UNCONDITIONALITY WITH RESPECT TO ORTHONORMAL SYSTEMS IN NONCOMMUTATIVE $L_2$ SPACES

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ABSTRACT. Orthonormal systems in commutative $L_2$ spaces can be used to classify Banach spaces. When the system is complete and satisfies certain norm condition the unconditionality with respect to the system characterizes Hilbert spaces. As a noncommutative analogue we introduce the notion of unconditionality of operator spaces with respect to orthonormal systems in noncommutative $L_2$ spaces and show that the unconditionality characterizes operator Hilbert spaces when the system is complete and satisfy certain norm condition. The proof of the main result heavily depends on free probabilistic tools such as contraction principle for $*$-free Haar unitaries, comparison of averages with respect to $*$-free Haar unitaries and $*$-free circular elements and $K$-cotype, type 2 and cotype 2 with respect to $*$-free circular elements.

1. Introduction

In Banach space theory orthonormal systems in classical $L_2$ spaces have been frequently used for many purposes. For an orthonormal system $\Phi = (\phi_i)_{i \geq 1}$ in $L_2(M, \mu)$, where $(M, \mu)$ is a measure space, we consider so called “$\Phi$-average” of a sequence $(x_i)_{i=1}^n$ in a Banach space $X$ as follows.

$$\left\| \sum_{i=1}^n \phi_i \otimes x_i \right\|_{L_2(\mu; X)} = \left[ \int_M \left\| \sum_{i=1}^n \phi_i(t) x_i \right\|_X^2 d\mu(t) \right]^{\frac{1}{2}}.$$ 

Many properties of Banach space are defined using this “averages”. For example, type 2 and cotype 2 conditions are defined by comparing averages with respect to rademacher systems $(\varepsilon_i)_{i \geq 1}$ in $L_2[0, 1]$ and unit vector systems $(e_i)_{i \geq 1}$ in $\ell_2$, where $\varepsilon_i(t) = \text{sign}(\sin(2^i \pi t))$, $t \in [0, 1]$ and $i = 1, 2, \ldots$.

Thus, it is natural to expect orthonormal systems in noncommutative $L_2$ spaces play a similar role in operator space theory. In this paper we focus on an operator space version of the result in [2]. In [2] another example of using averages, namely, the notion of $\Phi$-unconditionality was considered. We say that a Banach space $X$ is $\Phi$-unconditional if

$$\left\| \sum_{i=1}^n \varepsilon_i \phi_i \otimes x_i \right\|_{L_2(\mu; X)} \sim \left\| \sum_{i=1}^n \phi_i \otimes x_i \right\|_{L_2(\mu; X)}$$

for any $n \in \mathbb{N}$, $(x_i)_{i=1}^n \subseteq X$ and any $\varepsilon_i \in \{\pm 1\}$, where ‘$\sim$’ implies equivalent allowing constants. The authors showed that if $(\phi_i)_i$ is complete and

$$\left\| \sup_i |\phi_i| \right\|_{L_2(\mu)} = \left[ \int_M \left\| \sup_i |\phi_i| \right\|_X^2 d\mu(t) \right]^{\frac{1}{2}} < \infty,$$

then a Banach space is $\Phi$-unconditional if and only if it is isomorphic to a Hilbert space.

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In order to consider an operator space analogue of the above result we first need to define the right notion of unconditionality. Let $\mathcal{M}$ be a von Neumann algebra and $\Phi = (\phi_i)_{i \geq 1}$ be an orthonormal system in $L_2(\mathcal{M})$. We will define that an operator space $E$ is $\Phi$-unconditional if

$$\left\| \sum_{i=1}^n \phi_i \otimes u_i \otimes x_i \right\|_{L_2(\mathcal{M} \otimes M_n; E)} \sim \left\| \sum_{i=1}^n \phi_i \otimes x_i \right\|_{L_2(\mathcal{M}; E)}$$

for any $n, m \in \mathbb{N}$, $(x_i)_{i=1}^n \subseteq E$ and any unitaries $u_i \in M_n$. Actually, we need some restrictions on $\mathcal{M}$ and $E$ to make $L_2(\mathcal{M}; E)$ sense. Recall that a $C^*$-algebra $A$ has WEP (Lance’s weak expectation property) if the inclusion map $i_A : A \hookrightarrow A^{**}$ factors completely positively and completely contractively through $B(H)$ for some Hilbert space $H$. A $C^*$-algebra $B$ has QWEP if there is a WEP $C^*$-algebra $A$ and two sided ideal $I \subseteq A$ such that $B \cong A/I$. Let $C^*(\mathbb{F}_\infty)$ be the full group $C^*$-algebra of the free group with infinite number of generators. Then we say that an operator space $E$ is locally-$C^*(\mathbb{F}_\infty)$ (chapter 21. of [18] and [5]) if

$$d_f(E) := \sup_{F \subseteq E, \text{finite dimensional}} \inf \{ d_{\text{cb}}(F, G) : G \subseteq C^*(\mathbb{F}_\infty) \} < \infty.$$ 

If $\mathcal{M}$ has QWEP and $E$ is locally-$C^*(\mathbb{F}_\infty)$, then the above $L_2(\mathcal{M}; E)$ is well-defined. (See section 2 for the details.)

In order to impose a norm condition on $\Phi$ we need another concept. Recall that (see [3] for the details) for a sequence $(a_i)_{i \geq 1}$ in $L_p(\mathcal{M})$ ($1 \leq p < \infty$) we define

$$\left\| \sup_i |a_i| \right\|_p := \inf_{a_i = a b} \|a\|_{L_2(\mathcal{M})} \|b\|_{L_2(\mathcal{M})} \sup_i \|y_i\|_{\mathcal{M}}.$$ 

Note that the notation $\sup_i |a_i|$ was used in [3] instead of $\sup_i |a_i|$ but we will use the latter for the compatibility with commutative cases.

Then, the main result is as follows.

**Theorem.** Let $\mathcal{M}$ be a von Neumann algebra with QWEP which is not subhomogeneous, and $\Phi = (\phi_i)_{i \geq 1}$ be a complete orthonormal system in $L_2(\mathcal{M})$ with

$$\left\| \sup_i |\phi_i| \right\|_2 < \infty.$$ 

Then, a locally-$C^*(\mathbb{F}_\infty)$ operator space $E$ is $\Phi$-unconditional if and only if $E$ is completely isomorphic to an operator Hilbert space.

We exclude the case $\mathcal{M}$ is subhomogeneous since the unconditionality in this case does not use the whole operator space structure of $E$.

The essential tools for the proof of Banach space case is concerned with averages with respect to the Rademacher system and the standard gaussian variables. Thus, it is also natural to expect to their noncommutative analogues, $*$-free Haar unitaries and $*$-free circular elements would play a similar role in operator space case.

This paper is organized as follows. In section 2 we collect some basic materials about vector valued noncommutative $L_p$ spaces with respect to QWEP von Neumann algebras, $*$-free Haar unitaries and $*$-free circular elements. In section 3 we develop the following essential tools for the proof of the main results: contraction principle for $*$-free Haar unitaries, comparison of averages with respect to $*$-free Haar unitaries and $*$-free circular elements and $K$-covexity, type 2 and cotype 2 with respect to $*$-free circular elements. In the last section we define the unconditionality of operator spaces with respect to orthonormal systems in noncommutative $L_2$ spaces and prove the main result.

Throughout this paper, we assume that the reader is familiar with the general results of operator spaces ([11][18]), operator algebras ([20]), free probability ([10][22]) and completely $p$-summing maps ([17]). For a index set $I$ we denote the operator...
Hilbert space on $\ell_2(I)$ by $OH(I)$. When $I = \{1, \cdots, n\}, n \in \mathbb{N}$ we simply write $OH_n$.

2. Preliminaries

2.1. Vector valued noncommutative $L_p$ spaces with respect to QWEP von Neumann algebra. The theory of vector valued noncommutative $L_p$ spaces was initiated by G. Pisier ([17]) for the case that the underlying von Neumann algebra is injective and semifinite. For an operator space $E$ and an injective and semifinite von Neumann algebra $M$ we define

$$L_p(M; E) := [M \otimes_{\min} E, L_1(M) \hat{\otimes} E]_{\frac{1}{p}},$$

where $\otimes_{\min}$ and $\hat{\otimes}$ imply injective and projective tensor product of operator spaces and $[\cdot, \cdot]$ implies complex interpolation of operator spaces. Recently, M. Junge ([10]) extended this theory to the case that the underlying von Neumann algebra satisfies QWEP using the following characterization of QWEP von Neumann algebras.

