The effect of curvature on convexity properties of harmonic functions and eigenfunctions

Dan Mangoubi

Dedicated to Shmuel Agmon with admiration and gratitude
on the occasion of his 90th birthday

Abstract
We give a proof of the growth bound of Laplace–Beltrami eigenfunctions due to Donnelly and Fefferman which is probably the easiest and the most elementary one. Our proof also gives new quantitative geometric estimates in terms of curvature bounds which improve and simplify previous work by Garofalo and Lin. The proof is based on a generalization of a convexity property of harmonic functions in \( \mathbb{R}^n \) to harmonic functions on Riemannian manifolds following Agmon’s ideas.

1. Introduction
In their seminal paper [6], Donnelly and Fefferman found growth bounds (DF-growth bound) for eigenfunctions on compact Riemannian manifolds. Roughly, they showed that a \( \lambda \)-eigenfunction grows like a polynomial of order \( \sqrt{\lambda} \) at most. This result is central in the study of eigenfunctions. In [6], it was applied to prove Yau’s conjecture on real analytic manifolds. Namely, sharp upper and lower bounds on the size of the nodal set on real analytic manifolds were found. The proof of the growth bound in [6] went through a fine version of a Carleman-type inequality for the operator \( \Delta + \lambda \), with a careful geometric choice of the weight function.

Soon after, Lin [12], based on an earlier work with Garofalo [7], gave a simpler proof of the growth bound. This proof is based on properties of the spherical \( L^2 \)-norm, \( q(r) \) (defined in (2.1)), for harmonic functions. It had been known [1, 3] that in \( \mathbb{R}^n \), \( \log q \) is monotonically increasing and convex as a function of \( \log r \). Equivalently, \( rq'(r)/q(r) \) is monotonically increasing. Garofalo–Lin showed that for a harmonic function defined on a general Riemannian manifold \( e^{\Lambda r}rq'(r)/q(r) \) is monotonically increasing in \((0, R)\), where \( \Lambda \) and \( R \) are some positive constants depending on bounds on the Riemannian metric, on its first derivatives and on the ellipticity constant of the Riemannian metric. This result can be viewed as an approximated convexity result. The proof of this result was based on a non-trivial geometric variational argument which was first used by Almgren [3].

The first aim of this paper is to give new geometric estimates on \( \Lambda \) and \( R \) in terms of the curvature of the manifold. Namely, we find that all one needs is a lower and an upper bound on the sectional curvature in order to guarantee the existence of \( \Lambda \) and \( R \). Moreover, we show that in fact \( e^{C_1 r^2 K} rq'(r)/q(r) \) is monotonic in \((0, R)\), where \( K \) is an upper bound on the curvature, \( R \) is the minimum of \( C_2/\sqrt{K} \) and the injectivity radius, and \( C_1, C_2 \) depend only on the dimension of the manifold. We emphasize that our result distinguishes between negative and positive curvatures. This is the content of the main Theorem 2.2.
The second aim of this paper is to have a simple proof of the DF-growth bound for
eigenfunctions. Owing to the importance of this result, three simplifications to its proof had
previously been given by different authors in the course of the years, which we briefly survey.

The idea of Lin [9, 12] was to consider a conic manifold, \( N \), over \( M \) and to extend the
eigenfunction \( u_\lambda \) to a harmonic function on \( N \). Then, Lin applied the monotonicity property
of \( e^{N}r q'/q \) from [7] to the harmonic function obtained, and went back to the eigenfunction.

In [10], Jerison and Lebeau applied a similar extension of eigenfunctions. Then, they
could use standard Carleman-type inequalities for harmonic functions, instead of the original
approach taken by Donnelly and Fefferman in which a special and delicate Carleman-type
inequality for eigenfunctions was used.

In two dimensions, Nazarov–Polterovich–Sodin [13] took advantage of the conformal coor-
dinates, thus allowing them to simplify the problem by considering only the standard Laplace
operator in \( \mathbb{R}^2 \). Then, they extend the eigenfunction to a harmonic function on \( N = M \times \mathbb{R} \),
and apply the convexity argument to the harmonic function (in \( \mathbb{R}^3 \)) obtained. Their proof of
the convexity of \( \log q \) is considerably simpler than the variational approach taken in [7]. It is
close in spirit to Agmon’s approach. This gives the easiest proof of the DF-growth bound in two
dimensions, since there is no need for variational arguments or Carleman-type inequalities at all.

This paper extends the work started in [13], to dimensions greater than or equal to three,
where no conformal coordinates exist. We follow and generalize Agmon’s ideas in [1], where
a general approximated convexity theorem for second-order elliptic equations is proved by
considering them as an abstract second-order ordinary differential equation. Our contribution
here comes in adding the geometric point of view, clarifying the way curvature affects the
Euclidean result. Our proof also simplifies and improves Agmon’s results in [1]. In this way,
we are able to circumvent the need to use the non-trivial variational argument in [7] or any
Carleman-type inequality.

**Organization of the paper.** The main result is presented in Section 2. In Section 3, we
recall a way that eigenfunctions can be extended to harmonic functions and the translation of
the convexity property of harmonic functions to a local growth bound on eigenfunctions. In
Section 4, we conclude the proof of the DF-growth bound on compact manifolds and we outline
the proof of Yau’s conjecture in [6]. Sections 3 and 4 are strongly based on [13]. In Section 5,
we give the proof of the main theorem. In Section 6, we consider constant curvature manifolds
as examples to the main theorem and find a second proof in some of these cases. In Section 7,
we discuss several open questions.

**Notation.** Throughout this paper \( C_i \) and \( C_i(n) \) denote positive constants which depend only
on dimension. The positive constants \( C_g(\ldots) \) depend on bounds on the metric \( g \), its first
derivatives, its ellipticity constant and additional parameters appearing in parentheses.

### 2. Main Theorem: a perturbed log-convexity property of harmonic functions

Let \( u \) be a harmonic function in \( \mathbb{R}^n \). Let \( q(r) \) denote the square of the spherical \( L^2 \)-norm:

\[
q(r) := \int_{S_r} u^2 \, d\sigma_r,
\]

where \( S_r \) denotes the sphere of radius \( r \) centred at 0, and \( d\sigma_r \) is the standard area measure
on \( S_r \). It is easy to check that \( q \) is a convex function of \( \log r \). It turns out that even \( \log q \) is a
convex function of \( \log r \).

**Theorem 2.1** [1]. The function \( q \) has the following two properties:

(i) \( (\log q)'(r) \geq (n - 1)/r \);

(ii) \( (\log q)''(r) + (1/r)(\log q)'(r) \geq 0. \)
In two dimensions, this can be seen by a complex analysis argument. In higher dimensions, this fact goes back at least to Agmon [1], and it was rediscovered by Almgren [3]. Landis [11, Chapter II.2] also found several results close in spirit to that one. All these kinds of results were inspired by Hadamard’s three circles theorem (see [2, Chapter 6.2]), which shows log-convexity of the spherical $L^\infty$-norm for a holomorphic function.

Remark. It is somewhat surprising that the fundamental solution does not play a role here: $\log q$ is a convex function of $\log r$ in all dimensions. The weaker statement is that $\log q$ is a convex function of $G(r) = -1/r^{n-2}$, which is equivalent to $(\log q)''(r) + \frac{n-1}{r} (\log q)'(r) \geq 0$.

