COMPARISON OF STEKLOV EIGENVALUES AND LAPLACIAN EIGENVALUES ON GRAPHS

YONGJIE SHI¹ AND CHENGJIE YU²

Abstract. In this paper, we obtain a comparison of Steklov eigenvalues and Laplacian eigenvalues on graphs and discuss its rigidity. As applications of the comparison of eigenvalues, we obtain Lichnerowicz-type estimates and some combinatorial estimates for Steklov eigenvalues on graphs.

1. Introduction

On a compact Riemannian manifold with boundary, the Dirichlet-to-Neumann map or Steklov operator sends the Dirichlet boundary data of a harmonic function on the manifold to its Neumann boundary data. The eigenvalues of the Dirichlet-to-Neumann map or Steklov operator are called Steklov eigenvalues of the Riemannian manifold. This subject was first introduced by Steklov [10, 18] when considering liquid sloshing. It was later found useful in applied mathematics, especially in electrical impedance tomography for medical imaging (see [19]).

In the discrete setting, Dirichlet-to-Neumann maps and Steklov eigenvalues were introduced in [8, 7] and received attentions recently (see [6, 9, 14, 15, 16, 17]). In this paper, we obtain a comparison of Steklov eigenvalues and Laplacian eigenvalues on graphs. It seems that this is a major difference of Steklov eigenvalues on graphs with that on Riemannian manifolds. Such a comparison was also mentioned in [9] for graphs equipped with normalized weights. One of the application of this comparison of eigenvalues is in obtaining Lichnerowicz-type estimate for Steklov eigenvalues which in some sense extend Escobar’s conjecture (see [3]) from the smooth case to discrete case. Escobar’s conjecture was recently partially solved by Xiong-Xia [20].

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Let’s recall some notations and notions on graphs before stating our main results.

A weighted graph is a triple \((G, m, w)\) where \(G\) is a graph, \(m\) is the vertex measure which is a positive function on the set \(V(G)\) of the vertices of \(G\) and \(w\) is the edge weight which is a positive function on the set \(E(G)\) of the edges of \(G\). For convenience, we view \(w\) as a symmetric function on \(V \times V\) by zero extension and we simply write \(m(x)\) and \(w(x, y)\) as \(m_x\) and \(w_{xy}\) respectively. Throughout this paper, the graph \(G\) is assumed to be finite, simple and connected. We will also simply write \(V(G)\) and \(E(G)\) as \(V\) and \(E\) if no confusion was made.

For \(A, B \subset V\), \(E(A, B)\) denotes the set of edges in \(G\) with one end point in \(A\) and the other in \(B\). We call the weight with \(m \equiv 1\) and \(w \equiv 1\) a unit weight. For each \(x \in V\),

\[
\text{(1.1)} \quad \text{Deg}(x) := \frac{1}{m_x} \sum_{y \in V} w_{xy}
\]

is called the weighted degree of \(x\). If for any \(x \in V\), \(\text{Deg}(x) = 1\), the graph is called a graph with a normalized unit weight.

Let \((G, m, w)\) be a weighted graph. Let \(A^0(G)\) be the space of functions on \(V\) and \(A^1(G)\) be the space of skew-symmetric functions \(\alpha\) on \(V \times V\) such that \(\alpha(x, y) = 0\) when \(x \not\sim y\). Equip \(A^0(G)\) and \(A^1(G)\) with the natural inner products

\[
\text{(1.2)} \quad \langle u, v \rangle = \sum_{x \in V} u(x)v(x)m_x
\]

and

\[
\text{(1.3)} \quad \langle \alpha, \beta \rangle = \sum_{\{x, y\} \in E} \alpha(x, y)\beta(x, y)w_{xy} = \frac{1}{2} \sum_{x, y \in V} \alpha(x, y)\beta(x, y)w_{xy}
\]

respectively. For any \(u \in A^0(G)\), define the differential \(du\) of \(u\) as

\[
\text{(1.4)} \quad du(x, y) = \begin{cases} u(y) - u(x) & \{x, y\} \in E \\ 0 & \text{otherwise.} \end{cases}
\]

Let \(d^* : A^1(G) \to A^0(G)\) be the adjoint operator of \(d : A^0(G) \to A^1(G)\). The Laplacian operator on \(A^0(G)\) is defined as

\[
\text{(1.5)} \quad \Delta = -d^*d.
\]

By direct computation,

\[
\text{(1.6)} \quad \Delta u(x) = \frac{1}{m_x} \sum_{y \in V} (u(y) - u(x))w_{xy}
\]
Comparison of Steklov eigenvalues and Laplacian eigenvalues

for any $x \in V$. Moreover, by the definition of $\Delta$, it is clear that

\begin{equation}
\langle \Delta u, v \rangle = -\langle du, dv \rangle
\end{equation}

for any $u, v \in \mathbb{R}^V$. So $-\Delta$ is a nonnegative self-adjoint operator on $A^0(G) = \mathbb{R}^V$. Let

\begin{equation}
0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_{|V|}
\end{equation}

be the eigenvalues of $-\Delta$ on $(G, m, w)$. It is clear that $\mu_1 = 0$ because constant functions are the corresponding eigenfunctions and $\mu_2 > 0$ because we always assume that $G$ is connected.

