INVERSE MEAN CURVATURE FLOWS IN WARPED PRODUCT MANIFOLDS

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Abstract. We study inverse mean curvature flows of starshaped, mean convex hypersurfaces in warped product manifolds with a positive warping factor $h(r)$. If $h'(r) > 0$ and $h''(r) \geq 0$, we show that these flows exist for all times, remain starshaped and mean convex. Plus the positivity of $h''(r)$ and a curvature condition we obtain a lower positive bound of mean curvature along these flows independent of the initial mean curvature. We also give a sufficient condition to extend the asymptotic behavior of these flows in Euclidean spaces into some more general warped product manifolds.

1. Introduction

In this article we study inverse mean curvature flows in warped product manifolds. Our objective is to propose that for mean convex, starshaped hypersurfaces in warped product manifolds such flows have good evolution behaviors under certain weak conditions.

Fix $n \geq 3$. Throughout this paper we assume that $N$ is a $n - 1$ dimensional closed Riemannian manifold with a smooth metric $\sigma$ and $h(r)$ is a smooth positive warping function on $\mathbb{R}^+$. A warped product manifold is the set $\{(x, r)\} \times N$ for $x \in N, r \in \mathbb{R}^+$ endowed with the warped metric $dr^2 + h^2(r)\sigma$, written as $N \times_h \mathbb{R}^+$. This general definition includes Euclidean spaces, Hyperbolic spaces [12], Fuchsian manifolds [16], Anti-Desitter-Schwarzchild manifolds [5] as well as Schwarzschild-Kottler spaces [23] etc (also see [2]). A hypersurface is starshaped if it is a graph over $N$ in $N \times_h \mathbb{R}^+$.

Assume $\Sigma$ is a mean convex closed smooth hypersurface in a Riemannian manifold $M$. An inverse mean curvature flow of $\Sigma$ is a smooth map $F : \Sigma \times [0, T] \to M$ satisfying

$$\frac{\partial F}{\partial t} = \vec{v}$$

where $F(., 0)$ is the identity on $\Sigma$. Here $H$ and $\vec{v}$ are the mean curvature and normal vector of $\Sigma_t = F(., t)(\Sigma)$ respectively.

The investigation of inverse mean curvature flows are closely related to various warped product manifolds. Gerhardt [10] and Urbas [28] first study...
inverse mean curvature flows for initial mean convex, starshaped hypersurface in Euclidean spaces. These authors obtain long time existence and asymptotic behaviors that the flow converges uniformly to the round sphere after a metric scaling. A similar long time existence result is established in [11]. However Hung-Wang [20] show that no such asymptotic behaviors in Hyperbolic spaces by examples. Rationally symmetric spaces $S^n \times_h \mathbb{R}^+$ are investigated by Ding [7] and Scheuer [26].

On the other hand inverse mean curvature flow is a powerful tool to explore geometric properties of ambient manifolds. Some interesting applications are given by Huisken-Ilmanen [18] for Riemannian Pernose inequalities, by Bray-Neves [3] for the Poincaré conjecture for 3-manifolds with Yamabe invariant and by Brendle-Hung-Wang [5], Kwong-Miao [22] and Li-Wei [23] for various geometric inequalities in certain classes of warped product manifolds.

With delicate integral techniques Huisken-Ilmanen [19] obtain high regularity properties of inverse mean curvature flows in Euclideans space: a sharp lower bound for the mean curvature of starshaped hypersurfaces along the flows, independent of initial mean curvature. A central tool is the Sobolev inequality in [24]. Without this inequality Li-Wei [23] discover similar estimates in Schwarzschild-Kottler spaces. A consequence of high regularity properties is that an inverse mean curvature flow of bounded weakly mean convex, starshaped $C^1$ hypersurface will become smooth instantly in the settings of [19], [23]. Other recent progresses on inverse mean curvature flows can be found in [1], [13] and [6] etc.

The results mentioned above imply some nontrivial connections between inverse mean curvature flows and warped product manifolds. We are motivated to summarize such kinds of connections.

This paper is organized as follows. In section 2 we develop some preliminary facts for latter applications. Section 3 is devoted to a long time existence result regarding inverse mean curvature flows (Theorem 3.1). Provided \( h'(r) > 0 \) and \( h''(r) \geq 0 \) we obtain the long time existence and the preservation of starshaped, mean convex properties of inverse mean curvature flows for starshaped, mean convex initial hypersurfaces in warped product manifold \( N \times_h \mathbb{R}^+ \). Comparing to most previous results ([26],[23],[10],[28]) we discover the essential role of the condition \( h'(r) > 0 \) and \( h''(r) \geq 0 \) in the evolution of inverse mean curvature flows in warped product manifolds.

In section 4, we demonstrate a lower positive bound for mean curvature of starshaped inverse mean curvature flows in warped product manifolds under the assumption: \( h'(r) > 0, \ h'' > 0 \) and \( hh'' - h'^2 + \rho \geq 0 \) for all \( r > 0 \) where \( \text{Ric}_N \geq (n-2)\rho \sigma \) (see (4.1) and Theorem 4.1). This bound is independent of the curvature of the initial surface and only depends on the support function \( \omega = h(r)\langle \bar{v}, \partial_r \rangle \) of the initial surface, the time \( t \) and the positivity of \( h''(r) \). Comparing to [19], our proof does not rely on the Sobolev inequality [24] which is very restrictive in general Riemannian manifolds (see [15]).
Inspired by Li-Wei [23], our new ingredients here are the assumption mentioned above is sufficient to obtain Theorem 4.1. Similarly as in [19] and [23], inverse mean curvature flows can smooth a $C^1$ starshaped hypersurface with bounded, weak nonnegative mean curvature in the setting of Theorem 4.1 (see Theorem 4.3).

In section 5 we discuss the question: to what extent there exist similar asymptotic behaviors of starshaped, mean convex inverse mean curvature flows in warped product manifolds as these in Euclidean spaces. [20] and [23] give some negative and positive answers for this question respectively. In views of these existing examples, we answer this question positively in Theorem 5.1 provided that (a) $h' > 0$, $hh'' \geq 0$ and both of them are uniformly bounded; (b) the Ricci curvature of $N$ is positive; (c) the Ricci curvature and the metric $g$ of $N \times_h \mathbb{R}^+$ have the relation that

$$(1.2) \quad \text{Ric}(p) \leq \frac{C}{h^{2+\beta(r)}} g$$

for $p = (x, r)$ and two positive constants $C, \beta > 0$ where $r \geq r_0$ for a fixed $r_0$. Condition (a) is necessary due to the negative answer for hyperbolic spaces in [20]. These conditions are satisfied by Euclidean spaces and Schwarzschild-Kottler spaces in [23]. We expect future applications for this result due to the generality of condition (b) and (c).

In this paper we do not consider the case that $h'(r)$ goes to infinity as $r \to \infty$. First in [26] Scheuer has an excellent description on inverse mean curvature flows in rationally symmetric spaces for this case. Second we are working on the evolution of inverse curvature flows in warped product manifolds in a coming paper. We hope to deal this issue with some new ingredients.

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2. Preliminary Facts

In this section we develop preliminary facts needed for latter applications.

2.1. The geometry of graphs. In a warped product manifold $N \times_h \mathbb{R}^+$, a hypersurface is starshaped if it is a graph over $N$.

Let $\Sigma$ be a starshaped smooth hypersurface in $N \times_h \mathbb{R}^+$, namely, it is represented by $\{(x, r(x)) : x \in N\}$ where $r(x)$ is a positive function on $N$. Let $\{x_i\}$ be a local coordinate in $N$. We denote $\frac{\partial}{\partial x_i}$ by $\partial_i$ and $\frac{\partial}{\partial r}$ by $\partial_r$. The metric $\sigma$ on $N$ takes the form

$$(2.1) \quad \sigma = \sigma_{ij} dx_i dx_j$$
As in [5], we define a positive function $\varphi(x)$ on $N$ by
\begin{equation}
(2.2) \quad \varphi(x) = \Phi(r(x))
\end{equation}
where $\Phi(r)$ is a positive function satisfying $\Phi'(r) = \frac{1}{h(r)}$. The covariant derivatives of $\varphi$ are denoted by $\varphi_i$ and $\varphi_{ij}$. The upward normal vector of $\Sigma$ is
\begin{equation}
(2.3) \quad \vec{v} = \Theta(\partial_r - \frac{1}{h(r)}\varphi^i \partial_i) \quad \text{with} \quad \Theta = \langle \vec{v}, \partial_r \rangle = \frac{1}{\sqrt{1 + |D\varphi|^2}}
\end{equation}
Here $\varphi^i = \sigma^{ik}\varphi_k$ and $D\varphi$ is the gradient of $\varphi$ on $N$. $\Theta$ is called as the angle function of $\Sigma$.

