LARGE TIME EXISTENCE OF STRONG SOLUTIONS TO MICROPOLAR EQUATIONS IN CYLINDRICAL DOMAINS

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ABSTRACT. We examine the so-called micropolar equations in three dimensional cylindrical domains under Navier boundary conditions. These equations form a generalization of the ordinary incompressible Navier-Stokes model, taking the structure of the fluid into account. We prove that under certain smallness assumption on the rate of change of the initial data and the external data there exists a unique and strong solution for any finite time $T$.

1. Introduction

Micropolar equations, which were suggested and introduced by A. Eringen in 1966 (see [Eri66]), are a significant step toward generalization of the standard Navier-Stokes model, which describes motion of viscous and incompressible fluid. By their very nature, the equations suggested by C.L. Navier and shortly afterward complemented and mathematically formalized by G.G. Stokes, do not take into account the structure of the media they describe, although it plays crucial role in modeling for some well-known fluids, e.g. animal blood or liquid crystals (see [Pop69, PRU74]). The deviancy becomes highly apparent in microscales.

In this work we base on the model proposed by A. Eringen in [Eri66]. It takes into account that the molecules may rotate independently of the fluid rotation. Thus, the standard Navier-Stokes system is shortly afterward complemented and mathematically formalized by G.G. Stokes, do not take into account the structure of the media they describe, although it plays crucial role in modeling for some well-known fluids, e.g. animal blood or liquid crystals (see [Pop69, PRU74]). The deviancy becomes highly apparent in microscales.

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The domain. To describe the domain $\Omega$ consider a closed curve $\varphi: \mathbb{R}^2 \to \mathbb{R}$, $\varphi(x_1, x_2) = c^0$ which is at least of class $C^2$. We do not put any additional constraints on the properties or on the shape of $\varphi$. For convenience, denote by $S$ the boundary of $\Omega$. This set is composed of two sets, $S_1$ and $S_2$, $S = S_1 \cup S_2$, where by $S_1$ we denote the side boundary and by $S_2$ the top and the bottom of the cylinder. Thus

$$S_1 = \{x \in \mathbb{R}^3: \varphi(x_1, x_2) = c_0, -a < x_3 < a\}$$

and

$$S_2 = \{x \in \mathbb{R}^3: \varphi(x_1, x_2) < c_0, x_3 = a\}.$$

Date: August, 2011.

2000 Mathematics Subject Classification. 35Q35, 35D35, 76D03.

Key words and phrases. micropolar fluids, cylindrical domains, strong solutions, Navier boundary conditions.

The author is partially supported by Polish KBN grant N N201 393137.
We draw the distinction between $S_1$ and $S_2$, because some boundary conditions to auxiliary problems will be expressed by significantly different formulas.

By simple computation we immediately get
\begin{align}
  n|_{S_1} &= \frac{1}{|\nabla \varphi|}(\varphi_{x_1},0,0) \quad \tau_1|_{S_1} = \frac{1}{|\nabla \varphi|}(-\varphi_{x_2},\varphi_{x_1},0) \quad \tau_2|_{S_1} = (0,0,1),
  \\
  n|_{S_2} &= \left(0,0,\frac{a}{|a|}\right) \quad \tau_1|_{S_2} = (1,0,0) \quad \tau_2|_{S_2} = (0,1,0),
\end{align}
(1.2)
where $n$, $\tau_i$, $i = 1, 2$, are the unit outward normal and the unit tangent vectors respectively.

Let us now justify the choice of the domain. Since the problem of uniqueness (or regularity) of weak solutions for Navier-Stokes equations in three dimensions is open, several alternative approaches were taken. One of them is intensely focused upon search for such solutions that are close to two dimensional (see e.g. [RZ08], [Zaj05], [Zaj11]). It is also our case. Therefore, the solutions which are proved to exist, can be regarded as a slight perturbation of two dimensional micropolar flow along the perpendicular direction. This perturbation will be somehow measured by $\delta(t)$ (see (2.2)), which we introduce later. We shall emphasize that since we only require the initial rate of change of the flow and microrotation, as well as the derivatives of the external data with respect to $x_3$ to be small, the flow alongside the cylinder can be large, but close to constant.

**The boundary and initial conditions.** We supplement system (1.1) with the boundary conditions of the form
\begin{align}
  v \cdot n &= 0 \quad \text{on } S^\infty := S \times (t_0,t), \\
  \text{rot } v \times n &= 0 \quad \text{on } S^1, \\
  \omega &= 0 \quad \text{on } S^1_1, \\
  \omega' &= 0, \quad \omega_{3,3} = 0 \quad \text{on } S^2_2,
\end{align}
(1.3)
where $n$ is the unit outward vector.

Let us briefly justify our choice. Our proof of the existence of regular solutions to (1.1) uses an estimate for the third component of the vorticity field (see Lemmas 5.3 and 7.5) which is equal to $\tau_{2,x_1} - v_{1,x_2}$. Thus, the Dirichlet condition $v = 0$ is not admissible because it does not provide any information concerning derivatives of $v$ on the boundary. Therefore we decided to employ slip boundary condition $\text{rot } v \times n = 0$, which was already considered by C.L. Navier in 1827 (and is often referred as the Navier boundary condition; see [Nav27]) and satisfies the relation (see Lemma 6.5) $n \cdot \text{D}(v) \tau_\alpha + 2\kappa (2-\alpha) (v \cdot \tau_1) = \text{rot } v \times n \tau_\alpha$, where $n$ and $\tau_\alpha$, $\alpha \in \{1,2\}$, denote the unit normal and tangent vectors and $\text{D}(v)$ is a dilatation tensor equal to $\frac{1}{2} (\nabla v + \nabla^\top v)$. The function $\kappa$ represents the curvature of $S$. From the physical point of view it may be interpreted as tangential “slip” velocity being proportional to tangential stress with a factor of proportionality depending only on the curvature (see e.g. [Kel06], [CMR98]).

Nevertheless, under Dirichlet condition it is still possible to prove the existence of regular solution by adopting different technique (see e.g. [BDRM10]). From mathematical perspective, the boundary conditions which shall complement problem (1.1) have to provide the energy estimates. Although not for every available choice of these boundary conditions the existence of regular solutions has been proved, but it does not mean that such proofs will never appear.

As for the initial data we simply put
\begin{align}
  v|_{t=t_0} = v(t_0), \quad \omega|_{t=t_0} = \omega(t_0) \quad \text{in } \Omega.
\end{align}
(1.4)

2. Notation

Before we formulate the main result let us shortly clarify the notation deployed throughout this work.

The most frequently used notation in the sequel will be $\Omega^t$, which denotes the product $\Omega \times (t_0,t)$. Unless stated directly, we only assume that $0 \leq t_0 < t < \infty$.

By $c$ we denote a generic constant that may change from line to line. Additionally, such constants are subscripted with appropriate symbols, which indicate the dependence on the domain, embedding theorems, etc. The possible values are listed below:

- $c_{\alpha, \beta, \nu, \nu'}$: appears when the constant $c$ depends on the viscosity coefficients,
- $c_{1f}$: refers to embedding theorems (e.g. $H^1(\Omega) \hookrightarrow L^2(\Omega)$),
- $c_p$: refers to the Poincaré inequality,
- $c_{2p}$: indicates the direct dependence on the geometry of the domain $\Omega$.

Our motivation to keep the information which factors contribute to the constants is caused by the necessity of precise control of their dependence with respect to time. Time dependent constants would surely have a negative impact on the proof of global in time solutions. We do not say that such proof would not be
possible, but unquestionably much harder. Note also, that since the Poincaré constant and the embedding constant depend on the domain, we could write $\Omega$ instead of $P$ and $I$ every time they appear. We decided not to make such generalization in order to keep the passage from line to line readable and clear.

Throughout this work we shall use the following notation to simplify the formulas:

$$h = v_{x3}, \quad \theta = \omega_{x3}.$$  

In energy estimates the initial and the external data in $L_p$-norms will appear. Therefore we introduce the following quantities to shorten formulas:

$$E_{v, \omega}(t) := \|f\|_{L_2(t_0, t, L_2(\Omega))} + \|g\|_{L_2(t_0, t, L_2(\Omega))} + \|v(t_0)\|_{L_2(\Omega)} + \|\omega(t_0)\|_{L_2(\Omega)},$$  

$$E_{h, \theta}(t) := \|f_{x3}\|_{L_2(t_0, t, L_2(\Omega))} + \|g_{x3}\|_{L_2(t_0, t, L_2(\Omega))} + \|h(t_0)\|_{L_2(\Omega)} + \|\theta(t_0)\|_{L_2(\Omega)}.$$  

The following function will be of particular interest

$$\delta(t) := \|f_{x3}\|_{L_2(\Omega)}^2 + \|g_{x3}\|_{L_2(\Omega)}^2 + \|\text{rot} \ h(t_0)\|_{H_0^1(\Omega)}^2 + \|h(t_0)\|_{L_2(\Omega)}^2 + \|\theta(t_0)\|_{L_2(\Omega)}^2.$$  

This is the most important quantity since it expresses the smallness assumption which has to be made in order to prove the existence of global and regular solutions. It contains no $L_2$-norms of the initial or the external data but only $L_2$-norms of their derivatives alongside the axis of the pipe. In other words the data need not to be small but small must be their rate of change.

Also note, that if we considered Navier-Stokes equations only and the external data were missing, $h(t_0)$ had a gradient structure (i.e. $h(t_0) = F A$, $A : \mathbb{R}^3 \rightarrow \mathbb{R}$; problem for $h$ is demonstrated in Lemma 7.1), then $\delta(t)$ would be equal to zero. Although $h(t_0)$ is in $L_2$-norm appears it can be estimated by $\text{rot} h(t_0)$ (see Lemma 6.7). The same would also hold for $f_{x3}$ with a gradient structure.

From time to time we use the rotation of vector field. By rot $F$, where $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, we denote

$$\text{rot} F = [F_{3,x2} - F_{2,x3}, F_{1,x3} - F_{3,x1}, F_{2,x1} - F_{3,x2}].$$  

To define certain functions spaces that will be used frequently in the sequel we simply follow [Łuk99, Ch. 3, §1.1], [LSU67, Ch. 2, §3] and [Tem79, Ch. 1, §1.1]:

- $L_p(\Omega)$ is the set of all Lebesgue measurable function $u : \Omega \rightarrow \mathbb{R}^n$ with the norm
  $$\|u\|_{L_p(\Omega)} = \left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}},$$

- $W_p^m(\Omega)$, where $m \in \mathbb{N}$, $p \geq 1$, is the closure of $C^\infty(\Omega)$ in the norm
  $$\|u\|_{W_p^m(\Omega)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|_{L_p(\Omega)}^p\right)^{\frac{1}{p}},$$

- $H^k(\Omega)$, where $k \in \mathbb{N}$, is simply $W^k_2(\Omega)$,

- $W_p^{2,1}(\Omega)$, where $p \geq 1$, is the closure of $C^\infty(\Omega \times (t_0, t_1))$ in the norm
  $$\|u\|_{W_p^{2,1}(\Omega)} = \left(\int_{t_0}^{t_1} \int_\Omega \left(|u_{xx}(x, s)|^p + |u_{x,x}(x, s)|^p + |u_{x}(x, s)|^p + |u_{x}(x, s)|^p \right) dx \, ds\right)^{\frac{1}{p}},$$

- $H^1_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in the norm
  $$\|u\|_{H^1_0(\Omega)} = \left(\int_\Omega |\nabla u(x)|^2 \, dx\right)^{\frac{1}{2}},$$

- $L_0(t_0, t_1; X)$, where $q \geq 1$ and $X$ is a Banach space, is the set of all strongly measurable functions defined on the interval $[t_0, t_1]$ with values in $X$ with finite norm defined by
  $$\|u\|_{L_0(t_0, t_1; X)} = \left(\int_{t_0}^{t_1} \|u(s)\|_X^q \, ds\right)^{\frac{1}{q}},$$
  where $1 \leq p < \infty$ and by
  $$\|u\|_{L_\infty(t_0, t_1; X)} = \text{ess sup}_{t_0 \leq s \leq t_1} \|u(s)\|_X,$$
  for $q = \infty$,

- $V^2_2(\Omega)$, where $k \in \mathbb{N}$, is the closure of $C^\infty(\Omega \times (t_0, t_1))$ in the norm
  $$\|u\|_{V^2_2(\Omega)} = \text{ess sup}_{t \in (t_0, t_1)} \|u\|_{H^1_0(\Omega)} + \left(\int_{t_0}^{t_1} \|\nabla u\|_{H^1_0(\Omega)}^2 \, dt\right)^{1/2}.$$
3. Weak and strong solutions

**Definition 3.1.** Let \( v(t_0) \in L_2(\Omega) \) and \( \omega(t_0) \in L_2(\Omega) \). By a weak solution to problem (1.1) complemented with the boundary conditions (1.3) we mean a pair of functions \( (v, \omega) \) such that \( v, \omega \in V_2^0(\Omega^t) \) (for definition of \( V_2^0(\Omega^t) \) see Section 2) and \( \text{div} \, v = 0, \, v \cdot n|_{\partial \Omega} = 0 \), which satisfies the integral identities

\[
(3.1a) \quad \int_{\Omega^t} \left( -v \cdot \varphi_t + (\nu + \nu_r) \text{rot} \, v \cdot \text{rot} \varphi + (v \cdot \nabla)v \cdot \varphi \right) \, dx \, dt \\
+ \int_{\Omega} v \cdot \varphi|_{t=t_0} \, dx - \int_{\Omega} v \cdot \varphi|_{t=t_0} \, dx = \int_{\Omega^t} (2\nu_r \text{rot} \, \varphi + f \cdot \varphi) \, dx \, dt
\]

for any \( \varphi \in H^1(\Omega^t) \) such that \( \text{div} \varphi = 0, \, \varphi \cdot n = 0 \) and

\[
(3.1b) \quad \int_{\Omega^t} \left( -\omega \cdot \psi_t + \alpha \nabla \omega \cdot \nabla \psi + \beta \text{div} \omega \, \text{div} \psi + (v \cdot \nabla)\omega \cdot \psi + 4\nu_r \omega \cdot \psi \right) \, dx \, dt \\
+ \int_{\Omega} \omega \cdot \psi|_{t=t_0} \, dx - \int_{\Omega} \omega \cdot \psi|_{t=t_0} \, dx = \int_{\Omega^t} (2\nu_r \text{rot} \, \psi + g \cdot \psi) \, dx \, dt
\]

for any \( \psi \in H^1(\Omega^t) \) such that \( \psi|_{S_1} = 0 \) and \( \psi'|_{S_2} = 0, \, \psi_{3,x_3}|_{S_2} = 0 \).

The existence of weak solutions is assured by the following result:

**Lemma 3.2.** There exist at least one weak solution to problem (1.1) in the sense of the above definition.

**Proof.** The proof is quite standard and is based on Galerkin approximations, a priori estimates (see Lemma 3.1 and compactness method. It can be found in [RMB98] for general case or in [Łuk99, Ch. 4, Theorem 1.6.1], [Łuk01, Theorem 2.1]. □

For the strong (or regular) solutions there are various definitions, which are equivalent. For our purposes we will use the following

**Definition 3.3.** By a strong (or regular) solution to problem (1.1) we mean a pair of functions \( (v, \omega) \) such that \( (v, \omega) \in V_2^1(\Omega^t) \times V_2^1(\Omega^t) \) (for the definition of \( V_2^1(\Omega^t) \) see Section 2) satisfying (3.1a) and (3.1b).

4. Main result

Our main result is to demonstrate that under certain conditions there exists a unique solution to problem (1.1) for any \( t \) such that \( 0 \leq t \leq T < \infty \).

