Slow Integral Manifolds and Control Problems in Critical and Twice Critical Cases

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Abstract. We consider singularly perturbed differential systems in cases where the standard theory to establish a slow integral manifold existence does not work. The theory has traditionally dealt only with perturbation problems near normally hyperbolic manifold of singularities and this manifold is supposed to isolated. Applying transformations we reduce the original singularly perturbed problem to a regularized one such that the existence of slow integral manifolds can be established by means of the standard theory. We illustrate our approach by several examples.

1. Introduction
Singularly perturbed differential systems of the type
\begin{align}
\frac{dx}{dt} &= f(x, y, t, \varepsilon), \\
\varepsilon \frac{dy}{dt} &= g(x, y, t, \varepsilon)
\end{align}
(1)
play an important role as mathematical models of numerous nonlinear phenomena in biology, chemistry, control theory, and in other fields (see e.g. [1, 2, 3, 4, 6, 7]). A usual approach in the qualitative study of (1) is to consider first the degenerate system
\begin{align}
\frac{dx}{dt} &= f(x, y, t, 0), \\
0 &= g(x, y, t, 0)
\end{align}
(2)
and then to draw conclusions for the qualitative behavior of the full system (1) for sufficiently small \( \varepsilon \). A special case of this approach is the quasi–steady state assumption. A mathematical justification of that method can be given by means of the theory of integral manifolds for singularly perturbed systems (1) (see e.g. [3, 4, 8, 9]).

In order to recall a basic result of the geometric theory of singularly perturbed systems we introduce the following notation and assumptions.
Let \( I_i \) be the interval \( I_i := \{ \varepsilon \in \mathbb{R} : 0 < \varepsilon < \varepsilon_i \} \), where \( 0 < \varepsilon_i \ll 1, i = 0, 1, \ldots \).

\((A_1)\). \( f : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times I_0 \rightarrow \mathbb{R}^m, \ g : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \times I_0 \rightarrow \mathbb{R}^n \) are sufficiently smooth and uniformly bounded together with all their derivatives.
(A2). There are some region \( G \in \mathbb{R}^m \) and a map \( h : G \times \mathbb{R} \to \mathbb{R}^m \) of the same smoothness as \( g \) such that
\[
g(x, h(x, t), t, 0) \equiv 0 \quad \forall (x, t) \in G \times \mathbb{R}.
\]

(A3). The spectrum of the Jacobian matrix \( B(x, t) = g_y(x, h(x, t), t, 0) \) is uniformly separated from the imaginary axis for all \( (x, t) \in G \times \mathbb{R} \), i.e. the eigenvalues \( \lambda_i(x, t)(i = 1, \ldots, n) \) of the matrix \( B(x, t) \) satisfy the inequality
\[
|\text{Re}\lambda_i(x, t)| \geq \gamma
\]
for some positive number \( \gamma \).

Then the following result is valid (see e.g. [8, 9]):

**Proposition 1.1.** Under the assumptions (A1) − (A3) there is a sufficiently small positive \( \varepsilon_1, \varepsilon_1 \leq \varepsilon_0 \), such that for \( \varepsilon \in I_1 \) system (1) has a smooth integral manifold \( \mathcal{M}_\varepsilon \) (slow integral manifold) with the representation
\[
\mathcal{M}_\varepsilon := \{(x, y, t) \in \mathbb{R}^{m+n+1} : y = \psi(x, t, \varepsilon), (x, t) \in G \times \mathbb{R}\}
\]
and with the asymptotic expansion
\[
\psi(x, t, \varepsilon) = h(x, t) + \varepsilon\psi_1(x, t) + \ldots.
\]

The motion on this manifold is described by the slow differential equation
\[
\dot{x} = f(x, \psi(x, t, \varepsilon), t, \varepsilon).
\]

**Remark 1.1.** The global boundedness assumption in (A1) with respect to \( (x, y) \) can be relaxed by modifying \( f \) and \( g \) outside some bounded region of \( \mathbb{R}^n \times \mathbb{R}^m \).

**Remark 1.2.** In applications it is usually assumed that the spectrum of the Jacobian matrix \( g_y(x, y, t, 0) \) is located in the left half plane. Under this additional hypothesis the manifold \( \mathcal{M}_\varepsilon \) is exponentially attracting for \( \varepsilon \in I_1 \).

The case that assumption (A3) is violated is called critical. We distinguish three subcases:

(i) The Jacobian matrix \( g_y(x, y, t, 0) \) is singular on some subspace of \( \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \). In that case, system (1) is referred to as a singular singularly perturbed system [10]. This subcase has been treated in [3, 4, 7, 10, 11].

(ii) The Jacobian matrix \( g_y(x, y, t, 0) \) has eigenvalues on the imaginary axis with nonvanishing imaginary parts. A similar case has been investigated in [3, 4, 12].

(iii) The Jacobian matrix \( g_y(x, y, t, 0) \) is singular on the set \( \mathcal{M}_0 := \{(x, y, t) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} : y = h(x, t), (x, t) \in G \times \mathbb{R}\} \). In that case, \( y = h(x, t) \) is generically an isolated root of \( g = 0 \) but not a simple one.

