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About Lebesgue inequalities on the classes of generalized Poisson integrals

For the functions $f$, which can be represented in the form of the convolution $f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{k=1}^{\infty} e^{-\alpha kr} \cos(\beta k - \frac{\pi}{2}) \varphi(x - t) dt$, $\varphi \perp 1$, $\alpha > 0$, $r \in (0, 1)$, $\beta \in \mathbb{R}$, we establish the Lebesgue-type inequalities of the form

$$\|f - S_{n-1}(f)\|_{C} \leq e^{-\alpha n^r} \left( \frac{4}{\pi^2} \ln \frac{n^{1-r}}{\alpha r} + \gamma_n \right) E_n(\varphi)_C.$$ 

These inequalities take place for all numbers $n$ that are larger than some number $n_1 = n_1(\alpha, r)$, which constructively defined via parameters $\alpha$ and $r$. We prove that there exists a function, such that the sign "$\leq$" in given estimate can be changed for "$=$".

Keywords: Lebesgue inequalities, Fourier sums, classes of convolutions of periodic functions, best approximation.

1. Introduction

Let $L_p$, $1 \leq p < \infty$, be the space of $2\pi$–periodic functions $f$ summable to the power $p$ on $[0, 2\pi)$, in which the norm is given by the formula $\|f\|_p = \left( \int_0^{2\pi} |f(t)|^p dt \right)^{1/p}$; $L_\infty$ be the space of measurable and essentially bounded $2\pi$–periodic functions $f$ with the norm $\|f\|_\infty = \text{ess sup}_t |f(t)|$; $C$ be the space of continuous $2\pi$–periodic functions $f$, in which the norm is specified by the equality $\|f\|_C = \max_t |f(t)|$.

By $\rho_n(f; x)$ we denote the deviation of the function $f$ from its partial Fourier sum of order $n - 1$:

$$\rho_n(f; x) := f(x) - S_{n-1}(f; x),$$

where

$$S_{n-1}(f; x) = \frac{a_0}{2} + \sum_{k=1}^{n-1} (a_k \cos kx + b_k \sin kx),$$

$$a_k = a_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt dt, \quad b_k = b_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt dt,$$
and by $E_n(f)_C$ we denote the best uniform approximation of the function $f$ by elements of the subspace $\tau_{2n-1}$ of trigonometric polynomials $t_{n-1}(\cdot)$ of the order $n - 1$:

$$E_n(f)_C := \inf_{t_{n-1} \in \tau_{2n-1}} \|f - S_{n-1}(f)\|_C.$$ 

The norms $\|\rho_n(f; \cdot)\|_C$ can be estimated via $E_n(f)_C$, using the Lebesgue inequality

$$\|\rho_n(f; \cdot)\|_C \leq (1 + L_{n-1})E_n(f)_C, \quad n \in \mathbb{N}. \quad (1)$$

Here the sequence of numbers

$$L_{n-1} = \frac{1}{\pi} \int_{-\pi}^{\pi} |D_{n-1}(t)| dt = \frac{2}{\pi} \int_{0}^{2\pi} \frac{\sin(2n-1)t}{\sin t} dt,$$

where

$$D_{n-1}(t) := \frac{1}{2} + \sum_{k=1}^{\infty} \cos kt = \frac{\sin(n - \frac{1}{2})t}{2 \sin \frac{t}{2}},$$

are called the Lebesgue constants of the Fourier sums.

The asymptotic equality for Lebesgue constants $L_n$ was obtained in [1]:

$$L_n = \frac{4}{\pi^2} \ln n + O(1), \quad n \to \infty.$$ 

For more exact estimates for the differences $L_n - \frac{4}{\pi^2} \ln(n + a), a > 0$, as $n \in \mathbb{N}$ the reader can be referred to the works [2]–[8]. In particular, it follows from [5] (see also [7, p.97]) that

$$\left| L_{n-1} - \frac{4}{\pi^2} \ln n \right| < 1,271, \quad n \in \mathbb{N}.$$ 

Then, the inequality (1) can be written in the form

$$\|\rho_n(f; \cdot)\|_C \leq \left( \frac{4}{\pi^2} \ln n + R_n \right) E_n(f)_C, \quad (2)$$

where $|R_n| < 2,271$.

On the whole space $C$ the inequality (2) is asymptotically exact. At the same there exist subsets of functions from $C$ and for elements of these subsets the inequality (2) is not exact even by order (see, e.g., [9, p. 434]).
In the paper [11] the following estimate was proved
\[ \|\rho_n(f; \cdot)\|_C \leq K \sum_{\nu=n}^{2n-1} \frac{E_\nu(f)_C}{\nu - n + 1}, \quad f \in C, \quad n \to \infty, \]
(here \( K \) is some absolute constant) and it was proved that this constant is exact by the order on the classes \( C(\varepsilon) \) with a given majorant of the best approximations \( C(\varepsilon) := \{ f \in C : E_\nu(f)_C \leq \varepsilon_\nu, \quad \nu \in \mathbb{N} \} \), \( \{ \varepsilon_\nu \}_{\nu=0}^{\infty} \) is a sequence of nonnegative numbers, such that \( \varepsilon_\nu \downarrow 0 \) as \( \nu \to \infty \). This estimate sharpens Lebesgue classical inequality for "fast" decreasing \( E_\nu \).

