BRAID GROUP REPRESENTATIONS ARISING FROM THE 
YANG BAXTER EQUATION

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ABSTRACT. This paper aims to determine the images of the braid group under representations afforded by the Yang Baxter equation when the solution is a nontrivial $4 \times 4$ matrix. Making the assumption that all the eigenvalues of the Yang Baxter solution are roots of unity, leads to the conclusion that all the images are finite.

Using results of Turaev, we have also identified cases in which one would get a link invariant. Finally, by observing the group algebra generated by the image of the braid group sometimes factor through known algebras, in certain instances we can identify the invariant as particular specializations of a known invariant.

1. Introduction

Any invertible matrix which satisfies the Yang Baxter equation can be used to obtain representations of the braid group. This paper focuses on cases where the Yang Baxter solution is a $4 \times 4$ unitary matrix. One of the goals is to identify the image of the braid group under these representations. Our main technique is to study the pure braid group, $P_n$, a subgroup of the braid group, and apply the results of this analysis to the entire group. We show that restricting the eigenvalues of the solution results in the image of the braid group being finite.

1.1. The $n$-strand braid group, $B_n$, (for $n \geq 2$) is generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ with the following relations:

\begin{align*}
(B1) & : \sigma_i \sigma_j = \sigma_j \sigma_i, \ |i - j| \geq 2 \\
(B2) & : \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ 1 \leq i \leq n - 2
\end{align*}

Representations of the braid group often give rise to knot and link invariants and Turaev \cite{T} defined a criteria, called an enhancement, which if satisfied, would produce a Markov trace and hence lead to link invariants. One approach to obtaining representations of the braid group is through consideration of $R$ matrices of quantum groups. However, all that is required is a solution to the Yang Baxter equation, which can be systematically produced using the theory of quantum groups. But

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simple solutions of the Yang Baxter equation can also be found by directly solving the equation with the help of computers. Starting with a finite-dimensional vector space $V$, let $R$ be a linear map on the tensor product of $V$ with itself, then $R$ is said to satisfy the Yang Baxter equation if:

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R).$$

If $R$ satisfies the Yang Baxter equation and if it is an invertible linear map, it can be used to obtain a representation of the braid group as follows: $\pi(\sigma_i) = I^{\otimes(i-1)} \otimes R \otimes I^{\otimes(n-i-1)}$ where $I$ is the identity on $V$. Pictorially, if

$$R = \begin{pmatrix} & \big\uparrow & \\
& R & \big\uparrow \\
\end{pmatrix}$$

this corresponds to the (B2) relation in the braid group (and the third Reidemeister move).

All solutions of the form $R : V \otimes V \to V \otimes V$ for $V$ of dimension $= 2$ to the Yang Baxter equation has been listed in [H]. Dye found all unitary solutions of this form to the braid relations based on this list [D]. The particular importance of one of the solutions, $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 \\
0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
\end{pmatrix}$, was pointed out in the work of Kauffman and Lomonaco [KL], and the connection of $R$ with quantum computing was explored there. Additionally, in [FRW] it was shown that the image of the braid group under the representation afforded by that particular solution is a finite group. Furthermore, along with E. Rowell and Z. Wang, we showed there is an exact sequence:

$$1 \to E_{n-1}^{-1} \to Im(B_n) \to S_n \to 1$$

$\forall n \geq 2$, where the group $E_{m}^{-1}$ is an extraspecial 2 group. Obtaining representations of the braid group from representations of extraspecial 2-groups also has been studied in [GJ]. This paper handles the remaining solutions given by [D].

One method proposed to build quantum computers is based on topology. In a topological quantum computer, representations of the braid group can be used to describe the actions of the quantum bits, a.k.a. qubits. It has been conjectured that due to the topological stability of the system, this construction would be invulnerable to local errors, which is one of the formidable obstacles to physically realizing a quantum computer. The qubits themselves are encoded in the lowest energy states of quasi particles, called anyons, at some fixed positions in a plane and their use in topological models is described in [FKLW]. Trajectories of the quasi particles in the 3-dimensional space-time form braids. Braiding of anyons in a topological quantum computer change the encoded quantum information.
Representations of braid groups have been proposed as the fractional statistics of anyons [Wil].