The critical case (i) is considered in Section 2 as applied to the high-gain control problem, the case (ii) is considered in Section 3 as applied to the manipulator control, the case (iii) is considered in Section 4 as applied to the partially cheap control problem. Two critical cases, (i) and (ii) are combined in the optimal control problem which is analyzed in Section 5, and therefore it is possible to say that this Section is devoted to the consideration of the twice critical case. It is not inconceivable that combinations of other pairs of critical cases and even triple critical case are of interest as well and possibly they will be considered later.
2. Singular Singularly Perturbed Systems

Consider the system
\[
\varepsilon \dot{z} = Z(z, t, \varepsilon), \quad z \in \mathbb{R}^{m+n}, \quad t \in \mathbb{R},
\]
where \(0 \leq \varepsilon \ll 1\), and the vector-function \(Z\) is sufficiently smooth.

Formally, this system may be considered as system (1) in the case where \(x\) is absent and \(z = y\). From the other hand, system (1) can be reduced to the form (5) with \(z = \begin{pmatrix} x \\ y \end{pmatrix}\) by the presentation of the first equation in (1) of form \(\varepsilon \frac{dx}{dt} = \varepsilon f(x, y, t, \varepsilon)\).

Suppose that the limit system \(Z(z, t, 0) = 0\) \((\varepsilon = 0)\) has a family of solutions \(z = \psi(v, t)\), \(v \in \mathbb{R}^m\), \(t \in \mathbb{R}\), (6)

with a sufficiently smooth vector-function \(\psi\). We try to find a slow integral manifold
\[
z = P(v, t, \varepsilon),
\]
with a flow described by the equation
\[
\dot{v} = Q(v, t, \varepsilon).
\]

Suppose that the rank of the matrix \(\psi_v(v, t)\) is equal to \(m\), the rank of the matrix \(A(v, t) = Z_z(\psi(v, t), t, 0)\) is equal to \(n\) where \(Z_z = \left( \frac{\partial Z}{\partial z} \right) \), \(i, j = 1, \ldots, n + m\), and the matrix \(A(v, t)\) has an \(m\)-fold zero eigenvalue and \(n\) other eigenvalues \(\lambda_i(v, t)\) have have strictly negative real parts, i.e. they satisfy the inequality
\[
\text{Re}\lambda_i(v, t) \leq -\gamma < 0, \quad t \in \mathbb{R}, \quad v \in \mathbb{R}^m.
\]

We shall restrict our consideration to smooth integral surfaces situated in the \(\varepsilon\)-neighborhood of the slow surface \(z = \psi(v, t)\), i.e.
\[
P(v, t, 0) = \psi(v, t),
\]
a motion on which is described by differential equations of form (8) with a smooth right hand side.

The equation (5) describes motions with speeds of order \(O(\varepsilon^{-1})\), while (8) describes motions with speeds of order \(O(1)\). Thus, the integral manifold (7) is a manifold of slow motions or a slow integral manifold. The matrix \(A(v, t)\) is a singular one and therefore the system (5) is called as singular singularly perturbed [10].

2.1. High-gain control

Consider the control system
\[
\dot{x} = \zeta(x) + B(x)u, \quad x(0) = x_0,
\]
where \(x \in \mathbb{R}^m\), \(u \in \mathbb{R}^r\) and \(t \geq 0\). The vector function \(\zeta\) and the matrix function \(B\) are taken to be sufficiently smooth and bounded. The control vector \(u\) is to be selected in such a way as to transfer the vector \(x\) from \(x = x_0\) to a sufficiently small neighborhood of a smooth \(m\)-dimensional surface \(S(x) = 0\). A commonly employed feedback control is
\[
u = -\frac{1}{\varepsilon}KS(x),
\]
where \(K\) is a constant \(r \times m\)-matrix and \(\varepsilon\) is a small positive parameter, see [6] and references therein.
Suppose that we can choose the matrix $K$ in such a way that the matrix $-N(x,t) = -GBK$ is stable\(^1\) and its inverse matrix is bounded, and introduce the additional variable $y = S(x)$, then $x$ and $y$ satisfy the system

\[
\begin{align*}
\varepsilon \dot{x} &= \varepsilon \zeta(x) - B(x)Ky, \quad x(0) = x_0, \\
\varepsilon \dot{y} &= \varepsilon G(x)\zeta(x) - G(x)B(x)Ky, \quad y(0) = y_0 = S(x_0),
\end{align*}
\]

where $G(x) = \partial S/\partial x$. The reduced ($\varepsilon = 0$) algebraic problem possesses an $n$–parameter family of solutions $x = v$, $y = 0$. The role of $A$ is played by the singular matrix

\[
\begin{pmatrix}
0 & -BK \\
0 & -N
\end{pmatrix}.
\]

The latter singular singularly perturbed differential system possesses an $n$–dimensional slow integral manifold $x = v, \quad y = \varepsilon N^{-1}(v,t)G(v)\zeta(v) + O(\varepsilon^2)$. The flow on the manifold is governed by

\[
\dot{v} = [I - B(v)KN^{-1}(v)G(v)]\zeta(v) + O(\varepsilon).
\]

Introduce the new variable

\[
y = z + \varepsilon N^{-1}(x)G(x)\zeta(x).
\]

Then for $z$ we obtain the equations

\[
\varepsilon \dot{z} = -N(x)z + O(\varepsilon).
\]

It is now clear that the representations

\[
y = N^{-1}G\zeta + O(\varepsilon)
\]

are valid for for all $t > 0$. Thus, under the control law (10) the trajectory very quickly attains the $\varepsilon$–neighborhood of $S(x) = 0$.

