Concentration Inequalities and Optimal Number of Layers for Stochastic Deep Neural Networks

Michele Caprio¹ and Sayan Mukherjee²,³,⁴,⁵,⁶,⁷
¹PRECISE Center, Department of Computer and Information Science, University of Pennsylvania, Philadelphia, PA 19104, USA
²Center for Scalable Data Analytics and Artificial Intelligence, Universität Leipzig, 04105 Leipzig, Germany
³Max Planck Institute for Mathematics in the Sciences, 04103 Leipzig, Germany
⁴Department of Statistical Science, Duke University, Durham, NC 27708, USA
⁵Department of Mathematics, Duke University, Durham, NC 27708, USA
⁶Department of Computer Science, Duke University, Durham, NC 27708, USA
⁷Department of Biostatistics and Bioinformatics, Duke University, Durham, NC 27708, USA
Corresponding author: Michele Caprio (caprio@seas.upenn.edu)

The work of Michele Caprio was supported in part by the NSF under Grant CCF-1934964, and in part by the Army Research Office (ARO) under Grant MURI W911NF2010080. The work of Sayan Mukherjee was supported in part by the Human Frontier Science Program (HFSP) under Grant RGP005; in part by the NSF under Grant DMS 17-13012, Grant BCS 1552848, Grant DBI 1661386, Grant IIS 15-46331, and Grant DMS 16-13261; in part by the Alexander von Humboldt Foundation; and in part by the German Federal Ministry of Education and Research within the Project Competence Center for Scalable Data Analytics and Artificial Intelligence (ScaDS.AI), Dresden/Leipzig, under Grant BMBF 01IS18026B.

ABSTRACT We state concentration inequalities for the output of the hidden layers of a stochastic deep neural network (SDNN), as well as for the output of the whole SDNN. These results allow us to introduce an expected classifier (EC), and to give probabilistic upper bound for the classification error of the EC. We also state the optimal number of layers for the SDNN via an optimal stopping procedure. We apply our analysis to a stochastic version of a feedforward neural network with ReLU activation function.

INDEX TERMS Stochastic deep neural network, feedforward neural network, ReLU activation, concentration inequality, martingales, optimal stopping, optimal number of layers.

I. INTRODUCTION

Deep neural networks (DNNs) are used extensively in modern statistics and machine learning due to their prediction accuracy; they are applied across many different areas of artificial intelligence, computer vision, speech recognition, and natural language processing [4], [16], [23], [24], [26]. Nonetheless, it is common knowledge that the theoretical understanding of many of their properties is still incomplete (for an account on recent results, see [48, Section 1]).

In contrast to classical neural networks – which learn mappings from a set of inputs to a set of outputs – stochastic neural networks (SNNs) learn mappings from a set of inputs to a set of probability distributions over the set of outputs. The added complexity introduced by the stochasticity exacerbates the study of their theoretical properties. SNNs were introduced by [46] to generalize Hopfield networks [17]. Following [22], let us briefly describe SNNs; they are a particular type of artificial neural networks (ANNs). The goal of ANNs is to represent an arbitrary function \( y = \Phi(x) \). Traditional ANNs are built using one input layer \( v^{(0)}(x) = x \), a sequence of hidden layers \( (v^{(l)}(x))_{l=1}^{L-1} \), and an output layer \( v^{(L)}(x) \), for some \( L \in \mathbb{N} \). In the simplest architecture of feedforward neural networks, each layer is represented by a linear transformation followed by a nonlinear operation \( \sigma^{(l)} \) called activation function, \( v^{(0)}(x) = x \), \( v^{(l)}(x) = \sigma^{(l)}(A^{(l)}v^{(l-1)}(x) + b^{(l)}) \), \( l \in \{1, \ldots, L\} \), \( v^{(L)}(x) = y \). Let \( A = \{A^{(1)}, \ldots, A^{(L)}\} \) and \( b = \{b^{(1)}, \ldots, b^{(L)}\} \). Then, \( \theta = (A, b) \) represents the parameters of the network, where the \( A^{(l)} \)'s are weight matrices, and the \( b^{(l)} \)'s are bias vectors. We call \( \Theta \) the space that \( \theta \) belongs to.

Deep learning is the process of regressing parameters \( \theta \) from a training data \( D \) composed of inputs \( x \) and their corresponding outputs \( y \). Stochastic neural networks are a type of ANN built by introducing stochastic components to the network. This is achieved by giving the network either a stochastic activation or stochastic weights to simulate multiple possible models with their associated probability distribution. This can be summarized as \( \theta \sim p(\theta) \), \( y = \Phi_{\theta}(x) + \varepsilon \), where
\(\Phi\) depends on \(\theta\) to highlight the stochastic nature of the neural network, \(p\) is the density of a probability measure on \(\Theta\) with respect to some \(\sigma\)-finite dominating measure \(\mu\), and \(\varepsilon\) represents random noise to account for the fact that function \(\Phi_\theta\) is just an approximation. A well known example of SNN are Bayesian neural networks (BNNs), that are ANNs trained using a Bayesian approach \([10, \ 12, \ 22, \ 25, \ 42, \ 44, \ 45]\).

Stochastic neural networks have several advantages with respect to their deterministic counterpart: they have greater expressive power as they allow for multi-modal mappings, and the stochasticity can be seen as added regularization \([27]\). In \([22]\), the authors point out that the main goal of using SNNs is to obtain a better idea of the uncertainty associated with the underlying process.

The theoretical features of stochastic deep neural networks (SDNNs) have been the object of study of many recent papers. In \([32]\), the authors study universal approximation properties of deep belief networks, a class of stochastic feed-forward networks that were pivotal in the resurgence of deep learning. A key observation is that, for a finite number of inputs and outputs, the number of maps in the stochastic setting is infinite, unlike in the deterministic framework. This points to the massive increase in the approximation power of SDNNs. In \([5]\), SDNNs are framed as learning measures (LM); deep architectures are introduced to address issues that arise in LM such as permutation invariance or equivariance and variation in weights. Training stochastic feed-forward networks is significantly more challenging. To address this issue, \([40]\) proposed a stochastic feedforward network with hidden layers composed of both deterministic and stochastic variables with a novel generalized EM training procedure that allows the user to efficiently learn complicated conditional distributions.

**A. PREVIOUS WORK**

Concentration inequalities for SDNNs have been studied in the context of PAC-Bayes bounds and stochastic gradient descent (SGD) solutions. In particular, in \([18]\) the authors propose the Kronecker Flow, an invertible transformation-based method that generalizes the Kronecker product to a nonlinear formulation, and uses this construction to tighten PAC-Bayesian bounds. They show that the KL divergence in the PAC-Bayes bound can be estimated with high probability (they give a Hoeffding-type concentration result), and demonstrate the generalization gap can be further reduced and explained by leveraging structure in parameter space. In \([49]\) the authors study the concentration property of SGD solutions. They consider a very rich class of gradient noise – not imposing restrictive requirements such as boundedness or sub-Gaussianity – where only finitely-many moments are required, thus allowing heavy-tailed noises. In the present work, we focus on concentration inequalities (of the Chernoff type) for the output of the hidden layers and of the whole SDNN.

The other focus of our work is finding the number of layers of a generic SDNN that strikes a balance between computational cost and accuracy of the analysis. A similar problem was studied in \([43]\). There, the authors present Binary Stochastic Filtering (BSF), an algorithm for feature selection and neuron pruning in a special version of the classical deep neural network structure where stochastic neurons are mixed with deterministic ones. To the best of our knowledge, the present paper is the first to present an optimal stopping procedure to select the number of layers in a generic SDNN.

**B. CONTRIBUTIONS**

Inspired by \([32]\), this paper studies universal properties of stochastic deep neural networks. Our first goal is to give concentration inequalities for the outputs of the hidden layers of a generic SDNN. This problem has already been studied for particular types of SDNNs in [11] and [34]. In the former, the authors establish a framework for modeling non-causal random fields and prove a Hoeffding-type concentration inequality: it is especially important because it can be applied to the field of Natural Language Processing (NLP). In [34], the authors introduce a new stochastic model of infinite neuronal networks, for which they establish sharp oracle inequalities for Lasso methods and restricted eigenvalue properties for the associated Gram matrix with high probability. Their results hold even if the network is only partially observed; their use of stochastic chains inspired the martingale inequalities inspected in the supplementary material. Our results are very general because they require little mathematical structure: instead of focusing on non-causal random fields or on infinite neuronal networks, we only require that the output \(v^{(l)}(X)\) of the \(l\)-th hidden layer of the SDNN is a random vector. The \(v^{(l)}(X)\)'s can be correlated with each other, and also have different dimensions. The relevance of our findings is given by the lack of specific hypotheses: they apply to any SDNN. We show the following.

**Proposition 1:** Let \(\{v^{(l)}(X)\}_{l=1}^L\) denote the sequence of outputs of the hidden layers of a generic SDNN. If

- either \(v^{(l)}(X)\) is bounded
- or the sequence of centered outputs \(\{v^{(l)}(X) - \mathbb{E}[v^{(l)}(X)]\}_{l=1}^L\) is a smooth weak martingale,

then \(v^{(l)}(X)\) concentrates around its expected value.

Proposition 1 is stated informally. The result is proven assuming that the first bullet point holds in Corollary 1 in section II-A, and assuming that the second bullet point holds in Corollary 2 in appendix A, where we also state conditions that allow us verify the martingale hypothesis. Furthermore, we give an upper bound to the classification error of the expected classifier (EC). This latter is a classifier based on the expected value \(\mathbb{E}[s(v(X))]\) of the score function \(s(v(X))\), a concept introduced in section I-C.

**Proposition 2:** If the output \(s(v(X))\) of the score function that we choose for our analysis is bounded, then the score concentrates around its expected value \(\mathbb{E}[s(v(X))]\).
The classification error obtained using a classifier based on \( \mathbb{E}[s(\nu(X))] \) is bounded.

Proposition 2 is informal as well. It states that the mistake we make when classifying using the EC is not arbitrarily large; it will be smaller the closer \( s(\nu(X)) \) is to its expected value. The first part is proven in Proposition 4, and the second one in Proposition 5, both in section II-B.