Next recall the notion of graphs with boundary. A pair $(G, B)$ is said to be a graph with boundary if $G$ is graph and $B \subset V(G)$ such that (i) any two vertices in $B$ are not adjacent, (ii) any vertex in $B$ is adjacent to some vertex in $\Omega := V \setminus B$. The set $B$ is called the vertex-boundary of $(G, B)$ and the set $\Omega$ is called the vertex-interior of $(G, B)$. An edge joining a boundary vertex and an interior vertex is called a boundary edge. The induced graph of $G$ on $\Omega$ is denoted as $G|\Omega$.

Let $(G, m, w, B)$ be a weighted graph with boundary. For any $u \in \mathbb{R}^V$ and $x \in B$, define the normal derivative of $u$ at $x$ as:

\begin{equation}
\frac{\partial u}{\partial n}(x) := \frac{1}{m_x} \sum_{y \in V} (u(x) - u(y))w_{xy} = -\Delta u(x).
\end{equation}

Then, by (1.7), one has the following Green’s formula:

\begin{equation}
\langle \Delta u, v \rangle_{\Omega} = -\langle du, dv \rangle + \left\langle \frac{\partial u}{\partial n}, v \right\rangle_{B}.
\end{equation}

Here, for any set $S \subset V$,

\begin{equation}
\langle u, v \rangle_S := \sum_{x \in S} u(x)v(x)m_x.
\end{equation}

Similarly, for any $S \subset E$,

\begin{equation}
\langle \alpha, \beta \rangle_S := \sum_{\{x,y\} \in S} \alpha(x, y)\beta(x, y)w_{xy}.
\end{equation}

We are now ready to introduce the notions of Dirichlet-to-Neumann map and Steklov eigenvalues on graphs. For each $f \in \mathbb{R}^B$, let $u_f$ be the harmonic extension of $f$ into $\Omega$:

\begin{equation}
\begin{cases}
\Delta u_f(x) = 0 & x \in \Omega \\
u_f(x) = f(x) & x \in B.
\end{cases}
\end{equation}

Define the Dirichlet-to-Neumann map $\Lambda : \mathbb{R}^B \to \mathbb{R}^B$ as

\begin{equation}
\Lambda(f) = \frac{\partial u_f}{\partial n}.
\end{equation}
By (1.10),
\[ \langle \Lambda(f), g \rangle_B = \langle du_f, du_g \rangle \]
for any \( f, g \in \mathbb{R}^B \). This implies that \( \Lambda \) is a nonnegative self-adjoint operator on \( \mathbb{R}^B \). Let
\[ 0 = \sigma_1 < \sigma_2 \leq \cdots \leq \sigma_{|B|} \]
be the eigenvalues of \( \Lambda \). It is clear that \( \sigma_1 = 0 \) because constant functions are the corresponding eigenfunctions and \( \sigma_2 > 0 \) because we always assume that \( G \) is connected.

We are now ready to state the first main result of this paper, a comparison of the the Steklov eigenvalues and the Laplacian eigenvalues on graphs, and its rigidity.

**Theorem 1.1.** Let \((G, m, w, B)\) be a connected weighted finite graph with boundary. Then,
\[ \sigma_i \geq \mu_i \]
for \( i = 1, 2, \cdots, |B| \). If \( \sigma_i = \mu_i \) for some \( i = 2, 3, \cdots, |B| \), then there is an eigenfunction \( v_i \) of \( \mu_i \) such that \( v_i|_{\Omega} = \Delta v_i|_{\Omega} = 0 \). Moreover, the quality of (1.17) holds for all \( i = 1, 2, \cdots, |B| \) if and only if the following statements are true:

1. There is a nonnegative function \( \rho \in \mathbb{R}^\Omega \), such that
\[ w_{xy} = \rho_y m_x m_y \]
for each \( x \in B \) and \( y \in \Omega \). In particular, each interior vertex is either adjacent to all boundary vertices or adjacent to no boundary vertices.
2. For any \( u \in \mathbb{R}^\Omega \) with \( \langle u, 1 \rangle_{\Omega} = 0 \),
\[ \langle du, du \rangle_{\Omega} - \text{Deg} \langle u, u \rangle_{\Omega} + V_B \langle \rho u, u \rangle_{\Omega} - \frac{V_G}{\text{Deg}} \langle \rho, u \rangle_{\Omega}^2 \geq 0 \]
Here,
\[ \text{Deg} := \langle \rho, 1 \rangle_{\Omega} = \text{Deg}(x) \]
for any \( x \in B \), \( V_B = \sum_{x \in B} m_x, V_\Omega = \sum_{y \in \Omega} m_y, V_G = V_B + V_\Omega, \) and
\[ \langle du, du \rangle_{\Omega} := \langle du, du \rangle_{E(\Omega, \Omega)} \].

The eigenvalue comparison (1.17) is almost an obvious observation from definitions as follows. Note that the Laplacian eigenvalues \( \mu_i \)'s can be obtained by applying Courant’s min-max principle to the Rayleigh quotient:
\[ R[u] = \frac{\langle du, du \rangle}{\langle u, u \rangle} \]
and the Steklov eigenvalues $\sigma_i$'s can be obtained by applying Courant’s min-max principle to the Rayleigh quotient:

\begin{equation}
R_\sigma[u] = \frac{\langle du, du \rangle}{\langle u, u \rangle_B}.
\end{equation}

It is clear that

\begin{equation}
R_\sigma[u] \geq R[u].
\end{equation}

Then, the eigenvalue comparison (1.17) follows directly from Courant’s min-max principle. The comparison (1.17) was also mentioned in [9, Corollary 1.6] for normalized weighted graphs. Our contribution here is characterizing the rigidity of (1.17).