**Theorem 1** (large time existence). Let \( E_{v,\omega}(t) < \infty, \, E_{h,\theta}(t) < \infty \). Suppose that \( v(t_0), \omega(t_0) \in H^1(\Omega), \, f, g \in L_2(\Omega^t) \). Finally, assume that \( f_3|_{S_2} = 0, \, g'|_{S_2} = 0 \). Then, for \( \delta(t) \) sufficiently small there exists a unique solution \( (v, \omega) \in W_2^{1,1}(\Omega^t) \times W_2^{1,1}(\Omega^t) \) to problem (1.1) complemented with the boundary conditions (1.3) such that

\[
\|v\|_{W_2^{1,1}(\Omega^t)} + \|\nabla p\|_{L_2(\Omega^t)} \leq c_{\alpha, \nu, \nu_r, \gamma, \delta \Omega} \left( E_{v,\omega}(t) + E_{h,\theta}(t) + \|f\|_{L_2(\Omega^t)} + \|v(t_0)\|_{H^1(\Omega)} + 1 \right)^3
\]

and

\[
\|\omega\|_{W_2^{1,1}(\Omega^t)} \leq c_{\alpha, \nu, \nu_r, \gamma, \delta \Omega} \left( E_{v,\omega}(t) + E_{h,\theta}(t) + \|f\|_{L_2(\Omega^t)} + \omega(t_0) \right) + \|v(t_0)\|_{H^1(\Omega)} + \|\omega(t_0)\|_{H^1(\Omega)} + 1 \right)^3.
\]

Let us briefly outline how we prove the above theorem: The beginning point is the energy estimate for solutions to problem (1.1) on the time interval \([t_0, t_1]\). It ensures the existence of weak solutions ([Łuk99, Ch. 3, Theorem 1.6.1], [Łuk01, Theorem 2.1]) or in general case: [RMB98]. In the next step we introduce several auxiliary problems which provide us with better estimates for \( v \) and \( \omega \) but for the price of some smallness assumption on the rate of the change of the data (see (2.2)). Finally, the application of regularity results for the Stokes system (see [Ala95]) and general parabolic systems (see [Sol65]) leads to considerable improvement in the regularity of weak solutions. Next we utilize the Leray-Schauder fixed point theorem (see Lemma 3.3) to prove the existence of strong solution. Finally, we demonstrate its uniqueness.
5. State of the art

Since the 70’s dozens of results concerning the existence of weak or strong solutions, various conditional regularity criteria or qualitative properties of solutions, also for a generalized system, the so-called magneto-micropolar, are known. For a comprehensive summary we refer the reader to [Łuk99, Ch. 3, §5]. However, let us shortly outline these results which are closely related to our work.

One of the earliest results was established by Galdi and Rionero in [GR77] where they stated the boundary value problem with Dirichlet boundary conditions for micropolar flows belongs to the same class of evolution problems as Navier-Stokes equations. Therefore they were also able to formulate existence results for the micropolar equations which are similar to those obtained for the Navier-Stokes. The direct proofs of the existence and uniqueness of (global) strong solutions to (1.1) under the zero Dirichlet boundary conditions came later and were obtained by Łukaszewicz in [Łuk89]. He needed sufficiently large $\nu$ and small data in comparison to $\nu$.

In [OTRM97] Ortega-Torres and Rojas-Medar showed that under certain smallness assumption on the initial and external data there exists a unique, global and strong solution to problem (1.1) (in fact they considered the magneto-microhydrodynamic equations, but we can safely put the magnetic field $b$ to be equal to zero which yields problem (1.1)). In contrast to [Łuk89] they no longer assumed a decay for the external data as times goes to the infinity.

A couple of years later the same authors proved (see [RMOT05]) again the existence and uniqueness of strong solutions to (1.1) complemented with zero Dirichlet boundary conditions both for $v$ and $\omega$. Their new proof used an interactive approach and required certain smallness of $L^2$-norms of the external data and absence of the initial data.

Subsequently, Yamaguchi (see [Yam05]) proved the existence of global and strong solutions to (1.1) in bounded domains under zero Dirichlet boundary conditions. His proof is based on the semi-group approach and requires some smallness of the data.

By application of iterative scheme, Boldrini, Durán and Rojas-Medar proved (see [BDRM10]) the existence of local strong solutions $(v, \omega) \in W^{2,1}_p(\Omega') \times W^{2,1}_p(\Omega')$, $p > 3$, to problem (1.1) complemented with zero Dirichlet boundary conditions for both $v$ and $\omega$ in bounded and unbounded domains in $\mathbb{R}^3$ with compact $C^2$-boundaries.

To the best of our knowledge there are no results which are close to 2d solutions to (1.1) which would justify the choice of domains of cylindrical type. It is also clear that most authors considers only the homogeneous Dirichlet boundary condition both for $v$ and $\omega$. Therefore it makes us a case for this detailed study.

What needs to be particularly emphasized is the fact that in our work we do not assume any smallness on the initial or external data. Their $L^2$-norms can be large. However, the data cannot change significantly alongside the cylinder. Their derivatives in the $x_3$-directions must remain small all the time. Therefore the flow is somehow close to the constant with respect to one variable.

Note also that cylindrical domains or boundary conditions were considered for the standard Navier-Stokes equations (for a comprehensive summary we refer the reader to the Introduction in [Zaj11]).

6. Auxiliary results

Lemma 6.1 (Embedding theorem). Let $\Omega$ satisfy the cone condition and let $q \geq p$. Set

$$\kappa = 2 - 2r - s - 5 \left( \frac{1}{p} - \frac{1}{q} \right) \geq 0.$$  

Then for any function $u \in W^{2,1}_p(\Omega')$ the inequality

$$\|\partial_x D_x^i u\|_{L^q(\Omega')} \leq c_1 e^{\kappa s} \|u\|_{W^{2,1}_p(\Omega')} + c_2 e^{-\kappa + 2s - 2} \|u\|_{L^p(\Omega')}$$

holds, where the constants $c_1$ and $c_2$ depend only on $p$, $q$, $r$, $s$ and $\Omega$ but do not depend on $t$.

For the proof of the lemma we refer the reader to [LSU67, Ch. 2, §3, Lemma 3.3]. As before, we lay emphasis on the fact that the constant $c_1$ and $c_2$ do not depend on time.

Lemma 6.2. Suppose that $u \in V^0_q(\Omega')$. Then $u \in L_q(0,t;L^p(\Omega))$ and

$$\|u\|_{L_q(t_0,t;L^p(\Omega))} \leq c_{p,q,t} \|u\|_{V_q^2(\Omega')}$$

holds under the condition $\frac{3}{p} + \frac{3}{q} = \frac{3}{2}$, $2 \leq p \leq 6$.

Let us emphasize that the constant that appears on the right-hand side does not depend on time.

Proof. We want to show that

$$\|u\|_{L_q(t_0,t;L^p(\Omega))} \leq c_{p,q,t} \left( \|u\|_{L_q^2(t_0,t;H^1(\Omega))} + \|u\|_{L_q^2(t_0,t;L^2(\Omega))} \right).$$
For $L_p$-spaces we have the interpolation inequality
\[
\|u\|_{L^p(\Omega)} \leq \|u\|_{L^r(\Omega)}^{\frac{\theta}{r}} \|u\|_{L^s(\Omega)}^{1-\frac{\theta}{s}},
\]
where $s < p < r$ and $\frac{1}{p} = \frac{\theta}{r} + \frac{1-\theta}{s}$. Integrating with respect to time yields
\[
\|u\|_{L^q(t_0,t_1;L^p(\Omega))} = \left(\int_{t_0}^{t_1} \|u(\tau)\|_{L^p(\Omega)}^q \, d\tau\right)^{\frac{1}{q}} \leq \left(\int_{t_0}^{t_1} \|u(\tau)\|_{L^r(\Omega)}^q \|u(\tau)\|_{L^s(\Omega)}^{q(1-\theta)} \, d\tau\right)^{\frac{1}{q}} = \sup_{\tau \in (t_0,t_1)} \|u(\tau)\|_{L^q(\Omega)}^{\frac{q\theta}{q-\theta}} \left(\int_{t_0}^{t_1} \|u(\tau)\|_{L^q(\Omega)}^{q(1-\theta)} \, d\tau\right)^{\frac{1}{q-\theta}}.
\]
Using the Young inequality gives
\[
\|u\|_{L^q(t_0,t_1;L^p(\Omega))} \leq \theta \sup_{\tau \in (t_0,t_1)} \|u(\tau)\|_{L^r(\Omega)} + (1 - \theta) \left(\int_{t_0}^{t_1} \|u(\tau)\|_{L^q(\Omega)}^{q(1-\theta)} \, d\tau\right)^{\frac{1}{q(1-\theta)}}.
\]
Next we set $r = 2, s = 6$ and $q(1-\theta) = 2$. Then $\frac{1}{p} = \frac{2}{q} + \frac{1-\theta}{s} = \frac{1+2\theta}{6}$ and finally
\[
3 - \frac{2}{p} = \frac{1+2\theta}{6} + (1 - \theta) = \frac{1+2\theta+2-2\theta}{2} = \frac{3}{2}.
\]
For $p = 2, q = \infty$ or $p = 6, q = 2$ the estimate follows immediately from definition of the space $V^0_2(\Omega')$. This ends the proof. \hfill \Box

**Lemma 6.3** (Leray-Schauder fixed point principle). Let $X$ denote a Banach space. Suppose that $\Phi : X \times [0,1] \to X$ satisfies the following conditions:
- for any fixed $\lambda \in [0,1]$ the mapping $\Phi(\cdot, \lambda)$ : $X \to X$ is continuous,
- for any fixed $x \in X$ the mapping $\Phi(x, \cdot)$ : $[0,1] \to X$ is uniformly continuous,
- there exists a bounded subset $A \subset X$ such that every fixed point of the mapping $\Phi(\cdot, \lambda)$ : $X \to X$ for any choice of $\lambda \in [0,1]$ belongs to $A$,
- the mapping $\Phi(\cdot, 0)$ has only one fixed point.

Then, $\Phi(\cdot, 1)$ has at least one fixed point.

**Proof.** For the proof we refer the reader to [DH82]. \hfill \Box

In further considerations we will often integrate by parts. To avoid repetition of some calculations, we shall demonstrate the most general case in the below Lemma:

**Lemma 6.4** (On integration by parts). Let $u$ and $w$ belong to $H^1(\Omega)$. Then
\[
\int_{\Omega} \nabla u \cdot \nabla w \, dx = \int_{\Omega} w \cdot \nabla u \, dx + \int_{\partial \Omega} \vec{u} \times \vec{n} \cdot \nabla S = \int_{\Omega} w \cdot \nabla u \, dx - \int_{\partial \Omega} \vec{w} \times \vec{n} \cdot \nabla S
\]

**Proof.** It is an easy computation. \hfill \Box

**Lemma 6.5.** Let $\kappa$ and $\mathbb{D}(v)$ denote the curvature of the boundary $S$ and dilatation tensor, i.e. $\mathbb{D}(v) = \frac{1}{2} (\nabla v + \nabla v)$, respectively. Then
\[
n \cdot \mathbb{D}(v) \cdot \tau_\alpha + \kappa(2-\alpha)v \cdot \tau_1 = -\frac{1}{2} \nabla v \times n \cdot \tau_\alpha
\]
holds on $S$, where $\tau_\alpha, \alpha \in \{1,2\}$ is a tangent vector.

**Proof.** We have
\[
n \cdot \mathbb{D}(v) \cdot \tau_\alpha = \frac{1}{2} \sum_{i,j} n_i \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) \tau_{\alpha j} = \frac{1}{2} \sum_{i,j} n_i \left( \frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j} \right) \tau_{\alpha j} + \sum_{i,j} n_i \frac{\partial v_i}{\partial x_j} \tau_{\alpha j} =: I_1 + I_2.
\]
Clearly
\[
I_1 = -\frac{1}{2} \nabla v \times n \cdot \tau_\alpha.
\]
For $I_2$ we have
\[
I_2 = \sum_{i,j} n_i \left( \frac{\partial (n_i v_j)}{\partial x_j} - v_i \frac{\partial n_i}{\partial x_j} \tau_{\alpha j} \right) = \sum_{i,j} v_i \frac{\partial n_i}{\partial x_j} \tau_{\alpha j}.
\]
Since \( n|_{S_2} = [0, 0, 1] \), we obtain \( I_2|_{S_2} = 0 \). On \( S_1 \) we use (1.2). We see that \( n \) does not depend on \( x_3 \), which implies that \( i, j \in \{1, 2\} \) and \( \alpha = 1 \). By simple computation we get

\[
\sum_{i,j} \frac{\partial n_i}{\partial x_j} \tau_j = \frac{1}{|\nabla \phi|^2} \left\{ -\varphi_{,x_1} \varphi_{,x_2} + \varphi_{,x_1} \varphi_{,x_1} - \varphi_{,x_2} \varphi_{,x_2} \varphi_{,x_1} - \varphi_{,x_2} \varphi_{,x_2} \right\}.
\]

We can express \( v \) in the basis \( n, \tau_1, \tau_2 \) as follows:

\[
v = (v \cdot n)n + (v \cdot \tau_1)\tau_1 + (v \cdot \tau_2)\tau_2.
\]

Since \( v \cdot n|_S = 0 \), we get

\[
I_2 = \frac{(v \cdot \tau_1)}{|\nabla \phi|^2} \left( \varphi_{,x_1} \varphi_{,x_2}^2 - 2\varphi_{,x_1} \varphi_{,x_1} \varphi_{,x_2} + \varphi_{,x_2} \varphi_{,x_2} \varphi_{,x_1}^2 \right).
\]

For a curve given in an implicit form, i.e. by \( \varphi(x_1, x_2) = c \), the curvature is defined as

\[
\kappa = \frac{\begin{vmatrix} \varphi_{,x_1}^2 & \varphi_{,x_2} \varphi_{,x_1} \\ \varphi_{,x_2} & \varphi_{,x_2} \varphi_{,x_2} \end{vmatrix}}{|\nabla \phi|^2}.
\]

Thus,

\[
I_2 = -\kappa(v \cdot \tau_1),
\]

which ends the proof. \( \square \)

Lemma 6.6 (Boundary conditions on \( S_2 \)). Let (1.3) be satisfied. Then

(a) \( v_3|_{S_2} = 0, \quad v_{3,x_1}|_{S_2} = 0, \quad v_{3,x_2}|_{S_2} = 0 \),

(b) \( (\text{rot } v)|_{S_2} = 0, \quad \chi|_{S_2} = 0 \),

(c) \( h'|_{S_2} = 0, \quad h_{3,x_3}|_{S_2} = 0 \),

where \( (\text{rot } v)' = (\text{rot } v_1, (\text{rot } v_2, 0), \quad h' = (h_1, h_2, 0) \).

Proof. As immediate consequence of (1.3) we get

\[
v \cdot n|_{S_2} = v_3|_{S_2} = 0 \quad \Rightarrow \quad v_{3,x_1}|_{S_2} = 0 \quad \text{and} \quad v_{3,x_2}|_{S_2} = 0
\]

and

\[
\text{rot } v \times n|_{S_2} = 0 \Leftrightarrow \begin{cases} v_{3,x_2} - v_{2,x_3} = (\text{rot } v_1)|_{S_2} = 0 \\ v_{1,x_3} - v_{3,x_1} = (\text{rot } v_2)|_{S_2} = 0 \end{cases} \Rightarrow v_{1,x_3}|_{S_2} = v_{2,x_3}|_{S_2} = 0.
\]

This yields

\[
\chi|_{S_2} = v_{2,x_1} - v_{1,x_2}|_{S_2} = 0.
\]

and from \( \text{div } h = 0 \) we have

\[
v_{3,x_3}|_{S_2} = -v_{1,x_3} - v_{2,x_2}|_{S_2} = 0 \quad \Rightarrow \quad h_{3,x_3}|_{S_2} = 0.
\]

This ends the proof. \( \square \)

Lemma 6.7. Suppose that

\[
\begin{align*}
\text{rot } u &= \alpha & \text{in } \Omega, \\
\text{div } u &= \beta & \text{in } \Omega
\end{align*}
\]

with either \( u \cdot n|_S = 0 \) or \( u \times n|_S = 0 \). Then

\[
\|u\|_{H^{k+1}(\Omega)} \leq c_0 (\|\alpha\|_{H^k(\Omega)} + \|\beta\|_{H^k(\Omega)}).
\]

Proof. For the proof we refer the reader to (DLT72 Ch. 7, Thm. 6.1) (case \( k = 0 \)) and (Sol73) (case \( k \in \mathbb{N} \)). In the latter general overdetermined elliptic systems were examined. In particular, the case of tangent components of \( u \) was considered. \( \square \)

Lemma 6.8. Let \( \alpha \) and \( \beta \) be positive constants. Let us define the operator \( L \) by the formula

\[
L = -\alpha \Delta - \beta \nabla \text{div} = \sum_{i,j=1}^3 \alpha^{ij} \partial_{x_i} \partial_{x_j}.
\]

Next, consider the problem

\[
\begin{align*}
Lu &= f & \text{in } \Omega, \\
u &= 0 & \text{on } S_1, \\
u' &= 0, \quad u_{3,x_3} = 0 & \text{on } S_2.
\end{align*}
\]

Then \( L \) is uniformly elliptic and for any \( f \in L_2(\Omega) \) the estimate

\[
\|u\|_{H^2(\Omega)} \leq c_{\alpha, \beta, \Omega} \|f\|_{L_2(\Omega)}.
\]
We immediately deduce that the symmetric matrix \( A = (a^{ij}) \) of the form
\[
\begin{bmatrix}
\alpha + \beta & \beta & \beta \\
\beta & \alpha + \beta & \beta \\
\beta & \beta & \alpha + \beta 
\end{bmatrix}
\]
which corresponds to the operator \( L \), is positive definite and its smallest eigenvalue is greater than \( \theta \). To compute its eigenvalues we solve the equation \( \det (A - t \text{Id}) = 0 \) with respect to \( t \). We see that
\[
\det A - t \text{Id} = (\alpha + \beta - t)^3 + 2\beta^3 - 3\beta^2(\alpha + \beta) = (\alpha + \beta - t)^3 - 3\beta^3 = f(t).
\]
Since
\[
f'(t) = -3(\alpha + \beta - t)^2 < 0
\]
we immediately deduce that \( f \) is decreasing. Therefore there exists only one \( t^* \) such that \( f(t^*) = 0 \). Since \( f(0) = (\alpha + \beta)^3 - 3\beta^3 - 3\alpha^2\beta > 0 \) we infer that \( t^* > 0 \). This implies that the smallest eigenvalue of the matrix \( A \) is positive. Hence the operator \( L \) is uniformly elliptic.

