Singular and Non-singular Vortices and Their Excitation Spectra of Neutral Fermi Superfluids in Restricted Geometries

K. Machida, Y. Tsutsumi, T. Kawakami, T. Mizushima, M. Ichioka, and M. Takahashi
Department of Physics, Okayama University, Okayama 700-8530, Japan
E-mail: machida@mp.okayama-u.ac.jp

Abstract. We study stable vortices and their bound excitation spectra in superfluid $^3$He-$\Lambda$ phase under restricted geometries. By choosing appropriate geometry we can control the $\vec{l}$-vector and $\vec{d}$-vector of the $\Lambda$ phase order parameter (OP), making it possible to select a particular vortex among a variety of singular or non-singular vortices. This tunability through the boundary condition and external field is demonstrated to be useful. Here we take up two examples of the restricted geometries under rotation: A narrow cylinder whose radius is 50$\mu$m under field parallel to the rotation axis and thin film configuration sandwiched by parallel plates whose gap is an order of $\mu$m or less. We examine the ongoing experiment by rotating cryostat at ISSP, University of Tokyo, reporting on new vortex state for a cylinder in a slow rotation speed region near $T_c$. For parallel plates in a field parallel to the film surface we analyze the conditions under which singular vortex is stable. This study allows us to examine experimental realization of so-called Majorana zero energy state bound in a core in $^3$He-$\Lambda$ chiral phase.

1. Introduction
Much interest focuses on neutral Fermion superfluids with exotic pairings, in particular, on vortex structures created under rotation. A classical example of this can be found in superfluid $^3$He which exhibits two pairing phases ABM (A) and BW (B) phases. Various kinds of interesting vortices have been identified[1, 2]. The ABM (A) phase in superfluid $^3$He is characterized by OP $A_{\mu i} = \Delta_0 d_{\mu i}(\vec{n} + i\vec{m})_i$ $(i = x, y, z)$. The spin and orbital structure of $p$-wave pairing, which is generally defined as $\Delta = i \sum_{\mu i} (A_{\mu i} \sigma_\mu \hat{\lambda}_i) \sigma_y$, are characterized by $\vec{d}$ vector and $\vec{l}$ vector respectively ($\vec{l} = \vec{n} \times \vec{m}$, $\vec{m}$ and $\vec{n}$ are unit vectors, forming a triad)[1, 2]. It is known that $\vec{l}$ vector points always perpendicular to the wall surface and $\vec{d}$ vector is perpendicular to applied field direction if its magnitude is sufficiently strong ($H > H_d = 2.0mT$, $H_d$ dipole field[1]). We can control the pairing symmetry both by the confining geometry and the field. This kind of controllability in superfluid $^3$He is a great advantage over other research fields in addition to the decisive fact that the precise OP forms of the $p$-wave pairing are established for ABM (A) and BW (B) phases[1, 2]. In this paper, we present two cases to demonstrate this controllability to manipulate OP in $^3$He-$\Lambda$ phase and the associated vortices under rotation; (1) Narrow cylinder with the radius R=50$\mu$m[3] and (2) the parallel plate whose gap is 12.5$\mu$m[4]. These systems are now being investigated in ISSP, Univ. of Tokyo by using the rotating cryostat. Its maximum
rotation speed is 12 rad/sec.

2. Narrow cylinders

According to the experiments done in ISSP, Univ. of Tokyo, Izumina et al[3] report that under slow rotation speeds in temperature region $T \geq 0.9T_{c}$ (P=3.2MPa) unidentified vortex state is found. This state changes into the Mermin-Ho vortex (MH) upon increasing rotation speed above $\Omega \sim 1$ rad/sec for R=115\(\mu\)m system, which is followed by multiple continuous unlocked vortices (CUV) under higher rotation $\Omega \sim 4$ rad/sec.

In order to investigate this unknown vortex or texture at rest and low rotations, we employ a standard GL functional:

$$f = \alpha A_{i}^{3} A_{i} + \beta_{13} A_{i}^{3} A_{j} A_{j} + \beta_{245} A_{i}^{3} A_{j} A_{i} A_{j}$$

$$-K d_{\mu} \left[ A_{i}^{\mu} \partial_{i} \partial_{j} \left( d_{\mu} A_{j} \right) + A_{j}^{\mu} \partial_{i} \partial_{j} \left( d_{\mu} A_{i} \right) + A_{j}^{\mu} \partial_{j} \partial_{j} \left( d_{\mu} A_{i} \right) \right]$$

