Estimating Range Queries using
Aggregate Data with Integrity Constraints:
a Probabilistic Approach

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Abstract

The problem of recovering (count and sum) range queries over multidimensional data only on the basis of aggregate information on such data is addressed. This problem can be formalized as follows. Suppose that a transformation $\tau$ producing a summary from a multidimensional data set is used. Now, given a data set $D$, a summary $S = \tau(D)$ and a range query $r$ on $D$, the problem consists of studying $r$ by modelling it as a random variable defined over the sample space of all the data sets $D'$ such that $\tau(D') = S$. The study of such a random variable, done by the definition of its probability distribution and the computation of its mean value and variance, represents a well-founded, theoretical probabilistic approach for estimating the query only on the basis of the available information (that is the summary $S$) without assumptions on original data.

1 Introduction

In many application contexts, such as statistical databases, transaction recording systems, scientific databases, query optimizers, OLAP (On-line Analytical Processing), and many others, a multidimensional view of data is often

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adopted: Data are stored in multidimensional arrays, called datacubes [19,24], where every range query (computing aggregate values such as the sum of the values contained inside a range, or the number of occurrences of distinct values) can be answered by visiting sequentially a sub-array covering the range. In demanding applications, in order to both save storage space and support fast access, datacubes are summarized into lossy synopses of aggregate values, and range queries are executed over aggregate data rather than over raw ones, thus returning approximate answers. Approximate query answering is very useful when the user wants to have fast answers, thus avoiding waiting a long time to get a precision which is often not necessary.

Data aggregation and approximate answering have been first introduced many years ago for histograms [29] in the context of selectivity estimation (i.e. estimation of query result sizes) for query optimization in relational databases [8,10,32,34]. In this scenario, histograms are built on the frequency distribution of attribute values occurring in a relation, and are constructed by partitioning this distribution into a number of non-overlapping blocks (called buckets). For each of these blocks, a number of aggregate data are stored, instead of the detailed frequency distribution. The selectivity a query is estimated by interpolating the aggregate information stored in the histogram.

Later on, several techniques for compressing datacubes and allowing fast approximate answering have been proposed in the literature in the context of OLAP applications, where data to be summarized are called measure values (e.g., daily income of a shop, number of users accessing a service, etc.). Some of these approaches use either sampling [15,16,23,25] or complex mathematical transformations (such as wavelets) to compress data [14,36,38,39]. Indeed, the approach which turned out to be the most effective one (in terms of accuracy of the estimates) is the histogram-based one. In fact, both frequency distributions occurring in selectivity estimation and measure values in OLAP datacubes are multi-dimensional data distributions, which can be partitioned and aggregated adopting the same technique. Therefore, due to the increasing popularity of OLAP applications (which turned out to be particularly useful for the decision making process [7]), a renewed interest has been devoted to histogram-based compression techniques. Most of works on this topic mainly deal with either improving partitioning techniques in terms of efficiency and effectiveness [5,18,22,27], or maintaining the summary data up-to-date when the collected information changes continuously [11,20,21,37].

In this paper we address a different problem, which has been rather disregarded by previous research, and which is very relevant for an effective applicability of summarization techniques: We focus on the analysis of estimation errors which occur when evaluating range queries directly on summary data, without accessing original ones. Indeed, in all previous works dealing with histogram-based summarization techniques, either the estimation error is not studied at all, or only a rough evaluation of upper bounds of this error is given [28]. The lack of information on the estimation error reduces the scope of applicability
of approximate query answering: approximate results are really usable only if they are returned together with a detailed analysis of the possible error so that, if the user is not satisfied with the obtained precision, s/he may eventually decide to submit the query on the actual datacube.

In more detail, we study the problem of estimating count and sum range queries issued on a compressed datacube in a rather general framework: we assume that compression has been performed by first partitioning a given datacube into a number of blocks using any of the various proposed techniques, and then storing aggregate information for each block. This aggregate information mainly consists in the sum and the number of the elements belonging to each block. Moreover, we assume that some integrity constraints, which are expressible in a succinct format, are stored. Our approach is independent on the technique used to partition the datacube into blocks: its concern is estimating values and accuracy of range queries using aggregate data - no matter how they have been obtained - using just interpolation with no assumptions on the actual distribution of data into aggregation blocks.

The evaluation of the accuracy of estimates is based on a probabilistic framework where queries are represented by means of random variables, and is performed as follows. Given a datacube \( D \) and the compressed datacube \( S \) obtained from \( D \) by applying the histogram-based compression strategy introduced above, we denote the transformation from \( D \) to \( S \) by \( \tau \), thus \( S = \tau(D) \). Let now \( \mathcal{D}_S \) denote the set of all the datacubes \( \bar{D} \) such that \( \tau(\bar{D}) = S \). Observe that any datacube \( \bar{D} \in \mathcal{D}_S \) is a possible guess of the original datacube \( D \), done only on the basis of the knowledge of \( S \). So, if we are given a range query \( r \) on \( D \), estimating \( r \) from \( S \) can be thought of as guessing the response to \( r \) on \( D \) by applying the range query \( r \) to any datacube \( \bar{D} \) of \( \mathcal{D}_S \). According to this observation, we model the estimation of the range query \( r \) from \( S \) as the mean value of the random variable defined by \( r \) on the sample set \( \mathcal{D}_S \), representing all possible guesses of \( r \) compatible with the summary \( S \). In order to analyze the estimation error, we thus study this random variable by determining its probability distribution and its variance. Actually, our analysis considers a family of transformations \( \tau \) based on the partition of the datacube into blocks, where each transformation stores different aggregate information for each block and a number of integrity constraints. The introduction of integrity constraints allows us to take into account more detailed information than sum and count on a block, whose exploitation may bias significantly the estimation toward the actual value. Indeed, integrity constraints produce a restriction of the sample space and a reduction of the variance of the estimation w.r.t. to the case of absence of integrity constraints. The integrity constraints which have been considered in this work concern the minimum number of null or non-null tuples occurring in ranges of datacubes. Although more complex constraints could be considered, we have restricted our attention to this kind of constraint since they often arise in practice. For instance, given a datacube
whose dimensions are the time (in terms of days) and the products, and the
measure is the amount of daily product sales, realistic integrity constraints
are that the sales are null during the week-end, while at least 4 times a week
the sales are not null.

Plan of the paper. The paper is organized as follows. In Section 2, a simple
compression technique which will be used for explaining and applying our esti-
mation paradigm is introduced, and integrity constraints about the number of
null or non-null tuples in the datacube ranges are formally defined. In Section
3 the probabilistic framework for estimating count and sum range queries is
formalized. In Sections 4, 5 and 6 three different estimation paradigms (explo-
iting different classes of aggregate information) are introduced: in Section
4, sum queries are estimated by using only the information about the sum of
the values contained in each bucket, whereas count queries are evaluated by
exploiting only the information on the number of non null elements in each
bucket. Section 5 shows how the information on the number and the sum of
the non null values contained in each bucket can be used jointly to estimate
sum and count queries. In Section 6 the estimation using integrity constraints
is formalized and in the subsequent section we elaborate on the “positive”
influence of integrity constraints on the accuracy of query estimations, and
substantiate our claim with some experimental results obtained applying our
estimation techniques to real-life data distributions. Some interesting applica-
tions of our theoretical framework for the estimation of frequency distributions
inside a histogram are presented in Section 8. Finally, in Section 9 conclusions
and future research lines are discussed. In particular, we stress that our work
is not certainly conclusive, since a larger family of transformations can be
considered, by taking into account different aggregates and other integrity
constraints. Indeed the main contribution of the work is the definition of a
novel approach for modelling and studying the issue of approximate queries
from a theoretic point of view.

2 Datacubes, their Compressed Representation and Integrity Con-
straints

2.1 Preliminary Definitions

In this section we give some preliminary definitions and notations. Let \( \mathbf{i} = <i_1, \ldots, i_r> \) and \( \mathbf{j} = <j_1, \ldots, j_r> \) be two \( r \)-tuples of cardinals, with \( r > 0 \). We extend common operators for cardinals to tuples in the obvious way: \( \mathbf{i} \leq \mathbf{j} \) means that \( i_1 \leq j_1, \ldots, i_r \leq j_r \); \( \mathbf{i} + \mathbf{j} \) denotes the tuple \( <i_1 + j_1, \ldots, i_r + j_r> \) and
so on. Given a constant \( p \geq 0 \), \( p^r \) (or simply \( \mathbf{p} \), if \( r \) is understood) denotes
the \( r \)-tuple of all \( p \); for instance, if \( p = 1 \) and \( r = 5 \), the term \( \mathbf{1} \) denotes the
tuple \(<1,1,1,1>\). Finally, \([i..j] = [i_{1..j_1}, \ldots, i_{r..j_r}]\) denotes the range of all \(r\)-tuples from \(i\) to \(j\), that is \(\{q| 1 \leq q \leq j\}\).

**Definition 1** A multidi-dimensional relation \(R\) is a relation whose scheme consists of \(r > 0\) dimensions (also called functional attributes) and \(s > 0\) measure attributes. The dimensions are a key for the relation so that there are no two tuples with the same dimension value.

From now on consider given a multidimensional relation \(R\). For the sake of presentation but without loss of generality, we assume that:

- \(s = 1\), and the domain of the unique measure attribute is the set of cardinals,
- \(r \geq 1\), and the domain of each dimension \(q\), with \(1 \leq q \leq r\), is the range \([1..n_q]\), where \(n_q > 2\) (i.e., the projection of \(R\) on the dimensions is a subset of \([1..n]\), where \(n = <n_1, \ldots, n_r>\)).

Given any range \([i..j]\), \(1 \leq i \leq j \leq n\), we consider the following range queries on \(R\):

- **count query:** \(\text{count}^{i..j}(R)\) denotes the number of tuples of \(R\) whose dimension values are in \([i..j]\);
- **sum query:** \(\text{sum}^{i..j}(R)\) denotes the sum of all measure values for those tuples of \(R\) whose dimension values are in \([i..j]\).

Since the dimension attributes are a key, the relation \(R\) can be naturally viewed as a \([1..n]\) matrix \(M\) of elements with values in \(\mathcal{N}\). In the rest of the paper this matrix will be called *datacube*.

**Definition 2** The datacube \(M\) corresponding to the multidimensional relation \(R\) is the \([1..n]\) matrix of cardinals such that, for each \(i \in [1..n]\), \(M[i] = v\) if the tuple \(<i, v>\) is in \(R\) or \(M[i] = 0\), otherwise.

As a consequence of the above definition, \(1\) is a null element if either \(<1, 0>\) is in \(R\) or no tuple with dimension value \(1\) is present in \(R\).

The above range queries can be now re-formulated in terms of array operations as follows:

- \(\text{count}^{i..j}(R) = \text{count}(M[i..j]) = |\{q| q \in [i..j] \text{ and } M[q] > 0\}|\);
- \(\text{sum}^{i..j}(R) = \text{sum}(M[i..j]) = \sum_{q \in [i..j]} M[q]\),

where \([i..j]\) is any range such that \(1 \leq i \leq j \leq n\).

### 2.2 Compressed Datacubes

We next introduce a compressed representation of the relation \(R\) by dividing the corresponding datacube \(M\) into a number of blocks and by storing a
First we need to formalize the notion of compression factor:

**Definition 3** Given $\mathbf{m} = \langle m_1, \ldots, m_r \rangle$, $1 \leq \mathbf{m} \leq \mathbf{n}$, an $\mathbf{m}$-compression factor for $M$ is any tuple $\mathbf{F} = \langle f_1, \ldots, f_r \rangle$, such that for each dimension $q$, $1 \leq q \leq r$, $f_q$ is a $[0 \ldots m_q]$ array for which $0 = f_q[0] < f_q[1] < \cdots < f_q[m_q] = n_q$, i.e., $f_q$ divides the dimension $q$ into $m_q$ parts.

Observe that $\mathbf{F}$ partitions the range $[1 \ldots n]$ into $m_1 \times \cdots \times m_r$ blocks. Each of these blocks, denoted as $B_k$, corresponds to a tuple $\mathbf{k} = \langle k_1, \ldots, k_r \rangle$ in $[1..\mathbf{m}]$. Each block $B_k$ has range $[F^- (k) \ldots F^+(k)]$, where $F^+(k)$ and $F^-(k)$ denote the tuples $<f_1[k_1], \ldots, f_r[k_r]>$ and $<f_1[k_1-1]+1, \ldots, f_r[k_r-1]+1>$, respectively. The size of $B_k$ (i.e. the number of cells inside the range of $B_k$) is $(f_1[k_1] - f_1[k_1-1]) \times \cdots \times (f_r[k_r] - f_r[k_r-1])$.

As an example, consider the $[1..10,1..6]$ datacube $M$ in Figure 1(a), which is partitioned into 6 blocks as shown in Figure 1(b). We have that $\mathbf{m} = <3,2>$, $f_1[0] = 0$, $f_1[1] = 3$, $f_1[2] = 7$, $f_1[3] = 10$, and $f_2[0] = 0$, $f_2[1] = 4$, $f_2[2] = 6$. The block $B_{<1,1>}$ has size $3 \times 2$ and range $[1..3,1..4]$; the block $B_{<1,2>}$ has size $3 \times 2$ and range $[1..3,5..6]$, and so on.

**Definition 4** Given an $\mathbf{m}$-compression factor $\mathbf{F}$, a ($\mathbf{F}$-)compressed representation of the datacube $M$ is the pair of $[1..\mathbf{m}]$ matrices $M_{\text{count}, \mathbf{F}}$ and $M_{\text{sum}, \mathbf{F}}$ such that for each $\mathbf{k} \in [1..\mathbf{m}]$, $M_{\text{count}, \mathbf{F}}[\mathbf{k}] = \text{count}(M[F^-(\mathbf{k})..F^+(\mathbf{k})])$ and $M_{\text{sum}, \mathbf{F}}[\mathbf{k}] = \text{sum}(M[F^-(\mathbf{k})..F^+(\mathbf{k})])$.

The compressed representation of the datacube $M$ in Figure 1(a) is represented in Figure 1(c), where each block is associated to a triplet of values. These values indicate, respectively, the range, the number of non-null elements and the sum of the elements in the corresponding block. For instance, the block $B_{<1,1>}$ has range $[1..3,1..4]$ and contains 8 non-null elements with sum 26; the block $B_{<1,2>}$ has range $[1..3,5..6]$ and contains 5 non-null elements with sum 29, and so on.

![Fig. 1. A two-dimensional datacube and its compressed representation](image.png)
From now on, consider given an m-compression factor $F$ and the corresponding $F$-compressed representation of the datacube $M$.

### 2.3 Integrity Constraints

The aim of compressing a datacube is to reduce the storage space consumption of its representation, in order to make answering range queries more efficient to perform. In fact queries can be evaluated on the basis of the aggregate data stored in the compressed datacube without accessing the original one, and the amount of data that must be extracted from the compressed datacube to answer a query is generally smaller than the number of data that should be extracted from the original datacube. This approach introduces some approximation, which is tolerated in all those scenarios (such as selectivity estimation and OLAP services) where the efficiency of query answering is mandatory and the accuracy of the answers is not so relevant.

The estimation of queries could be improved (in terms of accuracy) if further information on the original data distribution inside the datacube is available. Obviously, this additional information should be easy to be exploited so that the efficiency of the estimation is not compromised.

In this section we introduce a class of integrity constraints which match these properties: they can be stored in a succinct form (thus they can be accessed efficiently), and provide some additional information (other than the aggregate data stored in the compressed datacube) which can be used in query answering, as will be explained in the following sections.