To prove the estimate we proceed in a standard way. First, we introduce a partition of unity \( \sum_{k=0}^{3} \zeta_k(x_3) = 1 \) on \( \Omega \). Let us denote \( \bar{u} = u\zeta_k \). For fixed \( k \) four cases may occur:

1. \( \text{supp} \zeta_k \cap S = \emptyset \): In this case we deal with the problem in the whole space
\[
Lu = f + [0, 0, 2\alpha \nabla u_3 \cdot \nabla \zeta_k + \alpha u_3 \Delta \zeta_k] + \beta \zeta_k, x_3 \nabla \omega_3 =: F_k \quad \text{in} \quad \text{supp} \zeta_k \cap \Omega,
\]
\[
\bar{u} = 0 \quad \text{on} \quad \partial (\text{supp} \zeta_k \cap \Omega).
\]
From the classical theory (see [LM65 Ch.2, §3.2, Thm. 3.1]) it follows that
\[
\|\bar{u}\|_{H^2(\text{supp} \zeta_k \cap \Omega)} \leq c_0 \left( \|F_k\|_{L^2(\text{supp} \zeta_k \cap \Omega)} + \|\bar{u}\|_{H^1(\text{supp} \zeta_k \cap \Omega)} \right).
\]

2. \( \text{supp} \zeta_k \cap S_1 \neq \emptyset \), \( \text{supp} \zeta_k \cap S_2 = \emptyset \): Since \( \bar{u}|_{S_1} = 0 \) we obtain that \( \bar{u}|_{\partial(\text{supp} \zeta_k \cap \Omega)} = 0 \). Next, we transform the set \( \text{supp} \zeta_k \cap \Omega \) into the half-space and apply the result from classical theory for the half-space (see [LM65 Ch.2, §4.5, Thm. 4.3]), which finally gives \([31]\). For the meticulous details we refer the reader to the proof of Theorem 5.1 in [LM65 Ch. 2, §5.1].

3. \( \text{supp} \zeta_k \cap S_1 \neq \emptyset \), \( \text{supp} \zeta_k \cap S_2 \neq \emptyset \): Let us recall that \( u'|_{S_2} = 0 \) and \( u_3, x_3|_{S_2} = u_3 \zeta_k, x_3 = 0 \), which follows from the fact that \( \zeta_k = \zeta_k(x_3) \). It allows us to reflect the function \( \bar{u} \) outside the cylinder according to the formula
\[
\bar{u}(x) = \begin{cases} 
\bar{u}(x) & x_3 \in \text{supp} \zeta_k \cap \Omega, \\
(u'(\tilde{x}), \bar{u}_3(\tilde{x}) - \bar{u}_3(\tilde{x})) & x_3 \leq -a, \\
(u'(\tilde{x}), -\bar{u}_3(\tilde{x})) & x_3 \geq a,
\end{cases}
\]
where \( \tilde{x} = (x', -2a - x_3) \) and \( \tilde{x} = (x', 2a - x_3) \). Note that \( \bar{u}|_{\partial(\text{supp} \zeta_k \cap \Omega)} = 0 \), so we may proceed as in Case 2.

4. \( \text{supp} \zeta_k \cap S_1 = \emptyset \), \( \text{supp} \zeta_k \cap S_2 \neq \emptyset \): This case does not differ from the previous one in any major way. We also reflect the function \( \bar{u} \) as described above and proceed as in Case 2.

Summing over \( k \) yields
\[
\|u\|_{H^2(\Omega)} \leq \sum_{k=1}^{N} \|u\zeta_k\|_{H^2(\Omega)} \leq c_0 \sum_{k=1}^{N} \left( \|F_k\|_{L^2(\text{supp} \zeta_k \cap \Omega)} + \|u\zeta_k\|_{H^1(\Omega)} \right) \leq c_{a, \beta, \Omega} \left( \|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} \right).
\]

It remains to eliminate the last term on the right-hand side. We shall prove that the inequality \( \|Lu\|_{L^2(\Omega)} = \|f\|_{L^2(\Omega)} \geq c_0 \|u\|_{H^1(\Omega)} \) holds. Conversely, suppose that it is not true. Then there would exist sequences \( u_i \) in \( H^1(\Omega) \) and \( f_i \) in \( L^2(\Omega) \) such that
\[
Lu_i = f_i \quad \text{in} \ \Omega,
\]
\[
u_i = 0 \quad \text{on} \ S_1,
\]
\[
u_i' = 0, \quad u_3, x_3 = 0 \quad \text{on} \ S_2
\]
and \( \|u_i\|_{H^1(\Omega)} \geq l \|f_i\|_{L^2(\Omega)} \). Let us define \( v_i = \frac{u_i}{\|u_i\|_{H^1(\Omega)}} \). Then
\[
\|Lv_i\|_{L^2(\Omega)} = \frac{\|Lu_i\|_{L^2(\Omega)}}{\|u_i\|_{H^1(\Omega)}} = \frac{\|f_i\|_{L^2(\Omega)}}{\|u_i\|_{H^1(\Omega)}} \leq \frac{1}{l}.
\]
From the above inequality we see that \( v_t \) is bounded in \( H^2(\Omega) \). The Rellich-Kondrachov Compactness Theorem implies that there exist a subsequence \( v_{k_\ell} \) which converges strongly in \( H^1(\Omega) \) to some element \( v \). Thus
\[
\begin{align*}
v_{k_\ell} &\to v \quad \text{in } H^2(\Omega), \\
v_{k_\ell} &\to v \quad \text{in } H^1(\Omega).
\end{align*}
\]
On the other hand we see that for every \( \phi \in C^\infty_0(\Omega) \)
\[
\int_{\Omega} v \cdot \phi_{x_3} \, dx = \lim_{k \to \infty} \int_{\Omega} v_k \cdot \phi_{x_3} \, dx = \lim_{k \to \infty} \int_{\Omega} v_{k_\ell} \cdot \phi_{x_3} \, dx = 0,
\]
wheras \( \|v\|_{H^1(\Omega)} = 1 \), which is a contradiction. \( \square \)

**Lemma 6.9.** Let us denote \( \Omega^t = \Omega \times (t_0, t) \). Consider the following initial-boundary value problem
\[
\begin{align*}
u_{t} - \alpha \Delta u - \beta \nabla \div u &= F \quad \text{in } \Omega^t, \\
u &= 0 \quad \text{on } S_t^1, \\
u' &= 0, \quad u_{3, x_3} = 0 \quad \text{on } S_t^2, \\
u|_{t=t_0} &= u(t_0) \quad \text{on } \Omega \times \{ t = t_0 \},
\end{align*}
\]
where \( \alpha \) and \( \beta \) are positive constants. Assume that \( F \in L_p(\Omega^t) \). Then there exist a unique solution \( u \) such that \( u \in W^{2,1}_p(\Omega^t) \) and
\[
\|u\|_{W^{2,1}_p(\Omega^t)} \leq c_\Omega \left( \|F\|_{L_p(\Omega)} + \|u(t_0)\|_{W^{1,1}_p(\Omega)} \right).
\]

**Proof.** Let us introduce a partition of unity \( \sum_{k=1}^N \zeta_k(x_3) = 1 \) and denote \( \bar{u} = u \zeta_k \). Then we can repeat the considerations from the proof of Lemma (Lemma 6.8). However, there is a slight difference:

1. sup\( \zeta_k \cap S = \emptyset \): In this case we have
   \[
   \begin{align*}
u_{t} + L \bar{u} &= F + [0, 0, 2\nabla u_3 \cdot \nabla \zeta_k + u_3 \Delta \zeta_k] =: F_k \quad \text{in } \zeta_k \cap \Omega, \\
u &= 0 \quad \text{on } \partial(\zeta_k \cap \Omega),
   \end{align*}
   \]
   which is seen as the problem in the whole space. From [Sol65, Thm. 1.1] it follows that

\[
\|\bar{u}\|_{W^{2,1}_p(\zeta_k \cap \Omega^t)} \leq c_\Omega \left( \|F_k\|_{L_p(\zeta_k \cap \Omega^t)} + \|\bar{u}(t_0)\|_{W^{1,1}_p(\zeta_k \cap \Omega)} \right).
\]

Originally, the constant which appears on the right-hand side may depend on time. However, since we have the energy estimates for solutions to (6.2), we can utilize the Rellich-Kondrachov Compactness to exclude the time dependence of the constant. The remaining cases are reduced to the ones presented in Lemma 6.8 analogously: we consider three cases when the supports of the cut-off functions \( \zeta_k \) touch the boundary and reduce every case to the problem in the half-space. Then we utilize Theorem 5.5 from [Sol65] to obtain (6.3) but in the half space. Summing over \( k \) yields
\[
\|u\|_{W^{2,1}_p(\Omega^t)} \leq c_\Omega \left( \|F\|_{L_p(\Omega)} + \|u\|_{L_p(t_0, t; W^{1,1}_p(\Omega))} + \|u(t_0)\|_{W^{1,1}_p(\Omega)} \right).
\]

To eliminate the second term on the right-hand side we use the energy estimate for solutions to (6.2). This ends the proof. \( \square \)

**Remark 6.10.** In the above proof we omitted certain details related to the estimates in the half space. In subsequent considerations they will also be omitted. We refer the interested reader to [Sol65, §6, proofs of Theorems 1.1 and 1.2].

It is worth mentioning that an alternative approach was presented in [BZ97, §3].

**Lemma 6.11.** Consider the Stokes problem
\[
\begin{align*}
u_{t} - (\nu + \nu_r) \Delta v + \nabla p &= F \quad \text{in } \Omega^t, \\
\div v &= 0 \quad \text{in } \Omega^t, \\
v \cdot n &= 0 \quad \text{on } S^t, \\
\rot v \times n &= 0 \quad \text{on } S^t, \\
v|_{t=t_0} &= v(t_0) \quad \text{on } \Omega \times \{ t = t_0 \}.
\end{align*}
\]
If \( F \in L_p(\Omega) \) then there exist a solution to the above problem such that \( v \in W^{2,1}_p(\Omega^t) \) and the estimate
\[
\|v\|_{W^{2,1}_p(\Omega^t)} + \|\nabla p\|_{L_p(\Omega)} \leq c_\Omega \left( \|F\|_{L_p(\Omega)} + \|v(t_0)\|_{W^{1,1}_p(\Omega)} \right).
\]
Lemma 7.3. Let $S$ be given. Then $\theta$ is solution to the problem
\[
\theta = 0 \quad \text{on } S_1^i, \\
\theta_3 = 0, \quad \theta'_x = -\frac{1}{\alpha}g' \quad \text{on } S_2^i, \\
\theta|_{t=t_0} = \theta(t_0) \quad \text{in } \Omega.
\]
Proof. Equation (7.2) follows directly from (1.1) by differentiating along $x_3$ direction. Analogously, we compute the boundary condition (7.2), because $x_3$ is the tangent direction.

To prove (7.2), we take two first components of (1.1):

$$
\omega'_t + v \cdot \nabla \omega' - \alpha \Delta \omega' - \alpha \theta_{x_3}^2 \omega' - \beta \nabla' \div \omega + 4 \nu_r \omega' = 2 \nu_r (\operatorname{rot} v)' + g'
$$

and project them onto $S_2$. From Lemma 6.6 we deduce immediately that

$$-\alpha s_{x_3} |s_2 = g'|s_2.$$

The initial condition (7.2) follows from (1.3). □

Remark 7.4. Let us notice that for $\theta$ the Poincaré inequality holds. For $\theta_3$, which vanishes on $S_2$ it is obvious. For $\theta'$ we simply calculate the mean value:

$$
\int_\Omega \theta' \, dx = \int_\Omega \omega'_{x_3} \, dx = \int_{S_2(x_3 = 0)} - \omega' \, dx - \int_{S_2(x_3 = 0)} \omega' \, dx' = 0,
$$

which follows from (1.3).

Lemma 7.5. Let $v$, $h$, $\omega$ and $F_3$ be given. Then the function $\chi$ is solution to the following set of equations:

$$
\begin{aligned}
\chi_{,t} + v \cdot \nabla \chi - h_3 \chi + h_2 v_{3,x_1} - h_1 v_{1,x_2} - (\nu + \nu_r) \Delta \chi &= F_3 + 2 \nu_r \left( (\operatorname{rot} \omega)_2^{,x_1} - (\operatorname{rot} \omega)_1^{,x_2} \right) \\
\chi &= 0 \\
\chi_{x_3} &= 0 \\
\chi|_{t = t_0} &= \chi(t_0)
\end{aligned}
$$

(7.3)

Proof. To deduce (7.3) we simply differentiate (1.1) with respect to $x_1$, subtract it from (1.1) differentiated with respect to $x_2$ and take the third component.

To get (7.3), we multiply (1.2) by $\tau_1$ (see (1.2)). It yields

$$(\operatorname{rot} v \times n) \cdot \tau_1 = 0 \quad \Rightarrow \quad \operatorname{rot} v \cdot \tau_1 = 0 \quad \Leftrightarrow \quad v_{2,x_1} - v_{1,x_2} = 0.$$

The condition (7.3) was derived in Lemma 6.6(b). The initial condition (7.3) follows from (1.3) in the same manner as (7.3) from (1.1). □

8. Energy estimates

The prime goal of this Section is to establish certain basic energy estimates, we formulate three lemmas. The first one presents an estimate for the velocity and microfield rates (see Lemma 8.1) in $V^0_2(\Omega')$ space. In the second we derive estimates for the functions $h$ and $\theta$ (see Lemma 8.2) and in the third for the function $\chi$ (see Lemma 8.3). Unlike for the functions $v$ and $\omega$, the inequalities for $h$, $\omega$ and $\chi$ in $V^0_2(\Omega')$ cannot be estimated on the right-hand side due to the data which is a priori unknown and cannot be estimated. This term, i.e. $\|h\|_{L^\infty(t_0, t; L^2(\Omega))}$, will therefore appear every time we make use of these inequalities. It will be particularly visible in the subsequent Section, where we pay attention to higher order derivatives of the functions $v$, $h$ and $\omega$. Finally, it will be absorbed by the left-hand side but at the cost of a smallness assumption. As a consequence we shall get an estimate for $v$ and $\omega$ in $W^{2,1}_2(\Omega')$ space in terms of the data only.

The below Lemma demonstrates the estimate for the functions $v$ and $\omega$.

