VOLUME APPROXIMATIONS OF STRICTLY PSEUDOCONVEX DOMAINS

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1. Introduction

1.1. Background. Let $\Omega \subset \mathbb{C}^d$ be a $C^2$-smooth strictly pseudoconvex domain. The Fefferman hypersurface measure on $\partial \Omega$ (first introduced by Fefferman in [10]) is the $(2d-1)$-form, $\sigma_\Omega$, satisfying

$$\sigma_\Omega \wedge d\rho = 4^d d^{d+1} M(\rho) \frac{1}{\pi^{d+1}} \omega_{\mathbb{C}^d},$$

where $\omega_{\mathbb{C}^d}$ is the standard volume form on $\mathbb{C}^d$, $\rho$ is a defining function for $\Omega$ with $\Omega = \{ \rho < 0 \}$, and $M(\rho) = -\det \begin{pmatrix} \rho & \rho z_j \\ \rho z_i & \rho z_j \end{pmatrix}_{1\leq i,j \leq d}$. $\sigma_\Omega$ does not depend on the choice of $\rho$ and satisfies the following transformation law:

$$F^* \sigma_{F(\Omega)} = |\det J_C(F)| \frac{2d}{\pi^{d+1}} \sigma_\Omega,$$

where $F$ is a biholomorphism on $\Omega$ that is $C^2$-smooth on $\Omega$.

The Fefferman hypersurface measure shares strong connections with the Blaschke surface area measure (explored in [3] and [4], for instance) studied in affine convex geometry. If $K \subset \mathbb{R}^d$ is a $C^2$-smooth convex body, the Blaschke surface area measure on $\partial K$ is given by

$$\tilde{\sigma}_K = \kappa \frac{1}{\pi^{d+1}} s,$$

where $\kappa$ and $s$ are the Gaussian curvature function and the Euclidean surface area measure on $\partial K$, respectively. Its resemblance to Fefferman measure is reflected in the following identity:

$$A^* \tilde{\sigma}_{A(K)} = |\det J_R(A)| \frac{d-1}{\pi^{d+1}} \tilde{\sigma}_K,$$

where $A$ is an affine transformation of $\mathbb{R}^d$. Since its introduction by Blaschke (in [5]), several mathematicians have extended the notion of affine surface area to arbitrary convex bodies (see [18] for details). As this measure is invariant under volume-preserving affine maps, it occurs naturally in volume approximations of convex bodies by polyhedra (see [12, Chap. 1.10] for a survey). The first complete asymptotic result was due to Gruber [11] who showed that if $K \subset \mathbb{R}^d$ is a $C^2$-smooth strictly convex body, then

$$\inf \{ \text{vol}(P \setminus \Omega) : P \in \mathcal{P}_{n}^\circ \} \sim \frac{1}{2} \text{div}_{d-1} \left( \int_{\partial K} \tilde{\sigma}_K \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}},$$

as $n \to \infty$, where $\mathcal{P}_{n}^\circ$ is the class of all polyhedra that circumscribe $K$ and have at most $n$ facets, and $\text{div}_{d-1}$ is a dimensional constant. Ludwig [19] later showed that, if the approximating polyhedra are from $\mathcal{P}_{n}$, the class of all polyhedra with at most $n$ facets, then

$$\inf \{ \text{vol}(\Omega \Delta P) : P \in \mathcal{P}_{n} \} \sim \frac{1}{2} \text{ldiv}_{d-1} \left( \int_{\partial K} \tilde{\sigma}_K \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}},$$

as $n \to \infty$, where $\Delta$ denotes the symmetric difference between sets and $\text{ldiv}_{d-1}$ is a dimensional constant. Later, Böröczky [15] proved both these formulae for general smooth convex bodies.
Similar asymptotics have been obtained using other notions of complexity for a polyhedra — such as the number of vertices.

In [3], Barrett asks whether such relations can be found between the Fefferman hypersurface measure on a pseudoconvex domain and the complexity of approximating analytic polyhedra. An analytic polyhedra in \( \Omega \) is a relatively compact subset that is a union of components of any set of the form

\[
P = \{ z \in \Omega : |f_j(z)| < 1, \ j = 1, \ldots, n \},
\]

where \( f_1, \ldots, f_n \) are holomorphic functions in \( \Omega \). The natural notion of complexity for an analytic polyhedron, \( P \), is its order — i.e., the number of inequalities that define \( P \). This setup, however, is not suited for our purpose as demonstrated by a result due to Bishop (Lemma 5.3.8 in [14]) which says that any pseudoconvex domain in \( \mathbb{C}^d \) can be approximated arbitrarily well (in terms of the volume of the gap) by analytic polyhedra of order at most \( 2d \). The following example indicates where the problem lies:

**Example 1.** Let \( \Omega = \mathbb{D} \) be the unit disc in \( \mathbb{C} \). Consider the lemniscate-bound domains

\[
P_n := \left\{ z \in \mathbb{D} : |f_n(z)| = \prod_{k=0}^{2n-1} |z - \exp(k\pi/n)| > \frac{\pi}{n} \right\}.
\]

Each \( P_n \) has order 1 and satisfies \( \{|z| < 1 - \pi/n\} \subset P_n \subset \{|z| < 1 - \sqrt{3}\pi/2n\} \). Thus, for all \( n \geq 1 \),

\[
\inf\{\text{vol}(\mathbb{D} \setminus P) : P \text{ is an analytic polyhedron of order at most } n\} = 0.
\]

If we, instead, declare the complexity of \( P_n \) to be \( 2n \) — i.e., the number of zeros of \( f_n \), then, since \( \lim_{n \to \infty} n \cdot \text{vol}(\mathbb{D} \setminus P_n) < \infty \), we can expect results similar to (1.1) and (1.2).

1.2. Statement of results. Hereafter, we work in \( \mathbb{C}^2 \). Example 1 leads us to a special class of polyhedral objects. For any fixed \( f \in \mathcal{C}(\overline{\Omega} \times \partial \Omega) \), let \( \mathcal{P}_n(f) \) be the collection of all relatively compact sets in \( \Omega \) of the form

\[
P = \{ z \in \Omega : |f(w^j, z)| > \delta_j, j = 1, \ldots, n \},
\]

where, \( w^1, \ldots, w^n \in \partial \Omega \) and \( \delta_1, \ldots, \delta_n > 0 \). We present a class of functions \( f \) for which asymptotic results such as (1.1) and (1.2) can be obtained:

**Theorem 1.1.** Let \( \Omega \subset \subset \mathbb{C}^2 \) be a \( \mathcal{C}^4 \)-smooth strictly pseudoconvex domain. Suppose \( f \in \mathcal{C}(\overline{\Omega} \times \partial \Omega) \) is such that

(i) \( f(z, w) = 0 \) if and only if \( z = w \in \partial \Omega \), and

(ii) there exist \( \nu \in \mathbb{N}_+ \), \( \eta > 1 \) and \( \tau > 0 \) such that

\[
(*) \quad f(z, w) = a(z, w)p(z, w)^\nu + O(p(z, w)^{\nu\tau})
\]

on \( \Omega_{\tau} := \{(z, w) \in \overline{\Omega} \times \partial \Omega : |z - w| \leq \tau\} \), where \( p \) is the Levi polynomial of some strictly plurisubharmonic defining function of \( \Omega \) (see Section 2) and \( a \) is some continuous non-vanishing function on \( \Omega_{\tau} \).

Then, there exists a constant \( l_{\text{kor}} > 0 \), independent of \( \Omega \), such that

\[
(1.3) \quad \inf\{\text{vol}(\Omega \setminus P) : P \in \mathcal{P}_n(f)\} \sim \frac{1}{2} l_{\text{kor}} \left( \int_{\partial \Omega} \sigma_{\Omega} \right)^{\frac{3}{2}} \frac{1}{\sqrt{n}},
\]

as \( n \to \infty \).

**Remark 1.2.** For \( \Omega \) as above, let \( \text{LP}(\Omega) \) denote the class of \( f \in \mathcal{C}(\overline{\Omega} \times \partial \Omega) \) that satisfy conditions (i) and (ii) of Theorem 1.1. Then, \( \text{LP}(\Omega) \) is invariant under biholomorphisms that extend \( (\mathcal{C}^2) \)-smoothly to the boundary.
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(*) is a natural condition when working with strictly pseudoconvex domains. We exhibit its scope by making special choices of \( f \in L^p(\Omega) \) that yield analytic polyhedra.

**Corollary 1.3.** Let \( \Omega \) be as in Theorem 1.1. Then, (1.3) holds when \( f \) is a Henkin-Ramirez generating map of \( \Omega \). (see Section 2).

**Remark 1.4.** It is natural to ask whether Theorem 1.1 can be obtained for holomorphic generating maps (of Cauchy-Fantappiè kernels) that satisfy condition (i) but don’t necessarily satisfy condition (*). The Cauchy-Leray map (see Section 2) on strongly convex domains is one such example. To understand this scenario, we define

\[
B_f(w, \delta) := \{ z \in \partial \Omega : |f(z, w)| < \delta \}, \quad w \in \partial \Omega, \quad \delta > 0,
\]

where \( f \) satisfies condition (i) in Theorem 1.1. Further, let

\[
\phi_f : w \mapsto \limsup_{\delta \to 0} \sup_{y \in B_p(w, \delta)} \inf_{\delta' \in B_y(w, \delta)} \{ \delta' : y \in B_f(w, \delta') \},
\]

where \( p \) is as in Theorem 1.1. The definition of \( \phi_f \) is inspired by the notion of quasiconformality (see [7, Section 6.5]), and captures the infinitesimal shape of the holomorphic discs \( \{ f(z, w) = \delta \} \cap \Omega \), as \( |\delta| \to 0 \). In particular, \( \phi_f \equiv 1 \) for \( f \) satisfying (*). Our proof of Theorem 1.1 indicates that for a general generating map, \( f \), the above procedure will yield a measure on \( \partial \Omega \) whose Radon-Nikodym derivative with respect to the Fefferman measure exists and is a continuous function of \( \phi_f \).

Well-known estimates on the Bergman kernel ([9]) yield a corollary to Theorem 1.1 that suggests a way of extending (1.3) to more general domains (see Section 7 for some elaboration).

**Corollary 1.5.** Let \( \Omega \) be a smooth strictly pseudoconvex domain and \( K_\Omega \) denote its Bergman kernel function. Let \( BP_n \) be the collection of all analytic polyhedra in \( \Omega \) of the form

\[
P = \{ z \in \Omega : |K_\Omega(w^j, z)| < m_j, j = 1, \ldots, n \},
\]

where \( w^1, \ldots, w^n \in \partial \Omega \) and \( m_1, \ldots, m_n > 0 \). Then,

\[
(1.4) \quad \inf\{ \text{vol}(\Omega \setminus P) : P \in BP_n \} \sim \frac{1}{2} \ker \left( \int_{\partial \Omega} \sigma_\Omega \right)^{\frac{1}{2}} \frac{1}{\sqrt{n}},
\]

as \( n \to \infty \).

In the same vein, the expansion for the Szegö kernel (see [6]) gives the following result.

**Corollary 1.6.** Corollary 1.5 holds when \( K_\Omega \) is replaced by \( S_\Omega \), the Szegö kernel function of \( \Omega \) with respect to any smooth multiple of the surface area measure.

1.3. **Plan of paper.** Definitions, notation and terminology that feature in multiple sections are collected in Section 2. The proof of Theorem 1.1 is spread over subsequent sections. A critical lemma allows us to pass from \( L^p(\Omega) \) to a single representative — this lemma and other technical issues are dealt with in Section 3. In Section 4, we address the problem for certain model domains and model polyhedra. The rate of decay and the relevant exponents in (1.3) become evident in this section. We move from the model to the general case (locally), and from the local to the global case in Sections 5 and 6, respectively. Section 6 also contains brief proofs of the corollaries, and some further questions are raised in Section 7. The appendix is devoted to a tiling problem on the Heisenberg group that emerged naturally in the course of this work, and seems indispensable in proving Theorem 1.1.

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2. Preliminaries

In this article, \( \mathbb{N}_+ \) denotes the set of all positive natural numbers. For \( D \subseteq \mathbb{R}^n \), \( C(D) \) is the set of all continuous functions on \( D \), and \( C^k(D) \), \( k \geq 1 \), denotes the set of all functions that are \( k \)-times continuously differentiable in some open neighborhood of \( D \). If \( A \subseteq B \subseteq \mathbb{R}^n \), \( \text{int}_B A \) is the interior of \( A \) in the relative topology of \( B \). When well defined, \( J_c(f)(x) \) and \( f_s(x) \) denote the real Jacobian matrix of the function \( f \) at \( x \), and \( J_C(f)(z) \) is the complex Jacobian of \( f \) at \( z \). For brevity, we often abbreviate \( \frac{\partial f}{\partial x} \) and \( \frac{\partial^2 f}{\partial x \partial y} \) to \( f_x \) and \( f_{xy} \), respectively. In \( \mathbb{C}^2 \), we employ the notation

- \( z = (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \), \( w = (w_1, w_2) = (u_1 + iv_1, u_2 + iv_2) \) for points;
- \( B_2(z; r) \) for the Euclidean ball centered at \( z \) and of radius \( r \);
- \( \langle \cdot , \cdot \rangle \) for the complex pairing between a co-vector and a vector;
- \( \langle \cdot \rangle \) to indicate projection onto \( \{ y_2 = 0 \} = \mathbb{C} \times \mathbb{R} \);
- \( A^{\text{res}} \) for \( \langle A \{y_2 = 0\} \rangle : \mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{R} \), where \( A : \mathbb{C}^2 \to \mathbb{C}^2 \);
- \( | \cdot | \) for the Lebesgue measure in \( \mathbb{C} \times \mathbb{R} \), and
- \( s \) for the standard Euclidean surface area measure on the boundary of a smooth domain.

We reintroduce the polyhedral objects of our study.

Definition 2.1. Let \( \Omega \subset \mathbb{C}^2 \) be a domain and \( f \in C(\overline{\Omega} \times \partial \Omega) \). Given a compact set \( J \subset \partial \Omega \), an \( f \)-polyhedron over \( J \) is any set of the form

\[
P = \{ z \in \Omega : |f(w^j, z)| > \delta_j, \, j = 1, \ldots, n \}, \quad (w^j, \delta_j) \in \partial \Omega \times (0, \infty),
\]

such that \( J \subset \partial \Omega \setminus P \) and for every \( j \in \{1, \ldots, n\} \), \( |f(w^j, z)| < \delta_j \) for some \( z \in J \). If \( \Omega \) is bounded, then an \( f \)-polyhedron over \( \partial \Omega \) is simply called an \( f \)-polyhedron. We call

- each \( (w^j, \delta_j) \) a source-size pair of \( P \);
- each \( C(w^j, \delta_j; f) := \{ z \in \overline{\Omega} : |f(w^j, z)| \leq \delta_j \} \) a cut of \( P \);
- each \( F(w^j, \delta_j; f) := \{ z \in \overline{\Omega} : |f(w^j, z)| = \delta_j, |f(w^l, z)| \geq \delta_l, \ l \neq j \} \) a facet of \( P \);
- \( (w^1, \ldots, w^n) \) and \( (\delta_1, \ldots, \delta_n) \) the source-tuple and size-tuple of \( P \), respectively.

We emphasize that, by definition, the cuts of an \( f \)-polyhedron over \( J \) cover \( J \), and each of its cuts intersects \( J \) non-trivially.

Remarks. When there is no ambiguity in the choice of \( f \), we drop any reference to it from our notation for cuts and facets. Repetitions are permitted when listing the sources of an \( f \)-polyhedron. Thus, \( P \) — as in Definition 2.1 — has at most \( n \) facets.

Notation. Let \( \Omega, f, P \) and \( J \) be as in Definition 2.1 above.

