HOMOTOPICAL MORITA THEORY FOR CORINGS

ALEXANDER BERGLUND AND KATHRYN HESS

ABSTRACT. A coring \((A, C)\) consists of an algebra \(A\) in a symmetric monoidal category and a coalgebra \(C\) in the monoidal category of \(A\)-bimodules. Corings and their comodules arise naturally in the study of Hopf-Galois extensions and descent theory, as well as in the study of Hopf algebroids. In this paper, we address the question of when two corings \((A, C)\) and \((B, D)\) in a symmetric monoidal model category \(\mathcal{V}\) are homotopically Morita equivalent, i.e., when their respective categories of comodules \(\mathcal{V}_A^C\) and \(\mathcal{V}_B^D\) are Quillen equivalent.

The category of comodules over the trivial coring \((A, A)\) is isomorphic to the category \(\mathcal{V}^A\) of \(A\)-modules, so the question above englobes that of when two algebras are homotopically Morita equivalent. We discuss this special case in the first part of the paper, extending previously known results to the case when the homotopy category of \(\mathcal{V}\) is not necessarily a triangulated category.

To approach the general question, we introduce the notion of a braided bimodule and show that adjunctions between \(\mathcal{V}_A^C\) and \(\mathcal{V}_B^D\) that lift to adjunctions between \(\mathcal{V}_A^C\) and \(\mathcal{V}_B^D\) correspond precisely to braided bimodules between \((A, C)\) and \((B, D)\). We then give criteria, in terms of homotopic descent, for when a braided bimodule induces a Quillen equivalence between \(\mathcal{V}_A^C\) and \(\mathcal{V}_B^D\). In particular, we obtain criteria for when a morphism of corings induces a Quillen equivalence, providing a homotopic generalization of results by Hovey and Strickland on Morita equivalences of Hopf algebroids.

To illustrate the general theory, we examine homotopical Morita theory for corings in the unstable model category \(\mathcal{C}h_{fin}^R\) of finite-type, non-negatively graded chain complexes over a field \(k\).

CONTENTS

1. Introduction 1
2. Homotopical Morita theory for algebras 3
3. Corings and braided bimodules 12
4. Homotopical Morita theory for corings 25
5. The case of finite-type, non-negatively graded chain complexes 29
Appendix A. Enriched model categories 36
Appendix B. Dualizability 38
References 40

1. Introduction

The study of equivalences between categories of comodules over \(k\)-coalgebras, where \(k\) is a field, was initiated by Takeuchi [33] and is commonly referred to as Morita-Takeuchi theory. The more general question of when categories of comodules over corings, \(\mathcal{V}_A^C\) and \(\mathcal{V}_B^D\), are strictly equivalent categories has also been studied, when \(A\) and \(B\) are \(R\)-algebras, where \(R\) is a commutative ring; see [5] and the references therein.

In this paper, we address a homotopical generalization of this question. Our setup will be very general, working with algebras and corings in a closed symmetric monoidal model category \(\mathcal{V}\) (see Appendix A). A morphism in the category \(\mathcal{V}_A\) of
right modules over an algebra $A$ in $\mathcal{V}$ is defined to be a weak equivalence (fibration) if its image under the forgetful functor $U: \mathcal{V}_A \to \mathcal{V}$ is a weak equivalence (fibration) in $\mathcal{V}$. Conditions for the existence of a model structure on $\mathcal{V}_A$ with these weak equivalences and fibrations have been studied by Schwede and Shipley \cite{schwede2003algebraic} (see Theorem 2.2 below). A morphism in the category $\mathcal{V}_A^c$ of right comodules over a coring $(A,C)$ in $\mathcal{V}$ is declared to be a weak equivalence (cofibration) if its image under the forgetful functor $U_A: \mathcal{V}_C^c \to \mathcal{V}_A^c$ is a weak equivalence (cofibration). In particular, a morphism in $\mathcal{V}_A^c$ is a weak equivalence if and only if it is a weak equivalence when viewed as a morphism in $\mathcal{V}$. Conditions for the existence of a model structure on $\mathcal{V}_A^c$ with these weak equivalences and cofibrations have been studied in \cite{berglund2013model}. We address the following general question (presuming we are in a particular, a morphism in $\mathcal{V}$ if its image under the forgetful functor $U$ is a weak equivalence if and only if it is a weak equivalence when viewed as a morphism in $\mathcal{V}$). Conditions for the existence of a model structure on $\mathcal{V}_A^c$ with these weak equivalences and cofibrations have been studied in \cite{berglund2013model}. We address the following general question (presuming we are in a situation where the model structures exist):

When are the model categories $\mathcal{V}_A^c$ and $\mathcal{V}_B^D$ Quillen equivalent?

Lacking a general classification of adjunctions between $\mathcal{V}_A^c$ and $\mathcal{V}_B^D$, we focus attention on adjunctions that lift a given adjunction between $\mathcal{V}_A$ and $\mathcal{V}_B$. We prove that these are governed precisely by what we call braided bimodules, i.e., bimodules $A_XB$ together with a ‘braiding’ $T: C \otimes_A X \to X \otimes_B D$ satisfying pentagon and counit axioms (Definition \ref{def:braided-bimodule}). Adjunctions governed by braided bimodules are not the most general kind of adjunctions, but they are enough for the applications to homotopic descent and homotopic Hopf-Galois extensions that motivated this work. These applications will appear in a sequel paper \cite{berglund2015homotopy}.

If $A_XB$ is a braided bimodule whose underlying right $B$-module is dualizable, then to every $A$-coring $C$ and $B$-coring $D$, we can canonically associate a $B$-coring $X.(C)$, called the descent coring, and a morphism of $B$-corings $g_T: X.(C) \to D$ (Proposition \ref{prop:descent-coring}). A special case of our main result (Theorem \ref{thm:main-result}) characterizes when a braided bimodule gives rise to a Quillen equivalence in terms of the morphism of corings $g_T$.

**Theorem 1.1.** Let $(A,C)$ be a flat coring in $\mathcal{V}$, and let $(X,T): (A,C) \to (B,D)$ be a braided bimodule. Suppose that $X$ is strictly dualizable and cofibrant as a right $B$-module, as well as homotopy flat and homotopy compact as a left $A$-module. The induced functor $T_\#: \mathcal{V}_C^c \to \mathcal{V}_B^D$ is a Quillen equivalence if and only if

1. $X$ satisfies effective homotopic descent with respect to $C$, and
2. $g_T: X.(C) \to D$ is a copure weak equivalence.

We refer the reader to Definitions \ref{def:effective-descent} and \ref{def:copure-weak-equivalence} for the notions of effective homotopic descent and copure weak equivalences and remark here only that it is possible to describe explicit classes of morphisms satisfying the required conditions. In Theorem \ref{thm:main-theorem} for example, we provide a relatively simple description of an interesting class of copure weak equivalences when the underlying category is that of finite-type non-negatively graded chain complexes over a field. Moreover, we generalize Grothendieck’s classical theorem on faithfully flat descent for homomorphisms of commutative rings and show that if $X$ is homotopy faithfully flat as a left $A$-module, then $X$ satisfies homotopic descent with respect to any coring $C$ (Theorem \ref{thm:grothendieck}).

The paper is structured as follows. In Section 2, we discuss homotopical Morita theory for algebras in $\mathcal{V}$. In Section 3 we introduce and study $\mathcal{V}$-categories of comodules over corings. Here we introduce the two main new concepts: braided bimodules (Section 3.2) and the canonical coring associated to a braided bimodule whose underlying bimodule is strictly dualizable (Section 3.3). Section 4 contains our main results on homotopical Morita theory for corings. In Section 5 we apply the general theory to the case when $\mathcal{V}$ is the category of finite-type non-negatively graded chain complexes over a field. In the appendices we recall necessary elements.
of the theory of enriched model categories and the theory of dualizable objects in
a symmetric monoidal category.

2. Homotopical Morita theory for algebras

Classical Morita theory provides criteria for equivalences of categories of modules
over rings. In this section we answer the corresponding question in the homotopi-
cal setting: for \( \mathcal{V} \) a symmetric monoidal model category and \( A \) and \( B \)
algebras (monoids) in \( \mathcal{V} \), when are the \( \mathcal{V} \)-model categories \( \mathcal{V}_A \) and
\( \mathcal{V}_B \) Quillen equivalent?

Homotopical Morita theory for unbounded differential graded algebras was studied
by Dugger-Shipley [7], and for ring spectra by Schwede and Shipley [30] and Shipley
[32]. For derived categories, see Rickard [28]. We give a self-contained and rela-
tively short account that subsumes known results in these settings. Our results also
apply to some new cases, such as the unstable model categories of non-negatively
graded differential graded algebras and topological spaces.

We begin this section by recalling and elaborating somewhat on the homotopy
theory of modules in a monoidal model category. We completely characterize en-
riched adjunctions between module categories, then use this characterization to
provide conditions under which such an adjunction is a Quillen adjunction. Fi-
ally, we prove a homotopical version of the usual Morita theorem, giving criteria
under which an adjunction between module categories is a Quillen equivalence.

2.1. \( \mathcal{V} \)-model categories of modules. Let \( (\mathcal{V}, \otimes, k) \) be a monoidal category.

An algebra (also known as a monoid) in \( \mathcal{V} \) is an object \( A \) together with two maps
\( \mu: A \otimes A \to A \) and \( \eta: k \to A \) that satisfy the usual associativity and unit axioms.

We let \( \text{Alg}_{\mathcal{V}} \) denote the category of algebras in \( \mathcal{V} \). Dually, the category of coalgebras
in \( \mathcal{V} \), which are endowed with a coassociative comultiplication and counit, is
denoted \( \text{Coalg}_{\mathcal{V}} \).

A right module over \( A \) is an object \( M \) in \( \mathcal{V} \) together with a map \( \rho: M \otimes A \to M \)
satisfying the usual axioms for a right action. We let \( \mathcal{V}_A \) denote the category of right
\( A \)-modules in \( \mathcal{V} \). We usually omit the multiplication and unit from the notation
for an algebra and the action map from the notation for an \( A \)-module.

**Proposition 2.1.** Let \( \mathcal{V} \) be a closed, symmetric monoidal category. The category
\( \mathcal{V}_A \) of right \( A \)-modules is a \( \mathcal{V} \)-category.

**Proof.** For an \( A \)-module \( (M, \rho) \) and an object \( K \) in \( \mathcal{V} \), the objects in \( \mathcal{V} \) underlying
the tensor product \( K \otimes M, \rho) \) and the cotensor product \( (M, \rho)^K \) are the tensor and
cotensor products of the underlying objects in \( \mathcal{V} \). The right \( A \)-action on \( K \otimes M \) is
given by

\[
1 \otimes \rho: K \otimes M \otimes A \to K \otimes M,
\]

while the right action

\[
M^K \otimes A \to M^K
\]
is adjoint to the composite

\[
K \otimes M^K \otimes A \xrightarrow{\text{ev} \otimes 1} M \otimes A \xrightarrow{\rho} M.
\]

The forgetful functor \( \mathcal{U}: \mathcal{V}_A \to \mathcal{V} \) therefore preserves the tensor and cotensor struc-
tures.

Given two \( A \)-modules \( (M, \rho_M) \) and \( (N, \rho_N) \), the enrichment \( \text{Map}_A(M, N) \) is
defined in terms of an equalizer diagram,

\[
\begin{array}{ccc}
\text{Map}_A(M, N) & \xrightarrow{\epsilon} & \text{Map}(M, N) \\
\downarrow & & \downarrow \\
\text{Map}(M \otimes A, N)
\end{array}
\]
where the top map is induced by $\rho_M : M \otimes A \to M$ and the bottom map is the composite

$$\text{Map}(M, N) \xrightarrow{- \otimes A} \text{Map}(M \otimes A, N \otimes A) \xrightarrow{(\rho_N) \cdot} \text{Map}(M \otimes A, N).$$

It is an easy exercise, which we leave it to the reader, to check that these structures are compatible. □

The $\mathcal{V}$-structure described above interacts well with model category structure, when the monoidal and model category structures are appropriately compatible, e.g., if $\mathcal{V}$ is a monoidal model category [31, Definition 3.1]. Recall that if $\mathcal{M}$ and $\mathcal{N}$ are model categories, the model category structure on $\mathcal{N}$ is right-induced by an adjunction

$$\mathcal{M} \xrightarrow{L} \mathcal{N} \xleftarrow{R}$$

if the right adjoint $R$ preserves and reflects both weak equivalences and fibrations.

**Theorem 2.2.** [31] Let $\mathcal{V}$ be a symmetric monoidal model category. If $\mathcal{V}$ is cofibrantly generated and satisfies the monoid axiom, and every object of $\mathcal{V}$ is small relative to the whole category, then the category $\mathcal{V}_A$ of right $A$-modules admits a model structure that is right induced from the adjunction

$$\mathcal{V} \xrightarrow{- \otimes A} \mathcal{V}_A.$$

**Remark 2.3.** The forgetful functor $U : \mathcal{V}_A \to \mathcal{V}$ is a tensor functor, so it follows from Proposition [A.4] that, when it exists, the right-induced model structure on $\mathcal{V}_A$ is $\mathcal{V}$-structured.

**Remark 2.4.** It is, of course, also true that the category $\mathcal{V}_A$ of left $A$-modules admits a right-induced model category structure under the hypotheses of the theorem above, because $\mathcal{V}_A$ is isomorphic to the category $\mathcal{V}_{A^{op}}$ of right modules over the opposite algebra $A^{op}$.

**Convention 2.5.** Henceforth, we assume always that $\mathcal{V}$ is a symmetric monoidal model category such that the adjunction

$$\mathcal{V} \xrightarrow{- \otimes A} \mathcal{V}_A$$

right-induces a model category structure on $\mathcal{V}_A$, for every algebra $A$ in $\mathcal{V}$, and similarly for $\mathcal{V}_A$. Whenever we refer to weak equivalences, fibrations, or cofibrations of $A$-modules, we mean with respect to this right-induced structure.

As we illustrate in Section 5, the convention above holds in many monoidal model categories of interest.

We recall the following, classical construction, as it occurs throughout this article.

**Definition 2.6.** Given right and left $A$-modules $M_A$ and $A_N$, with structure maps $\rho : M \otimes A \to M$ and $\lambda : A \otimes N \to N$, their tensor product over $A$ is the object $M \otimes_A N$ in $\mathcal{V}$ defined by the following coequalizer diagram:

$$M \otimes A \otimes N \xrightarrow{\rho \otimes 1} M \otimes N \xrightarrow{1 \otimes \lambda} M \otimes_A N.$$
Definition 2.7. Let \( V \) be a symmetric monoidal model category satisfying Convention 2.5. A left \( A \)-module \( M \) is called

- **homotopy compact** if for every finite category \( J \) and every functor \( \Phi: J \to V_A \), the natural map
  \[
  (\lim_J \Phi) \otimes_A M \to \lim_J (\Phi \otimes_A M)
  \]
  is a weak equivalence in \( V \);
- **homotopy flat** if \( - \otimes_A M: V_A \to V \) preserves weak equivalences;
- **homotopy faithful** if \( - \otimes_A M: V_A \to V \) reflects weak equivalences;
- **homotopy faithfully flat** if it is both faithful and flat;
- **homotopy projective** if \( \text{Map}_A(M, -): A V \to V \) preserves weak equivalences;
- **homotopy cofaithful** if \( \text{Map}_A(M, -): A V \to V \) reflects weak equivalences.

There is, of course, an analogous definition for right \( A \)-modules.

2.2. Bimodules and Quillen adjunctions. In this section we characterize completely adjunctions between enriched module categories and provide criteria under which these adjunctions are Quillen pairs.

Let \((V, \otimes, k)\) be a symmetric monoidal model category satisfying Convention 2.5. Given algebras \( A \) and \( B \) in \( V \) and a bimodule \( A X B \), there is a \( V \)-adjunction
\[
V_A - \otimes_A X \to V_B \quad \text{and} \quad - \otimes_A X \dashv \text{Map}_B(X, -),
\]
where \( V_A \) and \( V_B \) are endowed with the \( V \)-structures of Proposition 2.1. Let us say that a \( V \)-adjunction,
\[
(\text{2.1})
\]
is governed by a bimodule \( A X B \), if the \( V \)-functors \( F \) and \( - \otimes_A X \) are isomorphic.

The following is an enriched version of the classical Eilenberg-Watts theorem [9].

Proposition 2.8. Let \( V \) be a symmetric monoidal category that admits all reflexive coequalizers. Every \( V \)-adjunction between \( V_A \) and \( V_B \) is governed by an \( A-B \)-bimodule \( X \).

Proof. Given a \( V \)-adjunction as in (2.1), let \( X = F(A) \). A priori, \( X \) is only a right \( B \)-module, but since \( F \) is a tensor functor we can endow \( X \) with a left \( A \)-action \( \lambda: A \otimes X \to X \), equal to the composite
\[
A \otimes X = A \otimes F(A) \xrightarrow{\alpha_{A,A}^{-1}} F(A \otimes A) \xrightarrow{\mu} F(A) = X,
\]
where \( \alpha_{K,M}: F(K \otimes M) \cong K \otimes F(M) \) is the natural isomorphism of Proposition A.1 and \( \mu: A \otimes A \to A \) is the multiplication map. We leave it to reader to check that \( X \) is indeed an \( A-B \)-bimodule when endowed with this left \( A \)-action.

For any right \( A \)-module \( M \) the canonical isomorphism \( M \otimes_A A \cong M \) may be expressed as a coequalizer diagram in \( V_A \):
\[
M \otimes A \xrightarrow{\rho \otimes 1} M \otimes A \xrightarrow{\rho \mu} M.
\]
Being a left adjoint, the functor $F$ takes this to a coequalizer diagram in $\mathcal{V}_B$, which is the top row in the commuting diagram below.

\[
\begin{array}{cccccc}
F(M \otimes A \otimes A) & \xrightarrow{F(\rho \otimes 1)} & F(M \otimes A) & \xrightarrow{F(\rho)} & F(M) \\
M \otimes A \otimes X & \xrightarrow{\rho \otimes 1} & M \otimes X & \xrightarrow{1 \otimes \lambda} & M \otimes_A X.
\end{array}
\]

The two left-hand squares commute because $\alpha$ is a natural transformation and, in the case of the square involving $F(1 \otimes \mu_A)$ and $1 \otimes \lambda_X$, because $\alpha_{M \otimes A, A} = (1 \otimes \alpha_{A, A}) \alpha_{M, A \otimes A}$.

