On Some Properties of the Beta Normal Distribution

L. C. Rêgo*  R. J. Cintra†  G. M. Cordeiro‡

Abstract

The beta normal distribution is a generalization of both the normal distribution and the normal order statistics. Some of its mathematical properties and a few applications have been studied in the literature. We provide a better foundation for some properties and an analytical study of its bimodality. The hazard rate function and the limiting behavior are examined. We derive explicit expressions for moments, generating function, mean deviations using a power series expansion for the quantile function, and Shannon entropy.

Keywords

Beta normal distribution, bimodality, hazard function, generating function, quantile function, mean deviation, Shannon entropy.

1 Introduction

The beta normal (BN) distribution [4] contains as special sub-models the normal distribution and the normal order statistics. In fact, for the BN(α, β, µ, σ) distribution with parameters α > 0, β > 0, µ ∈ ℝ and σ > 0, the probability density function (pdf) is

\[
f(x) = \frac{1}{σB(α, β)} \left[ Φ\left(\frac{x-µ}{σ}\right)\right]^{α-1} \left[ 1 - Φ\left(\frac{x-µ}{σ}\right)\right]^{β-1} φ\left(\frac{x-µ}{σ}\right), \quad x ∈ ℝ,
\]

where Φ(·) and φ(·) are the cumulative distribution function (cdf) and the standard normal density function, respectively, and B(·, ·) is the beta function. The parameters α and β are shape parameters, that characterize the skewness, kurtosis and bimodality of (1), µ is a location parameter and σ is a dispersion parameter that stretches out or shrinks the distribution. The BN distribution can be both unimodal and bimodal. A few authors have provided important properties of the BN distribution [7].

Location and scale parameters are redundant in (1) since if Z ~ BN(α, β, 0, 1) then X = µ + σZ ~ BN(α, β, µ, σ). The random variable Z has the so-called beta standard normal (BSN) distribution. The parameters α and β control skewness through the relative tail weights. The BN distribution is symmetric if α = β; presents negative skewness when α < β, and positive skewness when α > β. Moreover, as β decreases, the skewness increases. Conversely, as α increases, the skewness increases.

For α > 1 and β > 1, the BN distribution has positive excess kurtosis, and as α and β increase the higher its peak. On the other hand, when α < 1 and β < 1, it has negative excess kurtosis, and as both α and β decrease until bimodality, the heavier are the tails.

* L. C. Rêgo was with the Departamento de Estatística, Universidade Federal de Pernambuco (UFPE), Brazil; currently he is with the Departamento de Estatística e Matemática Aplicada, Universidade Federal do Ceará, Brazil. E-mail: leandro@dema.ufc.br
† R. J. Cintra is with the Signal Processing Group, UFPE, Brazil. E-mail: rjdsc@de.ufpe.br
‡ G. M. Cordeiro is with the Departamento de Estatística, UFPE, Brazil.
As pointed out in [5], the bimodal distribution can occur in fields as diverse as: statistical processing of synthetic aperture radar imagery [3], neurological disorder assessment [9], distribution of radiation mechanisms [11], and quantification of atmospheric pressure [16].

Indeed, a careful choice of parameters allows the BN distribution to exhibit bimodality. Famoye and collaborators pioneered the study of the shape of this distribution [5]. Although their study addressed several issues about the bimodality properties, it was primarily numerical in nature, relying on estimation techniques, for instance. In particular, the study on the parameter range for which the BN distribution switches behavior from unimodality to bimodality was partial. It was asserted that “the BN distribution becomes bimodal for certain values of the parameters \(\alpha\) and \(\beta\) and the analytical solution of \(\alpha\) and \(\beta\), where the distribution becomes bimodal, cannot be solved algebraically” [5].

We examine some structural properties of the BN distribution that were not developed so far. The rest of the paper is organized as follows. In Section 2, we provide an analytical study of the bimodality region of the BN distribution. We propose an exact algebraic description for the critical bimodal parameter values. In Section 3, we investigate the hazard rate function and its limiting behavior. Explicit expressions for the moments are discussed in Section 4. Further, in Sections 5-8, we derive quantile measures, generating function, mean deviations and Shannon entropy.

2 Bimodality

The analysis of the critical points of the BN density function furnishes a natural path for characterizing the distribution shape and quantifying the number of modes. After considering the normalization \(z = \frac{x - \mu}{\sigma}\), we have

\[
\frac{\partial f(z)}{\partial z} = \frac{1}{\text{B}(\alpha, \beta)} \phi(z)\Phi(z)^{\alpha-2}(1 - \Phi(z))^{\beta-2} \\
\times \left\{ (\alpha - 1)\phi(z)(1 - \Phi(z)) - (\beta - 1)\phi(z)\Phi(z) - z\Phi(z)(1 - \Phi(z)) \right\}.
\]

Let us refer to the term in curly brackets as \(s(z)\). At the critical points, where \(\frac{\partial f(z)}{\partial z} = 0\), we have \(s(z) = 0\), since the remaining terms of \(\frac{\partial f(z)}{\partial z}\) are strictly positive. Hence, the critical points satisfy the following implicit equation

\[
z = (2 - \alpha - \beta)\frac{\phi(z)}{\Phi(z)} + (\alpha - 1)\frac{\phi(z)}{\Phi(z)(1 - \Phi(z))}.
\]

In [5], it is claimed that the solutions of (2) are the modes of the distribution. In fact, there are solutions of this equation that can be local minimum of the density function, thus they are not modes. For example, Figure 1(a) gives the plot of the solutions of (2) in terms of \(\alpha\) for a fixed value of \(\beta = 0.15\). For \(\alpha = 0.1\), there are three solutions indicated by filled dots (•). However, for \(\alpha = 0.2\), only one critical point is found (◦).