As a direct consequence of (1.19), we have the following sufficient condition for (1.17) to hold for all $i = 1, 2, \ldots, |B|$ on general weighted graphs.

**Corollary 1.1.** Let $(G, m, w, B)$ be a connected weighted finite graph, suppose that for any $x \in B$ and $y \in \Omega$, $w_{xy} = \rho_y m_x m_y$ for some nonnegative function $\rho \in \mathbb{R}^\Omega$, and

\begin{equation}
\mu_2(\Omega) \geq \text{Deg} - V_B \rho_{\text{min}} + \frac{V_G}{\text{Deg}} \langle \rho, \rho \rangle_\Omega
\end{equation}

where $\mu_2(\Omega)$ is the second Laplacian eigenvalue of the induced graph of $G$ on $\Omega$,

\begin{align*}
\text{Deg} := \langle \rho, 1 \rangle_\Omega = \text{Deg}(x)
\end{align*}

for any $x \in B$, and $\rho_{\text{min}} = \min_{y \in \Omega} \rho_y$. Then, $\sigma_i = \mu_i$ for all $i = 1, 2, \ldots, |B|$.

By using Corollary 1.1, one can construct many graphs such that equality of (1.17) holds for $i = 1, 2, \ldots, |B|$. For example, first fix a connected weighted graph $(\Omega, m_\Omega, w_\Omega)$. Then add the boundary $B$ to $\Omega$, joining $B$ to $\Omega$ and arranged the measure of the boundary vertices and weight of the boundary edges so that

\begin{align*}
w_{xy} = m_x m_y \rho_y
\end{align*}

holds for any $x \in B$ and $y \in \Omega$. Note that $\mu_2(G|_\Omega, m_\Omega, w_\Omega) > 0$ and

\begin{equation}
\mu_2(G|_\Omega, m_\Omega, \lambda w_\Omega) = \lambda \mu_2(G|_\Omega, m_\Omega, w_\Omega).
\end{equation}

So, if $\lambda$ is large enough, the equality (1.23) will hold. Then, by Corollary 1.1, the equality of (1.17) holds for $i = 1, 2, \ldots, |B|$ on the graph equipped with the re-scaled weight.

To study more explicitly the rigidity of (1.17), we first consider the case that $\rho$ is constant.
Corollary 1.2. Let \((G, m, w, B)\) be a connected weighted finite graph with boundary and suppose that there is a positive constant \(\rho\), such that for each \(x \in B\) and \(y \in \Omega\), \(w_{xy} = \rho m_x m_y\). Then \(\sigma_i = \mu_i\) for all \(i = 1, 2, \cdots, |B|\) if and only if
\[
\rho(V_\Omega - V_B) \leq \mu_2(\Omega).
\]
In particular, if the induced graph on \(\Omega\) is disconnected, then \(\sigma_i = \mu_i\) for all \(i = 1, 2, \cdots, |B|\) if and only if
\[
V_B \geq V_\Omega.
\]

Next, we consider the rigidity of (1.17) for graphs with unit weight.

Corollary 1.3. Let \((G, B)\) be a connected finite graph with boundary equipped with the unit weight. Then, the equality of (1.17) holds for all \(i = 1, 2, \cdots, |B|\) if and only if all the following statements are true:

1. Each interior vertex is either adjacent to all boundary vertices or adjacent to no boundary vertex;
2. Let \(\Omega_0\) be the set of interior vertices that are not adjacent to any boundary vertex and \(\Omega_1 = \Omega \setminus \Omega_0\), then \(E(\Omega_0, \Omega_1) = |\Omega_0||\Omega_1|\) which means for any \(x \in \Omega_0\) and \(y \in \Omega_1\), \(x\) and \(y\) are adjacent;
3. \(\mu_2(\Omega_1) \geq |\Omega_1| - |\Omega_0| - |B|\).

Moreover, by (1.17), a lower bound on \(\mu_i\) will automatically give us a lower bound for \(\sigma_i\). For example, by the Lichnerowicz estimate for \(\mu_2\) in [1, 11] with respect to Bakry-Émery curvature, one has the following Lichnerowicz estimate for \(\sigma_2\) directly.

Corollary 1.4. Let \((G, m, w, B)\) be a connected weighted finite graph with boundary. Suppose that \((G, m, w)\) satisfy the Bakry-Émery curvature-dimension \(CD(K, n)\) with \(K > 0\) and \(n > 1\). Then,
\[
\sigma_2 \geq \frac{nK}{n - 1}.
\]

In [17], we give a direct proof to (1.26) and discuss its rigidity.

Similarly, by the Licherowicz estimate for \(\mu_2\) with respect to Ollivier curvature [12], one can have a Licherowicz estimate for \(\sigma_2\) with respect to Ollivier curvature. Here, the Ollivier curvature we used is the most general one given by Münch and Wojciechowski [13] recently. Their definition is a natural extension of the definition by Lin-Lu-Yau [12] on general weighted graphs. In [13], Münch and Wojciechowski showed that the Ollivier curvature \(\kappa(x, y)\) they defined can be computed by the following formula:
\[
\kappa(x, y) = \inf_{f \in \text{Lip}(1), \nabla_x f = 1} \nabla_{xy} \Delta f
\]
for any two distinct vertices $x, y$, where
\begin{equation}
\nabla_{xy} f := \frac{f(x) - f(y)}{d(x, y)}.
\end{equation}

By substituting an eigenfunction $f$ of $\mu_2$ into (1.27), one obtain the Lichnerowicz estimate:
\begin{equation}
\mu_2 \geq \kappa
\end{equation}
for $\mu_2$ directly when the Ollivier curvature of $(G, m, w)$ has a positive lower bound $\kappa$. Combining this with (1.17), one has the following Lichnerowicz estimate for $\sigma_2$.

