The Running Coupling from Lattice Gauge Theory

C. Michael
DAMTP, University of Liverpool, Liverpool L69 3BX, U.K.

Abstract

From an accurate determination of the inter-quark potential, one can study the running coupling constant for a range of $R$-values and hence estimate the scale $\Lambda_{\overline{\text{MS}}}$. Detailed results are presented for $SU(2)$ pure gauge theory to illustrate the method.

1 Introduction

In the continuum the potential between static quarks is known perturbatively to two loops in terms of the scale $\Lambda_{\overline{\text{MS}}}$. For $SU(2)$ colour, the continuum force is given by

$$\frac{dV}{dR} = \frac{3}{4} \frac{\alpha(R)}{R^2}$$

with the effective coupling $\alpha(R)$ defined as

$$\alpha(R) = \frac{1}{4\pi \left[ b_0 \log(R\Lambda_R)^{-2} + \frac{b_1}{b_0} \log \log(R\Lambda_R)^{-2} \right]}$$

where $b_0 = 11/24\pi^2$ and $b_1 = 102 b_0^2/121$ are the usual coefficients in the perturbative expression for the $\beta$-function and, neglecting quark loops in the vacuum, $\Lambda_R = 2.055\Lambda_{\overline{\text{MS}}}$. Note that the usual lattice regularisation scale $\Lambda_L = 0.05045\Lambda_{\overline{\text{MS}}}$.

At large separation $R$, the potential behaves as $KR$ where $K$ is the string tension. Thus in principle knowledge of the potential $V(R)$ serves to determine the dimensionless ratio $\sqrt{K}/\Lambda$ which relates the perturbative scale $\Lambda$ to a non-perturbative observable such as the string tension $K$. This is the basis of the method we shall employ here.

It is worth recalling the lattice method that has been used previously: namely determining the dimensionless string tension $Ka^2$ directly from the large-$R$ potential and then using the perturbative relationship between $a$ and $\beta$ to find
\( \sqrt{K}/\Lambda \). This method requires that \( a(\beta) \) is given by the two-loop perturbative beta-function: a condition known as asymptotic scaling in the lattice gauge theory realm. Ample evidence exists that this condition is not satisfied at present \( \beta \)-values (up to \( \beta = 2.85 \) for \( SU(2) \) pure gauge theory for example the beta-function is only 82\% of the perturbative value \[2\]).

However, though asymptotic scaling is not yet manifest, the weaker scaling requirement is well satisfied. Thus the dimensionless ratios of physical quantities are found to be independent of \( \beta \). For example the potential \( V(R) \) scales \[2\] over a range of lattice spacing of a factor of 4 (from \( \beta = 2.4 \) to 2.85). That scaling but not asymptotic scaling is valid, implies that the bare coupling constant derived from \( \beta \) is inappropriate and that an effective coupling constant derived from some physical quantity is a better choice. This has been emphasized by Lepage and Mackenzie \[3\]. It is also the basis of the method proposed by Lüscher et al \[4\] to extract the running coupling constant. Here we use lattice simulation to determine the interquark potential between static quarks and so obtain the running coupling constant at small distance \( R \). One subtlety is that we require small \( R \) and hence large energy \( 1/R \) to make most precise contact with the perturbative expression but the lattice method implies the presence of lattice artefacts when \( R \approx a \). Our main concern will be to show how to cope with these lattice artefacts and the method will be to study potentials off-axis on the lattice as well as on-axis.

### 2 Lattice potentials

To explore the interquark potential precisely, we use as large a lattice as feasible and as large a value of \( \beta \) consistent with remaining in the large-volume vacuum. Previous work \[5\] has shown that \( \beta = 2.7 \) and a 32\(^4\) lattice is suitable. We use rather similar methods to those used previously to determine the string tension \[5\] but here we concentrate our attention on small \( R \). At small \( R \), the statistical errors will be rather small and the main uncertainty will be the systematic effects coming from lattice artefacts. Thus we measure the potentials at on-axis separations with \( R/a = 1, 2, 3, 4, 6, 8, 10, 12, 14, 16 \) and off-axis separations with \( R/a \) vectors \((1,1,0), (2,1,0), (2,2,0), (3,1,0), (3,2,0) \) and \((3,3,0)\).

