Integrable Structure of $5d$ $\mathcal{N} = 1$ Supersymmetric Yang-Mills and Melting Crystal

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Abstract

We study loop operators of $5d$ $\mathcal{N} = 1$ SYM in $\Omega$ background. For the case of $U(1)$ theory, the generating function of correlation functions of the loop operators reproduces the partition function of melting crystal model with external potential. We argue the common integrable structure of $5d$ $\mathcal{N} = 1$ SYM and melting crystal model.

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1 Introduction

It is shown in [1] that the Seiberg-Witten solutions [2] of $4d$ $\mathcal{N} = 2$ supersymmetric gauge theories emerge through random partition, where Nekrasov’s formulas [1, 3] for these gauge theories are understood as the partition functions of random partition. The integrable structure of random partition is elucidated in [4], and thereby the integrability of correlation functions among single-traced chiral observables is explained. Such an extension of the Seiberg-Witten geometries also becomes attractive to understand $4d$ $\mathcal{N} = 1$ supersymmetric gauge theories by providing a powerful tool [5].

Integrable structure of melting crystal model with external potential is clarified in [6]. Melting crystal model, known as random plane partition has a significant relation with $5d$ $\mathcal{N} = 1$ supersymmetric gauge theories. Nekrasov’s formula for these gauge theories can be retrieved from the partition function of melting crystal model [7], where the model is interpreted as a $q$-deformed random partition. It is argued [6] a relation between loop operators of $5d$ $\mathcal{N} = 1$ supersymmetric Yang-Mills (SYM) and external potentials of the melting crystal model.

We start Section 2 with providing a brief review about $5d$ $\mathcal{N} = 1$ SYM in $\Omega$ background [8]. We introduce loop operators of this theory. Computation of correlation functions among these operators is discussed. Generating function of the correlation functions of $U(1)$ theory reproduces the partition function of the aforementioned melting crystal model. In Section 3 we discuss a common integrable structure of $5d$ $\mathcal{N} = 1$ SYM in $\Omega$ background and melting crystal model for the case of the $U(1)$ theory. In Section 4 we present an extension of the Seiberg-Witten geometry of the $U(1)$ theory by using the loop operators.

2 Loop operators of $5d$ $\mathcal{N} = 1$ SYM in $\Omega$ background

We first consider an ordinary $5d$ $\mathcal{N} = 1$ SYM on $\mathbb{R}^4 \times S^1$. Let $E$ be the $SU(N)$-bundle on $\mathbb{R}^4$ with $c_2(E) = n \geq 0$. A gauge bundle of this theory is the $SU(N)$-bundle $\pi^*E$ on $\mathbb{R}^4 \times S^1$ pulled back from $\mathbb{R}^4$. $\pi$ is the projection from $\mathbb{R}^4 \times S^1$ to $\mathbb{R}^4$. All the fields in the vector multiplet are set to be periodic along $S^1$. The bosonic ingredients are a $5d$ gauge potential $A_M(x,t)dx^M$ and a scalar field $\varphi(x,t)$ taking the value in $su(N)$. These describe a $5d$ Yang-Mills-Higgs system. The gauge potential can be separated into two parts $A_\mu(x,t)dx^\mu$ and $A_t(x,t)dt$, respectively the
components of the $\mathbb{R}^4$- and the $S^1$-directions. Let $A_E$ be the infinite dimensional affine space consisting of all the gauge potentials on $E$. $A_\mu(x,t)dx^\mu$ describes a loop $A(t)$ in $A_E$, where the loop is parametrized by the fifth-dimensional circle. As for $A_t(x,t)$, together with $\varphi(x,t)$, the combination $A_t + i\varphi$ describes a loop $\phi(t)$ in $\Omega^0(\mathbb{R}^4, \text{ad}E \otimes \mathbb{C})$, the space of all the sections of $\text{ad}E \otimes \mathbb{C}$, where $\text{ad}E$ is the adjoint bundle on $\mathbb{R}^4$ with fibre $su(N)$. Taking account of the periodicity, the same argument is also applicable to the gauginos. The vector multiplet thereby describes a loop in the configuration space of the 4d theory. In the case of the Yang-Mills-Higgs system, the loop $A(t)$ gives a family of covariant differentials on $E$ as $d_{A(t)} = d + A(t)$. For the loop $\phi(t)$, since it involves $A_t(x,t)$, it becomes convenient to introduce the differential operator

$$\mathcal{H}(t) \equiv \frac{d}{dt} + \phi(t).$$

(2.1)

