Research Article

Multiple Positive Solutions for a System of Caputo Fractional $p$-Laplacian Boundary Value Problems

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In this work, we are pleased to investigate multiple positive solutions for a system of Caputo fractional $p$-Laplacian boundary value problems, and we also provide an example for illustrating our main results.

1. Introduction

In this work, we are pleased to discuss the positive solutions for the following system of Caputo fractional $p$-Laplacian boundary value problems:

\[
\begin{align*}
\begin{cases}
cD_t^\mu \left( p(x) \right) + g_1(t, u(t), v(t)) = 0, & 0 < t < 1, \\
cD_t^\nu \left( q(x) \right) + g_2(t, u(t), v(t)) = 0, & 0 < t < 1, \\
\alpha_i u(0) - \beta_i u(1) = 0, & i = 1, 2, \\
\gamma_i u'(0) - \delta_i u'(1) = 0, & i = 1, 2, \\
\alpha_i v(0) - \beta_i v(1) = 0, & i = 1, 2, \\
\gamma_i v'(0) - \delta_i v'(1) = 0, & i = 1, 2
\end{cases}
\end{align*}
\]

As a generalization of integer-order equations, fractional-order equations can effectively describe various materials and physical processes with memory and genetic properties. It has a large number of applications in our society, such as biology, chemical kinetics, electromagnetics, transmission and diffusion, and automatic control. Recently, there many researchers pay their attentions to studying the existence of solutions for various types of fractional-order equations by use of fixed point theorems, upper and lower solution methods, and monotone iterative techniques. For

where $\{cD_t^\mu, cD_t^\nu \}$ are the fractional derivatives of Caputo sense with $\mu, \nu \in (1, 2)$; $p(x)$ is the $p$-Laplacian, i.e., $p(x) = |x|^{p-2}x$ with $p > 1, x \in \mathbb{R}$; and the constants $\alpha_i, \beta_i, \gamma_i, \delta_i, b_i, \xi_i (i = 1, 2)$ and the functions $g_i (i = 1, 2)$ satisfy the following conditions:

- (H0) $\alpha_i > \beta_i > 0, \quad \gamma_i > \delta_i > 0, \quad b_i \geq 0, \quad \xi_i \in (0, 1)$ with $b_i^{p-1} \xi_i \in [0, 1)$
- (H1) $g_i (i = 1, 2)$ are nonnegative continuous functions on $[0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+$, where $\mathbb{R}^+ = [0, +\infty)$
instance, we refer the readers to [1–25] and references therein.

In [1], by using the method of upper and lower solutions and the Schauder fixed point theorem, Vong investigated the positive solutions for the following nonlocal fractional boundary value problem:

\[
\begin{aligned}
\mathcal{C}D_t^\alpha u(t) + f(t, u(t)) &= 0, \quad 0 < t < 1, \\
\mathcal{C}D_t^\alpha u(0) &= \cdots = u^{(m-1)}(0) = 0, \quad u(1) = \int_0^1 u(s) d\mu(s),
\end{aligned}
\]  

(2)

where \( n \geq 2, \alpha \in (n - 1, n) \), and \( \mu(s) \) is a function of bounded variation. \( f \) may be singular at \( t = 1 \).

In [2], Wang and Yang studied the integral boundary value problem of Caputo sense:

\[
\begin{aligned}
\mathcal{C}D_t^\alpha \varphi_p \left( D_{0^+}^\beta v(t) \right) + \lambda f(t, v(t), D_{0^+}^\beta v(t)) &= 0, \quad 0 < t < 1, \\
v(0) &= D_{0^+}^\beta v(0) = 0, \quad i = 1, \ldots, n-2, \\
D_{0^+}^\alpha v(1) &= \lambda_1 \int_0^1 h_1(s) D_{0^+}^\alpha v(s) dA_1(s) + \lambda_2 \int_0^1 h_2(s) D_{0^+}^\alpha v(s) dA_2(s) + \lambda_3 \sum_{j=1}^{\infty} \rho_j D_{0^+}^\alpha v(\eta_j),
\end{aligned}
\]  

(4)

where \( \mathcal{C}D_t^\alpha \) is the Caputo fractional derivative and \( D_{0^+}^\beta, D_{0^+}^\gamma, D_{0^+}^\delta, \) and \( D_{0^+}^\alpha \) are the Riemann–Liouville fractional derivatives.

Recently, coupled systems of fractional differential equations have also been investigated by many authors.

\[
\begin{aligned}
D_{0^+}^\beta \left( \varphi_p \left( \mathcal{C}D_t^\alpha \mathcal{C}D_t^\alpha u(t) \right) \right) + f_1(t, u(t), v(t)) &= 0, \quad 0 < t < 1, \\
D_{0^+}^\beta \left( \varphi_p \left( \mathcal{C}D_t^\alpha \mathcal{C}D_t^\alpha v(t) \right) \right) + f_2(t, u(t), v(t)) &= 0, \quad 0 < t < 1, \\
u'(0) &= u''(0) = \cdots = u^{(m-1)}(0) = 0, \\
v'(0) &= v''(0) = \cdots = v^{(m-1)}(0) = 0, \\
\mathcal{C}D_t^\alpha u(0) &= 0, \\
\mathcal{C}D_t^\alpha v(0) &= 0,
\end{aligned}
\]  

(5)

Some results on the direction can be found in a series of papers [11–25] and the references cited therein.