- \( \delta(P) := \max_{1 \leq j \leq n} \{ \delta_j : (\delta_1, \ldots, \delta_n) \text{ is the size-tuple of } P \} \).
- \( P_n(f) := \text{the collection of all } f \text{-polyhedra in } \Omega \text{ with at most } n \text{ facets.} \)
- \( P_n(J; f) := \text{the collection of all } f \text{-polyhedra over } J \text{ with at most } n \text{ facets.} \)
- \( P_n(J \subset H; f) := \{ P \in P_n(J; f) : \partial \Omega \setminus P \subset H \} \), where \( H \subset \partial \Omega \) is a compact superset of \( J \).
- \( v(\Omega; \mathcal{P}) := \inf \{ \text{vol}(\Omega \setminus P) : P \in \mathcal{P} \} \), for any sub-collection \( \mathcal{P} \subset P_n(J; f) \).

We now recall some standard concepts (see [13, Ch. 1]) in the theory of integral representation kernels in \( \mathbb{C}^d \) (focusing on \( d = 2 \)). For a bounded domain \( \Omega \subset \mathbb{C}^2 \), a \( C^1 \)-smooth function \( g(z, w) = (g_1(z, w), g_2(z, w)) \) on \( \Omega \times \partial \Omega \) is called a Leray map for \( \Omega \) if

\[
g(z, w) := g_1(z, w)(z_1 - w_1) + g_2(z, w)(z_2 - w_2) \neq 0 \quad \text{for all } (z, w) \in \Omega \times \partial \Omega.
\]

The Cauchy-Fantappié form generated by \( g \) is given by

\[
\text{CF}(g)(z, w) = \frac{g_1(z, w) \wedge \partial_w g_2(z, w) \wedge dw - g_2(z, w) \wedge \partial_z g_1(z, w) \wedge dw}{g(z, w)^2},
\]
where \( dw = dw_1 \wedge dw_2 \). Indulging in non-standard terminology, we call \( g \) the \emph{generating map} of \( \text{CF}(g) \).

Cauchy Fantappié forms act as reproducing kernels: if \( \Omega \) has piecewise \( C^1 \)-boundary, then
\[
 f(z) = \frac{1}{(2\pi i)^2} \int_{\partial \Omega} f(w) \wedge \text{CF}(g)(z, w), \quad z \in \Omega,
\]
where \( f \in \mathcal{C}(\overline{\Omega}) \) is holomorphic in \( \Omega \). It has been of interest to construct Leray maps such that \( \text{CF}(g)(z, w) \) is holomorphic in \( z \in \Omega \). For strictly pseudoconvex domains, it is enough to directly construct a generating map that is holomorphic in \( z \). Henkin and Ramirez constructed such maps (see [20, §3] for details) for \( C^2 \)-smooth strictly pseudoconvex domains, based on
\[
 p(z, w) = \sum_{j=1}^{2} \frac{\partial \rho}{\partial z_j}(w)(z_j - w_j) + \frac{1}{2} \sum_{j,k=1}^{2} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(w)(z_j - w_j)(z_k - w_k),
\]
where \( \rho \) is a defining function of \( \Omega \). \( p \) is called the \emph{Levi polynomial} of \( \rho \). The corresponding Cauchy Fantappié kernels are called Henkin-Ramirez reproducing kernels. If \( \Omega \) is a \( C^1 \)-smooth \( C \)-linearly convex domain, i.e., the complement of \( \Omega \) is a union of complex hyperplanes, a simpler holomorphic (in \( z \)) generating map is given by the \emph{Cauchy-Leray map} of \( z \).

\[ l(z, w) = \sum_{j=1}^{2} \frac{\partial \rho}{\partial z_j}(w)(z_j - w_j). \]

3. SOME TECHNICAL LEMMAS

Here, we restrict our attention to Jordan measurable domains \( \Omega \subset \mathbb{C}^2 \). \( J \) and \( H \) are compact subsets of \( \partial \Omega \) such that \( J \subset \text{int}_{\partial \Omega} H \). We will concern ourselves with \( f \)-polyhedra that lie above \( J \) but are constrained by \( H \). We first prove a lemma that will allow us to work locally.

**Lemma 3.1.** Let \( \Omega, J \) and \( H \) be as above. Suppose \( f \in \mathcal{C}(\overline{\Omega} \times H) \) satisfies

(a) \( \{ z \in \overline{\Omega} : f(z, w) = 0 \} = \{ w \} \), for any \( w \in H \),

(b) For some \( \delta_0 > 0 \) and \( c > 0 \), \( C(w, \delta; f) \supseteq C(w, c\delta; g) \), for all \( w \in H \) and \( \delta < \delta_0 \), where \( g \in \mathcal{C}(\overline{\Omega} \times H) \) satisfies (a) and \( C(w, \delta; g) \) is Jordan measurable for each \( w \in H \) and \( \delta < \delta_0 \).

Then, for \( P_m \in \mathcal{P}_m(J \subset H; f) \) such that \( \lim_{m \to \infty} \text{vol}(\Omega \setminus P_m) = 0 \), we have that \( \lim_{m \to \infty} \delta(P_m) = 0 \).

**Proof.** It suffices to show that for each \( \delta < \delta_0 \), there is a \( b > 0 \) such that \( \text{vol}(C(w, \delta; f)) > b \) for all \( w \in H \). By condition (b), it is enough to show this for the cuts of \( g \). As \( g \) satisfies condition (a), \( \text{vol}(C(w, \delta; g)) > 0 \) for each \( w \in H \). Therefore, if we can establish the continuity of \( w \mapsto \text{vol}(C(w, \delta; g)) \) on the compact set \( H \), we will be done.

Fix a \( \delta \in (0, \delta_0) \). Let \( \chi_w := \chi_{C(w, \delta; g)} \), where \( \chi_A \) denotes the indicator function of \( A \). For a given \( w \in H \), consider a sequence of points \( \{ w^n \}_{n \in \mathbb{N}} \subset H \) that converges to \( w \) as \( n \to \infty \). Then,
\[
\lim_{n \to \infty} \chi_{w^n}(z) = \chi_w(z) \quad \text{for a.e. } z \in \overline{\Omega}.
\]

To see this, consider a \( z \in \overline{\Omega} \) such that \( \chi_w(z) = 0 \). Suppose, there is a subsequence \( \{ w^{n_j} \}_{j \in \mathbb{N}} \subset \{ w^n \}_{n \in \mathbb{N}} \) such that \( \chi_{w^{n_j}}(z) = 1 \). Then, \( |g(w^{n_j}, z)| \leq \delta \) but \( \lim_{j \to \infty} |g(w^{n_j}, z)| = |g(w, z)| \geq \delta \). This is only possible if \( g(w, z) = \delta \). An analogous argument holds if \( \chi_w(z) = 1 \). Thus, \( z \in \partial C(w, \delta; g) \).

Due to assumption (b), this is a null set. Thus, (3.1) is true and we invoke Lebesgue’s dominated convergence theorem to conclude that
\[
\text{vol}(C(w, \delta; g)) = \int_{\overline{\Omega}} \chi_{w^n} \rightarrow \int_{\overline{\Omega}} \chi_w = v_g(w, \delta)
\]
as \( n \to \infty \). \( \square \)
Next, we prove a lemma that permits us to concentrate on a single representative of LP(Ω).

**Lemma 3.2.** Let Ω, J and H be as above. Suppose \( f, g \in C(\overline{\Omega} \times H) \) are such that

(i) \( \{ z \in \overline{\Omega} : f(z, w) = 0 \} = \{ z \in \overline{\Omega} : g(z, w) = 0 \} = \{ w \}, \) for any fixed \( w \in H, \) and

(ii) there exist constants \( \varepsilon \in (0, 1/3) \) and \( \tau > 0, \) such that

\[
|f(z, w) - g(z, w)| \leq \varepsilon(|g(z, w)| + |f(z, w)|) \tag{3.2}
\]
on \( \{(z, w) \in \overline{\Omega} \times H : |z - w| \leq \tau \}. \)

Further, assume that the cuts of \( g \) are Jordan measurable and satisfy a doubling property as follows

\( \circ \) there is a \( \delta_g > 0 \) and an \( \mathcal{E} \in C([0, 16]) \) such that, for any \( m \in \mathbb{N}_+, \) \((w^j, \delta_j) \in H \times (0, \delta_g), j = 1, \ldots, m, \) and \( t \in [0, 16], \)

\[
\text{vol} \left( \bigcup_{j=1}^{m} C(w^j, (1 + t)\delta_j) \right) \leq \mathcal{E}(t) \cdot \text{vol} \left( \bigcup_{j=1}^{m} C(w^j, \delta_j) \right). \tag{3.3}
\]

Then, for every \( \beta > 0, \)

\[
\limsup_{n \to \infty} n^{\beta} v_n(f) \leq D_1(\varepsilon) \limsup_{n \to \infty} n^{\beta} v_n(g); \tag{3.4}
\]

\[
\liminf_{n \to \infty} n^{\beta} v_n(f) \geq D_2(\varepsilon)^{-1} \liminf_{n \to \infty} n^{\beta} v_n(g), \tag{3.4}
\]

where \( v_n(h) := v(\Omega; P_n(J \subset H; h)), \) and \( D_1, D_2 \) depend on \( \mathcal{E} \) and satisfy \( \lim_{\varepsilon \to 0} D_j(\varepsilon) = \lim_{t \to 0} \mathcal{E}(t). \)

**Proof.** Observe that if \( \bar{\varepsilon} := \frac{1 + \varepsilon}{1 - \varepsilon}, \) then inequality (3.2) may be transcribed as

\[
|f(z, w)| \leq \bar{\varepsilon} |g(z, w)| \text{ and } |g(z, w)| \leq \bar{\varepsilon} |f(z, w)| \tag{3.6}
\]
on \( \{(z, w) \in \overline{\Omega} \times H : |z - w| \leq \tau \}. \) Hence, for any \( w \in H \) and \( \delta > 0, \)

\[
C(w, \delta; f) \subseteq B_2(w; \tau) \implies C(w, \delta; f) \subseteq C(w, \bar{\bar{\varepsilon}} \delta; g); \tag{3.7}
\]

\[
C(w, \delta; g) \subseteq B_2(w; \tau) \implies C(w, \delta; g) \subseteq C(w, \bar{\bar{\varepsilon}} \delta; f). \tag{3.8}
\]

We first show that

\[
\limsup_{n \to \infty} n^{\beta} v_n(f) \leq \mathcal{E} \left( \frac{(1 + \varepsilon)^2}{(1 - \varepsilon)^2} - 1 \right) \limsup_{n \to \infty} n^{\beta} v_n(g). \tag{3.9}
\]

Let \( \xi > 1. \) Assume that \( L_{\sup} := \limsup_{n \to \infty} n^{\beta} v_n(g), \) is finite. Then, there is an \( n_{\xi} \in \mathbb{N}_+ \) such that for each \( n \geq n_{\xi}, \) we can pick a \( Q_n \in P_n(J \subset H; g) \) satisfying

\[
\text{vol}(\Omega \setminus Q_n) \leq \xi n^{\beta}. \tag{3.10}
\]

As the cuts of \( g \) are Jordan measurable, Lemma 3.1 implies that \( \delta(Q_n) \to 0 \) as \( n \to \infty. \) Consequently, \( n_{\xi} \) can be chosen so that (3.9) continues to hold, and for all source-size pairs \((w, \delta)\) of \( Q_n, \) \( n \geq n_{\xi}, \) we have that

(a) \( \delta < \delta_g \) (see condition \( \circ \) on \( g); \)

(b) \( C(w, \delta; g) \subseteq B_2(w; \tau) \) and \( C(w, 4\delta; g) \cap \partial \Omega \subset H; \) and

(c) \( C(w, 2\delta; f) \subseteq B_2(w; \tau). \)

The second part of (b) is possible as each cut of \( Q_n \) is compelled to intersect \( J \) non-trivially, by definition. For a fixed source-size pair \((w, \delta)\) of \( Q_n, \) we have, due to (3.7) and (3.6),

\[
C(w, \delta; g) \subseteq C(w, \bar{\bar{\varepsilon}} \delta; f) \subseteq C(w, \bar{\bar{\varepsilon}}^2 \delta; g). \tag{3.11}
\]

The second inclusion is valid as \( \bar{\bar{\varepsilon}} \delta < 3\delta, \) thus permitting the use of (3.6), given (c).
We can now approximate $Q_n$ by an $f$-polyhedron by setting
\[
\bar{Q}_n := \{ z \in \Omega : |g(z, w)| > \varepsilon^2 \delta, (w, \delta) \text{ is a source-size pair of } Q_n \} ; \\
\bar{P}_n := \{ z \in \Omega : |f(z, w)| > \varepsilon \delta, (w, \delta) \text{ is a source-size pair of } Q_n \} .
\]

Our assumptions imply that $\bar{Q}_n$ and $\bar{P}_n$ are in $\mathcal{P}_n(J \subset H; g)$ and $\mathcal{P}_n(J \subset H; f)$, respectively. From the above inclusions, we have that $\bar{Q}_n \subseteq P_n \subseteq Q_n$, $n \geq n_\xi$. Hence, by property $\circ$ of $g$ and (3.9), we see that
\[
n^\beta v_n(f) \leq n^\beta \text{vol}(\Omega \setminus \bar{P}_n) \leq n^\beta \text{vol}(\Omega \setminus \bar{Q}_n) \leq \mathcal{E}(\varepsilon^2 - 1)n^\beta \text{vol}(J; \Omega \setminus Q_n) \leq \xi \mathcal{E}(\varepsilon^2 - 1) L_{\text{sup}},
\]
for $n \geq n_\xi$. As $\xi > 0$ was arbitrary and $\varepsilon = \frac{1 + \xi}{1 - \xi}$, (3.8) follows.

To complete this proof, we show that
\[
\liminf_{n \to \infty} n^\beta v_n(f) \geq \mathcal{E} \left( \frac{(1 + \varepsilon)^4}{(1 - \varepsilon)^4} - 1 \right)^{-1} \liminf_{n \to \infty} n^\beta v_n(g).
\]

For this, fix a $\xi > 1$, and assume that $L_{\text{inf}} := \liminf_{n \to \infty} n^\beta v_n(g)$, is finite. Thus, there is an $n_\xi \in \mathbb{N}_+$ such that
\[
v_n(g) \geq \frac{1}{\xi} L_{\text{inf}} n^{-\beta}; \quad \text{for } n \geq n_\xi.
\]

For each $n$, we pick an $R_n \in \mathcal{P}_n(J \subset H; f)$ that satisfies
\[
v(\Omega \setminus R_n) \leq \xi v_n(f).
\]

Now, we may also assume that $\liminf_{n \to \infty} n^\beta v_n(f) < \infty$ (else, there is nothing to prove), thus obtaining that $v_n(f) \to 0$ for infinitely many $n \in \mathbb{N}_+$. But, as $v_n(f)$ is decreasing in $n$, we get that $v_n(f) \to 0$ for all $n \in \mathbb{N}_+$. At this point, we wish to invoke Lemma 3.1. This can be done as, owing to (3.7), it is possible to choose $\delta$ small enough so that
\[
C \left( w, \frac{\delta}{\varepsilon}; g \right) \subseteq C(w, \delta; f),
\]
for each $w \in H$. Therefore, $\delta(R_n) \to 0$ as $n \to \infty$. As before, we find a new $n_\xi$ such that (3.11) continues to hold, and for all $n \geq n_\xi$ and all source-size pairs $(w, \delta)$ of $R_n$, we have
\[
(a') \delta < \delta_g \quad \text{(see condition } \circ \text{ on } g); \\
(b') C(w, 4\delta; f) \subseteq B_2(w; \tau) \text{ and } C(w, 4\delta; f) \cap \partial \Omega \subset H; \text{ and} \\
(c') C(w, 2\delta; g) \subseteq B_2(w; \tau).
\]

Then, as before
\[
C \left( w, \frac{\delta}{\varepsilon}; g \right) \subseteq C(w, \delta; f) \subseteq C \left( w, \varepsilon \delta; g \right) \subseteq C \left( w, \varepsilon^2 \delta; f \right) \subseteq C \left( w, \varepsilon^3 \delta; g \right).
\]

We now approximate $R_n$ with an $n$-faceted $g$-polyhedron, using
\[
\tilde{R}_n := \{ z \in \Omega : |f(z, w)| > \varepsilon^2 \delta, (w, \delta) \text{ is a source-size pair of } R_n \} ; \\
\tilde{S}_n := \{ z \in \Omega : |g(z, w)| > \varepsilon \delta, (w, \delta) \text{ is a source-size pair of } R_n \} .
\]

Our assumptions are designed to ensure that $\tilde{R}_n \in \mathcal{P}_n(J \subset H; f)$ and $\tilde{S}_n \in \mathcal{P}_n(J \subset H; g)$. From the above inclusions, we have that
\[
\tilde{R}_n \subseteq \tilde{S}_n \subseteq R_n, \quad n \geq n_\xi.
\]
Moreover, the first and last inclusions in (3.13) and the assumption $\otimes$ on $g$ (note that $\hat{\varepsilon}^4 < 16$) imply that
\begin{equation}
\begin{aligned}
&v(\Omega \setminus \tilde{R}_n) - v(\Omega \setminus R_n) \\
\leq & \ v \left( \bigcup_{(w,\delta) \in \Lambda_n} C \left( w, \varepsilon^3\delta; g \right) - \bigcup_{(w,\delta) \in \Lambda_n} C \left( w, \delta; g \right) \right) \\
\leq & \ E(\hat{\varepsilon}^4 - 1)v(\Omega \setminus R_n),
\end{aligned}
\end{equation}
(3.14)
where $\Lambda_n$ is the set of source-size pairs of $R_n$.