When considering harmonic functions on manifolds, one expects a perturbed version of Theorem 2.1 in small geodesic balls. However, it is not clear a priori how far from the centre this perturbation goes and how curvature controls it. Theorem 2.2 will give an answer to these questions. Let $u$ be a harmonic function defined in a small geodesic ball of a Riemannian manifold $N$. Let $q(r) := \int_{S(r)} u^2 \, dA_r,$ (2.1)

where $S(r)$ is a geodesic sphere centred at $p \in N$, and $dA_r$ is the area measure on $S(r)$. We let $\text{Sec}_N$ denote the sectional curvature of $N$, $K^+ = \max\{K, 0\}$, $\Theta(K)$ equal one if $K > 0$ and zero if $K \leq 0$, $\sin_K r = \begin{cases} \frac{\sin(\sqrt{K}r)}{\sqrt{K}}, & K > 0, \\ r, & K = 0, \\ \frac{\sinh(\sqrt{-K}r)}{\sqrt{-K}}, & K < 0, \end{cases}$

and $\cot_K r = (\sin_K r)'/(\sin_K r)$. We can now state our main result.

Theorem 2.2. Let $N$ be a Riemannian manifold of $n$ dimensions. Let $u$ be a harmonic function on a geodesic ball in $N$, and $q$ be defined as in (2.1). Let $\kappa, K \in \mathbb{R}$, $\kappa \leq K$. Let $R = \min(\text{inj}(M), \pi/(2\sqrt{K^+}))$. We have the following statements.

(i) If $\text{Sec}_N \leq K$, then $q(r)/(\sin_K r)^{n-1}$ is monotonically increasing for $r < R$. Equivalently, $(\log q)'(r) \geq (n-1)(\cot_K r)$.

(ii) If $\kappa \leq \text{Sec}_N \leq K$, then for $r < R$,

$$(\log q)''(r) + (\cot_K r)(\log q)'(r) + (n+1)(\cot_K r - \cot_K r)(\log q)'(r) \geq -K - (n-2)K^+ - (n-1)(K - \kappa) - \frac{n}{2}(n-1)(n-2)(K - \kappa)\Theta(K).$$

The proof of the theorem is given in Section 5.

Remarks.

(1) The result shows that in the case where $K < 0$ and where $K - \kappa$ is small we have actual convexity, since the right-hand side in Theorem 2.2(ii) is positive. One could state part (ii) of the theorem with the function $\tilde{q} = q/(\sin_K r)^{n-1}$ replacing $q$: then, the right-hand side becomes $-(n-2)K^- - c_1(n)(K - \kappa)$ which gives ‘advantage’ to positive curvature (see also the discussion in Section 7.2).
For the constant non-positive curvature case, we obtain an exact convexity statement for $\log q$.

Comparing with the result of Garofalo and Lin [7], from Theorem 2.2(ii), one deduces that $e^{c(n)r^2K}rq'(r)/q(r)$ is monotonically increasing for $r < R$. Observe that besides the explicit estimates of $\Lambda$ and $R$ mentioned in the introduction this also gives a correction of the result in [7] in the power of $r$ in the exponential term. Moreover, the statement here is more geometric in nature.

We would now like to have an integrated version of Theorem 2.2. We restrict our attention to the case $\kappa = -K$, $K > 0$. We obtain a local doubling estimate for harmonic functions (see proof in Section 5.5).

**Corollary 2.3.** Let $N$ be a complete Riemannian manifold of dimension $n$ with $|\text{Sec}_N| \leq K$. Then

$$\frac{q(2r)}{q(r)} \leq \left( \frac{q(2s)}{q(s)} \right)^{1+c_3(n)r^2K}$$

for all $r < s < 1/(4\sqrt{nK})$.

### 3. Harmonic extension of eigenfunctions

In this section, we recall a connection between harmonic functions and eigenfunctions found in [10, 12, 13]. Let $M$ be a Riemannian manifold of dimension $m$. Let $u_\lambda$ be a $\lambda$-eigenfunction on $M$. Consider the direct product Riemannian manifold $N = M \times \mathbb{R}$ of dimension $n = m + 1$, where the metric on $\mathbb{R}$ is the standard one. Let $H$ be the following function on $N$:

$$\forall x \in M, t \in \mathbb{R}, \quad H(x, t) := u_\lambda(x) \cosh(\sqrt{\lambda}t).$$

Observe that $H$ extends $u_\lambda$ to $N$ and is harmonic on $N$, since the Laplacian on $N$ can be written as

$$\Delta_N u = \Delta_M u + \frac{\partial^2 u}{\partial t^2}.$$

On $N$, we take geodesic coordinates $(r, \theta_1, \ldots, \theta_{n-1})$ in a neighbourhood of the point $(p, 0) \in N$. In these coordinates, the metric $g_N$ takes the following form:

$$g_N = dr^2 + r^2a_{ij} d\theta^i d\theta^j, \quad 1 \leq i, j \leq n - 1.$$

We let $\hat{\theta} = (\theta_1, \ldots, \theta_{m-1})$, and $b_{ij}(r, \hat{\theta}) := a_{ij}(r, \hat{\theta}, 0)$, then

$$g_M = dr^2 + r^2b_{ij} d\hat{\theta}^i d\hat{\theta}^j, \quad 1 \leq i, j \leq m - 1.$$

Accordingly, the equation $\Delta_N H = 0$ can be written in these coordinates as

$$H_{rr} + \left( \frac{n-1}{r} + \gamma(r, \theta) \right) H_r + \frac{1}{r^2} \Delta_{S(r)} H = 0,$$
where \( \gamma(r, \theta) = (\sqrt{a(r)} / \sqrt{a} \) with \( a = \det(a_{ij}) \), and \( \Delta_{S(r)} \) is the spherical Laplacian on the geodesic sphere of radius \( r \):

\[
\Delta_{S(r)} H := \frac{1}{\sqrt{a}} \frac{\partial}{\partial \theta^i} \left( \sqrt{a} a^{ij} \frac{\partial H}{\partial \theta^j} \right).
\]

The following lemma relates \( q(r) \), the spherical \( L^2 \)-norm of the harmonic function \( H \) on an \((n - 1)\)-dimensional sphere of radius \( r \), to \( M_r(u_\lambda) \), the \( L^\infty \)-norm of the eigenfunction \( u_\lambda \) on an \( m = (n - 1) \)-dimensional ball of radius \( r \). Let \( M_r(u_\lambda) := \max_{B(p, r)} |u_\lambda(x)| \).

**Lemma 3.1.** Suppose \( M \) is a complete Riemannian manifold with bounded geometry. Fix \( 0 < \alpha < 1 \). Then, for all \( 0 < r < \text{inj}_M \),

\[
C_\alpha r^m (1 + r \sqrt{\lambda})^{-m - 1} M_{ar}(u_\lambda)^2 \leq q(r) \leq C_2 r^m \epsilon^{2r \sqrt{\lambda}} M_r(u_\lambda)^2,
\]

where \( C_\alpha \) depends both on \( \alpha \) and the metric, and \( C_2 \) depends on the metric alone.