2.1 $5d$ $\mathcal{N} = 1$ SYM in $\Omega$ background

Via the standard dimensional reductions, $6d$ $\mathcal{N} = 1$ SYM gives lower dimensional Yang-Mills theories with 8 supercharges, including the above theory. Furthermore, the dimensional reductions in the $\Omega$ background provide powerful tools to understand these theories [8]. The $\Omega$ background is a 6d gravitational background on $\mathbb{R}^4 \times T^2$ described by a metric of the form:

$$ds^2 = \sum_{\mu=1}^{4}(dx^\mu - \sum_{a=5,6} V_\mu^a dx^a)^2 + \sum_{a=5,6}(dx^a)^2,$$

where two vectors $V_5^\mu, V_6^\mu$ generate rotations on two-planes $(x^1, x^2)$ and $(x^3, x^4)$ in $\mathbb{R}^4$. By letting $V_1 = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}$ and $V_2 = x^4 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^4}$, they are respectively the real part and the imaginary part of the combination

$$V_{\epsilon_1, \epsilon_2} \equiv \epsilon_1 V_1 + \epsilon_2 V_2, \quad \epsilon_1, \epsilon_2 \in \mathbb{C}.$$

(2.2)

The above combination is expressed in component as $V_{\epsilon_1, \epsilon_2} = \Omega^{\mu}_{\nu} x^\nu \frac{\partial}{\partial x^\mu}$. To see the dimensional reduction in the $\Omega$-background, we first consider the bosonic part of the $5d$ SYM. The corresponding Yang-Mills-Higgs system is modified from the previous one. However, the system is eventually controlled by replacing $\mathcal{H}(t)$ with

$$\mathcal{H}_{\epsilon_1, \epsilon_2}(t) \equiv \mathcal{H}(t) + \mathcal{K}_{\epsilon_1, \epsilon_2}(t).$$

(2.3)

Here $\mathcal{K}_{\epsilon_1, \epsilon_2}(t)$ is another differential operator of the form [9]

$$\mathcal{K}_{\epsilon_1, \epsilon_2}(t) \equiv V_{\epsilon_1, \epsilon_2}^{\mu} \partial A(t)_\mu + \frac{1}{2} \Omega^{\mu\nu} J_{\mu\nu},$$

(2.4)
where $\mathcal{J}_{\mu\nu}$ denote the $SO(4)$ Lorentz generators of the system. This operator generates a $T^2$-action by taking the commutators with $d_{A(t)}$ and $\mathcal{H}(t)$. For instance, we have

$$[d_{A(t)}, \mathcal{K}_{\epsilon_1, \epsilon_2}(t)] = -i\epsilon_{\epsilon_1, \epsilon_2} F_{A(t)}.$$ (2.5)

The right hand side is precisely the transformation brought about on $A_E$ by the infinitesimal rotation $\delta x^\mu = -V^\mu_{\epsilon_1, \epsilon_2}$.

The supercharges $Q_{\alpha a}$ and $\bar{Q}_{\dot{\alpha} a}$ are realized in a way different from the case of $\epsilon_1 = \epsilon_2 = 0$. Note that we use the $4d$ notation such that $\alpha, \dot{\alpha}$ and $a$ denote the indices of the Lorentz group $SU(2)_L \times SU(2)_R$ and the R-symmetry $SU(2)_I$. By the standard argument, we may interpret the $5d$ SYM as a topological field theory. Actually, by regarding the diagonal $SU(2)$ of $SU(2)_R \times SU(2)_I$ as a new $SU(2)_R$, we can extract a supercharge that behaves as a scalar under the new Lorentz symmetry. We write the scalar supercharge as $Q_{\epsilon_1, \epsilon_2}$. The gaugino acquires a natural interpretation as differential forms, $\eta(x,t), \psi_{\mu}(x,t)$ and $\xi_{\mu\nu}(x,t)$. These give fermionic loops, $\eta(t), \psi(t)$ and $\xi(t)$. The main part of the $Q$-transformation takes the forms

$$Q_{\epsilon_1, \epsilon_2} A(t) = \psi(t), \quad Q_{\epsilon_1, \epsilon_2} \psi(t) = [d_{A(t)}, \mathcal{H}_{\epsilon_1, \epsilon_2}(t)],$$ (2.6)

$$Q_{\epsilon_1, \epsilon_2} \mathcal{H}_{\epsilon_1, \epsilon_2}(t) = 0,$$ (2.7)

where $\psi(t)$ is a fermionic loop in $\Omega^1(\mathbb{R}^4, \text{ad}E)$.

