Representations of Higman-Thompson Groups from Cuntz Algebras

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Abstract. Every representation of the Cuntz algebra $O_n$ leads to a unitary representation of the Higman-Thompson group $V_n$. We consider the family $\{\pi_x\}_{x \in [0,1]}$ of permutative representations of $O_n$ that arise from the interval map $f(x) = nx \pmod{1}$ acting on the Hilbert space that underlies each orbit, and then study the unitary equivalence and the irreducibility of the corresponding family $\{\rho_x\}_{x \in [0,1]}$ of representations of Higman-Thompson group $V_n$, showing that that these representations are indeed irreducible and moreover $\rho_x$ and $\rho_y$ are equivalent if and only if the orbits of $x$ and $y$ coincide.

Keywords: Higman-Thompson groups, Cuntz algebras, Representations

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1. Introduction

In this paper we investigate the interplay between representations of Cuntz algebras (see [12]) and representations of Higman-Thompson groups $V_n$ (see [11, 14, 9]). In the context of the interval map $f(x) = nx \pmod{1}$, we explain how every point $x \in [0, 1]$ gives rise to a representation $\rho_x$ of the Higman-Thompson group $V_n$ acting on the Hilbert space associated to the orbit of $x$ (which shows up via a representation of the Cuntz algebra $O_n$ on the same Hilbert space). Then the main aim is to study the (unitary) equivalence and the irreducibility of this family $\{\rho_x\}_{x \in [0,1]}$ of representations of Higman-Thompson groups, leading to the main result: $\rho_x$ and $\rho_y$ are equivalent if and only if the orbits (under $f$) of $x$ and $y$ coincide. In order to prove this, we need to use the knowledge of the underlying representations of the Cuntz algebras together with both embeddings of Cuntz algebras and embeddings of Higman-Thompson groups which leads us to extend the results obtained in [3] (for the particular $n = 2$ case) to the general case $n$. The proofs for the general case typically required a different methods of the ones for the $n = 2$ case [3].
Our work is motivated by the recent renewed interest in the well developed theory of representations of the Cuntz algebras $\mathcal{O}_n$ and its application to the representation theory of the Higman-Thompson groups. New examples of concrete representations of these groups are regarded as being important not only to tackle some famous open problems (e.g. the (non)amenability of $F_2$) but also in pursuing the recent Jones’ machinery where subfactors, Higman-Thompson groups and conformal field theory become linked together, see e.g. [2, 8].

We remark that our representations are related to Pythagorean representations in the sense of [7]. Namely, the $n$ generators $t_1, \ldots, t_n$ of the $C^*$-algebra $P_n$ (put forward in [7] as part of Jones’ technology to produce representations of Higman-Thompson groups from those of $P_n$, the so-called Pythagorean representations) decomposes the space in $n$ subspaces

$$t_1^*t_1 + t_2^*t_2 + \cdots + t_n^*t_n = 1,$$

whereas the $n$ generators $s_1, \ldots, s_n$ of the Cuntz algebra $\mathcal{O}_n$ provides an orthogonal decompositions of the space

$$s_1^*s_1 + \cdots + s_n^*s_n = 1, \quad s_i^*s_i = \delta_{ij} 1.$$ 

Then, $P_n$ has many quotients, with $\mathcal{O}_n$ being one of them, mapping $t_i \to s_i^*$. So every representation of $\mathcal{O}_n$ gives rise to a representation of $P_n$, and thus of the Higman-Thompson groups.

In the 1960’s, R.J. Thompson introduced the so called Thompson groups $F_2 \subset T_2 \subset V_2$, see [11]. Then G. Higman generalized $V_2$ to a family $\{V_n\}_{n \in \mathbb{N}}$ of finitely presented discrete groups [14]. K.S. Brown extended the groups $V_n$ to triplets $F_n \subset T_n \subset V_n$, see [9]. These groups are now called Higman-Thompson groups. These are countable and discrete groups and fairly easy to define as certain piecewise linear maps from the interval $[0, 1]$ onto itself, as we shall review below. Almost every question related to these groups is a challenge, typically harder for the smaller groups, for example it is still an open problem whether $F_n$ is an amenable group or not whereas the others contain copies of free groups thus they are nonamenable, see e.g. [13]. We note that several approximation properties for groups are based on asymptotic behaviour of matrix coefficients of representations of the groups in question. Besides this, V.F.R. Jones has recently developed a machinery to produce certain unitary representations of Higman-Thompson groups leading him to yield (unoriented and oriented) knot and link invariants [2] by playing with the matrix coefficients of his representations. It is therefore of interest to determine as much information as possible about the representations theory of the Higman-Thompson groups (the amenability problem springs to mind).

On the other hand we have a fairly well developed theory of Cuntz algebra $\mathcal{O}_n$ representations with an enormous boost since the seminal work by Bratteli and Jorgensen [6]. Recall that a representations of $\mathcal{O}_n$ on a complex Hilbert space is a family of isometries $S_1, \ldots, S_n$ acting on $H$ with orthogonal ranges such that $H$ is subdivided into these ranges. The permutative representations of $\mathcal{O}_n$, where the isometries permute the vectors of a fixed Hilbert basis, have a host of applications, for example to fractals, wavelets, dynamical systems [6, 11, 18].

The link between these two subject, representations of Cuntz algebras and Higman-Thompson groups, was unveiled in [16, 4], where a realization of the Higman-Thompson group $V_n$ as a subgroup of the unitary group of $\mathcal{O}_n$ was fabricated. It is naturally expected that tools from the representation theory of Cuntz algebras can be used in the study of the representation theory of the Higman-Thompson groups. This interplay is what we investigate in this paper. Indeed we clarify why every representation $\pi$ of the Cuntz algebra $\mathcal{O}_n$, on a Hilbert space $H$ gives rise to a unitary representation $\rho_\pi$ of the Higman-Thompson group $V_n$ on the same Hilbert space:

$$\pi \in \text{Rep}(\mathcal{O}_n, H) \quad \mapsto \quad \rho_\pi \in \text{Rep}(V_n, H),$$

(1)
see Theorem 4.2. This is accomplished by regarding the elements of $V_n$ either as piecewise linear maps or as tables. Namely, the Higman-Thompson group $V_n$ is the set of piecewise linear functions $g : [0,1) \to [0,1)$ whose slopes are powers of $n$ and the non-differentiable points live in the $n$-adic set $\mathbb{Z} \left[ \frac{1}{n} \right]$ (with composition being the group operation). Every such function can be represented by a table
\[
\begin{bmatrix}
  a_1 & a_2 & \ldots & a_m \\
  b_1 & b_2 & \ldots & b_m
\end{bmatrix}
\]
where $a_i$ and $b_i$ are multi-indices in \{1, ..., $n$\} related with the $x$- and $y$-axis partitions of $f$, as we will review below. Then
\[
\begin{bmatrix}
  a_1 & a_2 & \ldots & a_m \\
  b_1 & b_2 & \ldots & b_m
\end{bmatrix} \mapsto s_{a_1}a_{a_1}^* + \ldots + s_{a_m}a_{a_m}^*
\]
gives an embedding of $V_n$ into the (unitaries) C*-algebra $\mathcal{O}_n$, where for example $s_{132} = s_1s_3s_2$. Hence every representation $\pi : \mathcal{O} \to B(H)$ on a Hilbert space $H$ gives rise to a representation of $V_n$ by taking the restriction of $\pi$ into the image of the above embedding of $V_n$. Therefore the representations $\rho_\pi : V_n \to B(H)$ can be seen as unitary operators on the Hilbert space $H$ as follows
\[
\begin{bmatrix}
  a_1 & a_2 & \ldots & a_m \\
  b_1 & b_2 & \ldots & b_m
\end{bmatrix} \mapsto \pi(s_{a_1}a_{a_1}^* + \ldots + s_{a_m}a_{a_m}^*)
\]

Then several questions can be asked, for example, if the unitary equivalence or irreducibility is preserved in $\mathcal{O}_n$. We study this problem by considering the permutative representations $\pi_x$ of $\mathcal{O}_n$ on the Hilbert space $H_x$ that encode the orbit of point $x \in [0,1]$ with respect to the interval map $f(x) = nx \pmod{1}$ as in [18], leading to the representation $\rho_x := \rho_{\pi_x}$ of $V_n$. It is usually difficult to distinguish equivalent classes and decide the irreducibility of the Higman-Thompson groups. We however succeeded in characterizing the inequivalent classes and irreducibility of this family $\{\rho_x\}_{x \in [0,1]}$ of representations.

The plan of the rest of the paper is as follows. In Sect. 2 we review some background on the representation theory of operator algebras and of group theory together with the definition of the Higman-Thompson groups as piecewise linear maps and as tables.

In Sect. 3 we first provide and embedding $\iota : \mathcal{O}_{k(n-1)+1} \to \mathcal{O}_n$ as in Lemma 3.1. This embedding together with the embedding $\Psi : V_n \to \mathcal{O}_n$ in [16] leads us to an embedding $E_{k,n} : V_{k(n-1)+1} \to V_n$, which was also proved by Higman [14]. This is proved in Theorem 3.4 where we also show that $E_{k,n}$ maps $T_{k(n-1)+1}$ into $T_n$ and $F_{k(n-1)+1}$ into $F_n$. In Subsect. 3.3 we clarify the Higman-Thompson group elements $g \in V_n$ as a piecewise map $g : [0,1] \to [0,1]$ and as a table $g = \begin{bmatrix}
  a_1 & a_2 & \ldots & a_m \\
  b_1 & b_2 & \ldots & b_m
\end{bmatrix}$, in particular we write explicitly the piecewise map $g \in V_n$ from a table, see Eq. (8).

In Sect. 4 we derive the main results of the paper, starting with the construction of the representation $\rho_\pi$ of the Higman-Thompson group $V_n$ from a Cuntz algebra $\mathcal{O}_n$ representation $\pi$. We remark that since $\pi : \mathcal{O}_n \to B(H)$ is a *-homomorphism, $\pi$ is automatically continuous w.r.t. the norm topologies, and since $V_n$ is a discrete group $\rho_\pi$ is automatically continuous. Then from Subsect. 4.1 we concentrate on the family of representations $\{\rho_x\}_{x \in [0,1]}$ of $V_n$ (with $\rho_x := \rho_{\pi_x}$) that arises from the 1-dimensional dynamical systems that underlies the interval maps $f(x) = nx \pmod{1}$, with $n \in \mathbb{N}$. Here the Hilbert space is attached to the (generalized) orbit $\text{orb}(x) = \bigcup_{k \in \mathbb{Z}} f^k(x)$. In Theorem 4.6 we show that the action $g.y := g(y)$ is well defined for $g \in V_n$ and $y \in \text{orb}(x)$ and moreover the underlying representation of $V_n$ coincides with that of $\rho_x$ – this is proved in several steps. Then we can use the embedding $E_{k,n} : V_{k(n-1)+1} \to V_n$ and obtain a representation $\rho_x^{(n)} \circ E_{k,n}$ of $V_{k(n-1)+1}$ where we denote by $\rho_x^{(n)}$ the already introduced representation $\rho_x$ of $V_n$ on $H_x^{(n)} := H_x$. The unitary operator $U : H_x^{(k(n-1)+1)} \to U(H_x^{(n)}) \subset H_x^{(n)}$ introduced in Definition 4.17 is used to prove that in fact $\rho_x^{(n)} \circ E_{k,n}$ and $\rho_x^{(k(n-1)+1)}$ are unitarily equivalent as in Theorem 4.19.
Using a natural embedding \( t' : \pi_x^{(k(n-1)+1)}(O_k(n-1)+1) \rightarrow B(H_n^2) \) as in Definition 4.15, then using [18] we show in Corollary 4.20 that the representations \( t' \circ \pi_x \) and \( t' \circ \pi_y \) of \( O_k(n-1)+1 \) are unitarily equivalent if and only if \( \text{orb}(x) = \text{orb}(y) \). We then study in Subsect. 4.3 the more involved unitary equivalence and irreducibility of the Higman-Thompson groups representations \( \{\rho_x\}_{x \in [0,1]} \); so that Theorem 4.23 shows that the C*-algebra \( C^*(\pi_x(n)V_n) \) generated by \( \rho_x(n)V_n \) equals the C*-algebra \( \pi_x(O_n) \) on \( H_x \), i.e.

\[
C^*(\pi_x(n)V_n) = \pi_x(O_n)
\]

where the non-amenability of the underlying actions plays a role as well as the recent embedding result of \( V_2 \) into \( V_n \) as in [5]. This leads to Theorem 4.24 where we prove that \( \rho_x \) and \( \rho_y \) are equivalent representations of \( V_n \) if and only if \( \text{orb}(x) = \text{orb}(y) \), i.e.

\[
\rho_x \sim \rho_y \text{ if and only if } \text{orb}(x) = \text{orb}(y),
\]

and moreover they are irreducible.

