Current Correlators and AdS/CFT Geometry

Edwin Barnes, Elie Gorbatov, Ken Intriligator, and Jason Wright

Department of Physics
University of California, San Diego
La Jolla, CA 92093-0354, USA

We consider current-current correlators in 4d $\mathcal{N} = 1$ SCFTs, and also 3d $\mathcal{N} = 2$ SCFTs, in connection with AdS/CFT geometry. The superconformal $U(1)_R$ symmetry of the SCFT has the distinguishing property that, among all possibilities, it minimizes the coefficient, $\tau_{RR}$ of its two-point function. We show that the geometric $Z$-minimization condition of Martelli, Sparks, and Yau precisely implements $\tau_{RR}$ minimization. This gives a physical proof that $Z$-minimization in geometry indeed correctly determines the superconformal R-charges of the field theory dual. We further discuss and compare current two point functions in field theory and AdS/CFT and the geometry of Sasaki-Einstein manifolds. Our analysis gives new quantitative checks of the AdS/CFT correspondence.
1. Introduction

This work is devoted to the geometry / gauge theory interrelations of the AdS/CFT correspondence \cite{1,2,3}, which has been much developed and checked over the past year (a sample of recent references is \cite{4,5,6,7,8,9,10,11}).

In the AdS/CFT correspondence \cite{1,2,3}, global currents $J^\mu_I$ ($I$ labels the various currents) of the $d$-dimensional CFT couple to gauge fields in the $AdS_{d+1}$ bulk. The current two-point functions of the CFT are of fixed form,

$$\langle J^\mu_I(x)J^\nu_J(y) \rangle = \frac{\tau_{IJ}}{(2\pi)^d} \left( \partial^2 \delta_{\mu\nu} - \partial_\mu \partial_\nu \right) \frac{1}{(x-y)^{2(d-2)}},$$

(1.1)

with only the coefficients $\tau_{IJ}$ depending on the theory and its dynamics. Unitarity restricts $\tau_{IJ}$ to be a positive matrix (positive eigenvalues). The coefficients $\tau_{IJ}$ map to the coupling constants of the corresponding gauge fields in $AdS_{d+1}$: writing their kinetic terms as

$$S_{AdS_{d+1}} = \int d^dz d_0 \sqrt{g} \left[ -\frac{1}{4} g_{IJ} F^I_{\mu\nu} F^{\mu\nu J} + \ldots \right],$$

(1.2)

the relation is \cite{12}:

$$\tau_{IJ} = \frac{2^{d-2} \pi^{\frac{d}{2}} \Gamma[d]}{(d-1)\Gamma[d/2]} L^{d-3} g_{IJ},$$

(1.3)

where $L$ is the $AdS_{d+1}$ length scale. Our main interest here will be in the quantities $\tau_{IJ}$, and comparing field theory results with the $AdS$ relation (1.3).

We will here consider 4d $\mathcal{N} = 1$ superconformal field theories, 3d $\mathcal{N} = 2$ SCFTs, and their AdS duals, coming, respectively, from IIB string theory on $AdS_5 \times Y_5$, 11d SUGRA or M-theory on $AdS_4 \times Y_7$. Supersymmetry requires $Y_5$ and $Y_7$ to be Sasaki-Einstein. In general, a Sasaki-Einstein space $Y_{2n-1}$ is the horizon of a non-compact local Calabi-Yau $n$-fold $X_{2n} = C(Y_{2n-1})$, with conical metric

$$ds^2(C(Y_{2n-1})) = dr^2 + r^2 ds^2(Y_{2n-1}).$$

(1.4)

The gauge theories come from $N$ $D3$ or $M2$ branes at the tip of the cone. In the large $N$ dual, the radial $r$ becomes that of $AdS_{d+1}$. The dual to 4d $\mathcal{N} = 1$ SCFTs is IIB on

$$AdS_5 \times Y_5 : \quad ds^2_{10} = \frac{r^2}{L^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{L^2}{r^2} dr^2 + L^2 ds^2(Y_5),$$

(1.5)

and the dual to 3d $\mathcal{N} = 2$ SCFTs is 11d SUGRA or M-theory with metric background

$$AdS_4 \times Y_7 : \quad ds^2_{11} = \frac{r^2}{L^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{L^2}{r^2} dr^2 + (2L)^2 ds^2(Y_7).$$

(1.6)
The SCFTs have a conserved, superconformal $U(1)_R$ current, in the same supermultiplet as the stress tensor. The scaling dimensions of chiral operators are related to their superconformal $U(1)_R$ charges by
\[ \Delta = \frac{d - 1}{2} R. \tag{1.7} \]
There are also typically various non-R flavor currents, whose charges we’ll write as $F_i$, with $i$ labeling the flavor symmetries. The superconformal $U(1)_R$ of RG fixed point SCFTs is then not determined by the symmetries alone, as the R-symmetry can mix with the flavor symmetries. Some additional dynamical information is then needed to determine precisely which, among all possible R-symmetries, is the superconformal one, in the stress tensor supermultiplet.

On the field theory side, we presented a new condition in [13], which, in principle, uniquely determines the superconformal $U(1)_R$: among all possible trial R-symmetries,
\[ R_t = R_0 + \sum_i s_i F_i, \tag{1.8} \]
the superconformal one is that which *minimizes* the coefficient $\tau_{R_t R_t}$ of its two point function (1.1). An equivalent way to state this is that the two-point function of the superconformal R-current with all non-R flavor symmetries necessarily vanishes:
\[ \tau_{R_i} = 0 \quad \text{for all non-R symmetries } F_i. \tag{1.9} \]
(Our notation will always be that capital $I$ runs over all symmetries, including the superconformal $U(1)_R$, and lower case $i$ runs over the non-R flavor symmetries.) We refer to the field theory condition of [13] as “$\tau_{RR}$ minimization”. The minimal value of $\tau_{R_t R_t}$ is then the coefficient, $\tau_{RR}$, of the superconformal $U(1)_R$ current two-point function, which is related by supersymmetry to the coefficient of the stress-tensor two-point function,
\[ \tau_{RR} \propto C_T. \tag{1.10} \]

For the case of 4d $\mathcal{N} = 1$ SCFTs, $a$-maximization [14] gives another way, besides $\tau_{RR}$ minimization, to determine the superconformal $U(1)_R$: the exact superconformal R-symmetry is that which (locally) maximizes the combination of ’t Hooft anomalies
\[ a_{trial}(R_t) = \frac{3}{32} (3\text{Tr}R^3 - \text{Tr}R). \tag{1.11} \]
Equivalently, the superconformal $U(1)_R$ satisfies the 't Hooft anomaly identity

$$9 \text{Tr} R^2 F_i = \text{Tr} F_i$$

for all flavor symmetries $F_i$. (1.12)

\(^{1}\) a-maximization does not apply for 3d SCFTs, as there are there no 't Hooft anomalies.

The global symmetries of the $SCFT_d$ map to the following gauge symmetries in the $AdS_{d+1}$ bulk:

1. The graviphoton, which maps to the superconformal $U(1)_R$, is a Kaluza-Klein gauge field, associated with the “Reeb” Killing vector isometry of Sasaki-Einstein $Y_{2n-1}$. The R-charge is normalized so that superpotential terms, which are related to the holomorphic $n$ form of $X_{2n}$, have charge $R = 2$.

2. Any other Kaluza-Klein gauge fields, from any additional isometries of $Y_{2n-1}$. These can be taken to be non-R symmetries, by taking the holomorphic $n$-form to be neutral. We refer to these as “mesonic, non-R, flavor symmetries,” because mesonic operators (gauge invariants not requiring an epsilon tensor) of the dual gauge theory can be charged under them. When $Y_{2n-1}$ is toric, there is always (at least) a $U(1)^{n-1}$ group of mesonic, non-R flavor symmetries.

3. Baryonic $U(1)^{b_3}$ gauge fields, from reducing Ramond-Ramond gauge fields on non-trivial cycles of $Y_{2n-1}$. In particular, for IIB on $AdS_5 \times Y_5$, there are $U(1)^{b_3}$ baryonic gauge fields come from reducing $C_4$ on the $b_3 = \text{dim}(H_3(Y_5))$ non-trivial 3-cycles of $Y_5$. These are also non-R symmetries. Baryonic $U(1)$ symmetries have the distinguishing property in the gauge theory that only baryonic operators, formed with an epsilon tensor, are charged under them. It was pointed out in [14] that 4d baryonic symmetries have another distinguishing property: their cubic 't Hooft anomalies all vanish, $\text{Tr} U(1)_B^3 = 0$, as seen from the fact that it’s not possible to get the needed Chern-Simons term $\frac{3}{2} A_B \wedge dA_B \wedge dA_B$ from reducing 10d string theory on $Y_5$.

In field theory, the superconformal $U(1)_R$ can, and generally does mix with the mesonic and baryonic flavor symmetries. The correct superconformal $U(1)_R$ can, in principle, be determined by $\tau_{RR}$ minimization. $\tau_{RR}$ minimization is not especially practical to
implement in field theory, because the coefficients (1.9) get quantum corrections. But, on the AdS dual side, $\tau_{RR}$ minimization becomes more useful and tractable, because the AdS duality gives a weakly coupled dual description of $\tau_{Roi}$ and $\tau_{ij}$, via (1.3).

The problem of determining the superconformal $U(1)_R$ in the field theory maps to a corresponding problem in the geometry: determining which $U(1)$, out of the $U(1)^n$ geometric isometries of toric Sasaki-Einstein spaces, is that of the Reeb vector. A solution of this mathematical problem was recently found by Martelli, Sparks, and Yau [9]: the correct Reeb vector is that which minimizes the Einstein-Hilbert action on $Y_{2n-1}$ – this is referred to as “Z-minimization,” [9]. The mathematical result of [9] was shown, on a case-by-case basis, to always lead to the same superconformal R-charges as found from a-maximization [14] in the corresponding field theory, but there was no general proof as to why Z-minimization in geometry implements a-maximization in field theory. In addition, Z-minimization applies to general $Y_{2n-1}$, whereas a-maximization is limited to 4d SCFTs, and hence the case of $AdS_5 \times Y_5$.

Our main result will be to show that the Z-minimization of Martelli, Sparks, and Yau [9] is precisely equivalent to ensuring that the $\tau_{RR}$ minimization conditions (1.9) of [13] are satisfied, i.e. $Z$-minimization $= \tau_{RR}$ minimization. This demonstrates that Z-minimization in the geometry indeed determines the correct superconformal R-symmetry of the dual SCFT, not only for 4d SCFTs, but also for 3d SCFTs with dual (1.6). We will also explain why it’s OK that the $U(1)_b$ baryonic $U(1)$ symmetries did not enter into the geometric Z-minimization of [9]: the condition (1.9) is automatically satisfied in the string theory constructions for all baryonic symmetries.

The outline of this paper is as follows. In sect. 2, we review relations in 4d $\mathcal{N} = 1$ field theory for the current two-point functions, and the ’t Hooft anomalies of the superconformal $U(1)_R$. We then show that these relations are satisfied by the effective $AdS_5$ bulk SUGRA theory, thanks to the structure of real special geometry. In particular, the kinetic terms in the $AdS_5$ bulk are related to the Chern-Simons terms, which yield the ’t Hooft anomalies of the dual SCFT. In the following sections, we discuss how these kinetic terms are obtained from the geometry of $Y$; it would be interesting to also directly obtain the Chern-Simons terms from the geometry of $Y$, but that will not be done here. In sect. 3, we discuss the contributions to the kinetic terms in the $AdS$ bulk. As usual, Kaluza-Klein gauge fields get a contribution, with coefficient $(g^{-2}_{ij})^{KK}$, from reducing the Einstein term in the action on $Y$. Because of the background flux in $Y$, there is also a contribution $(g^{-2}_{IJ})^{CC}$ from reducing the Ramond-Ramond $C$ field kinetic terms on $Y$. We
point out (closely following [16]) that these two contributions always have the fixed ratio:

\[(g_{IJ}^{-2})^{CC} = \frac{1}{2}(D_c - 1)(g_{IJ}^{-2})^{KK},\]

for any Einstein manifold \(Y\) of dimension \(D_c\). This relation will be used, and checked, in following sections. For the baryonic gauge fields, there is only the contribution \((g_{IJ}^{-2})^{CC}\), from reducing the Ramond-Ramond kinetic term on \(Y\).

