Symmetrized Robust Procrustes: Constant-Factor Approximation and Exact Recovery

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Abstract

The classical Procrustes problem is to find a rigid motion (orthogonal transformation and translation) that best aligns two given point-sets in the least-squares sense. The Robust Procrustes problem is an important variant, in which a power-1 objective is used instead of least squares to improve robustness to outliers. While the optimal solution of the least-squares problem can be easily computed in closed form, dating back to Schönemann (1966), no such solution is known for the power-1 problem. In this paper we propose a novel convex relaxation for the Robust Procrustes problem. Our relaxation enjoys several theoretical and practical advantages: Theoretically, we prove that our method provides a $\sqrt{2}$-factor approximation to the Robust Procrustes problem, and that, under appropriate assumptions, it exactly recovers the true rigid motion from point correspondences contaminated by outliers. In practice, we find in numerical experiments on both synthetic and real robust Procrustes problems, that our method performs similarly to the standard Iteratively Reweighted Least Squares (IRLS). However the convexity of our algorithm allows incorporating additional convex penalties, which are not readily amenable to IRLS. This turns out to be a substantial advantage, leading to improved results in high-dimensional problems, including non-rigid shape alignment and semi-supervised interlingual word translation.

1 Introduction

The (Rigid) Procrustes problem is the problem of finding a rigid motion that aligns two given point clouds as accurately as possible in the least squares sense. Formally, given two ordered sets of $n$ points in $\mathbb{R}^d$, denoted by $P = (p^{(1)}, \ldots, p^{(n)})$ and $Q = (q^{(1)}, \ldots, q^{(n)})$, the Procrustes problem is the optimization problem

$$\min_{R \in O(d), t \in \mathbb{R}^d} \sum_{i=1}^{n} \| R p^{(i)} - q^{(i)} + t \|^2,$$

with $O(d)$ denoting the set of $d \times d$ orthogonal matrices, and $\| \cdot \|$ denoting the $\ell_2$-norm throughout the paper. In low dimension, $d = 2, 3$, the Procrustes problems and its generalizations (unknown correspondences, robustness to outliers) are well-studied problems in computer vision, graphics, and robotics [1][2][3], with applications in scientific disciplines such as morphology [4] and chemistry.
The high-dimensional case $d > 3$ also has many applications, including translation tasks in NLP [6,7,8] or non-rigid correspondence problems in computer vision and graphics [9,10,11].

From an optimization viewpoint, the rigid Procrustes problem is convenient since it has a closed-form solution that is easy to compute, as shown by the author of [12]. In essence, the solution is computed by applying a Singular Value Decomposition (SVD) to a matrix created from $P$ and $Q$. A key pitfall of this approach, however, is that the least-squares objective $E_{\text{procr}}$ is sensitive to outliers. Due to known results on sparse signal recovery with the $\ell_1$ norm ([13,14,15]), a natural way to address this is to consider the Robust Procrustes problem, where the least-squares objective in (1) is replaced by a power-1 sum of $\ell_2(\mathbb{R}^d)$ norms

$$
\min_{R \in O(d), t \in \mathbb{R}^d} E(R,t) := \sum_{i=1}^{n} \left\| R p^{(i)} - q^{(i)} + t \right\|.
$$

To the best of our knowledge, no known algorithm to date is guaranteed to find the global minimum of the Robust Procrustes problem (2). Perhaps the most natural way to optimize this problem is by Iteratively Reweighted Least Squares (IRLS), whereby problem (2) is replaced by a sequence of weighted Procrustes problems (1) (see [16,17] and also [18,19,20]). While this method often works well in practice, it may fail to find a global minimizer since (in contrast with more classical IRLS applications) the domain of (2) is non-convex. In fact, it seems that there were no known algorithms with theoretical guarantees for this problem until the recent paper [22], where a RANSAC-like algorithm was proposed. Given enough running time, this algorithm was shown to find a solution whose energy is optimal up to a multiplicative constant $(1 + \sqrt{2})^d$ which depends exponentially on $d$. While in practice this algorithm works well for low-dimensional problems, its performance deteriorates as the dimension increases, which is common to RANSAC-type algorithms.

In this paper we propose a family of simple polynomial-time algorithms with strong theoretical guarantees for the Robust Procrustes problem, denoted by SRP$_p$, $1 \leq p \leq \infty$, focusing on $p = 2, \infty$. First, as we discuss in Section 3 our SRP$_2$ algorithm is guaranteed to find a solution to (2) that is optimal up to a multiplicative factor of $\sqrt{2}$. For SRP$_\infty$ we prove a weaker $2\sqrt{2}$ approximation factor, thought its performance is often slightly better in practice. We stress that these constants are universal, and in particular, independent of the number of points $n$ and their dimension $d$.

In addition, if a subset of the points in $P$ and $Q$ (the inliers) are exactly related by a rigid motion $(R_0,t_0)$, and they dominate the remaining points (the outliers), then our SRP algorithm is guaranteed to recover the true rigid motion $(R_0,t_0)$; see Section 3.2 for an exact formulation. Our algorithm and theoretical results are also applicable to the related Orthogonal Robust Procrustes problem

$$
\min_{R \in O(d)} E(R) := \sum_{i=1}^{n} \left\| R p^{(i)} - q^{(i)} \right\|,
$$

where one seeks an orthogonal transformation without translation.

The SRP algorithms we propose follow a ‘relax and project’ scheme. In the ‘relax’ step, a symmetrized relaxation of (2) is optimized over the space of affine transformations. This problem is convex and can be solved globally. Its minimal objective is a lower bound on the true minimum of (2), and the minimizers themselves need not be a rigid motion. To obtain a feasible solution to (2), in the ‘project’ step the solution is projected onto the set of rigid motions. The approximation results discussed above follow from a corresponding result on the ratio between the lower bound and the objective value of the projected solution.

One advantage of algorithms that provide lower bounds is that they can be used to estimate the accuracy of other algorithms; specifically, our method provides a lower bound on the objective obtainable by any method for the Robust Procrustes problem (2). Moreover, this bound is tight up to a factor of $\frac{1}{\sqrt{2}}$. In practice we often find it to be significantly tighter. Additionally, lower bounds may be useful for Branch-and-Bound algorithms, which rely on the availability of such bounds.

Another advantage our ‘relax and project’ approach is that it can easily accommodate additional convex energy terms and constraints; for example, if $E(A)$ is some convex energy on $\mathbb{R}^{d \times d}$, our algorithm can be readily extended to approximately solve

$$
\min_{R \in O(d)} \sum_{i=1}^{n} \left\| R p^{(i)} - q^{(i)} \right\| + E(R).
$$

2
This type of energy is particularly useful in high dimension, where the Robust Procrustes problem may be underdetermined \((n < d)\) or unstable without additional regularization. For example, such energy was used by the authors of [11] to recover non-rigid transformations of surfaces. In Section 2.1 we discuss incorporating their energy into our objective to recover an orthogonal transformation in underdetermined settings in which an additional set of unordered, unmapped points is available. In Section 4 we apply this approach to semi-supervised learning of interlingual word translation.

Our experiments further demonstrate that our ability to incorporate additional convex energies alongside the Robust Procrustes energy is a significant advantage. We show this advantage in Section 4 for synthetic experiments, non-rigid correspondence problems, and NLP. For example, for interlingual translation NLP tasks with small dictionaries, we obtain a 14% improvement compared to both standard Procrustes optimization and IRLS.

To summarize, the main contributions of this paper are:

1. We provide a polynomial-time algorithm that achieves a \(\sqrt{2}\)-factor approximation for the robust Procrustes problem.
2. We prove that our algorithm, under certain assumptions, is able to exactly recover a rigid motion from noiseless correspondences with outliers.
3. Our algorithm provides lower bounds that can be used to evaluate solutions of other methods, and for branch and bound algorithms.
4. Our algorithm can handle additional convex energies, and yields state-of-the-art performance for such problems.

2 Method

We first describe our method for the simpler orthogonal problem \((3)\) and then move to the rigid problem \((2)\). To this end, we define the family of relaxation to \((4)\), parametrized by \(p \in [1, \infty)\),

\[
\min_{A \in \mathbb{R}^{d \times d}} E_p(A) := \sum_{i=1}^{n} \left( \frac{\|A^p q^{(i)} - p^{(i)}\|^p + \|A^T q^{(i)} - p^{(i)}\|^p}{2} \right)^{1/p}, 1 \leq p < \infty \tag{5}
\]

with the objective \(E_p\) defined for any matrix \(A \in \mathbb{R}^{d \times d}\). Note that if \(A = R\) is orthogonal, then its relaxed objective \(E_p(R)\) for any \(p\) coincides with the original \(E(R)\) of \((5)\). Therefore, the optimal objective of \((5)\) is a lower bound on that of the original problem \((4)\). Problem \((5)\) is convex for any \(p \in [1, \infty)\). For \(p \in \{1, 2, \infty\}\), it can be formulated as a second order conic program (SOCP) \((23)\) and solved (globally) by standard solvers such as MOSEK \((24)\) or GUROBI \((25)\). In practice we use our own faster implementation, as discussed in Appendix B.2.

Since the optimal solution \(A^*\) to \((5)\) will generally not be an orthogonal matrix, we project it onto the nearest orthogonal matrix \(\Pi(A^*) = UV^T\), where \(U, V\) are taken from a singular value decomposition \(A^* = U\Sigma V^T\) of \(A\). Our SRP\(_p\) method for the orthogonal problem is summarized in Algorithm 1.

In this work we focus mostly on SRP\(_2\) and SRP\(_\infty\). By the generalized mean inequality, \(E_p(A)\) is monotonically increasing with respect to \(p\). Thus, \(p = \infty\) has the advantage of incurring the highest penalty on non-orthogonal matrices. On the other hand, we derive better approximation bounds for \(p = 2\), as detailed in the next section.
For the rigid problem (2) we redefine the objective $E_p$ to accommodate translations, and consider the relaxation

$$
\min_{A \in \mathbb{R}^{d \times d}, t, s \in \mathbb{R}^d} E_p(A, t, s) := \sum_{i=1}^{n} \left( \left\| Ap^{(i)} - q^{(i)} + t \right\|^p + \left\| AT q^{(i)} - p^{(i)} + s \right\|^p \right) \frac{1}{2}. \tag{6}
$$

Note that for any orthogonal matrix $R$ and vector $t \in \mathbb{R}^d$,

$$
E_p(R, t, -RT t) = E(R, t), \quad p \in [1, \infty]. \tag{7}
$$

Thus, similarly to the orthogonal case, the global minimum of problem (6) is a lower bound to that of the original problem (2).

To approximately solve the rigid problem (2), we first find an optimal solution $(A^*, t^*, s^*)$ to (6) and take $\hat{R} = \Pi(A^*)$ as an estimate for the orthogonal part. Then, as an estimate for the translation, we take $\hat{t}$ to be the minimizer of the Robust Procrustes problem (2) when $R = \hat{R}$ is fixed.

**Algorithm 2** Relax-and-project algorithm for the Rigid Robust Procrustes problem

- **Input:** $P, Q \in \mathbb{R}^{d \times n}, \quad p \in [0, \infty]$
- **Output:** Approximate solution $(\hat{R}, \hat{t})$ of Problem (2)
  - Find minimizer $(A^*, t^*, s^*)$ of Problem (6)
  - Set $\hat{R} = \Pi(A^*)$
  - Find minimizer $\hat{t}$ of Problem (2) with $R = \hat{R}$ fixed.

**Output:** Approximate solution $\hat{R}$ of Problem (3)

- Find minimizer $A^*$ of Problem (5)
  - Set $\hat{R} = \Pi(A^*)$

2.1 Covariance energy for semi-supervised Procrustes

As discussed in Section 1, it is straightforward to generalize our algorithm to optimize robust Procrustes energies with an additional convex energy $\tilde{E}$ as in (3). In the orthogonal case, this is achieved by minimizing the energy $E_p(A) + \tilde{E}(A)$ over $\mathbb{R}^{d \times d}$ and then projecting the solution to an orthogonal matrix as before. By a similar approach, not discussed in this paper, it is possible to add convex constraints and energy $\tilde{E}(R, t)$ to the rigid problem (2).

In this work we use a specific choice of $\tilde{E}$ that is useful for semi-supervised Procrustes problems. This choice is inspired by the operator commutativity constraint used for functional maps [1]. Suppose we are given a point cloud $\tilde{P} \in \mathbb{R}^{d \times n_p}$, sampled uniformly from a surface $S \subset \mathbb{R}^d$, and another point cloud $\tilde{Q} \in \mathbb{R}^{d \times n_Q}$ sampled from a transformed version of that surface $\tilde{S} = R \circ S$, related to $S$ by an orthogonal transformation $R$. Suppose that in addition we are given a small number $n \ll n_p, n_Q$ of points $P, Q \in \mathbb{R}^{d \times n}$ that are known to be corresponding pairs $q^{(i)} = Rp^{(i)}$, possibly with some outliers. Such settings arise in semi-supervised learning tasks, where two unlabelled sets are available and a small number of correspondences are found manually, possibly with some labelling errors. In this case the transformation $R$ that takes $S$ to $\tilde{S}$ satisfies

$$
R \cdot \text{cov}(p) = \text{cov}(q) R, \tag{8}
$$

where $\text{cov}(p), \text{cov}(q)$ are the non-centered covariance matrices of $p \sim \text{Unif}(S), q \sim \text{Unif}(\tilde{S})$ respectively; see derivation in Appendix C. We take the empirical covariance matrices $\text{cov}(\tilde{P}) = \frac{1}{n_p} \tilde{P} \tilde{P}^T$, $\text{cov}(\tilde{Q}) = \frac{1}{n_Q} \tilde{Q} \tilde{Q}^T$ as estimates of $\text{cov}(p), \text{cov}(q)$ and incorporate the following covariance energy into our objective,

$$
E_{\text{cov}}(R) = \left\| R \cdot \text{cov}(\tilde{P}) - \text{cov}(\tilde{Q}) R \right\|_F, \tag{9}
$$

with $\| \cdot \|_F$ being the Frobenius norm. In Section 3 we demonstrate the effectiveness of this approach in semi-supervised recovery of an orthogonal transformation from real and synthetic data.
3 Theoretical results

3.1 Approximation Guarantees

For motivation, consider the following question: when does our relaxation (5) of the orthogonal robust Procrustes problem (3) achieve zero energy? What about the simpler, non-symmetrized relaxation

\[ \min_{A \in \mathbb{R}^{d \times d}} \sum_{i=1}^{n} \left\| Ap^{(i)} - q^{(i)} \right\|? \]  

(10)

Clearly, the optimal energy of problem (10) is zero if \( P \) and \( Q \) are related by a linear map \( A \), even if it is not orthogonal. In contrast, it is straightforward to prove

**Proposition.** Let \( P, Q \in \mathbb{R}^{d \times n} \) such that the columns of \( P \) span \( \mathbb{R}^d \). Let \( E^*, E_p^* \) be the minimal objectives of (3) and (5) respectively, with \( p \in [1, \infty] \). Then \( E^* = 0 \) if and only if \( E_p^* = 0 \).

