1 INTRODUCTION AND MOTIVATION

We consider the problem in which a collection of $K$ networked agents, indexed $k = 1, 2, \ldots, K$, is interested in tracking the average of time-varying signals $\{r_{k,i}\}$ arriving at the agents, where $k$ is the agent index and $i$ is the time index. The objective is for the agents to attain tracking in a decentralized manner through local interactions among their neighbors. This type of problem is common in many applications. For example, consider the following distributed empirical risk minimization problem \[ w^* = \min_{w \in \mathbb{R}^d} J(w) = \frac{1}{K} \sum_{k=1}^{K} Q(w; X_k) \]

where $Q(w; X_k)$ is some loss function that depends on the data $X_k$ at location or agent $k$. If we let $w_{k,i}$ denote an estimate for the minimizer $w^*$ at agent $k$ at time $i$, and let $X_{k,i}$ denote the data received at that agent at the same time instant, then some solution methods to \[ (1) \] involve tracking the average gradient defined by \[ \tilde{r}_i = \frac{1}{K} \sum_{k=1}^{K} \nabla_w Q(w_{k,i}; X_{k,i}) \]

where each term inside the summation represents the signal $r_{k,i}$. Likewise, in learning problem formulations involving feature vectors and parameter models that are distributed over space, or loss functions that are expressed in the form of sums \[ (6) \], we encounter optimization problems of the form \[ w^* = \min_{w \in \mathbb{R}^d} J(w) = \frac{1}{N} \sum_{n=1}^{N} \left( \frac{1}{K} \sum_{k=1}^{K} f(w_k; X_{k,n}) \right) \]

where $f(w_k; X_{k,n})$ is some linear or nonlinear function that depends on the $n$–th feature set, $X_{k,n}$, available at agent $k$. Some solution methods to \[ (3) \] involve tracking the average quantity:

\[ \tilde{r}_i = \frac{1}{K} \sum_{k=1}^{K} \begin{bmatrix} f(w_{k,1}; X_{k,1}) \\ \vdots \\ f(w_{k,K}; X_{k,K}) \end{bmatrix} \]

where again each term inside the summation represents an $r_{k,i}$ signal.

There are several useful distributed algorithms in the literature for computing the average of static signals $\{r_k\}$ (i.e., signals that do not vary with the time index $i$), and which are distributed across a network \[ (1), (3), (5), (15), (17) \]. One famous algorithm is the consensus strategy which takes the form

\[ w_{k,i} = \sum_{\ell \in N_k} a_{\ell k} w_{\ell,i-1}, \quad \text{where } w_{k,0} = r_k \]

under some mild conditions on $A \{11, 12, 20\}$. When the static signals $\{r_k\}$ become dynamic and are replaced by $\{r_{k,i}\}$, a useful variation is the dynamic average consensus algorithm from \[ (21), (23) \]. It replaces \[ (5) \] by the recursion:

\[ w_{k,i} = \sum_{\ell \in N_k} a_{\ell k} w_{\ell,i-1} + r_{k,i} - r_{k,i-1}, \quad \text{where } w_{k,0} = r_k \]

where the difference $r_{k,i} - r_{k,i-1}$ is added as a driving term. In this case, it can be shown that if the signals $\{r_{k,i}\}$ converge to static values, i.e., if $r_{k,i} \rightarrow r_k$, then result \[ (6) \] continues to hold \[ (21), (22) \]. Recursion \[ (7) \] is motivated in \[ (21), (22) \] using useful but heuristic arguments.

Motivated by these considerations, in this work, we develop a dynamic average diffusion strategy for tracking the average of time-varying signals $\{r_{k,i}\}$ by formulating an optimization problem and showing how to solve it by applying the exact diffusion strategy from \[ (21), (22) \]. One of the main contributions relative to earlier approaches is that we are specifically interested in the case in which the dimension of the observation vectors $\{r_{k,i}\}$ may be too large, which means that a solution like \[ (7) \] will necessitate the sharing of long vectors $w_{k,i}$ among the agents resulting in an inefficient communication scheme. We are also interested in the case in which each agent $k$ can only observe one random entry of $r_{k,i}$ at each iteration (either by...
design or by choice). In this case, it will be wasteful to share the full vector \( w_{k,i} \) since only one entry of \( w_{k,i} \) will be affected by the new information. To handle these situations, we will need to incorporate elements of randomized coordinate-descent \([24, 25]\) into the operation of the algorithm in line with approaches from \([26, 28]\). Doing so, however, introduces one nontrivial complication: different agents may be selecting or observing different entries of their vectors \( r_{k,i} \), which raises a question about how to coordinate or synchronize their interactions. In order to facilitate the presentation, we shall assume initially that all agents select the same initial entry that all agents select the same entry of their observations vectors at each iteration. Subsequently, we will show how to employ push-sum ideas \([29, 30]\) to allow each agent to select its own local entry independently of the other agents.

2. ALGORITHM DEVELOPMENT

2.1. Review of Exact Diffusion Strategy

One effective decentralized method to solve problems of the form:

\[
J(w) = \frac{1}{K} \sum_{k=1}^{K} J_k(w)
\]

is the Exact diffusion strategy \([6, 31, 32]\). In \([8]\), each \( J_k(w) \) refers to the risk function at agent \( k \) and is generally convex or strongly-convex. For simplicity, we shall assume in this work that each \( J_k(w) \) is differentiable although the analysis can be extended to non-smooth risk functions by employing subgradient constructions, along the lines of \([7,33]\), or proximal constructions similar to \([3,34]\). To implement exact diffusion, we need to associate a combination matrix \( A = [a_{k,i}]_{k=1}^{K} \) with the network graph, where a positive weight \( a_{k,i} \) is used to scale data that flows from node \( k \) to \( i \) if both nodes happen to be neighbors. In this paper we assume that:

**Assumption 1 (TOPOLOGY)** The underlying topology is strongly connected, and the combination matrix \( A \) is symmetric and doubly stochastic, i.e.,

\[
A = A^T \text{ and } A \mathbb{1}_K = \mathbb{1}_K
\]

where \( \mathbb{1} \) is a vector with all unit entries. We further assume that \( a_{k,i} > 0 \) for at least one agent \( k \).

We further introduce \( \mu \) as the positive step-size parameter for all nodes. The exact diffusion algorithm is listed in \([10a, 10c]\). It is shown in \([31]\) that the local variables \( w_{k,i} \) converge to the exact minimizer of problem \([8]\), \( w^* \), at a linear convergence rate under relatively mild conditions.

