APPROXIMATION OF LINEAR DIFFERENTIAL EQUATIONS
WITH VARIABLE DELAY ON AN INFINITE INTERVAL

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Abstract. We study approximation of non-autonomous linear differential equations with variable delay over infinite intervals. We use piecewise constant argument to obtain a corresponding discrete difference equation. The study of numerical approximation over an unbounded interval is in correspondence with the problem of transference of qualitative properties between a continuous and the corresponding discrete dynamical systems. We use theory of differential equations with delay and theory of integral inequality to prove our main result. We state sufficient conditions for: a) uniform approximation of solutions over an unbounded interval and b) transference of uniform asymptotic stability of non-autonomous linear differential equations with variable delay to the corresponding discrete difference equations. We improve and extended the results of [Cooke and Győri (1994)].

Keywords: Linear functional-differential equations, Theoretical approximation of solutions, Stability theory, Numerical approximation of solutions.

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1. Introduction

We study approximation of non-autonomous functional differential equations over infinite intervals. Recently, several researchers have discretized systems of differential equations using piecewise constant argument, they obtained an Euler’s approximation of the solution of original system, see Mohamad and Gopalsamy (2003); Huang et al. (2006); Abbas and Yonghui (2013). These paper follow the ideas developed by Győri (1991), who was interested in the convergence of several approximation, by piecewise constant argument, to the actual solution of a class of delay linear differential equations. Actually, that work belongs to a series of papers about approximations of solutions of differential equations with delay (see Győri, 1988, 1991; Cooke and Győri, 1994; Győri et al., 1995; Győri and Hartung, 2002, 2008). Numerical methods for differential equations with delay are well-known, see Bellen and Zennaro (2013). Numerical analysis for the study of stability of delay differential equations have been developed recently, see Breda et al. (2015). However, the study of numerical approximation of a solution over an unbounded interval is in correspondence with the problem of transference of qualitative properties between a continuous dynamical system and the corresponding discrete dynamical system.

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In this work we study the approximation of the homogeneous non-autonomous linear differential equations with variable delay

\[
x'(t) = -a(t)x(t - r(t)),
\]
where \(a : [0, \infty) \to [0, \infty)\) and \(r : [0, \infty) \to [0, q]\), with initial condition

\[
x(t) = \varphi(t), \quad -q \leq t \leq 0, \quad \varphi \in \mathcal{C} = C([-q, 0], \mathbb{R}),
\]
where \(q := \sup_{t \in \mathbb{R}^+} \{r(t)\}\) is a positive real number.

Our aims are find conditions for:

1. the uniform convergence of approximate solutions over \([0, \infty)\); and
2. the transfer of uniform stability between solutions of linear differential equations with variable delay and its corresponding difference equation.

The study of transfer of uniform asymptotic stability between solutions of linear differential equations with delay and the corresponding discrete difference equation started with Cooke and Győri (1994). Cooke and Ivanov (2000) studied the dynamic of the solutions of a singular difference equation with delay which can be interpreted as an Euler discretizations of a singular differential equations with delay. They concluded that numerical approximation of solutions of singularly perturbed delay differential equations maybe showing dynamics which are irrelevant to the actual dynamics in these equations. These type of difficulties, namely spurious fixed point, are well-known for Runge-Kutta methods of numerical approximation for ordinary differential equations.

Liz and Ferreiro (2002) proved a discrete version of Halanay’s inequality and used it to obtain the transfer of asymptotic stability between the solutions of

\[
x'(t) = -ax(t) + b(t)f(t, x_t), \quad x_0 = \phi,
\]
and

\[
x_{n+1} - x_n \frac{h}{h} = -ax_n + b(t_n)f(t_n, \phi_n).
\]

Győri and Hartung (2002) studied the transfer of uniform asymptotic stability between solutions of linear neutral differential equations with constant delay and the corresponding discrete difference equation.

We use theory of differential equations with delay and an integral version of Halanay inequality to prove our main result. To our knowledge our result is the first, of this type, for non-autonomous linear differential equations with variable delay, we use Halanay’s inequality to obtain our main theorem, in this way we complement, extend and improve the results of Cooke and Győri (1994).

The rest of the paper is organized as follows. In the next section, Halanay Inequality, some definitions and preliminary results are presented. Section 3 is devoted to discretization using piecewise constant argument and approximation over compact interval. Section 4 we prove main results about transference of stability properties and approximation over non-compact interval. Finally, in Section 5 we examine some implications of our results.

2. Preliminaries and Halanay inequality

In this section we introduce usual notation and some definitions of theory of differential equations with delay. Suppose \(q \geq 0\) is a given real number, \(\mathbb{R} = (-\infty, \infty)\), \(C([a, b], \mathbb{R})\) is the Banach space of continuous functions mapping the interval \([a, b]\) into \(\mathbb{R}\) with the topology of uniform convergence. If \([a, b] = [-q, 0]\)
we let $C = C([-q,0],\mathbb{R})$ and designate the norm of an element $\varphi$ in $C$ by $\|\varphi\| = \sup_{-r \leq \theta \leq 0} |\varphi(\theta)|$.

