Direct construction of the effective action of chiral gauge fermions in the anomalous sector

L. L. Salcedo
Departamento de Física Atómica, Molecular y Nuclear,
Universidad de Granada, E-18071 Granada, Spain
(Dated: July 25, 2008)

The anomaly implies an obstruction to a fully chiral covariant calculation of the effective action in the abnormal parity sector of chiral theories. The standard approach then is to reconstruct the anomalous effective action from its covariant current. In this work we use a recently introduced formulation which allows to directly construct the non trivial chiral invariant part of the effective action within a fully covariant formalism. To this end we develop an appropriate version of Chan’s approach to carry out the calculation within the derivative expansion. The result to four derivatives, i.e., to leading order in two and four dimensions and next-to-leading order in two dimensions, is explicitly worked out. Fairly compact expressions are found for these terms.

PACS numbers: 11.30.Rd 11.15.Tk 11.10.Kk
Keywords: chiral fermions; chiral determinant; effective action; chiral anomaly; derivative expansion; gauge field theory

I. INTRODUCTION

This paper deals with fermions of both chiralities coupled to local external fields of spin 0 or 1. The physics of such system is contained in its effective action, formally the logarithm of the determinant of the Dirac operator \( \mathcal{D} \). A key feature of this system is that it enjoys local chiral symmetry at the classical level. At the quantum level the fermionic measure fails to be invariant \( \mathcal{D} \) and this gives rise to the presence of an anomaly \( \mathcal{D} \) in the abnormal parity sector of the effective action \( \mathcal{A} \). The imaginary part of the effective action contains also other interesting structure, such as many-valuation or topological terms, and for this reason has been thoroughly studied in the literature \( \mathcal{A} \). A good account on the subject can be found in \( \mathcal{A} \).

Clearly, the presence of the anomaly in the abnormal parity sector implies an impediment to a direct construction of the effective action using a chiral covariant formalism. The anomaly is saturated by the Wess-Zumino-Witten, so when this term is removed from the effective action what remains is a chiral invariant functional which can be written using simple chiral covariant elements, namely, the spin zero fields, the field strengths and their chiral covariant derivatives. However, even if this remainder is chiral invariant no direct chiral covariant construction of it was available in the literature. Instead, the best route was to compute the chiral covariant version of the current and then reconstruct the effective action from it \( \mathcal{A} \).

Recently we have shown \( \mathcal{A} \) that the chiral invariant remainder can in fact be expressed as the standard effective action of a local Klein-Gordon operator which moreover is manifestly chiral invariant. This opens the possibility to a direct covariant calculation of the invariant part of the effective action in the abnormal parity sector, along the same lines available to the real part. Such calculation is addressed here within the derivative expansion approach.

In Section II we present a background of previous results. In II A we briefly recall the concepts involved in this subject. The main result of \( \mathcal{A} \) is reviewed in II B Also the calculation in \( \mathcal{A} \) using the method of Chan \( \mathcal{A} \) is described in II C In II D we introduce some notational conventions taken from \( \mathcal{A} \). Such notation is quite transparent and allows to manipulate the expressions to appear subsequently suppressing redundant information. Moreover, it permits to work with formally vector covariant quantities using the original fields (rather than chirally rotated ones). II E summarizes the available result for the effective action in the abnormal parity sector at leading order in the covariant derivative expansion.

In Section III A we introduce an overcomplete basis of standard functions to be used subsequently which allow to express in a simple form the existing results for the effective action or the current. This is explicitly done in III B for the leading order term. Section III C is a digression which shows that the overall structure of the result for the effective action can be understood in some cases although no simple pattern is found for the general case.

In Section III V we apply the findings in \( \mathcal{A} \) to carry out a direct calculation the effective action at leading order in the abnormal parity sector. Chan’s approach, designed for bosonic theories, is discussed. This approach can be applied to the present problem but it implies a redefinition of the covariant derivative and the algebra quickly becomes quite involved. To bypass this problem we construct from scratch a completely new derivative expansion along the same lines of original Chan approach but tailored for the fermionic case in the abnormal parity sector. The analogous of Chan’s formulas for two derivative terms in two dimensions and four derivative terms in four dimensions are derived. (Unlike the bosonic Chan formula, the fermionic one depends on the dimension.) The calculation is fully worked out...
to obtain the effective action at leading order in two and four dimensions and the previous results of [17] are indeed reproduced. As a byproduct some structural properties of the result are found which were not easily visible in the calculation based on the current.

In Section IV we discuss the general form of imaginary part of the effective action to next-to-leading order in the two dimensional case. In order to compute it we extend our fermionic Chan formula to four derivatives in two dimensions in [18]. For comparison, the same calculation using the current method is presented in [19]. It is verified that both methods give the same result and that this is consistent with the calculation of the same quantity in [18], where the world-line approach is used.

Section IV summarizes our conclusions. The relation between the basis of functions introduced in III A and the usual momentum integrals is established in the Appendix.

II. BACKGROUND OF PREVIOUS RESULTS

A. The Dirac operator and the effective action

The Dirac operators we consider describe Dirac fermions coupled to general spin 0 and 1 fields without derivative couplings. These are of the form

\[ D = \gamma_\mu P \gamma_\nu L P + m_{LR} P_R + m_{RL} P_L, \]  

with \( D^{R,L}_{\mu} = \partial_\mu + v^{R,L}_\mu \), \( P_{R,L} = \frac{1}{2}(1 \pm \gamma_5) \). \( d \) is the (even) dimension of the Euclidean space-time and the background fields \( v^{R,L}_\mu(x) \) and \( m_{LR}(x) \) are matrices in some arbitrary internal space. Unitarity requires \( v^{LR}_\mu \) to be antihermitian and \( m^{1}_{RL} = m_{LR} \). In addition, we assume that the matrices \( m_{LR}, m_{RL} \) are not singular at any point. The space-time is flat and the temperature zero.

The fermionic effective action \( W \) is introduced through standard functional integration of the fermionic fields

\[ e^{-W} = \int D\bar{\psi} D\psi e^{-\int d^d x \bar{\psi} D \psi} = \text{Det} D \]

so formally

\[ W = -\text{Tr log} D \]

modulo ultraviolet (UV) ambiguities. (Tr represents the functional trace.)

\( W \) can be split into normal and abnormal parity components, \( W = W^+ + W^- \), \( W^+ \) is the component without Levi-Civita pseudo-tensor, is real (in Euclidean space) and even under the exchange \( R \leftrightarrow L \). \( W^- \) contains the Levi-Civita pseudo-tensor, is imaginary and odd under the exchange of chiral labels \( R \leftrightarrow L \). In this work we concentrate on the abnormal parity component, \( W^- \). In general we will follow the notation and conventions of [17, 19, 21] to which we refer for further details. In particular, the Dirac gammas are hermitian and

\[ \gamma_\mu \gamma_\nu = \delta_{\mu\nu} + \sigma_{\mu\nu}, \quad \gamma_5 = i^{d/2} \gamma_0 \cdots \gamma_{d-1}. \]

The fermionic action \( \int d^d x \bar{\psi} D \psi \) is invariant under local chiral transformations

\[ \bar{\psi}^{\Omega} = (\Omega^{-1}_R P_R + \Omega_L P_L) \psi, \quad \bar{\psi}^{\Omega} = \bar{\psi}(\Omega_L P_R + \Omega_R P_L) \]

\[ D^{\Omega} = D_R^{\Omega} P_R + D_L^{\Omega} P_L + m^{\Omega}_{LR} P_R + m^{\Omega}_{RL} P_L \]

with

\[ (v^{R,L}_\mu)^{\Omega} = \Omega^{-1}_{R,L} v^{R,L}_\mu \Omega_{R,L} + \Omega^{-1}_{R,L} [\partial_\mu, \Omega_{R,L}], \quad m^{\Omega}_{LR} = \Omega^{-1}_L m_{LR} \Omega_R, \quad m^{\Omega}_{RL} = \Omega^{-1}_R m_{RL} \Omega_L, \]

and so, \( (D^{R,L}_\mu)^{\Omega} = \Omega^{-1}_{R,L} D^{R,L}_\mu \Omega_{R,L} \). Chiral covariant derivatives and field strengths are defined correspondingly:

\[ (\hat{D}_\mu m)_{RL} = D_R^\mu m_{RL} - m_{RL} D^\mu_L, \quad (\hat{D}_\mu m)_{LR} = D_L^\mu m_{LR} - m_{LR} D^\mu_R, \quad F^{R,L}_{\mu\nu} = [D^{R,L}_\mu, D^{R,L}_\nu]. \]

In general the effective action cannot be determined in closed form and so several expansions are used to address its systematic calculation. Of special interest to us will be the derivative expansion. In this approach the terms are classified by the number of covariant derivatives they carry. Features of this expansion are that different orders do not mix under chiral rotations and also that UV ambiguities affect only terms with \( d \) derivatives or less. The derivative expansion starts at order \( d \) (\( d \) derivatives) in the abnormal parity sector.
B. The invariant factor of the chiral determinant

In this section we summarize the findings in [19].

