A New Spatio-Temporal Model Exploiting Hamiltonian Equations

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Stationarity, convergence of lagged correlations to zero and non-Gaussianity of the detrended Alaska data process

- Stationarity of the detrended Alaska data process
- Convergence of lagged spatio-temporal correlations to zero for the Alaska data
- Non-Gaussianity of the Alaska data
Abstract

The solutions of Hamiltonian equations are known to describe the underlying phase space of the mechanical system. In Bayesian Statistics, the only place, where the properties of solutions to the Hamiltonian equations are successfully applied, is Hamiltonian Monte Carlo. In this article, we propose a novel spatio-temporal model using a strategic modification of the Hamiltonian equations, incorporating appropriate stochasticity via Gaussian processes. The resultant spatio-temporal process, continuously varying with time, turns out to be nonparametric, nonstationary, nonseparable and no-Gaussian. Besides, the lagged correlations tend to zero as the spatio-temporal lag tends to infinity. We investigate the theoretical properties of the new spatio-temporal process, along with its continuity and smoothness properties. Considering the Bayesian paradigm, we derive methods for complete Bayesian inference using MCMC techniques. Applications of our new model and methods to two simulation experiments and two real data sets revealed encouraging performance.

Keywords: Continuously varying time and space; Hamiltonian dynamics; Markov Chain Monte Carlo; Non-stationarity; Non-Gaussianity; Spatio-temporal modeling.
1 Introduction

Modeling spatially and spatio-temporally dependent data drew much attention in the last few decades within the statistics community. Diverse areas of science, including but not restricted to meteorology [1, 2, 3, 4, 5, 6, 7, 8, 9], environment [10, 11, 12, 13, 14, 15, 16, 17] and ecology [18, 19], give rise to challenging spatio-temporal data. The goal of modeling spatio-temporal data is to predict values of the underlying spatio-temporal process at desired locations and future time points. A common technique for modeling spatio-temporal data is to assume separability of the covariance function in space and time (for definition of separability, see [20]) and stationarity, in particular, isotropic stationarity of the underlying spatio-temporal process [21]. A more general assumption than stationarity, often employed in practice, is covariance stationarity, that is, the covariance of the observations at any two locations and time points is a function of the separation vector between the two locations and time points (or function of the distance between the two locations and time points, for isotropic stationarity). The usual techniques, for example, universal or simple kriging, heavily rely upon these assumptions ([21]). However, in reality these assumptions can be very artificial if there is local influence on the correlation structure. Indeed, the scenario of local influence is not uncommon in practice. [22] and [23] showed that the PM10 pollution dataset, analyzed by [16], is not stationary. The same finding was established for a sea-temperature dataset by [24]. [25] demonstrated that assuming a stationary covariance function wrongly would result in an over-smoothed or under-smoothed process.

In recent times, many attempts are made to incorporate nonstationarity of the underlying spatio-temporal process. [26] first significantly contributed in capturing nonstationarity of the covariance function based on the idea of spatial deformation. This idea has been further exploited in the Bayesian paradigm by [27] and [28]. [29, 30, 31, 32] used kernel convolution to model nonstationarity of the underlying processes. With the help of Dirichlet process, [33] attempted to model the underlying spatial process nonparametrically along with “conditional” nonstationarity. A discretized version of a certain stochastic differential equation is considered by [34] to model spatio-temporal nonstationarity. Further, [35] proposed a nonparametric nonstationary model based on kernel processes mixing.

In all these above mentioned work, the correlation among the two spatio-temporal points was not shown to converge to 0 as the distance between the two points increases to infinity, the property that is naturally enjoyed by a stationary spatio-temporal process. It is natural to believe, and aptly demonstrated with real datasets in [23] and [24], that as the distance between two points, either in terms of spatial locations,
and/or in terms of time points, increases to infinity the correlation of the underlying process should go to 0. Very recently, [22] proposed a non-parametric, non-separable, non-stationary and non-Gaussian spatio-temporal model based on order-based dependent Dirichlet process, where they showed that the underlying covariance function goes to 0 as the distance between the two locations and/or time points increases to infinity. While modeling spatio-temporal data, [22] also assumed that time and the space vary continuously in their respective domains.

Although, [22] made a successful attempt building the non-parametric, non-stationary, non-Gaussian model with the desirable properties of covariance, this model does not impart dynamic properties to the temporal part, either directly or via any latent process. On the other hand, [36] introduces a non-parametric spatio-temporal model, which, through a non-parametric dynamic latent process, induces desirable dynamic properties in the temporal part. In addition, the model of [36] is non-stationary with the property that the covariance goes to 0 as distance between two time points and/or difference among two spatial points goes to infinity. However, time does not vary continuously on the respective domain.

In a nutshell, it is observed that so far to the best of our knowledge, there is no proposal of a non-parametric, non-separable, non-Gaussian dynamic spatio-temporal model which is continuous in time and space with an underlying structured latent process and with the property that the correlation between two spatio-temporal realizations goes to 0 as the spatial/temporal lag goes to infinity. To fill up this gap we propose a dynamic spatio-temporal non-parametric, non-Gaussian model, where the underlying process is non-stationary. A structured latent process is incorporated in the proposed model. The time and the space in our proposed spatio-temporal model vary continuously over their respective domains. Further, the underlying process enjoys the property that the covariance goes to 0 as the spatial/temporal lag tends to infinity.

For constructing the model proposal, we take help of the Hamiltonian dynamics from physics. The idea of Hamilton’s equations are applied to Bayesian statistics in formulating Hamiltonian Monte Carlo [37, 38]. However, formulation of the spatio-temporal model exploiting the idea of the Hamiltonian dynamics is not done earlier. Here we first briefly describe the Hamiltonian dynamics and the equations. Thereafter, we shall connect the idea of the spatio-temporal model to Hamiltonian dynamics. In the latter section (Section 2) we describe the mathematical formulation in detail.

Let \((\mathcal{M}, \mathcal{L})\) be a mechanical system, where \(\mathcal{M}\) is the configuration space and \(\mathcal{L}\) is the smooth Lagrangian. The coordinate system of \(\mathcal{M}\) is determined by \((\theta, \dot{\theta})\), where \(\theta\) is the position of a particle at
time $t$ and the $\dot{\theta}$ is the derivative vector with respect to time, thus representing the velocity. The partial derivative of $L$ with respect to $\dot{\theta}$, known as momenta, is denoted by $p$, which is a function of time $t$, the position $\theta$, and velocity $\dot{\theta}$. Now the Hamiltonian, a function of $p$, $\theta$ and $t$, is defined as

$$\mathcal{H}(p, \theta, t) = \sum_i p_i \dot{\theta}_i - L(\theta, \dot{\theta}, t),$$

which is the energy function of the mechanical system. Here $p_i$ and $\dot{\theta}_i$ are the $i$th component of $p$ and $\dot{\theta}$. The pair $(\theta, p)$ is called phase space coordinates. The phase space coordinates, which varies in time $t$ continuously, are the solution of the Hamiltonian equations

$$\frac{d\theta}{dt} = \frac{\partial \mathcal{H}}{\partial p}, \quad \text{and} \quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial \theta}.$$
latent variables and the parameters (Subsection 3.2 and Section S-2 of Supplementary Information) and then obtain the joint density of latent variables given the observed data and the parameters (Subsection 3.3 and Section S-3 of Supplementary Information), before presenting the complete likelihood of the parameter vector (Subsection 3.4). Section 4 deals with the choice of prior distributions for the parameter vectors. A plausible justification of the choice of the prior distributions is also discussed in this section. For applying Gibbs sampling, we evaluate the full conditional densities of all the parameters along with the latent variables in Section 5. The detailed calculation of the full conditional densities are given in the Section S-4 of Supplementary Information. Simulation studies and the real data analysis are given in the Sections 6 and 7, respectively. In Section 6, we provide the results of two simulations studies. Results of another another simulation experiment is presented in Section S-5 of Supplementary Information. In Section 7, two real data sets are analyzed to show the performance of the newly proposed spatio-temporal model. Among these two real data sets, one of the data sets corresponds to non-stationary and non-Gaussian, while another is associated with a stationary non-Gaussian spatio-temporal process. Finally, in Section 8, we summarize our contributions and make concluding remarks.

2 Modified Hamiltonian equations, proposed process and its properties

2.1 The key idea of a spatio-temporal process via modified Hamiltonian equations

Let the total energy $H(\theta)$, also known as Hamiltonian function, be defined as $V(\theta) + W(p)$, where $V(\theta)$ is the potential energy and $W(p) = \frac{1}{2}p^TM^{-1}p$, with $M$ being a chosen matrix (mass), is the kinetic energy, (see [38] for more details). Then the original Hamiltonian equations are given by

\[
\frac{dp}{dt} = -\frac{\partial H}{\partial \theta} = -\nabla V(\theta) \text{ and } \frac{d\theta}{dt} = \frac{\partial H}{\partial p} = M^{-1}p,
\]

where by $\nabla V(\theta)$ we mean gradient of $V$ with respect to $\theta$. The leap-frog algorithm for numerically solving the Hamiltonian equations is given by
\[ \theta(t + \delta t) = \theta(t) + \delta t M^{-1} \left\{ p(t) - \frac{1}{2} \delta t \nabla V(\theta(t)) \right\} \] and
\[ p(t + \delta t) = p(t) - \frac{1}{2} \delta t \{ \nabla V(\theta(t)) + \nabla V(\theta(t + \delta t)) \}. \]

We modify the original Hamiltonian equations to suit our purpose as follows:

\[
\frac{dp}{dt} = \alpha^* p - \nabla V(\theta) \text{ and } \frac{d\theta}{dt} = \beta^* \theta + M^{-1} p.
\]

The modified leap-frog algorithm associated with the modified Hamiltonian equations are as follows:

\[
p \left( t + \frac{\delta t}{2}\right) = p(t) + \alpha^* p(t) \frac{\delta t}{2} - \nabla V(\theta(t)) \frac{\delta t}{2},
\]

\[
= p(t) \left( 1 + \alpha^* \frac{\delta t}{2} \right) - \nabla V(\theta(t)) \frac{\delta t}{2}
\]

\[
= \alpha p(t) - \nabla V(\theta(t)) \frac{\delta t}{2}, \tag{1}
\]

where \( \alpha = (1 + \alpha^* \frac{\delta t}{2}) \). We set \(|\alpha| < 1\), that implies \(-2 < \alpha^* \frac{\delta t}{2} < 0\). Restricting \( \alpha \) on \((-1, 1)\) led to good MCMC mixing in our Bayesian applications. For \( \theta \), we have

\[ \theta(t + \delta t) = \theta(t) + \beta^* \theta(t) \delta t + \delta t M^{-1} p(t + \delta t/2) \]

\[ = \theta(t) (1 + \beta^* \delta t) + \delta t M^{-1} p(t + \delta t/2) \]

\[ = \beta \theta(t) + \delta t M^{-1} p(t + \delta t/2), \tag{2} \]

where \( \beta = (1 + \beta^* \delta t) \). As above, we set \(|\beta| < 1\), which implies \(-2 < \beta^* \delta t < 0\). As will be seen subsequently, this restriction is necessary for the lagged correlations between the observations to tend to zero as the space-time lag tends to infinity. Further, note that

\[ p(t + \delta t) = p(t + \delta t/2) + \alpha^* p(t + \delta t/2) \frac{\delta t}{2} - \nabla V(\theta(t + \delta t)) \frac{\delta t}{2} \]

\[ = p(t + \delta t/2) \left( 1 + \alpha^* \frac{\delta t}{2} \right) - \nabla V(\theta(t + \delta t)) \frac{\delta t}{2} \]

\[ = \alpha p(t + \delta t/2) - \nabla V(\theta(t + \delta t)) \frac{\delta t}{2} \]
\[ = \alpha \left( \alpha p(t) - \nabla V(\theta(t)) \frac{\delta t}{2} \right) - \nabla V(\theta(t + \delta t)) \frac{\delta t}{2} \]
\[ = \alpha^2 p(t) - \frac{\delta t}{2} \left\{ \alpha \nabla V(\theta(t)) + \nabla V(\theta(t + \delta t)) \right\}, \tag{3} \]

where the fourth equality of the equation (3) follows from equation (1). Finally, replacing the form of \( p(t + \delta t/2) \) from equation (1) to equation (2) we get

\[ \theta(t + \delta t) = \beta \theta(t) + \delta t M^{-1} \left( \alpha p(t) - \nabla V(\theta(t)) \frac{\delta t}{2} \right) \tag{4} \]

Equations (3) and (4) constitute the modified leap-frog equations and are the key ingredients of our proposed spatio-temporal process.

Indeed, for location \( s \in S \), where \( S \) is some index set, we replace \( \theta \) and \( p \) in the (modified) Hamiltonian equations with \( \theta_s \) and \( p_s \) respectively and with \( s_n = (s_1, \ldots, s_n)' \), let \( \theta_{s_n}(t) = (y(s_1, t), \ldots, y(s_n, t))' \), where \( y(s, t) \) is the observed value at location \( s \in S \) and time \( t \in [0, T] \) and let \( p_{s_n}(t) = (x(s_1, t), \ldots, x(s_n, t))' \), where \( x(s, t) \) is a latent unobserved process at location \( s \in S \) and time \( t \in [0, T] \). In the modified leap-frog algorithm defined by (3) and (4), we replace \( \theta(t) \) and \( p(t) \) with \( \theta_{s_n}(t) \) and \( p_{s_n}(t) \), respectively. For single location \( s \), these will be denoted by \( \theta_s(t) \) and \( p_s(t) \). Similarly, we shall replace \( M \) with \( M_s \) and \( M_{s_n} \), the latter being an appropriate diagonal matrix. To complete the specification of our spatio-temporal process, we need to model the function \( V(\cdot) \) as some appropriate stochastic process; we consider the Gaussian process for our purpose. We shall also model \( \theta_s(0) \) and \( p_s(0) \) as appropriate Gaussian processes indexed by \( s \). Complete specification of the process also requires an appropriate form for \( M_s \). Details on these, along with investigation of the theoretical properties of our spatio-temporal processes, are provided in the next subsection.

**Remark 1** Note that the equation (3) is not the equation for latent process since it involves the observed process as well. Integrating the conditional distribution of the latent process (equation (3)) over the observed process would give us the latent process distribution.

### 2.2 Completion of specification of the proposed spatio-temporal process and investigation of its theoretical properties

Let \( S \) be a compact subset of \( \mathbb{R}^d \); for our purpose we choose \( d = 2 \). We put the following assumptions on the processes \( p_s(0), \theta_s(0), s \in S \) and the random function \( V(\cdot) \).
A1. $p_s(0), s \in S$ is assumed to be a centered Gaussian process with a symmetric, positive definite covariance function with bounded partial derivatives. For instance, the Matérn covariance function with $\nu > 1$ has bounded partial derivatives, and could be employed. In particular, we consider the squared exponential covariance between $p_{s_1}(0)$ and $p_{s_2}(0)$, of the form $\sigma_p^2 \exp \{-\eta_1|s_1 - s_2|^2\}$, $s_1, s_2 \in S$.

A2. $\theta_s(0), s \in S$ is also assumed to be a centered Gaussian process with a symmetric, positive definite covariance function with bounded partial derivatives. Again, we consider the squared exponential covariance function of the form $\sigma_\theta^2 \exp \{-\eta_2|s_1 - s_2|^2\}$.

A3. The function $V(\cdot)$ is assumed to be a Gaussian random function with zero mean and covariance function $c_v(x_1, x_2) = \sigma^2 \exp \{-\eta_3|x_1 - x_2|^2\}$. As for the other cases, different choices of covariance structure can be assumed with the assumption that covariance function is continuously twice differentiable and the mixed partial derivatives are Lipschitz continuous. If the covariance function has third bounded partial derivatives then the function will be Lipschitz. In our example, the covariance function is infinitely differentiable and the derivatives are bounded. Other choices may include rational quadratic covariance function, Matérn covariance function with $\nu > 2$ among many others.

Remark 2 Above assumptions need well behaved covariance functions in the sense of smoothness. For a list of such covariance functions one may see [20]. Smoothness properties along with other important properties of Matérn covariance functions have been covered in [40] (Chapter 2).

Remark 3 The assumptions A1 and A2 imply that the covariance functions of $p_s(0)$ and $\theta_s(0)$ are symmetric, positive definite and Lipschitz continuous, and thus $p_s(0)$ and $\theta_s(0)$ will have continuous sample paths with probability 1. If the covariance functions are taken to be Matérn covariance function with $\nu > 1$ or squared exponential covariance function, then $p_s(0)$ and $\theta_s(0)$ will have differentiable sample paths in $s$.

Remark 4 Assumption A3 implies that the derivative process of $V(\cdot)$ is also a Gaussian process with continuous sample paths almost surely. In fact, if the squared exponential covariance function is assumed then all the derivatives of $V(\cdot)$ will be Gaussian processes. In particular, the differential of $V(\cdot)$ will have differentiable sample paths. Also note that for Matérn covariance function with $\nu > 2$, the random function $V(\cdot)$ is differentiable and the corresponding covariance function is the mixed partial derivative of the covariance function of $V(\cdot)$. Moreover, the differential of $V(\cdot)$ will have differentiable sample paths as $\nu$ is assumed to be more than 2 for Matérn covariance function (refer to [41] along with [40]).
Now we propose a form of $M_s$ which is continuous and infinitely differentiable when defined on a compact set $S \subset \mathbb{R}^2$.

**Definition 5 (Definition of $M_s$)** Let $S$ be a compact subset of $\mathbb{R}^2$. Define $M_s = \exp(\max\{||s^2 - u^2||^2 : u \in S\})$, where by $v^2$ we mean $v^2 = (v_1^2, v_2^2)^T$, for $v \in S$.

**Lemma 2.1** $M_s$ is infinitely differentiable in $s \in S$.

**Remark 6** It can be argued that $M_s \to \infty$ and $M_s' \to \infty$ as $||s - s'|| \to \infty$ in the following fashion. Let $||s - s'|| \to \infty$ as $S$ also grows in the sense $S_1 \subset S_2 \subset \ldots$, such that at each stage $i$, $S_i$ remains compact. Under this limiting situation, $M_s = \exp(\max\{||s^2 - u^2||^2 : u \in S\}) \geq \exp(||s^2 - s'^2||^2) \to \infty$ and $M_s' = \exp(\max\{||s'^2 - u^2||^2 : u \in S\}) \geq \exp(||s'^2 - s^2||^2) \to \infty$.

The next results shows that the conditional covariance function of $\theta_s(t)$ given $p$ goes to 0 as the distance between two time points and two spatial locations increase to infinity. The distance between two time points and two spatial locations are measured with respect to their corresponding distance metrics.

**Theorem 2.1** Under the assumptions A1 to A3, $\text{cov}(\theta_s(h\delta t), \theta_s'(h'\delta t) | p)$ converges to 0 almost surely, as $||s - s'|| \to \infty$ and $|h - h'| \to \infty$.