In sect. 4, we discuss generally how the gauge fields \(A_I\) alter Ramond-Ramond flux background, and thereby alter the Ramond-Ramond field at linearized level, as \(\delta C = \sum_I \omega_I \wedge A_I\), for some particular \(2n - 3\) forms \(\omega_I\) on \(Y\). We discuss how the \(A_I\) charges of branes wrapped on supersymmetric cycles can be obtained by integrating \(\omega_I\) over the cycle, and how the Ramond-Ramond contribution to the gauge kinetic terms is written as \(\sim \int_Y \omega_I \wedge \ast \omega_J\). In sect. 5, we review some aspects of Sasaki-Einstein geometry, and the analysis of [17] for how to determine the form \(\omega_R\) for the \(U(1)_R\) gauge field. In sect. 6, we generalize this to determine the forms \(\omega_I\) for the non-R isometry and baryonic gauge fields. In sect. 7, we give expressions for the gauge kinetic terms \(g_{IJ}^{-2}\), and thereby the current-current two-point function coefficients \(\tau_{IJ}\) that we are interested in, in terms of integrals \(\sim \int_Y \omega_I \wedge \ast \omega_J\) of these forms. We note that this immediately implies that there is never any mixing in the kinetic terms between Kaluza-Klein isometry gauge fields and the baryonic gauge fields, i.e. that

\[\tau_{IJ} = 0 \quad \text{automatically, for } I = \text{Kaluza-Klein and } J = \text{baryonic.} \quad (1.13)\]

This shows that our condition (1.9) for the \(U(1)_R\) is automatically satisfied, for all baryonic symmetries, by taking \(U(1)_R\) to be purely a Kaluza-Klein isometry gauge field, without any mixing with the baryonic symmetries. For the mesonic, non-R isometry gauge fields, the condition (1.9) becomes

\[\int_Y g_{ab}K^a K^b_{i} \text{vol}(Y) = 0, \quad (1.14)\]

which give conditions to determine the \(U(1)_R\) isometry Killing vector \(K^a\). The condition (1.14) must hold for every non-R isometry Killing vector of \(Y\), i.e. for every Killing vector \(K^a_{i}\) under which the the holomorphic \(n\) form of \(C(Y_{2n-1})\) is neutral.

In sect. 8, we summarize the results of Martelli, Sparks, and Yau [9] for toric \(C(Y)\). Then \(Y_{2n-1}\) always has at least \(U(1)^n\) isometry, associated with shifts of toric coordinates \(\phi_i\), and the \(U(1)_R\) Killing Reeb vector \(K^a\) is given by some components \(b_i, i = 1 \ldots n\), in this basis. The volume of \(Y\) and its supersymmetric cycles are completely determined by the \(b_i\), without needing to know the metric on \(Y\). And the \(b_i\) are themselves determined by \(Z\)-minimization [3], which is minimization of the Einstein-Hilbert action on \(Y\). In sect.
9, we point out that Z-minimization is precisely equivalent to \( \tau_{RR} \) minimization. We also discuss the flavor charges of wrapped branes. In sect. 10, we illustrate our results for the \( Y^{p,q} \) examples of \([4,5]\). We find the forms \( \omega_I \), and thereby use the flavor charges of wrapped branes. We also compute from the geometry of \( Y \) the gauge kinetic term coefficients, and thus the current-current two-point function coefficients \( \tau_{IJ} \). These quantities, computed from the geometry of \( Y \), match with those computed in the dual field theory of \([7]\); this gives new checks of the AdS/CFT correspondence for these theories.

In the final stages of writing up this paper, the very interesting work \([18]\) appeared, in which it was mathematically shown that the Z-function \([9]\) of 5d toric Sasaki-Einstein \( Y_5 \) and the \( a_{\text{trial}} \) function \([14]\) of the dual quiver 4d gauge theory are related by \( Z(x,y) = 1/a(x,y) \) (even before extremizing). The approach and results of our paper are orthogonal and complementary to those of \([18]\). Also in the final stages of writing up this paper, the work \([19]\) appeared, which significantly overlaps with the approach of section 2 of our paper, and indeed goes further along those lines than we did here.

2. 4d \( \mathcal{N} = 1 \) SCFTs and real special geometry

This section is somewhat orthogonal to the rest of the paper. The rest of this paper is devoted to deriving the AdS bulk gauge field kinetic terms \( g_{ij}^{-2} \) in (1.2) and (1.3) directly from the geometry of \( Y \). In the present section, without explicitly considering \( Y \), we will discuss how the various identities of 4d \( \mathcal{N} = 1 \) SCFTs are guaranteed to also show up in the effective AdS\(_5\) SUGRA theory, thanks to the structure of real, special geometry.

Because the superconformal R-current is in the same supermultiplet as the stress tensor, their two-point function coefficients are proportional, \( \tau_{RR} \propto C_T \). Also, in 4d \( C_T \propto c \), with \( c \) the conformal anomaly coefficient in

\[
\langle T_\mu^\nu \rangle = \frac{1}{120} \frac{1}{(4\pi)^2} \left( c(\text{Weyl})^2 - \frac{a}{4}(\text{Euler}) \right).
\]

(2.1)

So \( \tau_{RR} \propto c \); more precisely,

\[
\tau_{RR} = \frac{16}{3} c,
\]

(2.2)

with \( c \) normalized such that \( c = 1/24 \) for a free \( \mathcal{N} = 1 \) chiral superfield. Supersymmetry also relates \( a \) and \( c \) in (2.1) to the ’t Hooft anomalies of the superconformal \( U(1)_R \) \([20]\):

\[
a = \frac{3}{32} (3\text{Tr}R^3 - \text{Tr}R) \quad c = \frac{1}{32} (9\text{Tr}R^3 - 5\text{Tr}R).
\]

(2.3)
Combining (2.2) and (2.3), we have

\[ \tau_{RR} = \frac{3}{2} \text{Tr} R^3 - \frac{5}{6} \text{Tr} R, \]  

(2.4)

The flavor current two-point functions are also given by 't Hooft anomalies [20]:

\[ \tau_{ij} = -3 \text{Tr} RF_i F_j. \]  

(2.5)

There are precise analogs to the above relations in the effective 5d \( \mathcal{N} = 2 \) bulk gauged U(1) supergravity; this is not surprising given that, on both sides of the duality, these relations come from the same \( SU(2,2|1) \) superconformal symmetry group.

The bosonic part of the effective 5d Lagrangian is [21] (also see e.g. [22])

\[
L_{\text{bosonic}} = \sqrt{|g|} \left[ \frac{1}{2} R - \frac{1}{2} G_{ij} \partial \phi^i \partial \phi^j - \frac{1}{4} g_{IJ} F^I \cdot F^J - V(X) \right] + \frac{1}{48} C_{IJK} A^I \wedge F^J \wedge F^K \]  

(2.6)

where, to simplify expressions, we’ll set the 5d gravitational constant \( \kappa_5 = 1 \) in this section. There are \( n_V + 1 \) gauge fields, \( I = 1 \ldots n_V + 1 \), one of them being the graviphoton, which corresponds to the superconformal \( U(1)_R \) in the 4d SCFT. The \( n_V \) gauge fields correspond to the non-R (i.e. the gravitino is neutral under them) flavor symmetries, which reside in current supermultiplets \( J_i, i = 1 \ldots n_V \); the first component of this supermultiplet is a scalar, which couples to the scalars \( \phi^i \) in (2.6). The scalars of the \( n_V \) vector multiplets are constrained by real special geometry to the space

\[
\mathcal{N} \equiv \frac{1}{6} C_{IJK} X^I X^J X^K = 1. \]  

(2.7)

The kinetic terms are all determined by the Chern-Simons coefficients \( C_{IJK} \). In particular, the gauge field kinetic term coefficients \( g_{IJ}^{-2} \) are given by

\[
g_{IJ}^{-2} = -\frac{1}{2} \partial I \partial J \ln \mathcal{N} |_{\mathcal{N} = 1} = -\frac{1}{2} (C_{IJK} X^K - X_I X_J), \]  

(2.8)

where \( X_I \equiv \frac{1}{2} C_{IJK} X^J X^K \). In a given vacuum, where \( X^I \) has expectation values satisfying (2.7), the \( n_V \) scalars in (2.6) are given by the tangents \( X_i^I \) to the surface (2.7), which satisfy

\[
C_{IJK} X_i^I X^J X^K = 0. \]  

(2.9)

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2 The 5d SUGRA theory suffices for studying current two-point functions, and relations to ’t Hooft anomalies, even if there is no full, consistent truncation from 10d to an effective 5d theory.
This can be written as $X_I X^I_r = 0$. The vacuum expectation value $X^I$ picks out the direction of the graviphoton $A_R$, and the tangents $X^I_i$ pick out the direction of the non-R flavor gauge fields:

$$A^I = \alpha X^I A_R + X^I_i A_i,$$  \hfill (2.10)

with $\alpha$ a normalization factor, to ensure that the R-symmetry is properly normalized, to give the gravitinos charges $\pm 1$. The correct value is $\alpha = 2L/3$, where $L$ is the $\text{AdS}_5$ length scale, related to the value of the potential at its minimum by $\Lambda = -6/L^2$.

Using (2.10) and (2.8), we can compute the kinetic term coefficients for the graviphoton and non-R gauge fields. Using (1.3) to convert these into the current-current 2-point function coefficients, we have for the R-symmetry/graviphoton kinetic term

$$\tau_{RR} = 8\pi^2 L g_{RR}^{-2} = 8\pi^2 L \alpha^2 g_{IJ}^{-2} X^I X^J = 12\pi^2 L \alpha^2.$$  \hfill (2.11)

For the $n_V$ non-R gauge fields, we have

$$\tau_{ij} = 8\pi^2 L g_{ij}^{-2} = 8\pi^2 L g_{ij}^{-2} X^I_i X^J_j = -4\pi^2 L C_{IJK} X^I_i X^J_j X^K.$$  \hfill (2.12)

It also follows from (2.8) and (2.9), $X_I X^I = 0$, that there is no kinetic term mixing between the graviphoton and the non-R gauge fields:

$$\tau_{Ri} = 8\pi^2 L g_{Ri}^{-2} = 8\pi^2 L \alpha g_{IJ}^{-2} X^I_i X^J_j = 0 \quad \text{for all } i = 1 \ldots n_V.$$  \hfill (2.13)

This matches with the general SCFT field theory result (1.3) of [13].

The Chern-Simons terms for the graviphoton and flavor gauge fields are similarly found from (2.10). We’ll normalize them as $C_{IJK}/48 = k_{IJK}/96\pi^2$, where $k_{IJK}$ is the properly normalized 5d Chern-Simons coefficients, which map [3] to the ’t Hooft anomalies of the gauge theory:

$$\text{Tr} R^3 = k_{RRR} = 2\pi^2 \alpha^3 C_{IJK} X^I X^J X^K = 12\pi^2 \alpha^3,$$  \hfill (2.14)

$$\text{Tr} R^2 F_i = k_{RRj} = 2\pi^2 \alpha^2 C_{IJK} X^I X^J X^K_i = 0,$$  \hfill (2.15)

where we used (2.3), and also

$$\text{Tr} RF_i F_j = k_{Rij} = 2\pi^2 \alpha C_{IJK} X^I_i X^J_j X^K.$$  \hfill (2.16)

The field theories with (weakly coupled) AdS duals generally have $\text{Tr} R = 0$ and also $\text{Tr} F = 0$. The result (2.15) then reproduces the ’t Hooft anomaly identity (1.12) of [14].
For $\text{Tr}R = 0$, (2.4) becomes $\tau_{RR} = \frac{3}{2} \text{Tr} R^3$, which is reproduced by (2.11) and (2.14) for $\alpha = 2L/3$ in (2.10). Also the relation (2.3) of [23], which for $\text{Tr}R = 0$ is $a = c = \frac{9}{32} \text{Tr} R^3$, is also reproduced by (2.14) for $\alpha = 2L/3$, since the result of [24] is $a = c = L^3 \pi^2$ in $\kappa_5 = 1$ units. The relation (2.5) is also reproduced, for $\alpha = 2L/3$, by (2.12) and (2.16).

In later sections, we will be interested in computing the $AdS_5$ gauge field kinetic terms $\tau_{IJ}$ directly from IIB string theory on $AdS_5 \times Y_5$. To connect with the above expressions, we restore the factors of $\kappa_5$ via dimensional analysis, and convert using

$$\frac{L^3}{\kappa^2_5} = \frac{L^3}{8\pi G_5} = \frac{L^8 \text{Vol}(Y_5)}{8\pi G_{10}} = \frac{N^2}{4} \frac{\pi}{\text{Vol}(Y_5)},$$

(2.17)

where $\text{Vol}(Y_5)$ is the dimensionless volume of $Y_5$, with factors of its length scale, which coincides with the $AdS_5$ length scale $L$, factored out. The last equality of (2.17) uses the flux quantization / brane tensions relation (see [23] and references therein)

$$2\sqrt{\pi} \kappa^{-1} L^4 \text{Vol}(Y_5) = \frac{L^4 \text{Vol}(Y_5)}{\sqrt{2} G_{10}} = N\pi.$$ (2.18)

E.g. using (2.17) the result of [24] becomes [26]

$$a = c = \frac{L^3 \pi^2}{\kappa^2_5} = \frac{N^2}{4} \frac{\pi^3}{\text{Vol}(Y_5)},$$

(2.19)

and (2.11) for $\alpha = 2L/3$ becomes

$$\tau_{RR} = \frac{16\pi^2 L^3}{3 \kappa^2_5} = \frac{4N^2}{3} \frac{\pi^3}{\text{Vol}(Y_5)}.$$ (2.20)

In the following sections, we will directly compute the $\tau_{IJ}$ kinetic terms from reducing SUGRA on $Y$. One could also directly determine the Chern-Simons coefficients $C_{IJK}$ from reduction on $Y$, but doing so would require going beyond our linearized analysis, and we will not do that here. It would be nice to extend our analysis to compute the $C_{IJK}$ from $Y$, and explicitly verify that the special geometry relations reviewed in the present section are indeed satisfied.

