Leading-Order Actions of Goldstino Fields

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Abstract

This paper starts with a self-contained discussion of the so-called Akulov-Volkov action $S_{AV}$, which is traditionally taken to be the leading-order action of Goldstino field. Explicit expressions for $S_{AV}$ and its chiral version $S_{AV}^{ch}$ are presented. We then turn to the issue on how these actions are related to the leading-order action $S_{NL}$ proposed in the newly proposed constrained superfield formalism. We show that $S_{NL}$ may yield $S_{AV}/S_{AV}^{ch}$ or a totally different action $S_{KS}$, depending on how the auxiliary field in the former is integrated out. However, $S_{KS}$ and $S_{AV}/S_{AV}^{ch}$ always yield the same $S$-matrix elements, as one would have expected from general considerations in quantum field theory.

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Supersymmetry (SUSY) is arguably among the most attractive extensions of the standard model. It renders a reasonable framework to circumvent the hierarchy problem and has interesting phenomenological implications at the TeV scale. Tremendous efforts have been made on the subject in the last several decades. Hopefully, it is to be discovered in the coming LHC experiments.

To be consistent with existing experiments and to have certain predictive power, SUSY must be broken and broken spontaneously. According to the general theory of spontaneously global symmetry breaking, this would result in a massless neutral Nambu-Goldstone fermion, the Goldstino.\(^1\)

For its low energy physics, the Goldstino can be studied in the framework of nonlinear realization of SUSY. The leading-order action of Goldstino field was traditionally taken to be the so-called Akulov-Volkov action \(S_{AV}\) \([1]\) or its chiral version \(S_{AV}^{ch}\) \([2]\). Both actions are manifestly invariant under nonlinear SUSY transformations. In the newly proposed constrained superfield formalism, the leading-order action of Goldstino field is assumed to be one \(S_{NL}\) \([3]\). In this paper, we will show that \(S_{NL}\) may yield \(S_{AV}/S_{AV}^{ch}\) or a totally different action \(S_{KS}\), depending on how the auxiliary field in the former is integrated out. \(S_{KS}\) takes a particularly simple form, but does not have transparent properties under nonlinear SUSY transformations. However, \(S_{KS}\), \(S_{AV}\) and \(S_{AV}^{ch}\) always yield the same \(S\)-matrix elements, regardless how the auxiliary field is integrated out, as one would have expected from general considerations in quantum field theory.

In the standard (non-chiral) version of nonlinear realization of SUSY, the Goldstino field \(\lambda\) is assumed to change nonlinearly under SUSY transformations \([1, 4]\)

\[
\begin{align*}
\delta_\xi \lambda_\alpha &= \frac{1}{\kappa} \xi_\alpha - i\kappa(\lambda \sigma^\mu \bar{\xi} - \xi \sigma^\mu \bar{\lambda}) \partial_\mu \lambda_\alpha, \\
\delta_\xi \bar{\lambda}_{\dot{\alpha}} &= \frac{1}{\kappa} \bar{\xi}_{\dot{\alpha}} - i\kappa(\lambda \sigma^\mu \bar{\xi} - \xi \sigma^\mu \bar{\lambda}) \partial_\mu \bar{\lambda}_{\dot{\alpha}},
\end{align*}
\]

while matter fields \(\zeta\) are to change according to \([5, 6]\)

\[
\delta_\xi \zeta = -i\kappa(\lambda \sigma^\mu \bar{\xi} - \xi \sigma^\mu \bar{\lambda}) \partial_\mu \zeta.
\]

\(^1\) In supergravity, the Goldstino is absorbed by the gravitino particle and becomes the \(\pm 1/2\) helicity components of the latter. However, if the SUSY breaking scale is much smaller than the Planck scale, the lower energy physics of gravitino will be dominated by the Goldstino. In a sense, this provides a supersymmetric version of the equivalence theorem. Therefore, it makes sense to investigate the physics of Goldstino independently, as it may provide an interesting window to look into SUSY.
The Akulov-Volkov action assumes the following form [1, 4]

\[ S_{AV} = -\frac{1}{2\kappa^2} \int d^4x \det T, \]  

(3)

where \( T^\mu_\nu = \delta^\mu_\nu - ik^2 \partial_\mu \lambda \sigma^\nu \bar{\lambda} + ik^2 \bar{\lambda} \sigma^\nu \partial_\mu \lambda \). It is invariant under the SUSY transformation Eq (1) since the change of \( \det T \) is a total derivative

\[ \delta_\xi \det T = -i\kappa \partial_\mu [(\lambda \sigma^\mu \bar{\xi} - \xi \sigma^\mu \bar{\lambda}) \det T]. \]

