Canonical partition function for anomalous systems described by the $\kappa$-entropy

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Starting from the $\kappa$-distribution function, obtained by applying the maximal entropy principle to the $\kappa$-entropy [G. Kaniadakis, Phys. Rev. E 66 (2002), 056125], we derive the expression of the canonical $\kappa$-partition function and discuss its main properties. It is shown that all important macroscopical quantities of the system can be expressed employing only the $\kappa$-partition function. The relationship between the associated $\kappa$-free energy and the $\kappa$-entropy is also discussed.

§1. Introduction

Anomalous statistical systems which exhibit an asymptotic power law behavior in the probability distribution function (pdf) are ubiquitous in nature.$^{1-3}$ Although not in equilibrium these distributions characterize a metastable configuration in which the system remains for a very long period of time compared to the typical time scales of its underlying microscopical dynamics. The statistical properties of such systems can be investigated through the introduction of a generalization of the Boltzmann-Gibbs (BG) entropy $S^B = -\sum_i p_i \ln(p_i)$ with $p \equiv \{p_i\}_{i=1,\ldots,N}$ a discrete pdf, (throughout this paper we adopt units within $k_B = 1$) from which, by means of the maximal entropy principle, the corresponding pdf can be derived.

Recently, $^{4-6}$ it has been proposed a generalized statistical mechanics based on the deformed entropy

$$S_\kappa = -\sum_i p_i \ln_{(\kappa)}(p_i),$$

which preserves the structure of the BG statistical mechanics.

Eq. (1.1) mimics the well-known classical entropy by replacing the standard logarithm with the generalized version

$$\ln_{(\kappa)}(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa},$$

where $-1 < \kappa < 1$, $\ln_{(\kappa)}(x) = \ln_{(\kappa)}(x)$ and $\ln_{(\kappa)}(x) = -\ln_{(\kappa)}(1/x)$. Eq. (1.2) reduces to the classical logarithm in the $\kappa \to 0$ limit: $\ln_{(\kappa)}(x) = \ln(x)$, as well as, in

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the same limit, Eq. (1.1) reduces to the BG entropy. The $\kappa$-entropy (1.1) has many properties of the standard entropy like positivity, continuity, symmetry, expansibility, decisivity, maximality and concavity. Moreover, in\(^9\) it has been shown that the $\kappa$-entropy is also Lesche stable, an important property that must be fulfilled in order to represent a well defined physical observable. The maximization of Eq. (1.1) under the constraints on the normalization and on the mean energy

\[
\sum_i p_i = 1, \quad \sum_i E_i p_i = U, \tag{1.3}
\]

leads to the following $\kappa$-distribution function\(^6\)

\[
p_i = \alpha \exp_\kappa \left( -\frac{1}{\lambda} (\gamma + \beta E_i) \right), \tag{1.4}
\]

where $\gamma$ and $\beta$ are the Lagrange multipliers associates to the constraints (1.3). The deformed exponential $\exp_\kappa (x)$, the inverse function of $\ln_\kappa (x)$, is defined as

\[
\exp_\kappa (x) = \left( \kappa x + \sqrt{1 + \kappa^2 x^2} \right)^{1/\kappa}, \tag{1.5}
\]

and reduces to the standard exponential in the $\kappa \to 0$ limit: $\exp_0 (x) = \exp(x)$. The constants $\lambda$ and $\alpha$ in Eq. (1.4) are given by

\[
\lambda = \sqrt{1 - \kappa^2}, \quad \alpha = \left( \frac{1 - \kappa}{1 + \kappa} \right)^{1/2}, \tag{1.6}
\]

respectively, and are related each other through the relation $\ln_\kappa (\alpha) = -1/\lambda$.

