WHIPPLE-TYPE $3F_2$-SERIES AND SUMMATION FORMULAE INVOLVING GENERALIZED HARMONIC NUMBERS

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ABSTRACT. By means of the derivative operator and Whipple-type $3F_2$-series identities, two families of summation formulae involving generalized harmonic numbers are established.

1. Introduction

For a complex variable $x$, define the shifted factorial to be

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1) \cdots (x+n-1) \quad \text{with} \quad n \in \mathbb{N}.$$ 

Following Andrews, Askey and Roy [3, Chapter 2], define the hypergeometric series by

$$_{r}F_{s} \left[ \begin{array}{c} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{array} \mid z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \cdots (a_r)_k z^k}{(b_1)_k(b_2)_k \cdots (b_s)_k k!},$$

where $\{a_i\}_{i \geq 1}$ and $\{b_j\}_{j \geq 1}$ are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side. Then Whipple’s $3F_2$-series identity (cf. [3, p. 149]) can be stated as

$$3F_2 \left[ \begin{array}{c} a, 1-a, b \\ c \end{array} \mid 1 \right] = \frac{\Gamma \left( \frac{c+1}{2} \right) \Gamma \left( \frac{1-c}{2} \right) \Gamma (b+\frac{1-c}{2}) \Gamma (b+\frac{c-2}{2})}{\Gamma \left( \frac{a+c}{2} \right) \Gamma \left( \frac{1-a-c}{2} \right) \Gamma (b+\frac{1-a+c}{2}) \Gamma (b+\frac{a-2}{2})},$$

where $\text{Re}(b) > 0$ and $\Gamma(x)$ is the well-known gamma function

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad \text{with} \quad \text{Re}(x) > 0.$$ 

For a complex number $x$ and a positive integer $\ell$, define generalized harmonic numbers of $\ell$-order to be

$$H^{(\ell)}_0(x) = 0 \quad \text{and} \quad H^{(\ell)}_n(x) = \sum_{k=1}^{n} \frac{1}{(x+k)^\ell} \quad \text{with} \quad n \in \mathbb{N}.$$ 

When $x = 0$, they become harmonic numbers of $\ell$-order

$$H^{(\ell)}_0 = 0 \quad \text{and} \quad H^{(\ell)}_n = \sum_{k=1}^{n} \frac{1}{k^\ell} \quad \text{with} \quad n \in \mathbb{N}.$$
Fixing \( \ell = 1 \) in \( H^{(\ell)}_0(x) \) and \( H^{(\ell)}_n(x) \), we obtain generalized harmonic numbers

\[
H_0(x) = 0 \quad \text{and} \quad H_n(x) = \sum_{k=1}^{n} \frac{1}{x + k} \quad \text{with} \quad n \in \mathbb{N}.
\]

When \( x = 0 \), they reduce to classical harmonic numbers

\[
H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{with} \quad n \in \mathbb{N}.
\]

For a differentiable function \( f(x) \), define the derivative operator \( D^i_x \) by

\[
D^i_x f(x) = \frac{d^i}{dx^i} f(x).
\]

When \( i = 1 \), the corresponding sign can be simplified to \( D_x \).

In order to explain the relation of the derivative operator and generalized harmonic numbers, we introduce the following lemma.

**Lemma 1.** Let \( x \) and \( \{a_j, b_j, c_j, d_j\}_{j=1}^{s} \) be all complex numbers. Then

\[
D_x \prod_{j=1}^{s} \frac{a_j x + b_j}{c_j x + d_j} = \prod_{j=1}^{s} \frac{a_j x + b_j}{c_j x + d_j} \sum_{j=1}^{s} \frac{a_j d_j - b_j c_j}{(a_j x + b_j)(c_j x + d_j)}.
\]

**Proof.** It is not difficult to verify the case \( s = 1 \) of Lemma 1. Suppose that

\[
D_x \prod_{j=1}^{m} \frac{a_j x + b_j}{c_j x + d_j} = \prod_{j=1}^{m} \frac{a_j x + b_j}{c_j x + d_j} \sum_{j=1}^{m} \frac{a_j d_j - b_j c_j}{(a_j x + b_j)(c_j x + d_j)}
\]

is true. We can proceed as follows:

\[
D_x \prod_{j=1}^{m+1} \frac{a_j x + b_j}{c_j x + d_j} = D_x \left\{ \prod_{j=1}^{m} \frac{a_j x + b_j}{c_j x + d_j} \frac{a_{m+1} x + b_{m+1}}{c_{m+1} x + d_{m+1}} \right\}
\]

\[
= \frac{a_{m+1} x + b_{m+1}}{c_{m+1} x + d_{m+1}} D_x \prod_{j=1}^{m} \frac{a_j x + b_j}{c_j x + d_j} + \prod_{j=1}^{m} \frac{a_j x + b_j}{c_j x + d_j} D_x a_{m+1} x + b_{m+1} \\
= \frac{a_{m+1} x + b_{m+1}}{c_{m+1} x + d_{m+1}} \prod_{j=1}^{m} \frac{a_j x + b_j}{c_j x + d_j} \sum_{j=1}^{m} \frac{a_j d_j - b_j c_j}{(a_j x + b_j)(c_j x + d_j)} \\
+ \prod_{j=1}^{m} \frac{a_j x + b_j}{c_j x + d_j} \frac{a_{m+1} x + b_{m+1} - b_{m+1} c_{m+1}}{(c_{m+1} x + d_{m+1})^2}
\]

\[
= \prod_{j=1}^{m+1} \frac{a_j x + b_j}{c_j x + d_j} \left\{ \sum_{j=1}^{m} \frac{a_j d_j - b_j c_j}{(a_j x + b_j)(c_j x + d_j)} + \frac{a_{m+1} d_{m+1} - b_{m+1} c_{m+1}}{(a_{m+1} x + b_{m+1})(c_{m+1} x + d_{m+1})} \right\}
\]

\[
= \prod_{j=1}^{m+1} \frac{a_j x + b_j}{c_j x + d_j} \sum_{j=1}^{m+1} \frac{a_j d_j - b_j c_j}{(a_j x + b_j)(c_j x + d_j)}.
\]

This proves Lemma 1 inductively. \( \Box \)

Setting \( a_j = 1, b_j = r - j + 1, c_j = 0 \) and \( d_j = j \) in Lemma 1 it is easy to see that

\[
D_x \binom{x + r}{s} = \binom{x + r}{s} \{H_r(x) - H_{r-s}(x)\},
\]

where \( r, s \in \mathbb{N}_0 \) with \( s \leq r \). Besides, we have the following relation:

\[
D_x H^{(\ell)}_n(x) = -\ell H^{(\ell+1)}_n(x).
\]
Lemma 2. Let

\[ W_n(\alpha) = \sum_{k=0}^{n} \binom{n}{k} \alpha^k (1 + \alpha(n - 2k)H_k) \]

with \( \alpha = 1, 2, 3, 4, 5 \) by combining this way with Zeilberger’s algorithm for definite hypergeometric sums. In terms of the derivative operator and the hypergeometric form of Andrews’ \( q \)-series transformation, Krattenthaler and Rivoal [6] deduced general Paule-Schneider type identities with \( \alpha \) being a positive integer. More results from differentiation of binomial coefficients can be seen in the papers [11, 16, 17, 19, 20]. For different ways and related results, the reader may refer to [4, 7, 8, 10, 12, 15]. It should be mentioned that Sun [13, 14] showed recently some congruence relations concerning harmonic numbers to us.

Inspired by the work just mentioned, we shall explore, according to the derivative operator and Whipple-type \( \text{F}_2 \)-series identities, closed expressions for the following two families of series involving generalized harmonic numbers:

\[
\begin{align*}
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{n+k}{k} \right) x^k H_k^{(2)}(x), \\
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} x^k H_k(2), \\
\end{align*}
\]

where \( t \in \mathbb{N} \). When \( x \) is a nonnegative integer \( p \), they give closed expressions for the following two classes of series on harmonic numbers:

\[
\begin{align*}
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \left( \frac{p+n}{n-k} \right) x^k H_{p+k}, \\
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \left( \frac{p+n}{n-k} \right) x^k H_{p+k}^2, \\
\end{align*}
\]

Due to limit of space, our explicit formulae are offered only for \( t = 0, 1, 2 \) in this paper.