The method used to extract the potentials is to use spatially smeared links (APE smearing \[6\] with \( SU(2) \) projection of 2.5 \( \times \) straight link plus four spatial U-bends) to build up paths between the static sources. Recursive smearing with 30 and 60 levels is used, so providing two paths which gives a 2 \( \times \) 2 correlation matrix that can be used in the standard variational approach. For the on-axis separation we use straight paths, while for the off-axis separations we sum over two L-shaped paths. We measure potentials for the \( R \)-separations above and \( T \)-separations 0 to 5. The lattice is well equilibrated from previous work and we measure 40 blocks of 6 configurations with each block containing 150 update
The force $\Delta V/\Delta R$ and lattice artefact corrected force $\Delta V_c/\Delta R$ at average separation $R$. The running coupling $\alpha(R)$ derived from the corrected force is shown as well. The second error shown on $\alpha$ is 10% of the lattice artefact correction.

Table 1: The force $\Delta V/\Delta R$ and lattice artefact corrected force $\Delta V_c/\Delta R$ at average separation $R$. The running coupling $\alpha(R)$ derived from the corrected force is shown as well. The second error shown on $\alpha$ is 10% of the lattice artefact correction.

| $R/a$ | $\Delta V/\Delta R$ | $\Delta V_c/\Delta R$ | $\alpha(R)$ |
|-------|----------------------|------------------------|-------------|
| 1.2071 | 0.1745(2)            | 0.1339                 | 0.2525(4)(76) |
| 1.7071 | 0.0640(2)            | 0.0345                 | 0.3909(153)(161) |
| 2.1180 | 0.0829(6)            | 0.0570                 | 0.3774(28)(13) |
| 2.5322 | 0.0463(3)            | 0.0448                 | 0.3009(8)(60) |
| 2.9142 | 0.0203(14)           | 0.0347                 | 0.4384(132)(114) |
| 3.0811 | 0.0437(10)           | 0.0304                 | 0.4616(80)(27) |
| 3.3839 | 0.0321(5)            | 0.0340                 | 0.4913(144)(42) |
| 3.8028 | 0.0234(7)            | 0.0255                 | 0.5740(242)(90) |
| 4.1213 | 0.0294(11)           | 0.0204                 | 0.6526(57)(10) |
| 5.0000 | 0.0207(2)            | 0.0153                 | 0.9797(121)(4) |
| 7.0000 | 0.0154(2)            | 0.0132                 | 1.4128(215)(2) |
| 9.0000 | 0.0132(2)            | 0.0128                 | 2.0408(571)(1) |
| 11.0000| 0.0128(4)            | 0.0117                 | 2.6124(680)(1) |
| 13.0000| 0.0117(3)            | 0.0112                 | 3.3546(973)(1) |

The potentials are given by the extrapolation in $T$ of the ratio of generalised Wilson loops.

$$V(R) = \lim_{T \to \infty} V(R, T),$$

where

$$V(R, T) = -\log W(R, T)/W(R, T - a)$$

This is a monotonic decrease with $T$ and the rate of decrease can be estimated from the energy gap between the ground state (which is what we wish to determine) and the first excited state. We obtain estimates of this energy gap from our variational method in the 2:1 $T$-ratio basis and these estimates agree with previous work [5]. We then use those estimates to complete the $T$-extrapolation. We find consistency between such extrapolations based on $T$-values 2-4 and on $T$-values 3-5 which confirms the stability of the method. Error analyses use bootstrap from our 40 samples which we find to be consistent with being statistically independent. These results for the potentials agree within 1 $\sigma$ with those of ref. [5] at common $R$-values. The force derived from our potential measurements is shown in table 1.

The potential shows a lack of rotational invariance at small $R$. To lowest
order this can be attributed to the difference $\delta G(R)$ between the lattice one gluon exchange expression and the continuum expression.

$$\delta G(R) = \frac{4\pi}{a} \int_{-\pi}^{\pi} \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot R/a}}{4 \sum_{i=1}^{3} \sin^2(k_i/2)} - \frac{1}{R}$$

On a lattice, the next order of perturbation has been calculated [8] and the dominant effect is a change from the bare coupling to an effective coupling [3]. In that case, using the difference above but with an adjustable strength will correct for the small $R/a$ lack of rotational invariance. A test of this will be that a smooth interpolation of $V(R)$ versus $R$ is obtained with this one free parameter to the 6 off-axis potential values.

