THE “GHOST” SYMMETRY IN THE CKP HIERARCHY

JIPENG CHENG†, JINGSONG HE∗ ‡

† Department of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu 221116, P. R. China
‡ Department of Mathematics, Ningbo University, Ningbo, Zhejiang 315211, P. R. China

Abstract. In this paper, we systematically study the “ghost” symmetry in the CKP hierarchy through its actions on the Lax operator, dressing operator, eigenfunctions and the tau function. In this process, the spectral representation of the eigenfunction is developed and the squared eigenfunction potential is investigated.

Keywords: CKP hierarchy, squared eigenfunction symmetry, spectral representation, squared eigenfunction potential.

PACS: 02.30.Ik

2010 MSC: 35Q53, 37K10, 37K40

1. Introduction

In this paper, given any pseudo-differential operator

\[ A = \sum_{i \geq 0} a_i \partial^i + \sum_{i < 0} a_i \partial^i, \]

\[ \text{Res}(\sum_{i} a_i \partial^i) = a_{-1}, \quad (\sum_{i} a_i \partial^i)[k] = a_k, \quad A^* = \sum_{i} (-\partial)^i a_i, \]

and \( A(f) \) denotes the action of \( A \) on \( f \).

The Kadomtsev-Petviashvili (KP) hierarchy [1] is an important research object in the area of mathematical physics, which is defined by the following Lax equation

\[ \frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad n = 1, 2, 3, \cdots, \]

with the Lax operator \( L \) given by

\[ L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \cdots, \]

where the coefficient functions \( u_i \) are all the functions of the time variables \( t = (t_1 = x, t_2, t_3, \cdots) \).

The Lax operator (3) can be generated by the dressing operator \( \Phi = 1 + \sum_{k=1}^{\infty} a_k \partial^{-k} \) in the following way:

\[ L = \Phi \partial \Phi^{-1} \]

Then the Lax equation (4) can also be expressed as Sato’s equation

\[ \frac{\partial \Phi}{\partial t_n} = -(L^n)_- \Phi, \quad n = 1, 2, 3, \cdots. \]

*Corresponding author. Email: hejingsong@nbu.edu.cn.
Another important object is the Baker-Akhiezer (BA) wave function $\psi_{BA}(t, \lambda)$ defined via:

$$\psi_{BA}(t, \lambda) = \Phi(\xi(t, \lambda)) = \phi(t, \lambda)e^{\xi(t, \lambda)}$$

(6)

with $\xi(t, \lambda) \equiv \sum_{k=1}^{\infty} t_k \lambda^k$ and $\phi(t, \lambda) = 1 + \sum_{i=1}^{\infty} a_i(t) \lambda^{-i}$, which satisfies

$$L(\psi_{BA}(t, \lambda)) = \lambda \psi_{BA}(t, \lambda), \quad \partial_n \psi_{BA}(t, \lambda) = (L^n)_{+}(\psi_{BA}(t, \lambda)), \quad n = 1, 2, 3, \cdots.$$  

(7)

The adjoint BA function $\psi_{BA}^*(t, \lambda)$ is introduced through the following way

$$\psi_{BA}^*(t, \lambda) = \Phi^* - 1(e^{-\xi(t, \lambda)}).$$

(8)

The KP hierarchy can also be expressed in terms of a single function called the tau function $\tau(t)$ \cite{1}, which is related with the wave function in the way below,

$$\psi_{BA}(t, \lambda) = \tau(t_1, t_2 - \frac{1}{2}\lambda, t_3 - \frac{1}{3}\lambda, \cdots) \tau(t_1, t_2, t_3, \cdots) e^{\xi(t, \lambda)}.$$  

(9)

Because of the existence of the tau function, many important results in the KP hierarchy can be considered in terms of the tau function, such as the flow equation, Hirota's bilinear equation and algebraic constraint. KP hierarchy has two famous sub-hierarchies \cite{1,2}: the BKP hierarchy and the CKP hierarchy. Just like the KP hierarchy, the BKP hierarchy also owns one single tau function, which bring much convenience to the study of the BKP hierarchy.

The CKP hierarchy \cite{2} is a reduction of the KP hierarchy through the constraint on $L$ given by (3) as

$$L^* = -L,$$

(10)

then $L$ is called the Lax operator of the CKP hierarchy, and the associated Lax equation of the CKP hierarchy is

$$\frac{\partial L}{\partial t_n} = [(L^n)_{+}, L], \quad n = 1, 3, 5, \cdots,$$

(11)

which compresses all even flows, i.e., the Lax equation of the CKP hierarchy has only odd flows. The CKP constraint (10) on the correspond dressing operator $\Phi$ will be $\Phi^* = \Phi^{-1}$. And thus in the CKP hierarchy $\psi_{BA}^*(t, \lambda) = \Phi^* - 1(e^{-\xi(t, \lambda)}) = \Phi(e^{-\xi(t, \lambda)}) = \psi_{BA}(t, -\lambda)$. So in the CKP hierarchy, it is enough to only study the wave function $\psi_{BA}(t, \lambda)$. The CKP hierarchy (11) is equivalent to the following bilinear equation:

$$\int d\lambda \psi_{BA}(t, \lambda)\psi_{BA}(t', -\lambda) = 0,$$

(12)

where $\int d\lambda \equiv \oint_{\infty} d\lambda = \text{Res}_{\lambda=\infty}$. By now, the CKP hierarchy has attracted many researches \cite{3,4,5,6,7,8,9,10,11,12,13}.

