THE MODULI SPACE OF FLAT $G$-BUNDLES ON A COMPACT HYPER-KÄHLER MANIFOLD

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1. Introduction

1.1. Let $(X, g)$ be a compact connected oriented Riemannian manifold, and let $G$ be a compact connected Lie group with Lie algebra $\mathfrak{g}$. We study the moduli space of flat $G$-bundles on $X$, which can be identified with the quotient

$$\text{Loc}_G(X) = \text{Hom}(\pi_1(X), G)/G,$$

where $\text{Hom}(\pi_1(X), G)$ denotes the space of all group homomorphisms from the fundamental group $\pi_1(X)$ into $G$, and $G$ acts on this space by conjugation. Let $\phi : \pi_1(X) \to G$ be a homomorphism, and let $[\phi]$ denote the corresponding point of $\text{Loc}_G(X)$. Note that $\phi$ defines a $\pi_1(X)$-module structure on $\mathfrak{g}$ via the adjoint representation of $G$. Let us denote this $\pi_1(X)$-module by $\mathfrak{g}_\phi$, and let $E_\phi$ denote the associated flat vector bundle on $X$. It is well known (see, e.g., [Hi]) that if $[\phi]$ is a smooth point of $\text{Loc}_G(X)$, then there is a natural identification of the tangent space

$$T_{[\phi]} \text{Loc}_G(X) \cong H^1_{DR}(X, E_\phi).$$

The space on the right is the first de Rham cohomology of $X$ with coefficients in the flat bundle $E_\phi$.

Let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a fixed positive definite $G$-invariant inner product, which always exists since $G$ is compact. It defines a positive definite metric on $E_\phi$ which is compatible with the flat connection. Thus we have a notion of a harmonic form on $X$ with coefficients in $E_\phi$ ([De], Chapter VI, §3.3), and Hodge theory gives an isomorphism

$$\mathcal{H}^1(X, E_\phi) \cong H^1_{DR}(X, E_\phi),$$

where the space on the left is the space of harmonic 1-forms on $X$ with coefficients in $E_\phi$. (Cf. [De], Theorem VI.3.17. That result is stated for flat Hermitian vector bundles; however, the proof works just as well in our situation.) Let

$$\langle \cdot , \cdot \rangle : \mathcal{H}^1(X, E_\phi) \times \mathcal{H}^1(X, E_\phi) \to C^\infty(X)$$

denote the obvious pairing obtained by combining the metric on $\Omega^1_X$ induced by $g$ and the metric on $E_\phi$ induced by $B$. It allows us to define an $L^2$ inner product on $\mathcal{H}^1(X, E_\phi)$ by

$$(s, t) \mapsto \int_X \langle s, t \rangle \cdot \text{vol}_g,$$
where \( \text{vol}_g \) denotes the volume form associated to the metric \( g \). The corresponding metric on (the smooth locus of) \( \text{Loc}_G(X) \) induced by the identifications (1.1) and (1.2) is called the \( L^2 \) metric on the moduli space of flat \( G \)-bundles on \( X \).

1.2. Our main result is the following

**Theorem 1.1.** If \( X \) is hyper-Kähler, then the \( L^2 \) metric on the smooth locus of the moduli space of flat \( G \)-bundles on \( X \) is a hyper-Kähler metric.

This result has been known before in the case \( \dim X = 4 \). See, e.g., §7 of [Hi], where it is proved using the hyper-Kähler reduction technique. If \( \dim X > 4 \), it is not known to us how to exhibit \( \text{Loc}_G(X) \) as the hyper-Kähler reduction of another hyper-Kähler manifold. Thus we have to use a different approach, which was suggested by Karshon’s work [Ka]. Using her results we define three symplectic forms on the smooth locus \( \text{Loc}^\circ_G(X) \) of \( \text{Loc}_G(X) \) which are compatible with the \( L^2 \) metric, such that the associated complex structures produce the desired hyper-Kähler structure on \( \text{Loc}^\circ_G(X) \).

The details of the proof are presented in Section 2.

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2. Proof of Theorem 1.1

2.1. Let us define a pseudo-hyper-Kähler manifold to be a manifold \( X \) equipped with a pseudo-Riemannian metric \( g \) and an action of the algebra of quaternions \( \mathbb{H} \) on the tangent bundle \( TX \) which is parallel with respect to the Levi-Civita connection of \( g \). Then \( X \) is hyper-Kähler if and only if \( g \) is positive definite. The basic theory of hyper-Kähler manifolds does not use the positive definiteness condition; in particular, if \( X \) is a pseudo-hyper-Kähler manifold, then every imaginary unit quaternion \( I \in \mathbb{H} \) gives rise to an integrable almost complex structure on \( X \) and a real symplectic form \( \omega_I \) defined by \( \omega_I(v, w) = g(Iv, w) \) for all tangent vectors \( v, w \in TX \).

