Optimal Vertex Connectivity Oracles*

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Abstract

A \textit{k}-vertex connectivity oracle for undirected $G$ is a data structure that, given $u, v \in V(G)$, reports $\min\{k, \kappa(u, v)\}$, where $\kappa(u, v)$ is the pairwise vertex connectivity between $u, v$. There are three main measures of efficiency: construction time, query time, and space. Prior work of Izsak and Nutov [IN12] shows that a data structure of total size $\tilde{O}(kn)$ can even be encoded as a $\tilde{O}(k)$-bit labeling scheme so that vertex-connectivity queries can be answered in $\tilde{O}(k)$ time. The construction time is polynomial, but unspecified.

In this paper we address the top three complexity measures.

\textbf{Space.} We prove that any \textit{k}-vertex connectivity oracle requires $\Omega(kn)$ bits of space. This answers a long-standing question on the structural complexity of vertex connectivity, and gives a strong separation between the complexity of vertex- and edge-connectivity. Both Izsak and Nutov [IN12] and our data structure match this lower bound up to polylogarithmic factors.

\textbf{Query Time.} We answer queries in $O(\log n)$ time, independent of $k$, improving on $\tilde{\Omega}(k)$ time of [IN12]. The main idea is to build instances of \texttt{SetIntersection} data structures, with additional structure based on affine planes. This structure allows for optimum query time that is linear in the output size (This evades the general $k^{1/2-o(1)}$ and $k^{1-o(1)}$ lower bounds on \texttt{SetIntersection} from the 3SUM or OMv hypotheses, resp. [KPP16, HKNS15].)

\textbf{Construction Time.} We build the data structure in time roughly that of a max-flow computation on a unit-capacity graph, which is $m^{4/3+o(1)}$ using state-of-the-art algorithms [KLS20]. Max-flow is a natural barrier for many problems that have an all-pairs-min-cut flavor. The main technical contribution here is a fast algorithm for computing a $k$-bounded version of a Gomory-Hu tree for \textit{element connectivity}, a notion that generalizes edge and vertex connectivity.

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1 Introduction

Most measures of graph connectivity can be computed in polynomial time, and much of the recent work in graph algorithms aims at reducing these complexities to some natural barrier, either near-linear [Kar00, KP09, FNY+20, GMW20, GMW21, MN20, vdBLL+21, Sar21], max-flow-time [LP20, LNP+21], or matrix-multiplication-time [CLL13, AGI+19]. Graph connectivity has also been extensively studied from a structural perspective, where the aims are to understand the structure of some/all minimum cuts. This genre includes Gomory-Hu trees [GH61], the cactus representation [DKL76], block-trees, SPQR-trees, and extensions to higher vertex connectivity [BT96, KTBC91, CDBKT93, PY21], and many others [Ben95b, BG08, GN93, DV94, DV95, DV00, DN95, DN99a, DN99b, PQ80, GILP16, GILP18, FGI+16].

In this paper we study the data structural approach to understanding graph connectivity, which incorporates elements of the algorithmic and structural camps, but goes further in that we want to be able to efficiently query the connectivity, e.g., either ask for the size of a min-cut or the min-cut itself.

Suppose we are given an undirected graph $G$ and wish to be able to answer pairwise edge- and vertex-connectivities up to some threshold $k$. Let $\lambda(u,v)$ and $\kappa(u,v)$ be the maximum number of edge-disjoint and internally vertex-disjoint paths, resp., between $u$ to $v$. By Menger’s theorem, these are equal to the minimum number of edges and vertices, resp., necessary to disconnect $u$ and $v$.\footnote{If $\{u,v\} \in E$, then $\kappa(u,v)$ represents the size of a mixed cut separating $u, v$ consisting of $\{u,v\}$ and $\kappa(u,v) - 1$ other vertices.}

$$
e\text{-conn}(u,v) : \text{ Return min}\{\lambda(u,v), k\}.
$$

$$
e\text{-cut}(u,v) : \text{ If } \lambda(u,v) < k, \text{ return an edge-cut separating } u,v \text{ with size } \lambda(u,v).
$$

$$
v\text{-conn}(u,v) : \text{ Return min}\{\kappa(u,v), k\}.
$$

$$
v\text{-cut}(u,v) : \text{ If } \kappa(u,v) < k, \text{ return a vertex-cut separating } u,v \text{ with size } \kappa(u,v).
$$

The edge-connectivity problems are close to being solved. A Gomory-Hu tree $T$ [GH61] (aka cut equivalent tree) is an edge-weighted tree such that the bottleneck edge $e$ between any $u,v$ has weight $\lambda(u,v)$, and the partition defined by $T - \{e\}$ corresponds to a $\lambda(u,v)$-edge cut, which can be explicitly associated with $e$ if we wish to also answer e-cut queries. Bottleneck queries can be answered in $O(1)$ time with $O(n \log n)$ preprocessing, or $O(\alpha(n))$ time with $O(n)$ preprocessing; see [Pet06, DLW09, Cha87].\footnote{These upper and lower bounds are in the comparison model. When $G$ is unweighted, all min-cut values are integers in $[1, n]$, which can be sorted in linear time. In this case $O(n)$ preprocessing for $O(1)$-time queries is possible.}

The time to compute the full Gomory-Hu tree is\footnote{$\min\{f(u,v), f(v,u)\}$} in directed graphs, and in near-linear time [Schnorr 1979 Schnorr] proposed a cut-equivalent tree for roundtrip flow\footnote{$\min\{f(u,v), f(v,u)\}$} in directed graphs, and in 1990 Gusfield and Naor [GN90] used Schnorr’s result to build a Gomory-Hu-type tree for vertex connectivity. Benczur [Ben95a] found errors in Schnorr [Sch79] and Gusfield and Naor [GN90], and
Table 1: A history of vertex-connectivity oracles. By convention, space for centralized data structures is measured in $O(\log n)$-bit words and space for labeling schemes is measured in bits. With a couple exceptions, all data structures and lower bounds are for $\mathsf{v\text{-}conn}(u,v) = \min\{|\ell(u,v),k\}$ queries, where $k$ is an arbitrary parameter. The constructions based on block trees, SPQR trees [BT96], and [KTBC91] only work for $k \in \{2,3,4\}$, and Pettie and Yin [PY21] only works when $k = \kappa(G) + 1$ where $\kappa(G)$ is the global minimum vertex connectivity of $G$. Nutov’s [Nut21] data structure $\langle * \rangle$ determines whether $\kappa(u,v) \leq k$ and if so, returns a pointer to a cut of size at most $k$ in $O(1)$ time. The $\Omega(k \log(n/k^3))$ lower bound of Katz et al. [KKKP04] is for the worst-case (longest) label length; it implies nothing on average length. The new $\Omega(nk^3 + n^2)$ lower bound is for the total size of the data structure, and implies an $\Omega(k)$-bit lower bound on the average length of any labeling scheme.

proved more generally that there is no cut-equivalent tree for vertex connectivity. Benczur [Ben95a, pp. 505-506] suggested a way to get a flow-equivalent tree for roundtrip flow using a result of Cheng and Hu [CH91], which would yield a Gomory-Hu-type tree suitable for answering $\mathsf{v\text{-}conn}$ (but not $\mathsf{v\text{-}cut}$) queries. This too, turned out to be incorrect. Hassin and Levin [HL07] gave an example of a vertex-capacitated graph (integer capacities in $[1,n^{O(1)}]$) that has $\Omega(n^2)$ distinct pairwise vertex connectivities, which cannot be captured by a Gomory-Hu-type tree representation.\(^4\)

When the underlying graph has unit capacity, the counterexamples of [Ben95a, HL07] do not rule out a representation of vertex-connectivity using, say, $\tilde{O}(1)$ trees, nor do they rule out some completely different $O(n)$-space structure for answering $\mathsf{v\text{-}conn}$-queries, independent of $k$.

