An Offline and Online Algorithm for All Minimal $k|U|$ Parameter Subsets of a Soft Set Based on Integer Partition

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This work was supported in part by the National Science Foundation of China under Grant 61862055, in part by the Major Research and Development Project of China under Grant 2020YFC1523305, and in part by the Major Research and Development Project of Qinghai under Grant 2019-GX-162.

ABSTRACT This article investigates $k|U|$ parameter subsets of a soft set matrix whose column sums are integral multiples of $|U|$ (i.e., the number of objects in the soft set domain $U$). This kind of parameter subset represents an important data structure. Particularly, as a necessary condition, it has been shown to be useful in the parameter reduction problems of soft sets. This article focuses on the minimal $k|U|$ parameter subsets, whose any proper subset cannot be a $k|U|$ parameter subset. An offline and online algorithm for minimal $k|U|$ parameter subsets is proposed. Its basic function is based on integer partition in an offline way. When soft set data come online, the algorithm only needs to filter the factorization results according to the related constraints within the input soft set. We also bring in combinatorial formulas for computing the number of $k|U|$ parameter subsets and the approximate number of minimal $k|U|$ parameter subsets. As an application of $k|U|$ parameter subsets, the method of integer partition is also extended for normal parameter reduction problems of soft sets. The experimental results show that the proposed method does result in better performance.

INDEX TERMS $k|U|$ parameter subsets, minimal $k|U|$ parameter subsets, soft set, integer partition, normal parameter reduction.

I. INTRODUCTION

In 1999, Molodtsov introduced the theory of the soft set [1], which is a novel mathematical tool for dealing with uncertainties and vagueness. Many works have been conducted by researchers on soft set theory and its potential applications. Soft set theory has been investigated in combination with algebraic structure [2]–[4], topological structure [5]–[7] and partial order structure [7]–[9], where the abovementioned algebraic structure also includes logic algebraic structure [10]–[12]. Soft set theory is also combined and compared with other mathematical theories designed for modeling various types of vague concepts, such as fuzzy sets [13]–[18], rough sets [19]–[22] and probability theory [24].

There is also another important branch of soft set research. Quantitative research work has been performed with respect to the data of the soft set itself. Reference [25] studied a kind of soft set with specific measurement structure, i.e., the bijection soft set. In a bijection soft set, every object belongs to a unique parameter approximation. Reference [26] applied it to the reduction of large-scale high-dimensional information systems. In [27], the parameter mapping subsets are granted the semantics of concept extension. Combining with the three-way decision theories, the structural econometric research of POS, NEG and BND (i.e., positive domain, negative domain and boundary domain) of 0-1 information distribution for a soft set is given. The three-way decision ideas expand the connotation of the soft set and provide new ideas and directions for the study of soft set reduction [27]–[29]. In [30] and [31], the 0-1 or [0,1] interval-valued
modeling of the soft set is understood as the valuation in binary or fuzzy logic, respectively. And the approximate reasoning mode in the soft set framework is studied based on quantitative propositional logic [32].

The parameter reduction problems of soft sets actually deal with a kind of structure for the parameter domain. Different kinds of parameter reduction problems [33]–[38] correspond with different criterions. Most of them are required to maintain the same decision making results after a parameter reduction operation. For example, a normal parameter reduction is a parameter subset whose sums of rows are all equal. This means that if they are deleted from the parameter domain, the same choice values will be lost, the same rank of the objects will be obtained. Efforts have been devoted towards issues concerning parameter reduction of soft sets or fuzzy soft sets [34], [35], [37], [38].

The $k|U|$ parameter subsets of soft sets, whose sum of column values is an integral multiple of the number of rows (i.e., number of objects), is a kind of data structure of the soft set matrix and plays an important role in solving the normal parameter reduction problems. This kind of parameter subsets was first presented and proven as a necessary condition for normal parameter reduction by [39], in another word, a $k|U|$ parameter subset is a candidate for a normal parameter reduction of the soft set. In [39] a normal parameter reduction method of soft sets is given by using this property. Since the sum of each column (i.e., the number of elements in the corresponding parameter approximation of the soft set) can be calculated in advance, unnecessary repetition can be avoided from this point of view.

In this article we focus on the minimal $k|U|$ parameter subsets of soft sets. In [40], a hierarchical algorithm for computing all minimal $k|U|$ parameter subsets is studied, and then the normal parameter reduction problem of the soft set can be solved by testing the disjoint combinations of these minimal $k|U|$ parameter subsets. However, the hierarchical algorithm involves a high computing complexity. So we need to develop a much more efficient method for computing all minimal $k|U|$ parameter subsets. This will be beneficial for the knowledge mining and parameter reduction problems of soft sets.

Our idea is as follows. Since the column sum of each parameter (i.e., the number of 1 values) does not consider the specific distribution of the 0 or 1 values, it is possible to classify these parameters into different classes just according to their corresponding column sums. Hence it becomes a question of the integer partition problem. Furthermore, we can conduct integer factorization offline. For a series of soft sets with the same size, we only need to filter the factorization results according to the related constraints induced by the soft set data itself. This is the motivation of our paper. Since the algorithm proposed in [39] didn’t give an explicit method for computing all $k|U|$ parameter subsets. That’s one possible application of our method. We also intends to generate the integer partition method to the normal parameter reduction problems of soft sets.

The remainder of this article is organized as follows. Section 2 introduces basic concepts such as the soft set, $k|U|$ parameter subsets and the main problems to be investigated with respect to $k|U|$ parameter subsets. The theoretical foundation of this article will be provided in section 3. In section 4, four algorithms including the offline and online algorithm for minimal $k|U|$ parameter subsets will be given. Section 5 will list and analyze the experimental results for the algorithms proposed in section 4. Finally, we will reach a conclusion of this article, indicate the novelty and potential weakness of our methods, and offer our outlook for potential future work.

II. PRELIMINARIES

In this article, suppose that $U = \{u_1, u_2, \cdots, u_n\}$ is a finite set of objects, and $E$ is a set of parameters. For example, the attributes in information systems can be taken as parameters. $\wp(U)$ means the powerset of $U$, and $|A|$ means the cardinality of set $A$. By [1] and [41], we have basic concepts about soft sets shown in Definitions 2.1 and 2.2.

Definition 2.1 (Soft Set): A soft set on $U$ is a pair $S = (F, A)$, where

(i) $A$ is a subset of $E$;
(ii) $F : A \rightarrow \wp(U), \forall e \in A, F(e)$ means the subset of $U$ corresponding with parameter $e$. We also use $F(u, e) = 1$ ($F(u, e) = 0$) to indicate that $u$ is (not) an element of $F(e)$.

Definition 2.2 (Support set of Parameters for Objects): Let $S = (F, A)$ be a soft set over $U$. Define the support set of parameters for $u$ as the set $\{ e \in A | F(u, e) = 1 \}$, denoted by $supp(u)$.

Definition 2.3 (Choice Value Function): Let $S = (F, A)$ be a soft set over $U$. The function $\sigma_S : U \rightarrow \mathbb{N}$ defined by $\sigma_S(u) = |supp(u)| = \sum_{e \in A} F(u, e)$ is called the choice value function of $S$.