Figure 1(b) provides the associated density plots. For \(\alpha = 0.1\), only two of the marked points are indeed modes of the distribution but the remaining point characterizes a local minimum. For \(\alpha = 0.2\), the single mode is depicted. For given \(\beta\), the choice of \(\alpha\) determines the number of modes of \(f(z)\). Additionally, we note that the only parts of the implicit curve with probabilistic meaning are those situated in the region \(\alpha > 0\). Conversely, the above discussion also holds if \(\alpha\) is fixed and \(\beta\) varies.
In order to determine which critical points are modes of the distribution, we should consider the sign of the second derivative at the critical points. In particular, a mode of \( f(z) \) is a critical point with non-positive second derivative. At the critical points, we have

\[
\frac{\partial^2 f(z)}{\partial z^2} = \frac{1}{\text{B}(\alpha, \beta)} \phi(z)[\Phi(z)]^{\alpha-2}[1-\Phi(z)]^{\beta-2} \frac{\partial s(z)}{\partial z}.
\]

The sign of \( \frac{\partial^2 f(z)}{\partial z^2} \) is the same of \( \frac{\partial s(z)}{\partial z} \). Then, at a mode, the condition \( \frac{\partial s(z)}{\partial z} \leq 0 \) holds. Explicit evaluation of \( \frac{\partial s(z)}{\partial z} \) yields

\[
\frac{\partial s(z)}{\partial z} = -\Phi(z)[1-\Phi(z)] - z \phi(z) \left[ \alpha - (\alpha + \beta) \Phi(z) \right] + (2 - \alpha - \beta) \phi^2(z).
\]

We consider the variational behavior of the critical points of \( f(z) \) with respect to changes in the parameter \( \alpha \). From (2), the first derivative of \( z \) with respect to \( \alpha \) is

\[
\frac{\partial z}{\partial \alpha} = \frac{\phi(z)[1-\Phi(z)]}{\Phi(z)[1-\Phi(z)] + z \phi(z) \left[ \alpha - (\alpha + \beta) \Phi(z) \right] - (2 - \alpha - \beta) \phi^2(z)}.
\]

Since \( \phi(z)[1-\Phi(z)] > 0 \), the sign of \( \frac{\partial z}{\partial \alpha} \) depends entirely on the behavior of the denominator term. Moreover, except for the sign, this denominator is equal to \( \frac{\partial s(z)}{\partial z} \). Thus, for any \( z \), we have

\[
\frac{\partial z}{\partial \alpha} = \frac{\phi(z)[1-\Phi(z)]}{-\frac{\partial s(z)}{\partial z}} \quad \text{and} \quad \text{sign} \left( \frac{\partial z}{\partial \alpha} \right) = -\text{sign} \left( \frac{\partial s(z)}{\partial z} \right).
\]
where sign(·) is the sign function. Further, if \( \frac{\partial s(z)}{\partial z} \) is negative at a critical point \( z \), then \( z \) must be a mode which is an increasing function of \( \alpha \). However, nothing prevents the existence of a mode for which \( \frac{\partial s(z)}{\partial z} \) vanishes. Indeed, in the case \( \alpha = \beta = 1 - \pi/4 \), it can be shown that

\[
\frac{\partial f(0)}{\partial z} = \frac{\partial^2 f(0)}{\partial z^2} = \frac{\partial^3 f(0)}{\partial z^3} = 0 \quad \text{and} \quad \frac{\partial^4 f(0)}{\partial z^4} = \frac{4\sqrt{2}}{\pi} \frac{\Gamma(3/2 - \pi/4)}{\Gamma(1 - \pi/4)} \left\{ \frac{3}{\pi} - 1 \right\} < 0.
\]

Thus, \( z = 0 \) is a mode of the distribution. Numerical computations give evidence that this is the unique case for which the second derivative vanishes at the mode. In this situation, \( \frac{\partial z}{\partial \alpha} \) is undefined. Nevertheless, it is still true that \( z \) is an increasing function of \( \alpha \), as shown in Figure 2. We are then in position to state the following proposition.

**Proposition 1** If \( z \) is a mode location, then \( z \) is an increasing function of \( \alpha \).

Based on a similar analysis, we can show that every mode is a decreasing function of \( \beta \). These results were previously examined in [5, Corollary 3]. However, their proof relied on an inaccurate derivation of \( \frac{\partial z}{\partial \alpha} \) and \( \frac{\partial z}{\partial \beta} \), which were mistakenly shown to be strictly positive and negative, respectively.

Now, we consider the symmetric case when \( \alpha = \beta \). The following proposition was also stated in [5, Corollary 1]. However, the associated proof was partial. Indeed, it was only shown that if \( z_0 \) is a critical point, then \( -z_0 \) is also a critical point of the distribution; we complete the proof here.

**Proposition 2** Let \( \alpha = \beta \). If \( z_0 \) is a modal point, then \( -z_0 \) is also a modal point.

**Proof:** For \( \alpha = \beta \), \( f(z) \) is an even function and, consequently, \( \frac{\partial f(z)}{\partial z} \) is odd. Thus, if \( z_0 \) is a critical point of \( f(z) \), so is \( -z_0 \). Note that \( s(z) \) is also odd which implies that \( \frac{\partial s(z)}{\partial z} \) is even. Then, \( \frac{\partial s(z_0)}{\partial z} = \frac{\partial s(-z_0)}{\partial z} \), which assures that \( -z_0 \) is a modal point. \( \square \)

We also provide a more complete proof of the following result stated in [5, Corollary 2].
Proposition 3 If $BN(\alpha, \beta, \mu, \sigma)$ has a mode at $z_0$, then $BN(\beta, \alpha, \mu, \sigma)$ has a mode at $-z_0$.

