Waterfall field in hybrid inflation and curvature perturbation

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We study carefully the contribution of the waterfall field to the curvature perturbation at the end of hybrid inflation. In particular we clarify the parameter dependence analytically under reasonable assumptions on the model parameters. After calculating the mode function of the waterfall field, we use the $\delta N$ formalism and confirm the previously obtained result that the power spectrum is very blue with the index 4 and is absolutely negligible on large scales. However, we also find that the resulting curvature perturbation is highly non-Gaussian and hence we calculate the bispectrum. We find that the bispectrum is at leading order independent of momentum and exhibits its peak at the equilateral limit, though it is unobservably small on large scales. We also present the one-point probability distribution function of the curvature perturbation.

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I. INTRODUCTION

Currently, primordial inflation \[1\] is supposed to be the leading candidate to provide the necessary conditions for the successful big bang cosmology \[2\]. The simplest model of inflation driven by only a single inflaton field is consistent with most recent observations \[3\]. It is however expected that, in the context of theories beyond the standard model of particle physics e.g. supersymmetry, there is a number of multiple scalar fields which may contribute to the inflationary dynamics \[4\]. Furthermore, we may be able to observationally detect deviations from the predictions of single field models in the near future and to discuss interesting phenomenology, such as isocurvature perturbations and non-Gaussianity.

Hybrid inflation \[5\] is an interesting realization with two field contents, the usual inflaton field $\phi$ which drives slow-roll inflation and the waterfall field $\chi$ which terminates inflation by triggering an instability, a “waterfall” phase transition. Previously, it has been assumed that $\chi$ becomes momentarily massless only at the time of waterfall and very heavy otherwise, and thus does not contribute to the curvature perturbation $R_c$ on large scales: only the quantum fluctuations of $\phi$ contributes to $R_c$ and we can follow the well-known calculations of single field case, with the energy density of the universe being dominated by a non-zero vacuum energy.

This naive picture has been receiving a renewed interest \[6–8\] with the common qualitative results that the power spectrum of the curvature perturbation induced by the waterfall field is very blue and extremely small on large scales\[1\]. However, quantitatively it is not clear if they all agree or not. In particular, in Ref. \[8\] the $\delta N$ formalism, which takes account of fluctuations only on super-horizon scales by construction, was employed to derive the power spectrum, but the approach there was not quantitative enough and hence the dependence on the model parameters was not explicitly presented.

In this note, we provide another complementary view. We adopt a few reasonable assumptions on the model parameters and solve the mode functions of $\chi$ in terms of the number of $e$-folds analytically. Then using the $\delta N$ formalism \[10\] we calculate the corresponding $R_c$ induced by $\chi$ explicitly.

The result is consistent with the above references, i.e. the contribution of $\chi$ to the

\[1\] For early attempts, see e.g. Ref. \[9\].
large scale curvature perturbation is totally negligible. We also clarify the model parameter
dependence on the spectrum of the curvature perturbation. Furthermore, we calculate the
corresponding bispectrum, which shows its peak at the equilateral limit. We also compute
explicitly the one-point probability distribution function which clearly shows the highly
non-Gaussian nature of the curvature perturbation.

The outline of this note is as follows. In Section II, we find the mode function solution of
the waterfall field \( \chi \) valid both on super-horizon and sub-horizon scales. In Section III, we
calculate the corresponding curvature perturbation \( R_c \) induced by \( \chi \) using the \( \delta N \) formalism.
In Section IV we present the power spectrum and bispectrum of \( R_c \). In Section V we show
the explicit form of the one-point probability distribution function of \( R_c \) and discuss relates
issues. We conclude in Section VI. In Appendices, we discuss some technical details. In
Appendix A to check the consistency of the \( \delta N \) formalism with the standard perturbation
theory, we give an estimation of the curvature perturbation by using the linear perturbation
equation for \( R_c \). We find a good agreement with our result based on the \( \delta N \) formalism. In
Appendix B we reconsider the splitting of the super- and sub-horizon modes and compute
the average over the horizon scales. The results agree with the formulae we use in the main
text.

II. MODE FUNCTION SOLUTION OF WATERFALL FIELD

Before we begin explicit computations, first of all we make the physical picture clear. Our
purpose is to calculate the contribution of the waterfall field \( \chi \) to the curvature perturbation
\( R_c \). This is only possible when \( \chi \) becomes dynamically relevant. While \( \chi \) is well anchored
at its minimum during the phase of slow-roll inflation and hence does not participate in the
inflationary dynamics, it controls the physical processes from the moment of waterfall till
the end of inflation. Thus, in the context of the \( \delta N \) formalism, if we can find the evolution of
\( \chi \) during this phase as a function of the number of \( e \)-folds \( N \), it amounts to finding \( R_c \) by the
geometrical identity \( R_c = \delta N \). Therefore, our aim in this section is to calculate \( \chi = \chi(N) \)
starting from the moment of waterfall. We will directly use this result to calculate \( R_c \) in
the next section.
We consider the potential of the two fields, the inflaton $\phi$ and the waterfall field $\chi$, as

$$V(\phi, \chi) = \frac{\lambda}{4} \left( \frac{M^2}{\lambda} - \chi^2 \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} g^2 \phi^2 \chi^2.$$

We note that during the most period of inflation of our interest, it is assumed that the vacuum energy $V_0 = M^4/(4\lambda)$ dominates so that the Hubble parameter is effectively a constant, $H = H_0$. This is a good approximation even after the waterfall phase transition until the last moment of inflation. The slow-roll and the waterfall conditions are

\begin{align}
\frac{m^2}{H_0^2} &\ll 1, \\
\frac{M^2}{H_0^2} &\equiv \beta \gg 1,
\end{align}

respectively.

The equations of motion are given by

\begin{align}
\ddot{\phi} + 3H \dot{\phi} + \left( m^2 + g^2 \chi^2 \right) \phi &= 0, \\
\ddot{\chi} + 3H \dot{\chi} - \frac{1}{a^2} \nabla^2 \chi + \left( -M^2 + g^2 \phi^2 + \lambda \chi^2 \right) \chi &= 0,
\end{align}

where the spatial gradient term for $\phi$ is neglected as usual. Note that before waterfall, $\phi^2 > \phi_c^2 \equiv M^2/g^2$, $\chi$ is well anchored at its minimum $\chi = 0$ so it is itself the same as its fluctuation, $\chi = \delta \chi$. Thus we may regard (2.5) as the equation for $\delta \chi^2$, which arises from the vacuum fluctuations. Then after the waterfall transition, $\delta \chi$ becomes unstable and $\delta \chi^2$ starts to grow rapidly, and inflation ends when the inflaton starts to roll fast, which happens when the term $g^2 \delta \chi^2$ exceeds $m^2$ in (2.4). Here we adopt the mean field approximation, i.e. we replace $g^2 \delta \chi^2$ by its expectation value $g^2 \langle \delta \chi^2 \rangle$, which should be valid for the motion of the homogeneous inflaton field $\phi$. We also assume that the nonlinear term $\lambda \delta \chi^2$ in (2.5) can be neglected until the end of inflation. That is, we assume

$$M^2 \gg \frac{\lambda}{g^2} m^2 \gtrsim \lambda \langle \delta \chi^2 \rangle.$$

At the end of calculation, we must check if this condition is satisfied for the range of the parameters of our interest.