2. Preliminaries

We start by defining the Cuntz algebras, a set of simple universal C*-algebras first introduced by Cuntz in [12]. The Cuntz algebra \( O_n \) is defined as the (universal) C*-algebra generated by the \( n \) isometries \( \{s_1, s_2, \ldots, s_n\} \) satisfying:

\[
\sum_{j=1}^{n} s_js_j^* = 1, \quad s_is_j = \delta_{ij} 1 \text{ for any } i,j \in \{1,2,\ldots,n\}
\]

where \( 1 \) denotes identity.

Let \( A \) be a C*-algebra. Given a Hilbert space \( H \), we denote by \( B(H) \) the C*-algebra of linear bounded operators in \( H \), where the product of \( B(H) \) is the composition of operators and 1 also denotes the identity of \( B(H) \). A representation of \( A \) is a *-homomorphism \( \pi : A \rightarrow B(H) \) and thus satisfies \( \pi(a^*) = \pi(a)^* \) for all \( a \in A \). \( \pi \) is also automatically continuous for the norm topologies. The representation \( \pi \) is said to be irreducible if it has no invariant subspaces, that is, if there is no non trivial subspace \( K' \subset H \) such that \( \pi(K') \subset K' \). This is equivalent to the commutant of \( \pi \) being \( \mathbb{C}1 \), that is, \( k\pi(a) = \pi(a)k \) (for every \( a \in A \)) if and only if \( k \in \mathbb{C}1 \) (see for example [17]).

Given two Hilbert spaces \( H_1, H_2 \), we say that \( U : H_1 \rightarrow H_2 \) is unitary, if \( UU^* = 1 \) and \( U^*U = 1 \), where \( U^* \) denotes the adjoint operator of \( U \). This is equivalent to \( U \) being onto, and satisfying \( \langle Ux,Uy \rangle_{H_2} = \langle x,y \rangle_{H_1} \) for any \( x,y \in H_1 \). Given a C*-algebra \( A \), the representations \( \pi_1 : A \rightarrow B(H_1) \) and \( \pi_2 : A \rightarrow B(H_2) \) are said to be unitarily equivalent if there exists a unitary \( U : H_1 \rightarrow H_2 \) such that

\[
U \pi_1(a) = \pi_2(a)U \text{ for any } a \in A.
\]

Given a group \( G \) and a Hilbert space \( H \), there are analogous properties for a representation \( \rho \) of \( G \), that is, a group homomorphism from \( G \) to \( B(H) \). We say that \( \rho \) is irreducible if there is no non trivial subspace \( K \subset H \) such that \( \pi(g)(K) \subset K \) for all \( g \in G \). This is equivalent to \( \rho'(G) = \mathbb{C}1 \). We say that \( \rho \) is unitary, if for any \( g \in G \), \( \rho(g^{-1}) = \rho(g)^* \). Finally, given two Hilbert spaces \( H_1 \) and \( H_2 \) not necessarily different, we say that the representations \( \rho_1 : G \rightarrow B(H_1) \), \( \rho_2 : G \rightarrow B(H_2) \) are unitarily equivalent, and write \( \rho_1 \sim \rho_2 \), if there is a unitary operator \( U : H_1 \rightarrow H_2 \) such that \( U \rho_1(g) = \rho_2(g)U \) for every \( g \in G \). Given a representation \( \rho \) of a discrete group \( G \) in a Hilbert space \( H \), let \( C^*_\rho(G) \) denote the C*-subalgebra of \( B(H) \) generated by \( \rho(G) \), so that

\[
C^*_\rho(G) = \text{span}(\{\rho(g) : g \in G\})_{\|a\|_{B(H)}}.
\]
We now define the Higman-Thompson groups, see [11,14]. There are several equivalent realizations of this groups. In this paper, we will consider them as linear piecewise maps in the interval $[0,1]$, and as tables. The relation between these is made explicit in section I. Fix a certain $n \geq 2$, and let $M = \{ \frac{a}{n^k} : a, k \in \mathbb{N} \text{ and } 0 \leq a < n^k \}$. We now define the Higman-Thompson groups, as a particular case of the groups defined in [20].

**Definition 2.1.** The Higman-Thompson group $V_n$ is the group of piecewise linear maps $g : [0,1] \rightarrow [0,1]$ such that:

1. $g$ is bijective in $[0,1]$.
2. $g(M) = M$.
3. $g'(x) = n^k$ for some $k$ in the points where it is differentiable.
4. If $g$ is not differentiable in $x$, then $x \in M$.

The Higman-Thompson group $T_n$ is the subset of $V_n$ such that $g$ has at most one discontinuity. The Higman-Thompson group $F_n$ is the subset of $T_n$ such that $g$ has no discontinuities.

We will now describe the realization of the Higman-Thompson groups as tables, which can also be found in e.g. [16]. An alphabet $A$ is a finite set and its elements are called letters. We will denote by $A^*$ the free monoid generated by $A$, that is, the set of finite sequences $a_1a_2...a_m$ with $a_i \in A$. The length of a word $w \in A^*$ is the number of letters occurring in $w$. For example, $w = a_1a_2a_2$ has length 3. Given $a \in A^*$, we will denote by $aX$ the set $\{ax : x \in X\}$. The set of words on $A$ of infinite length will be denoted by $A^\omega$. Given $x \in A^*$, $y \in A^\omega \cup A^*$, we say that $x$ is a prefix of $y$, if there is a $z \in A^\omega \cup A^*$ such that $y = xz$.

Let $A$ be an alphabet with $n$ letters. An admissible language $L := \{a_1, \ldots, a_m\}$ is a subset of $A^*$ such that $A^\omega = \bigcup_{i=1}^m a_iA^\omega$ and no $a_i$ is a prefix for $a_j$, for all $i, j \in \{1, \ldots, m\}$. Symbolically, $a_iA^\omega \cap a_jA^\omega = \emptyset$, for pairwise different $a_i, a_j$.

We will be concerned with admissible transformations,

$$g = \left[\begin{array}{cccc}
    a_1 & a_2 & \cdots & a_m \\
    b_1 & b_2 & \cdots & b_m
\end{array}\right],$$

where $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$ are admissible languages.

Any admissible transformation induces a permutation of $A^\omega$ as follows: for all $w \in A^\omega$, $T(w) = T(a_iu) = T(a_i)u$. This is well defined because for every $w \in A^\omega$ there is one and only one $a_i \in \{a_1, \ldots, a_m\}$ and $u \in A^\omega$ such that $w = a_iu$.

The composition of two admissible transformations is admissible. Furthermore, the identity transformation on an admissible set induces the identity permutation on $A^\omega$, and switching the rows in $T$ yields the inverse permutation. It is worth observing that for any admissible transformation there is an infinite number of tables one can associate to it.

The Higman-Thompson group $V_n$, is the group of all admissible transformations of a $n$-letters alphabet. We may assume, without loss of generality, that $b_1 < b_2 < \ldots < b_m$, where $\leq$ is the lexicographic order. In order for a table to be associated to a map that belongs to $T_n$, we additionally need to have: $a_i < a_{i+1} < \ldots < a_m < a_1 < \ldots < a_{i-1}$ for some $i \in \{1, 2, \ldots, m\}$. For the table to be associated to a map that belongs to $F_n$, we need to have $i = 1$.

3. **Embeddings of the Higman-Thompson’s groups**

In this section, we use some known embeddings of the Cuntz Algebras to prove some embedding results for the Higman-Thompson groups thus retrieving a result by Higman (see [14]). These embeddings will be critical in Section II.2.

We now setup an embedding of the Cuntz algebras which can be traced back in [12]. For completeness we provide its proof.
Lemma 3.1. Given any integer \( k \geq 1 \), the Cuntz Algebra \( \mathcal{O}_{k(n-1)+1} \) is embedded in \( \mathcal{O}_n \). An embedding is the map \( \iota : \mathcal{O}_{k(n-1)+1} \rightarrow \mathcal{O}_n \) satisfying:

\[
\iota(\hat{s}_1) = s_1^k \quad \iota(\hat{s}_{1+i(n-1)+(j-1)}) = \iota(\hat{s}_{i(n-1)+j}) = s_1^{k-i}s_j
\]

for \( 0 \leq i < k, \ 2 \leq j \leq n \).

Proof. We must prove that the set \( \{ \iota(\hat{s}_i) : 1 \leq i \leq k(n-1)+1 \} \) satisfy the Cuntz relations. We start by proving that:

\[
s_1^{k}(s_1^*)^{k} + \sum_{i=1}^{k} \sum_{j=2}^{n} s_1^{k-i}s_j s_1^*(s_1^{*})^{k-i} = 1
\]

We prove the result by induction in \( k \). For \( k = 1 \) the result comes from the definition of \( \mathcal{O}_n \).

Now, suppose the result is true for \( k = 1 \). Then:

\[
\begin{align*}
\mathcal{O}_{k+1}(s_1^*)^{k+1} + \sum_{i=1}^{k+1} \sum_{j=2}^{n} s_1^{k+1-i}s_j s_1^*(s_1^{*})^{k+1-i} \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{O}_1(s_1^*)^{k} + \sum_{i=1}^{k} \sum_{j=2}^{n} s_1^{k-i}s_j s_1^*(s_1^{*})^{k-i} + \sum_{j=2}^{n} s_j s_1^* \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{O}_1(s_1^*)^{k} + \sum_{i=1}^{k} \sum_{j=2}^{n} s_1^{k-i}s_j s_1^*(s_1^{*})^{k-i} + \sum_{j=2}^{n} s_j s_1^* \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{O}_1(s_1^*)^{k} + \sum_{j=2}^{n} s_j s_1^* \\
\end{align*}
\]

One can easily verify that \( \iota(\hat{s}_i)\iota(\hat{s}_j) = \delta_{ij} \)

We are now going to use the Cuntz algebra embedding \( \iota \) of Lemma 3.1 to yield an embedding of the Higman-Thompson groups. To do this, we will use the map \( \Psi_n : V_n \rightarrow \mathcal{O}_n \) defined as follows

\[
\Psi_n(g) = \Psi \left[ \begin{array}{cccc}
a_1 & a_2 & \cdots & a_m \\
b_1 & b_2 & \cdots & b_m \\
\end{array} \right] = s_{a_1}s_{b_1}^* + s_{a_2}s_{b_2}^* + \cdots + s_{a_m}s_{b_m}^*. \\
\]

as defined in [10], where \( g \in V_n \) is represented by a table as explained in [5]. In [10], it is proven that \( \Psi \) is a faithful unitary representation of \( V_n \) in \( \mathcal{O}_n \). We now check that \( \iota \) maps \( \mathcal{O}_{k(n-1)+1}(V_{k(n-1)+1}) \) to \( \Psi_n(V_n) \).

