Scheme Dependence of the Wilsonian Effective Action and Sharp Cutoff Limit of the Flow Equation

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Abstract

The cutoff scheme dependence in the several formulations of the Exact Renormalization Group (ERG) is investigated. It is shown that the cutoff scheme dependence of the Wilsonian effective action is regarded as a certain coordinate transformation on the theory space. From this observation the Wilsonian effective actions are found to suffer from strong dependence on the schemes even in the infra-red asymptotic region for massive theories. However there is no such scheme dependence in the one particle irreducible parts of them, which is called the effective average actions. We also derive the explicit form of the Polchinski RG equation in the sharp cutoff limit. Finally this equation is shown to be identical with the Wegner-Houghton RG equation.

1 Introduction

The Exact renormalization group (ERG) \[1\] has been one of the analytical tools to investigate non-perturbative phenomena of field theories, (e.g. the chiral symmetry breaking \[2\] etc.). The ERG flow equations are the functional differential equations for the Wilsonian effective actions \(S_\Lambda[\phi]\), where \(\Lambda\) is an ultra-violet (infra-red) momentum cutoff of the low energy modes \(\phi(q)\) (the high energy modes which are already integrated out). The response of the effective action under variation of the cutoff is exactly represented as

\[
\frac{\partial}{\partial \Lambda}S_\Lambda[\phi] = F[S_\Lambda],
\]

where \(F[S_\Lambda]\) is a finite functional of the field \(\phi\). The explicit forms of \(F[S_\Lambda]\) are shown later. By solving the ERG flow equations toward to \(\Lambda = 0\) with certain bare actions as the initial conditions, we can obtain the generating functionals of the connected Green’s functions. There have been also known another type of the ERG equations for the cutoff Legendre effective action (or the effective average action) \(\Gamma_\Lambda[\phi]\). In this case the solutions
of the ERG lead to the ordinary effective actions, or the generating functionals of the 1PI Green’s functions.

Usually, we write down the ERG equations for the dimensionless parameters in the effective actions by scaling the parameters with the infra-red cutoff $\Lambda$, since the energy unit used to represent the theories does not have any physical significance. The functional space of the dimensionless effective action is called the theory space. Through this manipulation the beta-functional $F$ becomes free from the scale $\Lambda$. Among the RG flows of these dimensionless quantities, especially the fixed points, the critical surfaces and the renormalized trajectories, are of utility indispensable in investigating (statistical) continuum limit of field theories.

It is important for the practical analyses that the ERG admits non-perturbative as well as systematic approximations; e.g. the derivative expansion $[3]$, the momentum scale expansion $[4, 5]$. Though we used the word of ‘expansion’ here, the approximation schemes are not the series expansions with respect to some explicit small parameters. This is an essential distinction from the ordinary expansion schemes; $\epsilon$-expansion, $1/N$-expansion and perturbation, which lead to the asymptotic series. The solutions of the ERG equations are expected to converge smoothly with the improvement of approximations. Furthermore, we may obtain fairly good non-perturbative results within the simple approximation schemes $[6]$.

The ERG flow equation depends on the cutoff schemes. Here the cutoff scheme means the profile of the cutoff function in the propagator. It is convenient to perform the cutoff of the infra-red region $p^2 < \Lambda^2$ by adding a momentum dependent mass,

$$\Delta S[\phi] \equiv \int d^d x \frac{\Lambda^2}{2} C^{-1}(-\partial^2/\Lambda^2) \cdot \phi,$$

where $C$ is a proper cutoff function satisfying that $C(x) \to 0$ as $x \to 0$ and $d$ is the space-time dimensions. Then the propagator is multiplied by the cutoff function $\theta(p^2) = xC(x)/(1 + xC(x))$. In Fig. 1 the examples of the cutoff functions for various $C(x)$ are shown. The sharp cutoff scheme corresponds to the step function; $\theta(p^2) = 0$ for $p < \Lambda$ and $\theta(p^2) = 1$ for $p > \Lambda$.

Figure 1: The examples of the infra-red cutoff functions $\theta(x) = xC(x)/(1 + xC(x))$ for various $C(x)$. $C^{-1}(x)$ is the mass function introduced in Eq. (2).

The effective actions treated by the ERG equations themselves are cutoff scheme dependent even after the cutoff is removed; $\Lambda \to 0$. While the physical quantities obtained
from their continuum limit, or the renormalized trajectories, should not be affected by the regularization scheme. Therefore it will be important to see the cutoff scheme dependence of the RG flows, specially the renormalized trajectories, and to find out the scheme independent quantities at $\Lambda = 0$ which are the physical quantities obtained in the ERG approach. In this paper we discuss the basic structure of the cutoff scheme dependence of the ERG equations and of ther solutions; the effective actions. Especially we look into the scheme dependence of the low energy effective actions, or the renormalized trajectories, in the asymptotic regions for massive theories. In such regions the Wilsonian effective actions are found to suffer from strong scheme dependence, while the Legendre effective actions are free from such problems. Related with this we also examine the sharp cutoff limit of the so-called Polchinski equation by taking care of the singular limit of the cutoff profiles. Moreover it will be shown that the Polchinski equations turn out to be equivalent to the Wegner-Houghton equation in the sharp cutoff limit. This equivalence between these two formulations of the ERG has not been proved yet.

This paper is organized as follows. In Sec. 2 we briefly overview the derivation of the ERG flow equations; the Wegner-Houghton equation, the Polchinski equation and the flow equation for the cutoff Legendre effective action. In Sec. 3 we will examine the general aspects of the scheme dependence of the effective actions. At first it will be shown that the variation of the cutoff function; $\theta(x) \rightarrow \theta(x) + \delta \theta(x)$, can be reinterpreted as the coordinate transformation on the theory space. From this observation it immediately follows that the critical exponents obtained by the ERG method are independent of the cutoff scheme. We also discuss the formulations using the cutoff mass functions depending on the wave function renormalization from this point of view. We will study also the cutoff scheme dependence of the renormalized trajectories in the infra-red asymptotic region of massive theories. The scheme dependence of the coefficient function $V_{k}^{\Lambda}(\phi)$ of the Wilsonian effective action $S_{\Lambda}[\phi]$ behaves as $1/\Lambda^{2k}$ and, therefore, becomes so strong as to prevent from taking any physical information as the infra-red cutoff $\Lambda$ is lowered. While the cutoff Legendre effective actions are free from such a strong scheme dependence.

The sharp cutoff limit of the Polchinski equations and their equivalence to the Wegner-Houghton formulation will be clarified in Sec. 4 and in Sec. 5. Section 6 is devoted to the conclusions and some remarks. Throughout this paper, we restrict ourselves to the single scalar theories. This restriction does not loose the generality of the discussions.

2 ERG Equations

To fix our conventions and to make this paper self-contained, we briefly overview the derivations of the ERG flow equations. Let us start from the generator of the connected Green’s functions $W[J]$ given by

$$\exp (W[J]) = \int D\phi \exp \left( -\Delta S_{U.V.} - S_{\Lambda_0} + J \cdot \phi \right),$$

where $\Delta S_{U.V.}$ is the ultra-violet cutoff term regularizing the path-integral (3). We sometimes use the shorthand: $J \cdot \phi = \int d^d x J(x) \phi(x)$ etc., where $d$ is the (euclidean) space-time dimensions. Now, we introduce the intermediate scale $\Lambda < \Lambda_0$ and formally integrate out the high energy modes $\phi_{>}(q)$ ($\Lambda < q \leq \Lambda_0$).
Then we get the effective action for the low energy modes $\phi(q)$ ($q < \Lambda$),

$$\exp(-S_\Lambda[\phi,J]) = \int D\phi_> \exp\left(-\Delta S^\Lambda_{\Lambda_0}[\phi_>]-S_{\Lambda_0}[\phi+\phi_>]+J \cdot (\phi+\phi_>)\right),$$

where the cutoff action $\Delta S^\Lambda_{\Lambda_0}[\phi_>]$ is given by,

$$\Delta S^\Lambda_{\Lambda_0}[\phi] \equiv \frac{1}{2} \phi \cdot [P^\Lambda_{\Lambda_0}]^{-1} \cdot \phi. \quad (5)$$

The support of $P^\Lambda_{\Lambda_0} (q)$ is effectively restricted in the region $\Lambda < q < \Lambda_0$ by means of a certain smooth cutoff function. Furthermore, we set, $\Delta S_{\text{U.V.}} = \Delta S^\Lambda_{\Lambda_0}$ and $P^\Lambda_{\Lambda_0} (q) = P^\Lambda_0 (q) + P^{\Lambda_0}_{\Lambda_0} (q)$. This can be achieved by multiplying the partition of unity $\theta^\Lambda_0 (q)$ to the propagator $1/q^2$, i.e. $P^\Lambda_{\Lambda_0} (q) = \theta^\Lambda_0 (q)/q^2$. $\theta^\Lambda_0 (q)$ approximately vanishes when $q \gg \Lambda_0$ or $q \ll \Lambda$, and $\theta^\Lambda_0 (q) \approx 1$ for $\Lambda < q < \Lambda_0$. In the RG flow equation, we can safely forget the ultra-violet cutoff $\Lambda_0$ by taking the limit $\Lambda_0 \to \infty$, since $\partial P^{\Lambda_0=\infty} (q)/\partial \Lambda$ decays in both $q \to 0$ and $q \to \infty$ sufficiently fast.

