On well-posedness of the Muskat problem with surface tension

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ABSTRACT. We consider the Muskat problem with surface tension for one fluid or two fluids, with or without viscosity jump, with infinite depth or Lipschitz rigid boundaries, and in arbitrary dimension $d$ of the interface. The problem is nonlocal, quasilinear, and to leading order, is scaling invariant in the Sobolev space $H^{s_c}(\mathbb{R}^d)$ with $s_c = 1 + \frac{d}{2}$. We prove local well-posedness for large data in all subcritical Sobolev spaces $H^s(\mathbb{R}^d)$, $s > s_c$, allowing for initial interfaces whose curvatures are unbounded and, furthermore when $d = 1$, not square integrable. To the best of our knowledge, this is the first large-data well-posedness result that covers all subcritical Sobolev spaces for the Muskat problem with surface tension. We reformulate the problem in terms of the Dirichlet-Neumann operator and use a paradifferential approach to reduce the problem to an explicit parabolic equation, which is of independent interest.

1. Introduction

1.1. The Muskat problem. The Muskat problem [47] of practical importance in geoscience describes the dynamics of two immiscible fluids in a porous medium with different densities $\rho^\pm$ and different viscosities $\mu^\pm$. Let us denote the interface between the two fluids by $\Sigma$ and assume that it is the graph of a time-dependent function $\eta(x, t)$

$$\Sigma_t = \{(x, \eta(t, x)) : x \in \mathbb{R}^d\},$$

where $d \geq 1$ is the horizontal dimension. The associated time-dependent fluid domains are then given by

$$\Omega^+_t = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : \eta(t, x) < y < b^+(x)\},$$
$$\Omega^-_t = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : b^-(x) < y < \eta(t, x)\},$$

where $b^\pm$ are the parametrizations of the rigid boundaries $\Gamma^\pm = \{(x, b^\pm(x)) : x \in \mathbb{R}^d\}$.

The incompressible fluid velocity $u^\pm$ in each region is governed by Darcy’s law

$$\mu^\pm u^\pm + \nabla_x y p^\pm = -\rho^\pm g e_{d+1}, \quad \text{div}_{x,y} u^\pm = 0 \quad \text{in} \ \Omega^\pm_t,$$

where $g$ is the acceleration due to gravity and $e_{d+1}$ is the $(d + 1)$th vector of the canonical basis of $\mathbb{R}^{d+1}$.

At the interface $\Sigma$, the normal velocity is continuous

$$u^+ \cdot n = u^- \cdot n \quad \text{on} \ \Sigma_t$$

where $n = \frac{1}{\sqrt{1 + |\nabla \eta|^2}}(-\nabla \eta, 1)$ is the upward pointing unit normal to $\Sigma_t$. Then, the interface moves with the fluid

$$\partial_t \eta = \sqrt{1 + |\nabla \eta|^2} u^- \cdot n|_{\Sigma_t}.$$  

According to the Young-Laplace equation, the pressure jump at the interface is proportional to the mean curvature $H(\eta)$

$$p^- - p^+ = sH(\eta) := -s \text{div} \left(\frac{\nabla_x \eta}{\sqrt{1 + |\nabla_x \eta|^2}}\right) \quad \text{on} \ \Sigma_t,$$

where $s \geq 0$ denotes the surface tension coefficient.
Finally, at the two rigid boundaries, the no-penetration boundary conditions are imposed
\[ u^\pm \cdot \nu^\pm = 0 \quad \text{on } \Gamma^\pm, \]
where \( \nu^\pm = \pm \frac{1}{\sqrt{1 + \nabla h^\pm}} (\nabla h^\pm, 1) \) denotes the outward pointing unit normal to \( \Gamma^\pm \). We will also consider the case that at least one of \( \Gamma^\pm \) is empty (infinite depth) in which case the velocity \( u \) vanishes at infinity.

We shall refer to the system (1.2)-(1.9) as the two-phase Muskat problem. When the top phase corresponds to vacuum, i.e. \( \mu^+ = \rho^+ = 0 \), the two-phase Muskat problem reduces to the one-phase Muskat problem and (1.8) becomes
\[ p^- = sH(\eta) \quad \text{on } \Sigma_t. \]

1.2. Reformulation and main results. Our reformulation for the Muskat problem involves the Dirichlet-Neumann operators \( G^\pm(\eta) \) associated to \( \Omega^\pm \). For a given function \( f \), letting \( \phi^\pm \) solve
\[
\begin{cases}
\Delta_{x,y} \phi^\pm = 0 & \text{in } \Omega^\pm, \\
\phi^\pm = f & \text{on } \Sigma, \\
\frac{\partial \phi^\pm}{\partial n} = 0 & \text{on } \Gamma^\pm,
\end{cases}
\]
we define
\[ G(\eta)^\pm f := \sqrt{1 + |\nabla \eta|^2} \frac{\partial \phi^\pm}{\partial n}. \]

**Proposition 1.1 (Reformulation).** (i) If \( (u, p, \eta) \) solve the one-phase Muskat problem then \( \eta : \mathbb{R}^d \to \mathbb{R} \) obeys the equation
\[ \partial_t \eta = -\frac{1}{\mu^-} G^-(\eta)(sH(\eta) + \rho^- g\eta). \] Conversely, if \( \eta \) is a solution of (1.13) then the one-phase Muskat problem has a solution which admits \( \eta \) as the free surface.

(ii) If \( (u^\pm, p^\pm, \eta) \) is a solution of the two-phase Muskat problem then
\[ \partial_t \eta = -\frac{1}{\mu^-} G^-(\eta)f^-, \]
where \( f^\pm := p^\pm|\Sigma + \rho^\pm g\eta \) satisfy
\[
\begin{cases}
f^- - f^+ = sH(\eta) + [\rho] g\eta, & [\rho] = \rho^- - \rho^+,
\frac{1}{\mu^-} G^+(\eta)f^+ - \frac{1}{\mu^-} G^-(\eta)f^- = 0.
\end{cases}
\]

Conversely, if \( \eta \) is a solution of (1.14) where \( f^\pm \) solve (1.15) then the two-phase Muskat problem has a solution which admits \( \eta \) as the free interface.

We postpone the proof of Proposition 1.1 to Appendix B. The above reformulation contains as a special case the reformulation obtained in [49] in the absence of surface tension, i.e. \( s = 0 \). In this work, we are interested in the case that \( s \) is a fixed positive constant. To leading order, since \( sH(\eta) + \rho^- g\eta \sim -s\Delta_x \eta \), equation (1.13) behaves like
\[ \partial_t \eta = -\frac{s}{\mu^-} G^-(\eta)\Delta_x \eta. \]

It can be easily checked that if \( \eta(t, x) \) solves (1.16) then so is
\[ \eta_\lambda(t, x) = \lambda^{-1}\eta(\lambda^3 t, \lambda x) \quad \forall \lambda > 0, \]
and thus the \( (L^2\text{-based}) \) Sobolev space \( H^{1+\frac{d}{2}}(\mathbb{R}^d) \) is scaling invariant. Interestingly, the Muskat problem without surface tension also admits \( H^{1+\frac{d}{2}}(\mathbb{R}^d) \) as the scaling invariant Sobolev space ([49]).
results state that the Muskat problem with surface tension is locally well-posed for large data in all subcritical Sobolev spaces $H^s(\mathbb{R}^d)$, $s > 1 + \frac{d}{2}$, either for one fluid or two fluids, with or without viscosity jump, with infinite depth or with Lipschitz rigid boundaries, and in arbitrary dimension. Here well-posedness is obtained in the sense of Hadamard: existence, uniqueness and Lipschitz dependence on initial data.

Introducing the spaces
\[
\dot{W}^{1,\infty}(\mathbb{R}^d) = \left\{ v \in L^1_{\text{loc}}(\mathbb{R}^d) : \nabla v \in L^\infty(\mathbb{R}^d) \right\},
\]
\[
Z^s(T) = C([0, T]; H^s(\mathbb{R}^d) \cap L^2([0, T]; H^s+\frac{3}{2}(\mathbb{R}^d))).
\]
we state our main results in the following theorems.

**Theorem 1.2 (Well-posedness for the one-phase problem).** Let $\mu^- > 0$, $\rho^- > 0$ and $s > 0$. Let $s > 1 + \frac{d}{2}$ be a real number with $d \geq 1$. Consider either $\Gamma^- = \emptyset$ or $\bar{\Gamma}^- \in \dot{W}^{1,\infty}(\mathbb{R}^d)$. Let $\eta_0 \in H^s(\mathbb{R}^d)$ satisfy
\[
\text{dist}(\eta_0, \Gamma^-) > 2h > 0.
\]

Then there exist $T > 0$, depending only on $\|\eta_0\|_{H^s}$ and $(h, s, \frac{\rho^- - \rho^+}{\mu^-}, \frac{s}{\mu^-})$, and a unique solution $\eta \in Z^s(T)$ to (1.13) such that $\eta|_{t=0} = \eta_0$ and
\[
\inf_{t \in [0, T]} \text{dist}(\eta(t), \Gamma^-) > h.
\]

Moreover, if $\eta_1$ and $\eta_2$ are two solutions of (1.13) then the stability estimate
\[
\|\eta_1 - \eta_2\|_{Z^s(T)} \leq F(\|\eta_1 - \eta_2\|_{Z^s(T)}), T)\|\eta_1 - \eta_2\|_{t=0}\|_{H^s}
\]
holds for some function $F : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ depending only on $(h, s, \frac{\rho^- - \rho^+}{\mu^-}, \frac{s}{\mu^-})$.

**Theorem 1.3 (Well-posedness for the two-phase problem).** Let $\mu^\pm > 0$, $\rho^\pm > 0$ and $s > 0$. Let $s > 1 + \frac{d}{2}$ be a real number with $d \geq 1$. Consider any combination of $\Gamma^\pm = \emptyset$ and $\bar{\Gamma}^\pm \in \dot{W}^{1,\infty}(\mathbb{R}^d)$. Let $\eta_0 \in H^s(\mathbb{R}^d)$ satisfy
\[
\text{dist}(\eta_0, \Gamma^\pm) > 2h > 0.
\]

Then there exist $T > 0$, depending only on $\|\eta_0\|_{H^s}$ and $(h, s, \frac{\rho^- - \rho^+}{\mu^-}, \frac{s}{\mu^-}, [\rho]g)$, and a unique solution $\eta \in Z^s(T)$ to (1.13)-(1.15) such that $\eta|_{t=0} = \eta_0$ and
\[
\inf_{t \in [0, T]} \text{dist}(\eta(t), \Gamma^\pm) > h.
\]

Moreover, if $\eta_1$ and $\eta_2$ are two solutions of (1.14)-(1.15) then the stability estimate
\[
\|\eta_1 - \eta_2\|_{Z^s(T)} \leq F(\|\eta_1 - \eta_2\|_{Z^s(T)}), T)\|\eta_1 - \eta_2\|_{t=0}\|_{H^s}
\]
holds for some function $F : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ depending only on $(h, s, \frac{\rho^- - \rho^+}{\mu^-}, \frac{s}{\mu^-}, [\rho]g)$.

To the best of our knowledge, Theorems 1.2 and 1.3 are the first large-data well-posedness results that cover all subcritical Sobolev spaces for the Muskat problem with surface tension. The corresponding results in the absence of surface tension were obtained in the recent work [49]; see Subsection 1.3 for a discussion on prior results. In particular, Theorems 1.2 and 1.3 allow for initial interfaces whose curvatures are unbounded for $d \geq 1$ and not square integrable for $d = 1$.

Using results on paralinearization of the Dirichlet-Neumann operator obtained in [2, 49] we shall reduce both the one-phase and two-phase Muskat problems with surface tension to the following explicit parabolic paradifferential equation
\[
\partial_t \eta + \frac{s}{\mu^+ + \mu^-} T_{\lambda, \eta} \eta = g
\]
where $g$ satisfies
\[
\|g\|_{H^{s-\frac{3}{2}+\delta}} \leq F(\|\eta\|_{H^s})\|\eta\|_{H^{s+\frac{3}{2}}}
\]
provided that \( s > 1 + \frac{d}{2} \) and \( \delta \in (0, s - 1 - \frac{d}{2}) \), \( \delta \leq \frac{1}{2} \). We refer to Propositions \[3.1\] and \[4.4\] for the precise statements and to Appendix \[A\] for notation of paradifferential operators. Here \( \lambda(x, \xi) \) and \( \ell(x, \xi) \), defined by \[2.11\] and \[3.2\], are respectively the principal symbol of the Dirichlet-Neumann operator \( G^-(\eta) \) and the mean curvature operator \( H(\eta) \); moreover they are elliptic and of first and second order respectively. Consequently, \( T* \) is an elliptic paradifferential operator of third order and thus the solution \( \eta \) to \[1.19\] gains \( \frac{3}{2} \) derivatives when measured in \( L^2_L \). The estimate \[1.20\] then shows that for any subcritical data \( \eta_0 \in H^s(\mathbb{R}^d) \), the right-hand side \( g \) is smoothing which in turn allows one to close the energy estimate in \( L_t^\infty H^s_x \cap L^2_H^{s+\frac{3}{2}} \). The stability estimate is more delicate, especially for the two-phase problem.