In contrast to the KP and the BKP cases, there seems not a single tau function to describe the CKP hierarchy in the form of Hirota bilinear equations (that is, Hirota’s equations are no longer of the type $P(D)\tau \cdot \tau = 0$) \cite{2}. The existence of this kind of tau function for the CKP hierarchy is a long-standing problem. Much work has been done in this field \cite{3,4}. Since the existence of tau function of the CKP hierarchy is not proved, many important results on the Lax operator and dressing operator of the
CKP hierarchy can not be transferred to the tau function, until Chang and Wu [6] introduces a kind of tau function $\tau_c(t)$ for the CKP case, which is related with the wave function in the following way

$$
\psi_{BA}(t, \lambda) = \sqrt{\varphi(t, \lambda)} \frac{\tau_c(t - 2[\lambda^{-1}])}{\tau_c(t)} e^{\xi(t, \lambda)} \tau_c(t - 2[\lambda^{-1}]) e^{\xi(t, \lambda)},
$$

where $[\lambda^{-1}] = (\lambda^{-1}, \frac{1}{3} \lambda^{-3}, \cdots)$ and $\varphi(t, \lambda) = \phi(t, \lambda) \phi(t - 2[\lambda^{-1}], -\lambda)$. Note that the relation between the tau function and the wave function is different from the cases of KP and BKP, because there is a square-root factor depending on the tau function. And the relation of the new CKP tau function $\tau_c$ to the (C-reduced) KP tau function $\tau$ is showed in the Appendix A.

The “ghost” symmetry [14–18], sometimes called the squared eigenfunction symmetry [14], is one of the most important symmetries in the integrable system, which is defined through the squared eigenfunctions [14]. The usage of the squared eigenfunction to construct the symmetry flows can be traced back to [19–21], where operators $\psi \partial^{-1} \psi^*$ were introduced to construct $L - A$ pairs for symmetry flows ($\psi$ and $\psi^*$ being wave functions respectively of $L$ and $L^*$ operators). And similar symmetry flows are also studied from the Hamiltonian point of view [22]. The “ghost” symmetry can be used to define the new integrable system, such as the symmetry constraint [15, 17, 23–28] and the extended integrable systems [29, 30], and investigate the additional symmetry [10, 13, 31–35]. The “ghost” symmetry has attracted many researches recently. For example, recently the “ghost” symmetries for the BKP hierarchy [36], discrete KP hierarchy [37], the Toda lattice hierarchy [38] and its B type and C type cases [39] are all studied.

In this paper, firstly starting from the bilinear identity for the CKP hierarchy, the spectral representation for the eigenfunction is established. Upon the basis of the spectral representation, the expression of the squared eigenfunction potential (SEP) for the eigenfunction and the wave function is derived, and further all other SEPs are also obtained. Then the “ghost” symmetry of the CKP hierarchy is constructed by its action on the Lax operator and the dressing operator. At last, since the existence of the tau function $\tau_c(t)$, the action of the “ghost” symmetry is transferred to the tau function $\tau_c(t)$, which has never been studied before.

This paper is organized in the following way. In Section 2, the SEP for the CKP hierarchy is investigated. In Section 3, we study the “ghost” symmetry in the CKP hierarchy. At last, some conclusions and discussions are given in section 4.

## 2. SEP FOR THE CKP HIERARCHY

If the functions $q(t)$ and $r(t)$ satisfy

$$
\frac{\partial q}{\partial t_n} = L^+_n(q), \quad \frac{\partial r}{\partial t_n} = -(L^n)^+_n(r), \quad n = 1, 3, 5, \cdots, \tag{14}
$$

where $\partial t_n = \partial / \partial t_n$, $L_n^+$ is the Lax operator, and $\partial / \partial t_n$ is the dressing operator.
then they are called the eigenfunction and the adjoint eigenfunction of the CKP hierarchy respectively. Obviously according to (7), \( \psi_{BA}(t,\lambda) \) is the eigenfunction. By the CKP constraint (10), \( L^+_n = -(L^n)^*_+ \) for \( n \in \mathbb{Z}_{+}^{\text{odd}} \). Thus any adjoint eigenfunction \( r(t) \) can be viewed as the eigenfunction and vice versa.

For an arbitrary pair of eigenfunctions \( q_1(t) \) and \( q_2(t) \) of the CKP hierarchy, there exists the function \( S(q_1(t), q_2(t)) \), called the squared eigenfunction potential (SEP) \([14]\), which is determined by the following characteristics:

\[
\frac{\partial}{\partial t_n} S(q_1(t), q_2(t)) = \text{Res} \left( \partial^{-1} q_2(t) L^+_n q_1(t) \partial^{-1} \right), n = 1, 3, 5, \ldots \tag{15}
\]

These flows are compatible \([14]\), namely,

\[
\frac{\partial}{\partial t_n} \frac{\partial}{\partial t_m} S(q_1, q_2) = \frac{\partial}{\partial t_m} \frac{\partial}{\partial t_n} S(q_1, q_2), \quad m, n = 1, 3, 5, \ldots \tag{16}
\]

Thus this definition is reasonable. The predecessor of SEP is in fact the Cauchy-Baker-Akhiezer kernel introduced in \([40]\). Note that \( S(q_1(t), q_2(t)) \) can be up to a constant. In particular, for \( n = 1 \)

\[
\partial_x S(q_1(t), q_2(t)) = q_1(t)q_2(t). \tag{17}
\]

There are two important properties showed below.