Lemma 8.1. Let $E_{v,\omega}(t) < \infty$ hold (see (2.1)). Then for any $t_0 \leq t \leq t_1$ we have

$$
\| v \|^2_{L^2(\Omega')} + \| \omega \|^2_{L^2(\Omega')} \leq c_{v,\omega, t_0, t_1} E_{v,\omega}(t).
$$

Proof. First, recall that $-\alpha \Delta \omega = \alpha \operatorname{rot} \omega - \alpha \nabla \div \omega$. Thus, multiplying (1.1) by $v$, (1.1) by $\omega$, integrating over $\Omega$ and utilizing Lemma 6.4 yields

$$
\frac{1}{2} \frac{d}{dt} \| v \|^2_{L^2(\Omega)} + (\nu + \nu_r) \| \operatorname{rot} v \|^2_{L^2(\Omega)} + (\nu + \nu_r) \int_\Omega v \times n \cdot v \, dS = 2 \nu_r \int_\Omega v \cdot \omega \, dx + \int_\Omega f \cdot v \, dx,
$$

$$
\frac{1}{2} \frac{d}{dt} \| \omega \|^2_{L^2(\Omega)} + \alpha \| \operatorname{rot} \omega \|^2_{L^2(\Omega)} - \alpha \int_\Omega \omega \times n \cdot \operatorname{rot} \omega \, dS + (\alpha + \beta) \| \div \omega \|^2_{L^2(\Omega)} + 4 \nu_r \int_\Omega \omega^2 \, dx
$$

$$
= 2 \nu_r \int_\Omega v \cdot \omega \, dx + \int_\Omega g \cdot \omega \, dx.
$$
Observe that the term with the pressure vanished due to \( \text{div} v = 0 \) and \( v \cdot n|_{S} = 0 \). Because of (1.3), the boundary integrals above are equal to zero. Adding both equalities and utilizing Lemma 6.4 again gives

\[
\frac{1}{2} \frac{d}{dt} \|v\|_{L^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2(\Omega)}^2 + (\nu + \nu_r) \| \text{rot} v \|_{L^2(\Omega)}^2 + \alpha \| \text{rot} \omega \|_{L^2(\Omega)}^2 + (\alpha + \beta) \| \text{div} \omega \|_{L^2(\Omega)}^2 + 4\nu_r \int \omega^2 \, dx = 4\nu_r \int_{\Omega} v \cdot \omega \, dx + \int_{S} \omega \times n \cdot v \, dS + \int_{\Omega} \omega \cdot v \, dx + \int_{\Omega} g \cdot \omega \, dx.
\]

By the means of the Hölder and the Young inequalities with \( \epsilon \) we can estimate the right hand side in the following way

\[
4\nu_r \int_{\Omega} v \cdot \omega \, dx + \int_{\Omega} f \cdot v \, dx + \int_{\Omega} g \cdot \omega \, dx \leq 4\nu_r \epsilon_1 \| \text{rot} v \|_{L^2(\Omega)}^2 + \frac{\nu_r}{\epsilon_1} \| \omega \|_{L^2(\Omega)}^2 + \epsilon_2 \| v \|_{L^2(\Omega)}^2 + \frac{1}{4\epsilon_2} \| f \|_{L^6(\Omega)}^2 + \epsilon_3 \| \omega \|_{L^6(\Omega)}^2 + \frac{1}{4\epsilon_3} \| g \|_{L^6(\Omega)}^2.
\]

Taking now \( \epsilon_1 = \frac{1}{4} \), \( \epsilon_2 = \frac{\nu}{\epsilon_1} \), and \( \epsilon_3 = \frac{\nu}{\epsilon_1} \), where the constant \( c_I \) comes from the imbedding \( H^{1}(\Omega) \rightarrow L_6(\Omega) \), and using Lemma 6.4, we see that

\[
\frac{d}{dt} \|v\|_{L^2(\Omega)}^2 + \frac{d}{dt} \|\omega\|_{L^2(\Omega)}^2 + \frac{\nu}{\epsilon_1} \| v \|_{H^1(\Omega)}^2 + \frac{\alpha}{\epsilon_1} \| \omega \|_{H^1(\Omega)}^2 \leq \frac{c_I}{\nu \epsilon_1} \| f \|_{L^6(\Omega)}^2 + \frac{c_I}{\alpha \epsilon_1} \| g \|_{L^6(\Omega)}^2 + \| v(t_0) \|_{L^2(\Omega)}^2 + \| \omega(t_0) \|_{L^2(\Omega)}^2.
\]

Integrating with respect to time on \((t_0, t)\) yields

\[
\|v\|_{L^2(\Omega)}^2 + \|\omega\|_{L^2(\Omega)}^2 \leq \frac{1}{\min \left\{ 1, \frac{\nu}{\epsilon_1}, \frac{\alpha}{\epsilon_1} \right\}} \left( \frac{c_I}{\nu \epsilon_1} \| f \|_{L^2(t_0, t; L^6(\Omega))}^2 + \frac{c_I}{\alpha \epsilon_1} \| g \|_{L^2(t_0, t; L^6(\Omega))}^2 \right) + \| v(t_0) \|_{L^2(\Omega)}^2 + \| \omega(t_0) \|_{L^2(\Omega)}^2.
\]

This completes the proof. \(\square\)

In the second Lemma in this Section we are looking for an estimate for \( h \) and \( \theta \). As mentioned, we obtain only inequality with an a priori unknown term on the right-hand side, which is assumed to be finite. At this stage we are not provided with any appropriate means to control that term, thereby postponing its estimation till more conclusive results are derived (see Lemma 9.1 in the next Section).

Before we formulate Lemma, let us state the following remarks:

**Remark 8.2.** Observe that \( q|_{S_2} = f_3|_{S_2} \).

Indeed, taking the third component of (1.1), projecting it into \( S_2 \) and using Lemma 6.6 gives

\[
q|_{S_2} = -v \cdot \nabla v \cdot n|_{S_2} + f_3|_{S_2} + 2\nu_r \text{rot} \omega \cdot n|_{S_2} = f_3|_{S_2}.
\]

**Remark 8.3.** We will prove that

\[
\|h\|_{H^{k+1}(\Omega)} \leq c_3 \| \text{rot} h \|_{H^k(\Omega)}.
\]

To this end, let us first consider the following problem

\[
\begin{align*}
\text{rot} h &= \alpha \quad \text{in } \Omega, \\
\text{div} h &= 0 \quad \text{in } \Omega, \\
\ h \cdot n &= 0 \quad \text{on } S_1, \\
\ h \times n &= 0 \quad \text{on } S_2.
\end{align*}
\]

Introduce a partition of unity \( \sum_{k=1}^{N} \zeta_k(x_3) = 1 \). If we denote \( \tilde{h} = h \zeta_k \), then the above system becomes

\[
\begin{align*}
\text{rot} \tilde{h} &= \tilde{\alpha} + [-h_2 \zeta_k, x_3, h_1 \zeta_k, x_3, 0] \quad \text{in } \text{supp } \zeta_k \cap \Omega, \\
\text{div} \tilde{h} &= h_3 \zeta_k, x_3 \quad \text{in } \text{supp } \zeta_k \cap \Omega, \\
\tilde{h} \cdot n &= 0 \quad \text{on } \text{supp } \zeta_k \cap S_1, \\
\tilde{h} \times n &= 0 \quad \text{on } \text{supp } \zeta_k \cap S_2.
\end{align*}
\]

Next, we perform similar computations as we did in proof of Lemma 6.8.
Lemma 8.4. Let $E_{v, \omega}(t) < \infty$ and $E_{h, \theta}(t) < \infty$. Additionally, let $g'_|S_2 = 0$, $f_3|S_2 = 0$. Finally, assume that $\|h\|_{L_{\infty}(t_0, t_1, L^4(\Omega))} < \infty$. Then

$$\|h\|_{L^2(\Omega)}^2 + \|\theta\|_{L^2(\Omega)}^2 \leq c_{\alpha, \nu, L, \Omega} \left( E_{v, \omega}(t) \|h\|_{L_{\infty}(t_0, t_1, L^4(\Omega))} + E_{h, \theta}(t) \right).$$

Note that the conditions $g'_|S_2 = 0$ and $f_3|S_2 = 0$ can be dropped, but then we would have to deal with two non-trivial boundary integrals on $S_2$. We would be even capable to estimate them in suitable norms by application of the extension and the interpolation theorems, which would result in the appearance of the term $c_I \left( \|h\|^2_{L^2(\Omega)} + \|\theta\|^2_{L^2(\Omega)} \right)$ on the right-hand side. But we should not forget that the presented lemma serves only a supporting role in the proof of global existence of regular solutions to problem (1.1). Since we have no control over the magnitude of $c_I$ we would encounter insurmountable difficulties in further considerations (see the proof of Lemma 9.1).

Proof. Multiplying (7.1) and (7.2) by $h$ and $\theta$ respectively and integrating over $\Omega$ yields

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} h^2 \, dx - (\nu + \nu_r) \int_{\Omega} \Delta h \cdot h \, dx = - \int_{\Omega} \nabla q \cdot h \, dx - \int_{\Omega} v \nabla h \cdot h \, dx - \int_{\Omega} h \cdot \nabla v \cdot h \, dx + 2\nu_r \int_{\Omega} \text{rot} \theta \cdot h \, dx + \int_{\Omega} f_{x_5} \cdot h \, dx$$

and

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \theta^2 \, dx - \alpha \int_{\Omega} \Delta \theta \cdot \theta \, dx = \beta \int_{\Omega} \nabla \text{div} \theta \cdot \theta \, dx + 4\nu_r \int_{\Omega} \theta^2 \, dx$$

Consider first the terms containing the Laplace operator. From Lemma 6.4 it follows that

$$- \int_{\Omega} \Delta h \cdot h \, dx = \int_{\Omega} \text{rot} \, h \cdot h \, dx = \int_{\Omega} |\text{rot} h|^2 \, dx + \int_{\Omega} h \times n \cdot h \, dS = \int_{\Omega} |h|^2 \, dx,$$

because $\text{rot} h \times n|_{S_1} = 0$ and on $S_2$ the equality $(\text{rot} h \times n)|_3 = 0$ holds. Again, from Lemma 6.4 we have

$$- \int_{\Omega} \Delta \theta \cdot \theta \, dx = \int_{\Omega} \text{rot} \, \theta \cdot \theta \, dx - \int_{\Omega} \nabla \text{div} \theta \cdot \theta \, dx$$

$$= \int_{\Omega} |\text{rot} \theta|^2 \, dx + \int_{\Omega} \text{rot} \theta \times n \cdot \theta \, dS + \int_{\Omega} |\text{div} \theta|^2 \, dx - \int_{\Omega} (\theta \times n) \text{div} \theta \, dS$$

and

$$\int_{\Omega} \text{rot} \, \theta \cdot h \, dx = \int_{\Omega} \text{rot} \, h \cdot \theta + \int_{S} \theta \times n \cdot h \, dS = \int_{\Omega} \text{rot} \, h \cdot \theta \, dx$$

because of the boundary conditions (1.3). For the nonlinear terms the equalities

$$\int_{\Omega} v \cdot \nabla h \cdot h \, dx = - \frac{1}{2} \int_{\Omega} \text{div} v |h|^2 \, dx + \frac{1}{2} \int_{S} |h|^2 v \cdot n \, dS = 0,$$

$$\int_{\Omega} v \cdot \nabla \theta \cdot \theta \, dx = - \frac{1}{2} \int_{\Omega} \text{div} v |\theta|^2 \, dx + \frac{1}{2} \int_{S} |\theta|^2 v \cdot n \, dS = 0$$

hold because of (1.3). Finally, integration by parts yields

$$- \int_{\Omega} \nabla \text{div} \theta \cdot \theta \, dx = \int_{\Omega} |\text{div} \theta|^2 \, dx - \int_{\Omega} \text{div} \theta \cdot (\theta \times n) \, dS = \int_{\Omega} |\text{div} \theta|^2 \, dx,$$

$$\int_{\Omega} \nabla q \cdot h \, dx = - \int_{\Omega} q \, \text{div} h \, dx + \int_{S} q (h \cdot n) \, dS = \int_{S_{\Gamma}} f_{x_5} \, dS = 0,$$

where to justify the last equality we use Remark 8.2. Now, adding both sides in (8.1) and (8.2) and taking into account the above integration we obtain

$$\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} h^2 \, dx + \int_{\Omega} \theta^2 \, dx \right) + (\nu + \nu_r) \int_{\Omega} |\text{rot} h|^2 \, dx + \alpha \int_{\Omega} |\text{rot} \theta|^2 \, dx \right) + (\alpha + \beta) \int_{\Omega} |\text{div} \theta|^2 \, dx$$

$$+ 4\nu_r \int_{\Omega} \theta^2 \, dx = - h \cdot \nabla v \cdot h \, dx - \int_{\Omega} h \cdot \nabla \omega \cdot \theta + 4\nu_r \int_{\Omega} \text{rot} \, h \cdot \theta \, dx$$

$$+ \int_{S_{x_5}} f \, dx + \int_{S_{x_5}} g \, \theta \, dx = \sum_{k=1}^{5} f_k.$$
We estimate every term on the right-hand side by the means of the Hölder and the Young inequalities:

\[ I_1 \leq \|h\|_{L^6(\Omega)} \|\nabla v\|_{L^4(\Omega)} \|h\|_{L^3(\Omega)} \leq \epsilon_1 \|h\|_{L^6(\Omega)}^2 + \frac{1}{4\epsilon_1} \|\nabla v\|_{L^4(\Omega)}^2 \|h\|_{L^3(\Omega)}^2, \]

\[ I_2 \leq \|h\|_{L^6(\Omega)} \|\nabla \omega\|_{L^4(\Omega)} \|\theta\|_{L^4(\Omega)} \leq \epsilon_2 \|\theta\|_{L^6(\Omega)}^2 + \frac{1}{4\epsilon_2} \|\nabla \omega\|_{L^4(\Omega)}^2 \|h\|_{L^3(\Omega)}^2, \]

\[ I_3 \leq 4\nu \|\text{rot } h\|_{L^2(\Omega)} \|\theta\|_{L^2(\Omega)} \leq 4\nu \epsilon_3 \|\text{rot } h\|_{L^2(\Omega)}^2 + \frac{\nu}{\epsilon_3} \|\theta\|_{L^2(\Omega)}^2, \]

\[ I_4 \leq \|f_{x_3}\|_{L^4(\Omega)} \|h\|_{L^3(\Omega)} \leq \epsilon_5 \|h\|_{L^3(\Omega)}^2 + \frac{1}{4\epsilon_5^2} \|f_{x_3}\|_{L^2(\Omega)}^2, \]

\[ I_5 \leq \|g_{x_3}\|_{L^4(\Omega)} \|\theta\|_{L^3(\Omega)} \leq \epsilon_6 \|\theta\|_{L^3(\Omega)}^2 + \frac{1}{4\epsilon_6^2} \|g_{x_3}\|_{L^2(\Omega)}^2. \]

Now we set \( \epsilon_3 = \frac{1}{4} \). Then we estimate \( \nu \|\text{rot } h\|_{L^2(\Omega)}^2 \) from below by \( \frac{\nu}{\epsilon_3} \|h\|_{H^1(\Omega)}^2 \) (see Remark \[8.3\]) and \( \alpha \|\text{rot } \theta\|_{L^2(\Omega)}^2 + \|\text{div } \theta\|_{L^2(\Omega)}^2 \) from below by \( \frac{\alpha}{\epsilon_3} \|\theta\|_{H^1(\Omega)}^2 \) (see Lemma \[6.7\]). \( \theta \cdot n|_{S} = 0 \). Next we set

\[ \epsilon_1 c_I = \epsilon_5 c_I = \frac{\nu}{4\epsilon_3}, \quad \epsilon_2 c_I = \epsilon_6 c_I = \frac{\alpha}{4\epsilon_3}. \]

Summarizing,

\[
(8.3) \quad \frac{1}{2} \frac{d}{dt} \left( \int \Omega h^2 \, dx + \int \Omega \theta^2 \, dx \right) + \frac{\nu}{2\epsilon_3} \|h\|_{H^1(\Omega)}^2 + \frac{\alpha}{2\epsilon_3} \|\theta\|_{H^1(\Omega)}^2 \leq \frac{C_{f, \Omega}}{\nu} \|\nabla \nu\|_{L^2(\Omega)} \|h\|_{L^2(\Omega)} \|h\|_{L^3(\Omega)} + \frac{C_{f, \Omega}}{\alpha} \|f_{x_3}\|_{L^2(\Omega)} \|h\|_{L^3(\Omega)} + \frac{C_{f, \Omega}}{\alpha} \|g_{x_3}\|_{L^2(\Omega)} \|\theta\|_{L^3(\Omega)}.
\]