Algorithm 1 [Exact diffusion strategy for each node \( k \)] \([6,31]\)

**Initialize** \( w_{k,0} \) arbitrarily, and let \( \psi_{k,0} = w_{k,0} \).

**Repeat** iteration \( i = 1, 2, 3 \ldots \) until convergence

\[
\psi_{k,i} = w_{k,i-1} - \mu \nabla w_{r_k} (w_{k,i-1})
\]

\[
\phi_{k,i} = w_{k,i} - \psi_{k,i-1} - \psi_{k,i}
\]

\[
w_{k,i} = \sum_{\ell \in N_k} a_{k,\ell} \phi_{\ell,i}
\]

2.2. Dynamic Average Diffusion

Now, we consider a time-varying quadratic risk function of the form

\[
J_{k,i}(w) = \frac{1}{2} \| w - r_{k,i} \|^2
\]

and introduce the average cost

\[
J_i(w) \triangleq \frac{1}{K} \sum_{k=1}^{K} J_{k,i}(w)
\]

At every time instant \( i \), if we optimize \( J_i(w) \) over \( w \) then it is clear that the minimizer, denoted by \( \tilde{w}_i \), will coincide with the average of the observed signals:

\[
\tilde{w}_i = \bar{r}_i \triangleq \frac{1}{K} \sum_{k=1}^{K} r_{k,i}
\]

Therefore, one way to track the average of the signals \( \{ r_{k,i} \} \) is to track the minimizer of the aggregate cost \( J_i(w) \) defined by \([12]\). Apart from the time index, this cost has a form similar to \([8]\) especially when the observations signals \( \{ r_{k,i} \} \) approach steady-state values where they become static. This motivates us to apply the exact diffusion construction \([10a, 10c]\) to the risks defined by \([12]\). Doing so leads to the recursions:

\[
\psi_{k,i} = (1 - \mu)w_{k,i-1} + \mu r_{k,i}
\]

\[
\phi_{k,i} = \psi_{k,i} + w_{k,i-1} - \psi_{k,i-1}
\]

\[
w_{k,i} = \sum_{\ell \in N_k} a_{k,\ell} \phi_{\ell,i}
\]

Combining \((14a) - (14c)\) into a single recursion, we obtain:

\[
w_{k,i} = \sum_{\ell \in N_k} a_{k,\ell} \left( (1 - \mu)w_{\ell,i-1} + \mu r_{\ell,i} + w_{\ell,i-1} - (1 - \mu)w_{\ell,i-2} - \mu r_{\ell,i-1} \right)
\]

so that by selecting \( \mu = 1 \), the algorithm reduces to what we shall refer to as the dynamic average diffusion algorithm:

**Algorithm 2 [Dynamic average diffusion]**

**Initialize**: \( w_0 = r_{k,0} \).

**Repeat** iteration \( i = 1, 2, 3 \ldots \)

\[
w_{k,i} = \sum_{\ell \in N_k} a_{k,\ell} (w_{\ell,i-1} + r_{\ell,i} - r_{\ell,i-1})
\]

Other values for \( \mu \) are of course possible by using instead \([15]\). Comparing \([16]\) with the consensus version \([5]\), we see that the scaling weights \( \{ a_{k,\ell} \} \) in \([16]\) are multiplying the combined sum of the weight iterate \( w_{\ell,i-1} \) and the difference of the current and past observation vectors, \( r_{\ell,i} - r_{\ell,i-1} \). Moreover, and importantly, while in the consensus construction \([5]\) each agent \( k \) employs only its own observation vector, we see in \([16]\) that all observations vectors from the neighborhood \( N_k \) of agent \( k \) contribute to the update of \( w_{k,i} \). In this way, agents need to share their weight iterates along with the difference of their observation vectors. In a future section, we shall show how agents can only share single entries of their observations vectors chosen at random.

There are several interesting properties associated with the dynamic diffusion strategy \([16]\). First, at any time \( i \), the average of the
\( \{ w_{k,i} \} \) coincides with the average of the \( \{ r_{k,i} \} \), i.e.,

\[
\frac{1}{K} \sum_{k=1}^{K} w_{k,i} = \frac{1}{K} \sum_{k=1}^{K} r_{k,i}, \quad \forall i
\]  (17)

This property can be easily shown using mathematical induction. Second, when the signal is static, i.e., \( r_{k,i} = r_k \), the algorithm reduces to the classical consensus construction \([5]\). Third, when the signal \( r_{k,i} \) converges to some steady-state value \( r_k \), or their time variations become uniform after some time \( i_0 \), i.e.,

\[
r_{k,i} - r_{k,i-1} = r_{k',i} - r_{k',i-1}, \quad \forall k, k', i > i_0
\]  (18)

then it can be shown that

\[
\lim_{i \to \infty} \| w_{k,i} - r_i \| = 0
\]  (19)

This conclusion is a special case of later results in this paper and therefore its proof will follow by specializing the arguments used later in theorem \([1]\).

### 3. SYNCHRONIZED RANDOM UPDATES

Let us consider next the case in which each agent \( k \) can only access (either by design or by choice) one random entry within the vector \( r_{k,i} \). We denote the index of that entry by \( n_k \), at iteration \( i \); we use the boldface notation because \( n_k \) will be selected at random and boldface symbols denote random quantities in our notation. We shall first assume that all agents select the same \( n_k \); later we consider the case in which \( n_k \) varies among agents and replace the notation by \( n_k^i \), instead, with the superscript \( k \) referring to the agent. This situation will then enable a fully distributed solution.

When all agents select the same random index \( n_k \), one naive solution to update their weight iterates is to resort to coordinate-descent type constructions \([24,25]\). Namely, at iteration \( i \), the index \( n_k \) is selected uniformly and then only the \( n_k \)-th entry of \( w_{k,i} \) is updated, say, as:

\[
\begin{align*}
\mathbf{w}_{k,i}(n) &= \sum_{\ell \in N_k} a_{\ell k} \left( \mathbf{w}_{\ell,i-1}(n) + r_{\ell,i}(n) - r_{\ell,i-1}(n) \right) \\
\mathbf{w}_{k,i}(n) &= w_{k,i-1}(n), \quad n \neq n_k
\end{align*}
\]  (20)

where the notation \( w(n) \), for a vector \( w \), refers to the \( n \)-th entry of that vector. This iteration applies (19) to the \( n_k \)-th entry of \( w_{k,i} \) and keeps all other entries of this vector unchanged relative to \( w_{k,i-1} \). Although simple, this algorithm is not implementable for one subtle reason. This is because at time \( i - 1 \), agent \( \ell \) can only observe \( r_{\ell,i-1}(n_{k-1}) \) and not \( r_{\ell,i-1}(n_k) \). In other words, the variable \( r_{\ell,i-1}(n_k) \) is not available; this variable would be available if we allow agent \( \ell \) to save the entire vector \( r_{\ell,i-1} \) from the previous iteration and then select its \( n_k \)-th entry at time \( i \). However, doing so, defeats the purpose of a coordinate-descent solution where the purpose is to avoid working with long observation vectors and to work instead with scalar entries. We can circumvent this difficulty as follows. We let \( j \) refer to the most recent iteration from the past where the same index \( n_k \) was chosen the last time; the value of \( j \) clearly depends on \( n_k \). Then, we can replace (20) by:

\[
\begin{align*}
\mathbf{w}_{k,i}(n) &= \sum_{\ell \in N_k} a_{\ell k} \left( \mathbf{w}_{\ell,j}(n) + r_{\ell,i}(n) - r_{\ell,j}(n_k) \right) \\
\mathbf{w}_{k,i}(n) &= w_{k,i-1}(n), \quad n \neq n_k
\end{align*}
\]  (21)

where the index \( j \) appears in two locations on the right-hand side: within \( \mathbf{w}_{\ell,j} \) and \( r_{\ell,j} \). Note first that this implementation is now feasible because the scalar value \( r_{\ell,j}(n_k) \) from the past can be saved into a memory variable. Specifically, for every agent \( k \) we introduce a vector \( v_{k,i} \), which is updated with time. At every iteration \( i \), an index \( n_k \) is selected and the value of the observation entry \( r_{k,i}(n_k) \) is saved into the \( n_k \)-th location of \( v_{k,i} \), for later access the next time the index \( n_k \) is selected. It is also important to use \( w_{\ell,j}(n_k) \), with the same subscript \( j \), along with \( r_{\ell,j}(n_k) \) in (21) in order to maintain the mean property \([17]\). However, due the definition of \( j \) and the second line in (21), we know that \( w_{\ell,j}(n_k) = w_{\ell,i-1}(n_k) \). Hence, the resulting algorithm is:

\[
\begin{align*}
\mathbf{w}_{k,i}(n_k) &= \sum_{\ell \in N_k} a_{\ell k} \left( \mathbf{w}_{\ell,i-1}(n_k) + r_{\ell,i}(n_k) - w_{\ell,i-1}(n_k) \right) \\
\mathbf{w}_{k,i}(n) &= w_{k,i-1}(n), \quad n \neq n_k
\end{align*}
\]  (22)