We then turn our attention to the number of layers to select. When specifying the structure of a generic DNN \( \nu \), we face a trade-off between accuracy and computational efficiency. One of the main drivers of this trade-off is the number of layers: more layers may yield a more accurate analysis, but the deeper a neural network is, the more computationally intensive it is to train. To solve this problem, in [29] the authors introduce a new type of DNN called a Dynamic Deep Neural Network (D\(^2\)NN) that selects a subset of neurons to execute computations. Given an input, only a subset of D\(^2\)NN neurons are executed, and the particular subset is determined by the D\(^2\)NN itself. In [39], the authors introduce a (deterministic) DNN called Floor-Exponential-Step (FLES) network that only requires three hidden layers to achieve super approximation power. In [37], the author points out that the problem of selecting the correct number of layers has been studied also when stochasticity is featured in the architecture design of the neural network. In [19], for example, the authors study DNNs with stochastic depth: the number of layers is chosen randomly, while in [28] and [47] the authors use a random search algorithm to optimize over the space of neural network architectures. Given a generic SDNN, we adopt an optimal stopping approach to the problem.

Proposition 3: A backward induction approach to an optimal stopping problem can be used to find the number of layers \( \ell \) of a generic SDNN that strikes the perfect balance between accuracy of the analysis and computational cost.

Proposition 3, which is informal and that is built on results from [36, Section 1.2], is proven in Theorem 2 in section III. To the best of our knowledge, this is the first time such a problem is studied via an optimal stopping procedure for generic SDNNs.

Finally, we apply our findings to a stochastic feedforward neural networks with ReLU activation (SFNNRA). We generalize the setup in [48] – that builds a bridge between tropical geometry and deep neural networks – by letting the weight matrix and the bias vector in every layer be stochastic and possibly correlated with the ones in the previous layers. Although the tropical geometry of deep neural networks has been further investigated [1], [31], this is the first time probabilistic results are presented within the framework introduced in [48]. They are extremely significant: in Proposition 6 we show that – because the assumption in the first bullet point of Proposition 1 is easily satisfied in the context of SFNNRAs – the output of every layer concentrates around its expected value. In addition, in Proposition 8 we show that the upper bounds to the number of connected regions with value above and below the decision boundary in the SFNNRA concentrate too around their expected value. In the future, as more general results for SDNNs like the ones presented in this paper are discovered, applying such results to SFNNRAs will improve our understanding of their theoretical properties.

The paper is organized as follows. In section II we provide concentration inequalities for the hidden layers of a SDNN and for its output. In section III we provide results for the optimal number of layers. In section IV we apply our results to a stochastic version of the feedforward neural networks with ReLU activation presented in [48]. Section V is a discussion. In the supplementary material, we give martingale inequalities for the hidden layers of a SDNN and for its output, and we prove our results. In an effort to make the paper self-contained, we also provide background on sub-Gaussian random variables and norm-sub-Gaussian random vectors, martingales and filtrations, and tropical algebra.

C. NOTATION

We introduce the notation for deep neural networks (DNNs) that we use throughout the paper. An \( L \)-layered DNN is a map \( \nu : \mathbb{R}^d \rightarrow \mathbb{R}^p \). We denote the width of the \( l \)-th layer, that is, the number of nodes of the \( l \)-th layer, by \( n_l \), \( l \in \{0, \ldots, L \} \), where \( n_0 = d \) and \( n_L = p \), the dimensions of the input and the output of the network, respectively. The output of the \( l \)-th layer is given by \( \nu^{(l)} : \mathbb{R}^{n_{l-1}} \rightarrow \mathbb{R}^{n_l} \). We assume for convenience that \( \nu^{(0)}(x) \equiv x \). The final output \( \nu(x) \equiv \nu^{(L)}(x) \) of the neural network is fed into a score function \( s : \mathbb{R}^p \rightarrow \mathbb{R}^m \) that is application specific. We call \( \mathcal{X} \) the space of inputs and \( \mathcal{Y} \) the space of responses. We assume we collect data \((X, Y) \sim \mathcal{D}, \) a distribution on \( \mathcal{X} \times \mathcal{Y} \). In a SDNN, \((\nu^{(l)}(X))_{l=0}^L \) is a sequence of random vectors having possibly different dimensions and being possibly correlated. Notice that \( \nu^{(0)}(x) \) is the realization of random vector \( \nu^{(0)}(X) : \Omega \rightarrow \mathbb{R}^{n_0} \) whose elements may be correlated. Throughout this paper, we assume that \( \nu^{(l)}(X) \) has a finite first moment, for all \( l \in \{1, \ldots, L \} \).

II. CONCENTRATION INEQUALITIES

In this section we derive concentration inequalities for the hidden layers of a generic SDNN. Because we require the SDNN to have little mathematical structure, the results in this section are applications of standard tools in high-dimensional probability.

A. NORM-SUB-GAUSSIAN-BASED CONCENTRATION INEQUALITIES

The properties of norm-sub-Gaussian random vectors and sub-Gaussian random variables needed in this section are explored in appendix C. Let \( \| \cdot \| \) denote the Euclidean norm; the following is the first result.
Theorem 1: Suppose that \( v^{(1)}(X) \) is bounded so that 
\[ \|v^{(1)}(X)\|_2 \leq \xi_1, \] for some \( \xi_1 \in \mathbb{R} \). Then, for all \( t \in \mathbb{R} \),
\[
P_{(X,Y)\sim D}(\|v^{(1)}(X) - E_{(X,Y)\sim D}[v^{(1)}(X)]\|_2 \geq t) \leq 2 \exp \left(-\frac{t^2}{2\xi_1^2}\right). \tag{1}
\]

Theorem 1 tells us that the output \( v^{(1)}(X) \) of the first layer of our SDNN concentrates around its expected value \( E_{(X,Y)\sim D}[v^{(1)}(X)] \). The assumption that \( v^{(1)}(X) \) is bounded is mild: it can be interpreted as a safety check to ensure that output \( v^{(1)}(X) \) does not take on values that are too extreme.

We can use Theorem 1 to find a concentration inequality for the second layer of our neural network. In particular, if \( \|v^{(2)}(X)\|_2 \leq \xi_2 \), for some \( \xi_2 \in \mathbb{R} \) possibly different from \( \xi_1 \), we have that, for all \( t \in \mathbb{R} \),
\[
P_{(X,Y)\sim D}(\|v^{(2)}(X) - E_{(X,Y)\sim D}[v^{(2)}(X)]\|_2 \geq t) \leq 2 \exp \left(-\frac{t^2}{2\xi_2^2}\right). \tag{2}
\]
This tells us that the output \( v^{(2)}(X) \) of the second layer of our SDNN concentrates around its expected value \( E_{(X,Y)\sim D}[v^{(2)}(X)] \).

The following corollary tells us that the result in Theorem 1 holds for every layer of our SDNN.

Corollary 1: Pick any \( l \in \{1, \ldots, L\} \). Suppose that \( v^{(l)}(X) \) is bounded so that \( \|v^{(l)}(X)\|_2 \leq \xi_l \), for some \( \xi_l \in \mathbb{R} \), such that \( \xi_l \geq \xi_{l-1} \) for all \( l \), \( l \geq 1 \). Then, for all \( t \in \mathbb{R} \),
\[
P_{(X,Y)\sim D}(\|v^{(l)}(X) - E_{(X,Y)\sim D}[v^{(l)}(X)]\|_2 \geq t) \leq 2 \exp \left(-\frac{t^2}{2\xi_l^2}\right). \tag{3}
\]

As a result of Corollary 1, we have that the output \( v(X) = v^{(L)}(X) \) of our SDNN concentrates around its expected value \( E_{(X,Y)\sim D}[v(X)] = E_{(X,Y)\sim D}[v^{(L)}(X)] \). This proves Proposition 1 assuming the first bullet point holds. We defer to future work finding a method to explicitly compute parameter \( \xi_l \) that drives our concentration result.

B. CLASSIFICATION ACCURACY

In this section, we show that a classifier based on the expected value of score function \( s \) does not perform much worse than the stochastic neural network classifier. We focus on binary classification for the sake of exposition. For a discussion on multi-class classification, see e.g. [6, Chapters 4 and 5, especially section 5.7.3]. A DNN \( v : \mathbb{R}^d \to \mathbb{R}^p \), together with a choice of score function \( s : \mathbb{R}^p \to \mathbb{R} \), gives us a classifier. If the score function \( s \) computed at the output value \( v(x) \), \( s(v(x)) \), exceeds some decision threshold \( c \), then the neural network predicts \( x \) is from a certain class \( \mathcal{C}_1 \), otherwise \( x \) is from the other category \( \mathcal{C}_2 \). The input space is then partitioned into two disjoint subsets by the decision boundary \( B := \{ x \in \mathbb{R}^d : v(x) = s^{-1}(c) \} \). We call connected regions with value above the threshold positive regions, while those having value below the threshold negative regions.

For the sake of exposition suppose that \( n_L = 1 \), that is, \( v : \mathbb{R}^d \to \mathbb{R} \). Then, in our stochastic setting, we have that \( s(v(x)) \) is a random variable that maps into \( \mathbb{R} \); we assume it has finite first moment. We have the following important result.

Proposition 4: If \( a \leq s(v(x)) \leq b \) with probability \( 1 \), for some \( a, b \in \mathbb{R} \), \( a < b \), then for all \( t > 0 \),
\[
P_{(X,Y)\sim D}(\|s(v(X)) - E_{(X,Y)\sim D}[s(v(X))]\| \geq t) \leq 2 \exp \left(-\frac{2t^2}{(b-a)^2}\right).
\]

Proposition 4 tells us that \( s(v(X)) \) concentrates around its expected value. The fact that \( s(v(X)) \) is bounded is a mild condition; it simply amounts to the choice of the score function. When used as an \( m \)-category classifier, \( s \) may be chosen for example to be a soft-max or sigmoidal function (this was pointed out in [48]); these functions satisfy our assumption.

