MAGIC RECTANGLES, SIGNED MAGIC ARRAYS AND INTEGER $\lambda$-FOLD RELATIVE HEFFTER ARRAYS

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Abstract. Let $m, n, s, k$ be integers such that $4 \leq s \leq n$, $4 \leq k \leq m$ and $ms = nk$. Let $\lambda$ be a divisor of $2ms$ and let $t$ be a divisor of $\frac{2ms}{\lambda}$. In this paper we construct magic rectangles $MR(m, n; s, k)$, signed magic arrays $SMA(m, n; s, k)$ and integer $\lambda$-fold relative Heffter arrays $\lambda H_t(m, n; s, k)$ where $s, k$ are even integers. In particular, we prove that there exists an SMA($m, n; s, k$) for all $m, n, s, k$ satisfying the previous hypotheses. Furthermore, we prove that there exist an MR($m, n; s, k$) and an integer $\lambda H_t(m, n; s, k)$ in each of the following cases: (i) $s, k \equiv 0 \pmod{4}$; (ii) $s \equiv 2 \pmod{4}$ and $k \equiv 0 \pmod{4}$; (iii) $s \equiv 0 \pmod{4}$ and $k \equiv 2 \pmod{4}$; (iv) $s, k \equiv 2 \pmod{4}$ and $m, n$ both even.

1. Introduction

In this paper we study partially filled (p.f., for short) arrays, with entries in $\mathbb{Z}$ and whose rows and columns have prescribed sums. In particular, we construct magic rectangles, signed magic arrays and integer $\lambda$-fold relative Heffter arrays.

Definition 1.1. A signed magic array $SMA(m, n; s, k)$ is an $m \times n$ p.f. array with elements in $\Omega \subset \mathbb{Z}$, where $\Omega = \{0, \pm 1, \pm 2, \ldots, \pm (ms - 1)/2\}$ if $ms$ is odd and $\Omega = \{\pm 1, \pm 2, \ldots, \pm ms/2\}$ if $ms$ is even, such that

(a) each row contains $s$ filled cells and each column contains $k$ filled cells;
(b) every $x \in \Omega$ appears exactly once in the array;
(c) the elements in every row and column sum to 0.

The existence of an SMA($m, n; s, k$) has been settled out in the square case (i.e., when $m = n$) and in the tight case (i.e., when $k = m$ and $s = n$), by Khodkar, Schulz and Wagner.

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Theorem 1.2. [17] There exists an \( \text{SMA}(n, n; k, k) \) if and only if either \( n = k = 1 \) or \( 3 \leq k \leq n \).

Theorem 1.3. [17] There exists an \( \text{SMA}(m, n; n, m) \) if and only if one of the following cases occurs:

1. \( m = n = 1 \);
2. \( m = 2 \) and \( n \equiv 0, 3 \pmod{4} \);
3. \( n = 2 \) and \( m \equiv 0, 3 \pmod{4} \);
4. \( m, n > 2 \).

Also the cases when each column contains 2 or 3 filled cells have been solved.

Theorem 1.4. [13] There exists an \( \text{SMA}(m, n; s, 2) \) if and only if one of the following cases occurs:

1. \( m = 2 \) and \( n = s \equiv 0, 3 \pmod{4} \);
2. \( m, s > 2 \) and \( ms = 2n \).

Theorem 1.5. [16] There exists an \( \text{SMA}(m, n; s, 3) \) if and only if \( 3 \leq m, s \leq n \) and \( ms = 3n \).

In this paper we consider the case when \( s \) and \( k \) are both even, proving the following result.

Theorem 1.6. Let \( s, k \) be two even integers with \( s, k \geq 4 \). There exists an \( \text{SMA}(m, n; s, k) \) if and only if \( 4 \leq s \leq n \), \( 4 \leq k \leq m \) and \( ms = nk \).

This result will be obtained working in the more general context of the integer \( \lambda \)-fold relative Heffter arrays.

In [1] Dan Archdeacon introduced an important class of p.f. arrays, called Heffter arrays. One of the applications of these objects is that they allow, under suitable conditions, to construct pairs of cyclic cycle decompositions of the complete graph \( K_v \) on \( v \) vertices. With the aim to extend this application to complete multipartite graphs, in [8] the authors of the present paper, in collaboration with Costa and Pasotti, proposed a first generalization of Archdeacon’s idea introducing p.f. arrays called relative Heffter arrays. A further generalization, that allows to work with complete multipartite multigraphs, was introduced in [9] by Costa and Pasotti. These new objects are called \( \lambda \)-fold relative Heffter arrays. We recall here their definition, where we denote by \( E(A) \) the list of the entries of the filled cells of a p.f. array \( A \).

Definition 1.7. Let \( m, n, s, k, t, \lambda \) be positive integers such that \( \lambda \) divides \( 2ms \) and \( t \) divides \( \frac{2ms}{\lambda} \). Let \( J \) be the subgroup of order \( t \) of \( \mathbb{Z}_v \), where \( v = \frac{2ms}{\lambda} + t \). A \( \lambda \)-fold Heffter array over \( \mathbb{Z}_v \) relative to \( J \), denoted by \( \lambda \text{H}t(m, n; s, k) \), is an \( m \times n \) p.f. array \( A \) with elements in \( \Omega = \mathbb{Z}_v \setminus J \) such that:

- (a) each row contains \( s \) filled cells and each column contains \( k \) filled cells;
- (b) every element of \( \Omega \) appears exactly \( \lambda \) times in the list \( E(A) \cup -E(A) \);
- (c) the elements in every row and column sum to 0.

Item (b) of the previous definition requires some explications. The additive group \( \mathbb{Z}_v \) contains an involution if and only if \( v \) is even: in this case, the unique involution \( \iota \in \mathbb{Z}_v \) belongs to \( \Omega \) if and only if \( t \) is odd. We observe that the assumption \( v \) even and \( t \) odd implies that \( \lambda \) is even and does not divide \( ms \). So, we can write (b) as follows: if \( \Omega \) does not contain involutions, every \( x \in \Omega \) appears in \( A \), up to sign, exactly \( \lambda \) times; if \( \Omega \) contains the involution \( \iota \), then every \( x \in \Omega \setminus \{ \iota \} \) appears, up to sign, exactly \( \lambda \) times, while \( \iota \) appears exactly \( \frac{\lambda}{2} \) times.
Instead of working in a finite cyclic group, one can construct \( \lambda \)-fold relative Heffter arrays whose entries are rational integers. In this case, the previous definition becomes as follows.

**Definition 1.8.** Let \( m, n, s, k, t, \lambda \) be positive integers such that \( \lambda \) divides \( 2ms \) and \( t \) divides \( \frac{2ms}{\lambda} \). Let

\[
\Phi = \left\{ 1, 2, \ldots, \left\lfloor \frac{v}{2} \right\rfloor \right\} \setminus \left\{ \ell, 2\ell, \ldots, \left\lfloor \frac{t}{2} \ell \right\rfloor \right\} \subset \mathbb{Z}, \quad \text{where } v = \frac{2ms}{\lambda} + t \quad \text{and} \quad \ell = \frac{v}{t}.
\]

An integer \( ^\lambda \text{H}_t(m, n; s, k) \) is an \( m \times n \) p.f. array with elements in \( \Phi \) such that:

(a) each row contains \( s \) filled cells and each column contains \( k \) filled cells;

(b) if \( v \) is odd or if \( t \) is even, every element of \( \Phi \) appears, up to sign, exactly \( \lambda \) times in the array; if \( v \) is even and \( t \) is odd, every element of \( \Phi \setminus \left\{ \frac{v}{2} \right\} \) appears, up to sign, exactly \( \lambda \) times while \( \frac{v}{2} \) appears, up to sign, exactly \( \frac{\lambda}{2} \) times;

(c) the elements in every row and column sum to 0.

Observe that when \( \lambda = 1 \) one retrieves the concept of (integer) relative Heffter array. In particular, an (integer) \( ^1 \text{H}_1(m, n; s, k) \) is exactly a classical (integer) Heffter array, as defined by Archdeacon. The problem of the existence of square classical Heffter arrays has been completely solved in [3, 12] for the integer case, and in [5] for the general case. For the other cases (non-square or relative), partial results have been obtained in [2, 10, 18]. Applications of (relative) Heffter arrays to graph decompositions and biembeddings are described, for instance, in [4, 6, 7, 11].

Here, we prove the following result.

**Theorem 1.9.** Let \( m, n, s, k \) be integers such that \( 4 \leq s \leq n, \ 4 \leq k \leq m \) and \( ms = nk \). Let \( \lambda \) be a divisor of \( 2ms \) and let \( t \) be a divisor of \( \frac{2ms}{\lambda} \). There exists an integer \( ^\lambda \text{H}_t(m, n; s, k) \) in each of the following cases:

1. \( s, k \equiv 0 \pmod{4} \);
2. \( s \equiv 2 \pmod{4} \) and \( k \equiv 0 \pmod{4} \);
3. \( s \equiv 0 \pmod{4} \) and \( k \equiv 2 \pmod{4} \);
4. \( s, k \equiv 2 \pmod{4} \) and \( m, n \) both even.

Looking at Definitions 1.1 and 1.8 the reader can easily see that, when \( ms \) is even, a signed magic array is a particular integer 2-fold relative Heffter array. In fact, the integer \( ^\lambda \text{H}_1(m, n; s, k) \) we construct in the following sections is actually a signed magic array \( \text{SMA}(m, n; s, k) \). So, Theorem 1.9 will follow from Theorem 1.10 except when \( s, k \equiv 2 \pmod{4} \) and \( m, n \) are odd. Nevertheless, for these exceptional values, we will construct \( \text{SMA}(m, n; s, k) \) starting from square signed magic arrays, whose existence is assured by Theorem 1.2 and exploiting the flexibility of our constructions.

Our results on signed magic arrays allow us to build also magic rectangles.

**Definition 1.10.** A magic rectangle \( \text{MR}(m, n; s, k) \) is an \( m \times n \) p.f. array with elements in \( \Omega = \{0, 1, \ldots, ms - 1\} \subset \mathbb{Z} \) such that

(a) each row contains \( s \) filled cells and each column contains \( k \) filled cells;

(b) every \( x \in \Omega \) appears exactly once in the array;

(c) the sum of the elements in each row is a constant value \( c_1 \) and the sum of the elements in each column is a constant value \( c_2 \).
Clearly, in the previous definition we must have \( c_1 = \frac{a(mn-1)}{2} \) and \( c_2 = \frac{k(mn-1)}{2} \). The reader can find results on the existence of these objects in [13][15] and in the references within. Here, we prove the following.

**Theorem 1.11.** Let \( m, n, s, k \) be integers such that \( 4 \leq s \leq n, 4 \leq k \leq m \) and \( mn = nk \). There exists an \( \text{MR}(m, n; s, k) \) in each of the following cases:

1. \( s, k \equiv 0 \pmod{4} \);
2. \( s \equiv 2 \pmod{4} \) and \( k \equiv 0 \pmod{4} \);
3. \( s \equiv 0 \pmod{4} \) and \( k \equiv 2 \pmod{4} \);
4. \( s, k \equiv 2 \pmod{4} \) and \( m, n \) both even.

2. **Notations**

In this paper, the arithmetic on the row (respectively, on the column) indices is performed modulo \( m \) (respectively, modulo \( n \)), where the set of reduced residues is \( \{1, 2, \ldots, m\} \) (respectively, \( \{1, 2, \ldots, n\} \)), while the entries of the arrays are taken in \( \mathbb{Z} \). Given two integers \( a \leq b \), we denote by \([a, b]\) the interval consisting of the integers \( a, a+1, \ldots, b \). If \( a > b \), then \([a, b]\) is empty. We denote by \((i, j)\) the cell in the \( i \)-th row and \( j \)-th column of an array \( A \).

The **support** of \( A \), denoted by \( \text{supp}(A) \), is defined to be the set of the absolute values of the elements contained in \( A \).

If \( A \) is an \( m \times n \) p.f. array, for \( i \in [1, n] \) we define the \( i \)-th diagonal as

\[
D_i = \{(1, i), (2, i+1), \ldots, (m, i+m-1)\}.
\]

**Definition 2.1.** A p.f. array with entries in \( \mathbb{Z} \) is said to be **shiftable** if every row and every column contains an equal number of positive and negative entries.

Let \( A \) be a shiftable p.f. array and \( x \) be a nonnegative integer. Let \( A \pm x \) be the (shiftable) p.f. array obtained adding \( x \) to each positive entry of \( A \) and \(-x \) to each negative entry of \( A \). Observe that, since \( A \) is shiftable, the row and column sums of \( A \pm x \) are exactly the row and column sums of \( A \).

We denote by \( \tau_i(A) \) and \( \gamma_j(A) \) the sum of the elements of the \( i \)-th row and the sum of the elements of the \( j \)-th column, respectively, of a p.f. array \( A \).

For a block \( B \), we write \( \mu(B) = \mu \) if every element of \( \text{supp}(B) \) appears, up to sign, exactly \( \mu \) times in \( \mathcal{E}(B) \).

Given a sequence \( S = (B_1, B_2, \ldots, B_r) \) of shiftable p.f. arrays and a nonnegative integer \( x \), we write \( S \pm x \) for the sequence \( (B_1 \pm x, B_2 \pm x, \ldots, B_r \pm x) \). We set \( \mathcal{E}(S) = \cup_i \mathcal{E}(B_i) \) and \( \text{supp}(S) = \cup_i \text{supp}(B_i) \). We also write \( \mu(S) = \mu \) if \( \mu(B_i) = \mu \) for all \( i \).

If \( S_1 = (a_1, a_2, \ldots, a_r) \) and \( S_2 = (b_1, b_2, \ldots, b_u) \) are two sequences, by \( S_1 \# S_2 \) we mean the sequence \((a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_u)\) obtained by concatenation of \( S_1 \) and \( S_2 \). In particular, if \( S_1 \) is the empty sequence then \( S_1 \# S_2 = S_2 \). Furthermore, given the sequences \( S_1, \ldots, S_c \), we write \( \oplus_{i=1}^c S_i \) for \((\cdots ((S_1 \# S_2) \# S_3) \# \cdots) \# S_c \).