For any given model for quantum computing, it is important to understand whether or not the model is capable of carrying out any computation up to any given precision, i.e. whether or not the model is universal. In topological models, the universality issue is translated to a question about the closed images of the braid group representations. In particular, for the topological models in [FKLW], universality is equivalent to the closure of the images containing $SU(K_n)$. We do not have density here since we claim:

**Theorem.** For any unitary $4 \times 4$ solution to the Yang Baxter equation, $R$, there are representations of the braid group $\pi_R : B_n \to U(K_n)$. If all eigenvalues of $R$ are roots of unity, there is a short exact sequence

$$1 \to \text{Im}(P_n) \to \text{Im}(B_n) \to S_n \to 1$$

Moreover, $\text{Im}(B_n)$ is finite.

However, there are more elaborate adaptive models which make use of these cases for quantum computing see [FNW1] [FNW2].

2. Preliminaries

Although, any invertible solution to the Yang Baxter equation will yield a representation of the braid group, with an eye towards possible applications to quantum computing, we restrict ourselves to Dye’s list since we would need the unitarity condition for physical realizability. The following propositions indicate that a single solution to the Yang Baxter equation produces a class of solutions by conjugation and scalar multiplication. (See [Kas]).

It is not sufficient to determine which representations from [H] are unitary since $ARA^{-1}$ may be unitary when $R$ is not.

**Theorem.** [D] There are five families of $4 \times 4$ unitary matrix solutions to the Yang Baxter equation. Each has the form

$$kARA^{-1}$$

where $k$ is a scalar with norm 1, $A = Q \otimes Q$, and $Q$ is an invertible matrix such that $Q = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$

1. $R_0 = Id.$

2. $R_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$

The matrix $Q$ has the following restrictions: $c = \frac{-a}{d}$ and $|a| = |d|$. 
(3) \( R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix} \) where \( 1 = \alpha \overline{\alpha} = \beta \overline{\beta} = \gamma \overline{\gamma} \).

The variables in the matrix \( Q \) also has the following restriction: \( c = \frac{-a\overline{b}}{d} \).

(4) \( R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \beta & 0 & 0 & 0 \end{pmatrix} \) where \( |\alpha \beta| = 1 \), \( \alpha \overline{\alpha} = \frac{(ab+cd)(\overline{a}b+\overline{c}d)}{(ab+cd)(a\overline{b}+c\overline{d})} \) and \( \beta \overline{\beta} = \frac{(a\overline{b}+c\overline{d})(a\overline{b}+cd)}{(a\overline{b}(b\overline{c}+cd)} \).

The matrix \( Q \) has the restriction: \( c = \frac{-a\overline{b}}{d} \).

(5) \( R'_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \beta & 0 & 0 & 0 \end{pmatrix} \) where \( |\alpha \beta| = 1 \), \( \alpha = \frac{(ab+cd)(\overline{a}b+\overline{c}d)}{(a\overline{b}+c\overline{d})} \) and \( \beta = \frac{(a\overline{b}+c\overline{d})(a\overline{b}+cd)}{(a\overline{b}(b\overline{c}+cd)} \).

The matrix \( Q \) has the restriction: \( c \neq \frac{-a\overline{b}}{d} \).

As the identity case is clearly not interesting and since the restrictions are of little consequence when looking at images of the braid group, for the remainder of this paper we will content ourselves with analyzing the cases \( R_2 \) and \( R_3 \) since \( R_1 \) was covered in [FRW]. We mainly concern ourselves with

\[
R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}
\]

since, if we conjugate \( R_2 \) we get \( R_3 \) so as abstract groups, the closed images will be the same.

There are common notations used in both and we record some here:

Our convention for tensor products of matrices is to use “left into right,” that is, if

\[
X = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \text{ and } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then } X \otimes A = \begin{pmatrix} aX & bX \\ cX & dX \end{pmatrix}.
\]

We let

\[
\pi_n(\sigma_i) = I_2^{\otimes i-1} \otimes \sigma_i \otimes I_2^{\otimes n-i-1}
\]

be the representations of the braid group, \( B_n \) arising from \( R \).

