Large Generalized Books are $p$-Good

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Abstract

Let $B_q^{(r)} = K_r + qK_1$ be the graph consisting of $q$ distinct $(r + 1)$-cliques sharing a common $r$-clique. We prove that if $p \geq 2$ and $r \geq 3$ are fixed, then

$$r(K_{p+1}, B_q^{(r)}) = p(q + r - 1) + 1$$

for all sufficiently large $q$.

Keywords: Ramsey numbers; $p$-good; generalized books; Szemerédi lemma

1 Introduction

The title of this paper refers to the notion of goodness introduced by Burr and Erdős in [3] and subsequently studied by Burr and various collaborators. A connected graph $H$ is $p$-good if the Ramsey number $r(K_p, H)$ is given by

$$r(K_p, H) = (p - 1)(|V(H)| - 1) + 1.$$

In this paper we prove that for every $p \geq 3$ the generalized book $B_q^{(r)} = K_r + qK_1$ is $p$-good if $q$ is sufficiently large.

As much as possible, standard notation is used; see, for example, [2]. A set of cardinality $p$ is called a $p$-set. Unless explicitly stated, all graphs are defined on the vertex set $[n] = \{1, 2, \ldots, n\}$. Let $u$ be any vertex; then $N_G(u)$ and $d_G(u) = |N_G(u)|$ denote its neighborhood
and degree respectively. A graph with \( n \) vertices and \( m \) edges will be designated by \( G(n, m) \).

By an \( r \)-book we shall mean some number of independent vertices that are each connected to every vertex of an \( r \)-clique. The given \( r \)-clique is called the base of the \( r \)-book and the additional vertices are called the pages. The number of pages of an \( r \)-book is called its size; the size of the largest \( r \)-book in a graph \( G \) is denoted by \( bs^{(r)}(G) \). We shall denote the complete \( p \)-partite graph with each part having \( q \) vertices by \( K_{p}(q) \). The Ramsey number \( r(H_{1}, H_{2}) \) is the least number \( n \) such that for every graph \( G \) of order \( n \) either \( H_{1} \subset G \) or \( H_{2} \subset \overline{G} \).

## 2 The structure of subsaturated \( K_{p+1} \)-free graphs

We shall need the following theorem of Andrásfai, Erdős and Sós [1].

**Theorem 1** If \( G \) is a \( K_{p+1} \)-free graph of order \( n \) and
\[
\delta(G) > \left(1 - \frac{3}{3p-1}\right) n,
\]
then \( G \) is \( p \)-chromatic. \( \square \)

The celebrated theorem of Turán gives a tight bound on the maximum size of a \( K_{p} \)-free graph of given order. In the following theorem we show that if the size of a \( K_{p+1} \)-free graph is close to the maximum then we may delete a small portion of its vertices so that the remaining graph is \( p \)-chromatic. This is a particular stability theorem in extremal graph theory (see [9]).

**Theorem 2** For every \( p \geq 2 \) there exists \( c = c(p) > 0 \), such that for every \( \alpha \) satisfying \( 0 < \alpha \leq c \), every \( K_{p+1} \)-free graph \( G = G(n, m) \) satisfying
\[
m \geq \left(\frac{p-1}{2p} - \alpha\right) n^2
\]
contains an induced \( p \)-chromatic graph \( G_{0} \) of order at least \( (1 - 2\alpha^{1/3}) n \) and with minimum degree
\[
\delta(G_{0}) \geq \left(1 - \frac{1}{p} - 4\alpha^{1/3}\right) n.
\]
Proof Let $c_0$ be the smallest positive root of the equation

$$x^3 + \left(1 + \frac{3}{3p-1} \left(\frac{p-1}{p}\right)^2\right)x - \frac{1}{2(3p-1)p} = 0 \quad (1)$$

and set $c(p) = c_0^3$; then, for every $y$ satisfying $0 < y \leq c(p)$, we easily see that

$$y + \left(1 + \frac{3}{3p-1} \left(\frac{p-1}{p}\right)^2\right)y^{1/3} \leq \frac{1}{2(3p-1)p}. \quad (2)$$

A rough approximation of the function $c(p)$ is $c(p) \approx 6^{-3}p^{-6}$, obtained by neglecting the $x^3$ term in equation (1) and substituting the appropriate asymptotic (for large $p$) approximations for the remaining coefficients. This gives reasonable values even for small $p$. For all $p \geq 2,$

$$\frac{1}{(2p(3p+2))^3} < c(p) < \frac{1}{(2p(3p-1))^3}. \quad (3)$$

The upper bound is evident, and the lower bound follows from a simple computation.