As is well known the symmetry under chiral gauge transformations is broken by an anomaly which can be eliminated in \( W^+ \) but not in \( W^- \). The chiral anomaly is saturated by the gauged Wess-Zumino-Witten (WZW) action, so, in the abnormal parity sector, the effective action can be written as

\[
W^- = \Gamma_{gWZW} + W_c^- ,
\]

where \( \Gamma_{gWZW} \) is the gauged WZW action [10] and \( W_c^- \) is the chiral invariant remainder. As it stands there is an ambiguity in the separation between anomalous and non anomalous pieces. This ambiguity is resolved in [17, 19] showing that there is a natural choice of \( \Gamma_{gWZW} \). For instance in two dimensions:

\[
\Gamma_{gWZW} = -\frac{i}{24\pi} \int \epsilon_{\mu\nu\alpha\beta} m_{LR}^{-1} \partial_\mu m_{LR} \partial_\nu m_{LR} m_{LR}^{-1} \partial_\alpha m_{LR} - m_{RL}^{-1} \partial_\mu m_{RL} \partial_\nu m_{RL} m_{RL}^{-1} \partial_\alpha m_{RL} \) \(d^3 x\)
\[
+ \frac{i}{8\pi} \int \epsilon_{\mu\nu\alpha\beta} (\partial_\mu m_{RL} m_{LR}^{-1} v_\nu^R - \partial_\mu m_{LR} m_{LR}^{-1} v_\nu^L - m_{LR}^{-1} \partial_\nu m_{LR} v_\mu^R + m_{RL}^{-1} \partial_\nu m_{RL} v_\mu^L
\]
\[- m_{RL} v_\mu^R m_{RL}^{-1} v_\nu^L + m_{LR} v_\mu^L m_{LR}^{-1} v_\nu^R) \) \(d^3 x\).

\[
\tag{2.9}
\]

The functional \( W_c^- \) is chiral invariant and remarkably it can be expressed as the Tr log of a Klein-Gordon like operator. Indeed, as shown in [19]

\[
W_c = -\frac{1}{2} \text{Tr log } K, \tag{2.10}
\]

where

\[
K = K_L P_R + K_R P_L \tag{2.11}
\]

and

\[
K_R = m_{RL} m_{LR} - \psi_R m_{LR}^{-1} \partial_L m_{LR} , \quad K_L = m_{LR} m_{RL} - \psi_L m_{RL}^{-1} \partial_R m_{RL} . \tag{2.12}
\]

In (2.10) \( W_c \) refers to \( W^+ \), which is chiral invariant, plus \( W_c^- \). It implies

\[
W^+ = \frac{1}{4} \text{Tr log } K_R - \frac{1}{4} \text{Tr log } K_L , \tag{2.13}
\]

\[
W_c^- = \frac{1}{4} \text{Tr } (\gamma_5 \log K_R) - \frac{1}{4} \text{Tr } (\gamma_5 \log K_L) . \tag{2.14}
\]

Let us explain why (2.10) is of interest. As noted, the real part of the effective action \( W^+ \) is chiral invariant and can be computed by means of

\[
W^+ = -\frac{1}{2} \text{Tr log } (D^\dagger D) . \tag{2.15}
\]

\( D^\dagger D \) has all the good properties. It is a positive definite Klein-Gordon operator and so there is a variety of methods available in the bosonic case to work out the computation, in particular the heat kernel approach. Chiral invariance helps also since everything will depend on chiral covariant blocks, namely, \( m_{LR}, m_{RL}, F^R_{\mu\nu} \) and their covariant derivatives. For the imaginary part, one has instead

\[
W^- = -\frac{1}{2} \text{Tr log } (D^\dagger^{-1} D) . \tag{2.16}
\]

The operator \( D^{\dagger-1} D \) does not enjoy these nice features. In fact it is non local and non chiral covariant. An obvious approach is to use \( D^2 \) (assuming a suitable analytic continuation to turn \( D \) into an Hermitian matrix), so that

\[
W = -\frac{1}{2} \text{Tr log } (D^2) . \tag{2.17}
\]

\( D^2 \) has the virtue of being of the Klein-Gordon type. Unfortunately \( D^2 \) is far from transforming nicely under chiral rotations. As a consequence only vector gauge invariance is preserved and so calculations along this route are difficult.
A direct computation of Tr log(D) (rather than Tr log(D^2)) was undertook in [23] using the \(\zeta\)-function approach [24, 25]. Once again only vector gauge invariance is preserved in this approach.

At the root of these problems is the presence of the chiral anomaly in \(W^-\). The anomaly is an obstruction to a chiral invariant treatment. Any approach that tries to compute \(W^-\) (or \(W\)) without previous extraction of the anomaly cannot enjoy chiral invariance.

A chiral covariant approach is based on the current. Unlike the effective action, the current (the variation of \(W\) with respect to the gauge fields) admits a chiral covariant version. This is the so called covariant current. Such current is not directly consistent (i.e., it is not a true variation) but it can be turned into a consistent current by adding the appropriate counter-term [11]. The point is that the covariant current is amenable to direct chiral covariant computation. Then \(W^-\), constructed with covariant blocks, can be adjusted in order to reproduce the current. This is the approach introduced in [17] and applied there to compute \(W^-\) at leading order of the derivative expansion in two and four dimensions. The same method has been applied in [18] to compute the next-to-leading order in two dimensions.

The abovementioned obstruction induced by the anomaly is a consequence of the ambiguities introduced by the ultraviolet divergences, as is the mismatch between the covariant and consistent currents. Therefore there should be no obstruction in ultraviolet finite terms. This includes all terms beyond the lowest order one in the derivative expansion of \(W^-\). In this view, the current method of [17] is covariant but is not a direct computation of the effective action.

The relation (2.10), and in particular (2.14), provides us precisely with such a direct approach to \(W^-\) (to all orders actually). \(K\) is a manifestly chiral covariant operator of the Klein-Gordon type and this opens the possibility to address the computation of the imaginary part of the effective action \(W^-\) using the same efficient techniques available for the real part.

It should be noted that the calculation of \(W^-\) through Tr log\(K\) is still subject to UV ambiguities. These are chiral covariant polynomial counter-terms constructed with \(D_R^{\mu} m_{LR}(\hat{D}_\mu m)_{LR}\), and \(m_{RL}^{-1}(\hat{D}_\mu m)_{RL}\) in \(W^-\) and also \(m_{RL} m_{LR}, m_{LR} m_{RL}\) in \(W^+\). Some of these polynomial are spurious contributions to \(W^-\) and should be removed [19].

### C. Chan-like calculation

The operators \(K_{R,L}\) can be brought to a standard Klein-Gordon form

\[
K_R = \hat{M}_R - (\hat{D}_\mu)^2, \quad K_L = \hat{M}_L - (\hat{D}_\mu)^2
\]

(2.18)

with

\[
\hat{M}_R = m_{RL} m_{LR} - \frac{1}{2} \sigma_{\mu\nu} F_{\mu\nu}^R + (B_{\mu}^R)^2 - [D_{\mu}^R, B_{\mu}^R],
\]

\[
\hat{D}_{\mu}^R = D_{\mu}^R + B_{\mu}^R,
\]

\[
B_{\mu}^R = \frac{1}{2} \gamma_\mu \gamma_\nu Q_{\nu}^R.
\]

(2.19)

(and similarly for \(K_L\)) where

\[
Q_{\mu}^R = m_{LR}^{-1}(\hat{D}_\mu m)_{LR}, \quad Q_{\mu}^L = m_{RL}^{-1}(\hat{D}_\mu m)_{RL}.
\]

(2.20)

The standard form “\(M - D_\mu^2\)” allows to apply Chan’s method [20] straightforwardly. This calculation has been carried out in [19] for \(W^-\) to two derivatives in two dimensions and yields

\[
W^-_{c,d=2,LO} = -\frac{i}{2} \int \frac{d^2x d^2p}{(2\pi)^2} \epsilon_{\mu\nu\lambda\tau} \left( N_R^2 m_{RL}(\hat{D}_\mu m)_{LR} N_R m_{LR}(\hat{D}_\tau m)_{LR} + N_L^2 m_{LR}(\hat{D}_\lambda m)_{RL} N_L m_{LR}(\hat{D}_\mu m)_{RL} \right)
\]

(2.21)

where

\[
N_R = (p^2 + m_{RL} m_{LR})^{-1}, \quad N_L = (p^2 + m_{LR} m_{RL})^{-1}.
\]

(2.22)

The label LO refers to leading order in the derivative expansion. As shown in [19] (2.21) leads to the result previously established in [17].
D. Notational conventions

As we have just noted, it is possible to apply (2.14) to compute \( W^- \) using only chiral covariant quantities. It is also clear that as one considers more complicated cases (e.g., four derivatives) the formulas will become more involved. This leads us to use a number of notational conventions in order to simplify the expressions. Such conventions were already introduced in \([17, 21]\).

For convenience, in what follows we will use Lorentz indices to indicate covariant derivatives, e.g.,

\[
m^{LR}_{\mu\nu} = \hat{D}_\mu \hat{D}_\nu m^{LR}, \quad F^{R}_{\alpha\mu\nu} = \hat{D}_\alpha F^{R}_{\mu\nu}.
\]  
(2.23)

(Each new derivative adds an index to left.)

Under chiral transformations, the various quantities transform as \( LR \) (e.g. \( m_{LR} \)), \( RL \), \( RR \) (e.g., \( D^R_\mu \) or \( F^{R}_{\mu\nu} \)) or \( LL \). However, due to chiral symmetry, it is clear that the explicit writing of these labels is largely redundant. In fact, as can be seen in previous formulas, in functionals like the effective action, a field with a label already introduced in \([17, 21]\). This leads us to use a number of notational conventions in order to simplify the expressions. Such conventions were clear that as one considers more complicated cases (e.g., four derivatives) the formulas will become more involved.

Note also that inside the trace the total number of \( m \)'s (with or without derivatives) should always be even. This is because there should be as many \( LR \) fields as \( RL \) ones (since expressions in the trace start and end with the same label).