**Proposition 2.1.** A von Neumann algebra $M$ is QWEP if and only if there are a normal $\ast$-isomorphism

$$\pi : M \to \mathcal{M}_U := \left( \prod_{i \in I} L_1(M_i) \right)^\ast$$

for some injective and semifinite von neumann algebras $M_i$ and some free ultrafilter $U$ on an index set $I$ and a normal conditional expectation

$$\mathcal{E} : \mathcal{M}_U \to \pi(M).$$

Let $\mathcal{M}$ be a von Neumann algebra with QWEP and $\alpha = (\pi, \mathcal{E})$ as above. Then for $1 \leq p \leq \infty$ there are complete isometries

$$\pi_p : L_p(M) \to L_p(M_U) = \prod_U L_p(M_i)$$

and complete contractions

$$\mathcal{E}_p : L_p(M_U) \to L_p(\pi(M))$$

induced from $\pi$ and $\mathcal{E}$, respectively. The followings are basic properties of $L_p(M, \alpha; E)$ from ([10]) which we will need in the sequel.

**Proposition 2.2.** Let $\mathcal{M}$ be a von Neumann algebras with QWEP, $\alpha = (\pi, \mathcal{E})$ as above, $E_1$ and $E_2$ be operator spaces, $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

1. If $E_1 \subseteq E_2$ is a completely isometric embedding, then we have

$$L_p(M, \alpha; E_1) \subseteq L_p(M, \alpha; E_2)$$

is completely isometric.

2. If $T : E_1 \to E_2$ is a completely bounded map, then we have

$$\|I_{L_p(M)} \otimes T : L_p(M, \alpha; E_1) \to L_p(M, \alpha; E_2)\| \leq \|T\|_{cb}.$$
exactness implies “locally-$C^*(F_\infty)$”, and “locally-$C^*(F_\infty)$” is stable under duality and complex interpolation (\cite{4}), so that $S_p$ and $L_p(\mu)$ ($1 \leq p \leq \infty$) for some measure $\mu$ are all locally-$C^*(F_\infty)$.

Furthermore, the condition $E \subseteq C^*(F_\infty)$ guarantees the $E$-valued extension property of completely positive maps between noncommutative $L_p$-spaces. (\cite{10})

Recall that, for von Neumann algebras $\mathcal{M}_1$ and $\mathcal{M}_2$, $S : L_p(\mathcal{M}_1) \to L_p(\mathcal{M}_2)$ is called completely positive if for each $n \in \mathbb{N}$, $I_{\mathcal{M}_n} \otimes S$ maps the positive cone $L_p(\mathcal{M}_n \otimes \mathcal{M}_1)^+$ into the positive cone $L_p(\mathcal{M}_n \otimes \mathcal{M}_2)^+$.

**Proposition 2.3.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras with QWEP, $E \subseteq C^*(F_\infty)$ and $1 \leq p < \infty$. If $S : L_p(\mathcal{M}_1) \to L_p(\mathcal{M}_2)$ is a completely positive map, then we have

$$\|S \otimes I_E : L_p(\mathcal{M}_1; E) \to L_p(\mathcal{M}_2; E)\| \leq \|T\|.$$ 

We end this subsection with the following duality result.

**Proposition 2.4.** Let $\mathcal{M}$ be a von Neumann algebra with QWEP, $\alpha = (\pi, E)$ as above, $E$ be a locally-$C^*(F_\infty)$ operator space, $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then, we have

$$L_{p'}(\mathcal{M}, \alpha; E^*) \hookrightarrow L_p(\mathcal{M}, \alpha; E)^*$$

completely isomorphically. More precisely, we have

$$\|\xi\|_{L_p(\mathcal{M}, \alpha; E^*)} \leq \|\xi\|_{L_{p'}(\mathcal{M}, \alpha; E^*)} \leq d_f(E) \|\xi\|_{L_p(\mathcal{M}, \alpha; E)^*},$$

for any $\xi \in L_{p'}(\mathcal{M}, \alpha; E^*)$.

**Proof.** We consider $\xi \in L_{p'}(\mathcal{M}) \otimes E^*$ with $\pi_{p'} \otimes I_E(\xi) = (\xi_i)_{\mu} \in \prod_{\mu} L_{p'}(\mathcal{M}_i; E^*)$, then for $x \in L_p(\mathcal{M}) \otimes E$ with $\pi_p \otimes I_E(x) = (x_i)_{\mu} \in \prod_{\mu} L_p(\mathcal{M}_i; E)$ we have

$$|\langle \xi, x \rangle| = |\langle \pi_{p'}(\xi), \pi_p(x) \rangle| = \lim_{\mu} |\langle \xi_i, x_i \rangle|$$

$$\leq \lim_{\mu} \|\xi\|_{L_{p'}(\mathcal{M}_i; E^*)} \|x\|_{L_p(\mathcal{M}_i; E)} = \|\xi\|_{L_p(\mathcal{M}, \alpha; E^*)} \|x\|_{L_p(\mathcal{M}, \alpha; E)},$$

which implies $\|\xi\|_{L_p(\mathcal{M}, \alpha; E^*)} \leq \|\xi\|_{L_{p'}(\mathcal{M}, \alpha; E^*)}$.

For the converse inequality we choose $y_i \in L_p(\mathcal{M}_i; E)$ with

$$\langle \xi_i, y_i \rangle = \|\xi_i\|_{L_{p'}(\mathcal{M}_i; E^*)} \quad \text{and} \quad \|y_i\|_{L_p(\mathcal{N}_i)} \leq 1 + \frac{1}{i}.$$ 

Let $\pi_{p'} : L_{p'}(\mathcal{M}) \to \prod_{\mu} L_{p'}(\mathcal{M}_i)$ be the complete isometry derived from $\pi$, which is clearly completely positive. Then, its adjoint $\pi_{p'}^* : \prod_{\mu} L_p(\mathcal{M}_i) \to L_p(\mathcal{M})$ is also completely positive and

$$|\langle \xi, \pi_{p'}^* \otimes I_E([y_i]_{\mu}) \rangle| = |\langle \pi_{p'}^* \otimes I_E(\xi), (y_i)_{\mu} \rangle|$$

$$= \lim_{\mu} |\langle \xi_i, y_i \rangle| = \lim_{\mu} \|\xi_i\|_{L_{p'}(\mathcal{M}_i; E^*)}$$

$$= \|\xi\|_{L_{p'}(\mathcal{M}_i; E^*)}.$$ 

On the other hand we have by Proposition 2.3

$$\|\pi_{p'}^* \otimes I_E([y_m]_{\mu})\|_{L_p(\mathcal{M}; E)} \leq d_f(E) \|\pi_{p'}^*\| \|y_m\|_{L_p(\mathcal{M}_i; E)} \leq d_f(E),$$

which implies $\|\xi\|_{L_{p'}(\mathcal{N}; E^*)} \leq d_f(E) \|\xi\|_{L_p(\mathcal{N}; E)^*}$. 

By repeating the same argument for $S_n^p(E)$, $n \geq 1$ instead of $E$ we get the desired complete isometry. \(\square\)
2.2. ∗-free Haar unitaries and ∗-free circular elements. In this subsection we consider specific \(\alpha = (\pi, \mathcal{E})\)'s for free group von Neumann algebra obtained by using random matrix models for ∗-free Haar unitaries and ∗-free circular elements.

Let \(F_\infty\) be the free group with generators \((g_i)_{i \geq 1}\) and \(\lambda(g_i)\) be the left translation by \(g_i\) in \(\ell_2(F_\infty)\) for \(i \geq 1\).

Let \(\mathcal{H}\) be a Hilbert space with Hilbert space basis \((e_i)_{i \geq 1}\) and \((f_i)_{i \geq 1}\) is the random matrix models as follows. Let \(W_i := \ell(e_i) + \ell(f_i)\), where \(\ell(f) \in B(\mathcal{F}(\mathcal{H}))\), \(f \in \mathcal{H}\) is the left creation operator defined by

\[
\ell(f)(\Omega) := f
\]

and

\[
\ell(f)(f_1 \otimes \cdots \otimes f_n) := f \otimes f_1 \otimes \cdots \otimes f_n
\]

for \(n \geq 1\) and \(f_1, \cdots, f_n \in \mathcal{H}\), and \(\ell(f) \in B(\mathcal{F}(\mathcal{H}))\) is the adjoint of \(\ell(f)\).

\((\lambda(g_i))_{i \geq 1}\) and \((W_i)_{i \geq 1}\) are typical examples of ∗-free Haar unitaries and ∗-free circular elements, respectively, and they have the random matrix models as follows. Let \((\Omega, \mathcal{P})\) and \((\Omega', \mathcal{P}')\) be probability spaces, \(m \in \mathbb{N}\) and \(U(m)\) be the compact group of \(m \times m\) unitary matrices with the normalized Haar measure \(\gamma_m\). Now we consider the standard unitary random matrix

\[
U^m : (\Omega, \mathcal{P}) \to U(m)
\]

with distribution \(\gamma_m\) and the standard gaussian random matrix

\[
G^m : (\Omega', \mathcal{P}') \to M_m, \text{ where } G^m = \left(\frac{1}{\sqrt{m}}g_{ij}\right)_{i,j=1}^m
\]

and \(g_{ij}\)'s are i.i.d. complex valued standard gaussian random variables.

