Weighted and endpoint estimates for commutators of bilinear pseudo-differential operators

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Abstract: In this paper, by applying the accurate estimates of the Hörmander class, the authors consider the commutators of bilinear pseudo-differential operators and the operation of multiplication by a Lipschitz function. By establishing the pointwise estimates of the corresponding sharp maximal function, the boundedness of the commutators is obtained respectively on the products of weighted Lebesgue spaces and variable exponent Lebesgue spaces with σ ∈ BBS^1_{1,1}. Moreover, the endpoint estimate of the commutators is also established on L^∞ × L^∞.

Keywords: bilinear pseudo-differential operator; commutator; Lipschitz function; the product of variable exponent Lebesgue space

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1 Introduction and Main Results

Let T be a linear operator. Given a function a, the commutator [T, a] is defined by

[T, a](f) := T(af) − aT(f).

There is an increasing interest to the study of T being a pseudo-differential operator because of its theory plays an important role in many aspects of harmonic analysis and it has had quite a success in linear setting. As one of the most meaningful branches, the study of bilinear pseudo-differential operators was motivated not only as generalizations

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of the theory of linear ones but also its natural appearance and important applications. This topic is continuously attracting many researchers.

Let $a$ be a Lipschitz function and $1 < p < \infty$. The estimates of the form

$$\| [T, a](f) \|_{L^p} \lesssim \| a \|_{Lip} \| f \|_{L^p}, \quad \text{for all } f \in L^p(\mathbb{R}^n) \quad (1.1)$$

have been studied extensively. In particular, Calderón proved that (1.1) holds when $T$ is a pseudo-differential operator whose kernel is homogeneous of degree of $-n - 1$ in [1]. Coifman and Meyer showed (1.1) when $T = T_\sigma$ and $\sigma$ is a symbol in the Hörmander class $S^1_{1,0}$ go back to [2-3], this result was later extended by Auscher and Taylor in [4] to $\sigma \in BS^1_{1,1}$, where the class $BS^1_{1,1}$, which contains $S^1_{1,0}$ modulo symbols associated to smoothing operators, consists of symbols whose Fourier transforms in the first $n$-dimensional variable are appropriately compactly supported. The method in the proofs of [2-3] was mainly showed that, for each Lipschitz continuous functions $a$ on $\mathbb{R}^n$, $[T, a]$ is a Calderón-Zygmund singular integral whose kernel constants are controlled by $\| a \|_{Lip}$. For another thing, Auscher and Taylor proved (1.1) in two different ways: one method is based on the paraproducts while the other is based on the Calderón-Zygmund singular integral operator approach that relies on the $T(1)$ theorem. For a more systematic study of these (and even more general) spaces, we refer the readers to see [5-6].

Given a bilinear operator $T$ and a function $a$, the following two kinds commutators are respectively defined by

$$[T, a]_1(f, g) = T(af, g) - aT(f, g)$$

and

$$[T, a]_2(f, g) = T(f, ag) - aT(f, g).$$

In 2014, Bényi and Oh proved that (1.1) is also valid to this bilinear setting in [7]. More precisely, given a bilinear pseudo-differential operator $T_\sigma$ with $\sigma$ in the bilinear Hörmander class $BS^1_{1,0}$ and a Lipschitz function $a$ on $\mathbb{R}^n$, it was proved in [7, Theorem 1] that $[T, a]_1$ and $[T, a]_2$ are bilinear Calderón-Zygmund operators. The main aim of this paper is to study (1.1) of $[T_\sigma, a]_j (j = 1, 2)$ on the products of weighted Lebesgue spaces and variable exponent Lebesgue spaces with $\sigma \in BBS^1_{1,1}$. Before stating our main results, we need to recall some definitions and notations.

We say that a function $a$ defined on $\mathbb{R}^n$ is Lipschitz continuous if

$$\| a \|_{Lip} := \sup \frac{|a(x) - a(y)|}{|x - y|} < \infty.$$
Let $\delta \geq 0$, $\rho > 0$ and $m \in \mathbb{R}$. An infinitely differentiable function $\sigma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ belongs to the bilinear Hörmander class $BS^m_{\rho, \delta}$ if for all multi-indices $\alpha, \beta, \gamma \in \mathbb{N}^n_0$ there exists a positive constant $C_{\alpha, \beta, \gamma}$ such that
\[
|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \leq C((1 + |\xi| + |\eta|)^{m+\delta}|\alpha-\rho(\beta+\gamma)|).
\]
Given a $\sigma(x, \xi, \eta) \in BS^m_{\rho, \delta}$, the bilinear pseudo-differential operator associated to $\sigma$ is defined by
\[
T_\sigma(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi+\eta)} \, d\xi \, d\eta, \text{ for all } x \in \mathbb{R}^n, f, g \in \mathcal{S}(\mathbb{R}^n).
\]

In 1980, Meyer [8] firstly introduced the linear $BS^m_{1,1}$, and corresponding boundedness of $[T_{\sigma}, a]_j (j = 1, 2)$ is obtained by Bényi-Oh in [7], that is, given $m \in \mathbb{R}$ and $r > 0$, an infinitely differentiable function $\sigma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ belongs to $\mathcal{B}BS^m_{1,1}$ if
\[
\sigma \in BS^m_{1,1}, \text{ supp}(\hat{\sigma}^1) \subset \{(\tau, \xi, \eta) \in \mathbb{R}^{3n} : |\tau| \leq r(|\xi| + |\eta|)\},
\]
where $\hat{\sigma}^1$ denotes the Fourier transform of $\sigma$ with respect to its first variable in $\mathbb{R}^n$, that is, $\hat{\sigma}^1(\tau, \xi, \eta) = \sigma(\cdot, \xi, \eta)(\tau)$, for all $\tau, \xi, \eta \in \mathbb{R}^n$. The class $\mathcal{B}BS^m_{1,1}$ is defined as
\[
\mathcal{B}BS^m_{1,1} = \bigcup_{r \in (0, 1]} \mathcal{B}BS^m_{1,1}.
\]

Recently, many authors are interested in bilinear operators, which is a natural generalization of linear case. With the further research, Árpád Bényi and Virginia Naibo proved that boundedness for the commutators of bilinear pseudo-differential operators and Lipschitz functions with $\sigma \in \mathcal{B}BS^1_{1,1}$ on the Lebesgue spaces in [9]. In 2018, Tao and Li proved that the boundedness of the commutators of bilinear pseudo-differential operators was also true on the classical and generalized Morrey spaces in [10]. Motivated by the results mentioned above, a natural and interesting problem is to consider whether or not (1.1) is true on the weighted Lebesgue spaces and variable exponent Lebesgue spaces with $\sigma \in \mathcal{B}BS^1_{1,1}$. The purpose of this paper is to give an surely answer. And also, the endpoint estimate is obtained on $L^\infty \times L^\infty$. Our proofs are based on the pointwise estimates of the sharp maximal function proved in the next section.

Many results involving bilinear pseudo-differential operators theory have been obtained in parallel with the linear ones but some new interesting phenomena have also been observed. One aspect developed rapidly is the one related to the compactness of the bilinear
pseudo-differential operators, especially, the properties of compactness for the commutators of bilinear pseudo-differential operators and Lipschitz functions. As the commutators \([T_\sigma, a]_j (j = 1, 2)\) are bilinear Calderón-Zygmund operators if \(\sigma \in BBS^{1}_{1,1}\), similar to the proof of [11, Theorem A and Theorem 2.12], we can obtain easily that \([T_\sigma, a]_j\) and \([[T_\sigma, a]_j, b]_i \ (i, j = 1, 2)\) are compact operators on the Lebesgue spaces and the Morrey spaces. For the sake of convenience, there are no further details below.

Suppose that \(\sigma \in BBS^{1}_{1,1}\). Let \(K\) and \(K_j\) denote the kernel of \(T_\sigma\) and \([T_\sigma, a]_j (j = 1, 2)\), respectively. We have
\[
K(x, y, z) = \int \int e^{i\xi(x-y)}e^{i\eta(x-z)}\sigma(x, \xi, \eta)d\xi d\eta,
\]
\[
K_1(x, y, z) = (a(y) - a(x))K(x, y, z), \quad K_2(x, y, z) = (a(z) - a(x))K(x, y, z).
\]

Then the following consequences are true.

