1. Introduction

This paper is inspired by the talk of Yiannis Sakellaridis in the Simons Symposium held at the Schloss Elmau in April 2018. Let us begin by describing the relevant context for his talk.

The study of periods of automorphic forms has been an important theme in the Langlands program, beginning with the early work of Harder-Langlands-Rapoport and Jacquet. In particular, the nonvanishing of certain periods is known to characterize the image of certain Langlands functorial lifting and to be related to the analytic properties of certain automorphic $L$-functions. An effective approach for proving such results is the technique of relative trace formulae developed by Jacquet. Typically, such an approach involves the comparison of the geometric sides of two relative trace formulae, which results in a global spectral identity and an accompanying family of local relative character identities.

In [23], Sakellaridis and Venkatesh initiated a general framework for treating such period problems in the context of spherical varieties. In particular, to a spherical variety $X = H\backslash G$ over a local field $F$ or a global field $k$, they associated a Langlands dual group $X^\vee$ (at least when $G$ is split), together with a canonical (up to conjugacy) map

$$\iota : X^\vee \times \text{SL}_2(\mathbb{C}) \to G^\vee.$$  

They then conjectured, among other things, that representations of $G$ (in the automorphic dual) which have nonzero $H$-periods are those belonging to A-packets whose associated A-parameters factor through $\iota$. This means roughly that the $H$-distinguished representations of $G$ are those which are Langlands functorial lift via $\iota$ from a (split) group $G_X$ whose dual group $G_X^\vee$ is $X^\vee$. Experience shows that it is sometimes more pertinent to regard $H$-distinguished representations of $G$ as lifted from a strongly tempered spherical variety such as the Whittaker variety $(N_X, \psi)\backslash G_X$, as opposed to the group variety $G_X$ itself.

The conjecture of Sakellaridis-Venkatesh can be made on several fronts. We give a brief description of the various incarnations of their conjecture (at least a first approximation), under some simplifying hypotheses and without using the language of A-parameters.

(a) In the context of smooth representation theory of $G(F)$ over a local field $F$, one is interested in determining $\text{Hom}_H(\pi, \mathbb{C})$ for any $\pi \in \text{Irr}(G(F))$. One expects (in some instances) a map

$$\iota_* : \text{Irr}(G_X(F)) \to \text{Irr}(G(F)),$$

such that for any $\pi \in \text{Irr}(G(F))$, there is an isomorphism

$$f : \bigoplus_{\sigma, \iota_*(\sigma) = \pi} \text{Hom}_{N_X}(\sigma, \psi) \cong \text{Hom}_H(\pi, \mathbb{C}).$$

In the smooth setting, the Sakellaridis-Venkatesh conjecture thus gives a precise quantitative formulation of the expectation that $H$-distinguished representations of $G$ are lifted from $G_X$. 

If further \( \iota_* \) is injective, there is at most one term on the left hand side, and all these Hom spaces are at most one-dimensional (by the uniqueness of Whittaker models). This will be the favourable situation encountered in this paper. In such instances, if \( L \in \text{Hom}_{N_X}(\sigma, \psi) \), with corresponding \( f(L) \in \text{Hom}_H(\iota_*(\sigma), \mathbb{C}) \), one can define relative characters \( B_{\sigma,L} \) and \( B_{\iota_*(\sigma),f(L)} \) which are certain equivariant distributions on \((N_X, \psi) \setminus G_X \) and \( X \) respectively. In this case, one might expect a relative character identity relating \( B_{\sigma,L} \) and \( B_{\iota_*(\sigma),f(L)} \).

(b) In the context of \( L^2 \)-representation theory, one is interested in obtaining the spectral decomposition of the unitary representation \( L^2(X) \) of \( G \). By abstract results of functional analysis, one has a direct integral decomposition

\[
L^2(X) \cong \int_{\Omega} \pi_\omega \, d\mu_X(\omega)
\]

for some measure space \((\Omega, d\mu_X)\) and \( \pi : \omega \mapsto \pi_\omega \) a measurable map from \( \Omega \) to the unitary dual \( \hat{G} \) of \( G \). There is some fluidity in this direct integral decomposition; for example, given \( \Omega \), only the measure class of \( d\mu_X \) is well-defined.

In this \( L^2 \)-setting, the crux of the Sakellaridis-Venkatesh conjecture is to provide a canonical candidate for \((\Omega, d\mu_X, \pi)\). Namely, one expects a map

\[
\iota_* : \hat{G}_X \rightarrow \hat{G}
\]

associated to \( \iota \) from the unitary dual of \( G_X \) to that of \( G \), so that one has a (unitary) isomorphism

\[
L^2(X) \cong \int_{\hat{G}_X} m(\sigma) \cdot \iota_*(\sigma) \, d\mu_{G_X}(\sigma),
\]

where \( d\mu_{G_X} \) denotes the Plancherel measure of \( G_X \) and \( m(\sigma) \) is a multiplicity space which is typically isomorphic to the dual space of \( \text{Hom}_{N_X}(\sigma, \psi) \). In other words, one may take \((\Omega, d\mu_X, \pi)\) to be \((\hat{G}_X, m(-) \cdot d\mu_{G_X}, \iota_* )\). One can think of this as saying that the spectral decomposition of \( L^2(X) \) is obtained from the Whittaker-Plancherel theorem

\[
L^2(N_X, \psi \setminus G_X) \cong \int_{\hat{G}_X} m(\sigma) \cdot \sigma \, d\mu_{G_X}(\sigma)
\]

by applying \( \iota_* \). One consequence of this spectral decomposition is that it provides a canonical element \( L_\sigma \in \text{Hom}_H(\iota_*(\sigma), \mathbb{C}) \), as we explained in \([2]\) for \( m(-) \cdot d\mu_{G_X} \)-almost all \( \sigma \).

(c) Globally, when \( k \) is a global field with ring of adeles \( \mathbb{A} \), one considers the global period integral along \( H \):

\[
P_H : A_{\text{cusp}}(G) \rightarrow \mathbb{C}
\]

defined by

\[
P_H(\phi) = \int_{H(k) \setminus H(\mathbb{A})} \phi(h) \, dh
\]
on the space of cusp forms on \( G \). The restriction of \( P_H \) to a cuspidal representation \( \Pi = \otimes_v \Pi_v \) of \( G \) then defines an element \( P_{H,\Pi} \in \text{Hom}_{H(\mathbb{A})}(\Pi, \mathbb{C}) \). One is interested in two problems in the global setting:

(i) characterising those \( \Pi \) for which \( P_{H,\Pi} \) is nonzero as functorial lifts from \( G_X \) via the map \( \iota \);

(ii) seeing if \( P_{H,\Pi} \) can be decomposed as the tensor product of local functionals.

Such a factorization certainly exists in the instances discussed in this paper since the local Hom spaces \( \text{Hom}_{H(F_v)}(\Pi_v, \mathbb{C}) \) is at most 1-dimensional for all places \( v \). In (a), we have seen that these Hom spaces are nonzero precisely when \( \Pi_v = \iota_*(\sigma_v) \) for some \( \sigma_v \in \text{Irr}(G_X(k_v)) \). Thus, in the context of the
In this rank 1 setting, the group instructive to observe the crucial unifying role played by the typically ignored hyperboloid (or a sphere) in an n-dimensional quadratic space, and a more exotic example is Spin varieties the canonical basis elements in the relevant local Hom spaces for use in the factorization of the global relative trace formula of as a first step towards establishing local relative character identities and effecting a global comparison of the spaces of test functions may be larger than the space of compactly supported smooth functions; this necessitates that one defines a normalization of $L_{\Pi_v}$ by:

$$L_{\Pi_v}^\# = \frac{1}{|L_{X,v}(1/2, \Sigma_v)|^{1/2}} \cdot L_{\Pi_v}$$

Then the main issue with the second global problem is to determine the constant $c(\Pi)$ such that

$$|P_{H,\Pi}(\phi)|^2 = c(\Pi) \cdot L_X(1/2, \Pi) \cdot \prod_v |L_{\Pi_v}^\#(\varphi_v)|^2 \quad \text{for } \phi = \otimes_v \varphi_v \in \otimes_v \Pi_v.$$  

Here the global $L$-function $L_X(s, \Sigma)$ is defined by the Euler product $\prod_v L_{X,v}(s, \Sigma_v)$ for $\text{Re}(s) \gg 0$ and needs to be meromorphically continued so that one can evaluate it at $s = 1/2$.

This concludes our brief and simplified description of the Sakellaridis-Venkatesh conjecture. It is instructive to observe the crucial unifying role played by the typically ignored $L^2$-theory, which supplies the canonical basis elements in the relevant local Hom spaces for use in the factorization of the global periods.

We can now describe the content of Sakellaridis’ lecture at the Simons Symposium. In a series of recent papers [20, 21, 22], Sakellaridis examined aspects of the above program in the context of rank 1 spherical varieties $X$. There is a classification of such rank 1 $X$’s, but a standard example is $X \cong SO_{n-1} \backslash SO_n$, i.e. a hyperboloid (or a sphere) in an n-dimensional quadratic space, and a more exotic example is $Spin_6 \backslash F_4$. In this rank 1 setting, the group $G_X$ is $SL_2$ or its variants (such as $PGL_2$ or $Mp_2$). For example, for $X = SO_{n-1} \backslash SO_n$ with $n$ even, $X^\vee \cong PGL_2(\mathbb{C})$, so that $G_X \cong SL_2$ and the map $\iota$ is given by:

$$\iota : PGL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \xrightarrow{\text{Sym}^3 \times \text{Sym}^{n-4}} SO_3(\mathbb{C}) \times SO_{n-3}(\mathbb{C}) \rightarrow SO_n(\mathbb{C}).$$

On the other hand, if $n$ is odd, then $X^\vee \cong SL_2(\mathbb{C})$ and we take $G_X \cong Mp_2$, with the map $\iota$ given by

$$\iota : SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \xrightarrow{\text{Sym}^3 \times \text{Sym}^{n-4}} Sp_2(\mathbb{C}) \times Sp_{n-3}(\mathbb{C}) \rightarrow Sp_{n-1}(\mathbb{C}).$$

In such rank 1 setting, Sakellaridis developed a theory of transfer of test functions from $X$ to $(N, \psi) \backslash G_X$ as a first step towards establishing local relative character identities and effecting a global comparison of the relative trace formula of $X$ and the Kuznetsov trace formula for $G_X$. The formula for the transfer map he discovered was motivated by considering an analogous transfer for the boundary degenerations of $X$ and $(N_X, \psi) \backslash G_X$. For the hyperboloid $SO_{n-1} \backslash SO_n$, the boundary degeneration is simply the cone of nonzero null vectors in the underlying quadratic space. In any case, the transfer map he wrote down differs from the typical transfer map in the theory of endoscopy in two aspects:

• the spaces of test functions may be larger than the space of compactly supported smooth functions;
• the transfer map in endoscopy is carried out via an orbit-by-orbit comparison, whereas the transfer map in this relative setting is more global in nature, involving an integral kernel transformation reminiscent of the Fourier transform.

An ongoing work [12] of D. Johnstone and R. Krishna establishes the fundamental lemma for the basic functions in the space of test functions; this is necessary for the comparison of relative trace formulae. In the special case when \( n = 4 \), one has:

\[
X = \text{SO}_3 \backslash \text{SO}_4 \cong \text{PGL}_2 \backslash (\text{SL}_2 \times \text{SL}_2) / \mu_2^\Delta.
\]

The relative trace formula for this \( X \) is essentially the stable trace formula for \( \text{SL}_2 \). Thus, the expected comparison of relative trace formulae is between the stable trace formula for \( \text{SL}_2 \) and the Kuznetsov trace formula for \( \text{SL}_2 \). The local transfer in this case was first investigated in the thesis work of Z. Rudnick. The discussion of these results was the content of Sakellaridis’s lecture in the Simons Symposium.

On the other hand, the spectral analysis of \( L^2(X) \) when \( X = \text{SO}_{n-1} \backslash \text{SO}_n \) or the analysis of the \( \text{SO}_{n-1} \)-period for representations of \( \text{SO}_n \) (both locally and globally) is familiar from the theory of theta correspondence. The \( L^2 \)-theory was studied in the early work of Strichartz [16] and Howe [11]. In a paper [7] by the first author and R. Gomez, the \( L^2 \)-theory was treated using theta correspondence for essentially general rank 1 spherical varieties from the viewpoint of the Sakellaridis-Venkatesh conjecture. For the smooth theory, one can see the recent expository paper [6]. In the case of \( X = \text{SO}_{n-1} \backslash \text{SO}_n \), it was known that \( \text{SO}_{n-1} \)-distinguished representations of \( \text{SO}_n \) are theta lifts (of \( \psi \)-generic representations) from \( \text{SL}_2 \) or \( \text{Mp}_2 \) according to whether \( n \) is even or odd. Indeed, the theta lifting from \( \text{SL}_2 \) or \( \text{Mp}_2 \) to \( \text{SO}_n \) realises the functorial lifting (at least at the level of unramified representations) predicted by the map \( \iota : X^\vee \times \text{SL}_2(\mathbb{C}) \to \text{SO}_n(\mathbb{C}) \). As such, it is very natural to ask if the results discussed in Sakellaridis’ talk can be approached from the viewpoint of the theta correspondence.

This paper is the result of this investigation. In short, its main conclusion is that the theory of transfer developed by Sakellaridis can be very efficiently developed using the theta correspondence. More precisely,

• one can give a conceptual definition of the transfer and the relevant spaces of test functions (Definition 7.1), from which the fundamental lemma (for the basic function and its translate by the spherical Hecke algebra) follows readily (see Lemmas 7.5 and 7.6);

• one can establish the desired relative character identities highlighted in (a) above, without doing a geometric comparison; (see Theorem 8.1);

• one can express this conceptually defined transfer in geometric terms, from which one sees that it agrees with Sakellaridis’ formula (see Proposition 9.1);

• one can address the two global problems highlighted in (c) above (see Theorem 10.5).

We leave the precise formulation of the results to the main body of the paper. We would like to remark that, as far as we are aware, the paper [1] of Baruch-Lapid-Mao is the first instance where one finds a derivation of relative character identities using the theta correspondence. In addition, it has been known to practitioners that the theory of theta correspondence is useful for addressing period problems in the smooth local context, the global context, as well as in the local \( L^2 \)-context [6, 7], with similar computations and parallel treatment in the various settings. One goal of this paper is to demonstrate how the treatment of the 3 different threads can be synthesised into a rather coherent story.

Here is a short summary of the contents of this paper. In §2, we recall some foundational results of Bernstein [2] on spectral decomposition of \( L^2(X) \). These results provide the mechanism for us to navigate between the \( L^2 \)-setting and the smooth setting. We illustrate Bernstein’s general theory in the setting of
the Harish-Chandra-Plancherel formula and the Whittaker-Plancherel formula in §3. In §4, we recall the setup of theta correspondence, especially a recent result of Sakellaridis [19] on the spectral decomposition of the Weil representation when restricted to a dual pair. Using the theory of theta correspondence, we address in §5 the local problems (a) and (b), except for the part involving relative character identities. After recalling the notion of relative characters in §6, we come to the heart of the paper (§7-8), where we develop the theory of transfer and establish some of its key properties, culminating in the relative character identity. We verify that our transfer map is the same as that of Sakellaridis’ in §9, where we describe the transfer in geometric terms. The final §10 discusses and resolves the global problems.

Acknowledgments: The first author thanks Sug Woo Shin, Nicholas Templier and Werner Mueller for their kind invitation to participate in the Simons Symposium and the Simons Foundation for providing travel support. He also thanks Yiannis Sakellaridis for helpful conversations on the various topics discussed in this paper. The first author is partially supported by a Singapore government MOE Tier 2 grant R146-000-233-112, whereas the second author is supported by an MOE Graduate Research Scholarship.

2. Spectral Decomposition à la Bernstein

Let \( F \) be a local field and \( G \) a reductive group over \( F \) acting transitively on a variety \( X \). We fix a base point \( x_0 \in X(F) \), with stabilizer \( H \subset G \), so that \( g \mapsto g^{-1} \cdot x_0 \) gives an identification \( H \backslash G \cong X \). For simplicity, we shall write \( X = G(F) \cdot x_0 \cong H(F) \backslash G(F) \).

2.1. Direct integral decompositions. Suppose that there is a \( G \)-invariant measure \( dx \) on \( X \), in which case we may consider the unitary representation \( L^2(X) \) of \( G \), with \( G \)-invariant inner product

\[
\langle \phi_1, \phi_2 \rangle_X = \int_X \phi_1(x) \cdot \overline{\phi_2(x)} \, dx.
\]

Such a unitary representation admits a direct integral decomposition

\[
i : L^2(X) \cong \int_\Omega \sigma(\omega) \, d\mu(\omega).
\]

Here,

- \( \Omega \) is a measurable space, equipped with a measure \( d\mu(\omega) \);
- \( \sigma : \omega \mapsto \sigma(\omega) \) is a measurable map from \( \Omega \) to the unitary dual \( \hat{G} \) of \( G \) (equipped with the Fell topology and the corresponding Borel measurable structure).

In this section, we give an exposition of some results of Bernstein [2] which provide some useful ways of understanding the above direct integral decomposition. This viewpoint of Bernstein underpins the results of this paper.