We believe that the reduction \[1.19\]-\[1.20\] is of independent interest. It is worth remarking that unlike the case of zero surface tension \[49\], this reduction does not involve the trace of velocity on the interface.

This work emphasizes the strength of the paradifferential calculus approach in establishing (almost) sharp large-date well-posedness for free boundary problems in fluid dynamics. In the context of water waves, this approach was initiated in \[1, 2, 3\] with inspiration from \[4, 41\]. In the context of Muskat, this approach was independently employed in \[5, 49\] for the case without surface tension. In this work, by taking advantage of the strong dissipation mechanism of the Muskat problem with surface tension, we obtain well-posedness results that allow for curvature singularity of initial data. Such a result for the water waves problem with surface tension remain open in view of the recent works \[1, 29, 30, 48\].

**Remark 1.4.** Theorems \[1.2\] and \[1.3\] still hold in the following situations:

- Gravity is neglected \( (g = 0) \), as usually assumed for the Hele-Shaw problem.
- Periodic data \( \eta_0 \in H^s(\mathbb{T}^d) \) for any \( s > 1 + \frac{d}{2} \).

Since \( b^\pm \in \dot{W}^{1, \infty}(\mathbb{R}^d) \), the rigid boundaries \( \Gamma^\pm \) can be unbounded.

**Remark 1.5.** It is well known that the smoothing effect of surface tension bypasses the Rayleigh-Taylor stability condition required for well-posedness of free boundary problems in the absence of surface tension. In particular, Theorem \[1.3\] does not require that the less dense fluid is above the denser one, i.e., \( \rho^+ < \rho^- \).

We refer to \[15, 17, 41, 49, 52, 53\] for in-depth discussions on the Rayleigh-Taylor stability condition for Muskat and water waves.

**Remark 1.6.** For simplicity let us consider the infinite depth case, i.e., \( \Gamma = \emptyset \). As a consequence of the fact that the existence time \( T \) in Theorems \[1.2\] and \[1.3\] depends only on the \( H^s(\mathbb{R}^d) \) norm of initial data, if \( T^* \) is the maximal time of existence then

\[
\lim_{t \to T^*} \| \eta(t) \|_{H^s(\mathbb{R}^d)} = \infty. \tag{1.21}
\]

Since \( s > 1 + \frac{d}{2} \) we have \( H^s(\mathbb{R}^d) \subset W^{1+\varepsilon, \infty}(\mathbb{R}^d) \) if \( \varepsilon > 0 \) is sufficiently small. It is possible that by combing the techniques in the present paper with mixed Hölder-Sobolev estimates for the Dirichlet-Neumann operator in the spirit of \[3, 29\], one can prove that

\[
\lim_{t \to T^*} \| \eta(t) \|_{W^{1+\varepsilon, \infty}(\mathbb{R}^d)} = \infty \quad \forall \varepsilon > 0. \tag{1.22}
\]

It is an open problem for the Muskat problem (with or without surface tension) whether the maximal slope blows up at \( T^* \), i.e.

\[
\lim_{t \to T^*} \| \nabla_x \eta(t) \|_{L^\infty(\mathbb{R}^d)} = \infty. \tag{1.23}
\]

Any continuation criterion in terms of scaling invariant quantities should be interesting. For the 2D Muskat problem without surface tension and constant viscosity, it is known from \[20\] that the solution remains regular so long as the slope \( \partial_x \eta \) remains bounded and uniformly continuous.

**Remark 1.7.** The time interval \([0, T]\) in Theorems \[1.2\] and \[1.3\] shrinks to 0 as the surface tension coefficient \( s \) vanishes. The question of zero surface tension limit is interesting but will not be pursued in the present paper. We refer to \[9\] for a relating result.
1.3. Priori results. The Muskat problem and its mathematical analog – the Hele-Shaw problem have recently been the subject of intense study in analysis of PDEs and numerical analysis. The literature is vast and we will mostly discuss the topic of well-posedness. We refer to the recent surveys [33, 36] for discussions on other topics, and in particular [14, 15, 34] for interesting results on finite-time singularity formation.

Taking advantage of the parabolic nature of the Muskat problem, global strong solutions for small data have been considered in a large number of studies. We refer to [13, 16, 17, 18, 19, 20, 21, 24, 26, 50] for data in subcritical $L^2$-based and $L^\infty$-based Sobolev spaces, and to [35] for data in the critical Wiener space $F^{1,1}$. We note in particular that [17, 35] allow for viscosity jump and [26] allows for interfaces with large slopes. In the case of constant viscosity, by using maximum principles for the slope, global weak solutions were constructed in [19, 27].

We discuss in detail the issue of local well-posedness for large data. In the context of the Muskat problem, the case without surface tension is better understood. Early results on local well-posedness for large data in Sobolev spaces date back to [16, 31, 54, 7, 8]. Córdoba and Gancedo [24] introduced the contour dynamics formulation for the Muskat problem without viscosity jump and with infinite depth, and proved local well-posedness in $H^3(\mathbb{R})$ and $H^4(\mathbb{R}^2)$ when the interface is a graph. In [22, 23], Córdoba, Córdoba and Gancedo extended this result to the case of viscosity jump and non-graph interfaces satisfying the arc-chord and the Rayleigh-Taylor conditions. One of the main difficulties is to invert a highly nonlocal equation to express the vorticity amplitude in terms of the interface. Using an Arbitrary Lagrangian-Eulerian approach, Cheng, Granero and Shkoller [17] (see also [37]) proved local well-posedness for the one-phase problem with flat bottoms when the initial surface $\eta \in H^2(\mathbb{T})$, allowing for unbounded curvatures. This result was then extended by Matioc [44] to the case of viscosity jump (but infinite depth). For the case of constant viscosity, using nonlinear lower bounds, a technique developed for critical SQG, the authors in [20] obtained local well-posedness for $\eta \in W^{2,p}(\mathbb{R})$ for all $p \in (1, \infty)$. The space $W^{2,1}(\mathbb{R})$ is scaling invariant yet requires $\frac{1}{2}$ more derivative compared to $H^{\frac{3}{2}}(\mathbb{R})$. Matioc [43] sharpened the local well-posedness theory to $\eta \in H^{\frac{3}{2}+\varepsilon}(\mathbb{R})$ for the case of constant viscosity and infinite depth. This is the first result that covers all subcritical ($L^2$-based) Sobolev spaces for the given one-dimensional setting. By paralinearizing the nonlinearity in the contour dynamics formulation, Alazard and Lazar [5] obtained a simpler proof and extended the result in [43] to $\eta_0 \in H^{\frac{3}{2}+\varepsilon}(\mathbb{R})$. In the recent joint work [49] of the author, we reformulated the Muskat problem in terms of the Dirichlet-Neumann operator for the general setting: one fluid or two fluids, with or without viscosity jump, with or without rigid boundaries and in arbitrary dimension. Then employing a paradifferential calculus approach we proved local well-posedness for large data in all subcritical Sobolev spaces. Independently and at the same time, the work [6] obtained a similar result for the case of one fluid and without bottom.

Next we discuss results on large-data well-posedness for the Muskat and Hele-Shaw problems with surface tension, which is the problem considered in the present paper. Early results for the 2D case date back to Duchon and Robert [28], Chen [16] and Escher-Simonett [31] where the initial interface is smooth enough so that its curvature is at least bounded. In [9], the zero surface tension limit is established for the 2D Muskat problem with smooth ($H^0$) Sobolev data. The issue of low regularity well-posedness has been recently addressed for constant viscosity and viscosity jump respectively in [43] and [44] in which the initial one-dimensional interface is taken in $H^s(\mathbb{R})$ with $s \in (2, 3)$. These results are ($\frac{1}{2} + \varepsilon$)-derivative above scaling, i.e. $H^{2+\varepsilon}(\mathbb{R})$ versus $H^{\frac{3}{2}}(\mathbb{R})$, yet allows for unbounded curvatures. The same result for the periodic case was obtained in [45]. Our Theorems 1.2 and 1.3 appear to be the first large-data well-posedness results that cover all subcritical Sobolev spaces for the Muskat problem with surface tension in a general setting.

The paper is organized as follows. In Section 2 we recall results on the continuity, paralinearization and contraction estimates for the Dirichlet-Neumann operator, most of which are taken from [2] and [49].
Sections 5 and 4 are devoted to the proofs of Theorems 1.2 and 1.3. Appendix A provides a review of the paradifferential calculus machinery. Finally, we prove Proposition 1.1 in Appendix B.

**Notation 1.8.** Throughout this paper we use $\mathcal{F}$ to denote a continuous increasing positive nonlinear function which may change from line to line but its dependency on relevant parameters will be indicated.

## 2. Results on the Dirichlet-Neumann operator

We consider the Dirichlet-Neumann problem associated to the fluid domain $\Omega^-$ defined by (1.3) with the time variable being frozen. Regarding the bottom $\Gamma^-$, we assume either $\Gamma^- = \emptyset$ or $\Gamma^- = \{(x, \hat{b}^-(x)) : x \in \mathbb{R}^d\}$, where $\hat{b}^- \in \dot{W}^{1,\infty}(\mathbb{R})$ satisfying $\text{dist}(\Sigma, \Gamma^-) > h > 0$. Consider the elliptic problem

$$
\begin{aligned}
\begin{cases}
\Delta_{x,y} \phi = 0 & \text{in } \Omega^-,

\phi = f & \text{on } \Sigma,

\frac{\partial \phi}{\partial n} = 0 & \text{on } \Gamma^-,
\end{cases}
\end{aligned}
$$

where in the case of infinite depth ($\Gamma^- = \emptyset$), the Neumann condition is replaced by the decay condition

$$
\lim_{y \to -\infty} \nabla_{x,y} \phi = 0.
$$

The Dirichlet-Neumann operator associated to $\Omega^-$ is formally defined by

$$
G(\eta)^- f = \sqrt{1 + |\nabla \eta|^2} \frac{\partial \phi}{\partial n},
$$

where we recall that $n$ is the upward-pointing unit normal to $\Sigma$. Similarly, if $\phi$ solves the elliptic problem (2.1) with $(\Omega^-, \Gamma^-, \nu^-)$ replaced by $(\Omega^+, \Gamma^+, \nu^+)$ then we define

$$
G(\eta)^+ f = \sqrt{1 + |\nabla \eta|^2} \frac{\partial \phi}{\partial n}.
$$

Note that $n$ is inward-pointing for $\Omega^+$, making $G^+(\eta)$ a skew-adjoint operator, whereas $G^-(\eta)$ is self-adjoint. In the rest of this section, we only state results for $G^-(\eta)$ since corresponding results for $G^+(\eta)$ are completely parallel.

The Dirichlet data $f$ for (2.1) will be taken in the following screened fractional Sobolev space (see [42])

$$
\dot{H}^\frac{1}{2}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \cap L^2_{\text{loc}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \int_{B_{\text{ad}}(0,\Upsilon(x))} \frac{|f(x + x') - f(x)|^2}{|x'|^{d+1}} dx' dx < \infty \right\} / \mathbb{R},
$$

where $\Upsilon : \mathbb{R}^d \to (0, \infty]$ is a given lower semi-continuous function. For the bottom domain $\Omega^-$, we will choose

$$
\Upsilon(x) = \begin{cases}
\infty & \text{when } \Gamma^- = \emptyset,

\sigma_-(x) := \frac{\eta(x) - \hat{b}^-(x)}{2(\|\nabla_x \eta\|_{L^\infty} + \|\nabla_x \hat{b}^-\|_{L^\infty})} & \text{when } \hat{b}^- \in \dot{W}^{1,\infty}(\mathbb{R}^d).
\end{cases}
$$

Since $\text{dist}(\Sigma, \Gamma^-) > h$, we have

$$
\sigma_-(x) \geq \frac{h}{2(\|\nabla_x \eta\|_{L^\infty} + \|\nabla_x \hat{b}^-\|_{L^\infty})}.
$$

We also define the slightly-homogeneous Sobolev spaces

$$
H^{1,\sigma}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \cap L^1_{\text{loc}}(\mathbb{R}^d) : \nabla f \in H^{\sigma-1}(\mathbb{R}^d) \right\} / \mathbb{R},
$$

The continuous embeddings

$$
\dot{H}^\frac{1}{2}(\mathbb{R}^d) \subset \dot{H}^\frac{1}{2}_{\text{loc}}(\mathbb{R}^d) \subset \dot{H}^\frac{1}{2}_{\sigma^-}(\mathbb{R}^d) \subset \dot{H}^\frac{1}{2}(\mathbb{R}^d) = \dot{H}^\frac{1}{2}(\mathbb{R}^d)
$$

(2.7)
hold (see [49]). Here the embedding $\tilde{H}^\frac{1}{2}_\delta(R^d) \subset \tilde{H}^\frac{1}{2}_1(R^d)$ is due to the lower bound (2.5). In addition, if $b^- \in W^{1,\infty}(R^d)$ then $\tilde{H}^1_\delta(R^d) = \tilde{H}^1_1(R^d)$.