Most prior data structures supporting $\mathsf{v\text{-}conn}$ queries were actually labeling schemes. I.e., a vertex labeling $\ell: V \rightarrow \{0,1\}^*$ is created such that a $\mathsf{v\text{-}conn}(u,v)$ query is answered by inspecting $\ell(u), \ell(v)$. Katz, Katz, Korman, and Peleg [KKKP04] initiated this line of research into labeling for connectivity. They proved that the maximum label length to answer $\mathsf{v\text{-}conn}$ queries is $\Omega(k \log(n/k^3))$ and $O(2^k \log n)$. To be specific, they give a class of graphs for which a $\Theta(1/k^2)$-fraction of the

\(^4\)Hassin and Levin [HL07] pointed out that Benczur’s proposal yields a Gomory-Hu-type tree representation for vertex “separations” in a capacitated graph $G = (V,E,c: V \rightarrow \mathbb{R}^+)$. The minimum separation of $u,v$ is the minimum of $c(u), c(v)$, and the minimum $c(K)$ over all vertex cuts $K$ disconnecting $u,v$.  

| Citation | Space (Words) Labeling (Bits) | Query Time | Construction Time |
|----------|-----------------------------|------------|------------------|
| Block-tree $k = 2$ | $O(n)$ | $O(1)$ | $O(m)$ |
| Block-tree + SPQR-tree [BT96] $k = 3$ | $O(n)$ | $O(1)$ | $O(m)$ |
| Block-tree + SPQR-tree + [KTBC91] $k = 4$ | $O(n)$ | $O(1)$ | $O(m + n\alpha(n))$ |
| [KKKP04] | $O(n2^k)$ | $O(2^k \log n)$ | $O(2^k)$ | $\text{poly}(n)$ |
| [Kor10] | $O(nk^3)$ | $O(k^3 \log n)$ | $O(k \log k)$ | $\text{poly}(n)$ |
| [HL09] | $O(nk^3)$ | $O(k^3 \log n)$ | $O(k \log k)$ | $\text{poly}(n)$ |
| [IN12] | $O(nk \log n)$ | $O(k \log^3 n)$ | $O(k \log n)$ | $\text{poly}(n)$ |
| AGI+19 | $O(n^2)$ | $O(n \log n)$ | $O(1)$ | $\Omega(nk^2 \text{ avg.})$ |
| [PY21] $k = \kappa(G) + 1$ | $O(n \kappa(G))$ | $O(\kappa(G) \log n)$ | $O(1)$ | $O(m + n \cdot \text{poly}(\kappa(G)))$ |
| [Nut21] $\langle * \rangle$ | $O(nk^2 + n^2)$ | $O(k \log n)$ | $O(\text{flow}(m) \cdot \text{poly}(k, \log n))$ |
| New | $\Omega(nk/\log n)$ | $\Omega(k)$ | $\text{any}$ | $\text{any}$ |
vertices require $\Omega(k \log(n/k^3))$-bit labels. However, this does not imply any non-trivial bound on the average/total label length. Their upper bound was subsequently improved to $O(k^2 \log n)$ [Kor10] and then to $O(k^2 \log n)$ [HL09]. Using a different approach, Izsak and Nutov [IN12] gave an $O(k \log^2 n)$-bit labeling scheme for vertex connectivity. A centralized version of the data structure takes $O(kn \log n)$ total space and can be augmented to support vertex-cut queries with $O(k^2 n \log n)$ total space.

The labeling schemes [KKKP04, Kor10, HL09, IN12] focus on label-length, and do not discuss the construction time in detail, which is a large polynomial.

1.1 New Results

We resolve the long-standing question concerning the space complexity of representing vertex connectivity; cf. [Ben95a, HL07]. In particular, we prove that any data structure answering $v$-conn queries requires space $\Omega(kn/\log n)$ (i.e., $\Omega(kn)$ bits), and that the lower bound extends to threshold queries (decide whether $\kappa(u,v) \leq k$), equality queries (decide whether $\kappa(u,v) = k$), and approximate queries (distinguish $\kappa(u,v) < k - \sqrt{k}$ from $\kappa(u,v) > k + \sqrt{k}$). This implies that Izsak and Nutov’s [IN12] centralized data structures are optimal to within a $\log^2 n$-factor, and that even the average length of their labeling scheme cannot be improved by more than a $\log^3 n$-factor; cf. [KKKP04]. It also implies a strong separation between the space complexity of storing all edge-connectivities ($O(n)$ space) and all vertex-connectivities ($\Omega(n^2/\log n)$ space when $k = n$).

Although the Izsak-Nutov structure is space-optimal, its $O(k \log n)$ query time and polynomial construction time can be substantially improved. We design a version of this structure that allows for $O(\log n)$ query time independent of $k$. The key problem is to create random instances of SetIntersection that are still structured enough to answer intersection queries $S_i \cap S_j$ optimally, in time $O(|S_i \cap S_j|)$. Conditional lower bounds from 3SUM and OMv [KPP16, HKNS15] imply that this should be impossible for worst-case instances of SetIntersection.

Turning to construction time, we prove that the vertex connectivity oracle (or labeling scheme) can be constructed in time $T_{\text{uflow}}(m) \cdot \text{poly}(k, \log n)$, where $T_{\text{uflow}}(m)$ is the time for one max-flow computation on unit-capacity graphs with $m$ edges. The main subproblem solved in the construction algorithm is computing a Gomory-Hu tree for element connectivities up to $k$.

1.2 Organization

Section 2 covers some preliminary concepts such as vertex connectivity, element connectivity, Gomory-Hu trees, and the (vertex-cut version of) the isolating cuts lemma of [LP20, LNP+21].

Section 3 presents the lower bound on representations of vertex connectivity. Section 4 presents a space- and query-efficient vertex connectivity oracle based on Izsak and Nutov’s [IN12] labeling scheme. In order to build this oracle efficiently, we need to compute Gomory-Hu trees that capture all element connectivities up to $k$. In Section 5 we show how to do this in $T_{\text{uflow}}(m)\text{poly}(k, \log n)$ time.

Section 6 concludes with some outstanding open problems.

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5The labeling schemes of Katz et al. [KKKP04], Korman [Kor10], and Hsu and Lu [HL09] differentiate between $\kappa(u,v) \geq k_0$ and $\kappa(u,v) < k_0$ using labels of size $O(2^{k_0} \log n)$, $O(k_0 \log n)$, and $O(k_0 \log n)$, respectively. They can be used to answer $v$-conn queries by concatenating labels for all $k_0 = 1, 2, \ldots, k$, thereby introducing an $O(k)$-factor overhead in [Kor10, HL09].

6Following convention, the space of centralized data structures is measured in $O(\log n)$-bit words and labeling schemes are measured in bits.
2 Preliminaries

2.1 Vertex Connectivity, Element Connectivity, and Gomory-Hu Trees

Define $\kappa_G(u,v)$ to be the vertex connectivity of $u,v$ in $G = (V,E)$, i.e., the maximum number of internally vertex-disjoint paths between $u$ and $v$. By Menger’s theorem [Men27], this is the size of the smallest mixed cut $C \subseteq (E \cup (V-\{u,v\}))$ whose removal disconnects $u,v$. Element connectivity is a useful generalization of vertex- and edge-connectivity [Che15, CRX15, JMVW02, CK12]. Let $U \subseteq V$ be a set of terminals. Then for $u,v \in U$, $\kappa'_{G,U}(u,v)$ is the maximum number of paths between $u$ and $v$ that are $E$-disjoint and $(V-U)$-disjoint. By duality, this is equivalently the size of the smallest mixed cut $C \subseteq (E \cup (V-\{u,v\}))$ whose removal disconnects $u,v$. Observe that when $U = V$, $\kappa'_{G,U}(u,v) = \lambda_G(u,v)$ is the same as edge-connectivity, and when $U = \{u,v\}$, $\kappa'_{G,U}(u,v) = \kappa_G(u,v)$ captures vertex connectivity. More generally we have Lemma 2.1, which follows directly from the definitions.

**Lemma 2.1.** Fix an undirected graph $G = (V,E)$ and a terminal set $U \subseteq V$. For any $u,v \in U$, $\kappa'_{G,U}(u,v) \geq \kappa_G(u,v)$ with equality if and only if there exists a mixed $\kappa_G(u,v)$-cut $C$ disconnecting $u,v$ with $C \cap U = \emptyset$.

If $C$ is a mixed cut, the connected components of $G - C$ are called sides of $C$. Note that minimum edge-cuts have two sides but minimum vertex-cuts may have an unbounded number of sides. For any $A \subset V$, $\partial A$ is the set of vertices adjacent to $A$ in $G$, but not in $A$. Thus, $\partial A$ is a vertex cut disconnecting $A$ from $V - (A \cup \partial A)$.

It is well known that Gomory-Hu trees exist for element connectivity; see [Sch03, CRX15]. We use the following general definition:

**Definition 2.2.** A $k$-Gomory-Hu tree for element connectivity w.r.t. graph $G$ and terminal set $U$ is a triple $(T,f,C)$ satisfying

- (Flow equivalency) $T = (V_T,E_T,w : E_T \rightarrow [1,k])$ is a weighted tree and $f : U \rightarrow V_T$. If $f(u) = f(v)$ then $\kappa'_{G,U}(u,v) \geq k$ and otherwise, $\kappa'_{G,U}(u,v) = \min_{e \in T(f(u),f(v))} w(e)$, where $T(x,y)$ is the unique $T$-path between $x$ and $y$.

- (Cut equivalency) $C : E_T \rightarrow 2^E \cup (V-U)$. For any $e \in T(f(u),f(v))$, $C(e)$ is a $w(e)$-cut disconnecting $u,v$. (Each of the two connected components in $T - e$ represents the union of some subset of the sides of $C(e)$.)

Note that $f$ is unnecessary ($V_T = U$) if $k = \infty$ is unbounded, and that $C$ is unnecessary if we are only interested in $\min\{\kappa'_{G,U}(u,v),k\}$-queries. This definition can be extended to $(1+\epsilon)$-approximations by relaxing the first criterion to $\kappa'_{G,U}(u,v) \leq \min_{e \in T(f(u),f(v))} w(e) \leq (1+\epsilon)\kappa'_{G,U}(u,v)$, with trivial $f(V_T = U$ and $f$ maps trivially), and the second criterion similarly.