Let $2^{[1..n]}$ be the family of all subsets of indices in $[1..n]$. We analyze two types of integrity constraint:

- **number of elements that are known to be null**: we are given a function $LB_{=0} : 2^{[1..n]} \rightarrow \mathbb{N}$ returning, for any $D$ in $2^{[1..n]}$, a lower bound to the number of null elements occurring in $D$; the datacube $M$ satisfies $LB_{=0}$ if, for each $D$ in $2^{[1..n]}$, $\sum_{i \in D} count(M[i]) \leq |D| - LB_{=0}(D)$, where $|D|$ is the number of elements of $M$ in $D$;

- **number of elements that are known to be non-null**: we are given a function $LB_{>0} : 2^{[1..n]} \rightarrow \mathbb{N}$ returning, for any $D$ in $2^{[1..n]}$, a lower bound for the number of non-null elements occurring in $D$; the datacube $M$ satisfies $LB_{>0}$ if, for each $D$ in $2^{[1..n]}$, $\sum_{i \in D} count(M[i]) \geq LB_{>0}(D)$.

The two functions $LB_{=0}$ and $LB_{>0}$ are monotonic: for each $D', D''$ in $2^{[1..n]}$, if $D' \subset D''$ then both $LB_{=0}(D') \leq LB_{=0}(D'')$ and $LB_{>0}(D') \leq LB_{>0}(D'')$ hold. From now on, consider given the above two functions together with the compressed representation of $M$.

We point out that the integrity constraints expressed by $LB_{=0}$ and $LB_{>0}$ often occur in practice. For instance, consider the case of a temporal dimension with
granularity *day* and a measure attribute storing the amount of sales for every day. Given any temporal range, we can easily recognize a number of *certain* null values, corresponding to the holidays occurring in that range. In similar cases, the constraints provide additional information that can be efficiently computed with no overhead in terms of storage space on the compressed representation of $M$.

As an example on how $LB_{=0}$ and $LB_{>0}$ influences the estimation of range queries, consider the following case. Suppose that $LB_{=0}([4..6,1..3]) = 3$ and $LB_{>0}([4..6,1..3]) = 1$ for the two-dimensional datacube of Figure 1. From this, we can infer that the number of non-null elements in the range $[4..6,1..3]$ is between $1$ and $(6 - 4 + 1) \times (3 - 1 + 1) - 3 = 6$. Note that the compressed representation of $M$ in Figure 1(b) only contains the information that the block $[4..7,1..4]$ has $7$ non-nulls; so, without the knowledge about the above constraints, we could only derive that the bounds on the number of non-null elements in $[4..6,1..3]$ are $0$ and $7$.

### 3 The Probabilistic Framework for Range Query Estimation

We next introduce a probabilistic framework for estimating the answers of range queries (*sum* and *count*) by consulting aggregate data rather than the actual datacube. To this aim, we view queries as random variables and we give their estimation in terms of mean and variance.

A range query $Q$ on the datacube $M$ is modelled as a random variable $\overline{Q}$ defined by applying $Q$ on a datacube $\tilde{M}$ extracted from a datacube population *compatible* with $M$, thus consisting of datacubes whose $F$-compressed representations coincide (at least partially) with that of $M$.

More precisely, we have different random variables modelling $Q$, depending on what exactly we mean for ‘compatible’, and thus on the datacube population on which the query is applied. In particular, we consider the following populations:

- $M_{c,F}^{-1}$ is the set of all the $[1..n]$ matrixes $M'$ of elements in $\mathcal{N}$ for which $M'_{\text{count},F} = M_{\text{count},F}$;
- $M_{s,F}^{-1}$ is the set of all the $[1..n]$ matrixes $M'$ of elements in $\mathcal{N}$ for which $M'_{\text{sum},F} = M_{\text{sum},F}$;
- $M_{cs,F}^{-1}$ is the set of all the $[1..n]$ matrixes $M'$ of elements in $\mathcal{N}$ for which $M'_{\text{count},F} = M_{\text{count},F}$ and $M'_{\text{sum},F} = M_{\text{sum},F}$;
- $\Pi_{LB_{=0},LB_{>0}}(M_{cs,F}^{-1}) = \{M' | M' \in M_{cs,F}^{-1} \land M' \text{satisfies both } LB_{=0} \text{ and } LB_{>0} \}$ is the sub-population of $M_{cs,F}^{-1}$ which also satisfy the integrity constraints.
On the whole, given a range \([i..j]\), \(1 \leq i \leq j \leq n\), of size (i.e., number of elements occurring in it) \(b_{i..j}\), we study the following six random variables, grouped into three cases:

**Case 1:** For the estimation of \(\text{count}(M[i..j])\) we consider the population of all datacubes having the same number of non-nulls in each block as \(M\), and for that of \(\text{sum}(M[i..j])\) the population of all datacubes whose blocks have the same sum of the corresponding blocks in \(M\). Thus, we study the following two random variables:
- The random variable \(C_1(b_{i..j})\), computing \(\text{count}(\tilde{M}[i..j])\), where \(\tilde{M}\) is extracted from the population \(M^{-1}_{c,F}\).
- The random variable \(S_1(b_{i..j})\), computing \(\text{sum}(\tilde{M}[i..j])\), where \(\tilde{M}\) is extracted from the population \(M^{-1}_{s,F}\).

Note that, as will be clear in the following, both the random variables above are only function of the size \(b_{i..j}\) of the range size, and not of its boundaries \(i\) and \(j\).

**Case 2:** We estimate the number and the sum of the non-null elements in \(M[i..j]\) by considering the population of all the datacubes whose blocks have both the same sum and the same number of non-nulls as the corresponding blocks in \(M\). Then, the random variables are:
- The random variable \(C_2(b_{i..j})\), computing \(\text{count}(\tilde{M}[i..j])\), where \(\tilde{M}\) is extracted from the population \(M^{-1}_{cs,F}\).
- The random variable \(S_2(b_{i..j})\), computing \(\text{sum}(\tilde{M}[i..j])\), where \(\tilde{M}\) is extracted from the population \(M^{-1}_{cs,F}\).

Again, \(C_2\) and \(S_2\) depend only on the size of the range and not on the range itself.

**Case 3:** We consider the population of all datacubes having both the same sum and the same number of non-nulls in each block as \(M\), and, besides, satisfying the lower bound constraints on the number of null and non-null elements occurring in each range. Thus, we study the following two random variables:
- The random variable \(C_3([i..j])\), computing \(\text{count}(\tilde{M}[i..j])\), where \(\tilde{M}\) is extracted from the population \(\Pi_{LB_{=0},LB_{>0}}(M^{-1}_{cs,F})\).
- The random variable \(S_3([i..j])\), computing \(\text{sum}(\tilde{M}[i..j])\), where \(\tilde{M}\) is extracted from the population \(\Pi_{LB_{=0},LB_{>0}}(M^{-1}_{cs,F})\).

In this case, differently from the previous ones, the examined random variables are function of the range \([i..j]\) (not only of its size) as the value returned by \(LB_{=0}\) and \(LB_{>0}\) depend on the considered range.

We observe that Case 2 can be derived from the more general Case 3 but, for the sake of presentation, we first present the simpler case, and then we move to the general one. Actually the results of Case 2 will be stated as corollaries of the corresponding ones of Case 3 and their proofs will be postponed in the Appendix.
For each random variable above, say \( cs(\bar{M}[i..j]) \) (where \( cs \) stands for \textit{count} or \textit{sum}), we have to determine its probability distribution and then its mean and variance. Concerning the mean \( E(cs(\bar{M}[i..j])) \), due to the linearity of \( E \), we have:

\[
E(cs(\bar{M}[i..j])) = \sum_{B_q \in TB_F(i..j)} M_{cs,F}[q] + \sum_{B_k \in PB_F(i..j)} E(cs(\bar{M}[i_k..j_k]))
\]

where:

1. \( TB_F(i..j) \) returns the set of blocks \( B_q \) that are totally contained in the range \([i..j]\), i.e. every block \( B_q \) such that both \( i \leq F^-(q) \) and \( F^+(q) \leq j \),
2. \( PB_F(i..j) \) returns the set of blocks \( B_k \) that are partially inside the range, i.e. \( B_k \not\in TB_F(i..j) \) and either \( i \leq F^-(k) \leq j \) or \( 1 \leq F^+(k) \leq j \), and
3. for each \( B_k \in PB_F(i..j) \), \( s_k \) and \( j_k \) are the boundaries of the portion of the block \( B_k \) which overlaps the range \([i..j]\), i.e., \( [s_k..j_k] = [i..j] \cap [F^-(k)..F^+(k)] \).

For instance, consider the datacube in Figure 1(a) and the range \([i..j]\) whose boundaries are \( i = <4,3> \) and \( j = <8,6> \). Then the block \( B_{<2,2>} \) is totally contained in the \([i..j]\), the blocks \( B_{<2,1>}, B_{<3,1>}, B_{<3,2>} \) are partially contained in \([i..j]\), whereas the blocks \( B_{<1,1>}, B_{<1,2>} \) are outside \([i..j]\).

Concerning the variance, we assume statistical independence between the measure values of different blocks, so that its value is determined by summing the variances of all the partially overlapped blocks, thus introducing no covariance:

\[
\sigma^2(cs(\bar{M}[i..j])) = \sum_{B_k \in PB_F(i..j)} \sigma^2(cs(\bar{M}[i_k..j_k])).
\]

It turns out that we only need to study the estimation of a query inside one block, as all other cases can be easily re-composed from this basic case: the estimate of a query involving more than one block is the sum of the estimates for each of the blocks involved, and the same holds for the variance.

Therefore, from now on we assume that the query range \([i..j]\) is strictly inside one single block, say the block \( B_k \), i.e. \( F^-(k) \leq i \leq j \leq F^+(k) \). We use the following notations and assumptions:

1. \( b \) is the size of \( B_k \), that is the total number of null and non-null elements in \( B_k \);
2. \( b_{i..j} \) is the size of the query range \([i..j]\), that is the number of elements in the range \( 1 \leq b_{i..j} < b \);
3. \( t = M_{\text{count},F}[k] \) is the number of non-null elements in \( B_k \) \( (1 \leq t \leq b) \);
4. \( s = M_{\text{sum},F}[k] \) is the sum of the elements in \( B_k \).
Case 1: using the number and the sum of non-null elements separately

In this section we study the estimation of count and sum queries on the basis of the sum and count information given for each block (that is, the sum $s$ of the elements occurring in each block, and the number $t$ of non null elements in it).

Let us first perform the estimation of the range query $\text{count}(M[i..j])$. Notice that the random variable representing the answer of a count query depends on the size $b_{i..j}$ of the range $[i..j]$ involved in the query, rather than on the position of the range in the block.

**Theorem 1** Let $C_1(b_{i..j}) = \text{count}(\tilde{M}[i..j])$ be the integer random variable ranging from 0 to $t$ defined by extracting $\tilde{M}$ from the datacube population $M^{-1}_{c,F}$. Then:

1. the probability distribution $P(C_1(b_{i..j}) = t_{i..j})$ is:
   \[
P = \begin{cases} 
   \left( \begin{array}{c} b_{i..j} \\ t_{i..j} \\ \end{array} \right) \cdot \left( \begin{array}{c} b - b_{i..j} \\ t - t_{i..j} \\ \end{array} \right) & \text{if} \ max\{0, b_{i..j} - (b-t)\} \leq t_{i..j} \leq \min\{t, b_{i..j}\} \\
   0 & \text{otherwise}
   \end{cases}
   \]

2. mean and variance are, respectively:
   \[
   E(C_1(b_{i..j})) = (b_{i..j}/b) \cdot t \\
   \sigma^2(C_1(b_{i..j})) = t \cdot (b - t) \cdot b_{i..j} \cdot \frac{b - b_{i..j}}{b^2 \cdot (b - 1)}
   \]

**Proof.** It is easy to see that the probability that the number of non-null elements is $t_{i..j}$ corresponds to the probability of extracting $t_{i..j}$ times the value 1 in $b_{i..j}$ trials, using a binary variable (with values $\epsilon$, corresponding to a null value, and 1, corresponding to non-null) in a sample set composed of $b$ variables, with probability of finding 1 equal to $t/b$. This case is known to be characterized by the above expression, that is called hypergeometric distribution [13]. Thus, mean and variance are those of a random variable following a hypergeometric distribution.

The diagram in Fig. 2 shows how the variance of $C_1(b_{i..j})$ changes when we vary $t/b$ for a query of size $b_{i..j} = b/2$ and a block of size $b = 1000$. The estimated error is maximum for $t = b/2$ and behaves symmetrically for $t > b/2$ and $t < b/2$. This result can be explained by observing that, when $t = b/2$, the uncertainty in the distribution of null elements is maximum, since the probability that a fixed element inside the block is null is the same.
Fig. 2. $\sigma(C_1(b_{i..j}))$ versus $t/b$ for a block of size $b = 1000$ and a query of size $b_{i..j} = b/2$ as it is not null. The variance is symmetric w.r.t. $t = b/2$ as the error which occurs when we estimate $count([i..j])$ on a block of size $b$ containing $t$ non null elements is equal to the error of the estimate of the same range query over a block with the same size $b$, but containing $b - t$ non null elements.

The behavior of $\sigma(C_1(b_{i..j}))$ w.r.t. $b_{i..j}/b$ is analogous to the behavior of $\sigma(C_1(b_{i..j})$ w.r.t. $t/b$: The estimated error is maximum for $b_{i..j} = b/2$, and is symmetric for $b_{i..j} > b/2$ and $b_{i..j} < b/2$. The maximum uncertainty in the estimated result is reached when the size of the query is an half of the size of the whole block. The estimation becomes more accurate as $b_{i..j}$ gets near to $b$ or to 0: When $b_{i..j} = b$ the computed answer of the query is exact and is given by $t$, whereas if $b_{i..j} = 0$ the returned answer is zero.

The maximum estimation error which may occur when $E(C_1(b_{i..j}))$ is returned as the answer of the range query $count(b_{i..j})$, denoted by $err_{C_1}^{MAX}$, is quantified next.

**Proposition 1**

$$err_{C_1}^{MAX} = \max \left\{ \frac{b_{i..j}}{b} \cdot t - \max\{0, t - (b - b_{i..j})\}, \min\{t, b_{i..j}\} - \frac{b_{i..j}}{b} \cdot t \right\}$$

**Proof.** The maximum error can be obtained when the actual number of non null elements inside the range of the query is either minimum (i.e. $count(b_{i..j}) = \max\{0, t - (b - b_{i..j})\}$) or maximum (i.e. $count(b_{i..j}) = \min\{t, b_{i..j}\}$).

Let us now study the random variable $sum(\tilde{M}[i..j])$ representing the answer of a sum query on the range $[i..j]$ assuming only the knowledge of the sum $s$ of the elements occurring in the block $k$. Once again, the estimated value depends on the size of the range involved in the query and not on its actual position in the block.