**Notation 2.1.** We denote

$$\tilde{H}^\frac{1}{2}_\pm(R^d) = \begin{cases} \tilde{H}^\frac{1}{2}_\pm(R^d) & \text{if } \Gamma^\pm = \emptyset, \\ \tilde{H}^1_\delta(R^d) & \text{if } b^\pm \in W^{1,\infty}(R^d), \end{cases}$$

(2.8)

where $\partial_+$ is defined similarly to $\partial_-$ with $b^-$ replaced by $b^+$. For $s > \frac{1}{2}$, we set

$$\tilde{H}^s_\pm(R^d) = \tilde{H}^\frac{1}{2}_\pm(R^d) \cap H^{1,s}(R^d).$$

(2.9)

The Sobolev spaces $\tilde{H}^s_\pm$ are homogeneous and tailored to the boundaries $\Gamma^\pm$. This is crucial for the two-phase Muskat problem since the traces $f^\pm$ obtained by solving (1.15) are only determined up to additive constants. Employing the trace theories developed in [42, 51] for homogeneous Sobolev spaces, it was proved in [49] that for each $f \in \tilde{H}^\frac{1}{2}_\pm$, there exists a unique variational solution $\phi$ to (2.1). This in turn implies that $G^-(\eta)f \in H^{-\frac{1}{2}}_\pm$ provided that $\eta \in W^{1,\infty}$. For continuity estimates in higher Sobolev norms, we shall appeal to the following theorem.

**Theorem 2.2 ([2, 49]).** Let $d \geq 1$, $s > 1 + \frac{d}{2}$ and $\frac{1}{2} \leq \sigma \leq s$. Consider $f \in \tilde{H}^\sigma_\pm(R^d)$ and $\eta \in H^s(R^d)$ with $\text{dist}(\eta, \Gamma^-) \geq h > 0$. Then we have $G^-(\eta)f \in H^s(R^d)$ and

$$\|G^-(\eta)f\|_{H^s} \leq \mathcal{F}(\|\eta\|_{H^s}) \|f\|_{\tilde{H}^\sigma_\pm}$$

(2.10)

for some $\mathcal{F}: \mathbb{R}^+ \to \mathbb{R}^+$ depending only on $(s, \sigma, h)$.

It is well known that for smooth domains, the Dirichlet-Neumann operator is a first-order pseudo-differential operator whose principal symbol is given by

$$\lambda(x, \xi) = \left(1 + |\nabla \eta(x)|^2\right)|\xi|^2 - (\nabla \eta(x) \cdot \xi)^2 \right)^{\frac{1}{2}}.$$  

(2.11)

The one-dimensional case is special since $\lambda(x, \xi) = |\xi|$ is $x$-independent. The following result provides error estimates when paralinearizing $G^-(\eta)$ by $T_\lambda$, which will be the key tool for paralinearizing the Muskat problem with surface tension.

**Theorem 2.3 ([2, 49]).** Let $d \geq 1$, $s > 1 + \frac{d}{2}$ and $\sigma \in [\frac{1}{2}, s - \frac{1}{2}]$. Fix a real number $\delta \in (0, s - 1 - \frac{d}{2})$, $\delta \leq \frac{1}{2}$. If $f \in \tilde{H}^\sigma_\pm(R^d)$ and $\eta \in H^s(R^d)$ with $\text{dist}(\eta, \Gamma^-) > h > 0$ then

$$G^-(\eta)f = T_\lambda f + R^-(\eta)f,$$

(2.12)

$$\|R^-(\eta)f\|_{H^{s-\delta}} \leq \mathcal{F}(\|\eta\|_{H^s}) \|f\|_{\tilde{H}^\sigma_\pm}$$

(2.13)

for some $\mathcal{F}: \mathbb{R}^+ \to \mathbb{R}^+$ depending only on $(s, \sigma, \delta, h)$.

Theorems 2.2 and 2.3 were first obtained in [2] (see Theorem 3.12 and Proposition 3.13 therein) when $f \in H^\sigma$, and extended to $f \in \tilde{H}^\sigma_\pm$ as a special case of Theorem 3.15 in [49]. It surprisingly turns out that the case with surface tension requires a less precise paralinearization compared to the one needed in [49] for the case without surface tension. This is in contrast with the water waves problem [11, 2].

Finally, we will need contraction estimates for the Dirichlet-Neumann operator in order to obtain uniqueness and stability of solutions.
Theorem 2.4 ([49] Proposition 3.28)). Let $s > 1 + \frac{d}{2}$ with $d \geq 1$. Consider $f \in \dot{H}^{s-\frac{1}{2}}(\mathbb{R}^d)$ and $\eta_1$, $\eta_2 \in H^s(\mathbb{R}^d)$ with $\text{dist}(\eta_j, \Gamma^-) \geq h$ for $j = 1, 2$. Then there exists $F : \mathbb{R}^+ \to \mathbb{R}^+$ depending only on $(s, h)$ such that
\[
\|G^{-}(\eta_1)f - G^{-}(\eta_2)f\|_{H^{s-\frac{1}{2}}} \leq F\left(\|\eta_1 - \eta_2\|_{H^s}\right)\|\eta_1 - \eta_2\|_{H^s}f\|_{\dot{H}^{s-\frac{1}{2}}}.
\]  

(2.14)

3. Proof of Theorem 1.2

3.1. Paradifferential reduction. We assume that $\eta \in Z^s(T)$ with $s > 1 + \frac{d}{2}$ is a solution of (1.13) and satisfies
\[
\inf_{t \in [0, T]} \text{dist}(\eta(t), \Gamma^-) > h > 0.
\]  

(3.1)

The next proposition shows that equation (1.13) can be reduced to an explicit third-order parabolic equation with a smoothing right-hand side.

Proposition 3.1. Set
\[
\ell(x, \xi) = (1 + |\nabla \eta|^2)^{-\frac{1}{2}} \left( |\xi|^2 - \frac{(\nabla \eta \cdot \xi)^2}{(1 + |\nabla \eta|^2)} \right).
\]  

(3.2)

For $\delta \in (0, s - \frac{d}{2})$ and $\delta \leq \frac{1}{2}$, there exists $F : \mathbb{R}^+ \to \mathbb{R}^+$ depending only on $(h, s, \delta)$ such that
\[
\|g\|_{H^{s-\frac{3}{2}+\delta}} \leq F(\|\eta\|_{H^s})(\frac{\mu^-}{\mu^+} - \frac{\rho^-}{\rho^+} G^{-}(\eta)\eta).
\]  

(3.4)

Proof. Let us rewrite (1.13) as
\[
\partial_t \eta = -\frac{\mu^-}{\mu^+} G^{-}(\eta)H(\eta) + g.
\]  

(3.5)

Theorem 2.2 applied with $\sigma = s - \frac{1}{2} + \delta$ gives
\[
\|G^{-}(\eta)\eta\|_{H^{s-\frac{3}{2}+\delta}} \leq F(\|\eta\|_{H^s})\|\eta\|_{H^{s-\frac{1}{2}+\delta}}.
\]  

(3.6)

Regarding $G^{-}(\eta)H(\eta)$, we apply Theorem 2.2 with $\sigma = s - \frac{1}{2}$ and Theorem A.11 with $s = s + \frac{1}{2}$ to have
\[
G^{-}(\eta)H(\eta) = T_\lambda H(\eta) + R^{-}(\eta)H(\eta)
\]  

with
\[
\|R^{-}(\eta)H(\eta)\|_{H^{s-\frac{1}{2}+\delta}} \leq F(\|\eta\|_{H^s})\|H(\eta)\|_{H^{s-\frac{1}{2}} + \frac{1}{2}} \lesssim F(\|\eta\|_{H^s})\|\eta\|_{H^{s+\frac{1}{2}}}.
\]  

The rest of the proof is devoted to control the main term $T_\lambda H(\eta)$. We paralinearize the mean-curvature operator $H(\eta)$ by means of Theorem A.8 with $\mu = s + \frac{1}{2}$, $\tau = \delta$:
\[
\nabla \eta \frac{1}{\sqrt{1 + |\nabla \eta|^2}} = T_M \nabla \eta + f_1, \quad M = \frac{1}{(1 + |\nabla \eta|^2)^{\frac{1}{2}}} \text{Id} - \frac{\nabla \eta \otimes \nabla \eta}{(1 + |\nabla \eta|^2)^{\frac{3}{2}}}
\]  

(3.7)

where $f_1$ satisfies
\[
\|f_1\|_{H^{s+\frac{1}{2}+\delta}} \leq F(\|\eta\|_{H^s})\|\nabla \eta\|_{H^{s+\frac{1}{2}}} \|\nabla \eta\|_{C^2} \lesssim F(\|\eta\|_{H^s})\|\eta\|_{H^{s+\frac{1}{2}}}.
\]  

Consequently,
\[
H(\eta) = -\text{div} \left( \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) = T_M \xi \cdot \eta + T_{-i(\text{div} M)} \xi \cdot \eta - \text{div} f_1,
\]  

where we note that $M \xi \cdot \xi = \ell$. To estimate $T_{-i(\text{div} M)} \cdot \nabla \eta$ we use (A.8) and the fact that
\[
\|M - \text{Id}\|_{H^{s-1}} \leq F(\|\eta\|_{H^s}),
\]  

(3.8)
yielding
\[ \| T_{\text{div}} M \cdot \nabla \eta \|_{H^{s-\frac{1}{2}+\delta}} \leq \| \text{div} M \|_{H^{s-2}} \| \nabla \eta \|_{H^{s+\frac{1}{2}}} \leq \mathcal{F}(\| \eta \|_{H^{s}}) \| \eta \|_{H^{s+\frac{1}{2}}} . \]

We thus obtain
\[ \| H(\eta) - T_{\lambda} \eta \|_{H^{s-\frac{1}{2}+\delta}} \leq \mathcal{F}(\| \eta \|_{H^{s}}) \| \eta \|_{H^{s+\frac{1}{2}}} . \] \quad (3.9)

Since \( M_1^J(\lambda) + M_2^J(\ell) \leq \mathcal{F}(\| \eta \|_{H^{s}}) \), Theorem A.4 (ii) yields that \( T_{\lambda}T_{\delta} - T_{\lambda} \) is of order \( 3 - \delta \) and that
\[ \| (T_{\lambda}T_{\delta} - T_{\lambda}) \eta \|_{H^{s-\frac{1}{2}+\delta}} \leq \mathcal{F}(\| \eta \|_{H^{s}}) \| \eta \|_{H^{s+\frac{1}{2}}} . \]

Putting together the above considerations we arrive at
\[ \| G^- (\eta) H(\eta) - T_{\lambda} \eta \|_{H^{s-\frac{1}{2}+\delta}} \leq \mathcal{F}(\| \eta \|_{H^{s}}) \| \eta \|_{H^{s+\frac{1}{2}}} \] \quad (3.10)

which combined with (3.6) and (3.5) concludes the proof.

\[ \square \]

**Remark 3.2.** In view of (2.11) and (3.2) we have
\[ \lambda \ell = (1 + |\nabla \eta|^2)^{\frac{3}{2}} \geq \frac{|\xi|^3}{(1 + |\nabla \eta|_{L^\infty})^2} \] \quad (11.11)

which shows that \( \lambda \ell \) is elliptic so long as \( \eta \in \tilde{W}^{1,\infty} \).