Then \(U^m\) and \(G^m\) are noncommutative random variables in

\[
(L_\infty(\Omega; M_m), \tau_m) \text{ and } (\cap_{1 \leq p < \infty} L^p(\Omega'; M_m), \tau'_m),
\]

respectively, where

\[
\tau_m(x) = \int_{\Omega} \frac{1}{m} \text{tr}(x(\omega))d\mathcal{P}(\omega) \text{ and } \tau'_m(y) = \int_{\Omega'} \frac{1}{m} \text{tr}(y(\omega'))d\mathcal{P}'(\omega')
\]

for \(x : \Omega \to M_m\) and \(y : \Omega' \to M_m\), and it is well known (by Voiculescu) that \((U^m_i)_{i \geq 1}\) (resp. \((G^m_i)_{i \geq 1}\)), independent copies of \(U^m\) (resp. \(G^m\)), converges in distribution to \((\lambda(g_i))_{i \geq 1}\) (resp. \((W_i)_{i \geq 1}\)) as \(m\) goes to infinity.

Now we consider a free ultrafilter \(\mathcal{U}\) in \(\mathbb{N}\) and denote \(\mathcal{N}_m := L_\infty(\Omega; M_m)\). Since

\[
\left(\prod_{\mathcal{U}} L_1(\mathcal{N}_m)\right)^* \text{ coincide with the ultraproduct of } (\mathcal{N}_m)_{m \geq 1} \text{ in the sense of finite von Neumann algebras the above convergence implies that }
\]

\[
((U^m_i)_{\mathcal{U}})_{i \geq 1} \subseteq \left(\prod_{\mathcal{U}} L_1(\mathcal{N}_m)\right)^*
\]

has the same ∗-distribution with \((\lambda(g_i))_{i \geq 1}\). Thus, we get a normal ∗-isomorphism

\[
\pi_{\mathcal{U}} : VN(F_\infty) \to \left(\prod_{\mathcal{U}} L_1(\mathcal{N}_m)\right)^*
\]

extending the natural map

\[
P(\lambda(g_1), \lambda(g_2), \cdots) \to P(\left(\prod_{\mathcal{U}} U^m_i\right)_{\mathcal{U}}, \left(\prod_{\mathcal{U}} U^m_2\right)_{\mathcal{U}}, \cdots)
\]

for any noncommutative polynomial \(P\). (See Lemma 1 of \(\mathbb{U}\) for example) Moreover, since \(\pi_{\mathcal{U}}(VN(F_\infty)) \subseteq \left(\prod_{\mathcal{U}} L_1(\mathcal{N}_m)\right)^*\) are both finite von neumann algebras
with respect to the same trace we have the natural normal conditional expectation \( \mathcal{E}_U : \left( \prod_L L_1(N_m) \right)^* \to \pi_U(V N(\mathbb{F}_\infty)) \), which is the adjoint of the inclusion 
\[ \pi_U(V N(\mathbb{F}_\infty)) \hookrightarrow \left( \prod_L L_1(N_m) \right)^* . \]

For the gaussian case we need to truncate since gaussian variables are not bounded. Let 
\[ \tilde{G}^m : (\Omega', P') \to M_m, \text{ where } \tilde{G}^m = \left( \frac{1}{\sqrt{m}} \sum_{i,j=1}^m \left| g_{ij} \right|^2 \right)_{i,j=1}^m . \]

Then for \( N_m \coloneqq L_\infty(\Omega'; M_m) \) we have \( \tilde{G}^m \in (N_m, \tau'_m) \), and \((\tilde{G}^m_i)_{i \geq 1}\), independent copies of \( \tilde{G}^m \), has the same asymptotic \(*\)-distribution as \((G^m_i)_{i \geq 1}\). Indeed, for fixed \( k \in \mathbb{N} \) and 
\[ \hat{G}^m = G^m - \tilde{G}^m = \left( \frac{1}{\sqrt{m}} \hat{g}_{ij} \right)_{i,j=1}^m , \]
we have 
\[ \tau'_m((\hat{G}^m)^k) = \frac{1}{m^{k+1}} \sum_{1 \leq i_1, \ldots, i_k \leq m} \mathbb{E}(\hat{g}_{i_1 i_2} \hat{g}_{i_2 i_3} \cdots \hat{g}_{i_k i_1}) , \]
where \( \mathbb{E} \) implies the expectation with respect to \((\Omega', P')\). Since the cube 
\[ [-m^{m-\frac{1}{2}}, m^{m-\frac{1}{2}}]^m \subseteq \mathbb{R}^m \]
is contained in the ball centered at 0 with radius \( m^m \) we have 
\[ |\mathbb{E}(\hat{g}_{i_1 i_2} \hat{g}_{i_2 i_3} \cdots \hat{g}_{i_k i_1})| \leq \mathbb{E}\left(1_{|g_{i_1}| > m^{m-\frac{1}{2}}, 1 \leq i, j \leq m} |g_{i_1 i_2} g_{i_2 i_3} \cdots g_{i_k i_1}| \right) \]
\[ \leq \mathbb{E}\left(1_{|g_{i_1 i_2}| > m^{m-\frac{1}{2}}} |g_{i_1 i_2}| \cdots |g_{i_k i_1}| > m^{m-\frac{1}{2}} \cdot |g_{i_k i_1}| \right) \]
\[ \leq \left[ \mathbb{E}\left(1_{|g_{i_1 i_2}| > m^{m-\frac{1}{2}}} |g_{i_1 i_2}| \right) \right]^k \cdots \]
\[ = \mathbb{E}\left(1_{|g_{i_1}| > m^{m-\frac{1}{2}}} |g_{i_1}| \right) \leq C e^{-m^{m-\frac{1}{2}}} \]
for some constant \( C > 0 \) independent of \( m \). Thus, we have 
\[ \left| \tau'_m((\hat{G}^m)^k) \right| \leq C m^{m-\frac{1}{2} - 1} \]
\[ \leq C m^{m-\frac{1}{2}} \rightarrow 0 \text{ as } m \to \infty , \]
which implies \( \hat{G}^m \) converges to 0 in \(*\)-distribution, and consequently \( \tilde{G}^m \) converges to \( G^m \) in \(*\)-distribution. On the other hand since \( \tilde{G}^m \) is bi-unitarily invariant (i.e. \( V_1 \tilde{G}^m V_2 \) also has the same \(*\)-distribution as \( \tilde{G}^m \) for every \( m \times m \) unitary matrices \( V_1 \) and \( V_2 \) \((\tilde{G}^m_i)_{i \geq 1}\) is asymptotically \(*\)-free (Theorem 4.3.11 of \[3\]), so that we get the desired asymptotic \(*\)-distribution of \((\tilde{G}^m_i)_{i \geq 1}\).

Then we get a normal \(*\)-isomorphism 
\[ \pi_G : \{ \tilde{W}_i : i \geq 1 \}'' \to \left( \prod_L L_1(N_m) \right)^* \]
extending the natural map 
\[ P(\tilde{W}_1, \tilde{W}_2, \cdots) \to P((\tilde{G}^m_1)_{i \in \mathbb{N}}, (\tilde{G}^m_2)_{i \in \mathbb{N}}, \cdots) \]
for any noncommutative polynomial \( P \) and the natural normal conditional expectation \( \mathcal{E}_G : \left( \prod_L L_1(N_m) \right)^* \to \pi_G(V N(\mathbb{F}_\infty)) \), which is the adjoint of the inclusion 
\[ \pi_G(V N(\mathbb{F}_\infty)) \hookrightarrow \left( \prod_L L_1(N_m) \right)^* . \]
Combining the above discussions we have the following representations of vector valued noncommutative $L_p$ spaces with respect to

\[ \mathcal{N} := VN(\mathbb{F}_\infty) \text{ and } \mathcal{N} := \{\tilde{W}_i : i \geq 1\}'' \]

which are both satisfying $QWEP$. These $\mathcal{N}$, $N$, $N_m$ and $N_m$ will be fixed from now on.