The left and middle vertical maps are isomorphisms because $F$ is a tensor functor. The bottom row is the coequalizer that defines the tensor product $M \otimes A X$. The desired natural isomorphism $F(M) \cong M \otimes A X$ follows from the universal property of coequalizers.

When $\mathcal{V}$ is a monoidal model category, it is natural to ask when the $\mathcal{V}$-adjunction between $\mathcal{V}_A$ and $\mathcal{V}_B$ governed by a bimodule $A X B$ is a Quillen pair.

**Proposition 2.9.** Let $\mathcal{V}$ be a symmetric monoidal model category satisfying Convention 2.5. Let $A$ and $B$ be algebras in $\mathcal{V}$, and let $A X B$ be a bimodule. If $X$ is cofibrant as a right $B$-module, then the adjunction governed by $X$,

\[
\mathcal{V}_A \xrightarrow{- \otimes A X} \mathcal{V}_B ,
\]

is a Quillen adjunction. The converse holds if the unit $k$ is cofibrant in $\mathcal{V}$.

**Proof.** We have a commutative diagram of adjunctions

\[
\begin{array}{ccc}
\mathcal{V}_A & \xrightarrow{- \otimes A X} & \mathcal{V}_B \\
\mathcal{V} & \xrightarrow{- \otimes A} & \mathcal{V}_B \\
\mathcal{V} & \xrightarrow{- \otimes X} & \mathcal{V}_B \\
\end{array}
\]

Since the model structure on $\mathcal{V}_A$ is right induced from $\mathcal{V}$ via the vertical adjunction, the horizontal adjunction is a Quillen adjunction if and only if the diagonal one is. This in turn happens if and only if for every (trivial) cofibration $i: K \to L$ in $\mathcal{V}$, the induced map $i \otimes 1: K \otimes X \to L \otimes X$ is a (trivial) cofibration in $\mathcal{V}_B$. If $X$ is $B$-cofibrant, this condition is satisfied since $\mathcal{V}_B$ satisfies Axiom A.3. Conversely, if the unit $k$ in $\mathcal{V}$ is cofibrant, then $X \cong k \otimes X$ must be cofibrant as a right $B$-module.

**Example 2.10.** A morphism of algebras $\varphi: A \to B$ in $\mathcal{V}$ induces a natural $A$-$B$-bimodule structure on $B$. Since $\text{Map}_B(B, -) = \varphi^*: \mathcal{V}_B \to \mathcal{V}_A$, the restriction-of-coefficients functor, adjunction (2.2) for $X = B$ is exactly the extension/restriction-of-coefficients adjunction

\[
\mathcal{V}_A \xrightarrow{- \otimes A B} \mathcal{V}_B .
\]

The right adjoint $\varphi^*$ preserves and reflects weak equivalences and fibrations, because these are detected in the underlying category $\mathcal{V}$ by Convention 2.5, so $\varphi^*$ is always a right Quillen functor.
Example 2.11. The restriction of scalars functor \( \varphi^* \) also admits a right adjoint,

\[
\begin{array}{ccc}
\mathcal{V}_B & \xrightarrow{\varphi^*} & \mathcal{V}_A \\
\text{Map}_A(B, -) & \xleftarrow{} & \end{array}
\]

This adjunction is governed by the bimodule \( _B B_A \), where \( A \) acts on the right via \( \varphi \).

In particular, if \( B \) is cofibrant as a right \( A \)-module, then \( \varphi^* \) is a left Quillen functor by Proposition 2.9. As a special case, note that the category \( \mathcal{V} \) may be identified with the category \( \mathcal{V}_k \) of right \( k \)-modules.

The unit map \( \eta_A: k \to A \) governs the forgetful functor \( \eta^*_A: \mathcal{V}_A \to \mathcal{V} \).

In particular, if \( A \) is cofibrant as an object of \( \mathcal{V} \), then all cofibrant \( A \)-modules are also cofibrant as objects of \( \mathcal{V} \).

2.3. Dualizable bimodules and Quillen equivalences. We now address the question of when the Quillen adjunction governed by a bimodule is a Quillen equivalence.

We begin by analyzing when the restriction-of-scalars adjunction associated to a morphism of algebras induces a Quillen equivalence. To this end, we introduce the concept of a pure weak equivalence.

Definition 2.12. A morphism of left \( A \)-modules \( f: N \to N' \) is a pure weak equivalence if the induced map \( 1 \otimes f: M \otimes_A N \to M \otimes_A N' \) \( \forall \) is a weak equivalence for all cofibrant right \( A \)-modules \( M \).

Under reasonable conditions, all weak equivalences are pure.

Definition 2.13. We say that \( \mathcal{V} \) satisfies the CHF hypothesis if for every algebra \( A \) in \( \mathcal{V} \), every cofibrant right \( A \)-module is homotopy flat (cf. Definition 2.7).

As pointed out in \[31, \S 4\], the CHF hypothesis holds in many monoidal model categories of interest, such as the categories of simplicial sets, symmetric spectra, (bounded or unbounded) chain complexes over a commutative ring, and \( S \)-modules.

Proposition 2.14. Let \( \mathcal{V} \) be a symmetric monoidal model category satisfying Convention 2.5. If \( \mathcal{V} \) satisfies the CHF hypothesis, then the notions of pure weak equivalence and weak equivalence coincide.

Proof. If all cofibrant right \( A \)-modules are homotopy flat, then clearly every weak equivalence is pure. Conversely, let \( f: N \to N' \) be a pure weak equivalence. We may without loss of generality assume that \( N \) and \( N' \) are cofibrant. Indeed, by standard model category theory, we can find cofibrant resolutions \( q_N: QN \to N \) and \( q_{N'}: QN' \to N' \) and a lift \( Qf: QN \to QN' \) making the diagram

\[
\begin{array}{ccc}
QN & \xrightarrow{Qf} & QN' \\
\sim & \downarrow & \sim \\
N & \xrightarrow{f} & N'
\end{array}
\]

commute. Clearly, \( f \) is a weak equivalence if and only if \( Qf \) is. By tensoring the diagram above from the left with cofibrant (hence homotopy flat) right \( A \)-modules, one sees that \( Qf \) is a pure weak equivalence.

Assume now that \( f: N \to N' \) is a pure weak equivalence between cofibrant, and hence homotopy flat, \( A \)-modules. If \( q_A: QA \to A \) is a cofibrant resolution of \( A \) as a right \( A \)-module, then the commutative diagram

\[
\begin{array}{ccc}
QA \otimes_A N & \xrightarrow{QA \otimes_A f} & QA \otimes_A N' \\
\sim & \downarrow & \sim \\
N & \xrightarrow{f} & N'
\end{array}
\]

is a lift. Clearly, \( f \) is a weak equivalence if and only if \( QA \otimes_A f \) is.
shows that $f$ is a weak equivalence. Indeed, the top horizontal map is a weak equivalence because $f$ is a pure weak equivalence, and the vertical maps are weak equivalences because $N$ and $N'$ are homotopy flat.

The following result is a slight strengthening of [31, Theorem 4.3].

**Proposition 2.15.** Let $\mathcal{V}$ be a symmetric monoidal model category satisfying Convention 2.5, and let $\varphi: A \to B$ be a morphism of algebras in $\mathcal{V}$. The restriction/-extension-of-scalars adjunction,

\[
\mathcal{V}_A \xrightarrow{\sim \otimes_A B} \mathcal{V}_B,
\]

is a Quillen equivalence if and only if $\varphi: A \to B$ is a pure weak equivalence of right $A$-modules.

**Remark 2.16.** If all cofibrant modules are homotopy flat, then pure weak equivalences are the same as weak equivalences by Proposition 2.14, so the “if” direction of the above proposition recovers [31, Theorem 4.3].

**Proof of Proposition 2.15.** Since fibrations and weak equivalences are created in the underlying category $\mathcal{V}$ (by Convention 2.5), it is clear that the restriction-of-scalars functor $\varphi^*$ is a right Quillen functor and that it preserves and reflects all weak equivalences. Therefore, the adjunction is a Quillen equivalence if and only if the unit

\[
\eta_M: M \to \varphi^*(M \otimes_A B)
\]

is a weak equivalence for all cofibrant right $A$-modules $M$ [13, Corollary 1.3.16]. To conclude, note that the morphism in $\mathcal{V}$ underlying $\eta_M$ may be identified with $1 \otimes_A \varphi: M \otimes_A A \to M \otimes_A B$.

We now turn to the question of when the Quillen adjunction governed by a bimodule $A \otimes_B Y$ induces a Quillen equivalence between $\mathcal{V}_A$ and $\mathcal{V}_B$.

**Definition 2.17.** A bimodule $A \otimes_B Y$ is called right dualizable if there exists a bimodule $B \otimes_A X$ together with morphisms

\[
u: A \to X \otimes_B Y, \quad e: Y \otimes_A X \to B,
\]

in $\mathcal{V}_A$ and $\mathcal{V}_B$, respectively, such that the composites

\[
X \xrightarrow{\nu \otimes 1} X \otimes_B Y \otimes_A X \xrightarrow{1 \otimes e} X, \quad Y \xrightarrow{1 \otimes \nu} Y \otimes_A X \otimes_B Y \xrightarrow{e \otimes 1} Y,
\]

are the identity maps on $X$ and $Y$, respectively.

For a right $B$-module $N$, let

\[
\ell_N: N \otimes_B \text{Map}_B(X, B) \to \text{Map}_B(X, N)
\]

be the map of right $A$-modules that is right adjoint to the map

\[
N \otimes_B \text{Map}_B(X, B) \otimes_A X \xrightarrow{1 \otimes \text{ev}} N \otimes_B B \cong N,
\]

induced by the evaluation map $\text{ev}: \text{Map}_B(X, B) \otimes_A X \to B$. Recall that a bimodule $A \otimes_B X$ is right dualizable if and only if the map

\[
\ell_N: N \otimes_B \text{Map}_B(X, B) \to \text{Map}_B(X, N)
\]

is an isomorphism for all right $B$-modules $N$. In this case, every right dual $B \otimes_A X$ is isomorphic to $\text{Map}_B(X, B)$ as a $B$-$A$-bimodule (Lemma B.9).

**Example 2.18.** It is easy to prove that if $\mathcal{V}$ is the category of abelian groups, then a bimodule $A \otimes_B X$ is right dualizable if and only if it is finitely generated and projective as a right $B$-module.
To formulate the homotopical version of the Morita theorem, we will need a homotopical version of dualizability.

**Definition 2.19.** Let $A \otimes B$ be a bimodule that is fibrant and cofibrant as a right $B$-module. We call $A \otimes B$ homotopy right dualizable if the natural map

$$\ell_N: N \otimes_B \text{Map}_B(X, B) \to \text{Map}_B(X, N)$$

is a weak equivalence for all fibrant and cofibrant right $B$-modules $N$.

**Remark 2.20.** If $\mathcal{V}$ is a stable model category, then $X$ is homotopy right dualizable if and only if it is compact as an object of the triangulated homotopy category; see, e.g., [24, Theorem A.1]. For example, compact objects in the derived category of a ring correspond to perfect complexes, i.e., bounded complexes of finitely generated projective modules.

**Remark 2.21.** It is possible to express the notion of homotopy dualizability in terms of the left derived functors of the tensor product. However, unless additional hypotheses are imposed, e.g., that $A$ and $B$ are homotopy flat as objects of $\mathcal{V}$, then the derived tensor product of bimodules need not be balanced nor associative.

**Proposition 2.22.** Let $\mathcal{V}$ satisfy Convention [2.5]. If $\mathcal{V}$ also satisfies the CHF hypothesis, then for every bimodule $A \otimes B$, the functor $- \otimes_A X: \mathcal{V}_A \to \mathcal{V}_B$ preserves weak equivalences between homotopy flat right $A$-modules. In particular, it preserves weak equivalences between cofibrant right $A$-modules.

**Proof.** Let $f: N \to N'$ be a weak equivalence between homotopy flat right $A$-modules. If $q: QX \to X$ is a cofibrant resolution of $X$ as a left $A$-module, then in the diagram

$$\begin{array}{ccc}
N \otimes A QX & \overset{f \otimes 1}{\longrightarrow} & N' \otimes A QX \\
\downarrow 1 \circ q & & \downarrow 1 \circ q \\
N \otimes A X & \overset{f \otimes 1}{\longrightarrow} & N' \otimes A X,
\end{array}$$

the vertical maps are weak equivalences as $N$ and $N'$ are homotopy flat. The top horizontal map is a weak equivalence because $QX$ is cofibrant, whence homotopy flat. It follows that the bottom horizontal map is a weak equivalence. \hfill \square

We are now prepared to formulate and prove our homotopical analogue of the classical Morita theorem.

**Theorem 2.23** (Homotopical Morita theorem). Let $\mathcal{V}$ satisfy Convention [2.5] and the CHF hypothesis. Let $A$ and $B$ algebras in $\mathcal{V}$, and let $A \otimes B$ be a bimodule. If $B$ is fibrant in $\mathcal{V}$, and $X$ is fibrant and cofibrant as a right $B$-module, then the Quillen adjunction

$$\mathcal{V}_A \quad \quad \overset{- \otimes_A X}{\longrightarrow} \quad \mathcal{V}_B$$

is a Quillen equivalence if and only if

1. the map $\eta_A: A \to \text{Map}_B(X, X)$ is a weak equivalence;
2. the bimodule $X$ is homotopy cofaithful as a right $B$-module, i.e., the functor $\text{Map}_B(X, -)$ reflects weak equivalences between fibrant objects; and
3. the bimodule $X$ is homotopy right dualizable, i.e., the canonical map

$$\ell_N: N \otimes_B \text{Map}_B(X, B) \to \text{Map}_B(X, N)$$

is a weak equivalence for all fibrant and cofibrant right $B$-modules $N$. 


Remark 2.24. The theorem above recovers the classical Morita theorem. If $A$ and $B$ are ordinary associative unital rings, then a bimodule $A\times B$ induces an equivalence between $\mathcal{V}_A$ and $\mathcal{V}_B$ if and only if $A \cong \text{Hom}_B(X, X)$ and the right $B$-module $X$ is a finitely generated projective generator. A right $B$-module $X$ is finitely generated and projective if and only if it is right dualizable, and it is a generator if and only if the functor $\text{Hom}_B(X, -)$ reflects isomorphisms.

Remark 2.25. The fibrancy hypotheses on $B$ and $X$ are not essential. They may be removed at the expense of taking derived mapping spaces in the hypotheses.

Proof. Suppose first that conditions (1), (2) and (3) are fulfilled. Since the right $\mathcal{V}_B$-adjoint $\text{Map}_B(X, -)$ reflects weak equivalences between fibrant objects, we need only to show that the homotopy unit $\tilde{\eta}_M$ is a weak equivalence for all cofibrant objects $M$ in $\mathcal{V}_A$ [10, Corollary 1.3.16]. Let $r: M \otimes_A X \to (M \otimes_A X)^f$ be a fibrant replacement in $\mathcal{V}_B$, with $r$ a trivial cofibration, and consider the following commutative diagram in $\mathcal{V}_B$:

\[
\begin{array}{ccc}
M \otimes_A A & \overset{\text{Map}_B(X, X)}{\longrightarrow} & M \otimes_A \text{Map}_B(X, X) \\
\downarrow^{\eta_M} & & \downarrow^{\ell_M} \\
\text{Map}_B(X, (M \otimes_A X)^f) & \overset{\ell_{(M \otimes_A X)^f}}{\longrightarrow} & (M \otimes_A X)^f \otimes_B \text{Map}_B(X, B) \\
\end{array}
\]

The maps labeled $(a), (b), (c), (d)$ are weak equivalences, for the following reasons.

(a) By our hypothesis (1), the map $\eta_A$ is a weak equivalence. Since the right $A$-module $M$ is assumed to be cofibrant, it is also homotopy flat.

(b) Since $X$ is fibrant and cofibrant in $\mathcal{V}_B$, the map $\ell_X$ is a weak equivalence by (3). As we pointed out above, $M_A$ is homotopy flat.

(c) Since $M$ is cofibrant, and $- \otimes_A X$ is a left Quillen functor (Proposition 2.22), $M \otimes_A X$ is cofibrant in $\mathcal{V}_B$. Since $r$ is a weak equivalence and a cofibration with cofibrant source, it is in particular a weak equivalence between two cofibrant objects. Since we assume that all cofibrant modules are homotopy flat, it follows from Proposition 2.22 that $r \otimes 1$ is a weak equivalence.

(d) The right $B$-module $(M \otimes_A X)^f$ is fibrant and cofibrant, so the map $\ell_{(M \otimes_A X)^f}$ is a weak equivalence by (3).

It follows that $\tilde{\eta}_M$ is a weak equivalence.

Conversely, suppose that $\tilde{\eta}_M$ is a Quillen equivalence. Then clearly the right adjoint $\text{Map}_B(X, -)$ reflects weak equivalences between fibrant objects, i.e., (2) holds.

Moreover, even though $A$ is not necessarily cofibrant as a right $A$-module, the map $\eta_A$ represents the homotopy unit for $A \in \text{Ho} \mathcal{V}_A$, because $A$ is homotopy flat, and $X$ is fibrant. Indeed, if $q: QA \rightarrow A$ is a cofibrant replacement of $A$ in $\mathcal{V}_A$, then $q$ is a weak equivalence between homotopy flat objects, so $q \otimes_A X: QA \otimes_A X \to X$ is a weak equivalence by Proposition 2.22. It follows that we may take $X$ as a fibrant replacement of $QA \otimes_A X$, whence the diagonal map $\tilde{\eta}_QA$ in the commutative diagram

\[
\begin{array}{ccc}
QA & \overset{\text{Map}_B(X, QA \otimes_A X)}{\longrightarrow} & \text{Map}_B(X, X) \\
\downarrow^{q} & & \downarrow^{(q \otimes_A X)_*} \\
A & \overset{\eta_A}{\longrightarrow} & \text{Map}_B(X, X). \\
\end{array}
\]
is a weak equivalence because it represents the homotopy unit for \(QA\). Condition (1) follows immediately.

Finally, we check condition (3). Let \(N\) be a fibrant right \(B\)-module,
\[
p_N: \operatorname{Map}_B(X, N)^c \to \operatorname{Map}_B(X, N)
\]
a cofibrant replacement in \(\mathcal{T}_A\), and
\[
q: \overset{\circ}{c}X \to X
\]
a cofibrant replacement in \(A\mathcal{V}\). Consider the following commutative diagram in \(\mathcal{V}\), where the maps labeled by \(\sim\) are weak equivalences because cofibrant left or right \(A\)-modules are homotopy flat.