We now consider the expected classifier based on the expected decision boundary (EDB)
\[
B_{exp} := \{ x \in \mathbb{R}^d : E_{(X,Y)\sim D}[s(v(X))] = c \},
\]
for some decision threshold \( c \). That is,
(i) if \( E_{(X,Y)\sim D}[s(v(X))] > c \), then \( x \in \mathcal{C}_1 \);
(ii) if \( E_{(X,Y)\sim D}[s(v(X))] < c \), then \( x \in \mathcal{C}_2 \).

Note that being based on the value \( E_{(X,Y)\sim D}[s(v(X))] \), the expected classifier will most likely not be perfect. We can provide a probabilistic bound for the classification error of the expected classifier.

Proposition 5: If \( E_{(X,Y)\sim D}[s(v(X))] > c \), then there exists \( t_1 > 0 \) such that
\[
P_{(X,Y)\sim D}(s(v(X)) \leq c) \leq \exp \left(-\frac{2t_1^2}{(b-a)^2}\right). \tag{4}
\]
If instead \( E_{(X,Y)\sim D}[s(v(X))] < c \), then there exists \( t_2 > 0 \), possibly different from \( t_1 \), such that
\[
P_{(X,Y)\sim D}(s(v(X)) \geq c) \leq \exp \left(-\frac{2t_2^2}{(b-a)^2}\right). \tag{5}
\]

Call \( p_1 \) the bound in (4) and \( p_2 \) the one in (5). Then, Proposition 5 tells us that the expected classifier is correct (that is, its prediction coincides with the stochastic neural network classifier’s prediction) with probability \( p > 1 - \max\{p_1, p_2\} \). Together with Proposition 4, it proves Proposition 2.

III. OPTIMAL NUMBER OF LAYERS

We find the number of layers for a generic SDNN that strikes the perfect balance between accuracy of the analysis and computational cost. We do so using an optimal stopping technique.\footnote{We implicitly assume that all the expected and conditional expected values computed in this section are well defined.}

Let \( v = (v^{(L)}(X))_{l=1}^{L} \) denote the sequence of outputs of SDNNs having \( L \) many layers, \( L \in \{1, \ldots, \mathcal{L}\} \), where \( \mathcal{L} \in \mathbb{N} \) is an arbitrarily large number of layers we can select for our analysis. It induces a stochastic process (that is, a sequence of
random elements) \( y = (y^{(L)})_{L=1}^{\mathcal{L}}, \) where for every \( L, \) \( y^{(L)} \) is a real-valued random variable, \( y^{(L)} : \Omega \to \mathbb{R}. \) An example of function \( \gamma^{(\cdot)} \) is given in a few lines. We interpret \( y^{(L)} \) as the utility of choosing \( L \) many layers for the neural network, that is, of stopping the observation of \( y \) at index \( L. \) For example, consider the following situation. Assume that, for all \( L_1, L_2 \in \{1, \ldots, \mathcal{L}\}, \)

\[ L_1 \leq L_2 \implies \|v^{(L_1)}(X) - Y^*\|_2 \geq \|v^{(L_2)}(X) - Y^*\|_2. \]  

(6)

where \( Y^* \) is the correct response to input \( X. \) The assumption in equation (6) means that the deeper the neural network is – that is, the more hidden layers we select, captured by a larger \( L \in \{1, \ldots, \mathcal{L}\} \) – the closer its output is to the truth. This assumption reflects the “accuracy of the analysis” side of the problem.

Now, write the loss function used in the study as a function of the number of layers. So for example the mean squared error (MSE) loss is written as

\[ \text{LOSS}(L) = \frac{1}{p} \|v^{(L)}(X) - Y^*\|_2^2. \]  

(7)

Suppose then that for our analysis we choose a loss function that is positive and monotonically decreasing in \( L. \) The importance of the monotonicity assumption is discussed in a few lines. Given our assumption in (6), the MSE loss in (7) is a sensible choice. To be precise, in this example we are using the training loss, so the procedure described finds the optimal number of layers for the training network.

Let \( g \) be a generic functional on \( \mathbb{R} \) such that \( g \circ \text{LOSS} \) is a positive monotonically increasing function on \( L. \) In our example, we can let \( x \mapsto g(x) := 1/x; \) in view of our assumption (6), \( g \circ \text{LOSS} = 1/\text{LOSS} \) represents the utility gained by adding an extra layer to our neural network. To see this, notice that by (6) and (7), we have that for all \( L_1, L_2 \in \{1, \ldots, \mathcal{L}\}, L_1 \leq L_2 \implies 1/[\text{LOSS}(L_1)] \leq 1/[\text{LOSS}(L_2)]. \)

Finally, consider a generic functional \( h \) on \( \{1, \ldots, \mathcal{L}\} \) that is positive and monotonically decreasing in \( L. \) In our example, we can take \( h(L) := \frac{1}{\sqrt{L}}, \) for some \( c > 0. \)

Then, for all \( L \in \{1, \ldots, \mathcal{L}\}, y^{(L)} = (g \circ \text{LOSS})(L) \cdot h(L). \) In our example, then, \( y^{(L)} = 1/[\text{LOSS}(L)] \cdot c\sqrt{L}. \) This is a reasonable choice for \( y^{(L)} \) because the utility gain is larger as \( L \) increases, but at the same time values of \( L \) that are too high are penalized. The optimal number of layers gives us the balance between cost and accuracy that we are looking for.

In this example, for every \( L \in \{1, \ldots, \mathcal{L}\}, y^{(L)} \) is the product of two positive monotone functions in \( L \) defined on the finite domain \( \{1, \ldots, \mathcal{L}\}. \) This means that \( y^{(L)} \) itself is positive. Because \( g \circ \text{LOSS} \) is increasing and \( h \) is decreasing, \( y^{(\cdot)} \) is not guaranteed to be monotonic. In an applied setting, the scholar should choose \( g \) and \( h \) so that \( y^{(\cdot)} \) increases monotonically for some elements of the domain (locally monotonically increasing) and decreases monotonically for some other elements of the domain (locally monotonically decreasing). This is important because if \( y^{(\cdot)} \) were not to be locally monotonic, then all possible values of the number of layers would have to be explored to find the optimum. As a simple example, let \( (g \circ \text{LOSS})(L) \approx \log L, h(L) \approx 1/\sqrt{L}, \) and \( y^{(L)} = (g \circ \text{LOSS})(L) \cdot h(L), \) for all \( L. \) In this case, \( y^{(\cdot)} \) increases monotonically until \( L \approx 7, \) and then decreases monotonically. The choice of functions \( g \) and \( h \) gives the scholar control on the trade-off: if more accuracy is required, then \( g \) will be chosen so that \( g \circ \text{LOSS} \) increases faster than \( h \) decreases, and vice versa if the computational aspect is more important.

Then, we consider the natural filtration \( \mathcal{F}^y \) of \( \mathcal{F} \) with respect to \( y \) given by

\[ \mathcal{F}^y_L := \sigma \left( \left( (y^{(s)})^{-1}(A) : s \in \mathbb{N}, s \leq L, A \in \mathcal{B}(\mathbb{R}) \right) \right), \]

where \( \mathcal{B}(\mathbb{R}) \) denotes the Borel \( \sigma \)-algebra on \( \mathbb{R}. \) We assume that \( y \) is adapted to filtration \( \mathcal{F}^y \), that is, \( y^{(L)} \) is \( \mathcal{F}^y_L \)-measurable, for all \( L. \) We interpret \( \mathcal{F}^y_L \) as the information encoded in an \( L \)-layered SDNN. Our decision regarding whether to choose \( L \) many layers – that is, to stop observing \( y \) at index \( L – \) must be based on this information only (no anticipation is allowed).

Definition 1: Let \((\Omega, \mathcal{F})\) be the measurable space of interest. Given a generic filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_L)_{L \in \mathbb{N}}, P), 3\) a random variable \( \tau : \Omega \to \mathbb{N} \cup \{\infty\} \) is called a Markov time if for all \( \tau \leq L \in \mathcal{F}_L, \) for all \( L \in \mathbb{N}. \) A Markov time \( \tau \) is called a stopping time if \( \tau < \infty \) \( P \)-a.s.

We denote the family of all stopping times by \( \mathcal{M}, \) and the family of all Markov times by \( \mathcal{M}. \) A family that we are going to use later in this section is \( \mathcal{M}_L := \{ \tau \in \mathcal{M} : L \leq \tau \leq \mathcal{L} \}, 1 \leq L \leq \mathcal{L}. \)

The optimal stopping problem we study is the following

\[ V_* = \sup_{\tau \in \mathcal{M}_L} E_{(X,Y) \sim \mathcal{G}} \left[ y^{(\tau)} \right] \]  

(8)

where the supremum is taken over a family of stopping times. Note that (8) involves two tasks, that is computing the value function \( V_* \) as explicitly as possible, and finding an optimal stopping time \( \tau_* \) at which the supremum is attained.

