Generating Random Vectors in \((\mathbb{Z}/p\mathbb{Z})^d\) Via an Affine Random Process

Martin Hildebrand  
Department of Mathematics and Statistics  
University at Albany  
State University of New York  
Albany, NY 12222

Joseph McCollum  
Department of Mathematical Sciences  
Elms College  
Chicopee, MA 01013

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Abstract

This paper considers some random processes of the form \(X_{n+1} = TX_n + B_n \pmod{p}\) where \(B_n\) and \(X_n\) are random variables over \((\mathbb{Z}/p\mathbb{Z})^d\) and \(T\) is a fixed \(d \times d\) integer matrix which is invertible over the complex numbers. For a particular distribution for \(B_n\), this paper improves results of Asci to show that if \(T\) has no complex eigenvalues of length 1, then for integers \(p\) relatively prime to \(\det(T)\), order \((\log p)^2\) steps suffice to make \(X_n\) close to uniformly distributed where \(X_0\) is the zero vector. This paper also shows that if \(T\) has a complex eigenvalue which is a root of unity, then order \(p^b\) steps are needed for \(X_n\) to get close to uniformly distributed for some positive value \(b \leq 2\) which may depend on \(T\) and \(X_0\) is the zero vector.
1 Introduction

Previous work has looked at the following random processes on $\mathbb{Z}/p\mathbb{Z}$. These processes were of the form

$$X_{n+1} = a_nX_n + b_n \pmod{p}$$

where $X_0 = 0$ and $a_n$ and $b_n$ had certain probability distributions. See [2], [5], [6], and [7]. The study of these random processes was inspired by some pseudorandom number generators used by computers; see, for example, [8]. For many choices of the distributions for $a_n$ and $b_n$, the distribution for $X_n$ would be close to uniformly distributed for relatively small values of $n$ (e.g. order $(\log p)^2$ or order $(\log p) \log(\log p)$) provided that $p$ is chosen to avoid “parity” problems. For some choices of $a_n$ (e.g. $a_n = 1$ always), $n$ would have to be much larger (e.g. order $p^2$) for the distribution for $X_n$ to be close to uniformly distributed.

One can examine similar random processes on other finite structures. For example, Asci [1] considered random processes of the form

$$X_{n+1} = TX_n + B_n \pmod{p}$$

where $X_n, B_n$ are random variables on $(\mathbb{Z}/p\mathbb{Z})^d$ with $B_0, B_1, \ldots$ i.i.d., $d$ is constant, $X_0 = 0$, and $T$ is a fixed $d \times d$ integer matrix. The main results of [1] assumed that $T$ had non-zero integer eigenvalues. One result in [1] does not have such an assumption but only requires $n$ to be of order $p^2(\log p)$ for $X_n$ to be close to uniformly distributed on $(\mathbb{Z}/p\mathbb{Z})^d$. This paper will give significantly better upper bounds provided that the eigenvalues of $T$ are non-zero and all have length different than 1. Such bounds apply, for example, to the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$
which does not have integer eigenvalues. In this paper, we shall consider a specific distribution for $B_n$, but the arguments potentially can be extended to other distributions for $B_n$.

2 Definitions and Main Results

Throughout this paper, we shall assume $d$ is constant and $T$ is a fixed $d \times d$ integer matrix with non-zero determinant. We assume $X_0 = 0$ and $P(B_n = 0) = P(B_n = e_1) = \ldots = P(B_n = e_d) = 1/(d + 1)$ where $e_k$ is the vector in $(\mathbb{Z}/p\mathbb{Z})^d$ whose $k$-th coordinate is 1 and whose other coordinates are 0. We assume $B_0, B_1, \ldots$ are i.i.d. and

$$X_{n+1} = TX_n + B_n \pmod{p}.$$ 

We let $P_n(s) = \Pr(X_n = s)$. Recall the variation distance

$$\|P_n - U\| = \frac{1}{2} \sum_{s \in G} |P_n(s) - 1/|G||$$

$$= \max_{A \subseteq G} |P_n(A) - U(A)|$$

where $G$ is a finite group (here $G = (\mathbb{Z}/p\mathbb{Z})^d$) and $U$ is the uniform distribution on $G$ (i.e. $U(s) = 1/|G|$ for all $s \in G$).

One of our results is the following theorem.

**Theorem 1** Suppose $T$ has no eigenvalues of length 1 over $\mathbb{C}$. For some value $C > 0$ not depending on $p$, if $n \geq C(\log p)^2$, then $\|P_n - U\| \to 0$ as $p \to \infty$ provided that $p$ is restricted to integers which are relatively prime to $\det(T)$.