2.2 Isolating Cut Lemma and Max-Flows

Li and Panigrah’s [LP20] isolating cuts lemma finds, for a terminal set $I$, the minimum edge-cut separating $u \in I$ from $I - \{u\}$ in time proportional to $O(\log |I|)$ max-flows. It was generalized to

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7Sometimes only “pure” vertex cuts $C \subset V - \{u,v\}$ are considered and $\kappa_G(u,v)$ is left undefined or artificially defined to be $|V| - 1$ whenever $u,v$ are adjacent in $G$. This definition fails to distinguish highly connected and poorly connected pairs of vertices that happen to be adjacent.
vertex connectivity in [LNP+21] and here we generalize this lemma slightly further to be able to handle element connectivity.

**Lemma 2.3** (Isolating cuts, vertex version with forbidden terminals). Let $I \cup F$ be an independent set in graph $G = (V, E)$, and $I \cap F = \emptyset$. We want to find, for each $v \in I$, a set $S_v \subseteq V$ such that (i) $S_v \cap I = \{v\}$ and $(\partial S_v) \cap F = \emptyset$, (ii) it minimizes $|\partial S_v|$, (iii) subject to (ii), it minimizes $|S_v|$. Then $\{S_v\}_{v \in I}$ can be computed with $\log |I|$ calls to max-flow on graphs with $O(m)$ edges, $O(m)$ vertices, and capacities in $\{1, \infty\}$. If we do not care to minimize $|\partial S_v|$ if it is larger than $k$, then the max-flow instances can be unit-capacity multigraphs with $O(m + k|F|)$ edges and vertices.

**Proof.** (sketch) The proof follows [LNP+21, Lemma 4.2]. The authors called max-flow instances to compute the isolating cuts. Our lemma adds a new requirement, that $\partial S_v \cap F = \emptyset$. This can be effected by replacing $v \in F$ in the flow networks with two vertices and an edge $v_{in} \to v_{out}$ with capacity $\infty$. (Or in the unit-capacity case, with $k + 1$ parallel edges from $v_{in}$ to $v_{out}$.) This ensures vertices in $F$ never appear in the cuts $\partial S_v$ we are interested in.

### 3 Space Lower Bound on Vertex Connectivity Oracles

**Informal strategy and intuitions.** We use an error-correcting-code-type argument to derive an $\Omega(n^2)$-bit lower bound in the case that $k = n$. By taking the product of $n/k$ copies of this construction on $\Theta(k)$-vertex graphs, we derive the $\Omega(kn)$-bit lower bound for general $k \in [1, n]$. The idea is to show the existence of a codebook $\mathcal{T}$ of $n \times n$ Boolean matrices with the following property. Each $T \in \mathcal{T}$ can be represented by a certain graph $G[T]$ on $O(n)$ vertices. We compute a $b$-bit vertex connectivity oracle for this graph and query $\kappa_{G[T]}(u, v)$ for all pairs $u, v$. From these values we reconstruct a different Boolean matrix $\hat{T} \neq T$, which is within the decoding radius of $\mathcal{T}$, and therefore lets us deduce the identity of the original matrix $T$. In other words, it must be that $b \geq \log_2 |\mathcal{T}|$. A key technical idea is to show that each $T \in \mathcal{T}$ in the codebook can, in a certain sense, be approximately factored as the product of two rectangular Boolean matrices, with addition and multiplication over $\mathbb{Z}$.

The specifics of the construction are detailed in Theorem 3.1.

**Theorem 3.1.** There exists a constant $c \geq 5$ and a subset $\mathcal{T} \subseteq \{0, 1\}^{n \times n}$ of boolean matrices having the following properties.

1. (Code Properties) $|\mathcal{T}| = 2^{n^2/3}$, each row of each $T \in \mathcal{T}$ has Hamming weight exactly $n/2$, and every two $T, T' \in \mathcal{T}$ has Hamming distance at least $n^2/3$.

2. (Matrix Decomposition) For every $T \in \mathcal{T}$, there exists $A \in \{0, 1\}^{n \times cn}, B \in \{0, 1\}^{cn \times n}$ such that $C = AB$ encodes $T$ in the following way. Let $\hat{T} \in \{0, 1\}^{n \times n}$ be such that

   $$\hat{T}(i, j) = \begin{cases} 0 & \text{if } C(i, j) < 2n \\ 1 & \text{if } C(i, j) \geq 2n \end{cases}$$

   Then $T, \hat{T}$ have Hamming distance less than $n^2/6$. Moreover, $C(i, j) \leq 2.1n$ for all $i, j$.

3. (Vertex Connectivity) Let $A, B$ be the matrices corresponding to $T$ from Part 2. Let $G[T] = (X \cup Y \cup Z, E)$ be an undirected tripartite graph with $|X| = |Z| = n$ and $|Y| = cn$, where
\[E \cap (X \times Y) \text{ encodes } A \text{ and } E \cap (Y \times Z) \text{ encodes } B. \text{ Let } \tilde{T} \text{ be such that}\]

\[\tilde{T}(i, j) = \begin{cases} 
0 & \text{if } \kappa(x_i, z_j) < 4n - 2 \\
1 & \text{if } \kappa(x_i, z_j) \geq 4n - 2
\end{cases}\]

Then, \(T, \tilde{T}\) have Hamming distance less than \(n^2/6\).

**Proof.** For Part 1, a simple probabilistic construction shows the existence of \(T\). Pick a random codebook \(T_0\) containing \((1 + o(1))2^{n^2/3}\) matrices and discard a negligible fraction of codewords that are within distance \((2/5)n^2\) of another codeword. Obtain \(T\) from \(T_0\) as follows. Take each \(T_0 \in T_0\) and see if it is within Hamming distance \(O(n^{3/2})\) of a matrix \(T\), all of whose rows have Hamming weight \(n/2\). If so, include \(T \in T\) as a representative of \(T_0\). Only a negligible fraction of \(T_0\) matrices fail to satisfy this property. Thus, \(T\) has the requisite size and each pair of matrices in \(T\) has Hamming distance at least \((2/5)n^2 - O(n^{3/2}) > n^2/3\).

Turning to Part 2, the construction of \(A, B\) is probabilistic. We pick \(B\) uniformly at random such that each row has Hamming weight \(n/2\), then choose \(A\) depending on \(B\). We call \((i, k)\) eligible if the vector \(B(k, \cdot)\) has an unusually high agreement with \(T(i, \cdot)\), in particular, if

\[|\{j : B(k, j) = T(i, j)\}| \geq n/2 + \sqrt{n}.\]

For each row \(i\), we set \(A(i, k) = 1\) for the first \(4n\) values of \(k\) for which \((i, k)\) is eligible. Since the probability of \((i, k)\) being eligible is some constant \(p\), it is possible to build \(A\) with high probability if \(c\) is sufficiently large, say \(4p^{-1} + 1\).

Suppose \(T(i, j) = 1\) and consider the random variable \(C(i, j)\). By the definition of eligibility and the fact that rows of \(B\) and \(T\) have Hamming weight \(n/2\),

\[\Pr(B(k, j) = 1 \mid (i, k) \text{ eligible}) \geq \frac{1/2(n/2 + \sqrt{n})}{n/2} = \frac{1}{2} + \frac{1}{\sqrt{n}}\]

Thus \(C(i, j)\) is a random variable that dominates Binom\((4n, 1/2 + 1/\sqrt{n})\), hence \(E(C(i, j)) \geq 2n + 4\sqrt{n}\). The case when \(T(i, j) = 0\) is symmetric, in which case Binom\((4n, 1/2 - 1/\sqrt{n})\) dominates \(C(i, j)\). By a Chernoff bound, the probability that \(\tilde{T}(i, j) \neq T(i, j)\) is the probability that \(C(i, j)\) deviates from its expectation by at least \(4\sqrt{n}\), which is at most \(\exp(-2(4\sqrt{n}^2)/4n) = \exp(-8)\). The Hamming distance between \(\tilde{T}\) and \(T\) is therefore at most \(n^2\exp(-8) < n^2/6\) in expectation.

Lastly, by a Chernoff bound, the probability that \(C(i, j)\) deviates from \(2n\) by a constant factor is exponentially small. In particular, we have \(C(i, j) \leq 2.1n\) for all \(i, j\) with high probability. This proves the existence of matrices \(A, B\) for any \(T \in T\).

Part 3. Observe that \(C(i, j)\) reflects the maximum flow from \(x_i\) to \(z_j\) if \(G[T]\) were a unit-capacity network with all edges directed from \(X \rightarrow Y \rightarrow Z\). However, \(G[T]\) is undirected, and by Menger’s theorem \(\kappa(x_i, z_j)\) is the maximum number of internally vertex disjoint paths from \(x_i, z_j\). We consider paths of length 2 (corresponding to \(C(i, j)\)) and two types of paths of length 4.

For technical reasons we will modify the construction of \(A\) as follows. For each row \(i\), set \(A(i, k) = 1\) for \(r = O(c \log n)\) values of \(k\) chosen uniformly at random, then set \(A(i, k) = 1\) for \(4n - r\) additional values of \(k\) for which \((i, k)\) is eligible. This changes the expectation of \(C(i, j)\) by \(O(c \log n)/\sqrt{n} < 1\) but does not otherwise affect the analysis in Part 2.