**3.2. A priori estimates.** Using the reduction in Proposition 3.1 and symbolic calculus for paradifferential operators, we derive a closed a priori estimate for \( \eta \) in \( Z^s(T) \):

**Proposition 3.3.** Let \( s > 1 + \frac{d}{2} \). Assume that \( \eta \in Z^s(T) \) is a solution of (1.13) such that (3.1) is satisfied. There exists \( \mathcal{F} : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) depending only on \((h, s, \frac{\mu \delta - \frac{6}{\mu}}{\mu})\) such that
\[ \| \eta \|_{Z^s(T)} \leq \mathcal{F}(\| \eta(0) \|_{H^s} + T \mathcal{F}(\| \eta \|_{Z^s(T) \}) ) . \] \quad (3.12)

**Proof.** Denote \( \langle D_x \rangle = (\text{Id} - \Delta_x)^{\frac{3}{2}} \) and \( \eta_s = \langle D_x \rangle \eta \). Commuting equation (3.3) with \( \langle D_x \rangle \eta \) we obtain
\[ \partial_t \eta_s = -\frac{\delta}{\mu - T_{\lambda} \eta_s - \frac{6}{\mu - T_{\lambda} \eta_s} (\langle D_x \rangle \eta, T_{\lambda} \eta) L^2 \times L^2 } \]
which yields
\[ \frac{1}{2} \frac{d}{dt} \| \eta_s \|_{L^2}^2 = -\frac{\delta}{\mu - T_{\lambda} \eta_s} \| \eta_s \|_{L^2 \times L^2}^2 - \frac{6}{\mu - T_{\lambda} \eta_s} (\langle D_x \rangle \eta, T_{\lambda} \eta) L^2 \times L^2 + (\langle D_x \rangle \eta, \eta_s) L^2 \times L^2 \] \quad (3.13)

In view of (3.4),
\[ \| (\langle D_x \rangle \eta, \eta_s) L^2 \times L^2 \|_{L^2 \times L^2} \leq \| (\langle D_x \rangle \eta, \eta_s) \|_{H^{s-\frac{3}{2}+\delta}} \| \eta_s \|_{H^{s-\frac{3}{2}+\delta}} \]
\[ \leq (\frac{\delta}{\mu - T_{\lambda} \eta_s} + \frac{\rho}{\mu^2}) \mathcal{F}(\| \eta \|_{H^s}) \| \eta \|_{H^{s+\frac{1}{2}}} \| \eta \|_{H^{s+\frac{1}{2}+\delta}} \] \quad (3.14)

In light of Theorem A.4 (ii), \( \| \langle D_x \rangle \|_{L^2 \times L^2} \) is of order \( s + 3 - \delta \) and that
\[ \| (\langle D_x \rangle \eta, T_{\lambda} \eta) \|_{H^{s-\frac{3}{2}+\delta}} \leq \mathcal{F}(\| \eta \|_{H^s}) \| \eta \|_{H^{s+\frac{1}{2}}} \],
whence
\[ \| (\langle D_x \rangle \eta, T_{\lambda} \eta, \eta_s) L^2 \times L^2 \|_{L^2 \times L^2} \leq \mathcal{F}(\| \eta \|_{H^s}) \| \eta \|_{H^{s+\frac{1}{2}}} \| \eta \|_{H^{s+\frac{1}{2}+\delta}} \] \quad (3.15)

Next we write
\[ (T_{\lambda} \eta_s) L^2 \times L^2 = (T_{\sqrt{\lambda} \eta_s}) L^2 \times L^2 + (T_{\sqrt{\lambda} \eta_s} - T_{\sqrt{\lambda} \eta_s} \eta_s) L^2 \times L^2 \]
\[ + ((T_{\lambda} - T_{\sqrt{\lambda} \eta_s} \eta_s) L^2 \times L^2 \]
\[ = I + II + III . \]
Applying Theorem A.4 (ii) and (iii) we find that \( T_{\mathcal{M}} - T_{\sqrt{\mathcal{M}}} T_{\sqrt{\mathcal{M}}} \) and \( (T_{\sqrt{\mathcal{M}}})^* - T_{\sqrt{\mathcal{M}}} \) are respectively of order \( 3 - \delta \) and \( \frac{3}{2} - \delta \) and that
\[
\| (T_{\mathcal{M}} - T_{\sqrt{\mathcal{M}}} T_{\sqrt{\mathcal{M}}}) \eta_s \|_{H^{-\frac{3}{2}} + \delta} + \| (T_{\sqrt{\mathcal{M}}})^* - T_{\sqrt{\mathcal{M}}} \) \( \eta_s \|_{H^3} \leq \mathcal{F}(\|\eta\|_{H^3}) \|\eta\|_{H^{3+\frac{3}{2}}}.
\]
Consequently,
\[
|II| + |III| \leq \mathcal{F}(\|\eta\|_{H^3}) \|\eta\|_{H^{3+\frac{3}{2}}} \|\eta\|_{H^{3+\frac{3}{2}}}.
\]
(3.17)

As for \( I \) we first note that the lower bound (3.11) implies \( M_{\delta}^{\frac{3}{2}} \leq \mathcal{F}(\|\eta\|_{H^3}) \). Theorem A.4 (ii) then gives that \( T_{\sqrt{\mathcal{M}}}, T_{\sqrt{\mathcal{M}}} - \text{Id} \) is of order \( -\delta \) and that
\[
\|\eta_s\|_{H^{3\frac{1}{2}}} \leq \| T_{\sqrt{\mathcal{M}}} T_{\sqrt{\mathcal{M}}} \eta_s \|_{H^{3\frac{1}{2}}} + \| \text{Id} - T_{\sqrt{\mathcal{M}}^{-1}} T_{\sqrt{\mathcal{M}}} \|_{H^{3\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^3}) (\|\eta\|_{L^2} + \|\eta_s\|_{H^{3\frac{1}{2}}}).
\]
In other words,
\[
I = \| T_{\sqrt{\mathcal{M}}} \eta_s \|_{L^2}^2 \geq \frac{1}{\mathcal{F}(\|\eta\|_{H^3})} \|\eta_s\|_{H^{3\frac{1}{2}}}^2 - \mathcal{F}(\|\eta\|_{H^3}) \|\eta\|_{H^{3\frac{1}{2}}} \|\eta\|_{H^{3\frac{1}{2}}} \|\eta\|_{H^{3\frac{1}{2}}}.
\]
(3.18)

Combining (3.16), (3.17) and (3.18) leads to
\[
- (T_{\mathcal{M}} \eta_s, \eta_s)_{L^2 \times L^2} \leq - \frac{1}{\mathcal{F}(\|\eta\|_{H^3})} \|\eta_s\|_{H^{3\frac{1}{2}}}^2 + \mathcal{F}(\|\eta\|_{H^3}) \|\eta\|_{H^{3\frac{1}{2}}} \|\eta\|_{H^{3\frac{1}{2}}}
\]
(3.19)

for some \( \mathcal{F} \) depending only on \((h, s)\). From this, (3.13), (3.14) and (3.15) we arrive at
\[
\frac{1}{2} \frac{d}{dt} \|\eta\|_{H^3}^2 \leq - \frac{9}{\mu} \mathcal{F}(\|\eta\|_{H^3}) \|\eta\|_{H^{3\frac{1}{2}}}^2 + \frac{\eta}{\mu} \|\eta\|_{H^{3\frac{1}{2}}}^2 + \mathcal{F}(\|\eta\|_{H^3}) \|\eta\|_{H^{3\frac{1}{2}}} \|\eta\|_{H^{3\frac{1}{2}}}
\]
where \( \mathcal{F} \) depends only on \((h, s)\). The gain of \( \delta \) derivative in the second term allows one to interpolate
\[
\|\eta\|_{H^{3\frac{1}{2}}} \|\eta\|_{H^{3\frac{1}{2}}}^2 \leq \|\eta\|_{H^{3\frac{1}{2}}} \|\eta\|_{H^{3\frac{1}{2}}} \|\eta\|_{H^{3\frac{1}{2}}} \leq \|\eta\|_{H^{3\frac{1}{2}}} \|\eta\|_{H^{3\frac{1}{2}}} \|\eta\|_{H^{3\frac{1}{2}}}, \quad \theta \in (0, 1),
\]

leading to
\[
\frac{d}{dt} \|\eta\|_{H^3}^2 \leq - \frac{1}{\mathcal{F}(\|\eta\|_{H^3})} \|\eta\|_{H^{3\frac{1}{2}}}^2 + \mathcal{F}(\|\eta\|_{H^3}) \|\eta\|_{H^{3\frac{1}{2}}}^2.
\]
Finally, a Grönwall argument finishes the proof.

As the function \( \mathcal{F} \) in (3.12) depends on the distance between the surface and the bottom, we need an a priori estimate for this quantity.

**Lemma 3.4.** Under the assumptions of Proposition 3.2 there exist \( \theta \in (0, 1) \) and \( \mathcal{F} : \mathbb{R}^+ \to \mathbb{R}^+ \) depending only on \((h, s, \frac{3\cdot \theta}{\mu}, \frac{s}{\mu}, \frac{3\cdot \theta}{\mu})\) such that
\[
\inf_{t \in [0, T]} \text{dist}(\eta(t), \Gamma^-) \geq \text{dist}(\eta(0), \Gamma^-) - T^\theta \mathcal{F}(\|\eta\|_{Z^3(T)}).
\]
(3.20)

**Proof.** Using equation (3.13), Theorem 2.2 and the fact that \( s + \frac{3}{2} > 3 \), we have
\[
\|\eta(t) - \eta(0)\|_{L^2} \leq \int_0^t \|G^- (\eta) \left( \frac{5}{\mu} H(\eta) + \frac{\rho_q}{\mu} \eta \right)(r)\|_{L^2} dr \leq \int_0^t \mathcal{F}(\|\eta(r)\|_{H^3}) \|\eta(r)\|_{H^3} dr \leq t^{\frac{3}{2}} \mathcal{F}(\|\eta\|_{L^\infty([0, t]; H^3)}) \|\eta\|_{L^2([0, t]; H^{3+\frac{3}{2}})}.
\]
Fixing \( s' \in (1 + \frac{d}{2}, s) \) and using interpolation yields
\[
\|\eta(t) - \eta(0)\|_{H^{s'}} \leq \|\eta(t) - \eta(0)\|_{L^2}^{\theta} \|\eta(t) - \eta(0)\|_{H^{s'}}^{1-\theta} \leq t^{\frac{\theta}{2}} F(\|\eta\|_{Z^+(t)})
\]
for some \( \theta \in (0, 1) \). Then in view of the embedding \( H^{s'} \subset L^\infty \), this implies (3.20). \( \square \)

### 3.3. Contraction estimates

Our goal in this subsection is to prove the following contraction estimate for solutions of (1.13).

**Theorem 3.5.** Let \( s > 1 + \frac{d}{2} \). Assume that \( \eta_1 \) and \( \eta_2 \) are two solutions of (1.13) in \( Z^+(T) \) that satisfy (3.1). There exists \( F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) depending only on \( h, s, \frac{\theta - \delta}{\mu}^{-1} \) such that
\[
\|\eta_1 - \eta_2\|_{Z^+(T)} \leq F\big(\|\eta_1, \eta_2\|_{Z^+(T)}, T\big)\|\eta_1 - \eta_2\|_{t=0}^1 H^s.
\]  

We first prove a contraction estimate for the remainder in the paralinearization \( H(\eta) \sim T\ell \eta \).

**Lemma 3.6.** Set
\[
R_H(\eta) = H(\eta) - T\ell \eta
\]
where \( \ell \) is defined in terms of \( \eta \) as in (3.2). For \( \delta \in (0, s - 1 - \frac{d}{2}) \) and \( \delta \leq 1 \), there exists \( F \) depending only on \( s \) such that
\[
\|R_H(\eta_1) - R_H(\eta_2)\|_{H^{s-\frac{1}{2}}} \leq F\big(\|\eta_1, \eta_2\|_{H^s}\big)\big(\|\eta_1 - \eta_2\|_{H^{s+\frac{1}{2}}} + (1 + \|\eta_1, \eta_2\|_{H^{s+\frac{1}{2}}}^2)\|\eta_1 - \eta_2\|_{H^s}\big).
\]

**Proof.** We denote the Gâteaux derivative \( d_u F(\u) \) of a function \( F \) at \( u \) in the direction \( \u \) by
\[
d_u F(u) \u = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (F(u + \varepsilon \u) - F(u)).
\]

By virtue of the mean-value theorem for Gâteaux derivative, it suffices to prove that
\[
\|d_\eta R_H(\eta)\u\|_{H^{s-\frac{1}{2}}} \leq F\big(\|\eta\|_{H^s}\big)\big(\|\u\|_{H^{s+\frac{1}{2}}} + (1 + \|\eta\|_{H^{s+\frac{1}{2}}}^2)\|\eta\|_{H^s}\big).
\]

Setting \( f(z) = \frac{z}{\sqrt{1 + |z|^2}} \) for \( z \in \mathbb{R}^d \), we write \( H(\eta) = -\div (\nabla \eta) \). Since \( d_\eta f(\nabla \eta)\u = M \nabla \u \), where \( M = M(\nabla \eta) \) is given by (3.7), it follows that
\[
d_\eta R_H(\eta)\u = -\div (M) \nabla \u - M \nabla \cdot \nabla \u - T\ell \u - T\nu \u \eta.
\]