**Remark 7** Note that by Theorem 2.1 and the dominated convergence theorem, the unconditional correlation $\text{corr}(\theta_s(h\delta t), \theta_s'(h'\delta t)) \to 0$ as $||s - s'|| \to \infty$ and $|h - h'| \to \infty$.

**Remark 8** In Theorem 2.1, $\delta t$ in $h\delta t$ and $h'\delta t$ make the time points continuous for continuous $\delta t$. However, discrete values of $\delta$ are also allowed.

Now we show that $\theta_s(t)$ and $p_s(t)$ are continuous in $s$ with probability 1 and in the mean square sense. Since the (modified) Hamiltonian equations already imply that $\theta_s(t)$ and $p_s(t)$ are path-wise differentiable with respect to $t$, we focus on their smoothness properties with respect to $s$.

**Theorem 2.2** If the assumptions A1-A3 hold true, then $\theta_s(h\delta t)$ and $p_s(h\delta t)$ are continuous in $s$, for all $h \geq 1$, with probability 1.

**Theorem 2.3** Under assumptions A1-A3, $\theta_s(h\delta t)$ and $p_s(h\delta t)$ are continuous in $s$ in the mean square sense, for all $h \geq 1$.

The next two results deal with the differentiability of the processes $\theta_s(t)$ and $p_s(t)$.
**Theorem 2.4** Under assumptions A1-A3, \( \theta_s(h\delta t) \) and \( p_s(h\delta t) \) have differentiable sample paths with respect to \( s \), almost surely.

**Remark 9** Theorem 2.4 is about once differentiability, however, it can be extended to \( k \) times differentiability depending on the structure of the covariances assumed on the processes \( \theta_s(0) \), \( p_s(0) \) and the random function \( V(\cdot) \). For example, if we assume squared exponential covariance functions on each process then \( \theta_s(h\delta t) \) and \( p_s(h\delta t) \) will have \( k \) times differentiable sample paths in \( s \), for any \( k \in \mathbb{N} \).

For proving that the processes \( \theta_s(t) \) and \( p_s(t) \) are mean square differentiable in \( s \), we need a lemma, stated below, which may be of independent interest.

**Lemma 2.2** Let \( f : \mathbb{R} \to \mathbb{R} \) be a zero mean Gaussian random function with covariance function \( c_f(x_1, x_2) \), \( x_1, x_2 \in \mathbb{R} \), which is four times continuously differentiable. Let \( \{Z(s) : s \in S\} \) be a random process with the following properties

1. \( E(Z(s)) = 0 \),
2. covariance function \( c_Z(s_1, s_2), s_1, s_2 \in S \), where \( S \) is a compact subspace of \( \mathbb{R}^2 \), is four times continuously differentiable, and
3. \( \frac{\partial^2 Z(s)}{\partial s_i} \) has finite fourth moment.

Then the process \( \{g(s) : s \in S\} \), where \( g(s) = f(Z(s)) \), is mean square differentiable in \( s \).

**Theorem 2.5** Let A1-A3 hold true, with the covariance functions of all the assumed Gaussian processes being squared exponential. Then \( \theta_s(h\delta t) \) and \( p_s(h\delta t) \) are mean square differentiable in \( s \), for every \( h \geq 1 \).

### 3 Calculation of likelihood functions

Let the observations on \( Y \) be available for positions \( s_1, s_2, \ldots, s_n \) and time points \( 1, 2, \ldots, T \). That is, data set is given as

\[
\text{Data} = \{Y(s_1, 1), Y(s_2, 1), \ldots, Y(s_n, 1); Y(s_1, 2), Y(s_2, 2), \ldots, Y(s_n, 2); \ldots, Y(s_1, T), Y(s_2, T), \ldots, Y(s_n, T)\},
\]

and the corresponding latent variables are

\[
\text{Latent} = \{X(s_1, 1), X(s_2, 1), \ldots, X(s_n, 1); X(s_1, 2), X(s_2, 2), \ldots, X(s_n, 2); \ldots, X(s_1, T), X(s_2, T), \ldots, X(s_n, T)\}
\]
In this section we will derive the data model and the process model under assumptions A1-A3. Particularly, we assume here that the random function $V(\cdot)$ is a Gaussian process with mean 0 and squared exponential function as covariance function for simplicity of calculations. Other covariance functions (with required properties, see assumption A3) will work in the same manner. In particular, we assume that the covariance function of $V(\cdot)$ takes the form $\text{cov}(V(x), V(y)) = k(h) = \sigma^2 e^{-\eta_3 h^2}$, where $h = ||x - y||$. Then $V'(\cdot)$ will be a Gaussian random function with mean 0 and covariance function

$$\text{cov}(V'(x), V'(y)) = 2\eta_3 \sigma^2 e^{-\eta_3 h^2}(1 - 2\eta_3 h^2),$$

(see [40], Chapter 2).

### 3.1 Notation

We will use the following notation for all the future calculations regarding the different distribution. Let $i \in \{1, \ldots, n\}$, $m \in \{0, \ldots, T\}$ and $r \in \{1, \ldots, T\}$.

$$y_m = (y(s_1, m), y(s_2, m), \ldots, y(s_n, m))^T,$$

$$x_m = (x(s_1, m), x(s_2, m), \ldots, x(s_n, m))^T,$$

$$h_{ij}(m) = |y(s_i, m) - y(s_j, m)|,$$

$$\mu_m = (\mu_1(m), \ldots, \mu_n(m))^T,$$

$$\mu_i(m) = \beta y(s_i, m) + \frac{\alpha x(s_i, m)}{M_{s_i}},$$

$$W_r = (V'(y(s_1, r)), \ldots, V'(y(s_n, r)))^T,$$

$$\ell_{ik}(r - 1, r) = |y(s_i, r - 1) - y(s_k, r)|,$$

$$W_{r-1} = \begin{pmatrix} \alpha W_{r-1} \\ W_r \end{pmatrix},$$

$$\theta = (\alpha, \beta, \sigma^2, \sigma_\theta^2, \sigma_p^2, \eta_1, \eta_2, \eta_3).$$

By $a^T$, we mean transpose of a vector $a$ and by $f'$ we mean derivative of $f$. 
3.2 Joint conditional density of the observed data

The joint conditional density of the data given the latent variables \(x_0, x_1, \ldots, x_T\) and the parameter vector \(\theta\) is given by

\[
L = [\text{Data} \mid x_0; \ldots; x_{T-1}; y_0; \theta] \\
\propto [y_1 \mid y_0; x_0; \theta] \cdots [y_T \mid y_{T-1}; \ldots; y_0; x_{T-1}; \ldots; x_0; \theta] \\
\propto \frac{(\sigma^2)^{-nT/2}}{\prod_{t=1}^{T} |\Sigma_{t-1}|^{1/2}} e^{-\frac{1}{2} \sum_{t=1}^{T} (y_t - \mu_{t-1})^T \Sigma_{t-1}^{-1} (y_t - \mu_{t-1})},
\]

where, for \(j = 1, 2, \ldots, T\), \((k, \ell)\)th element of \(\Sigma_{j-1}\) is

\[
\frac{2\eta_3 e^{-\eta_3 h_{k\ell}^2(j-1)} (1 - 2\eta_3 h_{k\ell}^2(j-1))}{M_{k\ell}}.
\]

The details of the calculation of the joint density is provided in Section S-2 of Supplementary Information.

3.3 Joint conditional density of latent data

It can be shown that (see Section S-3 of Supplementary Information for the detailed calculation) \([x_1, \ldots, x_T \mid y_0; \ldots; y_T; x_0; \theta]\) is proportional to

\[
\frac{(\sigma^2)^{-nT/2}}{\prod_{t=1}^{T} |\Omega_t|^{1/2}} e^{-\frac{1}{2} \sum_{t=1}^{T} (x_t - \alpha^2 x_{t-1})^T \Omega_t^{-1} (x_t - \alpha^2 x_{t-1})}.
\]

Specifically,

\[
[x_1 \mid x_0; y_0; y_1; \theta] \cdots [x_T \mid x_{T-1}; \ldots; x_0; y_0; \ldots; y_T; \theta] \\
\propto \frac{(\sigma^2)^{-nT/2}}{\prod_{t=1}^{T} |\Omega_t|^{1/2}} e^{-\frac{1}{2} \sum_{t=1}^{T} (x_t - \alpha^2 x_{t-1})^T \Omega_t^{-1} (x_t - \alpha^2 x_{t-1})},
\]

where, for \(m \in \{1, 2, \ldots, T\}\), \(\Omega_t = \alpha^2 \Sigma_{t-1,t-1} + \alpha \Sigma_{t-1,t} + \alpha \Sigma_{t,t-1} + \Sigma_{t,t}\), where the \((i, k)\)th element of \(\Sigma_{jj}\), for \(j = t - 1, t\), is \(2\eta_3 e^{-\eta_3 h_{ik}^2(j-1)} (1 - 2\eta_3 h_{ik}^2(j))\), and \((i, k)\)th element of \(\Sigma_{t-1,t} = \Sigma'_{t-1}\) is \(2\eta_3 e^{-\eta_3 h_{ik}^2(t-1,t)} (1 - 2\eta_3 h_{ik}^2(t-1, t))\).
3.4 Complete likelihood combining observed and latent data

Next we will find the joint distribution of Data and Latent observations given \(x_0, y_0\) and \(\theta\). Finally, using the prior distributions on \(\theta_s(0)\) and \(p_s(0)\) as mentioned in A1-A2, we shall obtain the complete joint distribution of \((y_T, \ldots, y_1, y_0)\) and \((x_T, \ldots, x_1, x_0)\), given the parameter \(\theta\). The joint distribution of \((y_T, \ldots, y_1)\) and \((x_T, \ldots, x_1)\) given \((x_0, y_0, \theta)\), using equations (7) and (8), is given by

\[
\begin{align*}
[y_T, x_T, \ldots, y_1, x_1 | y_0, x_0, \theta] &= \left[ y_1 | x_0, y_0, \theta \right] \left[ x_1 | x_0, y_0, y_1, \theta \right] \ldots \left[ y_T | x_{T-1}, y_{T-1}, \theta \right] \\
&= \left[ x_T | x_{T-1}, y_{T-1}, y_T, \theta \right] \\
&\propto \frac{(\sigma^2)^{-Tn/2}}{T} e^{-\frac{1}{2\sigma^2} \sum_{t=1}^{T} (y_t - \mu_{t-1})^T \Sigma_{t-1}^{-1} (y_t - \mu_{t-1})} \\
&\times \prod_{t=1}^{T} |\Sigma_{t-1}|^{1/2} \\
&\times \frac{(\sigma^2)^{-nT/2}}{T} e^{-\frac{1}{2\sigma^2} \sum_{t=1}^{T} (x_t - \alpha^2 x_{t-1})^T \Omega_{t-1}^{-1} (x_t - \alpha^2 x_{t-1})}. 
\end{align*}
\]

(8)

Now we will find the full joint distribution of \([y_T, \ldots, y_1, y_0; x_T, \ldots, x_1, x_0|\theta]\) using the priors on \(\theta_s(0)\) and \(p_s(0)\). Again for simplicity we will assume that the \(\theta_s(0)\) & \(p_s(0)\) are zero mean Gaussian processes with squared exponential functions as their covariance functions. In particular, we will assume that \(\text{cov}(\theta_s(0), \theta_s(0)) = \sigma_\theta^2 \exp\{-\eta_2 k^2\}\) (see assumption A2), and \(\text{cov}(p_s(0), p_s(0)) = \sigma_p^2 \exp\{-\eta_2 k^2\}\) (see assumption A1), where \(k = ||s_1 - s_2||\). Therefore, \([y_0|\theta] \sim N_n(0, \sigma_\theta^2 \Delta_0)\) and \([x_0|\theta] \sim N_n(0, \sigma_p^2 \Omega_0)\), where the \((i, j)\)th element of \(\Delta_0\) and \(\Omega_0\) are \(\exp\{-\eta_2 k_{ij}^2\}\) and \(\exp\{-\eta_1 k_{ij}^2\}\), respectively, with \(k_{ij} = |s_i - s_j|\).

Thus,

\[
[y_T, x_T, \ldots, y_1, x_1, y_0, x_0 | \theta] = [y_T, x_T, \ldots, y_1, x_1 | y_0, x_0, \theta] [x_0, y_0 | \theta] \\
\propto \frac{(\sigma^2)^{-Tn}}{T} e^{-\frac{1}{2\sigma^2} \sum_{t=1}^{T} (y_t - \mu_{t-1})^T \Sigma_{t-1}^{-1} (y_t - \mu_{t-1}) + (x_t - \alpha^2 x_{t-1})^T \Omega_{t-1}^{-1} (x_t - \alpha^2 x_{t-1})} \\
\times \prod_{t=1}^{T} |\Sigma_{t-1}|^{1/2} |\Omega_{t-1}|^{1/2} \\
\times \frac{(\sigma^2)^{-nT}}{T} \left( \frac{\sigma_\theta^2}{\Delta_0} \right)^{1/2} \left( \frac{\sigma_p^2}{\Omega_0} \right)^{1/2} \exp\left\{-\frac{1}{2\sigma_\theta^2} y_0^T \Delta_0^{-1} y_0 \right\} \exp\left\{-\frac{1}{2\sigma_p^2} x_0^T \Omega_0^{-1} x_0 \right\}. 
\]

(9)
4 Prior distributions

In this section we will specify the prior distributions of the components of $\theta = (\alpha, \beta, \sigma^2, \sigma^2_\theta, \sigma^2_p, \eta_1, \eta_2, \eta_3)'$.

The parameter spaces of the each component of $\theta$ are the following: $|\alpha| < 1$, $|\beta| < 1$, and $0 < \sigma^2, \sigma^2_\theta, \sigma^2_p, \eta_1, \eta_2, \eta_3 < \infty$. We make the following transformations on $\alpha$, $\beta$, $\eta_i$, for $i = 1, 2, 3$ for better MCMC mixing. Define $\alpha^* = \log\left(\frac{1+\alpha}{1-\alpha}\right)$, $\beta^* = \log\left(\frac{1+\beta}{1-\beta}\right)$, and $\eta_i^* = \log(\eta_i)$, for $i = 1, 2, 3$, so that $\alpha^*, \beta^*, \eta_i^* \in \mathbb{R}$, for $i = 1, 2, 3$. This implies $\alpha = 1 - \frac{2e^{\alpha^*}}{1+e^{\alpha^*}}$, $\beta = 1 - \frac{2e^{\beta^*}}{1+e^{\beta^*}}$, $\eta_i = e^{\eta_i^*}$, $i = 1, 2, 3$, respectively.

We assume that the prior distributions are independent. Since the parameter space of $\alpha^*$ and $\beta^*$ are $\mathbb{R}$ and they are involved in the mean function of our proposed model, the prior for $\alpha^*$ and $\beta^*$ are taken as normal with the mean 0 and large variances (of order 100). Particular choices of the prior variances are discussed in Sections 6 and 7. Moreover, the parameter spaces of $\eta_i^*$, $i = 1, 2, 3$ are also $\mathbb{R}$ and they are involved in the covariance structure of our proposed model in the sense that they determine the amount of correlations between spatial and temporal points. So, we take the priors for $\eta_i^*$, $i = 1, 2, 3$, as normal with the means $\mu_{\eta_i}$ and variances 1. Larger variance of the $\eta_i^*$ made the variances of their posterior distributions unreasonably large due to huge data variability. Nevertheless, the prior variance for the $\eta_i$ turn out to be 4.671, which is not too small. It also turned out that in all the simulation and real data analyses, the choice of variance 1 (for $\eta_i^*$) rendered good mixing properties to our MCMC sampler. The exact value of the hyper-prior means depend upon the data under consideration and is discussed in Sections 6 and 7.

Finally, the prior distribution of variance parameters are taken to be inverse-gamma, as they are the conjugate priors, conditionally. It is expected that the variability of the spatio-temporal data is very large which might possibly render the posterior means and variances of $\sigma^2_\theta$, $\sigma^2_p$, $\sigma^2$ very large (specially for $\sigma^2$).

So we decided to choose the hyper-parameters in such a way that the prior means (with some exceptions in the mean in a few cases) and variances are both close to zero. The exact choices of the hyper-parameters of priors of $\sigma^2_p$, $\sigma^2_\theta$ and $\sigma^2$ are mentioned in Sections 6 and 7.

The general forms of the prior distributions of $\alpha^*$, $\beta^*$, $\sigma^2$, $\sigma^2_\theta$, $\sigma^2_p$, $\eta_i^*$, $i = 1, 2, 3$, are taken as follows:

$$[\alpha^*] \propto N(0, \sigma_\alpha), [\beta^*] \propto N(0, \sigma^2_\beta)$$

$$[\sigma^2] \propto IG(\alpha_p, \gamma_p/2), [\sigma^2_\theta] \propto IG(\alpha_\theta, \gamma_\theta/2), [\sigma^2_p] \propto IG(\alpha_p, \gamma_p/2)$$

$$[\eta_1^*] \propto N(\mu_{\eta_1}, 1), [\eta_2^*] \propto N(\mu_{\eta_2}, 1)[\eta_3^*] \propto N(\mu_{\eta_3}, 1)$$

where IG stands for inverse gamma distribution.
5 Full conditional distributions of the parameters and latent variables, given the observed data

In this section we will obtain the full conditional distributions of the parameters, which will be used for generating samples from posterior distributions of the parameters using Gibbs sampling steps when feasible or Metropolis-Hastings steps otherwise. The detailed calculations are provided in Section S-4 of Supplementary Information.

Full conditional distribution of $\beta^*$

The full conditional density of $\beta^*$, is given by

$$[\beta^* | \ldots] \propto \pi(\beta^*) g_1(\beta^*),$$

(10)

where $\pi(\beta^*) = e^{-\frac{\beta^2}{2\sigma^2}}$ and $g_1(\beta^*) = e^{-\frac{2\beta^2}{2\sigma^2} \sum_{t=1}^{T} (y_t - \Sigma_t y_{t-1}) + \frac{2\beta^2}{2\sigma^2} \sum_{t=1}^{T} y_t y_{t-1}}$, where $\beta = 1 - \frac{2e^{\beta^*}}{1+e^{\beta^*}}$. The closed form of the full conditional density for $\beta^*$ is not available and thus we shall update it using a random walk Metropolis step.

