Relativistic Generalized Uncertainty Principle

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The Generalized Uncertainty Principle and the related minimum length are normally considered in non-relativistic Quantum Mechanics. Extending it to relativistic theories is important for having a Lorentz invariant minimum length and for testing the modified Heisenberg principle at high energies. In this paper, we formulate a relativistic Generalized Uncertainty Principle. We then use this to write the modified Klein-Gordon, Schrödinger and Dirac equations, and compute quantum gravity corrections to the relativistic hydrogen atom, particle in a box, and the linear harmonic oscillator.

I. INTRODUCTION

Theories of Quantum Gravity, such as String Theory [1–3], Loop Quantum Gravity, [4–7] as well as Doubly-Special Relativity theories [8, 9], have one thing in common: they all predict a minimum measurable length or scale in spacetime. This is in direct contradiction with the Heisenberg uncertainty principle [10], since the latter allows for infinitely small uncertainties in position. Phenomenological approaches to Quantum Gravity can describe the existence of such a minimal length via the so-called Generalized Uncertainty Principle (GUP) [11–17]. While the minimum length, normally considered to be the Planck length, ℓ_P = 10^{-35} m, signifies the scale at which quantum gravity effects would become manifest, one is faced with the problem that theories incorporating a minimum length break Lorentz covariance, simply because a measured length is not a Lorentz invariant quantity. This effectively chooses a special frame of reference, bringing back the notion of an aether. Because of this difficulty, GUP models so far have been mainly considered in non-relativistic theories with few attempts in relativistic theories and quantum field theory [18–28]. In the present work, we study this aspect and propose a Lorentz covariant GUP which reduces to its familiar non-relativistic versions at low energies, and incorporates a minimum length.

This paper is organized as follows. In Section II, we start with the most general quadratic GUP (i.e. only quadratic terms in momenta), and do a relativistic generalization thereof, similar to [29]. We proceed to calculate the Poincaré algebra associated with the newly defined position and momentum, along with the corresponding Casimir operators. We then show that the corresponding algebras are unmodified for a certain range of parameters. In Section III, using the modified dispersion relation, we compute GUP corrections to the modified Klein-Gordon, relativistically corrected Schrödinger and the Dirac equations. Finally, we apply them to a set of experiments to impose bounds on the GUP parameters.

In Section IV, we present a summary of the results and a list future works.

II. RELATIVISTIC GENERALIZED UNCERTAINTY PRINCIPLE

We start from the algebra for position x^i and momentum p^j proposed in [11], giving us the well-known quadratic GUP

\[ [x^i, p^j] = i\hbar \delta^{ij} \left( 1 + f(p^2) \right). \]  

In the above, \( i, j \in \{1, 2, 3\} \), and Eq. (1) is non-relativistic. In [11], the authors show that the position operators obey the following commutation relation

\[ [x^i, x^j] = -i\hbar f'(p^2) (x^i p^j - x^j p^i). \]  

Then, considering \( f(p^2) = \beta_1 p^2 \), they proceed calculating the minimal uncertainty in position corresponding to that uncertainty relation, finding \( \Delta x_{\text{min}} = \hbar \sqrt{\beta_1} \).

Inspired by [29], we will expand this algebra to the full Minkowski spacetime, with the following signature \( \{-, +, +, +\} \). In particular, we will consider the following commutator

\[ [x^\mu, p^\nu] = i\hbar \left( 1 + (\varepsilon - \alpha) \gamma^2 p^\rho p_\rho \right) \eta^{\mu\nu} + i \hbar (\beta + 2\varepsilon) \gamma^2 p^\mu p^\nu, \]

where \( \mu, \nu \in \{0, 1, 2, 3\} \) and \( \alpha, \beta, \varepsilon, \) and \( \xi \) are dimensionless parameters which can be used to fix the particular model [1]. We will assume the parameter \( \gamma \), with dimensions of inverse momentum, to be \( \gamma = \frac{1}{\sqrt{M_{Pl}}} \), where \( M_{Pl} \) is the Planck mass. In this way, the minimal length will

\[ \ell_{min} = \frac{\hbar}{\gamma}, \]

The parameters used in [11] and [29] are related to the ones used here as follows: \( \beta_1 = (\alpha + \varepsilon)\gamma^2 \) and \( \beta_2 = (\beta + 2\varepsilon)\gamma^2 \).