In [9]–[14] (see also [5]) for the classes \( C^r_\beta C \) of the functions \( f \in C \), which are defined with a help of convolutions
\[ f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} \Psi_\beta(x - t)\varphi(t)dt, \quad \varphi \perp 1, \quad \varphi \in C, \quad a_0 \in \mathbb{R}, \quad (3) \]
with summable kernels \( \Psi_\beta(t) \) whose Fourier series has the form
\[ \Psi_\beta(t) \sim \sum_{k=1}^{\infty} \psi(k) \cos \left( kt - \frac{\beta \pi}{2} \right), \quad \psi(k) \geq 0, \quad \beta \in \mathbb{R}, \]
asymptotically best possible analogs of Lebesgue-type inequalities were found. In these inequalities the norms of deviations of Fourier sums \( \|\rho_n(f; \cdot)\|_C \) are expressed via the best approximations \( E_n(\varphi)_C \) of the function \( \varphi \) (the function \( \varphi \), which is connected with \( f \) with a help of equality (3) is called \((\psi, \beta)\)–derivative of the function \( f \) and is denoted by \( f^\psi_{\beta} \)).

Denote by \( C^r_\alpha, r C, \quad \alpha > 0, \quad r > 0 \), the set of all \( 2\pi \)–periodic functions, such that for all \( x \in \mathbb{R} \) can be represented in the form of convolution
\[ f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int_{-\pi}^{\pi} P_{\alpha, r, \beta}(x - t)\varphi(t)dt, \quad a_0 \in \mathbb{R}, \quad \varphi \perp 1, \quad (4) \]
where \( \varphi \in C \), and \( P_{\alpha, r, \beta}(t) \) is a generalized Poisson kernel of the form
\[ P_{\alpha, r, \beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k^r} \cos \left( kt - \frac{\beta \pi}{2} \right), \quad \alpha > 0, \quad r > 0, \quad \beta \in \mathbb{R}. \]

If \( f \) and \( \varphi \) are connected with a help of equality (4), then the function \( f \) in this equality is called the generalized Poisson integral of the function \( \varphi \) and is
denoted by $J_{\beta}^{\alpha,r}(\varphi)$. The function $\varphi$ in the equality (4) is called the generalized derivative of the function $f$ and is denoted by $f_{\beta}^{\alpha,r}$.

It is clear that the sets of generalized Poisson integrals $C_{\beta}^{\alpha,r}C$ are subsets of the sets $C_{\beta}^{\psi}C$, if to put $\psi(k) = e^{-\alpha kr}$, $\alpha > 0$, $r > 0$. In this case for all $t \in \mathbb{R}$ the equality holds $f_{\beta}^{\psi}(t) = f_{\beta}^{\alpha,r}(t)$.

It should be noticed that for any $r > 0$ the classes $C_{\beta}^{\alpha,r}C$ belong to set of infinitely differentiable $2\pi$–periodic functions $D^\infty$, i.e., $C_{\beta}^{\alpha,r}C \subset D^\infty$ (see, e.g., [5, p. 128]) For $r \geq 1$ the classes $C_{\beta}^{\alpha,r}C$ consist of functions $f$, admitting a regular extension into the strip $|\text{Im } z| \leq c$, $c > 0$ in the complex plane (see, e.g., [5, p. 141]), i.e., are the classes of analytic functions. For $r > 1$ the classes $C_{\beta}^{\alpha,r}C$ consist of functions regular on the whole complex plane, i.e., of entire functions (see, e.g., [5, p. 131]). Besides, it follows from the Theorem 1 in [10] that for any $r > 0$ the embedding holds $C_{\beta}^{\alpha,r}C \subset \mathcal{J}_{1/r}$, where $\mathcal{J}_a, a > 0$, are known Gevrey classes

$$\mathcal{J}_a = \left\{ f \in D^\infty : \sup_{k \in \mathbb{N}} \left( \frac{\|f^{(k)}\|_C}{(k!)^a} \right)^{1/k} < \infty \right\}.$$

In the paper of Stepanets [9] the general results were obtained. From them, in particular, it follows that for any $f \in C_{\beta}^{\alpha,r}C$, $r \in (0, 1)$, $\alpha > 0$, $\beta \in \mathbb{R}$, for any $n \in \mathbb{N}$ the following asymptotically best possible inequality holds

$$\|\rho_n(f; x)\|_C \leq e^{-\alpha n^r} \left( \frac{4}{\pi^2} \ln n^{1-r} + \mathcal{O}(1) \right) E_n(f_{\beta}^{\alpha,r}) C,$$

where $\mathcal{O}(1)$ is a quantity uniformly bounded with respect to $f \in C_{\beta}^{\alpha,r}C$, $n \in \mathbb{N}$ and $\beta \in \mathbb{R}$.

Herewith the behavior (speed of increasing) of the quantity $\mathcal{O}(1)$ in the inequality (5) with respect to values of parameters $\alpha$ and $r$ in the work [9] was not considered.