Full conditional distribution of $\alpha^*$

The full conditional density of $\alpha^*$ is given by

$$[\alpha^* | \ldots] \propto e^{-\frac{\alpha^2}{2\sigma^2}} g_2(\alpha^*),$$

(11)

where $g_2(\alpha^*) = \frac{1}{\prod_{t=1}^{T} |\Omega_t|^{1/2}} e^{-\frac{\alpha^2}{2\sigma^2} \sum_{t=1}^{T} (x_t - \alpha^2 x_{t-1})^T \Omega_t^{-1} (x_t - \alpha^2 x_{t-1})} \times e^{-\frac{2\alpha^2}{2\sigma^2} \sum_{t=1}^{T} (x_t^T D \Sigma_t^{-1} D x_{t-1} - 2y_t^T \Sigma_t^{-1} D x_{t-1})}$, and $\alpha = 1 - \frac{2e^{\alpha^*}}{1+e^{\alpha^*}}$. However, the closed form is not available and hence will be updated using random walk Metropolis, similar to $\beta^*$.

Full conditional distribution of $\sigma^2_\theta$

The full conditional distribution of $\sigma^2_\theta$ is $\text{IG} \left( \alpha_\theta + n/2, \frac{\gamma_\theta + y_0^T \Delta_0^{-1} y_0}{2} \right)$. Hence it is updated using a Gibbs sampling step.

Full conditional distribution of $\sigma^2_p$

The full conditional distribution of $\sigma^2_p$ is $\text{IG} \left( \alpha_p + n/2, \frac{\gamma_p + x_0^T \Delta_0^{-1} x_0}{2} \right)$. Therefore, $\sigma^2_p$ is updated using a Gibbs sampling step.
The full conditional distribution of $\sigma^2$ is inverse-Gamma with parameters $\alpha_v + Tm$ and $\gamma_v/2 + 2\zeta$, where $\zeta = \sum_{t=1}^{T} \left[ (\mathbf{y}_t - \mathbf{\mu}_t)^T \Sigma_{t-1}^{-1} (\mathbf{y}_t - \mathbf{\mu}_t) + (\mathbf{x}_t - \alpha^2 \mathbf{x}_{t-1})^T \Omega_t^{-1} (\mathbf{x}_t - \alpha^2 \mathbf{x}_{t-1}) \right]$. Thus, it is updated using a Gibbs sampling step.

None of the full conditional distributions of $\eta^*_1$, $\eta^*_2$, or $\eta^*_3$ have close form and hence they are updated using random walk Metropolis. The complete calculations of the full conditionals for $\eta^*_i$, for $i = 1, 2, 3$, are given in Section S-4 of Supplementary Information.

From the equation (7), we immediately see that $[\mathbf{x}_t | \ldots] \sim N_n \left( \alpha^2 \mathbf{x}_{t-1}, \frac{\sigma^2}{4} \Omega_t \right)$, for $t = 1, \ldots, T$.

With $A = \Omega_0^{-1} + \frac{4\sigma^2_0 \alpha^4}{\alpha^2} \Omega_1^{-1} + \frac{4\sigma^2_0 \alpha^2}{\alpha^2} D \Sigma_0^{-1} D$, $B = \frac{4\sigma^2_0 \alpha^2}{\alpha^2} \Omega_1^{-1}$ and $C = \frac{4\sigma^2_0 \alpha}{\alpha^2} D \Sigma_0^{-1}$, the full conditional density of $\mathbf{x}_0$ is found to be a $n$-variate normal with the mean $A^{-1}(B \mathbf{x}_1 + C(\mathbf{y}_1 - \beta \mathbf{y}_0))$ and the variance-covariance matrix $\sigma^2 p A^{-1}$. Hence, it is updated using a Gibbs sampling step.

6 Simulation Studies

We conducted three simulation experiments to evaluate the performance of our model and methods. In these experiments, we fitted our spatio-temporal model to data generated from a linear dynamic spatio-temporal model (LDSTM), a nonlinear dynamic spatio-temporal model (NLDSTM) and a general quadratic nonlinear model (GQM). For brevity, here we report the simulation experiments with data generated from NLDSTM and GQN only, while our experiments on fitting our model to LDSTM is reported in Section S-5 of Supplementary Information.

In all the simulation experiments, we chose the number of locations to be 50 and the number of time points to be 20. The last time point is kept aside for checking the performance of our proposed model for the purpose of prediction. We ran our MCMC sampler for 1,75,000 iterations with a burn-in period 1,50,000 iterations for each analysis. The MCMC computations were carried out in MATLAB R2018a. For each model-fitting it took about 3 hours 49 minutes in a desktop computer with the following specifications: 8GB RAM, 1 TB Hard Drive and 3.8 GHz core i5.
6.1 Simulation experiment with data from NLDSTM

We test our model performance on a data set simulated from NLDSTM; in particular, we choose the power-transform based NLDSTM given by (see also [36]):

\[
Y(s_i, t) = \begin{cases} 
X(s_i, t)^b & \text{if } X(s_i, t) > 0, \\
0 & \text{if } X(s_i, t) \leq 0
\end{cases}
\]

\[
X(s_i, t) = \gamma_0 + \gamma_1 X(s_i, t - 1) + \zeta(s_i, t)
\]

\[
\{X(s_i, 0) : i = 1, \ldots, n\} \sim N(0, \Sigma_0)
\]

\[
\{\zeta(s_i, t) : i = 1, \ldots, n\} \overset{iid}{\sim} N(0, \Sigma_\eta).
\]

The above model form has been used for rain-fall modelling by [8].

The covariance matrices \(\Sigma_0\) and \(\Sigma_\eta\) are generated by exponential covariance function of the form

\[
c(u, v) = \sigma^2 \exp(-\lambda \|u - v\|),
\]

with \(\sigma = 1\), \(\lambda = 1\). The choices of the other parameters are as follows:

\(\gamma_0 = 1\), \(\gamma_1 = -0.8\) and \(b = 3\).

We generate \(n = 50\) locations in \([0,1] \times [0,1]\) and take \(T = 20\) time points. Latent variables and observed variables, \(x_t = (x(s_1, t), \ldots, x(s_n, t))^T\) and \(y_t = (y(s_1, t), \ldots, y(s_n, t))^T\), respectively, for \(t = 1, \ldots, 20\), are simulated from the above model. Keeping the last time point for the purpose of prediction, the data set, which is used for Bayesian inference, is \(D = (y_1, \ldots, y_{19})\).

Although we have chosen a finite prior variance for \(\eta_3^*\), the posterior variance of \(\eta_3\) turned out to be very large, which is probably due to huge variability present in the spatio-temporal data. This large posterior variance of \(\eta_3\) makes the complete system unstable, so, we fix the value of \(\eta_3\) at its maximum likelihood estimate (7.0020) computed by simulated annealing. Note that it is not very uncommon to fix the decay parameter in Bayesian inference of spatial data analysis, see for example [42], [43] and the citations therein.

6.1.1 Choice of prior parameters

The particular choices of the hyper-prior parameters have been chosen using the leave-one-out cross-validation technique. Specifically, for \(t = 1, \ldots, 20\), we leave out \(y_t\) in turn and compute its posterior predictive distribution using relatively short MCMC runs. We then selected those hyperparameters that yielded the minimum average length of the 95% prediction intervals. As such, the complete prior specifi-
cations are provided as follows:

\[
\alpha^* \sim N(0, \sqrt{500}), \beta^* \sim N(0, \sqrt{300})
\]

\[
[\sigma^2] \propto IG(850000, 2/2), [\sigma_p^2] \propto IG(590, 780/2), [\sigma_p^2] \propto IG(90, 100/2),
\]

\[
[\eta_1^*] \propto N(-3, 1), [\eta_2^*] \propto N(-5, 1).
\]

6.1.2 MCMC convergence diagnostics and posterior analysis

The trace plots of the parameters are given in Figure S-6 of the Supplementary Information. Clearly, the plots exhibit strong evidence of convergence of our MCMC algorithm.

The posterior probability densities for the latent variables at 50 locations for 19 time points are shown in Figures S-7 and S-8 of the Supplementary Information, where higher probability densities are depicted by progressively intense colours. We observe that true latent time series at all the 50 locations always lie in the high probability density regions.

For the purpose of prediction, we kept \( y_{20} \) out of our data set \( D \). The posterior predictive densities for \( y_{20} \) at 50 locations are shown in Figure 1 (first 25 locations in Figure 1a and last 25 locations in Figure 1b). We observe that all the true values lie within the 95% predictive intervals at every location (see Figures 1a and 1b). Further, we obtained the posterior predictive densities for \( x_{20} \) given \( D \). The plots are provided in Figure 2. Here also we notice that the true values lie well within the 95% credible intervals (see Figures 2a and Figure 2b).
(a) Predictive densities for first 25 locations

(b) Predictive densities for last 25 locations

Figure 1: Predictive densities of the $y_{20}$ for 50 locations for data simulated from NLDSTM. The red horizontal lines denote the 95% predictive intervals. The vertical black lines indicate the true values. All the true values, except two (marginally outside), lie within the 95% predictive intervals.

(a) Predictive densities of latent variable for first 25 locations for NLDSTM.

(b) Predictive densities of latent variable for last 25 locations for NLDSTM.

Figure 2: Predictive densities of the $x_{20}$ for 50 locations for NLDSTM. The red horizontal lines denote the 95% predictive interval. The vertical black lines indicate the true values. All the true values lie within the 95% predictive interval.
6.2 Simulation details and results for data simulated from GQN

As an illustration of our proposed model and inference techniques on a non-linear model we choose to simulate data from the GQN model (see [44], [21], and [24] for the details on the GQN model). The model from which we simulated observations with $T = 20$ time points and $n = 50$ locations is as follows:

$$Y(s_i, t_k) = \phi_1(t_k, s_i) + \phi_2(t_k, s_i) \tan(X(s_i, t_k)) + \epsilon(s_i, t_k),$$

$$X(s_i, t_k) = \sum_{j=1}^{n} a_{ij} X(s_j, t_{k-1}) + \sum_{j=1}^{n} \sum_{l=1}^{n} b_{ijl} X(s_j, t_{k-1})[X(s_l, t_{k-1})]^2 + \eta(s_i, t_k),$$

where $i \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, T\}$. As earlier, we kept the last time point out of the data set, at each 50 locations, for the purpose of prediction. Thus, we use observations on 50 locations and 19 time points for fitting the model. The coefficients $\phi_1(t_k, \cdot), \phi_2(t_k, \cdot)$, random errors $\epsilon(t_k, \cdot), \eta(t_k, \cdot)$ and the initial latent variable $X(t_0, \cdot)$ are assumed to be independent, zero-mean Gaussian processes having covariance structure $c(s_1, s_2) = \exp(-\|s_1 - s_2\|)$, for $s_1, s_2 \in \mathbb{R}^2$, with $\|\cdot\|$ being the Euclidean norm. Moreover, it is assumed that, for $i, j, l \in \{1, \ldots, n\}$, $a_{ij}$ and $b_{ijl}$ have independent univariate 0 mean normal distributions with variance 0.001². As before, locations $s_i$ are simulated independently from $U(0, 1) \times U(0, 1)$.

In this case as well, we fixed $\eta_3$ at its maximum likelihood estimate 10.5853, obtained using simulated annealing.

6.2.1 Choice of hyper parameters

Using the same cross-validation technique as before, we obtain the following prior specifications:

$$\alpha^* \sim N(0, \sqrt{500}), \beta^* \sim N(0, \sqrt{300})$$

$$[\sigma^2] \propto IG(750000, 2/2), [\sigma_p^2] \propto IG(50000, 780/2), [\sigma_p^2] \propto IG(900, 100/2)$$

$$[\eta_1^*] \propto N(-3, 1), [\eta_2^*] \propto N(-5, 1).$$

6.2.2 MCMC convergence diagnostics and posterior analysis

The trace plots of the parameters are provided in the Figure S-9, and the posterior density plot of $x_t, t = 1, \ldots, 19$ are displayed in Figures S-10 and S-11 of Supplementary Information. The trace plots provide strong evidence of MCMC convergence. Further, the posterior density plots show that the posteriors of the latent variables contain the true latent variable time series at all the 50 locations in the high probability
Figure 3: Predictive densities of $y_{50}$ for 50 locations for data simulated from GQN. The red horizontal lines denote the 95% predictive intervals. The vertical black lines indicate the true values. All the true values, except one, lie within the 95% predictive intervals.

Density regions successfully.

The predictive density plots of $y_{20}$ are given in Figure 3 and that of $x_{20}$ are shown in Figures 4. Other than only one location, all the true values of $y$ fall well within the 95% credible intervals (see Figures 3a and 3b). Moreover, it is seen from Figures 4a and 4b that the true values of $x_{20}$ fall within the 95% credible intervals at each of the 50 locations.

Thus, our proposed model successfully captures the variability of the data generated from a complicated model like GQN.

7 Real data analysis

We evaluate our model performance on two real data sets on temperatures. The first spatio-temporal data is the temperature values taken around Alaska recorded for 65-70 years. The second data set is sea surface temperatures over a wide range of areas noted for 100 months. The details of the data sets are provided in Sections 7.1 and 7.2 respectively.

Before applying our model on these data sets, we first make the following transformation (Lambert projection, see [36]) of the locations so that the Euclidean distance makes more sense. Let $\phi$ be longitude...
(a) Predictive densities of latent variable for the first 25 locations.

Figure 4: Predictive densities of $x_{50}$ for 50 locations for data simulated from GQN. The red horizontal lines denote the 95% predictive intervals. The vertical black lines indicate the true values. All the true values lie within the 95% predictive intervals.

and $\psi$ be latitude in radian. Then the following transformation is made:

$$s_1 = 2 \sin \left( \frac{\pi}{4} - \frac{\psi}{2} \right) \sin \phi$$  \hspace{1cm} (12)

$$s_2 = -2 \sin \left( \frac{\pi}{4} - \frac{\psi}{2} \right) \cos \phi.$$  \hspace{1cm} (13)

7.1 Alaska temperature data

A real data analysis is done on the temperature data of Alaska and its surroundings. The data set is collected from [https://www.metoffice.gov.uk/hadobs/crutem4/data/download.html](https://www.metoffice.gov.uk/hadobs/crutem4/data/download.html) by clicking the link [CRUTEM.4.6.0.0.station_files.zip](https://www.metoffice.gov.uk/hadobs/crutem4/data/download.html) given under the heading Station data. The details of the data set can be read from [https://crudata.uea.ac.uk/cru/data/temperature/crutem4/station-data.htm](https://crudata.uea.ac.uk/cru/data/temperature/crutem4/station-data.htm).

A total of 30 locations are considered for the analysis. Annual average temperature data for the years 1950 to 2015 are taken after detrending. Of these 30 locations, at four locations many data were missing. So, we decided to construct the complete time series for these four locations. Among these 30 locations, data till 2021 were available for 16 positions. We thus have made a multiple time predictions for these 16 locations. The 26 spatial points are indicated in Figure 5 in red and 4 locations (for which the complete time series is reconstructed) are indicted in blue in the same graph. The latitudes and longitudes are provided in Table 1.
Table 1: Latitude and Longitude (in degrees) of 30 locations in and around Alaska. The locations corresponding to the serial numbers, indicated in bold, are used for multiple predictions. The serial numbers which are denoted in blue, for the corresponding spatial locations, complete time series are reconstructed. These blue serial numbers are indicated by \(L_{25}, L_{26}, L_{28}\) and \(L_{30}\) for future references.

| Sl No. | Latitude | Longitude |
|--------|----------|-----------|
| 1-15  | 71.3N    | 166.8W    |
| 16-30 | 61.6N    | 149.3W    |

Thus the data set considered here contains 26 locations and 65 time points from 1950 to 2014 (last time point is set aside for purpose of single time point prediction). We denote the data set by \(\mathbb{D} = \{y_1, \ldots, y_{65}\}\), where \(y_t, t \in \{1, \ldots, 65\}\), is a 26 dimensional vector. In this analysis, we have made both temporal and spatial predictions after detrending the data. At these 26 locations, we obtained the predictive densities for the year 2015 and for 16 locations (indicated by \(L_1, \ldots, L_7, L_{11}, \ldots, L_{13}, L_{18}, \ldots, L_{20}, L_{22}, L_{27}, L_{29}\) for references hereafter) we made multiple time point predictions. Along with these temporal predictions, we constructed the 95% predictive intervals for the complete time series for the 4 left out locations. To do this, we augmented the data set \(\mathbb{D}\) with initial values for 65 time points of the four locations and then updated the values by calculating the conditional densities given the data set \(\mathbb{D}\). That is to say, we start with \(\mathbb{D}^* = \{y_1^*, \ldots, y_{65}^*\}\), where \(y_t^*\) is a 30 dimensional vector, with the last four values augmented with...
$y_t$, for $t \in \{1, \ldots, 65\}$. The parameters of our model, including the latent variables, are updated given $D^*$, and then the last four values of $y_t^*$ are updated given the parameter values and latent variables. This is continued for the complete MCMC run.

In Section S-10 of Supplementary Information, we validate that the underlying spatio-temporal process that generated the Alaska data is non-Gaussian, strictly stationary, and the lagged correlations converge to zero as the lags tend to infinity. Besides, there is no reason to assume separability of the spatio-temporal covariance structure. Although our spatio-temporal process is nonstationary, it is endowed with the other desirable features, and the results of analysis of the Alaska data shows that it is an appropriate mode for this data.

### 7.1.1 Prior choices for the Alaska temperature data

With the same cross-validation technique as in the simulation experiments, the complete specifications of the priors are obtained as follows:

\[
\alpha^* \sim N(0, \sqrt{500}), \beta^* \sim N(0, \sqrt{300}),
\]

\[
[\sigma^2] \propto IG(450000, 2/2), [\sigma_p^2] \propto IG(700, 780/2), [\sigma_r^2] \propto IG(250, 100/2),
\]

\[
[\eta_1^*] \propto N(-3, 1), [\eta_2^*] \propto N(-5, 1).
\]

As before, we fix $\eta_3$ at its maximum likelihood estimate 5.0581.

### 7.1.2 Results of the Alaska temperature data

As for the case of the simulation studies, we implemented 1,75,000 MCMC iterations with the first 1,50,000 iterations as burn-in. The time taken was about 5 hours 43 minutes in our desktop computer. The MCMC trace plots for the parameters, except $\eta_3$, are given in Figure S-12 of Supplementary Information, which bear clear evidence of convergence in each case. The posterior densities of the latent variables are displayed in Figures S-13 of Supplementary Information.

The predictive densities for $y_{66}$ for 26 locations are depicted in Figure 6. All the true values (detrended temperature values), which are denoted by black bold vertical lines, lie well within the 95% predictive intervals (indicated as red bold horizontal line in the figure). The predictive densities for the years 2016-2021 at the 16 locations are shown in two figures (Figure 7 and Figure 8), each containing 8 locations. Except for one location, at one time point, in all the other scenarios, the true value is captured by the
Figure 6: Posterior predictive densities of the detrended temperatures for the year 2015 at 26 locations for the Alaska temperature data. The red horizontal lines denote the 95% predictive intervals. The vertical black lines indicate the true values. All the true values lie within the 95% posterior predictive intervals.

95% predictive interval associated with our model.

