TOV equation of state and bulk properties of astro-nuclear objects: an investigation

S H Mondal¹, M Alam¹, M Hasan¹, Md A Khan*¹

¹Department of Physics, Aliah University, IIA/27, Newtown, Kolkata-700160, India.
*drakhan.rsm.phys@gmail.com; drakhan.phys@aliah.ac.in

Abstract. This study of the common characteristics of compact celestial bodies is mostly inspired by the Potekhin group’s current findings on the evolution and structure of compact objects, exploiting the unified equation of state (EoS), proposed by the group reported as Brussels-Montreal (B-M) group. We used three parametric approaches to solve the TOV EoS numerically and obtain number density of baryon, internal stellar pressure, bulk coefficients, and some other quantities for a broad spectrum of mass densities. Numerically simulated results of these quantities are reported in terms of matter density in a tabular form. A schematic view of calculated quantities and their comparison with reference values wherever available are reported in the representative cases.

Keywords: EoS, bulk modulus, adiabatic index, baryon density, compressibility

1. Introduction

Nuclear astrophysics deals with different processes occurring in various stellar systems and it is one of the emerging branches of physics. A natural physical system called an astronuclear system involves both long and short-range interactions. There is ample evidence from various astronuclear phenomena indicating that ‘matter exists’ under rare situations. It is necessary to numerically solve the pressure-density equation, i.e. the EoS, to predict a significant stellar model and to carry out fruitful studies of the physical characteristics of interstellar objects, such as electrical and thermal conductivities, coefficient of rigidity modulus, enthalpy, entropy, and many more. Compact celestial objects provide exceptional cosmic laboratories for investigating materials of with extremely high densities. Solving the Tolman Oppenheimer-Volkoff (TOV) equations under suitable matching conditions will yield the compactness of such highly concentrated astronuclear bodies [1].

Existences of few remarkable conflicts are noted about the compressibility of nuclear stuff at the limit of saturation [2]. Existence of gigantic monopole resonance in nuclei at about 220 MeV [3, 4] is considered as the origin of these features, though Brown and Osnes claimed it to occur at some lower energy of the order of 100 MeV [5]. Nucleosynthesis in the primordial stage is significantly influenced by fusion process of low mass nuclei [6]. With increasing stellar hotness and condensation of nuclear constituents, the well-established Carbon-Nitrogen-Oxygen (CNO) cycle leads the synthesis process until iron (Fe⁵⁶) is produced. Because of the failure of the remnants of a stellar object to
produce enough heat pressure to withstand gravitational collapse, it contracts noticeably and reaches an incredibly high density. Due to fermionic degeneracy and interactions between fermions (such as those between nucleons, electrons, strange fermions, etc.) at this super dense state, non-thermal pressure is generated. Since the density of the matter in compact objects varies greatly, it ranges from a few gm per cubic centimeter (for Fe$^{56}$) at the outermost layer of a celestial body to nucleonic mass density ($\rho \sim 10^{15}$ gm per cubic centimeter) at the center. The prerequisite for internal structure exploration of any compact celestial body, simultaneous consideration of the strong short range nuclear interaction, nuclear weak interaction and long range electromagnetic interactions are all essential. Neutron stars (NS), one of the densest and smallest stars besides black holes, are among the most researched and fascinating compact astronuclear objects. They emerge from the collapse of a high mass star ($1.4\,M_\odot$ - $2M_\odot$), when it lacks sufficient consumable mass [7]. It stops producing heat after its birth, and over time it keeps getting colder. Owing to possibility of their accretion or mutual clash they may continue to evolve thereafter. As NSs are formed in astronuclear environments where electrons and protons interact to generate neutrons, the majority of available ideas about NS predict that they are mainly made of neutral nucleons. If the residual star's mass exceeds the TOV limit of around $2\,M_\odot$, the nuclear forces and degeneracy pressure due to Pauli’s blocking cannot support the NS against gravity, and it continues to collapse into a black hole [8, 9]. The estimated density of a NS fluctuates by around a fifth order of magnitude when one travels from its inner core to the crust, which may be relevant at the stage of the game. An EoS, which is unified in nature, is the prime key for collecting the structural data of the interstellar environment of NS. The current study has two dual-focused primary goals. Formulation of a set of suitable coupled nonlinear differential equations (CNDE) is the first step. It can smoothly be related to the structural attributes in the EoS. The second step is to build an efficient scheme based on the data in hand to explore the fundamental characteristics of a neutron star, such as its compressibility, super fluidity, both thermal and electric conductivity, etc., in the literature, which concentrates on the compactness of highly dense celestial objects, several significant studies have already been published. The pressure-density equation of nuclear medium in equilibrated conditions is another factor that is found in the literature study. Majority of them involve determination of extent of matter distribution in compact celestial bodies. In such cases, the radius is mainly driven by the degree of interstellar pressure. In the range of nuclear mass density, the pressure effectively depends on the nuclear symmetry energy. Generally, in the normal state, the sun ($M_\odot \sim 10^{30}$ kg, $R_\odot \sim 7 \times 10^8$ km) has a typical mass density of $1.41\,\text{gm per cubic centimeter}$, while its core stuff concentration is roughly $162.2 \,\text{gm per cubic centimeter}$ [10]. The density of the typical sun matter approaches the nuclear matter density ($\sim 5 \times 10^{14}$ gm per cubic centimeter) when it is compressed into a ball of 10 km radius, turning it into a super nuclear ball. While maintaining the same radius, if the mass of this ball is nearly doubled to that of the sun, the asymmetric energy of the nuclear substance begins to dominate, and the astro-nuclear object begins to lose its nuclear form, as a consequence of the trapping of negatively charged electrons by positively charged particles, protons to generate neutral neutrons. Thus, a neutron star is created as the substance within the star essentially converts into neutron-rich stuff. Keep in mind that dense material no longer follows the conventional mass density relation, which states that density = mass/volume. In this scenario, the matter density and the intervening pressure have a logarithmic relationship.

