The structure of all the supersymmetric solutions of ungauged $\mathcal{N} = (1, 0), d = 6$ supergravity

Pablo A Cano and Tomás Ortín

Instituto de Física Teórica UAM/CSIC C/ Nicolás Cabrera, 13–15, C.U. Cantoblanco, E-28049 Madrid, Spain

E-mail: pablo.cano@uam.es and Tomas.Ortin@csic.es

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Abstract

We characterize all the supersymmetric configurations and solutions of minimal ($\mathcal{N} = (1, 0)$) $d = 6$ supergravity coupled in the most general gauge-invariant way to an arbitrary number of tensor and vector multiplets and hypermultiplets.

Keywords: supersymmetry, supergravity, BPS, solution, unbroken supersymmetry

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Introduction

After the pioneering works of Gibbons, Hull and Tod \([1, 2]\) on the characterization of the supersymmetric solutions of pure (minimal) \(\mathcal{N} = 2, d = 4\) supergravity, a great effort (leading to a wealth of very important and useful results we will not try to review here) has been devoted to the characterization (a.k.a. ‘classification’) of the supersymmetric configurations and solutions of more general supergravity theories. This effort has been particularly intense and fruitful in the realm of the so-called \(\mathcal{N} = 2\) supergravity theories in \(d = 4, 5\) and \(6\) dimensions. These are theories with 8 supercharges that admit timelike supersymmetric solutions, which include black holes, and not just null supersymmetric solutions, which in 4 dimensions only include waves and ‘stringy cosmic strings’. These theories admit many different matter couplings but the amount of supersymmetry they have constrains their structure the right amount which is needed to endow them with interesting geometries and dualities.

In \(d = 4\), using the ‘bilinear method’ of \([3]\), the timelike case of the most general \(\mathcal{N} = 2\) theory was worked out in \([4]\), culminating a long series of works in which theories with more general matter couplings were studied \([5–11]\), in which the the null case (in absence of non-Abelian gaugings) was also solved. Only the null case for the most general non-Abelian-gauged theory remains to be worked out.

In \(d = 5\) dimensions, the timelike and null cases have been solved for the most general theory after another long sequence of works dealing with increasingly complicated matter couplings \([3, 12–18]\). An interesting aspect of the 5-dimensional case is that the supersymmetric solutions that admit an isometry which acts with no fixed points can be reduced to a supersymmetric solution of a \(\mathcal{N} = 2, d = 4\) supergravity.

Less is known about the six-dimensional case, which only admits null Killing spinors, which nonetheless has interesting applications. For example, supersymmetric solutions of \(d = 6\) supergravity have recently proven to be very useful in the context of the fuzzball proposal \([19]\), thanks to the construction horizonless microstate geometries \([20–23]\) that are able to account for a finite fraction of the black hole entropy.

In spite of the interest of this case, in \(d = 6\) dimensions, however, only two theories have been completely studied so far: pure supergravity, in \([24]\), and Fayet–Iliopoulos-gauged supergravity coupled to some vector multiplets in \([25]\). In between these two theories, there is a huge gap corresponding to ungauged theories coupled to arbitrary numbers of vector and tensor multiplets and also to hypermultiplets which are the theories we are going to consider...
here. This should be understood as a first step towards the complete characterization of all the supersymmetric solutions of the most general matter-coupled $N = (1, 0), d = 6$ supergravities: on the one hand, only if one considers vector multiplets can one gauge any symmetries of the ungauged theory. On the other hand, the symmetries that can be gauged are $R$-symmetry, which is gauged via Fayet–Iliopoulos terms (the case considered in [25]) and the isometries of the hyperscalar manifold and of the scalar manifold associated to the tensor multiplets. Thus, if one wants to take a step beyond what is already known, one is forced to consider, at least, tensor and vector multiplets together.

There is another reason for considering these two kinds of multiplets simultaneously: the existence of duality between ungauged theories with tensor and vector multiplets compactified in a circle discovered in [26]. More precisely, in that reference it was shown that the dimensional reduction of one of these theories with just one tensor multiplet ($n_T = 1$) and an arbitrary number of vector multiplets $n_V$ and that of a theory with $n'_T = 1 + n_V$ and $n'_V = 0$ give exactly the same $N = 1, d = 5$ theory with $n_{V5} = n_V + 2$ vector multiplets.

This situation is entirely analogous to the identity between the reductions on dual circles of the $N = 2A, d = 10$ and $N = 2B, d = 10$ theories [27] which signals, at the effective action level, the T-duality between type IIA and IIB superstring theories discovered in [28, 29]. In the case at hands, there is no known stringy/brany duality underlying the duality found at the supergravity level. Nevertheless, it is possible to derive a set of Buscher-type duality rules that transform solutions of the $n_T = 1, n_V$ theory with an isometry into a solutions of the $n'_T = n_V + 1, n'_V = 0$ theory and vice-versa.

Although this duality has been found in the bosonic equations of motion it is, most likely, a duality between the two complete supergravity theories and, therefore, it is to be expected that the supersymmetric solutions of both kinds of theories are related by it. A first step to check whether this is true is the characterization of all the solutions of the ungauged theories with arbitrary numbers of vector and tensor multiplets, which we are going to present here. The relation between the supersymmetric solutions will be studied elsewhere.

Early work on supersymmetric solutions of the theories that we are going to consider here can be found in [30], but this work is quite far from the systematic and exhaustive approach we aim to pursue here. More recently, Akyol and Papadopoulos in [31] solved the Killing spinor equations of these theories in the most general, gauged case identifying the geometry and the field strengths of supersymmetric field configurations. However, since the main goal of that paper was to study the different kinds of Killing spinors admissible by the supersymmetric configurations, they did not solve the Bianchi identities of the vector field strength nor did they impose the equations of motion on them. Thus, the supersymmetric configurations were not completely characterized and the supersymmetric solutions (the equations that they have to solve) were left unidentified.

It is known that supersymmetry ensures that some of the equations of motion of supersymmetric solutions are related among them or automatically solved but there is always a number of them which are independent and need to be solved. The relations between the equations of motion of supersymmetric field configurations can be obtained bia the so-called Killing spinor identities (KSI) [32, 33] or via the integrability conditions of the Killing spinor equations. In order to construct the KSIs one needs the locally supersymmetric action of the theory, which does not exist for the theories under consideration because they include 2-forms with (anti-) self-dual 3-form field strengths\textsuperscript{1}. Thus, in this work we will find these relations from the integrability conditions of the KSEs. We will identify the independent and non-trivial (for

\textsuperscript{1}It might be possible to derive the KSIs from the pseudo-action equation (1.15), but this requires further investigation.
supersymmetric configurations) equations of motion and we will impose them, together with the Bianchi identities of the vector fields, on the supersymmetric configurations, finding a reduced number of simplified differential equations to be solved. Our main result, summarized in section 4, will be this set of simplified differential equations and a recipe to construct supersymmetric solutions of theories with an arbitrary number of vector and tensor multiplets. We will also study some further simplifications of these equations for particular cases and make contact with the results on classifications of $\mathcal{N} = 1, d = 5$ supersymmetric solutions, partially confirming the results of this paper.

This paper is organized as follows: in section 1 we present the theories, their field content, (pseudo-)action, equations of motion and supersymmetry transformation rules. In section 2 we characterize the field configurations which, satisfying the equations of motion or not, admit at least one Killing spinor. In section 3 we impose the equations of motion on the supersymmetric configurations we have characterized. Only a few of them are actually independent and, therefore, as usual, the number of equations that have to be solved by the building blocks of a supersymmetric configuration is very reduced. At this stage, we have achieved all the goals we were aiming for and is simply rest to summarize our results. We do this, and conclude, in section 4.

1. Ungauged six-dimensional supergravity

Six-dimensional supergravity coupled to matter has been described in increasing levels of generality in [35–41]. For the sake of completeness we review here the most relevant results. We will mostly use the notation of [41]. The bosonic sector of six-dimensional supergravity contains the graviton, represented by the Vielbein $e^a_{\mu}$, a number $n_T$ of scalars $\phi^\alpha$, $n_T + 1$ two-forms, $B_{\mu\nu}^r$, with respective field strengths

\[ H^r = dB^r + \frac{1}{2} C_{ij}^r F^i \wedge A^j, \tag{1.1} \]

$n_V$ abelian vectors $A^i$, with field strengths

\[ F^i = dA^i, \tag{1.2} \]

and $4n_H$ hyperscalars $\phi^X$. The fermionic sector consists of the gravitino $\psi^A_{\mu}$, $n_T$ tensorinos $\chi^M A$, $n_V$ gauginos $\lambda^A$, and $2n_H$ hyperinos $\Psi^a$. Let us explain how these fields couple among them.

The scalars $\phi^\alpha$, $\alpha = 1, ..., n_T$, parametrize the coset $SO(n_T, 1)/SO(n_T)$. The indices $M, N = 1, ..., n_T$ belong to the fundamental representation of $SO(n_T)$, while $r, s = 0, 1, ..., n_T$ label the fundamental representation of $SO(n_T, 1)$. On the other hand, $A = 1, 2$ is an $Sp(1)$ index. Let us introduce a coset representative of $SO(n_T, 1)/SO(n_T)$ as a $(n_T + 1) \times (n_T + 1)$ matrix, $L^A_r$, which belongs to $SO(n_T, 1)$. It is useful to split its components in the form $L^A_r \equiv L^0_r$ and $L^A_r$, so that they satisfy

\[ After this work was already completed we learned about another work by het Lam and Vandoren [34] which studies the case of coupling to an arbitrary number of tensor multiplets only from the same point of view.

\[ The detailed comparison of our results with those of [31] is very complicated because we are not solving exactly the same problem: we consider solutions with at least one unbroken supersymmetry and, in [31], field configurations with exactly one, two etc unbroken supersymmetries are considered. Furthermore, we make some explicit choices of coordinates that lead to the definition of the, most useful, base space, in which many of the objects that we determine are defined. The definition of base space is not used in [31]. One can only compare the components of the 3-form field strengths and some general structures of the supersymmetric solutions and check that, indeed, they agree.

\[ P A Cano and T Ortín Class. Quantum Grav. 36 (2019) 125007 \]
\[ L_r L_s = L_r M L_s = \eta_{rs}, \quad (1.3) \]

where \( \eta_{rs} = \mathrm{diag}(+1, -1, -1, \ldots, -1) \). The indices \( r, s \) are lowered and raised with \( \eta_{rs} (\eta^{rs}) \) in the way \( L' = \eta^{rs} L_s, \quad L'^M = -\eta^{rs} L_r^M \). We also have the relations

\[ L'_r L_s = 1, \quad L'_r L'^M = 0, \quad L_r^M L'^N = \delta^{MN}. \quad (1.4) \]

We will also need the following symmetric but not constant tensor:

\[ G_{rs} = L_r L_s + L_r M L_s M. \quad (1.5) \]

The scalars parametrize the coset representative, and we have the following relations

\[ \partial_\alpha L_r = V_\alpha^M L_r^M, \quad (1.6) \]

\[ \partial_\alpha L_r^M = -A_{\alpha}^N L_r L^N + V_\alpha^M L_r, \quad (1.7) \]

where \( \partial_\alpha \equiv \partial/\partial \varphi_\alpha \) and \( V_\alpha^M \) is the Vielbein, which satisfies

\[ V_\alpha^M V_\beta^M = g_{\alpha \beta}. \quad (1.8) \]

where \( g_{\alpha \beta} \) is the metric of the scalar manifold associated to the tensor multiplets.

It is also convenient to define the 3-forms field strengths in the basis defined by the coset \( \mathcal{H}, \mathcal{H}^M \), because the supersymmetric transformation rules are written in terms of them. They are related to those defined in equation (1.1) by

\[ \mathcal{H} = L_r H_r, \quad \mathcal{H}^M = L_r^M H_r. \quad (1.9) \]

In this way, we can distinguish the several supermultiplets of this theory: we have the supergravity multiplet \( \{ e^a, \psi^A_{\mu}, H \} \), \( n_T \) tensor supermultiplets \( \{ \chi^M, \varphi^a, \mathcal{H}^M \} \), \( n_V \) vector supermultiplets \( \{ A^i, \lambda^A \} \) and \( n_H \) hypermultiplets \( \{ \phi^X, \Psi^a \} \).

