Nil Geodesic Triangles and Their Interior Angle Sums

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Abstract In this paper we study the interior angle sums of geodesic triangles in Nil geometry and prove that these can be larger, equal or less than $\pi$. We use for the computations the projective model of Nil introduced by Molnár (Beitr. Algebra Geom. 38(2):261–288, 1997).

Keywords Thurston geometries · Nil geometry · Geodesic triangles · Interior angle sum

Mathematics Subject Classification 53A20, 53A35, 52C35, 53B20

1 Introduction

A geodesic triangle in Riemannian geometry and more generally in metric geometry is a figure consisting of three different points together with the pairwise connecting geodesic curves. The points are called vertices, while the geodesic curves are called sides of the triangle.

In the geometries (of constant curvature) $E^3$, $H^3$, $S^3$ the well-known sums of the interior angles of geodesic triangles characterize the space. It is related to the Gauss-Bonnet theorem which states that the integral of the Gauss curvature on a compact 2-dimensional Riemannian manifold $M$ is equal to $2\pi \chi(M)$ where $\chi(M)$ denotes the Euler characteristic of $M$. This theorem has a generalization to any compact even-

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dimensional Riemannian manifold [see e.g. Chavel (2006), Kobayashi and Nomizu (1963)].

However, in the other 3-dimensional homogeneous maximal Riemann spaces (Thurston geometries) there are few results concerning the angle sums of geodesic triangles. Therefore, it is interesting to study similar question in the other five Thurston geometries, \( S^2 \times \mathbb{R}, H^2 \times \mathbb{R}, \text{Nil}, \text{SL}_2 \mathbb{R}, \text{Sol} \).

In Csima and Szirmai (2017) we investigated the angle sum of translation and geodesic triangles in \( \text{SL}_2 \mathbb{R} \) geometry and proved that the possible sum of the interior angles in a translation triangle must be greater or equal than \( \pi \). However, in geodesic triangles this sum is less, greater or equal to \( \pi \).

In this paper we consider the analogous problem in \( \text{Nil} \) geometry.

**Remark 1.1** We note here, that nowadays \( \text{Nil} \) geometry is a widely investigated space concerning the interesting surfaces, tilings, geodesic and translation ball packings (see e.g. Brodaczewska (2014), Inoguchi (2012), Molnár et al. (2009), Molnár and Szirmai (2006), Pallagi et al. (2011), Schultz and Szirmai (2012), Szirmai (2007), Szirmai (2012) and the references given there).

In Brodaczewska (2014) K. Brodaczewska showed, that the sum of the interior angles of translation triangles of \( \text{Nil} \) space is larger or equal than \( \pi \).

Now, we are interested in geodesic triangles in \( \text{Nil} \) space that is one of the eight Thurston geometries [see Scott (1983) and Thurston (1997)] on the base of Heisenberg matrix group. In Sect. 2 we describe the projective model of \( \text{Nil} \) and we shall use its standard Riemannian metric obtained by pull back transform to the infinitesimal arc-length-square at the origin. We also recall the isometry group of \( \text{Nil} \) and give an overview about geodesic curves.

In Sect. 3 we study the \( \text{Nil} \) geodesic triangles and prove that the interior angle sum of a geodesic triangle in \( \text{Nil} \) geometry can be larger, equal or less than \( \pi \).

## 2 Basic Notions of the Nil Geometry

In this section we summarize the significant notions and denotations [see Molnár (1997), Szirmai (2007)].

\( \text{Nil} \) geometry is a homogeneous 3-space derived from the famous real matrix group \( \text{L}(\mathbb{R}) \), used by W. Heisenberg in his electro-magnetic studies. The Lie theory with the method of projective geometry makes possible to describe this topic.