3. Kaluza-Klein gauge couplings: a general relation for Einstein spaces

Our starting point is the Einstein action in $D_t = D + D_c$ spacetime dimensions, along with the Ramond-Ramond gauge field kinetic terms:

$$\frac{1}{16\pi G_{D_t}} \int \left( R_{D_t} \ast 1 - \frac{1}{4} F \wedge F \right).$$ (3.1)
We’ll be interested in fluctuations of this action around a background solution of the form $M_D \times Y$, with $M_D$ non-compact and $Y$ compact, of dimension $D_c \equiv p + 2$, with flux

$$F^{bkgd}_{p+2} = (p + 1)m^{-(p+1)}vol(Y),$$

and metric

$$ds^2 = ds^2_M + m^{-2}ds^2_Y.$$  

(3.3)

Here $m^{-1}$ is the length scale of $Y$, which we’ll always factor out explicitly; $vol(Y)$ is the volume form of $Y$, with the length scale $m^{-1}$ again factored out. (We always use lower case $vol(Y)$ for a volume form, and upper case $Vol(Y)$ for its integrated volume.) Our units are such that the integrated flux is

$$\mu_p \int_Y F^{bkgd}_{p+2} \sim \mu_p m^{-(p+1)}Vol(Y) \sim N,$$

(3.4)

with $\mu_p$ the $p$-brane tension. Our particular cases of interest will be IIB on $AdS_5 \times Y_5$ and 11d SUGRA on $AdS_4 \times Y_7$, but we’ll be more general in this section.

Metric fluctuations along directions of Killing vectors $K^a_I$ of $Y$ lead to Kaluza-Klein gauge fields $A^\mu_I$ in $M$. Fluctuations of the Ramond-Ramond gauge field background, reduced on non-trivial cycles of $Y$ lead to additional, “baryonic” gauge fields that we’ll also discuss. In general, Kaluza-Klein reduction involves a detailed, and highly non-trivial, ansatz for how the Kaluza-Klein gauge fields affect the metric and background field strengths. But here we’re simply interested in the coefficients $g^{-2}_{IJ}$ of the gauge field kinetic term, and for these it’s unnecessary to employ the full Kaluza-Klein ansatz: a linearized analysis suffices.

The linearized analysis will be presented in the following section. In this section, we’ll note some general aspects, and discuss a useful relation that can be obtained by a generalization of an argument in [16], that was based on the non-trivial Kaluza-Klein ansatz for how the Kaluza-Klein gauge fields modify the backgrounds.

For Kaluza-Klein isometry gauge fields, both the Einstein term and the $C$ field kinetic terms in (3.1) contribute to their gauge kinetic terms:

$$g^{-2}_{IJ} = (g^{-2}_{IJ})^{KK} + (g^{-2}_{IJ})^{CC},$$

(3.5)

where $(g^{-2}_{IJ})^{KK}$ is the Kaluza-Klein contribution coming from the Einstein term in (3.1) and $(g^{-2}_{IJ})^{CC}$ is that coming from the Ramond-Ramond $C$ field kinetic terms in (3.1). On
the other hand, if either $I$ or $J$ is a baryonic gauge field, coming from $C$ reduced on a non-trivial cycle of $Y$, then only the $dC$ kinetic terms in (3.1) contribute

$$g_{IJ}^{-2} = (g_{IJ}^{-2})^{CC}, \quad \text{if } I \text{ or } J \text{ is baryonic.} \quad (3.6)$$

Let’s review how the Kaluza-Klein contribution in (3.5) is obtained, see e.g. [27]. Let $y^a$ be coordinates on $Y$, and $K^a_I(y)$ isometric Killing vectors ($I$ labels the isometry). The one-form $d\phi_I$ dual to $K_I$ is shifted by the 1-form gauge field $A_I(x) = A_I^\mu dx^\mu$, with $x^\mu$ coordinates on $M$. This variation of the metric leads to variation of the Ricci scalar

$$R \rightarrow R - \frac{m^{-2}}{4} g_{ab}(y) K^a_I(y) K^b_J(y) (F_I)_{\mu\nu} (F_J)^{\mu\nu}, \quad (3.7)$$

where $ds^2_Y = g_{ab} dy^a dy^b$ is the metric on $Y$, with the length scale $m^{-1}$ factored out. Since (3.7) is already quadratic in $A_I$, we don’t need to vary $\sqrt{|g|}$. The contribution to the Kaluza-Klein gauge field kinetic terms coming from the Einstein action is thus

$$(g_{IJ}^{-2})^{KK} = \frac{m^{-(D_c+2)}}{16\pi G_D} \int_Y g_{ab} K^a_I K^b_J \text{vol}(Y). \quad (3.8)$$

In [27], the Killing vectors are normalized so that the gauge fields have canonical kinetic terms, and then what we’re referring to as the “coupling” becomes the “charge” unit; here we’ll normalize $K^a_I$ and gauge fields so that the charge unit is unity, and then physical charges governing interactions are given by what we’re calling the couplings $g_{IJ}^{-2}$.

As an example, it was shown [27] that reducing the Einstein action on a $D_c$ dimensional sphere, $Y = S^{D_c}$ of radius $m^{-1}$ leads to $SO(D_c + 1)$ Kaluza-Klein gauge fields in the uncompactified directions, with coupling [27]

$$(g^{-2})^{KK} = \frac{1}{8\pi G_D(D_c+1)m^2} \quad \text{for } Y = S^{D_c}, \quad (3.9)$$

with $G_D = G_D, m^{D_c}/\text{Vol}(Y)$ the effective Newton’s constant in the uncompactified $M_D$.

In [16], it was pointed out that (3.9), applied to 11d SUGRA on $S^7$, with Freund-Rubin flux for the Ramond-Ramond gauge field, would be incompatible with the 4d $N = 8$ $SO(8)$ SUGRA of [28], but that properly including the additional contribution from the Ramond-Ramond fields fixes this problem. In our notation above, it was shown in [16] that the full coupling of the $SO(8)$ gauge fields in the $AdS_4$ bulk is

$$g^{-2} = (g^{-2})^{KK} + (g^{-2})^{CC} = 4g_K^{-2} = \frac{1}{16\pi G_A m^2}, \quad (3.10)$$
which is now perfectly compatible with the 4d $\mathcal N = 8$ theory of \[28\].

We here point out that, for general Freund-Rubin compactifications on any Einstein space $Y$ of dimension $D_c$, there is always a fixed proportionality between the Einstein and Ramond-Ramond contributions to the Kaluza-Klein gauge kinetic terms:

$$
(g_{IJ}^{-2})^{CC} = \frac{D_c - 1}{2} (g_{IJ}^{-2})^{KK}, \quad (3.11)
$$

of which (3.10) is a special case. Our relation (3.11) follows from a generalization of the argument in \[16\]. In a KK ansatz like that of (3.10), the contribution to $g_{IJ}^{-2}$ from the Ramond-Ramond kinetic term in (3.1) is

$$
(g_{IJ}^{-2})^{CC} = \frac{m^{- (D_c+2)}}{16\pi G_D} \int_Y \frac{1}{2} g_{ab} \nabla_c K_I^a \nabla^c K_J^b \text{vol}(Y) = \frac{D_c - 1}{2} (g_{IJ}^{-2})^{KK}. \quad (3.12)
$$

In the last step, there was an integration by parts, use of $-\nabla_c \nabla^c K_I^a = R^a_{bc} K_I^b$, use of $R_{ab} = (D_c - 1) m^2 g_{ab}$ since $Y$ is taken to be Einstein, and comparison with (3.8). We will check and verify the relation (3.11) more explicitly in the following sections.

As a quick application, we find from (3.9) and (3.11) that reducing 10d IIB SUGRA on $S^5$ leads to a theory in the $AdS_5$ bulk with $SO(6)$ gauge fields with coupling

$$
g_{SO(6)}^{-2} = (g_{SO(6)}^{-2})^{KK} + (g_{SO(6)}^{-2})^{CC} = 3 (g_{SO(6)}^{-2})^{KK} = \frac{L^2}{16\pi G_5}, \quad (3.13)
$$

where $m^{-1} = L$ is the radius of the $S^5$, and also the length scale of the $AdS_5$ vacuum. The result (3.13) agrees with that found in \[29\] for 5d $\mathcal N = 8$ SUGRA: the $SO(5)$ invariant vacuum in eqn. (5.43) of \[29\] has, in $4\pi G_5 = 1$ units, $R_{\mu\nu} = g_{\mu\nu}$; thus $g^{-2} = L^2/4 = L^2/16\pi G_5$, in agreement with (3.13). Using (2.17), with $Vol(S^5) = \pi^3$, gives $\tau_{SO(6)} = 8\pi^2 L g^{-2} = \pi L^3/2G_5 = N^2$. On the other hand, (2.20) here gives $\tau_{RR} = 4N^2/3$. We can also verify $\tau_{RR} = 4N^2/3$ by direct computation in the $\mathcal N = 4$ theory (where the free field value is not renormalized). The apparent difference with the above $\tau_{SO(6)}$ is because of the different normalization of the $U(1)_R$ vs. $SO(6)$ generators.

The relation (3.11) will prove useful in what follows, because the Ramond-Ramond contribution $(g_{IJ}^{-2})^{CC}$ is sometimes, superficially, easier to compute than the Kaluza-Klein contribution (3.8). Thanks to the general relation (3.11), the full coefficient of the kinetic terms for Kaluza-Klein gauge fields can be computed from $(g_{IJ}^{-2})^{CC}$ as

$$
g_{IJ}^{-2} = (g_{IJ}^{-2})^{KK} + (g_{IJ}^{-2})^{CC} = \frac{D_c + 1}{D_c - 1} (g_{IJ}^{-2})^{CC}. \quad (3.14)
$$
4. Gauge fields and associated $p$-forms on $Y$

The linearized fluctuations of the gauge fields modify the background as

$$F_{p+2}^{bgd} \rightarrow (p+1)m^{-(p+1)} vol(Y) + d\left(\sum_I \omega_I \wedge A_I\right),$$  \hspace{1cm} (4.1)

and hence, writing $F = dC$,

$$C_{p+1} \rightarrow C_{p+1}^{bgd} + \sum_I \omega_I \wedge A_I$$  \hspace{1cm} (4.2)

Here $A_I$ are all of the gauge fields, both Kaluza-Klein and the baryonic ones coming from reducing $C_{p+1}$ on non-trivial $p$ cycles of $Y$.

So every gauge field $A_I$ enters into $C_{p+1}$ at the linearized level, and we’ll here be interested in determining the associated form $\omega_I$ in (4.2). The $\omega_I$ associated with Kaluza-Klein gauge fields $A_I$ are found from the variation of $vol(Y)$ in (3.2) by the linearized shift of the 1-form, dual to the Killing vector isometry $K_I$, by $A_I$:

$$vol(Y) \rightarrow vol(Y) + d\left(\sum_I \tilde{\omega}_I \wedge A_I\right), \text{ with } \quad d\tilde{\omega}_I = i_{K_I} vol(Y).$$  \hspace{1cm} (4.3)

Using this in (4.1) gives (4.2), with associated $p$-form $\omega_I \equiv (p+1)m^{-(p+1)} \tilde{\omega}_I$ on $Y$.

Note that this definition of the $\omega_I$ is ambiguous under shifts of the $\omega_I$ by any closed $p$ form. Shifts of $\omega_I$ by any exact form will have no effect, so this ambiguity in defining the $\omega_I$ associated with Kaluza-Klein gauge fields is associated with the cohomology $H_p(Y)$ of closed, mod exact, $p$ forms on $Y$.

The baryonic gauge fields $A_I$ enter into (4.2) with $\omega_I$ running over a basis of the cohomology $H_p(Y)$ of closed, mod exact, $p$-forms on $Y$. The ambiguity mentioned above in the Kaluza-Klein gauge fields corresponds to the freedom in one’s choice of basis of the global symmetries, as any linear combination of a “mesonic” flavor symmetry and any “baryonic” flavor symmetry is also a valid “mesonic” flavor symmetry.