There are significant signs of advances in this domain, as evidenced by the detection of the giant stars of mass ~ $2M_\odot$ [11, 12, 13] and with the recent findings of NS merger events, GW170817 was discovered by LIGO and Virgo [14, 15]. The bulk properties of stellar objects, especially NS’s, are still poorly understood despite these numerous advancements in NS physics. Here, we limit our concentration to the bulk characteristics of the detected NS’s.

This study is highly inspired by the current investigation of Potekhin et al [16], where they have derived two EoS that correspond to FPS EoS [17] and SLy EoS [9] incorporating tabulated data. The theoretical background of the work is mentioned in section 2. Section 3 addresses the results and discussion, and concluding remarks will be placed in Section 4.
2. Theoretical formulation

The following data are frequently used to obtain an EoS in tabulated form: stellar pressure ($P$), baryon number density ($n_b$), and nucleonic matter density ($\rho$) [18]. For fully equilibrated nuclear stuff, at $T = 0 \text{K}$ the EoS is obtained by the following energy dependency relation,
$$\epsilon = \epsilon (n_b),$$
where $\epsilon$ represents the energy per unit baryon.

Matter density, $\rho$ of a celestial compact object is linked up to the energy density, $\epsilon$ by the relation
$$\rho = \epsilon c^2.$$ By incorporating the 1st law of thermodynamics, interstellar pressure, $P$ can be expressed at zero temperature (i.e., invoking the limits $T = 0$) in the following way,
$$P(n_b) = n_b^2 \frac{d}{dn_b} \epsilon(n_b) = n_b^2 c^2 \frac{d}{dn_b} \left( \frac{\rho}{n_b} \right)$$ (1)
The rigidity level of the EoS at a specific $\epsilon$ is measured by the fascinating dimensionless parameter adiabatic index ($\Gamma$), which is defined as
$$\Gamma = \frac{d(log P)}{d(log n_b)}$$ (2)
The stiffness of stellar matter has been investigated by Douchin and Haensel in terms of $\Gamma$[10]. Their findings showed a substantial rise in the stiffness index at the boundary of the crust-core interface of NS, where the values of $\Gamma$ sharply shift to 2.2 from 1.7 [19]. We estimate the acoustic wave velocity in the interstellar medium using Equations (1), (2), and (3) in the forms of the well-known Newtonian-Laplace formula,
$$v_s = \sqrt{\frac{\Gamma P}{\rho}}$$ (3)
In order to analyze the bulk coefficient ($B$) in terms of baryon number density, the aforementioned relations (Eqs. (1) and (2)) can also be used, and they can be written as
$$B(n_b) = n_p \frac{dP}{dn_b} = \left(1 + \frac{P}{\rho^2 c^2} \right) \frac{\rho}{P} \frac{d}{d\rho}$$ (4)
In a study published in 2013, Potekhin et al. gave numerical estimates for pressure ($P$) and gravitational mass density ($\rho$) in terms of baryon number density ($n_b$) [16]. The application of the authors' findings to the investigation of the fundamental aspects of cold, highly dense matter, like NS, was also shown. They studied the limiting masses, compactness, and electrical properties like conductivity, thermal properties like enthalpy (H), entropy(S) in the crust of NS by using various rigidity-dependent EoS like, Bsk19, Bsk20, and Bsk21. For this work, they use B-M EoS, which is based on 'energy-density functional theory (EDF)', [20, 21]. By incorporating Potekhin’s numerical recipe, we calculated the bulk coefficient ($B$) and baryon number density ($n_b$) following equation (4) for a diverse variety of nuclear stuff densities from the NS's sub-to supra nuclear regions.

