Schur Function Expansions and the Rational Shuffle Conjecture

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Abstract

Gorsky and Negut [9] introduced operators $Q_{m,n}$ on symmetric functions and conjectured that, in the case where $m$ and $n$ are relatively prime, the expansion of $Q_{m,n}(-1)^n$ in terms of the fundamental quasi-symmetric functions are given by polynomials introduced by Hikita [15]. Later, Bergeron, Garsia, Leven and Xin [3] extended and refined the conjectures of Gorsky and Negut to give a combinatorial interpretation of the coefficients that arise in the expansion of $Q_{m,n}(-1)^n$ in terms of the fundamental quasi-symmetric functions for arbitrary $m$ and $n$ which we will call the Rational Shuffle Conjecture. In the special case $Q_{n+1,n}(-1)^n$, the Rational Shuffle Conjecture becomes the Shuffle Conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov [12]. The Shuffle Conjecture was proved in 2015 by Carlsson and Mellit [4] and full Rational Shuffle Conjecture was later proved by Mellit [19]. The main goal of this paper is to study the combinatorics of the coefficients that arise in the Schur function expansion of $Q_{m,n}(-1)^n$ in certain special cases. Leven gave a combinatorial proof of the Schur function expansion of $Q_{2,2n+1}(-1)^{2n+1}$ and $Q_{2n+1,2}(-1)^2$ in [17]. In this paper, we explore several symmetries in the combinatorics of the coefficients that arise in the Schur function expansion of $Q_{m,n}(-1)^n$. Especially, we study the hook-shaped Schur function coefficients, and the Schur function expansion of $Q_{m,n}(-1)^n$ in the case where $m$ or $n$ equals 3.

Keywords: Macdonald polynomials, parking functions, Dyck paths, Rational Shuffle Conjecture

1 Introduction

The Rational Shuffle Conjecture, as a rational generalization of the Classical Shuffle Conjecture, comes from the study of the ring of diagonal harmonics. Let $X = x_1, x_2, \ldots, x_n$ and $Y = y_1, y_2, \ldots, y_n$ be two sets of $n$ variables. The ring of diagonal harmonics consists of those polynomials in $\mathbb{Q}[X,Y]$ which satisfy the following system of differential equations

$$\partial_{x_1}^a \partial_{y_1}^b f(X,Y) + \partial_{x_2}^a \partial_{y_2}^b f(X,Y) + \ldots + \partial_{x_n}^a \partial_{y_n}^b f(X,Y) = 0$$

for each pair of integers $a$ and $b$ such that $a+b > 0$. Haiman in [13] proved that the ring of diagonal harmonics has dimension $(n+1)^{n-1}$.

Further, Haiman [13] proved that the bigraded Frobenius characteristic of the $S_n$-module of diagonal harmonics, $DH_n(X;q,t)$ is given by

$$DH_n(X;q,t) = \nabla e_n,$$  \hspace{1cm} (1)

here $\nabla$ (nabla) is the symmetric function operator defined by Bergeron and Garsia [2], and $e_n$ is the elementary symmetric function of degree $n$. 

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Let \(n\) be a positive integer. An \(n \times n\) Dyck path \(\Pi\) is a lattice path from \((0,0)\) to \((n,n)\) which always remains weakly above the main diagonal \(y = x\). The cells that are cut through by the main diagonal is called diagonal cells. Let \(\text{area}\) be the number of cells between \(\Pi\) and the diagonal.

We can get an \(n \times n\) parking function by labeling the cells east of and adjacent to north steps of a Dyck path with numbers \(\{1, \ldots, n\}\) such that the labels (called cars) increase in each column. The set of parking functions of size \(n\) is denoted by \(\mathcal{PF}_n\). We define the rank of a car \(c\) in cell \((x,y)\) be \(\text{rank}(c) = (n+1)y - nx\). Let \(\text{area}\) of a parking function be the area of the underlying Dyck path, and \(\text{dinv}\) be the number of pairs of cars \((i < j)\) such that \(\text{rank}(i) < \text{rank}(j) \leq \text{rank}(i) + n\).

Figure 1 gives an example of a \(5 \times 5\) Dyck path and a \(5 \times 5\) parking function.

Figure 1: A \(5 \times 5\) Dyck path and a \(5 \times 5\) parking function.

Let the word of a parking function \(\mathcal{PF}\) be the permutation obtained by reading the cars from the highest rank to the lowest rank, and let \(\text{ides}(\mathcal{PF})\) be the descent set of the inverse of the word. Let \(F_S[x]\) denote the fundamental quasi-symmetric functions of Gessel [8] associated to 
\[F_S[x] = \sum_{1 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq n \atop i \in S \Rightarrow a_i < a_{i+1}} x_{a_1}x_{a_2} \cdots x_{a_n}.\] (2)

The Classical Shuffle Conjecture of Haglund, Haiman, Loehr, Remmel and Ulyanov [12] gives a well-studied combinatorial expression for the bigraded Frobenius characteristic of the ring of diagonal harmonics. The Shuffle Conjecture (or Shuffle Theorem) that has been proved by Carlson and Mellit [4] is that for all \(n \geq 0\),
\[\nabla e_n = \sum_{\mathcal{PF} \in \mathcal{PF}_n} t^{\text{area}(\mathcal{PF})} q^{\text{dinv}(\mathcal{PF})} F_{\text{ides}(\mathcal{PF})}[X].\] (3)

Gorsky and Negut [9] introduced the symmetric function operator \(Q_{m,n}\) and extended the algebraic side of the Shuffle Conjecture from \(\nabla e_n\) to \(Q_{m,n}\) applied to \((-1)^n\). On the other hand, Bergeron, Garsia, Leven and Xin [3] extended the combinatorial side of the Shuffle Conjecture to Hikita polynomial \(H_{m,n}[X; q, t]\). The Extended Rational Shuffle Conjecture is that, for any pair of positive integers \((m, n)\),
\[Q_{m,n}(-1)^n = H_{m,n}[X; q, t],\] (4)
which was also proved by Mellit [19].

A more important goal is to find the Schur function expansion of \(\nabla e_n\) since that would allows us to find the character generating function of the ring of diagonal harmonics, see [13]. More generally, we would like to find a combinatorial interpretation of the coefficients that arise in the Schur function expansion of \(Q_{m,n}(-1)^n\). The main goal of this paper is to find such Schur function expansions in the case where \(m\) or \(n\) equals 3. The Schur function expansion of \(Q_{m,n}(-1)^n\) in the
case where \( m \) and \( n \) are coprime and either \( m \) or \( n \) equals 2 was given by Leven [17]. That is, let 
\[
[n]_{q,t} = \frac{q^n - t^n}{q - t} = q^{n-1} + q^{n-2}t + \ldots + t^{n-1},
\]
then Leven [17] gave a combinatorial proof of the following theorem.

**Theorem 1 (Leven).** For any \( k \geq 0 \),
\[
Q_{2k+1,2} = H_{2k+1,2}[X; q, t] = [k]_{q,t} s_{2} + [k + 1]_{q,t} s_{1,1}
\]
and
\[
Q_{2,2k+1}(-1) = H_{2,2k+1}[X; q, t] = \sum_{r=0}^{k} [k + 1 - r]_{q,t} s_{2r} 1_{2k+1-2r}.
\]

By the combinatorial side of the Extended Rational Shuffle Conjecture formulated in [3], we can extend Leven’s theorem to compute the Schur function expansion of \( Q_{m,n}(-1)^n \) where either \( m \) or \( n \) is equal to 2, but \( m \) and \( n \) are not co-prime. That is, we can give a combinatorial proof of the following.

**Theorem 2.**
\[
Q_{2k,2} = H_{2k,2}[X; q, t] = ([k]_{q,t} + [k - 1]_{q,t}) s_{2} + ([k + 1]_{q,t} + [k]_{q,t}) s_{1,1}
\]
and
\[
Q_{2k,1} = H_{2,2k}[X; q, t] = \sum_{r=0}^{k} ([k + 1 - r]_{q,t} + [k - r]_{q,t}) s_{2r} 1_{2k+1-2r}.
\]

The coefficient at \( s_{1^n} \) in \( Q_{m,n}(-1)^n \) is known as the rational \( q,t \)-Catalan number, computed by Gorsky and Mazin [10] for the case \( n = 3 \) and studied by Lee, Li and Loehr [16] for the case \( n = 4 \). The coefficients at the hook-shaped Schur functions were discussed by Armstrong, Loehr and Warrington [1].

In this paper, we explore the combinatorics of the Schur function expansion of \( Q_{m,n}(-1)^n \) in several special cases.

In Section 2, we provide background of the problem in both combinatorial side (parking function side) and algebraic side (symmetric function side). Then in Section 3, we prove a number of symmetries of the coefficients of Schur functions. Let \( [s_{\lambda}]_{m,n} \) be the coefficient of Schur function \( s_{\lambda} \) in \( \text{H}_{m,n} \), then we can combinatorially prove

**Theorem 3.** For all \( m, n > 0 \) and \( \lambda \vdash (n - am) \),
\[
(a) \ [s_{1^n}]_{m,n} = [s_n]_{m+n,n},
(b) \ [s_{m^n \lambda}]_{m,n} = [s_{\lambda}]_{m,n-a \times m},
(c) \ [s_{k1^{n-k}}]_{m,n} = [s_{k1^{m-k}}]_{n,m}.
\]

In Section 4, we prove the following theorem to give explicit formulas for the Schur function expansion of \( Q_{m,3}(-1) \) from both symmetric function side and combinatorial side.
Theorem 4. For any $k \geq 0$,

$$Q_{3k+1,3}(-1) = H_{3k+1,3}[X; q, t] = \left( \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i+1]_{qt} \right) s_3$$

$$+ \left( \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i+2]_{qt} + [3i+3]_{qt}) \right) s_{2,1} + \left( \sum_{i=0}^{k} (qt)^{k-i}[3i+1]_{qt} \right) s_{1^3}, \quad (10)$$

$$Q_{3k+2,3}(-1) = H_{3k+2,3}[X; q, t] = \left( \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i+2]_{qt} \right) s_3$$

$$+ \left( \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i+3]_{qt} + [3i+4]_{qt}) \right) s_{2,1} + \left( \sum_{i=0}^{k} (qt)^{k-i}[3i+2]_{qt} \right) s_{1^3}, \quad (11)$$

$$Q_{3k,3}(-1) = H_{3k,3}[X; q, t] = \left( \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i-1]_{qt} + [3i]_{qt} + [3i+1]_{qt}) \right) s_3$$

$$+ \left( (qt)^{k+1}([3]_{qt} + [2]_{qt} + [1]_{qt}) + \sum_{i=1}^{k-1} (qt)^{k-1-i}([3i]_{qt} + 2[3i+1]_{qt} + 2[3i+2]_{qt} + 3[3]_{qt}) \right) s_{2,1}$$

$$+ \left( \sum_{i=0}^{k} (qt)^{k-i}([3i-1]_{qt} + [3i]_{qt} + [3i+1]_{qt}) \right) s_{1^3}. \quad (12)$$

Note that this independently proves the Shuffle Theorem when $n \leq 3$.

In Section 5, we study several Schur function coefficient formulas and symmetries in $Q_{3,n}(-1)^n$ (some of which are consequences of Theorem 3), and conjecture a concise recursive formula for any Schur function coefficients $[s_\lambda]_{3,n}$. In particular, we study a new symmetry that

$$[s_{2^a b}]_{3,n} = [s_{2^b a}]_{3,(a+b)-n}, \quad (13)$$

and a combinatorial action on parking functions called the switch map $S$.

2 Background

To state our results, we must first recall the Rational Shuffle Conjecture. This will require a series of definitions.

2.1 Combinatorial side

Let $m$ and $n$ be positive integers. An $(m, n)$-Dyck path is a lattice paths from $(0, 0)$ to $(m, n)$ which always remains weakly above the main diagonal $y = \frac{n}{m} x$. The cells that are cut through by the main diagonal will be called diagonal cells. Here Figure 2(a) gives an example of a $(5, 7)$-Dyck path, and Figure 2(b) gives an example of a $(4, 6)$-Dyck path, where the diagonal cells are the light blue cells.

For an $(m, n)$-Dyck path, we have the statistic area defined as follows.
Definition (area). The number of full cells between an \((m, n)\)-Dyck path \(\Pi\) and the main diagonal is denoted \(\text{area}(\Pi)\).