To simplify the notation, we introduce the indicator function:

\[
\mathbb{I}[\text{expression}] \triangleq \begin{cases} 1, & \text{if expression is true} \\ 0, & \text{if expression is false} \end{cases}
\]  (23)

and the selection matrix:

\[
\mathbb{S}_{n_k} \triangleq \begin{bmatrix} \mathbb{I}[n_k = 1] \\ \mathbb{I}[n_k = 2] \\ \vdots \\ \mathbb{I}[n_k = N] \end{bmatrix}
\]  (24)

This matrix is diagonal with a single unit entry on the diagonal at the location of the active index \( n_k \). All other entries are zero. We also introduce the complement matrix:

\[
\mathbb{S}_{n_k}^c \triangleq I_N - \mathbb{S}_{n_k}
\]  (25)

Using these matrices, the resulting algorithm is listed in Algorithm 3. The proof of the convergence is provided later in Sec. 5.1.

### 4. INDEPENDENT RANDOM UPDATES

#### 4.1. A first attempt at random indices

The previous algorithm requires all agents to observe the same “random” index \( n_k \) at iteration \( i \). In this section, we will allow \( n_k \) to be locally selected by the agents. To refer to this generality, we replace the notation \( n_k \) by \( n_k^i \), where \( n_k^i \) is selected uniformly from \([1, 2, \ldots, N]\) by agent \( k \).

In this way, agents now cannot share the same entries of their observation vectors. However, they will generally exist smaller groups of agents that end up selecting the same index (since indexes are chosen at random). We can represent this possibility by examining
replicas of the network topology, as illustrated by Fig. 1. In each layer, we highlight in blue the agents that selected the same index. For example, all four blue agents in the top layer selected the entry index \( n = 1 \); i.e., for these agents, \( n_k^1 = 1 \). Only one agent in the second layer selected index \( n = 2 \) and three agents in the bottom layer selected the index \( n = 3 \).

Motivated by the discussion that led to Algorithm 3, we can similarly start from the following recursions:

\[
\begin{align*}
\mathbf{w}_{k,i}(n_k^t) &= \sum_{\ell \in \mathcal{N}_k, n_{\ell}^t = n_k^t} a_{\ell k}(w_{\ell,i-1}(n_{\ell}^t) + r_{\ell,i}(n_{\ell}^t) - s_{\ell,i-1}(n_{\ell}^t)) \\
\mathbf{w}_{k,i}(n) &= \mathbf{w}_{k,i-1}(n) + \sum_{\ell \in \mathcal{N}_k, n_{\ell}^t = n} a_{\ell k}(w_{\ell,i-1}(n) + r_{\ell,i}(n) - s_{\ell,i-1}(n)) \\
\mathbf{v}_{k,i}(n) &= \begin{cases} 
\mathbf{r}_{k,i}(n), & \text{if } n = n_k^t \\
\mathbf{v}_{k,i-1}(n), & \text{if } n \neq n_k^t
\end{cases}
\end{align*}
\]

(27)

where the summation \( \sum_{\ell \in \mathcal{N}_k, n_{\ell}^t = n} \) refers to adding over the neighbor agents \( \ell \) whose selected random index \( n_{\ell}^t \) is equal to \( n \). In this implementation, agents that select the same index within the neighborhood of agent \( k \) are processed together in a manner similar to Algorithm 3. However, there is one important difficulty with this implementation, which does not work correctly. This is because

\[
\sum_{\ell \in \mathcal{N}_k, n_{\ell}^t = n} a_{\ell k} \neq \sum_{\ell \in \mathcal{N}_k} a_{\ell k} = 1
\]

(28)

In other words, the “effective” combination matrix for any of the layers (on the right side of Fig. 1) is not necessarily doubly-stochastic anymore. Even worse, the topology from one layer to another and from one iteration to another keeps changing due to the random selections at each agent. These facts bias the operation of the algorithm and prevent the agents from reaching consensus. We need to account for these difficulties.

4.2. Push-sum correction

We shall exploit some properties from the push-sum construction. Basically, recall that the original push-sum algorithm deals with the problem of seeking the mean \( \bar{r} \) of static signals \( \{r_k\} \). One appealing property of the algorithm is that it can be applied to time-varying row stochastic matrices, i.e., to graphs where outgoing scaling factors add up to one, say,

\[
\sum_{k=1}^{K} a_{\ell k}^{(i)} = 1, \quad \forall \ell, i
\]

(29)

where the superscript \( i \) is added to indicate time-variation. This condition only requires the outgoing weights \( a_{\ell k} \) (from agent \( \ell \) to agent \( k \)) to sum up to one; it does not require the incoming weights into agent \( k \) to add up to one. Moreover, it is common to assume that the topology satisfies the following condition.

**Assumption 2 (Time-Varying Topology Assumption)**

The sequence \( A^{(i)} = [a_{\ell k}^{(i)}] \) is a stationary and ergodic sequence of stochastic matrices with positive diagonal entries, and \( \mathbb{E} A^{(i)} \) is primitive.

If we apply the classical consensus algorithm \(^5\) under this condition:

\[
\mathbf{w}_{k,i} = \sum_{\ell \in \mathcal{N}_k} a_{\ell k}^{(i)} \mathbf{w}_{\ell,i-1}, \quad \text{where } \mathbf{w}_{k,0} = r_k
\]

(30)

then \( \mathbf{w}_{k,i} \) will not reach consensus \(^6\). In order to reach consensus under this time-varying row stochastic topology, the push-sum algorithm construction introduces a vector variable \( p_{k,i} \) to help correct for bias. The algorithm starts from \( \mathbf{w}_{k,0} = r_k \) and \( p_{0,k} = 1 \) is the vector with all entries equal one:

\[
\begin{align*}
\mathbf{w}_{k,i} &= \sum_{\ell \in \mathcal{N}_k} a_{\ell k}^{(i)} \mathbf{w}_{\ell,i-1} \\
p_{k,i} &= \sum_{\ell \in \mathcal{N}_k} a_{\ell k}^{(i)} p_{\ell,i-1} \\
x_{k,i} &= \mathbf{w}_{k,i}/p_{k,i}
\end{align*}
\]

(31)

where the last equality is used to mean that the individual entries of \( \mathbf{w}_{k,i} \) are divided by the corresponding entry in \( p_{k,i} \); it refers to an element-wise division. It can be shown under Assumption 2 \(^7\) that this algorithm leads to \(^{29,30}\):

\[
\lim_{i \to \infty} x_{k,i}^{a.s.} = \bar{r}
\]

(32)

Later in Sec. 5.2 we provide additional explanations that further clarify why this construction works correctly — see the explanation leading to (69).