### 2.2 Loop operators and their correlation functions

Taking account of the relation $\phi(x,t) = A_i(x,t) + i\varphi(x,t)$, the following path-ordered integral provides an analogue of a holonomy of the gauge potential.

$$W^{(0)}(x; t_1, t_2) = Pe^{-\int_{t_2}^{t_1} dt \phi(x,t)},$$ (2.8)

where the symbol means the path-ordered integration, more precisely, it is defined by the differential equation

$$\left(\frac{d}{dt_1} + \phi(x, t_1)\right)W^{(0)}(x; t_1, t_2) = 0, \quad \left.W^{(0)}(x; t_2, t_2) = 1.\right.$$ (2.9)

The trace of the holonomy along the circle defines a loop operator as

$$\mathcal{O}^{(0)}(x) = \text{Tr} W^{(0)}(x; R, 0),$$ (2.10)
where $R$ is the circumference of $S^1$. The above operator is an analogue of the Wilson loop along the circle. Unlike the case of $\epsilon_1 = \epsilon_2 = 0$, it is not $Q$-closed except at $x = 0$. To see this, note that the $Q$-transformations (2.6) and (2.7) imply $Q_{\epsilon_1, \epsilon_2} \phi(t) = -\iota_{\nabla_{\epsilon_1, \epsilon_2}} \psi(t)$. By using this, we find

$$Q_{\epsilon_1, \epsilon_2} O^{(0)}(x) = \int_0^R dt_1 \text{Tr} \left\{ W^{(0)}(x; R, t_1) \iota_{\nabla_{\epsilon_1, \epsilon_2}} \psi(x, t_1) W^{(0)}(x; t_1, 0) \right\}.$$  (2.11)

Since the right hand side of the above formula vanishes only at $x = 0$, this means that $O^{(0)}(x)$ becomes $Q$-closed only at $x = 0$.

The above property may be explained in terms of the equivariant de Rham theory. To see this, let us first generalize the path-ordered integral (2.8) by exponentiating the combination $F_A(t) - \psi(t) + \phi(t)$ in place of $\phi(t)$ as

$$W(x; t_1, t_2) = \mathcal{P} e^{-\int_{t_1}^{t_2} dt \left( F_A(t) - \psi(t) + \phi(t) \right)}(x),$$  (2.12)

where the right hand side is given by a differential equation similar to (2.9). This means that $W$ has the components, according to degrees of differential forms on $\mathbb{R}^4$, as $W = W^{(0)} + W^{(1)} + \cdots + W^{(4)}$, where the indices denote the degrees. We generalize the loop operator (2.10) as

$$O(x) = \text{Tr} W(x; R, 0).$$  (2.13)

This also has components as $O = O^{(0)} + O^{(1)} + \cdots + O^{(4)}$. Eq. (2.11) can be now expressed as $Q_{\epsilon_1, \epsilon_2} O^{(0)} = \iota_{\nabla_{\epsilon_1, \epsilon_2}} O^{(1)}$. This is actually the first equation among a series of the equations that $O^{(i)}$ obey. Such equations eventually show up by expanding the identity [10]

$$(d_{\epsilon_1, \epsilon_2} + Q_{\epsilon_1, \epsilon_2}) O(x) = 0,$$  (2.14)

where $d_{\epsilon_1, \epsilon_2} \equiv d - \iota_{\nabla_{\epsilon_1, \epsilon_2}}$ is the $T^2$-equivariant differential on $\mathbb{R}^4$.