In [11], Wang utilized the Guo–Krasnosel’skii fixed point theorem to investigate the multiple positive solutions for the mixed fractional boundary value problems involving the \( p \)-Laplacian:

\[
\begin{aligned}
\mathcal{C}D_t^\alpha \chi(t) &= f(t, \chi(t)), \quad 0 < t < 1, \\
\mathcal{C}D_t^\gamma \delta(t) &= \int_0^t h(t) \chi(t) dt, \\
\mathcal{C}D_t^\delta \gamma(t) &= \int_0^t g(t) \chi(t) dt.
\end{aligned}
\]  

(3)

By the Leggett–Williams fixed point theorem, they obtained multiple positive solutions when the nonlinearity \( f \) is bounded from below. Moreover, when \( f \) is asymptotically linear at infinity, they also obtained an existence theorem.

In [3], Wang et al. adopted the theory of mixed monotone operators to obtain a unique positive solution for the mixed fractional boundary value problems involving the \( p \)-Laplacian:

\[
\begin{aligned}
\mathcal{C}D_t^\alpha \chi(t) &= f(t, \chi(t)), \quad 0 < t < 1, \\
\mathcal{C}D_t^\gamma \delta(t) &= \int_0^t h(t) \chi(t) dt, \\
\mathcal{C}D_t^\delta \gamma(t) &= \int_0^t g(t) \chi(t) dt.
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\]  

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By the Leggett–Williams fixed point theorem, they obtained multiple positive solutions when the nonlinearity \( f \) is bounded from below. Moreover, when \( f \) is asymptotically linear at infinity, they also obtained an existence theorem.
When the nonlinearities $f_i$ satisfy some appropriate conditions, the author made use of the Avery–Henderson fixed point theorem and the six functionals’ fixed point theorem to obtain some existence theorems of multiple positive solutions.

Inspired by the aforementioned results, in this work, we study the solvability for (1) and establish the existence results of multiple positive solutions via the six functional fixed point theorem under some bounded conditions for $g_i$ $(i = 1, 2)$. Finally, we also provide an example to illustrate our main results.

2. Preliminaries

In this section, we only recall the definition of Caputo fractional derivative, for more details, see the book [26].

**Definition 1.** The fractional derivative of $f$ in the Caputo sense is defined as

\[
\begin{aligned}
\ C D^\mu_x x(t) + g_1(t, u(t), v(t)) &= 0, \quad 0 < t < 1, \\
\ C D^\mu_y y(t) + g_2(t, u(t), v(t)) &= 0, \quad 0 < t < 1, \\
x(0) &= 0, \\
y(0) &= 0,
\end{aligned}
\]

\[
\begin{aligned}
\ C D^\mu_x x(t) + g_1(t, u(t), v(t)) &= 0, \quad 0 < t < 1, \\
\ C D^\mu_y y(t) + g_2(t, u(t), v(t)) &= 0, \quad 0 < t < 1, \\
x(0) &= 0, \\
y(0) &= 0,
\end{aligned}
\]

We next translate (10) into an equivalent system of integral equations. By the similar arguments as in [2], we have the following result.

**Lemma 1.** Problem (10) is equivalent to the following system of Hammerstein-type integral equations:

\[
\begin{aligned}
\begin{pmatrix}
\ x(t) \\
\ y(t)
\end{pmatrix} &= \begin{pmatrix}
\int_0^1 H_1(t, s) g_1(s, u(s), v(s)) ds \\
\int_0^1 H_2(t, s) g_2(s, u(s), v(s)) ds
\end{pmatrix},
\end{aligned}
\]

where

\[
\begin{aligned}
H_1(t, s) &= \bar{H}_1(t, s) + \frac{b_1^{p-1} \xi_1}{1 - b_1^{p-1} \xi_1} \bar{H}_1(s, \xi_1), \\
\bar{H}_1(t, s) &= \frac{1}{\Gamma(y_1)} \begin{cases}
(t - s)^{y_1 - 1} - (t - s)^{y_1 - 1}, & 0 \leq s \leq t \leq 1, \\
(t - s)^{y_1 - 1}, & 0 \leq t \leq s \leq 1.
\end{cases}
\end{aligned}
\]