Let $0 < \alpha \leq c(p)$ and the graph $G = G(n,m)$ satisfy the hypothesis of the theorem. We shall prove first that

$$\sum_{u=1}^{n} d^2(u) \leq 2 \left(\frac{p-1}{p}\right)mn. \quad (4)$$

Indeed, writing $k_3(G)$ for the number of triangles in $G$, we have

$$3k_3(G) = \sum_{uv \in E} |N(u) \cap N(v)| \geq \sum_{uv \in E} (d(u) + d(v) - n) = \sum_{u=1}^{n} d^2(u) - mn.$$ 

Applying Turán’s theorem to the $K_p$-free neighborhoods of vertices of $G$, we deduce

$$3k_3(G) \leq \frac{p-2}{2(p-1)} \sum_{u=1}^{n} d^2(u).$$

Hence,

$$\sum_{u=1}^{n} d^2(u) - mn \leq \frac{p-2}{2(p-1)} \sum_{u=1}^{n} d^2(u)$$

and (4) follows.
Since $0 < \alpha \leq c(p)$, taking the upper bound in (8) for $p = 2$, we see that $\alpha \leq 20^{-3}$. Hence,

\[
(1 + 8\alpha) \frac{4m^2}{n} \geq 2 (1 + 8\alpha) \left( \frac{p - 1}{p} - 2\alpha \right) mn
\]

\[
= 2 \left( \frac{p - 1}{p} + \left( 6 - \frac{8}{p} \right) \alpha - 16\alpha^2 \right) mn
\]

\[
\geq 2 \left( \frac{p - 1}{p} + 2\alpha - 16\alpha^2 \right) mn > 2 \left( \frac{p - 1}{p} \right) mn,
\]

and from (5) we deduce

\[
\sum_{u=1}^{n} \left( d(u) - \frac{2m}{n} \right)^2 = \sum_{u=1}^{n} d^2(u) - \frac{4m^2}{n} \leq 2 \left( \frac{p - 1}{p} \right) mn - \frac{4m^2}{n}
\]

\[
< 8\alpha \frac{4m^2}{n} \leq 8\alpha \left( \frac{p - 1}{p} \right)^2 n^3.
\]

Set $V = V(G)$ and let $M_\varepsilon$ be the set of all vertices $u \in V$ satisfying $d(u) < 2m/n - \varepsilon n$. For every $\varepsilon > 0$, inequality (5) implies

\[
|M_\varepsilon| \varepsilon^2 n^2 < \sum_{u \in M_\varepsilon} \left( d(u) - \frac{2m}{n} \right)^2 \leq \sum_{u \in V} \left( d(u) - \frac{2m}{n} \right)^2 \leq 8\alpha \left( \frac{p - 1}{p} \right)^2 n^3,
\]

and thus,

\[
|M_\varepsilon| < 8\varepsilon^{-2\alpha} \left( \frac{p - 1}{p} \right)^2 n.
\]

Furthermore, setting $G_\varepsilon = G[V \setminus M_\varepsilon]$, for every $u \in V(G_\varepsilon)$, we obtain

\[
d_{G_\varepsilon}(u) \geq d(u) - |M_\varepsilon| \geq \frac{2m}{n} - \varepsilon n - |M_\varepsilon| > \frac{p - 1}{p} n - 2\alpha n - \varepsilon n - |M_\varepsilon|.
\]

For $\varepsilon = 2\alpha^{1/3}$ we claim that

\[
\frac{p - 1}{p} n - 2\alpha n - \varepsilon n - |M_\varepsilon| > \frac{3p - 4}{3p - 1} (n - |M_\varepsilon|) = \frac{3p - 4}{3p - 1} v(G_\varepsilon).
\]

Indeed, assuming the opposite and applying inequality (5) with $\varepsilon = 2\alpha^{1/3}$, we see that

\[
\left( \frac{1}{(3p - 1)} - 2\alpha - 2\alpha^{1/3} \right) n \leq \frac{3}{3p - 1} |M_{2\alpha^{1/3}}| < 2 \cdot \frac{3}{3p - 1} \left( \frac{p - 1}{p} \right)^2 \alpha^{1/3} n;
\]
hence,
\[
2\alpha + 2 \left( 1 + \frac{3}{3p - 1} \left( \frac{p - 1}{p} \right)^2 \right) \alpha^{1/3} - \frac{1}{(3p - 1)p} > 0,
\]
contradicting (2).