To ensure that the expected value in (8) exists, we need a further assumption, that is

\[ E_{(X,Y) \sim \mathcal{G}} \left[ \sup_{L \leq \mathcal{L}} \left| y^{(k)} \right| \right] < \infty. \]  

(9)

If (9) is satisfied, then \( E_{(X,Y) \sim \mathcal{G}}[y^{(\tau_*)}] \) is well defined for all \( \tau \in \mathcal{M}_L. \)

To each of the families \( \mathcal{M}_L \) we assign the following value function

\[ V_L^{\mathcal{L}} := \sup_{\tau \in \mathcal{M}_L} E_{(X,Y) \sim \mathcal{G}} \left[ y^{(\tau)} \right], \]  

(10)

where the suprema are taken over all \( \tau \) \( \leq \mathcal{L}. \)

We solve problem (10) using the backward induction approach outlined in [36, Section 1.2]. We assume \( \mathcal{L} < \infty, 2\)

\[ \text{Algorithm 1: Backward Induction Algorithm} \]

\[ \text{Input: } y, \quad \text{Output: } V_L^{\mathcal{L}} \]

1. For every \( L \in \{1, \ldots, \mathcal{L}\} \), do
   a. \( V_L^{\mathcal{L}} := \sup_{\tau \in \mathcal{M}_L} E_{(X,Y) \sim \mathcal{G}} \left[ y^{(\tau)} \right] \)

2. End for

3. Return \( V_L^{\mathcal{L}} \)

4. Note: As \( L \) increases, the set \( \mathcal{M}_L \) decreases, and the optimal stopping time \( \tau_* \) may decrease as well.
but this is without loss of generality. Notice that (10) can be rewritten as
\[ V^*_L = \sup_{L \leq L' \leq L} E_{(X,Y) \sim \mathcal{D}} \left[ y^{(\tau)} \right], \]  
where \( \tau \) is a stopping time and \( 1 \leq L \leq L' \). We solve the problem by letting time go backwards; we proceed recursively.\(^4\) Let \( L \) be a large number in \( \mathbb{N} \), e.g. 1000. It is going to represent the maximum number of layers we deem “usable” for our neural network. We consider an ancillary sequence of random variables \( (S^*_L)_{1 \leq L \leq L'} \) induced by \( y \) that is built as follows. For \( L = L' \) we stop, and our utility is \( S^*_L = y^{(\tau)} \), for \( L = L' - 1 \), we can either stop or continue. If we stop, our utility is \( S^*_{L-1} = y^{(\tau'-1)} \), while if we continue our utility is \( S^*_{L-1} = E_{(X,Y) \sim \mathcal{D}} \left[ S^*_L | \mathcal{F}^*_{L-1} \right] \). As it is clear from the latter conclusion, our decision about stopping at layer \( L = L' - 1 \) or continuing with an extra layer must be based on the information contained in \( \mathcal{F}^*_{L-1} \) only. So, if \( y^{(\tau'-1)} \geq E_{(X,Y) \sim \mathcal{D}} \left[ S^*_L | \mathcal{F}^*_{L-1} \right] \), we stop at layer \( L' - 1 \), otherwise we add an extra layer. For \( L \in \{L - 2, \ldots, 1\} \) the considerations are continued analogously.

By the backward induction method we just described, we have that the elements of the sequence \( (S^*_L)_{1 \leq L \leq L'} \) are defined recursively as
\[ S^*_L = y^{(\tau)}, \quad \text{for } L = L', \]
\[ S^*_L = \max \left\{ y^{(\tau)}, E_{(X,Y) \sim \mathcal{D}} \left[ S^*_{L+1} | \mathcal{F}^*_L \right] \right\}, \]
for \( L \in \{L - 1, \ldots, 1\} \). The method also suggests that we consider the following stopping time
\[ \tau^*_L = \inf_{1 \leq k \leq L} \left\{ \left( S^*_k = y^{(k)} \right) \right\}, \]
for all \( 1 \leq L \leq L' \). Notice that the infimum is always attained. The following is the main result of this section; it tells us that \( \tau^*_L \) is indeed the optimal stopping time for problem (11). It comes from [36, Theorem 1.2].

**Theorem 2**: Consider the optimal stopping problem (11), and assume that (9) holds. Then, for all \( L \in \{1, \ldots, L'\} \), we have that
\[ S^*_L \geq E_{(X,Y) \sim \mathcal{D}} \left[ y^{(\tau)} \right] | \mathcal{F}^*_L, \]  
for all \( \tau \in \mathcal{M}^*_L \), \[ S^*_L = E_{(X,Y) \sim \mathcal{D}} \left[ y^{(\tau')} \right], \]  
for all \( \tau' \in \mathcal{M}^*_L \).

In addition, fix any \( L \in \{1, \ldots, L'\} \). Then,
(i) the stopping time \( \tau^*_L \) is optimal for (11);
(ii) if \( \tau^*_L \) is any optimal stopping time for (11), then \( \tau^*_L \leq \tau^*_L \).

It follows immediately from Theorem 2.(i) that the possible number of layers for our neural network \( \nu \) is given by \( \tau^*_L \). This proves Proposition 3. In light of (14), a procedure to find \( \tau^*_L \) could be the following. Without loss of generality, assume that \( \{1, \ldots, L'\} \) is divisible into \( \mathcal{L} \) equidimensional batches of length \( l \). Then, starting from the first batch, for each of its elements compute \( S^*_L \) and \( y^{(k)} \). As soon as they coincide, we obtain the desired \( \tau^*_L \). Given the results in [39], we conjecture that in general the optimal number of hidden layers will be low. Notice that, although this task can be parallelized, it is a greedy – and possibly computationally expensive – algorithm. We defer finding the optimal way of implementing the procedure elicited in this section to future work.

**IV. APPLICATION: STOCHASTIC FEEDFORWARD NEURAL NETWORKS WITH ReLU ACTIVATIONS**

In this section we apply our results for a general SDNN to a stochastic version of the feedforward neural network with ReLU activation (FNNRA) introduced in [48]. The key insight in [48] is that ideas from tropical algebra can be used to study feedforward neural networks, especially with ReLU activations. The intuition is that the activation function of a feedforward neural network requires computing a maximum, which turns out to correspond to tropical addition.

We first present the notation we use for feedforward neural networks. A short summary of ideas from tropical algebra is given in appendix E.

**A. DETERMINISTIC FEEDFORWARD NEURAL NETWORKS**

We use this section to fix the notation for feedforward neural networks. We then introduce the FNNRA proposed in [48]. We restrict our attention to fully connected feedforward neural networks.

An \( L \)-layered feedforward neural network is a map \( \nu : \mathbb{R}^d \rightarrow \mathbb{R}^p \) given by a composition of functions
\[ \nu = \nu^{(L)} \circ \sigma^{(L)} \circ \cdots \circ \sigma^{(1)} \circ \rho^{(1)}. \]

The preactivation functions \( \rho^{(1)}, \ldots, \rho^{(L)} \) are affine transformations to be determined, and the activation functions \( \sigma^{(1)}, \ldots, \sigma^{(L)} \) are chosen and fixed in advance. Affine function \( \rho^{(l)} : \mathbb{R}^{n_{l-1}} \rightarrow \mathbb{R}^{n_l} \) is given by a weight matrix \( A^{(l)} \in \mathbb{R}^{n_l \times n_{l-1}} \) and a bias vector \( b^{(l)} \in \mathbb{R}^{n_l} \),
\[ \rho^{(l)}(v^{(l-1)}) := A^{(l)}v^{(l-1)} + b^{(l)}. \]

The \( (i,j) \)-th coordinate of \( A^{(l)} \) is denoted by \( a_{ij}^{(l)} \), and the \( i \)-th coordinate of \( b^{(l)} \) by \( b_i^{(l)} \). Collectively they form the parameters of the \( l \)-th layer. Notice that, for all \( l, A^{(l)} \) can be decomposed as a difference of two nonnegative integer valued matrices, \( A^{(l)} = A^{+}_l - A^{-}_l \), with \( A^{+}_l, A^{-}_l \in \mathbb{Z}^{n_l \times n_{l-1}} \), so that their entries are
\[ a_{ij}^{(l)} = \max(a_{ij}, 0), \quad a_{ij}^{(l)} = \max(-a_{ij}, 0). \]

For a vector input \( x \in \mathbb{R}^{n_l} \), \( \sigma^{(l)}(x) \) is understood to be in coordinatewise sense. We make the following assumptions on the architecture of our feedforward neural network:

(a) the weight matrices \( A^{(1)}, \ldots, A^{(L)} \) are integer-valued;
(b) the bias vectors \( b^{(1)}, \ldots, b^{(L)} \) are real-valued;
(c) the activation functions $\sigma^{(1)}, \ldots, \sigma^{(L)}$ take the form

$$\sigma^{(l)}(x) := \max \{x, t^{(l)}\} = (\max \{x_1, t^{(l)}_1\}, \ldots, \max \{x_n, t^{(l)}\}_n)^T,$$

where $t^{(l)} \in \mathbb{R} \cup \{-\infty\}$ is the threshold vector, and $x_j$ and $t^{(l)}_j$ denote the $j$-th element of $x$ and $t^{(l)}$, respectively.

We assume all the neural networks in this section to satisfy (a)-(c).

As pointed out in [48, Section 4], assumption (b) is general, and yields no loss of generality. The same goes for (a), since:

- real weights can be approximated arbitrarily closely by rational weights;
- one may “clear denominators” in these rational weights by multiplying them by the least common multiple of their denominators to obtain integer weights;
- scaling all weights and biases by the same positive constant does not influence the workings of a neural network.

The form of the activation function in (c) includes both ReLU activation ($t^{(l)} = 0I_{n_l}$) and identity map ($t^{(l)} = -\infty I_{n_l}$), so that $\sigma^{(l)}(x) = x$ as special cases, where $I_{n_l}$ is vector $(1, \ldots, 1)^T$ having $n_l$ entries. We only consider the ReLU activation function in this section as we are generalizing the model in [48] where the tropical algebra makes the most sense for ReLU activations. In addition, much of the theory literature on neural networks has focused on ReLU networks [3], [33], [48].