On the other hand, the hyperscalars \( \phi^X \) parametrize a quaternionic-Kahler manifold of holonomy \( \mathrm{Sp}(1) \times \mathrm{Sp}(n_H) \). For completeness, we give here certain formulas which are used throughout the calculations in the text. The Vielbein of the quaternion manifold is denoted as \( V_a^A X \), with \( X = 1, \ldots, 4n_H \), while \( a \) and \( A \) are indices of \( \mathrm{Sp}(n_H) \) and \( \mathrm{Sp}(1) \), respectively. We are interested in the \( \mathrm{Sp}(1) \) connection, which is denoted \( A^X_{\alpha} B \) and it is anti-hermitian in the \( A, B \) indices. Equivalently, we can write the components of the connection in the adjoint representation

\[ A^X_{\alpha} B = \frac{i}{2} (\sigma^*)_{\alpha}^A B X \Leftrightarrow A^X = -i(\sigma^*)_{\alpha}^A B A_X^B, \quad (1.10) \]

where \( x = 1, 2, 3 \) and \( \sigma^* \) are the Pauli matrices. The field-strength of the connection is defined as

\[ F_{XY}^A = \partial_X A^Y_{\alpha} B - \partial_Y A^X_{\alpha} B + [A_X, A_Y]^\alpha_{\alpha} B. \quad (1.11) \]

We also have the following relations [41, 42]\(^4\)

\[ V_a^X V_b^Y + V_a^Y V_b^X = g_{XY}^A \delta_a^b, \]
\[ V_a^X V_b^Y + V_a^Y V_b^X = g_{XY}^A \delta_a^b, \]
\[ V_a^X V_b^Y + V_a^Y V_b^X = F_{XY}^A. \quad (1.12) \]

\(^4\)We thank P. Vandoren for pointing to us a missprint in the first of these relations in [42].
The quaternionic structures are related to the field strength according to
\[ J^X Y = \frac{1}{2} F^{X Y} \equiv -\frac{1}{2} (\sigma^4)^B_A F^X Y A_B. \] (1.13)

They are covariantly constant with respect to the Sp(1) connection and they satisfy the quaternionic algebra
\[ J^X \cdot J^Y = -\delta^Y_X + \epsilon^{YZ} J^Z. \] (1.14)

11. Field equations and supersymmetry transformations

It is always most convenient to have an action principle from which the equations of motion can be derived. Precisely, one of the difficulties of \( \mathcal{N} = (1, 0), d = 6 \) supergravity is that it contains self-dual 3-forms whose equations of motion cannot be obtained from a covariant action functional unless one introduces PST-type auxiliary variables [43–46] and reformulates the theory using them. However, one can also use a ‘pseudo-action’ (which is not supersymmetric) from which, through its functional derivatives, one obtains equations that have to be supplemented by the duality constraints. For instance, this has been done in [47, 48] for the \( \mathcal{N} = 2B, d = 10 \) theory, whose Ramond–Ramond 4-form has a self-dual 5-form field strength, and for the case at hands (with no hypermultiplets) in [26]. The pseudo-action we need is\(^5\) [41]
\[ S = \frac{1}{16\pi G_N^{(6)}} \int d^6x \sqrt{|g|} \left\{ R - \partial_{\alpha} L' \partial^\alpha L_r + \frac{1}{3} G_{rs} H'^{r \mu \nu} H^{-\mu \nu} - L^c' y F^i_{\mu \nu} F^j_{\rho \sigma} ight. \]
\[ - \frac{1}{4} c_{r} \epsilon^{\mu \nu \rho \sigma \lambda \eta} B'^{\mu \nu} F_{\rho \sigma} F_{\lambda \eta} + 2 g_{XY} \partial_{\mu} \phi^X \partial_{\nu} \phi^Y \right\}, \] (1.15)

and has to be supplemented by the self-duality relations
\[ \star G_{rs} H'^{r} = -\eta_{rs} H'^{s}. \] (1.16)

Given the relations equations (1.5) and (1.3), these equations imply that the rotated field strengths \( H \) and \( H^M \) are, respectively, anti-self-dual and self-dual
\[ H^+ = 0, \quad H^- = 0. \] (1.17)

The theory is invariant under the gauge transformations
\[ A' \rightarrow A' + dA', \quad B' \rightarrow B' - \frac{1}{2} c'_{y} A' \wedge dA' + d\chi', \] (1.18)

for arbitrary 0– and 1–forms \( A' \) and \( \chi' \), providing that the constants \( c'_{y} \) satisfy the relation
\[ \eta^{(y)} c'_{(ij) c (k)} = 0, \] (1.19)

which we will assume to hold. Then the field equations are gauge-invariant and they read
\[ E_{\mu \nu} = R_{\mu \nu} + 8 \epsilon_{\mu \nu} \partial_{\mu} \phi^X \partial_{\nu} \phi^Y + G_{rs} H_{\mu \rho \sigma} H'^{\nu} \rho \sigma - 2 L_{\nu} c'_{y} F^i_{\mu \rho} F^j_{\nu \rho} 
+ \frac{1}{4} g_{XY} \partial_{\mu} \phi^X \partial_{\nu} \phi^Y, \] (1.20)

\(^5\)We follow the conventions of [49].
\[ \mathcal{E}_r = d(\ast G_r H^r) + \frac{1}{2} c_{r, g} F^i \wedge F^j, \]
\[ \mathcal{E}_i = d(\ast L_r c_{r, g} F^i) - 2c_{r, g} H^r \wedge F^j, \]
\[ \mathcal{E}_\alpha = D_\mu \partial^\mu \varphi^\alpha - \frac{2}{3} V^a M H^M \mu\nu\rho \mathcal{H}^{\mu\nu\rho} + \frac{1}{2} V^a M L_r M c_{r, g} F_{\mu\nu} F^i F^{j\mu\nu}, \]
\[ \mathcal{E}^X = D_\mu \partial^\mu \phi^X, \]

where \( D_\mu \) denotes the covariant derivative in space-time and in the corresponding scalar manifold.

Along with these equations we have the Bianchi identities of the vector fields\(^6\),
\[ dF^i = 0. \]

It is also convenient to write the equations of motion of the 1- and 2-forms in their dual form
\[ \ast \mathcal{E}^\mu = \nabla_\mu (G_r H^r \rho\mu\nu) + \frac{1}{8} c_{r, g} \frac{\epsilon^{\rho\mu\nu\sigma\alpha\beta}}{\sqrt{|g|}} F_{\rho\nu} F_{\sigma\alpha\beta}, \]
\[ -\frac{1}{8} \ast \mathcal{E}_i = \nabla_\mu (L_r c_{r, g} F^i + G_r c_{r, g} H^r \alpha\beta\nu F_{j}^{\alpha\beta}). \]

For vanishing fermions, the supersymmetry transformations of the fermion fields are given by\(^7\)
\[ \delta_\epsilon \psi^A = D_\mu \epsilon^A - \frac{1}{4} \mathcal{H}_\mu \epsilon^A, \]
\[ \delta_\epsilon \chi^{MA} = \frac{1}{2} \left[ \epsilon_\nu \omega V^\nu M + \frac{1}{6} \mathcal{H}^M \epsilon^A \right] e^A, \]
\[ \delta_\epsilon \lambda^A = -\frac{1}{2} \sqrt{2} \mathcal{F}^{A} \epsilon^A, \]
\[ \delta_\epsilon \Psi^A = i \partial_\mu \phi^X V^A X_{\epsilon^A}, \]

where \( D_\mu \) is the space-time and \( \text{Sp}(1) \) covariant derivative
\[ D_\mu e^A = \nabla_\mu e^A + A^A_{\mu B} e^B = \left( \partial_\mu - \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \right) e^A + A^A_{\mu B} e^B, \]

and \( A^A_{\mu B} \equiv \partial_\mu \phi^X A^X_{\epsilon^A} \) is the pullback of the connection.

2. Supersymmetric configurations

In this section we are going to identify all the supersymmetric configurations of the theories that we have just introduced. First, in section 2.1 we are going to find the necessary conditions

\(^6\) By (anti-) self-duality, the Bianchi identities of the 3-form field strengths are the equations of motion themselves.

\(^7\) Our conventions on the spinors and gamma matrices are explained in appendix A.
that a field configuration has to satisfy in order for the Killing spinor equations (KSEs) to admit at least a solution (a Killing spinor). In a second stage, in section 2.2 we will show that these conditions are also sufficient and we will explicitly determine the form of the Killing spinor.

2.1. Necessary conditions

We assume that we have certain purely bosonic field configuration which admits a Killing spinor \( \epsilon^A \). By definition, this means that every field is invariant under the supersymmetry transformation generated by \( \epsilon^A \) and, in particular,

\[
\delta \epsilon f = 0,
\]

(2.1)

for every fermion \( f \) of the theory and for \( \epsilon^A \). These equations are, by definition, the KSEs of the theory.

In order to find useful information from these fermionic equations, we will use the ‘bilinear method’ pioneered in [3]. Given a Killing spinor \( \epsilon^A \) we can construct an associated vector and a triplet of 3-form bilinears, as explained in appendix A:

\[
l_\mu \equiv \bar{\epsilon}^A \gamma_\mu \epsilon_A, \quad W^{\mu \nu \rho}_x \equiv i (\sigma^x)_A^B \bar{\epsilon}^A \gamma^{\mu \nu \rho} \epsilon_B.
\]

(2.2)

The properties of these bilinears are described in appendix A.2. In particular, the triplet \( W^x \) is anti-self dual and \( l \) is null and transverse to \( W^x \):

\[
\ast W^x = - W^x, \quad l_\mu l^\mu = 0, \quad l^\lambda W^x_{\mu \nu \lambda} = 0.
\]

(2.3)

We introduce an auxiliary null vector \( n_\mu \) satisfying

\[
n_\mu n^\mu = 0, \quad l_\mu n^\mu = 1,
\]

(2.4)

and define

\[
\mathcal{J}^x_{\mu \nu} \equiv n^\lambda W^x_{\mu \nu \lambda}.
\]

(2.5)

\( \mathcal{J}^x \) is transverse to \( l \) and \( n \), self-dual in the four-dimensional transverse space and, most importantly, with one index raised, \( \mathcal{J}^x \) \( l_\mu \), it satisfies the quaternionic algebra equation (A.53).

Our next task is to extract all the possible information from the KSEs by using these bilinears. The analysis is more or less independent for each equation and we dedicate one section to each of them.

2.1.1. Gravitino equation. The gravitino KSE, \( \delta \psi^A = 0 \), can be written as

\[
\nabla_\mu \epsilon^A = \frac{1}{4} \mathcal{H}_\mu \epsilon^A - A^A_{\mu} B \epsilon^B.
\]

(2.6)

We are going to translate this spinorial equation into equations for the spinor bilinears. By taking their covariant derivatives and using the above KSE, we find that \( l_\mu \) and \( W^x_{\mu \nu \rho} \) satisfy the following identities:

\[
\nabla_\mu l_\nu = l^\lambda \mathcal{H}_{\mu \nu \lambda},
\]

(2.7)

8 The supersymmetry transformations of the bosonic fields, being proportional to the fermionic fields, which vanish by assumption, are trivially satisfied.