Therefore, using (3.14) and (3.12), we see that
\begin{equation}
\frac{1}{\xi} L_{\inf} n^{-\beta} \leq v_n(g) \leq v(\Omega \setminus S_n) \leq v(\Omega \setminus \tilde{R}_n) \leq E(\hat{\varepsilon}^4 - 1)v(\Omega \setminus R_n) \leq E(\hat{\varepsilon}^4 - 1)\xi v_n(f).
\end{equation}
Therefore,
\begin{equation}
n^\beta v_n(f) \geq \xi^{-2} E(\hat{\varepsilon}^4 - 1)^{-1} L_{\inf}, \quad n \geq n_\xi.
\end{equation}
As $\xi > 0$ was arbitrary and $\hat{\varepsilon} = \frac{4}{1 + \varepsilon}$, (3.10) follows. \qed

**Remark 3.3.** In practice, $f$ and $g$ may only be defined on $(\overline{\Omega} \cap U) \times H$ for some open set $U \subset \mathbb{C}^2$ containing a $\tau$-neighborhood of $H$, while satisfying the analogous version of condition (i) there. As the remaining hypothesis (and indeed the result itself) depends only on the values of $f$ and $g$ on an arbitrarily thin tubular neighborhood of $H$ in $\overline{\Omega}$, we may replace $f$ (and, similarly, $g$) by $f_e$ to invoke Lemma 3.2, where
\begin{equation}
f_e := f(z, w)\varsigma(|z - w|^2) + |z - w|^2(1 - \varsigma(|z - w|^2))
\end{equation}
for some non-negative $\varsigma \in C^\infty(\mathbb{R})$ such that $\varsigma(x) = 1$ when $x \leq \tau^2/2$ and $\varsigma(x) = 0$ when $x \geq \tau^2$. We will do so without comment, when necessary.

4. Approximating Model Domains

As a first step, we examine volume approximations of the Siegel domain by a particular class of analytic polyhedra. This problem enjoys a connection with Laguerre-type tilings of the Heisenberg surface equipped with the Korányi metric (see the appendix for further details).

Let $\mathcal{S} := \{(z_1, x_2 + iy_2) \in \mathbb{C}^2 : y_2 > |z_1|^2 \}$ and $f_{\mathcal{S}}(z, w) = z_2 - \overline{w_2} - 2iz_1\overline{w_1}$. We view $\mathbb{C} \times \mathbb{R}$ as the first Heisenberg group, $\mathbb{H}$, with group law
\begin{equation}
(z_1, x_2, \overline{w_1}, w_2) \cdot (u_1, x_2, \overline{w_1}, u_2) = (z_1 + w_1, x_2 + u_2 + 2 \text{Im}(z_1\overline{w_1}))
\end{equation}
and the left-invariant Korányi gauge metric (see [7, Sec. 2.2])
\begin{equation}
d_{\mathbb{H}}((z_1, x_2), (w_1, u_2)) := ||(w_1, u_2)^{-1} \cdot (z_1, x_2)||_{\mathbb{H}},
\end{equation}
where $||((z_1, x_2)||_{\mathbb{H}} := |z_1|^4 + x_2^2$. Observe that, for any cut $C(w, \delta) = C(w, \delta; f_{\mathcal{S}})$, $w \in \partial \mathcal{S}$, $C(w, \delta)'$ is the set
\begin{equation}
K(w', \sqrt{\delta}) = \{(z_1, x_1) \in \mathbb{C} \times \mathbb{R} : |z_1 - w_1|^4 + (x_2 - u_2 + 2 \text{Im}(z_1\overline{w_1}))^2 \leq \delta^2 \},
\end{equation}
which is the ball of radius $\sqrt{\delta}$ centered at $w'$, in the Korányi metric.

**Notation.** We will use the following notation in this section:

- $I^r := \{(x_1 + iy_1, x_2) \in \mathbb{C} \times \mathbb{R} : 0 \leq x_1 \leq r, 0 \leq y_1 \leq r, 0 \leq x_2 \leq r^2 \}, r > 0$. 
• $\hat{I}^r := I^{2r} - \left( \frac{r}{2} + i \frac{r}{2}, 3r^2 \right)$, $r > 0$. $I^r \subset \hat{I}^r$ and they are concentric.
• $v_n(J \subset H) := v(S; P_n(J \subset H; f_s))$, for $J \subset H \subset \partial S$. If $J \subset H \subset \mathbb{C} \times \mathbb{R}$, $v_n(J \subset H)$ is meaningful in view of the obvious correspondence between $\mathbb{C} \times \mathbb{R}$ and $\partial S$.

**Lemma 4.1.** Let $I = I^1$ and $\hat{I} = \hat{I}^1$. There exists a positive constant $l_{kor} > 0$ such that

$$v_n(I \subset \hat{I}) \sim \frac{l_{kor}}{\sqrt{n}}$$

as $n \rightarrow \infty$.

**Proof.** Simple calculations show that

(4.2) $\quad \text{vol}(C(w, \delta)) = \frac{2\pi}{3} \delta^3$

(4.3) $\quad \text{vol}(K(w', \sqrt{\delta})) = \frac{\pi^2}{2} \delta^2$

for all $w \in \partial S$ and $\delta > 0$.

We utilize a special tiling in $\mathbb{C} \times \mathbb{R}$. Let $k \in \mathbb{N}_+$ and consider the following points in $\mathbb{C} \times \mathbb{R}$:

$$v_{pqr} := \left( \frac{p}{k} + \frac{q}{k}, \frac{r}{k^2} \right), \quad (p, q, r) \in \Sigma_k,$$

where $\Sigma_k := \{ (p, q, r) \in \mathbb{Z}^3 : -2q \leq r \leq k^2 - 1 + 2p, \ 0 \leq p, q \leq k - 1 \}$. Observe that card($\Sigma_k$) = $k^4 + 2k^3 - 2k^2$. Now, we set $E_{pqr} := v_{pqr} \cdot \hat{I}^1_k$, and note that $I \subset \cup \Sigma_k E_{pqr} \subset \hat{I}$, for all $k \in \mathbb{N}_+$.

![Figure 1. The 24 tiles $E_{pqr}$ when $k = 2$.](image)

1. We first show that there is a constant $\alpha_1 > 0$ such that

(4.4) $\quad v_n(I \subset \hat{I}) \leq \frac{\alpha_1}{\sqrt{n}}$

for all $n \in \mathbb{N}_+$. 
For this, let
\[ u_{pqr} := \text{center of } E_{pqr} = v_{pqr} \cdot \left( \frac{1}{2k} + i \frac{1}{2k^2}, \frac{1}{2k^2} \right), \quad (p, q, r) \in \Sigma_k, \quad k \in \mathbb{N}_+. \]

Then, the Korányi ball \( K \left( u_{pqr}, \frac{\sqrt{5}}{\sqrt{2k}} \right) \) (see (4.1)) contains \( E_{pqr} \) and is contained in \( \hat{I} \). Hence, if \( w_{pqr} \in \partial S \) is such that \( w'_{pqr} = u_{pqr} \), the cuts
\[
C \left( w_{pqr}, \frac{\sqrt{5}}{\sqrt{2k^2}}; f_S \right), \quad (p, q, r) \in \Sigma_k,
\]
define \( P_k \), an \( f_S \)-polyhedron over \( I \) with \( k^4 + 2k^3 - 2k^2 \) facets. In fact, \( P_k \in \mathcal{P}_{k^4+2k^3-2k^2}(I \subset \hat{I}; f_S) \), for all \( k \in \mathbb{N}_+ \). Therefore, using (4.2)
\[
v_{k^4+2k^3-2k^2}(I \subset \hat{I}) \leq \ vol(S \setminus P_k)
\]
\[
\leq \ vol \left( \bigcup_{\Sigma_k} C \left( w_{pqr}, \frac{\sqrt{5}}{\sqrt{2k^2}} \right) \right)
\]
\[
\leq \frac{2\pi}{3} \left( \frac{\sqrt{5}}{\sqrt{2k^2}} \right)^3 (k^4 + 2k^3 - 2k^2) = \frac{5\sqrt{5}\pi}{3\sqrt{2}} \frac{(k^4 + 2k^3 - 2k^2)}{k^6},
\]
k \in \mathbb{N}_+. Now, for a given \( n \in \mathbb{N}_+ \), choose \( k \) such that \( k^4 + 2k^3 - 2k^2 \leq n \leq (k + 1)^4 + 2(k + 1)^3 - 2(k + 1)^2 \). Then, one can easily find a \( \alpha_1 > 0 \) such that
\[
v_n(I \subset \hat{I}) \leq v_{k^4+2k^3-2k^2}(I \subset \hat{I}) \sqrt{(k + 1)^4 + 2(k + 1)^3 - 2(k + 1)^2}
\]
\[
\leq \frac{5\sqrt{5}\pi}{3\sqrt{2}} \frac{(k^4 + 2k^3 - 2k^2)}{k^6} \sqrt{(k + 1)^4 + 2(k + 1)^3 - 2(k + 1)^2}
\]
\leq \alpha_1.

2. Next, we show that there is an \( \alpha_2 > 0 \) such that
\[
(4.5)
\]
\[
v_n(I \subset \hat{I}) \geq \frac{\alpha_2}{\sqrt{n}}
\]
for \( n \in \mathbb{N}_+ \).

If finitely many Korányi balls of radii \( \sqrt{\rho_1}, \ldots, \sqrt{\rho_k} \) cover \( I \), then (4.3) yields
\[
(4.6)
(\sqrt{\rho_1})^4 + \cdots + (\sqrt{\rho_k})^4 \geq \frac{2}{\pi^2} |I| = \frac{2}{\pi^2}.
\]

We will also need the following mean inequality
\[
(4.7)
\left( \rho_1^{d+1} + \cdots + \rho_k^{d+1} \right)^{\frac{1}{d+1}} \geq \left( \rho_1^{d-1} + \cdots + \rho_k^{d-1} \right)^{\frac{1}{d+1}}
\]
for positive \( \rho_j \), \( 1 \leq j \leq k \), and \( d > 1 \).

Now, fix a \( \xi > 1 \). Let \( P_n \in \mathcal{P}_n(I \subset \hat{I}; f_S) \) be such that
\[
\text{vol}(S \setminus P_n) \leq \xi v_n(I \subset \hat{I}).
\]
Let \( C_j(n) \) and \( K_j(n) \), \( j = 1, \ldots, n \), be the cuts and their projections, respectively, of \( P_n \). Now, \( \mathcal{K}_n := \{ K_j(n), j = 1, \ldots, n \} \) is a finite covering of \( I \), so by the Wiener covering lemma (see [17, Lemma 4.1.1] for a proof that generalizes to metric spaces), we can find disjoint Korányi balls \( K_1, \ldots, K_k \in \mathcal{K}_n \), of radii \( \sqrt{\rho_1}, \ldots, \sqrt{\rho_k} \), such that \( \bigcup_{K \in \mathcal{K}_n} K \subset \bigcup_{1 \leq j \leq k} 3K_j \), where, for \( j = 1, \ldots, k \),
3 \cdot K_j \) has the same centre as \( K_j \) but thrice its radius. Let \( C_j \) denote the cut that projects to \( K_j \), \( j = 1, \ldots, k \). It follows from (4.2) and the inequalities (4.7) (for \( d = 5 \)) and (4.6) that

\[
v_n(I \subset \hat{I}) \geq \sqrt{n} \left( \sum_{j=1}^{k} \text{vol}(C_j) \right) \sqrt{k}
\]

\[
= \xi \left( \sum_{i=1}^{n} \text{vol}(C_j) \right) \sqrt{k} = \xi \frac{2\pi}{3} (\rho_1^3 + \cdots + \rho_k^3) \sqrt{k}
\]

\[
= \xi \frac{2\pi}{3} ((9\rho_1)^3 + \cdots + (9\rho_k)^3) \sqrt{k} = \xi \frac{2\pi}{3} ((3\sqrt{\rho_1})^6 + \cdots + (3\sqrt{\rho_k})^6) k^{\frac{3}{2}}
\]

\[
\geq \xi \frac{4\sqrt{2}}{\pi^2} |I|^\frac{3}{2} = \xi \frac{4\sqrt{2}}{\pi^2} > 0, \text{ for } n = n_0, n_0 + 1, \ldots
\]

As \( \xi > 1 \) was arbitrary, we have proved (4.5).

3. Define

\[
l_{kor} = \liminf_{n \to \infty} v_n(I \subset \hat{I}) \sqrt{n}.
\]

By (4.5) and (4.4), \( 0 < l_{kor} < \infty \). We now show that

\[
l_{kor} = \lim_{n \to \infty} v_n(I \subset \hat{I}) \sqrt{n}.
\]

For this, it suffices to show that for every \( \xi > 1 \), if \( n_0 \in \mathbb{N}_+ \) is chosen so that

\[
v_{n_0}(I \subset \hat{I}) \sqrt{n_0} \leq \xi l_{kor}
\]

then,

\[
v_n(I \subset \hat{I}) \sqrt{n} \leq \xi^4 l_{kor}
\]

for \( n \) sufficiently large.

Now, let \( P_{n_0} \in \mathcal{P}_{n_0}(I \subset \hat{I}; f_S) \) be such that

\[
\text{vol}(S \setminus P_{n_0}) \leq \xi v_{n_0}(I \subset \hat{I}).
\]

For any \( w \in \partial S \) and \( k \in \mathbb{N}_+ \), let \( A_{w,k} : \mathbb{C}^2 \to \mathbb{C}^2 \) be the biholomorphism

\[
(z_1, z_2) \mapsto \left( w_1 + \frac{1}{k} z_1, w_2 + \frac{1}{k^2} z_2 - \frac{2i}{k} z_1 \overline{w_1} \right).
\]

Then, \( A_{w,k} \) has the following properties:

- \( A_{w,k}(z') = w' \cdot z (1, \frac{1}{k} x_2) \);
- \( A_{w,k}(S) = S \);
- \( A_{w,k}(P_{n_0}) \in \mathcal{P}_{n_0}(w' \cdot z (1, \frac{1}{k} x_2) \subset w' \cdot z (1, \frac{1}{k} x_2) \hat{I}^\frac{1}{2} ; f_S) \); and
- \( \text{vol}(S \setminus A_{w,k}(P_{n_0})) \leq \xi v_{n_0}(I \subset \hat{I}) \frac{k}{k^2} \).