Given a positive integer \( n \) and a sequence \( S = (a_1, a_2, \ldots, a_r) \), we denote by \( n \ast S \) the sequence obtained concatenating \( n \) copies of \( S \).

Finally, we recall that the support of an integer \( \lambda_H(m, n; s, k) \) is the set

\[
\Phi = \left[ 1, \left\lceil \frac{t\ell}{2} \right\rceil \right] \setminus \left\{ \ell, 2\ell, \ldots, \left\lceil \frac{t}{2} \right\rceil \ell \right\}, \quad \text{where} \quad \ell = \frac{2mn}{\lambda_H} + 1 = \frac{v}{t}.
\]
Note that, if $\lambda$ divides $ms$, then

$$\Phi = \left[ 1, \frac{ms}{\lambda} + \left\lfloor \frac{t}{2} \right\rfloor \right] \setminus \{(t, 2t, \ldots, \left\lfloor \frac{t}{2} \right\rfloor)\}.$$  

Also, every element of $\Phi$ appears in $^tH_t(m, n; s, k)$, up to sign, exactly $\lambda$ times. If $\lambda$ does not divide $ms$, in order to obtain an integer $^tH_t(m, n; s, k)$, we have to construct a p.f. array $A$ such that

- if $t$ is even, every element of $\Phi$ appears in $A$, up to sign, exactly $\lambda$ times; otherwise, i.e, if $t$ is even and $t$ is odd, every element of $\Phi \setminus \{(t, 2t, \ldots, \left\lfloor \frac{t}{2} \right\rfloor)\}$ appears in $A$, up to sign, exactly $\lambda$ times, while the integer $\frac{t}{2}$ appears, up to sign, $\frac{t}{2}$ times.

### 3. The case $s, k \equiv 0 \pmod{4}$

In this section we prove the existence of an integer $^tH_t(m, n; s, k)$ when both $s$ and $k$ are divisible by 4. First of all, we set

$$d = \gcd(m, n), \quad m = d\tilde{n}, \quad n = d\tilde{n}, \quad s = 4\tilde{s}, \quad k = 4\tilde{k}.$$  

Note that from $ms = nk$ we obtain that $\tilde{n}$ divides $\tilde{s}$ and $\tilde{m}$ divides $\tilde{k}$. Hence, we can write $\tilde{s} = c\tilde{n}$ and $\tilde{k} = c\tilde{m}$.

Fix two integers $a, b \geq 0$ and consider the following shiftable p.f. array:

$$B = B_{a,b} = \begin{bmatrix} 1 & -(a+1) \\ -(b+1) & a+b+1 \end{bmatrix}.$$  

Note that the sequences of the row/column sums are $(-a, a)$ and $(-b, b)$, respectively. We use this $3 \times 2$ block for constructing p.f. arrays whose rows and columns sum to zero. Start taking an empty $m \times n$ array $A$, fix $m\tilde{n}$ nonnegative integers $y_0, y_1, \ldots, y_{m\tilde{n}-1}$, and arrange the blocks $B \pm y_j$ in such a way that the element $1 + y_j$ fills the cell $(j + 1, j + 1)$ of $A$ (recall that we work modulo $m$ on row indices and modulo $n$ on column indices). In this way, we fill the diagonals $D_{im-1}, D_{im}, D_{im+1}, D_{im+2}$ with $i \in [1, \tilde{n}]$. In particular, every row has $4\tilde{n}$ filled cells and every column has $4\tilde{n}$ filled cells.

Looking at the rows, the elements belonging to the diagonals $D_{im+1}, D_{im+2}$ sum to $-a$, while the elements belonging to the diagonals $D_{im-1}, D_{im}$ sum to $a$. Looking at the columns, the elements belonging to the diagonals $D_{im+1}, D_{im-1}$ sum to $-b$, while the elements belonging to the diagonals $D_{im+2}, D_{im}$ sum to $b$. Then $A$ has row/column sums equal to zero.

Applying this process $c$ times (working with the diagonals $D_{im+3}, D_{im+4}, D_{im+5}, D_{im+6}$, and so on), we obtain a p.f. array $A$, whose rows have exactly $4\tilde{n} \cdot c = s$ filled cells and whose columns have exactly $4\tilde{m} \cdot c = k$ filled cells.

**Example 3.1.** For $a = 2$ and $b = 5$, fixing the integers $0, 1, 10, 11, 20, 21, 30, 31, 40, 41, 50, 51$, we can fill the diagonals $D_1, D_2, D_5, D_6, D_7, D_8, D_{11}, D_{12}$ of the following $6 \times 12$ p.f. array, where we highlighted the block $B_{2,5}$:

|   | 1 | −3 |   | −26 | 28 | 31 | −32 |   | −56 | 58 |
|---|---|----|---|-----|----|----|-----|---|-----|----|
| 59 | 2 | −4 |   | −27 | 29 | 32 | −34 |   | −56 | 57 |
| −6 | 8 | 11 |   | −13 | 36 | 38 | 41 |   | −43 |   |
| 7 | 9 | 12 |   | −14 | −38 | 39 | 42 |   | −44 |   |
|    |   | −16 | 18 | 21 | −23 | −46 | 48 | 51 | −53 |   |
| −54 | −17 | 19 | 22 | −24 |   | −47 | 49 | 52 |   |   |

![MAGIC RECTANGLES, SIGNED MAGIC ARRAYS AND...](image-url)
Note that \( \text{supp}(A) = [1, 60] \setminus \{5j : j \in [1, 12]\} \). As the reader can verify, \( A \) is an integer \( ^1H_{24}(6, 12; 8, 4) \): in this case \( \ell = \frac{2 \cdot 5 \cdot 8}{24} + 1 = 5 \).

The constructions we present in this section are obtained following this procedure, so they all produce shiftable p.f. arrays of size \( m \times n \) whose rows and columns sum to zero.

Here we always assume that \( 4 \leq s \leq n, 4 \leq k \leq m, ms = nk \) and \( s, k \equiv 0 \pmod{4} \). Let \( \lambda \) be a divisor of \( 2ms \) and \( t \) be a divisor of \( \frac{2ms}{\lambda} \), set

\[
\ell = \frac{2ms}{\lambda} + 1.
\]

We first consider the case when \( \lambda \) divides \( ms \). To obtain an integer \( ^\lambda H_i(m, n; s, k) \) with \( s, k \equiv 0 \pmod{4} \), we only have to determine two integers \( a, b \geq 0 \) and a set \( X = \{x_0, x_1, \ldots, x_{f-1}\} \subset \mathbb{N} \) such that \( \mu(B_{a,b}) = \mu \) divides \( \lambda \) and \( \bigcup_{x \in X} \text{supp}(B_{a,b} \pm x) = \Phi \). Note that \( f = \frac{ms \mu}{4} \).

So we can take the sequence \( Y = \frac{1}{\mu} \ast (x_0, x_1, \ldots, x_{f-1}) \). Writing \( Y = (y_0, y_1, \ldots, y_{ms-1}) \) we construct \( A \) using the blocks \( B_{a,b} \pm y_j \). In this way, every element of \( \text{supp}(A) \) occurs, up the sign, \( \lambda \) times in \( A \). For instance, we can arrange the blocks in such a way that the element \( 1 + y_j \) fills the cell \( (j + 1, 4q_j + j + 1) \), where \( q_j \) is the quotient of the division of \( j \) by \( \text{lcm}(m, n) \).

**Lemma 3.2.** Let \( \lambda \) be a divisor of \( ms \) such that \( \lambda \equiv 0 \pmod{4} \). There exists an integer \( ^\lambda H_i(m, n; s, k) \) for any divisor \( t \) of \( \frac{2ms}{\lambda} \).

**Proof.** Let \( B = B_{0,0} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \). Note that \( \mu(B) = 4 \). An integer \( ^\lambda H_i(m, n; s, k) \), say \( A \), can be obtained following the construction described before, once we exhibit a suitable set \( X \) of size \( \frac{ms}{\lambda} \), in such a way that \( \text{supp}(A) = \Phi \). Consider the set \( X = \{i - 1 : i \in \Phi\} \) of size \( \frac{ms}{\lambda} \): clearly, \( \bigcup_{x \in X} \text{supp}(B \pm x) = \Phi \). Now we take \( \frac{4}{\lambda} \) copies of every block \( B \pm x \): the p.f. array \( A \) obtained following our procedure is an integer \( ^\lambda H_i(m, n; s, k) \). \( \square \)

For instance, to construct an integer \( ^8H_5(5, 10; 8, 4) \) we can follow the proof of the previous lemma. In fact, \( \lambda = 8 \) and \( t = 5 \) divides \( \frac{2 \cdot 5 \cdot 8}{8} \); note that \( \ell = 3 \) and \( Y = 2 \ast (0, 1, 3, 4, 6) \).

\[
^8H_5(5, 10; 8, 4) = \begin{array}{cccccccc}
1 & -1 & -5 & 5 & 1 & -1 & -5 & 5 \\
7 & 2 & -2 & 7 & 2 & -2 & 7 & 2 \\
-1 & 1 & 4 & -1 & 1 & 4 & -1 & 1 \\
-2 & 2 & 5 & -2 & 2 & 5 & 5 & -5 \\
-7 & -4 & 7 & -7 & -4 & 7 & -7 & -4
\end{array}
\]

**Lemma 3.3.** Let \( \lambda \) be a divisor of \( ms \) such that \( \lambda \equiv 2 \pmod{4} \). There exists an integer \( ^\lambda H_i(m, n; s, k) \) for any divisor \( t \) of \( \frac{2ms}{\lambda} \).

**Proof.** We first consider the case when \( \ell \) is odd, which means that \( t \) divides \( \frac{ms}{\lambda} \). Let \( B = B_{1,0} = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \); note that \( \mu(B) = 2 \). We start considering the set \( X_0 = \{0, 2, 4, \ldots, \ell - 3\} \) of size \( \frac{\ell - 1}{2} = \frac{ms}{\lambda \ell} \); it is easy to see that \( \bigcup_{x \in X_0} \text{supp}(B \pm x) = [1, \ell] \setminus \{\ell\} \). Similarly, for any \( i \in \mathbb{N} \),
if $X_i = \{i\ell, i\ell + 2, i\ell + 4, \ldots, (i + 1)\ell - 3\}$, then
$$\bigcup_{x \in X_i} \text{supp}(B \pm x) = [i\ell + 1, (i + 1)\ell] \setminus \{(i + 1)\ell\}$$
and $X_{i_1} \cap X_{i_2} = \emptyset$ if $i_1 \neq i_2$.

If $t$ is even, take $X = \bigcup_{i=0}^{t/2-1} X_i$: this is a set of size $\frac{t}{2} \cdot \frac{ms}{M} = \frac{ms}{2X}$, as required. Furthermore,

$$\bigcup_{x \in X} \text{supp}(B \pm x) = \bigcup_{i=0}^{t/2-1} ([i\ell + 1, (i + 1)\ell] \setminus \{(i + 1)\ell\}) = [1, \frac{t}{2\ell}] \setminus \{\ell, 2\ell, \ldots, \frac{t}{2}\ell\}. $$

Suppose now that $t$ is odd, which implies that $\ell \equiv 1 \pmod{4}$. Take $Z = \left\{ \left(\frac{t-1}{2}\right)\ell, \left(\frac{t-1}{2}\right)\ell + 2, \left(\frac{t-1}{2}\right)\ell + 4, \ldots, \left(\frac{t-1}{2}\right)\ell + 2\left(\frac{t-1}{4}\right) \right\}$. Then $|Z| = \frac{\ell + 1}{2} = \frac{ms}{2X}$ and $\bigcup_{z \in Z} \text{supp}(B \pm z) = [(\frac{t+1}{2})\ell + 1, (\frac{t+1}{2})\ell + \frac{t+1}{2}]$. So, we can take $X = \left( \bigcup_{i=0}^{(t-3)/2} X_i \right) \cup Z$: this is a set of size $\frac{t-1}{2} \cdot \frac{ms}{M} + \frac{ms}{2X} = \frac{ms}{2X}$, as required. In this case,

$$\bigcup_{x \in X} \text{supp}(B \pm x) = \bigcup_{i=0}^{(t-3)/2} ([i\ell + 1, (i + 1)\ell] \setminus \{(i + 1)\ell\}) \cup [(\frac{t-1}{2})\ell + 1, (\frac{t-1}{2})\ell + \frac{t-1}{2}] = [1, \frac{t}{2\ell}] \setminus \{\ell, 2\ell, \ldots, \frac{t}{2}\ell\}.$$ 

In both cases, considering $\frac{t}{2}$ copies of the distinct blocks $B \pm x$ with $x \in X$, the p.f. array $A$ obtained following our procedure is an integer $^\lambda H_s(m, n; s, k)$.

Finally, we consider the case when $\ell$ is even, which implies that $t \equiv 0 \pmod{4}$. Let $B = B_{\ell,0} = \begin{bmatrix} 1 & -\ell + 1 \\ -1 & \ell + 1 \end{bmatrix}$: note that $\mu(B) = 2$. We start considering the set $X_0 = [0, \ell - 2]$ of size $\ell - 1 = \frac{2ms}{M}$, it is easy to see that $\bigcup_{x \in X_0} \text{supp}(B \pm x) = [1, 2\ell] \setminus \{\ell, 2\ell\}$. Similarly, for any $i \in \mathbb{N}$, if $X_i = [2i\ell, (2i + 1)\ell - 2]$, then $\bigcup_{x \in X_i} \text{supp}(B \pm x) = [2i\ell + 1, (2i + 2)\ell] \setminus \{(2i + 1)\ell, (2i + 2)\ell\}$ and $X_{i_1} \cap X_{i_2} = \emptyset$ if $i_1 \neq i_2$. Take $X = \bigcup_{i=0}^{t/4-1} X_i$: this is a set of size $\frac{t}{4} \cdot (\ell - 1) = \frac{ms}{2X}$, as required. In this case,

$$\bigcup_{x \in X} \text{supp}(B \pm x) = \bigcup_{i=0}^{t/4-1} ([2i\ell + 1, (2i + 2)\ell] \setminus \{(2i + 1)\ell, (2i + 2)\ell\}) = [1, \frac{t}{2\ell}] \setminus \{\ell, 2\ell, \ldots, \frac{t}{2}\ell\}. $$

Now we take $\frac{1}{2}$ copies of every block $B \pm x$: the p.f. array $A$ obtained following our procedure is an integer $^\lambda H_s(m, n; s, k)$. $\square$
We now deal with the case \( \lambda \) odd. This implies that \( \lambda \) divides \( \frac{m_s}{4} \).