We can also consider the loop operators encircling the circle many times. Correspondingly we introduce

$$O_k(x) = \text{Tr} W(x; kR, 0), \quad k = 1, 2, \cdots$$  (2.15)

These satisfy

$$(d_{\epsilon_1, \epsilon_2} + Q_{\epsilon_1, \epsilon_2}) O_k(x) = 0.$$  (2.16)
Let us examine the correlation functions \( \langle \prod_a \int_{\mathbb{R}^4} O_{k_a} \rangle^{\epsilon_1,\epsilon_2} \). Since the integral \( \int_{\mathbb{R}^4} O_k = \int_{\mathbb{R}^4} O_k^{(4)} \) is Q-closed by virtue of the formula (2.16), these can be computed by a supersymmetric quantum mechanics (SQM) which is substantially equivalent to the 5d SYM as the topological field theory. Such a SQM turns to be \( T^2 \)-equivariant SQM on \( \tilde{\mathcal{M}}_n \[3\], where \( \tilde{\mathcal{M}}_n \) is the moduli space of the framed \( n \) instantons. The \( Q \)-transformation (2.6) is converted to the supersymmetry of the quantum mechanics

\[
Q_{\epsilon_1,\epsilon_2} m(t) = \chi(t), \quad Q_{\epsilon_1,\epsilon_2} \chi(t) = \frac{dm(t)}{dt} + \mathcal{V}_{\epsilon_1,\epsilon_2}(m(t)),
\]

where \( \mathcal{V}_{\epsilon_1,\epsilon_2} \) is the Killing vector induced by the variation \( \delta A = \iota_{V_{\epsilon_1,\epsilon_2}} F_A \) on \( \mathcal{A}_E \). The combination \( F_A(t) - \psi(t) + \phi(t) \) can be identified with a loop space analogue of the \( T^2 \)-equivariant curvature \( \mathcal{F}_{\epsilon_1,\epsilon_2} \) of the universal connection, where the universal bundle becomes equivariant by the \( T^2 \)-action on \( \mathcal{A}_E \times \mathbb{R}^4 \).

In the computation of the correlation function, by virtue of the supersymmetry (2.17), only the constant modes \( m_0, \chi_0 \) contribute to the observable, and the above combination precisely becomes \( \mathcal{F}_{\epsilon_1,\epsilon_2} \[3\]. This means that \( O_k(x) \) truncates to the equivariant Chern character \( \text{Tr} e^{-k R \mathcal{F}_{\epsilon_1,\epsilon_2}} \).

Thus we obtain the finite dimensional integral representation

\[
\langle \prod_a \int_{\mathbb{R}^4} O_{k_a} \rangle^{\epsilon_1,\epsilon_2}_{n\text{-instanton}} = \frac{1}{(2\pi i R)^{\dim \tilde{\mathcal{M}}_n}} \int_{\tilde{\mathcal{M}}_n} \hat{A}_{T^2}(R t_{\epsilon_1,\epsilon_2}, \tilde{\mathcal{M}}_n) \prod_a \int_{\mathbb{R}^4} \text{Tr} e^{-k_a R \mathcal{F}_{\epsilon_1,\epsilon_2}},
\]

where \( \hat{A}_{T^2}(\cdot, \tilde{\mathcal{M}}_n) \) is the \( T^2 \)-equivariant \( \hat{A} \)-genus of the tangent bundle of \( \tilde{\mathcal{M}}_n \), and \( t_{\epsilon_1,\epsilon_2} \) is a generator of \( T^2 \) that gives the Killing vector \( \mathcal{V}_{\epsilon_1,\epsilon_2} \).

Introducing the coupling constants \( t = (t_1, t_2, \cdots) \), the generating function of the correlation functions is given by \( Z_{\epsilon_1,\epsilon_2}(t) = \langle e^{\sum k_i t_k \int_{\mathbb{R}^4} O_k} \rangle^{\epsilon_1,\epsilon_2} \). Since \( n \)-instanton contributes with the weight \( (RA)^{2nN} \), where \( \Lambda \) is the dynamical scale, letting \( Q = (RA)^2 \), we can express the generating function as

\[
Z_{\epsilon_1,\epsilon_2}(t) = \sum_{n=0} Q^{nN} \langle e^{\sum k_i t_k \int_{\mathbb{R}^4} O_k} \rangle^{\epsilon_1,\epsilon_2}_{n\text{-instanton}}.
\]

### 2.3 Application of localization technique

The right hand side of the formula (2.18) is eventually replaced with a statistical sum over partitions. To see their appearance, note that the integration localizes to the fixed points of the \( T^2 \)-action. However, the fixed points in \( \tilde{\mathcal{M}}_n \) are small instanton singularities since the
variation $\delta A = -\epsilon_{\psi_1, \psi_2} F_A$ vanishes there. These can be resolved by instantons on a non-commutative $\mathbb{R}^4$. Applying such a regularization via the non-commutativity, the fixed points get isolated, so that they are eventually labelled by using partitions $[\Pi]$.

A partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a sequence of nonnegative integers satisfying $\lambda_i \geq \lambda_{i+1}$ for all $i \geq 1$. Partitions are identified with the Young diagrams in the standard manner. The size is defined by $|\lambda| = \sum_{i \geq 1} \lambda_i$, which is the total number of boxes of the diagram.

Let us describe the formula (2.18) for the $U(1)$ theory. The relevant computation of the localization can be found in [11, 12]. We truncate $\epsilon_{1,2}$ as $-\epsilon_1 = \epsilon_2 = i\hbar$, where $\hbar$ is a positive real parameter. Consequently, the formula becomes a $q$-series, where $q = e^{-R\hbar}$. The fixed points in $\tilde{\mathcal{M}}_n$ are labelled by partitions of $n$. The equivariant $\hat{A}$-genus takes the following form at the partition $\lambda$ of $n$:

$$
(2\pi i R)^{-2n} \hat{A}_{T^2}(R t_{-i\hbar, i\hbar} \tilde{\mathcal{M}}_n) \big|_{\lambda} = (-)^n \left( \frac{\hbar}{2\pi} \right)^{2n} \left( \prod_{s \in \lambda} h(s) \right)^2 q^{\frac{s(\lambda)}{2}} s_\lambda(q^\rho)^2,
$$

(2.20)

where $h(s)$ denotes the hook length of the box $s$ of the Young diagram $\lambda$, and $s_\lambda(q^\rho)$ is the Schur function $s_\lambda(x_1, x_2, \cdots)$ specialized to $x_i = q^{i-\frac{1}{2}}$. Similarly, the fixed points in $\tilde{\mathcal{M}}_n \times \mathbb{R}^4$ are labelled by partitions of $n$. Denoting them as $(\lambda, 0)$, the equivariant Chern character takes the form $\text{Tr} e^{-kR\mathcal{F}-i\hbar, i\hbar} |_{(\lambda, 0)} = O_k(\lambda)$, where $O_k(\lambda)$ is given by

$$
O_k(\lambda) = (1 - q^{-k}) \sum_{i=1}^{\infty} \left\{ q^{k(\lambda_i-i+1)} - q^{k(-i+1)} \right\} + 1.
$$

(2.21)

The above functions have been exploited in [4, 13] from the 4d gauge theory viewpoint. By taking account of (2.20) and (2.21), the formula (2.18) becomes eventually as

$$
\left\langle \prod_a \int_{\mathbb{R}^4} O_{k_a} \right\rangle_{n-instanton}^{-i\hbar, i\hbar} = (-)^n \sum_{|\lambda| = n} q^{\frac{s(\lambda)}{2}} s_\lambda(q^\rho)^2 \prod_a h^{-2} O_{k_a}(\lambda).
$$

(2.22)

Although we have not taken into account, the Chern-Simon term can be added to a 5d gauge theory, with the coupling constant being quantized, in particular, for the $U(1)$ theory, $m = 0, \pm 1$. It modifies the right hand side of (2.22) by giving a contribution of the form $(-)^m |\lambda| q^{-\frac{m(s(\lambda))}{2}}$, for each $\lambda$ [7]. Hereafter, we consider the case of the $U(1)$ theory having the Chern-Simon coupling, $m = 1$. The corresponding generating function becomes

$$
Z_{-i\hbar, i\hbar}^{U(1)}(t) = \sum_{\lambda} Q^{|\lambda|} s_\lambda(q^\rho)^2 e^{h^{-2} \sum_{k=1} t_k O_k(\lambda)}.
$$

(2.23)
3 Integrability of $5d \mathcal{N} = 1$ SYM in $\Omega$ background

We can view the generating function (2.23) as a $q$-deformed random partition. To see this, note that the $4d$ limit $R \to 0$ makes $q = e^{-Rh} \to 1$, the Boltzmann weight takes at this limit, the form $(\Lambda/h)^{2|\lambda|} (\prod_{s \in \lambda} h(s))^{-2}$, which is the standard weight of a random partition. It can be also viewed as a melting crystal model, known as random plane partition. The corresponding model is studied in [6] as a melting crystal model with external potential, where the Chern characters $O_k$ correspond precisely to the external potentials.