**Theorem 2** Let $S_1(b_{i..j}) = sum(\tilde{M}[i..j])$ be the integer random variable ranging from 0 to $s$ defined by extracting $M$ from the datacube population $M_{s,F}^{-1}$.
Then:

(1) the probability distribution of \( S_1(b_{i,j}) \) is:

\[
P(S_1(b_{i,j}) = s_{i,j}) = \begin{cases} 
\frac{b_{i,j} + s_{i,j} - 1}{s_{i,j}} \left( \frac{b - b_{i,j} + s - s_{i,j} - 1}{s - s_{i,j}} \right) & \text{if } 0 \leq s_{i,j} \leq s \\
0 & \text{otherwise}
\end{cases}
\]

(2) mean and variance are, respectively:

\[
E(S_1(b_{i,j})) = (b_{i,j}/b) \cdot s
\]

\[
\sigma^2(S_1(b_{i,j})) = b_{i,j} \cdot s \cdot \frac{(b - b_{i,j}) \cdot (b + s)}{b^2 \cdot (b + 1)}
\]

**Proof.** (1) We can see the block \( k \) as a vector \( V \) of \( b \) elements which can assume values between 0 and \( s \). Let \( V_{b_{i,j}} \) be the portion of \( V \) of size \( b_{i,j} \) containing the elements inside the range of the query, and let \( V_{b-b_{i,j}} \) be the remainder part of \( V \). The random variable \( S_1(b_{i,j}) \) represents the sum of the elements belonging to \( V_{b_{i,j}} \). The probability that \( S_1(b_{i,j}) \) assumes the value \( s_{i,j} \) can be obtained by considering all possible value assignments to the elements in \( V_{b_{i,j}} \) so that their sum is \( s_{i,j} \), combined with all possible value assignments to the elements of \( V_{b-b_{i,j}} \) so that the total sum is \( s \). The above considered assignments represent the cases of success. The number of possible cases can be similarly obtained by considering all possible value assignments to the \( b \) elements in \( V \) so that their sum is \( s \).

The number of all the possible assignments from the domain of cardinals to \( y \) elements whose sum is \( z \) is equal to the number of multisets with elements taken from the set \( \{1, \ldots, y\} \) and having cardinality \( z: \binom{y + z - 1}{z} \).

Thus, the number of possible assignments for the elements in the portion \( V_{b_{i,j}} \) is: \( A = \binom{b_{i,j} + s_{i,j} - 1}{s_{i,j}} \), whereas the assignments for \( V_{b-b_{i,j}} \) are: \( B = \binom{(b-b_{i,j}) + (s-s_{i,j}) - 1}{(s-s_{i,j})} \cdot \binom{b + s - 1}{s} \). Analogously, there are \( C = \binom{b + s - 1}{s} \) different assignments of cardinals to the elements in the whole \( V \) such that the sum is \( s \). Hence, the probability that the sum inside a range of size \( b_{i,j} \) is \( s_{i,j} \) is given by: \( \frac{A \cdot B}{C} \).

(2) Consider the vectors \( V, V_{b_{i,j}} \) and \( V_{b-b_{i,j}} \) defined above. The event \( (S_1([i..j]) = s_{i,j}) \) is equivalent to the following event: The sum of all the elements in \( V_{b_{i,j}} \) is \( s_{i,j} \). Let \( V[i] \) be a random variable corresponding to the \( i \)-th element of \( V \). From \( s = \sum_{1 \leq i \leq b} V[i] \), we derive \( s = \sum_{1 \leq i \leq b} E(V[i]) \) by linearity of the operator \( E \). The mean of the random variable \( V[i] \) is equal to the mean of the random variable \( V[j] \), for any \( i, j, 1 \leq i, j \leq b \). For symmetry, the probability that an element of \( V \) assumes a given value is independent on the position of
this element inside the vector. Let denote by $m$ this mean. From the above formula for $s$ it follows that $m \cdot b = s$, thus $m = s/b$. Consider now the vector $V_{b_{i,j}}$. Let $S'$ be the random variable representing the sum of all the elements of $V_{b_{i,j}}$. Then $E(S') = b_{i,j} \cdot m$. Hence, $E(S') = b_{i,j} \cdot s/b$. 

The variance can be obtained using its definition. The detailed proof is rather elaborated and, for the sake of presentation, is included in the Appendix as Claim 1.

The maximum estimation error which may occur while returning $E(S_1(b_{i,j}))$ as the answer of the range query $\text{sum}(b_{i,j})$, denoted by $err_{S_1}^{MAX}$, is quantified next.

**Proposition 2**

$$err_{S_1}^{MAX} = \max \left\{ \frac{b_{i,j}}{b} \cdot s, s - \frac{b_{i,j}}{b} \cdot s \right\}$$

**Proof.** The maximum error occur when the elements inside the range of the query are either all null or all non-null. 

In the diagrams of Fig. 3 we show how the standard deviation of $S_1(b_{i,j})$ changes, respectively, when we vary $b_{i,j}/b$ (with $s = 1000$), and when we vary $s$ (with $b_{i,j} = b/2$) for a query over a block of size $b = 100$. The behavior of the estimated error w.r.t. $b_{i,j}/b$ is the same as that of $\sigma(C_1(b_{i,j}))$: The standard deviation is maximum for $b_{i,j} = b/2$ and is symmetric for $b_{i,j} > b/2$ and $b_{i,j} < b/2$. As shown in the diagram on the right-hand side of Fig. 3, the estimated error increases as the sum of the elements contained in the block increases: this result is rather expected, as the variance can be thought of as an estimate of the absolute error.
5 Case 2: using the number and the sum of non-null elements jointly

We now perform the estimation of count and sum queries by exploiting sum and count aggregate information simultaneously. This issue consists in studying the conjunction of two events: The value of the sum (in a range of size \( b_{i..j} \)) is \( s_{i..j} \), and the number of non-nulls (in the same range) is \( t_{i..j} \). In this case, count and sum queries are evaluated on datacubes belonging to \( M^{-1}_{cs,F} \).

More precisely, we have to study the joint probability distribution of the two random variables (representing the answer of the count query and the sum query), in order to derive the two probability distributions. As this case can be viewed as a specialization of Case 3 (where also integrity constraints will be exploited to evaluate the estimates – see Section 6), results on this estimation strategy are formalized in the following corollaries, whose proofs are reported after the proofs of the corresponding theorems of Case 3.

**Corollary 1** Let \( C_2(b_{i..j}) = \text{count}(\tilde{M}[i..j]) \) and \( S_2(b_{i..j}) = \text{sum}(\tilde{M}[i..j]) \) be two integer random variables ranging, respectively, from 0 to \( t_{i..j} \) and from 0 to \( s_{i..j} \), defined by extracting \( \tilde{M} \) from the datacube population \( M^{-1}_{cs,F} \). Then the joint probability distribution \( P(C_2(b_{i..j}) = t_{i..j}, S_2(b_{i..j}) = s_{i..j}) \) is given by:

\[
P = \begin{cases} 
Q(b_{i..j}, t_{i..j}, s_{i..j}) \cdot \frac{Q(b - b_{i..j}, t - t_{i..j}, s - s_{i..j})}{Q(b, t, s)} & \text{if:} \begin{cases} 0 \leq t_{i..j} \leq b_{i..j}, \\ t_{i..j} \leq s_{i..j} \leq s \end{cases} \\
0 & \text{otherwise}
\end{cases}
\]

where \( Q(x, y, z) \) is equal to:

\[
Q(x, y, z) = \begin{cases} 
0 & \text{if } (y = 0 \land z > 0) \lor (y > 0 \land z < y) \lor y > x \\
1 & \text{if } y = 0 \land z = 0 \\
\left( \frac{x}{y} \right) \left( \frac{z - 1}{z - y} \right) & \text{otherwise}
\end{cases}
\]

With the next corollary we formalize a first result about the estimation of the count query using both count and sum information: that is, the estimation of the count query cannot exploit the aggregate information about the sum of the elements in a block. Therein, we derive the probability distribution of \( C_2(b_{i..j}) \), its mean and its variance. In particular, we obtain that the probability distribution of \( C_2(b_{i..j}) \) coincides with that of \( C_1(b_{i..j}) \), representing the answer of the count query when only the knowledge of \( t \) is given.

**Corollary 2** The probability distribution of the random variable \( C_2(b_{i..j}) \) defined in Corollary 1 is: \( P(C_2(b_{i..j}) = t_{i..j}) = P(C_1(b_{i..j}) = t_{i..j}) \), where \( C_1(b_{i..j}) \) is
the random variable defined in Theorem 1.

From the corollary above, it follows that also mean and variance of \( C_2(b_{i..j}) \) are the same as those of \( C_1(b_{i..j}) \), as well as the maximum estimation error which may occur while returning \( E(C_2(b_{i..j})) \) as the answer of the range query \( \text{count}(M[i..j]) \) is the same as that of Case 1 (Proposition 1).

Now, we derive mean and variance of the random variable \( S_2(b_{i..j}) \), representing the estimated answer of a sum query given the knowledge of \( t \) and \( s \). Its probability distribution is given by

\[
P(S_2(b_{i..j}) = s_{i..j}) = \sum_{0 \leq t_{i..j} \leq t} P(C_2(b_{i..j}) = t_{i..j}, S_2(b_{i..j}) = s_{i..j}),
\]

according to the definition of joint probability distribution.

**Corollary 3** Mean and variance of the random variable \( S_2(b_{i..j}) \) defined in Theorem 1 are, respectively:

\[
E(S_2(b_{i..j})) = \frac{b_{i..j}}{b} \cdot s
\]

\[
\sigma^2(S_2(b_{i..j})) = \frac{s \cdot b_{i..j} \cdot (b - b_{i..j})}{b^2 \cdot (b - 1) \cdot (t + 1)} \cdot [b \cdot (2 \cdot s - t + 1) - s \cdot (t + 1)].
\]

Next we derive the maximum error \( \text{err}_{S_2}^{MAX} \) produced by estimating the answer of the range query \( \text{sum}(M[i..j]) \) by means of \( E(S_2(b_{i..j})) \).

**Proposition 3**

\[
\text{err}_{S_2}^{MAX} = \max \left\{ \frac{b_{i..j}}{b} \cdot s - \max\{0, t - (b-b_{i..j})\}, \ s - \max\{0, t-b_{i..j}\} - \frac{b_{i..j}}{b} \cdot s \right\}
\]

**Proof.** As non-null elements have a value equal to or greater than 1, the minimum value of the sum inside \([i..j]\) is given by the minimum number of non null elements occurring in this range, that is \( \max\{0, t - (b-b_{i..j})\} \). The maximum value of \( \text{sum}(b_{i..j}) \) is reached when the number of elements outside the range of the query is minimum and all of them have minimum value (i.e. 1). As the minimum of \( \text{count}(i..j) \) is given by \( \max\{0, t - b_{i..j}\} \), it holds that the maximum value of \( \text{sum}(b_{i..j}) \) is given by: \( s - \max\{0, t - b_{i..j}\} \). The formula expressing the maximum error is obtained by considering the cases when \( \text{sum}(b_{i..j}) \) is either maximum or minimum. \( \square \)

The main consequence of Corollary 2 is that the knowledge of \( s \) does not influence the estimation of the answer of a count query: The probability distribution of \( C_2(b_{i..j}) \) coincides to that of \( C_1(b_{i..j}) \). On the other hand, the knowledge of the number of null elements in each block changes the estimation of the answer of a sum query: The probability distribution of \( S_2(b_{i..j}) \) is different w.r.t. that of \( S_1(b_{i..j}) \). Indeed, the two random variables have the same mean but
different variances. In Fig. 4 we show \( \sigma(S_2(b_{1..j})) \) (dashed line) and \( \sigma(S_1(b_{1..j})) \) (dotted line) versus \( t/b \) for a query of size 50 on a block of size 100 whose elements have sum 1000.

![Fig. 4. \( \sigma(S_1(b_{1..j})) \) and \( \sigma(S_2(b_{1..j})) \) versus \( t/b \) for \( b = 100 \) and \( s = 1000 \).](image)

The standard deviation \( \sigma(S_2(b_{1..j})) \) is a decreasing function of \( t \): as \( t \) gets near \( b \), \( \sigma(S_2(b_{1..j})) \) decreases, and reaches a minimum for \( t = b \). The influence of \( t \) on the value of the estimated error can be strong. For instance, if \( t = b/3 \) the value of \( \sigma(S_1(b_{1..j})) \) is approximately an half of the value of \( \sigma(S_2(b_{1..j})) \). The measure of the error provided by \( \sigma(S_2(b_{1..j})) \) is generally greater than that obtained by means of \( S_1(b_{1..j}) \), but is more truthful. For instance, if \( t \) has a ‘small’ value (w.r.t. \( b \)), we have that \( \sigma(S_2(b_{1..j})) \gg \sigma(S_1(b_{1..j})) \). In this scenario, \( \sigma(S_2(b_{1..j})) \) provides a better description of the case, since when \( t \ll b \) the block is very sparse, the sum is distributed among few elements and, as there is no information about the exact position of non null elements, there is no way to decide whether the non null elements are inside or outside the range of the query.

Note that for \( t \cong b \) , \( \sigma(S_2(b_{1..j})) < \sigma(S_1(b_{1..j})) \). Indeed, \( \sigma(S_1(b_{1..j})) \) is an increasing function of \( s \) and, when \( t = b \), evaluating \( \sigma(S_2(b_{1..j})) \) is the same as evaluating \( \sigma(S_1(b_{1..j})) \) over a block of the same size (i.e. \( b \) whose elements have sum \( s - b \)).

6 Case 3: using integrity constraints

In this section we show how the knowledge of both lower bounds and upper bounds on the number of non-null elements derived by the functions \( LB_{=0} \) and \( LB_{>0} \) can be exploited in the estimation process. We use the following additional notations:

1. \( t_{1..j}^U = b_{1..j} - LB_{=0}(1..j) \) and \( t_{1..j}^L = LB_{>0}(1..j) \) are respectively an upper bound and a lower bound on the number of non-null elements in the range \([1..j]\);

2. \( t_{1..j}^U = b_{1..j} - LB_{=0}(1..j) \) and \( t_{1..j}^L = LB_{>0}(1..j) \) are respectively an upper bound and a lower bound on the number of non-null elements in the block.
where $B_k$ outside the range $[i..j] - [i..j]$ denotes the set of elements that are in $B_k$ but not in the range $[i..j]$;

(3) $t^U = t^L = t^L_{i,j} + t^L_{i,j}$ is $LB_{>0}([i..j]) - LB_{=0}([i..j])$ and $t^L = t^L_{i,j} + t^L_{i,j}$ is $LB_{>0}([i..j]) + LB_{>0}([i..j])$, that is $t^U$ and $t^L$ are respectively an upper bound and a lower bound on the number of non-null elements in $B_k$.

We define the random variables $count(\tilde{M}[i..j])$ and $sum(\tilde{M}[i..j])$ by extracting $\tilde{M}$ from the population $\Pi_{LB_{=0},LB_{>0}}(M^{-1}_{cs,F})$. We point out that, differently from the previous cases, the random variable representing the answer of a query also depends on the position of the range $[i..j]$ in the block and not only on its size. This is because integrity constraints contain information about the position of null elements in the block, and two distinct ranges of the same size and belonging to the same block may have different upper bounds and lower bounds on the number of null and non-null elements.