$$+ g_{d} \left[ d_{\mu} d_{\nu} \left( A_{i}^{\mu} A_{\nu} + A_{j}^{\mu} A_{\nu} \right) - \frac{2}{3} A_{i}^{\mu} A_{j}^{\mu} \right]$$

$$+ g_{m} A_{i}^{3} A_{k} \left( \vec{d} \cdot \vec{H} \right)^{2} \tag{1}$$

where the A-phase OP is given by $A_{i}^{\mu} = \vec{d}_{\mu} A_{i}$. The GL parameters in this expression are well-known and documented[1, 2]. The current is written as

$$j_{i} = \frac{4m_{0}^{2} \mu_{0}}{h} Im \left[ A_{i}^{\mu} \partial_{j} A_{j} + A_{j}^{\mu} \partial_{i} A_{j} + A_{j}^{\mu} \partial_{j} A_{i} \right] \tag{2}$$

The experiment is done under $H=21.6$ mT applied parallel to the rotation axis $z$, which is larger than $H_{d}$. Thus it tends to orient $d$-vector in the $x,y$ plane. Note that the coherence length $\xi_{0}=10$ nm at $T=0$ and the dipole coherence length $\xi_{d}=8.64$ \(\mu\)m. The former (latter) characterizes the length scale of singular (non-singular) vortex core.

We have done extensive computations to determine the stable vortex structure at rest and under slow rotations. The radial disgyration (RD) vortex shown in Figs. 1 and 2 is found to be stable over MH (see Figs. 3 and 4) at rest and up to $\Omega_{c}=2.3$ rad/sec for R=50\(\mu\)m system. RD is a singular vortex with a core where $\vec{l}$-vector vanishes, that is, the polar state is realized. The associated $\vec{d}$-vector is in the plane at rest and forms a hyperbolic type configuration with two fold symmetry. The mutual interaction between $\vec{l}$ and $\vec{d}$-vectors makes $\vec{d}$-vector distort from perfect radial pattern. Under a finite rotation $\vec{l}$ and $\vec{d}$-vectors acquire the finite $z$-component to cancel the mass current induced by the external rotation. Above $\Omega_{c}=2.3$ rad/sec MH texture becomes stable as shown in Figs. 3 and 4. $\vec{l}$-vector pointing up-wards at the center flares out and becomes perpendicular to the wall. $\vec{d}$-vector pattern is similar to that in RD. It is noted that the radial movement of $\vec{l}$-vector is also distorted through the dipole interaction. The resulting mass current exhibits not only the in-plane component which mimics the rigid-body rotation, but also the radial movement of $\vec{l}$-vector. Under a finite rotation $\Omega \geq \frac{H}{\xi_{d}}$, the $\vec{l}$-vector becomes perpendicular to the wall.

3. Parallel plates

We consider a parallel plate geometry whose gap is comparable to $\xi_{d}$ or less and field $(H > H_{d})$ is applied along the plates. Since in this configuration the two vectors $\vec{l} \parallel \vec{d}$ are locked together
perpendicular to the plates, OP is $\mathbf{A}_\mu = \Delta_0 d_\mu (\hat{n} + i\hat{m})_i (i = x, y)$ with $\mu$ being fixed. Namely it is a “spinless” chiral superfluid, a situation ideal to the Majorana zero energy state. This parallel plate geometry is already realized experimentally[4]: The $^3$He-A phase sample[4] is confined in a thin cylindrical region with thickness $12.5\mu m$ and radius $R = 1.5mm$. The texture transition under rotation is monitored by NMR spectrum to identify each vortex.

Necessary conditions for Majorana bound state to exist are: (1) The bulk superfluid is a chiral $p$-wave symmetry. (2) The vortex is singular whose winding number is odd. The condition (1) is satisfied because the $^3$He-A phase is triplet. Since $\mathbf{d}$ and $\mathbf{l}$ vectors are locked together perpendicular to the plane, the superfluid state is in the chiral $p$-wave symmetry characterized by $p_\pm = \mp(p_x \pm ip_y)/\sqrt{2}$ whose spin freedom is frozen. We must carefully examine the condition (2) because there are several possible vortex states under this configuration. They are either the singular or non-singular vortices and compete each other. Thus it is quite important to sort out the possible stable parameter space in light of the realistic experimental situation[4]. We also study microscopic excitation spectrum of the Majorana state lived in three dimensional Fermi surface by solving the microscopic Bogoliubov-de Gennes (BdG) equation for the vortex.