In\(^6\),\(^7\) it has been shown that, starting from the deformed logarithm (1.2) and the deformed exponential (1.5), the $\kappa$-algebra can be developed in a way that many algebraic properties of the standard logarithm and exponential can be reproduced in the deformed version. For instance, through the definition of the $\kappa$-sum $x \oplus y$, given by

\[
x \oplus y = x \sqrt{1 + \kappa^2 y^2} + y \sqrt{1 + \kappa^2 x^2}, \tag{1.7}
\]

it is easy to verify the following useful relation

\[
\exp_\kappa (x \oplus y) = \exp_\kappa (x) \cdot \exp_\kappa (y). \tag{1.8}
\]

In\(^6\),\(^7\) it has been shown that the $\kappa$-statistical mechanics emerges within the special relativity. The physical mechanism introducing the $\kappa$-deformation is originated from the Lorentz transformations and the $\kappa \to 0$ limit, reproducing the ordinary statistical mechanics, corresponds to the classical limit $c \to \infty$.

The purpose of this work is to introduce the generalized canonical partition function $Z_\kappa$ in the framework of the $\kappa$-deformed statistical mechanics and to show that all the relevant relations, valid in the BG theory, still hold in the deformed version.
We start by recalling the useful relations. Firstly, the $\kappa$-logarithm fulfills the functional-differential equation $^8$

$$\frac{d}{dx}\left[x\Lambda(x)\right] = \lambda\frac{x}{\alpha}\Lambda\left(\frac{x}{\alpha}\right), \quad (1.9)$$

with the boundary conditions $\Lambda(1) = 0$ and $(d/dx)\Lambda(x)\big|_{x=1} = 1$. Another solution of Eq. (1.9), which follows from the boundary conditions $\Lambda(1) = 1$ and $(d/dx)\Lambda(x)\big|_{x=1} = 0$, is given by $^{11}$

$$u_{(\kappa)}(x) = \frac{x^\kappa + x^{-\kappa}}{2}. \quad (1.10)$$

It fulfills the relations $u_{(\kappa)}(x) = u_{(-\kappa)}(x)$, $u_{(\kappa)}(x) = u_{(\kappa)}(1/x)$ and $u_{(\kappa)}(1) = 1/\lambda$.

From the definitions (1.2) and (1.10) we obtain the useful equations

$${\ln}_{(\kappa)}(xy) = u_{(\kappa)}(x)\ln_{(\kappa)}(y) + \ln_{(\kappa)}(x)u_{(\kappa)}(y), \quad (1.11)$$

$${u}_{(\kappa)}(xy) = u_{(\kappa)}(x)u_{(\kappa)}(y) + \kappa^2\ln_{(\kappa)}(x)\ln_{(\kappa)}(y), \quad (1.12)$$

showing a deep link between both the functions $\ln_{(\kappa)}(x)$ and $u_{(\kappa)}(x)$.

Starting from Eq. (1.10), in analogy with Eq. (1.1), we introduce the function

$$I_{\kappa} = \sum_i p_iu_{(\kappa)}(p_i), \quad (1.13)$$

which can also be defined as the mean value of $u_{(\kappa)}(x)$ according to the relation $I_{\kappa} = \langle u_{(\kappa)}(p) \rangle$. The $\kappa$-entropy $S_{\kappa} = \langle \ln_{(\kappa)}(p) \rangle$ as well, can be defined as the mean value of $\ln_{(\kappa)}(x)$.

It is worth to observe that, by using the definitions (1.1) and (1.13), we obtain from Eq. (1.11) the equation $^{11}$

$$S_{\kappa}(A \cup B) = I_{\kappa}(A)S_{\kappa}(B) + S_{\kappa}(A)I_{\kappa}(B), \quad (1.14)$$

stating the additivity rule of the $\kappa$-entropy for two statistically independent systems $A$ and $B$, in the sense of $p_{A \cup B} = p_A \cdot p_B$. In the $\kappa \to 0$ limit $u_{(0)}(p_i) = 1$ and $I_0(p) = \sum_i p_i = 1$ so that Eq. (1.14) recovers the additivity rule of the BG entropy.