Set \( G_0 = G_{2\alpha^{1/3}} \); from (3), we see that \( G_0 \) satisfies the conditions of Theorem \( \mathbf{1} \) so it is \( p \)-chromatic.

Finally, from (3) and (4), we have
\[
\delta(G_0) \geq \frac{p - 1}{p} n - 2\alpha n - 2\alpha^{1/3}n - \left( \frac{p - 1}{p} \right)^2 \alpha^{1/3}n > \frac{p - 1}{p} n - 2\alpha n - 3\alpha^{1/3}n
\]
\[
> \left( 1 - \frac{1}{p} - 4\alpha^{1/3} \right) n,
\]
completing the proof. \( \square \)

3 A Ramsey property of \( K_{p+1} \)-free graphs

The main result of this section is the following theorem.

**Theorem 3** Let \( r \geq 2, p \geq 2 \) be fixed. For every \( \xi > 0 \) there exists an \( n_0 = n_0(p, r, \xi) \) such that every graph \( G \) of order \( n \geq n_0 \) that is \( K_{p+1} \)-free either satisfies \( bs^{(r)}(G) > n/p \), or contains an induced \( p \)-chromatic graph \( G_1 \) of order \( (1 - \xi)n \) and minimum degree
\[
\delta(G_1) \geq \left( 1 - \frac{1}{p} - 2\xi \right) n.
\]

Our main tool in the proof of Theorem 3 is the regularity lemma of Szemerédi (SRL for short); for expository matter on SRL see [2] and [7]. For the sake of completeness we formulate here the relevant basic notions.

Let \( G \) be a graph; if \( A, B \subset V(G) \) are nonempty disjoint sets, we write \( e(A, B) \) for the number of \( A \) - \( B \) edges and call the value
\[
d(A, B) = \frac{e(A, B)}{|A||B|}
\]
the density of the pair \((A, B)\).

Let \(\varepsilon > 0\); a pair \((A, B)\) of two nonempty disjoint sets \(A, B \subset V(G)\) is called \(\varepsilon\)-regular if the inequality

\[ |d(A, B) - d(X, Y)| < \varepsilon \]

holds whenever \(X \subset A, Y \subset B, |X| \geq \varepsilon |A|, \) and \(|Y| \geq \varepsilon |B|\).

We shall use SRL in the following form.

**Theorem 4 (Szemerédi’s Regularity Lemma)** Let \(l \geq 1, \varepsilon > 0\). There exists \(M = M(\varepsilon, l)\) such that, for every graph \(G\) of sufficiently large order \(n\), there exists a partition \(V(G) = \bigcup_{i=0}^k V_i\) satisfying \(l \leq k \leq M\) and:

(i) \(|V_0| < \varepsilon n, |V_1| = \ldots = |V_k|; \)

(ii) all but at most \(\varepsilon k^2\) pairs \((V_i, V_j), (i, j \in [k])\), are \(\varepsilon\)-uniform.

We also need a few technical results; the first one is a basic property of \(\varepsilon\)-regular pairs (see [7], Fact 1.4).

**Lemma 1** Suppose \(0 < \varepsilon < d \leq 1\) and \((A, B)\) is an \(\varepsilon\)-regular pair with \(e(A, B) = d|A||B|\). If \(Y \subset B\) and \((d - \varepsilon)^{r-1}|Y| > \varepsilon|B|\) where \(r > 1\), then there are at most \(\varepsilon r|A|^r\) \(r\)-sets \(R \subset A\) with

\[ \left| \left( \bigcap_{u \in R} N(u) \right) \cap Y \right| \leq (d - \varepsilon)^r |Y|. \]

The next lemma gives a lower bound on the number of \(r\)-cliques in a graph consisting of several dense \(\varepsilon\)-regular pairs sharing a common part.

**Lemma 2** Suppose \(0 < \varepsilon < d \leq 1\) and \((d - \varepsilon)^{r-2} > \varepsilon\). Suppose \(H\) is a graph and \(V(H) = A \cup B_1 \cup \cdots \cup B_t\) is a partition with \(|A| = |B_1| = \cdots = |B_t|\) and such that for every \(i \in [t]\) the pair \((A, B_i)\) is \(\varepsilon\)-regular with \(e(A, B_i) \geq d|A||B_i|\). If \(m\) is the number of the \(r\)-cliques in \(A\), then at least

\[ t|A| (m - \varepsilon r|A|^r) (d - \varepsilon)^r \]