**Lemma 1.** If \( q_1 \) and \( q_2 \) are two eigenfunctions of the CKP hierarchy, then

\[
S(q_1, q_2) = S(q_2, q_1) \tag{18}
\]

**Proof.** Assume \( n \in \mathbb{Z}_{+}^{\text{odd}} \), then

\[
\frac{\partial}{\partial t_n} S(q_1, q_2) = \text{Res} \left( \partial^{-1} q_2 L^+_n q_1 \partial^{-1} \right) = \text{Res} \left( \partial^{-1} q_1 L^n q_2 \partial^{-1} \right) = \text{Res} \left( \partial^{-1} q_1 L^+_n q_2 \partial^{-1} \right) = \frac{\partial}{\partial t_n} S(q_2, q_1),
\]

where in the second identity, the relation \( \text{Res}A = -\text{Res}A^* \) is used. And in the third identity, we have used the relation \( L^+_n = -(L^n)^*_+ \) derived by the CKP constraint \([10]\). \( \square \)

**Lemma 2.** Assume \( q(t) \) is the eigenfunction of the CKP hierarchy, then

\[
S(q(t), \psi_{BA}(t, \lambda)) = e^{\xi(t,\lambda)} \left( q(t)\lambda^{-1} + \mathcal{O}(\lambda^{-2}) \right) \tag{19}
\]

**Proof.** It can be proved in the same way as Lemma 3 in \([36]\). \( \square \)

Then the relation of the eigenfunction and the wave function can be found in the following proposition.
Proposition 3. For any eigenfunction $q(t)$ of the CKP hierarchy,

$$q(t) = - \int d\lambda \psi_{BA}(t, \lambda) S(q(t'), \psi_{BA}(t', -\lambda)).$$

(20)

In other words, $q(t)$ owns a spectral representation in the following form

$$q(t) = \int d\lambda \rho(\lambda) \psi_{BA}(t, \lambda),$$

(21)

with spectral densities given by $\rho(\lambda) = -S(q(t'), \psi_{BA}(t', -\lambda))$.

Proof. Denote the right hand side of (20) as $I(t, t')$. Then according to the bilinear identity (12) of the CKP hierarchy, one can find that $\partial_{t'} I(t, t') = 0$ for $m \in \mathbb{Z}_{+}^{\text{odd}}$. Thus $I(t, t') = f(t)$. By considering (19),

$$I(t, t' = t) = \int d\lambda \psi_{BA}(t, \lambda) e^{-\xi(t, \lambda)}(q(t)\lambda^{-1} + O(\lambda^{-2})) = q(t)$$

□

Remark 1: From (8) and (20), we can know that

$$q(t) = - \int d\lambda \psi_{BA}(t, \lambda) S(q(t'), \psi_{BA}^\ast(t', \lambda),)$$

(22)

which is consistent with the ordinary KP hierarchy [18].

Remark 2: Since any adjoint eigenfunction can be viewed as the eigenfunction in the CKP case, the spectral representation of the adjoint eigenfunction $r(t)$ can be obtained through (20),

$$r(t) = - \int d\lambda \psi_{BA}(t, \lambda) S(r(t'), \psi_{BA}(t', -\lambda))$$

$$= \int d\lambda \psi_{BA}(t, -\lambda) S(r(t'), \psi_{BA}(t', \lambda)) \text{ letting } \lambda \rightarrow -\lambda$$

$$= \int d\lambda \psi_{BA}^\ast(t, \lambda) S(\psi_{BA}(t', \lambda), r(t')) \text{ using (8) and (18),}$$

(23)

which is also consistent with the case of the ordinary KP hierarchy [18].

Because of (19), we can represent $S(q(t), \psi_{BA}(t, -\lambda))$ as $K(t, \lambda)e^{-\xi(t, \lambda)}$ (with $K(t, \lambda) = -q(t)\lambda^{-1} + O(\lambda^{-2})$). Then by exchanging $t$ and $t'$ and letting $t' = t + 2[k^{-1}]$, the relation (20) becomes,

$$q(t + 2[k^{-1}])$$

$$= - \int d\lambda \sqrt{\varphi(t + 2[k^{-1}], \lambda)} \frac{\tau_c(t + 2[k^{-1}] - 2[\lambda^{-1}])}{\tau_c(t + 2[k^{-1}] - 2[\lambda^{-1}])} K(t, \lambda) \left(-1 + \frac{2}{1 - \lambda/k}\right)$$

$$= -q(t) - 2k \sqrt{\varphi(t + 2[k^{-1}], k)} \frac{\tau_c(t)}{\tau_c(t + 2[k^{-1}] - 2[\lambda^{-1}])} K(t, k)$$

which leads to

$$K(t, k) = \frac{-1}{2k \sqrt{\varphi(t, -k)}} (q(t + 2[k^{-1}]) + q(t)) \frac{\tau_c(t + 2[k^{-1}])}{\tau_c(t)},$$

(24)
where we have used the relations \( \varphi(t, \lambda) = 1 + \mathcal{O}(\lambda^{-2}) \) and \( \varphi(t + 2[k^{-1}], k) = \varphi(t, -k) \) derived by the definition of \( \varphi(t, k) \). Thus

\[
S(q(t), \psi_{BA}(t, -\lambda)) = \frac{-1}{2\lambda\varphi(t, -\lambda)} (q(t + 2[\lambda^{-1}]) + q(t)) \psi_{BA}(t, -\lambda)
\]  
(25)

Further, all the expressions of the squared eigenfunction potential can be obtained in the following proposition.

