Research Article

Fekete-Szegő Problems for Quasi-Subordination Classes

Maisarah Haji Mohd and Maslina Darus
School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Malaysia

Correspondence should be addressed to Maslina Darus, maslina@ukm.my

Received 7 August 2012; Accepted 15 September 2012

Copyright © 2012 M. Haji Mohd and M. Darus. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

An analytic function \( f \) is quasi-subordinate to an analytic function \( g \), in the open unit disk if there exist analytic functions \( \phi \) and \( w \), with \( |\phi(z)| \leq 1 \), \( w(0) = 0 \) and \( |w(z)| < 1 \) such that

\[
f(z) = \phi(z)g(w(z)).
\]

Certain subclasses of analytic univalent functions associated with quasi-subordination are defined and the bounds for the Fekete-Szegő coefficient functional \( |a_3 - \mu a_2^2| \) for functions belonging to these subclasses are derived.

1. Introduction and Motivation

Let \( \mathbb{A} \) be the class of analytic function \( f \) in the open unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \) normalized by \( f(0) = 0 \) and \( f'(0) = 1 \) of the form \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \). For two analytic functions \( f \) and \( g \), the function \( f \) is subordinate to \( g \), written as follows:

\[
f(z) \prec g(z),
\]

if there exists an analytic function \( w \), with \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f(z) = g(w(z)) \). In particular, if the function \( g \) is univalent in \( \mathbb{D} \), then \( f(z) \prec g(z) \) is equivalent to \( f(0) = g(0) \) and \( f(\mathbb{D}) \subset g(\mathbb{D}) \). For brief survey on the concept of subordination, see [1].

Ma and Minda [2] introduced the following class

\[
S^*(\phi) = \left\{ f \in \mathbb{A} : \frac{zf'(z)}{f(z)} < \phi(z) \right\},
\]

(1.2)
where \( \phi \) is an analytic function with positive real part in \( \mathbb{D} \), \( \phi(\mathbb{D}) \) is symmetric with respect to the real axis and starlike with respect to \( \phi(0) = 1 \) and \( \phi'(0) > 0 \). A function \( f \in \mathcal{S}^*(\phi) \) is called Ma-Minda starlike (with respect to \( \phi \)). The class \( \mathcal{C}(\phi) \) is the class of functions \( f \in \mathcal{A} \) for which \( 1 + zf''(z)/f'(z) < \phi(z) \). The class \( \mathcal{S}^*(\phi) \) and \( \mathcal{C}(\phi) \) include several well-known subclasses of starlike and convex functions as special case.

In the year 1970, Robertson [3] introduced the concept of quasi-subordination. For two analytic functions \( f \) and \( g \), the function \( f \) is quasi-subordinate to \( g \), written as follows:

\[
f(z) \prec_q g(z),
\]

if there exist analytic functions \( \varphi \) and \( w \), with \( |\varphi(z)| \leq 1 \), \( w(0) = 0 \) and \( |w(z)| < 1 \) such that \( f(z) = \varphi(z)g(w(z)) \). Observe that when \( \varphi(z) = 1 \), then \( f(z) = g(w(z)) \), so that \( f(z) \prec g(z) \) in \( \mathbb{D} \). Also notice that if \( w(z) = z \), then \( f(z) = \varphi(z)g(z) \) and it is said that \( f \) is majorized by \( g \) and written \( f(z) \ll g(z) \) in \( \mathbb{D} \). Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization. See [4–6] for works related to quasi-subordination.

Throughout this paper it is assumed that \( \phi \) is analytic in \( \mathbb{D} \) with \( \phi(0) = 1 \). Motivated by [2, 3], we define the following classes.

**Definition 1.1.** Let the class \( \mathcal{S}^*_q(\phi) \) consists of functions \( f \in \mathcal{A} \) satisfying the quasi-subordination

\[
\frac{zf''(z)}{f'(z)} - 1 \prec_q \phi(z) - 1.
\]

**Example 1.2.** Since

\[
\frac{zf''(z)}{f'(z)} - 1 = z(\phi(z) - 1) \prec_q \phi(z) - 1,
\]

the function \( f : \mathbb{D} \to \mathbb{C} \) defined by the following:

\[
f(z) = z \exp\left(-z + \int_0^z d\xi \phi(\xi)\right)
\]

belongs to the class \( \mathcal{S}^*_q(\phi) \).