The complete time series for the four locations, which were indicated in blue in Figure 5, are reconstructed using our model. The Bayesian predictive densities at each time point for these locations are shown in Figure 9 using probability plot. Higher the intensity of the color higher is the density. The available true detrended temperature values at these locations are plotted and depicted by black stars in Figure 9. At one location (L30), three values lie away from the high density region. However, from the pattern of the data values, it seems that these values are outlying in comparison to the rest of the values. Other than these, all the available true detrended temperatures lie within high density regions (except one value at L28).

In a nutshell, we can claim that our model performs well in analyzing the temperature data of Alaska and its surroundings.
Figure 7: Predictive densities of the detrended temperatures for the years 2016-2021 at 8 locations for the Alaska temperature data. The eight locations are indicated by $L_1 \ldots, L_7$, and $L_{11}$ at the left side of the figure corresponding to the rows. The years are shown at the top of the figure corresponding to the columns. The red horizontal line in each plot denotes the 95% posterior predictive interval. The vertical black lines indicate the true values. All the true values, except one, lie within the 95% posterior predictive intervals.
Figure 8: Predictive densities of the detrended temperatures for the years 2016-2021 at other 8 locations for the Alaska temperature data. The eight locations are indicated by $L_{12}, L_{13}, L_{18}, L_{19}, L_{20}, L_{22}, L_{27}$ and $L_{29}$ at the left side of the figure corresponding to the rows. The years are shown at the top of the figure corresponding to the columns. The red horizontal line in each plot denotes the 95% posterior predictive interval. The vertical black lines indicate the true values. All the true values lie within the 95% posterior predictive interval.
Figure 9: Posterior predictive densities of the reconstructed detrended temperature data of Alaska and its surroundings at 8 locations. Higher the intensity of the colour, higher is the probability density. The black stars represent the available true temperature values (detrended). Except for 3 to 4 points at $L_{30}$, and one point at $L_{28}$, all the other points fall within the high probability density region.
7.2 Sea surface temperature data

The sea surface temperature data is obtained from [http://iridl.ldeo.columbia.edu/SOURCES/.CAC/](http://iridl.ldeo.columbia.edu/SOURCES/.CAC/).

For the purpose of illustration, we have taken spatio-temporal observations of the first 40 locations and the first 100 time points from the complete data set. The locations of the chosen data set varies from 56°W to 110°E and 19°S to 25°N. The locations are given in Table 2. Hundred monthly average temperatures are taken starting from the month of January 1970. Among these 40 locations and 100 time points, observations corresponding to 30 locations and 99 time points are taken as the learning set. The observations corresponding to the last 10 locations and the last time point are kept aside for the prediction purpose.

Only for the sake of convenience, the time series at each location is referenced to its mean, in the sense that the mean of the time series at each location is subtracted from the original data.

| Sl No. | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|--------|----|----|----|----|----|----|----|----|----|----|
| Latitude | 19S | 19N | 19S | 17N | 7N | 11S | 3S | 25S | 25N | 7N |
| Longitude | 108E | 20W | 42E | 58E | 2W | 76E | 2E | 28W | 100E | 26W |

| Sl No. | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|--------|----|----|----|----|----|----|----|----|----|----|
| Latitude | 1N | 19S | 13N | 27N | 5S | 17S | 7S | 23S | 21S | 1S |
| Longitude | 84E | 14E | 48E | 18W | 110E | 46W | 10W | 14E | 36W | 28W |

| Sl No. | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|--------|----|----|----|----|----|----|----|----|----|----|
| Latitude | 13N | 7S | 29N | 15S | 1N | 1S | 13S | 9N | 15N | 25S |
| Longitude | 30E | 36E | 104E | 24W | 72E | 46W | 64E | 12E | 24E | 52E |

| Sl No. | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
|--------|----|----|----|----|----|----|----|----|----|----|
| Latitude | 25N | 7N | 11N | 5N | 7N | 19S | 1S | 17S | 19S | 29N |
| Longitude | 4E | 52E | 18E | 16W | 56W | 106E | 20W | 28W | 0 | 48W |

Table 2: Latitude and Longitude (in degrees) of 40 locations for the sea surface temperature data. The locations corresponding to the serial numbers, indicated in bold, are used for complete time series prediction. These bold serial numbers are indicated as $L_{31}, \ldots, L_{40}$ for the future reference.

As in the Alaska temperature data analysis, we denote the learning data set by $\mathcal{D} = \{y_1, \ldots, y_{99}\}$, where $y_i$ is a 30 dimensional spatial observation at time point $t$. At these 30 locations, we obtain the posterior predictive densities for the 100th time point and find out the high probability density regions for the entire time series of the locations corresponding to the serial numbers indicated in bold within Table 2. These locations are referred to as $L_{31}, \ldots, L_{40}$ hereafter. For calculating the posterior predictive densities at 99 time points for the above mentioned 10 spatial locations, the data set $\mathcal{D}$ is first augmented with the initial values for 99 time points of the ten locations and then the values are updated by sampling from the corresponding conditional distributions given the data set $\mathcal{D}$. That is to say, we start with $\mathcal{D}^* = \{y^*_1, \ldots, y^*_{99}\}$, where $y^*_i = [y^*_i : z^*_i]^T$ is a 40 dimensional vector, with $z_i$ being the initial guess for the ten locations $L_{31}, \ldots, L_{40}$, $t \in \{1, \ldots, 99\}$. The parameters of our model, including the latent variables,
are then updated given $D^*$, and then the last ten values of $y_t^*$ are updated given the parameter values and the latent values. This continues for the entire MCMC run. Note that this sea surface temperature data arises from a non-stationary (both weakly and strongly), non-Gaussian spatio-temporal process with lagged correlations tending to zero (refer to [24] for further details).

7.2.1 Prior choices for sea surface temperature data

Following our cross-validation technique to obtain the hyper parameters, we completely specify the priors as follows:

$$
\alpha^* \sim N(0, \sqrt{500}), \beta^* \sim N(0, \sqrt{300}),
\sigma^2 \propto IG(35000, 2/2), [\sigma_p^2] \propto IG(1000, 780/2), [\sigma_p^2] \propto IG(90, 100/2)
$$

$$
[\eta^*_1] \propto N(-3, 1), [\eta^*_2] \propto N(-5, 1).
$$

As before, we fixed $\eta_3$ at its maximum likelihood estimate; here the values is 14.2981.

7.2.2 Controlling $M_s$

Since the locations are distributed very widely over the space, the value of $M_s$ becomes too large for this data for a given $s$. Now $M_s$ appears in the denominator of the variance covariance matrix of the predictive densities of the time series at a particular location (see equation (6)). Therefore, the variability becomes too less. So, to control the variability we modify the definition of $M_s$ as $M_s = \exp(c \max\{||s^2 - u^2||^2 : u \in S\})$, where $c$ is a positive small constant. Note that it does not hamper the properties of $M_s$ and hence all the theoretical properties of the processes remain unchanged. The constant $c$ controls the spatial variability which can be thought of as a distance scaling factor. The choice of $c$ was also done by cross-validation. It was found that $c = 0.25$ works reasonably well for the sea surface temperature data.

7.2.3 Results of the sea surface temperature data

Figure S-14 of Supporting Information, representing the MCMC trace plots of the parameters, exhibits no evidence of non-convergence of our MCMC algorithm. The posterior predictive colour density plots for the latent variables are shown in Figure S-15 of Supporting Information.

Next we provide the posterior predictive densities at the 100th month for each of the 30 locations. Thirty density plots are shown in Figure 10. As the plots indicate, the posterior predictive densities for
the future time point correctly contains the true values for all of these locations. The results encourage us to use the proposed model for the future time point prediction for a non-stationary spatio-temporal data.

Another important aspect of the spatio-temporal modeling is to predict the complete time series at the given locations. As mentioned in Section 7.2, we kept aside the complete temporal observations for the ten locations for evaluating the performance of the proposed model. As described in Section 7.2, we obtained the posterior predictive densities for each of the 99 time points for a given location. The plots are depicted in Figure 11. As one can see, except for the two locations, which are represented by \( L_{36} \) and \( L_{39} \), most of the true values, indicated by black stars in Figure 11, fall well within the high probability density regions. Next, we give a plausible explanation for the poor performance at the locations \( L_{36} \) and \( L_{39} \).

Figure 10: Predictive densities of averaged referenced temperatures for the 100th month at 30 locations for the sea surface temperature data. The red horizontal lines denote the 95% predictive interval. The vertical black lines indicate the true values. All the true values lie within the 95% predictive interval.
Figure 11: Posterior predictive densities of the reconstructed time series of sea surface temperature data (average referenced) at ten locations. Higher the intensity of the colour, higher is the probability density. The black stars represents the true temperature values (average referenced). Except for the locations denoted by $L_{36}$ and $L_{39}$, majority of the true values fall well within the high density regions.

7.2.4 Plausible explanation for the poor performance at the two locations

First we provide below a plot (Figure 12) which represents the locations of the sea temperature data after we make the Lambert projections (see equation (12)). The blue dots in Figure 12 are used as the test set locations for the complete time series prediction. Two points are specially indicated by black star and a green star. These two locations are also members of the test set and at these two locations the high probability density regions fail to capture majority of the true values. The black star corresponds to $L_{36}$ and the green star corresponds to $L_{39}$.

Figure 12: Locations of sea surface temperature data after the transformation given in equations (12). The red points represent the locations of the observations in the leaning set. The blue solid dots, the black and the green stars are the locations of the observations in the test set.
From the above plot, it is evident that the location which is indicated by black star, that is, $L_{36}$, is far away from the majority of the other locations. Thus, the value of $M_s$ for location $L_{36}$ becomes too large making the variability of the predictive densities for every time point very small. For this reason, the high posterior density region could not capture the true value well as compared to other locations (except $L_{39}$). On the other hand, the location indicated by green star is not far apart from the rest of the locations. However, the variability of the observations obtained across the time points are comparatively very large than the variability of the observations at other points. In fact, the variability across the time points of $L_{39}$ is found to be 8.4670 which is more than double the variability of all other locations present in the test set. In addition, at $L_{39}$ more than 70% data lie away from -2 and 2, while the mean remains close to 0. This makes $L_{39}$ unique among the other test locations. Furthermore, it is observed that the mean variability of the learning set locations is 2.0407 and more than 90% locations have data variability less than 5.3314. Therefore, from the learning data, the model did not get enough training to deal with such highly variable data. These are the reasons why the proposed model could not perform as expected for these two particular locations.

To end this discussion, we can note that even when the location $L_{36}$ is far away from the rest, thanks to our modification to $M_s$, the model performs better at this location than at location $L_{39}$. The large temporal variability compared to other locations including training and test sets, made this point ($L_{39}$) outlier and that is why the predictive densities fail to capture the true values.

8 Summary and conclusion

Although Hamiltonian equations are very well-known in physics, in statistics its importance is confined to Hamiltonian Monte Carlo for simulating approximately from posterior distributions. However, given the success of the Hamiltonian equations in phase-space modeling, it is not difficult to anticipate its usefulness in spatio-temporal statistics, if properly exploited. This key insight motivated us to build a new spatio-temporal model through the leap-frog algorithm of a suitably modified set of Hamiltonian equations, where stochasticity is induced through appropriate Gaussian processes. Our marginal, observed stochastic process is nonparametric, non-Gaussian, nonstationary, nonseparable, with appropriate dynamic temporal structure, with time treated as continuous. Additionally, the lagged correlations between the observations tend to zero as the space-time lag goes to infinity. Hence, compared to the existing spatio-temporal processes, our process seems to be the most realistic, and this is vindicated by the results of our
applications to two real data sets. The flexibility of our model is also corroborated by the results of two simulation experiments. Interesting continuity and smoothness properties add further elegance to our new process.

It is important to remark that this article essentially serves as the proof of concept that our novel spatio-temporal process based on modified Hamilton equations covers more ground than the existing processes and is much more flexible with respect to fitting stationary and nonstationary data alike. As such, in this article, we did not attempt to analyze very large datasets. In our future endeavors, we shall consider application of our ideas to very large datasets and with MCMC methods replaced with the iid simulation procedure developed in [45], [46], [47] and [48]. Comparison of our Bayesian model and methods with relevant existing Bayesian models and methods will also be of our interest. However, it is to be borne in mind that our spatio-temporal process attempts to cover a lot more ground than the existing ones, so that designing experiments for fair comparison seems to be a non-trivial task.
Supplemental Information

S-1 Proofs of the theorems

Lemma S-1.1 $M_s$ is infinitely differentiable in $s \in S$.

Proof: Since the exponential function is infinitely differentiable, it is sufficient to show that $\max\{||s^2-u^2||^2 : u \in S\}$ is infinitely differentiable with respect to $s$. We first prove the result in one dimension and then will generalize to higher dimension. Let $S$ be a compact set in $\mathbb{R}$. We consider different cases as follows.

Case 1: Let $S = [a, b]$, where $0 < a < b$. Then $\max\{||s^2-u^2||^2 : u \in [a, b]\} = s^4 + \max(a^4, b^4) - 2s^2a^2$, which is an infinitely smooth function of $s$.

Case 2: Let $S = [a, b]$, where $a < b < 0$. Then $\max\{||s^2-u^2||^2 : u \in [a, b]\} = s^4 + \max(a^4, b^4) - 2s^2b^2$, which is also infinitely differentiable function of $s$.

Case 3: Let $S = [-a, b]$, where $a, b > 0$. Then

$$\max\{||s^2-u^2||^2 : u \in [-a, b]\} = s^4 + \max(a^4, b^4) - 2s^2\min(a^2, b^2)$$

Clearly, $\max\{||s^2-u^2||^2 : u \in [-a, b]\}$ is infinitely differentiable function of $s$, for any $s \in [-a, b]$.

Now suppose that $S = [-a_1, b_1] \times [-a_2, b_2]$, where $a_i, b_i$ are positive for each $i = 1, 2$. Let $s \in S$. Define $f_s : S \to \mathbb{R}$, such that $f_s(u) = ||s^2-u^2||^2 = (s^2_1-u_1^2)^2 + (s^2_2-u_2^2)^2$. Define $f_{s_1}(u_1) = (s^2_1-u_1^2)^2$ and $f_{s_2}(u_2) = (s^2_2-u_2^2)^2$. Therefore, $f_s(u) = f_{s_1}(u_1)+f_{s_2}(u_2)$ and $\max_u f_s(u) = \max_{u \in S}[f_{s_1}(u_1)+f_{s_2}(u_2)] = \max_{u_1 \in [-a_1, b_1]} f_{s_1}(u_1)+\max_{u_2 \in [-a_2, b_2]} f_{s_2}(u_2)$. The equality follows as $u = (u_1, u_2) \in S$, with $u_1 \in [-a_1, b_1]$ and $u_2 \in [-a_2, b_2]$. We have already proved that (Case 3, equation (1)) $\max_{u \in [-a_1, b_1]} f_{s_1}(u_1) = s^4_1 + \max(a^4_1, b^4_1) - 2s^2_1\min(a^2_1, b^2_1)$ for $i = 1, 2$.

Hence $\max_{u \in S} f_s(u) = \sum_{i=1}^2 s^4_i + \max(a^4_1, b^4_1) - 2s^2_i\min(a^2_1, b^2_1)$, which is infinitely differentiable with respect to $s_1$ and $s_2$.

If $S = [a_1, b_1] \times [a_2, b_2]$ where (i) $0 < a_1 < b_1$ and $0 < a_2 < b_2$ or (ii) $a_1 < b_1 < 0$ and $a_2 < b_2 < 0$ or (iii) $0 < a_1 < b_1$ and $a_2 < b_2 < 0$ or (iv) $a_1 < b_1 < 0$ and $0 < a_2 < b_2$, the proof goes in a similar manner as above except that now we have to use Case 1 and Case 2 for one dimension, instead of Case 3. □

Theorem S-1.1 Under assumptions A1 to A3, $\cov\left(\theta_s(h\delta t), \theta_{s'}(h'\delta t)\right)|p$ converges to 0 as $||s-s'|| \to \infty$ and $|h-h'| \to \infty$. 
Proof: Without loss of generality let us assume that $h > h'$. Now

$$
cov \left( \theta_s(h \delta t), \theta_{s'}(h' \delta t) \big| p \right) \\
= \text{cov} \left( \beta \theta_s((h - 1) \delta t) + \frac{\delta t}{M_s} \left\{ \alpha p_\delta((h - 1) \delta t) - \frac{\delta t}{2} \frac{\partial}{\partial \theta_s} V(\theta_s((h - 1) \delta t)) \right\}, \theta_{s'}(h' \delta t) \big| p \right) \\
= \text{cov}(\beta \theta_s((h - 1) \delta t), \theta_{s'}(h' \delta t) \big| p) - \frac{1}{2} \frac{(\delta t)^2}{M_s} \text{cov} \left( \frac{\partial}{\partial \theta_s} V(\theta_s((h - 1) \delta t)), \theta_{s'}(h' \delta t) \big| p \right) \\
= \ldots \\
= \beta^{h - h'} \text{cov}(\theta_s(h' \delta t), \theta_{s'}(h' \delta t) \big| p) - \frac{1}{2} \frac{(\delta t)^2}{M_s} \sum_{k=1}^{h-h'} \beta^{k-1} \text{cov} \left[ \frac{\partial}{\partial \theta_s} V(\theta_s((h - k) \delta t)), \theta_{s'}(h' \delta t) \big| p \right]. \tag{2}
$$