**Theorem 3** Let $C_3([i..j]) = count(\tilde{M}[i..j])$ and $S_3([i..j]) = sum(\tilde{M}[i..j])$ be two integer random variables ranging, respectively, from 0 to $t$ and from 0 to $s$, and defined over the datacube population $\Pi_{LB_{=0},LB_{>0}}(M^{-1}_{cs,F})$. Then, for each $t_{i,j}$ and $s_{i,j}$, such that $t^L_{i,j} \leq t_{i,j} \leq t^U_{i,j}$, and $0 \leq s_{i,j} \leq s$, the joint probability distribution $P(C_3([i..j]) = t_{i,j}, S_3([i..j]) = s_{i,j})$ is equal to:

$$P = \begin{cases} 
\frac{N(t^U_{i,j}, t_{i,j}, s_{i,j}, t^L_{i,j}) \cdot N(t^U_{i,j}, t_{i,j}, s_{i,j}, t^L_{i,j})}{N(t^U, t, s, t^L)} & \text{where: } t^L_{i,j} \leq t_{i,j} \leq t^U_{i,j}, \\
0 & \text{otherwise}
\end{cases}$$

where $t_{i,j} = t - t_{i,j}$, $s_{i,j} = s - s_{i,j}$, and

$$N(\bar{t}_u, \bar{t}_i, \bar{t}_j) = \begin{cases} 
0 & \text{if } \bar{t}_u > \bar{t}_i \lor \bar{t}_j > \bar{t} \lor (\bar{t} = 0 \land \bar{t} > 0) \\
1 & \text{if } \bar{t} = 0 \land \bar{t} = 0 \\
(\bar{t}_u - \bar{t}_i) \cdot (\bar{t}_j - \bar{t}_i) & \text{otherwise}
\end{cases}$$

**Proof.** $N(x, y, z, v)$ represents the number of configurations of a vector of size $x$ containing $y$ non null elements with sum $z$ such that we know the exact position of $v$ of them. If $y = 0$ and $z = 0$ there is an unique configuration (all elements are null), and so $N(x, y, z, v) = 1$. Furthermore, it is not possible that $y = 0$ and $z > 0$ (if the sum is greater than 0 there must be at least one non null element), or that $y > 0$ and $z < y$ (each non null element has at least value 1), or that $y > x$ (the number of non null elements cannot be greater than the size of the vector): in such cases $N(x, y, z, v) = 0$.

Otherwise, $N(x, y, z, v)$ can be obtained by disposing on $x - v$ positions the $y - v$ non null elements of which we don’t know the exact position (that can be
accomplished in \( \left( \frac{x - v}{y - v} \right) \) different ways) and, for each of these configurations, by distributing the sum \( s \) on \( y \) elements. This can generate \( \left( \frac{z - 1}{z - y} \right) \) different configurations. The value of \( N(x, y, z, v) \) is given by the product of these two quantities.

The probability distribution does not change if we remove from the block \( B_k \) the elements which are certainly null, according to the constraints expressed by \( LB=0 \). The block \( B'_k \) we obtain removing such elements can be seen as a vector \( V \) of size \( t^U \), and the query re-formulated over \( B'_k \) defines a sub-vector \( V_{i,j} \) of \( V \) which has size \( t^U_{i,j} \).

In order to evaluate the total number of “successful” configurations for the entire vector \( V \), we have to observe that for each successful configuration for the portion \( V_{i,j} \) we have a number of configurations for the remainder portion of the vector, say \( V_{t^U_{i,j}} \), which is equal to the number of ways of disposing \( t - t_{i,j} \) non null elements on \( t^U - t^U_{i,j} \) places, having that their sum is \( s - s_{i,j} \). Thus, the cases of success are given by \( N(t^U_{i,j}, t_{i,j}, s_{i,j}, t^L_{i,j}) \cdot N(t^U_{i,j}, t_{i,j}, s_{i,j}, t^L_{i,j}) \) appearing as numerator in the expression of the statement. The denominator \( N(t^U, t, s, t^L) \) can be similarly obtained by considering that the number of possible cases are all the configurations of the vector \( V \) such that the number of non null elements is \( t \), the sum is \( s \), and satisfying both \( LB=0 \) and \( LB>0 \).

Results stated in Theorem 3 can be used to prove Corollary 1. We recall that Corollary 1 concerns the definition of the probability distribution of the random variables \( C_2(b_{i,j}) \) and \( S_2(b_{i,j}) \) defined in Case 2, where no integrity constraints were considered.

**Proof of Corollary 1** We first observe that Case 2 corresponds to Case 3 with trivial bounds, i.e., \( LB=0([i..j]) = LB>0([i..j]) = 0 \). Then \( t^U_{i,j} = b_{i,j} \), \( t^L_{i,j} = 0 \); so the expression for \( P(C_3([i..j]) = t_{i,j}, S_3([i..j]) = s_{i,j}) \) (see Theorem 3) reduces to the one of \( P(C_2([i..j]) = t_{i,j}, S_2([i..j]) = s_{i,j}) \).

**Theorem 4** Let \( C_3([i..j]) \) be the random variable defined in Theorem 3. Then:

(1) the probability distribution \( P(C_3([i..j]) = t_{i,j}) \) is:
\[
P = \begin{cases} 
    \left( \frac{t_{i,j}^U - t_{i,j}^L}{t_{i,j}^U - t_{i,j}^L} \right) \left( \frac{t_{i,j}^U - t_{i,j}^L}{t_i^U - t_i^L} \right) & \text{if } t_{i,j} \geq \max\{t_{i,j}^L, t - t_{i,j}^U\} \\
    0 & \text{otherwise}
\end{cases}
\]

(2) Mean and variance of the random variable \( C_3([i..j]) \) are:

\[
E(C_3([i..j])) = \begin{cases} 
    t_{i,j}^L + \frac{t_{i,j}^U - t_{i,j}^L}{t_i^U - t_i^L} \cdot (t - t_i^L) & \text{if } t_i^U > t_i^L \\
    t_{i,j}^L & \text{if } t_i^U = t_i^L
\end{cases}
\]

\[
\sigma^2(C_3([i..j])) = \begin{cases} 
    \frac{(t_{i,j}^U - t_{i,j}^L)^2 \cdot (t_i^U - t_i^L)}{(t_i^U - t_i^L)^2} & \text{if } t_i^U > t_i^L + 1 \\
    0 & \text{if } t_i^U \leq t_i^L \leq t_i^L + 1
\end{cases}
\]

**Proof.** (1) The probability distribution of \( C_3([i..j]) \) can be obtained by considering that \( P(C_3([i..j] = t_{i,j}^L) = \sum_{s=0}^s P(C_3([i..j]) = t_{i,j}, S_3([i..j]) = s_{i,j}) \), and applying the following equation:

\[
\sum_{s_i = t_{i,j}}^s \binom{s_i - 1}{t_{i,j} - t_i} \cdot \binom{s - s_i - 1}{s - s_j - (t - t_{i,j})} = \binom{s - 1}{s - t}
\]

which holds as both its left-hand side term and right-hand side term represent the number of sets containing \( t \) cardinals (strictly greater than 0) with sum \( s \).

(2: computation of the mean) If \( t^U = t^L \) it is the case that all null and non null elements are located by integrity constraints. Therefore, \( P(C_3([i..j] = t_{i,j}^L) = 1 \). Otherwise, if \( t^U > t^L \) we can reason as follows. The block \( k \) can be viewed as a vector \( V \) of \( b \) elements whose values range between 0 and \( s \). Let \( V_{i..j} \) be the portion of \( V \) corresponding to the range \([i..j]\), and let \( V_{i..j}^\prime \) be the remainder part of \( V \). The event \( (C_3([i..j]) = t_{i,j}) \) is equivalent to the following event: The sum of all elements in \( V_{i..j} \) is \( s_{i,j} \). Let \( V[i] \) be a random variable which assumes the value 1 if the \( i \)-th element of \( V \) is not null, the value 0 otherwise.

From \( t = \sum_{1 \leq i \leq b} V[i] \), we derive \( t = t^L + \sum_{1 \leq i \leq b \land LB_{>0}(i) = 0 \land LB_{=0}(i)} V[i] \cdot t^L + \sum_{1 \leq i \leq b \land LB_{>0}(i) = 0 \land LB_{=0}(i)} E(V[i]) \) by linearity of the operator \( E \). The mean of the random variable \( V[i] \) is equal to the mean of the random variable \( V[j] \), for any \( i, j \) s.t. \( 1 \leq i, j \leq b \) and \( LB_{>0}(i) = LB_{>0}(j) = LB_{=0}(i) = LB_{=0}(j) = 0 \): For symmetry, the positions which are not localized by the integrity constraints have the same probability of containing null or non null elements. Let \( m \) be the mean \( E(V[i]) \). From the above formula for \( t \), it follows that \( m \cdot (t^U - t^L) = t - t^L \).

Consider now the vector \( V_{i..j} \). Since \( C_3([i..j]) \) can be seen as the random variable representing the number of non null elements of \( V_{i..j} \), we have that:
\[ E(C_3([i..j])) = t_{i,j}^L + (t_{i,j}^U - t_{i,j}^L) \cdot m. \] Hence, \[ E(C_3([i..j])) = t_{i,j}^L + \frac{t_{i,j}^U - t_{i,j}^L}{t_{i,j}^U - t_{i,j}^L} \cdot (t - t^L). \]

(2. computation of the variance) If \( t^U = t^L \), as explained for part (1) of this proof, it holds that \( P \left( C_3([i..j]) = t_{i,j}^t \right) = 1 \), and therefore \( \sigma^2(C_3([i..j])) \). If \( t^U = t^L + 1 \), two cases can occur: 1) \( t = t^U \), or 2) \( t = t^L \). In the former case, \( P \left( C_3([i..j]) = t_{i,j}^t \right) = 1 \) holds, whereas in the latter one \( P \left( C_3([i..j]) = t_{i,j}^t \right) = 1 \). In both cases, we have that: \( \sigma^2(C_3([i..j])) = 0 \).

The formula expressing \( \sigma^2 \) for \( t^U > t^L + 1 \) can be obtained using the definition of variance. The detailed proof is rather elaborated and, for the sake of presentation, is reported in Appendix as Claim 2.

Results stated in Theorem 3 can be used to prove Corollary 2, as the random variable \( C_2(b_{i..j}) \) can be seen as a special case of \( C_3([i..j]) \).

**Proof of Corollary 2** As shown in the proof of Corollary 1, \( t_{i,j}^U = b_{i..j} \), \( t_{i,j}^L = 0 \). For the same reasons, \( t^U = b \) and \( t^L = 0 \). By performing these substitutions, the statement of Theorem 4 reduces to that of Corollary 2.

Let us now quantify the maximum estimation error \( err^{MAX}_{C_3} \) which may occur while returning \( E(C_3([i..j])) \) as the answer of the range query \( count([i..j]) \).

**Proposition 4**

\[
err^{MAX}_{C_3} = \max \left\{ E(C_3([i..j])) - \max \{t_{i,j}^L, t - (t^U - t_{i,j}^L)\}, \right. \\
\left. \min \{t_{i,j}^U, t - (t^L - t_{i,j}^L)\} - E(C_3([i..j])) \right\}
\]

**Proof.** The minimum number of non null elements which could be contained in the range of the query is given by: \( \max \{t_{i,j}^L, t - t_{i,j}^L\} = \max \{t_{i,j}^L, t - (t^U - t_{i,j}^U)\} \), whereas the maximum of \( count([i..j]) \) is: \( \min \{t_{i,j}^U, t - t_{i,j}^L\} = \min \{t_{i,j}^U, t - (t^L - t_{i,j}^L)\} \). The formula of the maximum error is obtained by considering the cases where the actual number of non null elements inside the range of the query is either minimum or maximum.

We now focus our attention on the random variable \( S_3([i..j]) \), whose mean and variance are computed in the following theorem. Results stated in this theorem will be used, in the following, to prove Corollary 3.

**Theorem 5** Mean and variance of the random variable \( S_3([i..j]) \) defined in Theorem 3 are:
Let us first prove the formula expressing $E(S_3([i..j]))$. We assume that $t^U > t^L + 1$, as the proof for the case $t^U = t^L + 1$ is trivial.

$$E(S_3([i..j])) = \sum_{s_{i,j}=0}^{s} \sum_{t_{i,j}=0}^{t} s_{i,j} \cdot P(C_3([i..j]) = t_{i,j}, S_3([i..j]) = s_{i,j}) =$$

$$= \sum_{t_{i,j}=t^L_{i,j}}^{t-(t-t^L_{i,j})} \sum_{s_{i,j}=t_{i,j}}^{t-(t-t^L_{i,j})} \frac{(t^U_{i,j} - t^L_{i,j}) \cdot (s_{i,j} - t_{i,j}) \cdot (s_{i,j} - t_{i,j}) \cdot (s_{i,j} - t_{i,j})}{(t^U_{i,j} - t^L_{i,j}) \cdot (s_{i,j} - t_{i,j}) \cdot (s_{i,j} - t_{i,j}) \cdot (s_{i,j} - t_{i,j})}$$

$$= \frac{t^U_{i,j} - t^L_{i,j}}{t - t^L_{i,j}} \cdot \sum_{s_{i,j}=t_{i,j}}^{s_{i,j}=t_{i,j}+1} \frac{(s_{i,j} - t_{i,j}) \cdot (s_{i,j} - t_{i,j}) \cdot (s_{i,j} - t_{i,j})}{(s_{i,j} - t_{i,j}) \cdot (s_{i,j} - t_{i,j}) \cdot (s_{i,j} - t_{i,j})}$$

The term:

$$\sum_{s_{i,j}=t_{i,j}}^{s_{i,j}=t_{i,j}+1} s_{i,j} \cdot (s_{i,j} - 1) \cdot (s_{i,j} - t_{i,j})$$

can be re-written, by replacing $s_{i,j}$ with $S_{i,j} + t_{i,j}$, as:

$$\sum_{s_{i,j}=t_{i,j}}^{s_{i,j}=t_{i,j}+1} s_{i,j} \cdot (s_{i,j} - 1) \cdot (s_{i,j} - t_{i,j}) =$$
\[= \sum_{S_{i,j}=0}^{s-t} (S_{i,j} + t_{i,j}) \cdot \left( \frac{t_{i,j} + S_{i,j} - 1}{S_{i,j}} \right) \cdot \left( \frac{-t_{i,j} + s - S_{i,j} - 1}{s - S_{i,j} - t} \right) = \]
\[= \sum_{S_{i,j}=0}^{s} (S_{i,j} + t_{i,j}) \cdot \left( \frac{t_{i,j} + S_{i,j} - 1}{S_{i,j}} \right) \cdot \left( \frac{t - t_{i,j} + S - S_{i,j} - 1}{S - S_{i,j}} \right) \]

where: \( S = s - t \).

Since \( \left( \begin{array}{c} x \\ y \end{array} \right) = \frac{x-y+1}{y} \cdot \left( \begin{array}{c} x \\ y-1 \end{array} \right) \), it results that:

\[= \sum_{S_{i,j}=0}^{s} S_{i,j} \cdot \left( \frac{t_{i,j} + S_{i,j} - 1}{S_{i,j}} \right) \cdot \left( \frac{t - t_{i,j} + S - S_{i,j} - 1}{S - S_{i,j}} \right) = \]
\[= \sum_{S_{i,j}=1}^{s} t_{i,j} \cdot \left( \frac{t_{i,j} + S_{i,j} - 1}{S_{i,j} - 1} \right) \cdot \left( \frac{t - t_{i,j} + S - S_{i,j} - 1}{S - S_{i,j}} \right) = \]
\[= \sum_{Q_{i,j}=0}^{s-1} t_{i,j} \cdot \left( \frac{(t_{i,j}+1)+Q_{i,j}-1}{Q_{i,j}} \right) \cdot \left( \frac{(t+1)-(t_{i,j}+1)+(S-1)-Q_{i,j}-1}{(S-1)-Q_{i,j}} \right) \]

where: \( Q_{i,j} = S_{i,j} - 1 \).