Branes that are electrically charged under $C_{p+1}$ have worldvolume coupling $\mu_p \int C_{p+1}$, with $\mu_p$ the brane tension. Wrapping these branes on the non-trivial cycles $\Sigma$ of $H^p(Y)$ yield particles in the uncompactified dimensions, and (4.2) implies that these wrapped branes carry electric charge

$$q_I(\Sigma) = \mu_p \int_{\Sigma} \omega_I$$  \hspace{1cm} (4.4)
under the gauge field $A_I$.

Plugging (4.2) into $F_{p+2}$ kinetic terms in (3.1) gives what we called the $(g_{IJ}^{-2})_{CC}$ contribution to the gauge field kinetic terms to be

$$(g_{IJ}^{-2})_{CC} = \frac{1}{16\pi G_D} \int_Y \omega_I \wedge *\omega_J \equiv \frac{(p+1)^2 m^{-(p+4)}}{16\pi G_D} \int_Y \tilde{\omega}_I \wedge *\tilde{\omega}_J,$$

(4.5)

where $\omega_I \equiv (p+1)m^{-(p+1)}\tilde{\omega}_I$ and $*\omega_I \equiv (p+1)m^{-3} *\tilde{\omega}_I$.

We will use (4.5), together with (3.14) for Kaluza-Klein gauge fields, or (3.6) for baryonic gauge fields, to compute the coefficients $g_{IJ}^{-2}$ of the gauge field kinetic terms in $AdS_{d+1}$. These are then related to the coefficients, $\tau_{IJ}$, of the current-current two-point functions in the gauge theory according to (1.3).

5. Sasaki-Einstein $Y$, and the form $\omega_R$ for the R-symmetry.

The modification (4.2) for the $U(1)_R$ gauge field, coming from the $U(1)_R$ isometry of Sasaki-Einstein spaces, was found in [17], which we’ll review in this section.

The metric of Sasaki-Einstein $Y_{2n-1}$ can locally be written as

$$ds^2(Y) = \left(\frac{1}{n}d\psi' + \sigma\right)^2 + ds^2_{2(n-1)},$$

(5.1)

with $ds^2_{2(n-1)}$ a local, Kahler-Einstein metric, and

$$d\sigma = 2J \quad d\Omega = n i \sigma \wedge \Omega,$$

(5.2)

with $J$ the local Kahler form and $\Omega$ the local holomorphic $(n-1, 0)$ form for $ds^2_{2(n-1)}$. In [17] the coordinate $\psi = \psi'/q$ was used, in order to have the range $0 \leq \psi < 2\pi$; $q$ is given by $n d\sigma = 2\pi q c_1$, with $c_1$ the first Chern class of the $U(1)$ bundle over the $n - 1$ complex dimensional Kahler-Einstein space with metric $ds^2_{2(n-1)}$. The $U(1)_R$ isometry is associated with the Reeb Killing vector

$$K = n \frac{\partial}{\partial \psi'}.$$

(5.3)

It is convenient to define the unit 1-form, dual to the Reeb vector, of the $U(1)_R$ fiber

$$e^\psi \equiv \frac{1}{n}d\psi' + \sigma.$$

(5.4)

Note that $de^\psi = d\sigma = 2J$. The volume form of $Y_{2n-1}$ is

$$vol(Y_{2n-1}) = \frac{1}{(n-1)!} e^\psi \wedge J^{n-1}.$$

(5.5)
Following [17], the linearized effect of the $U(1)_R$ isometry (5.3) Kaluza-Klein gauge field is found by shifting

$$e^\psi \rightarrow e^\psi + \frac{2}{n} A_R,$$

(5.6)

where the coefficient of $A_R$ is chosen so that the $U(1)_R$ symmetry is properly normalized: the holomorphic n-form on $C(Y)$, which leads to superpotential terms, has R-charge 2. The shift (5.6) affects the volume form (5.3) as

$$vol(Y_{2n-1}) \rightarrow vol(Y_{2n-1}) + \frac{2}{n!} A_R \wedge J^{n-1} - \frac{1}{n!} dA_R \wedge e^\psi \wedge J^{n-2},$$

(5.7)

where the last term in (5.7) was added to keep the form closed:

$$vol(Y_{2n-1}) \rightarrow vol(Y_{2n-1}) + \left( \frac{1}{n!} e^\psi \wedge J^{n-2} \wedge A_R \right).$$

(5.8)

The shift (5.8) alters the Ramond-Ramond flux background $F_{bkgd}^{2n-1}$ (4.1), and thus alters $C_{2n-2}$ as in (4.2), $\delta C_{2n-2} = \omega_R \wedge A_R$, with the $2n-3$ form $\omega_R$ given by

$$\hat{\omega}_R \equiv \frac{\omega_R}{(2n-2)m-(2n-2)} = \frac{1}{n!} e^\psi \wedge J^{n-2}.$$

(5.9)

In particular, for type IIB on $AdS_5 \times Y_5$, the background flux is

$$F_5^{bkgd} = 4L^4 (vol(Y_5) + *vol(Y_5)),$$

(5.10)

and (5.8) alters the $C_4$ on $Y_5$ as in (4.2), with 3-form $\omega_R$ given by [17]:

$$\hat{\omega}_R \equiv \frac{1}{4L^4} \omega_R = \frac{1}{6} e^\psi \wedge J, \quad \text{for } Y_5.$$

(5.11)

For 11d SUGRA on $AdS_4 \times Y_7$, the effect of (5.8) on the Ramond-Ramond flux

$$F_7 = 6(2L)^6 vol(Y_7)$$

(5.12)

leads to a shift as in (4.2) of $C_6$, by $\omega_R \wedge A_R$, with 5-form $\omega_R$ given by [17]

$$\hat{\omega}_R \equiv \frac{1}{6(2L)^6} \omega_R = \frac{1}{24} e^\psi \wedge J \wedge J.$$

(5.13)

Wrapping a brane on a supersymmetric $2n-3$ cycle $\Sigma$ of $Y$ yields a baryonic particle $B_\Sigma$ in the $AdS_{d+1}$ bulk, dual to a baryonic chiral operator in the gauge theory. It was verified in [17] that the R-charges assigned to such objects by the forms (5.11) and (5.13)
are compatible with the relation (1.7) in the dual field theory. Using (5.9), the R-charge assigned to such an object is related to the operator dimension $\Delta$ as

$$R[B_{\Sigma}] = \mu_{2n-3} \int_{\Sigma_{2n-3}} \omega_R = \frac{2}{n} \mu_{2n-3} m^{-(2n-2)} \int_{\Sigma} \frac{1}{(n-2)!} e^\psi \wedge J^{n-2}$$

$$= \frac{2}{n} \mu_{2n-3} m^{-(2n-2)} \text{Vol}(\Sigma_{2n-3}) = \frac{2m^{-1}}{nL} \Delta[B_{\Sigma}].$$  (5.14)

In going from the first to the second line of (5.14), we used the fact that the supersymmetric $2n-3$ cycles in $Y$ are calibrated, with $\text{vol}(\Sigma) = e^\psi \wedge J^{n-2}/(n-2)!$. For both IIB on $AdS_5 \times Y_5$ and M theory on $AdS_4 \times Y_7$, (5.14) matches with the relation (1.7) in the 4d and 3d dual, respectively [17]: in the former case, $m^{-1} = L$ and $n = 3$ in (5.14), and in the latter case $m^{-1} = 2L$ and $n = 4$.

The $\mu_{2n-3} m^{-(2n-2)}$ factor in (5.14) is proportional to $N/\text{Vol}(Y)$ by the flux quantization condition. For $AdS_5 \times Y_5$, using (2.18) then gives [17]

$$R(\Sigma_i) = \frac{2}{3} \mu_3 L^4 \text{Vol}(\Sigma_i) = \frac{\pi N}{3} \frac{\text{Vol}(\Sigma_i)}{\text{Vol}(Y_5)}.  \quad (5.15)$$

For M theory on $AdS_4 \times Y_7$, the flux quantization condition (see e.g. the recent work [30])

$$6(2L)^6 \text{Vol}(Y_7) = (2\pi \ell_{11})^6 N, \quad (5.16)$$

where $16\pi G_{11} = (2\pi)^8 \ell_{11}^9$. Using the M5 tension $\mu_5 = 1/(2\pi)^4 \ell_{11}^6$, (5.14) then gives

$$R(\Sigma_i) = \frac{\pi^2 N}{3} \frac{\text{Vol}(\Sigma_i)}{\text{Vol}(Y_7)}. \quad (5.17)$$

6. The forms $\omega_I$ for other symmetries

In this section, we find the forms entering in (4.2), for the non-R flavor symmetries. Those associated with non-R isometries are found in direct analogy with the discussion of [17], reviewed in the previous section, for $\omega_R$. We re-write (5.5) as

$$\text{vol}(Y_{2n-1}) = \frac{1}{2^{n-1}(n-1)!} e^\psi \wedge (de^\psi)^{n-1}. \quad (6.1)$$

Under a non-R isometry, the form $e^\psi$ (5.4) shifts by

$$e^\psi \rightarrow e^\psi + h_i(Y) A_{F_i}, \quad (6.2)$$
with the functions \( h_i(Y) \) obtained by contracting the 1-form \( \sigma \) in (5.4) with the Killing vector \( K_i \) for the flavor symmetry,

\[
h_i(Y) = i_{K_i} \sigma = g_{ab} K^a K^b_i.
\] (6.3)

The last equality follows from (5.1): \( i_{K_i} \sigma \) can be obtained by contracting the Reeb vector \( K^a_i \) and the general Killing vector \( K^b_i \), using the metric (5.1).

In the last section, for \( U(1)_R \), only the first \( e^\psi \) factor in (5.1) was shifted, as that \( e^\psi \) factor is associated with the \( U(1)_R \) fiber, where \( U(1)_R \) acts. Conversely, since non-R isometries do not act on the \( U(1)_R \) fiber, but rather in the Kahler Einstein base, we should not shift the first \( e^\psi \) factor in (5.1), but instead shift the \( n-1 \) factors of \( de^\psi \) in (5.1). Effecting this shift gives

\[
\delta \text{vol}(Y_{2n-1}) = \frac{1}{2^{n-1}(n-2)!} \left( e^\psi \wedge d(h_i(Y)A_{F_i}) \wedge (de^\psi)^{n-2} - de^\psi \wedge h_i(Y)A_{F_i} \wedge (de^\psi)^{n-2} \right),
\] (6.4)

where the last term was added to keep the form closed:

\[
\delta \text{vol}(Y_{2n-1}) = -d \left( \frac{1}{2(n-2)!} h_i(Y)e^\psi \wedge J^{n-2} \wedge A_{F_i} \right).
\] (6.5)

Effecting this shift in \( F^{bkgd} \) leads to \( \delta C_{2n-2} = \omega_{F_i} \wedge A_{F_i} \), with \( 2n-3 \) form \( \omega_{F_i} \):

\[
\tilde{\omega}_{F_i} \equiv \frac{\omega_{F_i}}{(2n-2)!} = -\frac{1}{2(n-2)!} h_i(Y)e^\psi \wedge J^{n-2} = -\frac{n(n-1)}{2} h_i(Y)\tilde{\omega}_R.
\] (6.6)

Aside from the factor of \(-\frac{1}{2} n(n-1) h_i(Y)\), \( \omega_{F_i} \) is the same as for \( \omega_R \), as given in (5.9).

In particular, for IIB on \( AdS_5 \times Y_5 \) we have

\[
\tilde{\omega}_{F_i} \equiv \frac{\omega_{F_i}}{4L^4} = -\frac{1}{2} h_i(Y_5)e^\psi \wedge J = -3 h_i(Y_5)\tilde{\omega}_R,
\] (6.7)

and for M theory on \( AdS_4 \times Y_7 \) we have

\[
\tilde{\omega}_{F_i} \equiv \frac{1}{6(2L)^6} \omega_{F_i} = -\frac{1}{4} h_i(Y_7)e^\psi \wedge J \wedge J = -6 h_i(Y_7)\tilde{\omega}_R.
\] (6.8)

As reviewed in (5.14), the R-charge of branes wrapped on supersymmetric cycles \( \Sigma \) is

\[
R[B_{\Sigma}] = \frac{2}{n} \mu_{2n-3} \int_{\Sigma} \text{vol}(\Sigma).
\] (6.9)
Using (6.6), the flavor charges of these wrapped branes can similarly be written as

\[ F_i[B_\Sigma] = \mu_{2n-3} \int_{\Sigma} \omega_{F_i} = -(n-1)\mu_{2n-3}m^{-(2n-2)} \int_{\Sigma} h_i \text{vol}(\Sigma) \]

\[ = -\frac{n(n-1)}{2} \cdot R[B_\Sigma] \cdot \frac{\int_{\Sigma} h_i \text{vol}(\Sigma)}{\int_{\Sigma} \text{vol}(\Sigma)} . \]  

(6.10)

In particular, for IIB on \( AdS_5 \times Y_5 \), we have

\[ F_i[B_\Sigma] = -\frac{\pi N}{\text{Vol}(Y)} \int_{\Sigma} h_i \text{vol}(\Sigma) = -3R[B_\Sigma] \frac{\int_{\Sigma} h_i \text{vol}(\Sigma)}{\int_{\Sigma} \text{vol}(\Sigma)} . \]  

(6.11)

The baryonic symmetries, coming from reducing \( C_{2n-2} \) on the non-trivial \((2n-3)\)-cycles of \( Y_{2n-1} \), also alter \( C_{2n-2} \) at linear order as in (4.2), \( \delta C_{2n-2} = \omega_{B_i} \wedge A_{B_i} \), where the \( 2n-3 \) forms \( \omega_{B_i} \) are representatives of the cohomology \( H_{2n-3}(Y, \mathbb{Z}) \). These can be locally written on \( Y_{2n-1} \) as

\[ \omega_{B_i} = k_i e^{\psi} \wedge \eta_i, \]  

(6.12)

where \( \eta_i \) are \( 2(n-2) \) forms on the Kähler-Einstein base, satisfying \( d\eta_i = 0 \), and \( \eta_i \wedge J = 0 \). The normalization constants \( k_i \) in (6.12) are chosen so that \( \mu_{2n-3} \int_{\Sigma} \omega_{B_i} \) is an integer for all \((2n-3)\)-cycles \( \Sigma \) of \( Y_{2n-1} \).