3. The Image of the Braid Group

3.1. Restriction to \( P_n \). Notice that

\[
R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix} = P \cdot D
\]

Where \( P \) is a permutation matrix and \( D \) is a diagonal matrix.

**Proposition 3.1.** \( \text{Image}(P_n) \) is abelian
Proof. First notice that the image of $\sigma^2_i$ is a diagonal matrix for each $i$. Now consider

$$
\pi_n(\sigma_{i+1})\pi_n(\sigma_i^2)\pi_n(\sigma_{i+1}^{-1}) \\
= (I^\otimes_i \otimes R \otimes I^{\otimes n-i-2})(I^\otimes_{i-1} \otimes R^2 \otimes I^{\otimes n-i-1})(I^\otimes_i \otimes R \otimes I^{\otimes n-i-2})^{-1} \\
= I^\otimes_{i-1}[(I \otimes R)(R^2 \otimes I)(I \otimes R^{-1})] \otimes I^{\otimes n-i-2} \\
= I^\otimes_{i-1} \otimes [(I \otimes PD)(R^2 \otimes I)(I \otimes (PD)^{-1})] \otimes I^{\otimes n-i-2} \\
= I^\otimes_{i-1} \otimes [(I \otimes PD)(R^2 \otimes I)(I \otimes D^{-1}P)] \otimes I^{\otimes n-i-2} \\
= I^\otimes_{i-1} \otimes [(I \otimes P)(I \otimes D)(R^2 \otimes I)(I \otimes D^{-1})(I \otimes P)] \otimes I^{\otimes n-i-2} \\
= I^\otimes_{i-1} \otimes [(I \otimes P)(R^2 \otimes I)(I \otimes P)] \otimes I^{\otimes n-i-2} \\
= D'
$$

where $D'$ is a diagonal matrix. We only used that $R^2 \otimes I$, $I \otimes D$ and $D \otimes I$ are a diagonal matrices and therefore commute. Since the pure braid group, $P_n$, is generated by all conjugates of $\sigma^2_i$ the above shows that the Image($P_n$) is a subset of the diagonal matrices and thus abelian. \qed

Remark 3.2. Since $R$ has arbitrary variables, it might be possible for $R$ itself to generate an infinite group. So for the remainder of this paper we restrict ourselves to the case where the eigenvalues of $R$ are roots of unity and thus $\exists k$ such that $p^k_i = I$, where $p_i = \pi_n(\sigma_i^2) = I_2^\otimes_i \otimes R^2 \otimes I_2^{\otimes (n-i-1)}$. This in combination with the preceding proposition allow us to conclude the following:

Proposition 3.3. Image($P_n$) is a finite abelian group.

Theorem 3.4. We have an exact sequence:

$$
1 \to \text{Im}(P_n) \to \text{Im}(B_n) \to S_n \to 1
$$

for all $n \geq 3$. In other words, $G_n$ is an extension of $H_n$ by $S_n$.

Proof. Notice $\pi_n(B_n)/\pi_n(P_n)$ is a homomorphic image of $S_n$ as $\pi_n$ induces a surjective homomorphism $\hat{\pi}_n : B_n/P_n \to \pi_n(B_n)/\pi_n(P_n)$ and $B_n/P_n \cong S_n$. We would like to know if $\hat{\pi}_n$ is an isomorphism in this case as well. Therefore, we must determine if Ker$(\hat{\pi}_n)$ is trivial. We note for $n \geq 4$ it is sufficient to check that the element (12)(34) is not in the kernel, while for $n = 3$ we should check that (123) is not in the kernel. We simply observe the corresponding elements $\sigma_1\sigma_3$ and $\sigma_2\sigma_1$ are not diagonal and note that in Proposition 3.1, we showed the image of the pure braid group is a subset of diagonal matrices. \qed

In particular, we have shown that the image of the braid group is finite as long as the eigenvalues are roots of unity.
3.2. Invariants. We now put our representation through the machinery given by Turæv [T] and uncover link invariants if and only if \( \gamma = \pm 1 \) where:

\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & \alpha & 0 \\
0 & \beta & 0 & 0 \\
0 & 0 & 0 & \gamma
\end{pmatrix}.
\]