9 These identities imply that \( l_\mu \) and \( W^x_{\mu \nu \rho} \) are covariantly constant with respect to the torsionful connection \( \Gamma(\sigma) + \mathcal{H} + A \).
\[ \nabla_{\mu} W^{\alpha}_{\nu\rho\sigma} = 3W^{\alpha}_{\lambda[\rho\sigma]} H_{\mu|\lambda} - e^{3/2} A^{\alpha}_{\mu} W^{\sigma}_{\nu\rho\sigma}. \]  

Equation (2.7) implies, in particular, that \( \nabla_{\mu}(l_{\nu}) = 0 \), so \( l_{\nu} \) is a null Killing vector. Then, let us characterize all the metrics which allow for a null Killing vector which in general is not covariantly constant. First, we introduce a coordinate \( \nu \) associated to \( l^{\nu} \) defined through

\[ l^{\nu} \partial_{\nu} = \partial_{t}. \]  

Hence, in this coordinate system, \( l^{\nu} = \delta^{\nu}_{\mu} \). On the other hand, \( l_{\mu} \) is transverse to \( l^{\nu} \). Let us write \( \hat{t} = l_{\nu} dx^{\nu} \). Since, in general, \( \hat{t} \neq 0 \), we cannot find a coordinate \( u \) such that \( \hat{t} = du \). In addition, generically \( \hat{t} \) is not hypersurface-orthogonal, \( \hat{t} \wedge \hat{t} \neq 0 \), so \( \hat{t} \neq f du \) for any function \( f \) and coordinate \( u \). Therefore, we must write in general

\[ \hat{t} = f(du + \beta), \quad \text{where} \quad \beta = \beta_{m} dx^{m}, \quad m = 1, 2, 3, 4, \]  

so\( \beta \) is a 1-form on the four-dimensional space transverse to \( l \) and \( n \) while \( f \) is just a function. Both \( f \) and \( \beta \) can depend on \( u \) and \( x^{m} \) but not on \( \nu \). Now, since \( l^{\mu} n_{\mu} = 1 \), the 1-form \( \hat{n} = n_{\mu} dx^{\mu} \) can be written as

\[ \hat{n} = d\nu + H du + \omega, \]  

where \( H \) is a function which again can depend on \( u \) and \( x^{m} \), and \( \omega \) is a 1-form in the four-dimensional space (which can also depend on \( u \)). Finally, since \( n \) and \( \hat{t} \) are null, the metric must be given by

\[ ds^{2} = 2\hat{t} \otimes \hat{n} - f^{-1} \delta_{mn} v^{m} \otimes v^{n}, \]  

where \( v^{m} \) is the Vielbein of the four-dimensional Euclidean space which we will call, as it is customary, ‘base space’. In coordinate form, the metric reads

\[ ds^{2} = 2f(du + \beta)(d\nu + H du + \omega) - f^{-1} \gamma_{mn} dx^{m} dx^{n}, \]  

where \( \gamma_{mn} \) is the metric of the base space\(^{10} \). No quantity in this metric depends on the isometric null coordinate \( \nu \). In order to make any further progress in our analysis, we introduce a null Vielbein \( e^{\alpha}_{\mu} \):

\[ e^{+} = f(du + \beta), \quad e^{-} = d\nu + H du + \omega, \quad e^{m} = f^{-1/2} v^{m}; \]  

and the inverse Vielbein is

\[ e_{+} = f^{-1} (\beta_{u} - H \partial_{u}), \quad e_{-} = \partial_{\nu}, \quad e_{m} = f^{1/2} v^{m} - f^{1/2} \beta_{m} \partial_{u} - f^{1/2} (\omega_{m} - \beta_{m}) \partial_{\nu}, \]  

where \( \beta_{m} \equiv v_{m} \beta / \beta_{u} \), and the same for \( \omega_{m} \). Note that \( e^{+} = \hat{l}, e^{-} = \hat{n} \) and \( e_{m}^{\mu} = n^{\mu} \), \( e_{-}^{\mu} = l^{\mu} \). The spin connection of this Vielbein is computed in appendix B.

From the condition equation (2.7) we have been able to find the generic form of the metric of a supersymmetric configuration. The next step is to exploit the rest of information contained there in (2.8) in order to find the general form of the field strength \( H \) in a supersymmetric configuration.

First, let us inspect the independent components of \( H_{abc} \) (in the null Vielbein basis). It has four kind of components:

\[ H_{+m}, \quad H_{+ma}, \quad H_{-ma}, \quad H_{mn}. \]  

\(^{10} \) It is related to the Vielbein by the usual expression

\[ \gamma_{mn} = \delta_{mn} v^{m} v^{n}. \]
On account on the anti-self-duality of $\mathcal{H}$:

$$\mathcal{H}_{mn} = \tilde{\varepsilon}^{mnpq} \mathcal{H}_{+-q}, \quad (2.18)$$

where the Levi-Civita symbol of the transverse space is given by

$$\tilde{\varepsilon}^{mnpq} = \varepsilon^{mnpq+}. \quad (2.19)$$

Furthermore, considered as 2-forms in this four-dimensional space, $\mathcal{H}_{+mn}$ and $\mathcal{H}_{-mn}$ are, respectively, self-dual and anti-self-dual:

$$\tilde{x} \mathcal{H}_{+mn} = \mathcal{H}_{+mn}, \quad \tilde{x} \mathcal{H}_{-mn} = -\mathcal{H}_{-mn}. \quad (2.20)$$

All these conditions reduce the list of independent components of $\mathcal{H}$. Let us proceed with the computation of the independent components. In flat indices, equation (2.7) can be written as

$$de^+ = \mathcal{H}_{ab} - \varepsilon^a \wedge \varepsilon^b, \quad (2.21)$$

and we get the following relation between $\mathcal{H}$ and the spin connection:

$$\omega_{ab-} + \mathcal{H}_{ab-} = 0. \quad (2.22)$$

Automatically, this relation gives us the components $\mathcal{H}_{-mn}$ and $\mathcal{H}_{+mn}$, and, by using the duality relation equation (2.18), we also obtain $\mathcal{H}_{mn}$. Hence, it only remains to find $\mathcal{H}_{+mn}$, or, equivalently, its self-dual part. In order to do so, let us note that equation (2.8) can be written as

$$D_a W^x_{bcd} = 3 W^x_{e[cd]} \mathcal{H}_{a|b]}^e. \quad (2.23)$$

where now $D_a$ is also the Sp(1) covariant derivative. Then, we take into account that, since $W^x$ is anti-self-dual and transverse to $e^+$, the only non-vanishing components are $W^x_{+mn} = \tilde{\mathcal{H}}^x_{mn}$. By using also equation (2.22) we see that equation (2.23) is equivalent to

$$D_a \tilde{\mathcal{H}}^x_{mn} = 2 \tilde{\mathcal{H}}^x_{[mp] H_{a|n]}^p}. \quad (2.24)$$

By using again (2.22), the component $a = -$, gives us $\partial_v \tilde{\mathcal{H}}^x_{mn} = 0^{11}$. On the other hand, the component $a = +$ gives us the following relation:

$$\mathcal{H}^+_{+mn} + \omega^+_{+mn} = \tilde{\mathcal{H}}^+_{mn} \left( \frac{1}{16} \varepsilon^{xyz} \partial_x \mathcal{H}^x_{yz} - \frac{1}{2} A^+_x \right). \quad (2.25)$$

Since $\mathcal{H}^+_{+mn} = \mathcal{H}^+_{+mn}$, we have determined the general form of all the components of $\mathcal{H}$.

From the gravitino KSE we can also obtain information about the base space. In order to simplify the notation, let us introduce the following derivative operator acting on the p-form $\alpha$ [24]

$$D\alpha \equiv \tilde{d}\alpha - \beta \wedge \dot{\alpha}, \quad (2.26)$$

where $\tilde{d}$ is the exterior derivative in the base space and where $\dot{\alpha}$ denotes the derivative with respect to $u$ in the coordinate basis. For example, for a 1-form:

$$\alpha = \alpha_\mu dx^\mu \Rightarrow \dot{\alpha} = \partial_\mu \alpha_\mu dx^\mu. \quad (2.27)$$

Note that the components of $\dot{\alpha}$ in the Vielbein basis are given by $\dot{\alpha}_a = e_a^\mu \partial_\mu \alpha_\mu$ (in the case of a 1-form). The operator $D$ satisfies the identity

$$D^2 \alpha = -D \beta \wedge \dot{\alpha}. \quad (2.28)$$

11 We are advancing that $A^+_x = 0$. 
By using this operator, the full exterior derivative is given by
\[ d\alpha = D\alpha + f^{-1} e^+ \wedge \dot{\alpha}. \] (2.29)

Coming back to the issue of interest, we know that the structures \( J^x \) satisfy the quaternionic algebra. However, these are not the natural quaternionic structures of the base space, since they must be defined with respect to the Vielbein \( v^a_\mu \). Therefore, we define the complex structures in the coordinate basis as
\[ J^x \equiv v^a_\mu \gamma^x \gamma^a v^\mu \gamma^x. \] (2.30)

Now the indices are raised and lowered with \( \gamma_{\mu\nu} \) instead of \( g_{\mu\nu} \):
\[ J^x = \gamma^m J^x_m v^m v^n J^x_n, \] (2.31)
These relations imply, in particular, that, as 2-forms, the \( J^x \) and \( J^x \) are related by
\[ J^x = -f J^x, \] (2.32)
while in the corresponding Vielbein basis their components are related by
\[ J^x \mid_\mu = -\tilde{J}^x \mid_\mu. \] (2.33)

Now, we can express \( W^x \) in terms of \( J^x \) as
\[ W^x = e^+ \wedge \tilde{J}^x = -f^{-1} e^+ \wedge J^x. \] (2.34)

From equation (2.8), it follows that
\[ dW^x + \epsilon_{xyz} A^y \wedge W^z = 0. \] (2.35)

Then, this equation implies that
\[ \tilde{\nabla} J^x + e^{\Sigma} A^y \wedge J^x = \partial_\beta (\beta \wedge J^x) + e^{\Sigma} A^x_\beta \wedge J^x, \quad (D\beta)^+ = 0, \] (2.36)
where \( A^y = \tilde{\nabla} \phi^X A^X \) is the pullback of the \( Sp(1) \) connection onto the base space, and \( A^x_\beta = \partial_\beta \phi^X A^X \). Note that in the cases \( \beta = 0 \) or \( u \)-independent, the first equation tells us that \( J^x \) is covariantly closed in the base space with respect to the \( Sp(1) \) connection. However, in a quaternionic-Kähler manifold the complex structures \( J^x \) must be not only covariantly closed, but covariantly constant. Indeed, if we use the equation (2.24) we see that
\[ \tilde{\nabla} J^x_\mu + e^{\Sigma} A^x_\nu J^x_\mu = \beta_\mu \epsilon^{\Sigma} A^x_\nu J^x_\mu + \beta_{\nu} \partial_\mu J^x_\nu - \delta_{\mu} \beta_\nu J^x_\nu + \beta_{\nu} \partial_\mu J^x_\nu - \delta_{\mu} \beta_\nu J^x_\nu + \beta_{\nu} \partial_\mu J^x_\nu - \delta_{\mu} \beta_\nu J^x_\nu + \beta_{\nu} \partial_\mu J^x_\nu - \delta_{\mu} \beta_\nu J^x_\nu, \] (2.37)
where
\[ U_{\mu\nu} \equiv -\delta_{\mu}[\beta_\nu] + \delta_{\nu}[\beta_\mu] - \delta_{\mu}[\beta_\nu], \] (2.38)
so that \( J^x \) is actually \( Sp(1) \) covariantly constant in the cases \( \beta = 0 \) or \( u \)-independent.

Observe that this does not mean that the base space is quaternionic-Kähler, because, precisely for \( d = 4 \) dimensions, the definition of a quaternionic-Kähler space is different\(^\text{12}\). In absence of hypermultiplets, the space is hyper-Kähler.