As a consequence,

\[
P := \bigcup_{\Sigma_k} A_{v_{por},k}(P_{n_0})
\]

satisfies the following conditions:

- \( P \in \mathcal{P}_{n_0}(k^4 + 2k^3 - 2k^2)(I \subset \hat{I} ; f_S) \)
- \( \text{vol}(S \setminus P) \leq \xi v_{n_0}(I \subset \hat{I}) k^4 + 2k^3 - 2k^2 \).
Hence, by assumption (4.9),

\[
v_{n_0(k^4+2k^3-2k^2)}(I \subset \hat{I}) \sqrt{n_0(k^4+2k^3-2k^2)} \leq \xi v_{n_0}(I \subset \hat{I}) \sqrt{n_0(k^4+2k^3-2k^2)^2} / k^6
\]

(4.11)

for sufficiently large \(k\). Choose \(k_0\) so that (4.11) holds and \(n_0(k^4+2k^3-2k^2) \leq \xi v_{n_0}(I \subset \hat{I}) \sqrt{n_0(k^4+2k^3-2k^2)^2} \leq \xi l_{kor} \) for \(k > k_0\).

For \(n \geq n_0(k_0^4+2k_0^3-2k_0^2)\), let \(k\) be such that \(n_0(k^4+2k^3-2k^2) \leq n \leq n_0((k+1)^4+2(k+1)^3-2(k+1)^2)\). Consequently,

\[
v_n(I \subset \hat{I}) \sqrt{n} \leq v_{n_0(k^4+2k^3-2k^2)}(I \subset \hat{I}) \sqrt{n_0((k+1)^4+2(k+1)^3-2(k+1)^2)} \leq \xi l_{kor} \sqrt{n_0(k^4+2k^3-2k^2)^2} \leq \xi^2 l_{kor},
\]

by (4.11). We have proved (4.10) and, therefore, our claim (4.8).

Our choice of the unit square in the above lemma facilitates the computation for polyhedra lying above more general Jordan measurable sets in the boundary of \(S\).

**Lemma 4.2.** Let \(J, H \subset \partial S\) be compact and Jordan measurable with \(J \subset \text{int}_{\partial S} H\). Then

\[
v_n(J \subset H) \sim |J|^\frac{3}{2} l_{kor} \frac{1}{\sqrt{n}}
\]

as \(n \to \infty\).

**Proof.** 1. We first show that

(4.12) \(\limsup_{n \to \infty} v_n(J \subset H) \sqrt{n} \leq l_{kor} |J|^\frac{3}{2}\).

Let \(\xi > 1\) be fixed. As \(J\) is Jordan measurable, we can find \(m\) points \(v_1, \ldots, v_m \in C \times \mathbb{R}\) and some \(r > 0\), such that

(4.13) \(J \subset \bigcup_{1}^{m} (v_j \cdot \mathbb{I}^r) \subset \bigcup_{1}^{m} (v_j \cdot \hat{\mathbb{I}}^r) \subset H\)

and

(4.14) \(m|I^r| \leq \xi |J|\).

Now, observe that

(4.15) \(v_n(v_j \cdot \mathbb{I}^r \subset v_j \cdot \hat{\mathbb{I}}^r) = v_n(I^r \subset \hat{I}^r) = r^6 v_n(I \subset \hat{I})\).

Thus, due to (4.13), Lemma 4.1 and (4.14), we have

\[
v_{km}(J \subset H) \sqrt{km} \leq \sum_{j=1}^{m} v_k(v_j \cdot \mathbb{I}^r) \sqrt{k} \sqrt{m} \leq \xi l_{kor} |I^r|^\frac{3}{2} m^\frac{3}{2} \leq \xi^2 l_{kor} |J|^\frac{3}{2}
\]

(4.16)
for \( k \) sufficiently large. Choose \( k_0 \in \mathbb{N}_+ \) such that for \( k \geq k_0 \), (4.16) holds and \( \sqrt{(k+1)/k} \leq \xi \). For sufficiently large \( n \), we can find a \( k \geq k_0 \) such that \( mk \leq n \leq m(k+1) \). Hence,

\[
v_n(J \subset H)\sqrt{n} \leq v_{km}(J \subset H)\sqrt{(k+1)m} \leq \xi^2 l_{kor}|J|^2 \left( \frac{k+1}{k} \right) \leq \xi^2 l_{kor}|J|^2.
\]

As \( \xi > 1 \) was arbitrarily fixed, we have proved (4.12).

2. It remains to show that

\[
\lim\inf_{n \to \infty} v_n(J \subset H)\sqrt{n} \geq l_{kor}|J|^2.
\]

Once again, fix a \( \xi > 1 \). The Jordan measurability of \( J \) ensures that there are pairwise disjoint sets \( I_1, \ldots, I_m \), where \( I_j = v_j \cap I^{r_j} \) for some \( r_j > 0 \) and \( v_j \in C \times \mathbb{R} \), \( 1 \leq j \leq m \), such that

\[
\bigcup_{j=1}^{m} I_j \subset J \quad \text{and} \quad \bigcup_{j=1}^{m} \hat{I}_j \subset J,
\]

where \( \hat{I}_j = v_j \cap \hat{I}^{r_j} \), and

\[
|J| \leq \xi \sum_{j=1}^{m} |I_j|.
\]

Choose a \( P_n \in \mathcal{P}(J \subset H; f_S) \) such that \( v(S \setminus P_n) \leq \xi v_n(J \subset H) \) and let \( n_j \) denote the number of cuts of \( P_n \) whose projections intersect \( I_j \) and are contained in \( \hat{I}_j \). By part 1., \( v_n(J \subset H) \to 0 \) as \( n \to \infty \). Thus, recalling (4.2), \( \delta(P_n) \to 0 \) as \( n \to \infty \). So, we may choose \( n \) so large that the projections of these \( n_j \) cuts, in fact, cover \( I_j \) and no two cuts of \( P \) whose projections intersect two different \( I_j \)’s intersect. Therefore,

\[
n_1 + \cdots + n_m \leq n.
\]

By Lemma 4.1 and (4.15), there is an \( n_0 \in \mathbb{N}_+ \) such that

\[
v_k(I_j) \geq \frac{1}{\xi} l_{kor}|I_j|^3 \frac{1}{\sqrt{k}}
\]

for \( k \geq n_0 \) and \( j = 1, \ldots, m \). We may further increase \( n \) to ensure that

\[
n_j \geq n_0 \quad \text{for} \quad j = 1, \ldots, m.
\]

Consequently, by (4.18) and (4.21), we have,

\[
v_n(J \subset H) \geq \frac{1}{\xi} \sum_{j=1}^{m} v_{n_j}(I_j) \geq \frac{l_{kor}}{\xi^2} \sum_{j=1}^{m} \frac{|I_j|^3}{\sqrt{n_j}}.
\]

Now, Hölder’s inequality yields,

\[
\sum_{j=1}^{m} |I_j| = \sum_{j=1}^{m} \left( \frac{|I_j|}{n_j^{2/6}} \right)^{n_j/6} \leq \left( \sum_{j=1}^{m} \left( \frac{|I_j|}{n_j^{2/4}} \right)^{\frac{3}{2}} \right)^{\frac{2}{3}} \left( \sum_{j=1}^{m} n_j \right)^{\frac{2}{3}}.
\]

Using this, (4.19) and (4.20), we obtain

\[
v_n(J \subset H) \geq \frac{l_{kor}}{\xi^2} \left( \sum_{j=1}^{m} |I_j| \right)^{\frac{6}{4}} \left( \sum_{j=1}^{m} n_j \right)^{\frac{2}{4}} \geq \frac{l_{kor}}{\xi^{7/2}} |J|^3 \frac{1}{\sqrt{n}}.
\]
for $n$ sufficiently large. As the choice of $\xi > 1$ was arbitrary, (4.17) now stands proved.

As a final remark, we extend the above lemma to a class of slightly more general model domains in order to illustrate the effect of the Levi-determinant on our asymptotic formula.

**Corollary 4.3.** Let $S_\lambda := \{(z_1, z_2) \in \mathbb{C}^2 : y > \lambda|z_1|^2\}$ and $f_{S_\lambda}(z, w) = \lambda(z - \overline{w}) - 2i\lambda^2(z_1\overline{w_1})$. Let $J, H \subset \partial S_\lambda$ be compact and Jordan measurable with $J \subset \text{int}_0 S_\lambda H$. Then

$$v_n(S_\lambda; J \subset H) := v(S; P_n(J \subset H; f_{S_\lambda})) \sim \lambda^{\frac{1}{2}}|J|^{\frac{3}{2}}l_{\text{kor}} \frac{1}{\sqrt{n}}$$

as $n \to \infty$.

**Proof.** Let $\Xi : \mathbb{C}^2 \to \mathbb{C}^2$ be the biholomorphism $\Xi : (z_1, z_2) \mapsto (\lambda z_1, \lambda z_2)$. Then, $S = \Xi(S_\lambda)$ and $f_{S_\lambda}(z, w) = f_{S}(\Xi(z), \Xi(w))$. Therefore, there is a bijective correspondence between $P_n(J \subset H; f_{S_\lambda})$ and $P_n(\Xi J \subset \Xi H; f_{S})$ given by $P \mapsto \Xi P$. Now, as $\text{det}(J_{\mathbb{R}} \Xi) \equiv \lambda^4$ and $\text{det}(J_{\mathbb{R}} \Xi^{\text{res}}) \equiv \lambda^3$, we have

$$\frac{v_n(S_\lambda; J \subset H)}{|J|^{\frac{3}{2}}} = \lambda^{-4}v_n(S; \Xi J \subset \Xi H) \sim \lambda^{\frac{1}{2}}l_{\text{kor}} \frac{1}{\sqrt{n}}.$$

□

5. **Local Estimates Via Model Domains**

Lemma 3.2 suggests a way to locally compare the volume-minimizing approximations drawn from two different classes of $f$-polyhedra which exhibit some comparability. In this section, we set up a local correspondence between $\Omega$ and a model domain $S_\lambda$, pull back the special cuts given by $f_{S_\lambda}$ (see Section 4) via this correspondence, and establish a (3.2)-type relationship between the pulled-back cuts and those coming from the Levi polynomial of a defining function of $\Omega$. First, we note a useful estimate on the Levi polynomial.

**Lemma 5.1.** Let $\Omega$ be a $C^2$-smooth strictly pseudoconvex domain. Suppose $\rho \in C^2(\mathbb{C}^2)$ is a strictly plurisubharmonic defining function of $\Omega$. Then, there exist constants $C > 0$ and $\tau > 0$ such that

\begin{equation}
|z - w|^2 \leq C|p(z, w)|,
\end{equation}

on $\Omega_\tau$, where $p$ is the Levi polynomial of $\rho$.

**Proof.** The second-order Taylor expansion of $\rho$ about $w \in \partial \Omega$ gives:

$$-2\text{Re } p(z, w) = -\rho(z) + \sum_{j,k=1}^{2} \frac{\partial^2 \rho(w)}{\partial z_j \partial \overline{z_k}} (z_j - w_j)(\overline{z_k} - \overline{w_k}) + o(|z - w|^2),$$

The strict plurisubharmonicity of $\rho$ implies the existence of a $c > 0$ so that

$$\sum_{j,k=1}^{2} \frac{\partial^2 \rho(w)}{\partial z_j \partial \overline{z_k}} (z_j - w_j)(\overline{z_k} - \overline{w_k}) \geq c|z - w|^2, \quad (z, w) \in \overline{\Omega} \times \overline{\Omega}.$$

The result follows quite easily from this. □

5.1. **Special Darboux Coordinates.**

**Notation.** As we are now going to construct a non-holomorphic transformation, we need to alternate between the real and complex notation. Here are some clarifications.

- We will use $z$ (and similarly $w$) to denote both $(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2$ and $(x_1, y_1, x_2, y_2) \in \mathbb{R}^4$. The usage will be clear from the context. In the same vein, by $z'$ we mean either $(z_1, x_2) = (x_1 + iy_1, x_2) \in \mathbb{C} \times \mathbb{R}$ or $(x_1, y_1, x_2) \in \mathbb{R}^3$. 
• Recall that $\langle \theta, z \rangle$ denoted the pairing between a complex covector and a complex vector. When $\theta$ is a real covector, we write $\langle \langle \theta, z \rangle \rangle$ to stress that $z$, here, is a tuple in $\mathbb{R}^4$.

Fix a $\lambda > 0$. For reasons that will become clear in the next section, we consider a special $C^4$-smooth strictly pseudoconvex domain $\Omega$ such that $0 \in \partial \Omega$ and for a neighborhood $U$ of the origin, there is a convex function function $\rho : U \to \mathbb{R}$ such that $\Omega \cap U = \{ z \in U : \rho(z) < 0 \}$ and

\[ \rho(z) = -\text{Im } z_2 + \lambda|z_1|^2 + 2\text{Re}(\bar{z}_1 z_2) + v|z_2|^2 + o(|z|^2). \]

We may shrink $U$ to find a convex function $F := F_\rho : U' \to \mathbb{R}$ that satisfies $\rho(z_1, x_2, F(z_1, x_2)) = 0$. $\rho$ and $F_\rho$ are both $C^4$-smooth and $-i(\partial \rho - \overline{\partial} \rho)$ is a $C^3$-smooth contact form on $\partial\Omega \cap U$. The domain $S_\lambda$ from Section 4 is such a domain with $\rho^\lambda(z) = -\text{Im } z_2 + \lambda|z_1|^2$ and $F_\rho^\lambda(z_1, x_2) = \lambda|z_1|^2$.

Darboux’s theorem in contact geometry (see [1, Appendix 4]) says that any two equi-dimensional contact structures are locally contactomorphic. We seek local diffeomorphisms between $\Omega$ and $\Omega$ such that $0 \in \partial \Omega$ and satisfy estimates essential to our goal. We carry out this construction over the next three lemmas, working initially on $\mathbb{R}^3$ instead of $\partial \Omega$. For this, if $\mathbf{gr}_\rho : U' \to U$ maps $(x_1, y_1, x_2)$ to $(x_1, y_1, x_2, F_\rho(x_1, y_1, x_2))$, we set

\[ \theta_\rho := (\mathbf{gr}_\rho)^* \left( \frac{\partial \rho - \overline{\partial} \rho}{i} \right) \]

\[ = \frac{-1}{\rho_{y_2}} \left( (\rho_{y_2} \rho_{x_1} + \rho_{x_2} \rho_{x_2})dx_1 - (\rho_{y_2} \rho_{x_1} - \rho_{y_1} \rho_{x_2})dy_1 + (\rho_{x_1}^2 + \rho_{x_2}^2)dx_2 \right), \]

where, by the partial derivatives of $\rho$ we mean their pull-backs to $U'$ via $\mathbf{gr}_\rho$.

**Lemma 5.2.** Let $\Omega$ be defined by (5.2). There is an open subset $(0 \in) V \subset U'$ and a $C^2$-smooth diffeomorphism $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3) : V \to \mathbb{R}^3$ with $\mathbf{d}(0) = 0$ satisfying

- $a(z')(\theta_\rho)(z') = (\mathbf{d}^* \theta_\rho^\lambda)(z')$ for all $z' \in V$, and some $a \in C(V)$ with $a(0) = 1$; and
- $|\det(\mathbf{d}_a(0))| = 1$.

**Proof.** We proceed with the understanding that when referring to functions defined a priori on $U$ (such as $\rho$ or its derivatives) we implicitly mean their pull-backs to $U'$ via $\mathbf{gr}_\rho$.