Returning to the $\kappa$-distribution, we pose

$$x_i = \gamma + \beta E_i, \quad (1.15)$$

so that Eq. (1.14) can be written in

$$p_i = \alpha \exp_{(\kappa)}\left(-\frac{x_i}{\lambda}\right) = \exp_{(\kappa)}\left(-\frac{1}{\lambda}\right) \cdot \exp_{(\kappa)}\left(-\frac{x_i}{\lambda}\right)$$

$$= \exp_{(\kappa)}\left(-\frac{1}{\lambda}\right) \oplus \left(-\frac{x_i}{\lambda}\right), \quad (1.16)$$
where Eq. (1.18) has been employed. By using the definition of the \( \kappa \)-sum the argument in Eq. (1.16) can be rewritten in
\[
\left( -\frac{1}{\lambda} \right) \oplus \left( -\frac{x}{\lambda} \right) = -\frac{1}{\lambda} \left( \frac{x_i}{\lambda} + \sqrt{1 + \frac{\kappa^2}{\lambda^2} x^2_i} \right),
\]
and Eq. (1.16) becomes
\[
p_i = \exp_{\{\kappa\}} \left[ -\frac{1}{\lambda} \left( \frac{x_i}{\lambda} + \sqrt{1 + \frac{\kappa^2}{\lambda^2} x^2_i} \right) \right].
\]
On the other hand, by using Eq. (1.12) with \( x = \alpha \) and \( y = \exp_{\{\kappa\}} \left( -x_i/\lambda \right) \), and taking into account the relation
\[
\exp_{\{\kappa\}} \left( \exp_{\{\kappa\}} \left( -x_i/\lambda \right) \right) = \sqrt{1 + \frac{\kappa^2}{\lambda^2} x^2_i},
\]
which follows from the definitions (1.5) and (1.10), we obtain
\[
u_{\{\kappa\}} (p_i) = \frac{1}{\lambda} \left( \frac{\kappa^2}{\lambda} x_i + \sqrt{1 + \frac{\kappa^2}{\lambda^2} x^2_i} \right),
\]
so that Eq. (1.18) can be easily written as
\[
p_i = \exp_{\{\kappa\}} \left( -\nu_{\{\kappa\}} (p_i) - \gamma - \beta E_i \right),
\]
which is an alternative but equivalent expression of the \( \kappa \)-distribution (1.4).

§2. Canonical partition function

We recall that in the classical theory the canonical partition function \( Z \) is an important quantity that encodes the statistical properties of a system. From the BG distribution
\[
p_i = e^{-1-\gamma} e^{-\beta E_i} = \frac{1}{Z} e^{-\beta E_i},
\]
it follows that the partition function can be introduced as
\[
\ln(Z) = 1 + \gamma.
\]
It depends firstly on the Lagrange multiplier \( \beta \) and secondly on the microstate energies \( E_i \) which are determined by other macroscopical quantities like the volume or the number of particles. Remarkably, most of the thermodynamical functions of the system can be expressed in terms of the partition function or its derivatives. For instance, the entropy \( S_{BG} = \ln(Z) + \beta U \), the total energy \( U = -d \ln(Z) / d \beta \) and the free energy \( F = -\ln(Z) / \beta \).
We observe that, by using the expression (1.21), from the definition of the $\kappa$-entropy we obtain the relation

$$
S_\kappa = -\sum_i p_i \ln(p_i) = \sum_i p_i \left( u_\kappa(p_i) + \gamma + \beta E_i \right) = \mathcal{I}_\kappa + \gamma + \beta U ,
$$

which reminds us the classical expression $S = 1 + \gamma + \beta U$, and it is recovered in the $\kappa \to 0$ limit.

According to Eq. (2.3), taking into account its limit for $\kappa \to 0$ and the classical relationship between the BG entropy and the canonical partition function $Z$, we are guided to define the canonical $\kappa$-partition function $Z_\kappa$ through

$$
\ln_{(\kappa)}(Z_\kappa) = \mathcal{I}_\kappa + \gamma .
$$

Eq. (2.4) reduces to the definition (2.2) in the $\kappa \to 0$ limit.