**Proof.** By definition \( 0 \leq E_p^* \leq E^* \) and thus \( E^* = 0 \Rightarrow E_p^* = 0 \). In the other direction, let \( A^* \) be a minimizer of \( E_p(A) \). If \( E_p(A^*) = 0 \), then \( (A^*)^T A^* p^{(i)} = (A^*)^T q^{(i)} = p^{(i)} \) for all \( i = 1, \ldots, n \). Since we assumed the columns of \( P \) span \( \mathbb{R}^d \), it follows that \( A^T A = I_d \) and so \( E^* = 0 \).

The following theorem shows that the assumption that the columns of \( P \) span \( \mathbb{R}^d \) is not necessary.

**Theorem 3.1.** For \( P, Q \in \mathbb{R}^{d \times n} \) and \( p \in [2, \infty] \), let \( A^* \) be a minimizer of \( E_p(A) \), and let \( \hat{R} = \Pi(A^*) \). Let \( E^* \) be the optimal objective of the orthogonal robust Procrustes problem (3). Then

\[ E^* \leq E(\hat{R}) \leq 2E_p(A^*) \leq 2E^*. \]  

Moreover, if \( p = 2 \) then

\[ E^* \leq E(\hat{R}) \leq \sqrt{2}E_p(A^*) \leq \sqrt{2}E^*. \]  

(11)  

(12)

This theorem particularly implies that for \( p \geq 2 \), \( E^* = 0 \) if and only if \( E_p^* = 0 \). More importantly, it shows that the ratio between the objective \( E(\hat{R}) \) obtained by Algorithm \( \Pi \) and the best possible objective \( E^* \) is uniformly bounded by a constant \((2) \) or \( \sqrt{2} \). The ratio between \( E^* \) and our lower bound \( E_p(A^*) \) is upper-bounded by this constant as well.

Theorem 3.1 thus provides a \( \sqrt{2} \)-factor approximation result for the orthogonal robust Procrustes problem. This factor is in fact the best achievable by any relax-and-project method; see discussion in Appendix \( \text{D.1} \).

The following theorem provides a similar result for the rigid problem.

**Theorem 3.2.** For \( P, Q \in \mathbb{R}^{d \times n} \) and \( p \in [2, \infty] \), let \( (A^*, t^*, s^*) \) be a minimizer of \( E_p(A,t,s) \). Let \( (\hat{R}, \hat{t}) \) be the output of Algorithm \( \Pi \) and let \( E^* \) be the optimal objective of Problem (1). Then

\[ E^* \leq E(\hat{R}, \hat{t}) \leq 2\sqrt{2}E_p(A^*, t^*, s^*) \leq 2\sqrt{2}E^*. \]  

Moreover, if \( p = 2 \) then

\[ E^* \leq E(\hat{R}, \hat{t}) \leq \sqrt{2}E_p(A^*, t^*, s^*) \leq \sqrt{2}E^*. \]  

(13)  

(14)

3.2 Recovery guarantees

We now study the problem of recovery: Suppose that a subset of the points in \( P, Q \), indexed by \( I \subseteq \{1, \ldots, n\} \), are related by an orthogonal transformation \( R_0 \) or rigid motion \( (R_0, t_0) \). Under what condition does the minimizer of the relaxed problems (5) and (6) yield a successful recovery of \( R_0 \) or \( (R_0, t_0) \)? Theorems 3.1 and 3.2 already guarantee exact recovery when \( P \) and \( Q \) are related by an orthogonal or rigid transformation and are not degenerate. We now show that this is true also when \( P, Q \) contain some outliers, provided that the inliers are more ‘dominant’, in the following sense:

**Definition.** We say that \( P, Q \in \mathbb{R}^{d \times n} \) satisfy the linear dominance-of-inliers (DIP) property with respect to \( (R_0, I) \) if \( d^{(i)} = R_0 p^{(i)} \) for \( i \in I \), and for any unit vector \( u \in \mathbb{R}^d \),

\[ \sum_{i \in I} |\langle u, p^{(i)} \rangle| > \sum_{i \in C} |\langle u, p^{(i)} \rangle| \quad \text{and} \quad \sum_{i \in I} |\langle u, q^{(i)} \rangle| > \sum_{i \in C} |\langle u, q^{(i)} \rangle|. \]  

(15)
Intuitively, the linear DIP requires that the inliers, indexed by $I$, be more dominant than the outliers along every axis. By the following theorem, this condition guarantees exact recovery in the orthogonal problem.

**Theorem 3.3.** Suppose that $P, Q \in \mathbb{R}^{d \times n}$ satisfy the linear DIP with respect to $(R_0, I)$. Then for any $p \in [1, \infty]$, the unique global minimizer of $E_p(A)$ is $R_0$.

We can achieve a similar result for the rigid problem by defining the following variant of DIP, which accommodates translations.

**Definition.** We say that $P$, $Q$ satisfy the affine DIP with respect to $(R_0, t, I)$ if $q^{(i)} = R_0 p^{(i)} + t_0$ for $i \in I$, and for any vector $u \in \mathbb{R}^d$ and any $\alpha \in \mathbb{R}$ such that $\|u\| + |\alpha| > 0$,

$$\sum_{i \in I} |\langle u, p^{(i)} \rangle + \alpha| > \sum_{i \in R} |\langle u, p^{(i)} \rangle + \alpha| \quad \text{and} \quad \sum_{i \in I} |\langle u, q^{(i)} \rangle + \alpha| > \sum_{i \in R} |\langle u, q^{(i)} \rangle + \alpha|.$$  \hspace{1cm} (16)

The affine DIP is a stronger condition than the linear DIP by definition. Intuitively, it requires the inliers to be more dominant than the outliers along every line, whereas the linear DIP only considers lines that go through the origin. Note that by setting $u = 0$ and $\alpha = 1$ in (16), it can be seen that the affine DIP requires the number of inliers to be greater than that of outliers.

The following theorem shows that the affine DIP guarantees successful recovery in the rigid problem.

**Theorem 3.4.** Suppose that $P, Q$ satisfy the affine DIP with respect to $(R_0, t, I)$. Then for any $p \in [1, \infty]$, the unique global minimizer of $E_p(A, t, s)$ is $(R_0, t, -R_0^T t_0)$.

In addition to the results stated here, we have several other theoretical results, stated in Appendix D; these results show that the constants in our theorems are optimal, in an appropriate sense. Proofs are in Appendix E.

### 4 Numerical experiments

In this section we provide empirical evaluation of our proposed method and theoretical results. First, in Section 4.1 we visualise the theoretical guarantees discussed in Section 3 and juxtapose them with empirical evidence. Then, in Section 4.2 we demonstrate the competitive performance of our method in various synthetic settings. Lastly, in Section 4.3 we evaluate our method in two applications: (a) recovery of non-rigid shape transformations by Functional Maps [11], and (b) learning how to translate words between natural languages in a semi-supervised setting.

#### 4.1 Demonstration of theoretical results

**Approximation bounds** We start by illustrating the approximation bounds of Section 3.1. Recall that the optimal objective value $E_2(A^*)$ of the relaxation (6) with $p = 2$ provides lower bounds to the true minimum $E^*$ of the Robust Procrustes problem. By Theorem 3.2, the ratio between the obtained objective $E(\tilde{R}, \tilde{t})$ and the aforementioned lower bound, is bounded by $\sqrt{2}$. In Figure 1 we see a numerical verification of this result. We further see that in practice most other methods also obtain energy which is lower than $\sqrt{2}E_2(A^*)$, with the exception of the algorithm of [22] in high dimensions.

Besides the Approx-Alignment (AA) algorithm of [22], methods plotted in Figure 1 are the standard Rigid Procrustes (1) (Procrustes); the non-symmetrized relaxation of (2),

$$\min_{A \in \mathbb{R}^{d \times d}, t \in \mathbb{R}^d} \sum_{i = 1}^n \|Ap^{(i)} + t - q^{(i)}\|,$$  \hspace{1cm} (17)

followed by orthogonal projection (NonSym); our proposed SRP$_2$ and SRP$_\infty$ outlined in Algorithm 2 and an IRLS scheme initialized with uniform weights. We also show the lower bounds obtained by our relaxations (6) with $p = 2$ and $p = \infty$ (SRP$_2$ lower bound, SRP$_\infty$ lower bound) and the energy of the rotation and translation used to generate the problem $(R_0, t_0)$ (Ground Truth). Details of the experimental setting are given in Appendix B.1.
Figure 1: Approximation bound vs. empirical results

Lower and upper bounds obtained by our method, together with the energies obtained by different methods for the Robust Procrustes problem \(\mathcal{P}^{\mathcal{R}}\) with 200 inliers and 2\% noise. Left panel: \(d = 3\). Right panel: \(d = 100\).

Figure 2: Exact recovery of \(R\) in a noiseless setting

We conclude our discussion of (6) by noting that the figure also verifies that the lower bound of \(\text{SRP}_\infty\) is tighter than that of \(\text{SRP}_2\), as discussed in Section 2, and that while most methods perform similarly in terms of their objective value; our next experiment will show that small relative difference in the energy often manifest as a significant difference in recovery performance.

**Exact recovery with outliers** Next, we demonstrate the capability of our method to exactly recover a rigid motion from noiseless correspondences contaminated by outliers, as implied by Theorem 3.4. In a setting similar to the above but with no noise, we evaluate several methods for their recovery of \((R_0, t_0)\). The error for \(R\) is measured by \(\|R - R_0\|\), with \(\|\cdot\|\) denoting the spectral norm \(\|A\| = \max_{\|v\|=1}\|Av\|\). The results appear in Figure 2. Recovery errors for \(t\) appear in Figure 4 in Appendix A.

It can be seen that all methods relying on a robust convex objective (NonSym, SRP_2, SRP_\infty) can tolerate outliers up to a certain threshold and yield an exact recovery, with \(\text{SRP}_\infty\) being the most resilient, followed by \(\text{SRP}_2\) – indicating the advantage of a symmetrized objective. It is also evident that IRLS initialized by \(\text{SRP}_2\) (\(\text{SRP}_2+\text{IRLS}\)), together with plain \(\text{IRLS}\), perform similarly and yield superior recovery compared to all other methods. In the next subsection we will see an example where IRLS initialized by \(\text{SRP}_2\) (\(\text{SRP}_2+\text{IRLS}\)) outperforms plain \(\text{IRLS}\).
Lastly, while AA succeeds in finding an exact recovery in a low dimension, in a high dimension it breaks down even with a small number of outliers; this is since it depends on finding at least \( d + 1 \) inliers at random – an event whose probability decreases exponentially with \( d \).

4.2 Synthetic exeriments

Recovery under noise

Here we demonstrate the ability of our method to recover a rigid motion from noisy point correspondences contaminated by outliers. Similarly to the above, we generate 200 inliers with 2\% noise and add outliers. We evaluate different methods for their recovery of \( R \).

As in the noiseless case, the symmetrized methods \( \text{SRP}_2, \text{SRP}_\infty \) have an overall advantage over the non-symmetrized \( \text{NonSym} \), with a greater advantage to \( \text{SRP}_\infty \). Notably, \( \text{SRP}_2+\text{IRLS} \) and \( \text{SRP}_\infty+\text{IRLS} \) outperform plain \( \text{IRLS} \) for \( d = 7 \), whereas for \( d = 100 \) the three methods perform similarly. Thus there exist settings where our refined method outperforms plain IRLS. However, we do find that typically IRLS yields similar results whether initialized by our methods or uniformly.

Semi-supervised Procrustes

The following experiment demonstrates how incorporating the covariance energy \( \mathcal{E}_{\text{cov}}(R) \) described in Section 2.1 into our SRP method enables the recovery of an orthogonal matrix from a small number of point correspondences \( P, Q \), less than the dimension \( d \), given larger, unordered sets of points \( \tilde{P}, \tilde{Q} \) that are not matched in corresponding pairs. We eval-
Table 1: Semi-supervised learning of word translation

| Method | n = 250 |  | n = 500 |  | n = 1000 |  |
|--------|--------|--------|--------|--------|--------|--------|
| En-It  | En-De  | En-It  | En-De  | En-It  | En-De  |
| Procrustes | 15.79 ± 0.92 | 22.90 ± 1.23 | 36.79 ± 0.84 | 40.93 ± 0.90 | 52.62 ± 0.47 | 52.91 ± 0.54 |
| IRLS   | 16.60 ± 0.94 | 23.82 ± 0.97 | 38.65 ± 0.93 | 42.51 ± 0.77 | 54.13 ± 0.49 | 53.71 ± 0.53 |
| NonSym | 26.12 ± 1.03 | 28.91 ± 0.64 | 42.24 ± 0.76 | 41.62 ± 0.75 | 54.17 ± 0.50 | 51.77 ± 0.51 |
| Sym-2  | 29.47 ± 0.85† | 31.94 ± 0.59† | 44.25 ± 0.72† | 43.48 ± 0.70 | 55.08 ± 0.50 | 53.24 ± 0.55 |

Translation accuracy in percents ± 95%-confidence radius. English-Italian (En–It) and English-German (En–De). The maximum in each is highlighted; significant differences (p < 0.05) are marked by †.

We consider the task of translating words between two natural languages. For each language we are given 200,000 words embedded in \( \mathbb{R}^d, d = 300 \) by Word2Vec [27], and a dictionary with small number of correspondences between some of the words, varying between \( n = 200 \) and \( 1000 \). In [29] this problem was approached by minimizing the Procrustes objective with \( P, Q \) being the embedded corresponding words. This approach is now quite popular, and is often used as a subroutine by unsupervised algorithms, which iteratively construct a dictionary and then solve a Procrustes problem [3] [6] [30]. In this experiment we compare the performance of the standard Procrustes algorithm (Procrustes); the IRLS robust Procrustes algorithm (IRLS); and relax-and-project using a non-symmetrized semi-supervised translation

SRP square, adapted to minimize the robust Procrustes objective with the additional objective in this setting. Functional maps

Functional Maps (FM) [11] is a popular approach for computing non-rigid isometries between surfaces. This problem is visualized by the two surfaces (cats) in Figure 5: the ground-truth mapping between the surfaces in this problem preserves geodesic distances but not Euclidean distances. The FM approach reduces the non-rigid isometry problem in \( \mathbb{R}^3 \) to a rigid-motion problem in high dimension, and then seeks a correspondence using the standard Procrustes energy with an additional covariance penalty term. This formulation is optimized in [11] by dropping the orthogonality constraint and solving the non-symmetrized relaxation of this problem. This solution is then refined using the ICP algorithm, which does enforce the orthogonality constraint.

In Figure 5 we show that replacing the non-symmetrized relaxation used in [11] by our symmetrized formulation typically leads to improved results in terms of average geodesic error. Note that since this experiment does not include outliers, we only consider the squared objective as in (18). The setup for this experiment is described in Appendix B.1. 3D models were taken from TOSCA [26].

Semi-supervised translation

We consider the task of translating words between two natural languages. For each language we are given 200,000 words embedded in \( \mathbb{R}^d, d = 300 \) by Word2Vec [27], and a dictionary with small number of correspondences between some of the words, varying between \( n = 200 \) and \( 1000 \). In [29] this problem was approached by minimizing the Procrustes objective with \( P, Q \) being the embedded corresponding words. This approach is now quite popular, and is often used as a subroutine by unsupervised algorithms, which iteratively construct a dictionary and then solve a Procrustes problem [3] [6] [30]. In this experiment we compare the performance of the standard Procrustes algorithm (Procrustes); the IRLS robust Procrustes algorithm (IRLS); and relax-and-project using a non-symmetrized method.

