Heavy-to-light form factors for non-relativistic bound states

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We investigate transition form factors between non-relativistic QCD bound states at large recoil energy. Assuming the decaying quark to be much heavier than its decay product, the relativistic dynamics can be treated according to the factorization formula for heavy-to-light form factors obtained from the heavy-quark expansion in QCD. The non-relativistic expansion determines the bound-state wave functions to be Coulomb-like. As a consequence, one can explicitly calculate the so-called “soft-overlap” contribution to the transition form factor.

1. Introduction

The factorization of high-energy hadronic processes into perturbative QCD scattering amplitudes and universal hadronic parameters is one of the most powerful tools in strong-interaction physics. In the case of massive $B$-meson decays into light energetic hadrons, this factorization is complicated due to the presence of three relevant scales: (i) a “hard” scale set by the mass $m_b$ of the decaying $b$-quark, (ii) a “soft” (“collinear”) scale $\Lambda \ll m_b$ set by the non-perturbative dynamics in the initial (final) hadronic bound states, and (iii) a “hard-collinear” scale $\mu_{hc} \sim \sqrt{m_b \Lambda}$ with $\Lambda \ll \mu_{hc} \ll m_b$ which appears due to interactions between the low-energetic particles in the initial and the high-energetic particles in the final state.

A systematic separation of these scales in exclusive two-body $B$-decays has been worked out in [1]. The quantum-field theoretical formalism has been developed in terms of a so-called “soft-collinear effective theory” (SCET) in [2,3]. In the following we consider the form factor for a $B$-meson transition into a single energetic light hadron, for instance a pion. Schematically, the factorization formula in the heavy-quark limit, $m_b \to \infty$, reads [2]

$$\langle \pi|\bar{\psi} \Gamma_i b|B\rangle \simeq T_i^1 \cdot \xi_\pi + T_i^1 \otimes \phi_B \otimes \phi_{\pi}$$

(1)

Here, the short-distance function $T_i^1$ contains dynamics from the hard scale, whereas $T_i^{1,1}$ contains hard and hard-collinear dynamics. The second term in (1) is thus completely factorized, and the light-cone distribution amplitudes $\phi_B$ and $\phi_{\pi}$, which provide the non-perturbative information, contain dynamics below $\mu_{hc}$ only. This is not the case for the first term in (1) where the so-called “soft” form factor $\xi_\pi$ still contains dynamics at the hard-collinear scale and cannot be factorized further. These non-factorizable contributions are related to endpoint divergences which appear when one of the partons in the perturbative scattering amplitude has almost vanishing (light-cone) energy, see the discussion in [5,6,7].

In (1) the non-factorizable form factor $\xi_\pi$ does not depend on the Dirac structure $\Gamma_i$ of the $b$-quark decay. As a consequence, heavy-to-light form factors obey symmetry relations [8] which are broken by calculable perturbative effects in $T_i^{1,1}$ in the heavy-quark limit [9,10].

The idea of the presented work (see also [11]) is to extend this formalism to non-relativistic bound states where, in the limit of small relative velocities, the non-perturbative dynamics is contained in the usual Coulomb wave function. Consider, for instance, the decay $B_c \to \eta_c \ell \nu$ where we assume the hierarchy of scales

$$\Lambda_{\text{QCD}} \ll m_c \ll \sqrt{m_c m_b} \ll m_b$$

(2)

The above formalism can be applied if one identifies the charm-quark mass $m_c$ with the soft scale.
The latter is larger than the QCD scale, and the dynamics at \( m_c \) is still perturbative, such that the soft (non-factorizable) form factor can be calculated explicitly. The non-relativistic approach may also be considered as a toy model for the \( B \to \pi \) case which may shed further light on the structure of the factorization formula (1).

2. Non-relativistic bound states

\[
ψ_C = \sum_{n=0}^{\infty} \frac{1}{n!} \cdots \frac{1}{n!} \gamma_5
\]

Figure 1. Resummation of ”potential” gluons into a non-relativistic Coulomb wave function.