**Theorem A**[7, Lemma 3] If \(x \neq y\) or \(x \neq z\), then we have
1. \(|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma K(x, y, z)| \lesssim (|x - y| + |x - z|)^{-2n-1-|\alpha| - |\beta| - |\gamma|},
2. \(|K_j(x, y, z)| \lesssim \|a\|_{\text{Lip}}^i (|x - y| + |x - z| + |y - z|)^{-2n}.

The statement of our main theorems will be presented in follows.

**Theorem 1.1** Let \(q' > 1, \ \sigma \in BBS^{1}_{1,1}\) and \(a\) be a Lipschitz function on \(\mathbb{R}^n\). Suppose for fixed \(1 \leq r_1, r_2 \leq q'\) with \(1/r = 1/r_1 + 1/r_2\), \([T_\sigma, a]_j (j = 1, 2)\) is bounded from \(L^{r_1} \times L^{r_2}\) into \(L^{r,\infty}\) with norm controlled by \(\|a\|_{\text{Lip}}^1\). If \(0 < \delta < 1/2\), then
\[
M^1_\delta([T_\sigma, a]_j(f, g))(x) \leq C\|a\|_{\text{Lip}}^1 M^{q'}_\delta(f)(x)M^{q'}_\delta(g)(x), \quad j = 1, 2
\]
for all \(f, g\) of bounded measurable functions with compact support.

**Theorem 1.2** Let \(q' > 1, \ \sigma \in BBS^{1}_{1,1}\) and \(a\) be a Lipschitz function on \(\mathbb{R}^n\). Suppose for fixed \(1 \leq r_1, r_2 \leq q'\) with \(1/r = 1/r_1 + 1/r_2\), \([T_\sigma, a]_j (j = 1, 2)\) is bounded from \(L^{r_1} \times L^{r_2}\) into \(L^{r,\infty}\) with norm controlled by \(\|a\|_{\text{Lip}}^1\). If \(b \in BMO, \ 0 < \delta < 1/2, \ \delta < \varepsilon < \infty, \ q' < s < \infty\), then
\[
M^1_\delta([[T_\sigma, a]_j, b]_i)(x) \leq C\|b\|_{BMO} ((M_s([T_\sigma, a]_j(f, g))(x) + \|a\|_{\text{Lip}}^1 (M_s(f)(x))(M_s(g)(x)))
\]
where \(i, j = 1, 2\) and above inequality is valid for all \(f, g\) of bounded measurable functions with compact support.
Theorem 1.3 Let \( q' > 1 \), \( \sigma \in BBS_{1,1}^1 \) and \( a \) be a Lipschitz function on \( \mathbb{R}^n \). Suppose for fixed \( 1 \leq r_1, r_2 \leq q' \) with \( 1/r = 1/r_1 + 1/r_2 \), \( [T_\sigma, a]_j(j = 1, 2) \) is bounded from \( L^{r_1} \times L^{r_2} \) into \( L^{r,\infty} \) with norm controlled by \( \|a\|_{\text{Lip}^1} \). If \( (\omega_1, \omega_2) \in (A_{p_1/q'}, A_{p_2/q'}) \) and \( \omega = \omega_1^{p} \omega_2^{q'} \), then for \( q' < p_1, p_2 < \infty \) with \( 1/p = 1/p_1 + 1/p_2 \), \( [T_\sigma, a]_j(j = 1, 2) \) is bounded from \( L^{p_1}(\omega) \times L^{p_2}(\omega) \) into \( L^p(\omega) \).

Theorem 1.4 Let \( q' > 1 \), \( \sigma \in BBS_{1,1}^1 \) and \( a \) be a Lipschitz function on \( \mathbb{R}^n \). Suppose for fixed \( 1 \leq r_1, r_2 \leq q' \) with \( 1/r = 1/r_1 + 1/r_2 \), \( [T_\sigma, a]_j(j = 1, 2) \) is bounded from \( L^{r_1} \times L^{r_2} \) into \( L^{r,\infty} \) with norm controlled by \( \|a\|_{\text{Lip}^1} \). If \( b \in \text{BMO} \), \( (\omega_1, \omega_2) \in (A_{p_1/q'}, A_{p_2/q'}) \) and \( \omega = \omega_1^{p} \omega_2^{q'} \), then for \( q' < p_1, p_2 < \infty \) with \( 1/p = 1/p_1 + 1/p_2 \), \( [T_\sigma, a], b(i, j = 1, 2) \) is bounded from \( L^{p_1}(\omega) \times L^{p_2}(\omega) \) into \( L^p(\omega) \).

Theorem 1.5 Let \( p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n) \) with \( 1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot) \), and \( q_0' \) be given as in Lemma 4.4 for \( p_j(\cdot), j=1,2 \). Suppose that \( \sigma \in BBS_{1,1}^1 \), \( a \) is a Lipschitz function on \( \mathbb{R}^n \) and \( 1 < q' \leq \min\{q_{01}', q_{02}'\} \). If for fixed \( 1 \leq r_1, r_2 \leq q' \) with \( 1/r = 1/r_1 + 1/r_2 \), \( [T_\sigma, a]_j(j = 1, 2) \) is bounded from \( L^{r_1} \times L^{r_2} \) into \( L^{r,\infty} \) with norm controlled by \( \|a\|_{\text{Lip}^1} \), then \( [T_\sigma, a]_j(j = 1, 2) \) is bounded from \( L^{p_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot)}(\mathbb{R}^n) \) into \( L^{p(\cdot)}(\mathbb{R}^n) \).

Theorem 1.6 Let \( p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n) \) with \( 1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot) \), and \( q_0' \) be given as in Lemma 4.4 for \( p_j(\cdot), j=1,2 \). Suppose that \( \sigma \in BBS_{1,1}^1 \), \( a \) is a Lipschitz function on \( \mathbb{R}^n \) and \( 1 < q' \leq \min\{q_{01}', q_{02}'\} \). If for fixed \( 1 \leq r_1, r_2 \leq q' \) with \( 1/r = 1/r_1 + 1/r_2 \), \( [T_\sigma, a]_j(j = 1, 2) \) is bounded from \( L^{r_1} \times L^{r_2} \) into \( L^{r,\infty} \) with norm controlled by \( \|a\|_{\text{Lip}^1} \), and \( b \in \text{BMO} \), then \( [T_\sigma, a], b(i, j = 1, 2) \) is bounded from \( L^{p_1(\cdot)}(\mathbb{R}^n) \times L^{p_2(\cdot)}(\mathbb{R}^n) \) into \( L^{p(\cdot)}(\mathbb{R}^n) \).

Theorem 1.7 Let \( \sigma \in BBS_{1,1}^1 \) and \( a \) be a Lipschitz function. Suppose for fixed \( 1 \leq r_1, r_2 \leq q' \) with \( 1/r = 1/r_1 + 1/r_2 \), \( [T_\sigma, a]_j(j = 1, 2) \) is bounded from \( L^{r_1} \times L^{r_2} \) into \( L^{r,\infty} \) with norm controlled by \( \|a\|_{\text{Lip}^1} \). Then \( [T_\sigma, a]_j(j = 1, 2) \) is bounded from \( L^\infty \times L^\infty \) into \( \text{BMO} \).

We use the following notation: for \( 1 \leq p \leq \infty \), \( p' \) is the conjugate index of \( p \), that is, \( 1/p + 1/p' = 1 \). \( B(x, R) \) denotes the ball centered at \( x \) with radius \( R > 0 \) and \( f_B = \)
\[ \frac{1}{|B(x,R)|} \int_{B(x,R)} f(y)dy. \] The paper is organized as follows. The pointwise estimates of the sharp maximal functions are presented in Section 2. The weighted boundedness is given in Section 3. The proofs of the boundedness on the product of variable exponent Lebesgue spaces are showed in Section 4. The endpoint estimate is proved in Section 5.