**Proposition 2.5.** Let $E \subseteq C^*(\mathbb{F}_\infty)$, $1 \leq p < \infty$ and $\mathcal{U}$ be a free ultrafilter in $\mathbb{N}$. Then, for any $n \in \mathbb{N}$ and $(x_i)_{i=1}^n \subseteq E$, we have

\[
\left\| \sum_{i=1}^n \lambda(g_i) \otimes x_i \right\|_{L_p(\mathcal{N};E)} = \lim_{\mathcal{U}} \left\| \sum_{i=1}^n U_i^m \otimes x_i \right\|_{L_p(N_m;E)}
\]

and

\[
\left\| \sum_{i=1}^n \tilde{W}_i \otimes x_i \right\|_{L_p(\mathcal{N};E)} = \lim_{\mathcal{U}} \left\| \sum_{i=1}^n G_i^m \otimes x_i \right\|_{L_p(N_m;E)}.
\]

It would be convenient for later use to fix $\alpha_U = (\pi_U, \mathcal{E}_U)$ and $\alpha_G = (\pi_G, \mathcal{E}_G)$ for $\mathcal{N}$ and $N$, respectively, with the same free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and use the notation $L_p(\mathcal{N};E)$ and $L_p(N;E)$ for $L_p(\mathcal{N}, \alpha_U;E)$ and $L_p(N, \alpha_G;E)$.

### 3. Free probabilistic tools

From now on, $\mathcal{M}$ and $E$ implies a von Neumann algebra satisfying $QWEP$ and a locally-$C^*(\mathbb{F}_\infty)$ operator space, respectively. The reasons for this restriction comes from the discussions in the subsection 2.4. Moreover, we fix $\alpha = (\pi, \mathcal{E})$ for $\mathcal{M}$.

In this section we collect operator space analogues of several probabilistic tools in Banach space theory. We start with the “contraction principle for $*$-free Haar Unitaries”

**Theorem 3.1.** Let $(\phi_i)_{i \geq 1}$ be a sequence in $L_p(\mathcal{M})$ $(1 \leq p < \infty)$ with

\[
\left\| \sup_i |\phi_i| \right\|_p < \infty.
\]

Then we have for any $n \in \mathbb{N}$ and $(x_i)_{i=1}^n \subseteq E$

\[
\left\| \sum_{i=1}^n \phi_i \otimes \lambda(g_i) \otimes x_i \right\|_{L_p(\mathcal{M} \otimes \mathcal{N};E)} \leq 2d_f(E) \left\| \sup_i |\phi_i| \right\|_p \left\| \sum_{i=1}^n \lambda(g_i) \otimes x_i \right\|_{L_p(\mathcal{N};E)}.
\]

**Proof.** We assume that $E \subseteq C^*(\mathbb{F}_\infty)$. By a usual density argument we can also assume that $\mathcal{M}$ is $\sigma$-finite. Then there is a normal faithful state $\varphi$ on $\mathcal{M}$. Now we suppose that

\[
\left\| \sup_i |\phi_i| \right\|_p = 1
\]

and fix $n \in \mathbb{N}$. For any given $\epsilon > 0$ we can find $a, b \in L_{2p}(\mathcal{M})$ and contractions $(x_i)_{i \geq 1} \subseteq \mathcal{M}$ such that

\[
\phi_i = ax_i b \text{ for all } i \geq 1 \text{ and } \|a\|_{L_{2p}(\mathcal{M})} = 1, \|b\|_{L_{2p}(\mathcal{M})} \leq 1 + \epsilon.
\]

By a standard argument we can find $D(\geq 0) \in L_1(\mathcal{M})$ and contractions $(u_i)_{i \geq 1} \subseteq \mathcal{M}$ such that

\[
\phi_i = D_{\pm} u_i D_{\pm} \text{ for all } i \geq 1 \text{ and } \|D\|_{L_1(\mathcal{M})} \leq 4(1 + \epsilon'),
\]

where $\epsilon' \to 0$ as $\epsilon \to 0$. Indeed, if we set $D = (|a|^p + |b|^p)^{1/2}$ we have $|a|, |b| 
\leq D_{\pm}$ so that there are contractions $V, W \in \mathcal{M}$ such that

\[
a = D_{\pm} V \text{ and } b = W D_{\pm}.
\]
Then we have \( \phi_i = ax_i b = D\overline{\varphi} V x_i W D\overline{\varphi} = D\overline{\varphi} u_i D\overline{\varphi} \) with \( u_i = V x_i W \), and 
\[
\|D\|_{L\overline{\varphi}}(\mathcal{M}) \leq \|a\|_{L1(\mathcal{M})}^{2p} + \|b\|_{L1(\mathcal{M})}^{2p} + 2 \|a\|_{L1(\mathcal{M})}^{p} \|b\|_{L1(\mathcal{M})}^{p} \leq 1 + (1 + \epsilon)^{2p} + 2(1 + \epsilon)^{2p}.
\]
By Russo-Dye theorem and the usual convexity argument it is enough to show the case that \( u_i \)'s are all unitaries.

With all these assumption for any \( (x_i)_{i=1}^{n} \subseteq E \) we have
\[
\sum_{i=1}^{n} \phi_i \otimes \lambda(g_i) \otimes x_i = \sum_{i=1}^{n} D\overline{\varphi} u_i D\overline{\varphi} \otimes \lambda(g_i) \otimes x_i = \Phi \circ \pi \otimes I_E \left( \sum_{i=1}^{n} \lambda(g_i) \otimes x_i \right),
\]
where
\[
\Phi : L_p(\mathcal{M} \otimes \mathcal{N}) \rightarrow L_p(\mathcal{M} \otimes \mathcal{N}), \quad z \mapsto d\overline{\varphi} z d\overline{\varphi}^*
\]
with \( d = D \otimes I_N \) and
\[
\pi : L_p(\mathcal{N}) \rightarrow L_p(\mathcal{M} \otimes \mathcal{N})
\]
is the map induced from the \(*\)-isomorphism
\[
\pi_\infty : \mathcal{N} \rightarrow \mathcal{M} \otimes \mathcal{N}, \quad \text{with } \pi_\infty(\lambda(g_i)) = u_i \otimes \lambda(g_i).
\]
Note that it is easy to check that \( (\lambda(g_i))_{i \geq 1} \) and \( (u_i \otimes \lambda(g_i))_{i \geq 1} \) have the same \(*\)-distribution on \( W \)-probability spaces with faithful states \( (\mathcal{N}, \tau) \) and \( (\mathcal{M} \otimes \mathcal{N}, \varphi \otimes \tau) \), respectively. Thus, \( \pi_\infty \) is a \(*\)-isomorphism by Lemma 1 of \([13]\).

Now we have \( \Phi \) and \( \pi \) are completely positive, and consequently so is the composition \( \Phi \circ \pi \) with
\[
\|\Phi \circ \pi\| = \|\Phi \circ \pi(I)\| = \left\| d\overline{\varphi} \right\|_{L2(\mathcal{M} \otimes \mathcal{N})} = \left\| D\overline{\varphi} \right\|_{L2(\mathcal{M})} = \|D\|_{L1(\mathcal{M})} \leq 2\sqrt{1 + \epsilon'}.
\]
Then, by Proposition \([22]\) we have
\[
\left\| \sum_{i=1}^{n} \phi_i \otimes \lambda(g_i) \otimes x_i \right\|_{L_p(\mathcal{M} \otimes \mathcal{N}; E)} \leq \|\Phi \circ \pi\| \left\| \sum_{i=1}^{n} \lambda(g_i) \otimes x_i \right\|_{L_p(\mathcal{N}; E)} \leq 2\sqrt{1 + \epsilon'} \left\| \sum_{i=1}^{n} \lambda(g_i) \otimes x_i \right\|_{L_p(\mathcal{N}; E)}.
\]

Now we consider a partial relationship between the average with respect to \(*\)-free Haar unitaries and the average with respect to \(*\)-free circular elements. In the proof we will use the random matrix model approach in the subsection \([22]\).