**Definition 1.3.** Let the class \( \mathcal{C}_q(\phi) \) consists of functions \( f \in \mathcal{A} \) satisfying the quasi-subordination

\[
\frac{zf''(z)}{f'(z)} \prec_q \phi(z) - 1.
\]
Example 1.4. The function \( f : \mathbb{D} \to \mathbb{C} \) defined by the following:

\[
f(z) = \int_0^z \exp \left( -\zeta + \int_0^\zeta \phi(\xi)d\xi \right) d\zeta
\]

belongs to the class \( C_q(\phi) \).

The classes \( S^*_q(\phi) \) and \( C_q(\phi) \) are analogous to the Ma-Minda starlike and convex classes defined in the form of quasi-subordination.

Definition 1.5. Let the class \( R_q(\phi) \) consist of functions \( f \in \mathcal{A} \) satisfying the quasi-subordination

\[
f'(z) - 1 \prec q \phi(z) - 1.
\]

Example 1.6. The function \( f : \mathbb{D} \to \mathbb{C} \) defined by the following:

\[
f(z) = z - \frac{z^2}{2} + \exp \left( \int_0^z \phi(\xi)d\xi \right)
\]

belongs to the class \( R_q(\phi) \).

It is known that a function \( f \in \mathcal{A} \) with \( \text{Re} f'(z) > 0 \) in \( \mathbb{D} \) is univalent. The above class of functions defined in terms of the quasi-subordination is associated with the class of functions with positive real part.

Functions in the following classes, \( \mathcal{M}_q(\alpha, \phi) \) and \( \mathcal{L}_q(\alpha, \phi) \) are analogous to the \( \alpha \)-convex functions of Miller et al. [7] and \( \alpha \)-logarithmically convex functions introduced by Lewandowski et al. [8] (see also [9]), respectively.

Definition 1.7. Let the class \( \mathcal{M}_q(\alpha, \phi) \), \( (\alpha \geq 0) \) consist of functions \( f \in \mathcal{A} \) satisfying the quasi-subordination

\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) - 1 \prec q \phi(z) - 1.
\]

Example 1.8. The function \( f : \mathbb{D} \to \mathbb{C} \) defined by the following:

\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) - 1 = z(\phi(z) - 1)
\]

belongs to the class \( \mathcal{M}_q(\phi) \).
Definition 1.9. Let the class $\mathcal{L}_q(\alpha, \phi)$, ($\alpha \geq 0$) consist of functions $f \in \mathcal{A}$ satisfying the quasi-subordination
\[
\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} - 1 \prec \phi(z) - 1.
\] (1.13)

Example 1.10. The function $f : \mathbb{D} \to \mathbb{C}$ defined by the following:
\[
\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} - 1 = z(\phi(z) - 1)
\] (1.14)

belongs to the class $\mathcal{L}_q(\phi)$.

It is well known (see [10]) that the $n$-th coefficient of a univalent function $f \in \mathcal{A}$ is bounded by $n$. The bounds for coefficient give information about various geometric properties of the function. Many authors have also investigated the bounds for the Fekete-Szegő coefficient for various classes [11–25]. In this paper, we obtain coefficient estimates for the functions in the above defined classes.

Let $\Omega$ be the class of analytic functions $w$, normalized by $w(0) = 0$, and satisfying the condition $|w(z)| < 1$. We need the following lemma to prove our results.

Lemma 1.11 (see [26]). If $w \in \Omega$, then for any complex number $t$
\[
|w_2 - tw_1^2| \leq \max\{1, |t|\}.
\] (1.15)

The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

2. Main Results

Although Theorems 2.1 and 2.4 are contained in the corresponding results for the classes $\mathcal{M}_q(\alpha, \phi)$ and $\mathcal{L}_q(\alpha, \phi)$, they are stated and proved separately here because of the importance of the classes.

Throughout, let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, $\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$, $\varphi(z) = c_0 + c_1z + c_2z^2 + c_3z^3 + \cdots$, $B_1 \in \mathbb{R}$ and $B_1 > 0$.