**Proof.** Let us denote by \( d\sigma(\hat{\theta}) \) the standard volume form on the unit sphere of dimension \( m - 1 \). Then

\[
q(r) = 2 \int_0^r \int_{S^{m-1}} u_\lambda(\rho, \hat{\theta})^2 \cosh^2(\sqrt{\lambda} \sqrt{\rho^2 - r^2}) \cdot \rho^{m-1} \sqrt{b(\rho, \hat{\theta})} \frac{r}{\sqrt{r^2 - \rho^2}} \ d\hat{\theta} d\rho
\]

\[
\leq CM_r(u_\lambda)^2 (3 + \epsilon^{2r \sqrt{\lambda}}) \int_0^r \int_{S^{m-1}} \rho^{m-1} \frac{r}{\sqrt{r^2 - \rho^2}} \ d\sigma(\hat{\theta}) d\rho
\]

\[
= C\omega_m r^m M_r(u_\lambda)^2 (3 + \epsilon^{2r \sqrt{\lambda}}),
\]

where we used the fact that the volume element is bounded from above by the metric [4, Chapter 11, Theorem 15].

On the other hand, we have

\[
q(r) \geq 2 \int_0^r \int_{S^{m-1}} u_\lambda(\rho, \hat{\theta})^2 \rho^{m-1} \sqrt{b(\rho, \hat{\theta})} \ d\hat{\theta} d\rho = \int_{B^m(p, r)} u_\lambda^2 d\text{Vol}_M.
\]

Hence, from elliptic regularity we get

\[
q(r) \geq C_\alpha M_{ar}(u_\lambda)^2 r^m (1 + r \sqrt{\lambda})^{-m},
\]

where \( C_\alpha \) depends on the metric and on \( \alpha \). \( \square \)

From Corollary 2.3 and Lemma 3.1 we obtain the following theorem.

**Theorem 3.2.** Let \( M \) be a complete Riemannian manifold of dimension \( m \) with \( |\text{Sec}_M| \leq K \). Then for all \( r \leq s < C / \sqrt{K} \),

\[
\frac{M_{3r}(u_\lambda)}{M_{2r}(u_\lambda)} \leq C_1 e^{C_2 s \sqrt{\lambda}} \left( \frac{M_{8s}(u_\lambda)}{M_{3r}(u_\lambda)} \right)^{1 + C_3 r^2 K},
\]

where the constants \( C_2 \) and \( C_3 \) denote positive constants which depend only on the injectivity radius of \( M \), while \( C_1 \) depends on bounds on the metric, its derivatives and its ellipticity constant.

**Remark.** The subindices \( 3r, 2r, 8s, \) and \( 3s \) can be replaced by \( \beta r, r, \gamma s, \) and \( s \), respectively, where \( 1 < \beta < 2 \) and \( \gamma > \beta \). The constants \( C_2 \) and \( C_3 \) can be taken to be independent of \( \beta \) and \( \gamma \), while \( C_1 \to \infty \) as \( \gamma / \beta \to 1 \).
4. Two global growth estimates

In this section, we deduce from the local inequality in Theorem 3.2 two global results in the compact case.

4.1. Large values on large balls

**Theorem 4.1.** Let $M$ be a compact Riemannian manifold of dimension $m$. Then for all eigenfunctions $u_\lambda$ and $r > 0$,

$$\max_{B(x,r)} |u_\lambda| \leq C_g (r, d_M) e^{-C_2 d_M \sqrt{\lambda}} \quad \forall x \in M,$$

where $d_M$ is the diameter of $M$.

**Proof.** Normalize $u_\lambda$ so $\max_M |u_\lambda| = 1$. Take $r = s$ in Theorem 3.2. We obtain

$$M_{3r}(u_\lambda)^{2+C_3 r^2 K} \leq C_1 e^{C_2 r \sqrt{\lambda}} M_{3r}(u_\lambda)^{1+C_3 r^2 K} M_{2r}(u_\lambda) \leq C_1 e^{C_2 r \sqrt{\lambda}} M_{2r}(u_\lambda). \quad (4.1)$$

Let $|u_\lambda(x_0)| = 1$. Fix $r_0 > 0$ small enough in order to apply Theorem 3.2. Take a point $x$ in $M$. There exists a sequence of points $x_0, x_1, \ldots, x_N = x$ such that $d(x_k, x_{k+1}) < r_0$ for $0 \leq k \leq N - 1$, where $N$ only depends on $r_0$ and the diameter of $M$. Inequality (4.1) gives

$$\max_{B(x_k,2r_0)} |u_\lambda| \geq C_1 e^{-C_2 r_0 \sqrt{\lambda}} \max_{B(x_k,3r_0)} |u_\lambda|^{2+C_2 r^2 K} \geq C_1 e^{-C_2 r_0 \sqrt{\lambda}} \max_{B(x_{k-1},2r_0)} |u_\lambda|^3. \quad (4.2)$$

Multiplying the inequalities (4.2) for $1 \leq k \leq N$ gives

$$\max_{B(x,2r_0)} |u_\lambda| \geq C_1^{-N} e^{-C_2 N r_0 \sqrt{\lambda}} \geq C_1^{-N} e^{-C_2 d \sqrt{\lambda}}. \quad \Box$$

4.2. Global DF growth bound

**Theorem 4.2 [6].** For all eigenfunctions $u_\lambda$, $x \in M$ and $r > 0$,

$$\max_{B(x,3r)} |u_\lambda| \leq C_g (d_M) e^{C_2 d_M \sqrt{\lambda}}.$$ 

**Proof.** Let $R > 0$ be as in Theorem 4.1. If $r \geq R$, the theorem follows from Theorem 4.1. Else, Theorems 3.2 and 4.1 tell us that

$$\frac{M_{3r}(u_\lambda)}{M_{2r}(u_\lambda)} \leq C_g e^{C_2 R \sqrt{\lambda}} \left( \frac{M_{3R}(u_\lambda)}{M_{3r}(u_\lambda)} \right)^2 \leq C_g (d_M) e^{2C_2 d_M \sqrt{\lambda}}. \quad \Box$$

4.3. Outline of the proof of Yau’s conjecture for real analytic manifolds

In this section, we illustrate the importance of Theorem 4.2. We recall the following result.

**Conjecture 4.3 [16].** Let $M$ be a $C^\infty$ closed compact Riemannian manifold. Let $u_\lambda$ be a $\lambda$-eigenfunction on $M$. Then

$$C_1 \sqrt{\lambda} \leq \text{Vol}_{n-1}(\{u_\lambda = 0\}) \leq C_2 \sqrt{\lambda},$$

where $C_1$ and $C_2$ are constants independent of $\lambda$. 


The conjecture was proved in the case of real analytic Riemannian metrics in [6]. A major ingredient in the proof of both bounds was Theorem 4.2. We outline here the main ideas involved.

**Lower bound.** Let \( B \subset M \) be a ball of radius \( r = C/\sqrt{\lambda} \) such that \( u_\lambda \) vanishes at the centre of \( B \). One can cover, say, \( 1/2 \) of the volume of \( M \) by a disjoint collection \( \mathcal{B} \) of such balls [5]. One observes that if the growth of \( u_\lambda \) in a ball \( B \) is smaller than, say, 20, then one can control from below the size of the nodal set in \( B \). This can be seen for harmonic functions in the unit ball using the mean value principle and the isoperimetric inequality, and can be adapted to eigenfunctions on balls of radius \( C/\sqrt{\lambda} \).