Proof: If $\alpha = \beta$, then the result follows from Proposition 2. We address the case $\alpha \neq \beta$. Let $t(z)$ be the result of interchanging the roles of $\alpha$ and $\beta$ in $s(z)$. Note that $t(-z) = -s(z)$ and $z_0$ is a critical point of $BN(\beta, \alpha, \mu, \sigma)$ if, and only if, $t(z_0) = 0$. Thus, if $z_0$ is a critical point of $BN(\alpha, \beta, \mu, \sigma)$, then $-z_0$ is a critical point of $BN(\beta, \alpha, \mu, \sigma)$. Moreover, since $\alpha \neq \beta$, it follows that $z_0$ is a mode of $BN(\beta, \alpha, \mu, \sigma)$ if, and only if, $\frac{\partial t(z_0)}{\partial z} < 0$. Since $\frac{\partial t(z_0)}{\partial z} = \frac{\partial s(-z_0)}{\partial z}$, the result follows. \[\square\]

Consider again the symmetric case $\alpha = \beta$. In Figure 3, we plot the critical points of $f(z)$ in terms of the parameter $\alpha$. This curve is obtained from (2). There is a critical value of $\alpha$ after which the BN density function exhibits a single critical point ($z = 0$), that is the unique mode of the distribution. This particular critical value can be determined as follows. From (2), we can express $\alpha$ in terms of $z$ as

$$\alpha = 1 + z \frac{\Phi(z)}{\phi(z) - 2\Phi(z)}, \quad \text{for } z \neq 0.$$ 

From Figure 3 we note that the exact critical value $\alpha^*$ can be obtained as the limit of the above function as $z$ tends to zero. Thus, using L’Hôpital rule, we obtain

$$\alpha^* = 1 + \lim_{z \to 0} z \frac{\Phi(z)}{\phi(z) - 2\Phi(z)} = 1 + \frac{\sqrt{2\pi}}{4} \lim_{z \to 0} \frac{z}{1 - 2\Phi(z)} = 1 + \frac{\sqrt{2\pi}}{4} \lim_{z \to 0} \frac{1}{-2\phi(z)} = 1 - \frac{\pi}{4} \approx 0.2146.$$ 

Further, in the symmetric case, we have $s(0) = 0$. Thus, $z = 0$ is always a critical point. We examine the
Figure 4: Implicit curve for the critical points of \( f(z) \) (a) as a function of \( \alpha \) for \( \beta \in \{0.15, 0.158896, 0.18, 1 - \pi/4, 0.28\} \) and (b) as a function of \( \beta \) for \( \alpha \in \{0.15, 0.158896, 0.18, 1 - \pi/4, 0.28\} \).

The sign of \( \frac{\partial^2 f(z)}{\partial z^2} \) at \( z = 0 \). Since its sign is the same as that of \( \frac{\partial s(z)}{\partial z} \), by (3), we obtain

\[
\left. \frac{\partial s(z)}{\partial z} \right|_{z=0} = -\frac{1}{4} + \frac{1 - \alpha}{\pi}.
\]

Hence, \( \left. \frac{\partial s(z)}{\partial z} \right|_{z=0} < 0 \) if, and only if, \( \alpha > 1 - \frac{\pi}{4} \). As discussed before, in the symmetric case, where \( \alpha = \beta = 1 - \pi/4 \), \( z = 0 \) is the unique mode of the distribution.

### 2.1 Modality Regions

In Figure 4(a), we plot the critical points \( z \) of the distribution in terms of \( \alpha \) for selected values of \( \beta \). Concerning the modality of the BN distribution, we can separate three regions: (i) \( 0 < \beta \leq 0.158896 \), (ii) \( 0.158896 < \beta < 1 - \pi/4 \), and (iii) \( \beta \geq 1 - \pi/4 \).

For \( 0 < \beta \leq 0.158896 \) and small positive values of \( \alpha \), the distribution is already bimodal, but there is a critical value of \( \alpha \) after which the distribution has a single critical point and then it is unimodal. For \( 0.158896 < \beta < 1 - \pi/4 \) and small positive values of \( \alpha \), the distribution is unimodal and then there is a critical value of \( \alpha \) after which the distribution has three critical points, where two of them are modes of the distribution. But, as in the previous case, when the value of \( \alpha \) increases, there is another critical value of \( \alpha \) after which the distribution has a single critical point and then it is unimodal. For the last case, \( \beta \geq 1 - \pi/4 \) and the distribution is always unimodal.

From (2), we can express \( \alpha \) in terms of \( z \) for a fixed value of \( \beta \). Let \( \alpha(\beta, z) \) denote this function of \( z \) by fixing
Table 1: Selected boundary coordinate pairs

| γ   | \( \alpha_\gamma^{*} = \beta_\gamma^{*} \) |
|-----|---------------------------------------------|
| 0   | 0.158896                                   |
| 0.01| 0.160179                                   |
| 0.05| 0.165872                                   |
| 0.10| 0.174668                                   |
| 0.15| 0.186511                                   |
| 0.20| 0.205147                                   |
| 1−\( \pi/4 \) | 1−\( \pi/4 \) |

\( \beta \). Hence,

\[
\alpha_\beta(z) = z \Phi(z) - \frac{(2-\beta)\Phi(z) - 1}{1-\Phi(z)}. \]

Moreover, for a fixed value of \( \beta \in (0, 1−\pi/4) \), let \( \alpha_\beta^{*} \) denote the local maximum of \( \alpha_\beta(z) \).

A similar analysis of the variation of the critical points in terms of \( \beta \) for several values of \( \alpha \) can be made. Figure 4(b) illustrates this analysis. Due to its symmetric behavior, the discussion follows mutatis mutandis.