Theorem 2.1. If $f \in \mathcal{A}$ belongs to $\mathcal{S}_q^0(\phi)$, then
\[
|a_2| \leq B_1,
\]
and, for any complex number $\mu$,
\[
|a_3 - \mu a_2^2| \leq \frac{1}{2} \left(B_1 + \max\left(B_1, B_1^2 + |B_2|\right)\right),
\] (2.1)

and, for any complex number $\mu$,
\[
|a_3 - \mu a_2^2| \leq \frac{1}{2} \left(B_1 + \max\left(B_1, |1 - 2\mu| B_1^2 + |B_2|\right)\right).
\] (2.2)
Proof. If $f \in S_2^c(\phi)$, then there exist analytic functions $\varphi$ and $w$, with $|\varphi(z)| \leq 1$, $w(0) = 0$ and $|w(z)| < 1$ such that

$$\frac{zf'(z)}{f(z)} - 1 = \varphi(z)(\phi(w(z)) - 1). \quad (2.3)$$

Since

$$\frac{zf'(z)}{f(z)} - 1 = a_2z + \left(-a_2^2 + 2a_3\right)z^2 + \cdots,$$  \quad (2.4)

$$\phi(w(z)) - 1 = B_1w_1z + \left(B_1w_2 + B_2w_1^2\right)z^2 + \cdots,$$

$$\varphi(z)(\phi(w(z)) - 1) = B_1c_0w_1z + \left(B_1c_1w_1 + c_0\left(B_2 + B_2w_1^2\right)\right)z^2 + \cdots,$$ \quad (2.5)

it follows from (2.3) that

$$a_2 = B_1c_0w_1$$

$$a_3 = \frac{1}{2}\left(B_1c_1w_1 + B_1c_0w_2 + c_0\left(B_2 + B_2^2c_0 - 2\mu B_1^2c_0\right)w_1^2\right). \quad (2.6)$$

Since $\varphi(z)$ is analytic and bounded in $\mathbb{D}$, we have [27, page 172]

$$|c_n| \leq 1 - |c_0|^2 \leq 1 \quad (n > 0). \quad (2.7)$$

By using this fact and the well-known inequality, $|w_1| \leq 1$, we get

$$|a_2| \leq B_1. \quad (2.8)$$

Further,

$$a_3 - \mu a_2^2 = \frac{1}{2}\left(B_1c_1w_1 + c_0\left(B_1w_2 + \left(B_2 + B_2^2c_0 - 2\mu B_1^2c_0\right)w_1^2\right)\right). \quad (2.9)$$

Then

$$|a_3 - \mu a_2^2| \leq \frac{1}{2}\left(|B_1c_1w_1| + \left|B_1c_0\left(w_2 - \left(2\mu B_1c_0 - B_1c_0 - \frac{B_2}{B_1}\right)w_1^2\right)\right|\right). \quad (2.10)$$

Again applying $|c_n| \leq 1$ and $|w_1| \leq 1$, we have

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2}\left(1 + \left|w_2 - \left(-(1 - 2\mu)B_1c_0 - \frac{B_2}{B_1}\right)w_1^2\right|\right). \quad (2.11)$$
Applying Lemma 1.11 to

\[ |w_2 - \left(-\left(1 - 2\mu\right)B_1c_0 - \frac{B_2}{B_1}\right)w_1^2| \]  

(2.12)
yields

\[ |a_3 - \mu a_2^2| \leq \frac{B_1}{2} \left(1 + \max\left\{1, \left|-\left(1 - 2\mu\right)B_1c_0 - \frac{B_2}{B_1}\right|\right\} \right). \]  

(2.13)

Observe that

\[ \left|-\left(1 + 2\mu\right)B_1c_0 + \frac{B_2}{B_1}\right| \leq B_1|c_0|1 - 2\mu + \left|\frac{B_2}{B_1}\right|, \]  

(2.14)

and hence we can conclude that

\[ |a_3 - \mu a_2^2| \leq \frac{B_1}{2} \left(1 + \max\left\{1, B_1|1 - 2\mu| + \left|\frac{B_2}{B_1}\right|\right\} \right). \]  

(2.15)

For \(\mu = 0\), the above will reduce to the estimate of \(|a_3|\). \(\square\)

**Remark 2.2.** For \(\varphi(z) \equiv 1\), Theorem 2.1 gives a particular case of the estimates in [13, Theorem 1] for \(p = 1\) and [14, Theorem 2.1] for \(k = 1\).

