
E|_{\ker D}: \ker D \rightarrow E_w
\]

are surjective. By Corollary 4.14, $D$ is surjective. Since $\dim \ker D = \text{ind } D = 2$ and $\dim_E E_w = 2$, the map $E|_{\ker D}$ is surjective if and only if it is injective. So, we show that $\ker (E|_{\ker D}) = \ker E \cap \ker D = \{0\}$. For this, let $\xi \in \ker E \cap \ker D$ and assume by contradiction that $\xi \neq 0$. Consider the quantities defined in Remark 4.15. We compute

\[
\begin{align*}
0 & = \text{ind } D - 2 & \text{[by assumption]} \\
& = 2c_1(E, A_z) & \text{[by Equation (15)]} \\
& = 2Z(\xi) + 2Z_\infty(\xi) & \text{[by Equation (16)]} \\
& \geq 0 & \text{[by definition of } Z \text{ and } Z_\infty],
\end{align*}
\]

which implies that $Z(\xi) = 0$. This gives the desired contradiction, because

\[
\begin{align*}
0 & = Z(\xi) & \text{[by the previous computation]} \\
& = \sum_{z \in \xi^{-1}(0)} \text{ord}(\xi; z) & \text{[by definition of } Z]\n& \geq \text{ord}(\xi; w) & \text{[since } \xi_w = E(\xi) = 0]\n& > 0 & \text{[by Remark 4.15].}
\end{align*}
\]

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4.5 Cauchy–Riemann operators as sections

In this subsection, we phrase the notion of a map $u: \hat{\Sigma} \to \hat{X}$ being holomorphic in terms of $u$ being in the zero set of a section $\mathcal{D}: \mathcal{T} \times \mathcal{B} \to \mathcal{E}$. The goal of this point of view is that we can then think of moduli spaces of holomorphic curves in $\hat{X}$ as the zero set of the section $\mathcal{D}$. To see if such a moduli space is regular near $(j, u)$, one needs to consider the linearization $L_{(j, u)}$ of $\mathcal{D}$ at $(j, u)$, and prove that it is surjective. We will see that a suitable restriction of $L_{(j, u)}$ is a real linear Cauchy–Riemann operator, and therefore we can use the theory from the last subsection to show that $L_{(j, u)}$ is surjective in some particular cases (Lemmas 4.19 and 4.20).

**Definition 4.17.** Let $(X, \omega, \lambda)$ be a symplectic cobordism, $J \in \mathcal{J}(X)$ be a cylindrical almost complex structure on $\hat{X}$, and $\Gamma^\pm = (\gamma_1^\pm, \ldots, \gamma_p^\pm)$ be tuples of Reeb orbits on $\partial^\pm X$. Consider the sphere $S^2$ together with a set of punctures $z^\pm = \{z^+_1, \ldots, z^+_p\} \subset S^2$ and a corresponding set of asymptotic markers $v^\pm = \{v^+_1, \ldots, v^+_p\}$ (i.e., $v^+_i \in (T^*_z S^2 \setminus \{0\})/\mathbb{R}_{>0}$). Define $\mathcal{M}^{S,J}(\Gamma^+, \Gamma^-)$ to be the moduli space of (equivalence classes of) pairs $(j, u)$, where $j$ is an almost complex structure on $S^2$ and $u: (\hat{S}^2, j) \to (\hat{X}, J)$ is an asymptotically cylindrical holomorphic curve such that

1. $u$ is positively/negatively asymptotic to $\gamma_i^\pm$ at $z_i^\pm$ for all $i$;
2. If $c$ is a path in $S^2$ with $c(0) = z_i^\pm$ and $\dot{c}(0) = v_i^\pm$ for some $i$, then $\lim_{t \to \pm \infty} u(c(t)) = (\pm \infty, \gamma_i^\pm(0))$.

**Remark 4.18.** There is a surjective map $\pi^s: \mathcal{M}^{S,J}_X(\Gamma^+, \Gamma^-) \to \mathcal{M}^J(\Gamma^+, \Gamma^-)$ given by forgetting the asymptotic markers. By [Wen16, Proposition 11.1], for every $u \in \mathcal{M}^J_X(\Gamma^+, \Gamma^-)$ the preimage $(\pi^s)^{-1}(u)$ contains exactly

$$\frac{\prod_{\gamma \in \Gamma^+ \cup \Gamma^-} m(\gamma)}{|\text{Aut}(u)|}$$

elements, where $m(\gamma)$ is the multiplicity of the Reeb orbit $\gamma$ and $\text{Aut}(u)$ is the automorphism group of $u = (\Sigma, j, z, u)$, i.e. an element of $\text{Aut}(u)$ is a biholomorphism $\phi: \Sigma \to \Sigma$ such that $u \circ \phi = u$ and $\phi(z_i^\pm) = z_i^\pm$ for every $i$.

We will work with the following assumptions. Let $\Sigma = S^2$ be the sphere, (without any specified almost complex structure). Let $z \in \Sigma$ be a puncture\(^3\) and $v \in (T_z \Sigma \setminus \{0\})/\mathbb{R}_{>0}$ be a corresponding asymptotic marker. There are cylindrical coordinates $(s, t)$ on $\Sigma$ near $z$, with the additional property that $v$ agrees with the direction $t = 0$. We will also assume that $\mathcal{T} \subset \mathcal{J}(\Sigma)$ is a Teichmüller slice as in [Wen10, Section 3.1], where $\mathcal{J}(\Sigma)$ denotes the set of almost complex structures on $\Sigma = S^2$. Let $(X, \lambda)$ be a nondegenerate Liouville domain of dimension $2n$ and $J \in \mathcal{J}(X)$ be an admissible almost complex structure on $\hat{X}$. Let $\gamma$ be a Reeb orbit in $\partial X$. Denote by $m$ the multiplicity of $\gamma$ and by $T$ the period of the simple Reeb orbit underlying $\gamma$ (so, the period of $\gamma$ is $mT$). Choose once and for all a parametrization $\phi: S^1 \times D^{2n-2} \to O$, where $O \subset \partial X$ is an open neighbourhood of $\gamma$ and

$$D^{2n-2} := \{(z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1} | |z_1| < 1, \ldots, |z_{n-1}| < 1\}$$

\(^3\)We point out that the results of this subsection can be stated in the case where $\Sigma$ has no negative punctures and any number of positive punctures. Since for our purposes it is enough to consider the case of one positive puncture, we will restrict ourselves to this case to keep the notation simpler.
Finally, we require that there exists $u$ of class $W^{k,p}$ such that
\[ \gamma \in W^{k,p} \text{ satisfies the following property.} \]
Let $\pi$ denote the restriction\[ \pi \colon \mathcal{E} \to \mathcal{T} \times \mathcal{B}. \]
The linearizations of $\mathcal{E}_{(j,u)}$ are sections $D_{(j,u)}$ of $\mathcal{E}_{(j,u)}$ with respect to the cylindrical coordinates $(\vartheta, \zeta)$ near $\gamma$ on the target and cylindrical coordinates $(s, t)$ on the domain:\[ u(s, t) = (\pi_R \circ u(s, t), \pi_{\partial X} \circ u(s, t)) = (\pi_R \circ u(s, t), \vartheta(s, t), \zeta(s, t)). \]
Finally, we require that there exists $a \in \mathbb{R}$ such that the map\[ (s, t) \mapsto (\pi_R \circ u(s, t), \vartheta(s, t), \zeta(s, t)) - (mTs + a, mTt, 0) \]
is of class $W^{k,p}$. The fibre, total space, projection and zero section are defined by\[ \mathcal{E}_{(j,u)} := W^{k-1,p_0}(\text{Hom}^{0,1}(\mathcal{T}^{\hat{\Sigma}}, j), (u^*T \hat{X}, J)), \quad \text{for every } (j, u) \in \mathcal{T} \times \mathcal{B}, \]
\[ \mathcal{E} := \prod_{(j,u) \in \mathcal{T} \times \mathcal{B}} \mathcal{E}_{(j,u)} = \{ (j, u, \xi) \mid (j, u) \in \mathcal{T} \times \mathcal{B}, \xi \in \mathcal{E}_{(j,u)} \}, \]
\[ \pi(j, u, \eta) := (j, u), \quad \vartheta(j, u) := (j, u, 0). \]