We begin with the following result, which is undoubtedly well known to the experts.

**Proposition 2.1.** Let \( X \) be a real manifold equipped with three symplectic forms \( \omega_I, \omega_J, \omega_K \).

(a) Assume that \( \omega_{I,J,K} \), when viewed as isomorphisms \( TX \to T^*X \), satisfy the conditions

\[
\begin{align*}
\omega_I^{-1} \circ \omega_J &= -\omega_J^{-1} \circ \omega_I, \\
\omega_I^{-1} \circ \omega_K &= -\omega_K^{-1} \circ \omega_I, \\
\omega_K^{-1} \circ \omega_I &= -\omega_I^{-1} \circ \omega_K.
\end{align*}
\]

Then the isomorphism \( g := \omega_I \circ \omega_J^{-1} \circ \omega_K : TX \to T^*X \) corresponds to a (possibly indefinite) nondegenerate metric on \( TX \), and \( X \) is pseudo-hyper-Kähler with respect to this metric.

(b) Consequently, if the following condition holds:

\[
\text{the isomorphism } g = \omega_I \circ \omega_J^{-1} \circ \omega_K : TX \to T^*X
\]

then \( X \) is hyper-Kähler.
Proof. It clearly suffices to prove (a). For any morphism of vector bundles $A : TX \to T^*X$, we will denote by $A^* : T^*X \to TX$ the adjoint map. Using the relations $\omega^i_{IJK} = -\omega_{IJK}$ and (2.1), it is easy to check that $g^* = g$, whence $g$ defines a symmetric bilinear form on $TX$, which is obviously nondegenerate. Next define $I, J, K : TX \to TX$ by $\omega_I(v, w) = g(Iv, w)$, etc. Again, using (2.1), it is straightforward to check that these automorphisms satisfy the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -1.$$  

(2.3)

Finally, it is well known that if a Riemannian manifold is equipped with three almost complex structures $I, J, K$ satisfying (2.3), then the manifold is hyper-Kähler if and only if the associated forms $\omega_{IJK}$ are closed. For a proof, see, e.g., [Ka], Lemma 3.37. This result does not use the positive definiteness assumption, thus it also applies in our situation, since $\omega_{IJK}$ are closed by assumption. This completes the proof of the proposition. \hfill $\square$

2.2. We now recall one of the main results of [Ka]. If $X$ is a compact connected Kähler manifold and $G$ is as in Section 1, then for every $\phi \in \text{Hom}(\pi_1(X), G)$, we have a well defined map

$$L : \mathcal{H}^*(X, E_{\phi}) \longrightarrow \mathcal{H}^{*+2}(X, E_{\phi}), \quad \tau \mapsto \omega \wedge \tau,$$

where $\omega$ is the Kähler form. The hard Lefschetz theorem holds in this situation and implies in particular that if $\dim \mathbb{R} X = 2n$, then the map

$$L^{n-1} : \mathcal{H}^1(X, E_{\phi}) \longrightarrow \mathcal{H}^{2n-1}(X, E_{\phi})$$

is an isomorphism. (Both the fact that $L$ is well defined, i.e., takes harmonic forms to harmonic forms, and the fact that the last map is an isomorphism, follow at once from the corresponding statements for harmonic forms with coefficients in $\mathbb{R}$ and the observation that a differential form on $X$ with coefficients in $E_{\phi}$ can be written locally as $\tau = \sum \tau_j \otimes e_j$, where $\{e_j\}$ is a flat local orthonormal frame for $E_{\phi}$. Then $\tau$ is harmonic if and only if the coefficients $\tau_j$ are harmonic forms on $X$. This follows trivially from the definitions of the Laplace operators on $\Omega^*_X$ and $\Omega^*_X \otimes E_{\phi}$, see [De], Chapter VI, §3.3.)

With this notation, we have

**Proposition 2.2** ([Ka], Theorem 5). The bilinear forms $\varpi_{\phi}$ on $\mathcal{H}^1(X, E_{\phi}) \cong T_{[\phi]} \text{Loc}^0_G(X)$ defined by

$$\varpi_{\phi}(\tau, \eta) = \int_X \tau \wedge L^{n-1}(\eta)$$

are nondegenerate, skew-symmetric, and together form a closed 2-form on $\text{Loc}^0_G(X)$.  