**Proof.** We only need to consider the case $i = 1$ (by the similar method, the case $i = 2$ can be easily proved). Using Lemma 2.5 of [2], we have

\[
x(t) = c_0 + c_1 t - \int_0^t \frac{(t - s)^{y_1 - 1}}{\Gamma(y_1)} g_1(s, u(s), v(s)) ds,
\]

where $c_i \in \mathbb{R}$, $i = 0, 1$, $x(0) = 0$ implies that $c_0 = 0$, and by $x(1) = b_1^{p-1} x(\xi_1)$, we have

\[
c_1 = \frac{1}{1 - b_1^{p-1} \xi_1} \int_0^1 \frac{(1 - s)^{y_1 - 1}}{\Gamma(y_1)} g_1(s, u(s), v(s)) ds - \frac{b_1^{p-1}}{1 - b_1^{p-1} \xi_1} \int_0^{\xi_1} \frac{(\xi_1 - s)^{y_1 - 1}}{\Gamma(y_1)} g_1(s, u(s), v(s)) ds.
\]
Therefore,

\[ x(t) = \frac{1}{1 - b_1^{-1}\xi_1} \int_0^1 \frac{t(1-s)^{p-1}}{\Gamma(p)} g_1(s, u(s), v(s))ds \]

\[ - \frac{b_1^{-1}t}{1 - b_1^{-1}\xi_1} \int_0^1 \frac{t(1-s)^{p-1}}{\Gamma(p)} g_1(s, u(s), v(s))ds - \frac{b_1^{-1}t}{1 - b_1^{-1}\xi_1} \int_0^1 \frac{t(1-s)^{p-1}}{\Gamma(p)} g_1(s, u(s), v(s))ds \]

\[ \right. \]

\[ = \frac{1}{1 - b_1^{-1}\xi_1} \int_0^1 \frac{t(1-s)^{p-1}}{\Gamma(p)} g_1(s, u(s), v(s))ds \]

\[ - \frac{b_1^{-1}t}{1 - b_1^{-1}\xi_1} \int_0^1 \frac{t(1-s)^{p-1}}{\Gamma(p)} g_1(s, u(s), v(s))ds - \frac{b_1^{-1}t}{1 - b_1^{-1}\xi_1} \int_0^1 \frac{t(1-s)^{p-1}}{\Gamma(p)} g_1(s, u(s), v(s))ds \]

\[ \right. \]

\[ = \frac{1}{1 - b_1^{-1}\xi_1} \int_0^1 \frac{t(1-s)^{p-1}}{\Gamma(p)} g_1(s, u(s), v(s))ds - \frac{b_1^{-1}t}{1 - b_1^{-1}\xi_1} \int_0^1 \frac{t(1-s)^{p-1}}{\Gamma(p)} g_1(s, u(s), v(s))ds \]

\[ = \int_0^1 \frac{t(1-s)^{p-1}}{\Gamma(p)} g_1(s, u(s), v(s))ds + \int_0^1 \frac{t(1-s)^{p-1}}{\Gamma(p)} g_1(s, u(s), v(s))ds \]

\[ = \int_0^1 \frac{t(1-s)^{p-1}}{\Gamma(p)} g_1(s, u(s), v(s))ds + \int_0^1 \frac{t(1-s)^{p-1}}{\Gamma(p)} g_1(s, u(s), v(s))ds \]

\[ \right. \]

\[ = \int_0^1 H_1(t, s)g_1(s, u(s), v(s))ds. \]

This completes the proof.

\[ \right. \]

Note from (8) and Lemma 1, we have

\[ \left\{ \begin{array}{l}
    \varphi_p(C D_t^\alpha u(t)) = x(t) = \int_0^1 H_1(t, s)g_1(s, u(s), v(s))ds, \quad 0 < t < 1, \\
    \varphi_p(C D_t^\beta v(t)) = y(t) = \int_0^1 H_2(t, s)g_2(s, u(s), v(s))ds, \quad 0 < t < 1,
\end{array} \right. \]  

(16)

and

\[ \left\{ \begin{array}{l}
    C D_t^\alpha u(t) = \varphi_p^{-1} \left( \int_0^1 H_1(t, s)g_1(s, u(s), v(s))ds \right), \quad 0 < t < 1, \\
    C D_t^\beta v(t) = \varphi_p^{-1} \left( \int_0^1 H_2(t, s)g_2(s, u(s), v(s))ds \right), \quad 0 < t < 1,
\end{array} \right. \]  

(17)

with the boundary conditions

\[ \left\{ \begin{array}{l}
    \alpha_1 u(0) - \beta_1 u(1) = 0, \quad \gamma_1 u'(0) - \delta_1 u'(1) = 0, \\
    \alpha_2 v(0) - \beta_2 v(1) = 0, \quad \gamma_2 v'(0) - \delta_2 v'(1) = 0.
\end{array} \right. \]  

(18)

By Lemma 2.5 of [2], we have the result: \qed

Lemma 2. Problems (17) and (18) is equivalent to the following Hammerstein-type integral equation:

\[ \left( \begin{array}{l}
    u(t) \\
    v(t)
\end{array} \right) = \left[ \begin{array}{l}
    \int_0^1 G_1(t, s)\varphi_p^{-1} \left( \int_0^1 H_1(s, r)g_1(r, u(r), v(r))dr \right)ds \\
    \int_0^1 G_2(t, s)\varphi_p^{-1} \left( \int_0^1 H_2(s, r)g_2(r, u(r), v(r))dr \right)ds
\end{array} \right]. \]  