Expanding \( \det T \) in terms of \( \kappa \) explicitly,

\[
\det T = 1 - i\kappa^2 (\partial_\mu \lambda \sigma^\mu \bar{\lambda} - \lambda \sigma^\mu \partial_\mu \bar{\lambda}) - \kappa^4 \left[ i\epsilon^{\mu\nu\gamma} \bar{\lambda} \partial_\mu \lambda \sigma_\gamma \partial_\nu \bar{\lambda} + \bar{\lambda}^2 \partial_\mu \lambda \sigma^{\mu\nu} \partial_\nu \bar{\lambda} + \lambda^2 \partial_\mu \bar{\lambda} \sigma^{\mu\nu} \partial_\nu \lambda \right] \\
- i\kappa^6 \lambda^2 \bar{\lambda} \left[ \sigma^{\gamma} \partial_\gamma \lambda \partial_\mu \bar{\lambda} \sigma^{\mu\nu} \partial_\nu \bar{\lambda} + 2\sigma^{\nu} \partial_\nu \lambda \partial_\mu \bar{\lambda} \sigma^{\mu\nu} \partial_\rho \lambda \right] \\
- i\kappa^6 \bar{\lambda}^2 \left[ \sigma^{\nu} \partial_\nu \bar{\lambda} \partial_\mu \lambda \sigma^{\mu\nu} \partial_\nu \lambda + 2\sigma^{\nu} \partial_\nu \bar{\lambda} \partial_\mu \sigma^{\nu\rho} \partial_\rho \lambda \right].
\]

Noticing that the \( \kappa^8 \) terms are absent in the above expression, in contrast with [1, 7]. This was first observed in [8] and reconfirmed recently in [9]. Here we provide another verification by a brute force calculation. According to [1, 7], the \( \kappa^8 \) terms are proportional to

\[ \partial_\mu \bar{\lambda} \sigma^{\mu\nu} \partial_\nu \partial_\rho \lambda \sigma^{\nu\rho} \partial_\gamma \lambda + \partial_\mu \bar{\lambda} \sigma^{\nu\gamma} \partial_\nu \lambda \sigma^{\mu\rho} \partial_\rho \gamma \lambda + 4\partial_\mu \bar{\lambda} \sigma^{\mu\rho} \partial_\rho \lambda \sigma^{\nu\partial} \partial_\gamma \lambda. \]

Since these terms come from the determinant of a \( 4 \times 4 \) matrix, possible nonvanishing terms are only those with spacetime derivatives of different Lorentz indices. We may take \( \partial_1 \bar{\lambda}, \partial_2 \bar{\lambda}, \partial_3 \lambda \) and \( \partial_0 \lambda \) for example. All relevant terms are in the following

\[
4\partial_1 \bar{\lambda} \sigma^{12} \partial_2 \bar{\lambda} \partial_3 \lambda \sigma^{30} \partial_0 \lambda + 4\partial_1 \bar{\lambda} \sigma^{20} \partial_2 \bar{\lambda} \partial_3 \lambda \sigma^{12} \partial_0 \lambda + 4\partial_1 \bar{\lambda} \sigma^{13} \partial_2 \bar{\lambda} \partial_3 \lambda \sigma^{02} \partial_0 \lambda \\
+ 4\partial_1 \bar{\lambda} \sigma^{10} \partial_2 \bar{\lambda} \partial_0 \lambda \sigma^{32} \partial_3 \lambda + 4\partial_2 \bar{\lambda} \sigma^{23} \partial_1 \bar{\lambda} \partial_0 \lambda \sigma^{01} \partial_3 \lambda + 4\partial_2 \bar{\lambda} \sigma^{20} \partial_1 \bar{\lambda} \partial_0 \lambda \sigma^{31} \partial_3 \lambda,
\]

which can be regrouped as

\[
\sum_{j=1,2,3} \left( i\partial_1 \bar{\lambda} \sigma^j \partial_2 \bar{\lambda} \partial_3 \lambda \sigma^j \partial_0 \lambda - i\partial_1 \bar{\lambda} \sigma^j \partial_2 \bar{\lambda} \partial_3 \lambda \sigma^j \partial_0 \lambda \right).
\]

It vanishes trivially. All other terms can be worked out similarly.

The action \( S_{AV} \) in Eq (3) can also be constructed with the help of superfield formalism by promoting the Goldstino field \( \lambda \) to a superfield \( \Lambda \) [4]

\[
\Lambda = \exp(\theta Q + \bar{\theta} \bar{Q}) \times \lambda.
\]  

(5)
An invariant action can be obtained by taking the $D$-component of $\bar{\Lambda}^2 \Lambda^2$ [4], namely

$$S_{\text{AV}} = -\frac{\kappa^2}{2} \int d^4 x d^4 \theta \bar{\Lambda}^2 \Lambda^2. \quad (6)$$