The main claim is that on at least, say, 10% of the balls in the collection \( \mathcal{B} \), the growth is bounded by 20. We sketch the main ideas involved in the proof of this claim.

1. Let us assume \( M = \{|x| < 30\} \subset \mathbb{R}^n \). One can continue the function \( u_\lambda \) to a holomorphic function \( F \) on \( M \times M \subset \mathbb{C}^n \), with \( F|_{M \times \{0\}} = u_\lambda \). We let \( Q \subset M \times \{0\} \) be a Euclidean real cube. A key fact is that due to Theorem 4.2, the growth of \( F \) in \( M \times M \) is controlled by \( \sqrt{\lambda} \).

2. We subdivide \( Q \) to small subcubes \( Q_\nu \) of size \( 1/\sqrt{\lambda} \). In order to bound the growth of \( u_\lambda \) in a cube \( Q_\nu \) by a constant independent of \( \lambda \), it is enough to say that \( u_\lambda (x) \) is close to the average of \( u \) on \( Q_\nu \), for most of the points \( x \in Q_\nu \).

3. The last property behaves well under averaging. Therefore, it can be reduced to a one-dimensional problem:

\[
Q = [-1, 1] \subset \mathbb{R}, \quad B = \{|z| < 2\} \subset \mathbb{C},
\]

where \( F \) is a holomorphic function defined on \( B \), \( F|_Q \) is real, and the growth of \( F \) in \( B \) is bounded by \( \sqrt{\lambda} \).

4. For simplicity, let us assume that \( F \) is a polynomial (of one complex variable) of degree \( \sqrt{\lambda} \). We subdivide \( Q \) into subintervals \( Q_\nu \) of size \( 1/\sqrt{\lambda} \). One has to show that \( F|_{Q_\nu} \) is close to its average on \( Q_\nu \) for at least 10% of these subintervals. Use the Hilbert transform to attack the one-dimensional problem.

**Upper bound.** As before, we assume \( M = \{|x| < 30\} \subset \mathbb{R}^n \). The size of the nodal set is estimated from above by an integral geometry argument. One needs to estimate from above the number of zeros of \( u_\lambda \) on intervals \( J \subset M \). As in the lower bound case, we let \( F \) be the holomorphic continuation of \( u_\lambda \) to \( M \times M \subset \mathbb{C}^n \). Given an interval \( J \subset M \), one can consider its complexification in \( M \times M \). This is a strip of real two dimensions, \( J \subset \mathbb{C}^n \), and \( F|_J \) is holomorphic. One uses Jensen’s formula on the function \( F|_J \) in order to estimate from above the number of zeros of \( F \) on \( J \). In order to use Jensen’s formula one must have a bound on the growth of \( F|_J \). This is where Theorem 4.2 comes in.

5. **Proof of Theorem 2.2**

5.1. **Preliminary geometric estimates**

Let \( N \) be a Riemannian manifold of dimension \( n \). Fix a point \( p \), and let \( r(x) = \text{dist}(x, p) \). Let

\[
\gamma_K = \Delta r - (n - 1) \cot_K r;
\]

\( \gamma_K \) is controlled by the curvature of \( N \).

**Lemma 5.1.** If \( \kappa \leq \text{Sec}_N \leq K \), then \( 0 \leq \gamma_K \leq (n - 1)(\cot_K r - \cot_K r) \).

**Proof.** Both parts directly follow from the Hessian comparison theorem [4, 15].
Lemma 5.2. Suppose \( \kappa \leq \Sec_N \leq K \). Then, we have
\[
\gamma_{K,r} \geq - (n-1)(K-\kappa).
\]

Proof. We know [14, Chapter 9.1]
\[
\gamma_{K,r} = (\Delta r)_r + \frac{n-1}{(\sin_K r)^2} - \text{Ric}(\partial_r, \partial_r) - ||\text{Hess}(r)||^2 + \frac{n-1}{(\sin_K r)^2}.
\]
By the Hessian comparison theorem [15],
\[
(\cot K r)||X||^2 \leq \text{Hess}(r)(X, X) \leq (\cot_\kappa r)||X||^2.
\]
Hence,
\[
||\text{Hess}(r)||^2 = \sum ||\text{Hess}(r)(e_i, e_i)||^2 \leq (n-1)(\cot_\kappa r)^2.
\]
Consequently,
\[
\gamma_{K,r} \geq -(n-1)K - (n-1)(\cot_\kappa r)^2 + \frac{n-1}{(\sin_K r)^2} = (n-1)(\cot_K r - \cot_\kappa r)
\]
\[
\geq -(n-1)(K-\kappa),
\]
where the last inequality follows from Lemma 5.3 (iii), (iv).

Lemma 5.3.

(i) \(-\frac{1}{2} \leq (\sqrt{x} \cot \sqrt{x})' \leq 0 \) for all \( 0 \leq x < (\pi/2)^2 \);
(ii) \(0 \leq (\sqrt{x} \coth \sqrt{x})' \leq \frac{1}{2} \) for all \( x \geq 0 \);
(iii) \(-1 \leq (x \cot^2 \sqrt{x})' \leq 0 \) for all \( 0 \leq x < (\pi/2)^2 \);
(iv) \(0 \leq (x \coth^2 \sqrt{x})' \leq 1 \) for all \( x \geq 0 \).

Proof. We prove the right inequality in (ii): since \( y \coth y \geq 1 \), we have \((3y + 2y \sinh^2 y)' \geq 3(\cosh y \sinh y)'\). Integrating, we conclude
\[
3y + 2y \sinh^2 y \geq 3 \cosh y \sinh y.
\]
Equivalently, \((y \coth y)' \leq 2y/3\). Hence, \((\sqrt{x} \coth \sqrt{x})' \leq \frac{1}{3}\).

We prove the left inequality in (iii):
\[
(x \cot^2 \sqrt{x})' = \cot^2 \sqrt{x} - \frac{\sqrt{x} \cot \sqrt{x}}{\sin^2 \sqrt{x}}. \tag{5.2}
\]
Observe that for \( 0 \leq y < \pi/2 \),
\[
y \cot y \leq 1. \tag{5.3}
\]
From (5.2) and (5.3), it follows that
\[
(x \cot^2 \sqrt{x})' \geq \cot^2 \sqrt{x} - \frac{1}{\sin^2 \sqrt{x}} = -1.
\]
The proofs of all the other inequalities in the lemma are omitted. \( \square \)
5.2. Choice of coordinates and notation

We take geodesic polar coordinates centred at \( p \in N \). Fix any \( K \in \mathbb{R} \). The metric can be written as

\[
g = dr^2 + (\sin_K r)^2 (a_K)_{ij} d\theta^i d\theta^j,
\]

where \( \theta^i \) are coordinates on the standard unit sphere \( S^{n-1} \subset \mathbb{R}^n \).