(19)
where

$$G_i(t, s) = \begin{cases} 
\frac{(t-s)^{\mu_i-1}}{\Gamma(\mu_i)} + \frac{\beta_i (1-s)^{\mu_i-1}}{(\alpha_i - \beta_i) \Gamma(\mu_i)} + \frac{\beta_i \delta_i (\mu_i - 1) (1-s)^{\mu_i-2}}{\Gamma(\mu_i)}, & 0 \leq s \leq t \leq 1, \\
\frac{\beta_i (1-s)^{\mu_i-1}}{(\alpha_i - \beta_i) \Gamma(\mu_i)} + \frac{\beta_i \delta_i (\mu_i - 1) (1-s)^{\mu_i-2}}{\Gamma(\mu_i)}, & 0 \leq t \leq s \leq 1.
\end{cases}$$

(20)

**Lemma 3** (see [2], Lemma 2.8). The functions $G_i, H_i, (i = 1, 2)$ have the following properties:

(i) $G_i, H_i \in C([0, 1] \times [0, 1], \mathbb{R}^+)$

(ii) $\mathcal{M}_i(s) \leq G_i(t, s) \leq \alpha_i / \beta_i \mathcal{M}_i(s), \forall t, s \in [0, 1], \text{ where } 
\mathcal{M}_i(s) = \left(\beta_i (1-s)^{\mu_i-1}/(\alpha_i - \beta_i) \Gamma(\mu_i)\right) + \left(\beta_i \delta_i (\mu_i - 1) (1-s)^{\mu_i-2}/(\alpha_i - \beta_i) (\gamma_i - \delta_i) \Gamma(\mu_i)\right)$

Then, we obtain that $T_i, (i = 1, 2): P \times P \rightarrow P, T: P \times P \rightarrow P \times P$ are completely continuous operators, and if there exists $(\pi, \nu) \in P \times P \setminus \{0\}$ such that $T(\pi, \nu) = (\pi, \nu)$, i.e., $T_1(\pi, \nu) = \pi$, $T_2(\pi, \nu) = \nu$, then $(\pi, \nu)$ is a positive solution for (1).

**Lemma 4** (see [27]). Let $P$ be a cone in a real Banach space $E$. Suppose that $\alpha, \psi, \zeta$ are nonnegative continuous concave functionals on $P$, $\beta, \sigma,$ and $\theta$ are nonnegative continuous convex functionals on $P$, and there are nonnegative constants $l, l', r, r', \Gamma,$ and $R'$ such that $A: \mathcal{Q}(\beta, R) \rightarrow P$ is a completely continuous operator and

1. $Q(\beta, R)$ is a bounded set
2. $Q(\alpha, l)$ and $Q(\alpha, \beta, r, R)$ are disjoint subset of $Q(\beta, R)$
3. $\{u \in P: \theta(u) < r', r < \alpha(u), R' < \psi(u), \beta(u) < R\} \neq \emptyset$
4. $\{u \in P: l' < \zeta(u), \sigma(u) < l\} \neq \emptyset$
5. $\{u \in P: l < \sigma(u), \alpha(u) < r\} \neq \emptyset$

If the following claims hold:

1. $\alpha(Au) > r, \forall u \in P$ with $\alpha(u) = r, \beta(u) \leq R$, and $r' < \theta(Au)$
2. $\alpha(Au) > r, \forall u \in P$ with $\alpha(u) = r, \beta(u) \leq R$, and $\beta(u) \leq r'$
3. $\beta(Au) < R, \forall u \in P$ with $r \leq \alpha(u), \beta(u) = R$, and $\psi(Au) < R'$
4. $\beta(Au) < R, \forall u \in P$ with $r \leq \alpha(u), \beta(u) = R$, and $R' \leq \psi(u)$
5. $\sigma(Au) < l, \forall u \in P$ with $\sigma(u) = l$ and $\zeta(Au) < l'$

(21)

Then, $A$ has at least three fixed points $u_1, u_2$, and $u_3$ in $Q(\beta, R)$ such that $\alpha(u_1) \leq l, r \leq l' \leq \sigma(u_2)$ with $\beta(u_2) \leq R$, and $l < \sigma(u_3) \leq \zeta(u)$.

3. Main Results

We first define some notations:

$$Q(\beta, R) = \{(u, v) \in P \times P: \beta(u, v) \leq R\},$$

$$Q(\alpha, \beta, r, R) = \{(u, v) \in P \times P: \beta(u) \leq R, \alpha(u, v) \leq r\},$$

$$\alpha(u, v) = \min_{t \in [0, 1]} \{u(t) + v(t)\},$$

$$\psi(u, v) = \min_{t \in I} \{u(t) + v(t)\},$$

$$\zeta(u, v) = \min_{t \in \tilde{I}} \{u(t) + v(t)\},$$

$$\theta(u, v) = \max_{t \in [0, 1]} \{u(t) + v(t)\},$$

$$\beta(u, v) = \max_{t \in \tilde{I}} \{u(t) + v(t)\},$$

$$\sigma(u, v) = \max_{t \in \tilde{I}} \{u(t) + v(t)\},$$

(22)

where $I = [(1/4), (3/4)]$ and $I_1 = [(1/8), (7/8)]$. Then, $\alpha, \psi,$ and $\zeta$ are the concave functionals on $P$, and $\theta, \beta,$ and $\sigma$ are the convex functionals on $P$. 