The Cauchy–Riemann operators are sections\[ \overline{\partial}_j \colon \mathcal{B} \to \mathcal{E}, \quad \overline{\partial}_j(u) := Tu + J \circ Tu \circ j \in \mathcal{E}_{(j,u)}, \]
\[ \overline{\partial} \colon \mathcal{T} \times \mathcal{B} \to \mathcal{E}, \quad \overline{\partial}(j, u) := \overline{\partial}_j(u). \]
Let $(j, u) \in \mathcal{T} \times \mathcal{B}$ be such that $\overline{\partial}(j, u) = 0$. There is a vertical projection map\[ P_{(j,u)} \colon T_{(j,u)}\mathcal{E} \to \mathcal{E}_{(j,u)} \]
which is given by\[ P_{(j,u)}(\eta) := (\text{id} - D(\vartheta)(j, u, 0))\eta. \]
The linearizations of $\overline{\partial}_j$ and $\overline{\partial}$ at $(j, u)$ are then given by\[ D_{(j,u)} := P_{(j,u)} \circ D(\overline{\partial}_j)(u) \colon T_u \mathcal{B} \to \mathcal{E}_{(j,u)}, \]
\[ L_{(j,u)} := P_{(j,u)} \circ D(\overline{\partial})(j, u) \colon T_j \mathcal{T} \oplus T_u \mathcal{B} \to \mathcal{E}_{(j,u)}. \]
Define also the restriction\[ F_{(j,u)} := L_{(j,u)}|_{T_j \mathcal{T}} : T_j \mathcal{T} \to \mathcal{E}_{(j,u)}. \]
Now choose a smooth function $f \colon \hat{\Sigma} \to \mathbb{R}$ such that $f(s, t) = \delta s$ on the cylindrical end of $\hat{\Sigma}$. Define the restriction of $D_{(j,u)}$, denoted $D_\delta$, and the conjugation of $D_{(j,u)}$, denoted $D_0$, to be the unique maps such that the diagram
\[
\begin{array}{ccc}
W^{k,p}(u^*T \hat{X}) & \xrightarrow{\xi \mapsto e^f \xi} & W^{k,p}(u^*T \hat{X}) \\
\downarrow D_\delta & & \downarrow D_0 \\
W^{k-1,p}(\text{Hom}^{0,1}(\mathcal{T}^\hat{\Sigma}, u^*T \hat{X})) & \xleftarrow{\eta \mapsto e^f \eta} & W^{k-1,p}(\text{Hom}^{0,1}(\mathcal{T}^\hat{\Sigma}, u^*T \hat{X}))
\end{array}
\]
commutes. The maps $D_\delta$ and $D_0$ are real linear Cauchy–Riemann operators.
Lemma 4.19. If \( n = 1 \) then \( L_{(j,u)} \) is surjective.

Proof. Let \( \tau_1 \) be a global complex trivialization of \( u^*T\hat{X} \) extending to an asymptotic unitary trivialization near \( z \). Let \( \tau_2 \) be the unitary trivialization of \( u^*T\hat{X} \) near \( z \) which is induced from the decomposition \( T_{(r,s)}(\mathbb{R} \times \partial X) = \langle \partial_r \rangle \oplus \langle R^{0X}_r \rangle \). It is shown in the proof of [Wen16, Lemma 7.10] that the operator \( D_0^\tau \) is asymptotic at \( z \) to \(-J\partial_t + \delta\), which is nondegenerate and has Conley–Zehnder index \( \mu_{\hat{X}}(\tau_2) = -1 \). Therefore, \( z \) is an odd puncture and \#\( \mathbb{Z}_0 = 0 \). We show that \( c_l^2(u^*TX) = m \), where \( m \) is the multiplicity of the asymptotic Reeb orbit \( \gamma \):

\[
c_l^2(u^*T\hat{X}) = c_l^2(u^*T\hat{X}) + \deg(\tau_1|_{E_z} \circ (\tau_2|_{E_z})^{-1}) \quad \text{[by [Wen16, Exercise 5.3]]}
\]

\[
= \deg(\tau_1|_{E_z} \circ (\tau_2|_{E_z})^{-1}) \quad \text{[by definition of } c_l^2]\n\]

where in the last equality we have used the fact that if \((s, t)\) are the cylindrical coordinates near \( z \), then for \( s \) large enough the map \( t \mapsto \tau_1|_{u(s,t)} \circ (\tau_2|_{u(s,t)})^{-1} \) winds around the origin \( m \) times. We show that \( \text{ind } D_0 \geq 2 \).

\[
\text{ind } D_0 = n\chi(\Sigma) + 2c_l^2(u^*T\hat{X}) + \mu_{\hat{X}}(\tau_2) \quad \text{[by Theorem 4.12]}
\]

\[
= 2m \quad \text{[since } n = 1 \text{ and } g = 0]\n\]

\[
\geq 2 \quad \text{[since } m \geq 1].
\]

By Corollary 4.14, this implies that \( D_0 \) is surjective. By Diagram (17), the operator \( D_{(j,u)} \) is also surjective. Therefore, \( L_{(j,u)} = F_{(j,u)} + D_{(j,u)} \) is also surjective. \( \square \)

From now until the end of this subsection, let \((X, \lambda^X)\) be a Liouville domain of dimension \( 2n \) and \((Y, \lambda^Y)\) be a Liouville domain of dimension \( 2n + 2 \) such that

1. \( X \subset Y \) and \( \partial X \subset \partial Y \);
2. the inclusion \( \iota: X \to Y \) is a Liouville embedding;
3. if \( x \in X \) then \( Z^X_x = Z^Y_x \);
4. if \( x \in \partial X \) then \( R^{0X}_x = R^{0Y}_x \).

In this case, we have an inclusion of completions \( \hat{X} \subset \hat{Y} \) as sets. By assumption, \( Z^X \) is \( \iota \)-related to \( Z^Y \), which implies that there is a map \( \hat{\iota}: \hat{X} \to \hat{Y} \) on the level of completions. Since in this case \( \hat{X} \subset \hat{Y} \), \( \hat{\iota} \) is the inclusion. Assume that \( J^X \in \mathcal{J}(X) \) and \( J^Y \in \mathcal{J}(Y) \) are almost complex structures on \( \hat{X} \) and \( \hat{Y} \) respectively, such that \( \hat{\iota}: \hat{X} \to \hat{Y} \) is holomorphic. As before, let \( \gamma \) be a Reeb orbit in \( \partial X \). Notice that \( \gamma \) can also be seen as a Reeb orbit in \( \partial Y \). Choose once and for all parametrizations \( \phi^X: S^1 \times D^{2n-2} \to O^X \) and \( \phi^Y: S^1 \times D^{2n} \to O^Y \) near \( \gamma \) with the properties as before, and also such that the diagram

\[
\begin{array}{ccc}
S^1 \times D^{2n-2} & \xrightarrow{\phi^X} & O^X \\
\downarrow & \cong & \downarrow \circ_{\phi^Y,\partial X} \\
S^1 \times D^{2n} & \xrightarrow{\phi^Y} & O^Y \\
\end{array}
\]

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commutes. We will consider the Cauchy–Riemann operator and its linearization for both $X$ and $Y$. We will use the notation

$$\pi^X: \mathcal{E}X \rightarrow \mathcal{T} \times B_X, \quad \overline{\partial}^X: \mathcal{T} \times B_X \rightarrow \mathcal{E}X, \quad L^X_{(j,u)}: T_j \mathcal{T} \oplus T_u B_X \rightarrow \mathcal{E}_{(j,u)} X,$$

$$\pi^Y: \mathcal{E}Y \rightarrow \mathcal{T} \times B_Y, \quad \overline{\partial}^Y: \mathcal{T} \times B_Y \rightarrow \mathcal{E}Y, \quad L^Y_{(j,u)}: T_j \mathcal{T} \oplus T_u B_Y \rightarrow \mathcal{E}_{(j,u)} Y$$

to distinguish the bundles and maps for $X$ and $Y$. Define maps

$$B_t: B_X \rightarrow B_Y, \quad B_t(u) := t \circ u,$$

$$\mathcal{E}t: \mathcal{E}X \rightarrow \mathcal{E}Y, \quad \mathcal{E}t(j, u, \eta) := (j, t \circ u, T \circ \eta).$$