Using Bony’s decomposition and the fact that \( M \xi \cdot \xi = \ell \), we obtain
\[
d_\eta R_H(\eta)\u = -T_{\div (M)} \nabla \u + T_{\div (M)} \xi \u \xi - g_0 - T\ell \u - T\nu \u \eta
\]
where \( g_0 = g_1 + g_2 \),
\[
g_1 = \left[ \div (M) \nabla \u - T_{\div (M)} \nabla \u \right], \quad g_2 = \left[ M \nabla \cdot \nabla \u - M \nabla \cdot \nabla \u \right].
\]

Since
\[
\|M - 1d\|_{H^{s-\frac{1}{2}}} \leq F\big(\|\eta\|_{H^s}\big)\|\nabla \eta\|_{H^{s+\frac{1}{2}}} \leq F\big(\|\eta\|_{H^s}\big)\|\eta\|_{H^{s+\frac{1}{2}}},
\]
(A.9) implies
\[
\|g_1\|_{H^{s-\frac{1}{2}}} \lesssim \|\div M\|_{H^{s-\frac{1}{2}}} \|\nabla \u\|_{H^{s-\frac{1}{2}}} \lesssim F\big(\|\eta\|_{H^s}\big)\|\eta\|_{H^{s+\frac{1}{2}}} \|\eta\|_{H^{s}},
\]
\[
\|g_2\|_{H^{s-\frac{1}{2}}} \lesssim (\|M - 1d\|_{H^{s-\frac{1}{2}}} + 1)\|\nabla^2 \u\|_{H^{s-\frac{1}{2}}} \lesssim F\big(\|\eta\|_{H^s}\big)(1 + \|\eta\|_{H^{s+\frac{1}{2}}}^2)\|\u\|_{H^s}.
\]

By means of (3.8) and (A.8) we get
\[
\|T_{\div (M)} \nabla \u\|_{H^{s-\frac{1}{2}}} \lesssim \|\div M\|_{H^{s-\frac{1}{2}}} \|\nabla \u\|_{H^{s+\frac{1}{2}}} \|\u\|_{H^{s+\frac{1}{2}}} \lesssim F\big(\|\eta\|_{H^s}\big)\|\u\|_{H^{s+\frac{1}{2}}},
\]
for \( \delta \in (0, s - 1 - \frac{d}{2}) \) and \( \delta \leq 1 \).
Finally, for $T_{d_1\ell(\eta)}\eta$ we note that $d_1\ell(\eta)\dot{\eta} = F(\nabla\eta, \xi)\nabla\dot{\eta}$ where $F$ is homogeneous of order 2 in $\xi$. Hence,

$$M_0^2(d_1\ell(\eta)\dot{\eta}) \leq \mathcal{F}(\|\eta\|_{H^s})\|\dot{\eta}\|_{H^s},$$

and thus applying Theorem A.4 (i) gives

$$\|T_{d_1\ell(\eta)}\eta\|_{H^{s-\frac{3}{2}}} \leq \mathcal{F}(\|\eta\|_{H^s})\|\dot{\eta}\|_{H^s}\|\eta\|_{H^{s+\frac{3}{2}}}.$$

Putting together the above estimates we arrive at (3.23) which completes the proof. \qed

**Proof of Theorem 3.3**

Setting $\eta_0 = \eta_1 - \eta_2$ we have

$$\partial_t \eta_0 = -\frac{\delta}{\mu}G^-(\eta_1)(H(\eta_1) - H(\eta_2)) - R_0,$$

$$R_0 := \frac{\rho_\theta \delta}{\mu}G^-(\eta_1)\eta_0 + [G^-(\eta_1) - G^-(\eta_2)](\frac{\delta}{\mu}H(\eta_2) + \frac{\rho_\theta \delta}{\mu} \eta_2).$$

(3.24)

(3.25)

According to Theorem 2.2

$$\|G^-(\eta_1)\eta_0\|_{H^{s+\frac{3}{2}}} \leq \mathcal{F}(\|\eta_1\|_{H^s})\|\eta_0\|_{H^{s+\frac{3}{2}}},$$

On the other hand, Theorem 2.4 applied with $f = \frac{\delta}{\mu}H(\eta_2) + \frac{\rho_\theta \delta}{\mu} \eta_2 \in H^{s+\frac{1}{2}}$ gives

$$\|G^-(\eta_1) - G^-(\eta_2)\|_{H^{s+\frac{3}{2}}}(\frac{\delta}{\mu}H(\eta_2) + \frac{\rho_\theta \delta}{\mu} \eta_2)\|_{H^{s+\frac{3}{2}}} \leq (\frac{\delta}{\mu} + \frac{\rho_\theta \delta}{\mu})\mathcal{F}(N_s)\|\eta_0\|_{H^s}\|\eta_2\|_{H^{s+\frac{3}{2}}}.$$
Since $M^2_0(\ell_1 - \ell_2) \leq \mathcal{F}(N_s)\|\eta_s\|_{H^s}$, Theorem A.4 (i) gives

\[\|T_{\lambda_1}T_{\ell_1 - \ell_2}\eta_2\|_{H^{s+\frac{3}{2}}} \leq \mathcal{F}(N_s)\|\eta_s\|_{H^s}\|\eta_2\|_{H^{s+\frac{3}{2}}} .\]

Finally, Theorem A.4 (ii) yields that $T_{\lambda_1}T_{\ell_1} - T_{\lambda_2}\delta_1$ is of order $3 - \delta$ and

\[\|(T_{\lambda_1}T_{\ell_1} - T_{\lambda_2}\delta_1)\|_{H^{s+\frac{3}{2}}} \leq \mathcal{F}(\|\eta_1\|_{H^s})\|\eta_2\|_{H^{s+\frac{3}{2}}}.\]  (3.30)

The above estimates together imply

\[T_{\lambda_1}(H(\eta_1) - H(\eta_2)) - e_1\eta_{\delta} + \mathcal{R}_3,\]

\[\|\mathcal{R}_3\|_{H^{s+\frac{3}{2}}} \leq \mathcal{F}(N_s)\left((\|\eta_\delta\|_{H^{s+\frac{3}{2}}} + (1 + N_s^{\frac{3}{2}}))\|\eta_s\|_{H^s}\right).\]  (3.31)

Therefore, we arrive at (3.28) with $\mathcal{R}_1 = \mathcal{R}_2 + \mathcal{R}_3$.

Now it follows from equations (3.24), (3.28) and the estimates (3.27), (3.29) that

\[\partial_t \eta_\delta = -\frac{s}{\mu - T_{\lambda_1}e_1\eta_\delta} + \mathcal{R}_1,\]  (3.32)

where $\mathcal{R}_1 = -\frac{s}{\mu - T_{\lambda_1}e_1\eta}$ satisfies

\[\|\mathcal{R}_1\|_{H^{s+\frac{3}{2}}} \leq \left(\frac{s}{\mu - T_{\lambda_1}e_1\eta}\right)\mathcal{F}(N_s)\left((\|\eta_\delta\|_{H^{s+\frac{3}{2}}} + (1 + N_s^{\frac{3}{2}}))\|\eta_s\|_{H^s}\right).\]

(3.33)

where $\mathcal{F}$ depends only on $(h, s)$. An $H^s$ energy estimate for (3.32) yields

\[\frac{1}{2}\frac{d}{dt}\|\eta_\delta\|_{H^s}^2 \leq -\frac{s}{\mu - T_{\lambda_1}e_1\eta_\delta} + \|\mathcal{R}_1\|_{H^{s+\frac{3}{2}}} \|\eta_\delta\|_{H^{s+\frac{3}{2}}}.\]  (3.34)

The argument leading to (3.19) gives

\[-(T_{\lambda_1}e_1\eta_\delta, H^{s+\frac{3}{2}}) \leq -\frac{1}{\mathcal{F}(\|\eta_1\|_{H^s})}\|\eta_\delta\|_{H^{s+\frac{3}{2}}}^2 + \mathcal{F}(\|\eta_1\|_{H^s})\|\eta_\delta\|_{H^{s+\frac{3}{2}}} \|\eta_\delta\|_{H^{s+\frac{3}{2}}} - \mathcal{F}(\|\eta_1\|_{H^s})\|\eta_\delta\|_{H^{s+\frac{3}{2}}} \|\eta_\delta\|_{H^{s+\frac{3}{2}}}.\]  (3.35)

Combining (3.34), (3.35) and (3.33) we obtain

\[\frac{d}{dt}\|\eta_\delta\|_{H^s}^2 \leq -\frac{1}{\mathcal{F}(N_s)}\|\eta_\delta\|_{H^{s+\frac{3}{2}}}^2 + \mathcal{F}(N_s)\|\eta_\delta\|_{H^{s+\frac{3}{2}}} \|\eta_\delta\|_{H^{s+\frac{3}{2}}} - \mathcal{F}(N_s)(1 + N_s^{\frac{3}{2}})\|\eta_\delta\|_{H^{s+\frac{3}{2}}} \|\eta_\delta\|_{H^{s+\frac{3}{2}}} .\]  (3.36)

for some function $\mathcal{F}$ depending only on $(h, s, \frac{\rho - 6}{\mu - 6}, \frac{s}{\mu - 6})$. By interpolation and Young’s inequality we have

\[\mathcal{F}(N_s)\|\eta_\delta\|_{H^{s+\frac{3}{2}}} \|\eta_\delta\|_{H^{s+\frac{3}{2}}} \leq -\frac{1}{10\mathcal{F}(N_s)}\|\eta_\delta\|_{H^{s+\frac{3}{2}}}^2 + \mathcal{F}_1(N_s)\|\eta_\delta\|_{H^s}^2 ,\]

\[\mathcal{F}(N_s)(1 + N_s^{\frac{3}{2}})\|\eta_\delta\|_{H^s} \|\eta_\delta\|_{H^{s+\frac{3}{2}}} \leq -\frac{1}{10\mathcal{F}(N_s)}\|\eta_\delta\|_{H^{s+\frac{3}{2}}}^2 + \mathcal{F}_2(N_s)(1 + N_s^{\frac{3}{2}})\|\eta_\delta\|_{H^s}^2 .\]

It follows that

\[\frac{d}{dt}\|\eta_\delta\|_{H^s}^2 \leq -\frac{1}{\mathcal{F}(N_s)}\|\eta_\delta\|_{H^{s+\frac{3}{2}}}^2 + \mathcal{F}(N_s)(1 + N_s^{\frac{3}{2}})\|\eta_\delta\|_{H^s}^2 \]  (3.37)

for some $\mathcal{F}$ depending only on $(h, s, \frac{\rho - 6}{\mu - 6}, \frac{s}{\mu - 6})$. Finally, since

\[\int_0^T N_s^{\frac{3}{2}}(t) dt \leq \|(\eta_1, \eta_2)\|_{L^2(T)}^2,\]

a simple Grönwall argument leads to (3.21).
3.4. Proof of Theorem 1.2 Consider an initial datum \( \eta_0 \in H^s(\mathbb{R}^d), s > 1 + \frac{d}{2}, \) satisfying \( \text{dist}(\eta_0, \Gamma^-) > 2h > 0. \) We construct the sequence of approximate solutions \( \eta_\varepsilon, \varepsilon \in (0, 1), \) that solve the ODE
\[
\partial_t \eta_\varepsilon = -\frac{1}{\mu} J_\varepsilon \left( G^-(J_\varepsilon \eta_\varepsilon) (\mathbf{s} H(J_\varepsilon \eta_\varepsilon) + \rho^- g J_\varepsilon \eta_\varepsilon) \right), \quad \eta_\varepsilon|_{t=0} = \eta_0, \tag{3.38}
\]
where \( J_\varepsilon \) denotes the usual mollifier that cut off frequencies of size greater than \( \varepsilon^{-1}. \) Each \( \eta_\varepsilon \) exists on some maximal time interval \([0, T_\varepsilon)\) in light of the Cauchy-Lipschitz theorem and Theorems 2.2 and 2.4 for the Dirichlet-Neumann operator. It is easy to check that the a priori estimates in Proposition 3.3 and Lemma 3.4 remain valid for \( \eta_\varepsilon. \) Consequently, a continuity argument guarantees the existence of a positive time \( T \) such that \( T < T_\varepsilon \) for all \( \varepsilon \in (0, 1) \) and that on \([0, T]\) the uniform estimates
\[
\|\eta_\varepsilon\|_{Z^s(T)} \leq \mathcal{F}(\|\eta_0\|_H^s), \quad \inf_{t \in [0, T]} \text{dist}(\eta_\varepsilon(t), \Gamma^-) > h \tag{3.39}
\]
hold for some \( \mathcal{F} \) depending only on \((h, s, \frac{\mu^-}{\mu}, \frac{\mu^-}{\mu}). \) Theorem 3.5 also holds for \( \eta_\varepsilon, \) giving that the sequence \((\eta_\varepsilon)\) is Cauchy in \( Z^s(T) \) and thus converges to some \( \eta \in Z^s(T). \) By virtue of Theorems 2.10 and 2.4 we can pass to the limit \( \varepsilon \to 0 \) and obtain that \( \eta \) is a solution of (1.13) with initial data \( \eta_0. \) Finally, uniqueness and stability follow at once from Theorem 3.5.