Now, Eq. (3) can be rewritten in terms of $S_\Lambda$ in Eq. (4), (See ref.4)

$$\exp (W[J]) = \int D\phi_< \exp \left(-\Delta S^\Lambda_0 - S_\Lambda[\phi,J]\right). \quad (6)$$

Putting $J = 0$, Eq. (6) is nothing but the definition of the Wilsonian effective action $\Delta S^\Lambda_0 + S_\Lambda[\phi,0]$. Evidently, if we put $\phi_< = 0$ in Eq. (4) then $S_\Lambda$ becomes the generator of the connected Green’s functions with the infra-red cutoff $W_\Lambda[J] = -S_\Lambda[0,J]$ [4]. By shifting $\phi_> \to \phi_> - \phi<$ and setting $J = 0$ in Eq. (4), we also find [4],

$$W_\Lambda[P^{-1}_\Lambda \cdot \phi_<] = \frac{1}{2} \phi_< \cdot P^{-1}_\Lambda \cdot \phi_< - S_\Lambda[\phi_<,0]. \quad (7)$$

Hereafter, we write $S_\Lambda[\phi] = S_\Lambda[\phi,0]$.

The above cutoff $\theta$ is called the ‘multiplicative cutoff’, because we multiplied it to the kinetic term of $\phi$. $\Delta S$ is called ‘additive’, since the inverse cutoff propagator is given by $C^{-1}(q^2)$ in Eq. (2) plus the ordinary kinetic term $q^2$. $\theta(x) \equiv \theta^\Lambda_0 (x)$ above is written in terms of $C(x)$ as $\theta(x) = xC(x)/(1 + xC(x))$. The relation between these two cutoff schemes: the multiplicative cutoff and the additive cutoff is given as follows. The bare actions of both schemes obviously satisfy $S^\Lambda_{\Lambda_0} = \frac{1}{2} \int d^d x (\partial \phi)^2 + S^\text{multi}_{\Lambda_0}$, where $S^\text{add}_{\Lambda_0}$ and $S^\text{multi}_{\Lambda_0}$ are the bare actions with the additive cutoff and with the multiplicative cutoff respectively. By the definition, the generator of the connected Green’s functions $W_\Lambda[J]$ is common for both schemes. Thus, it immediately follows,

$$S^\text{multi}_{\Lambda_0} [P_\Lambda \cdot J] - \frac{1}{2} J \cdot P_\Lambda \cdot J = S^\text{add}_{\Lambda_0}[C \cdot J] - \frac{1}{2} J \cdot C \cdot J. \quad (8)$$

We will employ the multiplicative cutoff scheme, since it is convenient to investigate the sharp cutoff limit of the RG flow equations.
2.1 Flow equations

Setting $\phi_\lambda = 0$ and differentiating the boths sides of Eq. (4) with respect to $\Lambda$, we get the RG flow equation for $W_\Lambda[J]$,

$$
\frac{\partial}{\partial \Lambda} W_\Lambda = -\frac{1}{2} W'_\Lambda \cdot \frac{\partial}{\partial \Lambda} P^{-1}_\Lambda \cdot W'_\Lambda - \frac{1}{2} \text{tr} \frac{\partial}{\partial \Lambda} P^{-1}_\Lambda \cdot W''_\Lambda,
$$

where the prime denotes the derivative with respect to the source $J$. By using Eq. (7), we also get,

$$
\frac{\partial}{\partial \Lambda} S_\Lambda = -\frac{1}{2} S'_\Lambda \cdot \frac{\partial}{\partial \Lambda} P \cdot S'_\Lambda + \frac{1}{2} \text{tr} \frac{\partial}{\partial \Lambda} P \cdot S''_\Lambda.
$$

This is the famous ‘Polchinski equation’ [1]. This equation may be represented diagrammatically as in Fig. 2.

![Figure 2: The diagrams of the Polchinski equation (10). The crossed circles and the filled circles correspond to $\partial P/\partial \Lambda$ and the vertices of $S_\Lambda$ respectively.](image)

Next, we scale all the dimensionful quantities in terms of the infra-red cutoff $\Lambda$, i.e. $\phi = \Lambda^d \phi$, $p = \Lambda \hat{p}$ and $L_\Lambda = \Lambda^d \hat{L}_t$, where $d_\phi = (d - 2)/2$ is the canonical dimension of the field, $t = \ln \Lambda_0/\Lambda$ is the cutoff scale factor and $\hat{L}_t$ is the Lagrangian density. We also write the dimensionless Wilsonian effective action as $\hat{S}_t(\hat{\phi}) = \int d^d \hat{x} \hat{L}_t(\hat{\phi})$. Then, we get

$$
\Lambda \frac{\partial}{\partial \Lambda} S_\Lambda = -\Lambda^d \left( \frac{\partial}{\partial t} + d_\phi \Delta_\phi + \Delta_\partial - d \right) \hat{S}_t,
$$

where $\Delta_\phi$ and $\Delta_\partial$ count the degree of the fields and that of the derivatives $\partial_\mu$ respectively. The initial boundary condition of Eq. (11) is given by the bare action, $S_{\Lambda_0}$.

We can derive the RG flow equation for the Legendre effective action with the infra-red cutoff $\Gamma_\Lambda[\phi]$ given by the Legendre transform of $W_\Lambda[J]$.

$$
W_\Lambda[J] = J \cdot \phi - \Gamma_\Lambda[\phi] + \frac{1}{2} \phi \cdot \left( P^{-1}_\Lambda - P^{-1}_{\Lambda=0} \right) \cdot \phi,
$$

where $\phi$ is given by $\phi = \delta W_\Lambda/\delta J$. After the Legendre transformation, the RG flow equation for $\Gamma_\Lambda[\phi]$ can be read,

$$
\frac{\partial}{\partial \Lambda} \Gamma_\Lambda[\phi] = \frac{1}{2} \text{tr} \frac{\partial}{\partial \Lambda} P^{-1}_\Lambda \cdot \left( P^{-1}_\Lambda - P^{-1}_{\Lambda=0} + \Gamma'' \right)^{-1}.
$$

The initial condition of Eq. (12) is given by $\Gamma_{\Lambda=0} = S_{\Lambda_0}$, because all quantum corrections vanish at $\Lambda = \Lambda_0$. Since $\Gamma_\Lambda[\phi]$ is composed of the one particle irreducible diagrams, the diagrams corresponding to Eq. (13) contain no tree diagrams, as is shown in Fig. 3. By the definition, $\Gamma_\Lambda[\phi]$ coincides with the ordinary Legendre effective action at $\Lambda = 0$, i.e. $\Gamma_{\Lambda=0}[\phi] = \Gamma[\phi]$. One can find the RG flow equation for the dimensionless effective action $\hat{\Gamma}_t[\hat{\phi}]$ by the same manner as we did for the Polchinski one.
2.2 Wegner-Houghton equation

We start from the following partition function,

\[ Z = \int_{p \leq \Lambda} D\phi \exp \left(-\frac{1}{2} \phi \cdot P^{-1} \cdot \phi - S_\Lambda[\phi] \right), \quad (14) \]

where the support of \( \phi \) is restricted to \( p \leq \Lambda \). We shift the quadratic part, the \( P^{-1} \) term, from the Wilsonian effective action, since it is subtracted also in \( S_\Lambda \) given by Eq. (4).

Let us integrate out the modes with momenta \( \Lambda - \delta \Lambda < p \leq \Lambda \), we call these modes the ‘shell modes’ and write as \( \phi_\delta \). Expanding the action \( S_\Lambda[\phi + \phi_\delta] \) in the shell modes \( \phi_\delta \), we have

\[ S_\Lambda[\phi + \phi_\delta] = S_\Lambda[\phi] + \phi_\delta \cdot S_\Lambda^{(1)}[\phi] + \frac{1}{2} \phi_\delta^2 \cdot S_\Lambda^{(2)}[\phi] + \cdots, \quad (15) \]

where the superscript \( (n) \) denotes the \( n \)-th functional derivative with respect to the shell mode. We can regard the Taylor coefficients \( S_\Lambda^{(n)}[\phi] \) to the field \( \phi \) dependent vertices. The quantum fluctuations of the shell modes can be incorporated by perturbative expansion.

For infinitesimal \( \delta \Lambda \), the leading corrections, i.e. of order \( \delta \Lambda \), come from less than or equal to one loop diagrams. The higher \( (n \geq 2) \) loops diagrams do not contribute to the leading order in \( \delta \Lambda \), since every loop integral introduces the factor \( \delta \Lambda \). Furthermore, each articulation line also brings the factor \( \delta \Lambda \). Hence, the leading corrections are found to be the Feynman diagrams with only one propagator which is either an articulation line or a loop one. Taking into account this constraint, the higher vertices \( S_\Lambda^{(n \geq 3)}[\phi] \) cannot appear and can be dropped in Eq. (15). After performing the Gaussian integration of the shell modes, we get a coarse-grained action \( S_{\Lambda-\delta \Lambda}[\phi] \) of the low energy modes \( \phi(p) : p \leq \Lambda - \delta \Lambda \),

\[ S_{\Lambda-\delta \Lambda}[\phi] = S_\Lambda[\phi] - \frac{1}{2} \delta \Lambda S_\Lambda' \cdot \left(P^{-1} + S_\Lambda''\right)^{-1} \cdot S_\Lambda' + \frac{1}{2} \delta \Lambda \text{tr} \ln \left(P^{-1} + S_\Lambda''\right), \quad (16) \]

where prime denotes the functional derivative with respect to \( \phi_\delta \). By letting \( \delta \Lambda \to 0 \), we find

\[ \frac{\partial}{\partial \Lambda} S_\Lambda = \frac{1}{2} S_\Lambda' \cdot \left(P^{-1} + S_\Lambda''\right)^{-1} \cdot S_\Lambda' - \frac{1}{2} \text{tr} \ln \left(P^{-1} + S_\Lambda''\right). \quad (17) \]

This is called the ‘Wegner-Houghton equation’ [4]. The diagrams corresponding to Eq. (17) are shown in Fig. 4.
3 General Aspects of the Cutoff Scheme Dependence

In this section, we discuss some general aspects on the cutoff scheme dependence of the ERG. In the first two subsections we show that the formal relation among the Wilsonian effective actions with different cutoff schemes can be regarded as the coordinate transformation on the theory space. Therefore it immediately follows that the critical exponents, which are the macroscopic physical quantities of the phase transition, do not suffer from the cutoff scheme dependence. In the remaining subsections the scheme dependence of the renormalized trajectories in the infra-red asymptotic region is examined. It is shown that the scheme dependence disappears form the cutoff Legendre effective action in this region, while not from the Wilsonian effective action.