Then, the diagrams

$$\begin{array}{ccc}
\mathcal{E}X & \xrightarrow{\pi^X} & \mathcal{T} \times B_X \\
\mathcal{E}Y & \xrightarrow{\mathcal{E}t} & \mathcal{T} \times B_Y \\
\end{array} \quad \begin{array}{ccc}
\mathcal{T} \times B X & \xrightarrow{z^X} & \mathcal{E}X \\
\mathcal{T} \times B Y & \xrightarrow{z^Y} & \mathcal{E}Y \\
\end{array}$$

commute. By the chain rule, the diagram

$$\begin{array}{ccc}
D^X_{(j,u)} & \circlearrowright & D^X_{(j,u)} \\
T_u B X & \xrightarrow{D \overline{\partial}^X_{T_j}(u)} & T_{(j,u,0)} \mathcal{E}X \\
D(\mathcal{E}t)(\overline{\partial}^X_{T_j}(u)) & \downarrow & D(\mathcal{E}t)(\overline{\partial}^X_{T_j}(u)) \\
T_{t \circ u} B Y & \xrightarrow{D \overline{\partial}^Y_{T_j}(t \circ u)} & T_{(j,t \circ u,0)} \mathcal{E}Y \\
& \downarrow & \downarrow \\
& & \mathcal{E}_{(j,t \circ u)} Y \\
& & \mathcal{E}_{(j,t \circ u)} Y \\
& & \mathcal{E}_{(j,t \circ u)} Y \\
\end{array}$$

is also commutative whenever $\overline{\partial}^X(j, u) = 0$. For simplicity, we will denote $t \circ u \in B_Y$ by $u$. Let $w \in \Sigma$ and define the evaluation map

$$\text{ev}^X: B X \rightarrow \hat{X}$$

$$u \mapsto u(w)$$

as well as its derivative $E^X_u := D(\text{ev}^X)(u): T_u B X \rightarrow T_{u(w)} \hat{X}$. In the following lemma, we show that if a holomorphic curve $u$ in $X$ is regular (in $X$) then the corresponding holomorphic curve $u$ in $Y$ is also regular. See also [MS22, Proposition A.1] for a similar result.
Lemma 4.20. Let \((j, u) \in \mathcal{T} \times \mathcal{B} X\) be such that \(\overline{D} X (j, u) = 0\). Assume that the normal Conley–Zehnder index of \(\gamma\) is 1.

1. If \(L^X_{(j, u)}\) is surjective then so is \(L^Y_{(j, u)}\).
2. If \(L^X_{(j, u)} \oplus E^X_u\) is surjective then so is \(L^Y_{(j, u)} \oplus E^Y_u\).

Proof. Consider the decomposition \(T_x \hat{Y} = T_x \hat{X} \oplus (T_x \hat{X})^\perp\) for \(x \in \hat{X}\). Let \(\tau\) be a global complex trivialization of \(u^* T \hat{Y}\), extending to an asymptotic unitary trivialization near the punctures, and such that \(\tau\) restricts to a trivialization of \(u^* T \hat{X}\) and \(u^* (T \hat{X})^\perp\). There are splittings

\[
T_u B X = T_u B X \oplus T_u^\perp B X, \quad \text{where} \quad T_u^\perp B X = W^{k, p, \delta}(u^* (T \hat{X})^\perp),
\]

\[
\mathcal{E}_{(j, u)} Y = \mathcal{E}_{(j, u)} X \oplus \mathcal{E}_{(j, u)}^\perp X, \quad \text{where} \quad \mathcal{E}_{(j, u)}^\perp X = W^{k-1, p, \delta}(\text{Hom}_{0,1}(T \hat{\Sigma}, u^* (T \hat{X})^\perp)).
\]

We can write the maps

\[
L^Y_{(j, u)} : T_j \mathcal{T} \oplus T_u B X \oplus T_u^\perp B X \rightarrow \mathcal{E}_{(j, u)} X \oplus \mathcal{E}_{(j, u)}^\perp X,
\]

\[
D^Y_{(j, u)} : T_u B X \oplus T_u^\perp B X \rightarrow \mathcal{E}_{(j, u)} X \oplus \mathcal{E}_{(j, u)}^\perp X,
\]

\[
L^X_{(j, u)} : T_j \mathcal{T} \oplus T_u B X \rightarrow \mathcal{E}_{(j, u)} X,
\]

\[
F^Y_{(j, u)} : T_j \mathcal{T} \rightarrow \mathcal{E}_{(j, u)} X \oplus \mathcal{E}_{(j, u)}^\perp X,
\]

\[
E^Y_u : T_u B X \oplus T_u^\perp B X \rightarrow T_{u(w)} \hat{X} \oplus (T_{u(w)} \hat{X})^\perp
\]

as block matrices

\[
L^Y_{(j, u)} = \begin{bmatrix} F^X_{(j, u)} & D^X_{(j, u)} & D^T N_{(j, u)} \\ 0 & 0 & D^T N_{(j, u)} \end{bmatrix}, \tag{19}
\]

\[
D^Y_{(j, u)} = \begin{bmatrix} D^X_{(j, u)} & D^T N_{(j, u)} \\ 0 & D^N_{(j, u)} \end{bmatrix}, \tag{20}
\]

\[
L^X_{(j, u)} = \begin{bmatrix} F^X_{(j, u)} & D^X_{(j, u)} \end{bmatrix}, \tag{21}
\]

\[
F^Y_{(j, u)} = \begin{bmatrix} F^X_{(j, u)} \\ 0 \end{bmatrix}, \tag{22}
\]

\[
E^Y_u = \begin{bmatrix} E^X_u \\ 0 \\ E^N_{(j, u)} \end{bmatrix}, \tag{23}
\]

where (23) follows by definition of the evaluation map, (22) is true since \(F^Y_{(j, u)}\) is given by the formula \(F^Y_{(j, u)}(y) = J \circ T_u \circ y\), (20) follows because Diagram (18) commutes, and (21) and (19) then follow by definition of the linearized Cauchy–Riemann operator. Let \(D^N_{\delta}\) be the restriction and \(D^N_{0,\delta}\) be the conjugation of \(D^N_{(j, u)}\) (as in Diagram (17)). Denote by \(B^N_{\gamma}\) the asymptotic operator of \(D^N_{(j, u)}\) at \(z\). Then the asymptotic operator of \(D^N_{0,\delta}\) at \(z\) is \(B^N_{\gamma} + \delta\), which by assumption has Conley–Zehnder index equal to 1. We show that \(\text{ind} D^N_{0,\delta} = 2\).

\[
\text{ind} D^N_{0,\delta} = \chi(\hat{\Sigma}) + 2c_1(u^* T \hat{X}) + \mu^\tau_{\text{CZ}}(B^N_{\gamma} + \delta) \tag{by Theorem 4.12}
\]

\[
= 2 \quad \text{[since \(\mu^\tau_{\text{CZ}}(B^N_{\gamma} + \delta) = 1\).}
\]

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We prove (1).

\[
\text{ind } D_{NN}^0 = 2 \implies D_{NN}^0 \text{ is surjective} \quad \text{[by Corollary 4.14]}
\]
\[
\implies D_{\delta}^{NN} \text{ is surjective} \quad [D_{NN}^0 \text{ and } D_{\delta}^{NN} \text{ are conjugate}]
\]
\[
\implies D_{(j,u)}^{NN} \text{ is surjective} \quad [D_{\delta}^{Y} \text{ is a restriction of } D_{(j,u)}^{Y}]
\]
\[
\implies L_{(j,u)}^{Y} \text{ is surjective} \quad [L_{(j,u)}^{X} \text{ is surjective by assumption}].
\]

We prove (2).

\[
\text{ind } D_{NN}^0 = 2
\]
\[
\implies D_{NN}^0 \oplus E_{u}^{NN} \text{ is surjective} \quad \text{[by Lemma 4.16]}
\]
\[
\implies D_{\delta}^{NN} \oplus E_{u}^{NN} \text{ is surjective} \quad [D_{NN}^0 \oplus E_{u}^{NN} \text{ and } D_{\delta}^{NN} \oplus E_{u}^{NN} \text{ are conjugate}]
\]
\[
\implies D_{(j,u)}^{NN} \oplus E_{u}^{NN} \text{ is surjective} \quad [D_{\delta}^{Y} \oplus E_{u}^{Y} \text{ is a restriction of } D_{(j,u)}^{Y} \oplus E_{u}^{Y}]
\]
\[
\implies L_{(j,u)}^{Y} \oplus E_{u}^{Y} \text{ is surjective} \quad [L_{(j,u)}^{X} \oplus E_{u}^{X} \text{ is surjective by assumption}].
\]

\section{4.6 Moduli spaces of curves in ellipsoids}

We now use the techniques explained in the past two subsections to compute the augmentation map of an ellipsoid (Theorem 4.34). The proof of this theorem consists in an explicit count of curves in the ellipsoid satisfying a tangency constraint (Proposition 4.32) together with the fact that the moduli space of such curves is transversely cut out (Propositions 4.29 to 4.31). Therefore, the explicit count agrees with the virtual count. We now state the assumptions for this subsection.