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be associated with the Planck length. Note that Eq. (3) reduces in the non-relativistic \((\epsilon \to \infty)\) limit to Eq. (1) and in the non-relativistic \((\gamma \to 0)\) limit to the standard Heisenberg algebra. Note also that, while \(x^\mu\) and \(p^\nu\) are the physical position and momentum, they are not canonically conjugate. Therefore, we introduce two new 4-vectors, \(x_0^\mu\) and \(p_0^\nu\) which are canonically conjugate, such that

\[
\begin{align*}
\frac{\partial}{\partial x_0^\mu} &= -i\hbar, \\
\frac{\partial}{\partial p_0^\nu} &= i\hbar p^{\mu\nu}.
\end{align*}
\]

and in terms of which the physical position and momentum of \(\epsilon\) in the parameter space, one has a non-trivial relativistic reduction in the non-relativistic \((\epsilon, \gamma \to \infty)\) limit and the quadratic part of Eq.(1) of \([12]\) (up to an unimportant numerical factor). It is interesting to note that algebras Eqs. (2) and (3) have similarities to the ones proposed in \([31]\). We will study further implications of the above in the following sections.

\section{Applications}

For the case \(\epsilon = \alpha\), Eqs. (5) and (6) take the following form,

\[
\begin{align*}
x^\mu &= x_0^\mu (1 - \alpha \gamma^2 \rho^0_0 \rho_0^0), \\
p^\mu &= p_0^\mu (1 + \alpha \gamma^2 \rho^0_0 \rho_0^0).
\end{align*}
\]

As we showed earlier, since now the the definition of \(M^{\mu\nu}\) in terms of the physical position and momentum, \(x^\mu\) and \(p^\nu\), as well as the Poincaré algebra, remains unchanged, the squared physical momentum \(p^\rho p^\rho\) is again a Casimir invariant, commuting with every other operator in the group. Using this, we can derive the dispersion relation and the Klein-Gordon (KG) equation. It takes the following form

\[p^\rho p_\rho = - (mc)^2,\]

or, in terms of the variables \(p_0^\rho\),

\[p_0^\rho p^\rho (1 + 2\alpha \gamma^2 \rho^0_0 \rho_0^0) = - (mc)^2,\]

where \(m\) is the mass of the particle. Observe that using Eq. (1), the KG equation now is a fourth order equation, with four linearly independent solutions. Solving Eq. (17) for \(p_0^\rho p_\rho\), we obtain

\[
p_0^\rho p^\rho = - \frac{1}{4\alpha \gamma^2} \sqrt{\frac{1}{(4\alpha \gamma^2)^2} - \frac{(mc)^2}{2\alpha \gamma^2}} \sim - \frac{(mc)^2}{2\alpha \gamma^2} - O(\gamma^4),
\]

where we have discarded the other solution since it does not reduce to \((mc)^2\) in the \(\gamma \to 0\) limit. In this process, we also lose two remaining solutions of the fourth order equation (17). As was shown in [12], including those solutions introduced very small corrections and we will ignore them in this paper. In the next three subsections, we study applications of the above equation as well as the GUP-modified Dirac equation for a number of quantum systems.

\section{Klein-Gordon equation}

Writing Eq. (19) as an operator equation on the wavefunction \(\Psi\) and using Eq. (20), we obtain the following GUP-modified Klein-Gordon equation:

\[
\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Psi - \nabla^2 \Psi + \frac{1}{\hbar^2} \left[ (mc)^2 + 2\alpha c^4 \gamma^2 m^4 \right] \Psi = 0.
\]
It is worth noticing that in the limit $\gamma \to 0$, the above equation reduces to the standard KG equation. Moreover, the solutions of the modified equation have the same form of the standard one but with modified parameters.