We write $\sigma_S$ as $\sigma$ for short if the underlying soft set $S$ is explicit.

Definition 2.4: Let $S = (F, A)$ be a soft set over $U$, and $B \subseteq A$ define $S_B(e) = \sum_{u \in U} F(u, e)$, where $F(e) = \{ u_1, u_4, u_5 \}$, $F(e_2) = \{ u_6 \}$, $F(e_3) = \{ u_1, u_2, u_4, u_5 \}$, $F(e_4) = \{ u_1, u_2, u_6 \}$, $F(e_5) = \{ u_1, u_2 \}$, $F(e_6) = \{ u_2, u_3, u_6 \}$, $F(e_7) = \{ u_2, u_3 \}$, and $\sigma_S$ is the choice value function of soft set $S$.

| $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $\sigma_S$ |
|-------|-------|-------|-------|-------|-------|-----------|
| 1     | 0     | 1     | 1     | 1     | 0     | 0         |
| 0     | 0     | 1     | 1     | 1     | 1     | 1         |
| 0     | 0     | 0     | 0     | 0     | 1     | 1         |
| 1     | 0     | 1     | 0     | 0     | 0     | 2         |
| 1     | 0     | 1     | 0     | 0     | 0     | 2         |
| 0     | 1     | 1     | 0     | 1     | 0     | 4         |

| $S_F(e_i)$ | 3 | 1 | 5 | 3 | 2 | 3 | 2 | $S_F(E) = 19$ |

TABLE 1. Tabular representation of a soft set $S = (F, A)$. | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $\sigma_S$ | $S_F(e_i)$ | $S_F(E)$ | 19 |
Definition 2.5 (\(k|U|\) Parameter Subsets): Let \(S = (F, A)\) be a soft set over \(U\), \(\emptyset \neq B \subseteq A\), we call \(B\) a \(k|U|\) parameter subset if \(\exists\) an integer \(k\) satisfying \(S_F(B) = k|U|\). For a particular \(k\), we denote the set of all \(k|U|\) parameter subsets as \(\mathbb{K}[\mathbb{I}]\). For all \(k\) related to our problem (note that \(S_F(B) = k|U|\) \(\iff S_F(A)\)), we denote the union of the sets of all \(k|U|\) parameter subsets as \(\mathbb{K}[\mathbb{I}]\).

Definition 2.6 (Minimal \(k|U|\) Parameter Subsets): Let \(S = (F, A)\) be a soft set over \(U\), \(\emptyset \neq B \subseteq A\), \(B \in k|U|\). We say is minimal if any nonempty proper subset of \(B\) is not in \(k|U|\).

Denote the set of all minimal \(k|U|\) parameter subsets as \(k|U|_{\text{minimal}}\).

Example 2.2: For the soft set \(S = (F, E)\) represented in TABLE 1, it is easy to see that \(\{e_1, e_2, e_3, e_4\}, \{e_1, e_4\}, \{e_2, e_3\}\) are all in \(\mathbb{K}[\mathbb{I}]\), and \(\{e_1, e_2, e_3, e_4\}\) is not minimal, while \(\{e_1, e_4\}\) and \(\{e_2, e_3\}\) are both minimal.

Aims (Problems) to be investigated in this article

- Given an arbitrary soft set \(S = (F, A)\) over \(U\), propose a combinatorial formula for computing the number of all minimal \(k|U|\) parameter subsets \(\mathbb{K}[\mathbb{I}]\).
- Given an arbitrary soft set \(S = (F, A)\) over \(U\), propose a combinatorial formula for computing the approximate number of all minimal \(k|U|\) parameter subsets.
- (Main Problem 2.1 in this article) Given an arbitrary soft set \(S = (F, A)\) over \(U\), develop an offline and online algorithm for computing the set of all minimal \(k|U|\) parameter subsets \(\mathbb{K}[\mathbb{I}]_{\text{minimal}}\) based on the integer partition method.
- Given an arbitrary soft set \(S = (F, A)\) over \(U\), investigate the application of \(k|U|\) parameter subsets in normal parameter reduction problems of \(S\). Compare our integer partition algorithm with the idea of Ma et al. based on numerical [39].

III. THEORETICAL FOUNDATIONS FOR ALL MINIMAL \(k|U|\) PARAMETER SUBSETS OF A SOFT SET

In this section, we first wish to investigate an important question:

Question 3.1: Suppose \(S = (F, A)\) is a soft set over \(U\), what are the differences of parameters \(e_i\) in \(A\) with respect to the Main Problem 2.1?

Question Analysis: for the Main Problem 2.1, we only use the information \(S_F(e_i)\). That is, we do not care what the exact distribution of values 1 is for \(e_i\). We thus have an answer for the above question as follows:

Theorem 3.1: Letting \(S = (F, A)\) be a soft set over \(U\), for any pair \(e_i, e_j\) of \(A\), if \(S_F(e_i) = S_F(e_j)\), then \(e_i\) and \(e_j\) make no difference with respect to the Main Problem 2.1.

Motivated by the above discussion, we point out the following work.

A. EQUIVALENT CLASSES OF PARAMETERS SET OF A SOFT SET

Definition 3.1: Let \(S = (F, A)\) be a soft set over \(U\), we then define a relation \(\mathcal{R}\) on parameters set \(A\) as follows: \(\forall e_i, e_j \in A,\)

\[e_i \mathcal{R} e_j \iff e_i \neq e_j \text{ if and only if } S_F(e_i) = S_F(e_j).\] (3.1)

Since values of \(S_F(e_i)\) belong to real numbers \(\mathbb{R}\), and the relation \(\approx\) on \(\mathbb{R}\) is an equivalent relation, it is easy to obtain:

Proposition 3.1: Let \(S = (F, A)\) be a soft set over \(U\), then the relation \(\mathcal{R}\) on parameters set \(A\) is an equivalent relation on \(A\).

With \(\mathcal{R}\), we can divide \(A\) into different equivalent classes. We use \([e_i]_{\mathcal{R}}\) to denote the equivalent class containing \(e_i\). That is, \([e_i]_{\mathcal{R}} = \{e \in A | e \mathcal{R} e_i\}\).

Example 3.1: For the soft set \(S = (F, A)\) represented in TABLE 1, we can see that \(S_F(e_1) = \omega_1, S_F(e_2) = \omega_2, S_F(e_3) = \omega_3, S_F(e_4) = \omega_4\). Thus, we have \([e_1]_{\mathcal{R}} = \{e_1\}, [e_2]_{\mathcal{R}} = \{e_2\}, [e_3]_{\mathcal{R}} = \{e_3\}, [e_4]_{\mathcal{R}} = \{e_4\}\).

We use \(\mathcal{U}\) to denote the set \(\{0, 1, 2, \ldots , 6\}\).

Example 3.2: For the soft set \(S = (F, A)\) represented in TABLE 1, \(|\mathcal{U}| = 6\), we have \(\mathcal{U} = \{0, 1, 2, \ldots, 6\}\).