Now observe that the following holds:

\[= \sum_{k=0}^{z} \left( \begin{array}{c} y + k - 1 \\ k \end{array} \right) \cdot \left( \begin{array}{c} x - y + z - k - 1 \\ z - k \end{array} \right) = \left( \begin{array}{c} x + z - 1 \\ z \end{array} \right) \]

(2)

since both the above terms represent the number of sets containing \( x \) naturals (including zero) such that their sum is \( z \).

Then, by applying formula (2) we obtain:

\[= \sum_{S_{i,j}=0}^{s} S_{i,j} \cdot \left( \frac{t_{i,j} + S_{i,j} - 1}{S_{i,j}} \right) \cdot \left( \frac{t - t_{i,j} + S - S_{i,j} - 1}{S - S_{i,j}} \right) = \]
\[= \sum_{Q_{i,j}=0}^{s-1} t_{i,j} \cdot \left( \frac{(t_{i,j}+1)+Q_{i,j}-1}{Q_{i,j}} \right) \cdot \left( \frac{(t+1)-(t_{i,j}+1)+(S-1)-Q_{i,j}-1}{(S-1)-Q_{i,j}} \right) = \]
\[= t_{i,j} \cdot \left( \frac{t + S - 1}{S - 1} \right) = t_{i,j} \cdot \frac{s - t}{t} \cdot \left( \frac{s - 1}{s - t} \right) \]
and:

\[
\sum_{s_{i,j}=0}^{s} t_{i,j} \cdot \left( \frac{t_{i,j} + S_{i,j} - 1}{S_{i,j}} \right) \cdot \left( \frac{t - t_{i,j} + S - S_{i,j} - 1}{S - S_{i,j}} \right) =
\]

\[
= t_{i,j} \cdot \left( \frac{t + S - 1}{S} \right) = t_{i,j} \cdot \left( \frac{s - 1}{s - t} \right)
\]

By replacing these two terms in (1), we obtain:

\[
\sum_{s_{i,j}=t_{i,j}}^{s-t+h_{i,j}} s_{i,j} \cdot \left( \frac{s_{i,j} - 1}{s_{i,j} - t_{i,j}} \right) \cdot \left( \frac{s - s_{i,j} - 1}{s - s_{i,j} - t + h_{i,j}} \right) = t_{i,j} \cdot \frac{s}{t} \cdot \left( \frac{s - 1}{s - t} \right)
\]

so that:

\[
E(S_3([i,j])) = \sum_{t_{i,j}=t_{i,j}^L}^{t-(t^L-t_{i,j}^L)} \left[ \left( \frac{t^U - t^L_{i,j}}{t_{i,j} - t^L_{i,j}} \right) \cdot \left( \frac{t^U - t^L_{i,j} + t^L_{i,j}}{t - t_{i,j} - t^L + t^L_{i,j}} \right) \right] \cdot t_{i,j} \cdot \frac{s}{t}
\]

(3)

Moreover, it holds that:

\[
\sum_{t_{i,j}=t_{i,j}^L}^{t-(t^L-t_{i,j}^L)} t_{i,j} \cdot \left( \frac{t^U - t^L_{i,j}}{t_{i,j} - t^L_{i,j}} \right) \cdot \left( \frac{t^U - t^L_{i,j} + t^L_{i,j}}{t - t_{i,j} - t^L + t^L_{i,j}} \right) =
\]

\[
= \sum_{h_{i,j}=0}^{m} (h_{i,j} + t^L_{i,j}) \cdot \left( \frac{l_{i,j}}{h_{i,j}} \right) \cdot \left( \frac{n-h_{i,j}}{m-h_{i,j}} \right)
\]

where: \( h_{i,j} = t_{i,j} - t^L_{i,j} \), \( l_{i,j} = t^U_{i,j} - t^L_{i,j} \), \( m = t - t^L \), and \( n = t^U - t^L \).

As \( \left( \frac{x}{y} \right) = \frac{x}{y} \cdot \left( \frac{x-1}{y-1} \right) \) we have that:

\[
\sum_{h_{i,j}=0}^{m} h_{i,j} \cdot \left( \frac{l_{i,j}}{h_{i,j}} \right) \cdot \left( \frac{n-h_{i,j}}{m-h_{i,j}} \right) = \sum_{h_{i,j}=0}^{m} l_{i,j} \cdot \left( \frac{l_{i,j} - 1}{h_{i,j} - 1} \right) \cdot \left( \frac{n-l_{i,j}}{m-h_{i,j}} \right) =
\]

\[
= \sum_{p_{i,j}=0}^{m-1} l_{i,j} \cdot \left( \frac{l_{i,j} - 1}{p_{i,j}} \right) \cdot \left( \frac{n-l_{i,j}}{m-1-p_{i,j}} \right) \quad \text{where: } p_{i,j} = h_{i,j} - 1
\]

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By applying the Vandermonde formula:
\[
\sum_{i=0}^{k} \binom{x}{i} \cdot \binom{y}{k-i} = \binom{x+y}{k}
\]  

we obtain:
\[
\sum_{p_{i,j}=0}^{m-1} l_{i,j} \cdot \binom{n-l_{i,j}}{m-1-p_{i,j}} = l_{i,j} \cdot \frac{m}{n} \cdot \binom{n}{m}
\]

and:
\[
\sum_{t_{i,j}=0}^{m} t_{i,j}^L \cdot \binom{l_{i,j}}{h_{i,j}} \cdot \binom{n-l_{i,j}}{m-h_{i,j}} = t_{i,j}^L \cdot \binom{n}{m}
\]

After replacing these terms in (3), we obtain:
\[
E(S_3([i..j])) = 2 \cdot \frac{(l_{i,j} \cdot \frac{m}{n} + t_{i,j}^L) \cdot \binom{n}{m} \cdot \frac{s}{t}}{(t_{i,j}^U - t_{i,j}^L) \cdot \frac{t - t_{i,j}^L}{t_{i,j}^U - t_{i,j}^L} + t_{i,j}^L} = \left(\frac{t_{i,j}^U - t_{i,j}^L}{t_{i,j}^U - t_{i,j}^L} \cdot \frac{t - t_{i,j}^L}{t_{i,j}^U - t_{i,j}^L} + t_{i,j}^L\right) \cdot \frac{s}{t}
\]

As regards the proof of the formula expressing the variance, for the sake of presentation, this proof is postponed in Appendix as Claim 3. □

**Proof of Corollary 3.** By applying the same arguments used in the proof of Corollaries 1 and 2, it is easy to see that the statement of Theorem 5 reduces to the statement of Corollary 3. □

The maximum estimation error \(err_{S_3}^{MAX}\) which may occur while returning \(E(S_3([i..j]))\) as the answer of the range query \(sum([i..j])\) is evaluated next:

**Proposition 5**

\[
err_{S_3}^{MAX} = \max \left\{ E(S_3([i..j])) - \max \{t_{i,j}^L, t - (t_{i,j}^U - t_{i,j}^L)\}, s - \min \{t_{i,j}^L, t - t_{i,j}^U\} - E(S_3([i..j])) \right\}
\]

**Proof.** The maximum error can be obtained when the actual sum inside the range of the query is either minimum or maximum. This sum is minimum if the number of non null elements inside \([i..j]\) is minimum, and each of these non null elements has the minimum value (i.e. 1). Thus, the minimum sum
inside [i..j] coincides with the minimum value of count([i..j]) and is given by
\[
\max\{t^L_{i..j}, t - LB_0([i..j])\},
\]
that is: \( t - t^L + t^L_{i..j} \). On the other hand, the value of sum([i..j]) is maximum if the number of non null elements outside [i..j] is minimum, and if all of non null elements in [i..j] have value 1. Therefore, the maximum value of the sum inside [i..j] is given by:
\[
max\{t^L_{i..j}, t - t^U_{i..j}\} = s - \max\{t^L - t^L_{i..j}, t - t^U_{i..j}\}.
\]
\( \Box \)

**Remark.** Note that, unlike the mean values of \( S_1(b_{i..j}) \) and \( S_2(b_{i..j}) \), the value of \( E(S_3([i..j])) \) generally depends on \( t \). That is, when the integrity constraints provided by \( LB_0 > 0 \) are exploited, the estimated answer of a sum query depends on the number of non null elements occurring in the whole block \( B_k \). This difference between Case 3 and previous cases can be explained as follows. In Case 1 and 2 (when the function \( LB_0 > 0 \) is not available or not exploited), no information about the exact position of non null elements inside \( B_k \) is provided. Now, the estimation of the sum query is made by considering all possible ways of distributing \( t \) non null elements in the block. Thus, if we partition \( B_k \) into two equal halves (by splitting \( B_k \) along one of its dimensions), for each configuration of \( B_k \) consisting of \( t' \) non null elements located inside the first half of \( B_k \) and \( t - t' \) non null elements in the other half, there exists a “symmetric” configuration where \( t - t' \) non null elements are in the first half of \( B_k \) and \( t' \) non null elements are in the second half. This implies that the only knowledge of \( t \) does not make the distribution of the sum \( s \) inside the block “unbalanced”. In contrast, the information encoded in the function \( LB_0 \) invalidates the symmetry condition described above. That is, given a consistent configuration of \( B_k \) containing \( t' \) non null elements inside the first half of \( B_k \) and \( t - t' \) non null elements in the other half, the “symmetric” configuration exists only if it is consistent according to the integrity constraint expressed by the function \( LB_0 > 0 \).

It should be pointed out that if \( LB_0 > 0 \) is not available or not used, the estimate provided using Case 3 does not depend on \( t \). In fact, when only \( LB_0 = 0 \) is exploited, the estimation process described in Case 3 works in the same way as Cases 1 and 2, after removing from \( B_k \) all elements which are certainly null according to \( LB_0 = 0 \). We can reach the same conclusion by extracting a formula for \( E(S_3([i..j])) \) from the one provided in Theorem 5, by substituting \( LB_0([i..j]) = 0 \) and \( LB_0([i..j]) = 0 \), thus obtaining: \( E(S_3([i..j])) = (t^U_{i..j} - t^L_{i..j}) \cdot \frac{s}{t}, \) which is independent on \( t \).

7 Influence of integrity constraints on accuracy: some experimental results

In the analysis of the accuracy of the estimated answers for the Cases 1 and 2, we focused our attention on discussing the dependence of variance on the
ratios \( b_{i,j}/b \) and \( t/b \). The introduction of integrity constraints makes both the estimated answer and variance to depend on the position of \([i..j]\) inside the block (since the values of \( t_{i,j}^U \) and \( t_{i,j}^L \) change as the boundaries of the range move), and on the maximum number \( t^U \) and minimum number \( t^L \) of non nulls inside the block. Therefore, it is relevant to check how much the variance change when we use the knowledge of \( LB_{=0}(i..j) \), \( LB_{=0}(\tilde{i}..\tilde{j}) \), \( LB_{>0}(i..j) \) and \( LB_{>0}(\tilde{i}..\tilde{j}) \), whose values determine \( t_{i,j}^U \), \( t_{i,j}^L \), \( t^U \) and \( t^L \). Next we perform this analysis but, for the sake of brevity, we shall only consider the presence of the constraints \( LB_{=0}(i..j) \) and \( LB_{=0}(\tilde{i}..\tilde{j}) \), thus assuming \( LB_{>0}(i..j) = 0 \) and \( LB_{>0}(\tilde{i}..\tilde{j}) = 0 \) — indeed the dependency of the estimates on the latter classes of constraints are quantitatively the same.

Consider a sum query of size \( b_{i,j} = 500 \) over a block with \( b = 1000 \) and \( t = 500 \). Fig. 5 shows the standard deviation of the random variable \( S_3([i..j]) \) versus the value of \( LB_{=0}(i..j) \), for different values of \( LB_{=0}(\tilde{i}..\tilde{j}) \): the solid line corresponds to the value 0 of \( LB_{=0}(i..j) \), the dotted line to the value 10, and the dash-dot line to the value 20. The diagram shows that, when \( LB_{=0}(\tilde{i}..\tilde{j}) = 0 \) is fixed, \( \sigma \) decreases from 84.31 to 70.44, as \( LB_{=0}(i..j) \) changes from 0 (which is equivalent to consider no integrity constraint) to 30. This change corresponds to a variation of 16% of the standard deviation.

![Fig. 5. \( \sigma(S_3([i..j])) \) versus \( LB_{=0}(i..j) \) for different values of \( LB_{=0}(\tilde{i}..\tilde{j}) \)](image)

The decrease of the standard deviation depicted in Fig. 5 corresponds to a “restriction” of the datacube population on which the random variable associated to the query is applied. In fact, evaluating a query of size \( b_{i,j} \) over a block of size \( b \) whose elements have sum \( s \) is equivalent to evaluating a query of size \( b_{i,j} - LB_{=0}(i..j) \) over a block containing \( b - LB_{=0}(i..j) - LB_{=0}(\tilde{i}..\tilde{j}) \) elements with the same value of \( s \). Thus, when \( LB_{=0}(i..j) > 0 \) or \( LB_{=0}(\tilde{i}..\tilde{j}) > 0 \), the population of datacubes which are compatible with the given aggregate data is restricted w.r.t. both the cases \( LB_{=0}(i..j) = 0 \) and \( LB_{=0}(\tilde{i}..\tilde{j}) = 0 \). This restricted population of datacubes corresponds to a lower “degree of uncertainty” in distributing the value of \( s \) among the elements inside the blocks.

The diagram in Fig. 6 reports \( \sigma(S_3([i..j])) \) versus \((LB_{=0}(i..j) + LB_{=0}(\tilde{i}..\tilde{j}))/ (b-t)\), that is the ratio between the number of the null elements localized by integrity constraints and the total number of null elements of the block (ac-
cording to the aggregate data (t). Fig. 6 shows that the larger the number \( LB_{=0}(\lfloor i..j \rfloor) + LB_{=0}(\lceil i..j \rceil) \) of null elements localized by integrity constraints (compared to the total number of nulls inside the block), the lower is the value of the estimated error. In Fig. 6 the sum \( LB_{=0}(\lfloor i..j \rfloor) + LB_{=0}(\lceil i..j \rceil) \) is denoted as \( LB_{=0} \).

![Graph showing \( \sigma(S_3(\lfloor i..j \rfloor)) \) versus \( LB_{=0}/(b-t) \) for \( b = 100, t = 50, b..j = 50 \) and \( s = 1000 \).] *Fig. 6. \( \sigma(S_3(\lfloor i..j \rfloor)) \) versus \( LB_{=0}/(b-t) \) for \( b = 100, t = 50, b..j = 50 \) and \( s = 1000 \).*

The same diagram shows that the estimated error is smaller when \( LB_{=0}(\lfloor i..j \rfloor) \) and \( LB_{=0}(\lceil i..j \rceil) \) are “unbalanced”, i.e. either \( LB_{=0}(\lfloor i..j \rfloor) > LB_{=0}(\lceil i..j \rceil) \) or \( LB_{=0}(\lceil i..j \rceil) < LB_{=0}(\lfloor i..j \rfloor) \). As it can be easily intuited, knowing that most of null elements are distributed either inside or outside the range of the query reduces the approximation in evaluating the distribution of \( s \) inside the block.

In sum, as expected, introducing integrity constraints on the number of null elements in each block influences “positively” the estimation process. We stress that our results are valid for random data samples so errors may be larger in real-world applications whose data distributions can be rather “biased”, so that the accuracy of the estimates evaluated using the framework can be far from being accurate. Next, we present the results of testing our estimation models to a sample consisting of ten real-life two-dimensional datacubes which confirm the positive influence of integrity constraints on the accuracy of estimations.