3. Results and discussion

For a continuously rising matter density, we evaluated stellar pressure ($P$), baryon number density ($n_b$), bulk coefficients ($B$), and speed of sound ($v_s$), and reported the results in columns 2, 4, 6, and 7, respectively, of Table 1. In addition, columns 3, 5, and 8 indicate the relevant reference data for each of the above-mentioned quantities. Here, we have represented the bulk coefficient ($B$) and stellar pressure ($P$) in S. I (i.e., MN/ fm$^2$) unit. Figure 1 depicts the calculated findings of the bulk coefficient ($B$) for various nuclear matter densities ($\rho$) on a logarithmic scale. The reference data from Baym et al. [23, 25] are shown on this plot and exhibit a fine match with our calculated values. Only typical data points are shown in Table 1. A huge collection of numerical data generated for mass densities from
$10^6 gm/cm^3$ to $10^{16} gm/cm^3$, are used to produce the graph. From Figure 1, we discovered that the bulk coefficient ($B$) varies seamlessly with matter density ($\rho$) up to around $\rho \approx 10^{16} gm/cm^3$. There is a kink in the graph at about $\rho \approx 5 \times 10^{12} gm/cm^3$ which is also observed in recent investigations by Baym [25] and may indicate a probable phase transition in the nuclear matter.

![Graph](image1)

**Figure 1.** Logarithmic change of bulk coefficient, $B$ with matter density, $\rho$ for various EoS.

![Graph](image2)

**Figure 2.** Logarithmic change of baryon number density, $n_b$ with matter density, $\rho$ for various EoS.

![Graph](image3)

**Figure 3.** Logarithmic change of baryon number density, $n_b$ with bulk coefficient, $B$ for various EoS.

This might result from a fast transformation of first-generation quarks, which are composed of the hadron components, into strange quarks, which raises the relative concentration of strange particles.
The entire procedure makes neutron star substance more stable up to $2M_{\odot}$, and a new phase forms that warrant more research. The drip line domain for initially formed quark matter (QM) may thus be represented as a step-like graph with densities between $10^{12}$ $gm/cm^3$ to $10^{14}$ $gm/cm^3$, the graph's slope gets noticeably steeper, which is consistent with the composition of subsequent generation QM. The conventional EoS of Baym et al. [23] estimated the density-volume of neutron, around $\rho_{n-drip}(\sim 4 \times 10^{13}$ $gm/cm^3)$[23]. Mass density, $\rho \leq \rho_{n-drip}$ describes stuff in the outer surface of NS, whereas $\rho > \rho_{n-drip}$ characterizes the substances near the innermost region of the NS. Figure 2 describes the dependency of baryon number density, $n_b$ on matter density, $\rho$. As anticipated, $n_b$ rises practically smoothly as the mass density approaches saturation above $10^{16}$ $gm/cm^3$. Beyond the superabundance level, $n_b$ may drop as a result of baryon disintegration and the ensuing rise in light constituents like quarks and leptons. Whereas, the logarithmic variation of baryon number density ($n_b$) with the bulk coefficient (B) has been presented in figure 3. The plot suggests that the lower stiffness end requires a slightly higher population expansion rate of baryon number density than the bigger zone of rigidity.