Expanding $\Lambda$ in terms of $\theta$ and $\bar{\theta}$, one reproduces Eq (3). On the other hand, one notices that the superfield Goldstino $\kappa \Lambda(x) = \theta' = \theta + \kappa \lambda(z)$, where $z = x - i\kappa \lambda(z) \sigma \bar{\theta} + i\kappa \theta \sigma \bar{\lambda}(z)$. This procedure of changing variables from $(x, \theta, \bar{\theta})$ to $(z, \theta', \bar{\theta}')$ was pioneered in [5, 6]. Changing the integration variables, one has [10]

$$S_{\text{AV}} = -\frac{\kappa^2}{2} \int d^4 z d^4 \theta' \det T \det M \left( \frac{\theta'}{\kappa} \right)^2 \left( \frac{\theta'}{\kappa} \right)^2 = -\frac{1}{2\kappa^2} \int d^4 z \det T, \quad (7)$$

where $\det T \det M$ is the Jacobian determinant of this transformation and $M'_{\mu} = \delta'_{\mu} - i\kappa \theta \sigma^\mu \bar{\lambda}_\mu + i\kappa \lambda \sigma^\mu \bar{\theta}$. Explicitly,

$$\det M = 1 + i\kappa (\lambda_\mu \sigma^\mu \bar{\theta} - \theta \sigma^\mu \bar{\lambda}_\mu) \quad (8)$$

$$-\kappa^2 \left[ i\epsilon^{\mu\nu\rho\gamma} \theta \sigma_\rho \lambda_\mu \sigma_\gamma \bar{\lambda}_\nu + \bar{\theta}^2 \lambda_\mu \sigma^{\mu\nu} \lambda_\nu + \theta^2 \bar{\lambda}_\mu \bar{\sigma}^{\mu\nu} \bar{\lambda}_\nu \right]$$

$$+i\kappa^3 \bar{\theta}^2 \bar{\theta} \left[ \bar{\sigma}^\rho \lambda_\mu \bar{\lambda}_\mu \sigma^{\mu\nu} \bar{\lambda}_\nu + 2\bar{\sigma}^\nu \lambda_\mu \bar{\lambda}_\nu \sigma^{\mu\lambda} \bar{\lambda}_\lambda \right]$$

$$+i\kappa^3 \bar{\theta}^2 \bar{\theta} \left[ \sigma^\rho \bar{\lambda}_\mu \lambda_\mu \sigma^{\mu\nu} \lambda_\nu + 2\sigma^\nu \bar{\lambda}_\mu \lambda_\nu \sigma^{\mu\lambda} \lambda_\lambda \right],$$

where $\lambda_\mu = (T^{-1})_{\mu}^\nu \partial_\nu \lambda$. In [7], there were $\kappa^4 \theta^2 \bar{\theta}^2$ terms proportional to

$$\bar{\lambda}_\mu \bar{\sigma}^{\mu\nu} \lambda_\nu \sigma^{\nu\gamma} \lambda_\gamma + \bar{\lambda}_\mu \sigma^{\nu\gamma} \bar{\lambda}_\mu \sigma^{\mu\lambda} \lambda_\lambda + 4 \bar{\lambda}_\mu \bar{\sigma}^{\mu\rho} \lambda_\rho \lambda_\nu \sigma^{\nu\lambda} \lambda_\lambda.$$

They can be shown to vanish by the same line of arguments for the $\kappa^8$ terms in $\det T$.

For discussions related to chiral superfields, it is convenient to introduce an alternative (chiral) Goldstino field $\tilde{\lambda}$ [11]. Under SUSY transformations, $\tilde{\lambda}$ is to change as

$$\left\{ \begin{array}{l}
\delta_t \tilde{\lambda}_\alpha = \frac{1}{\kappa} \xi_\alpha - 2i\kappa \lambda \sigma^\mu \bar{\xi} \partial_\mu \tilde{\lambda}_\alpha, \\
\delta_t \tilde{\lambda}_{\dot{\alpha}} = \frac{1}{\kappa} \bar{\xi}_{\dot{\alpha}} + 2i\kappa \bar{\xi} \sigma^\mu \bar{\lambda} \partial_\mu \tilde{\lambda}_{\dot{\alpha}}. 
\end{array} \right. \quad (9)$$

$\tilde{\lambda}$ is not a new nonlinear realization of SUSY. It is related to $\lambda$ via [2, 5]

$$\tilde{\lambda}_\alpha(x) = \lambda_\alpha(z), \quad \bar{\tilde{\lambda}}_{\dot{\alpha}}(x) = \bar{\lambda}_{\dot{\alpha}}(z^*), \quad z = x - i\kappa^2 \lambda(z) \sigma \bar{\lambda}(z), \quad z^* = x + i\kappa^2 \lambda(z^*) \sigma \bar{\lambda}(z^*). \quad (10)$$
Explicit relations between $\lambda$ and $\tilde{\lambda}$ can be obtained by iterations as

$$
\lambda_\alpha = \tilde{\lambda}_\alpha + ik^2\tilde{\nu}^\mu\partial_\mu\tilde{\lambda}_\alpha - \kappa^4\tilde{\nu}^\mu\partial_\mu\lambda^{\sigma\nu}\lambda^{\lambda\alpha} + \kappa^4\mu^\nu\lambda\tilde{\lambda}^{\nu}\partial_\mu\lambda^{\alpha}
- \frac{1}{2}\kappa^4\tilde{\nu}^\mu\lambda^{\sigma\nu}\partial_\mu\tilde{\lambda}_\alpha + \frac{i}{2}\kappa^6\tilde{\nu}^\mu\tilde{\nu}^\rho\partial_\mu\tilde{\lambda}_\alpha + i\kappa^6\tilde{\nu}^\mu\tilde{\nu}^\rho\partial_\mu\tilde{\nu}^\rho\partial_\mu\tilde{\lambda}_\alpha,
$$