We evaluate $\delta G(R)$ numerically using the limit of a very large lattice since we are not here concerned with long-range effects. Then we find the following empirical expression provides a good fit for $R > a$,

$$aV(R) = C - \frac{A}{R} + \frac{B}{R^2} + KR - Af\delta G(R),$$

with $\chi^2$ per degree of freedom 0.8 using the full correlation matrix. The fit parameters are shown in table 2. For our present purposes the detailed form of this fit at small $R$ is not relevant - what is needed is a confirmation that a good fit can be obtained. This then supports our prescription to correct the lattice artefacts responsible for the lack of rotational invariance. What is more difficult is to assign errors to this correction procedure. Since the correction coefficient $f$ is determined to a few per cent, the statistical errors are small. The fact that one parameter corrects 6 off-axis points simultaneously is very encouraging. The only way to be certain that lattice artefacts are eliminated is by the comparison of different $\beta$ values (with thus different $R/a$ values at the same physical $R$ value) and we shall see later that this test is satisfied. This leads us to use as an illustration a 10% systematic error to the correction itself (3 times the statistical error) with the proviso that for the lowest $R$ value ($R = a$) the smooth interpolation is less of a constraint so that we disregard that datum in the analysis. We then assume that an improved estimate for the continuum potential $V_c$ will be obtained by correcting the measured lattice values $V$ by $\delta G$ with the fitted coefficient. These values are shown in table 1.

| $A$       | $f$    | $Ka^2$ | $B/a$     |
|-----------|--------|--------|-----------|
| 0.261(5)  | 0.68(3)| 0.0103(2) | 0.054(5)  |

Table 2: Fit to force for $R > a$. 
Figure 1: The effective running coupling constant $\alpha(R)$ obtained from the force between static quarks at separation $R$. The scale is set by the string tension $K$. The dotted error bars represent an estimate of the systematic error due to lattice artefact correction as described in the text. The curves are the two-loop perturbative expression.

It is now straightforward to extract the running coupling constant by using

$$\alpha\left(\frac{R_1 + R_2}{2}\right) = \frac{4}{3} R_1 R_2 \frac{V_c(R_1) - V_c(R_2)}{R_1 - R_2}$$

where the error in using a finite difference is here negligible. This is shown in table 1 and is plotted in the figure versus $R/\sqrt{K}$ where $K$ is taken from the fit - see Table 2. The interpretation of $\alpha$ as defined above as an effective running coupling constant is only justified at small $R$ where the perturbative expression dominates. Also shown are the two-loop perturbative results for $\alpha(R)$ for different values of $\Lambda_L$.

The figure clearly shows a running coupling constant. Moreover the result is consistent with the expected perturbative dependence on $R$ at small $R$. There are systematic errors, however. At larger $R$, the perturbative two-loop expression will
not be an accurate estimate of the measured potentials, while at smaller $R$, the lattice artefact corrections are relatively big. Setting the scale using $\sqrt{K} = 0.44$ GeV implies $1/a(2.7) = 4.34$ GeV, so $R < 4a(2.7)$ corresponds to values of $1/R > 1$ GeV. This $R$-region is expected to be adequately described by perturbation theory. Another indication that perturbation theory is accurate at such $R$-values is that $\Delta V_c/\Delta R$ at small $R$ is found to be very much greater than the non-perturbative value $K$ at large $R$.

The best way to gain confidence that these systematic errors are under control, is to repeat the method at another $\beta$-value. The UKQCD data \cite{2} at $\beta = 2.85$ for the potential at $R/a(2.85) = 2, 4$ and 6 are processed in the same way to yield $\alpha(R)$ at $R/a(2.85) = 3$ and 5. Since only the on-axis values were measured, we fix $fA$ at the value found in the above fit at $\beta = 2.7$ for the lattice artefact correction. Since we have no cross-check from off-axis potentials, we assign a larger systematic error (30\%) to this lattice artefact correction. The results for the effective running coupling are shown in the figure and are seen to confirm nicely the result from $\beta = 2.7$.