\footnote{Note that during inflation $\delta \rho_\phi \sim \delta \phi$ while $\delta \rho_\chi \sim \delta \chi^2$, and thus the metric fluctuations are relatively second order with respect to $\delta \chi$ and does not appear in the equation of motion for $\delta \chi$. This situation is closely analogous to the case of false vacuum inflation\cite{11}. The correlation functions also show similar momentum dependence to those produced during false vacuum inflation\cite{11, 12}.}
We can rewrite (2.4) and (2.5) in a more convenient form by using the number of $e$-folds as the time variable, $dN = Hdt$. Denoting the derivative with respect to $N$ by a prime, we write
\[
\phi'' + 3\phi' + \left(\frac{m^2}{H_0^2} + g^2 \frac{\langle \delta \chi^2 \rangle}{H_0^2}\right) \phi = 0,
\]
(2.7)
\[
\delta \chi'' + 3\delta \chi' - \frac{1}{a^2 H_0^2} \nabla^2 \delta \chi + \left(-\beta + g^2 \frac{\phi^2}{H_0^2} + \lambda \frac{\delta \chi^2}{H_0^2}\right) \delta \chi = 0.
\]
(2.8)
Let $N_c$ be the time at which the waterfall transition occurs, $\phi(N_c) = \phi_c = M/g$. Before waterfall, since $\delta \chi$ is very massive, $g^2 \phi^2 \gg H^2$, it is dominated by the standard vacuum fluctuations and the bare expectation value $\langle \delta \chi^2 \rangle$ is ultraviolet divergent. Here we regularize it so that it vanishes before waterfall, $\langle \delta \chi^2 \rangle = 0$ at $N < N_c$. Then (2.7) is easily solved to give
\[
\phi = \phi_c e^{-rn},
\]
(2.9)
where $n = N - N_c$ is the number of $e$-folds measured relative to the time of the waterfall transition, and we have introduced the parameter $r$ by
\[
r \equiv \frac{3}{2} - \sqrt{\frac{9}{4} - \frac{m^2}{H_0^2}} \approx \frac{m^2}{3H_0^2} \ll 1.
\]
(2.10)
We note that we can write the scale factor $a$ and the conformal time $\eta = -1/(aH)$ using $n$ as
\[
a = a_c e^n = \frac{k_c}{H_0} e^n,
\]
(2.11)
\[
\eta = -e^{-n} - \frac{e^{-n}}{a_c H_0} = -\frac{e^{-n}}{k_c},
\]
(2.12)
respectively, where $a_c = a(N_c)$ and $k_c = a_c H_0$.

Inserting the background solution (2.9) for $\phi$ into (2.8) and neglecting the nonlinear term in accordance with the assumption (2.6), we obtain the equation for $\delta \chi$ in the Fourier space,
\[
\delta \chi''_k + 3\delta \chi'_k + \left[\frac{k^2}{k_c^2} e^{-2n} + \beta \left(e^{-2rn} - 1\right)\right] \delta \chi_k = 0.
\]
(2.13)

A. High frequency limit $k/a \to \infty$: WKB solution

In the high frequency limit, we can solve (2.13) in terms of the WKB approximation. In this limit the proper asymptotic behavior of the positive frequency function is given by
\[
\delta \chi_k \xrightarrow{k \to \infty} \frac{e^{-ik\eta}}{\sqrt{2ka}} = \frac{H_0}{\sqrt{2k_c^3} \sqrt{k/k_c}} \exp \left(-i \frac{k}{k_c} \int^n dne^{-n}\right).
\]
(2.14)
The WKB solution that has this asymptotic behavior is readily obtained as

\[ \delta\chi_k = \frac{e^{-n}H_0}{\sqrt{2\kappa^3 c^2 \left( (k/k_c)^2 + \beta e^{2n} \right)}} \left[ -i \int^n dne^{-n} \left( \frac{k}{k_c} \right)^2 + \beta e^{2n} \right]^{1/4} \exp \left[ -i \int^n dne^{-n} \sqrt{\left( \frac{k}{k_c} \right)^2 + \beta e^{2n}} \right], \]  
\begin{equation} \tag{2.15} \end{equation}

where for convenience we have defined \( \tilde{\beta} \) by

\[ \tilde{\beta} \equiv \beta(e^{-2rn} - 1). \]  
\begin{equation} \tag{2.16} \end{equation}

The above WKB solution is valid for any \( k \) at sufficiently early times, \(-n \gg 1\).  

### B. Low frequency limit \( k/a \to 0 \): Hankel function solution

In the large scale limit \( k \to 0 \), \( (2.13) \) becomes

\[ \delta\chi''_0 + 3\delta\chi'_0 + \beta( e^{-2rn} - 1 ) \delta\chi_0 = 0. \]  
\begin{equation} \tag{2.17} \end{equation}

Then, the solution is easily found to be

\[ \delta\chi_0(n) = e^{-3n/2} \left[ c_1 H^{(1)}(\nu \sqrt{\beta} e^{-rn}) + c_2 H^{(2)}(\nu \sqrt{\beta} e^{-rn}) \right], \]  
\begin{equation} \tag{2.18} \end{equation}

where \( H^{(1)}_\nu \) and \( H^{(2)}_\nu \) are the Hankel function of first and second kind, respectively, and are complex conjugate to each other, \( c_1 \) and \( c_2 \) are constants to be determined, and

\[ \nu \equiv \frac{\sqrt{\beta} + 9/4}{r} \approx \frac{\sqrt{\beta}}{r}. \]  
\begin{equation} \tag{2.19} \end{equation}

### C. Large scale modes: \( k \ll k_c \)

Now let us consider the long wavelength modes \( k \ll k_c \) which are already on super-horizon scales by the time of the waterfall transition. For these modes, we match the WKB solution to the Hankel function solution at some time well before the waterfall, \( n < 0 \) and \( |n| \gg 1 \).

In the limit \( k/k_c \to 0 \), the WKB solution \( (2.15) \) becomes

\[ \delta\chi_k \to e^{-3n/2} H_0 \left( \frac{\sqrt{\beta}}{r} e^{-rn} \right)^{-1/2} \exp \left( i \frac{\sqrt{\beta}}{r} e^{-rn} \right), \]  
\begin{equation} \tag{2.20} \end{equation}

where we have assumed \( e^{-2rn} \gg 1\).
As for the Hankel function solution, the argument is very large in the limit \(-n \gg 1\), \(\sqrt{\beta e^{-rn}}/r \approx \nu e^{-rn} \gg \nu\). Thus using the asymptotic form of the Hankel function,

\[ H^{(1)}_\nu(z) \xrightarrow{z \gg \nu} \sqrt{\frac{2}{\pi z}} \exp \left[ i \left( z - \frac{\nu}{2} - \frac{\pi}{4} \right) \right], \tag{2.21} \]

we find that (2.18) becomes

\[ \delta \chi_k \xrightarrow{-n \gg 1} c_1 e^{-3n/2} \sqrt{\frac{2}{\pi}} \left( \frac{\nu}{r} e^{-rn} \right)^{-1/2} e^{-i(\nu\pi/2 + \pi/4)} \exp \left( i \frac{\sqrt{\beta}}{r} e^{-rn} \right) + \cdots, \tag{2.22} \]

where for notational simplicity we have omitted the term proportional to \(H^{(2)}_\nu\) whose coefficient is \(c_2\).

Comparing (2.22) with (2.20), we see that \(H^{(1)}_\nu\) gives the correct phase factor dependence of (2.20) and thus we have \(c_2 = 0\) and

\[ c_1 = \sqrt{\frac{\pi}{2}} \frac{H_0}{\sqrt{2\pi k^3 c^3}} e^{i(\nu\pi/2 + \pi/4)}. \tag{2.23} \]

Thus, the long wavelength positive frequency function is given by

\[ \delta \chi_k \xrightarrow{k \ll k_c} e^{-3n/2} \sqrt{\pi} \frac{H_0}{2\sqrt{2\pi k^3 c^3}} e^{i(\nu\pi/2 + \pi/4)} H^{(1)}_\nu \left( \frac{\sqrt{\beta}}{r} e^{-rn} \right). \tag{2.24} \]

Here let us evaluate the mode function at the moment of waterfall \(n = 0\). At \(n = 0\), remembering that \(\beta \gg 1\), the Hankel function takes the form

\[ H^{(1)}_\nu \left( \frac{\sqrt{\beta}}{r} \right) \approx H^{(1)}_\nu(\nu), \tag{2.25} \]

with \(\nu \approx \sqrt{\beta}/r\). That is, the index and the argument of the Hankel function are the same. In this case, the Hankel function solution takes the form

\[ H^{(1)}_\nu(\nu) = \left( \frac{6}{\nu} \right)^{1/3} \frac{2}{3\Gamma(2/3)} e^{-i\pi/3}. \tag{2.26} \]

Then, denoting by a subscript \(L\) the long wavelength modes which are on super-horizon scales at \(n = 0\), we can write

\[ \delta \chi_L(n = 0) \xrightarrow{k \to 0} \frac{2\sqrt{\pi}}{3^{2/3}\Gamma(2/3)} \frac{H_0}{\sqrt{2k^3 c^3 \alpha^{1/3}}} \exp \left[ i \left( \nu - \frac{1}{6} \right) \frac{\pi}{2} \right], \tag{2.27} \]

where the numerical factor reads \(2\sqrt{\pi}/\left[3^{2/3}\Gamma(2/3)\right] \approx 1.25854\), and we have defined\(^3\)

\[ \alpha \equiv \sqrt{2r\beta}. \tag{2.28} \]

\(^3\) Our \(\alpha\) is equal to \(\epsilon_\psi\) in Ref. [7].
As we will see in the next section, we must require $\alpha \gg 1$. The above result (2.27) implies that all the super-horizon modes have the same amplitude at the moment of waterfall. The moment of waterfall will be taken as the “initial” time to estimate the contribution of $\delta \chi$ to the curvature perturbation $R_c$.