In what follows let $X_i$ denote the vector $\partial_i + r_i \partial_r$. We obtain a local frame $\{X_i\}_{i=1}^n$ along $\Sigma$. With this frame the metric matrix of $\Sigma$ is
\begin{equation}
(2.4) \quad g_{ij} = \langle X_i, X_j \rangle = h^2(r)(\sigma_{ij} + \varphi_i \varphi_j)
\end{equation}
with the inverse $g^{ij} = \frac{1}{h^2(r)}(\sigma^{ij} - \Theta^2 \varphi^i \varphi^j)$.

The second fundamental form of $\Sigma$ is computed by
\begin{equation}
(2.5) \quad h_{ij} = -\langle \vec{\nabla} X_i, X_j, \vec{v} \rangle
\end{equation}
where $\vec{\nabla}$ is the covariant derivative of $N \times h \mathbb{R}^+$. It is well-known that $V = h(r)\partial_r$ is a conformal vector field on $N \times h \mathbb{R}^+$ (see [25]) such that $\vec{\nabla} X = h'(r)X$ for any smooth tangent vector field $X$. A straightforward computation yields that
\begin{align}
(2.6) \quad & h_{ij} = -h(r)\Theta(\varphi_{ij} - h'(r)(\sigma_{ij} + \varphi_i \varphi_j)) \\
(2.7) \quad & h^i_j = g^{ik} h_{kj} = -\Theta h(r)(\sigma^{ik} \varphi_{kj} - h'(r)\delta_{ij})
\end{align}
where $\tilde{\sigma}^{ik} = \sigma^{ik} - \Theta^2 \varphi^i \varphi^k$. Since $H = \sum_{i=1}^{n-1} h^i_i$, we obtain that

**Proposition 2.1.** The mean curvature of $\Sigma = \{(x, r(x)) : x \in N\}$ in $N \times h \mathbb{R}^+$ is
\begin{equation}
(2.8) \quad H = \frac{\Theta}{h(r)}(\tilde{\sigma}^{ik} \varphi_{ij} - (n-1)h'(r))
\end{equation}
where $n-1$ is the dimension of $N$.

Our computation is similar to those in [5], [11] and [28] by replacing the sphere with any Riemannian manifold.

### 2.2. Some facts on curvature tensor.

Let $\nabla$ be the covariant derivative of a Riemannian manifold $N$. The Riemann curvature tensor of $N$ is given by
\begin{equation}
(2.9) \quad R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z
\end{equation}
We write $R(X, Y, Z, W)$ for $\langle R(X, Y)Z, W \rangle$. Now we introduce a commuting formula for covariant derivatives and skip its proof since it is a direct verification.
Lemma 2.2. Suppose $\varphi_i dx^i$ is a covector on $N$. Then
\[ \varphi_{ijk} = \varphi_{ikj} + R_{kijp} \varphi^p \]
where $R_{kijp} = R(\partial_k, \partial_j, \partial_i, \partial_p)$ and $\varphi^p = \sigma^{pk} \varphi_k$.

Let $\text{Ric}_N, \text{Ric}$ denote the Ricci curvature of $N$ and $N \times h \mathbb{R}^+$ respectively. Their relation is given as follows.

Proposition 2.3 (Proposition 9.106 in [2]). Let $n - 1$ be the dimension of $N$. Then
\[ \text{Ric} = \text{Ric}_N - (h(r)h''(r) + (n - 2)h'^2(r))\sigma - (n - 1) \frac{h''(r)}{h(r)} dr^2 \]

Remark 2.4. Suppose $\text{Ric}_N \leq (n - 2)h'^2(r)\sigma$ and $h''(r) \geq 0$, then $\text{Ric} \leq 0$. This gives a sufficient condition such that (1.2) holds.

2.3. Nonparametric form. Let $\Sigma$ be a smooth hypersurface in $N \times h \mathbb{R}^+$. The parametric form of the inverse mean curvature flow is the function $F_1 : \Sigma \times [0, T) \to N \times h \mathbb{R}^+$ as the solution in (1.1). The nonparametric form of the inverse mean curvature flow is the function $F_2 : \Sigma \times [0, T) \to N \times h \mathbb{R}^+$ satisfying
\[ \left( \frac{\partial}{\partial t} F_2 \right) \perp = \frac{\vec{v}}{H} \]
where $\perp$ is the projection into the normal bundle of $F_2(., t)$. In fact $F_1(x, t) = F_2(g(x, t), t)$ where $g(x, t)$ is a family of diffeomorphism on $\Sigma$. Similar derivations for mean curvature flows are given by Ecker-Huisken [8]. We will use two forms of inverse mean curvature flows interchangeably.

Suppose $\Sigma_t$ is a nonparametric, starshaped mean convex inverse mean curvature flow with a representation $(x, r(x, t))$. Then $r(x, t)$ satisfies
\[ \left( \frac{\partial}{\partial t} r \right) = \frac{1}{H\Theta} \]
In terms of $\varphi(x, t) = \Phi(r(x, t))$, this is equivalent to
\[ \left( \frac{\partial}{\partial t} \varphi \right) = \frac{1}{H h(r) \Theta} = \frac{1}{H \omega} \]
where $\omega$ is the support function $\langle h(r) \partial_r, \vec{v} \rangle$.

3. LONG TIME EXISTENCE

In this section we establish a long time existence of inverse mean curvature flows as follows.

Theorem 3.1. Let $h(r)$ be a positive function on $[0, \infty)$ satisfying
\[ h'(r) > 0, h''(r) \geq 0 \]
for all $r \in (0, \infty)$. Then in the warped product manifold $N \times h \mathbb{R}^+$ the inverse mean curvature flow (1.1) with any starshaped, mean convex initial hypersurface remains starshaped, mean convex and exists for all times.
Similar long time existence results in various rotationally symmetric spaces are obtained by [7], [23], [5] and [26]. Also see [11] and [28] for the case in Euclidean spaces.

3.1. Evolution Equations. Now we record some evolution equations for geometric quantities along the flows in warped product manifolds. A key point is the role of Ricci curvature of ambient Riemannian manifolds in these equations.

**Proposition 3.2.** Let $\Sigma_t$ be an inverse mean curvature flow in a Riemannian manifold $M$. Let $g_{ij}$ and $h_{ij}$ denote the metric and the second fundamental form on $\Sigma_t$ respectively. Then

\begin{align*}
(1) \quad & \partial_t g_{ij} = \frac{\sqrt{H}}{H} h_{ij} \quad \text{and} \quad \partial_t g^{ij} = -\frac{2}{\sqrt{H}} g^{ik} h_{kj} g^{lj}; \\
(2) \quad & \partial_t \bar{v} = \frac{\nabla H}{H} \quad \text{where} \quad \nabla \text{ is the gradient on } \Sigma_t; \\
(3) \quad & \partial_t d\mu = d\mu; \\
(4) \quad & \partial_t H = \frac{\Delta H}{H} - \frac{2(\nabla H)^2}{H} - \frac{1}{\sqrt{H}} (|A|^2 + \bar{\text{Ric}}(\bar{v}, \bar{v})); \\
(5) \quad & \partial_t h_{ij} = \frac{\Delta h_{ij}}{H^2} - \frac{2}{H^2} \nabla_i H \nabla_j H - 2 \frac{\bar{R}_{00ij}}{H} + \left( \frac{|A|^2}{H} + \bar{\text{Ric}} \right) h_{ij} - \frac{1}{H^2} (2 h_{ij}^l \bar{R}_{ljp} + h_{jk}^l \bar{R}_{ikp}) - \frac{1}{H} (\nabla_k \bar{R}_{0ji}^l + \nabla_i \bar{R}_{0kj}) ;
\end{align*}

Here $\bar{R}, \bar{\text{Ric}}$ are the Riemann curvature and Ricci curvature of $M$. $d\mu$ is the volume form of $\Sigma_t$.