**Theorem 3.2.** Let \( 1 \leq p < \infty \). Then there is a constant \( C_G > 0 \) such that we have for any \( n \in \mathbb{N} \) and \( (x_i)_{i=1}^{n} \subseteq E \)
\[
\left\| \sum_{i=1}^{n} \lambda(g_i) \otimes x_i \right\|_{L_p(\mathcal{N}; E)} \leq C_G \left\| \sum_{i=1}^{n} \overline{W}_i \otimes x_i \right\|_{L_p(\mathcal{N}; E)}.
\]

**Proof.** First, we assume that \( E \subseteq C^*(\mathbb{F}_\infty) \), and fix \( n \in \mathbb{N} \) and \( (x_i)_{i=1}^{n} \subseteq E \). Let \( (U_i^m)_{i=1}^{n} \) and \( (G_i^m)_{i=1}^{n} \) be the independent families of the standard unitary random matrices and the standard gaussian random matrices as in the subsection \([22]\). Note that \( (U_i^m|G_i^m)_{i=1}^{n} \) have the same \(*\)-distribution and
\[
\mathbb{E}(U_i^m|G_i^m) = U_i^m \mathbb{E}|G_i^m|,
\]
where \( \mathbb{E} \) implies the expectation with respect to \( (\Omega', P') \). Moreover, it is well known that (Remark 1.7. in p.80 of \([12]\))
\[
\mathbb{E}|G_i^m| = \delta(m)I_m,
\]
where $I_m$ is the identity matrix on $M_m$ and $\delta(m) \geq \delta$ for some constant $\delta > 0$. Thus, for $N_m = L_\infty(\Omega; M_m)$ and $N_m = L_\infty(\Omega'; M_m)$, we have

$$\left\| \sum_{i=1}^{n} U_i^m \otimes x_i \right\|_{L_p(N_m; E)} = \delta(m)^{-1} \left\| \sum_{i=1}^{n} U_i^m \mathbb{E} \left| G_i^m \right| \otimes x_i \right\|_{L_p(N_m; E)}$$

$$= \delta(m)^{-1} \left\| \sum_{i=1}^{n} \mathbb{E}(U_i^m \left| G_i^m \right|) \otimes x_i \right\|_{L_p(N_m; E)}$$

$$\leq \delta^{-1} \lim_{m,\mathcal{U}} \left\| \sum_{i=1}^{n} G_i^m \otimes x_i \right\|_{L_p(N_m; E)}$$

Let $\mathcal{U}$ be the free ultrafilter on $\mathbb{N}$ chosen for $\alpha_U = (\pi_U, \mathcal{E}_U)$ and $\alpha_G = (\pi_G, \mathcal{E}_G)$, then we have

$$\left\| \sum_{i=1}^{n} \lambda(g_i) \otimes x_i \right\|_{L_p(N; E)} = \lim_{m,\mathcal{U}} \left\| \sum_{i=1}^{n} U_i^m \otimes x_i \right\|_{L_p(N_m; E)}$$

$$\leq \delta^{-1} \lim_{m,\mathcal{U}} \left\| \sum_{i=1}^{n} G_i^m \otimes x_i \right\|_{L_p(N_m; E)}$$

$$= \delta^{-1} \left\| \sum_{i=1}^{n} \tilde{W}_i \otimes x_i \right\|_{L_p(N; E)}$$

\[\square\]

Another important ingredient is the concept of $K$-convexity with respect to circular elements.

**Definition 3.3.** Let $E$ be a locally-C$^*$-($\mathcal{F}_\infty$) operator space. Then we define the "$\tilde{W}K$-convexity" constant $\tilde{W}K(E)$ by $\tilde{W}K(E) = \sup_n \tilde{W}K_n(E)$, where

$$\tilde{W}K_n(E) = \left\| P_n \otimes I_E : L_2(N; E) \to L_2(N; E) \right\|$$

and

$$P_n : L_2(N) \to L_2(N), \ a \mapsto \sum_{i=1}^{n} \left\langle a, \tilde{W}_i \right\rangle \tilde{W}_i.$$ 

**Theorem 3.4.** Suppose $S_2(E)$ is $K$-convex as a Banach space. Then $E$ is "$\tilde{W}K$-convex" with

$$\tilde{W}K(E) \leq K(S_2(E)),$$

where $K(X)$ is the $K$-convexity constant of a Banach space $X$.

**Proof.** Using the isometric embedding

$$L_2(N; E) \hookrightarrow \prod_{m,\mathcal{U}} L_2(N_m; E),$$

where $N_m = L_\infty(\Omega'; M_m)$, we will take a detour down to $m$-th random matrix level.

Let

$$\mathcal{G}_m^n = \text{span}\{ g_{ij}^k : 1 \leq k \leq n, 1 \leq i, j \leq m \} \subseteq L_2(\Omega')$$

and

$$\tilde{\mathcal{G}}_m = \text{span}\{ G_i^m : 1 \leq k \leq n \} \subseteq L_2(N_m).$$

Let $Q_m$ be the orthogonal projection from $L_2(\Omega')$ onto $\mathcal{G}_m^n$ and $R_m : \mathcal{G}_m^n(L_2(M_m)) \to \tilde{\mathcal{G}}_m$.

\[\text{9}\]
be the linear map defined by
\[
R_m \left( \sum_{k=1}^{n} \sum_{i,j=1}^{m} g_{ij}^{k} \sum_{r,s=1}^{m} e_{rs} \otimes x_{rs}^{ijk} \right) = \sum_{k=1}^{n} \left( \sum_{i,j=1}^{m} g_{ij}^{k} e_{ij} \right) \otimes \text{Av}(x_{rs}^{ijk})
\]
for any \( x_{rs}^{ijk} \in \mathbb{C} \) (1 \( \leq i,j,r,s \leq m, 1 \leq k \leq n \)), where
\[
\text{Av}(x_{rs}^{ijk}) = \frac{1}{|\Sigma_m|^2} \sum_{\sigma,\rho \in \Sigma_m} g_{\sigma(i)\rho(j)}^{k} \cdot g_{ij}^{k} = x_{ij}^{ijk}
\]
and \( \Sigma_m \) is the set of all permutations of \( \{1, \cdots, m\} \). Note that \( \text{Av}(x_{rs}^{ijk}) \) depends only on \( k \).

Then we can recover \( P_n \) by taking ultraproduct of \( R_m \circ (Q_m \otimes I_{M_m}) \). More precisely, for any \( f \in L_2(N) \otimes E \) we can associate \( (f_m)_{m,\mathcal{U}} \in L_2(N_m) \) obtained from the isometric embedding \( L_2(N) \hookrightarrow \prod_{m,\mathcal{U}} L_2(N_m) \). Then we have
\[
P_n(f) = \lim_{m,\mathcal{U}} R_m \circ (Q_m \otimes I_{M_m})(f_m).
\]
Thus, it is enough to show that
\[
Q_m \otimes I_{M_m} \otimes I_E : L_2(\Omega')(M_m(E)) \to G_{\text{m}}^n(M_m(E))
\]
and
\[
R_m \otimes I_E : G_{\text{m}}^n(M_m; E) \to \mathcal{G}_m(E)
\]
are both bounded. The first one is obtained by the assumption of \( K \)-convexity of \( S_2(E) \) so that we have
\[
\|Q_m \otimes I_{M_m} \otimes I_E\| \leq K(S_{2m}^m(E)) \leq K(S_2(E)).
\]

Now we consider the map \( R_m \otimes I_E \). Let \((\varepsilon_i)_{i \geq 1}\) is the Rademacher sequences on \([0,1]\). Then since \( (g_{ij}^k) \) and \( (\varepsilon_i^k g_{ij}^k) \) have the same distribution for any \( \varepsilon_i^k \in \{\pm 1\} \) we have
\[
\left\| \sum_{k=1}^{n} \sum_{i,j=1}^{m} g_{ij}^{k} \sum_{r,s=1}^{m} e_{rs} \otimes x_{rs}^{ijk} \right\|_{L_2(N_m; E)} = \int_0^1 \left\| \sum_{k=1}^{n} \sum_{i,j,r,s=1}^{m} \varepsilon_i(t) \varepsilon_r(t) g_{ij}^{k} e_{rs} \otimes x_{rs}^{ijk} \right\|_{L_2(N_m; E)} dt
\]
\[
\geq \sum_{k=1}^{n} \sum_{i,j,r,s=1}^{m} \left[ \int_0^1 \varepsilon_i(t) \varepsilon_r(t) dt \right] g_{ij}^{k} e_{rs} \otimes x_{rs}^{ijk}
\]
\[
= \sum_{k=1}^{n} \sum_{i,j,s=1}^{m} g_{ij}^{k} e_{is} \otimes x_{is}^{ijk}
\]
By repeating the same procedure we get
\[
\left\| \sum_{k=1}^{n} \sum_{i,j=1}^{m} g_{ij}^{k} \sum_{r,s=1}^{m} e_{rs} \otimes x_{rs}^{ijk} \right\|_{L_2(N_m; E)} \geq \left\| \sum_{k=1}^{n} \sum_{i,j=1}^{m} g_{ij}^{k} e_{ij} \otimes g_{ij}^{k} \right\|_{L_2(N_m; E)}
\]
for \( g_{ij}^{k} = x_{ij}^{ijk} \).