\( \sum_{i=1}^{n} \left\| A p(i) + t - q(i) \right\|^2 + \lambda E_{\text{cov}}(A) \) based on the non-symmetrized relaxation (NonSym); and a relax-and-project method based on the squared variant of \( E_2(A) \),

\[
\min_{A \in \mathbb{R}^{d \times d}} \sum_{i=1}^{n} \frac{1}{2} \left( \left\| A p(i) - q(i) \right\|^2 + \left\| A^T q(i) - p(i) \right\|^2 \right) + \lambda E_{\text{cov}}^2(A),
\]

(SRP square). The results appear in Figure 4. Here the robust symmetrized methods SRP2, SRP∞ have a clear advantage over NonSym when \( d = 30 \) and 200. Moreover, for \( d = 30 \), SRP∞ is superior to SRP2, whereas both perform similarly at \( d = 200 \). The squared-symmetrized method SRP square breaks down even with a small number of outliers – indicating the importance of using a robust objective in this setting.

4.3 Applications

Functional maps

Functional Maps (FM) [11] is a popular approach for computing non-rigid isometries between surfaces. This problem is visualized by the two surfaces (cats) in Figure 5: the ground-truth mapping between the surfaces in this problem preserves geodesic distances but not Euclidean distances. The FM approach reduces the non-rigid isometry problem in \( \mathbb{R}^3 \) to a rigid-motion problem in high dimension, and then seeks a correspondence using the standard Procrustes energy with an additional covariance penalty term. This formulation is optimized in [11] by dropping the orthogonality constraint and solving the non-symmetrized relaxation of this problem. This solution is then refined using the ICP algorithm, which does enforce the orthogonality constraint.

In Figure 5 we show that replacing the non-symmetrized relaxation used in [11] by our symmetrized formulation (18) typically leads to improved results in terms of average geodesic error. Note that since this experiment does not include outliers, we only consider the squared objective as in (18). The setup for this experiment is described in Appendix B.1. 3D models were taken from TOSCA [26].

Semi-supervised translation

We consider the task of translating words between two natural languages. For each language we are given 200,000 words embedded in \( \mathbb{R}^d, d = 300 \) by Word2Vec [27], and a dictionary with small number of correspondences between some of the words, varying between \( n = 200 \) and \( 1000 \). In [29] this problem was approached by minimizing the Procrustes objective with \( P, Q \) being the embedded corresponding words. This approach is now quite popular, and is often used as a subroutine by unsupervised algorithms, which iteratively construct a dictionary and then solve a Procrustes problem [3] [6] [30].

In this experiment we compare the performance of the standard Procrustes algorithm (Procrustes); the IRLS robust Procrustes algorithm (IRLS); and relax-and-project using a non-symmetrized method.

†The data was downloaded from [28] [8] [27].
(NonSym) and symmetrized (SRP₂) relaxation. The results appear in Table 1. We see that for small values of \( n \), the methods that use the covariance energy (NonSym, SRP₂) have a significant advantage over the other methods, with our symmetrized method outperforming the non-symmetrized variant. As \( n \) increases, this gap diminishes, with a slight advantage to IRLS at \( n = 1000 \). Additional results appear in Appendix A and a full discussion of the experimental setup is given in Appendix B.1.

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A Additional numerical results

Here we present additional numerical results for the experiments described in Section 4. Figure 6 presents the recovery errors for $t$ in a noiseless setting. These results are qualitatively similar to those for $R$, shown in Figure 2, with the exception that in low dimension ($d = 3$) AA significantly outperforms the remaining methods in recovering $t$, whereas it performs similarly to IRLS and IRLS random in recovering $R$. Recovery errors are measured by $\|\hat{t} - t_0\|/\|t_0\|$.

Figure 7 shows the recovery errors for $t$ under noise, complementary to those for $R$ shown in Figure 3. While some of the differences between the methods are smaller here compared to the results for $R$, possibly indicating that recovering $t$ is easier than recovering $R$, these differences remain consistent.

Table 2 shows translation results for English-Spanish and English-Finnish. Plots for additional values of $n$, in the settings of Tables 1 and 2, appear in Figures 8 and 9 respectively.
Figure 8: Semi-supervised learning of word translations

Learning of word translations.

Figure 9: Semi-supervised learning of word translations

Learning of word translations.

Table 2: Semi-supervised learning of word translation

| Method | n = 250 | n = 500 | n = 1000 |
|--------|---------|---------|----------|
|        | En-Es   | En-Fi   | En-Es    | En-Fi    | En-Es    | En-Fi    |
| Procrustes | 12.87 ± 0.85 | 12.21 ± 0.86 | 31.64 ± 0.87 | 25.52 ± 0.93 | 47.75 ± 0.68 | 35.46 ± 0.39 |
| IRLS    | 13.62 ± 0.65 | 13.01 ± 0.99 | 33.17 ± 0.82 | 26.46 ± 0.79 | 49.45 ± 0.60 | 36.54 ± 0.48† |
| NonSym  | 15.08 ± 0.99 | 13.88 ± 0.81 | 31.21 ± 0.77 | 25.32 ± 0.73 | 46.20 ± 0.69 | 33.72 ± 0.28 |
| Sym-2   | **19.14 ± 1.00†** | **17.17 ± 1.04†** | **34.44 ± 0.82** | **27.20 ± 0.95** | **48.32 ± 0.69** | **35.59 ± 0.37** |

Translation accuracy in percents ± 95% confidence radius. English-Spanish (En-Es) and English-Finnish (En-Fi). The maximum in each is highlighted; significant differences (p < 0.05) are marked by †.
B Technical details

B.1 Experimental settings

Figures 1 to 3 We generated a random rotation matrix \( R_0 \in O(d) \) and translation vector \( t_0 \in \mathbb{R}^d \). We then generated 200 random pairs of inliers \( p^{(i)}, q^{(i)} \in \mathbb{R}^d \), related by \( q^{(i)} = R_0 p^{(i)} + t_0 \), to which noise was added as detailed below. We then add to \( P \) and \( Q \) a varying number of random outliers. All random vectors are drawn from scaled standard Gaussian distributions, with scaling parameters chosen such that for any index \( i \),

\[
E \left[ \| p^{(i)} \|^2 \right] = 1, \quad E \left[ \| q^{(i)} \|^2 \right] = \sigma_i^2, \quad E \left[ \| R_0 p^{(i)} + t - q^{(i)} \|^2 \right] = \sigma^2, \quad (19)
\]

and \( E \left[ \| q^{(i)} \|^2 \right] \) is equal for all \( i \), inlier or outlier. Thus, \( \sigma \) denotes the relative noise strength and \( \sigma_i \) denotes the proportion of \( t_0 \) compared to the points in \( P \). We used \( \sigma_i = 0.3 \) in all three figures. In the noiseless setting of Figure 2 we used \( \sigma = 0 \), and in Figures 1 and 3 we used \( \sigma = 0.02 \), making for 2% noise. In each setting we tested 200 independent instances and took the average result.

Random rotation matrices were drawn uniformly over \( O(d) \). Coordinates of random noise vectors \( \xi \) were drawn i.i.d. \( N(0, \frac{1}{\sqrt{d}}) \). When a nonzero translation vector \( t_0 \) was used, its coordinates were drawn i.i.d. \( N(0, \frac{1}{\sqrt{d}}) \). The coordinates of inlier and outlier \( p^{(i)} \) were drawn from \( N \left( 0, \frac{1}{\sqrt{d}} \right) \), and those of outlier \( q^{(i)} \) are \( N \left( 0, \frac{1}{\sqrt{d}} + \sigma_i^2 \right) \), thus satisfying (19). This also implies \( E \left[ \| q^{(i)} \|^2 \right] = 1 + \sigma_i^2 + \sigma^2 \) for any inlier or outlier \( i \).

Figure 4 Here we simulated a setting whereby a few point correspondences are chosen manually, possibly with some labelling errors, from larger, unordered point sets that are known to correspond. First, in the same manner as described above, we generated a random rotation \( R_0 \) with zero translation (\( \sigma_i = 0 \)). We then generated two point-sets of noisy corresponding inliers \( P, Q \) and \( \tilde{P}, \tilde{Q} \), of sizes \( n, \tilde{n} \) respectively. Outliers were created to simulate ‘mismatches’, caused by manual labelling; this was achieved by adding to \( P, Q \) a varying number \( \text{nout} \) of pairs \( p^{(j)}, q^{(j)} \) chosen randomly from \( \tilde{P}, \tilde{Q} \) respectively. We then reordered \( \tilde{Q} \) randomly so that \( \tilde{P}, \tilde{Q} \) are no longer ordered in corresponding pairs. We gave \( P, Q, \tilde{P}, \tilde{Q} \) as input to each of the evaluated methods.

In the left panel of Figure 4 we used \( d = 30, n = 16, \tilde{n} = 100 \) and \( \sigma = 0.01 \). In the right panel we used \( d = 200, n = 40, \tilde{n} = 400 \) and \( \sigma = 0.02 \).

The parameter \( \lambda \) was set to \( \lambda = \tilde{\lambda} \alpha \), with \( \tilde{\lambda} = 0.2 \), and \( \alpha \) being a balancing factor, used to balance the two terms in the objective. For \( \text{SRP}_2 \) we used \( \alpha = \sum_{i=1}^n \frac{1}{2} \sqrt{\| p^{(i)} \|^2 + \| q^{(i)} \|^2} / \max_{i,j \in [d]} | \sigma_i - \sigma_j | \), with \( \sigma, \tau \) denoting the singular values of \( \text{cov}(\tilde{P}) \), \( \text{cov}(\tilde{Q}) \) respectively. For \( \text{SRP}_\infty \) we used \( \alpha = \sum_{i=1}^n \max \left\{ \| p^{(i)} \|, \| q^{(i)} \| \right\} / \max_{i,j \in [d]} | \sigma_i - \sigma_j | \), for \( \text{NonSym} \) \( \alpha = \sum_{i=1}^n \| p^{(i)} \| / \max_{i,j \in [d]} | \sigma_i - \sigma_j | \) and for \( \text{SRP square} \) \( \alpha = \sum_{i=1}^n \frac{1}{2} \left( \| p^{(i)} \|^2 + \| q^{(i)} \|^2 \right) / \max_{i,j \in [d]} | \sigma_i - \sigma_j |^2 \).

Functional Maps For this experiment we used the original Matlab code provided by [11], and solved problems in \( d = 50 \) dimensions (corresponding to choosing the 50 smallest eigenvalues of the Laplace-Beltrami operator). We took eleven models of cats from the TOSCA dataset [26] and solved the one-sided and two-sided relaxations on all 55 possible pairings of these models.

Semi-supervised word translation For each of the languages English, Italian, German, Spanish and Finnish, we used a dataset of 200,000 word embeddings into \( \mathbb{R}^{300} \) by Word2Vec [27], along with dictionaries from English to each of the other languages, downloaded from [28,8]. In each problem instance we randomly chose between 200 and 1000 translated words from a training dictionary of size 1500, from which the point correspondences \( \tilde{P}, \tilde{Q} \) were created by taking the embedding of each
Table 3: Running times

| Figure | $d$ | SRP$_2$ | SRP$_\infty$ | IRLS |
|--------|-----|---------|--------------|------|
| Figure 1 left | 3 | 6.9e-3 | 1.9e-1 | 4.7e-3 |
| Figure 1 right | 100 | 2.1e-1 | 4.8 | 1.8e-1 |
| Figure 2 left | 3 | 1.5e-2 | 3.00e-1 | 1.0e-02 |
| Figure 2 right | 100 | 1.1 | 4.7 | 4.4e-1 |
| Figure 3 left | 7 | 6.8e-3 | 1.8e-1 | 7.9e-3 |
| Figure 3 right | 100 | 2.9e-1 | 5.1 | 2.6e-1 |
| Figure 4 left | 30 | 3.9e-2 | 3.1e-1 | - |
| Figure 4 right | 200 | 1.5 | 9.9 | - |

Average running times in seconds.

word in the source language $p^{(i)}$ and its embedded translation in the target language $q^{(i)}$. Together with these, we took the full set of 200,000 unmapped word embeddings from the source and target language, denoted $\tilde{P}, \tilde{Q}$ respectively. As a preprocessing step, $[[P, \tilde{P}]]$ and $[[Q, \tilde{Q}]]$ were centered, followed by point-normalization to the $\ell_2$ unit sphere.

The task was to learn a rigid motion $(R,t)$ such that given a new embedded word $p$ in the source language, $Rp + t$ should approximate the embedded translation $q$ of that word in the target language. English was used as the source language in all tests.

Evaluation was done on a separate test dictionary of 6000 words, using the Cross-domain Similarity Local Scaling (CSLS) measure of [6] with parameter $k = 10$, as used in [8]. The average result over 10 independent instances was taken, together with 95% confidence radii, estimated as two empirical standard errors of the mean:

$$\text{radius} = \frac{2}{\sqrt{10}} \sqrt{\frac{1}{10 - 1} \sum_{r=1}^{10} \left( \text{accuracy}_r - \frac{1}{10} \sum_{r=1}^{10} \text{accuracy}_r \right)^2}$$

For SRP$_2$, NonSym, we set $\lambda = \bar{\lambda} \alpha$ as above, and chose lambda for each method among $[0, 0.025, 0.05, 0.1, 0.2, 0.3, 0.4, 0.6]$ using 10-fold cross validation. For each method, the validation error for each $\bar{\lambda}$ was measured by the average error $\| \hat{R}p + \hat{t} - q \|$ of the solution obtained for that $\bar{\lambda}$. The range $[0.025, 0.6]$ was chosen based on a preliminary 7-fold, 2-instance cross-validation experiment using a wider range $[0.01, 0.8]$, in which none of the methods chose values below 0.025 or above 0.6.

We note that in the preliminary experiment mentioned above, the difference in performance between SRP$_\infty$ and SRP$_2$ was negligible, hence we omitted the slower SRP$_\infty$ from this experiment to save running time.

B.2 Implementation details

To minimize convex objectives $E_p(A,t,s)$ for $p = 2, \infty$, as well as the non-symmetrized relaxation of (17), we are using majorization-minimization schemes. Our code solves these problems significantly faster than CVXPY.

B.3 Resources and running times

**Hardware used** All our experiments were run on a Lenovo Legion 7-16ACH 82N600CXIV laptop computer with an AMD Ryzen 9 5900HX processor (8 cores, 3.30 to 4.60 GHz) and 32GB RAM. Our implemented solver for the robust Procrustes problem did not use a GPU.

**Software used** All experiments were run on Python 3.9.7.

**Running times** The running times of SRP$_2$ are comparable to those of IRLS, while SRP$_\infty$ is typically slower by an order of magnitude. Table 3 table summarizes the average running times of
our method together with those of IRLS for Figures 1 to 4. The times listed are the average times corresponding to the worst point of each graph. The running times for the functional maps test are around two seconds for our methods and less than a second for the non-symmetrized variant.