Since, for any $1 \leq \ell \leq h - h'$,

$$
\text{cov} \left( \frac{\partial}{\partial \theta_s} V(\theta_s((h - \ell) \delta t)), \theta_{s'}(h' \delta t) \big| p \right) \\
= \text{cov} \left( \frac{\partial}{\partial \theta_s} V(\theta_s((h - \ell) \delta t)), \beta \theta_{s'}((h' - 1) \delta t) + \frac{\delta t}{M_{s'}} \left\{ \alpha p_\delta((h' - 1) \delta t) - \frac{\delta t}{2} \frac{\partial}{\partial \theta_{s'}} V(\theta_{s'}((h' - 1) \delta t)) \right\} \big| p \right) \\
= \beta \text{cov} \left( \frac{\partial}{\partial \theta_s} V(\theta_s((h - \ell) \delta t)), \theta_{s'}((h' - 1) \delta t) \big| p \right) - \frac{1}{2} \frac{(\delta t)^2}{M_{s'}} \text{cov} \left( \frac{\partial}{\partial \theta_s} V(\theta_s((h - \ell) \delta t)), \frac{\partial}{\partial \theta_{s'}} V(\theta_{s'}((h - \ell) \delta t)) \big| p \right) \\
= \ldots \\
= \beta^{h'} \text{cov} \left( \frac{\partial}{\partial \theta_s} V(\theta_s((h - \ell) \delta t)), \theta_{s'}(0) \big| p \right) - \frac{(\delta t)^2}{2 M_{s'}} \sum_{k=1}^{h'} \beta^{k-1} \text{cov} \left[ \frac{\partial}{\partial \theta_s} V(\theta_s((h - \ell) \delta t)), \frac{\partial}{\partial \theta_{s'}} V(\theta_{s'}((h - k) \delta t)) \big| p \right], \\
$$

we have,

$$
\left| \text{cov} \left( \frac{\partial}{\partial \theta_s} V(\theta_s((h - \ell) \delta t)), \theta_{s'}(h' \delta t) \big| p \right) \right| \\
\leq \left| \beta^{h'} \text{cov} \left( \frac{\partial}{\partial \theta_s} V(\theta_s((h - \ell) \delta t)), \theta_{s'}(0) \big| p \right) \right| + \frac{(\delta t)^2}{2 M_{s'}} \sum_{k=1}^{h'} \left| \beta^{k-1} \text{cov} \left[ \frac{\partial}{\partial \theta_s} V(\theta_s((h - \ell) \delta t)), \frac{\partial}{\partial \theta_{s'}} V(\theta_{s'}((h - k) \delta t)) \big| p \right] \right| \\
\leq |\beta|^{h'} \sigma_{\theta \theta} + \frac{1 - |\beta| h'}{1 - |\beta|} \frac{\sigma^2}{M_{s'}} = \epsilon, \text{ say,} \tag{3}
$$

where $\sigma^2$ and $\sigma^2_{\theta}$ are the variance terms of the processes $\frac{\partial}{\partial \theta_s} V(\theta_s(h - \ell) \delta t)$ and $\theta_s(0)$, respectively (see A2...
and A3). Therefore, from Equation (2) we obtain

\[ | \text{cov} \left( \theta_s(h\delta t), \theta_{s'}(h'\delta t) \mid p \right) - \beta^{h-h'} \text{cov} \left( \theta_s(h'\delta t), \theta_{s'}(h'\delta t) \mid p \right) | \]

\[ \leq \frac{1}{2} \frac{(\delta t)^2}{M_s} \sum_{k=1}^{h'} |\beta|^{k-1} \left| \text{cov} \left( \frac{\partial}{\partial \theta_s} V \left( \theta_s((h-k)\delta t) \right), \theta_{s'}(h'\delta t) \mid p \right) \right| \]

\[ \leq \frac{1}{2} \frac{(\delta t)^2}{M_s} \frac{1 - |\beta|^{h-h'}}{1 - |\beta|^{-\epsilon}}, \]

(4)

using equation (3). Now from the first term of the right hand side of equation (2), we obtain

\[ \text{cov} \left( \theta_s(h\delta t), \theta_{s'}(h'\delta t) \mid p \right) \]

\[ = \text{cov} \left( \theta_s((h'-1)\delta t) + \frac{\delta t}{M_s} \left\{ \alpha p_s((h'-1)\delta t) - \frac{\delta t}{2} \frac{\partial}{\partial \theta_s} V \left( \theta_s((h'-1)\delta t) \right) \right\}, \right. \]

\[ \beta \theta_{s'}((h'-1)\delta t) + \frac{\delta t}{M_s} \left\{ \alpha p_s((h'-1)\delta t) - \frac{\delta t}{2} \frac{\partial}{\partial \theta_{s'}} V \left( \theta_s((h'-1)\delta t) \right) \right\} | p \right) \]

\[ = \beta^2 \text{cov} \left( \theta_s((h'-1)\delta t), \theta_{s'}((h'-1)\delta t) \mid p \right) - \beta \frac{1}{2} \frac{(\delta t)^2}{M_s} \text{cov} \left( \theta_s((h'-1)\delta t), \frac{\partial}{\partial \theta_s} V \left( \theta_s((h'-1)\delta t) \right) \mid p \right) \]

\[ - \beta \frac{1}{2} \frac{(\delta t)^2}{M_s} \text{cov} \left( \theta_{s'}((h'-1)\delta t), \frac{\partial}{\partial \theta_{s'}} V \left( \theta_s((h'-1)\delta t) \right) \mid p \right) \]

\[ + \frac{1}{4} \frac{(\delta t)^4}{M_s M_s'} \text{cov} \left( \frac{\partial}{\partial \theta_s} V \left( \theta_s((h'-1)\delta t) \right), \frac{\partial}{\partial \theta_{s'}} V \left( \theta_s((h'-1)\delta t) \right) \mid p \right), \]

which in turn implies

\[ | \text{cov} \left( \theta_s(h\delta t), \theta_{s'}(h'\delta t) \mid p \right) - \beta^{2k'} \text{cov} \left( \theta_s(0), \theta_{s'}(0) \mid p \right) | \]

\[ \leq \frac{(\delta t)^2}{2M_s} \sum_{k=1}^{h'} |\beta|^{2k-1} \left| \text{cov} \left( \theta_s((h-k)\delta t), \frac{\partial}{\partial \theta_s} V \left( \theta_s((h-k)\delta t) \right) \mid p \right) \right| \]

\[ + \frac{(\delta t)^2}{2M_s} \sum_{k=1}^{h'} |\beta|^{2k-1} \left| \text{cov} \left( \theta_{s'}((h-k)\delta t), \frac{\partial}{\partial \theta_{s'}} V \left( \theta_s((h-k)\delta t) \right) \mid p \right) \right| \]

\[ + \frac{1}{4} \frac{(\delta t)^4}{M_s M_s'} \sum_{k=1}^{h'} |\beta|^{2(k-1)} \left| \text{cov} \left( \frac{\partial}{\partial \theta_s} V \left( \theta_s((h-k)\delta t) \right), \frac{\partial}{\partial \theta_{s'}} V \left( \theta_s((h-k)\delta t) \right) \mid p \right) \right|, \]
\[
\frac{\partial}{\partial \theta_s} V(\theta_s((h' - k)\delta t)) \bigg|_p \leq \frac{(\delta t)^2}{2M_s} \sum_{k=1}^{K} |\beta|^{2k-1} + \frac{(\delta t)^2}{2M_s} \sum_{k=1}^{K} |\beta|^{2k} + \frac{(\delta t)^4}{4 M_s M_{s'}} \sum_{k=1}^{K} |\beta|^{2k-1} \epsilon + \frac{(\delta t)^4}{4 M_s M_{s'}} \sum_{k=1}^{K} |\beta|^{2k} \sigma^2.
\]

From equations (4) and (5), we get

\[
\text{Cov} \left[ \theta_s(h\delta t), \theta_s'(h'\delta t) \right] - \beta^{(h-h')} + 2h' \text{ Cov} \left( \theta_s(0), \theta_s'(0) \right) \leq \text{Cov} \left[ \theta_s(h\delta t), \theta_s'(h'\delta t) \right] - \beta^{h-h'} \text{Cov} \left[ \theta_s(h\delta t), \theta_s(h'\delta t) \right] + |\beta|^{h-h'} \text{Cov} \left[ \theta_s(h\delta t), \theta_s'(h'\delta t) \right] - \beta^{2h'} \text{Cov} \left( \theta_s(0), \theta_s'(0) \right) \leq \frac{(\delta t)^2}{2M_s} |\beta|^{1} \frac{1}{1 - |\beta|^2} + \frac{(\delta t)^2}{2M_s} |\beta|^{2} \frac{1}{1 - |\beta|^2} \epsilon + \frac{(\delta t)^4}{4 M_s M_{s'}} \sum_{k=1}^{K} |\beta|^{2k} \sigma^2,
\]

which goes to 0 as \( h-h' \to \infty \) and \( |s-s'| \to \infty \), under assumptions A1-A3 in conjunction with Remark 6. Writing

\[
\text{Cov} \left[ \theta_s(h\delta t), \theta_s'(h'\delta t) \right] \text{ as } \text{Cov} \left[ \theta_s(h\delta t), \theta_s(h'\delta t) \right] - \beta^{(h-h')} + 2h' \text{Cov} \left[ \theta_s(0), \theta_s'(0) \right] + \beta^{(h-h')} + 2h' \text{Cov} \left[ \theta_s(0), \theta_s'(0) \right] \quad \text{and noting that } |\beta|^{(h-h')} + 2h' \text{Cov} \left[ \theta_s(0), \theta_s'(0) \right] \text{ is less than } |\beta|^{(h-h')} \beta^{2h'} \sigma^2,
\]

we have our desired result. \( \square \)

**Theorem S-1.2** If assumptions A1-A3 hold true, then \( \theta_s(h\delta t) \) and \( p_s(h\delta t) \) are continuous in \( s \), for all \( h \geq 1 \), with probability 1.

Proof: Note that, for \( h \geq 1 \),

\[
\theta_s(h\delta t) = \beta \theta_s((h-1)\delta t) + \frac{\delta t}{M_s} \left\{ \alpha p_s((h-1)\delta t) - \frac{\delta t}{2} \frac{\partial}{\partial \theta_s} V(\theta_s((h-1)\delta t)) \right\} \quad \text{and} \quad (7)
\]

\[
p_s(h\delta t) = \alpha^2 p_s((h-1)\delta t) - \frac{\delta t}{2} \left\{ \alpha \frac{\partial}{\partial \theta_s} V(\theta_s((h-1)\delta t)) + \frac{\partial}{\partial \theta_s} V(\theta_s(h\delta t)) \right\}. \quad \text{(8)}
\]

Putting \( h = 1 \) in equations (7) and (8), we get

\[
\theta_s(\delta t) = \beta \theta_s(0) + \frac{\delta t}{M_s} \left\{ \alpha p_s(0) - \frac{\delta t}{2} \frac{\partial}{\partial \theta_s} V(\theta_s(0)) \right\} \quad \text{and}
\]

\[
p_s(\delta t) = \alpha^2 p_s(0) - \frac{\delta t}{2} \left\{ \alpha \frac{\partial}{\partial \theta_s} V(\theta_s(0)) + \frac{\partial}{\partial \theta_s} V(\theta_s(\delta t)) \right\}.
\]

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Now $p_s(0)$ and $\theta_s(0)$ are Gaussian processes with continuous sample paths with probability 1 under assumptions A1 and A2 (also see Remark 3), and Lemma 2.1 shows that $M_s$ is continuous in $s$. Furthermore, assumption A3 implies that the derivative of $V(\cdot)$ is also Gaussian process with continuous sample paths (see Remark 4). Since composition of two continuous function is also a continuous function therefore, \( \frac{\partial}{\partial s} V(\theta_s(0)) \) is also continuous in $s$ with probability 1. This implies $\theta_s(\delta t)$ is continuous in $s$ with probability 1 as it is a linear combination of three almost sure continuous functions in $s$. This immediately implies that $p_s(\delta t)$ is also continuous in $s$ with probability 1.

Assume that $\theta_s(h\delta t)$ and $p_s(h\delta)$ are continuous in $s$ with probability 1, for $h = k + 1$. We will show that $\theta_s(h\delta t)$ and $p_s(h\delta)$ are almost surely continuous in $s$ for $h = k + 2$. Now

\[
\begin{align*}
\theta_s((k + 2)\delta t) &= \beta \theta_s((k + 1)\delta t) + \frac{\delta t}{M_s} \left\{ p_s((k + 1)\delta t) - \frac{\delta t}{2} \frac{\partial}{\partial \theta_s} V(\theta_s((k + 1)\delta t)) \right\} \quad \text{and} \\
p_s((k + 2)\delta t) &= \alpha^2 p_s((k + 1)\delta t) - \frac{\delta t}{2} \left\{ \alpha \frac{\partial}{\partial \theta_s} V(\theta_s((k + 1)\delta t)) + \frac{\partial}{\partial \theta_s} V(\theta_s((k + 2)\delta t)) \right\} .
\end{align*}
\]

Since $\theta_s((k + 1)\delta t)$ and $p_s((k + 1)\delta t)$ are assumed to be continuous in $s$ with probability 1, derivative of a Gaussian process is also a Gaussian process, composition of two continuous functions is also a continuous function, and linear combinations of continuous functions is a continuous function, we have $\theta_s((k + 2)\delta t)$ is continuous in $s$ with probability 1. Similar arguments show that $p_s((k + 2)\delta t)$ is also continuous in $s$ with probability 1. Hence using the argument of induction, we claim that $\theta_s(h\delta t)$ and $p_s(h\delta t)$ are continuous in $s$ for any $h \geq 1$, with probability 1.

**Theorem S-1.3** Under assumptions A1-A3, $\theta_s(h\delta t)$ and $p_s(h\delta t)$ are continuous in $s$ in the mean square sense, for all $h \geq 1$.

**Proof:** We follow similar steps as in Lemma 2.2. From equation (7) we have

\[
\theta_s(\delta t) = \beta \theta_s(0) + \frac{\delta t}{M_s} \left\{ \alpha p_s(0) - \frac{\delta t}{2} \frac{\partial}{\partial \theta_s} V(\theta_s(0)) \right\}.
\]

Under assumptions A1 and A2, $p_s(0)$ and $\theta_s(0)$ are continuous in $s$ in the mean square sense, and by Lemma 2.1 $M_s$ is continuous in $s$. Now since the random function $V(\cdot)$ is twice differentiable under assumption A3 (see Remark 4 also), the partial derivative process of $V$ is Lipschitz and hence the composition function $\frac{\partial}{\partial \theta_s} V(\theta_s(0))$ is also mean square continuous in $s$ (see page 416 of [49]). Now using the fact the linear combination of mean square continuous processes is also mean-square continuous, we have $\theta_s(\delta t)$
is mean square continuous in \( s \). This, in turn implies that, 
\[
\frac{\partial}{\partial s} V(\theta_s(\delta t)) \]
is also mean square continuous in \( s \) using the same argument as above. Therefore,
\[
p_s(\delta t) = \alpha^2 p_s(0) - \frac{\delta t}{2} \left\{ \alpha \frac{\partial}{\partial \theta_s} V(\theta_s(0)) + \frac{\partial}{\partial s} V(\theta_s(\delta t)) \right\}
\]
is mean square continuous in \( s \).

Now applying the similar argument of induction as in Lemma 2.2, we have the desired result. \(\square\)

**Theorem S-1.4** Under assumptions A1-A3, \( \theta_s(h\delta t) \) and \( p_s(h\delta t) \) have differentiable sample paths with respect to \( s \), almost surely.

Proof: The proof goes in the same line as that of Lemma 2.2. We start with \( \theta_s(\delta t) \) which is a linear combination of \( \theta_s(0) \), \( p_s(0) \) and \( \frac{\partial}{\partial \theta_s} V(\theta_s(0)) \). Note that \( \theta_s(0) \), \( p_s(0) \) have differentiable sample paths in \( s \) by assumptions A1 and A2 (see Remark 3). Now using the fact that composition of two differentiable function is also differentiable (see the Lemma 3.4 of [30]), \( \frac{\partial}{\partial \theta_s} V(\theta_s(0)) \) has differentiable sample paths in \( s \) (refer to assumption A3 and Remark 4). Moreover, since \( M_s \) is differentiable (Lemma 2.1) and since the linear combination of differentiable functions is a differentiable function, \( \theta_s(\delta t) \) has differentiable sample paths in \( s \). The existence of differentiable sample paths of \( \theta_s(\delta t) \) implies that \( p_s(\delta t) \) will have differentiable sample paths in \( s \). The rest of the proof is similar to that of Lemma 2.2. \(\square\)

**Lemma S-1.2** Let \( f : \mathbb{R} \to \mathbb{R} \) be a zero mean Gaussian random function with covariance function \( c_f(x_1, x_2) \), \( x_1, x_2 \in \mathbb{R} \), which is four times continuously differentiable. Let \( \{ Z(s) : s \in S \} \) be a random process with the following properties

1. \( E(Z(s)) = 0 \),

2. The covariance function \( c_Z(s_1, s_2) \), \( s_1, s_2 \in S \), where \( S \) is a compact subspace of \( \mathbb{R}^2 \), is four times continuously differentiable, and

3. \( \frac{\partial Z(s)}{\partial s_i} \) has finite fourth moment.

Then the process \( \{ g(s) : s \in S \} \), where \( g(s) = f(Z(s)) \), is mean square differentiable in \( s \).

Proof: To show that \( \{ g(s) : s \in S \} \) is mean square differentiable in \( s \) we have to show that, for any \( p \in S \), there exists a function \( L_s(p) \), linear in \( p \), such that
\[
g(s + p) = g(s) + L_s(p) + R(s, p),
\]
where
\[
\frac{R(s, p)}{||p||} \xrightarrow{L_2} 0.
\]

Let \( s_0 \in S \) be any point in \( S \). Using multivariate Taylor series expansion we have
\[
g(s_0 + p) = g(s_0) + p^T \nabla g(s_0) + R(s_0, p),
\]
where \( \nabla g(s_0) = \left( \frac{\partial f(Z(s))}{\partial s_1}, \frac{\partial f(Z(s))}{\partial s_2} \right)^T \), with \( \frac{\partial f(Z(s))}{\partial s_i} = \frac{\partial f(Z(s))}{\partial Z_i} \frac{\partial Z_i}{\partial s_i} \), for \( i = 1, 2 \). Therefore, \( L_{s_0}(p) = p^T \nabla g(s_0) \), a linear function in \( p \). To complete the proof we note that, from multivariate Taylor series expansion, \( |R(s_0, p)| \leq M^* ||p||^2 \), where
\[
M^* = \max \left\{ \left| \frac{\partial^2 f(Z(s))}{\partial s_i^2} \right|, \left| \frac{\partial^2 f(Z(s))}{\partial s_1 \partial s_2} \right|, \left| \frac{\partial^2 f(Z(s))}{\partial s_2 \partial s_1} \right|, \left| \frac{\partial^2 f(Z(s))}{\partial s_2^2} \right| \right\},
\]
with
\[
\frac{\partial^2 f(Z(s))}{\partial s_i^2} = \frac{\partial^2 f(Z(s))}{d((Z(s)))^2} \left( \frac{\partial Z_i}{\partial s_i} \right)^2 + \frac{\partial f(Z(s))}{dZ(s)} \frac{\partial^2 Z_i}{\partial s_i^2}, \text{ for } i = 1, 2,
\]
and
\[
\frac{\partial^2 f(Z(s))}{\partial s_1 \partial s_2} = \frac{\partial^2 f(Z(s))}{dZ(s)^2} \frac{\partial Z_i}{\partial s_1} \frac{\partial Z_i}{\partial s_2} + \frac{\partial f(Z(s))}{dZ(s)} \frac{\partial^2 Z_i}{\partial s_1 \partial s_2}.
\]

Since we have assumed that \( Z(\cdot) \) and \( f(\cdot) \) have covariance functions which are four times continuously differentiable, \( Z(\cdot) \) and \( f(\cdot) \) are twice differentiable (in the mean square sense) and hence the above terms involving first and second derivatives of \( f(\cdot) \) and \( Z(\cdot) \) are well-defined.