If
\[
\sigma \in \mathbb{R}, \ A \geq 0, \ \text{and} \ x \in C([-q-q+A],\mathbb{R}),
\]
then for any $t \in [\sigma, \sigma + A]$, we let $x_t(\theta) = x(t + \theta), -q \leq \theta \leq 0$. If $D$ is a subset of $\mathbb{R} \times C$, $f : D \to \mathbb{R}$ is a given function and $\uparrow$ represents the right-hand derivative, we say that the relation
\[
x'(t) = f(t,x_t)
\]
is a differential equation with delay or a retarded functional differential equation on $D$. A function $x$ is said to be a solution of Equation (3) on $[\sigma - q, \sigma + A]$ if there are $\sigma \in \mathbb{R}$ and $A > 0$ such that $x \in C([\sigma - q, \sigma + A], \mathbb{R})$, $(t, x_t) \in D$ and $x(t)$ satisfies Equation (3) for $t \in [\sigma, \sigma + A]$. For given $\sigma \in \mathbb{R}$, $\phi \in C$, we say $x(\sigma, \phi, f)$ is a solution of equation (3) with initial value $\phi$ if there is an $A > 0$ such that $x(\sigma, \phi, f)$ is a solution of equation (3) on $[\sigma - q, \sigma + A]$ and $x_\sigma(\sigma, \phi, f) = \phi$. The theory of differential equations with delay can be found in books [Driver, 1977] and [Hale and Lunel, 1993].

2.1. Halanay Inequality. Halanay (1966) proved an asymptotic formula for the solutions of a differential inequality with delay, and applied it in the stability theory of linear systems with delay. Since beginning of the twenty-first century several authors have been extended, improved and called Halanay inequality to such inequality (see Baker and Tang, 2000; Mohamad and Gopalsamy, 2000; Liz and Trofimchuk, 2000; Pinto and Trofimchuk, 2000; Liz and Ferreiro, 2002; Liz et al., 2005; Ou et al., 2015).

**Theorem 1.** Let $t_0$ be a real number and $q$ be a non-negative number. If $v : [t_0 - q, \infty) \to \mathbb{R}^+$ satisfies
\[
\frac{d}{dt} v(t) \leq -\alpha v(t) + \beta \left[ \sup_{s \in [t-q,t]} v(s) \right]; \ t \geq t_0
\]
where $\alpha$ and $\beta$ are constants with $\alpha > \beta > 0$, then
\[
v(t) \leq \|v_{t_0}\|_q e^{-\eta(t-t_0)} \text{ for } t \geq t_0,
\]
where $\eta$ is the unique positive solution of
\[
\eta = \alpha - \beta e^{\eta q}.
\]
A statement of this theorem can be found in Driver (1977). Next we show an integral version of Halanay inequality.

**Lemma 1.** Consider
\[
v(t) \leq \|v_{t_0}\|_q e^{-\sigma(t-t_0)} + \int_{t_0}^{t} e^{-\sigma(t-s)} K \left[ \sup_{u \in [s-q,s]} v(u) \right] ds, \ t \geq t_0,
\]
where $\sigma, q$ and $K$ are positive real numbers, and $f \in C(\mathbb{R})$ is a positive non-decreasing function. If $\sigma > K > 0$, then there exist $\eta > 0$ and $M > 0$ such that
\[
v(t) \leq M \|v_{t_0}\|_q f(t)e^{-\eta(t-t_0)}, \ t \geq t_0,
\]
where $\eta$ is the real solution of
\[
\eta = -\sigma + Ke^{\eta q}.
\]
Proof. We define \( w(t) := \frac{v(t)}{\|v_{t_0}\|_q f(t)} \), so we have
\[
w(t) \leq e^{-\sigma(t-t_0)} + \int_{t_0}^{t} e^{-\sigma(t-s)} K \left[ \sup_{u \in [s,q]} \|v_{t_0}\|_q f(u) \right] \left[ \sup_{u \in [s,q]} w(u) \right] ds.
\]
Since the function \( f \) is non decreasing we have
\[
w(t) \leq e^{-\sigma(t-t_0)} + \int_{t_0}^{t} e^{-\sigma(t-s)} K \left[ \sup_{u \in [s,q]} w(u) \right] ds.
\]
Now define
\[
\mu(t) := \begin{cases} 
1 & \text{for } t_0 - q \leq t \leq t_0 \\
 e^{-\sigma(t-t_0)} + \int_{t_0}^{t} e^{-\sigma(t-s)} K \left[ \sup_{u \in [s,q]} w(u) \right] ds & \text{for } t_0 \leq t.
\end{cases}
\]
Then \( \mu \) is continuous and nonnegative, and \( w(t) \leq \mu(t) \) for \( t_0 - q \leq t \). Moreover, for \( t_0 \leq t \)
\[
\mu'(t) \leq -\sigma \mu(t) + K \left[ \sup_{u \in [t-q,t]} \mu(u) \right].
\]
Since \( \sigma > K > 0 \), by Halanay’s inequality, there exists \( \eta > 0 \) such that
\[
\mu(t) \leq e^{-\eta(t-t_0)}, \quad t \geq t_0.
\]
Therefore
\[
w(t) \leq \mu(t) \leq e^{-\eta(t-t_0)}, \quad t \geq t_0.
\]
and
\[
v(t) \leq \|v_{t_0}\|_q f(t)e^{-\eta(t-t_0)}, \quad t \geq t_0.
\]