2.2. Pointwise-defined morphisms. Let \( S \subset L^2(X) \) be a subspace which is \( G \)-stable. Following Bernstein [2, §1.3], one says that the inclusion \( S \hookrightarrow L^2(X) \) is pointwise-defined if there exists a family of \( G \)-equivariant morphisms \( \alpha_{\sigma(\omega)} : S \rightarrow \sigma(\omega) \) for \( \omega \in \Omega \) such that for each \( \phi \in S \), the function

\[
\omega \mapsto \alpha_{\sigma(\omega)}(\phi)
\]

represents the vector \( \phi \in S \subset L^2(X) \) under the isomorphism \( \iota \) in the direct integral decomposition \eqref{2.1}. Such a family \( \{\alpha_{\sigma(\omega)} : \omega \in \text{supp}(d\mu)\} \) is essentially unique, in the sense that any two such families differ only on a subset of \( \Omega \) with measure zero with respect to \( d\mu \).
2.3. The maps $\alpha_{\sigma(\omega)}$ and $\beta_{\sigma(\omega)}$. A basic result of Bernstein \cite{Bernstein} Prop. 2.3], obtained as an application of the Gelfand-Kostyuchenko method \cite{Bernstein} Thm. 1.5], is that the natural inclusion $C^\infty_c(X) \hookrightarrow L^2(X)$ is pointwise-defined. We let $\{\alpha_{\sigma(\omega)} : \omega \in \text{supp}(d\mu)\}$ be the associated family of $G$-equivariant morphisms as above.

When $F$ is $p$-adic, the elements in $C^\infty_c(X)$ are smooth vectors and so the image of each $\alpha_{\sigma(\omega)}$ is contained in the space $\sigma(\omega)^\infty$ of smooth vectors in $\sigma(\omega)$. The map $\alpha_{\sigma(\omega)}$ is nonzero for $d\mu$-almost all $\omega$, in which case the image is precisely $\sigma(\omega)^\infty$. To simplify notation, we shall write $\sigma(\omega)$ in place of $\sigma(\omega)^\infty$, trusting that the context will make it clear whether one is working with a unitary representation on a Hilbert space or a smooth representation. In particular, $\alpha_{\sigma(\omega)} \in \text{Hom}_G(C^\infty_c(X), \sigma(\omega))$.

If $\alpha_{\sigma(\omega)}$ is nonzero, then by duality, one obtains a $G$-equivariant embedding

$$\overline{\beta_{\sigma(\omega)}} : \sigma(\omega)^\vee \cong \sigma(\omega) \longrightarrow C^\infty(X).$$

Here, the isomorphism $\sigma(\omega)^\vee \cong \sigma(\omega)$ is induced by the fixed inner product $\langle -, - \rangle_{\sigma}$. Taking complex conjugate on $C^\infty(X)$, we obtain a $G$-equivariant linear map

$$\beta_{\sigma(\omega)} : \sigma(\omega) \longrightarrow C^\infty(X).$$

The maps $\alpha_{\sigma(\omega)}$ and $\beta_{\sigma(\omega)}$ are thus related by the adjunction formula:

$$\langle \alpha_{\sigma(\omega)}(\phi), v \rangle_{\sigma(\omega)} = \langle \phi, \beta_{\sigma(\omega)}(v) \rangle_X, \quad \text{for } \phi \in C^\infty_c(X) \text{ and } v \in \sigma(\omega).$$

If we compose $\beta_{\sigma(\omega)}$ with the evaluation-at-$x_0$ map $ev_{x_0}$, we obtain

$$\ell_{\sigma(\omega)} := ev_{x_0} \circ \beta_{\sigma(\omega)} \in \text{Hom}_H(\sigma(\omega), \mathbb{C}).$$

Thus the direct integral decomposition gives rise to a family of canonical elements $\ell_{\sigma(\omega)} \in \text{Hom}_H(\sigma(\omega), \mathbb{C})$ for $\omega \in \text{supp}(d\mu)$. This family depends on the isomorphism $\iota$ in \cite{Bernstein} 2.1]; changing $\iota$ will result in another family which differs from the original one by a measurable function $f : \text{supp}(d\mu) \longrightarrow S^1$. Thus, the family

$$\{\alpha_{\sigma(\omega)} \otimes \overline{\alpha_{\sigma(\omega)}} : \omega \in \text{supp}(d\mu)\}$$

is independent of the choice of the isomorphism $\iota$ in \cite{Bernstein} 2.1]. Likewise, the family

$$\{\beta_{\sigma(\omega)} \otimes \overline{\beta_{\sigma(\omega)}} : \omega \in \text{supp}(d\mu)\}$$

is independent of $\iota$.

2.4. Schwarz space of $X$. In \cite{Bernstein} Pg. 689], Bernstein showed that the space $X$ has a naturally associated Harish-Chandra Schwarz space $\mathcal{C}(X)$ which is $G$-stable and which contains $C^\infty_c(X)$. Moreover, $\mathcal{C}(X)$ has a natural topology, defined by a family of semi-norms, such that $C^\infty_c(X)$ is a dense subspace. More importantly, he showed in \cite{Bernstein} Thm. 3.2] that the inclusion $\mathcal{C}(X) \hookrightarrow L^2(X)$ is pointwise-defined. Hence, the maps $\alpha_{\sigma(\omega)} : C^\infty_c(X) \rightarrow \sigma(\omega)$ defined above extends continuously to the larger space $\mathcal{C}(X)$:

$$\alpha_{\sigma(\omega)} : \mathcal{C}(X) \longrightarrow \sigma(\omega).$$

The elements $\ell_{\sigma(\omega)} \in \text{Hom}_H(\sigma(\omega), \mathbb{C})$ are called $X$-tempered forms and the support of $d\mu$ consists precisely of those representations with nonzero $X$-tempered forms \cite{Bernstein} Pg. 689].
2.5. Inner Product. The direct integral decomposition \([2.1]\) leads to a spectral decomposition of the inner product \(\langle -,- \rangle_X\) of \(X\):

\[
\langle \phi_1, \phi_2 \rangle_X = \int_{\Omega} J_{\sigma(\omega)}(\phi_1, \phi_2) \, d\mu(\omega),
\]

where \(J_{\sigma(\omega)}\) is a \(G\)-invariant positive-semidefinite Hermitian form on \(C_c^\infty(X)\). To derive a formula for \(J_{\sigma(\omega)}\), we note that

\[
\langle \phi_1, \phi_2 \rangle_X = \int_X \phi_1(x) \overline{\phi_2(x)} \, dx
\]

\[
= \int_X \int_{\Omega} \beta_{\sigma(\omega)} \circ \alpha_{\sigma(\omega)}(\phi_1)(x) \cdot \overline{\phi_2(x)} \, d\mu(x) \, d\mu(\omega)
\]

\[
= \int_{\Omega} \int_X \beta_{\sigma(\omega)} \circ \alpha_{\sigma(\omega)}(\phi_1)(x) \cdot \overline{\phi_2(x)} \, dx \, d\mu(\omega)
\]

\[
= \int_{\Omega} \int \langle \beta_{\sigma(\omega)} \alpha_{\sigma(\omega)}(\phi_1), \phi_2 \rangle_X \, d\mu(\omega)
\]

This implies that, for \(d\mu\)-almost all \(\omega\), one has

\[
J_{\sigma(\omega)}(\phi_1, \phi_2) = \langle \beta_{\sigma(\omega)} \circ \alpha_{\sigma(\omega)}(\phi_1), \phi_2 \rangle_X = \langle \alpha_{\sigma(\omega)}(\phi_1), \alpha_{\sigma(\omega)}(\phi_2) \rangle_{\sigma(\omega)}.
\]

In particular, \(J_{\sigma(\omega)}\) factors as:

\[
J_{\sigma(\omega)} : C_c^\infty(X) \times C_c^\infty(X) \xrightarrow{\alpha_{\sigma(\omega)} \otimes \alpha_{\sigma(\omega)}} \sigma(\omega) \otimes \sigma(\omega) \xrightarrow{\langle -,- \rangle_{\sigma(\omega)}} \mathbb{C}.
\]

The crux of Bernstein’s viewpoint in \([2]\) is that to give the isomorphism \(\iota\) in the direct integral decomposition \([2.1]\) is equivalent to giving the family \(\{\alpha_{\sigma(\omega)} : \omega \in \Omega\}\) (satisfying appropriate properties), together with the measure \(d\mu\) on \(\Omega\). In the next section, we shall illustrate this in two basic examples.

3. Basic Plancherel Theorems

In this section, we describe two basic Plancherel theorems as an illustration of the abstract theory of Bernstein discussed in the previous section. These are the Harish-Chandra-Plancherel theorem and the Whittaker-Plancherel theorem.

We shall continue to work over a local field \(F\). However, we will implicitly be assuming that \(F\) is non-archimedean. In fact, the results of this paper will hold for archimedean local fields as well, but greater care is needed in introducing the various objects (such as various spaces of functions and the topologies they carry) and in formulating the results. Thus, there are analytic and topological considerations that need to be addressed in the archimedean case. We refer the reader to the papers \([3, 4]\) where these issues are formulated and dealt with carefully and elegantly and content ourselves with treating the nonarchimedean case in the interest of efficiency.

3.1. Harish-Chandra-Plancherel Theorem. The most basic example is the regular representation \(L^2(G)\) of a semisimple group \(G \times G\) (acting by left and right translation). Here, we have fixed a Haar measure \(dg\) on \(G\) which defines the inner product on \(L^2(G)\). Harish-Chandra’s Plancherel theorem \([24, 25]\) asserts that there is an (explicitly constructed) \(G \times G\)-equivariant isomorphism

\[
L^2(G) \cong \int_{\hat{G}} \overline{\sigma} \boxtimes \sigma \, d\mu_G(\sigma)
\]
for a specific measure $d\mu_G$ on $\hat{G}$ known as the Plancherel measure of $G$. The support of this measure is precisely the subset $\hat{G}_{\text{temp}}$ of irreducible tempered representations of $G$. Thus, in this case, one may take the measurable space $\Omega$ to be the unitary dual $\hat{G}$ and the map $\hat{G} \to \hat{G} \times \hat{G}$ is given by $\sigma \mapsto \sigma \otimes \sigma$.

Associated to the above direct integral decomposition (including the isomorphism), one has the family of maps for irreducible tempered $\sigma$:

$$\alpha_{\sigma \otimes \sigma} : C_c^\infty(G) \to \sigma \otimes \sigma \cong \sigma^\vee \otimes \sigma \cong \text{End}(\sigma),$$

given by

$$\alpha_{\sigma \otimes \sigma}(\phi) = \sigma(\phi) := \int_G \phi(g) \cdot \sigma(g) \, dg.$$

The (conjugate) dual map

$$\beta_{\sigma \otimes \sigma} : \sigma \otimes \sigma \to C^\infty(G)$$

is given by the formation of matrix coefficients. The associated inner product $J_\sigma$ is given by:

$$J_{\sigma \otimes \sigma}(\phi_1, \phi_2) = \text{Tr}(\sigma(\phi_1)\sigma(\phi_2^\vee)),$$

where

$$\phi_2^\vee(g) = \overline{\phi_2(g^{-1})}.$$  

### 3.2. Whittaker-Plancherel Theorem

Our second example is the Whittaker-Plancherel theorem (see [4, 5, 23, 24]), which is a variant of the setting discussed above. Let $G$ be a quasi-split semisimple group with $N$ the unipotent radical of a Borel subgroup. Fix a nondegenerate unitary character $\psi$ of $N$. We consider the Whittaker variety $(N, \psi) \backslash G$ and its associated unitary representation $L^2(N, \psi \backslash G)$ (which depends on fixed Haar measures $dg$ on $G$ and $dn$ on $N$). This extends the setting we discussed above, as one is considering $L^2$-sections of a line bundle on the spherical variety $N \backslash G$ instead of $L^2$-functions, but it is also covered in [2].

It has been shown that one has a direct integral decomposition

$$L^2(N, \psi \backslash G) \cong \int_{\hat{G}} \dim \text{Hom}_N(\sigma, \psi) \cdot \sigma \, d\mu_G(\sigma),$$

where we recall that $d\mu_G$ is the Plancherel measure of $G$. Thus, in this case, we are taking $\Omega$ to be $\hat{G}$ and the map $\Omega \to \hat{G}$ is the identity map. The spectral measure $d\mu_{G, \psi}$ is equal to $\dim \text{Hom}_N(\sigma, \psi) \cdot d\mu_G$, whose support is the subset $\hat{G}_{\text{temp}, \psi}$ of $\psi$-generic irreducible tempered representations.

Associated to this direct integral decomposition is the family of morphisms

$$\alpha_\sigma : C_c^\infty(N, \psi \backslash G) \to \sigma$$

for all $\sigma \in \hat{G}_{\text{temp}, \psi}$. Moreover, the map $\alpha_\sigma$ extends to the Harish-Chandra-Schwarz space $C(N, \psi \backslash G)$.

We can describe the (conjugate) dual map

$$\beta_\sigma \otimes \overline{\beta_\sigma} : \sigma \otimes \overline{\sigma} \to C^\infty(N \times N, \psi \otimes \overline{\psi} \backslash G \times G)$$

as follows (recall that this is independent of the choice of the isomorphism in the direct integral decomposition). Given $v_1, v_2 \in \sigma$, one has

$$\beta_\sigma \otimes \overline{\beta_\sigma}(v_1 \otimes v_2)(g_1, g_2) = \int_N^* \overline{\psi(n)} \cdot \langle \sigma(n \cdot g_1)(v_1), \sigma(g_2)(v_2) \rangle_\sigma \, dn$$
where the integral is a regularized one (see \[23, 3, 20\]). The composite of this with the evaluation-at-1 map is thus the Whittaker functional
\[
\ell_\sigma \otimes \overline{\ell_\sigma} : v_1 \otimes v_2 \mapsto \int_N^* \overline{\psi(n)} \cdot \langle \sigma(n)(v_1), v_2 \rangle_\sigma \, dn.
\]

The associated inner product \(J_\sigma\) is given by
\[
J_\sigma(f_1, f_2) = \sum_{\psi \in \mathcal{ONB}(\sigma)} \langle f_1, \beta_\sigma(v) \rangle_{N/\text{SL}_2} \cdot \langle \beta_\sigma(v), f_2 \rangle_{N/\text{SL}_2}
\]
\[
= \sum_{\psi \in \mathcal{ONB}(\sigma)} \int_{N \times N/\text{SL}_2 \times \text{SL}_2} f_1(g_1) \cdot \overline{f_2(g_2)} \cdot \ell_\sigma(g_2v) \cdot \overline{\ell_\sigma(g_1v)} \, dg_1 \, dg_2
\]
\[
= \sum_{\psi \in \mathcal{ONB}(\sigma)} \int_{N \times N/\text{SL}_2 \times \text{SL}_2} f_1(g_1) \cdot \overline{f_2(g_2)} \cdot \left( \int_N^* \overline{\psi(n)} \cdot \langle n g_2v, g_1v \rangle \, dn \right) \, dg_1 \, dg_2.
\]

3.3. **The case** \(\text{SL}_2\). Let us specialise to the case \(G = \text{SL}_2\) with Borel subgroup \(B = T \cdot N\) and maximal compact subgroup \(K\). An element \(f \in C^\infty(N, \psi/\text{SL}_2)\) is determined by its restriction to \(TK\), by the Iwasawa decomposition. The invariant measure on \(N/\text{SL}_2\) is given by
\[
\phi \mapsto \int_T \int_K \phi(tk) \cdot \delta_B(t)^{-1} \, dt \, dk.
\]
If we identity \(T\) with \(F^\times\), then \(\delta_B(t) = |t|^2\).

By the smoothness of \(f\), the function \(t \mapsto f(tk)\) on \(T \cong F^\times\) is necessarily rapidly decreasing at \(|t| \to \infty\) (indeed, it vanishes on some domain \(|t| > C\) in the p-adic case). Thus the analytic properties of \(f\) depend on its asymptotics as \(|t| \to 0\). The above discussion immediately implies the following lemma:

**Lemma 3.1.** Let \(f \in C^\infty(N, \psi/\text{SL}_2)\) and suppose that there exists \(C > 0\) and \(d > 0\) such that
\[
\sup_{k \in K} |f(tk)| \leq C \cdot |t|^d \quad \text{as } |t| \to 0.
\]

(i) If \(d > 1\), then \(f \in C(N, \psi/\text{SL}_2)\).

(ii) If \(d > 2\), then \(f \in L^1(N, \psi/\text{SL}_2)\).

3.4. **Continuity properties.** We conclude this section by considering the issue of continuity (in \(\sigma\)) for some of the quantities discussed above. We first need to say a few words about the Fell topology on \(\hat{G}_{\text{temp}}\).

The unitary dual \(\hat{G}\) is typically non-Hausdorff even though it is still a T1 space. The tempered dual \(\hat{G}_{\text{temp}}\) is still not necessarily Hausdorff, but can often be replaced by a substitute which is Hausdorff. Namely, one can work with the space of equivalence classes of induced representations \(\tau = \text{Ind}_P^G \pi\) where \(P\) is a parabolic subgroup of \(G\) and \(\pi\) a discrete series representation of its Levi factor \(M\). This space was variously denoted by \(\mathcal{T}\) in [19], \(\text{Temp}_{\text{ind}}(G)\) in [31] and \(\mathcal{X}_{\text{temp}}(G)\) in [31 27], so we are spoilt for choices! To add to this galore, we shall denote this space by \(\hat{G}_{\text{temp}}^{\text{ind}}\). Then \(\hat{G}_{\text{temp}}^{\text{ind}}\) has the structure of an orbifold.
(given by twisting $\pi$ by unramified unitary characters of $M$). There is a natural continuous finite-to-one surjective map

$$\hat{G}_{\text{temp}} \twoheadrightarrow \hat{G}_{\text{ind temp}}$$

sending a tempered irreducible representation $\sigma$ to the unique induced representation $\text{Ind}_{\pi}^{G}\sigma$ containing $\sigma$. This map is injective outside a subset of $\hat{G}_{\text{temp}}$ which has measure zero with respect to the Plancherel measure $d\mu_G$.