3.1 Coordinate Transformation on the Theory Space

The Polchinski RG equation\([1]\) for a single scalar theory can be rewritten,

\[
\frac{d}{d\Lambda} S_t(\phi) = - \sum \text{Ring} + \sum \text{Dambbell}
\]

Figure 4: The diagrams of Eq. (17). The filled circles correspond to the vertices of \(S_\Lambda\).

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3.1 Coordinate Transformation on the Theory Space

The Polchinski RG equation\([1]\) for a single scalar theory can be rewritten,

\[
\left( \frac{\partial}{\partial t} + d_\phi \cdot \frac{\delta}{\delta \phi} + \int \frac{d^d q}{(2\pi)^d} \phi(q) \frac{\partial}{\partial q_\mu} \frac{\delta}{\delta \phi(q)} \right) \exp(-S_t[\phi]) = \int \frac{d^d q}{(2\pi)^d} \frac{\delta}{\delta \phi(q)} \left( \frac{\partial}{\partial q^2} \theta(q^2) \right) \frac{\delta}{\delta \phi(-q)} \exp(-S_t[\phi]),
\]

where, for the convenience, we write the Fourier transform of the functional derivative with respect to \(\phi(x)\) by

\[
\frac{\delta}{\delta \phi(q)} \equiv \int d^d x e^{iq \cdot x} \frac{\delta}{\delta \phi(x)} = (2\pi)^d \frac{\delta}{\delta \phi(q)}.
\]

In Eq. (18), \(\theta(q^2)\) denotes the cutoff function, which is given in Fig. 1 for example. Note that, the momentum derivative operating to the effective action \(S_t[\phi]\) in the first line of Eq. (18) does not operate to the delta function \(\delta(\Sigma q_i)\) representing momentum conservation. Hence it operates to the effective action as

\[
\int \frac{d^d q}{(2\pi)^d} \phi(q) \frac{\partial}{\partial q_\mu} \frac{\delta}{\delta \phi(q)} S_t[\phi] = (\Delta_\phi - d) S_t[\phi],
\]

where \(\Delta_\phi\) counts the degree of derivatives.

Let us consider the coordinate transformation of the theory space: \(S_t[\phi] \rightarrow \tilde{S}_t[\phi]\), given by the following transformation:

\[
\exp(-\tilde{S}_t[\phi]) = \exp \left( \frac{1}{2} \delta D \right) \exp(-S_t[\phi]),
\]
where $\delta D$ is given by

$$
\delta D = \int \frac{d^d q}{(2\pi)^d} \frac{\delta}{\delta \phi(q)} \cdot \frac{1}{q^2} \delta \theta(q^2) \cdot \frac{\delta}{\delta \phi(-q)}.
$$

(22)

Since $\delta D$ is independent of the cutoff scale $t$, this transformation is in fact a coordinate transformation on the theory space \( t \). Therefore the critical exponents obtained by the RG technique are invariant under the transformation (21). (See Ref. [14].)

Indeed, by this transformation the cutoff function $\theta(q^2)$ is changed to $\theta(q^2) + \delta \theta(q^2)$ in the Polchinski RG equation. Operating $\exp(\delta D/2)$ to both sides of Eq. (18) and using

\[
\left[ d_\phi \cdot \frac{\delta}{\delta \phi} + \int \frac{d^d q}{(2\pi)^d} \phi(q) q^\mu \frac{\partial}{\partial q^\mu} \frac{\delta}{\delta \phi(q)} \cdot \frac{\delta D}{2} \right] = \int \frac{d^d q}{(2\pi)^d} \frac{\delta}{\delta \phi(q)} \left( \frac{\partial \delta \theta(q^2)}{\partial q^2} \right) \frac{\delta}{\delta \phi(-q)},
\]

(23)

one can realize that the effective action $\tilde{S}_t[\phi]$ just satisfies the Polchinski RGE with the cutoff scheme $\theta(q^2) + \delta \theta(q^2)$. Therefore, if $S_t[\phi]$ is a solution of Eq. (18) with cutoff scheme $\theta(q^2)$, then $\tilde{S}_t[\phi]$, defined by Eq. (21), is also a solution for another cutoff scheme $\theta(q^2) + \delta \theta(q^2)$. The transformation (21) maps the fixed point, the critical surface and the renormalized trajectories to those given in another scheme. (See Fig. 5.) However the critical exponents at the fixed points are scheme independent.

**3.2 Wave-function Renormalization and Additive cutoff**

In order to extract the anomalous dimension it is more convenient to employ the additive cutoff instead of the multiplicative one for following two reasons. 1). In the multiplicative case the part of the kinetic term of the Wilsonian effective action is stolen by the inverse cutoff propagator $q^2 (\theta(q^2))^{-1}$. 2). We can eliminate the wave-function renormalization.

\[\text{Figure 5: The coordinate transformation (21) changes the cutoff scheme (A) to another cutoff scheme (B) in the Polchinski equation.}\]
factor $Z_\phi$ in RG equation by rescaling the cutoff function $C(q^2)$ to $Z_\phi^{-1} C(q^2)$ in the additive case, and the anomalous dimension ($\eta$) of the field may be explicitly extracted.

The additive cutoff is introduced by Eq. (8) and the Wilsonian effective actions with two cutoff schemes are related by the relation (8), i.e.

$$S_{\text{multi}}[\phi] = S_{\text{add}}[(1 - \partial^2 C)\phi] + \frac{1}{2} \int d^d x \partial^2 (1 - \partial^2 C) \phi,$$

where $S_{\text{multi}}[\phi]$ is the effective action with the multiplicative cutoff. In the multiplicative cutoff case, we drop the part of the kinetic term of (the interaction part of) the effective action. It is rather convenient to include the kinetic term in the effective action completely in extracting the anomalous dimension. The additive cutoff $C(x)$ in Eq. (2) and the multiplicative cutoff $\theta(x)$ are related by $\theta(x) = x C(x)/(1 + x C(x))$.

Let us rescale the field $\phi$ to $\hat{\phi} = Z_\phi^{1/2} \phi$, where $Z_\phi$ is the wave-function renormalization factor. If we also rescale the cutoff function as,

$$C^{-1}(q^2/\Lambda^2) \rightarrow Z_\phi C^{-1}(q^2/\Lambda^2),$$

then the explicit $Z_\phi$ dependence of the RG flow equation can be eliminated. The RG flow equation depends on $Z_\phi$ only through the anomalous dimension $\eta$ defined by the consistency condition, i.e. the kinetic term should be unity at each scale.

Consequently, $\eta$ becomes the function of the coupling constants. It means that the beta-function of $Z_\phi$ is given by,

$$\frac{\partial}{\partial t} Z_\phi = \eta(g_i) Z_\phi,$$

where $\{g_i\}$ is a coordinate system on the theory space. In this coordinate system, the RG beta-functions have the following structure:

$$\Omega_{ij}(g) = \frac{\partial \beta_i}{\partial g_j}(g) = 0 \quad \text{for} \quad g_j = Z_\phi,$$

because the beta-functions have no $Z_\phi$ dependence except for $\beta_Z$. Such a parametrization is called the 'perfect coordinate' in Ref. [6]. For the dimensionless Wilsonian effective action, the RG flow equation becomes,

$$\left( \frac{\partial}{\partial t} + d_{\phi} \phi \cdot \frac{\delta}{\delta \phi} + \int \frac{d^d q}{(2\pi)^d} \delta \phi(q) \frac{\partial}{\partial q_{\mu}} \delta \phi(q) \right) \exp(-S_t[\phi])$$

$$= - \int \frac{d^d q}{(2\pi)^d} \delta \phi(q) \left( q^2 \frac{\partial}{\partial q^2} C + \frac{1}{2} (\eta - 2) C \right) \frac{\delta}{\delta \phi(-q)} \exp(-S_t[\phi]),$$

where $d_{\phi}$ and $\eta = \hat{Z}_\phi/Z_\phi$ are the physical scaling dimension of the field; $d_{\phi} = (d + \eta - 2)/2$ and the anomalous dimension of $\phi$ respectively. One can easily extend the scheme dependence relations given by Eq. (23) etc. to this type of RGE.