The map \(\bar{\varepsilon}_N\) is a map of right \(B\)-modules and represents the homotopy counit for \(N\), so it is a weak equivalence because \([2.4]\) is a Quillen equivalence. It follows that the map \(f_N\) is a weak equivalence for every fibrant right \(B\)-module \(N\). In particular, since \(B\) is fibrant, the map \(f_B\) is a weak equivalence. Note moreover that \(f_B\) is a map of left \(B\)-modules.

Next, let \(N\) be a right \(B\)-module that is both fibrant and cofibrant, and consider the following commutative diagram in \(\mathcal{V}\).

The composite of the top horizontal maps is equal to \(N \otimes_B f_B\). As we just noted, \(f_B\) is a weak equivalence. Since \(N_B\) is cofibrant, hence homotopy flat, \(N \otimes_B f_B\) is a weak equivalence. On the other hand, the composite of the bottom horizontal maps is equal to \(f_N\), which is a weak equivalence by the above, since \(N\) is fibrant. We deduce that the left vertical map \(\ell_N \otimes_A \overset{\circ}{c}X\) is a weak equivalence. Since \([2.4]\) is a Quillen equivalence, the left Quillen functor \(- \otimes_A \overset{\circ}{c}X\) reflects weak equivalences between cofibrant right \(A\)-modules. It follows that \(- \otimes_A \overset{\circ}{c}X\) reflects weak equivalences between any right \(A\)-modules. (Indeed, if \(g: M \to M'\) is a map in \(\mathcal{T}_A\), then one can take a cofibrant replacement \(Qg: QM \to QM'\) in \(\mathcal{V}\) and argue using the commutative diagram

observing that \(Qg\) is a weak equivalence if and only if \(g\) is.) Thus, \(\ell_N\) is a weak equivalence. \(\square\)
3. Corings and Braided Bimodules

Our goal in this section is to study and classify adjunctions between categories of comodules over corings in symmetric monoidal categories admitting appropriate limits and colimits. We begin by recalling the elementary theory of corings and their comodules, then introduce the notion of braided bimodules and show that every adjunction between categories of comodules over corings, relative to a fixed adjunction between the underlying module categories, is governed by a braided bimodule.

Throughout this section, \((\mathcal{V}, \otimes, k)\) denotes a symmetric monoidal category that admits all reflexive coequalizers and coreflexive equalizers.

3.1. Corings and their Comodules. If \(A\) is an algebra in \(\mathcal{V}\), then the tensor product \(- \otimes A-\) endows the category of \(A\)-bimodules \(A\mathcal{V}A\) with a (not necessarily symmetric) monoidal structure. The unit is \(A\), viewed as an \(A\)-bimodule over itself.

**Definition 3.1.** An \(A\)-coring is a coalgebra in the monoidal category \((A\mathcal{V}A, \otimes, A)\), i.e., an \(A\)-bimodule \(C\) together with maps of \(A\)-bimodules \(\Delta_C: C \to C \otimes_A C\) and \(\epsilon_C: C \to A\), such that the diagrams are commutative. Here, we tacitly make the identifications \(A \otimes_A C = C = C \otimes_A A\) in the lower right corner. A morphism of \(A\)-corings is a map of \(A\)-bimodules \(f: C \to D\) such that the diagrams commute.

We need to allow morphisms between corings to change the algebra as well. To this end, note that if \(\varphi: A \to B\) is a morphism of algebras, then there is an extension/restriction-of-scalars adjunction, 

\[ A\mathcal{V} \xrightarrow{\varphi} B\mathcal{V} \quad \varphi^* \dashv \varphi^*, \]

where \(\varphi^*(M) = B \otimes_A M \otimes_A B\). Moreover, \(\varphi^*\) is an op-monoidal functor, i.e., there is a natural transformation

\[ \varphi^*(M \otimes_A N) \to \varphi^*(M) \otimes_B \varphi^*(N), \]

which allows us to endow \(\varphi^*(C)\) with the structure of a \(B\)-coring whenever \(C\) is an \(A\)-coring.

**Definition 3.2.** A coring in \(\mathcal{V}\) is a pair \((A, C)\) where \(A\) is an algebra in \(\mathcal{V}\) and \(C\) is an \(A\)-coring. A morphism of corings \((A, C) \to (B, D)\) is a pair \((\varphi, f)\) where \(\varphi: A \to B\) is a morphism of algebras and \(f: \varphi^*(C) \to D\) is a morphism of \(B\)-corings. The category of corings in \(\mathcal{V}\) is denoted \(\text{Coring}_{\mathcal{V}}\).
There is no natural $A$-coring structure on $\varphi^*(D)$ in general, but if we let $f^\sharp: C \to \varphi^*(D)$ denote the adjoint of $f: \varphi_*(C) \to D$, then the condition that $f$ is a morphism of $B$-corings is equivalent saying that the diagrams $A$-bimodules

\begin{equation}
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes_A C \\
\downarrow{f^\flat} & & \downarrow{f^\flat} \\
D & \xrightarrow{\Delta_B} & D \otimes_B D
\end{array}
\quad
\begin{array}{ccc}
C & \xrightarrow{\epsilon_C} & A \\
\downarrow{f^\flat \otimes 1} & & \downarrow{\varphi} \\
D & \xrightarrow{\epsilon_D} & B
\end{array}
\end{equation}

are commutative.

**Definition 3.3.** Let $(A, C)$ be a coring in $\mathcal{V}$. A right $(A, C)$-comodule is a right $A$-module $M$ together with a morphism of right $A$-modules $\delta_M: M \to M \otimes_A C$ such that the diagrams

\begin{equation}
\begin{array}{ccc}
M & \xrightarrow{\delta_M} & M \otimes_A C \\
\downarrow{\delta_M} & & \downarrow{1 \otimes \Delta_C} \\
M \otimes_A C & \xrightarrow{\delta_M \otimes 1} & M \otimes_A C \otimes_A C
\end{array}
\end{equation}

are commutative. A *morphism of $(A, C)$-comodules* is a morphism $f: M \to N$ of right $A$-modules such that the diagram

\begin{equation}
\begin{array}{ccc}
M & \xrightarrow{\delta_M} & M \otimes_A C \\
\downarrow{f} & & \downarrow{f \otimes 1} \\
N & \xrightarrow{\delta_N} & N \otimes_A C
\end{array}
\end{equation}

commutes.

We let $\mathcal{V}_A^C$ denote the category of right $(A, C)$-comodules. There is an adjunction

$$
\mathcal{V}_A^C \xrightarrow{\mathcal{U}_A} \mathcal{V}_A, \quad \mathcal{U}_A \dashv \otimes_A C,
$$

where $\mathcal{U}_A$ is the forgetful functor. The category $\mathcal{V}_A^C$ of left $(A, C)$-comodules is defined analogously.

Next, we will give some examples of corings.

**Example 3.4** (Trivial coring). For every algebra $A$, we can form the trivial coring $(A, A)$. The comultiplication $A \to A \otimes_A A$ is the natural isomorphism and the counit $A \to A$ is the identity map. The forgetful functor $\mathcal{U}_A: \mathcal{V}_A^C \to \mathcal{V}_A$ is an isomorphism of categories.

**Example 3.5** (Coalgebras). For every coalgebra $C$ in $\mathcal{V}$, there is a coring $(k, C)$. The category $\mathcal{V}_k^C$ is isomorphic to the category of comodules over $C$.

These examples show that the study of comodules over corings englobes the study of modules over algebras and comodules over coalgebras.

**Example 3.6** (Descent coring). Let $\varphi: A \to B$ be a morphism of algebras in $\mathcal{V}$. The *descent coring* is the coring $(B, B \otimes_A B)$, where the comultiplication is the composite

$$
B \otimes_A B \cong B \otimes_A A \otimes_A B \xrightarrow{1 \otimes \varphi \otimes 1} B \otimes_A B \otimes_A B \cong (B \otimes_A B) \otimes_B (B \otimes_A B),
$$

and the counit $B \otimes_A B \to B$ is induced by the multiplication in $B$.

There is a morphism of corings $(\varphi, f): (A, A) \to (B, B \otimes_A B)$, where $f$ is the identity map on $B \otimes_A B$. 

Further important examples of corings arise in the theory of Hopf-Galois extensions (see [12]. Also, every Hopf algebroid gives rise to a coring by forgetting some structure (see [5]).

**Proposition 3.7.** Let \( \mathcal{V} \) be a closed, symmetric monoidal category that admits all reflexive coequalizers and coreflexive equalizers. The category \( \mathcal{V}_A^C \) of \((A, C)\)-comodules is a \( \mathcal{V} \)-category if it admits all coreflexive equalizers.

**Remark 3.8.** If \( \mathcal{V} \) is locally presentable, then \( \mathcal{V}_A \) is locally presentable, as it is the category of algebras for the monad on \( \mathcal{V} \) with underlying functor \(- \otimes A\) [1, 2.78]. The implies in turn that \( \mathcal{V}_A^C \) is locally presentable, and therefore complete, since it is the category of coalgebras for the comonad on \( \mathcal{V}_A \) with underlying functor \(- \otimes A C\) [6, Proposition A.1]. In particular, if \( \mathcal{V} \) is locally presentable, then \( \mathcal{V}_A^C \) admits all coreflexive equalizers.

On the other hand, by the dual of [22, Corollary 3], if \(- \otimes A C : \mathcal{V}_A \to \mathcal{V}_A \) preserves coreflexive equalizers, then \( \mathcal{V}_A^C \) admits all coreflexive equalizers.

**Proof.** For \( K \in \mathcal{V} \) and \( M \in \mathcal{V}_A^C \), the tensor product \( K \otimes M \in \mathcal{V}_A^C \) is defined as the tensor product of the underlying objects in \( \mathcal{V} \), together with the evident right \( A \)-module and \( C \)-comodule structures.

That \( \mathcal{V}_A^C \) is cotensored over \( \mathcal{V} \) is ensured by the (dual) Adjoint Lifting Theorem [4, §4.5], which we can apply to the diagram:

\[
\begin{array}{ccc}
\mathcal{V}_A^C & \xrightarrow{K \otimes -} & \mathcal{V}_A^C \\
\downarrow & & \downarrow \\
\mathcal{V}_A & \xrightarrow{(-)^K} & \mathcal{V}_A \\
\end{array}
\]

because \( \mathcal{U}_A \) and \( \mathcal{U}_B \) are comonadic, and \( \mathcal{V}_A^C \) admits coreflexive equalizers by hypothesis.

Explicitly, the cotensor product \( M^K \) can be defined as the equalizer of the following diagram in \( \mathcal{V}_A^C \):

\[
\text{Map}_\mathcal{V}(K, M) \otimes_A C \rightrightarrows \text{Map}_\mathcal{V}(K, M \otimes_A C) \otimes_A C.
\]

The top map is induced by \( \Delta_M \) and the bottom map is given by

\[
\text{Map}_\mathcal{V}(K, M) \otimes_A C \xrightarrow{\text{id} \otimes \Delta_C} \text{Map}_\mathcal{V}(K, M) \otimes_A C \otimes_A C \xrightarrow{\nu \otimes 1} \text{Map}_\mathcal{V}(K, M \otimes_A C) \otimes_A C,
\]

where \( \nu : \text{Map}_\mathcal{V}(K, M) \otimes_A C \to \text{Map}_\mathcal{V}(K, M \otimes_A C) \) is adjoint to the map

\[
K \otimes \text{Map}_\mathcal{V}(K, M) \otimes_A C \xrightarrow{\text{ev} \otimes 1} M \otimes_A C.
\]

Similarly, existence of the \( \mathcal{V} \)-enrichment \( \text{Map}_\mathcal{V}^C(M, N) \in \mathcal{V} \) is ensured by applying the (dual) Adjoint Lifting Theorem to the following diagram:

\[
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{- \otimes M} & \mathcal{V}_A^C \\
\downarrow & \swarrow \text{Map}_\mathcal{V}^C(M, -) & \downarrow \mathcal{U}_A \\
\mathcal{V} & \xrightarrow{\text{Map}_\mathcal{V}(M, -)} & \mathcal{V}_A \\
\end{array}
\]
Explicitly, \( \text{Map}^{C}(M, N) \) is the equalizer of the following diagram in \( V \):

\[
\text{Map}_{A}(M, N) \xrightarrow{\otimes_{A}C} \text{Map}_{A}(M \otimes_{A} C, N \otimes_{A} C) \xrightarrow{\Delta_{M}^{*}} \text{Map}_{A}(M, N \otimes_{A} C).
\]

The top map is \((\Delta_{N})_{*} : \text{Map}_{A}(M, N) \to \text{Map}_{A}(M, N \otimes_{A} C)\), and the bottom map is the composite

\[
\text{Map}_{A}(M, N) \xrightarrow{\otimes_{A}C} \text{Map}_{A}(M \otimes_{A} C, N \otimes_{A} C) \xrightarrow{\Delta_{M}^{*}} \text{Map}_{A}(M, N \otimes_{A} C).
\]

\[\square\]

**Remark 3.9.** The forgetful functor \( U_{A} \) is clearly a tensor functor, so the adjunction

\[
\text{Map}^{C}_{A} \rightleftharpoons U_{A} : V_{C} \to V_{A} \leftarrow U_{B} \rightleftharpoons \text{Map}^{D}_{B}
\]

is a \( V \)-adjunction with respect to the structures defined in the proof above, by Proposition A.1.

### 3.2. Braided bimodules.

By Proposition 2.8, every \( V \)-adjunction \((F, G)\) between \( V_{A} \) and \( V_{B} \) is governed by a bimodule \( A \times_{B} \). Our next goal is to investigate what extra structure on \( X \) is needed to lift \((F, G)\) to a \( V \)-adjunction \((\tilde{F}, \tilde{G})\) between \( V_{C} \) and \( V_{D} \), such that the diagram of left adjoints

\[
\text{(3.2)}
\]

commutes up to natural isomorphism. To this end, we introduce the notion of a braided bimodule.

**Definition 3.10.** Let \((A, C)\) and \((B, D)\) be corings in a monoidal category \( V \). A **braided** \((A, C)-(B, D)\)-bimodule is a pair \((X, T)\) where \( X \) is an \( A-B \)-bimodule and \( T \) is a morphism of \( A-B \)-bimodules

\[
T : C \otimes_{A} X \to X \otimes_{B} D
\]

satisfying the following axioms.

**Pentagon axiom**

The diagram

\[
\text{(3.3)}
\]

commutes.

**Counit axiom**

The diagram

\[
\text{(3.4)}
\]
commutes.

A morphism of braided \((A,C)-(B,D)\)-bimodules \((X,T) \rightarrow (X',T')\) is a morphism of bimodules \(f: A_XB \rightarrow A'_{X'}B\) such that the diagram

\[
\begin{array}{ccc}
C \otimes_A X & \xrightarrow{T} & X \otimes_B D \\
\downarrow 1 \otimes f & & \downarrow f \otimes 1 \\
C \otimes_A X' & \xrightarrow{T'} & X' \otimes_B D
\end{array}
\]

commutes.

**Notation 3.11.** We write \((X,T): (A,C) \rightarrow (B,D)\) to indicate that \((X,T)\) is a braided \((A,C)-(B,D)\)-bimodule.

**Remark 3.12.** The notion of a braided bimodule does not seem to have appeared in the literature before, but it is related to the notion of an entwining structure (see [5, §32]). Indeed, \((A,C)_\psi\) is an entwining structure if and only if \(X = \varepsilon A_k\) and \(T = \psi: C \otimes A \rightarrow A \otimes C\) define a braided bimodule from the coring \((k,C)\) to itself such that \(\psi\) is a morphism of right \(A\)-modules and \(\psi \circ (1 \otimes \eta_A) = \eta_A \otimes 1\).

Just as one may form the bicategory \(\text{ALG}_\mathcal{Y}\) of algebras and bimodules, we may define a bicategory \(\text{CORING}_\mathcal{Y}\) of corings and braided bimodules.

**Definition 3.13.** The bicategory \(\text{CORING}_\mathcal{Y}\) has as objects corings \((A,C)\) in \(\mathcal{Y}\). A 1-morphism from \((A,C)\) to \((B,D)\) is a braided bimodule \((X,T): (A,C) \rightarrow (B,D)\), while a 2-morphism is a morphism of braided bimodules, as in Definition 3.10.

The composition of 1-morphisms is given by tensoring, i.e., the composite of \((X,T): (A,C) \rightarrow (A',C')\) and \((X',T'): (A',C') \rightarrow (A'',C'')\) is the braided bimodule

\[
(X \otimes_{A'} X', (1 \otimes T')(T \otimes 1)) : (A,C) \rightarrow (A'',C'').
\]

The composition of 2-morphisms is simply the usual composition of morphisms of \(A\)-\(B\)-bimodules.

It is a straightforward exercise to prove first that \((X \otimes_{A'} X', (1 \otimes T')(T \otimes 1))\) is indeed a braided bimodule and then that \(\text{CORING}_\mathcal{Y}\) does satisfy the axioms of a bicategory. Note that forgetting the corings and braidings defines a bifunctor

\[
\text{CORING}_\mathcal{Y} \rightarrow \text{ALG}_\mathcal{Y}
\]

(cf. Example [15.7]).

Given an algebra \(A\), the trivial coring is \(A\) itself, with structure maps the natural isomorphisms. Every bimodule \(A_XB\) may be viewed as a braided bimodule between the trivial corings \((X,T): (A,A) \rightarrow (B,B)\), where the braiding is the natural isomorphism \(T: A \otimes_A X \cong X \otimes_B B\). This defines a bifunctor \(\text{ALG}_\mathcal{Y} \rightarrow \text{CORING}_\mathcal{Y}\), which is a section of the bifunctor [5.6].

**Remark 3.14.** If \(C\) is the trivial \(A\)-coring \(C = A\) itself, then a braided bimodule \((X,T): (A,A) \rightarrow (B,D)\) is the same thing as an \(A\)-\(B\) bimodule \(X\) together with a compatible right \(D\)-comodule structure on \(X\).