Using this, we explain how to define a group embedding \( E_{k,n} : V_{k(n-1)+1} \rightarrow V_n \) such that the following diagram commutes

\[
\begin{array}{ccc}
\Psi_{k(n-1)+1}(V_{k(n-1)+1}) & \xrightarrow{\iota} & \Psi_n(V_n) \\
\uparrow & & \downarrow \Psi_n^1 \\
V_{k(n-1)+1} & \xrightarrow{E_{k,n}} & V_n
\end{array}
\]

and then study the corresponding Higman-Thompson group embeddings.

To do this, we need to define some auxiliary maps. In this section, we will denote by \( X \) and \( Y \) the alphabets \( X = \{1, 2, \ldots, n\} \) and \( Y = \{1, 2, \ldots, k(n-1) + 1\} \). In these, we then define a lexicographic order as it should be expected. Given two letters, \( i \) and \( j \), we write \( i < j \) if the natural number \( i \) is less than the natural number \( j \). Then, given \( v, w \in X^* \), we write \( v < w \) if there is a \( j \) such that \( v_j < w_j \), and \( v_i = w_i \) for all \( i < j \). With this, we define our auxiliary functions.
Definition 3.2. Let $\gamma : Y \rightarrow X^*$ be the map such that, for all $y \in Y$:

$$\iota(\hat{s}_y) = s_{\gamma(y)}$$

Also, let $f : Y^* \rightarrow X^*$ be such that

$$f(u) = f(u_1u_2\ldots u_m) = \gamma(u_1)\gamma(u_2)\ldots\gamma(u_m).$$

That is, the map such that:

$$\iota(\hat{s}_u) = \iota(\hat{s}_{u_1u_2\ldots u_m}) = \iota(\hat{s}_{u_1}) \ldots \iota(\hat{s}_{u_m}) = s_{\gamma(u_1)} \ldots s_{\gamma(u_m)} = s_{\gamma(u_1)\gamma(u_2)\ldots\gamma(u_m)} = s_f(u)$$

So, for example, since $\iota(\hat{s}_1) = s_1 \ldots s_1$, we have $\gamma(1) = (1)^k \ldots (1)$. We are now in a position to describe our embedding.

We define $E_{k,n} = \Psi_n \circ \iota \circ \Psi_{k(n-1)+1}$ such that

$$E_{k,n} : \begin{bmatrix} u_1 & u_2 & \ldots & u_m \\ v_1 & v_2 & \ldots & v_m \end{bmatrix} \mapsto \begin{bmatrix} f(u_1) & f(u_2) & \ldots & f(u_m) \\ f(v_1) & f(v_2) & \ldots & f(v_m) \end{bmatrix}.$$  \tag{7}$$

In order to show that $\iota$ maps $\Psi_{k(n-1)+1}(V_{k(n-1)+1})$ to $\Psi_n(V_n)$, and thus that $E_{k,n}$ is well defined, we need to prove that $f$ maps admissible languages in $Y^*$ to admissible languages in $X^*$, that is

$$Y^\omega = \bigcup_{i=1}^m u_i Y^\omega = \bigcup_{i=1}^m v_i Y^\omega \Rightarrow X^\omega = \bigcup_{i=1}^m f(u_i) X^\omega = \bigcup_{i=1}^m f(v_i) X^\omega$$

and $u_i Y^\omega \cap u_j Y^\omega = \emptyset$ implies that $f(u_i) Y^\omega \cap f(u_j) Y^\omega = \emptyset$. Furthermore, in order to prove that $E_{k,n}$ is also an embedding of $T_{1+k(n-1)}$ and $F_{1+k(n-1)}$ in $T_n$ and $F_n$ respectively, one must show that $f$ preserves the lexicographic order, that is, $a < b \Rightarrow f(a) < f(b)$, since this implies that:

$$u_i < u_{i+1} < \ldots < u_m < u_1 < \ldots < u_{i-1} \Rightarrow f(u_i) < f(u_{i+1}) < \ldots < f(u_m) < f(u_1) < \ldots < f(u_{i-1})$$

for all $i$.

We prove an auxiliary result.

Lemma 3.3.

(1) $f$ is injective.

(2) If $f(a)$ is a prefix of $f(b)$, then $a$ is a prefix of $b$.

Proof. (1) We will prove that if $a \neq b$, then $f(a) \neq f(b)$. Start by noticing that for all $i, j \in Y$, $\gamma(i)$ is not a prefix of $\gamma(j)$ if $i \neq j$ and that $\gamma$ is injective. Suppose $a \neq b$. Then let $j$ be the first index such that $a_j \neq b_j$. It follows that $\gamma(a_j) \neq \gamma(b_j)$. We have that $f(a) = \gamma(a_1)\gamma(a_2)\ldots\gamma(a_j)\ldots$ and $f(b) = \gamma(b_1)\gamma(b_2)\ldots\gamma(b_j)\ldots$. If $f(a) = f(b)$, then we would need to have that $\gamma(a_j)$ is a prefix of $\gamma(b_j)$ or vice versa, which is impossible.

(2) We prove the result by induction on the length of $b$. If $b \in Y$, $f(a) = \gamma(a)$ and $f(b) = \gamma(b)$, so if $f(a)$ is a prefix of $f(b)$, $\gamma(a) = \gamma(b)$ and thus $a = b$, from which it follows that $a$ is a prefix of $b$. Now, suppose that $b$ has length $n$. We either have that $f(a) = f(b)$, or that $f(a)$ is a prefix of $f(b_1 \ldots b_{n-1})$. If $f(a) = f(b)$, then by part (1), we have that $a = b$. If $f(a)$ is a prefix of $f(b_1 \ldots b_{n-1})$, then by the induction hypothesis we have that $a$ is a prefix of $b_1 \ldots b_{n-1}$, and hence a prefix of $b$. \qed

We have now the tools to prove the requested Higman-Thompson group embeddings.

Theorem 3.4. The map $E_{k,n}$ is an embedding of $V_{1+k(n-1)}$, $T_{1+k(n-1)}$ and $F_{1+k(n-1)}$ in $V_n$, $T_n$ and $F_n$ respectively.
Proof. We start by showing that \( E_{k,n} \) is well defined by proving that \( \iota \) (see Lemma 3.1) maps \( \Psi_{k(n-1)+1}(V_{k(n-1)+1}) \) to \( \Psi_{n}(V_n) \).

Let \( r \) denote the length of the biggest \( u_i \). Let \( y \in Y^* \) be any word of length more than \( r \). To \( y \) add an infinite number of 1s. This new word must have as prefix one of the \( u_j \). As \( y \) is larger than any \( u_i \) it follows that \( u_j \) is a prefix of \( y \). Now, notice that

\[
X^\omega = \bigcup_{i=1}^{1+k(n-1)} f(i)X^\omega.
\]

Let \( x \in X^\omega \). By the result above, we have that \( x = f(w_1)t_1, t_1 \in X^\omega, w_1 \in Y \). Repeating the steps \( r \) times we conclude \( x = f(w_1) \ldots f(w_{r+1})t_{r+1} = f(w_1 \ldots w_{r+1})t_{r+1} = f(w)t_{r+1} \). Since \( w \in Y^* \) is a word of size \( r + 1 \), we conclude it must have a prefix \( u_i \). Thus, \( f(w) \) must have a prefix \( f(u_i) \) and thus \( x \in f(u_i)X^\omega \). It follows that

\[
X^\omega = \bigcup_{i=1}^{m} f(u_i)X^\omega.
\]

Now we claim that if \( i \neq j \), then \( f(u_i)X^\omega \cap f(u_j)X^\omega = \emptyset \). We have that \( u_iX^\omega \cap u_jX^\omega = \emptyset \) and that \( aX^\omega \cap bX^\omega \neq \emptyset \) is the same as saying that \( a \) is a prefix of \( b \) or vice versa. Suppose \( f(u_i)X^\omega \cap f(u_j)X^\omega \neq \emptyset \). Then, we must have that \( f(u_i) \) is a prefix of \( f(u_j) \), or vice versa. Suppose without loss of generality, that \( f(u_i) \) is a prefix of \( f(u_j) \). By Lemma 3.3 we conclude that \( u_i \) is a prefix of \( u_j \). But this implies that \( u_iX^\omega \cap u_jX^\omega = \emptyset \), a contradiction.

Given that every Cuntz algebra is simple, \( \iota \) must be an injective map. Given that \( \Psi_m \) is a faithful representation, it is also an injective map for any \( m \). Considering \( \iota \) as a group homomorphism, we conclude that \( E_{k,n} = \Psi_n \circ \iota \circ \Psi_{k(n-1)+1} \) is an injective homomorphism from \( V_{k(n-1)+1} \) to \( V_n \). It remains to show that \( f \) preserves the lexicographic order, and therefore \( E_{k,n} \) is also an embedding for \( T_n \) and \( F_n \). First we show that \( f \) preserves the lexicographic order of the letters, and then that of words. Suppose \( a \) and \( b \) are letters in \( Y \). Then, \( f(a) = \gamma(a) \). We have two cases. If \( a = i(n-1)+\alpha, b = j(n-1)+\beta \), then

\[
a < b \Rightarrow \alpha \beta \Rightarrow (1) \ldots (1)(\alpha < (1) \ldots (1)(\beta) \Rightarrow f(a) < f(b).
\]

Else, if \( a = i(n-1)+\alpha, b = j(n-1)+\beta \), then

\[
a < b \Rightarrow i < j \Rightarrow (1) \ldots (1)(\alpha < (1) \ldots (1)(\beta) \Rightarrow f(a) < f(b).
\]

Let \( w \) and \( z \) be words with letters in \( Y \). We can write \( w = w_1 \ldots w_m \) and \( z = z_1 \ldots z_l \). Then, by definition, \( w < z \) means that there is a \( j \) such that \( w_i = z_i \) for any \( i < j \), and that \( w_j < z_j \). Thus, \( \gamma(w_i) = \gamma(z_i) \) for all \( i < j \), and \( \gamma(w_j) < \gamma(z_j) \). Therefore, \( \gamma(w_1) \ldots \gamma(w_{j-1})\gamma(w_j) \ldots < \gamma(z_1) \ldots \gamma(z_{j-1})\gamma(z_j) \ldots \) which implies that \( f(w) < f(z) \). \( \square \)

For other proof of the embedding \( V_{k(n-1)+1} \) into \( V_n \), see Theorem 7.2 of [14]. If we use [16] to represent the elements of \( V_n \) as unitaries \( g \in \mathcal{O}_n \) in the algebraic form \( g = \sum_{i=1}^{m} s_\alpha s_\beta^* \) as in [7], then it is straightforward to see that \( V_{k(n-1)+1} \) into \( V_n \) when using the restriction of the morphism \( \iota \) of Lemma 3.1 to unitaries of \( \mathcal{O}_n \) in such algebraic form.

