AN ANALOGUE OF THE STRENGTHENED HANNA NEUMANN CONJECTURE FOR VIRTUALLY FREE GROUPS AND VIRTUALLY FREE PRODUCTS

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The Friedman–Mineyev theorem, earlier known as the (strengthened) Hanna Neumann conjecture, gives a sharp estimate for the rank of the intersection of two subgroups in a free group. We obtain an analogue of this inequality for any two subgroups in a virtually free group (or, more generally, in a group containing a free product of left-orderable groups as a finite-index subgroup).

0. Introduction

The Hanna Neumann Conjecture (1957), proved independently by Mineyev and Friedman is the following fact.

Friedman–Mineyev theorem[Mi12a], [Mi12b], [Fr14]. For any nontrivial subgroups $A$ and $B$ of a free group $F$,

$$\text{rank}(A \cap B) - 1 \leq (\text{rank}(A) - 1) \cdot (\text{rank}(B) - 1);$$

(the classical Hanna Neumann conjecture)

moreover, for any system of representatives $S$ of the double cosets $AsB$ in $F$,

$$\sum_{s \in S} \text{rank}(A \cap sBs^{-1}) \leq \text{rank}(A) \cdot \text{rank}(B),$$

(the strengthened Hanna Neumann conjecture)

where $\text{rank}(H) \overset{\text{def}}{=} \max(0, \text{rank}(H) - 1)$ is the reduced rank of a free group $H$.

Alternative proofs and generalisations of this result can be found, e.g., in [D12], [AMS14], [Za14], [ASS15], [Nos16], [HW16], [Iv17], [JZ17] and [KP20]. In particular, the following analogue of the classical Hanna Neumann conjecture for free subgroups of a virtually free group was obtained in [KP20]:

for any free subgroups $A$ and $B$ of a virtually free group $G$ containing a free finite-index subgroup $F$

$$\text{rank}(A \cap B) \leq |G:F| \cdot \text{rank}(A) \cdot \text{rank}(B).$$

This estimate strengthens earlier known inequalities [Za14], [ASS15] (and is already sharp). We generalise this fact in two directions:

– first, we obtain an analogue of the strengthened Hanna Neumann conjecture;
– and secondly, our estimate applies to arbitrary subgroups $A$ and $B$ of a virtually free group.

Theorem on intersection of subgroups in virtually free groups. For any subgroups $A$ and $B$ of a virtually free group $G$ containing a free group $F$ as a finite-index subgroup and for any system of representatives $S$ of the double cosets $AsB$ in $G$,

$$\sum_{s \in S} \overline{\text{rk}}(A \cap sBs^{-1}) \leq |G:F| \cdot \overline{\text{rk}}(A) \cdot \overline{\text{rk}}(B).$$

In particular, $\overline{\text{rk}}(A \cap B) \leq |G:F| \cdot \overline{\text{rk}}(A) \cdot \overline{\text{rk}}(B)$.

Here $\overline{\text{rk}}(H)$ is the virtual reduced rank of a virtually free group: $\overline{\text{rk}}(H) \overset{\text{def}}{=} \frac{1}{|H:F|} \cdot \max(0, \text{rank}(K) - 1)$, where $K$ is a finite-index free subgroup of $H$. It is easy to show that the virtual reduced rank is well-defined (i.e. it does not depend on the choice of a free subgroup $K$); and $\overline{\text{rk}}(H) = \text{rank}(H)$, if $H$ is free. Note that the virtual reduced rank of a virtually free group coincides with the rank gradient of this group [La05].

Actually, the theorem above is a special case of a more general main theorem of this paper (see the next section), which is about intersections of subgroups in virtually free products. In particular, our main theorem generalises the following known analogue of the strengthened Hanna Neumann conjecture.