**Proposition 3.15.** Let \(\mathcal{Y}\) be a closed, symmetric monoidal category that admits all reflexive coequalizers and coreflexive equalizers. If \((A,C)\) is a coring in \(\mathcal{Y}\) such that \(\mathcal{Y}_A^C\) admits all coreflexive equalizers, then every braided bimodule \((X,T): (A,C) \rightarrow (B,D)\) gives rise to a \(\mathcal{Y}\)-adjunction

\[
\begin{array}{ccc}
\mathcal{Y}_A^C & \xrightarrow{T_*} & \mathcal{Y}_B^D \\
\xrightarrow{T^*} & & \\
\mathcal{Y}_B^D & \xrightarrow{T_\ast} & \mathcal{Y}_A^C
\end{array}
\]
such that the following diagram of left adjoints commutes.

\[
\begin{array}{ccc}
\mathcal{Y}^C_A & \xrightarrow{T_*} & \mathcal{Y}^D_B \\
\downarrow{\mu_A} & & \downarrow{\mu_B} \\
\mathcal{Y}^A_A & \xrightarrow{\otimes_A X} & \mathcal{Y}^B_B.
\end{array}
\]

**Proof.** Given \( M \in \mathcal{Y}^T_B \), the braiding \( T \) allows us to define a right \( D \)-comodule structure on the right \( B \)-module \( M \otimes_A X \) to be the composite

\[(3.6) \quad M \otimes_A X \xrightarrow{\delta_M \otimes 1} M \otimes_A C \otimes_A X \xrightarrow{1 \otimes T} M \otimes_A X \otimes_B D.\]

Axioms (3.3) and (3.4) ensure that this morphism endows \( M \otimes_A X \) with the structure of a \( D \)-comodule. More precisely, the commutativity of the diagram

\[
\begin{array}{ccc}
M \otimes_A X \xrightarrow{\delta_M \otimes 1} M \otimes_A C \otimes_A X & \xrightarrow{1 \otimes T} & M \otimes_A X \otimes_B D \\
\downarrow{\delta_M \otimes 1} & & \downarrow{1 \otimes \Delta_B \otimes 1} \\
M \otimes_A C \otimes_A X & \xrightarrow{1 \otimes \Delta_C \otimes 1} & (M \otimes_A C) \otimes_A C \otimes_A X \xrightarrow{M \otimes_A (3.3)} M \otimes_A X \xrightarrow{1 \otimes \Delta_D} M \otimes_A X \otimes_B D \otimes_B D
\end{array}
\]

implies that the \( D \)-coaction \((3.6)\) is coassociative if and only if diagram \((3.3)\) commutes after applying the functor \( M \otimes_A - \) to it. Similarly, the commutativity of the diagram

\[
\begin{array}{ccc}
M \otimes_A X \xrightarrow{\delta_M \otimes 1} M \otimes_A C \otimes_A X & \xrightarrow{1 \otimes T} & M \otimes_A X \otimes_B D \\
\downarrow{1 \otimes \Delta_C \otimes 1} & & \downarrow{1 \otimes \Delta_D} \\
M \otimes_A C \otimes_A X \xrightarrow{\delta_M \otimes 1} M \otimes_A C \otimes_A X \otimes_B D \xrightarrow{M \otimes_A (3.4)} M \otimes_A X \otimes_B D \otimes_B D
\end{array}
\]

implies that the \( D \)-coaction \((3.6)\) is counital if and only if the diagram \((3.3)\) commutes after applying \( M \otimes_A - \) to it. We can therefore set

\[T_*(M, \delta_M) = (M \otimes_A X, (1 \otimes T)(\delta_M \otimes 1)).\]

The existence of the right adjoint \( T^* \) is ensured by the (dual) Adjoint Lifting Theorem, since \( \mathcal{U}_A \) and \( \mathcal{U}_B \) are comonadic, and \( \mathcal{Y}^T_B \) admits all coreflexive equalizers by hypothesis. For \( M \in \mathcal{Y}_B \) the value of \( T^* \) at the cofree \( D \)-comodule \( M \otimes_B D \) is the cofree \( C \)-comodule:

\[T^*(M \otimes_B D) = \text{Map}_B(X, M) \otimes_A C.\]

In particular, \( T^*(D) = \text{Map}_B(X, B) \otimes_A C \).

In general, the right adjoint \( T^*(N) \) can be calculated as the equalizer

\[(3.7) \quad \text{Map}_B(X, N) \otimes_A C \rightarrow \text{Map}_B(X, N \otimes_B D) \otimes_A C \]

where the top map is

\[(\delta_N)_* \otimes 1: \text{Map}_B(X, N) \otimes_A C \rightarrow \text{Map}_B(X, N \otimes_B D) \otimes_A C,\]

where \( \delta_N \) is the \( B \)-comodule structure on \( N \) induced by \( (\delta_N)_* \).

The bottom map is

\[(\epsilon_N)_* \otimes 1: \text{Map}_B(X, N) \otimes_A C \rightarrow \text{Map}_B(X, N \otimes_B D) \otimes_A C,\]

where \( \epsilon_N \) is the counit of the \( A \)-comodule structure on \( N \) induced by \( (\epsilon_N)_* \).
and the bottom map is the composite
\[
\text{Map}_B(X, N) \odot_A C \xrightarrow{(- \odot_B D) \odot 1} \text{Map}_B(X \otimes_B D, N \otimes_B D) \odot_A C
\]
\[
\xrightarrow{T^* \odot 1} \text{Map}_B(C \odot_A X, N \otimes_B D) \odot_A C
\]
\[
\xrightarrow{1 \otimes \Delta_C} \text{Map}_B(C \odot_A X, N \otimes_B D) \odot_A C \odot_A C
\]
\[
\xrightarrow{g \odot 1} \text{Map}_B(X, N \otimes_B D) \odot_A C,
\]
where the map \( g : \text{Map}_B(C \odot_A X, N \otimes_B D) \odot_A C \rightarrow \text{Map}_B(X, N \otimes_B D) \) is the adjoint to the evaluation map
\[
\text{Map}_B(C \odot_A X, N \otimes_B D) \odot_A C \odot_A X \rightarrow N \otimes_B D.
\]

\[\square\]

**Remark 3.16.** It is useful for computations later in this article to observe that for any right \( B \)-module \( M \), the \( M \otimes_B D \)-component of the counit of the \( T_* \dashv T^* \) adjunction is given by the composite
\[
\text{Map}_B(X, M) \odot_A C \odot_A X \xrightarrow{1 \otimes T} \text{Map}_B(X, B) \odot_A X \otimes_B D \xrightarrow{\text{ev} \odot 1} M \otimes_B D.
\]

**Remark 3.17.** If \( C \) is coaugmented, then \( A \) is a \( C \)-comodule, and by plugging in \( M = A \), the argument above shows that axioms (5.3) and (5.4) are equivalent to saying that (3.4) defines a \( D \)-comodule structure on \( M \otimes_A X \) for every \( M \in \mathcal{V}_{A}^{X} \). Note also that (5.3) may be interpreted as saying that \( T \) is a morphism of right \( D \)-comodules, when \( C \otimes_A X \) is given the \( D \)-comodule structure (3.4).

**Remark 3.18.** There is a natural bijection between \( C \)-\( D \)-brai

\[\square\]

**Example 3.19 (Forgetful functor).** Recall that for any algebra \( A \) in \( \mathcal{V} \), the canonical isomorphism \( A \cong A \odot_A A \) and the identity map on \( A \) endow \( A \) with the structure of a coring in \( \mathcal{V}_A \), which we call the trivial coring. Moreover, the forgetful functor \( U_A : \mathcal{V}_A \rightarrow \mathcal{V} \) is an isomorphism of categories. For any coring \( C \) in \( \mathcal{V}_A \) with counit \( \epsilon : C \rightarrow A \), the adjunction
\[
\mathcal{V}_A \xrightarrow{\epsilon} \mathcal{V}_A \odot_A C \xleftarrow{U_A} \mathcal{V}_A
\]
is governed by the braided bimodule \((X, T) : (A, C) \rightarrow (A, A)\), where \( X = A \) and \( T = \epsilon_C : C \rightarrow A \), i.e., \( U_A = (\epsilon)_* \), and \( - \odot_A C = (\epsilon)^* \).

**Example 3.20 (Morphisms of corings).** Just as morphisms of algebras give rise to bimodules, morphisms of corings give rise to braided bimodules. The braided bimodule associated to a morphism of corings,
\[
(\varphi, f) : (A, C) \rightarrow (B, D),
\]
has underlying $A$-$B$-bimodule $A_{\otimes B}$, where $A$ acts on $B$ through $\varphi$. The braiding
\[
T_{\varphi,f} : C \otimes_A B \to B \otimes_B D \cong D
\]
is defined to be the composite
\[
C \otimes_A B \cong A \otimes_A C \otimes_A B \xrightarrow{\varphi \otimes 1 \otimes 1} B \otimes_A C \otimes_A B \xrightarrow{1} D.
\]

We have thus constructed a braided bimodule $(B, T_{\varphi,f}) : (A, C) \to (B, D)$, inducing an adjunction
\[
\mathcal{V}_A^C \xrightarrow{(T_{\varphi,f})_*} \mathcal{V}_B^D,
\]
as long as $\mathcal{V}_A^C$ admits all coreflexive equalizers. It is important to observe that it is not necessary for the monoidal structure on $\mathcal{V}$ to be closed in order for this adjunction to exist, as we are lifting the adjunction
\[
\mathcal{V}_A \xrightarrow{\varphi_*} \mathcal{V}_B
\]
which exists even if $\mathcal{V}$ is not closed, unlike when $X \neq B$. Note that the $D$-component of the counit of the adjunction $(T_{\varphi,f})_* \dashv (T_{\varphi,f})^*$ may be identified with the morphism
\[
f : B \otimes_A C \otimes_A B \to D.
\]

**Example 3.21** (Change of corings). When $A = B$ and $\varphi = 1_A : A \to A$ in the example above, the braiding $T_{1_A,f} : C \otimes_A A \to A \otimes_A D$ is nothing but the morphism of $A$-corings $f : C \to D$, up to isomorphism in the source and target. We denote the induced adjunction
\[
(3.8) \quad \mathcal{V}_A^C \xrightarrow{f_*} \mathcal{V}_A^D
\]
and call it the coextension/corestriction-of-coefficients adjunction or change-of-corings adjunction associated to $f$. Note that the $D$-component of the counit of the $f_* \dashv f^*$ adjunction is $f$ itself and that for every $(A, C)$-comodule $(M, \delta)$,
\[
f_*(M, \delta) = (M, (1 \otimes f) \delta).
\]

We will now establish the general case.

**Proposition 3.22.** Let $\mathcal{V}$ be a closed monoidal category admitting all reflexive coequalizers and coreflexive equalizers.

If $(A, C)$ is a coaugmented coring such that $\mathcal{V}_A^C$ admits all coreflexive equalizers, and $(B, D)$ is any coring, then every $\mathcal{V}$-adjunction between $\mathcal{V}_A^C$ and $\mathcal{V}_B^D$, relative to a $\mathcal{V}$-adjunction between $\mathcal{V}_A$ and $\mathcal{V}_B$, is governed by a braided bimodule.

**Proof.** Consider a relative adjunction $(\tilde{F}, \tilde{G})$, as in [3.2]. By Proposition 2.8 the underlying adjunction $F : \mathcal{V}_A \rightleftarrows \mathcal{V}_B : G$ is governed by a bimodule $\Delta X$. We have to construct a braiding $T : C \otimes_A X \to X \otimes_B D$ and show that $(X, T) : (A, C) \to (B, D)$ governs the adjunction we started with.

Since the diagram [3.2] of left adjoints commutes, for every $C$-comodule $(M, \delta)$, the $B$-module underlying $\tilde{F}(M, \delta)$ is isomorphic to $M \otimes_A X$. Let
\[
\tilde{\delta} : M \otimes_A X \to M \otimes_A X \otimes_B D
\]
de note the right $D$-comodule structure on $\tilde{F}(M, \delta)$. Define $T : C \otimes_A X \to X \otimes_B D$ to be the composite
\[
C \otimes_A X \xrightarrow{\tilde{\delta}} C \otimes_A X \otimes_B D \xrightarrow{\delta \otimes 1} A \otimes_A X \otimes_B D \cong X \otimes_B D,
\]
where $\Delta : C \to C \otimes_A C$ is the comultiplication on $C$, seen as a right $C$-coaction on $C$, and $\epsilon : C \to A$ is the counit of $C$. We have to verify axioms (3.3) and (3.4) and show that the comodule structure $\delta$ on $M \otimes_A X$ agrees with the one induced from $T$ as in (3.0).

To check this last condition, consider the diagram

\[
\begin{array}{ccc}
M \otimes_A X & \xrightarrow{\delta} & M \otimes_A X \otimes_B D \\
\downarrow{\delta \otimes 1_1} & & \downarrow{\delta \otimes 1_1} \\
M \otimes_A C \otimes_A X & \xrightarrow{1 \otimes \Delta} & M \otimes_A C \otimes_A X \otimes_B D. \\
\end{array}
\]

Commutativity of the left square is equivalent to the fact that $\tilde{F}(\delta)$ is a morphism of $D$-comodules, since $\tilde{F}(M \otimes_A C) \cong M \otimes_A \tilde{F}(C)$ in $\mathcal{V}_B^D$. Commutativity of the right triangle is simply the counit axiom for the $C$-comodule structure on $M$. Axioms (3.3) and (3.4) hold automatically, because they are equivalent to saying that (3.6) defines a $D$-comodule structure on $A \otimes_A X$ (cf. Remark 3.17), and we know a priori that $\delta_A$ defines a $D$-comodule structure on $A \otimes_A X$.

\[\square\]

Remark 3.23. Not every adjunction between $\mathcal{V}_A^C$ and $\mathcal{V}_B^D$ is governed by a braided bimodule. For instance, this is usually not the case for adjunctions arising from twisting cochains, as discussed in [3].

3.3. Cotensor products. The right adjoint $T^*$ in the adjunction governed by a braided bimodule $(X, T)$ is difficult to describe in general. However, we will show that under appropriate conditions on the underlying (bi)modules $A_X B$ and $A_C$, it is possible to express $T^*$ as a cotensor product.

Definition 3.24. We call a left $A$-module $N$ flat if the functor $- \otimes_A N : \mathcal{V}_A \to \mathcal{V}$ preserves coreflexive equalizers.

Remark 3.25. If $\mathcal{V}$ is an abelian category, then it is easy to show that $- \otimes_A N$ preserves coreflexive equalizers if and only if $N$ is flat in the usual sense that $- \otimes_A N$ preserves monomorphisms.

Note that the notions of flatness and homotopy flatness (Definition 2.7) are different in general. For instance, if the weak equivalences in $\mathcal{V}$ are the isomorphisms, then every module is homotopy flat but not necessarily flat.

Definition 3.26. We will call a coring $(A, C)$ flat if $C$ is flat as a left $A$-module, i.e., if $\sim \otimes_A C : \mathcal{V}_A \to \mathcal{V}_A$ preserves coreflexive equalizers.

The next proposition follows directly from the observations Remark 3.25.

Proposition 3.27. If $(A, C)$ is a flat coring, then the forgetful functor $\mathcal{U}_A : \mathcal{V}_A^C \to \mathcal{V}_A$ creates coreflexive equalizers.

Definition 3.28. Suppose that $\mathcal{V}$ admits coreflexive equalizers. Let $(A, C)$ be a coring in $\mathcal{V}$, let $M$ be a right and $N$ a left $(A, C)$-comodule. The cotensor product $M \square_C N$ is defined as the coreflexive equalizer in $\mathcal{V}$:

\[
M \square_C N \quad M \otimes_A N \xrightarrow{\delta \otimes 1_1 \otimes 1_N} M \otimes_A C \otimes_A N.
\]

Proposition 3.29. Let $\mathcal{V}$ be a closed monoidal category admitting all reflexive coequalizers and coreflexive equalizers. Let $(A, C)$ be a flat coring in $\mathcal{V}$, and let $(X, T) : (A, C) \to (B, D)$ be a braided bimodule.

If the underlying bimodule $A_X B$ admits a strict right dual $X^\vee$, then $X^\vee \otimes_A C$ is a left $(B, D)$-comodule in $\mathcal{V}_A^C$, and the right adjoint of the adjunction governed by
\((X,T)\) is isomorphic to the cotensor product functor \(-\square_D(X^\vee \otimes_A C)\), i.e., there is an adjunction

\[
\begin{array}{c}
\mathcal{Y}_A^C \\
\downarrow^{T_*} \\
\downarrow_{-\square_D(X^\vee \otimes_A C)} \\
\mathcal{Y}_B^D
\end{array}
\]

**Proof.** The left \(D\)-comodule structure on \(X^\vee \otimes_A C\) is defined by the following composite:

\[
\begin{align*}
X^\vee \otimes_A C & \xrightarrow{1 \otimes \Delta} X^\vee \otimes_A C \otimes_A C \\
& \xrightarrow{1 \otimes \varepsilon \otimes 1} X^\vee \otimes_A C \otimes B X \otimes_B X^\vee \otimes_A C \\
& \xrightarrow{1 \otimes T \otimes 1} X^\vee \otimes_A X \otimes_B D \otimes_B X^\vee \otimes_A C \\
& \xrightarrow{e \otimes 1} D \otimes_B X^\vee \otimes_A C.
\end{align*}
\]

Here, \(\Delta\) is the comultiplication on \(C\), the map \(\varepsilon\) is the coevaluation, and \(e\) is the evaluation. We leave it to the reader to verify that the axioms for a \(C\)-comodule are satisfied. It follows from the natural isomorphism \(N \otimes_B X^\vee \cong \text{Map}_B(X,N)\), which holds because \(X\) is dualizable, that the coreflexive equalizer diagram \((3.7)\) defining \(T^\ast(N)\) may be identified with the equalizer diagram defining the cotensor product \(N \square_D(X^\vee \otimes_A C)\). Note that there is a subtlety in that the coreflexive equalizer defining the cotensor product should be calculated in \(\mathcal{Y}_C\), whereas the coreflexive equalizer defining the cotensor product should be calculated in \(\mathcal{Y}_A\). Since we assume that the coring \((A,C)\) is flat, the forgetful functor \(U_A: \mathcal{Y}_C \rightarrow \mathcal{Y}_A\) creates coreflexive equalizers, so we may identify the two. \(\square\)

Important special cases of braided bimodules for which Proposition \(3.29\) applies are the braided bimodules associated to morphisms of corings. Indeed, let \((\varphi,f): (A,C) \rightarrow (B,D)\) be a morphism of corings and let \((B,T_\varphi,f)\) be the associated braided bimodule (see Example \((3.20)\)). The underlying bimodule \(X = A_B B\) is dualizable, with right dual \(X^\vee = B_B A\). If the coring \((A,C)\) is flat, it follows that the adjunction governed by the morphism \((\varphi,f)\) can be written as

\[
\begin{array}{c}
\mathcal{Y}_A^C \\
\downarrow_{-\square_D(B \otimes_A C)} \\
\downarrow_{\varphi} \\
\mathcal{Y}_B^D
\end{array}
\]

Specializing further, if \(A = B\), \(\varphi: A \rightarrow B\) is the identity map, and \(f: C \rightarrow D\) is a morphism of \(A\)-corings, we recover the familiar change of corings adjunction

\[
\begin{array}{c}
\mathcal{Y}_A^C \\
\downarrow_{-\square_D C} \\
\downarrow_{f} \\
\mathcal{Y}_A^D
\end{array}
\]

### 3.4. The canonical coring.

In this section we introduce the **canonical coring** \(X\ast(C)\) associated to a coring \((A,C)\) and a right dualizable bimodule \(A_X B\). The canonical coring generalizes the descent coring associated to a morphism of algebras, and it will be useful for our analysis of Quillen equivalences between comodule categories in Section \(4\).