We further remark that given any \( n, m \geq 2 \), we have that there is an embedding from \( V_n \) to \( V_m \). This is so because Theorem 3.4 implies that \( V_n \) embeds in \( V_2 \) and then a recent result [5] shows that \( V_2 \) can be embedded in \( V_m \).

Given any \( n, m \geq 2 \) it is also known that there are quasi-isometric embeddings from \( F_n \) to \( F_m \), as proven in [10]. Similarly, in [19], it is proven that there is a quasi-isometric embedding from
Definition 3.5. Let \( \phi : X^* \rightarrow \mathcal{P} \) be:

\[
\phi(u) = \phi(u_1u_2\ldots u_m) = \left[ \sum_{i=1}^{m} \frac{u_i - 1}{n^i}, \frac{1}{n^m} \right]
\]

where \( \mathcal{P} = \left\{ \left[ \frac{a}{n^k}, \frac{a+1}{n^k} \right] : a, k \in \mathbb{N}, 0 \leq a < n^k \right\} \).

We start by proving the following lemma.

Lemma 3.6.

(1) \( \phi \) is bijective.

(2) Let \( a, b \in X^* \). Then, \( \phi(b) \subset \phi(a) \) if and only if \( a \) is a prefix of \( b \).

Proof. (1) First, we prove \( \phi \) is injective. Let \( a, b \in X^* \), and suppose \( \phi(a) = \phi(b) \). We can write \( a \) and \( b \) as \( a = a_1a_2\ldots a_k \) and \( b = b_1\ldots b_m \) for some \( k, m \). From the system of equations

\[
\begin{align*}
\sum_{i=1}^{k} \frac{a_i - 1}{n_i} &= \sum_{i=1}^{m} \frac{b_i - 1}{n_i} \\
\sum_{i=1}^{k} \frac{a_i - 1}{n_i} + \frac{1}{n^k} &= \sum_{i=1}^{m} \frac{b_i - 1}{n_i} + \frac{1}{n^m}
\end{align*}
\]

we get \( \frac{1}{n^k} = \frac{1}{n^m} \) and therefore \( k = m \), so that \( a \) and \( b \) have the same length. Thus

\[
\sum_{i=1}^{k} \frac{a_i - 1}{n_i} = \sum_{i=1}^{k} \frac{b_i - 1}{n_i} \Rightarrow \sum_{i=1}^{k} (a_i - 1)n^{k-i} = \sum_{i=1}^{k} (b_i - 1)n^{k-i} \\
\Rightarrow (a_1-1)\ldots(a_k-1)n^k = ((b_1-1)\ldots(b_k-1))n^k,
\]

where \( (x_1x_2\ldots x_k)_n \) is the numeric representation of number \( x = x_1x_2\ldots x_n \) (with \( x_1 > 0, x_i \in \{0,1,2,\ldots,(n-1)\} \)) in base \( n \). Since both words give the same representation in base \( n \) and this representation is unique except for zeroes in the beginning, we conclude that \( a = b \), since \( a \) and \( b \) have the same length.

Now, let \([\alpha, \beta] \in \mathcal{P} \). By the definition of \( \mathcal{P} \), we know that there exist \( k, l \in \mathbb{N} \) such that \( \alpha = kn^{-l} \). Let \( k = (k_1k_2\ldots k_l)_n = \sum_{i=1}^{l} k_i n^{l-i} \). Then:

\[
\phi((k_1+1)(k_2+1)\ldots(k_l+1)) = \left[ \sum_{i=1}^{l} \frac{k_i}{n^i}, \frac{1}{n^l} \right] = \left[ \sum_{i=1}^{l} \frac{k_i n^{l-i}}{n^l}, \sum_{i=1}^{l} \frac{k_i n^{l-i}}{n^l} + \frac{1}{n^l} \right] = [\alpha, \beta].
\]

(2) Let \( a \) be a prefix of \( b \). Then we can write \( a = a_1a_2\ldots a_m, b = a_1a_2\ldots a_m b_{m+1}\ldots b_k \). The result then follows from:
\[
\sum_{i=1}^{m} \frac{a_i - 1}{n^i} \leq \sum_{i=1}^{m} \frac{a_i - 1}{n^i} + \sum_{i=m+1}^{k} \frac{b_i - 1}{n^i} \leq \sum_{i=1}^{m} \frac{a_i - 1}{n^i} + \left( \sum_{i=m+1}^{k} \frac{b_i - 1}{n^i} + \frac{1}{n^k} \right)
\]

where the last inequality follows from the fact that for all \( i \), \( 0 \leq (b_i - 1) \leq n - 1 \) and also

\[
\sum_{i=m+1}^{k} \frac{b_i - 1}{n^i} + \frac{1}{n^k} \leq \sum_{i=m+1}^{k} \frac{n - 1}{n^i} + \frac{1}{n^k} = \sum_{i=m+1}^{k} \left( \frac{1}{n^{i-1}} - \frac{1}{n^i} \right) + \frac{1}{n^k} = \left( \frac{1}{n^m} - \frac{1}{n^k} \right) + \frac{1}{n^k} = \frac{1}{n^m}.
\]

We now prove the converse. Let \( a = a_1 \ldots a_m \), \( b = b_1 \ldots b_k \), and suppose that \( \phi(b) \subset \phi(a) \), but that \( a \) is not a prefix of \( b \). Since \( \phi(b) \subset \phi(a) \), we have that \( n^{-k} \leq n^{-m} \), and thus \( m \leq k \).

Notice that if we denote by \( X_l \) the set of words of size \( l \), then, for all \( l \), \( \phi(X_l) \) is a partition of \( [0,1] \). This follows from the fact that the restriction \( \phi : X_l \rightarrow \{ [\frac{a}{n^l}, \frac{a+1}{n^l}] : 0 \leq a \leq n^l \} \) is also a bijection.

Because \( m \leq k \), if \( a \) is not a prefix of \( b \), then there is a word \( u \neq a \) of size \( m \), such that \( u \) is the prefix of \( b \). But then, by the only if proof above, \( \phi(b) \subset \phi(u) \), and therefore \( \phi(u) \cap \phi(a) \neq \emptyset \). A contradiction! Thus \( a \) must be a prefix of \( b \).

Given \( g \in V_n \), we can write \( g \) as \( \{(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\} \), where \( (a_i, b_i) \in P^2 \), \( \bigcup_{i=1}^{k} a_i = \bigcup_{i=1}^{k} b_i = [0,1] \) and \( a_i \cap b_j = b_i \cap b_j = \emptyset \) if \( i \neq j \). Therefore the elements of \( V_n \) can be written as maps between two partitions \( A, B \) of \( [0,1] \), such that \( A, B \subset P \). Thus, given \( a_i = [\alpha_i, \alpha_i + n^{-k}] \), \( b_i = [\beta_i, \beta_i + n^{-l}] \) and \( x \in a_i \), the map

\[
g(x) = \beta_i + n^{k-l}(x - \alpha_i)
\]

is such that \( g \in V_n \) as in Definition 2.1. Given an element of the Higman-Thompson group \( V_n \), we can write \( \Psi(g) \) explicitly as

\[
\Psi(g) = \Psi\{(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\} = s_{\phi^{-1}(a_1)} s_{\phi^{-1}(b_1)} + s_{\phi^{-1}(a_2)} s_{\phi^{-1}(b_2)} + \ldots + s_{\phi^{-1}(a_k)} s_{\phi^{-1}(b_k)}.
\]

For this to be well defined we need to show that \( \{\phi^{-1}(a_1), \ldots, \phi^{-1}(a_k)\} \) and \( \{\phi^{-1}(b_1), \ldots, \phi^{-1}(b_k)\} \) are admissible sets. This is the aim of the next proposition.

**Proposition 3.7.** Let \( A = \{a_1, \ldots, a_k\} \) be a partition of \([0,1]\), such that \( a_i \in P \) for all \( i \). Then, for all \( i \neq j \):

\[
\bigcup_{m=1}^{k} \phi^{-1}(a_m) X^w = X^w \quad \text{and} \quad \phi^{-1}(a_i) X^w \cap \phi^{-1}(a_j) X^w = \emptyset.
\]

**Proof.** Let \( x \in X^w \). Suppose there is no \( i \) such that \( \phi^{-1}(a_i) \) is a prefix of \( x \). For any \( l \in \mathbb{N} \), we can write \( x \) as \( x_1 x_2 \ldots x_l u \), where \( u \in X^w \). Let \( w = \sum_{i=1}^{l} \frac{x_i - 1}{n^i} \). By definition, \( \phi(x_1 x_2 \ldots x_l) = [w, w + n^{-l}] \). Since \( A \) is a partition of \([0,1]\), we know there is an \( a_j \in A \) such that \( a_j = [a, b] \), and \( w \in a_j \). Since \( l \) is arbitrary, we can choose \( l \) such that \( n^{-l} < b - w \) which implies \( \phi(x_1 x_2 \ldots x_l) \subset a_j \).
By Lemma 3.6 (2), this implies that \( \phi^{-1}(a_j) \) is a prefix of \( x_1x_2\ldots x_l \), and thus of \( x \). A contradiction. It is proved that

\[
\bigcup_{m=1}^{k} \phi^{-1}(a_m)X^\omega = X^\omega.
\]

Now, suppose there are \( i \neq j \) such that \( \phi^{-1}(a_i)X^\omega \cap \phi^{-1}(a_j)X^\omega = \{x\} \) for some \( x \in X^\omega \). Then, both \( \phi^{-1}(a_i) \) and \( \phi^{-1}(a_j) \) will be prefixes of \( x \). By Lemma 3.6 (2), this implies that there exists an \( l \) big enough such that \( \phi(x_1x_2\ldots x_l) \subset a_i \) and \( \phi(x_1x_2\ldots x_l) \subset a_j \), a contradiction, since \( A \) is a partition. Thus, \( \phi^{-1}(a_i)X^\omega \cap \phi^{-1}(a_j)X^\omega = \emptyset \).

\( \square \)

4. Representations of \( V_n \) from the Cuntz algebra \( \mathcal{O}_n \)

Using \( \Psi \) as in (6), we can now associate a representation of the Higman-Thompson group \( V_n \) from every representation of the Cuntz algebra \( \mathcal{O}_n \), acting on the same Hilbert space \( H \).

**Definition 4.1.** Given a Hilbert space \( H \) and a representation \( \pi : \mathcal{O}_n \to B(H) \), we define the map \( \rho_\pi : V_n \to B(H) \) as

\[
\rho_\pi(g) = (\pi \circ \Psi)(g).
\]

We then have the following theorem.