The left (row–column) multiplication of Heisenberg matrices

\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix} =
\begin{pmatrix}
1 & a + x & c + xb + z \\
0 & 1 & b + y \\
0 & 0 & 1
\end{pmatrix}
\quad (2.1)
\]

defines “translations” \( \text{L}(\mathbb{R}) = \{(x, y, z) : x, y, z \in \mathbb{R}\} \) on the points of \( \text{Nil} = \{(a, b, c) : a, b, c \in \mathbb{R}\} \). These translations are not commutative, in general. The matrices \( \text{K}(z) \triangleleft \text{L} \) of the form

\[ \text{K}(z) = \begin{pmatrix} 1 & z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
constitute the one parametric centre, i.e. each of its elements commutes with all elements of $\mathbf{L}$. The elements of $\mathbf{K}$ are called fibre translations. \textbf{Nil} geometry of the Heisenberg group can be projectively (affinely) interpreted by the “right translations” on points as the matrix formula

$$
(1; a, b, c) \rightarrow (1; a, b, c) \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} = (1; x + a, y + b, z + bx + c)
$$

shows, according to (2.1). Here we consider $\mathbf{L}$ as projective collineation group with right actions in homogeneous coordinates. We will use the Cartesian homogeneous coordinate simplex $E_0(e_0), E_1\infty(e_1), E_2\infty(e_2), E_3\infty(e_3), \{e_i\} \subset \mathbf{V}^4$ with the unit point $E(e = e_0 + e_1 + e_2 + e_3)$ which is distinguished by an origin $E_0$ and by the ideal points of coordinate axes, respectively. Moreover, $y = cx$ with $0 < c \in \mathbf{R}$ (or $c \in \mathbf{R} \setminus \{0\}$) defines a point $(x) = (y)$ of the projective 3-sphere $\mathcal{P}^3$ (or that of the projective space $\mathcal{P}^3$ where opposite rays $(x)$ and $(-x)$ are identified). The dual system $\{e^i\} \subset \mathbf{V}^4$ describes the simplex planes, especially the plane at infinity $(e^0) = E_1\infty E_2\infty E_3\infty$. In general, $v = u^\perp$ defines a plane $(u) = (v)$ of $\mathcal{P}^3$ (or that of $\mathcal{P}^3$). Thus $0 = xu = yv$ defines the incidence of point $(x) = (y)$ and plane $(u) = (v)$. Thus \textbf{Nil} can be visualized in the affine 3-space $\mathbf{A}^3$ (so in $\mathbf{E}^3$) as well.

In this context Molnár (1997) has derived the well-known infinitesimal arc-length-square, invariant under translations $\mathbf{L}$ at any point of \textbf{Nil} as follows

$$
(dx)^2 + (dy)^2 + (-xdy + dz)^2 = (dx)^2 + (1 + x^2)(dy)^2 - 2x(dy)(dz) + (dz)^2 =: (ds)^2
$$

Hence we get the symmetric metric tensor field $g$ on \textbf{Nil} by components $g_{ij}$, furthermore its inverse:

$$
g_{ij} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + x^2 & -x \\ 0 & -x & 1 \end{pmatrix}, \quad g^{ij} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & x & 1 + x^2 \end{pmatrix}
$$

with $\det(g_{ij}) = 1$.

The translation group $\mathbf{L}$ defined by formula (2.3) can be extended to a larger group $\mathbf{G}$ of collineations, preserving the fibering, that will be equivalent to the (orientation preserving) isometry group of \textbf{Nil}. In Molnár (2010) E. Molnár has shown that a rotation trough angle $\omega$ about the $z$-axis at the origin, as isometry of \textbf{Nil}, keeping invariant the Riemann metric everywhere, will be a quadratic mapping in $x, y$ to $z$-image $\overline{z}$ as follows:
\[ \mathbf{r}(O, \omega) : (1; x, y, z) \rightarrow (1; \bar{x}, \bar{y}, \bar{z}); \]
\[ \bar{x} = x \cos \omega - y \sin \omega, \quad \bar{y} = x \sin \omega + y \cos \omega, \]
\[ \bar{z} = z - \frac{1}{2} xy + \frac{1}{4} (x^2 - y^2) \sin 2\omega + \frac{1}{2} xy \cos 2\omega. \] (2.6)

This rotation formula, however, is conjugate by the quadratic mapping
\[
\mathcal{M} : x \rightarrow x' = x, \quad y \rightarrow y' = y, \quad z \rightarrow z' = z - \frac{1}{2} xy 
\text{to}
\]
\[ (1; x', y', z') \rightarrow (1; x', y', z') \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \omega & \sin \omega & 0 \\
0 & -\sin \omega & \cos \omega & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = (1; x'', y'', z''), \]
with \( x'' \rightarrow \bar{x} = x'', \quad y'' \rightarrow \bar{y} = y'', \quad z'' \rightarrow \bar{z} = z'' + \frac{1}{2} x'' y''. \) (2.7)

i.e. to the linear rotation formula. This quadratic conjugacy modifies the \textbf{Nil} translations in (2.3), as well. This can also be characterized by the following important classification theorem.