The work of the first author was supported by the Russian Science Foundation, project no. 22-11-00075.
The work of the second author was supported by (Polish) Narodowe Centrum Nauki, grant UMO-2018/31/G/ST1/02681.
**Theorem AMS** [AMS14] (see also [Iv17]). For any subgroups $A$ and $B$ of a free product $G = \bigast_{i \in I} G_i$ of left-orderable groups $G_i$ and for any system of representatives $S$ of double cosets $AsB$ in $G$

\[
\sum_{s \in S} \text{rank}_K(A \cap sBs^{-1}) \leq \text{rank}_K(A) \cdot \text{rank}_K(B). \quad \text{In particular, } \text{rank}_K(A \cap B) \leq \text{rank}_K(A) \cdot \text{rank}_K(B).
\]

Here, $\text{rank}_K(H)$ is the *reduced Kurosh rank* of a subgroup $H \subseteq G = \bigast_{i \in I} G_i$, which is defined as follows: the subgroup $H$ decomposes (by the Kurosh theorem) into a free product $H = \left( \bigast_{j \in J} H_j \right) * F$, where each $H_j$ is nontrivial and conjugate to a subgroup of one of $G_i$, and $F$ is free and trivially intersects all conjugate to subgroups $G_i$; then $\text{rank}_K(H) \overset{\text{def}}{=} \max(0, |J| + \text{rank}(F) - 1)$.

Our proof of the main theorem is based on Mineyev’s approach [Mi12b], but our definitions are somewhat different; therefore, we prove everything “from scratch”; thus, this paper contains yet another alternative (simpler) proof of the Friedman–Mineyev theorem.

The main features of our argument are that, considering actions of groups on forests, we

- never refer explicitly to the quotient graph of this action
- and never require the cocompactness of the action.

The authors thank an anonymous referee for useful remarks. The first author thanks also the Theoretical Physics and Mathematics Advancement Foundation “BASIS”.

### 1. Main theorem

If a group $G$ contains a free product $F = \bigast_{i \in I} G_i$ of infinite groups $G_i$ as a finite-index subgroup, then, for any subgroup $H \subseteq G$, the *virtual reduced Kurosh rank* $\text{rk}(H)$ with respect to the family of subgroups $G_i$ is defined as

\[
\text{rk}(H) \overset{\text{def}}{=} \frac{\text{rank}_K(K)}{|H:K|},
\]

where $K$ is a subgroup of finite index in $H$ contained in $F$, and $\text{rank}_K(K)$ is the (usual) reduced Kurosh rank of a subgroup $K$ of $F = \bigast_{i \in I} G_i$. This value is well-defined, i.e. it does not depend on the choice of $K$ (because of an analogue of the Schreier formula for the Kurosh rank [Ku83]), but is not conjugation-invariant, i.e. the numbers $\text{rk}(H)$ and $\text{rk}(gHg^{-1})$ may differ. To remedy this unpleasant feature, we define the *total virtual reduced Kurosh rank* $\text{rk}(H)$ (with respect to the family $\{G_i \mid i \in I\}$) as: $\text{rk}(H) = \sum_{j=1}^{n} \text{rk}(g_jHg_j^{-1})$, where $\text{rk}$ is the virtual reduced Kurosh rank with respect to the given family of subgroups, and $g_1, \ldots, g_n$ are representatives of the right cosets of $F$ in $G$. It is easy to see that this value is conjugation-invariant and does not depend on the choice of representatives $g_j$. Note that $\text{rk}(H) = 0$ for finite $H$.

**Main theorem.** Let a group $G$ be a virtually free product of left-orderable groups, i.e. $G$ contains a finite-index subgroup $F = \bigast_{i \in I} G_i$, where all groups $G_i$ are left-orderable. Let $A$ and $B$ be subgroups of $G$, and let $S$ be a set of distinct representatives of double cosets $AgB$ in $G$. Then $\sum_{s \in S} \text{rk}(A \cap sBs^{-1}) \leq \text{rk}(A) \cdot \text{rk}(B)$, where $\text{rk}(H)$ is the total virtual reduced Kurosh rank of $H \subseteq G$ (with respect to the family $\{G_i \mid i \in I\}$).

In particular, $\text{rk}(A \cap B) \leq \text{rk}(A) \cdot \text{rk}(B)$.

This generalises earlier known results:

- the case, where $F = G$, of our theorem is Theorem AMS [AMS14] (if, in addition, all $G_i$ are infinite cyclic, then we obtain the Friedman–Mineyev theorem, earlier known, as the strengthened Hanna Neumann conjecture);
- in the case, where $A$ and $B$ are free groups trivially intersecting subgroups conjugate to free factors $G_i$, the assertion “In particular” is the main result of [KP20].