**Theorem 4.2.** \( \rho_\pi \) is a unitary representation of \( V_n \) in \( H \).

**Proof.** Since both \( \pi \) and \( \Psi \) are homomorphisms, we conclude that \( \rho_\pi \) is a representation of \( V_n \) in \( H \). In order to prove that \( \rho_\pi \) is unitary, we must show that

\[
\rho_\pi(g^{-1}) = (\rho_\pi(g))^*.
\]

Using the fact that given a map \( g = \{(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\} \) we can write \( g^{-1} \) as

\[
g^{-1} = \{(b_1, a_1), (b_2, a_2), \ldots, (b_k, a_k)\}
\]

we automatically have

\[
\rho_\pi(g^{-1}) = s_{\phi^{-1}(b_1)}s_{\phi^{-1}(a_1)}^* + \ldots + s_{\phi^{-1}(b_k)}s_{\phi^{-1}(a_k)}^* = (s_{\phi^{-1}(a_1)}s_{\phi^{-1}(b_1)}^* + \ldots + s_{\phi^{-1}(a_k)}s_{\phi^{-1}(b_k)}^*)^* = (\rho_\pi(g))^*.
\]

\( \square \)

4.1. Representations \( \rho_x \) of \( V_n \) on the orbits of the interval map \( f(x) = nx \pmod{1} \). In this subsection, we will study the representations of \( V_n \) in a specific set of Hilbert spaces. Fix \( x \in [0,1] \) and define its (generalized) orbit as

\[
\text{orb}(x) := \{f^z(x) : z \in \mathbb{Z}\}
\]

where \( f(y) = ny \pmod{1} \) with \( y \in [0,1] \). Sometimes we will use the notation \( \text{orb}_n(x) \) instead of \( \text{orb}(x) \). Note that \( \text{orb}(x) \) is the forward and backward orbit of \( x \), thus \( f(x) = nx \pmod{1} \), \( f^{-1}(x) = \{\frac{x}{n}, \frac{x+1}{n}, \ldots, \frac{x+(n-1)}{n}\} \),

\[
f^{-2}(x) = \bigcup_{y \in f^{-1}(x)} f^{-1}(y)
\]

and so on.

Let \( \sim \) be a binary relation in \( [0,1] \) such that \( y \sim x \) if and only if \( y \in \text{orb}(x) \). This is equivalent to existing \( p, k \in \mathbb{N} \) such that \( f^p(y) = f^k(x) \). One can easily prove that \( \sim \) is an equivalence relation.
relation whose equivalence classes are the different orbits. Denote by \( H_x = \ell^2(\text{orb}(x)) \). The set \( \{\delta_y : y \in \text{orb}(x)\} \), where \( \delta_y : \text{orb}(x) \to \mathbb{R} \) is the map

\[
\delta_y(z) = \begin{cases} 
1 & y = z \\
0 & y \neq z 
\end{cases}
\]

is an orthonormal basis of \( H_x \). Since \( \text{orb}(x) \) is a countable union of countable sets, it is a countable set. Therefore, \( H_x \) is a separable Hilbert space.

**Definition 4.3.** Let \( i \in \{1, \ldots, n\} \). Define \( S_i \in B(H_x) \) first on the vector basis

\[
S_i\delta_y = \delta_{y+i(i-1)}
\]

then extend it to the linear span and finally extended to \( H_x \) by density of the basis and continuity of \( S_i \) on the algebraic span of \( \{\delta_y : y \in \text{orb}(x)\} \).

For every \( i \), we easily check that the adjoint of \( S_i \) is as follows

\[
S_i^*\delta_y = \begin{cases} 
\delta_y & y \in \left\{ \frac{j}{n} \mod 1 : j \in \{i-1, i, \ldots, i+n-1\} \right\} \\
0 & \text{otherwise}
\end{cases}
\]

thus \( S_i \) is isometry \( S_i^*S_i = 1 \). One can verify that the operators \( S_i \ (i = 1, \ldots, n) \) satisfy the Cuntz relations. We therefore have the following result.

**Theorem 4.4.** The map \( \pi_x : \mathcal{O}_n \to B(H_x) \) such that \( \pi_x(s_i) = S_i \) is a representation of \( \mathcal{O}_n \) in \( H_x \).

**Notation 4.5.** If confusion arises, we will denote:

1. the representations \( \pi_x \) introduced in Theorem 4.4 by \( \pi_x^{(n)} \);
2. the representation \( \rho_{\pi_x} \) of \( V_n \) by \( \rho_x^{(n)} \) (see Definition 2.1);
3. the Hilbert space \( H_x = \ell^2(\text{orb}(x)) \) by \( H_x^{(n)} \).

The next result is the main result of this section: it shows that the action of \( V_n \) on the set \( \{\delta_y : y \in \text{orb}(x)\} \) is well defined and the underlying representation of \( V_n \) on \( H_x \) coincides with \( \rho_x \).

**Theorem 4.6.** Given \( g \in V_n \) and \( y \in \text{orb}(x) \), we have

1. \( g(y) \in \text{orb}(x) \),
2. \( \pi_x(g)(\delta_y) = \delta_{g(y)} \).

This theorem is going to be proved in a series of lemmas. The first claim is Lemma 4.9. The second claim is Lemma 4.11.

Our first goal is to prove Lemma 4.9. The idea is that for any \( y \), \( g(y) \) will be of the form \( an^k + yn^m \). Therefore, we need to show that multiplying an element of \( \text{orb}(x) \) by powers of \( n \) preserves the orbit, and that for each \( y \in \text{orb}(x) \), there is an element of the form \( an^k + yn^m \) in \( \text{orb}(x) \). We start by proving that powers of \( n \) preserve the orbit.

**Lemma 4.7.** Let \( k \in \text{orb}(x) \). Then, given \( m \in \mathbb{Z}, \ km^m \mod 1 \in \text{orb}(x) \)

**Proof.** The result is obvious for \( m \leq 0 \). For \( m > 0 \), we prove the result by induction on \( m \). We know that, for any given \( k \in \text{orb}(x) \), we have that \((nk) \mod 1 \in \text{orb}(x) \). The result is thus true for \( m = 1 \).

Suppose the result is true for \( m \). Notice that \( n^mk = (n^mk \mod 1) + [n^mk] \). Then, since \( n^{m+1}k \mod 1 = n(n^mk) \mod 1 \)

\[
= n((n^mk \mod 1) + [n^mk]) \mod 1 = n(n^mk \mod 1) \mod 1
\]

and by assumption, \((n^mk \mod 1) \in \text{orb}(x) \), we conclude that \( n^{m+1}k \mod 1 \in \text{orb}(x) \). \( \square \)
We now show that for any \( a \in \mathbb{N} \), and \( y \in \text{orb}(x) \), there is an element of the form \( an^k + yn^m \) in \( \text{orb}(x) \).

**Lemma 4.8.** Let \( k \in \text{orb}(x) \). Then for any \( a \in \mathbb{N} \) such that the number of digits in base \( n \) of \( a \) is \( m \), we have that \( kn^{-m} + an^{-m} \in \text{orb}(x) \).

**Proof.** Let \((a_1a_n \ldots a_m)_n\) be the representation of \( a \) in base \( n \). We claim that \( a_1 > 0 \) and \( a_i \in \{0, 1, 2, \ldots, (n-1)\} \).

We prove the result by induction on \( m \). If \( m = 1 \), we have \( a \in \{1, 2, \ldots, (n-1)\} \). Since, given \( y \in \text{orb}(x) \), and \( h \in \{0, 1, 2, \ldots, (n-1)\} \), \((y + h)/n \in \text{orb}(x) \) the result follows for \( m = 1 \).

Now, let \( j \) be the first number such that \( a_j > 0 \) and \( j > 1 \). We have \( a = a_1n^{m-1} + \ldots \). By the induction hypothesis, we have \( kn^{-m} + (a_1n^{m-1})n^{-m} \in \text{orb}(x) \). Multiplying by \( n^{-j-1} \) and using Lemma 4.7, we obtain \( n^{-j-1}(kn^{-m} + (a_1n^{m-1})n^{-m}) \in \text{orb}(x) \).

Thus \( kn^{-m} + (a_1n^{m-1})n^{-m} \in \text{orb}(x) \) and thus \( kn^{-m} + an^{-m} - a_1n^{-1} + a_1n^{-1} \).

which is in \( \text{orb}(x) \) and therefore \( kn^{-m} + an^{-m} \).

We can now prove Lemma 4.9. Observe that, given \( y \in [0, 1[ \), we have \( g(y) = \alpha n^b + yn^c \), with \( \alpha \in \mathbb{Z} \). Also, since \( g(k) \in [0, 1[ \), we have \( g(y) = \alpha n^b + yn^c \) mod 1. As \( \alpha n^b + yn^c \) mod 1 \( = (\alpha n^b \mod 1) + yn^c \mod 1 \), we can write \( g(y) = n^b + yn^c \mod 1 \), where \( an^b = (\alpha n^b \mod 1) \in [0, 1[ \).

**Lemma 4.9.** Let \( g \in V_n \), \( f(x) = nx \mod 1 \) and \( \text{orb}(x) = \bigcup_{m \in \mathbb{Z}} \{f^m(x)\} \). Then, for all \( y \in \text{orb}(x) \), \( g(y) \in \text{orb}(x) \).

**Proof.** Let \( y \in \text{orb}(x) \). We have that \( g(y) = \alpha n^b + yn^c \mod 1 \). Let \( n^{m-1} < a < n^m \). From Lemma 4.8, we have \( kn^{-m} + an^{-m} \in \text{orb}(x) \) for any \( k \in \text{orb}(x) \). By Lemma 4.7, \( g^m \mod 1 \in \text{orb}(x) \). Replacing \( k \) with \( g(n^{b}) \mod 1 \) we obtain \( gn^{b}) + an^{-m} mod 1 \in \text{orb}(x) \). Using Lemma 4.7, we conclude that \( n^m + g(n^{b}) + an^{-m} \) mod 1 \( = an^b + yn^c \mod 1 \), where \( g(y) \in \text{orb}(x) \).

The goal of the next sequence of lemmas is to prove Lemma 4.14. We start by giving an intuitive idea of the proof. We have that \( \rho_x(g) = S_{\phi^{-1}(a_1)}S_{\phi^{-1}(a_2)} \ldots + S_{\phi^{-1}(a_k)}S_{\phi^{-1}(a_k)} \). We know that there is one and only one \( i \) such that \( y \in b_i \). We will show that \( S_{\phi^{-1}(a)} \delta_y = 0 \) if \( j \neq i \). Then, we will show that \( S_{\phi^{-1}(a_i)}S_{\phi^{-1}(a_i)} \delta_y = \delta_{g(y)} \).

**Lemma 4.10.** Let \( y \in \text{orb}(x) \) and \( u \in X^* \), such that \( u = u_1 \ldots u_k \). Then \( Su \delta_y = Su_1 \ldots Su_k \delta_y = \delta_a \), where

\[
a = yn^k + \sum_{i=1}^{k} \frac{u_i - 1}{n^i}.
\]

**Proof.** We prove the result by induction in \( k \). For \( k = 1 \), we have that \( S_u \delta_y = S_{u_1} \delta_y = \delta_{x+(u_1-1)} \).

Suppose the result is true for \( k - 1 \). Then \( S_u \delta_y = S_{u_1} \ldots S_{u_{k-1}} \delta_y = S_{u_1} \delta_a \).

Then, \( a_{k-1} = yn^{-(k-1)} + \sum_{i=2}^{k} \frac{u_i - 1}{n^{i-1}} \), from which we obtain

\[
a = a_{k-1} + (u_1 - 1) = \frac{yn^{-(k-1)} + \sum_{i=2}^{k} \frac{u_i - 1}{n^{i-1}} + (u_1 - 1)}{n} = yn^k + \sum_{i=1}^{k} \frac{u_i - 1}{n^i}.
\]
Lemma 4.13. Let \( \phi \) be a word. We can apply Lemma 4.12 to conclude that \( S_v \) applies to \( \phi \).

Proof. The proof follows from the definition of \( \phi \): \( b \in \phi(v_{m+1}) \) if and only if \( b \in \phi(v_{m+1}) \) where

\[
\phi = \sum_{i=1}^{m+1} v_i - 1.
\]

The result follows.

Next, given a word \( v \), we want to see how \( S_v \) acts on \( H_x \). Yet, in order to do this, we will first need Lemma 4.11.