2. THE FIRST FAMILY OF SUMMATION FORMULAE INVOLVING GENERALIZED HARMONIC NUMBERS

**Lemma 2.** Let \( a, b \) and \( c \) be all complex numbers. Then

\[
\text{F}_2 \left[ a, 1-a, b+1 \mid \begin{array}{c} 1 \\ 1+c, 1+2b-c \end{array} \right] = \frac{1}{b} \frac{\Gamma(\frac{1+c}{2})\Gamma(\frac{2+b}{2})\Gamma(b+\frac{1+c}{2})\Gamma(b+\frac{2+b-c}{2})}{\Gamma(\frac{1+c}{2})\Gamma(\frac{2+b}{2})\Gamma(b+\frac{1+c}{2})\Gamma(b+\frac{2+b-c}{2}) + 1 \text{b}^{\frac{1+c}{2}}\Gamma(\frac{2+b}{2})\Gamma(b+\frac{1+c}{2})\Gamma(b+\frac{2+b-c}{2})}.
\]

provided that \( \text{Re}(b) > 0. \)

**Proof.** Perform the replacement \( c \to 1+c \) in (1) to get

\[
\text{F}_2 \left[ a, 1-a, b \mid \begin{array}{c} 1 \\ 1+c, 2b-c \end{array} \right] = \frac{1}{b} \frac{\Gamma(\frac{1+c}{2})\Gamma(\frac{2+b}{2})\Gamma(b+\frac{1+c}{2})\Gamma(b+\frac{2+b-c}{2})}{\Gamma(\frac{1+c}{2})\Gamma(\frac{2+b}{2})\Gamma(b+\frac{1+c}{2})\Gamma(b+\frac{2+b-c}{2}) + 1 \text{b}^{\frac{1+c}{2}}\Gamma(\frac{2+b}{2})\Gamma(b+\frac{1+c}{2})\Gamma(b+\frac{2+b-c}{2})}.
\]

(2)

It is routine to show the continuous relation

\[
\text{F}_2 \left[ a, 1-a, b \mid \begin{array}{c} 1 \\ 1+c, 1+2b-c \end{array} \right] = \frac{c}{2b} \text{F}_2 \left[ a, 1-a, b \mid \begin{array}{c} 1 \\ c, 1+2b-c \end{array} \right] + \frac{2b-c}{2b} \text{F}_2 \left[ a, 1-a, b \mid \begin{array}{c} 1 \\ 1+c, 2b-c \end{array} \right].
\]
Calculating, respectively, the two series on the right hand side by (1) and (2), we gain Lemma 2.

**Theorem 3.** Let $x$ be a complex number. Then

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} (2x-y+1) \binom{y}{k} \binom{x+k}{k} \binom{k}{k} \left( H_n^{(2)}(x) - H_n^{(2)}(-x) \right) + \frac{4n}{x(x-n)}
$$

$$
\left[ H_n(x) - H_n(-x) - H_n(\frac{x-n}{2}) \right] \left[ H_n(x) - H_n(-x) - H_n(\frac{x-n}{2}) - \frac{2(x+n)}{x(x-n)} \right].
$$

**Proof.** The case $a = -n$, $b = x$ and $c = y$ of Lemma 2 reads as

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} (2x-y+1) \binom{y}{k} \binom{x+k}{k} \binom{k}{k} \left( H_n(2x-y) - H_k(y) \right) = \Omega_n(x, y),
$$

where the symbol on the right hand side stands for

$$
\Omega_n(x, y) = \frac{2x-y}{2x} \frac{\binom{y}{k} \binom{x+k}{k} \binom{k}{k} \left( H_n^{(2)}(x) - H_n^{(2)}(-x) \right)}{\binom{y}{k} \binom{x+k}{k} \binom{k}{k}}
$$

$$
\times \left\{ \frac{1}{2} H_n(\frac{y-n}{2}) - \frac{1}{2} H_n(\frac{y-n}{2} - x) + H_{n+1}(y - 2x - 1) - H_n(y) \right\}
$$

$$
+ \frac{2x-y}{2x} \frac{\binom{y}{k} \binom{x+k}{k} \binom{k}{k} \left( H_n^{(2)}(x) - H_n^{(2)}(-x) \right)}{\binom{y}{k} \binom{x+k}{k} \binom{k}{k}}
$$

$$
\times \left\{ \frac{1}{2} H_{n+1}(\frac{y-n}{2}) - \frac{1}{2} H_{n+1}(\frac{y-n}{2} - x) + H_{n+1}(y - 2x - 1) - H_n(y) \right\}.
$$

The last equation can be reformulated as

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} (2x-y+1) \binom{k}{k} \sum_{i=1}^{k} \frac{1}{(2x-y+i)(y+i)} = \frac{\Omega_n(x, y)}{2(y-x)}.
$$

Finding the limit $y \to x$ of it by using the relation

$$
\lim_{y \to x} \frac{\Omega_n(x, y)}{2(y-x)} = \lim_{y \to x} \frac{\partial_y \Omega_n(x, y)}{2} = \frac{-1}{2} \frac{n}{(x+n)} \left\{ \left[ H_n^{(2)}(x) - H_n^{(2)}(-x) \right] + \frac{4n}{x(x-n)}
$$

$$
\left[ H_n(x) - H_n(-x) - H_n(\frac{x-n}{2}) \right] \left[ H_n(x) - H_n(-x) - H_n(\frac{x-n}{2}) - \frac{2(x+n)}{x(x-n)} \right] \right\},
$$

from L’Hôpital rule, we attain Theorem 3.

**Corollary 4** (Harmonic number identity).

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} H_k^{(2)} = \begin{cases}
(-1)^n \left\{ 2H_n^{(2)} - H_n^{(2)}(\frac{x}{2}) \right\}, & n \equiv 0 \mod 2; \\
(-1)^n \left\{ 2H_n^{(2)} - H_n^{(2)}(\frac{x}{2}) \right\}, & n \equiv 1 \mod 2.
\end{cases}
$$
Proof. When \( n = 0 \pmod{2} \), Theorem 3 can be manipulated as
\[
\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} \binom{n+k}{x+k}}{n} H_k^{(2)}(x) = \frac{(-1)^n}{2} \left( \frac{-x+n}{x+n} \right)
\]
\[
\times \left\{ \left[ H_n^{(2)}(x) - H_n^{(2)}(-x) \right] + \left[ H_n(x) - H_n(-x) - H_\frac{x}{2})(\frac{x}{2}) + H_\frac{x}{2}(-\frac{x}{2}) - \frac{2}{x-n} \right] \right\}
\times \left[ H_n(x) - H_n(-x) - H_\frac{x}{2})(\frac{x}{2}) + H_\frac{x}{2}(-\frac{x}{2}) + \frac{2}{x-n} \right] + \frac{4}{(x-n)}.
\]
Taking the limit \( x \to 0 \) of it by utilizing the relation
\[
\lim_{x \to 0} \frac{H_n(x) - H_n(-x) - H_\frac{x}{2})(\frac{x}{2}) + H_\frac{x}{2}(-\frac{x}{2})}{x} = H_n^{(2)}(0) - 2H_n^{(2)}(0)
\]
\[
\text{from L'Hôpital rule, we obtain}
\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} \binom{n+k}{x+k}}{n} H_k^{(2)}(x) = (-1)^n \left\{ 2H_n^{(2)}(0) - H_n^{(2)}(0) \right\}.
\]
When \( n = 1 \pmod{2} \), Theorem 3 can be restated as
\[
\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} \binom{n+k}{x+k}}{n} H_k^{(2)}(x)
\]
\[
= \frac{(-1)^n}{2} \left( \frac{-x+n}{x+n} \right) \left\{ \left[ H_n^{(2)}(x) - H_n^{(2)}(-x) \right] + \left[ H_n(x) - H_n(-x) - H_n(\frac{x-n}{2}) \right]^2 \right\}
\]
\[
+ \frac{2}{(n-x)^2} \frac{\left[ H_n(x) - H_n(-x) - H_n(\frac{x-n}{2}) + 2n \right]}{x}.
\]
Finding the limit \( x \to 0 \) of it by using the relation
\[
\lim_{x \to 0} \frac{(n^2 - x^2)[H_n(x) - H_n(-x) - H_n(\frac{x-n}{2})] + 2n}{x}
\]
\[
= \lim_{x \to 0} D_x \{ (n^2 - x^2)[H_n(x) - H_n(-x) - H_n(\frac{x-n}{2})] + 2n \}
\]
\[
= \frac{n^2}{2} \left[ H_n^{(2)}(0) - 4H_n^{(2)}(0) \right]
\]
\[
= n^2 \left[ 2H_n^{(2)}(0) - H_n^{(2)}(0) \right] - \frac{2}{n^2}
\]
from L'Hôpital rule, we get
\[
\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} \binom{n+k}{x+k}}{n} H_k^{(2)}(x) = (-1)^n \left\{ 2H_n^{(2)}(0) - H_n^{(2)}(0) \right\}.
\]
Then (4) and (5) are unified to Corollary 4. \( \square \)

**Corollary 5.** Let \( p \) be a positive integer satisfying \( 0 < p \leq n \). Then
\[
\sum_{k=0}^{n} \frac{(-1)^k \binom{n+k}{p+k}}{n-k} H_{p+k}^{(2)}(0) = \frac{(-1)^{n-p+1}}{p(n)}
\]
\[
\times \left\{ H_{n+p} - H_{n-p} - H_{\frac{n+p}{2}} + H_{\frac{n-p}{2}}, \quad n-p = 0 \pmod{2}; \right\}
\]
\[
\left\{ H_{n+p} - H_{n-p} - H_{\frac{n+p}{2}} + H_{\frac{n-p}{2}}, \quad n-p = 1 \pmod{2}. \right\}
\]
Taking the limit 