Definition 3.2: Let \(S = (F, A)\) be a soft set over \(U\), define a function \(\mathfrak{g} : U \rightarrow ([e_i]_{\mathcal{R}} | e_i \in A) \cup \{\emptyset\}\) by

\[\mathfrak{g}(k) = [e_i]_{\mathcal{R}} \text{ if and only if } S_F(e_i) = k, \quad (3.2)\]

otherwise, \(\mathfrak{g}(k) = \emptyset\).

According to Example 3.1, we have

Example 3.3: For the soft set \(S = (F, A)\) represented in TABLE 1, we can see \(\mathfrak{g}(1) = [e_1]_{\mathcal{R}}, \mathfrak{g}(2) = [e_1]_{\mathcal{R}} = [e_1]_{\mathcal{R}}, \mathfrak{g}(3) = [e_1]_{\mathcal{R}} = [e_4]_{\mathcal{R}} = [e_6]_{\mathcal{R}}, \mathfrak{g}(5) = [e_3]_{\mathcal{R}}, \mathfrak{g}(4) = \emptyset, \mathfrak{g}(6) = \emptyset\).

According to Definition 2.6, we have

Proposition 3.2: Letting \(S = (F, A)\) be a soft set over \(U\), \(\forall e_i \in A\), if \(S_F(e_i) = 0\) or \(S_F(e_i) = |\mathcal{U}|\), then \([e_i]_{\mathcal{R}} \in \mathbb{K}[\mathbb{I}]_{\text{minimal}}\).

Corollary 3.1: Let \(S = (F, A)\) be a soft set over \(U\). \(\forall e_i \in A\), if \(S_F(e_i) = 0\) or \(S_F(e_i) = |\mathcal{U}|\), then any subset \(B\) of \(A\) containing \(e_i\) and satisfying \(|B| > 2\) cannot be a minimal \(k|U|\) subset of parameters.

By Proposition 3.2, we need not consider these parameters satisfying the condition \(S_F(e_i) = 0\) or \(S_F(e_i) = |\mathcal{U}|\).

B. INTEGER PARTITION WITH RESPECT TO ADDITION OPERATION OF REAL NUMBERS UNDER CONSTRAINTS OF SOFT SET

Definition 3.3: Given a positive integer \(I = \{H_1, H_2, \ldots, H_k\}\) is a set of positive integers. If

\[I = \omega_1H_1 + \omega_2H_2 + \cdots + \omega_kH_k, \quad (3.3)\]

where \(\forall i = 1, 2, \ldots, k\), \(\omega_i\) is a nonnegative integer, we call the vector \(\omega = [\omega_1, \omega_2, \ldots, \omega_k]\) as a positive integer partition of \(I\) with respect to \(H = \{H_1, H_2, \ldots, H_k\}\) under the addition operation of real numbers. Denote the set of all positive integer partitions of \(I\) with respect to \(H = \{H_1, H_2, \ldots, H_k\}\) under the addition operation by \(W(I, H)\).

Example 3.4: Given the positive integer 6, \(H = \{1, 2, 3, 4, 5, 6\}\) is a set of positive integers. Then \(\{3, 0, 1, 0, 0, 0\}\) is a positive integer partition of 6 with respect to \(H = \{1, 2, 3, 4, 5, 6\}\). This is because 6 = \(3 \times 1 + 0 \times 2 + 1 \times 3 + 0 \times 4 + 0 \times 5 + 0 \times 6\). We can obtain the \(W(6, H)\) as:
\[[6, 0, 0, 0, 0, 0], [4, 1, 0, 0, 0, 0], [3, 0, 1, 0, 0, 0], [2, 2, 0, 0, 1, 0], [2, 0, 0, 1, 0, 0], [1, 1, 1, 0, 0, 0], [1, 0, 0, 1, 0, 0], [0, 3, 0, 0, 0, 0], [0, 0, 1, 1, 0, 0], [0, 0, 2, 0, 0, 0], [0, 0, 0, 0, 0, 0]]

Definition 3.3: has no constraints on the maximum value of coefficients \( \omega_i \). But for our problem, we do have some constraints on coefficients \( \omega_i \).

Definition 3.4: Let \( S = (F, A) \) be a soft set over \( U \). \( H_U \) denotes the set \( \{1, 2, \ldots, |U| - 1\} \). Given a positive integer \( I \), if

\[ I = \omega_1I_1 + \omega_2I_2 + \cdots + \omega_H_U I_{|H_U|}, \tag{3.4} \]

where \( I_i = i, 1 \leq i \leq |U| - 1 \), and \( \forall I = 1, 2, \ldots, |U| - 1 \), \( \omega_i \) is a nonnegative integer satisfying

\[ \omega_i \leq |\xi(i)|, \tag{3.5} \]

we then call the vector \( \omega = [\omega_1, \omega_2, \ldots, \omega_{|H_U|}] \) as an integer partition of \( I \) with respect to the addition operation and soft set \( S = (F, A) \). Denote the set of all positive integer partitions of \( I \) with respect to \( H \) under the addition operation by \( W_S(I, H_U) \).

Example 3.5: Consider the soft set \( S = (F, A) \) represented in Table 1. Given the positive integer 6, \( H_U = \{1, 2, 3, 4, 5\} \). Then \( [3, 0, 1, 0, 0, 0] \) does not satisfy the condition (3.5), because \( |\xi(1)| = 1 < 3 \). Similarly, we can obtain:

\[ W_S(6, H_U) = \{[1, 0, 0, 0, 1], [0, 0, 2, 0, 0], [1, 1, 1, 0, 0]\}. \]

C. TRANSFORMATION METHOD FROM INTEGER PARTITIONS TO COMBINATIONS OF PARAMETERS

In this subsection, we will connect the above theories with our problem: that is, how to transform an integer partition to a combination of parameters. This question is not difficult due to the function \( \xi \) (see expression (3.2)).

Definition 3.5: Let \( S = (F, A) \) be a soft set over \( U \). \( H_U \) denotes the set \( \{1, 2, \ldots, |U| - 1\} \). \( \xi \) is the function shown in expression (3.2). Given an \( \omega = [\omega_1, \omega_2, \ldots, \omega_{|H_U|}] \in W_S(I, H_U) \), construct one set \( B = \bigcup_{i=1}^{\omega_{|H_U|}} B_i \), where \( B_i \subseteq \xi(i) \) and \( |B_i| = \omega_i \). We call \( B = \bigcup_{i=1}^{\omega_{|H_U|}} B_i \) a combination of parameters with respect to the integer partition \( \omega = [\omega_1, \omega_2, \ldots, \omega_{|H_U|}] \in W_S(I, H_U) \). We also denote it as \( B \) \( \omega \).

Obviously, we have

Proposition 3.3: Let \( S = (F, A) \) be a soft set over \( U \), \( I = |U| \). Given an \( \omega = [\omega_1, \omega_2, \ldots, \omega_{|H_U|}] \in W_S(I, H_U) \), suppose \( \bigcup_{i=1}^{\omega_{|H_U|}} B_i \) is a combination of parameters with respect to the integer partition \( \omega = [\omega_1, \omega_2, \ldots, \omega_{|H_U|}] \in W_S(I, H_U) \), then \( \bigcup_{i=1}^{\omega_{|H_U|}} B_i \) satisfies the condition of \( k|U| \) parameter subsets where \( k = 1 \) by Definition 2.5.