3.1 Melting crystal model

A plane partition $\pi$ is an array of non-negative integers

\begin{align*}
\pi_{11} & \pi_{12} \pi_{13} \cdots \\
\pi_{21} & \pi_{22} \pi_{23} \cdots \\
\pi_{31} & \pi_{32} \pi_{33} \cdots \\
\vdots & \vdots \vdots
\end{align*}

(3.1)
satisfying $\pi_{ij} \geq \pi_{i+1,j}$ and $\pi_{ij} \geq \pi_{ij+1}$ for all $i, j \geq 1$. Plane partitions are identified with the $3d$ Young diagrams. The $3d$ diagram $\pi$ is a set of unit cubes such that $\pi_{ij}$ cubes are stacked vertically on each $(i,j)$-element of $\pi$. Diagonal slices of $\pi$ become partitions, as depicted in

![Figure 1: The 3d Young diagram (a) and the corresponding sequence of partitions (b).](image)

Fig.1. Denote $\pi(m)$ the partition along the $m$-th diagonal slice, where $m \in \mathbb{Z}$. In particular, $\pi(0) = (\pi_{11}, \pi_{22}, \cdots)$ is the main diagonal one. This series of partitions satisfies the condition

\[ \cdots \prec \pi(-2) \prec \pi(-1) \prec \pi(0) \succ \pi(1) \succ \pi(2) \succ \cdots, \]

(3.2)
where $\mu \succ \nu$ means the interlace relation; $\mu \succ \nu \iff \mu_1 \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \mu_3 \geq \cdots$.

The Hamiltonian picture emerges from the above relations, by viewing a plane partition as evolutions of partitions by the discrete time $m$. Eventually it is described \[14\] by using 2d free complex fermions $\psi, \psi^*$. We may separate the relations (3.2) into two parts, each describing the evolutions for $m \leq 0$ and $m \geq 0$. These two types of the evolutions are realized in the 2d CFT by using operators $G_\pm$ of the forms \[14\]

$$ G_\pm = e^{\sum_{k=1}^{\infty} \frac{k}{k(1-q^k)} J_{\pm k}}, $$

where $J_{\pm k} = \sum_{n=-\infty}^{\infty} : \psi_{\pm k-n} \psi_n^* :$ are the modes of the $U(1)$ current.

Using the free fermion description, one can express the generating function as

$$ Z_{U(1)}^{-1} e^{\frac{1}{2} \sum \gamma \hat{O}_k G_- |0\rangle}, $$

where $L_0 = \sum_{n=-\infty}^{\infty} n : \psi_{-n} \psi_n^* :$ is an element of the Virasoro algebra. The loop operators $O_k$ are converted to operators $\hat{O}_k$ in the above representation. They are fermion bilinears given by

$$ \hat{O}_k = (1 - q^{-k}) \sum_{n=-\infty}^{+\infty} q^{kn} : \psi_{-n} \psi_n^* : + 1, $$

3.2 The integrable structure

The fermion bilinears $\hat{O}_k$ can be regarded as a commutative sub-algebra of the quantum torus Lie algebra realized by the free fermions \[6\]. The adjoint actions of $G_\pm$ on the Lie algebra generate automorphisms of the algebra. Among them, taking advantage of the shift symmetry, the representation \[3.4\] can be eventually reformulated \[6\] to

$$ Z_{U(1)}^{-1} e^{\frac{1}{2} \sum \gamma \hat{O}_k G_- |0\rangle} = \langle 0 | e^{\frac{1}{2} \sum_{k=1}^{\infty} (-1)^k (1-q_k^k) t_k J_{\pm k}} | 0 \rangle, $$