**Proof.** The derivation of the first four identities is very classical (see [17]). So we skip the detail here. As for the last one, applying the definition of $h_{ij}$ in (2.5) and $\partial_t = H^{-1} \bar{v}$ one derives that

\begin{align*}
\partial_t h_{ij} &= \langle \nabla_{\partial_t} \bar{v}, \nabla, X_j \rangle + \langle \bar{v}, \nabla \partial_t X_j \rangle; \\
&= \bar{R}(X_i, \partial_t, \bar{v}, X_j) + \langle \bar{v}, \nabla \partial_t X_j \rangle + \langle \bar{v}, \nabla \partial_t \bar{v}, X_j \rangle; \\
(3.2) \quad &= -\frac{1}{H} \bar{R}_{00ij} + \frac{1}{H^2} \nabla_i \nabla_j H - \frac{2}{H^3} \nabla_i H \nabla_j H + \frac{1}{H} h_{ik} h_{jl} g^{kl}
\end{align*}

In the last line we have used conclusion (2). Thus conclusion (5) follows from the expression of $\nabla_i \nabla_j H$, referred as the Simons’ identity (see Lemma 2.1 in [17]). \qed

Following [19] for a starshaped hypersurface $\Sigma$ in $N \times \mathbb{R}^+$ we define the support function $\omega$ and the modified speed function $u$ as follows

\begin{align*}
(3.3) \quad \omega &= h(r) (\bar{v}, \partial_r) = h(r) \Theta \quad u = \frac{1}{H \omega} 
\end{align*}

where $\bar{v}$ is the upward normal vector of $\Sigma$.

**Proposition 3.3.** Let $\Sigma_t$ be the mean convex, starshaped inverse mean curvature flow in (1.1). Let $\Delta, \nabla$ be the Laplacian operator and the covariant
derivative on \( \Sigma_t \) respectively. Then \( \omega \) and \( u \) satisfy that

\begin{align}
(3.4) \quad \partial_t \omega &= \frac{\Delta \omega}{H^2} + \frac{|A|^2}{H^2} \omega - \frac{h}{H^2} \text{Ric}(\vec{v}, \partial_r^T); \\
(3.5) \quad \partial_t u &= \frac{\Delta u}{H^2} + \frac{|A|^2}{H^2} \omega + \omega \frac{(1 - \Theta^2)}{H^2 h^2} K_h \\
(3.6) \quad \partial_t u &= \frac{\Delta u}{H^2} - 2u^{-1} |\nabla u|^2 - \frac{2 \nabla_i \nabla_j u}{H^3} - (n - 1) \frac{h''}{h} u^3 \omega^2
\end{align}

where \( \partial_r^T \) is the component of \( \partial_r \) along \( \Sigma_t \) and \( K_h \) is

\begin{align}
(3.7) \quad (n - 2)(hh'' - h'^2) + \text{Ric}_N(v_N, v_N)
\end{align}

Here \( \Sigma_t \) has the representation \((x, r(x, t)), v_N = 0 \) if \( \Theta = 1 \) and \( v_N = \frac{Dr(x, t)}{|Dr(x, t)|} \) if \( \Theta \neq 1 \) where \( Dr \) is the gradient of \( r(x, t) \) on \( N \).

**Remark 3.4.** Notice that \( \sigma(v_N, v_N) = 1 \) if \( \Theta \neq 1 \). Assuming \( \text{Ric}_N \geq (n - 2) \rho \sigma \) (3.5) becomes

\begin{align}
(3.8) \quad \partial_t \omega &\geq \frac{\Delta \omega}{H^2} + \frac{|A|^2}{H^2} \omega + \omega \frac{(n - 2)}{H^2 h^2} (1 - \Theta^2)(hh'' - h'^2 + \rho)
\end{align}

Suppose \( hh'' - h'^2 + \rho \geq 0 \), then

\begin{align}
(3.9) \quad \partial_t \omega &\geq \frac{\Delta \omega}{H^2} + \frac{|A|^2}{H^2} \omega
\end{align}

A similar assumption \( hh'' - h'^2 + \rho > 0 \) appears as the condition (H4) of Theorem 1.1 by Brendle [4] regarding the rigidity of constant mean curvature hypersurfaces in warped product manifolds.

**Proof.** Since \( \nabla_X(h(r)\partial_r) = h'(r)X \) and \( \partial_r \vec{v} = \frac{\nabla H}{H^2} \) from conclusion (2) in Proposition 3.2, we compute

\begin{align}
(3.10) \quad \partial_t \omega &= \frac{1}{H^2}(\nabla H, h(r)\partial_r) + \frac{h'(r)}{H}
\end{align}

Let \( \{e_i\} \) be an orthonormal frame on \( \Sigma_t \). Then \( h_{ik} = \langle \nabla_{e_i} \vec{v}, e_k \rangle \) and

\[
\Delta \omega = \langle \nabla_{e_i} \nabla_{e_i} \vec{v}, h(r)\partial_r \rangle + 2\langle \nabla_{e_i} \vec{v}, \nabla_{e_i} h(r)\partial_r \rangle + \langle \vec{v}, \nabla_{e_i} \nabla_{e_i} (h(r)\partial_r) \rangle = \langle \nabla_{e_i} (h_{ik} e_k), h(r)\partial_r \rangle + 2h'(r)\langle \nabla_{e_i} \vec{v}, e_i \rangle + \langle \vec{v}, \nabla_{e_i} (h'(r)e_i) \rangle = h_{ik,i} (e_k, h(r)\partial_r) - |A|^2 \omega + h'(r)H;
\]

From the Codazzi identity, we have \( h_{ik,i} = h_{ii,k} + \bar{R}_{0iki} \). Notice that \( \partial_r^T = \langle \partial_r, e_k \rangle e_k \). Putting those facts together, we obtain

\begin{align}
(3.11) \quad \Delta \omega &= \langle \nabla H, h(r)\partial_r \rangle + h(r)\text{Ric}(\vec{v}, \partial_r^T) - |A|^2 \omega + h'(r)H
\end{align}

(3.4) follows from the combination of (3.10) and (3.11).

Next we compute \( \text{Ric}((\vec{v}, \partial_r^T)) \). From (2.3), we have the decomposition

\begin{align}
(3.12) \quad \vec{v} &= \Theta \partial_r - \Theta \frac{Dr}{h^2(r)} \quad \partial_r^T = (1 - \Theta^2)\partial_r + \Theta^2 \frac{Dr}{h^2(r)}
\end{align}
With these expressions and Proposition 2.3 we obtain

\[
\text{Ric}(\vec{v}, \partial_r^T) = \Theta(1 - \Theta^2)\text{Ric}(\partial_r, \partial_r) - \frac{\Theta^3}{h^4}\text{Ric}(Dr, Dr)
\]

\[
= -(n-1)\Theta(1 - \Theta^2) \frac{h''}{h} - \Theta(1 - \Theta^2) \frac{1}{h^2}\text{Ric}_N(v_N, v_N)
\]

\[
+ \frac{\Theta(1 - \Theta^2)}{h^2}(h(r)h''(r) + (n-2)h'(r)^2)
\]

\[
= -\frac{\Theta(1 - \Theta^2)}{h^2}((n-2)(hh'' - h^2) + \text{Ric}_N(v_N, v_N))
\]

(3.13)

where \(v_N = 0\) if \(\Theta = 1\) and \(v_N = \frac{Dr(x,t)}{|Dr(x,t)|}\) if \(\Theta \neq 1\). Putting these facts together, one establishes (3.5).

According to (3.4) and conclusion (4) in Proposition 3.2 we compute \(\partial_t u\) as follows.

\[
\partial_t u = -u^2(\partial_t H\omega + \partial_t \omega H);
\]

\[
= -\frac{u^2}{H^2}(\omega \Delta H + H \Delta \omega) + 2u^2\frac{\nabla H^2}{H^3}\omega
\]

\[
+ \frac{u^2}{H}(\text{Ric}(\vec{v}, \vec{v})\omega + h(r)\text{Ric}(\vec{v}, \partial_r^T))
\]

(3.14)

Proposition 2.3 implies that \(\text{Ric}(\partial_r, X) = 0\) for any \(X\) satisfying \(\langle X, \partial_r \rangle = 0\). By (3.12), \(\partial_r = \Theta \vec{v} + \partial_r^T\). Thus by Proposition 2.3 we have

\[
\text{Ric}(\vec{v}, \vec{v})\omega + h(r)\text{Ric}(\vec{v}, \partial_r^T) = h(r)\text{Ric}(\vec{v}, \partial_r);
\]

\[
= h(r)\Theta\text{Ric}(\partial_r, \partial_r);
\]

(3.15)

On the other hand,

\[
\Delta H\omega + H\Delta \omega = \Delta(\frac{1}{u}) - 2\nabla_i H\nabla_i \omega
\]

\[
= -\frac{\Delta u}{u^2} + 2\frac{|
abla u|^2}{u^3} + 2\frac{\nabla H^2}{H} \omega + \frac{2}{Hu^2}\nabla_i u \nabla_i H
\]

(3.16)

Plugging (3.15) and (3.16) into (3.14), one arrives (3.6). \(\square\)

3.2. Starshapedness. The aim of this subsection is to show that

**Theorem 3.5.** Assume \(h(r)\) satisfies \(h'(r) > 0\) and \(h''(r) \geq 0\) for all \(r > 0\).

Suppose \(\Sigma_t\) is an inverse mean curvature flow in (1.1) with a starshaped, mean convex initial hypersurface on \([0, T)\) for some finite \(T > 0\), then \(\Sigma_t\) remains mean convex and starshaped. Moreover

\[
0 < C_1 \leq H(p) \leq C_0
\]

for \(p \in \Sigma_t\). Here \(C_0\) and \(C_1\) are two positive constants depending on \(T\) and initial data.
The first step is to prove that the warping factor $h(r)$ has an exponential growth along the flow.