The collection of cells above a Dyck path \(\Pi\) forms a Ferrers diagram (in English notation) of a partition \(\lambda(\Pi)\). In the example of the rational Dyck path pictured in Figure 2(a), \(\lambda(\Pi) = (3, 3, 1, 1) = \mathcal{P}\).

For any partition \(\mu\) and any cell \(c\) in the Ferrers diagram (in English notation) of \(\mu\), we let \(\text{arm}(c)\) be the number of cells to the right of \(c\) in \(\mu\) and \(\text{leg}(c)\) be the number of cells below \(c\) in \(\mu\). Let \(\chi(-)\) denotes the function that takes value 1 if its argument is true, and 0 otherwise, then we can define the path dinv statistic of an \((m, n)\)-Dyck path.

Definition (path dinv). The path dinv (pdinv) of an \((m, n)\)-Dyck path \(\Pi\) is given by

\[
\text{pdinv}(\Pi) = \sum_{c \in \lambda(\Pi)} \chi \left( \frac{\text{arm}(c)}{\text{leg}(c) + 1} \leq \frac{m}{n} < \frac{\text{arm}(c) + 1}{\text{leg}(c)} \right).
\]

An \((m, n)\)-parking function \(PF\) is obtained by labeling the cells east of and adjacent to north steps of an \((m, n)\)-Dyck path with the integers \(1, \ldots, n\) in such a way that the numbers increase in each column as we read from bottom to top. We will refer to these labels as cars. The underlying Dyck path is denoted by \(\Pi(PF)\). The partition formed by the collection of cells above the path \(\Pi(PF)\) is denoted by \(\lambda(PF)\). The set of \((m, n)\)-parking functions is denoted by \(\mathcal{PF}_{m,n}\). Figure 2(c) pictures a \((5, 7)\)-parking function based on the \((5, 7)\)-Dyck path pictured in Figure 2(a).

Next we define statistics ides(PF) and pides(PF) for any parking function PF. For any pair of coprime positive integers \(m\) and \(n\), we define the rank of a cell \((x, y)\) in the \((m, n)\)-grid to be \(\text{rank}(x, y) = my - nx\). If \(m\) and \(n\) are not coprime, we shall generalize the rank to be \(\text{rank}(x, y) = my - nx + \left\lfloor \frac{x \text{gcd}(m, n)}{m} \right\rfloor\). Figure 2(d) shows the rank of the cars in Figure 2(c). \(\sigma(PF)\), the word (or diagonal word) of PF, is obtained by reading cars from highest to lowest ranks. In our example in Figure 2(c), \(\sigma(PF) = 7563412\). We define ides(PF) to be the descent set of \(\sigma(PF)^{-1}\). In other words,

Definition (ides).

\[
\text{ides}(PF) = \{i \in \sigma(PF) : i + 1 \text{ is to the left of } i \text{ in } \sigma(PF)\} \\
= \{i : \text{rank}(i) < \text{rank}(i + 1)\}.
\]

Then we define pides(PF) to be the composition set of ides(PF).

Definition (pides). If \(\text{ides}(PF) = \{i_1 < i_2 < \cdots < i_d\}\), then

\[
\text{pides}(PF) = \{i_1, i_2 - i_1, \ldots, n - i_d\}.
\]
In Figure 2(c), we have $\text{ides}(PF) = \text{ides}(7563412) = \{2, 4, 6\}$, and $\text{pides}(PF) = \{2, 2, 2, 1\}$. If, $i < j$ and $\text{rank}(i) > \text{rank}(j)$, then $i$ and $j$ cannot be in the same column, otherwise $j$ lies on top of $i$, which lead to a contradiction with $\text{rank}(i) > \text{rank}(j)$. Thus, 

**Remark 1.** Let $i < j$ be two cars in the parking function $PF$. If $i$ is to the left of $j$ in $\sigma(PF)$, then the cars $i, j$ must be in different columns.

If $M \in \text{pides}(PF)$ and $M > m$, then there do not exist $M$ cars $k, k + 1, \ldots, k + M - 1$ with decreasing ranks. Otherwise, by Remark 1, the $M$ cars are in different columns, which is impossible, and we have the following remark. 

**Remark 2.** The parts in the composition set $\text{pides}(PF)$ of a parking function $PF \in \mathcal{P}F_{m,n}$ are less than or equal to $m$.

In many papers, the statistic dinv of a parking function is defined by 3 components – path dinv ($\text{pdinv}$), max dinv ($\text{maxdinv}$) and temporary dinv ($\text{tdinv}$).

**Definition (tdinv).** Let $PF$ be any $(m, n)$-parking function, then

$$\text{tdinv}(PF) = \sum_{\text{cars} \ i < j} \chi(\text{rank}(i) < \text{rank}(j) < \text{rank}(i) + m).$$

In Figure 2(c), $\text{tdinv}(PF) = 7$ since the pairs of cars contributing to $\text{tdinv}$ are $(1, 3)$, $(1, 4)$, $(3, 5)$, $(3, 6)$, $(4, 6)$, $(5, 7)$ and $(6, 7)$. Then, the statistic max dinv of a path is defined as the maximum of temporary dinvs of parking functions on the path.

**Definition (maxdinv).**

$$\text{maxdinv}(PF) = \max\{\text{tdinv}(PF') : \Pi(PF') = \Pi(PF)\}.$$ 

Finally, the statistic dinv is defined as

**Definition (dinv).**

$$\text{dinv}(PF) = \text{tdinv}(PF) + \text{pdinv}(\Pi(PF)) - \text{maxdinv}(PF).$$

We shall use this definition of dinv in some combinatorial proofs in Section 3. However, our definition of dinv($PF$) in Section 4 will follow the formulation by Leven and Hicks [14], who gave a simplified formula for dinv($PF$) as the sum of two simpler statistics, tdinv and dinvcorr.

**Definition (dinvcorr).** Let $\Pi$ be any $(m, n)$-Dyck path and set $\frac{m}{n} = 0$ and $\frac{m}{n} = \infty$ for all $x \neq 0$, then

$$\text{dinvcorr}(\Pi) = \sum_{c \in \lambda(\Pi)} \chi\left(\frac{\text{arm}(c)}{\text{leg}(c)} < \frac{m}{n} < \frac{\text{arm}(c) + 1}{\text{leg}(c) + 1}\right) - \sum_{c \in \lambda(\Pi)} \chi\left(\frac{\text{arm}(c)}{\text{leg}(c)} < \frac{m}{n} < \frac{\text{arm}(c) + 1}{\text{leg}(c) + 1}\right).$$

**Definition (dinv, alt.).** Let $PF$ be any $(m, n)$-parking function, then

$$\text{dinv}(PF) = \text{tdinv}(PF) + \text{dinvcorr}(\Pi(PF)).$$

Note that the statistic dinvcorr is only depend on the path $\Pi$, and it is the difference of two sums

$$\sum_{c \in \lambda(\Pi)} \chi\left(\frac{\text{arm}(c)}{\text{leg}(c)} < \frac{m}{n} < \frac{\text{arm}(c)}{\text{leg}(c)} + 1\right) \quad \text{and} \quad \sum_{c \in \lambda(\Pi)} \chi\left(\frac{\text{arm}(c)}{\text{leg}(c)} < \frac{m}{n} < \frac{\text{arm}(c) + 1}{\text{leg}(c) + 1}\right),$$

of which at most one is nonzero. If $m = n$, then there is no dinvcorr. Otherwise, we count dinvcorr by all the cells in $\lambda(\Pi)$. Given a cell $c \in \lambda(\Pi)$, we high-light the vertical line segment $N$ which is a north step of the path $\Pi$ to the east of $c$, and the horizontal line segment $E$ which is a east step of $\Pi$ to the south of $c$. We draw two lines with slope $\frac{n}{m}$ from the north end and south end of $N$. 

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(1) If \( n > m \), the cells of type (a) and (b) in Figure 3 contribute \(-1\) to \( \operatorname{dinvcorr} \).

(2) If \( m > n \), the cells of type (c) and (d) in Figure 3 contribute 1 to \( \operatorname{dinvcorr} \).

Given \( S \subseteq \{1, 2, \ldots, n-1\} \), we let \( F_S[X] \) denote the fundamental quasi-symmetric function of Gessel associated to \( S \), where \( X = x_1 + x_2 + \cdots + x_n \). Then following Hikita, we define the Hikita polynomial \( H_{m,n}[X; q, t] \) where \( m \) and \( n \) are coprime by

\[
H_{m,n}[X; q, t] = \sum_{PF \in \mathcal{PF}_{m,n}} t^{\text{area}(PF)} q^{\operatorname{dinv}(PF)} F_{\operatorname{ides}(PF)}[X].
\]

(14)

For non-coprime case, we formulate Hikita polynomials as follows. Given \( m, n \) coprime and \( k \geq 1 \), we defined the return of a \((km, kn)\)-parking function \( PF \), \( \text{ret}(PF) \) to be the smallest positive integer \( i \) such that the supporting path of \( PF \) goes through the point \((im, in)\). Then following the formulation of Garsia, Leven, Wallach, and Xin, we define the extended Hikita polynomial to be

\[
H_{km,kn}[X; q, t] = \sum_{PF \in \mathcal{PF}_{km,kn}} [\text{ret}(PF)] t^{\text{area}(PF)} q^{\operatorname{dinv}(PF)} F_{\operatorname{ides}(PF)}[X].
\]

(15)

### 2.2 Algebraic side

For any partition \( \mu \) of \( n \), let \( \tilde{H}_\mu \) be the modified Macdonald symmetric function associated to \( \mu \), and let \( \nabla \) be the linear operator defined in terms of the modified Macdonald symmetric functions \( \tilde{H}_\mu(X; q, t) \) by

\[
\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu')} \tilde{H}_\mu,
\]

(16)

where \( \mu' \) is the conjugate of \( \mu \), and \( n(\mu) = \sum_i (i-1) \mu_i \).

The Classical Shuffle Conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov which was proved by Carlson and Mellit can be stated as follows.

**Theorem 5 (Carlson-Mellit).** For all \( n \geq 0 \),

\[
\nabla e_n = H_{n+1,n}[X; q, t].
\]

(17)

Gorsky and Negut introduced operators \( Q_{m,n} \) on symmetric functions in the case where \( m \) and \( n \) are coprime.

As shown in [3], the \( Q_{m,n} \) operators of the Gorsky-Negut conjecture can be defined in terms of the operators \( D_k \) which were introduced by Bergeron and Garsia. In plethystic notation, the
action of $D_k$ on a symmetric function $F[X]$ is defined as

$$D_k F[X] = F \left[ X + \frac{M}{z} \sum_{i \geq 0} (-z)^i e_i[X] \right]^{\cdot k}, \quad (18)$$

where $M = (1 - t)(1 - q)$.

Then one can construct a family of symmetric function operators $Q_{m,n}$ for any pair of coprime positive integers $(m, n)$ as follows. First for any $n \geq 0$, set $Q_{1,n} = D_n$. Next, one can recursively define $Q_{m,n}$ for $m > 1$ as follows. Consider the $m \times n$ lattice with diagonal $y = \frac{n}{m} x$. Let $(a, b)$ be the lattice point which is closest to and below the diagonal. Set $(c, d) = (m - a, n - b)$. In such a case, we will write

$$\text{Split}(m, n) = (a, b) + (c, d). \quad (19)$$

Note that the pairs $(a, b)$ and $(c, d)$ are coprime since any point of the form $(kx, ky)$ is further from the diagonal than the point $(x, y)$. Then we have the following recursive definition of the $Q_{m,n}$ operators:

$$Q_{m,n} = \frac{1}{M}[Q_{c,d}, Q_{a,b}] = \frac{1}{M}(Q_{c,d}Q_{a,b} - Q_{a,b}Q_{c,d}). \quad (20)$$

Figure 4 gives an example of Split(3, 5). Split(3, 5) = (2, 3) + (1, 2) so that

$$Q_{3,5} = \frac{1}{M}[Q_{1,2}, Q_{2,3}] = \frac{1}{M}[D_2, Q_{2,3}]. \quad (21)$$

![Figure 4: The geometry of Split(3, 5).](image)

The same procedure gives $Q_{2,3} = \frac{1}{M}[Q_{1,2}, Q_{1,1}] = \frac{1}{M}[D_2, D_1]$. Therefore

$$Q_{3,5} = \frac{1}{M^2}[D_2, [D_2, D_1]] = \frac{1}{M^2}(D_2 D_2 D_1 - 2D_2 D_1 D_2 + D_1 D_2 D_2). \quad (22)$$

For the non-coprime case, we can define the $Q_{km, kn}$ operator as follows. We choose one of the lattice points, $(a, b)$, in the $km \times kn$ lattice that are strictly below and closest to the diagonal, then we set

$$Q_{km, kn} = \frac{1}{M}[Q_{km-a, kn-b}, Q_{a,b}]. \quad (23)$$

This recursive definition is well-defined as it is proved in [3] that any choice of such point $(a, b)$ defines the same operation.
2.3 The Rational Shuffle Conjecture

Garsia, Leven, Wallach, and Xin \cite{GarsiaLevenWallachXin2003} and Gorsky and Negut \cite{GorskyNegut2009} conjectured the following theorem which was proved by Mellit \cite{Mellit2013}.