**Proposition 4.** If \( q(t), q_1(t) \) and \( q_2(t) \) are two eigenfunctions of the CKP hierarchy, then

\[
S(\psi_{BA}(t, \mu), \psi_{BA}(t, \lambda)) = \frac{1}{2\lambda\varphi(t, \lambda)} (\psi_{BA}(t - 2[\lambda^{-1}], \mu) + \psi_{BA}(t, \mu)) \psi_{BA}(t, \lambda),
\]

\[
S(q(t), \psi_{BA}(t, \lambda)) = \frac{1}{2\lambda\varphi(t, \lambda)} (q(t + 2[\lambda^{-1}]) + q(t)) \psi_{BA}(t, \lambda),
\]

\[
S(q_1(t), q_2(t)) = \int \int d\lambda d\mu \rho_1(\mu) \rho_2(\lambda) S(\psi_{BA}(t, \mu), \psi_{BA}(t, \lambda)) .
\]

3. The “ghost” symmetry of the CKP hierarchy

Given a set of eigenfunctions \( \{q_{1i}, q_{2i}\}_{i \in \alpha} \), the “ghost” flow of the CKP hierarchy is defined through its actions on \( L \) and \( \Phi \) as follows,

\[
\partial_\alpha L \equiv \{\sum_{i \in \alpha} (q_{1i} \partial^{-1} q_{2i} + q_{2i} \partial^{-1} q_{1i}), L\}, \quad \partial_\alpha \Phi \equiv \sum_{i \in \alpha} (q_{1i} \partial^{-1} q_{2i} + q_{2i} \partial^{-1} q_{1i}) \Phi
\]

and the corresponding action on the eigenfunction \( q(t) \) is

\[
\partial_\alpha q = \sum_{i \in \alpha} (q_{1i} S(q_{2i}, q) + q_{2i} S(q_{1i}, q)) .
\]

The examples of the “ghost” flows can be found in Appendix 3.

Next, we need to check the consistence of \( \partial_\alpha \) with the CKP constraint (10), and \([\partial_\alpha, \partial_{t_n}] = 0\), that is to say, show \( \partial_\alpha \) is indeed the symmetry of the CKP hierarchy.

**Proposition 5.** \( \partial_\alpha \) is consistent with the CKP constraint (10), that is, \((\partial_\alpha L)^* + \partial_\alpha L = 0\).

**Proof.** Firstly, denote \( A = \sum_{i \in \alpha} (q_{1i} \partial^{-1} q_{2i} + q_{2i} \partial^{-1} q_{1i}) \) for convenience. Then by (10)

\[
(\partial_\alpha L)^* + \partial_\alpha L = [A, L]^* + [A, L] = -[A^*, L^*] + [A, L] = [A^* + A, L] = 0,
\]

where \( A^* + A = 0 \) is obvious. \( \square \)

**Proposition 6.**

\[
[\partial_\alpha, \partial_{t_n}] = 0, \quad n \in \mathbb{Z}^{\text{odd}}
\]

**Proof.** Firstly, by (11) and (29)

\[
[\partial_\alpha, \partial_{t_n}] L = \partial_\alpha [L^n_+, L] - \partial_{t_n} [A, L] = [[A, L^n_+], L] + [L^n_+, [A, L]] - [\partial_{t_n} A, L] - [A, [L^n_+, L]]
\]
\[ [A, L^n]_+ - [A, L^n_+]_+ - \partial_{t_n} A, \]

where we have used the Jacobi relation in the third identity and \([A, L^n]_+ = [A, L^n_+]_+\) in the fourth identity. Thus it is only to show

\[ \partial_{t_n} A = -[A, L^n_+]_. \] (32)

In fact, according to (14)

\[ [A, L^n_+]_+ = \sum_{i\in\alpha} (q_{1i} \partial^{-1} q_{2i} + q_{2i} \partial^{-1} q_{1i}) L^n_+ \] \[ \left( \sum_{i\in\alpha} (q_{1i} \partial^{-1} q_{2i} + q_{2i} \partial^{-1} q_{1i}) L^n_+ \right) - \sum_{i\in\alpha} \left( L^n_+ (q_{1i}) \partial^{-1} q_{2i} + L^n_+ (q_{2i}) \partial^{-1} q_{1i} \right) \]

\[ = -\partial_{t_n} A, \]

where the following relations have been applied

\[ (F \partial^{-1})_- = F[0] \partial^{-1}, \quad (\partial^{-1} F)_- = (F^*)[0] \partial^{-1}, \] (33)

with \( F \) a pseudo differential operator.

From the propositions above, we can see that the squared eigenfunction flow \( \partial_\alpha \) is indeed a kind of symmetry for the CKP hierarchy. Next, let’s investigate the action of \( \partial_\alpha \) on the tau function. Before this, the lemma [6] below is needed.

**Lemma 7.**

\[ \text{Res} \Phi = a_1(t) = -2\partial_x \log \tau_c(t). \] (34)

**Proof.** From (6) and (13), we can obtain

\[ \phi(t, \lambda) = \sqrt{\varphi(t, \lambda)} \frac{\tau_c(t - 2[\lambda^{-1}])}{\tau_c(t)}. \] (35)

By noting \( \sqrt{\varphi(t, \lambda)} = 1 + O(\lambda^{-2}) \) and \( \phi(t, \lambda) = 1 + \sum_{i=1}^{\infty} a_i(t) \lambda^{-i} \), the comparison of coefficients of \( \lambda^{-1} \) for the both side of (35) will lead to (34).