Let us introduce the modified control

\[
u = -\frac{1}{\varepsilon} K \left[ S(x) - \varepsilon N^{-1}(x)G(x)\zeta(x) \right],
\]

with the stable matrix $-N(x) = -GBK$. Under this control for the variable $x$ we obtain the equation

\[
\varepsilon \dot{x} = \varepsilon \left[ I - B(x)K(GBK)^{-1}G(x) \right] \zeta(x) - B(x)KS(x),
\]

and for the variable $y = S(x)$ we obtain the equation

\[
\varepsilon \dot{y} = -N(x)y,
\]

\[
y = O(e^{-\nu \varepsilon^{-1}t}), \quad \nu > 0, \quad t > 0, \quad \varepsilon \to 0
\]

for some positive $\nu$. Note that the modified control law turns $y = 0$ invariant and exponentially attractive. This means that the modified control law (11) is more preferable than the usually used law (10). Surprisingly, there is no need for using of asymptotic expansion to design the control law since the modified control law gives the exact result.

\(^1\) A stable matrix is one whose eigenvalues all have strictly negative real parts.
3. Weakly Attractive Integral Manifolds

In this section we consider the system (1) when the matrix $B = g_y(x, \phi(x, t), t, 0)$ has eigenvalues on the imaginary axis with nonvanishing imaginary parts. If the eigenvalues at $\varepsilon = 0$ are pure imaginary but after taking into account the perturbations of higher order they move to the complex left half-plane, then the system under consideration has stable slow integral manifolds.

It seems reasonable to say that this kind of problem for gyroscopic systems was previously investigated by the integral manifolds method, see, for example, [12]. Note that in the eigenvalues of the matrix of the linearized fast subsystem in such problems have the form $-\alpha \pm i \beta / \varepsilon$ with positive $\alpha$ and the existence problem for slow integral manifolds has not any connections with the bifurcation problems when the real parts of the eigenvalues change their sign. Some problems of the mechanics of manipulators with high-frequency and weakly damped transient regimes are now discussed in this context. More results along this line can be found in [3, 4, 12].

3.1. Control of a One Rigid-Link Flexible-Joint Manipulator

Consider a simple model of a rigid-link flexible joint manipulator [13, 14], where $J_1$ is the link inertia, $M$ is the link mass, $l$ is the link length, $c$ is the damping coefficient, $k$ is the stiffness. The model is described by the equations:

$$J_1 \ddot{q}_1 + Mgl \sin q_1 + c(\dot{q}_1 - \dot{q}_m) + k(q_1 - q_m) = 0,$$

$$J_m \ddot{q}_m - c(\dot{q}_1 - \dot{q}_m) - k(q_1 - q_m) = u.$$

Here $q_1$ is the link angle, $q_m$ is the rotor angle, and $u$ is the torque input which is the controller.

The control problem under consideration consists of a tracking problem in which it is desired that the link coordinate $q_1$ follows a time-varying smooth and bounded desired trajectory $q_d(t)$ so that $|q_d(t) - q_1(t)| \to 0$ as $t \to \infty$ [13, 14].

If we rewrite the original system in the form

$$J_1 \ddot{q}_1 + J_m \ddot{q}_m + Mgl \sin q_1 = u,$$

$$\ddot{q}_1 - \dot{q}_m + \frac{Mgl}{J_1} \sin q_1 + k \left( \frac{1}{J_1} + \frac{1}{J_m} \right) (q_1 - q_m) + c \left( \frac{1}{J_1} + \frac{1}{J_m} \right) (\dot{q}_1 - \dot{q}_m) = -\frac{u}{J_m},$$

then the use of the small parameter $\varepsilon = 1/\sqrt{k}$ and new variables

$$x_1 = (J_1 q_1 + J_m q_m)/(J_1 + J_m), \quad x_2 = \dot{x}_1, \quad y_1 = q_1 - q_m, \quad y_2 = \varepsilon \dot{y}_1,$$

yield the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{Mgl}{J_1 + J_m} \sin \left( x_1 + \frac{J_m}{J_1 + J_m} y_1 \right) + \frac{u}{J_1 + J_m}, \tag{13}$$

$$\varepsilon \dot{y}_1 = y_2, \quad \varepsilon \dot{y}_2 = -\left( \frac{1}{J_1} + \frac{1}{J_m} \right) y_1 - \varepsilon c \left( \frac{1}{J_1} + \frac{1}{J_m} \right) y_2, \tag{14}$$

$$\varepsilon^2 \frac{Mgl}{J_1} \sin \left( x_1 + \frac{J_m}{J_1 + J_m} y_1 \right) - \varepsilon^2 \frac{u}{J_m}.$$