We will now show that the second moment of \( \frac{\partial^2 f(Z(s))}{d((Z(s)))^2} \), for \( i = 1, 2 \), and \( \frac{\partial^2 f(Z(s))}{\partial s_1 \partial s_2} \) are finite. To prove the above fact, we first show that \( \text{var} \left( \frac{\partial^2 f(Z(s))}{d((Z(s)))^2} \right) < \infty \). Note that
\[
\text{var} \left( \frac{\partial^2 f(Z(s))}{d((Z(s)))^2} \right) = \text{var} \left( E \left\{ \frac{\partial^2 f(Z(s))}{d((Z(s)))^2} \bigg| Z(s) \right\} \right) + E \left( \text{var} \left\{ \frac{\partial^2 f(Z(s))}{d((Z(s)))^2} \bigg| Z(s) \right\} \right).
\]
Moreover, since \( f''(x) \) is a Gaussian function with 0 mean and a constant variance, we have
\[
E \left\{ \frac{\partial^2 f(Z(s))}{d((Z(s)))^2} \bigg| Z(s) \right\} = 0 \text{ and } \text{var} \left\{ \frac{\partial^2 f(Z(s))}{d((Z(s)))^2} \bigg| Z(s) \right\} = \text{constant} < \infty.
\]
Thus
\[
\text{var} \left( E \left\{ \frac{\partial^2 f(Z(s))}{d((Z(s)))^2} \bigg| Z(s) \right\} \right) = 0 \text{ and } \text{var} \left( E \left\{ \frac{\partial^2 f(Z(s))}{d((Z(s)))^2} \bigg| Z(s) \right\} \right) = 0.
\]
\[ E \left( \text{var} \left\{ \frac{d^2 f(Z(s))}{d((Z(s))^2)} Z(s) \right\} \right) = \text{constant.} \] (11)

Therefore, combining equations (10) and (11), and using equation (9) we see that \( \text{var} \left( \frac{d^2 f(Z(s))}{d((Z(s))^2)} \right) < \infty. \) Similar argument shows that \( \text{var} \left( \frac{d f(Z(s))}{d s_i} \right) < \infty. \) Now to show that \( E \left( \frac{\partial^2 f(Z(s))}{\partial s_i^2} \right)^2 < \infty, \) we use \((a + b)^2 \leq 2(a^2 + b^2)\) and have

\[ E \left( \frac{\partial^2 f(Z(s))}{\partial s_i^2} \right)^2 \leq 2 \left\{ E \left( \frac{d^2 f(Z(s))}{d((Z(s))^2)} \right)^2 \left[ \frac{\partial Z(s)}{\partial s_i} \right]^4 \right\} + E \left( \frac{d f(Z(s))}{d Z(s)} \right)^2 \left[ \frac{\partial^2 Z(s)}{\partial s_i^2} \right]^2 \) 

\[ = 2E \left\{ E \left( \frac{d^2 f(Z(s))}{d((Z(s))^2)} \right)^2 \left[ Z(s) \right] \left[ \frac{\partial Z(s)}{\partial s_i} \right]^4 \right\} + 2E \left\{ E \left( \frac{d f(Z(s))}{d Z(s)} \right)^2 \left[ Z(s) \right] \left[ \frac{\partial^2 Z(s)}{\partial s_i^2} \right]^2 \right\}. \] (12)

Again using the fact that \( f'(x) \) and \( f''(x) \) are Gaussian with 0 mean and constant variance, we have

\[ E \left( \frac{d^2 f(Z(s))}{d((Z(s))^2)} \right)^2 \left| Z(s) \right) = \text{constant and } E \left( \frac{d f(Z(s))}{d Z(s)} \right)^2 \left| Z(s) \right) = \text{constant.} \]

Further, since the covariance function of \( Z(s) \) is assumed to be four times continuously differentiable (2nd assumption of the Lemma), the first two derivatives of \( Z(s), \frac{\partial Z(s)}{\partial s_i}, i = 1, 2, \) and \( \frac{\partial^2 Z(s)}{\partial s_i^2}, i = 1, 2 \) will also exist in the mean square sense, with zero means. Thus, \( E \left[ \frac{\partial^2 Z(s)}{\partial s_i^2} \right]^2 = \text{constant.} \) Also by the 3rd assumption of the Lemma, the fourth moment of \( \frac{\partial Z(s)}{\partial s_i}, \) for \( i = 1, 2, \) are finite. That is, \( E \left[ \frac{\partial Z(s)}{\partial s_i} \right]^4 = \text{constant.} \) Therefore, from equation (12) we see that \( E \left( \frac{\partial^2 f(Z(s))}{\partial s_i^2} \right)^2 < M_1 < \infty, \) for some \( M_1 \in \mathbb{R}. \)

Next we show that \( E \left( \frac{\partial^2 f(Z(s))}{\partial s_1 \partial s_2} \right)^2 < \infty. \) Using the same inequality, \((a + b)^2 \leq 2(a^2 + b^2),\) we have

\[ E \left( \frac{\partial^2 f(Z(s))}{\partial s_1 \partial s_2} \right)^2 \leq 2 \left\{ E \left( \frac{d^2 f(Z(s))}{d((Z(s))^2)} \right)^2 \left[ \frac{\partial Z(s)}{\partial s_1} \right] \left[ \frac{\partial Z(s)}{\partial s_2} \right] \right\} + E \left( \frac{d f(Z(s))}{d Z(s)} \right)^2 \left[ \frac{\partial^2 Z(s)}{\partial s_1 \partial s_2} \right]^2 \) 

\[ = 2E \left\{ E \left( \frac{d^2 f(Z(s))}{d((Z(s))^2)} \right)^2 \left[ Z(s) \right] \left[ \frac{\partial Z(s)}{\partial s_1} \right] \left[ \frac{\partial Z(s)}{\partial s_2} \right] \right\} + 2E \left\{ E \left( \frac{d f(Z(s))}{d Z(s)} \right)^2 \left[ Z(s) \right] \left[ \frac{\partial^2 Z(s)}{\partial s_1 \partial s_2} \right]^2 \right\}. \] (13)

Since \( f'(x) \) and \( f''(x) \) are Gaussian functions with mean 0 and constant variance, therefore,

\[ E \left( \frac{d^2 f(Z(s))}{d((Z(s))^2)} \right)^2 \left| Z(s) \right) = \text{constant and } E \left( \frac{d f(Z(s))}{d Z(s)} \right)^2 \left| Z(s) \right) = \text{constant.} \] Moreover, since by our assumption, the fourth moment of \( \frac{\partial Z(s)}{\partial s_i}, \) for \( i = 1, 2, \) are finite, we have \( E \left[ \left( \frac{\partial Z(s)}{\partial s_1} \right)^2 \left( \frac{\partial Z(s)}{\partial s_2} \right)^2 \right] \leq \)
\[
\sqrt{E\left(\frac{\partial^2 f(Z(s))}{\partial s_1^2}\right)^4 E\left(\frac{\partial^2 f(Z(s))}{\partial s_2^2}\right)^4} = \text{constant.}
\]
Since the covariance function of \( Z(s) \) is four times continuously differentiable, \( E\left(\frac{\partial^2 f(Z(s))}{\partial s_1^2}\right)^2 = \text{constant.} \) Thus we have, from equation (13), \( E\left(\frac{\partial^2 f(Z(s))}{\partial s_1^2}\right)^2 < M_2 \), for some \( M_2 \in \mathbb{R} \). Hence each term in \( M^* = \max \left\{ \left| \frac{\partial^2 f(Z(s))}{\partial s_1^2} \right|, \left| \frac{\partial^2 f(Z(s))}{\partial s_2^2} \right|, \left| \frac{\partial^2 f(Z(s))}{\partial s_1 \partial s_2} \right| \right\} \) has bounded second moment.

Next we have to show that \( E(M^*)^2 < \infty \). Denote \( M^* = \max\{A, B, C, D\} \), where \( A = \left| \frac{\partial^2 f(Z(s))}{\partial s_1^2} \right| \), \( B = \left| \frac{\partial^2 f(Z(s))}{\partial s_1 \partial s_2} \right| \), \( C = \left| \frac{\partial^2 f(Z(s))}{\partial s_2 \partial s_1} \right| \) and \( D = \left| \frac{\partial^2 f(Z(s))}{\partial s_2^2} \right| \). Note that it is sufficient to show that \( X = \max\{A, B\} \) has finite second moment, because then with the same argument \( Y = \max\{C, D\} \) will have finite second moment, and finally \( Z = \max\{X, Y\} \left( = \max\{A, B, C, D\} \right) \) will have finite second moment.

Now \( X = \max\{A, B\} = \frac{A+B+|A-B|}{2} \leq \frac{A+B+|A|+|B|}{2} \). Therefore,

\[
EX^2 \leq \frac{1}{4} \left[ E \left\{ (A+B)^2 + (|A|+|B|)^2 \right\} \right] \leq E\left\{ A^2 + B^2 + |A|^2 + |B|^2 \right\} = 2E(A^2 + B^2).
\]
Since \( EA^2 < \infty \) and \( EB^2 < \infty \), therefore, \( EX^2 < \infty \). Now exactly same arguments imply that \( EY^2 < \infty \) and \( EZ^2 < \infty \). Hence, \( \frac{R(s_0, p)}{|p|} \xrightarrow{L^2} 0 \). Since \( s_0 \) is any point in \( S \), the proof is complete. \( \square \)

**Theorem S-1.5** Let A1-A3 hold true, with the covariance functions of all the assumed Gaussian processes being square exponentials. Then \( \theta_s(h\delta t) \) and \( p_s(h\delta t) \) are mean square differentiable in \( s \), for every \( h \geq 1 \).

**Proof:** The proof will follow the similar argument of induction as done in Lemma 2.2. For \( h = 1 \), we have

\[
\theta_s(\delta t) = \beta \theta_s(0) + \frac{\delta t}{M_s} \left\{ \alpha p_s(0) - \frac{1}{2} \delta t \frac{\partial}{\partial \theta_s} V(\theta_s(0)) \right\}.
\]
Since, under the assumption A2 and the assumption of the theorem, \( \theta_s(0) \) is a centered Gaussian process with squared exponential covariance function, the fourth moment of \( \frac{\partial \theta_s(0)}{\partial s_i} \), \( i = 1, 2 \) are finite, and by assumption A3, \( V(\cdot) \) is a zero-mean Gaussian random function with squared exponential covariance. Therefore, using lemma 2.2, \( \frac{\partial}{\partial \theta_s} V(\theta_s(0)) \) is mean square differentiable. Under the assumptions A1 and A2, and using the fact the covariance functions are assumed to squared exponential, \( p_s(0) \) and \( \theta_s(0) \) are mean squared differentiable. Using the fact that \( M_s \) is differentiable in \( s \) (see Lemma 2.1), \( \theta_s(\delta t) \), being a linear combination of mean square differentiable functions, is also mean square differentiable.

Next we show that \( E(\theta_s(\delta t)) = 0 \) and \( \text{cov}(\theta_{s_1}(\delta t), \theta_{s_2}(\delta t)) \) is 4-times differentiable. Denoting the first
derivative of \( V \) by \( V' \), we obtain
\[
E(\theta_s(\delta t)) = \beta E(\theta_s(0)) + \frac{\delta t}{M_s} \left\{ \alpha E(\phi_s(0)) - \frac{1}{2} \delta t E(V'(\theta_s(0))) \right\} = 0,
\]
where we have used the fact that \( E(V'(\theta_s(0))) = E[E(V'(\theta_s(0))|\theta_s(0)] = E(0) = 0 \).

Therefore, \( \text{cov}(\theta_{s_1}(\delta t), \theta_{s_2}(\delta t)) = E(\theta_{s_1}(\delta t)\theta_{s_2}(\delta t)) \). Since the processes are assumed to be independent, we have
\[
E(\theta_{s_1}(\delta t)\theta_{s_2}(\delta t)) = \beta^2 \sigma_\theta^2 e^{-\eta_2||s_1-s_2||^2} + \langle \frac{\delta t}{M_1 M_2} \sigma_\theta^2 \rangle + \frac{(\delta t)^2}{4M_1 M_2} E[V'(\theta_{s_1}(0))V'(\theta_{s_2}(0))].
\]

Now according to our assumption, \( \text{cov}(V(x_1), V(x_2)) = \sigma_3^2 e^{-\eta_3||x_1-x_2||^2} \), that is, the covariance function of \( V(\cdot) \) can be written as \( \kappa(h) = \sigma_3^2 e^{-\eta_3 h^2} \), where \( h = ||x_1 - x_2|| \). The second derivative of \( \kappa(h) \) is given by \( -2\sigma_3^2 \eta_3 e^{-\eta_3 h^2} (1 - 2\eta_3 h^2) \). Hence the covariance function of \( V'(\cdot) \) will be \( \text{cov}(V'(x_1), V'(x_2)) = 2\sigma_3^2 \eta_3 e^{-\eta_3 h^2} (1 - 2\eta_3 h^2) \) (see [40], page 21). Therefore, the last term of the right hand side of equation (14) can be computed as

\[
E[V'(\theta_{s_1}(0))V'(\theta_{s_2}(0))] = E E[V'(\theta_{s_1}(0))V'(\theta_{s_2}(0))|\theta_{s_1}(0)\theta_{s_2}(0)]
= E \left[ 2\sigma_3^2 \eta_3 e^{-\eta_3 (\theta_{s_1}(0) - \theta_{s_2}(0))^2} (1 - 2\eta_3 (\theta_{s_1}(0) - \theta_{s_2}(0))^2) \right]
= 2\sigma_3^2 \eta_3 \left[ E e^{-\eta_3 (\theta_{s_1}(0) - \theta_{s_2}(0))^2} - 2\eta_3 E \left\{ (\theta_{s_1}(0) - \theta_{s_2}(0))^2 e^{-\eta_3 (\theta_{s_1}(0) - \theta_{s_2}(0))^2} \right\} \right].
\]

Since \( (\theta_{s_1}(0), \theta_{s_2}(0))^T \sim N_2(0, \begin{pmatrix} \sigma_\theta^2 & \sigma_\theta^2 e^{-\eta_2 ||s_1-s_2||^2} \\ \sigma_\theta^2 e^{-\eta_2 ||s_1-s_2||^2} & \sigma_\theta^2 \end{pmatrix}) \), therefore, \( \theta_{s_1}(0) - \theta_{s_2}(0) \sim N(0, 2\sigma_\theta^2 - 2\sigma_\theta^2 e^{-\eta_2 ||s_1-s_2||^2}) \), which in turn implies that
\[
\frac{(\theta_{s_1}(0) - \theta_{s_2}(0))^2}{\nu} \sim \chi_1^2,
\]
where \( \nu = 2\sigma_\theta^2 - 2\sigma_\theta^2 e^{-\eta_2 ||s_1-s_2||^2} \). Using the fact that the moment generating function (mgf) of a \( \chi_1^2 \) random variable is \( (1 - 2t)^{-1/2} \), we have
\[
E \left[ e^{-\eta_3 (\theta_{s_1}(0) - \theta_{s_2}(0))^2} \right] = \left( 1 + 4\eta_3 \sigma_\theta^2 \left( 1 - e^{\eta_2 ||s_1-s_2||^2} \right) \right)^{-1/2}.
\]
Differentiating equation (16) with respect to \( \eta_3 \) and cancelling the minus sign from both sides yield

\[
E \left[ (\theta_s(0) - \theta_s(0))^2 e^{-\eta_3(\theta_s(0) - \theta_s(0))^2} \right] = 2\sigma_\theta^2 (1 - e^{\eta_3^2\|s_1 - s_2\|^2}) \left( 1 + 4\eta_3^2\sigma_\theta^2 (1 - e^{\eta_3^2\|s_1 - s_2\|^2}) \right)^{-3/2}.
\] (17)

Combining equations (15), (16), and (17), we get

\[
E \left[ V'(\theta_s(0))V'(\theta_s(0)) \right] = 2\sigma_\theta^2 \eta_3 \left[ \frac{1}{(1 + 4\eta_3^2\sigma_\theta^2 (1 - e^{\eta_3^2\|s_1 - s_2\|^2}))^{1/2}} - \frac{4\eta_3^2\sigma_\theta^2 (1 - e^{\eta_3^2\|s_1 - s_2\|^2})}{(1 + 4\eta_3^2\sigma_\theta^2 (1 - e^{\eta_3^2\|s_1 - s_2\|^2}))^{3/2}} \right].
\] (18)

Then inserting the value of \( E \left[ V'(\theta_s(0))V'(\theta_s(0)) \right] \) from equation (18) to equation (14) we obtain

\[
E(\theta_s(\delta t)\theta_s(\delta t)) = \beta^2\sigma_\theta^2 e^{-\eta_3\|s_1 - s_2\|^2} + \frac{(\delta t)^2\alpha^2}{M_s M_{s_2}}\sigma_\theta^2 e^{-\eta_3\|s_1 - s_2\|^2} + \frac{(\delta t)^4}{4M_s M_{s_2}} 2\sigma_\theta^2 \eta_3 \left[ \frac{1}{(1 + 4\eta_3^2\sigma_\theta^2 (1 - e^{\eta_3^2\|s_1 - s_2\|^2}))^{1/2}} - \frac{4\eta_3^2\sigma_\theta^2 (1 - e^{\eta_3^2\|s_1 - s_2\|^2})}{(1 + 4\eta_3^2\sigma_\theta^2 (1 - e^{\eta_3^2\|s_1 - s_2\|^2}))^{3/2}} \right].
\]

Clearly, the covariance function of \( \theta_s(\delta t) \) is four times differentiable, provided \( M_s \) is also four times differentiable. To apply Lemma 1.6 on \( V'(\theta_s(\delta t)) \) we have to show that the fourth moment of \( \frac{\partial}{\partial s_1} \theta_s(\delta t) \) finitely exists. From

\[
\theta_s(\delta t) = \beta \theta_s(0) + \frac{\delta t}{M_s} \left\{ \alpha p_s(0) - \frac{\delta t}{2} \frac{\partial}{\partial \theta_s} V(\theta_s(0)) \right\}
\]

we obtain

\[
\frac{\partial \theta_s(\delta t)}{\partial s_1} = \beta \frac{\partial \theta_s(0)}{\partial s_1} + \frac{\delta t}{M_s} \left\{ \frac{\partial p_s(0)}{\partial s_1} - \frac{\delta t}{2} \frac{\partial}{\partial \theta_s} V'(\theta_s(0)) \right\} - \frac{\delta t}{2} \frac{\partial M_s}{\partial \theta_s} \left\{ p_s(0) - \frac{1}{2} \delta t \frac{\partial}{\partial \theta_s} V(\theta_s(0)) \right\}
\]

\[
= \beta \frac{\partial \theta_s(0)}{\partial s_1} + \frac{\delta t}{M_s} \left\{ \frac{\partial p_s(0)}{\partial s_1} - \frac{\delta t}{2} V''(\theta_s(0)) \frac{\partial \theta_s(0)}{\partial s_1} \right\} - \frac{\delta t}{2} \frac{\partial M_s}{\partial \theta_s} \left\{ p_s(0) - \frac{1}{2} \delta t \frac{\partial}{\partial \theta_s} V(\theta_s(0)) \right\}.
\] (19)