1. Energy spectrum for Relativistic Hydrogen atom

Introducing the minimal coupling \[31\], \[32\]
\[
\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + ie\frac{\gamma}{\hbar} A^\rho
\]
and the Hydrogen atom nuclear potential in Coulomb gauge
\[
A^\rho = \left\{ \frac{e}{4\pi \varepsilon_0 r}, 0, 0, 0 \right\},
\]
in Eq.\(20\), we obtain the GUP modified Klein-Gordon equation for the Hydrogen atom
\[
- \left( \frac{\partial}{\partial t} + \frac{\kappa^2}{r} \right)^2 \Psi - c^2 \nabla^2 \Psi
+ \frac{c^2}{\hbar^2} \left[ (mc)^2 + 2\alpha\gamma^2 (mc)^4 \right] \Psi = 0,
\]
(23)
In the above, $\kappa = e^2 / 4\pi \varepsilon_0 \hbar c$ is the fine structure constant. The solution of Eq.\(23\) in spherical coordinates is of the form \[33\]
\[
\Psi_{nlm} = R_{nl}(r) Y_{lm}(\theta, \phi) e^{-E_{nl} t / \hbar},
\]
(24)
where the $R_{nl}(r)$ are spherical Bessel functions and $Y_{lm}(\theta, \phi)$ the spherical harmonics. The energy levels for these solutions are given as
\[
E_{0nl} = (mc^2 + \alpha\gamma^2 mc^3)^{1/2} \left[ 1 + \frac{\kappa^2}{(n+\frac{1}{2})^2} \right]^{-1/2},
\]
(25)
where
\[
\eta = \frac{1}{2} \pm \sqrt{\left( \frac{l + \frac{1}{2}}{2} \right)^2 - \kappa^2},
\]
(26)
where the minus sign recovers the standard Hydrogen atom spectrum from Eq.\(25\) in the non-relativistic limit, with the identification $N \equiv n + 1 - \eta$ for the the hydrogen atom energy level. Expanding Eq.\(24\) in powers of $\kappa^2$, we find
\[
E_{0N} = (mc^2 + \alpha\gamma^2 mc^3) - \frac{\kappa^2 (mc^2)}{2N^2}
+ \frac{3\kappa^4 (mc^2)}{8N^4} + \frac{3\kappa^4 (\alpha\gamma^2 mc^3)}{8N^4} - \frac{\kappa^2 (\alpha\gamma^2 mc^3)}{2N^2},
\]
(27)
We now translate these corrections to $E_0$ (the 0th component of $p_0^\rho$) to corrections to $E$ (the 0th component of $p^\rho$ or the physical momentum), using the appropriate component of Eq.\(15\), namely
\[
E = E_0 (1 + \alpha\gamma^2 p_0^\rho p_0^\rho)
= E_0 [1 - \alpha\gamma^2 (mc)^2] + O(\gamma^4),
\]
(28)
to obtain
\[
E_N = (mc^2) - \frac{\kappa^2 (mc^2)}{2N^2} + \frac{\kappa^4 (mc^2)}{8N^4}
\]
(29)
As evident from the above equation, all the GUP corrections of $\gamma^2$ vanish. It is important to clarify that the Casimir still remains modified which will give modified results in field theory. Furthermore, including the remaining two solutions of the fourth order equation \[17\] may give rise to GUP corrections. We will investigate this elsewhere.

B. Schrödinger equation with relativistic and GUP corrections

We can write Eq.\(19\) as
\[
- E_0^2 + c^2 p_0^2 + (mc^2)^2 + 2\alpha\gamma^2 mc^6 = 0.
\]
(30)
We now expand it to fourth order in $\tilde{p}_0$ and second order in $\gamma$ to obtain
\[
E_0 = mc^2 (1 + \alpha\gamma^2 (mc)^2) + \frac{\tilde{p}_0^2}{2m} \left[ 1 - \frac{1}{2}\alpha\gamma^2 (mc)^2 \right]
- \frac{\tilde{p}_0^4}{8m^4 c^2} (1 - 3\alpha\gamma^2 (mc)^2) .
\]
(31)
Next, using Eq.\(28\), we get the expression for the physical energy
\[
E = mc^2 + \frac{\tilde{p}_0^2}{2m} \left[ 1 - \frac{3}{2}\alpha\gamma^2 (mc)^2 \right]
- \frac{\tilde{p}_0^4}{8m^4 c^2} (1 - 4\alpha\gamma^2 (mc)^2) ,
\]
(32)
which consists of the rest mass, non-relativistic kinetic energy, relativistic and GUP corrections. Next, the operator version of Eq.\(31\), using $E_0 = \hbar \frac{\partial}{\partial \tilde{x}_0}$ and $\tilde{p}_0 = -i\hbar \nabla_0$, and including a potential $V(\tilde{x})$, yields the modified Schrödinger equation with relativistic and GUP corrections
\[
\hbar \frac{\partial}{\partial \tilde{t}_0} \Psi(\tilde{t}_0, \tilde{x}_0) = \left[ mc^2 (1 + \alpha\gamma^2 mc^2) \right]
+ \frac{(-i\hbar)^2}{2m} \left[ 1 - \frac{1}{2}\alpha\gamma^2 mc^2 \right] \nabla_0^2
- \frac{(-i\hbar)^4}{8m^4 c^2} (1 - 3\alpha\gamma^2 mc^2) \nabla_0^4 + V(\tilde{x}) \Psi(\tilde{t}_0, \tilde{x}_0).
\]
(33)
Omitting the rest energy term, we apply the above equation to a couple of problems.
1. Corrections for particle in a box