Let \( a_1 < \cdots < a_n \in \mathbb{R}_{>0} \) be rationally linearly independent and consider the ellipsoid \( E(a_1, \ldots, a_n) \subset \mathbb{C}^n \). By [GH18, Section 2.1], \( \partial E(a_1, \ldots, a_n) \) has exactly \( n \) simple Reeb orbits \( \gamma_1, \ldots, \gamma_n \), which satisfy

\[
\gamma_j(t) = \sqrt{\frac{a_j}{\pi}} e^{\frac{2\pi i}{a_j}} e_j,
\]

\[
A(\gamma_j^m) = ma_j,
\]

\[
\mu_{CZ}(\gamma_j^m) = n - 1 + 2 \sum_{i=1}^{n} \left\lfloor \frac{ma_i}{a_j} \right\rfloor,
\]

where \( \gamma_j: \mathbb{R}/a_j\mathbb{Z} \rightarrow \partial E(a_1, \ldots, a_n) \) and \( e_j \) is the \( j \)th vector of the canonical basis of \( \mathbb{C}^n \) as a vector space over \( \mathbb{C} \). For simplicity, for every \( \ell = 1, \ldots, n \) denote \( E_\ell = E(a_1, \ldots, a_\ell) \subset \mathbb{C}^\ell \). Notice that \( \gamma_1 \) is a Reeb orbit of \( \partial E_1, \ldots, \partial E_n \). Define maps

\[
\iota_\ell: \mathbb{C}^\ell \rightarrow \mathbb{C}^{\ell+1}, \quad \iota_\ell(z_1, \ldots, z_\ell) := (z_1, \ldots, z_\ell, 0)
\]
\[
h_\ell: \mathbb{C}^\ell \rightarrow \mathbb{C}, \quad h_\ell(z_1, \ldots, z_\ell) := z_1.
\]

The maps \( \iota_\ell: E_\ell \rightarrow E_{\ell+1} \) are Liouville embeddings satisfying the assumptions in Section 4.5. Define also

\[
x_\ell := 0 \in \mathbb{C}^\ell,
\]
\[
D_\ell := \{(z_1, \ldots, z_\ell) \in \mathbb{C}^\ell \mid z_1 = 0\} = h_\ell^{-1}(0).
\]
Choose an admissible almost complex structure $J_\ell \in \mathcal{J}(E_\ell, D_\ell)$ on $\hat{E}_\ell$ such that $J_\ell$ is the canonical almost complex structure of $\mathbb{C}$ near 0. We assume that the almost complex structures are chosen in such a way that $\iota_\ell: \hat{E}_\ell \to \hat{E}_{\ell+1}$ is holomorphic and also such that there exists a biholomorphism $\varphi: \hat{E}_1 \to \mathbb{C}$ such that $\varphi(z) = z$ for $z$ near 0 ∈ $\mathbb{C}$ (see Lemma 4.21 below). Let $m \in \mathbb{Z}_{\geq 1}$ and assume that $ma_1 < a_2 < \cdots < a_n$.

Consider the sphere $S^2$, without any specified almost complex structure, with a puncture $z \in S^2$ and an asymptotic marker $v \in (T_z S^2 \setminus \{0\})/\mathbb{R}_{>0}$, and also a marked point $w \in \hat{S}^2 = S^2 \setminus \{z\}$. For $k \in \mathbb{Z}_{\geq 0}$, denote

$$\mathcal{M}_p^{\ell, (k)} := \mathcal{M}_{E_\ell}^{S, J_\ell}(\gamma^m_1) \langle T^{(k)}x_\ell \rangle_p$$

$$\quad := \left\{ (j, u) \bigg| j \text{ is an almost complex structure on } S^2, \right.$$

$$\left. u: (\hat{S}^2, j) \to (\hat{E}_\ell, J_\ell) \text{ is as in Definition 4.17,} \right.$$

$$\left. u(w) = x_\ell \text{ and } u \text{ has contact order } k \text{ to } D_\ell \text{ at } x_\ell \right\}.$$

Here, the subscript $p$ means that the moduli space consists of parametrized curves, i.e. we are not quotienting by biholomorphisms. Denote the moduli spaces of regular curves and of unparametrized curves by

$$\mathcal{M}_{p, \text{reg}}^{\ell, (k)} := \mathcal{M}_{E_\ell}^{S, J_\ell}(\gamma^m_1) \langle T^{(k)}x_\ell \rangle_{p, \text{reg}},$$

$$\mathcal{M}^{\ell, (k)} := \mathcal{M}_{E_\ell}^{S, J_\ell}(\gamma^m_1) \langle T^{(k)}x_\ell \rangle := \mathcal{M}_p^{\ell, (k)}/\sim.$$
\[ = \exp\left(\frac{\rho}{2}\right). \]

Therefore, \( \varphi(z) = z \) for \( z \) near 0 \( \in \mathbb{B}(a) \subset \mathbb{C} \), and in particular \( \varphi \) can be extended smoothly to a map \( \varphi: \hat{\mathbb{B}}(a) \to \mathbb{C} \). We show that \( \varphi \) is holomorphic.

\[
\begin{align*}
    j \circ D\varphi(\rho, w)(\partial_\rho) &= \left(\frac{\partial}{\partial \rho} \left(f(\rho) | w\right)\right) \left(\frac{\partial}{\partial r} \varphi(\rho, w)\right) \quad \text{[by definition of } \varphi \text{]} \\
    &= \frac{2\pi}{a} g(\rho) \left( f(\rho) | w\right) \left(\frac{\partial}{\partial r} \varphi(\rho, w)\right) \quad \text{[by definition of } f \text{]} \\
    &= \frac{2\pi}{a} g(\rho) \varphi(\rho, w) \left(\frac{\partial}{\partial r} \varphi(\rho, w)\right) \quad \text{[by definition of } \varphi \text{]} \\
    &= \frac{2\pi}{a} g(\rho) \left(\frac{\partial}{\partial \theta} \varphi(\rho, w)\right) \quad \text{[by definition of } j \text{]} \\
    &= g(\rho) D\varphi(\rho, w)(R^\mathbb{B}(a)) \quad \text{[by } [\text{GH18, Equation (2.2)}]} \\
    &= D\varphi(\rho, w) \circ J(\partial_\rho) \quad \text{[by definition of } J \text{]},
\end{align*}
\]

Where \((r, \theta)\) are the polar coordinates of \( \mathbb{C} \). Since \( \varphi \) is holomorphic and \( \varphi \) is the identity near the origin, we conclude that \( J \) is the canonical almost complex structure of \( \mathbb{C} \) near the origin. In particular, \( J \) can be extended smoothly to an almost complex structure on \( \hat{\mathbb{B}}(a) \), which proves (2). Finally, we show that \( \varphi \) is a diffeomorphism. For this, it suffices to show that \( \Phi^{-1} \circ \varphi: \mathbb{R} \times \partial B(a) \to \mathbb{R} \times \partial B(a) \) is a diffeomorphism. This map is given by \( \Phi^{-1} \circ \varphi(\rho, w) = (2 \ln(f(\rho)), w) \). Since

\[
\frac{d}{d\rho}(2 \ln(f(\rho))) = \frac{2f'(\rho)}{f(\rho)} = \frac{4\pi}{a} g(\rho) > 0,
\]

\( \varphi \) is a diffeomorphism. \( \square \)

**Lemma 4.22.** Let \( \text{inv}: \overline{\mathbb{C}} \to \overline{\mathbb{C}} \) be the map given by \( \text{inv}(z) = 1/z \) and consider the vector \( V := D\text{inv}(0) \partial_z \in T_\infty \mathbb{C} \). For every \( j \in \mathcal{T} \) there exists a unique biholomorphism \( \psi_j: (\mathbb{C}, j_0) \to (S^2, j) \) such that

\[
\begin{align*}
    \psi_j(0) &= w, \quad \psi_j(\infty) = z, \quad D\psi_j(\infty)V = \frac{v}{\|v\|},
\end{align*}
\]

where \( \|\cdot\| \) is the norm coming from the canonical Riemannian metric on \( S^2 \) as the sphere of radius 1 in \( \mathbb{R}^3 \).

**Proof.** By the uniformization theorem [dB16, Theorem XII.0.1], there exists a biholomorphism \( \phi: (S^2, j) \to (\overline{\mathbb{C}}, j_0) \). Since there exists a unique Möbius transformation \( \psi_0: (\overline{\mathbb{C}}, j_0) \to (\mathbb{C}, j_0) \) such that

\[
\begin{align*}
    \psi_0(0) &= \phi(w), \quad \psi_0(\infty) = \phi(z), \quad D\psi_0(\infty)V = D\phi(z) \frac{v}{\|v\|},
\end{align*}
\]

the result follows. \( \square \)

We will denote also by \( \psi_j \) the restriction \( \psi_j: (\mathbb{C}, j_0) \to (S^2, j) \). 

