MOVING BETWEEN WEIGHTS OF WEIGHT MODULES

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Abstract. In Lie theory the partial sum property says that for a root system in any Kac–Moody algebra, every positive root is an ordered sum of simple roots whose partial sums are all roots. In this paper, we present two generalizations of this property:
1) “Parabolic” generalization: if \( I \) is any nonempty subset of simple roots, then every root with positive \( I \)-height is an ordered sum of roots of \( I \)-height 1, whose partial sums are all roots. In fact, we show this on the Lie algebra level, by showing that every root space is spanned by the Lie words formed from root vectors of \( I \)-height 1. As an application, we provide a “minimal” description for the set of weights of every (non-integrable) simple highest weight module over any Kac–Moody algebra. This seems to be novel even in finite type.
2) Generalization to the set of weights of weight modules: the partial sum property gives a chain of roots between 0 (fixed) and any positive root. We generalize this phenomenon to the set of weights of weight modules to get a chain of weights between any two comparable weights. This was shown by S. Kumar, for any finite-dimensional simple module over a semisimple Lie algebra. In this paper, we extend this result to (i) a large class of highest weight modules over a general Kac–Moody algebra \( g \), which includes all the simple highest weight modules over \( g \); (ii) more generally, for non-highest weight modules such as \( g \) itself (adjoint representation) and arbitrary submodules of parabolic Verma modules over \( g \); (iii) arbitrary integrable modules (not necessarily highest weight) over semisimple \( g \). Additionally, we also prove the “parabolic” generalizations of this second generalization to the best possible extent.

As an application of the above two generalizations, we obtain the set of extremal rays of the convex hull of the weights of an arbitrary highest weight module over Kac–Moody \( g \), which was previously not known in the literature even in finite type.

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1. Introduction

In the root system $\Delta$ of a Kac–Moody or Borcherds Kac–Moody algebra, one of the standard results one studies is the partial sum property (PSP): every positive root is an ordered sum of simple roots such that each partial sum is also a root.

In this paper, we present two generalizations of this property:

A) Structure theory: We call this property as parabolic partial sum property (parabolic-PSP).

This strengthens the usual PSP in two ways:
1) We show that the parabolic-PSP holds over any Kac–Moody/ Borcherds Kac–Moody/more general Lie algebras $\mathcal{G}$ graded over free abelian semigroups.
2) Moreover, the parabolic-PSP holds at the level of Lie words in $\mathcal{G}$, not just on the level of roots/grades.

B) Representation theory: We show that an analogous version of the PSP, see Question 2 of Khare below, holds in the set of weights of (i) the adjoint representation, (ii) all the simple highest weight modules, and more generally (iii) all the submodules of parabolic Verma modules over any Kac–Moody algebra $\mathfrak{g}$.

We prove the first generalization (parabolic-PSP) to a “best possible extent” and level of generality. In this we were motivated by the following problem.

Problem: Let $\mathfrak{g}$ be a Kac–Moody algebra, with Cartan subalgebra $\mathfrak{h}$. Suppose $L(\lambda)$ denotes the simple highest weight $\mathfrak{g}$-module, with highest weight $\lambda \in \mathfrak{h}^*$. Find a “minimal” description of the set of weights of $L(\lambda)$. (Recall that the weights of non-integrable simple highest weight module $L(\lambda)$, were computed in [11].)

In this paper, we use the parabolic-PSP to obtain such a minimal description. To our knowledge, this result is novel even in finite type.

For the rest of this section, $\mathfrak{g}$ is a Kac–Moody algebra over $\mathbb{C}$. Observe that the partial sum property gives a chain of roots between 0 (fixed) and a positive root, or equivalently the weight 0 and any weight in $\text{wt}\mathfrak{g} = \Delta \sqcup \{0\}$ for the adjoint representation. Our second generalization of the PSP involves “moving between weights” of arbitrary representations to get a chain of weights between two comparable weights, and is natural in view of the previous line. This property was proved for all finite dimensional simple highest weight modules and parabolic Verma modules over semisimple $\mathfrak{g}$ by S. Kumar and A. Khare respectively, and to our knowledge there has not been further progress towards this problem beyond the semisimple case in the literature. In this paper, we solve this problem for: (i) a large class of highest weight modules over Kac–Moody $\mathfrak{g}$, including all simple highest weight modules over $\mathfrak{g}$; (ii) more generally, for non-highest weight modules such as $\mathfrak{g}$ itself (adjoint representation) and arbitrary submodules of parabolic Verma modules over Kac–Moody $\mathfrak{g}$; (iii) arbitrary integrable modules (not necessarily highest weight) over semisimple $\mathfrak{g}$. Additionally, we also prove/discuss in various remarks the further generalizations and limitations of the above two generalizations of the PSP.

As an application of the above two generalizations, we obtain the set of extremal rays of the convex hull of every highest weight module, and more generally, of the shape $\mathcal{P}(\lambda, J)$ over Kac–Moody algebra $\mathfrak{g}$, for $\lambda \in \mathfrak{h}^*$ and $J \subset J^\prime_\lambda$; see equation (2.10) for definitions. The shape $\mathcal{P}(\lambda, J)$
generalizes the convex hull of the parabolic Verma modules, and hence the convex hull of all highest weight modules, by [12, Theorem 3.13]. Closedness, polyhedrality, and the faces/weak faces of the convex hull of the set of weights of parabolic Verma modules, and hence of the simple highest weight modules, are well studied in [12, 16, 18]. This builds on previous work for finite dimensional modules by Borel–Tits [2], Casselman [3], Cellini–Marietti [5], Satake [20], Vinberg [21]. However, to our knowledge, the extremal rays of the set of weights of highest weight modules have not been studied, even in finite type.

The two generalizations of the PSP in this paper, which we completely resolved/studied over Kac–Moody algebras, also have additional applications to representation theory and combinatorics. In a forthcoming paper, we make use of these two generalizations of the PSP and obtain interesting results about certain combinatorial subsets of wtV studied in the literature, for a highest weight module V over a Kac–Moody algebra g. Namely, weak faces and weak-A-faces (for arbitrary additive subgroups A of R under addition). These subsets were introduced and studied by Chari and her co-authors in [6, 7, 8, 9, 16, 18].

2. Preliminaries and main results

In order to state and prove the results of this paper, we need the following notation.

2.1. Notation. In this paper, we denote the set of non-negative, positive, non-positive and negative real numbers by R≥0, R>0, R≤0 and R<0 respectively. Similarly, for any S ⊆ R we define S+ := S ∩ R+ for any of ≥ 0, > 0, ≤ 0 and < 0. For n ∈ N, we denote the set {1, . . . , n} by [n]. For B a subset of an R vector space, define

\[ \text{conv}_{\mathbb{R}} B := \left\{ \sum_{j=1}^{n} c_j b_j \mid n \in \mathbb{N}, b_j \in B, c_j \in \mathbb{R}_{\geq 0} \forall 1 \leq j \leq n \text{ and } \sum_{j=1}^{n} c_j = 1 \right\} \]

to be the convex hull of B over R. For any two subsets C and D of a real or complex vector space, C ± D := \{ c ± d \mid c \in C, d \in D \} denotes the Minkowski sum of C and ±D respectively. When C = \{ x \} is singleton, we denote C ± d by x ± D for simplicity.

Throughout the paper, we denote by I an indexing set and by Π = \{ α_i \mid i \in I \} the free generating set for the semigroup \( \mathbb{Z}_{\geq 0} \Pi \). For I ⊆ I, we denote I \ I by Ic.

For every Lie algebra g over a field k, we denote by [\cdot,\cdot] its Lie bracket and by U(g) its universal enveloping algebra. Let \( \xi := (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n \) an ordered (formal) sequence. Let \( x_{\xi_1}, \cdots, x_{\xi_n} \in g \) be a sequence of vectors indexed by \( \xi_i, 1 \leq i \leq n \). We define

\( \langle [x_{\xi_i}] \rangle_{i=1}^{n} = [\langle x_{\xi_i} \rangle_{\xi \in \mathcal{I}} := \left[ x_{\xi_1}, [\cdots, [x_{\xi_{n-1}}, x_{\xi_n}] \cdots] \right] \)

to be the right normed Lie word on \( x_{\xi_1}, \cdots, x_{\xi_n} \in g \)–i.e. the iterated Lie bracket of \( x_{\xi_1}, \cdots, x_{\xi_n} \in g \) in the same order. In this notation, when n = 1, we define \( \langle [x_{\xi_i}] \rangle_{i=1}^{1} = [\langle x_{\xi_i} \rangle_{\xi \in \{\xi_1\}} := x_{\xi_1} \).

Notation for the Kac–Moody setting. Let g = g(A) denote the Kac–Moody algebra over \( \mathbb{C} \) corresponding to a generalized Cartan matrix A with the realisation (h, Π, Π∨), triangular decomposition n+ ⊕ h ⊕ n−, and the root system Δ. Whenever we make additional assumptions, such as g being symmetrizable or semisimple or of finite/affine type, we will clearly mention it. Let Π = \{ α_i \mid i \in I \} be the simple system and Π∨ = \{ α_i∨ \mid i \in I \} be the simple co-root system, where I is a fixed indexing set for the simple roots. I also stands for the set of vertices/nodes in the Dynkin diagram for A or g. In the Kac–Moody setting, throughout, we assume I to be at most countable. Let e_i, f_i, \alpha_i∨, ∀ i ∈ I be the Chevalley generators for g, and W denote the Weyl group of g generated by simple reflections \{ s_i \mid i ∈ I \}. Let \( g^\vee := [g, g] \) be the derived subalgebra of g, which is generated by e_i, f_i, \alpha_i∨, ∀ i ∈ I. When g is symmetrizable, we fix a standard non-degenerate symmetric invariant bilinear form on h* and denote it by ⟨,⟩.
For $\emptyset \neq I \subseteq \mathcal{I}$, define $\Pi_I := \{\alpha_i \mid i \in I\}$ and $\Pi_I^\vee := \{\alpha^\vee_i \mid i \in I\}$. Define $g_I := g(A_{I \times I})$ to be the Kac–Moody algebra corresponding to the submatrix $A_{I \times I}$ of $A$ with realisation $(h^I, \Pi_I, \Pi_I^\vee)$, where $h^I \subset h$, and the Chevalley generators $e_i, f_i, \alpha_i^{\vee}, \forall \ i \in I$. By [13] Exercise 1.2], $g_I$ can be thought of as a subalgebra of $g$, and the subroot system $\Delta_I := \Delta \cap Z\Pi_I \subset Z$ coincides with the root system of $g_I$. Let $l_I = g_I + h$ and $p_I = g_I + h + n^+$ be the standard Levi and the parabolic Lie subalgebras of $g$ corresponding to $I$, respectively. Let $W_I$ denote the parabolic subgroup of $G$ generated by the simple reflections $\{s_i \mid i \in I\}$. When $I = \emptyset$, for completeness we define (i) $\Pi_I, \Pi_I^\vee$ and $\Delta_I$ to be $\emptyset$, (ii) $g_\emptyset$ and $h^\emptyset$ to be $\{0\}$, and (iii) $W_\emptyset$ to be the trivial subgroup $\{e\}$ of $W$. For $\alpha \in \Pi$, define $s_\alpha := g_{-\alpha} \oplus C\alpha^\vee \oplus g_\alpha \simeq s_2(C)$. We occasionally use $s_\alpha$ for the simple reflection about the hyperplane perpendicular to $\alpha$.

Let $\preceq$ be the usual partial order on $\mathcal{CPI}$, under which $x \prec y \in \mathcal{CPI} \iff y - x \in \mathbb{R}_{\geq 0}\Pi$. Fix $\emptyset \neq I \subset \mathcal{I}$, $J \subset \mathcal{I}$, $\alpha \in \Delta^+$, and a vector $x = \sum_{i \in I} c_i \alpha_i \in \mathcal{CPI}$ for some $c_i \in \mathbb{C}$. We define

$$
supp(x) := \{i \in I \mid c_i \neq 0\}, \quad (\text{supp}(0) := \emptyset),$$

$$ht(x) := \sum_{i \in I} c_i, \quad (\text{ht}(0) := 0),$$

$$\text{(2.2)} \quad (\text{supp}_I(x) := \{i \in I \mid c_i \neq 0\}, \quad \Delta_{\alpha,J} := \{\beta \in \Delta^+ \mid \text{supp}(\beta - \alpha) \subset J\}, \quad (\Delta_{\alpha,\emptyset} := \{\alpha\}).$$

In the notation as in equation (2.2), note that

$$\text{(2.3)} \quad \Delta_{\alpha,\emptyset} = \Delta_{\alpha,J} = \Delta_{\alpha,J} \cap Z\Pi_J, \quad \alpha \in \Delta_{\alpha,J} \forall J \subset \mathcal{I}, \quad \Delta_{\alpha,J(i)c} = \Delta_{\alpha,J} \cap ZJ(i) \forall i \in \mathcal{I}, \quad g_{\emptyset} = g_J^c.$$

Let $M$ be a $h$-module and $\mu \in h^*$. We denote the $\mu$-weight space and the set of weights of $M$, respectively, by

$$M_{\mu} = \{v \in M \mid h \cdot v = \mu(h)v \forall h \in h\}, \quad \text{wt}M = \{\mu \in h^* \mid M_{\mu} \neq \{0\}\}.$$

We say that $M$ is a weight module, if $M = \bigoplus_{\mu \in h^*} M_\mu$. When each weight space of a weight module $M$ is finite dimensional, we define $\text{char}M := \sum_{\mu \in \text{wt}M} \dim(M_{\mu})e^\mu$ to be the formal character of $M$. For $\lambda \in h^*$, let $M(\lambda)$ and $L(\lambda)$ denote the Verma module over $g$ with highest weight $\lambda$ and its unique simple quotient respectively. By $M(\lambda) \rightarrow V$, we denote a non-trivial highest weight $g$-module $V$ with highest weight $\lambda$.

We denote the set of dominant integral weights by $P^+ := \{\mu \in h^* \mid \langle \mu, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \forall i \in \mathcal{I}\}$, where $\langle \mu, h \rangle = \mu(h)$ denotes evaluation of $\mu$ at $h \in h$. For $\lambda \in h^*$, $M(\lambda) \rightarrow V$ and $I \subset \mathcal{I}$, we define

$$\text{(2.4)} \quad \begin{align*}
J_\lambda & := \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}\}, \quad J_0 := \{j \in J_\lambda \mid \langle \lambda, \alpha_j^\vee \rangle = 0\}, \\
J'_\lambda & := \{i \in I \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{R}_{\geq 0}\}, \quad wt_I V := \text{wt}V \cap (\lambda - Z_{\geq 0}\Pi_I).
\end{align*}$$

Fix $I \subset \mathcal{I}$, and suppose $V$ is a highest weight $p_I$ or $l_I$-module with highest weight $\lambda \in h^*$. Then $V$ becomes a highest weight $g_I$-module with highest weight $\lambda|_{h_I}$, the restriction of $\lambda$ to $h_I$, where $h_I$ is the Cartan subalgebra of $g_I$. But for simplicity, throughout the paper, we also denote the highest weight of $V$ for the $g_I$ action by $\lambda$.

**Notation for the graded setting.** In section [3] we fix an arbitrary field $\mathbb{F}$, an indexing set $\mathcal{I}$, and the abelian semigroup $\mathbb{Z}_{\geq 0}\Pi$ freely generated by $\Pi = \{\alpha_i \mid i \in \mathcal{I}\}$. We work with a more general $\mathbb{Z}_{\geq 0}\Pi$-graded $\mathbb{F}$-Lie algebra $G$ (that is, $G = \bigoplus_{\gamma \in \mathbb{Z}_{\geq 0}\Pi} G_\gamma$, and $[G_\alpha, G_\beta] \subset G_{\alpha + \beta} \forall \alpha, \beta \in \mathbb{Z}_{\geq 0}\Pi$), which is generated by subspaces $G_{\alpha_i}, \forall i \in \mathcal{I}$. We assume that each subspace $G_{\alpha_i} \subset G$ is non-zero, $\forall i \in \mathcal{I}$. We define $A := \{\alpha \in \mathbb{Z}_{\geq 0}\Pi \mid G_\alpha \neq 0\}$, the candidate for the set of positive roots $\Delta^+$ of
the root system $\Delta$. We do not assume the subspace $G_\alpha$, $\alpha \in A$, (in particular $G_{\alpha_i}$, $i \in I$) to be finite dimensional. We define the partial order $\prec$, and functions $\text{supp}(\cdot), \text{ht}(\cdot)$, $\text{supp}_J(\cdot)$ and $\text{ht}_J(\cdot)$, for $\emptyset \neq I \subset I$, analogously on $\Pi$. For $\emptyset \neq I \subset I$, and $n \in \mathbb{Z}$, analogously define

$$(2.5) \quad A_{I,n} := \{ \beta \in A \mid \text{ht}_I(\beta) = n \}, \quad G_{I,n} := \bigoplus_{\beta \in A_{I,n}} G_\beta.$$

**Note 1**: Through the paper, for any $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_{\geq 0}\Pi$, $n \geq 2$, and $U \subset \mathbb{R}_{\geq 0}\Pi$ (or respectively $\mathfrak{h}^*$ in place of $\mathbb{R}_{\geq 0}\Pi$, for the Kac–Moody setting), the notation

$$\lambda_1 \prec \cdots \prec \lambda_i \prec \cdots \prec \lambda_n \in U \forall i$$

denotes: (1) $\lambda_{i-1} \prec \lambda_i \forall i > 1$, and (2) $\lambda_i \in U \forall i \in [n]$.

### 2.2. Preliminaries and useful results.

We begin by noting the following basic facts about Lie algebras and highest weight modules over Kac–Moody algebras, which we use without mention. Throughout this subsection, $\mathfrak{g}$ stands for a Kac–Moody algebra.

(F1) Let $L$ be a Lie algebra generated by $X \subset L$. Then $L$ is spanned by the right normed Lie words on the elements in $X$. More precisely, there exists a basis of $L$ consisting of elements of the form $[x_1, \cdots, [x_{n-1}, x_n]] = ([x_i])_{i=1}^n$ in the notation as in equation (2.1), $n \in \mathbb{N}$, such that $x_1, \ldots, x_n \in X$.

(F2) Let $\mathfrak{g}$ be a Kac–Moody algebra, $\lambda \in \mathfrak{h}^*$, $M(\lambda) \rightarrow V$ and $0 \neq v \in V$ be a highest weight vector. Then the weight space $V_\mu$ corresponding to $\mu \in \text{wt}V$ is spanned by the weight vectors of the form $f_{i_1} \cdots f_{i_n}v$ such that $f_{i_j} \in \mathfrak{g}_{\alpha_{i_j}}$ and $\alpha_{i_j} \in \Pi$ for each $1 \leq j \leq n$, and

$$\sum_{j=1}^n \alpha_{i_j} = \lambda - \mu.$$

(F3) With the notation as in (F2), suppose $\mu \in \text{wt}V$ and $\alpha_i \in \Pi$ such that $\langle \mu, \alpha_i^\vee \rangle > 0$. Then $\mu, \mu - \alpha_i, \ldots, \mu - \lceil \langle \mu, \alpha_i^\vee \rangle \rceil \alpha_i \in \text{wt}V$ by the $\mathfrak{sl}_{\alpha_i}$ action on $V_\mu$, where $\lceil \cdot \rceil$ denotes the ceiling function. Note that if $\langle \mu, \alpha_i^\vee \rangle \in \mathbb{Z}_{>0}$, then $s_i(\mu) = \mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i$.

Most of the results of this paper hold true for a large class of highest weight modules – namely, the class of highest weight modules whose set of weights coincides with that of a parabolic Verma module. This class contains all Verma, parabolic Verma, simple highest weight modules and many more; by Theorem 2.3 below, proved by Dhillon and Khare [11]. Parabolic Verma modules, also known as generalized Verma modules, were introduced and studied by Lepowsky in a series of papers; see [19] and the references therein. For a detailed list of the properties of parabolic Verma modules, we refer the reader to [14] Chapter 9).

**Definition 2.1.**

(1) Given a generalized Cartan matrix $A$, let $\widehat{\mathfrak{g}}(A)$ denote the Lie algebra generated by $e_i, f_i, \mathfrak{h}$ modulo only the Serre relations, and let the Kac–Moody Lie algebra $\mathfrak{g} = \widehat{\mathfrak{g}}(A)$ be the further quotient of $\widehat{\mathfrak{g}}(A)$ by the largest ideal intersecting $\mathfrak{h}$ trivially.