As mentioned in sect. 4, this construction of the forms \( \omega_{F_i} \) involves integrating an expression for \( d\omega_{F_i} \), so there’s an ambiguity of adding an arbitrary closed form to \( \omega_{F_i} \). Since addition of an exact form would not affect the charges of branes wrapped on closed cycles, the interesting ambiguity corresponds precisely to the same cohomology class of forms as the \( \omega_{B_j} \). This is as it should be: there is an ambiguity in our basis for the mesonic flavor symmetries, as one can always re-define them by arbitrary additions of the baryonic flavor symmetries. The form (6.6) for \( \omega_{F_i} \) corresponds to some particular choice of the basis for the mesonic flavor symmetries. In the field theory dual, it may look more natural to call this a linear combination of mesonic and baryonic flavor symmetries.

7. Computing \( \tau_{IJ} \) from the geometry of \( Y \)

The expressions (4.3) for the Ramond-Ramond kinetic term contribution \( (g^{-2}_{IJ})^{CC} \) is

\[ (g^{-2}_{IJ})^{CC} = \frac{1}{16\pi G_D} \int_Y \omega_I \wedge * \omega_J \equiv \frac{(2n-2)^2 m^{-(2n+1)}}{16\pi G_D} \int_Y \hat{\omega}_I \wedge * \hat{\omega}_J \]  

(7.1)
and the Einstein action contribution (3.8) is
\[
(g_{IJ}^{-2})^{KK} = \frac{m^{-(2n+1)}}{16\pi G_D} \int_{Y_{2n-1}} g_{ab} K^a_I K^b_J \text{vol}(Y_{2n-1});
\]
whereas for baryonic symmetries there is no contribution from the Einstein action, so \( g_{IJ}^{-2} = (g_{IJ}^{-2})^{CC} \).

Our claimed general proportionality (3.11) here gives
\[
(g_{IJ}^{-2})^{CC} = (n-1)(g_{IJ}^{-2})^{KK},
\]
which implies that
\[
4(n-1) \int_{Y_{2n-1}} \hat{\omega}_I \wedge \ast \hat{\omega}_J = \int_{Y_{2n-1}} g_{ab} K^a_I K^b_J \text{vol}(Y_{2n-1}).
\]
As we’ll see, this relation can look non-trivial in the geometry.

To compute \((g_{IJ}^{-2})^{CC}\) from (5.1), we first note that (5.9) gives
\[
*\hat{\omega}_R \equiv \frac{*\omega_R}{(2n-2)m^{-3}} = \frac{1}{n!} * e^\psi \wedge J^{n-2} = \frac{n-2}{n!} J,
\]
and then, using (5.5), gives
\[
\hat{\omega}_R \wedge * \hat{\omega}_R = \frac{(n-2)}{n!n} \text{vol}(Y_{2n-1}).
\]
In particular, for the \(U(1)_R\) graviphoton, we obtain
\[
(g_{RR}^{-2})^{CC} = \frac{(2n-2)^2 m^{-(2n+1)}}{16\pi G_D} \frac{(n-2)}{n!n} \text{Vol}(Y_{2n-1}).
\]
For the mixed kinetic term between \(U(1)_R\) and non-R isometries \(U(1)_{F_i}\),
\[
(g_{RF_i}^{-2})^{CC} = \frac{(2n-2)^2 m^{-(2n+1)}}{16\pi G_D} \frac{(n-2)}{n!n} \left(-\frac{n(n-1)}{2}\right) \int_Y h_i(Y) \text{vol}(Y).
\]
For the \(U(1)_{F_i}\) and \(U(1)_{F_j}\) kinetic terms, we similarly obtain
\[
(g_{F_iF_j}^{-2})^{CC} = \frac{(2n-2)^2 m^{-(2n+1)}}{16\pi G_D} \frac{(n-2)}{n!n} \left(\frac{n(n-1)}{2}\right)^2 \int_Y h_i(Y) h_i(Y) \text{vol}(Y).
\]
For \(U(1)_{B_i}\) symmetries, we have

\[
\mathcal{g}^{-2}_{RB_i} = \frac{1}{16\pi G_{D_t}} \int_Y \omega_{B_i} \wedge \star \omega_R = \frac{(2n-2)m^{-(2n-2)}e^{2\psi}}{16\pi G_{D_t}} n! \int_Y k_i e^\psi \wedge \eta_i \wedge J = 0,\]

(7.10)

where we used (5.12) for \(\omega_{B_i}\), (7.5), and we get zero immediately from \(\eta_i \wedge J = 0\). Likewise,

\[
g^{-2}_{F_i B_i} = 0,\]

(7.11)

for any isometry symmetry \(F_i\), since (6.6) gives \(\omega_{F_j} \propto \omega_R\), so \(\star \omega_{F_i} \propto J\), and we immediately get zero in (7.11) again from \(\eta_i \wedge J = 0\). As mentioned in the introduction, there is thus never any kinetic term mixing between any of the isometry Kaluza-Klein gauge fields and any of the gauge fields coming from reducing the \(C\) fields on non-trivial homology cycles of \(Y\). Finally, for the baryonic kinetic terms, we have

\[
g^{-2}_{B_i B_j} = \frac{1}{16\pi G_{D_t}} \int_Y k_i k_j e^\psi \wedge \eta_i \wedge \star_B \eta_j,\]

(7.12)

where \(\star_B\) acts on the \(2n-2\) dimensional Kahler-Einstein base.

For the isometry (non-baryonic) gauge fields, we have to add the Kaluza-Klein contributions, \((g^{-2}_{IJ})^{KK}\), from the Einstein action, to the kinetic terms. These can either be explicitly computed, using (7.2), or one can just use our relation (7.4) to the above Ramond-Ramond contributions. It’s interesting to check that our relation (7.4) is indeed satisfied. For example, the Kaluza-Klein contribution \((g^{-2}_{RR})^{KK}\) is

\[
\frac{m^{-(2n+1)}}{16\pi G_{D_t}} \int_{Y_{2n-1}} g_{ab} K^a K^b \text{vol}(Y_{2n-1}) = \frac{m^{-(2n+1)}}{16\pi G_{D_t}} \frac{4}{n^2} \text{Vol}(Y_{2n-1}),\]

(7.13)

where we used the local form of the metric (5.1), and \(U(1)_R\) isometry Killing vector (5.3), rescaled by the factor in (5.6) to have \(U(1)_R\) properly normalized. Comparing with (7.7), our relation (7.4) is indeed satisfied for both of our cases of interest, \(n = 3\) and \(n = 4\), appropriate for IIB on \(AdS_5 \times Y_5\) and M theory on \(AdS_4 \times Y_7\), respectively.

Our main point will be that the \(\tau_{R_i R_t}\) minimization condition (1.9) of [13] requires (7.8) to vanish, \(\tau_{RF_i} = 0\), so we must have

\[
\int_Y h_i(Y) \text{vol}(Y) = \int_Y i_{K_i} \sigma \text{vol}(Y) = \int_Y g_{ab} K^a K^b_i = 0,\]

(7.14)

for every non-R isometry Killing vector \(K^a_i\). We know from the field theory argument of (1.3) that the conditions (7.14) must uniquely determine which, among all possible R-symmetries, is the superconformal R-symmetry. Correspondingly, (7.14) determines the
isometry $K$, from among all possible mixing with the $K_a$. As we’ll discuss in the following sections, the Z-minimization of [9] precisely implements (7.14) (in the context of toric $C(Y)$). Also, (7.12) implies that the condition $\tau_{Ri}$ of [13] is automatically satisfied for baryonic $U(1)_B$. This is the reason why the Z-minimization method of [9] did not need to include any mixing of $U(1)_R$ with the baryonic $U(1)_B$ symmetries.

For future reference, we’ll now explicitly write out the above formulae for our cases of interest. For IIB on $AdS_5 \times Y_5$, we have $n = 3$ and $m^{-1} = L$, so (7.1) is

$$\tau_{CC}^{IJ} = 8\pi^2 L (g_{IJ}^{-1})_{CC} = \frac{8\pi L^8}{G_{10}} \int_{Y_5} \hat{\omega}_I \wedge \ast \hat{\omega}_J = \frac{16N^2 \pi^3}{Vol(Y_5)^2} \int_{Y_5} \hat{\omega}_I \wedge \ast \hat{\omega}_J, \quad (7.15)$$

where we used (2.18) to write the result in terms of $N$. For $I$ or $J$ baryonic, this is the entire contribution:

$$\tau_{IJ} = \frac{16N^2 \pi^3}{Vol(Y_5)^2} \times \int_{Y_5} \hat{\omega}_I \wedge \ast \hat{\omega}_J, \quad \text{for } I \text{ or } J \text{ baryonic.} \quad (7.16)$$

For isometry gauge fields, we add this to

$$\tau_{KK}^{IJ} = \frac{8\pi^2 L^8}{16\pi G_{10}} \int_{Y_5} vol(Y_5) g_{ab} K_a^I K_b^J = \frac{N^2 \pi^3}{Vol(Y_5)^2} \int_{Y_5} vol(Y_5) g_{ab} K_a^I K_b^J, \quad (7.17)$$

or, using relation (3.11), we simply have

$$\tau_{IJ} = \frac{3}{2} \tau_{CC}^{IJ} = \frac{24N^2 \pi^3}{Vol(Y_5)^2} \int_{Y_5} \hat{\omega}_I \wedge \ast \hat{\omega}_J, \quad \text{for } I \text{ and } J \text{ Kaluza-Klein.} \quad (7.18)$$

In particular, for the $U(1)_R$ kinetic term we compute

$$\tau_{CC}^{RR} = \frac{16N^2 \pi^3}{Vol(Y_5)^2} \int_{Y_5} \omega_R \wedge \ast \omega_R = \frac{16N^2 \pi^3}{Vol(Y_5)^2} \int_{Y_5} \frac{1}{36} e^\psi \wedge J \wedge J = \frac{8N^2 \pi^3}{9Vol(Y_5)}, \quad (7.19)$$

and

$$\tau_{KK}^{RR} = \frac{N^2 \pi^3}{Vol(Y_5)^2} \int_{Y_5} \frac{4}{9} vol(Y_5) = \frac{4N^2 \pi^3}{9Vol(Y_5)}, \quad (7.20)$$

verifying (3.11). The total for the graviphoton kinetic term coefficient then gives

$$\tau_{RR} = \tau_{CC}^{RR} + \tau_{KK}^{RR} = \frac{4}{3} \frac{N^2 \pi^3}{Vol(Y_5)}, \quad (7.21)$$