Note that neglecting all terms of order $O(\varepsilon^2)$ in the r.h.s. of the last equation we obtain the independent subsystem

$$\dot{y}_1 = y_2,$$

$$\varepsilon \dot{y}_2 = -\left( \frac{1}{J_1} + \frac{1}{J_m} \right) y_1 - \varepsilon c \left( \frac{1}{J_1} + \frac{1}{J_m} \right) y_2.$$
solutions of which are characterized by high frequency $\approx \sqrt{(1/J_1 + 1/J_m)/\varepsilon}$ and relatively slow decay $c(1/J_1 + 1/J_m)/2$, since this differential system has the characteristic polynomial

$$\varepsilon^2 \lambda^2 + c \left( \frac{1}{J_1} + \frac{1}{J_m} \right) \lambda + \left( \frac{1}{J_1} + \frac{1}{J_m} \right)$$

which possesses complex zeros

$$\lambda_{1,2} = -\frac{c}{2} \left( \frac{1}{J_1} + \frac{1}{J_m} \right) \pm \frac{i}{\varepsilon} \sqrt{\left( \frac{1}{J_1} + \frac{1}{J_m} \right) - \varepsilon^2 \left( \frac{1}{J_1} + \frac{1}{J_m} \right)^2}.$$

Since the real part of these numbers is negative, for the analysis of the manipulator model under consideration it is possible to use the slow invariant manifold noting that the reducibility principle holds for this manifold (the exact statement may be found in [3]). The terms of $O(\varepsilon^2)$ of the subsystem (14) lead us to conclude that the slow invariant manifold may be found in the form $y_1 = \varepsilon^2 Y + O(\varepsilon^3)$ and $y_2 = O(\varepsilon^3)$, where

$$Y = - \left[ \frac{Mgl}{J_1} \sin(x_1) + \frac{u_0}{J_m} \right] \left( \frac{1}{J_1} + \frac{1}{J_m} \right)^{-1}. \tag{15}$$

Here we used the representation $u = u_0 + \varepsilon^2 u_1 + O(\varepsilon^3)$. Thus, the flow on this manifold is described by equations

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = - \frac{Mgl}{J_1 + J_m} \sin \left( x_1 + \varepsilon^2 \frac{J_m}{J_1 + J_m} Y \right) + \frac{u_0 + \varepsilon^2 u_1}{J_1 + J_m} + O(\varepsilon^3). \tag{16}$$

It is important to emphasize that due to (12) $q_1 = x_1 + \frac{J_m}{J_1 + J_m} y_1$, where $y_1 = \varepsilon^2 Y + O(\varepsilon^3)$, and on the slow invariant manifold we obtain the representation

$$q_1 = x_1 + \varepsilon^2 \frac{J_m}{J_1 + J_m} Y + O(\varepsilon^3). \tag{17}$$

This allows us to rewrite the system (16) on the slow invariant manifold using the original variable $q_1$ instead $x_1$ in the form

$$\dot{q}_1 - \varepsilon^2 \frac{J_m}{J_1 + J_m} \dot{Y} = - \frac{Mgl}{J_1 + J_m} \sin(q_1) + \frac{u_0 + \varepsilon^2 u_1}{J_1 + J_m} + O(\varepsilon^3). \tag{18}$$

The function $\dot{Y}$ will be calculated below.

Let $q_d$ be the desired trajectory, i.e., the goal of the controlled motion is $q_1 \to q_d$ as $t \to \infty$ [13, 14]. Unlike [13, 14] we do not use a fast term added to the control input to make the fast dynamics asymptotically stable to guarantee the fast decay of fast variables $y_1$ and $y_2$, but we use the slow component of the control function $u$ which is written as a sum $u_0 = (J_1 + J_m) u_d + Mgl \sin q_1$, where $u_d = \dot{q}_d - a_1(x_1 - q_d) - a_2(x_1 - \dot{q}_d)$ [13, 14]. The goal of this control law is to obtain an equation with decaying solutions for the difference between $q_1$ and $q_d$.

Setting $\varepsilon = 0$, using (18) and the definitions of $u_0$ and $u_d$ we obtain, to an accuracy of order $O(\varepsilon^2)$, 

$$\dot{q}_1 - \dot{q}_d + a_2(\dot{q}_1 - \dot{q}_d) + a_1(q_1 - q_d) = 0$$

for the difference $q_1 - q_d$, since $q_1 = x_1 + O(\varepsilon^2)$ on the slow invariant manifold by (17). Note that any solution of this second order linear homogeneous differential equation tends to zero as time increase for any positive constant coefficients $a_1$ and $a_2$. This differential equation allows us to
choose the coefficients $a_1$ and $a_2$ in the control function $u_d$ in such a way that the corresponding control function $u$ gives the possibility of realizing a desired motion. Let, for example [4, 13, 14], $M = 1$, $k = 100$, $l = 1$, $J_1 = 1$, $J_m = 1$, $g = 9.8$ and $c = 2$. Setting $a_1 = 3$, $a_2 = 4$ for the desired trajectory $q_d = \sin t$ we obtain the following control law for the original variables

$$u = (J_1 + J_m)u_d + Mgl \sin x_1 = (J_1 + J_m)u_d + Mgl \sin q_1 = 2u_d + 9.8 \sin q_1 =$$

$$2[- \sin t - 4(\dot{q}_1 - \cos t) - 3(q_1 - \sin t)] + 9.8 \sin q_1.$$

Note that the trajectory of $q_1$ tends to the desired trajectory $\sin t$ as $t$ increases.