Example 3.6: Consider the soft set \( S = (F, A) \) represented in Table 1. Given the positive integer 6 = \( |U| \), \( H_U = \{1, 2, 3, 4, 5\} \). By Example 3.5 we have already obtained \( W_S(6, H_U) = \{[1, 0, 0, 0, 1], [0, 0, 2, 0, 0], [1, 1, 1, 0, 0]\} \). Then, by Definition 3.5, we have \( |U| = \{[e_2, e_3], [e_1, e_4, e_5, e_6, e_7], [e_2, e_4, e_5, e_6, e_7], [e_2, e_4, e_5, e_6, e_7], [e_2, e_4, e_5, e_6, e_7], [e_2, e_4, e_5, e_6, e_7]\} \).

D. COMBINATORIAL CALCULATION FORMULA FOR NUMBER OF \( k|U| \) PARAMETER SUBSETS

In this subsection, we will focus on the number of all \( k|U| \) partitions satisfying the constraints induced by a soft set \( S \) over \( U \). The combinatorial calculation formula for number of \( k|U| \) parameter subsets will be proposed. We will also propose an approximate method for the number of minimal \( k|U| \) Parameter Subsets.

Definition 3.6: Let \( S = (F, A) \) be a soft set over \( U \). \( M \) is an integer. Denote the number of all parameter subsets corresponding to the integer partitions of \( M \) under the constraints of \( S \) by \( P_S(M) \).

Definition 3.7: Let \( S = (F, A) \) be a soft set over \( U \). \( M \) is an integer. \( K \) is an integer in \( \{0, 1, 2, \ldots, |U|\} \). Denote the number of all parameter subsets corresponding to the integer partitions with \( K \) as the maximal partition factor of \( M \) under the constraints of \( S \) by \( f_S(M, K) \).

Denote the descending sequence of \( S_F(e_i) \) as \( \mathcal{S} \). Denote \( \mathcal{S}(1) \), i.e., the maximal element in \( \mathcal{S} \), as \( \text{Max}(\mathcal{S}) \).

According to Definition 3.6 and Definition 3.7, we have

\[ P_{\xi}(M) = \sum_{K=0}^{\text{Max}(\mathcal{S})} f_{\xi}(M, K), \tag{3.6} \]

and

\[ P_{\xi}(0) = 2^{|\mathcal{S}(0)|}. \tag{3.7} \]

Here, \( f_{\xi}(M, K) \) is defined as

\[ f_{\xi}(M, K) = \sum_{J=1}^{J^*} C^J_{|\mathcal{S}(K)|} P_{\xi}(M - JK), \tag{3.8} \]

and

\[ f_{\xi}(0, 0) = 2^{|\mathcal{S}(0)|}, \tag{3.9} \]

where

\[ J^* = \text{argmax}_{J} J \]

s.t. \( J \cdot K \leq M, J \leq |\mathcal{S}(K)| \).

and if such \( J^* \) does not exist, then

\[ f_{\xi}(M, K) = 0. \]

- \( \mathcal{S}' \) in \( P_{\xi}(M - JK) \) denotes the part of \( \mathcal{S} \) with elements less than or equal to \( K - 1 \).
- \( C^J_{|\mathcal{S}(K)|} \) is the combinatorial number.

Note that the expressions (3.6), (3.7), (3.8) and (3.9) together actually provide a recursion algorithm for computing \( P_{\xi}(k|U|) \). Here we offer an example by using the soft set in Table 1 as follows:

Example 3.7: Consider the soft set \( S = (F, A) \) represented in Table 1. By Table 1, we have \( S_F(e_1) = 3, S_F(e_2) = 1, S_F(e_3) = 5, S_F(e_4) = 3, S_F(e_5) = 2, S_F(e_6) = 3, S_F(e_7) = 2, S_F(E) = 19, |U| = 6 \). Since \( 4 \times 6 = 24 > S_F(E) \), we have
Let \( S = \{3, 3, 3, 2, 2, 1\} \). Here, \( S \) represents a set of minimal parameter subsets, the cost will be high. We need to optimize the procedure of the computing process. Of course, we should guarantee that the final result will be obtained. Note that we are pursuing the minimal \( |U| \) subsets, so what if we execute some type of minimal operation on the set of partitions? If we can do that and lose no solutions, it will represent a good measure to obtain an optimal procedure.

**Definition 3.6 (Minimal Operation of Integer Partitions):** Suppose that vectors \( \omega_1 = [\omega_{11}, \omega_{12}, \cdots, \omega_{1K}] \) and \( \omega_2 = [\omega_{21}, \omega_{22}, \cdots, \omega_{2K}] \) are two integer partitions for \( k_1|U| \) and \( k_2|U| \), respectively. Then, if \( \omega_1 \prec \omega_2 \) is nonnegative, i.e., \( \forall i = 1, 2, \cdots, K, \omega_{1i} \geq \omega_{2i} \), delete \( \omega_1 \). We call this operation a minimal operation.

**Theorem 3.2:** Let \( S = (F, A) \) be a soft set over \( U \). We denote the set \( \{1, 2, \cdots, |A|\} \). Suppose that vectors \( \omega_1 = [\omega_{11}, \omega_{12}, \cdots, \omega_{1K}] \) and \( \omega_2 = [\omega_{21}, \omega_{22}, \cdots, \omega_{2K}] \) are two different partitions for \( k_1|U| \) and \( k_2|U| \), respectively. If \( \omega_1 \prec \omega_2 \) is nonnegative, i.e., \( \forall i = 1, 2, \cdots, K, \omega_{1i} \geq \omega_{2i} \), then \( \omega_1 \) cannot be transformed to any minimal solution in \( \mathbb{K} |U| \).

**Proof:** We prove this by the contrary. If \( B \) is in \( \mathbb{K} |U| \) such that \( B \prec \omega_1 \), i.e., \( B \) is obtained by transforming \( \omega_1 \) to parameter sets of soft set \( S \), then we see that \( \omega_1 \) must satisfy the constraint (3.5) with respect to \( S = (F, A) \). Therefore, due to the fact that \( \omega_1 \prec \omega_2 \) is nonnegative, \( \omega_2 \) must also satisfy the constraint (3.5). Then, \( B \prec \omega_2 \) must satisfy Definition 2.5. This is a contradiction since \( B \prec \omega_1 \) and \( B \) is minimal.

According to Theorem 3.2, we provide FIGURE 1 as follows. In FIGURE 1, the red way means that we transform all partitions to combinations of parameters, then we conduct the minimalization operation with respect to these combinations. By Theorem 3.2, this is not an optimal way. The green way in FIGURE 1 means that we first execute the minimalization operation to the partitions. As a result, we can reduce the complexity.