We claim that with high probability, the vertex connectivity of \(x_i, z_j\) is exactly

\[\kappa(x_i, z_j) = \min\{C(i, j) + 2(n - 1), 4n\}.\]
In particular there are $\kappa(x_i, z_j)$ paths $P = P_0 \cup P_1 \cup P_2$, where $|P_0| = C(i, j)$ are length-2 paths, $P_1$ are $n - 1$ paths internally disjoint from $P_0$ of the form $x_i \rightarrow Y \rightarrow X \rightarrow Y \rightarrow z_j$ and $P_2$ are up to $n - 1$ paths disjoint from $P_0 \cup P_1$ of the form $x_i \rightarrow Y \rightarrow Z \rightarrow Y \rightarrow z_j$.

We construct $P_1$ as follows. Choose $Y_{1,x}$ to be any $n - 1$ neighbors of $x_i$ that are not part of $P_0$ paths, and $Y_{1,z}$ to be any $n - 1$ neighbors of $z_j$ that are not part of $P_0$ paths. Clearly $Y_{1,x} \cap Y_{1,z} = \emptyset$. Let $G_{1,x}$, $G_{1,z}$ be the subgraphs of $G[T]$ induced by $X - \{x_i\} \cup Y_{1,x}$ and $X - \{x_i\} \cup Y_{1,z}$, respectively. These graphs contain many edges of derived from $Y$. Together with $Y_{1,x}$, $G_{1,x}$ and $G_{1,z}$ have degree $\Theta(\log n)$ with high probability. It is well known that such graphs have perfect matchings w.h.p. (see [Bol11] or [FK16]); call them $M_{1,x}, M_{1,z}$. Together with $x_i, z_j$, these form $n - 1$ paths $P_1$ internally disjoint from $P_0$. The construction of paths $P_2$ is symmetric. Let $Y_{2,x}$ be $\min\{n - 1, 4n - |P_0| - |P_1|\}$ neighbors of $x_i$ not already included in $P_0, P_1$, and $Y_{2,z}$ be $|Y_{2,x}|$ neighbors of $z_j$ not included in $P_0, P_1$. Note that both $Y_{2,x}$ and $Y_{2,z}$ have size at least $0.9n$ because $|P_0| \leq 2.1n$ and $|P_1| \leq n$. The graphs $G_{2,x}, G_{2,z}$ induced by $Z - \{z_j\} \cup Y_{2,x}$ and $Z - \{z_j\} \cup Y_{2,z}$ again contain perfect matchings $M_{2,x}, M_{2,z}$ with high probability. Together with $x_i$ and $z_j$ these generate $|Y_{2,x}|$ paths internally disjoint from $P_0, P_1$.

We conclude that with high probability, $C(i, j) < 2n$ if and only if $\kappa(x_i, z_j) < 2n + 2(n - 1)$ and, from Part 2, that there exist $A, B$ such that $T, T'$ have Hamming distance less than $n^2/6$.

**Corollary 3.2.** Suppose $\mathcal{D}(G, k)$ is a data structure for an undirected $n$-vertex graph $G$ that can determine whether $\kappa(u, v) < k$ or $\kappa(u, v) \geq k$. Then $\mathcal{D}$ requires $\Omega(kn)$ bits of space in the worst case.

**Proof.** Pick any $T \in \mathcal{T}$, and let $G = G[T]$ be the $(c + 2)n$-vertex graph encoding of $T$ from Theorem 3.1. Using $\mathcal{D}(G, k)$ we can generate the matrix $\tilde{T}$, where $k = 4n - 2$, and then deduce $T$ since $\tilde{T}$ is within its decoding radius. Thus, $\mathcal{D}$ requires at least $\log |T| = \Theta(n^2)$ bits of space. For general values of $k$, we can take $\Theta(n/k)$ disjoint copies of this construction, each having $\Theta(k)$ vertices and requiring $\Theta(k^2)$ space.

**Remark 3.3.** The lower bound of Corollary 3.2 also applies to data structures $\mathcal{D}(G, k)$ that distinguish between $\kappa(u, v) = k$ and $\kappa(u, v) \neq k$. For this case we would use the construction of Theorem 3.1 with $k = 4n$ and let $\tilde{T}(i, j) = 1$ iff $\kappa(x_i, z_j) = 4n$.

**Corollary 3.4.** Suppose $\mathcal{D}(G, k, \epsilon)$ is a data structure for an undirected $n$-vertex graph $G$ that can distinguish between $\kappa(u, v) < (1 - \epsilon)k$ or $\kappa(u, v) > (1 + \epsilon)k$. Then $\mathcal{D}$ requires $\Omega(nce^{-2})$ bits of space for some $\epsilon$.

**Proof.** The construction of Theorem 3.1 and Corollary 3.2 still works when $k = \Theta(n)$ and $\epsilon = \Theta(1/\sqrt{n})$.\footnote{The edges included by the eligibility criterion have subtle dependencies, which makes reasoning about them somewhat tricky. Including truly random edges in $A$ is a technical hack and surely not necessary.} \footnote{Note that $G_{2,x}, G_{2,z}$ correspond to submatrices of $B$, which was chosen randomly with density $1/2$. Due to the Hamming weight restriction, the entries of $B$ are slightly negatively correlated, which only improves the chance of finding perfect matchings.}


4 Vertex Connectivity Oracles

Lemma 2.1 says that for any terminal set \( U \), \( \kappa'_{G,U}(u,v) \) never underestimates the true value \( \kappa_G(u,v) \) and achieves the equality \( \kappa'_{G,U}(u,v) = \kappa_G(u,v) \) in the event that some minimum mixed \( u\,v \) cut \( C_{u,v} \) is disjoint from \( U \). Furthermore, \( \kappa'_{G,U} \) is superior than \( \kappa_G \) inasmuch as it can be succinctly represented as a Gomory-Hu tree (Definition 2.2) and queried via bottleneck edge queries [Pet06, DLW09, Cha87].

Izsak and Nutov’s [IN12] ingenious algorithm proceeds by sampling several terminal sets with probability \( 1/k \). Each terminal set includes both \( u \) and \( v \) with probability \( 1/k^2 \) and avoids \( C_{u,v} \), with constant probability if \( |C_{u,v}| \leq k \), hence \( O(k^2 \log n) \) terminal sets suffice to accurately capture all vertex connectivities up to \( k \) with high probability. The space for the centralized data structure is just that of storing \( O(k^2 \log n) \) Gomory-Hu trees and data structures for answering bottleneck-edge queries. Each tree is on \( O(n/k) \) terminals, for a total of \( O(kn \log n) \) space. A query \( v\text{-conn}(u,v) \) needs to examine all terminal sets containing both \( u \) and \( v \). This is a classic \texttt{SetIntersection} query. Each of \( u \) and \( v \) is in \( \Theta(k \log n) \) sets, and are jointly in \( \Theta(\log n) \) sets. In this section we show how to build appropriate \texttt{SetIntersection} instances such that queries are answered in (optimal) \( O(\log n) \) time. (If minimum cuts are associated with edges in the Gomory-Hu tree, a \( v\text{-cut}(u,v) \) query can be answered in \( O(1) \) additional time be returning a pointer to the appropriate cut. This increases the space to \( O(k^2 n \log n) \).

A terminal set \( U \subseteq V \) captures \( u,v \) if there exists a \( \kappa(u,v) \)-cut \( C \subseteq V \cup E \) such that

\[
U \cap (\{u,v\} \cup C) = \{u,v\}.
\]

We show how to find \( O(k^2 \log n) \) terminal sets that capture all pairs in \( V^2 \) using a construction based on affine planes and 3-wise independent hash functions.

**Lemma 4.1.** There is an algorithm using \( O(\log^2 n) \) random bits that generates terminal sets \( \mathcal{U} \) with the following properties.

- \( |\mathcal{U}| = O(k^2 \log n) \) and each \( U \in \mathcal{U} \) has \( |U| = O(n/k) \).
- Given vertices \( u,v \) with \( \kappa(u,v) \leq k \) we can find \( O(\log n) \) sets \( U_1, \ldots, U_{O(\log n)} \) such that each, independently, captures \( u,v \) with constant probability. As a consequence, w.h.p., \( \mathcal{U} \) captures all of \( V^2 \).