To take into account the terms of order $O(\varepsilon^2)$ we set $u_1 = -J_m\dot{Y}$ to obtain the same equation $\ddot{q}_1 - \ddot{q}_d + a_2(\dot{q}_1 - \dot{q}_d) + a_1(q_1 - q_d) = 0$ from (18). To calculate $Y$ we use (15) with $u_0 = (J_1 + J_m)u_d + Mgl \sin q_1$, where $u_d = \ddot{q}_d - a_1(x_1 - q_d) - a_2(x_1 - \dot{q}_d)$, and obtain

$$Y = -Mgl \sin q_1 - J_1 u_d.$$

It is easy now to obtain $\dot{Y}$

$$\dot{Y} = -Mgl \cos q_1 \frac{dq_1}{dt} - J_1 \frac{du_d}{dt}$$

$$= -Mgl \cos q_1 \frac{dq_1}{dt} - J_1 \left[ \frac{d^3 q_1}{dt^3} - a_1 \left( \frac{dq_1}{dt} - \frac{dq_d}{dt} \right) - a_2 \left( \frac{d^2 q_1}{dt^2} - \frac{d^2 q_d}{dt^2} \right) \right] =$$

$$-Mgl \cos q_1 \frac{dq_1}{dt} - J_1 \left[ \frac{d^3 q_1}{dt^3} - a_1 a_2 q_1 - (a_2 - a_1) \left( \frac{dq_1}{dt} - \frac{dq_d}{dt} \right) \right],$$

because $\ddot{q}_1 - \ddot{q}_d = -a_1(q_1 - q_d) - a_2(\dot{q}_1 - \dot{q}_d)$. Similarly we obtain

$$\dot{Y} = Mgl \sin q_1 \left( \frac{dq_1}{dt} \right)^2 - Mgl \cos q_1 \frac{dq_1}{dt}$$

$$-J_1 \left[ \frac{d^4 q_1}{dt^4} - a_1(a_2 - a_1)(q_1 - q_d) - a_2(a_2 - 2a_1) \left( \frac{dq_1}{dt} - \frac{dq_d}{dt} \right) \right],$$

and, finally, with $\varepsilon^2 = 1/k$,

$$u = (J_1 + J_m)u_d + Mgl \sin q_1 - J_m \dot{Y}/k.$$

4. The case of multiple root of the degenerate equation

We consider system (1) under the assumptions $(A_1)$ and $(A_2)$. Instead of hypothesis $(A_3)$ we suppose

$$\det Y_y(x, h(x, t), t, 0) \equiv 0 \quad \forall (x, t) \in \mathcal{G} \times R,$$  \hspace{1cm} (19)

that is, $y = h(x, t)$ is not a simple root of the degenerate equation

$$g(x, y, t, 0) = 0.$$ \hspace{1cm} (20)

Under this assumption we cannot apply Proposition 1.1 to system (1) in order to establish the existence of a slow integral manifold near $M_0$ for small $\varepsilon$. Our goal is to derive conditions which imply that for sufficiently small $\varepsilon$ system (1) has at least one integral manifold $M_\varepsilon$ with the representation

$$y = \psi_1(x, t, \varepsilon) = h(x, t) + \varepsilon^q h_{1_1}(x, t) + \varepsilon^{2q} h_{2_1}(x, t) + \ldots.$$
where \( q_i, 0 < q_i < 1 \), is a rational number.
The key idea to solve this problem consists in looking for scalings and transformations of the type
\[
\varepsilon = \mu^r, \quad y = \tilde{y}(\mu, z, x, t), \quad t = \tilde{t}(\mu, \tau)
\]
such that system (1) can be reduced to a system type
\[
\text{such that system (1) can be reduced to a system}
\]
\[
\frac{dx}{d\tau} = f(x, z, \tau, \mu), \quad \mu \frac{dz}{d\tau} = g(x, z, \tau, \mu)
\]
to which Proposition 1.1 can be applied.