For instance, without any loss of information \([2.16] \) and \([2.17] \) can be written as

\[
v^\Omega_\mu = \Omega^{-1} v_\mu \Omega + \Omega^{-1} \{D_\mu, \Omega\}, \quad m^\Omega = \Omega^{-1} m \Omega, \quad D^\Omega_\mu = \Omega^{-1} D_\mu \Omega,
\]

\[
\hat{D}_\mu m = D_\mu m - m D_\mu = [D_\mu, m] = m_\mu, \quad F_{\mu\nu} = [D_\mu, D_\nu].
\]  
(2.26)

Also \([2.12] \) and \([2.14] \) can be written as

\[
K = m^2 - \slashed{D} m^{-1} \slashed{D} m, \quad W_c^- = \frac{1}{2} \text{Tr} (\gamma_5 \log K),
\]  
(2.27)

Likewise \([2.9] \) becomes

\[
\Gamma_{gWZW} = -\frac{i}{12\pi} \int \text{tr} (m^{-1} d m)^3 + \frac{i}{4\pi} \int \text{tr} (d m m^{-1} \nu - m^{-1} d m \nu - m \nu m^{-1} \nu).
\]  
(2.28)

Here \( d m = \partial_\mu m d x_\mu, \nu = v_\mu d x_\mu \). In this formula we use differential forms. Differential forms are introduced in the formulas through the identity

\[
\epsilon_{\mu_1 \mu_2 \cdots \mu_d} d^d x = \partial^{\mu_1} x_1 \partial^{\mu_2} x_2 \cdots \partial^{\mu_d} x_d.
\]  
(2.29)

We will also use the differential forms

\[
m' = m_\mu d x_\mu, \quad F = \frac{1}{2} F_{\mu\nu} d x_\mu d x_\nu.
\]  
(2.30)

---

1 The gauged WZW term can also be written using this notation since it relies only on invariance under global chiral rotations.
The notation

\[ X' := dx_\mu \hat{D}_\mu X \]  

will be used throughout. Here \( X \) represents any matrix-valued \( n \)-form. Two useful results are \( X' = [F, X] \) and \( F' = 0 \). Finally, (2.21) becomes

\[ W_{c,d=2,LO}^- = -i \int \frac{d^2 p}{(2\pi)^2} \text{tr} \left( N^2 m m' N m' \right) \]  

where

\[ N = (p^2 + m^2)^{-1}. \]  

A further notational convention is that of labeled operators \[ \text{17, 21, 26, 27, 28} \]: we will use labels 1, 2, 3, \ldots in the quantity \( m \) (with no derivatives) to indicate its position in an expression. For instance

\[ m^2 F m m^{-1} m' m^3 = \frac{m_1^2 m_2 m_3^3}{m_3^2} F^2 m'. \]  

This is useful because the labeled \( m \)'s can be treated as \( c \)-numbers, and so compact expressions like \( f(m_1, m_2)F \) become meaningful (\( f(x, y) \) being an ordinary function). As discussed in \[ \text{17} \], an expression like \( f(m_1, m_2)F \) is well defined provided \( f(x, y) \) is regular in the coincidence limit \( x^2 - y^2 \to 0 \). (On the other hand, the use of labeled operators with functions violating the regularity condition in the coincidence limit may yield nonsensical expressions \[ \text{17} \).)

Using this notation (2.32) becomes

\[ W_{c,d=2,LO}^- = -i \int \frac{d^2 p}{(2\pi)^2} \text{tr} \left[ \frac{m_1 m_2}{(p^2 + m_1^2)(p^2 + m_2)^2} m' m' \right]. \]  

The point of labeling the operators is that the momentum integration is now straightforward and yields

\[ W_{c,d=2,LO}^- = \frac{i}{4\pi} \int \frac{m_1 m_2}{m_1^2 - m_2^2} \left[ \frac{1}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) - \frac{\log(m_1^2/m_2^2)}{m_1^2 - m_2^2} \right] m' m'. \]  

This coincides with (3.41) of \[ \text{19} \] where the expression is given using the eigen-basis method instead of labeled operators.

A noteworthy feature of our conventions is that within this notation chiral invariance becomes formally identical to vector invariance (cf. (2.26)). This is different from the standard approach of chirally rotating the background fields \( v_{\mu L}^R, m_{LR}, m_{RL} \) so that the rotated \( m_{LR}, m_{RL} \) become equal. With that choice the remaining freedom is vector gauge invariance. By construction, in that approach any vector invariant calculation becomes chiral invariant upon undoing the chiral rotation. However, the rotated expressions depend on three fields, \( V_\mu = (v_{\mu L}^R + v_{\mu R}^L)/2, A_\mu = (v_{\mu L}^R - v_{\mu R}^L)/2, \) and \( S = m_{LR} = m_{RL} \), instead of only two, \( v_\mu \) and \( m \). We achieve formal vector gauge invariance using the original fields, without any change of variables to rotated variables.

We have introduced three main conventions here, namely, (i) removing the redundant chiral labels, (ii) antisymmetrization with respect to \( R, L \) labels in traced quantities and (iii) labeling \( m \) with respect to fixed operators. Perhaps some readers may find this notation obscure. On the contrary, we think that our conventions highlight the underlying structure of the expressions, are quite natural and well suited to the present context and no ambiguity is introduced. In any case, it is a fact that the LO terms of \( W^- \) at \( d = 2, 4 \) for a generic theories were not obtained until this notation was used in \[ \text{17} \].

E. The effective action in the abnormal parity sector

At leading order, the remainder \( W_{c=2,4}^- \) vanishes identically when the scalar and pseudo-scalar fields satisfy a generalized chiral circle constraint, namely, when \( m_{RL} m_{LR} \) is a \( c \)-number. However, in general, \( W_{c=2,4}^- \) is a non trivial functional. \( W_{c=2,4}^- \) has been computed in \[ \text{17} \] for \( d = 2, 4 \) to leading order (LO), that is, to \( d \) covariant derivatives, and next-to-leading order (NLO) in \[ \text{18} \] for \( d = 2 \). The LO results take the form

\[ W_{c,LO}^- = \langle N_{12} m'^2 \rangle \]  

\[ W_{c,LO}^- = \langle N_{1234} m'^4 + N_{123} m'^2 F \rangle \]  

for \( d = 2, 4 \).
In these formulae the symbol $\langle \rangle$ is a shorthand for
\[
\langle X \rangle := \frac{i^{d/2}(d/2)!}{(2\pi)^{d/2}d!} \int \text{tr}(X)
\] (2.38)
($X$ being a $d$-form).

On the other hand $N_{12} = N(m_1, m_2)$, $N_{123} = N(m_1, m_2, m_3)$, $N_{1234} = N(m_1, m_2, m_3, m_4)$ are three known functions of the labeled $m$’s \textbf{[17]}. For instance,
\[
N_{12} = -\frac{m_1 m_2}{m_1^2 - m_2^2} \left( \frac{\log(m_1^2/m_2^2)}{m_1^2 - m_2^2} - \frac{1}{2} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \right).
\] (2.39)
(This is just the result quoted in \textbf{[2.30]}.)

Terms of the form $\langle f(m)F \rangle$ for $d = 2$ or $\langle f(m_1, m_2)F^2 \rangle$ for $d = 4$ do not appear in \textbf{[2.37]} because they can be removed by using integration by parts. On the other hand, there is an ambiguity in the functions $N_{123}$ and $N_{1234}$ of $d = 4$ due to the identity
\[
0 = \langle (H_{123}m^3) \rangle
\] (2.40)
and $m'' = [F, m]$, for any function $H_{123}$.

Cyclic symmetry of the trace allows to impose the conditions
\[
N_{12} = N_{21}, \quad N_{1234} = N_{2341}, \quad H_{123} = -H_{231},
\] (2.41)
where we use the short-hand $N_{21} = N(m_2, -m_1)$, etc. Further, unitarity guarantees the following mirror symmetry \textbf{[17]}:
\[
N_{12} = -N_{21}, \quad N_{123} = -N_{321}, \quad N_{1234} = -N_{4321}, \quad H_{123} = -H_{321}.
\] (2.42)
This ends our summary of previous results.

III. THE CHIRAL REMAINDER AT LO

A. Set of standard functions

In \textbf{[17]} the $d = 4$ functions $N_{123}$ and $N_{1234}$ were obtained by means of a rather tortuous procedure. If these functions were unique they would certainly satisfy the condition of regularity in the coincidence limit ($m_j^2 \to m_j^2$) noted in Section \textbf{[11]} for the use of labeled operators. However, the existence of the ambiguity (2.40) allowed spurious solutions violating that condition. As a consequence some ingenuity was required to properly fix the ambiguity in that approach. As we will see in Section \textbf{[14]} the method based on (2.14) directly yields acceptable results. Also the explicit results in \textbf{[17]} were complicated and lacked any systematics.

Here we present simple expressions for $N_{12}$, $N_{123}$ and $N_{1234}$ in which the regularity condition is manifestly checked. This is achieved by making use of the set of “standard” functions
\[
F_{\alpha, \ldots, \alpha} := \int \frac{dz}{2\pi i} z^\alpha \log(z) \prod_{j=1}^{n} \frac{1}{(z - m_j^2)^{r_j}}, \quad r_j \in \mathbb{Z}, \quad \alpha \in \mathbb{C},
\] (3.1)
where the integration is along a positive closed simple contour enclosing the poles at $m_j^2$ but excluding $z = 0$. In applications the $m_j^2$ are positive and we choose $\log(z)$ real on the real positive axis with the branch cut taken along the negative real axis. Besides $z^\alpha = \exp(\alpha \log(z))$. On their Riemann surfaces the functions $F_{\alpha, \ldots, \alpha}$ (as functions of the $m_j^2$) are regular everywhere, except at $m_j = 0$ where they present branching points or poles. In particular they are manifestly regular in the coincidence limits $m_j^2 \to m_j^2$. They are easy to compute by residues. In addition, different values of $n$ are related by recurrence relations and increasing values of $r_j$ can be obtained through derivatives with respect to $m_j^2$ \textbf{[28]}. For instance
\[
F_0 = \log(m_j^2),
\]
\[
F_0^{2,1} = \frac{1}{m_1^2} \frac{1}{m_1^2 - m_2^2} - \frac{\log(m_1^2) - \log(m_2^2)}{(m_1^2 - m_2^2)^2}.
\] (3.2)
These functions are introduced in the calculation in a natural manner due to the following relation, which holds provided \( s + d/2 - 1 \) is a non-negative integer (see the Appendix),

\[
\int \frac{d^dp}{(2\pi)^d} (p^2)^s \prod_{j=1}^n \frac{1}{(p^2 + m_j^2)^{r_j}} = \frac{(-1)^s + d/2 - 1 + \Sigma_j r_j}{(4\pi)^{d/2} \Gamma(d/2)} f_1^{s + d/2 - 1}, \quad s + d/2 = 1, 2, 3, \ldots
\]  

(3.3)

This relation holds modulo possible UV divergent contributions on the left-hand side; the right-hand side is always finite. A further convenient feature of the standard functions \( I_{r_1,\ldots,r_n}^a \) is that they do not depend on the space-time dimension \( d \).