**D. Small scale modes: $k \gg k_c$**

For the modes that are still on sub-horizon scales at the time of waterfall, $k \gg k_c$, the WKB solution is valid until $n = 0$. Denoting them by a subscript $S$, (2.13) readily gives

$$\delta \chi_S = \frac{H_0}{\sqrt{2kk_c}} e^{-n} \exp \left( \frac{k}{k_c} e^{-n} \right),$$

so that at the moment of waterfall

$$\delta \chi_S(n = 0) = \frac{H_0}{\sqrt{2kk_c}} e^{ik/k_c}.$$  

This is the “initial” amplitude of the sub-horizon modes.

Before we move on, we mention that the initial amplitudes of large scale limit (2.27) and that of small scale limit (2.30) do not match at $k = k_c$ if we extrapolate from both sides, but are different by a suppression factor $\alpha^{-1/3}$. This indicates that in the intermediate regime around $k = k_c$ these two extreme values are deviating from the limiting values and smoothly connected [8]. In particular, this implies that the sub-horizon modes with $k \gtrsim k_c$ have slightly different initial amplitudes from (2.30). However this will not affect our subsequent discussions because of the phase volume $\sim k^3$ that gives rise to a sharp peak in the spectrum at $k \approx \alpha k_c \gg k_c$, as we will see below. Hence we just use (2.27) for the initial amplitude of the large scale modes with $k < k_c$ and (2.30) for that of the small scale modes with $k > k_c$.

**E. Evolution of the relevant modes after waterfall**

Having found the “initial” amplitudes of both large scale and small scale modes, now we can calculate the subsequent evolution of the modes until the end of inflation.

Let us first consider the large scale modes. The solution is given by (2.24) and is valid
for \( n > 0 \) as well. Then, using the asymptotic form of the large \( \nu = \sqrt{\beta/r} \) we can find

\[
H^{(1)}_\nu(\nu e^{-rn}) = \sqrt{\frac{2r}{\pi \alpha}} \exp \left( \frac{2}{3} \alpha n^{3/2} - \frac{1}{4} \log n \right) e^{-i\pi/2}.
\] (2.31)

This is obtained with \( rn \ll 1 \), and is thus valid for \( n \ll 1/r \). For any sensible model of hybrid inflation \( r \ll 1 \), while the number of \( e \)-folds after waterfall until the end of inflation, \( n_f \), is \( \mathcal{O}(1) \) or at most a few. Hence this asymptotic form is valid until the end of inflation. Then, plugging (2.31) into (2.24), we can find that after waterfall the mode function on super-horizon scales evolves as

\[
|\delta \chi_L(n)| = \frac{H_0}{\sqrt{2\alpha k_c^3}} \exp \left( \frac{2}{3} \alpha n^{3/2} - \frac{3}{2} n - \frac{1}{4} \log n \right)
\]

\[
= |\delta \chi_L(n = 0)| \frac{3^{2/3} \Gamma(2/3)}{2\sqrt{\pi}} \alpha^{-1/6} \exp \left( \frac{2}{3} \alpha n^{3/2} - \frac{3}{2} n - \frac{1}{4} \log n \right),
\] (2.32)

where the initial amplitude of the large scale modes \( \delta \chi_L(n = 0) \) is given by (2.27). As the logarithmic term indicates, (2.32) does not hold precisely at \( n = 0 \) but is valid for, as mentioned above, some time after waterfall till the end of inflation. As we will evaluate \( \delta \chi \) at the end of inflation \( n_f = \mathcal{O}(1) \), we can justifiably use (2.32) to calculate the curvature perturbation.

Let us now turn to the small scale modes. An important point to calculate the evolution of sub-horizon modes is that the end of inflation is determined by the quanta of \( \chi \) which become tachyonic right after waterfall, and affect the effective mass of \( \phi \) in the form \( g^2 \langle \delta \chi^2 \rangle \). The modes which become tachyonic satisfy, by definition, \( (k/k_c)^2 < |\tilde{\beta}| \) in (2.13). Assuming \( n = \mathcal{O}(1) \), we have \( |\tilde{\beta}| \approx 2\beta rn \sim \alpha^2 \). Hence we find that the modes with

\[
\frac{k}{k_c} \lesssim \alpha
\] (2.33)

become tachyonic. Thus \( \alpha \) must be much greater than unity in order to have an effective tachyonic instability.

To summarize, the small scale modes of our interest, which contribute to the tachyonic instability and control the end of inflation, are those in the interval

\[
k_c \lesssim k \lesssim \alpha k_c.
\] (2.34)

\footnote{Note that the same dependence on the number of \( e \)-folds was found from the Airy function solutions in Refs. [6, 7]. But the corresponding equation solved in these references is a particular limit of the general equation (2.17) and thus so does the solution, as we show here explicitly.}
Since \((k/k_c)^2\) can be neglected in comparison with \(\beta\) at leading order approximation, they satisfy the same equation as the equation for the large scale modes, \((2.17)\). Hence the evolution of these modes at \(n > 0\) is the same as that given by \((2.32)\). That is,

\[
|\delta \chi_S(n)| = |\delta \chi_S(n = 0)| A \exp \left(\frac{2}{3} \alpha n^{3/2} - \frac{3}{2} n - \frac{1}{4} \log n \right),
\]

where we have set the overall coefficient as

\[
A \equiv \frac{3^{2/3} \Gamma(2/3)}{2\sqrt{\pi}} \alpha^{-1/6}.
\]

### III. CURVATURE PERTURBATION INDUCED BY WATERFALL FIELD

In this section, we calculate the curvature perturbation \(\mathcal{R}_c\) by using \((2.32)\) and \((2.35)\) in the context of the \(\delta N\) formalism. In the \(\delta N\) formalism the spacetime geometry is spatially smoothly varying over super-horizon scales while each Hubble horizon size region is regarded as a homogeneous and isotropic universe. Hence we first need to smooth over the horizon scale \(H_0^{-1}\),

\[
\delta \chi^2(n) = [\delta \chi_L^2(0) + \langle \delta \chi_S^2(0) \rangle] A^2 \exp \left(\frac{4}{3} \alpha n^{3/2} - 3n \right)
\]

\[
= \left[\delta \chi_L^2(0) + \int_{k_c}^{\alpha k c} \frac{d^3 k}{(2\pi)^3} \delta \chi_S^2(0) \right] A^2 \exp \left(\frac{4}{3} \alpha n^{3/2} - 3n \right),
\]

where \(\delta \chi_L^2(0)\) and hence \(\delta \chi^2(n)\) is spatially varying on super-horizon scales. Note that we have omitted the logarithmic dependence term on \(n\) in the exponent, which is sub-dominant when we evaluate at \(n = n_f = \mathcal{O}(1)\). We have also subtracted the contribution from the modes with \(k > \alpha k\) since they remain stable and behave in the same way as the flat Minkowski vacuum modes, in accordance with the regularization we adopted, i.e. \(\langle \delta \chi^2(n) \rangle = 0\) at \(n < 0\).