**Proof.** Let \( p_0 \) be the first prime larger than \( n = |V| \) and \( p \) be the first prime larger than \( 2k \). Let \( H = (P,L) \) be a subset of an affine plane, defined as follows. \( P = \{u_{i,j} \mid i \in [p], j \in [[p_0/p]]\} \) is a set of points arranged in a rectangular grid and \( L = \{\ell_{s,j} \mid s,j \in [p]\} \) is a set of lines, where \( \ell_{s,j} = \{u_{t,j+st \mod p} \mid t \in [[p_0/p]]\} \) is the line with slope \( s \) passing through \( u_{0,j} \). We pick a hash function \( h : V \rightarrow P \) of the form

\[
h(x) = (ax^2 + bx + c) \mod p_0
\]

where \( a, b, c \) are chosen uniformly at random from \( [p_0] \), then form the \( p^2 = \Theta(k^2) \) terminal sets \( \mathcal{U}[h] = \{U_{s,j}\} \) as follows.

\[
U_{s,j} = \{v \mid h(v) \in \ell_{s,j}\}.
\]

Now fix any two vertices \( u,v \) and a \( \kappa(u,v) \) cut \( C \) with \( \kappa(u,v) \leq k \). Then \( \mathcal{U}[h] \) will capture \( u,v \) if (i) \( h(u),h(v) \) differ in their 1st coordinates of the rectangular grid \( P \), and, assuming this happens,


\[
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\]
(ii) $C \cap U_{s,j} = \emptyset$ where $\ell_{s,j}$ is the unique line containing $h(u), h(v)$. The probability of (i) is 
$1 - p/p_0 = 1 - \Theta(k/n)$ and the probability of (ii) is 
\[
\Pr(C \cap U_{s,j} = \emptyset \mid h(u), h(v)) \geq 1 - \sum_{x \in C \cap V} \Pr(h(x) \in \ell_{s,j} \mid h(u), h(v)) \geq 1 - k [p_0/p] / p_0 \geq 1/2
\]
where the second last inequality is because $|\ell_{s,j}| \leq [p_0/p]$, $|C| \leq k$, and $h$ is sampled from a 3-wise independent hash family.

Let $\mathcal{U}$ be the union of $\mathcal{U}[h_1], \ldots, \mathcal{U}[h_{O(\log n)]}$ using independent hash functions $h_1, \ldots, h_{O(\log n)}$. It follows that $|\mathcal{U}| = O(p^2 \log n) = O(k^2 \log n)$, that given $u, v$ we can identify $U_1, \ldots, U_{O(\log n)} \in \mathcal{U}$ in $O(\log n)$ time such that at least one of these terminal sets captures $u, v$ w.h.p.\(^{10}\)

We now turn to the load balancing condition $|U| = O(n/k)$. Note that if the coefficients of $h$ satisfy $(a, b, c) \neq (0, 0, c)$ then $h$ is at most a 2-to-1 function (as the polynomial defining $h$ has degree 2), meaning that each $U \in \mathcal{U}$ has $|U| < 2 [p_0/p] = O(n/k)$. The performance of the algorithm is clearly quite bad when $a = b = 0$ (each terminal set $U$ is either $\emptyset$ or $V$) so we can remove these hash functions from the hash family and only improve the probability of success. \hfill \\checkmark

**Remark 4.2.** One way that $\mathcal{U}[h]$ can fail to capture $u, v$ is if $h(u)$ and $h(v)$ agree on their 1st coordinate. This could be rectified by including the lines with infinite slope in $L$, but this only seems to work when $k \leq \sqrt{n}$. When $k \geq \sqrt{n}$ doing so would violate the load balancing constraint $|U| = O(n/k)$ as some lines would necessarily have size $\Omega(k)$.

Now, by using the collection $\mathcal{U}$ of terminal sets with additional structure from Lemma 4.1 instead of the $O(k^2 \log n)$ random terminal sets generated independently as used in [IN12], we can speed up the query time from $O(k)$ to $O(\log n)$. Below, we state the guarantee of our oracle formally.

**Theorem 4.3.** Given an undirected graph $G = (V, E)$ and $k \in [1, n]$, a data structure with size $O(kn \log n)$ can be constructed in $O(m) + T_{uflow}(nk)\text{poly}(k, \log n) = O(m) + n^{1/3+o(1)}\text{poly}(k)$ time such that $\nu\text{-conn}(u, v) = \min\{\kappa_G(u, v), k\}$ queries can be answered in $O(\log n)$ time. Using space $O(k^2 n \log n)$, a $\nu\text{-cut}(u, v)$ query can be answered in $O(1)$ additional time; it returns a pointer to a $\kappa_G(u, v)$-size $u$-$v$ cut whenever $\kappa_G(u, v) < k$.

The claims of Theorem 4.3 concerning construction time are substantiated in Section 5. In the context of our vertex connectivity oracle, note that we can assume that our original graph has $O(nk)$ edges by applying Nagamochi-Ibaraki algorithm [NI92] with $O(m)$ running time to reduce the number of edges to $O(kn)$. So the construction time is $n^{1/3+o(1)}\text{poly}(k)$ using the max-flow algorithm for unit-capacity graphs by Kathuria, Liu, and Sidford [KLS20].

## 5 Gomory-Hu Trees for Element Connectivity

The goal in this section is to prove the following:

**Theorem 5.1.** A $k$-Gomory-Hu tree for element connectivity w.r.t. graph $G$ and terminal set $U$ can be constructed in $\tilde{O}(k \cdot T_{uflow}(m + k |U|))$ time.

---

\(^{10}\)Given points $h(u), h(v)$ differing in 1st coordinate, we can clearly identify $\ell_{s,j}$ containing them in $O(1)$ time using a table of inverses modulo $p$. 

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Note that, given the above theorem, we can indeed conclude Theorem 4.3 because there are \(O(k^2 \log n)\) terminal sets in the oracle construction and we just need to build a Gomory-Hu tree for each terminal set by calling Theorem 5.1. As \(m = O(nk)\) and \(|U| = O(n/k)\) by Lemma 4.1, this takes \(T_{\text{uflow}}(m + k|U|) = T_{\text{uflow}}(O(nk))\) for each terminal set.

**Obstacles in Adapting Algorithms of [LP21] for Element Connectivity.** The proof of Theorem 5.1 is obtained by adapting the Gomory-Hu tree construction for edge connectivity by Li and Panigrahi [LP21] to work for element connectivity. Although we use the same high-level approach, element connectivity introduces some extra complication that we need to deal with.

For example, given an input graph \(G\), all Gomory-Hu tree algorithms for edge connectivity proceed by finding a minimum edge cut \((A, B)\), contract one side, say \(B\), of the cut into a single vertex \(b\), and recurse on the contracted graph denoted by \(G'\). By submodularity of edge cuts, we have that the edge connectivity between any two vertices \(a_1, a_2 \in A\) are preserved in \(G'\). This is crucial for the correctness of the whole algorithm.

Unfortunately, the direct analog of this statement fails completely for element connectivity. For example, suppose \(p, q \notin B\) are disconnected by an element cut \(C\) of graph \(G\). Then in graph \(G'\), \(C' = \{b\} \cup (C \setminus B)\) becomes an element cut disconnecting \(p\) and \(q\). As long as \(C\) contains more than one element in \(B\) (an edge, or a non-terminal vertex), \(|C'| < |C|\), so \(\kappa'(p, q)\) decreased.

To bypass this complication, we actually exploit the generality of element connectivity. When we recurse in a contracted graph, the trick is to add the contracted node into a terminal set for element connectivity. That is, the terminal set will change throughout the recursion so that we can preserve element connectivity between vertices inside the subject graph.

In the rest of this section, we formally prove Theorem 5.1.

**5.1 The Approximate Element Connectivity Gomory-Hu Tree Algorithm**

Instead of proving Theorem 5.1 directly, it is more convenient to prove the following \((1 + \epsilon)\)-approximation version, which is precisely the element connectivity analog of the result in [LP21].

**Theorem 5.2.** A \((1 + \epsilon)\)-approximate Gomory-Hu tree for element connectivity w.r.t. graph \(G\) and terminal set \(U\) can be computed in \(O(\epsilon^{-1}T_{\text{flow}}(m))\) time, where \(T_{\text{flow}}(m)\) is the time to compute max flow in an \(m\)-edge graph.

Theorem 5.1 almost follows from Theorem 5.2 just by setting \(\epsilon = 1/k\). However, there is some minor things to take care of, and we give the formal proof in Section 5.2.

Before we give the proof of Theorem 5.2, observe that as a simplifying assumption, Lemma 2.3 (isolating cuts with forbidden terminals) required that \(I \cup F\) be an independent set. We can force any instance to satisfy this property by subdividing all edges in \(E \cap (I \cup F)^2\). As a consequence, from now we can assume that all element cuts in the modified graph consist solely of vertices.\(^{11}\)

**5.1.1 Algorithm**

The precise algorithm for Theorem 5.2 is described in Algorithm 1. For the reader to better understand, we first briefly explain how the algorithm works, how the input and the output of the algorithm relate to Definition 2.2.

\(^{11}\)If \((x, y)\) is subdivided into \((x, v_{xy}), (v_{x,y}, y)\), then any vertex-only element cut containing \(v_{x,y}\) in the modified instance contains \((x, y)\) in the original instance, and vice-versa.
The basic framework of the algorithm is a recursion. For a random set of terminals \( R^i \), we call Lemma 2.3 to compute its isolating cuts \( S^i_v \). We select those \( v \) such that \( S^i_v \leq (1 + \epsilon) \lambda \) and \( |S^i_v \cap U| \leq |U|/2 \) into set \( R^i_{\text{sm}} \). Now \( \partial S^i_v \) is a \((1 + \epsilon)\)-approximation to the minimum element cut between \( S^i_v \) and \( V \setminus (\mathcal{S}^i_v \cup \partial S^i_v) \), so we split the problem into sub-graphs \( G_v \) generated by \( S^i_v \) for each \( v \in R^i_{\text{sm}} \) and one large part, which we call \( G_{\text{lg}} \). We specify the new parameters added in the algorithm.