In present paper we establish the asymptotically best possible Lebesgue-type inequalities for the functions $f \in C_{\beta}^{\alpha,r}C$, in which for all $n$, starting from some number $n_1 = n_1(\alpha, r)$, an additional term is estimated by absolute constant. Herewith the number $n_1$ is defined constructively via parameters of the problem.
(the inequality (6)), and an absolute constant is written in an explicit form $20\pi^4$.

Obtained results complement the results of the papers [15]– [16], and also clarify the estimate (5), which was obtained in [9].

2. Main results

Let us formulate now the main results of the paper.

For arbitrary $\alpha > 0$, $r \in (0, 1)$ we denote by $n_1 = n_1(\alpha, r)$ the smallest integer $n \in \mathbb{N}$, such that

$$\frac{1}{\alpha r} \left(1 + \ln \frac{\pi n^{1-r}}{\alpha r}\right) + \frac{\alpha r}{n^{1-r}} \leq \frac{1}{(3\pi)^3}. \quad (6)$$

**Theorem 1.** Let $\alpha > 0$, $r \in (0, 1)$, $\beta \in \mathbb{R}$ and $n \in \mathbb{N}$. Then, for any function $f \in C^\alpha_{\beta} C$ and all $n \geq n_1(\alpha, r)$ the following inequality holds

$$\|\rho_n(f; \cdot)\|_C \leq e^{-\alpha n^r} \left(\frac{4}{\pi^2} \ln \frac{n^{1-r}}{\alpha r} + \gamma_n\right) E_n(f^\alpha_{\beta,r})_C. \quad (7)$$

Moreover, for arbitrary function $f \in C^\alpha_{\beta} C$ one can find a function $F(f, n, x)$ from the set $C^\alpha_{\beta} C$, such that $E_n(F^\alpha_{\beta,r})_C = E_n(f^\alpha_{\beta,r})_C$, such that for $n \geq n_1(\alpha, r)$ the equality holds

$$\|\rho_n(F; \cdot)\|_C = e^{-\alpha n^r} \left(\frac{4}{\pi^2} \ln \frac{n^{1-r}}{\alpha r} + \gamma_n\right) E_n(f^\alpha_{\beta,r})_C. \quad (8)$$

In (7) and (8) for the quantity $\gamma_n = \gamma_n(\alpha, r, \beta)$ the estimate holds $|\gamma_n| \leq 20\pi^4$.

**Proof.** Let $f \in C^\alpha_{\beta} C$. Then, for arbitrary $x \in \mathbb{R}$ the following integral representation takes place

$$\rho_n(f; x) = f(x) - S_{n-1}(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^\alpha_{\beta,r}(t) P^{(n)}_{\alpha,r,\beta}(x - t) dt, \quad (9)$$

where

$$P^{(n)}_{\alpha,r,\beta}(t) := \sum_{k=n}^{\infty} e^{-\alpha k^r} \cos\left(kt - \frac{\beta \pi}{2}\right), \quad 0 < r < 1, \quad \alpha > 0, \quad \beta \in \mathbb{R}. \quad (10)$$

Whereas the function $P^{(n)}_{\alpha,r,\beta}(t)$ is orthogonal to any trigonometric polynomial $t_{n-1} \in \tau_{2n-1}$, then because of (9)

$$\rho_n(f; x) = f(x) - S_{n-1}(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta_n(t) P^{(n)}_{\alpha,r,\beta}(x - t) dt, \quad (11)$$
where
\[ \delta_n(x) = \delta_n(\alpha, r, \beta; x) := f^{\alpha,r}_\beta(x) - t_{n-1}(x). \] (12)

By \( t^*_n \in \tau_{2n-1} \) we denote the polynomial of the best uniform approximation of the function \( f^{\alpha,r}_\beta \), namely, such that
\[ \| f^{\alpha,r}_\beta - t^*_n \|_C = E_n(f^{\alpha,r}_\beta)_C. \]

Then, in view of (11), we have
\[ \| f(\cdot) - S_{n-1}(f; \cdot) \|_C \leq \frac{1}{\pi} \| P^{(n)}_{\alpha,r,\beta} \|_1 E_n(f^{\alpha,r}_\beta)_C. \] (13)

As it follows from the formula (20) of the paper [17] (see also [18]) for arbitrary \( r \in (0, 1), \alpha > 0, \beta \in \mathbb{R}, 1 \leq s < \infty, \frac{1}{s} + \frac{1}{s'} = 1, n \in \mathbb{N} \) and \( n \geq n_0(\alpha, r, s') \) the relation holds
\[ \frac{1}{\pi} \| P^{(n)}_{\alpha,r,\beta} \|_s = e^{-\alpha n} \frac{1}{n^{1-r}} \left( \frac{\| \cos t \|_s}{\pi^{1+s} (\alpha r)^{s'}} \mathcal{I}_s \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \delta^{(1)}_{n,s} \left( \frac{1}{(\alpha r)^{1+s'}} \mathcal{I}_s \left( \frac{\pi n^{1-r}}{\alpha r} \right) \frac{1}{n^r} + \frac{1}{n^{1-r}} \right) \right), \] (14)