Let us now describe the deterministic ReLU neural network proposed in [48]; its architecture is depicted in appendix B. Define tropical sum and tropical multiplication as $a \oplus b := \max\{a, b\}$ and $a \odot b := a + b$, for all $a, b \in \mathbb{R} \cup \{-\infty\}$, respectively. In addition, tropical exponentiation is given by

$$a^\odot b := \begin{cases} a \cdot b & \text{if } b \in \mathbb{Z}_+ \\ (-a) \cdot (-b) & \text{if } b \in \mathbb{Z}_- \end{cases},$$

for all $a \in \mathbb{R} \cup \{-\infty\}$ and all $b \in \mathbb{Z}$. A more in-depth discussion of tropical algebra is deferred to appendix E. In [48, Section 5], the authors state that the $(l + 1)$-th layer $v^{(l+1)}(x)$ of an $L$-layered FNRNA $v(x)$ can be written as $v^{(l+1)}(x) = F^{(l+1)}(x) - G^{(l+1)}(x)$, for all $x \in \mathbb{R}^d$, where

$$F^{(l+1)}(x) = H^{(l+1)}(x) + G^{(l+1)}(x) \circ t^{(l+1)} = \max[H^{(l+1)}(x), G^{(l+1)}(x) + t^{(l+1)}],$$

$$G^{(l+1)}(x) = A^{(l+1)}_+ G^{(l)}(x) \circ A^{(l+1)} F^{(l)}(x) = A^{(l+1)}_+ G^{(l)}(x) + A^{(l+1)} F^{(l)}(x),$$

$$H^{(l+1)}(x) = A^{(l+1)}_- F^{(l)}(x) \circ A^{(l+1)} G^{(l)}(x) \circ b^{(l+1)} = A^{(l+1)}_+ F^{(l)}(x) + A^{(l+1)}_- G^{(l)}(x) + b^{(l+1)},$$

where $F^{(l)}(x)$ and $G^{(l)}(x)$ are vectors in $\mathbb{R}^{n_l}$ whose coordinates are tropical polynomials in $x$.

That is,

$$F^{(l)}(x) = \left( f^{(l)}_1(x), \ldots, f^{(l)}_{n_l}(x) \right)^T,$$

$$G^{(l)}(x) = \left( g^{(l)}_1(x), \ldots, g^{(l)}_{n_l}(x) \right)^T,$$

where

$$f^{(l)}_j(x) = \max\{c_{j1} x^{\odot a_1} + \ldots, c_{j_{n_l}} x^{\odot a_{n_l}} \},$$

$$g^{(l)}_j(x) = \max\{c_{j1} x^{\odot a_1} + \ldots, c_{j_{n_l}} x^{\odot a_{n_l}} \},$$

for all $j \in \{1, \ldots, n_l\}$ and some $r \in \mathbb{N}$, and

$$c_{j_{n_l}} x^{\odot a_{n_l}} := c_{j_{n_l}} + \alpha x_{n_l + 1} + \alpha x_{n_l + 2} + \ldots + \alpha x_{n_l + d},$$

for all $j \in \{1, \ldots, n_l\}$ and all $s \in \{1, \ldots, r\}$. Similarly,

$$c_{j_{n_l}} x^{\odot a_{n_l}} := c_{j_{n_l}} + \alpha x_{n_l + 1} + \alpha x_{n_l + 2} + \ldots + \alpha x_{n_l + d},$$

for all $j \in \{1, \ldots, n_l\}$ and all $s \in \{1, \ldots, r\}$. Notice that $c_{j_{n_l}}', c_{j_{n_l}}' \in \mathbb{R}$ for all $j$ and all $s$, and that $a_{j_{n_l}} = (\alpha x_{n_l + 1}, \ldots, \alpha x_{n_l + d})^T, a_{j_{n_l}}' = (\alpha x_{n_l + 1}', \ldots, \alpha x_{n_l + d}')^T \in \mathbb{N}^{d}$, for all $j$ and all $s$.

**B. STOCHASTIC FEEDFORWARD RELU NETWORKS**

In this section we derive a stochastic version of the FNNRA described in section IV-A. We introduce two sources of stochasticity: (1) the initialization or the starting values of the neural network are random (that is, $F^{(0)}(X)$ and $G^{(0)}(X)$ are random vectors), and (2) the parameters of all the layers are aleatory (that is, $A^{(l)}$ is a random matrix with integer entries, and $b^{(l)}$ is a random vector with real entries, for all $l \in \{1, \ldots, L\}$). We now state the update rules for the first layer, and the second layer given the first. By induction, this is enough to specify the stochastic feedforward network.

For the first layer we sample random vectors $F^{(0)}(X)$ and $G^{(0)}(X)$ from a distribution $\mathcal{R}$ on $\mathbb{R}^{d_0}$ that is built as follows. Sample

- $(c_{j1}, \ldots, c_{jp})^T, (c_{j1}', \ldots, c_{jp}')^T \sim \mathcal{S}_j$ on $\mathbb{R}^r$,
- $\alpha_{j1}, \ldots, \alpha_{jr}, \alpha_{j1}', \ldots, \alpha_{jr}' \sim \mathcal{S}_j$ on $\mathbb{R}^d$, $j \in \{1, \ldots, n_0\}$. Note that we do not require the $\alpha_j$’s and the $\alpha_j'$’s to be iid and we do not require the samples to be independent across the index $j$.

Then, $F^{(0)}(X)$ and $G^{(0)}(X)$ – the realizations of $F^{(0)}(X)$ and $G^{(0)}(X)$, respectively – are computed according to equations (18) and (19). Notice that $F^{(0)}(X)$ and $G^{(0)}(X)$ need not be independent.

To compute the subsequent layer we first sample $A^{(l+1)} \sim \mathcal{R}$ on $\mathbb{Z}^{n_{l+1} \times n_l}$ and $b^{(l+1)} \sim \mathcal{S}_l$ on $\mathbb{R}^{n_{l+1}}$, for $l \in \{0, \ldots, L - 5\}$. Notice that we can write the $i$-th entry of vector $A^{(l+1)}(x)$ in tropical notation as

$$\bigcirc_{j \in \{1 \ldots n_l\}} \left[ g^{(l)}_j(x) \odot a^{(l+1)}_{i,j} \right].$$

We can write similarly the $i$-th entries of $A^{(l+1)}_+(x), A^{(l+1)}_-(x), A^{(l+1)}_+ F^{(l)}(x), and A^{(l+1)}_+ F^{(l)}(x).$
1. After computing $F^{(l+1)}(x)$ and $G^{(l+1)}(x)$ from $F^{(l)}(x)$ and $G^{(l)}(x)$ by equation (17), we find $v^{(l+1)}(x) = F^{(l)}(x) - G^{(l)}(x)$. Note that $v^{(l+1)}(x)$ is the realization of random vector $v^{(l+1)}(X) : \Omega \rightarrow \mathbb{R}^{n_{l+1}}$ whose elements may be correlated and that we assume to have finite first moment. We now apply Corollary 1 to our stochastic FNNRA.

Proposition 6: If $\mathcal{P}, 2, \mathcal{J}_j$, and $\mathcal{J}_j, j \in \{1, \ldots, n_0\}$ are bounded, then for all $l \in \{1, \ldots, L\}$, there exists $\xi(l) \in \mathbb{R}$ such that $\|v^{(l)}(X)\| \leq \xi(l)$ and (3) holds.

The assumption that $\mathcal{P}, 2, \mathcal{J}_j$, and $\mathcal{J}_j, j \in \{1, \ldots, n_0\}$, are bounded should always be verified: even if the distributions we want to use are unbounded, we can always truncate them and use the truncated versions to obtain the concentration result in Proposition 6.

Remark 1: Notice that if $t^{(l)}$ were a random quantity distributed according to $\mathcal{Y}$ on $\mathbb{R}^{n_l}$, and possibly correlated with other $t^{(k)}$’s, Proposition 6 would still hold, provided that $\mathcal{Y}$ is bounded.\(^6\)

C. CONCENTRATION INEQUALITY FOR POSITIVE AND NEGATIVE REGIONS

Tropical rational functions (TRFs) are introduced formally in Appendix E. They are a (standard) difference between tropical polynomials; these latter were introduced in equation (19). TRFs are piecewise linear, so the notion of linear regions applies. A linear region of a TRF $f$ is a maximal connected subset of the domain on which $f$ is linear;\(^8\) the number of linear regions of $f$ is denoted by $N(f)$.

In [48, Corollary 5.3, Theorem 5.4, and Proposition 5.5], the authors show the equivalence of tropical rational functions, continuous piecewise linear functions with integer coefficients, and neural networks satisfying assumptions (a)-(c). Hence, they are able to link the number of linear regions of a tropical rational function to (bounds on) the number of positive and negative regions that the neural network divides the input space into.\(^9\)

Proposition 7: [48, Proposition 6.1] Let $v$ be an L-layer stochastic neural network satisfying (a)-(c) in section IV-A such that $t^{(l)} = -\infty$ and $p = n_l = 1$, that is, $v : \mathbb{R}^d \rightarrow \mathbb{R}$. Let the score function $s : \mathbb{R} \rightarrow \mathbb{R}$ be injective with decision threshold $c$ in its range. If $v = f \otimes g$, where $f$ and $g$ are tropical polynomials, then the number of connected positive regions is at most $N(f)$, while the number of connected negative regions is at most $N(g)$.

Suppose now all the assumptions of Proposition 7 hold. In our stochastic setting, we have that $f$ and $g$ are both random variables, so $N(f)$ and $N(g)$ are two stochastic quantities.

\(^6\)Notice that distributions $\mathcal{P}$ and $\mathcal{P}$ need not be the same for all $l$; if they change, we denote them by $\mathcal{P}_l$ and $\mathcal{P}_l$, for all $l$, respectively.

\(^7\)If $\mathcal{P} = \mathcal{Y}$ on $\mathbb{R}^{n_l}$, this means that we rule out the identity activation map. To include it, $\mathcal{Y}$ should be a distribution on $(\mathbb{R} \cup (-\infty, \infty))^{n_l}$.

\(^8\)Call $\mathcal{X} \subset X$ the domain on which $f$ is linear, and endow it with topology $t$. Set $A \subset \mathcal{X}$ is connected if it cannot be represented as the union of two or more disjoint non-empty open subsets. Set $A$ is maximal connected if no connected subset of $\mathcal{X}$ properly contains it.

\(^9\)Recall that positive and negative regions were introduced in section II-B.

V. CONCLUSION

In this paper we present concentration inequalities for the hidden layers and the output of a generic SDNN based on the idea of norm-sub-Gaussian distribution. We introduce the notion of expected classifier and give a probabilistic bound to its classification error. We also find – via an optimal stopping procedure – the number of layers for the SDNN that strikes the perfect balance between computational cost and accuracy of the analysis. Finally, we apply our findings to a stochastic version of the FNNRA of [48]. In future work, we plan to explicitly compute the value of parameter $\xi(l)$ in Corollary 1 that drives the concentration of layer $v^{(l)}(X)$ around its expectation, to find the optimal way of implementing the optimal stopping procedure elicted in section III, and to explore the geometric properties of SDNNs, in the spirit of [1], [14], and [31].