\((r + 1)\)-cliques of \(H\) have exactly \(r\) vertices in \(A\).
Proof. Set $a = |A| = |B_1| = \cdots = |B_t|$. For every $i \in [t]$, applying Lemma 1 to the pair $(A, B_i)$ with $Y = B_i$ we conclude that there are at most $\varepsilon ra^{r-1}$ $r$-sets $R \subset A$ with

$$\left| \left( \bigcap_{u \in R} N(u) \right) \cap B_i \right| \leq (d - \varepsilon)^r a,$$

and therefore, at least $(m - \varepsilon ra^r)$ $r$-cliques $R \subset A$ satisfy

$$\left| \left( \bigcap_{u \in R} N(u) \right) \cap B_i \right| > (d - \varepsilon)^r a.$$

Hence, at least $t(d - \varepsilon)^r (m - \varepsilon ra^r)a$ $(r + 1)$-cliques of $H$ have exactly $r$ vertices in $A$ and one vertex in $\bigcup_{i \in [t]} B_i$, completing the proof. \hfill \Box

The following consequence of Ramsey’s theorem has been proved by Erdős [5].

Lemma 3. Given integers $p \geq 2$, $r \geq 2$, there exist a $c_{p,r} > 0$ such that if $G$ is a $K_{p+1}$-free graph of order $n$ and $n \geq r(K_{p+1}, K_r)$ then $G$ contains at least $c_{p,r} n^r$ independent $r$-sets.

We need another result related to the regularity lemma of Szemerédi, the so-called Key Lemma (e.g., see [7], Theorem 2.1). We shall use the following simplified version of the Key Lemma.

Theorem 5. Suppose $0 < \varepsilon < d < 1$ and let $m$ be a positive integer. Let $G$ be a graph of order $(p + 1)m$ and let $V(G) = V_1 \cup \cdots \cup V_{p+1}$ be a partition of $V(G)$ into $p + 1$ sets of cardinality $m$ so that each of the pairs $(V_i, V_j)$ is $\varepsilon$-regular and has density at least $d$. If $\varepsilon \leq (d - \varepsilon)^p / (p + 2)$ then $K_{p+1} \subset G$.

Proof. Our proof is straightforward but rather rich in technical details, so we shall briefly outline it first. For some properly selected $\varepsilon$, applying SRL, we partition all but $\varepsilon n$ vertices of $G$ in $k$ sets $V_1, \ldots, V_k$ of equal cardinality such that almost all pairs $(V_i, V_j)$ are $\varepsilon$-regular. We may assume that the number of dense $\varepsilon$-regular pairs $(V_i, V_j)$ is no more than $\frac{p-1}{2p} k^2$, since otherwise, from Theorem 5 and Turán’s theorem, $G$ will contain a $K_{p+1}$.
Therefore, there are at least \((1/2p + o(1))k^2\) sparse \(\varepsilon\)-regular pairs \((V_i, V_j)\). From Lemma 3 it follows that the number of independent \(r\)-sets in any of the sets \(V_1, \ldots, V_k\) is \(\Theta(n^r)\). Consider the size of the \(r\)-book in \(G\) having for its base the average independent \(r\)-set in \(V_i\). For every sparse \(\varepsilon\)-regular pair \((V_i, V_j)\) almost every vertex in \(V_j\) is a page of such a book. Also each \(\varepsilon\)-regular pair \((V_i, V_j)\) whose density is not very close to 1 contributes substantially many additional pages to such books. Precise estimates show that either \(bs^{(r)}(G) > n/p\) or else the number of all \(\varepsilon\)-regular pairs \((V_i, V_j)\) with density close to 1 is \(\left(\frac{p-1}{2p} + o(1)\right)k^2\). Thus the size of \(G\) is \(\left(\frac{p-1}{2p} + o(1)\right)n^2\) and therefore, according to Theorem 2, \(G\) contains the required induced \(p\)-chromatic subgraph with the required minimum degree.