**Lemma 3.4.** Let \( \lambda \) be a positive odd integer. There exists an integer \( \lambda \mathcal{H}_t(m, n; s, k) \) for any divisor \( t \) of \( \frac{m_s}{4} \) such that \( t \equiv 0 \pmod{8} \).

**Proof.** Let \( B = B_{t,2\ell} = \begin{bmatrix} 1 & -2 \\ -\lambda & 2 \end{bmatrix} \), where \( \ell = \frac{2 ms}{\lambda} + 1 \). Note that \( \mu(B) = 1 \).

An integer \( \lambda \mathcal{H}_t(m, n; s, k) \), say \( A \), can be obtained following the construction described before, once we exhibit a suitable set \( X \) of size \( \frac{m_s}{4\lambda} \), in such a way that \( \text{supp}(A) = \left[ 1, \frac{m_s}{\lambda} + \frac{\ell}{2} \right] \setminus \{ \ell, 2\ell, \ldots, \ell \} \). We start considering the set \( X_0 = [0, \ell - 2] \) of size \( \ell - 1 = \frac{2ms}{\lambda} \): it is easy to see that

\[
\bigcup_{x \in X_0} \text{supp}(B \pm x) = [1, 4\ell] \setminus \{ \ell, 2\ell, 3\ell, 4\ell \}.
\]

Similarly, for any \( i \in \mathbb{N} \), if \( X_i = [4i\ell, (4i+1)\ell - 2] \), then

\[
\bigcup_{x \in X_i} \text{supp}(B \pm x) = [4i\ell + 1, (4i + 4)\ell] \setminus \{ (4i + 1)\ell, (4i + 2)\ell, (4i + 3)\ell, (4i + 4)\ell \}.
\]

Clearly, \( X_i \cap X_i = \emptyset \) if \( i_1 \neq i_2 \). So, take \( X = \bigcup_{i=0}^{t/8-1} X_i \): this is a set of size \( \frac{t}{8} \cdot (\ell - 1) = \frac{t}{8} \cdot \frac{2ms}{\lambda} = \frac{m_s}{4\lambda} \), as required. It is easy to see that

\[
\bigcup_{x \in X} \text{supp}(B \pm x) = \bigcup_{i=0}^{t/8-1} \left( [4i\ell + 1, (4i + 4)\ell] \setminus \{ (4i + 1)\ell, (4i + 2)\ell, (4i + 3)\ell, (4i + 4)\ell \} \right)
= \left[ 1, \frac{4\ell}{t} \right] \setminus \{ \ell, 2\ell, \ldots, \frac{\ell}{t} \} = \left[ 1, \frac{m_s}{\lambda} + \frac{\ell}{4} \right] \setminus \{ \ell, 2\ell, \ldots, \frac{\ell}{t} \}.
\]

Now we take \( \lambda \) copies of every block \( B \pm x \): the p.f. array \( A \) obtained following our procedure is an integer \( \lambda \mathcal{H}_t(m, n; s, k) \).

**Lemma 3.5.** Let \( \lambda \) be a positive odd integer. There exists an integer \( \lambda \mathcal{H}_t(m, n; s, k) \) for any divisor \( t \) of \( \frac{m_s}{\lambda} \) such that \( t \equiv 0 \pmod{4} \).

**Proof.** Let \( B = B_{1,\ell} = \begin{bmatrix} 1 & -2 \\ -\ell & \ell + 2 \end{bmatrix} \): note that \( \mu(B) = 1 \) and, since \( t \) divides \( \frac{m_s}{\lambda} \), \( \ell = \frac{2ms}{\lambda} + 1 \) is an odd integer. We start considering the set \( X_0 = \{ 0, 2, 4, \ldots, \ell - 3 \} \) of size \( \frac{\ell - 1}{2} = \frac{m_s}{4\lambda} \): it is easy to see that

\[
\bigcup_{x \in X_0} \text{supp}(B \pm x) = [1, \ell - 1] \cup [\ell + 1, 2\ell - 1] = [1, 2\ell] \setminus \{ \ell, 2\ell \}.
\]

Similarly, for any \( i \in \mathbb{N} \), if \( X_i = \{ 2i\ell, 2i\ell + 2, 2i\ell + 4, \ldots, (2i + 1)\ell - 3 \} \), then

\[
\bigcup_{x \in X_i} \text{supp}(B \pm x) = [2i\ell + 1, 2(i + 1)\ell] \setminus \{ (2i + 1)\ell, (2i + 2)\ell \}
\]

and \( X_i \cap X_i = \emptyset \) if \( i_1 \neq i_2 \). So, take \( X = \bigcup_{i=0}^{t/4-1} X_i \): this is a set of size \( \frac{t}{4} \cdot \frac{\ell - 1}{2} = \frac{t}{4} \cdot \frac{m_s}{4\lambda} = \frac{m_s}{4\lambda} \), as required. Hence,

\[
\bigcup_{x \in X} \text{supp}(B \pm x) = \bigcup_{i=0}^{t/4-1} \left( [2i\ell + 1, 2(i + 1)\ell] \setminus \{ (2i + 1)\ell, (2i + 2)\ell \} \right)
= \left[ 1, \frac{2\ell}{t} \right] \setminus \{ \ell, 2\ell, \ldots, \frac{\ell}{t} \} = \left[ 1, \frac{m_s}{\lambda} + \frac{\ell}{4} \right] \setminus \{ \ell, 2\ell, \ldots, \frac{\ell}{t} \}.
\]
Now we take $\lambda$ copies of every block $B \pm x$: the p.f. array $A$ obtained following our procedure is an integer $^{\lambda}H_t(m, n; s, k)$.

For instance, to construct an integer $^{5}H_{4}(5, 10; 8, 4)$ we can follow the proof of the previous lemma. In fact, $\lambda = 5$ and $t = 4$ divides $\frac{5 \cdot 8}{4}$; note that $\ell = 5$ and $Y = 5 \ast (0, 2)$.

$$^{5}H_{4}(5, 10; 8, 4) = \begin{bmatrix}
1 & -2 & -8 & 9 & 3 & -4 & -6 & 7 \\
9 & 3 & -4 & -6 & 7 & 1 & -2 & -8 \\
-6 & 7 & 1 & -2 & -8 & 9 & 3 & -4 \\
-8 & 9 & 3 & -4 & -6 & 7 & 1 & -2 \\
-4 & -6 & 7 & 1 & -2 & -8 & 9 & 3
\end{bmatrix}.$$ 

**Lemma 3.6.** Let $\lambda$ be a positive odd integer. There exists an integer $^{\lambda}H_t(m, n; s, k)$ for any divisor $t$ of $\frac{mn}{\lambda}$.

**Proof.** Let $B = B_{1, 2} = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$. Note that $\mu(B) = 1$ and $\ell = \frac{2ms}{\lambda} + 1 \equiv 1 \pmod{4}$ since $t$ divides $\frac{mn}{\lambda}$. We start considering the set $X_0 = \{0, 4, 8, \ldots, \ell - 5\}$ of size $\frac{t - 1}{4} = \frac{ms}{2\lambda}$; clearly, $\bigcup_{x \in X_0} \text{supp}(B \pm x) = [1, \ell] \setminus \{\ell\}$. Similarly, for any $i \in \mathbb{N}$, if $X_i = \{it, i\ell + 4i, i\ell + 8i, \ldots, (i + 1)\ell - 5\}$, then

$$\bigcup_{x \in X_i} \text{supp}(B \pm x) = [it + 1, (i + 1)\ell] \setminus \{(i + 1)\ell\}$$

and $X_{i_1} \cap X_{i_2} = \emptyset$ if $i_1 \neq i_2$.

If $t$ is even, take $X = \bigcup_{i=0}^{t/2 - 1} X_i$: this is a set of size $\frac{t}{2} \cdot \frac{t - 1}{4} = \frac{t}{2} \cdot \frac{ms}{2\lambda} = \frac{ms}{4\lambda}$, as required. Hence,

$$\bigcup_{x \in X} \text{supp}(B \pm x) = \bigcup_{i=0}^{t/2 - 1} ([il + 1, (i + 1)\ell] \setminus \{(i + 1)\ell\})$$

$$= [(1, \frac{t}{2}\ell) \setminus \{\ell, 2\ell, \ldots, \frac{t}{2}\ell\} = [1, \frac{ms}{\lambda} + \frac{t}{2}] \setminus \{\ell, 2\ell, \ldots, \frac{t}{2}\ell\}. $$

Suppose now that $t$ is odd. Notice that, in this case, $\ell \equiv 1 \pmod{8}$. Take

$$Z = \left\{ \left(\frac{t - 1}{2}\ell, \frac{t - 1}{2}\ell + 4, \frac{t - 1}{2}\ell + 8, \ldots, \frac{t - 1}{2}\ell + 4\ell - 9\ell \right) \right\}.$$ 

Then $|Z| = \frac{t - 1}{8} = \frac{ms}{4\lambda}$ and $\bigcup_{x \in Z} \text{supp}(B \pm x) = [\left(\frac{t - 1}{2}\ell + 1, \frac{t - 1}{2}\ell + \frac{t - 1}{2}\ell\right)]$. Take $X = \left(\bigcup_{i=0}^{(t - 3)/2} X_i\right) \cup Z$: this is a set of size $\frac{t - 1}{2} - \frac{t - 1}{4} + \frac{t - 1}{8} = \frac{t - 1}{2} \cdot \frac{ms}{2\lambda} + \frac{ms}{4\lambda} = \frac{ms}{4\lambda}$, as required. In this case,

$$\bigcup_{x \in X} \text{supp}(B \pm x) = \left[\bigcup_{i=0}^{(t - 3)/2} (il + 1, (i + 1)\ell) \setminus \{(i + 1)\ell\}\right] \cup \left[\left(\frac{t - 1}{2}\ell + 1, \frac{t - 1}{2}\ell + \frac{t - 1}{2}\ell\right)\right]$$

$$= \left[1, \frac{t - 1}{\lambda} + \frac{t - 1}{2}\ell\right] \setminus \{\ell, 2\ell, \ldots, \frac{t}{2}\ell\}. $$

In both cases, we construct the p.f. array $A$ using $\lambda$ copies of every block $B \pm x$; so, the p.f. array $A$ obtained following our procedure is an integer $^{\lambda}H_t(m, n; s, k)$. \qed
For instance, we can follow the proof of the previous lemma for constructing an integer $3H_3(9, 9; 8, 8)$. In fact, $\lambda = 3$ and $t = 3$ divides $\frac{9s}{2^3}$; note that $\ell = 17$ and $Y = 3 \cdot (0, 4, 8, 12, 17, 21).

\begin{array}{cccccccc}
1 & -2 & -20 & 21 & 13 & -14 & -7 & 8 \\
12 & 5 & -6 & -24 & 25 & 18 & -19 & -11 \\
-3 & 4 & 9 & -10 & -15 & 16 & 22 & -23 \\
-7 & 8 & 13 & -14 & -20 & 21 & 1 & -2 \\
9 & -10 & -15 & 16 & 22 & -23 & -3 & 4 \\
8 & 13 & -14 & -20 & 21 & 1 & -2 & -7 \\
-11 & 12 & 18 & -19 & -24 & 25 & 5 & -6 \\
-10 & -15 & 16 & 22 & -23 & -3 & 4 & 9 \\
\text{3H}_3(9, 9; 8, 8) = & & & & & & & \\
\end{array}

We now consider the case when $\lambda$ does not divide $ms$. We need to adjust our general strategy in order to satisfy (2.1).

Lemma 3.7. Suppose that $\lambda$ does not divide $ms$. There exists an integer $^\lambda H_k(m, n; s, k)$ for any divisor $t$ of $\frac{2ms}{\lambda}$.

Proof. Since $\lambda$ divides $2ms$ but does not divide $ms$, from $s \equiv 0 \pmod{4}$ we obtain $\lambda \equiv 0 \pmod{8}$. We can easily adapt the proof of Lemma 3.2, using the block $B = B_{0,0} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and considering two possibilities. In both cases, an integer $^\lambda H_k(m, n; s, k)$, say $A$, can be obtained following the construction given at the beginning of this section and using the blocks $B \pm y_0, B \pm y_1, \ldots, B \pm \frac{y_{m,n}}{\lambda}$ for a suitable sequence $Y = (y_0, y_1, \ldots, y_{\frac{m,n}{\lambda}})$ in such a way that condition (2.1) is satisfied.

Suppose that $\ell$ is odd or $t$ is even. It suffices to consider the sequence $X$ obtained by taking the natural ordering $\leq$ of $\{i - 1 \mid i \in \Phi \} \subset \mathbb{N}$, and define $Y = \frac{X}{\lambda} \ast X$.

Suppose that $\ell$ is even and $t$ is odd. Let $X_1$ be the sequence obtained by taking the natural ordering $\leq$ of $\{i - 1 \mid i \in \Psi \} \subset \mathbb{N}$, where $\Psi = \Phi \setminus \{i \ell \}$. Also, let $Y_1 = \frac{X}{\lambda} \ast X_1$ and let $Y_2$ be the sequence obtained by repeating $\frac{X_1}{\lambda}$ times the integer $\frac{t}{\lambda} - 1$. Define $Y = Y_1 + Y_2$ and note that $|Y| = \frac{s}{\lambda} \cdot \frac{2ms - \lambda}{2\lambda} + \frac{s}{\lambda} = ms$.