The wave function for a non-relativistic bound state can be obtained from the resummation of ”potential” gluon exchange, see Fig. 1. The solution of the corresponding Schrödinger equation with Coulomb potential describes a wave function which is sharply peaked around small three-momenta \( \vec{p} = O(m_r \alpha_s) \) where \( m_r \) is the reduced quark mass. The normalization of the wave function gives the meson decay constant.

In this picture, the \( B_c \) meson is described as a bound state of a heavy bottom quark with mass \( M = m_b \) and a lighter charm quark with mass \( m = m_c \). Consequently, to first approximation with respect to a simultaneous expansion in \( m/M \) and \( v = |\vec{p}|/m_r \), the \( B_c \) meson consists of a heavy quark with momentum \( mw_\mu \) and a light quark with momentum \( mw'_\mu \), where \( w_\mu \) is the four-velocity of the heavy meson \( (p_\mu \simeq Mw_\mu) \). The spinor degrees of freedom for the heavy meson in the initial state are represented by the Dirac projection \( \frac{1+w}{2} \gamma_5 \). Similarly, the \( \eta_c \) meson is represented as a \( c\bar{c} \) bound state where both constituents have approximately equal momenta \( mw'_\mu \), and the total momentum of the meson is \( p'_\mu = 2mw'_\mu \). The Dirac projection for the \( \eta_c \) meson in the final state is thus given by \( \frac{1-w'}{2} \gamma_5 \).

3. Relativistic dynamics

In the following, for simplicity, we will consider the case of maximum recoil energy, i.e. the momentum transfer is given by \( q_2^2 = (p - p')^2 = 0 \). Furthermore, we will concentrate on the form factor \( F_+ \), defined via

\[
\langle \eta_c | \bar{c} \gamma_\mu b | B_c \rangle = F_+(q^2) (p_\mu + p'_\mu) + F_-(q^2) q_\mu
\]

According to the general discussion, we have to consider hard, hard-collinear, collinear and soft gluon exchange in order to describe the relativistic dynamics of the transition form factor. Because of the large energy transfer, there is no direct overlap between the non-relativistic wave functions in the heavy-quark limit.

3.1. Spectator scattering at tree level

At tree level we have to calculate the two diagrams in Fig. 2 where a hard-collinear gluon is exchanged between the active quarks and the spectator. The result for the form factor reads

\[
F_+^{LO}(0) = \frac{4\pi \alpha_s(\mu) C_F}{N_c} \frac{3f_M f_m}{mM}
\]

where \( f_M \) and \( f_m \) denote the (non-relativistic) decay constants of the initial and final-state meson, respectively. Notice, that to this (fixed) order in perturbation theory, one does not encounter endpoint singularities, because, in contrast to the relativistic case, the light-cone wave functions of the non-relativistic bound states have vanishing support at the endpoints. As we will see below, the endpoint configurations require the exchange of at least another relativistic gluon (soft or collinear, respectively).
3.2. Spectator scattering at one-loop

In Feynman gauge one has momentum regions for the pentagon diagram. It is instructive to separate the various momentum regions that contribute when the diagram is calculated within dimensional regularization (\(D = 4 - 2\epsilon\)). In our example, the hard region is power-suppressed. The remaining regions contribute to the NLO corrections

\[
\Delta F_{\text{pent}}^{\text{NLO}} = F_{\text{pent}}^{\text{LO}} \frac{\alpha_s C_F}{4\pi} \{ I_{\text{hc}} + I_s + I_c \} .
\]

In fixed-order perturbation theory, one would like to resum these large logarithms into short-distance coefficient functions. In the context of the effective-theory approach such contributions are thus identified as non-factorizable.

In the soft and collinear momentum region we had to introduce an additional regularization procedure for the divergences related to light-cone momentum fractions of the internal loop momenta approaching the endpoints. We apply an analytic continuation of one of the gluon propagators, \(1/[k^2 + i\eta] \to (-\nu^2)^{\delta}/[k^2 + i\eta]^{1+\delta}\). In this way endpoint divergences show up as poles in \(1/\delta\).