**Theorem 2.1** [Molnár (2010) modified]

1. Any group of \textbf{Nil} isometries, containing a 3-dimensional translation lattice, is conjugate by the quadratic mapping in (2.5) to an affine group of the affine (or Euclidean) space \( \mathbb{A}^3 = \mathbb{E}^3 \) whose projection onto the \((x,y)\) plane is an isometry group of \( \mathbb{E}^2 \). Such an affine group preserves a plane \( \rightarrow \) point null-polarity.

2. Of course, the involutive line reflection about the \( y \) axis
\[ (1; x, y, z) \rightarrow (1; -x, y, -z), \]

preserving the Riemann metric, and its conjugates by the above isometries in 1 (those of the identity component) are also \textbf{Nil}-isometries. There does not exist orientation reversing \textbf{Nil}-isometry.

**Remark 2.2** We obtain a new projective model of \textbf{Nil} geometry from the above projective model, derived by the above quadratic mapping \( \mathcal{M} \). This is the \textit{linearized model of Nil space} (see Brodaczewska (2014), Molnár (2010)) that seems to be more advantageous to the future investigations. But we remain in the classical so called Heisenberg model in this paper.

### 2.1 Geodesic Curves

The geodesic curves of the \textbf{Nil} geometry are generally defined as having locally minimal arc length between their any two (near enough) points. The equation systems of the parametrized geodesic curves \( g(x(t), y(t), z(t)) \) in our model (now by (2.4)) can be determined by the Levy-Civita theory of Riemann geometry. We can assume, that
the starting point of a geodesic curve is the origin because we can transform a curve into an arbitrary starting point by translation (2.1):

\[ x(0) = y(0) = z(0) = 0; \quad \dot{x}(0) = c \cos \alpha, \quad \dot{y}(0) = c \sin \alpha, \]

\[ \dot{z}(0) = w; \quad -\pi \leq \alpha \leq \pi. \]

The arc length parameter \( s \) is introduced by

\[ s = \sqrt{c^2 + w^2 \cdot t}, \quad \text{where} \quad w = \sin \theta, \quad c = \cos \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \]

i.e. unit velocity can be assumed.

The equation systems of a helix-like geodesic curves \( g(x(t), y(t), z(t)) \) if \( 0 < |w| < 1 \):

\[ x(t) = \frac{2c}{w} \sin \frac{w t}{2} \cos \left( \frac{w t}{2} + \alpha \right), \quad y(t) = \frac{2c}{w} \sin \frac{w t}{2} \sin \left( \frac{w t}{2} + \alpha \right), \]

\[ z(t) = wt \cdot \left\{ 1 + \frac{c^2}{2w^2} \left[ (1 - \frac{\sin(2wt + 2\alpha) - \sin 2\alpha}{2wt}) - \left( 1 - \frac{\sin(wt + 2\alpha) - \sin 2\alpha}{2wt} \right) \right] \right\} \]

\[ = wt \cdot \left\{ 1 + \frac{c^2}{2w^2} \left[ (1 - \frac{\sin(wt)}{wt}) + \left( 1 - \cos(2wt) \right) \sin(wt + 2\alpha) \right] \right\}. \]

(2.8)

In the cases \( w = 0 \) the geodesic curve is the following:

\[ x(t) = c \cdot t \cos \alpha, \quad y(t) = c \cdot t \sin \alpha, \quad z(t) = \frac{1}{2} c^2 \cdot t^2 \cos \alpha \sin \alpha. \]

(2.9)

The cases \( |w| = 1 \) are trivial: \( (x, y) = (0, 0), \quad z = w \cdot t. \)

**Definition 2.3** The distance \( d(P_1, P_2) \) between the points \( P_1 \) and \( P_2 \) is defined by the arc length of geodesic curve from \( P_1 \) to \( P_2.\)

### 3 Geodesic Triangles

We consider 3 points \( A_1, A_2, A_3 \) in the Heisenberg model of \( \text{Nil} \) space (see Sect. 2). The geodesic segments \( a_k \) between the points \( A_i \) and \( A_j \) \( (i < j, \quad i, j, k \in \{1, 2, 3\}, \quad k \neq i, j) \) are called sides of the geodesic triangle with vertices \( A_1, A_2, A_3 \) (see Fig. 1).