where \( n_0 = n_0(\alpha, r, p) \) is a smallest number \( n \), such that
\[ \frac{1}{\pi} \| P^{(n)}_{\alpha,r,\beta} \|_s = e^{-\alpha n} \frac{1}{n^{1-r}} \left( \frac{\| \cos t \|_s}{\pi^{1+s} (\alpha r)^{s'}} \mathcal{I}_s \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \delta^{(1)}_{n,s} \left( \frac{1}{(\alpha r)^{1+s'}} \mathcal{I}_s \left( \frac{\pi n^{1-r}}{\alpha r} \right) \frac{1}{n^r} + \frac{1}{n^{1-r}} \right) \right), \]

where \( n_0 = n_0(\alpha, r, p) \) is a smallest number \( n \), such that
\[ \frac{1}{\pi} \| P^{(n)}_{\alpha,r,\beta} \|_s = e^{-\alpha n} \frac{1}{n^{1-r}} \left( \frac{\| \cos t \|_s}{\pi^{1+s} (\alpha r)^{s'}} \mathcal{I}_s \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \delta^{(1)}_{n,s} \left( \frac{1}{(\alpha r)^{1+s'}} \mathcal{I}_s \left( \frac{\pi n^{1-r}}{\alpha r} \right) \frac{1}{n^r} + \frac{1}{n^{1-r}} \right) \right), \]

where \( \chi(p) = p \) for \( 1 \leq p < \infty \) and \( \chi(p) = 1 \) for \( p = \infty \) and
\[ \mathcal{I}_s(v) := \begin{cases} \left( \int_0^v \frac{1}{(t^2+1)^s} dt \right)^{\frac{1}{s}}, & 1 \leq s < \infty, \\ \text{ess sup}_{t \in [0,v]} \frac{1}{\sqrt{t^2+1}} = 1, & s = \infty, \end{cases} \] (16)

and for the quantity \( \delta^{(1)}_{n,s} = \delta^{(1)}_{n,s}(\alpha, r, \beta) \) the following estimate holds
\[ |\delta^{(1)}_{n,s}| \leq (14\pi)^2. \]

Putting in the formula (14) \( s = 1 \), we get that for \( r \in (0, 1), \alpha > 0 \) and \( \beta \in \mathbb{R}, n \in \mathbb{N} \) and \( n \geq n_0(\alpha, r, \infty) \) the relation takes place
\[ \frac{1}{\pi} \| P^{(n)}_{\alpha,r,\beta} \|_1 = e^{-\alpha n} \left( \frac{4}{\pi^2} \mathcal{I}_1 \left( \frac{\pi n^{1-r}}{\alpha r} \right) + \delta^{(1)}_{n,1} \left( \frac{1}{\alpha r} \mathcal{I}_1 \left( \frac{\pi n^{1-r}}{\alpha r} \right) \frac{1}{n^r} + 1 \right) \right). \] (17)
According to the formula (112) of the work [18]

\[
\mathcal{L}_1\left(\frac{\pi n^{1-r}}{\alpha r}\right) = \int_0^1 \frac{dt}{\sqrt{t^2 + 1}} = \ln \frac{\pi n^{1-r}}{\alpha r} + \Theta_{\alpha, r, n},
\]

(18)

where \(0 < \Theta_{\alpha, r, n} < 1\). It is easy to show that for \(n \geq n_1(\alpha, r)\) the following inequality holds

\[
\frac{4}{\pi^2} (\ln \pi + \Theta_{\alpha, r, n}) + |\delta_{n, 1}^{(1)}| \left( \frac{1}{\alpha r n^r} \ln \frac{\pi n^{1-r}}{\alpha r} + \Theta_{\alpha, r, n} + 1 \right)
\]

\[
< \frac{4}{\pi^2} (\ln \pi + 1) + (14\pi)^2 \left( \frac{1}{(3\pi)^3} + 1 \right) < 1938,
\]

then formulas (17) and (18) imply that for \(n \geq n_1(\alpha, r)\)

\[
\| P^{(n)}_{\alpha, r, \beta} \|_1 = e^{-\alpha n^r} \left( \frac{4}{\pi^2} \ln \frac{n^{1-r}}{\alpha r} + \gamma_n^* \right),
\]

(19)

where for the quantity \(\gamma_n^* = \gamma_n(\alpha, r, \beta)\) the estimate is true \(|\gamma_n^*| < 1938\). The inequalities (13) and (19) prove the truth of (7).

To prove the second part of Theorem 1 it is enough to show that for any function \(\varphi \in C\) one can construct a function \(\Phi(\cdot) = \Phi(\varphi, \cdot) \in C\), such that \(E_n(\Phi)_C = E_n(\varphi)_C\) and for any \(n \geq n_1(\alpha, r)\) the equality holds

\[
\frac{1}{\pi} \left| \int_{-\pi}^{\pi} \Phi(t) P^{(n)}_{\alpha, r, \beta}(0 - t) dt \right| = e^{-an^r} \left( \frac{4}{\pi^2} \ln \frac{n^{1-r}}{\alpha r} + \gamma_n \right) E_n(\varphi)_C,
\]

(20)

where \(|\gamma_n| < 20\pi^4\).