As mentioned earlier, to make sure the element cuts represented by the sub-trees are still exact, or of good approximation in the original graph \( G \), we add a new parameter: the forbidden set \( F \). The vertices in \( F \) are terminals counted when computing the connectivity, but queries related to them are not supported. Accordingly, the output \( T \) is a \((1 + \epsilon)\)-approximate Gomory-Hu tree, that represents element connectivity in \( G \) with the terminal set \( U \cup F \), while the tree nodes only represent vertices from \( U \).\(^{12}\) For \( u, v \in U \), the bottleneck edge weight between \( f(u) \) and \( f(v) \) on \( T \) is \( \kappa'_{G, U \cup F}(u, v) \). In the top-level call \( F \) is set to \( \emptyset \), but may accumulate up to \( O(|U|) \) vertices in the recursion.

As will be proved in Lemma 5.3, the element connectivity between vertices inside \( U_{\text{lg}} \) are preserved exactly in the contracted graph \( G_{\text{lg}} \). For the small graphs \( G_v \), by Lemma 5.4, we accept a \((1 + \epsilon)\)-factor approximation to element connectivity, which accrues to \((1 + \epsilon)^{\log |U|} \) as a vertex can only appear in the small branch of the recursion \( \log |U| \) times.

To link the sub-trees at correct nodes and to compute \( f \), we added a tool function \( g : V \rightarrow V_T \cup \{ \perp \} \) that maps vertices of \( G \) into tree nodes of \( T \), or a symbol \( \perp \). \( g \) indicates at which tree node we should link two sub-trees. Taking \( G_{\text{lg}} \) as example again, as the recursion goes down, and each \( y_v \) would finally appear in the base case for some tree node \( t_v \), we would set \( g(y_v) = t_v \). From this construction, \( t_v \) is the only terminal that no element cut disconnect \( y_v \) and \( t_v \). Therefore, when linking the sub-tree of \( G_{\text{lg}} \) with \( G_v \), we should link at the tree node \( t_v = g(y_v) \), and the same holds for \( G_v \). Moreover, we will make sure that at the end, \( g(v) \neq \perp \) for every \( v \in U \). From the definition of \( g, g(v) \) with \( v \in U \) satisfies the properties of the embedding function \( f \) in Definition 2.2.

To summarize, to compute \((1 + \epsilon)\)-approximate Gomory-Hu tree, call Algorithm 1 with desired \( G, U, \epsilon, \) and \( F = \emptyset \). It outputs \((T, g, C)\). We set the embedding \( f(v) = g(v) \) for all \( v \in U \), and then return \((T, f, C)\), which satisfies Definition 2.2.

### 5.1.2 Correctness

We prove the results needed to show the correctness of Algorithm 1. Denote \( F_{\text{lg}} = F \cup \{ y_v | v \in R^i_{\text{sm}} \} \) and \( F_v = F \cup \{ x_v \} \).

**Lemma 5.3.** For any two vertices \( p, q \in U_{\text{lg}} \), we have \( \kappa'_{G_{\text{lg}}, U_{\text{lg}} \cup F_{\text{lg}}}(p, q) = \kappa'_{G, U \cup F}(p, q) \).

**Proof.** We first show that \( \kappa'_{G_{\text{lg}}, U_{\text{lg}} \cup F_{\text{lg}}}(p, q) \geq \kappa'_{G, U \cup F}(p, q) \). Suppose \( C \) is a minimum element cut disconnecting \( p \) and \( q \) in \( G_{\text{lg}} \) with terminal set \( U_{\text{lg}} \cup F_{\text{lg}} \). Then \( C \) does not contain any \( y_v \), so \( C \) is still an element cut for \( p \) and \( q \) in \( G \), and therefore \( \kappa'_{G_{\text{lg}}, U_{\text{lg}} \cup F_{\text{lg}}}(p, q) = |C| \geq \kappa'_{G, U \cup F}(p, q) \).

It remains to show that \( \kappa'_{G_{\text{lg}}, U_{\text{lg}} \cup F_{\text{lg}}}(p, q) \leq \kappa'_{G, U \cup F}(p, q) \). Suppose \( C \) is a minimum element cut disconnecting \( p \) and \( q \) in \( G \) with terminal set \( U \cup F \), and let \( A, B \) be the sides containing \( p, q \), respectively. Every \( v \in R^i_{\text{sm}} \) is by definition not in \( C \); let the side of \( C \) containing \( v \) be \( D_v \).

Without loss of generality we assume \( v \) and \( q \) are in different sides. For brevity, all the following unspecified element cuts are with respect to \( G \) and \( U \cup F \). Let \( \partial H = D_v \cup A \), we have that \( \partial H \) is a

\(^{12}\)I.e., \( f : U \rightarrow V(T) \) does not embed \( F \) in \( T \).
**Algorithm 1:** APPROXELEMCNNGHTREE\((G = (V, E), U, F, \epsilon)\)

**input:** The graph \(G = (V, E)\), the terminal set \(U\), the forbidden set \(F\), the approximation accuracy \(\epsilon\)

**output:** A \((1 + \epsilon)\)-approximate element-connectivity Gomory-Hu tree \((T, g, C)\)

1. If \(|U| = 1\) then // The Base Case
   1. Construct \(T\) with one node \(t\), \(g(u) \leftarrow t\) for all \(u \in V(G)\) and \(C\) an empty function
   2. Return \((T, g, C)\)

2. Let \(\lambda \leftarrow \) the global minimum element connectivity // See Remark 5.7
3. Call CUTTHRESHOLDSTEP\((G, U, F, (1 + \epsilon)\lambda)\) and store its output \(s, \{R^{i}_{sm}, R^{j}, S^{i}_{v}\}\)
4. Fix \(i \in \{0, 1, \ldots, \lfloor \log |U| \rfloor \}\) that maximizes \(\big| \cup_{v \in R^{i}_{sm}} (S^{i}_{v} \cap U) \big|\).
5. For each \(v \in R^{i}_{sm}\)
   1. Let \(G_v \leftarrow \) the graph with \(V \setminus (S^{i}_{v} \cup \partial S^{i}_{v})\) contracted into a vertex \(x_v\).
   2. Let \(U_v \leftarrow S^{i}_{v} \cap U\).
   3. Let \((T_v, g_v, C_v) \leftarrow \) APPROXELEMCNNGHTREE\((G_v, U_v, F \cup \{x_v\}, \epsilon)\).

6. Let \(G_{lg} \leftarrow \) the graph with \(S^{i}_{v}\) contracted into a vertex \(y_v\) for each \(v \in R^{i}_{sm}\).
7. Let \(U_{lg} \leftarrow U \setminus \cup_{v \in R^{i}_{sm}} (S^{i}_{v} \cap U)\).
8. Let \((T_{lg}, g_{lg}, C_{lg}) \leftarrow \) APPROXELEMCNNGHTREE\((G_{lg}, U_{lg}, F \cup \{y_v \mid v \in R^{i}_{sm}\}, \epsilon)\).
9. Initialize \(T \leftarrow T_{lg} \cup (\bigcup_{v \in R^{i}_{sm}} T_v)\), and then add edges \((g_v(x_v), g_{lg}(y_v))\) with weight \(\partial S^{i}_{v}\).
   1. Let \(g\) inherit values from \(g_{lg}\) or one of the \(\{g_v\}\). If the value for \(v\) is defined in more than one such function, set \(g(v) \leftarrow \perp\). Let \(C\) inherit the assignment of \(C_{lg}\) and \(\{C_v\}\), and for the new edges set \(C((g_v(x_v), g_{lg}(y_v))) \leftarrow \partial S^{i}_{v}\).
10. Return \((T, g, C)\).

**Algorithm 2:** CUTTHRESHOLDSTEP\((G = (V, E), U, F_{in}, W)\)

1. Set \(s \leftarrow\) a uniformly random vertex in \(U\).
2. \(R^{0} \leftarrow U\).
3. For \(j\) from 0 to \(\lfloor \log |U| \rfloor\)
   1. Call Lemma 2.3 to compute sets \(\{S^{j}_{v} : v \in R^{j}\}\), with \(I = R^{j}\) and \(F = (U \setminus R^{j}) \cup F_{in}\).
   2. Let \(R^{j+1}_{sm} \leftarrow \{v \in R^{j} \setminus \{s\} : |S^{j}_{v} \cap U| \leq |U|/2 \text{ and } |\partial S^{j}_{v}| \leq W\}\).
   3. \(R^{j+1} \leftarrow\) sampling each vertex of \(R^{j}\) with probability \(\frac{1}{2}\), but \(s\) with probability 1.
4. Return \(s\) and, for each \(j\), \(\{R^{j}_{sm}, R^{j}, \{S^{j}_{v} : v \in R^{j}_{sm}\}\}\).
minimum element cut between \{v, p\} and \{q\}, and \(\partial(H \cup S^i_v)\) is an element cut between \{v, p\} and \{q\}, so \(|\partial(H \cup S^i_v)| \geq |\partial H|\). Now that \(\partial S^i_v\) is also a minimum element cut between \{v\} and \(R^i \setminus \{v\}\), and \(\partial(S^i_v \cap H)\) is also an element cut between \{v\} and \(R^i \setminus \{v\}\), we have \(|\partial(H \cap S^i_v)| \geq |\partial S^i_v|\).