APPENDIX A MARTINGALE INEQUALITIES

Let us assume that the sequence $v = (v^{(l)}(X))_{l=0}^L$ of outputs of the hidden layers of the SDNN defined in section II of the main portion of the paper is a martingale; this allows us to state some interesting properties of SDNNs. Note that if $v$ is any kind of martingale (very-weak, weak, or strong, see appendix D), then we need to assume that the conditional expectation of layer $l$, given one or more other layers, is well defined (again, see appendix D). A way of guaranteeing that this assumption holds is to require that our SDNN is “rectangular”, that is, the width of the layers of the SDNN is constant throughout the neural network and equal to $p$. Neural networks with constant width are a well studied subject [41].

We now show how a martingale hypothesis allows us to weaken the already mild assumption in Theorem 1 and Corollary 1 in the main portion of the paper. The price to pay is that the martingale inequalities become looser as the number in \(N\). Assume their first moment is finite and notice that they are bounded below by 1. Then, the following proposition holds.

Proposition 8: If $N(f) \leq b_1$ and $N(g) \leq b_2$ with probability 1, for some possibly different natural numbers $b_1, b_2 > 1$, then for all $t > 0$,

$$P_{(X,Y) \sim \mathcal{P}}( |N(f) - \mathbb{E}_{(X,Y) \sim \mathcal{P}}(N(f)) | \geq t ) \leq 2 \exp\left( - \frac{2t^2}{(b_1 - 1)^2} \right)$$

and

$$P_{(X,Y) \sim \mathcal{P}}( |N(g) - \mathbb{E}_{(X,Y) \sim \mathcal{P}}(N(g)) | \geq t ) \leq 2 \exp\left( - \frac{2t^2}{(b_2 - 1)^2} \right).$$

The assumption that $N(f)$ and $N(g)$ have an upper bound is always verified: all the possible realizations of $f$ and $g$ have a finite number of linear regions. Proposition 8 tells us that the upper bounds for the number of positive and negative regions concentrate around their expected value.
l of layers increases.\textsuperscript{10} This entails that the results we find in this section better suit shallow networks, that is, networks for which \( L \) is small. The following theorem is an immediate result of [15, Theorem 1.8].\textsuperscript{11}

**Theorem 3:** If \( v \) is a weak martingale in \( \mathbb{R}^p \) such that for every \( l \in \{1, \ldots, L\} \), \( \|v^{(l)}(X) - v^{(l-1)}(X)\|_2 \leq M \), for some \( M > 0 \), then, for every \( a > 0 \),

\[
P_{(X,Y) \sim \mathcal{D}} \left( \|v^{(l)}(X)\|_2 \geq Ma \right) < 2 \exp \left( 1 - \frac{(Ma - 1)^2}{2l} \right). \tag{20}
\]

The fact that we demand the distance between the output of successive layers of the neural network to be bounded can be interpreted as the mild requirement for the SDNN to be “smooth”. That is, no steep jumps from the output of one layer to the output of the successive one are allowed.

Consider now the sequence \( Z = (Z^{(l)})_{l=0}^{\infty} \) such that \( Z^{(l)} = v^{(l)}(X) - \mathbb{E}_{(X,Y) \sim \mathcal{D}} [v^{(l)}(X)] \), for all \( l \in \{1, \ldots, L\} \). Then, we have the following.

**Corollary 2:** If \( Z \) is a weak martingale in \( \mathbb{R}^p \) such that for every \( l \in \{1, \ldots, L\} \), \( \|Z^{(l)} - Z^{(l-1)}\|_2 \leq M \), for some \( M > 0 \), then, for every \( a > 0 \),

\[
P_{(X,Y) \sim \mathcal{D}} \left( \|Z^{(l)}\|_2 \geq Ma \right) = P_{(X,Y) \sim \mathcal{D}} \left( \|v^{(l)}(X) - \mathbb{E}_{(X,Y) \sim \mathcal{D}} [v^{(l)}(X)]\|_2 \geq Ma \right) < 2 \exp \left( 1 - \frac{(Ma - 1)^2}{2l} \right). \tag{21}
\]

Requiring the distance between \( Z^{(l)} \) and \( Z^{(l-1)} \) to be bounded for all \( l \) retains the “smoothness” interpretation we have given before: we do not want jumps that are too steep between successive centered outputs of the layers. Corollary 2 tells us that \( v^{(l)}(X) \) concentrates around \( \mathbb{E}_{(X,Y) \sim \mathcal{D}} [v^{(l)}(X)] \), for all \( l \in \{1, \ldots, L\} \). In turn, this provides conditions under which the output of the neural network concentrates around its expected value, thus proving Proposition 1 in the main portion of the paper when the hypothesis in second bullet point holds.

There is one main difference between Corollary 2 and Corollary 1. In the latter we require \( v^{(l)}(X) \) to be bounded, while in the former we only require that the Euclidean distance between \( v^{(l)}(X) - \mathbb{E}_{(X,Y) \sim \mathcal{D}} [v^{(l)}(X)] \) and \( v^{(l-1)}(X) - \mathbb{E}_{(X,Y) \sim \mathcal{D}} [v^{(l-1)}(X)] \) is bounded. This is a milder assumption since it governs the dynamics of the sequence of layer outputs, rather than that of the outputs themselves. It comes at the cost of requiring that the network is shallow and of verifying that the sequence of outputs in each layer is a weak martingale.

We now provide two theorems that allow us to check whether sequences \( v \) or \( Z \) are weak or very-weak martingales. We first need a definition.

\textsuperscript{10}This type of inequalities are usually called tail bounds in the probability theory literature.

\textsuperscript{11} [15, Theorem 1.8] is summarized in appendix D.

**Definition 2:** Consider two generic p-dimensional random vectors \( X_1, X_2 \). If

\[
\mathbb{E}(\phi(X_1)) \leq \mathbb{E}(\phi(X_2))
\]

for all convex functions \( \phi : \mathbb{R}^p \rightarrow \mathbb{R} \), provided the expectations exist, then \( X_1 \) is smaller than \( X_2 \) in the convex order, denoted by \( X_1 \preceq_{cv} X_2 \).

The following results come from [38, Theorem 7.A.1].

**Theorem 4:** Pick any \( l \in \{1, \ldots, L\} \) and any \( j < l \). If \( v^{(l)}(X) \preceq_{cv} v^{(j)}(X) \), then \( v \) is a weak martingale. Similarly, if \( Z^{(l)} \preceq_{cv} Z^{(j)} \), then \( Z \) is a weak martingale.

**Theorem 5:** Pick any \( l \in \{1, \ldots, L\} \). If \( v^{(l-1)}(X) \preceq_{cv} v^{(j)}(X) \), then \( v \) is a very-weak martingale. Similarly, if \( Z^{(l-1)} \preceq_{cv} Z^{(j)} \), then \( Z \) is a very-weak martingale.

To the best of our knowledge, Theorems 4 and 5 are the only existing method for checking whether a vector-valued stochastic processes is a weak or a very-weak martingale. The closest previous procedure derives the distributions of test statistics required for testing the null hypothesis that a given univariate stochastic process is a very-weak martingale [35]. To apply it to our case, for every \( l \in \{1, \ldots, L\} \), we would need to assume that the entries of \( v^{(l-1)}(X) \) are independent; we would need to assume the same for \( Z^{(l-1)} \). This is an extremely strong assumption and unlikely to hold in practice.

**Remark 2:** For Theorem 3 and Corollary 2 to hold it is enough that \( v \) and \( Z \) are a very-weak martingales [15, Theorem 1.8]. However, we require them to be weak martingales for the following result to hold. Pick any \( l \in \{1, \ldots, L - 1\} \). Then, we have that

\[
\mathbb{E}_{(X,Y) \sim \mathcal{D}} \left[ \mathbb{E}_{(X,Y) \sim \mathcal{D}} [v(X) \mid v^{(l)}(X)] - \mathbb{E}_{(X,Y) \sim \mathcal{D}} [v^{(l)}(X)] \right]
\]

\[
= \mathbb{E}_{(X,Y) \sim \mathcal{D}} [v(X) \mid v^{(l)}(X) - \mathbb{E}_{(X,Y) \sim \mathcal{D}} [v^{(l)}(X)]] - \mathbb{E}_{(X,Y) \sim \mathcal{D}} [v(X)]
\]

\[
= v^{(l)}(X) - \mathbb{E}_{(X,Y) \sim \mathcal{D}} [v^{(l)}(X)], \tag{22}
\]

where the last equality comes from the weak martingale property of \( Z \). Then, the equalities in (22) imply that, for all \( a > 0 \),

\[
P_{(X,Y) \sim \mathcal{D}} \left( \|\mathbb{E}_{(X,Y) \sim \mathcal{D}} [v(X) \mid v^{(l)}(X)] - \mathbb{E}_{(X,Y) \sim \mathcal{D}} [v^{(l)}(X)]\|_2 \geq Ma \right)
\]

\[
= P_{(X,Y) \sim \mathcal{D}} \left( \|v^{(l)}(X) - \mathbb{E}_{(X,Y) \sim \mathcal{D}} [v^{(l)}(X)]\|_2 \geq Ma \right)
\]

\[
< 2 \exp \left( 1 - \frac{(Ma - 1)^2}{2l} \right). \tag{23}
\]

where the inequality comes from Corollary 2. Equation (23) tells us that we can approximate the value \( \mathbb{E}_{(X,Y) \sim \mathcal{D}} [v(X)] \)
that the output $\nu(X)$ of our SDNN concentrates around with the quantity $\nu(X) \sim \nu^{(l)}(X) - \nu^{(l)}(X)$ as $l \to \infty$.

The approximation is good for the first layers, and then becomes coarse, since as the number $l$ of layers increases, so does the bound. This estimate, then, is better in the context of shallow networks.