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Lemma 4.23. If $((j,u) \in M^{1,(0)}$ then $\varphi \circ u \circ \psi_j : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree $m$.

Proof. Since $u$ is positively asymptotic to $\gamma_1^m$, the map $\varphi \circ u \circ \psi_j$ goes to $\infty$ as $z$ goes to $\infty$. Therefore, $\varphi \circ u \circ \psi_j$ is a polynomial. Again using the fact that $u$ is positively asymptotic to $\gamma_1^m$, we conclude that for $r$ big enough the path $\theta \mapsto \varphi \circ u \circ \psi_j(r e^{i\theta})$ winds around the origin $m$ times. This implies that the degree of $\varphi \circ u \circ \psi_j$ is $m$. \qed

We now wish to compute the normal Conley–Zehnder index of $\gamma_1^m$. For this, we will use the following result.

Proposition 4.24 ([Gut12, Proposition 41]). Let $S$ be a symmetric, nondegenerate $2 \times 2$-matrix and $T > 0$ be such that $\exp(TJ_0S) \neq I$. Consider the path of symplectic matrices $A : [0,T] \to Sp(2)$ given by

$$A(t) := \exp(tJ_0S).$$

Let $a_1$ and $a_2$ be the eigenvalues of $S$ and sign $S$ be its signature. Then,

$$\mu_{\text{CZ}}(A) = \begin{cases} \left(\frac{1}{2} + \frac{\sqrt{a_1a_2T}}{2\pi}\right) \text{sign } S & \text{if } \text{sign } S \neq 0, \\ 0 & \text{if } \text{sign } S = 0. \end{cases}$$

Lemma 4.25. For every $\ell = 1,\ldots,n-1$, view $\gamma_1^m$ as a Reeb orbit of $\partial E \subset \partial E_{\ell+1}$. The normal Conley–Zehnder index of $\gamma_1^m$ is $1$.

Proof. By [GH18, Equation (2.2)], the Reeb vector field of $\partial E_{\ell+1}$ is given by

$$R_{\partial E_{\ell+1}} = \frac{2\pi}{a_{\ell+1}} \sum_{j=1}^{\ell+1} \frac{1}{a_j} \frac{\partial}{\partial \theta_j},$$

where $\theta_j$ denotes the angular polar coordinate of the $j$th summand of $\mathbb{C}^{\ell+1}$. Therefore, the flow of $R_{\partial E_{\ell+1}}$ is given by

$$\phi^t_{R_{\partial E_{\ell+1}}} : \partial E_{\ell+1} \to \partial E_{\ell+1}$$

$$(z_1,\ldots,z_{\ell+1}) \mapsto \left(e^{\frac{2\pi}{\alpha} t} z_1,\ldots,e^{\frac{2\pi}{\alpha(t+1)} t_{\ell+1}}\right).$$

The diagram

$$\begin{array}{cccc}
\xi_{\gamma_1^m(0)}^{\partial E_\ell} & \longrightarrow & \xi_{\gamma_1^m(0)}^{\partial E_{\ell+1}} & \xrightarrow{\phi^t_{R_{\gamma_1^m(0)}}} \mathbb{C} \\
D\phi^t_{R_{\gamma_1^m(0)}} & & D\phi^t_{R_{\gamma_1^m(0)}} & \times \exp\left(\frac{2\pi i}{\alpha(t+1)}\right) \\
\xi_{\gamma_1^m(t)}^{\partial E_\ell} & \longrightarrow & \xi_{\gamma_1^m(t)}^{\partial E_{\ell+1}} & \left(\xi_{\gamma_1^m(t)}^{\partial E_{\ell+1}}\right)^+ \\
\end{array}$$

commutes. Define a path $A_{\gamma_1^m} : [0,ma_1] \to Sp(2)$ by $A_{\gamma_1^m}(t) = \exp(tJ_0S)$, where

$$S = \frac{2\pi}{a_{\ell+1}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad 41$$
The only eigenvalue of $S$ is $2\pi/a_{\ell+1}$, which has multiplicity 2. Therefore, the signature of $S$ is $\text{sign } S = 2$. These facts allow us to compute $\mu_{CZ}^\perp(\gamma_1^m)$ using Proposition 4.24:

\[
\mu_{CZ}^\perp(\gamma_1^m) = \mu_{CZ}(A_{\gamma_1^m}) = \left(\frac{1}{2} + \sqrt{\frac{2\pi \cdot 2\pi}{a_{\ell+1}a_{\ell+1}}} \frac{ma_1}{2\pi}\right) \text{sign } S \quad \text{[by definition of } \mu_{CZ}^\perp]\]

\[
= \frac{1}{2} \text{sign } S \quad \text{[since } ma_1 < a_2 < \cdots < a_n]\]

\[
= 1 \quad \text{[by the discussion above]}.
\]

**Lemma 4.26.** If $\ell = 1, \ldots, n$ then $\gamma_1^m$ is the unique Reeb orbit of $\partial E_\ell$ such that $\mu_{CZ}(\gamma_1^m) = \ell - 1 + 2m$.

**Proof.** First, notice that

\[
\mu_{CZ}(\gamma_1^m) = \ell - 1 + 2 \sum_{j=1}^\ell \left\lfloor \frac{ma_1}{a_j} \right\rfloor \quad \text{[by equation (26)]}
\]

\[
= \ell - 1 + 2m \quad \text{[since } ma_1 < a_2 < \cdots < a_n]\]

Conversely, let $\gamma = \gamma_i^k$ be a Reeb orbit of $\partial E_\ell$ with $\mu_{CZ}(\gamma) = \ell - 1 + 2m$. By equation (26), this implies that

\[
m = \sum_{j=1}^\ell \left\lfloor \frac{ka_i}{a_j} \right\rfloor. \quad \text{(27)}
\]

We show that $i = 1$. Assume by contradiction otherwise. Then

\[
m = \sum_{1 \leq j \leq \ell} \left\lfloor \frac{ka_i}{a_j} \right\rfloor \quad \text{[by equation (27)]}
\]

\[
\geq \sum_{1 \leq j \leq i} \left\lfloor \frac{ka_i}{a_j} \right\rfloor \quad \text{[since every term in the sum is } \geq 0]\]

\[
= \left\lfloor \frac{ka_i}{a_1} \right\rfloor + \sum_{1 < j < i} \left\lfloor \frac{ka_i}{a_j} \right\rfloor + k \quad \text{[since by assumption, } i > 1]\]

\[
\geq (m + i - 1)k \quad \text{[since } ma_1 < a_2 < \cdots < a_i]\]

\[
> mk \quad \text{[since by assumption, } i > 1],
\]

which is a contradiction, and therefore $i = 1$. We show that $k = m$, using the fact that $m \geq \left\lfloor ka_i/a_1 \right\rfloor = k$.

\[
m = \sum_{1 \leq j \leq \ell} \left\lfloor \frac{ka_i}{a_j} \right\rfloor \quad \text{[by equation (27) and since } i = 1]\]

\[
= k + \sum_{2 \leq j \leq \ell} \left\lfloor \frac{ka_i}{a_j} \right\rfloor
\]

\[
= k \quad \text{[since } k \leq m \text{ and } ka_1 \leq ma_1 < a_1 < \cdots < a_n]. \quad \square
\]

Using the previous results, we can now compute the linearized contact homology of $E_n$. 42
Lemma 4.27. The module $CH_{n-1+2m}(E_n)$ is the free $\mathbb{Q}$-module generated by $\gamma_1^m$.