2 Pointwise Estimates for the Sharp Maximal Functions

In this section, we shall prove Theorems 1.1-1.2. In order to do this, let’s recall some definitions.

Given a function \( f \in L_{\text{loc}}(\mathbb{R}^n) \), the sharp maximal function is defined by

\[
M^\#(f)(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y) - f_B|dy \approx \sup_{x \in B} \inf_{a \in \mathbb{C}} \frac{1}{|B|} \int_B |f(y) - a|dx,
\]

where the supremum is taken over all balls \( B \) containing \( x \). Let \( 0 < \delta < \infty \). We denote by \( M^\#_\delta \) the operator

\[
M^\#_\delta(f) = [M^\#(|f|^{\delta})]^{1/\delta}.
\]

Similarly, we use \( M_\delta \) to denote the operator \( M^\delta(f) = [M(|f|^{\delta})]^{1/\delta} \), where \( M \) is the Hardy-Littlewood maximal function defined by

\[
Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_B f(y)dy.
\]

The operator \( M^\#_\delta \) was appeared implicitly in a paper by John [12] and was introduced by Strömberg [13]. The sharp maximal function \( M^\# \) and \( M^\#_\delta \) not only have close relation to BMO, but also are important tools to obtain pointwise inequalities regarding many operators in harmonic analysis (see [14-19]).

To prove the theorems 1.1-1.2, we need the following Kolmogorov’s inequality and the inequality regarding the BMO functions.

**Lemma 2.1** [20,21] Let \( 0 < p < q < \infty \). Then there is a constant \( C = C_{p,q} > 0 \), such that

\[
|Q|^{-1/p} \|f\|_{L^p(Q)} \leq C|Q|^{-1/q} \|f\|_{L^{p,\infty}(Q)}
\]

for all measurable functions \( f \).

**Lemma 2.2** [22] Let \( f \in \text{BMO}(\mathbb{R}^n) \). Suppose \( 1 \leq p < \infty \), \( r_1 > 0 \), \( r_2 > 0 \) and \( x \in \mathbb{R}^n \). Then

\[
\left( \frac{1}{|B(x,r_1)|} \int_{B(x,r_1)} |f(y) - f_{B(x,r_2)}|^{p}dy \right)^{1/p} \leq C \left( 1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|f\|_{\text{BMO}},
\]

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where \( C \) is a positive constant independent of \( f, x, r_1 \) and \( r_2 \).

**Lemma 2.3** If \( \sigma \in BBS^{1,1}_{1,1} \) and \( a \) is a Lipschitz function on \( \mathbb{R}^n \), then the commutators \( [T_\sigma, a]_j, j = 1, 2 \) are bilinear Calderón-Zygmund operators. In particular, \( [T_\sigma, a]_j, j = 1, 2 \) are bounded from \( L^{p_1} \times L^{p_2} \) into \( L^p \) for \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) and \( 1 < p_1, p_2 < \infty \) and verify appropriate end-point boundedness properties. Moreover, the corresponding norms of the operators are controlled by \( \|a\|_{\text{Lip}}^1 \).

**Proof of Theorem 1.1.** Let \( f, g \) be bounded measurable functions with compact support. Then for any ball \( B = B(x_0, r_B) \) containing \( x \), we decompose \( f \) and \( g \) as follows:

\[
f = f\chi_{16B} + f\chi_{(16B)^c} := f^1 + f^2, \quad g = g\chi_{16B} + g\chi_{(16B)^c} := g^1 + g^2.
\]

Choose a \( z_0 \in 3B \setminus 2B \). Then

\[
\left( \frac{1}{|B|} \left| \left| [T_\sigma, a]_j(f, g)(z) \right| \right|^\delta - \left| \left| [T_\sigma, a]_j(f^2, g^2)(z_0) \right| \right|^\delta \right)^{1/\delta} \leq C \left( \frac{1}{|B|} \left| [T_\sigma, a]_j(f, g)(z) - [T_\sigma, a]_j(f^2, g^2)(z_0) \right| \right)^{1/\delta}
\]

\[
\leq C \left( \frac{1}{|B|} \left| [T_\sigma, a]_j(f^1, g^1)(z) \right| \right)^{1/\delta} + C \left( \frac{1}{|B|} \int_B \left| [T_\sigma, a]_j(f^2, g^1)(z) \right| \right)^{1/\delta}
\]

\[
+ C \left( \frac{1}{|B|} \left| [T_\sigma, a]_j(f^1, g^2)(z) \right| \right)^{1/\delta} + C \left( \frac{1}{|B|} \int_B \left| [T_\sigma, a]_j(f^2, g^2)(z) - [T_\sigma, a]_j(f^2, g^2)(z_0) \right| \right)^{1/\delta}
\]

\[
:= \sum_{s=1}^4 I_s.
\]

For any \( 0 < \delta < r < \infty \), it follows from Lemma 2.1 that

\[
I_1 \leq C |B|^{-1/\delta} \left\| [T_\sigma, a]_j(f^1, g^1) \right\|_{L^s(B)}
\]

\[
\leq C |B|^{-1/r} \left\| [T_\sigma, a]_j(f^1, g^1) \right\|_{L^{r,\infty}(B)}
\]

\[
\leq C \|a\|_{\text{Lip}} M_{r_1}(f)(x) M_{r_2}(g)(x)
\]

\[
\leq C \|a\|_{\text{Lip}} M_{r'}(f)(x) M_{r'}(g)(x).
\]

If \( z \in B, \ y_1 \in (16B)^c, \ y_2 \in 16B \), noticing that \( |z - y_1| + |z - y_2| + |y_1 - y_2| \sim |z - y_1| + \)
\[ |z - y_2| \geq |z - y_1|, \] then we have by Theorem A,

\[
I_2 \leq C \left( \frac{1}{|B|} \int_B \left( \int_{16B} \int_{16B} |K(z, y_1, y_2)| \left| f(y_1) \right| |g(y_2)| \, dy_2 \, dy_1 \, dz \right)^{\frac{1}{\delta}} \leq C \left( \frac{1}{|B|} \int_B \left( \int_{16B} \int_{16B} |g(y_2)| \, dy_2 \right)^{\frac{1}{\delta}} \frac{\|a\|_{\text{Lip}}}{|z - y_1|^{2n}} \, dy_1 \, dz \right)^{\frac{1}{\delta}} \leq C \|a\|_{\text{Lip}} \left( \frac{1}{|B|} \int_B \left( \int_{16B} |g(y_2)| \, dy_2 \right)^{\frac{1}{\delta}} \sum_{k=4}^{\infty} 2^{-kn} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |f(y_1)| \, dy_1 \right)^{\frac{1}{\delta}} \leq C \|a\|_{\text{Lip}} M(f)(x) M(g)(x) \sum_{k=4}^{\infty} 2^{-kn} \leq C \|a\|_{\text{Lip}} M_q(f)(x) M_q(g)(x) \] .