### 4.1. Partially cheap linear-quadratic optimal control problem

Consider the linear-quadratic optimal control problem
\[
\dot{x} = A(t, \varepsilon)x + B_1(t, \varepsilon)u_1 + B_2(t, \varepsilon)u_2;
\]
\[
J = \frac{1}{2}x^T(1)Fx(1) + \frac{1}{2} \int_0^1 [x^T(t)Q(t, \varepsilon)x(t) + \varepsilon^2u_1^T(t)R_1(t, \varepsilon)u_1(t) + u_2^T(t)R_2(t, \varepsilon)u_2(t)]dt,
\]
where \( Q = Q^T \geq 0, \quad F = F^T \geq 0, \quad R_1 = R_1^T > 0, \quad R_2 = R_2^T > 0, \quad t \in [0, 1], \ \varepsilon \) is a small positive parameter. If the matrices \( B_1 \) and \( R_1 \) are zero matrices the control problem under consideration is known linear-quadratic optimal control problem. If the case \( B_2 \) and \( R_2 \) are zero matrices is considered, such a problem is called a cheap control problem because there is a small parameter multiplied by a control function in the cost functional [15]. The solution of this problem is given by the formula
\[
\mathbf{u} = \mathbf{u}_1 = -\varepsilon^{-2}R_1^{-1}B_1^T K_1 x,
\]
where \( K_1 \) is the solution of differential matrix Riccati equation
\[
\varepsilon^2(K_1 + A^T K_1 + K_1 A + Q) = K_1 S_1 K_1, \quad S_1 = B_1 R_1^{-1} B_1^T; \quad K_1(1) = F.
\]
The asymptotic solution of this equation which contains fractional exponents of \( \varepsilon \) can be found in [15]. Under the condition \( \varepsilon = 0 \), the degenerate equation has the multiple solution \( K_1 = 0 \) and the branching of slow integral manifolds takes place for this Riccati equation [17]. In this sense we obtain a critical case in the problem under consideration.

In the case of partially cheap control the control law is given by the formula
\[
\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = - \begin{pmatrix} \varepsilon^{-2}R_1^{-1}B_1^T \\ R_2^{-1}B_2^T \end{pmatrix} \mathbf{K} \mathbf{x},
\]
i.e., \( u_1 = -\varepsilon^{-2}R_1^{-1}B_1^T Kx \) and \( u_2 = -R_2^{-1}B_2^T Kx \).

The method of regularization [17] of the differential matrix Riccati equation is adaptable to the Riccati equation in this case:
\[
\varepsilon^2(\dot{K} + A^T K + KA + Q - KS_2 K) = KS_1 K, \quad S_1 = B_1 R_1^{-1} B_1^T, \quad S_2 = B_2 R_2^{-1} B_2^T; \quad K(1) = F,
\]
but below we shall restrict the consideration of the partially cheap linear-quadratic optimal control problem by a simple example.
4.2. Example 1
We investigate the optimal control problem
\[ \dot{x}_1 = u_1, \quad \dot{x}_2 = x_1 + x_2 + u_2 \]
with the cost functional
\[ J = \frac{1}{2} \int_0^T \left[ x_1^2(t) + \varepsilon x_2^2(t) + u_1^2(t) + \varepsilon^2 u_2^2(t) \right] dt \to \min. \]

This problem is a *partially cheap control problem* because one of the control terms in the cost functional is multiplied by a small parameter. The optimal control in this problem is given by the formula
\[ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}^{-1} K \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \]
where the matrix \( K \) is a nonnegative solution of matrix Riccati equation
\[ \frac{dK}{dt} = -K \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} K + K \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}^{-1} K - \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \]
satisfying the condition \( K(T) = 0 \). If we put
\[ K = \begin{pmatrix} k_1 & \varepsilon k_2 \\ \varepsilon k_2 & \varepsilon k_3 \end{pmatrix} \]
we obtain the differential system
\[ \begin{align*}
\frac{dk_1}{dt} &= k_1^2 + k_2^2 - 2\varepsilon k_2 - 1, \\
\varepsilon \frac{dk_2}{dt} &= \varepsilon k_1 k_2 + k_2 k_3 - \varepsilon (k_2 + k_3), \\
\varepsilon \frac{dk_3}{dt} &= k_3^2 + \varepsilon^2 k_2^2 - 2\varepsilon k_3 - \varepsilon.
\end{align*} \tag{22} \]

In this example the method of the Newton diagram or blow-up transformations may be useful but we prefer to use more simple gauge function method [16].

The corresponding degenerate system
\[ k_2 k_3 = 0, \quad k_3^2 = 0 \]
has the solution \( k_2 = k_3 = 0 \), but this solution is not simple. In order to get simple roots we apply the scaling
\[ k_3 = \mu \kappa_3, \quad \varepsilon = \mu^2. \tag{23} \]

Substituting (23) into (22) we obtain
\[ \begin{align*}
\frac{dk_1}{dt} &= k_1^2 + k_2^2 - 2\mu^2 k_2 - 1, \\
\mu \frac{dk_2}{dt} &= \mu k_1 k_2 + k_2 k_3 - \mu (k_2 + \mu \kappa_3), \\
\mu \frac{d\kappa_3}{dt} &= \kappa_3^2 - 1 + \mu^2 k_2^2 - 2\mu \kappa_3.
\end{align*} \tag{24} \]
The corresponding degenerate system

\[ k_2 \kappa_3 = 0, \; \kappa_3^2 = 1 \]

has the simple nonnegative solution \( k_2 = 0, \kappa_3 = 1 \). Applying Proposition 1.1 we get that the original system (22) has the invariant manifold

\[ k_2 = \mu \psi_1(k_1, \mu), \; \kappa_3 = \mu + \mu^2 \psi_2(k_1, \mu). \tag{25} \]

Substituting (25) into (22) leads to the initial value problem

\[ \frac{dk_1}{dt} = k_2 - 1 + O(\mu^2), \; k_1(T) = 0. \]

By this way, the elements of the matrix \( K \) can be determined approximately.