Evaluating the series on the right hand side by (3), we have

When \( n - p = 0 \pmod{2} \), Theorem [3] can be written as

\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} H_k^{(2)}(x) = \frac{(-1)^{n+p} x - p}{2} \binom{x+n}{n} \binom{p}{n}
\]

\[
\times \left\{ H_n^{(2)}(x) - H_{p-1}^{(2)}(x) - H_{n-p}^{(2)}(p-x) \right\} + \frac{4n}{x(x-n)^2}
\]

\[
+ \left[ H_n(x) + H_{p-1}(x-p) - H_{n-p}(p-x) - H_{n+p} \left( \frac{x-n}{2} + H_{n-p} \right) \right]
\]

\[
\times \left[ H_n(x) + H_{p-1}(x-p) - H_{n-p}(p-x) - H_{n+p} \left( \frac{x-n}{2} + H_{n-p} \right) - \frac{2(x+n)}{x(x-n)} \right]
\]

\[
- 2\left[ H_n(x) + H_{p-1}(x-p) - H_{n-p}(p-x) - H_{n+p} \left( \frac{x-n}{2} + H_{n-p} \right) - \frac{x+n}{x(x-n)} \right]
\}

Taking the limit \( x \to p \) of it, we gain

\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} H_k^{(2)}
\]

\[
= H_p^{(2)} \sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k}
\]

\[
+ \frac{(-1)^{n-p+1}}{p^p} \left\{ H_{n+p} - H_{n-p} - H_{n+p} + H_{n-p} \right\}.
\]

Evaluating the series on the right hand side by (3), we have

\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} H_k^{(2)}
\]

\[
= \frac{(-1)^{n-p+1}}{p^p} \left\{ H_{n+p} - H_{n-p} - H_{n+p} + H_{n-p} \right\}.
\]

When \( n - p = 1 \pmod{2} \), Theorem [3] can be reformulated as

\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} H_k^{(2)}(x) = \frac{(-1)^{n+p} x - p}{2} \binom{x+n}{n} \binom{p}{n}
\]

\[
\times \left\{ H_n^{(2)}(x) - H_{p-1}^{(2)}(x) - H_{n-p}^{(2)}(p-x) \right\} + \frac{4n}{x(x-n)^2}
\]

\[
+ \left[ H_n(x) + H_{p-1}(x-p) - H_{n-p}(p-x) - H_{n+p} \left( \frac{x-n}{2} + H_{n-p} \right) \right]
\]

\[
\times \left[ H_n(x) + H_{p-1}(x-p) - H_{n-p}(p-x) - H_{n+p} \left( \frac{x-n}{2} + H_{n-p} \right) - \frac{2(x+n)}{x(x-n)} \right]
\]

\[
+ \frac{2\left[ H_n(x) + H_{p-1}(x-p) - H_{n-p}(p-x) - H_{n+p} \left( \frac{x-n}{2} + H_{n-p} \right) - \frac{x+n}{x(x-n)} \right]}{x-p}
\}

\[
\}

Finding the limit \( x \to p \) of it, we achieve

\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} H_k^{(2)}
\]

\[
= H_p^{(2)} \sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k}
\]

\[
+ \frac{(-1)^{n-p+1}}{p^p} \left\{ H_{n+p} - H_{n-p} - H_{n+p-1} + H_{n-p-1} \right\}.
\]
Calculating the series on the right hand side by \( (3) \), we attain
\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} H_{p+k}^{(2)} = \frac{(-1)^{n-p+1}}{p(n)} \left\{ H_{n+p} - H_{n-p} - H_{\frac{n+1}{2}} + H_{\frac{n-1}{2}} \right\},
\]
(7)

Then \( (6) \) and \( (7) \) are unified to Corollary \( 5 \).

**Corollary 6** \((p = n)\) in Corollary \( 5 \).
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} H_{n+k}^{(2)} = \frac{1}{n(2n)} \left\{ H_n - H_{2n} \right\}.
\]

**Corollary 7.** Let \( p \) be a positive integer with \( p > n \). Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} H_{p+k}^{(2)} = \frac{1}{2} \binom{p-1}{n} \left\{ H_{p+n}^{(2)} + H_{p-n}^{(2)} + A(p, n) \right\},
\]
where the expression on the right hand side is
\[
A(p, n) = \begin{cases} 
(H_{p+n} - H_{p-n} - H_{\frac{n+1}{2}} + H_{\frac{n-1}{2}}) \\
\times(H_{p+n} - H_{p-n} - H_{\frac{n+1}{2}} + H_{\frac{n-1}{2}}), & p - n = 0 \pmod{2}; \\
(H_{p+n} - H_{p-n} - H_{\frac{n+1}{2}} + H_{\frac{n-1}{2}}) \\
\times(H_{p+n} - H_{p-n} - H_{\frac{n+1}{2}} + H_{\frac{n-1}{2}} + \frac{2}{p-n}), & p - n = 1 \pmod{2}.
\end{cases}
\]

**Proof.** When \( p - n = 0 \pmod{2} \), the case \( x = p \) of Theorem \( 3 \) can be manipulated as
\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} H_{p+k}^{(2)} = H_{p}^{(2)} \sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k}
+ \frac{1}{2} \binom{p-1}{n} \left\{ H_{p+n}^{(2)} - 2H_{p}^{(2)} + H_{p-n}^{(2)} + (H_{p+n} - H_{p-n} - H_{\frac{n+1}{2}} + H_{\frac{n-1}{2}}) \\
\times(H_{p+n} - H_{p-n} - H_{\frac{n+1}{2}} + H_{\frac{n-1}{2}}) \right\}.
\]

Evaluating the series on the right hand side by \( (3) \), we obtain
\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} H_{p+k}^{(2)} = \frac{1}{2} \binom{p-1}{n} \\
\times \left\{ H_{p+n}^{(2)} + H_{p-n}^{(2)} + (H_{p+n} - H_{p-n} - H_{\frac{n+1}{2}} + H_{\frac{n-1}{2}}) \\
\times(H_{p+n} - H_{p-n} - H_{\frac{n+1}{2}} + H_{\frac{n-1}{2}}) \right\}.
\]
(8)

When \( p - n = 1 \pmod{2} \), the case \( x = p \) of Theorem \( 3 \) can be restated as
\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} H_{p+k}^{(2)} = H_{p}^{(2)} \sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k}
+ \frac{1}{2} \binom{p-1}{n} \left\{ H_{p+n}^{(2)} - 2H_{p}^{(2)} + H_{p-n}^{(2)} + (H_{p+n} - H_{p-n} - H_{\frac{n+1}{2}} + H_{\frac{n-1}{2}}) \\
\times(H_{p+n} - H_{p-n} - H_{\frac{n+1}{2}} + H_{\frac{n-1}{2}} + \frac{2}{p-n}) \right\}.
\]
Calculating the series on the right hand side by (3), we get
\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} H^{(2)}_{p+k} = \frac{1}{2} \binom{p-1}{n} \times \left\{ H^{(2)}_{p+n} + H^{(2)}_{p-n} + (H_{p+n} - H_{p-n} - H_{p+n-1} + H_{p-n-1}) \times (H_{p+n} - H_{p-n} - H_{p+n-1} + H_{p-n-1}) \right\}. \tag{9}
\]

Then (8) and (9) are unified to Corollary (7). \qed

**Lemma 8.** Let \(a, b\) and \(c\) be all complex numbers. Then
\[
\sum_{k=0}^{\infty} \frac{(a)_k(1-a)_k(1+b)_k}{k!(1+c)_k(1+2b-c)_k} = \frac{a^2 - a + bc - c^2}{b(1-b)} \times \frac{\Gamma\left(\frac{1+c}{2}\right)\Gamma\left(\frac{2+a+c}{2}\right)\Gamma\left(b + \frac{1-c}{2}\right)\Gamma\left(b + \frac{2-c}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+a-c}{2}\right)\Gamma\left(b + \frac{1-a}{2}\right)\Gamma\left(b + \frac{1-a-c}{2}\right)} \times \frac{a^2 - a - 2b^2 + 3bc - c^2}{b(1-b)} \times \frac{\Gamma\left(\frac{1+c}{2}\right)\Gamma\left(\frac{2+a+c}{2}\right)\Gamma\left(b + \frac{1-c}{2}\right)\Gamma\left(b + \frac{2-c}{2}\right)}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+a-c}{2}\right)\Gamma\left(b + \frac{1-a}{2}\right)\Gamma\left(b + \frac{1-a-c}{2}\right)}
\]

provided that \(Re(b-1) > 0\).

