THE STANDARD MODEL AND THE
GENERALIZED COVARIANT DERIVATIVE

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Abstract

The generalized covariant derivative, that uses both scalar and vector bosons, is defined. It is shown how a grand unified theory of the Standard Model can be constructed using a generalized Yang-Mills theory.

1 The generalized covariant derivative

Last year the authors developed a generalized Yang-Mills theory (GYMT) that used a covariant derivative that included not only vector bosons but scalar fields as well. The motivation at the time was to simplify the writing of the multiple terms of the Glashow-Weinberg-Salam (GWS) model using $U(3)$. Our inspiration was an old idea by Fairlie and Ne’eman. The idea is that the Higgs bosons fit neatly in the adjoint of $SU(2/1)$, along with the gauge vector bosons, and that the hypercharges of the leptons are given correctly by one of the diagonal generators of that graded group. This model has two main problems, reviewed by us. We then proposed that the problems of the old model could be resolved if one switched to the $U(3)$ Lie group, since it is possible to obtain the correct quantum numbers for all the particles of the GWS if instead of the usual Gell-Mann representation a different one is used. This new representation is a linear combination of generators of the usual one. In this model an extra scalar boson makes its appearance, but it decouples from all the other particles.

Here we study the application of a GYMT to the building of a grand unified theory (GUT) of the Standard Model at the rank 5 level. It turns out that, at this rank, there is only one possible GUT, and it is based on the group $SU(6)$. The grand unification group has to contain two $SU(3)$’s, one to represent flavodynamics and another to represent chromodynamics. The algebra of $SU(6)$ has $SU(3) \otimes SU(3) \otimes U(1)$ as the group associated with one of its maximal subalgebras. It turns out that this group
gives, using a generalized covariant derivative, all the correct quantum numbers of all the fermions, vector bosons, GWS Higgs and GUT Higgs. Again, as in the GWS case, this does not happen in the usual representation, but in a different one that is a linear combination of the generators of other. It is in this new representation that the particles of the Standard Model appear with their correct quantum numbers.

In what follows we will give a description of how the GUT is constructed and its overall structure. Certain dynamical details are, for the moment, left out, since at this time we are not finished with our calculations.

2 Quick review of the generalized covariant derivative

We define the generalized covariant $D$ to transform as a four-vector contracted with Dirac matrices. Assume we have an expression that is invariant under a Lorentz transformation and contains a contracted 4-vector $\tilde{A}$. If now we were to substitute this contracted vector by $\gamma^5 \varphi$, where $\varphi$ is a scalar field, then, from the properties of the Lorentz spinorial representation the expression would still be Lorentz-invariant.

Consider a Lie group with $N$ generators. Associate $N_V$ generators with an equal number of vector gauge fields $A^a_\mu$, and $N_S$ generators with an equal number of scalar gauge fields $\varphi^b$, with $N = N_V + N_S$. Then the covariant derivative $D$ is defined by

$$D \equiv \partial + \tilde{A} + \Phi,$$

with

$$\tilde{A} \equiv \gamma^\mu A_\mu \equiv ig\gamma^\mu A^a_\mu T^a, \quad a = 1, \ldots, N_V,$$

$$\Phi \equiv \gamma^5 \varphi \equiv -g\gamma^5 \varphi^b T^b, \quad b = N_V + 1, \ldots, N.$$

We take the gauge transformation for these fields to be

$$\tilde{A} + \Phi \rightarrow U(\tilde{A} + \Phi)U^{-1} - (\partial U)U^{-1},$$

therefore, we can have

$$D \rightarrow UDU^{-1}.$$

If the theory is going to contain fermions they must be placed in an irrep that is either the fundamental or at least can be constructed from products involving the fundamental or its conjugate, to assure gauge invariance. This point will be illustrated later. The non-abelian lagrangian is constructed based on the requirements that it should contain only fermion fields and covariant derivatives, and possess both Lorentz and gauge invariance:

$$\mathcal{L}_{NA} = \overline{\psi}iD\psi + \frac{1}{2g^2} \tilde{\text{Tr}} \left( \frac{1}{8} \text{Tr}^2 D^2 - \frac{1}{2} \text{Tr} D^4 \right),$$

