Functions of Pairs of Unbounded Noncommuting Self-Adjoint Operators under Perturbation

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Abstract—For a pair $(A, B)$ of not necessarily bounded and not necessarily commuting self-adjoint operators and for a function $f$ on the Euclidean space $\mathbb{R}^2$ that belongs to the inhomogeneous Besov class $B^{p,1}_{p,1}(\mathbb{R}^2)$, we define the function $f(A, B)$ of these operators as a densely defined operator. We consider the problem of estimating the functions $f(A, B)$ under perturbations of the pair $(A, B)$. It is established that if $A_1 - A_2$ and $B_1 - B_2$ belong to the Schatten–von Neumann class $S_p$ with $p \in [1, 2]$ and $f \in B^{p,1}_{p,1}(\mathbb{R}^2)$, then the following Lipschitz type estimate holds: $\|f(A_1, B_1) - f(A_2, B_2)\|_{S_p} \leq \|f\|_{B^{p,1}_{p,1}} \max\{\|A_1 - A_2\|_{S_p}, \|B_1 - B_2\|_{S_p}\}$.

Keywords: unbounded self-adjoint operators, Schatten–von Neumann classes, Besov classes, double operator integrals, triple operator integrals, Haagerup tensor products, functions of pairs of noncommuting self-adjoint operators

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1. INTRODUCTION

The results of this note extend the results of [1] to the case of pairs of unbounded noncommuting self-adjoint operators. Recall (see, e.g., [1]), that for a pair $(A, B)$ of not necessarily bounded self-adjoint operators and for a complex-valued function $f$ on $\mathbb{R} \times \mathbb{R}$, being a Schur multiplier with respect to arbitrary Borel spectral measures, the function $f(A, B)$ of $A$ and $B$ is defined as the double operator integral

$$f(A, B) \overset{\text{def}}{=} \iint_{\mathbb{R} \times \mathbb{R}} f(x, y) dE_A(x) dE_B(y)$$

$$= \iint_{\mathbb{R} \times \mathbb{R}} f(x, y) dE_A(x) I dE_B(y),$$

where $I$ is the identity operator while $E_A$ and $E_B$ are spectral measures of the operators $A$ and $B$. Then $f(A, B)$ is a bounded operator. We refer the reader to [5–7] for the definition and basic properties of double operator integrals.

We also refer the reader to [2] and [10] for the definition of Schur multipliers with respect to spectral measures. Recall (see [2, 10]) that a function $\Phi$ is a Schur multiplier with respect to spectral measures $E_1$ and $E_2$ if and only if $\Phi$ belongs to the Haagerup tensor product $L^\infty(E_1) \otimes_h L^\infty(E_2)$, i.e., $\Phi$ admits a representation of the form

$$\Phi(x, y) = \sum_{n} \varphi_n(x) \psi_n(y),$$

where $\varphi_n, \psi_n \in L^\infty(E_1), L^\infty(E_2)$ and

$$\left( \sum_{n} |\varphi_n(x)|^2 \right)^{1/2} \left( \sum_{n} |\psi_n(y)|^2 \right)^{1/2} < \infty.$$
The norm of $\Phi$ in $L^p(E_1) \otimes_h L^q(E_2)$ is the infimum of the left-hand side of (1.3) over all representations of the form (1.2). In this case
$$\int \int \Phi(x,y) dE_1(x)Q dE_2(y) = \sum_n (\int \Phi_n dE_1) Q(\int \Psi_n dE_2);$$
the series on the right converges in the weak operator topology and
$$\left\| \int \int \Phi dE_1 Q dE_2 \right\| \leq \|\Phi\|_{L^p(E_1) \otimes_h L^q(E_2)} \|Q\|$$
(see, e.g., [2]).

In this note we define functions $f(A,B)$ of unbounded noncommuting operators for certain functions $f$ that do not belong to the Haagerup tensor product of the spaces of bounded functions. In this case $f(A,B)$ turns out to be a densely defined unbounded operator.