As an application, we write in this basis the functions introduced in Eqs. (34) and (36) of [17], corresponding to the covariant current at LO in two and four dimensions:

\[
\begin{align*}
A_{12} &= -2m_1 I_{2,1}^1 + 2m_2 I_{1,2}^1, \\
A_{123} &= 6m_1 I_{2,1,1}^2 + 6m_2 I_{1,2,1}^2 - 6m_3 I_{1,1,2}^2, \\
A_{1234} &= -6m_1 I_{2,1,1,1}^2 + 6m_2 I_{1,2,1,1}^2 - 6m_3 I_{1,1,2,1}^2 + 6m_4 I_{1,1,1,2}^2.
\end{align*}
\]  

(3.4)

B. LO terms using standard functions

In order to use the basis \( I_{r_1,\ldots,r_n}^a \) to express the functions \( N_{12}, N_{123} \) and \( N_{1234} \), we first decompose the latter in components with well-defined parity under \( m_j \to -m_j \), for each \( j \). For instance,

\[
N_{12} = N_{12}^{++} + m_1 N_{12}^{-+} + m_2 N_{12}^{+-} + m_1 m_2 N_{12}^{--},
\]  

(3.5)

with components \( N_{12}^{\pm\pm} \) depending on \( m_1^2 \) and \( m_2^2 \). Then the \( d = 2 \) result in (2.3)); this can be written as

\[
N_{12}^{--} = \frac{1}{2} (I_{0,1,2}^0 - I_{1,2}^0), \quad N_{12}^{++} = N_{12}^{+-} = N_{12}^{--} = 0.
\]  

(3.6)

Note that the functions \( I_{r_1,\ldots,r_n}^a \) are not linearly independent and a single function, such as \( N_{12}^{--} \), can be written in different ways using them.

In the case \( d = 4 \), the functions \( N_{123} \) and \( N_{1234} \) are the solutions of the Eqs. (90) of [17]. As noted, these functions are not unique due the ambiguity introduced by (2.39). One of the solutions is presented in [17]; however, we have not tried to express this particular solution in terms of the basis \( I_{r_1,\ldots,r_n}^a \). Instead, we have directly used the defining equations and rewritten them in terms of the components \( N_{123}^{++} \) and \( N_{1234}^{++} \).

Because the functions \( N_{123} \) and \( N_{1234} \) are even under \( m \to -m \), all the odd components (e.g. \( N_{123}^{---} \)) vanish. The ambiguity introduced by (2.40) can be used to set \( N_{123}^{--} \) to zero and in this case \( N_{123}^{++} \) also turns out to be zero. The remaining components of \( N_{123} \) are related by mirror symmetry (2.42), namely, \( N_{123}^{--} = -N_{123}^{++++} \). For the latter we find the following valid choice (the ambiguity was not completely fixed by our previous choice \( N_{123}^{++} = 0 \))

\[
N_{123}^{--} = 6I_{1,1,2}^1 + \frac{1}{2} m_3^2 (I_{0,1,2,1}^0 - 3I_{1,2,1,1}^0 + 5I_{1,1,2,1}^0) + 5I_{1,1,1,2}^0.
\]  

(3.7)

Once the ambiguity has been settled for \( N_{123} \), the function \( N_{1234} \) is completely fixed. All odd components of \( N_{1234} \) vanish. The equations imply that \( N_{1234}^{++++} \), and \( N_{1234}^{-----} \) are also zero (and so \( N_{1234}^{----} \), by mirror symmetry). The non-vanishing components can be written as

\[
\begin{align*}
N_{1234}^{---} &= \frac{1}{2} (I_{2,1,1,1}^2 - I_{1,2,1,1}^1), \\
N_{1234}^{---} &= \frac{1}{4} (I_{2,1,1,1}^2 - I_{1,2,1,1}^2 + I_{1,1,2,1}^0 - I_{1,1,1,2}^0),
\end{align*}
\]  

(3.8)

together with \( N_{1234}^{----} = N_{1234}^{---} = N_{1234}^{-----} = -N_{1234}^{----} \).

The expressions (3.3), (3.6), (3.7) and (3.8) have been obtained from \( N_{123}^{++} \), \( N_{123}^{+++} \) and \( N_{1234}^{+++} \) by fitting the numerical coefficients in an expansion in terms of the \( I_{r_1,\ldots,r_n}^a \). The fit is not unique and we have tried to select the simplest ones.\(^2\) In the next section we show that, by construction, the effective action can always be written using

\(^2\) In \( N_{123}^{--} \) we have allowed a factor \( m_3^2 \) in order to obtain a simpler expression. Of course, this is not mandatory since positive powers of \( m_j^2 \) can be reabsorbed in the functions \( I_{r_1,\ldots,r_n}^a \).
the $I_{r_1,\ldots, r_n}$ times (possibly negative) integer powers of $m^2$. It is not clear why these negative powers are actually not needed in the final expressions for the effective action. It is noteworthy that this puzzle does not exist for the current; when the method of covariant symbols is used [17], at no place $m^{-1}$ appears in the calculation and by construction the final result involves only functions $I_{r_1,\ldots, r_n}$, as in (3.4).

C. Chern character-like ideas

In this Section we take a small digression before going to the main results of the present work.

It is tempting to try to find a simple systematics in the form of the functions $N_{123}$, $N_{1234}$ and $N_{12345}$ just quoted. However, there is a number of problems to do that. First, there is a huge ambiguity in how the functions are expressed in terms of the overcomplete basis $I_{r_1,\ldots, r_n}$. Second, in $d = 4$ there is another ambiguity due to integration by parts (we have selected $N_{123}^{-1} = 0$). Finally, the expressions could perhaps display a simple pattern if the redundant operators $F$ (in $d = 2$) or $\tilde{F}^2$ (in $d = 4$) were allowed in (2.37). In any case, we have been unable to find any systematics in that expansion. Nevertheless, there is a remarkable exception: the expression

$$\Gamma = \langle \log(m^2 + mm') \rangle,$$

when expanded through order $d$, correctly reproduces the terms of the type $(mm')^d$ of $W_c^-$, that is, the functions $N_{12}^{-1}$ of $d = 2$ and $N_{1234}^{-1}$ of $d = 4$. Indeed,

$$\Gamma = \left\langle \int \frac{dz}{2\pi i} \log(z) \frac{1}{z - m^2 - mm'} \right\rangle = \langle I_{12}^0 + I_{1,1}^0 mm' + I_{1,1,1}^0 (mm')^2 + \cdots \rangle = \langle I_{12}^0 + m_1 I_{12,0}^0 m' + m_1 m_2 I_{12,1,0}^0 m'^2 + \cdots \rangle. \tag{3.10}$$

Picking up the term which is a two-form, for $d = 2$, and rewriting it so that cyclic symmetry of the trace is manifest, produces

$$\Gamma_{d=2} = \left\langle \frac{1}{2} m_1 m_2 (I_{12}^0 - I_{1,2}^0) m'^2 \right\rangle, \tag{3.11}$$

which is (3.6). In two dimensions this is the full result. The corresponding result in $d = 4$

$$\Gamma_{d=4} = \left\langle \frac{1}{4} m_1 m_2 m_3 m_4 (I_{12,1,1,1}^0 - I_{12,1,2,1}^0 + I_{12,1,1,2}^0 - I_{12,1,1,1}^0) m'^4 \right\rangle, \tag{3.12}$$

is the full result when $v^0_{\mu} = v^1_{\mu} = 0$ and $d_{n_{RL}} = 0$ (a case studied in [17]), but not in general. It is noteworthy that (3.4) has some resemblance with the Chern character of algebraic topology [29, 30] which finds direct application in anomalies and WZW actions [14].

In an attempt to reproduce all the LO components of $W_c^-$ we have considered

$$\Gamma = \langle \log(m^2 + mm') + O_2 + O_3 + O_4 + \cdots \rangle,$$  

where the $O_n$ are general $n$-forms of the LO type:

$$O_2 = f_{123}^1 m'^2 + f_{12}^1 F,$$  

$$O_3 = f_{1234}^1 m'^3 + f_{123}^1 m' F + f_{12}^1 F m',$$  

$$O_4 = f_{12345}^1 m'^4 + f_{1234}^1 m'^2 F + f_{123}^1 m' F m' + f_{12}^1 F m'^2 + f_{12}^{10} F^2,$$  

etc. Here, the $f^{k}$ are functions of $m$ to be chosen suitably so that $W_c^{{\text{LO}}} = \Gamma$. Unfortunately, an analysis of the case $d = 4$ shows that no such functions exist if one requires them to be (i) one-valued (no logarithms) and (ii) regular in the coincidence limits $m^2_1 \to m^2_j$. This statement holds regardless of how the ambiguity in the functions $N_{123}$, $N_{1234}$ is fixed.
IV. DIRECT COMPUTATION OF THE CHIRAL REMAINDER AT LO

As discussed in Section II C the elegant method of Chan [20] provides the derivative expansion of \( \text{Tr} \log K \) when the operator \( K \) is of the form \( M - D_\mu^2 \), \( M \) and \( D_\mu \) being non-abelian in general. Chan’s technique is based on the symbols method, which allows to write

\[
\langle x | f(M, D_\mu) | x \rangle = \int \frac{d^d p}{(2\pi)^d} \langle x | f(M, D_\mu + p_\mu) | 0 \rangle. \tag{4.1}
\]

\( f(M, D_\mu) \) denotes an operator constructed with \( D_\mu \) (of the form \( \partial_\mu + A_\mu \)) and \( M \), a multiplicative operator, and we assume \( f \) to be sufficiently UV convergent. On the other hand, \( | 0 \rangle \) is the state such that \( \langle x | 0 \rangle = 1 \) (and so \( \partial_\mu | 0 \rangle = 0 \)). Also by convenience, in order to avoid the proliferation of i’s in the formulas, we use a purely imaginary \( p_\mu \), however, \( p^2 := -p_\mu^2 \) and \( d^d p \) are the usual ones.