2
where the trace with the tilde is over the Lie group matrices and the one without it is over matrices of the spinorial representation of the Lorentz group. The additional factor of 1/2 that the traces of \( \tilde{\text{Tr}} \) have comes from the usual normalization in the non-abelian case
\[
\tilde{\text{Tr}} \ T^a T^b = \frac{1}{2} \delta_{ab}.
\] (5)

If we expand the covariant derivative into its component fields, the lagrangian shows to be made of terms that are traditional in Yang-Mills theories:
\[
\mathcal{L}_{N \Lambda} = \bar{\psi} i (\partial + A) \psi - g \bar{\psi} i \gamma^5 \varphi b T^b \psi + \frac{1}{2g^2} \tilde{\text{Tr}} (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])^2
\]
\[
+ \frac{1}{g^2} \tilde{\text{Tr}} (\partial_\mu \varphi + [A_\mu, \varphi])^2,
\] (6)
where the first term on the right looks like the usual matter term of a gauge theory, the second like a Yukawa term, the third like the kinetic energy of vector bosons in a Yang-Mills theory and the fourth like the gauge-invariant kinetic energy of scalar bosons in the non-abelian adjoint representation.

We call the differential operators in equations (3) and (4) \textit{unrestrained}, because they keep acting indefinitely to the right. However, in the expanded form of (6), after having done all the algebra, the differential operators there are \textit{restrained}, that is, the partial derivatives acts only on the immediately succeeding functions to the right.

Notice that in GYMTs the gauge invariance is given by the full Lie group. That is, the lagrangian is completely invariant under transformation (3). This does not mean that the form of the transformed covariant derivative has to remain exactly the same. Similarly, there are transformations in the group that will mix chiralities. But after doing all the transformations the final result is invariant. Interestingly enough, in both the case of the GWS model of Ref. 1 and the GUT theory studied here, the maximal subgroup maintains the chiralities of the sectors of fermions, and the reason is very clear: it is a Yang-Mills theory. In other words, the GYMTs we have studied contain a typical Yang-Mills theory that uses as Lie group the maximal subgroup of the GYMT.

3 The group generators

The choice of our unifying group is to be guided by the requirement that it should contain $SU(3)_C \otimes U(3)$, where the “C” stands for color. The $U(3)$ is necessary because it contains the GWS model using GYMTs. Therefore the candidate group should be at least rank 5, (= rank 2 due to $SU(3)_C$ plus rank 3 due to $U(3)$.) The smallest such group is $SU(6)$, which contains $SU(3) \otimes SU(3) \otimes U(1)$ as a maximal subgroup, so that the first $SU(3)$ is a color subgroup and $SU(3) \otimes U(1)$ can be identified with $U(3)$, because their Lie algebras are the same and the connection between triality of
color and electric charge can be established through the $SU(3) \otimes U(1)$’s embedding in $SU(6)$ that unifies the coupling constants.

The $SU(6)$ diagonal generators in the $SU(3)_C \otimes SU(3) \otimes U(1)$ decomposition (normalized by (4) are:

$$T^C_3 = \frac{1}{2} \text{diag}(1, -1, 0, 0, 0, 0).$$
$$T^C_8 = \frac{1}{2\sqrt{3}} \text{diag}(1, 1, -2, 0, 0, 0).$$
$$T_3 = \frac{1}{2} \text{diag}(0, 0, 0, 1, -1, 0).$$
$$T_8 = \frac{1}{2\sqrt{3}} \text{diag}(0, 0, 0, 1, 1, -2).$$
$$T_{35} = \frac{1}{2\sqrt{3}} \text{diag}(1, 1, 1, -1, -1, -1).$$

The two first generators are the diagonal ones of QCD, the third represents the electrically neutral component of the $SU(2)$ that is included in the subgroup $SU(3)$ and the last two are related to the hypercharge as we shall soon see.

Similarly to what happens in our model for GWS, the two diagonal generators would seem to be the assignments of the isospin $T_3$ and the hypercharge $Y$, but this would give the wrong value for the hypercharge of the Higgs boson. To correct this problem we changed the group and used $U(3)$ instead. With the help of the extra generator we could obtain correctly all the quantum numbers of the GSW model through a linear combination of it and the original hypercharge generator. Consequently the two last generators in (4) must be related to the hypercharge and to the new scalar boson.