We can proceed further using permutations. Note that \( (g_{ij}^k) \) and \( (g_{ij}^{k-1}(i,j)) \) have the same distribution for any permutation \( \sigma \in \Sigma_m \), and also interchanging \( i \)-th row

10
into $\sigma^{-1}(i)$-th row does not affect the norm. Thus we have
\[
\left\| \sum_{k=1}^n \sum_{i,j=1}^m g_{ij}^k e_{ij} \otimes \frac{1}{|\Sigma_m|} \sum_{\sigma \in \Sigma_m} y_{\sigma(i)}^k \right\|_{L^2(N_m;E)} \leq \frac{1}{|\Sigma_m|} \sum_{\sigma \in \Sigma_m} \left\| \sum_{k=1}^n \sum_{i,j=1}^m g_{ij}^k e_{ij} \otimes y_{\sigma(i)}^k \right\|_{L^2(N_m;E)} = \frac{1}{|\Sigma_m|} \sum_{\sigma \in \Sigma_m} \left\| \sum_{k=1}^n \sum_{i,j=1}^m g_{ij}^k e_{\sigma^{-1}(i)j} \otimes y_{ij}^k \right\|_{L^2(N_m;E)} = \frac{1}{|\Sigma_m|} \sum_{\sigma \in \Sigma_m} \left\| \sum_{k=1}^n \sum_{i,j=1}^m g_{ij}^k e_{ij} \otimes y_{ij}^k \right\|_{L^2(N_m;E)} = \left\| \sum_{k=1}^n \sum_{i,j=1}^m g_{ij}^k e_{ij} \otimes y_{ij}^k \right\|_{L^2(N_m;E)}.
\]

By repeating the same procedure we get
\[
\left\| \sum_{k=1}^n \sum_{i,j=1}^m g_{ij}^k e_{ij} \otimes y_{ij}^k \right\|_{L^2(N_m;E)} \geq \left\| \sum_{k=1}^n \sum_{i,j=1}^m g_{ij}^k e_{ij} \otimes \text{Av}(x_{ij}^k) \right\|_{L^2(N_m;E)},
\]
and combining these results we get that $R_m \otimes I_E$ is a contraction.

In Banach space theory, one of the useful characterizations of Hilbert spaces is being type 2 and cotype 2 simultaneously. We will need similar characterization of operator Hilbert spaces.

First, we define a noncommutative analogue of type 2 and cotype 2 using completely 2-summing norm and a variant of "$\ell^2$-norm". Note that this is a variant of $S_2$-type and $S_2$-cotype defined in \[\text{I.}\] Recall that for a linear map $u : E \to F$ between operator spaces we define
\[
\pi_2^n(u) = \sup \left\{ \| (ux_{ij}) \|_{S_2^2(F)} : \left\| \sum_{i,j=1}^n e_{ij} \otimes x_{ij} \right\|_{S_2^2 \otimes \min E} \leq 1, \, n \in \mathbb{N} \right\}.
\]

**Definition 3.5.** Let $u : \ell_2^n \to E$, $n \in \mathbb{N}$. Then we define "$\ell_{W^n}$-norm" of $u$ by
\[
\ell_{W^n}(u) := \left\| \sum_{i=1}^n W_i \otimes u e_i \right\|_{L^2(N_m;E)}.
\]

**Definition 3.6.** We say that $E$ has $W$-type 2 if there is a constant $C > 0$ such that
\[
\ell_{W^n}(u) \leq C \pi_2^n(u^*)
\]
for any $n \in \mathbb{N}$ and $u : OH_n \to E$.

We say that $E$ has $W$-cotype 2 if there is a constant $C' > 0$ such that
\[
\pi_2^n(u) \leq C' \ell_{W^n}(u)
\]
for any $n \in \mathbb{N}$ and $u : OH_n \to E$. We denote $\tilde{W}T^\text{oh}_2(E)$ and $\tilde{WC}^\text{oh}_2(E)$ for the infimum of such $C$ and $C'$, respectively.

**Proposition 3.7.** $E$ is completely isomorphic to an operator Hilbert space if and only if $E$ has $\tilde{W}$-type 2 and $\tilde{W}$-cotype 2. In this case we have

$$d_{oh}(E, OH(I)) \leq \tilde{W}T^\text{oh}_2(E)\tilde{WC}^\text{oh}_2(E)$$

for some index set $I$.

**Proof.** By applying trace duality to (iii) of Theorem 6.5 it is enough to show that

$$\pi^0_2(u) \leq \tilde{W}T^\text{oh}_2(E)\tilde{WC}^\text{oh}_2(E)\pi^0_2(u^*)$$

for any $n \in \mathbb{N}$ and $u : OH_n \to E$. Indeed, we have

$$\pi^0_2(u) \leq \tilde{WC}^\text{oh}_2(E)\ell^*_W(u) \leq \tilde{W}T^\text{oh}_2(E)\tilde{WC}^\text{oh}_2(E)\pi^0_2(u^*).$$

\(\square\)

**Proposition 3.8.** Let $u : \ell^2_n \to E$, $v : E \to \ell^2_n$ and $A : \ell^2_n \to \ell^2_n$ for $n \in \mathbb{N}$. Then we have

1. $\ell^*_W(uA) \leq \ell^*_W(u)\|A\|.$
2. $\ell^*_W(v^*) \leq d_f(E)\tilde{WK}(E)\ell^*_W(v),$

where $\ell^*_W(\cdot)$ refers to the trace dual norm of $\ell_W(\cdot)$.

**Proof.** (1) We consider the random matrix model again. By the usual convexity argument it is enough to show that for any $m \in \mathbb{N}$

$$\left\| \sum_{i=1}^n G^m_i \otimes uAe_i \right\|_{L_2(N;m;E)} = \left\| \sum_{i=1}^n G^m_i \otimes ue_i \right\|_{L_2(N;m;E)}$$

whenever $A = (a_{ij})$ is unitary. Indeed, we have

$$\sum_{i=1}^n G^m_i \otimes uAe_i = \sum_{i=1}^n G^m_i \otimes u \left( \sum_{j=1}^n a_{ji}e_j \right) = \sum_{j=1}^n \left( \sum_{i=1}^n a_{ji}G^m_i \right) \otimes ue_j$$

and since $(G^m_i)_{j=1}^n$ and $(\sum_{i=1}^n a_{ji}G^m_i)_{j=1}^n$ have the same $*$-distribution we are done.

(2) Consider $\sum_{i=1}^n \tilde{W}_i \otimes v^*e_i \in L_2(N;E^*)$. Since $L_2(N;E^*) \to L_2(N;E^*)^*$ completely isomorphically (Proposition 2.4) for any $\epsilon > 0$ we can choose $f \in L_2(N;E)$ with

$$\|f\|_{L_2(N;E)} = 1$$

and

$$\left| \left\langle \sum_{i=1}^n \tilde{W}_i \otimes v^*e_i, f \right\rangle \right| \geq d_f(E)^{-1}(1 - \epsilon)\ell^*_W(v^*).$$

Then for $w : \ell^2_n \to E$, $e_i \to \langle \tilde{W}_i, f \rangle$ we have

$$d_f(E)^{-1}(1 - \epsilon)\ell^*_W(v^*) \leq \left| \left\langle \sum_{i=1}^n \tilde{W}_i \otimes v^*e_i, f \right\rangle \right| = |\text{tr}(vw)|$$

$$\leq \ell^*_W(w)\ell^*_W(v) \leq \tilde{WK}(E)\|f\|_{L_2(N;E)}\ell^*_W(w)$$

$$= \tilde{WK}(E)\ell^*_W(w).$$

Since $\epsilon > 0$ is arbitrary we get the desired result.

\(\square\)
Moreover, \( L \) is algebra not subhomogeneous, \( \Phi = (\text{Proposition 4.1.}) \). Then by (1) of Proposition 3.8, we can easily check that \( E \) has \( \tilde{W} \)-type 2 if and only if there is a constant \( C > 0 \) such that

\[
\left\| \sum_{i,j=1}^{n} \tilde{W}_{ij} \otimes x_{ij} \right\|_{L_{2}(N;E)} \leq C \left\| (x_{ij}) \right\|_{S_{2}^{\Phi}(E)}
\]

for any \( n \in \mathbb{N} \) and \( x_{ij} \in E \).

Similarly, \( E \) has \( \tilde{W} \)-cotype 2 if and only if there is a constant \( C' > 0 \) such that

\[
\left\| (x_{ij}) \right\|_{S_{2}^{\Phi}(E)} \leq C' \left\| \sum_{i,j=1}^{n} \tilde{W}_{ij} \otimes x_{ij} \right\|_{L_{2}(N;E)}
\]

for any \( n \in \mathbb{N} \) and \( x_{ij} \in E \).

As in the Banach space case we have a duality of \( W \)-type 2 for “\( \widetilde{WK} \)-convex” operator space.