However one may wonder whether the coordinate transformation induced by Eq. (25) is well-defined or not. Suppose that $C(x)$ is a polynomial i.e. $C(x) = x^k$. Since the cutoff $\Lambda$ appears only though the cutoff function,

$$e^{-S_\Lambda[\phi]} \equiv \int D \phi \exp \left\{ - \frac{Z_\phi}{2} \phi^2 C^{-1}(\partial^2/\Lambda^2) \phi > - S[\phi > + \hat{\phi}] \right\},$$
we can eliminate $Z_\phi$ by shifting the cutoff; $Z_\phi \Lambda^2 C^{-1} \left( \frac{p^2}{\Lambda^2} \right) = \Lambda^2 C^{-1} \left( \frac{p^2}{\Lambda'^2} \right)$ where $\Lambda = Z_\phi^{1/(2k-2)} \Lambda'$. Therefore the Wilsonian effective action $S_\Lambda[\phi]$ with a cutoff scheme $C^{-1}$ can be written in terms of $S_\Lambda[\phi]$,

$$
S_\Lambda[\phi] = S_\Lambda[\phi] = S_\Lambda[\phi] + \int_\Lambda^\Lambda d\Lambda' \frac{\partial}{\partial \Lambda} S_\Lambda[\phi],
$$

$$
= S_\Lambda[\phi] + \delta f(Z_\phi) N' \frac{\partial}{\partial N} S_\Lambda[\phi] + \frac{1}{2!} (\delta f(Z_\phi) N')^2 \frac{\partial^2}{\partial N^2} S_\Lambda[\phi] + \cdots,
$$

(30)

where $\delta f(Z_\phi)$ is given by $\Lambda - \Lambda' = \delta f(Z_\phi) \Lambda'$. In our case, $\delta f(Z_\phi) = Z_\phi^{1/(2k-2)} - 1$. The derivative with respect to $\Lambda'$ in Eq. (30) will be eliminated by the RG flow equation. Thus one can find the coordinate transformation between $S_\Lambda[\phi]$ and $S_\Lambda[\phi]$. Since all the loop momentum integrals are regularized in the both regions of infra-red and ultra-violet, Eq. (30) gives the well-defined coordinate transformation at all orders in the Taylor expansion.

In the more general case, we cannot eliminate $Z_\phi$ by shifting the cutoff. However, since the change of $\theta(x) = xC(x)/(1 + xC(x))$ induced by the change of the cutoff function $C^{-1}(x)$ is concentrated in the finite region of the momentum $x = p^2$, it also gives the well-defined coordinate transformation equivalent to Eq. (21).

Consequently, the critical exponents given by Eq. (18) and by Eq. (28) are completely the same, since these formulations can be understood as the difference of the coordinate systems on the theory space.

### 3.3 Asymptotic Region of the Renormalized Trajectory

In this and the next subsections we discuss the cutoff scheme dependence of the renormalized trajectories in the ‘asymptotic region’ $\Lambda \ll M_R$, where $M_R$ is the renormalized mass of $\phi$. We note that Eq. (21) may be rewritten as follows. Let $S^{(n)}_t$ be the vertices of the effective action $S_t[\phi]$, and $W_t$ be sum of the connected diagrams composed of the propagator $(\delta P^{-1} + S^{(2)}_t)^{-1}$ and the vertices $S^{(n)}_t$ ($n > 2$), where $\delta P(q^2)$ is the cutoff propagator i.e. $\delta P(q^2) = \frac{1}{q^2} \delta \theta(q^2)$. Then it is easily found that (See appendix),

$$
\exp \left( \frac{1}{2} \delta D[\delta/\delta \phi] \right) \exp (-S_t[\phi]) = \exp \left( -\frac{1}{2} \phi \cdot \delta P^{-1} \cdot \phi + W_t[\delta P^{-1} \cdot \phi] \right).
$$

(31)

The cutoff scheme dependence of the renormalized trajectory is given by Eq. (31). (See Fig. 4)

If the RG flows of the dimensionful quantities freeze when $t \to \infty$, then all dimensionless coupling $g_i(t)$ should behave $g_i(t) \sim g_i^R e^{d_i t}$ as $t \to \infty$ with some finite dimensionful coupling $g_i^R$, where $d_i$ is the canonical dimension of $g_i$. Here, we take $M_R$ to be a unit of the mass scale: $t = \ln M_R/\Lambda$. Hereafter we call such a region ‘the freezing region’, if it exists. As is seen below the asymptotic region is not always the freezing region.

In the asymptotic region, it can be realized that the loop diagrams in Eq. (31) are found to be suppressed compared with the tree diagrams. It is seen by the following arguments. Let us consider the Feynman diagram with $N_I$ internal lines, $N_E$ external legs and $N_V$ vertices. By comparing the canonical dimension of each operator in the both
sides of Eq. (31), we find the following factor,

$$\exp\{\Delta t\} \equiv \exp \left\{ \left( dN_V - d_\phi(N_E + 2N_I) - N_D^{(1)} - 2N_I - (d - d_\phi N_E) + N_D^{(2)} \right) t \right\},$$  \hspace{1cm} (32)

where $N_D^{(1)}$ is the total degree of the external momenta (or the derivatives) of the $N_V$ vertices in the Feynman diagram, and $N_D^{(2)}$ is that of the subset of $N_D^{(1)}$ derivatives which operate to the field $\phi(x)$. The first three terms of $\Delta$ come from the canonical dimensions of the vertices, the next one is from propagators and the last two terms are the dimension of $N_E$ point vertex with $N_D^{(2)}$ derivatives. $N_D^{(1)}$ and $N_D^{(2)}$ satisfy the relation $N_D^{(1)} \geq N_D^{(2)}$, since the number of derivatives only decreases by the loop integration \[\text{[3]}\]. We do not expand the propagators with respect to the external momenta, since we are interested only in the scaling behavior of the Feynman diagrams in which each propagator gives a negative power of the external momenta and brings the factor $\exp(-2t)$ \[\text{[4]}\]. The loop integration does not change the factor $\Delta$, because the support of $\delta\theta(p^2)$ is concentrated in the small region around $\hat{p}^2 \sim 1$. The factor $\Delta$ describes the response of the shrinkage of the loop momentum integral region. The diagrams with $\Delta < 0$ are dropped in Eq. (31). \[\text{Nota Bene}\] the massive field decouples in the asymptotic region not due to its large (dimensionless) mass but due to the shrinkage of the loop momentum integral region.

One can rewrite $\Delta$ by using the number of the loops $L$,

$$\Delta[L] = -dL - (N_D^{(1)} - N_D^{(2)}) \leq -dL,$$  \hspace{1cm} (33)

where we used the relation,

$$N_I - N_V + 1 = L.$$  \hspace{1cm} (34)

Hence all loop corrections i.e. ‘quantum’ corrections are relatively suppressed by the factor $\exp(-dLt)$ compared with the tree diagrams. Since the number of derivatives is decreased by only loop integration, $N_D^{(1)} = N_D^{(2)}$ in the tree diagrams. Therefore all the tree diagrams survive. Now we can conclude that in the asymptotic region $\tilde{W}_t$ in Eq. (31) is consist of tree diagrams only.

In Fig. 6 we show examples of the Feynman diagrams and their suppression factors. In these examples, the factor $\Delta$ of first tree diagram is $\Delta = 2(d - 4d_\phi) - 2 - (d - 6d_\phi) = 0$

![Figure 6](image_url)

Figure 6: The examples of the factor $\Delta$ are shown. In this figure, $d_\phi$ is the canonical dimension of the field i.e. $d_\phi = (d - 2)/2$. The dots denote the higher order corrections.

and that of the second loop diagram is $\Delta = 2d - (4 + 6)d_\phi - 4 - (d - 6d_\phi) = -d < 0$. The later diagram is suppressed in comparison with the six points vertex itself.

\[\text{2}\]Some of the derivatives will be replaced by the loop momenta $q \sim 1$ which does not contribute to $\Delta$.

\[\text{3}\]Namely we do not perform the derivative expansion here. The asymptotic behavior in the derivative expansion will be discussed in § 6.
The above observation holds also for the RG flows, in which $\delta P$ is induced by lowering the cutoff $\Lambda$. Hence the RG flows of the dimensionful couplings of the 1PI building blocks of the Wilsonian effective action freeze in the asymptotic region, however the Wilsonian effective action itself does not. The later conflicts to our assumption made before, i.e. $g_i(t) \sim g_i^R e^{\delta t}$ with fixed $g_i^R$. Thus there is no freezing region on the RG flow diagram of the Polchinski equation.

The coordinate transformation $\Delta_{\delta P}[S_t]$ in the asymptotic region is written by,
\begin{equation}
\Delta_{\delta P}[S_t[\tilde{\phi}]] = \frac{1}{2} \tilde{\phi} \cdot \delta P^{-1} \cdot \phi - \tilde{W}_{\text{tree}}[\delta P^{-1} \phi],
\end{equation}
where $\tilde{W}_{\text{tree}}$ is the tree part of the connected diagrams. It can be easily realized that $\tilde{W}_{\text{tree}}$ is given by the Legendre transform of the effective action $S_t$, since the 1PI part of $\tilde{W}_{\text{tree}}$ is nothing but the ‘Legendre effective action’ $\tilde{\Gamma}_t$. ($\tilde{\Gamma}_t$ should not be confused with $\Gamma_\Lambda$ given in Eq. (12), which is equal to the effective action $S_t$ in the asymptotic region \[\] ) Therefore we find,
\begin{equation}
\tilde{W}_{\text{tree}}[J] = \tilde{\phi} \cdot J - S_t[\tilde{\phi}] - \frac{1}{2} \tilde{\phi} \cdot \delta P^{-1} \cdot \tilde{\phi}.
\end{equation}
We add the last term of r.h.s. of Eq. (36) to the effective action, because $\tilde{W}_{\text{tree}}$ in Eq. (31) consists of the connected diagrams with bare action $S_t[\phi] + \frac{1}{2} \phi \cdot \delta P^{-1} \cdot \phi$. Here $J$ and $\phi$ satisfy the relations,
\begin{equation}
J - \delta P^{-1} \tilde{\phi} = \frac{\delta}{\delta \tilde{\phi}} S_t, \quad \tilde{\phi} = \frac{\delta}{\delta J} \tilde{W}_{\text{tree}}.
\end{equation}
One can find the coordinate transformation $\Delta_{\delta P}$ as
\begin{equation}
\Delta_{\delta P}[S_t[\tilde{\phi}]] = \frac{1}{2} S_t'[\tilde{\phi}] \cdot \delta P \cdot S_t'[\tilde{\phi}] + S_t[\tilde{\phi}],
\end{equation}
where the prime denotes the functional derivative with respect to $\tilde{\phi} \left( \equiv \phi - \delta PS_t'[\tilde{\phi}] \right)$. One can also easily check the following relations,
\begin{align}
\Delta_{\delta P_1}[\Delta_{\delta P_2}[S_t]] &= \Delta_{\delta P_1 + \delta P_2}[S_t], \quad \text{(39)} \\
\Delta_{\delta P}[\Delta_{-\delta P}[S_t]] &= S_t. \quad \text{(40)}
\end{align}