In the setting of the Harish-Chandra-Plancherel theorem of §3.1, one could safely replace the integral over $\hat{G}_{\text{temp}}$ in (3.1) by an integral over $\hat{G}_{\text{ind temp}}$. Moreover, we have the Hermitian form $J_{\tau \otimes \sigma}(\phi_1, \phi_2)$ for $\phi_i \in C^\infty_c(G)$ (or more generally $\mathcal{C}(G)$) and $\sigma \in \hat{G}_{\text{temp}}$. We can similarly define $J_{\tau \otimes \tau}(\phi_1, \phi_2)$ for $\tau \in \hat{G}_{\text{ind temp}}$.

Then we have [4, §2.13]:

**Lemma 3.2.** For fixed $\phi_1$ and $\phi_2$, the map

$$\tau \mapsto J_{\tau \otimes \sigma}(\phi_1, \phi_2)$$

is continuous as a $\mathbb{C}$-valued function on $\hat{G}_{\text{ind temp}}$. In particular, the map $\sigma \mapsto J_{\sigma \otimes \sigma}(\phi_1, \phi_2)$ is continuous on the subset of $\hat{G}_{\text{temp}}$ which maps injectively into $\hat{G}_{\text{ind temp}}$.

In the context of the Whittaker-Plancherel theorem, we are working with the subset $\hat{G}_{\text{temp}, \psi}$. Because each $\tau = \text{Ind}_{\pi}^{G}\sigma$ in $\hat{G}_{\text{ind temp}}$ can have at most one irreducible constituent which is $\psi$-generic, we see that the composite map

$$\hat{G}_{\text{temp}, \psi} \twoheadrightarrow \hat{G}_{\text{temp}} \twoheadrightarrow \hat{G}_{\text{ind temp}}$$

is injective (and continuous). As a consequence of [4, §2.14], we have:

**Lemma 3.3.** In the context of the Whittaker-Plancherel theorem, for fixed $f_1$ and $f_2$ in $S(N, \psi \backslash G)$, the map

$$\sigma \mapsto J_\sigma(f_1, f_2)$$

is a continuous $\mathbb{C}$-valued function on $\hat{G}_{\text{temp}, \psi}$. Likewise, for fixed $v_1$ and $v_2$, the map

$$\sigma \mapsto \int_N \overline{\psi(n)} \cdot \langle \sigma(n)(v_1), v_2 \rangle_\sigma \, dn.$$

is a continuous $\mathbb{C}$-valued map on $\hat{G}_{\text{temp}, \psi}$.

4. **Theta Correspondence**

In this section, we recall the setup of the theta correspondence and recall some results of Sakellaridis on the spectral decomposition of the Weil representation for a dual pair.

4.1. **Weil representation.** If $W$ is a symplectic vector space and $(V, q)$ a quadratic space over a local field $F$, then one has a dual reductive pair

$$\text{Sp}(W) \times \text{O}(V) \rightarrow \text{Sp}(V \otimes W).$$

In this paper, we shall only consider the case where $W = F \cdot e \oplus F \cdot f$ is 2-dimensional with $\langle e, f \rangle_W = 1$. With the Witt basis $\{e, f\}$, we may identify $\text{Sp}(W)$ with $\text{SL}_2(F)$, and we let $B = T \cdot N$ be the Borel subgroup which stabilises the line $F \cdot e$. 

Attached to a fixed nontrivial additive character $\psi$ of $F$ and other auxiliary data, this dual pair has a distinguished representation $\Omega_\psi$ known as the Weil representation. To be precise, if $\dim V$ is odd, we need to work with the metaplectic double cover $Mp_2(F)$ of $SL_2(F)$. To simplify notation, we shall ignore this issue; the reader may assume $\dim V$ is even.

The unitary representation $\Omega_\psi$ can be realised on $L^2(f \otimes V) = L^2(V)$ (where we have fixed an $O(V)$-invariant Haar measure on $V$). The action of various elements of $SL_2(F) \times O(V)$ via $\Omega_\psi$ is given as follows:

$$
\begin{cases}
  h \cdot \Phi(v) = \Phi(h^{-1} \cdot v), & \text{for } h \in O(V); \\
  n(b) \cdot \Phi(v) = \psi(b \cdot q(v)) \cdot \Phi(v), & \text{for } n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in N; \\
  t(a) \cdot \Phi(v) = |a|^{\frac{1}{2} \dim V} \chi_{\text{disc}(V)}(a) \cdot \Phi(a^2 v), & \text{for } t(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in T.
\end{cases}
$$

This describes $\Omega_\psi$ as a representation of $B \times O(V)$. To describe the full action of $SL_2(F)$, one needs to give the action of a nontrivial Weyl group element, which acts by a normalized Fourier transform $F$. We omit the precise formula here.

One may consider the underlying smooth representation $\Omega_\psi^\infty$ which is realized on the subspace $S(V)$ of Schwarz-Bruhat functions on $V$. Following our convention, we shall use $\Omega_\psi$ to denote the Weil representation in both the smooth and $L^2$-setting when there is no cause for confusion.

4.2. Smooth Theta correspondence. The theory of theta correspondence concerns the understanding of the representation $\Omega_\psi$ of $SL_2(F) \times O(V)$. One can consider this question on the level of smooth representation theory or $L^2$-representation theory. In this subsection, we recall the setup of the smooth theory.

For $\pi \in \text{Irr}(O(V))$, the (smooth) big theta lift of $\pi$ to $SL_2$ is:

$$
\Theta(\pi) := (\Omega_\psi^\infty \otimes \pi^\vee)_{O(V)}.
$$

Likewise, if $\sigma \in \text{Irr}(SL_2)$, then its big theta lift to $O(V)$ is:

$$
\Theta(\sigma) := (\Omega_\psi^\infty \otimes \sigma^\vee)_{SL_2}.
$$

It is known [9] that the representations $\Theta(\pi)$ and $\Theta(\sigma)$ are finite length representations which have unique irreducible quotients (if nonzero).

If $\sigma \in \text{Irr}(SL_2)$ is tempered and $V$ is split with $\dim V \geq 4$, then it is known that $\Theta(\sigma)$ is nonzero irreducible unitary. Even when $\dim V = 3$, $\Theta(\sigma)$ is nonzero unitary if $\sigma$ is $\psi$-generic. In these cases, with $\pi = \Theta(\sigma)$, it is also the case that $\Theta(\pi)$ is irreducible and isomorphic to $\sigma$.

4.3. $L^2$-theta correspondence. On the other hand, we may consider the theta correspondence in the $L^2$-setting. Though we are not exactly in the setting discussed in §2, Bernstein’s theory continues to apply here (see [19]). When $\dim V \geq 4$, one has a direct integral decomposition of $SL_2(F) \times O(V)$-representations (see [7, 19]):

$$
\Omega_\psi \cong \int_{SL_2} \sigma \boxtimes \Theta(\sigma) \, d\mu_{SL_2}(\sigma),
$$

(4.1)
where $d\mu_{\text{SL}_2}$ is the Plancherel measure of $\text{SL}_2(F)$. Hence the spectral measure of $\Omega_\psi$ as an $\text{SL}_2$-module is absolutely continuous with respect to the Plancherel measure. Indeed, when $V$ is split and $\dim V \geq 4$, the support of $\Omega_\psi$ as an $\text{SL}_2$-module is precisely $\hat{\text{SL}}_2_{\text{temp}}$.

By the theory of spectral decomposition à la Bernstein, the isomorphism in the direct integral decomposition gives rise to a family
\begin{equation}
\theta_\sigma : \Omega_\psi^\infty \cong S(V) \longrightarrow \sigma \otimes \Theta(\sigma)
\end{equation}
for $\sigma \in \hat{\text{SL}}_2_{\text{temp}}$. This implies that (for almost all $\sigma$), the smooth representation underlying the unitary representation $\Theta(\sigma)$ is precisely the small theta lift of $\sigma$ in the smooth setting. Hence, the use of the notation $\Theta(\sigma)$ is consistent for the smooth and $L^2$-setting.

One also has the inner product
\begin{equation}
J_\sigma^\theta(\Phi_1, \Phi_2) := \langle \theta_\sigma(\Phi_1), \theta_\sigma(\Phi_2) \rangle_{\sigma \otimes \Theta(\sigma)}
\end{equation}
where $\text{ONB}(\sigma)$ stands for an orthonormal basis of $\sigma$. Here, $Z(-)$ is the local doubling zeta integral given by:
\begin{equation}
Z_\sigma(\Phi_1, \Phi_2, v_1, v_2) = \int_{\text{SL}_2} \langle g \cdot \Phi_1, \Phi_2 \rangle_{\Omega} \cdot \langle \sigma(g) \cdot v_1, v_2 \rangle_{\sigma} \, dg,
\end{equation}
which converges for tempered $\sigma$ when $\dim V \geq 3$.

Now for $\tau \in \hat{\text{SL}}_{\text{2temp}}^\text{ind}$, one can define $Z_\tau(\Phi_1, \Phi_2, v_1, v_2)$ by the same formula as above and then define $J_\tau^\theta$ by the formula (4.3). Then it is useful to note [27, Lemma 3.4.2]:

**Lemma 4.1.** For fixed $\Phi_i$ and $v_i$, the $C$-valued function
\begin{equation}
\tau \mapsto Z_\tau(\Phi_1, \Phi_2, v_1, v_2)
\end{equation}
is continuous in $\tau \in \hat{\text{SL}}_{\text{2temp}}^\text{ind}$. Further, for fixed $\Phi_1$, the $C$-valued function $\tau \mapsto J_\tau^\theta(\Phi_1, \Phi_2)$ is continuous in $\tau \in \hat{\text{SL}}_{\text{2temp}}^\text{ind}$.

4.4. **The maps $A_\sigma$ and $B_{\theta(\sigma)}$.** We will introduce some formal variants of the map $\theta_\sigma : \Omega_\psi \longrightarrow \sigma \otimes \Theta(\sigma)$ given in (4.2). By duality, we obtain $\text{SL}_2$-invariant and $O(V)$-equivariant map
\begin{equation}
A_\sigma : \Omega_\psi \otimes \sigma^\vee \cong \Omega_\psi \otimes \sigma^\vee \longrightarrow \theta(\sigma),
\end{equation}
characterized by
\begin{equation}
A_\sigma(\Phi, v) = \langle \theta_\sigma(\Phi), v \rangle_{\sigma}.
\end{equation}
Likewise, we have a $O(V)$-invariant and $\text{SL}_2$-equivariant map
\begin{equation}
B_{\theta(\sigma)} : \Omega_\psi \otimes \theta(\sigma) \longrightarrow \sigma
\end{equation}
characterized by
\begin{equation}
B_{\theta(\sigma)}(\Phi, w) = \langle \theta_\sigma(\Phi), w \rangle_{\theta(\sigma)}.
\end{equation}
The two maps are related by:
\begin{equation}
\langle A_\sigma(\Phi, v), w \rangle_{\theta(\sigma)} = \langle \theta_\sigma(\Phi), v \otimes w \rangle_{\sigma \otimes \theta(\sigma)} = \langle B_{\theta(\sigma)}(\phi, w), v \rangle_{\sigma}
\end{equation}
for \( \Phi \in \Omega_{v}, \; v \in \sigma \) and \( w \in \sigma(\sigma) \). Moreover, the inner product \( J_{\sigma}^0 \) can be expressed in terms of \( A_{\sigma} \) and \( B_{\theta(\sigma)} \) as follows:

\[
J_{\sigma}^0(\Phi_1, \Phi_2) = \langle \theta_\sigma(\Phi_1), \theta_\sigma(\Phi_2) \rangle_{\sigma \otimes \theta(\sigma)} = \sum_{v \in \ONB(\sigma)} \langle A_\sigma(\Phi_1, v), A_\sigma(\Phi_2, v) \rangle_{\theta(\sigma)}.
\]

and

\[
J_{\sigma}^0(\Phi_1, \Phi_2) = \sum_{w \in \ONB(\theta(\sigma))} \langle B_{\theta(\sigma)}(\Phi_1, w), B_{\theta(\sigma)}(\Phi_2, w) \rangle_{\sigma}.
\]

Comparing with the formula (4.3), we deduce that

\[
(4.5) \quad Z_\sigma(\Phi_1, \Phi_2, v_1, v_2) = \langle A_\sigma(\Phi_1, v_1), A_\sigma(\Phi_2, v_2) \rangle_{\theta(\sigma)}.
\]

This identity may be considered as the local analog of the Rallis inner product formula [8]. The reason for introducing the maps \( A_\sigma \) and \( B_{\theta(\sigma)} \) is that they relate better to the theory of global theta lifting in the global setting considered in \([10]\).

5. Periods

It is a basic principle that theta correspondence frequently allows one to transfer periods on one member of a dual pair to the other member. For an exposition of this in the setting of smooth theta correspondence, the reader can consult [6]. On the other hand, in the setting of \( L^2 \)-theta correspondence, this principle has been exploited in [7] to establish low rank cases of the local conjecture of Sakellaridis-Venkatesh on the unitary spectrum of spherical varieties.

In this section, we shall consider the dual pair \( SL_2 \times O(V) \) and show how the spectral decomposition à la Bernstein allows one to refine the results of [6] and [7].

5.1. Transfer of periods. We first consider periods in smooth representation theory. For \( a \in F^\times \), fix a vector \( v_a \in V \) with \( q(v_a) = a \) (if it exists), so that \( V = F \cdot v_a \oplus v_a^\perp \). Set

\[
X_a = \{ v \in V : q(v) = a \} \subset V,
\]

which is a Zariski closed subset of \( V \). By Witt’s theorem, \( O(V) \) acts transitively on \( X_a \) and the stabilizer of \( v_a \) in \( O(V) \) is \( O(v_a^\perp) \). Hence

\[
X_a \cong O(v_a^\perp) \setminus O(V)
\]

via \( h \mapsto h^{-1} \cdot v_a \). If \( v_a \) does not exist, we understand \( X_a \) to be empty (i.e., the algebraic variety has no \( F \)-points). To fix ideas, we shall assume that \( v_1 \) exists; this is not a serious hypothesis. We also set \( \psi_a(x) = \psi(ax) \).

The following proposition essentially resolves the local problem (a) in the smooth setting for the Sakellaridis-Venkatesh conjecture highlighted in the introduction, except for the part about relative character identities. It is essentially a folklore result and a proof has been written down in [6] in a more general setting. We recount the proof here to explicate a particular point.

**Proposition 5.1.** Let \( \pi \) be an irreducible smooth representation of \( O(V, q) \) and let \( \Theta_{\psi}(\pi) \) be its big theta lift to \( SL_2 \) (or \( Mp_2 \) if \( \dim V \) is odd). For \( a \in F^\times \), there is a composite isomorphism

\[
f_a : \Hom(\Theta(\pi)_{N, \psi_a, \C}) \cong \Hom_{O(V)}(C_c^\infty(X_a), \pi) \cong \Hom_{O(S_\psi)}(\pi^\vee, \C),
\]

where the second isomorphism is by Frobenius reciprocity. Here, the right hand side is understood to be 0 if \( X_a \) is empty. In particular, suppose that \( \pi \) is such that \( \sigma := \Theta(\pi) \) is irreducible tempered, we see that
\[ \sigma \text{ is } \psi_a\text{-generic if and only if } \pi \text{ is } O(v_+^\perp)\text{-distinguished, in which case } \dim \text{Hom}_{O(v_+^\perp)}(\pi^\vee, \mathbb{C}) = 1. \]

**Proof.** We describe the proof when \( F \) is nonarchimedean. The archimedean case is based on the same ideas, and the reader can consult [10, 28] for a careful treatment.

We prove the proposition by computing the space

\[ \text{Hom}_{N \times O(V)}(\Omega_\psi, \psi_a \boxtimes \pi) \]

in two different ways.