Now the fourth moment of \( \frac{\partial \theta_s(\delta t)}{\partial s_1} \) will be finite if individually each term of equation (19) has finite fourth moment, because

\[
\left( \frac{\partial \theta_s(\delta t)}{\partial s_1} \right)^4 \leq \kappa \left[ \beta^4 \left( \frac{\partial \theta_s(0)}{\partial s_1} \right)^4 + \left( \frac{\delta t}{M_s} \right)^4 \left\{ \left( \frac{\partial p_s(0)}{\partial s_1} \right)^4 + \left( \frac{\delta t}{2} \right)^4 \left( V''(\theta_s(0)) \frac{\partial \theta_s(0)}{\partial s_1} \right)^4 \right\} + \left( \frac{\delta t}{M_s^2} \frac{\partial M_s}{\partial \theta_s} \right)^4 \left\{ (p_s(0))^4 + \left( \frac{1}{2} \delta t \right)^4 \left( \frac{\partial}{\partial \theta_s} V(\theta_s(0)) \right)^4 \right\} \right],
\]

where \( \kappa \) is a suitable constant. Note that due to assumptions A1-A2 and squared exponential covariance functions, \( \frac{\partial \theta_s(0)}{\partial s_1}, \frac{\partial p_s(0)}{\partial s_1}, \) and \( p_s(0) \) are Gaussian processes with 0 mean and constant variances. Hence they
have finite fourth moments. Therefore, we just need to show that $V^{\prime\prime}(\theta_s(0)) \frac{\partial \theta_s(0)}{\partial s_1}$ and $\frac{\partial}{\partial \theta_s} V(\theta_s(0))$ have finite 4th moments. Observe that

$$E \left[ V^{\prime\prime}(\theta_s(0)) \frac{\partial \theta_s(0)}{\partial s_1} \right]^4 = EE \left\{ \left[ V^{\prime\prime}(\theta_s(0)) \frac{\partial \theta_s(0)}{\partial s_1} \right]^4 \bigg| \theta_s(0) \right\}$$

$$= E \left\{ E \left\{ \left[ V^{\prime\prime}(\theta_s(0)) \right]^4 \bigg| \theta_s(0) \right\} \left( \frac{\partial \theta_s(0)}{\partial s_1} \right)^4 \right\}$$

(20)

Now since $V^{\prime\prime}(\theta_s(0))$, given $\theta_s(0)$, is Gaussian with mean 0 and constant variance, $E \left\{ \left[ V^{\prime\prime}(\theta_s(0)) \right]^4 \bigg| \theta_s(0) \right\}$ is constant (independent of $\theta_s(0)$), say $\kappa_1$. Hence from equation (20) we obtain

$$E \left[ V^{\prime\prime}(\theta_s(0)) \frac{\partial \theta_s(0)}{\partial s_1} \right]^4 = \kappa_1 E \left( \frac{\partial \theta_s(0)}{\partial s_1} \right)^4.$$  

(21)

Now note that $\frac{\partial \theta_s(0)}{\partial s_1}$ is also Gaussian with mean 0 and constant variance, so that $E \left( \frac{\partial \theta_s(0)}{\partial s_1} \right)^4$ is also constant. Thus, combining equations (20) and (21) we have

$$E \left[ V^{\prime\prime}(\theta_s(0)) \frac{\partial \theta_s(0)}{\partial s_1} \right]^4 < \infty.$$  

(22)

Next to show that $\frac{\partial}{\partial \theta_s} V(\theta_s(0))$ has finite 4th moment, we notice that

$$E(V^{\prime}(\theta_s(0)))^4 = E \left[ E(V^{\prime}(\theta_s(0)))^4 | \theta_s(0) \right] = \text{constant} = \kappa_2,$$ 

say.

The last equality follows because $V^{\prime}(\theta_s(0))$, given $\theta_s(0)$, is a Gaussian process with 0 mean and constant variance.

Therefore, the fourth moment of $\frac{\partial}{\partial s_1} \theta_s(\delta t)$ exists finitely. So, by Lemma 2.2, $V^{\prime}(\theta_s(\delta t))$ is mean square differentiable, and hence under assumptions A1, A2, for $h = 1$,

$$p_4(\delta t) = \alpha^2 p_4(0) - \frac{\delta t}{2} \left\{ \alpha \frac{\partial}{\partial \theta_s} V(\theta_s(0)) + \frac{\partial}{\partial \theta_s} V(\theta_s(\delta t)) \right\},$$

a linear combination of mean square differentiable function, is also mean square differentiable. Before, applying the steps of induction, we show that $\frac{\partial p_{4i}(\delta t)}{\partial \theta_s}$, $i = 1, 2$, have finite 4th moment as it has to be used
for the next step of induction. Note that, for $i = 1, 2$,

$$\frac{\partial p_s(\delta t)}{\partial s_i} = \alpha^2 \frac{\partial p_s(0)}{\partial s_i} - \frac{\delta t}{2} \left\{ \alpha V''(\theta_s(0)) \frac{\partial \theta_s(0)}{\partial s_i} + V''(\theta_s(\delta t)) \frac{\partial \theta_s(\delta t)}{\partial s_i} \right\}.$$  

From equation (22), $E \left( V''(\theta_s(0)) \frac{\partial \theta_s(0)}{\partial s_i} \right)^4 < \infty$. We have already shown that $\frac{\partial \theta_s(\delta t)}{\partial s_i}$ has finite fourth moment, for $i = 1, 2$. Thus, each term in the expression of $\frac{\partial p_s(\delta t)}{\partial s_i}$ has finite fourth moment. Hence $E \left( \frac{\partial p_s(\delta t)}{\partial s_i} \right)^4 < \infty$.

Thereafter, using the argument of induction as in the proof of Lemma 2.2, we have the desired result.

□

S-2 Calculation of joint conditional density of the observed data

Here we will find the data model, that is, the conditional distribution of Data given Latent, $y_0$, $x_0$ and the parameter $\theta$.

First, we will find the conditional distribution of $y_1$ given $y_0$, $x_0$ and the parameter $\theta$. We have, for $i = 1, 2, \ldots, n$,

$$y(s_i, 1) = \beta y(s_i, 0) + \frac{\alpha x(s_i, 0)}{M_{s_i}} - \frac{1}{2} \frac{V'(y(s_i, 0))}{M_{s_i}},$$

and

$$\frac{1}{2} \left[ \frac{V'(y(s_1, 0))}{M_{s_1}}, \ldots, \frac{V'(y(s_n, 0))}{M_{s_n}} \right]' \sim N_n \left( 0, \frac{\sigma^2}{4} \Sigma_0 \right),$$

where $\Sigma_0$ is the $n \times n$ covariance matrix with $\text{(i,j)}$th element $\frac{2\eta e^{-\eta h_{ij}(0)}(1-2\eta h_{ij}(0))}{M_{s_i}M_{s_j}}$. Therefore,

$$\begin{bmatrix} y_1 | y_0; x_0; \theta \end{bmatrix} \sim N_n \left( \mu_0, \frac{\sigma^2}{4} \Sigma_0 \right).$$

Similarly,

$$\begin{bmatrix} y_2 | y_1; x_1; y_0; x_0; \theta \end{bmatrix} \sim N_n \left( \mu_1, \frac{\sigma^2}{4} \Sigma_1 \right),$$

where the $(i, j)$th element of the $n \times n$ covariance matrix $\Sigma_1$ is $\frac{2\eta e^{-\eta h_{ij}^2(1)}(1-2\eta h_{ij}^2(1))}{M_{s_i}M_{s_j}}$. Following the same argument, we can write down the likelihood as

$$L = [\text{Data} | x_0; \ldots; x_{T-1}; y_0; \theta]$$
\[ \propto [y_1 | y_0; x_0; \theta] \cdots [y_T | y_{T-1}, \ldots, y_0; x_{T-1}, \ldots, x_0; \theta] \]

\[ \propto \frac{(\sigma^2)^{-nT/2}}{\prod_{t=1}^T |\Sigma_{t-1}|^{1/2}} e^{-\frac{2}{\sigma^2} \sum_{i=1}^T (y_i - \mu_{t-1})^T \Sigma_{t-1}^{-1} (y_i - \mu_{t-1})}, \]

where, for \( j = 1, 2, \ldots, T \), the \((k, \ell)\)th element of \( \Sigma_{j-1} \) is

\[ \frac{2\eta_3 e^{-\eta_3 h_{k\ell}^2 (j-1)} (1 - 2\eta_3 h_{k\ell}^2 (j-1))}{M_{k\ell} M_{k\ell}}. \]

### S-3 Calculation of the conditional joint density of latent data

Here we will derive the conditional distribution of the latent variables

\[ \text{Latent} = \{X(s_1, 1), X(s_2, 1), \ldots, X(s_n, 1); X(s_1, 2), X(s_2, 2), \ldots, X(s_n, 2); \ldots; X(s_1, T), X(s_2, T), \ldots, X(s_n, T)\} \]

given Data, \( y_0, x_0 \) and the parameter \( \theta \).

First, we shall find the conditional distribution of \( x_1 \) given \( x_0, y_0, \) and \( y_1 \). For \( i = 1, \ldots, n \), we have

\[ x(s_i, 1) = \alpha^2 x(s_i, 0) - \frac{1}{2} \{ \alpha V'(y(s_i, 0)) + V'(y(s_i, 1)) \}. \]

Since \( V'(\cdot) \) is a random Gaussian function with zero mean and covariance function given in equation [5] and \( \mathbb{W}_0 = (W'_0, W'_1)' \) is a 2\( n \times 1 \) vector, \( \mathbb{W}_0 \sim N_{2n}(0, \sigma^2 \Sigma) \), where \( \Sigma \) is the 2\( n \times 2n \) covariance matrix partitioned as

\[ \begin{pmatrix} \alpha^2 \Sigma_{00} & \alpha \Sigma_{01} \\ \alpha \Sigma_{10} & \Sigma_{11} \end{pmatrix}, \]

where the \((i, k)\)th element of \( \Sigma_{jj} \) is \( 2\eta_3 e^{-\eta_3 h_{ik}^2 (j)} (1 - 2\eta_3 h_{ik}^2 (j)) \), for \( j = 0, 1 \), and the \((i, k)\)th element of \( \Sigma_{01} = \Sigma_{10} \) is \( 2\eta_3 e^{-\eta_3 h_{ik}^2 (0, 1)} (1 - 2\eta_3 h_{ik}^2 (0, 1)) \). Therefore,

\[ W_0 + W_1 = [I_n; I_n] \mathbb{W}_0 \sim N_n(0, \sigma^2 (\alpha^2 \Sigma_{00} + \alpha \Sigma_{01} + \alpha \Sigma_{10} + \Sigma_{11})). \]

Let \( \Omega_1 = \alpha^2 \Sigma_{00} + \alpha \Sigma_{01} + \alpha \Sigma_{10} + \Sigma_{11} \). Then

\[ [x_1 | x_0; y_0; y_1; \theta] \sim N_n(\alpha^2 x_0, \sigma^2 \frac{\Omega_1}{4}). \]
Similarly,

$$[x_2|x_1; x_0; y_0; y_1; y_2; \theta] \sim N_n(\alpha^2 x_1, \frac{\sigma^2}{4} \Omega_2),$$

(26)

where \( \Omega_2 = \alpha^2 \Sigma_{11} + \alpha \Sigma_{12} + \alpha \Sigma_{21} + \Sigma_{22} \). For \( j = 1, 2 \), the \((i, k)\)th element of \( \Sigma_{jj} \) is \( 2\eta_\beta e^{-\eta h_{ik}^2(j)} (1 - 2\eta_\beta h_{ik}^2(j)) \), and the \((i, k)\)th element of \( \Sigma_{12} = \Sigma_{21} \) is \( 2\eta_\beta e^{-m_{ik}^2(1, 2)} (1 - 2\eta_\beta \ell_{ik}^2(1, 2)) \). Now with the help of equations (25) and (26) we write down the conditional latent process model as

$$\begin{bmatrix} \text{Latent} | y_0; \ldots; y_T; x_0; \theta \\ \end{bmatrix}$$

$$\propto [x_1|x_0; y_0; y_1; \theta] \ldots [x_T|x_{T-1}; \ldots; x_0; y_0; \ldots; y_T; \theta]$$

$$\propto \left(\frac{\alpha^2}{\sigma^4}\right)^{-nT/2} e^{-\frac{1}{2} \sum_{t=1}^{T} (x_t-\alpha^2 x_{t-1})' \Omega_t^{-1} (x_t-\alpha^2 x_{t-1})}$$

$$\prod_{t=1}^{T} (\Omega_t)^{1/2},$$

where, for \( m \in \{1, 2, \ldots, T\} \), \( \Omega_t = \alpha^2 \Sigma_{t-1, t-1} + \alpha \Sigma_{t-1, t} + \alpha \Sigma_{t, t-1} + \Sigma_{t, t} \), where the \((i, k)\)th element of \( \Sigma_{jj} \), for \( j = t-1, t \), is \( 2\eta_\beta e^{-\eta h_{ik}^2(j)} (1 - 2\eta_\beta h_{ik}^2(j)) \), and the \((i, k)\)th element of \( \Sigma_{t-1, t} = \Sigma_{t, t-1} \) is \( 2\eta_\beta e^{-m_{ik}^2(t-1, t)} (1 - 2\eta_\beta \ell_{ik}^2(t-1, t)) \).

S-4 Calculation of full conditional distributions of the parameters and the latent variables, given the observed data

S-4.1 Full conditional distribution of \( \beta^* \)

Before finding the full conditional distribution of \( \beta \) (hence \( \beta^* \)), we note that the only term that depends on \( \beta \) in equation (9) is \( \exp \left\{ -\frac{2}{\sigma^2} \sum_{t=1}^{T} [\mu_{t-1}^{-1} \Sigma_{t-1}^{-1} \mu_{t-1} - 2y_t T \Sigma_{t-1}^{-1} \mu_{t-1}] \right\} \). Further notice that \( \mu_t = \beta y_{t-1} + \text{constant} \) (with respect to \( \beta \)). Therefore, the term which depends on \( \beta \) (hence on \( \beta^* \)) simplifies to

$$e^{-\frac{2\beta^2}{\sigma^2} \sum_{t=1}^{T} y_{t-1} \Sigma_{t-1}^{-1} y_{t-1} + \frac{4\beta^2}{\sigma^2} \sum_{t=1}^{T} y_{t} \Sigma_{t-1}^{-1} y_{t-1}},$$

where \( \beta = -1 + \frac{2\beta^*}{1 + 2\sigma^2} \). The full conditional density of \( \beta^* \), therefore, is given by

$$[\beta^*] \propto [\beta^*] e^{-\frac{2\beta^2}{\sigma^2} \sum_{t=1}^{T} y_{t-1} \Sigma_{t-1}^{-1} y_{t-1} + \frac{4\beta^2}{\sigma^2} \sum_{t=1}^{T} y_{t} \Sigma_{t-1}^{-1} y_{t-1}}$$

$$\propto e^{-\frac{2\beta^2}{\sigma^2} \sum_{t=1}^{T} y_{t-1} \Sigma_{t-1}^{-1} y_{t-1} + \frac{4\beta^2}{\sigma^2} \sum_{t=1}^{T} y_{t} \Sigma_{t-1}^{-1} y_{t-1}} e^{\frac{4\beta^2}{\sigma^2} \sum_{t=1}^{T} y_{t} \Sigma_{t-1}^{-1} y_{t-1}}$$

$$= \pi(\beta^*) g_1(\beta^*),$$

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where \(\pi(\beta^\ast) = e^{-\frac{\beta^\ast}{2\sigma^\ast^2}}\) and \(g_1(\beta^\ast) = e^{-\frac{2\beta^\ast}{2\sigma^2} \sum_{t=1}^{T} y_t' \Sigma_t^{-1} y_{t-1} + \frac{4\beta^\ast}{\sigma^2} \sum_{t=1}^{T} y_t' \Sigma_t^{-1} y_{t-1}}\), as mentioned in equation (10).

### S-4.2 Full conditional distribution of \(\alpha^\ast\)

The full conditional density of \(\alpha^\ast\) will be given by \([\alpha^\ast] \times [\text{Data,Latent}|\theta]\). Now the term that depends on \(\alpha\) (hence on \(\alpha^\ast\)) in \([\text{Data,Latent}|\theta]\), that is, in equation (9), is given by

\[
g_2(\alpha^\ast) = \frac{1}{\prod_{t=1}^{T} |\Omega_t|^{1/2}} e^{-\frac{2\alpha}{\sigma^2} \sum_{t=1}^{T} (\alpha_0^\ast y_0^\ast y_0^\ast)^T (\alpha_0^\ast y_0^\ast y_0^\ast - 1) + \frac{2\alpha}{\sigma^2} \sum_{t=1}^{T} \alpha_0^\ast y_0^\ast y_0^\ast \Sigma_t^{-1} y_{t-1}^\ast \Sigma_t^{-1} y_{t-1}^\ast} \times e^{-\frac{2\alpha}{\sigma^2} \sum_{t=1}^{T} \alpha_0^\ast y_0^\ast y_0^\ast \Sigma_t^{-1} D \Sigma_t^{-1} D x_t \Sigma_t^{-1} D x_t \Sigma_t^{-1}}
\]

where \(D\) is the \(n \times n\) diagonal matrix containing the diagonal elements \(\frac{1}{M_{i\ast}}, i = 1, \ldots, n\). In the above calculation we use \(\mu_t = \beta y_t + \alpha D x_t\). Thus the full conditional density of \(\alpha^\ast\) is given by

\[
[\alpha^\ast | \ldots] \propto e^{-\frac{\alpha^\ast}{2\sigma^\ast^2}} g_2(\alpha^\ast).
\]

### S-4.3 Full conditional distribution of \(\sigma_\theta^2\)

The only term that depends on \(\sigma_\theta^2\) in equation (9) is \((\frac{1}{\sigma_\theta^2})^{n/2} \exp \left\{-\frac{1}{2\sigma_\theta^2} y_0^\ast \Delta_0^{-1} y_0\right\}\), and therefore, the full conditional distribution of \(\sigma_\theta^2\) is

\[
[\sigma_\theta^2 | \ldots] \propto [\sigma_\theta^2] \left(\frac{1}{\sigma_\theta^2}\right)^{n/2} \exp \left\{-\frac{1}{2\sigma_\theta^2} y_0^\ast \Delta_0^{-1} y_0\right\} \\
\times \left(\frac{1}{\sigma_\theta^2}\right)^{\alpha_\theta + n/2 + 1} \exp \left\{-\frac{1}{\sigma_\theta^2} \left(\frac{\gamma_\theta + y_0^\ast \Delta_0^{-1} y_0}{2}\right)\right\}.
\]

That is, the full conditional distribution of \(\sigma_\theta^2\) is \(\text{IG}\left(\alpha_\theta + n/2, \frac{\gamma_\theta + y_0^\ast \Delta_0^{-1} y_0}{2}\right)\).

