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The standard $\mathcal{PT}$-symmetric dimer is a linearly-coupled two-site discrete nonlinear Schrödinger equation with one site losing and the other one gaining energy at the same rate. We show that despite gain and loss, the standard $\mathcal{PT}$-dimer is a Hamiltonian system. We also produce a Lagrangian formulation for the dimer.

I. INTRODUCTION

The Schrödinger dimer is a two-site discrete nonlinear Schrödinger equation of the form

\[ i \frac{du}{dz} + v + F(u, u^*, v, v^*) = 0, \]
\[ i \frac{dv}{dz} + u + G(u, u^*, v, v^*) = 0. \]

The physically interesting situation pertains to the functions $F$ and $G$ being covariant under the simultaneous phase shifts in $u$ and $v$; that is,

\[ F(e^{i\alpha} u, e^{-i\alpha} u^*, e^{i\alpha} v, e^{-i\alpha} v^*) = e^{i\alpha} F(u, u^*, v, v^*), \]
\[ G(e^{i\alpha} u, e^{-i\alpha} u^*, e^{i\alpha} v, e^{-i\alpha} v^*) = e^{i\alpha} G(u, u^*, v, v^*) \]

for any real $\alpha$.

The Schrödinger dimers with various polynomial nonlinearities are workhorses of photonics, where they serve to model stationary light beams in coupled optical waveguides. The variables $u$ and $v$ represent the corresponding complex beam amplitudes, and $z$ measures the distance along the parallel cores.

Dimers also occur in the studies of the Bose-Einstein condensate trapped in a double-well potential. Here, $u$ and $v$ are the complex amplitudes of the mean-field condensate wave functions localized in the left and right well, respectively, or the amplitudes of the ground and the first excited state. The nonlinear Schrödinger dimers were employed, extensively, in the context of electric lattices.

With the advent of the parity-time ($\mathcal{PT}$) symmetry, the studies of the optical couplers expanded to include structures consisting of a waveguide with loss and a guide with an equal amount of optical gain. While a variety of cubic nonlinearities were considered, the most frequently used nonlinear model has the form

\[ i \frac{du}{dz} + v + |u|^2 u = i\gamma u, \]
\[ i \frac{dv}{dz} + u + |v|^2 v = -i\gamma v. \]  

This discrete nonlinear Schrödinger equation is occasionally referred to as the standard $\mathcal{PT}$-symmetric dimer.
The aim of this note is to establish the Hamiltonian formulation for the standard dimer. We also provide the Lagrangian framework for this system.

II. THE HAMILTONIAN FORMALISM

The canonical formulation for the standard lossless dimer is well known (see e.g. \[5, 21, 22\]). The system \[1\] with \(\gamma = 0\) can be written as

\[
\begin{aligned}
\frac{du}{dz} &= \frac{\partial \mathcal{H}}{\partial u^*}, \\
\frac{dv}{dz} &= -\frac{\partial \mathcal{H}}{\partial v^*},
\end{aligned}
\]

where the Hamilton function is

\[
\mathcal{H} = -(uv^* + u^*v) - \frac{|u|^4 + |v|^4}{2}.
\]

The two pairs of canonical variables are \(u\) and \(u^*\), \(v\) and \(v^*\). We emphasise the difference between Eq. \[6\] and the Hamiltonian structure of the nonlinearly coupled dimer, Eq. \[3\], where the canonical pairs were \(u\) and \(v\), \(v\) and \(v^*\).

Unlike Eq. \[3\], the structure \[6\] does not survive the addition of the gain-loss terms — that is, \(u\) and \(u^*\), \(v\) and \(v^*\) are not the canonical variables for the system \[1\] with \(\gamma \neq 0\). That being the case, we attempt to transform both dimers to a new set of coordinates where the similarity of the two systems would allow to model the Hamiltonian structure of the standard \(\mathcal{P}\mathcal{T}\)-symmetric dimer on the canonical formulation of the system \[2\].

A natural choice of coordinates is furnished by the Stokes variables,

\[
X = \frac{u^*v + uv^*}{2}, \quad Y = i \frac{u^*v - uv^*}{2}, \quad Z = \frac{|u|^2 - |v|^2}{2}.
\]

We also introduce the notation

\[
\phi = \phi_1 + \phi_2,
\]

where \(\phi_1\) and \(\phi_2\) are the arguments of the complex variables \(u\) and \(v\):

\[
u = |v|e^{i\phi_2}.
\]

In terms of \(X, Y, Z\) and \(\phi\), the standard dimer \[1\] acquires the form

\[
\dot{\phi} = \left(\frac{r + X}{r}\right) \cosh \psi, \quad \dot{X} = -YZ, \quad \dot{Y} = (X - 1)Z, \quad \dot{Z} = \gamma R + Y.
\]

In Eq. \[9a\], \(r\) is the magnitude of the two-component vector \((X, Y)\):

\[
r = \sqrt{X^2 + Y^2},
\]

whereas in \[9b\], \(R\) denotes the length of the vector \((X, Y, Z)\):

\[
R = \sqrt{X^2 + Y^2 + Z^2}.
\]

The hyperbolic angle \(\psi\) is defined by

\[
R = r \cosh \psi, \quad Z = r \sinh \psi,
\]

and the overdot stands for the derivative with respect to \(t = 2z\).