(2) Let $\lambda \in \mathfrak{h}^*$, $\emptyset \neq J \subset I$ and $M(\lambda)$ denote the Verma module over $\mathfrak{g}$ with highest weight $\lambda$. Recall that the parabolic Lie subalgebra of $\mathfrak{g}$ corresponding to $J$ is denoted by $\mathfrak{p}_J$. Assume $J \subset J_\lambda$. Let $L^\text{max}_J(\lambda)$ denote the largest integrable highest weight module over $\mathfrak{g}_J$ (or equivalently $\mathfrak{p}_J$, via the natural action of $\mathfrak{h}$ and the trivial action of $\bigoplus_{\alpha \in \Delta^+ \setminus \Delta^+_J} \mathfrak{n}^+_\alpha$) with highest weight $\lambda$. $L^\text{max}_J(\lambda)$ has the property that $L^\text{max}_J(\lambda) \rightarrow L'_J(\lambda)$ for any integrable highest weight $\mathfrak{g}_J$-module $L'_J(\lambda)$ whose highest weight is $\lambda$. Let $L_J(\lambda)$ denote the simple highest weight module over $\mathfrak{g}_J$ (or equivalently $\mathfrak{p}_J$) with highest weight $\lambda$. Note that when $\mathfrak{g}_J$ is symmetrizable, by [15] Corollary 10.4 $L^\text{max}_J(\lambda)$ is simple.
(3) For $J \subset J_\lambda$, denote the parabolic Verma module over $\mathfrak{g}$ corresponding to $J$ with highest weight $\lambda$ by

$$M(\lambda, J) := U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_J)} L_{J}^{\max}(\lambda) \simeq M(\lambda)/\left(\sum_{j \in J} U(n^-)f_j^{(\lambda, c_j^\vee)+1} M(\lambda_\lambda)\right).$$

Note that $L_{J}^{\max}(\lambda)$ is the parabolic Verma module over $\mathfrak{g}_J$, corresponding to $J$ and with highest weight $\lambda$.

Given $J \subset \mathcal{I}$, a weight module $M$ of $\mathfrak{g}$ is $\mathfrak{g}_J$-integrable if $e_j$ and $f_j$ act locally nilpotently on $M \forall j \in J$. If $M$ is $\mathfrak{g}_J$-integrable, then $\text{wt} M$ is $W_J$-invariant and every submodule of $M$ is also $\mathfrak{g}_J$-integrable.

Let $M(\lambda) \to V$ and $J \subset J_\lambda$. Then $V$ is $\mathfrak{g}_J$-integrable if $f_j$ acts locally nilpotently on $V \forall j \in J$.

**Lemma 2.2.**

1. Let $j \in \mathcal{I}$. Then $f_j$ acts locally nilpotently on $V$ if and only if $f_j$ acts nilpotently on the highest weight space $V_\lambda$.
2. $V$ is $\mathfrak{g}_J$-integrable if and only if $\text{char} V$ is $W_J$-invariant.

Define $I_V := \{i \in \mathcal{I} \mid \langle \lambda, c_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ and } f_i^{(\lambda, c_i^\vee)+1} V_\lambda = \{0\}\}$ to be the integrability of $V$. Note that $V$ is $\mathfrak{g}_{I_V}$-integrable.

In our results on the parabolic Verma module $M(\lambda, J)$ or precisely its weights, we make use of the $\mathfrak{g}_J$-integrability of $M(\lambda, J)$ and the following formulas for its weights. See e.g. [11, Proposition 3.7 and Section 4] for the proofs and consequences of these formulas.

- **(2.6)** Minkowski decomposition: $\text{wt} M(\lambda, J) = \text{wt} L_J(\lambda) - \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_J^+)$.  
- **(2.7)** Integrable slice decomposition: $\text{wt} M(\lambda, J) = \bigcup_{\xi \in \mathbb{Z}_{\geq 0} \Pi_{J\mathfrak{c}}} \text{wt} L_J(\lambda - \xi)$.

The results of this paper on $\text{wt} M(\lambda, J)$, respectively on $\text{conv}_\mathbb{R} \text{wt} M(\lambda, J)$ hold true for almost all, respectively all the highest weight modules by parts (a) and (b) of the following Theorem. See [11 Theorem 2.9] and [12 Theorem 3.13] for the proofs.

**Theorem 2.3** (Dhillon–Khare). Fix $\lambda \in \mathfrak{h}^*$ and $J \subset I_{L(\lambda)} = J_\lambda$.

- **(a)** Every highest weight module $V$ of highest weight $\lambda$ and integrability $J$ has the same weights if and only if the Dynkin subdiagram on $I_{L(\lambda)} \setminus J$ is complete, i.e. $\langle \alpha_i, \alpha_j^\vee \rangle \neq 0, \forall i, j \in I_{L(\lambda)} \setminus J$. In particular,

$$\text{wt} L(\lambda) = \text{wt} M(\lambda, I_{L(\lambda)}).$$

- **(b)** For $M(\lambda) \to V$, the following data are equivalent:

  1. $I_V$, the integrability of $V$.
  2. $\text{conv}_\mathbb{R} \text{wt} V$, the convex hull of the set of weights of $V$.
  3. The stabilizer of $\text{conv}_\mathbb{R} \text{wt} V$ in $W$.

     In particular, the convex hull in (2) is always that of the parabolic Verma module $M(\lambda, I_V)$, and the stabilizer in (3) is always the parabolic subgroup $W_{I_V}$.

**Definition 2.4.** Let $C$ be a convex subset of a real vector space, and fix a point $x \in C$ and a vector $y$. Then $x + \mathbb{R}_{\geq 0}y$ denotes the ray at/through the point $x$ in the direction of $y$. We say $x + \mathbb{R}_{\geq 0}y$ is an extremal ray of $C$ if whenever $x + ry = \sum_{i=1}^n d_i x_i$ for some $r, d_i \in \mathbb{R}_{\geq 0}$ such that $\sum_{i=1}^n d_i = 1$, and $x_i \in C, 1 \leq i \leq n$, we have $x_i \in x + \mathbb{R}_{\geq 0}y$ whenever $d_i > 0$.

We now begin stating the main results of this paper.
2.3. Main results. Throughout this subsection, unless specified, assume \( g \) to be a Kac–Moody algebra over \( \mathbb{C} \). \( I \) denotes a fixed indexing set for the set of simple roots. We first state the parabolic partial sum property (parabolic-PSP).

**Definition 2.5 (Parabolic-PSP).** Let \( \Delta \) be the root system of a Kac–Moody algebra \( g \) and \( \emptyset \neq I \subset \mathcal{I} \). Define \( \Delta_{I,1} := \{ \alpha \in \Delta \mid h_t(\alpha) = 1 \} \). Suppose \( \beta \in \Delta^+ \) is such that \( h_I(\beta) > 1 \). Then there exists a root \( \gamma \in \Delta_{I,1} \) such that \( \beta - \gamma \in \Delta \). Equivalently, there exists a sequence of roots \( \gamma_1 \in \Delta_{I,1}, 1 \leq i \leq n = h_I(\beta) \), such that
\[
\gamma_1 < \cdots < \sum_{j=1}^{i} \gamma_j < \cdots < \sum_{j=1}^{n} \gamma_j = \beta \in \Delta \quad \forall \beta.
\]
(See Note \( \boxed{1} \) for the notation.)

**Question 1 (Khare).** Does the parabolic-PSP hold true in the root system of a Kac–Moody algebra?

The first main result of this paper positively answers this question, together with an application to the set of weights of non-integrable highest weight modules over \( g \). For \( \lambda \in \mathfrak{h}^* \), recall from equation (2.4) that \( J_\lambda := \{ j \in \mathcal{I} \mid \langle \lambda, \alpha_j^\vee \rangle \in \mathbb{Z}_{\geq 0} \} \).

**Theorem A.** Let \( g \) be a Kac–Moody algebra with root system \( \Delta \). Then:

(A1) The parabolic-PSP holds true in \( \Delta \) for any \( \emptyset \neq I \subset \mathcal{I} \).

(A2) For \( \lambda \in \mathfrak{h}^* \), \( J \subset J_\lambda \) and for any \( M(\lambda) \to V \) such that \( wtV = wtM(\lambda, J) \), we have the following description:
\[
Z_{\geq 0}(\Delta^+ \setminus \Delta^+_j) = Z_{\geq 0}(\Delta_{J^c,1}) \quad \text{wt}V = wtL_J(\lambda) - Z_{\geq 0}(\Delta_{J^c,1}) = wtL_J(\lambda) + Z_{\geq 0}(\Delta_{J^c,-1}) - Z_{\geq 0}(\Delta_{J^c,0}).
\]
In particular,
\[
wtL(J) = wtL_J(\lambda) - Z_{\geq 0}(\Delta_{J^c,1}) = wtL_J(\lambda) + Z_{\geq 0}(\Delta_{J^c,-1}) = wtL_J(\lambda).
\]
The description of \( \text{wt}V \) in equation (2.9) is minimal, in that \( \Delta_{J^c,1} \) cannot be reduced.

**Remark 2.6.** To our knowledge, neither the parabolic-PSP, nor its application in representation theory to \( \text{wt}V \)—for example any simple non-integrable highest weight module \( V = L(\lambda) \)—was known even for \( g \) of finite type.

In fact, we prove a more general version of the parabolic-PSP for any \( \mathbb{Z}_{\geq 0}\Pi \)-graded \( \mathbb{F} \)-Lie algebra \( \mathcal{G} \) generated by \( \mathcal{G}_\Pi \). We also prove a “going up” version of the parabolic-PSP for a Kac–Moody algebra, see Theorem 3.6 below. This also generalizes the well-known basic property \( \boxed{15} \) Proposition 4.9.

Having found a minimal description for \( wtM(\lambda, J) \), we study \( \text{conv}_R wtM(\lambda, J) \) through studying the convex hulls of certain \( W_J \)-invariant subsets of \( \Delta \), see subsection 5.2. Let \( g \) be a Kac–Moody algebra, \( \lambda \in \mathfrak{h}^* \), \( J \subset J_\lambda' := \{ j \in \mathcal{I} \mid \langle \lambda, \alpha_j^\vee \rangle \in \mathbb{R}_{\geq 0} \} \) and \( W_J \) be the parabolic subgroup of \( W \) generated by \( \{ s_j \mid j \in J \} \). Then
\[
\mathcal{P}(\lambda, J) := \text{conv}_R W_J \lambda - \mathbb{R}_{\geq 0}(\Delta^+ \setminus \Delta^+_j)
\]
is a \( W_J \)-invariant convex subset of \( \lambda - \mathbb{R}_{\geq 0}\Pi \). When \( J \subset J_\lambda \), note by Theorem 2.3 above that \( \mathcal{P}(\lambda, J) = \text{conv}_R wtV \) for any \( M(\lambda) \to V \) such that \( I_V = J \). The sets \( \mathcal{P}(\lambda, J) \) were introduced, and the faces and inclusion relations among its faces studied, in \( \boxed{12} \) \( \boxed{17} \).

The second main result of this paper gives the description of the set of extremal rays of \( \mathcal{P}(\lambda, J) \) over Kac–Moody \( g \). By Theorem 2.3 (b), we hence get the extremal rays of \( \text{conv}_R wtV \) for any \( M(\lambda) \to V \) such that \( I_V = J \) over Kac–Moody \( g \) (again, this was not known even in finite type). Additionally, we also discuss a maximal property of \( \mathcal{P}(\lambda, J) \), see Maximal property \( \boxed{5.2} \) below.
Theorem B. Let $\mathfrak{g}$ be a Kac–Moody algebra, $\lambda \in \mathfrak{h}^*$ and $J \subseteq J'_\lambda$. Define $J_0 := \{ j \in J \mid s_j(\lambda) = \lambda \}$. Then $P(\lambda, J)$ is a $W_J$-invariant convex subset of $\lambda - \mathbb{R}_{\geq 0}\Pi$, with the set of extremal rays $W_J(\lambda - \mathbb{R}_{\geq 0}W_{J'}\Pi)$. In particular, the extremal rays at $\lambda$ are along the directions given by $-W_{J_0}\Pi_{J'}$.

The key tool in the proof of Theorem B is the third main result of this paper, Theorem C. This is the second generalization of the PSP (in representation theory) as mentioned in the introduction. Theorem C positively answers the following question of Khare in several prominent cases.

Question 2 (Khare). Let $\mathfrak{g}$ be a Kac–Moody algebra, $\lambda \in \mathfrak{h}^*$, and $M(\lambda) \rightarrow V$. Suppose $\mu_0 \leq \mu \in \text{wt}V$. Then does there exist a sequence of weights $\mu_i \in \text{wt}V$, $1 \leq i \leq n = \text{ht}(\mu - \mu_0)$, such that

$$\mu_0 < \cdots < \mu_i < \cdots < \mu_n = \mu \in \text{wt}V \quad \text{and} \quad \mu_i - \mu_{i-1} \in \Pi \forall i.$$  

See Note 1 for the notation.

This question was answered positively in the cases when $V$ is a finite dimensional simple highest module or a parabolic Verma module over semisimple $\mathfrak{g}$, by S. Kumar and Khare respectively. See Appendix in [16].

Theorem C. Let $\mathfrak{g}$ be a Kac–Moody algebra and $V$ be a $\mathfrak{g}$-module. Suppose

(C1) $V = \mathfrak{g}$ (adjoint representation) or
(C2) $\lambda \in \mathfrak{h}^*$, $J \subseteq J'_\lambda$ and $V$ is a submodule of $M(\lambda, J)$, and $\mu_0 \not\leq \mu \in \text{wt}V$. Then there exists a sequence of weights $\mu_i \in \text{wt}V$, $1 \leq i \leq n = \text{ht}(\mu - \mu_0)$, such that

$$\mu_0 < \cdots < \mu_i < \cdots < \mu_n = \mu \in \text{wt}V \quad \text{and} \quad \mu_i - \mu_{i-1} \in \Pi \forall i.$$  

Theorem C applies to the above and other class of modules:

- On one hand, note that the result holds for the set of weights of parabolic Verma modules. Hence Theorem C holds true for a bigger class of modules, by Theorem 2.3 (a).
- In particular, this proves the theorem for all simple highest weight modules. By the previous line, additionally, we are also able to extend Theorem C for any highest weight $\mathfrak{g}$-module with dominant highest weight and length at most two. See Theorem 6.1 in subsection 6.1.
- On the other hand, observe that many of the submodules of parabolic Verma modules need not be highest weight modules. Thus, we are able to prove Theorem C not only for the above highest weight modules but also for certain “non-highest weight” modules.
- Motivated by the previous point, we also extend Theorem C for any integrable module (not necessarily highest weight or has finite dimensional weight spaces) over semisimple $\mathfrak{g}$, see Lemma 4.4 below.

We also prove more (parabolic) generalizations of Theorem C to the “best possible extent”, see subsection 1.2.

Remark 2.7 (Results over related Kac–Moody algebras). While the above results—and results in the later sections—are stated and proved over the Kac–Moody algebra $\mathfrak{g}(A)$, they hold equally well over the Kac–Moody algebra $\overline{\mathfrak{g}}(A)$ (see Definition 2.1 (1)) and hence uniformly over any ‘intermediate’ algebra $\overline{\mathfrak{g}}(A) \rightarrow \mathfrak{g}' \rightarrow \mathfrak{g}(A)$. This is because as was clarified in [11], the results there hold over all such Lie algebras $\mathfrak{g}'$; similarly, the root system and the weights of highest weight modules (of a fixed highest weight $\lambda$) remain unchanged over all $\mathfrak{g}'$.

3. PARABOLIC-PSP AND ANALOGUES

3.1. Proof of Theorem A. We begin this subsection with the proof of parabolic-PSP in the more general setting of a $\mathbb{Z}_{\geq 0}\Pi$-graded $\mathbb{F}$-Lie algebra $\mathcal{G}$ generated by non-zero subspaces $\mathcal{G}_\alpha$, $\forall \alpha \in \mathcal{I}$. Here, $\mathbb{F}$ is an arbitrary field, $\Pi$ freely generates the abelian semigroup $\mathbb{Z}_{\geq 0}\Pi$ and $\mathcal{I}$ is a fixed
indexing set for \( \Pi \). Recall, \( A := \{ \alpha \in \mathbb{Z}_{\geq 0} \Pi \mid G_\alpha \neq 0 \} \). Throughout this subsection, we neither assume \( \Pi \) to be finite, nor \( G_\beta \) for \( \beta \in A \) (in particular, \( G_\alpha, \forall i \in I \)) to be finite dimensional. For \( \eta \in \mathbb{Z}_{\geq 0} \Pi \) and \( 0 \neq x \in G_\eta \), we define \( \text{gr}(x) := \eta \), the grade of \( x \). Recall from equation (2.5) that \( A_{I,1} := \{ \gamma \in A \mid \text{ht}(\gamma) = 1 \} \).

**Theorem 3.1.** Let \( G \) be a \( \mathbb{Z}_{\geq 0} \Pi \)-graded \( \mathbb{F} \)-Lie algebra generated by non-zero subspaces \( G_{\alpha_i}, \forall i \in I \). Suppose \( \emptyset \neq I \subset I \), and \( \beta \in A \) such that \( \text{ht}(\beta) > 0 \). Then \( G_\beta \) is spanned by the Lie words of the form \( \left[ x_{\gamma_1}, \ldots, x_{\gamma_{n-1}}, x_{\gamma_n} \right] \) such that \( \gamma_j \in A_{I,1} \) and \( 0 \neq x_{\gamma_j} \in G_{\gamma_j} \) for each \( 1 \leq j \leq n = \text{ht}(\beta) \), and \( \sum_{j=1}^{n} \gamma_j = \beta \).

**Proof.** We use the following notation for convenience (only) in this proof: let \( i.e. \text{gr}(\theta_t) \in A_{I,1} \) for each \( t \in \mathcal{J} \). For a finite ordered sequence \( \hat{a} \) with terms in \( \mathcal{J} \), we define \( \theta_{\hat{a}} := \left[ [\theta_{a}]_{a \in \hat{a}} \right] \), see equation (2.1), where this notation was introduced.

We prove this theorem by induction on \( \text{ht}(\beta) \). Base step: \( \text{ht}(\beta) = 1 \), then \( \beta \in \Pi_I \), in which case the theorem is immediate.

Induction step: \( \text{ht}(\beta) > 1 \), once again there is nothing to prove if \( \text{ht}(\beta) = 1 \). So we assume \( \text{ht}(\beta) > 1 \). Pick

\[
0 \neq X = \left[ e_{i_1}, \ldots, e_{i_k} \right] = \left[ \left[ e_{i_j} \right]_{j=1}^{k} \right] \in G_\beta
\]

such that \( i_j \in I \) and \( 0 \neq e_{i_j} \in G_{\alpha_{i_j}} \) for each \( 1 \leq j \leq k := \text{ht}(\beta) \), and \( \sum_{j=1}^{k} \alpha_{i_j} = \beta \).

Note that \( G_\beta \) is spanned by Lie words of the form \( X \), as \( G \) is \( \mathbb{Z}_{\geq 0} \Pi \)-graded and generated by \( G_{\Pi} \). We will show that \( X \) is a linear combination of the Lie words as in the statement of the theorem. The theorem then follows immediately. By the induction hypothesis applied to \( \beta - \alpha_{i_1} \), we have

\[
\left[ \left[ e_{i_j} \right]_{j=2}^{k} \right] = \sum_{\hat{u}} d_\hat{u} \theta_{\hat{u}} \quad \text{and} \quad X = \sum_{\hat{u}} d_\hat{u} [e_{i_1}, \theta_{\hat{u}}],
\]

where both the sums are over some finitely many finite sequences \( \hat{u} \), each with terms in \( \mathcal{J} \), and \( d_\hat{u} \in \mathbb{F} \). If \( i_1 \in I \), then we are done, as the second summation for \( X \) in equation (3.1) is in the desired form. Else if \( i_1 \notin I \), consider a non-zero summand \( [e_{i_1}, \theta_{\hat{u}}] \) of \( X \) in equation (3.1) for some ordered sequence \( \hat{a} = (a_l)_{l=1}^{\text{ht}(\beta)} \). Let \( \hat{b} = (a_l)_{l=2}^{\text{ht}(\beta)} \), and recall that we assumed \( \text{ht}(\beta) \geq 2 \). By the Jacobi identity we have

\[
[e_{i_1}, [\theta_{a_1}, \theta_{b_1}]] = \left[ [e_{i_1}, \theta_{a_1}], \theta_{b_1} \right] + \left[ \theta_{a_1}, [e_{i_1}, \theta_{b_1}] \right].
\]

Observe that \( [e_{i_1}, \theta_{a_1}] \) can be expressed as \( \sum_{t} p_t \theta_t \), where the sum is over some finitely many \( \theta_t \in \Theta \), \( t \in \mathcal{J} \), such that \( \text{gr}(\theta_t) = \alpha_{i_1} + \text{gr}(\theta_{a_1}) \) and \( p_t \in \mathbb{F} \). By the induction hypothesis, we can express \( [e_{i_1}, \theta_{b_1}] \) as \( \sum_{\hat{c}} q_{\hat{c}} \theta_{\hat{c}} \), where the sum is over some finitely many finite sequences \( \hat{c} \) each with terms in \( \mathcal{J} \) such that \( \text{gr}(\theta_{\hat{c}}) = \alpha_{i_1} + \text{gr}(\theta_{b_1}) \) and \( q_{\hat{c}} \in \mathbb{F} \). By the previous two lines we get the following equation.

\[
[e_{i_1}, \theta_{\hat{a}}] = [e_{i_1}, [\theta_{a_1}, \theta_{b_1}]] = \sum_{t} p_t [\theta_t, \theta_{b_1}] + \sum_{\hat{c}} q_{\hat{c}} [\theta_{a_1}, \theta_{\hat{c}}].
\]

Note that each summand on the right hand side of the equation just above is a Lie word similar to the ones in the statement of the theorem. So every non-zero summand of \( X \), and hence \( X \) can be expressed as a linear combination of the Lie words as in the statement. Hence, the proof is complete. \( \square \)
Observe that if \( \Pi \) is not free, then we cannot define the functions \( \text{ht}(.) \) and \( \text{ht}_I(.) \). In view of this, Theorem 3.1 cannot be further extended to the Lie algebras graded over arbitrary semigroups.