**Proposition 3.30.** Let \(A\) and \(B\) be algebras in \(\mathcal{Y}\), and let \(A_X B\) be a right dualizable bimodule. For every \(A\)-comodule \(C\), there is a \(B\)-comodule \(X\ast(C)\) and a braided bimodule \((X,T^\ast(C))\) \((A,C) \rightarrow (B,X\ast(C))\) satisfying the following universal property: for every \(B\)-comodule \(D:\) \((A,C) \rightarrow (B,D)\) with \(Z = X \otimes_B Y\) for some bimodule \(B Y_B\), there is a unique braided bimodule \((Y,S)\)
such that the following diagram in CORING\textsubscript{V} commutes.

\[
\begin{array}{ccc}
(A,C) & (X,T_{\text{univ}}) & (B,X_*(C)) \\
\downarrow & \downarrow & \downarrow \\
(X \otimes_B Y,T) & (Y,S) & (B',D)
\end{array}
\]

In particular, for every braiding \(T: C \otimes_A X \to X \otimes_B D\), there is a unique morphism of \(B\)-corings

\[g_T: X_*(C) \to D\]

making the following diagram in CORING\textsubscript{V} commute:

\[
\begin{array}{ccc}
(A,C) & (X,T_{\text{univ}}) & (B,X_*(C)) \\
\downarrow & \downarrow & \downarrow \\
(X,T) & (B,T_1,O_T) & (B,D)
\end{array}
\]

**Remark 3.31.** Proposition 3.30 may be formulated succinctly by saying that \((X,T_{\text{univ}}): (A,C) \to (B,X_*(C))\) is a co-cartesian morphism over \(X: A \to B\), under the forgetful functor CORING\textsubscript{V} \to ALG\textsubscript{V}. In other words, the pullback of the forgetful functor CORING\textsubscript{V} \to ALG\textsubscript{V} along the subcategory of ALG\textsubscript{V} consisting of right dualizable morphisms is a co-cartesian fibration. In particular, since the underlying bimodule of the braided bimodule associated to a morphism of corings is always right dualizable, the category Coring\textsubscript{V} is cofibered over Alg\textsubscript{V}.

**Proof.** Let \(u: A \to X \otimes_B X^\vee\) and \(e: X^\vee \otimes_A X \to B\) denote the coevaluation and evaluation maps, and let \(\Delta: C \to C \otimes_A C\) and \(\epsilon: C \to A\) denote the comultiplication and counit of the \(A\)-coring \(C\). We only define \(X_*(C)\) and the structure maps, leaving the straightforward verification of their properties to the reader.

Define \(X_*(C)\) to be the \(B\)-bimodule

\[X_*(C) = X^\vee \otimes_A C \otimes_A X,
\]

and define the comultiplication as the composite

\[X^\vee \otimes_A C \otimes_A X \xrightarrow{1 \otimes \Delta \otimes 1} X^\vee \otimes_A C \otimes_A C \otimes_A X \xrightarrow{1 \otimes 1 \otimes u \otimes 1 \otimes 1} (X^\vee \otimes_A C \otimes_A X) \otimes_B (X^\vee \otimes_A C \otimes_A X).
\]

The counit is defined to be the composite

\[X^\vee \otimes_A C \otimes_A X \xrightarrow{1 \otimes \epsilon \otimes 1} X^\vee \otimes_A X \xrightarrow{e} B.
\]

The universal braiding is defined to be

\[T_{\text{univ}} = u \otimes 1: C \otimes_A X \to X \otimes_B (X^\vee \otimes_A C \otimes_A X).
\]

Finally, given a coring \((B',D)\) and a braided bimodule

\[(X \otimes_B Y,T): (A,C) \to (R,D),\]

we define the braided bimodule \((Y,S)\) by letting \(S\) be the composite

\[X^\vee \otimes_A C \otimes_A X \otimes_B Y \xrightarrow{1 \otimes T} X^\vee \otimes_A X \otimes_B Y \otimes_R D \xrightarrow{c \otimes 1} Y \otimes_R D.
\]
In the special case of a $B$-coring $D$ and a braiding $T: C \otimes_A X \to X \otimes_B D$, the morphism $g_T$ is the composite

$$X^\vee \otimes_A C \otimes_A X \xrightarrow{1 \otimes T} X^\vee \otimes_A X \otimes_B D \xrightarrow{c \otimes 1} D.$$  

□

**Definition 3.32.** We will refer to the $B$-coring $X_*(C)$ introduced in Proposition 3.30 as the canonical coring associated to $X$ and $(A, C)$.

Furthermore, we define the canonical adjunction associated to $X$ and $C$ to be the adjunction governed by the universal braided bimodule $(X, T^\text{univ}_C)$,

$$
\begin{array}{c}
\mathcal{V}_A^C \xrightarrow{(T^\text{univ}_C)^*} \mathcal{V}_B^X(C) \\
\mathcal{V}_B^X(C) \xleftarrow{(T^\text{univ}_C)^*} \mathcal{V}_A^C
\end{array}
$$

(3.10)

**Remark 3.33.** The canonical coring $X_*(C)$ generalizes the descent coring associated to a morphism of algebras $\varphi: A \to B$. Indeed, if $C$ is the trivial $A$-coring $A$, and $X$ is the bimodule $A \otimes_A B$, then the canonical coring $B_*(A)$ is isomorphic to the descent coring $\text{Desc}(\varphi)$ associated to $\varphi$ [23, 13], which has underlying $B$-bimodule $B \otimes_A B$. Moreover, the canonical adjunction is the adjunction

$$
\begin{array}{c}
\mathcal{V}_A^B \xrightarrow{\text{Can}_\varphi} \mathcal{V}_B^\text{Desc}(\varphi) \\
\mathcal{V}_B^\text{Desc}(\varphi) \xleftarrow{\text{Prim}_\varphi} \mathcal{V}_A^B
\end{array}
$$

familiar from descent theory. Note that $B$ itself is an object in $\mathcal{V}_B^\text{Desc}(\varphi)$, where the left $\text{Desc}(\varphi)$-coaction is given by

$$(B \otimes_A A 1 \otimes_\varphi f) \otimes_B B \cong (B \otimes_A A 1 \otimes_\varphi 1) \otimes_B B$$

and the right $\text{Desc}(\varphi)$-coaction by

$$B \cong A \otimes_A B \xrightarrow{\varphi \otimes 1} B \otimes_A B \cong B \otimes_B (B \otimes_A B).$$

**Notation 3.34.** Motivated by the remark above, we write henceforth

$$\text{Can}_X = (T^\text{univ}_C)^*: \mathcal{V}_A^C \to \mathcal{V}_B^{X_*(C)}$$

and

$$\text{Prim}_X = (T^\text{univ}_C)^*: \mathcal{V}_B^{X_*(C)} \to \mathcal{V}_A^C$$

for every universal braided bimodule $(X, T^\text{univ}_C)$.

**Example 3.35.** Let $(\varphi, f): (A, C) \to (B, D)$ be a morphism of corings with associated braided bimodule $(B, T_{\varphi, f})$ (Example 3.20). A straightforward calculation shows that

$$g_{T_{\varphi, f}} = f: B \otimes_A C \otimes_A B \to D.$$
Let us make this more explicit.

**Proposition 3.37.** A braided bimodule \((X,T) : (A,C) \to (B,D)\) is left dual to \((X^\vee,T^\vee) : (B,D) \to (A,C)\) if and only if the underlying bimodules are strictly dual (cf. Example B.7), via a coevaluation \(u : A \to X \otimes_B X^\vee\) and an evaluation \(e : X^\vee \otimes_A X \to B\), and the diagrams

\[
\begin{align*}
(3.11) & \quad C \otimes_A A \cong A \otimes_A C \\
& \quad C \otimes_A X \otimes_B X^\vee \\
& \quad X \otimes_B D \otimes_B X^\vee \\
& \quad X \otimes_B X^\vee \otimes_A C \\
\end{align*}
\]

\[
\begin{align*}
(3.12) & \quad X^\vee \otimes_A C \otimes_A X \\
& \quad D \otimes_B Y \otimes_A X \\
& \quad D \otimes_B B \\
\end{align*}
\]

commute.

**Proof.** The proof simply amounts to unwinding the definition of an adjunction in a bicategory. The commutativity of the diagrams \((3.11)\) and \((3.12)\) corresponds exactly to requirement that the coevaluation \(u : A \to X \otimes_B X^\vee\) and evaluation \(e : X^\vee \otimes_A X \to B\) be morphisms of braided bimodules. □

**Remark 3.38.** Let \(\mathcal{V}\) be a closed, symmetric monoidal category. Let \(\text{CAT}_\mathcal{V}\) denote the bicategory (in fact, 2-category) of \(\mathcal{V}\)-categories, \(\mathcal{V}\)-functors, and \(\mathcal{V}\)-natural transformations. There is a bifunctor

\[ \mathcal{V} : \text{Coring}_\mathcal{V} \to \text{CAT}_\mathcal{V} \]

that sends a coring \((A,C)\) to the \(\mathcal{V}\)-category \(\mathcal{V}_A^C\), a braided bimodule \((X,T)\) to the associated \(\mathcal{V}\)-functor \(T_* : \mathcal{V}_A^C \to \mathcal{V}_B^D\), and a morphism \(f : (X,T) \to (X',T')\) of braided bimodules to the obvious natural transformation that sends an object \((M,\delta)\) in \(\mathcal{V}_A^C\) to the morphism

\[ 1 \otimes f : M \otimes_A X \to M \otimes_A X' , \]

where \((X,T), (X',T') : (A,C) \to (B,D)\). Since bifunctors preserve adjunctions, it follows that if \((X,T) : (A,C) \to (B,D)\) is a dualizable braided bimodule with dual \((X^\vee,T^\vee) : (B,D) \to (A,C)\), then

\[ \mathcal{V}_A^C \xrightarrow{T_*} \mathcal{V}_B^D \]

is a \(\mathcal{V}\)-adjunction, whence there exists a natural isomorphism

\[ (T^\vee)_* \cong T^* : \mathcal{V}_B^D \to \mathcal{V}_A^C , \]

by uniqueness up to isomorphism of right adjoints.
Remark 3.39. The bimodule $A B$ arising from a morphism of algebras $\varphi: A \to B$ is dualizable. Unfortunately, it is not true in general that the braided bimodule associated to a morphism of corings $(\varphi, f): (A, C) \to (B, D)$ is dualizable. Indeed, let $\varphi: A \to B$ be a morphism of algebras, and consider the induced morphism of corings $(A, A) \to (B, B \otimes_A B)$. The induced adjunction is the descent adjunction

$$\mathcal{V}_A \xrightarrow{\text{Can}_\varphi} \mathcal{V}_B^{B \otimes_A B}.$$ 

The braided bimodule that governs this adjunction is $X = A B$, where $B$ is viewed as a left $A$-module via the morphism $\varphi$. The braiding is given by the canonical right $B \otimes_A B$-comodule structure on $B$ (cf. Remark 3.14), $T: A \otimes_A B \to B \otimes_B (B \otimes_A B)$. The underlying bimodule is dualizable with right dual $X = B B$. However, the braided bimodule $(B, T)$ is dualizable if and only if $\text{Prim}_\varphi$ is isomorphic to the forgetful functor, which rarely happens.

4. Homotopical Morita theory for corings

In this section we elaborate a homotopical version of Morita-Takeuchi theory for corings (cf. [5, §23]), providing conditions under which two $\mathcal{V}$-model categories of comodules over corings are Quillen equivalent. In particular, we provide criteria in terms of homotopic descent under which a morphism of corings induces a Quillen equivalence between the associated comodule categories.

Convention 4.1. Throughout this section, $\mathcal{V}$ denotes a symmetric monoidal model category [31]. Moreover, for every coring $(A, C)$ that we consider here, we suppose that $\mathcal{V}$ admits the model category structure right-induced from $\mathcal{V}$ and that $\mathcal{V}_A$ admits the model category structure left-induced from $\mathcal{V}_A$, via the adjunction

$$\mathcal{V}_C \xrightarrow{\mathcal{U}_A} \mathcal{V}_A.$$ 

Remark 4.2. Conditions on $\mathcal{V}$ under which the convention above holds can be found in [15], [2], and [14], where a number of concrete examples are also elaborated. See also Section 5.

Remark 4.3. Since we assume henceforth that $\mathcal{V}$, $\mathcal{V}_A$, and $\mathcal{V}_C$ are model categories, they are in particular complete and cocomplete and thus admit all reflexive coequalizers and coreflexive equalizers.

Remark 4.4. By definition of the $\mathcal{V}$-tensor structure on $\mathcal{V}_A$ (see Proposition 3.7), the left adjoint $\mathcal{U}_A$ is a tensor functor, so it follows from Proposition [3.7] that the adjunction is $\mathcal{V}$-structured. By Proposition [A.4] it follows that the left-induced model structure on $\mathcal{V}_A$ is $\mathcal{V}$-structured, when it exists.

4.1. Towards Quillen equivalences of comodule categories. We begin our study of the homotopy theory of comodules over corings by providing conditions under which an adjunction governed by a braided bimodule is a Quillen adjunction.

Proposition 4.5. Let $(X, T): (A, C) \to (B, D)$ be a braided bimodule. If

$$\mathcal{V}_A \xrightarrow{\text{Map}_B(X, -)} \mathcal{V}_B$$

is a Quillen adjunction, then so is

$$\mathcal{V}_A \xrightarrow{T_*} \mathcal{V}_B.$$
In particular, this holds if \( X \) is cofibrant as a right \( B \)-module. The converse holds if \((A,C)\) is coaugmented.

Proof. This follows readily from the fact that the model structure on \( \mathcal{V}_A^C \) is left induced via the forgetful functor \( U_A : \mathcal{V}_A^C \to \mathcal{V}_A \).

The next result, which is a sort of dual to Proposition 2.15, is a first step towards understanding Quillen equivalences of comodule categories.

Proposition 4.6. Let \( f : C \to D \) be a morphism of \( A \)-corings. The change-of-corings adjunction,

\[
\begin{array}{ccc}
\mathcal{V}_A^C & \xrightarrow{f_*} & \mathcal{V}_A^D \\
\downarrow f^* & & \downarrow f^* \\
\mathcal{V}_A^D & \xleftarrow{f^*} & \mathcal{V}_A^C
\end{array}
\]

is a Quillen equivalence if and only if the counit of the adjunction,

\[
\epsilon_M : f_* f^*(M) \to M,
\]

is a weak equivalence for all fibrant right \( D \)-comodules \( M \). If \( A \) is fibrant in \( \mathcal{V} \), and the change-of-corings adjunction is a Quillen equivalence, then \( f \) is a weak equivalence.

Remark 4.7. If the coring \((A,C)\) is flat, then \( f^*(M) = M \square_D C \) by Proposition 3.29. In this case, if \( f \) is a weak equivalence, and the functor \( M \square_D - : \mathcal{V}_A^D \to \mathcal{V} \) preserves weak equivalences for all fibrant right \( D \)-comodules \( M \), then the adjunction above is a Quillen equivalence.

Proof. The functor \( f_* \) preserves and reflects weak equivalences, because these are created in the underlying category \( \mathcal{V}_A \), and \( f_* \) does not change the underlying \( A \)-module. It follows that the adjunction is a Quillen equivalence if and only if the counit of the adjunction is a weak equivalence for all fibrant objects \( M \) in \( \mathcal{V}_A^D \).

If \( A \) is fibrant in \( \mathcal{V} \) and therefore in \( \mathcal{V}_A \), it follows that \( D \) is fibrant as an object of \( \mathcal{V}_A \) because it is the image of \( A \) under the right Quillen functor \(- \otimes_A D : \mathcal{V}_A \to \mathcal{V}_A^D \). Thus, if the counit \( \epsilon_M \) is a weak equivalence for all fibrant \( M \), then \( f \) is necessarily a weak equivalence, because the component of the counit at \( D \) is exactly \( f \) (cf. Example 3.20). □

Since the condition on \( f \) in Proposition 4.6 recurs throughout the rest of this article, we give it a name, dual to that for the condition that arose in the module case (Proposition 2.15). In Section 5 we provide concrete examples of chain maps satisfying this condition.

Definition 4.8. Let \( \mathcal{V} \) be a symmetric monoidal model category satisfying Convention 4.4. We say that a weak equivalence \( f : C \to D \) of \( A \)-corings is copure if

\[
\epsilon_M : f_* f^*(M) \to M
\]

is a weak equivalence for all fibrant right \( D \)-comodules \( M \).

4.2. Homotopic descent over corings. Our description of the canonical coring associated to a dualizable \( A \)-\( B \)-bimodule (Definition 3.32) hints at an interesting generalization of the usual notion of homotopic descent [13], which turns out to be important for our discussion of Quillen equivalences of comodule categories. Recall Notation 3.34.

Definition 4.9. Let \((A,C)\) be a coring in \( \mathcal{V} \) and \( B \) an algebra in \( \mathcal{V} \). A strictly dualizable \( A \)-\( B \)-bimodule \( X \) satisfies effective homotopic descent with respect to \( C \)
if the canonical adjunction

\[
\begin{array}{c}
\mathcal{V}^\mathcal{X}(-) \\
\downarrow \text{Prim}_X \\
\mathcal{V}^X(-)
\end{array}
\]

(4.1)

\[
\text{Can}_X 
\]

is a Quillen equivalence.

**Remark 4.10.** If the strictly dualizable bimodule \( AB \) satisfies effective homotopic descent with respect to the coring \((A, A)\), then the algebra morphism \( \varphi : A \to B \) satisfies effective homotopic descent, in the sense of [13].

The following theorem generalizes Grothendieck’s classical theorem on faithfully flat descent for morphism of commutative rings. Recall the definitions of the special classes of modules in Definition 2.7 and the definition of a flat coring from Definition 3.26.