4.3. Dynamic diffusion with independent random updates

We can exploit the push-sum construction in the dynamic diffusion scenario when random indexes are selected at each iteration. As we mentioned before, the implementation (27) will not reach the desired consensus since the incoming weights \( \{a_{\ell k}\} \) do not add up to one. However, we assumed the underlying matrix \( A \) is doubly-stochastic, which implies that the outgoing weights still add up to one. Hence, the push-sum construction can be utilized to solve the bias introduced by (27). One important property to enforce is that the entries in \( p_{k,i} \) and \( \mathbf{w}_{k,i} \) should undergo similar updates. Doing so leads to Algorithm 4.

Comparing (27) with (33b), there are two main modifications. One is that the updated index is allowed to vary at different locations. Another is that the output is \( x_{k,i} \) instead of \( \mathbf{w}_{k,i} \), i.e., the value after correction by \( p_{k,i} \). On the other hand, if we force \( n_k^i = n_k^{i'} \) for all \( k \) and \( i' \), the Algorithm 4 will reduce to Algorithm 3 by noting that \( p_{k,i} = 1 \) for any \( i, k \). The proof of the convergence of Algorithm 3 is provided later in the Sec. 5.3.
have shown that the output in this case it holds that \( w \) gradients in empirical risk minimization problems. It can be verified has been used in the recent push-pull type algorithm \([8, 35]\), which the desired mean value without the push-sum correction. This case

\[ 4.4. \text{ Special case without push-sum correction} \]

There is one special case where \( w_{k,i} \) from \([27]\) can still converge to the desired mean value without the push-sum correction. This case has been used in the recent push-pull type algorithm \([8, 35]\), which use pull-network for consensus \( \{ w_{k,i} \} \) and pus-network for aggregate the gradient over agents. The special case is when

\[ \lim_{i \to \infty} \frac{1}{K} \sum_{k=1}^{K} r_{k,i} = 0 \] (34)

This scenario is quite common in the case of tracking the sum of gradients in empirical risk minimization problems. It can be verified that in this case it holds that \( w_{k,i} \to 0 \), i.e., with or without division by \( p_{k,i} \). To shed some intuition on this statement, assume that we have shown that the output \( x_{k,i} \) in Algorithm 4 has converged to the desired consensus value 0. Then, we also know \( p_{k,i} \) is non-zero due to the non-zero initial value of \( p_{k,0} \) and the fact \( A \) is primitive. Combining these two facts and \( x_{k,i} = w_{k,i}/p_{k,i} \), we can infer that \( w_{k,i} \) must be zero. The detailed proof of this statement is provided in the next section.

5. CONVERGENCE ANALYSIS

In this section, we establish the convergence of Algorithms 3 and 4 for both case of synchronous and independent random entry updates.

5.1. Convergence of algorithm 3

First, we verify that recursions \([26a, 26b]\) can reach consensus if the observation signals \( r_{k,i} \) converge to \( r_k \). Then we consider the case that the signals have a small perturbation.

**Theorem 1 (Mean-Square Convergence of Algorithm 3)**

Suppose the underlying topology \( A \) satisfy the Assumption \([7]\) and each signal \( r_{k,i} \) converges to a limiting value \( r_k \). It then holds that the algorithm converges in the mean-square-error sense, namely,

\[ \lim_{t \to \infty} E \| w_{k,i} - \bar{r}_{i} \|^2 = 0, \quad \forall k \] (35)
When \( \lambda = 0 \), which implies full-connectivity, we can end the proof quickly since (55) becomes:
\[
E \| \mathbf{w}_i - \mathbf{v}_i \|^2 \leq \frac{N - 1}{N} E \| \mathbf{w}_{i-1} - \mathbf{v}_{i-1} \|^2
\]
(48)

Hence, in the following argument, we exclude the trivial case \( \lambda = 0 \).

We continue with (45) to get:
\[
E \| \mathbf{w}_i - \mathbf{v}_i \|^2 \leq \frac{N - 1}{N} E \| \mathbf{w}_{i-1} - \mathbf{v}_{i-1} \|^2
+ \frac{\lambda^2}{N} E \| \mathbf{w}_i - \mathbf{w}_{i-1} + \mathbf{r}_i - \mathbf{v}_{i-1} \|^2
= \frac{N - 1}{N} E \| \mathbf{w}_{i-1} - \mathbf{v}_{i-1} \|^2
+ \frac{\lambda^2}{N} E \left( \frac{\lambda}{\lambda} \| \mathbf{w}_i - \mathbf{v}_{i-1} \| + \frac{1 - \lambda}{1 - \lambda} \| \mathbf{r}_i - \mathbf{v}_{i-1} \| \right)^2
\leq \frac{N - 1}{N} E \| \mathbf{w}_{i-1} - \mathbf{v}_{i-1} \|^2
+ \frac{\lambda}{N} E \| \mathbf{w}_i - \mathbf{w}_{i-1} \|^2 + \frac{\lambda^2}{N(1 - \lambda)} E \| \mathbf{r}_i - \mathbf{v}_{i-1} \|^2
\]
(49)

where the second inequality is due to Jensen’s inequality. Similarly, we execute the same procedure on (59):
\[
E \| \mathbf{v}_i - \mathbf{r}_{i+1} \|^2
= E \| S_{\mathbf{v}_i}(n) \mathbf{v}_{i-1} + S_{\mathbf{r}_i}(n) \mathbf{r}_i - \mathbf{r}_{i+1} \|^2
= E \| S_{\mathbf{v}_i}(n) (\mathbf{v}_{i-1} - \mathbf{r}_i) + \mathbf{r}_i - \mathbf{r}_{i+1} \|^2
= E \left( \frac{1}{t} E \| S_{\mathbf{v}_i}(n) (\mathbf{v}_{i-1} - \mathbf{r}_i) + \frac{1 - t}{1 - t} \| \mathbf{r}_i - \mathbf{r}_{i+1} \| \right)^2
\leq \frac{1}{t} E \| S_{\mathbf{v}_i}(n) \|^2 E \| \mathbf{v}_{i-1} - \mathbf{r}_i \|^2 + \frac{1 - t}{1 - t} \| \mathbf{r}_i - \mathbf{r}_{i+1} \|^2
\]
(50)

where the inequality rely on Jensen’s inequality and we choose \( t = \frac{N - 1}{N - 1} \) in last equality. If \( \mathbf{r}_i \) converges, it means that
\[
\| \mathbf{r}_i - \mathbf{r}_{i+1} \|^2 \to 0
\]
(51)
so that due to (50), we conclude
\[
E \| \mathbf{v}_i - \mathbf{r}_{i+1} \|^2 \to 0
\]
(52)
Combining with (49), we get
\[
E \| \mathbf{w}_i - \mathbf{v}_i \|^2 \to 0
\]
(53)
Hence, we have proven that the algorithm reaches the consensus if the observation signals converge. Lastly, we show that the consensus value is actually the desired \( \mathbf{r}_i \). Let \( \bar{\mathbf{v}}_i = \mathbf{\Gamma}^T \mathbf{w}_i / K \) and \( \bar{\mathbf{v}}_i = \mathbf{\Gamma}^T \mathbf{v}_i / K \). It follows from (38) and (39) that
\[
\bar{\mathbf{w}}_i = S_{\mathbf{w}_i}^c (\bar{\mathbf{w}}_{i-1} + \bar{\mathbf{v}}_{i-1} - \bar{\mathbf{v}}_{i-1})
\]
\[
\bar{\mathbf{v}}_i = S_{\mathbf{v}_i} (\bar{\mathbf{v}}_{i-1} + \bar{\mathbf{r}}_i)
\]
(54)
(55)
Subtracting (55) from (54), we have
\[
\bar{\mathbf{w}}_i - \bar{\mathbf{v}}_i = S_{\mathbf{w}_i}^c (\bar{\mathbf{w}}_{i-1} - \bar{\mathbf{v}}_{i-1}) + S_{\mathbf{w}_i} (\bar{\mathbf{w}}_{i-1} - \bar{\mathbf{v}}_{i-1})
= \bar{\mathbf{w}}_{i-1} - \bar{\mathbf{v}}_{i-1}
\]
(56)
so that
\[
\bar{\mathbf{w}}_i - \bar{\mathbf{v}}_i = \mathbf{w}_0 - \bar{\mathbf{v}}_0 = 0
\]
(57)
and we conclude \( \bar{\mathbf{w}}_i \) is always the same as \( \bar{\mathbf{v}}_i \). Recall that \( \mathbf{v}_{k,i} \) is a vector that stores the past state of \( r_{k,i} \) and it is easy to see that \( \bar{\mathbf{v}}_i \to \bar{r}_i \) if \( r_i \) converges, which completes the proof.