**Theorem 2.3.** If \(f \in \mathcal{A}\) satisfies

\[ \frac{zf'(z)}{f(z)} - 1 \ll \varphi(z) - 1, \]  

(2.16)

then the following inequalities hold:

\[ |a_2| \leq B_1, \]  

\[ |a_3| \leq \frac{1}{2} \left(B_1 + B_1^2 + |B_2|\right), \]  

(2.17)

and, for any complex number \(\mu\),

\[ |a_3 - \mu a_2^2| \leq \frac{1}{2} \left(B_1 + |1 - 2\mu|B_1^2 + |B_2|\right). \]  

(2.18)

**Proof.** The result follows by taking \(w(z) = z\) in the proof of Theorem 2.1. \(\square\)
Abstract and Applied Analysis

**Theorem 2.4.** If $f \in \mathcal{A}$ belongs to $\mathcal{C}_q(\varphi)$, then

\[
|a_2| \leq \frac{B_1}{2},
\]

\[
|a_3| \leq \frac{1}{6} \left( B_1 + \max \left\{ B_1, B_1^2 + |B_2| \right\} \right),
\]  \hspace{1cm} (2.19)

and, for any complex number $\mu$,

\[
|a_3 - \mu a_2^2| \leq \frac{1}{6} \left( B_1 + \max \left\{ B_1, \left| 1 - \frac{3}{2} \mu \right| B_1^2 + |B_2| \right\} \right).
\]  \hspace{1cm} (2.20)

**Proof.** Observe that when $zf' \in S^*_q$, equality (2.3) becomes

\[
\frac{z(zf''(z))'}{zf'(z)} - 1 = \varphi(z) \left( \phi(\omega(z)) - 1 \right),
\]  \hspace{1cm} (2.21)

or equally

\[
\frac{zf''(z)}{f'(z)} \ll \phi(\omega(z)) - 1,
\]  \hspace{1cm} (2.22)

and the converse can be verified easily. By the Alexander relation, that is $f \in \mathcal{C}_q$ if and only if $zf' \in S^*_q$, we can obtain the required estimates. \qed

**Theorem 2.5.** If $f \in \mathcal{A}$ satisfies

\[
\frac{zf''(z)}{f'(z)} \ll \phi(z) - 1,
\]  \hspace{1cm} (2.23)

then the following inequalities hold:

\[
|a_2| \leq \frac{B_1}{2},
\]  \hspace{1cm} (2.24)

\[
|a_3| \leq \frac{1}{6} \left( B_1 + B_1^2 + |B_2| \right),
\]

and, for any complex number $\mu$,

\[
|a_3 - \mu a_2^2| \leq \frac{1}{6} \left( B_1 + \left| 1 - \frac{3}{2} \mu \right| B_1^2 + |B_2| \right).
\]  \hspace{1cm} (2.25)
Theorem 2.6. If \( f \in \mathcal{A} \) belongs to \( \mathcal{R}_q(\phi) \), then

\[
|a_2| \leq \frac{B_1}{2},
\]

\[
|a_3| \leq \frac{1}{3} (B_1 + \max\{B_1, |B_2|\}),
\]

and, for any complex number \( \mu \),

\[
|a_3 - \mu a_2^2| \leq \frac{1}{3} \left( B_1 + \max\{B_1, \frac{3}{4} |\mu| B_2^2 + |B_2|\} \right).
\]

Proof. For \( f \in \mathcal{R}_q(\phi) \), we know that by Definition 1.5 there exist analytic functions \( \varphi \) and \( w \), with \( |\varphi(z)| \leq 1 \), \( w(0) = 0 \) and \( |w(z)| < 1 \) such that

\[
f'(z) - 1 = \varphi(z) \left( \varphi(w(z)) - 1 \right).
\]

Since

\[
f'(z) - 1 = 2a_2 z + 3a_3 z^2 + \cdots,
\]

it follows from (2.28) and (2.5) that

\[
a_2 = \frac{1}{2} B_1 c_0 w_1,
\]

\[
a_3 = \frac{1}{3} \left( B_1 c_1 w_1 + c_0 \left( B_1 w_2 + B_2 w_2^2 \right) \right).
\]

Following the same argument as in Theorem 2.1, where \( |c_0| \leq 1 \) and \( |c_1| \leq 1 \), we can deduce that

\[
|a_2| \leq \frac{B_1}{2},
\]

\[
|a_3 - \mu a_2^2| \leq \frac{B_1}{3} \left( 1 + \left| w_2 - \left( \frac{3B_1 c_0}{4} \mu - \frac{B_2}{B_1} \right) w_1^2 \right| \right).
\]