We shall rename the fourth $T_8$ in (4) $T_Y'$, because it stands for the original hypercharge in our $SU(3)$ electroweak model, and the fifth $T_{35}$ as the generator $T_{Z'}$, because it represents the extra one in our $U(3)$ model that permitted us to avoid the wrong hypercharge assignment of the Higgs boson. So we discover here the origin of the trick we had to do in our first paper.

### 4 The generators and the fermions quantum numbers

Take $SU(6)$ in the $SU(3)_C \otimes SU(2) \otimes U(1)$ decomposition, that will give us the quantum numbers of the particles in the large irreps. The expected results are the quantum numbers of the fermions of the Standard Model

$$L_e = (\nu e)^T_L : (1, 2)_1 , \quad L_u = (u d)^T_L : (3, 2)_{-1/3}$$
$$e_R : (1, 1)^2 , \quad u_R : (3, 1)_{-4/3}$$
$$\nu_R : (1, 1)_0 , \quad d_R : (3, 1)_{2/3}.$$
where the “T” stands for “transposed”. We are using the Gell-Mann-Nishijima relation in the form $Q = T_3 - \frac{1}{2} Y$. We are looking for a 15-dimensional representation to accommodate the fermions.

The branching rule [4] for this representation into the fundamental ones of $SU(3)_C \otimes SU(3) \otimes U(1)_{Z'}$ is

$$15 = (\bar{3}, 1)_{-2} + (1, \bar{3})_{-2} + (3, 3)_{0}. \quad (9)$$

In these entries for the 15, call them $(x, y)_{Z'}$, generically, the $x$ and $y$ irreps belong to $SU(3)_C$ and $SU(3)$, respectively. The subindex $Z'$ is the value of the $U(1)_{Z'}$ generator when acting on the states given by the irreps. Therefore in the second term of this equation we expect to accommodate the antileptons; thus we shall work with the 15-dimensional representation. We need to decompose the $SU(3)$ in terms of its maximal subgroup $SU(2) \otimes U(1)_{Y'}$, in order to recognize the fermion fields in the $SU(3)_C \otimes SU(3) \otimes U(1)_{Z'}$ branching rule. Employing $3 = (2)_1 + (1)_{-2}$ for $SU(3)$, the 15-dimensional representation of $SU(6)$ can be broken up into irreps of $SU(3)_C \otimes SU(2) \otimes U(1)_{Y'} \otimes U(1)_{Z'}$:

$$\bar{15} = (3, 1)_{0, -2} + (1, 2)_{1, 2} + (1, 1)_{-2, 2} + (\bar{3}, \bar{2})_{1, 0} + (\bar{3}, 1)_{2, 0}. \quad (10)$$

Notice that neither $Y'$ nor $Z'$ represent the correct hypercharge for $L_e$ and $e_R$ as given in (8), but we use them to find the correct linear combination that gives the hypercharge.

Take

$$T_Y = \alpha T_{Y'} + \beta T_{Z'}. \quad (11)$$

From the normalization condition we get the condition for the coefficients $\alpha$ and $\beta$ in order that $T_Y$ is normalized directly by the above equation, i.e.

$$1 = \alpha^2 + \beta^2. \quad (12)$$

From inspection, the correct combination to obtain the quantum numbers of the leptons is

$$T_Y = -\frac{1}{\sqrt{5}} T_{Y'} + \frac{2}{\sqrt{5}} T_{Z'}. \quad (13)$$

The accompanying generator $T_Z = \gamma T_{Y'} + \delta T_{Z'}$ is obtained from the orthogonality of the generators, with the result

$$T_Z = \frac{2}{\sqrt{5}} T_{Y'} + \frac{1}{\sqrt{5}} T_{Z'}. \quad (14)$$

Explicitly, these new generators become

$$T_Y = \frac{1}{2} \sqrt{\frac{3}{5}} \text{diag} \left( \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, -1, -1, 0 \right).$$

$$T_Z = \frac{1}{2} \sqrt{\frac{1}{15}} \text{diag} (1, 1, 1, 1, 1, -5). \quad (15)$$

5
We shall denote them as the hypercharge and the ultracharge generator, respectively. All the other generators of $SU(6)$ are left unchanged. From now on we use this new representation in all calculations.