Let $E = C[0, 1], \|u\| = \max_{t \in [0, 1]}|u(t)|$ and $P = \{u \in E: u(t) \geq 0, \forall t \in [0, 1]\}$. Then, $(E, \|\cdot\|)$ is a real Banach space and $P$ is a cone on $E$. Moreover, $E \times E$ is a Banach space with the norm $\|\langle u, v \rangle\| = \|u\| + \|v\|$, and $P \times P$ a cone on $E \times E$. From Lemmas 1 and 2, we define operators $T_1, (i = 1, 2)$ and $T$ as follows:
Theorem 1. Suppose that there exist positive real numbers \( l, l', r, r', R, \) and \( R' \) such that \( g_i (i = 1, 2) \) satisfy the following conditions:

\[
\begin{align*}
(\text{H2}) & \quad g_1 (t, u(t), v(t)) < \phi_p (l \beta_1 / 2\alpha_1 \rho_1), g_2 (t, u(t), v(t)) < \phi_p (l \beta_2 / 2\alpha_2 \rho_2) \quad \text{for all } t \in [0,1] \text{ and } (u, v) \in [0, (r/3)]. \\
(\text{H3}) & \quad g_1 (t, u(t), v(t)) < \phi_p (l \beta_1 / 2\alpha_1 \rho_1), g_2 (t, u(t), v(t)) < \phi_p (l \beta_2 / 2\alpha_2 \rho_2) \quad \text{for all } t \in [0,1] \text{ and } (u, v) \in [0, R]. \\
(\text{H4}) & \quad g_1 (t, u(t), v(t)) < \phi_p (l \beta_1 / 2\alpha_1 \rho_1), g_2 (t, u(t), v(t)) < \phi_p (l \beta_2 / 2\alpha_2 \rho_2) \quad \text{for all } t \in [0,1] \text{ and } (u, v) \in [0, (r/3)].
\end{align*}
\]

Then, (1) has at least three positive solutions \((u_1, v_1), (u_2, v_2), \) and \((u_3, v_3)\) in \(Q(\beta, R)\).

Proof. Note, from a standard calculus argument, we obtain the set \(Q(\beta, R)\) is bounded. Since if \((u, v) \in Q(\beta, R),\) and then \(\max_{t \in [0,1]} (u(t) + v(t)) < R.\) Let \(l' = (r/\eta), R = (R'/\eta), \) \(l = (l'/\eta), \) and \(l = (r/3),\) with \(R' \in ((2n^2r'/(\eta + 1)), (2n^2)/(\eta + 1)).\) Then, if \((u, v) \in Q(\beta, R),\) we have max_{t \in [0,1]} (u(t) + v(t)) < l. This means that

\[
\alpha(u, v) = \min_{t \in [0,1]} \{ T_1 (u, v)(t) + T_2 (u, v)(t) \} = \min_{t \in [0,1]} \left\{ \int_0^1 \! G_1 (t, s) \phi_p \left( \int_0^1 \! H_1 (s, r) g_1 (r, u(r), v(r)) \, dr \right) \, ds + \int_0^1 \! G_2 (t, s) \phi_p \left( \int_0^1 \! H_2 (s, r) g_2 (r, u(r), v(r)) \, dr \right) \, ds \right\} \geq \int_0^1 \! \mathcal{M}_1 (s) \phi_p \left( \int_0^1 \! H_1 (s, r) g_1 (r, u(r), v(r)) \, dr \right) \, ds + \int_0^1 \! \mathcal{M}_2 (s) \phi_p \left( \int_0^1 \! H_2 (s, r) g_2 (r, u(r), v(r)) \, dr \right) \, ds \geq \eta \left( \begin{array}{c} \int_0^1 \! G_1 (t, s) \phi_p \left( \int_0^1 \! H_1 (s, r) g_1 (r, u(r), v(r)) \, dr \right) \, ds + \int_0^1 \! G_2 (t, s) \phi_p \left( \int_0^1 \! H_2 (s, r) g_2 (r, u(r), v(r)) \, dr \right) \, ds \end{array} \right) = \eta \theta(T(u, v)) > \eta r' = r.
\]

Claim 2. \(\alpha(T(u, v)) > r',\) for all \((u, v) \in Q(\alpha, \beta, R),\) \(\theta(u, v) \leq r'\) with \(\alpha(u, v) = r.\) At this case, we have \(r \leq u(r) + v(r) \leq (r/\eta),\) for \(r \in [0,1],\) and from (H4), we obtain

\[
\alpha(u, v) = \min_{t \in [0,1]} \{ u(t) + v(t) \} \leq \min_{t \in [0,1]} \{ u(t) + v(t) \} \leq \max_{t \in [0,1]} \{ u(t) + v(t) \} \quad \text{for } r \leq u(r) + v(r) \leq (r/\eta).
\]

\[
\alpha(u, v) = \min_{t \in [0,1]} \{ u(t) + v(t) \} \leq \min_{t \in [0,1]} \{ u(t) + v(t) \} \leq \max_{t \in [0,1]} \{ u(t) + v(t) \} \quad \text{for } r \leq u(r) + v(r) \leq (r/\eta).
\]
$$\alpha(T(u, v)) = \min_{t \in [0,1]} \{ T_1(u, v)(t) + T_2(u, v)(t) \}$$