Note that $p$ need not be prime.

Another result deals with some cases where an eigenvalue of $T$ has length 1.
Theorem 2 Suppose that $T$ has an eigenvalue which is a root of unity over $\mathbb{C}$. There exists a positive value $b \leq 2$ not depending on $p$ such that if $n \leq p^b$, then $\|P_n - U\| \to 1$ as $p \to \infty$ provided that $p$ is restricted to the prime numbers.

3 Background for Proofs of the Theorems

The proofs of these theorems will involve the Fourier transform of $P_n$. A more extensive description of this area appears in Diaconis [3]. A representation $\rho$ on a finite group $G$ is a map from $G$ to $GL_n(\mathbb{C})$ such that $\rho(s)\rho(t) = \rho(st)$ for all $s, t \in G$. The value $n$ is called the degree of the representation and is denoted $d_\rho$. A representation $\rho$ with degree $n$ is said to be irreducible if whenever $W$ is a subspace of $\mathbb{C}^n$ with $\rho(s)W \subseteq W$ for all $s \in G$, either $W = \{0\}$ or $W = \mathbb{C}^n$. The trivial representation is given by $\rho(s) = (1)$ for all $s \in G$. If $\rho_2(s) = M\rho_1(s)M^{-1}$ for some invertible complex matrix $M$, then the representations $\rho_1$ and $\rho_2$ are said to be equivalent. Every irreducible representation on a finite group $G$ is equivalent to a unitary representation, i.e., a representation $\rho$ such that $(\rho(s))^{-1} = (\rho(s))^*$ for all $s \in G$ where $(\rho(s))^*$ is the conjugate transpose of $\rho(s)$. If $\rho$ is an irreducible representation on a finite group $G$ and $P$ is a probability on $G$, we define the Fourier transform

$$\hat{P}(\rho) = \sum_{s \in G} P(s)\rho(s).$$

The irreducible representations of $(\mathbb{Z}/p\mathbb{Z})^d$ all have degree 1 and are given by

$$\rho_c(b) := q^{\sum_{i=1}^d b_i c_i}$$

where $q := q(p) := e^{2\pi i/p}$ and $b := \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{pmatrix}$ and $c := \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_d \end{pmatrix}$ with $b_i, c_i \in \mathbb{Z}/p\mathbb{Z}$.
The representation \( \rho_c \) where \( c \) is the zero vector is the trivial representation.

We shall use the following lemma of Diaconis and Shahshahani. This lemma, known as the Upper Bound Lemma, is proved in [3].

**Lemma 1** Let \( P \) be a probability on a finite group \( G \), and let \( U \) be the uniform distribution on \( G \). Then

\[
\| P - U \|^2 \leq \frac{1}{4} \sum_{\rho}^* d_{\rho} \text{Tr}(\hat{P}(\rho)\hat{P}(\rho)^*)
\]

where the sum is over non-trivial irreducible representations \( \rho \) such that \( \rho \) is unitary and exactly one member of each equivalence class of non-trivial irreducible representations is included in the sum, \( d_{\rho} \) is the degree of \( \rho \), and \(*\) of a matrix denotes its conjugate transpose.

### 4 Proof of Theorem \[1\]

First we shall develop a recurrence relation relating the Fourier transform of \( P_{n+1} \) to the Fourier transforms of \( P_n \) and \( P_1 \). In doing so, we shall let \( \hat{P}_n(c) \) denote \( \hat{P}_n(\rho_c) \). The relation is given by the following lemma, due to Asci \[1\].

**Lemma 2** If \( p \) and \( \det(T) \) are relatively prime, then

\[
\hat{P}_{n+1}(c) = \frac{1}{d+1} \left( \hat{P}_n(T^t c) \right) \left( 1 + \sum_{r=1}^d q^{c_r} \right)
\]

where \( T^t \) is the transpose of \( T \).

We shall show that if \( c \neq 0 \), then in the sequence \( c, T^t c, (T^t)^2 c, \ldots \), at least one of the coordinates of at least one of the first \( C_2 \log p \) terms (where \( C_2 \) is a constant) will not be “near” 0 mod \( p \).
Lemma 3 Suppose $T$ has no eigenvalues of length 1 over $\mathbb{C}$. Suppose $c \neq 0$.
Write $c \in (\mathbb{Z}/p\mathbb{Z})^d$ as an element of $\mathbb{Z}^d$ with its entries as close to 0 as possible (that is $c \in [-p/2,p/2]^d$). Then for some positive values $C_1$ and $C_2$ (depending on $T$ but not $p$ or $c$), for sufficiently large $p$, $(T^t)c$ has a coordinate (viewed in $\mathbb{Z}^d$) of length at least $C_1p$ but no more than $p/2$ for some non-negative $\ell \leq C_2 \log p$.