**Corollary 3.2.** Let \( \mathcal{G} \) be as in Theorem 3.1 and \( I \subset \mathcal{I} \). Then \( U(\mathcal{G}) \) is spanned by the non-zero monomials all of the form either \( \prod_{\beta_j} \prod_{i} x_{i}^{e_{i}} \) or \( \prod_{i} x_{i}^{e_{i}} \), where \( e_{i} \in \mathcal{A}_{I,0}, \gamma_{i} \in \mathcal{A}_{I,1}, x_{\beta_j} \in \mathcal{G}_{\beta_j}, x_{\gamma_{i}} \in \mathcal{G}_{\gamma_{i}}, p_{j}, q_{i} \in \mathbb{Z}_{\geq 0} \forall 1 \leq j \leq n, 1 \leq i \leq m. \)

**Proof.** We assume that \( \emptyset \neq I \subsetneq \mathcal{I} \) throughout the proof, as there is nothing to prove in the extreme cases \( I = \emptyset \) or \( I = \mathcal{I} \). Fix an ordered basis \( \mathcal{B} \) for \( \mathcal{G} \) consisting of homogeneous elements such that the elements corresponding to \( \mathcal{A}_{I,0} \) always occur either before or after those corresponding to \( \mathcal{A} \setminus \mathcal{A}_{I,c} \) in \( \mathcal{B} \). By the PBW theorem, (ordered) monomials on the elements of \( \mathcal{B} \) span \( U(\mathcal{G}) \). Apply Theorem 3.1 to the elements of \( \mathcal{B} \) corresponding to \( \{\alpha \in \mathcal{A} \mid \text{ht}_I(\alpha) > 1\} \). Upon re-writing the Lie brackets in terms of commutators in \( U(\mathcal{G}) \), observe that each element in \( \mathcal{B} \) corresponding to \( \{\alpha \in \mathcal{A} \mid \text{ht}_I(\alpha) > 1\} \) is further a “polynomial” on the elements of \( \mathcal{B} \) corresponding to \( \mathcal{A}_{I,1} \). Observe that the corollary follows by the previous line.

**Proof of Theorem A** The proof of (A1) follows by Theorem 3.1 applied to \( \mathcal{G} = \mathcal{G}_{I,1} \) as the base of \( \Delta \). (A1) immediately implies (A2) once we set \( I = \mathcal{I} \setminus J \). Observe that \( \Delta_{I,1} \) is the minimal generating set for the semigroup \( \mathbb{Z}_{\geq 0}(\Delta_{I,1}^+ \setminus \Delta_{I,c}^+) \), as a root in \( \Delta_{I,1} \) cannot be further written as a sum of roots in \( \Delta_{I,1} \). This justifies the term “minimal” description. Hence the proof of Theorem A is complete.

The following remark addresses some remarks related to the “free-ness” of the subset \( \Delta_{I,1} \) in generating the cones \( \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_{I,c}^+) \) and \( \mathbb{R}_{\geq 0}(\Delta^+ \setminus \Delta_{I,c}^+) \).

**Remark 3.3.**

1. Given \( \emptyset \neq I \subset \mathcal{I} \), \( \Delta_{I,1} \) needn’t “freely” generate the semigroup \( \mathbb{Z}_{\geq 0}(\Delta^+ \setminus \Delta_{I,c}^+) \), as the following example shows. Let \( \mathcal{G} = \mathfrak{sl}_4(\mathbb{C}) \) and \( \mathcal{I} = \{1, 2, 3\} \) where node 2 is not a leaf in the Dynkin diagram, and suppose \( I = \{2\} \). Then we have \( (\alpha_1 + \alpha_2) + (\alpha_2 + \alpha_3) = (\alpha_1 + \alpha_2 + \alpha_3) + (\alpha_2) \).

2. Note that \( \Delta_{I,1} \) is a generating set for the cone \( \mathbb{R}_{\geq 0}(\Delta^+ \setminus \Delta_{I,c}^+) \), but it need not be minimal, as the following example shows. Let \( \mathcal{G} \) be of type \( \tilde{B}_2 \) and \( \mathcal{I} = \{1, 2\} \) where node 2 corresponds to the long simple root, and suppose \( I = \{2\} \). Then we have \( \frac{1}{2}(\alpha_2) + \frac{1}{2}(\alpha_2 + 2\alpha_1) = \alpha_2 + \alpha_1 \).

3. (In the above spirit, we find the minimal generating (over \( \mathbb{R}_{\geq 0} \)) set for the cone \( \mathbb{R}_{\geq 0}(\Delta_{I,1}^+ \setminus \Delta_{I,c}^+) \) in Lemma 3.3 in subsection 5.2.)

Let \( \mathcal{G} \) be as in Theorem 3.1 and \( I \subset \mathcal{I} \). We end this subsection by exhibiting (1) a spanning set for \( \mathcal{G}_{I,1} \) and thereby (2) a lower bound on the size of \( \mathcal{A}_{I,1} \), see Lemma 3.4. Lemma 3.4 was proved when \( \mathcal{G} \) is a Borcherds Kac–Moody algebra by Arunkumar et al in [11] Lemma 4.6. We now prove it in the more general graded setting of this paper.

**Lemma 3.4.** Let \( \mathcal{G} \) be as in Theorem 3.1. Fix an \( i \in \mathcal{I} \), and suppose \( \beta \in \mathcal{A} \) such that \( \text{ht}_{\{i\}}(\beta) > 0 \). Then \( \mathcal{G}_\beta \) is spanned by the Lie words of the form \([e_{i_1}, \ldots, [e_{i_1}, e_{i}]]\) such that \( i_j, i \in \mathcal{I}, 0 \neq e_{i_j} \in \mathcal{G}_{\alpha_{i_j}}, 0 \neq e_{i} \in \mathcal{G}_{\alpha_{i}}, \text{ for each } 1 \leq j \leq n, \text{ and } \alpha_{i} + \sum_{j=1}^{n} \alpha_{i_j} = \beta \).

**Proof.** We prove this lemma by induction on \( \text{ht}(\beta) \). In the base step, \( \text{ht}(\beta) = 1 \), there is nothing to prove.

**Induction step:** Given \( \beta \) with \( \text{ht}(\beta) > 1 \), pick a non-zero Lie word

\[
X = [e_{\nu_1}, \ldots, [e_{\nu_{k-1}}, e_{\nu_k}]] \in \mathcal{G}_\beta
\]

such that \( \nu_j \in \Pi_{\text{supp}(\beta)} \) and \( 0 \neq e_{\nu_j} \in \mathcal{G}_{\nu_j} \) for each \( 1 \leq j \leq k \), and \( \sum_{j=1}^{k} \nu_j = \beta \).
Fix $s \in [k]$ largest such that $\nu_s = \alpha_i$; such an $s$ exists as $\text{ht}_{\{i\}}(\beta) > 0$. Note that $k = \text{ht}(\beta) \geq 2$. When $k = 2$, check that the lemma trivially holds, so we assume that $k \geq 3$. If $s > 1$, then $[[[\nu_{j-1}], \nu_{j}]] \in G_{\beta - \nu_1}$ (see equation (2.11) for the notation) can be expressed as a linear combination of the Lie words of the desired form by the induction hypothesis applied to $\beta - \nu_1$, and hence so can be $X$. Else if $s = 1$, then observe by the Jacobi identity that

$$
\begin{align*}
[e_{\nu_1}, \cdots & [e_{\nu_{k-1}}, e_{\nu_k}] \cdots ] = \sum_{j=2}^{k-1} (-1)^j [e_{\nu_j}, [[[e_{\nu_{j-p}}]]_{p=1}^{j-1}, \left( [[[e_{\nu_q}]_{q=j+1}^{k-1}]]) + \right. \\
& \left. (-1)^k [[[e_{\nu_{-p}}]]_{p=1}^{k-1}, e_{\nu_k}].
\end{align*}
$$

(3.2)

Each non-zero $\left( [[[e_{\nu_{j-p}}]]_{p=1}^{j-1}, \left( [[[e_{\nu_q}]_{q=j+1}^{k-1}]]) \in G_{\beta - \nu_j}$ can be expressed as the desired linear combination by the induction hypothesis applied to $\beta - \nu_j$, $2 \leq j \leq k$. Hence, every term in the summation on the right hand side of equation (3.2) above can be expressed as a linear combination of the Lie words of the desired form. Notice that the last term outside the summation on the right hand side of equation (3.2) is already in the desired form (once we reverse the order of the outermost Lie bracket). As $X$ is arbitrary in $G_\beta$ and $G$ is generated by $G_{\Pi}$, the proof is complete. 

Lemma 3.4 immediately proves the following Corollary. Notice that Corollary 3.5 (1) is a special case of Theorem C (1), more generally, in the graded setting not just general Kac–Moody.

**Corollary 3.5.** Let $G$ and $A$ be as in Lemma 3.4

1. Fix an $i \in I$. Suppose $\beta \in A$ such that $\beta \supsetneq \alpha_i$. Then there exists a sequence of grades $\beta_j \in A$, $1 \leq j \leq n = \text{ht}(\beta)$, such that

$$
\alpha_i = \beta_1 < \cdots < \beta_j < \cdots < \beta_n = \beta \in A \quad \text{and} \quad \beta_{j+1} - \beta_j \in \Pi \ \forall j.
$$

(See Note 4 for the notation.) Notice that $\text{ht}_{\{i\}}(\beta) > 0 \implies \text{ht}_{\{i\}}(\beta_j) > 0 \ \forall j$. In particular, $\text{ht}_{\{i\}}(\beta) = 1 \implies \text{ht}_{\{i\}}(\beta_j) = 1 \ \forall j$, due to which we get a lower bound on the size of $A_{I,1}$ as in the next part.

2. Let $G$ and $A$ be as in Lemma 3.4, and $\emptyset \neq I \subset \mathcal{I}$. For each $i \in I$, fix a grade $\beta_i \in A_{I,1}$ such that $\text{supp}_I(\beta_i) = \{i\}$. Then we have

$$
\#A_{I,1} \geq \sum_{i \in I} \text{ht}(\beta_i).
$$

3.2. Parabolic-PSP-going up version. In the rest of the paper, we work only with Kac–Moody algebras, which we denote by $g$.

In this subsection, we prove a “going up” version of the parabolic-PSP for an indecomposable Kac–Moody $g$, using Theorem 3.1. This generalizes a basic result in the theory, see e.g. Proposition 4.9 in Kac’s book [15].

**Theorem 3.6.** Let $g$ be an indecomposable Kac–Moody algebra, and $\emptyset \neq I \subset \mathcal{I}$. Suppose $\beta \in \Delta^+$ such that $\text{ht}_I(\beta) < \sup \{ \text{ht}_I(\alpha) \mid \alpha \in \Delta \}$ and $0 \neq x \in g_\beta$. Then there exists a root $\gamma \in \Delta_{I,1}$ and $0 \neq x_\gamma \in g_\gamma$ such that $[x_\gamma, x] \neq 0$, which implies $\beta + \gamma \in \Delta$.

**Proof.** Fix $\beta$ as in the statement of the theorem, $0 \neq x \in g_\beta$ and a root $\beta'$ such that $\text{ht}_I(\beta') > \text{ht}_I(\beta)$. Consider the ideal $L := [g, x] \oplus \mathbb{C}x$. By [13] Proposition 1.7, we must have either a) $L \subset Z(g)$ the center of $g$, or b) $L \supset g' \supset \bigoplus_{\alpha \in \Delta} g_\alpha$, where $g'$ is the subalgebra generated by the Chevalley generators of $g$. Note that a) is not possible, as $Z(g) \subset h$ by [15] Proposition 1.6. So, b) holds, and we must
have a non-zero Lie word of the form
\[ [x_{\gamma_1}, [\cdots, [x_{\gamma_k}, x] \cdots] ] \in \mathfrak{g}_{\beta'} \subset L \]
(3.3) such that \( \gamma_j \in \Delta \cap \{0\}, \) 0 \( \neq x_{\gamma_j} \in \mathfrak{g}_{\gamma_j} \) \( \forall j \in [k], \) and \( \sum_{j=1}^{k} \gamma_j = \beta' - \beta. \)

Assume without loss of generality that the above Lie word has least “length” \( k \)—length here denotes the number of elements occurring in the iterated Lie bracket—among all the Lie words satisfying all the conditions and of the form in equation (3.3). We proceed in two cases below.

(1) \( k = 1: \) Firstly, note that \( 0 \neq [x_{\gamma_1}, x] \in \mathfrak{g}_{\beta'} \implies \text{ht}(\gamma_1) > 0. \) So, \( x_{\gamma_1} \in \mathfrak{g}_{\gamma_1} \) is some linear combination of the Lie words as in the statement of Theorem 3.1 (for Kac–Moody \( \mathfrak{g} \)). Observe then that in such a linear combination for \( x_{\gamma_1}, \) there must exist a non-zero Lie word
\[ Y = [x_{\eta_1}, [\cdots [x_{\eta_2}, x_{\eta_1}] \cdots] ] \] such that \( [Y, x] \neq 0, \)
where \( \eta_j \in \Delta_{I, 1} \) and 0 \( \neq x_{\eta_j} \in \mathfrak{g}_{\eta_j} \) for each \( 1 \leq j \leq t = \text{ht}(\gamma_1), \) and \( \sum_{j=1}^{t} \eta_j = \gamma_1. \)

As ‘ad \( x \)’ acts on \( Y \) by the derivation rule, we must have \( [x_{\eta_i}, x] \neq 0 \) for some \( i \in [t]. \) Hence the proof is complete in this case.

(2) \( k > 1: \) By the derivation rule, for each \( 2 \leq i \leq k \) we have
\[ [x_{\gamma_k}, [\cdots, [x_{\gamma_i}, \cdots, [x_{\gamma_1}, x] \cdots] \cdots] ] = \]
(3.4) \( \left( \prod_{\ell \in [k], \ell \neq i} \text{ad} x_{\gamma_\ell} \right) ([x_{\gamma_i}, x]) + \sum_{j=1}^{i-1} \left( \prod_{\ell=1}^{i-1} \text{ad} x_{\gamma_\ell} \right) \cdot \left( \prod_{\ell=j}^{i+1} \text{ad} x_{\gamma_\ell} \right) \cdot \text{ad} [x_{\gamma_i}, x_{\gamma_j}] \cdot \left( \prod_{\ell=j}^{i+1} \text{ad} x_{\gamma_\ell} \right) \right)(x). \)
(Treat each product over increasing indices on the right hand side of equation (3.4) as the identity map on \( \mathfrak{g}. \))
Recall, \( k \) is assumed to be the least length of all the Lie words of the form in equation (3.4). So, each term appearing in the summation on the right hand side of equation (3.4) is zero. Note that \( [x_{\gamma_i}, x] \neq 0, \) and from above we get \( [x_{\gamma_i}, x] \neq 0 \) for each \( i \in [k]. \) In particular, \( [x_{\gamma_i}, x] \neq 0 \) while \( \text{ht}(\gamma_i) > 0 \) puts us in case (1) (with \( \gamma_i + \beta \) in place of \( \beta' \) in case (1)). Hence, the proof is complete.

We end this subsection with the following Lemma which generalizes another fundamental result on Kac–Moody algebras—see e.g. Lemma 1.5 in Kac’s book [15]. Note that this also gives an alternate proof of (a relatively stronger version of) the parabolic-PSP for Kac–Moody algebra \( \mathfrak{g}. \) For \( \emptyset \neq I \subset \mathcal{I}, \) define \( \Delta_{I, \mathbb{Z}_{>0}} := \{ \alpha \in \Delta \mid \text{ht}(\alpha) \in \mathbb{Z}_{>0} \}. \)

**Lemma 3.7.** Let \( \mathfrak{g} \) be a Kac–Moody algebra and \( \emptyset \neq I \subset \mathcal{I}. \) Suppose \( x \in \bigoplus_{\nu \in \Delta_{I, \mathbb{Z}_{>0}}} \mathfrak{g}_\nu \) is such that
\[ [x, f_{\eta}] = 0 \quad \forall \eta \in \Delta_{I, -1}, \] \( f_{\eta} \in \mathfrak{g}_\eta. \)
Then \( x = 0. \)

**Proof.** Firstly, observe that it is enough to prove the lemma when \( x \) is a root vector in the root space of some \( \beta \in \Delta_{I, \mathbb{Z}_{>0}}. \)

By induction on \( \text{ht}(\beta) \) we prove: given \( \beta \in \Delta_{I, \mathbb{Z}_{>0}} \) and \( 0 \neq x \in \mathfrak{g}_\beta \) there exists a root \( \gamma \in \Delta_{I, -1} \) and a root vector \( f_{\gamma} \in \mathfrak{g}_\gamma \) such that \( \beta + \gamma \in \Delta^+ \cup \{0\} \) and \( [x, f_{\gamma}] \neq 0. \) This implies the lemma. Fix
an $\alpha \in \Pi$ and $f_\alpha \in g_{-\alpha}$ such that $\beta - \alpha \in \Delta^+ \cup \{0\}$ and $[x, f_\alpha] \neq 0$, which exist by [15] Lemma 1.5. Observe that in the base step, $ht(\beta) = 1$, so the proof trivially follows by the previous line. Induction step: $ht(\beta) > 1$, so we are done if $\alpha \in \Pi_f$. Else, by the induction hypothesis applied to $\beta - \alpha$, we have $[[x, f_\alpha], f_\eta] \neq 0$ and $\beta - \alpha + \eta \in \Delta^+ \cup \{0\}$ for some $\eta \in \Delta_{I,-1}$ and $f_\eta \in g_\eta$. By the Jacobi identity we have
$$[[x, f_\alpha], f_\eta] = [x, [f_\alpha, f_\eta]] - [f_\alpha, [x, f_\eta]].$$
As one of the two terms on the right hand side of the above equation must be non-zero, we are finally done. \hfill $\square$

3.3. Parabolic-PSP in the set of short roots in finite type. In this subsection, we assume $\Delta$ to be a finite type root system, i.e. the root system of some finite dimensional simple Lie algebra $g$. We prove the analogous parabolic-PSP and its “going up” version in the set of short roots. We came across these phenomena while proving the parabolic-PSP for affine Kac–Moody algebras in a case by case manner. In view of Remarks 3.10 and 3.11 below, these phenomena cannot be further extended.

In this subsection $(\cdot, \cdot)$ denotes the usual Killing form on $h^*$. The length of a root $\alpha$ is denoted by $(\alpha, \alpha)$. Note that the simplicity of $g$ implies that $\Delta$ is indecomposable. Recall that there can be at most two lengths in any finite type root system, by [13, §10.4 Lemma C]. Short roots in $\Delta$ are the roots of shortest length.

In this subsection, we denote the set of short, positive short, long and positive long roots in $\Delta$ by $\Delta_s, \Delta_{s,+}, \Delta_l$ and $\Delta_{l,+}$ respectively, and the highest (also dominant) short root by $\theta_s$. Recall that $\langle \beta, \alpha \rangle = \frac{2\langle \beta, \alpha \rangle}{(\alpha, \alpha)} \forall \alpha, \beta \in \Delta$. Recall also that in finite type, the Killing form is positive definite, i.e. $(x, x) > 0 \forall x \neq 0 \in h^*$.

**Proposition 3.8.** Let $\Delta$ be a finite type root system and $\emptyset \neq I \subset \mathcal{I}$. Suppose $\beta \in \Delta_{s,+}$ such that $ht_I(\beta) > 1$. Then there exists a sequence of roots $\gamma_i \in \Delta_{I,1}$, $1 \leq i \leq n = ht_I(\beta)$, such that

$$\gamma_1 \prec \cdots \prec \sum_{j=1}^i \gamma_j \prec \cdots \prec \sum_{j=1}^n \gamma_j = \beta \in \Delta_s \ \forall i. \quad (See \ Note \ [\square] \ for \ the \ notation.)$$

**Proof.** We show: given $\beta$ as in the statement, there exists a root $\gamma \in \Delta_{I,1}$ such that $\beta - \gamma \in \Delta_s$.

This implies the proposition. By the parabolic-PSP, we have a root $\eta \in \Delta_{I,1}$ such that $\beta - \eta \in \Delta^+$. If $\beta - \eta \in \Delta_{s,+}$, then we are done. Else if $(\beta - \Delta_{I,1}) \cap \Delta^+ \subset \Delta_{l,+}$, then

$$(\beta - \eta, \beta - \eta) = (\beta, \beta) + (\eta, \eta)(1 - \langle \beta, \eta \rangle^\vee) = 2(\beta, \beta) \text{ or } 3(\beta, \beta)$$

$$\implies (\eta, \eta)(1 - \langle \beta, \eta \rangle^\vee) = (\beta, \beta) \text{ or } 2(\beta, \beta)$$

$$\implies \langle \beta, \eta \rangle^\vee = 0 \text{ or } -1 \implies (\eta, \eta) \leq 0.$$ 

Again by the parabolic-PSP, we can write $\beta = \eta_1 + \cdots + \eta_m$, for some $\eta_1, \ldots, \eta_m \in \Delta_{I,1}$ where $m = ht_I(\beta)$. As $(\beta, \beta^\vee) > 0$, we must have $(\eta_k, \beta^\vee) > 0$ for some $1 \leq k \leq m$. This implies $\beta - \eta_k \in \Delta^+$. Observe that this contradicts the above calculations with $\eta = \eta_k$. Hence $(\beta - \Delta_{I,1}) \cap \Delta_{s,+}$ must be non-empty. \hfill $\square$

We now prove the “going up” version of Proposition 3.8.