By the similar way, we can get that

\[
I_3 \leq C \|a\|_{\text{Lip}} M_q(f)(x) M_q(g)(x) .
\]

As \( z \in B \) and \( y_1, y_2 \in (16B)^c \), then \( |y_1 - z_0| \geq 2|z - z_0|, |y_2 - z_0| \geq 2|z - z_0| \) and \( r_B \leq |z - z_0| \leq 4r_B \). It follows from Hölder’s inequality that

\[
I_4 \leq C \left( \frac{1}{|B|} \int_B \left( \int_{R^n} \int_{R^n} |K(z, y_1, y_2) - K(z_0, y_1, y_2)| |f(y_1)| |g(y_2)| \, dy_1 \, dy_2 \right)^{\frac{1}{\delta}} \right)^{\frac{1}{\delta}} \leq C \left( \frac{1}{|B|} \int_B \left( \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \int_{2^{k_1}B} \int_{2^{k_2}B} |K(z, y_1, y_2) - K(z_0, y_1, y_2)| |f(y_1)| |g(y_2)| \, dy_1 \, dy_2 \right)^{\frac{1}{\delta}} \right)^{\frac{1}{\delta}} \nu^{\frac{1}{\delta}} \nu^{\frac{1}{\delta}} \leq C \left( \frac{1}{|B|} \int_B \left( \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \left( \int_{2^{k_1}B} |f(y_1)| |g(y_2)| \, dy_1 \right)^{\frac{1}{\delta}} \right)^{\frac{1}{\delta}} \nu^{\frac{1}{\delta}} \nu^{\frac{1}{\delta}} \leq C \left( \frac{1}{|B|} \int_B \left( \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \left( \int_{2^{k_1+4}B} |f(y_1)| |g(y_2)| \, dy_1 \right)^{\frac{1}{\delta}} \right)^{\frac{1}{\delta}} \nu^{\frac{1}{\delta}} \nu^{\frac{1}{\delta}} \leq C \left( \frac{1}{|B|} \int_B \left( \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \left( \int_{2^{k_1+4}B} |f(y_1)| |g(y_2)| \, dy_1 \right)^{\frac{1}{\delta}} \right)^{\frac{1}{\delta}} \nu^{\frac{1}{\delta}} \nu^{\frac{1}{\delta}} \right)^{\frac{1}{\delta}} .
\]
\[-K(z_0, y_1, y_2)|^q d(\frac{dy_1}{dy_2})^\frac{1}{q}dz\]
\[\leq C||a||_{\text{Lip}} \left( \frac{1}{|B|} \int_B \left( \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \left( \frac{1}{|2^{k_1}+1|B} \int_{2^{k_1}+1B} |f(y_1)|^q dy_1 \right)^{\frac{1}{q}} \times \left( \frac{1}{|2^{k_1}+1|B} \int_{2^{k_1}+1B} |g(y_2)|^q dy_2 \right)^{\frac{1}{q}} \times |2^{k_1+1}B|^{1/q} |2^{k_2+1}B|^{1/q} |z - z_0|^{\frac{2n}{q} C_{k_1} 2^{-k_1} C_{k_2} 2^{-k_2} d} dz \right)^{1/d} \]
\[\leq C||a||_{\text{Lip}} M_q(f(x)) M_q(g(x)) \left( \sum_{k_1=1}^{\infty} C_{k_1} \right) \left( \sum_{k_2=1}^{\infty} C_{k_2} \right) \]
\[\leq C||a||_{\text{Lip}} M_q(f(x)) M_q(g(x)), \quad (2.1)\]

where we use the fact of a weaker size condition of standard m-linear Calderón-Zygmund kernel than its classical size condition given in [23], that is: for any \(k_1, \ldots, k_m \in \mathbb{N}_+,\) there are positive constant \(C_{k_i}, \ i = 1, \ldots, m,\) such that
\[
\left( \int_{2^{k_1} |y_1 - y_0| \leq |y_m - y_0| \leq 2^{k_2} |y_0 - z_0| \right) \cdots \left( \int_{2^{k_1} |y_1 - y_0| \leq |y_1 - y_0| \leq 2^{k_1+1} |y_1 - z_0| \right) |K(y_0, y_1 \cdots y_m) - K(y'_0, y_1 \cdots y_m)|^q dy_1 \cdots dy_m \right)^{\frac{1}{q}} \leq C\|y_0 - y'_0\|^{-\frac{m}{q} \prod_{i=1}^{m} C_{k_i} 2^{-\frac{n}{q} k_i}}, \quad (2.2)\]

where \(\prod_{k_i=1}^{\infty} C_{k_i} < \infty, \ i = 1, 2, 1 < q < \infty.\) Together with the commutators \([T_\sigma, a]_j, j = 1, 2\) are bilinear Calderón-Zygmund operators and Theorem A, then we obtain the fact that
\[
\left( \int_{2^{k_1} |y_1 - y_0| \leq |y_1 - y_0| \leq 2^{k_2} |y_0 - z_0|} \int_{2^{k_1} |y_1 - y_0| \leq |y_1 - y_0| \leq 2^{k_1+1} |y_1 - z_0|} |K(y_0, y_1, y_2) - K(y'_0, y_1, y_2)|^q dy_1 dy_2 \right)^{\frac{1}{q}} \leq C\|a\|_{\text{Lip}} |y_0 - y'_0|^{-\frac{2n}{q} \prod_{i=1}^{2} C_{k_i} 2^{-\frac{n}{q} k_i}}, \quad (2.3)\]

Thus, we have
\[
M_q([T_\sigma, a]_j(f, g))(x) \approx \sup_{x \in B} \inf_{a \in \mathbb{C}} \left( \frac{1}{|B|} \int_B \left|[T_\sigma, a]_j(f, g)(z)\right|^q dz \right)^{1/q} \leq \sup_{x \in B} \left( \frac{1}{|B|} \int_B \left|[T_\sigma, a]_j(f, g)(z)\right|^q dz \right)^{1/q} \leq C\|a\|_{\text{Lip}} M_q(f(x)) M_q(g(x)).
\]

Thus we finish the proof of Theorem 1.1.
Proof of Theorem 1.2. Without loss of generality, we consider the case \( i = 1 \), the proof of the case \( i = 2 \) is similar. Let \( f_1, f_2 \) be bounded measurable functions with compact support. As in the proof of Theorem 1.1, we write \( f \) and \( g \) as

\[
f = f\chi_{16B} + f\chi_{16B}^c := f^1 + f^2, \quad g = g\chi_{16B} + g\chi_{16B}^c := g^1 + g^2.
\]

Then

\[
[[T_\sigma, a], j]_1(f, g)(z) = (b(z) - b_{16B})[T_\sigma, a]_j(f, g)(z) - [T_\sigma, a]_j((b - b_{16B})f, g)(z)
\]

\[
= (b(z) - b_{16B})[T_\sigma, a]_j(f, g)(z) - [T_\sigma, a]_j((b - b_{16B})f^1, g^1)(z)
\]

\[
- [T_\sigma, a]_j((b - b_{16B})f^1, g^2)(z) - [T_\sigma, a]_j((b - b_{16B})f^2, g^1)(z)
\]

\[
- [T_\sigma, a]_j((b - b_{16B})f^2, g^2)(z),
\]

where \( b_{16B} = \frac{1}{|16B|} \int_{16B} b(z) \, dz \). Therefore, for any fixed \( z_0 \in 3B \setminus 2B \), we have

\[
\left( \frac{1}{|B|} \int_B \left| [[T_\sigma, a], j]_1(f, g)(z) + [T_\sigma, a]_j((b - b_{16B})f^2, g^2)(z_0) \right|^\delta \, dz \right)^\frac{1}{\delta}
\]

\[
\leq C \left( \frac{1}{|B|} \int_B \left| (b(z) - b_{16B})[T_\sigma, a]_j(f, g)(z) \right|^\delta \, dz \right)^\frac{1}{\delta}
\]

\[
+ C \left( \frac{1}{|B|} \int_B \left| [T_\sigma, a]_j((b - b_{16B})f^1, g^1)(z) \right|^\delta \, dz \right)^\frac{1}{\delta}
\]

\[
+ C \left( \frac{1}{|B|} \int_B \left| [T_\sigma, a]_j((b - b_{16B})f^1, g^2)(z) \right|^\delta \, dz \right)^\frac{1}{\delta}
\]

\[
+ C \left( \frac{1}{|B|} \int_B \left| [T_\sigma, a]_j((b - b_{16B})f^2, g^1)(z) \right|^\delta \, dz \right)^\frac{1}{\delta}
\]

\[
+ C \left( \frac{1}{|B|} \int_B \left| [T_\sigma, a]_j((b - b_{16B})f^2, g^2)(z) - [T_\sigma, a]_j((b - b_{16B})f^2, g^2)(z_0) \right|^\delta \, dz \right)^\frac{1}{\delta}
\]

\[
:= \sum_{i=1}^{5} II_i.
\]