Using the same set of variables, the anharmonically coupled dimer \[2\] becomes

\[
\begin{aligned}
\dot{\phi} &= \left[2r + (1 + 2X)\frac{X}{r}\right] \cosh \psi, \\
\dot{X} &= 0, \quad \dot{Y} = -(1 + 2X)Z, \quad \dot{Z} = \gamma R + (1 + 2X)Y.
\end{aligned}
\]

Consider, first, the system \[2\] which admits the Lagrangian and Hamiltonian formulation. The Lagrange function for the dimer \[2\] is

\[
\mathcal{L} = \frac{i}{4} (u_z^*v - u_z^*v + v_z^*v - v_z^*u) + (1 + uv^* + u^*v)(|u|^2 + |v|^2) + \frac{\gamma}{2} (u^*v - uv^*).
\]

The corresponding Lagrangian for the system in the form \[12\] results by transforming to \(X, Y, Z, \text{and} \phi\). Dropping a total derivative, we have

\[
\mathcal{L} = \dot{X}\phi + \dot{Y}\psi + (1 + 2X)R + \gamma Y.
\]

We choose \(X\) and \(Y\) as the coordinates of the fictitious classical particle described by the Lagrangian \[13\]. The canonical momenta are then

\[
\begin{aligned}
P_X = \frac{\partial \mathcal{L}}{\partial \dot{X}} = \phi, \quad P_Y = \frac{\partial \mathcal{L}}{\partial \dot{Y}} = \psi,
\end{aligned}
\]

and the Hamiltonian of the particle results by the Legendre transform:

\[
\mathcal{H} = -(1 + 2X)R - \gamma Y,
\]

where

\[
R = \sqrt{X^2 + Y^2} \cosh \psi.
\]

The Hamilton equations

\[
\begin{aligned}
\dot{X} &= \frac{\partial \mathcal{H}}{\partial P_X}, \quad \dot{P}_X = -\frac{\partial \mathcal{H}}{\partial X}; \\
\dot{Y} &= \frac{\partial \mathcal{H}}{\partial P_Y}, \quad \dot{P}_Y = -\frac{\partial \mathcal{H}}{\partial Y}
\end{aligned}
\]

reproduce equations \[12\].

The Hamiltonian \[14\] is reminiscent of the expression for an integral of motion of the standard dimer, \(I = -\mathcal{R} - \gamma \theta\), with the role of \(\theta\) being taken over by \(Y\). Here \(\theta\) is one of the two polar coordinates on the \((X, Y)\)-plane defined by

\[
X = 1 + \rho \sin \theta, \quad Y = \rho \cos \theta.
\]
we define

$$P_θ = ψ$$

and propose the following expression for the Hamiltonian of the standard dimer:

$$H = -√ρ^2 + 1 + 2ρ sin θ. = 0, \quad (18a)$$

$$P_θ = -H = ρ + cos ρ ψ, \quad (18b)$$

$$P_ρ = -H = ρ + sin ρ ψ. \quad (18c)$$

The formulas (18a)-(18b) are equivalent to equations (18a, b) while Eq. (18c) can be used to define the momentum $P_θ$. Namely, comparing (18c) to (9a) and using $d\dot{r}$ yields

$$P_ρ = -\frac{\dot{φ}}{ρ} + 2\int_0^t \mathcal{R}(τ)dτ. \quad (19)$$

III. THE LAGRANGIAN FORMALISM

The aim of this section is to propose a Lagrangian formulation for the standard $PT$-symmetric dimer. The Lagrangian formulation complements the Hamilton equations and offers a number of advantages, e.g. the freedom in the coordinate transformations and access to Noether’s theorem.

Letting

$$φ_1 - φ_2 = χ,$$

the equations of the standard dimer (11) acquire the form

$$\dot{r} = -r sin χ sinh ψ,$$

$$\dot{χ} = (r - cos χ) sinh ψ,$$

$$\dot{φ} = (r + cos χ) cosh ψ,$$

$$\dot{ψ} = γ + sin χ cosh ψ. \quad (20)$$

Here $φ$, $r$, and $ψ$ are defined by (8), (10), and (11); we remind that $r$ admits a simple expression in polar coordinates, Eq. (13).