**Proposition 3.9.** Let $\Delta$ be a finite type root system and $\emptyset \neq I \subset \mathcal{I}$. Suppose $\beta \in \Delta_{s,+}$ such that $ht_I(\beta) < ht_I(\theta_s)$. Then there exists a sequence of roots $\gamma_i \in \Delta_{I,1}$, $1 \leq i \leq n = ht_I(\theta_s) - \beta$, such that

$$\beta \prec \cdots \prec \beta + \sum_{j=1}^i \gamma_j \prec \cdots \prec \beta + \sum_{j=1}^n \gamma_j \in \Delta_s \ \forall i.$$
Proof. We show: given $\beta$ as in the statement, there exists a root $\gamma \in \Delta_{I,1}$ such that $\beta + \gamma \in \Delta_s$. This implies the proposition.

Firstly, let $\mathfrak{g}$ be of type $G_2$ and $I = \{1, 2\}$. Assume that $\alpha_2$ is long; then $\theta_s = \alpha_2 + 2\alpha_1$. We give a choice for $\gamma$ in a few cases below, and it can be easily checked that considering these cases is enough.

1. If $\beta = \alpha_2 + \alpha_1$ and $1 \in I$, then $\gamma = \alpha_1$.
2. If $\beta = \alpha_1$ and $2 \in I$, then $\gamma = \alpha_2$.
3. If $\beta = \alpha_1$ and $I = \{1\}$, then $\gamma = \alpha_1 + \alpha_2$.

Now let $\mathfrak{g}$ be of any type $\neq G_2$. Theorem 3.6 yields a root $\eta \in \Delta_{I,1}$ such that $\beta + \eta \in \Delta^+$. If $\beta + \eta \in \Delta^+_s$, then we are done. Else if $\text{ht}_f(\beta) = 0$, then pick an $\omega \in W_{\text{supp}(\beta)}$ such that $\omega \beta = \alpha_t$, for some short simple root $\alpha_t$, $t \in I$. Choose a vertex in $I$ whose distance in the Dynkin diagram to $t$ is minimal, and let $T$ denote the path joining these nodes. Check that $\gamma := \omega^{-1}(\sum_{j \in T} \alpha_j) - \beta$ works, where the summation is over all nodes in the path $T$.

Else if $\text{ht}_f(\beta) > 0$ and $(\beta + \Delta_{I,1}) \cap \Delta^+ \subset \Delta^+_T$, then

$$(\beta + \eta, \beta + \eta) = (\beta, \beta) + (\eta, \eta)(1 + \langle \beta, \eta^\vee \rangle) = 2(\beta, \beta) \quad \Rightarrow \quad (\eta, \eta)(1 + \langle \beta, \eta^\vee \rangle) = (\beta, \beta) \quad \Rightarrow \quad (\beta, \beta) = (\eta, \eta) \quad \text{and} \quad (\beta, \eta^\vee) = (\eta, \beta^\vee) = 0.$$

Finally, a similar argument as in the end of proof of Proposition 3.8 gives a contradiction to $(\beta, \beta) > 0$. 

The following remarks discuss the limitations to further extend the above propositions.

Remark 3.10. (1) The analogous statements to Propositions 3.8 and 3.9 with $\beta \in \Delta^+_s$ are not true, as the following example shows. Let $\Delta$ be the root system of type $B_2$, $I = \{1, 2\}$ where node 2 corresponds to the long simple root. Let $\beta' = \alpha_2 + 2\alpha_1$, $\beta = \alpha_2$ and $I = \{1\}$.

Then, we can neither come down from $\beta'$, nor go up from $\beta$, to a root in $\Delta^+_T$.

(2) Note that when $\mathfrak{g}$ is semisimple, the above propositions work for each indecomposable component of $\Delta$.

Remark 3.11. Let $\Delta$ be an affine root system of type $C_2^{(1)}$, $I = \{0, 1, 2\}$. Assume that node 1 is not a leaf in the Dynkin diagram and it corresponds to a short simple root (see [15 Table Aff 1]). Let $\beta = \alpha_0 + 3\alpha_1$ and $\alpha_2 \in \Delta_s$, then observe that the only simple root that can be subtracted from $\beta$ to get again a root is $\alpha_1$. But then $\beta - \alpha_1$ is imaginary. In view of this, Propositions 3.8 and 3.9 cannot be extended beyond finite type.

4. Moving between comparable weights of representations

4.1. Proof of Theorem (C) In this subsection, we prove Theorem (C) by proving (C1) and (C2) separately for a Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$.

Throughout this subsection, we use without further mention the following fact.

Fact: Let $\mathfrak{g}$ be a Kac–Moody algebra and $M$ be a weight module of $\mathfrak{g}$. Fix a weight $\mu \in \text{wt}M$ and a real root $\alpha$. Suppose $M$ is $\mathfrak{sl}_\alpha$-integrable, where $\mathfrak{sl}_\alpha = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_\alpha^\vee$. Then

$$\langle \mu, \alpha^\vee \rangle > 0 \quad \Rightarrow \quad \mu - \alpha \in [s_\alpha \mu, \mu] \subset \text{wt}M,$$

similarly, $\langle \mu, \alpha^\vee \rangle < 0 \quad \Rightarrow \quad \mu + \alpha \in [\mu, s_\alpha \mu] \subset \text{wt}M$.

Proof of (C1) (for adjoint representation). Relabel $\mu_0$ and $\mu$ by $\beta_0$ and $\beta$ respectively for convenience. We prove (C1) by induction on $\text{ht}(\beta - \beta_0)$. In the base step $\text{ht}(\beta - \beta_0) = 1$, there is nothing to prove.

Induction step: $\text{ht}(\beta - \beta_0) > 1$, observe that if there exists an $\bar{\alpha} \in \pm \Pi \cup \{0\}$ such that $\beta_0 \not\leq \bar{\alpha} \not\leq \beta$, then...
then we will be immediately done by the induction hypothesis applied to $\beta_0 \not\leq \alpha$ and $\alpha \not\leq \beta$. So, we assume throughout the proof that both $\beta_0$ and $\beta$ are positive roots (by symmetry, the proof for the leftover case where both $\beta_0$ and $\beta$ are negative just follows from this). We show in steps that there exists an $\alpha \in \Pi$ such that $\alpha \prec \beta - \beta_0$, and either $\beta - \alpha$ or $\beta_0 + \alpha$ is a root. The induction hypothesis then completes the proof.

Write $\beta = \sum_{i \in I} a_i \alpha_i$ and $\beta_0 = \sum_{i' \in I_0} b_{i'} \alpha_\beta$ for some $I_0 \subseteq I \subseteq I$, and positive integers $a_i$ and $b_{i'} \leq a_{i'}$, $\forall i \in I$ and $i' \in I_0$. We proceed in several steps below.

**Step 1:** If $I_0 \subseteq I$, then, as $I = \text{supp}(\beta)$ is connected, we must have a simple root $\alpha' \in \Pi_I \setminus \Pi_{I_0}$ such that $\langle \beta_0, \alpha'^{\vee} \rangle < 0 \implies \beta_0 + \alpha' \prec \beta \in \Delta_I$.

So, we assume now that $I_0 = I$. Define $J := \text{supp}(\beta - \beta_0) \subseteq I$ and write $\beta - \beta_0 = \sum_{j \in J} (a_j - b_j) \alpha_j$.

Note that $b_j < a_j \forall j \in J$.

If $J = I$, then by the PSP there exists an $\alpha'' \in \Pi$ such that $\beta - \alpha'' \in \Delta$.

So, we also assume now that $J \subsetneq I$. Write $\beta = \sum_{i \in I \setminus J} a_i \alpha_i + \sum_{j \in J} a_j \alpha_j$ and $\beta_0 = \sum_{i \in I_0 \setminus J} a_i \alpha_i + \sum_{j \in J} b_j \alpha_j$.

**Step 2:** Consider the decomposition of the submatrix $A_{J \times J}$ of $A$ into indecomposable blocks.

Pick an indecomposable block $A_{K \times K}$ of $A_{J \times J}$ for some $K \subset J$.

If $A_{K \times K}$ (equivalently $A_{K \times K}^{I}$) is of finite type, then by [15, Theorem 4.3] there exists a vector $Y = \sum_{k \in K} p_k \alpha_k \in \mathfrak{h}$ such that $p_k > 0$ and $\langle \alpha_k, Y \rangle > 0 \forall k \in K$.

Check that $\langle \alpha_j, Y \rangle \geq 0 \forall j \in J$ and $\langle \beta - \beta_0, Y \rangle > 0$. By the previous line, there exists some $k_0 \in K \subset J$ such that $\langle \beta - \beta_0, \alpha_{k_0}^{\vee} \rangle > 0$. This implies either $\langle \beta, \alpha_{k_0}^{\vee} \rangle > 0$ or $\langle \beta_0, \alpha_{k_0}^{\vee} \rangle < 0$, yielding $\beta - \alpha_{k_0} \succ \beta_0 \in \Delta$ or respectively $\beta_0 + \alpha_{k_0} \prec \beta \in \Delta$, as required.

So, we assume now that $A_{K \times K}$ is not of finite type. This means $A_{K \times K}$ (equivalently $A_{K \times K}^{I}$) must be of affine or indefinite type, and once again by [15, Theorem 4.3] there exists a vector $X = \sum_{k \in K} q_k \alpha_k \in \mathfrak{h}$ such that $q_k > 0$ and $\langle \alpha_k, X \rangle \leq 0 \forall k \in K$.

Check that $X$ also satisfies $\langle \alpha_j, X \rangle \leq 0 \forall j \in J$. Note that the Dynkin subdiagram on $I$ is connected, and whenever $K \subsetneq J$, the subdiagrams on $K$ and $J \setminus K$ are disconnected. Thus, in view of the previous line, there must exist a pair of nodes $t_1 \in K$ and $t_2 \in I \setminus J$ such that $t_1$ and $t_2$ are connected by at least one edge in the Dynkin diagram, i.e. $\langle \alpha_{t_1}, \alpha_{t_2}^{\vee} \rangle < 0$. This yields

$$\langle \sum_{i \in I \setminus J} a_i \alpha_i, X \rangle < 0 \implies \langle \sum_{i \in I \setminus J} a_i \alpha_i + \sum_{j \in J} b_j \alpha_j, X \rangle = \langle \beta_0, X \rangle < 0$$

$$\implies \exists k' \in K \subset J \text{ such that } \langle \beta_0, \alpha_{k'}^{\vee} \rangle < 0 \implies \beta_0 + \alpha_{k'} \in \Delta.$$ 

Hence the proof of (C1) is complete. \(\square\)

Before proving (C2), we prove the following corollary of (C1) which is similar to Corollary 3.5. Recall from equation (22) that $\Delta_{\alpha,J} := \{ \beta \in \Delta^+ \mid \text{supp}(\beta - \alpha) \subset J \}$, for $\alpha \in \Delta^+$ and $J \subset \mathcal{I}$.

**Corollary 4.1.** Let $\mathfrak{g}$ be a Kac–Moody algebra, and fix a real root $\alpha \in \Delta^+$ such that $J := \text{supp}(\alpha) \subset \mathcal{I}$ and $\emptyset \neq J \subset \mathcal{I}$. Suppose $\beta \in \Delta_{\alpha,J}$ such that $\beta - \alpha \in \Delta^+$. Then there exists a sequence of roots $\beta_i \in \Delta_{\alpha,J}$, $0 \leq i \leq n = \text{ht}(\beta - \alpha)$, such that

$$\alpha = \beta_0 < \cdots < \beta_i < \cdots < \beta_n = \beta \in \Delta_{\alpha,J} \forall i \text{ and also } \beta_i - \alpha \in \Delta^+_\alpha \forall i \geq 1.$$ 

**Proof.** We prove this Corollary by induction on $\text{ht}(\beta - \alpha)$. In the base step $\text{ht}(\beta - \alpha) = 1$, there is nothing to prove. Induction step: $\text{ht}(\beta - \alpha) > 1$, pick $j \in \text{supp}(\beta - \alpha) \subset J$ such that $\langle \alpha, \alpha_j^{\vee} \rangle < 0$ (such a $j$ exists as the Dynkin subdiagram on $\text{supp}(\beta)$ is connected). By (C1) applied
to $\alpha_j \prec \beta - \alpha \in \Delta^+$, we get a root $\beta' \in \Delta^+_j$ such that $(\beta - \alpha) - \beta' \in \Pi_J$ and $\alpha_j \prec \beta'$. Observe that we must have $(\beta', \alpha_j^\vee) < 0$ (as $j \in \supp(\beta')$). This implies $\beta' + \alpha \prec \beta \in \Delta_{\alpha,J}$, and the induction hypothesis applied to $\beta' + \alpha$ now finishes the proof. \hfill \square

**Proof of (C2) (for submodules of $M(\lambda, J)$).** Let $V$ be a submodule of $M(\lambda, J)$ and $\mu_0 < \mu \in \text{wt} V$. We prove (C2) by induction on $\text{ht}(\mu - \mu_0)$. In the base step $\text{ht}(\mu - \mu_0) = 1$, there is nothing to prove.

Induction step: $\text{ht}(\mu - \mu_0) > 1$, let $J_1 := \supp(\mu - \mu_0)$, $I := \supp(\lambda - \mu) \setminus J_1$, and $m_\lambda$ span $M(\lambda, J)$. When $I \neq \emptyset$, write $\mu = \lambda - \sum_{i \in I} c_i \alpha_i - \sum_{j \in J_1} c'_j \alpha_j$ for some $c_i \in \mathbb{Z}_{>0}$ and $c'_j \in \mathbb{Z}_{\geq 0}$. Firstly, note the following.

(a) Let $p_J$ be the parabolic Lie subalgebra of $\mathfrak{g}$ corresponding to $J \subset J_\lambda \subset I$. Then $M(\lambda, J) \cong U(\bigoplus_{\beta \in \Delta - \Delta_J^+} \mathfrak{g}_\beta) \otimes L_{J}^{\text{max}}(\lambda)$, where $L_{J}^{\text{max}}(\lambda)$ is the largest integrable highest weight $p_J$-module with highest weight $\lambda$.

(b) $M(\lambda, J)$ is torsion free over $U(\bigoplus_{\beta \in \Delta - \Delta_J^+} \mathfrak{g}_\beta)$.

We show via several cases that there exists a simple root $\alpha \prec \mu - \mu_0$ such that either $\mu - \alpha$ or $\mu_0 + \alpha \in \text{wt} V$. The induction hypothesis then completes the proof.

If $J_1 \cap J^c \neq \emptyset$, then by (a) and (b) we must have $f_i V_i \neq 0 \forall i \in J_1 \cap J^c \implies \mu - \alpha_i > \mu_0 \in \text{wt} V \forall i \in J_1 \cap J^c$. So, we assume for the rest of the proof that $J_1 \subset J$. If $\langle \mu - \mu_0, \alpha_j^\vee \rangle > 0$ for some $j' \in J_1$, then we must have $\langle \mu, \alpha_j^\vee \rangle > 0$ or $\langle \mu_0, \alpha_j^\vee \rangle < 0$, in which case we are done by the existence of the injective mappings $V_\mu \xrightarrow{f_j} V_{\mu - \alpha_j}$ or respectively $V_{\mu_0} \xrightarrow{e_j^\vee} V_{\mu_0 + \alpha_j}$ (by [15] Proposition 3.6) as $M(\lambda, J)$ and hence $V$ is g$_J$-integrable.

So, we also assume now that $\langle \mu - \mu_0, \alpha_j^\vee \rangle \leq 0 \forall j \in J_1$. Fix an indecomposable block $A_{K \times K}$ of $A_{J_1 \times J_1}$ for some $K \subset J_1$, and note that (by the assumption in the previous line and [15] Theorem 4.3) $A_{K \times K}$ (equivalently $A_{J_1 \times J_1}^J$) must be of either affine or indefinite type. Thus, we have a vector

$$X = \sum_{k \in K} c_k \alpha_k^\vee \in \mathfrak{h}$$

such that $c_k \in \mathbb{R}_{>0}$ and $\langle \alpha_k, X \rangle \leq 0 \forall k \in K$.

Note that $X$ also satisfies $\langle \alpha_j, X \rangle \leq 0 \forall j \in J_1$.

We proceed in two cases below. We show in both the cases that $\langle \mu, X \rangle > 0$. This implies that $\langle \mu, \alpha_{j_0}^\vee \rangle > 0$ for some $j_0 \in K \subset J_1$, and then we will be done as above.

1) $J_1 \neq \emptyset$, and $J_1$ and $I$ are connected by at least one edge in the Dynkin diagram: Firstly, without loss of generality, we may assume that the subset $K$ of $J_1$ we started with has the property that the Dynkin subdiagram on $K \cup I$ is connected. Thus, $\langle \sum_{i \in I} c_i \alpha_i, X \rangle < 0$ as $c_i > 0 \forall i \in I$. Note that $\langle \lambda, \alpha_j^\vee \rangle \geq 0 \forall j \in J_1 \subset J$, in particular $\langle \lambda, X \rangle \geq 0$. The observation that $\langle \alpha_j, X \rangle \leq 0 \forall j \in J_1$ and the previous two lines together imply that $\langle \mu, X \rangle = \langle \lambda - \sum_{i \in I} c_i \alpha_i - \sum_{j \in J_1} c'_j \alpha_j, X \rangle > 0$, as required.

2) $J_1$ and $I$ are not connected by any edge in the Dynkin diagram (with $I$ possibly empty): Hence the subdiagrams on $K$, $J_1 \setminus K$ and $I$ are pairwise disconnected, so any vector in $V_{\mu_0} \neq 0$ can be expressed in the form $\sum_{p=1}^l F_p G_p H_p m_\lambda$ for some $F_p \in U(\mathfrak{n}_I^-$), $G_p \in U(\mathfrak{n}_{J_1 \setminus K}^-$), and $H_p \in U(\mathfrak{n}_K^-)$, $1 \leq p \leq l$. As $\text{ht}(\lambda - \mu_0) > 0$, in the previous line each $H_p$ may be assumed to be an element of the direct sum of the graded pieces $U(\mathfrak{n}_K^-) \forall v \in \mathbb{Z}_{\leq 0} \Pi_K \setminus \{0\}$ of $U(\mathfrak{n}_K^-)$. Next, if $\langle \lambda, \alpha_j^\vee \rangle = 0$ for some $j'' \in J_1$, then $f_{j''} m_\lambda = 0$, as $M(\lambda, J)$ is $g_J$-integrable.

Observe then that $\langle \lambda, X \rangle > 0$. Finally, $\langle \alpha_j, X \rangle \leq 0 \forall j \in J_1$, $\langle \alpha_i, X \rangle = 0 \forall i \in I$ and the previous
Remark 4.2. Let $\mathfrak{g}, \lambda, J$ and $M$ be as in the proof of (C2). Observe that in the proof of (C2) we only made use of the $\mathfrak{g}_J$-integrability of $M(\lambda, J)$, which is a consequence of the $\mathfrak{g}_J$-integrability of $L_j^{\text{max}}(\lambda)$. Recall that it is not known if $L_j^{\text{max}}(\lambda)$ over a general Kac–Moody algebra $\mathfrak{g}_J$ is simple. Let $L'_J(\lambda)$ be an integrable highest weight $\mathfrak{g}_J$-module with highest weight $\lambda$, and define $N := U(\bigoplus_{\beta \in \Delta^- \setminus \Delta_j^+} \mathfrak{g}_\beta) \otimes L'_J(\lambda)$. Note that $N$ is a $\mathfrak{g}$-module, and $L'_J(\lambda)$ is a quotient of $L_j^{\text{max}}(\lambda)$. Recall by the explicit description of the set of weights of an integrable highest weight module over a Kac–Moody algebra in [13, Proposition 11.2], that $\text{wt} L'_J(\lambda) = \text{wt} L_j^{\text{max}}(\lambda)$. This implies $\text{wt} N = \text{wt} M(\lambda, J)$. Thus, the above proof of (C2) also holds for the module $\mathfrak{g}$-module $N$.

It is natural to ask if Theorem C holds true “at the level of weight vectors”, made precise below, and similar to the parabolic-PSP. The following remark provides a negative answer.

Remark 4.3. A strengthened version of Theorem C one would naturally expect at the level of weight vectors does not hold true. More precisely, if $\mu_0 \not\leq \mu \in \text{wt} V$, then there need not exist a non-zero $\nu \in V_\mu$, and $f_{i,j} \in \mathfrak{g}_{-\alpha_{ij}}$, $1 \leq j \leq n = \text{ht}(\mu - \mu_0)$, such that

$$\mu - \sum_{j=1}^n \alpha_{ij} = \mu_0 \text{ and } 0 \neq \prod_{j=1}^n f_{i,j} \nu \in V_{\mu_0}.$$ 

This can be easily checked for $V$ the adjoint representation when $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$ and $I = \{1, 2, 3\}$, where nodes 1, 3 are the leaves in the Dynkin diagram, for the pair $(-\alpha_1) \prec \alpha_3 \in \text{wt} \mathfrak{g}$.