On one hand, let us fix an equivariant projection

\[ \theta : \Omega_\psi \longrightarrow \Theta(\pi) \boxtimes \pi. \]

Then by the universal property of \( \theta \), one has an isomorphism

\[ \theta^* : \text{Hom}_N(\Theta(\pi), \psi_a) \cong \text{Hom}_{N \times O(V)}(\Omega_\psi, \psi_a \otimes \pi). \]

On the other hand, for \( a \in F^\times \), consider the surjective restriction map

\[ \text{rest} : \Omega_\psi = C^\infty_c(V) \longrightarrow C^\infty_c(X_a). \]

This map induces an equivariant isomorphism

\[ \text{rest} : \Omega_{N, \psi_a} \cong C^\infty_c(X_a). \]

Hence, we have an induced isomorphism

\[ \text{rest}^* : \text{Hom}_{O(V)}(C^\infty_c(X_a), \pi) \cong \text{Hom}_O(V)(\Omega_{N, \psi_a}, \pi) \cong \text{Hom}_{N \times O(V)}(\Omega_\psi, \psi_a \otimes \pi). \]

Since

\[ C^\infty_c(X_a) \cong \text{ind}_{O(v_+^\perp)}^O(V) \mathbb{C}, \]

it follows by Frobenius reciprocity that one has the desired isomorphism:

\[ f_a^{-1} : \text{Hom}_{O(v_+^\perp)}(\pi^\vee, \mathbb{C}) \xrightarrow{\text{Frob}} \text{Hom}_{O(V)}(C^\infty_c(X_a), \pi) \xrightarrow{(\theta^*)^{-1} \circ \text{rest}^*} \text{Hom}_N(\Theta(\pi), \psi_a). \]

This proves the proposition. \( \square \)

The purpose of recounting the proof of the proposition is to bring forth the point that the isomorphism

\[ f_a : \text{Hom}(\Theta(\pi)_{N, \psi_a}, \mathbb{C}) \cong \text{Hom}_{O(v_+^\perp)}(\pi^\vee, \mathbb{C}) \]

essentially depends only on the choice of the projection map

\[ \theta : \Omega_\psi \longrightarrow \Theta(\pi) \boxtimes \pi. \]

On the other hand, when \( \sigma = \Theta(\pi) \) is irreducible and tempered, we have seen that the spectral decomposition of the unitary representation \( \Omega_\psi \) in (1.1) gives rise to a (essentially canonical) element

\[ \theta_\sigma : \Omega_\psi \longrightarrow \sigma \boxtimes \Theta(\sigma) = \sigma \boxtimes \pi. \]

In this context, we shall use this element \( \theta_\sigma \) to effect the isomorphism in the proposition.
5.2. **Decomposition of** \( L^2(X) \). One may also consider the local problem (b) in the Sakellaridis-Venkatesh conjecture, i.e. in the \( L^2 \)-setting. This has been done in [7] and the following proposition is a special case of the results in [7]. We recount the proof here to explicate certain isomorphisms used in the course of the proof.

**Proposition 5.2.** We have:
\[
L^2(\Omega(\nu_a^+) \setminus O(V)) \cong \int_{SL_2} \Theta(\sigma) d\mu_{SL_2,\psi_a}(\sigma).
\]

**Proof.** We shall exploit the spectral decomposition of the unitary Weil representation \( \Omega_\psi \) of \( SL_2(F) \times O(V) \) on \( S(V) \). More precisely, we shall consider its restriction to \( B \times O(V) \). With \( a \in F^\times \), we have seen that
\[
X_a = \{ v \in V : q(v) = a \} \cong O(\nu_a^+) \setminus O(V),
\]
under \( h^{-1}\nu_a \leftrightarrow h \) and we would like to understand the representation \( L^2(X_a) \) of \( O(V) \). We first note that \( L^2(X_a) \) is in fact a representation of \( O(\nu_a) \times O(V) \), where \( O(\nu_a) \cong \mu_2 \) acts by left translation. This gives a decomposition
\[
L^2(X_a) = L^2(X_a)^+ \oplus L^2(X_a)^-
\]
into \( O(V) \)-submodules which are the \( \pm \)-eigenspaces of the \( O(\nu_a) \)-action.

Now, on one hand, since
\[
\bigcup_{a \in F^\times \setminus F^\times} F^\times \cdot X_a \subset V
\]
is open dense (with complement of measure 0), one has
\[
L^2(V) \cong \bigoplus_{a \in F^\times \setminus F^\times} L^2(F^\times \cdot X_a)
\]
as \( B \times O(V) \)-modules. Moreover, with \( T \cong F^\times \) acting on \( V \) by scaling, we see that \( T \times O(V) \) acts transitively on \( F^\times \cdot X_a \) and the stabilizer of \( \nu_a \in X_a \) is the subgroup
\[
\mu_2^A \times O(\nu_a^+) \subset Z \times O(\nu_a) \times O(\nu_a^+),
\]
where \( Z = \mu_2 \) is the center of \( SL_2(F) \) and \( O(\nu_a) = \mu_2 \) is the orthogonal group of the 1-dimensional quadratic space \( F \cdot \nu_a \). Thus, as explained in [7, §3.4], one has
\[
L^2(F^\times \cdot X_a) \cong \bigoplus_{\epsilon = \pm} (\text{ind}_{ZN}^B \epsilon \otimes \psi_a) \otimes L^2(X_a)^\epsilon,
\]
as a \( B \times O(V) \)-module, so that
\[
\Omega \cong \bigoplus_{a \in F^\times \setminus F^\times} \bigoplus_{\epsilon = \pm} (\text{ind}_{ZN}^B \epsilon \otimes \psi_a) \otimes L^2(X_a)^\epsilon.
\]

On the other hand, one has:
\[
\iota : L^2(V) \cong \int_{SL_2} \sigma \otimes \theta(\sigma) d\mu_{SL_2}(\sigma)
\]
as \( SL_2 \times O(V) \)-modules. Restricting from \( SL_2 \) to \( B \), one has: for any tempered irreducible \( \sigma \),
\[
\iota_\sigma = \bigoplus_{a \in F^\times \setminus F^\times} \iota_{\sigma,a} : \sigma \vert_B \cong \bigoplus_{a \in F^\times \setminus F^\times} \dim \sigma_{N,\psi_a} \cdot \text{ind}_{ZN}^B \omega_\sigma \otimes \psi_a,
\]
where \( \omega_\sigma = \pm \) denotes the central character of \( \sigma \). Hence, one has

\[
\Omega_\psi \cong \bigoplus_{a \in F \times \mathbb{Z} \setminus \mathbb{F} \times \epsilon = \pm} \left( \text{ind}^B_{Z \times \mathbb{Z}^2} \otimes \psi_a \right) \otimes \int_{\text{SL}_2} \theta(\sigma) \cdot 1(\omega_\sigma = \epsilon) \, d\mu_{\text{SL}_2, \psi_a}(\sigma).
\]

Comparing the two descriptions of \( \Omega = L^2(V) \) as a \( B \times \text{O}(V) \)-module, one obtains an isomorphism

\[
L^2(X_\alpha)^\epsilon \cong \int_{\text{SL}_2} \theta(\sigma) \cdot 1(\omega_\sigma = \epsilon) \, d\mu_{\text{SL}_2, \psi_a}(\sigma).
\]

for \( \epsilon = \pm \). Summing over \( \epsilon \), we obtain the desired isomorphism in the proposition. \( \square \)

The following lemma, which is the content of [4, Prop. 2.14.3], gives a more explicit description of \( ev_1 \circ \tau_{\sigma,a} \):

**Lemma 5.3.** Up to an element of \( S^1 \), the Whittaker functional \( ev_1 \circ \tau_{\sigma,a} \) is equal to the Whittaker functional \( \ell_{\sigma,a} = ev_1 \circ \beta_{\sigma,a} \) given by the Whittaker-Plancherel theorem for \( (N, \psi_a) \setminus \text{SL}_2 \).

What this lemma says is that in the proof of the proposition above, the isomorphisms \( \tau_\sigma \) can be chosen to be that induced by \( \ell_\sigma \) or equivalently \( \beta_\sigma \), namely we could have taken

\[
\tau_{\sigma,1} = \text{rest}_{\text{SL}_2, B} \circ \beta_\sigma
\]

where \( \text{rest}_{\text{SL}_2, B} \) is the restriction of functions from \( \text{SL}_2 \) to \( B \).

### 5.3. A commutative diagram

The point of going over the proof of the proposition is to point out that the isomorphism in the proposition (for \( a = 1 \) say) depends on the following choices:

1. the choice of the isomorphism \( \tau \) in the direct integral decomposition of \( \Omega_\psi \), and hence the family \( \{ \theta_\sigma : \sigma \in \text{SL}_2 \} \);
2. the choice of the isomorphism \( \tau_{\sigma,1} \) or equivalently, by Lemma 5.3, the family of

\[
\{ \ell_\sigma \in \text{Hom}_N(\sigma, \psi) : \sigma \in \text{SL}_2\text{temp,}\psi \}
\]

of Whittaker functionals.

In particular, having fixed the above choices, the proposition gives a direct integral decomposition of \( L^2(X_1) \) (recall that we are assuming that \( X_1 \) is nonempty). Associated to this is the family

\[
\alpha_{\theta(\sigma)} : C^\infty_c(X_1) \rightarrow \theta(\sigma)
\]

for \( \sigma \in \text{supp}(d\mu_{\text{SL}_2, \psi}) \). An examination of the proof of Proposition 5.2 thus gives:

**Proposition 5.4.** For each \( \sigma \in \text{SL}_2\text{temp,}\psi \), there is a commutative diagram:
This proposition gives a precise relation between the transfer of periods in the smooth setting and the spectral decomposition of $L^2(N, \psi \backslash SL_2)$, $L^2(X_1)$ and $\Omega_\psi$ in the $L^2$-theory. Indeed, it is fairly clear that one has a commutative diagram as in the proposition up to scalars. The point of the proposition is to explicate the scalar. More precisely, one has:

**Corollary 5.5.** Under the isomorphism

$$f_\sigma : \text{Hom}_N(\sigma, \psi) \cong \text{Hom}_{\hat{O}(V)}(C^\infty_c(X_1), \theta(\sigma))$$

given in Proposition 5.4, which is induced by the map $\theta_\sigma$ intervening in the spectral decomposition of the Weil representation $\Omega_\psi$, one has:

$$f_\sigma(\ell_\sigma) = \alpha_{\theta(\sigma)}$$

where the Whittaker functional $\ell_\sigma$ is the one which intervenes in the Whittaker-Plancherel theorem for $SL_2$ and the morphism $\alpha_{\theta(\sigma)}$ is the one which intervenes in the spectral decomposition of $L^2(X_1)$ obtained in Proposition 5.4.

The commutative diagram in Proposition 5.4 gives an identity in $\theta(\sigma)$. If we pair both sides of the identity with a vector in $\theta(\sigma)$, using the inner product on $\theta(\sigma)$, we obtain:

**Corollary 5.6.** For any $\Phi \in \Omega_\psi$ and $w \in \theta(\sigma)$, one has

$$\ell_\sigma(B_{\theta(\sigma)}(\Phi, w)) = \langle \Phi|X, \beta_{\theta(\sigma)}(w)\rangle_X,$$

where $B_{\theta(\sigma)}$ was defined in §4.4.

**Proof.** We have

$$\langle \Phi|X, \beta_{\theta(\sigma)}(w)\rangle_X = \langle \alpha_{\theta(\sigma)}(\Phi|X), w\rangle_{\theta(\sigma)} = \langle \ell_\sigma(\theta_\sigma(\Phi)), w\rangle_{\theta(\sigma)} = \ell_\sigma \left( (\theta_\sigma(\Phi)), w\rangle_{\theta(\sigma)} \right) = \ell_\sigma(B_{\theta(\sigma)}(\Phi, w)).$$

We may also "double-up" the commutative diagram in Proposition 5.4 and contract the resulting doubled identity using the inner product on $\theta(\sigma)$. This gives:

**Corollary 5.7.** For $\Phi_1, \Phi_2 \in \Omega_\psi$, one has:

$$J_{\theta(\sigma)}(\Phi_1|X, \Phi_2|X) = \int_N^* \overline{\psi(n)} \cdot J^\theta_{\sigma}(n \cdot \Phi_1, \Phi_2) \ dn.$$

For fixed $\Phi_1|X$ and $\Phi_2|X$, the $\mathbb{C}$-valued function $\sigma \mapsto J_{\theta(\sigma)}(\Phi_1|X, \Phi_2|X)$ is continuous in $\sigma \in \hat{SL}_{2\text{temp}, \psi}$.

**Proof.** We have

$$J_{\theta(\sigma)}(\Phi_1|X, \Phi_2|X) = \langle \alpha_{\theta(\sigma)}(\Phi_1|X), \alpha_{\theta(\sigma)}(\Phi_2|X)\rangle_{\theta(\sigma)}$$

(by definition of $J_{\theta(\sigma)}$)

$$= \langle \ell_\sigma(\theta_\sigma(\Phi_1)), \ell_\sigma(\theta_\sigma(\Phi_2))\rangle_{\theta(\sigma)}$$

(by Proposition 5.4)

$$= \ell_\sigma \otimes \overline{\ell_\sigma} \left( (\theta_\sigma(\Phi_1), \theta_\sigma(\Phi_2))_{\theta(\sigma)} \right)$$

(clear)

$$= \int_N^* \overline{\psi(n)} \cdot \langle n \cdot \theta_\sigma(\Phi_1), \theta_\sigma(\Phi_2)\rangle_{\sigma \otimes \theta(\sigma)} \ dn$$

(formula for $\ell_\sigma \otimes \overline{\ell_\sigma}$)

$$= \int_N^* \overline{\psi(n)} \cdot J^\theta_{\sigma}(n \cdot \Phi_1, \Phi_2) \ dn$$

(definition of $J^\theta_{\sigma}$)

The continuity of $\sigma \mapsto J_{\theta(\sigma)}(\Phi_1|X, \Phi_2|X)$ follows from the above formula, together with Lemma 8.3 and Lemma 4.1.
The last two corollaries thus give different variants of the identity in Proposition 5.4.

6. Relative Characters

In this section, we briefly recall the notion of the relative character associated to a period in its various incarnations.

6.1. Relative characters. Suppose that, for \( i = 1 \) or 2, \( H_i \subset G \) is a subgroup of \( G \) and \( \chi_i : H_i(F) \to S^1 \) a unitary character of \( H_i(F) \). For any \( \pi \in \hat{G} \) and \( L_i \in \text{Hom}_{H_i}(\pi, \chi_i) \), one can associate a distribution on \( G \) as follows. Given \((f_1, f_2) \in C_c^\infty(G) \times C_c^\infty(G)\), one sets:

\[
B_{\pi, L_1, L_2}(f_1, f_2) = \sum_{v \in \text{ONB}(\pi)} L_1(\pi(f_1)(v)) \cdot L_2(\pi(f_2)(v))
\]

where the sum runs over an orthonormal basis of \( \pi \). The sum defining \( B_{\pi, L_1, L_2}(f_1, f_2) \) is independent of the choice of the orthonormal basis and is in fact a finite sum for fixed \((f_1, f_2)\). It gives a linear map

\[
B_{\pi, L_1, L_2} : C_c^\infty(G) \otimes C_c^\infty(G) \to \mathbb{C}
\]

which is \( G(F)^\Delta \)-invariant.

The distribution \( B_{\pi, L_1, L_2} \) is called the relative character of \( \pi \) with respect to \((L_1, L_2)\). Note that, in the literature, it is frequent to find a different convention in the definition of the relative character, using instead the sum

\[
\sum_{v \in \text{ONB}(\pi)} L_1(\pi(f_1)(v)) \cdot \overline{L_2(\pi(f_2)(v))}.
\]

The difference between the two conventions is merely one of form rather than substance, and it is easy to convert from one convention to the other using complex conjugation. We choose the normalisation given above so as to avoid the appearance of multiple complex conjugations in later formulae.

Now a short computation gives

\[
L_i(\pi(f_i)v) = \int_{H_i \backslash G} L_i(\pi(g_i)(v)) \cdot (f_i)_{H_i, \chi_i}(g) \, dg
\]

with

\[
(f_i)_{H_i, \chi_i}(g) = \int_{H_i} f(h_i g) \cdot \overline{\chi_i(h_i)} \, dh_i.
\]

Hence one deduces that the linear form \( B_{\pi, L_1, L_2} \) factors as

\[
C_c^\infty(G) \otimes C_c^\infty(G) \to C_c^\infty(H_1, \chi_1 \backslash G) \otimes C_c^\infty(H_2, \chi_2 \backslash G) \to \mathbb{C}.
\]

We may think of \( C_c^\infty(H_1, \chi_1 \backslash G) \) as the space of compactly supported smooth sections of the line bundle on \( X_i = H_i \backslash G \) determined by \( \chi_i \) and denote this space by the alternative notation \( C_c^\infty(X_i, \chi_i) \). Then we shall think of \( B_{\pi, L_1, L_2} \) as a \( G^\Delta \)-invariant linear form on \( C_c^\infty(X_1, \chi_1) \otimes C_c^\infty(X_2, \chi_2) \).

Let us write:

\[
f_{L,v}(g) = L(\pi(g)v)
\]

for the matrix coefficient associated to \( L \in \pi^* \) and \( v \in \pi \). Then the distribution \( B_{\pi, L_1, L_2} \) is given by the formula

\[
B_{\pi, L_1, L_2}(\phi_1, \phi_2) = \sum_{v \in \text{ONB}(\pi)} \langle \phi_1, f_{L_1,v} \rangle_{X_1} \cdot \langle f_{L_2,v}, \phi_2 \rangle_{X_2}.
\]
6.2. **Alternative incarnation.** We can also give an alternative formulation of the notion of relative characters. Continuing with the context of §6.1, it is not difficult to verify that

\[
B_{\pi, L_1, L_2}(f_1, f_2) = \sum_{v \in \text{ONB}(\pi)} L_1(\pi(f_1 \ast f_2^\vee)(v)) \cdot L_2(v),
\]

where

\[
f_2^\vee(g) = \overline{f_2(g^{-1})},
\]

and

\[
(f_1 \ast f_2)(g) = \int_G f_1(gx^{-1}) \cdot f_2(x) \, dx
\]

is the convolution of \(f_1\) and \(f_2\).