In order to verify the consistency of the definition (2.4), we derive its main properties in the framework of the $\kappa$-statistical mechanics. Firstly, it is trivial to verify that the entropy (2.3) becomes

$$
S_\kappa = \ln_{(\kappa)}(Z_\kappa) + \beta U ,
$$

which mimics the corresponding classical relationship.

Successively, we compute the derivative of the entropy $S_\kappa$ w.r.t the mean energy

$$
\frac{d S_\kappa}{dU} = -\sum_i \frac{d}{dp_i} \left[ p_i \ln_{(\kappa)}(p_i) \right] \frac{dp_i}{dU} = -\lambda \sum_i \ln_{(\kappa)} \left( \frac{p_i}{\alpha} \right) \frac{d p_i}{dU}
$$

$$
= \sum_i (\gamma + \beta E_i) \frac{d p_i}{dU} ,
$$

where we have taken into account Eqs. (1.4) and (1.9). Under the no-work condition $\sum_i p_i dE_i = 0$ (consequently $dU = \sum_i E_i dp_i$), and recalling that $\sum_i dp_i = 0$, which follows from the normalization condition on the pdf, we obtain

$$
\frac{d S_\kappa}{dU} = \beta .
$$

On the other hand, starting from Eq. (2.5) we have

$$
\frac{d S_\kappa}{dU} = \frac{d}{dU} \ln_{(\kappa)}(Z_\kappa) + U \frac{d \beta}{dU} + \beta ,
$$

and by comparing this relation with Eq. (2.7) it follows that, the following important property

$$
\frac{d}{d\beta} \ln_{(\kappa)}(Z_\kappa) = -U ,
$$

must hold.

Eq. (2.9) can be proved directly from the definition (2.4). In fact, by using Eq. (1.9)
we obtain
\[
\frac{d}{d \beta} \ln_{\kappa} (Z_\kappa) = \frac{d}{d \beta} (I_\kappa + \gamma) \\
= \sum_i \frac{d}{dp_i} \left[ p_i u_{\kappa} (p_i) \right] \frac{dp_i}{d \beta} + \frac{d \gamma}{d \beta} \\
= \lambda \sum_i u_{\kappa} \left( \frac{p(x_i)}{\alpha} \right) \frac{dp(x_i)}{dx_i} \frac{dx_i}{d \beta} + \frac{d \gamma}{d \beta},
\]
(2.10)
where \( p(x_i) \equiv p_i \). Using in Eq. (2.10) the relation
\[
p(x_i) = -\lambda u_{\kappa} \left( \frac{p(x_i)}{\alpha} \right) \frac{dp(x_i)}{dx_i},
\]
(2.11)
we finally obtain
\[
\frac{d}{d \beta} \ln_{\kappa} (Z_\kappa) = -\sum_i p_i \left( \frac{d \gamma}{d \beta} + E_i \right) + \frac{d \gamma}{d \beta} = -U.
\]
(2.12)
Eq. (2.9) is the dual relation of Eq. (2.5). They state, from one hand that both \( \beta \) and \( U \) are canonically conjugate variables, and on the other hand that \( S_\kappa \) is a function of \( U \) whereas \( Z_\kappa \) is a function of \( \beta \), like in the BG theory.

Accounting for the standard relationships between the free energy and the partition function, we introduce the \( \kappa \)-free energy
\[
F_\kappa = -\frac{1}{\beta} \ln_{\kappa} (Z_\kappa) .
\]
(2.13)
It is trivial to observe that the definition (2.13) can be obtained by means of a Legendre transformation on the mean energy
\[
F_\kappa = U - \frac{dU}{dS_\kappa} S_\kappa = U - \frac{1}{\beta} S_\kappa ,
\]
(2.14)
as it follows through Eqs. (2.5) and (2.7). Finally, we observe that the \( \kappa \)-free energy is a function of \( 1/\beta \) as it follows from the relation
\[
\frac{dF_\kappa}{d(1/\beta)} = -S_\kappa ,
\]
(2.15)
which mimics the classical relationship between the free energy and the BG entropy.