(11)

$$
\tilde{\lambda}_\alpha = \lambda_\alpha - ik^2\nu^\mu\partial_\mu\lambda_\alpha - \frac{1}{2}\kappa^4\nu^\mu\nu^\nu\partial_\mu\partial_\nu\lambda_\alpha - \kappa^4\nu^\mu\partial_\mu\nu^\nu\partial_\nu\lambda_\alpha
+ i\kappa^6\nu^\mu\nu^\nu\partial_\mu\nu^\nu\partial_\nu\lambda_\alpha + \frac{i}{2}\kappa^6\nu^\mu\nu^\nu\partial_\mu\nu^\nu\partial_\nu\lambda_\alpha,
$$

(12)

where $\nu^\mu = \lambda^{\sigma\nu}\tilde{\lambda}$ and $\tilde{\nu}^\mu = \tilde{\lambda}^{\sigma\nu}\lambda$. Eq (12) agrees with the expression in [2] but differs from the one in [12] by a factor of 2 in the last term. Similar to $\Lambda$, a superfield $\tilde{\Lambda}$ could be constructed from $\tilde{\lambda}$ via [2]

$$
\tilde{\Lambda} = \exp(\theta Q + \bar{\theta}\bar{Q}) \times \tilde{\lambda},
$$

(13)

out of which one can construct an invariant action of $\tilde{\lambda}$ [2]

$$
S^{ch}_{AV} = -\frac{\kappa^2}{2} \int d^4x d^4\theta \bar{\tilde{\Lambda}}^2 \tilde{\Lambda}^2.
$$

(14)

Expanding $S^{ch}_{AV}$ in terms of $\kappa$,

$$
S^{ch}_{AV} = -\frac{1}{2\kappa^2} \int d^4x[1 - ik^2(\partial_\mu\tilde{\lambda}^{\sigma\nu}\bar{\tilde{\lambda}} - \tilde{\lambda}^{\sigma\nu}\partial_\mu\bar{\tilde{\lambda}})
+ \kappa^4(2\bar{\tilde{\lambda}}^2\partial_\mu\bar{\tilde{\lambda}}^{\nu\mu}\partial_\nu\bar{\tilde{\lambda}} + 2\bar{\tilde{\lambda}}^{\nu\mu}\partial_\nu\bar{\tilde{\lambda}}^{\mu\nu}\partial_\mu\bar{\tilde{\lambda}} - \frac{1}{4}\bar{\tilde{\lambda}}^{\nu\mu}\partial_\nu\bar{\tilde{\lambda}}^{\mu\nu} - \frac{1}{4}\bar{\tilde{\lambda}}^{\mu\nu}\partial_\mu\bar{\tilde{\lambda}}^{\nu\mu})
- 2\partial_\mu\tilde{\lambda}\tilde{\lambda}^{\sigma\nu}\partial_\nu\bar{\tilde{\lambda}} - 2\partial_\mu\tilde{\lambda}^{\sigma\nu}\partial_\nu\tilde{\lambda}\partial_\rho\bar{\tilde{\lambda}} + 4\tilde{\lambda}^{\sigma\nu}\partial_\nu\tilde{\lambda}\partial_\rho\bar{\tilde{\lambda}}]
+ i\kappa^6\bar{\tilde{\lambda}}^2(\bar{\tilde{\lambda}}^2\partial_\nu\tilde{\lambda}^{\sigma\nu}\partial_\mu\bar{\tilde{\lambda}} - \partial_\nu\bar{\tilde{\lambda}}^2\partial_\mu\tilde{\lambda}^{\sigma\nu}\partial_\rho\bar{\tilde{\lambda}} - 4\tilde{\lambda}^{\sigma\nu}\partial_\nu\tilde{\lambda}\partial_\rho\bar{\tilde{\lambda}}
- i\kappa^6\bar{\tilde{\lambda}}^2(\bar{\tilde{\lambda}}^2\partial_\nu\tilde{\lambda}^{\sigma\nu}\partial_\mu\bar{\tilde{\lambda}} - \partial_\nu\bar{\tilde{\lambda}}^2\partial_\mu\tilde{\lambda}^{\sigma\nu}\partial_\rho\bar{\tilde{\lambda}} - 4\tilde{\lambda}^{\sigma\nu}\partial_\nu\tilde{\lambda}\partial_\rho\bar{\tilde{\lambda}})
+ 16\kappa^8\bar{\tilde{\lambda}}^2\partial_\mu\tilde{\lambda}^{\sigma\nu}\partial_\nu\bar{\tilde{\lambda}}^2\partial_\rho\tilde{\lambda}^{\sigma\rho}\partial_\sigma\bar{\tilde{\lambda}}^2).
$$