For instance, we can follow the proof of the previous lemma for constructing an integer $16H_3(10, 10; 4, 4)$. In fact, $\lambda = 16$ does not divide $ms = 40$; note that $\ell = 2$, $X_1 = (0, 2)$ and $Y = (0, 2, 0, 2, 0, 2, 0, 2, 0, 2, 0, 2, 4, 4).

\begin{array}{cccccccc}
1 & -1 & & & & & & -5 & 5 \\
5 & 3 & 3 & & & & & -5 & 5 \\
-1 & 1 & -1 & & & & & -5 & 5 \\
-3 & 3 & 3 & 3 & & & & -5 & 5 \\
-1 & 1 & 1 & 1 & & & & -5 & 5 \\
-3 & 3 & 3 & 3 & & & & -5 & 5 \\
-1 & 1 & 1 & -1 & & & & -5 & 5 \\
-3 & 3 & 3 & -3 & & & & -5 & 5 \\
-1 & -1 & -1 & -1 & & & & -5 & 5 \\
-3 & 3 & 3 & -3 & & & & -5 & 5 \\
-1 & 1 & 5 & 5 & & & & -5 & 5 \\
-3 & 3 & 5 & 5 & & & & -5 & 5 \\
\text{16H}_3(10, 10; 4, 4) = & & & & & & & \\
\end{array}
Proposition 3.8. Suppose $4 \leq s \leq n$, $4 \leq k \leq m$, $ms = nk$ and $s, k \equiv 0 \pmod{4}$. Let $\lambda$ be a divisor of $2ms$. There exists a shiftable integer $^H_B H_t(m, n; s, k)$ for every divisor $t$ of $2ms$.

Proof. If $\lambda$ does not divide $ms$, the statement follows from Lemma 3.7. So, suppose that $\lambda$ divides $ms$. If $\lambda \equiv 0 \pmod{4}$ or $\lambda \equiv 2 \pmod{4}$, then we can apply Lemma 3.2 or Lemma 3.3, respectively. Now we assume $\lambda$ odd. If $t \equiv 0 \pmod{8}$, we apply Lemma 3.4. If $t \equiv 4 \pmod{8}$, then $t$ divides $\frac{2ms}{\lambda}$ and hence we can apply Lemma 3.5. Finally, if $t \not\equiv 0 \pmod{4}$, then $t$ divides $\frac{ms}{2\lambda}$ and so the existence of an integer $^H_B H_t(m, n; s, k)$ follows from Lemma 3.6. In all these cases, the integer $\lambda$-fold Heffter array that we construct is shiftable.

4. The case $s \equiv 2 \pmod{4}$, $k$ and $m$ even

In this section, we will assume that $s, m, k$ are positive even integers with $s \equiv 2 \pmod{4}$ and $s \geq 6$. We need to distinguish two cases, according to the divisibility of $ms$ by $\lambda$. In fact, if $\lambda$ does not divide $ms$, from $ms \equiv 0 \pmod{4}$ we obtain $\lambda \equiv 0 \pmod{4}$ in this case, we have to construct p.f. arrays that satisfy (2.1).

If $\lambda$ divides $ms$ we write
\[(4.1) \quad \lambda = \lambda_1 \lambda_2, \quad \text{where } \lambda_1 \text{ divides } \frac{m}{2} \text{ and } \lambda_2 \text{ divides } 2s.\]

Let $t$ be a divisor of $\frac{2ms}{\lambda}$ and set
\[\ell = \frac{2ms}{\lambda t} + 1.\]

4.1. Construction of nice pairs of sequences. To obtain an integer $^H_B H_t(m, n; s, k)$, we first construct pairs of sequences, satisfying the following properties.

Definition 4.1. A pair $(B_1, B_2)$ of sequences is said to be nice if, for a fixed positive integer $b$, we have:

- the sequence $B_1$ consists of blocks satisfying this condition:

  \[(4.2) \quad \text{there exist } b \text{ integers } \sigma_1, \ldots, \sigma_b \text{ such that the elements of } B_1 \text{ are shiftable blocks } B \text{ of size } 2 \times 2b \text{ with } \tau_1(B) = \tau_2(B) = 0, \text{ and } \gamma_{2i-1}(B) = -\gamma_{2i}(B) = \sigma_i \text{ for all } i \in [1, b];\]

- the sequence $B_2$ consists of blocks satisfying this condition:

  \[(4.3) \quad \text{there exist } 2b \text{ integers } \sigma'_1, \ldots, \sigma'_{2b} \text{ with } \sum_{i=1}^{2b} \sigma'_{2i-1} = \sum_{i=1}^{2b} \sigma'_{2i} = 0, \text{ such that the elements of } B_2 \text{ are shiftable blocks } B' \text{ of size } 2 \times 2b \text{ with } \tau_1(B') = \tau_2(B') = 0 \text{ and } \gamma_i(B') = \sigma'_i \text{ for all } i \in [1, 2b];\]

- the sequences $B_1$ and $B_2$ have the same length and, writing $B_1 = (B_1, B_2, \ldots, B_e)$ and $B_2 = (B'_1, B'_2, \ldots, B'_e)$, then $E(B_1) = E(B'_1)$ for all $i \in [1, e]$.

Observe that the sequences $B_1, B_2$ in the previous definition do not need to be distinct.

We construct these nice pairs of sequences, starting with the case when $\lambda$ divides $ms$. In particular, our sequences $B_i$, consisting of shiftable blocks of size $2 \times s$, are of length $\frac{ms}{\lambda t}$ and such that $\mu(B_i) = \lambda_2$. We begin with the case when $\lambda_2$ is odd. Note that this implies that $\lambda_2$ divides $\frac{m}{2}$. 
Lemma 4.2. [13] Corollary 4.10 and Lemma 5.1 Let \(a\) and \(c\) be even integers with \(a \geq 2\), \(c \geq 6\) and \(c \equiv 2 \pmod{4}\). Let \(u\) be a divisor of \(2ac\) and set \(\rho = \frac{2ac}{u} + 1\). There exists a nice pair \((\tilde{B}_1, \tilde{B}_2)\) of sequences of length \(\frac{u}{2}\), where \(\tilde{B}_1\) and \(\tilde{B}_2\) consist of blocks of size \(2 \times c\), \(\mu(\tilde{B}_1) = \mu(\tilde{B}_2) = 1\) and
\[
\text{supp}(\tilde{B}_1) = \text{supp}(\tilde{B}_2) = [1, ac + \lfloor u/2\rfloor] \setminus \{j \rho : j \in [1, \lfloor u/2\rfloor]\}.
\]

Corollary 4.3. Let \(\lambda = \lambda_1 \lambda_2\) be as in (4.1). If \(\lambda_2 \neq \frac{\rho}{2}\) is odd, there exists a nice pair \((B_1, B_2)\) of sequences of length \(\frac{m}{2\lambda_1}\), where \(B_1\) and \(B_2\) consist of blocks of size \(2 \times s\), \(\mu(B_1) = \mu(B_2) = \lambda_2\) and
\[
\text{supp}(B_1) = \text{supp}(B_2) = \left[1, \frac{ms}{\lambda} + \left\lfloor \frac{t}{2} \right\rfloor\right] \setminus \left\{\ell, 2\ell, \ldots, \left\lfloor \frac{t}{2} \right\rfloor \ell\right\} = \Phi.
\]

Proof. Take \(a = \frac{m}{\lambda_1}, c = \frac{\rho}{2}\) and \(u = t\). Since \(\lambda_1\) divides \(\frac{m}{2}\), \(a\) is a positive even integer; since \(\lambda_2 \neq \frac{\rho}{2}\) is odd and divides \(2s\), then \(c\) is an even integer such that \(c \geq 6\) and \(c \equiv 2 \pmod{4}\).

Note that \(t\) divides \(2ac = \frac{2ms}{\lambda_1}\) and \(\rho = \frac{2ac}{u} + 1 = \frac{2ms}{u} + 1 = \ell\). Hence, we can apply Lemma 4.2 obtaining a nice pair \((\tilde{B}_1, \tilde{B}_2)\) of sequences of length \(\frac{m}{2\lambda_1}\) consisting of blocks of size \(2 \times \frac{\rho}{2}\) such that \(\mu(\tilde{B}_1) = \mu(\tilde{B}_2) = 1\) and \(\text{supp}(\tilde{B}_1) = \text{supp}(\tilde{B}_2) = \Phi\). Now, replace every block \(\tilde{B}\) of \(\tilde{B}_i\), \(i = 1, 2\), with the block \(B\) obtained juxtaposing \(\lambda_2\) copies of \(\tilde{B}\). So, \(B\) is a block of size \(2 \times s\) and \(\mu(B) = \lambda_2\). Call \(B_1, B_2\) the two sequences so obtained. It follows that the pair \((B_1, B_2)\) satisfies the required properties.

Now we consider the case when \(\lambda_2 = \frac{\rho}{2}\).

Lemma 4.4. Let \(\lambda = \lambda_1 \lambda_2\) be as in (4.1) with \(\lambda_2 = \frac{\rho}{2}\). There exists a nice pair \((B_1, B_2)\) of sequences of length \(\frac{m}{2\lambda_1}\), where \(B_1\) and \(B_2\) consist of blocks of size \(2 \times s\), \(\mu(B_1) = \mu(B_2) = \frac{\rho}{2}\) and \(\text{supp}(B_1) = \text{supp}(B_2) = \Phi\).

Proof. We first consider the case when \(\ell\) is odd. Consider the following shiftable blocks:

\[
A = \begin{pmatrix}
1 & -2 & -3 & 1 \\
1 & -1 & 2 & 3 \\
-2 & -1 & 3 & -4 \\
-1 & 2 & -1 & -4
\end{pmatrix}, \quad F = \begin{pmatrix}
1 & -2 & -4 & 5 \\
1 & -1 & 2 & 4 \\
-2 & -1 & 4 & -4 \\
-1 & 2 & -4 & -4
\end{pmatrix}
\]

\[
E = \begin{pmatrix}
1 & -1 & 3 & -4 & -3 & 4 \\
2 & -2 & 2 & -1 & 2 & -4 \\
2 & -4 & 4 & -1 & 2 & -4 \\
-2 & -1 & 4 & -3 & -4 & -4
\end{pmatrix}, \quad G = \begin{pmatrix}
4 & 2 & -2 & 2 & -1 & -5 \\
4 & 2 & 2 & -1 & 1 & 5 \\
-5 & 1 & 4 & -4 & -1 & 1 \\
-5 & 4 & -1 & -4 & 1 & 5
\end{pmatrix}
\]

\[
E' = \begin{pmatrix}
1 & 3 & -1 & -4 & -3 & 4 \\
-2 & -1 & 2 & 3 & -4 & -4 \\
-2 & -1 & 2 & 3 & -4 & -4 \\
-2 & -1 & 2 & 3 & -4 & -4
\end{pmatrix}, \quad G' = \begin{pmatrix}
4 & -2 & 2 & -1 & -5 \\
-5 & 1 & 4 & -4 & -1 & 1 \\
-5 & 4 & -1 & -4 & 1 & 5
\end{pmatrix}
\]

Note that \(A\) and \(F\) satisfy both (4.2) and (4.3); \(E\) and \(G\) satisfy (4.2); \(E'\) and \(G'\) satisfy (4.3).

We first construct the sequence \(B_1\). To this purpose, take the block \(B\) obtained juxtaposing the block \(E\) and \(\frac{\rho}{6}\) copies of the block \(A\). We obtain a block of size \(2 \times s\) such that \(\text{supp}(B) = [1, 4]\) and \(\mu(B) = \frac{\rho}{2}\). Also, let \(C\) be the block obtained juxtaposing the block \(G\) and \(\frac{\rho}{6}\) copies of the block \(F\). Then \(C\) is a block of size \(2 \times s\) such that \(\text{supp}(C) = \{1, 2, 4, 5\}\) and \(\mu(C) = \frac{\rho}{2}\).

Assume \(\ell \equiv 1 \pmod{4}\). Let \(S = (B, B \pm 4, B \pm 8, \ldots, B \pm 4\frac{\ell - 5}{4})\). Then \(|S| = \frac{\ell - 1}{4}\) and \(\text{supp}(S) = [1, \ell] \setminus \{\ell\}\). If \(t\) is even, take
\[
B_1 = S \# (S \pm \ell) \# (S \pm 2\ell) \# \ldots \# \left(S \pm \frac{t - 2}{2}\ell\right).
\]
If \( t \) is odd, then \( \ell - 1 = 8 \frac{m}{2M} \equiv 0 \) (mod 8). Let

\[
Y = \left( B, B \pm 4, B \pm 8, \ldots, B \pm \left( 4 \frac{\ell - 9}{8} \right) \right)
\]

and

\[
B_1 = S \# (S \pm \ell) \# (S \pm 2\ell) \# \ldots \# \left( S \pm \frac{t - 3}{2} \ell \right) \# \left( Y \pm \frac{t - 1}{2} \ell \right).
\]

In both cases, \( B_1 \) is a sequence of length \( \frac{(\ell - 1)t}{8} = \frac{m}{2M} \) such that \( \mu(B_1) = \frac{t}{2} \) and \( \text{supp}(B_1) = \Phi \).

The sequence \( B_2 \) is obtained by replacing in \( B_1 \) the block \( E \) with the block \( E' \).

Assume \( \ell \equiv 3 \) (mod 4). Note that, in this case, \( 8 \frac{m}{2M} \equiv 2 \) (mod 4) and so \( t \equiv 0 \) (mod 4).

Let \( S = (B, B \pm 4, B \pm 8, \ldots, B \pm \frac{6t}{4}, C \pm (\ell - 3), B \pm (\ell + 2), B \pm (\ell + 6), B \pm (\ell + 10), \ldots, B \pm (2\ell - 5)) \). Then \( |S| = \frac{\ell}{4} \) and \( \text{supp}(S) = [1, 2\ell] \setminus \{\ell, 2\ell\} \).

Define

\[
B_1 = S \# (S \pm 2\ell) \# (S \pm 4\ell) \# \ldots \# \left( S \pm \frac{t - 4}{4} \ell \right).
\]

So, \( B_1 \) is a sequence of length \( \frac{(\ell - 1)t}{8} = \frac{m}{2M} \) such that \( \mu(B_1) = \frac{t}{2} \) and \( \text{supp}(B_1) = \Phi \).

The sequence \( B_2 \) is obtained by replacing in \( B_1 \) the block \( G \) with the block \( G' \).