Let us write the branching rule for the $\mathbf{15}$-dimensional irrep into irreps of $SU(3)_C \otimes SU(2) \otimes U(1)_Y \otimes U(1)_Z$, using the values of $Y$ and $Z$ as subindices, in that order:

$$\mathbf{15} = (3, 1)_{-4/3,-1} + (1, 2)_{1,2} + (1, 1)_{2,-1} + (\bar{3}, \bar{2})_{1/3,-1} + (\bar{3}, 1)_{-2/3,2}.$$  

The last two terms are the antiquarks. Explicitly, the $\mathbf{15}$, which is the antisymmetric tensor product of two fundamentals, can be written as

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & u_R^3 & -u_R^2 & -d_R^{1c} & u_R^{1c} & d_R^{1c} \\
-u_R^2 & 0 & u_R^1 & -d_R^{2c} & u_R^{1c} & d_R^{2c} \\
u_R^{1c} & -u_R^{2c} & -u_R^{3c} & 0 & -e_R & \nu_L \\
d_R^{1c} & d_R^{2c} & d_R^{3c} & 0 & e_R & 0 \\
-u_R^1 & u_R^{2c} & u_R^{3c} & e_R & 0 & e_L \\
d_R^1 & -d_R^2 & -d_R^3 & -\nu_L & -e_L & 0
\end{pmatrix}.$$  

The quark colors have been denoted 1, 2 and 3. The identification of $(d - u)^T$ as a $\bar{2}$ of $SU(2)$ follows from the assignments of $(u \; d)^T$ as a $2$.

5 The quantum numbers of the gauge fields

The gauge bosons belong to the adjoint representation, which in our case is the $SU(6)$’s $\mathbf{35}$. To identify them, we first decompose the $\mathbf{35}$-dimensional representation with respect to $SU(3)_C \otimes SU(3) \otimes U(1)_{Z'}$ and obtain

$$\mathbf{35} = (1, 1)_0 + (8, 1)_0 + (1, 8)_0 + (3, \bar{3})_2 + (\bar{3}, 3)_2.$$  

Secondly, using the branching rules for the 3- and 8-representations of $SU(3)$ into irreps of $SU(2) \otimes U(1)_{Y'}$, we expand the previous equation as

$$\mathbf{35} = (1, 1)_{0,0} + (8, 1)_{0,0} + (1, 1)'_{0,0} + (1, 2)_{3,0} + (1, \bar{2})_{-3,0} + (1, 3)_{0,0} + (3, 1)_{2,2} + (\bar{3}, \bar{2})_{-1,2} + (\bar{3}, 3)_{-2,2}.$$  

Finally we rewrite the $Y'$ and $Z'$ in terms of the new quantum numbers, using linear combinations $[\mathbf{3}]^3$ and $[\mathbf{14}]$, for the same decomposition as before, arriving at

$$\mathbf{35} = (1, 1)_{0,0} + (8, 1)_{0,0} + (1, 1)'_{0,0} + (1, 2)_{-1,1} + (1, \bar{2})_{1,-1} + (1, 3)_{0,0} + (3, 1)_{2/3,1} + (3, \bar{2})_{5/3,0} + (\bar{3}, 1)_{-2/3,1} + (\bar{3}, 2)_{-5/3,0}.$$  

We identify the gauge bosons, as follows: the $(8, 1)_{0,0}$ is the adjoint representation of $SU(3)_C$; that is, the gluons $C^a_{\mu}$ ($i = 1, 2 \ldots 8$); the $(1, 1)'_{0,0}$ and $(1, 3)_{0,0}$ belong to the adjoint representation of $SU(2)_W \otimes U(1)_W$, and result in the bosons of the GWS model, $A^a_{\mu}$ ($a = 1, 2, 3$) and $B_{\mu i}$; $(1, 2)_{-1,1}$ and its hermitian conjugate (h.c.)
are color singlets and $SU(2)$ doublets, and are the Higgs boson of the GSW model, $\hat{\phi}$. The irreps $(3, 1)_{2/3, 1}$ and $(\bar{3}, 2)_{-5/3, 0}$ with their h.c. are the leptoquarks, with mixed quantum numbers and thus mediating transitions between quarks and leptons. Within our present understanding, there does not seem to exist, in principle, any particular reason to insist that these bosons be either scalar or vector. Finally the irrep $(1, 1)_{0, 0}$, with null quantum numbers and representation $\propto \text{diag}(1, 1, 1, 1, 1, -5)$ is naturally identified with the Higgs whose VEV gives the large-mass GUT scale, since its VEV would not produce a vacuum charged in any way.