**Details of the proof.** Let \(c(p)\) be as in Theorem 2 and \(c_{p,r}\) be as in Lemma 3. Select

\[
\delta = \min \left\{ \frac{c^3}{32}, \frac{c(p)}{4} \right\},
\]

(9)

set

\[
d = \min \left\{ \left(\frac{\delta}{2}\right)^{r+1} \left(\frac{r}{c_{p,r}} + 2r + 1 + 2p\right)^{-1}, \frac{p\delta}{1 + p\delta} \left(\frac{r}{c_{p,r}} + 2r + 1\right)^{-1} \right\},
\]

(10)

and let

\[
\varepsilon = \min \left\{ \delta, \frac{d^p}{2 (p + 1)} \right\}.
\]

(11)

These definitions are justified at the later stages of the proof. Since \(c_{p,r} < r\) we easily see that \(0 < 2\varepsilon < d < \delta < 1\). Hence, Bernoulli’s inequality implies

\[
(d - \varepsilon)^p \geq d^p - p\varepsilon d^{p-1} > d^p - p\varepsilon = 2(p+1)\varepsilon - p\varepsilon = (p+2)\varepsilon.
\]

(12)

Applying SRL we find a partition \(V(G) = V_0 \cup V_1 \cup \cdots \cup V_k\) so that \(|V_0| < \varepsilon n\), \(|V_1| = \cdots = |V_k|\) and all but \(\varepsilon k^2\) pairs \((V_i, V_j)\) are \(\varepsilon\)-regular. Without loss of generality we may assume \(|V_i| > r(K_{p+1}, K_r)\) and \(k > 1/\varepsilon\). Consider the graphs \(H_{\text{irr}}, H_{\text{lo}}, H_{\text{mid}}\) and \(H_{\text{hi}}\) defined on the vertex set \([k]\) as follows:

(i) \((i, j) \in E(H_{\text{irr}})\) iff the pair \((V_i, V_j)\) is not \(\varepsilon\)-regular,

(ii) \((i, j) \in E(H_{\text{lo}})\) iff the pair \((V_i, V_j)\) is \(\varepsilon\)-regular and

\[
d(V_i, V_j) \leq d,
\]

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(iii) \((i, j) \in E(H_{\text{mid}})\) iff the pair \((V_i, V_j)\) is \(\varepsilon\)-regular and
\[
d < d(V_i, V_j) \leq 1 - \delta,
\]
(iv) \((i, j) \in E(H_{\text{hi}})\) iff the pair \((V_i, V_j)\) is \(\varepsilon\)-regular and
\[
d(V_i, V_j) > 1 - \delta.
\]
Clearly, no two of these graphs have edges in common; thus
\[
e(H_{\text{irr}}) + e(H_{\text{lo}}) + e(H_{\text{mid}}) + e(H_{\text{hi}}) = \binom{k}{2}.
\]
Hence, from \(d > 2\varepsilon\) and \(k > 1/\varepsilon\), we see that
\[
e(H_{\text{lo}}) + e(H_{\text{mid}}) + e(H_{\text{hi}}) \geq \binom{k}{2} - \varepsilon k^2 = \frac{k^2}{2} - \frac{k}{2} - \varepsilon k^2
\]
\[
\geq \frac{k^2}{2} - \varepsilon k^2 - \varepsilon k^2 > \left(\frac{1}{2} - d\right) k^2.
\]
(13)
Since \(G\) is \(K_{p+1}\)-free, from \([12]\), we have \(\varepsilon \leq (d - \varepsilon)^p / (p + 2)\); applying Theorem \([5]\) we conclude that the graph \(H_{\text{mid}} \cup H_{\text{hi}}\) is \(K_{p+1}\)-free. Therefore, from Turán’s theorem,
\[
e(H_{\text{mid}}) + e(H_{\text{hi}}) \leq \left(\frac{p - 1}{2p}\right) k^2,
\]
and from inequality \([13]\) we deduce
\[
e(H_{\text{lo}}) > \left(\frac{1}{2p} - d\right) k^2.
\]
(14)
Next we shall bound \(bs^{(r)}(G)\) from below. To achieve this we shall count the independent \((r + 1)\)-sets having exactly \(r\) vertices in some \(V_i\) and one vertex outside \(V_i\). Fix \(i \in [k]\) and let \(m\) be the number of independent \(r\)-sets in \(V_i\). Observe that Lemma \([3]\) implies \(m \geq c_{p,r} |V_i|^r\).

Set \(L = N_{H_{\text{lo}}}(i)\) and apply Lemma \([2]\) with \(A = V_i, B_j = V_j\), for all \(j \in L\), and
\[
H = \overline{G} \left[ A \cup \left( \bigcup_{j \in L} B_j \right) \right].
\]
Since, for every $j \in L$, the pair $(V_i, V_j)$ is $\varepsilon$-regular and
\[ e_H(V_i, V_j) \geq (1 - d)|V_i||V_j|, \]
we conclude that there are at least
\[ d_{H_{lo}}(i)|V_i|(m - \varepsilon r|V_i|^r)(1 - d - \varepsilon)^r \]
independent $(r + 1)$-sets in $G$ having exactly $r$ vertices in $V_i$ and one vertex in $\cup_{j \in L} B_j$.