### S-4.4 Full conditional distribution of \(\sigma_p^2\)

Since the only term that depends on \(\sigma_p^2\) in equation (9) is \((\frac{1}{\sigma_p^2})^{n/2} \exp \left\{-\frac{1}{2\sigma_p^2} x_0^\ast \Omega_0^{-1} x_0\right\}\) and we have assumed inverse gamma with parameters \(\alpha_p\) and \(\gamma_p\),

\[
[\sigma_p^2 | \ldots] \propto \left(\frac{1}{\sigma_p^2}\right)^{\alpha_p + n/2} \left(\frac{1}{\sigma_p^2}\right)^{n/2} \exp \left\{-\frac{1}{\sigma_p^2} \left(\frac{\gamma_p + x_0^\ast \Omega_0^{-1} x_0}{2}\right)\right\}.
\]

This implies that the full conditional distribution of \(\sigma_p^2\) is \(\text{IG}\left(\alpha_p + n/2, \frac{\gamma_p + x_0^\ast \Omega_0^{-1} x_0}{2}\right)\).
S-4.5 Full conditional distribution of $\sigma^2$

Note that $[\text{Data|Latent}, x_0, y_0, \theta]$ and $[\text{Latent|Data}, x_0, y_0, \theta]$ depend on $\sigma^2$. Therefore, the full conditional distribution of $\sigma^2$ can be achieved as follows:

$$[\sigma^2|\ldots] \propto [\sigma^2] \prod_{t=1}^{T} [y_t|y_{t-1}, x_{t-1}, \theta][x_t|y_t, y_{t-1}, x_{t-1}, \theta]$$

$$\propto [\sigma^2](\sigma^2)^{-Tn} \exp \left\{ -\frac{2}{\sigma^2} \sum_{t=1}^{T} \left[ (y_t - \mu_t)^T \Sigma_{t-1}^{-1} (y_t - \mu_t) + (x_t - \alpha^2 x_{t-1})^T \Omega_t^{-1} (x_t - \alpha^2 x_{t-1}) \right] \right\}$$

$$\propto \left( \frac{1}{\sigma^2} \right)^{\alpha_v + Tn + 1} \exp \left\{ -\frac{1}{\sigma^2} \left[ \frac{\gamma_v}{2} + 2\zeta \right] \right\},$$

where $\zeta = \sum_{t=1}^{T} \left[ (y_t - \mu_t)^T \Sigma_{t-1}^{-1} (y_t - \mu_t) + (x_t - \alpha^2 x_{t-1})^T \Omega_t^{-1} (x_t - \alpha^2 x_{t-1}) \right]$. Hence the full conditional distribution of $\sigma^2$ is inverse-Gamma with parameters $\alpha_v + Tm$ and $\gamma_v/2 + 2\zeta$.

S-4.6 Full conditional distributions of $\eta_1^*$, $\eta_2^*$, and $\eta_3^*$

We observe that only $[x_0|\theta]$ depends $\eta_1$ (hence on $\eta_1^*$) and $[Y_0|\theta]$ depends on $\eta_2$ (hence on $\eta_2^*$). Therefore, the full conditional densities of $\eta_1^*$ and $\eta_2^*$ are given by

$$[\eta_1^*|\ldots] \propto [\eta_1^*][x_0|\theta]$$

$$\propto e^{-\eta_1^2/2} \frac{1}{|\Omega_0|^{1/2}} e^{-\frac{1}{2\sigma_p} x_0^T \Omega_0^{-1} x_0}$$

$$= \pi(\eta_1^*) g_3(\eta_1^*)$$

(27)

and

$$[\eta_2^*|\ldots] \propto [\eta_2^*][y_0|\theta]$$

$$\propto e^{-\eta_2^2/2} \frac{1}{|\Delta_0|^{1/2}} e^{-\frac{1}{2\sigma_y} y_0^T \Delta_0^{-1} y_0}$$

$$= \pi(\eta_2^*) g_4(\eta_1^*),$$

(28)

respectively, where $\eta_1 = e^{\eta_1^*}$, $\eta_2 = e^{\eta_2^*}$, $\pi(\eta_1^*) = e^{-\eta_1^2/2}$, $\pi(\eta_2^*) = e^{-\eta_2^2/2}$, $g_3(\eta_1^*) = \frac{1}{|\Omega_0|^{1/2}} e^{-\frac{1}{2\sigma_p} x_0^T \Omega_0^{-1} x_0}$ and $g_4(\eta_2^*) = \frac{1}{|\Delta_0|^{1/2}} e^{-\frac{1}{2\sigma_y} y_0^T \Delta_0^{-1} y_0}$.

On the other hand, in the joint conditional distribution of (Data, Latent) given $x_0$, $y_0$, $\theta$ depends on
\( \eta_3 \) (hence on \( \eta_3^* \)). Thus, the full conditional distribution of \( \eta_3^* \) is given by

\[
[\eta_3^*] \propto [\eta_3^*] \prod_{l=1}^T [y_l|y_{l-1}, x_{l-1}, \theta][x_l|y_{l}, y_{l-1}, x_{l-1}, \theta] \\
\propto e^{-\eta_3^2/2} \prod_{l=1}^T \frac{1}{|\Sigma_l|^{1/2}} e^{-\frac{1}{2\sigma^2} \sum_{l=1}^T [(y_l-\mu_l)^T \Sigma_l^{-1} (y_l-\mu_l) + (x_l-\alpha^2 x_{l-1})^T \Omega_l^{-1} (x_l-\alpha^2 x_{l-1})]} \\
= \pi(\eta_3^*) g_5(\eta_3^*), \tag{29}
\]

where \( \eta_3 = e^{\eta_3^*} \), \( \pi(\eta_3^*) = e^{-\eta_3^2/2} \) and

\[
g_5(\eta_3^*) = \frac{1}{\prod_{l=1}^T |\Sigma_l|^{1/2} |\Omega_l|^{1/2}} e^{-\frac{1}{2\sigma^2} \sum_{l=1}^T [(y_l-\mu_l)^T \Sigma_l^{-1} (y_l-\mu_l) + (x_l-\alpha^2 x_{l-1})^T \Omega_l^{-1} (x_l-\alpha^2 x_{l-1})]}.
\]

**S-4.7  Full conditional distribution of \( x_0 \)**

Using the fact that only \( x_1 \) and \( y_1 \) depend on \( x_0 \) and writing \( \mu_0 = \beta y_0 + \alpha D x_0 \), we have

\[
[x_0|\ldots] \propto [x_0|\theta][x_1|y_1, y_0, x_0, \theta][y_1|y_0, x_0, \theta] \\
\propto e^{-\frac{1}{2\sigma^2} \Sigma_0^{-1} x_0} e^{-\frac{1}{2\sigma^2} (x_1-\alpha^2 x_0)^T \Sigma_1^{-1} (x_1-\alpha^2 x_0)} e^{-\frac{1}{2\sigma^2} (y_1-\beta y_0-\alpha D x_0)^T \Sigma_0^{-1} (y_1-\beta y_0-\alpha D x_0)} \\
\propto e^{-\frac{1}{2\sigma^2} \left[ x_0^T \Omega_0^{-1} x_0 + \frac{4\sigma^2}{\sigma^2 \alpha^2} x_0^T \Omega_0^{-1} x_0 + \frac{4\sigma^2}{\sigma^2} x_0^T D \Sigma_0^{-1} D x_0 - \frac{8\sigma^2}{\sigma^2 \alpha^2} x_0^T \Omega_0^{-1} x_0 - \frac{8\sigma^2}{\sigma^2} x_0^T D \Sigma_0^{-1} (y_1-\beta y_0) \right]} \\
\propto e^{-\frac{1}{2\sigma^2} \left[ x_0^T A x_0 - 2 x_0^T B x_1 - 2 x_0^T y_c (1-\beta y_0) \right]} \\
\propto e^{-\frac{1}{2\sigma^2} \left[ x_0^T A (x_0-A^{-1} B x_1 - A^{-1} C (y_1-\beta y_0)) (x_0-A^{-1} B x_1 - A^{-1} C (y_1-\beta y_0)) \right]} \tag{30},
\]

where \( A = \Omega_0^{-1} + \frac{4\sigma^2}{\sigma^2 \alpha^2} \Omega_0^{-1} + \frac{4\sigma^2}{\sigma^2} D \Sigma_0^{-1} D, B = \frac{4\sigma^2}{\sigma^2 \alpha^2} \Omega_0^{-1}, C = \frac{4\sigma^2}{\sigma^2} D \Sigma_0^{-1} \). We note here that \( D \) (= \( \text{diag}(1/M_i^n) \)) is a positive definite matrix as all the diagonal entries are strictly positive, \( A \) being a sum of three positive definite matrices is also positive definite and hence invertible. Thus, from equation (30), we get \( [x_0|\ldots] \sim N_n(A^{-1}(B x_1 + C (y_1-\beta y_0)), \sigma^2 p A^{-1}) \).
S-5 Simulation details and results for data simulated from LDSTM

S-5.1 Data generation

Now we apply our model and methodology on a data set which is simulated from linear dynamic spatio-temporal model (LDSTM). The LDSTM is described below:

\[ Y(s_i, t) = X(s_i, t) + \epsilon(s_i, t), \]
\[ X(s_i, t) = \rho X(s_i, t - 1) + \eta(s_i, t), \]
\[ \{X(s_i, 0)\}_{i=1}^{n} \sim N(0, \Sigma_0), \]

where the random errors \(\{\epsilon(s_i, t)\}\) and \(\{\eta(s_i, t)\}\) are independently and identically distributed with respect to time as \(N(0, \Sigma_\epsilon)\) and \(N(0, \Sigma_\eta)\). The associated variance-covariance matrices are constructed using exponential covariance functions with the form \(c(u, v) = \sigma^2 \exp\{-\lambda ||u - v||\}\), where \(||\cdot||\) denotes the Euclidean norm. For the purpose of simulation, we took \(\rho = 0.8\), \((\sigma_\epsilon, \lambda_\epsilon) = (1, 0.25)\) and \((\sigma_\eta, \lambda_\eta) = (1, 1) = (\sigma_0, \lambda_0)\) for \(\{\epsilon(s_i, t)\}\), \(\{\eta(s_i, t)\}\) and \(\{X_0(s_i, 0)\}\), respectively.

Let \(x_t = (x(s_1, t), \ldots, x(s_m, t))^T\) and \(y_t = (y(s_1, t), \ldots, y(s_m, t))^T\) denote the latent vector and observed vector at time \(t\). We generated \((x_t, y_t)\), for \(t = 1, \ldots, 20\), as in the previous simulation study and assumed that only \(y_t\), \(t = 1, \ldots, 19\) are known to us. Hence the data set is given by \(D = \{y_1, \ldots, y_{19}\}\).

S-5.2 Choice of hyper parameters

As before, based on cross-validation, the priors are specified, completely, as follows:

\[ \alpha^* \sim N(0, \sqrt{500}), \beta^* \sim N(0, \sqrt{300}) \]
\[ [\sigma^2] \propto IG(75000, 2/2), [\sigma_\epsilon^2] \propto IG(900, 780/2), [\sigma_\eta^2] \propto IG(250, 100/2) \]
\[ [\eta_1^*] \propto N(3, 1), [\eta_2^*] \propto N(-5, 1). \]

Here we fixed \(\eta\) at the value 18.5067, the maximum likelihood estimate computed by simulated annealing.
S-5.3 Posterior analysis

The posterior densities of other parameters involved in the model, and that of $x_t$, for $t = 1, \ldots, 19$ given $D$ are obtained using the method described in Section 5. Further, for checking the predictive performance, we have calculated the posterior predictive densities of $y_{20}$ and the latent future variable $x_{20}$, given $D$.

The MCMC trace plots of all the parameters except $\eta_3$ are provided in the Figure S-1, all of which indicate good convergence properties of our MCMC algorithm. Figures S-2 and S-3 display the posterior densities of the latent variables at 50 locations. Observe that the true values lie in high posterior density regions for each locations. Finally, the posterior predictive densities for $y_{20}$ and $x_{20}$ are shown in Figures S-4 and S-5, of which Figure S-4a and Figure S-4a depict the plots for the first and the last 25 locations, respectively, for $y_{20}$. The posterior predictive density plots for the first 25 locations of the latent variable $x_{20}$ are given in Figure S-5a, whereas Figure S-5b provides the predictive density plots for the last 25 locations of the latent variable $x_{20}$. Evidently, all the true (future) values for $y$ (except two, marginally) and $x$ fall well within the 95% posterior predictive interval.
Figure S-2: Posterior predictive densities of the latent variables for the first 25 locations for LDSTM. Higher the intensity of the colour, higher is the probability density. The black line represents the true values of latent variables. $L_i, i = 1, \ldots, 25$ denote the locations.
Figure S-3: Predictive densities of the latent variables for the last 25 locations for LDSTM. Higher the intensity of the colour, higher is the probability density. The black line represents the true values of latent variables. \( L_i, i = 26, \ldots, 50 \) denote the locations.
Figure S-4: Predictive densities of the $y_{20}$ for 50 locations for LDSTM. The red horizontal lines denote the 95% predictive interval. The vertical black lines indicate the true values. All the true values, except two (marginally outside), lie within the 95% predictive interval.

Figure S-5: Predictive densities of the $x_{20}$ for 50 locations for LDSTM. The red horizontal lines denote the 95% predictive interval. The vertical black lines indicate the true values. All the true values lie within the 95% predictive interval.
Trace plot of the parameters and the posterior densities of the complete time series of the latent variables for simulated data NLDSTM.

Figure S-6: Trace plots for the parameters for data simulated from NLDSTM.
Figure S-7: Predictive densities of latent variables for the complete times series at first 25 locations for data simulated from NLDSTM. Higher the intensity of the colour, higher is the probability density. The black line represents the true values of the latent variables. $L_i, i = 1, \ldots, 25$, denote the locations.
Figure S-8: Predictive densities of latent variables for the complete time series at the last 25 locations for daat simulated from NLDSTM. Higher the intensity of the colour, higher is the probability density. The black line represents the true values of the latent variables. $L_i, i = 26, \ldots, 50$, denote the locations.
Trace plot of the parameters and the posterior densities of the complete time series of the latent variables for simulated data GQN.

Figure S-9: Trace plots for the parameters except for $\eta_3$ for GQN.
Figure S-10: Posterior densities of latent variables for the first 25 locations for data simulated from GQN. Higher the intensity of the colour, higher is the probability density. The black line represents the true values of the latent variables. $L_i, i = 1, \ldots, 25$, denote the locations.
Figure S-11: Posterior densities of latent variables for the last 25 locations for data simulated from GQN. Higher the intensity of the colour, higher is the probability density. The black line represents the true values of the latent variables. $L_i$, $i = 26, \ldots, 50$, denote the locations.
Trace plot of the parameters and the posterior densities of the complete time series of the latent variables for Alaska temperature data.

Figure S-12: Trace plot for the parameters except for $\eta_3$ corresponding to the Alaska temperature data.
Figure S-13: Posterior densities of the latent variables at 26 locations in Alaska and its surroundings corresponding to the annual detrended temperature data. Higher the intensity of the colour, higher is the probability density.
Trace plot of the parameters and the posterior densities of the complete time series of the latent variables for Sea temperature data.

Figure S-14: Trace plots of the parameters except for $\eta_3$ for the sea surface temperature data.
Figure S-15: Posterior densities of the latent variables at 30 locations of the sea surface temperature data. Higher the intensity of the colour, higher is the probability density.
S-10 Stationarity, convergence of lagged correlations to zero and non-Gaussianity of the detrended Alaska data process

S-10.1 Stationarity of the detrended Alaska data process

To check if the data arrived from a stationary or nonstationary process, we resorted to the recursive Bayesian theory and methods developed by [23]. In a nutshell, their key idea is to consider the Kolmogorov-Smirnov distance between distributions of data associated with local and global space-times. Associated with the $j$-th local space-time region is an unknown probability $p_j$ of the event that the underlying process is stationarity when the observed data corresponds to the $j$-th local region and the Kolmogorov-Smirnov distance falls below $c_j$, where $c_j$ is any non-negative sequence tending to zero as $j$ tends to infinity. With suitable priors for $p_j$, [23] constructed recursive posterior distributions for $p_j$ and proved that the underlying process is stationary if and only if for sufficiently large number of observations in the $j$-th region, the posterior of $p_j$ converges to one as $j \to \infty$. Nonstationarity is the case if and only if the posterior of $p_j$ converges to zero as $j \to \infty$.

In our implementation of the ideas of [23], we set the $j$-th local region to be the entire time series for the spatial location $s_j$, for $j = 1, \ldots, 29$. Thus, the size of each local region is 65. We choose $c_j$ to be of the same nonparametric, dynamic and adaptive form as detailed in [23]. The dynamic form requires an initial value for the sequence. In practice, the choice of the initial value usually has significant effect on the convergence of the posteriors of $p_j$, and so, the choice must be carefully made. However, in our case, for all initial values that we experimented with, lying between 0.05 and 1, the recursive Bayesian procedure led to the conclusion of stationarity of the underlying spatio-temporal process.

We implemented the idea with our parallelised C code on 29 parallel processors of a VMWare of Indian Statistical Institute; the time taken is less than a second. For the initial value 0.05, Figure S-16 displays the means of the posteriors of $p_j; j = 1, \ldots, 29$, showing convergence to 1. The respective posterior variances are negligibly small and hence not shown. Thus, the detrended spatio-temporal process that generated the Alaska data, can be safely regarded as stationary.
Recall that one major purpose of our Hamiltonian spatio-temporal model is to emulate the property of most real datasets that the lagged spatio-temporal correlations tend to zero as the spatio-temporal lag tends to infinity, irrespective of stationarity or nonstationarity. Here we compute the lagged correlations on 30 parallel processors on our VMWare, each processor computing the correlation for a partitioned interval of lag $\|h\|$ such that the interval is associated with sufficient data making the correlation well-defined. The time taken for this exercise are a few seconds. Figure S-17 demonstrates convergence of the lagged spatio-temporal correlations to zero; with larger amount of data such demonstration would have been more convincing.
S-10.3  Non-Gaussianity of the Alaska data

Simple quantile-quantile plots (not shown for brevity) revealed that the distributions of the time series data at the spatial locations, distributions of the spatial data at the time points, and the overall distribution of the entire dataset, are far from normal. Thus, traditional Gaussian process based models of the underlying spatio-temporal process are ruled out. Since the temporal distributions at the spatial locations and the spatial distributions at different time points are also much different, it does not appear feasible to consider parametric stochastic process models for the data. These seem to make the importance of our nonparametric Hamiltonian process more pronounced.

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