$S^{ch}_{AV}$ seems to differ from $S_{AV}$ drastically. In particular, there is a $\kappa^8$ term in $S^{ch}_{AV}$. However, one notices that $\bar{\tilde{\lambda}}^2\tilde{\lambda}^2 = \bar{\tilde{\lambda}}^2\tilde{\lambda}^2$, by a close inspection of Eq (11). One thus has

$$
\bar{\tilde{\lambda}}^2\tilde{\lambda}^2 = \bar{\tilde{\lambda}}^2\tilde{\lambda}^2.
$$

(16)

By taking the $\theta^2\bar{\theta}^2$ term on both side of this equation, one readily gets $S^{ch}_{AV} = S_{AV}$.

In the newly proposed constrained superfield formalism [3], the Goldstino field is assumed
to reside in the chiral superfield\(^2\)

\[
\hat{X}_{\text{NL}} = \frac{\hat{G}^2}{2F} + \sqrt{2}\theta\hat{G} + \hat{F}\theta^2,
\]

which satisfies the constraint \(\hat{X}_{\text{NL}}^2 = 0\). As shown in [7, 13], this constraint on \(\hat{X}_{\text{NL}}\) and other constraints on matter superfields can all be reformulated in the language of the standard realization of nonlinear SUSY, provided that one makes the following identification [13]

\[
\hat{\lambda} = \frac{\hat{G}}{\sqrt{2}\kappa F}.
\]

Of course, \(\lambda\) can then be constructed according to Eq (11).

As shown in [5–7, 14], spontaneously broken linear SUSY theories can always reformulated nonlinearly if the Goldstino field is identified [10, 13]. This is based upon the following observation: a linear superfield \(\hat{\Omega}(x, \theta, \bar{\theta})\) can always be converted to a set of nonlinear matter fields, via

\[
\Omega(x, \theta, \bar{\theta}) = \exp[-\kappa \lambda(x)Q - \kappa \bar{\lambda}(x)\bar{Q}] \times \hat{\Omega}(x, \theta, \bar{\theta}),
\]

where \(\Omega(x, \theta, \bar{\theta})\) transforms under SUSY transformations according to

\[
\delta_{\xi}\Omega = -i\kappa(\lambda\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\lambda})\partial_\mu\Omega.
\]

In particular, the non-linearized \(\hat{X}_{\text{NL}}\) is \(X_{\text{NL}} = \exp\{i\theta\sigma^\mu\partial_\mu \hat{\Delta}^+\}F\theta^2\) [13], where

\[
F = -\kappa^4\bar{\lambda}^2\partial^2\bar{\lambda}^2\hat{F} - 2i\kappa^2\hat{F}\partial_\mu\bar{\lambda}\sigma^\mu\bar{\lambda} + 2\kappa^4\bar{\lambda}\partial^2\bar{\lambda}^2\hat{F}
\]

and

\[
\bar{\lambda} = \bar{\lambda} - 2i\kappa^2\bar{\lambda}\sigma^\mu\partial_\mu\bar{\lambda} - 2\kappa^4\bar{\lambda}^2\bar{\lambda}\sigma^\nu\sigma^\mu\partial_\mu\partial_\nu\bar{\lambda} + \kappa^4\bar{\lambda}\bar{\lambda}\partial^2\bar{\lambda}.
\]

In [3], the leading-order action for the Goldstino field is proposed to be\(^3\)

\[
S_{\text{NL}} = \int d^4x d^2\theta \hat{X}_{\text{NL}}^\dagger \hat{X}_{\text{NL}} + \frac{1}{\sqrt{2}\kappa} \int d^4x d^2\theta \hat{\Delta}^+ \hat{X}_{\text{NL}} + \frac{1}{\sqrt{2}\kappa} \int d^4x d^2\theta \hat{\Delta}^+ \hat{X}_{\text{NL}}^\dagger.
\]

\(^2\) In this paper, superfields and their components in the linear SUSY are hatted while their counterparts in the nonlinear SUSY are not. Other notations and conventions conform to those of [4]. All symbols can be found in [7], if not explicitly defined in this paper.