The datacubes for our experiments contain the daily incomes corresponding to the products sold in a chain store during periods of two months belonging to ten different years. Each datacube consists of a matrix made of 7580 rows (corresponding to all store products) and 60 columns (corresponding to the working days). Both count and sum queries over all the ranges of size 100 × 20 have been evaluated for each datacube, comparing the exact answers to the approximate ones. In particular, different compressed representations of every datacube have been examined, corresponding to different sizes of the summary blocks; for each compressed structure, both the actual and the estimated errors obtained with and without the use of integrity constraints have been evaluated. For sum queries, the influence of using the parameter \( t \) on the query estimation result has been studied too.

In our experiments, the integrity constraints consist of “macro-blocks” which
delimit portions of the cube consisting of all null elements or of all non-null elements. These macro-blocks do not identify all null [resp., non null] elements inside the cube, but only those null [resp., non null] elements which are inside a portion of the cube containing at least 20 null [resp., non null] elements. Macro-blocks do not overlap, and can be efficiently stored and retrieved using traditional indexing methods for spatial access. On the average, the adopted constraints located 40% of null values and 10% of non null elements inside the examined samples.

In the tables of Figures 7 and 8, results obtained for count and sum queries are reported. The tables represent the intervals where the actual error for queries of size $100 \times 20$ are contained, considering all datacubes. That is, each entry of the table shows, in percentage terms, the number of estimates whose actual error is less than $3 \times \sigma$, $4 \times \sigma$, and $5 \times \sigma$, for each of the estimation techniques proposed in Case 1, Case 2 and Case 3.

| Block size | Without constraints (Case 1) | Using constraints (Case 3) |
|------------|------------------------------|---------------------------|
|            | $3 \times \sigma$ | $4 \times \sigma$ | $5 \times \sigma$ | $3 \times \sigma$ | $4 \times \sigma$ | $5 \times \sigma$ |
| 10x10      | 63.9% | 74.1% | 81.8% | 87.4% | 95.2% | 98.6% |
| 12x12      | 72.4% | 84.4% | 92.1% | 91.4% | 97.9% | 99.6% |
| 14x14      | 77.4% | 87.8% | 93.5% | 91.9% | 98.1% | 99.6% |
| 16x16      | 59.2% | 72.8% | 81.9% | 87.8% | 95.7% | 98.8% |
| 18x18      | 51.3% | 62.7% | 71.8% | 84.1% | 93.1% | 97.5% |
| 20x20      | 58.1% | 70.1% | 78.8% | 86.2% | 94.6% | 98.1% |

Fig. 7. Number of count queries whose actual error is less than $3\sigma$, $4\sigma$, and $5\sigma$

Results reported in the tables show that:

(1) the use of $t$ makes the estimation of the error for sum queries more accurate;
(2) for both count and sum queries, the accuracy of estimates benefits from the use of integrity constraints. In particular, a smaller coefficient to "correct" effectively the estimate provided by $\sigma$ is needed, and the value of this coefficient is almost independent from the particular compressed representation of the datacube. For instance, without using integrity constraints, the number of estimated count queries whose actual error is less than $5 \times \sigma$ is between 71.8% and 93.5%, depending on the block size. On the other hand, when integrity constraints are used, the number of estimated count queries whose actual error is less than $5 \times \sigma$ is greater
| Block size | Without $t$ (Case 1) | Without constraints (Case 2) | Using constraints (Case 3) |
|------------|----------------------|-----------------------------|--------------------------|
|           | $3 \times \sigma$ | $4 \times \sigma$ | $5 \times \sigma$ | $3 \times \sigma$ | $4 \times \sigma$ | $5 \times \sigma$ |
| 10×10     | 34.4%                | 43.7%                       | 51.8%                    | 70.4%                | 79.9%                | 86.9%                    | 81.7%                | 90.5%                | 95.7%                    |
| 12×12     | 33.1%                | 44.1%                       | 54.2%                    | 78.8%                | 90.1%                | 95.7%                    | 88.6%                | 96.1%                | 98.4%                    |
| 14×14     | 28.1%                | 37.1%                       | 45.7%                    | 69.1%                | 81.9%                | 89.4%                    | 73.3%                | 86.1%                | 92.8%                    |
| 16×16     | 22.1%                | 29.2%                       | 60.9%                    | 73.6%                | 82.1%                | 89.4%                    | 84.5%                | 91.5%                | 95.4%                    |
| 18×18     | 23.6%                | 30.7%                       | 36.9%                    | 60.6%                | 72.1%                | 80.2%                    | 76.9%                | 86.9%                | 92.8%                    |
| 20×20     | 30.2%                | 38.9%                       | 46.8%                    | 69.3%                | 79.1%                | 85.9%                    | 79.6%                | 88.9%                | 94.1%                    |

Fig. 8. Number of sum queries whose actual error is less than 3\(\sigma\), 4\(\sigma\), and 5\(\sigma\) than 90% for every block size.

8 Estimation of Range Queries on Histograms

In this section we apply our framework to derive some results about mono-dimensional histograms. Mono-dimensional histograms are constructed to summarize the frequency distribution of the values of a single attribute in a database relation, and can be exploited to estimate query result sizes [26,34,33]. The estimation is accomplished on the basis of the knowledge of both the number $t$ of non-null frequencies and the total frequency sum $s$ in each block $B_k$ (called bucket in the histogram terminology). As mentioned in the Introduction, a crucial point for providing good estimations is the way the frequency distributions for original values are partitioned into buckets. Here we assume that the buckets have been already arranged using any of the known techniques, and we therefore focus on the problem of estimating the frequency distribution inside a bucket.

8.1 A theory for the Continuous Value Assumption

The most common approach to estimate frequency distribution inside a bucket is the continuous value assumption [35]: The sum of frequencies in a range of a bucket is estimated by linear interpolation. It corresponds to equally distributing the overall sum of frequencies of the bucket to all attribute values occurring in it.

Corollary 3 (where both $t$ and $s$ are used to estimate sum range queries) provides a theoretical foundation of the continuous value assumption, as it
states that the mean value of the random variable $S_2(b_{i..j})$ is $\frac{b_{i..j}}{b} \cdot s$. Thus our approach gives a model to explain the linear interpolation and, besides, allows to evaluate the error of the estimation, thus exploiting the knowledge about the number $t$ of non-nulls in a block — instead $t$ is not mentioned in the computation of the mean.

We point out that, in order to provide a more elaborated interpolation scheme, in [33,34] another method for estimating sum of frequencies inside a block is proposed, based on the uniform spread assumption: The $t$ non-null attribute values in each bucket are assumed to be located at equal distance from each other, and the overall frequency sum is therefore equally distributed among them. This method does not give a correct estimation unless we assume that non-nulls are scattered on the block in some particular, biased way. Next, using our theoretical framework, we propose an unbiased estimation inside a block which takes into account the number $t$ of non-null values.

8.2 The $1/2$-Biased Assumption

We first recall that the classical definition of histogram requires that both lowest and highest elements (or at least one of them) of any block are not null [34] (i.e. they are attribute values occurring in the relation). We call $2$-biased a block for which the extreme elements are not null; if only the lowest (or the highest) element is not null then the block is called $1$-biased.

So far linear interpolation is also used for biased blocks, thus producing a wrong estimation — it is the case to say a “biased” estimation. We next show the correct formulas, that are derived from Theorem 5.

**Corollary 4** Let $B_k$ be a block of a histogram, and let $S_4([i..j]) = \text{sum}(\bar{M}[i..j])$ be an integer random variable ranging from $0$ to $s$, defined by taking $\bar{M}$ in the population $\Pi_{LB>0}(M^{-1}_{cs,F})$. Then

$(1)$ if the block $B_k$ is $1$-biased and $i$ is the lowest element of the block then mean and variance of $S_4([i..j])$ are, respectively:

$$E(S_4([i..j])) = \frac{s}{t} + (b_{i..j} - 1) \cdot \frac{s}{t} \cdot \frac{t - 1}{b - 1},$$

$$\sigma^2(S_4([i..j])) = \alpha \cdot (b_{i..j} - 1) \cdot \frac{t - 1}{b - 1} \cdot \left[ 1 + (b_{i..j} - 2) \cdot \frac{t - 2}{b - 2} \right] + \left( \beta + 2 \cdot \alpha \right) \cdot (b_{i..j} - 1) \cdot \frac{t - 1}{b - 1} + (\alpha + \beta) - E(S_4([i..j]))^2$$

$(2)$ if the block $B_k$ is $1$-biased and $i$ is not the lowest element of the block then
mean and variance of $S_4([1..j])$ are, respectively:

$$E(S_4([1..j])) = b_{i,j} \cdot \frac{s \cdot t - 1}{t \cdot (b - 1)} ,$$

$$\sigma^2(S_4([1..j])) = \alpha \cdot b_{i,j} \cdot \frac{t - 1}{b - 1} \cdot \left[ 1 + (b_{i,j} - 1) \cdot \frac{t - 2}{b - 2} \right] + \beta \cdot b_{i,j} \cdot \frac{t - 1}{b - 1} - E(S_4([1..j]))^2$$

(3) if the block $B_k$ is 2-biased and either 1 or $j$ is an extreme element of the
block then mean and variance of $S_4([1..j])$ are, respectively:

$$E(S_4([1..j])) = \frac{s}{t} + (b_{i,j} - 1) \cdot \frac{s \cdot t - 2}{t \cdot (b - 2)} ,$$

$$\sigma^2(S_4([1..j])) = \alpha \cdot (b_{i,j} - 1) \cdot \frac{t - 2}{b - 2} \cdot \left[ 1 + (b_{i,j} - 2) \cdot \frac{t - 3}{b - 3} \right] + (\beta + 2 \cdot \alpha) \cdot (b_{i,j} - 1) \cdot \frac{t - 2}{b - 2} + (\alpha + \beta) - E(S_4([1..j]))^2$$

(4) if the block $B_k$ is 2-biased, and neither 1 nor $j$ is an extreme element of
the block, then mean and variance of $S_4([1..j])$ are, respectively:

$$E(S_4([1..j])) = b_{i,j} \cdot \frac{s \cdot t - 2}{t \cdot (b - 2)} ,$$

$$\sigma^2(S_4([1..j])) = \alpha \cdot b_{i,j} \cdot \frac{t - 2}{b - 2} \cdot \left[ 1 + (b_{i,j} - 1) \cdot \frac{t - 3}{b - 3} \right] + \beta \cdot b_{i,j} \cdot \frac{t - 2}{b - 2} - E(S_4([1..j]))^2$$

where:

$$\alpha = \frac{s \cdot (s + 1)}{t \cdot (t + 1)} , \text{ and } \beta = \frac{s \cdot (s - t)}{t \cdot (t + 1)} .$$

Proof.

(1) ($B_k$ is 1-biased and 1 is the lowest element of the block). In this case,
$E(S_4([1..j]))$ and $\sigma^2(S_4([1..j]))$ coincide to $E(S_3([1..j]))$ and $\sigma^2(S_3([1..j]))$,
respectively, computed in Theorem 5, by considering $LB_{=0}([1..j]) = 0$,
$LB_{=0}(1..j) = 0$, $LB_{>0}([1..j]) = 1$, and $LB_{>0}(1..j) = 0$. The statement of
the corollary is thus obtained by considering that $t^U = b$, $t^U_{i,j} = b_{i,j}$,
$t^L_{i,j} = 1$ and $t^L = 1$.

(2) ($B_k$ is 1-biased and 1 is not the lowest element of the block). In this case,
$E(S_4([1..j]))$ and $\sigma^2(S_4([1..j]))$ coincide with $E(S_3([1..j]))$ and $\sigma^2(S_3([1..j]))$,
respectively, computed in Theorem 5, by considering $LB_{=0}([1..j]) = 0$,
The above formulas have been used in [4] to replace the continuous value assumption inside one of the most efficient methods for histogram representation (the maxdiff method [33]), and have produced some meaningful improvements in the performance of the method.

9 Conclusion and Future Work

In this paper we have defined a probabilistic framework for estimating range queries on a compressed datacube obtained by partitioning the original datacube into a number of non-overlapping blocks and then storing, for each block, some aggregate information on its data distribution. The proposed estimation paradigm allows us to provide an approximate answer of range queries (more specifically, sum and count queries) together with an estimate of the error of the returned answer, by accessing only the compressed representation of the datacube. The estimates of both the answer and the error depend on the aggregate data and integrity constraints which are exploited, without any a priori assumption on the particular data distribution inside the original datacube. We have investigated how the values of the answer and the estimated error depend on the available aggregate data and integrity constraints, by performing both an analytical and experimental study.

We remark that the idea of introducing integrity constraints is crucial to improve the accuracy of estimations in real applications. In fact, the need of integrity constraints is due to the fact that the real-life datacubes are rather
biased” with respect to the virtual population of datacubes on which the theoretical estimation process is performed. The effectiveness of the estimates improves as integrity constraints are introduced because the estimation process is accomplished on a restricted population of datacubes, and the examined samples are more representative of this population than the more general one.

Therefore, further types of constraints are needed in order to catch the actual distribution of data inside a datacube and improve the accuracy of the estimates. Thus, extensions of this work will follow the directions below:

- extending the framework by considering further aggregate data on the blocks of the datacube, other than the sum and the number of non null values inside each block (for instance, the maximum and the minimum value inside the blocks);
- taking into account data skew: this issue can be accomplished by storing some information regarding the number of distinct values inside each block, or the values with the maximum number of occurrences in the blocks.