This agrees perfectly with the relation (2.2) and (2.4), given (2.19).
For the kinetic terms for two mesonic non-R symmetries, (7.18) gives
\[
\tau_{F_i F_j} = \frac{12N^2\pi^3}{Vol(Y_5)^2} \times \int_{Y_5} h_i h_j vol(Y_5). \quad (7.22)
\]
The relation (3.11), \(\tau_{KK}^{IJ} = \frac{1}{2} \tau_{CC}^{IJ}\), which was already used in (7.22) can be written as
\[
\int_{Y_5} g_{ab} K^a_{K_i} K^b_{K_j} vol(Y_5) = 4 \int_{Y_5} h_i h_j vol(Y_5) = 4 \int_{Y_5} g_{ac} g_{bd} K^c_{K_i} K^d_{K_j} vol(Y_5). \quad (7.23)
\]
Likewise, using (7.16), the kinetic terms for two baryonic flavor symmetries are
\[
\tau_{B_i B_j} = \frac{16N^2\pi^3}{Vol(Y_5)^2} \times \int_{Y_5} e^\psi \wedge \eta_i \wedge *(e^\psi \wedge \eta_j). \quad (7.24)
\]
For \(M\) theory on \(AdS_4 \times Y_7\), we set \(n = 4\) for \(Y_7\), and \(m^{-1} = 2L\) for its length scale, in the above expressions. Then we obtain from (7.1), using also (1.3) with \(d = 3\),
\[
\tau_{CC}^{IJ} \equiv 4\pi(g_{IJ}^{-2})^{CC} = 4\pi(6)^2(2L)^9 \frac{1}{16\pi G_{11}} \int Y_7 \hat{\omega}_I \wedge \hat{\omega}_J. \quad (7.25)
\]
Using the flux quantization relation (5.16), (7.25) becomes
\[
\tau_{CC}^{IJ} = \frac{48\pi^2 N^{3/2}}{\sqrt{6 Vol(Y_7))^{3/2}}} \int Y_7 \hat{\omega}_I \wedge \hat{\omega}_J. \quad (7.26)
\]
Using (3.8) we can also write the Kaluza-Klein contribution, as
\[
\tau_{KK}^{IJ} \equiv 4\pi(g_{IJ}^{-2})^{KK} = \frac{4\pi^2 N^{3/2}}{3\sqrt{6 Vol(Y_7))^{3/2}}} \int Y_7 g_{ab} K^a_{K_i} K^b_{K_j} vol(Y_7). \quad (7.27)
\]
For \(\tau_{RR}\), (7.4) gives
\[
\tau_{RR}^{CC} = \frac{\pi^2 N^{3/2}}{\sqrt{6 Vol(Y_7)}}. \quad (7.28)
\]
The Kaluza-Klein contribution is given by (7.2), with \(g_{ab} K^a_{K_i} K^b_{K_j} = (1/2)^2\) from (5.6), so
\[
\tau_{KK}^{RR} = \frac{\pi^2 N^{3/2}}{3\sqrt{6 Vol(Y_7))}}. \quad (7.29)
\]
Comparing (7.28) and (7.29), we verify that \(\tau_{RR}^{CC} = 3\tau_{RR}^{KK}\), in agreement with our general expression (3.12) (specializing \(Y_7 = S^7\) gives the case analyzed in [10]). The total is
\[
\tau_{RR} = \frac{4\pi^2 N^{3/2}}{3\sqrt{6 Vol(Y_7))}}. \quad (7.30)
\]
We can compare (7.30) with the 3d $\mathcal{N} = 2$ gauge theory proportionality relation

$$\tau_{RR} = \frac{\pi^3}{3} C_T \quad \text{in } d = 3,$$

(7.31)

where $C_T$ is the coefficient of the stress tensor two-point function. Along the lines of [24,26], the central charge $C_T$ is determined in the dual, from the Einstein term of $M$ theory on $AdS_4 \times Y_7$, to be

$$C_T = \frac{(2N)^{3/2}}{\pi \sqrt{3Vol(Y_7)}}, \quad (7.32)$$

so (7.30) indeed satisfies (7.31). As a special case, for $Y_7 = S^7$, $Vol(S^7) = \pi^4/3$ and (7.30) gives $\tau_{RR} = (2N)^{3/2}/3$.

For two non-R isometries, we have from (7.25) and (7.3), for $AdS_4 \times Y_7$:

$$\tau_{F_i F_j} = \frac{4}{3} \tau_{F_i F_j}^{CC} = \frac{\pi^2 (2N)^{3/2}}{3 \sqrt{3(Vol(Y_7))^{3/2}}} \int_{Y_7} (6)^2 h_i h_j vol(Y)^2.$$

(7.33)

8. Toric Sasaki-Einstein Geometry and Z-minimization

In this section, we’ll briefly summarize some of the results of [9]. Consider a Sasaki-Einstein manifold $Y_{2n-1}$, of real dimension $2n-1$, whose metric cone $X = C(Y)$ (1.4) is a local Calabi-Yau $n$-fold. The condition that (1.4) be Kahler is equivalent to $Y = X|_{r=1}$ being Sasaki, which is needed for the associated field theory to be supersymmetric. The complex structure of $X$ pairs the Euler vector $r \partial/r$ with the Reeb vector $K$, $K = T(r \partial/r)$. This is the AdS dual version of the pairing, by supersymmetry, between the dilatation generator and the superconformal R-symmetry, respectively. The physical problem of determining the superconformal R-symmetry among all possibilities (1.8) maps to the mathematical problem of determining the Reeb vector among all $U(1)$ isometries of $Y$.

When $X = C(Y)$ is toric, it can be given local coordinates $(y^i, \phi_i)$, $i = 1 \ldots n$, and both $C(Y)$ and $Y$ have a $U(1)^n$ isometry group, associated with the torus coordinates $\phi_i \sim \phi_i + 2\pi$. It is useful to introduce both symplectic coordinates $(y^i, \phi_i)$ and complex coordinates $(x_i, \phi_i)$. In the symplectic coordinates, the symplectic Kahler form is simply $\omega = dy^i \wedge d\phi_i$, and the metric with toric $U(1)^n$ isometry takes the form

$$ds^2 = G_{ij} dy^i dy^j + G^{ij} d\phi_i d\phi_j,$$

(8.1)
with $G^{ij}$ the inverse to $G_{ij}(y)$, and $G_{ij} = \partial^2 G/\partial y^i \partial y^j$ for some convex symplectic potential function $G(y)$. In the complex coordinates, $z_i = x_i + i\phi_i$, the metric is

$$ds^2 = F^{ij} dx_i dx_j + F^{ij} d\phi_i d\phi_i,$$

(8.2)

and $F^{ij} = \partial^2 F(x)/\partial x_i \partial x_j$, with $F(x)$ the Kahler potential. The two coordinates are related by a Legendre transform, $y^i = \partial F(x)/\partial x_i$ and $F^{ij}(x) = G^{ij}(y = \partial F/\partial x)$, with $F(x) = (y_i \partial G/\partial y_i - G)(y)$. The holomorphic n-form of the cone $X = C(Y)$ is

$$\Omega_n = e^{x_1 + i\phi_1} (dx_1 + id\phi_1) \wedge \ldots \wedge (dx_n + id\phi_n).$$

(8.3)

The Reeb vector can be expanded as

$$K = b_i \frac{\partial}{\partial \phi_i},$$

(8.4)

and its symplectic pairing with $r \frac{\partial}{\partial r}$ implies that

$$b_i = 2G_{ij} y^j, \quad \text{note: } b_i = \text{constant.}$$

(8.5)

The problem of determining the superconformal R-symmetry maps to that of determining the coefficients $b_i$, $i = 1 \ldots n$. The component $b_1$ is fixed to $b_1 = n$ by the condition that $L_K \Omega_n = in\Omega_n$, which is the condition that $U(1)_R$ in the field theory is properly normalized to give the superpotential charge $R(W) = 2$. The remaining $n - 1$ components $b_i$ are unconstrained by symmetry conditions, corresponding to the field theory statement that $U(1)_R$ can mix with an $U(1)^{n-1}$ group of non-R flavor symmetries.

The space $X = C(Y)$ is mapped by the moment map, $\mu$, where one forgets the angular coordinates $\phi_i$, to $C = \{y|(y, v_a) \geq 0\}$, where $v_a \in \mathbb{Z}^n$, for $a = 1 \ldots d$, are the “toric data”. The supersymmetric divisors $D_a$ of $X$ are mapped by $\mu$ to the subspaces $(y, v_a) = 0; \text{here } a = 1 \ldots d$ label the divisors ($d$ here, of course, is unrelated to the spacetime dimension $d$ of our other sections). The Sasaki-Einstein $Y$ is given by $X|_{r=1}$, and $r = 1$ gives $1 = b_ib_jG^{ij} = 2(b, y)$. It is also useful to define $X_1 \equiv X|_{r \leq 1}$, with $\mu(X_1) = \Delta_b \equiv \{y|(y, v_a) \geq 0, \text{ and } (y, b) \leq \frac{1}{2}\}$. The supersymmetric $2n - 3$ dimensional cycles $\Sigma_a$ of $Y$, for $a = 1 \ldots d$, have cone $D_a = C(\Sigma_a)$ which are the divisors of $X$, and $\mu(\Sigma_a)$ is the subspace $\mathcal{F}_a$ of $\Delta_b$ with $(y, v^n) = 0$.

The volume of $Y$ and its supersymmetric cycles $\Sigma_a$ are found from considering their cones in $X_1$, which are calibrated by the Kahler form $\omega = dy^i \wedge d\phi_i$. This gives

$$Vol_b(Y) = 2n(2\pi)^n Vol(\Delta_b), \quad Vol_b(\Sigma_a) = (2n - 2)(2\pi)^{n-1} \frac{1}{|v_a|} Vol_b(\mathcal{F}_a).$$

(8.6)
As shown in [9], \( \sum_a \frac{1}{|v_a|} Vol_b(\mathcal{F}_a)(v_a)_i = 2nVol(\Delta_b)b_i \), from which it follows that these volumes satisfy \( \pi \sum_a Vol(\Sigma_a) = n(n-1)Vol(Y) \). (This ensures that superpotential terms, associated in the geometry with the holomorphic n-form, have \( R(W) = 2 \).)

The key point [9] is that the full information of the Sasaki-Einstein metric on \( Y \) is not needed to determine the volumes [8.6]; the weaker information of the Reeb vector \( b_i \) and the toric data \( v_a \) suffice.

Moreover, the Reeb vector \( b_i \) can be determined from the toric data [9]. This fits with the fact that the toric data determines the dual quiver gauge theory (see e.g. [10] and references cited therein), from which the superconformal R-charges can be determined.

The Z-minimization method of [9] for determining the Reeb vector is to start with the 2\( n-1 \) dimensional Einstein-Hilbert action for the metric \( g \) on \( Y_{2n-1} \):

\[
S[g] = \int_Y (R_g + 2(n-1)(3-2n))vol(Y),
\]

including the needed cosmological constant term associated with the added flux. Though (8.7) appears to be a functional of the metric, it was shown in [9] that it’s actually only a function of only the Reeb vector:

\[
S[g] = S[b] = 4\pi \sum_a Vol_b(\Sigma_a) - 4(n-1)^2Vol_b(Y).
\]

The full information of the metric is not needed, the weaker information of the Reeb vector suffices to evaluate the action.

As shown in [9], the condition that \( b \) be the correct Reeb vector, associated with a Sasaki-Einstein metric, is precisely the condition that the action (8.8) be extremal:

\[
\frac{\partial}{\partial b_i} S[b] = 0.
\]

Defining

\[
Z[b] \equiv \frac{1}{4(n-1)(2\pi)^n}S[b] = (b_1 - (n-1))2nVol(\Delta_b),
\]

the equation (8.9) for \( i = 1 \) gives \( b_1 = n \), which is just the condition that the holomorphic n-form transforms as appropriate for a \( U(1)_R \) symmetry. Following [9], define

\[
\tilde{Z}[b_2, \ldots b_n] = Z|_{b_1=n} = 2nVol_b(\Delta)|_{b_1=n}.
\]

The equations (8.9) for \( i \neq 1 \) give, upon setting \( b_1 = n \),

\[
0 = \frac{\partial}{\partial b_i} \tilde{Z}[b] = -2(n+1) \int_{\Delta_b} y_i dy_1 \ldots dy_n \quad \text{for} \ i \neq 1.
\]
These are the equations that determine the components $b_i$, for $i = 2 \ldots n$, of the Reeb vector, i.e. that pick out the superconformal $U(1)_R$ from the $U(1)^n$ isometry group \[9\]. The correct Reeb vector minimizes $\tilde{Z}$, since the matrix of second derivatives is positive \[9\]

$$\frac{\partial^2 \tilde{Z}}{\partial b_i \partial b_j} \propto \int_H y_i y_j d\sigma > 0.$$  \hspace{1cm} (8.13)

9. $Z$-minimization = $\tau_{RR}$ minimization.

Let’s write (8.11) and (8.6) as

$$\tilde{Z}[b_2, \ldots b_n] = 2n Vol_b(\Delta) = \frac{1}{(2\pi)^n} Vol_b(Y)|_{b_1=n},$$  \hspace{1cm} (9.1)

so $Z$ minimization corresponds to minimizing the volume of $Y$, over the choices of $b_2, \ldots, b_n$, subject to $b_1 = n$. This can be directly related to $\tau_{RR}$ minimization \[13\], i.e. minimization of the $U(1)_R$ graviphoton’s coupling, since

$$\tau_{RR} = C_n \frac{L^{d-3} m^{-(2n+1)}}{16\pi G_D t} Vol(Y).$$  \hspace{1cm} (9.2)

The constant $C_n$ is obtained from adding the contributions (7.7) and (7.13) and using the relation (1.3). Let us now consider the quantity (9.2), but with $Vol(Y)$ promoted to the function $Vol_b(Y)$, depending on components $b_2, \ldots, b_n$ of the Reeb vector:

$$\tilde{\tau}_{R_t R_t}[b_2, \ldots, b_n] \equiv C_n \frac{L^{d-3} m^{-(2n+1)}}{16\pi G_D t} Vol_b(Y) = C_n (2\pi)^n \frac{L^{d-3} m^{-(2n+1)}}{16\pi G_D t} \cdot \tilde{Z}[b_2, \ldots, b_n].$$  \hspace{1cm} (9.3)

For the superconformal $U(1)_R$ values of $b_2, \ldots, b_n$, $\tilde{\tau}_{R_t R_t} = \tau_{RR}$.