$$= \min_{t \in [0,1]} \left\{ \int_0^1 G_1(t, s) \varphi_p^{-1} \left( \int_0^1 H_1(s, \tau) \varphi_p \left( \int_0^1 H_2(s, \tau) g_1(r, u(r), v(r)) dr \right) d\tau \right) ds + \int_0^1 G_2(t, s) \varphi_p^{-1} \left( \int_0^1 H_2(s, \tau) g_2(r, u(r), v(r)) dr \right) ds \right\}$$

$$\geq \int_0^1 \mathcal{M}_1(s) \varphi_p^{-1} \left( \int_0^1 H_1(s, \tau) \varphi_p \left( \frac{r}{2\rho_1} \right) d\tau \right) ds + \int_0^1 \mathcal{M}_2(s) \varphi_p^{-1} \left( \int_0^1 H_2(s, \tau) \varphi_p \left( \frac{r}{2\rho_2} \right) d\tau \right) ds$$

$$= \frac{r}{2} + \frac{r}{2}$$

$$= r.$$  \hfill (27)

Claim 3. $\beta(T(u, v)) < R$, for all $(u, v) \in Q(\alpha, \beta, r, R)$, $\beta(u, v) = R$, and $\psi(T(u, v)) < R'$. Then, we have

$$\beta(T(u, v)) = \max_{t \in \mathbb{I}} \{ T_1(u, v)(t) + T_2(u, v)(t) \}$$

$$= \max_{t \in \mathbb{I}} \left\{ \int_0^1 G_1(t, s) \varphi_p^{-1} \left( \int_0^1 H_1(s, \tau) \varphi_p \left( \int_0^1 H_2(s, \tau) g_1(r, u(r), v(r)) dr \right) d\tau \right) ds + \int_0^1 G_2(t, s) \varphi_p^{-1} \left( \int_0^1 H_2(s, \tau) g_2(r, u(r), v(r)) dr \right) ds \right\}$$

$$\leq \int_0^1 \mathcal{M}_1(s) \varphi_p^{-1} \left( \int_0^1 H_1(s, \tau) \varphi_p \left( \frac{r}{2\rho_1} \right) d\tau \right) ds + \int_0^1 \mathcal{M}_2(s) \varphi_p^{-1} \left( \int_0^1 H_2(s, \tau) \varphi_p \left( \frac{r}{2\rho_2} \right) d\tau \right) ds$$

$$\leq \frac{1}{\eta} \left\{ \int_0^1 \mathcal{M}_1(s) \varphi_p^{-1} \left( \int_0^1 H_1(s, \tau) \varphi_p \left( \frac{r}{2\rho_1} \right) d\tau \right) ds + \int_0^1 \mathcal{M}_2(s) \varphi_p^{-1} \left( \int_0^1 H_2(s, \tau) \varphi_p \left( \frac{r}{2\rho_2} \right) d\tau \right) ds \right\}$$

$$\leq \frac{1}{\eta} \min_{t \in \mathbb{I}} \left\{ \int_0^1 G_1(t, s) \varphi_p^{-1} \left( \int_0^1 H_1(s, \tau) \varphi_p \left( \frac{r}{2\rho_1} \right) d\tau \right) ds + \int_0^1 G_2(t, s) \varphi_p^{-1} \left( \int_0^1 H_2(s, \tau) \varphi_p \left( \frac{r}{2\rho_2} \right) d\tau \right) ds \right\}$$

$$= \frac{1}{\eta} \psi(u, v)$$

$$< \frac{R'}{\eta}$$

$$= R.$$  \hfill (28)
Lemma 3 (ii) and (H3), we have

\[ \frac{\beta(T(u,v))}{R} < R, \text{ for all } (u,v) \in \{(u,v) \in Q(\alpha, \beta, \rho, R) : R' \leq \psi(u,v)\} \text{ with } \beta(u,v) = R. \text{ Therefore, by } \]

Claim 4. \( \beta(T(u,v)) < R \), for all \( (u,v) \in (u,v) \in Q(\alpha, \beta, \rho, R) \) with \( \beta(u,v) = R \). Therefore, by Lemma 3 (ii) and (H3), we have

\[
\beta(T(u,v)) = \max\{T_1(u,v)(t) + T_2(u,v)(t)\}
\]

\[
= \max\{\int_0^1 G_1(t,v)\phi(t)\left(\int_0^1 H_1(s,t)g_1(s,\tau)\phi(\tau)d\tau\right)ds + \int_0^1 G_2(t,v)\phi(t)\left(\int_0^1 H_2(s,t)g_2(s,\tau)\phi(\tau)d\tau\right)ds\}
\]

\[
\leq \int_0^1 \frac{\alpha_1}{\beta_1} M_1(s)\phi(t)\left(\int_0^1 H_1(s,t)g_1(s,\tau)\phi(\tau)d\tau\right)ds + \int_0^1 \frac{\alpha_2}{\beta_2} M_2(s)\phi(t)\left(\int_0^1 H_2(s,t)g_2(s,\tau)\phi(\tau)d\tau\right)ds
\]

\[
= \frac{R}{2} + \frac{R}{2}
\]

\[
= R.
\]