In Riemannian geometries the metric tensor (see (2.5)) is used to define the angle \( \theta \) between two geodesic curves. If their tangent vectors in their common point are \( \mathbf{u} \) and \( \mathbf{v} \) and \( g_{ij} \) are the components of the metric tensor then

\[ \cos(\theta) = \frac{u_i g_{ij} v_j}{\sqrt{u_ig_{ij}u_j v_i g_{ij}v_j}} \]

(3.1)
Fig. 1 Two different views of geodesic triangle with vertices $A_1 = (1, 0, 0, 0)$, $A_2 = (1, 1/2, -1, 1)$, $A_3 = (1, 1/3, 2, 1)$ where its interior angle sum is $\approx 3.45294 > \pi$.

It is clear by the above definition of the angles and by the metric tensor (2.5), that the angles are the same as the Euclidean ones at the origin by pull back translation.

We note here that the angle of two intersecting geodesic curves depend on the orientation of the tangent vectors. We will consider the interior angles of the triangles that are denoted at the vertex $A_i$ by $\omega_i$ ($i \in \{1, 2, 3\}$).

### 3.1 Fibre-Like Right-Angled Triangles

A geodesic triangle is called fibre-like if one of its edges lies on a fibre line. In this section we study the right-angled fibre-like triangles. We can assume without loss of generality that the vertices $A_1, A_2, A_3$ of a fibre-like right-angled triangle (see Fig. 2.a) have the following coordinates:

$$
\begin{align*}
A_1 &= (1, 0, 0, 0), \\
A_2 &= (1, 0, 0, z^2), \\
A_3 &= (1, x^3, 0, z^3 = z^2)
\end{align*}
$$

The geodesic segment $A_2A_3$ lies on a straight line, parallel to $x$ axis. The geodesic segment $A_1A_2$ lies on the $z$ axis and their angle is $\omega_2 = \frac{\pi}{2}$ in the Nil space (this angle is in Euclidean sense also $\frac{\pi}{2}$) (see Fig. 2).

In order to determine the further interior angles, we define translations $T_{A_i}$, ($i \in \{2, 3\}$) as elements of the isometry group of Nil, that maps the origin $E_0$ onto $A_i$ (see Fig. 2). E.g. the isometry $T_{A_3}$ and its inverse (up to a positive determinant factor) can be given by:

$$
T_{A_3} = \begin{pmatrix}
1 & x^3 & 0 & z^3 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & x^3 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad T_{A_3}^{-1} = \begin{pmatrix}
1 & -x^3 & 0 & -z^3 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -x^3 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

and the images $T_{A_3}(A_i)$ of the vertices $A_i$ ($i \in \{1, 2, 3\}$) are the following:
Our aim is to determine angle sum $\sum_{i=1}^{3}(\omega_i)$ of the interior angles of the above right-angled fibre-like geodesic triangle $A_1A_2A_3$. We have seen that $\omega_2 = \frac{\pi}{2}$ and the angle of geodesic curves with common point at the origin $E_0$ is the same as the Euclidean one. Therefore it can be determined by usual Euclidean sense. Hence, $\omega_1$ is equal to the angle $\angle(g(E_0, A_3), g(E_0, A_2))$ (see Fig. 2.a) where $g(E_0, A_3)$ and $g(E_0, A_2)$ are oriented geodesic curves. Moreover, the translation $T_{A_3}$ is isometry in Nil geometry thus $\omega_3$ is equal to the angle $\angle(g(A_3^3, A_1^3), g(A_3^3, A_2^3))$ (see Fig. 2.b) where $g(A_3^3, A_1^3)$ and $g(A_3^3, A_2^3)$ are also oriented geodesic curves ($E_0 = A_3^3$).