Lemma 4.11. Let \( y \in \text{orb}(x) \) and \( v \in X^* \), such that \( v = v_1 \ldots v_m \). Then \( y \in \phi(v) \) if and only if \( b \in \phi(v_{m+1}) \), where

\[
b = n^m \left( y - \sum_{i=1}^{m} \frac{v_i - 1}{n^i} \right).
\]

Proof. The proof follows from the definition of \( \phi \): \( b \in \phi(v_{m+1}) \) if and only if \( b \in \phi(v_{m+1}) \) where

\[
\phi = \sum_{i=1}^{m+1} v_i - 1.
\]

The result follows.

Lemma 4.12. Let \( y \in \text{orb}(x) \) and \( v \in X^* \), such that \( v = v_1 \ldots v_m \) and \( y \in \phi(v) \). Then \( S_v^* \delta_y = \delta_b \), where

\[
b = n^m \left( y - \sum_{i=1}^{m} \frac{v_i - 1}{n^i} \right).
\]

Proof. We will prove the result by induction on \( m \). For \( m = 1 \), we can write \( v = v_1 = i \). If \( y \in \phi(v) = \left[ \frac{i-1}{n}, \frac{i}{n} \right] \), we have that \( S_v^* \delta_y = \delta_{\phi(v)} \).

Now, suppose that the result is true for \( m \). We can use Lemma 3.6 (2) to conclude that, since \( v_1 \ldots v_m \) is a prefix of \( v \), we have that \( \phi(v) \subset \phi(v_1 \ldots v_m) \), and thus \( y \in \phi(v_1 \ldots v_m) \). We can thus apply the induction hypothesis to conclude that \( S_v^* \delta_y = S_{v_{m+1}}^* (S_{v_1}^* \ldots S_{v_m}^* \delta_y) = S_{v_{m+1}}^* \delta_{b_{m+1}} \) where

\[
b_{m+1} = n^{m+1} \left( y - \sum_{i=1}^{m+1} \frac{v_i - 1}{n^i} \right).
\]

By Lemma 4.11 \( b_{m+1} \in \phi(v_{m+1}) \) and thus \( S_{v_{m+1}}^* \delta_{b_{m+1}} = \delta_{\phi(v_{m+1})^{-b_{m+1}}} = \delta_b \) where

\[
b = n^{m+1} \left( y - \sum_{i=1}^{m+1} \frac{v_i - 1}{n^i} \right).
\]

Knowing how \( S_v^* \) acts on \( H_x \), we can now show that if \( y \notin \phi(v) \), \( S_v^* \delta_y = 0 \).

Lemma 4.13. Let \( v \in X^* \). If \( y \notin \phi(v) \) then \( S_v^* \delta_y = 0 \).

Proof. Let \( v \in X^* \), then we can write \( v = v_1 v_2 \ldots v_m \). We will prove by induction on \( m \) that if \( y \notin \phi(v) \), then \( S_v^* \delta_y = 0 \). For \( m = 1 \), we have that \( v \in \{1, 2, \ldots, n\} \), and thus \( \phi(v) = \left[ \frac{i-1}{n}, \frac{i}{n} \right] \). Then \( y \notin \phi(v) \) is equivalent to \( y \notin \left[ \frac{i-1}{n}, \frac{i}{n} \right] \) and thus \( S_v^* \delta_y = 0 \).

Now, suppose the result is true for \( m \) and let \( v = v_1 v_2 \ldots v_{m+1} \). Suppose that \( y \notin \phi(v) \). We have that either \( y \in \phi(v_1 v_2 \ldots v_m) \) or \( y \notin \phi(v_1 v_2 \ldots v_m) \). In the latter case, since \( S_v^* \delta_y = S_{v_{m+1}}^* (S_{v_1}^* \ldots S_{v_m}^* \delta_y) \), we have by the induction hypothesis that \( S_v^* \delta_y = 0 \). In the former case, we can apply Lemma 4.12 to conclude that \( S_v^* \delta_y = S_{v_{m+1}}^* \delta_b \). By Lemma 4.11 \( y \notin \phi(v) \) implies that \( b \notin \phi(v_{m+1}) \). Thus \( S_v^* \delta_y = S_{v_{m+1}}^* \delta_b = 0 \).
The next lemma completes the proof of Theorem 4.6.

Lemma 4.14. Let $y \in \text{orb}(x)$. Then, $\rho_x(g)\delta_y = \delta_{g(y)}$.

Proof. We have that $\rho_x(g) = S_{c_1}S_{d_1}^* + S_{c_2}S_{d_2}^* \ldots + S_{c_i}S_{d_i}^*$ where $c_i, d_i \in X^*$ for all $i$. Let $y \in \text{orb}(x)$. Since there is a bijective correspondence between the $d_i$'s, and a partition of $[0, 1[$, we have that there is a unique $j$ such that $y \in \phi(d_j)$. By Lemma 4.13 we conclude that

$$\rho_x(g)\delta_y = (S_{c_1}S_{d_1}^* + S_{c_2}S_{d_2}^* \ldots + S_{c_i}S_{d_i}^*)\delta_y = S_{c_j}S_{d_j}^*\delta_y.$$  

Let us denote $u = c_j$, and $v = d_j$. By Lemma 4.12 $S_v\delta_y = \delta_b$ where

$$b = n^m \left(y - \sum_{i=1}^m \frac{v_i - 1}{n^i}\right)$$

and by Lemma 4.10 $S_u\delta_b = \delta_{g(y)}$, where

$$g(y) = bn^{-k} + \sum_{i=1}^k \frac{u_i - 1}{n^i} = n^{-k} \left(y - \sum_{i=1}^m \frac{v_i - 1}{n^i}\right) + \sum_{i=1}^k \frac{u_i - 1}{n^i}.$$  

A linear transformation mapping $\phi(u)$ to $\phi(v)$. \qed

4.2. Cuntz algebra embeddings and Higman-Thompson representations. Let $n \in \mathbb{N}$ and $x \in [0, 1[$. Recall the definition of orbit $\text{orb}_n(x)$ from [9] and let $H_{\ast}^{(n)} = l^2(\text{orb}_n(x))$. Our goal in this section is to study the family $\{\rho_x\}_{x \in [0, 1[}$ of representations of the Higman-Thompson group $V_n$ introduced in the previous one.

We start by observing that for all $n > 1$, the Cuntz algebra $O_n$ is simple, as proven by Cuntz in [12]. Hence, every *-homomorphism of $O_n$ is injective. In particular, the maps $\pi_x$ (see Theorem 4.4) and $\iota$ (see Lemma 3.1) are injective. Recall from Notation 4.5 that $\pi_x^{(k(n-1)+1)}$ as the representation of $O_{(n-1)+1}$ in $H_x^{(k(n-1)+1)}$ such that the images of the generators are $S_1$, $S_2$, $S_3$, ..., $S_k$, $S_{(n-1)+1}$. In this fixed context, we also use $S_1$, $S_2$, $S_3$, ..., $S_n$ as the images of the generators of $O_n$ in the representation $\pi_x^{(n)}$ on $H_x^{(n)}$ (see Notation 4.3).

Definition 4.15. Let $\iota': \pi_x^{(k(n-1)+1)}(O_{(n-1)+1}) \to B(H_x^{(n)})$ be defined as

$$\iota'(\hat{S}_1) = S_1^k \quad \iota'(\hat{S}_j) = S_1^{k-j}S_j$$

for $0 \leq i < k$, $2 \leq j \leq n$.

Since all this maps are injective, we automatically get that $\iota'$ is injective. Furthermore, since $\iota$ is a *-homomorphism from $O_{(n-1)+1}$ to $O_n$, it is continuous. Hence, due to the continuity of $\pi_x^{(k(n-1)+1)}$ and $\pi_x^{(n)}$, $\iota'$ is continuous. We obtain the following commutative diagram:

$$\begin{array}{ccc}
O_{(n-1)+1} & \xrightarrow{(\pi_x^{(k(n-1)+1)})^{-1}} & \pi_x^{(k(n-1)+1)}(O_{(n-1)+1}) \\
\downarrow \iota & & \downarrow \iota' \\
O_n & \xrightarrow{\pi_x^{(n)}} & \pi_x'(O_n).
\end{array}$$

Proposition 4.16. Let $y \in \text{orb}_m(x)$ for some $m \geq 2$. Then $\delta_y = T_1T_2 \ldots T_k\delta_x$ for some $T_1, T_2, \ldots, T_k \in \{\hat{S}_1, \ldots, \hat{S}_m, \hat{S}_1^*, \ldots, \hat{S}_m^*\}$.
that for any $z \in \mathbb{Z}$, $f(x) = mx \mod 1$. We can represent the orbit by a graph whose vertices are the elements of $\text{orb}_m(x)$, and $y - z$ if there is an $i \in \{0, \ldots, (m-1)\}$ such that $y = \frac{z + ri}{m}$, or $z = \frac{y + ri}{m}$. The graph must be connected, since it is inductively constructed from $x$ adding the vertices corresponding to the operations $y \mapsto \frac{y}{m}$ and $y \mapsto ym \mod 1$. Let us denote by $x_0, x_1, x_2, \ldots, x_k$ a path starting in $x_0 = x$ and ending on $x_k = y$. Given $x_j$ and $x_{j+1}$, we either have that $x_{j+1} = \frac{x_j + 1}{m}$ or $x_j = \frac{x_{j+1} + 1}{m}$. In the first case, we conclude that $\delta_{x_{j+1}} = \hat{S}_{j+1} \delta_{x_j}$. In the second case, we conclude that $\delta_{x_{j+1}} = \hat{S}_j \delta_{x_j}$. Repeating the process $k$ times, we conclude that there exist $T_1, T_2, \ldots, T_k \in \{\hat{S}_1, \ldots, \hat{S}_m, \hat{S}_1^*, \ldots, \hat{S}_m^*\}$ such that $\delta_y = T_1 T_2 \ldots T_k \delta_x$. □

We now use Proposition 4.16 in the following definition.

**Definition 4.17.** We define $U : H_x^{(k(n-1)+1)} \to H_x^{(n)}$ as the map such that, given $\delta_y \in H_x^{(k(n-1)+1)}$

$$U(\delta_y) = U(T_1 \ldots T_k \delta_x) = \iota'(T_1 \ldots T_k) \delta_x.$$ 

We start by showing that $U$ is well defined. Suppose that $\delta_y = T_1 \ldots T_k \delta_x$ and $\delta_y = L_1 \ldots L_l \delta_x$. Then, because $\iota'$ is a map, $\iota'(L_1 \ldots L_l) = \iota'(T_1 \ldots T_k)$, and thus $U(T_1 \ldots T_k \delta_x) = U(L_1 \ldots L_l \delta_x)$. Having defined $U$ on the basis of $H_x^{(k(n-1)+1)}$, we can extend it to the linear span and then by continuity and density to the whole space $H_x^{(k(n-1)+1)}$.

We claim that $U : H_x^{(k(n-1)+1)} \to U(H_x^{(k(n-1)+1)})$ is unitary. In fact, $U$ is injective and linear since $\iota'$ is injective and linear. Therefore, given $\delta_y, \delta_z \in H_x^{(k(n-1)+1)}$, we have

$$\langle U\delta_y, U\delta_z \rangle_{H_x^{(n)}} = \begin{cases} 1 & ; U\delta_y = U\delta_z \\ 0 & ; U\delta_y \neq U\delta_z \end{cases} = \begin{cases} 1 & ; \delta_y = \delta_z \\ 0 & ; \delta_y \neq \delta_z \end{cases} = \langle \delta_y, \delta_z \rangle_{H_x^{(k(n-1)+1)}}.$$ 

Hence, $U$ is unitary in $H_x^{(k(n-1)+1)}$ because it is unitary on the linear span of its basis. The claim follows.