In fixed-order perturbation theory, one would simply add up all three momentum regions at a common renormalization scale \(m \leq \mu \leq M\). This reproduces the divergences of the full QCD result, whereas the endpoint divergences and the dependence on the ad-hoc parameters \(\nu^2\) and \(\delta\) vanish.

It is instructive to separate the contributions related to the endpoint divergences in the sum of soft and collinear momentum regions

\[
I_{s+c}\big|_{\text{endpoint}} = \frac{1}{3} \ln \frac{M}{2m} \left( \frac{1}{\epsilon} + \frac{\ln^2 m^2}{m^2} + 1 \right) \quad \text{(7)}
\]

The point to notice is that the result contains large logarithms, \(\ln M/m\), which involve the ratio of the two distinct physical scales. In renormalization-group-improved perturbation theory one would like to resum these large logarithms into short-distance coefficient functions. Whether this is possible can be read off the individual momentum regions:

The hard-collinear integral \(I_{\text{hc}}\) in (6) does not contain large logarithms if one chooses a renormalization scale \(\mu \sim \sqrt{mM}\). For the soft and collinear integrals \(I_{s,c}\) the choice \(\mu \sim m\), which refers to the typical virtualities of soft and collinear fields, leads to small logarithms \(\ln \mu^2/m^2\). Still, one would be left with the large logarithm \(\ln M/m\) in (7). This implies that, in general, one cannot resum all large logarithms \(\ln m/M\) into short-distance (hard-collinear) coefficient functions. In the context of the effective-theory approach such contributions are thus identified as non-factorizable.

In the \(B \to \pi\) case, the light quark mass enters as an IR regulator of the order of \(\Lambda_{\text{QCD}}\), and the sum of soft and collinear momentum regions define a non-perturbative form factor which still depends on the heavy-quark mass in an unknown, non-analytic way. In contrast, for the non-relativistic description of the \(B_c \to \eta_c\) tran-
sition, the light charm quark mass is a physical parameter, and the soft and collinear momentum regions still refer to the perturbative QCD sector.

4. Numerical applications

For the numerical estimate we stick to fixed-order perturbation theory, summing up all diagrams at order $\alpha_s$. We do not attempt to resum large logarithms (a more thorough discussion will be given in [12]). Without resummation our result shows a sizeable but moderate dependence on the renormalization scale $\mu$. We emphasize that this dependence would become out of control in the $B \rightarrow \pi$ case where the large logarithms have to be counted as $1/\alpha_s$.

![Figure 4. Renormalization-scale dependence of $F_+(0)$ for $B_c \rightarrow \eta_c$ transitions at LO and NLO.](image)

In Fig. 4 we compare the renormalization-scale dependence of the LO and NLO result, using $M = 5$ GeV, $m = 1.5$ GeV, $\alpha_s(M) = 0.2$, $\ln [\alpha_s(m)/\alpha_s(M)] \approx 0.4$ as an example. We observe that the NLO corrections stabilize the perturbative expansion, with the scale-uncertainty improving from about 30% to 15%.

It is possible to disentangle the non-factorizable and factorizable form factor contributions, using the definitions in [3]. It turns out that, by accident (i.e. not due to an obvious symmetry), the non-factorizable part of the form factor $F_+(0)$ numerically dominates over the factorizable one by about a factor of 5. Notice that in the non-relativistic set-up both, the factorizable and non-factorizable contributions, are of order $\alpha_s(\mu)$.

5. Summary

We have shown how to describe the dynamics of non-relativistic bound states in heavy-to-light quark decays at large recoil, using the example of $B_c \rightarrow \eta_c \ell \nu$, where we consider $m_c \ll m_b$. We have worked to leading order in the non-relativistic and the $m_c/m_b$ expansion, and took into account NLO effects from relativistic gluon exchange. We found that (without resummation of large logarithms) the perturbative uncertainties of the NLO result amount to about 15%. This is already below the expected size of power-corrections (about 30%) which should be included for reliable phenomenological estimates.

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