**Lemma 3.6.** Suppose $\Sigma$ is a mean convex, starshaped hypersurface with

\begin{equation}
R_2 \geq \omega \geq R_1 > 0;
\end{equation}

and $h(r)$ satisfies $h'(r) > 0, h''(r) \geq 0$. Let $\Sigma_t$ be a starshaped, mean convex inverse mean curvature flow in (1.1) on $[0, T)$ in $N \times h^R$. Then

1. $h(r)$ satisfies

\begin{equation}
e^{\frac{t}{n-1}} R_2 \geq h(r) \geq e^{\frac{t}{n-1}} R_1
\end{equation}

for $(x, r) \in \Sigma_t$.

2. for any $t \in [0, T)$ $\Sigma_t$ lies in the region $N \times h^{-1}([R_1, R_2 e^{\frac{t}{n-1}}])$.

**Remark 3.7.** Our proof follows from Gehardt’s idea [10].

**Proof.** Conclusion (2) is obvious from conclusion (1). It is sufficient to prove (1).

Since $h(r)$ is strictly increasing, $\Theta = 1$ at the points where $h(r)$ achieves its local extreme on $\Sigma$. By (3.17) on $\Sigma$ we have

$$R_1 \leq h(r) \leq R_2$$

Suppose $\Sigma_t$ have graphical representations $(x, r(x,t))$. Define an auxiliary function $\tilde{h}$ on the cylinder $Q_T = N \times [0, T)$ as follows

$$\tilde{h}(x,t) = (\ln h(r(x,t))) - \frac{t}{n-1} - \ln R_2) e^{-\alpha t}$$

where $\alpha$ is a fixed positive constant. Then by (2.11) $r(x,t)$ satisfies

\begin{equation}
\frac{\partial r}{\partial t} = \frac{1}{H\Theta} \quad (\Leftrightarrow \frac{\partial \varphi}{\partial t} = \frac{1}{H h(\sqrt{1 + |D\varphi|^2})})
\end{equation}

and thus

\begin{equation}
\frac{\partial \tilde{h}}{\partial t} = -\alpha \tilde{h} + (\frac{h'(r)}{H h(r) \Theta} - \frac{1}{n-1}) e^{-\alpha t};
\end{equation}

Let $Q_{\tilde{T}}$ be the cylinder $N \times [0, \tilde{T})$ for some $\tilde{T} \in (0, T]$. Suppose the supremum of $\tilde{h}$ on $Q_{\tilde{T}}$ is obtained at a point $(x_0, t_0)$ for some $t_0 \leq \tilde{T}$. Then at this point we have

\begin{equation}
D\varphi = 0, \quad D^2 \varphi \leq 0, \quad \Theta = 1, \quad \frac{\partial \tilde{h}}{\partial t} \geq 0;
\end{equation}

Therefore at $(x_0, t_0)$ Proposition 2.1 implies that

$$H h(r) \Theta \geq (n-1)h'(r);$$

Together with (3.20) and (3.21), this yields that

\begin{equation}
0 \geq \alpha \tilde{h} - (\frac{h'(r)}{H h(r) \Theta} - \frac{1}{n-1}) e^{-\alpha t} \geq \alpha \tilde{h}
\end{equation}
Since \(\alpha\) is strictly positive and \(\tilde{h} \leq 0\) at time \(t = 0\), then \(\tilde{h} \leq 0\) on \(Q_{\tilde{T}}\). Due to the arbitrariness of \(\tilde{T}\), then \(\tilde{h} \leq 0\) on \(Q_T\). Hence

\[
h(r) \leq e^{\frac{t}{n-1}} R_2
\]

This is the second inequality in (3.18). The first one in (3.18) follows from a similar derivation just by considering

\[
\tilde{h}_1(r) = (\ln h(r) - \frac{t}{n-1} - \ln R_1)e^{-\alpha t}
\]

where \(\alpha\) is a positive constant. We skip the details. \(\square\)

Now we are ready to show Theorem 3.5. Although our estimate about \(H\) is very rough, it is sufficient for our purpose.

**Proof.** Let \(\Sigma_t\) be the inverse mean curvature flow of \(\Sigma\) in (1.1) existing smoothly on a time interval \([0, T)\) for \(T < \infty\).

Suppose the conclusion in Theorem 3.5 is not correct. By continuity, we can assume \(\Sigma_t\) remains mean convex and starshaped on the maximal interval \([0, T')\) where \(T' < T\).

Assume \(t \in [0, T')\) for a while. By Lemma 3.6, \(\Sigma_t\) lies in a compact domain \(N \times h^{-1}([R_1, R_2 e^{-\frac{t}{n-1}}])\) only depending on \(T\). Thus

(3.23)

\[
|Ric(v, v)| \leq C
\]

for some constant \(C\) depending on \(T\).

Since \(|A|^2 \geq \frac{|H|^2}{n-1}\) and \(H > 0\), equation (4) in Proposition 3.2 becomes

(3.24)

\[
\partial_t H^2 \leq \frac{\Delta H^2}{H^2} - 4 \frac{\nabla H^2}{H^2} - 2 \frac{H^2}{n-1} + C;
\]

By the comparison principle, we have \(H(p) \leq C_0\) for \(C_0\) only depending on \(T\) and the initial hypersurface \(\Sigma\). Since \(h''(r) \geq 0\), (3.6) is simplified as

\[
\partial_t u \leq \frac{\Delta H}{H} - 2 \frac{\nabla_i H \nabla_i u}{H^3};
\]

for all \(t \in [0, T')\). Applying the comparison principle and (3.18) we have

\[
HR_2e^{\frac{t}{n-1}} \geq Hh(r) \geq Hh(r)\Theta = u^{-1} \geq \min_{\Sigma} u^{-1};
\]

for all \(t \in [0, T')\). As a result \(H \geq C_1 > 0\) where \(C_1 = R_2^{-1} e^{-\frac{t}{n-1}} \min_{\Sigma} u^{-1}\).

From conclusion (1) of Lemma 3.6 and \(H \geq C_1 > 0\), we observe that

\[
\frac{K_h}{H^2 h^2} (1 - \Theta^2) \geq C_3
\]

where \(K_h\) is from (3.5) and \(C_3\) is a constant depending on \(T\) and \(\Sigma\). Thus (3.5) becomes

(3.25)

\[
\partial_t \omega \geq \frac{\Delta \omega}{H^2} + \frac{1}{n-1} \omega - C_3 \omega;
\]
for all \( t \in [0, T') \).

In summary \( \omega \geq e^{-C_tT}e^{-\frac{t^2}{2}}R_1 \) and \( C_0 \geq H \geq C_1 > 0 \) for all \( t \in [0, T') \). Since \( T' < T \), by continuity \( \Sigma_t \) is still starshaped and mean convex at time \( T' \). The maximality of \( T' \) gives a contradiction. Thus \( T' = T \) and the starshapedness and mean convex property are preserved along \( \Sigma_t \). \( \square \)

3.3. The proof of Theorem 3.1. It is well-known that for inverse mean curvature flows, the positive lower bound of mean curvature implies its regularity. This fact is firstly shown by Smoczyk [27] in dimension 2, and pointed by Huisken-Ilmanen [19] for general Riemannian manifolds. For the convenience of readers and completeness, we give its proof here.

Lemma 3.8. Let \( M \) be a complete Riemannian manifold. Suppose \( \Sigma_t \) is a smooth mean convex solution in (1.1) on \([0, T)\) such that \( 0 < H_0 \leq H \leq H_1 \) and the set \( \{\Sigma_t\}_{t \in [0, T)} \) lies in a compact set \( \Omega_T \). Then the second fundamental form \( A \) of \( \Sigma_t \) satisfies the estimate

\[
|A| \leq C(H_0, H_1, \Omega_T)
\]

for any \( t \in [0, T) \) where \( C(H_0, H_1, \Omega_T) \) is a positive constant depending on \( H_0, H_1 \) and \( \Omega_T \).