The scheme dependence of the Wilsonian effective action may be understood as follows. As discussed in Ref. [4], the Wilsonian effective action is consist of two different elements. In the high energy region, the vertices of the effective action give the connected Green’s functions. In the low energy region, they coincide with those of the 1PI effective action, since all the articulation lines carry the infra-red cutoff.[4] In the boundary of these regions, two quantities are connected to each other by the cutoff function. Therefore, the Wilsonian effective action is scheme dependent even after removing the cutoff. This scheme dependence turns out to be an obstacle in the approximated analyses. (See Sec. [3])

\[\text{All the loop (quantum) corrections for the 1PI vertices are dropped in the asymptotic region. It means } \tilde{\Gamma}_t = S_t.\]
3.4 Scheme independence of the cutoff Legendre effective action

For the infinitesimal transformation $\delta P \ll 1$, the scheme dependence of the generator of the connected Green’s functions $W_t[J]$ becomes simpler. Now $W_t[J]$ is written in terms of the Wilsonian effective action $S_t[\phi]$ by

$$W_t[J] = \frac{1}{2} J \cdot P_{\Lambda(t)} \cdot J - S_t[P_{\Lambda(t)} J]. \quad (41)$$

By using Eq. (38) we find

$$\delta W_t = \frac{1}{2} W_t' \cdot \delta P_{\Lambda(t)} \cdot W_t'. \quad (42)$$

Since all the loop corrections are suppressed, the 1PI parts of the cutoff connected Green’s functions in the asymptotic region are completely scheme independent. Indeed, by the Legendre transformation

$$W_t[J] = J \cdot \phi - \Gamma_t[\phi] = \frac{1}{2} \phi \cdot \left( P_{\Lambda(t)}^{-1} - P_{\Lambda(t)}^{-1} \right) \cdot \phi, \quad (43)$$

and Eq. (42), one can find $\delta \Gamma_t[\phi] = 0$. Namely, the renormalized trajectories of the 1PI vertices defined in the different cutoff schemes approaches to each other in the asymptotic region, as is schematically shown in Fig. 7.

![Figure 7: The scheme dependence of the renormalized trajectories of the 1PI vertices (the solid lines). The dotted arrows are the coordinate transformation (21) and the shadow region is the asymptotic (freezing) region. The dimensionful cutoff Legendre effective action is frozen in this region and the renormalized trajectories with the different cutoff schemes come close to each other.](image)

The coordinate transformation on the functional space of the 1PI effective action $\Gamma_t[\phi]$ induced by Eq. (21) maps the fixed points, the renormalized trajectories and the critical
surfaces to those of the another scheme. Here we write this coordinate transformation as \( \Delta_{(A \rightarrow B)} \). Once the effective action of the scheme A (\( \Gamma^A_t[\phi] \)) and that of the scheme B (\( \Gamma^B_t[\phi] \)) satisfy the relation,
\[
\Gamma^B_t[\phi] = \Delta_{(A \rightarrow B)}[\Gamma^A_t],
\]
at a certain scale \( t_1 \), then it holds also for the each scale \( t_2 \). (See Fig. 7.) Solving the RG flow equation, the cutoff Legendre effective actions finally arrive at the asymptotic region with maintaining the same relation \( \Delta_{(A \rightarrow B)} \). As discussed in the last subsection \( \Delta_{(A \rightarrow B)} \) reduces to the identical mapping in the asymptotic region. Note that the RG flow of the dimensionful cutoff Legendre effective action is frozen in the asymptotic region.

The continuum limit of the field theories are found by tuning the initial boundary condition of the RG equation close to the critical surface or the fixed point and are discribed by the renormalized trajectories. As is seen above, the renormalized trajectories are cutoff scheme independent in the freezing region. This converging property of the RG flows of the cutoff Legendre effective action ensures that the solutions of the RG flow equations become cutoff scheme independent in the continuum limit. Needless to say, each theory must be specified by imposing the renormalization conditions for the renormalized couplings. Then other couplings are determined scheme independently. This structure should be compared with the scheme dependence of the Wilsonian effective action (or the Polchinski RGE). It is an advantageous feature of the Legendre flow equations that the physically meaningful results can be obtained directly.

4 Sharp cutoff limit of Polchinski equation

In this and next section, we confirm the equivalence between the sharp cutoff limit of the Polchinski equation and the Wegner-Houghton equation. It seems that the sharp cutoff RG equation, the Wegner-Houghton RG is quite different from the smooth cutoff one; the Polchinski RG. At first, we clarify the sharp cutoff limit of the Polchinski RGE in this section, and confirm the equivalence to the Wegner-Houghton equation in next section.

Since we would like to consider the sharp cutoff limit, it is more convenient to write the cutoff propagator \( P_\Lambda \) in terms of the cutoff function \( \theta_\varepsilon(q, \Lambda) \). Here \( \theta_\varepsilon \) is a smooth function with respect to the momentum \( q \), the cutoff \( \Lambda \) and a smoothness parameter \( \varepsilon \). In the limit of \( \varepsilon \rightarrow 0 \), \( \theta_\varepsilon(q, \Lambda) \) becomes a step function \( \theta(q - \Lambda) \), therefore,
\[
P_\Lambda(q) = \frac{1}{q^2} \theta_\varepsilon(q, \Lambda) \xrightarrow{\varepsilon \rightarrow 0} \frac{1}{q^2} \theta(q - \Lambda). \tag{45}
\]
If we also introduce \( \delta_\varepsilon(q, \Lambda) \) denoting the derivative of \( \theta_\varepsilon \) with respect to the cutoff \( \Lambda \), which satisfies
\[
-\frac{\partial}{\partial \Lambda} \theta_\varepsilon(q, \Lambda) = \delta_\varepsilon(q, \Lambda) \xrightarrow{\varepsilon \rightarrow 0} \delta(q - \Lambda). \tag{46}
\]

It is pointed out in Ref. [3] that one has to be careful for the behavior of \( \theta_\varepsilon \) and \( \delta_\varepsilon \) in the sharp cutoff limit. The non-trivial and universal behavior of \( \theta_\varepsilon \) and \( \delta_\varepsilon \) is
\[
\delta_\varepsilon(q, \Lambda)f(\theta_\varepsilon(q, \Lambda), q, \Lambda) \xrightarrow{\varepsilon \rightarrow 0} \delta(\Lambda - q) \int_0^1 dt f(t, q, \Lambda). \tag{47}
\]
For the derivation of the above formula, see Ref. [4]. Note that, \( \theta \) with another momentum \( q' \neq q \) does not behave as Eq. (47).

Let \( S_{\Lambda}^\varepsilon = 0 \) be the effective action for the sharp cutoff case and \( S_{\Lambda}^\varepsilon \) be one for the smooth cutoff case with \( \theta_{\varepsilon} \). These two effective actions should be related to each other by the formula (21),

\[
\exp \left( -S_{\Lambda}^\varepsilon [\phi] \right) = \exp \left( \frac{1}{2} \delta D \right) \exp \left( -S_{\Lambda}^{\varepsilon=0} [\phi] \right),
\]

where

\[
\delta D = \int \frac{d^d q}{(2\pi)^d} \frac{\delta}{\delta \phi(q)} \cdot \frac{1}{q^2} \left( \theta_{\varepsilon}(q^2) - \theta(q - \Lambda) \right) \cdot \frac{\delta}{\delta \phi(-q)}.
\] (49)

Here, \( S_{\Lambda}^\varepsilon [\phi] \) has the dependence of \( \theta_{\varepsilon} \) so that we have to take into account of Eq. (46). For the sake of simplicity, we write \( \theta_{\varepsilon}(q^2_i) - \theta(q_i - \Lambda) \equiv \Delta_i \).

The cutoff functions \( \theta_{\varepsilon} \) contributing to the non-trivial limit (47) should have common argument \( q \) with that of \( \delta_{\varepsilon} \) in the RG flow equation. They lie only on the external legs. Diagrammatically, they can be found in Fig. 8. In this diagram, all the momentum of \( \theta_{\varepsilon} \) are the same as that of \( \delta_{\varepsilon} \). Here, the filled circles in Fig. 8 correspond to the self energy \( \Sigma_{\Lambda}^\varepsilon(q) \). They are summed up and construct the 'Full propagator' times the inverse cutoff propagator i.e. \( (q^2/\Delta)/(q^2/\Delta + \Sigma_{\Lambda}^\varepsilon(q)) \). The pre-factor \( q^2/\Delta \), the inverse cutoff propagator, is required, since the argument of \( \tilde{W}_\Lambda \) in Eq. (31) is \( \delta P^{-1}\phi \).