**Corollary 1.5.** Let $(G, m, w, B)$ be a connected weighted finite graph with boundary. Suppose that the Ollivier curvature of $(G, m, w)$ is not less than a positive constant $\kappa$. Then,
\begin{equation}
\sigma_2 \geq \kappa.
\end{equation}

Furthermore, by using the lower bound of $\mu_2$ by Fiedler [5], we have the following lower bounds for $\sigma_2$.

**Theorem 1.2.** Let $G$ be a connected finite graph equipped with the unit weight. Then,
\begin{equation}
\sigma_2 \geq 2e(G) \left(1 - \cos \frac{\pi}{|V(G)|}\right) \geq 2v(G) \left(1 - \cos \frac{\pi}{|V(G)|}\right)
\end{equation}
where $e(G)$ is edge connectivity of $G$ and $v(G)$ is the vertex connectivity of $G$. That is, the least number of edges and least number of vertices in $G$ required to be deleted to make $G$ becoming disconnected respectively.

Finally, by using the lower bounds of $\mu_i$'s obtained by Friedman in [4], one has the following lower bounds for Steklov eigenvalues. For the definition of a star, see [4, P.1].

**Theorem 1.3.** Let $(G, B)$ be a connected finite graph with boundary equipped with the unit weight and $i \geq 2$. Then,
\begin{equation}
\sigma_i \geq 2 - 2 \cos \frac{\pi}{2k + 1}
\end{equation}
where $k = \left\lceil \frac{|V(G)|}{i} \right\rceil$. When $|V(G)| \equiv 1(\text{mod } i)$, the equality of (1.32) holds if and only if $k = 1$, i.e. $|V(G)| = i + 1$, and $(G, B)$ is a star of degree $i$ with each arm of length 1, and with the end points of each arm the boundary vertices.
When \( i \mid |V(G)| \),
\begin{equation}
\sigma_i > \mathcal{P}(k, \lambda_i).
\end{equation}

Here \( k = \frac{|V(G)|}{i} \), \( \lambda_i \) is the largest eigenvalue of the path \( P_i \) on \( i \) vertices equipped with the unit weight, and \( \mathcal{P}(k, \lambda) \) is the first Dirichlet eigenvalue of \( (P_{k+1}, m, w, B) \) where \( P_{k+1} \) is a path with vertices \( 0, 1, 2, \ldots, k \) and \( B = \{0\} \), and moreover \( m_j = 1 \) for \( j = 0, 1, \ldots, k \), \( w_{12} = w_{23} = \cdots = w_{k-1,k} = 1 \) and \( w_{01} = \lambda \).

When \( |V(G)| \equiv s \pmod{i} \) with \( 2 \leq s \leq i - 1 \), we lack of rigidity for (1.32) because we lack of rigidity for the corresponding estimate for \( \mu_i \) in [4]. The same as in [4], in this case, the equality of (1.32) at least holds for two different graphs. For example, when \( |V(G)| = 5 \), \( i = 3 \), the equality of (1.32) holds for a star of degree 4 with all arms of length 1, and with the end points of three arms as the boundary vertices or with all the four end points of the four arms as boundary vertices. It also holds for a star of degree 3 with two arms of length 1 and one arm of length 2 and with the the end points of the two arms of length 1 and the middle point the arm of length 2 as the three boundary vertices.

The rest of the paper is organized as follows.

2. Comparison of \( \sigma_i \) and \( \mu_i \)

Although (1.17) comes from an almost obvious observation via Courant’s min-max principle as mentioned in the last section, we will present a detailed proof below for convenience of handling the rigidity of (1.17).

Proof of Theorem 1.1. Let \( f_1 = 1, f_2, \ldots, f_{|B|} \in \mathbb{R}^B \) be the eigenfunctions of \( \sigma_1 = 0, \sigma_2, \ldots, \sigma_{|B|} \) respectively, such that
\begin{equation}
\langle f_i, f_j \rangle_B = 0
\end{equation}
when \( i \neq j \). Moreover, let \( u_1 = 1, u_2, \ldots, u_{|V|} \in \mathbb{R}^V \) be the eigenfunctions of \( \mu_1 = 0, \mu_2, \ldots, \mu_{|V|} \) respectively, such that
\begin{equation}
\langle u_i, u_j \rangle = 0
\end{equation}
when \( i \neq j \). For \( i = 2, 3, \ldots, |B| \), let
\[ v_i = c_1 u_{f_1} + c_2 u_{f_2} + \cdots + c_i u_{f_i} \]
with \( c_1, c_2, \ldots, c_i \) not all zero, be such that
\begin{equation}
\langle v_i, u_j \rangle = 0 \text{ for } j = 1, 2, \ldots, i - 1.
\end{equation}
This can be done because (2.3) is a homogeneous linear system with \( i - 1 \) equations and \( i \) unknowns \( c_1, c_2, \ldots, c_i \) which certainly has nonzero
solutions. Then
\begin{equation}
\mu_i \leq \frac{\langle dv_i, dv_i \rangle}{\langle v_i, v_i \rangle} \leq \frac{\langle dv_i, dv_i \rangle}{\langle v_i, v_i \rangle_B} \leq \sigma_i.
\end{equation}

It is clear that the equality \( \sigma_i = \mu_i \) holds only when \( v_i|_\Omega = \Delta v_i|_\Omega = 0 \) and \( v_i \) is simultaneously an eigenfunction of \( \mu_i \) and \( \sigma_i \).