In this case for arbitrary function \(f \in C^{\alpha, r}_\beta C\) there exists a function \(\Phi(\cdot) = \Phi(f^{\alpha, r}_\beta, \cdot)\), such that \(E_n(\Phi)_C = E_n(f^{\alpha, r}_\beta)_C\) and for \(n \geq n_1(\alpha, r)\) formula (20) is true, where as function \(\varphi\) we take the function \(f^{\alpha, r}_\beta\). Let us assume \(F(\cdot) = J^{\alpha, r}_\beta(\Phi(\cdot) - \frac{a_0}{2})\), where \(a_0 = a_0(\Phi) = \frac{1}{\pi} \int_{-\pi}^{\pi} \Phi(t) dt\). The function \(F\) is the function, which we have looked for, because \(F \in C^{\alpha, r}_\beta C\), and \(E_n(F^{\alpha, r}_\beta)_C = E_n(\Phi - \frac{a_0}{2})_C = E_n(\Phi)_C = E_n(f^{\alpha, r}_\beta)_C\) and moreover, formulas (7), (9), (11) and (20) yield (8).

To prove (20) we need more detailed information about the character of oscillation of the kernel \(P^{(n)}_{\alpha, r, \beta}(t)\). We denote by \(n_* = n_*(\alpha, r)\) the smallest number, for which the inequality holds

\[
\frac{1}{\alpha r n^r} + \frac{\alpha r}{n^{r-1}} < \frac{117}{784\pi^2}.
\]

(21)
Lemma 1. Let $\alpha > 0$, $r \in (0, 1)$, $\beta \in \mathbb{R}$ and $n \in \mathbb{N}$. For $n \geq n_*$ the function $P_{\alpha, r, \beta}^{(n)}(t)$ has exactly $2n$ simple zeros $z_k$ on the period $[0, 2\pi)$, where the function $P_{\alpha, r, \beta}^{(n)}(t)$ takes values with alternating signs.

Proof. According to formulas (44) and (47) of the work [18] we can write

$$P_{\alpha, r, \beta}^{(n)}(t) = g_{\alpha, r, n}(t) \cos \left( nt - \frac{\beta \pi}{2} \right) + h_{\alpha, r, n}(t) \sin \left( nt - \frac{\beta \pi}{2} \right)$$

$$= \sqrt{g_{\alpha, r, n}^2(t) + h_{\alpha, r, n}^2(t)} \cos \left( nt - \frac{\beta \pi}{2} - \arctg \frac{h_{\alpha, r, n}(t)}{g_{\alpha, r, n}(t)} \right)$$

$$= \sqrt{g_{\alpha, r, n}^2(t) + h_{\alpha, r, n}^2(t)} \cos(n \cdot y(t)), \quad (22)$$

where

$$g_{\alpha, r, n}(t) = \sum_{k=0}^{\infty} e^{-\alpha(k+n)r} \cos kt, \quad (23)$$

$$h_{\alpha, r, n}(t) = \sum_{k=0}^{\infty} e^{-\alpha(k+n)r} \sin kt, \quad (24)$$

$$y(t) = y(\alpha, r, n; t) = t - \frac{\beta \pi}{2n} - \frac{1}{n} \arctg \frac{h_{\alpha, r, n}(t)}{g_{\alpha, r, n}(t)}. \quad (25)$$

Lemma 1 will be proved, if one can show that for $n \geq n_*$ the function $y(t)$ of the form (25) increasing on $[0, 2\pi]$ from a value $y(0) = -\frac{\beta \pi}{2}$ to a value $y(2\pi) = 2\pi - \frac{\beta \pi}{2}$. In this case the function $\cos(n \cdot y(t))$, and also the function $P_{\alpha, r, \beta}^{(n)}(t)$ (taking into account (22) and also the strict inequality $\sqrt{g_{\alpha, r, n}^2(t) + h_{\alpha, r, n}^2(t)} > 0$ (see (47) from [18])) have on $[0, 2\pi)$ exactly $2n$ simple zeros $z_k$ of the form

$$z_k = y^{-1} \left( \frac{\pi}{2} + \frac{k \pi}{n} \right), \quad k = 0, ..., 2n - 1, \quad (26)$$

where $y^{-1}(\cdot)$ is inverse function to $y(\cdot)$. In points $z_k$ the function $\cos(ny(t))$ (and also the function $P_{\alpha, r, \beta}^{(n)}(t)$) takes values with alternating signs.

Let us consider the derivative of the function $y(t)$:

$$y'(t) = 1 - \frac{1}{n} \left( \frac{h_{\alpha, r, n}(t)}{g_{\alpha, r, n}(t)} \right)' = 1 + \frac{1}{n} \left( \frac{-h_{\alpha, r, n}'(t)g_{\alpha, r, n}(t) + h_{\alpha, r, n}(t)g_{\alpha, r, n}'(t)}{g_{\alpha, r, n}(t)^2 + h_{\alpha, r, n}(t)^2} \right). \quad (27)$$
Let us estimate the absolute value of the last term in formula (27)