Proof: Note that since $T$ is invertible over $\mathbb{C}$, then so is $T^t$. Thus, since $c \neq 0$, we conclude $(T^t)c \neq 0$, $(T^t)^2c \neq 0$, etc.

Let’s write $T^t$ in Jordan block diagonal form over $\mathbb{C}$:
$$T^t = M^{-1} \begin{pmatrix} J_1 & 0 & \cdots \\ 0 & J_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} M$$
for some invertible complex matrix $M$ where
$$J_i = \begin{pmatrix} a_i & 1 & 0 & \cdots & 0 \\ 0 & a_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_i \end{pmatrix}.$$ 

By the assumption on the eigenvalues of $T$, $|a_i| \neq 1$. Since $T$ is invertible over $\mathbb{C}$, $a_i \neq 0$. We can write
$$(T^t)^\ell = M^{-1} \begin{pmatrix} J_1^\ell & 0 & \cdots \\ 0 & J_2^\ell & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} M.$$

The following propositions will be used in the proof of Lemma 3.

Proposition 1 There exist positive constants $C_3$ and $C_4$ such that if $v \in \mathbb{C}^d$ has $v_{\text{large}}$ as the largest length of a coordinate, then the coordinate of $M^{-1}v$ with largest length has length at least $C_3v_{\text{large}}$ but no more than $C_4v_{\text{large}}$ while the coordinate of $Mv$ with largest length has length at least $(1/C_4)v_{\text{large}}$ but no more than $(1/C_3)v_{\text{large}}$. 

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Proposition 2 If $M^{-1}v \in \mathbb{Z}^d$ and $M^{-1}v \neq 0$, then at least one of the coordinates of $v$ must have length at least $C_5$ for some constant $C_5 > 0$.

Proposition 3 For some constant $C_6 > 0$, if $\ell > C_6 \log p$, then all coordinates of

$$
\begin{pmatrix}
J_1^\ell & 0 & \cdots \\
0 & J_2^\ell & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix} M c
$$

which correspond to an eigenvalue of length less than 1 will, for sufficiently large $p$, all have length less than $C_5$.

Proposition 4 Suppose $C_1$ is such that $(C_1/C_3)(|a| + 1) < 1/(2C_4)$ where $a$ is the eigenvalue of $T$ with the largest length. Suppose all coordinates of $M c$ have length no more than $(C_1/C_3)p$. Suppose some coordinate of

$$
\begin{pmatrix}
J_1^{\ell'} & 0 & \cdots \\
0 & J_2^{\ell'} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix} M c
$$

which corresponds to an eigenvalue of length greater than 1 has length greater than $C_5$ for some $\ell' \leq (C_6 \log p) + 1$. Then for some constant $C_2 > 0$, there exists a value $s < C_2 \log p$ such that for sufficiently large $p$, the coordinate with the largest length of

$$
\begin{pmatrix}
J_1^s & 0 & \cdots \\
0 & J_2^s & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix} M c
$$

has length at least $(C_1/C_3)p$ but no more than $p/(2C_4)$.

Proof of Proposition 1: Let $A$ be the largest length of the entries of $M^{-1}$. Then all coordinates of $M^{-1}v$ will have length at most $Ad_{\text{large}}$. Now let $B$ be the largest length of the entries of $M$. The largest coordinate of
$M^{-1}v$ must have length at least \((1/(Bd))v_{\text{large}}\); otherwise all coordinates of $v = M(M^{-1}v)$ would have length under $v_{\text{large}}$. The statement about $Mv$ follows directly. \(\square\)

**Proof of Proposition 2:** Let $C_5 = 1/C_4$, and note that $M^{-1}v$ must have at least one coordinate of length at least $1$. Then use Proposition 1. \(\square\)

To prove Propositions 3 and 4, we shall use the following proposition.