In general, one can imagine other diagrams like Fig. 9 which the momentum \( p \) flows in. In this figure, the filled circle denotes the multi-point 1PI vertex. Since the cutoff function \( \theta_{\varepsilon}(p + q) \) behaves differently at \( p = 0 \) in the sharp cutoff limit, the discontinuous momentum dependence should appear in the beta-functional. If the field \( \phi(p) \) is a smooth function, there are no finite contributions since the measure of the point \( p = 0 \) is zero. However since \( \phi(p) \) may have the distribution like a VEV \( \varphi(2\pi)^d \delta^d(p) \), we separate such distributions explicitly. Furthermore, since we would like to claim the equivalence between
the Wegner-Houghton equation and the Polchinski equation in the sharp cutoff limit including these discontinuous momentum dependence, we also introduce the singularities like \( \delta^d (q - q_i) \),

\[
\phi (q) \rightarrow (2\pi)^d \delta^d (q - q_i) \varphi_i + \phi (q).
\]  
(50)

Now, the effective action acquires \( \varphi_i \) dependence i.e. \( \hat{S}^e_{\Lambda} = \hat{S}^e_{\Lambda} [\phi, \varphi] \) and also satisfies the same formula (48). In this case, the Feynman diagrams like Fig. 9 contribute to the sharp cutoff flow equation via the combination \( \prod \varphi_i^{n_i} \) with \( \sum n_i p_i = 0, n_i \in \mathbb{N} \), that is a point \( p = 0 \) of the diagram in Fig. 9. The corrections from \( \prod \varphi_i^{n_i} \) terms can be absorbed by redefinition of the self energy \( \Sigma^e_{\Lambda} (q) \). Consequently we regard the self energy \( \Sigma^e_{\Lambda} \) as a \( \varphi_i \) dependent function.

One may wonder if the self energy \( \Sigma^e_{\Lambda} (q) \) has the discontinuous momentum dependence, since there are \( \theta_{\varepsilon}(q) \) with momenta in common with \( \delta_{\varepsilon}'s \) in the loop integrals giving \( \Sigma^e \). However, since the regions of the loop momentum integration have the vanishing measure, the above discontinuous momentum dependence does not contribute to Eq. (47) at all. Similarly, all the 1PI building blocks smoothly approach to those for the sharp cutoff. Hence, what we must check is only articulation lines. Furthermore, the internal lines with momenta \( q + p \) vanish, since \( \Delta (q + p) \) goes to zero in the limit \( \varepsilon \rightarrow 0 \). Therefore we need to care the external legs only.

Finally, one can conclude that the relevant scheme dependence of \( n(> 2) \)-point functions comes only from the external legs. Diagrammatically, they can be illustrated as in Fig. 10. The scheme dependence of the two-point function \( S^{(2)}_{\Lambda} \) is given by \( q^2 / \Delta \) minus the ‘Full propagator’ times \( (q^2 / \Delta)^2 \) and is different from those of other vertices. Therefore we separate the two point function in the effective action and write as

\[
S_{\Lambda} [\phi] = \frac{1}{2} \phi \cdot S^{(2)}_{\Lambda} \cdot \phi + \hat{S}_{\Lambda} [\phi],
\]  
(51)

where \( \hat{S}_{\Lambda} [\phi] \) is the part composed of \( n(> 2) \)-point functions.

\[\begin{array}{c}
S^{e}_{\Lambda} \\
S^{e=0}_{\Lambda} \quad \text{trivial contribution}
\end{array}\]

\[
\text{Full propagator}
\]

Figure 10: These kinds of diagrams behave as Eq. (10). \( S^{e=0}_{\Lambda} \) is the vertex with the sharp cutoff and the ‘full propagator’ is the propagator with the smooth one. Other contributions shown by the ‘trivial contribution’ vanish in the limit \( \varepsilon \rightarrow 0 \).

Taking account of Eq. (11), we can extract the relevant scheme (\( \Delta \)) dependence as follows. For the two-point function \( S^{(2)}_{\Lambda} \),

\[
\frac{\delta^2 S^e_{\Lambda}}{\delta \phi(p) \delta \phi(q)} \bigg|_{\phi=0} = \left( \frac{q^2}{\Delta} \right)^2 \left( \frac{\Delta}{q^2} - \frac{1}{q^2 / \Delta + \Sigma^e_{\Lambda}(q)} \right) \cdot (2\pi)^d \delta^d(p + q),
\]  
(52)
and for $n(>2)$-point functions $\hat{S}_\Lambda$,

$$
\hat{S}_\Lambda^{\varepsilon=0}[\phi] = \sum_{n\neq 2} \frac{1}{n!} \prod_{i=1}^{n} \int \frac{d^dq_i}{(2\pi)^d} \left( \frac{q_i^2 / \Delta_i \cdot \phi(q_i)}{q_i^2 / \Delta_i + \Sigma_\Lambda^\varepsilon} \right) S_\Lambda^{\varepsilon=0}(q_1, \ldots, q_n) + \ldots,
$$

(53)

where dots '⋯' have no significant dependence on $\Delta$ and vanish in the sharp cutoff limit.

As mentioned above $\Sigma_\Lambda^\varepsilon$ and $S_\Lambda^\varepsilon$ depend on $\varphi_i$, e.g. $\Sigma_\Lambda^\varepsilon(q) = \Sigma(q, \Pi \varphi_i^{\text{ann}})$ and $\hat{S}_\Lambda[\phi, \Pi \varphi_i^{\text{ann}}]$.

Let us first see the sharp cutoff limit of Eqs. (52) and (53). In this limit, $\Delta$ vanishes and $\Sigma_\Lambda^\varepsilon(q)$ can be replaced by $\Sigma_\Lambda^{\varepsilon=0}(q)$ safely. For $n > 2$ we easily find

$$
\hat{S}_\Lambda^{\varepsilon=0}[\phi] = \sum_{n\neq 2} \frac{1}{n!} \prod_{i=1}^{n} \int \frac{d^dq_i}{(2\pi)^d} \phi(q_i) S_\Lambda^{\varepsilon=0}(q_1, \ldots, q_n).
$$

(54)

For the two-point function, we can rewrite Eq. (52) as,

$$
\Sigma_\Lambda^{\varepsilon=0}(q) = q^2 \Sigma_\Lambda^\varepsilon / \left( q^2 + \Delta \cdot \Sigma_\Lambda^\varepsilon \right),
$$

(55)

where $\Sigma_\Lambda^{\varepsilon=0}$ corresponds to the two point function of the sharp cutoff effective action $S_\Lambda^{\varepsilon=0}[\phi]$,

$$
\frac{\delta^2 S_\Lambda^{\varepsilon=0}}{\delta \phi(p) \delta \phi(q)} |_{\phi=0} = \Sigma_\Lambda^{\varepsilon=0}(q, \Pi \varphi_i^{\text{ann}})(2\pi)^d \delta^d(p + q).
$$

(56)

We must fix the $\theta(0)$ ambiguity before letting $\varepsilon \to 0$ in the RG flow equation, since $\Delta$ in Eqs. (52) and (53) is given by $\Delta(q) = \theta_\varepsilon(q/\Lambda) - \theta(q - \Lambda)$ and satisfy

$$
\delta_\varepsilon(q/\Lambda)f(\Delta(q)) \xrightarrow{\varepsilon \to 0} \delta(q - \Lambda) \int_0^1 dt f(t - \theta(0)).
$$

(57)

We simply set $\theta(0) = 0$ here. It is different from the ordinary convention; $\theta(0) = 1/2$. This is because, we implicitly used $\theta(0) = 0$ to derive the Wegner-Houghton equation. The ‘shell modes’ $\phi_s$ integrated out by the RG transformation have momenta $\Lambda - \delta \Lambda < q \leq \Lambda$, lower than the scale $\Lambda$ of the effective action $S_\Lambda$. In the limit $\delta \Lambda \to 0$, the shell momentum $q$ reaches to $\Lambda$ from below. To make this limit well-defined, we should employ the left semi-open interval $\Lambda - \delta \Lambda < q \leq \Lambda$. Hence we can say that the fluctuations with $q > \Lambda$ are incorporated in the Wilsonian effective action $S_{\Lambda}[\phi]$, while that with $q = \Lambda$ are not. It means that our infra-red cutoff $\theta(q - \Lambda)$ satisfies $\theta(0) = 0$!

Since the ‘delta’-function $\delta_\varepsilon(q, \Lambda)$ lies on the $\Lambda$ derivative of the cutoff propagator, what we must check are the following two terms. One is a field $\phi(q)$ dependent term,

$$
- \delta S_\Lambda / \delta \phi(q) \cdot \left( \partial / \partial \Lambda P_\Lambda \right) \cdot \delta S_\Lambda / \delta \phi(-q) + \partial / \partial \Lambda P_\Lambda \cdot \delta^2 \hat{S}_\Lambda / \delta \phi(q) \delta \phi(-q),
$$

(58)

where the first term corresponds to the ‘dumbbell’ diagram and the second term corresponds to the ‘ring’ diagram in Fig. [1]. One can easily realize that the above equation is proportional to

$$
\frac{1}{q^2} \delta_\varepsilon(q, \Lambda) \left( \frac{q^2 / \Delta}{q^2 / \Delta + \Sigma_\Lambda^\varepsilon(q)} \right)^2 \xrightarrow{\varepsilon \to 0} \frac{\delta(q - \Lambda)}{q^2 + \Sigma_\Lambda^{\varepsilon=0}(q)},
$$

(59)
where we were taking account of the following relations,
\[
\begin{align*}
\frac{\bar{\delta}S_\Lambda}{\partial \phi(q)} & \sim \frac{q^2}{q^2 + \Delta \cdot \Sigma_{\Lambda = 0}} \cdot \frac{\bar{\delta}S_{\Lambda = 0}}{\partial \phi(q)} + \text{no significant terms}, \\
\text{and} \\
\frac{\bar{\delta}^2 S_\Lambda}{\partial \phi(q) \partial \phi(-q)} & \sim \left(\frac{q^2}{q^2 + \Delta \cdot \Sigma_{\Lambda = 0}(q)}\right)^2 \cdot \frac{\bar{\delta}^2 S_{\Lambda = 0}}{\partial \phi(q) \partial \phi(-q)} + \text{no significant terms},
\end{align*}
\]
for the functional derivative with respect to the field \(\phi(q)\).