In [1] for pairs $(A_1, B_1)$ and $(A_2, B_2)$ of noncommuting bounded self-adjoint operators $A$ and $B$ and for functions $f$ in the homogeneous Besov class $B_{\infty,1}(\mathbb{R}^2)$ the operators $f(A_1, B_1)$ and $f(A_2, B_2)$ were defined and the following Lipschitz type estimate in the Schatten–von Neumann classes $S_p$ with $1 \leq p \leq 2$ was given:
$$\|f(A_1, B_1) - f(A_2, B_2)\|_{S_p} \leq \text{const} \|f\|_{B_{\infty,1}} \times \max\{\|A_1 - B_1\|_{S_p}, \|A_2 - B_2\|_{S_p}\}.$$  

In the same paper [1] it was shown that the same inequality is false for $p > 2$ and is false in the operator norm.

Recall also that in the case of functions of a single self-adjoint operator, such Lipschitz type estimates hold true for $1 \leq p \leq \infty$, see [10, 11].

The main purpose of this note is to establish this inequality for pairs of unbounded noncommuting self-adjoint operators and for functions $f$ in the homogeneous Besov class $B_{\infty,1}(\mathbb{R}^2)$. We refer the reader to [9] for the definition and basic properties of Besov spaces.

As in the case of bounded noncommuting operators, the key role is played by triple operator integrals. We refer the reader to [1, 3] for triple integral operators.

2. TRIPLE OPERATOR INTEGRALS, HAAGERUP AND HAAGERUP-LIKE TENSOR PRODUCTS

Triple operator integrals are expressions of the form
$$\int \int \int \Psi(x_1, x_2, x_3) dE_1(x_1)T dE_2(x_2) R dE_3(x_3), \quad (2.1)$$
where $\Psi$ is a bounded measurable function on $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$; $E_1$, $E_2$, and $E_3$ are spectral measures on Hilbert space while $T$ and $R$ are bounded linear operators. Such operator integrals can be defined under certain assumptions on $\Psi$, $T$, and $R$.

In [12] integrals of the form (2.1) are defined for arbitrary bounded operators $T$ and $R$ and for functions $\Psi$ in the integral projective tensor product $L^p(E_1) \otimes_h L^q(E_2) \otimes_h L^r(E_3)$. In this case the following inequality holds:
$$\left\| \int \int \int \Psi dE_1 T dE_2 R dE_3 \right\|_{S_p} \leq \left\| \Psi \right\|_{L^p(E_1) \otimes_h L^q(E_2) \otimes_h L^r(E_3)} \left\| T \right\|_{S_p} \left\| R \right\|_{S_q}, \quad T \in S_p, \quad R \in S_q,$$
$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$
under the assumption $\frac{1}{p} + \frac{1}{q} \leq 1$.

Later in [8] triple operator integrals were defined for functions $\Psi$ in the Haagerup tensor product $L^p(E_1) \otimes_h L^q(E_2) \otimes_h L^r(E_3)$. We refer the reader to [3] for the definition and basic properties of such Haagerup tensor products. Note here that for functions $\Psi$ in $L^p(E_1) \otimes_h L^q(E_2) \otimes_h L^r(E_3)$, the following estimates hold:
$$\left\| \int \int \int \Psi dE_1 T dE_2 R dE_3 \right\|_{S_p} \leq \left\| \Psi \right\|_{L^p(E_1) \otimes_h L^q(E_2) \otimes_h L^r(E_3)} \left\| T \right\|_{S_p} \left\| R \right\|_{S_q};$$
in the case of bounded operators $T$ and $R$, and
$$\left\| \int \int \int \Psi dE_1 T dE_2 R dE_3 \right\|_{S_p} \leq \left\| \Psi \right\|_{L^p(E_1) \otimes_h L^q(E_2) \otimes_h L^r(E_3)} \left\| T \right\|_{S_p} \left\| R \right\|_{S_q},$$
in the case when $T \in S_p$, $R \in S_q$, $1/r = 1/p + 1/q$, and $p, q \in [2, \infty]$.

However, it turned out that for Lipschitz type estimates for functions of pairs of noncommuting operators, we need triple operator integrals with integrands in so-called Haagerup-like tensor products of the first kind $L^p(E_1) \otimes_h L^q(E_2) \otimes_h L^r(E_3)$ and the second kind $L^p(E_1) \otimes_h L^q(E_2) \otimes_h L^r(E_3)$. Such tensor products were introduced in [1] and were studied in more detail in [3].