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APPENDIX

Claim 1  \[
\sigma^2(S_1(b_{i,j})) = b_{i,j} \cdot s \cdot \frac{(b - b_{i,j}) \cdot (b + s)}{b^2 \cdot (b + 1)}
\]

Proof. We start from the definition of variance:

\[
\sigma^2(S_1(b_{i,j})) = \sum_{s_{i,j}=0}^{s} \left( s_{i,j} - \frac{b_{i,j}}{b} \cdot s \right)^2 \cdot P(S_1(b_{i,j}) = s_{i,j}) =
\]

\[
= \sum_{s_{i,j}=0}^{s} s_{i,j}^2 \cdot P(S_1(b_{i,j}) = s_{i,j}) - \left( \frac{b_{i,j}}{b} \cdot s \right)^2 =
\]

\[
= \sum_{s_{i,j}=0}^{s} s_{i,j}^2 \cdot \left( \frac{b_{i,j} + s_{i,j} - 1}{s_{i,j}} \right) \cdot \left( \frac{b - b_{i,j} + s - s_{i,j} - 1}{s - s_{i,j}} \right) - \left( \frac{b_{i,j}}{b} \cdot s \right)^2
\]

As \(\begin{pmatrix} x \\ y \end{pmatrix} = \frac{x-y+1}{y} \cdot \begin{pmatrix} x \\ y-1 \end{pmatrix}\), the term:

\[
\sum_{s_{i,j}=0}^{s} s_{i,j}^2 \cdot \left( \frac{b_{i,j} + s_{i,j} - 1}{s_{i,j}} \right) \cdot \left( \frac{b - b_{i,j} + s - s_{i,j} - 1}{s - s_{i,j}} \right)
\]

can be re-written as:

\[
\sum_{s_{i,j}=0}^{s} b_{i,j} \cdot s_{i,j} \cdot \left( \frac{b_{i,j} + s_{i,j} - 1}{s_{i,j} - 1} \right) \cdot \left( \frac{b - b_{i,j} + s - s_{i,j} - 1}{s - s_{i,j}} \right) =
\]

\[
= \sum_{s_{i,j}=0}^{s} b_{i,j} \cdot (s_{i,j} - 1) \cdot \left( \frac{b_{i,j} + s_{i,j} - 1}{s_{i,j} - 1} \right) \cdot \left( \frac{b - b_{i,j} + s - s_{i,j} - 1}{s - s_{i,j}} \right) +
\]

\[
+ \sum_{s_{i,j}=0}^{s} b_{i,j} \cdot \left( \frac{b_{i,j} + s_{i,j} - 1}{s_{i,j} - 1} \right) \cdot \left( \frac{b - b_{i,j} + s - s_{i,j} - 1}{s - s_{i,j}} \right) =
\]

\[
= \sum_{S_{i,j}=1}^{S_{i,j} - 1} b_{i,j} \cdot S_{i,j} \cdot \left( \frac{b_{i,j} + S_{i,j}}{S_{i,j}} \right) \cdot \left( \frac{b - b_{i,j} + (s-1) - S_{i,j} - 1}{(s-1) - S_{i,j}} \right) +
\]

\[
+ \sum_{S_{i,j}=0}^{S_{i,j} - 1} b_{i,j} \cdot \left( \frac{(b_{i,j} + 1) + S_{i,j} - 1}{S_{i,j}} \right) \cdot \left( \frac{(b+1)-(b_{i,j} + 1) + (s-1)-S_{i,j} - 1}{(s-1) - S_{i,j}} \right)
\]

(6)

where: \(S_{i,j} = s_{i,j} - 1\).
Then, by applying formula (2), the latter becomes:

\[
\sum_{S_{i,j}=1}^{s-1} b_{i,j} \cdot S_{i,j} \cdot \frac{b_{i,j}+1}{S_{i,j}} \cdot \left( \frac{b_{i,j} + S_{i,j}}{S_{i,j} - 1} \right) \cdot \left( \frac{b - b_{i,j} + (s-1) - S_{i,j} - 1}{(s-1) - S_{i,j}} \right) + \\
+ b_{i,j} \cdot \left( \frac{b + s - 1}{s - 1} \right) = \\
= \sum_{S_{i,j}=1}^{s-1} b_{i,j} \cdot (b_{i,j} + 1) \cdot \left( \frac{b_{i,j} + S_{i,j}}{S_{i,j} - 1} \right) \cdot \left( \frac{b - b_{i,j} + (s-1) - S_{i,j} - 1}{(s-1) - S_{i,j}} \right) + \\
+ b_{i,j} \cdot \frac{s}{b} \cdot \left( \frac{b + s - 1}{s} \right) = [\text{where } \alpha = S_{i,j} - 1] \\
= b_{i,j} \cdot (b_{i,j} + 1) \cdot \left( \frac{b + s - 1}{s - 2} \right) + b_{i,j} \cdot \frac{s}{b} \cdot \left( \frac{b + s - 1}{s} \right) = \\
= b_{i,j} \cdot (b_{i,j} + 1) \cdot \frac{s \cdot (s - 1)}{b \cdot (b + 1)} \cdot \left( \frac{b + s - 1}{s} \right) + b_{i,j} \cdot \frac{s}{b} \cdot \left( \frac{b + s - 1}{s} \right) = \\
= b_{i,j} \cdot \frac{s}{b} \cdot \left( \frac{b + s - 1}{s} \right) \cdot \left[ (b_{i,j} + 1) \cdot \frac{s - 1}{b + 1} + 1 \right] \\
\]

(7)

By substituting (7) in (5) we obtain:

\[
\sigma^2(S_1(b_{i,j})) = b_{i,j} \cdot \frac{s}{b} \cdot \left[ (b_{i,j} + 1) \cdot \frac{s - 1}{b + 1} + 1 \right] - \left( \frac{b_{i,j}}{b} \cdot s \right)^2 = \\
= b_{i,j} \cdot \frac{s}{b^2} \cdot \frac{b \cdot (b_{i,j} + 1) \cdot (s - 1) + b \cdot (b + 1) - b_{i,j} \cdot s \cdot (b + 1)}{b + 1} = \\
= b_{i,j} \cdot s \cdot \frac{(b - b_{i,j}) \cdot (b + s)}{b^2 \cdot (b + 1)}
\]


\[
\]
Claim 2

$$\sigma^2(C_3([i..j])) = \begin{cases} \frac{t^{U}_{i..j} - t^{L}_{i..j}}{t^{U} - t^{L}} \cdot (t - t^{L}) \cdot \frac{[(t^{U} - t^{L}) - (t^{U}_{i..j} - t^{L}_{i..j})] \cdot (t^{U} - t)}{(t^{U} - t^{L}) \cdot (t^{U} - t^{L} - 1)} & \text{if } t^{U} > t^{L} + 1 \\ 0 & \text{if } t^{L} \leq t^{U} \leq t^{L} + 1 \end{cases}$$

Proof. We consider the case that $t^{U} > t^{L} + 1$. We start from the definition of variance:

$$\sigma^2(C_3([i..j])) = \sum_{t_{i..j}=0}^{t} (t_{i..j} - E(C_3([i..j])))^2 \cdot \sum_{s_{i..j}=0}^{s} P(C_3([i..j]) = t_{i..j}, S_3([i..j]) = s_{i..j}) =$$

$$= \sum_{t_{i..j}=0}^{t} t^2_{i..j} \cdot \sum_{s_{i..j}=0}^{s} P(C_3([i..j]) = t_{i..j}, S_3([i..j]) = s_{i..j}) - (E(C_3([i..j])))^2 =$$

$$= \sum_{t_{i..j}=t^{L}_{i..j}}^{t-(t^{U} - t^{L}_{i..j})} t^2_{i..j} \cdot \left[ \frac{(t^{U}_{i..j} - t^{L}_{i..j})}{(t^{U} - t^{L}_{i..j})} \cdot \frac{(t^{U} - t^{U}_{i..j} - (t^{L} - t^{L}_{i..j}))}{(t - t^{U})} \right] \cdot \left[ \sum_{s_{i..j}=t_{i..j}}^{s-t+t_{i..j}} \frac{s_{i..j} - 1}{s_{i..j} - t_{i..j}} \cdot \left( \frac{s - s_{i..j} - 1}{s - s_{i..j} - t + t_{i..j}} \right) \right] - (E(C_3([i..j])))^2$$

By applying the substitutions: $t_{i..j} = s_{i..j} - t_{i..j}$, and: $S = s - t$, the previous expression can be rewritten as:

$$= \sum_{t_{i..j}=t^{L}_{i..j}}^{t-(t^{U} - t^{L}_{i..j})} t^2_{i..j} \cdot \left[ \frac{(t^{U}_{i..j} - t^{L}_{i..j})}{(t^{U} - t^{L}_{i..j})} \cdot \frac{(t^{U} - t^{U}_{i..j} - (t^{L} - t^{L}_{i..j}))}{(t - t^{U})} \right] \cdot \left[ \sum_{S_{i..j}=0}^{S+t-(S_{i..j}+t_{i..j})-1} \frac{S_{i..j}+t_{i..j}-1}{S_{i..j}} \cdot \left( \frac{S+t-S_{i..j}-1}{S-S_{i..j}} \right) \right] - (E(C_3([i..j])))^2 =$$

$$= \sum_{t_{i..j}=t^{L}_{i..j}}^{t-(t^{U} - t^{L}_{i..j})} t^2_{i..j} \cdot \left[ \frac{(t^{U}_{i..j} - t^{L}_{i..j})}{(t^{U} - t^{L}_{i..j})} \cdot \frac{(t^{U} - t^{U}_{i..j} - (t^{L} - t^{L}_{i..j}))}{(t - t^{U})} \right] - (E(C_3([i..j])))^2 \quad (8)$$
which holds since, from (2), we have that:

\[
\sum_{s_{i,j}=0}^{s} \left( \frac{s_{i,j} + t_{i,j} - 1}{s_{i,j}} \right) \cdot \left( \frac{s + t - (s_{i,j} + t_{i,j}) - 1}{s - s_{i,j}} \right) = 1.
\]

Let: \( h_{i,j} = t_{i,j} - t_{i,j}^{L} \), \( l_{i,j} = t_{i,j}^{U} - t_{i,j}^{L} \), \( m = t - t^{L} \), and: \( n = t^{U} - t^{L} \).

By applying these substitutions in (8) we obtain:

\[
\sigma^{2}(C_{3}(\lfloor i \cdot j \rfloor)) = \sum_{h_{i,j}=0}^{m} (h_{i,j} + t_{i,j}^{L})^{2} \cdot \left( \frac{l_{i,j}}{h_{i,j}} \right) \cdot \left( \frac{n - l_{i,j}}{m - h_{i,j}} \right) - (E(C_{3}(\lfloor i \cdot j \rfloor)))^{2}
\]

Finally, by substituting (11), (12) and (13) in the above expression, we obtain:

\[
\sigma^{2}(C_{3}(\lfloor i \cdot j \rfloor)) = \frac{t_{i,j}^{U} - t_{i,j}^{L}}{t^{U} - t^{L}} \cdot (t - t^{L}) \cdot \frac{[(t^{U} - t^{L}) - (t_{i,j}^{U} - t_{i,j}^{L})] \cdot (t^{U} - t)}{(t^{U} - t^{L}) \cdot (t^{U} - t^{L} - 1)}
\]

\( \Box \)

Claim 3

\[
\sigma^{2}(S_{3}(\lfloor i \cdot j \rfloor)) = \begin{cases} 
\alpha \cdot (t_{i,j}^{U} - t_{i,j}^{L}) \cdot \frac{t - t^{L}}{t^{U} - t^{L}} \cdot \left[ 1 + (t_{i,j}^{U} - t_{i,j}^{L} - 1) \cdot \frac{t - t^{L} - 1}{t^{U} - t^{L} - 1} \right] + 
\beta \cdot (2 \cdot \alpha) \cdot (t_{i,j}^{U} - t_{i,j}^{L}) \cdot \frac{t - t^{L}}{t^{U} - t^{L}} + \gamma \cdot t_{i,j}^{L} + \beta \cdot t_{i,j}^{L} - \gamma^{2} & \text{if } t^{U} > t^{L} + 1 \\
\frac{s \cdot t_{i,j}^{L} \cdot (t - t^{L} - (s - t))}{t^{U} \cdot (t + 1)} & \text{if } t^{U} = t^{L} \land t = t^{L} \\
\frac{s \cdot t_{i,j}^{U} \cdot (t - t^{L})}{t^{U} \cdot (t + 1)} & \text{if } t^{U} = t^{L} + 1 \land t = t^{U}
\end{cases}
\]

where:

\[
\alpha = \frac{s \cdot (s + 1)}{t \cdot (t + 1)}, \quad \beta = \frac{s \cdot (s - t)}{t \cdot (t + 1)}, \quad \text{and } \gamma = E(S_{3}(\lfloor i \cdot j \rfloor)).
\]

Proof. We consider the case that \( t^{U} > t^{L} + 1 \). We start from the definition of
variance:

\[ \sigma^2(S_3([1,.j])) = \sum_{s_{i,j}=0}^{\mathcal{S}} (s_{i,j} - E(S_3([1,.j])))^2 \cdot \sum_{t_{i,j}=0}^{t} P(C_4([1,.j]) = t_{i,j}, S_3([1,.j]) = s_{i,j}) = \]

\[ = \sum_{s_{i,j}=0}^{\mathcal{S}} s_{i,j}^2 \cdot \sum_{t_{i,j}=0}^{t} P(C_4([1,.j]) = t_{i,j}, S_3([1,.j]) = s_{i,j}) - (E(S_3([1,.j])))^2 = \]

\[ t-(t^L - t^L_{i,j}) s-t+t_{i,j} \right]

\[ = \sum_{t_{i,j}=t^L_{i,j}}^{t-(t^L - t^L_{i,j}) s-t+t_{i,j}} \left[ \frac{(t^U - t^U_{i,j} - (t^L - t^L_{i,j}))(s - s_{i,j} - 1)}{(s - s_{i,j} - t + t_{i,j})} \right] - (E(S_3([1,.j])))^2 \]

\[ = \sum_{t_{i,j}=t^L_{i,j}}^{t-(t^L - t^L_{i,j}) s-t+t_{i,j}} \left[ \frac{(t^U - t^U_{i,j} - (t^L - t^L_{i,j}))(s - s_{i,j} - 1)}{(s - s_{i,j} - t + t_{i,j})} \right] - (E(S_3([1,.j])))^2 \]

\[ \text{(9)} \]

The term: \[ \sum_{s_{i,j}=t^L_{i,j}}^{s+t_{i,j}} s_{i,j}^2 \cdot \frac{(s_{i,j} - 1)}{(s_{i,j} - t_{i,j})} \cdot \frac{(s - s_{i,j} - 1)}{(s - s_{i,j} - t + t_{i,j})} \]

can be re-written, by replacing \( s_{i,j} \) with \( S_{i,j} + t_{i,j} \), obtaining:

\[ \sum_{s_{i,j}=t^L_{i,j}}^{s+t_{i,j}} s_{i,j}^2 \cdot \frac{(S_{i,j} + t_{i,j} - 1)}{(s_{i,j} - t_{i,j})} \cdot \frac{(s - S_{i,j} - 1)}{(s - S_{i,j} - t + t_{i,j})} = \]

\[ = \sum_{s_{i,j}=0}^{s-t} (S_{i,j} + t_{i,j})^2 \cdot \frac{(S_{i,j} + t_{i,j} - 1)}{S_{i,j}} \cdot \frac{(-t_{i,j} + s - S_{i,j} - 1)}{s - S_{i,j}} = \]

\[ = \sum_{s_{i,j}=0}^{S} (S_{i,j} + t_{i,j})^2 \left( \frac{S_{i,j} + t_{i,j} - 1}{S_{i,j}} \right) \left( \frac{t - t_{i,j} + S - S_{i,j} - 1}{S - S_{i,j}} \right) \text{ where: } S = s - t \]

Since: \[ \left( \begin{array}{c} x \\ y \end{array} \right) = \frac{x - y + 1}{y} \cdot \left( \begin{array}{c} x \\ y - 1 \end{array} \right), \text{ it results that:} \]

\[ \sum_{s_{i,j}=0}^{S} (S_{i,j})^2 \cdot \left( \frac{S_{i,j} + t_{i,j} - 1}{S_{i,j}} \right) \cdot \left( \frac{t - t_{i,j} + S - S_{i,j} - 1}{S - S_{i,j}} \right) = \]

\[ 41 \]
= \sum_{s_{i,j}=1}^{s} t_{i,j} \cdot s_{i,j} \cdot \left( t_{i,j} + s_{i,j} - 1 \right) \cdot \left( t - t_{i,j} + S - s_{i,j} - 1 \right)

= \sum_{s_{i,j}=2}^{s} t_{i,j} \cdot (s_{i,j} - 1) \cdot \left( t_{i,j} + s_{i,j} - 1 \right) \cdot \left( t - t_{i,j} + S - s_{i,j} - 1 \right) +

+ \sum_{s_{i,j}=1}^{s} t_{i,j} \cdot \left( t_{i,j} + s_{i,j} - 1 \right) \cdot \left( t - t_{i,j} + S - s_{i,j} - 1 \right) =

= \sum_{q_{i,j}=1}^{s-1} t_{i,j} \cdot q_{i,j} \cdot \left( t_{i,j} + q_{i,j} \right) \cdot \left( t - t_{i,j} + (S - 1) - q_{i,j} - 1 \right) +

+ \sum_{q_{i,j}=0}^{s-1} t_{i,j} \cdot \left( t_{i,j} + q_{i,j} - 1 \right) \cdot \left( (t+1) - (t_{i,j} + 1) + (S - 1) - q_{i,j} - 1 \right)

where: \( q_{i,j} = s_{i,j} - 1 \).