We denote the oriented unit tangent vectors of the geodesic curves $g(E_0, A_i^j)$ with $t_i^j$ where $(i, j) \in \{(1, 3), (2, 3), (3, 0), (2, 0)\}$ and $A_3^0 = A_3, A_2^0 = A_2$. The Euclidean coordinates of $t_i^j$ (see Sect. 2.1) are:

$$t_i^j = (\cos(\theta_i^j), \cos(\alpha_i^j), \cos(\theta_i^j) \sin(\alpha_i^j), \sin(\theta_i^j)).$$

(3.5)

Lemma 3.1 The sum of the interior angles of a fibre-like right-angled geodesic triangle is greater or equal to $\pi$.

Proof It is clear, that $t_2^0 = (0, 0, 1)$ and $t_2^3 = (-1, 0, 0)$. Moreover, the points $A_3$ and $A_4^3$ are antipodal to the origin $E_0$ (see Fig. 3) therefore the equation $|\theta_2^0| = |\theta_2^3|$ holds i.e. the angle between the vector $t_2^0$ and $[x, y]$ plane are equal to the angle between the vector $t_2^3$ and the $[x, y]$ plane (see Fig. 4). Moreover, we have seen, that $\omega_2 = \frac{\pi}{2}$. That means, that $\omega_1 = \frac{\pi}{2} - |\theta_3^0| = \frac{\pi}{2} - |\theta_3^3|$. The vector $t_3^2$ lies in the $[x, y]$ plane therefore
The points $A_3$ and $A_3^1$ are antipodal related to the origin $E_0$ ($A_1 = (1, 0, 0, 0)$, $A_2 = (1, 0, 0, \frac{1}{2})$, $A_3 = (1, 4, 0, \frac{1}{2})$.)

The angle between the vector $t_0^3$ and the $[x, y]$ plane are equal to the angle between the vector $t_1^3$ and the $[x, y]$ plane, $|\theta_0^3| = |\theta_1^3|$. The angle $\omega_3$ is greater or equal than $|\theta_0^3| = |\theta_1^3|$. Finally we obtain, that

$$\sum_{i=1}^{3} (\omega_i) = \frac{\pi}{2} + \frac{\pi}{2} - |\theta_0^3| + \omega_3 \geq \pi.$$

**Conjecture 3.2** *The sum of the interior angles of any fibre-like geodesic triangle is greater or equal to $\pi$.***

We fix the coordinates the $z_2 = z_3 \in \mathbf{R}$ of $A_2$ and $A_3$ and study the interior angle sum $\sum_{i=1}^{3} (\omega_i(x^3))$ of the right-angled geodesic triangle $A_1A_2A_3$ if $x^3$ coordinate of $A_3$ tends to zero or infinity. E.g. if $x_3 \to 0$ then $\lim_{x_3 \to 0} (\omega_1(x^3)) = 0$ because the
geodesic line \( g(E_0,A_3) \) tends to the geodesic line \( g(E_0,A_2) \) therefore their angle \( \omega_1 \) tends to the zero, and \( \omega_3 \) tends to \( \frac{\pi}{2} \) (see Fig. 2.a). Similarly to this from the system of equations (2.8) we obtain the following results

**Lemma 3.3** If the coordinates \( z^2 = z^3 = 1/2 \)

\[
\lim_{x^3 \to 0} (\omega_1(x^3)) = 0, \quad \lim_{x^3 \to 0} (\omega_3(x^3)) = \frac{\pi}{2} \Rightarrow \lim_{x^3 \to 0} \left( \sum_{i=1}^{3} (\omega_i(x^3)) \right) = \pi,
\]

\[
\lim_{x^3 \to \infty} (\omega_1(x^3)) = \frac{\pi}{2}, \quad \lim_{x^3 \to \infty} (\omega_3(x^3)) = 0 \Rightarrow \lim_{x^3 \to \infty} \left( \sum_{i=1}^{3} (\omega_i(x^3)) \right) = \pi.
\]

In the Table 1 we summarize some numerical data of fibre-like geodesic triangles for given parameters.