Finally, assume that \( \ell \) is even. Note that, in this case, \( t \equiv 0 \) (mod 8). Consider the shiftable blocks:

\[
\begin{array}{c|cccc}
H & 1 & -(\ell + 1) & -(2\ell + 1) & 3\ell + 1 \\
\hline
1 & \ell + 1 & 2\ell + 1 & -(3\ell + 1) \\
\hline
-1 & -(\ell + 1) & -(2\ell + 1) & 2\ell + 1 & -1 & -(3\ell + 1) \\
\hline
\end{array}
\]

\[
\begin{array}{c|cccc}
L & 1 & 3\ell + 1 & -(\ell + 1) & \ell + 1 \\
\hline
1 & \ell + 1 & 2\ell + 1 & -(3\ell + 1) \\
\hline
-1 & -(\ell + 1) & -(2\ell + 1) & 2\ell + 1 & -1 & -(3\ell + 1) \\
\hline
\end{array}
\]

Note that the blocks \( H \) and \( L \) satisfy both (1.2) and (4.3). Let \( K \) be the block obtained juxtaposing the block \( L \) and \( \frac{t}{4} \) copies of the block \( H \). Then \( K \) is a block of size \( 2 \times s \) such that \( \text{supp}(K) = \{1, \ell + 1, 2\ell + 1, 3\ell + 1\} \) and \( \mu(K) = \frac{t}{2} \). Let \( S = (K, K \pm 1, K \pm 2, \ldots, K \pm (\ell - 2)) \).

Then \( |S| = \ell + 1 \) and \( \text{supp}(S) = [1, 4\ell] \setminus \{\ell, 2\ell, 3\ell, 4\ell\} \).

Define

\[
B_1 = B_2 = S \# (S \pm 4\ell) \# (S \pm 8\ell) \# \ldots \# \left( S \pm 4 \frac{t - 8}{8} \ell \right).
\]

So, \( B_1 \) is a sequence of length \( \frac{(\ell - 1)t}{8} = \frac{m}{2M} \) such that \( \mu(B_1) = \frac{t}{2} \) and \( \text{supp}(B_1) = \Phi \).

For instance, taking in the previous lemma, \( m = 30, s = 10, \lambda_1 = 3 \) and \( t = 5 \), we have \( \ell = 9 \). The sequence \( B_1 \) consists of the following five shiftable blocks:

\[
\begin{array}{c|cccccccc}
B_1 & 1 & -1 & 3 & -4 & -3 & 4 & 1 & -2 & -3 & 4 \\
\hline
-2 & -2 & -1 & 2 & 3 & -4 & -1 & 2 & 3 & -4 \\
\hline
-5 & 7 & -8 & -7 & 8 & 5 & -6 & -7 & 8 \\
\hline
-6 & 6 & -5 & 6 & 7 & -8 & -5 & 6 & 7 & -8 \\
\hline
\end{array}
\]

\[
\begin{array}{c|cccccccc}
B_2 & 10 & -10 & 12 & -13 & -12 & 13 & 10 & -11 & -12 & 13 \\
\hline
11 & -11 & -10 & 11 & 12 & -13 & -10 & 11 & 12 & -13 \\
\hline
14 & -14 & 16 & -17 & -16 & 17 & 14 & -15 & -16 & 17 \\
\hline
-15 & 15 & -14 & 15 & 16 & -17 & -14 & 15 & 16 & -17 \\
\hline
19 & -19 & 21 & -22 & -21 & 22 & 19 & -20 & -21 & 22 \\
\hline
-20 & 20 & -19 & 20 & 21 & -22 & -19 & 20 & 21 & -22 \\
\hline
\end{array}
\]
We now deal with the case $\lambda_2 \equiv 2 \pmod{4}$.

**Lemma 4.5.** Let $\lambda = \lambda_1\lambda_2$ be as in (4.1) with $\lambda_2 \equiv 2 \pmod{4}$ and $\lambda_2 \geq 6$. There exists a nice pair $(B, B)$, where $B$ is a sequence of length $m_2 s$ consisting of blocks of size $2 \times s$ such that $\mu(B) = \lambda_2$ and $\text{supp}(B) = \Phi$.

**Proof.** We first consider the case when $\ell$ is odd. Consider the following shiftable blocks:

\[
A = \begin{pmatrix}
1 & -1 & 2 & -2 \\
-1 & 1 & -2 & 2
\end{pmatrix}, \quad E = \begin{pmatrix}
1 & 2 & -1 & -1 \\
-2 & -1 & -2 & 1
\end{pmatrix}.
\]

Note that $A$ and $E$ satisfy both (1.2) and (1.3). To construct the sequence $B$, first take the block $C$ obtained juxtaposing the block $E$ and $\frac{\lambda_2-6}{2}$ copies of the block $A$. We obtain a block of size $2 \times \lambda_2$ such that $\text{supp}(C) = \{1, 2\}$ and $\mu(C) = \lambda_2$. Consider the sequence $S = (C, C \pm 2, C \pm 4, \ldots, C \pm 2\ell - 3)$. Then $|S| = \frac{\ell - 1}{2}$, $\mu(S) = \lambda_2$ and $\text{supp}(S) = [1, \ell] \setminus \{\ell\}$. If $\ell$ is even, take

\[
\tilde{B} = S \#(S \pm \ell) \#(S \pm 2\ell) \# \ldots \# \left(S \pm \frac{t - 2}{2}\ell\right).
\]

If $\ell$ is odd, then $\ell - 1 = 4 \frac{m_2 - s}{t} \equiv 0 \pmod{4}$. Let

\[
Y = \left(C, C \pm 2, C \pm 4, \ldots, C \pm \left(2\ell - \frac{5}{4}\right)\right)
\]

and

\[
\tilde{B} = S \#(S \pm \ell) \#(S \pm 2\ell) \# \ldots \# \left(S \pm \frac{t - 3}{2}\ell\right) \# \left(Y \pm \frac{t - 1}{2}\ell\right).
\]

In both cases, $\tilde{B}$ is a sequence of length $\frac{(\ell - 1)t}{4} = \frac{m_2}{2A}$ such that $\mu(\tilde{B}) = \lambda_2$ and $\text{supp}(\tilde{B}) = \Phi$.

Suppose now that $\ell$ is even. Note that, in this case, $t \equiv 0 \pmod{4}$. Consider the shiftable blocks:

\[
F = \begin{pmatrix}
1 & -1 & \ell + 1 & -(\ell + 1) \\
-1 & 1 & -(\ell + 1) & \ell + 1
\end{pmatrix},
\]

\[
G = \begin{pmatrix}
1 & \ell + 1 & -1 & 1 \\
-\ell + 1 & 1 & -\ell + 1 & 1
\end{pmatrix}.
\]

Note that the blocks $F$ and $G$ satisfy both (1.2) and (1.3). Take the block $H$ obtained juxtaposing the block $G$ and $\frac{\lambda_2-6}{2}$ copies of the block $F$. We obtain a block of size $2 \times \lambda_2$ such that $\text{supp}(H) = \{1, \ell + 1\}$ and $\mu(H) = \lambda_2$. Consider the sequence $S = (H, H \pm 1, H \pm 2, \ldots, H \pm (\ell - 2))$. Then $|S| = \ell - 1$, $\mu(S) = \lambda_2$ and $\text{supp}(S) = [1, 2\ell] \setminus \{\ell, 2\ell\}$. Take

\[
\tilde{B} = S \#(S \pm 2\ell) \#(S \pm 4\ell) \# \ldots \# \left(S \pm \frac{t - 4}{4}\ell\right).
\]

Hence, $\tilde{B}$ is a sequence of length $\frac{(\ell - 1)t}{4} = \frac{m_2}{2A}$ such that $\mu(\tilde{B}) = \lambda_2$ and $\text{supp}(\tilde{B}) = \Phi$.

Finally, for every $\ell$, writing $\tilde{B} = (K_1, K_2, \ldots, K_{\frac{m_2}{2A}})$ and $q = \frac{m_2}{\lambda_2}$, for every $i \in \left[1, \frac{m_2}{2A}\right]$ we construct the block $B_i$ juxtaposing the $q$ blocks $K_{1+i(-1)q}, K_{2+i(-1)q}, \ldots, K_{iq}$. The blocks $B_i$ are of size $2 \times q\lambda_2$, that is, of size $2 \times s$. So, we can set $B = (B_1, B_2, B_3, \ldots, B_{\frac{m_2}{2A}})$. \qed
For instance, taking in the previous lemma, \( m = 84, s = 10, \lambda_1 = 7, \lambda_2 = 10 \) and \( t = 8 \), we have \( \ell = 4 \). The sequence \( \mathcal{B} \) consists of the following six shiftable blocks:

\[
\begin{align*}
B_1 &= \begin{pmatrix}
1 & 5 & -1 & 1 & -1 & -5 & 1 & -1 & 5 & -5 \\
-5 & -1 & 5 & -5 & 1 & 5 & -1 & 1 & -5 & 5
\end{pmatrix}, \\
B_2 &= \begin{pmatrix}
2 & 6 & -2 & 2 & -2 & -6 & 2 & -2 & 6 & -6 \\
-6 & -2 & 6 & -6 & 2 & 6 & -2 & 2 & -6 & 6
\end{pmatrix}, \\
B_3 &= \begin{pmatrix}
3 & 7 & -3 & 3 & -3 & -7 & 3 & -3 & 7 & -7 \\
-7 & -3 & 7 & -7 & 3 & 7 & -3 & 3 & -7 & 7
\end{pmatrix}, \\
B_4 &= \begin{pmatrix}
9 & 13 & -9 & 9 & -9 & -13 & 9 & -9 & 13 & -13 \\
-13 & -9 & 13 & -13 & 9 & -13 & 9 & -13 & 13 & 13
\end{pmatrix}, \\
B_5 &= \begin{pmatrix}
10 & 14 & -10 & 10 & -10 & -14 & 10 & -10 & 14 & -14 \\
-14 & -10 & 14 & -14 & 10 & -14 & 10 & -14 & 14 & 14
\end{pmatrix}, \\
B_6 &= \begin{pmatrix}
11 & 15 & -11 & 11 & -11 & -15 & 11 & -11 & 15 & -15 \\
-15 & -11 & 15 & -15 & 11 & -15 & 11 & -15 & 15 & 15
\end{pmatrix}.
\]

We now deal with the case \( \lambda_2 = 2 \).

**Lemma 4.6.** Let \( \lambda = \lambda_1\lambda_2 \) be as in (1.1) with \( \lambda_2 = 2 \). Suppose that \( t \) divides \( \frac{ms}{\lambda_1} \). There exists a nice pair \((B_1, B_2)\) of sequences of length \( \frac{m}{2\lambda_1} \), where \( B_1 \) and \( B_2 \) consist of blocks of size \( 2 \times s \), \( \mu(B_1) = \mu(B_2) = 2 \) and \( \text{supp}(B_1) = \text{supp}(B_2) = \Phi \).

**Proof.** Write \( s = 4q + 6 \) where \( q \geq 0 \) and take the following shiftable blocks:

\[
\begin{align*}
U_3 &= \begin{pmatrix} 1 & -2 & -4 & 5 \\ -1 & 2 & 4 & -5 \end{pmatrix}, & U_5 &= \begin{pmatrix} 1 & -2 & -3 & 4 \\ -1 & 2 & 3 & -4 \end{pmatrix}, \\
V_1 &= \begin{pmatrix} 2 & -2 & -5 & -6 & 4 & 7 \\ -3 & 3 & 6 & 5 & -4 & -7 \end{pmatrix}, & V_3 &= \begin{pmatrix} 1 & -1 & -5 & -6 & 4 & 7 \\ -2 & 2 & 6 & 5 & -4 & -7 \end{pmatrix}, \\
V_5 &= \begin{pmatrix} 6 & -6 & -2 & -3 & 1 & 4 \\ -7 & 7 & 3 & 2 & -1 & -4 \end{pmatrix}, & V_7 &= \begin{pmatrix} 1 & -1 & -4 & -5 & 3 & 6 \\ -2 & 2 & 5 & 4 & -3 & -6 \end{pmatrix}, \\
Z &= \begin{pmatrix} 1 & -1 & 4 & -5 & -7 & -8 \\ -2 & 2 & -4 & 5 & 7 & -8 \end{pmatrix}, & Z' &= \begin{pmatrix} 1 & 4 & -1 & -5 & -7 & 8 \\ -2 & -4 & 2 & 5 & 7 & -8 \end{pmatrix}.
\]

Note that, since \( t \) divides \( \frac{ms}{\lambda_1} \), \( \ell \) is an odd integer.

If \( \ell = 4x + 1 \geq 5 \), take \( \hat{S} = (U_5, U_5 \pm 4, U_5 \pm 8, \ldots, U_5 \pm 4(x - 1)) \). Then \( |\hat{S}| = x, \mu(\hat{S}) = 2 \) and \( \text{supp}(\hat{S}) = [1, \ell] \setminus \{\ell\} \). Let \( \hat{B} \) be the sequence obtained by taking the first \( \frac{ms}{2\lambda_1} \) blocks in \( \hat{S} \pm \ell c \). If \( \ell = 4x + 3 \geq 3 \), take \( \hat{S} = (U_5, U_5 \pm 4, U_5 \pm 8, \ldots, U_5 \pm 4(x - 1), U_3 \pm 4x, U_5 \pm (4x + 5), U_5 \pm (4x + 9), \ldots, U_5 \pm (8x + 1)) \). Then \( |\hat{S}| = 2x + 1, \mu(\hat{S}) = 2 \) and \( \text{supp}(\hat{S}) = [1, 2\ell] \setminus \{\ell, 2\ell\} \).

Let \( \hat{B} \) be the sequence obtained taking the first \( \frac{ms}{2\lambda_1} \) blocks in \( \hat{S} \pm 2\ell c \). In both cases we obtain a sequence \( \hat{B} \) of blocks of size \( 2 \times 4 \) that satisfy both (4.2) and (4.3) and such that \( \text{supp}(\hat{B}) = [1, N] \) where \( N = \frac{ms}{2\lambda_1} + \eta \) with \( \eta = \left\lfloor \frac{2t}{2} \right\rfloor \).