We next come to the rigidity part. If the equalities of (1.17) holds for \( i = 1, 2, \cdots, |B| \), we first claim that there is a sequence \( \tilde{v}_1 = 1, \tilde{v}_2, \cdots, \tilde{v}_{|B|} \) of nonzero functions on \( V \) such that
\begin{enumerate}
  \item \( \tilde{v}_i|_\Omega = \Delta \tilde{v}_i|_\Omega = 0 \) for \( i = 2, \cdots, |B| \);
  \item \( \frac{\partial \tilde{v}_i}{\partial n} = \sigma_i \tilde{v}_i \) for \( i = 1, 2, \cdots, |B| \);
  \item \( \Delta \tilde{v}_i = \mu_i \tilde{v}_i \) for \( i = 1, 2, \cdots, |B| \);
  \item \( \langle \tilde{v}_i, \tilde{v}_j \rangle_B = \langle \tilde{v}_i, \tilde{v}_j \rangle = 0 \) when \( 1 \leq j < i \leq |B| \).
\end{enumerate}

We now construct the sequence \( \tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_{|B|} \) by induction. For \( i \geq 2 \), suppose \( \tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_{i-1} \) satisfying (i),(ii),(iii) and (iv) has been constructed. Let
\[ \tilde{v}_i = c_1 u_{f_1} + c_2 u_{f_2} + \cdots + c_{i-1} u_{f_{i-1}} + c_i u_{f_i} \]
with \( c_1, c_2, \cdots, c_i \) not all zero, be such that
\begin{equation}
\langle \tilde{v}_i, \tilde{v}_j \rangle = 0
\end{equation}
for all \( j = 1, 2, \cdots, i-1 \). This can be done because of the same reason as before. Then, by replacing the function \( v_i \) by \( \tilde{v}_i \) in (2.4), and noting that \( \mu_i = \sigma_i \), we know that \( \tilde{v}_i \) must satisfy (i),(ii),(iii) and (iv).

Note that for any \( f \in \mathbb{R}^B \) with
\begin{equation}
\sum_{x \in B} f(x) m_x = \langle f, 1 \rangle_B = 0,
\end{equation}
one has
\begin{equation}
f = \sum_{i=2}^{\lfloor B \rfloor} c_i \tilde{v}_i|_B
\end{equation}
for some \( c_2, \cdots, c_{|B|} \). Then
\begin{equation}
u_f = \sum_{i=2}^{\lfloor B \rfloor} c_i \tilde{v}_i
\end{equation}
because \( \Delta \tilde{v}_i|_\Omega = 0 \). Moreover, by that \( \tilde{v}_i|_\Omega = 0 \) for \( i = 2, 3, \cdots, |B| \), \( u_f(y) = 0 \) for all \( y \in \Omega \). So, for any \( y \in \Omega \), we have
\begin{equation}0 = m_y \Delta u_f(y) = \sum_{x \in B} f(x) w_{xy},\end{equation}
Comparing this to (2.6), we know that \( w_{xy} = \kappa_y \cdot m_x \) for any \( x \in B \) and \( y \in \Omega \) for some nonnegative function \( \kappa \in \mathbb{R}_\Omega \). Let \( \rho_y = \frac{\kappa_y}{m_y} \) for \( y \in \Omega \). Then, we get (1) of the theorem.

Conversely, when \( w_{xy} = \rho_y m_x m_y \) for any \( x \in B \) and \( y \in \Omega \), for any nonzero \( f \in \mathbb{R}^B \) with \( \langle f, 1 \rangle_B = 0 \), one has \( u_f(y) = 0 \) for any \( y \in \Omega \). So,

\[
(2.10) \quad \Lambda f(x) = \frac{\partial u_f}{\partial n}(x) = \frac{1}{m_x} \sum_{y \in \Omega} f(x)w_{xy} = \text{Deg} \cdot f(x)
\]

for any \( x \in B \), and

\[
(2.11) \quad -\Delta u_f = \text{Deg} \cdot u_f
\]

This implies that \( \sigma_2 = \sigma_3 = \cdots = \sigma_{|B|} = \text{Deg} \), and equality of (1.17) holds for any \( i = 1, 2, \cdots, |B| \) if and only if

\[
(2.12) \quad \langle dv, dv \rangle \geq \text{Deg} \cdot \langle v, v \rangle.
\]

for any nonzero \( v \in \mathbb{R}^V \) with \( \langle v, u_{f_i} \rangle = 0 \) for \( i = 1, 2, \cdots, |B| \).

Note that \( u_{f_i}|_\Omega = 0 \) for \( i = 2, 3, \cdots, |B| \), so

\[
(2.13) \quad \langle v, u_{f_i} \rangle_B = \langle v, u_{f_i} \rangle = 0
\]

for \( i = 2, 3, \cdots, |B| \). This implies that \( v|_B \) must be constant.