The diagonal matrix element at the left-hand side of (4.1) is gauge covariant, but the matrix element at the right is not. The expression becomes gauge covariant after taking the momentum integration. This is easily understood as follows. The momentum integral is obviously invariant under the shift \( p_\mu \to p_\mu - a_\mu \), where \( a_\mu \) is an arbitrary constant c-number. This implies that, after momentum integration, the expression is invariant under the shift \( D_\mu \to D_\mu + a_\mu \). This implies that the operator \( D_\mu \) appears only through commutators, in the form \([D_\mu, \cdot \cdot \cdot]\), thus the expression is gauge covariant. (It is noteworthy that the method of covariant symbols [31, 32] provides gauge covariant expressions prior to momentum integration.)

In Chan’s method, (4.1) is applied to \( \log(M - D_\mu^2) \). This gives [20]

\[
\text{Tr} \log(M - D_\mu^2) = \int \frac{d^d x \, d^d p}{(2\pi)^d} \text{Tr} \left[ \log N + \frac{p^2}{d} N^2 \right]. \tag{4.2}
\]

where \( N = 1/(p^2 + M) \) and the dots refer to higher orders in the derivative expansion. The method was extended to sixth order in \( (p^2 + M) \) and to curved space-time in [33].

As shown in [33, 19] and Section II C Chan’s method combined with (2.27) allows to compute \( W_c^- \) with explicit chiral covariance at every step. However, the use of Chan’s formula there implies a redefinition of the covariant derivative, by the term \( B_\mu \), cf. [2,19], which moreover involves two Dirac matrices. Technically, this is quite inconvenient as the algebra quickly produces rather long expressions. To sort this problem we undertake here the task of developing an expansion from scratch along Chan’s ideas but specifically adapted to \( W_c^- \).

To this end let us express (2.27) in the form

\[
W_c^- = \frac{1}{2} \text{Tr} \left[ \gamma_5 \log(m^2 - \slashed{D} \slashed{D}) \right], \tag{4.3}
\]

with

\[
D_\mu = m^{-1} D_\mu m = D_\mu + Q_\mu, \quad Q_\mu = m^{-1} m_\mu. \tag{4.4}
\]

An application of the method of symbols then gives

\[
W_c^- = \frac{1}{2} \int \frac{d^d x \, d^d p}{(2\pi)^d} \text{Tr} \left[ \gamma_5 \log(m^2 - (\slashed{D} + \slashed{a}) (\slashed{D} + \slashed{a})) \right]. \tag{4.5}
\]

Following [20], the idea is to expand the logarithm in terms ordered by the number of derivatives, using formal cyclic symmetry of the trace, and then bring the expression to a manifestly covariant form, that is, one where all \( D_\mu \) operators (including that in \( D_\mu \)) appear only in commutators, \([D_\mu, \cdot \cdot \cdot]\). Of course, in order to guarantee that this works, UV divergences have to be treated adequately. To this end the integrals over \( p_\mu \) will be dealt with using standard dimensional regularization. On the other hand, because in the derivation of (2.27) it was assumed that \( \slashed{D} \) anticommutes with \( \gamma_5 \) [19], the Dirac gammas will be kept in the original integer dimension, \( d = 2, 4, \ldots \) (they are

\[3\] Thus, \( \langle x | f(M, D_\mu + p_\mu) | 0 \rangle \) is just the standard symbol of the pseudo-differential operator \( f(M, D_\mu) \).
derivatives gives bring the expression to a more homogenous form integrating by parts in momentum space. In the present case a non-vanishing result, and due to relations of type constant c-number. Therefore it is gauge invariant. Indeed, it can be cast in a manifestly covariant form:

\[
W_{c,2}^- = -\frac{1}{2} \int \frac{d^4x d^4p}{(2\pi)^d} \text{tr} \left[ \gamma_5 \left( N \not{D} \not{D} \not{D} + \frac{1}{2} \not{N} (\not{D} \not{D} + \not{p} \not{D} \not{D}) \not{N} (\not{D} \not{D} + \not{p} \not{D} \not{D}) \right) \right] \quad (4.6)
\]
where \( N = 1/(p^2 + m^2) \). Rotational invariance of the integral over \( p_\mu \) then implies that an angular average can be taken, \( p_\mu p_\nu \to -p^2 \delta_{\mu\nu}/d \),

\[
W_{c,2}^- = -\frac{1}{2} \int \frac{d^d x d^d p}{(2\pi)^d} \text{tr} \left[ \gamma_5 \left( N \not{D} \not{D} - \frac{p^2}{2d} N (\not{D} \not{D} + \not{p} \not{D} \not{D}) \not{N} (\not{D} \not{D} + \not{p} \not{D} \not{D}) \right) \right]. \quad (4.7)
\]

At this point we can proceed to take the Dirac trace. Because the trace with \( \gamma_5 \) requires at least \( d \) gammas to give a non-vanishing result, and due to relations of type \( \gamma_\lambda \gamma_\alpha = d \) and \( \gamma_\lambda \gamma_\alpha \gamma_\lambda = (2 - d) \gamma_\alpha \), it is clear that this second order contribution is identically zero when \( d > 2 \). This is as expected for the second order term. In two dimensions, using

\[
\gamma_\lambda \gamma_\alpha \gamma_\lambda = 0, \quad \text{tr} \gamma_5 \gamma_\mu \gamma_\nu = -2i \epsilon_{\mu\nu} \quad (d = 2)
\]
gives

\[
W_{c,\text{LO}, d=2}^- = \left\langle N \not{D} \not{D} - \frac{2p^2}{d} N \not{D} \not{D} \not{D} \not{D} \right\rangle_p, \quad (4.9)
\]
where

\[
D = D_\mu \, dx_\mu, \quad \bar{D} = \bar{D}_\mu \, dx_\mu \quad (4.10)
\]
are 1-forms and \( \left\langle \right\rangle_p \) is short-hand for \( i^d/2 \int d^d p/(2\pi)^d \text{tr} \left\langle \right\rangle_p \) (including \( \not{x} \)-integration of the \( d \)-form). Following [20], we bring the expression to a more homogenous form integrating by parts in momentum space. In the present case

\[
0 = \int \frac{d^d p}{(2\pi)^d} \frac{\partial}{\partial p_\mu} (\not{p} N) = \int \frac{d^d p}{(2\pi)^d} (d N - 2p^2 N^2) \quad (4.11)
\]
allows to write

\[
W_{c,\text{LO}, d=2}^- = \left\langle p^2 \left( N^2 \not{D} \not{D} - N \not{D} \not{D} \not{D} \not{D} \right) \right\rangle_p. \quad (4.12)
\]

Now, it can be verified that the expression is invariant under the shift \( D_\mu \to D_\mu + a_\mu, \bar{D}_\mu \to \bar{D}_\mu + a_\mu \) \((a_\mu \text{ being a constant c-number)}\). Therefore it is gauge invariant. Indeed, it can be cast in a manifestly covariant form:

\[
W_{c,\text{LO}, d=2}^- = \left\langle \frac{p^2}{2} \left( N^2 \{D, \bar{D} \} - [D, N][\bar{D}, N] \right) \right\rangle_p. \quad (4.13)
\]

Let us note that \( \int \text{tr} \{D, \bar{D} \} \) vanishes,\(^4\) hence the momentum integral is actually convergent. Although \( W_c^- \) is formally logarithmically divergent, the requirement of covariance completely fixes the UV ambiguities.\(^5\) This was also the case in the calculation of the covariant current [17].

Eq. \( 4.13 \) is the analogous of Chan’s formula at second order for the abnormal parity sector. The same expression can be written using the more standard \( F_{\mu\nu}, Q_{\mu\nu}, N \) and their derivatives

\[
W_{c,\text{LO}, d=2}^- = \left\langle p^2 \left( N^2 F + Q NN' \right) \right\rangle_p. \quad (4.14)
\]

---

\(^4\) We neglect topological contributions, such as \( \int \text{tr} F \) and assume the validity of formal integration by parts throughout.

\(^5\) For this reason ambiguities in the procedure followed are not relevant in the final expression. For instance, taking the Dirac trace in \( p_\mu p_\nu \gamma_\mu \gamma_\nu \) gives \(-2p^2\), whereas taking the angular average first and then the trace gives \(-4p^2/d\). The difference, of order \( d - 2 \), vanishes in UV convergent expressions after \( d \to 2 \).
where

$$Q = Q_\mu dx_\mu = m^{-1} m', \quad N' = N_\mu dx_\mu.$$  \hspace{1cm} (4.15)

In order to bring (4.14) to the standard form in (2.37) we remove $F$ using the identity $\langle 2p^2 N^2 F \rangle_p = \langle -N m' N m' \rangle_p$. Straightforward manipulations yield then

$$W_{c,LO,d=2}^- = \left\langle \left( -\frac{1}{2} N_1 N_2 + p^2 (1 - m_1^{-1} m_2) N_1 N_2 \right) m'^2 \right\rangle_p$$
$$= \left\langle \left( -\frac{1}{2} F_{1,1}^0 + (1 - m_1^{-1} m_2) I_{1,2}^1 \right) m'^2 \right\rangle.$$  \hspace{1cm} (4.16)

Upon symmetrization using the cyclic property to enforce explicit mirror symmetry, this result is easily shown to be equivalent to the known result (2.36).