Proof. As in [19] and [23] we define a new tensor

\[
\eta^i_j = H h^i_j
\]

By Proposition 3.2 tedious computations yield the evolution equation of \( \eta^i_j \) as follows.

\[
\partial_t \eta^i_j = \frac{\Delta \eta^i_j}{H^2} - 2 \frac{\langle \nabla H, \nabla \eta^i_j \rangle}{H^3} - 2 \frac{\nabla^i H \nabla_j H}{H^2} - 2 \overline{\partial_0 \partial_j} - 2 \frac{\eta^i_k \eta^j_k}{H^2} - \frac{1}{H} (\overline{\partial^i_l} \eta^j_l h^j_l + 2 \overline{\partial^i_l} j^j_l h^j_l + \overline{\partial^i_l} \bar{R}^j_l j^j_l + \overline{\partial^i_l} \bar{R}^j_l j^j_l)
\]

Since \( \Omega_T \) is a compact set and \( 0 < H_0 \leq H \leq H_1 < \infty \), for any \( t \in [0, T) \) \( \eta^i_j \) satisfies that

\[
\partial_t \eta^i_j \leq \frac{\Delta \eta^i_j}{H^2} - 2 \frac{\langle \nabla H, \nabla \eta^i_j \rangle}{H^3} - 2 \frac{\nabla^i H \nabla_j H}{H^2} - 2 \frac{\eta^i_k \eta^j_k}{H^2} + C(|\eta| + 1)
\]

where \( C \) is a positive constant depending on \( \Omega_T, H_0 \) and \( H_1 \).

Let \( k_n \) be the maximum eigenvalues of \( (\eta^i_j) \). Since \( \Sigma_t \) are mean convex and \( k_n \) is positive, \( |\eta| \leq n^2 k_n \). Notice that \( (H^{-2} \nabla^i H \nabla_j H) \) is a nonnegative definite matrix. According to Hamilton’s maximal principle about tensor fields [14], \( k_n \) is bounded above by \( \phi \) as the positive solution of

\[
\frac{d\phi}{dt} = -2 \frac{\phi^2}{H^2} + C_3 (\phi + 1);
\]

where \( C_3 \) is some different constant depending on \( \Omega_T, H_0, H_1 \) and \( n \). If \( \phi \geq \max\{1, 2C_3 H_1^2\} \), then \( \phi \) satisfies that \( \partial_t \phi \leq -\frac{\phi^2}{H^2} \). In this case, \( \phi \leq \frac{H_1}{4} \).
Therefore
\[(3.27) \quad k_n \leq \phi \leq \max\{1, 2C_3H_1^2, \frac{H_1^2}{t}\};\]

Let $\lambda_n$ be the maximal eigenvalue of $(h_j^i)$. Again because that $\Sigma_t$ are mean convex, $|A| \leq n^2\lambda_n$. Lemma 3.8 follows since $\eta_j^i = Hh_j^i$, $\lambda_n \leq \frac{1}{H_0} \max\{1, 2C_3H_1^2, \frac{H_1^2}{t}\}$. \hfill \Box

Now we conclude Theorem 3.1 as follows.

**Proof.** Let $\Sigma$ be a starshaped and mean convex hypersurface in $N \times_{h} \mathbb{R}^+$. Suppose its inverse mean curvature flow $\Sigma_t$ exists smoothly on the maximal interval $[0, T)$ where $T$ is finite. All constants and compact sets mentioned below shall depend on $T$ and the initial surface $\Sigma$. From conclusion (2) of Lemma 3.6, the flow $\Sigma_t$ lies in a compact set for $t \in [0, T)$. Theorem 3.5 says that $\Sigma_t$ remains mean convex and starshaped. Moreover $C_0 \geq H \geq C_1 > 0$ for two constants $C_1$ and $C_0$. Thus by Lemma 3.8 all principle curvatures are bounded above by a positive finite constant. As a result (2.11) (also see (2.12), (2.13)) is a uniformly parabolic equation for $t \in [0, T)$. From the regularity theory of Krylov [21], its solution can be extended smoothly over time $T$. This is a contradiction to the definition of $T$. Thus we conclude that the inverse mean curvature flow $\Sigma_t$ exists for all times. Moreover it preserves starshaped and mean convex properties by Theorem 3.5. The proof is complete. \hfill \Box

4. High regularity when $h''(r) > 0$

In this section we study the high regularity property of inverse mean curvature flows in warped product manifolds which is first proposed in [19]. It is a lower bound of mean curvature along starshaped inverse mean curvature flows as follows.

**Theorem 4.1.** Suppose $h(r)$ is a smooth positive function satisfying
\[(4.1) \quad h'(r) > 0 \quad h''(r) > 0 \quad hh'' - (h')^2 + \rho \geq 0\]

for all $r > 0$ where $\text{Ric}_N \geq (n-2)\rho \sigma$. Let $\Sigma$ be a mean convex and starshaped smooth hypersurface in $N \times_{h} \mathbb{R}^+$ with the property
\[(4.2) \quad 0 < R_1 \leq \omega = \langle \vec{v}, h(r)\partial_r \rangle \leq R_2;\]

and let $\Sigma_t$ be its inverse mean curvature flow. For $t \in [0, T)$ it holds that
\[(4.3) \quad H \geq e^{-\frac{1}{n-1}} \sqrt{h_0(n-1)}R_1R_2^{-1} \min\{\frac{1}{\sqrt{2}}, 1\}\]

for all $t \in (0, T]$. Here $h_0 > 0$ is the infimum of $\frac{h''(r)}{h(r)}$ on the interval $h^{-1}[R_1, R_2e^{\frac{T}{n-1}}]$. 

Remark 4.2. Similar estimates are given in [19] and [23]. The former depends heavily on the Michael-Simon-Sobolev inequality (see [24]). The latter takes advantage of the positivity of $h''(r)$ in Schwarzchild-Kottler spaces. Inspired by Lemma 4.1 in [23] we observe that the following two estimates

$$R_2 e^{n-t} \geq \omega \geq e^{n-t} R_1$$

$$\partial_t u \leq \frac{\Delta u}{H^2} - 2 u^{-1} |\nabla u|^2 \frac{H^2}{H^2} - 2 \frac{\nabla_i H \nabla_i u}{H^3} - (n-1) h_0 u^3 \omega^2$$

are key ingredients to establish (4.3) (see (4.4) and (4.5)). Both of them are guaranteed by condition (4.1).

Proof. From Theorem 3.1 the inverse mean curvature flow $\Sigma_t$ exists for all $t$, remains starshaped and mean convex. By (4.2), $h(r) \leq R_2$ for any point $p = (x, r) \in \Sigma$. By condition (4.1) the conclusion (1) of Lemma 3.6 implies that $h(r) \leq e^{n-t} R_2$ on $\Sigma_t$. Thus

$$\omega \leq h(r) \leq e^{n-t} R_2$$

on each $\Sigma_t$. On the other hand by equation (3.8) and condition (4.1), we have

$$\partial_t \omega \geq \frac{\Delta \omega}{H^2} + \frac{\omega}{n-1}$$

Here we use $(n-1)|A|^2 \geq H^2$ and $hh'' - h^2 + \rho \geq 0$ in (4.1). By (4.2) the comparison principle implies $\omega \geq e^{n-t} R_2$. In summary we have the estimate

(4.4) $$R_2 e^{n-t} \geq \omega \geq e^{n-t} R_1$$

on any $\Sigma_t$.