Corollary 1 (Small Perturbations) Suppose each entry in the signal \( r_{k,i} \) satisfies after sufficient iterations \( i_o \):
\[
\| r_{k,i}(n) - r_{k,i-1}(n) \| ^2 \leq \epsilon / N, \quad \forall i_i, k
\]
(58)
This property implies that \( \| r_{k,i} - r_{k,i-1} \|^2 \leq \epsilon \), where \( \epsilon \) is a small positive value. It then holds that
\[
\limsup_{i \to \infty} \frac{1}{K} \sum_{k=1}^{K} E \| \mathbf{w}_{k,i}(n) - \bar{r}_i(n) \|^2 \leq \frac{2 \lambda^2(2N - 1)}{(1 - \lambda)^2} \epsilon
\]
(59)
Proof: Substituting (58) into (50), for sufficiently large \( i \), we have:
\[
E \| \mathbf{v}_i - \mathbf{r}_{i+1} \|^2 \leq \frac{N - 1}{N} E \| \mathbf{w}_{i-1} - \mathbf{r}_i \|^2 + \frac{2N - 1}{N} K \epsilon
\]
(60)
We omit (n) again. Taking the limit over \( i \), we get
\[
\limsup_{i \to \infty} E \| \mathbf{w}_{i-1} - \mathbf{r}_i \|^2 \leq 2(2N - 1) K \epsilon
\]
(61)
Similarly, from (49), we have
\[
\limsup_{i \to \infty} E \| \mathbf{w}_i - \mathbf{v}_i \|^2 \leq \frac{\lambda^2}{(1 - \lambda)^2} \limsup_{i \to \infty} E \| \mathbf{v}_{i-1} - \mathbf{r}_i \|^2
\]
\[
\leq \frac{2 \lambda^2(2N - 1)}{(1 - \lambda)^2} K \epsilon
\]
(62)

5.2. Time-varying push-sum algorithm

Before we continue with the convergence proofs, we provide some useful intuition for the push sum construction. First, we note that the push-sum algorithm can be written in the following vector form for the \( n \)-th entry of the weight vectors (where we continue to drop the index \( n \)):
\[
\mathbf{w}_i = [A^{(i)}]^T \mathbf{w}_{i-1}
\]
(63a)
\[
\mathbf{p}_i = [A^{(i)}]^T \mathbf{p}_{i-1}
\]
(63b)
\[
\mathbf{x}_i = \mathbf{w}_i / \mathbf{p}_i
\]
(63c)
where the last division is element-wise and
\[
\mathbf{p}_i(n) \triangleq \begin{bmatrix}
\mathbf{p}_{1,i}(n) \\
\mathbf{p}_{2,i}(n) \\
\vdots \\
\mathbf{p}_{K,i}(n)
\end{bmatrix}
\]
(64)
Recall that the combination matrix \( A \) is row stochastic, which is equivalent to \( [A^{(i)}]^T \) is column stochastic, i.e.,
\[
1^T [A^{(i)}]^T = 1^T
\]
(65)
and satisfies Assumption 2. It is shown in [29][30][35], that, for sufficient large \( i \),
\[
\left( \prod_{i=1}^{\infty} [A^{(i)}]^T \right) \triangleq [A^{(1)}]^T [A^{(1)-1}]^T \cdots [A^{(1)}]^T \to \phi_1^0 1^T
\]
(66)
where the stochastic vector \( \phi^i_0 \) (whose entries add up to one) keeps changing with time no matter how large \( i \) is. Then, it is easy to see when \( i \) is sufficiently large:
\[
\nu_i \rightarrow \mathbf{1}^T \nu_0 \phi^i_1 \\
\rho_i \rightarrow \mathbf{1}^T \mathbb{1} \phi^i_1 \\
\chi_i \rightarrow \omega_0 \mathbb{1}
\]
Although \( \phi^i_0 \) keeps changing with time, the push-sum algorithm can reach consensus.

Before ending this section, we introduce a lemma that will be used in the convergence proof of the next section.

**Lemma 1 (Weak Ergodicity)** Suppose the sequence of stochastic matrices \( \{A^{(i)}\} \) satisfies Assumption 2 and for any \( l \) and \( l' \)
\[
\mathbb{E} A^{(l)} = \mathbb{E} A^{(l')} \triangleq A_E
\]
Then, there exists a unit vector \( \phi^i_1 \) such that for any time index \( i \geq j \):
\[
\mathbb{E} \left\| \prod_{l=j}^i [A^{(l)}]^T - \phi^i_1 \mathbb{1}^T \right\|_{\text{max}} \leq C \lambda^{i-j}
\]
where \( \left\| \cdot \right\|_{\text{max}} \) means the element-wise maximum, and \( C \) and \( \lambda < 1 \) are constants that depend on the graph structure. This means when \( i - j \) is sufficiently large, the matrix converges to a rank-1 matrix, whose rows are identical.