Let $\rho_x^{(n)} : V_n \to B(H_x^{(n)})$ be the representation of $V_n$ in $H_x^{(n)}$ (see Notation 4.5). We want to study the relation between $\rho_x^{(k(n-1)+1)}$ and $\rho_x^{(n)}(E_{k,n}(g))$, the restriction of $\rho_x^{(n)}$ to the elements of a subgroup of $V_n$ isomorphic to $V_{k(n-1)+1}$. (Recall that $E_{k,n}$ is the embedding of $V_{k(n-1)+1}$ in $V_n$, introduced in Theorem 3.4.) The following theorem gives us the relation between these maps.

**Theorem 4.18.** For $g \in V_{k(n-1)+1}$ we have $\rho_x^{(n)}(E_{k,n}(g)) = \iota'(\rho_x^{(k(n-1)+1)}(g)).$

Proof. We will represent $g$ as a table. From Theorem 3.4 we have that

$$\rho_x^{(n)}(E_{k,n}(g)) = \rho_x^{(n)} \left( E_{k,n} \left( \begin{array}{cccc} a_1 & \ldots & a_m \\ b_1 & \ldots & b_m \end{array} \right) \right) = \rho_x^{(n)} \left( \begin{array}{cccc} f(a_1) & \ldots & f(a_m) \\ f(b_1) & \ldots & f(b_m) \end{array} \right) = S_{f(a_1)} S_{f(b_1)}^* + \ldots + S_{f(a_m)} S_{f(b_m)}^*.$$

On the other hand, from definition 3.2, we have

$$\iota'(\rho_x^{(k(n-1)+1)}(g)) = \iota'(\hat{S}_{a_1} \hat{S}_{b_1}^* + \ldots + \hat{S}_{a_m} \hat{S}_{b_m}^*) = \iota'(\hat{S}_{a_1}) \iota'(\hat{S}_{b_1}^*) + \ldots + \iota'(\hat{S}_{a_m}) \iota'(\hat{S}_{b_m}^*) = S_{f(a_1)} S_{f(b_1)}^* + \ldots + S_{f(a_m)} S_{f(b_m)}^*.$$ 

□

The operator $U$ turns out to be an unexpectedly powerful tool. In fact, Theorem 4.18 implies that for any $g \in V_{k(n-1)+1}$ and some $U(\xi) \in U(H_x^{(k(n-1)+1)})$,\n
$$\rho_x^{(n)}(E_{k,n}(g))U(\xi) = \iota'(\rho_x^{(k(n-1)+1)}(g))U(\xi) = U(\rho_x^{(k(n-1)+1)}(g)(\xi))$$

(10)
where $\xi \in H_x^{(k(n-1)+1)}$. Therefore, $U(H_x^{(k(n-1)+1)})$ is a proper subset of $H_x^{(n)}$ and invariant under $\rho_x^{(n)}(E_{k,n}(g))$ for any $g$. Therefore, the representation $\rho_x^{(n)} \circ E_{k,n}$ of $V_{k(n-1)+1}^{(n)}$ in $H_x^{(n)}$ is not irreducible. In fact, we will prove that $\rho_x^{(n)} \circ E_{k,n}$ is an irreducible representation of $V_{k(n-1)+1}^{(n)}$ on $U(H_x^{(k(n-1)+1)})$ instead. In order to do this, we will need Theorem 4.19 which will allow us to relate what happens in $H_x^{(k(n-1)+1)}$ to what happens in $U(H_x^{(k(n-1)+1)})$.

We can adapt the proof in Eq. (10) and check that $(t' \circ \pi_x)(a)(U(\xi)) \in U(H_x^{(k(n-1)+1)})$ for any $a \in \mathcal{O}_{k(n-1)+1}$ and $\xi \in H_x^{(k(n-1)+1)}$. This means that besides the representation $\pi_x : \mathcal{O}_{k(n-1)+1} \to B(H_x^{(k(n-1)+1)})$ we also have a well defined representation $t' \circ \pi_x : \mathcal{O}_{k(n-1)+1} \to B(U(H_x^{(k(n-1)+1)}))$. We now relate these two families of Cuntz algebras representations of the Cuntz algebra $\mathcal{O}_{k(n-1)+1}$.

**Theorem 4.19.** Let $x, y \in [0, 1]$. Then:

1. $\pi_x$ and $t' \circ \pi_x$ are unitarily equivalent;
2. $\pi_x$ is unitarily equivalent to $\pi_y$ if and only if $t' \circ \pi_x$ is unitarily equivalent to $t' \circ \pi_y$;
3. $\pi_x$ is irreducible in $H_x^{(k(n-1)+1)}$ if and only if $t' \circ \pi_x$ is irreducible in $U(H_x^{(k(n-1)+1)})$.

**Proof.** (1) Since we already proved that $U : H_x^{(k(n-1)+1)} \to U(H_x^{(k(n-1)+1)})$ is unitary, it remains to show that given $a \in \mathcal{O}_{k(n-1)+1}, y \in \text{orb}_{k(n-1)+1}(x)$, we have $U(\pi_x(a)\delta_y) = (t'(\pi_x(a)))U(\delta_y)$, where $U$ was introduced in Definition 4.17. Thus

\[
(t'(\pi_x(a))U)\delta_y = t'(\pi_x(a))(U(T_1 \ldots T_1 \delta_x)) \quad \text{(by Proposition 4.16)}
\]

\[
= t'(\pi_x(a))U(T_1 \ldots T_1 \delta_x) \quad \text{(by Definition 4.17)}
\]

\[
= t'(\pi_x(a))T_1 \ldots T_1 \delta_x \quad \text{(Since $t'$ is linear)}
\]

\[
= U(\pi_x(a)T_1 \ldots T_1 \delta_x) \quad \text{(by Definition 4.17)}
\]

\[
= U(\pi_x(a)\delta_y).
\]

(2) Suppose that $\pi_x \sim \pi_y$, that is, that $\pi_x$ and $\pi_y$ are unitarily equivalent. Then, there is a unitary $K : H_x^{(k(n-1)+1)} \to H_y^{(k(n-1)+1)}$ such that $K\pi_x(a) = \pi_y(a)K$, for any $a \in \mathcal{O}_{k(n-1)+1}$. Using part (1) we obtain that $t'(\pi_y(a))(U_2^*KU_1^*) = (U_2^*KU_1^*)t'(\pi_x(a))$ which corresponds to the commutative diagram

\[
\begin{array}{cccccc}
U_1(H_x^{(k(n-1)+1)}) & \mathcal{U}_1 & H_x^{(k(n-1)+1)} & K & H_y^{(k(n-1)+1)} & U_2(H_y^{(k(n-1)+1)}) \\
\downarrow t'(\pi_x(a)) & & \downarrow \pi_x(a) & & \downarrow \pi_y(a) & \downarrow t'(\pi_y(a)) \\
U_1(H_x^{(k(n-1)+1)}) & \mathcal{U}_1 & H_x^{(k(n-1)+1)} & K & H_y^{(k(n-1)+1)} & U_2(H_y^{(k(n-1)+1)})
\end{array}
\]

Hence $t' \circ \pi_x \sim t' \circ \pi_y$. Now, suppose that $t' \circ \pi_x \sim t' \circ \pi_y$. Then, there is a unitary $K : U_1(H_x^{(k(n-1)+1)}) \to U_2(H_x^{(k(n-1)+1)})$ such that $K(t' \circ \pi_x)(a) = (t' \circ \pi_y)(a)K$, for any $a \in \mathcal{O}_{k(n-1)+1}$. Using Theorem 4.19 we get $\pi_y(a)(U_2^*KU_1) = (U_2^*KU_1)\pi_x(a)$. Hence $\pi_x \sim \pi_y$.

(3) Suppose that $\pi_x$ is not irreducible in $H_x^{(k(n-1)+1)}$. Then, there is a $T \not\in \mathbb{C}1$ such that $\pi_x(a)T = T\pi_x(a)$, for any $a \in \mathcal{O}_{k(n-1)+1}$. By the previous diagram, this means that $t'(\pi_x(a))(UTU^*) = (UTU^*)t'(\pi_x(a))$. Suppose that $(UTU^*) = t$, where $t \in \mathbb{C}1$. Then, $U = U^*tU = t(U^*U) = t$, since U is unitary. Therefore $(UTU^*) \not\in \mathbb{C}1$, and so $t'(\pi_x)$ is not irreducible in $U(H_x^{(k(n-1)+1)})$. An analogous argument implies that $t'(\pi_x)$ is not irreducible in $U(H_x^{(k(n-1)+1)})$ if $\pi_x$ is not irreducible in $H_x^{(k(n-1)+1)}$. The result follows.

Theorem 6 in [13] implies that $\pi_x \sim \pi_y$ if and only if $x \sim y$, and that $\pi_x$ is irreducible in $H_x^{(k(n-1)+1)}$. Note that $t' \circ \pi_x : \mathcal{O}_{k(n-1)+1} \to B(U(H_x^{(k(n-1)+1)}))$ for every $x$. Therefore, using Theorem 4.19 we immediately obtain the following additional corollary.

\[\square\]
Corollary 4.20. Let \( x, y \in [0, 1] \). We have that \( t' \circ \pi_x \sim t' \circ \pi_y \) if and only if \( x \sim y \). Also, \( t' \circ \pi_x \) is irreducible in \( U(H_x^{(k(n-1)+1)}) \).

Given \( g \in V_{k(n-1)+1} \), we have by Definition [4.11] that \( \rho_x^{(k(n-1)+1)}(g) = \pi_x(\Psi(g)) \), where \( \Psi(g) \in \mathcal{O}_{k(n-1)+1} \). Given \( g \in V_{k(n-1)+1} \), Theorem 4.18 tells us then that \( \rho_x^{(n)}(E_{k,n}(g)) = t'(\rho_x^{(k(n-1)+1)}) = t'(\pi_x(\Psi(g))) \). Furthermore, by considering the restriction of \( \pi_x \) to \( \Psi(V_{k(n-1)+1}) \) (a subset of \( \mathcal{O}_{k(n-1)+1} \)), Theorem 4.19 gives us this corollary.

Corollary 4.21. Let \( x \in [0, 1] \). Then:

1. \( \rho_x^{(k(n-1)+1)} \) and \( \rho_x^{(n)} \circ E_{k,n} \) are unitarily equivalent of \( V_{k(n-1)+1} \) on \( H_x^{(k(n-1)+1)} \) and \( U(H_x^{(k(n-1)+1)}) \), respectively;
2. \( \rho_x^{(k(n-1)+1)} \sim \rho_y^{(k(n-1)+1)} \) if and only if \( \rho_x^{(n)} \circ E_{k,n} \sim \rho_y^{(n)} \circ E_{k,n} \). Also, \( \rho_x^{(k(n-1)+1)} \) is an irreducible representation in \( H_x^{(k(n-1)+1)} \) if and only if \( \rho_x^{(n)} \circ E_{k,n} \) is irreducible in \( U(H_x^{(k(n-1)+1)}) \).

4.3. Unitarily equivalence and irreducibility of \( \{ \rho_x \}_{x \in [0,1]} \). We now study the unitarily equivalence and irreducibility of the Higman-Thompson groups representations \( \{ \rho_x \}_{x \in [0,1]} \).