Let us summarize our results so far: in a supersymmetric configuration, the metric and the 3-form field strength \( \mathcal{H} \) are given by

\text{12 The holonomy is } \text{Sp}(1) \times \text{Sp}(1) \sim \text{SO}(4) \text{ and is not special anymore. Therefore, it cannot be used to characterize these spaces. Instead, it is required that they are Einstein and a self-dual Weyl tensor.}
\[ ds^2 = 2f(du + \beta)(dv + Hdu + \omega) - f^{-1}\gamma_{\mu\nu}dx^\mu dx^\nu, \quad (2.39) \]

\[
\mathcal{H} = \frac{1}{2}f^{-1}\epsilon^+ \wedge \epsilon^- \wedge \left( Df - f\dot{\beta} \right) + \frac{1}{2}f\epsilon^- \wedge D\beta \\
- \frac{1}{2}\epsilon\left( Df^{-1} + f^{-1}\dot{\beta} \right) + \epsilon^+ \wedge \left[ f^{-2}\left( -\dot{\psi} + \frac{1}{2}J^\alpha \right) - \frac{1}{2}G^+ \right], \quad (2.40)\]

where \( G \) is the 2-form

\[
G = D\omega - \partial H \wedge \beta, \quad (2.41)\]

and

\[
\psi = \frac{1}{16}\epsilon^\alpha j^\alpha \wedge J^\beta, \quad (2.42)\]

and where all the objects that appear in these expressions are \( \nu \)-independent. In addition, \( \beta \) satisfies the equation

\[
(D\beta)^+ = 0, \quad (2.43)\]

and \( \gamma_{\mu\nu} \) is the metric manifold with self-dual complex structures \( J^\alpha \) which satisfy the quaternionic algebra, and whose covariant derivative is given by equation (2.37).

### 2.1.2. Tensorino equation.

The tensorino KSE \( \delta \chi^M \epsilon^A = 0 \) reads

\[
\left[ \partial_\alpha \epsilon \chi^M + \frac{1}{6} \epsilon^A \right] \epsilon^A = 0. \quad (2.44)\]

If we contract it with \( \bar{\epsilon}^A \), we get:

\[
0 = f^A \partial_\alpha \varphi^A = \partial_\nu \varphi^A. \quad (2.45)\]

Therefore, the scalars do not depend on the isometric coordinate \( \nu \). Another useful identity is obtained if we contract equation (2.44) with \( \bar{\epsilon}^A \gamma^{ab} \). In this case, we obtain:

\[
\mathcal{H}_M^{abc \lambda} e^\lambda F = \left[ \partial_\nu \varphi^A \right] \epsilon \chi^M \epsilon^A. \quad (2.46)\]

Decomposing the fields in this equation in their components, we find that

\[
\mathcal{H}_M^{m+n} = -\frac{1}{2} e_m \varphi^A \epsilon \chi^M \epsilon^A. \quad \mathcal{H}_M^{m-n} = 0. \quad (2.47)\]

This is all the information that we can get directly from (2.44). However, if we now make use of the self-duality of \( \mathcal{H}_M \), we find

\[
\mathcal{H}_M^{mn} = \frac{1}{2} \epsilon^m \varphi^A \epsilon \chi^M \epsilon^A. \quad (2.48)\]

On the other hand, we have not found any condition on \( \mathcal{H}_M^{+}, \) but the self-duality of \( \mathcal{H}_M \) implies that it must be anti-self-dual in the base space:

\[
\mathcal{H}_M^{+} = \mathcal{H}_M^{-} = (\mathcal{H}_M^{-})^+. \quad (2.49)\]

Hence, we can write the self-dual 3-forms \( \mathcal{H}_M \) as

\[
\mathcal{H}_M = -\frac{1}{2}\epsilon^+ \wedge \epsilon^- \wedge D\varphi^A \epsilon \chi^M \epsilon^A + \frac{1}{2}f^{-1}\epsilon D\varphi^A \epsilon \chi^M \epsilon^A + \epsilon^+ \wedge I^M. \quad (2.50)\]
By taking into account the results of the previous section, we can write the physical field strengths by using the relation
\[ H^r = L^r H + L^r M H^M. \] (2.51)

And we find
\[ H^r = \frac{1}{2} f^{-1} e^+ \wedge e^- \wedge [D(fL^r) - \hat{\beta} fL^r] + \frac{1}{2} fL^r e^- \wedge D\beta - \frac{1}{2} \hat{\beta} [D(f^{-1} L^r) + f^{-1} L^r \hat{\beta}] + e^+ \wedge \left\{ \chi^r + L^r \left[ f^{-2} \left( -\psi + \frac{1}{2} J^A A^A_u \right) - \frac{1}{2} G^+ \right] \right\}, \] (2.52)

where the quantities \( L^r \) satisfy
\[ \partial_v L^r = 0, \] (2.53)
and the anti-self dual 2-forms \( \chi^r = L^r M \mu M \) satisfy
\[ L^r \chi^r = 0. \] (2.54)

2.1.3. Gaugino equation. Let us now consider the KSE of the gauginos
\[ \delta_{\epsilon^A} \chi^A = -\frac{1}{2\sqrt{2}} f^A e^A = 0. \] (2.56)

By contracting this equation with \( \bar{\epsilon} B^A \), we obtain these two equations:
\[ \bar{\epsilon} B^A = 0, \] (2.57)
\[ W^A_{abc} F^i_{bc} = 0. \] (2.58)

The first equation simply tells us that
\[ F^i_{a-} = 0. \] (2.59)

Then, the second equation can be rewritten as
\[ F^i_{mn} F^i_{mn} = 0, \] (2.60)
which means that \( F^i_{mn} \) is anti-self-dual in the base space. Therefore, we can write the field strength \( F^i \) as
\[ F^i = e^+ \wedge \theta^i + \tilde{F}^i, \] (2.61)

where \( \theta^i \) and \( \tilde{F}^i \) are, respectively, 1- and 2-forms in the base space, and \( \tilde{F}^i \) is anti-self-dual in the base space
\[ \tilde{\star} \tilde{F}^i = -\tilde{F}^i. \] (2.62)

2.1.4. Hyperino equation. Finally, let us analyze the supersymmetric configurations for the hyperscalars. The hyperino KSE reads
\[ \delta_{\epsilon^A} \Psi^a = i \hat{\beta} \phi^A \chi^A \epsilon_A = 0. \] (2.63)
Contraction this equation with $\bar{\epsilon}^B$ just tells us that
\[ \partial_c \phi^X = 0, \] (2.64)
as expected. On the other hand, if we contract with $\bar{\epsilon}^B \gamma^{ab}$ we obtain
\[ i \bar{\epsilon}^B [\partial_b \phi^X V^{ab}_X + \partial_c \phi^X v^{aX} W^B_A e^c = 0. \] (2.65)

Now, by contracting with the inverse Vielbein $V^Y_A B$ and by using equations (1.12) and (1.13), one can see that this equation is equivalent to
\[ \partial_m \phi^X = J^X Y \partial_n \phi^Y J^{mn}, \] (2.66)
which characterizes $\phi^X$ as a ‘quaternionic map’. This is similar to what happens in $d = 5$ [17], with the difference that in our case the hyperscalars $\phi^X$ can depend on the null coordinate $u$.

2.2. Sufficient conditions

In the previous subsection we have determined the necessary conditions for a field configuration to be supersymmetric and we have obtained the general form of the fields. However, this does not imply that these configurations are actually supersymmetric and one has to make sure that there is a solution to the KSEs. We are going to show that, when the fields take the form described in the preceding section, there exists always a Killing spinor.

To begin with, let us consider a spinor $\epsilon^A$ satisfying the following conditions
\[ \gamma^+ \epsilon^A = 0, \quad \Pi^A B \epsilon^B = 0, \] (2.67)
where
\[ \Pi^A B \equiv \frac{1}{2} \left[ \delta^A_B + \frac{i}{4} \tilde{J}^X Y \tilde{\partial}_n \phi^Y J^{mn} \right], \] (no summation on $x$).
(2.68)

Once the condition $\gamma^+ \epsilon^A = 0$ is imposed, it follows that the $\Pi^A B$ are projectors. Moreover, the set of all these quantities $\{ \gamma^+, \Pi^A B \}$ is closed under commutation, so the conditions are consistent. We also have the relation
\[ \Pi^A B \Pi^A C \epsilon^C = \frac{1}{2} \left[ \Pi^A C + \Pi^A C - |\epsilon^C| \Pi^A C \right] \epsilon^C, \] (2.69)
so that once two of the three conditions are imposed, the third is automatically satisfied. Since each projector reduces in 1/2 the dimension of the space of allowed spinors, it follows that the dimension of the space of spinors satisfying equations (2.67) is 1/8 of the total and there is only one independent spinor. If these conditions guarantee that the KSEs are satisfied, this will imply that, in general, these configurations have 1/8 of the total supersymmetry. Also, note that the second condition in (2.67) is equivalent to
\[ i (\sigma^+)^A B \epsilon^B = \frac{1}{4} \tilde{J}^X Y \epsilon^A, \quad i (\sigma^+)^A B \epsilon_A = -\frac{1}{4} \tilde{J}^X Y \epsilon_B. \] (2.70)

In addition, the first condition in (2.67) fixes a chirality in the base space, and we obtain duality relations like
\[ \gamma^{mn} \epsilon^A = \frac{1}{2} \epsilon^m r \gamma_{\alpha} \epsilon^A, \] (2.71)
thus implying that $\gamma^{mn} \epsilon^A$ is self-dual.
Now we are going to prove that, indeed, there is always a Killing spinor fulfilling these properties for the configurations that satisfy the necessary conditions identified in the previous subsection.

We can start with the hyperino equation, \( \delta \psi^a = 0 \). First, since \( \gamma^+ e^A = 0 \) and \( \partial_- \phi^X = 0 \), this equation only involves derivatives \( \partial_m \phi^A \). Contracting this equation with \( V^Y_{ab} \), we obtain

\[
V^Y_{ab} \delta \gamma^m \left[ \partial_m \phi^Y \delta^A_B + \partial_m \phi^X J^{YX}_A \right] \epsilon_A = 0.
\]  
(2.72)

Now we make use of equation (2.70) and we obtain

\[
V^Y_{ab} \delta \gamma^m \left[ \partial_m \phi^Y - \partial_n \phi^X J^{YX} X^m n \right] \epsilon_A = 0.
\]  
(2.73)

Hence, on using equation (2.66) we see that the rhs vanishes, so that \( \delta \psi^a = 0 \) for the spinor \( \epsilon^A \). The gaugino equation is also satisfied for this spinor:

\[
\delta \lambda^A = - \frac{1}{2 \sqrt{2}} F^i \epsilon^A = - \frac{1}{2 \sqrt{2}} \left( 2 F^i_{mn} \gamma^{mn} e^A \right) e^A = 0,
\]  
(2.75)

where we have used that \( F^i_{a-} = 0 \), that \( \gamma^+ e^A = 0 \) and that \( \gamma^m e^A \) and \( \gamma^{mn} e^A \) have opposed chirality, so that their contraction is zero. On the other hand, for the tensorino equation we see that

\[
\delta \chi^M A = \frac{1}{2} \left[ \phi^2 V^M A + \frac{1}{6} H^M \right] e^A = 0.
\]  
(2.76)

In the second equality we have used \( \partial_- \phi^A = 0 \), \( H^M_{mn} = 0 \) and the projection \( \gamma^+ e^A = 0 \), in the third equality we have used the fact that \( H^M \) is self-dual and in the last equality we have used equation (2.50).

Hence, for every spinor satisfying the projections equations (2.67), and for the configurations obtained in the previous section, the KSEs \( \delta \psi^a = \delta \lambda^A = \delta \chi^M A = 0 \) always hold.

Finally, we have to prove that the gravitino KSE is also satisfied by a spinor constrained by equations (2.67). We can write the equation \( \delta \psi^A = 0 \) as

\[
\partial_a e^A = \frac{1}{4} (\omega_{abc} + H_{abc}) \gamma^b e^A - A^A_{\mu B} e^B.
\]  
(2.77)
By using equation (2.22) and $\gamma^+ e^A = 0$, we can simplify the rhs of this equation to

$$\partial_a e^A = \frac{1}{4} \left( \omega_{ama} + \mathcal{H}_{ama} \right) \gamma^{mn} e^A - A_a^A B e^B. \quad (2.78)$$

Taking now into account equation (2.24), this expression can be rewritten as

$$\partial_a e^A = \frac{1}{64} \left[ -e^{xyz} \partial_a J^x_{mn} J^y_{mn} + 8 A^z_a \right] e^A - A_a^A B e^B. \quad (2.79)$$

Then, using equation (2.70), we get

$$\partial_a e^A = -\frac{1}{64} e^{xyz} \partial_a J^x_{mn} J^y_{mn} e^A. \quad (2.80)$$

The $\nu$-independence of $J^x$ implies that of the Killing spinor. On the other hand, we can always find a basis of the tangent space such that the quaternionic structures take the form

$$J^1 = \nu^1 \wedge \nu^2 - \nu^3 \wedge \nu^4, \quad J^2 = \nu^1 \wedge \nu^3 + \nu^2 \wedge \nu^4, \quad J^3 = \nu^1 \wedge \nu^4 - \nu^2 \wedge \nu^3.$$ \quad (2.81)

In particular, the components in this basis are constant [24] $\partial_a J^x_{mn} = 0$. Therefore, in this basis any constant spinor satisfying the constraints equation (2.67) also solves the gravitino equation.