We denote the determinant of the matrix \( (a_K)_{ij} \) by \( a_K \). The Laplacian on \( N \) can be written as

\[
(\Delta f)(r, \theta) = f_{rr}(r, \theta) + ((n - 1) \cot_K r + \gamma_K) f_r(r, \theta) + \frac{1}{(\sin_K r)^2} (\Delta_S f(r, \cdot))(\theta),
\]

where \( \Delta_S \) is the following operator acting on functions \( g \) defined on \( S^{n-1} \):

\[
(\Delta_S g)(\theta) := \frac{1}{\sqrt{a_K}} \frac{\partial}{\partial \theta^i} \left( a_{ij}^{\gamma_K} \sqrt{a_K} \frac{\partial g}{\partial \theta^j} \right).
\]

We emphasize that the definition of \( \Delta_S \) depends on our choice of \( K \). With these definitions, we also have

\[
\gamma_K = \frac{((\sqrt{a_K})_r)}{\sqrt{a_K}}.
\]

5.3. Proof of Theorem 2.2(i)

We observe

\[
q(r) = \int u^2 (\sin_K r)^{n-1} \sqrt{a_K} d\theta,
\]

where the integration is understood to be performed over the parameter space \([0, \pi]^{n-2} \times [0, 2\pi]\) for \( S^{n-1} \) in \( \mathbb{R}^{n-1} \). A straightforward computation shows the following.

**Lemma 5.4.**

\[
q'(r) = \int 2uu_r (\sin_K r)^{n-1} \sqrt{a_K} d\theta + \int u^2 \gamma_K (\sin_K r)^{n-1} \sqrt{a_K} d\theta + (n - 1)(\cot_K r) \int u^2 (\sin_K r)^{n-1} \sqrt{a_K} d\theta.
\]

**Lemma 5.5.**

\[
\int 2uu_r (\sin_K r)^{n-1} \sqrt{a_K} d\theta \geq 0.
\]

**Proof.** By Green’s formula and the harmonicity of \( u \)

\[
\int 2uu_r (\sin_K r)^{n-1} \sqrt{a_K} d\theta = \int_{\partial B(p, r)} \frac{\partial (u^2)}{\partial n} dA_r = \int_{B(p, r)} \Delta(u^2) d\text{Vol} = \int_{B(p, r)} 2|\nabla u|^2 d\text{Vol}. \]

**Proof of Theorem 2.2(i).** Part (i) of the theorem follows directly from Lemmas 5.4, 5.5 and 5.1.
5.4. Proof of Theorem 2.2 (ii)

Let \( w = (\sin K r)^l u \), where \( l = (n - 2)/2 \), then \( w \) satisfies the equation

\[
w_{rr} + (\cot K r + \gamma_K)w_r + l(l + 1)Kw - \frac{l^2 w}{(\sin K r)^2} + \frac{\Delta_S w}{(\sin K r)^2} = 0. \tag{5.4}\]

Let

\[
Q(r) = \int w(r, \theta)^2 \sqrt{a_K} d\theta = \frac{q(r)}{\sin K(r)}. \tag{5.5}
\]

Let us also set

\[
\nabla_S w := (\sin K r) \left( \nabla w - w_r \frac{\partial}{\partial r} \right) = \frac{1}{\sin K r} a_{ij} \frac{\partial w}{\partial \theta^i} \frac{\partial}{\partial \theta^j};
\]

note that \( \nabla_S \) is defined in this way in order to have Green’s formula:

\[
\int f(\theta)(\Delta_S g)(\theta)\sqrt{a_K} d\theta = -\int \langle \nabla_S f, \nabla_S g \rangle \sqrt{a_K} d\theta. \tag{5.6}
\]

Note also that \( \langle \nabla_S w, \partial_r \rangle = 0 \).

**Lemma 5.6.**

(i) \( Q'(r) = \int 2w(w_r + \gamma_K w/2)\sqrt{a_K} d\theta; \)

(ii) \( Q'(r) \geq (n - 2)(\cot K r)Q(r) \geq 0. \)

**Proof.** Part (i) is a direct calculation. Part (ii) is just another formulation of Theorem 2.2(i). \( \square \)

A second direct calculation using equation (5.4) and formula (5.6) gives the following lemma.

**Lemma 5.7.**

\[
Q''(r) + (\cot K r)Q'(r) = \int 2 \left( w_r + \frac{\gamma_K}{2} w \right)^2 \sqrt{a_K} d\theta + \frac{2}{(\sin K r)^2} \int |\nabla_S w|^2 d\theta
\]

\[
+ \frac{2l^2}{(\sin K r)^2} \int w^2 \sqrt{a_K} d\theta - 2l(l + 1)K \int w^2 \sqrt{a_K} d\theta
\]

\[
+ \int w^2 \left( \gamma_{K,r} + \gamma_K \cot K r + \frac{\gamma_K^2}{2} \right) \sqrt{a_K} d\theta.
\]

**Lemma 5.8.**

\[
Q''(r) + (\cot K r)Q'(r) \geq 2 \left( w_r + \frac{\gamma_K}{2} w \right)^2 \sqrt{a_K} d\theta
\]

\[
+ \frac{2}{(\sin K r)^2} \int |\nabla_S w|^2 \sqrt{a_K} d\theta + \frac{2l^2}{(\sin K r)^2} Q
\]

\[
- 2l(l + 1)KQ - (n - 1)(K - \kappa)Q.
\]
Proof. This estimate is due to Lemma 5.7 and the estimates on $\gamma_K$ and $\gamma_{K,r}$ in Lemmas 5.1 and 5.2, respectively.

Immediately, we obtain the following result.

Lemma 5.9.

\[
Q''(r) + (\cot r)Q'(r) - \frac{Q'(r)^2}{Q(r)} \geq 2 \int \left( w_r + \frac{\gamma_K}{2}w \right)^2 \sqrt{a_K} d\theta \\
+ \frac{2}{(\sin r)^2} \int |\nabla_S w|^2 \sqrt{a_K} d\theta \\
+ \frac{2l^2}{(\sin r)^2} Q - 2l(l + 1)KQ - (n - 1)(K - \kappa)Q
\]

Lemma 5.10.

\[
Q''(r) + (\cot r)Q'(r) - \frac{Q'(r)^2}{Q(r)} \geq \frac{\varphi(r)}{(\sin r)^2} + \frac{2l^2}{(\sin r)^2} Q - 2l(l + 1)KQ - (n - 1)(K - \kappa)Q,
\]

where

\[
\varphi(r) = -2(\sin r)^2 \int w_r^2 \sqrt{a_K} d\theta + 2 \int |\nabla_S w|^2 \sqrt{a_K} d\theta.
\]

Proof.