3. Discretization by piecewise constant argument and approximation over compact interval

The set of DEPCA corresponding to equation (1) is:
\[
z_h'(t) = a(t)z_h \left( t - \left[ \frac{t}{h} \right] \right);
\]
where \( \left[ \frac{t}{h} \right] = \frac{\lfloor \frac{t}{h} \rfloor}{h} \) with \( \lfloor \cdot \rfloor \) the usual greatest integer function. The initial condition for differential equation (10) is
\[
z_h(nh) = \varphi(nh), \quad n = -k, \ldots, 0,
\]
where \( h \) is a positive number in the interval \((0,q]\). In fact we can consider \( h = \frac{q}{k} \) where \( k \geq 1 \) is an integer. By a solution of (10)-(11) we mean a function \( z_h \) defined on \( \{ih : i = -k, \ldots, 0\} \) by (11), which satisfy the following properties on \( \mathbb{R}^+ \):

\begin{enumerate}
  \item The function \( z_h \) is continuous on \( \mathbb{R}^+ \),
  \item the derivative \( z_h'(t) \) exists at each point \( t \in \mathbb{R}^+ \) with the possible exception of the points \( ih(i = 0, 1, 2, \cdots) \) where finite one-sided derivatives exist, and
  \item the function \( z_h \) satisfies (10) on each interval \( I_{(j,h)} := [jh,(j+1)h) \) for \( j = 0,1,2,\cdots \).
\end{enumerate}
Note that for every positive $h$ close to zero, it is expected that solutions of (10)-(11) have similar qualitative features to the solutions of (1)-(2), since $[t]_{h} \rightarrow t$ uniformly on $\mathbb{R}$, as $h \rightarrow 0$. But this can be false, even for Euler’s method in the setting of singular functional differential equations, see Cooke and Ivanov (2000).

If we denote 
\[ \gamma_h(t, r) := \left[ \frac{t}{h} - \left\lfloor \frac{r}{h} \right\rfloor \right] h, \]
then equation (10) can be written like
\[ z'(t) = a(t)z_h(\gamma_h(t, r)). \]

For $t \in I(i, h)$ the function $r(\lfloor \frac{t}{h} \rfloor)$ takes just one value, therefore the function $\left\lfloor \frac{r}{h} \right\rfloor := k_i$ is a fixed integer for $t \in I(i, h)$. It follows that
\[ \left[ \frac{t}{h} - \left\lfloor \frac{r}{h} \right\rfloor \right] = i - k_i. \]

It is follows that
\[ \gamma_h(t, r) = h(i - k_i), \quad \text{for } t \in I(i, h). \]

Therefore (10) is equivalent to
\[ z'(t) = a(t)z_h(h(i - k_i)), \quad t \in I(i, h), \quad i \geq 0. \]

Note that, for $ih \leq t \leq (i + 1)h$, we integrate (12) and obtain
\[ z_h(t) = z_h(ih) + \int_{ih}^{t} a(s)ds z_h(h(i - k_i)). \]

Making $t \rightarrow (i + 1)h$, from the continuity of $z_h$, we obtain
\[ z_h((i + 1)h)ds = z_h(ih) + \int_{ih}^{(i+1)h} a(s)ds z_h(h(i - k_i)). \]

Therefore the sequence $z_h(i) := z_h(ih)$ satisfy the linear difference equation with variable delay
\[ z_h(n + 1) = z_h(n) + \int_{nh}^{(n+1)h} a(s)ds z_h(n - k_n), \]
with initial conditions
\[ z_h(n) = \varphi(nh), \quad n = 0, 1, \cdots, \quad -q \leq -nh \leq 0. \]

Note that (13)-(14) is a discretization of the differential equations with variable delay (1)-(2) that coincides with Euler’s approximation method for autonomous
differential equations with variable delay. From the recurrence relation (13) and initial conditions, we have
\[ \begin{align*}
\delta_h(0) &= \varphi(0) \\
\delta_h(1) &= \delta_h(0) + \int_0^h a(s)ds \delta_h(0 - k_0), \\
\delta_h(2) &= \delta_h(1) + \int_h^{2h} a(s)ds \delta_h(1 - k_1) \\
&= \delta_h(0) + \int_0^h a(s)ds \delta_h(0 - k_0) + \int_h^{2h} a(s)ds \delta_h(1 - k_1).
\end{align*} \]

Therefore the sequence \( \delta_h \) solution of (13)-(14) is well-defined, and satisfy
\[ \delta_h(n) = \varphi(0) + \sum_{i=0}^{n-1} \int_{ih}^{(i+1)h} a(s)ds \delta_h(i - k_1), \quad k \geq 0. \tag{15} \]

From (15), it follows that the solution of (10)-(11) for \( t \geq 0 \) can be written
\[ z_h(t) = \delta_h(n) + \int_{nh}^{t} a(s)ds \delta_h(n - k_n), \tag{16} \]
or
\[ z_h(t) = \varphi(0) + \sum_{i=0}^{n-1} \int_{ih}^{(i+1)h} a(s)ds \delta_h(i - k_1) + \int_{nh}^{t} a(s)ds \delta_h(n - k_n), \tag{17} \]
where \( n = n(t) \) is such that \( nh \leq t < (n + 1)h \). Thus, we have proved
\[ \text{Theorem 2. The initial value problem (10)-(11) has a unique solution in the form} \]
\[ z_h(t) = \varphi(0) + \sum_{i=0}^{n-1} \int_{ih}^{(i+1)h} a(s)ds \delta_h(i - k_1) + \int_{nh}^{t} a(s)ds \delta_h(n - k_n) \]
for \( t \geq 0 \) and the sequence \( \delta_h(t) \) satisfies the nonlinear difference equations (13) with initial conditions (14).