**Theorem 4.11.** Let \( \mathcal{V} \) be a symmetric monoidal model category satisfying Convention 4.1. Let \((A, C)\) be a flat coring, let \( B \) be an algebra, and let \( AX_B \) be a right dualizable bimodule. If \( X \) is homotopy compact and homotopy faithfully flat as a left \( A \)-module, then \( X \) satisfies effective homotopic descent with respect to \((A, C)\).

**Proof.** The object in \( \mathcal{V} \) underlying \( \text{Can}_X(M) \) is simply \( M \otimes_A X \). Since \( X \) is homotopy faithfully flat as a left \( A \)-module, and since weak equivalences are detected in \( \mathcal{V} \), it follows that the functor \( \text{Can}_X : \mathcal{V}^C \to \mathcal{V}^X(-) \) preserves and reflects weak equivalences. Consequently, (4.1) is a Quillen equivalence if and only if the counit of the adjunction is a weak equivalence for every fibrant object \( M \).

Since \((A, C)\) is flat, and \( X \) is right dualizable, the right adjoint in (4.1) may be expressed as a cotensor product (Proposition 3.29). It follows that the counit of the adjunction may be identified with the map

\[
(M \sq X_{X}(C)(X^Y \otimes A C)) \otimes_A X \longrightarrow M \sq X_{X}(C)X_{X}(C) \cong M
\]

induced by the universal property of the cotensor product. Since \( X \) is homotopy compact, it follows that the counit is a weak equivalence. \( \square \)

The corollary below is an important special case of Theorem 4.11.

**Corollary 4.12.** Let \( \varphi : A \to B \) be a morphism of algebras in \( \mathcal{V} \). If \( B \) is homotopy compact and homotopy faithfully flat as a left \( A \)-module, then \( \varphi \) satisfies effective homotopic descent.

The classical theorem is recovered by taking \( \mathcal{V} \) to be the category of abelian groups, \( C \) to be the trivial \( A \)-coring \( A \), and \( X \) to be the bimodule \( AB \), where \( B \) is viewed as a left \( A \)-module via the morphism of algebras \( \varphi : A \to B \).

**Remark 4.13.** Homotopical faithful flatness is not necessary for homotopic effective descent. In fact, the analogous condition is already not necessary for ordinary effective descent for commutative rings. A morphism of commutative rings satisfies effective descent if and only it is pure. Pure morphisms are necessarily faithful, but not necessarily flat, see [4, 19].

4.3. The homotopical Morita-Takeuchi theorem. Assembling our results on Quillen equivalences induced by copure coring maps (Proposition 4.6) and on effective homotopic descent over corings (Theorem 4.11), we can now answer the question formulated in the introduction and provide conditions under which two categories of comodules over corings are Quillen equivalent. Recall that if \( AX_B \) is strictly right dualizable, then every braided module \((X, T) : (A, C) \to (B, D)\) determines a morphism of corings \( g(T) : (B, X_{X}(C)) \to (B, D) \) (Proposition 3.30).
Theorem 4.14. Let \( \mathcal{V} \) be a symmetric monoidal model category satisfying Convention 4.7. Let \((A, C)\) be a coring in \( \mathcal{V} \). Let \((X,T) : (A,C) \to (B,D)\) be a braided bimodule in \( \mathcal{V} \) such that \( X \) is strictly dualizable and cofibrant as a right \( B \)-module. Let \( g_T : X_*(C) \to D \) denote the associated morphism of \( B \)-corings.

If \( X \) satisfies effective homotopic descent with respect to \( C \), and \( g_T : X_*(C) \to D \) is a copure weak equivalence of corings, then the Quillen adjunction governed by \((X,T)\),

\[
\begin{array}{ccc}
\mathcal{V}_A^C & \xleftarrow{T_*} & \mathcal{V}_B^D \\
\xrightarrow{T^*} & & \\
\end{array}
\]

is a Quillen equivalence.

Conversely, if \( B \) is fibrant in \( \mathcal{V} \), the coring \((A, C)\) is flat, and \( A X \) is homotopy compact and homotopy flat, then \((A, C)\) is a Quillen equivalence only if \( X \) satisfies effective homotopic descent with respect to \( C \), and \( g_T : X_*(C) \to D \) is a copure weak equivalence of corings.

Proof. Since \( X \) is cofibrant as a right \( B \)-module, the adjunction \((A, C)\) is a Quillen adjunction by Proposition 4.3. Since \( X \) is right dualizable, it follows from Proposition 3.29 that the adjunction \((A, C)\) factors as a generalized descent adjunction followed by a change-of-corings adjunction,

\[
\begin{array}{ccc}
\mathcal{V}_A^C & \xrightarrow{\text{Can}_X} & \mathcal{V}_B^{X_*(C)} \\
\xleftarrow{\text{Prim}_X} & & \xrightarrow{(g_T)_*} \mathcal{V}_B^D. \\
\end{array}
\]

If \( X \) satisfies effective homotopic descent with respect to \( C \), then the first adjunction in the factorization \((A, C)\) is a Quillen equivalence (Definition 4.9). If \( g_T : X_*(C) \to D \) is a copure weak equivalence, then the second adjunction in \((A, C)\) is a Quillen equivalence by Proposition 4.6. It follows that \((A, C)\) is a Quillen equivalence.

Conversely, suppose that \( B \) is fibrant in \( \mathcal{V} \), \( (A, C) \) is flat, \( A X \) is homotopy compact and homotopy flat, and the adjunction \((A, C)\) is a Quillen equivalence. Proposition 4.24 implies that the component of counit of the adjunction \((A, C)\) at \( M \in \mathcal{V}_B^D \) may be identified with the composite

\[
(M \Box_D (X^\vee \otimes_A C)) \otimes_A X \to M \Box_D (X^\vee \otimes_A C \otimes_A X) \xrightarrow{M \Box_D g_T} M \Box_D D \cong D,
\]

which represents the homotopy counit when \( M \) is fibrant, since \( A X \) is homotopy flat, and is therefore a weak equivalence, since \((A, C)\) is a Quillen equivalence. Moreover, since \( A X \) is homotopy compact, the first map in the composite is a weak equivalence. Hence, \( M \Box_D g_T : M \Box_D X_*(C) \to M \Box_D D \) is a weak equivalence for all fibrant \( M \in \mathcal{V}_B^D \) and in particular for \( M = D = B \otimes_B D \), which is fibrant since \( B \) is fibrant in \( \mathcal{V} \) and therefore in \( \mathcal{V}_B \). In other words, \( g_T : X_*(C) \to D \) is a copure weak equivalence.

It follows from Proposition 4.6 that the second adjunction in the factorization \((A, C)\) is a Quillen equivalence. By the 2-out-of-3 property for Quillen equivalences, the first adjunction must be a Quillen equivalence as well, i.e., \( X \) satisfies effective homotopic descent with respect to \( C \).

The special case of adjunctions induced by a morphism of corings is worth singling out. Recall Examples 3.20 and 3.35.

Corollary 4.15. Let \( \mathcal{V} \) be a symmetric monoidal model category satisfying Convention 4.7. Let \((\varphi, f) : (A, C) \to (B, D)\) be a morphism of corings in \( \mathcal{V} \). Suppose that \( B \) is cofibrant as a right \( B \)-module.

If the morphism \( \varphi : A \to B \) satisfies effective homotopic descent with respect to \( C \), and the morphism of \( B \)-corings \( f : B_*(C) \to D \) is a copure weak equivalence,
then the adjunction governed by \((\varphi, f)\),

\[
\mathcal{Y}_A^C \xrightarrow{(T_{\varphi,f})^*} \mathcal{Y}_B^D,
\]

is a Quillen equivalence.

Conversely, if \((A, C)\) is flat, and \(B\) is fibrant, homotopy compact, and homotopy flat as a left \(A\)-module, then the adjunction governed by \((\varphi, f)\) is a Quillen equivalence only if \(\varphi\) satisfies effective homotopic descent with respect to \(C\), and \(f: B_*(C) \to D\) is a copure weak equivalence.

**Remark 4.16.** When specialized to Hopf algebroids in the category of graded modules over a commutative ring (with weak equivalences being the isomorphisms), Corollary 4.15 recovers [18, Theorem 6.2] and [17, Theorem D].

5. The case of finite-type, non-negatively graded chain complexes

In this section we apply the theory of the previous sections to the category of finite-type, non-negatively graded chain complexes of vector spaces. Since this category is unstable, the duality theory is quite limited, but we can still describe reasonable conditions ensuring that an adjunction of comodule categories is a Quillen equivalence. We provide some concrete examples below.

Let \(k\) be a field, and let \(\text{Ch}_k\) denote the category of non-negatively graded chain complexes of \(k\)-vector spaces. The category \(\text{Ch}_k\) admits a model structure where the weak equivalences are the quasi-isomorphisms, the cofibrations are the degreewise injections, and the fibrations are the maps that are surjective in positive degrees \(8\). This model category structure is closed monoidal with respect to the usual graded tensor product of chain complexes, where the internal hom is a truncated version of the unbounded hom-complex.

Let \(\text{Ch}^\text{fin}_k\) denote the full monoidal subcategory of \(\text{Ch}_k\), the objects of which are chain complexes that are of finite type, i.e., degreewise finite dimensional. Note that \(\text{Ch}^\text{fin}_k\) is neither complete nor cocomplete, but does admit all degreewise-finite limits and colimits. The monoidal model category structure of \(\text{Ch}_k\) therefore restricts to a monoidal model category structure on \(\text{Ch}^\text{fin}_k\), with the same distinguished classes of morphisms, but where, as in Quillen’s original definition \(27\), one requires only finite completeness and cocompleteness, which suffices to define and study the associated homotopy category.

5.1. Homotopical Morita theory of differential graded modules. Let \(A\) be an algebra in \(\text{Ch}_k\). As shown in \(31\) Section 4), the category \((\text{Ch}_k)_A\) also admits a model category structure right-induced by the adjunction

\[
\text{Ch}_k \xrightarrow{- \otimes A} (\text{Ch}_k)_A.
\]

It is clear that if \(A\) is itself of finite type, then this structure restricts to \((\text{Ch}^\text{fin}_k)_A\), so that the adjunction

\[
\text{Ch}^\text{fin}_k \xrightarrow{- \otimes A} (\text{Ch}^\text{fin}_k)_A.
\]

right-induces a model category structure on \((\text{Ch}^\text{fin}_k)_A\). Observe that the functor \(U\) preserves cofibrant objects, since all objects in \(\text{Ch}^\text{fin}_k\) are cofibrant. The analogous result obviously holds for left modules as well.

Since the chain complexes we consider here are non-negatively graded, most bimodules are not strictly dualizable. It is clear, however, that for any morphism
of algebras \( \varphi : A \to B \) and for any \( n \geq 1 \), the bimodule \( \bigoplus_{i=1}^{n} AB_{B} \) is strictly dualizable, with dual \( \bigoplus_{i=1}^{n} B A \).

The following class of \( A \)-modules is particularly important in homotopy theory.

**Definition 5.1.** An object \( N \) in \( _{A}(\text{Ch}_{k}) \) is \( A \)-semifree \([10]\) Section 6] if it admits an increasing filtration

\[
0 = F_{-1}N \subseteq F_{0}N \subseteq \cdots F_{p-1}N \subseteq F_{p}N \subseteq \cdots N,
\]
such that for all \( p \geq 0 \), there is a graded \( k \)-vector space \( V(p) \) such that

\[
F_{p}N/F_{p-1}N \cong A \otimes V(p)
\]
as differential graded \( A \)-modules. Semifree right \( A \)-modules are defined analogously.

If \( B \) is another algebra in \( \text{Ch}_{k} \), then an \( A \)-\( B \)-bimodule is \( A \)-semifree as a right \( B \)-module if, in addition, \( V(p) \) is a right \( B \)-module for all \( p \) in such a way that the recursively induced right \( B \)-module structure on \( F_{p}N \) is compatible with that of \( N \).

**Remark 5.2.** The definition above is actually a specialization of the notion of \( A \)-semifree modules in the context of unbounded chain complexes \([10]\) Section 6]. When we refer to \( \text{Map}_{A}(N,A) \) below as \( A \)-semifree, we mean this more general notion, since \( \text{Map}_{A}(N,A) \) is non-positively graded if \( N \) is non-negatively graded.

Semi-free \( A \)-modules play the role of CW-complexes. They provide “enough” cofibrant objects, in the sense of the following proposition.

**Lemma 5.3.** An \( A \)-module is cofibrant if and only if it is a retract of a semi-free \( A \)-module.

**Proof.** It is easy to show that \( A \)-semifree modules and their retracts are cofibrant in the model category structure defined above \([10]\) Section 6, Ex.4]. Conversely, since every \( A \)-module admits a semi-free resolution \([10]\) Proposition 6.6], it follows by standard arguments that every cofibrant \( A \)-module is a retract of a semi-free module.

Note in particular that \( A \) itself is always cofibrant as a right or left \( A \)-module.

Before formulating homotopical Morita theory in \( \text{Ch}_{k}^{\text{fin}} \), we establish a few useful technical results concerning semifree modules. Recall Definitions \([2.7]\) and \([3.21]\).

**Lemma 5.4.** Let \( A \) be an algebra in \( \text{Ch}_{k}^{\text{fin}} \).

1. If \( N \) is a cofibrant object in \( _{A}(\text{Ch}_{k}^{\text{fin}}) \), then it is homotopy projective and homotopy flat. In particular, the category \( _{A}(\text{Ch}_{k}^{\text{fin}}) \) satisfies the CHF hypothesis (Definition \([2.7])]). If \( N \) is \( A \)-semifree, then it is homotopy faithful.
2. If \( N \) is a semifree left \( A \)-module of finite type, then \( - \otimes_{A} N \) preserves all finite limits. In particular, every \( A \)-semifree module in \( _{A}(\text{Ch}_{k}^{\text{fin}}) \) is flat and homotopy compact.
3. Every algebra quasi-isomorphism \( A \cong B \) is homotopy pure.
4. Let \( B \) be an algebra, \( M \) a right \( B \)-module, and \( N \) an \( A \)-\( B \)-bimodule in \( \text{Ch}_{k}^{\text{fin}} \). If \( N \) is \( A \)-semifree as a right \( B \)-module, then \( \text{Map}_{B}(M,N) \) is left \( A \)-semifree.

**Proof.** (1) By \([10]\) Proposition 6.7], every semifree \( A \)-module is homotopy projective and homotopy flat and therefore every retract of a semifree \( A \)-module is homotopy projective and homotopy flat as well.

It is not hard to see that every \( A \)-semifree \( A \)-module that is of finite type is homotopy faithful. Indeed, suppose that \( f : M \to M' \) is a map of finite-type, right \( A \)-modules such that

\[
f \otimes_{A} 1 : M \otimes_{A} N \cong M' \otimes_{A} N
\]
Proof. (1) This follows immediately from Proposition 2.9, since retracts of semifree modules are cofibrant. Observe that as a right $B$-module.

Note that any finite direct sum of copies of Remark 5.6. induces increasing filtrations of $M$ which are bounded since all objects are of finite type. The filtrations give rise to first quadrant spectral sequences converging to $H^*(M \otimes_A N)$ and $H^*(M' \otimes_A N)$, and $f \otimes_A 1$ induces a morphism of spectral sequences between them. The $E_2$-terms of the spectral sequences are $H_*(M) \otimes (\bigoplus_{p \geq 0} V(p))$ and $H_*(M') \otimes (\bigoplus_{p \geq 0} V(p))$, so that the Zeeman comparison theorem [36, Corollary] allows us to conclude.

(2) To show that any finite-type, $A$-semifree module $N$ is homotopy compact, it suffices to show that the functor $- \otimes_A N : (\text{Ch}^\text{fin}_k)_A \to \text{Ch}^\text{fin}_k$ preserves finite products and kernels. It is immediate that $- \otimes_A N$ preserves finite products, since they are the same as finite sums in $\text{Ch}^\text{fin}_k$. On the other hand, there is an isomorphism $M \otimes_A N \cong M \otimes (\bigoplus_{p \geq 0} V(p))$ of graded $k$-vector spaces for every right $A$-module $M$, whence $- \otimes_A N$ preserves kernels as well.

(3) Since the CHF hypothesis is satisfied, this follows from 2.14.

(4) Let $0 = F_{-1}N \subseteq F_0N \subseteq \cdots \subseteq F_{p-1}N \subseteq F_pN \subseteq \cdots$ be the relevant increasing filtration of $N$, where for each $p \geq 0$

$$F_pN/F_{p-1}N \cong A \otimes V(p)$$

for some graded $k$-vector space $V(p)$. Since this is a filtration as right $B$-modules, $\text{Map}_B(M, N)$ admits an increasing filtration with

$$F_p \text{Map}_B(M, N) = \text{Map}_B(M, F_pN)$$

for all $p$. Moreover, because of the finite-type condition, there are isomorphisms of graded vector spaces

$$F_p \text{Map}_B(M, N)/F_{p-1} \text{Map}_B(M, N) \cong (F_pN \otimes_B \text{Map}_B(M, B))/(F_{p-1}N \otimes_B \text{Map}_B(M, B)) \cong (A \otimes V(p)) \otimes_B \text{Map}_B(M, B) \cong A \otimes \text{Map}_B(M, V(p))$$

We can now formulate homotopical Morita theory for finite-type, non-negatively graded chain complexes of vector spaces.

Theorem 5.5. Let $A$ and $B$ be algebras in $\text{Ch}^\text{fin}_k$, and let $X$ be an $A$-$B$-module of finite type.

1. If $X$ is a retract of a right $B$-semifree module, then the adjunction governed by $X$,

$$(\text{Ch}^\text{fin}_k)_A \xrightarrow{\sim \otimes_A X} (\text{Ch}^\text{fin}_k)_B,$$

is a Quillen pair.

2. If $X$ is $B$-semifree and homotopy cofaithful as a right $B$-module and homotopy right dualizable, and the morphism $\varphi : A \to \text{Map}_B(X, X)$ that encodes the left $A$-module structure of $X$ is a weak equivalence, then this adjunction is a Quillen equivalence.

Remark 5.6. Note that any finite direct sum of copies of $B$ is homotopy cofaithful as a right $B$-module.

Proof. (1) This follows immediately from Proposition 2.9, since retracts of semifree modules are cofibrant.