| Matter density, $\rho$ $(gm/cm^3)$ | Baryon number density, $n_b$ (fm$^{-3}$) | Pressure, P (MN/m$^2$) | Bulk Coefficient, B (MN/m$^2$) | Sound speed, $V_s$ (in terms of c) |
|-----------------------------------|----------------------------------------|-------------------------|-------------------------------|---------------------------------|
| Comp. Val.                        | Ref. Val. $^{*5}$                      | Comp. Val.              | Ref. Val. $^{*5}$             | Comp. Val.                      | Ref. Val. $^{*5}$                |
| 1x10$^6$                          | 6x10$^{-10}$ 6.02x10$^{-10}$ 2.2x10$^{-15}$ 3.63x10$^{-15}$ 3.45x10$^{-15}$ 0.0056c 0.0063c |
| 3x10$^6$                          | 1.81x10$^{-9}$ 1.19x10$^{-14}$ 1.8x10$^{-14}$ 0.0076c |
| 9x10$^6$                          | 5.49x10$^{-8}$ 6.85x10$^{-14}$ 8.8x10$^{-14}$ 0.0106c |
| 1x10$^7$                          | 6.01x10$^{-9}$ 6.02x10$^{-9}$ 7.04x10$^{-14}$ 6.81x10$^{-14}$ 1.02x10$^{-13}$ 0.0102c 0.0107c |
| 5x10$^7$                          | 3x10$^{-8}$ 3.01x10$^{-8}$ 6.85x10$^{-13}$ 6.84x10$^{-13}$ 9.48x10$^{-13}$ 0.0143c 0.0146c |
| 1x10$^8$                          | 6.0x10$^{-8}$ 6.02x10$^{-8}$ 1.76x10$^{-12}$ 1.78x10$^{-12}$ 2.41x10$^{-12}$ 0.0163c 0.0169c |
| 6x10$^8$                          | 3.6x10$^{-6}$ 1.94x10$^{-11}$ 2.56x10$^{-11}$ 0.0218c |
| 1x10$^9$                          | 6.03x10$^{-7}$ 6.02x10$^{-7}$ 3.84x10$^{-11}$ 3.91x10$^{-11}$ 4.96x10$^{-11}$ 0.0235c 0.0225c |
| 5x10$^9$                          | 3.05x10$^{-6}$ 3.01x$^{-6}$ 3.06x10$^{-10}$ 3.03x10$^{-10}$ 3.94x10$^{-10}$ 0.0100c 0.0101c |
| 1x10$^{10}$                       | 6.01x10$^{-6}$ 6.01x$^{-6}$ 7.42x10$^{-10}$ 7.24x10$^{-10}$ 9.54x10$^{-10}$ 0.0333c 0.0337c |
| 5x10$^{10}$                       | 2.98x10$^{-5}$ 3.0x$^{-5}$ 5.7x10$^{-9}$ 5.63x10$^{-9}$ 7.28x10$^{-9}$ 0.0412c 0.0377c |
| 1x10$^{11}$                       | 6x10$^{-5}$ 5.99x10$^{-5}$ 1.37x10$^{-8}$ 1.4x10$^{-8}$ 1.68x10$^{-8}$ 0.114c 0.0447c |
| 1x10$^{12}$                       | 5.98x10$^{-4}$ 5.97x10$^{-4}$ 1.25x10$^{-7}$ 1.26x10$^{-7}$ 8.56x10$^{-8}$ 0.0431c 0.034c |
| 6x10$^{12}$                       | 3.56x10$^{-3}$ 8.17x10$^{-7}$ 1.07x10$^{-6}$ 0.045c |
| 1x10$^{13}$                       | 5.94x10$^{-3}$ 5.95x10$^{-3}$ 1.57x10$^{-6}$ 1.16x10$^{-6}$ 2.12x10$^{-6}$ 0.0486c 0.0444c |
| 1x10$^{14}$                       | 5.9x10$^{-2}$ 5.92x10$^{-2}$ 2.88x10$^{-5}$ 3.93x10$^{-5}$ 3.81x10$^{-5}$ 0.067c 0.096c |
| 6x10$^{14}$                       | 3.41x10$^{-1}$ 5.12x10$^{-3}$ 1.63x10$^{-2}$ 0.3567c |
| 1x10$^{15}$                       | 5.3x10$^{-1}$ 5.5x10$^{-1}$ 1.91x10$^{-2}$ 1.52x10$^{-2}$ 5.63x10$^{-2}$ 0.5367c 0.7667c |
| 1x10$^{16}$                       | 1.32x10$^{1}$ 9.89x10$^{-1}$ 2.71 |

$^{*}$Baym et al. [23], $^{*5}$Baym [25]

4. Conclusion

Instead of compressibility ($\chi$), which corresponds to the inverse of bulk coefficient, B (i.e. $\chi = 1/B$), incompressibility is more frequently utilized in nuclear physics applications. Therefore, the
analysis of $B$ provides a clearer insight into the features of interstellar stuff that frequently shows unusual behaviours. It should be observed that the calculated values of speed of sound ($v_s$, expressed in terms of speed of light in vacuum, $c$) shown in column 7 of Table 1 reflect haphazard variation, which may have snuck in because of volatility in the adiabatic index ($\Gamma$) used in the computation. The magnitude of $B$ in a neutron star is $10^{23}$ times greater than the same parameter's value for the solar inner matter. So it makes intuitive sense that star materials would have extremely different characteristics from those of terrestrial materials. For instance, it is predicted that an acoustic wave, such as sound, travels at a speed of a few percent of $'c$ (speed of light) in NS! [26] Last but not least, it should be noted that a thorough investigation into the general characteristics of super dense celestial matter may provide highly significant data, and that, with the right parameterization, it may even be possible to deduce the gravitational wave's impact.

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