3.1. Approximation over compact interval. In this subsection we address the problem of approximation of solutions of initial value problem (1)-(2) over compact interval. We follow some ideas of Győri (1991) to obtain.

\[ \text{Theorem 3. If } r : [0, \infty) \to [0, q] \text{ is a continuous function then, for any } \varphi \in C, \text{ the solutions } x(\varphi)(t) \text{ and } z_h(\varphi)(t) \text{ of the initial value problems (1)-(2) and (10)-(11), respectively, satisfy the following relations for all } T > 0 \]
\[ \lim_{h \to 0} \max_{0 \leq t \leq T} \|x(\varphi)(t) - z_h(\varphi)(t)\| = 0, \tag{18} \]

namely
\[ \max_{0 \leq t \leq T} \|x(\varphi)(t) - z_h(\varphi)(t)\| \leq \left[ e \int_0^T a(s)ds \int_0^T a(s)ds \right] w_x \left( w_r(h; T) + 2h; T \right), \tag{19} \]

where \( w_r(h; T) \) and \( w_x(h; T) \) are defined by
\[ w_r(h; T) = \max \{ |r(t_2) - r(t_1)| : 0 \leq t_1, t_2 \leq T, |t_2 - t_1| \leq h \}, \]
\[ w_x(h; T + 2h; T) = \max \{ |x(t_2) - x(t_1)| : -q \leq t_1, t_2 \leq T, |t_2 - t_1| \leq 2h + w_r(h, t) \}. \]
Proof. Consider the solutions $x(t) = x(\varphi)(t)$ and $z_h(t) = z_h(\varphi)(t)$ of initial value problems (11), respectively. Then from (1) and (10) we find

$$x'(t) - z'_h(t) = -a(t) [x(t) - r(t)] - z_h(\gamma_h(t, r)),$$

for all $t \geq 0$. Thus the function $\varepsilon_h(t) = x(t) - z_h(t)$ satisfies

$$\varepsilon'_h(t) = -a(t) \varepsilon_h(\gamma_h(t, r)) - a(t) [x(t) - r(t)] - x(\gamma_h(t, r)),$$

for all $t \geq 0$ with $\varepsilon_h(0) = 0$. We integrate over $[0, t]$ and obtain

$$|\varepsilon_h(t)| \leq \int_0^t a(s) |\varepsilon_h(\gamma_h(s, r))| ds + \int_0^t a(s) |x(s) - r(s) - x(\gamma_h(s, r))| ds$$

$$\leq \int_0^t a(s) |\varepsilon_h(s, r)| ds + f_h(t),$$

where

$$f_h(t) := \int_0^t a(s) |x(s) - r(s) - x(\gamma_h(s, r))| ds, \quad t \geq 0.$$

On the other hand,

$$\gamma_h(s, r) \leq s$$

for all $s \geq 0$, and from initial conditions we have that

$$|\varepsilon_h(\gamma_h(s, r))| = |\varphi(\gamma_h(s, r)) - x_h(\gamma_h(s, r))| = 0,$$

for all $s \geq 0$ such that $\gamma_h(s - r) \leq 0$. Therefore we find that the function $\xi(t) = \max_{0 \leq s \leq t} |\varepsilon_h(s, r)|$ satisfies the inequality

$$\xi(t) \leq \int_0^t a(s) \xi(\gamma_h(s, r)) ds + f_h(t) \leq \int_0^t a(s) \xi(s) ds + f_h(t), \quad t \geq 0,$$

where we used that the integral term and $f_h(t)$ are monotone increasing functions. By Gronwall-Bellman inequality we find

$$\xi(t) \leq f_h(t) e^{\int_0^t a(s) ds}, \quad t \in [0, T].$$

Now, we note that $|t - r(t) - \gamma_h(t, r)| = |t - r(t) - (i - k_i)h|$ where $i = \left\lfloor \frac{t}{h} \right\rfloor$ and $k_i = \left\lceil \frac{r(t)}{h} \right\rceil$, so

$$|t - r(t) - \gamma_h(t, r)| \leq |t - ih| + |r(t) - k_i h|$$

$$= \left| t - \left\lfloor \frac{t}{h} \right\rfloor h \right| + \left| r(t) - \left\lfloor \frac{r(t)}{h} \right\rfloor h \right|$$

$$\leq h + \left| r(t) - r\left(\frac{t}{h} \right) h \right| + \left| r\left(\frac{t}{h} \right) h - \left\lfloor \frac{r(t)}{h} \right\rceil h \right|$$

$$\leq h + \left| r(t) - r\left(\frac{t}{h} \right) h \right| + h$$

$$\leq 2h + w_r(h; t),$$

where $w_r(h; t) = \max \{|r(t_2) - r(t_1)| : 0 \leq t_1, t_2 \leq t, |t_2 - t_1| \leq h\}$. Note that for uniformly continuous function $r$, $w_r(h, t)$ tends to zero as $h$ tends to $0$. Set

$$w_x(w_r(h; t) + 2h; t) = \max \{|x(t_2) - x(t_1)| : -q \leq t_1, t_2 \leq t, |t_2 - t_1| \leq 2h + w_r(h, t)\}.$$