So, from (2), we can express \( \beta \) in terms of \( z \) for a fixed value of \( \alpha \). Let \( \beta_\alpha(z) \) denote such function given by

\[
\beta_\alpha(z) = \frac{\alpha - 1}{\Phi(z)} - z \frac{1-\Phi(z)}{\phi(z)} + 2 - \alpha. 
\]

Moreover, for a fixed value of \( \alpha \in (0, 1−\pi/4) \), let \( \beta_\alpha^{*} \) denote the local maximum of \( \beta_\alpha(z) \). Using the symmetry properties of \( \phi(\cdot) \) and \( \Phi(\cdot) \), we obtain

\[
\beta_\alpha(-z) = \frac{\alpha - 1}{1-\Phi(z)} + z \frac{\Phi(z)}{\phi(z)} + 2 - \alpha. 
\]

After simple manipulations, for a fixed quantity \( \gamma \), we have \( \beta_\gamma(-z) = \alpha_\gamma(z) \). For any real function \( f(z) \), the set of values of the local maxima of \( f(z) \) and \( f(-z) \) are exactly the same. Therefore, for a given \( 0 < \gamma < 1−\pi/4 \), we have \( \beta_\gamma^{*} = \alpha_\gamma^{*} \). Table 1 lists the numerical values of \( \beta_\gamma^{*} = \alpha_\gamma^{*} \) for several values of \( \gamma \in (0, 1−\pi/4) \). Thus, the region of the parameters \( \alpha \) and \( \beta \) for which the distribution is bimodal reduces to

\[
\{(\alpha, \beta) : 0 < \alpha < \min \{1 - \frac{\pi}{4}, \alpha^{*}_\beta\}, 0 < \beta < \min \{1 - \frac{\pi}{4}, \beta^{*}_\alpha\}\}. 
\]

The curves \( \{(\alpha, \beta^{*}_\alpha) : 0 < \alpha < 1−\pi/4\} \) and \( \{(\alpha^{*}_\beta, \beta) : 0 < \beta < 1−\pi/4\} \) which delimit the above region are symmetric with respect to the line \( \alpha = \beta \).

In [5], a numerical estimate of these curves was approximated by means of linear regression techniques. However, such reported curves were not symmetric with respect to the line \( \alpha = \beta \). More accurately, Figure 5 shows the modality regions of the BN distribution.
3 Hazard Function

The BN hazard rate function takes the form

\[
h(x) = \frac{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{a-1} \left[1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1} \phi\left(\frac{x-\mu}{\sigma}\right)}{\sigma \beta B(a, \beta) \left[1 - I_{\Phi\left(\frac{x-\mu}{\sigma}\right)}(a, \beta)\right]},
\]

where \(I_x(a, \beta) = B(a, \beta)^{-1} \int_0^x w^{a-1}(1-w)^{\beta-1} dw\) denotes the incomplete beta function ratio. At \(x = \mu\), it gives

\[
h(\mu) = \frac{2^{2-a-\beta}}{\sqrt{2\pi} \sigma B(a, \beta) \left[1 - I_{\Phi\left(\frac{\mu-\mu}{\sigma}\right)}(a, \beta)\right]}
\]

We study the asymptotic behavior of the hazard function as \(x \to \infty\). Let \(h_N(x)\) be the hazard rate function of the normal distribution. We show that \(h_N(x) \sim x/\sigma^2\) as \(x \to \infty\). We have:

\[
\lim_{x \to \infty} \frac{h_N(x)}{x} = \lim_{x \to \infty} \frac{\phi\left(\frac{x-\mu}{\sigma}\right)\frac{1}{2}}{\sigma \phi\left(\frac{x-\mu}{\sigma}\right)}.
\]

Direct evaluation of this limit gives an indeterminate form. However, using L'Hôpital rule, straightforward manipulations show that the above limit is simply \(1/\sigma^2\). We are now able to show that \(h(x) \sim (\beta/\sigma^2)x\) as \(x \to \infty\).
Indeed, we have:

\[
\lim_{x \to \infty} \frac{h(x)}{x} = \frac{1}{B(\alpha, \beta)} \lim_{x \to \infty} \left[ \Phi\left( \frac{x - \mu}{\sigma} \right) \right]^{\alpha-1} \lim_{x \to \infty} \left[ 1 - \Phi\left( \frac{x - \mu}{\sigma} \right) \right]^{\beta-1} \phi\left( \frac{x - \mu}{\sigma} \right) \frac{1}{x} \\
= \frac{1}{B(\alpha, \beta)} \lim_{x \to \infty} \left[ 1 - \Phi\left( \frac{x - \mu}{\sigma} \right) \right]^{\beta-1} \phi\left( \frac{x - \mu}{\sigma} \right) \frac{1}{x}.
\]

Once again another indeterminate form arises. An application of L'Hôpital rule gives

\[
\lim_{x \to \infty} \left[ 1 - \Phi\left( \frac{x - \mu}{\sigma} \right) \right]^{\beta-1} \phi\left( \frac{x - \mu}{\sigma} \right) \frac{1}{x} = \sigma B(\alpha, \beta) \left\{ \lim_{x \to \infty} (\beta - 1) \frac{h_N(x)}{x} + \lim_{x \to \infty} \frac{1}{x^2} \right\} = B(\alpha, \beta) \frac{\beta}{\sigma}.
\]

Thus, \( h(x) \sim (\beta/\sigma^2)x \) as \( x \to \infty \). Combining both previous results, one can easily obtain \( h(x) \sim \beta h_N(x) \) as \( x \to \infty \). The limit of \( h(x) \) as \( x \to -\infty \) is zero. However, we show that \( h(x) \sim \frac{1}{\sigma B(\alpha, \beta)} \left( \frac{x - \mu}{\sigma} \right)^{-1-a} \phi\left( \frac{x - \mu}{\sigma} \right) \) when \( x \to -\infty \).