When \( v|_B \equiv 0 \), let \( u = v|_\Omega \). Then \( \langle u, 1 \rangle_\Omega = \langle v, 1 \rangle = 0 \) and

\[
(2.14) \quad \langle dv, dv \rangle = \langle du, du \rangle_\Omega + V_B \langle \rho u, u \rangle_\Omega.
\]

So, by (2.12), we have

\[
(2.15) \quad \langle du, du \rangle_\Omega - \text{Deg} \langle u, u \rangle_\Omega + V_B \langle \rho u, u \rangle_\Omega \geq 0
\]

for any \( u \in \mathbb{R}^\Omega \) with \( \langle u, 1 \rangle_\Omega = 0 \).

When \( v|_B \) is nonzero, without loss of generality, we assume that \( v|_B \equiv 1 \). Let \( u \in \mathbb{R}^\Omega \) be such that \( u = v|_\Omega + \frac{V_B}{V_\Omega} \). Then

\[
(2.16) \quad \langle u, 1 \rangle_\Omega = \left\langle v + \frac{V_B}{V_\Omega}, 1 \right\rangle_\Omega = \langle v, 1 \rangle - \langle v, 1 \rangle_B + V_B = 0.
\]

Moreover,

\[
(2.17) \quad \langle dv, dv \rangle = \langle du, du \rangle_\Omega + V_B \left\langle \rho \left( u - \frac{V_G}{V_\Omega} \right), u - \frac{V_G}{V_\Omega} \right\rangle_\Omega
\]

\[
= \langle du, du \rangle_\Omega + V_B \langle \rho u, u \rangle_\Omega - \frac{2V_B V_G}{V_\Omega} \langle \rho, u \rangle_\Omega + \frac{\text{Deg}V_B V_G^2}{V_\Omega^2}.
\]

On the other hand,

\[
(2.18) \quad \langle v, v \rangle = \langle v, v \rangle_\Omega + \langle v, v \rangle_B = \langle u, u \rangle_\Omega + \frac{V_B V_G}{V_\Omega}.
\]
Substituting (2.17) and (2.18) into (2.12), we have
(2.19)\[
\langle du, du \rangle_{\Omega} - \text{Deg} \langle u, u \rangle_{\Omega} + V_B \langle \rho u, u \rangle_{\Omega} - \frac{2V_B V_G \langle \rho, u \rangle_{\Omega}}{V_{\Omega}} + \frac{\text{Deg} V_B^2 V_G}{V_{\Omega}^2} \geq 0
\]
for any $u \in \mathbb{R}^\Omega$ with $\langle u, 1 \rangle_{\Omega} = 0$. Note that for any constant $\lambda$, $\langle \lambda u, 1 \rangle_{\Omega} = 0$, replacing $u$ by $\lambda u$ in (2.19), one has
(2.20)\[
(\langle du, du \rangle_{\Omega} - \text{Deg} \langle u, u \rangle_{\Omega} + V_B \langle \rho u, u \rangle_{\Omega}) \lambda^2 - \frac{2V_B V_G \langle \rho, u \rangle_{\Omega}}{V_{\Omega}} \lambda + \frac{\text{Deg} V_B^2 V_G}{V_{\Omega}^2} \geq 0
\]
for all $\lambda \in \mathbb{R}$. This is equivalent to
(2.21)\[
\left(\frac{2V_B V_G \langle \rho, u \rangle_{\Omega}}{V_{\Omega}}\right)^2 \leq 4 \left(\langle du, du \rangle_{\Omega} - \text{Deg} \langle u, u \rangle_{\Omega} + V_B \langle \rho u, u \rangle_{\Omega}\right) \frac{\text{Deg} V_B^2 V_G}{V_{\Omega}^2}.
\]
Conversely, it is not hard to see that (2.21) implies (2.19). Simplifying (2.21), we get
(2.22)\[
\langle du, du \rangle_{\Omega} - \text{Deg} \langle u, u \rangle_{\Omega} + V_B \langle \rho u, u \rangle_{\Omega} - \frac{V_G}{\text{Deg} \langle \rho, u \rangle_{\Omega}} \geq 0
\]
for any $u \in \mathbb{R}^\Omega$ with $\langle u, 1 \rangle_{\Omega} = 0$. Because (2.22) is stronger than (2.15), we only require (2.22).

Conversely, it is not hard to see that (2.22) implies (2.12) because (2.21) implies (2.19). This completes the proof of the theorem. \hfill \square

We next come to prove Corollary 1.1, a sufficient condition for (1.17) to hold on general weighted graphs.

**Proof of Corollary 1.1.** Note that, for any $u \in \mathbb{R}^\Omega$ with $\langle u, 1 \rangle_{\Omega} = 0$,
\[
\langle du, du \rangle_{\Omega} - \text{Deg} \langle u, u \rangle_{\Omega} + V_B \langle \rho u, u \rangle_{\Omega} - \frac{V_G}{\text{Deg} \langle \rho, u \rangle_{\Omega}} \geq 0
\]
(2.23)\[
\geq \left(\mu_2(\Omega) - \text{Deg} + V_B \rho_{\min} - \frac{V_G}{\text{Deg} \langle \rho, \rho \rangle_{\Omega}}\right) \langle u, u \rangle_{\Omega}
\]
\[
\geq 0.
\]
So, by Theorem 1.1, we get the conclusion. \hfill \square