**Proposition 8.**

\[ \partial_\alpha \tau(t) = -\sum_{i\in\alpha} S(q_{1i}(t), q_{2i}(t)) \tau(t). \] (36)
**Proof.** By taking the residue for the both sides of \( \partial_{\alpha} \Phi = \sum_{i \in \alpha} (q_{1i} \partial^{-1} q_{2i} + q_{2i} \partial^{-1} q_{1i}) \Phi \), one can obtain

\[
\partial_{\alpha} a_1(t) = 2 \sum_{i \in \alpha} q_{1i} q_{2i}.
\]

(37)

Then the applications of (34) and (17) will lead to (36).

We conclude this section with two Remarks.

**Remark 3:** Since the generating function of the additional symmetries for the CKP hierarchy is represented as [10]

\[
Y(\lambda, \mu) = \psi_{BA}(t, -\lambda) \partial^{-1} \psi_{BA}(t, \mu) + \psi_{BA}(t, \mu) \partial^{-1} \psi_{BA}(t, -\lambda),
\]

(38)

thus the “ghost” symmetry generated by \( \psi_{BA}(t, \mu) \) and \( \psi_{BA}(t, -\lambda) \) can be viewed as the generating function of the additional symmetries.

**Remark 4:** The constrained CKP hierarchy [7] is just to identify \( \partial_{\alpha} \) with \( -\partial_{2k+1} \), i.e.

\[
(L_{2k+1})_\sigma = \sum_{i \in \alpha} (q_{1i} \partial^{-1} q_{2i} + q_{2i} \partial^{-1} q_{1i}),
\]

(39)

or

\[
\partial_{2k+1} \tau(t) = \sum_{i \in \alpha} S(q_{1i}(t), q_{2i}(t)) \tau(t).
\]

(40)

This observation provides a simple mathematical explanation of the symmetry constraint of the CKP hierarchy. In Appendix [13] we investigate the example of \( \partial_{\alpha} = -\partial_x \). Besides the above reduction of the “ghost” symmetry to 1 + 1 dimensional equations, there is another important reduction from rational symmetry [42]. And the relation of these two approaches was considered in [22].

## 4. Conclusion and Discussion

In this paper, in order to get the expression of the SEPs, we start from the bilinear identity of the CKP hierarchy, and establish the spectral representation of the eigenfunction in Proposition 3. The expression of the SEP for the eigenfunction and the wave function is derived from the spectral representation. And further all the other expressions in Proposition 4. Then the “ghost” symmetry in the CKP hierarchy is constructed by its action on the Lax operator and the dressing operator (see (29)). At last, the corresponding action on the tau function is obtained in Proposition 7. The possible applications of the “ghost” symmetry in the additional symmetry and the symmetry constraint are also discussed in Remark 3 and 4.

Though the existence of a sole tau function for the CKP hierarchy is obtained, there are still many results on the Lax operator and the dressing operator which can not be shifted to the new tau function because of the complicated square root in eq. (13). For example, the algebraic constraints [6] on the new tau function are not thoroughly solved and worthy of further study. In Appendix [A] the relation of the new CKP tau function \( \tau_c \) to the (C-reduced) KP tau function \( \tau \) is obtained (see (47)).
In Appendix [13] the mKdV equation arises in the constrained CKP hierarchy [7, 20] (see (63) and (64)) when considering the reduction of $\partial_\alpha = -\partial_x$. Since the mKdV equation is the well-studied object, it will be very interesting to compare with the results about mKdV equation in the research of the “ghost” symmetry for the CKP hierarchy. And further reduction may lead to the Newton’s equation for a pair of particles [11]. These problems are very interesting in the study of the CKP hierarchy, and we will consider these problems in the later paper.

**Appendix A. The Relation between $\tau_c$ and $\tau$**

There are two kinds of tau functions for the CKP hierarchy: one is $\tau$ inherited from the KP hierarchy, the other is $\tau_c$ introduced by Chang and Wu in [9]. These two tau functions relate the wave function $\psi_{BA}(t, \lambda)$ of the CKP hierarchy in the following way,

$$\psi_{BA}(t, \lambda) = \left(1 + \frac{1}{z} \partial_z \log \frac{\tau_c(t - 2[\lambda^{-1}])}{\tau_c(t)}\right)^{1/2} \frac{\tau_c(t - 2[\lambda^{-1}])}{\tau_c(t)} e^{\xi(t, \lambda)}$$

$$= \frac{\tau(t_1 - \frac{1}{3\lambda}, -\frac{1}{2\lambda^2} t_3 - \frac{1}{3\lambda^3}, \cdots)}{\tau(t_1, 0, t_3, \cdots)} e^{\xi(t, \lambda)} \quad \text{(41)}$$