We start out with the same GL functional (1) by appropriately modifying the relevant terms so as to satisfy the parallel plate geometry. We introduce $A_\mp = \mp(A_x \mp iA_y)/\sqrt{2}$. Here we can drop out the spin degrees of freedom. Let us start out to discuss the stable state at rest. Our OP is chiral axial states $A_\pm$, which are degenerate at $\Omega = 0$. The external rotation removes it. We denote the state with the winding number combination $(w_+, w_-)$ where $A_\pm = |A_\pm|e^{i\omega_\pm\theta}$ ($\theta$ is the azimuthal angle). The $(0,2)$ state is stabilized over the $(2,0)$, that is, the dominant
component is $A_+$ with the winding $w_+ = 0$. The $A_-$ with the winding $w_- = 2$ is induced at the boundary. Under rotation the mass current flows less for $(0, 2)$ than $(2, 0)$. Thus at rest and lower rotation regions $(0, 2)$ is the ground state. The main component $A_+$ has no phase winding while the minor component $A_-$ is $4\pi$ phase winding, which leads to the spontaneous mass flow around the outer boundary even at rest. Under rotation, there are a variety of possible vortex states: The general forms of the winding number combination in the axis symmetry confined in a cylinder is given by $(w_+ = n, w_- = n \pm 2)$ with $n$ integer, meaning that through the gradient term two components with $w_+ = n$ and $w_- = n \pm 2$ are coupled. This includes $(-1, 1)$, $(1, -1)$, $(0, 2)$, $(1, 3)$, $(-3, -1)$, $(2, 4)$, etc. Note that in general vortices with higher winding number are less plausible. The $(1, 3)$ vortex is quite comparable in energy with the $(1, 3)$ vortex. The former is stable than the latter for systems with smaller radii $R = 2.5, 5.0, 10 \mu m$ in lower rotation region. Since in that region the $(2, 0)$ state, however, is in the lowest energy, the $(-1, 1)$ is not realized. The $(1, -1)$ vortex is another candidate, but it is not stabilized in our calculation. The higher winding vortex states, such as $(2, 4)$ become stabler than the $(1, 3)$ vortex in higher rotations. From these arguments which are confirmed by our detailed numerical calculations we are left with only one stable state $(1, 3)$ next to $(0, 2)$ upon increasing rotation. In the $(1, 3)$ vortex the dominant (minor) component is $A_+ (A_-)$ with $w_+ = 1$ $(w_- = 3)$. As for $(0, 2)$ versus $(1, 3)$, it is now obvious under rotation that the energy gain for $(0, 2)$ due to $\vec{j} \cdot \vec{\Omega}$ is less than that in $(1, 3)$, thus leading eventually to the phase transition from $(0, 2)$ at rest to a single vortex with $(1, 3)$ upon increasing $\Omega$. We performed extensive calculations to find the critical rotation speed $\Omega_c$ as a function of $R$ up to $R = 100 \mu m$ at a fixed temperature $T/T_c = 0.95$. It is found that at $R=15mm$, $\Omega_c=0.06rad/sec$ is estimated by an extrapolation[5]. This is feasible to realize it experimentally.

We analyze the spectral structure of the core localized bound states in $(1, 3)$ vortex with the 3D Fermi surface. The system is three dimensional. We solve the BdG equation under these conditions to examine the Majorana zero energy state. The analytical and numerical calculations for the obtained singular vortex $(1, 3)$ show that (1) There exists the zero energy bound state localized at the core. (2) The corresponding quasi-particle is Majorana particle in the sense that the particle creation and annihilation are self-Hermitian. (3) Along the third direction $k_z$ in the momentum space the energy is dispersionless, staying always at zero energy.

4. Conclusion
We have examined stable vortices of superfluid $^3$He confined in restricted geometries; a narrow cylinder and parallel plates. As for the former, we found that the singular vortex with the polar core, namely, radial disgyration texture is most stable at rest and slow rotations. This texture gives rise to the perpendicular mass current to the texture plane. This spontaneous symmetry breaking is due to the in-plane $\vec{l}$ vector bending. As for the parallel plate case, we found a stable singular vortex, which posses the Majorana zero energy state above the critical rotation. This may provide a new avenue to critically investigate the Majorana particle.

References
[1] A.J. Leggett, Rev. Mod. Phys. 47, 331 (1975).
[2] D. Vollhardt and P. Wölfle, The Superfluid phase of Helium 3 (Taylor and Francis, London, 1990).
[3] K. Izumina, et al private communication.
[4] M. Yamashita, K. Izumina, A. Matsubara, Y. Sasaki, O. Ishikawa, T. Takagi, M. Kubota and T. Mizusaki, AIP Conference Proceedings, 850, 185 (2005).
[5] Y. Tsutsuki, et al, private communication.