With \(\alpha \gg 1\), from the initial amplitudes \((2.27)\) and \((2.30)\) we can see that the contribution of sub-horizon modes is much bigger than the one from super-horizon modes if the average is taken. At the end of inflation we have \(\langle \delta \chi^2(n_f) \rangle = m^2/g^2\), so that using

\[
\langle \delta \chi_S^2(0) \rangle = \frac{\alpha^2 H_0^2}{8\pi^2},
\]

which follows from \((2.30)\), we have

\[
\frac{m^2}{g^2} = \langle \delta \chi^2(n_f) \rangle = \left[\langle \delta \chi_L^2(0) \rangle + \langle \delta \chi_S^2(0) \rangle \right] A^2 \exp \left(\frac{4}{3} \alpha n_f^{3/2} - 3n_f \right)
\]
\[
\frac{\alpha^2 H_0^2}{8\pi^2} A^2 \exp \left( \frac{4}{3} \alpha n_f^{3/2} - 3 n_f \right),
\]
we find
\[
\exp \left( \frac{4}{3} \alpha n_f^{3/2} - 3 n_f \right) = \frac{8\pi^2 m^2}{g^2 \alpha^2 A^2 H_0^2}.
\]
(3.4)

Now let us rephrase the above discussion in a form more convenient for the \(\delta N\) formalism. With coordinate dependence explicit, \(\delta \chi^2\) given by (3.1) is recast as
\[
\delta \chi^2(n, \mathbf{x}) = \delta \chi^2_L(n, \mathbf{x}) + \langle \delta \chi^2_S(n) \rangle .
\]
(3.5)

As mentioned in the first paragraph of this section, since the smoothing is done over the horizon scales \(H_0^{-1}\), there remains no spatial coordinate dependence in \(\langle \delta \chi^2_S \rangle\). Meanwhile, we do have a spatial coordinate dependence for the modes with wavelengths longer than \(H_0^{-1}\), which is what we should take care of in the context of the \(\delta N\) formalism. Neglecting \(-3n\) in the exponential for simplicity since \(\alpha \gg 1\), splitting \(n = \bar{n} + \delta n\) and expanding in terms of \(\delta n\), (3.1) is written as
\[
\delta \chi^2(\bar{n} + \delta n) = \left[ 1 + \frac{\delta \chi^2_L(0)}{\delta \chi^2_S(0)} \right] \langle \delta \chi^2_S(0) \rangle A^2 \exp \left( \frac{4}{3} \alpha \bar{n}^{3/2} \right) \left( 1 + 2 \alpha \bar{n}^{1/2} \delta n + \cdots \right) ,
\]
(3.6)

where \(\langle \delta \chi^2_S \rangle \gg \delta \chi^2_L\) as discussed above.

Now we evaluate \(\delta n\) at a later time, say, at the end of inflation \(n = n_f\). Here it is important to note that the end of inflation is controlled by the value of \(\delta \chi^2\) at each spatial point, namely,
\[
\delta \chi^2(n_f, \mathbf{x}) = \frac{m^2}{g^2} = \langle \delta \chi^2(n_f) \rangle .
\]
(3.7)

Analogous to the case when the value of the inflaton field determines the end of inflation hypersurface, this condition determines the end of inflation hypersurface on which the energy density is uniform (at leading order approximation where the contribution of the inflaton to the energy density is negligible). Then, using (3.2) and (3.4), we find
\[
1 \approx \left[ 1 + \frac{\delta \chi^2_L(0)}{\delta \chi^2_S(0)} \right] \left( 1 + 2 \alpha n_f^{1/2} \delta n \right) ,
\]
(3.8)

where we have truncated at linear order in \(\delta n\). Inverting this relation, we can write the curvature perturbation generated between the moment of phase transition and the end of inflation as
\[
R_c(\mathbf{x}) = \delta n \approx -\frac{1}{2\alpha n_f^{1/2}} \frac{\delta \chi^2_L(0, \mathbf{x})}{\langle \delta \chi^2_S(0) \rangle} .
\]
(3.9)
This explicitly shows that the spectrum of $R_c$ is determined by the spectrum of $\delta\chi^2_L$.

From the result obtained previously in Sec. [II C], the mode function is $k$-independent for $k < k_c$. This implies that the power spectrum of $\delta\chi_L$ is white: $P_{\delta\chi_L}(k)$ is constant (in the conventional terminology used in cosmology, it is blue with the spectral index of 4: $P_{\delta\chi_L}(k) \equiv k^3/(2\pi^2)P_{\delta\chi_L}(k) \propto k^{n-1}$ with $n = 4$. See (5.4) for example). Assuming that the spectrum has a ultraviolet cutoff at $k = k_c$, this implies that $\delta\chi^2_L$ also has the same white spectrum, since the convolution of two white spectra is white. Thus apart from the amplitude which we will calculate below, we can already conclude that $P_R \propto k^3$, so that the spectral index is strongly blue with $n_{R_c} = 4$, indicating that the curvature perturbation is strongly suppressed on large scales.

Before we move to the computation of the power spectrum, let us also observe that $R_c$ seems to be always negative. This can be also read from (3.8): although $\delta\chi_L$ may be positive or negative, it appears in the form of a square in (3.8). So irrespective of the sign of $\delta\chi_L$ its contribution is always positive. Meanwhile, the left hand side of (3.8) is a constant. Thus, to compensate the positive contribution of $\delta\chi^2_L$ to make the left hand side a constant, $\delta n$ is always negative. Also we note that the average value of $R_c$ is not zero,

$$\langle R_c \rangle = -\frac{1}{2\alpha n_\ell^{1/2}} \frac{\langle \delta\chi^2_L(0) \rangle}{\langle \delta\chi^2_S(0) \rangle}. \tag{3.10}$$

We will consider these issues a little further later.

Finally, before closing this section, let us discuss constraints on the model parameters. First we consider the condition that comes from the fact that the initial value of $\delta\chi^2$ must be smaller than the final value of it. From (3.2) and (3.7), we find

$$g^2 \ll \frac{24\pi^2r}{\beta} = \frac{12\pi^2}{\beta}. \tag{3.11}$$

On the other hand, for this hybrid inflation model to be viable, the amplitude of the curvature perturbation due to the inflaton field $\phi$ must not exceed the observed value, $P^{(\phi)}_R \lesssim 10^{-9}$,

$$10^{-9} \lesssim \frac{P^{(\phi)}_R}{2\pi H_0^2} \approx \left( \frac{3H_0^2g}{2\pi m^2M} \right)^2 = \frac{g^2}{(2\pi)^2r^2\beta}, \tag{3.12}$$

hence

$$g^2 \lesssim (2\pi)^2 10^{-9}r^2\beta. \tag{3.13}$$

We see that both (3.11) and (3.13) can be safely satisfied for reasonable values of the parameters. As a typical example, consider the case $r = m^2/(3H_0^2) = 1/10$ and $\beta =
\(M^2/H_0^2 = 100\), which implies \(\alpha^2 = 20\). In this case (3.11) gives \(g^2 \ll 1\) while we have \(g^2 \lesssim 4 \times 10^{-8}\) from (3.13). Thus the condition (3.11) is well satisfied in this typical case.

Let us also consider the other conditions on the model parameters. The requirement (2.6) that the linear approximation to the equation of motion for \(\chi\) is valid implies the condition on \(\lambda\) as, using (3.13),

\[
\lambda \ll \frac{g^2 M^2}{m^2} \lesssim \frac{(2\pi)^2}{3} 10^{-9} r \beta^2 .
\]

(3.14)

This gives \(\lambda \ll 10^{-5}\) for \(r = 1/10\) and \(\beta = 100\). Another condition of \(\lambda\) comes from the observational constraint on the amplitude of tensor perturbations, \(H^2/m_{Pl}^2 \lesssim 10^{-10}\). In the present model, since \(H^2/m_{Pl}^2 = V_0/(3m_{Pl}^4) = M^4/(12\lambda m_{Pl}^4)\), this gives the condition

\[
\frac{M^4}{m_{Pl}^4} \lesssim 10^{-9} \lambda .
\]

(3.15)

On the other hand, from \(\beta = M^2/H_0^2\) we have \(M^2/m_{Pl}^2 = 12\lambda/\beta\). Therefore we must have

\[
\lambda \lesssim 10^{-11} \beta^2 .
\]

(3.16)

Comparing with (3.14), we see that this condition is also well satisfied for typical values of the model parameters.

**IV. CORRELATION FUNCTIONS**

**A. Power spectrum**

In this section, we drop the subscript \(L\) for notational simplicity. Since \(\langle R_c \rangle \neq 0\), it is more relevant to consider \(R_c - \langle R_c \rangle\) rather than \(R_c\) itself given by (3.9). Nevertheless, the difference becomes irrelevant in the Fourier space as long as we focus on a finite wavenumber. We will discuss this point in the next section.