(2) Observe that
• \((\text{Ch}_{\mathbb{k}}^{\text{fin}})_{A}\) satisfies the CHF hypothesis by Lemma 5.4 (1); and
• \(X\) is cofibrant in \((\text{Ch}_{\mathbb{k}}^{\text{fin}})_{B}\), as it is a retract of a \(B\)-semifree module, and all \(B\)-modules are fibrant.

We can apply the Homotopical Morita Theorem (Theorem 2.23) and conclude. \(\square\)

Example 5.7. The theorem above implies a homotopical version of the usual Morita equivalence between a ring \(R\) and the ring of \((n \times n)\)-matrices with coefficients in \(R\). Let \(B\) be any algebra in \(\text{Ch}_{\mathbb{k}}^{\text{fin}}\), and let \(X = B^{\otimes n}\) for some \(n \in \mathbb{N}\). For any weak equivalence of algebras \(\varphi: A \xrightarrow{\sim} \operatorname{Map}_{B}(X, X)\), the adjunction governed by \(X\)

\[
(\text{Ch}_{\mathbb{k}}^{\text{fin}})_{A} \xrightarrow{\otimes A X} \operatorname{Map}_{B}(X, -) \xleftarrow{\otimes A C} (\text{Ch}_{\mathbb{k}}^{\text{fin}})_{B},
\]

is a Quillen equivalence, where the \(A\)-module structure on \(X\) is encoded by \(\varphi\).

5.2. Homotopical Morita theory of differential graded comodules. The existence of model category structure for categories of comodules over corings is somewhat delicate to establish. The next result is a special case of [15, Theorem 6.2].

Theorem 5.8. Let \(A \to k\) be an augmented algebra in \(\text{Ch}_{\mathbb{k}}^{\text{fin}}\) such that \(H_{1} A = 0\). If \(C\) is a finite-type \(A\)-coring that is semifree as left \(A\)-module, \(H_{0}(k \otimes_{A} C) = \mathbb{k}\), and \(H_{1}(k \otimes_{A} C) = 0\), then the category \((\text{Ch}_{\mathbb{k}}^{\text{fin}})_{A}^{C}\) of \((A, C)\)-comodules admits a model category structure left-induced by the adjunction

\[
(\text{Ch}_{\mathbb{k}}^{\text{fin}})_{A}^{C} \xrightarrow{\otimes AC} (\text{Ch}_{\mathbb{k}}^{\text{fin}})_{A}
\]

from the model structure on \((\text{Ch}_{\mathbb{k}}^{\text{fin}})_{A}\) defined above.

Remark 5.9. As established in the course of the proof of Theorem 5.8, all limits in \((\text{Ch}_{\mathbb{k}}^{\text{fin}})_{A}^{C}\) are in fact created in \((\text{Ch}_{\mathbb{k}}^{\text{fin}})_{A}\) and thus in \(\text{Ch}_{\mathbb{k}}^{\text{fin}}\) [15, Lemma 6.8].

It follows immediately from [15, Theorem 5.8] and its proof that we can characterize the fibrations in the left-induced model structure on \((\text{Ch}_{\mathbb{k}}^{\text{fin}})_{A}^{C}\) in a computationally useful way. Recall from [12] the following definition, which dualizes the definition of a relative cell complex in a model category.

Definition 5.10. Let \(X\) be a class of morphisms in a complete category \(C\). Let \(Y: \mathbb{N} \to C\) be a functor. If for all \(n \geq 0\), there is a pullback

\[
\begin{array}{ccc}
Y^{n+1} & \xrightarrow{\delta} & \Rightarrow X^{n+1} \\
\downarrow & & \downarrow \\
Y^{n} & \xrightarrow{\alpha \in X} & X^{n+1},
\end{array}
\]

then the composition of the tower

\[
\lim_{n} Y^{n} \to Y^{0},
\]

is an \(X\)-Postnikov tower of countable height. The class of all \(X\)-Postnikov towers of countable height is denoted \(\text{Post}_{X}^{\omega}\).

Proposition 5.11. [15, Theorem 5.8] Let \(\text{Fib}\) denote the class of fibrations in the right-induced model category structure on \((\text{Ch}_{\mathbb{k}}^{\text{fin}})_{A}\). Every fibration in the left-induced model category structure on \((\text{Ch}_{\mathbb{k}}^{\text{fin}})_{A}^{C}\) is a retract of an element of \(\text{Post}_{\text{Fib}}^{\omega} A C\).
This characterization of fibrations in \((\text{Ch}^{\text{fin}}_{k})_{A}\) enables us to establish the existence of an interesting class of copure morphisms of corings. Observe that Lemma 5.10 (2) and Proposition 5.29 together imply that for any morphism \(f : C \to D\) of \(A\)-corings, the right adjoint to \(f_{*} : (\text{Ch}^{\text{fin}}_{k})_{A} \to (\text{Ch}^{\text{fin}}_{k})_{A}\) is the cotensor functor \(- \otimes_{p} C : (\text{Ch}^{\text{fin}}_{k})_{A} \to (\text{Ch}^{\text{fin}}_{k})_{A}\).

**Theorem 5.12.** If \(f : C \to D\) is a morphism of \(A\)-corings that is a retract of a quasi-isomorphism of semifree modules, then it is copure.

**Proof.** Let \(N\) be a fibrant object in \((\text{Ch}^{\text{fin}}_{k})^{D}_{A}\). Since the class of weak equivalences is closed under retracts, Proposition 5.11 implies that we can assume without loss of generality that there is a sequence of morphism of \((A,D)\)-comodules

\[
\cdots \to N^n p^n \to N^{n-1} \to \cdots \to N^1 p^1 \to N^0
\]

such that

- there is a fibrant object \(M^0\) in \((\text{Ch}^{\text{fin}}_{k})_{A}\) such that \(N^0 = M^0 \otimes_{A} D\);
- for all \(n \geq 0\), there exist a fibration \(g^{n+1} : M^{n+1} \to M^n\) and a morphism \(g^n : M^n \to M^n\) in \((\text{Ch}^{\text{fin}}_{k})_{A}\) such that

\[
\text{(5.1)}
\]

\[
\begin{array}{cccc}
N^{n+1} & \to & M^{n+1} & \to_{A} D \\
\downarrow & & \downarrow & \\
N^n & \to & M^n & \to_{(g^n)^{t}} D
\end{array}
\]

is a pullback diagram in \((\text{Ch}^{\text{fin}}_{k})^{D}_{A}\), where \((g^n)^{t}\) denotes the transpose of \(g^n\); and
- \(N \cong \lim_{n} N^n\)

We prove below by induction that

\[
\text{(5.2)}
1 \square D f : N^n \square D C \to N^n \square D D \cong N^n
\]

is a quasi-isomorphism for all \(n\), which implies that

\[
1 \square D f : N \square D C \to N
\]

is a quasi-isomorphism as well, by the following argument.

The sequences

\[
\cdots \to N^n p^n \to N^{n-1} \to \cdots \to N^1 p^1 \to N^0
\]

and

\[
\cdots \to N^n \square D C p^n \square D \to N^{n-1} \square D C \to \cdots \to N^1 \square D C p^1 \to N^0 \square D C
\]

both satisfy the Mittag-Leffler condition, since all of the maps are surjections. By [35] Theorem 3.5.8, there is therefore a commuting diagram of exact sequences

\[
\begin{array}{cccc}
0 & \to & \lim_{n} H_{k+1}(N^n \square D C) & \to & H_k(N \square D C) & \to & \lim_{n} H_k(N^n \square D C) & \to & 0 \\
\downarrow & & \downarrow & & h_k(1 \square D f) & & \downarrow & & \downarrow \\
0 & \to & \lim_{n} H_{k+1}(N^n) & \to & H_k(N) & \to & \lim_{n} H_k(N^n) & \to & 0
\end{array}
\]

for all \(k \geq 0\), whence \(H_k(1 \square D f)\) is an isomorphism for all \(k\), as desired.

The inductive proof that (5.2) is a quasi-isomorphism for all \(n\) proceeds as follows. Observe first that \(N^n \square D C \cong M^n \otimes_{A} C\), whence \(1 \square D f : N^n \square D C \to N\) can be identified with \(1 \otimes f : M^n \otimes_{A} C \to M^n \otimes_{A} D\), which is a quasi-isomorphism since \(f\) is a retract of a quasi-isomorphism of semifree modules, cf. [10] Proposition 6.7(ii)].
Suppose (5.2) holds for some $n \geq 0$. Applying the right adjoint $- \Box^D C$ to the pullback diagram (5.1), we obtain a pullback diagram

$$
\begin{array}{c}
N^{n+1} \Box^D C \\
\downarrow^d \\
N^n \Box^D C
\end{array}
\quad
\begin{array}{c}
\overset{(g^n)^D C}{\longrightarrow} M^{n+1} \otimes_A C \\
\downarrow^{q^n+1 \otimes_A C} \\
M^n \otimes_A C
\end{array}
\quad
\begin{array}{c}
\overset{(g^n)^D C}{\longrightarrow} M^{n+1} \otimes_A C \\
\downarrow^{q^n+1 \otimes_A C} \\
M^n \otimes_A C
\end{array}
$$

in $(\text{Ch}_k)^G$. Moreover, there is a commuting diagram of chain maps

$$
\begin{array}{c}
N^n \Box^D C \\
\downarrow^{1 \Box^D f} \\
N^n
\end{array}
\quad
\begin{array}{c}
\overset{(g^n)^D C}{\longrightarrow} M^{n+1} \otimes_A C \\
\downarrow^{1 \otimes f} \\
M^n \otimes_A D
\end{array}
\quad
\begin{array}{c}
\overset{(g^n)^D C}{\longrightarrow} M^{n+1} \otimes_A C \\
\downarrow^{1 \otimes f} \\
\hat{M}^{n+1} \otimes_A C
\end{array}
\quad
\begin{array}{c}
\overset{(g^n)^D C}{\longrightarrow} M^{n+1} \otimes_A C \\
\downarrow^{1 \otimes f} \\
\hat{M}^{n+1} \otimes_A C
\end{array}
$$

where all three vertical maps are quasi-isomorphisms, and the horizontal maps labeled $q^n+1 \otimes_A 1$ are surjections and therefore fibrations. The Cogluing Lemma [11, Lemma II.8.10] therefore implies that the induced map from the pullback of the top row to the pullback of the bottom row is a quasi-isomorphism, i.e.,

$$1\Box^D f : N^{n+1} \Box^D C \to N^{n+1}$$

is a quasi-isomorphism, as desired. Note that we have used here that limits of $(A,D)$-comodules are created in the category of chain complexes (Remark 5.9). □

The next theorem provides an illustration of homotopy faithfully flat descent in the context of finite-type, non-negatively graded chain complexes of vector spaces.

**Theorem 5.13.** Let $A$ and $B$ be algebras in $\text{Ch}_k^\text{fin}$, and let $X$ be an $A$-$B$-module of finite type. If $X$ is semifree as a left $A$-module and strictly right dualizable, then $X$ satisfies effective homotopic descent with respect to any $A$-coring $C$. In particular, if $B$ is semifree as a left $A$-module, then any morphism of algebras $\varphi : A \to B$ satisfies effective homotopic descent.

**Proof.** Parts (1) and (2) of Lemma 5.4 together imply that this result follows from Theorem 4.11. □

We are now ready to formulate homotopical Morita theory for corings in finite-type, non-negatively graded chain complexes of vector spaces.

**Theorem 5.14.** Let $(A,C)$ and $(B,D)$ be corings in $\text{Ch}_k^\text{fin}$. Let $(X,T) : (A,C) \to (B,D)$ be a braided bimodule.

1. If $X$ is a retract of a right $B$-semifree module, then the adjunction governed by $(X,T)$,

$$
\begin{array}{c}
(\text{Ch}_k)^C \\
\overset{T}{\longrightarrow} \\
(\text{Ch}_k)^D
\end{array}
$$

is a Quillen adjunction.

2. If, in addition, $X$ is left $A$-semifree and strictly right dualizable with left $B$-semifree dual $X^\vee$, $C$ is left $A$-semifree, and the left $B$-module underlying $D$ is a retract of a semifree module, then this adjunction is a Quillen equivalence if and only if the associated morphism of corings $g_T : X_* (C) \to D$ is a quasi-isomorphism.

**Proof.** (1) This follows immediately from Proposition 4.5 since $X$ is cofibrant as a right $B$-module.

(2) To see that Theorem 4.14 implies the desired equivalence, observe that
all objects in $\text{Ch}_k^{\text{fin}}$ are fibrant, and every algebra is semifree and therefore cofibrant as a module over itself;

- $X$ is cofibrant, homotopy compact, and homotopy flat as a left $A$-module by Lemma 5.3 and Lemma 5.4 (1) and (2);

- by Theorem 5.13, $X$ satisfies effective descent with respect to $C$;

- the coring $(A, C)$ is flat by Lemma 5.4 (2); and

- by Theorem 5.12, $g_T$ is copure, since the fact that $X^\vee$ is left $B$-semifree and $C$ is left $A$-semifree implies that $X_*(C)$ is left $B$-semifree.

□

We distinguish the following important special case of the theorem above. Recall Examples 3.20 and 3.35.

**Corollary 5.15.** Let $\varphi: A \to B$ be a morphism of algebras in $\text{Ch}_k^{\text{fin}}$ such that $B$ is semifree as a left $A$-module. Let $(\varphi, f): (A, C) \to (B, D)$ be a morphism of corings of finite type such that $C$ is left $A$-semifree and $D$ is left $B$-semifree.

The adjunction

$$
\left(\text{Ch}_k^{\text{fin}}\right)^C_A \xrightarrow{(T_{\varphi, f})_*} (\text{Ch}_k^{\text{fin}})_B^D
$$

is a Quillen equivalence if and only if $f: B_*(C) \to D$ is a quasi-isomorphism.

**Remark 5.16.** In particular, if we let $f$ be the identity map of $B_*(C)$, then the adjunction

$$
(\text{Ch}_k^{\text{fin}})_A^C \xrightarrow{(T_{\varphi, f})_*} (\text{Ch}_k^{\text{fin}})_B^{B_*(C)}
$$

is always a Quillen equivalence, as long as $B$ is left $A$-semifree.

**Proof.** It suffices to recall that $AB_B$ is always strictly right dualizable, with dual $BB_A$, which is obviously left $B$-semifree. Since $B_*(C)$ is also clearly left $B$-semifree, we can conclude by applying Theorem 5.14. □

**Remark 5.17.** The hypothesis on $\varphi: A \to B$ in the Corollary 5.15, requiring that $B$ be left $A$-semifree, is only mildly restrictive. For example, the $KS$-extensions (also known as relative Sullivan algebras) of rational homotopy theory [10] are classical examples of such algebra morphisms. More generally, any algebra morphism in $\text{Ch}_k^{\text{fin}}$ can be replaced up to homotopy by an algebra morphism such that the codomain in semifree over the domain. Indeed, every morphism $\varphi: A \to B$ admits a factorization

$$
(A, d) \xrightarrow{\varphi} (B, d),

\xrightarrow{j} (A \coprod TV, D)

\xrightarrow{p} (A \coprod TV, D)
$$

where $j$ is the inclusion into a free extension, and $p$ is a quasi-isomorphism, and [20] Proposition 4.3.11, Remark 4.3.12 implies that $(A \coprod TV, D)$ is left $A$-semifree.

**Example 5.18.** For any algebra $A$, right $A$-module $M$, and left $A$-module $N$, let $\text{Bar}(M; A; N)$ denote the usual two-sided bar construction. An easy computation shows that for all algebra maps $A' \to A$, the map of graded vector spaces

$$
\text{Bar}(A; A'; A) \to \text{Bar}(A; A'; A) \otimes_A \text{Bar}(A; A'; A)
$$

$$
\sum_{i=0}^n (a \otimes s_{a_1} \otimes \cdots s_{a_i} \otimes 1) \otimes (1 \otimes s_{a_{i+1}} \otimes \cdots s_{a_n} \otimes a')
$$
commutes with the differentials and defines an $A$-coring structure on $\text{Bar}(A; A'; A)$. Note that $\text{Bar}(A; A'; A)$ is both right and left $A$-semifree.

Let $A' \xrightarrow{\varphi'} B' \xrightarrow{\psi} A \xrightarrow{\varphi} B$ be a composable sequence of algebra maps in $\text{Ch}^\text{fin}_k$ such that $B$ is semifree as a left $A$-module. Let $C = \text{Bar}(A; A'; A)$, $D = \text{Bar}(B; B'; B)$, and $f = \text{Bar}(\varphi; \varphi'; \varphi) : C \to D$. The pair $(\varphi, f) : (A, C) \to (B, D)$ is then a map of corings, to which we can apply Corollary 5.15 concluding that

$$
(\text{Ch}^\text{fin}_k)_A \xrightarrow{} \text{(T}_{\varphi,f}) \xrightarrow{} (\text{Ch}^\text{fin}_k)_B$$

is a Quillen equivalence if and only if $\varphi' : A' \to B'$ is quasi-isomorphism, since $B_*(C) \cong \text{Bar}(B; A'; B)$.

In [3] we provide further concrete applications of Corollary 5.15 related to the theory of homotopic Hopf-Galois extensions.

**Appendix A. Enriched model categories**

In this appendix we review some elementary aspects of enriched model category theory. For a thorough treatment, we refer to Riehl [29].

Let $(V, \otimes, k)$ be a closed symmetric monoidal category. A category $\mathcal{C}$ has a $V$-structure if it is tensored, cotensored and enriched in $V$. For objects $X, Y \in \mathcal{C}$ and $K \in V$, we use the notation $K \otimes X \in \mathcal{C}$, $\text{Map}_\mathcal{C}(X, Y) \in V$, $Y^K \in \mathcal{C}$, for the tensor product, enrichment and cotensor product, respectively. We assume that the structures are compatible in the sense that there are natural bijections

$$\mathcal{C}(K \otimes X, Y) \cong V(K, \text{Map}_\mathcal{C}(X, Y)) \cong \mathcal{C}(X, Y^K).$$

An adjunction between $\mathcal{V}$-categories that preserves all relevant structure is called a $\mathcal{V}$-adjunction. As is well known, the following proposition characterizes $\mathcal{V}$-adjunctions.

**Proposition A.1.** The following are equivalent for an adjunction

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\xleftarrow{G} & \mathcal{D} & \xleftarrow{G} \\
\end{array}$$

between $\mathcal{V}$-categories.

1. The left adjoint $F$ is a tensor functor, i.e., there is a natural isomorphism $\alpha_{K,X} : F(K \otimes X) \cong K \otimes F(X)$ for $K \in \mathcal{V}$ and $X \in \mathcal{C}$, such that

$$\alpha_{K \otimes L,X} = (1 \otimes \alpha_{L,X}) \alpha_{K,L \otimes X},$$

for all $K, L \in \mathcal{V}$ and $X \in \mathcal{C}$.