Proof. By equation (26), every Reeb orbit of $\partial E_n$ is good. We claim that the differential $\partial: CC(E_n) \to CC(E_n)$ is zero. Assume by contradiction that there exists a Reeb orbit $\gamma$ such that $\partial \gamma \neq 0$. By definition of $\partial$, this implies that there exist Reeb orbits $\eta, \alpha_1, \ldots, \alpha_p$ such that

$$0 \neq \#\text{vir} M^{\ell_n}_{\partial E_n}(\gamma; \eta, \alpha_1, \ldots, \alpha_p),$$

$$0 \neq \#\text{vir} M^{\ell_n}_{E_n}(\alpha_j), \text{ for } j = 1, \ldots, p.$$ 

By assumption on the virtual perturbation scheme,

$$0 = \text{virdim } M^{\ell_n}_{E_n}(\alpha_j) = n - 3 + \mu_{CZ}(\alpha_j) \quad \text{for every } j = 1, \ldots, p,$$

$$0 = \text{virdim } M^{\ell_n}_{\partial E_n}(\gamma; \eta, \alpha_1, \ldots, \alpha_p)$$

$$= (n - 3)(2 - (2 + p)) + \mu_{CZ}(\gamma) - \sum_{j=1}^{p} \mu_{CZ}(\alpha_j) - 1$$

$$= \mu_{CZ}(\gamma) - \mu_{CZ}(\eta) - 1$$

$$\in 1 + 2\mathbb{Z},$$

where in the last line we used equation (26). This gives the desired contradiction, and we conclude that $\partial: CC(E_n) \to CC(E_n)$ is zero. Therefore, $CH(E_n) = CC(E_n)$ is the free $\mathbb{Q}$-module generated by the Reeb orbits of $\partial E_n$. By Lemma 4.26, $\gamma_1^m$ is the unique Reeb orbit of $\partial E_n$ with $\mu_{CZ}(\gamma_1^m) = n - 1 + 2m$, from which the result follows. \qed

Lemma 4.28. If $\ell = 1, \ldots, n$ and $k \in \mathbb{Z}_{\geq 1}$ then $M^{\ell,(k)} = M^{1,(k)}_{\ell}$ and $M^{\ell,(k)} = M^{1,(k)}$.

Proof. It suffices to show that $M^{\ell,(k)}_{\ell} = M^{\ell+1,(k)}_{\ell}$ for every $\ell = 1, \ldots, n - 1$. The inclusion $M^{\ell,(k)}_{\ell} \subset M^{\ell+1,(k)}_{\ell}$ follows from the fact that the inclusion $\hat{E}_{\ell} \hookrightarrow \hat{E}_{\ell+1}$ is holomorphic and the assumptions on the symplectic divisors. To prove that $M^{\ell+1,(k)}_{\ell} \subset M^{\ell,(k)}_{\ell}$, it suffices to assume that $(j, u) \in M^{\ell+1,(k)}_{\ell}$ and to show that the image of $u$ is contained in $\hat{E}_\ell \subset \hat{E}_{\ell+1}$. Since $u$ has contact order $k$ to $D_{\ell+1}$ at $x_{\ell+1} = t_\ell(x_\ell)$, we conclude that $u$ is not disjoint from $\hat{E}_\ell$. By Lemma 3.26, $u$ is contained in $\hat{E}_\ell$. \qed

We now prove that the moduli spaces $M^{\ell,(k)}_{\ell}$ are regular. The proof strategy is as follows.

1. Proposition 4.29 deals with the moduli spaces $M^{1,(0)}$. We show that the linearized Cauchy–Riemann operator is surjective using Lemma 4.19.

2. Proposition 4.30 deals with the moduli spaces $M^{\ell,(1)}$. Here, we need to consider the linearized Cauchy–Riemann operator together with an evaluation map. We show inductively that this map is surjective using Lemma 4.20.

3. Finally, Proposition 4.31 deals with the moduli spaces $M^{\ell,(k)}$. We now need to consider the jet evaluation map. We prove inductively that this map is surjective by writing it explicitly.

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Proposition 4.29. The moduli spaces $\mathcal{M}_p^{1,(0)}$ and $\mathcal{M}_1^{1,(0)}$ are transversely cut out.

Proof. It is enough to show that $\mathcal{M}^{1,(0)}_p$ is transversely cut out, since this implies that $\mathcal{M}^{1,(0)}_1$ is transversely cut out as well. Recall that $\mathcal{M}^{1,(0)}_p$ can be written as the zero set of the Cauchy–Riemann operator $\bar{\partial}^1: \mathcal{T} \times \mathcal{B}E_1 \rightarrow \mathcal{E}E_1$. It suffices to assume that $(j, u) \in (\bar{\partial}^1)^{-1}(0)$ and to prove that the linearization

$$L_{(j, u)}^1: T_j \mathcal{T} \oplus T_u \mathcal{B}E_1 \rightarrow \mathcal{E}_{(j, u)}E_1$$

is surjective. This follows from Lemma 4.19. \qed

Proposition 4.30. If $\ell = 1, \ldots, n$ then $\mathcal{M}^{\ell,(1)}_p$ and $\mathcal{M}^{\ell,(1)}_1$ are transversely cut out.

Proof. We will use the notation of Section 4.5 with $X = E_\ell$ and $Y = E_{\ell + 1}$. We will show by induction on $\ell$ that $\mathcal{M}^{\ell,(1)}_p$ is transversely cut out. This implies that $\mathcal{M}^{\ell,(1)}_1$ is transversely cut out as well.

We prove the base case. By Proposition 4.29, $\mathcal{M}^{1,(0)}_p$ is a smooth manifold. Consider the evaluation map

$$ev^1: \mathcal{M}^{1,(0)}_p \rightarrow \hat{E}_1 \quad (j, u) \mapsto u(w).$$

Notice that $\mathcal{M}^{1,(1)}_p = (ev^1)^{-1}(x_1)$. We wish to show that the linearized evaluation map $E^1_{(j, u)} = D(ev^1)(j, u): T_{(j, u)}\mathcal{M}^{1,(0)}_p \rightarrow T_{u(w)}\hat{E}_1$ is surjective whenever $u(w) = ev^1(j, u) = x_1$. There are commutative diagrams

$$\begin{array}{ccc}
\mathcal{M}^{1,(0)}_p & \xrightarrow{\Phi} & \mathcal{M} \\
\downarrow ev^1 & & \downarrow ev_M \\
\hat{E}_1 & \xleftarrow{\varphi} & \mathcal{C}
\end{array} \quad \begin{array}{ccc}
T_{(j, u)}\mathcal{M}^{1,(0)}_p & \xrightarrow{D\Phi(j, u)} & T_f\mathcal{M} \\
\downarrow E^1_{(j, u)} & & \downarrow E_M \\
T_x\hat{E}_1 & \xleftarrow{D\varphi(x_1)} & \mathcal{C}
\end{array}$$

where

$$\mathcal{M} := \{ f: \mathbb{C} \rightarrow \mathbb{C} \mid f \text{ is a polynomial of degree } m \},$$

$$\mathcal{C} := \{ (a_0, \ldots, a_m) \in \mathbb{C}^{m+1} \mid a_m \neq 0 \},$$

$$\Phi(j, u) := \varphi \circ u \circ \psi_j,$$

$$ev_M(f) := f(0),$$

$$ev_C(a_0, \ldots, a_m) := a_0,$$

$$P(a_0, \ldots, a_m)(z) := a_0 + a_1 z + \cdots + a_m z^m,$$

and the diagram on the right is obtained by linearizing the one on the left. The map $\Phi$ is well-defined by Lemma 4.23. Since $E_{\mathbb{C}}(a_0, \ldots, a_m) = a_0$ is surjective, $E^1_u$ is surjective as well. This finishes the proof of the base case.

We prove the induction step, i.e. that if $\mathcal{M}^{\ell,(1)}_p$ is transversely cut out then so is $\mathcal{M}^{\ell + 1,(1)}_p$. We prove that $\mathcal{M}^{\ell,(1)}_{p,\text{reg}} \subset \mathcal{M}^{\ell + 1,(1)}_{p,\text{reg}}$. For this, assume that $(j, u) \in \mathcal{M}^{\ell,(1)}_p$ is such that $L^\ell_{(j, u)} \oplus E^1_u: T_j \mathcal{T} \oplus T_u \mathcal{B}E_{\ell} \rightarrow \mathcal{E}_{(j, u)}E_{\ell} \oplus T_{x_{\ell + 1}}\hat{E}_{\ell + 1}$ is surjective. By Lemma 4.20,

$$L^{\ell + 1}_{(j, u)} \oplus E^{\ell + 1}_u: T_j \mathcal{T} \oplus T_u \mathcal{B}E_{\ell + 1} \rightarrow \mathcal{E}_{(j, u)}E_{\ell + 1} \oplus T_{x_{\ell + 1}}\hat{E}_{\ell + 1}$$

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is also surjective, which means that \((j, u) \in \mathcal{M}_p^{\ell+1, (1)}\). This concludes the proof of \(\mathcal{M}_p^{\ell, (1)} \subseteq \mathcal{M}_p^{\ell+1, (1)}\). Finally, we show that \(\mathcal{M}_p^{\ell+1, (1)} = \mathcal{M}_p^{\ell+1, (1)}\).

\[
\mathcal{M}_p^{\ell, (1)} \subseteq \mathcal{M}_p^{\ell+1, (1)} \quad [\text{since regular curves form a subset}]
\]

\[
= \mathcal{M}_p^{\ell, (1)} \quad [\text{by Lemma 4.28}]
\]

\[
= \mathcal{M}_p^{\ell, (1)} \subseteq \mathcal{M}_p^{\ell+1, (1)} \quad [\text{by the induction hypothesis}]
\]

\[
\subseteq \mathcal{M}_p^{\ell+1, (1)} \quad [\text{proven above}]. \quad \Box
\]

**Proposition 4.31.** If \(\ell = 1, \ldots, n\) and \(k = 1, \ldots, m\) then \(\mathcal{M}_p^{\ell, (k)}\) and \(\mathcal{M}_p^{\ell, (k)}\) are transversely cut out.