Now, we have to construct a sequence \( S' \) of shiftable blocks of size \( 2 \times 6 \) satisfying condition (4.2) in such a way that \( |S'| = \frac{m}{2\lambda_1} \) and

\[
\text{supp}(S') = \left[N + 1, \frac{ms}{2\lambda_1} + \left\lfloor \frac{t}{2} \right\rfloor \right] \setminus \left\{j\ell : j \in \left[\eta + 1, \left\lfloor \frac{t}{2} \right\rfloor \right] \right\}.
\]
If \( \ell = 3 \), then \( t = \frac{m}{2\lambda_1} \) and \( N = 3\frac{m}{2\lambda_1} \equiv 0 \) (mod 3). We can take \( S' = \sum_{c=0}^{\frac{m}{2\lambda_1} - 1} (Z + (N + 9c)) \). If \( \ell = 5 \), then \( t = \frac{m}{4\lambda_1} \) and \( N = 5\frac{m}{2\lambda_1} \equiv 0 \) (mod 5). Define \( T = (V_5, V_3 + 7) \). If \( \frac{m}{2\lambda_1} \) is even, we can take \( S' = \sum_{c=0}^{\frac{m}{2\lambda_1} - 1} (T + (N + 15c)) \). If \( \frac{m}{2\lambda_1} \) is odd, we can take \( S' = \left( \sum_{c=0}^{\frac{m}{2\lambda_1} - 1} (T + (N + 15c)) \right) + (V_5 \pm \left( \frac{m}{2\lambda_1} + \frac{c-\ell}{2} \right)) \).

Suppose now that \( \ell \geq 7 \): in this case, any set of 6 consecutive integers contains at most one multiple of \( \ell \). We start considering the interval \([N + 1, N + 6]\) and the first multiple of \( \ell \) belonging to the interval \([N + 1, \frac{m}{2\lambda_1} + \lfloor t/2 \rfloor]\). So, if \((\eta + 1)\ell\) is an element of \([N + 1, N + 6]\) we take the block \( V_r \) where \( r \) must be chosen in such a way that \( \text{supp}(V_r \pm N) \) does not contain \((\eta + 1)\ell\). Otherwise, we take the block \( V_7 \) and repeat this process considering the interval \([N + 7, N + 12]\).

It will be useful to define, for all \( b \geq 1 \), the sequence
\[
H(b) = (V_7, V_7 \pm 6, V_7 \pm 12, \ldots, V_7 \pm 6(b - 1)).
\]
Also, we set \( H(0) \) to be the empty sequence: so, for all \( b \geq 0 \) the sequence \( H(b) \) contains \( b \) elements and \( \text{supp}(H(b)) = [1, 6b] \).

Write \((\eta + 1)\ell - N = 6h_0 + r_0\), where \( 0 \leq r_0 < 6 \), and define the sequence
\[
S'_0 = (H(h_0), V_{r_0} \pm 6h_0).
\]
Note that \( r_0 \) is odd, since \( \ell \) is odd and \((\eta + 1)\ell - N \equiv (\eta + 1)\ell + \eta \equiv 1 \) (mod 2). Furthermore, \( \text{supp}(S'_0 \pm N) = [N + 1, N + 6h_0 + 7] \setminus \{(\eta + 1)\ell\} \).

Now, for all \( j \in [1, \lfloor t/2 \rfloor - \eta] \), write \( \ell - 7 + r_{j-1} = 6h_j + r_j \), where \( 0 \leq r_j < 6 \), and define the sequence
\[
S'_j = \left( H(h_j) \pm \left( 7j + 6\sum_{i=0}^{j-1} h_i \right), V_{r_j} \pm \left( 7j + 6\sum_{i=0}^{j} h_i \right) \right) \). \]
Note that \((\eta + j + 1)\ell - N = 6\sum_{i=0}^{j} h_i + 7j + r_j\) and
\[
\text{supp}(S'_j \pm N) = \left[ N + 1 + 7j + 6\sum_{i=0}^{j-1} h_i, N + 7(j + 1) + 6\sum_{i=0}^{j} h_i \right] \setminus \{(\eta + j + 1)\ell\}.
\]
The elements of \( S' \) are the first \( \frac{m}{2\lambda_1} \) blocks in \( \left\lfloor \frac{t/2 - \eta}{\ell} \right\rfloor \) \( \{(S'_c \pm N) \}\).

Finally, writing \( \tilde{B} = (A_1, \ldots, A_{\frac{m}{2\lambda_1}}) \) and \( S' = (G_1, \ldots, G_{\frac{m}{2\lambda_1}}) \), for all \( i = 1, \ldots, \frac{m}{2\lambda_1} \), let \( B_i \) be the block of size \( 2 \times s \) obtained by juxtaposing the \( q \) blocks
\[
A_{(i-1)q+1}, A_{(i-1)q+2}, A_{(i-1)q+3}, \ldots, A_{iq}
\]
and the block \( G_i \). By construction, the sequence \( B_1 = (B_1, \ldots, B_{\frac{m}{2\lambda_1}}) \) satisfies condition \( 4.2 \), has cardinality \( \frac{m}{2\lambda_1}, \mu(B_1) = 2 \) and \( \text{supp}(B_1) = \text{supp}(S) \cup \text{supp}(S') = \Phi \).

The sequence \( \tilde{B}_2 \) is obtained by \( B_1 \) replacing the block \( Z \) with \( Z' \) (case \( \ell = 3 \)).

\[ \square \]

**Lemma 4.7.** Let \( \lambda = \lambda_1 \lambda_2 \) be as in (4.1) with \( \lambda_2 = 2 \). Let \( p \) be an odd prime dividing \( s \) and suppose that \( t \) is a divisor of \( \frac{m}{2\lambda_1} \) such that \( t \equiv 0 \) (mod 4p). There exists a nice pair \( (B, \tilde{B}) \),
where $B$ is a sequence of length $\frac{m}{2}\lambda_1^2$ consisting of blocks of size $2 \times s$ such that $\mu(B) = 2$ and $\text{supp}(B) = \Phi$.

Proof. Take the following blocks:

$$
W_4 = \begin{pmatrix}
1 & -2 & -1 & 3 \\
2 & 1 & -3 & 0 \\
3 & 4 & 2 & -1 \\
4 & 5 & 3 & 2
\end{pmatrix},
$$

$$
W_6 = \begin{pmatrix}
1 & -2 & -1 & 3 & 5 & 7 \\
2 & 1 & -3 & 0 & 5 & 3 \\
3 & 4 & 2 & -1 & 3 & 5 \\
4 & 5 & 3 & 2 & 1 & -3 \\
5 & 7 & 3 & 5 & 1 & -2 \\
6 & 4 & 5 & 3 & 2 & 1
\end{pmatrix}.
$$

Then $W_4$ and $W_6$ satisfy both properties (1.2) and (4.3) with column sums $(0, 0, 0, 0)$ and $(-\ell, \ell, -\ell, 0, 0)$, respectively. Furthermore, $\mu(W_4) = \mu(W_6) = 2$

$$
\text{supp}(W_4) = \{j\ell + 1 : j \in [0, 3]\} \quad \text{and} \quad \text{supp}(W_6) = \{j\ell + 1 : j \in [0, 5]\}.
$$

Let $V$ be the following $2 \times 2p$ block:

$$
V = \begin{pmatrix}
W_6 & W_4 + 6\ell & W_4 + 10\ell & \ldots & W_4 + (2p - 4)\ell \\
\end{pmatrix}.
$$

Clearly, also $V$ satisfies both (4.2) and (4.3) and its support is $\text{supp}(V) = \{j\ell + 1 : j \in [0, 2p - 1]\}$. We can use this block $V$ for constructing our sequence $B$: the $2 \times s$ blocks of $B$ are obtained simply by juxtaposing $h = \frac{m}{2\lambda_1^2}$ blocks of type $V \times x$, for $x \in X \subset \mathbb{N}$, following the natural order of $(X, \leq)$. So, we are left to exhibit a suitable set $X$ of size $\frac{m\lambda_1}{2\lambda_1^2}$ such that the support of the corresponding sequence $B$ is $\Phi$.

Let first $X_0 = [0, \ell - 2]$. Then $\text{supp}(V \pm x_{i_1}) \cap \text{supp}(V \pm x_{i_2}) = \emptyset$ for each $x_{i_1}, x_{i_2} \in X_0$ such that $x_{i_1} \neq x_{i_2}$. Furthermore,

$$
\bigcup_{x \in X_0} \text{supp}(V \pm x) = [1, 2p\ell] \setminus \{j\ell : j \in [1, 2p]\}.
$$

Similarly, for any $i \in \mathbb{N}$, if $X_i = [2pi\ell, (2pi + 1)\ell - 2]$ then

$$
\bigcup_{x \in X_i} \text{supp}(V \pm x) = [1 + 2pi\ell, 2p\ell + 2pi\ell] \setminus \{j\ell : j \in [1 + 2pi, 2p + 2pi]\}.
$$

Clearly, $X_{i_1} \cap X_{i_2} = \emptyset$ if $i_1 \neq i_2$. Therefore, take $X = \bigcup_{i=0}^{\lambda_1 - 1} X_i$; this is a set of size $\frac{1}{4p} \cdot (\ell - 1) = \frac{m\lambda_1}{2\lambda_1^2}$. It follows that the sequence $B$ obtained, as previously described, from the blocks $V \times x$, with $x \in X$, has support equal to

$$
\text{supp}(B) = \bigcup_{i=0}^{\lambda_1 - 1} ([1 + 2pi\ell, 2p\ell + 2pi\ell] \setminus \{j\ell : j \in [1 + 2pi, 2p + 2pi]\})
= [1, \frac{\ell}{2}] \setminus \{j\ell : j \in [\frac{1}{2}, \frac{\ell}{2}]\} = \frac{m\lambda_1}{2\lambda_1^2} + \frac{\ell}{2} \setminus \{\ell, 2\ell, \ldots, \frac{\ell}{2}\},
$$

as required. \qed

Example 4.8. Taking in the previous lemma, $m = 18$, $s = 10$, $\lambda_1 = 3$ and $t = 20$, we can choose $p = 5$ so that $t \equiv 0 \pmod{20}$. Hence $\ell = 4$ and $B$ consists of the following three shiftable blocks:

$$
B_1 = \begin{pmatrix}
1 & -1 & -13 & -17 & 9 & 21 & -29 & -33 & 37 \\
-5 & 5 & 17 & 13 & -9 & -21 & -25 & 29 & 33 & -37
\end{pmatrix}.
$$
Let $\lambda = \lambda_1 \lambda_2$ be as in (4.1) with $\lambda_2 = 2$. Let $p$ be an odd prime $p$ dividing $s$ and suppose that $t$ is a divisor of $\frac{2m}{\lambda_1}p$ such that $t \equiv 0 \pmod{4}$. There exists a nice pair $(B_1, B_2)$ of sequences of length $\frac{2m}{\lambda_1}$, where $B_1$ and $B_2$ consist of blocks of size $2 \times s$, $\mu(B_1) = \mu(B_2) = 2$ and $\text{supp}(B_1) = \text{supp}(B_2) = \Phi$.

**Proof.** By hypothesis we can write $\ell = py + 1$. Consider the following blocks:

\[
B_2 = \begin{pmatrix}
2 & -2 & -14 & -18 & 10 & 22 & -30 & -34 & 38 \\
-6 & 6 & 18 & 14 & -10 & -22 & -26 & 30 & -34 & -38
\end{pmatrix},
\]

\[
B_3 = \begin{pmatrix}
3 & -3 & -15 & -19 & 11 & 23 & -27 & -31 & -35 & 39 \\
-7 & 7 & 19 & 15 & -11 & -23 & -27 & 31 & 35 & -39
\end{pmatrix}.
\]

**Lemma 4.9.**

Note that the block $W_4$ satisfies both conditions (4.2) and (4.3), while $W_6$ satisfies condition (4.2) and $W'_6$ satisfies condition (4.3). Furthermore,

\[
\text{supp}(W_4) = \{(jp + 1)y + j + 1, (jp + 2)y + j + 1 : j \in [0, 1]\},
\]

\[
\text{supp}(W_6) = \text{supp}(W'_6) = \{(jp + j)y + j + 1, (jp + 2)y + j + 1 : j \in [0, 1]\}.
\]

Let $V$ be the following $2 \times 2p$ block:

\[
V = \begin{pmatrix} W_0 & W_4 & W_4 & \cdots & W_4 & W_4 & (p - 3)y \end{pmatrix}.
\]

Clearly, $V$ satisfies (4.2) and its support is

\[
\text{supp}(V) = \{iy + 1, (p + i)y + 2 : i \in [0, p - 1]\}
\]

\[
= \{iy + 1, \ell + i(y + 1) : i \in [0, p - 1]\}.
\]

We can use this block $V$ for constructing the sequence $B_1$ as done in Lemma 4.7; it suffices to exhibit a suitable set $X$ of size $\frac{mp}{2\lambda_1}$, where $h = \frac{2m}{\lambda_1}$, such that the support of the corresponding sequence $B_1$ is $\Phi$.

Let first $X_0 = [0, y - 1]$. Then $\text{supp}(V \pm x_{i_1}) \cap \text{supp}(V \pm x_{i_2}) = \emptyset$ for each $x_{i_1}, x_{i_2} \in X_0$ such that $x_{i_1} \neq x_{i_2}$. Furthermore,

\[
\bigcup_{x \in X_0} \text{supp}(V \pm x) = [1, py] \cup [\ell + 1, \ell + py] = [1, 2\ell] \setminus \{\ell, 2\ell\}.
\]

Similarly, for any $i \in \mathbb{N}$, if $X_i = [2i\ell, 2i\ell + y - 1]$ then

\[
\bigcup_{x \in X_i} \text{supp}(V \pm x) = [1 + 2i\ell, (2i + 1)\ell] \setminus \{(2i + 1)\ell, (2i + 2)\ell\}.
\]

Clearly, $X_{i_1} \cap X_{i_2} = \emptyset$ if $i_1 \neq i_2$. Therefore, take $X = \bigcup_{i=0}^{p-1} X_i$: this is a set of size $\frac{p}{4} \cdot y = \frac{\ell - 1}{p} = 4 \cdot \frac{2m}{\lambda_1} = \frac{mh}{2\lambda_1}$. It follows that the sequence $B_1$ obtained from the blocks $V \pm x$, with
$x \in X$, has support equal to
\[
\text{supp}(B_i) = \bigcup_{i=0}^{t-1} ([1 + 2i\ell, 2\ell(i+1)] \setminus \{(2i+1)\ell, (2i+2)\ell\})
= [1, \frac{r}{\ell}] \setminus \{\ell, 2\ell, \ldots, \frac{r}{2}\ell\} = \Phi,
\]
as required. The sequence $B_2$ is obtained using $W'_6$ instead of $W_6$. □

The last case we need is when $\lambda_2 \equiv 0 \pmod{4}$.