It will be interesting to investigate if Theorem C holds true, even in finite type, for: (1) non-integrable highest weight modules whose set of weights are not those of any parabolic Verma module and (2) more general weight modules which are not necessarily highest weight modules, for example modules in the category $\mathcal{O}$ over $\mathfrak{g}$.

For instance, Corollary 4.5 below proves Theorem C for arbitrary integrable modules (which need not be highest weight modules or even need not have finite dimensional weight spaces) over semisimple $\mathfrak{g}$.

We end this subsection by extending Theorem C (C2), which is for the set of weights of highest weight modules, to any saturated subset $U$ over semisimple $\mathfrak{g}$. Let $\Lambda$ be the weight lattice. Recall that $U \subset \Lambda$ is a saturated subset of $\Lambda$ if for every $\mu \in U$ and $\alpha \in \Pi$, $\mu - t\alpha \in U \forall t = 0, \ldots, \langle \mu, \alpha^\vee \rangle$, see [13, §13.4].

Lemma 4.4. Let $\mathfrak{g}$ be semisimple and $U$ be a saturated subset, and suppose $\mu_0 < \mu \in U$. Then there exists a sequence of weights $\mu_i \in U$, $1 \leq i \leq n = \text{ht}(\mu - \mu_0)$, such that

$$\mu_0 < \cdots < \mu_i \prec \cdots < \mu_n = \mu \in U \text{ and } \mu_i - \mu_{i-1} \in \Pi \forall i.$$ 

Proof. We prove the lemma by induction on $\text{ht}(\mu - \mu_0)$. In the base step $\text{ht}(\mu - \mu_0) = 1$, there is nothing to prove. To show the induction step, recall that when $\mathfrak{g}$ is semisimple the symmetric invariant (Killing) form on $\mathfrak{h}^*$ is positive definite, i.e. $(x, x) > 0 \forall \neq x \in \mathfrak{h}^*$. Now let $\mu - \mu_0 = \sum_{i \in I} c_i \alpha_i$ for some $c_i \in \mathbb{Z}_{\geq 0}$ and consider $0 \prec (\mu - \mu_0, \mu - \mu_0) = (\sum_{i \in I} c_i \alpha_i, \mu) - (\sum_{i \in I} c_i \alpha_i, \mu_0)$. By the previous line, we must have

either $(\sum_{i \in I} c_i \alpha_i, \mu) > 0$ or $(\sum_{i \in I} c_i \alpha_i, \mu_0) < 0$

$$\implies \langle \mu, \alpha^\vee \rangle > 0 \text{ or resp. } \langle \mu_0, \alpha^\vee \rangle < 0 	ext{ for some simple root } \alpha < \mu - \mu_0$$

$$\implies \mu_0 < \mu - \alpha \in U \text{ or resp. } \mu_0 + \alpha < \mu \in U \text{ (by the definition of } U).$$
The induction hypothesis then completes the proof. \hfill \square

**Corollary 4.5.**

1) *Theorem C* for finite dimensional simple modules, proved by S. Kumar, now holds true for any integrable module over semisimple \( \mathfrak{g} \).

2) Let \( \mathfrak{g} \) be a Kac–Moody algebra, \( J \subset \mathcal{I} \). Suppose \( \mathfrak{g}_J \) is semisimple and \( M \) is a finite dimensional \( \mathfrak{g}_J \) (or \( \mathfrak{p}_J \))-module. Then:

(a) The \( \mathfrak{g} \)-module \( N := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_J)} M \), defined analogous to \( M(\lambda, J) \), is \( \mathfrak{g}_J \)-integrable.

(b) *Theorem C* holds true for \( \text{wt} \, N \).

**Proof.** 1) holds by Lemma 4.4, as the set of weights of an integrable module is a saturated subset. 2) (a) is a trivial check. 2) (b) can be proved by combining the proofs of (C2) and Lemma 4.4. \hfill □

### 4.2. Moving between comparable weights in steps of \( \Delta_{I,1} \) in the semisimple case.

In this subsection, we discuss the “parabolic” generalizations of (C1) and (C2) of *Theorem C*—i.e. moving between comparable roots and weights in steps of \( \Delta_{I,1} \). See Proposition 4.7 below. As a “warmup” to our main result in this subsection, first note the following immediate consequence of Corollary 3.2 applied to \( U(n^-) \).

**Proposition 4.6.** Let \( \mathfrak{g} \) be a Kac–Moody algebra, \( \lambda \in \mathfrak{h}^* \), \( \emptyset \neq I \subset \mathcal{I} \), and \( M(\lambda) \rightarrow V \). Suppose \( \mu \prec \lambda \in \text{wt} \, V \) such that \( n := \text{ht}_I(\lambda - \mu) > 0 \). Then there exist sequences of weights \( \mu_i \) and \( \mu_i' \in \text{wt} \, V \) such that

\[
a) \quad \mu = \mu_0 < \cdots < \mu_i < \cdots < \mu_n \leq \lambda \quad \text{and} \quad \mu_i - \mu_{i-1} \in \Delta_{I,1} \forall 1 \leq i \leq n.
\]

\[
b) \quad \mu \preceq \mu_0' < \cdots < \mu_i' < \cdots < \mu_n' = \lambda \quad \text{and} \quad \mu_i' - \mu_{i-1}' \in \Delta_{I,1} \forall 1 \leq i \leq n.
\]

Notice that Proposition 4.6 discusses “moving between comparable weights in steps of \( \Delta_{I,1} \)” for arbitrary for \( M(\lambda) \rightarrow V \) over Kac–Moody \( \mathfrak{g} \) for the pair \( \mu_0 < \lambda \in \text{wt} \, V \), and this cannot be generalized to \( \mu_0 < \mu \in \text{wt} \, V \) when \( \mu \npreceq \lambda \). Importantly, note the inequalities in the starting/ending of the chains in the statement of Proposition 4.6 and also that of all the results of this subsection. These two lines will be justified in parts (ii) and (i) of Remark 4.10 (1), respectively.

We now state the main result of this subsection, which is also an improvement of the above proposition in the semisimple case.

**Proposition 4.7.** Let \( \mathfrak{g} \) be semisimple, \( \emptyset \neq I \subset \mathcal{I} \), and \( \beta \prec \beta' \in \text{wt} \, \mathfrak{g} = \Delta \sqcup \{0\} \) such that \( n := \text{ht}_I(\beta' - \beta) > 0 \).

a) If either \( \beta \geq 0 \), or \( \beta \prec 0 \) and \( \text{ht}_I(\beta) < 0 \), then there exists \( \beta_i \in \text{wt} \, \mathfrak{g} \) such that

\[
\beta = \beta_0 < \cdots < \beta_i < \cdots < \beta_n \leq \beta' \in \text{wt} \, \mathfrak{g} \quad \text{and} \quad \beta_i - \beta_{i-1} \in \Delta_{I,1} \forall 1 \leq i \leq n.
\]

b) If either \( \beta' \leq 0 \), or \( \beta' > 0 \) and \( \text{ht}_I(\beta') > 0 \), then there exists \( \beta_i' \in \text{wt} \, \mathfrak{g} \) such that

\[
\beta \leq \beta_0' < \cdots < \beta_i' < \cdots < \beta_n' = \beta' \in \text{wt} \, \mathfrak{g} \quad \text{and} \quad \beta_i' - \beta_{i-1}' \in \Delta_{I,1} \forall 1 \leq i \leq n.
\]

We first prove two preliminary lemmas needed in the proof of Proposition 4.7. In the rest of this subsection, when \( \mathfrak{g} \) is semisimple, we fix \( \{e_\alpha \in \mathfrak{g}_\alpha \mid e_\alpha \neq 0 \ \text{and} \ \alpha \in \Delta \} \sqcup \{\alpha_i^\vee\}_{i \in \mathcal{I}} \) to be the Chevalley basis of \( \mathfrak{g} \), and also we repeatedly use without mention the following fact.

**Fact:** Let \( \mathfrak{g} \) be semisimple, \( \alpha \in \Pi \), and \( \gamma \in \Delta \) such that \( \alpha + \gamma \in \Delta \). Then \( [e_\alpha, \mathfrak{g}_\gamma] = \mathfrak{g}_{\alpha + \gamma} \).

Observe that this fact holds true by [15] Proposition 3.6 (iv) and the basic fact that the root spaces of \( \mathfrak{g} \) are one dimensional.

**Lemma 4.8.** Let \( \mathfrak{g} \) be semisimple, and \( \emptyset \neq I \subset \mathcal{I} \). Suppose \( \beta \prec \beta' \in \Delta^+ \) such that \( \text{ht}_I(\beta' - \beta) > 0 \). Then there exists a sequence of roots \( \gamma_i \in \Delta_{I,1} \), \( 1 \leq i \leq n = \text{ht}_I(\beta' - \beta) \), such that

\[
[e_{\gamma_1}, \cdots, e_{\gamma_i}, e_{\beta}, \cdots] \neq 0 \quad \text{and} \quad \beta < \cdots < \beta + \sum_{j=1}^i \gamma_j < \cdots < \beta + \sum_{j=1}^n \gamma_j \leq \beta' \in \Delta \ \forall i.
\]
Proof. We prove the lemma by induction on $\text{ht}(\beta' - \beta)$. Base step: $\text{ht}(\beta' - \beta) = 1$ forces $\beta' - \beta \in \Pi_I$, and we have $[e_{\beta' - \beta}, g_\beta] = g_{\beta'}$ as desired.

Induction step: $\text{ht}(\beta' - \beta) > 1$, consider $0 < (\beta' - \beta, \beta' - \beta) = (\beta' - \beta, \beta') - (\beta' - \beta, \beta)$, giving the following two cases.

1. $(\beta' - \beta, \beta') > 0$: There must exist a simple root $\alpha' < \beta' - \beta$ such that $(\alpha', \beta') > 0$, implying $\beta' - \alpha' \in \Delta^+$.

If $\text{ht}_I(\beta' - \alpha') > \text{ht}_I(\beta)$, then apply the induction hypothesis to the pair $\beta < \beta' - \alpha'$ to get a chain of roots $\beta < \cdots < \beta(1) \leq \beta' - \alpha' < \beta$ as in the statement. We are done if $\alpha' \notin \Pi_I$. Otherwise, by applying the induction hypothesis further to the pair $\beta(1) < \beta'$, we will be done.

So, we assume now that $\text{ht}_I(\beta' - \alpha') = \text{ht}_I(\beta)$, which implies that $\text{ht}_I(\beta' - \beta) = 1$ and $\alpha' \in \Pi_I$. Consider $\beta < \beta' - \alpha' \in \Delta$. By Theorem C [C1] and [15, Proposition 3.6], we get a sequence of simple roots $\alpha_1, \ldots, \alpha_m \in \Pi_I$, $m = \text{ht}(\beta' - \beta) - 1$, such that

$$y := [e_{\alpha'}, [e_{\alpha_m}, [\cdots, [e_{\alpha_1}, e_{\beta}] \cdots]]] \in g_{\beta'}$$

and $y \neq 0$.

If $\beta' - \alpha_j \in \Delta$ for some $j \in [m]$, then we are done by the induction hypothesis applied to the pair $\beta < \beta' - \alpha_j$. Else, observe by the Jacobi identity that

$$[e_{\alpha'}, [e_{\alpha_m}, [\cdots, [e_{\alpha_1}, e_{\beta}] \cdots]]] = -[e_{\alpha_m}, [e_{\alpha'}, [\cdots, [e_{\alpha_1}, e_{\beta}] \cdots]]] + [e_{\alpha_m}, [e_{\alpha'}, [\cdots, [e_{\alpha_1}, e_{\beta}] \cdots]]]_0$$

$$= + [e_{\alpha_{m-1}}, [e_{\alpha_m}, [e_{\alpha'}, [\cdots, [e_{\alpha_1}, e_{\beta}] \cdots]]] + 0$$

$$\vdots$$

$$= (-1)^m [e_{\alpha_1}, [\cdots, [e_{\alpha_m}, e_{\alpha'}] \cdots]], e_{\beta}] + 0.$$

By the assumption $\beta - \alpha_j \notin \Delta \forall j \in [m]$, all the second terms in each line on the right hand side of the above equations are zero. Observe that $z$ (defined in the last line of the above equations) belongs to $g_{\beta' - \beta}$. As $y \neq 0$, we must have $[z, e_{\beta}] \neq 0 \iff z \neq 0 \iff \beta' - \beta \in \Delta$. Now $\text{ht}_I(\beta' - \beta) = 1$ implies $\beta' - \beta \in \Delta_{I,1}$. Putting all of this together, we have $\beta' - \beta \in \Delta_{I,1}$ and $[g_{\beta' - \beta}, e_{\beta}] \neq 0$ as desired, completing the proof in this case.

2. $(\beta' - \beta, \beta') \leq 0$: In this case we must have $(\beta' - \beta, \beta) < 0$. This implies there exists a simple root $\alpha < \beta' - \beta$ such that $(\alpha, \beta) < 0 \iff \beta + \alpha \in \Delta \iff [e_{\alpha}, e_{\beta}] \neq 0$.

If $\alpha \in \Pi_I$ and $\text{ht}_I(\beta' - \beta) = 1$, then we are done by $[e_{\alpha}, e_{\beta}] \neq 0$. Else if $\alpha \in \Pi_I$ and $\text{ht}_I(\beta' - \beta) > 1$, then we are done by $[e_{\alpha}, e_{\beta}] \neq 0$, and by the induction hypothesis applied to the pair $\beta + \alpha < \beta'$. So, we assume now that $\alpha \notin \Pi_I$. By the induction hypothesis applied to the pair $\beta + \alpha < \beta'$, we get a root $\gamma \in \Delta_{I,1}$ such that $\gamma < \beta' - (\beta + \alpha)$ and $[e_{\gamma}, [e_{\alpha}, e_{\beta}]] \neq 0$. By the Jacobi identity, we have

$$0 \neq [e_{\gamma}, [e_{\alpha}, e_{\beta}] = [e_{\gamma}, e_{\alpha}], e_{\beta}] + [e_{\alpha}, [e_{\gamma}, e_{\beta}]] =: z_1 + z_2.$$

If $\text{ht}_I(\beta' - \beta) = 1$, then we are done as either $[z_1, e_{\beta}]$ or $z_2$ in the above equation must be non-zero. Else, for the same reason, we are done by the induction hypothesis applied to either $\beta + \alpha + \gamma < \beta'$ or respectively $\beta + \gamma < \beta'$.

$\Box$
Lemma 4.9. Let \( g \) be semisimple and \( \emptyset \neq I \subset \mathcal{I} \). Suppose \( \beta < \beta' \in \Delta^+ \) such that \( \text{ht}_I(\beta' - \beta) > 0 \). Then there exists a sequence of roots \( \gamma'_i \in \Delta_{I,1}, 1 \leq i \leq n = \text{ht}_I(\beta' - \beta) \), such that

\[
\beta'' := \beta' - \sum_{j=1}^{n} \gamma'_j \in \Delta, \quad \left[ e_{\alpha'}, \cdots, [e_{\gamma'_1}, e_{\beta'}] \cdots \right] \neq 0
\]

and \( \beta \leq \beta'' \prec \cdots \prec \beta' - \sum_{j=1}^{i} \gamma'_j \prec \cdots \prec \beta' \in \Delta \ \forall i \).

Proof. We prove the lemma by induction on \( \text{ht}(\beta' - \beta) \). Base step: \( \text{ht}(\beta' - \beta) = 1 \) forces \( \beta' - \beta \in \Pi_I \), and we have \( [e_{\beta' - \beta}, g_\beta] = g_{\beta'} \) as desired.

Induction step: \( \text{ht}(\beta' - \beta) > 1 \). Now \( 0 < (\beta' - \beta, \beta' - \beta) = (\beta' - \beta', -\beta') - (\beta' - \beta, \beta) \), which leads to the following two cases.

1. \( (\beta' - \beta, \beta) < 0 \): There must exist a simple root \( \alpha < \beta' - \beta \) such that \( (\beta, \alpha) < 0 \). This implies \( [e_\alpha, e_\beta] \neq 0 \), implying \( \beta + \alpha \in \Delta \). If \( \text{ht}_I(\beta + \alpha) < \text{ht}_I(\beta') \), then apply the induction hypothesis to the pair \( \beta + \alpha < \beta' \) to get a chain of roots \( \beta + \alpha < (\beta + \alpha)^{(1)} < \cdots < \beta' \) as in the statement. We are done if \( \alpha \notin \Pi_I \). Otherwise, by applying the induction hypothesis further to the pair \( \beta < (\beta + \alpha)^{(1)} \), we will be done.

So, we assume now that \( \text{ht}_I(\beta + \alpha) = \text{ht}_I(\beta') \), which implies that \( \text{ht}_I(\beta' - \beta) = 1 \) and \( \alpha \in \Pi_I \). Consider \( \beta + \alpha < \beta' \). By Theorem C(1) and [15, Proposition 3.6], we get a sequence of simple roots \( \alpha_1, \ldots, \alpha_m \in \Pi_I^c \), where \( m = \text{ht}(\beta' - \beta) - 1 \), such that

\[
y := \left[ e_{\alpha_m}, \cdots, [e_{\alpha_1}, [e_\alpha, e_\beta]] \cdots \right] \in g_{\beta'} \text{ and } y \neq 0.
\]

If \( \beta + \alpha_j \in \Delta \) for some \( j \in [m] \), then we are done by the induction hypothesis applied to the pair \( \beta + \alpha_j \prec \beta' \). Else if \( \beta + \alpha_j \notin \Delta \ \forall j \in [m] \), then \( [e_\alpha, e_\beta] = 0 \ \forall j \in [m] \), and by the Jacobi identity we have

\[
\left[ e_{\alpha_m}, \cdots, [e_{\alpha_1}, [e_\alpha, e_\beta]] \cdots \right] = [e_{\alpha_m}, \cdots, [e_{\alpha_1}, e_\beta]] \cdots = \left[ e_{\alpha_m}, \cdots, [e_{\alpha_1}, e_\beta] \right] e_\beta
\]

Observe that \( z \) (defined in the last line of the above equation) belongs to \( g_{\beta' - \beta} \). As \( y \neq 0 \), we must have \( [z, e_\beta] \neq 0 \implies z \neq 0 \implies \beta' - \beta \in \Delta \). Moreover, \( \text{ht}_I(\beta' - \beta) = 1 \implies \beta' - \beta \in \Delta_{I,1} \). Thus, we have \( \beta' - \beta \in \Delta_{I,1} \) and \( [g_{\beta' - \beta}, e_\beta] \neq 0 \) as desired, completing the proof in this case.

2. \( (\beta' - \beta, \beta) \geq 0 \): In this case we must have \( (\beta' - \beta, \beta') > 0 \). Thus, there exists a simple root \( \alpha' < \beta' - \beta \) such that \( (\beta', \alpha') > 0 \). This implies \( \beta' - \alpha' \in \Delta \), implying \( [e_{\alpha'}, e_{\beta' - \alpha}] \neq 0 \).

If \( \alpha' \in \Pi_I \) and \( \text{ht}_I(\beta' - \beta) = 1 \), then we are done by \( [e_{\alpha'}, e_{\beta' - \alpha}] \neq 0 \) once we set \( \beta'' = \beta' - \alpha' \).

Else if \( \alpha \in \Pi_I \) and \( \text{ht}_I(\beta' - \beta) > 1 \), then we are done by \( [e_{\alpha'}, e_{\beta' - \beta}] \neq 0 \), and by the induction hypothesis applied to the pair \( \beta < \beta' - \alpha' \).

So, we assume now that \( \alpha' \notin \Pi_I \). By the induction hypothesis applied to the pair \( \beta < \beta' - \alpha' \) we get a root \( \gamma' \in \Delta_{I,1} \) such that \( \beta < \beta' - \alpha' - \gamma' \in \Delta^+ \) and \( [e_{\gamma'}, e_{\beta' - \alpha' - \gamma'}] \neq 0 \). By the Jacobi identity, we have

\[
0 \neq [e_{\alpha'}, [e_{\gamma'}, e_{\beta' - \alpha' - \gamma'}]] = \left[ e_{\gamma'}, e_{\beta' - \alpha' - \gamma'} \right] + \left[ e_{\gamma'}, [e_{\beta' - \alpha'}, e_{\beta' - \alpha' - \gamma'}] \right].
\]
If \( h_I(\beta' - \beta) = 1 \), then as either \([z_1, e_{\beta' - \alpha' - \gamma'}]\) or \([e_{\gamma'}, z_2]\) in the above equation must be non-zero, we will be done once we set \( \beta'' = \beta - \alpha' - \gamma' \) or respectively \( \beta' - \gamma' \). Else, as either \([z_1, e_{\beta' - \alpha' - \gamma'}]\) or \([e_{\gamma'}, z_2]\) is non-zero, we are done by the induction hypothesis applied to either \( \beta < \beta' - \alpha' - \gamma' \) or respectively \( \beta < \beta' - \gamma' \).