Moving to the Fourier space, we can write

\[
R_c(k) = -\frac{1}{2\alpha n_f^{1/2}} \frac{(\delta \chi_k^2)}{\langle \delta \chi_S^2 \rangle} ,
\]

(4.1)

so that the power spectrum is written as

\[
\langle R_c(k) R_c(q) \rangle \equiv (2\pi)^3 \delta^{(3)}(k + q) P_R(k) = \frac{1}{4\alpha^2 n_f H_o^3} \left\langle \left( \delta \chi_k^2 \right) \left( \delta \chi_q^2 \right) \right\rangle ,
\]

(4.2)
where we have used (3.2) in the last equality.

Before waterfall, $\delta \chi$ is purely quantum and it can be expressed in terms of the creation and annihilation operators $a_k^\dagger$ and $a_k$ as

$$
\delta \chi = \int \frac{d^3k}{(2\pi)^3} e^{i k \cdot x} \delta \chi_k = \int \frac{d^3k}{(2\pi)^3} e^{i k \cdot x} \left( a_k \chi_k + a_k^\dagger \chi_k^* \right),
$$

(4.3)

where $a_k^\dagger$ and $a_k$ satisfy the canonical commutation relations

$$
[a_k, a_q^\dagger] = (2\pi)^3 \delta^{(3)}(k - q),
$$

(4.4)

otherwise zero, and the mode function $\chi_k$ follows the same equation as that of $\delta \chi$. Since the Fourier component of $\delta \chi^2$ is written as a convolution

$$
(\delta \chi^2)_k = \int \frac{d^3q}{(2\pi)^3} \delta \chi_q \delta \chi_{k-q},
$$

(4.5)

we have to correlate four creation and annihilation operators with different momenta,

$$
\langle (\delta \chi^2)_k (\delta \chi^2)_q \rangle = \int \frac{d^3p d^3l}{(2\pi)^3} \langle (\delta \chi_p \delta \chi_{k-p}) (\delta \chi_l \delta \chi_{q-l}) \rangle .
$$

(4.6)

To calculate the above, we should note that what we are interested in are connected graphs, correlating different $(\delta \chi^2)_k$'s. Thus the meaningful contractions are

$$
\langle (\delta \chi^2)_k (\delta \chi^2)_q \rangle = \langle (\delta \chi_p \delta \chi_{k-p}) (\delta \chi_l \delta \chi_{q-l}) \rangle + \langle (\delta \chi_p \delta \chi_{k-p}) (\delta \chi_l \delta \chi_{q-l}) \rangle ,
$$

(4.7)

while the remaining possible contractions are within the same $(\delta \chi^2)_k$'s and hence are irrelevant. Then, we can easily find

$$
\langle (\delta \chi^2)_k (\delta \chi^2)_q \rangle \equiv \chi_{p|k-p|} \chi_{q|q-l|} (2\pi)^3 \times [\delta^3(p + q - l) \delta^3(k - p + l) + \delta^3(p - l) \delta^3(k - p + q - l)].
$$

(4.8)

Thus, eliminating one of the momenta using the delta functions, and using the remaining delta function $\delta^3(k + q)$ to replace $q$ with $-k$, we find

$$
\langle (\delta \chi^2)_k (\delta \chi^2)_q \rangle = 2 \int d^3p |\chi_p|^2 |\chi_{k-p}|^2 \delta^3(k + q).\n$$

(4.9)

However, from (2.27), we have already seen that the super-horizon mode $\chi_k$ is independent of $k$, and thus can be pulled out of the integral. Hence, we only have to integrate over the
relevant super-horizon scale momentum, for which the upper limit is \( k = k_c \). Therefore, using (2.27), we finally obtain

\[
\langle (\delta \chi^2)_k (\delta \chi^2)_q \rangle = (2\pi)^3 \delta^{(3)}(k + q) \frac{4}{3^{11/3} [\Gamma(2/3)]^3} \alpha^{-4/3} \frac{H_0^4}{k_c^5}.
\] (4.10)

Since this expression has, as it should, the correct delta function dependence, we can readily extract the power spectrum \( P_R \). Noting (4.2) we find

\[
P_R \equiv \frac{k^3}{2\pi^2} \frac{32\pi^2}{3^{11/3} [\Gamma(2/3)]^4} \frac{\alpha^{-22/3}}{n_f} \left( \frac{k}{k_c} \right)^3,
\] (4.11)

where the numerical coefficient reads \( 32\pi^2 / \left\{ 3^{11/3} [\Gamma(2/3)]^4 \right\} \approx 1.67255 \). Thus, with \( n_f = \mathcal{O}(1) \), the maximum amplitude is found at \( k = k_c \) as \( P_R \sim \alpha^{-22/3} \) which is already much smaller than unity for \( \alpha \gg 1 \). For larger scales, it is exponentially suppressed and thus becomes absolutely negligible: for example, for a scale that exited the horizon at 50 e-folds before waterfall, it is suppressed by a factor \( (e^{-50})^3 \approx 10^{-65} \). As already discussed in the previous section, setting \( P_R \propto k^{n_R-1} \), the spectrum is very blue with the index \( n_R = 4 \).

### B. Bispectrum

Having found the curvature perturbation and the solution of the mode function, it is now straightforward to calculate the three-point correlation function. We can start from the

\footnote{Note, however, that mathematically there seems no apparent reason to set the upper limit of the integral at \( k = k_c \). It seems reasonable to extend the range of integration into sub-horizon scales up to an arbitrary ultraviolet cutoff at \( k = k_{UV} \) with \( k_{UV} \gg k_c \). If proceeding with the sub-horizon mode function solution (2.30), one finds that the squared mode function \( |\chi_k|^2 \) is suppressed by a factor of \( k_c / k \) relative to super-horizon modes. But this suppression factor is not strong enough to make the integral independent of the ultraviolet cutoff. Since the integrand \( |\chi_p|^2 \left| \chi_{(k-p)} \right|^2 \) is proportional to \( p^{-2} \), the integral will be dominated by the contribution from the ultraviolet cutoff, leading to the result in proportional to \( k_{UV} \). Of course there is a natural choice for the cutoff in the present case: \( k_{UV} = \alpha k_c \), up to which the modes become tachyonic, as advocated in Ref. [6, 14]. If we are to take this choice, then the resulting amplitude of curvature perturbations will be substantially enhanced, though the qualitative result will not change. Nevertheless, this strong dependence of super-horizon fluctuations on the ultraviolet cutoff deep inside the horizon looks physically strange because it seems to imply the violation of causality. In fact if this were indeed the case, then we would have a first example in which the \( \delta N \) formalism fails even for the curvature perturbation on super-horizon scales. This may be originated from our assumption of the knowledge of the entire universe beyond the horizon scale in the Fourier transformation. We discuss this point of maintaining causality regarding the horizon scale patches in the inflating universe in Appendix B, justifying (3.1) and (4.9) which are the very foundation of our computation of the correlation functions.}
\[
\langle \mathcal{R}_c(k_1)\mathcal{R}_c(k_2)\mathcal{R}_c(k_3) \rangle = (2 \pi)^3 \delta^{(3)}(k_1 + k_2 + k_3) B \mathcal{R}(k_1, k_2, k_3)
\]
\[
= \left( \frac{-1}{2 \alpha n_f^{1/2} (\delta \chi S^2)} \right)^3 \left\langle (\delta \chi^2)_{k_1} (\delta \chi^2)_{k_2} (\delta \chi^2)_{k_3} \right\rangle
\]
\[
= \left( \frac{-1}{2 \alpha n_f^{1/2} (\delta \chi S^2)} \right)^3 \int \frac{d^3q_1d^3q_2d^3q_3}{(2\pi)^3} \\
\quad \times \left\langle (\delta \chi_{q_1}\delta \chi_{k_1-q_1})(\delta \chi_{q_2}\delta \chi_{k_2-q_2})(\delta \chi_{q_3}\delta \chi_{k_3-q_3}) \right\rangle.
\]