Applying Lemma 1.11, we get

\[
|a_3 - \mu a_2^2| \leq \frac{B_1}{3} \left( 1 + \max\left\{1, \left| \frac{3B_1 c_0}{4} \mu - \frac{B_2}{B_1} \right| \right\} \right).
\]
Since
\[ \left| \frac{3B_1c_0}{4} - \frac{B_2}{B_1} \right| \leq \frac{3B_1}{4} |\mu| |c_0| + \frac{B_2}{B_1}, \tag{2.33} \]
and $|c_0| \leq 1$ we can conclude the hypothesis.

**Theorem 2.7.** If $f \in \mathcal{A}$ satisfies
\[ f'(z) - 1 \ll \phi(z) - 1, \tag{2.34} \]
then the following inequalities hold:
\[ |a_2| \leq \frac{B_1}{2}, \tag{2.35} \]
\[ |a_3| \leq \frac{1}{3} (B_1 + |B_2|), \]
and, for any complex number $\mu$,
\[ |a_3 - \mu a_2^2| \leq \left( B_1 + \frac{3}{4} |\mu| B_1^2 + |B_2| \right). \tag{2.36} \]

Let the class $\mathcal{R}_\rho^\phi$ consist of functions $f \in \mathcal{A}$ satisfying the quasi-subordination
\[ \frac{1}{\rho} (f'(z) - 1) \prec \phi(z) - 1, \tag{2.37} \]
where $\rho \in \mathbb{C} \setminus \{0\}$. The following corollary gives the results for $f \in \mathcal{R}_\rho^\phi$.

**Corollary 2.8.** Let $\rho \in \mathbb{C} \setminus \{0\}$. If $f \in \mathcal{A}$ belongs to $\mathcal{R}_\rho^\phi$, then
\[ |a_2| \leq \frac{|\rho|}{2} B_1, \tag{2.38} \]
\[ |a_3| \leq \frac{|\rho|}{3} (B_1 + \max\{B_1, |B_2|\}), \]
and, for any complex number $\mu$,
\[ |a_3 - \mu a_2^2| \leq \left( B_1 + \max\left\{ B_1, \frac{3}{4} |\mu| B_1^2 + |B_2| \right\} \right). \tag{2.39} \]
Remark 2.9. (1) For \( p(z) \equiv 1 \), Corollary 2.8 gives a particular case of the estimates in [13, Theorem 3] for \( p = 1 \) and [14, Theorem 2.3] for \( k = 1 \).

(2) For \( \varphi(z) \equiv 1 \) and \( \phi(z) = (1 + Az)/(1 + Bz), (-1 \leq B < A \leq 1) \), Corollary 2.8 reduces to the results in [19, Theorem 4].

**Theorem 2.10.** Let \( \alpha \geq 0 \). If \( f \in \mathcal{A} \) belongs to \( \mathcal{M}_{q}(\alpha, \phi) \), then

\[
|a_2| \leq \frac{B_1}{1 + \alpha},
\]

\[
|a_3| \leq \frac{1}{2(1 + 2\alpha)} \left( B_1 + \max \left\{ B_1, \frac{1 + 3\alpha}{(1 + \alpha)^2} B_1^2 + |B_2| \right\} \right),
\]

and, for any complex number \( \mu \),

\[
|a_3 - \mu a_2^2| \leq \frac{1}{2(1 + 2\alpha)} \left( B_1 + \max \left\{ B_1, \frac{|2\mu(1 + 2\alpha) - (1 + 3\alpha)|}{(1 + \alpha)^2} B_1^2 + |B_2| \right\} \right).
\]

**Proof.** If \( f \in \mathcal{M}_{q}(\alpha, \phi) \), for \( \alpha \geq 0 \) then there are analytic functions \( \varphi \) and \( w \), with \( |\varphi(z)| \leq 1 \), \( w(0) = 0 \) and \( |w(z)| < 1 \) such that

\[
(1 - \alpha) \frac{zf''(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) - 1 = \varphi(z) (\phi(w(z)) - 1).
\]

A computation shows that

\[
(1 - \alpha) \frac{zf''(z)}{f(z)} = (1 - \alpha) + (1 - \alpha)a_2z + (1 - \alpha) \left( -a_2^2 + 2a_3 \right) z^2 + \cdots,
\]

\[
\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) = \alpha + 2\alpha a_2 z + 2\alpha \left( -2a_2^2 + 3a_3 \right) z^2 + \cdots.
\]