**Proof.** It is not difficult to verify the continuous relation
\[
3F_2 \left[ \begin{array}{c} a, -a, b \\ c, 1 + 2b - c \end{array} \right] = \frac{1}{2} 3F_2 \left[ \begin{array}{c} a, 1 - a, b \\ c, 1 + 2b - c \end{array} \right] + \frac{1}{2} 3F_2 \left[ \begin{array}{c} 1 + a, -a, b \\ c, 1 + 2b - c \end{array} \right].
\]

Evaluating the series on the right hand side by (11), we gain
\[
3F_2 \left[ \begin{array}{c} a, -a, b \\ c, 1 + 2b - c \end{array} \right] = \frac{1}{2} \Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{1+c}{2}\right)\Gamma\left(b + \frac{1-c}{2}\right)\Gamma\left(b + \frac{2-c}{2}\right) \times \frac{1}{2} \Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{1+c}{2}\right)\Gamma\left(b + \frac{1-c}{2}\right)\Gamma\left(b + \frac{2-c}{2}\right)
\]

By means of Kummer’s transformation formula (cf. [3, p. 142]):
\[
3F_2 \left[ \begin{array}{c} a, b, c \\ d, e \end{array} \right] = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-a)\Gamma(d+e-b-c)} 3F_2 \left[ \begin{array}{c} a, d-b, d-c \\ d, d+e-b-c \end{array} \right] 3F_2 \left[ \begin{array}{c} a, d-b, d-c \\ d, d+e-b-c \end{array} \right] \tag{11}
\]

we have
\[
3F_2 \left[ \begin{array}{c} a, b, c \\ 1 + a - b, a - c \end{array} \right] = \frac{\Gamma(1 + a - b)\Gamma(1 + a - 2b - 2c)}{\Gamma(1 + a - b - c)\Gamma(1 + a - 2b - c)} \times 3F_2 \left[ \begin{array}{c} c, -c, a - b - c \\ a - c, 1 + a - 2b - c \end{array} \right].
\]

Calculating the series on the right hand side by (12), we achieve
\[
3F_2 \left[ \begin{array}{c} a, b, c \\ 1 + a - b, a - c \end{array} \right]
\]

\[
= \frac{1}{21^c} \Gamma(1 + a - b)\Gamma(c - a - b)\Gamma(1 + a - b - c) \times \frac{1}{21^c} \Gamma(1 + a - b)\Gamma(c - a - b)\Gamma(1 + a - b - c)
\]

\[
+ \frac{1}{21^c} \Gamma(1 + a - b)\Gamma(c - a - b)\Gamma(1 + a - b - c)
\]

\[
\tag{12}
\]
It is routine to show the continuous relation

\[
\begin{align*}
4F_3 \left[ \frac{a, b, c, 1 + x}{2 + a - b, 1 + a - c, x} \bigg| 1 \right] &= \frac{a(x + b - a - 1)}{x(b - 1)} 4F_2 \left[ \frac{1 + a, b, c}{2 + a - b, 1 + a - c} \bigg| 1 \right] \\
&\quad + \frac{1 + a - b)(a - x)}{x(b - 1)} 4F_2 \left[ \frac{a, b, c}{1 + a - b, 1 + a - c} \bigg| 1 \right].
\end{align*}
\]

Evaluating, respectively, the two series on the right hand side by (12) and Dixon’s \(3F_2\)-series identity (cf. [3, p. 72]):

\[
3F_2 \left[ \frac{a, b, c}{1 + a - b, 1 + a - c} \bigg| 1 \right] = \frac{\Gamma \left( \frac{3 + a}{2} \right) \Gamma (1 + a - b) \Gamma (1 + a - c) \Gamma \left( \frac{3 + a}{2} - b - c \right)}{\Gamma (1 + a) \Gamma \left( \frac{3 + a}{2} - b \right) \Gamma \left( \frac{3 + a}{2} - c \right) \Gamma (1 + a - b - c)},
\]

we attain

\[
\begin{align*}
4F_3 \left[ \frac{a, b, c, 1 + x}{2 + a - b, 1 + a - c, x} \bigg| 1 \right] &= \frac{a(1 + a - b - 2c) - x(2 + a - 2b - 2c)}{2x(b - 1)} \\
&\quad \times \frac{\Gamma \left( \frac{3 + a}{2} \right) \Gamma (2 + a - b) \Gamma (1 + a - c) \Gamma \left( \frac{3 + a}{2} - b - c \right)}{\Gamma (1 + a) \Gamma \left( \frac{3 + a}{2} - b \right) \Gamma \left( \frac{3 + a}{2} - c \right) \Gamma (2 + a - b - c)} \\
&\quad + \frac{a(x + b - a - 1) \Gamma \left( \frac{1 + a}{2} \right) \Gamma (2 + a - b) \Gamma (1 + a - c) \Gamma \left( \frac{3 + a}{2} - b - c \right)}{2x(b - 1)} \frac{\Gamma (1 + a) \Gamma \left( \frac{3 + a}{2} - b \right) \Gamma \left( \frac{3 + a}{2} - c \right) \Gamma (2 + a - b - c)}{\Gamma \left( \frac{3 + a}{2} - b \right) \Gamma \left( \frac{3 + a}{2} - c \right) \Gamma (2 + a - b - c)}.
\end{align*}
\]

When the parameter \(x\) is specified, the last equation can produce the following two results:

\[
\begin{align*}
3F_2 \left[ \frac{a, b, c}{2 + a - b, 1 + a - c} \bigg| 1 \right] &= \frac{1 - b}{1 - b \Gamma (1 + a) \Gamma \left( \frac{3 + a}{2} - b \right) \Gamma \left( \frac{3 + a}{2} - c \right) \Gamma (2 + a - b - c)} \\
&\quad \times \frac{\Gamma \left( \frac{3 + a}{2} \right) \Gamma (2 + a - b) \Gamma (1 + a - c) \Gamma \left( \frac{3 + a}{2} - b - c \right)}{\Gamma (1 + a) \Gamma \left( \frac{3 + a}{2} - b \right) \Gamma \left( \frac{3 + a}{2} - c \right) \Gamma (2 + a - b - c)} \\
&\quad + \frac{1 - b - c}{2(1 - b) \Gamma (a) \Gamma \left( \frac{3 + a}{2} - b \right) \Gamma \left( \frac{3 + a}{2} - c \right) \Gamma (2 + a - b - c)},
\end{align*}
\]

It is routine to show the continuous relation

\[
\begin{align*}
3F_2 \left[ \frac{a, b, c}{3 + a - b, a - c} \bigg| 1 \right] &= \frac{(2 + a - b)(c)}{(a - c)(b - 2)} 3F_2 \left[ \frac{a, b, c}{2 + a - b, 1 + a - c} \bigg| 1 \right] \\
&\quad + \frac{(a - c)(b - 2)}{(a - c)(b - 2)} 3F_2 \left[ \frac{a, b, c}{3 + a - b, 1 + a - c} \bigg| 1 \right] + \frac{(a - c)(b - 2)}{(a - c)(b - 2)} 3F_2 \left[ \frac{a, b, c}{3 + a - b, 1 + a - c} \bigg| 1 \right].
\end{align*}
\]
Calculating, respectively, the two series on the right hand side by (14) and (15), we obtain

\[
3F2 \left[ \begin{array}{c} a, b, c \\ 3 + a - b, a - c \\ 1 \end{array} \right] = \frac{2 + b^2 - 3b + bc - ac - c}{(b - 1)(b - 2)} \Gamma(\frac{3 + a}{2} - b + b)\Gamma(3 + a - b)\Gamma(\frac{3 + a}{2} - b - c) \\
\Gamma(1 + a)\Gamma(b + \frac{1}{2} - a)\Gamma(\frac{3 + a}{2} - c)\Gamma(3 + a - b - c)
\]

\[+ \frac{2 + b^2 - 3b - bc + ac + c - 2c^2}{2(b - 1)(b - 2)} \Gamma(\frac{3 + a}{2} + a - b)\Gamma(3 + a - b)\Gamma(\frac{3 + a}{2} - b - c)\frac{\Gamma(3 + a - b - c)}{\Gamma(1 + a)\Gamma(b + \frac{1}{2} - a)\Gamma(\frac{3 + a}{2} - c)\Gamma(3 + a - b - c)} \quad (16)
\]

In accordance with (11), we get

\[
3F2 \left[ \begin{array}{c} a, 3 - a, b \\ c, 2b - c \\ 1 \end{array} \right] = \frac{\Gamma(2b - c)\Gamma(b - 3)}{\Gamma(2b - a - c)\Gamma(a + b - 3)} 3F2 \left[ \begin{array}{c} a, c - b, a + c - 3 \\ c, a + b - 3 \\ 1 \end{array} \right].
\]

Evaluating the series on the right hand side by (16), we gain

\[
3F2 \left[ \begin{array}{c} a, 3 - a, b \\ c, 2b - c \\ 1 \end{array} \right] = \frac{4(2 - 3a + a^2 - 2b + 2bc + 2c - c^2)}{(a - 1)(a - 2)(b - 1)(b - 2)(b - 3)} \times \frac{\Gamma(\frac{1}{2} + a - c)\Gamma(b - \frac{a}{2} + c)}{\Gamma(\frac{3 + a}{2} - c)\Gamma(b - \frac{3 + a}{2} + c)} + \frac{4(2 - 3a + a^2 + 2b - 2b^2 + 3bc - 2c - c^2)}{(a - 1)(a - 2)(b - 1)(b - 2)(b - 3)} \times \frac{\Gamma(\frac{1}{2} + a - c)\Gamma(b - \frac{a}{2} + c)}{\Gamma(\frac{3 + a}{2} - c)\Gamma(b - \frac{3 + a}{2} + c)} \quad (17)
\]

It is not difficult to verify that

\[
\sum_{k=0}^{\infty} \frac{(a)_k(1 - a)_k(1 + b)_k}{k!(1 + c)_k(1 + 2b - c)_k} = \sum_{k=1}^{\infty} \frac{(a)_k(1 - a)_k(1 + b)_k}{(k - 1)!(1 + c)_k(1 + 2b - c)_k} = \sum_{k=0}^{\infty} \frac{(a)_{k+1}(1 - a)_{k+1}(1 + b)_{k+1}}{k!(1 + c)_{k+1}(1 + 2b - c)_{k+1}} = \frac{a(1 - a)(1 + b)}{(1 + c)(1 + 2b - c)} 3F2 \left[ \begin{array}{c} 1 + a, 2 - a, 2 + b \\ 2 + c, 2 + 2b - c \\ 1 \end{array} \right].
\]

Calculating the series on the right hand side by (17), we achieve Lemma \[8\] \[9\].