In fact, considering that \( \lim_{x \to -\infty} \Phi\left( \frac{x - \mu}{\sigma} \right) = \lim_{x \to -\infty} I_{\Phi\left( \frac{x - \mu}{\sigma} \right)}(\alpha, \beta) = 0 \), we obtain:

\[
\lim_{x \to -\infty} \frac{h(x)}{\left( \frac{x - \mu}{\sigma} \right)^{1-a} \phi\left( \frac{x - \mu}{\sigma} \right)} = \frac{1}{\sigma B(\alpha, \beta)} \lim_{x \to -\infty} \left[ \phi\left( \frac{x - \mu}{\sigma} \right) \right]^{\alpha-1} \phi\left( \frac{x - \mu}{\sigma} \right) \left( \frac{x - \mu}{\sigma} \right)^{-1-a} \phi\left( \frac{x - \mu}{\sigma} \right) \frac{1}{\sigma}.
\]

Applying L'Hôpital rule, we can show that \( \Phi(x) \sim (-x)^{-1} \phi(x) \). Then, we obtain that the above limit is \( 1/\sigma B(\alpha, \beta) \).

In order to relate the asymptotic behavior of \( h(x) \) and \( h_N(x) \) as \( x \to -\infty \), let us show that \( h_N(x) \sim \phi\left( \frac{x - \mu}{\sigma} \right) / \sigma \) as follows

\[
\lim_{x \to -\infty} \frac{h_N(x)}{\phi\left( \frac{x - \mu}{\sigma} \right)} = \frac{1}{\sigma} \lim_{x \to -\infty} \phi\left( \frac{x - \mu}{\sigma} \right) \frac{1}{\phi\left( \frac{x - \mu}{\sigma} \right)} = \frac{1}{\sigma} \lim_{x \to -\infty} \frac{1}{1 - \Phi\left( \frac{x - \mu}{\sigma} \right)} = \frac{1}{\sigma}.
\]

So, the asymptotic behavior of \( h(x) \) can be related to that one of \( h_N(x) \) when \( x \to -\infty \) by:

\[
h(x) \sim \frac{1}{B(\alpha, \beta)\sigma^{1-a}} \left( \frac{x - \mu}{\sigma} \right)^{1-a} [h_N(x)]^a.
\]

### 4 Moments

We can work with the BSN distribution in generality, since the moments of \( X = \mu + \sigma Z \sim BN(\alpha, \beta, \mu, \sigma) \) follow from the moments of \( Z \sim BN(\alpha, \beta, 0, 1) \) using \( E(X^n) = E((\mu + \sigma Z)^n) = \sum_{r=0}^{n} \binom{n}{r} \mu^{n-r} \sigma^{r} E(Z^r) \). For \( s \) and \( r \) non-negative integers, let \( t_{s,r} = \int_{-\infty}^{\infty} x^s \phi(x) \phi(x)^r \) be the \((s,r)\)th probability weighted moment (PWM) of the standard normal distribution. For \( \alpha \) integer and \( \alpha \) real non-integer, the \( s \)th moment of \( Z \) can be expressed in
terms of linear combinations of these PWMs as \[2\]

\[
E(Z^s) = \sum_{r=0}^{\infty} w_r(\alpha, \beta) \tau_{s,r+a-1} \quad \text{and} \quad E(Z^t) = \sum_{i,j=0}^{l} w_{i,j,r}(\alpha, \beta) \tau_{s,r},
\]

respectively, where

\[
w_i(\alpha, \beta) = \frac{(-1)^i(\beta-1)}{B(\alpha, \beta)} \quad \text{and} \quad w_{i,j,r}(\alpha, \beta) = \frac{(-1)^{i+j+r}(\alpha+i-1)(\beta-1)^j}{B(\alpha, \beta)}.
\]

For \(s + r - l\) even, they demonstrated that

\[
\tau_{s,r} = 2^{s/2} \pi^{-(r+1/2)} \sum_{l=0}^{r} \binom{r}{l} 2^{-l} \Gamma \left( \frac{s + r - l + 1}{2} \right) \times F_{A}^{(r-l)} \left( \frac{s + r - l + 1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{3}{2}, \frac{3}{2}, \ldots, -1, \ldots, -1 \right),
\]

where \(F_{A}^{(r-l)}(\cdot)\) is the Lauricella function of type A [11]. Expressions for terms in \(\tau_{s,r}\) vanish when \(s + r - l\) is odd. Equations (4) and (5) are given as infinite weighted sums of Lauricella functions for which numerical routines for computation are available, for example, Mathematica [14]. They extend some previously known results which are valid only for both \(\alpha\) and \(\beta\) integers and can be more efficient than computing the moments by writing some codes in SAS or R. In the next section, we provide an alternative formula for the moments of the BSN distribution based on the quantile function.

Plots of the skewness of the BSN distribution as a function of parameter \(\alpha\) (for selected values of \(\beta\)), and as a function of parameter \(\beta\) (for selected values of \(\alpha\)), are given in Figure 6. Figure 7 does the same for the kurtosis of the BSN distribution. These plots show that the BSN skewness increases when \(\alpha\) increases (for selected values of \(\beta\)) and decreases when \(\beta\) increases (for selected values of \(\alpha\)). On the other hand, the BSN kurtosis first decreases steadily to a minimum value and then increases when \(\alpha\) increases for fixed \(\beta\) or when \(\beta\) increases for fixed \(\alpha\).