In the above formula, $g_{5dU(1)}$ is an element of $GL(\infty)$ of the form

$$ g_{5dU(1)} = q^W G_- G^Q G_- G^Q q^W, $$

where $W = W_{(3)}^0 = \sum_{n=-\infty}^{\infty} n^2 : \psi_{-n} \psi_n^* :$ is a special element of $W_\infty$ algebra. The loop operators $O_k$ are converted to $J_k$ or $J_{-k}$ in (3.6). These two are actually equivalent in the formula, since $g_{5dU(1)}$ satisfies \[6\]

$$ J_k g_{5dU(1)} = g_{5dU(1)} J_{-k}, \quad \text{for } k \geq 0. $$

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Viewing the coupling constants $t$ as a series of time variables, the right hand side of (3.6) is the standard form of a tau function of 2-Toda hierarchy [13]. However, by virtue of (3.8), the two-sided time evolutions of 2-Toda hierarchy degenerate to one-sided time evolutions. This precisely gives the reduction to 1-Toda hierarchy. Thus the generating function becomes a tau function of 1-Toda hierarchy.

4 Extended Seiberg-Witten geometry of 5d theory

We consider the field theory limit of the $U(1)$ theory, which is achieved by letting $\hbar \to 0$ and amounts to the thermodynamic limit of the melting crystal model. The system is described by the prepotential $\mathcal{F}^{(0)}(t; \Lambda, R)$. From the integrable system viewpoint, $\mathcal{F}^{(0)}$ may be interpreted as a dispersion-less tau function, since the generating function is substantially a tau function of 1-Toda hierarchy and $\mathcal{F}^{(0)}$ gives the leading order part of the $\hbar$ expansion of $\log Z^{U(1)}_{-\hbar, \hbar}(t)$. To obtain the semi-classical solution, one actually needs to solve the related variational problem, which is reformulated as a Riemann-Hilbert problem. This issue is treated in [10].

4.1 Seiberg-Witten curve of $U(1)$ theory

Let us present the Seiberg-Witten curve for the $U(1)$ theory. We first employ the following curve [16, 17]:

$$\mathcal{C}_\beta : \quad y + y^{-1} = \frac{1}{R \Lambda} (e^{-Rz} - \beta), \quad z \in \mathbb{C},$$

(4.1)

where $\beta$ is a real parameter. $\mathcal{C}_\beta$ is a double cover of the cylinder $\mathbb{C}^* = \mathbb{C}/\mathbb{Z}$, with a cut $I$ along the real axis on the Riemann sheet. The coupling constants $t$ determine $\beta$ as $\beta = \beta(t)$. To see this, let us introduce a meromorphic differential of the form

$$d\Psi = \left\{1 - \frac{1}{2} R^2 \sum_{k=1}^{\infty} k^3 t_k M_k(z)\right\} d\log y,$$

(4.2)

where $M_k(z) = \sum_{n=0}^{k} d_{k-n}(\beta)e^{-nRz}$. The coefficients $d_n(\beta)$ are given in the asymptotic expansion

$$\sqrt{(1 - \beta e^{Rz})^2 - (2R \Lambda e^{Rz})^2} = \sum_{n=0}^{\infty} d_n(\beta)e^{nRz}, \quad \Re z \to -\infty,$$

(4.3)
Finally, solving the Riemann-Hilbert problem, \( \beta \) is determined by the condition \([10]\)

\[
\oint_C zd\Psi = 0 \quad (C: \text{a contour encircling } I \text{ anticlockwise}).
\] (4.4)

### 4.2 Vevs of the loop operators

The vev of the loop operators \( O_k \) can be represented by using an analogue of the Seiberg-Witten differential. Eventually, the vev can be organized to the contour integral

\[
\frac{\partial \mathcal{F}^{(0)}(t; \Lambda, R)}{\partial t_k} = \lim_{\hbar \to 0} \langle O_k \rangle = \frac{-kR}{2\pi i} \oint_C e^{-kRz}dS,.
\]

where \( dS = S'(z)dz \) is an analogue of the Seiberg-Witten differential. \( S'(z) \) is given by the indefinite integral

\[
S'(z) = \int^z d\Psi.
\]

The contour integral in the right hand side of (4.5) can be converted to a residue integral. Actually, by using coordinate \( Z = e^{-Rz} \), we obtain

\[
\frac{\partial \mathcal{F}^{(0)}(t; \Lambda, R)}{\partial t_k} = \lim_{\hbar \to 0} \langle O_k \rangle = kR \text{Res}_{Z=\infty} \left( Z^k dS \right).
\]

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