Then from (21) it follows that

$$|x(s) - r(s) - x(\gamma_h(s, r))| \leq w_x(w_r(h; t) + 2h; t),$$
for all $0 \leq s \leq t$ and for all $r$. Also,

$$f_h(t) \leq \int_0^t a(s)ds \, w_x(w_r(h; t) + 2h; t), \quad t \geq 0,$$

and clearly (20) yields

$$\xi(t) \leq e^{\int_0^t a(s)ds} \int_0^t a(s)ds \, w_x(w_r(h; T) + 2h; T), \quad t \in [0, T].$$

Since (22) and $|\varepsilon_h(t)| = |x(t) - z_h(t)| = |x(\varphi)(t) - z_h(\varphi)(t)| \leq \xi(t)$ we obtain (19) for all $h = \frac{q}{k} > 0$ and $t \in [0, T]$. So, for all $T > 0$

$$\max_{0 \leq t \leq T} |x(\varphi)(t) - z_h(\varphi)(t)| \leq \left[ e^{\int_0^T a(s)ds} \int_0^T a(s)ds \right] w_x(w_r(h; T) + 2h; T) \to 0,$$

as $h \to 0$, from the uniform continuity of the functions $x$ and $r$ on $[0, T]$. \hfill \Box

4. TRANSFER OF ASYMPTOTIC STABILITY

In this section we obtain a sufficient conditions for: a) uniform approximation of solutions over an unbounded interval and b) transference of uniform asymptotic stability of the zero solution of non-autonomous linear differential equations with variable delay of (1) to the zero solution of the corresponding discrete difference equations (13).

(A1) The zero solution of (1) is uniformly asymptotically stable,

(A2) the function $a(t)$ is bounded, namely

$$a_0 = \sup_{t \geq 0} |a(t)| < \infty.$$

(A3) The function $r(t)$ is uniformly continuous on $[0, \infty)$.

Next we will obtain an estimate for the distance between the solutions of initial value problems (1)-(2) and (10)-(11) on $[0, \infty)$. We define the error function

$$E_h(\cdot) := x(\cdot) - z_h(\cdot),$$

and for all $t \geq 0$, $E_h(t) = -a(t) [x(t - r(t)) - z_h(\gamma_h(t, r))],

for all $t \geq 0$. Adding and substracting $a(t)z(t - r(t))$ we obtain

$$E_h'(t) = -a(t)E_h(t - r(t)) - a(t) [z_h(t - r(t)) - z_h(\gamma_h(t, r))].$$
and, by fundamental theorem of calculus, we have
\[ E'_h(t) = -a(t)E_h(t - r(t)) - a(t) \int_{\gamma_h(t,r)}^{t-r(t)} z_h'(\xi) d\xi. \]

Now, from (10), we obtain
\[ E'_h(t) = -a(t)E_h(t - r(t)) - a(t) \int_{\gamma_h(t,r)}^{t-r(t)} a(\xi)z_h(\gamma_h(\xi,r)) d\xi, \]
it follows that
\[ (24) \quad g_h(t) := -a(t) \int_{\gamma_h(t,r)}^{t-r(t)} a(\xi)x(\gamma_h(\xi,r)) d\xi. \]

Since variation-of-constants formula, (see Driver, 1977, pp. 334), we have
\[ E_h(t) = U(t; \tau, E_{h_0}) + \int_{\tau}^{t} U\left( t; s, \left[ a(s) \left( \int_{\gamma_h(s,r)}^{s-r(s)} a(\xi)E_h(\gamma_h(\xi,r)) d\xi \right) + g_h(s) \right] u \right) ds, \]
where \( U(t; \tau, E_{h_0}) \) is the unique solution of Equation (1) with initial value \( E_{h_0} \) at \( \tau \), and \( u \) is the unit step function \( u : [-q, 0] \to \mathbb{R} \) defined by
\[ u(t) = \begin{cases} 0, & \text{for } -q \leq t < 0, \\ 1, & \text{for } t = 0. \end{cases} \]

Thus \( |E_h(t)| \) for all \( t \geq t_0 \) satisfies
\[ (26) \quad |E_h(t)| \leq |U(t; t_0, E_{h_{t_0}})| + \int_{t_0}^{t} |U\left( t; s, \left[ a(s) \left( \int_{\gamma_h(s,r)}^{s-r(s)} a(\xi)E_h(\gamma_h(\xi,r)) d\xi \right) + g_h(s) \right] u \right) | ds. \]

Since we assume that zero solution of (1) is uniformly asymptotically stable, there are constants \( \sigma > 0 \) and \( K > 0 \), (see Hale and Lunel, 1993, pp. 185), such that for each \( \phi \in C \) we have
\[ (27) \quad |U(t; s, \phi)| \leq K \| \phi \|_q e^{-\sigma(t-s)}, \quad t \geq s. \]

Moreover, there exists a constant \( M_0 \) such that
\[ (28) \quad |x(t)| \leq M_0 \| \varphi \|_q e^{-\sigma t}, \quad t \geq 0. \]

In order to use (28) to estimate \( g_h(t) \) we need find a positive real number \( t_0 \) such that for \( t \geq t_0 \) then
\[ 0 \leq \gamma_h(\xi, t), \text{ whenever } \xi \geq \gamma_h(t, r). \]