Denote $g(t) = \log \tau_c(t) = \log \tau_c(t_1, t_3, t_5, \cdots)$ and $f(t; \lambda) = \log \tau(t_1, -\frac{\lambda^2}{2}, t_3, -\frac{\lambda^4}{4}, t_5, \cdots)$, then by [11],

$$\frac{1}{2} \log \left(1 + \lambda \partial_x g(t_1 - 2\lambda, t_3 - \frac{2}{3}\lambda^3, \cdots) - \lambda \partial_x g(t)\right) + g(t_1 - 2\lambda, t_3 - \frac{2}{3}\lambda^3, \cdots) - g(t)$$

$$= f(t_1 - \lambda, t_3 - \frac{1}{3}\lambda^3, \cdots; \lambda) - f(t; 0). \quad \text{(42)}$$

Next if let $t_j \rightarrow t_j + \frac{\lambda^i}{j}$, eq. (42) will become into

$$\frac{1}{2} \log \left(1 + \lambda \partial_x g(t_1 - \lambda, t_3 - \frac{\lambda^3}{3}, \cdots) - \lambda \partial_x g(t_1 + \lambda, t_3 + \frac{\lambda^3}{3}, \cdots)\right)$$

$$+ g(t_1 - \lambda, t_3 - \frac{\lambda^3}{3}, \cdots) - g(t_1 + \lambda, t_3 + \frac{\lambda^3}{3}, \cdots)$$

$$= f(t; \lambda) - f(t_1 + \lambda, t_3 + \frac{1}{3}\lambda^3, \cdots; 0). \quad \text{(43)}$$

Further if let $\lambda \rightarrow -\lambda$, then eq. (43) is transferred to

$$\frac{1}{2} \log \left(1 - \lambda \partial_x g(t_1 + \lambda, t_3 + \frac{\lambda^3}{3}, \cdots) + \lambda \partial_x g(t_1 - \lambda, t_3 - \frac{\lambda^3}{3}, \cdots)\right)$$

$$+ g(t_1 + \lambda, t_3 + \frac{\lambda^3}{3}, \cdots) - g(t_1 - \lambda, t_3 - \frac{\lambda^3}{3}, \cdots)$$

$$= f(t; \lambda) - f(t_1 - \lambda, t_3 - \frac{1}{3}\lambda^3, \cdots; 0). \quad \text{(44)}$$
where note that \( f(t; \lambda) = f(t; -\lambda) \).

At last the subtraction of (44) from (43) will lead to

\[
2 \left( g(t_1 - \lambda, t_3 - \frac{\lambda^3}{3}, \cdots) - g(t_1 + \lambda, t_3 + \frac{\lambda^3}{3}, \cdots) \right) \\
= f(t_1 - \lambda, t_3 - \frac{1}{3} \lambda^3, \cdots; 0) - f(t_1 + \lambda, t_3 + \frac{1}{3} \lambda^3, \cdots; 0).
\]

(45)

From (45), we can know that

\[
2g(t) = f(t; 0) + \text{const}
\]

(46)

Therefore

\[
\tau_c^2(t_1, t_3, \cdots) = \text{const} \cdot \tau(t_1, 0, t_3, 0, t_5, \cdots).
\]

(47)

**APPENDIX B. EXAMPLES OF THE “GHOST” FLOWS**

Here we list some examples of the “ghost” flows \( \partial_\alpha \) generated by the eigenfunctions \( q_1 \) and \( q_2 \) for the CKP hierarchy, that is,

\[
\partial_\alpha L = [q_1 \partial^{-1} q_2 + q_2 \partial^{-1} q_1, L],
\]

(48)

where \( L \) is the Lax operator of the CKP hierarchy given by (3), and satisfies the CKP constraint (10).

The actions of \( \partial_\alpha \) on the the first few terms of the Lax operator \( L \) are showed below.

\[
\begin{align*}
\partial_\alpha u_1 &= -2(q_1 q_2)_x, \\
\partial_\alpha u_2 &= (q_1 q_2)_{xx}, \\
\partial_\alpha u_3 &= 2(q_1 q_2)_x u_1 - 2q_1 q_2 u_{1x} - (q_1 q_{2xx} + q_2 q_{1xx})_x, \\
\partial_\alpha u_4 &= 4(q_1 q_2)_x u_2 - 2q_1 q_2 u_{2x} - 3(q_2 q_{1xx} + q_1 q_{2xx}) u_1 \\
&\quad - 6q_1 q_2 u_{1x} + 2(q_1 q_2 u_{1x})_x + (q_1 q_{2xxx} + q_2 q_{1xxx})_x, \\
\partial_\alpha u_5 &= (q_1 q_2)_x (6 u_3 + 2 u_{2x} - 3 u_{1xx}) - (q_2 q_{1xx} + q_1 q_{2xx})(8 u_2 + 3 u_{1x}) \\
&\quad - 16q_1 q_2 u_2 + 2q_1 q_2 (u_{2xx} - u_{3x}) + 4(q_2 q_{1xxx} + q_1 q_{2xxx}) u_1 \\
&\quad + 10(q_1 q_2)_{xx} u_1 - (q_2 q_{1xxxx} + q_1 q_{2xxxx})_x.
\end{align*}
\]

(53)

Note that the CKP constraint (10) is equivalent to

\[
\begin{align*}
u_2 &= -\frac{1}{2} u_{1x}, \\
u_4 &= \frac{1}{4} u_{1xxx} - \frac{3}{2} u_{3x}, \\
u_6 &= -\frac{1}{2} u_{1xxxx} + \frac{5}{2} u_{3xx} - \frac{5}{2} u_{5x}, \\
&\vdots
\end{align*}
\]