In [1, 3] it was shown that if $\Psi \in L^p(E_1) \otimes_h L^q(E_2) \otimes_h L^r(E_3)$, $1 \leq p \leq 2$, $T \in S_p$, and $R$ is a bounded linear operator, one can define the triple operator integral of the form (2.1); moreover, the following estimate holds:
$$\left\| \int \int \int \Psi dE_1 T dE_2 R dE_3 \right\|_{S_p} \leq \left\| \Psi \right\|_{L^p(E_1) \otimes_h L^q(E_2) \otimes_h L^r(E_3)} \left\| T \right\|_{S_p} \left\| R \right\|_{S_q}, \quad 1 \leq p \leq 2.$$

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On the other hand, if \( \Psi \in L^p(E_1) \otimes_b L^p(E_2) \otimes_h L^p(E_3) \), \( 1 \leq p \leq 2 \), \( T \) is a bounded linear operator and \( R \in S_p \), then
\[
\left\| \int \int \int \Psi dE_T dE_R dE_p \right\|_{S_p} \leq \left\| \Psi \right\|_{L^p(\mathbb{R}^\infty \otimes \mathbb{R}^\infty \otimes \mathbb{R}^\infty)} \| T \| \| R \|_{S_p}.
\]

Moreover,
\[
\left\| \Psi \right\|_{S_p} \leq \left\| \Psi \right\|_{L^p(\mathbb{R}^\infty \otimes \mathbb{R}^\infty \otimes \mathbb{R}^\infty)} \| T \| \| R \|_{S_p}. \tag{2.2}
\]

Note that in [1] more general Schatten–von Neumann estimates for triple operator integrals were obtained in the case when the integrand belongs to Haagerup-like tensor products of \( L^\infty \) spaces. Later in [3] the estimates obtained in [1] were extended to even a more general case.

Note also that in the same way one can define Haagerup-like tensor products \( \mathcal{B}^\infty \otimes_h \mathcal{R}^\infty \otimes_b \mathcal{R}^\infty \) and \( \mathcal{B}^\infty \otimes_b \mathcal{R}^\infty \otimes_h \mathcal{R}^\infty \), where \( \mathcal{R}^\infty \) is the space of bounded Borel functions on \( \mathbb{R} \).

Consider now a continuously differentiable function \( f \) on \( \mathbb{R}^2 \) and define the divided differences \( \Sigma^{[1]} f \) and \( \Sigma^{[2]} f \) by
\[
(\Sigma^{[1]} f)(x_1, x_2, y) \overset{\text{def}}{=} \frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2}, \quad x_1 \neq x_2
\]
and
\[
(\Sigma^{[2]} f)(x, y_1, y_2) \overset{\text{def}}{=} \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2}, \quad y_1 \neq y_2.
\]

In the case when \( x_1 = x_2 \) or \( y_1 = y_2 \), in the definition of \( \Sigma^{[1]} f \) and \( \Sigma^{[2]} f \) one has to replace divided differences with the corresponding partial derivatives.

We define the class \( \mathcal{E}_\sigma^\infty(\mathbb{R}^2) \) for \( \sigma > 0 \) as follows:
\[
\mathcal{E}_\sigma^\infty(\mathbb{R}^2) = \{ f \in L^\infty(\mathbb{R}^2) : \supp \Psi f \subset \{(x, y) \in \mathbb{R}^2 : \| f \|_2 \leq \sigma \} \};
\]
here we use the notation \( \overline{\| \|} \) for Fourier transform.

It was established in [1] that for \( \sigma > 0 \) and \( f \in \mathcal{E}_\sigma^\infty(\mathbb{R}^2) \) the following estimates hold:
\[
\left\| \Sigma^{[1]} f \right\|_{\mathcal{E}_\sigma^\infty(\mathbb{R}^2)} \leq \text{const} \sigma \| f \|_{L^2(\mathbb{R}^\infty)}, \quad \tag{2.3}
\]
\[
\left\| \Sigma^{[2]} f \right\|_{\mathcal{E}_\sigma^\infty(\mathbb{R}^2)} \leq \text{const} \sigma \| f \|_{L^1(\mathbb{R}^\infty)}, \quad \tag{2.4}
\]