Thus, we may alternatively define \(B_{\pi, L_1, L_2}\) as a linear form

\[
B_{\pi, L_1, L_2} : C^\infty_c(G) \to \mathbb{C}
\]

given by the formula

\[
B_{\pi, L_1, L_2}(f) = \sum_{v \in \text{ONB}(\pi)} L_1(\pi(f)(v)) \cdot L_2(v).
\]

As in §6.1 this linear form factors as:

\[
B_{\pi, L_1, L_2} : C^\infty_c(G) \to C^\infty_c(X_1, \chi_1) \to \mathbb{C},
\]

so that we may regard it as a linear form on \(C^\infty_c(X_1, \chi_1)\), given by the formula

\[
(6.3) \quad B_{\pi, L_1, L_2}(\phi) = \sum_{v \in \text{ONB}(\pi)} \langle \phi, f_{L_1, v} \rangle_{X_1} \cdot f_{L_2, v}(1).
\]

In fact, it further factors as:

\[
B_{\pi, L_1, L_2} : C^\infty_c(X_1, \chi_1) \to C^\infty_c(X_1, \chi_1)_{H_2, \chi_2} \to \mathbb{C}.
\]

From this alternative description of the relative character, we can recover the previous version discussed in the previous subsection by using the fact that any \(f \in C^\infty_c(G)\) can be expressed as \(f_1 \ast f_2\). This is clear in the nonarchimedean case and is a result of Dixmier-Malliavin in the archimedean case.

6.3. **J_\sigma as a relative character.** We shall now relate the notion of relative character with the theory of direct integral decomposition.

We shall focus on the case when \(H_1 = H_2 = H\) and \(\chi_1 = \chi_2 = \chi\) and such that \(\dim \text{Hom}_H(\pi, \chi) \leq 1\).

With \(X = H \backslash G\), suppose one has a direct integral decomposition:

\[
L^2(X, \chi) := L^2(H, \chi \backslash G) = \int_\Omega \sigma(\omega) \, d\mu(\omega)
\]

with associated families of maps \(\{\alpha_{\sigma(\omega)}\}\) and \(\{\beta_{\sigma(\omega)}\}\) and associated decomposition of inner product

\[
\langle -, - \rangle_{X_1} = \int_\Omega J_{\sigma(\omega)}( -, - ) \, d\mu(\omega).
\]

Observe that the positive semidefinite Hermitian form \(J_{\sigma(\omega)}\) is a \(G^\Delta\)-invariant linear form

\[
J_{\sigma(\omega)} : C^\infty_c(X, \chi) \otimes C^\infty_c(X, \chi) \to \mathbb{C}.
\]

This suggests that \(J_{\sigma(\omega)}\) may be regarded as a relative character according to our definition in §6.1. Indeed, one has:
Lemma 6.1. One has $J_\sigma(\omega) = B_\sigma(\omega), \ell_\sigma(\omega), \ell_\sigma(\omega)$, where $\ell_\sigma(\omega) = ev_1 \circ \beta_\sigma(\omega) \in \text{Hom}_H(\sigma(\omega), \chi)$.

Proof. Since $\omega$ is fixed in the proposition, we shall write $\sigma = \sigma(\omega)$ for simplicity. Now we have:

$$J_\sigma(\phi_1, \phi_2) = \langle \alpha_\sigma(\phi_1), \alpha_\sigma(\phi_2) \rangle_\sigma = \sum_{v \in \text{ONB}(\sigma)} \langle \alpha_\sigma(\phi_1), v \rangle_\sigma \cdot \langle v, \alpha_\sigma(\phi_2) \rangle_\sigma = \sum_{v \in \text{ONB}(\sigma)} \langle \phi_1, \beta_\sigma(v) \rangle_X \cdot \langle \beta_\sigma(v), \phi_2 \rangle_X.$$

Noting that $\beta_\sigma(v)(g) = ev_1 \circ \beta_\sigma(\sigma(g)(v)) = \ell_\sigma(\sigma(g)(v)) = f_\ell_\sigma,v(g)$, we see that the lemma follows by the equation (6.2). □

We can also work with the alternative context of §6.2. Set $I(X, \chi) := C_c^\infty(X, \chi)_{H,\chi}$. We think of this as the space of “orbital integrals” on $X$. Indeed an element of this space is represented by a test function pushed forward to the quotient $(H, \chi)\backslash G/(H, \chi)$, and given a $H$-orbit on $X$, the associated orbital integral factors to $I(X, \chi)$ and is given by point-evaluation. The relative character $B_\sigma(\omega), \ell_\sigma(\omega), \ell_\sigma(\omega)$ is thus a linear form on $I(X, \chi)$. In this incarnation, one has:

Lemma 6.2. As a linear form on $C_c^\infty(X, \chi)$, one has

$$B_\sigma(\omega), \ell_\sigma(\omega), \ell_\sigma(\omega)(\phi) = \ell_\sigma(\omega)(\alpha_\sigma(\phi)) = \beta_\sigma(\omega) \alpha_\sigma(\phi)(1).$$

Proof. We write $\sigma = \sigma(\omega)$ for simplicity. Then we have

$$\alpha_\sigma(\phi) = \sum_{v \in \text{ONB}(\sigma)} \langle \alpha_\sigma(\phi), v \rangle_\sigma \cdot v = \sum_{v} \langle \phi, f_\ell_\sigma,v \rangle_X \cdot v.$$

Hence

$$\ell_\sigma(\alpha_\sigma(\phi)) = \sum_{v} \langle \phi, f_\ell_\sigma,v \rangle_X \cdot \ell_\sigma(v),$$

so that the lemma follows by equation (6.3). □

Corollary 6.3. Let $C(X, \chi)$ be the Harish-Chandra-Schwarz space of $X = (H, \chi)\backslash G$. Then the relative character $B_{\sigma, \ell_\sigma, \ell_\sigma}$ extends to $C(X, \chi)$.

7. Transfer of Test Functions

If two periods on the two members of a dual pair are related by theta correspondence as in Proposition 5.1, then one might ask if the associated relative characters are related in a precise way. Such a relation is called a relative character identity. To compare the two relative characters in question, which are distributions on different spaces, we first need to define a correspondence of the relevant spaces of test functions.
7.1. A Correspondence of Test Functions. The considerations of the previous sections suggest that one considers the following maps. Set

\[ p : C_c^\infty(V) = \Omega^\infty \longrightarrow C_c^\infty(N, \psi \backslash SL_2) \]

given by

\[ p(\Phi)(g) := F_\Phi(g) = (g \cdot \Phi)(v_1). \]

This map is \(O(v_1^\perp)\)-invariant and \(SL_2\)-equivariant. Let us set

\[ S(N, \psi \backslash SL_2) := \text{image of } p, \]

noting that it is a \(SL_2\)-submodule. Likewise, consider the \(O(V) \times (N, \psi)\)-equivariant restriction map

\[ q = \text{rest} : C_c^\infty(V) \longrightarrow C_c^\infty(X_1) \]

which is surjective. We have already seen and used the map \(q\) in the setting of smooth theta correspondence, seeing that it induces an \(O(V)\)-equivariant isomorphism

\[ q : C_c^\infty(V)_{N, \psi} \cong C_c^\infty(X_1). \]

Hence we have the diagram:

\[
\begin{array}{ccc}
C_c^\infty(V) & \xrightarrow{p} & S(N, \psi \backslash SL_2) \\
\downarrow{q} & & \downarrow{q} \\
C_c^\infty(X_1) & & C_c^\infty(X_1)
\end{array}
\]

We now make a definition:

**Definition 7.1.** Say that \(f \in S(N, \psi \backslash SL_2)\) and \(\phi \in C_c^\infty(X_1)\) are in correspondence (or are transfers of each other) if there exists \(\Phi \in C_c^\infty(V)\) such that \(p(\Phi) = f\) and \(q(\Phi) = \phi\).

Our goal in this section is to establish some basic properties of the spaces of test functions and the transfer defined above. We start with the following simple observation.

**Proposition 7.2.** Every \(f \in S(N, \psi \backslash SL_2)\) has a transfer \(\phi \in C_c^\infty(X_1)\) and vice versa.

**Proof.** This is simply because the maps \(p\) and \(q\) above are surjective. \(\square\)

We also note:

**Lemma 7.3.** The space \(S(N, \psi \backslash SL_2)\) is contained in the Harish-Chandra-Schwarz space of the Whittaker variety \((N, \psi) \backslash SL_2\). In particular, for any \(\sigma \in \widetilde{SL_2}_{\text{temp}, \psi}\), the associated relative character \(B_{\sigma, \ell, \sigma}\) extends to a linear form on \(S(N, \psi \backslash SL_2)\).

**Proof.** From the formula defining the Weil representation, we see that for \(f = p(\Phi)\),

\[ f(t(a)k) = |a|^\frac{\dim V}{2} \cdot \chi_{\text{disc}(V)}(a) \cdot (k \cdot \Phi)(a \cdot v_1). \]

It follows by Lemma 3(i) that \(f \in \mathcal{C}(N, \psi \backslash SL_2)\) if \(\dim V \geq 3\). \(\square\)
7.2. Basic function and fundamental lemmas. Assume that $V$ is a quadratic space such that $\text{SO}(V)$ is an unramified group. In this case, the Weil representation $\Omega^\infty$ has a canonical unramified vector $\Phi_0$ (see [15]).

**Definition 7.4.** Set

$$f_0 = p(\Phi_0) \quad \text{and} \quad \phi_0 = q(\Phi_0).$$

We call these the basic functions in the relevant space of test functions.

It is immediate from definition that one has the following “fundamental lemma”::

**Lemma 7.5.** The basic functions $f_0$ and $\phi_0$ correspond.

Let $K \subset \text{SL}_2(F)$ and $K' \subset \text{O}(V)$ be hyperspecial maximal compact subgroups which fixes the unramified vector $\Phi_0$ and consider the corresponding spherical Hecke algebras $\mathcal{H}(\text{SL}_2, K)$ and $\mathcal{H}(G, K')$. It was shown by Howe (see [15, Chap. 5, Thm. I.4, Pg 103] and [15, Pg 107]) that

$$(\Omega^\infty)_K = C_c(G) \cdot \Phi_0 \quad \text{and} \quad (\Omega^\infty)_{K'} = C_c(\text{SL}_2) \cdot \Phi_0.$$ 

It follows that

$$(\Omega^\infty)^K \times K' = \mathcal{H}(\text{SL}_2, K) \cdot \Phi_0 = \mathcal{H}(G, K') (\Phi_0).$$

It also follows from (7.3) that if one has a nonzero equivariant map

$$\Omega^\infty \colon \sigma \otimes \pi \in \text{Irr}(\text{SL}_2 \times O(V)),$$

then $\sigma$ is $K$-unramified if and only if $\pi = \theta(\sigma)$ is $K'$-unramified. Since $\dim V \geq 4$, one can show that every unramified $\sigma$ does have nonzero theta lift to $O(V)$. Moreover, one can determine the corresponding map of Satake parameters; this follows, for example, from the results of Kudla [13] about how supercuspidal support behaves under the theta correspondence. Together, these basic results in theta correspondence imply that there is an algebra morphism

$$t : \mathcal{H}(G, K') \rightarrow \mathcal{H}(\text{SL}_2, K)$$

such that for any $f \in \mathcal{H}(G, K')$, one has

$$f \cdot \Phi_0 = t(f) \cdot \Phi_0.$$

From this, one easily deduces the following “fundamental lemma for spherical Hecke algebras”:

**Lemma 7.6.** For any $f \in \mathcal{H}(G, K')$, the element $f \cdot \phi_0 \in C_c(X_1)$ corresponds to the element $t(f) \cdot f_0 \in S(N, \psi \backslash \text{SL}_2)$.

7.3. Relation with Adjoint L-factors. We shall see that the space $S(N, \psi \backslash \text{SL}_2)$ is intimately related to the standard (degree 3) L-factor of irreducible representations of $\text{SL}_2$. If we view an L-packet of $\text{SL}_2$ as the restriction of an irreducible representation $\tilde{\sigma}$ of $\text{GL}_2$, then this L-factor is the adjoint L-factor of $\tilde{\sigma}$, or its twisted $\text{Sym}^2$ L-factor. Let us recall a certain Rankin-Selberg local zeta integral for this particular L-factor, due to Gelbart-Jacquet (following Shimura). It requires the following 3 pieces of data:

- a $\psi$-generic $\sigma \in \text{Irr}(\text{SL}_2)$ with $\psi$-Whittaker model $W_\sigma$,
- the Weil representation $\omega_\psi$ of $\text{Mp}_2$ acting on the space $C_c^\infty(F)$ (regarding $F$ as a 1-dimensional quadratic space equipped with the quadratic form $x \mapsto x^2$);
- a principal series representation $I_\psi(\chi, s)$ of $\text{Mp}_2$, consisting of functions $\phi_s : N \backslash \text{Mp}_2 \rightarrow \mathbb{C}$ such that $\phi_s(t(a)g) = \chi_\psi(a) \cdot \chi(a) \cdot |a|^{1+s} \cdot \phi_s(g)$ (where $\chi_\psi$ is a genuine character of the diagonal torus of $\text{Mp}_2$ defined using the Weil index).
Then for $f \in \mathcal{W}_\sigma$, $\varphi \in C_c^\infty(F)$ and a section $\phi_s \in I(s)$, one can consider the local zeta integral

$$Z(f, \varphi, \phi_s) = \int_{N \setminus SL_2} f(g) \cdot (g \cdot \varphi)(1) \cdot \phi_s(g) \, dg.$$ 

This converges when $Re(s) \gg 0$, and when $\sigma$ is tempered, it converges for $Re(s) > 0$. Moreover, it represents the $L$-factor $L(s + \frac{1}{2}, \sigma, Ad \times \chi) = L(s + \frac{1}{2}, \sigma, std \times \chi)$.

Hence, the (twisted) adjoint $L$-value $L(s + \frac{1}{2}, \sigma, Ad \times \chi)$ is obtained by considering the integrals of $f \in \mathcal{W}_\sigma$ against a space of functions $S_s(N, \psi SL_2)$ of the form

$$g \mapsto \phi_s(g) \cdot (g \cdot \varphi)(1).$$

Moreover, as an $SL_2$-module, $S_s(N, \psi SL_2)$ is a quotient of $\omega_\psi \otimes I_\psi(\chi, s)$.

Now let us return to our space of test functions $S(N, \psi SL_2)$. Let us write

$$V = \langle v_1 \rangle \oplus U, \quad \text{with } U = v_1^\perp.$$ 

Then $C_c^\infty(V) = C_c^\infty(Fv_1) \otimes C_c^\infty(U)$. Here, $C_c^\infty(Fv_1)$ affords the Weil representation $\omega_\psi$ of $Mp_2 \times O_1$ whereas $C_c^\infty(U)$ afford a Weil representation of $Mp_2 \times O(U)$. If $\Phi \in C_c^\infty(V)$ is of the form $\Phi_1 \otimes \Phi'$, then

$$p(\Phi)(g) = (g \cdot \Phi_1)(v_1) \cdot (g \cdot \Phi')(0).$$

The function

$$g \mapsto \phi_{\Phi'}(g) = (g \cdot \Phi')(0)$$

belongs to the principal series $I_\psi(\chi\text{disc}(U), \frac{1}{2}(\dim V - 3))$. Indeed, by a result of Rallis, the map

$$\Phi' \mapsto \phi_{\Phi'}$$

gives an $O(U)$-invariant, $Mp_2$-equivariant injective map

$$0 \neq C_c^\infty(U)_{O(U)} \hookrightarrow I_\psi(\chi\text{disc}(U), \frac{1}{2}(\dim V - 3)).$$

This result of Rallis underlies the theory of the doubling seesaw and the Siegel-Weil formula. When $\dim V \geq 4$, this injective map is surjective as well. Indeed, when $\dim V \neq 4$, the relevant principal series $I_\psi(\chi\text{disc}(U), \frac{1}{2}(\dim V - 3))$ is irreducible. If $\dim V = 4$, the principal series $I_\psi(\chi\text{disc}(U), 1/2)$ has length 2 with unique irreducible quotient the even Weil representation $\omega_\psi^{\epsilon}(\chi\text{disc}(U))$ and unique submodule a (twisted) Steinberg representation. The above map is nonetheless surjective, as the small theta lift of the trivial representation of $O(U)$ is equal to $\omega_\psi^{\epsilon}(\chi\text{disc}(U))$.

To summarise, we have more or less shown:

**Proposition 7.7.** When $\dim V \geq 4$, the map $p$ induces a surjective map

$$C_c^\infty(V) \twoheadrightarrow C_c^\infty(V)_{O(U)} \cong \omega_\psi \otimes I_\psi(\chi\text{disc}(U), \frac{1}{2}(\dim V - 3)) \twoheadrightarrow S(N, \psi SL_2),$$

so that

$$S(N, \psi SL_2) = S_\frac{1}{4}(\dim V - 3)(N, \psi SL_2).$$

**Proof.** Observe that

$$C_c^\infty(V)_{O(U)} = C_c^\infty(Fv_1) \otimes C_c^\infty(U)_{O(U)} \cong \omega_\psi \otimes I_\psi(\chi\text{disc}(U), \frac{1}{2}(\dim V - 3)).$$

The rest of the proposition follows from our preceding discussion.
Question: Is the surjective map in Proposition 7.7 in fact an isomorphism \( C^\infty_c(V)_{O(U)} \cong S(N, \psi \backslash SL_2) \)?