Multiplying by 2 and integrating with respect to \( t \in (t_0, t_1) \) yields

\[
\|h(t)\|_{L^2(\Omega)}^2 + \|\theta(t)\|_{L^2(\Omega)}^2 \leq \frac{C_{f, \Omega}}{\nu} \int_{t_0}^t \|\nabla v(s)\|_{L^2(\Omega)} \|h(s)\|_{L^2(\Omega)} \|h(s)\|_{L^3(\Omega)} \, ds + \frac{C_{f, \Omega}}{\alpha} \int_{t_0}^t \|\nabla \omega(s)\|_{L^2(\Omega)} \|h(s)\|_{L^3(\Omega)} \|\theta(s)\|_{L^2(\Omega)} \, ds
\]

\[
+ \frac{C_{f, \Omega}}{\nu} \|f_{x_3}(t)\|_{L^2(\Omega)} + \frac{C_{f, \Omega}}{\alpha} \|g_{x_3}(t)\|_{L^2(\Omega)} \|h(t)\|_{L^2(\Omega)} + \|\theta(t)\|_{L^2(\Omega)}^2 \geq \frac{\alpha}{2\epsilon_3} \|\theta\|_{H^1(\Omega)}^2 + \|\text{div } \theta\|_{L^2(\Omega)}^2 + \|h(t)\|_{L^2(\Omega)}^2 + \|\theta(t)\|_{L^2(\Omega)}^2.
\]

Next we estimate the left-hand side from below by

\[
\min \left\{ \frac{\nu}{C_{f, \Omega}}, \frac{\alpha}{\epsilon_3} \right\} \left( \|h\|_{L^2(\Omega)}^2 + \|\theta\|_{L^2(\Omega)}^2 \right).
\]

In view of Lemma \[8.1\] we get that

\[
\min \left\{ \frac{\nu}{C_{f, \Omega}} \frac{\alpha}{\epsilon_3} \right\} \left( \|h\|_{L^2(\Omega)}^2 + \|\theta\|_{L^2(\Omega)}^2 \right)
\]

\[
\leq \frac{2C_{f, \Omega}}{\min \{\alpha, \nu\}} \left( \|h\|_{L^2(\Omega)}^2 + \|\theta\|_{L^2(\Omega)}^2 \right)
\]

which concludes the proof. \( \Box \)

Remark 8.5. In the above Lemma we used Lemma \[6.7\] which implied that \( \frac{\alpha}{\epsilon_3} \|\theta\|_{H^1(\Omega)}^2 \leq \alpha \|\text{rot } \theta\|_{L^2(\Omega)}^2 + \|\text{div } \theta\|_{L^2(\Omega)}^2 \). In further considerations we need slightly different estimate. Since

\[
\alpha \|\text{rot } \theta\|_{L^2(\Omega)}^2 + (\alpha + \beta) \|\text{div } \theta\|_{L^2(\Omega)}^2 \geq \frac{\alpha}{2} \left( \|\text{rot } \theta\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|\text{rot } \theta\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|\text{rot } \theta\|_{L^2(\Omega)}^2 \right)
\]

we infer from Lemma \[6.7\] that

\[
\alpha \|\text{rot } \theta\|_{L^2(\Omega)}^2 + (\alpha + \beta) \|\text{div } \theta\|_{L^2(\Omega)}^2 \geq \frac{\alpha}{2\epsilon_3} \|\theta\|_{H^1(\Omega)}^2 + \frac{\alpha}{2} \|\text{rot } \theta\|_{L^2(\Omega)}^2.
\]

By the Poincaré inequality (see Remark \[7.2\]) and by the interpolation inequality we obtain

\[
\|h\|_{L^3(\Omega)} \leq \|h\|_{L^2(\Omega)}^{\frac{3}{2}} \|h\|_{L^\infty(\Omega)}^{\frac{1}{2}} \leq \epsilon_{L, \nu} \|h\|_{H^1(\Omega)}.
\]

Thus, (8.3) can be written in a following form

\[
\frac{1}{2} \frac{d}{dt} \left( \int \Omega h^2 \, dx + \int \Omega \theta^2 \, dx \right) + \frac{\nu}{2\epsilon_3} \|h\|_{H^1(\Omega)}^2 + \frac{\alpha}{2} \|\text{rot } \theta\|_{L^2(\Omega)}^2 \leq \frac{C_{f, \Omega, \nu}}{\nu} \|\nabla v\|_{L^2(\Omega)} \|h\|_{L^2(\Omega)} \|h\|_{L^3(\Omega)} \|\theta\|_{L^2(\Omega)} + \frac{C_{f, \Omega}}{\nu} \|f_{x_3}\|_{L^2(\Omega)} + \frac{C_{f, \Omega}}{\alpha} \|g_{x_3}\|_{L^2(\Omega)} \|\theta\|_{L^2(\Omega)}.
\]
Utilizing Remark \ref{5.3} and multiplying the above inequality by \( \frac{2\nu_r}{\alpha} \) leads to

\[
(8.4) \quad \frac{\nu_r}{\alpha} \frac{d}{dt} \left( \int_{\Omega} h^2 \, dx + \int_{\Omega} \theta^2 \, dx \right) + \nu_r \| \text{rot} \theta \|_{L^2(\Omega)}^2 \\
\leq c_{\alpha, \nu, \nu_r, 1, P, \Omega} \| h \|_{L^2(\Omega)}^2 \left( \| \nabla v \|_{L^2(\Omega)}^2 + \| \nabla \nu \|_{L^2(\Omega)}^2 \right) \\
+ c_{\alpha, \nu, \nu_r, 1, \Omega} \left( \| f \|_{L^2(\Omega)}^2 + \| g \|_{L^2(\Omega)}^2 \right).
\]

For now the above inequality seems to be useless, but later we shall see that it is necessary to obtain an estimate for \( h \) in \( V^2(\Omega) \) (see Lemma \ref{7.3}).

Finally we present the last Lemma of this Section.

**Lemma 8.6.** Let \( E_{t, \nu}(t) < \infty \). Suppose that \( \| h \|_{L^\infty(t_0, t, L^3(\Omega))} < \infty \). Assume that \( f' = (f_1, f_2) \in L^2(\Omega') \) and \( v(t_0) \in H^1(\Omega) \). Then

\[
\| \chi \|_{V^2(\Omega)} \leq c_{\alpha, \nu, \nu_r, 1, P, \Omega} E_{t, \nu}(t) \| h \|_{L^\infty(t_0, t, L^3(\Omega))} + c_{\alpha, \nu, \nu_r, 1, \Omega} E_{t, \nu}(t) + \| v(t_0) \|_{H^1(\Omega)}.
\]

**Proof.** Multiplying (7.3) by \( \chi \) and integrating over \( \Omega \) yields

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \chi^2 \, dx + (\nu + \nu_r) \int_{\Omega} |\nabla \chi|^2 \, dx = \int_{\Omega} h_3 \chi \, dx - \int_{\Omega} h_2 v_{3, x_1} \chi \, dx + \int_{\Omega} h_1 v_{3, x_2} \chi \, dx \\
+ \int_{\Omega} F_3 \chi \, dx + 2\nu_r \int_{\Omega} (\text{rot} \, \omega)_{2, x_1} - (\text{rot} \, \omega)_{1, x_2} \chi \, dx.
\]

The first three integrals on the right-hand side we estimate in the same way by the means of the H"older and the Young inequalities:

\[
\int_{\Omega} h_3 \chi \, dx \leq \| h_3 \|_{L^3(\Omega)} \| \chi \|_{L^2(\Omega)} \| \chi \|_{L^6(\Omega)} \leq \epsilon_1 \| \chi \|_{L^2(\Omega)}^2 + \frac{1}{4\epsilon_1} \| h_3 \|_{L^3(\Omega)}^2 \| \chi \|_{L^2(\Omega)}^2,
\]

\[
\int_{\Omega} h_2 v_{3, x_1} \chi \, dx \leq \| h_2 \|_{L^3(\Omega)} \| v_{3, x_1} \|_{L^2(\Omega)} \| \chi \|_{L^6(\Omega)} \leq \epsilon_2 \| \chi \|_{L^2(\Omega)}^2 + \frac{1}{4\epsilon_2} \| h_2 \|_{L^3(\Omega)}^2 \| v_{3, x_1} \|_{L^2(\Omega)}^2,
\]

and

\[
\int_{\Omega} h_1 v_{3, x_2} \chi \, dx \leq \| h_1 \|_{L^3(\Omega)} \| v_{3, x_2} \|_{L^2(\Omega)} \| \chi \|_{L^6(\Omega)} \leq \epsilon_3 \| \chi \|_{L^2(\Omega)}^2 + \frac{1}{4\epsilon_3} \| h_1 \|_{L^3(\Omega)}^2 \| v_{3, x_2} \|_{L^2(\Omega)}^2.
\]

In the last two integrals we first integrate by parts and then use the H"older and the Young inequalities. Note that the boundary integrals are equal to zero due to the boundary conditions for \( \chi \).

\[
\int_{\Omega} F_3 \chi \, dx = -\int_{\Omega} f_2 \chi_{, x_1} - f_1 \chi_{, x_2} \, dx + \int_{\Omega} f_2 \chi_{, x_1} - f_1 \chi_{, x_2} \, dS \\
\leq \| f' \|_{L^2(\Omega)} \| \nabla \chi \|_{L^2(\Omega)} \leq \epsilon_4 \| \nabla \chi \|_{L^2(\Omega)}^2 + \frac{1}{4\epsilon_4} \| f' \|_{L^2(\Omega)}^2.
\]

and

\[
\int_{\Omega} (\text{rot} \, \omega)_{2, x_1} - (\text{rot} \, \omega)_{1, x_2} \chi \, dx = -\int_{\Omega} (\text{rot} \, \omega)_{2, x_1} - (\text{rot} \, \omega)_{1, x_2} \chi \, dx \\
+ \int_{\Omega} (\text{rot} \, \omega)_{2} \chi_{, x_1} - (\text{rot} \, \omega)_{1} \chi_{, x_2} \, dS \\
\leq \| (\text{rot} \, \omega)' \|_{L^2(\Omega)} \| \nabla \chi \|_{L^2(\Omega)} \leq \epsilon_5 \| \nabla \chi \|_{L^2(\Omega)}^2 + \frac{1}{4\epsilon_5} \| (\text{rot} \, \omega)' \|_{L^2(\Omega)}^2.
\]

Since \( \chi \big|_{S_1} = 0 \), we can use the Poincaré inequality \( \| \chi \|_{L^6(\Omega)} \leq c_\ell \| \chi \|_{H^1(\Omega)} \leq c_{\ell, P} \| \nabla \chi \|_{L^6(\Omega)} \). Hence, we put \( \epsilon_1 = \epsilon_2 = \epsilon_3 = \frac{3\epsilon_4^2}{\nu + \nu_r} \) and \( \epsilon_4 = 2\nu_r \epsilon_5 = \frac{1}{4\epsilon_5} \). Then we see that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \chi^2 \, dx + \frac{\nu + \nu_r}{2} \int_{\Omega} |\nabla \chi|^2 \, dx \\
\leq \frac{3\epsilon_4^2}{\nu + \nu_r} \left( \| h_3 \|_{L^3(\Omega)} \| \chi \|_{L^2(\Omega)}^2 + \| h_2 \|_{L^3(\Omega)} \| v_{3, x_1} \|_{L^2(\Omega)}^2 + \| h_1 \|_{L^3(\Omega)} \| v_{3, x_2} \|_{L^2(\Omega)}^2 \right) \\
+ \frac{2}{\nu + \nu_r} \| f' \|_{L^2(\Omega)}^2 + \frac{4\nu_r}{\nu + \nu_r} \| (\text{rot} \, \omega)' \|_{L^2(\Omega)}^2.
\]
Next we multiply by 2, integrate with respect to \( t \in (t_0, t_1) \) and use the energy estimate (Lemma 5.1). It gives

\[
\min \{ 1, \nu + \nu_r \} \| \chi \|_{V^1_2(\Omega')}^2 \leq \frac{6c_2 \| h \|_{L_2(\Omega; t, L_2(\Omega))}^2}{\nu + \nu_r} \left( \| \chi \|_{L_2^2(\Omega')}^2 + \| v_{3, x_1} \|_{L_2(\Omega')}^2 + \| v_{3, x_2} \|_{L_2(\Omega')}^2 \right) + \frac{4}{\nu + \nu_r} \| f' \|_{L_2(\Omega')}^2 + \frac{8\nu_r}{\nu + \nu_r} \| (\text{rot} \omega') \|_{L_2(\Omega')}^2 + \| \chi(t_0) \|_{L_2(\Omega')}^2.
\]

To complete the proof we observe that \( \| \chi(t_0) \|_{L_2(\Omega')} \leq \| v(t_0) \|_{H^1(\Omega)} \).

9. Higher order estimates

In this Section we confine our attention to find estimates for \( v \) and \( \omega \) in the norm of the space \( W^{2,1}_2(\Omega') \) in terms of data only. Of course we cannot expect such a result without an additional smallness assumption. Therefore we shall make use of \( \delta(t) \) which was introduced (2.2). We emphasize that the smallness condition does not involve \( L_2 \)-norms of the initial velocity and the initial microrotation fields. Below we briefly outline the reason behind that.

The central aim is to estimate the nonlinear terms \( v \cdot \nabla v \) and \( v \cdot \nabla \omega \) in \( L_2(\Omega') \). To accomplish it we first consider the problem for \( h \) and seek for estimate of its solution in \( V^2_2(\Omega') \). The estimate has a form

\[
\| h \|_{V^2_2(\Omega')} \leq c \left( \| \nabla v \|_{L_2(t_0, t, L_6(\Omega))}^2 \right) + 1 \cdot \delta(t)
\]

(see Lemma 9.1). Its direct consequence is

\[
\| h \|_{L_2(t_0, t; H^1(\Omega'))} \leq c \left( \| \nabla v \|_{L_2(t_0, t, L_6(\Omega))}^2 \right) + 1 \cdot \delta(t),
\]

which we use to bound \( \| h \|_{L_2(t_0, t, L_6(\Omega))} \) by the means of the interpolation between \( L_p \) spaces and the Poincaré inequality. The reason becomes apparent if we take into account that the inequalities for \( \| h \|_{V^2_2(\Omega')} \) and \( \| \chi \|_{V^2_2(\Omega')} \) (see Lemmas 5.3 and 5.5) are dependent on that norm. Therefore we can write

\[
\| h \|_{V^2_2(\Omega')} + \| \chi \|_{V^2_2(\Omega')} \leq c \left( \| \nabla v \|_{L_2(t_0, t, L_6(\Omega))}^2 \right) + 1 \cdot \delta(t) + \text{data}.
\]

Applying the above inequality and (9.1) and using the structure of the domain, which is a Cartesian product of two sets, we are able to estimate \( \nabla v \) in the norm \( V^0_2(\Omega') \) (see Lemma 9.2). Since the estimate has the form

\[
\| \nabla v \|_{V^0_2(\Omega')} \leq c \left( \| \nabla v \|_{L_2(t_0, t, L_6(\Omega))}^2 \right) + 1 \cdot \delta(t) + \text{data}
\]

we finally write

\[
\| \nabla v \|_{V^0_2(\Omega')} \leq \text{data}
\]

if only \( \delta(t) \) is small enough. The above inequality in connection with embedding theorem guarantees that the nonlinear terms can be estimated in the \( L_2 \)-norm with respect to \( \Omega' \), thereby providing estimates for \( v \) and \( \omega \) in \( W^{2,1}_2(\Omega') \), which is demonstrated in Lemmas 9.4 and 9.5.

Let us now move on to the details of the proof:

**Lemma 9.1.** Suppose that \( f_3|_{S_2} = 0, \ g'|_{S_2} = 0 \) and \( \nabla v \in L_2(t_0, t; L_6(\Omega)) \). Let \( E_{v, \omega}(t) < \infty \) and \( \delta(t) < \infty \). Then

\[
\| h \|_{V^1_2(\Omega')} \leq c_{a, \nu, \nu_r, t, p, \Omega} \left( \| \nabla v \|_{L_2(t_0, t, L_6(\Omega))}^2 + E_{v, \omega}^2(t) \right) + 1 \cdot \delta(t).
\]

In the proof of the above Lemma, we use

\[
\| h \|_{H^2(\Omega)} \leq c\| \Delta h \|_{L_2(\Omega)},
\]

which we prove in analogous manner as in Remark 5.3.