Since \( 0 < \delta < 1/2 \) and \( \delta < \varepsilon < \infty \), there exists an \( l \) such that \( 1 < l < \min\{\frac{\varepsilon}{\delta}, \frac{1}{1-\delta}\} \). Then \( \delta l < \varepsilon \) and \( \delta l' > 1 \). By Hölder’s inequality, we have

\[
II_1 \leq C \left( \frac{1}{|B|} \int_B \left| (b(z) - b_{16B})^{\delta l'} \right| \, dz \right)^\frac{1}{\delta l'} \left( \frac{1}{|B|} \int_B \left| [T_\sigma, a]_j(f, g)(z) \right|^{\delta l} \, dz \right)^\frac{1}{\delta l}
\]

\[
\leq C \|b\|_{\text{BMO}} \left( \frac{1}{|B|} \int_B \left| [T_\sigma, a]_j(f, g)(z) \right|^{\delta} \, dz \right)^\frac{1}{\delta}
\]

\[
\leq C \|b\|_{\text{BMO}} M_{\varepsilon}([T_\sigma, a]_j(f, g))(x).
\]
Since \( q' < s < \infty \), denoting \( t = s/q' \), then \( 1 < t < \infty \). Noticing that \( 0 < \delta < r < \infty \), it follows from Lemmas 2.1 and 2.3 that

\[
II_2 \leq C|B|^{-1/\delta}||T_\sigma, a_j((b - b_{16B})f^1, g^1)||_{L^s(B)} \\
\leq C|B|^{-1/r}||T_\sigma, a_j((b - b_{16B})f^1, g^1)||_{L^r, \infty(B)} \\
\leq C||a||_{\text{Lip}^1}(\frac{1}{|16B|} \int_{16B} |b(y_1) - b_{16B}| ||f(y_1)||_r dy_1) \frac{1}{r} \left( \frac{1}{|16B|} \int_{16B} |g(y_2)||_{r'} dy_2 \right)^{\frac{1}{r'}} \\
\times (\frac{1}{|16B|} \int_{16B} |g(y_2)||^2 dy_2)^{\frac{1}{2}} \\
\leq C||a||_{\text{Lip}^1} ||b||_{\text{BMO}}(\frac{1}{|16B|} \int_{16B} |f(y_1)||^s dy_2)^{\frac{1}{s}} \times (\frac{1}{|16B|} \int_{16B} |g(y_2)||^2 dy_2)^{\frac{1}{2}} \\
\leq C||a||_{\text{Lip}^1} ||b||_{\text{BMO}} M_s(f)(x) M_s(g)(x).
\]

By Theorem A, as \( z \in B, y_1 \in (16B), y_2 \in 16B^c \), noticing that \( |z - y_1| + |z - y_2| + |y_1 - y_2| \sim |z - y_1| + |z - y_2| \geq |z - y_2| \), then we have

\[
II_3 \leq C\left( \frac{1}{|B|} \int_B \left( \int_{(16B)^c} \int_{(16B)} |K(z, y_1, y_2)||b(y_1 - b_{16B})||f(y_1)||g(y_2)||dy_1 dy_2 dz \right)^{1/\delta} \\
\leq C||a||_{\text{Lip}^1}(\frac{1}{|B|} \int_B (\int_{(16B)^c} (\int_{(16B)} |b(y_1 - b_{16B})||f(y_1)||dy_1) \frac{f(y_2)}{|z - y_2|^{2n}} dy_2) dz)^{1/\delta} \\
\leq C||a||_{\text{Lip}^1} \left( \int_{(16B)} |b(y_1 - b_{16B})||f(y_1)||dy_1 \right) \sum_{k=1}^{\infty} \int_{(2^{k+1}B)^c} \frac{f(y_2)}{|z_0 - y_2|^{2n}} dy_2 \\
\leq C||a||_{\text{Lip}^1} ||b||_{\text{BMO}} M_s^r(f)(x) M_s(g)(x) \sum_{k=1}^{\infty} 2^{-kn} \\
\leq C||a||_{\text{Lip}^1} ||b||_{\text{BMO}} M_s(f)(x) M_s(g)(x).
\]

Similar to estimate \( II_3 \), by Lemma 2.2, we can get that

\[
II_4 \leq C||a||_{\text{Lip}^1} \left( \int_{(16B)} |g(y_2)||dy_2 \right) \sum_{k=1}^{\infty} \int_{(2^{k+1}B)^c} \frac{|b(y_1 - b_{16B})||f(y_1)||}{|x_0 - y_2|^{2n}} dy_1 \\
\leq C||a||_{\text{Lip}^1} M_s(g)(x) \sum_{k=1}^{\infty} 2^{-kn} \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |b(y_1 - b_{16B})||f(y_1)||dy_1
\]
\[ \begin{align*}
\leq & \quad C \|a\|_{\text{Lip}} M(g)(x) \sum_{k=1}^{\infty} 2^{-kn} \left( \frac{1}{|2^{k+1}B|} |b(y_1 - b_{16B})|^q dy_1 \right)^{1/q} \\
& \quad \times \left( \frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |f(y_1)|^q dy_1 \right)^{\frac{1}{q}} \\
\leq & \quad C \|a\|_{\text{Lip}} \|b\|_{\text{BMO}} M_{\delta}(f)(x) M(g)(x) \sum_{k=1}^{\infty} 2^{-kn} \\
\leq & \quad C \|a\|_{\text{Lip}} \|b\|_{\text{BMO}} M_{\delta}(f)(x) M_{\delta}(g)(x).
\end{align*} \]

As \( z \in B \) and \( y_1, y_2 \in (16B)^c \), then \( |y_1 - z_0| \geq 2 |z - z_0| \), \( |y_2 - z_0| \geq 2 |z - z_0| \) and \( r_B \leq |z - z_0| \leq 4r_B \). Noticing that \( \frac{1}{q} + \frac{1}{2q} + \frac{1}{r_q} = 1 \). It follows from Hölder’s inequality, Theorem A and the fact (2.3) that

\[ H_5 \leq C \left( \frac{1}{|B|} \int_B \left( \int_{R^n} \int_{R^n} |K(z, y_1, y_2) - K(z_0, y_1, y_2)| |b(y_1) - b_{16B}| \right) \times |f^2(y_1)| g(y_2) dy_1 dy_2 \right)^{1/\delta} \\
\leq C \left( \frac{1}{|B|} \int_B \left( \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \int_{2^{k_1}B} |z - z_0| \leq |y_2 - z_0| \leq 2^{k_2+1} |z - z_0| \right. \times |K(z, y_1, y_2) - K(z_0, y_1, y_2)| \left. dy_2 \right)^{1/\delta} \\
\leq C \left( \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \left( \int_{2^{k_1}B} |b(y_1) - b_{16B}| \right)^{1/\delta} \times \left( \int_{2^{k_1}B} |g(y_2)| dy_2 \right)^{1/\delta} \\
\leq C \left( \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \left( \int_{2^{k_1}B} |f(y_1)| dy_1 \right)^{1/\delta} \times \left( \int_{2^{k_1}B} |g(y_2)| dy_2 \right)^{1/\delta} \\
\leq C \|a\|_{\text{Lip}} M_{\delta}(f)(x) M_{\delta}(g)(x) \].
This finishes the proof of Theorem 1.2 also obtained by using Theorem 1.1 and Theorem 1.2. BMO bilinear commutators with sharp maximal function, that is, Theorem 1.1. The boundedness of the corresponding boundedness of product of weighted Lebesgue spaces owning to the pointwise estimate of its bilinear pseudo-differential operators and Lipschitz functions, we will establish its boundedness on Product of Weighted Lebesgue Spaces and \( \omega \in A_p \) if for every cube \( Q \) in \( \mathbb{R}^n \), there exists a positive \( C \) independent of \( Q \) such that

\[
\left( \frac{1}{|Q|} \int_Q \omega(x) dx \right) \left( \frac{1}{|Q|} \int_Q \omega(x)^{1-\frac{n}{p}} dx \right)^{p-1} \leq C.
\]