To cast the system (20) in the form of the Lagrange-Euler equations for some functional $S = \int Ldt$, we start with introducing a new variable $μ$ such that $\dot{μ} = \mathcal{R}$. The constraint $\dot{μ} - \mathcal{R} = 0$ can be incorporated in the system by means of a Lagrange multiplier; call it $λ$. Thus we consider the Lagrangian

$$L = \frac{2μ - φ}{ρ} ρ + ψ + γ - λ(μ - \mathcal{R}). \quad (21)$$

Here $ρ, θ, φ$, and $ψ$ — as well as $μ$ and $λ$ — are regarded as independent variables, whereas $\mathcal{R}$ is a function of $ρ, θ$, and $ψ$ given by Eq. (15).

A slightly modified version of (21) is arrived at by dropping a total derivative:

$$\tilde{L} = -(2μ - φ) ln ρ + ψ + γ - λ(μ - \mathcal{R}). \quad (22)$$

The formulation (22) makes it obvious that the variable $φ$ is cyclic; this is a consequence of the $U(1)$ phase invariance of the dimer (11). Therefore, $∂\tilde{L}/∂φ = ln ρ$ is a conserved quantity:

$$\dot{ρ} = 0. \quad (23a)$$

The coordinate $μ$ is also cyclic; hence

$$\frac{∂\tilde{L}}{∂μ} = -2 ln ρ - λ$$

is another integral of motion — and so is $λ$.

The variation with respect to the remaining four independent coordinates gives

$$\dot{θ} = -λτ sinh ψ, \quad (23b)$$

$$\dot{ψ} = 2μ - λρ sinh ψ, \quad (23c)$$

$$\dot{φ} = 2μ - λρ sinh ψ, \quad (23d)$$

and $μ = \mathcal{R}$. The constant $λ$ may be chosen arbitrarily; different choices of $λ$ are equivalent up to a rescaling of $t$ and redefinition of $γ$. Choosing $λ = 1$, one can readily verify that four equations (23) (with $\mathcal{R}$ substituted for $μ$) are equivalent to the system (20).

IV. CONCLUDING REMARKS

In this note, we have revealed the Hamiltonian structure of the standard $PT$-symmetric dimer, equation (11). The Hamilton function is given by (17); the canonical coordinates are $ρ$ and $θ$, with the canonical momenta defined by (19) and (16), respectively.

We have also proposed the Lagrangian formulation for the standard dimer. The Lagrange function is in (21) (with $\mathcal{R}$ substituted for $μ$). Unlike its Hamiltonian formulation, the Lagrangian description requires introduction of an auxiliary degree of freedom [accounted for by the variables $μ$ and $λ$ in (21)].

We conclude with two remarks. First, we would like to acknowledge the importance of the Stokes coordinates.
that were crucial for our construction. Mathematically, the transformation (7) is an example of the Hopf fibration mapping a 3-sphere (a hypersphere in the four-dimensional space formed by the real and imaginary components of \(u\) and \(v\)) onto the 2-sphere in the \((X, Y, Z)\)-space. In physics, the Hopf map was used to establish the equivalence of two field-theoretic models on the plane, the \(\mathbb{CP}^2\) model and the \(O(3)\) \(\sigma\)-model. The same transformation is employed in the studies of quantum two-level systems where it was pioneered by Feynman and co-authors. (Accordingly, the \(X\), \(Y\), and \(Z\) are occasionally referred to as the Feynman variables.) A closely related object is the Bose-Hubbard dimer; in that context, the \(X, Y, Z\) triplet is known as the Bloch vector. The name Stokes variables hails from optics where the \(X\), \(Y\), and \(Z\) parameters are used to describe the polarization state of electromagnetic radiation. Jensen exploited the Stokes parameters for the analysis of his two-waveguide optical coupler.

Our second remark is on the integrability of the standard \(\mathcal{P}\mathcal{T}\)-symmetric dimer. The fact that a system with two degrees of freedom has two integrals of motion is generally insufficient to claim that the system is Liouville integrable. Indeed, assigning particular values to the two integrals reduces the motion to a two-dimensional manifold, e.g. a genus-two Riemann surface, where the flow may happen not to be integrable. However if the system is known to be Hamiltonian, the existence of the second integral of motion (which is obviously in involution with the Hamilton function) implies the complete integrability of the system. Thus, uncovering the Hamiltonian structure of the standard \(\mathcal{P}\mathcal{T}\)-symmetric dimer completes the proof of its integrability that was suggested when two conserved quantities were found.

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