**Lemma 4.10.** Let $\lambda = \lambda_1 \lambda_2$ be as in [4.1] with $\lambda_2 \equiv 0 \pmod{4}$. There exists a nice pair $(B, B)$, where $B$ is a sequence of length $\frac{n}{2\lambda_1}$ consisting of blocks of size $2 \times s$ such that $\mu(B) = \lambda_2$ and $\text{supp}(B) = \Phi$.

**Proof.** Let $Q$ be the $2 \times \frac{\lambda_2}{2}$ block obtained juxtaposing $\frac{\lambda_2}{4}$ copies of the shiftable block
\[
\begin{array}{ccc}
1 & -1 & -1 \\
-1 & 1 & 1
\end{array}
\]
Clearly, $Q$ satisfies both conditions [4.2] and [4.3]. Furthermore, $\text{supp}(Q) = \{1\}$ and $\mu(Q) = \lambda_2$. Take a partition of $\Phi$ into $\frac{n}{2\lambda_1}$ subsets $X_i$, each of cardinality $\frac{\lambda_2}{\lambda_2}$. Writing, for all $i \in [1, \frac{n}{2\lambda_1}]$, $X_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i, \frac{\lambda_2}{\lambda_2}}\}$, let $B_i$ the block
\[
B_i = \begin{bmatrix}
Q \pm (x_{i,1} - 1) & Q \pm (x_{i,2} - 1) & Q \pm (x_{i,3} - 1) & \cdots & Q \pm (x_{i, \frac{\lambda_2}{\lambda_2}} - 1)
\end{bmatrix}.
\]
Then each $B_i$ is a block of size $2 \times s$ such that $\text{supp}(B_i) = X_i$ and $\mu(B_i) = \lambda_2$. Finally, it suffices to take the sequence $B = (B_1, B_2, \ldots, B_{\frac{n}{2\lambda_1}})$. □

**Example 4.11.** Taking in the previous lemma, $m = 16$, $s = 10$, $\lambda_1 = 2$, $\lambda_2 = 4$ and $t = 5$, we have $\ell = 9$ and $\Phi = [1, 22] \setminus \{9, 18\}$. So, can take $X_1 = [1, 5]$, $X_2 = [6, 11] \setminus \{9\}$, $X_3 = [12, 16]$ and $X_4 = [17, 22] \setminus \{18\}$. Hence, the sequence $B$ consists of the following four shiftable blocks:
\[
B_1 = \begin{bmatrix}
1 & -1 & 2 & -2 & 3 & -3 & 4 & -4 & 5 & -5 \\
\hline
-1 & 1 & -2 & 2 & -3 & 3 & -4 & 4 & -5 & 5
\end{bmatrix},
B_2 = \begin{bmatrix}
-6 & -6 & 7 & -7 & 8 & -8 & 10 & -10 & 11 & -11 \\
\hline
-6 & 6 & -7 & 7 & -8 & 8 & -10 & 10 & -11 & 11
\end{bmatrix},
B_3 = \begin{bmatrix}
12 & -12 & 13 & -13 & 14 & -14 & 15 & -15 & 16 & -16 \\
\hline
-12 & 12 & -13 & 13 & -14 & 14 & -15 & 15 & -16 & 16
\end{bmatrix},
B_4 = \begin{bmatrix}
17 & -17 & 19 & -19 & 20 & -20 & 21 & -21 & 22 & -22 \\
\hline
-17 & 17 & -19 & 19 & -20 & 20 & -21 & 21 & -22 & 22
\end{bmatrix}.
\]

**Proposition 4.12.** Suppose that $\lambda$ divides $ms$ and write $\lambda = \lambda_1 \lambda_2$ be as in [4.1]. There exists a nice pair $(B_1, B_2)$ of sequences of length $\frac{ms}{2\lambda_1}$, where $B_1$ and $B_2$ consist of blocks of size $2 \times s$, $\mu(B_1) = \mu(B_2) = \lambda_2$ and
\[
\text{supp}(B_1) = \text{supp}(B_2) = \left[1, \frac{ms}{\lambda_1} + \left\lfloor \frac{t}{2} \right\rfloor \right] \setminus \left\{\ell, 2\ell, \ldots, \left\lfloor \frac{t}{2} \right\rfloor \ell\right\} = \Phi.
\]

**Proof.** If $\lambda_2 = \frac{s}{2}$, the statement follows from Lemma 4.4. If $\lambda_2 \neq \frac{s}{2}$ is odd, we apply Corollary 4.3. If $\lambda_2 \equiv 0 \pmod{4}$, we use Lemma 4.10. So, we may assume $\lambda_2 = 2 \pmod{4}$. If $\lambda_2 \geq 6$, the statement follows from Lemma 4.5. Finally, suppose $\lambda_2 = 2$. Since $s \geq 6$ and $s \equiv 2 \pmod{4}$, there exists an odd prime $p$ that divides $s$. Now, our analysis depends on $t$: recall
that $t$ is a divisor of $\frac{ms}{n_i}$, we apply Lemma 4.6. Otherwise, we must have $t \equiv 0 \pmod{4}$. If $t$ divides $\frac{ms}{n_i p}$, the result follows from Lemma 4.9. If $t$ does not divide $\frac{ms}{n_i p}$, then $t$ is divisible by $p$. In particular, $t \equiv 0 \pmod{4p}$ and so we can apply Lemma 4.7. □

**Proposition 4.13.** Suppose that $\lambda$ does not divide $ms$. There exists a nice pair $(B, B)$, where $B$ is a sequence of length $\frac{m}{t}$ consisting of blocks of size $2 \times s$, such that $\text{supp}(B) = \Phi$ and condition (2.1) is satisfied.

**Proof.** As previously observed, we have $\lambda \equiv 0 \pmod{8}$. Let $Q$ be the following shiftable block:

\[
Q = \begin{bmatrix}
1 & -1 & -1 \\
-1 & 1 & 1
\end{bmatrix}
\]

Clearly, $Q$ satisfies both conditions (4.2) and (4.3). Furthermore, $\text{supp}(Q) = \{1\}$ and $\mu(Q) = 4$.

Suppose that $\ell$ is odd or $t$ is even. Consider the sequence $X$ obtained by taking the natural ordering $\leq$ of $\{i - 1 \mid i \in \Phi\} \subset \mathbb{N}$, and define $Y = \frac{1}{\lambda} \times X$.

Suppose that $\ell$ is even and $t$ is odd. Let $X_1$ be the sequence obtained by taking the natural ordering $\leq$ of $\{i - 1 \mid i \in \Psi\} \subset \mathbb{N}$, where $\Psi = \Phi \setminus \{\frac{m}{\lambda}\}$. Also, let $Y_1 = \frac{1}{\lambda} \times X_1$ and let $Y_2$ be the sequence obtained by repeating $\frac{20}{\lambda}$ times the integer $\frac{20}{\lambda} - 1$. Define $Y = Y_1 + Y_2$ and note that $|Y| = \frac{ms}{\lambda}$.

In both cases, write $Y = (y_1, y_2, \ldots, y_{\frac{m}{\lambda}})$. For all $i \in \left[1, \frac{m}{\lambda}\right]$, let $B_i$ the block

\[
B_i = \begin{bmatrix}
Q \pm y_1(i-1)i \quad Q \pm y_2(i-1)i \quad \cdots \quad Q \pm y_{\frac{m}{\lambda}}i
\end{bmatrix}
\]

Then each $B_i$ is a block of size $2 \times s$: it suffices to take the sequence $B = (B_1, B_2, \ldots, B_{\frac{m}{\lambda}})$. □

4.2. **The subcase** $k \equiv 0 \pmod{4}$. Assuming $k \equiv 0 \pmod{4}$, from $ms = nk$ it follows that $m$ must be even. We now explain how to arrange the blocks of the sequences previously constructed, in order to build an integer $\lambda H_e(m, n; s, k)$. To this purpose, we define a ‘base unit’ that we will fill with the elements of the blocks.

Let $\mathcal{G} = (G_1, \ldots, G_d)$ be a sequence of blocks such that the following property is satisfied:

\[
\text{there exist } b \text{ integers } \sigma_1, \ldots, \sigma_b \text{ such that the elements of } \mathcal{G} \text{ are blocks } G_r, \text{ of size } 2 \times 2b \text{ with } \gamma_{2i-1}(G_r) = -\gamma_{2i}(G_r) = \sigma_i \text{ for all } i \in [1, b].
\]

So, let $\mathcal{G}$ be a sequence satisfying (4.4), where the blocks \(G_r = (g_{i,j}^{(r)})\) are all of size $2 \times 2b$, with $2b \leq d$. Let $P = P(\mathcal{G})$ be the p.f. array of size $2d \times d$ so defined. For all $i \in [1, b]$ and all $j \in [1, 2b]$, the cell $(i, i + j - 1)$ of $P$ is filled with the element $g_{i,j}^{(i)}$ and the cell $(d + i, i + j - 1)$ is filled with the element $g_{2b+i,j}^{(i)}$; here, the column indices are taken modulo $d$. The remaining cells of $P$ are empty. An example of such construction is given in Figure 1.

We prove that $P$ is a p.f. array whose columns all sum to zero. Observe that every row of $P$ contains exactly $2b$ filled cells and every column contains exactly $4b$ elements. The elements of the $i$-th column of $P$ are

\[
g_{1,1}^{(i)}, \ g_{1,2}^{(i-1)}, \ \cdots, \ g_{1,2b}^{(i+2b)}, \ g_{2,1}^{(i)}, \ g_{2,2}^{(i-1)}, \ \cdots, \ g_{2,2b}^{(i+2b)},
\]

where the exponents must be read modulo $d$, with residues in $[1, d]$. Since the sequence $\mathcal{G}$ satisfies (4.4), we obtain

\[
\gamma_i(P) = \sum_{j=1}^{2b} \gamma_j(G_{i+1-j}) = \sum_{j=1}^{2b} \gamma_j(G_i) = \sum_{u=1}^{b} (\sigma_u - \sigma_u) = 0.
\]
Furthermore, notice that \( \tau_j(P) = \tau_1(G_j) \) and \( \tau_{d+j}(P) = \tau_2(G_j) \) for all \( j \in [1, d] \).

**Proposition 4.14.** Suppose \( 4 \leq s \leq n, 4 \leq k \leq m \) and \( ms = nk \). Let \( \lambda \) be a divisor of \( 2ms \) and let \( t \) be a divisor of \( \frac{2ms}{\lambda} \). There exists a shiftable integer \( \lambda H_t(m, n; s, k) \) in each of the following cases:

1. \( s \equiv 2 \pmod{4} \) and \( k \equiv 0 \pmod{4} \);
2. \( s \equiv 0 \pmod{4} \) and \( k \equiv 2 \pmod{4} \).

**Proof.** (1) If \( \lambda \) divides \( ms \), let \( (B_1, B_2) \) be the nice pair of sequences constructed in Proposition 4.12, and set \( B = \lambda_1 \ast B_1 \). If \( \lambda \) does not divide \( ms \), let \( B \) be the sequence constructed in Proposition 4.13. Write \( d = \gcd(mn, n) \) and \( a = \frac{mn}{d} \). Note that \( a \) is even integer. In fact, write \( m = 2\bar{m}d \) and \( n = \bar{n} \bar{a} \). Since \( k \equiv 0 \pmod{4} \), from \( \frac{s}{\lambda} \equiv \frac{2}{\lambda} \pmod{\frac{2ms}{\lambda}} \) we obtain that \( \bar{n} \) divides \( \frac{\bar{a}}{2} \).

Given a block \( B_h \in B \), define for every \( j \in [1, \bar{n}] \) the block \( T_j(B_h) \) of size \( 2 \times a \) consisting of the columns \( C_i \) of \( B_h \) with \( i \in [a(j-1) + 1, a j] \). So, the block \( B_h \) of size \( 2 \times s \) is obtained juxtaposing the blocks \( T_1(B_h), T_2(B_h), \ldots, T_{\bar{n}}(B_h) \). Furthermore, for all \( i \in [1, \bar{n}] \) and each of the sequences

\[
T_j(B_{\bar{n}i+1}), T_j(B_{\bar{n}i+d+1}), \ldots, T_j(B_{\bar{n}d})
\]

of cardinality \( d \), satisfies condition (4.1).

Let \( A \) be an empty array of size \( \bar{m} \times \bar{n} \). For every \( i \in [1, \bar{m}] \) and \( j \in [1, \bar{n}] \), replace the cell \((i, j)\) of \( A \) with the block \( P(T_j(B_{\bar{n}i+1}), T_j(B_{\bar{n}i+d+1}), \ldots, T_j(B_{\bar{n}d})) \), according to the previous definition. Note that, for all \( r \in [1, \frac{\bar{m}}{d}] \), we have \( \tau_r(A) = \tau_1(B_r) = 0 \) and \( \tau_{\bar{m}+r}(A) = \tau_2(B_r) = 0 \).

By construction, \( A \) is a p.f. array of size \( m \times n \), \( \text{supp}(A) = \Phi \) and the rows and columns of \( A \) sum to zero. If \( \lambda \) divides \( ms \), then every element of \( \Phi \) appears, up to sign, exactly \( \lambda \) times. If \( \lambda \) does not divide \( ms \), condition (2.1) is satisfied. Furthermore, each row contains \( \bar{a} \bar{n} = s \) elements and each column contains \( 2a \bar{m} = k \) elements. We conclude that \( A \) is a shiftable integer \( \lambda H_t(m, n; s, k) \).