Now, consider the following $C^3$-smooth vector field in $\ker \theta_\rho$ on $U'$:

\[ v = \frac{\partial \rho}{\partial x_2} \frac{\partial}{\partial x_2} \frac{\partial}{\partial x_1} - \frac{\partial \rho}{\partial y_2} \frac{\partial}{\partial y_1} - \frac{\partial \rho}{\partial x_1} \frac{\partial}{\partial x_2}. \]

We let $\gamma^s(z') := \gamma(z'; s) = (\gamma_1(z', s), \gamma_2(z', s), \gamma_3(z', s))$ be the flow of $v$ such that $\gamma(z'; 0) = z'$. Note that $\gamma(z'; s)$ is $C^3$-smooth in $z'$ and $C^4$-smooth in $s$. Differentiating the initial value problem for the flow, we have

\[ (\gamma^0)_* \equiv \text{Id} \quad \text{and} \quad (\gamma^0)_{**} \equiv 0. \]

Observe that the map

\[ \Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) : z' = (x_1, y_1, x_2) \mapsto \gamma(x_1, 0, x_2; y_1), \]

is defined on some neighborhood, $U'_1 \subset U'$, of the origin. Moreover, dropping the arguments, switching to our shorthand notation and denoting $f \circ \Gamma$ by $\tilde{f}$, we have

\[ \Gamma_* = \begin{pmatrix} \Gamma_{1x_1} & \tilde{\rho}_{x_2} & \Gamma_{1x_2} \\ \Gamma_{2x_1} & -\tilde{\rho}_{y_2} & \Gamma_{2x_2} \\ \Gamma_{3x_1} & -\tilde{\rho}_{x_1} & \Gamma_{3x_2} \end{pmatrix}, \]
and
\[
\Gamma_*^{-1} = \left(\begin{array}{ccc}
\frac{\rho x_1 \Gamma_{1x_2} - \rho y_2 \Gamma_{1x_2}}{\det \Gamma_*} & \frac{-\rho x_1 \Gamma_{1x_2} - \rho y_2 \Gamma_{1x_2}}{\det \Gamma_*} & \frac{\rho y_2 \Gamma_{1x_2} + \rho x_1 \Gamma_{1x_2}}{\det \Gamma_*} \\
\frac{\Gamma_{2x_2} \Gamma_{1x_1} - \Gamma_{1x_2} \Gamma_{2x_1}}{\det \Gamma_*} & \frac{-\Gamma_{1x_2} \Gamma_{1x_1} + \Gamma_{1x_1} \Gamma_{2x_1}}{\det \Gamma_*} & \frac{\Gamma_{1x_1} \Gamma_{2x_1} - \Gamma_{1x_2} \Gamma_{2x_2}}{\det \Gamma_*} \\
\frac{\rho x_2 \Gamma_{1x_1} - \rho x_1 \Gamma_{1x_1}}{\det \Gamma_*} & \frac{\rho x_2 \Gamma_{1x_1} - \rho y_1 \Gamma_{1x_1}}{\det \Gamma_*} & \frac{-\rho y_2 \Gamma_{1x_1} - \rho x_1 \Gamma_{1x_1}}{\det \Gamma_*}
\end{array}\right),
\]
wherever \( \Gamma_* \) is invertible. In particular, \( \Gamma_*(0) = \Gamma_*^{-1}(0) = \text{Id} \). We may, therefore, locally invert \( \Gamma \) (as a \( C^3 \)-smooth function) in some neighborhood \( W_1 \subset U'_1 \) of 0. Let
\[
(X_1, Y_1, X_2) = \Gamma^{-1}(x_1, y_1, x_2).
\]
\( \Gamma \) is constructed to ‘straighten’ \( v \) — i.e., \( \Gamma_* \left( \frac{\partial}{\partial Y_1} \right) = v \). So, if we view \( X_1 \) and \( X_2 \) as \( C^3 \)-smooth functions on \( W := \Gamma(W_1) \cap U' \), they are linearly independent and \( v(X_1) = v(X_2) \equiv 0 \). Thus, \( dX_1 \wedge dX_2 \neq 0 \) everywhere on \( W \) and \( dX_1(v) \equiv dX_2(v) \equiv 0 \) on \( W \). So, it must be the case that
\[
\theta(\cdot) = w_1(\cdot)dX_1(\cdot) + w_2(\cdot)dX_2(\cdot),
\]
for some \( w_1, w_2 \in C^2(W) \). Substituting the expressions for \( \theta(\cdot), dX_1 \) and \( dX_2 \) (the latter two can be read off the matrix \( \Gamma_*^{-1} \) above), we get
\[
w_1 = -\Gamma_{1x_1} \tilde{\rho}_{y_2} (\rho y_1 \rho y_2 + \rho x_1 \rho x_2) + \Gamma_{2x_1} \tilde{\rho}_{y_2} (\rho y_1 \rho y_2 + \rho x_1 \rho x_2) - \tilde{\rho}_{y_2} (\rho y_1 \rho y_2 + \rho x_1 \rho x_2) \]
and
\[
w_2 = -\Gamma_{1x_1} \tilde{\rho}_{y_2} (\rho y_1 \rho y_2 + \rho x_1 \rho x_2) + \Gamma_{2x_1} \tilde{\rho}_{y_2} (\rho y_1 \rho y_2 + \rho x_1 \rho x_2) - \tilde{\rho}_{y_2} (\rho y_1 \rho y_2 + \rho x_1 \rho x_2),
\]
where, once again, \( \tilde{f} := f \circ \Gamma \). Observe that \( w_1(0) = 0 \) and \( w_2(0) = 1 \). Thus, for some neighborhood, \( V \subset W \), of the origin, \( w_2 \neq 0 \) and
\[
\theta(\cdot) = w_2(Y_1 dX_1 + dX_2),
\]
where \( Y_1 := w_1/w_2 \). Finally, set
\[
a := \frac{1}{w_2}, \quad d_1 := X_1, \quad d_2 := -\frac{Y_1}{4\lambda} \quad \text{and} \quad d_3 := X_2 + \frac{X_1 Y_1}{2}.
\]
Then, on \( V \),
\[
a \theta(\cdot) = -2\lambda d_2 d_1 + 2\lambda d_1 d_2 + d d_3 = \theta(\cdot)\lambda
\]
and \( a(0) = 1 \).
}

Refering to (5.3) and the formulae for \( w_1, w_2 \) and \( \Gamma^{-1}_* \), we get
\[
d_*(0) = \left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & -\frac{\text{Im} \mu}{2\lambda} \\
0 & 0 & 1
\end{array}\right).
\]
We have, thus, constructed the required map. \( \Box \)

We now show that the contact transformation constructed above satisfies an estimate crucial to our analysis.
Lemma 5.3. Let $\partial$ and $V$ be as in the proof of Lemma 5.2 and $V \Subset V$ be a neighborhood of the origin. Then, there is an $\varepsilon_1 \in C(V)$ with $\lim_{w' \to 0} \varepsilon_1(w') = 0$ and a $\varepsilon_1 > 0$ such that, for all $w' \in V$ and $z' \in \mathbb{R}^3$,

\[
|(z' - w')^T \cdot \text{Hess}_R \partial_3(w') \cdot (z' - w')| \leq \varepsilon_1(w')|z' - w'|^2 + \varepsilon_1(|z_1 - w_1||x_2 - u_2| + |x_2 - u_2|^2).
\]

(5.6)

Proof. Recall that $\partial_3 = X_2 + \frac{X_1 \lambda}{2}$. We refer to the construction in the proof of Lemma 5.2 and collect the following data:

\[
(X_1)_{x_1}(0) = 1, \quad (X_1)_{y_1}(0) = 0; \quad (Y_1)_{x_1}(0) = 0, \quad (Y_1)_{y_1}(0) = -4\lambda; \quad (X_2)_{x_1x_1}(0) = 0, \quad (X_2)_{x_1y_1}(0) = 2\lambda = (X_2)_{y_1x_1}(0), \quad (X_2)_{y_2y_2}(0) = 0.
\]

Next, we write out the relevant terms.

\[
(z' - w')^T \cdot \text{Hess}_R \partial_3(w') \cdot (z' - w') = (X_2x_1x_1(w') + X_1x_1(w')Y_1x_1(w') + \frac{1}{2}Y_1(w')X_1x_1x_1(w')(x_1 - u_1)^2
\]

\[
+ (2X_2x_1y_1(w') + X_1x_1(w')Y_1y_1(w') + X_1x_1y_1(w')Y_1x_1(w')(x_1 - u_1)(y_1 - v_1)
\]

\[
+ (Y_1(w')X_1x_1y_1(w') + X_1(w')Y_1x_1x_1(w')(x_1 - u_1)(y_1 - v_1)
\]

\[
+ (X_2y_1y_1(w') + X_1y_1(w')Y_1y_1(w') + \frac{1}{2}Y_1(w')X_1y_1y_1(w')(y_1 - v_1)^2
\]

\[
+ 2\partial_{3x_1x_2}(w')(x_1 - u_1)(x_2 - u_2) + 2\partial_{3y_1y_2}(w')(y_1 - v_1)(x_2 - u_2) + \partial_{3x_2x_2}(w')(x_2 - u_2)^2.
\]

Now, the coefficients of $(x_1 - u_1)^2$, $(x_1 - u_1)(y_1 - v_1)$ and $(y_1 - u_1)^2$ in the above expansion all vanish at the origin (see data listed above). Thus, we have that the estimate (5.6).

All that remains is to extend the above transformation to $\Omega$. For this, let $V$ be as in Lemma 5.2 and $G_\rho : \mathbb{R}^3 \to \mathbb{C}^2$ be the map

\[
(x_1, y_1, x_2, y_2) \mapsto (x_1, y_1, x_2, F_\rho(x_1, y_1, x_2) + y_2).
\]

$G_\rho$ is evidently a $C^4$-smooth diffeomorphism with $G(V \times (0, t]) \subset \Omega$ for some $t > 0$. We note the following facts about $G_\rho$:

- $(G_\rho)_*(0) = \text{Id}.$ and $(G_\rho^{\text{res}})_*(0) = \text{Id}.$

- $(G_\rho)^*(\partial \rho + \bar{\partial} \rho) = \left( \frac{\partial G_\rho}{\partial y_2} \circ G_\rho \right) dy_2$ and $(G_\rho)^*(\frac{\partial G_\rho}{\partial y_2} \circ G_\rho) = \theta_\rho$ on $V \times \{0\}$.

Lemma 5.4. There is a neighborhood $\mathcal{U}$ of the origin and a $C^2$-smooth diffeomorphism $\Psi : \mathcal{U} \to \mathbb{C}^2$ such that

- $\Psi(0) = 0$, $\Psi(\partial \mathcal{U}) = S_\lambda \cap \Psi(\mathcal{U})$ and $\Psi(\partial \mathcal{U}) \cap \Psi(\mathcal{U}) = \partial S_\lambda \cap \Psi(\mathcal{U})$;

- $\det(\Psi_*(0)) = 1$ and $\det(\Psi^{\text{res}}_*(0)) = 1$; and

- if $\iota_\rho$ and $\iota_\lambda$ denote the Cauchy-Leray map of $\rho$ and $\rho^\lambda$, respectively, then

\[
|\iota_\rho(z, w) - \iota_\lambda(\Psi(z), \Psi(w))| \leq (\mathcal{E}(w) + \mathcal{D}(z - w)) (|\iota_\lambda(\Psi(z), \Psi(w))| + |z - w|) + \mathcal{E}|\iota_\lambda(\Psi(z), \Psi(w))|^2,
\]

on $\{(z, w) \in \overline{\Omega} \times \mathcal{U} : |z - w| \leq \tau\}$, for some choice of $\mathcal{E} \in C(\mathcal{U})$ with $\lim_{w \to 0} \mathcal{E}(w) = 0$, $\mathcal{D}(\zeta) = o(1)$ as $|\zeta| \to 0$, and constants $\mathcal{E}, \tau > 0$. 

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Proof. Let \( \Psi = (\Psi_1, \Psi_2) := G_T \circ (\partial, \text{Id.}) \circ G_{\rho}^{-1} \), where \( \text{Id.} \) is the identity map on \( \mathbb{R} \), and \( \mathcal{U} \subseteq G_{\rho}(V \times [-t, t]) \) is a neighborhood of the origin. We use the notation \( (\Psi_1, \Psi_2) = (\psi_1 + i\bar{\psi}_2, \psi_3 + i\bar{\psi}_4) \). The regularity and mapping properties of \( \Psi \) follow from its definition. Since \( \text{id.}^{*}(-dy_2) = -dy_2 \) and \( \delta^{*}(\theta_{\rho}) = a\theta_{\rho} \) on \( \{y_2 = 0\} \),

\[
\Psi^{*}(\partial^\rho + \overline{\partial} \rho^\lambda) = a_1(\partial^\rho + \overline{\partial} \rho)
\]

and

\[
\Psi^{*}\left(\frac{\partial^\rho - \overline{\partial} \rho^\lambda}{i}\right) = a_2\left(\frac{\partial^\rho - \overline{\partial} \rho}{i}\right),
\]

on \( \partial \Omega \cap \mathcal{U} \), where \( a_1(x_1, y_1, x_2, y_2) = -\frac{\partial^\rho}{\partial y_2}(G_{\rho}(x_1, y_1, x_2, y_2)) \) and \( a_2(x_1, y_1, x_2, y_2) = a(x_1, y_1, x_2) \). Therefore, for all \( w \in \partial \Omega \cap \mathcal{U} \) and \( z \in \mathbb{C}^2 \),

\[
2\left\langle \partial^\rho \Psi(w), \Psi^*_w(z - w) \right\rangle (z - w) = 2 \Re \left\langle \partial^\rho \Psi(w), \Psi^*_w(z - w) \right\rangle + 2i \Im \left\langle \partial^\rho \Psi(w), \Psi^*_w(z - w) \right\rangle
\]

\[
= \left\langle \left\langle \left(\partial^\rho + \overline{\partial} \rho^\lambda\right)(\Psi(w)), \Psi^*_w(z - w) \right\rangle \right\rangle + i \left\langle \left\langle \left(\partial^\rho - \overline{\partial} \rho^\lambda\right)(\Psi(w)), \Psi^*_w(z - w) \right\rangle \right\rangle
\]

\[
= \left\langle \left\langle \Psi^*(\partial^\rho + \overline{\partial} \rho^\lambda)(w), z - w \right\rangle \right\rangle + i \left\langle \left\langle \Psi^*(\partial^\rho - \overline{\partial} \rho^\lambda)(w), z - w \right\rangle \right\rangle
\]

\[
= a_1(w)\left\langle \left\langle \left(\partial^\rho + \overline{\partial} \rho\right)(w), z - w \right\rangle \right\rangle + i a_2(w)\left\langle \left\langle \left(\partial^\rho - \overline{\partial} \rho\right)(w), z - w \right\rangle \right\rangle
\]

\[
= 2a_1(w) \Re \left\langle \partial^\rho(w), z - w \right\rangle + 2i a_2(w) \Im \left\langle \partial^\rho(w), z - w \right\rangle.
\]

Now, since \( \rho^\lambda := \lambda |z_1|^2 - y_2, \frac{\partial^\rho}{\partial z_1}(\Psi(z)) = \lambda \overline{\psi}_1(z) \) and \( \frac{\partial^\rho}{\partial z_2}(\Psi(z)) = \frac{i}{2} \). Therefore, there is a \( \tau_1 > 0 \) such that on \( \{(z, w) \in \mathbb{R}^4 \times \mathcal{U} : |z - w| \leq \tau_1\} \),

\[
\left\| \left\langle \partial^\rho \Psi(w), \Psi(z) - \Psi(w) - \Psi^*_w(z - w) \right\rangle \right\| \leq c|\Psi_1(w)| \cdot |z - w|^2 + \frac{1}{2}R_1(z - w) + R_2(z - w),
\]

where, \( c > 0, R_1(z - w) = |(z - w)^r \cdot (\text{Hess}_R \psi_3(w) + \text{Hess}_R \psi_4(w)) \cdot (z - w)| \), and \( R_2(\zeta) = o(|\zeta|^2) \) as \( |\zeta| \to 0 \). Observe that \( \psi_3(z, y_2) = \mathcal{O}_3(z') \) and \( \psi_4(z', y_2) = \mathcal{O}_1(z')^2 + \mathcal{O}_2(z')^2 + y_2 - F(z') \). As,

\[
\psi_{x_1 x_1}(w) = 2 \sum_{j=1}^2 (d_{jx_1}(w')^2 + d_j(w') d_{jx_1}(w')) - F_{x_1 x_1}(w'),
\]

\[
\psi_{y_1 y_1}(w) = 2 \sum_{j=1}^2 (d_{jy_1}(w')^2 + d_j(w') d_{jy_1}(w')) - F_{y_1 y_1}(w') \text{ and}
\]

\[
\psi_{x_1 y_1}(w) = 2 \sum_{j=1}^2 (d_{jx_1}(w') d_{jy_1}(w') + d_j(w') d_{jx_1 y_1}(w')) - F_{x_1 y_1}(w')
\]

all vanish at \( w = 0 \), we have, for all \( (z, w) \in \mathbb{R}^4 \times \mathcal{U} \),

\[
|(z - w)^r \cdot \text{Hess}_R \psi_4(w) \cdot (z - w)| \leq \mathcal{O}_2(w)|z - w|^2 + \mathcal{O}_2(|z_1 - w_1||z_2 - w_2| + |z_2 - w_2|^2),
\]