4. Proof of Theorem 1.3

4.1. Regularity of \( f^\pm. \) We first recall the well-posedness of variational solutions to (1.15).

Proposition 4.1 ([29 Proposition 4.8 and Remark 4.9]). Let \( \eta \in W^{1, \infty}(\mathbb{R}^d) \cap H^\frac{1}{2}(\mathbb{R}^d) \) satisfy \( \text{dist}(\eta, \Gamma^\pm) > h > 0. \) Then there exists a unique variational solution \( f^\pm \in \tilde{H}^\pm(\mathbb{R}^d) \) to the system (1.15). Moreover, \( f^\pm \) satisfy
\[
\|f^\pm\|_{\tilde{H}^\pm} \leq C(1 + \|\eta\|_{W^{1, \infty}})\|\mathbf{s} H(\eta)\| + \|\rho\|\|\eta\|_{H^\frac{1}{2}} \tag{4.1}
\]
where the constant \( C \) depends only on \((h, \mu^\pm). \)

It follows from (4.1) and Theorem [A.7] that
\[
\|f^\pm\|_{\tilde{H}^\pm} \leq \mathcal{F}(\|\eta\|_{W^{1, \infty}})(\|\mathbf{s}\|\|\eta\|_{H^\frac{1}{2}} + \|\rho\|\|\eta\|_{H^\frac{1}{2}}) \tag{4.2}
\]
for some function \( \mathcal{F} \) depending only on \((h, \mu^\pm). \) Using the variational estimate (4.2) and the paralinearization Theorem 2.3, we prove that higher Sobolev regularity for \( f^\pm \) can be transferred from \( \eta. \)

Proposition 4.2. Let \( f^\pm \) be the solution of (1.15) as given by Proposition 4.1. If \( \eta \in H^{s+\frac{1}{2}}(\mathbb{R}^d) \) with \( s > 1 + \frac{d}{2} \) then \( f^\pm \in \tilde{H}^{s-\frac{1}{2}}(\mathbb{R}^d) \) and
\[
\|f^\pm\|_{\tilde{H}^{s-\frac{1}{2}}} \leq \mathcal{F}(\|\eta\|_{H^s})(\|\mathbf{s}\|\|\eta\|_{H^{s+2}} + \|\rho\|\|\eta\|_{H^{s+2}}) \tag{4.3}
\]
for all \( r \in [\frac{1}{2}, s - \frac{1}{2}], \) where \( \mathcal{F} \) depends only on \((h, s, r, \mu^\pm). \)

Proof. Fix \( \delta \in (0, s - 1 - \frac{d}{2}) \) and \( \delta \leq \frac{1}{2}. \) First, we claim that for \( \sigma \in [\frac{1}{2}, s - \frac{1}{2} - \delta], \) if \( f^\pm \in \tilde{H}^\pm \) then there exists \( \mathcal{F} \) depending only on \((h, s, \sigma, \delta, \mu^\pm) \) such that
\[
\|T_\lambda f^\pm\|_{H^{s-1+\delta}} \leq \mathcal{F}(\|\eta\|_{H^s})\|f^\pm\|_{\tilde{H}^\pm} + \mathcal{F}(\|\eta\|_{H^s})(\|\mathbf{s}\|\|\eta\|_{H^{s+2+\delta}} + \|\rho\|\|\eta\|_{H^{s+2+\delta}}). \tag{4.4}
\]
Indeed, according to Theorem 2.3, there exists \( \mathcal{F} \) depending only on \((h, s, \sigma, \delta) \) such that
\[
G^\pm(\eta)f^\pm = \pm T_\lambda f^\pm + R^\pm(\eta)f^\pm,
\]
\[
\|R^\pm(\eta)f^\pm\|_{H^{s-1+\delta}} \leq \mathcal{F}(\|\eta\|_{H^s})\|f^\pm\|_{H^{s-1+\delta}}. \]
Then using the system (1.15) we obtain after rearranging terms that
\[
T_\lambda f^- = \frac{\mu^-}{\mu^+ + \mu^-} T_\lambda (sH(\eta)) + \frac{\mu^-}{\mu^+ + \mu^-} R^+(\eta) f^+ - \frac{\mu^+}{\mu^+ + \mu^-} R^-(\eta) f^- \tag{4.5}
\]
which together with the fact that
\[
\|sH(\eta) + [\rho] g\|_{H^{s-1+\delta}} \leq \mathcal{F}(\|\eta\|_{H^s})(s\|\eta\|_{H^{s+1+\delta}} + [\rho]\|\eta\|_{H^{s+1+\delta}})
\]
proves the claim (4.4). Note that \(\sigma + 2 + \delta \in [\frac{s}{2} + \delta, s + \frac{3}{2}]\).

We now bootstrap the regularity for \(f^\pm\) using (4.4) and the inequality
\[
\|u\|_{H^{1, \mu}} \lesssim \|u\|_{H^{1, \frac{s}{2}}} + \mathcal{F}(\|\eta\|_{H^s})(\|T_\lambda u\|_{H^{s-1}} + \|u\|_{H^{1, \mu-\delta}}), \quad \mu \geq \frac{1}{2}. \tag{4.6}
\]
Applying this with \(\mu = \sigma + \delta\), \(\sigma = \frac{1}{2}\) and invoking (4.2) and (4.4) we deduce that
\[
\|f^\pm\|_{H^{1, \frac{s}{2}+\delta}} \leq \mathcal{F}(\|\eta\|_{H^s})(s\|\eta\|_{H^{s+2}} + [\rho]\|\eta\|_{H^s})
\]
where \(\mathcal{F}\) depends only on \((h, s, \sigma, \delta, \mu^\pm)\). We have thus bootstrapped the regularity of \(f^\pm\) from \(H^{1, \frac{s}{2}}\) to \(H^{1, \frac{s}{2}+\delta}\) by using (4.4) with \(\sigma = \frac{1}{2}\). Since (4.4) holds for \(\sigma \in [\frac{s}{2}, s - \frac{1}{2} - \delta]\), an induction argument leads to
\[
\|f^\pm\|_{H^{1, r}} \leq \mathcal{F}(\|\eta\|_{H^s})(s\|\eta\|_{H^{s+2}} + [\rho]\|\eta\|_{H^s})
\]
for all \(r \in [\frac{s}{2}, s - \frac{1}{2}]\). In conjunction with (4.2), this yields (4.3).

It remains to prove (4.6). By virtue of Theorem A.4 (ii) and Remark A.5 we have for \(\mu \in \mathbb{R},\)
\[
\|\Psi(D_x) u\|_{H^\mu} = \|T_1 u\|_{H^\mu} \leq \|T_{\lambda-1} T_\lambda u\|_{H^\mu} + \|T_1 - T_{\lambda-1} T_\lambda u\|_{H^\mu}
\]
\[
\leq \mathcal{F}(\|\eta\|_{H^s})(\|T_\lambda u\|_{H^{s-1}} + \|u\|_{H^{1, \mu-\delta}}), \tag{4.7}
\]
where the cut-off \(\Psi\) removing the low frequency part is defined by (A.3). On the other hand, for \(\mu \geq \frac{1}{2}\) we have
\[
\|u\|_{H^{1, \mu}} \lesssim \|u\|_{H^{1, \frac{s}{2}}} + \|\Psi(D_x) u\|_{H^\mu}
\]
which combined with (4.7) yields (4.6).

\[\text{Remark 4.3:}\] The estimate (4.3) shows that \(f^\pm\) behave like \(sH(\eta) + [\rho] g\).

**4.2. Paradifferential reduction and a priori estimates.** Assume that \(\eta \in Z^s(T)\) with \(s > 1 + \frac{d}{2}\) solves (1.14) and satisfies
\[
\inf_{t \in [0, T]} \text{dist}(\eta(t), \Gamma^\pm) > h > 0. \tag{4.8}
\]
Moreover, let \(f^\pm \in \overline{H}^{s-\frac{1}{2}}\) be the solution of (1.15) as given by Propositions 4.1 and 4.2.

**Proposition 4.4.** For \(\delta \in (0, s - 1 - \frac{d}{2})\), \(\delta \leq \frac{1}{2}\), there exists \(\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}^+\) depending only on \((h, s, \delta, \mu^\pm)\) such that
\[
\partial_t \eta = \frac{-\delta}{\mu^+ + \mu^-} T_{\lambda} \xi \eta + g, \tag{4.9}
\]
\[
\|g\|_{H^{s-\frac{1}{2}+\delta}} \leq \mathcal{F}(\|\eta\|_{H^s})(s\|\eta\|_{H^{s+\frac{1}{2}}} + [\rho]\|\eta\|_{H^{s+\frac{1}{2}+\delta}}). \tag{4.10}
\]

**Proof.** We rewrite (4.5) as
\[
T_\lambda f^- = \frac{\delta \mu^-}{\mu^+ + \mu^-} T_{\lambda} \xi \eta - \frac{\delta \mu^-}{\mu^+ + \mu^-} (T_{\lambda} T_\lambda \xi \eta - T_{\lambda} \xi \eta) + \frac{\delta \mu^-}{\mu^+ + \mu^-} T_{\lambda} (H(\eta) - T_{\lambda} \xi \eta)
\]
\[
+ \frac{[\rho] \mu^-}{\mu^+ + \mu^-} T_{\lambda} \xi \eta + \frac{\mu^-}{\mu^+ + \mu^-} R^+(\eta) f^+ - \frac{\mu^+}{\mu^+ + \mu^-} R^-(\eta) f^-,
\]

\[\text{15}\]
where by virtue of Theorem 2.3 and Proposition 4.2,
\[ \|R^\pm(\eta)f^\pm\|_{H^{s-\frac{1}{2}+\delta}} \leq \mathcal{F}(\|\eta\|_{H^s})\|f^\pm\|_{H^{s-\frac{1}{2}}} \]
\[ \lesssim \mathcal{F}(\|\eta\|_{H^s})(\|\eta\|_{H^{s+\frac{1}{2}}} + [\rho] g \|\eta\|_{H^{s-\frac{1}{2}}}). \]  \hfill (4.11)

Using (4.9) and Theorem A.4 (i) (ii), we can bound
\[ \|\langle T\lambda T\ell - T\ell\lambda\rangle\eta\|_{H^{s-\frac{1}{2}+\delta}} + \|T\lambda(\langle H(\eta) - T\ell\eta\rangle)\|_{H^{s-\frac{1}{2}+\delta}} \leq \mathcal{F}(\|\eta\|_{H^s})\|\eta\|_{H^{s+\frac{1}{2}}}, \]
\[ \|T\lambda\eta\|_{H^{s-\frac{1}{2}+\delta}} \leq \mathcal{F}(\|\eta\|_{H^s})\|\eta\|_{H^{s-\frac{1}{2}+\delta}}. \]

We thus obtain
\[ T\lambda f^- = \frac{\delta\mu^-}{\mu^+ + \mu^-} T\lambda\eta + g_0, \]
\[ \|g_0\|_{H^{s-\frac{1}{2}+\delta}} \leq \mathcal{F}(\|\eta\|_{H^s})(\|\eta\|_{H^{s+\frac{1}{2}}} + [\rho] g \|\eta\|_{H^{s-\frac{1}{2}+\delta}}), \]
for some \( \mathcal{F} \) depending only on \((h, s, \delta, \mu^\pm)\). Plugging this into the paralinearization
\[ G^-(\eta)f^- = T\lambda f^- + R^-(\eta)f^- \]
and using (4.11) and (1.14) we conclude the proof. \qed

It follows from (4.10) that
\[ \|g\|_{H^{s-\frac{1}{2}+\delta}} \leq (\delta + [\rho] g) \mathcal{F}(\|\eta\|_{H^s})\|\eta\|_{H^{s+\frac{1}{2}}}. \]  \hfill (4.12)

We have thus reduced the two-phase Muskat problem to the paradifferential parabolic equation (4.9) which is of the same form as equation (3.3) for the one-phase problem. Therefore, the proofs of Proposition 3.3 and Lemma 3.4 yield the following a priori estimates.