\[
\frac{1}{n} \left| \frac{-h'_{\alpha,r,n}(t)g_{\alpha,r,n}(t) + h_{\alpha,r,n}(t)g'_{\alpha,r,n}(t)}{g_{\alpha,r,n}(t)^2 + h_{\alpha,r,n}(t)^2} \right| \\
= \frac{1}{n} \sqrt{\frac{g'_{\alpha,r,n}(t)^2 + h'_{\alpha,r,n}(t)^2}{g_{\alpha,r,n}(t)^2 + h_{\alpha,r,n}(t)^2}} \left| \frac{-h'_{\alpha,r,n}(t)}{g'_{\alpha,r,n}(t)^2 + h'_{\alpha,r,n}(t)^2} \right| \frac{g_{\alpha,r,n}(t)}{g_{\alpha,r,n}(t)^2 + h_{\alpha,r,n}(t)^2} \\
+ \frac{h_{\alpha,r,n}(t)}{\sqrt{g_{\alpha,r,n}(t)^2 + h_{\alpha,r,n}(t)^2}} \frac{g'_{\alpha,r,n}(t)}{\sqrt{g'_{\alpha,r,n}(t)^2 + h'_{\alpha,r,n}(t)^2}} \\
\leq \frac{1}{n} \sqrt{\frac{g'_{\alpha,r,n}(t)^2 + h'_{\alpha,r,n}(t)^2}{g_{\alpha,r,n}(t)^2 + h_{\alpha,r,n}(t)^2}} \leq \frac{M_n(\alpha, r)}{n},
\]  

(28)

where

\[
M_n(\alpha, r) := \sup_{t \in \mathbb{R}} \frac{\sqrt{g'_{\alpha,r,n}(t)^2 + h'_{\alpha,r,n}(t)^2}}{\sqrt{g_{\alpha,r,n}(t)^2 + h_{\alpha,r,n}(t)^2}}.
\]  

(29)

From formula (99) from [18] we have

\[
M_n \leq \frac{784\pi^2}{117} \left( \frac{n^{1-r}}{\alpha r} + \alpha n^r \right).
\]

This and inequality (21) yield that \(y'(t) > 0\) for \(n \geq n_*(\alpha, r)\), so the function \(y(t)\) strictly increasing. Lemma 1 is proved.

Let us prove now the estimate (20). Let \(\varphi \in C\). Denote by \(\Phi_\delta(t)\) the 2\(\pi\)-periodic function, which coincides with the function

\[
\Phi_0(t) = E_n(\varphi)_{\text{sign}}P_{\alpha,r,\beta}^{(n)}(-t)
\]

everywhere, except \(\delta\)-neighborhoods (\(\delta < \frac{1}{2} \min\{z_{k+1} - z_k\}\)) of points \(z_k\), where it is linear function and its graph connects the points with coordinates \((z_k - \delta, \Phi_0(z_k - \delta))\) and \((z_k + \delta, \Phi_0(z_k + \delta))\).

The function \(\Phi_\delta(\cdot)\) is continuous. As the condition (6) is more strong than the condition (21), then \(n_1(\alpha, r) \geq n_*(\alpha, r)\). On the basis of Lemma 1 for \(n \geq n_1(\alpha, r)\) the function \(\Phi_\delta\) has on \([0, 2\pi]\) exactly \(2n\) zeros \(z_k\) of the form (26), where it takes values with alternating signs, and in the middle of each interval \((z_k, z_{k+1})\) it takes the maximum absolute values with alternating signs
\[ \pm E_n(\varphi)_C. \]

Then, by Chebyshev theorem about alternance, the polynomial \( t_{n-1}^* \) of the best approximation of the function \( \Phi_\delta \) in the uniform metric will be identically equal to zero and \( E_n(\Phi_\delta)_C = E_n(\varphi)_C. \) Therefore

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \Phi_\delta(t)P^{(n)}_{\alpha,r,\beta}(0 - t)\,dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \Phi_0(t)P^{(n)}_{\alpha,r,\beta}(-t)\,dt + R_n(\delta),
\]

where

\[
R_n(\delta) = R_n(\alpha, r, \beta, \delta) = \frac{1}{\pi} \int_{-\pi}^{\pi} (\Phi_\delta(t) - \Phi_0(t))P^{(n)}_{\alpha,r,\beta}(-t)\,dt.
\]

As,

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \Phi_0(t)P^{(n)}_{\alpha,r,\beta}(-t)\,dt = E_n(\varphi)_C \int_{-\pi}^{\pi} \text{sign}P^{(n)}_{\alpha,r,\beta}(-t)\,dt
\]

\[= E_n(\varphi)_C \int_{-\pi}^{\pi} |P^{(n)}_{\alpha,r,\beta}(-t)|\,dt = E_n(\varphi)_C \int_{-\pi}^{\pi} |P^{(n)}_{\alpha,r,\beta}(t)|\,dt = E_n(\varphi)_C \|P^{(n)}_{\alpha,r,\beta}\|_1,
\]

then on the basis of (30) and (19) we have that

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \Phi_\delta(t)P^{(n)}_{\alpha,r,\beta}(0 - t)\,dt = e^{-\alpha n r} \left( \frac{4}{\pi^2} \ln \frac{n^{1-r}}{\alpha r} + \gamma_n^* \right) E_n(\varphi)_C + R_n(\delta),
\]

where \( |\gamma_n^*| < 1938. \)