As before, we are interested in the connected graphs. This means we only take contractions between those coming from different \((\delta \chi)_k\)'s. It is immediately seen that there are 8 possible contractions: for one of the two \(\delta \chi\)'s in \(\mathcal{R}_c(k_1)\), there are four choices of contractions to one of \(\delta \chi\)'s in \(\mathcal{R}_c(k_2)\) and \(\mathcal{R}_c(k_3)\), and for the remaining \(\delta \chi\) in \(\mathcal{R}_c(k_1)\), there are two ways of contraction to either \(\mathcal{R}_c(k_2)\) or \(\mathcal{R}_c(k_3)\) which are not chosen by the first contraction. This gives the total number of \(4 \times 2 = 8\) different contractions. These are explicitly written as

\[
\begin{align*}
&\langle (\delta \chi_{q_1}\delta \chi_{k_1-q_1})(\delta \chi_{q_2}\delta \chi_{k_2-q_2})(\delta \chi_{q_3}\delta \chi_{k_3-q_3}) \rangle + \langle (\delta \chi_{q_1}\delta \chi_{k_1-q_1})(\delta \chi_{q_2}\delta \chi_{k_2-q_2})(\delta \chi_{q_3}\delta \chi_{k_3-q_3}) \rangle \\
&+ \langle (\delta \chi_{q_1}\delta \chi_{k_1-q_1})(\delta \chi_{q_2}\delta \chi_{k_2-q_2})(\delta \chi_{q_3}\delta \chi_{k_3-q_3}) \rangle + \langle (\delta \chi_{q_1}\delta \chi_{k_1-q_1})(\delta \chi_{q_2}\delta \chi_{k_2-q_2})(\delta \chi_{q_3}\delta \chi_{k_3-q_3}) \rangle \\
&+ \langle (\delta \chi_{q_1}\delta \chi_{k_1-q_1})(\delta \chi_{q_2}\delta \chi_{k_2-q_2})(\delta \chi_{q_3}\delta \chi_{k_3-q_3}) \rangle + \langle (\delta \chi_{q_1}\delta \chi_{k_1-q_1})(\delta \chi_{q_2}\delta \chi_{k_2-q_2})(\delta \chi_{q_3}\delta \chi_{k_3-q_3}) \rangle \\
&+ \langle (\delta \chi_{q_1}\delta \chi_{k_1-q_1})(\delta \chi_{q_2}\delta \chi_{k_2-q_2})(\delta \chi_{q_3}\delta \chi_{k_3-q_3}) \rangle + \langle (\delta \chi_{q_1}\delta \chi_{k_1-q_1})(\delta \chi_{q_2}\delta \chi_{k_2-q_2})(\delta \chi_{q_3}\delta \chi_{k_3-q_3}) \rangle.
\end{align*}
\]

Each of these terms exactly corresponds to the term with \(\delta^{(3)}(k_1 + k_2 + k_3)\). Indeed, we find these 8 contractions give

\[
8\delta^{(3)}(k_1 + k_2 + k_3) \int d^3q \left| \chi_{k_1-q} \right|^2 \left| \chi_{k_2+q} \right|^2 \left| \chi_{k_3+q} \right|^2.
\]

Again noting that \(\left| \chi_k \right|^2\) is independent of momentum, and has a cut-off at \(k = k_c\), we obtain

\[
\int \frac{d^3q_1d^3q_2d^3q_3}{(2\pi)^3} \left\langle (\delta \chi_{q_1}\delta \chi_{k_1-q_1})(\delta \chi_{q_2}\delta \chi_{k_2-q_2})(\delta \chi_{q_3}\delta \chi_{k_3-q_3}) \right\rangle
\]
\[
= 8\delta^{(3)}(k_1 + k_2 + k_3) \frac{4\pi k_c^3}{3} \left[ \frac{2\sqrt{\pi}}{3^{2/3}\Gamma(2/3)} \frac{H_0}{\sqrt{2k_c^3\alpha^{1/3}}} \right]^6.
\]
Comparing this expression with the definition of the bispectrum, we find

\[ B_{R}(k_1, k_2, k_3) = -\frac{16(2\pi)^7}{3^5 [\Gamma(2/3)]^6} \frac{\alpha^{-11}}{n_f^{3/2} k_c^3}, \]  

(4.16)

where the numerical coefficient reads \(16(2\pi)^7 / \{3^5[\Gamma(2/3)]^6\} \approx 4128.89\). To leading order, the bispectrum has no momentum dependence, and thus the dimensionless shape function \((k_1k_2k_3)^2 B_{R}(k_1, k_2, k_3)\) exhibits its maximum amplitude at the equilateral limit \(k_1 = k_2 = k_3\). This is anticipated, since the curvature perturbation produced by the waterfall field is \textit{intrinsically} highly non-Gaussian. Note, however, that this bispectrum is completely unobservable on large scales: in the equilateral limit, multiplying \(k_6\), we see for example that it is exponentially suppressed by a factor of \((e^{-50})^6 \approx 10^{-130}\) for a scale that exited the horizon at 50 e-folds before the waterfall. Thus this bispectrum is totally hopeless to be detected on large scales.

V. DISTRIBUTION OF CURVATURE PERTURBATION

In this section, we consider the one-point probability distribution function of \(\mathcal{R}_c\). Basically, we can guess the form of the probability distribution function. At leading order \(\mathcal{R}_c\) is proportional to the square of \(\delta \chi_L\) which is very close to Gaussian. Thus, the probability distribution of \(\mathcal{R}_c \sim \delta \chi_L^2\) is expected to be very close to the chi-squared distribution.

The fully nonlinear distribution function can be obtained from (3.1). By setting \(n = n_f + \delta n\) and the left hand side of it to be \(m^2/g^2\), and regarding \(\delta n = \mathcal{R}_c\) as a function of \(\delta \chi_L\), the distribution function \(\mathbb{P}\) of \(\mathcal{R}_c\) is given as

\[ \mathbb{P}(\mathcal{R}_c) = \mathbb{P}_\chi(\delta \chi_L) \frac{d\delta \chi_L}{d\mathcal{R}_c}. \]  

(5.1)

Here, we already know that \(\mathbb{P}_\chi(\delta \chi_L)\) is a Gaussian distribution with zero mean, i.e.

\[ \mathbb{P}_\chi(\delta \chi_L) = \frac{1}{\sqrt{2\pi \sigma_{\delta \chi_L}}} \exp \left( -\frac{\delta \chi_L^2}{2\sigma_{\delta \chi_L}^2} \right), \]  

(5.2)

and the variance \(\sigma_{\delta \chi_L}^2 \equiv \langle \delta \chi_L^2 \rangle - \langle \delta \chi_L \rangle^2\), with \(\langle \delta \chi_L \rangle = 0\), is given by

\[ \sigma_{\delta \chi_L}^2 = \langle \delta \chi_L^2 \rangle = \int d\log k \mathcal{P}_{\delta \chi_L}(k), \]  

(5.3)

where the power spectrum \(\mathcal{P}_{\delta \chi_L}(k)\) of the fluctuations \(\delta \chi_L\) can be found from (2.27) as

\[ \mathcal{P}_{\delta \chi_L}(k) = \frac{k^3}{2\pi^2} |\delta \chi_k|^2 = \frac{\alpha^{-2/3}H_0^2}{3^{4/3}\pi [\Gamma(2/3)]^2} \left( \frac{k}{k_c} \right)^3. \]  

(5.4)
Noting that the large scale modes has a cut-off at \( k = k_c \), we obtain

\[
\sigma_{\chi L}^2 = \frac{\alpha^{-2/3}H_0^2}{3^{7/3}\pi \left[ \Gamma(2/3) \right]^2}.
\] (5.5)

Now, we evaluate (3.1) at \( n = n_f + \delta n = n_f + \mathcal{R}_c \) to write

\[
\frac{m^2}{g^2} = \left( \delta \chi_L^2 + \frac{\alpha^2 H_0^2}{8\pi^2} \right) A^2 \exp \left[ 4 \frac{1}{3}\alpha(n_f + \mathcal{R}_c)^{3/2} - 3(n_f + \mathcal{R}_c) \right].
\] (5.6)

This equation can be easily solved for \( \delta \chi_L \) as a function of \( \mathcal{R}_c \),

\[
\delta \chi_L = \sqrt{\frac{m^2}{A^2g^2} \exp \left[ -4 \frac{1}{3}\alpha(n_f + \mathcal{R}_c)^{3/2} + 3(n_f + \mathcal{R}_c) \right] - \frac{\alpha^2 H_0^2}{8\pi^2}}.
\] (5.7)

Thus from (5.1) we can immediately find the probability distribution of \( \mathcal{R}_c \) as

\[
\mathbb{P}(\mathcal{R}_c) = \frac{1}{\sqrt{2\pi}\sigma_{\chi L}} \exp \left\{ -\frac{1}{2\sigma_{\chi L}^2} \left[ \frac{m^2}{A^2g^2} e^{-4 \frac{1}{3}\alpha(n_f + \mathcal{R}_c)^{3/2} + 3(n_f + \mathcal{R}_c)} - \frac{\alpha^2 H_0^2}{8\pi^2} \right] \right\} \\
\times \left\{ \frac{m^2}{A^2g^2} \exp \left[ -4 \frac{1}{3}\alpha(n_f + \mathcal{R}_c)^{3/2} + 3(n_f + \mathcal{R}_c) \right] - \frac{\alpha^2 H_0^2}{8\pi^2} \right\}^{-1/2} \left( \alpha \sqrt{n_f + \mathcal{R}_c} - \frac{3}{2} \right)^2.
\] (5.8)

This is a fairly complex probability distribution function, and is very different from the Gaussian one. We plot it in Fig. 1.