§3. Mean value of an observable

Another interesting relation involving the partition function (2.3) can be obtained by evaluating the derivative of \( \ln_{\kappa} (Z_\kappa) \) w.r.t the energy levels \( E_i \), for an equilibrium (meta-equilibrium) state with \( \beta \simeq \text{const} \). We have
\[
\frac{d}{dE_i} \ln_{\kappa} (Z_\kappa) = \frac{d}{dE_i} (I_\kappa + \gamma)
\]
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\[\begin{align*}
\sum_j \frac{d}{dp_j} \left[ p_j u_\kappa (p_j) \right] \frac{dp_j}{dE_i} + \frac{d\gamma}{dE_i} \\
= \lambda \sum_j u_\kappa \left( \frac{p(x_j)}{\alpha} \right) \frac{dp_j}{dx_j} \frac{dx_j}{dE_i} + \frac{d\gamma}{dE_i} \\
= -\sum_j p_j \left( \beta \delta_{ij} + \frac{d\gamma}{dE_i} \right) + \frac{d\gamma}{dE_i} \\
= -\beta p_i .
\end{align*}\]

Thus, if the partition function is known, we can compute the distribution function as

\[p_i = -\frac{1}{\beta} \frac{d}{dE_i} \ln_\kappa (Z_\kappa) .\]

All the mean values of the macroscopical quantities associated with the system can be expressed employing the function \(Z_\kappa:\)

\[\langle A \rangle = \sum_i p_i A_i = -\frac{1}{\beta} \sum_i A_i \frac{d}{dE_i} \ln_\kappa (Z_\kappa) .\]

We recall that the microstate energies depend on other macroscopical variables like the volume or the number of particles. By following the standard literature\(^{12}\), let us introduce the canonically conjugate variables through the relation

\[A_i = -\frac{dE_i}{dA} .\]

Then, Eq. (3.2) assumes the expression

\[\langle A \rangle = \frac{1}{\beta} \frac{d}{dA} \ln_\kappa (Z_\kappa) ,\]

which, taking into account the definition of \(\kappa\)-free energy \(^{[2]}\), becomes

\[\langle A \rangle = -\frac{dF_\kappa}{dA} .\]

Let us observe that if the energies \(E_i\) depend on a parameter \(\varepsilon\) as

\[E_i = E_i^{(0)} + \varepsilon A_i ,\]

then the mean value of \(A\) is given by

\[\langle A \rangle = -\frac{1}{\beta} \frac{d}{d\varepsilon} \ln_\kappa \left( Z_\kappa (\beta, \varepsilon) \right) ,\]

or equivalently

\[\langle A \rangle = \frac{d}{d\varepsilon} F_\kappa (\beta, \varepsilon) .\]
This provides us with an alternative, useful trick for calculating the expected values of an observable. In fact, by adding the eigenvalues $A_i$ of the observable $A$ to the energy levels $E_i$, we can calculate the new partition function $Z_\kappa(\beta, \varepsilon)$ and the mean value $\langle A \rangle$ according to Eq. (3.7), and then, we set $\varepsilon$ to zero in the final expression. Remark that this is analogous to the source field method used in the path integral formulation of quantum field theory.

§4. Conclusions

In the framework of the generalized statistical mechanics based on $\kappa$-entropy we have derived the canonical partition function $Z_\kappa(\beta)$ and studied its main properties. This function plays a relevant role in the formulation of $\kappa$-deformed thermostatistics theory based on the entropy $S_\kappa(U)$. It has been shown that most of the thermodynamic variables of the system, like the total energy $U$, the free energy $F_\kappa(1/\beta)$ and the entropy $S_\kappa(U)$ can be expressed in terms of the partition function $Z_\kappa(\beta)$ or its derivatives. We have shown an algorithm to derive, starting from the expression of the partition function, the mean value of all macroscopical observables associated to the system.

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