**Proof:** The lemma is slightly different from the prior literature and we therefore provide a sketch of the proof. The main idea is similar to [30]. First, the Dobrushin coefficient \( \delta(A) \) of the column stochastic matrix \( A \) is defined as:
\[
\delta(A) \triangleq \frac{1}{2} \max_{k,k'} \sum_{k=1}^K |a_{ik} - a_{ik'}|
\]
To lighten the notation, we let
\[
Q^{(j,i)} \triangleq \prod_{l=j}^i [A^{(l)}]^T
\]
It is easy to verify that:
\[
\mathbb{E} \left\| Q^{(j,i)} - \phi^i_1 \mathbb{1}^T \right\|_{\text{max}} \leq \mathbb{E} \delta \left( Q^{(j,i)} \right)
\]
so that we can focus on \( \mathbb{E} \delta \left( Q^{(j,i)} \right) \) instead. It is shown in [37] that the Dobrushin coefficient has two useful properties:
\[
\delta(A^{(j)}) A^{(i)} \leq \delta(A^{(i)}) \delta(A^{(j)})
\]
and
\[
\delta(A^{(j)}) \leq 1 - \max_k a_{ik}^{(j)} \leq 1
\]
Since we assume \( \mathbb{E} A^{(j)} \) is primitive in Assumption 2 there exists a constant \( T \) such that
\[
\mathbb{E} Q^{(i-j,i)} = \mathbb{E} \prod_{j=i-T}^i [A^{(l)}]^T = \prod_{j=i-T}^i \mathbb{E} [A^{(l)}]^T = (A_E^T)^T \geq 0
\]
where the notation \( X \succ 0 \) means every entry of matrix \( X \) is strictly larger than 0. The strictly positive property is correct due to the primitive property on \( A_E \) [19,20]. Hence, assuming \( i - j \geq T \), we obtain:
\[
\mathbb{E} \delta \left( Q^{(j,i)} \right) \leq \mathbb{E} \delta \left( Q^{(j,i+j)} \right) \delta \left( Q^{(j-j,i)} \right) = \mathbb{E} \delta \left( Q^{(j,i+j)} \right) \mathbb{E} \delta \left( Q^{(j-i)} \right)
\]
Due to (77), there is at least one realization where all elements in one column of \( Q^{(j-i)} \) are strictly larger than 0, i.e.
\[
P \left( \delta \left( Q^{(j-i)} \right) < 1 \right) > 0
\]
Combining the fact that \( \delta \left( Q^{(j-i)} \right) \leq 1 \), we conclude that
\[
\mathbb{E} \delta \left( Q^{(j,i)} \right) \leq \lambda^T \mathbb{E} \delta \left( Q^{(j+j+i)} \right)
\]
Lastly, for the case \( i - j < T \), we have
\[
\mathbb{E} \delta \left( Q^{(j,i)} \right) \leq 1 - \lambda^T \mathbb{E} \delta \left( Q^{(j+j+i)} \right)
\]
where we let \( C = (1/\lambda)^T \). Substituting (83) into (84), we have:
\[
\mathbb{E} \delta \left( Q^{(j,i)} \right) \leq \lambda^T . C \lambda^T = C \lambda^{i-j}
\]

**5.3. Convergence of algorithm 4**

**Theorem 2 (Convergence of Algorithm 4)** Suppose the underlying topology \( A \) satisfy the Assumption 2 and each signal \( r_{k,i} \) converges to \( r_{k,i} \), then the algorithm converges in the mean-square-error sense meaning that
\[
\lim_{i \rightarrow \infty} \mathbb{E} \left\| x_{k,i} - \bar{r}_i \right\|^2 = 0, \quad \forall k
\]

**Proof:** Again, it is sufficient to focus on one entry/coordinate of the recursions. We have
\[
\nu_i(n) = \mathbb{K}_n(n) \nu_{i-1}(n) + A^T \mathbb{K}_n(n) (v_{i-1}(n) + \xi(n) - v_{i-1}(n))
\]
where
\[
\mathbb{K}_n(n) \triangleq \begin{bmatrix} \mathbb{I}[n_1^i = n] \\ \mathbb{I}[n_2^i = n] \\ \vdots \\ \mathbb{I}[n_K^i = n] \end{bmatrix}
\]
\[
\bar{r}_i(n) \triangleq I_K - \mathbb{K}_n(n)
\]
\[
[A^{(j)}]^T \triangleq A^{(n)} + A^T \mathbb{K}_n(n)
\]
It is not hard to verify that \( [A^{(j)}]^T \) is a time-varying column stochastic matrix as in (65), and
\[
\mathbb{E} [A^{(j)}]^T = \frac{N-1}{N} I + \frac{1}{N} A^T
\]
Obviously, $A^{(i)}$ satisfies Assumption 2. Similarly, we have
\[
\begin{align*}
\mathbf{p}_i(n) &= K_i(n)\mathbf{p}_i(n) + A^T K_i(n)\mathbf{p}_{i-1}(n) \\
&= [A^{(i)^T}]^T\mathbf{p}_{i-1}(n) \\
\mathbf{v}_i(n) &= K_i(n)\mathbf{v}_{i-1}(n) + K_i(n)\mathbf{v}_i(n) \\
&= \mathbf{v}_{i-1}(n) + K_i(n)(\mathbf{v}_i(n) - \mathbf{v}_{i-1}(n))
\end{align*}
\] (90)
Substituting (91) into (89), we have
\[
\mathbf{w}_i(n) = [A^{(i)^T}]^T\mathbf{w}_{i-1}(n) + A^T(\mathbf{v}_i(n) - \mathbf{v}_{i-1}(n))
\] (92)
Next, we establish the same result as (86) by computing the sum of (93).
\[
1^T\mathbf{w}_i(n) = 1^T\mathbf{w}_{i-1}(n) + 1^T(\mathbf{v}_i(n) - \mathbf{v}_{i-1}(n)) \\
= 1^T\mathbf{w}_{i-2}(n) + 1^T(\mathbf{v}_i(n) - \mathbf{v}_{i-1}(n)) \\
&+ 1^T(\mathbf{v}_i(n) - \mathbf{v}_{i-2}(n)) \\
&+ 1^T\mathbf{v}_i(n) - 1^T\mathbf{v}_0(n) \\
= 1^T\mathbf{v}_i(n)
\] (93)
where the last equality is because in the algorithm, we use $w_{k,0} = v_{k,0}$ so that $1^T\mathbf{w}_0(n) = 1^T\mathbf{v}_0(n) = 0$. Similarly, we have
\[
1^T\mathbf{p}_i(n) = 1^T[A^{(i)^T}]^T\mathbf{p}_{i-1}(n) \\
= 1^T\mathbf{p}_{i-1}(n) \\
= 1^T\mathbf{p}_0(n) \\
= N
\] (94)
Let
\[
\alpha_i(n) \triangleq \frac{1^T\mathbf{w}_i(n)}{N} = \frac{1^T\mathbf{v}_i(n)}{N}
\] (95)
Starting below, we will ignore $(n)$ for simplicity:
\[
\mathbf{w}_i - \alpha_i\mathbf{p}_i = [A^{(i)^T}](\mathbf{w}_{i-1} - \alpha_i\mathbf{p}_{i-1}) + A^T(\mathbf{v}_i - \mathbf{v}_{i-1}) \\
= [A^{(i)^T}]\mathbf{w}_{i-1} - [A^{(i)^T}]\alpha_i\mathbf{p}_{i-1} + A^T(\mathbf{v}_i - \mathbf{v}_{i-1}) \\
+ [A^{(i)^T}]\alpha_i\mathbf{p}_i + A^T(\mathbf{v}_i - \mathbf{v}_{i-1})
\] (96)
Let
\[
z_i \triangleq (\alpha_i - 1)^T[A^{(i)^T}]^T\mathbf{p}_i + A^T(\mathbf{v}_i - \mathbf{v}_{i-1})
\] (97)
then recursion (96) becomes:
\[
\mathbf{w}_i - \alpha_i\mathbf{p}_i = [A^{(i)^T}](\mathbf{w}_{i-1} - \alpha_i\mathbf{p}_{i-1}) + z_i
\] (98)
It is easy to verify that
\[
1^Tz_i = (\alpha_i - 1 - \alpha_i)1^T[A^{(i)^T}]^T\mathbf{p}_i + 1^TA^T(\mathbf{v}_i - \mathbf{v}_{i-1}) \\
= (\alpha_i - 1 - \alpha_i)1^T\mathbf{p}_i + 1^T(\mathbf{v}_i - \mathbf{v}_{i-1}) + (\alpha_i - \alpha_i)N
\] (99)
To give some intuition, if $\mathbf{v}_i$ converges, then we have:
\[
\alpha_i - 1 - \alpha_i \to 0, \quad \mathbf{v}_i - \mathbf{v}_{i-1} \to 0, \quad z_i \to 0
\] (100)
Recall that $\mathbf{v}_i$ is the history record of signal $r_i$, which implies that if the signals gradually converge, then iteration (98) will eventually be equal to the consensus algorithm. Now, expanding (98) with respect to $i$, we get
\[
\mathbf{w}_i - \alpha_i\mathbf{p}_i = \left(\prod_{i=1}^{i}[A^{(i)^T}]^T\right)(\mathbf{w}_0 - \alpha_0\mathbf{p}_0) + \sum_{j=1}^{i}\left(\prod_{i=j+1}^{i}[A^{(i)^T}]^T\right)\mathbf{z}_j
\] (101)
Recalling the definition in Eq. (95) and the property in Eq. (99), we have
\[
\phi_i^01^T(\mathbf{w}_0 - \alpha_0\mathbf{p}_0) = 0, \quad \forall i
\] (102)
\[
\phi_i^j1^T\mathbf{z}_j = 0 \quad \forall i, j
\] (103)
so that expression (101) is equivalent to
\[
\mathbf{w}_i - \alpha_i\mathbf{p}_i = \left(\prod_{i=1}^{i}[A^{(i)^T}]^T - \phi_i^01^T\right)(\mathbf{w}_0 - \alpha_0\mathbf{p}_0) + \sum_{j=1}^{i}\left(\prod_{i=j+1}^{i}[A^{(i)^T}]^T - \phi_i^j1^T\right)\mathbf{z}_j
\] (104)
We further introduce the notation
\[
[B_{i,j}^+] \triangleq \prod_{i=1}^{i}[A^{(i)^T}] - \phi_i^j1^T
\] (105)
\[
[B_{i,k}^-] \triangleq \text{the $k$-th row of } \prod_{i=1}^{i}[A^{(i)^T}] - \phi_i^j1^T
\] (106)
It is straightforward to show that the sequence of matrices $\{A^{(i)}\}$ defined in (88) satisfies all the assumptions in Lemma 1. Hence, for some constant $C$ we have
\[
\mathbb{E}\||B_{i}^+|| \leq C\lambda^{i-j}
\] (107)
\[
\mathbb{E}\||B_{i,k}^-|| \leq C\lambda^{i-j}, \quad \forall k
\] (108)
Taking the infinity norm of (104) and expectation, we have
\[
\mathbb{E}\||\mathbf{w}_i - \alpha_i\mathbf{p}_i||_\infty \leq \mathbb{E}\left[\left\|\sum_{j=1}^{i}[B_{i,j}^+]\mathbf{z}_j\right\|_\infty + \left\|\mathbf{w}_0 - \alpha_0\mathbf{p}_0\right\|_\infty + \mathbb{E}\left[\sum_{j=1}^{i}[B_{i,j}^-]\mathbf{z}_j\right]\right]
\] (109)
where step (a) exploits the Holder inequality [38, 39]:
\[
|x^Ty| \leq ||x|| \cdot ||y||
\] (110)
Finally, supposing for any $\delta > 0$, there exists an $N$ such that $||\mathbf{z}_i|| < \delta, \quad i > N$ due to (109), we have
\[
||\mathbf{w}_{i+1} - \alpha_{i+1}\mathbf{p}_{i+1}||_\infty \leq C\lambda^{i-1}||\mathbf{w}_0 - \alpha_0\mathbf{p}_0||_1 + \lambda^{i-N}C\lambda^{i-j}||\mathbf{z}_j||_1 + \frac{1}{1 - \lambda}
\] (111)
Letting $\delta \to 0$, we have
\[
\lim_{i \to \infty} ||w_i - \alpha_i \bar{P}_i||_{\infty} \leq \frac{1}{1 - \lambda} \delta
\] (112)