We recall (21), i.e., \( |t - r(t) - \gamma_h(t, r)| = |\gamma_h(t, r) - t + r(t)| \leq 2h + w_r(h) \), it follows
\[ t - r(t) - w_r(h) - 2h \leq \gamma_h(t, r) \leq t - r(t) + w_r(h) + 2h. \]

Since \( h \in [0, q] \) and \( r \) is uniformly continuous on \([0, \infty)\) it follows
\[ (29) \quad t - 3q - w_r(q) \leq t - r(t) - w_r(q) - 2q \leq t - r(t) - w_r(h) - 2h \leq \gamma_h(t, r). \]
Now we use (28) and (23) in (24) to estimate \( g_h(t) \) for \( t \geq t_0 := 3q + w_r(q) \) and obtain:

\[
|g_h(t)| \leq a_0^2 \int_{\gamma_h(t,r)}^{t-r(t)} |x(\gamma_h(\xi,r))| d\xi \\
\leq a_0^2 \int_{\gamma_h(t,r)}^{t-r(t)} M_0 \|\varphi\|_q e^{-\sigma \gamma_h(\xi,r)} d\xi \\
= a_0^2 \int_{\gamma_h(t,r)}^{t-r(t)} M_0 \|\varphi\|_q e^{-\sigma \xi} e^{\sigma [\gamma_h(\xi,r) - \gamma_h(\xi,r)]} d\xi,
\]

since (21) and uniform continuity of \( r \) we have

\[
|g_h(t)| \leq a_0^2 \int_{\gamma_h(t,r)}^{t-r(t)} M_0 \|\varphi\|_q e^{-\sigma \xi} e^{\sigma [w_r(h) + 2h]} d\xi \\
\leq e^{-\sigma \gamma_h(t,r)} a_0^2 \int_{\gamma_h(t,r)}^{t-r(t)} M_0 \|\varphi\|_q e^{\sigma [w_r(h) + 2h]} d\xi. \tag{30}
\]

Next, using (21), we estimate \( s - \gamma(s,r) \)

\[
s - \gamma(s,r) = s - r(s) - \gamma(s,r) + r(s) \\
\leq q + 2h + w_r(h) \\
\leq 3q + w_r(q). \tag{31}
\]

Since \( t \geq t_0 := 3q + w_r(q) \), (21), (29) and (31) inequality (30) become into

\[
|g_h(t)| \leq e^{-\sigma t} a_0^2 M_1 \|\varphi\|_q [2h + w_r(h)], \tag{32}
\]

where

\[
M_1 := M_0 e^{\sigma [5q + 2w_r(q)]}.
\]
Using estimations (27) and (32) for $t_0 := 3q + w_r(q)$, in (26) we obtain for $t \geq t_0$

$$|E_h(t)| \leq |U(t; t_0, E_{h,t_0})| + \int_{t_0}^{t} |U(t; s, E_{h,s})| \left( a(s) \left( \int_{h(s,r)}^{s-r(s)} a(\xi) E_h(\gamma_h(\xi, r)) d\xi \right) + g_h(s) \right) u(s) ds$$

$$\leq K \|E_{h,t_0}\|_q e^{-\sigma(t-t_0)} + \int_{t_0}^{t} K \left| a(s) \left( \int_{h(s,r)}^{s-r(s)} a(\xi) E_h(\gamma_h(\xi, r)) d\xi \right) + g_h(s) \right| e^{-\sigma(t-s)} ds$$

$$\leq K \|E_{h,t_0}\|_q e^{-\sigma(t-t_0)} + a_0^2 \int_{t_0}^{t} K \left( \sup_{s-r(s) \leq \gamma_h(s,r)} |E_h(s)| \right) \left( \int_{h(s,r)}^{s-r(s)} e^{-\sigma(t-s)} ds \right)$$

$$+ a_0^2 M_1 \|\varphi\| \left( 2h + w_r(h) \right) e^{-\sigma t}$$

$$= K \|E_{h,t_0}\|_q e^{-\sigma(t-t_0)} + a_0^2 \left( 2h + w_r(h) \right) \int_{t_0}^{t} K \left( \sup_{s-r(s) \leq \gamma_h(s,r)} |E_h(s)| \right) \left( s - r(s) - \gamma_h(s,r) \right) e^{-\sigma(t-s)} ds$$

$$+ a_0^2 M_1 \|\varphi\| \left( 2h + w_r(h) \right) e^{-\sigma t}$$

$$\leq K \|E_{h,t_0}\|_q e^{-\sigma(t-t_0)} + a_0^2 \left( 2h + w_r(h) \right) \left( \int_{t_0}^{t} K \left( \sup_{s \leq t} |E_h(s)| \right) e^{-\sigma(t-s)} ds \right)$$

$$+ a_0^2 M_1 \|\varphi\| \left( 2h + w_r(h) \right) e^{-\sigma t}$$

$$= e^{-\sigma(t-t_0)} \left[ K \|E_{h,t_0}\|_q + K_1(h)M_1 \|\varphi\| e^{\sigma t_0} \right] + K_1(h) \int_{t_0}^{t} e^{-\sigma(t-s)} \sup_{s-t_0 \leq \gamma_h(s,r)} |E_h(s)| ds,$$