From now on we restrict ourself to the time interval $(0, T]$. By the definition of $h_0$ equation (3.6) becomes

(4.5) $$\partial_t u \leq \frac{\Delta u}{H^2} - 2 u^{-1} |\nabla u|^2 \frac{H^2}{H^2} - 2 \frac{\nabla_i H \nabla_i u}{H^3} - (n-1) h_0 u^3 \omega^2$$

for all $t \in (0, T]$. Similarly as in ([19],[23]), we define

$$v = (t - t_0)^{1/2} u;$$

where $t_0 \in (0, T)$ is arbitrary but fixed. Then $v(x, t_0) = 0$ on $\Sigma_{t_0}$. Notice that $\frac{\Delta v}{H^2} - 2 \frac{\nabla_i H \nabla_i v}{H^3} = div\left(\frac{\nabla v}{H^2}\right)$. By equation (4.5) the following estimate holds for $v$

$$\partial_t v \leq \frac{1}{2} (t - t_0)^{-1} v + div\left(\frac{\nabla v}{H^2}\right) - 2 \frac{|\nabla v|^2}{H^2} v^{-1}$$

$$- (n-1) h_0 (t - t_0)^{-1} v^3 \omega^2.$$
We define \( v_k = \max\{v - k, 0\} \) for all \( k \geq 0 \) and \( A(k) := \{ x \in \Sigma_t : v(x, t) \geq k \} \). One derives that
\[
\partial_t \int_{\Sigma_t} v_k^2 d\mu_t \leq (t - t_0)^{-1} \int_{A(k)} vv_k + \int_{A(k)} v_k^2 d\mu - 2 \int_{A(k)} \frac{|\nabla v|^2}{H^2} d\mu_t

- 4 \int_{A(k)} v_k^2 \frac{|\nabla v|^2}{H^2} d\mu_t - 2(n - 1) h_0 (t - t_0)^{-1} \int_{A(k)} v_k^2 \omega^2 d\mu_t
\]
\[
\leq (t - t_0)^{-1} \int_{A(k)} vv_k + \int_{A(k)} v_k^2 d\mu - 2(n - 1) h_0 (t - t_0)^{-1} \int_{A(k)} v_k^2 \omega^2 d\mu_t
\]
Since \( \omega \geq e^{\frac{r}{n-1}} R_1 \) in (4.4) and \( v \geq k \) on \( A(k) \), we have the inequality
\[
\partial_t \int_{\Sigma_t} v_k^2 d\mu_t \leq (t - t_0)^{-1} \int_{A(k)} vv_k + \int_{A(k)} v_k^2 d\mu

- 2(n - 1) h_0 (t - t_0)^{-1} k^2 R_1^2 e^{\frac{2n}{n-1}} \int_{A(k)} vv_k d\mu_t
\]
To make the right hand side above nonpositive, we can choose
\[
k^2(t) = \frac{1}{(n - 1) h_0} R_1^{-2} e^{-\frac{2n}{n-1}} \max\{t - t_0, 1\}
\]
Choosing \( k = k(t^*) \) we have
\[
\partial_t \int_{\Sigma_t} v_k^2 d\mu_t \leq 0
\]
for \( t \in [t_0, t^*] \). By definition \( v_k(x, t_0) \equiv 0 \), the above inequality implies that \( v_k(x, t) \equiv 0 \) for all \( t \in [t_0, t^*] \). Therefore \( v(x, t) \leq k(t^*) \) on \( \Sigma_t \) for all \( t \in [t_0, t^*] \).

We divide \( t^* \in (0, T) \) into two cases. The first case is \( t^* \leq 2 \). We choose \( t_0 = \frac{t^*}{2} \leq 1 \). Hence \( k^2(t^*) \) takes the form
\[
k^2(t^*) = \frac{1}{(n - 1) h_0} R_1^{-2} e^{-\frac{2n}{n-1}} \leq \frac{1}{(n - 1) h_0} R_1^{-2} e^{-\frac{2n}{n-1}} e^{\frac{2}{n-1}}
\]
since \( 2t_0 \geq 2t^* - 2 \) indicates that \( \max\{t^* - t_0, 1\} = 1 \). By the definition of \( v(x, t) \), one conclude that
\[
\sup_{\Sigma_{t^*}} u \leq t_0^{-\frac{1}{2}} k(t^*) \leq \sqrt{2}(t^*)^{-\frac{1}{2}} h_0^{-\frac{1}{2}} (n - 1)^{-\frac{1}{2}} R_1^{-1} e^{-\frac{t^*}{n-1}} e^{\frac{1}{n-1}}
\]
The second case is \( t^* \geq 2 \). By choosing \( t_0 = t^* - 1 \) then \( k_0 \) satisfies that
\[
k^2(t^*) = \frac{1}{(n - 1) h_0} R_1^{-2} e^{-2t_0} = \frac{1}{(n - 1) h_0} R_1^{-2} e^{-\frac{2t^*}{n-1}} e^{\frac{2}{n-1}}
\]
Similarly as in the first case, one has that
\[
\sup_{\Sigma_{t^*}} u \leq k(t^*_*) = h_0^{-1} (n - 1)^{-\frac{1}{2}} R_1^{-1} e^{-\frac{t^*}{n-1}} e^{\frac{1}{n-1}}
\]
In summary for any \( t \in (0, T] \) we have

\[
\sup_{\Sigma_t} u \leq \max \{ \sqrt{2 t^{-\frac{1}{2}}} h_0 \frac{1}{2} (n-1)^{\frac{1}{2}} R_1^{-1} e^{-\frac{t}{n-1}} e^{\frac{1}{p-1}} \}
\]

The estimate in (4.3) follows from \( H = \frac{1}{\omega} \) and \( \omega \leq R_2 e^{\frac{t}{n-1}} \) in (4.4). The proof is complete. \( \square \)

Now we give an application of Theorem 4.1. Our method follows closely from Theorem 2.5 in \([19]\) and Theorem 1.2 in \([23]\).

Let \( \Sigma \) be a \( C^1 \) and oriented hypersurface in a Riemannian manifold. A measurable function \( H \) is called as the weak mean curvature of \( \Sigma \) if it holds that

\[
\int_{\Sigma} \text{div} X d\mu = \int_{\Sigma} H \langle X, \vec{v} \rangle d\mu
\]

for any smooth vector filed \( X \) with compact support. Here \( \vec{v} \) is the normal vector of \( \Sigma \).

**Theorem 4.3.** Let \( h(r) \) and \( N \) be given by Theorem 4.1. Let \( \Sigma_0 : \Sigma \to N \times h \mathbb{R}^+ \) be a starshaped hypersurface of class \( C^1 \) with measurable, bounded, nonnegative weak mean curvature \( H \geq 0 \) and

\[
0 < R_1 \leq \omega \leq R_2;
\]

for two positive constants \( R_1, R_2 \). Then there is a starshaped, mean convex inverse mean curvature flow \( \Sigma_t \) in \( N \times h \mathbb{R}^+ \) on \([0, \infty)\) such that \( \Sigma_t \) converges to \( \Sigma_0 \) uniformly in \( C^0 \) as \( t \to 0 \).

First we need an approximation lemma for starshaped hypersurfaces in general warped product manifolds. It follows from the spirit in (Lemma 2.6, \([19]\)) and (Lemma 4.2, \([23]\)). We record it here for completeness.

**Lemma 4.4.** Let \( N \times h \mathbb{R}^+ \) be a warped product manifold with \( h'(r) > 0 \) for all \( r > 0 \). Let \( \Sigma_0 \) be a closed \( C^1 \) starshaped hypersurface in \( M \) with measurable, bounded weak nonnegative mean curvature.

Then \( \Sigma_0 \) is of class \( C^{1, \beta} \cap W^{2, p} \) for all \( 0 < \beta < 1, 1 \leq p < \infty \) and can be approximately locally uniformly in \( C^{1, \beta} \cap W^{2, p} \) by a family of smooth hypersurfaces \( \Sigma_\varepsilon \) satisfying \( H > 0 \) where \( \varepsilon \in (0, \varepsilon_0) \) for some constant \( \varepsilon_0 \).

**Proof.** Since \( \Sigma_0 \) is \( C^1 \) and the weak mean curvature \( H \) is uniformly bounded, standard regularity results of Allard imply that \( \Sigma_0 \) is of class \( C^{1, \beta} \cap W^{2, p} \) for all \( \beta, 1 \leq p < \infty \). By mollification, we can pick up a sequence of starshaped surfaces \( \Sigma_i = (x, r_i(x)) \) in \( N \times h \mathbb{R}^+ \) converging locally uniformly to \( \Sigma_0 \) in \( C^{1, \beta} \cap W^{2, p} \). Now we consider mean curvature flows \( F_i(p, \varepsilon) \) starting from \( \hat{\Sigma}_i^n \) in \( N \times h \mathbb{R}^+ \) which satisfy the mean curvature flow equation

\[
\frac{\partial F_i(p, \varepsilon)}{\partial \varepsilon} = \hat{H}
\]

(4.10)

\[
F(p, 0) = p \quad p \in \hat{\Sigma}_i^n
\]

(4.11)
In a neighborhood of $\Sigma_0$, $N \times (a, b)$ can be written as a product manifold $\Sigma_0 \times (a, b)$ if we take the Gaussian adapted coordinate. Each $\Sigma_i^n$ remains a graph over $\Sigma_0$. By the interior estimate in [9] mean curvature flows $F_t(p, \varepsilon)$ exist smoothly on a uniform time interval $[0, \varepsilon_0)$ independent of $i$.