2. The right adjoint $G$ is a cotensor functor, i.e., there is a natural isomorphism $\beta_{K,Y} : G(Y)^K \cong G(Y^K)$ for $K \in \mathcal{V}$ and $Y \in \mathcal{D}$, satisfying a similar associativity relation.

3. The adjunction is $\mathcal{V}$-enriched, i.e., there is a natural isomorphism $\phi^{X,Y} : \text{Map}_\mathcal{D}(F(Y), X) \cong \text{Map}_\mathcal{C}(X, Y)$ for $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. 


Proof. Use the Yoneda Lemma and the diagram

\[
\begin{array}{ccc}
\mathcal{D}(K \otimes FX,Y) & \xrightarrow{\sim} & \mathcal{V}(K,\text{Map}_\mathcal{D}(FX,Y)) \\
\Downarrow \alpha_{K,X} & & \Downarrow \phi_{X,Y}^K \\
\mathcal{D}(F(K \otimes X),Y) & \xrightarrow{\sim} & \mathcal{V}(K,\text{Map}_\mathcal{D}(FX,Y)) \\
\Downarrow \phi_{K,X,Y} & & \Downarrow \beta_{X,Y}^K \\
\mathcal{E}(K \otimes X,GY) & \xrightarrow{\sim} & \mathcal{E}(X,(GY)^K) \\
\end{array}
\]

\[\mathcal{E}(K \otimes X,GY) \xrightarrow{\sim} \mathcal{E}(X,GY)^K \]

to prove the proposition. \(\square\)

A.1. \(\mathcal{V}\)-model categories. Suppose that \((\mathcal{V}, \otimes, k)\) is a closed symmetric monoidal category that also admits the structure of a Quillen model category.

**Definition A.2.** A \(\mathcal{V}\)-category \(\mathcal{C}\) with a model structure is called a \(\mathcal{V}\)-model category if the following axiom is satisfied.

**Axiom A.3.** Given morphisms

\[i: A \rightarrow B \in \mathcal{C}, \quad j: K \rightarrow L \in \mathcal{V}, \quad p: X \rightarrow Y \in \mathcal{C},\]

such that \(i\) and \(j\) are cofibrations, \(p\) is a fibration, and at least one out of \(i, j, p\) is a weak equivalence, the following lifting problems can be solved.

1. (In terms of the \(\mathcal{V}\)-enrichment of \(\mathcal{C}\))

\[
\begin{array}{c}
K \\
\downarrow j \\
L \\
\end{array} \rightarrow 
\begin{array}{c}
\text{Map}_\mathcal{E}(B,X) \\
\downarrow i \circ p \\
\text{Map}_\mathcal{E}(A,X) \times_{\text{Map}_\mathcal{E}(A,Y)} \text{Map}_\mathcal{E}(B,Y);
\end{array}
\]

2. (In terms of the \(\mathcal{V}\)-tensor structure on \(\mathcal{C}\))

\[
\begin{array}{c}
K \otimes B \sqcup_{K \otimes A} L \otimes A \\
\downarrow j \circ i \\
L \otimes B \\
\end{array} \rightarrow 
\begin{array}{c}
X \\
\downarrow p \\
Y;
\end{array}
\]

3. (In terms of the \(\mathcal{V}\)-cotensor structure on \(\mathcal{C}\))

\[
\begin{array}{c}
A \\
\downarrow j \circ p \\
B \\
\end{array} \rightarrow 
\begin{array}{c}
X^L \\
\downarrow \circ j \circ p \\
X^K \times_{Y^K} Y^L.
\end{array}
\]

It is an exercise in adjunctions to show that the lifting problems are equivalent: a solution to one yields solutions to the two other upon taking appropriate adjoints.

The symmetric monoidal category \(\mathcal{V}\), with its given model structure and its canonical \(\mathcal{V}\)-structure, is called a symmetric monoidal model category if it is a \(\mathcal{V}\)-model category itself.

A.2. \(\mathcal{V}\)-enrichment of induced model structures. Consider an adjunction between model categories

\[\mathcal{C} \xrightarrow{F} \mathcal{D}, \quad \mathcal{D} \xleftarrow{G} \mathcal{C}\]

where \(F\) is the left adjoint. We say that the model structure on \(\mathcal{D}\) is right induced from \(\mathcal{C}\) if a map \(f\) in \(\mathcal{D}\) is a weak equivalence (fibration) if and only if \(Gf\) is a weak
equivalence (fibration) in \( C \). Dually, we will say that the model structure on \( C \) is left induced from \( D \) if a map \( f \) in \( C \) is a weak equivalence (cofibration) if and only if \( Ff \) is a weak equivalence (cofibration) in \( D \).

The following proposition is presumably well-known (see e.g. [25] Lemma 2.25) for the case of left-induced structures, but we indicate the proof for the reader’s convenience.

**Proposition A.4.** A model structure induced from a \( \mathcal{V} \)-model structure along a \( \mathcal{V} \)-adjunction is itself a \( \mathcal{V} \)-model structure.

More precisely, suppose given an adjunction between model categories as in \( \text{(A.1)} \), where \( C \) and \( D \) have \( \mathcal{V} \)-structures and the adjunction has a \( \mathcal{V} \)-structure.

1. If the model structure on \( D \) is right induced from \( C \), and \( C \) satisfies Axiom \( A.3 \), then so does \( D \).
2. If the model structure on \( C \) is left induced from \( D \), and \( D \) satisfies Axiom \( A.3 \), then so does \( C \).

**Proof.** Given a cofibration \( j: K \to L \) in \( \mathcal{V} \) and a fibration \( p: X \to Y \) in \( D \), we need to show that \( j \circ p: X^L \to X^K \times_{Y^K} Y^L \) is a fibration and that it is a weak equivalence if either \( j \) or \( p \) is a weak equivalence. The map \( Gp \) is a fibration in \( C \), and since \( C \) satisfies Axiom \( A.3 \), the map \( j \circ Gp \) is a fibration. Since \( G \) is a cotensor functor, there is an isomorphism of morphisms \( G(j \circ p) \cong j \circ Gp \), whence \( G(j \circ p) \) is a fibration. Since the model structure on \( D \) is right induced from the model structure on \( C \), this means that \( j \circ p \) is a fibration. We leave the rest of the proof to the reader.

\[ \square \]

**Appendix B. Dualizability**

The notion of dualizability plays an important role in our study of those (braided) bimodules that induce Quillen equivalences between model categories of modules over algebras and of comodules over corings. We recall here this classical notion and some of its elementary properties, expressed in terms of adjunctions in bicategories, and refer the reader to [26] for further details and references. We do not recall the definition of a bicategory, which the reader can find at [25].

The following definition generalizes the usual notion of an adjunction of categories.

**Definition B.1.** Let \( C \) be a bicategory. An adjunction in \( C \) consists of a pair of objects \( A \) and \( B \), a pair of 1-morphisms \( l: A \to B \) and \( r: B \to A \), and a pair of 2-morphisms \( \eta: 1_A \to rl \) and \( \epsilon: lr \to 1_B \) satisfying the triangle identities

\[
(r \circ \epsilon)(\eta \circ r) = 1_r \quad \text{and} \quad (\epsilon \circ l)(l \circ \eta) = 1_l,
\]

where \( \ast \) denote the usual whiskering of 2-morphisms by a 1-morphism. We call \( l \) the left adjoint, \( r \) the right adjoint, \( \eta \) the unit, and \( \epsilon \) the counit of the adjunction, and write \( l \dashv r \).

**Remark B.2.** The right adjoint of a 1-morphism is unique up to isomorphism if it exists. Moreover, bifunctors clearly preserve adjunctions.

**Definition B.3.** Let \( (\mathcal{V}, \otimes, \text{id}) \) be a monoidal category. Its delooping bicategory \( \mathcal{B}\mathcal{V} \) is the bicategory with exactly one object \( \bullet \) and such that the category \( \mathcal{B}\mathcal{V}(\bullet, \bullet) \) is \( \mathcal{V} \), where composition of 1-morphisms in \( \mathcal{B}\mathcal{V} \) is given by the tensor product of objects in \( \mathcal{V} \), and composition of 2-morphisms in \( \mathcal{B}\mathcal{V} \) is the same as composition of morphisms in \( \mathcal{V} \).

**Definition B.4.** Let \( (\mathcal{V}, \otimes, \text{id}) \) be a monoidal category. An object \( X \) in \( \mathcal{V} \) is right dualizable if, seen as a 1-morphism in \( \mathcal{B}\mathcal{V} \), it admits a right adjoint \( Y \), which we call a right dual of \( X \), while \( X \) is a left dual to \( Y \).
Remark B.5. Unraveling the definition above, we see that $Y$ is a right dual to $X$ if there are morphisms

$$u: k \to X \otimes Y \quad \text{and} \quad e: Y \otimes X \to k,$$

which we call the coevaluation and evaluation in order to distinguish them from the numerous other units and counits of adjunctions with which we work in this paper, such that the composites

$$X \overset{u \otimes 1}{\to} X \otimes Y \otimes X \overset{1 \otimes e}{\to} X \quad \text{and} \quad Y \overset{1 \otimes u}{\to} Y \otimes X \otimes Y \overset{e \otimes 1}{\to} Y$$

are both identities.

Example B.6. It is easy to prove that if $\mathcal{V}$ is the category of abelian groups, then a bimodule $A \otimes B$ is right dualizable if and only if it is finitely generated and projective as a right $B$-module.

If $\mathcal{V}$ is a stable model category, then $X$ is dualizable if and only if it is compact as an object of the triangulated homotopy category, see, e.g., [21, Theorem A.1]. For example, compact objects in the derived category of a ring are precisely bounded complexes of finitely generated, projective modules.

The following well known example of a bicategory, in which the notion of adjunction is a many-object generalization of dualizability, is important in this paper.

Example B.7. Let $\mathcal{V}$ be a monoidal category. The bicategory $\text{ALG}_\mathcal{V}$ of bimodules in $\mathcal{V}$ has as objects all algebras in $\mathcal{V}$, while a 1-morphism from $A$ to $B$ is an $A$-$B$-bimodule $A \otimes B$, and a 2-morphism from $A \otimes B$ to $A \otimes B$ is a morphism of $A$-$B$-bimodules. If $X: A \to B$ and $Y: B \to C$ are 1-morphisms, then their composite is defined to be $X \otimes_B Y: A \to C$. Note that for any object $A$, the identity morphism on $A$ is itself, seen as an $A$-$A$-bimodule.

If there is an adjunction

$$A \overset{X}{\leftarrow} B; \quad u: A \to X \otimes_B Y, \quad e: Y \otimes_A X \to B,$$

then we say that $X$ is right dualizable and call $Y$ a right dual of $X$ and $X$ a left dual of $Y$. Notice that the composites

$$X \overset{u \otimes 1}{\to} X \otimes_B Y \otimes_A X \overset{1 \otimes e}{\to} X \quad \text{and} \quad Y \overset{1 \otimes u}{\to} Y \otimes_A X \otimes_B Y \overset{e \otimes 1}{\to} Y$$

are both identities.

To emphasize the difference between this definition and that of section 2.3, in which homotopy of bimodules is taken into account, we sometimes say that a bimodule is strictly dualizable if it is dualizable in the sense of this example.

Motivated by the fact that the usual notion of dual is a special case of the notion of adjoint in a bicategory, we introduce the following notation.

Notation B.8. Let $C$ be a bicategory, and let $l \dashv r$ be an adjunction in $C$. We then write

$$l^\vee := r.$$

The next lemma is well known to many category theorists and homotopy theorists; one proof can be found in [21, §III.1, Proposition 1.3].

Lemma B.9. Let $\mathcal{V}$ be a closed monoidal category. Let $A$ and $B$ be algebras in $\mathcal{V}$. The following are equivalent for any $A$-$B$-bimodule $X$.

1. $X$ is right dualizable.
2. For all right $B$-modules $N$, the map $\ell_N : N \otimes_B \text{Map}_B(X, B) \to \text{Map}_B(X, N)$ that is adjoint to $N \otimes_B \text{Map}_B(X, B) \otimes_A X \overset{1 \otimes \text{ev}}{\to} N$ is an isomorphism.
When $X$ is dualizable, the $B$-$A$-bimodule $\text{Map}_B(X, B)$ is a right dual to $X$.

Proof. Since the proof can be found elsewhere, we simply remind the reader how to prove the last statement. The right adjoint of a functor is uniquely determined up to isomorphism. The functor $-\otimes_A X: \mathcal{A} \to \mathcal{Y}$ always has right adjoint $\text{Map}_B(X, -)$. Hence, $Y$ is right dual to $X$ if and only if there is a natural isomorphism of $A$-modules

$$N \otimes_B Y \cong \text{Map}_B(X, N).$$

Plugging in $N = B$, we see that $Y$ must be isomorphic to $\text{Map}_B(X, B)$. \hfill \qed

References

1. Jiří Adámek and Jiří Rosický, Locally presentable and accessible categories, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, Cambridge, 1994. MR 1294136 (95j:18001)
2. Marzieh Bayeh, Kathryn Hess, Varvara Karpova, Magdalena Kedziorek, Emily Riehl, and Brooke Shipley, Left-induced model category structures on diagram categories, available on the arXiv, to appear in Contemporary Mathematics, 2014.
3. Alexander Berglund and Kathryn Hess, Homotopic descent, Koszul duality, and Hopf-Galois extensions, in preparation.
4. Francis Borceux, Handbook of categorical algebra. 2, Encyclopedia of Mathematics and its Applications, vol. 51, Cambridge University Press, Cambridge, 1994, Categories and structures. MR 1313497 (96g:18001b)
5. Tomasz Brzeziński and Robert Wisbauer, Corings and comodules, London Mathematical Society Lecture Note Series, vol. 309, Cambridge University Press, Cambridge, 2003. MR 2012570 (2004k:16093)
6. Michael Ching and Emily Riehl, Coalgebraic models for combinatorial model categories, available on the ArXiv, 2014.
7. Daniel Dugger and Brooke Shipley, Topological equivalences for differential graded algebras, Adv. Math. 212 (2007), no. 1, 37–61. MR 2319762 (2008e:55025)
8. William G. Dwyer and Jan Spalinski, Homotopy theories and model categories, Handbook of Algebraic Topology (I.M. James, ed.), 1995, pp. 73–126.
9. Samuel Eilenberg, Abstract description of some basic functors, J. Indian Math. Soc. (N.S.) 24 (1960), 231–234 (1961). MR 0125148 (23 #A2454)
10. Yves Félix, Stephen Halperin, and Jean-Claude Thomas, Rational homotopy theory, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001. MR 1802847 (2002d:55014)
11. Paul G. Goerss and John F. Jardine, Simplicial homotopy theory, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999. MR 1711612 (2001d:55012)
12. Kathryn Hess, Morita theory for Hopf-Galois extensions: foundations and examples, New topological contexts for Galois theory and algebraic geometry (BIRS 2008), Geom. Topol. Monogr., vol. 16, Geom. Topol. Publ., Coventry, 2009, pp. 79–132. MR 2544387 (2010j:55010)
13. Kathryn Hess, Morita theory for Hopf algebroids and presheaves of groupoids, Amer. J. Math. 124 (2002), no. 6, 1289–1318. MR 1939787 (2003k:16053)
14. Mark Hovey and Neil Strickland, Comodules and Landweber exact homology theories, Adv. Math. 192 (2005), no. 2, 427–456. MR 2128706 (2006e:55007)
15. G. Janelidze and W. Tholen, Facets of descent. III. Monadic descent for rings and algebras, Appl. Categ. Structures 12 (2004), no. 5-6, 461–477. MR 2107397 (2005i:18019)
16. Varvara Karpova, Homotopic Hopf-Galois extensions of commutative differential graded algebras, Ph.D. thesis, EPFL, 2014.
17. L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure, Equivariant stable homotopy theory, Lecture Notes in Mathematics, vol. 1213, Springer-Verlag, Berlin, 1986, With contributions by J. E. McClure. MR 866482 (88e:55002)
22. F. E. J. Linton, *Coequalizers in categories of algebras*, Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67), Springer, Berlin, 1969, pp. 75–90. MR 0244341 (39 #5656)

23. Bachuki Mesablishvili, *Monads of effective descent type and comonadicity*, Theory Appl. Categ. 16 (2006), No. 1, 1–45 (electronic). MR MR2210664 (2006m:18002)

24. Amnon Neeman, *Derived categories and Grothendieck duality*, Triangulated categories, London Math. Soc. Lecture Note Ser., vol. 375, Cambridge Univ. Press, Cambridge, 2010, pp. 290–350. MR 2681711 (2012h:14038)

25. nLab contributors, *2-category*, http://ncatlab.org/nlab/show/2-category.

26. *Dualizable object*, http://ncatlab.org/nlab/show/dualizable+object.

27. Daniel G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967. MR 0223432 (36 #6480)

28. Jeremy Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. (2) 43 (1991), no. 1, 37–48. MR 1099084 (92b:16043)

29. Emily Riehl, *Categorical homotopy theory*, New Mathematical Monographs, Cambridge University Press, 2014.

30. Stefan Schwede and Brooke Shipley, *Stable model categories are categories of modules*, Topology 42 (2003), no. 1, 103–153. MR 1928647 (2003g:55034)

31. Stefan Schwede and Brooke E. Shipley, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. (3) 80 (2000), no. 2, 491–511. MR 1734325 (2001c:18006)

32. Brooke Shipley, *Morita theory in stable homotopy theory*, Handbook of tilting theory, London Math. Soc. Lecture Note Ser., vol. 332, Cambridge Univ. Press, Cambridge, 2007, pp. 393–411. MR 2384618 (2009c:16006)

33. Mitsuhiro Takeuchi, *Morita theorems for categories of comodules*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 24 (1977), no. 3, 629–644. MR 0472967 (57 #12646)

34. Charles E. Watts, *Intrinsic characterizations of some additive functors*, Proc. Amer. Math. Soc. 11 (1960), 5–8. MR 0118757 (22 #9528)

35. Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR 1269324 (95f:18001)

36. E. C. Zeeman, *A proof of the comparison theorem for spectral sequences*, Proc. Cambridge Philos. Soc. 53 (1957), 57–62. MR 0084769 (18,918f)

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden
E-mail address: alexb@math.su.se

MATHGEOM, École Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland
E-mail address: kathryn.hess@epfl.ch