Set now $M = N_{H_{mid}}(i)$, and apply Lemma 2 with $A = V_i$, $B_j = V_j$ for all $j \in M$ and
\[ H = G \left[ A \cup \left( \bigcup_{j \in M} B_j \right) \right]. \]
Since, for every $j \in M$, the pair $(V_i, V_j)$ is $\varepsilon$-regular and
\[ e_H(V_i, V_j) \geq \delta|V_i||V_j|, \]
we conclude that there are at least
\[ d_{H_{mid}}(i)|V_i|(m - \varepsilon r|V_i|^r)(\delta - \varepsilon)^r \]
independent $(r + 1)$-sets in $G$ having exactly $r$ vertices in $V_i$ and one vertex in $\cup_{j \in L} B_j$. Since
\[ \left( \bigcup_{j \in L} B_j \right) \cap \left( \bigcup_{j \in M} B_j \right) = \emptyset, \]
there are at least
\[ d_{H_{lo}}(i)|V_i|(m - \varepsilon r|V_i|^r)(1 - d - \varepsilon)^r + d_{H_{mid}}(i)|V_i|(m - \varepsilon r|V_i|^r)(\delta - \varepsilon)^r \]
independent $(r + 1)$-sets in $G$ having exactly $r$ vertices in $V_i$ and one vertex outside $V_i$. Thus, taking the average over all $m$ independent $r$-sets in $V_i$, we conclude
\[ bs^{(r)}(G) \geq |V_i| \left( 1 - \frac{\varepsilon r}{c_{p,r}} \right) (d_{H_{lo}}(i)(1 - d - \varepsilon)^r + d_{H_{mid}}(i)(\delta - \varepsilon)^r) \]
\[ \geq n \left( \frac{1 - \varepsilon}{k} \right) \left( 1 - \frac{\varepsilon r}{c_{p,r}} \right) (d_{H_{lo}}(i)(1 - d - \varepsilon)^r + d_{H_{mid}}(i)(\delta - \varepsilon)^r). \]
Summing this inequality for all $i = 1, \ldots, k$ we obtain
\[
\frac{bs^{(r)}(G)}{n} \geq (1 - \varepsilon) \left(1 - \frac{\varepsilon r}{c_{p,r}}\right) \left(\frac{2e(H_{lo})}{k^2} (1 - d - \varepsilon)^r + \frac{2e(H_{mid})}{k^2} (\delta - \varepsilon)^r\right)
\]
\[
> \left(1 - \left(\frac{r}{c_{p,r}} + 1\right) d\right) \left(\frac{2e(H_{lo})}{k^2} (1 - r (d + \varepsilon)) + \frac{2e(H_{mid})}{k^2} (\delta - \varepsilon)^r\right)
\]
\[
> \left(1 - \left(\frac{r}{c_{p,r}} + 1\right) d\right) \frac{2e(H_{lo})}{k^2} (1 - 2rd) + \frac{2e(H_{mid})}{k^2} \left(\frac{\delta}{2}\right)^r
\]
\[
\geq \left(1 - \left(\frac{r}{c_{p,r}} + 2r + 1\right) d\right) \frac{2e(H_{lo})}{k^2} + \left(1 - \left(\frac{r}{c_{p,r}} + 1\right) d\right) \left(\frac{\delta}{2}\right)^r \frac{2e(H_{mid})}{k^2}.
\]
(15)

Assume the assertion of the theorem false and suppose
\[
bs^{(r)}(G) \leq \frac{n}{p}.
\]
(16)

We shall prove that this assumption implies
\[
e(H_{lo}) < \left(\frac{1}{2p} + \frac{\delta}{2}\right) k^2,
\]
(17)
\[
e(H_{mid}) < \delta k^2.
\]
(18)

Disregarding the term $e(H_{mid})$ in (15), in view of (16) and (10), we have
\[
e(H_{lo}) < \left(1 - \left(\frac{r}{c_{p,r}} + 2r + 1\right) d\right)^{-1} \frac{bs^{(r)}(G)}{2n} k^2
\]
\[
\leq \left(1 - \left(\frac{r}{c_{p,r}} + 2r + 1\right) d\right)^{-1} \frac{k^2}{2p}
\]
\[
\leq \left(1 - \frac{p\delta}{1 + p\delta}\right)^{-1} \frac{k^2}{2p} = \left(\frac{1}{2p} + \frac{\delta}{2}\right) k^2,
\]
and inequality (17) is proved.