Hence from (2.43), we have

\[
(1 - \alpha) \frac{zf''(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) - 1 = (1 + \alpha)a_2 z + (1 + 3\alpha)a_2^2 + 2(1 + 2\alpha)a_3 z^2 + \cdots.
\]

It then follows from relation (2.42) and (2.5) that

\[
a_2 = \frac{B_1 c_0 w_1}{1 + \alpha},
\]

\[
a_3 = \frac{1}{2(1 + 2\alpha)} \left( B_1 c_1 w_1 + B_1 c_0 w_2 + \left( B_2 c_0 + \frac{1 + 3\alpha}{(1 + \alpha)^2} B_1^2 c_0^2 \right) w_1^2 \right).
\]

We can then conclude the proof by proceeding similarly as previous theorems. \( \blacksquare \)
Remark 2.11. (1) When \( \alpha = 0 \), Theorem 2.10 reduces to Theorem 2.1.

(2) When \( \alpha = 1 \), Theorem 2.10 reduces to Theorem 2.4.

(3) For \( \varphi(z) = 1 \), Theorem 2.10 gives a particular case of the estimates in [14, Theorem 2.9] for \( k = 1 \).

**Theorem 2.12.** Let \( \alpha \geq 0 \). If \( f \in \mathcal{A} \) satisfies
\[
(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) - 1 \ll \varphi(z) - 1,
\]
then the following inequalities hold:
\[
|a_2| \leq \frac{B_1}{1 + \alpha},
\]
\[
|a_3| \leq \frac{1}{2(1 + 2\alpha)} \left( B_1 + \frac{1 + 3\alpha}{(1 + \alpha)^2} B_1^2 + |B_2| \right),
\]
and, for any complex number \( \mu \),
\[
|a_3 - \mu a_2^2| \leq \frac{1}{2(1 + 2\alpha)} \left( B_1 + \frac{2\mu(1 + 2\alpha) - (1 + 3\alpha)\beta}{(1 + \alpha)^2} B_1^2 + |B_2| \right).
\]

**Theorem 2.13.** Let \( \alpha \geq 0 \) and \( \beta = 1 - \alpha \). If \( f \in \mathcal{A} \) belongs to \( \mathcal{L}_4(\alpha, \varphi) \), then
\[
|a_2| \leq \frac{B_1}{\alpha + 2\beta},
\]
\[
|a_3| \leq \frac{1}{2|\alpha + 3\beta|} \left( B_1 + \max \left\{ B_1, \frac{(\alpha + 2\beta)^2 - 3(\alpha + 4\beta)}{2(\alpha + 2\beta)^2} B_1^2 + |B_2| \right\} \right),
\]
and, for any complex number \( \mu \),
\[
|a_3 - \mu a_2^2| \leq \frac{1}{2|\alpha + 3\beta|} \left( B_1 + \max \left\{ B_1, \frac{(\alpha + 2\beta)^2 - 3(\alpha + 4\beta) - 4\mu(\alpha + 3\beta)}{2(\alpha + 2\beta)^2} B_1^2 + |B_2| \right\} \right).
\]

**Proof.** If \( f \in \mathcal{L}_4(\alpha, \varphi) \), for \( \alpha \geq 0 \) and \( \beta = 1 - \alpha \) then there are analytic functions \( \varphi \) and \( w \), with \( |\varphi(z)| \leq 1 \), \( w(0) = 0 \) and \( |w(z)| < 1 \) such that
\[
\left( \frac{zf'(z)}{f(z)} \right)^a \left( 1 + \frac{zf''(z)}{f'(z)} \right)^\beta - 1 = \varphi(z)(\phi(w(z)) - 1).
\]
A computation shows that

\[
\left( \frac{zf'(z)}{f(z)} \right)^a = 1 + a a_2 z + \frac{1}{2} \left( (a^2 - 3a) a_2^2 + 4a a_3 \right) z^2 + \cdots ,
\]

\[
\left( 1 + \frac{zf''(z)}{f'(z)} \right)^\beta = 1 + 2\beta a_2 z + \left( 2(\beta^2 - 3\beta) a_2^2 + 6\beta a_3 \right) z^2 + \cdots .
\]