## 5 Quantile Function

Quantile functions are in widespread use in general statistics and often find representations in terms of lookup tables for key percentiles. Without loss of generality, we can work with the BSN density function given by

\[
f(x) = \frac{1}{B(\alpha, \beta)} \phi(x) \Phi(x) \alpha^{-1} [1 - \Phi(x)]^{\beta-1},
\]

and the corresponding cumulative function reduces to

\[
F(x) = I_{\phi(x)}(\alpha, \beta) = \frac{1}{B(\alpha, \beta)} \int_0^{\Phi(x)} \omega^{\alpha-1} (1 - \omega)^{\beta-1} d\omega.
\]

By inverting (7), the BSN quantile function, say \(Q(u)\), can be obtained from the quantile functions of the standard normal and beta distributions denoted by \(Q_{SN}(u)\) and \(Q_B(u)\), respectively. We readily have
Figure 6: Plots of the BSN skewness as functions of (a) $\alpha$ for $\beta = 2.5$ (solid curve) and $\beta = 3.5$ (dashed curve), and (b) $\beta$ for $\alpha = 2.5$ (solid curve) and $\alpha = 3.5$ (dashed curve).

Figure 7: Plots of the BSN kurtosis as functions of (a) $\alpha$ for $\beta = 2.5$ (solid curve) and $\beta = 3.5$ (dashed curve), and (b) $\beta$ for $\alpha = 2.5$ (solid curve) and $\alpha = 3.5$ (dashed curve).
5.1 Quantile Measures

The effect of the shape parameters $\alpha$ and $\beta$ on the skewness and kurtosis of the BSN distribution can be based on quantile measures. One of the earliest skewness measures to be suggested is the Bowley skewness \[ B = \frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)} \]
defined by the average of the quartiles minus the median, divided by half the interquartile range, namely

On the other hand, the Moors kurtosis is based on octiles \[ M = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)} \].

The measures $B$ and $M$ are less sensitive to outliers and they exist even for distributions without moments. Because $M$ is based on the octiles, it is not sensitive to variations of the values in the tails or to variations of the values around the median. Clearly, $M > 0$ and there is a good agreement with the usual kurtosis measures for some distributions. For the normal distribution, $B = M = 0$.

Figures 8 and 9 show the measures $B$ and $M$ as functions of $\alpha$ and $\beta$ for some parameter values of the BSN distribution, respectively. These plots really suggest that both measures are very sensitive on the shape parameters, thus indicating the importance of the model (1).
Figure 9: Bowley skewness (a) and Moors kurtosis (b) of the BSN distribution for $0 < \beta \leq 4$ and $\alpha \in \{1/8, 1/2, 3/4, 1, 5, 10\}$ (solid, dashed, dash-dotted, dotted, bold solid, and bold dashed curves, respectively).

5.2 Power Series Expansion

Power series methods are at the heart of many aspects of applied mathematics and statistics. In this section, we provide a power series expansion for the quantile function that can be useful to obtain mathematical properties of the BSN distribution. First, the following expansion is available for the beta quantile function [15]

$$x = Q_B(u) = \Gamma^{-1}_u(\alpha, \beta) = \sum_{i=1}^{\infty} d_i u^{i/\alpha}. \tag{8}$$

Here, $d_i = [\alpha B(\alpha, \beta)]^{i/\alpha} a_i$ for $i \geq 1$, $a_0 = 0$, $a_1 = 1$, and the quantities $a_i$ for $i \geq 2$ can be derived from a cubic recurrence equation

$$a_i = \frac{1}{[i^2 + (\alpha - 2)i + (1 - \alpha)]} \left\{ (1 - \delta_{i,2}) \sum_{r=2}^{i-1} a_r a_{i+1-r} [r(1-\alpha)(i-r) - r(r-1)] + \sum_{r=1}^{i-1} \sum_{s=1}^{i-r} a_r a_s a_{i+1-r-s} [r(r-\alpha) + s(\alpha + \beta - 2)] \times (i+1-r-s) \right\},$$

where $\delta_{i,2} = 1$ if $i = 2$ and $\delta_{i,2} = 0$ if $i \neq 2$. In the last equation, the quadratic term only contributes for $i \geq 3$. We have $a_2 = (\beta - 1)/(\alpha + 1)$, $a_3 = [(\beta - 1)(\alpha^2 + 3\beta\alpha - a + 5\beta - 4)]/[2(\alpha + 1)^2(\alpha + 2)]$, and so on. Following Steinbrecher [13], the standard normal quantile function can be expanded as $Q_{SN}(u) = \sum_{k=0}^{\infty} b_k w^{2k+1}$, where
\[ w = \sqrt{2\pi(u - 1/2)} \] and the quantities \( b_k \) can be calculated recursively from

\[
b_{k+1} = \frac{1}{2(2k+3)} \sum_{r=0}^{k} \frac{(2r+1)(2k - 2r + 1)b_r}{(r+1)(2r+1)} b_{k-r}.
\]

Here, \( b_0 = 1, b_1 = 1/6, b_2 = 7/120, b_3 = 127/7560, \ldots \) The function \( Q_{SN}(u) \) can be written as a power series given by

\[
Q_{SN}(u) = \sum_{i=0}^{\infty} c_k (u - 1/2)^k,
\]

whose \( c_k \) is defined by \( c_k = 0 \) for \( k = 0, 2, 4, \ldots \) and \( c_k = (2\pi)^{k/2} b_{(k-1)/2} \) for \( k = 1, 3, 5, \ldots \) Combining \( \text{(8)} \) and \( \text{(9)} \), \( Q(u) = Q_{SN}(Q_B(u)) \) can be reduced to

\[
Q(u) = \sum_{k=0}^{\infty} c_k \left( \sum_{i=0}^{\infty} d_i u^{i/a} \right)^k,
\]

where \( d_0 = -1/2 \) and (as before) \( d_i = a^{i/a} B(\alpha, \beta)^{i/a} a_i \) for \( i \geq 1 \).