where $K_1(h) := a_0^2 \left( 2h + w(r; h) \right) K$. If $h$ is small enough such that:

$$\sigma > K_1(h),$$

then by Halanay type inequality (Lemma 1) there exists $\eta > 0$ such that

$$|E_h(t)| \leq \left[ K \|E_{h,t_0}\|_q + K_1(h)M_1 \|\varphi\| e^{\sigma t_0} \right] e^{-\eta(t-t_0)}, \quad t \geq t_0,$$

where $\eta$ is the positive solution of

$$\eta = \sigma - K_1(h)e^{\eta}.$$

Thus we have proved the fundamental theorem of this chapter. There are similar results to our results, however the novelty of our theorem lies in the considered delayed differential equation and the technique used in the proof. In [Cooke and Győri (1994) and Győri and Hartung (2002)] used Gronwall-Bellman inequality to obtain an estimating exponential decay. Using Gronwall-Bellman inequality we shall obtain:

$$|E_h(t)| \leq \left[ K \|E_{h,t_0}\|_q + K_1(h)M_1 \|\varphi\| e^{\sigma t_0} \right] e^{-\sigma(t-t_0)}, \quad t \geq t_0,$$
where $\sigma_0 = \sigma - K_1(h)e^{\sigma q}$. Therefore we need $h$ small enough such that
\[ \sigma > K_1(h)e^{\sigma q} = a_0^2 [2h + w(r; h)] K e^{\sigma q}. \]

On the other hand, the necessary condition to use Halanay type inequality is: $h$ small enough such that
\[ \sigma > K_1(h) = a_0^2 [2h + w(r; h)] K, \]
then
\[ |E_h(t)| \leq \left[ K \left\| E_{h_{\to}} \right\|_q + K_1(h)M_1 \left\| \phi \right\|_q e^{\sigma t_0} \right] e^{-\eta(t-t_0)}, \quad t \geq t_0, \]
where $\eta$ is the positive real solution of
\[ \eta = \sigma - K_1(h)e^{\eta}. \]
We note that the size of $h$ is independent of the delay size $q$, and the number $-\eta$ is the unique real solution of the characteristic equation
\[ (34) \quad \lambda = -\sigma + K_1(h)e^{-\lambda q}, \]
corresponding to the differential equations with delay
\[ y'(t) = -\sigma y(t) + K_1(h)y(t-q). \]
In fact $-\eta$ is the eigenvalue of the characteristic equation (34) with the greatest real part.

Now we can prove that the solutions of (13)-(14) approximate uniformly the solutions of (1)-(2), and also that zero solution of (13)-(14) is uniformly asymptotically stable.

**Corollary 1.** Under the conditions of the Theorem 4, we have that:
1. $|x(t) - z_h([t]_h)| \to 0$ as $h \to 0$, for $t > 0$;
2. the zero solution of the difference equations with delay (13)-(14) is uniform asymptotically stable.

**Proof.** In section 3 we have shown that (13)-(14) correspond to a discrete version of differential equation (1)-(2). We recall that $z_h(hn) = y_h(n)$ for $n$ any positive integer. If $t < t_0$ we use Theorem 3. If $t \geq t_0$ then
\[ |x(t) - z_h([t]_h)| \leq |x(t) - x([t]_h)| + |x([t]_h) - z_h([t]_h)|. \]
Then, from inequality (33), we have
\[ |x(t) - z_h([t]_h)| \leq |x(t) - x([t]_h)| + \left[ K \left\| E_{h_{\to}} \right\|_q + K_1(h)M_1 \left\| \phi \right\|_q [t]_h \right] e^{-\eta([t]_h-t_0)}. \]
Next, for $\varepsilon > 0$ there are positive constants $h_1, h_2$ and $h_3$ such that:
If $h < h_1$ then $|x(t) - x([t]_h)| < \frac{\varepsilon}{3}$, since continuity of $x$. If $h < h_2$ then $\|E_{h_{\to}}\|_q < \frac{\varepsilon}{3K}$, from Theorem 3. If $h < h_3$ then $K_1(h) < \frac{\varepsilon}{3M_1\|\phi\|_q} e^{\frac{\eta}{\varepsilon} t_0}$. Therefore, for $h < \min\{h_1, h_2, h_3\}$ it is follows
\[ |x(t) - z_h([t]_h)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \]
We have shown part 1. Since inequality (33) we have also
\[ |x(nh) - y_h(n)| \leq \left[ K \left\| E_{h_{\to}} \right\|_q + K_1(h)M_1 \left\| \phi \right\|_q nh \right] e^{-\eta(nh-t_0)} \to 0 \text{ as } n \to \infty. \]
Since the zero solution of (1)-(2) is uniformly asymptotically stable, and $\|x(nh) - y_h(n)\|$ decay rate exponentially to zero, it is follows that $y_h(n)$ tends to zero, for every
initial conditions $\varphi$. Therefore the zero solution of (13)-(14) is uniformly asymptotically stable too.

Thus we have shown that under the hypotheses of Theorem 4 the numerical approximations of equation (13) are good for all $T > 0$, independent of the size of $T$. Moreover the corresponding discrete difference equation is uniformly asymptotically stable also. We use piecewise constant argument, theory of functional differential equations and Halanay-type inequality in the proof. Furthermore, our result is independent of delay size, this was possible thanks to our use of inequality Halanay. Thus we extend and improve the results of Cooke and Győri (1994).