(5.10)
where \( \mathcal{E}_1 \in \mathcal{C}(\mathcal{U}) \) with \( \lim_{w \to 0} \mathcal{E}_1(w) = 0 \), and \( \mathcal{C}_1 > 0 \) is a constant. Combining (5.9), (5.6) and (5.10) (and adding \( c \mathcal{C}_1, \mathcal{E}_1 \) and \( \mathcal{E}_2 \)), we have that

\[
A := \left| \left( \partial \rho^4(\Psi(w)), \Psi(z) - \Psi(w) - \Psi_*(w)(z - w) \right) \right|
\]

\[
\leq (\mathcal{E}_3(w) + \mathcal{D}_3(z - w)) |z - w|^2 + \mathcal{C}_3(|z_1 - w_1| |z_2 - w_2| + |z_2 - w_2|^2),
\]
on \( \{(z, w) \in \mathbb{R}^4 \times \mathcal{U} : |z - w| \leq \tau_3\} \), for some \( \mathcal{E}_3 \in \mathcal{C}(\mathcal{U}) \) with \( \lim_{w \to 0} \mathcal{E}_3(w) = 0 \), \( \mathcal{D}_3(\zeta) = o(1) \) as \( |\zeta| \to 0 \), and constants \( \mathcal{C}_3, \tau_3 > 0 \).

Next, we have that

\[
|\Psi_2(z) - \Psi_2(w)| = 2 \left| \left( \partial \rho^4(\Psi(w)), \Psi(z) - \Psi(w) \right) - \mathcal{C}_1(z)(\Psi_1(z) - \Psi_1(w)) \right|
\]

\[
\leq \mathcal{C}_4 \left| \left( \partial \rho^4(\Psi(w)), \Psi(z) - \Psi(w) \right) \right| + \mathcal{E}_4(w)|z - w|,
\]
on \( \{(z, w) \in \mathbb{R}^4 \times \mathcal{U} : |z - w| \leq \tau_4\} \), for some choice of \( \mathcal{E}_4, \mathcal{C}_4 \) and \( \tau_4 \) as before. Also, if \( \Psi^{-1} = (\psi_1, \psi_2, \psi_3, \psi_4) \), then \( (\psi_3)_*(0) = (0, 0, 1, 0) \) and \( (\psi_4)_*(0) = (0, 0, 0, 1) \). So, we are permitted to conclude that

\[
|z_2 - w_2| \leq \mathcal{C}_4|\Psi_2(z) - \Psi_2(w)| + (\mathcal{E}_5(z) + \mathcal{D}_5(z - w)) |z - w|,
\]
on \( \{(z, w) \in \mathbb{R}^4 \times \mathcal{U} : |z - w| \leq \tau_5\} \), for some \( \mathcal{E}_5, \mathcal{C}_5, \mathcal{D}_5 \) and \( \tau_5 \) as before.

Finally, as \( a_1(0) = a_2(0) = 1 \), (5.8), (5.11), (5.12) and (5.13) combine to give an \( \mathcal{E}, \mathcal{C}, \mathcal{D} \) and \( \tau \) with the required properties, such that

\[
|I_\rho(z, w) - I_\rho(\Psi(z), \Psi(w))| \\
\leq \left| \left( \partial \rho(w), z - w \right) - \left( \partial \rho^4(\Psi(w)), \Psi_*(w)(z - w) \right) \right| + A,
\]

\[
\leq (\mathcal{E}(w) + \mathcal{D}(z - w)) (|I_\rho(\Psi(z), \Psi(w))| + |z - w|^2) + \mathcal{C}|I_\rho(\Psi(z), \Psi(w))|^2.
\]
on \( \{(z, w) \in \mathbb{R}^4 \times \mathcal{U} : |z - w| \leq \tau\} \). \( \Box \)

5.2. Convexification. In this section, we return to general strictly pseudoconvex domains. Assume \( 0 \in \partial \Omega \) and the outward unit normal vector to \( \partial \Omega \) at 0 is \((0, -i)\). Let \( \rho \) be a \( C^2 \)-smooth strictly plurisubharmonic defining function of \( \Omega \) such that \( |\nabla \rho(0)| = 1 \). Now, \( \rho \) has the following second-order Taylor expansion about the origin:

\[
\rho(w) = \text{Im} \left( -w_2 + i \sum_{j,k=1}^2 \frac{\partial^2 \rho(0)}{\partial z_j \partial z_k} w_j w_k \right) + \sum_{j,k=1}^2 \frac{\partial^2 \rho(0)}{\partial z_j \partial z_k} w_j \overline{w_k} + o(|w|^2).
\]

Using an old trick, attributed to Narasimhan, we convexify \( \Omega \) near the origin via the map \( \Phi \) given by:

\[
w_1 \mapsto \Phi_1(w) = w_1
\]
\[
w_2 \mapsto \Phi_2(w) = w_2 - i \sum_{j,k=1}^2 \frac{\partial^2 \rho(0)}{\partial z_j \partial z_k} w_j w_k.
\]

Owing to the inverse function theorem, \( \Phi \) is a local biholomorphism on some neighborhood \( U \) of 0. We may further shrink \( U \) so that the strong convexity of \( \Phi(\partial \Omega \cap U) \) at 0 propagates to all of \( \Phi(\partial \Omega \cap U) \). We collect the following key observations:

- \( \Phi_*(0) = \text{Id} \) and \( \Phi^{\text{res}}_*(0) = \text{Id} \);
- \( \text{If } \hat{\rho} := \rho \circ \Phi^{-1}, \text{ then } \hat{\rho}(w) = -\text{Im} w_2 + \sum_{j,k=1}^2 \frac{\partial^2 \rho(0)}{\partial z_j \partial z_k} w_j \overline{w_k} + o(|w|^2). \)
Lemma 5.5. \textbf{Main Local Estimate.} For sufficiently large $n$, let \( M \) be the strictly plurisubharmonic defining function of \( \Omega \). Assume that $0 \in \partial \Omega$, $\nabla \rho(0) = (0, 0, 0, -1)$ and $M(\rho)(0) = \lambda$. Then, there exists a neighborhood $U$ of the origin, a $C^2$-smooth origin-preserving diffeomorphism $\Theta$ on $U$ that carries $\Omega \cap U$ onto $\bar{\Theta}(\Omega \cap U)$, and a constant $\tau > 0$ such that

\begin{align*}
|p(z, w) - l_\lambda(\Theta(z), \Theta(w))| &\leq \left| \langle \partial \rho(w), (z - w) \rangle - \langle \partial \rho(\Theta(w)), \Phi(w)(z - w) \rangle \right| \\
&+ \frac{1}{2} \left( \sum_{j,k=1}^2 \left( \frac{\partial^2 \rho(w)}{\partial z_j \partial z_k} + 2i \frac{\partial \rho(\Theta(w))}{\partial w_2} \frac{\partial^2 \rho(0)}{\partial z_j \partial z_k} \right) (z_j - w_j)(z_k - w_k) \right) \\
&\leq \left( \langle \partial \rho(w), (z - w) \rangle - \langle \Phi^*(\partial \rho)(w), (z - w) \rangle \right) \\
&+ \frac{1}{2} \left( \sum_{j,k=1}^2 \left( \frac{\partial^2 \rho(0)}{\partial z_j \partial z_k} + o(1) + (-1 + o(|w|)) \frac{\partial^2 \rho(0)}{\partial z_j \partial z_k} \right) (z_j - w_j)(z_k - w_k) \right) \\
&\leq \mathcal{E}(w)|z - w|^2,
\end{align*}

for some $\mathcal{E} \in C(U)$ with $\lim_{w \to 0} \mathcal{E}(w) = 0$.

5.3. Main Local Estimate. We combine the maps constructed above:

\textbf{Lemma 5.5.} Fix an $\varepsilon > 0$. Let $\Omega \subset \mathbb{C}^2$ be a $C^4$-smooth strictly pseudoconvex domain and $\rho$ a strictly plurisubharmonic defining function of $\Omega$. Assume that $0 \in \partial \Omega$, $\nabla \rho(0) = (0, 0, 0, -1)$ and $M(\rho)(0) = \lambda$. Then, there exists a neighborhood $U$ of the origin, a $C^2$-smooth origin-preserving diffeomorphism $\Theta$ on $U$ that carries $\Omega \cap U$ onto $\bar{\Theta}(\Omega \cap U)$, and a constant $\tau > 0$ such that

\begin{itemize}
  \item $1 - \varepsilon \leq \frac{\text{vol}(\Theta(V))}{\text{vol}(V)} \leq \frac{1}{1 - \varepsilon}$, for every Jordan measurable $V \subset U$;
  \item $1 - \varepsilon \leq \frac{|\Theta(J)|}{|J|} \leq \frac{1}{1 - \varepsilon}$, for every Jordan measurable $J \subset \partial \Omega \cap U$; and
  \item if $P$ is the Levi polynomial of $\rho$ and $l_\lambda$ is the Cauchy-Leray map of $\rho^\lambda$, then
    \[
    |p(z, w) - l_\lambda(\Theta(z), \Theta(w))| \leq \varepsilon(|p(z, w)| + |l_\lambda(\Theta(z), \Theta(w))|)
    \]
    on $\{(z, w) \in (\bar{\Omega} \cap U) \times H : |z - w| \leq \tau\}$, where $H \subset \partial \Omega \cap U$ is compact.
\end{itemize}

\textbf{Proof.} The needed map is $\Psi \circ \Phi$ (see Sections 5.1 and 5.2). The mapping and volume distortion properties follow from those of $\Psi$ and $\Phi$. The estimate is a combination of (5.14), (5.7) and (5.1). \hfill \Box

The following lemma is an application of Lemma 3.2 and gives us a local version of our main theorem.

\textbf{Lemma 5.6.} Let $\Omega$, $f$ and $\rho$ be as in Theorem 1.1. Fix an $\varepsilon \in (0, 1/3)$ and a point $q \in \partial \Omega$. Then, there exists a neighborhood $U_{q,\varepsilon}$ of $q$ such that for every Jordan measurable pair $J, H \subset \partial \Omega \cap U_{q,\varepsilon}$ such that $J \subset \interior \partial \Omega H$,

\[
(1 - \varepsilon)^{31} \, l_{\text{kn}} \frac{\lambda(q)^{\frac{1}{2}} s(J)^{\frac{1}{2}}}{\sqrt{n}} \leq v(\Omega; \mathcal{P}_n(J \subset H; f)) \leq (1 - \varepsilon)^{-19} \, l_{\text{kn}} \frac{\lambda(q)^{\frac{1}{2}} s(J)^{\frac{1}{2}}}{\sqrt{n}}
\]

for sufficiently large $n$, where $\lambda(q) := \frac{4M(\rho)(q)}{|
abla \rho(q)|^2}$. 

Proof. Let \( \rho \) be the strictly plurisubharmonic defining function of \( \Omega \) for which \((\ast)\) in Theorem 1.1 holds. Let \( A : \mathbb{C}^2 \to \mathbb{C}^2 \) be an isometry that takes \( q \) to the origin and the outer unit normal at \( q \) to \((0, -i \nabla \rho(q))\). Set \( \tilde{\rho}(z) := |\nabla \rho(q)|^{-1} \rho(A^{-1}z) \). Then, \( \tilde{\rho} \) satisfies the hypotheses of Lemma 5.5, with \( M(\tilde{\rho})(0) = \lambda(q) \). Moreover, the Levi polynomial \( \tilde{p} \) of \( \tilde{\rho} \) satisfies

\[
|\nabla \rho(q)||\tilde{p}(Az, Aw) = p(z, w). \tag{5.15}
\]

Suppose \( \Theta \) and \( U \) are the map and neighborhood, respectively, granted by Lemma 5.5. Set \( V_q := A^{-1}(U) \) and \( \Theta_q := \Theta \circ A \). Note that \( \Theta_q \) maps \( \Omega \) to \( \overline{\Omega} \) locally near \( q \). We define

\[
\tilde{f}(z, w) := \frac{f(z, w)}{|\nabla \rho(q)|};
\]
\[
g(z, w) := fS_{\lambda(q)}(\Theta_q z, \Theta_q w); \quad \text{and}
\]
\[
\tilde{g}(z, w) := a(w, w) \left( \frac{2i}{\lambda(q)} fS_{\lambda(q)}(\Theta_q z, \Theta_q w) \right)^\nu = a(w, w)\lambda(q)(\Theta_q z, \Theta_q w)^\nu \text{ (see Section 4).}
\]

Observe that, when defined,

\[
C(w, \delta, \tilde{f}) = C(w, |\nabla \rho(q)|\nu \delta; f) \quad \text{and}
\]
\[
C(w, \delta, \tilde{g}) = C(w, \lambda(q) \frac{\delta}{|a(w, w)|} \nu; g). \tag{5.17}
\]

Thus, for our point of interest, there is little difference between \( f \) and \( \tilde{f} \) (and \( g \) and \( \tilde{g} \)).

