TM-wave propagation in a graded waveguide structure

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Abstract

We investigate TM-wave propagation in a hollow waveguide with a graded dielectric layer, described using a hyperbolic tangent function. General formulae for the electric field components of the TM-waves, applicable to hollow waveguides with arbitrary cross sectional shapes, are presented. We derive the analytical results for the reflection and transmission coefficients valid for waveguides of arbitrary cross sectional shapes. Thereby, we show that the obtained reflection and transmission coefficients are in exact asymptotic agreement with those obtained for a very thin homogeneous dielectric layer using mode-matching and cascading. The proposed method is tractable since it gives analytical results that are directly applicable without the need of mode-matching, and it has the ability to model realistic, smooth transitions.

1 Introduction

Recent studies of radio frequency absorption and optimal plasmonic resonances in gold nanoparticle (GNP) suspensions [1, 2, 3] have given rise to an interest in plasmonic resonances in layered waveguide structures. In particular, the scattering on a single thin layer, modeled as a thin dielectric layer in a straight waveguide, with perfectly electrically conducting (PEC) boundaries and a homogeneous cross section with material parameters $\varepsilon$ and $\mu$, is reported in [4]. Following a number of previous studies by one of the present authors, (see e.g. [5, 6, 7, 8, 9, 10]), in this paper the surrounding homogeneous straight waveguide medium with a single thin layer is modeled as a stratified medium with frequency-dependent permittivity $\varepsilon(z)$ given by the following function of the waveguide axis direction (chosen to be the $z$-direction),

$$
\varepsilon(z) = \varepsilon_0 \varepsilon_R(z) = \varepsilon_0 \left\{ \varepsilon_L(z) - \left[ \varepsilon_L(z) - \varepsilon_G(z) \right] \tanh^2 \left( \frac{z}{2z_0} \right) \right\},
$$

where $\varepsilon_R(z)$ denotes the relative permittivity, and $2z_0$ determines the size of the inserted layer about the plane $z = 0$, as indicated in Fig. 1. A geometry with a very thin single layer with rapid smooth transition from $\varepsilon_G(z)$ to $\varepsilon_L(z)$ and back to $\varepsilon_G(z)$ is then obtained in the limit $z_0 \to 0$. A few examples of permittivity functions for different values of $z_0$, are shown in Fig. 2.

Figure 1. Hollow waveguide with a dielectric layer

asymptotically approaching such partial solutions in different materials.

2 Problem formulation

The geometry of the problem is illustrated in Fig. 1. In the surrounding non-magnetic lossy homogeneous straight waveguide medium with complex relative permittivity $\varepsilon_G(\omega)$, a single lossy non-magnetic thin layer with complex relative permittivity $\varepsilon_L(\omega)$ is inserted about the plane $z = 0$, as shown in Fig. 1. The proposed model is applicable to any complex permittivities of the two media, including negative values in chiral metamaterials, as long as they satisfy the Kramers-Kronig relations. Mathematically, the waveguide medium can be modeled as a stratified medium with frequency-dependent permittivity $\varepsilon = \varepsilon(\omega, z)$ given by the following function of the waveguide axis direction (chosen to be the $z$-direction),

$$
\varepsilon(\omega, z) = \varepsilon_0 \varepsilon_R(z) = \varepsilon_0 \left\{ \varepsilon_L(z) - \left[ \varepsilon_L(z) - \varepsilon_G(z) \right] \tanh^2 \left( \frac{z}{2z_0} \right) \right\},
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The wave equation for the longitudinal component of the electric field $E_z$ is

$$\nabla^2 E_z + \frac{\partial}{\partial z} \left( \frac{1}{\varepsilon_0 \varepsilon_R} \frac{d \varepsilon_R}{dz} E_z \right) + k^2 \varepsilon_R(z) E_z = 0,$$

(2)

where $k^2 = \omega^2 \varepsilon_0 \mu_0 = \omega^2 / c^2$. By means of separation of variables $E_z = T(x,y)Z(z)$, the above equation can be split into an equation for $T(x,y)$ and an equation for $Z(z)$

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T + k_T^2 T = 0, \quad \frac{d^2Z}{dz^2} + \frac{\partial}{\partial z} \left( \frac{1}{\varepsilon_0 \varepsilon_R} \frac{d \varepsilon_R}{dz} \right) Z + (k^2 \varepsilon_k(z) - k_T^2) Z = 0,$$