If \( \gamma = 1 \), then \( \mu = xy \cdot Id \) and the invariant is:

\[
T_{R,x} = x^{n - e(\sigma)} \text{Trace}(\pi_n(\sigma)).
\]

If \( \gamma = -1 \), then \( \mu = xy \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \) and the invariant is:

\[
T_{R,x} = x^{n - e(\sigma)} \text{Trace}(\mu \otimes \mu \circ \pi_n(\sigma)).
\]

We would like to know if these invariants correspond to some known invariant. We proceed by breaking into cases with the number of eigenvalues of \( R \). Note that if \( R \) has only 2 eigenvalues they must be \( \pm 1 \) and we get \( \pi_n(B_n) \cong S_n \). If we let \( \gamma = 1 \) then the other eigenvalues of \( R \) are \( \pm \sqrt{\alpha \beta} \) so we will have exactly 3 eigenvalues if \( \alpha \beta \neq 1 \). If we let \( \gamma = -1 \) and assume that \( \alpha \beta \neq 1 \) we will have 4 distinct eigenvalues.

4. Different Eigenvalues

4.1. Three eigenvalues. Let \( A_n \) denote the group algebra of the image of the braid group afforded by:

\[
R = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & \alpha & 0 \\
0 & \beta & 0 & 0 \\
0 & 0 & 0 & \gamma
\end{pmatrix}.
\]

The main result of what follows will hold for \( R_3 \) and the arguments are analogous. Recall that the BMW algebras, \( C_n(r, q) \), are complex group algebras with generators \( g_i \) satisfying the braid relations and

\[
(R1) \ e_i g_i = r^{-1} e_i \\
(R2) \ e_i g_{i-1} = r^\pm e_i
\]

where \( e_i = 1 - (g_i - g_i^{-1})/(q - q^{-1}) \).

These are quotients of the braid group and were originally defined by Birman and Wenzl [BW] and independently by Murakami [M]. However, we are following the definition given in [Wen] where Wenzl shows if \( r = \pm q^k \), \( C_n(r, q) \) is not semisimple, but there is a trace on this algebra (corresponding to the Kauffman polynomial) and moding out by the annihilator of this trace recovers semisimplicity [Wen].

We wish to make a connection between these algebras and \( A_n \) so we take the following steps:

1. We rescale our matrix \( R \) and by abuse of notation let \( R = QR \)
2. We next suppose that \( \alpha, \beta = Q^{-2} \).
Then

\[ R = \begin{pmatrix} Q & 0 & 0 & 0 \\ 0 & 0 & Q^{-1} & 0 \\ 0 & Q^{-1} & 0 & 0 \\ 0 & 0 & 0 & Q \end{pmatrix} \]

Now we define a map \( \phi : C_n(q, q) \to A_n \). To avoid confusion, we denote elements in \( C_n(q, q) \) as lower case letters and their image in capital letters: \( \phi(g_i) = G_i \) where \( G_i = \pi_n(\sigma_i) \) and \( q = Q \). Notice that \( G_1 = R \) and we have a nice form for

\[ E_i = I^{\otimes (i-1)} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes I^{\otimes (n-i-1)} \]

**Lemma 4.1.** The \( G_i \)'s satisfy (R1) and (R2) so \( \phi \) is a homomorphism.

**Proof.** For (R1) notice that

\[
E_i G_i = \left( I^{\otimes (i-1)} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes I^{\otimes (n-i-1)} \right) \left( I^{\otimes (i-1)} \otimes \begin{pmatrix} Q & 0 & 0 & 0 \\ 0 & 0 & Q^{-1} & 0 \\ 0 & Q^{-1} & 0 & 0 \\ 0 & 0 & 0 & Q \end{pmatrix} \otimes I^{\otimes (n-i-1)} \right)
\]

\[
= I^{\otimes (i-1)} \otimes \begin{pmatrix} Q & 0 & 0 & 0 \\ 0 & 0 & Q^{-1} & 0 \\ 0 & Q^{-1} & 0 & 0 \\ 0 & 0 & 0 & Q \end{pmatrix} \otimes I^{\otimes (n-i-1)}
\]

\[
= I^{\otimes (i-1)} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes I^{\otimes (n-i-1)}
\]

\[
= Q^{-1} \left( I^{\otimes (i-1)} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes I^{\otimes (n-i-1)} \right)
\]

\[
= Q^{-1} E_i
\]