At this point, the remainder of the proof in ([19], Lemma 2.6) can be carried here directly to get the conclusion. The reason is that the evolution equations of $H$ and $|A|^2$ has lower order additional curvature terms comparing to Euclidean spaces [17]. Also see the proof of (Lemma 4.2, [23]).

We conclude Theorem 4.3 as follows.

**Proof.** Let $\Sigma_0$ be a $C^1$ starshaped hypersurface with weakly nonnegative bounded mean curvature. By Lemma 4.4, we can construct a family of starshaped hypersurfaces $\Sigma_\varepsilon$ for $0 < \varepsilon < \varepsilon_0$ approaching $\Sigma_0$ as $\varepsilon \to 0$ locally uniformly in $C^1, B \cap W^{2, p}$ such that

$$0 < R_1 \leq \omega \leq R_2$$

on each $\Sigma_\varepsilon$. Let $\Sigma_{\varepsilon, t}$ denote a family of inverse mean curvature flows with initial data $\Sigma_\varepsilon$ for $t \in [0, 1)$. Moreover, Theorem 4.1 gives the estimate

$$(4.12) \quad H \geq \exp(-\frac{1}{n-1}R_2^{-1}R_1\sqrt{(n-1)h_0 \min(\frac{1}{\sqrt{2}}, 1})$$

Notice that smooth hypersurfaces $\{\Sigma_\varepsilon\}$ have $C^1$ uniform gradients. Let $\varepsilon$ approach to 0. According to (4.12) after choosing a subsequence we obtain a starshaped, mean convex inverse mean curvature flow $\Sigma_t$ for $t \in (0, 1)$ such that $\Sigma_t$ converges to $\Sigma_0$ in $C^0$ as $t \to 0$. From the uniqueness of inverse mean curvature flows with initial data, Theorem 3.1 implies that we can uniquely extend $\Sigma_t$ on $(0, 1)$ into $(0, \infty)$. The proof is complete. □

5. LONG TIME BEHAVIORS WHEN $h'(r)$ IS BOUNDED

In this section, we study long time behaviors of inverse mean curvature flows in warped product manifolds when $h'(r)$ becomes uniformly bounded as $r \to \infty$. We shall seek assumptions such that these flows will have similar asymptotic behaviors as those in Euclidean spaces [10, 28] and Schwarzschild-Kottler spaces [23].

**Theorem 5.1.** Let $(N, \sigma)$ be a closed Riemannian manifold and $C, \rho_0, c, \alpha, \beta, r_0$ are positive constants. Suppose it holds that

$$C \geq h'(r) > 0, \quad C \geq h(r)h''(r) \geq 0 \quad \text{for all } r > 0$$

$$(5.2) \quad \text{Ric}_N \geq \rho_0 \sigma$$

$$Ric(p) \leq \frac{C}{h^{2+\beta}(r)}g \quad \text{where } p = (x, r) \quad \text{for } r \geq r_0 > 0$$
where $\text{Ric}$ and $g$ are the Ricci curvature and the metric of $N \times_h \mathbb{R}^+$ respectively. Given any smooth, mean convex hypersurface $\Sigma$ in $N \times_h \mathbb{R}^+$, its inverse mean curvature flow $\Sigma_t$ exists smoothly for all times, remains mean convex and starshaped. Moreover
\begin{align*}
\text{(a)} & \quad e^{-\frac{1}{n-1}t} g_t \to c^2 \sigma; \\
\text{(b)} & \quad |h^j_i - \frac{k}{t}\delta_{ij}| = \frac{k}{t} O(e^{-\alpha t});
\end{align*}
as $t \to \infty$. Here $g_t$ and $h^i_j$ are the metric and the second fundamental form of $\Sigma_t$ respectively.

Special similar results in rationally symmetric spaces are given in Qi [7] and Li-Wei [23], Gerhardt [10] and Urbas [28]. Remark 5.2 gives a sufficient condition such that $\text{Ric}(p)$ is negative and thus condition (5.3) holds.

**Remark 5.2.** Condition (5.1) implies that $h'(r) > 0$ and $h''(r) \geq 0$ for all $r > 0$. Thus the long time part can be obtained from Theorem 3.1.

Notice that it is impossible to find a positive smooth function $h(r)$ such that
\[
C \geq h'(r) > 0 \quad hh''(r) \geq C_0 > 0
\]
for some positive constant $C_0$. If such function exists, a direct verification shows that $h''(r) \geq \frac{C_0}{r^2}$. This gives a contradiction since $h'(r) \leq C$.

**Remark 5.3.** In the proof of §5.2 two crucial facts are
\[
|D\varphi|^2 = O(e^{-\alpha t}) \quad \text{and} \quad e^{-\frac{i}{n-1}t} h(r) \to c
\]
along $\Sigma_t$ as $t \to \infty$ where $c, \alpha$ are constants.

Conclusion (a) implies the scaling metric $|\Sigma_t|^{-\frac{2}{n-1}} g$ on $\Sigma_t$ converges to the standard metric $\sigma$ on $N$. Here $|\Sigma_t|$ is the area of $\Sigma_t$. Condition (5.1) is necessary to obtain conclusion (a). In [20] Hung-Wang construct an example such that the limiting scaling metric along an inverse mean curvature flow in $\mathbb{H}^3$ is not a round metric on the sphere. In their case $\lim_{r \to \infty} h'(r)$ is infinity.

**5.1. An evolution equation.** Next we investigate the evolution of $|\varphi|^2$ along inverse mean curvature flows. For abbreviation we define
\begin{align*}
\text{(5.4)} & \quad F = Hh\Theta = \Theta^2((n-1)h'(r) - \hat{\sigma}^{ij}\varphi_{ij}) \\
\text{(5.5)} & \quad G_k = (F \varphi^k + \Theta^2 \sigma^{ik}\varphi_{ij} - \Theta^4 \varphi^k \varphi^l \varphi_{ij})
\end{align*}
where $\varphi^i = \sigma^{ik}\varphi_k, \varphi^k = \sigma_{ij} \varphi^i$ and $\hat{\sigma}^{ij} = (\sigma^{ij} - \Theta^2 \varphi^i \varphi^j)$. A direct computation yields that $\frac{\partial F}{\partial \varphi^k} = -2\Theta^2 G_k$. In the sequel $\tilde{\omega}$ denotes $\frac{1}{2}|D\varphi|^2$.

**Proposition 5.4.** Let $\Sigma_t$ be a starshaped, mean convex inverse mean curvature flow in $N \times_h \mathbb{R}^+$ with the representation $(x, r(x, t))$. Then $\tilde{\omega}$ satisfies
\begin{align*}
\frac{\partial \tilde{\omega}}{\partial t} & = \frac{1}{H^2 h^2} \{ \hat{\sigma}^{ij} \tilde{\omega}_{ij} + 2G_i \tilde{\omega}_i - \hat{\sigma}^{ij} \varphi_{ki} \varphi^k_j \\
& \quad - 2 \tilde{\omega}(\text{Ric}_N(v_N, v_N) + (n-1)h(r)h''(r)) \}
\end{align*}
where \( v_N = 0 \) if \( \tilde{\omega} = 0 \) or \( v_N = \frac{Dr(x,t)}{|Dr(x,t)|} \) if \( \tilde{\omega} \neq 0 \). Here \( Ric_N \) is the Ricci curvature of \( N \).