Having the distribution function of \( \mathcal{R}_c \) at hand, let us consider the mean value \( \langle \mathcal{R}_c \rangle \). We can formally write it as

\[
\langle \mathcal{R}_c \rangle = \int \mathcal{R}_c \mathbb{P}(\mathcal{R}_c) d\mathcal{R}_c = \int \mathcal{R}_c \delta \chi_L \mathbb{P}(\delta \chi_L) d\delta \chi_L,
\] (5.9)

where \( \mathcal{R}_c \) is now regarded as a function of \( \delta \chi_L \). Although we cannot invert (5.7) to find \( \mathcal{R}_c(\delta \chi_L) \) exactly, we can obtain an approximate expression by assuming \( |\mathcal{R}_c| \ll 1 \) as

\[
\mathcal{R}_c = \left\{ \frac{3}{4\alpha} \log \left[ \frac{m^2}{g^2} \left( \delta \chi_L^2 + \frac{\alpha^2 H_0^2}{8\pi^2} \right)^{-1} \right] \right\}^{2/3} - n_f \approx -\frac{(2\pi)^2}{\alpha^3 n_f^{1/2}} (\frac{\delta \chi_L}{H_0})^2 + \cdots,
\] (5.10)

where we have expanded in the limit \( \delta \chi_L^2 \ll \alpha^2 H_0^2/(8\pi^2) \). It is trivial to find that for a Gaussian distribution \( \mathbb{P}(x) \),

\[
\int x^2 \mathbb{P}(x) dx = \sigma_x^2,
\] (5.11)

and thus the average value of \( \mathcal{R}_c \) is found, using (5.3), as

\[
\langle \mathcal{R}_c \rangle = -\frac{4\pi}{3^{7/3}\pi \left[ \Gamma(2/3) \right]^2} \alpha^{-11/3} n_f^{1/2}.
\] (5.12)
FIG. 1: The probability distribution of $|\mathcal{R}_c|$ \[(5.8)\], with $r \approx 0.01$ and $\alpha \approx 8$. As $|\mathcal{R}_c|$ becomes larger, the probability drops down extremely sharply. For comparison, we also show the chi-squared distribution (dotted line) with appropriate normalization. At small value of $|\mathcal{R}_c|$, the two distribution functions behave in the same manner but they become different at larger $|\mathcal{R}_c|$. The overall numerical factor is $4\pi/\left\{3^{7/3} \left[\Gamma(2/3)\right]^2\right\} \approx 0.527976$. If we take $n_f = \mathcal{O}(1)$, the most important factor is its dependence on $\alpha$: for $\alpha \gg 1$, it is indeed very small. If we could have $\alpha$ of order unity, the mean value could become large. But as we have discussed in Sec. II E, this cannot be the case because of the condition for an efficient tachyonic instability.

As we have mentioned before, the relevant curvature perturbation is not $\mathcal{R}_c$ itself but $\mathcal{R}_c - \langle \mathcal{R}_c \rangle$. Therefore, although $\mathcal{R}_c$ is always negative, the true fluctuations from the mean value can become positive. Nevertheless, since the mean value $\langle \mathcal{R}_c \rangle$ turns out to be very small, there is no chance to have a large positive fluctuation: the fluctuation is bounded from above as

$$\mathcal{R}_c - \langle \mathcal{R}_c \rangle \leq |\langle \mathcal{R}_c \rangle|.$$ \[(5.13)\]

Recalling that $\mathcal{R}_c$ is negative of the gravitational potential, we can see that the curvature perturbation induced by the waterfall field repels matter around rather than attract. This implies that there would be no primordial black hole formation even on scales as small as the Hubble horizon scale, but rather bubbles of void may appear. But this is a highly qualitative
argument and we need more explicit calculations, which we do not pursue in this note.

VI. CONCLUSION

In this paper, we have examined the contribution of the waterfall field $\chi$ to the curvature perturbation. The waterfall field $\chi$ can change the final curvature perturbation during the period between the moment of waterfall and the end of inflation, because $\chi$ controls the physical processes during this time: the waterfall phase transition occurs as soon as the effective mass squared of $\chi$ becomes negative, and the end of inflation is determined by the mean square fluctuations of the sub-horizon modes of $\chi$ which became tachyonic after the waterfall transition.

By solving the equation of $\chi$, we have obtained for both the super- and sub-horizon modes the amplitudes at the moment of waterfall and time dependence until the end of inflation in terms of the number of $e$-folds. Using the $\delta N$ formalism, we have calculated both the power spectrum and bispectrum of the curvature perturbation induced by the waterfall field $\chi$. The power spectrum is steeply blue with $n_{R_c} = 4$, and the bispectrum exhibits the maximum amplitude at the equilateral limit. This indicates that the distribution of the curvature perturbation is intrinsically non-Gaussian, and we have presented the explicit form of the distribution function. On large scales, however, both the power spectrum and bispectrum are exponentially suppressed and totally negligible.

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Appendix A: Evaluation of $\mathcal{R}_c$ from linear perturbation equation

Here to check if our result based on the $\delta N$ formalism is consistent with the standard perturbation theory, we evaluate the curvature perturbation by using the linear perturbation equation for $\mathcal{R}_c$. See also Ref. [6] for this approach.

In linear theory, on super-horizon scales, it is known that the curvature perturbation on comoving slices $\mathcal{R}_c$ satisfies

$$\dot{\mathcal{R}}_c = -H \frac{\delta P_c}{\rho + P},$$

(A1)

where $\delta P_c$ is the pressure perturbation on comoving slices. The comoving slice is defined by $\delta T^0_0 = 0$. In the present case, this means

$$\delta T^0_0 = -\left( \dot{\phi} \partial_i \delta \phi + \dot{\chi} \partial_i \chi \right) \approx -\phi \partial_i \delta \phi = 0,$$

(A2)

where we have used the fact that $\chi = \delta \chi_L$ and $\mathcal{P}_{\delta \chi_L}(k) \propto k^3$, namely the fact that on super-horizon scales the contribution from the waterfall field to $\delta T^0_0$ is negligible compared to that from the inflaton field. That is, on super-horizon scales, the comoving slices are defined solely in terms of the inflaton as those on which the inflaton field is homogeneous.