**Theorem 6** (Mellit). For all pairs of coprime positive integers \((m, n)\) and any \(k \in \mathbb{Z}^+\), we have

\[
Q_{km, kn}(-1)^{kn} = H_{km, kn}[X; q, t].
\]  

(24)

The original Rational Shuffle Conjecture of Gorsky and Negut in the case where \(m\) and \(n\) are relatively prime is the special case when \(k = 1\) in Theorem 6. In this case, \(\text{ret}(PF)\) is always 1 and the theorem can be described as follows.

**Theorem 7** (Mellit). For all pairs of coprime positive integers \((m, n)\), we have

\[
Q_{m, n}(-1)^n = H_{m, n}[X; q, t].
\]  

(25)

The main goal of this paper is to study the combinatorics of the Schur function expansion of \(Q_{m, n}(-1)^n\). Given that Mellit has proved the Rational Shuffle Conjecture, we can find the Schur function expansion in one of two ways. That is, we can use the properties of \(Q_{m, n}\) to find the Schur function expansion of \(Q_{m, n}(-1)^n\) which we will refer to as working on the symmetric function side of the Rational Shuffle Conjecture. Second, one could start with the Hikita polynomial \(H_{m, n}[X; q, t]\) and expand that polynomial into Schur functions which we will call working on the combinatorial side of the Rational Shuffle Conjecture.

Since it is proved that \(Q_{m, n}(-1)^n = H_{m, n}[X; q, t]\), we let \([s_\lambda]_{m, n}\) be the coefficient of Schur function \(s_\lambda\) in both polynomials \(Q_{m, n}(-1)^n\) and \(H_{m, n}[X; q, t]\).

2.4 An alternative expression for the combinatorial side

Hikita \cite{Hikita2012} in 2012 proved that the Hikita polynomials \(H_{m, n}[X; q, t]\) are symmetric (in \(X\)) for any coprime \(m, n\).

A weak composition of \(n\) is a sequence of non-negative integers summing up to \(n\). Suppose that \(\gamma = (\gamma_1, \ldots, \gamma_n)\) is a weak composition of \(n\) into \(n\) parts. We let \(X = (x_1, \ldots, x_n)\) and

\[
\Delta_\gamma(X) = \det ||x_i^{\gamma_i+n-j}|| = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma)s(x_1^{\gamma_1+n-1} \cdots x_n^{\gamma_n+n-n}).
\]

We let \(\Delta(X) = \det ||x_i^{n-j}||\) be the Vandermonde determinant. Then the Schur function \(s_\gamma(X)\) associated to \(\gamma\) is defined to be

\[
s_\gamma(X) = \frac{\Delta_\gamma(X)}{\Delta(X)}.
\]  

(26)

It is well known that for any such weak composition \(\gamma\), either \(s_\gamma(X) = 0\) or there is a partition \(\lambda\) of \(n\) such that \(s_\gamma(X) = \pm s_\lambda(X)\). In fact, there is a well-known straightening relation which allows us to prove that fact. Namely, if \(\gamma_i+1 > 0\), then

\[
s_{(\gamma_1, \ldots, \gamma_i+1, \ldots, \gamma_n)}(X) = -s_{(\gamma_1, \ldots, \gamma_i+1-1, \gamma_i+1, \ldots, \gamma_n)}(X).
\]  

(27)
Suppose \( \alpha = (\alpha_1, \ldots, \alpha_k) \) is a composition of \( n \) with \( k \) parts. We associate a subset \( S(\alpha) \) of \( \{1, \ldots, n-1\} \) with \( \alpha \) by setting 
\[
S(\alpha) = \{ \alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1} \}.
\]
We let \( \tilde{\alpha} \) be the weak composition of \( n \) with \( n \) parts by adding a sequence of \( n-k \) 0's at the end of \( \alpha \). For example, if \( \alpha = (2,3,2,1) \), then \( S(\alpha) = \{2, 5, 7\} \) and \( \tilde{\alpha} = (2,3,2,1,0,0,0,0) \). We can then associate one Gessel’s fundamental quasi-symmetric function with each composition \( \alpha \) by 
\[
F_{\alpha}(X) = \sum_{\substack{1 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq n \\
i \in S(\alpha) \rightarrow a_i < a_{i+1}}} x_{a_1} x_{a_2} \cdots x_{a_n}.
\] (28)
The \( F_{\alpha}(X) \)'s as \( \alpha \) ranges over the compositions of \( n \) for a basis for the space of quasi-symmetric functions \( Q_n(x_1, \ldots, x_n) \) of degree \( n \).

In a remarkable and important paper, Egge, Loehr and Warrington [5] gave a combinatorial description of how to start with the a quasi-symmetric function expansion of a homogeneous symmetric function of degree \( n \), 
\[
P(X) = \sum_{\alpha | n} a_{\alpha} F_{\alpha}(X),
\]
and transform it into an expansion in terms of Schur functions 
\[
P(X) = \sum_{\lambda \vdash n} b_{\lambda}s_{\lambda}(X).
\]
The following theorem due to Garsia and Remmel [7] is implicit in the work of [5], but is not explicitly stated and it allows one to find the Schur function expansion by using the straightening laws.

**Theorem 8 (Garsia-Remmel).** Suppose that \( P(X) \) is a symmetric function which is homogeneous of degree \( n \) and 
\[
P(X) = \sum_{\alpha | n} a_{\alpha} F_{\alpha}(X).
\] (29)
Then 
\[
P(X) = \sum_{\alpha | n} a_{\alpha}s_{\tilde{\alpha}}(X).
\] (30)

Recall that \( \text{pides}(\sigma) \) is the composition set of \( \text{ides}(\sigma) \), then Theorem [3] and the straightening action allow us to transform \( H_{m,n} [X; q, t] \) into Schur function expansion that 
\[
H_{m,n} [X; q, t] = \sum_{\text{PF} \in \mathcal{P}_m, n} \left[ \text{ret} \langle \text{PF} \rangle \right] t^{\text{area}(\text{PF})} q^{\text{dinv}(\text{PF})} F_{\text{ides} \langle \text{PF} \rangle} [X]
\]
\[
= \sum_{\text{PF} \in \mathcal{P}_m, n} \left[ \text{ret} \langle \text{PF} \rangle \right] t^{\text{area}(\text{PF})} q^{\text{dinv}(\text{PF})} s_{\text{pides} \langle \text{PF} \rangle}.
\] (31)

From Section 3, we shall use this expression for \( H_{m,n} [X; q, t] \) to prove several facts about \( [s_{\lambda}]_{m,n} \), \( [s_{\lambda}]_{m,3} \) and \( [s_{\lambda}]_{3,n} \).
3 Combinatorial Results about Schur Basis Expansion of \((m, n)\) Case

In \((m, n)\) case, we have \(n\) cars, i.e. the word of an \((m, n)\)-parking function is a permutation of \([n]\).

By Remark 2, \([s_\lambda]_{m,n} \neq 0\) implies that \(\lambda\) must be of the form \(m^{\alpha_m} \cdots 1^{\alpha_1}\) with \(\sum_{i=1}^m i\alpha_i = n\), i.e. \([s_\lambda]_{m,n} \neq 0\) only if the partition \(\lambda\) only has parts of size less than or equal to \(m\). In this section, we shall prove the 3 symmetries about \([s_\lambda]_{m,n}\) described in Theorem 3.

Result. \([s_1^n]_{m,n} = [s_n]_{m+n,n}\).

Note that a parking function with pides \(n\) must have word \(12\cdots n\), and a parking function with pides \(1^n\) must have word \(n\cdot\cdots 21\).

![Figure 5: Bijection between \(\mathcal{PF}_{m,3}\) with word 123 and \(\mathcal{PF}_{m+3,3}\) with word 321.](image)

A parking function in \(\mathcal{PF}_{m,n}\) with word \(n\cdots 21\) is correspond with a unique \((m, n)\)-Dyck path, and a parking function in \(\mathcal{PF}_{m+n,n}\) with word \(12\cdots n\) is correspond with an \((m+n, n)\)-Dyck path with no two consecutive north steps. As shown is Figure 5, we can obtain a parking function \(\mathcal{PF}_{m+n,n}\) with word \(12\cdots n\) by pushing a staircase into a \(\mathcal{PF} \in \mathcal{PF}_{m,n}\) with word \(n\cdot\cdots 21\). Given a \(\mathcal{PF} \in \mathcal{PF}_{m,n}\) with word \(n\cdots 21\), let \(\lambda = \lambda(\mathcal{PF})\), we define \(hstr(\mathcal{PF}) \in \mathcal{PF}_{m+n,n}\), the horizontal stretch of \(\mathcal{PF}\), to be the parking function with word \(12\cdots n\) and \(\lambda(hstr(\mathcal{PF})) = (\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_n + 1)\), then

**Theorem 9.**

\[
hstr : \{\mathcal{PF} \in \mathcal{PF}_{m,n} : \text{word}(\mathcal{PF}) = n\cdots 21\} \rightarrow \{\mathcal{PF} \in \mathcal{PF}_{m+n,n} : \text{word}(\mathcal{PF}) = 12\cdots n\},
\]

\[
\mathcal{PF} \mapsto hstr(\mathcal{PF})
\]

is a bijection, and

\[
\text{area}(hstr(\mathcal{PF})) = \text{area}(\mathcal{PF}),
\]

\[
\text{dinv}(hstr(\mathcal{PF})) = \text{dinv}(\mathcal{PF}).
\]

**Proof.** The bijectivity of map \(hstr\) is clear since the map is invertible. Comparing the coarea of both parking functions immediately proves equation (32). To prove equation (33), recall that \(\text{dinv}(\mathcal{PF}) = \text{tdinv}(\mathcal{PF}) + \text{dinvcorr}(\mathcal{PF})\), we shall compare the two components of \(\text{dinv}\), i.e. \(\text{tdinv}\) and \(\text{dinvcorr}\).

For a \(\mathcal{PF} \in \mathcal{PF}_{m,n}\) with word \(\text{word}(\mathcal{PF}) = n\cdots 21\), \(\text{tdinv}(\mathcal{PF})\) reaches the maximum, i.e. any two north step with rank difference less than \(m\) will contribute 1 to \(\text{tdinv}\). For any two north steps, we fire two lines from the two end points of the upper north step, then the rank difference less than \(m\) means that either the upper line or the lower line intersects the lower north step. The two cases are pictured in Figure 6.

On the other hand, the parking function \(hstr(\mathcal{PF}) \in \mathcal{PF}_{m+n,n}\) always has no \(\text{tdinv}\) since \(\text{word}(hstr(\mathcal{PF})) = 12\cdots n\). We want to show that the increase of \(\text{dinvcorr}\) makes up for the missing \(\text{tdinv}\).
Suppose that there are \( j \) cells in row \( r \) of \( \mathcal{PF} \in \mathcal{PF}_{m,n} \) with leg \( i \), and their arms are \( a, a + 1, \ldots, a + j - 1 \), pictured in Figure 7(a). We fire two lines with slope \( \frac{m}{n} \) from the two end points of the north step (called \( N_1 \)) in row \( r \), then they intersect the east steps (called \( EEs \)) below the \( j \) cells at points \( A, B \) which have horizontal distances \( \frac{mi}{n} \) and \( \frac{m(i+1)}{n} \) to \( N_1 \). 

In the parking function \( hstr(\mathcal{PF}) \in \mathcal{PF}_{m+n,n} \), there are \( j + 1 \) cells in row \( r \) with leg \( i \), and their arms are \( a + i, a + i + 1, \ldots, a + i + j \), pictured in Figure 7(b). We again fire two lines with slope \( \frac{n}{m+n} \) from the two end points of the north step \( N_1 \) in row \( r \), then they intersect the east steps below the \( j + 1 \) cells at points \( A, B \) which have horizontal distances \( \frac{mi}{n} + i \) and \( \frac{m(i+1)}{n} + i + 1 \) to \( N_1 \). 