We are now able to discuss “moving between comparable roots in steps of \( \Delta_{I,1} \)” when both the roots are not necessarily positive.

**Proof of Proposition 4.7** We prove the proposition in cases below.

1) \( \beta = 0 \) or \( \beta' = 0 \): In this case the proposition follows by the parabolic-PSP applied to \( \beta' \) or \( \beta \) respectively.

2) \( \beta \geq 0 \): In this case a) and b) directly follow by Lemmas 4.8 and 4.9 respectively.

3) \( \beta' \geq 0 \): In this case a) and b) once again directly follow by respectively applying Lemmas 4.9 and 4.8 to the pair \( \gamma, \lambda \) under some finiteness conditions, see Maximal property 5.2 below. We begin by recalling

4) \( \beta \leq 0 \) \( \leq \beta' \) and \( h_I(\beta) < 0 \): We prove a); the proof of b) is similar. By the parabolic-PSP we get a chain of roots between \( \beta < 0 \). When \( h_I(\beta') = 0 \), we are done by the previous line. Else when \( h_I(\beta') > 0 \), by the parabolic-PSP further applied to \( \lambda < \beta' \) we will be done. Notice that only in this case do we use the assumption made in part a), and this assumption is made in view of Remark 4.10 (1) part (ii) below.

In view of Remark 4.10 below, observe that Propositions 4.7 and 4.6 prove the parabolic-generalizations of (C1) and (C2) to the “best possible” extent.

**Remark 4.10.** (1) Let \( g \) be of type \( A_3 \), and \( I = \{1, 2, 3\} \) where nodes 1, 3 are leaves in the Dynkin diagram. Note that \( g \simeq L(\alpha_1 + \alpha_2 + \alpha_3) \) as \( g \)-modules.

(i) Let \( I = \{1\} \) and consider \( \alpha_1 + \alpha_2 + \alpha_3 \geq \alpha_2 \). Check that there does not exist a root \( \gamma \in \Delta_{I,1} \) such that \( \alpha_1 + \alpha_2 + \alpha_3 - \gamma = \alpha_2 \in wt_g \) or \( \alpha_1 + \alpha_2 + \alpha_3 = \alpha_2 + \gamma \in wt_g \).

(ii) Let \( I_1 = \{1, 2\} \) and \( I_2 = \{2, 3\} \), and consider \( \alpha_3 \geq \alpha_1 \). Check that there does not exist a root \( \gamma_1 \in \Delta_{I,1} \) such that \( \alpha_3 - \gamma_1 \geq - \alpha_1 \in wt_g \). Similarly, check that there does not exist a root \( \gamma_2 \in \Delta_{I,1} \) such that \( \alpha_3 \geq - \alpha_1 + \gamma_2 \in wt_g \).

(2) Observe that one can construct similar examples as in (1) when \( g \) is any semisimple Lie algebra of rank \( \geq 3 \), and even when \( g \) is of type \( A_1 \times A_1 \).

(3) Let \( g \) be an affine Kac–Moody algebra of type \( X^{(r)}_\ell \) with rank \( \ell \geq 3 \), \( r \in \{1, 2, 3\} \) and \( I = \{0, 1, \cdots, \ell\} \), see [13 Table Aff 1–3]. Let \( i, j \in I \setminus \{0\} \) such that \( i \) and \( j \) are not connected by any edge in the Dynkin diagram. Set \( I_1 = \{j\} \) and \( I_2 = \{i\} \). Let \( \delta \) be the smallest positive imaginary root.

Consider \( r\delta + \alpha_i > r\delta - \alpha_j \in \Delta^+ \), note that they are real roots by [15 Proposition 6.3]. This result also implies that there does not exist a root \( \gamma_1 \in \Delta_{I,1} \) such that \( r\delta + \alpha_i - \gamma_1 > r\delta - \alpha_j \). Similarly, check that there does not exist a root \( \gamma_2 \in \Delta_{I,1} \) such that \( r\delta + \alpha_i > r\delta - \alpha_j + \gamma_2 \). Thus, Proposition 4.7 cannot be extended to affine root systems, even to move between two comparable positive roots.

5. DESCRIPTION OF \( \mathcal{P}(\lambda, J) \)

5.1. A maximal property of \( \mathcal{P}(\lambda, J) \). In this subsection, we prove a “maximal property” satisfied by \( \mathcal{P}(\lambda, J) \) under some finiteness conditions, see Maximal property 5.2 below. We begin by recalling the relevant notation.

Fix \( \lambda \in h^* \) and \( J \subset J_\lambda \) throughout this subsection. Recall from equations (2.4) and (2.10) that

\[
J_\lambda := \{ j \in I \mid \langle \lambda, \alpha_j' \rangle \in \mathbb{Z}_{\geq 0}\}, \quad W_J := \langle s_j \mid j \in J \rangle, W_0 := \{e\}, \quad J_\lambda' := \{ j \in I \mid \langle \lambda, \alpha_j' \rangle \in \mathbb{R}_{\geq 0}\}, \quad \mathcal{P}(\lambda, J) := \text{conv}_\mathbb{R} W_J \lambda - \mathbb{R}_{\geq 0}(\Delta^+ \setminus \Delta_J^+) .
\]
Lemma 5.3. Let \( \mathfrak{g} \) be a Kac–Moody algebra, and fix \( \lambda \in \mathfrak{h}^* \) and \( J \subset J' \). Suppose \( W_J \) is finite. Then \( \mu \in \mathcal{P}(\lambda, J) \iff \omega \mu < \lambda \forall \omega \in W_J \).

Proof. The proposition holds true if \( J = \emptyset \) as \( \mathcal{P}(\lambda, \emptyset) = \lambda - \mathbb{R}_{\geq 0} \Delta^+ = \lambda - \mathbb{R}_{\geq 0} \Pi \). So, we assume that \( J \neq \emptyset \). Observe by Corollary 5.7 below that the finiteness of \( W_J \) implies that \( \Delta_{J',1} \) is finite. It can be easily seen that the finiteness of \( W_J \) and \( \Delta_{J',1} \), and the Minkowski difference formula for \( \mathcal{P}(\lambda, J) \) together imply the closedness of \( \mathcal{P}(\lambda, J) \) in the usual Euclidean topology on \( \mathfrak{h}^* \).

Fix \( \mu \not\leq \lambda \in \mathfrak{h}^* \). As \( \mathfrak{g}_1 \) is semisimple, assume without loss of generality that \( \mu \) is the \( J \)-dominant element in \( W_J \mu \), i.e. \( (\mu, \alpha_J^i) \geq 0 \forall j \in J \). Observe that the forward implication of the proposition is obvious, as \( \mathcal{P}(\lambda, J) \) is \( W_J \)-invariant. To show the reverse implication, consider

\[
\epsilon := \inf \{ \text{ht}(\lambda' - \mu) \mid \mu \prec \lambda' \in \mathcal{P}(\lambda, J) \}. 
\]

As \( \eta \in \mathcal{P}(\lambda, J) \iff \eta - \sum_{i \in J^c} b_i \alpha_i \in \mathcal{P}(\lambda, J) \forall b_i \in \mathbb{R}_{\geq 0} \), we also have

\[
\epsilon = \inf \{ \text{ht}(\lambda' - \mu) \mid \mu \prec \lambda' \in \mathcal{P}(\lambda, J) \text{ and } \lambda' - \mu \in \mathbb{R}_{\geq 0} \Pi_J \}. 
\]

Fix a sequence \( \xi_n > \mu \in \mathcal{P}(\lambda, J) \) such that \( \xi_n - \mu \in \mathbb{R}_{\geq 0} \Pi_J \) and \( \text{ht}(\xi_n - \mu) \to \epsilon \). As \( \xi_n \) lie in the compact set \( \lambda - [0, \text{ht}(\lambda - \mu)\Pi] \cap \mathcal{P}(\lambda, J) \), we get a subsequence \( \xi_{n_k} \) and a point \( \xi \in \mathcal{P}(\lambda, J) \) such that \( \xi_{n_k} \to \xi \in \mathcal{P}(\lambda, J) \). Observe that \( \text{ht}(\xi - \mu) = \epsilon \). Observe that the result follows once we prove \( \epsilon = 0 \). On the contrary, suppose \( \epsilon > 0 \).

Let \( (\cdot, \cdot) \) denote the positive definite (Killing) form on \( \mathfrak{h}_J^* \times \mathfrak{h}_J^* \). Write \( \xi - \mu = \sum_{j \in J} c_j \alpha_j \), and define

\[
(\xi - \mu)^\vee := \sum_{j \in J} c_j^\prime \alpha_j^\vee, \quad \text{where } c_j \text{ and } c_j^\prime = \frac{(\alpha_j, \alpha_j)}{\langle \xi, \xi \rangle} c_j \in \mathbb{R}_{\geq 0} \forall j \in J. 
\]

Note that

\[
\text{for each } j \in J \text{ } c_j^\prime = 0 \iff c_j = 0, \quad 2 = \langle \xi - \mu, (\xi - \mu)^\vee \rangle = \langle \xi, (\xi - \mu)^\vee \rangle - \langle \mu, (\xi - \mu)^\vee \rangle. 
\]

As \( \mu \) is \( J \)-dominant, we have \( \langle \mu, (\xi - \mu)^\vee \rangle \geq 0 \), forcing \( \langle \xi, (\xi - \mu)^\vee \rangle > 0 \). This implies that there exists \( j_0 \in J \) such that \( c_{j_0} > 0 \) (equivalently \( c_{j_0}^\prime > 0 \)) and \( \langle \xi, \alpha_{j_0}^\vee \rangle > 0 \). Now, define \( r := \min\{1, \frac{c_{j_0}}{\langle \xi, \alpha_{j_0}^\vee \rangle} \} \).

Observe that

\[
(1 - r)\xi + rs_{j_0}(\xi) \in \mathcal{P}(\lambda, J) \quad \text{and} \quad \text{ht}(\text{ht}(1 - r)\xi + rs_{j_0}(\xi) - \mu) < \text{ht}(\xi - \mu) = \epsilon.
\]

This contradicts the minimality of \( \epsilon \). Hence, \( \epsilon = 0 \) must be 0.

\[\square\]

Maximal property 5.2. Let \( \mathfrak{g}, \lambda, J \in \mathcal{P}(\lambda, J) \) be as in Proposition 5.1. Suppose \( X \subset \mathfrak{h}^* \) such that \( X \subset \lambda - \mathbb{R}_{\geq 0} \Pi \) and \( W_J X = X \). Then \( X \subset \mathcal{P}(\lambda, J) \).

Note 2: Notice that when either (i) \( \mathfrak{g} \) is of affine type and \( |J^c| \geq 1 \) or (ii) \( \mathfrak{g} \) is hyperbolic and \( |J^c| \geq 2 \), \( \mathcal{P}(\lambda, J) \) defined over \( \mathfrak{g} \) has the above maximal property.

It will be interesting to check the extent to which the results of this subsection may be extended.

5.2. Extremal rays of \( \mathcal{P}(\lambda, J) \): proof of Theorem B

In this subsection, we prove Theorem B which gives the complete description of the extremal rays of \( \mathcal{P}(\lambda, J) \) or equivalently the extremal rays at \( \lambda \) of \( \mathcal{P}(\lambda, J) \). We first prove the following Lemma 5.3 which gives the minimal generating set for \( \text{conv}_{\mathbb{R}_{\geq 0}} \Delta_{J,1} \), and which is also needed in the proof of Theorem B. We say in this subsection that a convex set \( C \) in a real vector space is generated by a subset \( S \subseteq C \) if \( \text{conv}_{\mathbb{R}}(S) = C \).

For \( I \subset \mathcal{I} \) and \( n \in \mathbb{Z}_{\geq 0} \), define

\[
S_{I,n} := \{ \beta \in \Delta_{I,n} \mid \exists \alpha \in \Delta_{I,n} \text{ such that } \alpha \preceq \beta \}. 
\]

We call the elements of \( S_{I,n} \) the minimal elements of \( \Delta_{I,n} \).

Lemma 5.3. Let \( \mathfrak{g} \) be a Kac–Moody algebra, and fix \( I \subset \mathcal{I} \) and \( n \in \mathbb{Z}_{\geq 0} \). Then:

(a) \( \text{conv}_{\mathbb{R}} \Delta_{I,n} \) is generated by \( W_I S_{I,n} \).
Lemma 5.4. Observe then that we can write \( \beta \subseteq W \). Throughout the proof, we only deal with the extremal rays of \( \Delta \). (c) follows from (b) by noting: (i) \( \Delta \) is the only minimal element in \( \Delta \), we must have \( \omega \alpha \leq \alpha \), thus \( \omega \alpha \subseteq W \), contradicting the choice of \( \beta \). By Theorem C (Cl) applied to \( \omega \alpha \), we have \( \beta = \alpha \omega \alpha \subseteq \Delta \), for some \( j_0 \in I^c \). If \( \langle \beta, \alpha_{j_0} \rangle > 0 \), then as \( \text{ht}(s_{j_0} \beta) < \text{ht}(\beta) \) we have \( s_{j_0} \beta \in \text{conv}_R W_{I^n}S_{I,n} \). This implies \( \beta = s_{j_0} \beta \subseteq \text{conv}_R W_{I^n}S_{I,n} \) (by the \( W_{c} \)-invariance of \( \text{conv}_R W_{I^n}S_{I,n} \)), contradicting the choice of \( \beta \). So, we assume now \( \langle \beta, \alpha_{j_0} \rangle \leq 0 \). This implies \( \epsilon' := -1 - \langle \beta, \alpha_{j_0} \rangle \geq 1 \). As \( \text{ht}(\beta - \alpha_{j_0}) < \text{ht}(\beta) \) we have

\[
\beta - \alpha_{j_0} = \text{conv}_R W_{I^n}S_{I,n} \implies s_{j_0} \beta \subseteq \text{conv}_R W_{I^n}S_{I,n}.
\]

Observe then that we can write

\[
\beta = \frac{\epsilon'}{\epsilon' + 1}(\beta - \alpha_{j_0}) + \frac{1}{\epsilon' + 1}s_{j_0} \beta.
\]

This implies \( \beta \subseteq \text{conv}_R W_{I^n}S_{I,n} \), once again contradicting the choice of \( \beta \). Thus, \( \text{conv}_R \Delta_{I,n} = \text{conv}_R W_{I^n}S_{I,n} \).

We now prove (b). It can be similarly proved as in (a) that \( W_{I^n} \alpha \subseteq \text{conv}_R \Delta_{\alpha,J} \), by noting \( \alpha \subseteq \text{conv}_R \Delta_{\alpha,J} \). So, we only have to show that \( W_{I^n} \alpha \subseteq \text{conv}_R \Delta_{\alpha,J} \).

Suppose \( \omega \alpha = r_1 \omega_1 \alpha + \cdots + r_k \omega_k \alpha \) for some \( r_1, \ldots, r_k \in \mathbb{R}_{>0} \) and \( \omega, \omega_1, \ldots, \omega_k \in W \).

Then we have

\[
\alpha = r_1 \omega_1 \alpha + \cdots + r_k \omega_k \alpha \implies \text{supp}(\omega \alpha) \subseteq \text{supp}(\alpha) \forall 1 \leq i \leq k, \text{ as } \omega \omega_k = \Delta_{\alpha,J}.
\]

Observe then that, as \( \omega \omega_i \alpha \leq \alpha \), we must have \( \omega \alpha = \omega_i \alpha \forall 1 \leq i \leq k \). This proves the minimality.

(c) follows from (b) by noting: (i) \( \Delta_{I,1} = \bigcup_{i \in I} \Delta_{\alpha,J} \) respectively, and (ii) if

\[
\omega_i \alpha_i = r_1 \omega_i \alpha_i + \cdots + r_l \omega_i \alpha_i \text{ for some } r_1, \ldots, r_l \in \mathbb{R}_{>0}, \text{ and } \omega', \omega_1', \ldots, \omega_l' \in W_{I^n},
\]

then \( i = i_0 \) and \( \omega_i \alpha_i = \omega_i \alpha_i \forall 1 \leq t \leq l \). This also proves that the cone \( \mathbb{R}_{>0}(\Delta^+ \setminus \Delta^+_I) = \mathbb{R}_{>0}\Delta_{I,1} \) is minimally generated by \( W_{I^n} \).

We are now able to prove Theorem B.

Proof of Theorem B. We assume that \( \emptyset \not= J \subseteq I \), as the theorem obviously holds otherwise. Throughout the proof, we only deal with the extremal rays of \( \mathcal{P}(\lambda, J) \) at \( \lambda \), as all other extremal rays are \( W_{J^n} \)-conjugates of these. Observe that each ray \( \lambda - \mathbb{R}_{>0} \alpha_i \) is clearly an extremal ray \( \forall i \not\in I \setminus J \). We first prove the following useful lemma about the orbits under the parabolic subgroups of Weyl groups. For \( \lambda \in \mathfrak{h}^* \) and \( J \subseteq J_\lambda \), recall from equation (2.4) that \( J_0 := \{ j_0 \in J \mid \langle \lambda, \alpha_{j_0} \rangle = 0 \} \).

Lemma 5.4. Let \( \lambda \not= \mu \in \text{conv}_R W_{I^n}\lambda \). Then:

(a) \( \text{ht}_{I^n,J_0}(\lambda - \mu) > 0 \).

(b) The ray originating from \( \lambda \) and containing \( \mu \), which is precisely \( \lambda - \mathbb{R}_{>0}(\lambda - \mu) \), cannot be an extremal ray of \( \text{conv}_R W_{I^n}\lambda \).
Proof. For (a), we prove that \( \omega \in W_j \setminus W_{j_0} \implies \text{ht}_{J \setminus J_0}(\lambda - \omega \lambda) > 0 \) by induction on \( \ell(\omega) \). Observe that this proves (a). Base step: \( \ell(\omega) = 1 \), and we must have \( \omega = s_j \) for some \( j \in J \setminus J_0 \). By the definition of \( J_0 \), the assertion is immediate.

Induction step: Let \( k := \ell(\omega) > 1 \), and let \( s_{j_1} \cdots s_{j_k} = \prod_{t=1}^k s_{j_t} \) be a reduced expression of \( \omega \) for some \( j_1, \ldots, j_k \in J \) not necessarily distinct. As \( \omega \in W_j \setminus W_{j_0} \), there must exist some \( t \in [k] \) such that \( j_t \in J \setminus J_0 \). Assume without loss of generality that \( j_k \in J \setminus J_0 \) and write

\[
\omega \lambda = (\prod_{t=1}^{k-1} s_{j_t}) \lambda - (\lambda, \alpha_{j_k}^\vee)(\prod_{t=1}^{k-1} s_{j_t}) \alpha_{j_k}.
\]

If \( j_1, \ldots, j_{k-1} \in J_0 \), then \( \omega \lambda = \lambda - (\prod_{t=1}^{k-1} s_{j_t}) \alpha_{j_k} \in \lambda - \Delta_{(j_k),1} \), and we are done. Else, by the induction hypothesis applied to \( \ell(\prod_{t=1}^{k-1} s_{j_t}) = \ell(\omega) - 1 \), we have \( \text{ht}_{J_0}(\lambda - (\prod_{t=1}^{k-1} s_{j_t}) \lambda) > 0 \). As \( \prod_{t=1}^k s_{j_t} \) is reduced, by \([15]\) Lemma 3.11, we also have \( (\prod_{t=1}^{k-1} s_{j_t}) \alpha_{j_k} > 0 \). Thus, \( \text{ht}_{J_0}(\lambda - w \lambda) > 0 \) as required.

Let \( \mu = \lambda - \gamma \) for some \( 0 \neq \gamma \in \mathbb{R}_{\geq 0} \Pi_J \). For (b), we assume \( \lambda - \mathbb{R}_{\geq 0} \gamma \) to be an extremal ray of \( \text{conv}_R W_J \lambda \) and exhibit a contradiction. If

\[
\mu = c_1 w_1 \lambda + \cdots + c_r w_r \lambda \text{ for some } w_i \in W_J \text{ and } c_i \in \mathbb{R}_{\geq 0} \text{ such that } \sum_{i=1}^r c_i = 1,
\]

then \( w_i \lambda \in \lambda - \mathbb{R}_{\geq 0} \gamma \forall 1 \leq i \leq r \) (by the definition of an extremal ray). By the previous line, we may assume that the element \( \mu \) we started with lies in \( W_J \lambda \). Let \( \mu = w \lambda \) for some \( w \in W_J \lambda \) and consider \( \lambda - 2\gamma \in \lambda - \mathbb{R}_{\geq 0} \gamma \subset \text{conv}_R W_J \lambda \). Observe that

\[
(5.2) \quad \mu = w \lambda = \frac{1}{2}(\lambda) + \frac{1}{2}(\lambda - 2\gamma) \implies \lambda = \frac{1}{2}(w^{-1}\lambda) + \frac{1}{2}w^{-1}(\lambda - 2\gamma).
\]

Recall that \( x < \lambda \forall x \in \text{conv}_R W_J \lambda \), and \( \text{conv}_R W_J \lambda \) is \( W_J \)-invariant. Observe that the previous line and equation \((5.2)\) force \( w^{-1}\lambda = w^{-1}(\lambda - 2\gamma) = \lambda \). This implies \( \mu = \lambda \) which is a contradiction. \(\square\)

We now continue with the proof of Theorem \([13]\) in steps below. Let \( I := \mathcal{I} \setminus J \).