Therefore the contribution to the pressure perturbation $\delta P_c$ comes totally from the waterfall field $\chi$,

$$\delta P_c = \frac{1}{2} H_0^2 \delta \chi_L^2 + \frac{1}{2} (M^2 - g^2 \phi^2) \delta \chi_L^2.$$

(A3)

As for $\rho + P$, we have

$$\rho + P = H_0^2 \left[ \phi^2 + \langle \delta \chi_S^2 \rangle \right].$$

(A4)
Let us evaluate $\phi'^2$ and $\langle \delta \chi_S'^2 \rangle$ to see which term dominates during waterfall. For $\phi'^2$ we have

$$\phi'^2(n) = (r\phi_c)^2 e^{-2rnn} = H_0^2 r^2 \beta g^2 e^{-2rnn}. \quad (A5)$$

For $\langle \delta \chi_S'^2 \rangle$ we have at $n \gtrsim 1$

$$\langle \delta \chi_S'^2(n) \rangle \approx \alpha^2 n \langle \delta \chi_S^2(n) \rangle \approx H_0^2 \alpha^4 A^2 \exp \left( \frac{4}{3} \alpha n^{3/2} - 3n \right). \quad (A6)$$

Hence the time dependent ratio of $\langle \delta \chi_S'^2 \rangle$ to $\phi'^2$ is written as

$$R(n) \equiv \frac{\langle \delta \chi_S'^2(n) \rangle}{\phi'^2(n)} \approx 6n_f. \quad (A7)$$

Using (3.7) and (A6), at the end of inflation we have

$$R(n_f) = \frac{\langle \delta \chi_S'^2(n_f) \rangle}{\phi'^2(n_f)} \approx 6n_f. \quad (A8)$$

Therefore, for $n_f \gtrsim 1$, $\langle \delta \chi_S'^2 \rangle$ becomes dominant toward the end of inflation. Using this result, we can rewrite (A7) as

$$R(n) = R(n_f) \frac{R(n)}{R(n_f)} \approx 6n_f \exp \left[ \frac{4}{3} \alpha (n^{3/2} - n_f^{3/2}) - 3(n - n_f) \right]. \quad (A9)$$

Let $n_{eq} \equiv n_f - \Delta n$ be the time at which $\langle \delta \chi_S'^2 \rangle$ begins to dominate over $\phi'^2$. Since the growth rate of $\langle \delta \chi_S'^2 \rangle$ is very fast and the ratio $R(n_f)$ at the end of inflation (A8) is not so large, $\sim 10$ or so, the $\langle \delta \chi_S'^2 \rangle$-dominated stage appears only at the very near the end of inflation, $\Delta n \ll 1$. Specifically, setting $R(n_{eq}) = 1$, we find

$$\Delta n \approx \frac{\ln(6n_f)}{2\alpha n_f^{1/2}} \approx \frac{1}{2\alpha n_f^{1/2}}. \quad (A10)$$

Therefore, $\phi'^2$, which is almost constant in time, dominates over $\langle \delta \chi_S'^2 \rangle$ almost all the stage of the waterfall $n \lesssim n_{eq}$.

With the above result in mind, we rewrite (A11) as

$$\frac{dR_c}{dn} = -\frac{\delta P_c}{\rho + P} \approx -\frac{\delta \chi_L^2(n)}{\langle \delta \chi_S^2(n) \rangle} \frac{R(n)}{1 + R(n)}. \quad (A11)$$

Since $\delta \chi_L^2(n)/\langle \delta \chi_S^2(n) \rangle$ is time-independent, we can just replace it by that evaluated at $n = 0$. Then (A11) can be expressed as

$$\frac{dR_c}{dn} = -\frac{\delta \chi_L^2(0)}{\langle \delta \chi_S^2(0) \rangle} \frac{R(n)}{1 + R(n)}. \quad (A12)$$
The last factor on the right hand side is negligible for $n < n_{eq}$ and approximately equal to one for $n_{eq} < n < n_f$. Therefore, with the initial condition that $R_c(0) = 0$, it can be easily integrated to give

$$R_c(n_f) \approx -\frac{\delta \chi^2_0(0)}{\langle \delta \chi^2_0(0) \rangle} \Delta n . \quad (A13)$$

With the identification that $\Delta n \approx 1/(2\alpha n_f^{1/2})$ as evaluated in (A10), this agrees with our result using the $\delta N$ formalism$^6$.

**Appendix B: Short wavelength modes**

In this section, we justify (3.1) and the integration of (4.9), and argue why we do not go beyond the horizon scale. We consider a scalar field $\phi(x)$ and decompose it into Fourier modes $\tilde{\phi}(k)$. Let us call the modes with wavelengths smaller than the horizon size $L_H = 2\pi/H$ the short wavelength modes and those larger than $L_H$ the long wavelength modes. We assume the universe is inflating.

When we decompose $\phi(x)$, usually we assume we have the knowledge of the whole (infinitely large) universe. That is,

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \tilde{\phi}(k) e^{ikx} \leftrightarrow \tilde{\phi}(k) = \int d^3x \phi(x) e^{-ikx} . \quad (B1)$$

If we divide the above into those composed of long wavelength modes and short wavelength modes,

$$\phi(x) = \phi_L(x) + \phi_S(x) = \int_{k<H} \frac{d^3k}{(2\pi)^3} \tilde{\phi}(k)e^{ikx} + \int_{k>H} \frac{d^3k}{(2\pi)^3} \tilde{\phi}(k)e^{ikx} , \quad (B2)$$

then this will naturally induce a non-zero correlation between $\phi_S(x)$ and $\phi_S(y)$ even if the two points are separated at a distance larger than the horizon size,

$$\langle \phi_S(x)\phi_S(y) \rangle \neq 0 \quad (B3)$$

for $|x-y| > L_H$. Since each horizon size region should be causally unrelated during inflation, this result is acausal. This is apparently due to our assumption that we, i.e. the observers belonging to different regions of horizon size, have the knowledge of the whole universe.

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$^6$ Note that if we faithfully integrate (A12) using (A9), we can even recover the logarithmic correction factor $\ln(6n_f)$ in (A10) as the leading order approximation of the integral.
Therefore, in stead of (B2), it is more reasonable to divide the field in such a way that \( \phi_S(x) \) and \( \phi_S(y) \) will not be correlated if \(|x - y| > L_H\). To incorporate this prescription, we proceed as follows. We introduce two boxes of different size, a very large box \( L^3 \) where \( L = NL_H \) with \( N \gg 1 \) being a very large integer and the horizon size box \( L_H^3 \). The large box would correspond to the present horizon size of the universe.

We define \( \chi_S(x) \) for each horizon size box as

\[
\phi_{S(i)}(x) = \theta_{(i)}(x) \sum_k \tilde{\phi}_{(i)}(k)e^{ik(x-x_i)} \quad \left( k = \frac{2\pi}{L_H}n \right),
\]

(B4)

where \( n = (n_1, n_2, n_3) \) (\( n_i \) are integers), \( x_i \) is the center of \( i \)-th box, and \( \theta_{(i)}(x) = 1 \) if \( x \) is in the \( i \)-th horizon size region and zero otherwise. The long wavelength part is defined by

\[
\phi_L(x) = \sum_{|n| \leq N} \tilde{\phi}_L(k)e^{ikx} \quad \left( k = \frac{2\pi}{L}n = \frac{2\pi}{L_H}n \right).
\]

(B5)

Thus we have the decomposition,

\[
\phi(x) = \phi_L(x) + \phi_S(x),
\]

(B6)

\[
\phi_S(x) = \sum_i \phi_{S(i)}(x).
\]

(B7)

This guarantees that there is no correlation of between two short wavelength modes that belong to two different horizon size regions: for \(|x - y| > L_H\),

\[
\langle \phi_S(x)\phi_S(y) \rangle = 0.
\]

(B8)

Now we take the square of \( \phi(x) \) and average over the horizon scale. We obtain

\[
\langle \phi^2(x) \rangle_{L_H} = \phi_L^2(x) + \langle \phi_S^2(x) \rangle = \phi_L^2(x) + \sum_i \langle \phi_{S(i)}^2(x) \rangle + \sum_i \theta_{(i)}(x) \sum_k |\tilde{\phi}_{(i)}(k)|^2.
\]

(B9)

It is reasonable to assume that \( |\tilde{\phi}_{(i)}(k)|^2 \) is independent of the region \( (i) \). Hence we may set \( |\tilde{\phi}_{(i)}(k)|^2 = |\tilde{\phi}_S(k)|^2 \). Then since \( \sum_i \theta_{(i)}(x) = 1 \), we obtain

\[
\langle \phi^2(x) \rangle_{L_H} = \phi_L^2(x) + \sum_i \theta_{(i)}(x) \sum_k |\tilde{\phi}_S(k)|^2 = \phi_L^2(x) + \sum_k |\tilde{\phi}_S(k)|^2.
\]

(B10)

This agrees with (3.1).
Also, if we consider the sum on the short wavelength modes,

\[
\sum_p \left| \tilde{\phi}_i(p) \right|^2 \left| \tilde{\phi}_i(p + k) \right|^2 = \sum_p \left| \tilde{\phi}_S(p) \right|^2 \left| \tilde{\phi}_S(p + k) \right|^2 ,
\]

which appears in (4.9), it is apparent that this is non-vanishing only for \( |k| \geq 2\pi / L_H \), because there exists no sum for \( |k| < 2\pi / L_H \) by definition. This means there will be no contribution from the short wavelength modes to the curvature perturbation on super-horizon scales.

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