Another is the \(\phi(q)\) independent term which corresponds to the part evaluated in the Local Potential Approximation (LPA),
\[
-\frac{\partial}{\partial \Lambda} P_\Lambda \cdot S^{(2)}_\Lambda(q) = \frac{1}{2} q^2 \delta_\varepsilon(q, \Lambda) \cdot \frac{q^2 \Sigma_{\Lambda}}{q^2 + \Delta \cdot \Sigma_{\Lambda}} \cdot (2\pi)^d \delta^d(p + q).
\]
By taking \(\varepsilon \to 0\), we find the sharp cutoff limit of this as
\[
\delta(q - \Lambda) \ln \left(1 + \frac{\Sigma_{\Lambda} = 0}{q^2}\right) \cdot (2\pi)^d \delta^d(p + q).
\]
They correspond to the diagrams given in Fig. [1].

![Figure 11](image_url)

Figure 11: These diagrams correspond to Eqs. (59) and (63). The third graph has no dependence on the field \(\phi(q)\) but on the VEV \(\varphi\). It contributes to the LPA flow equation, and should be compared with the LPA Wegner-Houghton RGE. [1]

Using these results, the sharp cutoff limit of the Polchinski RG equation becomes,
\[
\frac{\partial}{\partial \Lambda} S^{\varepsilon = 0}_\Lambda = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \delta(q - \Lambda) \left\{ \frac{\bar{\delta}S_{\Lambda = 0}}{\partial \phi(q)} \cdot \frac{\bar{\delta}S_{\Lambda = 0}}{\partial \phi(-q)} - \frac{\bar{\delta}^2 S_{\Lambda = 0}}{\partial \phi(q) \partial \phi(-q)} \right\} - \frac{1}{2} (2\pi)^d \delta^d(0) \int \frac{d^d q}{(2\pi)^d} \delta(q - \Lambda) \ln \left(q^2 + \Sigma_{\Lambda = 0}(q, \Pi \varphi^{n_i})\right).
\]
The canonical scaling of the momentum \(p_\mu \partial / \partial p_\mu\) does not affect these results, since \(\partial \Delta / \partial p_\mu \to 0\) as \(\varepsilon \to 0\).
5 Comparison with the Wegner-Houghton equation

To confirm the equivalence between Eq. (64) and the Wegner-Houghton RG equation, we substitute Eq. (50) to the Wegner-Houghton equation. Let us start from the following formula which gives the effective action $S_{\Lambda-\delta\Lambda}$ up to $O(\delta\Lambda^2)$.

$$S_{\Lambda-\delta\Lambda} = S_{\Lambda} - \frac{1}{2} \Lambda \frac{\delta S_{\Lambda}}{\delta \phi_{s}} \left( P_{\Lambda}^{-1} + \frac{\delta^2 S_{\Lambda}}{\delta \phi_{s} \delta \phi_{s}} \right)^{-1} \cdot \frac{\delta S_{\Lambda}}{\delta \phi_{s}} + \frac{1}{2} \Lambda \left| \left( P_{\Lambda}^{-1} + \frac{\delta^2 S_{\Lambda}}{\delta \phi_{s} \delta \phi_{s}} \right) \right|, \quad (65)$$

where $\phi_{s}$ denotes the ‘shell mode’ whose support is given by the condition $p^2 = \Lambda^2$. Dot $(\cdot)$ denotes the integral $\int_{\Lambda-\delta\Lambda}^{\Lambda} dq/(2\pi)^d$. It can be realized that Eq. (64) involves the higher contribution of $O(\delta\Lambda^2)$. First, we define $\hat{S}_{\Lambda}$ by,

$$\left. \frac{\delta^2 S_{\Lambda}}{\delta \phi_{s}(p) \delta \phi_{s}(q)} \right|_{\phi_{s}=0} \equiv \Sigma_{\Lambda}(q, \prod \phi_{i}^{n_{i}})(2\pi)^d \delta^d(p + q) + \left. \frac{\delta^2 \hat{S}_{\Lambda}}{\delta \phi_{s}(p) \delta \phi_{s}(q)} \right|_{\phi_{s}=0}. \quad (66)$$

Here, $\Sigma_{\Lambda}$ is the same as $\Sigma_{\Lambda=0}$ given by Eq. (60) before. The second term of the r.h.s. of Eq. (66) is regular at $q = -p$.

Let us rewrite Eq. (65) in matrix notation. We define a matrix $M_{p,q}$ by,

$$M_{p,q} \equiv \left. \frac{\delta^2 \hat{S}_{\Lambda}}{\delta \phi_{s}(p) \delta \phi_{s}(q)} \right|_{\phi_{s}=0}. \quad (67)$$

The matrix $M_{p,q}$ may have off-diagonal singularities, e.g. $\delta(p + q + k)$ due to $\phi_{i}$. The first derivative of the effective action with respect to the field corresponds to a ‘vector’ $v$;

$$v_{q} \equiv \left. \frac{\delta S_{\Lambda}}{\delta \phi_{s}(q)} \right|_{\phi_{s}=0}. \quad (68)$$

Using these notations, the r.h.s. of Eq. (65) can be expressed by the following equation,

$$\frac{1}{2} v^{T} \cdot \left( (P^{-1} + \Sigma) \mathbf{1} + M \right)^{-1} \cdot v \left. - \frac{1}{2} \text{Tr} \ln \left( (P^{-1} + \Sigma) \mathbf{1} + M \right) \right|, \quad (69)$$

where the unit matrix $\mathbf{1}$ corresponds to $(2\pi)^d \delta^d(p + q)$. We expand this with respect to the matrix $M$. Since one momentum integral $\int_{\Lambda-\delta\Lambda}^{\Lambda} dq$ brings a factor $\delta\Lambda$, only the first few terms can contribute to the RG flow equation, therefore

$$\frac{1}{2} v^{T} \cdot \left( (P^{-1} + \Sigma) \mathbf{1} \right)^{-1} \cdot v \left. - \frac{1}{2} \text{Tr} \ln \left( (P^{-1} + \Sigma) \mathbf{1} \right) \right. \left. - \frac{1}{2} \text{Tr} \left( (P^{-1} + \Sigma) \mathbf{1} \right)^{-2} \cdot M \right) + \cdots, \quad (70)$$

where dots $\cdots$ are the higher order in $\delta\Lambda$. One of the higher order contribution is written as follows,

$$\frac{\delta S_{\Lambda}}{\delta \phi_{s}} \cdot P_{s} \cdot \frac{\delta^2 \hat{S}_{\Lambda}}{\delta \phi_{s} \delta \phi_{s}} \cdot P_{s} \cdot \frac{\delta S_{\Lambda}}{\delta \phi_{s}} + \cdots, \quad (71)$$

where $P_{s}(q)$ is the propagator of the shell mode $\phi_{s}$ whose support is restricted to the region $\Lambda - \delta\Lambda < q \leq \Lambda$. Eq. (74) corresponds to Fig. 12.
Since the cross section of the integral region of $p$ and that of $q$ is $O(\delta \Lambda^2)$, the contribution from the diagram given in Fig. 12 becomes $O(\delta \Lambda^2)$. If $k$ vanishes, the volume of the integral region above becomes the first order of $\delta \Lambda$, because two ‘spheres’ completely coincide with each other. Hence the value of the RG beta-function jumps at the point $k = 0$. However since the field $\phi$ is a smooth function of the momentum, not the distribution, we can drop it. It contributes through the distribution given in Eq. (50) by the combinations $\prod_\mathcal{N} \varphi_i^{n_i}$ with $\sum n_i p_i = 0$. They are already taken in the beta-function by the $\prod_\mathcal{N} \varphi_i^{n_i}$ dependence in the self energy $\Sigma_\Lambda$.

Consequently, the Wegner-Houghton equation can be found as,

$$S_\Lambda - S_{\Lambda-\delta\Lambda} = \frac{\delta \Lambda}{2} \int \frac{d^d q}{(2\pi)^d} \frac{\delta (q - \Lambda)}{q^2 + \Sigma_\Lambda(q, \prod_\mathcal{N} \varphi_i^{n_i})} \left\{ \frac{\delta S_\Lambda}{\delta \phi(q)} \cdot \frac{\delta S_\Lambda}{\delta \phi(-q)} - \frac{\delta^2 S_\Lambda}{\delta \phi(q) \delta \phi(-q)} \right\} - \frac{\delta \Lambda}{2} \frac{d^d q}{(2\pi)^d} \delta^d(0) \int \frac{d^d q}{(2\pi)^d} \delta(q - \Lambda) \ln \left( q^2 + \Sigma_\Lambda(q, \prod_\mathcal{N} \varphi_i^{n_i}) \right) + O(\delta \Lambda^2).$$

This flow equation is nothing but the sharp cutoff limit of the Polchinski equation (64).