**Step 1.** Observe firstly that, by the minimal description \( Z_{\geq 0}(\Delta^+ \setminus \Delta_j^+) = Z_{\geq 0} \Delta_{J,1} \) (by Theorem \([13](A2)\)) and by the proof of Lemma \([13](c)\), the extremal rays of \( \lambda - \mathbb{R}_{\geq 0} (\Delta^+ \setminus \Delta_{J}^+) \) are precisely \( \lambda - \mathbb{R}_{\geq 0} W_J \Pi_J \). Let \( \mu = \mu_1 - \mu_2 \in \mathcal{P}(\lambda, J) \) for some \( \mu_1 \in \text{conv}_R W_J \lambda \) and \( \mu_2 \in \mathbb{R}_{\geq 0} \Delta_{J,1} \). We show that if \( \mu_1 \neq \lambda \), then \( \lambda - \mathbb{R}_{\geq 0}(\lambda - \mu) \) cannot be an extremal ray. Note that if \( \mu_1 \neq \lambda \) and \( \mu_2 = 0 \), this just follows by Lemma \([13](b)\). If \( \mu_1 \neq \lambda \) and \( \mu_2 \neq 0 \), then by the non-trivial relation

\[
\lambda - \frac{1}{2}(\lambda - \mu_1 + \mu_2) = \frac{1}{2}(\mu_1) + \frac{1}{2}(\lambda - \mu_2),
\]

observe that \( \lambda - \mathbb{R}_{\geq 0}(\lambda - \mu) \) cannot be an extremal ray. So, all we are left is to check and cut down the further redundancies in \( \lambda - \mathbb{R}_{\geq 0} W_J \Pi_I \) in \( \mathcal{P}(\lambda, J) \).

**Step 2.** Fix \( \beta \in \Delta_{I,1} \) such that \( \text{ht}_{J \setminus J_0}(\beta) > 0 \), we will show that \( \lambda - \mathbb{R}_{\geq 0} \beta \) is not an extremal ray. Define \( I_0 := I \cup (J \setminus J_0) \) and note that \( \text{ht}_{I_0}(\beta) \geq 2 \). By the parabolic-PSP (with \( I_0 \subset \mathcal{I} \)), we can write

\[
\beta = \gamma_1 + \cdots + \gamma_m \text{ for some } \gamma_1, \ldots, \gamma_m \in \Delta_{I_0,1}, \text{ where } m = \text{ht}_{I_0}(\beta).
\]

Here, we do not need the partial sums of \( \sum_{t=1}^m \gamma_t \) to be roots. So, we may assume that \( \gamma_1 \in \Delta_{I,1} \) and \( \gamma_2, \ldots, \gamma_m \in \Delta_{J \setminus J_0,1} \cap \Delta_{J}^+ \). Note that \( \gamma_2, \ldots, \gamma_m \in \text{conv}_R (W_{J_0} \Pi_J \setminus J_0) \) by Lemma \([13](c)\). So, we get
a system of equations

\[ \gamma_t = \sum_{r=1}^{r(t)} \epsilon(r, t) \omega(r, t) \alpha(r, t) \]

where \( 2 \leq t \leq m \), \( r(t) \in \mathbb{N} \), \( r(t) \) depends on \( t \),

\( (r, t) \) vary in a finite set \( \Omega := \bigcup_{t=2}^{m} [1, r(t)] \times \{ t \} \subset \mathbb{N} \times \mathbb{N} \),

\( \epsilon(r, t) > 0 \) and \( \sum_{r=1}^{r(t)} \epsilon(r, t) = 1 \) for each \( 2 \leq t \leq m \),

\( \omega(r, t) \in W_{J_0} \), and \( \alpha(r, t) \in \Pi_{J \setminus J_0} \forall (r, t) \in \Omega \).

Pick \( \delta > 0 \) such that \( \delta < \langle \lambda, \omega(r, t) \alpha(r, t) \rangle = \langle \lambda, \alpha(r, t) \rangle \) \( \forall (r, t) \in \Omega \). Such a positive \( \delta \) exists as \( \alpha(r, t) \in \Pi_{J \setminus J_0} \) and \( \Pi_{J \setminus J_0} \) is finite. Note that

\[ \lambda - \delta \omega(r, t) \alpha(r, t) \in \text{conv}_{\mathbb{R}} W_{J} \lambda \forall (r, t) \in \Omega, \]

as \( \lambda - \delta \omega(r, t) \alpha(r, t) = \frac{\langle \lambda, \omega(r, t) \alpha(r, t) \rangle - \delta}{\langle \lambda, \omega(r, t) \alpha(r, t) \rangle} - \frac{\delta}{\langle \lambda, \omega(r, t) \alpha(r, t) \rangle} \langle \omega(r, t) \alpha(r, t) \rangle. \)

Similarly, note that

\[ \lambda - \delta \epsilon(r, t) \omega(r, t) \alpha(r, t) \in \text{conv}_{\mathbb{R}} W_{J} \lambda \forall (r, t) \in \Omega \] as \( 0 < \epsilon(r, t) < 1 \) \( \forall (r, t) \in \Omega \).

Notice also that \( \lambda - \mathbb{R}_{>0} \gamma_1 \subset \mathcal{P}(\lambda, J) \). Observe then, by the non-trivial convex combination

\[ \lambda - \frac{\delta}{|\Omega| + 1} \beta = \sum_{(r, t)\in\Omega} \frac{1}{|\Omega| + 1} (\lambda - \delta \epsilon(r, t) \omega(r, t) \alpha(r, t)) + \frac{1}{|\Omega| + 1} (\lambda - \delta \gamma_1), \]

that \( \lambda - \mathbb{R}_{>0} \beta \) cannot be an extremal ray.

**Step 3.** We now show that if \( \lambda - \mathbb{R}_{>0} \gamma' \) is an extremal ray of \( \mathcal{P}(\lambda, J) \) for some \( \gamma' \in W_{J} \Pi_{I} \), then \( \gamma' \in W_{J_0} \Pi_{I} \). Fix an \( i \in I \), and let \( \gamma' = w \alpha_i \) for some \( w \in W_{J} \). Assume that \( \lambda - \mathbb{R}_{>0} \gamma' \) is an extremal ray of \( \mathcal{P}(\lambda, J) \). By step 2 we have \( \text{ht}_{J \setminus J_0}(\gamma') = 0 \) or equivalently \( \gamma' \in \Delta_{\alpha_i, J_0} \). By Lemma 5.3 (b) applied to \( \text{conv}_{\mathbb{R}} \Delta_{\alpha_i, J_0} \), there exist \( u_1, \ldots, u_M \in W_{J_0} \) such that

\[ \gamma' = w \alpha_i = h_1 u_1 \alpha_i + \cdots + h_M u_M \alpha_i \] for some \( h_1, \ldots, h_M \in \mathbb{R}_{>0} \) summing up to 1, \( M \in \mathbb{N} \).

This gives \( \alpha_i = h_1 w^{-1} u_1 \alpha_i + \cdots + h_M w^{-1} u_M \alpha_i \). As \( w^{-1} u_x \alpha_i \in \Delta_{\alpha_i, J} \forall 1 \leq x \leq M \), by the previous we must have

\[ \text{supp}(w^{-1} u_x \alpha_i) \subset \{ i \} \forall x \in [M] \implies w^{-1} u_x \alpha_i = \alpha_i \forall x \in [M]. \]

This implies \( \gamma' \in W_{J_0} \alpha_i \).

**Step 4.** Finally, we show that if \( \xi' \in W_{J_0} \Pi_{I} \), then \( \lambda - \mathbb{R}_{>0} \xi' \) is an extremal ray of \( \mathcal{P}(\lambda, J) \). Fix \( i' \in I \), \( \xi \in W_{J_0} \alpha_{i'} \) and \( r \in \mathbb{R}_{>0} \). Suppose

\[ \lambda - r \xi = \sum_{l=1}^{p} c_l w_{l} \lambda - \sum_{l=1}^{q} t_{l} \xi_{l} \]

for some \( c_l, t_l \in \mathbb{R}_{>0} \) such that \( \sum_{l=1}^{p} c_l = 1, w_{l} \in W_{J} \),

\( \xi_{l} \in \Delta_{l, 1} \forall 1 \leq l' \leq p \) and \( 1 \leq l \leq q \).

Note that \( \text{supp}_{J \setminus J_0}(\xi) = \emptyset \) and \( \text{supp}_{J}(\xi) = \{ i' \} \). Note also by the proof of Lemma 5.3 that if \( w_{l} \in W_{J} \setminus W_{J_0} \), then \( \text{ht}_{J \setminus J_0}(\lambda - w_{l} \lambda) > 0 \). In view of the previous line, as \( \text{ht}_{J \setminus J_0}(\xi) = 0 \), we must have \( w_{l} \in W_{J_0} \forall l' \). This implies \( w_{l} \lambda = \lambda \) and \( \xi_{l} \in \Delta_{l, 1} \forall l' \) for some \( l \). So, we have \( \xi = \sum_{l=1}^{p} \frac{c_l}{r} \xi_{l}. \) By Lemma 5.3 (c), we may assume that \( \xi_{l} \in W_{J_0} \alpha_{i'} \). Observe then by a similar argument as at the end of the proof of Lemma 5.3 (b) (which proves the minimality of \( W_{J} \alpha \) in Lemma 5.3 (b)), that...
we must have \( \xi_l = \xi \forall l \). This proves that \( \lambda - \mathbb{R}_{\geq 0} \xi \) is an extremal ray of \( \mathcal{P}(\lambda, J) \) at \( \lambda \).
Hence the proof of Theorem 5.4 is complete.

5.3. Finiteness of \( \Delta_{I,1} \). In this subsection, we give necessary and sufficient conditions for the sets \( \Delta_{\alpha, J} \) and \( \Delta_{I,1} \) to be finite, see Proposition 5.6 and Corollary 5.7 below. In this we invoke an interesting result of Deodhar [10], which proves the equivalence of the finiteness of a Coxeter group (in our situation a parabolic subgroup of the Weyl group of the Kac–Moody algebra \( \mathfrak{g} \)) and that of the quotients by its parabolic subgroups.
We first note the following observation which motivates, and proves (independently), Proposition 5.6 below in the special case when \( \mathfrak{g} \) is an affine Kac–Moody algebra.

**Observation 5.5.** Let \( \mathfrak{g} \) be an affine Kac-Moody algebra, \( \alpha \in \Delta^+ \), \( J \subset \mathcal{I} \setminus \text{supp}(\alpha) \), and \( \emptyset \neq I \subset \mathcal{I} \). Then:

1. \( \mathfrak{g}_I \) and \( \mathfrak{g}_J \) are semisimple, or equivalently every connected component of the Dynkin subdiagram on \( I \) and respectively on \( J \) is of finite type.
2. It can be easily checked by the explicit description of \( \Delta \) in Proposition 6.3 of Kac’s book [15] that \( \Delta \cap \mathbb{R}_{\geq 0} \xi \) is finite and the previous line imply that \( \Delta \cap \mathbb{R}_{\geq 0} \xi \) is finite.

**Proposition 5.6.** Let \( \mathfrak{g} \) be a Kac–Moody algebra, \( \alpha \in \Delta^+ \), \( J \subset \mathcal{I} \setminus \text{supp}(\alpha) \). Then the following are equivalent:

1. \( \Delta_{\alpha, J} \) is finite.
2. \( W_J \alpha \) is finite.
3. The quotient \( W_J \alpha \) modulo the parabolic subgroup \( \langle s_j \mid j \in J \text{ and } \langle \alpha, \alpha_j^\vee \rangle = 0 \rangle \) is finite.
4. There exists \( J' \subset J \) such that \( \mathfrak{g}_{J'} \) is semisimple and \( \Delta_{\alpha, J} = \Delta_{\alpha, J'} \).

When the Dynkin subdiagram on \( J \) is connected, \( J' \) in (4) is equal to \( J \).

**Proof.** Observe that the proposition trivially holds true if \( J = \emptyset \). So, we assume throughout the proof that \( J \neq \emptyset \). (1) \( \implies \) (2) is obvious as \( W_J \alpha \subset \Delta_{\alpha, J} \). It can be easily checked that (2) \( \iff \) (1) follows by Lemma 5.3 part (b) and the fact that \( \Delta_{\alpha, J} \) is a discrete subset of \( \text{conv}_{\mathbb{R}} W_J \alpha \). For (2) \( \iff \) (3), consider the action of \( W_J \alpha \) on \( \mathcal{J} \). Observe that \(-\alpha \in \Delta_{\alpha, J} \) is a fundamental chamber of the (dual) J-Tits cone—i.e. \( \langle \alpha, \alpha_i^\vee \rangle \geq 0 \forall j \in J \). So, by [15] Proposition 3.12 we must have that the isotropy group \( (W_J \alpha)_{\alpha} = \{ w \in W_J \alpha \mid w(-\alpha) = -\alpha \} \) (which is same as \( (W_J \alpha)_{\alpha} \) equals the parabolic subgroup \( \langle s_j \mid j \in J \text{ and } \langle \alpha, \alpha_j^\vee \rangle = 0 \rangle \). By the bijection which exists between the set of left cosets \( W_J / (W_J \alpha) \) and the orbit \( W_J \alpha \), (2) \( \iff \) (3) follows.

For (3) \( \implies \) (4), let \( J = J_1 \sqcup \cdots \sqcup J_l \) give the decomposition of the Dynkin subdiagram on \( J \) into connected components \( (l = 1 \text{ when the Dynkin subdiagram on } J \text{ is connected}) \). Let \( K \) be the subset of \( J \) such that \( W_K = (W_J)_{\alpha} \). Note that \( W_J \cong W_{J_1} \times \cdots \times W_{J_l} \text{ and } W_K \cong W_{J_1 \cap K} \times \cdots \times W_{J_l \cap K} \) (direct products of groups). Thus, there exists a bijection between \( W_J / (W_J \alpha) \) and the Cartesian product \( \prod_{t=1}^l (W_{J_t} / W_{J_t \cap K}) \) of the sets of left cosets \( W_{J_t} / W_{J_t \cap K} \forall 1 \leq t \leq l \). The assumption in (3) that \( W_{J_t} / W_{J_t \cap K} \) is finite and the previous line imply that \( W_{J_t} / W_{J_t \cap K} \) is finite \( \forall 1 \leq t \leq l \). By [10] Proposition 4.2 the Dynkin subdiagram on \( J_t \) must be of finite type whenever \( K \cap J_t \subseteq J_t \). Set \( J' \) to be the union of all the subsets \( J_t \) such that \( K \cap J_t \subseteq J_t \). Set \( \Delta_{\alpha, J_t} = \Delta_{\alpha, J'} \).

As in (4), suppose \( \Delta_{\alpha, J} = \Delta_{\alpha, J'} \) for some \( J' \subset J \) such that \( \mathfrak{g}_{J'} \) is semisimple. Then \( W_{J'} \alpha \) is finite. This implies \( W_{J'} \alpha \) is finite. By (2) \( \implies \) (1) (for \( J' \) in place of \( J \)) we get that \( \Delta_{\alpha, J'} \) is finite. Thus, \( \Delta_{\alpha, J} \) is finite. This proves (4) \( \implies \) (1), completing the proof.

**Corollary 5.7.** Let \( \mathfrak{g} \) be a Kac–Moody algebra, and \( \emptyset \neq I \subset \mathcal{I} \). Then the following are equivalent:

1. \( \Delta_{I,1} \) is finite
(2) $W_I \Pi_I$ is finite.

(3) For each $i \in I$ the quotient $W_I$ modulo the parabolic subgroup \langle $s_t \mid t \in I^c$ and $\langle \alpha_i, \alpha_J \rangle = 0$ \rangle is finite.

(4) For each $i \in I$ there exists $J_i \subset I^c$ such that $g_{J_i}$ is semisimple and $\Delta_{\alpha_i, I^c} = \Delta_{\alpha_i, J_i}$.

When the Dynkin subdiagram on $I^c$ is connected, $J_i$ in (4) is equal to $I^c \forall i \in I$.

Proof. The proof immediately follows from Proposition 5.6 by noting that $\Delta_{I, 1} = \bigsqcup_{i \in I} \Delta_{\alpha_i, I^c}$. □

In the rest of this subsection, we give some insight of the sets $\Delta_{\alpha, J}$ when $g$ is symmetrizable.

**Lemma 5.8.** Let $g$ be a symmetrizable Kac–Moody algebra with symmetric invariant form $(.,.)$. Fix a real root $\alpha \in \Delta^+$ and $J \subset I \setminus \supp(\alpha)$. Suppose $\beta \in \Delta_{\alpha, J}$. Then:

(a) $(\beta, \beta) \leq (\alpha, \alpha)$

(b) $(\beta, \beta) = (\alpha, \alpha) \Rightarrow \exists \omega \in W_{\supp(\beta - \alpha)}$ such that $\omega \alpha = \beta$.

Proof. We prove (a) by induction on $\text{ht}(\beta - \alpha)$. In the base step, $\text{ht}(\beta - \alpha) = 0$, and (a) is immediate as $\beta = \alpha$.

Induction step: $\text{ht}(\beta - \alpha) \geq 1$, let $\beta = \alpha + \sum_{j \in J} c_j \alpha_j$ for some $c_j \in \mathbb{Z}_{\geq 0}$. If $\beta$ is an imaginary root, then, as $\alpha$ is real, $(a)$ just follows by $(\beta, \beta) \leq 0 < (\alpha, \alpha)$. So, we assume throughout the proof that $\beta$ is real. If there exists some $j' \in J$ such that $(\beta, \alpha_{j'}) > 0$, then $s_{j'} \beta \not\in \Delta_{\alpha, J}$. As $\text{ht}(s_{j'} \beta - \alpha) < \text{ht}(\beta - \alpha)$ and $(s_{j'} \beta, s_{j'} \beta) = (\beta, \beta)$, we are done by the induction hypothesis. So, we also assume now that $(\beta, \alpha_{j'}) \leq 0 \forall j \in J$. The assumption in the previous line forces $(\beta, \alpha_{j'}) > 0$ as $(\beta, \beta) > 0$, $\beta$ being a real root. Thus $s_{\alpha}(\beta) = \beta - (\beta, \alpha_{j'}) \alpha \in \Delta^+$, and this implies that $(\beta, \alpha_{j'}) = 1$ (as $\alpha$ can be subtracted from $\beta$ at most once). Thus $\sum_{j \in J} c_j \alpha_j = \beta - \alpha = s_{\alpha}(\beta)$ which implies that $\beta - \alpha$ is real. By the previous line we have $\beta(\beta, \beta - \alpha) > 0$. Observe that (i) $(\beta - \alpha, \beta - \alpha) + (\beta - \alpha, \alpha) = (\sum_{j \in J} c_j \alpha_j, \beta) \leq 0$ and (ii) $(\alpha, \alpha) + (\alpha, \beta) - (\alpha, \beta - \alpha) > 0$. Finally, by the previous line observe that $(\alpha, \alpha) > (\beta - \alpha, \beta - \alpha) = (\beta, \beta)$. This completes the proof of (a).

(b) was proved by Carbone et al in [3, Proposition 6.4]. □

**Corollary 5.9.** Let $\mathfrak{g}, \alpha$ and $J$ be as in Lemma 5.8. Then $\Delta_{\alpha, J} = W_J \alpha$ if and only if all the roots in $\Delta_{\alpha, J}$ are real and $(\alpha, \alpha)$ is least among the lengths of all the roots in $\Delta_{\alpha, J}$.

6. Further extensions of Theorem C

In this section, we extend Theorem C further in two cases: (1) for highest weight modules with dominant highest weights and length 2 over Kac–Moody $g$; (2) for highest weight modules over semisimple $g$ with some assumptions on the integrabilities of the Jordan–Hölder factors. Recall that for a highest weight $g$-module $V$, $I_V$ denotes the integrability (the set of integrable directions) of $V$.

6.1. Moving between weights of modules of length 2. Throughout this subsection, we assume that $\lambda \in P^+$. The main goal of this subsection is to extend Theorem C for $M(\lambda) \rightarrow V$ of length 2 over Kac–Moody g, see Theorem 6.1 below.