\(^3\) \(\kappa^{-1} = \sqrt{2}f\), to conform with notations in [3].
Following the general procedure in [7, 14], this action can be reexpressed as

\[ S_{\text{NL}} = \int d^4x d^4\theta \det T \det M e^{-i\theta \sigma^{\mu} \tilde{\gamma}_{\mu}} F^\dagger \tilde{\theta}^2 e^{i\theta \sigma^{\mu} \tilde{\gamma}_{\mu}} F \theta^2 \]

\[ + \frac{1}{\sqrt{2\kappa}} \int d^4x dB \det T \det M_+ F \theta^2 + \frac{1}{\sqrt{2\kappa}} \int d^4x dB \det T \det M_- F^\dagger \tilde{\theta}^2, \]

where

\[
\begin{align*}
\det M_+ &= 1 - 2i\kappa \theta \sigma^{\mu} \tilde{\gamma}_{\mu} + 4\kappa^2 \theta^2 \tilde{\gamma}_{\mu} \sigma^{\mu \nu} \tilde{\gamma}_{\nu}, \\
\det M_- &= 1 + 2i\kappa \lambda_\mu \sigma^{\mu \tilde{\nu}} + 4\kappa^2 \bar{\theta}^2 \lambda_\mu \sigma^{\mu \nu} \lambda_\nu.
\end{align*}
\]

Integrating out the \( \theta \)'s, one has

\[ S_{\text{NL}} = \int d^4x \det T \left( F^\dagger F + \frac{1}{\sqrt{2\kappa}} F + \frac{1}{\sqrt{2\kappa}} F^\dagger \right). \]

Being a nonlinear matter field, the auxiliary fields \( F \) can be integrated out without breaking the nonlinear SUSY, via its equation of motion

\[ F = -\frac{1}{\sqrt{2\kappa}}. \]

Substituting this \( F \) back into Eq (24), one recovers the Akulov-Volkov action \( S_{\text{AV}} \) in Eq (3).

On the other hand, substituting this \( F \) into Eq (20), one gets by iterations an expression of the linear auxiliary field \( \hat{F} \) in terms of \( \tilde{\lambda} \) solely\(^4\)

\[ \hat{F} = -\frac{1}{\sqrt{2\kappa}} + \sqrt{2i}\kappa \partial_{\mu} \tilde{\lambda} \sigma^{\mu} \tilde{\sigma} + 2\sqrt{2\kappa} \hat{\gamma}_3 (\partial_{\mu} \tilde{\lambda} \sigma^{\mu} \partial_{\nu} \tilde{\lambda} \sigma^{\nu} \tilde{\lambda} + \tilde{\lambda}^2 \partial_{\mu} \tilde{\lambda} \sigma^{\mu \nu} \partial_{\nu} \tilde{\lambda}) \]

\[ + \sqrt{2i}\kappa (\hat{\gamma}^5 \tilde{\lambda}^2 \partial_{\mu} \tilde{\lambda} \sigma^{\mu} \partial_{\nu} \tilde{\lambda} - 2\tilde{\lambda}^2 \partial_{\mu} \tilde{\lambda} \sigma^{\mu} \partial_{\nu} \tilde{\lambda} \tilde{\lambda} \sigma^{\nu} \partial_{\nu} \tilde{\lambda} + 4\tilde{\lambda}^2 \tilde{\lambda} \sigma^{\mu \nu} \partial_{\mu} \tilde{\lambda} \tilde{\lambda} \sigma^{\nu} \partial_{\nu} \tilde{\lambda}) \]

\[ - 8\sqrt{2\kappa} \tilde{\lambda} \sigma^{\mu \nu} \partial_{\mu} \tilde{\lambda} \tilde{\lambda} \sigma^{\nu \rho} \partial_{\rho} \tilde{\lambda}. \]

Integrating out the \( \theta \)'s directly in Eq. (21), one has [3]

\[ S_{\text{NL}} = \int d^4x \left[ i\partial_{\mu} \hat{G} \sigma^{\mu} \tilde{\lambda} g^2 \frac{\hat{G}^2}{2\hat{F}} \partial^2 \hat{G}^2 + \hat{F}^\dagger \hat{F} + \frac{1}{\sqrt{2\kappa}} \hat{F} + \frac{1}{\sqrt{2\kappa}} \hat{F}^\dagger \right]. \]

Reexpress this in terms of the nonlinear Goldstino field \( \tilde{\lambda} \) via (18)

\[ S_{\text{NL}} = \int d^4x \left[ 2i\kappa^2 \hat{F}^\dagger \partial_{\mu} (\hat{F} \tilde{\lambda}) \sigma^{\mu} \tilde{\lambda} + \kappa^4 \hat{F}^\dagger \hat{\lambda}^2 \partial^2 (\hat{F} \tilde{\lambda}^2) + \hat{F}^\dagger \hat{F} + \frac{1}{\sqrt{2\kappa}} \hat{F} + \frac{1}{\sqrt{2\kappa}} \hat{F}^\dagger \right]. \]

One obtains then the equation of motion for the auxiliary field \( \hat{F} \)

\[ 2i\kappa^2 \partial_{\mu} (\hat{F} \tilde{\lambda}) \sigma^{\mu} \tilde{\lambda} + \kappa^4 \hat{\lambda}^2 \partial^2 (\hat{F} \tilde{\lambda}^2) + \hat{F} + \frac{1}{\sqrt{2\kappa}} = 0. \]