C Derivation of covariance energy

Here we provide the derivation of the covariance energy discussed in 2.1. Suppose that $S, \tilde{S}$ are two surfaces in $\mathbb{R}^d$ such that $\tilde{S} = R \circ S$ for some orthogonal matrix $R$. Let $p, q$ be two random vectors uniformly distributed on $S, \tilde{S}$ respectively. Then,

$$R \cdot \mathbb{E}[pp^T] = \mathbb{E}[Rpp^TR^T] R = \mathbb{E}[(Rp)(Rp)^T] R = \mathbb{E}[qq^T] R,$$

with the last equality holding since $R$ is an orthogonal bijection from $S$ to $\tilde{S}$, and thus $Rp$ has the same distribution as $q$. Thus, under these assumptions, equation (5) is satisfied. Since in most applications the true matrices $\mathbb{E}[pp^T], \mathbb{E}[qq^T]$ are not known, we use instead their empirical estimates $\text{cov}(\hat{P}) = \frac{1}{n_\hat{P}} \hat{P} \hat{P}^T, \text{cov}(\hat{Q}) = \frac{1}{n_\hat{Q}} \hat{Q} \hat{Q}^T$.

We note that if the points in $\hat{P}, \hat{Q}$ are related by an orthogonal transformation $R$ followed by a permutation $\Pi$, then equation (5) holds also with the true covariances replaced by their empirical estimates, namely

$$R\text{cov}(\hat{P}) = \text{cov}(\hat{Q})R. \tag{20}$$

To see this, suppose that $\hat{P}, \hat{Q} \in \mathbb{R}^{d \times \tilde{n}}$ and that for all $i \in [\tilde{n}]$, $\hat{q}^{(i)} = R\tilde{p}^{(\Pi(i))}$. Then, $\tilde{Q} = R\tilde{P}\Pi$ and

$$\frac{1}{n} \hat{Q} \hat{Q}^T = \frac{1}{n} (R\tilde{P}\Pi) (R\tilde{P}\Pi)^T = \frac{1}{n} (R\tilde{P}\Pi) (\Pi^T \tilde{P}^T R^T) R = R \left( \frac{1}{n} \tilde{P} \tilde{P}^T \right),$$

and thus (20) is satisfied.

D Additional theoretical results

In this section we state theoretical results omitted from the main text.

D.1 Approximation guarantees

The approximation guarantee for the orthogonal case, stated in Theorem 3.1, is based on bounding the ratio $\text{E}(\Pi(A^*)) / \text{E}_p(A^*)$ for $p \geq 2$ by $2$ or $\sqrt{2}$, assuming that $A^*$ is optimal for $\text{E}_p(A)$. The following lemma shows that the optimality assumption is not necessary in order to bound this ratio by 2. Proofs are in Appendix E.

**Lemma D.1.** For any $A \in \mathbb{R}^{d \times d}$, $p \geq 2$,

$$\text{E}(\Pi(A)) \leq 2\text{E}_p(A). \tag{21}$$

**Optimality of approximation factors** We now discuss the optimality of the approximation factors of Section 3.1. First, the following lemma shows that the factors of Theorem 3.1 and Lemma D.1 cannot be improved.

**Lemma D.2.** For any $d \geq 1$ and $p \in [1, \infty]$,

1. There exist $P, Q \in \mathbb{R}^{d \times n}$ and $A \in \mathbb{R}^{d \times d}$ such that

$$\text{E}(\Pi(A)) = 2\text{E}_p(A) > 0.$$

2. There exist $P, Q \in \mathbb{R}^{d \times n}$ and $A \in \mathbb{R}^{d \times d}$ such that $A$ is optimal for $\text{E}_p$, and

$$\text{E}(\Pi(A)) = \hat{\text{E}}_p(A) > 0.$$

In particular, Lemma D.2 shows that the approximation factor of $E_p$ with $p < 2$ is at best $2^{1/p}$, which is inferior to the factor $\sqrt{2}$ of $E_2$. We do not know whether a $\sqrt{2}$-factor approximation guarantee holds for $p > 2$. However, the factors observed in practice are often much smaller.
We now show that for the orthogonal problem $\ell_2$, $\sqrt{2}$-factor approximation in the best achievable by any relax-and-project method; namely, any algorithm for (3) that returns the projected minimizer $\Pi(A^*)$ of a convex objective $G(A \mid P, Q)$ cannot achieve a better universal approximation factor than $\sqrt{2}$. To this end, we make the following definition.

**Definition.** A point mismatch function is a real function $G(A \mid P, Q)$, defined for any $P, Q \in \mathbb{R}^{d \times n}$, $n \geq 1$ and $A \in \mathbb{R}^{d \times d}$, such that

1. $G(A \mid P, Q)$ is convex as a function of $A$.
2. $G$ is invariant to the order of the points. Namely, for any permutation matrix $\Pi \in \mathbb{R}^{n \times n}$,
   
   
   $G(A \mid P \Pi, Q \Pi) = G(A \mid P, Q)$.
3. $G$ is invariant to global reflections:

   
   $G(-A \mid -P, Q) = G(-A \mid P, -Q) = G(A \mid P, Q)$.

Note that assumption 3 is satisfied by any function that is based on metrics of the form

\[ d(A p^{(i)}, q^{(i)}), \quad d(p^{(i)}, A^T q^{(i)}), \quad i = 1, \ldots, d \]

such that $d(-x, -y) = d(x, y)$.

By the following theorem, there exists a problem instance $P, Q$ at which any point mismatch function $G(\cdot \mid P, Q)$ has a minimizer $A^*$ whose best orthogonal approximation $\Pi(A^*)$ is $\sqrt{2}$-suboptimal in Problem (3).

**Theorem D.3.** For $d \geq 2$ there exist points $P, Q \in \mathbb{R}^{d \times n}$ such that for any point mismatch function $G(A \mid P, Q)$,

1. The zero matrix $A^* = 0_d$ is a minimizer of $G(\cdot \mid P, Q)$.
2. For $R = \Pi(A^*)$,

   
   $E(R) = \sqrt{2} \min_{R \in O(d)} E(R) > 0$.

**D.2 Recovery guarantees**

As the following theorem shows, the linear DIP condition of Theorem 4.3 is optimal, in the sense that it cannot be improved by a multiplicative constant and still guarantee exact, or even approximate, recovery of an orthogonal transformation.

**Theorem D.4.** For any $d \geq 1$ and $\varepsilon \in (0, 1)$ there exist a rotation matrix $R_0 \in SO(d)$, points $P, Q \in \mathbb{R}^{d \times n}$ and index set $I \subseteq [n]$ such that:

1. $q^{(i)} = R_0 p^{(i)}$ for $i \in I$.
2. For any unit vector $u \in \mathbb{R}^d$,

   
   \[ \sum_{i \in I} \left| \langle u, p^{(i)} \rangle \right| \geq (1 - \varepsilon) \sum_{i \in I} \left| \langle u, p^{(i)} \rangle \right| \quad \text{and} \quad \sum_{i \in I} \left| \langle u, q^{(i)} \rangle \right| \geq (1 - \varepsilon) \sum_{i \in I} \left| \langle u, q^{(i)} \rangle \right|. \] \hspace{1cm} \(22\)

3. There exists a rotation matrix $R_1$ that is the unique global minimizer of $E_p$ for any $p \geq 1$, and $\|R_1 - R_0\|_F = \sqrt{2}d$, with $\|\cdot\|_F$ denoting the Frobenius norm.

Note that Theorem D.4 implies that even when optimizing over the set of orthogonal matrices or rotation matrices, a weaker condition than the linear DIP cannot guarantee successful recovery.

**E Proofs**

**E.1 Approximation guarantees**

We start by proving Lemma D.1.
Lemma D.1. For any $A \in \mathbb{R}^{d \times d}$, $p \geq 2$,
\[ E(\Pi(A)) \leq 2E_p(A). \quad (21) \]

**Proof of Lemma D.1.** It is enough to prove the lemma for $p = 2$, since $E_p(A)$ increases monotonically with $p$. Let $A = U\Sigma V^T$ be an SVD of $A$ such that $\Pi(A) = UV^T$. Let $a = (a_k)_{k=1}^d$ be the vector in $\mathbb{R}^d$ consisting of the main diagonal entries of $\Sigma$, namely $a_k = \Sigma_{kk}$ for $k = 1, \ldots, d$. Denote
\[ u^{(i)} = V^T p^{(i)}, \quad v^{(i)} = U^T q^{(i)}, \quad i = 1, \ldots, n. \quad (23) \]
Then,
\[ E_2(A) = \frac{1}{\sqrt{2}} \sum_{i=1}^n \sqrt{\|Ap^{(i)} - q^{(i)}\|^2 + \|A^T q^{(i)} - p^{(i)}\|^2} \]
\[ = \frac{1}{\sqrt{2}} \sum_{i=1}^n \sqrt{\|U\Sigma V^T p^{(i)} - q^{(i)}\|^2 + \|V\Sigma U^T q^{(i)} - p^{(i)}\|^2} \]
\[ = \frac{1}{\sqrt{2}} \sum_{i=1}^n \sqrt{\|\Sigma(v^{(i)}) - U^T q^{(i)}\|^2 + \|\Sigma U^T q^{(i)} - V^T p^{(i)}\|^2} \]
\[ = \frac{1}{\sqrt{2}} \sum_{i=1}^n \sqrt{\|\Sigma u^{(i)} - v^{(i)}\|^2 + \|\Sigma v^{(i)} - u^{(i)}\|^2} \]
where $\odot$ denotes the Hadamard (entrywise) product. By a similar derivation it can be shown that
\[ E(\Pi(A)) = \sum_{i=1}^n \|u^{(i)} - v^{(i)}\|. \quad (25) \]
By (24) and (25) it is enough to show that
\[ \frac{1}{\sqrt{2}} \sum_{i=1}^n \sqrt{\|a \odot u^{(i)} - v^{(i)}\|^2 + \|a \odot v^{(i)} - u^{(i)}\|^2} \geq \frac{1}{\sqrt{2}} \sum_{i=1}^n \|u^{(i)} - v^{(i)}\|. \quad (26) \]
We shall show that the inequality (26) holds for each summand individually; namely, that
\[ \frac{1}{\sqrt{2}} \sqrt{\|a \odot u^{(i)} - v^{(i)}\|^2 + \|a \odot v^{(i)} - u^{(i)}\|^2} \geq \frac{1}{2} \|u^{(i)} - v^{(i)}\| \]
for each $i \in [n]$. Since the coordinates of $a$ are the singular values of $A$, they are nonnegative. Hence, it is enough to prove that for any $x \in \mathbb{R}^d$ with nonnegative entries, and any $u, v \in \mathbb{R}^d$,
\[ \frac{1}{\sqrt{2}} \sqrt{\|x \odot u - v\|^2 + \|x \odot v - u\|^2} \geq \frac{1}{2} \|u - v\|. \]
Taking the square, we need to show that
\[ 2 \left(\|x \odot u - v\|^2 + \|x \odot v - u\|^2\right) \geq \|u - v\|^2. \quad (27) \]
Both sides in (27) are separable sums over the $d$ coordinates of $x, u, v$. Hence, it is enough to prove (27) for the case $d = 1$. Let $u, v \in \mathbb{R}$ be two scalars. Let $g(x) : \mathbb{R} \to \mathbb{R}$ be given by
\[ g(x) = 2 \left((ux - v)^2 + (vx - u)^2\right) - (u - v)^2. \quad (28) \]
We shall now show that $g(x) \geq 0$ for any $x \geq 0$. Rearranging terms in (28), we get
\[ g(x) = (u^2 + v^2)(1 + 2x^2) + (1 - 4x)2uv. \quad (29) \]
If $uv \geq 0$, then using the inequality $u^2 + v^2 \geq 2uv$ on (29), we get
\[ g(x) \geq (2uv)(1 + 2x^2) + (1 - 4x)2uv \]
\[ = 2uv(2x^2 - 4x + 2) = 4uv(x - 1)^2 \geq 0. \]
On the other hand, if $uv < 0$, consider the following expression of $g(x)$, equivalent to (29).

\[ g(x) = 2(u^2 + v^2)x^2 - (8uv)x + (u + v)^2. \]

If $u^2 + v^2 = 0$, then $g$ is identically zero. Otherwise, by Vieta’s formula $g$ is minimized at $x^* = \frac{-2uv}{u^2 + v^2}$, which is negative by assumption. Thus, since $g$ is convex, it attains its minimum over $[0, \infty)$ at $x = 0$. Therefore, for any $x \geq 0$,

\[ g(x) \geq g(0) = (u + v)^2 \geq 0, \]

which concludes the proof of the lemma.

We now prove Theorem 3.1. Lemma E.1 will be used to prove the theorem in the case $p > 2$.

**Theorem 3.1.** For $P, Q \in \mathbb{R}^{d \times n}$ and $p \in [2, \infty]$, let $A^*$ be a minimizer of $E_p(A)$, and let $\hat{R} = \Pi(A^*)$. Let $E^*$ be the optimal objective of the orthogonal robust Procrustes problem (5). Then

\[ E^* \leq E(\hat{R}) \leq 2E_p(A^*) \leq 2E^*. \tag{11} \]

Moreover, if $p = 2$ then

\[ E^* \leq E(\hat{R}) \leq \sqrt{2}E_p(A^*) \leq \sqrt{2}E^*. \tag{12} \]

**Proof of Theorem 3.1.** The inequality $E^* \leq E(\hat{R})$ is by definition of $E^*$, and $E_p(A^*)$ is a lower bound on $E^*$ as discussed in Section 2. Since $\hat{R} = \Pi(A^*)$, from Lemma D.1 with $A = A^*$ we have $E(\hat{R}) \leq 2E_p(A^*)$. Hence, the first part of the theorem is proven.

To prove the second part, we state the following lemma, to be proven below.

**Lemma E.1.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be given by

\[ f(x) = \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \sqrt{||x \odot u^{(i)} - v^{(i)}||^2 + ||x \odot v^{(i)} - u^{(i)}||^2}. \tag{30} \]

Suppose that all coordinates of $a$ are nonnegative, and that $a$ is a global minimizer of $f$. Let $1 \in \mathbb{R}^d$ be the vector whose entries all equal 1. Then

\[ f(1) \leq \sqrt{2}f(a). \]

The second part of the theorem follows from Lemma E.1. To see this, let $u^{(i)}, v^{(i)}$ be as in (23). By the proof of Lemma D.1 specifically (24) and (25), $f(a) = E_2(A)$ and $f(1) = E_2(\Pi(A))$. Since $A$ is optimal for $E_2$, $a$ is optimal for $f$. Otherwise, suppose by contradiction there exists $\bar{a} \in \mathbb{R}^d$ such that $f(\bar{a}) < f(a)$. Let $\bar{A} = U \text{diag}(\bar{a}) V^T$. It can be verified in (23) and (30) that $E_2(\bar{A}) = f(\bar{a})$, and thus $E_2(\bar{A}) = f(\bar{a}) < f(a) = E_2(A)$, contradicting the optimality of $A$. Thus, by Lemma E.1

\[ E_2(A) = f(a) \geq \frac{1}{\sqrt{2}} f(1) = \frac{1}{\sqrt{2}} f(\Pi(A)). \]

Let us now prove Lemma E.1.