To derive the theorem on intersections of subgroups in virtually free groups (see Introduction) from the main theorem, it suffices to note that the virtual reduced rank $\text{rk}(H)$ of a virtually free subgroup $H \subseteq G$ coincides with the virtual reduced Kurosh rank with respect to any family of infinite cyclic subgroups whose free product is $F$. Therefore, all terms in the definition of the total virtual rank $\text{rk}(H)$ are equal (and their number is the index of $F$), i.e. $\text{rk}(H) = |G:F| \cdot \text{rk}(H)$ in this case.
Strange as it may seem, we have not succeeded in finding the following simple lemma in the literature.

**Orbit-intersection lemma.** Let $A$ and $B$ be subgroups of a group $G$ that acts freely on a set $X$, and let $D$ be a set of distinct representatives of double cosets $AgB$. Then

$$\sum_{d \in D} (\text{the number of } (A^d \cap B)\text{-orbits}) \leq (\text{the number of } A\text{-orbits}) \cdot (\text{the number of } B\text{-orbits}).$$

Moreover, for any $A$-invariant set $Y \subseteq X$ and any $B$-invariant set $Z \subseteq X$,

$$\sum_{d \in D} (\text{the number of } (A^d \cap B)\text{-orbits in } (d^{-1} \circ Y) \cap Z) \leq (\text{the number of } A\text{-orbits in } Y) \cdot (\text{the number of } B\text{-orbits in } Z).$$

**Proof.** Suppose that $G \times X \overset{\sigma}{\to} X$ is a free action, and $X/H$ is the set of orbits of the action of a subgroup $H \subseteq G$. Consider the mapping

$$\Phi: \{(d, U) \mid d \in D, \ U \in \left( (d^{-1} \circ Y) \cap Z \right)/\left( A^d \cap B \right) \} \to Y/A \times Z/B, \quad (d, (A^d \cap B) \circ x) \mapsto (A \circ d \circ x, B \circ x).$$

The assertion of the lemma follows immediately from the following observations: this mapping is

- well-defined,
- i.e., it does not depend on the choice of $x$ in the $(A^d \cap B)$-orbit (obviously),
- and injective:

indeed, $(A \circ d \circ x, B \circ x) = (A \circ d' \circ x', B \circ x')$ means that $d' \circ x' \in A \circ d \circ x$ and $x' \in B \circ x$, i.e. $(d' \circ x') \cap (Ad) \neq \emptyset$ (because the action is free) and, hence, $d' = d$ (by the definition of $D$); so, $x' \in (A^d \circ x) \cap (B \circ x) = (A^d \cap B) \circ x$, as required.

3. **Actions on forests**

All graphs in this paper are directed. Let a group $G$ act on a forest $\Gamma$ freely on edges (i.e. the stabiliser of each edge is trivial). A set $E$ of orbits of edges of $\Gamma$ is called *maximal essential* if $E$ is an inclusion maximal set such that each component of the forest $\Gamma \setminus \bigcup E$ (i.e. each component of the forest obtained from $\Gamma$ by deleting all edges from each orbit of the orbit-set $E$), which is not a component of $\Gamma$, has a nontrivial stabiliser. The following lemma is a simple (and probably known) fact on groups acting on trees.

Note that, actually, no component of $\Gamma$ has trivial stabiliser if the number of components is finite and the group is infinite (but we assume neither these conditions to hold by default).

**Kurosh-rank lemma.** A group $G$ acting on a tree $\Gamma$ freely on edges decomposes into a free product: $G = F \ast \left( \bigstar_{i \in I} G_i \right)$, where $F$ is a free group acting on $\Gamma$ freely, and $G_i \neq \{1\}$ are stabilisers of some vertices; if the Kurosh rank of this decomposition is finite (i.e. $\text{rank}(F) + |I| < \infty$), then the cardinality of any maximal essential set $E$ equals the reduced Kurosh rank of this decomposition: $|E| = \max(0, \text{rank}(F) + |I| - 1)$.

**Sketch of a proof.** The first assertion is well-known. To prove the second assertion, for any edge $e$, consider the components $X$ and $Y$ of the forest $\Gamma \setminus (G \circ e)$ connected by $e$. The ping-pong lemma implies immediately that

$$G = \begin{cases} \text{St}(X) \ast \langle g \rangle, & \text{if } g \circ X = Y \text{ for some } g \in G \text{ (which necessarily acts freely on } \Gamma) ; \\ \text{St}(X) \ast \text{St}(Y), & \text{if } g \circ X \neq Y \text{ for any } g \in G . \end{cases}$$

An obvious induction completes the proof (as the Kurosh rank is finite). This lemma also follows from [AMS14] (Theorem 2.4, using the arguments of Proposition 3.4).