**Remark 5.5.** Notice that the term \( \tilde{\sigma}^{ij} \varphi_{kij}\varphi_j^k \) is always nonnegative.

**Proof.** For \( r(x,t) \) satisfies that \( \frac{\partial r(x,t)}{\partial t} = \frac{1}{H} \) (see (2.11)), \( \frac{\partial \varphi}{\partial t} = \frac{1}{F} \). We differentiate \( \tilde{\omega} \) with respect to \( t \). Then

\[
\frac{\partial \tilde{\omega}}{\partial t} = \varphi^k(\frac{\partial \varphi}{\partial t})_k = -\frac{1}{F^2} \varphi^k(F)_k
\]

(5.7)

\[
= \frac{\Theta^2}{F^2}(\tilde{\sigma}^{ij}\varphi_{ijk}\varphi_j^k + 2G_{ij}\varphi_{ik}\varphi_j^k - (n-1)h(r)h''(r)\varphi_k^k)
\]

(5.8)

We notice that

\[
\tilde{\sigma}^{ij}\varphi_{ij} = \tilde{\sigma}^{ij}\varphi_{kij}\varphi_j^k + \tilde{\sigma}^{ij}\varphi_{ki}\varphi_j^k
\]

(5.9)

where the covariant derivatives are taken with respect to \((N, \sigma)\). In the last line we apply Lemma 2.2. Moreover,

\[
\tilde{\sigma}^{ij}\varphi_{ijk}\varphi_j^k = (\sigma^{ij} - \Theta^2\varphi^i\varphi^j)R_{jkiq}\varphi_q^k \varphi^k;
\]

\[= \sigma^{ij}R_{jkiq}\varphi_q^k \varphi^k - \Theta^2\varphi^i\varphi^j R_{jkiq}\varphi_q^k \varphi^k;
\]

\[= Ric_N(\partial_k, \partial_q)\varphi_q^k \varphi^k
\]

(5.10)

\[= Ric_N(D\varphi, D\varphi)
\]

where the third line is from \( \varphi^i\varphi^j R_{kijq}\varphi_q^k \varphi^k = 0 \). Here \( R \) is the Riemann curvature tensor of \( N \). Notice that \( Ric_N(D\varphi, D\varphi) = 2\tilde{\omega}Ric(v_N, v_N) \) if \( \Theta \neq 1 \). The proposition follows from combining (5.8),(5.10) and (5.9) together. \( \square \)

### 5.2. The proof of Theorem 5.1.

By Theorem 3.1 we can assume that \( \Sigma_t \) is a starshapd, mean convex inverse mean curvature flow in \( N \times [0, \infty) \). All functions are evaluated along this flow. Our proof is divided into three steps.

#### 5.2.1. The first step.

First we show that the mean curvature of \( \Sigma_t \) satisfies

(5.11) \[ C_2 \geq H(p)h(r) \geq C_1 > 0 \]

on \( \Sigma_t \). Here \( C_1, C_2 \) are two positive constants independent of \( t \) and \( p = (x,r) \in \Sigma_t \).

We start with the lower positive bound of \( HH \). By (3.6) and \( h''(r) \geq 0 \) in (5.1), the modified speed function \( u \) has the estimate

\[
\partial_t u \leq \frac{\Delta u}{H^2} - 2u^{-1}\frac{\nabla u^2}{H^2} - 2\frac{\nabla_i H \nabla_i u}{H^3}
\]

(5.12)
From the comparison principle, we have
\begin{equation}
(5.13) \quad \frac{1}{Hh\Theta} \leq \min_{\Sigma} u
\end{equation}
on any $\Sigma_t$. Let $C_1$ denote $\frac{1}{\min_{\Sigma} u}$. Since $\Theta \leq 1$, we have $Hh \geq C_1 > 0$ from (5.13). As for the upper bound of $Hh(r)$, we shall notice that $h(r) = O(e^{\frac{t}{n-1}})$ by the conclusion (1) of Lemma 3.6. Combining (4) in Proposition 3.2 with condition (5.3) we obtain
\begin{equation}
\partial_t H \leq \Delta H - 2|\nabla H|^2 - \frac{|A|^2}{H} + \frac{C}{H} e^{-\frac{(2+\beta)\tau}{n-1}}
\end{equation}
For $|A|^2(n-1) \geq H^2$ and $\Sigma$ are mean convex, it is equivalent that
\begin{equation}
(5.14) \quad \partial_t H^2 \leq -2H \Delta \left(\frac{1}{H}\right) - 2\frac{1}{n-1}|H|^2 + Ce^{-\frac{2+\beta}{n-1}t}
\end{equation}
Applying the comparison principle, we have $e^{\frac{t}{n-1}} H^2 \leq C$ for a uniformly constant $C$. This gives the estimate $Hh \leq C_2$ since $h(r) = O(e^{\frac{t}{n-1}})$ along the flow. We obtain (5.11).

5.2.2. The second step. The second step is to show that
\begin{equation}
(5.15) \quad |D\varphi|^2 = O(e^{-2\alpha t})
\end{equation}
for some positive constant $\alpha$. Since $h'' \geq 0$ and $\tilde{\sigma}_{ij} \varphi_i \varphi_j \geq 0$, $\tilde{\omega} = \frac{1}{2}|D\varphi|^2$ satisfies
\begin{equation}
(5.16) \quad \frac{\partial \tilde{\omega}}{\partial t} \leq \frac{1}{H^2 h^2} (\tilde{\sigma}_{ij} \tilde{\omega}_{ij} + 2G_i \tilde{\omega}_i) - 2\frac{\tilde{\omega}}{H^2 h^2} \text{Ric}_N(v_N, v_N)
\end{equation}
Let $s(t)$ denote the maximum of $\tilde{\omega}$ on $\Sigma_s$ for $s \in [0, t)$. Without loss of generality we assume that $s(t) > 0$. Then there exist $t_0 \leq t$ and $p_0 \in \Sigma_{t_0}$ such that
\[ \tilde{\omega}(p_0) = s(t) \]
Moreover, at this point $v_N \neq 0$ since $\Theta \neq 1$ by $s(t) > 0$. By (5.3), $\text{Ric}_N(v_N, v_N) \geq \rho_0 > 0$. According to (5.16) and $Hh \leq C_2$, the comparison principle implies that
\[ \partial_t s(t) \leq -\frac{\rho_0}{C_2} s(t) \]
Let $\alpha$ be the positive number $\frac{\rho_0}{C_2}$. This implies our claim in (5.15).

5.2.3. The last step. We consider an auxiliary function
\[ \tilde{h}(r) = h(r)e^{-\frac{t}{n-1}} \]
Then $\tilde{h}$ satisfies
\begin{equation}
(5.17) \quad \partial_t \tilde{h} = \frac{h'}{H\Theta} e^{-\frac{t}{n-1}} - \frac{1}{n-1} \tilde{h}(\tilde{F})
\end{equation}
where we use the nonparametric form of inverse mean curvature flows $\frac{\partial}{\partial t} = \frac{1}{H\Theta}$. Moreover let $\tilde{F}$ be the right hand side of (5.17). By the conclusion
(1) of Lemma 3.6, $\tilde{h}$ is uniformly bounded. From (5.11), $Hh$ is uniformly bounded. Thus
$$\frac{h'}{H\Theta}e^{-\frac{t}{n-1}}$$
is uniformly bounded because $|h'| \leq C$ and $\frac{1}{\Theta} = \sqrt{1 + |D\varphi|^2} \leq C$ by (5.15). Therefore $\partial_t \tilde{h}$ is uniformly bounded. On the other hand
$$\tilde{h}_{ij} = (h''h^2 + hh'^2)e^{-\frac{t}{n-1}}\varphi_i\varphi_j + hh'e^{-\frac{t}{n-1}}\varphi_{ij}$$
From the expression of $H$ in (2.8) we derive that
$$\frac{\partial \tilde{F}}{\partial \tilde{h}_{ij}} = \frac{1}{H^2h^2}(\sigma_{ij} - \frac{\varphi_i\varphi_j}{1 + |D\varphi|^2})$$
By (5.11) and (5.15), $(\frac{\partial \tilde{F}}{\partial \tilde{h}_{ij}})$ is strictly definite and bounded from above and below. Therefore (5.17) is a strictly parabolic equation. Considering the gradient of $\tilde{h}$, we see that
$$D\tilde{h} = h'(r)D\varphi h(r)e^{-\frac{t}{n-1}} = O(e^{-\alpha t})$$
decays exponentially fast. Thus $\tilde{h}$ converges to a positive constant $c$ uniformly.

From the regularity theory of Krylov (§5.5 in [21]), higher derivatives of $\tilde{h}$ are uniformly bounded. Applying the interpolation theorem (Lemma 6.1, [11] or page 371, [28]) upon $D^2\tilde{h}$, we obtain that $D^2\tilde{h} = O(e^{-\alpha t})$. In view of (5.18), we have $D^2\varphi = O(e^{-\alpha t})$ by $h''h \leq C$ from condition (5.1) and $he^{-\frac{t}{n-1}} \geq C_1 > 0$ from Lemma 3.6.

From (2.4) one derives that
$$e^{-\frac{t}{n-1}}(gt)_{ij} = e^{-\frac{t}{n-1}}h^2(r)(\varphi_i\varphi_j + \sigma_{ij})$$
converges to $c^2\sigma_{ij}$ since $\tilde{h}$ converges to $c$ and $D\varphi = O(e^{-\alpha t})$.

Due to the expression of $h^2_i$ in (2.7), we have
$$|h^2_i - \frac{h'}{h}G_i| \leq \frac{h'}{h}(O(D^2\varphi) + (\Theta - 1)) \leq \frac{h'}{h}O(e^{-\alpha t})$$
The proof is complete.

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