**Example 3.9:** With Example 3.4 and Example 3.8, consider the soft set \( S = (F, A) \) represented in TABLE 1. \( H_U = \{1, 2, 3, 4, 5\} \). We know that \( [0, 1, 0, 1, 0] \in W(6, H_U) \). By Definition 3.6, we know that the following vectors in \( W(12, H_U) \) can be deleted: \([6, 1, 0, 1, 0, 0], [4, 2, 0, 1, 0, 0], [2, 2, 0, 1, 0, 0], [2, 1, 0, 1, 0, 0], [2, 0, 1, 0, 0], [2, 0, 0, 1, 0, 0], [1, 3, 0, 1, 0], [1, 2, 1, 1, 0], [1, 1, 3, 0, 0], [1, 1, 1, 0, 0], [1, 1, 0, 1, 1], [1, 0, 2, 0, 1], [1, 0, 1, 2, 0], [1, 0, 0, 0, 1], [0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [0, 1, 0, 0, 0], [0, 2, 1, 1, 0], [0, 2, 0, 1, 1], [0, 1, 2, 1, 0], [0, 1, 1, 0, 0], [0, 1, 0, 1, 0], [0, 0, 0, 0, 0] \). That is, \( |W(12, H_U)| = 58 \). Due to space limitation, we omit the \( W(18, H_U) \) here, but we have \( |W(18, H_U)| = 199 \).

We can see that there are many partitions for integrals 12 and 18, not to mention the other larger integers. Thus, if we choose to transform these partitions direct to sets of parameter subsets, the cost will be high. We need to optimize the procedure of the computing process. Of course, we should guarantee that the final result will be obtained. Note that we are pursuing the minimal \( |U| \) subsets, so what if we execute some type of minimal operation on the set of partitions? If we can do that and lose no solutions, it will represent a good measure to obtain an optimal procedure.
 Similarly, we have these minimal elements of $W(6, H_U)$, $W(12, H_U)$, $W(18, H_U)$. See Table 2 for results.

**TABLE 2.** The minimal elements of $W(6, H_U)$, $W(12, H_U)$, $W(18, H_U)$.

| A | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 6 | 0 | 0 | 0 | 0 |
| 2 | 4 | 1 | 0 | 0 | 0 |
| 3 | 3 | 0 | 1 | 0 | 0 |
| 4 | 2 | 2 | 0 | 0 | 0 |
| 5 | 2 | 0 | 0 | 1 | 0 |
| 6 | 1 | 1 | 1 | 0 | 0 |
| 7 | 1 | 0 | 0 | 0 | 1 |
| 8 | 0 | 3 | 0 | 0 | 1 |
| 9 | 0 | 1 | 0 | 1 | 0 |
| 10 | 0 | 0 | 2 | 0 | 0 |
| 11 | 1 | 0 | 1 | 2 | 0 |
| 12 | 2 | 1 | 0 | 2 | 1 |
| 13 | 0 | 1 | 0 | 2 | 2 |
| 14 | 0 | 1 | 1 | 1 | 1 |
| 15 | 0 | 0 | 3 | 0 | 2 |
| 16 | 0 | 1 | 0 | 3 | 1 |
| 17 | 0 | 0 | 2 | 2 | 2 |

According to Definition 3.4, due to the real situation of soft set $S = (F, A)$ in Table 1, we have Table 3.

We now transform Table 3 to $\mathbb{K}[U]_{\text{minimal}}$:

**TABLE 3.** Elements of $W(6, A)$, $W(12, A)$, $W(18, A)$ remain after minimal operation and the constraints of $S = (F, A)$ in Table 1.

| A | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 6 | 1 | 1 | 1 | 0 | 0 |
| 7 | 1 | 0 | 0 | 0 | 1 |
| 10 | 0 | 0 | 2 | 0 | 0 |
| 12 | 2 | 1 | 0 | 1 | 1 |

- By row 6, we have $\{e_1, e_2, e_5\}$, $\{e_1, e_2, e_7\}$, $\{e_2, e_4, e_5\}$, $\{e_2, e_4, e_7\}$, $\{e_2, e_5, e_6\}$, $\{e_2, e_6, e_7\}$.
- By row 7, we have $\{e_2, e_3\}$.
- By row 10, we have $\{e_1, e_4\}$, $\{e_1, e_6\}$, $\{e_4, e_6\}$.
- By row 12, we have $\{e_1, e_3, e_5, e_7\}$, $\{e_3, e_4, e_5, e_7\}$, $\{e_3, e_5, e_6, e_7\}$.

Thus, we have $\mathbb{K}[U]_{\text{minimal}} = \{e_1, e_4, \{e_1, e_6\}, \{e_2, e_4, e_5, e_6\}, \{e_2, e_4, e_7\}, \{e_2, e_5, e_6, e_7\}, \{e_1, e_3, e_5, e_7\}, \{e_3, e_4, e_5, e_6, e_7\}\}$.

**IV. ALGORITHMS FOR $k|U|$ PARAMETER SUBSETS**

**A. ALGORITHMS FOR NUMBER OF ALL $k|U|$ PARAMETER SUBSETS AND APPROXIMATE NUMBER OF ALL MINIMAL $k|U|$ PARAMETER SUBSETS**

By combining expression (3.6) and expression (3.7) together, we have

$$P_{\mathbb{E}}(M) = \sum_{K=0, |\mathbb{E}(K)|>0} J_K \sum_{K=0, |\mathbb{E}(K)|>0} C_{|\mathbb{E}(K)|}^M P_{\mathbb{E}}(M - J_K K),$$

(3.10)

where $J_K$ means that $J$ is related with $K$.

We thus determine an iterative formula (3.10) which involves only one function $P_{\mathbb{E}}(K)$. This enables us to compute the number of integer partitions of arbitrarily generated integer $M$ and partition factors set $\mathbb{E}$.

**Algorithm 1** The Number of All $k|U|$ Parameter Subsets

1: Input soft set $S$ over object domain $U$ and parameter domain $A$.
2: Compute the column sum $K$ corresponding to each parameter, $\mathbb{E}$, $S_F(A)$, and $\mathbb{F}(K)$.
3: For each $M = k|U|$ satisfying $M \leq S_F(A)$,

$$P_{\mathbb{E}}(M) = \sum_{K=0, |\mathbb{E}(K)|>0} J_K \sum_{K=0, |\mathbb{E}(K)|>0} C_{|\mathbb{E}(K)|}^M P_{\mathbb{E}}(M - J_K K),$$

(3.11)

4: Output the sum of all $P_{\mathbb{E}}(M)$ satisfying $0 \leq M = k|U| \leq S_F(A)$.

**Algorithm 2** Approximate Number of All Minimal $k|U|$ Parameter Subsets

1: Input soft set $S$ over object domain $U$ and parameter domain $A$.
2: Compute the column sum $K$ corresponding to each parameter, $\mathbb{E}$, $S_F(A)$, $\mathbb{F}(0)$.
3: Delete the elements 0 in $\mathbb{E}$, then compute $\mathbb{F}(K)$.
4: For $M = 1|U|$, compute $P_{\mathbb{E}}(M)$ by Algorithm 1.
5: For $M = k|U|$, $2|U| \leq k|U| \leq S_F(A)$,

$$P_{\mathbb{E}}(M) = \sum_{K=0, |\mathbb{E}(K)|>0} f_{\mathbb{E}}(M, K),$$

(3.11)

and $f_{\mathbb{E}}(M, K)$ is defined as

$$f_{\mathbb{E}}(M, K) = \sum_j C_{|\mathbb{E}(K)|}^M P_{\mathbb{E}}(M - JK),$$

(3.12)

where

- $J$ in expression (3.12) satisfies:

$$J \leq M, \quad s.t. \quad J \leq |\mathbb{E}(K)|, \quad m \neq J \times K.$$

- $\mathbb{E}'$ in $P_{\mathbb{E}}(M - JK)$ means the part of $\mathbb{E}$ with elements $t$ satisfying the following two conditions:

(i) $t$ is less than or equal to $K - 1$.