Combining the estimate of \( H_j, j = 1, 2, 3, 4, 5 \), we get

\[
\left( \frac{1}{|B|} \int_B \left| [T_{\sigma}, a]_j, b_1 (f, g) (z) + [T_{\sigma}, a]_j ((b - b_{16}) f^2, g^2) (z_0) \right|^\delta dz \right)^{\frac{1}{\delta}} \leq C \| b \|_{BMO} (M_\sigma (T_{\sigma}, a)_j (f, g)) (x) + \| a \|_{\text{Lip}^3} M_\sigma (f) (x) M_\sigma (g) (x)).
\]

Similarly, for the case \( i = 2 \), we can obtain that

\[
\left( \frac{1}{|B|} \int_B \left| [T_{\sigma}, a]_j, b_2 (f, g) (z) + [T_{\sigma}, a]_j ((b - b_{16}) f^2, g^2) (z_0) \right|^\delta dz \right)^{\frac{1}{\delta}} \leq C \| b \|_{BMO} (M_\sigma (T_{\sigma}, a)_j (f, g)) (x) + \| a \|_{\text{Lip}^3} M_\sigma (f) (x) M_\sigma (g) (x)).
\]

Thus,

\[
M_\delta^2 \left( [T_{\sigma}, a]_j, b_1 (f, g) (x) \right) \approx \sup_{x \in B \in \mathcal{C}} \left( \frac{1}{|B|} \int_B \left| [T_{\sigma}, a]_j, b_1 (f, g) (z) \right|^\delta dz \right)^{\frac{1}{\delta}} \leq C \| b \|_{BMO} (M_\sigma (T_{\sigma}, a)_j (f, g)) (x) + \| a \|_{\text{Lip}^3} M_\sigma (f) (x) M_\sigma (g) (x)).
\]

This finishes the proof of Theorem 1.2.

3 Boundedness on Product of Weighted Lebesgue Spaces

The theory of weighted estimates has played very important roles in modern harmonic analysis with lots of extensive applications in the others fields of mathematics, which has been extensively studied (see [24-27], for instance). In this section, for the commutators of bilinear pseudo-differential operators and Lipschitz functions, we will establish its boundedness of product of weighted Lebesgue spaces owning to the pointwise estimate of its sharp maximal function, that is, Theorem 1.1. The boundedness of the corresponding bilinear commutators with BMO function on the product of weighted Lebesgue spaces is also obtained by using Theorem 1.1 and Theorem 1.2.

Let us recall the definition of the class of Muckenhoupt weights \( A_p \) before proving Theorems 1.3-1.4. Let \( 1 < p < \infty \) and \( \omega \) be a non-negative measurable function. We say \( \omega \in A_p \) if for every cube \( Q \) in \( \mathbb{R}^n \), there exists a positive \( C \) independent of \( Q \) such that

\[
\left( \frac{1}{|Q|} \int_Q \omega(x) dx \right) \left( \frac{1}{|Q|} \int_Q \omega(x)^{1-\frac{n}{p}} dx \right)^{p-1} \leq C.
\]
Denote by $A_\infty = \bigcup_{p \geq 1} A_p$. It is well known that if $\omega \in A_p$ with $1 < p < \infty$, then $\omega \in A_r$ for all $r > p$, and $\omega \in A_p$ for some $q$, $1 < q < p$.

To prove Theorems 1.3-1.4, we need the following inequality regarding maximal functions which is a version of the classical ones due to Fefferman and Stein in (see [28]), and a property of $A_p$.

**Lemma 3.1**[28] Let $0 < p, \delta < \infty$, and $\omega \in A_\infty$. Then there exists a positive constant $C$ depending on the $A_\infty$ constant of $\omega$ such that
\[
\int_{\mathbb{R}^n} |M_\delta(f)(x)|^p \omega(x)dx \leq C \int_{\mathbb{R}^n} |M_{\delta\delta}(f)(x)|^p \omega(x)dx,
\]
for every function $f$ such that the left-hand side is finite.

**Lemma 3.2**[29] For $(\omega_1, \cdots, \omega_m) \in (A_{p_1}, \cdots, A_{p_m})$ with $1 \leq p_1, \cdots, p_m < \infty$, and for $0 < \theta_1, \cdots, \theta_m < 1$ such that $\theta_1 + \cdots + \theta_m = 1$, we have $\omega_1^{\theta_1} \cdots \omega_m^{\theta_m} \in A_{\max\{p_1, \cdots, p_m\}}$.

**Proof of Theorem 1.3.** It follows from Lemma 3.2 that $\omega \in A_{\max\{p_1/q, p_2/q\}} \subset A_\infty$.

Take a $\delta$ such that $0 < \delta < 1/2$. Then by Lemma 3.1 and Theorem 1.1, we get
\[
\| [T_\sigma, a]_j(f, g) \|_{L^p(\omega)} \leq \| M_\delta([T_\sigma, a]_j(f, g)) \|_{L^p(\omega)}
\leq C \| a \|_{\text{Lip}} \| M_{\delta}(f)M_{\delta}(g) \|_{L^p(\omega)}
\leq C \| a \|_{\text{Lip}} \| M_{\delta}(f) \|_{L^{p_1}(\omega_1)} \| M_{\delta}(g) \|_{L^{p_2}(\omega_2)}
= C \| a \|_{\text{Lip}} \| M(|f|^{q'})^{1/q'}_{L^{p_1/q'}(\omega_1)} \| M(|g|^{q'})^{1/q'}_{L^{p_2/q'}(\omega_2)}
\leq C \| a \|_{\text{Lip}} \| f \|^{1/q'}_{L^{p_1/q'}(\omega_1)} \| g \|^{1/q'}_{L^{p_2/q'}(\omega_2)}
= C \| a \|_{\text{Lip}} \| f \|_{L^{p_1}(\omega_1)} \| g \|_{L^{p_2}(\omega_2)}.
\]

We complete the proof of the **Theorem 1.3**.

**Proof of Theorem 1.4.** It follows from Lemma 3.2 that $\omega \in A_\infty$. Take $\delta$ and $\varepsilon$ such that $0 < \delta < \varepsilon < 1/2$. Then by Lemma 3.1 and Theorem 1.1, let $f = (f_1, f_2)$, we get
\[
\| M_\varepsilon([T_\sigma, a]_j(f)) \|_{L^p(\omega)} \leq C \| M_\varepsilon([T_\sigma, a]_j(f)) \|_{L^p(\omega)}
\leq C \| a \|_{\text{Lip}} \left( \prod_{t=1}^{2} M_{\varepsilon}(f_t) \right) \|_{L^p(\omega)}.
\]

Since $\omega_t \in A_{p_t/q'}$, $t = 1, 2$, there exists an $l_t$ such that $1 < l_t < p_t/q'$ and $\omega_t \in A_{l_t}$. 
It follows from $q' < p_t/l_t$ that there is an $s_t$ such that $q' < s_t < p_t/l_t < p_t$. Let $s = \min\{s_1, s_2\}$. Then $s > q'$ and $s < p_t$.

Since $l_t < p_t/s_t \leq p_t/s$, then $\omega_i \in A_{l_t} \subset A_{p_t/s}$, $t = 1, 2$. It follows from Lemma 3.1 and Theorem 1.2 that

$$\|\{|T_\sigma, a_j, b_j|(f_i)\}|_{L^p(\omega)} \leq \|M_{BMO}(\|T_\sigma, a_j, b_j|(f_i)\|_{L^p(\omega)}) \leq C\|b\|_{BMO} \left( \|a\|_{Lip^1} \prod_{t=1}^{2} \|M_{s}(f_t)\|_{L^p(\omega)} \right) \leq C\|b\|_{BMO} \|a\|_{Lip^1} \prod_{t=1}^{2} \|M_{s}(f_t)\|_{L^p(\omega)} \leq C\|b\|_{BMO} \|a\|_{Lip^1} \prod_{t=1}^{2} \|M_{s}(f_t)\|_{L^p(\omega)} \leq C\|b\|_{BMO} \|a\|_{Lip^1} \prod_{t=1}^{2} \|M_{s}(f_t)\|_{L^p(\omega)} \leq C\|b\|_{BMO} \|a\|_{Lip^1} \prod_{t=1}^{2} \|M_{s}(f_t)\|_{L^p(\omega)} \leq C\|b\|_{BMO} \|a\|_{Lip^1} \prod_{t=1}^{2} \|M_{s}(f_t)\|_{L^p(\omega)}.$$

We complete the proof of the Theorem 1.4.