(54-56)
Using these examples, we can find that $\partial_\alpha$ is consistent with the CKP constraint (10) for the first few terms. In fact, by (49)-(52) and (54)-(55),

$$\partial_\alpha u_2 = -\frac{1}{2} (\partial_\alpha u_1)_x,$$

$$\partial_\alpha u_4 = \frac{1}{4} (\partial_\alpha u_1)_{xxx} - \frac{3}{2} (\partial_\alpha u_3)_x. \quad (57)$$

According to (14) and (54) and (55)

$$\partial_t q_i = q_{ixxx} + 3u_1 q_{ix} + \frac{3}{2} u_{1x} q_i, \quad (58)$$

$$\partial_t q_{1i} = q_{ixxxx} + 5u_1 q_{ixxx} + \frac{15}{2} u_{1x} q_{ixx} + (5u_3 + 10u_1^2 + 5u_{1xx}) q_{ix} + \left(\frac{5}{4} u_{1xxx} + \frac{5}{2} u_{3x} + 10u_1 u_{1x}\right) q_i, \quad i = 1, 2 \quad (59)$$

If letting $\partial_\alpha = -\partial_x$, then from (49) (51) (59) and (60), one can get

$$\partial_t q_1 = q_{1xxx} + 9q_1 q_2 q_{1x} + 3q_1^2 q_2, \quad (61)$$

and

$$\partial_t q_2 = q_{2xxx} + 9q_1 q_2 q_{2x} + 3q_2^2 q_1. \quad (62)$$

Let $q_1 = q_2 = q$. (61) implies the mKdV equation

$$\partial_t q = q_{xxx} + 12q^2 q_x, \quad (63)$$

and (61) leads to the 5th order mKdV equation

$$\partial_t q = q_{xxxxx} + 20q^2 q_{xxx} + 80q_{xx} q_x q + 20q_{2x}^2 + 120q^4 q_x. \quad (64)$$

Note that the mKdV and the 5th order mKdV equations are also reduced from 1-constrained CKP hierarchy through the third flow and the fifth flow respectively [43].

Acknowledgments

This work is supported by the NSFC (Grant Nos. 11301526 and 11371361) and the Fundamental Research Funds for the Central Universities (Grant No. 2012QNA45). We thank anonymous referee for his/her useful suggestions on appendices and several references.
References

[1] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Transformation groups for soliton equations, in Nonlinear integrable systems - classical theory and quantum theory, ed. by M. Jimbo and T. Miwa, (World Scientific, Singapore, 1983)pp.39-119.

[2] E. Date, M. Jimbo, M. Kashiwara, T. Miwa. Transformation groups for soliton equations. VI. KP hierarchies of orthogonal and symplectic type. J. Phys. Soc. Japan 50 (1981) 3813-3818.

[3] H. Aratyn and J. Van de Leur. The CKP hierarchy and the WDVV prepotential, in Bilinear integrable systems: from classical to quantum, continuous to discrete, NATO Science Series 201 (2006) 1-11. Also see arXiv: nlin/0302004

[4] H. Aratyn and J. Van de Leur. The symplectic Kadomtsev-Petviashvili hierarchy and rational solutions of Painleve VI. arXiv: nlin/0406038

[5] J. Van de Leur, A. Yu. Orlov and T. Shiota. CKP hierarchy, Bosonic Tau function and Bosonization Formulae. SIGMA 8 (2012) 036.

[6] L. Chang and C. Z. Wu. Tau function of the CKP hierarchy and non-linearizable Virasoro symmetries. Nonlinearity 26 (2013) 2577-2596 (arXiv:1208.1374).

[7] I. Loris. On reduced CKP equations. Inverse Problems 15(1999) 1099-1109.

[8] I. Loris. Dimensional reductions of BKP and CKP hierarchies. J. Phys. A: Math. Gen. 34 (2001) 3447-3459.

[9] J. S. He, Y. Cheng and R. A. Roemer. Solving bi-directional soliton equations in the KP hierarchy by gauge transformation. J. High Energy Phys. 03(2006) 103. arXiv:math-ph/0503028

[10] J. S. He, K. L. Tian, A. Foerster and W. X. Ma. Additional symmetries and string equation of the CKP hierarchy. Lett. Math. Phys. 81 (2007) 119-134.

[11] J. S. He, Z. W. Wu and Y. Cheng. Gauge transformations for the constrained CKP and BKP hierarchies. J. Math. Phys. 48 (2007) 113519.

[12] J. S. He and X. D. Li. Solutions of the (2 + 1)-dimensional KP, SK and KK equations generalized by gauge transformations from nonzero seeds. J. Nonlin. Math. Phys. 2 (2009) 179-194.

[13] K. L. Tian, J. S. He, J. P. Cheng and Y. Cheng. Additional symmetries of constrained CKP and BKP hierarchies. Sci. China Math. 54 (2011) 257-268.

[14] W. Oevel. Darboux theorems and Wronskian formulas for integrable system I: constrained KP flows. Physica A 195 (1993) 533-576.

[15] W. Oevel and W. Schief. Squared eigenfunctions of the (modified) KP hierarchy and scattering problems of Loewner Type. Rev. Math. Phys 6 (1994) 1301 - 1308.

[16] W. Oevel and S. Carillo. Squared eigenfunction symmetries for soliton equations: Part I. J. Math. Anal. Appl. 217 (1998) 161-178.

[17] W. Oevel and S. Carillo. Squared eigenfunction symmetries for soliton equations: Part II. J. Math. Anal. Appl. 217 (1998) 179-199.