(ii) $t \neq m - K$, i.e., $t + K \neq m$.

6: Output the sum of $|\mathbb{E}(0)|$ and all $P_{\mathbb{E}}(M)$ with $0 \leq M = k|U| \leq S_F(A)$.

As to the number of all minimal integer partitions, i.e., $|\mathbb{K}[U]_{\text{minimal}}|$, it becomes much more complicated. This is the case because we must filter those combinations which cannot be minimal solutions. Based on Algorithm 1, we propose Algorithm 2 which can plane part of nonminimal parameter subsets.
### B. Offline and Online Algorithms for All Minimal \(|k|U\) Parameter Subsets of a Soft Set

We have constructed the preliminary foundation and introduced the basic idea for our problem in the above subsections. In this subsection, we will present an offline and online algorithm for all minimal \(|k|U\) parameter subsets of a soft set based on integer partition. We separate our algorithm into two stages. One is called the offline stage, in which we attempt to fulfill the common work for soft sets with the same \(|U|\). The other is the online stage, which can handle a series of soft sets with different \(|A|\). See FIGURE 2 and Algorithm 3.

**Algorithm 3** An Offline and Online Algorithm for All Minimal \(|k|U|\) Parameter Subsets

**Offline stage:**
1. Set up \([U]_1\), i.e., the number of objects in the soft sets to be processed.
2. Compute \(K = |U| - 1\).
3. Create the integer partition to \(k|U|\) under additional operation with factors as \([1, 2, \ldots, |U|]\), where \(k = 1, 2, \ldots, K\).
4. Perform minimal operation on the partitions obtained in Step 3.

**Online stage:**
5. Input soft set \(S = (F, A)\) with the same \(|U|\).
6. Compute the sums of columns of the input soft set, and classify the parameter set into different parts \([e]_\mathcal{R} \in \mathcal{R}\) generated in Step 6.
7. Filter the minimal partitions obtained in Step 4, which satisfy the constraints of \(S\).
8. Conduct the transformation operation for these partitions with the results obtained in Step 7.
9. Output the final results.

### C. Algorithms for Normal Parameter Reduction Problems

In this section, we attempt to implement the idea of the partition into the normal parameter reduction problems of soft sets.

According to [42], we have the following concepts about parameter reduction of soft sets.

**Definition 4.1 (Normal Parameter Reduction):** For soft set \(S = (F, A)\) over \(U, B \subseteq A, B \neq \emptyset\), if the constraint \(\sum_{e \in A - B} F(u_1, e) = \cdots = \sum_{e \in A - B} F(u_n, e)\) (denoted by \(C_{normal}\)) is satisfied, then \(B\) is called a normal parameter reduction of \(S\).

Note that we do not require the minimality condition for normal parameter reductions as defined in [33].

See FIGURE 3 for a sketch map of Algorithm 4. Inspired by the integer partition idea, we propose Algorithm 4 for normal parameter reduction of soft sets. Its novelty lies in the fact that we first figure out all the integer partitions with respect to possible \(k|U|\), where we use the sums of different columns as factors.

**Algorithm 4** Our Normal Parameter Reduction Algorithm Based on the Integer Partition Method

1. Input soft set \(S\) over object domain \(U\) and parameter domain \(E\).
2. Compute the column sum \(K\) corresponding to each parameter, \(\mathcal{G}, \mathcal{F}\) and \(S_P(A)\).
3. Compute the integer partitions of \(k|U|\) (we adopt the depth-first search method in this paper).
4. Execute the transformation operation for combinations of parameter subsets, check and output all normal parameter reductions.

**Example 4.1:** TABLE 4 represents a soft set \(S = (F, A)\) over objects domain \(U = \{u_1, u_2, \ldots, u_4\}\) and parameters domain \(E = \{e_1, e_2, \ldots, e_6\}\), where \(F(e_1) = \{u_1, u_2, u_4\}, F(e_2) = \{u_1, u_4\}, F(e_3) = \{u_4\}, F(e_4) = \{u_1\}, F(e_5) = \{u_2, u_3\}, F(e_6) = U\), and \(\sigma_S\) can be regarded as the choice value function of soft set \(S\).

Step 1 \(\mathcal{G} = [4, 3, 2, 1, 1], |\mathcal{G}(0)| = 0, \mathcal{G}(1) = \{e_3, e_4\}, \mathcal{G}(1) = 2, \mathcal{G}(2) = \{e_2, e_5\}, |\mathcal{G}(2)| = 2, \mathcal{G}(3) = \{e_1\}, \mathcal{G}(3) = 1, \mathcal{G}(4) = \{e_6\}, |\mathcal{G}(4)| = 1\).

Step 2 Performs the integer partition operation in depth-first search method under the data structure shown in FIGURE 4, then we have

(i) \(1|U| = \{[4], [3, 1], [2, 2], [2, 1, 1]\}\);  
(ii) \(2|U| = \{[4, 3, 1], [4, 2, 2], [4, 2, 1, 1], [3, 2, 2, 1]\}\);  
(iii) \(3|U| = \{[4, 3, 2, 2, 1]\}\).

Step 3 Transform these integer partitions into combinations of parameter subsets, then by Step 1 we have

\[K|k| = \{[e_6], [e_1, e_3], [e_1, e_4], [e_2, e_3, e_4], [e_3, e_4, e_5], [e_1, e_3, e_6], [e_1, e_4, e_6], [e_2, e_3, e_4, e_5], [e_3, e_4, e_5, e_6], [e_1, e_2, e_3, e_4, e_5], [e_1, e_2, e_3, e_4, e_5, e_6]\}\].

**TABLE 4. Tabular representation of a soft set \(S = (F, A)\).**

| \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) | \(\sigma_S\) |
|-------------|-------------|-------------|-------------|-------------|-------------|---------------|
| \(u_1\)    | 1           | 1           | 0           | 1           | 0           | 1             | 4             |
| \(u_2\)    | 1           | 0           | 0           | 0           | 1           | 1             | 3             |
| \(u_3\)    | 0           | 0           | 0           | 0           | 1           | 1             | 2             |
| \(u_4\)    | 1           | 1           | 0           | 0           | 0           | 1             | 4             |

**Algorithm 5** A Normal Numeration Parameter Reduction Algorithm Based on Enumeration Method of Ma et al. in [39]

1. Input soft set \(S\) over object domain \(U\) and parameter domain \(E\).
2. Compute the column sum corresponding to each parameter.
3. For each subset of \(E\), check whether the total value of the related sums of parameters in this subset is a multiple of \(|U|\). If not, turn to the next subset; else, check whether it is a normal parameter reduction.
4. Output all normal parameter reductions.
At last we can check and obtain all normal parameter subsets \(A - \{e_6\}, A - \{e_2, e_5\}, A - \{e_3, e_4, e_5\}\). \(A - \{e_2, e_5, e_6\}\) is obviously an optimal normal parameter reduction, because it deletes the maximum number of parameters with each object losing the value of 2.