7.4. Orbital Integrals. Let us set 
\[
\mathcal{I}(N, \psi \backslash SL_2) := S(N, \psi \backslash SL_2)_{N, \psi}
\]
so that by definition, a linear functional on \( \mathcal{I}(N, \psi \backslash SL_2) \) is a \((N, \psi)\)-equivariant linear form on \( S(N, \psi \backslash SL_2) \). One may think of \( \mathcal{I}(N, \psi \backslash SL_2) \) as the space of orbital integrals (with respect to the \((N, \psi)\)-period) and write \( \mathcal{I}(f) \) for the image of \( f \) in \( \mathcal{I}(N, \psi \backslash SL_2) \). Likewise, we set 
\[
\mathcal{I}(X_1) = C^\infty_c(X_1)_{O(U)}.
\]
This may be regarded as the space of orbital integrals with respect to the \( O(U) \)-period and we write \( \mathcal{I}(\phi) \) for the image of \( \phi \) in \( \mathcal{I}(X_1) \).

The following proposition summaries the properties of the transfer of test functions:

**Proposition 7.8.** The composite map
\[
C^\infty_c(V) \xrightarrow{p} S(N, \psi \backslash SL_2) \xrightarrow{q} \mathcal{I}(N, \psi \backslash SL_2)
\]
factors through \( q \), i.e., it induces a linear map
\[
C^\infty_c(X_1) \rightarrow \mathcal{I}(N, \psi \backslash SL_2).
\]
Hence the transfer correspondence descends to a linear map when one passes to the space of orbital integrals in the target. Indeed, it further descends to give a surjective linear map
\[
t_\psi : \mathcal{I}(X_1) \rightarrow \mathcal{I}(N, \psi \backslash SL_2).
\]

**Proof.** The composite map in question factors through \( C^\infty_c(V)_{N, \psi} \cong C^\infty_c(X_1) \). It also factors through \( C^\infty_c(V)_{O(U)} \) as we have observed. Hence, it factors through
\[
(C^\infty_c(V)_{N, \psi})_{O(U)} = \mathcal{I}(X_1)
\]
as desired. \( \square \)

Likewise, one may consider the composite
\[
C^\infty_c(V) \xrightarrow{q} C^\infty_c(X_1) \xrightarrow{p} \mathcal{I}(X_1)
\]
which as above factors through \( C^\infty_c(V)_{O(U)} \). But now we do not know if \( C^\infty_c(V)_{O(U)} \cong S(N, \psi \backslash SL_2) \); see the Question at the end of the previous subsection. If the answer to that question is Yes, then we will likewise conclude that the above composite map induces a linear map
\[
S(N, \psi \backslash SL_2) \rightarrow \mathcal{I}(X_1),
\]
which descends further to
\[
\mathcal{I}(N, \psi \backslash SL_2) \rightarrow \mathcal{I}(X_1),
\]
In that case, this linear map will be inverse to the one in the proposition, and hence we will have an isomorphism of vector spaces:
\[
t_\psi : \mathcal{I}(N, \psi \backslash SL_2) \cong \mathcal{I}(X_1).
\]
In other words, the transfer correspondence would give an isomorphism of the space of orbital integrals (for the relevant spaces of test functions). As it stands, we only have the surjective transfer map
\[
t_\psi : \mathcal{I}(X_1) \rightarrow \mathcal{I}(N, \psi \backslash SL_2)
\]
given in the above proposition.

8. Relative Character Identities

Finally, we are ready to establish the following relative character identity, which is the main local result of this paper.

**Theorem 8.1.** Suppose that

- \( f \in S(N, \psi \setminus \text{SL}_2) \) and \( \phi \in C_c^\infty(X_1) \) are in correspondence;
- \( \sigma \in \hat{\text{SL}}_2^{\text{temp}, \psi} \) with (nonzero) theta lift \( \theta(\sigma) \in \hat{\text{O}}(V) \);
- \( \ell_\sigma \in \text{Hom}_N(\sigma, \psi) \) is the canonical element determined by the Whittaker-Plancherel theorem;
- \( \ell_{\theta(\sigma)} = f_\sigma(\ell_\sigma) \in \text{Hom}_{\text{O}(V)}(\theta(\sigma), \mathbb{C}) \) the canonical element determined by the spectral decomposition in Proposition 5.2, which is in turn determined by \( \ell_\sigma \) and \( \theta_\sigma \).

Then one has the character identity:

\[
B_{\sigma, \ell_\sigma}(f) = B_{\theta(\sigma), \ell_{\theta(\sigma)}}(\phi).
\]

More succinctly, one has the identity

\[
B_{\sigma, \ell_\sigma} \circ t_\psi = B_{\theta(\sigma), \ell_{\theta(\sigma)}}
\]

of linear forms on \( \mathcal{I}(X_1) \) or equivalently the identity

\[
B_{\sigma, \ell_\sigma} \circ p = B_{\theta(\sigma), \ell_{\theta(\sigma)}} \circ q
\]

of linear forms on \( C_c^\infty(V) \).

**8.1. Proof of Theorem 8.1.** This subsection is devoted to the proof of the theorem. With \( f \) and \( \phi \) as given in the theorem, choose \( \Phi \in C_c^\infty(V) \) such that \( f = p(\Phi) \) and \( \phi = q(\Phi) \). We shall now find two different expressions for \( \Phi(v_1) \).

On one hand, by the direct integral decomposition given in Proposition 5.2 one has

\[
\Phi(v_1) = \phi(v_1) = \int_{\text{SL}_2} \beta_{\theta(\sigma)} \alpha_{\theta(\sigma)}(\phi)(v_1) \, d\mu_{\text{SL}_2, \psi}(\sigma).
\]

On the other hand, by the Whittaker-Plancherel theorem for \( (N, \psi) \setminus \text{SL}_2 \), one has

\[
\Phi(v_1) = f(1) = \int_{\text{SL}_2} \beta_{\sigma} \alpha_{\sigma}(f)(1) \, d\mu_{\text{SL}_2, \psi}(\sigma).
\]

Comparing the two expressions, we deduce that

\[
\int_{\text{SL}_2} B_{\theta(\sigma), \ell_{\theta(\sigma)}}(\phi) \, d\mu_{\text{SL}_2, \psi}(\sigma) = \int_{\text{SL}_2} B_{\sigma, \ell_\sigma}(f) \, d\mu_{\text{SL}_2, \psi}(\sigma).
\]

We would like to remove the integral sign in the above identity. For this, we will apply a Bernstein center argument.
Given an arbitrary element \( z \) in the Bernstein center of \( \text{SL}_2 \times O(V) \), the element \( z \) acts on the irreducible representation \( \sigma \boxtimes \theta(\sigma) \) by a scalar \( z(\sigma \boxtimes \theta(\sigma)) \). This implies that one has a commutative diagram

\[
\begin{array}{ccc}
\Omega_\psi & \xrightarrow{z} & \Omega_\psi \\
\downarrow^{\theta_\sigma} & & \downarrow^{\theta_\sigma} \\
\sigma \boxtimes \theta(\sigma) & \xrightarrow{z(\sigma \boxtimes \theta(\sigma))} & \sigma \boxtimes \theta(\sigma) \\
\downarrow^{\lambda} & & \downarrow^{\lambda} \\
\sigma & \xrightarrow{z(\sigma \boxtimes \theta(\sigma))} & \sigma
\end{array}
\]

(8.2) for any linear form \( \lambda \) on \( \theta(\sigma) \). One has an analogous commutative diagram where one takes \( \lambda \) to be any linear form on \( \sigma \) (so the last row of the commutative diagram has \( \theta(\sigma) \) in place of \( \sigma \)).

Now what we would like to show is that there are commutative diagrams

\[
\begin{array}{ccc}
\Omega_\psi & \xrightarrow{z} & \Omega_\psi \\
p \downarrow & & q \downarrow \\
S(N, \psi \backslash \text{SL}_2) & \xrightarrow{\alpha_\sigma} & S(N, \psi \backslash \text{SL}_2) \\
\downarrow^{\sigma} & & \downarrow^{\theta(\sigma)} \\
\sigma & \xrightarrow{z(\sigma \boxtimes \theta(\sigma))} & \theta(\sigma)
\end{array}
\]

(8.3) and

\[
\begin{array}{ccc}
\Omega_\psi & \xrightarrow{z} & \Omega_\psi \\
\downarrow^{\alpha_\sigma} & & \downarrow^{\alpha_\sigma} \\
C^\infty_c(X_1) & \xrightarrow{\alpha_\sigma} & C^\infty_c(X_1) \\
\downarrow^{\theta(\sigma)} & & \downarrow^{\theta(\sigma)} \\
\theta(\sigma) & \xrightarrow{z(\sigma \boxtimes \theta(\sigma))} & \theta(\sigma)
\end{array}
\]

We shall explain how the commutativity of the diagram on the left follows from the commutativity of the diagram in (8.2); a similar argument works for the diagram on the right.

Since the map \( \alpha_\sigma \circ p \) is \( \text{SL}_2 \)-equivariant, it factors through \( \theta_\sigma : \Omega_\psi \longrightarrow \sigma \boxtimes \theta(\sigma) \), i.e. there is a \( \lambda : \theta(\sigma) \rightarrow \mathbb{C} \) such that

\[
\alpha_\sigma \circ p = \lambda \circ \theta_\sigma.
\]

Using this, we see that the desired commutativity of the left diagram in (8.3) is reduced to the commutativity of the diagram in (8.2).

Now we shall apply the identity (8.1) to the pair of test functions arising from \( z \cdot \Phi \). Note that

\[
B_{\sigma,\ell_\sigma}(p(z \cdot \Phi)) = \beta_\sigma \alpha_\sigma(p(z \Phi))(1) = z(\sigma \boxtimes \theta(\sigma)) \cdot B_{\sigma,\ell_\sigma}(f)
\]

and

\[
B_{\theta(\sigma),\ell_{\theta(\sigma)}} = \beta_{\theta(\sigma)} \alpha_{\theta(\sigma)}(q(z \cdot \Phi))(x_1) = z(\sigma \boxtimes \theta(\sigma)) \cdot B_{\theta(\sigma),\ell_{\theta(\sigma)}}(\phi).
\]

Hence the identity (8.1), when applied to \( z \cdot \Phi \), reads:

\[
\int_{\text{SL}_2} z(\sigma \boxtimes \theta(\sigma)) \cdot \left( B_{\sigma,\ell_\sigma}(f) - B_{\theta(\sigma),\ell_{\theta(\sigma)}}(\phi) \right) d\mu_{\text{SL}_2,\psi}(\sigma) = 0.
\]

(8.4)

Now note that there is a natural homomorphism from the Bernstein center of \( \text{SL}_2 \) to the Bernstein center for \( \text{SL}_2 \times O(V) \). Hence we may take \( z \) to be an element in the (tempered) Bernstein center of \( \text{SL}_2 \). Then \( z(\sigma \boxtimes \theta(\sigma)) = z(\sigma) \). When regarded as \( \mathbb{C} \)-valued functions on \( \hat{\text{SL}}_{2\text{temp},\psi} \), the elements \( z \) of the (tempered) Bernstein center of \( \text{SL}_2 \), are dense in the space of all Schwartz functions on \( \hat{\text{SL}}_{2\text{temp},\psi} \). Hence, (8.4) implies that for \( d\mu_{\text{SL}_2,\psi} \)-almost all \( \sigma \), one has

\[
B_{\sigma,\ell_\sigma}(f) = B_{\theta(\sigma),\ell_{\theta(\sigma)}}(\phi).
\]
To obtain the equality for all $\sigma \in \widehat{SL}_{2\text{temp},\psi}$, we note that both sides of continuous as functions of $\sigma \in \widehat{SL}_{2\text{temp},\psi}$ by Lemma [3.3] and Corollary [5.7]. This completes the proof of Theorem 8.1.

8.2. Some consequences. We shall now give some consequences of the relative character identity shown in Theorem 8.1. Let us consider the following diagram:

\[
\begin{array}{ccc}
C^\infty_c(V) & \xrightarrow{\theta_\sigma} & C^\infty_c(X_1) \\
\downarrow{q} & & \downarrow{\sigma \boxtimes \theta(\sigma)} \\
S(N, \psi \backslash SL_2) & \xrightarrow{\sigma} & \alpha_\sigma \\
\end{array}
\]

In this diagram, the rhombus at the bottom is clearly commutative. Now the parallelogram at the upper left side is precisely the commutative diagram in Proposition 5.4. On the other hand, the parallelogram at the upper right side is commutative up to a scalar since

$$\dim \text{Hom}_{O(v_1^+)}(\Omega_{\psi}, \mathbb{C} \boxtimes \sigma) = 1 \quad \text{for } \sigma \in \widehat{SL}_{2\text{temp},\psi}$$

and both $\alpha_\sigma \circ p$ and $\ell_{\theta(\sigma)} \circ \theta_\sigma$ are nonzero elements of this space. We would like to show that it is in fact commutative.

To deduce this, we observe that the composite of the three maps along the left boundary of the hexagon is simply the relative character $B_{\theta(\sigma)} \circ q$, whereas the composite of the three maps along the right boundary of the hexagon is the relative character $B_\sigma \circ p$. The relative character identity of Theorem 8.1 says that the boundary of the diagram is commutative! From this, we deduce the following counterpart of Proposition 5.4:

**Proposition 8.2.** The following diagram is commutative:

\[
\begin{array}{ccc}
C^\infty_c(V) & \xrightarrow{\theta_\sigma} & C^\infty_c(X_1) \\
\downarrow{q} & & \downarrow{\sigma \boxtimes \theta(\sigma)} \\
S(N, \psi \backslash SL_2) & \xrightarrow{\sigma} & \alpha_\sigma \\
\end{array}
\]

Pairing the above identity with an element $v \in \sigma$, we obtain the following counterpart of Corollary 5.6:

**Corollary 8.3.** For any $\Phi \in \Omega_{\psi}$ and $v \in \sigma$, one has:

$$\ell_{\theta(\sigma)}(A_\sigma(\Phi, v)) = \langle p(\Phi), \beta_\sigma(v) \rangle_{N \backslash SL_2}$$

where $A_\sigma$ is the map defined in §4.4.
9. Transfer in Geometric Terms

We have defined the transfer of test functions and established a relative character identity without making any geometric comparison. This is not so surprising, as the theta correspondence is a means of transferring spectral data from one group to another. Nonetheless, one can ask for an explicit formula for the transfer map $t\psi$, for example as an integral transform. We shall derive such a formula in this section, assuming the $F$ is nonarchimedean (with ring of integers $\mathcal{O}_F$ and uniformizer $\varpi$). We also assume for simplicity that the conductor of the additive character $\psi$ is $\mathcal{O}_F$ and the discriminant of $V$ is 1.

Given $\Phi \in C_c^\infty(V)$, we would like to compute the $(N,\psi)$-orbital integral of $p(\Phi) = f$. We consider this orbital integral as a function on the open Bruhat cell $NwB$ which is $(N,\psi)$-invariant on both sides. Hence it is determined by its value on $wT$ and we are interested in explicating the function on $F^\times$ defined by

$$I(f)(a) = \int_F f(wt(a)n(b)) \cdot \overline{\psi(n)} \, db = \int_F (wt(a)n(b) \cdot \Phi)(v_1) \cdot \overline{\psi(b)} \, db.$$

We should perhaps say a few words about the convergence of this integral. Let us identify $N\setminus SL_2$ with $W^* = F^2 \setminus O$ (where $O$ is the origin of $F^2$) via $g \mapsto (0, 1) \cdot g$. Then $|f|$ is a function on $W^*$ which vanishes on a neighbourhood of $O$. Now the element $Nwt(a)n(b) \in N\setminus SL_2$ corresponds to the element $(-a, -ab) \in W^*$. For fixed $a \in F^\times$, the function

$$b \mapsto f(wt(a)n(b))$$

is thus not necessarily compactly supported on $F$. However, if we had assumed that $f \in C_c^\infty(N,\psi\setminus SL_2)$ (which is a dense subspace of $\mathcal{C}(N,\psi\setminus SL_2)$), then $|f|$ would in addition vanish outside a compact set of $W$, so that the above function of $b$ is compactly supported on $F$ and the integral defining $I(f)(a)$ would have been convergent. This suggests that if we let $U_n = \varpi^{-n}\mathcal{O}_F$ and set

$$I_n(f)(a) = \int_{U_n} (wt(a)n(b) \cdot \Phi)(v_1) \cdot \overline{\psi(b)} \, db,$$

then the value $I_n(f)(a)$ should stabilize for sufficiently large $n$ (and this does happen for $f \in C_c^\infty(N,\psi\setminus SL_2)$). With this motivation, we shall define

$$I(f)(a) := \lim_{n \to \infty} I_n(f)(a)$$

and shall show below that the right hand side indeed stabilizes.
Now we will perform an explicit computation:

\[ \mathcal{I}_n(f)(a) = \int_{U_n} (\omega t(a)n(b) \cdot \Phi(v_1)) \cdot \overline{\psi(b)} \, db \]
\[ = \int_{U_n} \mathcal{F}(t(a)n(b) \cdot \Phi(v_1)) \cdot \overline{\psi(b)} \, db \]
\[ = \int_{U_n} \int_V (t(a)n(b) \cdot \Phi)(y) \cdot \psi(\langle v_1, y \rangle) \cdot \overline{\psi(b)} \, dy \, db \]
\[ = \int_{U_n} \int_V |a|^{\frac{1}{2} \dim(V)} \cdot (n(b) \cdot \Phi)(ay) \cdot \psi(\langle v_1, y \rangle) \cdot \overline{\psi(b)} \, dy \, db \]
\[ = \int_{U_n} \int_V \frac{|a|^{\frac{1}{2} \dim(V)} \cdot \Phi(ay) \cdot \psi(a^2bq(y)) \cdot \psi(\langle v_1, y \rangle) \cdot \overline{\psi(b)}}{a} \, dy \, db \]
\[ = \int_{U_n} |a|^{\frac{1}{2} \dim(V)} \cdot \Phi(x) \cdot \psi(\langle v_1, a^{-1}x \rangle) \cdot \left( \int_{U_n} \psi(b(q(x) - 1)) \, db \right) \, dx \]

where \( \omega = \text{supp}(\Phi) \) is compact and we have made the substitution \( x = ay \) in the last step. Recall also that \( \mathcal{F} \) is a normalized Fourier transform giving the action of the standard Weyl group element \( w \) on the Weil representation. We have however omitted the normalising factor (which is a Weil index) in writing down the effect of \( \mathcal{F} \) so that the third equality above really holds up to some scalar.