(3)

where $k_T$ denotes the transverse wave number of the waveguide. The solutions of the first of the equations in (3) are the standard solutions obtained for a particular waveguide cross sectional shape, and are unaffected by the graded material transition in the z-direction. As an example, for a rectangular waveguide the first of the equations (3) has the well-known solution

$$T(x,y) = A \sin \left( \frac{m \pi x}{a} \right) \sin \left( \frac{n \pi y}{b} \right), \quad k_T^2 = \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2,$$

(4)

where the constant $A$ is determined from the incoming field intensity $E_0 = E(-\infty)$. The present approach is quite general, and upon choosing any other hollow waveguide (e.g., parallel-plate waveguide, coaxial waveguide etc.) we can reuse the existing results for the transverse functions, whenever they are available in closed form. With our assumptions, the waves can only propagate if their frequency is higher than the maximum cutoff frequency

$$f_{c,\text{max}} = \frac{c}{k_{c,\text{max}}} = \frac{c}{\sqrt{\varepsilon_{c,\text{max}}}},$$

3 Results

Using analogous solution procedures to the ones used in our previous work on graded metamaterials (see e.g. [5, 6]), we readily obtain the solutions of the second of the equations (3), in the form

$$Z(z) = \frac{\exp \left( \frac{2p}{z_0} \right)}{\sqrt{1 + \exp \left( \frac{2}{z_0} \right)}} \frac{\Gamma \left( 2p + \frac{1}{2} + \sqrt{r^2 + \frac{1}{4}} \right)}{\Gamma \left( 2p + \frac{1}{2} + \sqrt{r^2 + \frac{1}{4}} \right)},$$

(5)

where $T$ is a constant to be determined from the asymptotic behavior of the solution (5) far away from the layer ($z \rightarrow \pm \infty$) and $Z_1(a,b,c;u) = F(a,b,c;u)$ is the ordinary Gaussian hypergeometric function defined by Gauss hypergeometric series [11]

$$F(a,b,c;u) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{u^n}{n!},$$

(6)

where $\Gamma$ is the Gamma function [11], and we define two dimensionless constants $p$ and $r$, as follows

$$p = \frac{z_0}{2} \sqrt{k^2 \varepsilon_G - k_T^2}, \quad r = \frac{k^2}{z_0^2}(\varepsilon_L - \varepsilon_G) \left[ 1 - \left( \frac{1}{\varepsilon_L} + \frac{1}{\varepsilon_G} \right) \frac{k_T^2}{k^2} \right],$$

(7)

with $k_{c,\text{max}} = \sqrt{k^2 \varepsilon_G - k_T^2}$ being the $z$-component of the wave vector of the asymptotic waves for $z \rightarrow \pm \infty$. On the other hand, the $z$-component of the wave vector about the origin, where the thin dielectric layer is situated, is denoted by $k_{c,\text{max}} = \sqrt{k^2 \varepsilon_G - k_T^2}$. Let us now investigate the asymptotic behavior of the solution (5) for $z \rightarrow +\infty$, when the argument of the hypergeometric function becomes zero. Using the series (6), we see that $F(a,b,c;0) = 1$, and we then obtain from (5),

$$Z(z) \rightarrow T \exp \left( -\frac{j k_{c,\text{max}} z}{z_0} \right) \quad \text{for} \quad z \rightarrow +\infty,$$

(8)

being a transmitted forward-propagating wave, as required. Next, we investigate the asymptotic behavior of the solution (5) for $z \rightarrow -\infty$, when the argument $u$ of the hypergeometric function becomes equal to one, and where it is convenient to use the following transformation formula for hypergeometric functions [11]

$$F(a,b,c;u) = \frac{\Gamma(c) \Gamma(a-b)}{\Gamma(a) \Gamma(b)} \left[ F(a,b,a+b-c+1;1-u) + \right.$$

$$\left. + (1-u)^{c-a-b} \frac{\Gamma(c) \Gamma(a+b-c)}{\Gamma(c) \Gamma(a+b-c)} \right],$$