By the submodularity of element-connectivity,

\[
|\partial H| + |\partial S^i_v| \geq |\partial(H \cap S^i_v)| + |\partial(H \cup S^i_v)|.
\]

Hence this inequality holds with equality, so \(|\partial(H \cap S^i_v)| = |\partial S^i_v|\). But by Lemma 2.3, \(S^i_v\) is minimum in size among minimum isolating cuts, so \(H \cap S^i_v = S^i_v\), and therefore \(S^i_v \subseteq H\).

If \(D_v = A\), we already have \(S^i_v \subseteq D_v\). If \(D_v \neq A\), then noticing that in \(G - C\), \(D_v\) is still connected to \(A\), while \(S^i_v\) is connected to \(v\), we know that \(S^i_v \cap A = \emptyset\) and \(S^i_v \subseteq D_v\).

We have shown that \(S^i_v \subseteq D_v\). When contracting \(S^i_v\) into a vertex \(x_v\), the cut \(\kappa\) is still an element cut between \(p\) and \(q\) in \(G_{lk}\) with terminal set \(U_{lk} \cup F_{lk}\), and \(\kappa'_{G_{lk},U_{lk} \cup F_{lk}}(p,q) \leq |\kappa| = \kappa'_{G, U \cup F}(p,q)\).

**Lemma 5.4.** For any two vertices \(p, q \in U_v\), we have \(1 \leq \kappa'_{G_v, U_v \cup F_v}(p,q)/\kappa'_{G, U \cup F}(p,q) \leq 1 + \epsilon\).

**Proof.** We first show that \(\kappa'_{G_v, U_v \cup F_v}(p,q) \geq \kappa'_{G, U \cup F}(p,q)\). Suppose \(C\) is a minimum element cut disconnecting \(p\) and \(q\) in \(G_v\) with terminal set \(U_v \cup F_v\). Then \(C\) does not contain \(x_v\), so \(C\) is still an element cut disconnecting \(p\) and \(q\) in \(G\), so \(\kappa'_{G, U \cup F}(p,q) \leq |\kappa| = \kappa'_{G_v, U_v \cup F_v}(p,q)\).

Next, we show that \(\kappa'_{G_v, U_v \cup F_v}(p,q) \leq (1 + \epsilon)\kappa'_{G, U \cup F}(p,q)\). Suppose \(C\) is a minimum element cut disconnecting \(p\) and \(q\) in \(G\) with terminal set \(U \cup F\) and let \(A, B\) be the sides of \(p, q\). Without loss of generality, suppose \(A\) does not contain \(s\). The following unspecified element cuts are with respect to \(G\) and \(U \cup F\).

We have that \(\partial(S^i_v \cup A)\) is an element cut disconnecting \(p\) and \(s\). Therefore, \(|\partial(S^i_v \cup A)| \geq \lambda\). Furthermore, by the definition of \(S^i_v\), \(|\partial S^i_v| \leq (1 + \epsilon)\lambda\). Therefore, by the submodularity of element cuts,

\[
(1 + \epsilon)\lambda + |\partial A| \geq |\partial S^i_v| + |\partial A| \geq \left|\partial(S^i_v \cup A)\right| + \left|\partial(S^i_v \cap A)\right| \geq \lambda + \left|\partial S^i_v \cap A\right|.
\]

Now that \(S^i_v \cap A\) contains \(p\) but not \(q\), \(\partial(S^i_v \cap A)\) is an element cut disconnecting \(p\) and \(q\), and since it is contained in \(G_v\), it is still an element cut in \(G_v\) with terminal set \(U_v \cup F_v\), so \(|\partial(S^i_v \cap A)| \geq \kappa'_{G_v, U_v \cup F_v}(p,q)\). And by definition \(|\partial A| = \kappa'_{G, U \cup F}(p,q) \geq \lambda\).

Therefore,

\[
\kappa'_{G_v, U_v \cup F_v}(p,q) \leq \epsilon \lambda + |\partial A| \leq (1 + \epsilon)\kappa'_{G, U \cup F}(p,q).
\]

**Lemma 5.5.** The assignment \(g(v) = \perp\) occurs if and only if \(v\) appears in \(C(e)\) for some edge. Therefore, \(g\) value of vertices in \(U \cup F\) never equals \(\perp\).

**Proof.** From the construction of \(G_{lk}\) and \(G_v\), it can be seen that only the vertices in \(\partial S^i_v = C((g_v(x_v), g_{lk}(y_v)))\) are defined twice (or more), so their \(g\)-value are set to \(\perp\). These are all the vertices such that \(g(v) = \perp\).

**Remark 5.6.** From Lemma 5.3, Lemma 5.4 and Lemma 5.5, in line 17 of Algorithm 1, the \(g\)-value of \(x_v, y_v\) are not \(\perp\), so linking the sub-trees would be successful. And \(g(v)\) for \(v \in U\) equals to \(\perp\).
Remark 5.7. As stated the algorithm computes the global element connectivity \( \lambda \). In reality \( \lambda \) is a lower bound on this quantity, which is increased once we are sure it has increased by a \( 1 + \epsilon \) factor. As we show in the proof of Theorem 5.2, with high probability, the global element connectivity increases by a factor of \( 1 + \epsilon \) every \( O(\log^2 n) \) steps taken in the “\( G_{ig} \)” branch of the recursion tree. Therefore, what the algorithm actually does is initialize \( \lambda = 1 \), record the recursion depth on the \( G_{ig} \) branch, and update \( \lambda \leftarrow (1 + \epsilon)\lambda \) whenever this depth is a multiple of \( \Theta(\log^3 n) \). In this way, \( \lambda \) never exceeds the global element connectivity w.h.p., which suffices to establish correctness.

5.1.3 Running Time Analysis

The running time analysis in this section closely follows Li and Panigrahi [LP21].

Lemma 5.8. Keeping definitions of \( R^i, R_{sm}^i, S_v^i \) as in line 4, 5, 6 of Algorithm 2. Keeping definitions of \( G_{ig}, U_{ig}, F_{ig} \) as in line 13, 14, 15 of Algorithm 1. Define \( P \subseteq U^2 \) to be

\[
P = \{ (u, v) : \kappa_{G, U \cup F}^i(u, v) \leq W, \text{ and if } A \text{ is the side of the minimum } u-v \text{ element cut containing } u, \text{ then } |A \cap U| \leq |U|/2 \}.
\]

Similarly define \( P_{ig} \) with \( G_{ig}, U_{ig}, F_{ig} \) and \( W \). Then

\[
\mathbb{E}(|P_{ig}|) \leq \left( 1 - \Omega \left( \frac{1}{\log^2 |U|} \right) \right) |P|.
\]

Proof. We need to lower bound the size of \( Q = P \setminus P_{ig} \). Define \( U_{sm}^i = \bigcup_{v \in R_{sm}^i} (S_{v}^i \cap U) \), \( U_{sm} = \bigcup_{i=1}^{\log |U|} U_{sm}^i \), and \( U^* = \{ v \mid (v, s) \in P \} \). We will show that \( \mathbb{E}(|Q|) \geq \Omega \left( \frac{1}{\log^2 n} \right) |P| \) follows from the following three claims.

1. For each \( u \in U^* \), there are at least \( |U|/2 \) vertices \( v \) such that \( (u, v) \in P \).
2. \( \mathbb{E}(|U_{sm}|) \geq \Omega \left( |U^*|/\log |U| \right) \).
3. For each \( (u, v) \in P, u \in U^* \) with probability at least \( 1/2 \);

For Claim (1), consider the minimum element cut disconnecting \( s \) and \( u \in U^* \). There are at least \( |U|/2 \) terminals \( v \) not in the same side as \( u \), and each \( (u, v) \in P \). Claim (2) is proved in Lemma 5.9. Claim (3) holds because \( s \) is randomly chosen from \( U \) and there are at least \( |U| - |U|/2 = |U|/2 \) terminals not in the side of \( u \). When \( s \) is such a terminal, \( u \in U^* \).

The algorithm fixes \( i \in \{1, 2, \cdots, \log |U| \} \) that maximizes \( |U_{sm}^i| \). Then,

\[
\mathbb{E}(|Q|) \geq \frac{|U|}{2} \mathbb{E}(|U_{sm}^i|) \geq \frac{|U|}{2 \log |U|} \mathbb{E}(|U_{sm}|) \geq \frac{|U|}{2 \log |U|} \frac{|U^*|}{\log |U|} \geq \Omega \left( \frac{1}{\log^2 |U|} \right) |P|.
\]

For the first inequality, Claim (1) implies that each \( v \in U_{sm}^* \) is involved in \( |U|/2 \) pairs in \( P \), all of which do not appear in \( P_{ig} \) whenever \( v \in U_{sm}^i \). The second inequality follows from the choice of \( i \). The third inequality follows from Claim (2). The fourth inequality follows from Claim (3) and the following bound on the size of \( P \).

\[
|P|/2 \leq \mathbb{E}(\{|(u, v) \in P : u \in U^*\}|) \leq |U| \cdot |U^*|.
\]

Note each \( u \) appears in at most \( |U| \) pairs of \( P \).
Lemma 5.9 (Claim (2) restated). $\mathbb{E}(|U_{\text{sm}}|) = \Omega(|U^*| / \log |U|)$.