In conclusion, we have proven that all the configurations found in the section 2.1 are indeed supersymmetric and they admit a constant Killing spinor satisfying equation (2.67).

### 3. Supersymmetric solutions

In the previous section we have characterized the supersymmetric configurations of $d = 6$ ungauged supergravity in terms of a number of elementary building blocks (functions, forms, metric) satisfying certain first-order equations. Our goal, now, is to find under which conditions they are solutions of the equations of motion of the theory as well. The naive answer would be to say that those conditions are, precisely, the equations of motion; all of them. However, once we assume that the field configuration is supersymmetric, many of the equations of motion are equivalent or automatically solved and only a reduced number of them remain independent and nontrivial. This is precisely the magic that one is seeking for and the reason why finding supersymmetric solutions is, indeed, simpler.

In order to identify these independent equations of motion one can use the integrability equations of the KSEs, which are typically proportional to combinations of the equations of motion, or the so called Killing spinor identities (KSI) [32, 33]. These can be understood as projections over the supersymmetric configurations of the gauge identities associated to the local supersymmetry invariance of the theory. They are typically derived from the supergravity action assuming invariance under local supersymmetry transformations. In this case, doing this is not possible and we have just worked out the integrability conditions of the KSEs[^13].

[^13]: We suspect that the KSI may be derived from the pseudo-action, assuming that it can be supersymmetrized up to terms proportional to the self-duality constraints or by some other trick. However, it is not clear how to prove that the KSI-obtained in this way are indeed correct, except by direct comparison with those obtained from the integrability conditions, since nobody is actually going to find the required supersymmetrization of the pseudoaction.
3.1. Killing spinor identities

Let us start with the gravitino KSE, which we can write as

\[ D_\mu \epsilon^A = \frac{1}{4} \overline{\epsilon} \partial_\mu \epsilon^A. \] (3.1)

Its integrability condition, which must hold for the supersymmetric configurations that we have determined in section 2.1, are

\[ [D_\mu, D_\nu] \epsilon^A = \frac{1}{4} D_\mu (\overline{\epsilon} \partial_\nu \epsilon^A) - \frac{1}{4} D_\nu (\overline{\epsilon} \partial_\mu \epsilon^A). \] (3.2)

The commutator in the lhs of this equation takes the value

\[ [D_\mu, D_\nu] \epsilon^A = -\frac{1}{4} R_{\mu \nu}^{ab} \gamma^{ab} \epsilon^A + \partial_\mu \phi^X \partial_\nu \phi^Y F_{XY}^A \epsilon^B. \] (3.3)

leading to the identity

\[ \frac{1}{4} \left[ R_{\mu \nu}^{ab} \gamma^{ab} + 2 \overline{\gamma} \phi^X \phi^Y g_{XY} + \gamma^{\nu} \left( 2 \overline{\gamma} \phi^X \phi^Y + \frac{1}{2} \overline{\epsilon} \partial_\mu \partial_\nu \epsilon^A \right) \right] e^A - \partial_\mu \phi^X \partial_\nu \phi^Y F_{XY}^A \epsilon^B = 0. \] (3.4)

If we contract on it with \( \overline{\gamma}\nu \), we get

\[ \frac{1}{4} \left[ -2 R_{\mu \nu}^{ab} \gamma^a - 4 \partial_\mu \phi^X \phi^Y g_{XY} + \gamma^\nu \left( 2 \overline{\gamma} \phi^X \phi^Y + \frac{1}{2} \overline{\epsilon} \partial_\mu \partial_\nu \epsilon^A \right) \right] e^A = 0. \] (3.5)

Then, after a long computation in which we make use of the gaugino KSE (2.56), we rewrite this identity as

\[ \frac{1}{2} \left[ -E_{ab} \gamma^b - \overline{\gamma} \phi^X \phi^Y g_{XY} + \gamma^\nu \left( 2 \overline{\gamma} \phi^X \phi^Y + \frac{1}{2} \overline{\epsilon} \partial_\mu \partial_\nu \epsilon^A \right) \right] e^A = 0, \] (3.6)

where we recall that the different \( E \)-tensors represent the equations of motion as defined in equations (1.20)–(1.24), and

\[ C_a \equiv \frac{1}{2} L_\epsilon e^0 \left( 2 l_a f^f_{+} f^f_{+} + n_a f^f_{-} f^f_{-} - 4 f^f_{-} f^f_{-} \right). \] (3.7)

From equation (3.6) we can obtain several interesting relations among the equations of motion: if we contract it with \( \gamma^a \) and we assume that the 2-form equations are already satisfied we obtain

\[ E_r = 0 \Rightarrow E_a (l) = 0. \] (3.8)

Then, by taking into account that \( l_c = \delta^+ c \) and that \( E_{ab} \) is symmetric, we find that

\[ E_{a-} = E_{-a} = E_{mn} = E_{mn} = 0. \] (3.9)

Hence, once the 2-form equations of motion are satisfied, so are all the components of the Einstein equation, except for the \( ++ \) one.

Less interesting relations can be derived for the equations of the 2-forms, \( \mathcal{E}_r \). For example,

\[ L' \star \mathcal{E}_r \sigma \omega_{abc} = 0, \Rightarrow L' \star \mathcal{E}_r - m = 0, \quad L' \star \mathcal{E}_r mn = 0, \] (3.10)

but we shall not need them since we will compute the full equations of the 2-forms explicitly.

Let us next consider the tensorino KSE (2.44). Its derivative must also also vanish, and in particular, \( \overline{\mathcal{P}} \delta X^M = 0 \), where \( \overline{\mathcal{P}} = \gamma^a \partial_a \) is the space-time and Sp(1)-covariant derivative.
After some computations in which we make use of the different KSEs we get the following result:

\[ 0 = \mathcal{D}_\phi \chi^M = \frac{1}{2} \left[ V_\phi^M \mathcal{E}^\alpha + L^M \star \mathcal{E}_{r^a \gamma_\alpha} \right] \epsilon^A. \]  

(3.11)

From this equation it is evident that once the equations for the 2-forms are satisfied, the equations for the scalars are also satisfied,

\[ \mathcal{E}_r = 0, \Rightarrow \mathcal{E}_\alpha = 0. \]  

(3.12)

In addition, we can obtain the identities \( L^M \star \mathcal{E}_{r - m} = 0 \), and \( L^M \star \mathcal{E}_{r,m} J^r m \) = 0, implying together with equation (3.10) that

\[ \star \mathcal{E}_{r - m} = 0, \quad \star \mathcal{E}_{r,m} J^r m = 0. \]  

(3.13)

An explicit computation shows that \( \star \mathcal{E}_{r,m} = 0 \) identically for supersymmetric configurations. Thus, the only non-vanishing components of the 2-form equations are \( \star \mathcal{E}_{r, +m} \) and \( \star \mathcal{E}_{r, -m} \).

From the gaugino KSE (2.56) we can get the following interesting identity relating some components of the vector field equations and of the Bianchi identities,

\[ 0 = \tilde{e}_a \mathcal{D}_c \left( \gamma^{abc} L_{c'} \delta_b \chi^A \right) = \frac{1}{4 \sqrt{2}} \left[ \mathcal{J}^{[a} \star \mathcal{E}^{b]} - 2 L_{c'} c''_{c} \epsilon^{abcde} \partial_{[c} F_{d e]} F_d \right]. \]  

(3.14)

In obtaining this identity we have made use of several results of section (2.1).

This equation implies that \( \star \mathcal{E}_{r - m} = 0 \) and that, once the Bianchi identity \( (dF^r)_{mnr} = 0 \) is satisfied, then \( \star \mathcal{E}_{r, m} = 0 \) is also satisfied. In addition, since \( F^r_{a -} = 0 \) and there is no dependence in \( v \), one can see that the non-vanishing components of the Bianchi identities are \( (dF^r)_{mnr} \) and \( (dF^r)_{+mn} \). Hence, the only independent equations that one needs to impose are

\[ (dF^r)_{mnr} = 0, \quad (dF^r)_{+mn} = 0, \quad \star \mathcal{E}_{i} = 0. \]  

(3.15)

Finally, we have to determine whether the equation for the hyperscalars,

\[ \mathcal{E}^X = \mathcal{D}^\rho \partial_\rho \phi^X = 0, \]  

(3.16)

is satisfied. If we take into account that \( \partial_\rho \phi^X = 0 \), we can write it as

\[ \mathcal{E}^X = -2 \omega_{-m} \partial_m \phi^X + \mathcal{D}_m \partial_m \phi^X. \]  

(3.17)

On the other hand, we have the relation equation (2.66). By taking the covariant derivative there and by using that \( \mathcal{J}^X Y_i \) is covariantly constant we obtain

\[ \mathcal{D}_\rho \partial_m \phi^X = \mathcal{J}^{X,Y_i} \mathcal{D}_\rho \partial_{m} \phi^Y J^{n}_{n + m} + \mathcal{J}^{X,Y_i} \partial_{m} \phi^Y \mathcal{D}_\rho J^{n}_{n + m}. \]  

(3.18)

Now the covariant derivative of \( J^X \) can be read from equation (2.24) and we get

\[ \mathcal{D}_\rho \partial_m \phi^X = \mathcal{J}^{X,Y_i} \mathcal{D}_\rho \partial_{m} \phi^Y J^{n}_{n + m} + \mathcal{J}^{X,Y_i} \partial_{m} \phi^Y \left( J^{n}_{n + m} \mathcal{H}_{pm \rho} + J^{n}_{m \rho} \mathcal{H}_{p \rho \gamma} \right). \]  

(3.19)

Then, if we contract \( m \) and \( p \) we obtain

\[ \mathcal{D}_m \partial^m \phi^X = -\delta_{nm} \mathcal{D}_\rho \partial_m \phi^X = -\mathcal{J}^{X,Y_i} \partial_{m} \phi^Y J^{n}_{n + m} \mathcal{H}_{cm \rho} + \mathcal{J}^{X,Y_i} \partial_{m} \phi^Y J^{n}_{n + m} \mathcal{H}_{cm \rho} \]

\[ = 2 \mathcal{J}^{X,Y_i} \partial_{m} \phi^Y J^{n}_{n + m} \mathcal{H}_{cm \rho} - 2 \delta^{rn} \phi^Y \mathcal{H}_{+ - m} = -2 \delta^{rn} \phi^X \mathcal{H}_{+ - m}. \]  

(3.20)

where we used that \( \mathcal{D}_\rho \partial_{m} \phi^Y = 0 \) and we recall that the indices of \( J^{n}_{m \rho} \) are raised and lowered with \( + \delta^{rn} \) instead of \( - \delta^{rn} \) by definition.
Finally, we take into account equation (2.22) from where we get \( \mathcal{H}_{+-m} = \omega_{+m} = -\omega_{-+m} \). Hence, we have proven that \( D_m \partial^m \phi^N = 2 \partial^m \phi^L \omega_{+-m} \), thus implying that the equation of the hyperscalars is automatically satisfied.