For (R2) notice that
\[ E_i G_{i-1} E_i \]

\[
= \left( I^\otimes (i-1) \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes I^\otimes (n-i-1) \right) (I^\otimes (i-1) \otimes R \otimes I^{(n-i)})
\]

\[
= \left( I^\otimes (i-2) \otimes \left( I \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes I^\otimes (n-i-1) \right) \right)
\]

\[
= I^\otimes (i-2) \otimes \left( I \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes I^\otimes (n-i-1) \right)
\]

\[
= I^\otimes (i-2) \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & Q & Q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes I^\otimes (n-i-1)
\]

\[
= Q I^\otimes (i-1) \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & Q & Q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes I^\otimes (n-i-1)
\]

\[
= Q \cdot E_i
\]

So we have:

\[
\xymatrix{ C_n(q, q) \ar[rr]^\phi & & A_n \ar@/_/[dl]^{\pi_n} \\
& B_n &}
\]

Where the map from \( B_n \to C_n(q, q) \) is given by \( \sigma_i \mapsto g_i \).

We also note here that by taking the standard trace of the matrices \( E_i \) and \( G_i \) we have that \( \text{Trace}(E_i) = \frac{1}{2} \cdot 2^n \) and \( \text{Trace}(G_i) = Q \cdot \frac{1}{2} \cdot 2^n \).

**Proposition 4.2.** [Wen] Let \( x = 1 + \frac{r - r^{-1}}{q - q^{-1}} \). There exists a functional trace, \( tr \), on \( C_\infty (r, q) \) uniquely defined inductively by:

1. \( tr(1) = 1 \)
2. \( tr(ab) = tr(ba) \)
3. \( tr(e_i) = \frac{1}{x} \)
(4) \( \text{tr}(g_t^{\pm 1}) = \frac{\pm 1}{x} \)

(5) \( \text{tr}(a\chi b) = \text{tr}(\chi)\text{tr}(ab) \) for \( a, b \in C_{n-1}(r, q), \chi = g_{n-1} \) or \( e_{n-1} \)

We are in the case \( r = q \) and so \( x = 2 \). Note that the standard Trace on matrices does not satisfy these conditions, but if we let \( \text{Tr}_n = \frac{1}{x} \text{Trace}_n \) where Trace is the standard trace, we have all but the last condition. That is, we wish to show \( \text{Tr}(A\chi B) = \text{Tr}(\chi)\text{Tr}(AB) \) for \( \chi \in \{E_n, G_n\} \) and \( A, B \in A_n \). Note that \( A_n \hookrightarrow A_{n+1} \) by \( X \hookrightarrow X \otimes I_2 \), so since \( A, B \in A_n \) and \( \chi \in \{E_n, G_n\} \subset A_{n+1} \), we actually consider \( A \otimes I_2 \) and \( B \otimes I_2 \) so that \( A\chi B \) makes sense. Let \( \chi = G_n = I^{\otimes n-1} \otimes R \). Then we have:

\[
\text{Tr}_{n+1}(A\chi B) = \text{Tr}_{n+1}((A \otimes I)\chi(B \otimes I))
= \text{Tr}_{n+1}((A \otimes I)(I^{\otimes n-1} \otimes R)(B \otimes I))
= \text{Tr}_{n+1}(I^{\otimes n-1} \otimes R)(B \otimes I)(A \otimes I)
= \text{Tr}_{n+1}(I^{\otimes n-1} \otimes R)\begin{pmatrix} BA & 0_{2^n} \\ 0_{2^n} & BA \end{pmatrix}
= \text{Tr}_n\begin{pmatrix} Q^{-1}I^{\otimes n-1} & 0_{2^n-1} \\ 0_{2^n-1} & Q^{-1}I^{\otimes n-1} \end{pmatrix} BA + \text{Tr}_n\begin{pmatrix} 0_{2^n-1} & 0_{2^n-1} \\ 0_{2^n-1} & 0_{2^n-1} \end{pmatrix} Q^{-1}I^{\otimes n-1} BA
= \text{Tr}_n\begin{pmatrix} Q^{-1}I^{\otimes n-1} & 0_{2^n-1} \\ 0_{2^n-1} & Q^{-1}I^{\otimes n-1} \end{pmatrix} BA
= \text{Tr}_n(Q \cdot I^{\otimes n} \cdot BA) = Q \cdot \text{Tr}_n(BA)
= \frac{1}{2} Q \cdot \text{Tr}_{n+1}(BA) = \frac{1}{2} \cdot \text{Tr}_{n+1}(AB) = \text{Tr}_{n+1}(G_n) \cdot \text{Tr}_{n+1}(AB)