**Proof of Lemma 9.1.** We multiply (7.1) by \( -\Delta h \) and integrate over \( \Omega \), which yields

\[
(9.3) \quad - \int_{\Omega} h \cdot \Delta h \, dx + (\nu + \nu_r) \int_{\Omega} |\Delta h|^2 \, dx - \int_{\Omega} \nabla q \cdot \Delta h \, dx = \int_{\Omega} v \cdot \nabla h \cdot \Delta h \, dx + \int_{\Omega} h \cdot \nabla v \cdot \Delta h \, dx - 2\nu_r \int_{\Omega} \text{rot} \theta \cdot \Delta h \, dx - \int_{\Omega} f_{3, x_3} \cdot \Delta h \, dx.
\]
For the first term on the left-hand side we have
\[
- \int_\Omega h_{,t} \cdot \Delta h \, dx = \int_\Omega h_{,t} \cdot \text{rot} \, h \, dx
= \frac{1}{2} \frac{d}{dt} \int_\Omega |\text{rot} \, h|^2 \, dx + \int_{S_1} h_{,t} \cdot \text{rot} \, h \times n \, dS_1 + \int_{S_2} h_{2,t} \, (\text{rot} \, h)_1 - h_{1,t} \, (\text{rot} \, h)_2 \, dS_2
= \frac{1}{2} \frac{d}{dt} \int_\Omega |\text{rot} \, h|^2 \, dx,
\]
where the boundary integrals vanish due to the boundary conditions (7.1, 3.4).

The third term on the left-hand side in (9.3) is equal to
\[
\int_\Omega \nabla q \cdot \text{rot} \, h \, dx = \int_\Omega \text{rot} \, \nabla q \cdot \text{rot} \, h \, dx + \int_{S_1} \nabla q \cdot \text{rot} \, h \times n \, dS_1
+ \int_{S_2} q_{,x_2} \, (\text{rot} \, h)_1 - q_{,x_1} \, (\text{rot} \, h)_2 \, dS_2 = 0,
\]
which follows from the boundary condition (7.1), Remark 3.2 and the assumption that \(f_3|_{S_2} = 0\).

Consider next the first term on the right-hand side in (9.3). Since \(\text{div} \, v = 0\) we may integrate by parts, which yields
\[
\int_\Omega v \cdot \nabla h \cdot \Delta h \, dx \leq \|\nabla v\|_{L^6(\Omega)} \|\nabla h\|_{L^3(\Omega)} \|\text{rot} \, h\|_{L_2(\Omega)} + \int_\Omega (v \cdot \nabla) \, (\text{rot} \, h \times n) \, dS
\leq \epsilon_1 c_I \|\nabla h\|_{L^2(\Omega)} + \frac{1}{4 \epsilon_1} \|\nabla v\|_{L^6(\Omega)} \|\text{rot} \, h\|_{L_2(\Omega)}
\leq \epsilon_1 c_I, \|\Delta h\|_{L_2(\Omega)} + \frac{1}{4 \epsilon_1} \|\nabla v\|_{L^6(\Omega)} \|\text{rot} \, h\|_{L_2(\Omega)},
\]
where we used that \(\|\nabla h\|_{L_4(\Omega)} \leq \frac{1}{3} \|\nabla h\|_{L_2(\Omega)} \|\text{rot} \, h\|_{L_4(\Omega)} \leq c_I \|\nabla h\|_{H^1(\Omega)}\). The last inequality above is justified in light of (9.2).

For the second term on the right-hand side in (9.3) we simply have
\[
\|h\|_{L_3(\Omega)} \|\nabla v\|_{L_6(\Omega)} \|\Delta h\|_{L_2(\Omega)} \leq \epsilon_2 \|\Delta h\|_{L_2(\Omega)}^2 + \frac{1}{4 \epsilon_2} \|h\|_{L_4(\Omega)} \|h\|_{L_6(\Omega)} \|\nabla v\|_{L_6(\Omega)}^2.
\]
The third term is estimated as follows
\[
2 \nu \epsilon_3 \int_\Omega \text{rot} \, \theta \cdot \Delta h \, dx \leq 2 \nu \epsilon_3 \|\text{rot} \, \theta\|_{L_2(\Omega)} \|\Delta h\|_{L_2(\Omega)} \leq 2 \nu \epsilon_3 \|\Delta h\|_{L_2(\Omega)}^2 + \frac{\nu}{2 \nu_3} \|\text{rot} \, \theta\|_{L_2(\Omega)}^2.
\]
Finally, for the fourth term we have
\[
\int_\Omega \|f_{,x_3}\| \, \Delta h \, dx \leq \|f_{,x_3}\|_{L_2(\Omega)} \|\Delta h\|_{L_2(\Omega)} \leq \epsilon_4 \|\Delta h\|_{L_2(\Omega)}^2 + \frac{1}{4 \epsilon_4} \|f_{,x_3}\|_{L_2(\Omega)}^2.
\]
Setting \(\epsilon_1 c_I, = \epsilon_2 = \epsilon_4 = \frac{\nu}{6} \) and \(\epsilon_3 = \frac{1}{2}\) yields
(9.4) \[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\text{rot} \, h|^2 \, dx + \frac{\nu}{2} \|\Delta h\|_{L_2(\Omega)}^2 \leq \frac{3 \epsilon_1 c_I}{2 \nu} \|\nabla v\|_{L_6(\Omega)} \|\text{rot} \, h\|_{L_2(\Omega)}
+ \frac{3}{2 \nu} \|h\|_{L_4(\Omega)} \|h\|_{L_6(\Omega)} \|\nabla v\|_{L_6(\Omega)}^2 + \nu \epsilon_3 \|\text{rot} \, \theta\|_{L_2(\Omega)}^2 + \frac{3}{2 \nu} \|f_{,x_3}\|_{L_2(\Omega)}^2
\leq \frac{3 \epsilon_1 c_I}{\nu} \|\nabla v\|_{L_6(\Omega)} \|\text{rot} \, h\|_{L_2(\Omega)} + \nu \epsilon_3 \|\text{rot} \, \theta\|_{L_2(\Omega)} + \frac{3}{2 \nu} \|f_{,x_3}\|_{L_2(\Omega)}^2,
\]
where in the last inequality we used \(\|h\|_{L_3(\Omega)} \|h\|_{L_6(\Omega)} \leq c_I \|h\|_{H^1(\Omega)}^2\) and subsequently utilized Lemma 6.7. Next we add inequality (8.4) (see Remark 8.3), which leads to
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\text{rot} \, h|^2 \, dx + \nu \frac{d}{dt} \left( \int_\Omega h^2 \, dx + \int_\Omega \theta^2 \, dx \right) + \nu \|\text{rot} \, \theta\|_{L_2(\Omega)}^2 + \frac{\nu}{2} \|\Delta h\|_{L_2(\Omega)}^2
\leq \frac{3 \epsilon_1 c_I}{\nu} \|\nabla v\|_{L_6(\Omega)} \|\text{rot} \, h\|_{L_2(\Omega)} + \nu \epsilon_3 \|\text{rot} \, \theta\|_{L_2(\Omega)}^2 + \frac{3}{2 \nu} \|f_{,x_3}\|_{L_2(\Omega)}^2
+ c_{\alpha, \nu, \nu_3, f_{,x_3}} \|\text{rot} \, h\|_{L_2(\Omega)} \left( \|\nabla v\|_{L_2(\Omega)}^2 + \|\nabla \omega\|_{L_2(\Omega)}^2 \right)
+ c_{\alpha, \nu, \nu_3, f_{,x_3}} \left( \|f_{,x_3}\|_{L_2(\Omega)}^2 + \|g_{,x_3}\|_{L_2(\Omega)}^2 \right).
\]
Multiplying by 2 and integrating with respect to $t \in (t_0, t)$ yields

$$
(9.5) \quad \|\text{rot} h(t)\|_{L^2(\Omega')}^2 + \frac{2\nu}{\alpha} \|h(t)\|_{L^2(\Omega')}^2 + \frac{2\nu}{\alpha} \|\theta(t)\|_{L^2(\Omega')}^2 + \nu \|\Delta h\|_{L^2(\Omega')}^2
\leq \frac{6c_{0,1}}{\nu} \int_{t_0}^{t} \|\nabla v(s)\|_{L^2(\Omega')}^2 \|\text{rot} h(s)\|_{L^2(\Omega')}^2 \, ds + \frac{3}{\nu} \|f(s)\|_{L^2(\Omega')}^2
+ c_{\alpha, \nu, \theta, I, \Omega} \left( \|f\|_{L^2(\Omega, t; L^2(\Omega'))}^2 + \|g\|_{L^2(\Omega, t; L^2(\Omega'))}^2 \right) \int_{t_0}^{t} \|\text{rot} h(s)\|_{L^2(\Omega')}^2 \left( \|\nabla v(s)\|_{L^2(\Omega')}^2 + \|\nabla \omega(s)\|_{L^2(\Omega')}^2 \right) \, ds
+ c_{\alpha, \nu, \theta, I, \Omega} \left( \|f\|_{L^2(\Omega, t; L^2(\Omega'))}^2 + \|g\|_{L^2(\Omega, t; L^2(\Omega'))}^2 \right) + 2\nu \|\theta(t)\|_{L^2(\Omega')}^2.
$$

From the Gronwall inequality and the energy estimates (Lemma 8.1) it follows that

$$
\|\text{rot} h\|_{L^\infty(0, t; L^2(\Omega))}^2 \leq c_{\alpha, \nu, \theta, I, \Omega, \Omega} \exp \left( \|\nabla v\|_{L^2(0, t; L^2(\Omega))}^2 + E\|v\|_{L^2(\Omega')}^2 \right)
\left( \|f\|_{L^2(0, t; L^2(\Omega'))}^2 + \|g\|_{L^2(0, t; L^2(\Omega'))}^2 + \|\theta(0)\|_{L^2(\Omega')}^2 \right).
$$

By the Hölder inequality we write

$$
\|\text{rot} h\|_{L^\infty(0, t; L^2(\Omega))}^2 \leq c_{\alpha, \nu, \theta, I, \Omega, \Omega} \exp \left( \|\nabla v\|_{L^2(0, t; L^2(\Omega))}^2 + E\|v\|_{L^2(\Omega')}^2 \right)
\left( \|f\|_{L^2(0, t; L^2(\Omega'))}^2 + \|g\|_{L^2(0, t; L^2(\Omega'))}^2 + \|\theta(0)\|_{L^2(\Omega')}^2 \right) + \|\text{rot} h(0)\|_{L^2(\Omega')}^2 + \|h(0)\|_{L^2(\Omega')}^2 + \|\theta(0)\|_{L^2(\Omega')}^2.
$$

Putting the above estimate in (9.5) and using the Hölder inequality yields

$$
\|\text{rot} h(t)\|_{L^2(\Omega')}^2 + \nu \|\Delta h\|_{L^2(\Omega')}^2
\leq c_{\alpha, \nu, \theta, I, \Omega} \|\text{rot} h\|_{L^\infty(0, t; L^2(\Omega))} \left( \|\nabla v\|_{L^2(0, t; L^2(\Omega))}^2 + \|\nabla \omega\|_{L^2(\Omega')}^2 \right)
+ c_{\alpha, \nu, \theta, I, \Omega} \left( \|f\|_{L^2(0, t; L^2(\Omega'))}^2 + \|g\|_{L^2(0, t; L^2(\Omega'))}^2 + \|\theta(0)\|_{L^2(\Omega')}^2 \right).
$$

In light of Remark 8.3 and 9.2 we estimate the left-hand side from below by $\min\{1, \nu\} \|h\|_{V^2_0(\Omega')}^2$. Thus

$$
\|h\|_{V^2_0(\Omega')}^2 \leq c_{\alpha, \nu, \theta, I, \Omega} \exp \left( \|\nabla v\|_{L^2(0, t; L^2(\Omega))}^2 + E\|v\|_{L^2(\Omega')}^2 \right) \cdot \|\Delta h\|_{L^2(\Omega')}^2 + c_{\alpha, \nu, \theta, I, \Omega} \|\theta(0)\|_{L^2(\Omega')}^2.
$$

An obvious inequality $e^{ax} \leq e^{ax}$ for $x \geq 0$ ends the proof.

**Lemma 9.2.** Let $\chi, h_3 \in V^2_0(\Omega')$, $f' = (f_1, f_2) \in L^2(\Omega')$ and $v(t_0) \in H^1(\Omega)$. Let $E\|v\|_{L^2(\Omega')} < \infty$, $E\|h\|_{L^2(\Omega')} < \infty$. Suppose that $\delta(t)$ is small enough. Then $\nabla v \in V^2_0(\Omega')$ and the estimate

$$
\|\nabla v\|_{V^2_0(\Omega')}^2 \leq c_{\alpha, \nu, \theta, I, \Omega} \left( \|E\|_{L^2(\Omega')}^2 + \|h\|_{L^2(\Omega')}^2 + \|f\|_{L^2(\Omega')}^2 + \|v(t_0)\|_{H^1(\Omega')}^2 \right).
$$

**Proof.** Consider the problem

\[
\begin{aligned}
v_{2, x_1} - v_{1, x_2} &= \chi && \text{in } \Omega', \\
v_{1, x_1} + v_{2, x_2} &= -h_3 && \text{in } \Omega', \\
v' \cdot n' &= 0 && \text{on } S'_1,
\end{aligned}
\]

where by $\Omega'$ and $S'$ we understand the sets $\Omega \cap \{ -a < x_3 < a : x_3 = \text{const} \}$ and $S' = S_1 \cap \{ -a < x_3 < a : x_3 = \text{const} \}$. For solutions to this problem we have estimates

\[
\begin{aligned}
\|\nabla v(x_3)\|_{L^2(\Omega')}^2 &\leq c_{\Omega} \left( \|\chi(x_3)\|_{L^2(\Omega')}^2 + \|h_3(x_3)\|_{L^2(\Omega')}^2 \right), \\
\|\nabla v(x_3)\|_{H^1(\Omega')}^2 &\leq c_{\Omega} \left( \|\chi(x_3)\|_{H^1(\Omega')}^2 + \|h_3(x_3)\|_{H^1(\Omega')}^2 \right).
\end{aligned}
\]

Integrating both inequalities with respect to $x_3 \in (-a, a)$ ant taking the $L^2$- and $L^\infty$-norms with respect to $t \in (t_0, t_1)$ yields

$$
\|\nabla v\|_{L^\infty(\Omega')}^2 \leq c_{\Omega} \left( \|\chi\|_{L^\infty(\Omega')}^2 + \|h_3\|_{L^\infty(\Omega')}^2 \right).
$$
Using the estimate (see Lemma 8.4)
\[ \|h\|^2_{V^2(\Omega')} \leq c_{\alpha,v,\nu,\Omega} \left( E_{\nu,w}(t) \|h\|^2_{L^\infty(t_0,t;L^3(\Omega))} + E_{h,\theta}(t) \right) \]
we obtain
\[ \|\nabla v\|^2_{V^2(\Omega')} \leq c_{\alpha,v,\nu,\Omega} \left( E_{\nu,w}(t) \|h\|^2_{L^\infty(t_0,t;L^3(\Omega))} + E_{h,\theta}(t) \right) \]
\[ + c_{\nu,v,\alpha,\Omega,p} E_{\nu,w}(t) \|h\|^2_{L^\infty(t_0,t;L^3(\Omega))} + c_{\nu,v,\nu,\Omega} f'_{L^2(\Omega')} + c_{\alpha,v,\nu,\Omega} E_{\nu,w}(t) + \|v(t_0)\|^2_{H^1(\Omega)}. \]
The interpolation and the Poincaré inequalities (see Remark 7.2) imply that
\[ \|h\|_{L^\infty(t_0,t;L^3(\Omega))} \leq \|h\|^2_{L^\infty(t_0,t;L^2(\Omega))} \|h\|^2_{L^\infty(t_0,t;L^6(\Omega))} \leq c_{I,P} \|h\|_{L^\infty(t_0,t;H^1(\Omega))}. \]
From Lemma 6.2 we infer that
\[ \|\nabla v\|_{L^2(t_0,t;L^6(\Omega))} \leq c_{I} \|\nabla v\|_{V^2(\Omega')}. \]
Finally
\[ \|\nabla v\|^2_{V^2(\Omega')} \leq c_{\alpha,v,\nu,\nu,\Omega} c_{I} E_{\nu,w}(t) \left( \exp \left( \|\nabla v\|^2_{V^2(\Omega')} + E_{\nu,w}(t) \right) + 1 \right) \cdot \delta(t) \]
\[ + E_{\nu,w}(t) + E_{h,\theta}(t) + \|f'\|^2_{L^2(\Omega')} + \|v(t_0)\|^2_{H^1(\Omega)}. \]
Taking \( \delta(t) \) sufficiently small ends the proof. \( \square \)