4 Boundedness on Product of Variable Exponent Lebesgue Spaces

The spaces with variable exponent have been widely studied in recent ten years. The results show that they are not only the generalized forms of the classical function spaces with invariable exponent, but also there are some new breakthroughs in the research techniques. These new real variable methods help people further understand the function spaces. Due to the fundamental paper [30] by Kovôčik and Rákosník, Lebesgue spaces with variable exponent $L^p(\cdot)(\mathbb{R}^n)$ becomes one of the important class function spaces. The theory of the variable exponent function spaces have been applied in fluid dynamics, elasticity dynamics, calculus of variations and differential equations with non-standard growth conditions(for example, see [31-33]). In [34], authors proved the extrapolation theorem which leads the boundedness of some classical operators including the commutators on $L^p(\cdot)(\mathbb{R}^n)$. Karlovich and Lerner also obtained the boundedness of the singular integral commutators in [35]. The boundedness of some typical operators is being studied with
keen interest on spaces with variable exponent (see [36-40]).

In this section, we will establish the boundedness of $[T_\sigma,a]_j$ and $[[T_\sigma,a],b]_i(i,j = 1,2)$ on the product of variable exponent Lebesgue spaces, that is, we shall prove Theorems 1.5-1.6.

Denote $\mathcal{P}(\mathbb{R}^n)$ to be the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ with

$$p_- = \text{ess inf}_{x \in \mathbb{R}^n} p(x) > 1 \text{ and } p_+ = \text{ess sup}_{x \in \mathbb{R}^n} p(x) < \infty,$$

and $\mathcal{B}(\mathbb{R}^n)$ to be the set of all functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the condition that the Hardy-littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

**Definition 4.1** [35] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{ f \text{ measurable} : \int_{\mathbb{R}^n} \left( \frac{|f(x)|^p}{\eta} \right)^{p(x)} \, dx < \infty \text{ for some constant } \lambda > 0 \right\}.$$

As $p(\cdot) = p$ is a constant, then $L^{p(\cdot)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ coincides with the usual Lebesgue space. It is pointed out in [35] that $L^{p(\cdot)}(\mathbb{R}^n)$ becomes a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \eta > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\eta} \right)^{p(x)} \, dx \leq 1 \right\}.$$

**Lemma 4.2** [41] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if $M_{q_0}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ for some $1 < q_0 < \infty$, where $M_{q_0}(f) = [M(|f|^{q_0})]^{1/q_0}$.

**Lemma 4.3** [42] Let $p(\cdot), p_1(\cdot), \ldots, p_m(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ so that $1/p(x) = 1/p_1(x) + \cdots + 1/p_m(x)$. Then for any $f_j \in L^{p_i}(\mathbb{R}^n)$, $j = 1,2,\ldots,m$, there has

$$\| \prod_{j=1}^m f_j \|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq 2^{m-1} \prod_{j=1}^m \| f_j \|_{L^{p_j(\cdot)}(\mathbb{R}^n)}.$$

**Lemma 4.4** [43] Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose for some fixed $0 < p_0 < \infty$, every $(f,g) \in \mathcal{F}$ and every $\omega \in A_1$,

$$\int_{\mathbb{R}^n} |f(x)|^{p_0} \omega(x) \, dx \leq C_0 \int_{\mathbb{R}^n} |g(x)|^{p_0} \omega(x) \, dx.$$

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $p_0 \leq p_-$. If $\left( \frac{p(\cdot)}{p_0} \right)^{\rho} \in \mathcal{B}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that for all $(f,g) \in \mathcal{F}$, $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \|g\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. 

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Lemma 4.5\textsuperscript{[43]} If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then $C_0^\infty$ is dense in $L^p(\mathbb{R}^n)$.

Lemma 4.6\textsuperscript{[41]} Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then the following conditions are equivalent.

1. $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.
2. $p'(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.
3. $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$ for some $1 < p_0 < p_-$.
4. $(p(\cdot)/p_0)' \in \mathcal{B}(\mathbb{R}^n)$ for some $1 < p_0 < p_-$.

Proof of Theorem 1.5. Here we note $\tilde{f} = (f_1, f_2)$, where $f_1$ and $f_2$ are bounded measurable functions with compact support. Since $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then by Lemma 4.6, there exists a $p_0$ such that $1 < p_0 < p_-$ and $(p(\cdot)/p_0)' \in \mathcal{B}(\mathbb{R}^n)$. Take a $\delta$ such that $0 < \delta < 1/2$. For any $\omega \in A_1$, it follows from Lemma 3.1 and Theorem 1.1 that

$$
\int_{\mathbb{R}^n} |T_\sigma, a_j(\tilde{f})|^p_0 \omega(x) dx \leq C \int_{\mathbb{R}^n} |M_\delta([T_\sigma, a]_j(\tilde{f}))(x)|p_0 \omega(x) dx \\
\leq C \int_{\mathbb{R}^n} [M_\delta([T_\sigma, a]_j(\tilde{f}))(x)|p_0 \omega(x) dx \\
\leq C\|a\|_{\text{Lip}^1} \int_{\mathbb{R}^n} \left[ \prod_{t=1}^{2} M_{q_t}(f_t)(x) \right]^{p_0} \omega(x) dx \\
\leq C\|a\|_{\text{Lip}^1} \int_{\mathbb{R}^n} \left[ \prod_{t=1}^{2} M_{q_t}(f_t)(x) \right] \omega(x) dx.
$$

Applying Lemma 4.4 to the pair $([T_\sigma, a]_j(\tilde{f}), \prod_{t=1}^{2} M_{q_t}(f_t))$, we can get

$$
\|T_\sigma, a_j(\tilde{f})\|_{L^p(\mathbb{R}^n)} \leq C\|a\|_{\text{Lip}^1} \left[ \prod_{t=1}^{2} M_{q_t}(f_t) \right] \leq C\|a\|_{\text{Lip}^1} \prod_{t=1}^{2} M_{q_t}(f_t).
$$

Then by Lemma 4.2 and 4.3, we have

$$
\|T_\sigma, a_j(\tilde{f})\|_{L^p(\mathbb{R}^n)} \leq C\|a\|_{\text{Lip}^1} \left[ \prod_{t=1}^{2} M_{q_t}(f_t) \right] \leq C\|a\|_{\text{Lip}^1} \prod_{t=1}^{2} M_{q_t}(f_t).
$$

This completes the proof of the Theorem 1.5.