We have shown the case that \( \chi = G_n \), but the proof for \( \chi = E_n \) is virtually the same and left to the reader. Therefore, we have that the traces correspond.
Since the trace on $C_n(q,q)$ computes the 2 variable Kauffman polynomial, we have shown in the 3 eigenvalue case if we let $\alpha, \beta = q^{-2}$ and rescale our representation, the invariant specified by Turaev is related to a specific specialization of the 2 variable Kauffman polynomial at $q$.

**Proposition 4.3.** \textit{\[Wen\]} $C_n(q,q) := C_n(q,q)/Ann(q,q)$ is semisimple where $Ann(q,q)$ is the annihilator of the trace. $Ann(q,q) = \{ b \in C_n(q,q) \mid tr(ab) = 0 \ \forall a \in C_n(q,q) \}$

**Lemma 4.4.** The induced map
\[
\overline{\phi}: C_n(q,q) \to \phi(C_n(q,q))/\phi(Ann(q,q))
\]
is injective.

**Proof.** It suffices to show $ker(\phi) \subset Ann(q,q)$. But since we have shown $tr$ corresponds to $Tr$, this is not difficult. Let $a \in ker(\phi)$ and $b \in C_n(q,q)$, then $tr(ab) = Tr(ab) = 0$, that is, $a \in Ann(q,q)$ \hfill \Box

We note the above proof was inspired by [LR].

An immediate corollary is that $C_n(q,q)$ is isomorphic to a quotient of $A_n$ which we know is finite and thus $C_n(q,q)$ is finite.

**Remark 4.5.** The question of identifying the closed image of the braid groups in unitary representations associated with the 2 variable Kauffman polynomial at specific roots of unity was begun in [LRW] and continued in [LR]. The focus of [LRW] is on the cases where the closed images are infinite; in particular, they contain the special unitary group. There are open questions about the cases where the images are finite. The BMW algebras are indexed by $q$ and $r$ and when the image of the braid group representations are known to be finite $q$ and $r$ are related in some way. The case $r = q$ was covered above which only leaves the case $r = q^{\pm1/2}$.

### 4.2. Four eigenvalues.

In [FRW] it was shown that after normalization certain solutions of the matrices from $[D]$ are related to the Jones polynomial which has a skein relation where $\sigma_i$ is the sum of 2 terms. This paper has shown similar results for the Kauffman polynomial which has a skein relation where $\sigma_i$ is the sum of 3 terms. We now note that the $G_2$ invariant defined by Kuperberg [K] has a skein relation in which $\sigma_i$ is the sum of 4 terms. When $\gamma = -1$ by [T] we get an invariant and with the aid of computers we can show in some cases our matrices correspond to particular specializations the $G_2$ invariant. We list them now for completeness:

**Theorem.** Let $x = \sqrt{\alpha \beta}$ and let $t$ be the variable in Kuperberg’s invariant. Then for the following specializations of $x$ and $t$ our matrices satisfy Kuperberg’s skein relation:

1. $t = 1, x$ arbitrary
These were found by using the skein relation as follows: First, we noted that the minimal polynomial of $R$ gave us:

$$(R^2 - \alpha \beta)(R^2 - 1) = 0$$

$$\Rightarrow R^4 - (\alpha \beta + 1)R^2 = \alpha \beta = 0$$

$$\Rightarrow R^4 = (\alpha \beta + 1)R^2 - \alpha \beta$$

Then we identified $R$ with a crossing and using Kuperberg’s skein relation resolved each crossing. Finally, we used a computer to equate like coefficients and found the preceding solutions. These were found using Maple and checked by the author by hand, but further calculations have proven fruitless.

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