The easiest way to describe the value of the running coupling constant $\alpha$ is in terms of a scale or $\Lambda$ value with the understanding that we are only determining $\alpha$ for a range of energy scales $1/R$ - namely 1 to 3 GeV. The final estimate of $\Lambda$ is made from the figure, weighting smaller $R$ more heavily since the perturbative expression is more accurate as $\alpha(R)$ becomes smaller. We exclude the lowest $R$ point since the lattice artefact correction for $R = a$ is untested. Remembering that the systematic errors due to lattice artefact correction are estimates only and since these systematic errors are dominant, we do not attempt a fit but we can conclude that our results are consistent with values of $\Lambda$ lying in the range shown by the two curves plotted. From the data at $\beta = 2.7$, these curves have $a(2.7)\Lambda_R = 0.0619$ and 0.0688. Using the value of the string tension from the fit, we get $\sqrt{K}/\Lambda_L = 31.9(1.7)$. Moreover, this value is consistent with the evaluation at both $\beta = 2.7$ and 2.85.

3 Conclusions

Using the bare coupling $g$ derived from $\beta = 4/g^2$ and the two-loop perturbative relationship $a(g)$ in terms of the scale $\Lambda_L$ gave \cite{3,4} the following slowly decreasing values of $\sqrt{K}/\Lambda_L = 53.3(3)$, 49.1(4) and 44.1(6) at $\beta = 2.5$, 2.7 and 2.85 respectively. Clearly, the $\beta \rightarrow \infty$ limit lies below these values. Our present method which does not rely on the bare coupling, gives the scaling result which should be independent of $\beta$. Our estimate is $\sqrt{K}/\Lambda_L = 31.9(1.7)$. This is sufficiently far below the values extracted from the bare coupling to imply that asymptotic scaling to two-loop perturbation theory is not “just around the corner” but will only be satisfied accurately at larger $\beta$-values than those currently accessible to lattice simulation.
Our method corresponds to an estimate of the continuum ratio $\sqrt{K}/\Lambda_{\overline{\text{MS}}} = 1.61(9)$ for pure SU(2) theory. Setting the scale using $\sqrt{K} = 0.44 \text{ GeV}$, then gives $\Lambda_{\overline{\text{MS}}} = 273(15) \text{ MeV}$. These results are obtained for rather modest energies ($1/R \approx 1 − 3 \text{ GeV}$) and it is important to extend the lattice methods to higher energies too. From lattice results for ratios of other non-perturbative quantities (glueball masses, critical temperature, etc) to the string tension, one can then determine their value in terms of $\Lambda_{\overline{\text{MS}}}$ as well.

As well as the case of $SU(2)$, we can apply the same method directly to $SU(3)$. Using published data [7] at $\beta = 6.2$ gives an estimate of $\Lambda_{\overline{\text{MS}}} = 250 − 300$ MeV. In order to improve on this determination, it will be necessary to study the small-$R$ on- and off-axis potential accurately at larger $\beta$ for $SU(3)$. This method will then determine the running coupling accurately in terms of any other physical quantity measurable on the lattice (such as the string tension $K$). This lattice method gives an accuracy which is competitive with that of experimental determinations of the running coupling for modest energy scales. The difference, however, is that these lattice methods are feasible for pure gauge simulation but we have yet to achieve similar results for full QCD.

I wish to acknowledge the suggestion of Rainer Sommer who emphasized to me the feasibility of extracting the running coupling from the small-$R$ potentials.

References

[1] A. Billoire, Phys. Lett. 104B (1981) 472.

[2] UKQCD Collaboration, S. P. Booth et al., Phys. Lett. B (in press).

[3] P. Lepage and P. Mackenzie, Nucl. Phys. B (Proc. Suppl.) 20 (1991) 173.

[4] M. Lüscher, P. Weisz and U. Wolff, Nucl. Phys. B359 (1991) 221.

[5] S. Perantonis and C. Michael, Nucl. Phys. B (Proc. Suppl.) 20 (1991) 177; and Liverpool preprint LTH 262.

[6] M. Albanese et al., Phys. Lett. 192B (1987) 163.

[7] C. Michael and M. Teper, Nucl. Phys. B 314 (1989) 347; S. Perantonis and C. Michael, Nucl. Phys. B 347 (1990) 854.

[8] U. Heller and F. Karsch, Nucl. Phys. B 251 (1985) 254.