Using Eq. (31), we can write the Schrödinger equation with relativistic and GUP corrections. Let us first consider a 1+1-dimensional case with the following potential

\[ V(x) = \begin{cases} V_0 & \text{for } 0 < x < L, \\ \infty & \text{for } x \leq 0 \cup x \geq L. \end{cases} \]  \hspace{1cm} (34)

Including the potential in Eq. (31), we obtain the Schrödinger equation for one dimensional particle in a box plus small perturbations. Consider the wave function for the unperturbed (\( \gamma = 0, c \to \infty \)) case

\[ \Psi_n(t_0, \vec{x}_0) = \sqrt{\frac{2}{L_0}} \sin \left( \frac{n\pi x_0}{L_0} \right) e^{-i\omega_{n} t_0}. \]  \hspace{1cm} (35)

From Eq. (14), one can read-off the physical dimensions of the box to be

\[ L = L_0(1 + \alpha \gamma^2(mc)^2 + O(\gamma^4)). \]  \hspace{1cm} (36)

The corrected spectrum of \( E_0 \) is

\[ E_{0n} = -\frac{1}{2m} \left( \frac{n\pi \hbar}{L_0} \right)^2 \left[ 1 - \frac{1}{2} \alpha \gamma^2(mc)^2 \right] + \frac{\hbar^4}{8m^3c^2} \left( \frac{n\pi}{L_0} \right)^4 \left[ 1 - 3\alpha \gamma^2(mc)^2 \right]. \]  \hspace{1cm} (37)

Using Eq. (28), we translate this to the following expression for the physical energy

\[ E_n = -\frac{1}{2m} \left( \frac{n\pi \hbar}{L} \right)^2 \left[ 1 + \frac{3}{2} \alpha \gamma^2(mc)^2 \right] - \frac{\hbar^4}{8m^3c^2} \left( \frac{n\pi}{L} \right)^4. \]  \hspace{1cm} (38)

The first term corresponds to the non-relativistic energy with GUP-corrections, while the last to relativistic corrections.

The results of this section can be applied to experiments measuring directly the energy levels of quantum dots. Comparing GUP corrections to the unperturbed energy term found above and equating to the accuracy of experiments measuring the energy levels of single quantum dot \( [28] \), we get

\[ \frac{\Delta E_n}{E_n} = \frac{3}{4} \alpha \gamma^2 m^2 c^2 \sim \alpha \cdot 10^{-42} \ll 10^{-1}. \]  \hspace{1cm} (39)

From this, one gets an upper bound on \( \alpha \)

\[ \alpha \lesssim 10^{41}. \]  \hspace{1cm} (40)

2. Corrections to the linear harmonic oscillator

Next, we include the harmonic oscillator potential in Eq. (31)

\[ V(x) = \frac{1}{2} m \omega^2 x^2. \]  \hspace{1cm} (41)

This gives rise to the following \( E_0 \)

\[ E_0 = \frac{p_0^2}{2m} + \frac{1}{2} m \omega^2 x_0^2 \left[ 1 + 2\alpha \gamma^2(mc)^2 \right] - \frac{p_0^2}{4m} (\alpha \gamma^2 m^2 c^2)^2 - \frac{p_0^4}{8m^3c^2} + \frac{3p_0^4}{8m^3c^2}. \]  \hspace{1cm} (42)

For the first order correction in perturbation theory, we obtain

\[ E_{on} = \hbar \omega \left[ n + \frac{1}{2} \right] \left( 1 + \frac{1}{2} \alpha \gamma^2 m^2 c^2 \right) - \frac{\hbar^2 \omega^2}{32mc^2} \left[ 5n(n + 1) + 3 \right]. \]  \hspace{1cm} (43)

From this, using Eq. (28), we find for the physical energy

\[ E_n = \hbar \omega \left[ n + \frac{1}{2} \right] \left( 1 - \frac{1}{2} \alpha \gamma^2 m^2 c^2 \right) - \frac{\hbar^2 \omega^2}{32mc^2} \left[ 4n(n + 1) + 3 \right]. \]  \hspace{1cm} (44)

Similar to the calculations for Landau levels done in \( [16] \), we use experiments to put bounds on \( \alpha \)

\[ \frac{\Delta E_n}{E_n} = \frac{3}{4} \alpha \gamma^2 m^2 c^2 \sim \alpha \cdot 10^{-44}. \]  \hspace{1cm} (45)

Equating this to the accuracy of direct measurements of Landau levels \( [34, 35] \), we get

\[ \alpha \cdot 10^{-44} \ll 10^{-3} \Rightarrow \alpha \lesssim 10^{41}. \]  \hspace{1cm} (46)

It is worth noting that this is several orders smaller than that was obtained in \( [16] \).