We want to generalise this simple fact to the case, where $\Gamma$ is a forest consisting of finitely many trees: $\Gamma = T_1 \sqcup \ldots \sqcup T_n$. In this case, the *virtual reduced Kurosh rank* of the (action of) the group $G$ is naturally defined: choose in $G$ a finite-index subgroup $H$ that stabilises a tree $T_j$ and, therefore, decomposes into a free product $H = F \ast \left( \bigstar_{i \in I} G_i \right)$, where $F$ is a free group acting on $T_j$ freely, and $G_i \neq \{1\}$ are stabilisers of some vertices of $T_j$; the reduced Kurosh rank of this subgroup (with respect to the given action on $T_j$) is $\text{rk}(H) \overset{\text{def}}{=} \max(0, \text{rank}(F) + |I| - 1)$, and the virtual reduced Kurosh rank of $G$ (with respect to the given action on $\Gamma$ and given component $T_j$ of $\Gamma$) is naturally defined as:

$$\text{vrk}_j(G) = \frac{1}{|G:H|} \cdot \text{rk}(H).$$

It is easy to see that this value does not depend on the choice of the subgroup $H$ (if nontrivial vertex stabilisers are infinite), but can depend on $j$. We call the value $\sum_j \text{vrk}_j(G)$ the *total virtual reduced Kurosh rank* of this action.
Virtual-Kurosh-rank lemma. Suppose that a group $G$ acts freely on edges on a forest $\Gamma = T_1 \sqcup \ldots \sqcup T_n$ consisting of trees $T_j$, the stabiliser of each $T_j$ has finite Kurosh rank, and nontrivial vertex stabilisers are infinite. Then

$$\sum_{j=1}^n \overline{\text{rk}}_j(G) = |E|$$

for each maximal essential set $E$.

Proof. Suppose that $\Gamma = \Gamma_1 \sqcup \ldots \sqcup \Gamma_k$ and, on each $G$-invariant forest $\Gamma_i$, the action of $G$ is transitive on components (i.e., for any components $T_1, T_m \subseteq \Gamma_i$, there exists $g \in G$ such that $g \cdot T_1 = T_m$). Then $E = E_1 \sqcup \ldots \sqcup E_k$, where $E_i = \{ G \circ e \in E \mid G \circ e \subseteq T_i \}$ is a maximal essential set of orbits of edges of the forest $\Gamma_i$. Therefore, it suffices to prove assertion for the case, where the action of $G$ on $\Gamma$ is transitive on components of $\Gamma$.

In this case, all stabilisers $H_j = \text{St}(T_j)$ of trees $T_j$ are conjugate and, hence, isomorphic and act on the correspondingly trees similarly. In particular, $\overline{\text{rk}}(H_j)$ does not depend on $j$. Moreover, $|G:H_j| = n$ for all $j$ (because the length of an orbit equals the index of the stabiliser). Therefore,

$$\sum_{j=1}^n \overline{\text{rk}}_j(G) = \sum_{j=1}^n \frac{1}{|G:H_j|} \cdot \overline{\text{rk}}(H_j) = \sum_{j=1}^n \frac{1}{n} \cdot \overline{\text{rk}}(H_j) = \overline{\text{rk}}(H_1).$$

On the other hand, the set of $H_1$-orbits of edges $E' = \{ G \circ e \cap T_1 \mid G \circ e \subseteq E \}$ is, obviously, maximal essential with respect to the action of $H_1$ on $T_1$. Therefore, $|E| = |E'| = \overline{\text{rk}}(H_1)$ (the latter equality follows from the Kurosh-rank lemma). This completes the proof.

4. Actions on ordered forests

We say that a graph is ordered if it is equipped with a partial order on the set of edges inducing a linear order on the set of edges of each connected component.

Induced-action lemma [KP20]. If a group $G$ has a subgroup $F$ of a finite index $n$, which acts on an ordered tree $T$ preserving the order, then $G$ can act preserving the order on an ordered forest consisting of $n$ trees; the stabilisers of vertices and edges under this action are conjugate to the stabilisers of vertices and edges under the initial action of $F$ on $T$.