Now recall the definition of the dinv correction. The dinvcorr contribution of \( N_1 \) in each picture is equal to the whole east steps contained in line segment \( AB \). The line segment \( AB \) in \( hstr(\mathcal{PF}) \) contains one more east step than \( AB \) in \( \mathcal{PF} \) in the following 2 cases:

1. In \( \mathcal{PF} \), \( A \) is not on \( EEs \) but \( B \) is on \( EEs \).
2. In \( \mathcal{PF} \), \( A \) is on \( EEs \).

In case (1), the car in row \( r \) of \( \mathcal{PF} \) cause a tdinv with the car in the row immediately below \( EEs \); in case (2), the car in row \( r \) of \( \mathcal{PF} \) cause a tdinv with the car in the row of the next north step that the upper line fired from \( N_1 \) intersects. Thus, the increase of dinv correction is equal to tdinv(\( \mathcal{PF} \)), which proves the theorem. 

Since Theorem 9 is an (area,dinv)-preserving bijection, \([s_{1^n}]_{m,n} = [s_n]_{m+n,n}\) follows immediately.

Result. \([s_{m^\omega m \ldots 1^\omega 1}]_{m,n} = [s_{m^\omega m + 1^\omega 1}]_{m,n+m}\)

This is a rewording of Theorem 4 (b). For a parking function \( \mathcal{PF} \in \mathcal{PF}_{m,n} \), we define a map \( vstr \), vertical stretch, that we push a staircase down to \( \mathcal{PF} \), then replace the car \( i \) in \( \mathcal{PF} \) by \( i + m \),
and fill the bottom of the \( m \) columns of the new parking function with cars 1, \ldots, \( m \) in a rank decreasing way to get \( vstr(PF) \), as shown in Figure 8.

Figure 8: Bijection between \( \mathcal{PF}_{3,n} \) with pides \( 3^a2^b1^c \) and \( \mathcal{PF}_{3,n+3} \) with pides \( 3^{a+1}2^b1^c \).

Similar to Theorem 9, we have the following theorem about the vertical stretch action.

**Theorem 10.**

\[
vstr : \{PF \in \mathcal{PF}_{m,n} : \text{pides}(PF) = m^{a_m} \ldots 1^{a_1}\} \to \{PF \in \mathcal{PF}_{m+n,n} : \text{pides}(PF) = m^{a_m+1} \ldots 1^{a_1}\},
\]

\[
PF \mapsto vstr(PF)
\]

is a bijection, and

\[
\text{area}(vstr(PF)) = \text{area}(PF) \tag{34}
\]

\[
\text{dinv}(vstr(PF)) = \text{dinv}(PF) \tag{35}
\]

**Proof.** The bijectivity is true since the map is invertible. Equation (34) is true for the same reason as equation (32). The proof of equation (35) is based on the same idea as the proof of (33): the action \( vstr \) is changing each car \( i \) in \( PF \) into \( i + m \), and the rank is also increased by \( m \), thus the temporary dinv of \( PF \) is equal to the temporary dinv of the cars \( m + 1, \ldots, m + n \) in \( vstr(PF) \). Since the dinv correction is negative, we can match each tdinv between cars 1, 2, \ldots, \( m \) and \( m + 1, \ldots, m + n \) with a new negative dinv correction, showing that the change of dinv is zero. \( \square \)

**Result.** \( [s_{k1-n-k}]_{m,n} = [s_{k1-m-k}]_{n,m} \).

We shall prove the special case when \( k = 1 \) first. That is, we first show \( [s_{1n}]_{m,n} = [s_{1m}]_{n,m} \). The bijection for this identity is that we can transpose the path of \( PF \in \mathcal{PF}_{m,n} \) and fill the word \( (m, m-1, \ldots, 1) \) to get \( PF' \in \mathcal{PF}_{m,n} \). It’s easy to verify that \( PF' \) has the same area as \( PF \). For the statistic \( \text{dinv} \), recall that the tdinv of a parking function with word \( (n, n-1, \ldots, 1) \) is equal to the \( \text{maxdinv} \) of the path, thus

\[
\text{dinv}(PF) = \text{tdinv}(PF) + \text{pdinv}(\Pi(PF)) - \text{maxdinv}(PF)
\]

\[
= \text{pdinv}(\Pi(PF))
\]

\[
= \sum_{c \in \lambda(\Pi)} \chi \left( \frac{\text{arm}(c)}{\text{leg}(c)} + 1 \leq \frac{m}{n} < \frac{\text{arm}(c) + 1}{\text{leg}(c)} \right). \tag{36}
\]

From the formula, we see that \( \text{dinv} \) is symmetric about \( m \) and \( n \), and preserved by the transpose action. Figure 9 shows an example of this bijection.

Then we consider the equality \( [s_{k1-n-k}]_{m,n} = [s_{k1-m-k}]_{n,m} \). This bijective proof is similar to that of \( [s_{1n}]_{m,n} = [s_{1m}]_{n,m} \).
That is, given a PF $\in \mathcal{PF}_{m,n}$ with pides $k_{1^{n-k}}$, one transposes the path and labels the path to produce pides $k_{1^{m-k}}$. If there are only $k$ peaks (which means $k$ different columns) in the Dyck paths, then the filling of cars in both $(m,n)$ and $(n,m)$ cases are unique since the cars $1, \ldots, k$ must be filled in a rank-decreasing way at bottom of each column in the two parking functions, while the rest cars should be filled in a rank-increasing way in the rest north steps. One can check that they have the same area and dinv statistics.

Otherwise, in any rational $(m,n)$-Dyck path $\Pi$ with $j > k$ peaks, the car $k$ must be in the first row since it has the smallest rank, and there are $\binom{j-1}{k-1}$ ways to choose columns to place the cars $1, \ldots, k-1$ in the north steps of both $\Pi$ and its transpose, while the rest cars should be filled in a rank-increasing way in the rest north steps. Using the idea of analyzing dinv in the proof of Theorem 9, we are able to match the $\binom{j-1}{k-1}$ possible positions of cars $1, \ldots, k-1$ in both $(m,n)$ and $(n,m)$ cases by the dinv statistic, thus prove the result.

Note that Theorem 3 (c) is a result about the hook-shaped Schur functions. As we proved within this result, Theorem 3 (c) implies the following corollary.

**Corollary 11.** For all $m, n > 0$,

$$[s_1^m]_{m,n} = [s_1^n]_{n,m}.$$  

# 4 Schur Basis Expansion of $(m, 3)$ Case

In this section, we give two proofs of Theorem 4 by both working on the symmetric function side and the combinatorial side of the Rational Shuffle Conjecture. Our proofs independently prove the Shuffle Theorem when $n \leq 3$.

## 4.1 Algebraic proof — $Q_{m,3}(-1)$

In this proof, we use Leven’s method in [17] to prove by induction. We use the following lemma about $q, t$-anologue integers to simplify the formulas.

**Lemma 1.**

$$[n]_{q,t}[k]_{q,t} = [n + k - 1]_{q,t} + qt[k - 1]_{q,t}[n - 1]_{q,t}. \quad (37)$$

**Proof.**

$$[n]_{q,t}[k]_{q,t} = (q^{n-1} + q^{n-2}t + \ldots + qt^{n-2} + t^{n-1})(q^{k-1} + q^{k-2}t + \ldots + qt^{k-2} + t^{k-1})$$

$$= q^{n-1}(q^{k-1} + q^{k-2}t + \ldots + qt^{k-2} + t^{k-1}) + (q^{n-2}t + \ldots + qt^{n-2} + t^{n-1})t^{k-1}$$

$$+(q^{n-2}t + \ldots + qt^{n-2} + t^{n-1})(q^{k-1} + q^{k-2}t + \ldots + qt^{k-2})$$

$$= [n + k - 1]_{q,t} + qt[k - 1]_{q,t}[n - 1]_{q,t}. \quad \Box (38)$$
We need the following lemma from [3] to prove the symmetric function side of the theorem.

**Lemma 2.** For any positive integers \( m, n \),

\[
\nabla Q_{m,n} \nabla^{-1} = Q_{m+n,n}.
\]

(39)

This allows us to get a recursion for \( Q_{m,n} \) operator that

\[
Q_{m+n,n}(-1)^n = \nabla Q_{m,n} \nabla^{-1}(-1)^n = \nabla Q_{m,n}(-1)^n.
\]

(40)

Using the recursion, we can prove Theorem 4 by inducting on \( m \). We shall give the complete algebraic proof of equation (10) in Theorem 4 and omit the algebraic proof of (11) and (12).

### Table 1: Coefficients of \( s_{\lambda} \) in \( Q_{3k+1,3}(-1) \).

| \( Q_{3k+1,3}(-1) \) | \( s_{\lambda} \) | \( s_3 \) | \( s_{21} \) | \( s_{13} \) |
|----------------------|-----------------|----------|----------|----------|
| \( Q_{1,3}(-1) \)    | 0               | 0        | [1]      |
| \( Q_{4,3}(-1) \)    | [1] \( q,t \)  | [2] \( q,t + [3] \) | [1] \( q,t \) + \( qt[4] \) |
| \( Q_{7,3}(-1) \)    | [4] \( q,t \)  | [5] \( q,t + [6] \) + \( qt([2] \( q,t + [3] \)) \) | [7] \( q,t \) + \( qt[4] \) + \( (qt)^2[1] \) |
| \( Q_{10,3}(-1) \)   | [7] \( q,t \) + \( qt[4] \) + \( (qt)^2[1] \) + \( (qt)^3[1] \) | [8] \( q,t + [9] \) + \( qt([5] \( q,t + [6] \)) \) + \( (qt)^2([2] \( q,t + [3] \)) \) | [13] \( q,t \) + \( qt[10] \) + \( (qt)^3[1] \) |

### Proof of equation (10). When \( k = 0 \), we can obtain by direct computation that

\[
Q_{1,3}(-1) = s_{13},
\]

(41)

which satisfies equation (10). Then we induct on \( k \) to prove equation (10) that suppose the Schur
function coefficients of $Q_{3k+1,3}$ are the following,

$$[s_3]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i + 1]_{q,t},$$  \hspace{1cm} (42)

$$[s_{21}]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i + 2]_{q,t} + [3i + 3]_{q,t}),$$  \hspace{1cm} (43)

$$[s_{13}]_{3k+1,3} = \sum_{i=0}^{k} (qt)^{k-1-i}[3i + 1]_{q,t},$$  \hspace{1cm} (44)

we want to show that

$$[s_3]_{3(k+1)+1,3} = \sum_{i=0}^{k} (qt)^{k-1-i}[3i + 1]_{q,t},$$  \hspace{1cm} (45)

$$[s_{21}]_{3(k+1)+1,3} = \sum_{i=0}^{k} (qt)^{k-1-i}([3i + 2]_{q,t} + [3i + 3]_{q,t}),$$  \hspace{1cm} (46)

$$[s_{13}]_{3(k+1)+1,3} = \sum_{i=0}^{k+1} (qt)^{k-1-i}[3i + 1]_{q,t}.$$  \hspace{1cm} (47)

One can directly compute that

$$\nabla s_3 = (qt)^2 s_{21} + (qt)^2 [2]_{q,t} s_{13},$$  \hspace{1cm} (48)

$$\nabla s_{21} = (qt) [2]_{q,t} s_{21} - (qt) [3]_{q,t} s_{13},$$  \hspace{1cm} (49)

$$\nabla s_{13} = s_3 + ([2]_{q,t} + [3]_{q,t}) s_{21} + (qt + [4]_{q,t}) s_{13}.$$  \hspace{1cm} (50)