**Theorem 9.** Let \( x \) be a complex number. Then

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(n + k)}{(x + k)} H_k^{(2)}(x) = (-1)^n n(n + 1) \frac{(-x + n)}{2(1 - x)} \frac{(x + n)}{(x + n)} \times \left\{ H_n^{(2)}(x) - H_n^{(2)}(-x) + \left[ H_n(x) - H_n(-x) \right] \right\} \times \left[ H_n(x) - H_n(-x) - 2H_{n+1}(\frac{x - n}{2}) - 2(x^2 - n^2) \right] + H_n(\frac{x - n}{2}) \left[ H_{n+1}(\frac{x - n - 2}{2}) + \frac{2(x^3 - n^3)}{x(n + 1)} \right] + \frac{4(x^2 - n^2 + n^2)}{x(n + 1)} \right\}.
\]
Proof. The case \( a = -n, b = x \) and \( c = y \) of Lemma \([8]\) reads as

\[
\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{n+k}{k} \binom{x+k}{k} \binom{y+n-1+n}{n} \binom{y-2x+n}{n+1} \binom{y-2x+n}{y+n} \binom{y-2x+n}{y+n} \binom{y-2x+n}{y+n} 
\]

\[
= \frac{(2x-y)(n^2 + n + xy - y^2) \left( \frac{y-n-1}{n} + \frac{y-2x+n}{n+1} \right) \left( \frac{y-2x+n}{y+n} \right)^2}{2x(1-x)}
\]

\[
+ \frac{(2x-y)(y-n)(n^2 + n - 2x^2 + 3xy - y^2) \left( \frac{y-n-1}{n} + \frac{y-2x+n}{n+1} \right) \left( \frac{y-2x+n}{y+n} \right)^2}{2x(1-x)(2x-y+n)}
\]

Applying the derivative operator \( D_y \) to both sides of it, we attain

\[
\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{n+k}{k} \binom{x+k}{k} \binom{y+n-1+n}{n} \binom{y-2x+n}{n+1} \binom{y-2x+n}{y+n} \binom{y-2x+n}{y+n} 
\]

where the symbol on the right hand side stands for

\[\Phi_n(x, y) = \frac{(2x-y)(n^2 + n + xy - y^2) \left( \frac{y-n-1}{n} + \frac{y-2x+n}{n+1} \right) \left( \frac{y-2x+n}{y+n} \right)^2}{2x(1-x)} - \frac{1}{2} H_n(x) + H_{n+1}(y) - H_n(y) + \frac{x-2y}{n^2+n+xy-y^2} \]

\[
+ \frac{(2x-y)(y-n)(n^2 + n - 2x^2 + 3xy - y^2) \left( \frac{y-n-1}{n} + \frac{y-2x+n}{n+1} \right) \left( \frac{y-2x+n}{y+n} \right)^2}{2x(1-x)(2x-y+n)} - \frac{1}{2} H_{n+1}(\frac{y-n-2}{2}) + H_{n+1}(y) - H_n(y) + \frac{3x-2y}{n^2+n-2x^2+3xy-y^2} \]

The last equation can be written as

\[
\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{n+k}{k} \binom{x+k}{k} \binom{y+n-1+n}{n} \binom{y-2x+n}{n+1} \binom{y-2x+n}{y+n} \binom{y-2x+n}{y+n} 
\]

\[
= \frac{\Phi_n(x, y)}{2(y-x)}.
\]

Taking the limit \( y \to x \) of it by utilizing the relation

\[
\text{Lim}_{y \to x} \frac{\Phi_n(x, y)}{2(y-x)}
\]

\[
= \frac{(-1)^n n(n+1) \left( \frac{y+n}{n} \right)^2}{2(1-x) \left( \frac{y+n}{n} \right)}
\]

\[
\times \left[ H_n^{(2)}(x) - H_n^{(2)}(-x) + \left( H_n(x) - H_n(-x) \right) \right]
\]

\[
\times \left[ H_n(x) - H_n(-x) - 2H_{n+1}(\frac{x-n-2}{2}) - \frac{2(x^2-n-n^2)}{x(n(n+1))} \right]
\]

\[
+ H_n(\frac{x-n}{2}) \left[ H_{n+1}(\frac{x-n-2}{x-n}) + \frac{2(x^3-n^2+n^2+n^3)}{x(x-n)n(n+1)} \right] + \frac{4(x^2-n+n^2+n^2)}{x(x-n)^2(n+1)} \}
\]

from l'Hôpital rule, we obtain Theorem \([9]\).

When \( x \to p \), where \( p \) a nonnegative integer, Theorem \([9]\) can give the following four corollaries.
Corollary 10 (Harmonic number identity).
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} k H_k^{(2)} = (-1)^n n(n+1)
\]
\[
\times \begin{cases} 
2 H_n^{(2)} - H^{(2)}_n, & n = 0 \pmod{2} \\
2 H_n^{(2)} - H^{(2)}_n - \frac{2}{n(n+1)}, & n = 1 \pmod{2}.
\end{cases}
\]

Corollary 11. Let \( p \) be a positive integer satisfying \( 0 < p \leq n \). Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} k H_x^{(2)} = \frac{n(n+1)(-1)^{n-p}}{p(p-1)} \binom{n}{p}
\]
\[
\times \begin{cases} 
H_n^{(p)} - H_{n-p}^{(p)} - \frac{p}{n(n+1)}, & n-p = 0 \pmod{2} \\
H_n^{(p)} - H_{n-p}^{(p)} - \frac{p}{n(n+1)}, & n-p = 1 \pmod{2}.
\end{cases}
\]

Corollary 12 (Harmonic number identity).
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} k H_n^{(2)} = \frac{n+1}{n-1} \left\{ H_n - H_{n-1} \right\}.
\]

Corollary 13. Let \( p \) be a positive integer with \( p > n \). Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} k H_x^{(2)} = \frac{n+n^2}{2(1-p)} \left\{ H_n^{(2)} + H_{n-1}^{(2)} + B(p,n) \right\},
\]
where the symbol on the right hand side stands for
\[
B(p,n) = \begin{cases} 
\left( \frac{H_{p+n} - H_{p-n} - H_{n-1}^{(p)} + H_{n-1}^{(p)}}{n^{(p+1)}} - \frac{3}{p} - \frac{2p}{n+n^2} \right) & p-n = 0 \pmod{2} \\
\left( \frac{H_{p+n} - H_{p-n} - H_{n-1}^{(p)} + H_{n-1}^{(p)}}{n^{(p+1)}} + \frac{2(p+n^2)}{(p+n^2)(p-n^2)} \right) & p-n = 1 \pmod{2}.
\end{cases}
\]

Lemma 14. Let \( a, b \) and \( c \) be all complex numbers. Then
\[
\sum_{k=0}^{\infty} k^2 \frac{(a)_{k}(1-a)_{k}(1+b)_{k}}{k!(1+c)_{k}(1+2b-c)_{k}} = \frac{\beta(a,b,c)}{b(b-1)(b-2) \Gamma(\frac{1+c}{2} \Gamma(\frac{2+c}{2}) \Gamma(b + \frac{1-c}{2}) \Gamma(b + \frac{2-c}{2})}
\]
\[
+ \frac{\gamma(a,b,c)}{b(b-1)(b-2) \Gamma(\frac{1+c}{2} \Gamma(\frac{2+c}{2}) \Gamma(b + \frac{1-c}{2}) \Gamma(b + \frac{2-c}{2})},
\]
where the convergence condition is Re\((b-2) > 0\) and
\[
\beta(a,b,c) = c(c-b)(1+b-2bc+c^2) + a(b-3bc+b^2+2c^2)
\]
\[
+ a^2(1-b+3bc-b^2-2c^2) - 2a^3 + a^4,
\]
\[
\gamma(a,b,c) = (c-b)(c-2b)(1+b-2bc+c^2) + a(b-5bc+3b^2+2c^2)
\]
\[
+ a^2(1-b+5bc-3b^2-2c^2) - 2a^3 + a^4.
\]