By applying formula (2), we obtain that:

\[ \sum_{q_{i,j}=0}^{s-1} t_{i,j} \cdot \left( \frac{(t_{i,j}+1)+q_{i,j}-1}{q_{i,j}} \right) \cdot \left( (t+1) - (t_{i,j}+1) + (S - 1) - q_{i,j} - 1 \right) = \]

\[ = t_{i,j} \cdot \left( \frac{t+S-1}{S-1} \right) \]

On the other hand,

\[ \sum_{q_{i,j}=1}^{s-1} t_{i,j} \cdot q_{i,j} \cdot \left( t_{i,j} + q_{i,j} \right) \cdot \left( t - t_{i,j} + (S - 1) - q_{i,j} - 1 \right) = \]

\[ = \sum_{q_{i,j}=1}^{s-1} t_{i,j} \cdot q_{i,j} \cdot \left( \frac{t_{i,j} + 1}{q_{i,j}} \right) \cdot \left( t_{i,j} + q_{i,j} \right) \cdot \left( t - t_{i,j} + (S - 1) - q_{i,j} - 1 \right) \]

Substituting \( r_{i,j} = q_{i,j} - 1 \) the latter becomes:

\[ \sum_{r_{i,j}=0}^{s-2} t_{i,j} \cdot (t_{i,j}+1) \cdot \left( \frac{(t_{i,j}+2)+r_{i,j}-1}{r_{i,j}} \right) \cdot \left( (t+2) - (t_{i,j}+2) + (S - 2) - r_{i,j} - 1 \right) = \]

\[ = t_{i,j} \cdot (t_{i,j} + 1) \cdot \left( \frac{t+S-1}{S-2} \right) \]
Thus we obtain that:
\[
\sum_{S_{i,j}=0}^{S} (S_{i,j})^2 \cdot \left( \frac{t_{i,j} + S_{i,j} - 1}{S_{i,j}} \right) \cdot \left( \frac{t - t_{i,j} + S - S_{i,j} - 1}{S - S_{i,j}} \right) =
\]
\[
= t_{i,j} \cdot (t_{i,j} + 1) \cdot \left( \frac{t + S - 1}{S - 2} \right) + t_{i,j} \cdot \left( \frac{t + S - 1}{S - 1} \right) =
\]
\[
= t_{i,j} \cdot \frac{S}{t} \left( \frac{t + S - 1}{S - 1} \right) \cdot \left[ (t_{i,j} + 1) \cdot \frac{S - 1}{t + 1} + 1 \right]
\]

It also holds that:
\[
\sum_{S_{i,j}=1}^{S} 2 \cdot t_{i,j} \cdot S_{i,j} \cdot \left( \frac{t_{i,j} + S_{i,j} - 1}{S_{i,j}} \right) \cdot \left( \frac{t - t_{i,j} + S - S_{i,j} - 1}{S - S_{i,j}} \right) =
\]
\[
= 2 \cdot t_{i,j} \cdot \sum_{S_{i,j}=1}^{S} S_{i,j} \cdot \frac{1}{S_{i,j}} \left( \frac{t_{i,j} + S_{i,j} - 1}{S_{i,j} - 1} \right) \cdot \left( \frac{t - t_{i,j} + S - S_{i,j} - 1}{S - S_{i,j}} \right) =
\]
\[
= 2 \cdot t_{i,j} \cdot \sum_{Q_{i,j}=0}^{s-1} \left( \frac{t_{i,j} + Q_{i,j}}{Q_{i,j}} \right) \cdot \left( \frac{t - t_{i,j} + (S - 1) - Q_{i,j}}{(S - 1) - Q_{i,j}} \right) =
\]
\[
= 2 \cdot t_{i,j} \cdot \left( \frac{t + S - 1}{S} \right) \cdot \frac{t_{i,j}}{t} \cdot S
\]

and, from (2):
\[
\sum_{S_{i,j}=1}^{S} t_{i,j}^2 \cdot \left( \frac{t_{i,j} + S_{i,j} - 1}{S_{i,j}} \right) \cdot \left( \frac{t - t_{i,j} + S - S_{i,j} - 1}{S - S_{i,j}} \right) = t_{i,j}^2 \cdot \left( \frac{t_{i,j} + S_{i,j} - 1}{S_{i,j}} \right)
\]

Thus, we have that:
\[
\sum_{S_{i,j}=t_{i,j}}^{s-t+t_{i,j}} S_{i,j}^2 \cdot \left( \frac{s_{i,j} - 1}{s_{i,j} - t_{i,j}} \right) \cdot \left( \frac{s - s_{i,j} - 1}{s - s_{i,j} - t_{i,j}} \right) =
\]
\[
= \sum_{S_{i,j}=0}^{S} (S_{i,j} + t_{i,j})^2 \cdot \left( \frac{t_{i,j} + S_{i,j} - 1}{S_{i,j}} \right) \cdot \left( \frac{t - t_{i,j} + S - S_{i,j} - 1}{S - S_{i,j}} \right) =
\]
\[
= \left( \frac{t + S - 1}{S} \right) \cdot \left[ \left( \frac{S \cdot (S - 1)}{t \cdot (t + 1)} + 1 + 2 \cdot \frac{S}{t} \right) \cdot t_{i,j}^2 + \frac{S \cdot (S + t)}{t \cdot (t + 1)} \cdot t_{i,j} \right] =
\]
\[
= \left( \frac{t + S - 1}{S} \right) \cdot \left[ \frac{s \cdot (s + 1)}{t \cdot (t + 1)} \cdot t_{i,j}^2 + \frac{s \cdot (s - t)}{t \cdot (t + 1)} \cdot t_{i,j} \right] =
\]

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\[ = \left( \frac{t+S-1}{S} \right) (\alpha \cdot t_{i,j}^2 + \beta \cdot t_{i,j}), \text{ where: } \alpha = \frac{s \cdot (s+1)}{t \cdot (t+1)}, \quad \beta = \frac{s \cdot (s-t)}{t \cdot (t+1)} \]

Substituting this term in (9) we obtain:

\[ \sigma^2(S_3[1..j]) = \frac{1}{(t^U-t^L)} \left[ \alpha \cdot \sum_{t_{i,j}=t^L}^{t^U-t^L} t_{i,j}^2 \cdot \left( \frac{t_{i,j}-t_{i,j}}{t_{i,j}-t_{i,j}} \right) \cdot \left( t^U-t^L-t_{i,j}+t_{i,j} \right) + \beta \cdot \sum_{t_{i,j}=t^L}^{t^U-t^L} t_{i,j} \cdot \left( \frac{t_{i,j}-t_{i,j}}{t_{i,j}-t_{i,j}} \right) \cdot \left( t^U-t^L-t_{i,j}+t_{i,j} \right) \right] - (E(S_3([1..j])))^2 = \]

\[ = \frac{1}{(t^U-t^L)} \left[ \alpha \cdot \sum_{h_{i,j}=0}^{m} (h_{i,j}+t_{i,j})^2 \cdot \left( \frac{h_{i,j}}{h_{i,j}} \right) \cdot \left( t^U-t^L-h_{i,j} \right) + \beta \cdot \sum_{h_{i,j}=0}^{m} (h_{i,j}+t_{i,j}) \cdot \left( \frac{h_{i,j}}{h_{i,j}} \right) \cdot \left( t^U-t^L-h_{i,j} \right) \right] - (E(S_3([1..j])))^2 \]

that is:

\[ \sigma^2(S_3([1..j])) = \frac{1}{(t^U-t^L)} \left[ \alpha \cdot \sum_{h_{i,j}=0}^{m} h_{i,j}^2 \cdot \left( \frac{h_{i,j}}{h_{i,j}} \right) \cdot \left( n-h_{i,j} \right) + \left( \beta+2\cdot t_{i,j} \cdot \alpha \right) \cdot \sum_{h_{i,j}=0}^{m} h_{i,j} \cdot \left( \frac{h_{i,j}}{h_{i,j}} \right) \cdot \left( n-h_{i,j} \right) + \left( \alpha \cdot t_{i,j}^2 + \beta \cdot t_{i,j} \right) \cdot \sum_{h_{i,j}=0}^{m} h_{i,j} \cdot \left( \frac{n-h_{i,j}}{h_{i,j}} \right) \right] - (E(S_3([1..j])))^2 \]

where: \( h_{i,j} = t_{i,j} - t^L_{i,j}, \quad l_{i,j} = t^U_{i,j} - t^L_{i,j}, \quad m = t - t^L, \quad \text{ and: } n = t^U - t^L. \)

Since: \( \left( \begin{array}{c} x \\ y \end{array} \right) = \frac{x}{y} \cdot \left( \begin{array}{c} x - 1 \\ y - 1 \end{array} \right), \) we have that:

\[ \sum_{h_{i,j}=0}^{m} h_{i,j}^2 \cdot \left( \frac{h_{i,j}}{h_{i,j}} \right) \cdot \left( n-h_{i,j} \right) = \sum_{h_{i,j}=0}^{m} l_{i,j} \cdot h_{i,j} \cdot \left( \frac{l_{i,j} - 1}{h_{i,j} - 1} \right) \cdot \left( n-h_{i,j} \right) = \]
By applying the Vandermonde formula (4) we obtain:

\[
= \sum_{h_{i,j}=1}^{m} l_{i,j}(h_{i,j}-1) \cdot \left( \frac{l_{i,j}-1}{h_{i,j}-1} \right) \left( \frac{n-l_{i,j}}{m-h_{i,j}} \right) + \sum_{h_{i,j}=1}^{m} l_{i,j} \cdot \left( \frac{l_{i,j}-1}{h_{i,j}-1} \right) \left( \frac{n-l_{i,j}}{m-h_{i,j}} \right) =
\]

\[
= \sum_{p_{i,j}=0}^{m-1} l_{i,j} \cdot p_{i,j} \cdot \left( \frac{l_{i,j}-1}{p_{i,j}} \right) \cdot \left( \frac{n-l_{i,j}}{m-1-p_{i,j}} \right) + \sum_{p_{i,j}=0}^{m-1} l_{i,j} \cdot \left( \frac{l_{i,j}-1}{p_{i,j}} \right) \cdot \left( \frac{n-l_{i,j}}{m-1-p_{i,j}} \right)
\]

where: \( p_{i,j} = h_{i,j} - 1 \).

By applying the Vandermonde formula (4) we obtain:

\[
= \sum_{h_{i,j}=0}^{m} \left( \frac{l_{i,j}}{h_{i,j}} \right) \cdot \left( \frac{n-l_{i,j}}{m-h_{i,j}} \right) = \left( \frac{n}{m} \right)
\]

(11)

and:

\[
= \sum_{p_{i,j}=0}^{m-1} l_{i,j} \cdot \left( \frac{l_{i,j}-1}{p_{i,j}} \right) \cdot \left( \frac{n-l_{i,j}}{m-1-p_{i,j}} \right) = l_{i,j} \cdot \left( \frac{n-1}{m-1} \right)
\]

On the other hand:

\[
= \sum_{p_{i,j}=0}^{m-1} l_{i,j} \cdot p_{i,j} \cdot \left( \frac{l_{i,j}-1}{p_{i,j}} \right) \cdot \left( \frac{n-l_{i,j}}{m-1-p_{i,j}} \right) =
\]

\[
= \sum_{p_{i,j}=1}^{m-1} l_{i,j} \cdot (l_{i,j}-1) \cdot \left( \frac{l_{i,j}-2}{p_{i,j}-1} \right) \cdot \left( \frac{n-l_{i,j}}{m-1-p_{i,j}} \right) =
\]

\[
= \sum_{U_{i,j}=0}^{m-2} l_{i,j} \cdot (l_{i,j}-1) \cdot \left( \frac{l_{i,j}-2}{U_{i,j}} \right) \cdot \left( \frac{n-l_{i,j}}{m-2-U_{i,j}} \right) = l_{i,j} \cdot (l_{i,j}-1) \cdot \left( \frac{n-2}{m-2} \right)
\]

Thus:

\[
= \sum_{h_{i,j}=0}^{m} h_{i,j}^2 \cdot \left( \frac{l_{i,j}}{h_{i,j}} \right) \cdot \left( \frac{n-h_{i,j}}{m-h_{i,j}} \right) = l_{i,j} \cdot \left( \frac{n-1}{m-1} \right) + l_{i,j} \cdot (l_{i,j}-1) \cdot \left( \frac{n-2}{m-2} \right) =
\]

\[
= l_{i,j} \cdot \frac{m}{n} \left( \frac{n}{m} \right) + l_{i,j} \cdot (l_{i,j}-1) \cdot \frac{m}{n} \cdot \frac{m-1}{n-1} \cdot \left( \frac{n}{m} \right) =
\]

\[
= h_{i,j} \cdot \frac{t-t^L}{tU-t^L} \left( \frac{tU-t^L}{t-t^L} \right) + l_{i,j} \cdot (l_{i,j}-1) \cdot \frac{t-t^L}{tU-t^L} \cdot \frac{t-t^L-1}{tU-t^L-1} \cdot \left( \frac{tU-t^L}{t-t^L} \right)
\]

(12)

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It also holds that:

\[
\sum_{h_{i,j}=0}^{m} h_{i,j} \cdot \left( \frac{l_{i,j}}{h_{i,j}} \right) \cdot \left( n - l_{i,j} \right) = \sum_{T_{i,j}=0}^{n-1} t_{i,j}^{U} \cdot \left( \frac{t_{i,j}^{U} - 1}{T_{i,j}} \right) \cdot \left( n - l_{i,j} \right) = \\
= h_{i,j} \cdot \frac{m}{n} \cdot \left( \frac{n}{m} \right) \cdot (t_{i,j}^{U} - t_{i,j}^{L}) \cdot \left( \frac{t_{i,j}^{U} - t_{i,j}^{L}}{t_{i,j}^{U} - t_{i,j}^{L}} \right) \tag{13}
\]

where: \( T_{i,j} = h_{i,j} - 1 \).

Finally, substituting (11), (12) and (13) in (10) we obtain:

\[
\sigma^2(S_3([i..j])) = \alpha \cdot (t_{i,j}^{U} - t_{i,j}^{L}) \cdot \frac{t_{i,j}^{U} - t_{i,j}^{L}}{t_{i,j}^{U} - t_{i,j}^{L}} \cdot \left[ 1 + (t_{i,j}^{U} - t_{i,j}^{L} - 1) \cdot \frac{t_{i,j}^{U} - t_{i,j}^{L} - 1}{t_{i,j}^{U} - t_{i,j}^{L}} \right] + \\
+ (\beta + 2 \cdot \alpha \cdot t_{i,j}^{L}) \cdot (t_{i,j}^{U} - t_{i,j}^{L}) \cdot \frac{t_{i,j}^{U} - t_{i,j}^{L}}{t_{i,j}^{U} - t_{i,j}^{L}} + (\alpha \cdot t_{i,j}^{L} + \beta \cdot t_{i,j}^{L}) + \\
-(E(S_3([i..j])))^2
\]