By equation (40), we have

$$Q_{3(k+1)+1,3}(-1) = [s_3]_{3k+4,3}s_3 + [s_{21}]_{3k+4,3}s_{21} + [s_{13}]_{3k+4,3}s_{13}$$

$$= \nabla Q_{3k+1,3}(-1)$$

$$= \nabla ([s_3]_{3k+1,3}s_3 + [s_{21}]_{3k+1,3}s_{21} + [s_{13}]_{3k+1,3}s_{13})$$

$$= [s_3]_{3k+1,3} \nabla s_3 + [s_{21}]_{3k+1,3} \nabla s_{21} + [s_{13}]_{3k+1,3} \nabla s_{13}$$

$$= [s_3]_{3k+1,3}s_3$$

$$+ ([qt]^2 [s_3]_{3k+1,3} - qt [2]_{q,t} [s_{21}]_{3k+1,3} + ([2]_{q,t} + [3]_{q,t}) [s_{13}]_{3k+1,3}] s_{21}$$

$$+ ([qt]^2 [2]_{q,t} [s_3]_{3k+1,3} - qt [3]_{q,t} [s_{21}]_{3k+1,3} + (qt + [4]_{q,t}) [s_{13}]_{3k+1,3}] s_{13},$$  \hspace{1cm} (51)

which implies that

$$[s_3]_{3k+4,3} = [s_{13}]_{3k+1,3},$$  \hspace{1cm} (52)

$$[s_{21}]_{3k+4,3} = (qt)^2 [s_3]_{3k+1,3} - qt [2]_{q,t} [s_{21}]_{3k+1,3} + ([2]_{q,t} + [3]_{q,t}) [s_{13}]_{3k+1,3},$$  \hspace{1cm} (53)

$$[s_{13}]_{3k+4,3} = (qt)^2 [2]_{q,t} [s_3]_{3k+1,3} - qt [3]_{q,t} [s_{21}]_{3k+1,3} + (qt + [4]_{q,t}) [s_{13}]_{3k+1,3}.$$  \hspace{1cm} (54)
By the recursions above, we can apply Lemma 1 and verify equation (45) \( \sim (47) \) inductively that

\[
[s_{13}]_{3k+4,3} = [s_{13}]_{3k+1,3} = \sum_{i=0}^{k} (qt)^{k-i}[3i+1]_{q,t}, \quad (55)
\]

\[
[s_{21}]_{3k+4,3} = (qt)^2 \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i+1]_{q,t} - qt[2]_{q,t} \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i+2]_{q,t} + [3i+3]_{q,t})
\]

\[
+ ([2]_{q,t} + [3]_{q,t}) \sum_{i=0}^{k} (qt)^{k-i}[3i+1]_{q,t}
\]

\[
= (qt)^2 \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i+1]_{q,t} - qt \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i+3]_{q,t} + qt[3i+1]_{q,t})
\]

\[
- qt[2]_{q,t} \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i+3]_{q,t}
\]

\[
+ \sum_{i=0}^{k} (qt)^{k-i}([3i+2]_{q,t} + [3i+3]_{q,t}) + qt[3]_{q,t}[3i]_{q,t} + qt[2]_{q,t}[3i]_{q,t}
\]

\[
= \sum_{i=0}^{k} (qt)^{k-i}([3i+2]_{q,t} + [3i+3]_{q,t}), \quad \text{and} \quad (56)
\]

\[
[s_{13}]_{3k+4,3} = (qt)^2[2]_{q,t}[s_{3}]_{3k+1,3} - qt[3]_{q,t}[s_{21}]_{3k+1,3} + (qt + [4]_{q,t})[s_{13}]_{3k+1,3}
\]

\[
= (qt)^2[2]_{q,t} \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i+1]_{q,t} - qt \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i+4]_{q,t} + qt[2]_{q,t}[3i+1]_{q,t})
\]

\[
- qt[3]_{q,t} \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i+3]_{q,t} + \sum_{i=0}^{k} (qt)^{k-i}(qt[3i+1]_{q,t} + [3i+4]_{q,t} + qt[3]_{q,t}[3i]_{q,t})
\]

\[
= \sum_{i=0}^{k+1} (qt)^{k+1-i}[3i+1]_{q,t}. \quad (57)
\]

### 4.2 Combinatorial Side — \( H_{m,3}[X; q, t] \)

Now we consider the Hikita polynomial defined by

\[
H_{m,n}[X; q, t] = \sum_{PF \in \mathcal{PF}_{m,n}} [ret(PF)]_{t}^1 t^{area(PF)} q^{dinv(PF)} s_{pides(PF)}. \quad (58)
\]

Any parking function \( PF \in \mathcal{PF}_{m,3} \) has 3 rows, thus only has 3 cars: 1, 2, 3, and the word \( \sigma(PF) \) can be any permutation \( \sigma \in S_3 \). Table 2 shows the \( s_{pides} \) contribution of the 6 permutations in \( S_3 \).

By our notation, \( H_{m,3}[X; q, t] = [s_{3}]_{m,3} s_{3} + [s_{21}]_{m,3} s_{21} + [s_{13}]_{m,3} s_{13} \). We can work out the combinatorial side of the Rational Shuffle Conjecture in the case where \( n = 3 \) by (58).
Table 2: $s_{\text{pides}}$ contribution of permutations in $S_3$.

| $\sigma \in S_3$ | 123 | 132 | 213 | 231 | 312 | 321 |
|-------------------|-----|-----|-----|-----|-----|-----|
| $s_{\text{pides}}$ | $s_3$ | $s_{21}$ | $s_{12} = 0$ | $s_{21}$ | $s_{12} = 0$ | $s_{13}$ |

### 4.2.1 Combinatorics of $H_{3k+1,3}[X; q, t]$

We show the combinatorics of $H_{3k+1,3}[X; q, t]$ by enumerating the parking functions on the $(3k + 1) \times 3$ lattice to prove the following coefficients of the polynomial $H_{3k+1,3}[X; q, t]$.

\[
[s_3]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i + 1]_{q,t},
\]

\[
[s_{21}]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i + 2]_{q,t} + [3i + 3]_{q,t}),
\]

\[
[s_{13}]_{3k+1,3} = \sum_{i=0}^{k} (qt)^{k-1-i}[3i + 1]_{q,t},
\]

Given a $\text{PF} \in \mathcal{PF}_{3k+1,3}$, we let $\Pi = \Pi(\text{PF})$ be the path of $\text{PF}$. Since $3k + 1 > 3$ for $k \geq 1$, the $dinv$ correction by definition is nonnegative that

\[
dinvcorr(\text{PF}) = \sum_{c \in \lambda(\Pi)} \chi \left( \frac{\text{arm}(c) + 1}{\text{leg}(c) + 1} \right) \leq \frac{m}{n} < \frac{\text{arm}(c)}{\text{leg}(c)}.\]

The partition corresponding to the Dyck path $\Pi$ of $\text{PF}$ has at most 2 parts, so the leg($c$) of a cell $c \in \lambda(\Pi)$ is either 0 or 1. Taking Figure 10 for reference, we have:

(a) $c \in \lambda(\Pi)$ with leg($c$) = 0 and $1 \leq \text{arm}(c) < k$ contributes 1 to $dinv$ correction, marked ○ in Figure 10.

(b) $c \in \lambda(\Pi)$ with leg($c$) = 1 and $k < \text{arm}(c) \leq 2k - 1$ contributes 1 to $dinv$ correction, marked △ in Figure 10.

Figure 10: The $dinv$ correction of a $(3k + 1) \times 3$ Dyck path when $k = 4$.

Further, we can directly count the statistics area and $dinv$ correction ($dinvcorr$) from the partition $\lambda(\Pi)$ of path $\Pi$. The path $\Pi$ corresponds with a partition $\lambda = (\lambda_1, \lambda_2) \subseteq \lambda_0 = (2k, k)$, i.e. $\lambda_1 \leq 2k$ and $\lambda_2 \leq k$. Clearly, the area is $|\lambda_0| - |\lambda|$ that

\[
\text{area}(\Pi) = 3k - \lambda_1 - \lambda_2,
\]
We can also write the formula for \( \text{dinv correction} \) according to the partition \( \lambda \):

\[
\text{dinvcorr}(\Pi) = \begin{cases} 
\lambda_1 - 1 & \text{if } \lambda_2 = 0 \text{ and } \lambda_1 \leq k, \\
\lambda_1 - 1 & \text{if } \lambda_2 = \lambda_1 \geq 1, \\
\lambda_1 - 2 & \text{if } \lambda_2 \geq 1, 1 \leq \lambda_1 - \lambda_2 \leq k, \text{ and } \lambda_1 \leq k \\
2\lambda_1 - k - 3 & \text{if } \lambda_2 \geq 1, 1 \leq \lambda_1 - \lambda_2 \leq k, \text{ and } \lambda_1 \geq k + 1 \\
2\lambda_2 + k - 2 & \text{if } \lambda_2 \geq 1 \text{ and } \lambda_1 - \lambda_2 \geq k + 1.
\end{cases}
\]  

(64)

Note that the return statistic is always 1 since \( 3k + 1 \) and 3 are coprime. We shall compute \([s_3]_{3k+1,3}\) first.

From Table 2 we see that only the parking functions in \( \mathcal{PF}_{3k+1,3} \) with word 123 contribute to the coefficient of \( s_3 \). We also notice that the 3 cars should be in different columns, otherwise there are cars \( i < j \) with \( \text{rank}(i) < \text{rank}(j) \), contradicting that the word of the parking function is 123. Thus we have one \( \Pi \in \mathcal{PF}_{3k+1,3} \) with word 123 on each \((3k + 1,3)\) Dyck path which has no two consecutive north steps.

Let \( \lambda(\Pi) = \{\lambda_1, \lambda_2\} \) be the partition associated with the Dyck path \( \Pi(\Pi) \), then \( \text{area}(\Pi) \) is counted by equation (63). Since the ranks of cars 1, 2, 3 are decreasing, there is always no \( tdinv \), thus \( \text{dinv}(\Pi) = \text{dinvcorr}(\Pi) \), which is counted by the later 3 cases of equation (64).

![Figure 11: Example: a PF ∈ \( \mathcal{PF}_{7,3} \) with word 123.](image)

For \([s_3]_{3k+1,3} = \sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+1]_{q,t} \), we construct each term \((qt)^{k-1-i}[3i+1]_{q,t} \) as a sequence of parking functions. Since each parking function corresponds to a unique partition \( \lambda \subset (2k,k) \) with 2 distinct parts, we shall use partitions to represent parking functions with diagonal word 123. For each \( i \), we have 3 branches of partitions (parking functions) to obtain \((qt)^{k-1-i}[3i+1]_{q,t} \):

\[
\Lambda_1 = \{(k+i+1,k),(k+i,k-1),\ldots,(k+2,k-i+1)\},
\Lambda_2 = \{(2k,i),(2k-1,i-1),\ldots,(2k+1-i,1)\},
\Lambda_3 = \{(k+1,k-i),(k,i+1),\ldots,(k+i+1,k-i)\}.
\]

The branch \( \Lambda_1 \) contains \( \lambda \)’s such that \( \lambda_1 - \lambda_2 = i + 1 \leq k \) with \( \lambda_2 > k - i \), the branch \( \Lambda_2 \) contains all \( \lambda \)’s such that \( \lambda_1 - \lambda_2 = 2k - i > k \), and the branch \( \Lambda_3 \) contains \( \lambda \)’s such that \( \lambda_2 = i + 1 \) and \( \lambda_1 - \lambda_2 \leq k - i \). Notice that \(|\Lambda_1| = |\Lambda_2|\). As shown in Figure 12, the construction begins with \( \text{alternatively} \) taking partitions from \( \Lambda_1 \) and \( \Lambda_2 \), ending with the last partition of \( \Lambda_2 \). Then continue the chain by taking partitions in \( \Lambda_3 \) and end the chain with the last partition \((k-i+1,k-i)\) in \( \Lambda_3 \). The weights of the parking functions are \((qt)^{k-1-i}q^{3i},(qt)^{k-1-i}q^{3i-1}t,\ldots,(qt)^{k-1-i}q^{3i}t\) following the order of the chain.
To be more precise, it is easy to check that each parking function with diagonal word 123 is contained in \( \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \) for some \( i \), and the parking function weights are

\[
\sum_{\text{PF}\in \Lambda_1} t^{\text{area}(\text{PF})} q^{\text{dinv}(\text{PF})} = (qt)^{k-i-1} q^{i+1} t[i] q^2 t^2, \tag{65}
\]

\[
\sum_{\text{PF}\in \Lambda_2} t^{\text{area}(\text{PF})} q^{\text{dinv}(\text{PF})} = (qt)^{k-i-1} q^{i+1} t[i] q^2 t^2, \tag{66}
\]

\[
\sum_{\text{PF}\in \Lambda_3} t^{\text{area}(\text{PF})} q^{\text{dinv}(\text{PF})} = (qt)^{k-i-1} t^{2i}[i+1] q^t, \tag{67}
\]

which sum up to \((qt)^{k-1-i}[3i+1]_{q,t}\). This proves that \([s_3]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i+1]_{q,t}\). Figure 13 shows the combinatorics of the coefficient \([s_3]_{10,3}\).