Let us choose \( \delta \) small enough, that the following inequality holds

\[
\delta < \frac{13\pi(10\pi^4 - 969)\alpha \tau n^r}{14 n^2}.
\]

For values \( \delta \), which satisfy the condition (33), for \( n \geq n_1(\alpha, r) \) the following estimate holds

\[
|R_n(\delta)| < (20\pi^4 - 1938)e^{-\alpha n r} E_n(\varphi)_C.
\]

Indeed, according to (31)

\[
|R_n(\delta)| \leq \frac{1}{\pi} \|P^{(n)}_{\alpha,r,\beta}\|_C \int_{-\pi}^{\pi} |\Phi_\delta(t) - \Phi_0(t)|\,dt \leq \frac{1}{\pi} \sum_{k=0}^{\infty} e^{-\alpha(k+n)^r} \int_{-\pi}^{\pi} |\Phi_\delta(t) - \Phi_0(t)|\,dt
\]
and, as follows from the formula (91) of the work [18], for \( n \geq n_1(\alpha, r) \), we derive the inequality
\[
\sum_{k=0}^{\infty} e^{-\alpha(k+n)r} < \frac{14}{13} e^{-\alpha n^r n^{1-r}}. 
\]
(36)

Then, whereas according to definitions of the functions \( \Phi_\delta \) and \( \Phi_0 \), the following equality holds
\[
\int_{-\pi}^{\pi} |\Phi_\delta(t) - \Phi_0(t)| dt = E_n(\varphi)C \sum_{k=0}^{2n-1} \int_{z_k}^{z_k+\delta} \frac{|-t + z_k - \delta|}{2\delta} dt = 2n\delta E_n(\varphi)_C, 
\]
then from (33), (35) and (36) we obtain the inequalities
\[
|R_n(\delta)| < \frac{28}{13\pi} e^{-\alpha n^r n^{2-r}} \delta E_n(\varphi)_C < 2(10\pi^4 - 969) e^{-\alpha n^r} E_n(\varphi)_C.
\]
This proves (34).

Hence, let us put \( \Phi(t) = \Phi_\delta(t) \), choosing some \( \delta \), such that \( \delta < \frac{1}{2} \min_{k \in \mathbb{Z}} \{z_{k+1} - z_k\} \). It should be noticed that for such choice of \( \delta \), the condition (33) is satisfied. Then, because of (32) and (34) for \( n \geq n_1(\alpha, r) \) for the function \( \Phi(t) \) the estimate (20) is true. Theorem 1 is proved.

Notice that the statement of Lemma 1 takes place not only for \( n \geq n_* \), but for all \( n \in \mathbb{N} \). To be sure in it, we use the following statement of the work [19].

**Proposition 1.** Let the coefficients \( a_k \) of the trigonometric series
\[
\sum_{k=n}^{\infty} a_k \sin(kx + \gamma), \quad \gamma \in \left(0, \frac{\pi}{2}\right], \quad n \in \mathbb{N}, 
\]
(37)
satisfy the conditions
\[
\Delta_m a_k := a_k - ma_{k+1} + \frac{m(m-1)}{1 \cdot 2} a_{k+2} - \ldots + (-1)^m a_{k+m} > 0, \quad m \in \mathbb{Z}_+, \quad k \in \mathbb{N},
\]
(38)
\[
\lim_{k \to \infty} a_k = 0, 
\]
(39)
\[
\sum_{k=n}^{\infty} a_k < \infty. 
\]
(40)
Then the function-sum (37) has exactly 2n simple zeros in the interval (0, 2π), which are alternately located inside of respective intervals

\[
\left(\frac{\pi - 2\gamma}{2n - 1}, \frac{\pi - \gamma}{n}\right), \left(\frac{3\pi - 2\gamma}{2n - 1}, \frac{2\pi - \gamma}{n}\right), \ldots, \left(\frac{(2n - 1)\pi - 2\gamma}{2n - 1}, \frac{n\pi - \gamma}{n}\right),
\]

\[
\left(\frac{(n + 1)\pi - \gamma}{n}, \frac{(2n + 1)\pi - 2\gamma}{2n - 1}\right), \ldots, \left(\frac{(2n - 1)\pi - \gamma}{n}, \frac{(4n - 3)\pi - 2\gamma}{2n - 1}\right), \ldots,
\]

\[
\left(\frac{2n\pi - \gamma}{2n - 1}, 2\pi\right).
\]

The condition (38) of Proposition 1 can be written in the form

\[
(-1)^m \Delta^m a_k > 0, \quad k \in \mathbb{Z}_+, \ m \in \mathbb{N}, \quad (41)
\]

(so called condition of absolutely monotonicity of the sequence \(a_k\)), where the difference operator \(\Delta^m\) is defined by induction with a help of equalities

\[
\Delta^0 a_k = a_k, \quad \Delta^1 a_k = a_{k+1} - a_k, \quad \Delta^2 a_k = \Delta^1(\Delta^1 a_k), \ldots,
\]

\[
\Delta^m a_k = \Delta^1(\Delta^{m-1} a_k) = \sum_{v=0}^{m} \binom{m}{v} (-1)^{m+v} a_{k+v}.
\]