**APPENDIX B FNNRA ARCHITECTURE**

The following is a replica of [48, Figure A.1], and it represents the architecture of the deterministic FNNRA $\nu : \mathbb{R}^d \rightarrow \mathbb{R}^p$ with $L$ layers described in section IV-A of the main portion of the paper.

![FNNRA Architecture Diagram]

**APPENDIX C SUB-GAUSSIAN RANDOM VARIABLES AND NORM-SUB-GAUSSIAN RANDOM VECTORS**

Sub-Gaussian random variables and norm-sub-Gaussian random vectors are crucial concepts for the results in section II-A. We first introduce the former [21].

**Definition 3:** A generic random variable $X \in \mathbb{R}$ is sub-Gaussian, written $SG(\sigma)$, if there exists $\sigma \in \mathbb{R}$ such that, for all $\theta \in \mathbb{R}$,

$$
\mathbb{E} \left[ e^{\theta (X - \mathbb{E}(X))} \right] \leq \exp \left( \frac{\theta^2 \sigma^2}{2} \right).
$$

The following is an important characterization of sub-Gaussian random variables.

**Proposition 9:** If $X$ is $SG(\sigma)$, then for all $t > 0$

$$
P(\|X - \mathbb{E}(X)\| \geq t) \leq 2 \exp \left( -\frac{t^2}{2\sigma^2} \right).
$$

If a random variable is bounded, then it is sub-Gaussian.

**Proposition 10:** Let $X$ be a generic random variable in $\mathbb{R}$ such that $a \leq X \leq b$ with probability 1, for some $a, b \in \mathbb{R}$, $a < b$. Then, for all $\theta \in \mathbb{R}$,

$$
\mathbb{E} \left[ e^{\theta (X - \mathbb{E}(X))} \right] \leq \exp \left( \frac{\theta^2(b-a)^2}{4} \right),
$$

that is, $X$ is $SG \left( \frac{b-a}{2} \right)$.

We now give the random vector counterpart of a sub-Gaussian random variable.

**Definition 4:** A generic random vector $X \in \mathbb{R}^d$ is norm-sub-Gaussian, written $nSG(\sigma)$, if there exists $\sigma \in \mathbb{R}$ such that, for all $t \in \mathbb{R}$,

$$
P(\|X - \mathbb{E}(X)\| \geq t) \leq 2 \exp \left( -\frac{t^2}{2\sigma^2} \right).
$$

Definition 4 is closely related to the characterization in Proposition 9. The following comes from [21, Lemma 1], and it is the random vector counterpart of Proposition 10.

**Proposition 11:** Let $X$ be a generic bounded random vector in $\mathbb{R}^d$ such that $\|X\| \leq \sigma$. Then, $X$ is $nSG(\sigma)$.

**APPENDIX D MARTINGALES AND FILTRATIONS**

We first define very-weak, weak, and strong martingales. They were introduced in [15, Definitions 1.2–1.4], and are extremely important for appendix A. We denote by $E$ any real Euclidean space (of finite or infinite dimension), and by $1_E$ the vector $(1, 1, \ldots, 1)^T$ having $\#E$ entries, where $\#$ denotes the cardinality operator.

**Definition 5:** Let $X = (X_j : \Omega \rightarrow E)$ be a sequence of random vectors taking values in $E$, such that $X_0 = 01_E$, and for every $j \geq 1$, $\mathbb{E}(\|X_j\|_2) < \infty$ and $\mathbb{E}(X_j | X_0, \ldots, X_{j-1}) = X_{j-1}$. Then we call $X$ a strong martingale in $E$.

**Definition 6:** Let $X = (X_j : \Omega \rightarrow E)$ be a sequence of random vectors taking values in $E$, such that $X_0 = 01_E$, and for every $j < i$, $\mathbb{E}(\|X_j\|_2) < \infty$ and $\mathbb{E}(X_j | X_i) = X_j$. Then we call $X$ a weak martingale in $E$.

If $X$ is a strong martingale, then it is also a weak martingale. The converse need not hold.

**Definition 7:** Let $X = (X_j : \Omega \rightarrow E)$ be a sequence of random vectors taking values in $E$, such that $X_0 = 01_E$, and for every $j \geq 1$, $\mathbb{E}(\|X_j\|_2) < \infty$ and $\mathbb{E}(X_j | X_{j-1}) = X_{j-1}$. Then we call $X$ a very-weak martingale in $E$.

If $X$ is a weak martingale, then it is also a very-weak martingale. The converse need not hold.

**Example 1:** A simple example for $\nu$ in appendix A to be a strong martingale is the following. For all $l$, let $\nu^{(l)}(X) = \nu^{(l-1)}(X) + N_l$, where $N_l$ is a zero-mean random vector such that $N_1, \ldots, N_L$ are independent. Then, it is immediate to see that $\nu$ satisfies the requirements of Definition 5. Since a strong martingale is also weak, then the $\nu$ introduced in this simple illustration allows to circumvent the use of Theorem 4 to verify the weak martingality of $\nu$.

Now that we introduced the concept of strong martingale, it is worth to point out that in [36, Theorem 1.2], the authors show that the stopped sequence $(S_{k\wedge L}^{\nu^L}_{k\leq L})_{k\geq L} \leq L$ is a strong martingale, where $k \wedge L := \min(k, L)$, and $S_{k\wedge L}^{\nu^L}, k \in \{1, \ldots, L\}$, was defined in section III.

In appendix A we implicitly assumed $\nu^{(0)}(X) = 01_E$. This is just a convention, even if it differs from the one in section II-A where we require $\nu^{(0)}(X)$ to be equal to $X$. We now state [15, Theorem 1.8], which we used extensively to derive the results in appendix A.

**Theorem 6:** [15, Theorem 1.8] Let $X = (X_i)_{i=1}^n$ be a very-weak martingale taking values in $E$ such that $X_0 = 01_E$. 

Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
and, for every $j \geq 1$, $\|X_j - X_{j-1}\|_2 \leq 1$. Then, for every $a > 0$,

$$P(\|X_\ell\|_2 \geq a) < 2 \exp \left(1 - \frac{(a - 1)^2}{2j}\right).$$

We then introduce the concepts of filtration and natural filtration. They are crucial for section III.

**Definition 8:** Let $(\Omega, F, P)$ be a probability space and let $I$ be an index set with a total order $\leq$. For every $i \in I$, let $F_i$ be a sub-$\sigma$-algebra of $F$. Then, $F = (F_i)_{i \in I}$ is a filtration if, for all $k \leq \ell$, $F_k \subseteq F_\ell$.

**Definition 9:** Let $(\Omega, F, P)$ be a probability space and let $I$ be an index set with a total order $\leq$. Let $(S, \Sigma)$ be a measurable space and let $X : I \times \Omega \to S$ be a stochastic process. Then the natural filtration of $F$ with respect to $X$ is defined to be the filtration $F^X = (F^X_i)_{i \in I}$ given by

$$F^X_i := \sigma\left(\left\{X^{-1}(A) : j \in I, j \leq i, A \in \Sigma\right\}\right).$$

That is, $F^X_i$ is the smallest $\sigma$-algebra on $\Omega$ that contains all pre-images of $\Sigma$-measurable subsets of $S$ for “times” $j$ up to $i$.

**APPENDIX E: TROPICAL ALGEBRA**

A detailed introduction to tropical algebra and tropical geometry is provided in [20] and [30]. The fundamental element of tropical algebra, the tropical semiring, is given by

$$\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot).$$

The two operations $\oplus$ and $\odot$ are called **tropical addition** and **tropical multiplication**, respectively, and are such that $a \oplus b := \max(a, b)$ and $a \odot b := a + b$, for all $a, b \in \mathbb{R} \cup \{\infty\}$. The distributive law holds for tropical addition and multiplication, the identity element of tropical addition is $-\infty$, and the identity element of tropical multiplication is 0. The tropical semiring is idempotent in the sense that $a \oplus a \oplus \cdots \oplus a = a$, for all $a \in \mathbb{R} \cup \{-\infty\}$. Because of this, there is no tropical subtraction, but tropical division is well defined

$$a \odot b := a - b,$$

for all $a, b \in \mathbb{R} \cup \{-\infty\}$. Tropical exponetiation is well defined as well, for all $a \in \mathbb{R} \cup \{-\infty\}$, we have that

$$a^{\odot b} := \begin{cases} a \cdot b & \text{if } b \in \mathbb{Z}_+, \\ (-a) \cdot (-b) & \text{if } b \in \mathbb{Z}_- \end{cases}.$$

As we can see, tropical exponetiation is well defined only for integer exponents. We also have that

$$-\infty^{\odot a} := \begin{cases} -\infty & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ \text{undefined} & \text{if } a < 0 \end{cases}.$$  

Notice that the tropical semiring is a (non-Diophantine) abstract prearithmetical, where the partial order is defined on the extended reals [8], [9]. We can now define tropical polynomials and tropical rational functions.

A **tropical monomial** in $d$ variables $x_1, \ldots, x_d$ is an expression of the form

$$c \odot x_1^{a_1} \odot x_2^{a_2} \odot \cdots \odot x_d^{a_d},$$

where $c \in \mathbb{R} \cup \{-\infty\}$ and $a_1, \ldots, a_d \in \mathbb{N}$. As a notational shorthand, we can write it in multi-index notation as $c x^a$, where $a = (a_1, \ldots, a_d)^T \in \mathbb{N}^d$ and $x = (x_1, \ldots, x_d)^T$.

A **tropical polynomial** $f(x) = f(x_1, \ldots, x_d)$ is a finite tropical sum of tropical monomials,

$$f(x) = c_1 x^{a_1} + \cdots + c_r x^{a_r},$$

where $a_1 = (a_{i1}, \ldots, a_{id})^T \in \mathbb{N}^d$ and $c_i \in \mathbb{R} \cup \{-\infty\}$, $i \in \{1, \ldots, r\}$. We assume that $a_i \neq a_j$, for all $i \neq j$.