![Figure 12: The construction of \((qt)^{k-1-i}[3i+1]_{q,t}\).](image)

![Figure 13: The construction of \([s_3]_{10,3} = [7]_{q,t} + (qt)[4]_{q,t} + (qt)^2[1]_{q,t}\).](image)

We can combinatorially prove \([s_{21}]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i+2]_{q,t} + [3i+3]_{q,t})\) in a similar way. In this case, we have 2 possible diagonal words: 132 and 312. In both case, the car 2 has the smallest rank, which means the label of the first (lowest) row must be 2. Thus, the pair of cars (1, 2) does not cause a tdinv. Now we still let \(\lambda = (\lambda_1, \lambda_2)\) be the partition correspond with path \(\Pi\), and let the labels of row 1, row 2, row 3 (counting from bottom to top) be \(\ell_1, \ell_2, \ell_3\), then we
have the following formula for temporary dinv:

\[
\text{tdinv}(PF) = \begin{cases} 
\chi(\ell_3 > \ell_2) & \text{if } \lambda_2 = 0 \text{ and } \lambda_1 \leq k, \\
\chi(\ell_3 > \ell_1) + \chi(\ell_2 > \ell_3) & \text{if } \lambda_2 = 0 \text{ and } \lambda_1 > k, \\
\chi(\ell_2 > \ell_1) + \chi(\ell_3 > \ell_2) & \text{if } \lambda_2 = \lambda_1 \geq 1, \\
\chi(\ell_2 > \ell_1) + \chi(\ell_3 > \ell_1) + \chi(\ell_3 > \ell_2) & \text{if } \lambda_2 \geq 1, 1 \leq \lambda_1 - \lambda_2 \leq k, \text{ and } \lambda_1 \leq k \\
\chi(\ell_2 > \ell_1) + \chi(\ell_3 > \ell_1) + \chi(\ell_2 > \ell_3) & \text{if } \lambda_2 \geq 1, 1 \leq \lambda_1 - \lambda_2 \leq k, \text{ and } \lambda_1 \geq k + 1 \\
\end{cases}
\]

(68)

For \([s_{21}]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i+2]_{q,t} + [3i+3]_{q,t})\), we construct each term \((qt)^{k-1-i}[3i+2]_{q,t}\) or \((qt)^{k-1-i}[3i+3]_{q,t}\) as a sequence of parking functions. First, we have 3 branches of parking functions to obtain \((qt)^{k-1-i}[3i+3]_{q,t}\) for each \(i\):

\[
\begin{align*}
\Lambda_1 &= \{PF : \lambda(PF) \in \{(2k, i + 1), (2k - 1, i), \ldots, (2k - i, 1)\}, (\ell_1, \ell_2, \ell_3) = (2,1,3)\}, \\
\Lambda_2 &= \{PF : \lambda(PF) \in \{(2k, i), (2k - 1, i - 1), \ldots, (2k - i, 0)\}, (\ell_1, \ell_2, \ell_3) = (2,3,1)\}, \\
\Lambda_3 &= \{PF : \lambda(PF) \in \{(k, k - i - 1), \ldots, (k - i, k - i - 1)\}, (\ell_1, \ell_2, \ell_3) = (2,3,1)\}.
\end{align*}
\]

With the 3 branches defined, the construction is similar to the construction of \((qt)^{k-1-i}[3i+1]_{q,t}\) as a term of \([s_{31}]_{3k+1,3}\). We alternatively take parking functions from \(\Lambda_1\) and \(\Lambda_2\), ending with the last partition of \(\Lambda_2\). Then continue the chain by taking partitions in \(\Lambda_3\) and end the chain with the last parking function corresponding to partition \((k - i, k - i - 1)\) with labels \((\ell_1, \ell_2, \ell_3) = (2,3,1)\) in \(\Lambda_3\). The weights of the parking functions are \((qt)^{k-1-i}q^{3i+2}, \ldots, (qt)^{k-1-i}q^{3i+2}\).

Second, we have another three branches of parking functions to obtain \((qt)^{k-1-i}[3i+2]_{q,t}\) for each \(i\):

\[
\begin{align*}
\Lambda_4 &= \{PF : \lambda(PF) \in \{(k + i + 1, k), (k + i, k - 1), \ldots, (k + 1, k - i)\}, (\ell_1, \ell_2, \ell_3) = (2,3,1)\}, \\
\Lambda_5 &= \{PF : \lambda(PF) \in \{(k + i, k), (k + i - 1, k - 1), \ldots, (k, k - i)\}, (\ell_1, \ell_2, \ell_3) = (2,1,3)\}, \\
\Lambda_6 &= \{PF : \lambda(PF) \in \{(k - 1, k - i), \ldots, (k - i, k - i)\}, (\ell_1, \ell_2, \ell_3) = (2,1,3)\}.
\end{align*}
\]

The construction is the same as that of \((qt)^{k-1-i}[3i+3]_{q,t}\), and the weights of the parking functions are \((qt)^{k-1-i}q^{3i+1}, \ldots, (qt)^{k-1-i}q^{3i+1}\).

Thus we have \([s_{21}]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i+2]_{q,t} + [3i+3]_{q,t})\). Figure 14 shows the combinatorics of the coefficient \([s_{21}]_{7,3}\).

The equality that \([s_{13}]_{3k+1,3} = \sum_{i=0}^{k} (qt)^{k-i}[3i+1]_{q,t} = [s_{3}]_{3k+4,3}\) follows immediately from the following corollary of Theorem 4(a).

**Corollary 12.** For any \(m > 0\), \([s_{13}]_{m,3} = [s_{3}]_{m+3,3}\).
2.1 \[ q,t \]

Figure 14: The construction of \([s_{21}]_{7,3} = [6]_{q,t} + [5]_{q,t} + (qt)[3]_{q,t} + [2]_{q,t} \).}

4.2.2 Combinatorics of \( H_{3k+2,3}[X; q, t] \)

We show the combinatorics of \( H_{3k+2,3}[X; q, t] \) in a similar manner by enumerating the parking functions on \((3k + 2) \times 3\) lattice to prove that

\[
[s_3]_{3k+2,3} = \sum_{i=0}^{k-1} (qt)^{k-i}[3i + 2]_{q,t}, \tag{69}
\]

\[
[s_{21}]_{3k+2,3} = \sum_{i=-1}^{k-1} (qt)^{k-i}([3i + 3]_{q,t} + [3i + 4]_{q,t}), \quad \text{and} \tag{70}
\]

\[
[s_1^3]_{3k+2,3} = \sum_{i=0}^{k} (qt)^{k-i}[3i + 2]_{q,t}. \tag{71}
\]

Given a parking function \( \text{PF} \in \mathcal{PF}_{3k+2,3} \), let \( \Pi = \Pi(\text{PF}) \), we can tell the \( \text{dinv} \) correction contribution of a cell \( c \in \lambda(\Pi) \). Taking Figure 15 for reference,

(a) \( c \in \lambda(\Pi) \) with \( \text{leg}(c) = 0 \) and \( 1 \leq \text{arm}(c) < k \) contributes 1 to \( \text{dinv} \) correction, marked \( \bigcirc \) in Figure 15

(b) \( c \in \lambda(\Pi) \) with \( \text{leg}(c) = 1 \) and \( k < \text{arm}(c) \leq 2k \) contributes 1 to \( \text{dinv} \) correction, marked \( \triangle \) in Figure 15

Figure 15: The \( \text{dinv} \) correction of a \((3k + 2) \times 3\) Dyck path for \( k = 4 \).

Further, we can directly count the statistics area and \( \text{dinv} \) correction (\( \text{dinvcorr} \)) from the partition \( \lambda(\Pi) \) of a \( 3k + 2 \times 3 \) path \( \Pi \). Similar to equation (63), we have

\[
\text{area}(\Pi) = 3k + 1 - \lambda_1 - \lambda_2, \tag{72}
\]
The dinv correction formula is the same as equation (64), and the return statistic is still always equal to 1.

To prove \( [s_3]_{3k+2,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i+2]_{q,t} \), we shall construct 3 branches of partitions (parking functions) for each term \((qt)^{k-1-i}[3i+2]_{q,t}\), which are

\[
\begin{align*}
\Lambda_1 &= \{ (2k+1, i+1), (2k, i), \ldots, (2k+1-i, 1) \}, \\
\Lambda_2 &= \{ (k+i+1, k), (k+i, k-1), \ldots, (k+1, k-i) \}, \\
\Lambda_3 &= \{ (k, k-i), (k-1, i+1), \ldots, (k-i+1, k-i) \}.
\end{align*}
\]

Then we can follow the same construction to get all parking functions with word 123 and weights \((qt)^{k-1-i}q^{3i+1}, \ldots, (qt)^{k-1-i}q^{3i+1}\).

To prove \([s_{21}]_{3k+2,3} = \sum_{i=-1}^{k-1} (qt)^{k-1-i}([3i+3]_{q,t} + [3i+4]_{q,t})\), we have similar 6 branches of parking functions as follows:

\[
\begin{align*}
\Lambda_1 &= \{ (PF : \lambda(PF) \in \{ (k+i+2, k), \ldots, (k+2, k-i) \}, (\ell_1, \ell_2, \ell_3) = (2, 3, 1) \}, \\
\Lambda_2 &= \{ (PF : \lambda(PF) \in \{ (k+i+1, k), \ldots, (k+1, k-i) \}, (\ell_1, \ell_2, \ell_3) = (2, 1, 3) \}, \\
\Lambda_3 &= \{ (PF : \lambda(PF) \in \{ (k+1, k-i-1), \ldots, (k-i, k-i-1) \}, (\ell_1, \ell_2, \ell_3) = (2, 3, 1) \}, \\
\Lambda_4 &= \{ (PF : \lambda(PF) \in \{ (2k+1, i+1), \ldots, (2k-i+1, 1) \}, (\ell_1, \ell_2, \ell_3) = (2, 1, 3) \}, \\
\Lambda_5 &= \{ (PF : \lambda(PF) \in \{ (2k+1, i), \ldots, (2k-i+1, 0) \}, (\ell_1, \ell_2, \ell_3) = (2, 3, 1) \}, \\
\Lambda_6 &= \{ (PF : \lambda(PF) \in \{ (k-k-i), \ldots, (k-i, k-i) \}, (\ell_1, \ell_2, \ell_3) = (2, 1, 3) \}.
\end{align*}
\]

Then, the total weight of parking functions in the first 3 branches is \((qt)^{k-1-i}[3i+4]_{q,t}\), and the total weight of parking functions in the last 3 branches is \((qt)^{k-1-i}[3i+3]_{q,t}\).

The proof of \([s_{11}]_{3k+2,3} = [s_3]_{3(k+1)+2,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}[3i+2]_{q,t}\) follows Corollary 12.

### 4.2.3 Combinatorics of \(H_{3k,3}[X; q, t]\)

Notice that the area and dinv of parking functions in \(PF_{3k,3}\) are equal to that of the parking functions in \(PF_{3k+1,3}\). Given a \(PF \in PF_{3k+1,3}\), let \(\lambda = \lambda(PF)\), then the corresponding return statistic of PF is formulated as

\[\text{ret}(PF) = \chi(\lambda_1 = 2k) + \chi(\lambda_2 = k).\]

By Theorem 3 of the Rational Shuffle Conjecture in non-coprime case,

\[
\begin{align*}
H_{3k,3}[X; q, t] &= \sum_{PF \in PF_{3k,3}} [\text{ret}(PF)]_t q^{\text{area}(PF)} t^{\text{dinv}(PF)} F_{\text{des}(PF)}[X] \\
&= \sum_{PF \in PF_{3k+1,3}} [\text{ret}(PF)]_t q^{\text{area}(PF)} t^{\text{dinv}(PF)} s_{\text{pides}(PF)}.
\end{align*}
\]

(73)
To prove that

\[
[s_3]_{3k,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i-1]_{q,t} + [3i]_{q,t} + [3i+1]_{q,t}), \quad (74)
\]

\[
[s_{21}]_{3k,3} = (qt)^{k+1}([3]_{q,t} + 2[2]_{q,t} + [1]_{q,t})
+ \sum_{i=1}^{k-1} (qt)^{k-1-i}([3i]_{q,t} + 2[3i+1]_{q,t} + 2[3i+2]_{q,t} + [3i+3]_{q,t}), \quad (75)
\]

\[
[s_{1^3}]_{3k,3} = \sum_{i=0}^{k} (qt)^{k-i}([3i-1]_{q,t} + [3i]_{q,t} + [3i+1]_{q,t}), \quad (76)
\]

we use the constructions of $[s_3]_{3k+1,3}$, $[s_{21}]_{3k+1,3}$, $[s_{1^3}]_{3k+1,3}$ in Section 4.2.1 and modify the weight of parking functions with nonzero returns.