Proof. It is easy to see the continuous relation
\[
_{3}F_{2} \left[ \begin{array}{c}
\frac{a,b,c}{3 + a-b,1+a-c} \\
\end{array} \right] = \frac{2 + a - b}{2 - b} \left( \begin{array}{c}
\frac{a,b,c}{2 + a - b,1+a-c} \\
\end{array} \right)
\]
\[
+ \frac{a}{b-2} \left( \begin{array}{c}
\frac{1 + a,b,c}{3 + a-b,1+a-c} \\
\end{array} \right).
\]
Evaluating, respectively, the two series on the right hand side by \([14]\) and \([15]\), we get
\[
\begin{aligned}
3F_2 \left[ \begin{array}{c} a, b, c \\ 3 + a - b, 1 + a - c \\ 1 \\ \end{array} \right] & = \frac{a^2 - a(2b + 2c - 3) + 2(b + c - 2)}{2(b - 1)(b - 2)} \frac{\Gamma(\frac{2 + a}{2}) \Gamma(3 + a - b) \Gamma(1 + a - c) \Gamma(\frac{4 + a}{2} - b - c)}{(1 + a) \Gamma(4 + a - b) \Gamma(\frac{2 + a}{2} - c) \Gamma(3 + a - b - c) - 1} \\
& \frac{1}{(b - 1)(b - 2)} \frac{\Gamma(\frac{1 + a}{2}) \Gamma(3 + a - b) \Gamma(1 + a - c) \Gamma(\frac{5 + a}{2} - b - c)}{\Gamma(a) \Gamma(\frac{5 + a}{2} - b) \Gamma(\frac{4 + a}{2} - c) \Gamma(3 + a - b - c)}.  \\
\end{aligned}
\] (19)

It is routine to show the relation
\[
\begin{aligned}
3F_3 \left[ \begin{array}{c} a, b, c, 1 + x \\ 4 + a - b, 1 + a - c, x \\ 1 \\ \end{array} \right] & = \frac{a(3 + a - b - x)(2 - 3b + 2c + b^2 + ac - bc - 2c^2) + (x - a)}{2x(1 - b)(2 - b)(3 - b)} \\
& \times \frac{\Gamma(\frac{2 + a}{2}) \Gamma(4 + a - b) \Gamma(1 + a - c) \Gamma(\frac{4 + a}{2} - b - c)}{\Gamma(1 + a) \Gamma(\frac{4 + a}{2} - b) \Gamma(\frac{2 + a}{2} - c) \Gamma(4 + a - b - c)} \\
& \times \frac{(a - x)(3 + a - 2b)(3 + a - b - c) + (3 + a - b - x)(2 - 3b - 2ac + 2b^2 - ac + 2bc)}{2x(1 - b)(2 - b)(3 - b)} \\
& \times \frac{\Gamma(\frac{5 + a}{2}) \Gamma(4 + a - b) \Gamma(1 + a - c) \Gamma(\frac{4 + a}{2} - b - c) - 1}{\Gamma(a) \Gamma(\frac{5 + a}{2} - b) \Gamma(\frac{4 + a}{2} - c) \Gamma(4 + a - b - c)}.
\end{aligned}
\]

Calculating, respectively, the two series on the right hand side by \([16]\) and \([19]\), we gain
\[
\begin{aligned}
4F_3 \left[ \begin{array}{c} a, b, c, 1 + x \\ 4 + a - b, 1 + a - c, x \\ 1 \\ \end{array} \right] & = \frac{3 + 2a + a^2 - 4b + b^2 - c - ab - 2ac + bc}{(1 - b)(2 - b)(3 - b)} \\
& \times \frac{\Gamma(\frac{2 + a}{2}) \Gamma(4 + a - b) \Gamma(1 + a - c) \Gamma(\frac{6 + a}{2} - b - c)}{\Gamma(1 + a) \Gamma(\frac{4 + a}{2} - b) \Gamma(\frac{2 + a}{2} - c) \Gamma(4 + a - b - c)} \\
& \times \frac{11 + 6a + a^2 - 12b + 3b^2 + 2b - ac - 6bc}{2(1 - b)(2 - b)(3 - b)} \\
& \times \frac{\Gamma(\frac{4 + a}{2}) \Gamma(4 + a - b) \Gamma(1 + a - c) \Gamma(\frac{5 + a}{2} - b - c)}{\Gamma(a) \Gamma(\frac{5 + a}{2} - b) \Gamma(\frac{4 + a}{2} - c) \Gamma(4 + a - b - c)}.  \\
\end{aligned}
\]

When the parameter \(x\) is specified, the last equation can offer the following two results:
\[
\begin{aligned}
3F_2 \left[ \begin{array}{c} a, b, c \\ 4 + a - b, 1 + a - c \\ 1 \\ \end{array} \right] & = \frac{b^3 + 2b^2(c - 3) + b(11 - 7c - 2ac + c^2) + c(5 + 3a + a^2 - c - 2ac)}{(b - 1)(b - 2)(b - 3)} \\
& \times \frac{\Gamma(\frac{2 + a}{2}) \Gamma(4 + a - b) \Gamma(1 + a - c) \Gamma(\frac{4 + a}{2} - b - c)}{\Gamma(1 + a) \Gamma(\frac{4 + a}{2} - b) \Gamma(\frac{2 + a}{2} - c) \Gamma(4 + a - b - c)}.
\end{aligned}
\] (20)
In terms of (11), we achieve

\[ b^3 - 2b^2(c + 3) + b(11 + 7c + 2ac - 3c^2) + c^2(5 + 2a) - c(5 + 3a + a^2) - 6 \]

\[ \frac{2(b - 1)(b - 2)(b - 3)}{\Gamma(\frac{4 + a}{2})\Gamma(4 + a - b)\Gamma(a - c)\Gamma(\frac{5 + a}{2} - b - c)} \]

\[ \times \frac{\Gamma(\frac{4 + a}{2})\Gamma(4 + a - b)\Gamma(a - c)\Gamma(\frac{5 + a}{2} - b - c)}{\Gamma(a)\Gamma(\frac{4 + a}{2} - b)\Gamma(\frac{5 + a}{2} - b)\Gamma(4 + a - b - c)} \]  \( (21) \)

It is not difficult to verify the continuous relation

\[ \sum_{F_2} \left[ \begin{array}{c} a, b, c \\ 5 + a - b, a - c \end{array} \right] 1 = \frac{(4 + a - b)c}{(a - c)(b - 4)} \sum_{F_2} \left[ \begin{array}{c} a, b, c \\ 4 + a - b, 1 + a - c \end{array} \right] 1 \]

\[ + \frac{a(b - c - 4)}{(a - c)(b - 4)} \sum_{F_2} \left[ \begin{array}{c} a, b, c \\ 5 + a - b, 1 + a - c \end{array} \right] 1 \]  \( (20) \) and \( (21) \), we have

\[ \sum_{F_2} \left[ \begin{array}{c} a, b, c \\ 5 + a - b, a - c \end{array} \right] 1 \]

\[ = \left( \frac{(4 + a - b)c}{(a - c)(b - 4)} \right) \sum_{F_2} \left[ \begin{array}{c} a, b, c \\ 4 + a - b, 1 + a - c \end{array} \right] 1 \]

\[ \times \sum_{F_2} \left[ \begin{array}{c} a, b, c \\ 5 + a - b, 1 + a - c \end{array} \right] 1 \]

\[ \times \frac{2(b - 1)(b - 2)(b - 3)(b - 4)}{\Gamma(5 + a - b)\Gamma(a - c)\Gamma(\frac{5 + a}{2} - b - c)} \]

\[ \times \frac{\Gamma(\frac{5 + a}{2} - b)\Gamma(\frac{5 + a}{2} - c)\Gamma(5 + a - b - c)}{\Gamma(4 + a - b)\Gamma(\frac{5 + a}{2} - b)\Gamma(\frac{5 + a}{2} - c)\Gamma(5 + a - b - c)} \]  \( (22) \)

In terms of \( (11) \), we achieve

\[ \sum_{F_2} \left[ \begin{array}{c} a, 5 - a, b \\ c, 2b - c \end{array} \right] 1 \]

\[ = \sum_{F_2} \left[ \begin{array}{c} a, c - b, a + c - 5 \\ c, a + b - 5 \end{array} \right] 1 \]  \( (23) \)

Calculating the series on the right hand side by \( (22) \), we attain

\[ \sum_{F_2} \left[ \begin{array}{c} a, 5 - a, b \\ c, 2b - c \end{array} \right] 1 \]

\[ \left( \frac{5a^3 - 35a^2 - 50a + 24 - (b - c)(36 - 15a - 12b - 18c)}{(a - 1)(a - 2)(a - 3)(a - 4)(b - 1)(b - 2)(b - 3)(b - 4)(b - 5)} \right) \]

\[ \times \sum_{F_2} \left[ \begin{array}{c} a, c - b, a + c - 5 \\ c, a + b - 5 \end{array} \right] 1 \]

\[ \times \sum_{F_2} \left[ \begin{array}{c} a, 5 - a, b \\ c, 2b - c \end{array} \right] 1 \]

\[ + \left( \frac{5a^3 - 35a^2 - 50a + 24 - (b - c)(36 - 15a - 12b - 18c)}{(a - 1)(a - 2)(a - 3)(a - 4)(b - 1)(b - 2)(b - 3)(b - 4)(b - 5)} \right) \]

\[ \times \sum_{F_2} \left[ \begin{array}{c} a, c - b, a + c - 5 \\ c, a + b - 5 \end{array} \right] 1 \]  \( (23) \)
It is easy to see that
\[
\sum_{k=0}^{\infty} k^2 \frac{(a)_k(1-a)_k(1+b)_k}{k!(1+c)_k(1+2b-c)_k}
= \sum_{k=1}^{\infty} \frac{(a)_k(1-a)_k(1+b)_k}{(k-1)!k(1+c)_k(1+2b-c)_k}
= \sum_{k=0}^{\infty} (1+k) \frac{(a)_{k+1}(1-a)_{k+1}(1+b)_{k+1}}{k!(1+c)_{k+1}(1+2b-c)_{k+1}}
= \frac{a(1-a)(1-b)}{(1+c)(1+2b-c)} F_2 \left( \begin{array}{c} 1+a, 2-a, 2+b \\ 2+c, 2+2b-c \end{array} \biggm/ 1 \right)
+ \frac{a(1+a)(1-a)(2-a)(1+b)(2+b)}{(1+c)(2+c)(1+2b-c)(2+2b-c)} F_2 \left( \begin{array}{c} 2+a, 3-a, 3+b \\ 3+c, 3+2b-c \end{array} \biggm/ 1 \right).
\]
Evaluating, respectively, the two series on the right hand side by (17) and (23), we obtain Lemma [14] □