5. Applications and examples

In our result we assume that the zero solution of (1) is uniformly asymptotically stable, the problem of find necessary condition for uniform stability of non-autonomous differential equations with variable delay called the attention of several authors because its difficulty. Next we recall some stability criteria for equation (1) and apply our result to obtain stability criteria for difference equation.

A classic result of stability for functional differential equations can be found in Yorke (1970). A consequence of Yorke’s theorem is:

Theorem A (Yorke). If the function $a(\cdot)$ satisfy

\[ 0 < a(t) \leq \alpha, \quad t \geq 0; \]

for a positive constant $\alpha$ such that

\[ 0 < \alpha_2 < \frac{3}{2}. \]

Then the zero solution of (1) is uniformly asymptotically stable.

Example 1. We consider the non-autonomous linear differential equations with delay

\[ x'(t) = - \left[ 1 + \frac{\sin(t)}{3} \right] x(t - |\cos(t)|). \]

Since $0 < 1 + \frac{\sin(t)}{3} < \frac{4}{3}$ and $0 \leq |\cos(t)| \leq 1$, it follows that $\alpha_2 = \frac{4}{3} < \frac{3}{2}$, therefore from Theorem A the zero solution of (35) is uniformly asymptotically stable so (A1) and (A2) holds. Since the function $\cos(x)$ is uniformly continuous on $\mathbb{R}$, (A3) holds. So Theorem 4 and Corollary 1 are valid. Therefore we can approximate the solution of (35) by the family of difference equations (13) corresponding to (35)

\[ \tilde{z}_h(n + 1) = \tilde{z}_h(n) - a_h(n)\tilde{z}_h(n - k_n), \]

where

\[ a_h(n) = \int_{nh}^{(n+1)h} \left[ 1 + \frac{\sin(s)}{3} \right] ds = h - \cos((n+1)h) - \cos(nh) - \frac{3}{3}, \]

and

\[ k_n = \left[ \frac{\cos(nh)}{h} \right]. \]

It follows that the zero solution of

\[ \tilde{z}_h(n + 1) = \tilde{z}_h(n) - \left( h - \frac{\cos((n+1)h) - \cos(nh)}{3} \right) \tilde{z}_h \left( n - \left[ \frac{\cos(nh)}{h} \right] \right), \]
is uniformly asymptotically stable. We note that, since mean value theorem, (36)
is equivalent to
\[
\begin{align*}
\Delta h(n + 1) - \Delta h(n) & = -h \left\{ h - \frac{\cos((n + 1)h) - \cos(nh)}{3} \right\} \Delta h(n) \left( n - \left\lfloor \frac{\cos(nh)}{h} \right\rfloor \right) \\
& = -h \left\{ 1 - \frac{\cos((n + 1)h) - \cos(nh)}{3} \right\} \Delta h(n) \left( n - \left\lfloor \frac{\cos(nh)}{h} \right\rfloor \right) \\
& = -h \left\{ 1 + \frac{\sin(c_{n+1})}{3} \right\} \Delta h(n) \left( n - \left\lfloor \frac{\cos(nh)}{h} \right\rfloor \right),
\end{align*}
\]
for some \( c_{n+1} \in (nh, nh + h) \).

(37) \[ \frac{\Delta h(n + 1) - \Delta h(n)}{h} = -\left[ 1 + \frac{\sin(c_{n+1})}{3} \right] \Delta h(n) \left( n - \left\lfloor \frac{\cos(nh)}{h} \right\rfloor \right). \]

**Figure 1.** Approximation solution of (1) with initial function \( \phi \equiv 5 \) with \( h = 0.5 \)

![Figure 1](image1.png)

**Figure 2.** Approximation solution of (1) with initial function \( \phi \equiv 5 \) with \( h = 0.3 \)

![Figure 2](image2.png)
6. Conclusion

In order to get our overall goal we need to identify the main assumptions, techniques and methods used in papers about approximation and transference of stability properties between solutions of differential equations with delay and the corresponding difference equations.

About the techniques and methods used, we note that the results of Cooke and Győri (1994) and Győri and Hartung (2002) rest on the functional differential equations that error \( E_h(t) \) satisfy. Actually \( E_h(t) \) satisfy a linear non-homogeneous or semi-linear functional differential equations. If the linear homogeneous differential equations with delay is uniformly asymptotically stable, then it is possible to conclude the exponential decay rate of the error by using an integral inequality. In the previous work they consider Gronwall-Bellman inequality, however we use Halanay inequality because it is more appropriate for delay differential equations. We also note that Mohamad and Gopalsamy (2000) and Liz and Ferreiro (2002) used Halanay inequality as a key technique to prove that the zero solution of both continuous functional differential equations and discrete equations are exponentially stable. However, this technique does not state any estimation of the error.

Our Theorem 4 states the approximation and transference of exponential stability properties of solutions of non-autonomous differential equations with variable delay and retarded functional differential equation with feedback, respectively, to the corresponding difference equation by using piecewise constant argument, the techniques and methods above mentioned.

Natural applications of our results can be found in the systems of differential equations used to model cellular neural networks and identification of parameters in functional differential equations see, for instance, Mohamad and Gopalsamy (2003); Abbas and Yonghui (2013) and Hartung and Turi (1997); Hartung et al. (1998, 2000) respectively.

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