**Proof.** To prove the lemma, we seek an upper bound on $f(1)$. To this end, define for $i \in [n], x \in \mathbb{R}^d$,

\[ \alpha_i(x) = \left\| \frac{x \odot u^{(i)} - v^{(i)}}{x \odot v^{(i)} - u^{(i)}} \right\|^2 = \sqrt{\sum_{k=1}^{d} \left( x_k u^{(i)} - v^{(i)} \right)^2 + \left( x_k v^{(i)} - u^{(i)} \right)^2}. \tag{31} \]

Using this definition, we have

\[ f(x) = \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \alpha_i(x). \tag{32} \]

The following proposition shall come in handy when dealing with terms $\alpha_i(a)$ that equal zero.
Proposition E.2. Let \(a, u, v \in \mathbb{R}^d\) such that
\[
\|a \odot u - v\|^2 + \|a \odot v - u\|^2 = 0 \tag{33}
\]
and suppose that all coordinates of \(a\) are nonnegative. Then \(u = v\). Moreover, for any \(k \in [d]\), if \(u_k^2 + v_k^2 > 0\), then \(a_k = 1\).

Proof. Let \(k \in [d]\). Then by (33),
\[
a_ku_k - v_k = 0, \quad a_kv_k - u_k = 0, \tag{34}
\]
and thus
\[
a_k(u_k - v_k) = v_k - u_k. \tag{35}
\]
Suppose by contradiction that \(u_k \neq v_k\). Then, by (35), \(a_k = -1\), which is a contradiction. Therefore, \(u = v\). Now, if \(u_k^2 + v_k^2 > 0\), suppose W.L.O.G. that \(u_k \neq 0\). Then by (34),
\[
a_ku_k = v_k = u_k
\]
and thus \(a_k = 1\). □

Define \(\Lambda \subseteq [n]\) to be the index set
\[
\Lambda = \{i \in [n] \mid \alpha_i(a) \neq 0\}.
\]

It follows from Proposition E.2 that if \(i \in [n] \setminus \Lambda\), then \(u^{(i)} = v^{(i)}\). Therefore,
\[
f(1) = \sum_{i=1}^n \|u^{(i)} - v^{(i)}\| = \sum_{i \in \Lambda} \|u^{(i)} - v^{(i)}\| = \sum_{i \in \Lambda} \sqrt{\alpha_i(a)^2 + \|u^{(i)} - v^{(i)}\|^2 - \alpha_i(a)^2}
\]
\[
= \sum_{i \in \Lambda} \alpha_i(a) \sqrt{1 + \frac{\|u^{(i)} - v^{(i)}\|^2 - \alpha_i(a)^2}{\alpha_i(a)^2}}. \tag{36}
\]

According to Bernoulli’s inequality, for any real number \(x \geq -1\),
\[
\sqrt{1 + x} \leq 1 + \frac{1}{2}x. \tag{37}
\]

Inserting (37) with \(x = \frac{\|u^{(i)} - v^{(i)}\|^2 - \alpha_i(a)^2}{\alpha_i(a)^2}\) into (36) yields
\[
f(1) \leq \sum_{i \in \Lambda} \alpha_i(a) \left(1 + \frac{1}{2} \frac{\|u^{(i)} - v^{(i)}\|^2 - \alpha_i(a)^2}{\alpha_i(a)^2}\right).
\]
\[
= \sum_{i \in \Lambda} \alpha_i(a) + \frac{1}{2} \sum_{i \in \Lambda} \frac{\|u^{(i)} - v^{(i)}\|^2 - \alpha_i(a)^2}{\alpha_i(a)}.
\]
\[
= \sqrt{2f(a)} + \frac{1}{2} \sum_{i \in \Lambda} \frac{\|u^{(i)} - v^{(i)}\|^2 - \alpha_i(a)^2}{\alpha_i(a)},
\]
where the last equality is by (32). Expanding the above by the definition of \(\alpha_i(a)\) in (31), we get
\[
f(1) \leq \sqrt{2f(a)} + \frac{1}{2} \sum_{i \in \Lambda} \sum_{k=1}^d \frac{-a_k^2 \left(u_k^{(i)} v_k^{(i)} \right) + (4a_k - 2)u_k^{(i)} v_k^{(i)}}{\alpha_i(a)}.
\]

To prove the lemma, we shall show that
\[
\sum_{i \in \Lambda} \sum_{k=1}^d \frac{-a_k^2 \left(u_k^{(i)} v_k^{(i)} \right) + (4a_k - 2)u_k^{(i)} v_k^{(i)}}{\alpha_i(a)} \leq 0. \tag{38}
\]
To prove (38), we wish to express the directional derivatives of \( f(x) \). Recall that for a function \( g : \mathbb{R}^d \to \mathbb{R} \), the directional derivative of \( g(x) \) at \( x = x_0 \) in the direction \( w \in \mathbb{R}^d \) is defined as the limit

\[
\nabla_w g(x_0) = \lim_{t \to 0^+} \frac{g(x_0 + tw) - g(x_0)}{t}.
\]

Using elementary calculus, it can be shown that for any \( i \) with \( \alpha_i(a) \neq 0 \), and any vector \( w \in \mathbb{R}^d \), the directional derivative of \( \alpha_i(a) \) in the direction \( w \) is given by

\[
\nabla_w \alpha_i(a) = \left[ a \odot u^{(i)} - v^{(i)} \right] \sum_{k=1}^d \left[ a \odot u^{(i)} - v^{(i)} \right]^T \left[ \text{diag}\left(u^{(i)}\right) \right] w = \left[ a \odot u^{(i)} - v^{(i)} \right] \sum_{k=1}^d \left[ a \odot u^{(i)} - v^{(i)} \right]^T \left[ \text{diag}\left(v^{(i)}\right) \right] w = \sum_{k=1}^d \alpha_k w_k \left( u_k^{(i)} - v_k^{(i)} \right)^2 - 2 \alpha_k w_k v_k^{(i)}.
\]

To prove the theorem, we consider the directional derivative of \( R \) in the direction \( a - 1 \). Setting \( w = a - 1 \) in the above equation yields that

\[
\nabla_{a - 1} \alpha_i(a) = \sum_{k=1}^d \left( a_k^2 - a_k \right) \left( u_k^{(i)} - v_k^{(i)} \right)^2 + (2 - 2a_k) u_k^{(i)} v_k^{(i)}.
\]

Let us now calculate \( \nabla_{a - 1} \alpha_i(a) \) for \( i \in [n] \setminus \Lambda \). Suppose that \( \alpha_i(a) = 0 \). Then

\[
\nabla_{a - 1} \alpha_i(a) = \lim_{t \to 0^+} \frac{\alpha_i(a + t(a - 1)) - \alpha_i(a)}{t} = \lim_{t \to 0^+} \left[ \frac{\left[ a \odot u^{(i)} - v^{(i)} \right] + t(a - 1) \odot u^{(i)}}{t} \right] = \left[ \frac{a \odot u^{(i)} - v^{(i)} + t(a - 1) \odot u^{(i)}}{t} \right] = \left[ \frac{t(a - 1) \odot u^{(i)}}{t} \right] = \left[ t(a - 1) \odot u^{(i)} \right] = 0.
\]

By Proposition E.2, for any \( i \in [n] \) such that \( \alpha_i(a) = 0 \), and any \( k \in [d] \), \( a_k = 1 \) or \( u_k^{(i)} = v_k^{(i)} = 0 \). Therefore, by (40)

\[
\nabla_{a - 1} \alpha_i(a) = 0, \quad i \in [n] \setminus \Lambda.
\]

In conclusion, from (39) and (41) it follows that

\[
\nabla_{a - 1} f(a) = \frac{1}{\sqrt{2}} \sum_{i=1}^n \nabla_{a - 1} \alpha_i(a) = \frac{1}{\sqrt{2}} \sum_{i=1}^n \frac{\left( a_k^2 - a_k \right) \left( u_k^{(i)} - v_k^{(i)} \right)^2 + (2 - 2a_k) u_k^{(i)} v_k^{(i)}}{\alpha_i(a)}.
\]

Recall that \( a \) is optimal for \( f \) by assumption. Therefore, all directional derivatives of \( f(x) \) at \( x = a \) are nonnegative. In particular,

\[
\nabla_{a - 1} f(a) \geq 0.
\]

Inserting (43) into (42) and rearranging terms yields

\[
\sum_{i \in \Lambda} \sum_{k=1}^d \frac{-a_k \left( u_k^{(i)} - v_k^{(i)} \right)^2}{\alpha_i(a)} \leq \sum_{i \in \Lambda} \sum_{k=1}^d \frac{-a_k \left( u_k^{(i)} - v_k^{(i)} \right)^2}{\alpha_i(a)} = \sum_{i \in \Lambda} \sum_{k=1}^d \frac{-a_k \left( u_k^{(i)} - v_k^{(i)} \right)^2}{\alpha_i(a)} \leq 0,
\]

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Recall that \( \hat{\mathbf{v}} \). Therefore, from (45) with bound (14) for \( p \).

Namely, \( F \) with (13) for a general \( E \). We first prove (13) for a general \( E \).



We shall now prove Theorem 3.2.

**Theorem 3.2.** For \( P, Q \in \mathbb{R}^{d \times n} \) and \( p \in [2, \infty] \), let \( (A^*, t^*, s^*) \) be a minimizer of \( \mathbf{E}_p(A, t, s) \). Let \( (\hat{R}, \hat{t}) \) be the output of Algorithm 2, and let \( \mathbf{E}^* \) be the optimal objective of Problem (1). Then

\[
\mathbf{E}^* \leq \mathbf{E}(\hat{R}, \hat{t}) \leq 2\sqrt{2} \mathbf{E}_p(A^*, t^*, s^*) \leq 2\sqrt{2} \mathbf{E}^*.
\]

Moreover, if \( p = 2 \) then

\[
\mathbf{E}^* \leq \mathbf{E}(\hat{R}, \hat{t}) \leq \sqrt{2} \mathbf{E}_2(A^*, t^*, s^*) \leq \sqrt{2} \mathbf{E}^*.
\]

**Proof of Theorem 3.2.** We first prove (13) for a general \( p \in [2, \infty] \) and then prove the improved bound (14) for \( p = 2 \).

Define the translated objective \( \mathbf{F}_p(A, p(0), q(0)) \) for \( A \in \mathbb{R}^{d \times d}, p(0), q(0) \in \mathbb{R}^d \) by

\[
\mathbf{F}_p(A, p(0), q(0)) = \sum_{i=1}^{n} \left( \left\| A(p^{(i)} - p(0)) - (q^{(i)} - q(0)) \right\|_p + \left\| A^T(q^{(i)} - q(0)) - (p^{(i)} - p(0)) \right\|_p \right)^{\frac{1}{p}}. \tag{44}
\]

Namely, \( \mathbf{F}_p(A, p(0), q(0)) \) is similar to the objective \( \mathbf{E}_p(A) \), with the points \( P, Q \) translated by \(-p(0), -q(0)\) respectively. Then for any \( p \in [1, \infty] \),

\[
\mathbf{E}(\hat{R}, \hat{t}) \overset{(a)}{=} \min_{t \in \mathbb{R}^d} \mathbf{E}(\hat{R}, t) = \min_{t \in \mathbb{R}^d} \sum_{j=1}^{n} \left\| \hat{R}p^{(j)} - q^{(j)} + t \right\|_p \overset{(b)}{\leq} \min_{t \in [n]} \sum_{j=1}^{n} \left\| \hat{R}p^{(j)} - q^{(j)} - (\hat{R}p^{(j)} - q^{(j)}) \right\|_p \leq \min_{t \in [n]} \sum_{j=1}^{n} \left\| \hat{R}(p^{(j)} - p^{(i)}) - (q^{(j)} - q^{(i)}) \right\|_p = \min_{t \in [n]} \mathbf{F}_p(\hat{R}, p^{(j)}, q^{(i)}) \tag{45},
\]

where (a) is by the definition of \( \hat{t} \), and (b) can be seen by taking \( t = -\left( \hat{R}p^{(i)} - q^{(i)} \right) \).

Recall that \( \hat{R} = \Pi(A^*) \). Invoking Theorem \ref{thm:5.1} for each \( i \in [n] \) on the translated objective \( \mathbf{F}_p \left( \cdot, p^{(j)}, q^{(i)} \right) \) with \( p = 2 \) yields

\[
\mathbf{F}_2(\hat{R}, p^{(j)}, q^{(i)}) \leq 2 \mathbf{F}_2(A^*, p^{(j)}, q^{(i)}), \quad i \in [d]. \tag{46}
\]

Therefore, from (45) with \( p = 2 \) and (46), we have

\[
\mathbf{E}(\hat{R}, \hat{t}) \leq 2 \min_{t \in [n]} \mathbf{F}_2(A^*, p^{(j)}, q^{(i)}) = 2 \min_{t \in [n]} \sum_{j=1}^{n} \frac{1}{\sqrt{2}} \left\| A^* \left( p^{(j)} - p^{(i)} \right) - (q^{(j)} - q^{(i)}) \right\| \overset{(a)}{=} 2 \min_{t \in [n]} \sum_{j=1}^{n} \frac{1}{\sqrt{2}} \left\| A^*p^{(j)} - q^{(j)} \right\| - \left( A^*p^{(i)} - q^{(i)} \right) = 2 \min_{t \in [n]} \sum_{j=1}^{n} \frac{1}{\sqrt{2}} \left\| A^*q^{(j)} - p^{(j)} \right\| - \left( A^*q^{(i)} - p^{(i)} \right) \tag{47},
\]

where (a) is by the definition of \( \hat{t} \), and (b) can be seen by taking \( t = -\left( \hat{R}p^{(i)} - q^{(i)} \right) \).
where (a) is by a reformulation of (44) with \( p = 2 \).

Let us now state a lemma, to be proven below.

**Lemma E.3.** Let \( \left( x^{(i)} \right)_{i=1}^{n} \) be points in \( \mathbb{R}^d \). Then

\[
\min_{i \in [n]} \sum_{j=1}^{n} \left\| x^{(j)} - x^{(i)} \right\| \leq \sqrt{2} \min_{v \in \mathbb{R}^d} \sum_{j=1}^{n} \left\| x^{(j)} - v \right\|. \tag{48}
\]

Using Lemma E.3 with \( x^{(i)} = \left[ A^* p^{(i)} - q^{(i)} \right] \), \( i \in [n] \) and \( v = - \left[ \frac{t}{s} \right] \) yields

\[
\min_{i \in [n]} \sum_{j=1}^{n} \frac{1}{\sqrt{2}} \left\| A^* p^{(j)} - q^{(j)} \right\| - \left\| A^* p^{(i)} - q^{(i)} \right\| 
\leq \sqrt{2} \min_{t,s \in \mathbb{R}^d} \sum_{j=1}^{n} \frac{1}{\sqrt{2}} \left\| A^* q^{(j)} - p^{(j)} \right\| + \left\| t \right\| \tag{49}
\]

\[
\leq \sqrt{2} \min_{t,s \in \mathbb{R}^d} E_2(A^*, t, s),
\]

where (a) is by a reformulation of (6). Hence, by (47) and (49),

\[
E(\hat{R}, \hat{t}) \leq 2\sqrt{2} \min_{t,s \in \mathbb{R}^d} E_2(A^*, t, s)
\]

\[
\leq 2\sqrt{2} \min_{t,s \in \mathbb{R}^d} E_p(A^*, t, s)
\]

\[
\leq 2\sqrt{2}E_p(A^*, t^*, s^*)
\]

\[
\leq 2\sqrt{2} \min_{A \in \mathbb{R}^{d \times d}} \min_{t,s \in \mathbb{R}^d} E_p(A, t, s)
\]

\[
\leq 2\sqrt{2} \min_{R \in O(d)} \min_{t \in \mathbb{R}^d} E(R, t, -R^T t)
\]

\[
\leq 2\sqrt{2} \min_{R \in O(d)} \min_{t \in \mathbb{R}^d} E(R, t) = 2\sqrt{2}E^*,
\tag{50}
\]

where (a) is since \( p \geq 2 \); (b), (c) are by the definition of \( (A^*, t^*, s^*) \); and (d) is by the identity (7).