V. EXPERIMENTAL RESULTS

- The experiments conducted in this section
  (i) The combinatorial property for the number of \(k\vert U\) and approximate number of minimal \(k\vert U\) parameter subsets. This part involves Algorithm 1, Algorithm 2 and Algorithm 3.
  (ii) Experimental results of the Offline and Online Algorithms for all minimal \(k\vert U\) parameter subsets. This part involves Algorithm 3.
  (iii) Comparison of experimental results between Algorithm 4 and Algorithm 5 in solving normal parameter reduction problems.
S. Geng et al.: Offline and Online Algorithm for All Minimal $k|U|$ Parameter Subsets

**A. EXPERIMENTAL RESULTS FOR COMBINATORIAL PROPERTY OF $k|U|$ PARAMETER SUBSETS**

(i) Let $|A| = 20$, and $|U|$ takes values from 5, 10, 15, 20, where the ratio of 1 equal to 0.1, 0.2, 0.3, 0.4 and 0.5. We consider the average time cost of 100 iterations of **Algorithm 1**. The results list in FIGURE 5. We see that the larger the ratio of 1, the more time it used with respect to the same $|U|$. When the ratio of 1 is equal to 0.4 and 0.5, the average times are very close to each other.

(ii) Let $|U| = 10$, and $|A|$ takes values from 8, 10, ..., 22, 24, where the ratio of 1 is equal to 0.1, 0.2, 0.3, 0.4 and 0.5. We consider the average time cost for 100 iterations of **Algorithm 1**. The results list in FIGURE 6. We see that the larger $|U|$ is, the smaller the number is with respect to the same ratio of 1. When the ratio of 1 is equal to 0.2 to 0.5, the average numbers of $k|U|$ parameter subsets are very close with respect to the same $|U|$.

(iii) Let $|A| = 16$, and $|A|$ takes values from 8, 10, ..., 24, where the ratio of 1 is equal to 0.3. $|U|$ takes values from 5, 10, 15, 20, 25. We compare the average number of minimal $k|U|$ parameter subsets in 100 iterations of **Algorithm 1**, **Algorithm 2** and **Algorithm 3**. The results list in FIGURE 8. We see that there is little change with respect to the exact

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**FIGURE 5.** The average time used for computing the number of $k|U|$ subsets with $|A|=20$ and $|U|=5, 10, 15, 20$, where the ratio of 1 is equal to 0.1, 0.2, 0.3, 0.4 and 0.5.

**FIGURE 6.** The average time used for computing the number of $k|U|$ subsets with $|U|=10$ and $|A|=8, 12, ..., 22, 24$, where the ratio of 1 is equal to 0.1, 0.2, 0.3, 0.4 and 0.5.

**FIGURE 7.** The average number of $k|U|$ subsets with $|A|=20$ and $|U|=5, 10, 15, 20$, where the ratio of 1 is equal to 0.1, 0.2, 0.3, 0.4 and 0.5.
number of minimal $k|U|$ parameter subsets. As far as our experiments, the approximate number approach the total number of $k|U|$ parameter subsets. We also observe that there is substantial room for improvement of Algorithm 2.

(vi) Let $|U| = 15$, and $|A|$ takes values from 8, 10, …, 18, where the ratio of 1 is equal to 0.3. We compare the average number of minimal $k|U|$ parameter subsets in 100 iterations of Algorithm 1, Algorithm 2 and Algorithm 3. The results list in FIGURE 10. It is shown that the differences among these three numbers increase with the growth of $|A|$.

(vi) Let $|U| = 15$ and $|A| = 14$. The ratio of 1 takes values from 0.1 to 0.5. We compare the average number of minimal $k|U|$ parameter subsets in 100 iterations of Algorithm 1, Algorithm 2 and Algorithm 3. The results list in FIGURE 11. It is shown that the differences between Algorithm 1 and Algorithm 2 became smaller when the ratio of 1 approaches 0.5. The total number of minimal parameter subsets change little when the ratio of 1 is equal to 0.2 to 0.5.

B. EXPERIMENTAL RESULTS OF THE OFFLINE AND ONLINE ALGORITHM 3 FOR ALL MINIMAL $k|U|$ PARAMETER SUBSETS

1) MAXIMAL $K$ EFFICIENCY FOR OFFLINE PROCESS
In the offline stage, with $k$ becoming larger, we see that the larger $k$ is, the more possibly the partial solutions of $k|U|$ will be larger (from the order of inclusion among sets) than those induced by $k_1|U|$, where $k_1 < k$. In other words, for the minimal operation, $k|U|$ contributes no new minimal solutions. Therefore, we can actually execute the partition operation to $|U|$, $2|U|$, …, $k|U|$. The identity of $K$ is very interesting. What is the relationship between $K$ and $|U|$? In order to answer this question, in this article we choose to conduct experiments. According to TABLE 5, it is shown that $K = |U| - 1$. In TABLE 6, we list the numerical results of all minimal solutions induced by different $k|U|$, where $k \leq |U| - 1$. 
TABLE 5. The maximum effective $K$ for minimal $k|U|$ partitions by experimental results.

| $|U|$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|
| $K$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

TABLE 6. The number of results of all minimal solutions by different $k|U|$, where the set $H$ of factors is $\{1, 2, \cdots, |U|\}$.

| $|U|$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|
| Number of effective minimal solutions | 2 | 4 | 7 | 15 | 20 | 48 | 65 | 119 | 166 |

Contributions (proportion) of different $k|U|$ to the total solutions

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|---|---|---|---|---|---|---|---|---|
| $k=1$ | 2 | 3 | 5 | 7 | 11 | 15 | 22 | 30 | 42 |
| $k=2$ | 1 | 1 | 5 | 5 | 16 | 19 | 40 | 50 |
| $k=3$ | 1 | 2 | 2 | 10 | 14 | 23 | 42 |
| $k=4$ | 1 | 1 | 4 | 5 | 14 | 15 |
| $k=5$ | 1 | 2 | 3 | 6 | 8 |
| $k=6$ | 1 | 1 | 3 | 4 |
| $k=7$ | 1 | 2 | 3 |
| $k=8$ | 1 | 1 |
| $k=9$ | 1 |

FIGURE 12. The average time cost of the online filtering operation in Algorithm 3 with $|U|=8$ and $|A|=8, 10, \cdots, 30$, where the ratio of 1 is equal to 0.1, 0.2, 0.3, 0.4 and 0.5.

FIGURE 13. The average time cost of the online expanding operation in Algorithm 3 with $|U|=8$ and $|A|=8, 10, \cdots, 30$, where the ratio of 1 is equal to 0.1, 0.2, 0.3, 0.4 and 0.5.