Since $\delta$ can be arbitrarily close to 0, we conclude that:
\[
\lim_{i \to \infty} ||w_i/\bar{P}_i - \alpha_i \hat{E}||_{\infty} = 0
\] (113)

Lastly, we show that if \(\sum_{k=1}^{K} r_{k,i} \to 0\), then \(w_{k,i} \to 0\) for all \(k\). Using the triangle inequality and (112), we obtain
\[
\lim_{i \to \infty} ||w_i||_{\infty} \leq \lim_{i \to \infty} ||w_i - \alpha_i \bar{P}_i||_{\infty} + \lim_{i \to \infty} ||\alpha_i \bar{P}_i||_{\infty}
\]
\[
\leq \frac{1}{1 - \lambda} \delta + ||\alpha_i|| \lim_{i \to \infty} ||\bar{P}_i||_{\infty}
\] (114)

Because $\alpha_i$ is the desired average value, we have $\alpha_i \to 0$. Therefore, we conclude all $w_{k,i}$ across the agents will converge to zero.  

6. NUMERICAL SIMULATION

We illustrate the results of Theorems 1 and 2 by means of numerical simulations. We generate a network with 25 agents, as shown in Fig. 2. The dimension $N$ is set to $N = 100$ and each entry of $r_{k,i}$ is generated according to the following model:
\[
r_{k,i}(n) = a_{k}(n) \exp(-\alpha i \sin(\beta i)) + b_{k}(n) + 1
\] (115)

where $a_{k}(n)$ and $b_{k}(n)$ are zero-mean Gaussian distributed with variance 1. The parameters $\alpha$ and $\beta$ are set to $\alpha = 0.01$ and $\beta = 0.1$. It is seen that, as the iteration index $i$ increases, the signals $r_{k,i}(n)$ converge to $b_{k}(n) + 1$. We generate 2000 samples according to model (115) and at iteration $i = 2000$, we replace +1 by −1 in (115) in order to shift the mean of the signals. This will enable us to observe the tracking mechanism by the dynamic-average diffusion strategy.

We plot in Fig. 4 the error measure $\frac{1}{K} \Sigma_{k=1}^{K} ||w_{k,i}(n) - \bar{r}_i(n)||^2$ only for the first entry, i.e., for $n = 1$ for illustration purposes. It is seen that this measure decreases, as expected, and that the network is able to track the new average value after the perturbation at $i = 2000$. The figure shows two curves: one corresponding to synchronous updates where all agents select the same entry of the iterates to communication, and the other corresponding to asynchronous updates where different agents may select randomly different entries. We observe that the curve for synchronous updates has a typical stair-like shape while the one for independent updates does not. The stair-like shape is due to the coordinate-wise updates, which imply that the error would stay constant until the coordinate is selected again.

7. CONCLUSION

In summary, this work derives and analyzes an online learning strategy for tracking the average of time-varying distributed signals by relying on randomized coordinate-descent updates. We proposed two dynamic-average diffusion algorithms: in one case all agents select the same entry from the observations, and in the second case all agents may select different entries from their observations. Auxiliary variables and push-sum ideas are utilized to avoid bias and ensure convergence.
8. REFERENCES

[1] A. H. Sayed, “Adaptation, learning, and optimization over networks,” Foundations and Trends in Machine Learning, vol. 7, no. 4–5, pp. 311–801, 2014.

[2] A. Nedić, A. Olshevsky, and W. Shi, “Achieving geometric convergence for distributed optimization over time-varying graphs,” SIAM Journal on Optimization, vol. 27, no. 4, pp. 2597–2633, 2017.