The relation analogous to (4.12) for the LO in $d = 4$ dimensions is

$$W_{c,LO,d=4}^- = \left\langle \rho^4 \left( -\frac{1}{3} N^2 D^2 N^2 D^2 - \frac{2}{3} N^3 D N D D + \frac{1}{3} N D N D N D N D \right. \right.$$  
$$\left. - \frac{2}{3} D N D^2 N D D - \frac{1}{3} N D N D N^3 D - \frac{1}{3} N D N^2 D N D - \frac{2}{3} N D N^2 D N D N D - \frac{1}{3} N D N D D^2 N D - \frac{2}{3} N D N D D N D N D - \frac{1}{3} N D N D D D N D - \frac{2}{3} N D N D D D N D - \frac{1}{3} N D N D D D N D - \frac{2}{3} N D N D D D N D - \frac{1}{3} N D N D D D N D \right) \rangle_p.$$  \hspace{1cm} (4.17)

Once again this turns out to be covariant. Explicitly, the equation analogous to (4.14) is

$$W_{c,LO,d=4}^- = \left\langle \rho^4 \left( -\frac{2}{3} Q N^2 F N' - \frac{1}{3} N^2 Q' N^2 F - \frac{1}{3} N^2 F N^2 F - \frac{2}{3} N^3 F N F \right. \right.$$  
$$\left. - \frac{2}{3} Q N Q N^2 F' - \frac{1}{3} Q N Q N^2 F' - \frac{2}{3} Q N F N^2 N' - \frac{1}{3} Q N Q N^2 F - \frac{1}{3} N Q N^2 F N F \right.$$  
$$\left. + \frac{1}{3} N Q N Q N F N' - \frac{1}{3} N Q N Q N F N' - \frac{1}{3} N Q N F N N' - \frac{1}{3} N Q N F N N' - \frac{1}{3} N Q N^2 Q N Q' \right.$$  
$$\left. + \frac{1}{3} N Q N Q N' N' - \frac{2}{3} Q N Q N' N' + \frac{1}{3} Q N Q N Q N' - \frac{2}{3} N Q N Q N Q' \right.$$  
$$\left. + \frac{1}{3} N Q N Q N' - \frac{1}{3} Q N Q N Q N' \right) \rangle_p.$$  \hspace{1cm} (4.18)

A technical comment is in order. Formal expressions of the type (4.12) and (4.14) are essentially unique (the only freedom being cyclic permutation). On the other hand, the gauge invariant expressions (4.14) and (4.18) are by no means unique, due to integration by parts and Bianchi identities. So it is straightforward to go from the gauge invariant form to the formal one by undoing all commutators but not the other way around. To find a gauge invariant form as short as possible becomes a major issue.\[18\,33\,33\]

To obtain (4.18) from (4.17) we have found it convenient to write down the possible covariant terms and fit their coefficients in order to reproduce (4.17), trying to minimize the number of terms. (4.18) is not yet in the standard form (2.37) but it can be brought to that form by eliminating terms with $P^2$ by integration by parts. Then, labeling of operators allows to directly use the basis functions $I_{r_1 \ldots r_v}$. We have verified that this procedure precisely reproduces the results of Sec. 11 obtained from integration of the current.\[6\,17\,17\] It should be noted that the right-hand side of (2.27) is subject to UV ambiguities which could in principle produce a spurious “polynomial” term of the form $\langle FQ^2 \rangle$ \[19\]. Such term has not appeared in the present calculation.

\[6\] To avoid errors, these manipulations and the similar ones at NLO in the next section, have been carried out with help of symbolic algebra software. The manipulations required are greatly simplified with the help of our notational conventions.
To finish this section, we note that these constructions, starting from (4.3), show that $W^-_c$ can be written (to all orders) using as building blocks $N$, $Q_\mu$, $E_\mu$, and their covariant derivatives. Moreover, there is a very definite pattern, characteristic of Chan’s form, which is illustrated by (4.11) and (4.18) at LO and by (5.8) at NLO in next section, namely, at $n$-th order ($n$ derivatives) there are exactly $n$ blocks $N$. Compared to the bosonic expansions in [21, 33], in the fermionic case the structure is complicated due to the presence of $Q_\mu$, which was absent in the bosonic case. (Although some simplification is gained due to the $d$-form structure as well as $F'=0$ and $X'='=[F,X]$.) Because of the very restrictive pattern allowed in expansions in Chan’s form, the proliferation of terms is avoided and this gives rise to compact expressions.

Another consequence of this constructive method, as compared with that based on the current, is that it guarantees that $W^-_c$ can be written using only terms of the form local operator (i.e., derivatives of $m$ and $F_\mu$) times $I_{\mu_1...\mu_n}$ times integer powers of $m_j$ (the positive powers coming from derivatives of $m^2$ and the negative powers coming from $m^{-1}$ in $Q_\mu$).

V. NLO IN TWO DIMENSIONS

A. General considerations

In this section we compute $W^-_c$ at NLO in $d=2$. Because the NLO is UV finite, this coincides with $W^-_c$ at the same order. The general form of this term is

$$W^-_{\text{NLO},d=2} = \langle N_{12}^{1}\alpha\alpha F + N_{13}^{2}\alpha F + N_{123}^{3}m_{\alpha\alpha}m^2 + N_{1234}^{4}m_{\alpha\alpha}m^2 \rangle, \tag{5.1}$$

where $N^1, N^2, N^3, N^4$ are four functions of the labeled $m$. Mirror symmetry of $W^-_c$ implies the relations

$$N_{12}^{1} = -N_{21}^{1}, \quad N_{123}^{2} = -N_{321}^{2}, \quad N_{123}^{3} = -N_{213}^{3}, \quad N_{1234}^{4} = -N_{3214}^{4}. \tag{5.2}$$

In addition, $N^1$ and $N^3$ are odd functions of $m$, whereas $N^2$ and $N^4$ are even:

$$N_{12}^{1} = -N_{12}^{1}, \quad N_{123}^{2} = N_{213}^{2}, \quad N_{123}^{3} = -N_{213}^{3}, \quad N_{1234}^{4} = N_{1234}^{4}. \tag{5.3}$$

All other possible structures not present in (5.1) are redundant, either by using integration by parts or by rearrangement of indices. The rearrangement comes from the identities $X_{\alpha\beta} = X_{\beta\alpha} + [F_{\alpha\beta},X]$, and

$$\delta_{\alpha\beta}\epsilon_{\mu_1\mu_2...\mu_d} = \delta_{\alpha\mu_1}\epsilon_{\beta\mu_2...\mu_d} + \delta_{\alpha\mu_2}\epsilon_{\beta\mu_1...\mu_d} + \cdots + \delta_{\alpha\mu_d}\epsilon_{\beta\mu_1...\mu_{d-1}} \tag{5.4}$$

This latter relation allows to transform any “metric” index into a “differential form” index. E.g.

$$\epsilon_{\mu\nu}F_{\alpha\mu}m_{\alpha\nu} = \epsilon_{\mu\nu}(F_{\alpha\mu}m_{\alpha\nu} + F_{\nu\mu}m_{\alpha\alpha}) = -\frac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu}m_{\alpha\alpha}. \tag{5.5}$$

Because integration by parts and rearrangement of indices preserve the regularity condition, the functions $N^1, N^2, N^3, N^4$ are regular in the coincidence limit. Unlike the LO of $d=4$, there are no integration by parts ambiguities in (5.1). The only remaining ambiguity comes from the two-dimensional identity

$$m_{\alpha\alpha}m^2 = m'_{\alpha\alpha}m' - m^2_{\alpha\alpha} - m_{\alpha\alpha}m^2 \quad (d=2). \tag{5.6}$$

This implies that the further condition

$$N_{1234}^{4} - N_{4123}^{4} + N_{4213}^{4} - N_{3214}^{4} = 0, \tag{5.7}$$

can be imposed on $N^4$. (This choice is compatible with manifest mirror symmetry (5.2).) After $N^4$ is projected to comply with this condition all functions are uniquely fixed.

To determine the functions $N^1, N^2, N^3, N^4$ one can use the method of fixing them by reproducing the NLO current. The calculation of the current can be done using the technique of covariant symbols, along the lines explained in [17] for the LO. Alternatively, the algebra involved in computing the covariant current can be dealt with by using the world-line method [33, 36]. This latter approach has been applied in [18] to reproduce the LO results of [17] and to the first computation of the NLO current and of the effective action in $d=2$. To carry out the same NLO calculation we have applied two methods: first, one based on the current, but computed using the method of covariant symbols. This is described in Section VC. And second, a direct computation of the effective action using our version of Chan’s approach in the abnormal parity sector. This approach is discussed subsequently. We have verified that the two methods yield identical results and moreover they are fully consistent with the results presented in [18].
B. Direct computation of the effective action

The Chan’s -like derivative expansion technique of Sec. [IV] applies immediately to NLO in \( d = 2 \). In analogy with (4.14) and (4.18), the manifestly covariant expression in the present case is as follows:

\[
W_{\text{NLO, }d=2} = \left( p^A \left( NQ'N^2N_{\alpha\alpha} + N^2N_{\alpha\alpha}NF + \frac{1}{2}N^2Q'N^2Q_{\alpha\alpha} + N^2FN_{\alpha\alpha} \right. \right.
\]
\[
+ N^3FNQ_{\alpha\alpha} + \frac{1}{2}N^2N'N_{\alpha}N_{\alpha} + N_{\alpha\alpha}NF - NQ'N'N_{\alpha\alpha}
\]
\[
\left. \left. - \frac{1}{2}NQ'N_{\alpha}N_{\alpha} + \frac{1}{2}NQ'N_{\alpha}N_{\alpha} - NQ'N^2N_{\alpha} + \frac{1}{2}NQ'N^2Q_{\alpha\alpha} \right) \right)
\]
\[
+ QN_{\alpha\alpha}N'N_{\alpha\alpha} - \frac{1}{2}N^2QN'N_{\alpha\alpha} - NQ^2FN_{\alpha\alpha} + N^2FN_{\alpha\alpha}N_{\alpha}
\]
\[
+ NN_{\alpha\alpha}N'N_{\alpha\alpha} - \frac{1}{2}NN_{\alpha\alpha}N'N_{\alpha\alpha} + \frac{1}{2}QNN_{\alpha\alpha}N_{\alpha}N_{\alpha}
\]
\[
- \frac{1}{2}NN_{\alpha\alpha}N_{\alpha}N_{\alpha} + NN_{\alpha\alpha}NN_{\alpha} + \frac{1}{2}NN_{\alpha\alpha}N_{\alpha}N_{\alpha} - \frac{1}{2}QNN_{\alpha\alpha}N_{\alpha}N_{\alpha}
\]
\[
- \frac{1}{2}QQNN_{\alpha}N_{\alpha}N_{\alpha}' + QNQ_{\alpha\alpha}N_{\alpha\alpha} \right) \right)_p.
\]

(5.8)

Here we have adopted the convention that derivatives with differential form indices always act before derivatives with metric indices, that is,

\[
N_{\alpha}' := (N')_{\alpha} = \hat{D}_{\alpha}N' = N_{\alpha\mu}dx_\mu.
\]

(5.9)

We have selected the covariant terms in (5.8) in order to obtain an expression as short as possible. In a sense, (5.8) is already the result, in compact form. Using integration by parts and rearrangement of indices, the same functional can be brought to its unique standard form (5.11). A virtue of the present approach (compared to that based on the current) is that it immediately provides expressions for the functions \( N^k \) that are manifestly regular in the coincidence limit. Moreover they are linear combinations of the type powers of \( M_j \) times \( I_{r_1,...,r_n} \).