For $[s_3]_{3k,3}$, the first parking function in each set $\Lambda_1$ and $\Lambda_2$ has return statistic 1, except that the first parking function in $\Lambda_1$ when $i = k - 1$ has return 2. All the remaining parking functions have return 3. Then we prove equation (74) by summing up the parking function weights.

For $[s_{21}]_{3k,3}$, the first parking function in each set $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5$ has return statistic 1, except that the first parking function in $\Lambda_1$ and $\Lambda_4$ when $i = k - 1$ has return 2. All the remaining parking functions have return 3. Then again we obtain equation (75) by direct computation.

The proof of $[s_{1^3}]_{3k,3} = [s_3]_{3(k+1),3}$ follows Corollary 12.

5 Combinatorial Results about Schur Basis Expansion of $(3, n)$ Case

5.1 Recursive Formula for $[s_\lambda]_{3,n}$

In $(3, n)$ case, we have $n$ cars, i.e. the word of a $(3, n)$ parking function is a permutation of $[n]$. By Remark 2, $[s_\lambda]_{3,n} \neq 0$ implies that $\lambda$ must be of the form $3^a2^b1^c$ with $3a + 2b + c = n$, i.e. $[s_\lambda]_{3,n} \neq 0$ only if the partition $\lambda$ only has parts of size less than or equal to 3.

We have the following corollary of Theorem 3 summarizing some symmetries about $[s_\lambda]_{3,n}$.

Corollary 13. For all $m, n > 0$ and $a, b, c \geq 0$,

(a) $[s_{1^3}]_{m-3,3} = [s_3]_{m,3}$,

(b) $[s_{3^a2^b1^c}]_{3,n} = [s_{2^b1^c}]_{3,n-3a}$,

(c) $[s_{1^3}]_{n,3} = [s_{1^n}]_{3,n}$,

(d) $[s_{21}]_{n,3} = [s_{21^{n-2}}]_{3,n}$,

Further, we conjecture another important symmetry.

Conjecture 1. For all $a, b, n > 0$,

$[s_{2^a1^b}]_{3,n} = [s_{2^a1^b}]_{3,3(2a+b)-n}$. 

24
We have found the straightening action in parking functions combinatorially from parking functions with pides \(\{\cdots, 1, 3, \cdots\}\) to parking functions with pides \(\{\cdots, 2, 2, \cdots\}\), which is an involution whose fixed points are the coefficients of \(s_{2a1b}\|3,n\). Further, we have conjectured a bijection between the fixed parking functions with pides \(2^a1^b\) and the fixed parking functions with pides \(2^b1^a\), mapping the 2 cars (or 1 car) causing part 2 (or 1) in pides \(2^a1^b\) to 1 car (or 2 cars) causing part 1 (or 2) in pides \(2^b1^a\). We will state the details later.

We can see that the results above show that the problem of computing the Schur function expansion of \(Q_{3,n}(-1)^n\) can be reduced to the problem finding the coefficients of Schur functions of the form \(s_{2a1b}\) where \(a < b\) in \(Q_{3,n}(-1)^n\). Finally, we conjecture a recursive formula for such coefficients \(s_{2a1b}\|3,n\) where \(a < b\).

**Conjecture 2.** Let \(a < b\), then

\[
[s_{2a1b}\|3,n] = \sum_{i=0}^{n} [b+i]q,t + (qt)[s_{2a1b-3}\|3,n-3].
\]

We verified this formula by Maple for \(n < 27\). If these conjectures are true, then we have solved the Schur function expansion in the \((3, n)\) case.

### 5.2 The Symmetry \([s_{2a1b}\|3,n] = [s_{2b1a}\|3,3(a+b)−n]\)

For this symmetry, we shall first introduce an involution of \(3 \times n\) parking functions whose pides contain 1, 3 or 2, 2. The fixed points of the involution is a subset of parking functions whose pides do not contain 1, 3. Then, we give a bijection between the fixed parking functions with pides \(2^a1^b\) and the fixed parking functions with pides \(2^b1^a\).

#### 5.2.1 The Involution

Note that by the straightening action on Schur functions, we have

\[
s_{\lambda 13 \mu} = -s_{\lambda 22 \mu}. \tag{77}
\]

We give the involution of parking functions in \(\mathcal{PF}_{3,n}\) with pides \(\lambda 13 \mu\) and parking functions in \(\mathcal{PF}_{3,n}\) with pides \(\lambda 22 \mu\).

Without loss of generality, we suppose that the cars that are causing pides 13 are 1, 2, 3, 4, i.e. \(\text{rank}(1) < \text{rank}(2) > \text{rank}(3) > \text{rank}(4)\). Then, there are 3 possible words for the 4 cars (subsequences of the words of the parking functions), which are

\[2341, \ 2314, \ 2134.\]

On the other hand, the cars 2, 3, 4 are in different columns since \(\text{rank}(2) > \text{rank}(3) > \text{rank}(4)\). Since there are only 3 columns, we have three possible placement of the four cars:

(I) Cars 1 and 4 are in the same column.

(II) Cars 1 and 2 are in the same column.

(III) Cars 1 and 3 are in the same column.

If the four cars form word 2341, then (I), (II), (III) are all possible; if the four cars form word 2314, then only (II), (III) are possible; if the four cars form word 2134, then only (II) is possible.
Next, we consider the case when the cars 1, 2, 3, 4 cause pides 22, i.e. \( \text{rank}(1) > \text{rank}(2) < \text{rank}(3) > \text{rank}(4) \). The possible words are

\[
3412, 3142, 3124, 1324, 1342.
\]

On the other hand, the cars 1, 2 and the cars 3, 4 should be in different columns since \( \text{rank}(1) > \text{rank}(2) \) and \( \text{rank}(3) > \text{rank}(4) \). We have 5 possible placement of the four cars:

(i) Both cars 1 and 2 are in the same column.

(ii) Only cars 1 and 4 are in the same column.

(iii) Only cars 2 and 4 are in the same column.

(iv) Only cars 1 and 3 are in the same column.

(v) Only cars 2 and 3 are in the same column.

If the four cars form word 3412, then (i), (ii), (iii), (iv) and (v) are all possible; if the four cars form word 3142, then only (i), (iii), (iv) and (v) are possible; if the four cars form word 3124, then only (iv) and (v) are possible; if the four cars form word 1324, then only (v) is possible; if the four cars form word 1342, then only (iii) and (v) are possible.

For any permutation \( \sigma \in S_n \) and any \( \text{PF} \in \mathcal{PF}_{3,n} \), we let \( \sigma \cdot \text{PF} \) be the parking function obtained by using \( \sigma \) to permute the cars of \( \text{PF} \). We also let \( \text{word}(\text{PF}) \) be the word of cars 1, 2, 3, 4. Then we can define the following injection \( \Phi \) from parking functions in \( \mathcal{PF}_{3,n} \) with pides \( \lambda_13\mu \) to parking functions in \( \mathcal{PF}_{3,n} \) with pides \( \lambda_22\mu \) (the words and the placements of the images are recorded in each case):

\[
\Phi(\text{PF}) = \begin{cases} 
(1,2)\text{PF} & \text{if } \text{word}(\text{PF}) = 2341 \text{ and placement is (I). } \Phi(\text{PF}) \text{ has word } 1342 \text{ (iii).} \\
(1,2,3)\text{PF} & \text{if } \text{word}(\text{PF}) = 2341 \text{ and placement is (II). } \Phi(\text{PF}) \text{ has word } 3142 \text{ (v).} \\
(1,2)\text{PF} & \text{if } \text{word}(\text{PF}) = 2314 \text{ and placement is (III). } \Phi(\text{PF}) \text{ has word } 1342 \text{ (v).} \\
(1,2,3)\text{PF} & \text{if } \text{word}(\text{PF}) = 2314 \text{ and placement is (II). } \Phi(\text{PF}) \text{ has word } 3124 \text{ (v).} \\
(1,2)\text{PF} & \text{if } \text{word}(\text{PF}) = 2134 \text{ and placement is (II). } \Phi(\text{PF}) \text{ has word } 3124 \text{ (v).} \\
(2,3)\text{PF} & \text{if } \text{word}(\text{PF}) = 2134 \text{ and placement is (II). } \Phi(\text{PF}) \text{ has word } 3124 \text{ (iv).}
\end{cases}
\]

Then we define map \( \Psi \) from parking functions in \( \mathcal{PF}_{3,n} \) with pides \( \lambda_22\mu \) to either parking functions in \( \mathcal{PF}_{3,n} \) with pides \( \lambda_13\mu \) or parking functions in \( \mathcal{PF}_{3,n} \) with pides \( \lambda_22\mu \):

\[
\Psi(\text{PF}) = \begin{cases} 
\text{PF} & \text{if } \text{word}(\text{PF}) = 3412, \text{ or } \text{word}(\text{PF}) = 3142 \text{ and placement is (i), (iii), (iv). } \Phi(\text{PF}) \text{ has word } 2341 \text{ (II).} \\
(1,3,2)\text{PF} & \text{if } \text{word}(\text{PF}) = 3142 \text{ and placement is (v). } \Phi(\text{PF}) \text{ has word } 2341 \text{ (II).} \\
(2,3)\text{PF} & \text{if } \text{word}(\text{PF}) = 3124 \text{ and placement is (iv). } \Phi(\text{PF}) \text{ has word } 2314 \text{ (II).} \\
(1,3,2)\text{PF} & \text{if } \text{word}(\text{PF}) = 3124 \text{ and placement is (v). } \Phi(\text{PF}) \text{ has word } 2314 \text{ (II).} \\
(1,2)\text{PF} & \text{if } \text{word}(\text{PF}) = 1324 \text{ and placement is (v). } \Phi(\text{PF}) \text{ has word } 2314 \text{ (III).} \\
(1,2)\text{PF} & \text{if } \text{word}(\text{PF}) = 1342 \text{ and placement is (iii). } \Phi(\text{PF}) \text{ has word } 2341 \text{ (I).} \\
(1,2)\text{PF} & \text{if } \text{word}(\text{PF}) = 1342 \text{ and placement is (v). } \Phi(\text{PF}) \text{ has word } 2341 \text{ (III).}
\end{cases}
\]

It is easy to check that the maps \( \Phi \) and \( \Psi \) do not change area and dinv statistic since the maps do not change the Dyck path of the parking functions, and they also do not affect cars other than
the 4 consecutive cars. Then, the maps \( \Phi \) and \( \Psi \) form a sign-reversing involution for the parking functions in \( \mathcal{P} \mathcal{F}_{3,n} \) with either pides \( \lambda 13 \mu \) or pides \( \lambda 22 \mu \). The set of fixed points of this involution is

\[
\{PF \in \mathcal{P} \mathcal{F}_{n,3} : \text{pides}(PF) = \lambda 22 \mu, \text{word}(PF) = 3412, \text{or word}(PF) = 3142 \text{ (i), (iii), (iv)}\}.
\]

If we apply the involution to all the parking functions in \( \mathcal{P} \mathcal{F}_{3,n} \), at the first non-fixed position, then the remaining fixed points are the parking functions that contribute to the coefficients of the Schur function basis. Thus,

\[
[s_{2^{a+1}b}]_{3,n} = \sum_{PF \in \mathcal{P} \mathcal{F}_{3,n}, \text{pides}(PF) = 2^a 1^b, \text{PF fixed by } \Psi} \sum_{i} \ell^{\text{area}(PF)} q^{\text{dinv}(PF)}.
\]  \hspace{1cm} (80)

### 5.2.2 The Bijection of \([s_{2^{a+1}b}]_{3,n} = [s_{2^{a+1}a}]_{3,3(a+b)−n}\)

For any parking function \( PF \in \mathcal{P} \mathcal{F}_{3,n} \) with pides(PF) = \( 2^a 1^b \), we are interested in the placement of the \( a \) pairs of numbers, \((1, 2), \ldots, (2a−1, 2a), \) and the \( b \) singletons \( 2a+1, \ldots, 2a+b \). Note that any two elements in the same pair cannot be placed in the same column since the rank of the smaller car is bigger than the rank of the bigger car.