Claim 5. \( \sigma(T(u,v)) < l \), for all \( (u,v) \in Q(\alpha, l) \) with \( \sigma(u,v) = l \), and \( \xi(T(u,v)) < l' \). Then, we obtain

\[
\sigma(T(u,v)) = \max\{T_1(u,v)(t) + T_2(u,v)(t)\}
\]

\[
= \max\{\int_0^1 G_1(t,v)\phi(t)\left(\int_0^1 H_1(s,t)g_1(s,\tau)\phi(\tau)d\tau\right)ds + \int_0^1 G_2(t,v)\phi(t)\left(\int_0^1 H_2(s,t)g_2(s,\tau)\phi(\tau)d\tau\right)ds\}
\]

\[
\leq \int_0^1 \frac{\alpha_1}{\beta_1} M_1(s)\phi(t)\left(\int_0^1 H_1(s,t)g_1(s,\tau)\phi(\tau)d\tau\right)ds + \int_0^1 \frac{\alpha_2}{\beta_2} M_2(s)\phi(t)\left(\int_0^1 H_2(s,t)g_2(s,\tau)\phi(\tau)d\tau\right)ds
\]

\[
\leq \frac{1}{\eta}\{\int_0^1 G_1(t,v)\phi(t)\left(\int_0^1 H_1(s,t)g_1(s,\tau)\phi(\tau)d\tau\right)ds + \int_0^1 G_2(t,v)\phi(t)\left(\int_0^1 H_2(s,t)g_2(s,\tau)\phi(\tau)d\tau\right)ds\}
\]

\[
\leq \frac{1}{\eta}\min\{\int_0^1 G_1(t,v)\phi(t)\left(\int_0^1 H_1(s,t)g_1(s,\tau)\phi(\tau)d\tau\right)ds + \int_0^1 G_2(t,v)\phi(t)\left(\int_0^1 H_2(s,t)g_2(s,\tau)\phi(\tau)d\tau\right)ds\}
\]

\[
= \frac{1}{\eta}\xi(T(u,v))
\]

\[
< \frac{l'}{\eta}
\]

\[
= l.
\]
Claim 6. $\sigma(T(u, v)) < l$, for all $(u, v) \in \{Q(\sigma, l) : l' \leq \zeta(u, v)\}$ with $\sigma(T(u, v)) < l$. Therefore, by Lemma 3 (ii) and (H2), we have

$$
\sigma(T(u, v)) = \max_{t \in I} \left\{ T_1(u, v)(t) + T_2(u, v)(t) \right\}
$$

$$
\leq \int_0^1 G_1(t, s) \varphi_p^{-1} \left( \int_0^1 H_1(s, r) g_1(r, u(r), v(r)) dr \right) ds + \int_0^1 G_2(t, s) \varphi_p^{-1} \left( \int_0^1 H_2(s, r) g_2(r, u(r), v(r)) dr \right) ds
$$

$$
\leq \frac{1}{\beta_1} \int_0^1 \mathcal{M}_1(s) \varphi_p^{-1} \left( \int_0^1 H_1(s, r) \varphi_p \left( \frac{f_1}{2 \alpha_1 \beta_1} \right) dr \right) ds + \frac{1}{\beta_2} \int_0^1 \mathcal{M}_2(s) \varphi_p^{-1} \left( \int_0^1 H_2(s, r) \varphi_p \left( \frac{f_2}{2 \alpha_2 \beta_2} \right) dr \right) ds
$$

$$
= \frac{1}{2} + \frac{l}{2}
$$

$$
= l.
$$

(31)

Remark 1. Our Green’s functions $G_i (i = 1, 2)$ satisfy some special inequalities (see Lemma 3 (iii)). So, in the Claims 1 and 2 of Theorem 1, $\alpha(T(u, v)) = \min_{t \in [0, 1]} \{ T_1(u, v)(t) + T_2(u, v)(t) \}$ cannot reach zero. Therefore, we use the Caputo derivative for the considered scheme. This also explains the reason that we do not use other forms of fractional derivatives (including the Riemann–Liouville type and the Hadamard type).

Example 1. Consider the example:

$$
\begin{cases}
\frac{d}{dt}^{\frac{3}{2}}(\varphi_p \left( \frac{d}{dt}^{\frac{3}{2}} u(t) \right)) + g_1(t, u(t), v(t)) = 0, & 0 < t < 1, \\
\frac{d}{dt}^{\frac{3}{2}}(\varphi_p \left( \frac{d}{dt}^{\frac{3}{2}} v(t) \right)) + g_2(t, u(t), v(t)) = 0, & 0 < t < 1, \\
2u(0) - u(1) = 0, 6u'(0) - 5u'(1) = 0, & cD_{1}^{\frac{3}{2}} u(0) = 0, cD_{1}^{\frac{3}{2}} u(1) = cD_{1}^{\frac{3}{2}} u_0 \left( \frac{1}{4} \right), \\
4v(0) - 3v(1) = 0, 8v'(0) - 7v'(1) = 0, & cD_{1}^{\frac{3}{2}} v(0) = 0, cD_{1}^{\frac{3}{2}} v(1) = 2cD_{1}^{\frac{3}{2}} v \left( \frac{1}{8} \right),
\end{cases}
$$

(32)

where $p = 2$.