In general, such expressions can be further simplified, as we do now. Using the decomposition in components of well defined parity, as in (5.5), we find (we use \( N^1_{++} \) for \( N_{12}^{1,-,+,} \), etc):

\[
N^1_{-+} = -2I^2_{3,2},
\]
\[
N^2_{++} = 2I^3_{2,1,2} - 2I^3_{3,1,2},
\]
\[
N^2_{-+} = -I^3_{2,2,2} - 4I^3_{3,1,2},
\]
\[
N^2_{+-} = 2I^3_{2,1,3} - 2I^3_{3,1,2},
\]
\[
N^3_{++} = 2I^2_{3,2,2},
\]
\[
N^3_{+-} = 0, 
\]
\[
N^3_{-+} = -2I^3_{2,1,3,1} + 2I^3_{3,1,2,1},
\]
\[
N_4_{++} = -I^3_{3,2,2} - \frac{1}{2}I^2_{2,3,1} + \frac{1}{2}I^2_{2,1,2} + 3I^2_{3,1,2,1},
\]
\[
N_4_{-+} = \frac{1}{2}I^2_{2,1,3} - \frac{1}{2}I^2_{2,1,3,1} - \frac{1}{2}I^2_{3,2,2,1} + I^2_{3,1,2,1},
\]
\[
N_4_{+-} = -I^3_{2,1,3} + \frac{1}{2}I^2_{2,1,2,2} - \frac{1}{2}I^2_{2,2,1,2} - I^2_{3,1,2,1},
\]
\[
N_4_{-+} = -I^3_{1,1,1,1} + I^2_{3,1,2,2} + \frac{3}{2}I^2_{2,1,2,2} + \frac{1}{2}I^2_{2,2,2,1},
\]
\[
N_4_{--} = 0.
\]

(5.10)

All other components not quoted follow from mirror symmetry (5.2) or vanish by overall parity (5.3). Once again we find that negative powers of \( m^2_0 \) are not required. We do not have an explanation for this, but the simple expressions obtained after simplification suggest that a more direct route could exist.
The vanishing of $N^3_{---}$ and $N^4_{---}$ is easy to understand. It follows from the observation in \[17\] (p.179) that, for $d > 0$, $W^-$ must vanish identically when one of the scalar fields, e.g. $m_{RL}$, happens to be constant and there are no gauge fields present:

$$W^- = 0, \quad \text{whenever} \quad v^R = v^L = dm_{RL} = 0 \quad (d > 0). \quad (5.11)$$

At LO, this property dictates the form of $N^2_{--}^{-}$ in $d = 2$ and of $N^4_{1234}^{-}$ in $d = 4$, in order to cancel the contribution of $\Gamma_{gWZW} \[17\]$. At NLO, $W_c^-$ should vanish by itself. As is easy to see, when $v^R = v^L = dm_{RL} = 0$, the only surviving contributions in \[5.1\] would be those coming from $N^3_{---}$ and $N^4_{---}$ and so these functions must vanish.

C. Two dimensional NLO from the current

For comparison we present here the calculation of the effective action to NLO in two dimensions using the method based on the current.

The consistent current is defined as the variation of the effective action under an infinitesimal change of the gauge field, $\delta v_\mu$. Specifically,

$$\delta W^- = \langle J^- \delta v \rangle. \quad (5.12)$$

Using the identity \[5.4\], the Lorentz index in $\delta v_\mu$ can be taken as a differential form one so that $J^-$ is a $(d - 1)$-form. The covariant current at NLO takes the form

$$J^-_{\text{NLO},d=2} = A^1_{12} D_\alpha + A^2_{12} m'_\alpha + A^3_{123} m_\alpha D_\alpha + A^4_{321} D_\alpha m_\alpha$$

$$+ A^5_{12} m'_\alpha m_\alpha + A^6_{123} m_\alpha m'_\alpha + A^7_{321} m'_\alpha m_\alpha$$

$$+ A^8_{123} m_\alpha m'_\alpha + A^9_{1234} m_\alpha m'_\alpha m_\alpha - A^{10}_{321} m_\alpha m'_\alpha m_\alpha.$$  \( (5.13) \)

In this expression

$$D_\alpha = \delta_\alpha D = F_{\alpha\mu} dx_\mu, \quad D_{\alpha\alpha} = F_{\alpha\mu\alpha} dx_\mu,$$  \( (5.14) \)

and once again our convention is that derivatives with differential form indices act before derivatives with metric indices, so

$$m'_\alpha := \delta_\alpha m' = m_{\alpha\mu} dx_\mu, \quad m'_\alpha := m_{\nu\alpha\mu} dx_\mu.$$  \( (5.15) \)

In \[5.13\] we have used mirror symmetry to relate some of the functions $A^k$. In addition,

$$A^1_{12} = A^1_{21}, \quad A^2_{12} = -A^2_{21}, \quad A^6_{1234} = -A^6_{321}.$$  \( (5.16) \)

The explicit form of these functions is given below. They have been obtained using the technique of covariant symbols \[31, 32\] applied in \[17\] to obtain the LO current. In \[18\] this NLO current have been computed using the word-line approach. Note that the functions in \[18\] are not identical to the $N$ and $A$ here due to the different choice in the order of the derivatives.

Identifying the current \[5.13\] with the variation of $W^-_{\text{NLO}}$ in \[5.1\] yields the following set of equations:

$$A^1_{12} = (m_2 - m_1)N^1_{12} - (m_1 + m_2)N^2_{12},$$

$$A^2_{12} = -N^1_{12},$$

$$A^3_{123} = -2N^3_{123} + (m_1 + m_3)(\nabla_2 N^1)_{312} + (m_3 - m_2)N^2_{123} - (m_1 + m_3)N^2_{312},$$

$$A^4_{123} = -(\nabla_1 N^1)_{123} + (m_1 + m_3)N^3_{312} - (m_1 + m_3)N^3_{212},$$

$$A^5_{123} = -2N^2_{123} + (m_1 + m_3)N^3_{12},$$

$$A^6_{1234} = -(\nabla_2 N^2)_{1234} - N^3_{312} + N^3_{213} - (m_1 + m_4)(\nabla_1 N^3)_{312} - (m_1 + m_4)(\nabla_2 N^3)_{213}$$

$$+ (m_1 + m_4) N^4_{112} + (m_1 + m_4) N^4_{123},$$

$$A^7_{1234} = -(\nabla_3 N^3)_{1234} - N^3_{323} + (m_1 + m_4)(\nabla_2 N^3)_{123} + (m_1 + m_4)(\nabla_3 N^3)_{123}$$

$$- (m_1 + m_4)N^4_{1234} - (m_1 + m_4)N^4_{213}. \quad (5.17)$$
In these equations $\nabla$ represents a variation operator that increments the number of arguments by one [17, 28]. Explicitly,

$$ (\nabla_j f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_j, \hat{x}_{j+1}, \ldots, x_n) - f(x_1, \ldots, \hat{x}_j, x_{j+1}, \ldots, x_n) / x_j - x_{j+1}, $$

(5.18)

(where the hat indicates that the variable is missing). Note that $\nabla_j$ represents the variation with respect the $j$-th argument of $f$, so e.g. $(\nabla_2 N^3)_{4123}$ is not the variation with respect to $m_2$, but the variation with respect the (abstract) second argument of $N^3$, and the resulting function with four arguments is then evaluated at $(-m_4, m_1, m_2, m_3)$.

The equations (5.17) have to be supplemented with mirror symmetry of $W_{\text{NLO}}$, (5.2), overall parity (5.3) and the condition (5.7). The full set of equations is to be solved with respect to the unknowns $N^1, N^2, N^3, N^4$ in terms of the known functions $A^k$.

$N^1$ is immediately obtained from $A^2$. The equation for $A^1$ is automatically satisfied. $N^2$ and $N^3$ are obtained from $A^5$. Indeed, exchanging the labels 1, 3 and using mirror symmetry produces the new equation

$$ A^5_{321} = N^2_{123} + (m_1 + m_3)N^3_{312}. $$

(5.19)

The two $A^5$ equations provide algebraic solutions for $N^2_{123}$ and $N^3_{312}$. The equations $A^3$ and $A^4$ are automatically satisfied.

Similarly, exchanging the labels 1, 4 and 2, 3 in $A^7$ and using mirror symmetry gives a new $A^7$ equation. The two $A^7$ equations, together with $A^6$ and the condition (5.7) provide algebraic solutions to $N^3_{1234}, N^4_{2134}, N^4_{2314}, N^4_{3214}$. One verifies that the four functions $N^4$ so obtained are identical.