We need to introduce some concepts and an auxiliary result. A probability measure \( \mu \) on a set \( Y \) is said to be finitely additive, if for any collection of finite pairwise disjoint subsets of \( Y \), we have

\[
\mu \left( \bigcup_{i=1}^{m} A_i \right) = \sum_{i=1}^{m} \mu(A_i).
\]

A group action of a discrete group \( G \) is said to be non-amenable, if there is no finitely additive probability measure \( \mu \) in \( Y \) such that, for any subset \( A \) of \( Y \), and some \( g \in G \), we have

\[
\mu(gA) = \mu(A).
\]

Given a group \( G \) with a subgroup \( H \), if the action of \( H \) in \( Y \) is non-amenable, then so is the action of \( G \) in \( Y \) since a \( G \)-invariant probability measure \( \mu \) would be \( H \)-invariant. We now quote a result from [13].

Theorem 4.22.

1. The action \( g \cdot y = g(y) \) of \( V_2 \) on \( [0,1] \) is non-amenable.
2. Suppose that \( G \) is a discrete group acting on a set \( X \), and let \( \rho \) denote the induced representation on \( L^2(X) \). Then the action of \( G \) on \( X \) is non-amenable if and only if there exist \( g_1, \ldots, g_m \) in \( G \) so that

\[
\frac{1}{m} \left\| \sum_{k=1}^{m} \rho(g_k) \right\| < 1.
\]

Let \( \pi_x: \mathcal{O}_n \rightarrow B(H_x^{(n)}) \) be the representation of \( V_n \) as in Notation 4.5. The following result is the last ingredient needed for the proof of Theorem 4.24.

Theorem 4.23. Let \( n \geq 2 \). Then

\[
C_{\rho_x^{(n)}}^*(V_n) = \pi_x(\mathcal{O}_n).
\]

Proof. Since \( \rho_x^{(n)}(g) = \pi_x(\Psi(g)) \), we have that \( \rho_x^{(n)}(V_n) \subset \pi_x(\mathcal{O}_n) \). Therefore, \( C_{\rho_x^{(n)}}^*(V_n) \subset \pi_x(\mathcal{O}_n) \).

In order to prove that \( \pi_x(\mathcal{O}_n) \subset C_{\rho_x^{(n)}}^*(V_n) \), we will show that the set \( \{S_1, S_2, \ldots, S_n\} \) is in \( C_{\rho_x^{(n)}}^*(V_n) \).

We will start by proving that \( \{S_1 S_1^*, S_2 S_2^*, \ldots, S_n S_n^*\} \subset C_{\rho_x^{(n)}}^*(V_n) \). Let \( J_i \) denote the set of \( h \in V \) such that \( h(z) = z \) for all \( z \in [\frac{i-1}{n}, \frac{i}{n}] \). If \( a, b \) are maps such that \( a(y) = b(y) = y \) for all
where we recall that by Theorem 4.6 the action coincides with the representation \( v \) is injective. Finally, given \( g \in V_n \), the map \( p \in K_2 \) such that

\[
p(x) = \begin{cases} (h^{-1}gh)(x) & x \in \left[0, \frac{1}{n}\right] \\ x & x \in \left[\frac{1}{n}, 1\right] \end{cases}
\]

satisfies \( v(p) = g \). We conclude that \( v \) is an isomorphism, and that \( K_2 \) is isomorphic to \( V_n \). One can prove that \( K_1 \) is isomorphic to \( V_n \) by considering the map

\[
h : [0, 1] \to \left[\frac{n-1}{n}, 1\right], \quad h(x) = (n - 1 + x)/n
\]

instead. Therefore, \( V_n \) is embedded in \( J_i \) for every \( i \). Since by [5], \( V_2 \) is embedded in \( V_n \), we conclude that \( V_2 \) is embedded in \( J_i \).

We have that the elements of \( J_i \) preserve the elements of the set \( \mathcal{X}_i \), where \( \mathcal{X}_i \) is the set

\[
\mathcal{X}_1 = \left[0, \frac{1}{n}\right] \cap \text{orb}_n(x), \quad \mathcal{X}_i = \left[i - \frac{1}{n}, i\right] \cap \text{orb}_n(x).
\]

The action \( g \cdot y = g(y) \) of \( V_2 \) on \([0, 1]\) is non-amenable (Theorem 4.22). Since \( V_2 \) is embedded in \( J_i \), the action \( g \cdot y = g(y) \) of \( J_i \) on \( \mathcal{X}_i \) is also non-amenable. By Theorems 4.6 and 4.22, there exist \( g_1, \ldots, g_m \in J_i \) such that

\[
\frac{1}{m}\|\sum_{k=1}^{m} \rho_x^{(n)}(g_k)\| < 1,
\]

where we recall that by Theorem 4.6 the action coincides with the representation \( \rho_x^{(n)} \). Let

\[
t = \frac{1}{m} \sum_{k=1}^{m} \rho_x^{(n)}(g_k).
\]

Observe that \( 1 = S_1S_1^* + S_2S_2^* + \ldots + S_nS_n^* \), and that, given \( a, b \in \{S_1S_1^*, S_2S_2^*, \ldots, S_nS_n^*\} \), we have \( ab = \delta_{ab}a \) and \( ta = at \), so that for any \( k \in \mathbb{N} \)

\[
t^k = (t1)^k = (t(S_1S_1^* + \ldots + S_2S_2^*))^k = (tS_1S_1^*)^k + \ldots + (tS_nS_n^*)^k.
\]

On the other hand, since for any \( g \in J_i \), we have \( \rho_x^{(n)}(g) = \ldots + S_iS_i^* + \ldots \), we conclude that \( (tS_iS_i^*) = S_iS_i^* \), and thus \( (tS_iS_i^*)^k = S_iS_i^* \). Since \( \|t\| < 1 \) and \( \|S_iS_i^*\| = 1 \), we conclude that \( t^k \) converges to \( S_iS_i^* \). Thus, \( S_iS_i^* \in C_{\rho_x^{(n)}}(V_n) \) for all \( i \). We will now use this to prove that \( S_i \in C_{\rho_x^{(n)}}(V_n) \) for all \( i \).

First notice that if \( S_n \) is in \( C_{\rho_x^{(n)}}(V_n) \), then so is \( S_i \) for any \( i \). In fact, consider the following table

\[
g = \begin{bmatrix} 1 & \ldots & i1 & i2 & \ldots & in & \ldots & a & \ldots & n1 & n2 & \ldots & nn \\ 1 & \ldots & n1 & n2 & \ldots & nn & \ldots & a & \ldots & i1 & i2 & \ldots & in \end{bmatrix} \in V_n
\]

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Then, noticing that for any $j \neq n$, $S_j S_n = 0$, and that only the words starting with $i$ are mapped to words starting with $n$, we obtain $\rho_x^{(n)}(g)S_n =$

$$(S_1 S_1^* S_n^* + S_1 S_2 S_2^* S_n^* + \ldots + S_n S_n S_n^* S_n) S_n = S_i (S_1 S_1^* + S_2 S_2^* + \ldots + S_n S_n^*) = S_i$$

Therefore $\rho_x^{(n)}(g)S_n = S_i$. Thus, all we need to do now is to prove that $S_n \in C_{\rho_x^{(n)}}(V_n)$. We start by proving that $S_i S_i^* \in C_{\rho_x^{(n)}}(V_n)$ for all $i$. To do this, consider the table on $\nu_n$ defined by $k =$

$$\begin{bmatrix}
1 & \ldots & i & \ldots & n \\
1 & \ldots & n & \ldots & i
\end{bmatrix}$$

Then, $\rho(\nu)(S_i^* S_i^*) = (S_i S_i^*) = S_i S_i^*$. Therefore, $S_i S_i^* \in C_{\rho_x^{(n)}}(V_n)$. Let us consider the following table

$$l = \begin{bmatrix}
1 & \ldots & i & \ldots & n \\
1 & \ldots & n & \ldots & i
\end{bmatrix} \in V_n$$

We have that $\rho(l)(S_i S_i^*) = (S_i S_i^*) = S_i S_i^*$. Hence $S_n S_i S_i^* \in C_{\rho_x^{(n)}}(V_n)$ for all $i$. Therefore $S_n S_i S_i^* + S_n S_2 S_2^* + \ldots + S_n S_n S_n^* = S_n (S_1 S_1^* + S_2 S_2^* + \ldots + S_n S_n^*) = S_n(1) = S_n$

which is in $\in C_{\rho_x^{(n)}}(V_n)$. This concludes the proof.

We will now use Corollary 4.20 and Theorem 4.23 to prove Theorem 4.24 (1).

**Theorem 4.24.** Let $n \geq 2$. Then:

1. $\rho_x^{(n)} \sim \rho_y^{(n)}$ if and only if $x \sim y$;
2. $\rho_x^{(n)} \circ E_{k,n} \sim \rho_y^{(n)} \circ E_{k,n}$ if and only if $x \sim y$;
3. $\rho_x^{(n)}$ is an irreducible representations of $V_n$ on $H_x^{(n)}$.

**Proof.** (1) Suppose $x \sim y$. Then, $H_x^{(n)} = H_y^{(n)}$, and thus, $\rho_x^{(n)} = \rho_y^{(n)}$. Now, suppose that $\rho_x^{(n)} \sim \rho_y^{(n)}$. By definition, there must exist a unitary operator $K : H_x^{(n)} \to H_y^{(n)}$ such that, given $g \in V_n$, $\rho_x^{(n)}(g) = K \rho_y^{(n)}(g) K^*$ which can be rewritten as $\pi_x(\Psi(g)) = K \pi_y(\Psi(g)) K^*$. Our goal is to show that, given $a \in O_n$, $\pi_x(a) = K \pi_y(a) K^*$. In order to do this, let us consider the following subset of $O_n$: $B = \text{span}(\{\Psi(g) : g \in V_n\})$. Then, given $b \in B$, we have that $b = \sum_{i=1}^m c_i \Psi(g)$ for some $c_i \in \mathbb{C}, m \in \mathbb{N}$. Furthermore, we have that $K \pi_y(b) K^* = K \pi_y(\sum_{i=1}^m c_i \Psi(g)) K^* = \sum_{i=1}^m c_i K(\pi_y(\Psi(g))) K^* = \sum_{i=1}^m c_i \pi_x(\Psi(g)) = \pi_x(b)$.

Let $a \in O_n$. By Theorem 4.23 there is a sequence $a_n$ in $B$ that converges to $a$. By continuity of $\pi_x$ and $\pi_y$, we conclude that $\pi_x(a) = K \pi_y(a) K^*$ and thus, $\pi_x \sim \pi_y$, which implies that $x \sim y$ by Corollary 4.20.

(2) It follows from Corollary 4.21 and part (1) of this theorem.

(3) Theorem 4.23 also gives us that $\rho_x^{(n)}$ is irreducible in $H_x^{(n)}$, given that

$$(\rho_x^{(n)}(V_n))' = (\text{span}(\rho_x^{(n)}(V_n)))' = (\text{span}(\{\rho_x^{(n)}(g) : g \in V_n\}))' = C_{\rho_x^{(n)}}(V_n)' = \pi_x(\mathcal{O}_n)'$$

and then, since Theorem 6 in [18] implies that $\pi_x$ is irreducible in $H_x^{(n)}$, $\rho_x^{(n)}$ is irreducible in $H_x^{(n)}$. \qed

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