2.3. Now let $X$ be a compact connected hyper-Kähler manifold with underlying Riemannian metric $g$. We choose three particular complex structures $I, J, K$ satisfying the quaternionic relations (2.3). They give rise to the three corresponding Kähler forms $\omega_{IJK}$. If $\phi \in \text{Hom}(\pi_1(X), G)$, we let

$$L_I, L_J, L_K : \mathcal{H}^*(X, E_{\phi}) \longrightarrow \mathcal{H}^{*+2}(X, E_{\phi})$$

denote the corresponding operators on the cohomology of $X$ with coefficients in $E_{\phi}$. As in Proposition 2.2, they induce three nondegenerate forms

$$\varpi_{I,\phi}, \varpi_{J,\phi}, \varpi_{K,\phi} : \mathcal{H}^1(X, E_{\phi}) \times \mathcal{H}^1(X, E_{\phi}) \longrightarrow \mathbb{R}$$

which define three 2-forms $\varpi_{IJK}$ on $\text{Loc}^0_G(X)$ that are closed by Karshon’s result. It is now obvious that our Theorem 1 follows from the following more precise
Proposition 2.3. The three symplectic forms \( \varpi_{I,J,K} \) on \( \text{Loc}^G_0(X) \) satisfy the relations \( 2.4 \). Moreover, the isomorphism \( \varpi_I \circ \varpi_J^{-1} \circ \varpi_K : T\text{Loc}^G_0(X) \rightarrow T^* \text{Loc}^G_0(X) \) corresponds to the \( L^2 \) metric on \( \text{Loc}^G_0(X) \).

2.4. The proof of Proposition 2.3 reduces immediately to linear algebra, since it only involves statements about harmonic forms on \( X \). Thus, for simplicity, we will change our setup and notation as follows. Let \( V \) be a hyper-Kähler vector space, i.e., a finite dimensional real vector space equipped with a positive definite inner product \( g \) and three endomorphisms \( I, J, K \) satisfying \( 2.3 \). The symplectic forms \( \omega_{I,J,K} \) and the operators \( L_{I,J,K} : \bigwedge^n V^* \rightarrow \bigwedge^{n+2} V^* \) are defined in the same way as before. It is clear that in the proof of Proposition 2.3 the bundle \( E_\phi \) can be ignored completely, since all the relations that we need to check hold trivially “in the direction of \( E_\phi \).” Now \( \varpi_{I,J,K} \) become symplectic forms on the dual space \( V^* \), and it is straightforward to check that the relations \( 2.4 \) in this situation reduce to

\[
(L_{I}^{-1}n)^{-1} \circ L_{J}^{-1} = -(L_{J}^{-1})^{-1} \circ L_{I}^{-1}, \quad \text{etc.,}
\]

where \( \dim_{\mathbb{R}} V = 2n \). (Of course, by symmetry, it suffices to check the first equation only.) We will show that, in fact, the automorphism \( (L_{I}^{-1})^{-1} \circ L_{J}^{-1} \) of \( V^* \) coincides with \( K^*: V^* \rightarrow V^* \), the adjoint of the automorphism \( K \). By symmetry, it will follow that \( (L_{J}^{-1})^{-1} \circ L_{I}^{-1} = -K^* \), which will imply \( 2.4 \). Moreover, these equations will also imply that \( \varpi_I \circ \varpi_J^{-1} \circ \varpi_K : V^* \rightarrow V^{**} \) corresponds to the dual metric \( g^* \) on \( V^* \), and the proof of Proposition 2.3 will be complete.

Let \( \star : \bigwedge^n V^* \rightarrow \bigwedge^{2n-n} V^* \) denote the Hodge star operator, see \[De\], Chapter VI, §3.1. We observe that \( \star^{-1} \circ L_{I}^{-1} = n! \cdot I^* \) on \( V^* \). Indeed, let \( v, w \in V^* \). Then, by definition,

\[
g^*(v, \star^{-1} L_{I}^{-1} w) \cdot \text{vol}_{g} = v \wedge L_{I}^{-1} w = \omega_{I}^{-1} v \wedge w = \omega_{I}(v^\flat, w^\flat) \cdot \omega_{I},
\]

where \( v^\flat \in V \) corresponds to \( v \) under the isomorphism \( V \rightarrow V^* \) defined by \( g \). But it is well known that \( \omega_{I}^n = n! \cdot \text{vol}_{g} \); on the other hand,

\[
\omega_{I}(v^\flat, w^\flat) = g(Iv^\flat, w^\flat) = g^*(v, I^* w).
\]

This shows that \( \star^{-1} \circ L_{I}^{-1} = n! \cdot I^* \), and similarly \( \star^{-1} \circ L_{J}^{-1} = n! \cdot J^* \), as endomorphisms of \( V^* \). Finally, this implies that

\[
(L_{I}^{-1}n)^{-1} \circ L_{J}^{-1} = (\star^{-1} \circ L_{I}^{-1})^{-1} \circ \star^{-1} \circ L_{J}^{-1} = (n! \cdot I^*)^{-1} \circ (n! \cdot J^*) = (I^*)^{-1} \circ J^* = (J \circ I^{-1})^*.
\]

which finishes the proof of Proposition 2.3.

References

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