Furthermore, observe that equality (10) implies
\[
\left(\frac{r}{c_{p,r}} + 1\right) d < \left(\frac{r}{c_{p,r}} + 2r + 1\right) d \leq \frac{p\delta}{1 + p\delta} \leq p\delta < \frac{1}{2},
\]

and consequently,
\[
\left(1 - \left(\frac{r}{c_{p,r}} + 1\right) d\right) > \frac{1}{2}.
\]

Hence, from (15), taking into account (16) and (14), we find that
\[
e\left(\frac{H_{\text{mid}}}{2}\right) \left(\delta \frac{r}{2}\right) < e\left(\frac{H_{\text{mid}}}{2}\right) \left(\delta \frac{r}{2}\right) \left(1 - \left(\frac{r}{c_{p,r}} + 1\right) d\right)
\]
\[
\leq \frac{bs^{(r)}(G) k^2}{2n} - \left(1 - \left(\frac{r}{c_{p,r}} + 2r + 1\right) d\right) e\left(H_{\text{low}}\right)
\]
\[
< \left(\frac{1}{2p} - \left(1 - \left(\frac{r}{c_{p,r}} + 2r + 1\right) d\right) \left(\frac{1}{2p} - d\right)\right) k^2
\]
\[
= \left(1 + \left(\frac{r}{c_{p,r}} + 2r + 1\right) \left(\frac{1}{2p} - d\right)\right) dk^2
\]
\[
< \frac{1}{2p} \left(\frac{r}{c_{p,r}} + 2r + 1 + 2p\right) dk^2 < \left(\frac{\delta}{2}\right)^{r+1} k^2.
\]

Therefore, inequality (18) holds also.

Furthermore, inequality (13), together with (17) and (18), implies
\[
e\left(\frac{H_{\text{hi}}}{2}\right) > \left(\frac{1}{2} - d\right) k^2 - \left(\frac{1}{2p} + \frac{\delta}{2}\right) k^2 - \delta k^2 = \left(\frac{p - 1}{2p} - \frac{5\delta}{2}\right) k^2,
\]
and consequently, from the definition of \(H_{\text{hi}}\), we obtain
\[
e\left(G\right) \geq e\left(H_{\text{hi}}\right) \left(\frac{(1 - \varepsilon) n}{k}\right)^2 (1 - \delta) > \left(\frac{p - 1}{2p} - \frac{5\delta}{2}\right) (1 - 2\varepsilon) (1 - \delta) n^2
\]
\[
= \frac{p - 1}{2p} \left(1 - \frac{5p\delta}{p - 1}\right) (1 - 2\varepsilon) (1 - \delta) n^2 >
\]
\[
> \frac{p - 1}{2p} \left(1 - \left(\frac{5p}{p - 1} + 3\right) \delta\right) n^2 > \left(\frac{p - 1}{2p} - 4\delta\right) n^2.
\]

Hence, by (9), applying Theorem 2, it follows that \(G\) contains an induced \(p\)-chromatic graph with the required properties.

Following the basic idea of the proof of Theorem 3 but applying the complete Key Lemma instead of Theorem 5, we obtain a more general result, whose proof, however, is considerably
easier than the proof of Theorem 3.

**Theorem 6** Suppose $H$ is a fixed $(p + 1)$-chromatic graph. For every $H$-free graph $G$ of order $n$,

$$bs^{(r)} (G) > \left( \frac{1}{p + o(1)} \right) n.$$  

\[\square\]

Note that the graph $K_p (q + r - 1)$ is $p$-chromatic and its complement has no $B_q^{(r)}$, so for every $(p + 1)$-chromatic graph $H$ and every $r, q$ we have

$$r (H, B_q^{(r)}) \geq p (q + r - 1) + 1.$$  

Hence, from Theorem 3 we immediately obtain the following theorem.

**Theorem 7** For every fixed $(p + 1)$-chromatic graph $H$ and fixed integer $r > 1$,

$$r (H, B_q^{(r)}) = pq + o(q).$$  

\[\square\]

Note that it is not possible to avoid the $o(q)$ term in Theorem 7 without additional stipulations about $H$, since, as Faudree, Rousseau and Sheehan have shown in [6], the inequality

$$r (C_4, B_q^{(2)}) \geq q + 2\sqrt{q}$$

holds for infinitely many values of $q$. However, when $H = K_{p+1}$ and $q$ is large we can prove a precise result.