This concludes the proof of (43) for \( p \in [2, \infty) \).

We shall now prove the improved bound (43) for \( p = 2 \). The following lemma shows that if \( (A^*, t^*, s^*) \) is a minimizer of \( E_2 \), then \( A^* \) can be completed to a minimizer \( (A^*, p^*, q^*) \) of \( F_2 \).

**Lemma E.4.** Let \( (A^*, t^*, s^*) \) be a minimizer of \( E_2(A, t, s) \). Then there exist \( p^*, q^* \in \mathbb{R}^d \) such that

\[
E_2(A^*, t^*, s^*) = F_2(A^*, p^*, q^*) = \min_{A \in \mathbb{R}^{d \times d}} \min_{p^{(0)}, q^{(0)} \in \mathbb{R}^d} F_2(A, p^{(0)}, q^{(0)}).
\]

The lemma is proven below. We continue the proof of the theorem.

Let \( (A^*, t^*, s^*) \) be a minimizer of \( E_2(A, t, s) \), and let \( p^*, q^* \) be as in Lemma E.4. Then

\[
E^* = \min_{R \in O(d)} \min_{t \in \mathbb{R}^d} E(R, t) = \min_{R \in O(d)} \min_{t \in \mathbb{R}^d} E_2(R, t, -R^T t) \geq \min_{A \in \mathbb{R}^{d \times d}} \min_{t \in \mathbb{R}^d} E_2(A, t, s)
\]

\[
= E_2(A^*, t^*, s^*) = F_2(A^*, p^*, q^*).
\tag{51}
\]

Since \( A^* \) is optimal for the translated objective \( F_2(\cdot, p^*, q^*) \), Lemma D.1 implies that

\[
F_2(\hat{R}, p^*, q^*) \leq \sqrt{2}F_2(A^*, p^*, q^*).
\]
Inserting the above inequality to (51) yields
\[ E^* \geq \frac{1}{\sqrt{2}} F_2(\hat{R}, p^*, q^*). \] (52)

Note that
\[ F_2(\hat{R}, p^*, q^*) = \sum_{i=1}^{n} \left\| \hat{R}(p^{(i)} - p^*) - (q^{(i)} - q^*) \right\| \]
\[ = \sum_{i=1}^{n} \left\| \hat{R}p^{(i)} - q^{(i)} - (\hat{R}p^* - q^*) \right\| \]
\[ \geq \min_{\hat{R} \in \mathbb{R}^d} \sum_{i=1}^{n} \left\| \hat{R}p^{(i)} - q^{(i)} + \hat{R} \right\| \]
\[ = \sum_{i=1}^{n} \left\| \hat{R}p^{(i)} - q^{(i)} + \hat{R} \right\| \]
\[ = E(\hat{R}, \hat{R}). \]

Combining (52) and (53) yields
\[ E(\hat{R}, \hat{R}) \leq \sqrt{2} E^* , \]
which concludes the proof of (14) for \( p = 2 \).

To complete the proof of the theorem, we shall now prove Lemmas 5.3 and 5.4

Proof of Lemma 5.3 Let \( v^* \in \mathbb{R}^d \) be a minimizer of
\[ g(v) = \sum_{j=1}^{n} \left\| x^{(j)} - v \right\|. \] (54)

If there exist \( i \in [n] \) for which \( v^* = x^{(i)} \), then the claim of the lemma clearly holds. Otherwise, \( g(v) \) is differentiable at \( v^* \), and since \( v^* \) is a minimizer of \( g \), we have
\[ \nabla g(v^*) = - \sum_{j=1}^{n} \frac{x^{(j)} - v^*}{\left\| x^{(j)} - v^* \right\|} = 0. \] (55)

Let
\[ i = \arg\min_{j \in [n]} \left\| x^{(j)} - v^* \right\|. \]

Then for any \( j \in [n] \),
\[ \left\| x^{(j)} - x^{(i)} \right\| = \left\| x^{(j)} - v^* - \left( x^{(i)} - v^* \right) \right\| \]
\[ = \sqrt{\left\| x^{(j)} - v^* \right\|^2 + \left\| x^{(i)} - v^* \right\|^2 - 2\left\langle x^{(j)} - v^*, x^{(i)} - v^* \right\rangle} \]
\[ \leq \sqrt{2\left\| x^{(j)} - v^* \right\|^2 - 2\left\langle x^{(j)} - v^*, x^{(i)} - v^* \right\rangle} \]
\[ = \sqrt{2} \left\| x^{(j)} - v^* \right\| \sqrt{1 - \left\langle \frac{x^{(j)} - v^*}{\left\| x^{(j)} - v^* \right\|}, x^{(i)} - v^* \right\rangle}. \]

Using Bernoulli’s inequality (57) with \( x = -\left\langle \frac{x^{(j)} - v^*}{\left\| x^{(j)} - v^* \right\|}, x^{(i)} - v^* \right\rangle \), we have
\[ \left\| x^{(j)} - x^{(i)} \right\| \leq \sqrt{2} \left\| x^{(j)} - v^* \right\| \left( 1 - \frac{1}{2} \left\langle \frac{x^{(j)} - v^*}{\left\| x^{(j)} - v^* \right\|}, x^{(i)} - v^* \right\rangle \right). \]
\[ = \sqrt{2} \left\| x^{(j)} - v^* \right\| - \frac{1}{\sqrt{2}} \left\langle \frac{x^{(j)} - v^*}{\left\| x^{(j)} - v^* \right\|}, x^{(i)} - v^* \right\rangle. \]
for \( j \in [d] \). Therefore,
\[
\sum_{j=1}^{n} \|x^{(j)} - x^{(i)}\| \leq \sqrt{2} \sum_{j=1}^{n} \|x^{(j)} - v^*\| - \frac{1}{\sqrt{2}} \left( \sum_{j=1}^{n} \|x^{(j)} - v^*\|, x^{(i)} - v^* \right)
\]
\[
= \sqrt{2} \sum_{j=1}^{n} \|x^{(j)} - v^*\| - \frac{1}{\sqrt{2}} (0, x^{(i)} - v^*),
\]
\[
= \sqrt{2} \sum_{j=1}^{n} \|x^{(j)} - v^*\|.
\]
where (a) is by (55). Thus, the lemma holds. \(\square\)

\textbf{Proof of Lemma E.3} Let \((A^*, t^*, s^*)\) be a minimizer of \(E_2(A, t, s)\). Then
\[
\begin{align*}
\min_{A \in \mathbb{R}_d, p^{0}, q^{0} \in \mathbb{R}^d} F_2 \left( A, p^{0}, q^{0} \right) \\
\quad a = \min_{A \in \mathbb{R}_d, p^{0}, q^{0} \in \mathbb{R}^d} \sum_{i=1}^{n} \left\| A p^{(i)} - q^{(i)} - (A p^{0} - q^{0}) \right\|^2 + \left\| A^T q^{(i)} - p^{(i)} - (A^T q^{0} - p^{0}) \right\|^2 \\
\quad b = \min_{A \in \mathbb{R}_d, t, s \in \mathbb{R}^d} \sum_{i=1}^{n} \left\| A p^{(i)} - q^{(i)} + t \right\|^2 + \left\| A^T q^{(i)} - p^{(i)} + s \right\|^2 \\
\quad c = \min_{A \in \mathbb{R}_d, t, s \in \mathbb{R}^d} E_2(A, t, s) = E_2(A^*, t^*, s^*),
\end{align*}
\]
where (a), (c) are by the definitions of \(F_2\), \(E_2\) respectively, and (b) can be shown by taking \(t = - (A p^{0} - q^{0}), s = - (A^T q^{0} - p^{0})\). If there exist \(p^*, q^* \in \mathbb{R}^d\) such that
\[
F_2(A^*, p^*, q^*) = E_2(A^*, t^*, s^*),
\]
then by (56),
\[
\min_{A \in \mathbb{R}_d, p^{0}, q^{0} \in \mathbb{R}^d} F_2 \left( A, p^{0}, q^{0} \right) \leq F_2(A^*, p^*, q^*) = E_2(A^*, t^*, s^*) \leq \min_{A \in \mathbb{R}_d, p^{0}, q^{0} \in \mathbb{R}^d} F_2 \left( A, p^{0}, q^{0} \right)
\]
and thus \((A^*, p^*, q^*)\) is a minimizer of \(F_2\). Therefore, to prove the lemma, it is enough to show that there exist \(p^*, q^* \in \mathbb{R}^d\) that satisfy (57).

Now, suppose there exist \(p^*, q^* \in \mathbb{R}^d\) that satisfy the matrix equation
\[
\begin{bmatrix} -A^* & I_d \\ I_d & -A^T \end{bmatrix} \begin{bmatrix} p^* \\ q^* \end{bmatrix} = \begin{bmatrix} p^* \\ q^* \end{bmatrix},
\]
\(\quad (58)\)
where \(I\) is the \(d \times d\) identity matrix. Equation (58) implies that for any \(i \in [n]\),
\[
A^* p^{(i)} - q^{(i)} + t^* = A^* \left( p^{(i)} - p^* \right) - \left( q^{(i)} - q^* \right),
\]
\[
A^T q^{(i)} - p^{(i)} + s^* = A^T \left( q^{(i)} - q^* \right) - \left( p^{(i)} - p^* \right),
\]
\(\quad (59)\)
In turn, equation (59) implies that (57) holds, as can be verified in the definition of \(E_2\) and \(F_2\). Therefore it is enough to prove that there exist \(p^*, q^* \in \mathbb{R}^d\) that satisfy (58).

We first prove the existence of such \(p^*, q^*\) under two assumptions: (i) \(A^*\) is diagonal with nonnegative entries, and (ii) \(t_k^* = -s_k^*\) for all \(k \in [d]\) such that \(d_{kk} = 1\). We then release these assumptions in two steps.
First, suppose that assumptions (i) and (ii) hold. Let \( p^*, q^* \in \mathbb{R}^d \) be given by

\[
\begin{bmatrix}
p_k^* \\
q_k^*
\end{bmatrix} = \begin{cases}
-\frac{a_{kk}^*}{1} \begin{bmatrix}
t_k^* \\
s_k^*
\end{bmatrix} & a_{kk}^* \neq 1 \\
\begin{bmatrix}
t_k^* \\
s_k^*
\end{bmatrix} & a_{kk}^* = 1,
\end{cases} \quad k \in [d].
\] (60)

Note that \( \det \left( \begin{bmatrix}
-\frac{a_{kk}^*}{1} \\
1
\end{bmatrix} \right) = a_{kk}^* - 1. \) Since \( a_{kk}^* \geq 0 \) by assumption (i), if \( a_{kk}^* \neq 1 \) then the matrix \( \begin{bmatrix}
-\frac{a_{kk}^*}{1} \\
1
\end{bmatrix} \) is invertible. Thus, \( p^* \) and \( q^* \) of (60) are well defined. Note that if \( a_{kk}^* = 1 \) then

\[
\begin{bmatrix}
-\frac{a_{kk}^*}{1} \\
1
\end{bmatrix} \begin{bmatrix}
p_k^* \\
q_k^*
\end{bmatrix} = \begin{bmatrix}
-1 \\
1
\end{bmatrix} = \begin{bmatrix}
p_k^* - p_k^* \\
q_k^* - q_k^*
\end{bmatrix} = \begin{bmatrix}
t_k^* \\
s_k^*
\end{bmatrix},
\]

where (a) is by assumption (ii). Thus, for any \( k \in [d] \),

\[
\begin{bmatrix}
-\frac{a_{kk}^*}{1} \\
1
\end{bmatrix} \begin{bmatrix}
p_k^* \\
q_k^*
\end{bmatrix} = \begin{bmatrix}
t_k^* \\
s_k^*
\end{bmatrix}. \quad (61)
\]

The combination of (61) for all \( k \in [d] \), in conjunction with the fact that \( A^* \) is diagonal, implies that (58) holds. Therefore the lemma holds under assumptions (i) and (ii).

Second, to release assumption (ii), suppose that assumption (i) holds. Let us define the vectors \( \tilde{r}^*, \tilde{s}^* \in \mathbb{R}^d \) by

\[
\begin{bmatrix}
\tilde{r}_k^* \\
\tilde{s}_k^*
\end{bmatrix} = \begin{cases}
\begin{bmatrix}
t_k^* \\
s_k^*
\end{bmatrix} & a_{kk}^* \neq 1 \\
\begin{bmatrix}
t_k^* - s_k^* \\
s_k^* - t_k^*
\end{bmatrix} & a_{kk}^* = 1,
\end{cases} \quad k \in [d].
\] (62)

We shall now show that \( (A^*, \tilde{r}^*, \tilde{s}^*) \) is a minimizer of \( E_2 \) that satisfies both assumptions (i), (ii). Assumption (i) is satisfied since \( A^* \) is unmodified. Assumption (ii) is satisfied by definition in (62).