Proof. Let $S \ni 1$ be a system of representatives of the left cosets of $F$ in $G$ (i.e. $|S| = n$). Thus, each element $g \in G$ decomposes uniquely into a product $g = s(g) f(g)$ of an element $s(g) \in S$ and an element $f(g) \in F$.

Take the ordered forest $L = \bigcup_{s \in S} sT$ consisting of $n$ copies $sT$ of the ordered tree $T$ (edges from different copies are incomparable) and consider the usual induced action of $G$ on $L$: $g \circ st = s(g) \cdot f(g) \circ t$. Clearly, this action satisfies all requirements. This completes the proof.

An edge $e$ of an ordered forest with an order-preserving action of a group $H$ is called important (or $H$-important) if it is the maximal edge on an bi-infinite line $T(e)$ intersecting only finitely many $H$-orbits of edges. Note that if $K \subseteq H$, then any $K$-important edge is $H$-important.

Important-edge lemma. If a group $G$ acts on an ordered forest $T$ preserving the order and freely on edges, then

- the set $\mathcal{E}$ of orbits of important edges contains a maximal essential set;
- each finite subset $\mathcal{E}' \subseteq \mathcal{E}$ is contained in a maximal essential set.

In particular, the total virtual reduced Kurosh rank of this action

- equals $|\mathcal{E}|$ if $|\mathcal{E}| < \infty$,
- is infinite if $\mathcal{E}$ is infinite.

Proof. The assertion “In particular” follows from the main assertion by the virtual-Kurosh-rank lemma. It remains to prove the main assertion. Take a finite subset $\mathcal{E}'$ of $\mathcal{E}$ and put $E = \bigcup \mathcal{E}$ and $E' = \bigcup \mathcal{E}'$ (i.e. $e \in E$ if and only if $G \circ e \in \mathcal{E}$; and similarly $e \in \mathcal{E}'$ if and only if $G \circ e \in \mathcal{E}'$; so $E$ and $E'$ are sets of edges, while $\mathcal{E}$ and $\mathcal{E}'$ are sets of orbits of edges). We have to establish two facts:

1) the stabiliser $\text{St}(K)$ of each component $K$ of the forest $T \setminus E$ has either a fixed point or an invariant line in $K$;
2) but the stabiliser of each component $K$ of the forest $T \setminus E'$ is nontrivial if there exists an important edge $e \in E'$ in $T$ incident to a vertex of $K$.

The both facts are easy to prove.

1) If 1) does not hold, then the stabiliser of a component $K$ of $T \setminus E$ contains a rank-two free subgroup $F(x, y) \subseteq \text{St}(K)$ acting freely on $K$ (because each non-dihedral group nontrivially decomposable into a free product contains a free subgroup that trivially intersects the free factors; the group $G$ cannot be dihedral, because $G$ is torsion-free if $T$ has at least one edge). Let $l_x$ and $l_y$ be invariant lines in $K$ for elements $x$ and $y$, respectively. The intersection of these lines is a finite graph: either an interval, a point, or the empty set (it cannot be a ray, as is known). Let us connect the lines $l_x$ and $l_y$ by a path $\pi$. Let us choose finite intervals $p_x$ and $p_y$ such that $l_x = \bigcup_{k \in \mathbb{Z}} x^k \circ p_x$
and \( l_y = \bigcup_{k \in \mathbb{Z}} y^k \circ p_x \); and let us take the maximal edges \( e_x \) and \( e_y \) on the intervals \( p_x \) and \( p_y \). Without loss of generality, we can assume that

- \( x \circ e_x < e_x \) and \( y \circ e_y < e_y \) (replace \( x \) with \( x^{-1} \) and/or \( y \) on \( y^{-1} \), if this is not the case);
- \( \left( \bigcup_{k=0}^{\infty} x^k \circ p_x \right) \cap (l_y \cup \pi) = \emptyset = \left( \bigcup_{k=0}^{\infty} y^k \circ p_y \right) \cap (l_x \cup \pi) \) (replace \( p_x \) with \( x^n \circ p_x \) and/or \( p_y \) with \( y^n \circ p_y \) for sufficiently large \( n \in \mathbb{N} \), if this not the case).

Let us connect now \( p_x \) and \( p_y \) by a path \( p \supset (p_x \cup p_y) \). The maximal edge \( e \) of \( p \) the maximal edge on the line \( p \cup \left( \bigcup_{k=0}^{\infty} x^k \circ p_x \right) \cup \left( \bigcup_{k=0}^{\infty} y^k \circ p_y \right) \). Thus the edge \( e \) is important (Fig. 1). This contradiction completes the proof of 1).