FIGURE 14. The average time cost of the online filtering operation and expanding operation in Algorithm 3 with $|U|=8$ and $|A|=30$, where the ratio of 1 is equal to 0.1, 0.2, 0.3, 0.4 and 0.5.

2) EXPERIMENTAL RESULTS OF THE OFFLINE AND ONLINE ALGORITHM 3 FOR ALL MINIMAL $k|U|$ PARAMETER SUBSETS

(i) Let $|U|=8$, $|A|=8, 10, \cdots, 30$. In the online stage, we compute the average time cost for filtering operation. The results with respect to different ratios of 1 are shown in FIGURE 12. There exists no clear order among different curves.

(ii) Let $|U|=8$, $|A|=8, 10, \cdots, 30$. In the online stage, we compute the average time cost for expanding operation. The results with respect to different ratios of 1 are shown in FIGURE 13. With respect to the same ratio of 1, the time cost escalates when $|A|$ increase.

(iii) Let $|U|=8$, $|A|=20$. In the online stage, we compare the average time cost for the filtering and expanding operation. The results with respect to different ratios of 1 are shown in FIGURE 14. It is shown that the time cost for the filtering operation is larger than that of the expanding process.

(iv) Let $|U|=8$, $|A|=8, 10, \cdots, 30$. We compute the average number of solutions. The results with respect to different ratios of 1 are shown in FIGURE 15. When the ratio of 1 is equal to 0.2, the number is the largest when $|A|$ is larger than 14.

(v) Let $|U|=8$, $|A|=20$. We compute the average number of solutions. The results with respect to different ratios of 1 are shown in FIGURE 16. When the ratio of 1 is equal to 0.2, the number is the largest.
FIGURE 15. The average number of solutions with $|U|=8$ and $|A|=8, 10, \ldots, 30$, where the ratio of 1 is equal to 0.1, 0.2, 0.3, 0.4 and 0.5.

FIGURE 16. The average number of solutions with $|U|=8$ and $|A|=20$, where the ratio of 1 is equal to 0.1, 0.2, 0.3, 0.4 and 0.5.

(vi) Let $|A|=20$, where the ratio of 1 is equal to 0.3. We compare the average time cost for the filtering and expanding operation. The results with respect to different $|U|$ are shown in FIGURE 14.

(vii) Let $|A|=20$, $|U|=5, 6, 7, 8, 9$. We compute the average number of solutions. The results with respect to different ratios of 1 are shown in FIGURE 18.

C. COMPARISON OF EXPERIMENTAL RESULTS BETWEEN ALGORITHM 4 AND ALGORITHM 5 WITH RESPECT TO SOLVING NORMAL PARAMETER REDUCTION PROBLEMS

(i) Let $|A|=20$, and $|U|$ takes values from 10, 15, 20, 25, where the ratio of 1 is equal to 0.3. We compare the average time cost of performing Algorithm 4 and Algorithm 5. The results list in FIGURE 12. It is shown that Algorithm 4 exhibits an obvious advantage over Algorithm 5.

(ii) Let $|U|=10$, and $|A|$ takes values from 8, 10, \ldots, 20, and the ratio of 1 is equal to 0.3. We compare the average time cost of performing Algorithm 4 and Algorithm 5. The results list in FIGURE 13. It is shown that Algorithm 4 exhibits an obvious advantage over Algorithm 5 when the number of parameters increase.
The main contributions of this article are as follows:

(i) We have explored the combinatorial property of $k|U|$ combinations and developed a recursive algorithm for computing the total number of $k|U|$ subsets of parameters.

(ii) Based on (i), we propose a method for computing an upper bound of the minimal number of $k|U|$ subsets of parameters.

(iii) We have proposed an offline and online algorithm for minimal $k|U|$ parameter subsets. Its novelty is based on offline integer partition. We need only to check the partition results according to the related constraints, which are induced by soft set data to come online.

(iv) As an application of $k|U|$ parameter subsets, the method of integer partition has also been extended for normal parameter reduction problems of soft sets.

B. LIMITATIONS OF THE PROPOSED METHOD

(i) We have not yet provided a combinatorial method for computing all $k|U|$ combinations and all minimal $k|U|$ combinations. There is still much room for improving the algorithm: an upper bound of the number of minimal $k|U|$ subsets of parameters requires improvement.

(ii) For the application of our method in normal parameter reduction, since excessive integer partitions exist, the efficiency of the proposed algorithm declines when the size increases (especially with respect to the growth in the number of columns).

(iii) When the number of input soft sets is small, or their scales are different, the benefits produced by offline operation will be offset by the integer partition workload. In this case, the effect of a direct online integer partition based on the constraints of each soft set will be better.

(iv) It should be mentioned that the minimal $k|U|$ partition only considers the information of column sums and does not use the information of row distribution. As a result, minimal $k|U|$ partition cannot guarantee a normal parameter reduction. We must consider all $k|U|$ reductions when we use the integer partition method.

C. FUTURE WORK

(i) It is possible that while we compute the number of the $k|U|$ subsets, we can output the exact solutions by recording variables such as $[K, J]$ in expressions (3.8) and (3.9). This therefore represents potential future work for a real combinatorial method.

(ii) Our work offers a potential way to propose an alternative value for the integers to become partitions. In other words, we need to consider a revised necessary condition for normal parameter reduction of soft sets. The idea of integer partition enables us to work on this direction.

(iii) For the normal parameter reduction problems, the linear programming method in [42] can be combined with the integer partition method proposed in this article. On the one hand, the linear constraints in the linear programming method can constrain the occurrence times of partition factors or the combination of multiple partition factors in the process of integer partition, so the solving process can be simplified. On the other hand, the combination of $k|U|$ constraints and the existing linear constraints in [42] can be used to mine normal parameter reductions with relevant features.
(iv) It is very interesting to combine our theory of 0–1 valued information systems with other kinds of information systems, such as fuzzy-valued information systems and Pythagorean fuzzy-valued information [43]. We also want to explore the potential connections of our algorithms for the normal parameter reduction problems with other fields such as the field of biology [44], [45].

REFERENCES

[1] D. Molodtsov, “Soft set theory—First results,” Comput. Math. Appl., vol. 37, nos. 4–5, pp. 19–31, Feb. 1999.
[2] Q. M. Sun, Z. L. Zhang, and J. Liu, “Soft sets and soft modules,” in Rough Sets and Knowledge Technology (Lecture Notes in Computer Science), vol. 5009, Berlin, Germany: Springer-Verlag, 2008, pp. 403–409.
[3] F. Feng, Y. B. Jun, and X. Zhao, “Soft semirings,” Comput. Math. Appl., vol. 56, pp. 2621–2628, Nov. 2008.
[4] K. Y. Qin and Z. Y. Kong, “On soft equality,” J. Comput. Appl. Math., vol. 281, pp. 107–116, Dec. 2015.
[5] L. A. Zadeh, “Fuzzy sets,” Inf. Control, vol. 8, no. 3, pp. 338–353, Jun. 1965.
[6] X. Ma, Q. Fei, H. Qin, X. Zhou, and H. Li, “New improved normal parameter reduction method for fuzzy soft sets,” IEEE Access, vol. 7, pp. 154912–154921, 2019.

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