**Proposition 5** If

\[
J = \begin{pmatrix}
a & 1 & 0 & \cdots & 0 \\
0 & a & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a
\end{pmatrix},
\]

then

\[
(J^\ell)_{ij} = \begin{cases} 
a^\ell & \text{if } i = j \\
\binom{j}{i} a^{\ell-(j-i)} & \text{if } i < j \\
0 & \text{if } i > j
\end{cases}
\]

Note that \(\binom{m}{n}\) is 0 by convention if $m$ and $n$ are non-negative integers with $m < n$.

**Proof:** We shall proceed by induction on $\ell$. Note that the result is true if $\ell = 0$ or $\ell = 1$. Suppose $J^\ell$ satisfies the proposition. We wish to show that $J^{\ell+1}$ satisfies the analogous result.

If $i > j$, note that since $J^\ell$ is an upper triangular matrix by the induction hypothesis and $J$ is also an upper triangular matrix, then $J^{\ell+1}$ is upper triangular and $(J^{\ell+1})_{ij} = 0$.

If $i = j$, then $(J^{\ell+1})_{ij} = \sum_k (J^\ell)_{ik}J_{kj} = (J^\ell)_{ii}J_{ii} = a^\ell a = a^{\ell+1}$.

If $i < j$, then

\[
(J^{\ell+1})_{ij} = \sum_k (J^\ell)_{ik}J_{kj} = (J^\ell)_{i,j-1}J_{j-1,j} + (J^\ell)_{ij}J_{jj}
\]
\[ \ell_{j-1-i} - \ell_{j-i} a_{\ell-1} + (j-i) \ell_{j-i} a_{\ell-1} = (j-i + 1) a_{\ell-i} = \ell_{j-i+1} a_{\ell-i}. \]

**Proof of Proposition 3** Note that the largest length of the coordinates of \( M_c \) is at most \( C_8 p \) where \( C_8 = 1/(2C_3) > 0 \) is a value depending on \( M \) (and hence \( T \)) but not \( p \).

Suppose \( a \) is the eigenvalue (with length less than 1) being considered, \( J \) is a Jordan block for the eigenvalue \( a \), and \( \text{proj}(M_c) \) is the projection of \( M_c \) onto the coordinates corresponding to this Jordan block. Then the coordinates of \( J^\ell \text{proj}(M_c) \) have length at most
\[
d_{\text{max}} \left( |a| \ell, \left( \frac{\ell}{1} \right) |a|^{\ell-1}, \ldots, \left( \frac{\ell}{d-1} \right) |a|^{\ell-d+1} \right) C_8 p \leq d \ell^d |a|^{\ell-d+1} C_8 p \to 0
\]
as \( p \to \infty \) if \( (\log |a|)C_6 < -1 \) and \( \ell > C_6 \log p \). The proposition follows.

**Proof of Proposition 4** In a Jordan block of size 1 and corresponding eigenvalue \( b \) with \( |b| > 1 \), the result should be straightforward since \( C_5 |b|^m \geq (C_1/C_3)p \) for some \( m \) no more than a multiple of \( \log p \) and since \( (C_1/C_3)p |b| < p/(2C_4) \).

In a Jordan block \( J \) of size \( c \) corresponding to eigenvalue \( b \) with \( |b| > 1 \), let’s consider the coordinates of
\[
J^\ell \text{proj}(M_c) = \begin{pmatrix} b & 1 & 0 & \cdots & 0 \\ 0 & b & 1 & \cdots & 0 \\ 0 & 0 & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b \end{pmatrix}^\ell \text{proj}(M_c)
\]
where $\text{proj}(M^c)$ is the projection of $M^c$ onto the coordinates corresponding to this Jordan block. Let $y = \begin{pmatrix} y_1 \\ \vdots \\ y_c \end{pmatrix} = J^c \text{proj}(M^c)$. If $|y_i| \geq C_5$ for some $i \in \{1, \ldots, c\}$, then at least one of the following statements fails:

Statement 1: $|y_c| < C_5/p^{c-1}$.
Statement 2: $|y_{c-1}| < C_5/p^{c-2}$.

etc.

Statement $c$: $|y_1| < C_5$.

Suppose statement $e$ is the first of these statements to fail. Suppose $r \geq 0$.