[3] J. Chen and A. H. Sayed, “Distributed pareto optimization via diffusion strategies,” IEEE Journal of Selected Topics in Signal Processing, vol. 7, no. 2, pp. 205–220, 2013.

[4] S. Kar and J. M. F. Moura, “Distributed consensus algorithms in sensor networks with imperfect communication: Link failures and channel noise,” IEEE Transactions on Signal Processing, vol. 57, no. 1, pp. 355–369, 2009.

[5] W. Shi, Q. Ling, G. Wu, and W. Yin, “A proximal gradient algorithm for decentralized composite optimization,” IEEE Transactions on Signal Processing, vol. 63, no. 22, pp. 6013–6023, 2015.

[6] K. Yuan, B. Ying, X. Zhao, and A. H. Sayed, “Exact diffusion for distributed optimization and learning – Part I: Algorithm development,” to appear in IEEE Trans. Signal Processing. Also available as arXiv:1702.05122, Feb. 2017.

[7] A. Nedić and A. Ozdaglar, “Distributed subgradient methods for multi-agent optimization,” IEEE Transactions on Automatic Control, vol. 54, no. 1, pp. 48–61, 2009.

[8] R. Xin and U. Khan, “A linear algorithm for optimization over directed graphs with geometric convergence,” IEEE Control Systems Letters, vol. 2, no. 3, pp. 325–330, 2018.

[9] J. Chen, Z. J. Towlie, and A. H. Sayed, “Dictionary learning over distributed models,” IEEE Transactions on Signal Processing, vol. 63, no. 4, pp. 1001–1016, 2015.

[10] S. Sundhar, A. Nedić, and V. V. Veeravalli, “A new class of distributed optimization algorithms: Application to regression of distributed data,” Optimization Methods and Software, vol. 27, no. 1, pp. 71–88, 2012.

[11] J. F. Mota, J. M. Xavier, P. M. Aguiar, and M. Puschel, “Distributed basis pursuit,” IEEE Transactions on Signal Processing, vol. 60, no. 4, pp. 1942–1956, 2012.

[12] B. Ying and A. H. Sayed, “Diffusion gradient boosting for networked learning,” in Proc. ICASSP, New Orleans, US, April 2017, pp. 2512–2516.

[13] B. Ying, K. Yuan, and A. H Sayed, “An exponentially convergent algorithm for learning under distributed features,” in IEEE Data Science Workshop, Lausanne, Switzerland, 2018, pp. 185–189.

[14] B. Ying, K. Yuan, and A. H Sayed, “Learning under distributed features,” arXiv preprint arXiv:1805.11384, 2018.

[15] S. Kar and J. M. F. Moura, “Convergence rate analysis of distributed gossip (linear parameter) estimation: Fundamental limits and tradeoffs,” IEEE Journal of Selected Topics in Signal Processing, vol. 5, no. 4, pp. 674–690, 2011.

[16] W. Shi, Q. Ling, G. Wu, and W. Yin, “EXTRA: An exact first-order algorithm for decentralized consensus optimization,” SIAM Journal on Optimization, vol. 25, no. 2, pp. 944–966, 2015.

[17] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, “Randomized gossip algorithms,” IEEE transactions on information theory, vol. 52, no. 6, pp. 2508–2530, 2006.

[18] A. H. Sayed, “Adaptive networks,” Proceedings of the IEEE, vol. 102, no. 4, pp. 460–497, April 2014.

[19] R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge University Press, 1990.

[20] S. U. Pillai, T. Suel, and S. Cha, “The perron-frobenius theorem: Some of its applications,” IEEE Signal Processing Magazine, vol. 22, no. 2, pp. 62–75, 2005.

[21] R. A. Freeman, P. Yang, and K. M. Lynch, “Stability and convergence properties of dynamic average consensus estimators,” in Proc. IEEE CDC, San Diego, CA, 2006, pp. 338–343.

[22] M. Zhu and S. Martinez, “Discrete-time dynamic average consensus,” Automatica, vol. 46, no. 2, pp. 322–329, 2010.

[23] Y. Cao, W. Yu, W. Ren, and G. Chen, “An overview of recent progress in the study of distributed multi-agent coordination,” IEEE Transactions on Industrial Informatics, vol. 9, no. 1, pp. 427–438, 2013.

[24] P. Tseng, “Convergence of a block coordinate descent method for nondifferentiable minimization,” Journal of optimization theory and applications, vol. 109, no. 3, pp. 475–494, 2001.

[25] Z.-Q. Luo and P. Tseng, “On the convergence of the coordinate descent method for convex differentiable minimization,” Journal of Optimization Theory and Applications, vol. 72, no. 1, pp. 7–35, 1992.

[26] R. Arablouei, S. Werner, Y.-F. Huang, and K. Dogancay, “Distributed least mean-square estimation with partial diffusion,” IEEE Transactions on Signal Processing, vol. 62, no. 2, pp. 472–484, 2014.

[27] A. Defazio, F. Bach, and S. Lacoste-Julien, “SAGA: A fast incremental gradient method with support for non-strongly convex composite objectives,” in Proc. Advances in Neural Information Processing Systems (NIPS), Montreal, Canada, 2014, pp. 1646–1654.

[28] C. Wang, Y. Zhang, B. Ying, and A. H. Sayed, “Coordinate-descent diffusion learning by networked agents,” IEEE Transactions on Signal Processing, vol. 66, no. 2, pp. 352–367, 2016.

[29] A. Nedić and A. Olshevsky, “Distributed optimization over time-varying directed graphs,” IEEE Transactions on Automatic Control, vol. 60, no. 3, pp. 601–615, 2015.

[30] F. Bénetzit, V. Blondel, P. Thiran, J. Tsitsiklis, and M. Vertesi, “Weighted gossip: Distributed averaging using non-doubly stochastic matrices,” in Proc. International Symposium on Information Theory Proceedings, Austin, Texas, 2010, pp. 1753–1757.

[31] K. Yuan, B. Ying, X. Zhao, and A. H. Sayed, “Exact diffusion for distributed optimization and learning – Part II: Convergence analysis,” to appear in IEEE Trans. Signal Processing. Also available as arXiv:1702.05122, Feb. 2017.
[32] K. Yuan, B. Ying, X. Zhao, and A. H. Sayed, “Exact diffusion strategy for optimization by networked agents,” in Proc. EUSIPCO, Kos Island, Greece, Aug. 2017, pp. 141–145.

[33] B. Ying and A. H. Sayed, “Performance limits of stochastic sub-gradient learning, part ii: Multi-agent case,” Signal Processing, vol. 144, pp. 253–264, 2018.

[34] S. Vlaski, L. Vandenberghe, and A. H. Sayed, “Diffusion stochastic optimization with non-smooth regularizers,” in Proc. ICASSP, Shanghai, China, Mar. 2016, pp. 4149–4153.

[35] S. Pu, W. Shi, J. Xu, and A. Nedic, “A push-pull gradient method for distributed optimization in networks,” available at arXiv:1803.07588, 2018.

[36] A. Tahbaz-Salehi and A. Jadbabaie, “A necessary and sufficient condition for consensus over random networks,” IEEE Transactions on Automatic Control, vol. 53, no. 3, pp. 791–795, 2008.

[37] P. Brémaud, Markov Chains: Gibbs Fields, Monte Carlo Simulation, and Queues, Springer, 2013.

[38] A. N. Kolmogorov and S. Fomin, Elements of the Theory of Functions and Functional Analysis, Courier Corporation, 1999.

[39] S. Boyd and L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.