2) Suppose that an important edge \( e \in E' \) ends at a vertex of \( K \). If the edge \( g \circ e \) ends also at a vertex of \( K \), then \( g \in \text{St}(K) \) and, therefore, \( \text{St}(K) \neq \{1\} \) if \( g \neq 1 \). Hence, it suffices to consider the case, where there are only finitely many (at most \( 2|E'| \)) important edges from \( E' \) incident to vertices of \( K \). Let \( e \in E' \) be the minimal edge of \( E' \) incident to a vertex of \( K \). Then an infinite ray of \( T(e) \) (from the definition of the importance) must lie in \( K \) (because of the minimality of \( e \)). Since this ray can intersect only finitely many orbits of edges (by the definition of importance), we obtain an infinite set of edges of \( K \) lying in the same orbit. Hence, \( \text{St}(K) \neq \{1\} \) as required.

5. Proof of the main theorem

Put \( n = |G:F| \) and let \( T \) be the (Bass–Serre) tree for the decomposition \( F = \ast_{i \in I} G_i \), i.e., \( F \) acts on \( T \) freely on edges and in such a way that the stabiliser of each vertex is conjugate to one of the factors \( G_i \). The tree \( T \) can be ordered: the order on the set of edges of \( T \) is induced by a left-invariant order on group \( F \) (which, as is known, exists [Vi49], [DS20]). Thus, the action of \( F \) on \( T \) preserves the order and is free on edges. By the induced-action lemma, the group \( G \) acts on an ordered forest \( \Gamma = T_1 \cup \ldots \cup T_n \) consisting of \( n \) trees \( T_j \) transitively on components, freely on edges, and preserving the order. Moreover, \( \text{St}(T_i) = F \) and \( T_j = g_j T_1 \), where \( g_1 = 1, g_2, \ldots, g_n \) are representatives of the left cosets of \( F \) in \( G \).

The groups \( A \) and \( B \) act on the forest \( \Gamma \) freely on edges and preserving the order. Then

\[
\sum_{s \in S} \left( \text{the number of } (A^s \cap B)\text{-orbits } (A^s \cap B)\text{-important edges} \right) \leq \sum_{s \in S} \left( \text{the number of } (A^s \cap B)\text{-orbits of edges that are both } A^s\text{-important and } B\text{-important} \right) = \sum_{s \in S} \left( \text{the number of } (A^s \cap B)\text{-orbits in the set } (s^{-1} \circ \{A\text{-important edges}\}) \cap \{B\text{-important edges}\} \right) \leq (\text{the number of } A\text{-orbits of } A\text{-important edges}) \cdot (\text{the number of } B\text{-orbits of } B\text{-important edges}),
\]

where

- the first inequality holds because any \( H\)-important edge is \( G\)-important if \( H \subseteq G \);
- the equality holds because an edge \( e \) is \( A\)-important if and only if \( s^{-1} \circ e \) is \( A^s\)-important;
- the last inequality is the orbit-intersection lemma applied to

\[
Y = \{A\text{-important edges of } \Gamma\} \subseteq X = \{\text{edges of } \Gamma\} \supseteq Z = \{B\text{-important edges of } \Gamma\}.
\]

By the important-edge lemma, the set of orbits of important edges is maximal essential (if a maximal essential set is finite). Thus, the number of \( C\)-orbits of \( C\)-important edges in \((*)\) is the cardinality of the maximal essential set with
respect to the action of the group $C$ on $\Gamma$ (where $C$ is $A$, $B$, or $A^s \cap B$). Therefore, by the virtual-Kurosh-rank lemma

$$\sum_{s \in S} \tau(A \cap sBs^{-1}) \leq \tau(A) \cdot \tau(B), \quad \text{where } \tau(H) = \sum_{j=1}^{n} \tau_{j}(H)$$

and $\tau_{j}(H)$ is the virtual reduced Kurosh rank with respect to (the corresponding decomposition of) $St(T_j)$. It remains to note that the “corresponding decomposition” of the stabiliser of the $j$th tree has form $St(T_j) = g_j F g_j^{-1} = \bigast_{i \in I} g_j G_i g_j^{-1}$.

This completes the proof.

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