(2.52)

Thus (2.52) give

\[
\left( \frac{zf'(z)}{f(z)} \right)^a \left( 1 + \frac{zf''(z)}{f'(z)} \right)^\beta - 1
\]

\[
= (\alpha + 2\beta) a_2 z + \frac{1}{2} \left( (\alpha + 2\beta)^2 - 3(\alpha + 4\beta) \right) a_2^2 + 4(\alpha + 3\beta) a_3 \right) z^2 + \cdots .
\]

(2.53)

By using the above equation and (2.5) in (2.51) we have

\[
a_2 = \frac{B_1 c_0 w_1}{\alpha + 2\beta}
\]

\[
a_3 = \frac{B_1}{2(\alpha + 3\beta)} \left( B_1 c_1 w_1 + B_1 c_0 w_2 + \left( B_2 c_0 - \frac{(\alpha + 2\beta)^2 - 3(\alpha + 4\beta)}{2(\alpha + 2\beta)^2} B_1 \right) w_1^2 \right).
\]

(2.54)

We can proceed similarly as previous theorems and proof the hypothesis.

Remark 2.14. (1) When \( \alpha = 0 \), Theorem 2.13 reduces to Theorem 2.4.

(2) When \( \alpha = 1 \), Theorem 2.13 reduces to Theorem 2.1.

(3) For \( \phi(z) \equiv 1 \), Theorem 2.13 gives a particular case of the estimates in [14, Theorem 2.7] for \( k = 1 \).

Theorem 2.15. Let \( \alpha \geq 0 \) and \( \beta = 1 - \alpha \). If \( f \in \mathscr{A} \) satisfies

\[
\left( \frac{zf'(z)}{f(z)} \right)^a \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-a} ≪ \phi(z) - 1,
\]

(2.55)

then the following inequalities hold:

\[
|a_2| \leq \frac{B_1}{|\alpha + 2\beta|},
\]

\[
|a_3| \leq \frac{1}{2|\alpha + 3\beta|} \left( B_1 + \frac{|(\alpha + 2\beta)^2 - 3(\alpha + 4\beta)|}{2(\alpha + 2\beta)^2} B_1^2 + |B_2| \right),
\]

(2.56)
Abstract and Applied Analysis

and, for any complex number $\mu$,

$$\left| a_3 - \mu a_2^2 \right| \leq \frac{1}{2|\alpha + 3\beta|}\left[ B_1 + \frac{\left( (\alpha + 2\beta)^2 - 3(\alpha + 4\beta) - 4\mu(\alpha + 3\beta) \right)}{2(\alpha + 2\beta)^2}B_1^2 + |B_2| \right].$$

(2.57)

Acknowledgment

The work presented here was supported in part by research Grant LRGS/TD/2011/UKM/ICT/03/02. The authors are thankful to the referees for their useful comments.

References

[1] P. Duren, “Subordination,” in Complex Analysis, vol. 599 of Lecture Notes in Mathematics, pp. 22–29, Springer, Berlin, Germany, 1977.

[2] W. C. Ma and D. Minda, “A unified treatment of some special classes of univalent functions,” in Proceedings of the Conference on Complex Analysis (Tianjin ’92), vol. 1 of Conference Proceedings Lecture Notes Analysis, pp. 157–169, International Press, Cambridge, Mass, USA, 1994.

[3] M. S. Robertson, “Quasi-subordination and coefficient conjectures,” Bulletin of the American Mathematical Society, vol. 76, pp. 1–9, 1970.

[4] O. Altıntaş and S. Owa, “Majorizations and quasi-subordinations for certain analytic functions,” Proceedings of the Japan Academy A, vol. 68, no. 7, pp. 181–185, 1992.

[5] S. Y. Lee, “Quasi-subordinate functions and coefficient conjectures,” Journal of the Korean Mathematical Society, vol. 12, no. 1, pp. 43–50, 1975.

[6] F. Y. Ren, S. Owa, and S. Fukui, “Some inequalities on quasi-subordinate functions,” Bulletin of the Australian Mathematical Society, vol. 43, no. 2, pp. 317–324, 1991.

[7] S. S. Miller, P. T. Mocanu, and M. O. Reade, “All $\alpha$-convex functions are starlike,” Revue Roumaine de Mathématiques Pures et Appliquées, vol. 17, pp. 1395–1397, 1972.