A **tropical rational function** is a standard difference, that is, a tropical quotient of two tropical polynomials $f(x)$ and $g(x)$,

$$f(x) - g(x) = f(x) \odot g(x).$$

We denote a tropical rational function by $f \odot g$, where $f$ and $g$ are tropical polynomial functions. A tropical polynomial $f$ can be seen as a tropical rational function, indeed $f = f \odot 0$. Hence, any result holding for tropical rational functions hold also for tropical polynomials.

A **$d$-variate tropical polynomial** $f(x)$ defines a function $f : \mathbb{R}^d \to \mathbb{R}$ that is a convex function, in that taking max and sum of convex functions preserves convexity [7]. So, a tropical rational function $f \odot g : \mathbb{R}^d \to \mathbb{R}$ is a difference of convex function [2], [13].

A function $F : \mathbb{R}^d \to \mathbb{R}^p$, $x = (x_1, \ldots, x_d)^T \mapsto (f_1(x), \ldots, f_p(x))^T$, is called a **tropical polynomial map** if each $f_i : \mathbb{R}^d \to \mathbb{R}$ is a tropical polynomial, for all $i \in \{1, \ldots, p\}$, and a **tropical rational map** if $f_1(x), \ldots, f_p(x)$ are tropical rational functions.

Tropical polynomials and tropical rational functions are piecewise linear functions. Hence, a tropical rational map is a piecewise linear map and the notion of a linear region applies. A **linear region** of a tropical rational map $F$ is a maximal connected subset of the domain on which $F$ is linear. The number of linear regions of $F$ is denoted by $\mathcal{N}(F)$.

**APPENDIX F: PROOFS**

We first prove the results in the main portion of the paper.

**Proof of Theorem 1:** Immediate from Definition 4 and Proposition 11. □

**Proof of Corollary 1:** Immediate from Theorem 1 and (2). □

**Proof of Proposition 4:** Immediate from Propositions 9 and 10. □

**Proof of Proposition 5:** Let us first assume that the following is true: $E_{(X,Y) \sim \mathcal{D}}[s(v(X))] < c$. By Proposition 4, we have that, for all $t > 0$,

$$P_{(X,Y) \sim \mathcal{D}}(s(v(X)) \geq E_{(X,Y) \sim \mathcal{D}}[s(v(X))] + t) \leq \exp \left( \frac{-2t^2}{(b-a)^2} \right).$$  

(24)
Then, since $\mathbb{E}_{\mathcal{X}, \mathcal{Y}}[s(v(X))] < c$, there exists $t_0 > 0$ such that $\mathbb{E}_{\mathcal{X}, \mathcal{Y}}[s(v(X))] + t_2 = c$. The bound in (24) implies that
\[
P_{\mathcal{X}, \mathcal{Y}}(s(v(X)) \geq \mathbb{E}(s(v(X))) + t_2) = P_{\mathcal{X}, \mathcal{Y}}(s(v(X)) \geq c)
\leq \exp\left(-\frac{2t_2}{(b-a)^2}\right).
\]
Assume now that $\mathbb{E}_{\mathcal{X}, \mathcal{Y}}[s(v(X))] > c$. By Proposition 4, we have that, for all $t > 0$,
\[
P_{\mathcal{X}, \mathcal{Y}}(s(v(X)) \leq \mathbb{E}(s(v(X))) - t) \leq \exp\left(-\frac{2t^2}{(b-a)^2}\right). \tag{25}
\]
Then, since $\mathbb{E}_{\mathcal{X}, \mathcal{Y}}[s(v(X))] > c$, there exists $t_1 > 0$ such that $\mathbb{E}_{\mathcal{X}, \mathcal{Y}}[s(v(X))] - t_1 = c$. The bound in (25) implies that
\[
P_{\mathcal{X}, \mathcal{Y}}(s(v(X)) \leq \mathbb{E}(s(v(X))) - t_1) = P_{\mathcal{X}, \mathcal{Y}}(s(v(X)) \leq c)
\leq \exp\left(-\frac{2t_1^2}{(b-a)^2}\right).
\]

**Proof of Proposition 6:** If $\mathcal{F}$ and $\mathcal{G}$ are bounded for all $j \in \{1, \ldots, n_0\}$, then $F^{(1)}(X)$ and $G^{(1)}(X)$ are bounded. In addition, if $\mathcal{F}$ and $\mathcal{G}$ are bounded, then $\mathcal{A}^{(1)}$ and $\mathcal{B}^{(1)}$ are bounded, for all $l \in \{1, \ldots, L\}$. In turn, this entails that $\nu^{(1)}(X)$ is bounded, that is, there exists $\xi(0) \in \mathbb{R}$ such that $\|\nu^{(1)}(X)\|_2 \leq \xi(0)$, for all $l \in \{1, \ldots, L\}$. Then, the fact that (3) holds comes immediately from Corollary 1. □

**Proof of Proposition 8:** We prove the claim only for $N(f)$, as the proof for $N(g)$ is analogous. Notice that $N(f) \in \mathbb{N}$, so given our assumption that for some $b_1 > 1$, $N(f) \leq b_1$ with probability 1, we can write that $1 \leq N(f) \leq b_1$ with probability 1. Then, by Proposition 10, we have that $N(f)$ is $SG(b_1^{-1})$. So, by Proposition 9, for all $t > 0$,
\[
P_{\mathcal{X}, \mathcal{Y}}(\mathbb{E}(N(f) - N(f)) \geq t) \leq 2\exp\left(-\frac{t^2}{2(b_1 - 1)^2}\right)
\]
\[
= 2\exp\left(-\frac{2t^2}{(b_1 - 1)^2}\right),
\]
concluding the proof. □

We now provide the proofs to Corollary 2 and to the results in appendix C.

**Proof of Corollary 2:** Immediate from Theorem 3. □

**Proof of Proposition 9:** Let $X$ be $SG(\sigma)$. We have that, for all $s > 0$,
\[
P(X - \mathbb{E}(X) \geq t) \leq P\left(e^{s(X - \mathbb{E}(X))} > e^{st}\right)
\leq \frac{\mathbb{E}(e^{s(X - \mathbb{E}(X))})}{e^{st}},
\]
where the first inequality comes from using Markov’s inequality, and the second one from using Chernoff’s bound. Then, since $X$ is $SG(\sigma)$, we have that
\[
P(X - \mathbb{E}(X) \geq t) \leq \exp\left(\frac{s^2\sigma^2}{2} - st\right).
\]
The above inequality holds for any $s > 0$ so to make it the tightest possible, we minimize with respect to $s > 0$. In particular, we solve $\phi'(s) = 0$, where $\phi(s) = \frac{s^2\sigma^2}{2} - st$, and we find $\inf_{s>0} \phi(s) = -\frac{t^2}{2\sigma^2}$. We complete the proof by repeating this process for $P(X - \mathbb{E}(X) \leq -t)$. □

**Proof of Proposition 10:** Without loss of generality, let $\mathbb{E}(X) = 0$. Then, let $P$ denote the probability distribution of $X$. Pick any $\theta \in \mathbb{R}$ and define $\varphi(\theta) := \log \mathbb{E}(e^{\theta X})$. Let then $Q_0$ be the distribution of $X$ defined by
\[
dQ_0(x) := \frac{\mathbb{E}(e^{\theta X})}{\mathbb{E}(e^{\theta X})} dP(x).
\]
We have that
\[
\varphi'(\theta) = \frac{\mathbb{E}(X e^{\theta X})}{\mathbb{E}(e^{\theta X})} = \int x \frac{e^{\theta x}}{\mathbb{E}(e^{\theta X})} dP(x) = \mathbb{E}_{Q_0}(X)
\]
and that
\[
\varphi''(\theta) = \frac{\mathbb{E}(X^2 e^{\theta X})}{\mathbb{E}(e^{\theta X})} - \frac{\mathbb{E}(X e^{\theta X})^2}{\mathbb{E}(e^{\theta X})^2}
\leq \mathbb{E}_{Q_0}(X^2) - \mathbb{E}_{Q_0}(X)^2 = \mathbb{V}_{Q_0}(X),
\]
where $\mathbb{V}$ denotes the variance operator. Now, by Popoviciu’s inequality, the following holds: $\mathbb{V}_{Q_0}(X) \leq \frac{(b-a)^2}{4}$. Then, by the fundamental theorem of calculus, we have that
\[
\varphi(\theta) = \int_0^\theta \int_0^{\mu} \varphi''(\rho) d\mu d\rho \leq \frac{(b-a)^2}{4} \theta^2 \tag{26}
\]
using $\varphi(0) = \log 1 = 0$ and $\varphi'(0) = \mathbb{E}_{Q_0}(X) = \mathbb{E}(X) = 0$. The proof is concluded by exponentiating both sides of the inequality in (26). □

**ACKNOWLEDGMENT**

The authors would like to thank Federico Ferrari, Vittorio Orlandi, Alessandro Zito, and two anonymous referees for their helpful comments.

**REFERENCES**

[1] M. Alfarrar, A. Bibi, H. Hammoud, M. Gaafar, and B. Ghanem, “On the decision boundaries of neural networks: A tropical geometry perspective,” 2020, arXiv:2002.08838.

[2] L. T. H. An and P. D. Tao, “The DC (difference of convex functions) programming and DCA revisited with DC models of real world nonconvex optimization problems,” Ann. Oper. Res., vol. 133, nos. 1-4, pp. 23-46, 2005.

[3] R. Arora, A. Basu, P. Mianjy, and A. Mukherjee, “Understanding deep neural networks with rectified linear units,” in Proc. Int. Conf. Learn. Represent., 2018, pp. 1–17.

[4] D. Bahdanau, K. Cho, and Y. Bengio, “Neural machine translation by jointly learning to align and translate,” 2014, arXiv:1409.0473.

[5] G. D. Bie, G. Peyre, and M. Cuturi, “Stochastic deep networks,” in Proc. Int. Conf. Mach. Learn., vol. 97, 2019, pp. 1556–1565.

[6] C. M. Bishop and N. M. Nasrabadi, Pattern Recognition and Machine Learning. New York, NY, USA: Springer, 2006.