In order to obtain the constants in Theorem 1, we need to calculate

$$
\int_0^1 H_1(t, s) ds = \frac{1}{\Gamma(\gamma_1)} \int_0^1 \frac{t (1 - s)^{\nu - 1}}{\Gamma(\gamma_1)} ds - \frac{1}{\Gamma(\gamma_1)} \int_0^1 (t - s)^{\nu - 1} ds
$$

$$
= \frac{t}{\gamma_1 \Gamma(\gamma_1)} \left[ -(1 - s)^{\nu} \right]_0^t + \frac{1}{\gamma_1 \Gamma(\gamma_1)} \int_0^t (t - s)^{\nu} ds
$$

$$
= \frac{t - t^{\nu}}{\Gamma(\gamma_1 + 1)}.
$$

(33)

and we have

$$
\int_0^1 H_1(t, s) ds = \frac{\Gamma(\gamma_1 + 1)}{\Gamma(\gamma_1)} \left[ -(1 - s)^{\nu} \right]_0^t + \frac{1}{\Gamma(\gamma_1)} \int_0^t (t - s)^{\nu} ds
$$

$$
= \frac{t - t^{\nu}}{\Gamma(\gamma_1)} \left[ -(1 - s)^{\nu} \right]_0^t + \frac{1}{\Gamma(\gamma_1)} \int_0^t (t - s)^{\nu} ds
$$

$$
= \frac{t - t^{\nu}}{\Gamma(\gamma_1 + 1)}.
$$

(34)
Moreover, we also obtain that \( \eta = 0.5, b_1^p \xi_1 = 0.25, b_2^p \xi_1 = 0.125, b_2^p \xi_2 = 0.088, \int_0^1 H_1(t,s)ds = 0.878t - 0.752t^{1.5}, \int_0^1 H_2(t,s)ds = 0.915t - 0.752t^{1.5}, \mathcal{M}_1(s) = 1.13(1-s)^{0.5} + 2.82(1-s)^{-0.5}, \mathcal{M}_2(s) = 3.39(1-s)^{0.5} + 11.85(1-s)^{-0.5}, \rho_1 = 0.9, \rho_2 = 4.29. \) If we put \( l = 2, l' = 1, r = 6, r' = 12, R = 15.6, R' = 7.8, \) and 

\[
g_1(t, u, v) = \begin{cases} \frac{t}{60} + 0.25(u + v), & 0 \leq u + v \leq 2, \ t \in [0, 1], \\ \frac{t}{60} + 0.875(u + v) - 1.25, & 2 \leq u + v \leq 6, \ t \in [0, 1], \\ \frac{t}{60} + 4, & 6 \leq u + v \leq 12, \ t \in [0, 1], \\ \frac{t}{60} - 0.14(u + v) + 5.68, & 12 \leq u + v \leq 15.6, \ t \in [0, 1], \\ \end{cases}
\]

\[
g_2(t, u, v) = \begin{cases} \frac{t}{70} + 0.08(u + v), & 0 \leq u + v \leq 2, \ t \in [0, 1], \\ \frac{t}{70} + 0.25(u + v) - 0.26, & 2 \leq u + v \leq 6, \ t \in [0, 1], \\ \frac{t}{70} + 1, & 6 \leq u + v \leq 12, \ t \in [0, 1], \\ \frac{t}{70} + 0.056(u + v) + 0.328, & 12 \leq u + v \leq 15.6, \ t \in [0, 1]. \\ \end{cases}
\]

Then, \( g_i (i = 1, 2) \) satisfies the following conditions:

(i) \( g_1(t, u(t), v(t)) < 0.56, g_2(t, u(t), v(t)) < 0.18 \) for all \( t \in [0, 1] \) and \( u, v \in [0, 2] \)

(ii) \( g_1(t, u(t), v(t)) < 4.33, g_2(t, u(t), v(t)) < 1.37 \) for all \( t \in [0, 1] \) and \( u, v \in [0, 15.6] \)

(iii) \( g_1(t, u(t), v(t)) > 3.33, g_2(t, u(t), v(t)) > 0.7 \) for all \( t \in [0, 1] \) and \( u, v \in [6, 12] \)

Therefore, all assumptions in Theorem 1 are satisfied, and the system of (32) has at least triple positive solutions.

4. Conclusion

In this paper, we use the six functional fixed point theorem to study the system of (1) under some bounded conditions for \( g_i (i = 1, 2) \). We first transform the original fractional differential system into the equivalent system of Hammerstein-type integral equations, and then with the help of properties of Green’s functions, we provide some sufficient conditions to guarantee the existence of multiple positive solutions for our system. Finally, an example is presented to illustrate the effectiveness of the main result.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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