**Theorem 15.** Let \( x \) be a complex number. Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(n+k)_k}{(x+k)_k} k^2 H^{(2)}_k(x) = (-1)^n \frac{(n+n^2)(n+n^2-x)}{(2(1-x)(2-x))} \frac{(-x+n)}{(x+n)}
\times \left\{ H_n^{(2)}(x) - H_n^{(2)}(-x) + \left[ H_n(x) - H_n(-x) - 2H_{n+1}(\frac{x-n}{2}) + U_n(x) \right] \right\}
\times \left[ H_n(x) - H_n(-x) \right] + H_n(\frac{x-n}{2}) \left[ H_n(\frac{x-n}{2}) - V_n(x) \right] + W_n(x),
\]
where the expressions on the right hand side are
\[
U_n(x) = 2[(n+n^2)(n+n^2-x) + (1-n-n^2)x^2 + x^3 - x^4],
\]
\[
V_n(x) = 2n^3(1+n)(1+n+x) - n^2(2+n)x^2 - (1-2n-n^2)x^3 - (1+n)x^4 + x^5],
\]
\[
W_n(x) = 2n^2(1+n)^2 - n^2(5+3n)x - 2(1-3n-2n^2)x^2 - 3(1+n)x^3 + 2x^4].
\]

**Proof.** The case \( a = -n, b = x \) and \( c = y \) of Lemma [14] reads as
\[
\sum_{k=0}^{n} (-1)^k k^2 \binom{n}{k} \frac{(n+k)_k}{(y+k)_k} \frac{(x+k)_k}{(2x-y+k)_k} = \Psi_n(x,y),
\]
where the symbol on the right hand side stands for
\[
\Psi_n(x,y) = \frac{y(y-x)(1+x-2xy+y^2) - n(x-3xy + x^2 + 2y^2)}{2x(x-1)(x-2)}
\times (2x-y) \frac{(y-n)_{n-1}}{n} \frac{(y-2x+n)}{n} + \frac{(y-n)_{n-1}}{n} \frac{(y-2x+n)}{n} \frac{(2x-y)(y-n)}{n}
\left( x-y \right) (2x-y) (x-2) + \frac{n^2(1-x+5xy-3x^2-2y^2)}{2x(x-1)(x-2)(2x-y+n)}.
\]
Applying the derivative operator \( D_y \) to both sides of it, we get
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \binom{x+k}{k} k^2 \{ H_k(2x - y) - H_k(y) \} = D_y \Psi_n(x,y).
\]

The last equation can be reformulated as
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \binom{x+k}{k} k^2 \sum_{i=1}^{k} \frac{1}{(2x - y + i)(y + i)} = \frac{D_y \Psi_n(x,y)}{2(y - x)}.
\]

Finding the limit \( y \to x \) of it by using the relation
\[
\lim_{y \to x} \frac{D_y \Psi_n(x,y)}{2(y - x)} = \lim_{y \to x} \frac{D^2 \Psi_n(x,y)}{2}
\]
\[
= (-1)^n \frac{(n + n^2)(n + n^2 - x) (-x+n)}{2(1-x)(2-x)} \frac{n+n}{x+n}
\]
\[
\times \left\{ H_n^{(2)}(x) - H_n^{(2)}(-x) + \left[ H_n(x) - H_n(-x) - 2H_{n+1}(\frac{x-n}{2}) + U_n(x) \right] \right.
\]
\[
\times \left[ H_n(x) - H_n(-x) \right] + H_n\left(\frac{x-n}{2}\right) \left[ H_n\left(\frac{x-n}{2}\right) - V_n(x) \right] + W_n(x) \right\}
\]

from L’Hôpital rule, we gain Theorem 15. \( \square \)

When \( x \to p \), where \( p \) a nonnegative integer, Theorem 15 can produce the following four corollaries.

**Corollary 16 (Harmonic number identity).**
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \binom{x+k}{k} k^2 H_k^{(2)} = (-1)^n \frac{n^2(n+1)^2}{2}
\]
\[
\times \left\{ \begin{array}{ll}
2H_n^{(2)} - H_n^{(2)} - \frac{1}{n+1}, & n = 0 \pmod{2}; \\
2H_n^{(2)} - H_n^{(2)} - \frac{x-n}{n+1}, & n = 1 \pmod{2}.
\end{array} \right.
\]

**Corollary 17.** Let \( p \) be a positive integer satisfying \( 0 < p \leq n \). Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} k^2 H_k^{(2)} p_k = \frac{(n + n^2)(n + n^2 - p)}{p(p-1)(p-2)} \binom{n^2-p}{p}
\]
\[
\times \left\{ \begin{array}{ll}
H_{n+p} - H_{n-p} - H_{n+p-n} + H_{n-p-n} + \frac{p(1-n-n^2+p-p^2)}{(n+n)(n+n-p)}, & n - p = 0 \pmod{2}; \\
H_{n+p} - H_{n-p} - H_{n+p-n} + H_{n-p-n} - \frac{p(1-n-n^2+p-p^2)}{(n+n)(n+n-p)}, & n - p = 1 \pmod{2}.
\end{array} \right.
\]

**Corollary 18 (Harmonic number identity).**
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^2 H_k^{(2)} p_k = \frac{n^2(n+1)}{2(n-1)(n-2)} \frac{1}{\binom{n}{p}} \left\{ H_{n+1} - H_{2n} + \frac{n-1}{n^2} \right\}.
\]

**Corollary 19.** Let \( p \) be a positive integer with \( p > n \). Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} k^2 H_k^{(2)} p_k = \frac{(n + n^2)(n + n^2 - p)}{2(p-1)(p-2)} \binom{p-1}{n}
\]
\[
\times \left\{ \begin{array}{ll}
H_{p+n}^{(2)} + H_{p-n}^{(2)} + C(p, n) [C(p, n) + D(p, n)] + \frac{2(p-2)}{(n+n^2-p)(n-p)} \right\},
\end{array} \right.
\]
where the symbols on the right hand side stand for

\[ C(p, n) = \begin{cases} 
H_{p+n} - H_{p-n} - H_{\frac{p-n}{2}} + H_{\frac{p-n}{2}}, & p - n = 0 \pmod{2}; \\
H_{p-n} - H_{p+n} - H_{\frac{p-n}{2}} + H_{\frac{p+n}{2}}, & p - n = 1 \pmod{2},
\end{cases} \]

\[ D(p, n) = \frac{2[n^2(1+n)^2 - n^2(2+n)p - (1-2n-n^2)p^2 - (1+n)p^3 + p^2]}{(n+n^2)(n+n^2-p)(n-p)}. \]

3. The second family of summation formulae involving generalized harmonic numbers

**Theorem 20.** Let \( x \) be a complex number. Then

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k+1} H_k^2(x) = \frac{(-1)^n}{2} \binom{-x+n}{x+n} \left\{ H_n^{(2)}(x) - H_n^{(2)}(-x) \right\} + 2 [H_n(x) + H_n(-x)]^2 - \left[ H_n(x) - H_n(-x) - H_n(\frac{x-n}{2}) \right] 
\times \left[ H_n(x) - H_n(-x) - H_n(\frac{x-n}{2}) - \frac{2(x+n)}{x-2n} \right] - \frac{4n}{x(x+n)}. \]

**Proof.** Fix \( y = x \) in (33) to achieve

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k+1} H_k^2(x) = (-1)^n \frac{(-x+n)}{(x+n)}. \tag{25}
\]

Applying the derivative operator \( D_y^2 \) to both sides of it, we attain

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k+1} \left\{ H_k^2(x) + H_k^{(2)}(x) \right\} = (-1)^n \frac{n}{x+n} \left\{ [H_n(x) + H_n(-x)]^2 + H_n^{(2)}(x) - H_n^{(2)}(-x) \right\}.
\]

The difference of the last equation and Theorem 3 gives Theorem 20. \qed

When \( x \to p \), where \( p \) a nonnegative integer, we can derive the following four corollaries from Theorem 20.