It is left to show that \( (A^*, \tilde{r}^*, \tilde{s}^*) \) is indeed a minimizer of \( E_2 \). For this it is enough to show that

\[
E_2(A^*, \tilde{r}^*, \tilde{s}^*) \leq E_2(A^*, r^*, s^*). \quad (63)
\]

It can be shown by a convexity argument that for any three real numbers \( a, x, y \),

\[
(a + x)^2 + (a - y)^2 \geq 2 \left( a + \frac{x - y}{2} \right)^2. \quad (64)
\]

Let \( k \in [d] \) such that \( a_{kk}^* = 1 \). Then for any \( i \in [n] \),

\[
\begin{align*}
&\left( a_{kk}^* p_k^{(i)} - q_k^{(i)} + t_k^* \right)^2 + \left( a_{kk}^* q_k^{(i)} - p_k^{(i)} + s_k^* \right)^2 \\
&= \left( p_k^{(i)} - q_k^{(i)} + t_k^* \right)^2 + \left( q_k^{(i)} - p_k^{(i)} + s_k^* \right)^2 \\
&= \left( p_k^{(i)} - q_k^{(i)} + t_k^* \right)^2 + \left( p_k^{(i)} - q_k^{(i)} - s_k^* \right)^2 \\
&\geq 2 \left( p_k^{(i)} - q_k^{(i)} + \frac{t_k^* - s_k^*}{2} \right)^2 = 2 \left( p_k^{(i)} - q_k^{(i)} + \frac{t_k^*}{2} \right)^2.
\end{align*}
\] (65)
where (a) follows from (64) with $a = p^{(i)}_k - q^{(i)}_k$, $x = t_k^*$ and $y = s_k^*$; and (b),(c) follow from the definition of $\tilde{F}$, $\tilde{s}$ in (62). Recall that for any $k \in [d]$ such that $a_{kk}^{(i)} \neq 1$, $t_k^* = t_k^i$ and $s_k^* = s_k^i$. This, together with (65), imply that for any $k \in [d],$

\[
\begin{align*}
(a_{kk}^{(i)} p_k^{(i)} - q_k^{(i)} + t_k^*)^2 + (a_{kk}^{(i)} q_k^{(i)} - p_k^{(i)} + s_k^*)^2 \\
\geq (a_{kk}^{(i)} p_k^{(i)} - q_k^{(i)} + t_k^*)^2 + (a_{kk}^{(i)} q_k^{(i)} - p_k^{(i)} + s_k^*)^2.
\end{align*}
\]

Therefore,

\[
\begin{align*}
E_2(A^*, t^*, s^*) = \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \sqrt{||A^* p^{(i)} - q^{(i)} + t^*||^2 + ||A^T q^{(i)} - p^{(i)} + s^*||^2} \\
= \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \sum_{k=1}^{d} ((a_{kk}^{(i)} p_k^{(i)} - q_k^{(i)} + t_k^*)^2 + (a_{kk}^{(i)} q_k^{(i)} - p_k^{(i)} + s_k^*)^2) \\
\geq \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \sum_{k=1}^{d} ((a_{kk}^{(i)} p_k^{(i)} - q_k^{(i)} + t_k^*)^2 + (a_{kk}^{(i)} q_k^{(i)} - p_k^{(i)} + s_k^*)^2) \\
= E_2(A^*, \tilde{F}^*, \tilde{s}^*),
\end{align*}
\]

where (a) is by the diagonal structure of $A^*$. Hence, (65) holds and thus $(A^*, \tilde{F}^*, \tilde{s}^*)$ is a minimizer of $E_2$. By the previous part of the proof, there exist $p^*, q^* \in \mathbb{R}^d$ with $F_2(A^*, p^*, q^*) = E_2(A^*, \tilde{F}^*, \tilde{s}^*)$. Since $(A^*, t^*, s^*)$, $(A^*, \tilde{F}^*, \tilde{s}^*)$ are both minimizers of $F_2,$

\[
F_2(A^*, p^*, q^*) = F_2(A^*, \tilde{F}^*, \tilde{s}^*) = E_2(A^*, t^*, s^*).
\]

Therefore $p^*, q^*$ satisfy (67), and thus the lemma holds under assumption (i).

Finally, to release assumption (i), let $(A^*, t^*, s^*)$ be a minimizer of $E_2$. Let $A^* = U\Sigma V^T$ be an SVD of $A^*$, and let

\[
u^{(i)} = V^T p^{(i)}, \quad v^{(i)} = U^T q^{(i)}, \quad i = 1, \ldots, n.
\]

Define the modified objectives $\tilde{E}_p(B, x, y)$ and $\tilde{F}_p(B, u^{(0)}, v^{(0)})$ by

\[
\begin{align*}
\tilde{E}_p(B, x, y) &= \sum_{i=1}^{n} \left( \frac{||B u^{(i)} - v^{(i)} + x||^p + ||B^T v^{(i)} - u^{(i)} + y||^p}{2} \right)^{\frac{1}{p}} \\
\tilde{F}_p(B, u^{(0)}, v^{(0)}) &= \sum_{i=1}^{n} \left( \frac{||B (u^{(i)} - u^{(0)}) - (v^{(i)} - v^{(0)})||^p + ||B^T (v^{(i)} - v^{(0)}) - (u^{(i)} - u^{(0)})||^p}{2} \right)^{\frac{1}{p}}
\end{align*}
\]

for $B \in \mathbb{R}^{d \times d}$ and $x, y, u^{(i)}, v^{(0)} \in \mathbb{R}^d$. By a similar derivation to (24) it can be shown that for any matrix $B \in \mathbb{R}^{d \times d}$ and vectors $x, y, u^{(0)}, v^{(0)} \in \mathbb{R}^d,$

\[
\begin{align*}
\tilde{E}_p(B, x, y) &= E_p(UBV^T, Ux,Vy), \\
\tilde{F}_p(B, u^{(0)}, v^{(0)}) &= F_p(UBV^T, Vu^{(0)}, Vu^{(0)}).
\end{align*}
\]

Let

\[
x^* = U^T t^*, \quad y^* = V^T s^*.
\]

Since $(A^*, t^*, s^*)$ is a minimizer of $E_2$, then by (67), $(\Sigma, x^*, y^*)$ is a minimizer of $\tilde{E}_2$. Since $\Sigma$ is diagonal with nonnegative entries, then $(\Sigma, x^*, y^*)$ satisfies assumption (i). Therefore, by the previous part of the proof, there exist $u^*, v^* \in \mathbb{R}^d$ such that

\[
\tilde{F}_2(\Sigma, u^*, v^*) = \tilde{E}_2(A^*, x^*, y^*).
\]

Let

\[
p^* = Vu^*, \quad q^* = Uv^*.
\]

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Then
\[ F_2(A^*, p^*, q^*) = F_2(A^*, Vu^*, Uv^*) \]
\[ = F_2(U^T A^* V, V^T V u^*, U^T U v^*) \]
\[ = \hat{F}_2(\Sigma, u^*, v^*) \]
\[ = \hat{F}_2(\Sigma, U^T t^*, V^T s^*) \]
\[ \leq E_2(U^T U t^*, V V^T s^*) \]
\[ = E_2(A^*, t^*, s^*) \],

where (a),(c) are by (67) and (b) is by (68). Therefore \( p^*, q^* \) satisfy (57), and thus the lemma is proven. \( \square \)

This concludes the proof of Theorem D.2. \( \square \)

**Lemma D.2.** For any \( d \geq 1 \) and \( p \in [1, \infty] \),

1. There exist \( P, Q \in \mathbb{R}^{d \times n} \) and \( A \in \mathbb{R}^{d \times d} \) such that
   \[ E(\Pi(A)) = 2E_p(A) > 0. \]

2. There exist \( P, Q \in \mathbb{R}^{d \times n} \) and \( A \in \mathbb{R}^{d \times d} \) such that \( A \) is optimal for \( E_p \), and
   \[ E(\Pi(A)) = \sqrt[p]{2} E_p(A) > 0. \]

**Proof of Lemma D.2.** Set \( n = 1 \) and let \( p^{(1)} \in \mathbb{R}^d \) be an arbitrary nonzero vector. Let \( A = 0_d \) be the zero matrix. For part 1, set \( q^{(1)} = -\Pi(A)p^{(1)} \). Then for any \( p \in [1, \infty] \),

\[ E_p(A) = E_p(0_d) = \left( \frac{\|q^{(1)}\|^p + \|p^{(1)}\|^p}{2} \right)^\frac{1}{p} = \|p^{(1)}\|. \]

However,

\[ E_p(\Pi(A)) = \left( \frac{\|\Pi(A)p^{(1)} - q^{(1)}\|^p + \|\Pi(A)^T q^{(1)} - p^{(1)}\|^p}{2} \right)^\frac{1}{p} \]
\[ = \left( \frac{2\|\Pi(A)p^{(1)}\|^p + \|2p^{(1)}\|^p}{2} \right)^\frac{1}{p} = 2\|p^{(1)}\| = 2E_p(A). \]

For part 2, set \( q^{(1)} = 0 \). Then

\[ E_p(A) = \left( \frac{\|Ap^{(1)}\|^p + \|p^{(1)}\|^p}{2} \right)^\frac{1}{p} \geq \frac{1}{2^p} \|p^{(1)}\| = E_p(0_d). \]

Therefore, \( 0_d \) is a global minimizer of \( E_p \), with an objective value of \( \frac{1}{2^p} \|p^{(1)}\| \). However, for any orthogonal matrix \( R \),

\[ E_p(R) = \left( \frac{\|Rp^{(1)}\|^p + \|p^{(1)}\|^p}{2} \right)^\frac{1}{p} = \|p^{(1)}\|. \]

Thus, for \( A = 0_d \),

\[ E_p(\Pi(A)) = \|p^{(1)}\| = 2^\frac{1}{p} E_p(A). \]

\( \square \)
Theorem D.3. For \( d \geq 2 \) there exist points \( P, Q \in \mathbb{R}^{d \times n} \) such that for any point mismatch function \( G(\cdot | P, Q) \),

1. The zero matrix \( A^* = 0_d \) is a minimizer of \( G(\cdot | P, Q) \).
2. For \( \hat{R} = \Pi(A^*) \),

\[
E(\hat{R}) = \sqrt{2} \min_{R \in O(d)} E(R) > 0.
\]

Proof of Theorem D.3. Let \( u, v \in \mathbb{R}^d \) be two vectors such that \( u \perp v \) and \( \|u\| = \|v\| = 1 \). Let \( R_0 \in O(d) \) such that \( R_0u = v \). Let \( \hat{R} = \Pi(0_d) \). Define \( P, Q \in \mathbb{R}^{d \times n} \) for \( n = 2 \) by

\[
p^{(1)} = p^{(2)} = \hat{R}^T u, \\
qu^{(1)} = v, \quad q^{(2)} = -v.
\]

Let \( \Pi \in \mathbb{R}^{2 \times 2} \) be the permutation matrix

\[
\Pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

and note that \( Q\Pi = -Q \) and \( P\Pi = P \). Let \( A \in \mathbb{R}^{d \times d} \) be an arbitrary matrix. Then

\[
G(-A | P, Q) = G(-A | P\Pi, -Q\Pi) \overset{\text{(a)}}{=} G(A | P, Q) + \overset{\text{(b)}}{=} G(A | P, Q),
\]

where (a), (b) are by Assumptions 3 and 2 respectively. Combining (69) with the assumption that \( G(\cdot | P, Q) \) is convex, we have

\[
G(0_d | P, Q) = G\left(\frac{A + (\frac{1}{2})}{2} \right) P, Q \leq \frac{1}{2} G(A | P, Q) + \frac{1}{2} G(-A | P, Q)
\]

\[
= \frac{1}{2} G(A | P, Q) + \frac{1}{2} G(A | P, Q) = G(A | P, Q).
\]

Since \( A \) is arbitrary, (70) implies that \( A^* = 0_d \) is a minimizer of \( G(\cdot | P, Q) \). Therefore,

\[
\hat{R} = \Pi(0_d) = \Pi(A^*).
\]

Since \( u \) and \( v \) are perpendicular unit vectors,

\[
\|u - v\| = \|u + v\| = \sqrt{\|u\|^2 + \|v\|^2} = \sqrt{2}
\]

and thus

\[
E(\hat{R}) = \|\hat{R}p^{(1)} - q^{(1)}\| + \|\hat{R}p^{(2)} - q^{(2)}\|
\]

\[
= \|\hat{R}(\hat{R}^T u) - v\| + \|\hat{R}(\hat{R}^T u) + v\|
\]

\[
= \|u - v\| + \|u + v\| = 2\sqrt{2}.
\]

However,

\[
E(R_0\hat{R}) = \|R_0\hat{R}p^{(1)} - q^{(1)}\| + \|R_0\hat{R}p^{(2)} - q^{(2)}\|
\]

\[
= \|R_0\hat{R}(\hat{R}^T u) - v\| + \|R_0\hat{R}(\hat{R}^T u) + v\|
\]

\[
= \|R_0u - v\| + \|R_0u + v\| = 2\|v\| = 2.
\]

Also note that for any \( R \in O(d) \),

\[
E(R) = \|R(\hat{R}^T u) - v\| + \|R(\hat{R}^T u) + v\|
\]

\[
\overset{\text{(a)}}{\geq} \sqrt{\|R(\hat{R}^T u) - v\|^2 + \|R(\hat{R}^T u) + v\|^2}
\]

\[
\overset{\text{(b)}}{=} \sqrt{2} \|R(\hat{R}^T u)\|^2 + \|v\|^2 = 2.
\]

where (a) is by the \( \ell_1 - \ell_2 \) norm inequality and (b) is by the parallelogram law. Therefore,

\[
E(\hat{R}) = \sqrt{2}E(R_0\hat{R}) = \sqrt{2} \min_{R \in O(d)} E(R) > 0.
\]

\( \square \)
Proof of Theorem 3.3. Define the advantage function $\phi : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}$,

$$\phi(A) = \frac{1}{2} \sum_{i \in I} \left( \|Ap^{(i)}\| + \|ATq^{(i)}\| \right) - \frac{1}{2} \sum_{i \in F} \left( \|Ap^{(i)}\| + \|ATq^{(i)}\| \right).$$  \hfill (71)

Using $\phi(A)$, the following lemma provides a lower bound on the advantage of the true $R_0$ as a solution of (5) over other potential solutions. A proof appears below.

**Lemma E.5.** For any $A \in \mathbb{R}^{d \times d}$ and any $p \in [1, \infty]$, 

$$E_p(R_0 + A) - E_p(R_0) \geq \phi(A).$$ \hfill (72)

It follows from Lemma E.5 that if $\phi(A) > 0$ for all $A \neq 0$, then $R_0$ is the unique global minimizer of $E_p$. The following lemma, proven below, shows that if $P, Q$ satisfy the linear DIP, then $\phi(A)$ is indeed positive for any $A \neq 0$.

**Lemma E.6.** Suppose that $P, Q$ satisfy the linear DIP. Then for any matrix $A \neq 0$, $\phi(A) > 0$.

We shall now prove Lemmas E.5 and E.6.

**Proof of Lemma E.5.** Let $A \in \mathbb{R}^{d \times d}$. Since $E_p(A) \geq E_1(A)$ and $E_p(R_0) = E(R_0) = E_1(R_0)$, we have $E_p(R_0 + A) - E_p(R_0) = E_p(R_0 + A) - E_1(R_0)$.

Therefore it is enough to prove (72) for $p = 1$. Expanding the left-hand side of (72), we get

$$2(E_1(R_0 + A) - E_1(R_0)) = \sum_{i=1}^{n} \left\| (R_0 + A)p^{(i)} - q^{(i)} \right\| + \left\| (R_0 + A)^T q^{(i)} - p^{(i)} \right\|$$

$$-\sum_{i=1}^{n} \left\| R_0 p^{(i)} - q^{(i)} \right\| + \left\| R_0^T q^{(i)} - p^{(i)} \right\|. \hfill (73)$$

Let us split the sums in (73) to inlier and outlier terms. First consider the inliers. Since $q^{(i)} = R_0 p^{(i)}$ for $i \in I$, we have

$$\sum_{i \in I} \left\| (R_0 + A)p^{(i)} - q^{(i)} \right\| + \left\| (R_0 + A)^T q^{(i)} - p^{(i)} \right\| - \sum_{i \in I} \left\| R_0 p^{(i)} - q^{(i)} \right\| + \left\| R_0^T q^{(i)} - p^{(i)} \right\|$$

$$= \sum_{i \in I} \left\| (R_0 + A)p^{(i)} - q^{(i)} \right\| + \left\| (R_0 + A)^T q^{(i)} - p^{(i)} \right\| \hfill (74)$$

$$= \sum_{i \in I} \left\| Ap^{(i)} \right\| + \left\| ATq^{(i)} \right\|.$$Second, consider the outliers. By the triangle inequality, for each $i$,

$$\left\| (R_0 + A)p^{(i)} - q^{(i)} \right\| + \left\| (R_0 + A)^T q^{(i)} - p^{(i)} \right\| + \left\| Ap^{(i)} \right\| + \left\| ATq^{(i)} \right\| \geq \left\| R_0 p^{(i)} - q^{(i)} \right\| + \left\| R_0^T q^{(i)} - p^{(i)} \right\|.$$Therefore,

$$\sum_{i \in F} \left\| (R_0 + A)p^{(i)} - q^{(i)} \right\| + \left\| (R_0 + A)^T q^{(i)} - p^{(i)} \right\| - \sum_{i \in I} \left\| R_0 p^{(i)} - q^{(i)} \right\| + \left\| R_0^T q^{(i)} - p^{(i)} \right\|$$

$$\geq - \sum_{i \in F} \left\| Ap^{(i)} \right\| + \left\| ATq^{(i)} \right\|. \hfill (75)$$

Inserting inequalities (74) and (75) into (73) yields

$$2(E_1(R_0 + A) - E_1(R_0)) \geq \sum_{i \in I} \left\| Ap^{(i)} \right\| + \left\| ATq^{(i)} \right\| - \sum_{i \in I} \left\| Ap^{(i)} \right\| + \left\| ATq^{(i)} \right\|$$

$$= 2\phi(A),$$

which proves the lemma. \qed
Let us now prove Lemma E.6.