By application of an equation of [6] for a power series raised to a positive integer \( k \), we have

\[
\left( \sum_{i=0}^{\infty} d_i u^{i/a} \right)^k = \sum_{i=0}^{\infty} e_{k,i} u^{i/a},
\]

whose coefficients \( e_{k,i} \) (for \( i = 1, 2, \ldots \)) can be determined numerically from the quantities \( d_i \) using the recurrence equation (with \( e_{k,0} = d_0^{k} \))

\[
e_{k,i} = (i d_0)^{-1} \sum_{m=1}^{i} [m(k+1) - i]d_m e_{k,i-m}.
\]

The coefficient \( e_{k,i} \) can be calculated from \( e_{k,0}, \ldots, e_{k,i-1} \) and hence from the quantities \( d_0, \ldots, d_i \). Further, it can also be given explicitly in terms of the coefficients \( d_i \), although it is not necessary for programming numerically these expansions in any algebraic or numerical software. From \( \text{(10)} \)–\( \text{(12)} \), we can write

\[
Q(u) = \sum_{i=0}^{\infty} f_i u^{i/a},
\]

where \( f_i = \sum_{k=0}^{\infty} c_k e_{k,i} \), for \( i = 0, 1, \ldots \) The power series expansion \( \text{(13)} \) is the main result of this section. Some mathematical properties of the BSN distribution (such as ordinary moments, generating function and mean deviations) can be directly obtained from \( \text{(13)} \). For example, if \( Z \) has the BSN distribution, we can use \( \text{(11)} \) and \( \text{(12)} \), to obtain after integration the \( s \)th moment of \( Z \) as

\[
E(Z^s) = \int_0^1 \left( \sum_{i=0}^{\infty} f_i u^{i/a} \right)^s du = \sum_{i=0}^{\infty} g_{s,i} \frac{g_{s,0}}{i/a + 1},
\]

where \( g_{s,0} = f_0^s \) and \( g_{s,i} = (i f_0)^{-1} \sum_{m=1}^{i} [m(s+1) - i]f_m g_{s,i-m} \) for \( s \geq 1 \).
6 Generating Function

Let \( X \) be a random variable with BSN density function and \( M(t) = E[\exp(tX)] \) be the moment generating function (mgf) of \( X \). Here, we give two representations for \( M(t) \). The first representation follows from (11)–(13) as

\[
M(t) = \sum_{s=0}^{\infty} \int_0^1 t^s \left( \sum_{i=0}^{\infty} \frac{f_i u^{i/a}}{s!} \right) du = \sum_{s,i=0}^{\infty} \frac{g_{s,i} t^s}{(i/a + 1)s!},
\]

(15)

where \( g_{s,i} \) was defined in Section 5.2.

The second representation follows, by expanding the binomial in (11), as

\[
f'(x) = \phi(x) \sum_{k=0}^{\infty} v_k \Phi(x)^{k+a-1},
\]

(16)

where \( v_k = v_k(\alpha, \beta) = (-1)^k \binom{\beta - 1}{k} [B(\alpha, \beta)] \). For \( \delta > 0 \) real non-integer, we can write

\[
\Phi(x)^\delta = \sum_{r=0}^{\infty} s_r(\delta) \Phi(x)^r,
\]

(17)

where \( s_r(\delta) = \sum_{j=0}^{\infty} (-1)^{r+j} \binom{j+\delta}{r} \). Combining (16) and (17), we have

\[
f(x) = \phi(x) \sum_{r=0}^{\infty} \pi_r \Phi(x)^r.
\]

(18)

where \( \pi_r = \sum_{k=0}^{\infty} v_k s_r(k + a - 1) \). From (18), we obtain

\[
M(-t) = \frac{1}{\sqrt{2\pi}} \sum_{r=0}^{\infty} \pi_r \int_{-\infty}^{\infty} \Phi(x)^r \exp \left( -tx - \frac{x^2}{2} \right) dx.
\]

The standard normal cdf \( \Phi(x) \) can be written as a power series expansion \( \Phi(x) = \sum_{j=0}^{\infty} a_j x^j \), where \( a_0 = (1 + \sqrt{2/\pi})^{-1} / 2 \), \( a_{2j+1} = (\frac{1}{\sqrt{2(2j+1)}}) \) for \( j = 0, 1, 2 \ldots \) and \( a_{2j} = 0 \) for \( j = 1, 2, \ldots \). We can write \( \Phi(x)^r = \sum_{j=0}^{\infty} c_{r,j} x^j \), whose coefficients \( c_{r,j} \) can be determined from (11) and (12) by \( c_{r,0} = a_0^r \) and \( c_{r,j} = (j a_0)^{-1} \sum_{m=1}^{j} (m(r+1) - j) \alpha_m c_{r,j-m} \).

Hence,

\[
M(-t) = \frac{1}{\sqrt{2\pi}} \sum_{r,j=0}^{\infty} \pi_r c_{r,j} \int_{-\infty}^{\infty} x^j \exp \left( -tx - \frac{x^2}{2} \right) dx.
\]

By equation (2.3.15.8) in (12), the integral becomes

\[
J(t,j) = \int_{-\infty}^{\infty} x^j \exp \left( -tx - \frac{x^2}{2} \right) dx = (-1)^j \sqrt{2\pi} \frac{\partial^j}{\partial t^j} \left\{ \exp \left( \frac{t^2}{2} \right) \right\}
\]

and thus

\[
M(-t) = \frac{1}{\sqrt{2\pi}} \sum_{r,j=0}^{\infty} \pi_r c_{r,j} J(t,j).
\]

(19)
Equations (15) and (19) are the main results of this section.