**Proposition 4.5.** There exist \( \theta \in (0, 1) \) depending only on \( s \) and \( \mathcal{F} : \mathbb{R}^+ \to \mathbb{R}^+ \) depending only on \((h, s, s, \mu^\pm, [\rho] g)\) such that
\[ \|\eta\|_{Z^s(T)} \leq \mathcal{F}(\|\eta(0)\|_{H^s} + T\mathcal{F}(\|\eta\|_{Z^s(T)})) \]  \hfill (4.13)
and
\[ \inf_{t \in [0,T]} \text{dist}(\eta(t), \Gamma^-) \geq \text{dist}(\eta(0), \Gamma^-) - T^\theta \mathcal{F}(\|\eta\|_{Z^s(T)}). \]  \hfill (4.14)

### 4.3. Contraction estimates.

Considering two solutions \( \eta_1 \) and \( \eta_2 \) in \( Z^s(T) \) of (1.14) that satisfy condition (4.8), we prove a contraction estimate in \( Z^s(T) \) for the difference \( \eta_1 - \eta_2 \).

**Theorem 4.6.** There exists \( \mathcal{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) depending only on \((h, s, s, \mu^\pm, [\rho] g)\) such that
\[ \|\eta_1 - \eta_2\|_{Z^s(T)} \leq \mathcal{F}(\|(\eta_1, \eta_2)\|_{Z^s(T)}, T)\|(\eta_1 - \eta_2)\|_{t=0}H^s. \]  \hfill (4.15)

#### 4.3.1. Contraction estimates for \( f^\pm \).

For \( j = 1, 2 \) let \( f^\pm_j \) solve
\[ \begin{cases} f^-_j - f^+_j &= k_j := \delta H(\eta_j) + [\rho] g \eta_j, \\ \frac{1}{\mu^+}G^+\eta_j f^+_j - \frac{1}{\mu^-}G^-\eta_j f^-_j &= 0. \end{cases} \]  \hfill (4.16)

We set \( f^\pm_0 = f^\pm_1 - f^\pm_2, k_0 = k_1 - k_2, \eta_0 = \eta_1 - \eta_2 \), where the subscript \( \delta \) only signifies the difference. We also recall the notation (3.26)
\[ N_r = \|(\eta_1, \eta_2)\|_{H^s}. \]
Lemma 4.7. Let \( \delta \in (0, s - 1 - \frac{d}{2}) \) and \( \delta \leq \frac{1}{2} \).

1) For each \( r \in \left( \frac{1}{2}, s - \frac{1}{2} \right] \), there exists \( F \) depending only on \((h, s, r, \mu^\pm)\) such that

\[
\| f_\delta^+ \|_{H^s_{\pm}} \leq F(N_s)(s\|\eta\|_{H^{s+2}} + \|\rho\|\|\eta\|_{H^r}) \\
+ F(N_s)(s\|\eta\|_{H^s(s_{s+\frac{3}{2}})} + \|\rho\|\|\eta\|_{H^{s-\frac{1}{2}}}). \tag{4.17}
\]

2) For each \( \sigma \in \left[ \frac{1}{2}, s - \frac{1}{2} - \delta \right] \), there exists \( F \) depending only on \((h, s, \sigma, \mu^\pm)\) such that

\[
T_{\lambda_1} f_\delta^- = \frac{\mu^-}{\mu^+ + \mu^-} T_{\lambda_1} k_\delta + g_-
\tag{4.18}
\]

with \( g_- \) satisfying

\[
\| g_- \|_{H^{s-1+\delta}} \leq F(N_s)(s\|\eta\|_{H^{s+2}} + \|\rho\|\|\eta\|_{H^r}) \\
+ F(N_s)(s\|\eta\|_{H^s(s_{s+\frac{3}{2}})} + \|\rho\|\|\eta\|_{H^{s-\frac{1}{2}}}). \tag{4.19}
\]

Proof. Taking the difference of the second equation in (4.16) for \( j = 1 \) and \( j = 2 \) we find that

\[
\frac{1}{\mu^-} G^- (\eta_1) f_\delta^- - \frac{1}{\mu^-} G^+ (\eta_1) f_\delta^+ = \frac{1}{\mu^+} [G^+ (\eta_1) - G^+ (\eta_2)] f_2^+ - \frac{1}{\mu^-} [G^- (\eta_1) - G^- (\eta_2)] f_2^-.
\]

Since \( G^\pm (\eta_1) f_\delta^\pm = \mp T_{\lambda_1} f_\delta^\pm + R^\pm (\eta_1) f_\delta^\pm \) and \( f_\delta^+ = f_\delta^- - k_\delta \), this gives

\[
T_{\lambda_1} f_\delta^- = \frac{\mu^-}{\mu^+ + \mu^-} T_{\lambda_1} k_\delta + \frac{\mu^+ \mu^-}{\mu^+ + \mu^-} F
\tag{4.20}
\]

where

\[
F = \frac{1}{\mu^+} R^+ (\eta_1) f_\delta^+ - \frac{1}{\mu^-} R^- (\eta_1) f_\delta^- + \frac{1}{\mu^+} [G^+ (\eta_1) - G^+ (\eta_2)] f_2^+ - \frac{1}{\mu^-} [G^- (\eta_1) - G^- (\eta_2)] f_2^-.
\]

Theorems A.3 (i) and A.7 together imply that for \( \nu \in [-1, s - \frac{3}{2}] \),

\[
\| T_{\lambda_1} k_\delta \|_{H^\nu} \leq F(N_s)(s\|\eta\|_{H^{s+2}} + \|\rho\|\|\eta\|_{H^{s+1}}).
\]

In light of Theorem 2.3 we have that for \( \sigma \in \left[ \frac{1}{2}, s - \frac{1}{2} \right] \),

\[
\| R^\pm (\eta_1) f_\delta^\pm \|_{H^{s-1+\delta}} \leq F(N_s)\| f_\delta^\pm \|_{H^s_{\pm}}.
\]

Finally, a combination of Theorem 2.14 and Proposition 4.2 yields

\[
\|[G^\pm (\eta_1) - G^\pm (\eta_2)] f_\delta^\pm \|_{H^{s-\frac{1}{2}}} \leq F(N_s)\|\eta\|_{H^s(s_{s+\frac{3}{2}})} + \|[\rho]\|\|\eta\|_{H^{s-\frac{1}{2}}}.
\]

Consequently, for \( \sigma \in \left[ \frac{1}{2}, s - \frac{1}{2} - \delta \right] \) we have

\[
\| F \|_{H^{s-1+\delta}} \leq F(N_s)\| f_\delta^\pm \|_{H^s_{\pm}} + F(N_s)\|\eta\|_{H^s(s_{s+\frac{3}{2}})} + \|[\rho]\|\|\eta\|_{H^{s-\frac{1}{2}}} \tag{4.21}
\]

and

\[
\| T_{\lambda_1} f_\delta^- \|_{H^{s-1+\delta}} \leq F(N_s)\| f_\delta^\pm \|_{H^s_{\pm}} + F(N_s)\|\eta\|_{H^{s+2+\delta}} + \|[\rho]\|\|\eta\|_{H^{s+4}}
\]

\[
+ F(N_s)\|\eta\|_{H^s(s_{s+\frac{3}{2}})} + \|[\rho]\|\|\eta\|_{H^{s-\frac{1}{2}}}.
\]

Invoking the relation \( f_\delta^- = f_\delta^- - k_\delta \) leads to the same bound for \( \| T_{\lambda_1} f_\delta^+ \|_{H^{s-1+\delta}} \) and thus

\[
\| T_{\lambda_1} f_\delta^+ \|_{H^{s-1+\delta}} \leq F(N_s)\| f_\delta^\pm \|_{H^s_{\pm}} + F(N_s)\|\eta\|_{H^{s+2+\delta}} + \|[\rho]\|\|\eta\|_{H^{s+4}}
\]

\[
+ F(N_s)\|\eta\|_{H^s(s_{s+\frac{3}{2}})} + \|[\rho]\|\|\eta\|_{H^{s-\frac{1}{2}}}.
\]
for $\sigma \in [\frac{1}{2}, s - \frac{1}{2} - \delta]$. Now we can apply (4.6) and use the definition of $\tilde{H}_{\pm}$ (see (2.9)) to have

$$
\|f^\pm\|_{H_{\pm}^1, \sigma + \delta} \leq \mathcal{F}(N_s)(\|f^\pm\|_{H_{\pm}^n, \sigma} + \|f^\pm\|_{H_{\pm}^1, \sigma})
+ \mathcal{F}(N_s)(\|\sigma\|_{\mathcal{H}_{\pm}} + \|\rho\|\mathcal{G}\|\eta_0\|_{\mathcal{H}_{\pm}^1, \sigma + \delta})
+ \mathcal{F}(N_s)(\|\eta_0\|_{\mathcal{H}_{\pm}^1, \sigma + \delta} + \|\rho\|\mathcal{G}\|\eta_0\|_{\mathcal{H}_{\pm}^1, \sigma + \delta})
$$

(4.22)

for $\sigma \in [\frac{1}{2}, s - \frac{1}{2} - \delta]$. Next we note that by using the variational form of (1.15) derived in Proposition 4.8 [49] it can be proved that the following $H_{\pm}^1$ contraction estimate holds

$$
\|f^\pm\|_{H_{\pm}^1} \leq \mathcal{F}(N_s)(\|k_\delta\|_{H_{\pm}^n} + \|\eta_0\|_{H_{\pm}^1}, (k_1, k_2)\|_{H_{\pm}^n}).
$$

(4.23)

It follows that

$$
\|f^\pm\|_{H_{\pm}^1} \leq \mathcal{F}(N_s)(\|\sigma\|_{\mathcal{H}_{\pm}} + \|\rho\|\mathcal{G}\|\eta_0\|_{H_{\pm}^1})
+ \mathcal{F}(N_s)(\|\eta_0\|_{\mathcal{H}_{\pm}}, \|\rho\|\mathcal{G}\|\eta_0\|_{H_{\pm}^1}).
$$

(4.24)

Then combining (4.22), (4.24) and an induction argument we arrive at

$$
\|f^\pm\|_{H_{\pm}^1} \leq \mathcal{F}(N_s)(\|\sigma\|_{\mathcal{H}_{\pm}} + \|\rho\|\mathcal{G}\|\eta_0\|_{H_{\pm}^1})
+ \mathcal{F}(N_s)(\|\eta_0\|_{\mathcal{H}_{\pm}}, \|\rho\|\mathcal{G}\|\eta_0\|_{H_{\pm}^1}).
$$

(4.25)

for all $r \in [\frac{1}{2}, s - \frac{1}{2}]$. This proves (4.17). Finally, (4.18)–(4.19) follow from (4.20), (4.21) and (4.25). \[
\]

### 4.3.2. Proof of Theorem 4.6

From equation (1.14) we see that $\eta_0 = \eta_1 - \eta_2$ satisfies

$$
\partial_t \eta_0 = -\frac{1}{\mu}G^-(\eta_1)f^-_\delta - \frac{1}{\mu}[G^-(\eta_1) - G^-(\eta_2)]f^-_\delta.
$$

According to Theorem 2.14

$$
\|[G^-(\eta_1) - G^-(\eta_2)]f^-\|_{H_{\pm}^1, r - \frac{\delta}{2}} \leq (s + \|\rho\|\mathcal{G})\mathcal{F}(N_s)\|\eta_0\|_{H_{\pm}^n, N_s + \frac{\delta}{2}}.
$$

Applying Theorem 2.3 and the estimate (4.17) for $f^-_\delta$ (with $r = s - \frac{1}{2} - \delta$) yields

$$
\|R^-(\eta_1)f^-\|_{H_{\pm}^1, r - \frac{\delta}{2}} \leq \mathcal{F}(N_s)(\|f^-\|_{H_{\pm}^n, r - \frac{\delta}{2}})
\lesssim (s + \|\rho\|\mathcal{G})\mathcal{F}(N_s)(\|\eta_0\|_{H_{\pm}^n, N_s + \frac{\delta}{2}}).
$$

Thus, for some $\mathcal{F}$ depending only on $(h, s, \delta, \mu_{\pm})$ we have

$$
\partial_t \eta_0 = -\frac{1}{\mu}T_{\lambda_1}f^-_\delta + \mathcal{R}_1,
$$

$$
\|\mathcal{R}_1\|_{H_{\pm}^1, r - \frac{\delta}{2}} \leq (s + \|\rho\|\mathcal{G})\mathcal{F}(N_s)(\|\eta_0\|_{H_{\pm}^n, N_s + \frac{\delta}{2}}).
$$

By virtue of (4.18)–(4.19) with $\sigma = s - \frac{1}{2} - \delta$,

$$
T_{\lambda_1}f^-_\delta = \frac{s\mu^-}{\mu^+ + \mu^-}T_{\lambda_1}(H(\eta_1) - H(\eta_2)) + \frac{[\rho]_\mathcal{G}\mu^-}{\mu^+ + \mu^-}T_{\lambda_1}\eta_\delta + g_-, \quad \|g_\|_{H_{\pm}^1, r - \frac{\delta}{2}} \leq (s + \|\rho\|\mathcal{G})\mathcal{F}(N_s)(\|\eta_0\|_{H_{\pm}^n, N_s + \frac{\delta}{2}}).
$$

Clearly,

$$
\|T_{\lambda_1}\eta_\delta\|_{H_{\pm}^1, r - \frac{\delta}{2}} \leq \mathcal{F}(N_s)\|\eta\|_{H_{\pm}^1}. \quad \Box
$$
Then in view of (3.31) we deduce that
\[
\frac{\partial_t \epsilon_\delta}{\mu^+ + \mu^-} = T_{\lambda_1} \epsilon_1 \eta_\delta - R_2, \\
\|R_2\|_{H^{s+\frac{3}{2}}} \leq (s + [\rho] g) F(N_s) \left( \|\eta_\delta\|_{H^{s+\frac{3}{2}}} + (1 + N_s + \frac{3}{2}) \|\eta\|_{H^s} \right),
\]
where \(F\) depends only on \((h, s, \delta, \mu^\pm)\). This reduction is of the same form as (3.32)-(3.33) in the proof of Theorem 3.5. Thus, we can conclude similarly.