C. Dirac equation and GUP corrections

Starting from Eq. (18), working in the following signature \( \{+, +, +, +\} \) signature, and considering the following Dirac matrices

\[ \tau^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \]  \hspace{1cm} (47)

we get the following GUP-modified Dirac equation

\[ i\hbar \tau^\mu \frac{\partial}{\partial x^0} \Psi - \tau^0 \sqrt{\frac{1}{4\alpha \gamma^2} - \sqrt{\frac{1}{4\alpha \gamma^2} - \frac{(mc)^2}{2\alpha \gamma^2}}} \Psi = 0, \]  \hspace{1cm} (48)

Truncating to \( O(\gamma^2) \), we get

\[ i\hbar \tau^\mu \partial_\mu \Psi - \tau^0 (mc + \alpha \gamma^2 m^3 c^3) \Psi = 0. \]  \hspace{1cm} (49)
1. **Hydrogen atom**

Using Eqs. (21) and (22) for minimal coupling and Coulomb potential, from Eq. (19) we get

\[
[\gamma^\mu \frac{\partial}{\partial x^\nu} - \frac{e}{\hbar} \gamma^\mu A_\mu - \sigma^0 \left( \frac{mc + \alpha \gamma^2 m^3 c^3}{\hbar} \right) ] \Psi = 0, \quad (50)
\]

with the solution

\[
\Psi(t, r, \theta, \phi) = T(t) \frac{1}{r} \left( F(r) Y_{jm}(\theta, \phi) \right) \cdot (51)
\]

In the above, \( F(r) \) and \( G(r) \) are spherical Bessel functions, and \( Y_{jm} \) and \( Y'_{jm} \) are the spherical spinors. Following Eq. (30), we find the energy spectrum

\[
E_{\text{phys}} = (mc^2 + \alpha \gamma^2 m^3 c^4) \left[ 1 + \frac{(\kappa E)}{(n - j - \frac{1}{2})^2} \right]^{-1/2}, \quad (52)
\]

where again \( \kappa = e^2/4\pi\varepsilon_0\hbar c \) is the fine structure constant. Expanding the above equation in Taylor series in \( \kappa \), we get

\[
E_{\text{phys}} = (mc^2 + \alpha \gamma^2 m^3 c^4) - \frac{mc^2 \kappa^2}{2N^2} - \frac{\alpha \gamma^2 m^3 c^4 \kappa^2}{2N^2} + \frac{3mc^2 \kappa^4}{8N^4} + \frac{\alpha \gamma^2 m^3 c^4 \kappa^4}{8N^4}. \quad (53)
\]

As before, using Eq. (28), we get for the physical energy spectrum \( E_N \)

\[
E_N = mc^2 - \frac{mc^2 \kappa^2}{2N^2} + \frac{3mc^2 \kappa^4}{8N^4}, \quad (54)
\]

Once again, we can see that the GUP modifications vanish for the physical energy.

### IV. CONCLUSION

In this article, we proposed a method of incorporating minimal length in relativistic quantum mechanics. We achieved this by expanding proposing a Generalized Uncertainty Principle to include both space and time. As a result, we obtained a non-commutative spacetime. For a set of GUP parametes, the Poincaré algebra remained unchanged, albeit with a modified Casimir operator. We use this Casimir to write down GUP-modified quantum mechanical wave equations and applied them to several examples estimate upper bounds on the GUP parameter. We found GUP corrections in the case of the relativistically corrected Schrödinger equation, but did not find GUP corrections in the few applications of the Klein-Gordon and Dirac equations. We believe GUP corrections will result in the case of the last two for other problems. Furthermore, this work is an important step in the application of a covariant GUP to Quantum Field Theory. The GUP-modified action therein will predict corrections to various scattering amplitudes. We hope to report on this elsewhere.

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