Proof of Theorem 1.6. Denote $q_0 = \min\{q_0^1, q_0^2\}$, then $q' < q_0 < \infty$. Let $\tilde{f} = (f_1, f_2)$, where $f_1$ and $f_2$ are bounded measurable functions with compact support. Since $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then by Lemma 4.6, there exists a $p_0$ such that $1 < p_0 < p_-$ and $(p(\cdot)/p_0)' \in \mathcal{B}(\mathbb{R}^n)$. Take $\delta$ and $\epsilon$ such that $0 < \delta < \epsilon < 1/2$. For any $\omega \in A_1$, it follows from Lemma 3.1,
Theorem 1.1 and Theorem 1.2 that

\[ \int_{\mathbb{R}^n} \| \cdot \|_{L^p} \omega(x) \, dx \leq C \int_{\mathbb{R}^n} \left[ M_a \left( \| \cdot \|_{L^p} \omega(x) \right) \right] \omega(x) \, dx \]

\[ \leq C \| \cdot \|_{BMO} \int_{\mathbb{R}^n} \left( M_a \left( \| \cdot \|_{L^p} \omega(x) \right) + \| \cdot \|_{\text{Lip}} \right) \omega(x) \, dx \]

\[ \leq C \| \cdot \|_{BMO} \left( \int_{\mathbb{R}^n} \left[ M_a \left( \| \cdot \|_{L^p} \omega(x) \right) \right] \omega(x) \, dx + \int_{\mathbb{R}^n} \left[ \| \cdot \|_{\text{Lip}} \right] \omega(x) \, dx \right) \]

\[ \leq C \| \cdot \|_{BMO} \left( \int_{\mathbb{R}^n} \left[ \| \cdot \|_{\text{Lip}} \right] \omega(x) \, dx \right) \]

Applying Lemma 4.4 to the pair \(||[T_\sigma, a, j, b, i, (\tilde{f})]\|_{L^p}||\), \(|\prod_{t=1}^{2} M_{q_t} (f_t)\|), we can get

\[ ||[T_\sigma, a, j, b, i, (\tilde{f})]||_{L^p} \leq C \| a \|_{\text{Lip}} \left( \prod_{t=1}^{2} M_{q_t} (f_t) \right) \|_{L^p} \]

Then by Lemmas 4.2-4.3, we have

\[ ||[T_\sigma, a, j, b, i, (\tilde{f})]||_{L^p} \leq C \| a \|_{\text{Lip}} \left( \prod_{t=1}^{2} M_{q_t} (f_t) \right) \|_{L^p} \]

We complete the proof of Theorem 1.6.

5 Endpoint Estimate

In this section, we will show the endpoint estimate for the \([T_\sigma, a, j]\) with \(j = 1, 2\), that is, we will give the proof of Theorem 1.7.

Proof of Theorem 1.7. Take \(p_1, p_2\) such that \(\max\{q', 2\} < p_1, p_2 < \infty\). Let \(1/p = 1/p_1 + 1/p_2\). Then \(1 < p < \infty\). It follows from Lemma 2.3 that \([T_\sigma, a, j] (j = 1, 2)\) is bounded from \(L^{p_1} \times L^{p_2}\) into \(L^p\).

Let \(f, g \in L^\infty\). Then for any ball \(B = B(x_0, r_B)\) with \(r_B > 0\), we decompose \(f\) and \(g\) as follows

\[ f = f \chi_{2B} + f \chi_{(2B)^c} := f^1 + f^2, \quad g = g \chi_{2B} + g \chi_{(2B)^c} := g^1 + g^2. \]

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Then

\[
\frac{1}{|B|} \int_B |[T_\sigma, a_j](f, g)(z) - [T_\sigma, a_j](f^2, g^2)(x_0)| \, dz
\]
\[
\leq \frac{1}{|B|} \int_B |[T_\sigma, a_j](f^1, g^1)(z)| + \frac{1}{|B|} \int_B |[T_\sigma, a_j](f^2, g^1)(z)| \, dz
\]
\[
+ \frac{1}{|B|} \int_B |[T_\sigma, a_j](f^1, g^2)(z)| \, dz
\]
\[
+ \frac{1}{|B|} \int_B |[T_\sigma, a_j](f^2, g^2)(z) - [T_\sigma, a_j](f^2, g^2)(x_0)| \, dz
\]
\[
:= \sum_{s=1}^4 J_s.
\]

It follows from the Hölder’s inequality and Lemma 2.3 that

\[
J_1 \leq \left( \frac{1}{|B|} \int_B |[T_\sigma, a_j](f^1, g^1)(z)|^p \right)^{1/p}
\]
\[
\leq C|B|^{-1/p}||a||_{\text{Lip}^1} ||f^1||_{L^p} ||g^1||_{L^p}
\]
\[
\leq ||a||_{\text{Lip}^1} ||f||_{\infty} ||g||_{\infty}.
\]

By the size conditions in Theorem A of the kernel, we have

\[
J_2 \leq \frac{1}{|B|} \int_B \left( \int_{(2B)^c} \int_{2B} |K(z, y_1, y_2)| f(y_1) |g(y_2)| \, dy_2 \, dy_1 \right) \, dz
\]
\[
\leq C||a||_{\text{Lip}^1} \frac{1}{|B|} \int_B \left( \int_{(2B)^c} \int_{2B} |g(y_2)| \, dy_2 \frac{|f(y_1)|}{|z - y_1|^{2n}} \, dy_1 \right) \, dz
\]
\[
\leq C||a||_{\text{Lip}^1} ||f||_{\infty} ||g||_\infty \left( \int_{2B} |dy_2| \right) \left( \int_{(2B)^c} \frac{1}{|x_0 - y_1|^{2n}} \, dy_1 \right)
\]
\[
\leq C||a||_{\text{Lip}^1} ||f||_{\infty} ||g||_\infty.
\]

Similarly, we can obtain that

\[
J_3 \leq C||a||_{\text{Lip}^1} ||f||_{\infty} ||g||_\infty.
\]

Noting that as \( z \in B \), and \( y_1, y_2 \in (2B)^c \), then \(|y_1 - x_0| \geq 2|z - x_0| \) and \(|y_2 - x_0| \geq 2|z - x_0| \).

It follows from the Hölder’s inequality and (2.3) that

\[
J_4 \leq \frac{1}{|B|} \int_B \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |K(z, y_1, y_2)| f(y_1) - K(x_0, y_1, y_2)| f^2(y_1) |g^2(y_2)| \, dy_1 \, dy_2 \right) \, dz
\]
\[
\leq C \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \frac{1}{|B|} \int_B \int_{2^{k_1} |z - x_0| \leq |y_2 - x_0| \leq 2^{k_2} |z - x_0|} \int_{2^{k_1} |z - x_0| \leq |y_1 - x_0| \leq 2^{k_1+1} |z - x_0|} \, dy_1 \, dy_2
\]
\[
\leq C \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \frac{1}{|B|} \int_B \int_{2^{k_1} |z - x_0| \leq |y_2 - x_0| \leq 2^{k_2} |z - x_0|} \int_{2^{k_1} |z - x_0| \leq |y_1 - x_0| \leq 2^{k_1+1} |z - x_0|} \, dy_1 \, dy_2
\]
\[
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\]
Thus,  
\[ |K(z, y_1, y_2) - K(x_0, y_1, y_2)| |f(y_1)||g(y_2)|dy_1dy_2|dy_1dy_2dz \]
\[ \leq C\|a\|_{\text{Lip}^1}\|f\|\|g\| \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \frac{1}{|B|} \int_B \int_{2^{k_2}|z-z_0| \leq |y_2-z_0| \leq 2^{k_1+1}|z-z_0|} (2^{k_1}|z-x_0|)^{\frac{\gamma}{\sigma}} \]
\[ \times \left( \int_{2^{k_2}|z-z_0| \leq |y_2-z_0| \leq 2^{k_1+1}|z-z_0|} |K(z, y_1, y_2) - K(x_0, y_1, y_2)|^q dy_1 \right)^{1/q} dy_2dz \]
\[ \leq C\|a\|_{\text{Lip}^1}\|f\|\|g\| \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} 2^{\frac{k_1n}{\sigma}} 2^{\frac{k_2n}{\sigma}} (C_{k_1} 2^{-\frac{k_1n}{\sigma}})(C_{k_2} 2^{-\frac{k_2n}{\sigma}}) \]
\[ \leq C\|a\|_{\text{Lip}^1}\|f\|\|g\|. \]

Thus,
\[ ||[T_{\sigma}, a]_j(f, g)||_{\text{BMO}} = \sup_B \frac{1}{|B|} \int_B ||[T_{\sigma}, a]_j(f, g)(z) - ([T_{\sigma}, a]_j(f, g))_B||dz \]
\[ \leq \sup_B \frac{1}{|B|} \int_B ||[T_{\sigma}, a]_j(f, g)(z) - [T_{\sigma}, a]_j(f^2, g^2)(x_0)||dz \]
\[ \leq C\|a\|_{\text{Lip}^1}\|f\|\|g\|. \]

Which completes the proof of the **Theorem 1.7**.

**Conflict of interest**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Authors’ contributions**

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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