\[
Q''(r) + (\cot r)Q'(r) - \frac{Q'(r)^2}{Q(r)} \geq 2 \int \left( w_r + \frac{\gamma_K}{2}w \right)^2 \sqrt{a_K} d\theta + \frac{2}{(\sin r)^2} \int |\nabla_S w|^2 \sqrt{a_K} d\theta \\
+ \frac{2l^2}{(\sin r)^2} Q - 2l(l + 1)KQ - (n - 1)(K - \kappa)Q
\]

\[
\geq 2 \left( \int w_r^2 \sqrt{a_K} d\theta + \frac{2}{(\sin r)^2} \int |\nabla_S w|^2 \sqrt{a_K} d\theta \right) \\
+ \frac{2l^2}{(\sin r)^2} Q - 2l(l + 1)KQ
\]

\[
- (n - 1)(K - \kappa)Q - 2 \left( \int w_r \sqrt{a_K} d\theta + \int \frac{\gamma_K}{2}w \sqrt{a_K} d\theta \right)^2 \\
\int w^2 \sqrt{a_K} d\theta
\]

\[
\geq \frac{2}{(\sin r)^2} \int |\nabla_S w|^2 \sqrt{a_K} d\theta + \frac{2l^2}{(\sin r)^2} Q - 2l(l + 1)KQ
\]

\[
- (n - 1)(K - \kappa)Q - 2 \left( \int w_r \sqrt{a_K} d\theta + \int \frac{\gamma_K}{2}w \sqrt{a_K} d\theta \right)^2 \\
\int w^2 \sqrt{a_K} d\theta
\]

\[
\geq \frac{\varphi(r)}{(\sin r)^2} + \frac{2l^2}{(\sin r)^2} Q - \frac{\gamma_K}{2}w \sqrt{a_K} d\theta - \frac{\gamma_K}{2}w \sqrt{a_K} d\theta + \left( \int \frac{\gamma_K}{2}w \sqrt{a_K} d\theta \right)^2 \\
\int w^2 \sqrt{a_K} d\theta
\]

\[
- 2l(l + 1)KQ - (n - 1)(K - \kappa)Q
\]
The first inequality is just a rewriting of Lemma 5.9. In the second inequality, we applied the Cauchy–Schwarz inequality to the last term. In the third inequality, we unfolded the parentheses in the last term and applied the Cauchy–Schwarz inequality to the term \(\int w w_r \sqrt{\alpha_K} \, d\theta\). In the last inequality, we used the fact that \(Q' \geq 0\) (Lemma 5.6) and the estimates on \(\gamma_K\) in Lemma 5.1.

It remains to control the function \(\varphi\) in terms of \(Q\) and \(Q'\). We would first like to calculate the derivative of \(\varphi\). To that end, we recall the definition and some of the properties of the Hessian as a bilinear form:

\[
\text{Hess}(X, Y) := XYf - (\nabla_X Y)f = \langle Y, \nabla_X \text{grad} f \rangle = \langle X, \nabla_Y \text{grad} f \rangle.
\]

In a geodesic ball centred at \(p\), we have a radial field \(r = \partial_r\), tangent to the geodesics emanating from \(p\). Since \(\partial_r\) is tangent to a geodesic, we have \(\nabla_{\partial_r} \partial_r = 0\). As a consequence, \((\text{Hess})_r) (\partial_r, Y) = 0\) for all vectors \(Y\). When computing the derivative of \(\varphi\), it is convenient to have the following formula.

**Lemma 5.11.**

\[
(|\nabla Sf|^2)_r = 2(\nabla Sf, \nabla Sf_r) - 2 \text{Hess}(r)(\nabla Sf, \nabla Sf) + 2(\cot_K r) |\nabla Sf|^2.
\]

**Proof.**

\[
2 \text{Hess}(r)(\nabla Sf, \nabla Sf) = 2(\sin_K r)^2 \text{Hess}(r)(\nabla f, \nabla f)
\]

\[
= 2(\sin_K r)^2 \langle \nabla f, \nabla \nabla f \partial_r \rangle = 2(\sin_K r)^2 \langle \nabla f, \nabla \nabla f \rangle + |\nabla f, \partial_r |
\]

\[
= \langle \sin_K r \rangle ^2 (|\nabla f|^2)_r + 2(\sin_K r)^2 \langle \nabla f, \nabla f \rangle
\]

\[
= -\langle \sin_K r \rangle ^2 (f_r^2 + (\sin_K r)^{-2} |\nabla Sf|^2)_r + 2(\sin_K r)^2 f_r f_{rr}
\]

\[
+ 2\langle \nabla Sf, \nabla Sf_r \rangle
\]

\[
= -(|\nabla Sf|^2)_r + 2(\cot_K r) |\nabla Sf|^2 + 2\langle \nabla Sf, \nabla Sf_r \rangle.
\]

Using the formula in Lemma 5.11, we can readily compute the derivative of \(\varphi(r)\) (defined in Lemma 5.10).

**Lemma 5.12.**

\[
\varphi'(r) = -4 \int \text{Hess}(r)(\nabla S w, \nabla S w) \sqrt{\alpha_K} \, d\theta
\]

\[
+ 4(\cot_K r) \int |\nabla S w|^2 \sqrt{\alpha_K} \, d\theta + 2l(l + 1)K(\sin_K r)^2 Q' - 2l^2 Q'
\]

\[
+ 2(\sin_K r)^2 \int |\nabla w|^2 \gamma_K \sqrt{\alpha_K} \, d\theta
\]

\[
+ 2l^2 \int w^2 \gamma_K \sqrt{\alpha_K} \, d\theta - 2l(l + 1)K \sin_K^2 r \int w^2 \gamma_K \sqrt{\alpha_K} \, d\theta.
\]
Lemma 5.13.

\[ \varphi'(r) \geq -4(\cot \kappa r - \cot K r) \left| \nabla_S w \right|^2 \sqrt{a_K} d\theta + 2l(l + 1)K(\sin K r)^2Q' - 2l^2Q' - 2l(l + 1)(\cot \kappa r - \cot K r)K^+(\sin K r)^2Q. \]

Proof. This is due to inequality (5.1) and Lemma 5.12. \qed

In Lemma 5.17, we integrate the inequality in Lemma 5.13. We need a few lemmas before that.

Lemma 5.14.

\[ \left( 1 - \frac{2}{n} \right) (\sin K r)^2 Q(r) \leq \int_0^r (\sin K \rho)^2 Q'(\rho) d\rho \leq (\sin K r)^2 Q(r). \]

Proof. The right-hand side follows from the fact that \( \sin K \rho \) is monotonically increasing in \( \rho \), and \( Q' \geq 0 \). By differentiating the left-hand side, we see that it is enough to prove

\[ \left( 1 - \frac{2}{n} \right) (\sin K r)^2 (2(\cot K r)Q(r) + Q'(r)) \leq (\sin K r)^2 Q'(r). \]  \hfill (5.7)

Inequality (5.7) is equivalent to Lemma 5.6(ii). \qed

Lemma 5.15.

\[ \int_0^r \left| \nabla_S w \right|^2 \sqrt{a_K} d\theta d\rho \leq \frac{(\sin K r)^2}{2} (Q'(r) - (n - 2)(\cot K r)Q). \]

Proof.

\[
\begin{align*}
\int_0^r \left| \nabla_S w \right|^2 \sqrt{a_K} d\theta d\rho &= \int_0^r \left| \nabla_S u \right|^2 (\sin K \rho)^n \sqrt{a_K} d\theta d\rho \\
&\leq \int_0^r \left| \nabla u \right|^2 (\sin K \rho)^n \sqrt{a_K} d\theta d\rho \\
&\leq \sin K r \int_0^r \left| \nabla u \right|^2 (\sin K \rho)^{n-1} \sqrt{a_K} d\theta d\rho \\
&= \sin K r \int_{B(p,r)} \left| \nabla u \right|^2 \text{Vol} = \sin K r \int_{u_r} \left| \nabla u \right|^2 (\sin K r)^{n-1} \sqrt{a_K} d\theta \\
&= (\sin K r)^2 \int w(w_r + \gamma K w/2) \sqrt{a_K} d\theta \\
&= (\sin K r)^2 \int w(w_r + \gamma K w/2) \sqrt{a_K} d\theta \\
&\leq \frac{(\sin K r)^2}{2} (Q'(r) - (n - 2)(\cot K r)Q). \quad \square
\end{align*}
\]
Lemma 5.16.\[
\int_0^r (\sin_K \rho)^2 Q(\rho) \, d\rho \leq r (\sin_K r)^2 Q(r).
\]