**Proposition 3.10.** \( \tilde{W}T_{2}^{oh}(E) \leq d_{f}(E)\tilde{WK}(E)\tilde{W}C_{2}^{oh}(E^{*}) \).

**Proof.** Note that \( \pi^{\phi}_{2} \) is self-dual in the sense of trace duality. (See [11] for example)

By applying (2) of Proposition 3.8 and trace duality we have

\[
\tilde{W}T_{2}^{oh}(E) = \sup \left\{ \frac{\pi^{\phi}_{2}(v^{*})}{\ell_{W}^{*}(v)} : n \in \mathbb{N} \text{ and } v : E \to OH_{n} \right\}
\]

\[
\leq d_{f}(E)\tilde{WK}(E) \cdot \sup \left\{ \frac{\pi^{\phi}_{2}(v^{*})}{\ell_{W}^{*}(v)} : n \in \mathbb{N} \text{ and } v : E \to OH_{n} \right\}
\]

\[
= d_{f}(E)\tilde{WK}(E)\tilde{W}C_{2}^{oh}(E^{*}).
\]

\[\square\]

4. Unconditionality with respect to complete orthonormal systems in noncommutative \( L_{2} \) spaces

We start this section with an equivalent formulation of completely 2-summing norm. The following proposition tells us that we can replace \( (e_{ij}) \) into complete orthonormal systems in some noncommutative \( L_{2} \)-space.

**Proposition 4.1.** Let \( F \) be a locally-\( C^{*}(\mathbb{F}_{\infty}) \) operator space, \( M \) be a von Neumann algebra not subhomogeneous, \( \Phi = (\phi_{i})_{i \geq 1} \) be a complete orthonormal system in \( L_{2}(M) \) and \( u : E \to F \). Then \( u \) is completely 2-summing if and only if there is a constant \( C > 0 \) such that

\[
(4.1) \quad \left\| \sum_{i,j=1}^{n} \phi_{ij} \otimes u_{x_{ij}} \right\|_{L_{2}(M;F)} \leq C \left\| \sum_{i,j=1}^{n} e_{ij} \otimes x_{ij} \right\|_{S_{2}^{\Phi}(E)}
\]

for any \( n \in \mathbb{N} \) and \( (x_{ij}) \in M_{n}(E) \), where \( (\phi_{ij})_{i,j \geq 1} \) be a re-indexing of \( (\phi_{i})_{i \geq 1} \).

Moreover,

\[
\frac{1}{d_{f}(F)} \cdot \inf C \leq \pi^{\phi}_{2}(u) \leq d_{f}(F) \cdot \inf C,
\]

where the infimum runs over the \( C \)’s satisfying \( (4.1) \).

**Proof.** First, we assume that \( F \subseteq C^{*}(\mathbb{F}_{\infty}) \).

Now suppose \( (4.1) \) holds, and fix \( n \in \mathbb{N} \) and \( (x_{ij}) \in M_{n}(E) \). Since \( M \) is not subhomogeneous for any \( \epsilon > 0 \) there are completely positive maps

\[
\rho : M_{n} \to M \text{ and } \sigma : M \to M_{n} \text{ such that } \left\| \sigma_{\rho} - I_{M_{n}} \right\|_{cb} \leq \epsilon
\]

where \( \epsilon \) is as small as desired.
by Lemma 2.7 of [19] and [7]. Thus by Proposition 2.4 we can replace matrix units $(e_{ij})_{i,j=1}^n \in M_n$ into for some $(f_{ij})_{i,j=1}^n \in M$ allowing constant $1 + \epsilon$. Since $\Phi$ is complete in $L_2(M)$ we can choose $(y_{kl})_{k,l=1}^m \subseteq E$ such that

$$ \left| \sum_{i,j=1}^n f_{ij} \otimes x_{ij} - \sum_{k,l=1}^m \phi_{kl} \otimes y_{kl} \right|_{L_2(M; F)} \leq \epsilon. $$

Then we have

$$ \left\| (ux_{ij}) \right\|_{S_2^n (F)} \leq (1 + \epsilon) \left\| \sum_{i,j=1}^n f_{ij} \otimes ux_{ij} \right\|_{L_2(M; E)} $$

$$ \leq (1 + \epsilon) \left\| u \right\|_{cb} \left\| \sum_{i,j=1}^n f_{ij} \otimes x_{ij} - \sum_{k,l=1}^m \phi_{kl} \otimes y_{kl} \right\|_{L_2(M; F)} $$

$$ + (1 + \epsilon) \left\| \sum_{k,l=1}^m \phi_{kl} \otimes uy_{kl} \right\|_{L_2(M; F)} $$

$$ \leq C(1 + \epsilon) \left\| u \right\|_{cb} + C(1 + \epsilon) \left\| (y_{kl}) \right\|_{S_2^n \otimes_{min} E} $$

and

$$ \left\| (y_{kl}) \right\|_{S_2^n \otimes_{min} E} = \left\| E^* \rightarrow S_2^n, e^* \mapsto \langle (e^*, y_{kl}) \rangle_{cb} \right\| $$

$$ = \left\| S_2(E^*) \rightarrow S_2(S_2^n), e^*_{rs} \mapsto \langle (e^*_{rs}, y_{kl}) \rangle_{1 \leq r,s, 1 \leq k,l \leq m} \right\| $$

$$ = \sup \left\| (e^*_{rs}, y_{kl}) \right\|_{S_2(S_2^n)} $$

$$ = \sup \left\| (e^*_{rs}) \right\|_{S_2(E^*) \leq 1} \left\| I_{L_2(M) \otimes \Psi(e^*_{rs})} \left( \sum_{k,l=1}^m \phi_{kl} \otimes y_{kl} \right) \right\|_{L_2(M; S_2)} $$

where

$$ \Psi(e^*_{rs}) : E \rightarrow S_2, z \mapsto \langle e^*_{rs}, z \rangle_{r,s}. $$

Note that we have by (2) of Proposition 2.4 and Lemma 5.14. of [17] that

$$ \left\| I_{L_2(M) \otimes \Psi(e^*_{rs})} \right\|_{cb} \leq \left\| \Psi(e^*_{rs}) \right\|_{cb} \leq \pi_{2} (\Psi(e^*_{rs})) \leq \left\| (e^*_{rs}) \right\|_{S_2(E^*)}. $$

Since $(f_{ij})$ is orthonormal we have

$$ \left\| (y_{kl}) \right\|_{S_2^n \otimes_{min} E} $$

$$ \leq \sup \left\| I_{L_2(M) \otimes \Psi(e^*_{rs})} \left( \sum_{i,j=1}^n f_{ij} \otimes x_{ij} - \sum_{k,l=1}^m \phi_{kl} \otimes y_{kl} \right) \right\|_{L_2(M; S_2)} $$

$$ + \sup \left\| I_{L_2(M) \otimes \Psi(e^*_{rs})} \left( \sum_{i,j=1}^n f_{ij} \otimes x_{ij} \right) \right\|_{L_2(M; S_2)} $$

$$ \leq \left\| \sum_{i,j=1}^n f_{ij} \otimes x_{ij} - \sum_{k,l=1}^m \phi_{kl} \otimes y_{kl} \right\|_{L_2(M; E)} + \sup \left\| (e^*_{rs}, x_{ij}) \right\|_{S_2(S_2^n)} $$

$$ \leq C(1 + \epsilon) \left\| u \right\|_{cb} + \epsilon + \left\| (y_{kl}) \right\|_{S_2^n \otimes_{min} E}. $$

By combining all these results we get

$$ \left\| (ux_{ij}) \right\|_{S_2^n (F)} \leq (1 + \epsilon) \left\| u \right\|_{cb} + C(1 + \epsilon) + C(1 + \epsilon) \left\| (x_{ij}) \right\|_{S_2^n \otimes_{min} E}, $$

\[14\]
and by letting $\epsilon \rightarrow 0$ we get $\|ux_{ij}\|_{S_2^o(E)} \leq C \|x_{ij}\|_{S_2^o \otimes_{\text{min}} E}$ and consequently

$$\pi_2^o(u) \leq C.$$ 

For the converse we consider a completely 2-summing map $u : E \rightarrow F$. By the factorization theorem (Proposition 6.1. of [17]) we have

$$A : E \rightarrow OH(I) \text{ and } B : OH(I) \rightarrow F$$

for some index set $I$ such that $u = BA$ with $\pi_2^o(A) \leq 1$ and $\|B\|_{cb} \leq \pi_2^o(u)$.

Then by (2) of Proposition 4.3. we have

$$\left\| \sum_{i,j=1}^n \phi_{ij} \otimes ux_{ij} \right\|_{L_2(M;E)} \leq \|B\|_{cb} \left\| \sum_{i,j=1}^n \phi_{ij} \otimes Ax_{ij} \right\|_{L_2(M;E)}$$

$$\leq \pi_2^o(u) \|Ax_{ij}\|_{S_2^o(OH(I))}$$

$$\leq \pi_2^o(u) \pi_2^o(A) \|x_{ij}\|_{S_2^o \otimes_{\text{min}} E}$$

$$\leq \pi_2^o(u) \|x_{ij}\|_{S_2^o \otimes_{\text{min}} E}$$

for any $n \in \mathbb{N}$ and $(x_{ij}) \subseteq E$.