**Proof.** By Proposition 4.30, \(\mathcal{M}_p^{\ell, (1)}\) is a smooth manifold. Consider the jet evaluation map

\[
j^{\ell, (k)} : \mathcal{M}_p^{\ell, (1)} \longrightarrow \mathbb{C}^{k-1}
\]

\[
(j, u) \longmapsto ((h_\ell \circ u \circ \psi_j)^{(1)}(0), \ldots, (h_\ell \circ u \circ \psi_j)^{(k-1)}(0)).
\]

The moduli space \(\mathcal{M}_p^{\ell, (k)}\) is given by \(\mathcal{M}_p^{\ell, (k)} = (j^{\ell, (k)})^{-1}(0)\). We will prove by induction on \(\ell\) that \(\mathcal{M}_p^{\ell, (k)}\) is transversely cut out. This shows that \(\mathcal{M}_p^{\ell, (k)}\) is transversely cut out as well. Define \(J^{\ell, (k)}_{(j, u)} := D(j^{\ell, (k)})(j, u) : T_{(j, u)} \mathcal{M}_p^{\ell, (1)} \longrightarrow \mathbb{C}^{k-1}\).

We prove the base case, i.e. that \(\mathcal{M}_p^{1, (k)}\) is transversely cut out. For this, it suffices to assume that \((j, u) \in \mathcal{M}_p^{1, (1)}\) is such that \(j^{1, (k)}(j, u) = 0\) and to prove that \(J^{1, (k)}_{(j, u)}\) is surjective. There are commutative diagrams

\[
j^{1, (k)} \quad \Phi \quad \mathcal{M} \quad \leftarrow \quad C \quad \leftarrow \quad \mathcal{M}
\]

\[
\mathcal{M}_p^{1, (1)} \quad \Phi \quad \mathcal{M} \quad \leftarrow \quad C \quad \leftarrow \quad \mathcal{M}
\]

\[
T_{(j, u)} \mathcal{M}_p^{1, (1)} \quad D\Phi_{(j, u)} \quad \mathcal{M} \quad \leftarrow \quad C \quad \leftarrow \quad \mathcal{M}
\]

\[
J^{1, (k)}_{(j, u)} \quad \mathcal{M}_p^{1, (1)} \quad \leftarrow \quad C \quad \leftarrow \quad \mathcal{M}
\]

where

\[
\mathcal{M} := \{ f : C \longrightarrow \mathbb{C} \mid f \text{ is a polynomial of degree } m \text{ with } f(0) = 0 \},
\]

\[
C := \{(a_1, \ldots, a_m) \in \mathbb{C}^m \mid a_m \neq 0 \},
\]

\[
\Phi(j, u) := \varphi \circ u \circ \psi_j,
\]

\[
j^{\ell, (1)}_M(f) := (f^{(1)}(0), \ldots, f^{(k-1)}(0)),
\]

\[
j^{\ell, (k)}_C(a_1, \ldots, a_m) := (a_1, \ldots, (k-1)!a_{k-1}),
\]

\[
P(a_1, \ldots, a_m)(z) := a_1 z + \cdots + a_m z^m,
\]

and the diagram on the right is obtained by linearizing the one on the left. The map \(\Phi\) is well-defined by Lemma 4.23. Since \(J^{\ell, (k)}_{(j, u)}(a_1, \ldots, a_m) = (a_1, \ldots, (k-1)!a_{k-1})\) is surjective, \(J^{\ell, (k)}_{(j, u)}\) is surjective as well. This finishes the proof of the base case.

We prove the induction step, i.e. that if \(\mathcal{M}_p^{\ell, (k)}\) is transversely cut out then so is \(\mathcal{M}_p^{\ell+1, (k)}\). We show that \(\mathcal{M}_p^{\ell, (k)} \subseteq \mathcal{M}_p^{\ell+1, (k)}\). For this, it suffices to assume that \((j, u) \in \mathcal{M}_p^{\ell, (k)}\) is
such that $J^{\ell,(k)}_{(j,u)}$ is surjective, and to prove that $J^{\ell+1,(k)}_{(j,u)}$ is surjective as well. This follows because the diagrams

$$
\begin{array}{ccc}
\mathcal{M}^{\ell,(1)}_{p} & \xrightarrow{j^{\ell,(k)}} & \mathcal{M}^{\ell+1,(1)}_{p} \\
\downarrow & & \downarrow \\
\mathcal{M}^{\ell+1,(1)}_{p} & \xrightarrow{j^{\ell+1,(k)}} & \mathbb{C}^{k-1} \\
\end{array}
\begin{array}{ccc}
T^{(j,u)}\mathcal{M}^{\ell,(1)}_{p} & \xrightarrow{j^{\ell,(k)}} & T^{(j,u)}\mathcal{M}^{\ell+1,(1)}_{p} \\
\downarrow & & \downarrow \\
T^{(j,u)}\mathcal{M}^{\ell+1,(1)}_{p} & \xrightarrow{j^{\ell+1,(k)}} & \mathbb{C}^{k-1} \\
\end{array}
$$

commute. Finally, we show that $\mathcal{M}^{\ell+1,(k)}_{p,\text{reg}} = \mathcal{M}^{\ell+1,(k)}_{p}$. 

$$
\begin{align*}
\mathcal{M}^{\ell+1,(k)}_{p,\text{reg}} & \subset \mathcal{M}^{\ell+1,(k)}_{p} \quad \text{[since regular curves form a subset]} \\
& = \mathcal{M}^{\ell,(k)}_{p} \quad \text{[by Lemma 4.28]} \\
& = \mathcal{M}^{\ell,(k)}_{p,\text{reg}} \quad \text{[by the induction hypothesis]} \\
& \subset \mathcal{M}^{\ell+1,(k)}_{p,\text{reg}} \quad \text{[proven above].}
\end{align*}
$$

Proposition 4.32. If $\ell = 1, \ldots, n$ then $\#^\text{vir}\mathcal{M}^{\ell,(m)} = \#\mathcal{M}^{\ell,(m)} = 1$.

Proof. By assumption on the perturbation scheme and Proposition 4.31, $\#^\text{vir}\mathcal{M}^{\ell,(m)} = \#\mathcal{M}^{\ell,(m)}$. Again by Proposition 4.31, the moduli space $\mathcal{M}^{\ell,(m)}$ is transversely cut out and

$$
\dim \mathcal{M}^{\ell,(m)} = (n - 3)(2 - 1) + \mu_{\text{CZ}}(\gamma^m_1) - 2\ell - 2m + 4 = 0,
$$

where in the second equality we have used Lemma 4.26. This implies that $\mathcal{M}^{\ell,(m)}$ is compact, and in particular $\#\mathcal{M}^{\ell,(m)} = \#\mathcal{M}^{\ell,(m)}$. By Lemma 4.28, $\#\mathcal{M}^{\ell,(m)} = \#\mathcal{M}^{1,(m)}$. It remains to show that $\#\mathcal{M}^{1,(m)} = 1$. For this, notice that $\mathcal{M}^{1,(m)}$ is the set of equivalence classes of pairs $(j,u)$, where $j$ is an almost complex structure on $\Sigma = S^2$ and $u: (\tilde{\Sigma},j) \rightarrow (\tilde{E}_1,J_1)$ is a holomorphic map such that

1. $u(w) = x_1$ and $u$ has contact order $m$ to $D_1$ at $x_1$;
2. if $(s,t)$ are the cylindrical coordinates on $\tilde{\Sigma}$ near $z$ such that $v$ agrees with the direction $t = 0$, then

$$
\begin{align*}
\lim_{s \to +\infty} \pi_\mathbb{R} \circ u(s,t) &= +\infty, \\
\lim_{s \to +\infty} \pi_{\partial\tilde{E}_1} \circ u(s,t) &= \gamma_1(a_1mt).
\end{align*}
$$

Here, two pairs $(j_0,u_0)$ and $(j_1,u_1)$ are equivalent if there exists a biholomorphism $\phi: (\Sigma,j_0) \rightarrow (\Sigma,j_1)$ such that

$$
u_0 = u_1 \circ \phi, \quad \phi(w) = w, \quad \phi(z) = z, \quad D\phi(z)v = v.$$