[18] H. Aratyn, E. Nissimov and S. Pacheva. Method of squared eigenfunction potentials in integrable hierarchies of KP type. Comm. Math. Phys. 193 (1998) 493-525.

[19] A. Yu. Orlov. Vertex Operators, ∂ Problem, Symmetries, Hamiltonian and Lagrangian Formalism of (2 + 1) Dimensional Integrable Systems, in Plasma Theory and Nonlinear and Turbulent Processes in Physics, Proc. III Kiev. Intern. Workshop (1987), ed. by V.G.Bar’yakhtar, V.E.Zakharov, (World Scientific, Singapore, 1988) vol. I, pp. 116–134.

[20] A. Yu. Orlov. Symmetries for unifying different soliton systems into a single integrable hierarchy, preprint IINS/Oce-04/03, March 1991; the report at the NEEDS workshop, Gallipoli 1991.
[21] A. Yu. Orlov. Volterra Operator Algebra for Zero Curvature Representation. University of KP, in *Nonlinear Processes in Physics, Proceedings of the III Potsdam V Kiev Workshop held 1-11 August, 1991 at Clarkson University, Potsdam, NY.*, ed. by A.S. Fokas, D.J. Kaup, A.C. Newell, and V.E. Zakharov, (Springer, Berlin, 1993) pp.126-131.

[22] B. Enriquez, A. Yu. Orlov and V. N. Rubtsov. Dispersionful analogues of Benney’s equations and N-wave systems. Inverse Problems 12 (1996) 241-250.

[23] Y. Cheng and Y. S. Li. The constraint of the KP equation and its special solutions. Phys. Lett. A 157 (1991) 22-26.

[24] J. Sidorenko and W. Strampp. Symmetry constraints of the KP hierarchy. Inverse Problems 7(1991) L37-L43.

[25] B. Konopelchenko and W. Strampp. New reductions of the Kadomtsev-Petviashvili and two dimensional Toda lattice hierarchies via symmetry constraints. J. Math. Phys. 33(1992), 3676-3686.

[26] Y. Cheng. Constraints of the KP hierarchy. J. Math. Phys. 33(1992) 3774-3782.

[27] I. Loris and R. Willox. Symmetry reductions of the BKP hierarchy. J. Math. Phys. 40(1999) 1420-1431.

[28] H. F. Shen and M. H. Tu. On the constrained B-type Kadomtsev-Petviashvili hierarchy: Hirota bilinear equations and Virasoro symmetry. J. Math. Phys. 52(2011) 032704.

[29] X. J. Liu, Y. B. Zeng, and R. L. Lin. A new extended KP hierarchy. Phys. Lett. A 372 (2008)3819-3823.

[30] X. J. Liu, Y. B. Zeng and R. L. Lin. An extended two-dimensional Toda lattice hierarchy and two-dimensional Toda lattice with self-consistent sources. J. Math. Phys. 49 (2008) 093506.

[31] A. Yu. Orlov and E. I. Schulman. Additional symmetries for integrable systems and conformal algebra representation. Lett. Math. Phys. 12 (1993) 171-179.

[32] M. Adler, T. Shiota and P. van Moerbeke. A Lax representation for the vertex operator and the central extension. Comm. Math. Phys. 171 (1995) 547-588.

[33] L. A. Dickey. On additional symmetries of the KP hierarchy and Sato’s Bäcklund transformation. Comm. Math. Phys. 167 (1995) 227-233.

[34] K. Takasaki. Toda lattice hierarchy and generalized string equations. Comm. Math. Phys. 181 (1996) 131-156.

[35] M. H. Tu. On the BKP hierarchy: Additional symmetries, Fay identity and Adler-Shiota- van Moerbeke formula. Lett. Math. Phys. 81 (2007) 91-105.

[36] J. P. Cheng, J. S. He and S. Hu. The “ghost” symmetry of the BKP hierarchy. J. Math. Phys. 51(2010) 053514.

[37] C. Z. Li, J. P. Cheng, K. L. Tian, M. H. Li and J. S. He. Ghost symmetry of the discrete KP hierarchy. arXiv:1201.4419

[38] J. P. Cheng and J. S. He. On the squared eigenfunction symmetry of the Toda lattice hierarchy. J. Math. Phys. 54 (2013) 023511.

[39] J. P. Cheng and J. S. He. Squared eigenfunction symmetries for the BTL and CTL hierarchies. Commun. Theor. Phys. 59 (2013) 131-136

[40] P. G. Grinevich and A. Yu. Orlov. Virasoro action on Riemann surfaces, Grassmannians, det $\partial$, and Segal–Wilson tau-function, in: *Problems of Modern Quantum Field Theory*, ed. by A. A. Belavin, A. U. Klimyk, and A. B. Zamolodchikov, (Springer, Berlin, 1989) pp.86-106(also see arXiv:math-ph/9804019).

[41] A. Yu. Orlov and S. Rauch-Wojciechowski. Dressing method, Darboux transformation and generalized restricted flows for the KdV hierarchy. Phys. D 69 (1993) 77-84

[42] I. M. Krichever. General rational reductions of the KP hierarchy and their symmetries. Funct. Anal. Appl. 29 (1995) 75-80.

[43] C. Z. Li, K. L. Tian, J. S. He and Y. Cheng. The recursion operator for a constrained CKP hierarchy. Acta Math. Sci. Ser. B 31(2011) 1295-1302 arXiv:1004.0478.