**Remark 9.3.** From the above Lemma two natural embeddings follow (see Lem. 3.7 in [Zaj04]). In view of Lemma 6.2 we see that \( \nabla v \in L^q(t_0,t;L^p(\Omega)) \) and
\[ \|\nabla v\|_{L^q(t_0,t;L^p(\Omega))} \leq c_{I} \|\nabla v\|_{V^2(\Omega')} \]
holds for \( p, q \) satisfying \( \frac{3}{p} + \frac{2}{q} = \frac{3}{2} \). Setting \( p = q \) we get \( \frac{3}{2} = \frac{3}{p} \Rightarrow p = \frac{4}{3} \). Hence
\[ \|\nabla v\|_{L^\infty(\Omega')} \leq c_{I} \|\nabla v\|_{V^2(\Omega')} \]
From the Sobolev embedding lemma we know that \( W^1_p(\Omega) \hookrightarrow L^\infty(\Omega) \) for \( p < 3 \). Thus,
\[ \|v\|_{L^q(t_0,t;L^p(\Omega))} \leq c_{I} \|\nabla v\|_{L^q(t_0,t;L^p(\Omega))} \leq c_{I} \|\nabla v\|_{V^2(\Omega')} \]
for \( \frac{3}{p} + \frac{2}{q} = \frac{3}{2} \). Let \( r = \frac{3p}{3q-3p} \) and \( q = r \). Then \( p = \frac{3q}{3q+1} \) and
\[ \frac{3}{r} + \frac{2}{q} = \frac{3}{2} \Rightarrow \frac{3}{r} + \frac{2}{2} = \frac{3}{2} \Rightarrow r = 10. \]
Hence
\[ \|v\|_{L^\infty(\Omega')} \leq c_{I} \|\nabla v\|_{V^2(\Omega')} \]
**Lemma 9.4.** Let \( E_{\nu,w}(t) < \infty \) and \( E_{h,\theta}(t) < \infty \). Assume that \( f,g \in L^2(\Omega^t) \), \( v(t_0), \omega(t_0) \in H^1(\Omega) \).
Then, for \( \delta(t) \) sufficiently small \( \omega \in W^2_{1,1}(\Omega^t) \) and the inequality
\[ \|\omega\|_{W^2_{1,1}(\Omega^t)} \leq c_{\alpha,v,\nu,\nu,\Omega} \left( E_{\nu,w}(t) + E_{h,\theta}(t) + \|f\|^2_{L^2(\Omega^t)} + \|v(t_0)\|^2_{H^1(\Omega)} + \|\omega(t_0)\|^2_{H^1(\Omega)} + 1 \right)^3 \]
holds.
**Proof.** Let us rewrite (1.1) and (1.3) in the form
\[ \omega_{t} - \alpha \Delta \omega - \beta \nabla \text{div} \omega = -v \cdot \nabla \omega - 4\nu_{v} \omega + 2\nu_{\nu} \text{rot} v + g \quad \text{in} \ \Omega^t, \]
\[ \omega = 0 \quad \text{on} \ S_{1}^t, \]
\[ \omega' = 0, \quad \omega_{t=0} = 0 \quad \text{on} \ S_{2}^t, \]
\[ \omega_{t=t_0} = \omega(t_0) \quad \text{in} \ \Omega \times \{ t = t_0 \}. \]
Then, from Lemma 6.3 it follows that
\[ (9.6) \quad \|\omega\|_{W^2_{1,1}(\Omega^t)} \leq c_{I} \left( \|v \cdot \nabla \omega\|^2_{L^2(\Omega^t)} + 4\nu_{v} \|\omega\|^2_{L^2(\Omega^t)} + 2\nu_{\nu} \|\text{rot} v\|^2_{L^2(\Omega^t)} \right. \]
\[ \left. \quad + \|g\|_{L^2(\Omega^t)} + \|\omega(t_0)\|^2_{H^1(\Omega)} \right). \]
From the Hölder inequality we get
\[ \|v \cdot \nabla \omega\|_{L^2(\Omega^t)} \leq c_{I} \|v\|_{L^1(\Omega^t)} \|\nabla \omega\|_{L^2(\Omega^t)}. \]
To estimate $\|\nabla \omega\|_{L_2^2(\Omega')}^2$ we use the interpolation theorem (see Lemma 6.1) with $p = 2$, $q = \frac{5}{2}$. It gives

$$\|\nabla \omega\|_{L_2^2(\Omega')}^2 \leq c_1 \epsilon^\frac{2}{5}^{\frac{3}{2}} \|\omega\|_{W_{1,1}^2(\Omega')}^5 + c_2 \epsilon^{-\frac{2}{5}} \|\omega\|_{L_2(\Omega')}^2.$$ 

Setting $c_1 \epsilon^\frac{2}{5} = \frac{1}{2c_1 \|\nabla \omega\|_{L_2(\Omega')}}$ yields

$$\|v \cdot \nabla \omega\|_{L_2(\Omega')} \leq \frac{1}{2} \|\nabla \omega\|_{L_2^2(\Omega')} + 2c_1 c_2 \epsilon \|v\|_{L_2(\Omega')}^2 \|\omega\|_{L_2(\Omega')}.$$ 

To estimate the second and third terms on the right-hand side in (9.7) we use Lemma 8.1. Finally, the inequality

$$\|\omega\|_{W_{2,1}^2(\Omega')} \leq c_{\alpha, \nu, \gamma, \nu, \Omega}(E_{v,\omega}(t) + E_{\gamma, \nu}(t) + \|f\|_{L_2(\Omega')}^3 + \|\nabla \omega\|_{L_2(\Omega')}^3 + \|\nabla \omega\|_{L_2(\Omega')} + \|\omega(t_0)\|_{H^1(\Omega)}^3),$$

holds, which ends the proof.

**Lemma 9.5.** Let $E_{v, \omega}(t) < \infty$, $E_{\gamma, \nu}(t) < \infty$. Assume that $f \in L_2(\Omega')$, $v(t_0) \in H^1(\Omega)$. Then for $\delta(t)$ sufficiently small $v \in W_{2,1}^2(\Omega')$ and

$$\|v\|_{W_{2,1}^2(\Omega')} + \|\nabla \omega\|_{L_2(\Omega')} \leq c_{\alpha, \nu, \gamma, \nu, \Omega}(E_{v,\omega}(t) + E_{\gamma, \nu}(t) + \|f\|_{L_2(\Omega')} + \|\nabla \omega\|_{L_2(\Omega')} + \|\omega(t_0)\|_{H^1(\Omega)} + 1)^3.$$ 

**Proof.** Let us rewrite (1.1), (1.3) and (1.4) in the following form

$$v_{\epsilon} = (v + \nu_r) \Delta v + \nabla p = -v \cdot \nabla v + 2\nu_r \text{rot} \omega + f \quad \text{in} \; \Omega',$$

$$\text{div} v = 0 \quad \text{in} \; \Omega',$$

$$v \cdot n = 0 \quad \text{on} \; S^\prime,$$

$$\text{rot} v \times n = 0 \quad \text{on} \; S^\prime,$$

$$v|_{t=0} = v(t_0) \quad \text{on} \; \Omega \times \{t = t_0\}.$$ 

For solution to the above problem we have the following estimate (see Lemma 6.11)

$$\|v\|_{W_{2,1}^2(\Omega')} + \|\nabla \omega\|_{L_2(\Omega')} \leq c_\alpha(E_{v,\omega}(t) + E_{\gamma, \nu}(t) + \|\nabla \omega\|_{L_2(\Omega')} + \|f\|_{L_2(\Omega')} + \|\omega(t_0)\|_{H^1(\Omega)}).$$ 

To estimate the first term on the right-hand side we use Lemma 9.2 which implies that for $\delta(t)$ small enough we have

$$\|v\|_{L_2(\Omega')} \leq c_{\alpha, \nu, \gamma, \nu, \Omega}(E_{v,\omega}(t) + E_{\gamma, \nu}(t) + \|f\|_{L_2(\Omega')} + \|\omega(t_0)\|_{H^1(\Omega)}).$$

Subsequently, from the Hölder inequality we get

$$\|v \cdot \nabla \omega\|_{L_2(\Omega')} \leq c_\gamma \|v\|_{L_2(\Omega')} \|\nabla \omega\|_{L_2(\Omega')}.$$ 

To estimate $\|\nabla \omega\|_{L_2(\Omega')}$ we use the interpolation theorem (see Lemma 6.1) with $p = 2$, $q = \frac{5}{2}$. It gives

$$\|\nabla \omega\|_{L_2(\Omega')} \leq c_1 \epsilon^\frac{2}{5} \|v\|_{W_{2,1}^2(\Omega')}^2 + c_2 \epsilon^{-\frac{2}{5}} \|v\|_{L_2(\Omega')}^2.$$ 

Setting $c_1 \epsilon^\frac{2}{5} = \frac{1}{2c_1 \|v\|_{L_2(\Omega')}}$ yields

$$\|v \cdot \nabla \omega\|_{L_2(\Omega')} \leq \|v\|_{W_{2,1}^2(\Omega')}^2 + 2c_1 c_2 \epsilon \|v\|_{L_2(\Omega')}^2 \|\omega(t_0)\|_{L_2(\Omega')}.$$ 

To estimate the second and third term on the right-hand side in (9.7) we use Lemma 8.1. Finally

$$\|v\|_{W_{2,1}^2(\Omega')} + \|\nabla \omega\|_{L_2(\Omega')} \leq c_{\alpha, \nu, \gamma, \nu, \Omega}(E_{v,\omega}(t) + E_{\gamma, \nu}(t) + \|f\|_{L_2(\Omega')}^3 + \|\nabla \omega\|_{L_2(\Omega')}^3 + \|\omega(t_0)\|_{L_2(\Omega')}^3 + \|\nabla \omega\|_{L_2(\Omega')} + \|\omega(t_0)\|_{H^1(\Omega)}).$$

This concludes the proof.
10. Existence of solutions

Now we prove the existence of regular solutions. The proof is based on the fixed-point principle (see Lemma 6.3) and makes use of the inequalities we derived in previous Sections.

First we rewrite (1.1) in following way

\[\begin{align*}
v_t - (v + \nu r) \Delta v + \nabla p &= -\lambda (\bar{v} \cdot \nabla \bar{v} + 2\nu r \text{ rot } \bar{\omega}) + f \quad \text{in } \Omega', \\
\text{div } v &= 0 \quad \text{in } \Omega', \\
\omega_t - \alpha \Delta \omega - \beta \nabla \text{ div } \omega + 4\nu r \omega &= \lambda (\bar{v} \cdot \nabla \bar{v} + 2\nu r \text{ rot } \bar{\omega}) + g \quad \text{in } \Omega', \\
\text{rot } v \times n &= 0, \quad v \cdot n = 0 \quad \text{on } S', \\
\omega &= 0 \quad \text{on } S'_1, \\
\omega' &= 0, \quad \omega_{3,3} = 0 \quad \text{on } S'_2, \\
v|_{t=t_0} = v(t_0), \quad \omega|_{t=t_0} = \omega(t_0) \quad \text{on } \Omega \times \{t = t_0\},
\end{align*}\]

(10.1)

where \(\lambda \in [0, 1]\) and \(\bar{v}, \bar{\omega}\) are considered as given functions.

Let us introduce space

\[\mathcal{M}(\Omega') := \left\{ u : \|u\|_{L^2_{x_3}(\Omega')} < \infty, \|\nabla u\|_{L^2(\Omega')} < \infty \right\}.
\]

Problem (10.1) determines the mapping

\[\Phi : \mathcal{M}(\Omega') \times \mathcal{M}(\Omega') \times [0, 1] \to \mathcal{M}(\Omega') \times \mathcal{M}(\Omega'),
\]

\[\Phi(\bar{v}, \bar{\omega}, \lambda) = (v, \omega).
\]

In Section 9 we found a priori estimate for a fixed point of \(\Phi\) when \(\lambda = 1\) (see Lemmas 9.3 and 9.4). For \(\lambda = 0\) we check if the uniqueness of solution is ensured.

Lemma 10.1. Suppose that \(\lambda = 0\). Then, problem (10.1) possesses a unique solution.

Proof. Let \((\nu^1, \omega^1)\) and \((\nu^2, \omega^2)\) be two different solutions to (10.1). Then, the pair \((V, \Theta)\), \(V = \nu^1 - \nu^2\), \(\Theta = \omega^1 - \omega^2\) is a solution to the problem

\[\begin{align*}
V_t - (\nu + \nu r) \Delta V + \nabla P &= 0 \quad \text{in } \Omega', \\
\text{div } V &= 0 \quad \text{in } \Omega', \\
\Theta_t - \alpha \Delta \Theta - \beta \nabla \text{ div } \Theta + 4\nu r \Theta &= 0 \quad \text{in } \Omega, \\
\text{rot } V \times n &= 0, \quad V \cdot n = 0 \quad \text{on } S', \\
\Theta &= 0 \quad \text{on } S'_1, \\
\Theta' &= 0, \quad \Theta_{3,3} = 0 \quad \text{on } S'_2, \\
V|_{t=t_0} = 0, \quad \Theta|_{t=t_0} = 0 \quad \text{on } \Omega \times \{t = t_0\},
\end{align*}\]

(10.2)

where we set \(P = p^1 - p^2\). Multiplying the first equation by \(V\), the third by \(\Theta\) and integrating over \(\Omega\) yields

\[\frac{d}{dt} \int_{\Omega} |V|^2 + |\Theta|^2 \, dx - (\nu + \nu r) \int_{\Omega} \Delta V \cdot V \, dx - \alpha \int_{\Omega} \Delta \Theta \cdot \Theta \, dx - \beta \int_{\Omega} \nabla \text{ div } \Theta \cdot \Theta \, dx - \int_{\Omega} \nabla P \cdot V \, dx + 4\nu r |\Theta|^2_{L^2(\Omega)} = 0.
\]

From Lemma 6.4 it follows

\[\begin{align*}
- \int_{\Omega} \Delta V \cdot V \, dx &= \int_{\Omega} \text{ rot } V \cdot V \, dx = \int_{\Omega} |\text{ rot } V|^2 \, dx + \int_{\Sigma} \text{ rot } V \times n \cdot V \, dS = ||\text{ rot } V||^2_{L^2(\Omega)}, \\
- \int_{\Omega} \Delta \Theta \cdot \Theta \, dx &= \int_{\Omega} \text{ rot } \Theta \cdot \Theta \, dx = \int_{\Omega} |\text{ rot } \Theta|^2 \, dx - \int_{\Sigma} \Theta \times n \cdot \text{ rot } \Theta \, dS = ||\text{ rot } \Theta||^2_{L^2(\Omega)},
\end{align*}\]

where the boundary integrals vanished due to (10.2). Integration by parts yields

\[\begin{align*}
- \int_{\Omega} \nabla \text{ div } \Theta \cdot \Theta \, dx &= \int_{\Omega} |\text{ div } \Theta|^2 \, dx - \int_{\Sigma} \text{ div } \Theta \cdot (\theta - n) \, dS = ||\text{ div } \Theta||^2_{L^2(\Omega)},
\end{align*}\]
where the boundary integral also vanishes due to (10.2)5,6. Thus
\[
\frac{d}{dt} \int_{\Omega} \left( |V|^2 + |\Theta|^2 \right) dx + (\nu + \nu_r) \left( \| \text{rot} \, V \|^2 + \alpha \left( \| \text{rot} \, \Theta \|^2_{L_2(\Omega)} + \| \text{div} \, \Theta \|^2_{L_2(\Omega)} \right) + \beta \| \text{div} \, \Theta \|^2_{L_2(\Omega)} + 4\nu_r \| \Theta \|^2_{L_2(\Omega)} = 0.
\]
After integrating with respect to \( t \in (t_0, t_1) \) we obtain
\[
\| V(t) \|^2_{L_2(\Omega)} + \| \Theta(t) \|^2_{L_2(\Omega)} \leq \| V(t_0) \|^2_{L_2(\Omega)} + \| \Theta(t_0) \|^2_{L_2(\Omega)} = 0.
\]
Thus, we have proved the uniqueness. The existence of solution follow from Lemmas 6.9 and 6.11. This ends the proof.