The solution found in this way coincides with that obtained in Sec. [17] through direct computation of the effective action.
For completeness we give below the functions $A^k$. The missing components are related by mirror symmetry.

\[
A^1_{++} = -2I_{2,2}^2,
A^1_{+-} = 0,
A^2_{++} = 2I_{2,2}^2,
A^3_{++} = -2m_3^2I_{2,1,2}^1,
A^3_{+-} = -2m_3^2m_1^2I_{2,1,2},
A^3_{--} = -2I_{1,1,2}^1,
A^3_{---} = 0,
A^5_{+++} = 6m_3^2I_{1,1,4}^2 + 2m_3^2I_{1,2,3}^2,
A^5_{++-} = -2m_3^2I_{1,2,3}^1,
A^5_{-+-} = -2m_3^2I_{2,1,3}^1,
A^5_{---} = -2I_{1,2,3}^1 - 2I_{1,3,2}^1 - 2I_{1,2,3}^1 - I_{2,2,2}^1,
A^5_{+++} = -4m_3^2I_{2,1,3}^2,
A^5_{++-} = -4m_3^2I_{1,2,3}^3,
A^5_{-+-} = -4I_{2,1,3}^1,
A^5_{---} = -4I_{2,1,3}^1,
A^9_{+++} = -2I_{2,1,1,3}^3 - \frac{2}{3}I_{2,1,2,2}^3 + \frac{2}{3}I_{2,1,3,1}^3 + \frac{4}{3}I_{2,2,1,2}^3 + \frac{2}{3}I_{2,2,2,1}^3 + \frac{2}{3}I_{3,2,1,1}^3 + \frac{8}{3}I_{3,1,1,2}^3
+ \frac{4}{3}I_{3,2,1,1}^1 + \frac{4}{3}I_{3,2,1,1}^1 + 2I_{1,1,1,1}^1,
A^9_{++-} = -\frac{2}{3}I_{2,1,1,3}^3 - \frac{2}{3}I_{2,2,1,2}^3 - \frac{2}{3}I_{1,2,3,1}^3 - \frac{4}{3}I_{1,3,1,2}^3 + \frac{4}{3}I_{1,3,2,1}^3 - 2I_{1,4,1,1}^3 - \frac{8}{3}I_{3,2,1,1}^3
- \frac{4}{3}I_{3,2,1,2}^3 - \frac{2}{3}I_{3,2,1,1}^3 + \frac{4}{3}I_{3,2,1,1}^3 - \frac{2}{3}I_{3,2,1,1}^3,
A^9_{-+-} = -4I_{2,1,1,3}^3 - 2I_{2,1,2,2}^3,
A^9_{---} = -4I_{2,1,1,3}^3 - 2I_{2,1,2,2}^3,
A^{10}_{+++} = -\frac{2}{3}I_{2,1,1,3}^3 + \frac{2}{3}I_{2,1,2,2}^3 + 2I_{2,1,3,1}^3 + \frac{4}{3}I_{2,2,1,2}^3 + \frac{2}{3}I_{2,2,2,1}^3 - \frac{2}{3}I_{3,2,1,1}^3
+ \frac{8}{3}I_{3,1,1,2}^3 + \frac{4}{3}I_{3,1,2,1}^3 - \frac{4}{3}I_{3,2,1,1}^1 - 2I_{4,1,1,1}^1,
A^{10}_{++-} = \frac{2}{3}I_{2,1,1,3}^3 + \frac{2}{3}I_{2,1,2,2}^3 + \frac{2}{3}I_{2,2,2,1}^3 + \frac{4}{3}I_{2,3,1,2}^3 + \frac{4}{3}I_{3,1,2,1}^3 + 2I_{4,1,1,1}^3 + \frac{4}{3}I_{3,2,1,2}^3
+ \frac{8}{3}I_{3,2,1,1}^3 + \frac{8}{3}I_{3,2,1,2}^3 + 2I_{2,2,2,1}^3 + \frac{4}{3}I_{2,3,1,1}^3 + 4I_{3,1,2,1}^3 + \frac{8}{3}I_{3,2,1,1}^3 + \frac{8}{3}I_{3,2,1,1}^3,
A^{10}_{-+-} = -\frac{2}{3}I_{2,1,1,3}^3 - \frac{4}{3}I_{2,1,3,2}^3 - 2I_{1,1,4,1}^3 - \frac{2}{3}I_{1,2,2,2}^3 - \frac{4}{3}I_{1,2,3,1}^3 - \frac{2}{3}I_{1,3,2,1}^3 - \frac{4}{3}I_{2,1,1,3}^3
+ \frac{4}{3}I_{3,2,1,1}^3 + \frac{4}{3}I_{3,2,1,2}^3 + \frac{2}{3}I_{3,2,2,1}^3 + \frac{8}{3}I_{3,3,1,2}^3 + 2I_{3,3,1,1}^3,
A^{10}_{---} = 2I_{1,1,1,4}^3 + \frac{4}{3}I_{1,1,3,3}^3 + \frac{2}{3}I_{1,1,3,2}^3 + \frac{4}{3}I_{1,2,1,3}^3 + \frac{2}{3}I_{1,2,2,2}^3 + \frac{2}{3}I_{1,3,1,2}^3
- \frac{8}{3}I_{3,2,1,1}^3 - \frac{4}{3}I_{3,2,1,2}^3 + \frac{2}{3}I_{3,2,2,1}^3 + \frac{2}{3}I_{3,3,1,2}^3,
A^{10}_{+++} = -4I_{2,1,1,3}^3 - 2I_{2,2,1,2}^3,
A^{10}_{++-} = -4I_{2,1,1,3}^3 - 2I_{2,1,2,2}^3,
A^{10}_{-+-} = -4I_{2,1,1,3}^3 - 2I_{2,1,2,2}^3,
A^{10}_{---} = 2I_{2,1,2,2}^3 + 4I_{2,1,3,1}^3 + 2I_{2,2,1,2}^3 + 2I_{2,2,2,1}^3 + 4I_{3,1,1,2}^3 + 4I_{3,1,2,1}^3.
\] (5.20)
VI. SUMMARY AND CONCLUSIONS

We have shown by direct calculation that, once the anomaly saturating WZW term is subtracted from the effective action, the chiral invariant remainder can be computed using a covariant formalism. Such a result was available in the literature for the real part but not for the imaginary part of the effective action.

The basic relation (2.27) holds to all orders, dimensions and topologies and presumably can be extended to include gravitational backgrounds. In particular, it should apply at finite temperature in the imaginary time approach. As is known, there is a thermal chiral invariant remainder [37] (the chiral anomaly is temperature independent [38]).

To carry out the calculations it has been extremely convenient to use the notation introduced in [17, 21]. The notation does not involve a rotation of the original fields appearing in the Dirac operator. In this notation, chiral covariant expressions behave formally as vector covariant ones and the number of structures involved, and so the algebra, diminishes considerably.

In order to carry out a calculation within the derivative expansion, we have found convenient to develop a new technique, along the lines of Chan’s approach for the bosonic case [20]. This approach yields manageable expressions for the effective action. Contributions to four derivatives are worked out explicitly and shown to agree with the results obtained by integration of the current.

Finally, a suitable basis of functions is introduced in terms of which the expressions, obtained by any method, are compactly packed while being easily translatable to explicit form (explicit rational functions with logarithms). This basis appears naturally in our calculation and avoids the need of fine-tuning required in the current method to satisfy the regularity conditions noted in [17].

Acknowledgments

I thank C. García-Recio for suggestions on the manuscript. This work is supported in part by funds provided by the Spanish DGI and FEDER funds with grant FIS2005-00810, Junta de Andalucía grants FQM225, FQM481 and P06-FQM-01735 and EU Integrated Infrastructure Initiative Hadron Physics Project contract RII3-CT-2004-506078.

APPENDIX A: MOMENTUM INTEGRALS

Let \( I \) be the integral on the left-hand side of the relation (3.3). We assume \( I \) to be UV convergent and \( s + d/2 - 1 \) to be a non negative integer. In addition, the \( r_j \) are integer and \( m_j^2 > 0 \). After angular integration

\[
I = \frac{1}{(4\pi)^{d/2} \Gamma(d/2)} \int_0^\infty dx x^{s+d/2-1} \prod_{j=1}^n \frac{1}{(x + m_j^2)^{r_j}}.
\]  

(A1)

This can be rewritten as

\[
I = \frac{(-1)^{s+d/2-1+\sum_j r_j}}{(4\pi)^{d/2} \Gamma(d/2)} \int_0^\infty dx \int_\gamma \frac{dz}{2\pi i} \left( \frac{1}{z_0 + x} - \frac{1}{z + x} \right) \\
\times z^{s+d/2-1} \prod_{j=1}^n \frac{1}{(z - m_j^2)^{r_j}}.
\]  

(A2)

Here \( \gamma \) is a contour that starts at \(-\infty \) (real) follows a path just above the real negative axis reaching zero and then goes back to \(-\infty \) following a path just below the real negative axis. For each \( x \), this is equivalent to a closed negative contour enclosing only the pole at \( z = -x \). The term with \( z_0 \) has no pole and so it gives no contribution.

Because the integral is UV convergent we can close the contour by adding the contour at infinity. This closed path can then be deformed to \( \Gamma \), which encloses only the poles at \( m_j^2 \) (by assumption there is no singularity at \( z = 0 \)),

\[
I = \frac{(-1)^{s+d/2-1+\sum_j r_j}}{(4\pi)^{d/2} \Gamma(d/2)} \int_\Gamma \frac{dz}{2\pi i} \log(z/z_0) z^{s+d/2-1} \prod_{j=1}^n \frac{1}{(z - m_j^2)^{r_j}}.
\]  

(A3)

In the UV convergent case this expression does not depend on the subtraction point \( z_0 \). (In the term with \( \log(z_0) \) \( \Gamma \)}
encloses all the singularities and so it is equivalent to the contour at infinity.) This is the right-hand side of (3.3).