### 4 Ramsey numbers $r \left( K_p, B_q^{(r)} \right)$ for large $q$

In this section we determine $r \left( K_p, B_q^{(r)} \right)$ for fixed $p \geq 3$, $r \geq 2$ and large $q$.

**Theorem 8** For fixed $p \geq 2$ and $r \geq 2$, $r (K_{p+1}, B_q^{(r)}) = p(q + r - 1) + 1$ for all sufficiently large $q$.  

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Proof Since $K_p(q+r-1)$ contains no $K_{p+1}$ and its complement contains no $B_q^{(r)}$, we have

$$r(K_{p+1}, B_q^{(r)}) \geq p(q + r - 1) + 1.$$  

Let $G$ be a $K_{p+1}$-free graph of order $n = p(q+r-1)+1$. Since $n/p > q$, either we’re done or else $G$ contains an induced $p$-chromatic subgraph $G_1$ of order $pq + o(q)$ with minimum degree

$$\delta(G_1) \geq \left(1 - \frac{1}{p} + o(1)\right) n.$$  

Using this bound on $\delta(G_1)$ we can easily prove by induction on $p$ that $G_1$ contains a copy of $K_p(r)$. Fix a copy of $K_p(r)$ in $G_1$ and let $A_1, A_2, \ldots, A_p$ be its vertex classes. Let $A = A_1 \cup \cdots \cup A_p$ and $B = V(G) \setminus A$. If some vertex $i \in B$ is adjacent to at least one vertex in each of the parts $A_1, A_2, \ldots, A_p$ then $G$ contains a $K_{p+1}$. Otherwise for each vertex $u \in B$ there is at least one $v$ so that $u$ is adjacent in $\overline{G}$ to all members of $A_v$. It follows by the pigeonhole principle that $bs^{(r)}(\overline{G}) = s$ where

$$s \geq \left\lceil \frac{n - p(r-1)}{p} \right\rceil = \left\lceil q - 1 + \frac{1}{p} \right\rceil = q,$$

and we really are done. \qed

The proof using the regularity lemma that $r(K_{p+1}, B_q^{(r)}) = p(q + r - 1) + 1$ if $q$ is sufficiently large does indeed require that $q$ increase quite rapidly as a function of the parameters $p$ and $r$. This raises the question of what growth rate is actually required. The following simple calculation shows that polynomial growth in $p$ is not sufficient.

**Theorem 9** For arbitrary fixed $k$ and $r$,

$$\frac{r(K_m, B_{m^k}^{(r)})}{m^{k+r-1}} \to \infty$$

as $m \to \infty$.

Proof We shall prove that $r(K_m, B_{m^k}^{(r)}) > cm^{k+r}/(\log m)^r$ for all sufficiently large $m$. Let $N = \left\lceil cm^{k+r}/(\log m)^r \right\rceil$ where $c$ is to be chosen, and set $p = (C/m) \log m$ where $C =$

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2(k + r − 1). Let $G$ be the random graph $G = G(N, 1 - p)$. The probability that $K_m \subset G$

$$P(K_m \subset G) \leq \left( \frac{N}{m} \right) (1 - p)^{\binom{n}{2}} \leq \left( \frac{N}{m} \right) e^{-pm(m-1)/2} \leq \left( \frac{Ne}{m} \right)^m e^{pm/2} m^{-(k+r-1)m}$$

$$= \left( \frac{Ne^{1+p/2}m^{-(k+r-1)}}{m} \right)^m = o(1), \ m \to \infty.$$

To bound the probability that $B_m^{(r)} \subset \overline{G}$, we use the following simple consequence of Chernoff’s inequality [4]: if $X = X_1 + X_2 + \cdots + X_n$ where independently each $X_i = 1$ with probability $p$ and $X_i = 0$ with probability $1 - p$ then

$$P(X \geq M) \leq \left( \frac{np e - M}{M} \right)^M$$

for any $M \geq np$. Thus we find

$$P(B_m^{(r)} \subset \overline{G}) \leq \left( \frac{N}{r} \right) p^{r(r-1)/2} \left( \frac{(N - r)p^r e}{m^k} \right)^{m^k}.$$

Since the product of the first two factors has polynomial growth in $m$, to have $P(B_m^{(r)}) = o(1)$ when $m \to \infty$, it suffices to take $c = 1/(3C^r)$, so that

$$\frac{(N - r)p^r e}{m^k} \leq \frac{(cm^k r/(\log m)^r)((C/m) \log m)^r e}{m^k} = \frac{e}{3},$$

making the last factor approach 0 exponentially. □

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