6 Conclusion and remarks

In this article, we investigated the cutoff scheme dependence of the Wilsonian effective action. It can be reinterpreted as the coordinate transformation on the theory space. It is written formally by Eq. (21). We have studied it in two limiting cases. One is in the asymptotic region i.e. $t \to \infty$, and another is in the sharp cutoff limit i.e. $\varepsilon \to 0$. In the both cases, we could write down the cutoff scheme dependence so simple as to explore the RG flows.
As we have shown in Sec. 3, the scheme dependence of the renormalized trajectories in the asymptotic region \( t \to \infty \) remains for the Wilsonian effective action. Besides, the RG flow of the Wilsonian effective action does not freeze in \( t \to \infty \). The origin is as follows. The vertices of the Wilsonian effective action consist of two different quantities; the connected Green’s function at high energy region (\( p > \Lambda \)) and the 1PI vertices at the low energy region (\( p < \Lambda \)). The boundary between these regions are connected scheme dependently. (See also Ref. [4].) Therefore the Wilsonian effective action itself is not a physical quantity.

Moreover, we have also shown the scheme independence of the Legendre effective action,[7, 4] or equivalently the 1PI building blocks of the Wilsonian effective action, on the renormalized trajectories. Recalling the statements in Sec. 3, we can say that if the RG flow of the dimensionful Legendre effective action \( \Gamma_\Lambda[\phi] \) freezes on the renormalized trajectory in the asymptotic region, i.e. in the statistical continuum limit \( \Lambda_0 \to \infty \), then our \( \Gamma_\Lambda[\phi] \) should be scheme independent.

In the perturbation theory, it can be easily realized. Indeed, all the cutoff scheme dependent contributions, i.e. the coefficients of the divergences, are completely absorbed into certain counterterms order by order, and remaining finite terms are scheme independent in the limit \( \Lambda_0 \to \infty \). Of course, needless to say, we should insist on the common renormalization condition (or equivalently the subtraction rule). In the non-perturbative case, however, the problem will be more complicated, because the ordinary renormalization procedure will not work in general, e.g. for the theory around a non-Gaussian fixed point. Hence, the cancellation of divergences, and therefore the cutoff scheme dependent constants can be confirmed only case by case if possible.

Turning to the Exact Renormalization Group, we can recapture it from another point of view. The scheme dependence of the counterterms corresponds to that of the fixed point and/or of the critical surface, and the scheme independence of the total solution can be appreciated by that of the renormalized trajectory in the asymptotic region. All these are described by a coordinate transformation (21). For our purpose, it is sufficient to investigate the asymptotic region of Eq. (21) without using the explicit solutions, since we have expected the asymptotic behavior \( g_i(t) \sim g_i^R e^{dt} \) and do not need the explicit value \( g_i^R \). Once we assume existence of the asymptotic region, then the scheme independence of \( \Gamma_\Lambda[\phi] \) as \( \Lambda \to 0 \) is confirmed. (Recall the discussion in Sec. 3.) For massive theories, the scheme dependence of \( \Gamma_\Lambda[\phi] \) decays like \( \exp(-dt) \sim (\Lambda/M_R)^t \) as \( t \to \infty \). Instead, for the massless case, one may start from a massive case and then letting \( M_R \to 0 \).

We also confirm the equivalence between the Wegner-Houghton equation and the Polchinski equation in the sharp cutoff limit. It seems that these equations are much different from each other even though they are expected to be equivalent. We can prove equivalence of these two ERGs by help of Eq. (21) which describes the scheme dependence of the Wilsonian effective action. The superficial difference occurs by the strong scheme dependence of the Wilsonian effective action. As we showed, the crucial cutoff scheme dependence of the Wilsonian effective action lies in the external legs.

Finally, we would like to comment on the cutoff scheme dependence of the approximate solutions. The ERG flow equations are approximated by projecting them onto smaller dimensional subspaces of the original theory space. In the derivative expansion, for example, we may employ these subspaces as the space of a finite number of the coefficient
functions \(\{V_0, V_2, \ldots, V_k^i\}\) defined by the following equation,

\[
S_\Lambda[\phi] = \int d^d x \left\{ V_0(\phi) + \frac{1}{2}(\partial_\mu \phi)^2 V_2(\phi) + \frac{1}{2}(\Box \phi)^2 V_4^1(\phi) + \cdots \right\},
\]

(73)

The subscript \(k\) of the coefficient function denotes the degrees of the derivatives and the superscript \(i\) of it labels the independent \(k\)-th derivative vertices. Then, the ERG flow equations are reduced to the coupled partial differential equations for the coefficient functions \(V_i^k(\phi)\). One can easily improve the approximation systematically by enlarging the subspace \(\{V_0, V_2, \ldots, V_k^i\}\) step by step. Especially, the approximation with \(k = 0\) is called the ‘local potential approximation’ (LPA). This procedure preserves the non-perturbative nature of the ERG flow equations.

The scheme dependence given by Eq. (38) are infinitely enhanced in the derivative expansion. By dimensional analysis, the Taylor expansion of \(\delta P(q)\), whose value changes rapidly near the infra-red cutoff \(q \sim \Lambda\), is the expansion with respect to the combination \(q^2/\Lambda^2 \gg 1\). Therefore the scheme dependence of the coefficient functions \(V_i^k(\phi)\) in Eq. (73) become stronger and stronger as the infra-red cutoff \(\Lambda\) decreases. The scheme dependence of \(V_i^k(\phi)\) behaves as \(1/\Lambda^{2k}\). At last it diverges in the limit \(\Lambda \to 0\). It means that the derivative expansion and the limit \(\Lambda \to 0\) do not commute. Hence the cutoff scheme dependence of \(V_i^k(\phi)\) is strong enough to prevent the physical predictions. The physical information is completely washed off except for the potential part \(V_0(\phi)\) which is the 1PI effective potential.

The RG beta-functionals of the coefficient functions of \(\Gamma_\Lambda[\phi]\) like Eq. (73) are also cutoff scheme dependent in the region \(t \to \infty\), since by the dimensional analysis, the expanding parameter there is \(\partial/\varepsilon \sim \partial/\Lambda\) where \(\varepsilon\) stands for the smoothness parameter given in Sec. 4. The higher derivative contributions finally overcome the suppression factor \((\Lambda/M_R)^{\delta}\). Hence, the RG beta-functionals of the higher derivative operators blow up to infinity. In the limit \(t \to \infty\), the cutoff scheme approaches towards to the sharp one, since \(\varepsilon \sim \Lambda \to 0\). It is known that these diverging series can be summed up and lead to non-analytical momentum dependence of the vertices, e.g. \(\sqrt{P_\mu P_\mu}\). This spurious scheme dependence can be avoided if we work on the sharp cutoff Legendre flow equation [4, 5].

Appendix

Equation (31) in Sec. 3 is driven as follows. First, let us consider the positive definite deviation of the cutoff propagator \(\delta P(q) = \delta P_+(q) > 0\), since we call for the Gaussian integral with positive \(\delta P(q)\). Then one can find

\[
\exp \left( \frac{1}{2} \delta D_+ [\delta/\delta \phi] \right) \exp \left( -S_t [\phi] \right) = \exp \left( \frac{1}{2} \delta D_+ [\delta/\delta \phi] \right) \exp \left( -S_t [\phi] + J \cdot \phi \right) \bigg|_{J=0}
\]

\[
= \exp \left( \frac{1}{2} \delta D_+ [\delta/\delta \phi] \right) \exp \left( -S_t [\delta/\delta J] \right) e^{J \cdot \phi} \bigg|_{J=0}
\]

\[
= \exp \left( -S_t [\delta/\delta J] \right) \exp \left( \frac{1}{2} \delta D_+ [J] + J \cdot \phi \right) \bigg|_{J=0}
\]
\[ \alpha \exp \left( -S_t [\delta / \delta J] \right) \int D\phi' \exp \left( -\frac{1}{2} \phi' \cdot \delta P_+^{-1} \cdot \phi' + J \cdot (\phi + \phi') \right) \bigg|_{J=0} \]

\[ = \int D\phi' \exp \left( -\frac{1}{2} \phi' \cdot \delta P_+^{-1} \cdot \phi' - S_t [\phi + \phi'] \right) \]

\[ = \exp \left( -\frac{1}{2} \phi \cdot \delta P_+^{-1} \cdot \phi \right) \int D\phi' \exp \left( -\frac{1}{2} \phi' \cdot \delta P_+^{-1} \cdot \phi' - S_t [\phi'] + \phi' \cdot \delta P_+^{-1} \cdot \phi \right) \]

\[ = \exp \left( -\frac{1}{2} \phi \cdot \delta P_+^{-1} \cdot \phi + W_t [J = \delta P_+^{-1} \cdot \phi] \right), \quad (74) \]

where \( \delta D_+[J] \) is given by,

\[ \delta D_+[J] = \int \frac{d^d q}{(2\pi)^d} J(q)\delta P_+(q)J(-q). \quad (75) \]

The negative part of the deviation \( \delta P_-(q) \) needs the special care, since the path-integral in the fourth line does not converge. However our final result can be hold also for the negative part \( \delta P_-(q) \), since Eq. (31) is the identity of \( \delta P \). In other words, Eq. (31) means the graph by graph correspondence of the Feynman diagrams. It does not restrict our observations in Secs. 3-5, since we need the diagramatical representation of Eq. (74).

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