Since there are 3 columns, we have \( \binom{3}{2} \) ways to choose columns for each pair \((2i−1, 2i)\). We call the 3 columns from left to right \( \ell, c, r \). Once we determine the 2 columns of the pair, the filling is fixed since rank\((2i−1) > \text{rank}(2i)\). Now, we define the notation for the placement of a pair \((2i−1, 2i)\):

1. \( L \) means \((2i−1, 2i)\) are in the left 2 columns \( \ell, c \),
2. \( R \) means \((2i−1, 2i)\) are in the right 2 columns \( c, r \),
3. \( C \) means \((2i−1, 2i)\) are in columns \( \ell, r \).

Similarly, we have \( \binom{3}{1} \) ways to choose a column for each singleton. For a singleton \( j \), we define the notation for the placement that,

1. \( L \) means \( j \) is in the left column \( \ell \),
2. \( R \) means \( j \) is in the right column \( r \),
3. \( C \) means \( j \) is in column \( c \).

Now we state our conjectured bijection between fixed parking functions in \( \mathcal{P} \mathcal{F}_{3,n} \) with pides(PF) = \( 2^a 1^b \) and fixed parking functions in \( \mathcal{P} \mathcal{F}_{3,n'} \) with pides(PF) = \( 2^b 1^a \).

Given a PF \( \in \mathcal{P} \mathcal{F}_{3,n} \) with pides(PF) = \( 2^a 1^b \) fixed by the involution, we track the placements of the \( a \) pairs of numbers, \((1, 2), \ldots, (2a−1, 2a), \) and the \( b \) singletons \( 2a+1, \ldots, 2a+b \). Let the \( a+b \) placements be \( p_1, \ldots, p_a, p_{a+1}, \ldots, p_{a+b} \) (here \( p_i \) is one of \( L, R \) or \( C \)).

Then we consider \( b \) pairs of numbers, \((1, 2), \ldots, (2b−1, 2b), \) and the \( a \) singletons \( 2b+1, \ldots, 2b+a \). We assign the \( b+a \) placements \( p_{a+b}, \ldots, p_1 \), then build the new parking function by first counting how many cars in each column, then making the path fit the number of cars. At last, fill from first pair \((1, 2)\) to last singleton \( 2b+a \) based on the column choice and the rule that rank\((2i−1) < \text{rank}(2i)\) for each \( i \leq b \). We call this map the switch map \( S \). Figure 11 shows an example that we can construct a parking function in \( \mathcal{P} \mathcal{F}_{3,5} \) with pides \( 21^3 \) from a parking function in \( \mathcal{P} \mathcal{F}_{3,7} \) with pides \( 2^3 1 \).
In order to show that the map $S$ is a bijection, we need to prove several properties of this map. It is not obvious that the image of a parking function is still above the diagonal, thus we shall show that

**Theorem 14.** If $PF$ is a $3 \times n$ parking function with pides $2^a 1^b$, then $S(PF)$ is also a parking function.

**Proof.** Suppose that there are $\ell_1, c_1, r_1$ placements of the first $a$ pairs of cars which are $L$, $R$ and $C$, and $\ell_2, c_2, r_2$ placements of the last $b$ singleton cars which are $L$, $R$ and $C$. Without loss of generality, we suppose that $n = 3k + 1$. Then we have that

\[
\ell_1 + c_1 + r_1 = a, \tag{81}
\]
\[
\ell_2 + c_2 + r_2 = b, \tag{82}
\]
\[
2a + b = 3k + 1. \tag{83}
\]

Since $PF$ is a parking function, the path of the parking function should be above the diagonal, thus the number of cars in the left column is at least $k + 1$ and the number of cars in the left two columns is at least $2k + 1$.

Note that a $L$ placement of a pair contribute 1 left car and 1 center car, a $C$ placement of a pair contribute 1 left car and 1 right car, and a $R$ placement of a pair contribute 1 right car and 1 center car. The contribution of the singleton cars are obvious. Thus the number of cars in the left column is $\ell_1 + c_1 + \ell_2$, and the number of cars in the left 2 columns is $2\ell_1 + r_1 + c_1 + \ell_2 + c_2 = a + \ell_1 + \ell_2 + c_2$, and we have that

\[
\ell_1 + \ell_2 \geq k + 1, \tag{84}
\]
\[
a + \ell_1 + \ell_2 + c_2 \geq 2k + 1. \tag{85}
\]

Next, for $S(PF)$, it has $\ell_2, c_2, r_2$ placements of the first $b$ pairs of cars, and $\ell_1, c_1, r_1$ placements of the last $a$ singleton cars. The total number of cars is equal to $2a + b = 3(a + b) - (2b + a) = 3(a + b) - 3k - 1 = 3(a + b - k - 1) + 2$, and the number of cars in the left column should be at least $a + b - k = (2a + b) - a - k = 3k + 1 - a - j = 2k + 1 - a$ and the number of cars in the left two columns should be at least $2a + 2b - 2k = b + (3k + 1) - 2k = b + k + 1$. $S(PF)$ is a parking function if the following is true.

\[
\ell_1 + c_2 + \ell_2 \geq 2k + 1 - a, \tag{86}
\]
\[
b + \ell_1 + \ell_2 + c_1 \geq b + k + 1. \tag{87}
\]

Clearly, (84) implies (87), (85) implies (86).

\[
\square
\]
Next, we have the formula for area.

**Theorem 15.** Let PF be a $3 \times n$ parking function with pides $2^a1^b$. Using the definition of $\ell_1,c_1,r_1,\ell_2,c_2,r_2$ in the proof of Theorem 14. Let $L = \ell_1 + \ell_2, R = r_1 + r_2, C = c_1 + c_2$, then

$$\text{area}(PF) = L - R - 1.$$  \hspace{1cm} (88)

**Proof.** We want to compute the area of a parking function as the difference of its coarea and the maximum coarea of a $3 \times (2a + b)$ parking function. The maximum coarea of a $3 \times (2a + b)$ parking function is equal to $(2a + b - 1)(3 - 1) = 2a + b - 1$.

Notice that the cars in the right column contribute 2 to coarea, and the cars in the center column contribute 1 to coarea, thus coarea is

$$\ell_1 + 3r_1 + 2c_1 + 2r_2 + c_2 = a + 2(r_1 + r_2) + (c_1 + c_2) = a + 2R + C.$$ \hspace{1cm} (89)

Then,

$$\text{area}(PF) = 2a + b - 1 - (a + 2R + C) = a + (L + R + C) - 1 - (a + 2R + C) = L - R - 1. \quad (90)$$

It follows immediately from Theorem 15 that

**Theorem 16.** For any $PF \in \mathcal{PF}_{3,n}$ with pides(PF) = $2^a1^b$,

$$\text{area}(PF) = \text{area}(S(PF)). \quad (91)$$

We have not yet proved, but verified all parking functions with less than or equal to 7 rows and less than or equal to 10 columns for the following conjecture.

**Conjecture 3.** For any $PF \in \mathcal{PF}_{3,n}$ with pides(PF) = $2^a1^b$,

1. $\text{dinv}(PF) = \text{dinv}(S(PF))$.
2. If $PF$ is a fixed point of the map $\Psi$, then so is $S(PF)$, and pides($S(PF)$) = $2^b1^a$.

By (80), it follows from Conjecture 3 immediately that

$$[s_{2^a1^b}]_{3,n} = [s_{2^b1^a}]_{3,3(a+b)-n}.$$

### 5.2.3 Switch Map in $m$ Columns Case

We haven’t completely understood how to use straightening to compute the coefficients of $s_\lambda$ for general $(m, n)$ case, but computations in Maple have led us to conjecture the following.

**Conjecture 4.** For all $m, n > 0$ and $\alpha_i \geq 0$,

$$[s_{(m-1)^{\alpha_m-1}(m-2)^{\alpha_{m-2}}\cdots1^{\alpha_1}}]_{m,n} = [s_{(m-1)^{\alpha_1}(m-2)^{\alpha_2}\cdots1^{\alpha_m-1}}]_{m,\sum_{i=1}^{m-1} \alpha_i - n}. \quad (92)$$

On the other hand, the switch map $S$ that we defined for the three column case can be naturally generalized to $m$ columns case, which conjecturally has many nice properties and is considered to be useful in proving Conjecture 4. The definition of an $m$ columns switch map will need some new definitions.
Given any parking function \( PF \in \mathcal{PF}_{m,n} \), we suppose that \( s, s+1, \ldots, s+r-1 \) is an increasing subsequence of the word \( \sigma(PF) \), then by Remark \( 1 \) the cars \( s, s+1, \ldots, s+r-1 \) must be placed in \( r \) different columns in a rank decreasing way.

There are \( \binom{m}{r} \) possible choices to pick \( r \) columns for such cars \( s, s+1, \ldots, s+r-1 \). Let \( p = \{s_1, s_2, \ldots, s_r\} \subset \{1, \ldots, m\} \) be a possible placement, then we define the reverse complement of \( p \) is \( p^c = \{1, \ldots, m\} \setminus \{m+1-s_r, \ldots, m+1-s_1\} \), which is a placement for \( m-r \) cars.

Given \( \mu = \mu_1 \cdots \mu_k \models n \). Now suppose that \( \sigma(PF) \), the word of \( PF \), is a shuffle of increasing sequences \((1, \ldots, \mu_1), (\mu_1+1, \ldots, \mu_1+\mu_2), \ldots, (n-\mu_k+1, \ldots, n)\), and the placement of \( \mu \) is \((1, \ldots, \nu_k)\). We make the word of \( S(PF) \) to be a shuffle of \((1, \ldots, \nu_1), (\nu_1+1, \ldots, \nu_1+\nu_2), \ldots, (n-\nu_k+1, \ldots, n)\), and the placement of \((\nu_1+1, \ldots, \nu_1+\nu_2, \ldots, \nu_1+\nu_2)\) is \( p_1 \). This construction is well defined, and for each given composition \( \mu \), we can invert the the map easily. Like the 3 column, we have

**Theorem 17.** \( PF \) is an \((m, n)\)-parking function if and only if \( S(PF) \) is a parking function. Further, \[ \text{area}(PF) = \text{area}(S(PF)). \]

**Theorem 18.** The switch map is a bijection between \((m, n)\)-parking functions whose words are shuffle of \( \mu = \mu_1 \cdots \mu_k \) and \((m, mk - n)\)-parking functions whose words are shuffle of \( \nu = (m-\mu_k) \cdots (m-\mu_1) \).

The switch map of \( m \) columns case still keeps the \( \text{dinv} \) statistic experimentally, which we are not able to prove.

**Conjecture 5.** For any \( PF \in \mathcal{PF}_{m,n} \) where \( \sigma(PF) \) is a shuffle of \( \mu \models n \), \[ \text{dinv}(PF) = \text{dinv}(S(PF)). \]

Thus, conjecturally, the switch map \( S \) is a bijective map between \((m, n)\)-parking functions whose words are shuffle of \( \mu \) and \((m, mk - n)\)-parking functions whose words are shuffle of \( \nu \). In the end, we shall discuss a consequence of Conjecture \( [5] \) and the switch map \( S \).

For any parking function whose word is a shuffle of \( \mu = \mu_1 \cdots \mu_k \models n \), we can replace the cars \( \mu_1 + \cdots + \mu_{i-1} + 1, \ldots, \mu_1 + \cdots + \mu_i \) with number \( i \) to obtain a parking function with cars \( 1^{\mu_1} \cdots k^{\mu_k} \) with the same area and \( \text{dinv} \) statistics. More generally, \[ Q_{m,n}(-1)^n \bigg|_{m,\mu} = H_{m,n}(X; q, t) \bigg|_{m,\mu} = \sum_{PF \in \mathcal{PF}_{m,n}, \sigma(PF) \text{ is a shuffle of } \mu} \text{area}(PF) q^{\text{dinv}(PF)}. \] (93)

We have the following identity about the Hall scalar product that for any symmetric function \( f \), \[ \langle f, h_{\mu} \rangle = f \bigg|_{m,\mu}, \] (94)
then the properties of the switch map \( S \) (i.e. Theorem \( [17]-[18] \) and Conjecture \( [5] \)) implies the following identities.

**Conjecture 6.** For \( m, n > 0 \), \( \mu = \mu_1 \cdots \mu_k \models n \) and \( \mu' = (m-\mu_k) \cdots (m-\mu_1) \),
\[ \langle Q_{m,n}(-1)^n, h_{\mu} \rangle = \langle Q_{m,mk-n}(1)^{mk-n}, h_{\mu'} \rangle, \]
(95)
\[ \langle H_{m,n}(X; q, t), h_{\mu} \rangle = \langle H_{m,mk-n}(X; q, t), h_{\mu'} \rangle. \] (96)
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