If we hold $L^{d-3} m^{-(2n+1)}/G_D t$ fixed, (9.3) suggests a direct relation between $Z$ and $\tau_{RR}$ minimization. Physically, we should hold the number of flux units $N$ fixed, i.e. use the flux quantization relation to eliminate $L^{d-3} m^{-(2n+1)}/G_D t$ in favor of $N/Vol(Y)$. In particular, for IIB on $AdS_5 \times Y_5$ and M theory on $AdS_4 \times Y_4$,

$$AdS_5 \times Y_5: \quad C_n \frac{L^{d-3} m^{-(2n+1)}}{16\pi G_D t} = \frac{4\pi^3}{3} \left( \frac{N}{Vol(Y)} \right)^2,$$

$$AdS_4 \times Y_7: \quad C_n \frac{L^{d-3} m^{-(2n+1)}}{16\pi G_D t} = \frac{4\pi^2}{3\sqrt{6}} \left( \frac{N}{Vol(Y)} \right)^{3/2}. $$ \hspace{1cm} (9.4)

Using these in (9.2) shows that, for fixed $N$, $\tau_{RR}$ is actually inversely related to $Vol(Y)$. From that perspective, it would seem that $Z$ minimization instead maximizes $\tau_{RR}$, which
is opposite to the result of [13] that the exact superconformal $U(1)_R$ minimizes $\tau_{RR}$. To avoid this, we do not promote the constant $Vol(Y)$ in the flux relations \eqref{9.4} to the function $Vol_b(Y)$ of the Reeb vector, but instead there hold it fixed to its true, physical value. Then the function $\tilde{\tau}_{R_i R_i} [b]$ \eqref{4.3} is simply a constant times the function $\tilde{Z} [b]$ of \[3\].

To use the formulae of our earlier sections, consider the Killing vectors

$$\chi = \chi_i \frac{\partial}{\partial \phi_i} \quad (9.5)$$

for the $U(1)^n$ isometries of toric $Y_{2n-1}$. R-symmetries, and in particular the Reeb vector, have $\chi_1 = n$, and non-R isometries have $\chi_1 = 0$. As we discussed in sections 5 and 6, the isometry $d\phi \rightarrow d\phi + A \chi$ has an associated 2n − 3 form, which is found from the associated shift $e^\psi \rightarrow e^\psi + h_\chi (Y) A \chi$. For the R-symmetry, this comes from the shift of $d\psi'$, and for non-R flavor symmetries the shift is via $h_\chi = i \chi \sigma$. Using the second equality in \eqref{6.3}, we have

$$h_\chi (Y) = F^{ij} b_i \chi_j = G^{ij} b_i \chi_j = 2 y^i \chi_i = 2 \langle r^2 \theta, \chi \rangle, \quad (9.6)$$

with the inner product with $r^2 \theta$ as in \[9\]. For the Reeb vector, \eqref{9.6} gives $h_K = 1$, since the cone $r = 1$ has $1 = b_i b_j G^{ij} = 2 (b, y)$ \[9\].

For the non-R isometries, we can take as our basis of Killing vectors e.g. $\chi^{(i)} = \frac{\partial}{\partial \phi_i}$, so $\chi^{(i)}_j = \delta_{ij}$, for $i = 2 \ldots n$. Then \eqref{9.6} gives simply

$$h_\chi^{(i)} = 2 y^i. \quad (9.7)$$

In this basis, where $U(1)_{F_i}$ is associated with Killing vector $\frac{\partial}{\partial \phi_i}$, the $F_i$ charge of a brane wrapped on cycle $\Sigma$ is

$$F_i [B_\Sigma] = -(n-1) \mu_{2n-3} m^{-(2n-2)} \int_\Sigma 2 y^i vol (\Sigma)$$

$$= -n (n-1) \cdot R [B_\Sigma] \cdot \frac{\int_\Sigma y_i vol (\Sigma)}{\int_\Sigma vol (\Sigma)} . \quad (9.8)$$

In particular, for IIB background $AdS_5 \times Y_5$, we have

$$F_i [B_\Sigma] = -\frac{2 \pi N}{Vol(Y_5)} \int_{\Sigma_3} y_i vol (\Sigma) , \quad (9.9)$$

and for M theory background $AdS_4 \times Y_7$ we have

$$F_i [B_\Sigma] = -\frac{4 \pi^2 N}{Vol(Y_7)} \int_{\Sigma_5} y_i vol (\Sigma) . \quad (9.10)$$
Using our formulae from sect. 7, we can determine the kinetic terms $g_{ij}^{-2}$, and hence $\tau_{ij}$ in terms of the geometry of $Y$. In particular, using (7.8) and (9.7), we have

$$\tau_{RF_i} = C_n \frac{L^{d-3} m^{-(2n+1)}}{16\pi G_D t} \left(-n(n-1)\right) \int_Y y^i vol(Y),$$

(9.11)

with $C_n$ the same constant appearing in (9.2). Note that

$$\int_Y y^i vol(Y) = 2(n + 1) \int_{X_1} y^i vol(X_1) = 2(n + 1)(2\pi)^n \int_{\Delta_b} y^i dy^1 \ldots dy^n,$$

(9.12)

(2(n + 1) accounts for the extra $r$ integral in $X_1$). Moreover, eqn. (3.21) of [9] gives

$$\int_{\Delta_b} y^i dy^1 \ldots dy^n = -\frac{1}{2(n + 1)} \frac{\partial}{\partial b_i} Vol_b(\Delta).$$

(9.13)

So (9.11) gives

$$\tau_{RF_i} = C_n \frac{L^{d-3} m^{-(2n+1)}}{16\pi G_D t} (2\pi)^n \frac{n(n - 1)}{2} \frac{\partial}{\partial b_i} \tilde{Z}[b_2, \ldots b_n].$$

(9.14)

As discussed, we take the factors in (9.4) to be $b_i$ independent constants, so (9.14) can be written as

$$\tau_{RF_i} = \frac{(n - 1)}{2} \frac{\partial}{\partial b_i} \tau_{R_t R_t}[b_2, \ldots b_n].$$

(9.15)

The relation (9.14) shows that the $\tau_{R_t R_t}$ minimization equations, $\tau_{RF_i} = 0$, are indeed equivalent to the $Z$ minimization equations (8.12) of [9].

We can similarly use our formula (7.8) and (9.6) to obtain the coefficient $\tau_{FiF_j}$ for two flavor currents:

$$\tau_{FiF_j} = C_n \frac{L^{d-3} m^{-(2n+1)}}{16\pi G_D t} (n(n - 1))^2 \int_Y y^i y^j vol(Y),$$

(9.16)

with $C_n$ the same constant appearing in (9.2). Note now that

$$\int_Y y^i y^j vol(Y) = 2(n + 2) \int_{X_1} y^i y^j vol(X_1) = 2(n + 2)(2\pi)^n \int_{\Delta_b} y^i y^j dy^1 \ldots dy^n.$$

(9.17)

Moreover, in analogy with the derivation of (9.13), in eqn. (3.21) of [9], we find:

$$\int_{\Delta_b} y^i y^j dy^1 \ldots dy^n = \frac{1}{4(n + 1)(n + 2)} \frac{\partial^2}{\partial b_i \partial b_j} Vol_b(\Delta).$$

(9.18)
We can then write (9.16) as
\[
\tau_{F_i F_j} = \frac{n(n-1)^2}{4(n+1)} \frac{\partial^2}{\partial b_i \partial b_j} \tilde{\tau}_{R_t R_t}[b_2 \ldots b_n],
\] (9.19)
where again we take (9.4) as \(b\) independent.

Since \(\tilde{\tau}_{R_t R_t}\) is proportional to \(\bar{Z}\), (9.19) provides a way to evaluate the current two-point function coefficients \(\tau_{F_i F_j}\) entirely in terms of the Reeb vector and the toric data, without needing to know the metric.

In [13], we discussed the trial function \(\tau_{R_t R_t}(s_i)\), which is quadratic in the parameters \(s_i\), and satisfies
\[
\tau_{R_t R_t}|_{s^*} = \tau_{RR}, \quad \frac{\partial}{\partial s_i} \tau_{R_t R_t}|_{s^*} = 2\tau_{R_t} = 0, \quad \frac{\partial^2}{\partial s_i \partial s_j} \tau_{R_t R_t}(s) = 2\tau_{ij}.
\] (9.20)
This can be compared with the function \(\tilde{\tau}_{R_t R_t}(b_i)\) defined above, which coincides with \(\tau_{RR}\) for the minimizing values \(b_i^\star\), which are determined by setting the derivatives to zero, (9.15), and the second derivatives (9.19) are proportional to \(\tau_{ij}\), as in (9.20). The relation between \(s_i\) and \(b_i\) can be chosen to convert the coefficients in (9.19) to equal those of (9.20).

Let us now consider further the expression (9.8), or more explicitly (9.9) and (9.10), for the flavor charges of branes wrapped on cycles. We would like to evaluate these for the supersymmetric cycles \(\Sigma_a \subset Y\), i.e. to evaluate
\[
\int_{\Sigma_a} y^i vol(\Sigma)
\] (9.21)
in terms of the toric data and Reeb vector. Note that
\[
\int_{\Sigma_a} y^i vol(\Sigma) = 2n \int_{C(\Sigma_a)} y^i vol(C(\Sigma_a)) = 2n(2\pi)^{n-1} \int_{F_a} y^i d\sigma_a,
\] (9.22)
where the \(2n\) factor is from the extra \(r\) integral in going from \(\Sigma_a\) to \(C(\Sigma_a)\), and \(d\sigma_a\) is the measure on \(F_a\), from \(\int \delta((y, v_a))dy^1 \ldots dy^n\). In analogy with the derivation of eqn. (3.21) in [4], it seems likely that the \(y^i\) in (9.21) and (9.22) can be obtained from the volume \(Vol_b(\Sigma_a)\) in (8.6) by differentiating w.r.t. \(b_i\). But completing this argument, accounting for all the potential new boundary terms, seems potentially subtle (to us).

Let us, instead, note a different way to compute the charges from the toric data. Consider the expression for \(Vol_b(Y)\), as a function of both \(b\) and the toric data \((v_a)_i\). In the integral leading to \(Vol_b(Y) = 2n(2\pi)^n Vol(\Delta_b)\) in (8.6), the vectors \((v_a)_i\) appear via
the boundary of \( \Delta_b \), which has \((y,v_a) \geq 0\). Thinking of them as variables, taking the derivative w.r.t. \( v_a \) then gives a contribution only on the boundary \((y,v_a) = 0\):

\[
\frac{\partial}{\partial (v_a)_i} \text{Vol}(\Delta_b) = - \int_{\mathcal{F}_a} y_i d\sigma_a. \tag{9.23}
\]

Using (9.22) and (8.0) then gives

\[
\int_{\Sigma} y^i \text{vol}(\Sigma) = - \frac{1}{2\pi} \frac{\partial}{\partial (v_a)_i} \text{Vol}_b(Y). \tag{9.24}
\]

In the above expressions for \( \tilde{\tau}_{RR} \) and \( \tau_{RF_i} \) and \( \tau_{F,F_J} \), the Ramond-Ramond and Kaluza-Klein contributions to \( g^{-2}_{IJ} \) were summed together, in the coefficient \( C_n \). Using the relation (3.12), which here gives \( (g^{-2}_{IJ})^{CC} = (n-1)(g^{-2}_{IJ})^{KK} \), those two contributions have a fixed ratio. Let us now examine that relation in the present context. For general Killing vectors \( \chi^{(I)} \) and \( \chi^{(J)} \), the contribution (4.13) to their mixed kinetic term is

\[
(g^{-2}_{IJ})^{CC} \propto \int_Y 4 y^i y^j \chi^{(I)}_i \chi^{(J)}_j \text{vol}(Y). \tag{9.25}
\]

The contribution (3.8) of the Einstein term is similarly

\[
(g^{-2}_{IJ})^{KK} \propto \int_Y G^{ij} \chi^{(I)}_i \chi^{(J)}_j \text{vol}(Y). \tag{9.26}
\]