Summarizing, we have found that the only equations that we have to solve are

\[
\mathcal{E}_{++} = 0, \quad *\mathcal{E}_{+-} = 0, \quad *\mathcal{E}_{+m} = 0, \\
*\mathcal{E}_{i+} = 0, \quad (dF^i)_mnr = 0, \quad (dF^i)_{+mn} = 0. \tag{3.21}
\]

3.2. Equations of motion

In the previous section we learned that we only have to solve the 2-form and vector field equations, the component ++ of the Einstein equation and the Bianchi identities of the vector field strengths and that some components of these equations are already satisfied. We are now going to find the form that the remaining equations have in terms of the building blocks of the supersymmetric solutions.

The 2-form field equations only have two non-trivially-satisfied components:

\[
\ddt \left\{ L_r \left[ fG^- + f^{-1} (J^r A^r_u - 2 \psi) \right] + 2f\chi_r \right\} \\
- \partial_u \left[ \beta \wedge \left[ L_r \left( fG^- + f^{-1} (J^r A^r_u - 2 \psi) \right) + 2f\chi_r \right] \right] \\
- \delta_r [D(f^{-1} L_r) + f^{-1} L_r \beta] + 2fe_r \theta^i \wedge F^i = 0. \tag{3.22}
\]

The Bianchi identities for the vectors have another two non-trivial sets of components:

\[
\ddt F^i + f \beta \wedge \theta^i - D(f \theta^i) = 0, \tag{3.24}
\]

\[
D\ddt F^i + f D\beta \wedge \theta^i = 0. \tag{3.25}
\]

The equation of the vector fields only have one non-trivial component:

\[
c^r_{ij} \left[ \ddt D(f^{-1} L^r) \wedge \theta^j - f^{-1} D(\ddt L^r \theta^j) + (\ddt L^r + 2 \chi_r) \wedge L^j \right] = 0, \tag{3.26}
\]

and, finally, the only non-trivial component of the Einstein equations reads\(^\text{14}\)

\[
- \ddt^2 H + \ddt^m \omega_m - \beta_m (\ddt^m - \partial^m H) - (\ddt^m - \partial^m H) \left( 2\beta_m + 2\psi_n \beta_n + \ddt^m \psi_m \theta \right) \\
- \frac{1}{4} f^2 G^{-2} + f^2 \left( \psi - \frac{1}{2} \ddt J^r A^r_u \right)^2 + \left( \ddt^m - \frac{1}{2} \ddt J^m A^m_u \right) G^+_{mn} + 5f^{-4} \ddt^2 \\
- 2f^{-3} \ddt^2 + \partial_u \left( f^{-2} \ddt^m \psi_m \right) + f^{-2} \ddt^m \ddt^m \ddt^2 \ddt \ddt^r + f^{-2} \ddt^m \ddt^m \ddt^2 \ddt^2 \ddt + 2\psi_n \ddt^m \ddt^m \ddt^m + 2g_{xy} f^{-2} \ddt^m \ddt^m \ddt^m = 0. \tag{3.27}
\]

\(^14\) The component \( R_{++} \) of the Ricci tensor is computed in appendix B.
where, for a 2-form \(G = \frac{1}{2} G_{\mu\nu} dx^\mu \wedge dx^\nu\) we define \(G^2 = G_{\mu\nu} G^{\mu\nu}\). Also, in the previous expressions, the four-dimensional indices are raised with \(+\delta^{\mu\nu}\), for example, \(\omega^m \beta^n = +\delta^{mn} \omega_n \beta_m\) and so on.

3.3. Solving the equations

The preceding equations are highly coupled and non-linear, and solving them is a considerably hard task. Nonetheless, we can sketch a possible procedure which one would ideally use in order to solve the equations. Since the main source of complication comes from the \(u\)-dependence, by demanding independence from this coordinate the system of equations can be recast in a triangular form and it is possible to construct explicit solutions upon choice of a base space.

3.3.1. Base space. The first thing we must do is to find a base space, for which we have to find a solution to the system of equations (2.43) and (2.37). The latter can be written in a more suggestive way as follows

\[
\hat{\nabla}_r J^x_{mn} + \epsilon^{xyz} \hat{A}_y^r J^z_{mn} = 0,
\]

(3.28)

where

\[
\hat{A}^r_x = A^r_x - \beta_r A^r_u + \frac{1}{8} \epsilon^{xyz} \partial_u J^y_{rs} J^z_{rs}.
\]

(3.29)

and \(\hat{\nabla}\) is a torsionful connection whose components \(\hat{\omega}_{mun}\) are determined by

\[
\left( \frac{d - \beta \wedge \partial_u}{+\frac{1}{2} \beta \wedge} \right) \tau^m + \hat{\omega}_m^m \wedge \tau^a = 0.
\]

(3.30)

Note that in the \(u\)-independent and \(\beta = 0\) cases this is just the usual spin connection. Now, in a frame in which the components \(J^x_{mn}\) are constant, the equation (3.28) becomes an algebraic relation between the \(\text{Sp}(1)\) connection \(A^r_x\) and the self-dual part of the connection \(\hat{\omega}^r\):

\[
\hat{\omega}^r_{mun} = -\frac{1}{6} (A^r_x - \beta_r A^r_u) J^x_{mn}.
\]

(3.31)

In the case of vanishing hyperscalars, this equation simply tells us that the connection \(\hat{\omega}\) is anti-self-dual. Moreover, if there is no dependence on \(u\), \(\hat{\omega}\) coincides with the spin connection and therefore the space is hyper-Kähler.

3.3.2. Simplification of the equations. We can in general simplify the equations of motion by introducing auxiliary quantities and further decompositions, regardless of whether we have determined the base space. The following decomposition of the vector fields is useful:

\[
\theta^I = f^{-1} (Dz^I - z^I \beta), \quad \tilde{F}^I = d\tilde{A}^I - z^I D\beta.
\]

(3.32)

In addition, we introduce the following auxiliary quantities

\[
\Sigma_r \equiv f (L_r G^- + 2 \chi_r) + 2 c_r g^r \tilde{d} \tilde{A}^l - c_r g^r \tilde{z} D^l \beta + J^r A^z_u = 2 \psi,
\]

(3.33)

\[
L \equiv \tilde{d} H - \omega - d \left( f^{-1} L_r c_r g^r \tilde{z} z^l \right) + \partial_u \left( \beta f^{-1} L_r c_r g^r \tilde{z} z^l \right) .
\]

(3.34)

We will treat \(\Sigma_r\) and \(L\) as independent fields which once determined can be used to obtain \(\chi_r\), \(G^-\) and \(dH\). The equations of motion are simplified and take the form
\[ \partial_u \tilde{A}^i = 0, \quad (\tilde{A}^i)^+ = 0, \] 
\[ \Sigma_r^+ - J^+ A^r_u + 2\psi = 0, \] 
\[ \tilde{d}\Sigma_r + \partial_u \left[ \tilde{\ast}(D(f^{-1}L_r) + f^{-1}L_r\beta) - \beta \wedge \Sigma_r \right] = 0, \] 
\[ \tilde{d} \left[ \tilde{\ast}(D(f^{-1}L_r) + f^{-1}L_r\beta) - \beta \wedge \Sigma_r + c_{r ij} \tilde{A}^i \wedge \tilde{d}\tilde{A}^j \right] = 0, \] 
\[ c_{r ij} \left[ -\tilde{d}\ast D(f^{-1}L_r') + \partial_{u} \left( \beta \wedge \ast D(f^{-1}L_r') \right) + \Sigma' \wedge \tilde{d}\tilde{A}^i \right] = 0, \] 
\[ \chi_r = \frac{1}{2} \left[ f^{-1} \Sigma_r^+ - L_r G^- \right], \] 
\[ G^- = f^{-1} L' \left[ \Sigma_r^+ - 2c_{r ij} \tilde{A}^i + c_{r ij} z^j D\beta \right], \] 
while \( \omega \) and \( H \) should be determined by solving the equations 
\[ \tilde{d}H - \omega = L + d \left( f^{-1} L' c_{r ij} z^j \right) - \partial_u \left( \beta f^{-1} L' c_{r ij} z^j \right) , \] 
\[ (d\omega)^- = G^- + \left( L + d \left( f^{-1} L' c_{r ij} z^j \right) - \partial_u \left( \beta f^{-1} L' c_{r ij} z^j \right) \right) \wedge \beta , \] 
where the right hand side is supposed to be known.

We see that the main difficult of solving the equations above, apart from determining the base space, \( \beta \), and the complex structures \( J' \). In the presence of hyperscalars this step gets coupled with the rest of equations but if we truncate the hyperscalars, it can be carried out independently.

1. First, one determines the base space, \( \beta \), and the complex structures \( J_x \). In the presence of hyperscalars this step gets coupled with the rest of equations but if we truncate the hyperscalars, it can be carried out independently.

2. Second, one solves the system of equations above. The equations are supposed to be solved in the given order, so, ideally, one would find in sequence \( \tilde{A}^i, \Sigma_r, f^{-1}L_r, z^j \) and \( L \). However, as we can see, these equations are not in a triangular form, so finding a solution is not straightforward.

3. Finally, one has to extract the information from the auxiliary fields \( \Sigma_r \) and \( L \): the 2-forms \( \chi_r \) can be obtained by using
In the \( u \)-independent case, the base space reduces to a hyper-Kähler space when the hyperscalars are truncated. The equations of motion get the form of a triangular system:

\[
(\tilde{\partial}A)^+ = 0, \quad (3.46)
\]

\[
\tilde{\partial}\Sigma_r = 0, \quad (3.47)
\]

\[
\Sigma^+_r = 0, \quad (3.48)
\]

\[
\tilde{\partial}\left[ s\tilde{\partial}(f^{-1}L_r) - \beta \wedge \Sigma_r + c_r y \tilde{\partial} \wedge \tilde{d}A^i \right] = 0, \quad (3.49)
\]

\[
c_r \tilde{\partial} \left[ -\tilde{d}s\tilde{\partial}(f^{-1}L'_r \tilde{\partial}^i) + \Sigma'_r \wedge \tilde{d}A^i \right] = 0, \quad (3.50)
\]

\[
\tilde{\partial}s\tilde{\partial} \left( H - f^{-1}L' c_r \tilde{\partial} \tilde{\partial}^i \tilde{\partial}^j \right) + \frac{1}{2} \Sigma'_r \wedge \Sigma'_r = 0, \quad (3.51)
\]

\[
(\tilde{\partial}\omega)^- = f^{-1}L' \left[ \Sigma_r - 2c_r \tilde{\partial} \tilde{\partial} A^i + c_r y \tilde{\partial} \tilde{\partial} \beta \right] - (\tilde{\partial}H \wedge \beta)^- = 0, \quad (3.52)
\]

\[
\chi_r - \frac{1}{2f} \left[ \Sigma_r - L_s L' \left( \Sigma_r - 2c_s \tilde{\partial} \tilde{\partial} A^i + c_s y \tilde{\partial} \tilde{\partial} \beta \right) \right] = 0. \quad (3.53)
\]

These equations can be solved step-by-step (in the given order) once the base space is determined. In the case of a hyper-Kähler base space (vanishing hyperscalars) a common technique consists in assuming the existence of one isometry, which allows to write the metric \( \gamma_{mn} \) in a Gibbons–Hawking form. With this choice it is possible to solve the preceding equations explicitly, as it was originally done in [3]. For a sake of completeness we do that next, but before so let us note that all these solutions can be obtained by uplifting 5-dimensional solutions. Indeed, since the coordinate \( u \) is isometric in these solutions, there always exists a space-like isometry which is a combination of the null isometry and the \( u \)-isometry. Dimensional reduction along this space-like direction will produce time-like solutions of 5-dimensional supergravity. Hence, all of the \( u \)-independent solutions can be obtained by uplifting the 5-dimensional time-like solutions, which are already known [15–18], using the map derived in [26].