[8] Z. Lewandowski, S. Miller, and E. Złotkiewicz, “Gamma-starlike functions,” Annales Universitatis Mariae Curie-Skłodowska A, vol. 28, pp. 53–58, 1976.

[9] M. Darus and D. K. Thomas, “$\alpha$-logarithmically convex functions,” Indian Journal of Pure and Applied Mathematics, vol. 29, no. 10, pp. 1049–1059, 1998.

[10] L. de Branges, “A proof of the Bieberbach conjecture,” Acta Mathematica, vol. 154, no. 1-2, pp. 137–152, 1985.

[11] H. R. Abdel-Gawad, “On the Fekete-Szegő problem for alpha-quasi-convex functions,” Tamkang Journal of Mathematics, vol. 31, no. 4, pp. 251–255, 2000.

[12] O. P. Ahuja and M. Jahangiri, “Fekete-Szego problem for a unified class of analytic functions,” Panamerican Mathematical Journal, vol. 7, no. 2, pp. 67–78, 1997.

[13] R. M. Ali, V. Ravichandran, and N. Seenivasagan, “Coefficient bounds for $p$-valent functions,” Applied Mathematics and Computation, vol. 187, no. 1, pp. 35–46, 2007.

[14] R. M. Ali, S. K. Lee, V. Ravichandran, and S. Supramaniam, “The Fekete-Szegő coefficient functional for transforms of analytic functions,” Bulletin of the Iranian Mathematical Society, vol. 35, no. 2, pp. 119–142, 2009.

[15] N. E. Cho and S. Owa, “On the Fekete-Szegő problem for strongly $\alpha$-logarithmic quasiconvex functions,” Southeast Asian Bulletin of Mathematics, vol. 28, no. 3, pp. 421–430, 2004.

[16] J. H. Choi, Y. C. Kim, and T. Sugawa, “A general approach to the Fekete-Szegő problem,” Journal of the Mathematical Society of Japan, vol. 59, no. 3, pp. 707–727, 2007.

[17] M. Darus and N. Tuneski, “On the Fekete-Szegő problem for generalized close-to-convex functions,” International Mathematical Journal, vol. 4, no. 6, pp. 561–568, 2003.

[18] M. Darus, T. N. Shanmugam, and S. Sivasubramanian, “Fekete-Szegő inequality for a certain class of analytic functions,” Mathematica, vol. 49(72), no. 1, pp. 29–34, 2007.
[19] K. K. Dixit and S. K. Pal, “On a class of univalent functions related to complex order,” *Indian Journal of Pure and Applied Mathematics*, vol. 26, no. 9, pp. 889–896, 1995.

[20] S. Kanas, “An unified approach to the Fekete-Szegő problem,” *Applied Mathematics and Computation*, vol. 218, pp. 8453–8461, 2012.

[21] S. Kanas and H. E. Darwish, “Fekete-Szegő problem for starlike and convex functions of complex order,” *Applied Mathematics Letters*, vol. 23, no. 7, pp. 777–782, 2010.

[22] S. Kanas and A. Lecko, “On the Fekete-Szegő problem and the domain of convexity for a certain class of univalent functions,” *Zeszyty Naukowe Politechniki Rzeszowskiej. Matematyka i Fizyka*, no. 10, pp. 49–57, 1990.

[23] O. S. Kwon and N. E. Cho, “On the Fekete-Szegő problem for certain analytic functions,” *Journal of the Korea Society of Mathematical Education B*, vol. 10, no. 4, pp. 265–271, 2003.

[24] V. Ravichandran, M. Darus, M. H. Khan, and K. G. Subramanian, “Fekete-Szegő inequality for certain class of analytic functions,” *The Australian Journal of Mathematical Analysis and Applications*, vol. 1, no. 2, article 2, 7 pages, 2004.

[25] V. Ravichandran, A. Gangadharan, and M. Darus, “Fekete-Szegő inequality for certain class of Bazilevic functions,” *Far East Journal of Mathematical Sciences*, vol. 15, no. 2, pp. 171–180, 2004.

[26] F. R. Keogh and E. P. Merkes, “A coefficient inequality for certain classes of analytic functions,” *Proceedings of the American Mathematical Society*, vol. 20, pp. 8–12, 1969.

[27] Z. Nehari, *Conformal Mapping*, Dover, New York, NY, USA, 1975, Reprinting of the 1952 edition.