We claim that any two pairs $(j_0,u_0)$ and $(j_1,u_1)$ are equivalent. By Lemma 4.23, the maps $\varphi \circ u_0 \circ \psi_{j_0}$ and $\varphi \circ u_1 \circ \psi_{j_1}$ are polynomials of degree $m$:

$$
\varphi \circ u_0 \circ \psi_{j_0}(z) = a_0 + \cdots + a_m z^m,
$$

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\[ \varphi \circ u_1 \circ \psi_{j_1}(z) = b_0 + \cdots + b_m z^m. \]

Since \(u_0\) and \(u_1\) have contact order \(m\) to \(D_1\) at \(x_1\), for every \(\nu = 0, \ldots, m - 1\) we have

\[ 0 = (\varphi \circ u_0 \circ \psi_{j_0})^{(\nu)}(0) = \nu! a_{\nu}, \]
\[ 0 = (\varphi \circ u_1 \circ \psi_{j_1})^{(\nu)}(0) = \nu! b_{\nu}. \]

Since \(u_0\) and \(u_1\) have the same asymptotic behaviour, \(\arg(a_m) = \arg(b_m)\). Hence, there exists \(\lambda \in \mathbb{R}_{>0}\) such that \(\lambda^m b_m = a_m\). Then,

\[ u_1 \circ \psi_{j_1}(\lambda z) = u_0 \circ \psi_{j_0}(z). \]

Therefore, \((j_0, u_0)\) and \((j_1, u_1)\) are equivalent and \(\#\mathcal{M}^{1,(m)} = 1\). \qed

**Remark 4.33.** In [CM18, Proposition 3.4], Cieliebak and Mohnke show that the signed count of the moduli space of holomorphic curves in \(\mathbb{C}P^n\) in the homology class \([\mathbb{C}P^1]\) which satisfy a tangency condition \(\langle T^{(m)}x \rangle\) equals \((n - 1)!\). It is unclear how this count relates to the one of Proposition 4.32.

Finally, we will use the results of this subsection to compute the augmentation map of the ellipsoid \(E_n\).

**Theorem 4.34.** The augmentation map \(\epsilon_m: CH_{n-1+2m}(E_n) \to \mathbb{Q}\) is an isomorphism.

**Proof.** By Proposition 4.32, Remark 4.18 and definition of the augmentation map, we have \(\epsilon_m(\gamma_1^m) \neq 0\). By Lemma 4.27, \(\epsilon_m\) is an isomorphism. \qed

### 4.7 Computations with contact homology

Finally, we use the tools developed in this section to prove Conjecture 3.11 (see Theorem 4.37). The proof we give is the same as that of Theorem 3.28, with the update that we will use the capacity \(\tilde{g}_k\) to prove that

\[ \tilde{g}_k(X) \leq g_k(X) = c_k^{\text{GH}}(X) \]

for any nondegenerate Liouville domain \(X\). Notice that in Theorem 3.28, \(\tilde{g}_k(X) \leq c_k^{\text{GH}}(X)\) held because by assumption \(X\) was a 4-dimensional convex toric domain. We start by showing that \(\tilde{g}_k(X) \leq g_k(X)\). This result has already been proven in [MS22, Section 3.4], but we include a proof for the sake of completeness.

**Theorem 4.35 ([MS22, Section 3.4]).** If \(X\) is a Liouville domain then

\[ \tilde{g}_k(X) \leq g_k(X). \]

**Proof.** By Remark 3.13, we may assume that \(X\) is nondegenerate. Choose a point \(x \in \text{int} X\) and a symplectic divisor \(D\) through \(x\). Let \(J \in \mathcal{J}(X, D)\) be an almost complex structure on \(\hat{X}\) and consider the complex \(CC(X)\), computed with respect to \(J\). Suppose that \(a > 0\) is such that the augmentation map

\[ \epsilon_k: CH^a(X) \to \mathbb{Q} \]

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is nonzero. By Definition 3.21, it is enough to show that there exists a Reeb orbit \( \gamma \) such that \( \mathcal{A}(\gamma) \leq a \) and \( \mathcal{M}_X^I(\gamma)(T^k)x \neq \emptyset \). Choose a homology class \( \beta \in CH^a(X) \) such that \( \epsilon_k(\beta) \neq 0 \), and write \( \beta \) as a finite linear combination of Reeb orbits \( \beta = \sum_{i=1}^n a_i \rho_i \), where every Reeb orbit has action \( \mathcal{A}(\rho_i) \leq a \). One of the orbits in this linear combination, which we denote by \( \rho \), is such that \( \#_{\text{vir}} \mathcal{M}_X^I(\rho)(T^k)x \neq 0 \). By assumption on the virtual perturbation scheme, \( \mathcal{M}_X^I(\rho)(T^k)x \) is nonempty. Choose \( F = (F^1, \ldots, F^N) \in \mathcal{M}_X^I(\rho)(T^k)x \) and denote by \( C \) the component of \( F \) which inherits the tangency constraint. Then, \( C \in \mathcal{M}_X^I(\gamma)(T^k)x \) for some Reeb orbit \( \gamma \) satisfying \( \mathcal{A}(\gamma) \leq \mathcal{A}(\rho) \).

**Theorem 4.36.** If \( X \) is a Liouville domain such that \( \pi_1(X) = 0 \) and \( 2c_1(TX) = 0 \) then

\[
\Phi_k(X) = c^\text{GH}_k(X).
\]

**Proof.** By Remark 3.13, we may assume that \( X \) is nondegenerate. Let \( E = E(a_1, \ldots, a_n) \) be an ellipsoid as in Section 4.6 such that there exists a strict exact symplectic embedding \( \phi: E \rightarrow X \). In [BO16], Bourgeois–Oancea define an isomorphism between linearized contact homology and positive \( S^1 \)-equivariant contact homology, which we will denote by \( \Phi_{BO} \). This isomorphism commutes with the Viterbo transfer maps and respects the action filtration. In addition, the Viterbo transfer maps in linearized contact homology commute with the augmentation maps. Therefore, there is a commutative diagram

\[
\begin{array}{ccc}
SH_{n-1+2k}^{S^1, [a]}(X) & \xrightarrow{\phi_{BO}^a} & SH_{n-1+2k}^{S^1, +}(X) \\
\downarrow \Phi_{BO} & & \downarrow \Phi_{BO} \\
CH_{n-1+2k}^a(X) & \xrightarrow{\epsilon^a} & CH_{n-1+2k}(X) \\
\downarrow \epsilon_k & & \downarrow \epsilon_k \Phi_{BO} \\
CH_{n-1+2k}^a(X) & \xrightarrow{\epsilon^a} & CH_{n-1+2k}(X) \\
\end{array}
\]

Here, the map \( \epsilon_k^E \) is nonzero, or equivalently an isomorphism, by Theorem 4.34. Then,

\[
c^\text{GH}_k(X) = \inf \{ a > 0 \mid \phi_{BO}^{S^1} \circ \epsilon^{S^1, a} \neq 0 \} \quad \text{[by Lemma 3.15]} \\
= \inf \{ a > 0 \mid \epsilon_k^X \circ \epsilon^a \neq 0 \} \quad \text{[since the diagram commutes]} \\
= \Phi_k(X) \quad \text{[by Definition 4.8].}
\]

**Theorem 4.37.** Under Assumption 4.1, if \( X_\Omega \) is a convex or concave toric domain then

\[
c_L(X_\Omega) = \delta_\Omega.
\]

**Proof.** Since \( X_\Omega \) is concave or convex, we have \( X_\Omega \subset N(\delta_\Omega) \). For every \( k \in \mathbb{Z}_{\geq 1} \),

\[
\delta_\Omega \leq c_P(X_\Omega) \leq c_L(X_\Omega) \leq \frac{\Phi_k(X_\Omega)}{k} \quad \text{[by Theorem 2.27]}
\]

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\[
\leq \frac{g_k(X_{\Omega})}{k} \quad \text{[by Theorem 4.35]}
\]
\[= \frac{c_k^{GH}(X_{\Omega})}{k} \quad \text{[by Theorem 4.36]}
\]
\[\leq \frac{c_k^{GH}(N(\delta_{\Omega}))}{k} \quad \text{[since } X_{\Omega} \subset N(\delta_{\Omega})]\]
\[= \frac{\delta_{\Omega}(k + n - 1)}{k} \quad \text{[by Lemma 3.16].}
\]

The result follows by taking the infimum over \( k \). \qed
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