(9)
Thus, in the limit \( z \to -\infty \), with \( c - a - b = -2p \), we obtain
\[
Z(z) \to \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \exp(+j k_G z) + T \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \exp(-j k_G z) \quad \text{for} \quad z \to -\infty ,
\]
where we require the solution to be a combination of an incident TM-wave and a reflected TM-wave, as follows
\[
Z(z) \to \exp(-j k_G z) + R \exp(+j k_G z) \quad \text{for} \quad z \to -\infty ,
\]
with the notation
\[
a = 2p + \frac{1}{2} + \sqrt{r^2 + \frac{1}{4}} \quad b = 2p + \frac{1}{2} - \sqrt{r^2 + \frac{1}{4}} \quad c = 2p + 1 .
\]
Comparing the equations (10) and (11), we readily obtain the general expressions for the transmission coefficient (\( T \)) and the reflection coefficient (\( R \)), in the form
\[
T = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(a+b-c)} \quad R = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c-a)\Gamma(c-b)\Gamma(a+b-c)} .
\]
The results (13) are the most general analytic results for transmission and reflection coefficients for TM-waves propagating over a graded dielectric layer in a straight hollow waveguide with any cross sectional shape. It is now of interest to study the transmission and reflection coefficients (13) in the case of a thin dielectric layer (\( z_0 \to 0 \)), when both constants \( p \) and \( r \) approach zero. Using the properties of the Gamma function [11] and definitions (12), with the assumption \( z_0 \to 0 \), we obtain from the results (13)
\[
R = \frac{r^2}{p} = kj_G z_0 \left( \frac{\epsilon_L - \epsilon_G}{\epsilon_L} \right) \left[ 1 - \left( \frac{1}{\epsilon_L} + \frac{1}{\epsilon_G} \right) \frac{z_0^2}{r^2} \right] + \mathcal{O}\left( \frac{z_0^2}{r^2} \right) ,
\]
\[
T = 1 + \frac{r^2}{p} = 1 + R .
\]
From the results (14), we readily see that there is no reflection (\( R = 0 \)) whenever the two materials have the same relative permittivity (\( \epsilon_L(\omega) = \epsilon_G(\omega) \)). Furthermore, we see that for \( z_0 = 0 \), which implies that the dielectric layer is removed, there is no reflection (\( R = 0 \)), as expected.

4 Asymptotic analysis

Finally, it is of interest to compare the asymptotic result (14) for the reflection coefficient with the corresponding result obtained in [4] for a non-graded layered waveguide structure using mode-matching and cascading methods for hollow waveguides. The reflection coefficient reported in [4] is denoted by \( T_{11}^{(2)} \), and using the notation employed in the present paper, has the form
\[
T_{11}^{(2)} = -2kj_G(2z_0) \frac{S_{11}^{(1)} + \mathcal{O}\left( z_0^2 \right)}{1 - \left( S_{11}^{(1)} \right)^2} ,
\]
with
\[
S_{11}^{(1)} = \frac{\epsilon_L k_{L3} - \epsilon_G k_{G3}}{\epsilon_L k_{L3} + \epsilon_G k_{G3}} ,
\]
where \( 2z_0 = d_L \) is the thickness of the thin dielectric layer and \( \mu_L = \mu_G = 1 \). From the two equations (15) and (16) together with the definitions of \( k_G \) and \( k_{3L} \), stated earlier in this paper, after some algebra, we obtain
\[
T_{11}^{(2)} = -2kj_G(2z_0) \frac{\epsilon_L k_{L3} - \epsilon_G k_{G3}}{4\epsilon_L \epsilon_G k_{L3} k_{G3}} = \frac{(\epsilon_L - \epsilon_G) \left[ 1 - \left( \frac{1}{\epsilon_L} + \frac{1}{\epsilon_G} \right) \frac{z_0^2}{r^2} \right]}{\sqrt{\epsilon_L - \frac{z_0^2}{r^2} \epsilon_G - \frac{z_0^2}{r^2}}} + \mathcal{O}\left( z_0^2 \right) = R .
\]
From the results (17) and (14), we see that the scattering matrix parameters reported in [4] for a homogeneous dielectric layer in a hollow waveguide structure, have the same thin layer asymptotics as the scattering parameters obtained here using the graded dielectric layer based on (1). This confirms that the graded permittivity function (1) can be employed to obtain useful scattering parameters for the waveguide without any need of mode matching and cascading techniques.

5 Conclusion

In conclusion, we have investigated TM-wave propagation in a hollow waveguide with a graded dielectric layer, described using a hyperbolic tangent function. We obtained useful analytical results for reflection and transmission coefficients valid for waveguides with arbitrary cross sectional shapes. Finally, we showed that the obtained reflection and transmission coefficients are in exact asymptotic agreement with those obtained in [4] for a very thin homogeneous dielectric layer using mode-matching and cascading. The proposed method is tractable, since it gives analytical results that are directly applicable without the need of mode-matching. At the same time, our method has the ability to model realistic, smooth transitions.

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