With the help of the adjoint constructed as the tensor product of the fundamental and its conjugate the gauge bosons can be written out in the form

$$A + \Phi = \frac{ig}{2} \begin{pmatrix} \mathcal{G} \cdot \lambda & \mathcal{X} & \tilde{X} \\ X^\dagger & \mathcal{A} \cdot \sigma & i\sqrt{2}\gamma^5 \tilde{\phi} \\ \tilde{X}^\dagger & i\sqrt{2}\tilde{\phi}^\dagger \gamma^5 & 0 \end{pmatrix}$$

$$+ i\frac{g}{2}\sqrt{\frac{3}{5}} \mathcal{B} \begin{pmatrix} \frac{2}{3} 1_{3 \times 3} & 0 & 0 \\ 0 & -1_{2 \times 2} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{g}{2}\sqrt{\frac{1}{15}} \tilde{\gamma}^5 \gamma \begin{pmatrix} 1_{3 \times 3} & 0 & 0 \\ 0 & 1_{2 \times 2} & 0 \\ 0 & 0 & -5 \end{pmatrix},$$

where $g$ is the $SU(6)$ coupling constant, the $\sigma^a, a = 1, 2, 3$, are the Pauli matrices, the $\lambda^i, i = 1, 2 \ldots 8$, are the Gell-Mann matrices, $1_{2 \times 2}$ is the $2 \times 2$ unit matrix, $1_{3 \times 3}$ is the $3 \times 3$ unit matrix, and

$$\tilde{\phi} = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi^1 - i\varphi^2 \\ \varphi^3 - i\varphi^4 \end{pmatrix},$$

the GWS Higgs fields.

There is still the open question as to what is the integer spin of the leptoquarks. The following argument, apparently inescapable, seems to leave no doubt that they must be vector bosons. To understand the physical behavior of a GYMT one must always go back to expansion \(6\). The interactions of the vector bosons among themselves are given by the third term on the right of this equation, and the interactions of the vector with the scalar bosons are given by the fourth term. One of the most peculiar facts about GYMT is that scalar bosons do not interact among themselves: there is no term that does this in the equation. Now, the leptoquarks interact with the vector bosons of the Standard Model, so that, in order not to contradict its phenomenology, it is necessary that they be given a large mass. But this immediately implies that they have to be vector bosons, since, if they were scalars, they would not interact with the scalar Higgs. In conclusion, in order not to contradict phenomenology, the leptoquarks must be vectorial.

6 Final comments

It is very satisfactory to see that, after the linear transformation of the generators, the quantum numbers of the Standard Model appear naturally using the ideas of
GYMTs. This means that not only does the model predict correctly all the quantum numbers for both fermions and vector bosons, but that it also predicts the correct numbers for both the GWS and GUT Higgs in the same grand unified irrep of the vector bosons.

While the GYMT lagrangian is both gauge and Lorentz invariant, a gauge transformation of the fermions by itself may mix different chiralities. However, transforming at the same time the generalized covariant derivative results in an invariant lagrangian. A maximal subgroup that maintains unchanged the chiral structure of the fermion multiplet is a usual Yang-Mills theory in this multiplet. This is the reason why the GYMTs look like Yang-Mills theories.

There is a detail that does not seem to be working correctly, and has to do with the conservation of the number of degrees of freedom before and after the GUT symmetry breaking. The problem is that the GUT Higgs gives mass to 18 leptoquarks, which means that there is an increase of 18 dynamical degrees of freedom. Where do they come from? In the usual GUTs the unitary gauge eliminates degrees of freedom from the Higgs bosons irrep, but here it is not clear what is happening.

References

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