(2) It follows from (1). In fact, if \( s \equiv 0 \pmod{4} \) and \( k \equiv 2 \pmod{4} \), an integer \( \lambda H_t(m, n; s, k) \) can be obtained simply by taking the transpose of an integer \( \lambda H_t(n, m; k; s) \).
The integer $^6H_{20}(18, 15; 10, 12)$ shown in Figure 2 (on the left) has been obtained repeating $\lambda_1 = 3$ times each of the blocks of Example 4.8. In the same figure (on the right) we also give an integer $^8H_5(16, 20; 10, 8)$, obtained repeating $\lambda_1 = 2$ times each of the blocks of Example 4.11.

4.3. The subcase $k \equiv 2 \pmod{4}$. Here we only solve the case $m$ even, which implies that also $n$ is even.

**Proposition 4.15.** Suppose $6 \leq s \leq n$, $6 \leq k \leq m$, $ms = nk$ and $s, k \equiv 2 \pmod{4}$. Let $\lambda$ be a divisor of $2ms$ and let $t$ be a divisor of $\frac{2ms}{\lambda}$. If $m$ is even, there exists a shiftable integer $^4H_t(m, n; s, k)$.

**Proof.** Without loss of generality, we may assume $m \geq n$ (and so $s \leq k$). If $\lambda$ divides $ms$, let $(B_1, B_2)$ be the nice pair of sequences constructed in Proposition 4.12. Take $B_1 = \lambda_1 * B_1$ and $B_2 = \lambda_2 * B_2$. So, $B_1^\ast$ and $B_2^\ast$ have length $\frac{ms}{\lambda}$ and $\mu(B_1^\ast) = \mu(B_2^\ast) = \lambda$. If $\lambda$ does not divide $ms$, let $(B_1^\ast, B_2^\ast)$ be the nice pair of sequences constructed in Proposition 4.13. In both cases, write $B_1 = (b_1, \ldots, B_{\frac{m}{2}})$ and $B_2 = (b_1', \ldots, B_{\frac{m}{2}}')$, where $B_1^\ast$ satisfies (4.2), $B_2^\ast$ satisfies (4.3) and 

$$\text{supp}(B_1^\ast) = \text{supp}(B_2^\ast) = \left[ \left\lfloor \frac{t\ell}{2} \right\rfloor \right] \setminus \{ j\ell : j \in [1, \lfloor t/2 \rfloor] \}$$

with $\ell = \frac{2ms}{\lambda t} + 1$.

Set $B_1 = (B_{\frac{m}{2}+1}, \ldots, B_{\frac{m}{2}})$ and $B_2 = (B_1', \ldots, B_{\frac{m}{2}}')$.

Since, by construction, $E(B_i) = E(B_i')$ for all $i \in [1, \frac{m}{2}]$, it follows that $E(B_2 + B_1) = E(B_1^\ast) = E(B_2^\ast)$ and $\text{supp}(B_2 + B_1) = \left[ \left\lfloor \frac{t\ell}{2} \right\rfloor \right] \setminus \{ j\ell : j \in [1, \lfloor t/2 \rfloor] \}$. Furthermore, if $\lambda$ divides $ms$ then $\mu(B_2 + B_1) = \lambda$; the same holds if $\lambda$ does not divide $ms$, and $\ell$ is odd or $t$ is even; if $\lambda$ does not divide $ms$, $\ell$ is even and $t$ is odd, then every element of $\Phi \setminus \{ \frac{t\ell}{2} \}$ appears in $E(B_2 + B_1)$, up to sign, exactly $\lambda$ times, while the integer $\frac{t\ell}{2}$ appears, up to sign, $\frac{t}{\lambda}$ times.

Using the blocks of the sequence $B_2$, we first construct a square shiftable p.f. array $A_1$ of size $n$ such that each row and each column contains $s$ filled cells and such that the elements in every row and column sum to zero. Hence, take an empty array $A_1$ of size $n \times n$ and arrange the $\frac{n}{2}$ blocks $B_i^\ast = (b_{i,j}^\ast)$ of $B_2$ in such a way that the element $b_{1,1}^\ast$ fills the cell $(2r-1, 2r-1)$ of $A_1$. This process makes $A_1$ a p.f. array with $s$ filled cells in each row and in each column. Since the rows of the blocks $B_i^\ast$ sum to zero, also the rows of $A_1$ sum to zero. Looking at the columns, the $s$ elements of a column of $A_1$ are

$$b_{1,s}^\ast, b_{2,s}^\ast, b_{1,s-2}^\ast, b_{2,s-2}^\ast, b_{1,s-4}^\ast, b_{2,s-4}^\ast, \ldots, b_{1,2}^\ast, b_{2,2}^\ast$$

or

$$b_{1,s-1}^\ast, b_{2,s-1}^\ast, b_{1,s-3}^\ast, b_{2,s-3}^\ast, b_{1,s-5}^\ast, b_{2,s-5}^\ast, \ldots, b_{1,1}^\ast, b_{2,1}^\ast,$$

where the exponents $r, \ldots, r + s/2$ must be read modulo $\frac{n}{2}$. Since $B_2$ satisfies condition (4.3), the sum of these elements is

$$\sum_{j=1}^{s/2} \sigma_{2j} = 0 \quad \text{or} \quad \sum_{j=1}^{s/2} \sigma_{2j-1} = 0,$$

respectively.

By construction, $E(A_1) = E(B_2)$. 

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Figure 2. An integer $H_6(18,15;10,12)$ (on the left) and an integer $H_8(16,20;10,8)$ (on the right).

| 1 | -1 | -13 | -17 | 9 | 21 | 25 | -29 | -33 | 37 |
|---|----|-----|-----|---|----|----|-----|-----|----|
| 2 | -2 | -14 | -18 | 10 | 22 | 26 | -30 | -34 | 38 |
| -3 | 3 | -19 | -15 | 23 | 11 | -31 | 27 | 39 | -35 |
| -5 | 5 | 17 | 13 | -9 | -21 | -25 | 29 | 33 | -37 |
| -6 | 6 | 18 | 14 | -10 | -22 | -26 | 30 | 34 | -38 |

| 1 | -1 | -13 | -17 | 9 | 21 | 25 | -29 | -33 | 37 |
|---|----|-----|-----|---|----|----|-----|-----|----|
| 2 | -2 | -14 | -18 | 10 | 22 | 26 | -30 | -34 | 38 |
| -3 | 3 | -19 | -15 | 23 | 11 | -31 | 27 | 39 | -35 |
| -5 | 5 | 17 | 13 | -9 | -21 | -25 | 29 | 33 | -37 |
| -6 | 6 | 18 | 14 | -10 | -22 | -26 | 30 | 34 | -38 |

| 1 | -1 | -13 | -17 | 9 | 21 | 25 | -29 | -33 | 37 |
|---|----|-----|-----|---|----|----|-----|-----|----|
| 2 | -2 | -14 | -18 | 10 | 22 | 26 | -30 | -34 | 38 |
| -3 | 3 | -19 | -15 | 23 | 11 | -31 | 27 | 39 | -35 |
| -5 | 5 | 17 | 13 | -9 | -21 | -25 | 29 | 33 | -37 |
| -6 | 6 | 18 | 14 | -10 | -22 | -26 | 30 | 34 | -38 |
Now, if \( m = n \), then \( A_1 \) is actually a shiftable integer \( ^\lambda H_1(m, n; k, s) \). Suppose that \( m > n \). If we arrange the blocks of the sequence \( \bar{B}_1 \) mimicking what we did for the construction of an integer \( ^1 H_1(m - n, n; s, k - s) \) in the proof of Proposition 4.14 we obtain a shiftable p.f. array \( A_2 \) of size \( (m - n) \times n \) such that \( \mathcal{E}(A_2) = \mathcal{E}(\bar{B}_1) \), rows and columns sum to zero, each row contains \( s \) filled cells and each column contains \( k - s \) filled cells. Let \( A \) be the p.f. array of size \( m \times n \) obtained taking

\[
A = \begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}
\]

Each row of \( A \) contains \( s \) filled cells and each of its columns contains \( s + (k - s) = k \) filled cells. By the previous properties of \( \bar{B}_2 + \bar{B}_1 \), it follows that \( A \) is a shiftable integer \( ^\lambda H_4(m, n; s, k) \).

An integer \( ^{28} H_4(16, 16; 14, 14) \) is shown in Figure 5 (on the left), choosing \( \lambda_1 = 2 \) and \( \lambda_2 = 14 \). In the same figure (on the right) we also give an integer \( ^{10} H_3(20, 12; 6, 10) \), where \( \lambda_1 = 5 \) and \( \lambda_2 = 2 \).

5. Conclusions

Thanks to the constructions of Sections 3 and 4 we can prove Theorem 1.9. In fact, case (1) follows from Proposition 3.3; cases (2) and (3) follow from Proposition 4.14; case (4) follows from Proposition 4.15. Unfortunately, we are not able to solve the existence of an integer \( ^\lambda H_4(m, n; s, k) \) when \( s, k \equiv 2 \pmod{4} \) and \( m, n \) are odd. However, we can prove the existence of an SMA(\( m, n; s, k, k \)) for this choice of \( m, n, s, k \).

Proof of Theorem 1.6 If \( s, k \equiv 0 \pmod{4} \), the integer \( ^2 H_1(m, n; s, k) \) we construct in Lemma 3.3 is actually a (shiftable) SMA(\( m, n; s, k \)). Similarly, if \( s \equiv 2 \pmod{4} \) and \( m \) is even, the integer \( ^2 H_1(m, n; s, k) \) constructed in Propositions 4.14 and 4.15 are (shiftable) signed magic arrays. So, we are left to consider the case \( s, k \equiv 2 \pmod{4} \) with \( m, n \) odd.

Without loss of generality, we may assume \( m \geq n \) (and so \( s \leq k \)). Let \( A_1 \) be an SMA(\( n, n; s, s \)), whose existence is assured by Theorem 1.2. Clearly if \( m = n \) we have nothing to prove. So, suppose \( m > n \). Since \( m - n \geq 2 \) is even and \( k - s \equiv 0 \pmod{4} \) with \( k - s \geq 4 \), by Proposition 4.14 there exists a shiftable SMA(\( m - n, n; k, k - s \)), say \( A_2 \). Let \( A \) be the p.f. array of size \( m \times n \) obtained taking

\[
A = \begin{bmatrix}
A_1 \\
A_2 \pm ns/2
\end{bmatrix}
\]

Each row of \( A \) contains \( s \) filled cells and each of its columns contains \( s + (k - s) = k \) filled cells. Also, note that \( \mathcal{E}(A_1) = \{\pm 1, \pm 2, \ldots, \pm ns/2\} \) and \( \mathcal{E}(A_2 \pm ns/2) = \{\pm (1 + ns/2), \pm (2 + ns/2), \ldots, \pm ms/2\} \). Since \( \mathcal{E}(A) = \mathcal{E}(A_1) \cup \mathcal{E}(A_2) = \{\pm 1, \pm 2, \ldots, \pm ms/2\} \), \( A \) is an SMA(\( m, n; s, k) \).

We can now prove the existence of magic rectangles.

Proof of Theorem 1.11 Let \( A \) be a shiftable SMA(\( m, n; s, k \)), whose existence was proved in Theorem 1.6 and let \( A^* \) be the p.f. array obtained by replacing every negative entry \( x \) of \( A \) with \( x + ms \) and by replacing every positive entry \( y \) with \( y + ms - 1 \). Since \( \mathcal{E}(A) = \{-1, -2, \ldots, -\frac{ms}{2}\} \cup \{1, 2, \ldots, \frac{ms}{2}\} \), we obtain \( \mathcal{E}(A^*) = \{0, 1, \ldots, \frac{ms}{2} - 1\} \cup \{\frac{ms}{2}, \frac{ms}{2} + 1, \ldots, ms - 1\} \). This means that every element of \( [0, ms - 1] \) appears just once in \( A^* \). Obviously,
Example 5.1. Take the following shiftable SMA(5,10,8,4), whose construction is given in Lemma 3.3.

$$A = \begin{array}{ccccccc}
1 & -2 & -7 & 8 & 11 & -12 & -17 & 18 \\
20 & 3 & -4 & -9 & 10 & 13 & -14 & -19 \\
-1 & 2 & 5 & -6 & -11 & 12 & 15 & -16 \\
-3 & 4 & 7 & -8 & -13 & 14 & 17 & -18 \\
-20 & -5 & 6 & 9 & -10 & -15 & 16 & 19
\end{array}$$

Figure 3. An integer $^{28}H_4(16, 16; 14, 14)$ (on the left) and an integer $^{10}H_3(20, 12; 6, 10)$ (on the right).

every row of $A^*$ contains exactly $s$ filled cells and every column of $A^*$ contains exactly $k$ filled cells. Now, since $A$ is shiftable, every row of $A$ contains $\frac{s}{2}$ negative entries and $\frac{s}{2}$ positive entries. So, the elements of every row of $A^*$ sum to $\frac{k}{2} \left( \frac{ms}{2} + \frac{ms}{2} - 1 \right) = \frac{s(m-1)}{2}$. Analogously, the elements of every column of $A^*$ sum to $\frac{k(m-1)}{2}$. We conclude that $A^*$ is an MR($m, n; s, k$).

Example 5.1. Take the following shiftable SMA(5,10,8,4), whose construction is given in Lemma 3.3.
Proceeding as described in the proof of Theorem 1.11 we obtain the following MR(5, 10; 8, 4):

\[
A^* = \begin{array}{cccccccc}
20 & 18 & 13 & 27 & 30 & 8 & 3 & 37 \\
39 & 22 & 16 & 11 & 29 & 32 & 6 & 1 \\
19 & 21 & 24 & 14 & 9 & 31 & 34 & 4 \\
17 & 23 & 26 & 12 & 7 & 33 & 36 & 2 \\
0 & 15 & 25 & 28 & 10 & 5 & 35 & 38
\end{array}
\]

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