5. Twice Critical Case

Consider the control system

\[ \varepsilon \dot{x} = A(t, \varepsilon)x + B(t, \varepsilon)u, \; x \in \mathbb{R}^{n+m}, \; x(0) = x_0 \tag{26} \]

with the cost functional

\[ J = \frac{1}{2} x^T(1)F x(1) + \frac{1}{2} \int_0^1 (x^T(t)Q(t)x(t) + u^T(t)R(t)u(t))dt. \tag{27} \]

The solution of this problem is the optimal linear feedback control law

\[ u = -\varepsilon^{-1}R^{-1}B^TP(t, \varepsilon)x, \]

where \( P \) satisfies the differential matrix Riccati equation

\[ \varepsilon \dot{P} = -PA - A^TP + PSP - Q, \; P(1, \varepsilon) = F. \]

Let the matrices in (26) (27) may be represented as

\[ A = \begin{pmatrix} \varepsilon A_1 & \varepsilon A_2 \\ A_3 & A_4 \end{pmatrix}, \; B = \begin{pmatrix} \varepsilon B_1 \\ B_2 \end{pmatrix}, \]

\[ F = \begin{pmatrix} F_1 & F_2 \\ F^T_2 & F_3 \end{pmatrix}, \; Q = \begin{pmatrix} Q_1 & Q_2 \\ Q^T_2 & Q_3 \end{pmatrix}, \]

where \( A_1, \; F_1, \; Q_1 \) are \( (m \times m) \)-matrices, \( A_2, \; F_2, \; Q_2 \) are \( (m \times n) \)-matrices, \( A_3, F_3, Q_3 \) are \( (n \times n) \)-matrices, \( B_1 \) is \( (m \times r) \)-matrix, \( B_2 \) is \( (n \times r) \)-matrix, and \( R \) is \( (r \times r) \)-matrix. Suppose that all these matrices have the following asymptotic presentations with respect to \( \varepsilon \):

\[ A_i(t, \varepsilon) = \sum_{j \geq 0} \varepsilon^j A_{ij}(t), \; i = 1, 4, \; B_i(t, \varepsilon) = \sum_{j \geq 0} \varepsilon^j B_{ij}(t), \; i = 1, 2, \; Q_i(t, \varepsilon) = \sum_{j \geq 0} \varepsilon^j Q_{ij}(t), \; i = 1, 3, \]

\[ R(t, \varepsilon) = \sum_{j \geq 0} \varepsilon^j R_{ij}(t), \; F_i(\varepsilon) = \sum_{j \geq 0} \varepsilon^j F_{ij}, \; i = 1, 3 \]

with smooth on \( t \) matrix coefficients, \( t \in [0, 1] \). The first \( m \) components of vector \( x \) are slow
variables, and the other $n$ components are fast variables. The control law can be found in the block form
\begin{equation}
    u = -R^{-1}(B_1^T \varepsilon^{-1} B_2^T)\left( \begin{array}{c} P_1 \\ \varepsilon P_2 \\ \varepsilon P_3 \end{array} \right) x,
\end{equation}

or
\begin{equation}
    u = -R^{-1}(B_1^T P_1 + B_2^T P_3^T) \varepsilon B_1^T P_2 + B_2^T P_3)x.
\end{equation}

For $P_1$, $P_2$, $P_3$ we obtain the following differential system
\begin{align}
    \dot{P}_1 &= -P_1 A_1 - A_1^T P_1 - P_2 A_3 - A_3^T P_2^T + P_1 S_1 P_1 + \\
    &\quad + P_1 S_2 P_1^T + P_1 S_3 P_3^T + Q_1,
    \\
    \varepsilon \dot{P}_2 &= -P_1 A_2 - P_2 A_4 - \varepsilon A_1^T P_2 - A_3^T P_3 + \\
    &\quad + P_1 S_2 P_3 + P_2 S_3 P_3 + \varepsilon(P_1 S_1 P_2 + P_2 S_1^T P_2) - Q_2,
    \\
    \varepsilon \dot{P}_3 &= -P_3 A_4 - A_2^T P_3 + P_3 S_3 P_3 + \varepsilon(-P_2^T A_2 - A_2^T P_2 + \\
    &\quad + \varepsilon P_2^T S_1 P_2 + P_2^T S_2 P_3 + P_3 S_2^T P_2) - Q_3,
\end{align}

with additional conditions
\begin{equation}
    P_1(1, \varepsilon) = F_1, \quad P_2(1, \varepsilon) = \varepsilon^{-1} F_2, \quad P_3(1, \varepsilon) = \varepsilon^{-1} F_3.
\end{equation}