### 3.1.1 Hyperbolic-Like Right Angled Geodesic Triangles

A geodesic triangle is hyperbolic-like if its vertices lie in the base plane (i.e. \([x, y]\) coordinate plane) of the model. In this section we analyse the interior angle sum of the right-angled hyperbolic-like triangles. We can assume without loss of generality that the vertices \( A_1, A_2, A_3 \) of a hyperbolic-like right-angled triangle (see Fig. 5) \( T_g \) have the following coordinates:

\[
A_1 = (1, 0, 0, 0), \quad A_2 = (1, 0, y^2, 0), \quad A_3 = (1, x^3, y^2 = y^3, 0) \quad (3.6)
\]

The geodesic segment \( A_1A_2 \) lies on the \( y \) axis, the geodesic segment \( A_2A_3 \) lies parallel to the \( x \) axis containing the point \( A_2 \). It is clear that \( \omega_2 = \frac{\pi}{2} \) in the \( \text{Nil} \) space (this angle is in Euclidean sense also \( \frac{\pi}{2} \)).

**In order to determine the further interior angles of the fibre-like geodesic triangle** \( A_1A_2A_3 \) similarly to the fibre-like case we define the translation \( T_{A_3} \), [see (2.6)] that maps the origin \( E_0 \) onto \( A_3 \) that can be given by:
Fig. 5  a Hyperbolic-like geodesic triangle $A_1A_2A_3$, where $A_1 = (1, 0, 0, 0)$, $A_2 = (1, 0, 3, 0)$, $A_3 = (1, \frac{3}{2}, 3, 0)$.  b Its translated image $A_1^3A_2^3A_3^3$ where $A_1^3 = (1, -\frac{1}{2}, -3, \frac{3}{2})$, $A_2^3 = (1, -\frac{1}{2}, 0, 0)$, $A_3^3 = (1, 0, 0, 0)$.

\[
T_{A_3} = \begin{pmatrix}
1 & x^3 & y^2 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & x^3 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad T_{A_3}^{-1} = \begin{pmatrix}
1 & -x^3 & -y^2 & x^3y^2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -x^3 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]  

(3.7)

We obtain that the images $T_{A_3}^{-1}(A_i)$ of the vertex $A_i$ ($i \in \{1, 2, 3\}$), are the following (see also Fig. 4):

\[
T_{A_3}^{-1}(A_1) = A_1^3 = (1, -x^3, -y^2, x^3y^2), \quad T_{A_3}^{-1}(A_2) = A_2^3 = (1, -x^3, 0, 0), \quad T_{A_3}^{-1}(A_3) = A_3^3 = E_0 = (1, 0, 0, 0).
\]

(3.8)

We study similarly to the above fibre-like case the sum $\sum_{i=1}^{3} (\omega_i)$ of the interior angles of the above right-angled hyperbolic-like geodesic triangle $A_1A_2A_3$.

It is clear, that the angle of geodesic curves with common point at the origin $E_0$ is the same as the Euclidean one therefore it can be determined by usual Euclidean sense. The translation $T_{A_3}$ preserve the measure of angles $\omega_i$ ($i \in \{2, 3\}$) therefore (see Fig. 5) $\omega_3 = \angle(g(A_3^3, A_1^3), g(A_3^3, A_2^3))$ ($A_3^3 = E_0$ and $g(E_0, A_i^3)$ ($i, j \in \{(1, 3), (2, 3)\}$) are oriented geodesic curves).

Similarly to the fibre-like case the Euclidean coordinates of the oriented unit tangent vector $t_i^j$ of the oriented geodesic curves $g(E_0, A_i^j)$ ($i, j \in \{(2, 3), (3, 2), (1, 2), (1, 3)\}$) is given by (3.5).