Now we define the unconditionality with respect to orthonormal systems in non-commutative $L_2$ spaces.

**Definition 4.2.** Let $\Phi = (\phi_i)_{i \geq 1}$ be an orthonormal system in $L_2(M)$. We say that $E$ is $\Phi$-unconditional if there is a constant $C > 0$ such that

$$\frac{1}{C} \left\| \sum_{i=1}^n \phi_i \otimes x_i \right\|_{L_2(M;E)} \leq \left\| \sum_{i=1}^n \phi_i \otimes u_i \otimes x_i \right\|_{L_2(M \otimes M_m;E)} \leq C \left\| \sum_{i=1}^n \phi_i \otimes x_i \right\|_{L_2(M;E)}$$

for any $n, m \in \mathbb{N}$, $(x_i)_{i=1}^n \subseteq E$ and any unitaries $u_i \in M_m$. We denote $\Phi_{\text{unc}}(E)$ by the infimum of such $C$.

Before we prove our main result we observe that unconditionality implies $\tilde{W}$-cotype 2.

**Proposition 4.3.** Let $M$ be a von Neumann algebra not subhomogeneous, and $\Phi = (\phi_i)_{i \geq 1}$ be a complete orthonormal system in $L_2(M)$ with

$$\left\| \sup_i \phi_i \right\|_2 < \infty.$$

If $E$ is $\Phi$-unconditional, then $E$ has $\tilde{W}$-cotype 2 with

$$\tilde{WC}_{2}^{sh}(E) \leq 2C_G d_f(E)^2 \left\| \sup_i \phi_i \right\|_2 \Phi_{\text{unc}}(E),$$

where $C_G$ is the constant in Theorem 3.4.1.

**Proof.** Let's fix $k, n \in \mathbb{N}$, $(x_{ij})_{i,j=1}^n \subseteq OH_k$ and $u : OH_k \rightarrow E$ and let $(\phi_{ij})_{i,j \geq 1}$ be a re-indexing of $(\phi_i)_{i \geq 1}$.

It is clear from the definition that

$$\left\| \sum_{i,j=1}^n \phi_{ij} \otimes ux_{ij} \right\|_{L_2(M;E)} \leq \Phi_{\text{unc}}(E) \left\| \sum_{i=1}^n \phi_{ij} \otimes U_{ij}^m \otimes ux_{ij} \right\|_{L_2(M \otimes N_m;E)}$$

for independent family of standard random unitaries $U_{ij}^m$, $m \in \mathbb{N}$, then by taking limit over the ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and applying Theorem 3.4.1, Theorem 3.4.2 and (1) of
Proposition 3.8 we have

\[ \left\| \sum_{i,j=1}^{n} \phi_{ij} \otimes u x_{ij} \right\|_{L_2(M;E)} \leq \Phi_{unc}(E) \left\| \sum_{i=1}^{n} \phi_{ij} \otimes (g_{ij}) \otimes u x_{ij} \right\|_{L_2(M \otimes N;E)} \]

\[ \leq 2d_f(E) \sup_i |\phi_i| \left\| \Phi_{unc}(E) \left( \sum_{i=1}^{n} \phi_{ij} \otimes (g_{ij}) \otimes u x_{ij} \right) \right\|_{L_2(N;E)} \]

\[ \leq 2C_Gd_f(E) \sup_i |\phi_i| \left\| \Phi_{unc}(E) \left( \sum_{i=1}^{n} \phi_{ij} \otimes u x_{ij} \right) \right\|_{L_2(N;E)} \]

\[ = 2C_Gd_f(E) \sup_i |\phi_i| \left\| \Phi_{unc}(E) \ell_{\tilde{W}}(u) \right\|_2 \]

\[ \leq 2C_Gd_f(E) \sup_i |\phi_i| \left\| \Phi_{unc}(E) \| \ell_{\tilde{W}}(u) \right\|_2 \]

\[ = 2C_Gd_f(E) \sup_i |\phi_i| \left\| \Phi_{unc}(E) \right\|_2, \]

where \( v : S_2^n \to OH_k, e_{ij} \mapsto x_{ij} \).

Thus, by Proposition 4.1 we have

\[ \pi_2^0(u) \leq 2C_Gd_f(E)^2 \left\| \sup_i |\phi_i| \right\|_2 \Phi_{unc}(E) \ell_{\tilde{W}}(u) \]

and consequently

\[ \tilde{W}C_2^{oh}(E) \leq 2C_Gd_f(E)^2 \left\| \sup_i |\phi_i| \right\|_2 \Phi_{unc}(E). \]

\[ \square \]

Finally we prove our main theorem. We state the theorem again with further comment on the constant.

**Theorem 4.4.** Let \( M \) and \( \Phi = (\phi_i)_{i \geq 1} \) be the same as in Proposition 4.2. Then, \( E \) is \( \Phi \)-unconditional if and only if \( E \) is completely isomorphic to an operator Hilbert space. Moreover, we have

\[ \Phi_{unc}(E) \leq d_{ch}(E, OH(I)) \leq C_1C_2^2(1 + \log C_2) \]

for some universal constant \( C_1 > 0 \) and \( C_2 = d_f(E)^4 \Phi_{unc}(E) \| \sup_i |\phi_i| \|_2 \).

**Proof.** Let \( F \) be any finite dimensional subspace of \( E \). Then \( F \) is clearly \( \Phi \)-unconditional with \( \Phi_{unc}(F) \leq \Phi_{unc}(E) \). Thus, by Proposition 5.7 and 5.10 we have

\[ d_{ch}(F, OH_{dim}F) \leq d_f(F)\tilde{W}T_2^{oh}(F)\tilde{W}C_2^{oh}(F) \]

\[ \leq d_f(E)^2\tilde{W}K(F)\tilde{W}C_2^{oh}(F^*)\tilde{W}C_2^{oh}(F). \]

\( \tilde{W}C_2^{oh}(F) \) is estimated by Proposition 6.6 and a similar estimate for \( \tilde{W}C_2^{oh}(F^*) \) can be done as follows.
For fixed $n \in \mathbb{N}$, $(y_i)_{i=1}^n \subseteq F^*$ and unitaries $(u_i)_{i=1}^n \subseteq M_m$ we have by Proposition 2.1 and the orthonormality of $\Phi$ that

$$\left\| \sum_{i=1}^n \phi_i \otimes y_i \right\|_{L_2(\mathcal{M}; F^*)} \leq d_f(F) \left\| \sum_{i=1}^n \phi_i \otimes y_i \right\|_{L_2(\mathcal{M}; E)^*}$$

$$= d_f(E) \sup \left\{ \left\| \sum_{i=1}^n \phi_i \otimes u_i \otimes y_i, \sum_{j=1}^k \phi_j \otimes x_j \right\|_{L_2(\mathcal{M}; F)} \right\}$$

$$\leq d_f(E) \omega_{unc}(E) \sup \left\{ \left\| \sum_{i=1}^n \phi_i \otimes u_i \otimes y_i, \sum_{j=1}^k \phi_j \otimes x_j \right\|_{L_2(\mathcal{M}; M_m; F)} \right\}$$

$$\leq d_f(E) \omega_{unc}(E) \sup \left\{ \left\| \sum_{i=1}^n \phi_i \otimes u_i \otimes y_i \right\|_{L_2(\mathcal{M}; M_m; F)^*} \right\}.$$

By the same argument as in the proof of Proposition 13 we get

$$\widetilde{W}_2^2(F^*) \leq 2C_G d_f(E) \omega_{unc}(E) \left\| \sup_1 |\phi_i| \right\|_2.$$

Using this estimate and Proposition 12.4. of [21] we get

$$d_{cb}(F, OH_{dimF}) \leq C \widetilde{W}_2^2(F) \leq CK(S_2(F)) \leq C_K C \log(1 + d(S_2(F), OH))$$

$$= C_K C \log(1 + d_{cb}(F, OH_{dimF})),$$

where $C = 4C_G^2 d_f(E)^8 \omega_{unc}(E)^2 \left\| \sup_1 |\phi_i| \right\|_2^2$ and $C_K$ is the universal constant in Proposition 12.4. of [21]. Thus, we have $d_{cb}(F, OH_{dimF}) \leq C_K C \log(2C_K C + 1)$ for any finite dimensional subspace $F$ of $E$, and consequently

$$d_{cb}(E, OH(I)) \leq C_K C \log(2C_K C + 1)$$

for some index set $I$.

\[\square\]

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