It follows that if a function \( f \) belongs to the homogeneous Besov class \( B^1_{\infty,1}(\mathbb{R}^2) \), then \( \Sigma^{[1]} f \in \mathcal{E}_\sigma^\infty \otimes_h \mathcal{R}^\infty \otimes_b \mathcal{R}^\infty \) and \( \Sigma^{[2]} f \in \mathcal{E}_\sigma^\infty \otimes_b \mathcal{R}^\infty \otimes_h \mathcal{R}^\infty \); moreover,
\[
\left\| \Sigma^{[1]} f \right\|_{\mathcal{E}_\sigma^\infty \otimes_h \mathcal{R}^\infty \otimes_b \mathcal{R}^\infty} \leq \text{const} \| f \|_{B^1_{\infty,1}},
\]
\[
\left\| \Sigma^{[2]} f \right\|_{\mathcal{E}_\sigma^\infty \otimes_b \mathcal{R}^\infty \otimes_h \mathcal{R}^\infty} \leq \text{const} \| f \|_{B^1_{\infty,1}}.
\]

The main result of [1] is the following fact.

**Theorem 2.1.** Let \( 1 \leq p \leq 2 \), \( f \in B^1_{\infty,1}(\mathbb{R}^2) \), and \( (A_i, B_i) \) and \( (A_2, B_2) \) are pairs of bounded noncommuting self-adjoint operators such that \( A_1 - A_2 \in S_p \) and \( B_2 - B_1 \in S_p \). Then:
\[
f(A_1, B_1) - f(A_2, B_2) = \left\{ \int \int \int \frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2} dE_A(x_1) \right\}_1 - (A_1 - A_2) dE_A(x_2) dE_B(y) \]
\[
+ \left\{ \int \int \int \frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} dE_A(x) \right\}_1 \times dE_B(y_1)(B_1 - B_2) dE_B(y_2).
\]

Moreover, the following estimate holds:
\[
\| f(A_1, B_1) - f(A_2, B_2) \|_{S_p} \leq \text{const} \| f \|_{B^1_{\infty,1}} \max\{ \| A_1 - A_2 \|_{S_p}, \| B_1 - B_2 \|_{S_p} \}.
\]

The main objective of this note is to establish the same inequality in the case of unbounded noncommuting pairs of operators under the assumption that the function \( f \) belongs to the inhomogeneous Besov class \( B^1_{\infty,1}(\mathbb{R}^2) \).

### 3. FUNCTIONS OF PAIRS OF UNBOUNDED NONCOMMUTING SELF-ADJOINT OPERATORS

Recall that we have defined functions of not necessarily commuting self-adjoint operators by (1.1) in the case when the function \( f \) belongs to the Haagerup tensor product \( \mathcal{B}^\infty \otimes_h \mathcal{R}^\infty \). Moreover, the following estimate holds:
\[
\| f(A, B) - f(A, B) \|_{S_p} \leq \| f \|_{\mathcal{E}_\sigma^\infty \otimes_h \mathcal{R}^\infty} \quad f \in \mathcal{E}_\sigma^\infty \otimes_h \mathcal{R}^\infty.
\]

Let \( f \) be a function of two variables and let \( f_i \) be the function defined by the equality \( f_i(s, t) = (1 - it)^{-1} f(s, t) \).

Suppose that \( f_i \in \mathcal{B}^\infty \otimes_h \mathcal{R}^\infty \). We define the operator \( f(A, B) \) by
\[
f(A, B) \overset{\text{def}}{=} f_i(A, B)(I - iB)
\]
\[
= \left( \int \int f_i(s, t) dE_A(s) dE_B(t) \right)(I - iB).
\]

Then \( f(A, B) \) is a densely defined operator whose domain coincides with the domain \( D(B) \) of \( B \). It does not have to be bounded but the operator \( f(A, B)(I - iB)^{-1} \) is bounded.

Note that if \( f \in \mathcal{E}_\sigma^\infty \otimes_h \mathcal{R}^\infty \), \( \sigma > 0 \), then \( f_i \in \mathcal{B}^\infty \otimes_h \mathcal{R}^\infty \). This was established in Corollary 7.3 of [4] for func-
4. INTEGRAL FORMULAE FOR OPERATOR DIFFERENCES AND LIPSCHITZ TYPE ESTIMATES

In this section we state the main result of the note. We obtain a formula for the operator difference in terms of triple operator integrals and we establish a Lipschitz type estimate in the $S_p$ norm for $p \in [1, 2]$. We are going to deal with not necessarily bounded and not necessarily commuting self-adjoint operators.