We next come to prove Corollary 1.2, the rigidity of (1.17) for the case that $\rho$ is constant.
Proof of Corollary 1.2. Note that, for any \( u \in \mathbb{R}^\Omega \) with \( \langle u, 1 \rangle_\Omega = 0 \),
\[
\langle du, du \rangle_\Omega - \text{Deg} \langle u, u \rangle_\Omega + V_B \langle \rho u, u \rangle_\Omega - \frac{V_G}{\text{Deg}} \langle \rho, u \rangle^2_\Omega
\]
(2.24)
\[
= \langle du, du \rangle_\Omega - \text{Deg} \langle u, u \rangle_\Omega + \rho V_B \langle u, u \rangle_\Omega
\]
\[
= \langle du, du \rangle_\Omega - \rho (V_\Omega - V_B) \langle u, u \rangle_\Omega
\]
since \( \text{Deg} = \rho V_\Omega \) in this case. So, (1.19) holds for any \( \langle u, 1 \rangle_\Omega = 0 \) if and only if
\[
\mu_2(\Omega) \geq \rho (V_\Omega - V_B).
\]
This completes the proof of the corollary. □

We next prove Corollary 1.3 the rigidity of (1.17) for graphs with unit weight.

Proof of Corollary 1.3. By Theorem 1.1, it is clear that (i) is true. Moreover, it is clear that \( \Omega_1 \neq \emptyset \). If \( \Omega_0 \) is empty, it reduces to case that \( \rho \equiv 1 \) and by Corollary 1.2, we know that the conclusion is true. So, in the following, we assume that \( \Omega_0 \neq \emptyset \).

Because the graph is of unit weight, we know that \( \rho_x = 0 \) when \( x \in \Omega_0 \) and \( \rho_y = 1 \) when \( y \in \Omega_1 \). So \( \text{Deg} = |\Omega_1| \).

For each \( u \in \mathbb{R}^\Omega \), let \( \bar{u}_0 = \langle u, 1 \rangle_\Omega_0 / |\Omega_0| \) and \( \bar{u}_1 = \langle u, 1 \rangle_\Omega_1 / |\Omega_1| \), \( v_0 \in \mathbb{R}^{\Omega_0} \) with \( v_0 = u|\Omega_0 - \bar{u}_0 \) and \( v_1 \in \mathbb{R}^{\Omega_1} \) with \( v_1 = u|\Omega_1 - \bar{u}_1 \). Then
\[
\langle du, du \rangle_\Omega - \text{Deg} \langle u, u \rangle_\Omega + V_B \langle \rho u, u \rangle_\Omega - \frac{V_G}{\text{Deg}} \langle \rho, u \rangle^2_\Omega
\]
(2.26)
\[
= \langle dv_0, dv_0 \rangle_{\Omega_0} - |\Omega_1| \langle v_0, v_0 \rangle_{\Omega_0} + \sum_{x \in \Omega_0} v^2_0(x) \text{deg}_{\Omega_1}(x) + 2(\bar{u}_0 - \bar{u}_1) \sum_{x \in \Omega_0} v_0(x) \text{deg}_{\Omega_1}(x)
\]
\[- 2 \sum_{x \in \Omega_0} \sum_{y \in \Omega_1} v_0(x)v_1(y)w_{xy} + \langle dv_1, dv_1 \rangle_{\Omega_1} - (|\Omega_1| - |B|) \langle v_1, v_1 \rangle_{\Omega_1}
\]
\[+ \sum_{y \in \Omega_1} v^2_1(y) \text{deg}_{\Omega_0}(y) + 2(\bar{u}_1 - \bar{u}_0) \sum_{y \in \Omega_1} v_1(y) \text{deg}_{\Omega_0}(y)
\]
\[+ (\bar{u}_0 - \bar{u}_1)^2 |E(\Omega_0, \Omega_1)| - \bar{u}_1^2(2|\Omega_1| + |\Omega_0|)|\Omega_1| - \bar{u}_0^2|\Omega_0||\Omega_1|,
\]
where
\[
\text{deg}_{\Omega_1}(x) = \sum_{y \in \Omega_1} w_{xy}
\]
for any \( x \in \Omega_0 \) and
\[
\text{deg}_{\Omega_0}(y) = \sum_{x \in \Omega_0} w_{xy}
\]
Comparison of Steklov eigenvalues and Laplacian eigenvalues

for any $y \in \Omega_1$. So (1.19) holds for any $u \in \mathbb{R}^\Omega$ with $\langle u, 1 \rangle_{\Omega} = 0$ if and only if (2.20) is nonnegative for any $v_0 \in \mathbb{R}^{\Omega_0}$, $v_1 \in \mathbb{R}^{\Omega_1}$, $\bar{u}_0, \bar{u}_1 \in \mathbb{R}$ with $\langle v_0, 1 \rangle_{\Omega_0} = \langle v_1, 1 \rangle_{\Omega_1} = 0$ and $\bar{u}_0|_{\Omega_0} + \bar{u}_1|_{\Omega_1} = 0$.