Now let us consider the inner integral

\[ \int_{U_n} \psi(b(q(x) - 1)) \, db \]

If \( q(x) - 1 \notin \pi^n \mathcal{O}_F \), then the integrand is a nontrivial character of \( U_n \) and hence the integral is 0. On the other hand, if \( q(x) - 1 \in \pi^n \mathcal{O}_F \), the integral gives the volume of \( U_n \). We may normalise the measure \( db \) so that this volume is \( q^n \) (where \( q \) is the size of the residue field of \( F \)). Hence

\[ \int_{U_n} \psi(b(q(x) - 1)) \, db = q^n \cdot 1(q(x) \in 1 + \pi^n \mathcal{O}_F) \]

and so

\[ \mathcal{I}_n(f)(a) = q^n \cdot |a|^{-\frac{1}{2} \dim(V)} \int_V \Phi(x) \cdot \psi(\langle v_1, a^{-1}x \rangle) \cdot 1(q(x) - 1 \in \pi^n \mathcal{O}_F) \, dx \]
\[ = q^n \cdot |a|^{-\frac{1}{2} \dim(V)} \int_{q^{-1}(1 + \pi^n \mathcal{O}_F)} \Phi(x) \cdot \psi(\langle v_1, a^{-1}x \rangle) \, dx. \]

Now this last expression is a quantity which appears in the theory of local densities in the theory of quadratic forms over local fields. Indeed, consider the map

\[ q : q^{-1}(1 + \pi^n \mathcal{O}_F) \longrightarrow 1 + \pi^n \mathcal{O}_F \]

of \( p \)-adic manifolds. Since every point in the base is a regular value of the map \( q \), or equivalently \( q \) is submersive at every point of the domain, the integral of the compactly supported and locally constant integrand over \( q^{-1}(1 + \pi^n \mathcal{O}_F) \) can be performed by first integrating over the fibers of \( q \) followed by integration over the base. In other words for any locally constant compactly supported \( \varphi \),

\[ \int_{q^{-1}(1 + \pi^n \mathcal{O}_F)} \varphi(x) \, dx = \int_{1 + \pi^n \mathcal{O}_F} q_*(\varphi)(z) \, dz \]
where
\[
q_*(\varphi)(z) = \int_{q^{-1}(z)} \varphi(x) \, dx.
\]
But \(q_*(\varphi)\) is also a locally constant function on the base. Hence for \(n\) sufficiently large, the above integral is simply equal to
\[
\text{Vol}(\varpi^n \mathcal{O}_F) \cdot q_*(\varphi)(1) = q^{-n} \cdot \int_{X_1} \varphi(x) \, dx.
\]
Applying this to the integral of interest, we thus deduce that the sequence \(\mathcal{I}_n(f)(a)\) stabilizes for large \(n\) and
\[
\mathcal{I}(f)(a) = |a|^{-\frac{1}{2} \dim(V)} \cdot \int_{X_1} \Phi(x) \cdot \psi(\langle v_1, a^{-1} x \rangle) \, dx
\]
Now observe that the map
\[
\gamma : X_1 = O(U) \backslash O(V) \rightarrow F
\]
given by
\[
x = h^{-1} v_1 \mapsto \langle v_1, x \rangle = \langle v_1, h^{-1} v_1 \rangle
\]
is \(O(U)\)-invariant (on the right). Moreover, for \(\xi \in F\), the preimage of \(\xi\) is equal to
\[
\gamma^{-1}(\xi) = \{x = \xi \cdot v_1 + v : v \in U \text{ and } q(v) = 1 - \xi^2\}.
\]
If \(\xi^2 \neq 1\), it follows by Witt’s theorem that this is a homogeneous space under \(O(U)\). For \(x_\xi \in \gamma^{-1}(\xi)\), \(\xi^2 \neq 1\), its stabilizer in \(O(U)\) is \(O(U \cap x_\xi^\perp)\). Thus, if \(F^\diamond = F \setminus \{\pm 1\}\) and \(X_1^\diamond = \gamma^{-1}(F^\diamond)\), then
\[
X_1^\diamond / O(U) \cong F^\diamond,
\]
and orbital integrals of smooth compactly supported functions on \(X_1\) can be identified with functions on \(F^\diamond\), via integration on the fibres of \(\gamma\).

Hence, continuing with our computation, we have:
\[
\mathcal{I}(f)(a) = |a|^{-\frac{1}{2} \dim(V)} \cdot \int_{X_1^\diamond} \Phi(x) \cdot \psi(a^{-1} \gamma(x)) \, dx
\]
\[
= |a|^{-\frac{1}{2} \dim(V)} \cdot \int_{F^\diamond} \int_{\gamma^{-1}(x_\xi)} \Phi(y) \cdot \psi(a^{-1} \xi) \, dy \, d\xi
\]
\[
= |a|^{-\frac{1}{2} \dim(V)} \cdot \int_{F^\diamond} \mathcal{I}(\phi)(\xi) \cdot \psi(a^{-1} \xi) \, d\xi
\]
Hence we have shown:

**Proposition 9.1.** The transfer map \(t_\psi : \mathcal{I}(X_1) \rightarrow \mathcal{I}(N, \psi \backslash \text{SL}_2)\) is given by:
\[
t_\psi(\phi)(a) = |a|^{-\frac{1}{2} \dim(V)} \cdot \int_{F^\diamond} \phi(\xi) \cdot \psi(a^{-1} \xi) \, d\xi.
\]
where we have regarded \(\mathcal{I}(X_1)\) and \(\mathcal{I}(N, \psi \backslash \text{SL}_2)\) as spaces of functions on \(F^\diamond\) and \(F^\times\) respectively.

### 10. Factorization of Global Periods

In this section, we examine the question of factorisation of global period integrals, in the context of the periods considered in the earlier sections. We first need to introduce the global analogs of various constructions encountered in the local setting.
10.1. **Automorphic Forms.** Let $k$ be a number field with ring of adeles $\mathbb{A}$. For a reductive group $G$ defined over $k$, we shall write $[G]$ for the quotient $G(k)\backslash G(\mathbb{A})$ and equip it with its Tamagawa measure $dg$. Let $\mathcal{A}(G)$ denote the space of (smooth) automorphic forms on $G$ and let $\mathcal{A}_{\text{cusp}}(G)$ denote the subspace of cusp forms. There is a canonical projection map $\mathcal{A}(G) \to \mathcal{A}_{\text{cusp}}(G)$. Moreover, on $\mathcal{A}_{\text{cusp}}(G)$, we have the Petersson inner product $\langle - , - \rangle_G$ (defined using the Tamagawa measure $dg$). In fact, this defines a pairing between $\mathcal{A}(G)$ and $\mathcal{A}_{\text{cusp}}(G)$.

In particular, for an irreducible cuspidal representation $\Sigma \subset \mathcal{A}_{\text{cusp}}(G)$, we have a projection $\text{pr}_\Sigma : \mathcal{A}(G) \to \Sigma$, and we denote the restriction of the Petersson inner product on $\Sigma$ by $\langle - , - \rangle_\Sigma$.

10.2. **Global periods.** Let $H \subset G$ be a subgroup so that $X = H \backslash G$ is quasi-affine. Fix a unitary Hecke character $\chi$ of $H$. Then we may consider the global $(H, \chi)$-period:

$$P_{H, \chi} : \mathcal{A}_{\text{cusp}}(G) \to \mathbb{C}$$

defined by

$$P_{H, \chi}(\phi) = \int_{[H]} \overline{\chi(h)} \cdot \phi(g) \, dg.$$ 

For a cuspidal representation $\Sigma \subset \mathcal{A}_{\text{cusp}}(G)$, we may thus consider the restriction of $P_{H, \chi}$ to $\Sigma$, denoting it by $P_{H, \chi, \Sigma}$.

10.3. **The maps $\alpha_{\text{Aut}}$ and $\beta_{\text{Aut}}$.** We shall now introduce the global analog of the maps $\alpha_\sigma$ and $\beta_\sigma$ introduced in §2.3 in the local setting. With $X_\mathbb{A} = H(\mathbb{A}) \backslash G(\mathbb{A})$, we have a $G(\mathbb{A})$-equivariant map

$$\theta : C^\infty_c(X_\mathbb{A}, \chi) \to \mathcal{A}(G)$$

defined by

$$\theta(f)(g) = \sum_{x \in X_F} f(x \cdot g)$$

The map $\theta$ is called the formation of theta series. Hence, we may define a composite map

$$\alpha^\text{Aut}_\Sigma : C^\infty_c(X_\mathbb{A}) \xrightarrow{\theta} \mathcal{A}(G) \xrightarrow{\text{pr}_\Sigma} \Sigma.$$ 

Concretely, we have:

$$\alpha^\text{Aut}_\Sigma(f) = \sum_{\phi \in \text{ONB}(\Sigma)} \langle \theta(f), \phi \rangle_G \cdot \phi.$$

On the other hand, we have the $G(\mathbb{A})$-equivariant map

$$\beta^\text{Aut}_\Sigma : \Sigma \to C^\infty(X_\mathbb{A})$$

defined by

$$\beta^\text{Aut}_\Sigma(\phi)(g) = P_{H, \chi}(g \cdot \phi).$$

One has the following lemma which connects $\alpha^\text{Aut}_\Sigma$ and $\beta^\text{Aut}_\Sigma$ and which is the global analog of (2.2):

**Lemma 10.1.** For $f \in C^\infty_c(X_\mathbb{A}, \chi)$ and $\phi \in \Sigma$, one has

$$\langle \alpha^\text{Aut}_\Sigma(f), \phi \rangle_G = \langle f, \beta^\text{Aut}_\Sigma(\phi) \rangle_X.$$
Proof. We have:
\[
\langle \alpha^\text{Aut}_\Sigma(f), \phi \rangle_G = \int_{[G]} \theta(f)(g) \cdot \overline{\phi(g)} \, dg = \int_{[G]} \sum_{\gamma \in H(k) \backslash G(k)} f(\gamma g) \cdot \overline{\phi(g)} \, dg
\]
\[
= \int_{H(k) \backslash G(k)} f(g) \cdot \overline{\phi(g)} \, dg = \int_{X_\lambda} \int_{[H]} f(hg) \cdot \overline{\phi(hg)} \, dh \, dg
\]
\[
= \int_{X_\lambda} f(x) \cdot \overline{P_{H,\chi}(\phi)}(x) \, dx = \langle f, \beta^\text{Aut}_\Sigma(\phi) \rangle_X,
\]
as desired. \(\square\)

10.4. Global Relative Characters. We may also introduce the global analog of the inner product \(J_\sigma\):
\[
J^\text{Aut}_\Sigma(\phi_1, \phi_2) := \langle \alpha^\text{Aut}_\Sigma(\phi_1), \alpha^\text{Aut}_\Sigma(\phi_2) \rangle = \langle \beta^\text{Aut}_\Sigma \alpha^\text{Aut}_\Sigma(\phi_1), \phi_2 \rangle_X.
\]
Then
\[
J^\text{Aut}_\Sigma(\phi_1, \phi_2) = \sum_{f \in \text{ONB}(\Sigma)} \langle \alpha^\text{Aut}_\Sigma(\phi_1), f \rangle \Sigma \cdot \langle f, \alpha^\text{Aut}_\Sigma(\phi_2) \rangle \Sigma
\]
\[
= \sum_{f \in \text{ONB}(\Sigma)} \langle \phi_1, \beta^\text{Aut}_\Sigma(f) \rangle_X \cdot \langle \beta^\text{Aut}_\Sigma(f), \phi_2 \rangle_X.
\]
By analog with the local case, we may introduce the global relative character \(B^\text{Aut}_\Sigma\) as an equivariant distribution on \(C^\infty_c(X_\lambda, \chi)\), defined by
\[
B^\text{Aut}_\Sigma(f) = \beta^\text{Aut}_\Sigma(\alpha^\text{Aut}_\Sigma(f)) (1) = \sum_{\phi \in \text{ONB}(\Sigma)} \langle f, \beta^\text{Aut}_\Sigma(\phi) \rangle_X \cdot P_{H,\chi}(\phi)
\]
for \(f \in C^\infty_c(X_\lambda, \chi)\). When pulled back to give a distribution on \(G(\mathbf{A})\), one has
\[
B^\text{Aut}_\Sigma(f) = \sum_{\phi \in \text{ONB}(\Sigma)} \overline{P_{H,\chi}(\Sigma(f))} \phi \cdot P_{H,\chi}(\phi),
\]
for \(f \in C^\infty_c(G(\mathbf{A}))\).

10.5. Global theta lifting. We specialise now to the setting of the dual pair \(\text{SL}_2 \times \text{O}(V)\) and recall the global theta correspondence.

Let \(\psi : k \backslash \mathbf{A} \to S^1\) be a nontrivial additive character and let \(\Omega_\psi\) be the associated Weil representation of the dual pair \(\text{SL}_2(\mathbf{A}) \times \text{O}(V)(\mathbf{A})\), realised on the space \(S(V_\mathbf{A})\) of Schwarz-Bruhat functions and equipped with its automorphic realization
\[
\theta : \Omega_\psi \to C^\infty([\text{SL}_2 \times \text{O}(V)]).
\]

For an irreducible cuspidal representation \(\Sigma\) of \(\text{SL}_2\), we may consider its global theta lift to \(\text{O}(V)\). More precisely, given \(\Phi \in \Omega_\psi\) and \(f \in \Sigma\), one defines the \(\text{SL}_2(\mathbf{A})\)-invariant and \(\text{O}(V)(\mathbf{A})\)-equivariant map
\[
A^\text{Aut}_\Sigma : \Omega_\psi \otimes \Sigma \to \mathcal{A}(\text{O}(V))
\]
by
\[
A^\text{Aut}_\Sigma(\Phi, f)(h) = \int_{[\text{SL}_2]} \theta(\Phi)(gh) \cdot \overline{f(g)} \, dg.
\]
The image of \(A^\text{Aut}_\Sigma\) is the global theta lift of \(\Sigma\), which we denote by \(\Pi = \Theta(\Sigma)\). If \(\Pi\) is cuspidal and nonzero, then it follows by the Howe duality conjecture that \(\Pi\) is an irreducible cuspidal representation.
Conversely, assume that $\Pi \subset \mathcal{A}_{cusp}(O(V))$ is cuspidal and nonzero. Then we may consider the global theta lift of $\Pi$ to $SL_2$. More precisely, given $\Phi \in \Omega_\psi$ and $\phi \in \Pi$, one defines the $O(V)(\mathbb{A})$-invariant and $SL_2(\mathbb{A})$-equivariant map
\[ B_{\Pi}^{Aut} : \Omega_\psi \otimes \Pi \rightarrow \Sigma \subset \mathcal{A}(G) \]
by
\[ B_{\Pi}^{Aut}(\Phi, \phi)(g) = \int_{[O(V)]} \theta(\Phi)(gh) \cdot \overline{\phi(h)} \, dh. \]
By computing constant term, one can show that the image of $B_{\Pi}^{Aut}$ is necessarily cuspidal.

10.6. The maps $A_{\Sigma}^{Aut}$ and $B_{\Theta(\Sigma)}^{Aut}$. We continue with the setting of the previous subsection. If $\Pi = \Theta(\Sigma)$ is nonzero cuspidal, then the image of $B_{\Theta(\Sigma)}^{Aut}$ is $\Sigma$. In this case, the maps $A_{\Sigma}^{Aut}$ and $B_{\Theta(\Sigma)}^{Aut}$ are global analogs of the maps $A_{\sigma}$ and $B_{\Theta(\sigma)}$ introduced in (3.4). By an exchange of the order of integration, we have the following global analog of (4.4):
\[ (A_{\Sigma}^{Aut}(\Phi, f), \phi)_{\Theta(\Sigma)} = (f, B_{\Theta(\Sigma)}^{Aut}(\Phi, \phi))_{\Sigma} \]
for $\Phi \in \Omega_\psi$, $f \in \Sigma$ and $\phi \in \Theta(\Sigma)$.

10.7. Global transfer of periods. For $\Phi \in \Omega_\psi$ and $\phi \in \Pi$, we may compute the $\psi$-Whittaker coefficient of $B_{\Pi}^{Aut}(\Phi, \phi)$. One has the following global analog of Corollary 5.6.

**Proposition 10.2.** For $\Phi \in \Omega_\psi$ and $\phi \in \Pi$,
\[ P_{N, \psi}(B_{\Pi}^{Aut}(\Phi, \phi)) = \int_{H(\mathbb{A}) \backslash G(\mathbb{A})} \Phi(g^{-1}v_1) \cdot P_{O(V^\perp)}(\phi)(g) \, dg = \langle \Phi|_{\chi}, \beta_{\Pi}^{Aut}(\phi) \rangle_{X_\mathbb{A}}. \]
In particular, a cuspidal representation $\Pi$ of $O(V)$ has nonzero $O(v^\perp_1)$-period if and only if its global theta lift to $SL_2$ is globally $\psi$-generic.