First we fix the $x^3 \in \mathbb{R}$ coordinate of the vertex $A_3$ and study the the interior angle sum $\sum_{i=1}^{3} (\omega_i(y^2 = y^3))$ of the right-angled geodesic triangle $A_1A_2A_3$ if $y^2 = y^3$ coordinates of vertices $A_2$ and $A_3$ tend to zero or infinity (see Table 2). From the system of equations (2.8) we obtain the following results

\[\text{Springer}\]
Table 2 $x^3 = 1/2$

| $y^2 = y^3$ | $|\theta_3^0| = |\theta_3^1|$ | $d(A_1A_3)$ | $\omega_1$ | $\omega_3$ | $\sum_{i=1}^{3} (\omega_i)$ |
|------------|----------------------------|-------------|--------|--------|-----------------|
| $\rightarrow 0$ | $\rightarrow \pi/2$ | $1/2$ | $\rightarrow \pi/2$ | $\rightarrow 0$ | $\rightarrow \pi$ |
| 1/100 | 0.00490 | 0.50011 | 1.54958 | 0.01940 | 3.13977 |
| 1/3 | 0.13378 | 0.60651 | 0.94882 | 0.56204 | 3.08166 |
| 3 | 0.13700 | 3.09310 | 0.14449 | 1.19813 | 2.91342 |
| 6 | 0.06132 | 6.06701 | 0.11959 | 1.30229 | 2.99268 |
| 20 | 0.00726 | 20.02442 | 0.04828 | 1.47308 | 3.09216 |
| 100 | 0.00030 | 100.00500 | 0.00999 | 1.55082 | 3.13160 |
| $\rightarrow \infty$ | $\rightarrow 0$ | $\rightarrow \infty$ | $\rightarrow 0$ | $\rightarrow \pi/2$ | $\rightarrow \pi$ |

**Lemma 3.4** If the coordinate $x^3 \in \mathbb{R}$ is fixed then

$$\lim_{y^2= y^3 \to 0} \left( \omega_1(y^2) \right) = \frac{\pi}{2}, \quad \lim_{y^2= y^3 \to 0} \left( \omega_3(y^2) \right) = 0 \Rightarrow \lim_{y^2= y^3 \to 0} \left( \sum_{i=1}^{3} (\omega_i(y^2)) \right) = \pi,$$

$$\lim_{y^2= y^3 \to \infty} \left( \omega_1(y^2) \right) = 0, \quad \lim_{y^2= y^3 \to \infty} \left( \omega_3(y^2) \right) = \frac{\pi}{2} \Rightarrow \lim_{y^2= y^3 \to \infty} \left( \sum_{i=1}^{3} (\omega_i(y^2)) \right) = \pi.$$

Secondly we fix the $y^2 = y^3 \in \mathbb{R}$ coordinates of the vertices $A_2$ and $A_3$ and study the internal angle sum $\sum_{i=1}^{3} (\omega_i(x^3))$ of the right-angled geodesic triangle $A_1A_2A_3$ if $x^3$ coordinate of vertex $A_3$ tends to zero or infinity (see Table 3). From the system of equations (2.8) we obtain the following

**Lemma 3.5** If the coordinates $y^2 = y^3 \in \mathbb{R}$ are fixed then

$$\lim_{x^3 \to 0} \left( \omega_1(x^3) \right) = 0, \quad \lim_{x^3 \to 0} \left( \omega_3(x^3) \right) = \frac{\pi}{2} \Rightarrow \lim_{x^3 \to 0} \left( \sum_{i=1}^{3} (\omega_i(x^3)) \right) = \pi,$$

$$\lim_{x^3 \to \infty} \left( \omega_1(x^3) \right) = \frac{\pi}{2}, \quad \lim_{x^3 \to \infty} \left( \omega_3(x^3) \right) = 0 \Rightarrow \lim_{x^3 \to \infty} \left( \sum_{i=1}^{3} (\omega_i(x^3)) \right) = \pi.$$

We can determine the interior angle sum of arbitrary hyperbolic-like geodesic triangle similarly to the fibre-like case. In the following table we summarize some numerical data of geodesic triangles for given parameters:

Finally, we get the following Lemma

**Lemma 3.6** The interior angle sums of hyperbolic-like right-angled geodesic triangles can be less or equal to $\pi$.

**Conjecture 3.7** The sum of the interior angles of any hyperbolic-like geodesic triangle is less or equal to $\pi$. 
Theorem 3.9
The sum of the interior angles of a geodesic triangle can be greater, less or equal to $\pi$.

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