3.3.3. Base space with one isometry. Further simplification of the equations can be achieved in absence of hyperscalars (so the base space is hyper-Kähler) by assuming, further, that the base space has a triholomorphic isometry\(^{15}\). The metric of the base space, then, is a Gibbons–Hawking metric of the form

\[
\gamma_{mn} dx^m dx^n = h^{-1} (d\phi + \chi)^2 + h dx^i dx^i, \quad i = 1, 2, 3, \quad (3.54)
\]

\(^{15}\) Since, as we have explained, the 6-dimensional \( u \)-independent supersymmetric solutions are the lift of the 5-dimensional timelike supersymmetric solutions, this case is equivalent to the timelike 5-dimensional case with one additional triholomorphic isometry in the base space or, depending on the choice of compact dimension, to the \( u \)-independent null 5-dimensional case. The interest of this exercise is that it facilitates the comparison between 6- and 5-dimensional supersymmetric solutions.
where the function $h$ and the 1-form $\chi$ satisfy\textsuperscript{16}
\[ \star_3 d h = - d \chi. \] (3.55)
In order to simplify the equations, let us first note that equation (3.47) implies that, locally,
\[ \Sigma_r = d \sigma_r \] (3.56)
for some 1-forms $\sigma_r$. Then, if we further decompose the fields as
\[ \beta = - \beta_6 h^{-1}(d \varphi + \chi) + \bar{\beta}, \] (3.57)
\[ \bar{A}^i = - \phi^i h^{-1}(d \varphi + \chi) + \bar{A}^i, \] (3.58)
\[ \sigma^r = - \phi^r h^{-1}(d \varphi + \chi) + \bar{\sigma}^r \] (3.59)
\[ f^{-1} L_r = h^{-1} \left[ c_{ij} \phi^i \phi^j - \beta_6 \phi_r \right] + \psi_r, \] (3.60)
\[ c_{ij} f^{-1} L_z^{ij} = - c_{ij} h^{-1} \phi^i \phi^j + \xi, \] (3.61)
\[ H = f^{-1} L' c_{ij} \phi^i \phi^j + \frac{1}{2} h^{-1} \phi_r \phi^r + \Lambda, \] (3.62)
we obtain the following set of equations for the scalars $\beta_6, \phi^i, \phi^r, \psi_r, \xi$ and $\Lambda$ and the 3-dimensional 1-forms $\bar{\beta}, \bar{A}^i$ and $\bar{\sigma}^r$:
\[ d \bar{\beta} + \star_3 d \beta_6 = 0, \] (3.63)
\[ d \bar{A}^i + \star_3 d \phi^i = 0, \] (3.64)
\[ d \bar{\phi} + \star_3 d \phi' = 0, \] (3.65)
\[ d \star_3 d \psi_r = 0, \] (3.66)
\[ d \star_3 d \xi = 0, \] (3.67)
\[ d \star_3 d \Lambda = 0. \] (3.68)

The functions $\psi_r, \xi$ and $\Lambda$ are harmonic on $E^3$, and the integrability conditions of equations (3.55) and equations (3.63)-(3.65) imply that the functions $h, \beta_6, \phi^i$ and $\phi'$ are also harmonic. Now we must determine $\omega$. It is useful to decompose it in the following way
\[ \omega = \omega_6 (d \varphi + \chi) + \tilde{\omega}, \] (3.69)
where $\tilde{\omega}$ is a 1-form in $E^3$. In the process of simplifying equation (3.52) one finds useful the following decomposition of $\omega_6$
\[ \omega_6 = \tilde{\omega}_6 - 2 c_{ij} \phi^i \phi^j \frac{1}{h^2} + \frac{\beta_6 \phi_r \phi^r}{h^2} - \frac{H h_6}{h}, \] (3.70)

\textsuperscript{16} The reason for the negative sign is that, in the conventions we are using, the complex structures $J_{mn}$ are self-dual. This implies that the spin connection must be anti-self-dual, and this is achieved if $\star_3 dh = - d \chi$. For the sake of clarity, in this subsection the symbol $d$ denotes the exterior derivative in the 3-dimensional Euclidean space $E^3$ with metric $dt^i dt^i$. 

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and, in terms of $\dot{\omega}_6$, equation (3.52) reads

$$d \left( \dot{\omega} - H \tilde{\beta} \right) = \star_3 \left[ -h^2 d \left( \frac{\dot{\omega}_6}{h} \right) - 2h \psi d \left( \frac{\phi}{h} \right) + 4h \xi d \left( \frac{\phi'}{h} \right) + 2h \lambda d \left( \frac{\beta_6}{h} \right) \right].$$

(3.71)

The integrability condition of this equation provides us with the equation for $\dot{\omega}_6$:

$$0 = h \mathbf{d} \star_3 d \left[ -\dot{\omega}_6 - \frac{\psi' \phi}{h} + 2 \xi \phi' h + \Lambda \beta_6 h \right],$$

(3.72)

whose solution can be written as

$$\dot{\omega}_6 = M - \frac{\psi' \phi}{h} + 2 \xi \phi' h + \frac{\Lambda \beta_6}{h},$$

(3.73)

where $d \star_3 dM = 0$.

Taking this result into account we may rewrite equation (3.71) as

$$d \left( \dot{\omega} - \frac{1}{2} \tilde{\beta} H \right) = \star_3 \left[ M \mathbf{d}h - h \mathbf{d}M + \phi d \psi' - \psi' d \phi + 2 \xi \mathbf{d} \phi' - 2 \phi' d \xi + \Lambda d \beta_6 - \beta_6 d \lambda \right],$$

(3.74)

whose integrability condition is now manifestly satisfied.

Therefore, the complete solution is determined by specifying the set of harmonic functions $h, \beta_6, \phi, \phi', \psi, \xi, \Lambda, M$, from which it is straightforward to obtain the 1-forms $\chi, \tilde{\beta}, \tilde{\Lambda}', \phi^r$ and $\tilde{\omega}$ by simple integration of equations (3.55), (3.63)–(3.65) and (3.74). We get the functions $L_r$ and $f$ from equation (3.60) and the condition $L_r L' = 1$. Explicitly, we have for $f$

$$f^{-2} = \left\{ h^{-1} \left[ c_{\mathbf{r}_y} \phi'^j - \beta_6 \phi \right] + \psi \right\} \left\{ h^{-1} \left[ c_{\mathbf{r}_y} \phi'^j - \beta_6 \phi \right] + \psi \right\} \phi'^{\prime j}. $$

(3.75)

Also, from equation (3.61), we can obtain $z'$, for which we need to solve a linear system of equations once an specific $c_{\mathbf{r}_y}$ has been chosen. We can write symbolically

$$z' = \left[ c_{\mathbf{r}_y} f^{-1} L' \right]^{-1} \left( -c_{\mathbf{r}_y} h^{-1} \phi'^j + \xi \right),$$

(3.76)

where $\left[ c_{\mathbf{r}_y} f^{-1} L' \right]^{-1}$ denotes the inverse matrix in $ij$ indices. Once the scalars $z'$ are determined, $H$ is given by equation (3.62), and one can compute the anti-self-dual 2-forms $\chi_r$ from equation (3.53). Thus, we have determined all the building blocks of the fields, and these can be written explicitly. In particular, let us note that the vectors $\tilde{\Lambda}'$ are given by

$$\tilde{\Lambda}' = -d u z' + (d \phi + \chi) h^{-1} (\beta_6 z' - \phi') + \tilde{\Lambda}' - z' \tilde{\beta}. $$

(3.77)

4. Summary

In a supersymmetric configuration, the fields are $\nu$-independent and have the form

$$d s^2 = 2 f (d u + \beta) (d v + H d u + \omega) - f^{-1} \gamma_{\text{max}} d x^a d x^5, $$

(4.1)

As it is well known [50, 51], the harmonic functions generically considered have singularities and the integrability condition will not be automatically satisfied there: additional conditions on the integration parameters of the harmonic functions have to be met.
\[ H' = \frac{1}{2} f^{-1} e^+ \wedge e^- \wedge (D(fL') - \beta fL') + \frac{1}{2} fL' e^- \wedge D\beta \]
\[- \frac{1}{2} \ddot{s}(D(f^{-1}L') + f^{-1}L'\dot{\beta}) \]
\[+ e^+ \wedge \left\{ \chi' + L' \left[ f^{-2} \left( -\dot{\psi} + \frac{1}{2} J^u A^u \right) - \frac{1}{2} G' \right] \right\}, \quad (4.2) \]
\[ F^i = e^+ \wedge \theta^i + \tilde{F}^i, \]
\[ \partial_m \phi^X = J^{AX} \partial_n \phi^Y J^n_m, \quad (4.4) \]
where
\[ G = D\omega - \bar{\partial}H \wedge \beta, \quad \text{and} \quad \psi = \frac{1}{16} e^{\bar{\omega}} J^u \tilde{J}^u \tilde{J}^u, \quad (4.5) \]
and where the derivative \( D \) is defined as
\[ D\alpha = \bar{\partial} \alpha - \beta \wedge \dot{\alpha}. \quad (4.6) \]
In addition, the quantities that appear in these expressions (the building blocks of the supersymmetric configurations) satisfy the following properties:
\[ \check{s} J^s = + J^s, \quad \check{s} D\beta = - D\beta, \quad \check{s} \tilde{F}^i = - \tilde{F}^i, \quad \check{s} \chi' = - \chi', \quad L_n \chi' = 0, \quad (4.7) \]
\[ \check{\nabla} r J^s_{mn} + e^{\bar{\omega}} A^Y r J^F_{mn} = \beta \epsilon e^{\bar{\omega}} A_u J^F_{mn} + \beta_i \partial_n J^s_{mn} - \delta_{[m} J^s_{n]} \beta, \]
\[ + J^s_{r[m]} \beta_n - 2 J^s_{r[m]} U_{n]} j\epsilon, \quad (4.8) \]
where
\[ U_{mn} \equiv - \check{\nabla}_{[n} \beta_{|m]} + \check{\nabla}_{[m} \beta_{|n]} - \check{\nabla}_{[m} \beta_{|n]}, \quad (4.9) \]
\[ A^Y = A^Y \check{\partial} \phi^X \] is the pullback of the Sp(1) connection onto the base space, and \( A^Y_u = A^Y \check{\partial} \phi^X \).
Moreover, the complex structures \( J^s \) satisfy the quaternionic algebra. In the cases \( \beta = 0 \) or \( u \)-independent these complex structures are Sp(1)-covariantly constant.
The previous configurations allow for one Killing spinor \( \epsilon^A \) which is constant in the basis in which the complex structures \( J^F_{mn} \) are constant, and which satisfies
\[ \gamma^+ \epsilon^A = 0, \quad \Pi^{\lambda A} B^B = 0, \quad (4.10) \]
where
\[ \Pi^{\lambda A} B^B = \frac{1}{2} \left[ \delta^A_B + \frac{i}{4} \check{\gamma} (\sigma^+)^B \right]. \quad (4.11) \]
Finally, the field equations that must be solved for these configurations are
\[ \check{\partial} \left\{ L_r \left( fG^+ + f^{-1}(J^A A^u - 2\psi) \right) + 2fX_r \right\} \]
\[- \partial_n \left\{ \beta \wedge \left[ L_r \left( fG^+ + f^{-1}(J^A A^u - 2\psi) \right) + 2fX_r \right] \right\} \]
\[- \dot{s} \left\{ D(f^{-1}L_r) + f^{-1}L_r \beta \right\} \right\} + 2f c_{r} \sigma \wedge \tilde{F}^r = 0, \quad (4.12) \]
\[ D\tilde{\bar{\chi}} (f^{-1} \Gamma_{\tilde{\bar{\omega}}}) + f^{-1} \left[ \Gamma_{\tilde{\bar{\omega}}} \right] + (\Gamma_{\tilde{\bar{\omega}}} G - 2 \chi) + c_{r ij} (\tilde{\bar{\omega}}^i \wedge \tilde{\bar{\omega}}^j ) = 0. \]  

(4.13)

\[ \tilde{\bar{\omega}}^i + f \tilde{\bar{\omega}}^i \wedge \theta^i - D(f \theta^i) = 0. \]  

(4.14)

\[ D\tilde{\bar{\omega}}^i + fD\tilde{\bar{\omega}}^i \wedge \theta^i = 0. \]  

(4.15)

\[ c_{r ij} \left[ \tilde{\bar{\omega}}^i (f^{-1} \Gamma_{\tilde{\bar{\omega}}}) \wedge \theta^j - f^{-1} \left[ \tilde{\bar{\omega}}^i \right] \wedge \Gamma_{\tilde{\bar{\omega}}} + (\Gamma_{\tilde{\bar{\omega}}} G + 2 \chi) \wedge \tilde{\bar{\omega}}^i \right] = 0. \]  

(4.16)

A simplification of these equations is explained in section 3.3.2.