**Theorem 4.1.** Let $f \in \mathcal{E}_o^\sigma(\mathbb{R}^3)$, and $A_1$, $A_2$, and $B$ are self-adjoint operators such that $A_1 - A_2 \in S_2$. Then

$$f(A_1, B) - f(A_2, B) = \int \int \int \frac{f(x_1, y) - f(x_2, y)}{x_1 - x_2} dE_A(x_1) \times (A_1 - A_2) dE_A(x_2) dE_B(y)$$

and so

$$\|f(A_1, B) - f(A_2, B)\|_{S_p} \leq \text{const} \|f\|_{L_\infty(\mathbb{R}^3)} \|A_1 - A_2\|_{S_p}.$$ 

Recall that $\mathcal{E}^{[1]} f \in \mathcal{B}^\sigma \otimes_h \mathcal{B}^\sigma \otimes_h \mathcal{B}^\sigma$ (see (2.3)), and so the triple operator integral on the right is defined.

**Corollary 4.2.** Let $f \in B_{1,p}^\sigma(\mathbb{R}^3)$ and $1 \leq p \leq 2$. Suppose that $A_1$, $A_2$, and $B$ are self-adjoint operators such that $A_1 - A_2 \in S_p$. Then the following inequality holds:

$$\|f(A_1, B) - f(A_2, B)\|_{S_p} \leq \text{const} \|f\|_{L_\infty(\mathbb{R}^3)} \|A_1 - A_2\|_{S_p}.$$ 

**Theorem 4.3.** Let $f \in \mathcal{E}_o^\sigma(\mathbb{R}^3)$. Suppose that $A_1$, $B_1$, and $B_2$ are self-adjoint operators such that $B_2 - B_1 \in S_2$. Then the following equality holds:

$$f(A_1, B_1) - f(A_2, B_2) = \int \int \int \frac{f(x_1, y_1) - f(x_2, y_2)}{y_1 - y_2} dE_A(x) \times dE_B(y_1)(B_1 - B_2) dE_B(y_2).$$

Again, $\mathcal{S}^{[2]} f \in \mathcal{B}^\sigma \otimes_h \mathcal{B}^\sigma \otimes_h \mathcal{B}^\sigma$ (see (2.4)), and so the triple operator integral on the right is defined.

**Corollary 4.4.** Let $f \in \mathcal{E}_o^\sigma(\mathbb{R}^3)$ with $p \in [1, 2]$. Suppose that $A$, $B_1$, and $B_2$ are self-adjoint operators such that $B_2 - B_1 \in S_p$. Then

$$\|f(A, B_1) - f(A, B_2)\|_{S_p} \leq \text{const} \|f\|_{L_\infty(\mathbb{R}^3)} \|B_1 - B_2\|_{S_p}.$$ 

**Theorem 4.5.** Let $f \in B_{1,p}^\sigma(\mathbb{R}^3)$ with $p \in [1, 2]$. Suppose that $A_1$, $A_2$, $B_1$, and $B_2$ are self-adjoint operators such that $A_2 - A_1 \in S_p$ and $B_2 - B_1 \in S_p$. Then

$$\|f(A_1, B_1) - f(A_2, B_2)\|_{S_p} \leq \text{const} \|f\|_{L_\infty(\mathbb{R}^3)} \max \{\|A_1 - A_2\|_{S_p}, \|B_1 - B_2\|_{S_p}\}.$$ 

**Theorem 4.6.** Let $f \in B_{1,p}^\sigma(\mathbb{R}^3)$. Suppose that $A_1$, $A_2$, $B_1$, and $B_2$ are self-adjoint operators such that $A_1 - A_2 \in S_2$ and $B_1 - B_2 \in S_2$. Then the following identity holds:

$$f(A_1, B_1) - f(A_2, B_2) = \int \int \int \frac{f(x_1, y_1) - f(x_2, y_2)}{y_1 - y_2} dE_A(x) \times (A_1 - A_2) dE_A(x_2) dE_B(y) \times dE_B(y_1)(B_1 - B_2) dE_B(y_2).$$

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**CONFLICT OF INTEREST**

The authors declare that they have no conflicts of interest.

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