Proof. The functions $\sin_K \rho$ and $Q(\rho)$ are monotonically increasing in $\rho$. \hfill \Box

Lemma 5.17.\[
\frac{\varphi(r)}{(\sin_K r)^2} \geq -2(\cot_\kappa r - \cot_K r) (Q' - (n - 2)(\cot_K r)Q) + \frac{n(n-2)}{2} KQ - (n-2)K^+Q - \frac{(n-2)^2 Q}{2(\sin_K r)^2} - \frac{n}{2}(n-1)(n-2)(\cot_\kappa r - \cot_K r)rK^+Q.
\]

Proof. Observe that the functions $\cot_\kappa r - \cot_K r$ and $\sin_K r$ are both monotonically increasing. Hence, integrating Lemma 5.13, while applying Lemmas 5.14–5.16, we obtain \[
\varphi(r) \geq -4(\cot_\kappa r - \cot_K r) \int_0^r \int |\nabla_{Sw}|^2 \sqrt{a_K} \, d\theta \, d\rho + 2l(l+1)K \int_0^r (\sin_K \rho)^2 Q'(\rho) \, d\rho - 2l^2 Q \\
- 2l(\cot_\kappa r - \cot_K r)K^+ \int_0^r (\sin_K \rho)^2 Q(\rho) \, d\rho \geq -2(\sin_K r)^2(\cot_\kappa r - \cot_K r) (Q' - (n - 2)(\cot_\kappa r - \cot_K r)(\cot_K r)Q) + 2l(l+1)K(\sin_K r)^2 Q(r) - 2l(l+1)K^+(2/n)(\sin_K r)^2 Q - 2l^2 Q \\
- (n-2)(\cot_\kappa r - \cot_K r)rK^+(\sin_K r)^2 Q(r). \hfill \Box
\]

Proof of Theorem 2.2(ii). From Lemmas 5.10 and 5.17, we obtain \[
Q''(r) + (\cot_\kappa r)Q'(r) - \frac{Q'(r)^2}{Q(r)} + (n-1)(\cot_\kappa r - \cot_K r)Q'(r) \geq -2(\cot_\kappa r - \cot_K r)Q'(r) + 2(n-2)(\cot_\kappa r - \cot_K r)(\cot_K r)Q + \frac{n(n-2)}{2} KQ(r) - (n-2)K^+Q - \frac{(n-2)^2 Q}{2(\sin_K r)^2} Q \\
- (n-2)(\cot_\kappa r - \cot_K r)rK^+Q - \frac{n(n-2)}{2} KQ + \frac{(n-2)^2}{2(\sin_K r)^2} Q - (n-1)(K-\kappa)Q \\
= -2(\cot_\kappa r - \cot_K r)Q'(r) + 2(n-2)(\cot_\kappa r - \cot_K r)(\cot_K r)Q - (n-2)K^+Q - (n-2)(\cot_\kappa r - \cot_K r)rK^+Q - (n-1)(K-\kappa)Q. \quad (5.8)
\]
where we applied Lemma 5.3(i),(ii). Recall \( q = 5.5. \)

**Proof of Corollary**

Finally, summing (5.9) and (5.10) gives the statement in the theorem.

\[ l < t \]

**5.5. Proof of Corollary 2.3**

**Proof of Corollary 2.3.** From Theorem 2.2,

\[ q''(r) + (\cot_K r)q' + (n + 1)(\cot_{-K} r - \cot_K r)q'(r) \geq -(5n - 7)Kq. \]  

(5.11)

From Lemma 5.3 and from the fact that \( q' \geq 0 \) (Theorem 2.2(i)) we know

\[ (n + 1)(\cot_{-K} r - \cot_K r)q' \leq 2(n + 1)rKq' / 3. \]  

(5.12)

From Theorem 2.2(i), we know

\[ -(5n - 7)Kq \geq -5Kq'(r) / \cot_K r. \]  

(5.13)

It is easy to check \( 1 / \cot_K r \leq 2r \) for \( r \leq \pi / (3\sqrt{K}) \).

Hence, from inequalities (5.11)–(5.13) we obtain

\[ q''(r) + \frac{1 + 8nr^2K}{r}q' - \frac{q'(r)^2}{q(r)} \geq 0 \]  

(5.14)

for \( r \sqrt{K} < \pi / 3. \) If we define \( l(t) = q(e^t) \), then (5.14) is equivalent to

\[ l''(t) + 8nKe^{2t}l'(t) \geq 0 \]  

(5.15)

for \( t < -(\log K) / 2 + \log(\pi / 3). \) We will now integrate inequality (5.15).

Inequality (5.15) can be rewritten as \( (e^{4nK(e^{2t} - e^{2s})})')' \geq 0 \), from which we see that for \( s_2 < s_1 \),

\[ l'(s_2) \leq e^{4nK(e^{2s_1} - e^{2s_2})}l'(s_1) \leq e^{4nK(e^{2s_1} - e^{2t_1})}l'(s_1), \]  

(5.16)

where the last inequality is true since \( l'(s) \geq 0 \) from Theorem 2.2(i). Hence, for \( t_2 < t_1 \) such that \( 16nKe^{2t_1} < 1 \), and \( 0 \leq h \leq \log 2 \),

\[ l(t_2 + h) - l(t_2) = \int_0^h l'(t_2 + s) ds \leq \int_0^h e^{4nK(e^{2t_1 + 2h} - e^{2s_2})}l'(t_1 + s) ds \leq e^{4nKe^{2t_1 + 2h}(l(t_1 + h) - l(t_1))} \]

\[ \leq (1 + 32nKe^{2t_1})(l(t_1 + h) - l(t_1)). \]

The last inequality follows from \( e^x \leq 1 + 2x \) for \( 0 \leq x \leq 1. \)

Going back from the variable \( t \) to the variable \( r \), we obtain the stated corollary.
6.  The case of constant curvature manifolds

We give a new proof of Theorem 2.1 and a second proof of Theorem 2.2 in the case of constant non-zero curvature in two dimensions.

6.1. Zero curvature

Let \( u_l(r, \theta) = r^l \cos(l \theta) \) and \( v_l = r^l \sin(l \theta) \); then \( q_{u_l}(r) = q_{v_l}(r) = \pi r^{2l+1} \). It is obvious that \( \log q_l \) is a convex function of \( \log r \).

Now, any harmonic function can be written as

\[
 u = a_0 + \sum_{l=1}^{\infty} a_l u_l(r, \theta) + b_l v_l(r, \theta).
\]

The functions \( u_l(r, \cdot) \) and \( v_l(r, \cdot) \) are pairwise orthogonal as functions on the unit circle for all fixed \( r \). For any two orthogonal functions \( f \) and \( g \) on the unit circle for all fixed \( r \), we have \( q_{f+g}(r) = q_f(r) + q_g(r) \). We also know that the sum of log-convex functions is log-convex and the pointwise limit of log-convex functions is log-convex. These considerations give a short new proof of Theorem 2.1.