**Theorem 6.1.** Let $g$ be a Kac–Moody algebra, $\lambda \in P^+$, and $M(\lambda) \rightarrow V$. Suppose $V$ satisfies a non-split exact sequence $0 \rightarrow L(\lambda') \rightarrow V \rightarrow L(\lambda) \rightarrow 0$ for some $\lambda' \prec \lambda$, where $L(\lambda)$ and $L(\lambda')$ are the simple highest weight $g$-modules with highest weights $\lambda$ and $\lambda'$ respectively, and suppose $\mu_0 \not\geq \mu \in wtV$. Then there exists a sequence of weights $\mu_i \in wtV$, $1 \leq i \leq n = \text{ht}(\mu - \mu_0)$, such that

$$\mu_0 < \cdots < \mu_i < \cdots < \mu_n = \mu \in wtV \quad \text{and} \quad \mu_i - \mu_{i-1} \in \Pi \forall i.$$ 

To prove Theorem 6.1 we first prove certain relations between the sets of weights of the simple modules $L(\lambda)$ and $L(s_i \lambda - \alpha_i)$, $i \in I$, over a Kac–Moody algebra $g$. Note that $s_i \lambda - \alpha_i = s_i \lambda$ when $g$ is of finite type, where ‘•’ denotes the dot-action of the Weyl group on $h^*$. 
**Proposition 6.2.** Let \( \mathfrak{g} \) be a Kac–Moody algebra, and \( \lambda \in P^+ \). Fix \( i \in I \), and define \( m_i := \langle \lambda, \alpha_i^\vee \rangle + 1 \), so that \( s_i \lambda - \alpha_i = \lambda - \langle \lambda, \alpha_i^\vee \rangle = 0 \). Then:

(a) If \( \mu \in \text{wt} L(\lambda) \) such that \( \text{ht}_{\{i\}}(\lambda - \mu) < m_i - 1 \), then \( \mu - t \alpha_i \in \text{wt} L(\lambda) \cap \{ \lambda - t \alpha_i \}_{t=0}^{\infty} \).

(b) \( \text{wt} L(\lambda) - m_i \alpha_i \subset \text{wt} L(\lambda - m_i \alpha_i) \).

(c) If \( \mu \in \text{wt} L(\lambda) \) such that \( \text{ht}_{\{i\}}(\lambda - \mu) < m_i \), then \( \mu - (m_i - \text{ht}_{\{i\}}(\lambda - \mu)) \alpha_i \in \text{wt} L(\lambda - m_i \alpha_i) \).

(d) If \( \mu \in \text{wt} L(\lambda) \) such that \( \text{ht}_{\{i\}}(\lambda - \mu) \geq m_i \), then \( \mu \in \text{wt} L(\lambda - m_i \alpha_i) \).

**Proof.** Fix an \( i \in I \). Firstly, note that \( L(\lambda) \) is integrable and \( L(\lambda - m_i \alpha_i) \) is \( \mathfrak{g}_{\{i\}} \)-integrable. Recall by Theorem 1.3 and equation (2.8) that

\[
\text{wt} L(\lambda - m_i \alpha_i) = \text{wt} M(\lambda - m_i \alpha_i, I_{L(\lambda - m_i \alpha_i)}) = \text{wt} M(\lambda - m_i \alpha_i, \{i\}^c) = \text{wt} L(\lambda - m_i \alpha_i - Z_{\geq 0} \Delta_{\{i\}, 1}),
\]

where \( L(\lambda - m_i \alpha_i) \) is the simple highest weight \( \mathfrak{g}_{\{i\}} \)-module with highest weight \( \lambda - m_i \alpha_i \).

We prove (a) by induction on \( \text{ht}(\lambda - \mu) \). Throughout the proof of (a) assume that \( m_i \geq 2 \).

Base step: \( \mu = \lambda \), and as \( L(\lambda) \) is integrable we must have \([\lambda - (m_i - 1) \alpha_i, \lambda] \subset \text{wt} L(\lambda) \) as required.

Induction step: Let \( \mu \in \text{wt} L(\lambda) \) be as in the assertion (a), and satisfy \( \text{ht}(\lambda - \mu) > 0 \). Fix \( j \in I \) such that \( \mu + \alpha_j \in \text{wt} L(\lambda) \). If \( j = i \), then by the induction hypothesis we have

\[
[\mu + \alpha_i - (m_i - 1 - \text{ht}_{\{i\}}(\lambda - \mu)\alpha_i, \mu + \alpha_i] = [\mu - (m_i - 1 - \text{ht}_{\{i\}}(\lambda - \mu)) \alpha_i, \mu + \alpha_i] \subset \text{wt} L(\lambda),
\]

immediately implying the result.

So, we assume now that \( j \neq i \). As \( L(\lambda) \) is \( \mathfrak{sl}_{\alpha_j} \)-integrable and \( \mu + \alpha_j \in \text{wt} L(\lambda) \), observe that there must exist a weight \( \mu' \in \text{wt} L(\lambda) \) such that

\[
(i) \, \mu' - \mu \in \mathbb{Z}_{\geq 0} \alpha_j, \quad (ii) \, \langle \mu', \alpha_j^\vee \rangle > 0, \quad (iii) \, \mu \in [s_j \mu', \mu'] \subset \text{wt} L(\lambda).
\]

Note that \( \text{ht}_{\{i\}}(\lambda - \mu') = \text{ht}_{\{i\}}(\lambda - \mu) \). By the induction hypothesis applied to \( \mu' \) we have \([\mu' - (m_i - 1 - \text{ht}_{\{i\}}(\lambda - \mu)) \alpha_i, \mu'] \subset \text{wt} L(\lambda) \). Note that \( \langle \mu' - t \alpha_i, \alpha_j^\vee \rangle \geq \langle \mu', \alpha_j^\vee \rangle > 0 \forall t \in \mathbb{Z}_{\geq 0} \). As \( L(\lambda) \) is \( \mathfrak{sl}_{\alpha_j} \)-integrable, we finally have

\[
\mu - t \alpha_i \in [s_j(\mu' - t \alpha_i), \mu' - t \alpha_i] \subset \text{wt} L(\lambda) \cap \{ \lambda - t \alpha_i \}_{t=0}^{\infty} \forall 0 \leq t \leq m_i - 1 - \text{ht}_{\{i\}}(\lambda - \mu).
\]

Hence, the proof of (a) is complete.

For (b), we prove by induction on \( \text{ht}(\lambda - \mu) \) that \( \mu \in \text{wt} L(\lambda) \) implies \( \mu - m_i \alpha_i \in \text{wt} L(\lambda - m_i \alpha_i) \) trivially.

Base step: \( \mu = \lambda \), and \( \lambda - m_i \alpha_i \in \text{wt} L(\lambda - m_i \alpha_i) \).

Induction step: Let \( \mu \in \text{wt} L(\lambda) \) with \( \text{ht}(\lambda - \mu) > 0 \). Fix \( j \in I \) such that \( \mu + \alpha_j \in \text{wt} L(\lambda) \). By the induction hypothesis \( \mu + \alpha_j - m_i \alpha_i \in \text{wt} L(\lambda - m_i \alpha_i) \). If \( j = i \), then by equation (6.1) we get

\[
\mu - m_i \alpha_i \in \text{wt} L(\lambda - m_i \alpha_i).
\]

So, we assume now that \( j \neq i \). If \( \langle \mu + \alpha_j - m_i \alpha_i, \alpha_j^\vee \rangle > 0 \), then

\[
\mu - m_i \alpha_i \in [s_j(\mu + \alpha_j - m_i \alpha_i), \mu + \alpha_j - m_i \alpha_i] \subset \text{wt} L(\lambda - m_i \alpha_i)
\]

by the \( \mathfrak{g}_{\{i\}} \)-integrability of \( L(\lambda - m_i \alpha_i) \). Else if \( \langle \mu + \alpha_j - m_i \alpha_i, \alpha_j^\vee \rangle \leq 0 \), then we must have \( \langle \mu + \alpha_j, \alpha_j^\vee \rangle \leq m_i \langle \alpha_i, \alpha_j^\vee \rangle \leq 0 \). This implies \( \langle \mu, \alpha_j^\vee \rangle \leq -2 \), and thereby \( \text{ht}(\lambda - s_j \mu) < \text{ht}(\lambda - \mu) \).

Note that \( s_j \mu \in \text{wt} L(\lambda) \) by \( W \)-invariance. By the induction hypothesis applied to \( s_j \mu \) we have \( s_j \mu - m_i \alpha_i \in \text{wt} L(\lambda - m_i \alpha_i) \). Note that \( \langle s_j \mu - m_i \alpha_i, \alpha_j^\vee \rangle \geq \langle s_j \mu, \alpha_j^\vee \rangle > 0 \). Hence, once again by the \( \mathfrak{g}_{\{i\}} \)-integrability of \( L(\lambda - m_i \alpha_i) \), we have

\[
\mu - m_i \alpha_i \in [s_j(\mu - m_i \alpha_i), s_j \mu - m_i \alpha_i] \subset \text{wt} L(\lambda - m_i \alpha_i),
\]

completing the proof of (b).

We prove (c) by induction on \( \text{ht}(\lambda - \mu) \).

Base step: \( \mu = \lambda \), and \( \lambda - m_i \alpha_i \in \text{wt} L(\lambda - m_i \alpha_i) \) trivially.
Induction step: Let $\mu$ be as in the assertion (c), and satisfy $\text{ht}(\lambda - \mu) > 0$. Fix $j \in \mathcal{I}$ such that $\mu + \alpha_j \in \text{wt}L(\lambda)$. If $j = i$, then by the induction hypothesis

$$
\mu + \alpha_i - (m_i - \text{ht}_{i,j}(\lambda - \mu - \alpha_i))\alpha_i = \mu - (m_i - \text{ht}_{i,j}(\lambda - \mu))\alpha_i \in \text{wt}L(\lambda - m_i\alpha_i)
$$

as required. So, we assume now that $j \neq i$. As $L(\lambda)$ is $\mathfrak{sl}_{\alpha_j}$-integrable and $\mu + \alpha_j \in \text{wt}L(\lambda)$, observe that there must exist a weight $\mu' \in \text{wt}L(\lambda)$ such that

$$(i) \ \mu' - \mu \in \mathbb{Z}_{>0}\alpha_j, \quad (ii) \ \langle \mu', \alpha_j^\vee \rangle > 0, \quad (iii) \ \mu \in [s_j\mu', \mu'] \subset \text{wt}L(\lambda).$$

Note that $\text{ht}_{i,j}(\lambda - \mu') = \text{ht}_{i,j}(\lambda - \mu) < m_i$. By the induction hypothesis applied to $\mu'$ we have $\mu' - (m_i - \text{ht}_{i,j}(\lambda - \mu))\alpha_i \in \text{wt}L(\lambda - m_i\alpha_i)$. Note that $\langle \mu' - (m_i - \text{ht}_{i,j}(\lambda - \mu))\alpha_i, \alpha_j^\vee \rangle \geq \langle \mu', \alpha_j^\vee \rangle > 0$. As $L(\lambda - m_i\alpha_i)$ is $\mathfrak{g}_{i,j}$-integrable, we finally have

$$
\mu - (m_i - \text{ht}_{i,j}(\lambda - \mu))\alpha_i \in [s_j(\mu' - (m_i - \text{ht}_{i,j}(\lambda - \mu))\alpha_i), \mu' - (m_i - \text{ht}_{i,j}(\lambda - \mu))\alpha_i] \subset \text{wt}L(\lambda - m_i\alpha_i).
$$

Hence, the proof of (c) is complete.

For (d), we prove by induction on $\text{ht}(\lambda - \mu)$ that $\mu \in \text{wt}L(\lambda)$, $\text{ht}_{i,j}(\lambda - \mu) \geq m_i$ implies $\mu \in \text{wt}L(\lambda - m_i\alpha_i)$.

Base step: $\mu = \lambda - m_i\alpha_i$, and $\mu \in \text{wt}L(\lambda - m_i\alpha_i)$ trivially.

Induction step: Let $\mu \in \text{wt}L(\lambda)$ such that $\text{ht}_{i,j}(\lambda - \mu) \geq m_i$ and $\text{ht}(\lambda - \mu) > m_i$. Fix $j \in \mathcal{I}$ such that $\mu + \alpha_j \in \text{wt}L(\lambda)$. If $j \neq i$, then clearly $\text{ht}_{i,j}(\lambda - \mu - \alpha_j) \geq m_i$. As $L(\lambda)$ is $\mathfrak{sl}_{\alpha_j}$-integrable and $\mu + \alpha_j \in \text{wt}L(\lambda)$, observe that there must exist a weight $\mu'' \in \text{wt}L(\lambda)$ such that

$$(i) \ \mu'' - \mu \in \mathbb{Z}_{>0}\alpha_j, \quad (ii) \ \langle \mu'', \alpha_j^\vee \rangle > 0, \quad (iii) \ \mu \in [s_j\mu'', \mu''] \subset \text{wt}L(\lambda).$$

Note that $\text{ht}_{i,j}(\lambda - \mu'') = \text{ht}_{i,j}(\lambda - \mu)$. By the induction hypothesis applied to $\mu''$ we have $\mu'' \in \text{wt}L(\lambda - m_i\alpha_i)$. As $L(\lambda - m_i\alpha_i)$ is $\mathfrak{g}_{i,j}$-integrable and $\langle \mu'', \alpha_j^\vee \rangle > 0$, we finally have $\mu \in [s_j\mu'', \mu'] \subset \text{wt}L(\lambda - m_i\alpha_i)$.

So, we assume now that $j = i$. If $\text{ht}_{i,j}(\lambda - \mu) > m_i$, then by the induction hypothesis applied to $\mu + \alpha_i$ we have $\mu + \alpha_i \in \text{wt}L(\lambda - m_i\alpha_i)$. Equation (6.1) then yields $\mu \in \text{wt}L(\lambda - m_i\alpha_i)$. Else if $\text{ht}_{i,j}(\lambda - \mu) = m_i$, then by part (c) we have

$$
\mu + \alpha_i - (m_i - \text{ht}_{i,j}(\lambda - \mu - \alpha_i))\alpha_i = \mu + \alpha_i - (m_i - m_i + 1)\alpha_i = \mu \in \text{wt}L(\lambda - m_i\alpha_i)
$$

as required. Hence, the proof of the proposition is complete. \hfill \Box

We are now able to prove Theorem 6.1.

**Proof of Theorem 6.1.** We prove the theorem by induction on $\text{ht}(\mu - \mu_0)$. In the base step, $\text{ht}(\mu - \mu_0) = 1$, there is nothing to prove.

Induction step: $\text{ht}(\mu - \mu_0) > 1$. Recall that it is not known if $L^{\text{max}}(\lambda)$ over general Kac–Moody algebra $\mathfrak{g}$ is simple. Nevertheless, by [15 Proposition 11.2], we have $\text{wt}L'(\lambda) = \text{wt}L^{\text{max}}(\lambda) = \text{wt}L(\lambda)$ for any integrable highest weight $\mathfrak{g}$-module $L'(\lambda)$ with highest weight $\lambda$. If $\text{wt}V = \text{wt}L(\lambda)$, then Theorem 6.1 holds true by Theorem [C]. So, we assume that $\text{wt}L(\lambda) \not\subseteq \text{wt}V$. By the previous line observe that $V$ cannot be integrable, in other words the integrability $I_V$ is not equal to $\mathcal{I}$. This implies that $\lambda' = \lambda - m_i\alpha_i$ for some $i \in \mathcal{I}$ (recall the length two non-split exact sequence satisfied by $V$ in the statement of the theorem) where $m_i = \langle \lambda, \alpha_i^\vee \rangle + 1$. Thus, we must have $\text{wt}V = \text{wt}L(\lambda) \cup \text{wt}L(\lambda - m_i\alpha_i)$.

Note that if both $\mu$ and $\mu_0$ belong to either $\text{wt}L(\lambda)$ or $\text{wt}L(\lambda - m_i\alpha_i)$, then we are once again done by Theorem [C]. So, we now proceed in the following two cases.

(1) $\mu \in \text{wt}L(\lambda)$ and $\mu_0 \in \text{wt}L(\lambda - m_i\alpha_i)$: If $\text{ht}_{i,j}(\lambda - \mu) \geq m_i$, then by Proposition 6.2 (d) we have $\mu \in \text{wt}L(\lambda - m_i\alpha_i)$, and the result follows as $\mu_0 \in \text{wt}L(\lambda - m_i\alpha_i)$. If $\text{ht}_{i,j}(\lambda - \mu) = m_i - 1$, then by Proposition 6.2 (c) we have $\mu - \alpha_i \in \text{wt}L(\lambda - m_i\alpha_i)$, and the result once again follows. Else if $\text{ht}_{i,j}(\lambda - \mu) < m_i - 1$, then by Proposition 6.2 (a) we have $\mu - \alpha_i \in \text{wt}L(\lambda)$, and the induction
hypothesis applied to the pair \( \mu_0 < \mu - \alpha_i \) now finishes the proof.

(2) \( \mu_0 \in \text{wt} L(\lambda) \) and \( \mu \in \text{wt} L(\lambda - m_i \alpha_i) \): As \( \mu \in \text{wt} L(\lambda - m_i \alpha_i) \) we must have \( \text{ht}_{\{i\}} (\lambda - \mu_0) \geq \text{ht}_{\{i\}} (\lambda - \mu) \geq m_i \). The previous line and Proposition 6.2 (d) yield \( \mu_0 \in \text{wt} L(\lambda - m_i \alpha_i) \), which implies the result. \( \square \)

Remark 6.3. For \( \lambda \in P^+ \), note that Theorem 6.1 in particular, extends Theorem C in the semisimple case to \( M(\lambda) \to V \) such that the Jordan–Hölder length of \( V \) is at most 2.

It will be interesting to investigate the applications and possible generalizations of Proposition 6.2.

6.2. Some more possible extensions of Theorem C in the semisimple case. In this subsection, we extend Theorem C to an arbitrary highest weight module \( V \) over semisimple \( g \) under some assumptions on the integrabilities of the Jordan–Hölder factors of \( V \).

Proposition 6.4. Let \( g \) be semisimple, and \( V \) be a highest weight \( g \)-module. Fix two factors \( L(\lambda') \) and \( L(\lambda'') \) in the Jordan–Hölder series of \( V \), for some \( \lambda' \) and \( \lambda'' \in h^* \), such that \( I_{L(\lambda')} \subset I_{L(\lambda'')} \). Suppose \( \mu \in \text{wt} L(\lambda') \) and \( \mu_0 \in \text{wt} L(\lambda'') \) such that \( \mu_0 \not\subseteq \mu \). Then there exists a sequence of weights \( \mu_i \in \text{wt} V, 1 \leq i \leq n = \text{ht}(\mu - \mu_0), \) such that

\[
\mu_0 < \cdots < \mu_i < \cdots < \mu_n = \mu \in \text{wt} V \quad \text{and} \quad \mu_i - \mu_{i-1} \in \Pi \ \forall i.
\]

Proof. We prove the proposition by induction on \( \text{ht}(\mu - \mu_0) \). In the base step, \( \text{ht}(\mu - \mu_0) = 1 \), there is nothing to prove.

Induction step: Let \( \mu_0 \) and \( \mu \) be as in the statement, and satisfy \( \text{ht}(\mu - \mu_0) > 1 \). If \( \text{supp}(\mu - \mu_0) \cap (I \setminus I_{L(\lambda')}) \neq \emptyset \), then \( \mu - \alpha_i \in \text{wt} L(\lambda') \) for every \( i \in \text{supp}(\mu - \mu_0) \cap (I \setminus I_{L(\lambda')}), \) by the formulas for \( \text{wt} L(\lambda') \) given by equations (2.8) and (2.6). The induction hypothesis then completes the proof. So, we assume that \( \text{supp}(\mu - \mu_0) \subset I_{L(\lambda')} \). By the assumption \( I_{L(\lambda')} \subset I_{L(\lambda'')} \), we also have \( \text{supp}(\mu - \mu_0) \subset I_{L(\lambda'')} \). Note that \( 0 < (\mu - \mu_0, \mu - \mu_0) \), where \( (\langle \cdot, \cdot \rangle) \) is the Killing form on \( h^* \). Thus, there exists a simple root \( \alpha_j, j \in \text{supp}(\mu - \mu_0) \) such that \( (\mu - \mu_0, \alpha_j) > 0 \). This implies either \( (\mu, \alpha_j) > 0 \) or \( (\mu_0, \alpha_j) < 0 \), implying \( \mu - \alpha_j \in \text{wt} L(\lambda') \) or respectively \( \mu_0 + \alpha_j \in \text{wt} L(\lambda'') \) by the \( \text{supp}(\mu - \mu_0) \)-integrability of both \( L(\lambda') \) and \( L(\lambda'') \). The induction hypothesis then completes the proof. \( \square \)

Corollary 6.5. Let \( g \) be semisimple, \( \lambda \in P^+ \), and \( M(\lambda) \to V \). Suppose \( \mu_0 \not\subseteq \mu \in \text{wt} V \) and \( \mu_0 \in \text{wt} L(\lambda) \). Then there exists a sequence of weights \( \mu_i \in \text{wt} V, 1 \leq i \leq n = \text{ht}(\mu - \mu_0), \) such that

\[
\mu_0 < \cdots < \mu_i < \cdots < \mu_n = \mu \in \text{wt} V \quad \text{and} \quad \mu_i - \mu_{i-1} \in \Pi \ \forall i.
\]

In view of Proposition 6.4, it will be interesting to find some examples of highest weight modules which possess some intermediate factors as in Proposition 6.4. We conclude the paper with the following remark.

Remark 6.6. Observe that in this paper we have answered Question 2 of Khare for a large class of modules, which contains all simple highest weight modules, over general Kac–Moody algebras. However, Question 2 still stands open for general highest weight modules even in the semisimple case. In this paper, we essentially looked at the integrability of a module, and made use of the formulas for the set of weights of simple highest weight modules given by [11]. Working with the integrabilities of the Jordan–Hölder factors, it might be possible to extend Theorem C for all/more highest weight modules at least in the semisimple case.

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