String tension from smearing and Wilson flow methods

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Last year, we proposed a new method to extract string tension from 4 dimensionally smeared Creutz ratios (PL B718 (2013) 1524).

After reviewing this method, we first show that the same physical results can be obtained replacing the 4-d smearing technique by Wilson flow for sufficiently small time steps $\Delta t$.

We then demonstrate the practical advantage of our method by applying it to the calculation of the Creutz ratio of SU(3) Yang-Mills theory in the continuum limit.
Creutz ratio from 4-dimensional smearing method

It is well known that Wilson loops $W(R,T)$ with large $R, T$ are quite noisy.

Our proposal is to calculate Creutz ratio with 4-dimensional Ape smearing

$$U_{\text{smeared}}^{n,\mu} = \text{Proj}_{SU(3)} \left[ (1 - f)U_{n,\mu} + \frac{f}{6} \sum_{v \neq \mu = \pm 1}^{\pm 4} U_{n,v}U_{n+v,\mu}U_{n+\mu,v}^\dagger \right]$$

This form of Ape smearing has been introduced by Narayanan and Neuberger, JHEP 0603 (2006) 064.

Let $t = f * n_s / 6$ with $n_s$ the number of smearing steps.
Wilson loop and potential have huge t dependences, which makes 4-d Ape smearing almost useless for these quantities. See, however, Lohmayer and Neuberger, JHEP 1208 (2012) 102.

It is crucial to consider Creutz ratio

\[
\chi(R, T) = -\log \frac{W(R+1/2, T+1/2)W(R-1/2, T-1/2)}{W(R+1/2, T-1/2)W(R-1/2, T+1/2)}
\]

which is free from ultraviolet divergences and its t dependence is quite well fitted by

\[
\chi(t) = a \left(1 - \exp\left(\frac{-b}{t+c}\right)\right)
\]

This is the method proposed in Phys. Lett. B718 (2013) 1524.

We demonstrate this method using SU(3) LGT on a 32^4 lattice at \( \beta = 6.17 \) with \( f = 0.1 \).
dependence of Wilson loop $W(R,T)$

$t = f \times n_s / 6$ dependence of Wilson loop $W(R,T)$

$W(6,6), \ 32^4, \ \beta = 6.17, \ f = 0.1$
dependence of Potential

\[ V(R,T) = -\log \frac{W(R,T + 1/2)}{W(R,T - 1/2)} \]

\[ V(6,5.5), \; 32^4, \; \beta = 6.17, \; f = 0.1 \]
$t = f * n_s / 6$  dependence of Creutz ratio

$$\chi(R,T) = -\log \frac{W(R+1/2,T+1/2)W(R-1/2,T-1/2)}{W(R+1/2,T-1/2)W(R-1/2,T+1/2)}$$

$\chi(5.5,5.5), 32^4, \beta = 6.17, f = 0.1$

$$\chi(t) = a \left( 1 - \exp \left( -b / (t + c) \right) \right)$$
4-d Ape smearing

\[ U_{n,\mu}^{\text{smeared}} = \text{Proj}_{SU(3)} \left[ (1 - f)U_{n,\mu} + \frac{f}{6} \sum_{\nu \neq \mu = \pm 1}^{\pm 4} U_{n,\nu}U_{n+\nu,\mu}U_{n+\mu,\nu}^* \right] \]

with \( t = f \times n_s / 6 \) is equivalent to Wilson flow

\[ \frac{dV_{n,\mu}(t)}{dt} = -g^2_0 \{ \partial_{n,\mu} S_W \} V_{n,\mu}(t), \quad V_{n,\mu}(t = 0) = U_{n,\mu} \]

provided \( f \) is sufficiently small.

In our method, we use data typically at \( t \leq \frac{(R - 1/2)^2}{25} \) for \( \chi(R, R) \) with \( f \sim 0.1 \).

For Wilson flow, we use 3\(^{rd}\) order Runge-Kutta with \( \Delta t = 0.01 \).
For small $t$, 4-d smearing and Wilson flow results are essentially identical.

$\chi(5.5, 5.5), \ \beta = 6.17$

$4$-d smearing $f = 0.1$

Wilson flow
Actually, for small $t$, 4-d smearing with $f = 0.1, 0.2, 0.4$ gives essentially the same result for Creutz ratios.

$$\chi(5.5,5.5), \quad \beta = 6.17$$
For large $t$, we need some care. \[ \left\{ t^2 \left\langle E \right\rangle \right\}_{t=t_0} = 0.3 \]
Creutz ratio in the continuum limit

We made simulations at three lattices having almost same physical volume \((La)^4\)

| Lattice | \(\beta\) | \(N_{\text{cnfg}}\) |
|---------|-----------|----------------|
| 24\(^4\) | 5.96      | 1200           |
| 32\(^4\) | 6.17      | 600            |
| 48\(^4\) | 6.42      | 100            |

In this talk, we concentrate on the diagonal \(\chi(R,R)\), although there are a lot of interesting physics in off-diagonal \(\chi(R,T)\).
Introducing a scale $\bar{r}$ and writing $r = Ra$, dimensional analysis implies that there should be $O(a^2)$ lattice artifact in the $1/r^4$ term.

$$\left(\frac{\bar{r}}{a}\right)^2 \chi(R, R) = \sigma \bar{r}^2 + 2\gamma \left(\frac{\bar{r}}{r}\right)^2 + 4 \left(\frac{\bar{r}}{r}\right)^4 \left(c + d \left(\frac{a}{\bar{r}}\right)^2\right)$$

$\tilde{F}(r)$ defined by

$$\bar{r}^2 \tilde{F}(r) \equiv \sigma \bar{r}^2 + 2\gamma \left(\frac{\bar{r}}{r}\right)^2 + 4c \left(\frac{\bar{r}}{r}\right)^4$$

is the Creutz ratio in the continuum limit.