Next we show that \( \Phi \) is a compact and continuous mapping.

**Lemma 10.2.** The mapping \( \Phi \) is compact and continuous.

**Proof.** Assume that \( \bar{v} \in \mathfrak{M}(\Omega^t) \). Then
\[
\| \bar{v} \cdot \nabla \bar{v} \|^2_{L_2(\Omega^t)} \leq \| \bar{v} \|_{L_\infty(\Omega^t)} \| \nabla \bar{v} \|_{L_\infty(\Omega^t)} \leq \| \bar{v} \|_{\mathfrak{M}(\Omega^t)}.
\]
In the same way we get that
\[
\| \bar{v} \cdot \nabla \bar{w} \|^2_{L_2(\Omega^t)} \leq \| \bar{v} \|_{L_\infty(\Omega^t)} \| \nabla \bar{w} \|_{L_\infty(\Omega^t)} \leq \| \bar{v} \|_{\mathfrak{M}(\Omega^t)} \| \bar{w} \|_{\mathfrak{M}(\Omega^t)}.
\]
In view of Lemmas 6.9 and 6.11 we get that the solution to \( 10.1 \) belongs to \( W^{2,1}_2(\Omega^t) \). From Lemma 6.1 we deduce that the embeddings
\[
W^{2,1}_2(\Omega^t) \hookrightarrow L^2(\Omega^t) \iff \kappa = 2 - 5 \left( \frac{1}{2} - \frac{3}{20} \right) = \frac{1}{4} > 0,
\]
\[
W^{2,1}_2(\Omega^t) \hookrightarrow L^2(0, t; W^{2,1}_2(\Omega)) \iff \kappa = 2 - 1 - 5 \left( \frac{1}{2} - \frac{7}{20} \right) = \frac{1}{4} > 0
\]
are compact. This proves the compactness of \( \Phi \).

In order to prove the continuity we consider problem (10.1) in a form
\[
v^k - (\nu + \nu_r) \Delta v + \nabla p^k = -\lambda (\bar{v} \cdot \nabla \bar{v}^k - 2\nu_r \, \text{rot} \, \bar{w}^k) + f \quad \text{in } \Omega^t,
\]
\[
\omega^k - \alpha \Delta \omega - \beta \nabla \text{div} \, \omega^k + 4\nu_r \omega^k = -\lambda (\bar{v} \cdot \nabla \bar{w}^k + 2\nu_r \, \text{rot} \, \bar{v}^k) + g \quad \text{in } \Omega^t,
\]
\[
\text{div} \, v^k = 0 \quad \text{in } \Omega^t,
\]
\[
\text{rot} \, v^k \times n = 0, \quad v^k \cdot n = 0 \quad \text{on } S^t,
\]
\[
\omega^k = 0 \quad \text{on } S^t_1,
\]
\[
(\omega^k)' = 0, \quad \omega^k_{3,xx} = 0 \quad \text{on } S^t_2,
\]
\[
v^k|_{t = t_0} = v(t_0), \quad \omega^k|_{t = t_0} = \omega(t_0) \quad \text{on } \Omega \times \{ t = t_0 \},
\]
where \( k = 1, 2 \). Let
\[
V = v^1 - v^2, \quad P = p^1 - p^2, \quad \Theta = \omega^1 - \omega^2.
\]
Then \((V, \Theta)\) is solution to the problem
\[
V_t - (\nu + \nu_r) \Delta V + \nabla P = -\lambda (\bar{V} \cdot \nabla \bar{v}^1 + \bar{v}^1 \cdot \nabla \bar{V} + 2\nu_r \, \text{rot} \, \bar{\Theta}) \quad \text{in } \Omega^t,
\]
\[
\text{div} \, V = 0 \quad \text{in } \Omega^t,
\]
\[
\Theta_t - \alpha \Delta \Theta - \beta \nabla \text{div} \, \Theta + 4\nu_r \Theta = -\lambda (\bar{v} \cdot \nabla \bar{\Theta} + \bar{\Theta} \cdot \nabla \bar{v}^1 + 2\nu_r \, \text{rot} \, \bar{V}) \quad \text{in } \Omega^t,
\]
\[
\text{rot} \, V \times n = 0, \quad V \cdot n = 0 \quad \text{on } S^t,
\]
\[
\Theta = 0 \quad \text{on } S^t_1,
\]
\[
\Theta' = 0, \quad \Theta_{3,xx} = 0 \quad \text{on } S^t_2,
\]
\[
V|_{t = t_0} = 0, \quad \Theta|_{t = t_0} = 0 \quad \text{on } \Omega \times \{ t = t_0 \}.
\]
Assume that \( \lambda \neq 0 \). Then the estimates
\[
\| V \|_{\mathfrak{M}(\Omega^t)} \leq C \| V \|_{W^{2,1}_2(\Omega^t)} \leq C \| \lambda \left( | \bar{V} \cdot \nabla \bar{v}^1 + \bar{v}^1 \cdot \nabla \bar{V} + 2\nu_r \, \text{rot} \, \bar{\Theta} \right) \|_{L_2(\Omega^t)}
\]

...
and
\[ \|\Theta\|_{3\mathcal{R}(\Omega')} \leq c_{\Omega} \|\Theta\|_{W^{2,1}_c(\Omega')} \leq c_{\Omega} \lambda \left( \|v^1 \cdot \nabla \Theta + V \cdot \nabla \omega^2 + 2\nu_r \text{rot } V\|_{L^2(\Omega')} \right). \]

follow from Lemmas 6.11 and 6.9. Next we add both inequalities and utilize the Hölder inequality. This yields
\[ \|V\|_{3\mathcal{R}(\Omega')} + \|\Theta\|_{3\mathcal{R}(\Omega')} \leq c_{\nu,\lambda,\Omega}(\|\tilde{V}\|_{L^\infty_{t}(\Omega')} \|\nabla v^1\|_{L^\infty_{t}(\Omega')} + \|\tilde{V}\|_{L^\infty_{t}(\Omega')} \|\nabla \omega^2\|_{L^\infty_{t}(\Omega')} + \|\nabla \Theta\|_{L^\infty_{t}(\Omega')} + \|\text{rot } \Theta\|_{L^\infty_{t}(\Omega')} + \|\text{rot } V\|_{L^2_{t}(\Omega')}). \]

Thus
\[ \|V\|_{3\mathcal{R}(\Omega')} + \|\Theta\|_{3\mathcal{R}(\Omega')} \leq c_{\nu,\lambda,\Omega}(\|\tilde{V}\|_{L^\infty_{t}(\Omega')} + \|\tilde{V}\|_{L^\infty_{t}(\Omega')} \|\nabla \omega^2\|_{L^\infty_{t}(\Omega')} + \|\nabla \Theta\|_{L^\infty_{t}(\Omega')} + \|\text{rot } \Theta\|_{L^\infty_{t}(\Omega')} + \|\text{rot } V\|_{L^2_{t}(\Omega')}). \]

In view of (10.3) and Lemmas 9.4, 9.5 and 5.1 we can estimate all norms in the last bracket in terms of data only. This observation results in the following inequality
\[ \|V\|_{3\mathcal{R}(\Omega')} + \|\Theta\|_{3\mathcal{R}(\Omega')} \leq c_{\nu,\lambda,\Omega, \text{data}}(\|\tilde{V}\|_{3\mathcal{R}(\Omega')} + \|\tilde{V}\|_{3\mathcal{R}(\Omega')} \|\nabla \omega^2\|_{3\mathcal{R}(\Omega')} + \|\nabla \Theta\|_{3\mathcal{R}(\Omega')} + \|\text{rot } \Theta\|_{3\mathcal{R}(\Omega')} + \|\text{rot } V\|_{3\mathcal{R}(\Omega')}), \]

where \(c_{\text{data}}\) indicates the dependence on the data. It proves the uniform continuity of \(\Phi\). The continuity of \(\Phi\) with respect to \(\lambda\) is evident. This ends the proof. \(\square\)

**Lemma 10.3.** Let \(E_{v,\omega}(t) < \infty\), \(E_{h,\rho}(t) < \infty\). Assume that \(f, g \in L^2(\Omega')\), \(v(t_0), \omega(t_0) \in H^1(\Omega)\). Then, for \(\delta(t)\) small enough problem (11.1) admits a solution \((v, \omega) \in W^{2,1}_c(\Omega') \times W^{2,1}_c(\Omega')\) such that
\[ \|v\|_{W^{2,1}_c(\Omega')} + \|\nabla p\|_{L^2(\Omega')} \leq c_{\alpha,\nu,\omega,\lambda,\Omega,\delta,\rho}(E_{v,\omega}(t) + E_{h,\rho}(t) + \|f\|_{L^2(\Omega')} + \|v(t_0)\|_{H^1(\Omega)}^3) \]

and
\[ \|\omega\|_{W^{2,1}_c(\Omega')} \leq c_{\alpha,\nu,\omega,\lambda,\Omega,\delta,\rho}(E_{v,\omega}(t) + E_{h,\rho}(t) + \|f\|_{L^2(\Omega')} + \|v(t_0)\|_{H^1(\Omega)} + \|\omega(t_0)\|_{H^1(\Omega)}^3). \]

**Proof.** Lemmas 10.3 and 10.2 ensure that the assumptions of the Leary-Schauder theorem (see Lemma 6.3) are met. Hence, we obtain the existence of solution \((v, \omega) \in W^{2,1}_c(\Omega') \times W^{2,1}_c(\Omega')\) and by Lemma 6.11 also the existence of \(\nabla p \in L^2(\Omega')\). The estimates follow from Lemmas 9.4 and 9.5. \(\square\)

### 11. Uniqueness of regular solutions

In the previous Section we have proved the existence of regular solutions to problem (11.1). Now we would like to ask about their uniqueness. Like for the ordinary Navier-Stokes equations, the answer is positive (see below).

**Lemma 11.1.** Suppose that \(\delta(t)\) is small enough. Then, problem (11.1) admits a unique solution.

**Proof.** The proof is straightforward and is based on the Gronwall inequality for difference of solutions. Let \((v^1, \omega^1)\) and \((v^2, \omega^2)\) be two different solutions to problem (11.1). Let us denote \(V = v^1 - v^2, \Theta = \omega^1 - \omega^2\) and \(P = p^1 - p^2\). Then, the pair \((V, \Theta)\) is a solution to the problem

\[
\begin{align*}
V_t &= (v^1 + \nu^1) \Delta V + \nabla P = -V \cdot \nabla v^1 - v^2 \cdot \nabla V + 2\nu_r \text{rot } \Theta & \text{in } \Omega', \\
\text{div } V &= 0 & \text{in } \Omega', \\
\Theta_t &= \alpha \Delta \Theta - \beta \text{div } \Theta + 4\nu_r \Theta = -v^1 \cdot \nabla \Theta - V \cdot \nabla \omega^2 + 2\nu_r \text{rot } V & \text{in } \Omega', \\
\text{rot } V \times n &= 0, & V \cdot n = 0 & \text{on } S', \\
\Theta &= 0 & \text{on } S_1', \\
\Theta_t &= 0, & \text{on } S_2', \\
v|_{t=t_0} &= 0, & \Theta|_{t=t_0} = 0 & \text{on } \Omega \times \{t = t_0\}. 
\end{align*}
\]
Now we multiply the first equation by \( V \) and the third by \( \Theta \). Next we integrate and since \( V \) and \( \Theta \) satisfy the boundary conditions (1.3) and the assumption (A) from Lemma 6.4 we easily see that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |V|^2 + |\Theta|^2 \, dx + (\nu + \nu_r) |\text{rot} \, V|^2_{L^2(\Omega)} + \alpha \left( |\text{rot} \, \Theta|^2_{L^2(\Omega)} + ||\text{div} \, \Theta||^2_{L^2(\Omega)} \right) + \beta ||\text{div} \, \Theta||^2_{L^2(\Omega)} + 4\nu_r ||\Theta||^2_{L^2(\Omega)}
\]

\[
= - \int_{\Omega} V \cdot \nabla v^1 \cdot V \, dx + 4\nu_r \int_{\Omega} \text{rot} \, V \, \Theta \, dx + \int_{\Omega} V \cdot \nabla \omega^2 \cdot \Theta \, dx.
\]

From the Hölder and the Young inequalities it follows that

\[
- \int_{\Omega} V \cdot \nabla v^1 \cdot V \, dx \leq \epsilon_1 ||V||^2_{L^6(\Omega)} + \frac{1}{4\epsilon_4} ||\nabla v^1||^2_{L^6(\Omega)} ||V||^2_{L^2(\Omega)},
\]

\[
4\nu_r \int_{\Omega} \text{rot} \, V \cdot \Theta \, dx \leq 4\nu_r \epsilon_2 |\text{rot} \, V|^2_{L^2(\Omega)} + \frac{\nu_r}{\epsilon_2} ||\Theta||^2_{L^2(\Omega)},
\]

\[
\int_{\Omega} V \cdot \nabla \omega^2 \cdot \Theta \, dx \leq \epsilon_3 ||V||^2_{L^6(\Omega)} + \frac{1}{4\epsilon_3} ||\nabla \omega^2||^2_{L^6(\Omega)} ||\Theta||^2_{L^2(\Omega)}.
\]

Now we set \( \epsilon_2 = \frac{1}{3} \Rightarrow \epsilon_2 = 4\nu_r \) and utilize Lemma 6.7 which implies that

\[
\frac{\nu}{\epsilon_2} ||V||^2_{H^{1/2}(\Omega)} \leq \nu |\text{rot} \, V|^2_{L^2(\Omega)}.
\]

From the embedding \( H^1 \hookrightarrow L^6 \) and for \( \epsilon_1 = \epsilon_3 = \frac{\nu}{4\epsilon_4} \) we deduce that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |V|^2 + |\Theta|^2 \, dx + \frac{\nu}{2\epsilon_4} |\text{rot} \, V|^2_{L^2(\Omega)} \leq \frac{C \Omega}{\nu} \left( ||\nabla v^1||^2_{L^6(\Omega)} ||V||^2_{L^2(\Omega)} + ||\nabla \omega^2||^2_{L^6(\Omega)} ||\Theta||^2_{L^2(\Omega)} \right).
\]

From the Gronwall inequality we infer that

\[
\sup_{t_0 \leq t \leq t_1} \left( ||V(t)||^2_{L^2(\Omega)} + ||\Theta(t)||^2_{L^2(\Omega)} \right)
\]

\[
\leq c \nu \epsilon \exp \left( \frac{c_\Omega ||\nabla v^1||^2_{L^2(\Omega)} + ||\nabla \omega^2||^2_{L^2(\Omega)}}{2} \right) \left( ||V(t_0)||^2_{L^2(\Omega)} + ||\Theta(t_0)||^2_{L^2(\Omega)} \right),
\]

Since \( H^2(\Omega) \hookrightarrow W^{1,2}_0(\Omega) \hookrightarrow W^{1,2}_0(\Omega) \) we see that

\[
||\nabla v^1||^2_{L^2(\Omega)} \leq c_\Omega ||v^1||^2_{W^{2,1}(\Omega)},
\]

\[
||\nabla \omega^2||^2_{L^2(\Omega)} \leq c_\Omega ||\omega^2||^2_{W^{2,1}(\Omega)},
\]

which justifies that the right-hand side is finite. In our case \( V(t_0) = \Theta(t_0) = 0 \). This implies that \( V(t) = \Theta(t) \equiv 0 \), which proves the uniqueness of solutions. \( \square \)

12. Proof of Theorem 1

\textit{Proof of Theorem 1}. The existence of solutions follows from Lemma 10.3 whereas uniqueness from Lemma 11.1. \( \square \)

13. Final remarks

We emphasize that the proof of the existence of regular solutions is free of constants that depend on time. As it can be regarded as a proof of global in time regular solutions because there are no restriction on \( t \). If we have a solution on \( (0, t) \) we can always extend it to \( (0, t + 1) \). But we cannot simply put \( t_0 = 0, t_1 = \infty \) because we would be confronted with improper integrals with respect to time. Among other things we adopt another approach which will allow us to consider the interval \( (0, \infty) \) but we demonstrate it in forthcoming paper.

The author wishes to express his thanks to Professor Wojciech Zajączkowski for many stimulating conversations during preparation of this work.

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