Let $w = \begin{pmatrix} w_1 \\ \vdots \\ w_c \end{pmatrix} = J^{e+r} \text{proj}(M^c) = J^r y$. Then

$$w_{c-e+1} = b^r y_{c-e+1} + \binom{r}{1} b^{r-1} y_{c-e+2} + \ldots + \binom{r}{e-1} b^{r-(e-1)} y_c$$

and

$$|w_{c-e+1}| \geq \frac{C_5}{p^{c-e}} |b|^r - \sum_{i=1}^{e-1} \binom{r}{i} |b|^{r-i} \frac{C_5}{p^{c-i+1}} \geq \frac{C_5 |b|^r}{p^{c-1}} \left( 1 - \frac{cr^e}{|b|^p} \right),$$

which for sufficiently large $p$ is at least $(C_1/C_3)p$ for some value of $r$ no more than a multiple of $\log p$. (Choose $r = \lfloor (c + 1) \log p/ \log |b| \rfloor$.) This multiple depends on $c$ and $b$ but does not depend on $p$. However, for a given matrix $T$, there are finitely many choices for $c$ and $b$. So for some value $s$ no more than a multiple of $\log p$ (where the multiple only depends on $T$), some coordinate of $J^s \text{proj}(M^c)$ has length at least $(C_1/C_3)p$ provided that $p$ is sufficiently large. Furthermore, if this value $s$ is the smallest non-negative value for which this statement holds, then this coordinate has length no more than $(1/2C_4)p$ since $(C_1/C_3)(|b| + 1) < 1/(2C_4)$. We can suppose that this coordinate is the coordinate with the largest length.

$\square$
To complete the proof of Lemma 3, note that if some coordinate of \( \mathbf{c} \) has length at least \( C_1 p \) (where \( C_1 \) is defined in Proposition 4), then the lemma follows directly with \( \ell = 0 \). Otherwise all entries in \( \mathbf{c} \) have length less than \( C_1 p \). Thus by Proposition 4 all entries of \( M \mathbf{c} \) must have length less than \( (C_1/C_3)p \). By Proposition 3 if \( \ell > C_6 \log p \), then all coordinates of

\[
\begin{pmatrix}
J_{1}^\ell & 0 & \cdots \\
0 & J_{2}^\ell & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

which correspond to an eigenvalue of length less than 1 will, for sufficiently large \( p \), all have length less than \( C_5 \). By Proposition 2 some coordinate of

\[
\begin{pmatrix}
J_{1}^s & 0 & \cdots \\
0 & J_{2}^s & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

will have length at least \( C_5 \). Thus by Proposition 4 there exists a value \( s < C_2 \log p \) such that for sufficiently large \( p \), the coordinate with the largest length of

\[
\begin{pmatrix}
J_{1}^s & 0 & \cdots \\
0 & J_{2}^s & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

has length at least \( (C_1/C_3)p \) but no more than \( p/(2C_4) \), and so by Proposition 1

\[
(T^t)^s \mathbf{c} = M^{-1}\begin{pmatrix}
J_{1}^s & 0 & \cdots \\
0 & J_{2}^s & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix} M \mathbf{c}
\]

has a coordinate of length at least \( C_1 p \) but no more than \( p/2 \) if \( p \) is sufficiently large.

To prove Theorem 1 note that if \( C_1 p \leq |c_r| \leq p/2 \) for some \( r \) in \( \{1, \ldots, d\} \), then

\[
\left| 1 + \sum_{r=1}^{d} q^{c_r} \right| \leq C_9
\]
for some constant $C_9 < d + 1$ and

$$|\hat{P}_1(c)| = \left| \frac{1 + \sum_{r=1}^{d} q^{cr}}{d + 1} \right| \leq \frac{C_9}{d + 1} < 1;$$

otherwise if $c \neq 0$, then

$$|\hat{P}_1(c)| \leq 1.$$

Suppose $p$ is sufficiently large and $s = \lceil C_2 \log p \rceil$. Lemma 2, Lemma 3 and the previous observation imply

$$|\hat{P}_{s+1}(c)| = \prod_{j=0}^{s} |\hat{P}_1((T^t)^{j}c)| \leq \frac{C_9}{d + 1}$$

for all $c \neq 0$. Thus if $r$ is a positive integer and $c \neq 0$, then

$$|\hat{P}_{r(s+1)}(c)| = \prod_{i=0}^{r-1} \prod_{j=0}^{s} |\hat{P}_1((T^t)^{j+i(s+1)}c)| \leq \left( \frac{C_9}{d + 1} \right)^r.$$