**Proof of Lemma E.6** We first make the following claim.

**Claim E.7.** For any \( x \in \mathbb{R}^d \),
\[
\int_{S^{d-1}} |\langle v, x \rangle| dv = C \|x\|, \tag{76}
\]
where \( S^{d-1} \) is the unit sphere in \( \mathbb{R}^d \), and \( C \) is a positive constant that depends only on the dimension \( d \).

Claim E.7 essentially states that the expected magnitude of a random projection of \( x \) on a line that goes through the origin is proportional to \( \|x\| \). This claim can be easily proven by symmetry considerations.

Now suppose that \( P, Q \) satisfy the linear DIP. Let \( A \in \mathbb{R}^d \) such that \( A \neq 0 \). Since the nullspace of \( A^T \) is at most \( (d-1) \)-dimensional, it is of measure zero in \( \mathbb{R}^d \). It follows that for almost any \( v \in S^{d-1} \) we have that \( A^T v \neq 0 \). For such \( v \) we can invoke the linear DIP (15) with \( u = A^T v \) and get
\[
\sum_{i \in J} |\langle A^T v, p^{(i)} \rangle| > \sum_{i \in I^c} |\langle A^T v, p^{(i)} \rangle|,
\]
or equivalently
\[
\sum_{i \in J} \left| \langle v, A p^{(i)} \rangle \right| - \sum_{i \in I^c} \left| \langle v, A p^{(i)} \rangle \right| > 0.
\]
By the above discussion, the left-hand side of the above inequality is positive for almost any \( v \in S^{d-1} \). Therefore its integral is also positive:
\[
\int_{S^{d-1}} \left( \sum_{i \in J} |\langle v, A p^{(i)} \rangle| - \sum_{i \in I^c} |\langle v, A p^{(i)} \rangle| \right) dv > 0.
\]
Applying Claim E.7 to the above yields
\[
\sum_{i \in J} \|A p^{(i)}\| - \sum_{i \in I^c} \|A p^{(i)}\| > 0.
\]
Using a similar argument on the right-hand side of (15) with \( u = Av \), it can be shown that the linear DIP implies that
\[
\sum_{i \in J} \|A^T q^{(i)}\| - \sum_{i \in I^c} \|A^T q^{(i)}\| > 0.
\]
The two above inequalities combined imply the claim of the lemma. \(\square\)

This concludes the proof of Theorem 3.4. \(\square\)

**Theorem 3.4.** Suppose that \( P, Q \) satisfy the affine DIP with respect to \( (R_0, t_0, I) \). Then for any \( p \in [1, \infty] \), the unique global minimizer of \( E_p(A, t, s) \) is \( (R_0, t_0, -R_0^T t_0) \).

**Proof of Theorem E.5.** The proof is similar to that of Theorem 3.3 We first redefine the advantage function \( \phi \) of (71) to accommodate translations:
\[
\phi(A, t, s) = \frac{1}{2} \sum_{i \in J} \left( \|A p^{(i)} + t\| + \|A^T q^{(i)} + s\| \right) - \frac{1}{2} \sum_{i \in I^c} \left( \|A p^{(i)} + t\| + \|A^T q^{(i)} + s\| \right). \tag{77}
\]
Using \( \phi(A, t, s) \), the following lemma lower-bounds the advantage of \( (R_0, t_0, -R_0^T t_0) \) over other solutions of (6).

**Lemma E.8.** For any \( p \in [1, \infty] \), any \( A \in \mathbb{R}^{d \times d} \) and any \( t, s \in \mathbb{R}^d \),
\[
E_p(R_0 + A, t_0 + t, -R_0^T t_0 + s) - E_p(R_0, t_0, -R_0^T t_0) \geq \phi(A, t, s). \tag{78}
\]
The following lemma shows that if \( P, Q \) satisfy the affine DIP, then \( \phi(A, t, s) \) is positive for any \( A, t, s \) that are not all equal to zero.
Lemma E.9. Suppose that $P, Q$ satisfy the affine DIP. Then for any $A, t, s$ such that $\|A\| + \|t\| + \|s\| > 0$, $\phi(A, t, s) > 0$.

From Lemmas E.8 and E.9, it follows that $\{R_0, t_0, -R_0^T t_0\}$ is the unique global minimizer of $E_p(R, t, s)$. To complete the proof of the theorem, let us now prove these lemmas.

Proof of Lemma E.8 Let $A \in \mathbb{R}^{d \times d}$ and $t, s \in \mathbb{R}^d$. Since $E_p$ is monotone-increasing with respect to $t$, $E_p(R_0, t_0, -R_0^T t_0) = E(R_0, t_0)$ for all $p \in [1, \infty]$, it is enough to prove (78) for $p = 1$. Expanding the left-hand side of (78), we get

$$2 \left( E_1(R_0 + A, t_0, -R_0^T t_0) - E_1(R_0, t_0, -R_0^T t_0) \right)$$

$$= \sum_{i=1}^n \left( \| (R_0 + A)p(i) - q(i) + t_0 + t \| + \| (R_0 + A)^T q(i) - p(i) - R_0^T t_0 + s \| \right. \right.$$  \[ (79) \]

Let us split (79) to inlier and outlier terms. For $i \in I$, $q(i) = R_0 p(i) + t_0$, and thus

$$\sum_{i \in I} \left( \| (R_0 + A)p(i) - q(i) + t_0 + t \| + \| (R_0 + A)^T q(i) - p(i) - R_0^T t_0 + s \| \right.$$ \[ (80) \]

Second, consider the outliers. By the triangle inequality, for each $i \in [n]$,

$$\| (R_0 + A)p(i) - q(i) + t_0 + t \| + \| (R_0 + A)^T q(i) - p(i) - R_0^T t_0 + s \| + \| A p(i) + t \| + \| A^T q(i) + s \|$$

$$\geq \| R_0 p(i) - q(i) + t_0 \| + \| R_0^T q(i) - p(i) - R_0^T t_0 \|.$$  \[ (81) \]

Therefore,

$$\sum_{i \in I} \left( \| (R_0 + A)p(i) - q(i) + t_0 + t \| + \| (R_0 + A)^T q(i) - p(i) - R_0^T t_0 + s \| \right.$$ \[ (80) \]

Inserting inequalities (80) and (81) into (79) yields

$$2 \left( E_1(R_0 + A, t_0, -R_0^T t_0 + s) - E_1(R_0, t_0, -R_0^T t_0) \right)$$

$$\geq \sum_{i \in I} \| A p(i) + t \| + \| A^T q(i) + s \| - \sum_{i \in I} \| A p(i) + t \| + \| A^T q(i) + s \| = 2 \phi(A, t, s),$$

which proves the lemma. \[ \square \]

Let us now prove Lemma E.9.

Proof of Lemma E.9 Let $A \in \mathbb{R}^{d \times d}$ and $t \in \mathbb{R}^d$ that are not both equal to zero. Let $v \in \mathbb{R}^d$. Denote

$$u(v) = A^T v, \quad \alpha(v) = \langle v, t \rangle.$$
For any $p \in \mathbb{R}^d$, 
\[ \langle v, Ap + t \rangle = \langle u(v), p \rangle + \alpha(v). \]

Therefore,
\[
\sum_{i \in I} \left| \langle v, Ap^{(i)} + t \rangle \right| = \sum_{i \in I} \left| \langle u(v), p^{(i)} \rangle + \alpha(v) \right|, \\
\sum_{i \in I} \left| \langle v, Ap^{(i)} + t \rangle \right| = \sum_{i \in I} \left| \langle u(v), p^{(i)} \rangle + \alpha(v) \right|.
\]

(82)

If $\|u(v)\| + |\alpha(v)| > 0$, then by the affine DIP (16) and (82),
\[
\sum_{i \in I} \left| \langle v, Ap^{(i)} + t \rangle \right| > \sum_{i \in I} \left| \langle v, Ap^{(i)} \rangle \right|.
\]

(83)

Since $A$ and $t$ are not both equal to zero, the set $\{ v \in S^{d-1} : u(v) = 0 \land \alpha(v) = 0 \}$ is of measure zero in the unit sphere $S^{d-1}$. Therefore, the inequality (83) is satisfied for almost any $v \in S^{d-1}$. This implies that
\[
\int_{\mathbb{S}^{d-1}} \left| \sum_{i \in I} \langle v, Ap^{(i)} + t \rangle \right| dv > \int_{\mathbb{S}^{d-1}} \left| \sum_{i \in I} \langle v, Ap^{(i)} \rangle + t \right| dv.
\]

Applying Claim E.7 to the above inequality yields
\[
\sum_{i \in I} \left| Ap^{(i)} + t \right| > \sum_{i \in I} \left| Ap^{(i)} \right|.
\]

(84)

By a similar argument it can be shown that for any $A \in \mathbb{R}^{d \times d}$, $s \in \mathbb{R}^d$ that are not both equal zero, the affine DIP implies that
\[
\sum_{i \in I} \left| A^T q^{(i)} + s \right| > \sum_{i \in I} \left| A^T q^{(i)} \right|.
\]

(85)

To finalize the proof, let $A \in \mathbb{R}^{d \times d}$, $t, s \in \mathbb{R}^d$ such that not all three equal zero. We need to show that $\phi(A, t, s) > 0$. If $A \neq 0$, or if $A = 0$ and $t, s \neq 0$, then we can use (84) and (85) and get
\[
\sum_{i \in I} \left| Ap^{(i)} + t \right| + \left| A^T q^{(i)} + s \right| > \sum_{i \in I} \left| Ap^{(i)} + t \right| + \left| A^T q^{(i)} + s \right|.
\]

If $A = 0$ and exactly one of $t, s$ equals zero, suppose W.L.O.G that $s = 0$. Then $t \neq 0$, and by (84),
\[
\sum_{i \in I} |t| = \sum_{i \in I} \left| Ap^{(i)} + t \right| > \sum_{i \in I} \left| Ap^{(i)} \right| = \sum_{i \in I} |t|.
\]

Therefore,
\[
\sum_{i \in I} \left| Ap^{(i)} + t \right| + \left| A^T q^{(i)} + s \right| = \sum_{i \in I} |t| > \sum_{i \in I} |t| = \sum_{i \in I} \left| Ap^{(i)} + t \right| + \left| A^T q^{(i)} + s \right|.
\]

The case $A, t = 0, s \neq 0$ is handled similarly. Thus the lemma is proven.

This concludes the proof of Theorem 3.4.

\[ \square \]

**Theorem D.4.** For any $d \geq 1$ and $\varepsilon \in (0, 1)$ there exist a rotation matrix $R_0 \in SO(d)$, points $P, Q \in \mathbb{R}^{d \times n}$ and index set $I \subseteq [n]$ such that:

1. $q^{(i)} = R_0 p^{(i)}$ for $i \in I$.
2. For any unit vector $u \in \mathbb{R}^d$, 
\[
\sum_{i \in I} \left| \langle u, p^{(i)} \rangle \right| \geq (1 - \varepsilon) \sum_{i \in I} \left| \langle u, p^{(i)} \rangle \right| \quad \text{and} \quad \sum_{i \in I} \left| \langle u, q^{(i)} \rangle \right| \geq (1 - \varepsilon) \sum_{i \in I} \left| \langle u, q^{(i)} \rangle \right|.
\]

(22)

3. There exists a rotation matrix $R_1$ that is the unique global minimizer of $E_p$ for any $p \geq 1$, and $\|R_1 - R_0\|_F = \sqrt{2d}$, with $\| \cdot \|_F$ denoting the Frobenius norm.

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Proof of Theorem D.4. Let $d \geq 1$ and $\varepsilon \in (0, 1)$. Let $\{e_i\}_{i=1}^d$ be the standard unit vectors in $\mathbb{R}^d$. Let $R_0 = I_d$ be the $d \times d$ unit matrix, and let $R_1 \in \mathbb{R}^{d \times d}$ be defined by

$$
R_1 e_i = e_{i+1}, \quad i = 1, \ldots, d - 1, \quad R_1 e_d = \sigma e_1,
$$

where $\sigma \in \{1, -1\}$ is chosen such that $\det(R_1) = 1$. It is easy to check that $\|R_0 - R_1\|_F = 2d$.

Define the points $P = \left( p^{(i)} \right)_{i=1}^n, Q = \left( q^{(i)} \right)_{i=1}^n$ for $n = 2d$ by

$$
p^{(i)} = (1 - \varepsilon)e_i, \quad q^{(i)} = R_0 p^{(i)}, \quad i = 1, \ldots, d,
$$

$$
p^{(i)} = e_{i-d}, \quad q^{(i)} = R_1 p^{(i)}, \quad i = d + 1, \ldots, 2d. \tag{86}
$$

Set $I = \{1, \ldots, d\}$, then $P, Q$ satisfy condition no. 1 of the theorem. It follows from (86) that $R_0$ and $R_1$ act as permutations on the points $P$, up to a possible multiplication by $-1$ (in the case $q^{(2d)} = \sigma e_1$).

Hence, the following equalities of unordered sets hold:

$$
\left\{ p^{(i)} \right\}_{i \in I} = \left\{ q^{(i)} \right\}_{i \in I} = \left\{ (1 - \varepsilon)e_i \right\}_{i=1}^{d},
$$

$$
\left\{ p^{(i)} \right\}_{i \in I^C} = \left\{ \sigma q^{(i)} \right\}_{i \in I^C} = \left\{ e_i \right\}_{i=1}^{d}. \tag{87}
$$

where $\sigma_i = \sigma$ for $i \neq 2d$ and $\sigma_{2d} = -1$.

Let $u \in \mathbb{R}^d$ be an arbitrary vector. Then by (87),

$$
\sum_{i \in I} \left| \langle u, p^{(i)} \rangle \right| = (1 - \varepsilon) \sum_{i \in I^C} \left| \langle u, p^{(i)} \rangle \right|, \tag{88}
$$

$$
\sum_{i \in I} \left| \langle u, q^{(i)} \rangle \right| = (1 - \varepsilon) \sum_{i \in I^C} \left| \langle u, q^{(i)} \rangle \right|.
$$

and thus $P$ and $Q$ satisfy condition no. 2 of the theorem. However, (88) also implies that $P, Q$ satisfy the linear DIP with respect to $(R_1, I^C)$. Therefore, by Theorem 3.3, the unique global minimizer of $E_p$ is $R_1$ for any $p \in [1, \infty]$. \qed