10.8. Factorisation of periods. Suppose now that $\Sigma$ is a tempered $\psi$-generic cuspidal representation of $SL_2$ and $\Pi = \Theta(\Sigma)$ is a nonzero cuspidal representation of $O(V)$. In this case, $\Pi = \Theta(\Sigma)$ is globally $O(v_1^\perp)$-distinguished. We shall fix
• isomorphisms of representations $\Sigma \cong \otimes_v \Sigma_v$ and $\Theta(\Sigma) \cong \otimes_v \Theta(\Sigma_v)$;
• decompositions of the Tamagawa measures $dg = \prod_v dq_v$ for $SL_2$ and $dh = \prod_v dh_v$ for $O(V)$;
• decompositions of Peterson inner products $\langle -, - \rangle_{\Sigma} = \prod_v \langle -, - \rangle_{\Sigma_v}$ and $\langle -, - \rangle_{\Theta(\Sigma)} = \prod_v \langle -, - \rangle_{\Theta(\Sigma_v)}$.

We have introduced various global quantities associated to $\Sigma$ and $\Theta(\Sigma)$, namely
• the maps $\alpha_{\Sigma}^{Aut}$, $\beta_{\Sigma}^{Aut}$, $P_{\Pi, \psi, \Sigma}$ and $J_{\Sigma}^{Aut}$ related to the global Whittaker period with respect to $(N, \psi)$;
• the maps $\alpha_{\Theta(\Sigma)}^{Aut}$, $\beta_{\Theta(\Sigma)}^{Aut}$, $P_{O(v_1^\perp), \Theta(\Sigma)}$ and $J_{\Theta(\Sigma)}^{Aut}$ related to the $O(v_1^\perp)$-period;
• the maps $A_{\Sigma}^{Aut}$, $B_{\Theta(\Sigma)}^{Aut}$ related to global theta lifting.

Now all the above global objects have local counterparts. Namely, for each place $v$ of $k$, we have:
• the maps $\alpha_{\Sigma_v}$, $\beta_{\sigma}$, $\ell_{\Sigma_v}$ and $J_{\Sigma_v}$ given by the Whittaker-Plancherel theorem;
• the maps $\alpha_{\Theta(\Sigma_v)}$, $\beta_{\Theta(\sigma)}$, $\ell_{\Theta(\Sigma_v)}$ and $J_{\Theta(\Sigma_v)}$ given by the spectral decomposition of $L^2(X_v)$;
• the maps $A_{\Sigma_v}$ and $B_{\Theta(\Sigma_v)}$ given by the spectral decomposition of the Weil representation $\Omega_{\psi_v}$.
We may take the Euler product of the above local quantities. As an example, we set:

\[
\alpha^A_\Sigma := \prod_v^\ast \alpha_{\Sigma_v} \quad \beta^A_\Sigma := \prod_v^\ast \beta_{\Sigma_v} \quad \text{and} \quad \ell^A_\Sigma := \prod_v^\ast \ell_{\Sigma_v}.
\]

Here, the Euler product has to be understood as a regularized product, via meromorphic continuation if necessary, as discussed in the introduction. Since both \(\alpha^A_\Sigma\) and \(\alpha^{\text{Aut}}_\Sigma\) are nonzero elements of the 1-dimensional space \(\text{Hom}_{\text{SL}_2(\mathbb{A})}(C^\infty_c(N(\mathbb{A}), \psi \setminus G(\mathbb{A})), \Sigma)\), there is a constant \(c(\Sigma) \in \mathbb{C}^\times\) such that

\[
\alpha^{\text{Aut}}_\Sigma = c(\Sigma) \cdot \alpha^A_\Sigma.
\]

By duality, it follows that

\[
\beta^{\text{Aut}}_\Sigma = \overline{c(\Sigma)} \cdot \beta^A_\Sigma
\]

and thus

\[
P_{\Sigma, N, \psi} = c(\Sigma) \cdot \ell^A_\Sigma \quad \text{and} \quad J_{\Sigma}^{\text{Aut}} = |c(\Sigma)|^2 \cdot J^{A}_\Sigma.
\]

Likewise, we have \(c(\Theta(\Sigma)) \in \mathbb{C}^\times\) such that

\[
\alpha^{\text{Aut}}_{\Theta(\Sigma)} = c(\Theta(\Sigma)) \cdot \alpha^A_{\Theta(\Sigma)} \quad \text{and} \quad \beta^{\text{Aut}}_{\Theta(\Sigma)} = c(\Theta(\Sigma)) \cdot \beta^A_{\Theta(\Sigma)},
\]

so that

\[
P_{\Theta(\Sigma), O(v_1^\perp)} = c(\Theta(\Sigma)) \cdot \ell^A_{\Theta(\Sigma)} \quad \text{and} \quad J_{\Theta(\Sigma)}^{\text{Aut}} = |c(\Theta(\Sigma))|^2 \cdot J^{A}_{\Theta(\Sigma)}.
\]

Similarly, we have \(a(\Sigma)\) and \(b(\Theta(\Sigma))\) \(\in \mathbb{C}^\times\) such that

\[
A_{\Sigma}^{\text{Aut}} = a(\Sigma) \cdot A^{A}_{\Sigma} \quad \text{and} \quad B_{\Theta(\Sigma)}^{\text{Aut}} = b(\Theta(\Sigma)) \cdot B^{A}_{\Theta(\Sigma)}.
\]

10.9. **Global result.** The main global problem is to determine the constant \(c(\Theta(\Sigma))\). We shall resolve this by relating \(c(\Theta(\Sigma))\) to the other constants \(c(\Sigma), a(\Sigma),\) and \(b(\Theta(\Sigma))\).

**Proposition 10.3.** We have:

\[
c(\Theta(\Sigma)) = c(\Sigma) \cdot \overline{b(\Theta(\Sigma))}.
\]

**Proof.** This follows by combining the global Proposition 10.2 and the local Corollary 5.6. \(\square\)

It remains then to compute \(c(\Sigma)\) and \(b(\Theta(\Sigma))\).

**Proposition 10.4.** We have:

\[
a(\Sigma) = \overline{b(\Theta(\Sigma))} \quad \text{and} \quad |a(\Sigma)| = 1.
\]

Moreover,

\[
c(\Sigma) = \begin{cases} 
1, & \text{if } \Sigma \text{ is non-endoscopic;} \\
1/2, & \text{if } \Sigma \text{ is an endoscopic lift from } O_2 \times O_1; \\
1/4, & \text{if } \Sigma \text{ is an endoscopic lift from } O_1 \times O_1 \times O_1.
\end{cases}
\]

**Proof.** The equality of \(a(\Sigma)\) and \(\overline{b(\Theta(\Sigma))}\) follows by the global equation (10.1) and the local equation (4.4). On the other hand, the Rallis inner product formula \[8\] gives:

\[
\langle A^{\text{Aut}}_{\Sigma}(\Phi, f), A^{\text{Aut}}_{\Sigma}(\Phi, f) \rangle = Z_{\Sigma}(\Phi, \Phi, f, f)
\]

where the right hand side is the global doubling zeta integral (evaluated at the point \(s = (\dim V - 3)/2\), where it is holomorphic). Combining this with the local equation (11.5), we deduce that \(|a(\Sigma)| = 1\).

Finally, the value of \(c(\Sigma)\) was determined in \[14\ Cor. 4.3\]. \(\square\)
As a consequence:

**Theorem 10.5.** Let $\Sigma$ be a globally $\psi$-generic cuspidal representation of $\text{SL}_2$ such that $\Pi = \Theta(\Sigma) \subset \mathcal{A}_{\text{cusp}}(O(V))$. Then $|c(\Theta(\Sigma))| = |c(\Sigma)|$, so that

$$|P_{\Pi, O(V)}(\phi)|^2 = |c(\Sigma)|^2 \cdot |f_{\Pi}(\phi)|^2$$

for all $\phi \in \Pi$. Equivalently, for $\phi_1, \phi_2 \in C_c^\infty(X_{A_k})$, we have

$$J_{\Pi}^{\text{Aut}}(\phi_1, \phi_2) = |c(\Sigma)|^2 \cdot J_{\Pi}^{\text{Aut}}(\phi_1, \phi_2).$$

10.10. **Global Relative Character Identity.** We can also establish the global analog of the relative character identity. One has the diagram

$$S(V_A) \xrightarrow{p} S(N(A), \psi \backslash \text{SL}_2(A)) \xrightarrow{q} S(X_A)$$

which is the adelic analog of the diagram (7.2). The space $S(N(A), \psi \backslash \text{SL}_2(A))$ is the restricted tensor product of the local spaces $S(N(k_v), \psi_v \backslash \text{SL}_2(k_v))$ of test functions defined in (7.1), where the restricted tensor product is with respect to the family of basic functions $\{f_0, v\}$ given in Definition 7.4. Likewise, the space $S(X_A)$ is the restricted tensor product of $S(X_{k_v})$ (which is $C_c^\infty(X_{k_v})$ at finite places) with respect to the family of basic functions $\{\phi_0, v\}$ in Definition 7.4.

As in the local case, one says that $f \in S(N(A), \psi \backslash \text{SL}_2(A))$ and $\phi \in S(X_A)$ are in correspondence or are transfers of each other if there exists $\Phi \in S(V_A)$ such that $f = p(\Phi)$ and $\phi = q(\Phi)$. The fundamental Lemma 7.5 ensures that every $f$ has a transfer $\phi$ and vice versa. One now has the following global relative character identity.

**Theorem 10.6.** If $f \in S(N(A), \psi \backslash \text{SL}_2(A))$ and $\phi \in S(X_A)$ are transfer of each other, then for a cuspidal representation $\Sigma$ of $\text{SL}_2$ with cuspidal theta lift $\Theta(\Sigma)$ on $O(V)$, one has:

$$B_{\Sigma}^{\text{Aut}}(f) = B_{\Theta(\Sigma)}^{\text{Aut}}(\phi).$$

**Proof.** One has

$$B_{\Sigma}^{\text{Aut}}(f) = |c(\Sigma)|^2 \cdot B_{\Sigma}^k(f)$$

and

$$B_{\Theta(\Sigma)}^{\text{Aut}}(\phi) = |c(\Theta(\Sigma))|^2 \cdot B_{\Theta(\Sigma)}^k(\phi).$$

Since $|c(\Sigma)| = |c(\Theta(\Sigma))|$, the result follows from the local relative character identity of Theorem 8.1. \qed

10.11. **Avoiding Rallis inner product.** In proving Theorem 10.5, we have pulled the Rallis inner product formula out of the hat to deduce that $|a(\Sigma)| = 1$ in Proposition 10.4. In fact, it is possible to avoid the Rallis inner product formula, as we briefly sketch in this subsection.

Just as Proposition 10.2 is the global analog of the local Corollary 5.6, one can establish a global analog of Corollary 8.3, namely:

**Proposition 10.7.** For $\Phi \in \Omega_\psi$ and $f \in \Sigma$, one has

$$P_{O(V^\perp)}(A_{\Sigma}^{\text{Aut}}(\Phi, f)) = \langle p(\Phi), \beta_{\Sigma}^{\text{Aut}}(f) \rangle_{N \backslash \text{SL}_2}.$$
Proof. The proof relies on the see-saw diagram

\[
\begin{array}{ccc}
\text{Mp}_2 \times \text{Mp}_2 & \rightarrow & \text{O}(V) \\
\text{SL}_2 & \leftrightarrow & \text{O}(v_1) \times \text{O}(v_1^\perp)
\end{array}
\]

which gives rise to a global see-saw identity. More precisely, if we take
\[
\Phi = \Phi_1 \otimes \Phi' \in \mathcal{S}(A_{v_1}) \otimes \mathcal{S}((v_1^\perp)_\lambda),
\]
then the see-saw identity reads:
\[
P_{\Pi, O(v_1^\perp)}(A^\text{Aut}_\Sigma(\Phi, f)) = \int_{[\text{SL}_2]} f(g) \cdot \theta(\Phi_1)(g) \cdot I(\Phi')(g) \, dg
\]
where
- \(\theta(\Phi_1)\) is the theta function associated to \(\Phi_1 \in \mathcal{S}(A_{v_1})\) which affords the Weil representation (associated to \(\psi\)) for \(\text{Mp}_2 \times \text{O}(v_1)\);
- \(I(\Phi')\) is the theta integral
\[
I(\Phi')(g) := \int_{[\text{O}(v_1^\perp)]} \theta(\Phi')(gh) \, dh
\]
which belongs to the global theta lift of the trivial representation of \(\text{O}(v_1^\perp)\) to \(\text{Mp}_2\).

The theta integral converges absolutely when \(\dim V > 4\) (it is in the so-called Weil’s convergence range) or when \(\text{O}(v_1^\perp)\) is anisotropic. When \(\dim V = 4\) and \(\text{O}(v_1^\perp)\) is split, one needs to regularise the theta integral following Kudla-Rallis (see \([8, \S 3]\)). Since our intention here is to indicate an alternative approach to a result which we have shown, we will ignore this analytic complication in the following exposition.

In \([7, \S 3]\) we have seen that the map \(\phi' \mapsto \phi_{\Phi'}\), where
\[
\phi_{\Phi'}(g) = (g \cdot \Phi')(0),
\]
gives an isomorphism
\[
\mathcal{S}((v_1^\perp)_\lambda)_{\text{O}(v_1^\perp)} \rightarrow I_\psi(\chi_{\text{disc}(v_1^\perp)})(\dim V - 3)/2
\]
of the \(\text{O}(v_1^\perp)\)-coinvariant of the Weil representation of \(\text{O}(v_1^\perp) \times \text{Mp}_2\) to a principal series representation of \(\text{Mp}_2\). Now the Siegel-Weil formula shows that
\[
I(\Phi') = E(\phi_{\Phi'})
\]
where \(E(\phi_{\Phi'})\) is the Eisenstein series associated to \(\phi_{\Phi'}\). Again, when \(\dim V > 4\), the sum defining the Eisenstein series is convergent, but when \(\dim V = 4\), it is defined by meromorphic continuation. Further, if \(\text{O}(v_1^\perp)\) is also split, then the Eisenstein series does have a pole at the point of interest, and we need to invoke the second term identity of the Siegel-Weil formula \([8]\). As mentioned before, we omit these extra (though interesting) details in this proof.

Hence, we have the following identity:
\[
P_{\Pi, O(v_1^\perp)}(A^\text{Aut}_\Sigma(\Phi, f)) = \int_{[\text{SL}_2]} f(g) \cdot \theta(\Phi_1)(g) \cdot E(\phi_{\Phi'}) \, dg.
\]
Now the right hand side is the value at \(s = (\dim V - 3)/2\) of the global zeta integral
\[
Z(f, \Phi_1, \phi_s) = \int_{[\text{SL}_2]} f(g) \cdot \theta(\Phi_1)(g) \cdot E(\phi_s) \, dg
\]
for $\phi_s \in I_\psi(x_{\text{disc}}(v_1^+), s)$. This is the global analog of the local zeta integrals we discussed in [7,3] and represents the (twisted) adjoint L-function of $\Sigma$. The unfolding of this global zeta integral, for $\text{Re}(s)$ sufficiently large, gives:

$$Z(s, f, \Phi_1, \phi) = \int_{N(\mathbb{A}) \backslash SL_2(\mathbb{A})} P_{N, \psi}(g \cdot f) \cdot (g \cdot \Phi_1)(v_1) \cdot \phi_s(g) \, dg.$$ 

Specializing to $s = (\dim V - 3)/2$ gives the Proposition.

By combining Proposition 10.7 and Corollary 8.3 we deduce:

**Corollary 10.8.** One has:

$$c(\Sigma) = c(\Theta(\Sigma)) \cdot a(\Sigma).$$

Combining Corollary 10.8 with Propositions 10.3 and 10.4, we see that

$$c(\Sigma) = c(\Theta(\Sigma)) \cdot a(\Sigma) = c(\Sigma) \cdot b(\Theta(\Sigma)) \cdot a(\Sigma) = c(\Sigma) \cdot |a(\Sigma)|^2$$

from which we deduce that

$$|a(\Sigma)| = 1.$$

There is a good reason for avoiding the use of the Rallis inner product formula in the treatment of the global problem. Indeed, the viewpoint and techniques developed in this paper should carry over to essentially all the low rank spherical varieties treated in [7]. Many of these (such as $\text{Spin}_9 \setminus \text{F}_4$, $G_2 \setminus \text{Spin}_8$ or $F_4 \setminus E_6$ to name a few) would involve the exceptional theta correspondence. Unfortunately, in the setting of the exceptional theta correspondence, an analog of the Rallis inner product formula is not known. The argument in this subsection, however, shows that this lack need not be an obstruction in the exceptional setting.

**10.12. End remarks.** We end this paper with some comparisons with the relative trace formula approach. The spectral side of a relative trace formula is essentially a sum of the relevant global relative characters over all cuspidal representations. One then hopes to separate the different spectral contributions by using the action of the spherical Hecke algebra at almost all places. The main global output of a comparison of (the geometric side of) two such relative trace formulae is typically a global relative character identity as in Theorem 10.6 as a consequence of which one deduces Proposition 10.2 and the local relative character identities in Theorem 8.1, which in turn implies Proposition 5.1. It is interesting to compare this with the approach via theta correspondence which we have pursued in this paper.

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