If $r$ is a sufficiently large multiple of $\log p$, then

$$\left( \frac{C_9}{d + 1} \right)^r < \frac{1}{p^{d+1}},$$

and

$$\sum_{c \neq 0} |\hat{P}_{r(s+1)}(c)|^2 < \frac{1}{p^2}.$$

Note that $r(s+1) < C(\log p)^2$ for some constant $C > 0$. Then, from Lemma 1 and Lemma 2, if $n \geq C(\log p)^2$, then

$$\|P_n - U\| \leq \|P_{r(s+1)} - U\| \leq \frac{1}{2} \sqrt{\sum_{c \neq 0} |\hat{P}_{r(s+1)}(c)|^2} < \frac{1}{2p} \to 0$$

as $p \to \infty$.  \[\square\]
5 Proof of Theorem 2

Suppose $T$ has an eigenvalue (over $\mathbb{C}$) which is an $m$-th root of unity. Then $T^m$ has an eigenvalue 1 over $\mathbb{C}$. So the characteristic polynomial of $T^m$ over $\mathbb{C}$ has 1 as a root. Thus the characteristic polynomial of $T^m$ over $\mathbb{Z}/p\mathbb{Z}$ has 1 as a root, and 1 is an eigenvalue of $T^m$ over $\mathbb{Z}/p\mathbb{Z}$. We can write $(T^m)^t = MJM^{-1}$ for some matrix $J$ in Jordan block form over $\mathbb{Z}/p\mathbb{Z}$ and so $T^m = (M^{-1})^t J^t M^t$. Thus there exists a basis $b_1, \ldots, b_d$ for $(\mathbb{Z}/p\mathbb{Z})^d$ over $\mathbb{Z}/p\mathbb{Z}$ such that if $p(v) = a_1$ when $v = a_1 b_1 + \ldots + a_d b_d$, we have $p(T^m b_1) = 1$ and $p(T^m b_j) = 0$ if $j = 2, \ldots, d$. Note that for all $n$,

\[ X_{n+m} = T^m X_n + T^{m-1} B_n + \ldots + TB_{n+m-2} + B_{n+m-1} \quad (\text{mod } p), \]

and so $p(X_{n+m}) = p(T^m X_n) + p(T^{m-1} B_n) + \ldots + p(TB_{n+m-2}) + p(B_{n+m-1})$. Note that $p(T^m X_n) = p(X_n)$. There are at most $d+1$ possible values of $p(T^m B_n)$, $d+1$ possible values of $p(TB_{n+m-2})$, and $d+1$ possible values of $p(B_{n+m-1})$. So there are at most $(d+1)^m$ possible values of $p(T^m B_n) + \ldots + p(TB_{n+m-2}) + p(B_{n+m-1})$. Thus $p(X_0), p(X_m), p(X_{2m}), \ldots$ forms a random walk on $\mathbb{Z}/p\mathbb{Z}$ with support of size $u$ which is at most $(d+1)^m$.

By Greenhalgh [4], given $\epsilon > 0$, there exists a value $C_2 > 0$ such that if $c < C_2 p^{2/(u-1)}$, then the variation distance of $p(X_{cm})$ from uniform in $\mathbb{Z}/p\mathbb{Z}$ is at least $1 - \epsilon$. Thus, in the random process on $(\mathbb{Z}/p\mathbb{Z})^d$, we get $\|P_{cm} - U\| > 1 - \epsilon$. Results of Asci [1] imply that $b$ is no larger than 2. The theorem follows. \hfill \Box

6 Questions for Further Study

One question to consider is whether the order $(\log p)^2$ rate of convergence can be improved to order $(\log p) \log(\log p)$ similar to Theorem 2 in [6]. For
diagonal matrices with non-zero eigenvalues all different from ±1, Theorem 4.5 of Asci [1] confirms this, but for more general cases, the question is open. Another question is analogous to one explored in Chung, Diaconis, and Graham [2]: Are order \( \log p \) steps sufficient for typical values of \( p \)?

Another question for further study is to see how large the value \( b \) in Theorem 2 can be. Also, given \( \epsilon > 0 \), will there exist a \( C > 0 \) such that if \( n \leq Cp^2 \), then \( \|P_n - U\| > 1 - \epsilon \) for sufficiently large \( p \)?

Another area to explore involves random processes on fields with \( p^d \) elements where \( p \) is a prime; these random processes would be of the form \( X_{n+1} = AX_n + b_n \) where \( b_n \) and \( X_n \) are random variables over this field and \( A \) is a fixed element of this field. Since elements of this field can be represented as a \( d \times d \) matrix over \( \mathbb{Z}/p\mathbb{Z} \) (see, for example, pp. 64-65 of [9]), perhaps some natural connections can be made between the random processes studied here and such random processes on finite fields.

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