**Remark.** A similar argument also carries in dimensions greater than or equal to three.

6.2. Positive curvature, two dimensions

The metric on the two-dimensional sphere of constant curvature \( K > 0 \) is given by

\[
 ds^2 = dr^2 + (\sin K r)^2 d\theta^2.
\]

Here, \( 0 < r < \pi / \sqrt{K} \), and \( 0 \leq \theta \leq 2\pi \). Hence,

\[
 q^K_u(r) = \int_0^{2\pi} u(r, \theta)^2 (\sin K r) \, d\theta.
\]

We also define \( q^0_f(r) = \int_0^{2\pi} f(r, \theta)^2 r \, d\theta \) for functions defined on \( \mathbb{R}^2 \).

Let \( f(r, \theta) \) be defined on \( \mathbb{R}^2 \) by \( u(r, \theta) = f(\tan(r \sqrt{K} / 2), \theta) \); then \( f \) is related to \( u \) by a stereographic projection. Since harmonic functions are preserved under conformal transformations in two dimensions, \( f(r, \theta) \) is harmonic if and only if \( u(r, \theta) \) is harmonic. We also note the relation

\[
 q^K_u(r) = \frac{q^0_f(\tan(r \sqrt{K} / 2))}{\tan(r \sqrt{K} / 2)} \sin K r.
\]

Suppose now \( f \) is harmonic. Then, from the fact that \( \log q^0_f \) is a convex function of \( \log r \), we obtain the following.

**Theorem 6.1.** If \( K > 0 \), then

\[
 (\log q^K_u)^\prime\prime(r) + (\cot K r)(\log q^K_u)^\prime(r) \geq -K.
\]
6.3. **Negative curvature**

In the spherical example one can replace all trigonometric functions by the corresponding hyperbolic functions and obtain the following result.

**Theorem 6.2.** If $K < 0$, then
\[
(\log q^K_u)'(r) + (\cot_K r)(\log q^K_u)'(r) \geq -K > 0.
\]

7. **Discussion**

We raise several questions which we find interesting to pursue.

7.1. **Beyond the injectivity radius**

It would be interesting to understand whether Theorem 2.2 remains true beyond the injectivity radius as long as $r\sqrt{K^+} < \pi / 2$ in the spirit of Bishop–Gromov’s volume comparison theorem [8].

7.2. **Proof by an orthogonal basis of functions**

In a manifold of constant curvature $K \neq 0$, of dimension greater than or equal to three, we would like to have a simple proof, inspired by the proof presented in Section 6 for the case $K = 0$. This would also shed light on the sharpness of Theorem 2.2 in dimensions $n \geq 3$.

7.3. **Ricci curvature**

Can one of the bound assumptions on the sectional curvature in Theorem 2.2 be relaxed to a bound on the Ricci curvature?

7.4. **Eigenfunctions on negatively curved manifolds**

Can we replace the extension procedure described in Section 3 by a procedure which will give us more information on the growth of eigenfunctions on negatively curved manifolds?

7.5. **A comparison theorem for positive harmonic functions**

Let $f(\theta)$ be a $2\pi$-periodic non-negative function. Let $u$ be a solution of the Dirichlet problem in the unit disk: $\Delta u = 0$ with $u(1, \theta) = f(\theta)$. Now, suppose we consider the unit geodesic disk in a Riemannian manifold with non-positive variable curvature, and solve the Dirichlet problem there. We obtain a solution $v(r, \theta)$. Can we compare the values of $u$ to the values of $v$? Or equivalently, can we compare the Poisson kernels involved?

**Acknowledgements.** I am grateful to Leonid Polterovich and Misha Sodin for encouraging me to write this paper and for several discussions concerning it. I am thankful to Józef Dodziuk for pleasurable relevant discussions. I owe my gratitude to Shing-Tung Yau for his support and for stimulating questions which left their imprint on this paper. I thank the anonymous referee for his valuable comments.

**References**

1. S. Agmon, *Unicité et convexité dans les problèmes différentiels*, Séminaire de Mathématiques Supérieures, No. 13 (Été, 1965), Les Presses de l’Université de Montréal, Montréal, Quebec, 1966.
CONVEXITY PROPERTIES OF HARMONIC FUNCTIONS

2. L. V. Ahlfors, An introduction to the theory of analytic functions of one complex variable, Complex analysis, 3rd edn., International Series in Pure and Applied Mathematics (McGraw-Hill, New York, 1978).
3. F. J. Almgren Jr., Dirichlet’s problem for multiple valued functions and the regularity of mass minimizing integral currents, Minimal submanifolds and geodesics (Proc. Japan-United States Sem., Tokyo, 1977), North-Holland, Amsterdam, 1979, pp. 1–6.
4. R. L. Bishop and R. J. Crittenden, Geometry of manifolds, Pure and Applied Mathematics XV (Academic Press, New York, 1964).
5. J. Brüning, ‘Über Knoten von Eigenfunktionen des Laplace–Beltrami-Operators’, Math. Z. 158 (1978) 15–21.
6. H. Donnelly and C. Fefferman, ‘Nodal sets of eigenfunctions on Riemannian manifolds’, Invent. Math. 93 (1988) 161–183.
7. N. Garofalo and F.-H. Lin, ‘Monotonicity properties of variational integrals, $A_p$ weights and unique continuation’, Indiana Univ. Math. J. 35 (1986) 245–268.
8. M. Gromov, Structures métriques pour les variétés riemanniennes, Textes Mathématiques [Mathematical Texts], vol. 1 (Cédic, Paris, 1981).
9. Q. Han and F.-H. Lin, ‘Nodal sets of solutions of elliptic differential equations’, available at http://nd.edu/~qhan/nodal.pdf.
10. D. Jerison and G. Lebeau, ‘Nodal sets of sums of eigenfunctions’, Harmonic analysis and partial differential equations, Chicago Lectures in Mathematics (University of Chicago Press, Chicago, IL, 1999) 223–239.
11. E. M. Landis, ‘Some questions in the qualitative theory of second-order elliptic equations (case of several independent variables)’, Uspehi Mat. Nauk 18 (1963) 3–62; English transl., Russian Math. Surveys 18 (1963) 1–62.
12. F.-H. Lin, ‘Nodal sets of solutions of elliptic and parabolic equations’, Comm. Pure Appl. Math. 44 (1991) 287–308.
13. F. Nazarov, L. Polterovich and M. Sodin, ‘Sign and area in nodal geometry of Laplace eigenfunctions’, Amer. J. Math. 127 (2005) 879–910.
14. P. Petersen, Riemannian geometry, 2nd edn. Graduate Texts in Mathematics 171 (Springer, New York, 2006).
15. R. Schoen and S.-T. Yau, ‘Lectures on differential geometry’, Conference Proceedings and Lecture Notes in Geometry and Topology, I (International Press, Cambridge, MA, 1994).
16. S.-T. Yau, ‘Problem section’, Seminar on differential geometry, Annals of Mathematical Studies 102 (Princeton University Press, Princeton, NJ, 1982) 669–706.

Dan Mangoubi
Einstein Institute of Mathematics
Hebrew University, Givat Ram
Jerusalem 91904
Israel
mangoubi@math.huji.ac.il