Keeping this observation in mind, we will apply Lemma 3.2 to \( \tilde{f}, \tilde{g} \in C(\overline{\Omega} \times (V_q \cap \partial \Omega)) \) (see Remark 3.3). By \((\ast)\), there exist \( \tau_1 \in (0, \tau] \) and \( l > 0 \) such that

\[
|p(z, w)|^\nu \leq l|\tilde{f}(z, w)| \quad \text{on } \Omega_{\tau_1}. \tag{5.18}
\]

Now, fix an \( \varepsilon \in (0, 1/3) \). Let \( \tilde{\varepsilon} :=
\]
\[
\frac{\varepsilon}{2} \min \left\{ \frac{|\nabla \rho(q)|^\nu}{l}, \frac{2\nu|\nabla \rho(q)|^\nu \max_{\Omega} \{\{a(z, w)\}\}^{-1}}{l}, \left( 2\nu \max_{\partial \Omega} \{\{a(z, w)\}\}^{-1}, \min_{\partial \Omega} \{\{a(w, w)\}\} \right) \right\}. \tag{5.19}
\]

By \((\ast)\), we can find a \( \tau_2 \in (0, \tau] \) such that

\[
|\tilde{f}(z, w) - a(z, w)\tilde{p}(Az, Aw)^\nu| = \frac{|f(z, w) - a(z, w)p(z, w)^\nu|}{|\nabla \rho(q)|^\nu} \leq \frac{\tilde{\varepsilon}}{|\nabla \rho(q)|^\nu} |p(z, w)|^\nu \quad \text{on } \Omega_{\tau_2}. \tag{5.20}
\]

By Lemma 5.5, \(5.15\), and the continuity of \( a \) on \( \overline{\Omega} \), we shrink \( \tau_2 \) so that on \( \Omega_{\tau_2} \cap (\overline{\Omega} \times V_q) \),

\[
|a(z, w)|\tilde{p}(Az, Aw)^\nu - a(z, w)\lambda(q)(\Theta_q z, \Theta_q w)^\nu| \\
\leq |a(z, w)| (|\tilde{p}(Az, Aw) - \lambda(q)(\Theta_q z, \Theta_q w)|) \nu \max_{\partial \Omega} \{\{a(z, w)\}\} |\tilde{p}(Az, Aw)| |\lambda(q)(\Theta_q z, \Theta_q w)|^{-\nu-1} \\
\leq |a(z, w)| \tilde{\varepsilon} (|\tilde{p}(Az, Aw)| + |\lambda(q)(\Theta_q z, \Theta_q w)|) \nu \max_{\partial \Omega} \{\{a(z, w)\}\} |\tilde{p}(Az, Aw)| |\lambda(q)(\Theta_q z, \Theta_q w)|^{-\nu-1} \\
\leq 2\nu \tilde{\varepsilon} |a(z, w)| (|\tilde{p}(Az, Aw)|^\nu + |\lambda(q)(\Theta_q z, \Theta_q w)|^\nu) \\
\leq \tilde{\varepsilon} \left( 2\nu \max_{\partial \Omega} \{\{a(z, w)\}\} \right) |p(z, w)|^\nu + \tilde{\varepsilon} \left( 2\nu \min_{\partial \Omega} \{\{a(w, w)\}\} \right) |\tilde{g}(z, w)|, \tag{5.21}
\]

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and
\[
|a(z, w)l_{\lambda(q)}(\Theta_q z, \Theta_q w) - \tilde{g}(z, w)| = |a(z, w) - a(w, w)\cdot l_{\lambda(q)}(\Theta_q z, \Theta_q w)| \leq \frac{\hat{\varepsilon}}{\min_{\partial \Omega} |a(w, w)|} |\tilde{g}(z, w)|.
\]

Adding (5.20), (5.21) and (5.22), and recalling (5.19) and (5.18), we get
\[
|\tilde{f}(z, w) - \tilde{g}(z, w)| \leq \varepsilon \left( |\tilde{f}(z, w)| + |\tilde{g}(z, w)| \right) \quad \text{on } \Omega_{\tau_2} \cap (\overline{\Omega} \times \mathbb{V}_q).
\]

We now need to show that \( \tilde{g} \) satisfies the remaining hypotheses of Lemma 3.2. But these are conditions on the cuts of \( \tilde{g} \), which are identical to the cuts of \( g \) (by (5.17)). So, we work with \( g \) instead. Let \( U_{q, \varepsilon} \subset \mathbb{V}_q \) be an open neighborhood of \( q \), and \( \delta_0 > 0 \) be such that \( C(w, \delta; g) \subset \mathbb{V}_q \) for all \( w \in U_{q, \varepsilon} \cap \partial \Omega \) and \( \delta < \delta_0 \). Then,

\[
(5.23) \quad \Theta_q = \Theta \circ A : C(w, \delta; g) \to C(\Theta_q w, \delta; f_{S_q})
\]

for \( w \in U_{q, \varepsilon} \cap \partial \Omega \) and \( \delta < \delta_0 \). Therefore, exploiting Lemma 8.1, we get

1. \( C(w, \delta; g) \) is Jordan measurable for all \( w \in U_{q, \varepsilon} \cap \partial \Omega \) and \( \delta < \delta_0 \);
2. If \( w^1, \ldots, w^m \in U_{q, \varepsilon} \cap \partial \Omega \), \( m \in \mathbb{N}_+ \), then

\[
\text{vol} \left( \bigcup_{j=1}^m C(w^j, (1 + t)\delta; g) \right) \leq \frac{1}{1 - \varepsilon} \text{vol} \left( \bigcup_{j=1}^m C(\Theta_q w^j, (1 + t)\delta; f_{S_q}) \right) \leq \frac{(1 + t)^3}{1 - \varepsilon} \text{vol} \left( \bigcup_{j=1}^m C(\Theta_q w^j, \delta; f_{S_q}) \right) \leq \frac{1}{(1 - \varepsilon)^2} (1 + t)^3 \text{vol} \left( \bigcup_{j=1}^m C(w^j, \delta; g) \right),
\]

for all \( t \in (0, 16) \) and \( \delta_j \leq \delta_0/16 \), \( j = 1, \ldots, m \). Thus, \( g \) satisfies the doubling property \( \heartsuit \) with quantifiers \( \delta_g = \delta_0/16 \) and \( \mathcal{E}(t) = (1 - \varepsilon)^{-2}(1 + t)^3 \).

Lastly, we further shrink \( U_{q, \varepsilon} \) — if necessary — to ensure that

(‡) for any \( s \)-measurable set \( J \subset (U_{q, \varepsilon} \cap \partial \Omega) \),

\[
1 - \varepsilon \leq \frac{s(J)}{|J''|} \leq \frac{1}{1 - \varepsilon},
\]

where \( J'' \) denotes the projection of \( J \) onto the tangent plane to \( \partial \Omega \) at \( q \) and \( |J''| = |A(J)'| \).

We are now ready to estimate. Consider Jordan measurable compact sets \( J \subset H \subset (U_{q, \varepsilon} \cap \partial \Omega) \) such that \( J \subset \text{int}_{\partial \Omega} H \). By (5.16), (3.8), (5.17), the volume-distortion properties of \( \Theta_q \) — see
Lemma 5.5 and recall that $A$ is an isometry — and property (‡), we have that
\[
\limsup_{n \to \infty} \sqrt{n} \ v(\Omega; \mathcal{P}_n(J \subset H; f)) = \limsup_{n \to \infty} \sqrt{n} \ v(\Omega; \mathcal{P}_n(J \subset H; \tilde{f}))
\leq \frac{1}{(1-\varepsilon)^2} \left( 1 + \frac{(1+\varepsilon)^2}{(1-\varepsilon)^2} - 1 \right)^3 \limsup_{n \to \infty} \sqrt{n} \ v(\Omega; \mathcal{P}_n(J \subset H; \tilde{g}))
\leq \frac{1}{(1-\varepsilon)^2} \left( 1 + \frac{(1+\varepsilon)^2}{(1-\varepsilon)^2} - 1 \right)^3 \limsup_{n \to \infty} \sqrt{n} \ v(\Omega; \mathcal{P}_n(J \subset H; g))
\leq (1-\varepsilon)^{-14} \limsup_{n \to \infty} \sqrt{n} \ (1-\varepsilon)^{-1} v(\mathcal{S}_{\lambda(q)}; \mathcal{P}_n(\Theta_q J \subset \Theta_q H; \tilde{f} \mathcal{S}_{\lambda(q)}))
\leq (1-\varepsilon)^{-15} l_{\text{kor}} \lambda(q)^{\frac{1}{2}} |(\Theta_q J)'|^{\frac{3}{2}}
\leq (1-\varepsilon)^{-15} l_{\text{kor}} \lambda(q)^{\frac{1}{2}} (|J''|^{\frac{3}{2}} \leq (1-\varepsilon)^{-18} l_{\text{kor}} \lambda(q)^{\frac{1}{2}} s(J)^{\frac{3}{2}}.
\]
By a similar argument, but now using (3.10) from the proof of Lemma 3.2, we get that
\[
\lim_{n \to \infty} \sqrt{n} \ v(\Omega; \mathcal{P}_n(J \subset H; f)) \geq (1-\varepsilon)^{30} l_{\text{kor}} \lambda(q)^{\frac{1}{2}} s(J)^{\frac{3}{2}}.
\]
Therefore, for large enough $n$, we get the desired estimates. \hfill \Box

6. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. Fix an $\varepsilon \in (0, 1/3)$. There exists a tiling $\{L_j\}_{1 \leq j \leq m}$ of $\partial \Omega$ consisting of Jordan measurable compact sets with non-empty interior such that

- for each $j = 1, \ldots, m$, there is a $q_j \in L_j$ for which $L_j \subset U_{q_j, \varepsilon}$, where the latter comes from Lemma 5.6;
- $(1-\varepsilon)\lambda(q) \leq \lambda(q_j) \leq (1-\varepsilon)^{-1}\lambda(q)$, for all $q \in L_j$.

Then, recalling that $\lambda(q) = \frac{4M(\rho)(q)}{|\nabla \rho(q)|^3}$, we obtain estimates as follows:
\[
4^{-\frac{3}{2}} \int_{\partial \Omega} \sigma_\Omega = \int_{\partial \Omega} 4^{\frac{3}{2}} M(\rho)(q)^{\frac{1}{2}} |d \rho(q)| \frac{ds(q)}{|d \rho(q)|} = \sum_{j=1}^{m} \int_{L_j} \lambda(q)^{\frac{1}{2}} ds(q)
\leq (1-\varepsilon)^{-1} \sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{2}} s(L_j)
\geq (1-\varepsilon) \sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{2}} s(L_j).
\]

(6.1)

We extend this tiling to a thin tubular neighborhood $N$ of $\partial \Omega$ in the obvious way, denoting the tile corresponding to $L_j$ by $\tilde{L}_j$. Lastly, for all $j = 1, \ldots, m$, we choose compact Jordan measurable sets $J_j$ and $H_j$ such that $J_j \subset \text{int}_{\partial \Omega} L_j \subset \text{int}_{\partial \Omega} H_j \subset U_{q_j, \varepsilon}$ and
\[
s(L_j) \geq (1-\varepsilon) s(L_j).
\]

(6.2)

1. We first estimate $v(\Omega; \mathcal{P}_n(f))$ from above. For $j = 1, \ldots, m$, choose $P^j \in \mathcal{P}_{n_j}(L_j \subset H_j; f)$ such that $\text{vol}(\Omega \setminus P^j) \leq (1-\varepsilon)^{-1} v(\Omega; \mathcal{P}_{n_j}(L_j \subset H_j; f))$. Let $P$ denote the intersection of all these $P^j$’s. Then, $P$ is an $f$-polyhedron with at most $n_1 + \cdots + n_m$ facets. Thus, by Lemma 5.6, for sufficiently
large \(n_1, \ldots, n_m,\)

\[
\text{vol}(\Omega \setminus P) \leq (1 - \varepsilon)^{-1} \sum_{j=1}^{m} \text{vol}(\Omega; \mathcal{P}_{n_j}(L_j \subset H_j; f))
\]

\[
\leq (1 - \varepsilon)^{-20} l_{\text{kor}} \sum_{j=1}^{m} \frac{\lambda(q_j)^{\frac{1}{2}} s(L_j)^{\frac{3}{2}}}{\sqrt{n_j}}
\]

(6.3)

\[
= (1 - \varepsilon)^{-20} l_{\text{kor}} \sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{2}} s(L_j) \left( \frac{\lambda(q_j)^{\frac{1}{2}} s(L_j)}{n_j} \right)^{\frac{1}{2}}.
\]

Now, fix an \(n \in \mathbb{N}_+.\) Suppose, we set

(6.4)

\[
n_j = \left\lfloor \frac{\lambda(q_j)^{\frac{1}{2}} s(L_j)}{\sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{2}} s(L_j)} n \right\rfloor, \quad j = 1, \ldots, m.
\]

Then,

(6.5)

\[
n_1 + \cdots + n_m \leq n;
\]

and

(6.6)

\[
(1 - \varepsilon) \frac{\lambda(q_j)^{\frac{1}{2}} s(L_j)}{\sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{2}} s(L_j)} n \leq n_j
\]

if \(n\) is large. We use (6.5), substitute (6.6) in (6.3) and invoke (6.1) to get

\[
v(\Omega; \mathcal{P}_n(f)) \leq (1 - \varepsilon)^{-21} l_{\text{kor}} \left( \sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{2}} s(L_j) \right)^{\frac{3}{2}} \frac{1}{\sqrt{n}}
\]

(6.7)

\[
\leq (1 - \varepsilon)^{-24} l_{\text{kor}} \left( \int_{\partial \Omega} \sigma_{\Omega} \right)^{\frac{3}{2}} \frac{1}{\sqrt{n}},
\]

for \(n\) sufficiently large.

2. Next, we produce a lower bound for \(v(\Omega; \mathcal{P}_f(n)).\) Choose a \(P_n \in \mathcal{P}_n(f)\) such that \(\text{vol}(\Omega \setminus P_n) \leq (1 - \varepsilon)^{-1} v(\Omega; \mathcal{P}_f(n)).\) Let \(n_{j}\) be the number of cuts of \(P_n\) that cover \(J_j.\) As \(\lim_{n \to \infty} \delta(P_n) = 0\) due to Lemma 3.1 and the upper bound on \(v(\Omega; \mathcal{P}_f(n))\) obtained above, we can choose \(n\) sufficiently large so that

- The \(n_{j}\) cuts that cover \(J_j\) lie in \(\hat{L}_j.\)
- Each \(n_{j}\) is large enough so that the bounds in Lemma 5.6 hold.

Thus, invoking Lemma 5.6 and using (6.2), we have that

\[
\text{vol}(\Omega \setminus P_n) \geq \sum_{j=1}^{m} \text{vol}(\hat{L}_j \setminus P_n) \geq \sum_{j=1}^{m} v(\Omega; \mathcal{P}_{n_j}(J_j \subset L_j; f))
\]

\[
\geq (1 - \varepsilon)^{31} l_{\text{kor}} \sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{2}} s(J_j)^{\frac{3}{2}} \frac{1}{\sqrt{n_j}}
\]

\[
\geq (1 - \varepsilon)^{33} l_{\text{kor}} \sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{2}} s(L_j)^{\frac{3}{2}} \frac{1}{\sqrt{n_j}}.
\]
Now, Hölder’s inequality gives
\[ \sum_{j=1}^{m} \lambda(q)^{1/2} s(L_j) = \sum_{j=1}^{m} \left( \frac{\lambda(q)s(L_j)}{n_j} \right)^{1/2} n_j^{1/2} \leq \left( \sum_{j=1}^{m} \frac{\lambda(q)s(L_j)^{3/2}}{\sqrt{n_j}} \right)^{1/2} \left( \sum_{j=1}^{m} n_j \right)^{1/2}. \]

Thus, using one of the estimates in (6.1),
\[ \text{vol}(\Omega \setminus P_n) \geq (1 - \varepsilon)^{33} l_{\text{kor}} \left( \frac{1}{4^{1/3}} \int_{\partial \Omega} \sigma_{\Omega} \right)^{1/2} \frac{1}{\sqrt{n}}, \]

By our choice of \( P_n \),
\[ v(\Omega; P_n(f)) \geq (1 - \varepsilon)^{36} \frac{1}{2} l_{\text{kor}} \left( \int_{\partial \Omega} \sigma_{\Omega} \right)^{1/2} \frac{1}{\sqrt{n}}, \]
for all \( n \) sufficiently large.

Finally, we combine (6.8) and (6.7), and recall that \( \varepsilon \in (0, 1/3) \) was arbitrary, to declare the proof of Theorem 1.1 complete.

**Proof of Corollary 1.3.** Let \( \rho \) be a \( C^4 \)-smooth strictly plurisubharmonic defining function of \( \Omega \). A Henkin-Ramirez generating map enjoys the following properties (see [20, Prop. 3.1] for a complete description):

1. \( g \) is defined and \( C^3 \)-smooth on some neighborhood of \( \Omega \times \partial \Omega \);
2. \( g(\cdot, w) \) is holomorphic in \( \Omega \) for each \( w \in \partial \Omega \);
3. for \( (z, w) \in \Omega \times \partial \Omega \), \( g(z, w) = 0 \) if, and only if, \( z = w \); and
4. there is a \( \tau > 0 \) and a function \( a \in C^3(\Omega_{\tau}) \) with \( |a| \geq \frac{2}{3} \) so that \( g = ap \) on \( \Omega_{\tau} \), where \( p \) is the Levi polynomial of \( \rho \).

This is precisely the set-up needed to invoke Theorem 1.1.