By setting $v_0 = v_1 = 0$ and $\bar{u}_0 = -\bar{\Omega}_1$ and $\bar{u}_1 = \bar{\Omega}_0$ in (2.26), we know that

$$ |E(\Omega_0, \Omega_1)| \geq |\Omega_0||\Omega_1| $$

if (1.19) holds. Because we always assume that the graph is simple, so

$$ E(\Omega_0, \Omega_1) = |\Omega_0||\Omega_1|. $$

Moreover, $w_{xy} = 1, \deg_{\Omega_1}(x) = |\Omega_1|$ and $\deg_{\Omega_0}(y) = |\Omega_0|$ for all $x \in \Omega_0$ and $y \in \Omega_1$. Substituting all these into (2.26), we know that the nonnegativity of (2.26) reduces to that

$$ \langle dv_0, dv_0 \rangle_{\Omega_0} + \langle dv_1, dv_1 \rangle_{\Omega_1} - (|\Omega_1| - |B| - |\Omega_0|)\langle v_1, v_1 \rangle_{\Omega_1} \geq 0 $$

for any $v_0 \in \mathbb{R}^{\Omega_0}$ and $v_1 \in \mathbb{R}^{\Omega_1}$ with $\langle v_0, 1 \rangle_{\Omega_0} = \langle v_1, 1 \rangle_{\Omega_1} = 0$. This is clearly equivalent to

$$ \mu_2(\Omega_1) \geq |\Omega_1| - |\Omega_0| - |B|. $$

This completes the proof of the corollary.

Finally, we come to prove Theorem 1.3.

Proof of Theorem 1.3. (1) By [4, Theorem 1.2],

$$ \mu_i \geq 2 - 2 \cos \frac{\pi}{2k+1}. $$

Combining this with (1.17), we get (1.32).

When the equality of (1.32) holds, we know that the equality of (2.30) holds. When $|V(G)| = ik + 1$, by the rigidity in [4, Theorem 1.2], $G$ is a star of degree $i$ with each arm of length $k$. Denote the center of the star as $o$, and the vertices of the $j$th arm as $v_{j1}, v_{j2}, \ldots, v_{jk}$ for $j = 1, 2, \cdots, k$ (with $v_{jk}$ the end point of the arm). Then,

$$ \mu_2 = \mu_3 = \cdots = \mu_i $$

and the corresponding eigenspace is generated by

$$ f_j(x) = \begin{cases} 
0 & x = o \\
\frac{f(s)}{x = v_{1s}} & x = v_{js} \\
-\frac{f(s)}{x = v_{js}} & 0 \text{ otherwise}
\end{cases} $$

for $j = 2, 3, \cdots, i$. Here $f$ is the first Dirichlet eigenfunction of the path on $k + 1$ vertices: $0, 1, 2, \cdots, k$ with $0$ the boundary vertex equipped with the unit weight. Without loss of generality, we can assume that $f(i) > 0$ when $i = 1, 2, \cdots, k$. 
Moreover, by theorem 1.1 because $\sigma_i = \mu_i$ with $i \geq 2$, there must be an eigenfunction $v_i$ of $\mu_i$ such that $v_i|_{\Omega} = 0$. Because the eigenspace of $\mu_i$ is generated by the functions listed in (2.32), we know that every eigenfunction of $\mu_i$ must be positive on at least one arm (except the center). Combining this with the fact that we require two boundary vertices not adjacent to each other, one has $k = 1$ and the conclusion follows.

(2) By [4, Theorem 1.4],

\begin{equation}
\mu_i \geq \mathcal{P}(k, \lambda_i). \tag{2.33}
\end{equation}

Combining this with (1.17), one has

\begin{equation}
\sigma_i \geq \mathcal{P}(k, \lambda_i). \tag{2.34}
\end{equation}

If the equality holds, then by (1.17), the equality of (2.33) holds. Then, when $k > 1$ or $|V(G)|$ is even, by the rigidity in [4, Theorem 1.4], we know that $G$ is a comb of degree $i$ with each tooth of length $k - 1$ (For the definition of a comb, see [4, P. 2]). Let the path on $v_{11}, v_{21}, \ldots, v_{i1}$ be the base of the comb and the path on $v_{j1}, v_{j2}, \ldots, v_{jk}$ be the tooth on $v_{j1}$ for $j = 1, 2, \ldots, i$. Let $g$ be the first Dirichlet eigenfunction of $(P_{k+1}, m, w, B)$ which is positive except on the boundary vertex and $h$ be a top eigenfunction for the base of the comb. Then, $\mu_i$ is a simple eigenvalue with eigenfunction

\begin{equation}
f(v_{rs}) = g(s)h(r) \tag{2.35}
\end{equation}

for $r = 1, 2, \ldots, i$ and $s = 1, 2, \ldots, k$. Moreover, by Theorem 1.1 $f$ must be vanish on $\Omega$. This implies that $\Omega$ is empty because $f$ is everywhere non-vanished which violates that $\Omega$ is not empty. Hence, the equality of (2.34) can not hold.

When $k = 1$ and $|V(G)|$ is odd, then $G$ is a path or a cycle. Because both of the top eigenfunction of a path and a cycle will not be vanish on any vertex, the same reason as before implies that the equality (2.34) can not hold. Thus, we complete the proof of the conclusion.

Remark 2.1. In [4], the author claim that the equality of (2.33) holds only when the graph is a comb with $i$ teeth and with each tooth of length $k - 1$. However, this is not true when $i = |V(G)|$ and $|V(G)|$ is odd. In fact, it is not hard to see that in this case $G$ can be either a path or a cycle because the top eigenvalues of the path and the cycle with an odd number of vertices are the same (see [2, P. 9]). By the argument in [4], it is not hard to see that these are the only two graphs that the equality of (2.33) holds in this case.
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Department of Mathematics, Shantou University, Shantou, Guangdong, 515063, China
Email address: yjshi@stu.edu.cn

Department of Mathematics, Shantou University, Shantou, Guangdong, 515063, China
Email address: cyyu@stu.edu.cn