**Corollary 21** (Harmonic number identity).

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} H_k^2 = \begin{cases} 
(-1)^n \left\{ 4H_n^2 - 2H_n^{(2)} + H_n^2 \right\}, & n = 0 \pmod{2}; \\
(-1)^n \left\{ 4H_n^2 - 2H_n^{(2)} + H_{n-1}^2 \right\}, & n = 1 \pmod{2}.
\end{cases}
\]

**Corollary 22.** Let \( p \) be a positive integer satisfying \( 0 < p \leq n \). Then

\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} H_{p+k}^2 = \frac{(-1)^{n-p+1}}{p} \left\{ \begin{array}{ll}
H_{n+p} + 3H_{n-p} - 2H_{p-1} + H_{a_{n+p}} - H_{a_{n-p}}, & n - p = 0 \pmod{2}; \\
H_{n+p} + 3H_{n-p} - 2H_{p-1} + H_{a_{n+p-1}} - H_{a_{n-p-1}}, & n - p = 1 \pmod{2}.
\end{array} \right.
\]

**Corollary 23** (Harmonic number identity).

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} H_{n+k}^2 = \frac{1}{n} \left\{ \begin{array}{ll}
H_n - H_{2n} - \frac{2}{n}, & n = 0 \pmod{2}; \\
H_n - H_{2n} - \frac{2}{n}, & n = 1 \pmod{2}.
\end{array} \right.
\]
Corollary 24. Let $p$ be a positive integer with $p > n$. Then

$$
\sum_{k=0}^{n} (-1)^{k} \binom{n+k}{k} \binom{p+n}{n-k} H_{p+k}^2 = \frac{1}{2} \binom{p-1}{n} + 2 \frac{H_{n}^{(2)} + H_{p-n}^{(2)} - 2H_p^{(2)} + 2H_p^2}{p(p-n)} H_p - \frac{4n}{p(p-n)} H_p - \frac{4n}{p^2(p-n)}.
$$

where the expression $A(p, n)$ on the right hand side has appeared in Corollary 7.

Theorem 25. Let $x$ be a complex number. Then

$$
\sum_{k=0}^{n} (-1)^{k} \binom{n+k}{k} \binom{p+n}{n-k} k H_k^2(x) = (-1)^n \frac{n(n+1)}{1-x} \binom{x+n}{n} - 2 \frac{H_n(x) + H_n(-x)}{x} - 2 \frac{H_{n+1}(x-\frac{x-n}{2}) - 2(x^2-n-n^2)}{x(n+1)}
$$

The difference of the last equation and Theorem 24 offers Theorem 25. $\Box$

When $x \to p$, where $p$ a nonnegative integer, we can deduce the following four corollaries from Theorem 25.

Corollary 26 (Harmonic number identity).

$$
\sum_{k=0}^{n} (-1)^{k} \binom{n+k}{k} \binom{p+n}{n-k} k H_k^2 = (-1)^n n(n+1)
$$

$$
\times \left\{ \begin{array}{lr} 4H_n^2 - 4H_n - 2H_n^{(2)} + H_n^{(2)} + 2, & n \equiv 0 \mod 2; \\
4H_n^2 - 4H_n - 2H_{n+1}^{(2)} + H_{n+1}^{(2)} + \frac{2(n^3+2n^2+n+1)}{n(n+1)^2}, & n \equiv 1 \mod 2. \end{array} \right.
$$

Corollary 27. Let $p$ be a positive integer satisfying $0 < p \leq n$. Then

$$
\sum_{k=0}^{n} (-1)^{k} \binom{n+k}{k} \binom{p+n}{n-k} k H_{p+k}^2 = \frac{n(n+1)}{p(p-1)} \binom{1-n-p}{n}
$$

$$
\times \left\{ \begin{array}{lr} H_{n+p} + 3H_{n-p} - 2H_{p-2} + \frac{2p}{n(n+1)}, & n \equiv 0 \mod 2; \\
H_{n+p} + 3H_{n-p} - 2H_{p-2} + \frac{2p}{n(n+1)} - \frac{p}{n(n+1)}, & n \equiv 1 \mod 2. \end{array} \right.
$$
Proof. Theorem 15.

Applying the derivative operator \(D\) where the expressions

When

Corollary 31

from Theorem 30.

Let

where the symbol \(B(p, n)\) on the right hand side can be seen in Corollary 28.

Corollary 29.

Let

\[ \sum_{k=0}^{n} (-1)^{k} \binom{n+k}{k} kH_{n+k}^2 = \frac{n+1}{2} \sum_{n=1}^{\frac{2n^2+n-2}{n^3-n}} \left\{ H_{2n} - H_n + \frac{2n^2+n-2}{n^3-n} \right\}. \]

The difference of the last equation and Theorem 15 produces Theorem 25.

Corollary 28 (Harmonic number identity).

\[ \sum_{k=0}^{n} (-1)^{k} \binom{n+k}{k} kH_{n+k}^2 = \frac{n+1}{2} \left( H_{2n} - H_n + \frac{2n^2+n-2}{n^3-n} \right). \]

Corollary 29. Let \( p \) be a positive integer with \( p > n \). Then

\[ \sum_{k=0}^{n} (-1)^{k} \binom{n+k}{k} \left( \binom{p+n}{k} - \binom{n}{k} \right) kH_{p+k}^2 = \frac{n+1}{2} \left( \binom{p+n}{k} - \binom{n}{k} \right) \left( H_{p+n} - H_{p-n-1} - H_p \right)^2 \]

\[ + H_{p+n}^{(2)} + H_{p-n-1}^{(2)} - 2H_{p-2}^{(2)} - B(p, n) \]

where the expressions \( U_n(x), V_n(x) \) and \( W_n(x) \) on the right hand side have appeared in Theorem 17.

Theorem 30. Let \( x \) be a complex number. Then

\[ \sum_{k=0}^{n} (-1)^{k} \binom{n+k}{k} k^2H_{n+k}^2(x) = (-1)^{n} \binom{n}{k} (n + n^2)(n + n^2 - x) \frac{(-x+n)}{2(1-x)(2-x)} \frac{(x+n)}{n}. \]

Applying the derivative operator \(D^2\) to both sides of it, we attain

\[ \sum_{k=0}^{n} (-1)^{k} \binom{n+k}{k} k^2 \left\{ H_{n+k}^2(x) + H_k^{(2)}(x) \right\} = (-1)^{n} \binom{n}{k} \left( n + n^2 \right) \left( n + n^2 - x \right) \left( -x+n \right) \frac{1}{2-3x+x^2} \frac{1}{n}. \]

\[ \times \left\{ [H_n(x) + H_{n-2}(2-x)] [H_n(x) + H_{n-2}(2-x) + \frac{2}{n+n^2-x}] \right\} \]

\[ + H_n^{(2)}(x) - H_{n-2}^{(2)}(2-x). \]

The difference of the last equation and Theorem 17 produces Theorem 25.

When \( x \to p \), where \( p \) a nonnegative integer, we can derive the following four corollaries from Theorem 30.

Corollary 31 (Harmonic number identity).

\[ \sum_{k=0}^{n} (-1)^{k} \binom{n+k}{k} k^2H_k^{(2)} = (-1)^{n} \frac{n^2(n+1)^2}{2} \]

\[ \times \left\{ 4H_n^2 + \frac{6n^2+6n-4}{n(n+1)} H_n - 2H_n^{(2)} + H_n^{(2)} \right\} \frac{1}{2} \frac{7n^2+7n-4}{2n(n+1)} + 2, \quad n \equiv 0 \pmod{2}. \]

\[ 4H_{n+1}^2 - \frac{6n^2+6n-4}{n(n+1)} H_n - 2H_n^{(2)} + H_n^{(2)} \right\} \frac{1}{2} \frac{7n^2+7n-4}{2n^2(n+1)^2}, \quad n \equiv 1 \pmod{2}. \]
Corollary 32. Let $p$ be a positive integer satisfying $0 < p \leq n$. Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} k^2 H_{p+k}^2 = \frac{(n+n^2)(n+n^2-p)}{p(p-1)(p-2)} \\
\times \left\{ H_{n+p} - H_{n-p} - \frac{p(1+p-p^2)-(2+p)(n+n^2)}{(n+n^2)(n+n^2-p)} \right\}
\times \left\{ H_{n+p-1} - H_{n-p-1} + \frac{p(1+p-p^2)+(2+p)(n+n^2)}{(n+n^2)(n+n^2-p)} \right\},
\]
where the expressions on the right hand side can be seen in Corollary 33 (Harmonic number identity).

Corollary 33. Let $p$ be a positive integer with $p > n$. Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} k^2 H_{n+k}^2 = \frac{n^2(n+1)}{(n-1)(n-2)} \left\{ H_{n} - H_{2n} - \frac{3n+1}{n^2} - \frac{5n^2-5n-4}{n^3-2n^2-2n+1} \right\}.
\]

Corollary 34. Let $p$ be a positive integer with $p > n$. Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n+k}{k} \binom{p+n}{n-k} k^2 H_{p+k}^2 = \frac{(n+n^2)(n+n^2-p)}{2(p-1)(p-2)} \left\{ H_{p+n} - H_{p-n} - 2H_{p-2}^2 + 2\left( H_{p+n} + H_{p-n-1} - H_{p-3} + \frac{2}{n+n^2-p} \right) \right\}
\times \left\{ H_{p+n} + H_{p-n-1} - H_{p-3} - C(p,n) \left[ C(p,n) + D(p,n) \right] \right\}
\times \left\{ 2(1-2n(1+n)+(2+n)p-p^2) \right\}
\times \left\{ (1+n)(n+n^2-p)(n-p) - \frac{2}{(p-n)^2} \right\},
\]
where the expressions $C(p,n)$ and $D(p,n)$ on the right hand side can be seen in Corollary 32.

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