**Proof of Corollary 1.5.** By Theorem 2 in [9], there is a \( \tau > 0 \) and a non-zero \( a \in C(\partial \Omega) \), such that
\[ K_{\Omega}(z, w) = \frac{a(w)}{p(z, w)^3} + O(p(z, w)^{-\nu}), \quad \nu \in (0, 3), \]
on \( \Omega_{\tau} \), where \( p \) is the Levi polynomial of some strictly plurisubharmonic defining function of \( \Omega \). One would like to apply Theorem 1.1 to \( f = K_{\Omega}^{-1} \). As \( K_{\Omega} \) may vanish when \( (z, w) \notin \Omega_{\tau} \), we use a cut-off function (see Remark 3.3) to obtain a \( \Phi \in C(\Omega \times \partial \Omega) \) such that \( \Phi = 0 \) precisely on the set \( \{ (z, w) : z = w \in \partial \Omega \} \) and \( \Phi = K_{\Omega}^{-1} \) on \( \Omega_{\tau} \). Then, there is an \( m > 0 \), such that for \( n \) sufficiently large,
\[ \{ z \in \Omega : |K_{\Omega}(z, w)| < m_j, \quad j = 1, \ldots, n \} = \{ z \in \Omega : |\Phi(z, w)| > 1/m_j, \quad j = 1, \ldots, n \} \in P_n(\Phi), \]
where \( w^1, \ldots, w^n \in \partial \Omega \) and \( m_1, \ldots, m_n > m \). But, by Lemma 3.1, if \( n \) is sufficiently large,
\[ \inf \{ \text{vol}(\Omega \setminus P) : P \in P_n(\Phi) \} = \inf \{ \text{vol}(\Omega \setminus P) : P \in P_n(\Phi), \delta(P) < 1/m \}. \]
Thus, \( \inf \{ \text{vol}(\Omega \setminus P) : P \in P \} \leq v(\Omega; P_n(\Phi)) \). The reverse inequality follows from a similar argument. As Theorem 1.1 applies to \( \Phi \)-polyhedra (due to (6.9)), the claimed asymptotic result holds.
Alternately, we can avoid constructing $\mathfrak{K}$ by observing that the statement and proof of Theorem 1.1 are not adversely affected if we allow $f$ to be a $\mathbb{P}^1$-valued function.

Proof of Corollary 1.6. This proof follows along the same lines as the proof of Corollary 1.5, with (6.9) replaced by the following formula (which can be deduced from Boutet de Monvel and Sjöstrand’s formulae in [6]):

\[(6.10) \quad S_{\Omega}(z, w) = \frac{a(z, w)}{p(z, w)^2} + O(p(z, w)^{-\nu}), \quad \nu \in (0, 2),\]
on $\Omega_{\tau}$, where $p$ is the Levi polynomial of some strictly plurisubharmonic defining function of $\Omega$.

\[\square\]

7. Concluding Remarks

Although the techniques used in the proof of Theorem 1.1 are exclusive to strictly pseudoconvex domains, we suspect that the result can be generalized to a larger class of domains. As evidence, we mention three situations for which Corollary 1.5 holds.

• Suppose $\Omega \subset \mathbb{C}^2$ is a smooth domain which is strictly pseudoconvex at all but $m$ points in $\partial \Omega$. Further, suppose that $K_{\Omega} : \overline{\Omega} \times \overline{\Omega} \to \mathbb{P}^1$ is a continuous function that takes the value $\infty$ precisely on the diagonal of $\partial \Omega \times \partial \Omega$. Let $\varepsilon \in (0, 1/3)$, and $C(\varepsilon)$ be a collection of $m$ disjoint $K_{\Omega}$-cuts that contain a weakly pseudoconvex point each, and $\text{vol}(C) < \varepsilon/m^3/2$ for each $C \in C(\varepsilon)$. Let $\partial \Omega(\varepsilon) := \partial \Omega \setminus \bigcup_{C(\varepsilon)}(C \cap \partial \Omega)$. We construct a tiling of $\partial \Omega(\varepsilon)$ as in Section 6. Repeating the computations in Section 6, (6.7) and (6.8) yield

\[ (1 - \varepsilon)^{36} \frac{1}{2} l_{\text{kor}} \left( \int_{\partial \Omega(\varepsilon)} \sigma_{\Omega} \right)^{3/2} \frac{1}{\sqrt{n + m}} \]

\[ \leq n(\Omega; \mathcal{B}P_{n+m}) \leq (1 - \varepsilon)^{-24} l_{\text{kor}} \left( \int_{\partial \Omega(\varepsilon)} \sigma_{\Omega} \right)^{3/2} \frac{1}{\sqrt{n}} + \frac{\varepsilon}{\sqrt{m}}, \quad n \text{ large}. \]

Applying the Cauchy-Schwartz inequality, and shrinking $\varepsilon$ to zero, we get the result.

• $\Omega = \{z_1, z_2\} \subset \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1$, $p > 1$. The locus of weakly pseudoconvex points is the curve $\mathcal{W} = \{(\exp(i\theta), 0) : \theta \in [0, 2\pi]\}$. As in the previous example, it suffices to cover $\mathcal{W}$ by $n$ cuts with total volume at most $O(\varepsilon)/\sqrt{n}$. The Bergman kernel of $\Omega$ (see [8]) is

\[ K_{\Omega}(z, w) = \frac{2}{p\pi^2} \sum_{j=1}^{2} c_j \frac{(1 - z_1 \overline{w_1})^{2+\frac{i}{p}}}{(1 - z_1 \overline{w_1})^{\frac{i}{p}} - (z_2 \overline{w_2})^{1+\frac{i}{p}}}, \quad c_1 = (p - 1)/2, \quad c_2 = 1. \]

Thus, $C = \{z \in \Omega : |K_{\Omega}(z, (1, 0))| > m\} = \left\{z \in \Omega : |z - 1| < cm^{-\frac{p}{2p+1}}\right\}$, for some constant $c$ depending only on $p$. Now, $\text{vol}(C) \sim m^{-1}$, and using the symmetry $(z_1, z_2) \mapsto (\exp(i\theta)z_1, z_2)$, we can cover $\mathcal{W}$ by $O\left([m^{p/(2p+1)}]\right)$ many such cuts. Hence, the claim.

• $\Omega = \mathbb{D}^2$. Then,

\[ \lim_{n \to \infty} \sqrt{n} \inf \{\text{vol}(\Omega \setminus P : P \in \mathcal{B}P_n)\} = 0. \]

Although $\Omega$ is non-smooth, since its boundary is Levi-flat almost everywhere, we can interpret $\int_{\partial \Omega} \sigma_{\Omega}$ as zero (see [3, Sec. 4] for more evidence).

Thus, for a smooth domain, it is reasonable to ask whether a control on the size of the locus of weakly pseudoconvex points is enough to grant (1.3)-type results. For non-smooth domains, the existence of the limit on the left-hand side of (1.3) is of interest.
8. Appendix: Power Diagrams in the Heisenberg Group

8.1. The Euclidean Plane. Let \( T(a; r) \subset \mathbb{R}^2 \) be a circle of radius \( r \) centered at \( a \in \mathbb{R}^2 \). The power of a point \( z = (x, y) \in \mathbb{R}^2 \) with respect to \( T = T(a; r) \) is the function
\[
\text{pow}(z, T) = |z - a|^2 - r^2.
\]
Note that if \( z \) is outside the disk bounded by \( T \), then \( \text{pow}(z, T) \) is the square of the length of a line segment from \( P \) to a point of tangency with \( T \). Thus, it is a generalized distance between \( z \) and \( T \). For a collection, \( \mathcal{T} \), of circles in the plane, the power diagram or Laguerre diagram of \( \mathcal{T} \) is the collection of all
\[
\text{cell}(T) = \{ z \in \mathbb{R}^2 : \text{pow}(T, z) < \text{pow}(T^*, z), \forall T^* \in \mathcal{T} \setminus \{ T \} \}, \quad T \in \mathcal{T}.
\]
If \( \mathcal{T} \) consists of equiradial circles, the power diagram reduces to the Dirichlet-Voronoi diagram of the centers of the circles. In general, the power diagram of any \( \mathcal{T} \) gives a convex tiling of the plane.

![Figure 2. A power diagram in the plane.](image)

Power diagrams occur naturally and have found several applications (see [2], for instance). From the point of view of polyhedral approximations, power diagrams (in \( \mathbb{R}^{d-1} \)) are intimately related to the constant \( \mathrm{ldiv}_{d-1} \) in (1.2) (see [19] and [16] for explicit details).

8.2. The Heisenberg Group. Let \( G(0; \delta) = \{ z' \in \mathbb{H} : |z_1|^4 + (x_2)^2 < \delta^4 \} \) be a Korányi sphere in \( \mathbb{H} \) (see (4.1)). We define the horizontal power of a point \( z' \in \mathbb{H} \) with respect to \( G = G(0; \delta) \) as
\[
\text{hpow}(G, z') = \begin{cases} 
|z_1|^2 - \sqrt{\delta^4 - (x_2)^2}, & \text{if } |x_2|^2 \leq \delta; \\
\infty, & \text{otherwise}.
\end{cases}
\]
Note that \( G_c := G \cap \{ x_2 = c \} \) is a (possibly empty) circle in the \( \{ x_2 = c \} \) plane, and \( \text{hpow}(G, (z_1, x_2)) = \text{pow}(G_{x_2}, z_1) \), where the right-hand side — being a generalized distance — is set as \( \infty \) when \( G_{x_2} \) is empty. \( \text{hpow} \) is then extended to all Korányi spheres to be left-invariant under \( \gamma_{\mathbb{H}} \) (defined in Section 4). For a collection \( \mathcal{G} \) of Korányi spheres in \( \mathbb{H} \), let
\[
K_\mathcal{G} := \bigcup_{\partial K \in \mathcal{G}} K,
\]
i.e., the union of all the Korányi balls bounded by the spheres in \( \mathcal{G} \). We define the horizontal power diagram of \( \mathcal{G} \) to be the collection of all
\[
\Delta(G) = \{ z' \in K_\mathcal{G} : \text{hpow}(G, z') < \text{hpow}(G^*, z'), \forall G^* \in \mathcal{G} \setminus \{ G \} \}, \quad G \in \mathcal{G}.
\]
Then, \( \Delta(G) \subset \) the Korányi ball bounded by \( G \), for all \( G \in \mathcal{G} \).

\[ \text{Figure 3. A \( \{x_1 = 0\} \)-slice of a horizontal power diagram in} \ H. \]

We now give two reasons why this concept is useful for us. Let
\[
\begin{align*}
\operatorname{dil}_\xi : (z_1, x_2) &\mapsto (\xi z_1, \xi^2 x_2), \\
\operatorname{dil}_{w', \xi} : z' &\mapsto w' \cdot \operatorname{dil}_\xi(-w' \cdot z')
\end{align*}
\]
be the dilations in \( H \) centered at the origin and \( w' \), respectively. Then,
\[
\begin{align*}
(1) \quad &\operatorname{dil}_{w', \xi}(\Delta(w', \delta)) = \Delta(w', \xi \delta), \\
(2) \quad &\operatorname{hpow}(\operatorname{dil}_{w', \xi}(z')) = \xi^2 \operatorname{hpow}(\Delta(w', \xi^{-1} \delta), z'), \\
(3) \quad &\text{if} \ \mathcal{G} \ \text{is given by the center-radius pairs} \ \{(a_1, \delta_1), \ldots, (a_m, \delta_m)\}, \ \text{then,} \ \operatorname{dil}_{a_j, \xi} \Delta(\mathcal{G}(a_j; \delta_j)) \cap \operatorname{dil}_{a_k, \xi} \Delta(\mathcal{G}(a_k; \delta_k)) = \emptyset, \ \text{for all} \ 1 \leq j \neq k \leq m \ \text{and} \ \xi \leq 1.
\end{align*}
\]

Now, consider the Siegel domain \( S \) and the function \( f_S \) studied in Section 4. The cuts of any \( f_S \)-polyhedron \( P \) over \( J \subset \partial S \) project to a collection of Korányi balls in \( C \times \mathbb{R} \) that form a covering of \( J' \). The (open) facets of \( P \) project to the horizontal power diagram of the corresponding set of spheres \( \mathcal{G}_P \). This perspective facilitates the proof of

**Lemma 8.1.** The cuts of \( f_{S_{\lambda}} \), \( \lambda > 0 \), are Jordan measurable and satisfy the doubling property \( \Theta \) for any \( \delta_{f_S_{\lambda}} > 0 \) and \( \mathcal{E}(t) = (1 + t)^3 \).

**Proof.** The Jordan measurability of the cuts is obvious. Now, without loss of generality, we may assume \( \lambda = 1 \) (the map \((z, w) \mapsto (\lambda z, \lambda w)\) can be used to handle the other cases). Let \( H \subset \partial S \) be a compact set, \( \{w^j\}_{1 \leq j \leq m} \subset H \), \( \{\delta_j\}_{1 \leq j \leq m} \subset (0, \infty) \) and \( t > 0 \). For \( j = 1, \ldots, m \), let
\[
\begin{align*}
C_j(t) &:= C(w_j, (1 + t)\delta_j; f_S), \\
v^j &:= (w^j)' = (w^j_1, w^j_2),
\end{align*}
\]
and (see (4.1))
\[
K_j(t) := C_j(t)' = K\left(v^j; \sqrt{(1 + t)\delta_j}\right).
\]
Consider \( \mathcal{G} = \{\partial K_j(t) : 1 \leq j \leq m\} \) and the corresponding horizontal power diagram \( \{\Delta_j(t) = \Delta(\partial K_j(t)) : 1 \leq j \leq m\} \). Then, setting \( dz' = dx_1 dy_1 dx_2 \), we have, by a change of variables and
(1), (2) and (3) above, that

\[
\begin{align*}
\text{vol} \left( \bigcup_{j=1}^{m} C_j(t) \right) &= \int_{\bigcup_{j=1}^{m} K_j(t)} \max_{1 \leq j \leq m} \left\{ \Re \sqrt{\delta_j^2 - (x_2 - u_j^1)^2 + 2 \text{Im} z_1 \overline{w_j^1}} - |z_1 - w_j^1|^2 \right\} \, dz' \\
&= \int_{\bigcup_{j=1}^{m} K_j(t)} \max_{1 \leq j \leq m} \left\{ -\text{hpow}(\partial K_j(t), z') \right\} dz'
\end{align*}
\]

\[
= -(1 + t)^2 \sum_{j=1}^{m} \int_{\Delta_j(t)} \text{hpow}(\partial K_j(t), z') \, dz'
\]

\[
= -(1 + t)^3 \sum_{j=1}^{m} \int_{\text{dil}_{\text{vol}}, \frac{1}{\sqrt{1+t}}}(\Delta_j(t)) \text{hpow}(\partial K_j(0), z') \, d\zeta
\]

\[
\leq (1 + t)^3 \int_{\bigcup_{j=1}^{m} K_j(0)} \max \left\{ -\text{hpow}(\partial K_j(0), \zeta) : 1 \leq j \leq m \right\} \, d\zeta
\]

\[
= (1 + t)^3 \text{vol} \left( \bigcup_{j=1}^{m} C_j(0) \right), \; \forall t \geq 0.
\]

\[ \square \]

The computations in the above proof also show that

\[ l_{\text{kor}} = \lim_{n \to \infty} \inf \left\{ -\sum_{G \in \mathcal{G}} \int_{\Delta(G)} \text{hpow}(G, z')dz' : I \subset K_\mathcal{G}, \#(\mathcal{G}) \leq n \right\}, \]

where \( I \) is the unit square in \( \mathbb{C} \times \mathbb{R} \) (see Section 4). Our proof of Lemma 4.1 yields bounds for \( l_{\text{kor}} \) as follows:

\[ \frac{4\sqrt{2}}{\pi^2 3^3} \leq l_{\text{kor}} \leq \frac{5\sqrt{7}}{3\sqrt{2}}. \]

It would be interesting to know if computations, similar to the ones carried out by Böröczky and Ludwig in [16] for \( \text{ldiv}_2 \), can be done to find the exact value of \( l_{\text{kor}} \).

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