4.4. Proof of Theorem 1.3 Let \(\eta_0 \in H^s\) be an initial datum satisfying \(\text{dist}(\eta_0, \Gamma^\pm) > 2h > 0\). For each \(\varepsilon \in (0, 1)\), let \(\eta_\varepsilon\) solve the ODE
\[
\frac{\partial_t \eta_\varepsilon}{\mu^+ + \mu^-} = T_\Lambda \epsilon_1 \eta_\varepsilon - R_2, \\
\eta_\varepsilon|_{t = 0} = \eta_0,
\]
where \(f_\varepsilon^\pm\) solve
\[
\begin{aligned}
f_\varepsilon^- - f_\varepsilon^+ &= \frac{s}{\mu} H(\eta_\varepsilon) + \frac{[\rho] g}{\mu} \eta_\varepsilon, \\
\frac{1}{\mu} G^+(\eta_\varepsilon) f_\varepsilon^+ - \frac{1}{\mu} G^-(\eta_\varepsilon) f_\varepsilon^- &= 0.
\end{aligned}
\]
Note that the solvability and regularity of \(f_\varepsilon^\pm\) are guaranteed by Propositions 4.1 and 4.2. Since the a priori estimates in Proposition 4.5 and the contraction estimate in Theorem 4.6 remain true for \(\eta_\varepsilon\), the existence, uniqueness and stability of solutions to (1.14)-(1.15) can be deduced as in the proof of Theorem 1.2.

Appendix A. A review of paradifferential calculus

We provide a review of basic features of Bony’s paradifferential calculus (see e.g. [10],[12],[40],[46]).

**Definition A.1.** 1. **(Symbols)** Given \(\rho \in [0, \infty)\) and \(m \in \mathbb{R}\), \(\Gamma^m_\rho(\mathbb{R}^d)\) denotes the space of locally bounded functions \(a(x, \xi)\) on \(\mathbb{R}^d \times (\mathbb{R}^d \setminus 0)\), which are \(C^\infty\) with respect to \(\xi\) for \(\xi \neq 0\) and such that, for all \(\alpha \in \mathbb{N}^d\) and all \(\xi \neq 0\), the function \(x \mapsto \partial_\xi^\alpha a(x, \xi)\) belongs to \(W^{\rho, \infty}(\mathbb{R}^d)\) and there exists a constant \(C_\alpha\) such that,
\[
\forall |\xi| \geq 2, \quad \|\partial_\xi^\alpha a(\cdot, \xi)\|_{W^{\rho, \infty}(\mathbb{R}^d)} \leq C_\alpha(1 + |\xi|)^{m-|\alpha|}.
\]

Let \(a \in \Gamma^m_\rho(\mathbb{R}^d)\), we define the semi-norm
\[
M^m_\rho(a) = \sup_{|\alpha| \leq 2(d+2)+\rho} \sup_{|\xi| \geq 2} \|\partial_\xi^\alpha a(\cdot, \xi)\|_{W^{\rho, \infty}(\mathbb{R}^d)}.
\]

2. **(Paradifferential operators)** Given a symbol \(a\), we define the paradifferential operator \(T_a\) by
\[
\widehat{T_a u}(\xi) = (2\pi)^{-d} \int \chi(\xi - \eta) \widehat{a}(\xi - \eta, \eta) \Psi(\eta) \widehat{u}(\eta) d\eta,
\]
where \(\widehat{a}(\theta, \xi) = \int e^{-ix \cdot \theta} a(x, \xi) dx\) is the Fourier transform of \(a\) with respect to the first variable; \(\chi\) and \(\Psi\) are two fixed \(C^\infty\) functions such that,
\[
\Psi(\eta) = 0 \quad \text{for } |\eta| \leq \frac{1}{5}, \quad \Psi(\eta) = 1 \quad \text{for } |\eta| \geq \frac{1}{4},
\]
and \(\chi(\theta, \eta)\) satisfies, for \(0 < \varepsilon_1 < \varepsilon_2\) small enough,
\[
\chi(\theta, \eta) = 1 \quad \text{if } |\theta| \leq \varepsilon_1|\eta|, \quad \chi(\theta, \eta) = 0 \quad \text{if } |\theta| \geq \varepsilon_2|\eta|,
\]
and such that,
\[
\forall (\theta, \eta), \quad |\partial_\theta^\alpha \partial_\eta^\beta \chi(\theta, \eta)| \leq C_{\alpha, \beta}(1 + |\eta|)^{-|\alpha|-|\beta|}.
\]

**Remark A.2.** The cut-off \(\chi\) can be appropriately chosen so that when \(a = a(x)\), the paradifferential operator \(T_a u\) becomes the usual paraproduct.
DEFINITION A.3. Let $m \in \mathbb{R}$. An operator $T$ is said to be of order $m$ if, for all $\mu \in \mathbb{R}$, it is bounded from $H^\mu$ to $H^{\mu-m}$.

Symbolic calculus for paradifferential operators is summarized in the following theorem.

THEOREM A.4. (Symbolic calculus) Let $m \in \mathbb{R}$ and $\rho \geq 0$.

(i) If $a \in \Gamma_0^m(\mathbb{R}^d)$, then $T_a$ is of order $m$. Moreover, for all $\mu \in \mathbb{R}$ there exists a constant $K$ such that
\[
\|T_a\|_{H^{\mu} \to H^{\mu-m}} \leq KM_0^m(a). \tag{A.4}
\]

(ii) If $a \in \Gamma_\rho^m(\mathbb{R}^d)$, $b \in \Gamma_\rho^{m'}(\mathbb{R}^d)$ then $T_aT_b - T_{ab}$ is of order $m + m' - \rho$, where
\[
a_{\#}b := \sum_{|\alpha|<\rho} (-i)^{\alpha/\alpha!} \partial_\xi^\alpha a(x, \xi) \partial_x^\alpha b(x, \xi).
\]

Moreover, for all $\mu \in \mathbb{R}$ there exists a constant $K$ such that
\[
\|T_aT_b - T_{ab}\|_{H^{\mu} \to H^{\mu-m-m'+\rho}} \leq K(M_\rho^m(a)M_\rho^{m'}(b) + M_0^m(a)M_\rho^{m'}(b)). \tag{A.5}
\]

(iii) Let $a \in \Gamma_\rho^m(\mathbb{R}^d)$. Denote by $(T_a)^*$ the adjoint operator of $T_a$ and by $\overline{a}$ the complex conjugate of $a$. Then $(T_a)^* - T_{\overline{a}}$ is of order $m - \rho$ where
\[
a^* = \sum_{|\alpha|<\rho} \frac{1}{i|\alpha|\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{a}.
\]

Moreover, for all $\mu$ there exists a constant $K$ such that
\[
\|(T_a)^* - T_{\overline{a}}\|_{H^{\mu} \to H^{\mu-m-\rho}} \leq KM_\rho^m(a). \tag{A.6}
\]

REMARK A.5. In the definition (A.2) of paradifferential operators, the cut-off $\Psi$ removes the low frequency part of $u$. Consequently, when $a \in \Gamma_\rho^m$ we have
\[
\|T_au\|_{H^s} \leq CM_\rho^m(a)\|\nabla u\|_{H^{s+m-1}}.
\]

Next we recall several useful product and paraproduct rules.

THEOREM A.6. Let $s_0$, $s_1$ and $s_2$ be real numbers.

1. For any $s \in \mathbb{R}$,
\[
\|T_au\|_{H^s} \leq C\|a\|_{L^\infty}\|u\|_{H^s}. \tag{A.7}
\]

2. If $s_0 \leq s_2$ and $s_0 < s_1 + s_2 - \frac{d}{2}$, then
\[
\|T_au\|_{H^{s_0}} \leq C\|a\|_{H^{s_1}}\|u\|_{H^{s_2}}. \tag{A.8}
\]

3. If $s_1 + s_2 > 0$, $s_0 < s_1 + s_2 - \frac{d}{2}$ then
\[
\|au - T_au\|_{H^{s_0}} \leq C\|a\|_{H^{s_1}}\|u\|_{H^{s_2}}. \tag{A.9}
\]

4. If $s_1 + s_2 > 0$, $s_0 \leq s_1$, $s_0 \leq s_2$ and $s_0 < s_1 + s_2 - \frac{d}{2}$ then
\[
\|u_1u_2\|_{H^{s_0}} \leq C\|u_1\|_{H^{s_1}}\|u_2\|_{H^{s_2}}. \tag{A.10}
\]

THEOREM A.7. Consider $F \in C^\infty(\mathbb{C}^N)$ such that $F(0) = 0$. For $s \geq 0$, there exists a non-decreasing function $F : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $U \in H^s(\mathbb{R}^d)^N \cap L^\infty(\mathbb{R}^d)^N$,
\[
\|F(U)\|_{H^s} \leq F(\|U\|_{L^\infty})\|U\|_{H^s}. \tag{A.11}
\]

THEOREM A.8 ([10] Theorem 2.92 and [46] Theorem 5.2.4). (Paralinearization for nonlinear functions) Let $\mu$, $\tau$ be positive real numbers and let $F \in C^\infty(\mathbb{C}^N)$ be a scalar function satisfying $F(0) = 0$. If $U = (u_j)_{j=1}^N$ with $u_j \in H^\mu(\mathbb{R}^d) \cap C^\tau(\mathbb{R}^d)$ then we have
\[
F(U) = \sum_{j=1}^N T_{\partial_j F(U)}u_j + R, \tag{A.12}
\]
\[
\|R\|_{H^{\mu+\tau}} \leq F(\|U\|_{L^\infty})\|U\|_{C^\tau} \|U\|_{H^\mu}. \tag{A.13}
\]
Appendix B. Proof of Proposition 1.1

Setting \( q^\pm = p^\pm + \rho^\pm g y \) we deduce from the Darcy law (1.5) that
\[
\Delta_{x,y} q^\pm = 0, \quad \mu^\pm u^\pm = -\nabla_{x,y} q^\pm \quad \text{in} \quad \Omega^\pm. \tag{B.1}
\]

The one-phase problem. Then boundary condition (1.10) gives \( q^-|\Sigma = s H(\eta) + [\rho] g \eta \). Consequently, by the definition of \( G^-(\eta) \) we have
\[
\sqrt{1 + |\nabla \eta|^2} \nabla_{x,y} q^- \cdot n|\Sigma = -\frac{1}{\mu^-} G^-(\eta)(s H(\eta) + \rho^- g \eta)
\]
which in conjunction with (B.1) yields
\[
\sqrt{1 + |\nabla \eta|^2} u^- \cdot n|\Sigma = -\frac{1}{\mu^-} G^-(\eta)(s H(\eta) + \rho^- g \eta).
\]

Combining this and the kinematic boundary condition (1.7) we obtain equation (1.13).

The two-phase problem. Set \( f^\pm = q^\pm |\Sigma = p^\pm |\Sigma + \rho^\pm g \eta \). In view of the pressure jump condition (1.8) we have
\[
f^- - f^+ = s H(\eta) + [\rho] g \eta, \quad [\rho] = \rho^- - \rho^+
\]
which gives the first equation in (1.15). On the other hand, since
\[
G^\pm(\eta) f^\pm = \sqrt{1 + |\nabla \eta|^2} \nabla_{x,y} q^\pm \cdot n|\Sigma,
\]
(B.1) implies that
\[
\mu^\pm \sqrt{1 + |\nabla \eta|^2} u^\pm \cdot n|\Sigma = -G^\pm(\eta) f^\pm. \tag{B.2}
\]
The second equation in (1.15) thus follows from (B.2) and the continuity (1.6) of \( u \cdot n \). Finally, (1.14) is a consequence of (1.7) and (B.2).

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