Taking both \( I \) and \( J \) to be the R-symmetry, with \( \chi_I \) and \( \chi_J \) the Reeb vector, the relation from \( (g^{-2}_{IJ})^{CC} = (n-1)(g^{-2}_{IJ})^{KK} \) is

\[
\int_Y G^{ij} b_i b_j dy_1 \ldots dy_n = 4 \int_Y (y^i b_i)^2 dy_1 \ldots dy_n; \tag{9.27}
\]

which is clearly satisfied, since \( 2b_i y^i = G^{ij} b_i b_j = 1 \). For non-R flavor symmetries, the identity is less trivial. For general \( Y_{2n-1} \) it states that

\[
\int_{Y_{2n-1}} G^{ij} \text{vol}(Y) = 4(n-1)^2 \int_{Y_{2n-1}} y^i y^j \text{vol}(Y) \quad i,j \neq 1. \tag{9.28}
\]

The extra factor of \((n-1)^2\), as compared with (9.27), is as in (7.8), coming from writing the volume form as \( \sim e^\psi \wedge (d e^\psi)^{n-1} \) and the fact that \( \omega_R \) is found from the shift of the first \( e^\psi \) factor, whereas the non-R isometries are obtained by shifting the \( n-1 \) factors of \( d(e^\psi) \). The relation (9.28) can indeed be verified to hold in the various examples. It can
also be written in terms of integrals over $\Delta_b$, by extending to $X_1$ and doing the extra $r$ integrals, as

$$(n + 1) \int_{\Delta_b} G^{ij} dy^1 \ldots dy^n = 4(n - 1)^2(n + 2) \int_{\Delta_b} y^i y^j dy^1 \ldots dy^n. \quad (9.29)$$

10. Examples and checks of AdS/CFT: $Y^{p,q}$

The metric of \[4,5\] is simply written in the basis of unit one-forms

$$e^\psi = \frac{1}{3}(d\psi' - \cos \theta d\phi + y(d\beta + \cos \theta d\phi))$$

$$e^\theta = \sqrt{\frac{1 - y}{6}} d\theta, \quad e^\phi = \sqrt{\frac{1 - y}{6}} \sin \theta d\phi, \quad e^y = \frac{1}{\sqrt{wv}} dy, \quad e^\beta = \frac{\sqrt{wv}}{6} (d\beta + \cos \theta d\phi), \quad (10.1)$$

as $ds_Y^2 = (e^\theta)^2 + (e^\phi)^2 + (e^y)^2 + (e^\beta)^2 + (e^\psi)^2$. The coordinate $y$ lives in the range $y_1 \leq y \leq y_2$, where $y_1$ and $y_2$ are the two smaller roots of $v(y) = 0 \[5\]$

$$y_1 = \frac{1}{4p} \left(2p - 3q - \sqrt{4p^2 - 3q^2}\right), \quad y_2 = \frac{1}{4p} \left(2p + 3q - \sqrt{4p^2 - 3q^2}\right). \quad (10.2)$$

The local Kahler form of the 4d base is

$$J = e^\theta \wedge e^\phi + e^y \wedge e^\beta. \quad (10.3)$$

The gauge symmetries in $AdS_5$ of IIB on $Y_{p,q}$, and the global symmetries of the dual SCFTs \[7\], are $U(1)_R \times SU(2) \times U(1)_F \times U(1)_B$. The first three factors are associated with isometries of the metric, and $U(1)_B$ comes from the single representative of $H_3(Y_{p,q}, Z)$ (topologically, all are $S^2 \times S^3$). As usual, the superconformal $U(1)_R$ symmetry is associated with the shift in $e^\psi$: $\frac{1}{3}d\psi' \rightarrow \frac{1}{3}d\psi' + \frac{2}{3}A_R$, and the associated 3-form is that of \[17\]:

$$\hat{\omega}_R \equiv \frac{1}{4L^4} \omega_R = \frac{1}{6} e^\psi \wedge J. \quad (10.4)$$

The $SU(2)$ is symmetry is an non-R isometry, associated with rotations of the spherical coordinates $\theta$ and $\phi$. Finally, the $U(1)_F$ isometry is associated with shifts $d\beta + \cos \theta d\phi \rightarrow d\beta + \cos \theta d\phi + A_F$. $U(1)_\phi \subset SU(2)$ and $U(1)_F$ form a basis for the $U(1)^2$ non-R isometries,
expected from the fact that $Y_{p,q}$ is toric \cite{5}. The 3-forms associated with these flavor $U(1)^2$ are found from (6.3) and (6.7) to be

$$\hat{\omega}_\phi \equiv \frac{1}{4L^4} \omega_\phi = -\cos \theta \hat{\omega}_R \quad \text{and} \quad \hat{\omega}_F \equiv \frac{1}{4L^4} \omega_F = -y \hat{\omega}_R. \quad (10.5)$$

The 3-form associated with the $U(1)_B$ baryonic symmetry was already constructed in \cite{8}, restricting their form $\Omega_2$ on $C(Y_{p,q})$ to $Y_{p,q}$ by setting $r = 1$:

$$\mu_3 \omega_B = \frac{9}{8\pi^2} (p^2 - q^2) e^\psi \wedge \eta \quad \eta \equiv \frac{1}{(1 - y)^2} (e^\theta \wedge e^\phi - e^y \wedge e^\beta), \quad (10.6)$$

where the normalization constant is to keep the periods of $\mu_3 \int C_4$ properly integral.

D3 branes wrapped on the various supersymmetric 3-cycles $\Sigma_a$ of $Y$ map to the di-baryons of the dual gauge theory \cite{7} as:

$$\Sigma_1 \leftrightarrow \det Y, \quad \Sigma_2 \leftrightarrow \det Z, \quad \Sigma_3 \leftrightarrow \det U_\alpha, \quad \Sigma_4 \leftrightarrow \det V_\alpha. \quad (10.7)$$

The cycles $\Sigma_1$ and $\Sigma_2$ are given by the coordinates at $y = y_1$ and $y = y_2$ respectively \cite{8}. The cycle $\Sigma_3$ is given by fixing $\theta$ and $\phi$ to constant values, which yields the $SU(2)$ collective coordinate of the di-baryon \cite{8}. The cycle $\Sigma_4 \cong \Sigma_2 + \Sigma_3$.

As in \cite{17}, the R-charges of the wrapped D-3 branes, computed from $\mu_3 \int_{\Sigma_i} \omega_R$, are

$$R(\Sigma_i) = \frac{\pi N}{3Vol(Y_5)} \int_{\Sigma_i} vol(\Sigma) = \frac{\pi N Vol(\Sigma_i)}{3 Vol(Y_5)}. \quad (10.8)$$

It was verified in \cite{5,6,7,8} that the R-charges computed from the cycle volumes as in (10.8) agree perfectly with the map (10.7) and the superconformal R-charges, computed in the field theory dual by using the a-maximization \cite{14} method.

We can similarly verify that integrating the $U(1)_\phi$, $U(1)_F$ and $U(1)_B$ 3-forms (10.3) and (10.6) over the 3-cycles $\Sigma_a$ agree with the map (10.7) and the corresponding charges of the dual field theory \cite{7}. For $U(1)_B$ we have

$$B(\Sigma_i) = \mu_3 \int_{\Sigma_i} \omega_B = \frac{9}{8\pi^2} (p^2 - q^2) \int_{\Sigma_i} e^\psi \wedge \frac{1}{(1 - y)^2} (e^\theta \wedge e^\phi - e^y \wedge e^\beta), \quad (10.9)$$

and, as already computed in \cite{8}, this gives (reversing $\Sigma_1$’s orientation)

$$B(\Sigma_1) = (p - q), \quad B(\Sigma_2) = (p + q), \quad B(\Sigma_3) = p, \quad (10.10)$$

in agreement with the $U(1)_B$ charges of \cite{7} for $Y$, $Z$, and $U_\alpha$, respectively. One minor difference is that we normalize the $U(1)_B$ charges for the bi-fundamentals with a factor of
only the \( \tau \) of \( [7] \), up to the ambiguity that we have mentioned for redefining \( \tau \) and the contributions \( \tau \).

Using the metric \([4,5]\), we can explicitly compute the contributions \( \tau_{ij}^{CC} \) in \((7.15)\) and the contributions \( \tau_{ij}^{KK} \) in \((7.17)\), and verify that \( \tau_{ij}^{CC} = 2\tau_{ij}^{KK} \), as expected from \((3.11)\), for the \( U(1)_R \) and \( U(1)_\phi \) and \( U(1)_F \) isometry gauge fields. For \( U(1)_B \), there is only the \( \tau_{ij}^{CC} \) contribution to \( \tau_{ij} \). For the superconformal \( U(1)_R \), we find, as expected \( \tau_{RR}^{KK} = 4N^2\pi^3/9\text{vol}(Y_{p,q}) \) and \( \tau_{RR}^{CC} = 8N^2\pi^3/9\text{vol}(Y_{p,q}) \), with \([3]\)

\[
\text{Vol}(Y_{p,q}) = \frac{q^2[2p + (4p^2 - 3q^2)^{1/2}]}{3p^2[3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}]}p^3. 
\]

For \( \tau_{KK}^{KK} \), the metric \([4,5]\) gives \( g_{ab}K_a^bK_F = \frac{1}{36}wq + \frac{1}{9}y^2 = \frac{1}{36}w(y) \), so \((7.17)\) yields

\[
\tau_{KK}^{KK} = \frac{N^2\pi^3}{36\text{Vol}(X_5)} \int \frac{dyw(y)(1-y)}{dy(1-y)} = \frac{N^2\pi^3}{18\text{Vol}(X_5)} \frac{\sqrt{4p^2 - 3q^2}}{p^2} \left(2p - \sqrt{4p^2 - 3q^2}\right). 
\]

Using \( \tilde{\omega}_F \) of \((10.5)\) in \((7.15)\) we can also compute

\[
\tau_{CC}^{CC} = \frac{N^2\pi^3}{6\text{Vol}(Y_{p,q})} \sqrt{4p^2 - 3q^2} \left(2p - \sqrt{4p^2 - 3q^2}\right). 
\]
This result for $\tau_{FF}$ can be compared with the field theory prediction. The $U(1)_F$ charges of the bifundamentals are found from the $U(1)_F$ charges (10.12) of the dibaryons, and the map (10.7) (so the factor of $N$ from (10.8) is eliminated), e.g. $F(Z) = -y_2 R(Z) = -y_2 \pi Vol(\Sigma_2)/3Vol(Y_5)$, which looks rather ugly when written out in terms of $p$ and $q$.

From these charges and the $U(1)_R$ charges, we can compute the ’t Hooft anomalies, and thereby compute $\tau_{FF}$ on the field theory side by using the relation $\tau_{FF} = -3Tr RFF$. The result is found to agree perfectly with (10.16).

Let us now consider $\tau_{RF}$. The Kaluza-Klein contribution is given as in (7.17), with $g_{ab} K^a_R K^b_F = y/9$, and the integral over $y$ vanishes, so $\tau_{RF}^{KK} = 0$. Likewise, $\tau_{RF}^{CC} = 0$, because $\int y(1-y)$ vanishes. So, as expected, $\tau_{RF} = 0$.

As we discussed in the previous section, the $F_i[\Sigma_a]$ charges and $\tau_{IJ}$ can also be computed entirely from the toric data and Z-function of [9]. In the toric basis of [9],

$$v_1 = (1,0,0), \quad v_2 = (1,p-q-1,p-q), \quad v_3 = (1,p,p), \quad v_4 = (1,1,0).$$

(10.17)

The Z-function is, with $(b_1, b_2, b_3) \equiv (x,y,t)$, [9]

$$Z[x,y,t] = \frac{(x-2)p(p(p-q)x + q(p-q)y + q(2-p+q)t)}{2t(px - py + (p-1)t)((p-q)y + (1-p+q)t)(px + qy - (q+1)t)},$$

(10.18)

which, imposing $x = 1$, is minimized for [9]:

$$b_{min} = \left( 3, \frac{1}{2}(3p-3q+\ell^{-1}), \frac{1}{2}(3p-3q+\ell^{-1}) \right), \quad \ell^{-1} = \frac{1}{q} \left( 3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2} \right).$$

(10.19)

Our formula (10.19), for example, gives $\tau_{F_iF_j}$, for the $F_i$ associated with the $\sim \frac{\partial}{\partial \phi_i}$ Killing vectors, in terms of the Hessian of second derivatives of the function (10.18), evaluated at (10.19). To connect the results in the toric basis for the flavor symmetries to those discussed above, we note that the Killing vector for shifting $\beta$ can be related to those for shifting $\phi_1$ and $\phi_2$ as $\frac{\partial}{\partial \beta} = \frac{\ell^{-1}}{6} \left( \frac{\partial}{\partial \phi_2} + \frac{\partial}{\partial \phi_3} \right)$, so $U(1)_F = \frac{\ell^{-1}}{6} (U(1)_2 + U(1)_3)$.

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