\(^4\) Similar expressions in two dimensions were presented in [15].
This results in an explicit expression for $\hat{F}$ by tedious iterations, which is identical to the one given in Eq (26). Substituting $\hat{F}$ back into Eq (28), we find

$$S_{NL} = -\frac{1}{2\kappa^2} \int d^4x [1 - 2i\kappa^2 \partial_\mu \lambda \sigma^\mu \ddot{\lambda} - 4 \kappa^4 (\partial_\mu \lambda \sigma^\mu \partial_\nu \dot{\lambda} \sigma^\nu \ddot{\lambda} + \dddot{\lambda}^2 \partial_\mu \lambda \sigma^\mu \partial_\nu \ddot{\lambda})$$

$$-2i \kappa^6 (\dddot{\lambda}^2 \partial_\mu \lambda \sigma^\mu \partial^2 \ddot{\lambda} - 2 \ddot{\lambda}^2 \partial_\mu \lambda \sigma^\mu \partial^2 \dot{\lambda} \dddot{\lambda} + 2 \dddot{\lambda} \dddot{\lambda} \lambda \sigma^\mu \partial_\nu \ddot{\lambda} \lambda \sigma^\mu \partial_\nu \ddot{\lambda} + 4 \dddot{\lambda} \dot{\lambda} \lambda \sigma^\mu \partial_\nu \ddot{\dot{\lambda}} \lambda \sigma^\mu \partial_\nu \ddot{\lambda})$$

$$+ 16 \kappa^8 \dddot{\lambda}^2 \dddot{\lambda} \partial_\mu \dot{\lambda} \sigma^\mu \partial_\nu \ddot{\dot{\lambda}} \dot{\lambda} \sigma^\mu \partial_\nu \ddot{\dot{\lambda}}],$$

which is identical to $S_{AV}$ in Eq (15) up to total derivative terms.

Notice that $\hat{F}$ does not have definite transformation properties in the formalism of nonlinear SUSY, an invariant action under nonlinear SUSY transformations is not guaranteed if $\hat{F}$ is integrated out in Eq (28). The nonlinear SUSY invariance of Eq (30) may be largely due to the fact that Eq (28) is quadratic in $\hat{F}$. If $\hat{F}$ is integrated out via the equation of motion obtained from Eq. (27) directly, one would have [3]

$$\hat{F}_{KS} = -\frac{1}{\sqrt{2} \kappa} - \frac{\kappa^3}{\sqrt{2}} \hat{G}^2 \partial^2 \hat{G}^2 + \frac{3 \kappa^7 \sqrt{2}}{\sqrt{2}} \hat{G}^2 \partial^2 \hat{G}^2 \partial^2 \hat{G}^2.$$

In this case, one obtains a particularly simple action from Eq (27)

$$S_{KS} = \int d^4x \left( -\frac{1}{2\kappa^2} + i \partial_\mu \hat{G} \sigma^\mu \hat{G} + \frac{\kappa^2}{2} \hat{G}^2 \partial^2 \hat{G}^2 - \frac{\kappa^6}{2} \hat{G}^2 \partial^2 \hat{G}^2 \partial^2 \hat{G}^2 \right).$$

Similar to $\hat{F}$, $\hat{G}$ does not have definite transformation properties in the formalism of nonlinear SUSY either. So, it is not transparent how $S_{KS}$ changes under nonlinear SUSY transformations. Naively, one may use Eq (18) to convert the $\hat{G}$ field to nonlinear Goldstino field $\ddot{\lambda}$. For example, one may take $\hat{G} = \sqrt{2} \kappa \ddot{\lambda} \hat{F}_{KS}$, which can be easily solved by

$$\hat{G} = -\ddot{\lambda} - \kappa^4 \dddot{\lambda} \dddot{\lambda} \partial^2 \ddot{\lambda}. \tag{33}$$

Substituting this into $S_{KS}$, one has

$$-\frac{1}{2\kappa^2} \int d^4x [1 + 2i \kappa^2 \ddot{\lambda} \partial_\mu \ddot{\lambda} - \kappa^4 \dddot{\lambda} \dddot{\lambda} \partial^2 \ddot{\lambda} - 2i \kappa^6 \dddot{\lambda} \dddot{\lambda} \partial_\mu \dddot{\lambda} \partial^2 \ddot{\lambda}^2$$

$$+ 2i \kappa^6 \dddot{\lambda} \dddot{\lambda} \partial^2 \ddot{\dot{\lambda}} \dddot{\lambda} \partial_\mu \ddot{\dot{\lambda}} - 3 \kappa^8 \dddot{\lambda} \dddot{\lambda} \partial^2 \ddot{\lambda} \partial^2 \ddot{\dot{\lambda}}].$$