To apply Proposition 1 to the sequence \(a_k = e^{-\alpha k^r}, \alpha > 0, \ r \in (0, 1)\) it is enough to be sure that (41) holds, because the verification of (39) and (40) is trivial. For \(\alpha > 0, \ r \in (0, 1)\) the function \(\psi(t) = \psi(\alpha, r, t) = e^{-\alpha t^r}\) is absolutely monotonic, namely, the condition holds

\[
(-1)^m \psi^{(m)}(t) > 0, \quad m \in \mathbb{N}, \ t > 0.
\]

It follows from the fact that the function \(\psi(t)\) is a superposition of absolutely monotonic function \(\exp(-t)\) and positive function \(g(t) = g(\alpha, r, t) = \alpha t^r, \alpha > 0, \ r \in (0, 1),\) which has absolutely monotonic derivative (see [20, Ch. 4, §4 ]).

Hence, the sequence \(a_k = e^{-\alpha k^r}\) satisfies the condition (38)–(40) of Proposition 1 and herefrom the following statement holds.

**Corollary 1.** Let \(\alpha > 0, \ r \in (0, 1), \ \beta \in [0,1)\) and \(n \in \mathbb{N}\). Then, on \([0, 2\pi]\) the function \(P_{\alpha, r, \beta}^{(n)}(t)\) has exactly 2n simple zeros, which are alternately located inside of respective intervals

\[
\left(\frac{\beta \pi}{2n - 1}, \frac{\pi - (1 - \beta)\pi/2}{n}\right), \left(\frac{(2 + \beta)\pi}{2n - 1}, \frac{2\pi - (1 - \beta)\pi/2}{n}\right), \ldots,
\]
\[
\left( \frac{(2n - 2 + \beta)\pi}{2n - 1}, \frac{n\pi - (1 - \beta)\frac{\pi}{2}}{n} \right), \\
\left( \frac{(n + 1)\pi - (1 - \beta)\frac{\pi}{2}}{n}, \frac{2(n + \beta)\pi}{2n - 1} \right), ..., \left( \frac{(2n - 1)\pi - (1 - \beta)\frac{\pi}{2}}{n}, \frac{(4n - 1 + \beta)\pi}{2n - 1} \right), \\
\left( \frac{2n\pi - (1 - \beta)\frac{\pi}{2}}{2n - 1}, 2\pi \right).
\]

In the same way as it was done in the works [9]–[13] we consider the classes \( C_{\alpha,r}^C(\varepsilon) \) of \( 2\pi \)-periodic functions \( f \) of the form (3), where \( \varphi = f_{\alpha,r}^\alpha \) belongs to the class \( C(\varepsilon) \), where as earlier \( \varepsilon = \{\varepsilon_\nu\}_{\nu=0}^\infty \) is monotonically decreasing to zero the sequence of nonnegative numbers.

The following statement gives an example that the inequality (7) is best possible not only on the set \( C_{\alpha,r}^C \), but also on such important subsets \( C_{\alpha,r}^C(\varepsilon) \) of the set \( C_{\alpha,r}^C \).

**Theorem 2.** Let \( \alpha > 0, r \in (0, 1), \beta \in \mathbb{R} \) and \( \varepsilon = \{\varepsilon_\nu\}_{\nu=0}^\infty \) is an arbitrary monotonically decreasing to zero the sequence of nonnegative real numbers. Then, for arbitrary class \( C_{\beta}^{\alpha,r}C(\varepsilon) \) and all numbers \( n \geq n_1(\alpha, r) \) the equalities hold

\[
E_n(C_{\beta}^{\alpha,r}C(\varepsilon))_C = \sup_{f \in C_{\beta}^{\alpha,r}C(\varepsilon)} \| f(\cdot) - S_{n-1}(f, \cdot) \|_C = e^{-\alpha n^r} \left( \frac{4}{\pi^2} \ln \frac{n^{1-r}}{\alpha r} + \gamma_n \right) \varepsilon_n,
\]

where \( |\gamma_n| \leq 20\pi^4 \).

**Proof.** Let \( f \in C_{\beta}^{\alpha,r}C(\varepsilon) \). Then, the function \( \varphi = f_{\beta}^{\alpha,r} \) is continuous and \( E_n(f_{\beta}^{\alpha,r}) \leq \varepsilon_n \). Then, taking into account (7), we obtain that for \( n \geq n_1(\alpha, r) \)

\[
\| \rho_n(f; \cdot) \|_C \leq e^{-\alpha n^r} \left( \frac{4}{\pi^2} \ln \frac{n^{1-r}}{\alpha r} + \gamma_n \right) \varepsilon_n \quad \forall f \in C_{\beta}^{\alpha,r}C(\varepsilon),
\]

where \( |\gamma_n| \leq 20\pi^4 \).

On other hand, from Theorem 1 it follows that for the function \( F(x) \), which is constructed for the function \( \varphi = f_{\beta}^{\alpha,r} \in C(\varepsilon) \), and such that \( E_n(f_{\beta}^{\alpha,r}) = \varepsilon_n \), the inequality (43) becomes an equality for \( n \geq n_1(\alpha, r) \). Herefrom we get (42). Theorem 2 is proved.
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