We can fix the scale a la Sommer as

$$\bar{r}^2 \tilde{F}(\bar{r}) = \sigma \bar{r}^2 + 2\gamma + 4c = 1.65$$
Eliminating \( c \) from the expression and replacing \( r \) to \( Ra \), we finally find a fitting function

\[
\left( \frac{r}{a} \right)^2 \chi(R, R) = \sigma r^2 + 2\gamma \left( \frac{r}{a} \right)^2 \frac{1}{R^2} + 4 \left( \frac{r}{a} \right)^4 \frac{1}{R^4} \left( c + d \left( \frac{a}{r} \right)^2 \right)
\]

\[
4c = 1.65 - \sigma r^2 - 2\gamma
\]

with 6 fitting parameters

\[
\sigma r^2, \gamma, d, \frac{a(\beta = 5.96)}{r}, \frac{a(\beta = 6.17)}{r}, \frac{a(\beta = 6.42)}{r}
\]

The resultant fit has \( \chi^2 / (\text{# of freedom}) = 1.05 \) with

\[
\sigma r^2 = 1.159(6) \quad \gamma = 0.250(3), \quad d = 0.24(2)
\]

\[
a(\beta = 5.96) / r = 0.2117(5),
\]

\[
a(\beta = 6.17) / r = 0.1499(2),
\]

\[
a(\beta = 6.42) / r = 0.1052(3)
\]
\[ \frac{r^2}{\alpha^2} \chi(R,R) \]
\[ \bar{r}^2 \tilde{F}(r) = \sigma \bar{r}^2 + 2\gamma \left( \frac{\bar{r}}{r} \right)^2 + 4c \left( \frac{\bar{r}}{r} \right)^4 \]
relation between $\bar{r}$ and $t_0$

| $\beta$ | $a / \bar{r}$ | $t_0 / a^2$ | $\sqrt{8t_0} / \bar{r}$ |
|---------|----------------|-------------|-------------------------|
| 5.96    | 0.2117(5)      | 2.794(3)    | 1.001(2)                |
| 6.17    | 0.1499(2)      | 5.506(7)    | 0.995(2)                |
| 6.42    | 0.1052(3)      | 11.17(3)    | 0.994(3)                |

In the continuum limit

$$\sqrt{8t_0} / \bar{r} = 0.990(3)$$

$$\sigma r^2 = 1.159(6)$$

$$\therefore \sqrt{8t_0\sigma} = 1.066(4)$$
Comparison with 3-d smearing methods

In the continuum limit, \( \sqrt{8t_0} / r_0 = 0.948(6) \). Lüscher JHEP08(2010) 071.

Then, our result is converted to \( r_0 \sqrt{\sigma} = 1.124(8) \)

Previous results of \( r_0 \sqrt{\sigma} \) derived from 3-d smeared Potential
are almost consistent with the value \( r_0 \sqrt{\sigma} = \sqrt{1.65 - \pi / 12} = 1.178 \)

It is not clear how to interpret this 5% difference, however,
they are obtained from quite different geometries of Wilson loops!

3-d smearing, \( W(r,t) \) with finite \( r \) and \( t = \infty \)

\[
r_0^2 F(r) = r_0^2 \sigma + \frac{\pi r_0^2}{12 r^2} \approx r_0^2 \sigma + 0.2612 \frac{r_0^2}{r^2}
\]

4-d smearing, \( W(r,t) \) with finite \( r \approx t \)

\[
\bar{r}^2 \tilde{F}(r) = \bar{r}^2 \sigma + 2\gamma \frac{\bar{r}^2}{r^2} + 4c \frac{\bar{r}^4}{r^4} \approx \bar{r}^2 \sigma + 0.499 \frac{\bar{r}^2}{r^2} - 0.008 \frac{\bar{r}^4}{r^4}
\]
- relation between $\bar{r}$ and $\Lambda_{\overline{MS}}$

We use $g_E^2 = 3(1-u_P)$ for

$$a\Lambda_{\overline{MS}} = a\Lambda_E (\Lambda_{\overline{MS}} / \Lambda_E)$$

| $\beta$    | $a / \bar{r}$ | $a\Lambda_{\overline{MS}}$ | $\bar{r}\Lambda_{\overline{MS}}$ |
|------------|---------------|-----------------------------|---------------------------------|
| 5.96       | 0.2117(5)     | 0.11523                     | 0.5443(13)                      |
| 6.17       | 0.1499(2)     | 0.08520                     | 0.5684(8)                       |
| 6.42       | 0.1052(3)     | 0.06105                     | 0.5803(17)                      |

In the continuum limit

$$\bar{r}\Lambda_{\overline{MS}} = 0.5924(16)$$

$$\sigma\bar{r}^2 = 1.159(6)$$

$$\therefore \frac{\Lambda_{\overline{MS}}}{\sqrt{\sigma}} = 0.550(2)$$
● Conclusion

Creutz ratios in the continuum limit can be evaluated precisely by 4-d Ape smearing, giving rather reliable determination of string tension.

Wilson flow technique gives the same physical results once the time steps $\Delta t$ is sufficiently small.

It is worth calculating Wilson loops during Wilson flow updating.

They should give fruitful physics!