7 Mean Deviations

If $X$ has the BN distribution, we can derive the mean deviations about the mean $\nu = E(X)$ and about the median $m$ from $\delta_1 = \int_{-\infty}^{\infty} |x - \nu| f(x)dx$ and $\delta_2 = \int_{-\infty}^{\infty} |x - m| f(x)dx$, respectively. The mean $\nu$ can be obtained from (14) with $s = 1$ and the median $m$ is the solution of the non-linear equation $I_{\Phi\left(\frac{x - \mu}{\sigma}\right)}(\alpha, \beta) = 1/2$. The deviations $\delta_1$ and $\delta_2$ can be expressed as

$$\delta_1 = 2\left[\nu F(\nu) - J(q)\right] \quad \text{and} \quad \delta_2 = \nu - 2J(m),$$

where $J(q) = \int_{-\infty}^{q} x f(x)dx$. We provide an explicit expression for $J(q)$ based on the quantile function expansion. From (13), we have

$$J(q) = \int_0^{I_{\Phi\left(\frac{x - \mu}{\sigma}\right)}(\alpha, \beta)} Q(u)du = \sum_{i=0}^{\infty} f_i \frac{I_{\Phi\left(\frac{x - \mu}{\sigma}\right)}(\alpha, \beta)}{i/\alpha + 1}.$$
As a consequence, the previous integrals reduce to

$$\int_{-\infty}^{\infty} f(x)(\alpha - 1) \ln \left\{ \Phi \left( \frac{x - \mu}{\sigma} \right) \right\} \, dx = \frac{1 - \alpha}{B(\alpha, \beta)} \sum_{n=1}^{\infty} \frac{B(n, \alpha + \beta)}{n}$$

and

$$\int_{-\infty}^{\infty} f(x)(\beta - 1) \ln \left\{ 1 - \Phi \left( \frac{x - \mu}{\sigma} \right) \right\} \, dx = \frac{1 - \beta}{B(\alpha, \beta)} \sum_{n=1}^{\infty} \frac{B(n + \alpha, \beta)}{n}.$$  

Now, we can write the last integral as

$$\int_{-\infty}^{\infty} f(x) \ln \left\{ \phi \left( \frac{x - \mu}{\sigma} \right) \right\} \, dx = \int_{-\infty}^{\infty} f(x) \left[ \ln \left( \frac{1}{\sqrt{2\pi}} \right) + \ln \left\{ \exp \left( \frac{(x - \mu)^2}{2\sigma^2} \right) \right\} \right] \, dx$$

$$= - \ln \left( \sqrt{2\pi} \right) - \frac{1}{2\sigma^2} \left[ E(X^2) - 2\mu E(X) + \mu^2 \right],$$

where \(E(X^2)\) and \(E(X)\) follow from (14). These moments are also given in [7]. Finally, we conclude that

$$H(X) = \ln \left\{ \sqrt{2\pi\sigma} B(\alpha, \beta) \right\} + \frac{1}{2\sigma^2} \left[ E(X^2) - 2\mu E(X) + \mu^2 \right]$$

$$+ \frac{1}{B(\alpha, \beta)} \sum_{n=1}^{\infty} \frac{1}{n} \left[ (\alpha - 1)B(n, \alpha + \beta) + (\beta - 1)B(n + \alpha, \beta) \right].$$

References

[1] R. M. Aarts, Lauricella functions. MathWorld – A Wolfram Web Resource, created by Eric W. Weisstein, Dec. 2010.
[2] G. M. Cordeiro and S. Nadarajah, Closed-form expressions for moments of a class of beta generalized distributions, Braz. J. Probab. Stat., 25 (2011), pp. 14–33.
[3] A. El-Zaart and D. Zhou, Statistical modelling of multimodal SAR images, International Journal of Remote Sensing, 28 (2007), pp. 2277–2294.
[4] N. Eugene, C. Lee, and F. Famoye, Beta-normal distribution and its applications, Communications in Statistics: Theory and Methods, 31 (2002), pp. 497–512.
[5] F. Famoye, C. Lee, and N. Eugene, Beta-normal distribution: Bimodality properties and application, Journal of Modern Applied Statistical Methods, 3 (2004), pp. 85–103.
[6] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, 6th ed., 2000.
[7] A. K. Gupta and S. Nadarajah, On the moments of the beta normal distribution, Communications in Statistics: Theory and Methods, 33 (2004), pp. 1–13.
[8] J. F. Kenney and E. S. Keeping, Mathematics of Statistics, Van Nostrand, Princeton, NJ, 3 ed., 1962, ch. The \(k\)-Statistics, pp. 99–100.
[9] E. D. Louisa and O. Dogue, Does age of onset in essential tremor have a bimodal distribution? Data from a tertiary referral setting and a population-based study, Neuroepidemiology, 29 (2007), pp. 208–212.
[10] J. J. A. Moors, A quantile alternative for kurtosis, Journal of the Royal Statistical Society. Series D (The Statistician), 37 (1988), pp. 25–32.
[11] M. Nardini, G. Ghisellini, and G. Ghirlanda, Optical afterglows of gamma-ray bursts: a bimodal distribution?, Monthly Notices of the Royal Astronomical Society, 383 (2008), pp. 1049–1057.
[12] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, Integrals and Series: Elementary Functions, Gordon & Breach Science Publishers, New York, NY, 1986.
[13] G. Steinbrecher, *Taylor expansion for inverse error function around origin*, Jan. 2002. working paper.

[14] Wolfram Research, *Mathematica 7*, 2009.

[15] ——, *Inverse beta regularized*. Wolfram Functions Site, Jan. 2011.

[16] A. Zangvil, D. H. Portis, and P. J. Lamb, *Investigation of the large-scale atmospheric moisture field over the Midwestern United States in relation to summer precipitation. Part I: Relationships between moisture budget components on different timescales*, Journal of Climate, 14 (2001), pp. 582–597.