Proof. Root the element connectivity Gomory-Hu tree $T$ at $s$. For each vertex $v \in U$, let $U_v$ be the set of terminals in the subtree rooted at $v$. For a terminal $v \in U$, we find the edge $e(v)$ along the path from $s$ to $v$ with minimum weight, and when not unique, the one with maximum depth. Let $r(v)$ be the deeper endpoint of $e(v)$. By the definition of $U^*$, a terminal $v \in U^*$ if and only if $w(e(v)) \leq W$ and $|U_{r(v)}| \leq |U|/2$.

We say that a vertex $v \in U^*$ is active if $v \in R^{i(v)}$ where $i(v) = \lfloor \log |U_{r(v)}| \rfloor$. In addition, if $U_{r(v)} \cap R^{i(v)} = \{v\}$, then we say that $v$ hits all of the vertices in $U_{r(v)}$, including itself. For completeness, we define vertices in $U \setminus U^*$ to be inactive; they do not hit other vertices. Now we show that

(a) each vertex that is hit is in $U_{\text{sm}}$;
(b) the total number of pairs $(u, v)$ for which $v \in U^*$ hits $u$ is $\Omega(|U^*|)$ in expectation;
(c) each vertex $u$ is hit by at most $O(\log |U|)$ vertices in $v \in U^*$.

For (a), suppose $u$ is hit by $v$. Then by definition, $U_{r(v)} \cap R^{i(v)} = \{v\}$. The isolating cut for $v$ returned by Lemma 2.3 corresponds to the edge joining $r(v)$ to its parent, so all vertices in $U_{r(v)}$ are on $v$’s side of the cut, and appear in $U_{\text{sm}}$, because $v \in R^{i(v)}_{\text{sm}}$, since $|S^{i(v)}_v \cap U| = |U_{r(v)}| \leq |U|/2$ and $w(e(v)) \leq W$.

For (b), the probability that $v \in R^{i(v)}$ and $v$ is the only such vertex is $(1 - 2^{-i(v)})|U_{r(v)}|-12^{-i(v)} = \Theta(1/2^i)$, and when it happens, it hits $|U_{r(v)}| = \Omega(2^{i(v)})$ vertices, so the contribution in the expectation is $\Omega(1)$. Since each $v \in U^*$ contributes $\Omega(1)$ in expectation, their sum is $\Omega(|U^*|)$.

For (c), we first show that for any different vertices $v, w \in U^*$ that both hit $u$, $i(v) \neq i(w)$. Since $u \in U_{r(v)}$ and $u \in U_{r(w)}$, without loss of generality we assume $r(v) \subseteq U_{r(w)}$, so $U_{r(v)} \subseteq U_{r(w)}$. From the definition of $R^i, R^0 \subseteq R^1 \subseteq \cdots R^{\lfloor \log |U| \rfloor}$, so $R^{i(v)} \cap R^{i(w)} = R^{\max(i(v),i(w))}$. Then,

$$\emptyset = \{v\} \cap \{w\} = (R^{i(v)} \cap U_{r(v)}) \cap (R^{i(w)} \cap U_{r(w)}) = R^{\max(i(v),i(w))} \cap U_{r(v)},$$

because $R^{i(v)} \cap U_{r(v)} = \{v\} \neq \emptyset$, we conclude that $\max(i(v),i(w)) > i(v)$, so $i(v) < i(w)$. Then, since $i(v) \in [1, \log |U|]$ has at most $O(\log |U|)$ kinds of choices, $u$ is hit by at most $O(\log |U|)$ vertices.

Finally, the proof follows from

$$\mathbb{E}[U_{\text{sm}}] \geq \mathbb{E}[\{|u : u \text{ is hit}\}] \geq \frac{\mathbb{E}[\{(u, v) : v \in U^*, u \text{ is hit by } v\}]}{O(\log |U|)} \geq \Omega(\frac{\mathbb{E}[U^*]}{\log |U|}).$$

The first inequality is because claim (a); the second inequality is because claim (c); and the third inequality is because claim (b).

5.2 Proof of Theorem 5.2 and Theorem 5.1

Now we are ready to give the proof for Theorem 5.2 and Theorem 5.1.
Proof of Theorem 5.2. The recursion makes progress in one of two ways. In the “non-$G_{lg}$” branches \( \{G_v\} \), each \( G_v \) contains at most half the number of terminals. Suppose we follow the “$G_{lg}$” branch \( \Theta(\log^3 n) \) times, yielding \( G', U', F' \) and \( P' \). By Lemma 5.8, with \( W = (1 + \epsilon)\lambda \), \( E(|P'|) \leq (1 - \Omega(1/\log^2 n))^{\Theta(\log^3 n)}|P| = n^{-\Omega(1)} \), meaning \( P' = \emptyset \) is empty w.h.p. and the global minimum element cut of \( G', U' \cup F' \) has increased to at least \((1 + \epsilon)\lambda \) and we can update \( \lambda \) accordingly.

This implies the total depth of recursion is \( O(\epsilon^{-1} \log^4 n) \) w.h.p. The total size of all graphs on each layer of recursion is \( O(m) \), hence by Lemma 2.3, the total time is \( O(\epsilon^{-1} \log^4 n \cdot T_{\text{flow}}(m) \log n) = \tilde{O}(\epsilon^{-1} T_{\text{flow}}(m)) \).

As for the correctness, by Lemma 5.3 the $G_{lg}$-branch preserves the exact value of \( \kappa'_{G,U,F} \), and all the non-$G_{lg}$ branches \( \{G_v\} \) introduce a \((1 + \epsilon)\)-factor approximation to the element connectivity. Since the depth of recursion in the non-$G_{lg}$ branches is at most \( \log |U| \), the tree returned is a \((1 + \epsilon)^{\log |U|} = 1 + \epsilon_0 \) approximate Gomory-Hu tree, for \( \epsilon = \epsilon_0 / \log |U| \). Expressed in terms of \( \epsilon_0 \), the running time is still \( \tilde{O}(\epsilon_0^{-1} T_{\text{flow}}(m)) \).

Proof of Theorem 5.1. The proof follows from the proof of Theorem 5.2 by setting \( \epsilon = \Theta(\frac{1}{k}) \), and we only address the difference of \( k \)-Gomory-Hu tree.

To prove correctness, a new base case is added\(^{13}\): if \( \lambda > k \), stop recursion, construct \( T \) with one node \( t \), set \( g(u) \leftarrow t \) for all \( u \in G \) and \( C \) an empty function and return \( (T, g, C) \). At the final output, construct \( f \) as \( f(v) \leftarrow g(v) \) for all \( v \in U \). Then \( (T, f, C) \) works exactly as in Definition 2.2.

To bound running time, since we are not concerned with cut-values exceeding \( k \), when calling Lemma 2.3 we can use any unit capacity flow algorithm, which runs in \( O(T_{\text{uflow}}(m + k |U|)) \) time, so the total time bound is \( \tilde{O}(kT_{\text{uflow}}(m + k |U|)) \).

6 Conclusion

In this paper we proved that \( \Omega(\min(kn/\log n)) \) space is necessary for encoding vertex connectivity information up to \( k \). This establishes the optimality of several previous results. For example, Nagamochi-Ibaraki [NI92] sparsifiers encode all vertex connectivities up to \( k \), but their space cannot be improved much, even if the format of the representation is not constrained to be a graph. It also implies that even the average length of the Izsak-Nutov [IN12] labeling scheme cannot be improved much. We improved [IN12] to have near-optimal query time \( O(\log n) \), independent of \( k \), and improved its the construction time to nearly max-flow time.

Here we highlight a few open problems.

- There is a trivial \( \Omega(kn) \) space lower bound for data structures answering \( v \)-cut queries. Our data structure (and Izsak-Nutov [IN12]) can be augmented to support fast \( v \)-cut queries with \( O(k^2 n \log n) \) space. Is this necessary? Note that if it is, the lower bound cannot be purely information-theoretic; it must hinge on the requirement that queries be answered efficiently.\(^{14}\)

- A special case of the vertex connectivity oracle problem is answering \( v \)-\text{conn}(u, v) \) queries when \( k = \kappa(G) + 1 \). In other words, decide whether \( u, v \) are separated by a globally minimum cut. Globally minimum vertex cuts have plenty of structure [PY21, CDBKT93], but it is still not clear whether \( \tilde{O}(kn) \) bits are necessary to answer such queries.

\(^{13}\)It can be inserted between line 5 and 6 in Algorithm 1

\(^{14}\)There are two natural ways to answer \( v \)-cut queries “optimally,” either enumerate their elements in \( O(k) \) time, or return a pointer in \( O(1) \) time to a pre-stored list of elements. The latter model was recently advocated by Nutov [Nut21] and seems to be easier to characterize from a lower bound perspective.
• Can a \(k\)-Gomory-Hu tree for element connectivity be constructed in \(\tilde{O}(m + n\text{poly}(k))\) time, perhaps by extending the methods of \([FNY^{+}20, PY21]\)?

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