Unfortunately, this new form is not invariant under nonlinear SUSY transformations. On the other hand, one may take $\hat{G} = \sqrt{2} \kappa \ddot{\lambda} \hat{F}$ with $\hat{F}$ in (26). But this does not yield an invariant action either. This makes the point. Integrating out the $\hat{F}$ directly from $S_{NL}$ does not necessarily generate an invariant action under nonlinear SUSY transformations. Consequently, $S_{KS}$ cannot be identified with $S_{AV}$ in a straightforward manner.
However, $S_{AV}$, $S_{AV}^{ch}$, and $S_{KS}$ are linked intrinsically via $S_{NL}$ and Eq (18). They should generate the same S-matrix elements, since S-matrix does not change under field redefinitions and how auxiliary fields are integrated [16]. This can be easily verified at the tree level, though complications arise at loop levels due to change of measures in path integrals [17]. Given its simple structure, it could be advantageous to use $S_{KS}$ in practical calculations.

For illustrations, we list below the S-matrix elements of several elementary processes, which are obtained from any of these actions. For processes involving four Goldstinos, the S-matrix elements can be read off from the effective operator

$$\mathcal{D}_4 = i\kappa^2 \int d^4x : \bar{\psi}_\text{in}^2(x) \partial_\mu \psi_\text{in}(x) \partial^\mu \psi_\text{in}(x) : .$$

(35)

Here the $:$ denotes normal ordering of operators and $\psi_\text{in}$ stands for the in-state operators of $\lambda$, $\lambda$ and $\hat{G}$ when the actions $S_{AV}$, $S_{AV}^{ch}$ and $S_{KS}$ are used respectively. Specifically, $\psi_\text{in}$ is the solution of the massless Dirac equation

$$i\bar{\sigma}^\mu \partial_\mu \psi_\text{in} = 0,$$

from which one can also obtain the free propagator

$$D_{\alpha\beta}^F(x-y) = <0|T\psi^\text{in}_\alpha(x)\bar{\psi}^\text{in}_\beta(y)|0> = \int \frac{d^4p}{(2\pi)^4} \frac{i p \cdot \sigma_{\alpha\beta}}{p^2} e^{ip(x-y)},$$

(36)

$$\bar{D}^{\alpha\beta}(x-y) = <0|T\bar{\psi}^\text{in}_\alpha(x)\psi^\text{in}_\beta(y)|0> = \int \frac{d^4p}{(2\pi)^4} \frac{i p \cdot \bar{\sigma^{\alpha\beta}}}{p^2} e^{ip(x-y)}.$$

(37)

For processes involving six Goldstinos, the S-matrix elements can be read off from

$$\mathcal{D}_6 = -4i\kappa^4 \int d^4xd^4y : \bar{\psi}_\text{in}^2(x) \frac{\partial \psi_\text{in}(x)}{\partial x^\mu} \frac{\partial D^F(x-y)}{\partial x^\mu} \bar{\psi}_\text{in}(y) \frac{\partial \psi_\text{in}(y)}{\partial y^\nu} \frac{\partial \psi_\text{in}(y)}{\partial y_\nu} : .$$

(38)

while for processes involving eight Goldstinos

$$\mathcal{D}_8 = -4i\kappa^6 \int d^4xd^4yd^4z$$

$$: [4\bar{\psi}_\text{in}^2(x) \frac{\partial \psi_\text{in}(x)}{\partial x^\mu} \frac{\partial D^F(x-y)}{\partial x^\mu} \bar{\psi}_\text{in}(y) \frac{\partial \psi_\text{in}(y)}{\partial y^\nu} \frac{\partial \psi_\text{in}(y)}{\partial y_\nu} \frac{\partial \psi_\text{in}(z)}{\partial z^\rho} \frac{\partial \psi_\text{in}(z)}{\partial z_\rho}$$

$$+\bar{\psi}_\text{in}^2(x) \bar{\psi}_\text{in}(y) \frac{\partial D^F(x-z)}{\partial x^\mu} \frac{\partial \psi_\text{in}(y)}{\partial y^\nu} \frac{\partial \psi_\text{in}(y)}{\partial y_\nu} \frac{\partial \psi_\text{in}(z)}{\partial z^\rho} \frac{\partial \psi_\text{in}(z)}{\partial z_\rho}$$

$$+\bar{\psi}_\text{in}^2(x) \bar{\psi}_\text{in}(y) \frac{\partial \psi_\text{in}(x)}{\partial x^\mu} \frac{\partial D^F(x-z)}{\partial x^\mu} \frac{\partial \psi_\text{in}(y)}{\partial y^\nu} \frac{\partial \psi_\text{in}(y)}{\partial y_\nu} \frac{\partial \psi_\text{in}(z)}{\partial z^\rho} \frac{\partial \psi_\text{in}(z)}{\partial z_\rho} : .$$

Note added: Since the first version of this paper listed on the arXiv, there have been more discussions on the subject [18–21]. The last two of these showed explicitly equivalences of all these actions.
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