Setting $\varepsilon = 0$ we obtain from (29) the matrix algebraic equation
\begin{equation}
    -MA_{40} - A_{40}^T M + MS_{30} M - Q_{30} = 0,
\end{equation}

where $S_{30} = B_{20} R_{0}^{-1} B_{20}^T$. If the last equation has a positive definite solution $M(t)$ and eigenvalues of $D_{40} = A_{40} - S_{30} M$ have negative real parts for $t \in [0, 1]$ then it is possible to use the boundary functions method [1, 2] or the integral manifolds method [6, 9, 18] setting $K_2 = h_2(K_1, t, \varepsilon) = h_{20}(K_1, t) + \varepsilon \ldots , K_3 = h_3(K_1, t, \varepsilon) = M(t) + \varepsilon \ldots$. However for systems with low energy dissipation the matrices $S_{30}$ and $Q_{30}$ are equal to zero and the main role plays the linear operator
\begin{equation}
    LX = X A_{40} + A_{40}^T X.
\end{equation}

For this class of systems the eigenvalues of $A_{40}$ are pure imaginary and the spectrum of the linear operator $L$ has a nontrivial kernel, since sums $(\lambda_i(t) + \lambda_j(t))$, $i, j = 1, \ldots, n$, form its spectrum. This means that the third equation of (29) is singular singularly perturbed. Thus, the dimension of the slow integral manifold of (29) is greater than the dimension of the matrix $P_1$ and the problem under consideration is critical in this sense. Moreover, under taking into account that all other, nonzero eigenvalues of $L$, are pure imaginary, it is possible to say that this problem is twice critical.

5.1. Example 2

Let
\begin{equation}
    A = \begin{pmatrix} -\varepsilon & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}, \quad R = (1), \quad Q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}.
\end{equation}
Consider the corresponding differential system

\[
\begin{align*}
\varepsilon \dot{p}_1 &= 2p_2 + 2\varepsilon p_1 + \varepsilon p_2^2 - 2\varepsilon, \\
\varepsilon \dot{p}_2 &= \varepsilon p_2 - p_1 + p_3 + \varepsilon p_2 p_3, \\
\varepsilon \dot{p}_3 &= -2p_2 + \varepsilon p_3^2 - \varepsilon.
\end{align*}
\] (30)

First, we need to separate it into a slow and a fast subsystem. At first glance, all three equations are singularly perturbed. However, we should consider the matrix of leading terms on the right hand side of the system, which has the form

\[
\begin{pmatrix}
0 & 2 & 0 \\
-1 & 0 & 1 \\
0 & -2 & 0
\end{pmatrix}
\]

Obviously, this matrix has a zero eigenvalue and two pure imaginary eigenvalues, i.e. the problem under consideration is twice critical.

Introducing the new variables

\[q_1 = p_1 + p_3 - 2, \quad q_2 = p_2, \quad q_3 = p_3 - 1,\]

we obtain the differential system

\[
\begin{align*}
\dot{q}_1 &= 2q_1 + q_2^2 + q_3^2, \\
\varepsilon \dot{q}_2 &= 2\varepsilon q_2 + 2q_3 - q_1 + \varepsilon q_2 q_3, \\
\varepsilon \dot{q}_3 &= -2q_2 + 2\varepsilon q_3 + \varepsilon q_3^2
\end{align*}
\] (31)

with the slow variable \(q_1\) and two fast variables \(q_2, q_3\).

The last system possesses one-dimensional slow invariant manifold which is weakly attractive with respect to argument \(1 - t\) because the main matrix of the fast subsystem is

\[
\begin{pmatrix}
2\varepsilon & 2 \\
-2 & 2\varepsilon
\end{pmatrix}
\]

Thus, the dimension of the system of Riccati differential equations can be reduced from three to one. Let us construct the slow integral manifold using the fact that it can be asymptotically expanded in powers of the small parameter. Setting

\[q_2 = \varphi(q_1, \varepsilon) = \varepsilon \varphi_1(q_1) + \varepsilon^2 \ldots,\]

\[q_3 = \psi(q_1, \varepsilon) = \psi_0(q_1) + \varepsilon \psi_1(q_1) + \varepsilon^2 \ldots,\]

we obtain

\[\psi_0(q_1) = q_1/2, \quad \varphi_1(q_1) = q_1^2/16, \quad \psi_1(q_1) = 0.\]

Thus we obtain the slow invariant manifold

\[q_2 = \varepsilon q_1^2/16 + O(\varepsilon^2), \quad q_3 = q_1/2 + O(\varepsilon^2),\]

with the equation on the integral manifold

\[\dot{q}_1 = 2q_1 + q_2^2/4 + O(\varepsilon^2).\]
6. Conclusion
Critical cases for singularly perturbed differential systems are studied in the paper. We have considered singularly perturbed control problems as applications. It has been shown that the reduction of dimensions of these problems can be done by means of the integral manifold method. The slow integral manifolds for the matrix Riccati equation of linear-quadratic control problem are constructed and it is shown that the method of integral manifolds allows us to reduce the dimension of control problems. This approach was used for the investigation of optimal filtering problems in [19, 20].

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