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Appendix A. Gamma matrices, spinors and bilinears

A.1. Gamma matrices and spinors in \( d = 6 \)

We choose the mostly \(-\) signature for the Minkowski metric

\[ \eta_{ab} = \text{diag}(+1, -1, -1, -1, -1, -1). \]  

(A.1)

The gamma matrices are defined through the relation

\[ \{ \gamma_a, \gamma_b \} = 2\eta_{ab}. \]  

(A.2)

In addition to this, we will choose that they are antisymmetric:

\[ \gamma^a = -\gamma_a. \]  

(A.3)

As usual, it is also define

\[ \gamma^a = \gamma_0 \gamma^1 \gamma^2 \gamma^3 \gamma^4 \gamma^5 = \frac{1}{6!} \epsilon^{abcdef} \gamma_a \gamma_b \gamma_c \gamma_d \gamma_e \gamma_f. \]  

(A.4)
which satisfies that $\gamma_7^2 = 1$ and $\gamma_7^T = -\gamma_7$. Thus, it is Hermitian and purely imaginary. We will use the following notation for the antisymmetrized product of gamma matrices

$$\gamma_{(n)} = \gamma_{a_1 a_2 ... a_n} = \gamma_{[a_1 a_2 ... a_n]}.$$  \hfill (A.5)

and the duality relation reads

$$\gamma_{b_1 ... b_n} = (-1)^{[n/2]} \frac{(6 - n)!}{n!} \epsilon_{b_1 ... b_n a_1 ... a_n} \gamma.$$  \hfill (A.6)

The following useful identities are satisfied:

$$\gamma_{abc} \gamma_{d} \gamma_{abc} = 0, \quad \gamma_{a} \gamma_{bcd} \gamma_{a} = 0, \quad \gamma_{abc} \gamma_{def} \gamma_{abc} = 0.$$  \hfill (A.7)

A.1.1. Reduction to five dimensions. We want to relate the previous antisymmetric representation of the six-dimensional $\gamma$-matrices to the five dimensional gammas. In $d = 5$ there is a unitary representation of gamma matrices, which we call $\tilde{\gamma}_i$, $i = 0, 1, 2, 3, 4$, that satisfies

$$\tilde{\gamma}_0^\dagger = \tilde{\gamma}_0, \quad \tilde{\gamma}_i^\dagger = -\tilde{\gamma}_i, i \neq 0.$$  \hfill (A.8)

Moreover, $\tilde{\gamma}_4$ is real and the rest purely imaginary. Now, from these matrices we can construct the six-dimensional ones by means of the following definitions

$$\hat{\gamma}_i = \tilde{\gamma}_i \otimes \sigma^1, \quad i = 0, ..., 4$$  \hfill (A.9)

$$\hat{\gamma}_5 = 1 \otimes i\sigma^2,$$  \hfill (A.10)

$$\hat{\gamma}_7 = \tilde{\gamma}_0 \hat{\gamma}_i \hat{\gamma}_4 \tilde{\gamma}_0 \otimes \sigma^1.$$  \hfill (A.11)

where $\sigma^i$ are the Pauli matrices. The matrices $\hat{\gamma}_a$, $a = 0, ..., 5$, satisfy the six-dimensional Clifford algebra; However, not all of them are antisymmetric. In order to get an antisymmetric representation we perform the following similarity transformation

$$\gamma^a = S \hat{\gamma}^a S^{-1},$$  \hfill (A.12)

where $S$ and its inverse are given by

$$S = \frac{1}{\sqrt{2}} (\tilde{\gamma}_0 \otimes \sigma^1 + \tilde{\gamma}_4 \otimes \sigma^3), \quad S^{-1} = \frac{1}{\sqrt{2}} (\tilde{\gamma}_0 \otimes \sigma^1 - \tilde{\gamma}_4 \otimes \sigma^3).$$  \hfill (A.13)

Then the $\gamma^a$'s are an antisymmetric representation of the six-dimensional Clifford algebra: $\gamma_7^a = -\gamma_a$. The explicit relation with the five-dimensional gammas is

$$\gamma^0 = \tilde{\gamma}_0 \otimes \sigma^1,$$  \hfill (A.14)

$$\gamma^i = \tilde{\gamma}_i \gamma^0 \otimes \sigma^3, \quad i = 1, 2, 3,$$  \hfill (A.15)

$$\gamma^4 = -\tilde{\gamma}_4 \otimes \sigma^1,$$  \hfill (A.16)

$$\gamma^5 = 1 \otimes i\sigma^2 + \tilde{\gamma}_0 \tilde{\gamma}_4 \otimes 1,$$  \hfill (A.17)

$$\gamma_7 = \tilde{\gamma}_0 \gamma^0 \otimes \sigma^1.$$  \hfill (A.18)

Note that $\gamma_7^2 = 1$ and it is imaginary and antisymmetric, as it should be.
A.1.2. Majorana–Weyl symplectic spinors. In $\mathcal{N} = (1, 0), d = 6$ supergravity we use Majorana–Weyl symplectic spinors. These are a pair spinors, $\chi^A, A = 0, 1$, such that they satisfy the Weyl condition
\[ \gamma_7 \chi^A = s \chi^A, \] (A.19)
where $s = \pm 1$ is the chirality, and a reality condition
\[ (\chi^A)^T = \bar{\chi}^A, \] (A.20)
where the Dirac conjugate, $\bar{\chi}^A$, is defined by
\[ \bar{\chi}^A = (\chi^A)^\dagger \gamma_0. \] (A.21)

The $\text{Sp}(1)$ indices $A, B$, can be raised and lowered in the following way
\[ \chi^A = \epsilon^{AB} \chi^B, \quad \chi_B = \chi^A \epsilon_{BA}, \quad \epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \] (A.22)

If $\gamma_7 \chi^A = s \chi^A$, then its Dirac conjugate satisfies
\[ \bar{\chi}^A \gamma_7 = (\chi^A)^\dagger \gamma_0 \gamma_7 = - (\gamma_7 \chi^A)^\dagger \gamma_0 = -s \bar{\chi}^A. \] (A.23)

Therefore, if $\chi^A$ and $\lambda^A$ have, respectively, chiralities $s_1$ and $s_2$, we get a relation for the bilinears formed with these two spinors:
\[ \bar{\lambda}^A \gamma_{(n)} \chi^B = (-1)^{n+1} s_1 s_2 \bar{\lambda}^A \gamma_{(n)} \chi^B. \] (A.24)

Now, by using the identities (A.6), we see that the bilinears with $n > 3$ are related to those with $n \leq 3$. In particular, if both spinors have the same chirality, the only non-vanishing bilinears are those with $n = 1, 3$, and if the spinors have opposed chirality, then $n = 0, 2$ are the only bilinears.

A.1.3. Fierz identities. The antisymmetrized product of gamma matrices provide a basis for the space of $8 \times 8$ matrices. Moreover, by using the identities (A.6), we can construct the following basis:
\[ \{ O^i \} = \{ 1, \gamma^a, i\gamma_{ab}, i\gamma_{abc}, \gamma^{a\gamma_{ab}}, \gamma^{a\gamma_{abc}} \}, \] (A.25)
and the dual basis
\[ \{ O_J \} = \{ 1, \gamma_{(a}, i\gamma_{ab}, i\gamma_{a\gamma_{abc}}, \gamma_{(a\gamma_{ab})}, \gamma_{(a\gamma_{abc})} \}. \] (A.26)

The scalar product of two matrices $M, N$ is defined as the trace of $MN$, and with respect to this scalar product, the basis is orthogonal:
\[ \text{Tr}(O_J O^I) = 8 \delta^I_J. \] (A.27)

This allows us to expand every matrix $M$ in this basis. In particular, if $\chi$ and $\psi$ are spinors with the same chirality, $s$, and $M$ and $N$ are matrices, we get the following identity for the product of bilinears
\[ (\bar{\lambda} M \chi) (\bar{\psi} N \varphi) = \frac{1}{8} \left[ \bar{\lambda} M \gamma^a (1 - s \gamma) N \varphi \right] (\bar{\psi} \gamma_a \chi) - \frac{1}{48} \left( \bar{\lambda} M \gamma^{abc} N \varphi \right) (\bar{\psi} \gamma_{abc} \chi). \] (A.28)
A.2. Spinor bilinears

Given a spinor $\kappa^A$, we want to construct all the possible bilinears with it and to determine their properties. Here we will assume that $\kappa^A$ has positive chirality $\gamma^7 \kappa^A = + \kappa^A$. According to our previous discussion, the only bilinears that can be constructed are a matrix of vectors and a matrix of 3-forms. Let us define them more precisely:

$$ V^A_{\ B \ a} \equiv \bar{\kappa}^A \gamma_a \kappa_B, \quad W^A_{\ B \ abc} \equiv i \bar{\kappa}^A \gamma_{abc} \kappa_B. $$

Now, let us take both indices down in the matrix of vectors and let us use the reality condition equation (A.20) and the antisymmetry of gamma matrices:

$$ V_{AB \ a} = \bar{\kappa}^A \gamma_a \kappa_B = (\bar{\kappa}^A \gamma_a \kappa_B)^T = \kappa^T_B \gamma_a \bar{\kappa}^T_A = -\kappa_B \gamma_a \kappa^A = -V_{BA \ a}. $$

This implies that $V_{AB \ a} = \frac{1}{2} \eta_{BA} l_a$, or

$$ V^A_{\ B \ a} = \frac{1}{2} \delta^A_B l_a. \quad l_a = \bar{\kappa}^A \gamma_a \kappa_A. $$

Hence, we only have one vector field. On the other hand, it can be shown that, with both indices down, $W_{AB \ abc}$ is symmetric:

$$ W_{AB \ abc} = W_{BA \ abc}. $$

Now, since the indices are raised with $\epsilon_{AB}$, this implies that the matrix of 3-forms is traceless, $W^A_{\ A \ abc} = 0$. Moreover, we can compute the complex conjugate of $W$:

$$ (W^A_{\ B \ abc})^* = (i \bar{\kappa}^A \gamma_{abc} \kappa_B)^\dagger = W^B_{\ A \ abc}. $$

This means that the matrix $W^A_{\ B }$ is hermitian and traceless. Therefore, we can expand it as a linear combination of Pauli matrices

$$ W^A_{\ B \ abc} = \frac{1}{2} (\sigma^x)^A_{\ B} W^x_{\ abc}, \quad \Leftrightarrow \quad W^x_{\ abc} = (\sigma^x)^B_{\ A} W^A_{\ B \ abc}. $$

Notice that the components $W^x$ are real, despite $W^A_{\ B }$ can be complex. Finally, we can compute the dual of the 3-form matrix

$$ \star W^A_{\ B \ abc} = \frac{1}{3!} \epsilon_{abc}^{\ \ \ \ \ def} W^A_{\ B \ def} = \frac{1}{3!} \epsilon_{abc}^{\ \ \ \ \ def} \gamma_{def} \kappa_B = -i \kappa^A \gamma_{abc} \gamma^7 \kappa_B = -W^A_{\ B \ abc}. $$

Therefore, $W$ is a matrix of anti-self-dual 3-forms. We have concluded our analysis: from a spinor $\kappa^A$ we can construct one vector $l_a$ and three real anti-self-dual 3-forms $W^x_{\ abc}$.

Now let us apply the Fierz identity equation (A.28) to products of $l_a$ and $W^A_{\ B \ abc}$ in order to extract relations between them. For example, we apply this identity to $l_a l_b$, and we get

$$ l_a l_b = \frac{1}{8} \bar{\kappa}^A \gamma_a \gamma_b \kappa_A F + \frac{1}{48} \bar{\kappa}^A \gamma_a \gamma^{cde} \gamma_b \kappa_b W^B_{\ A \ cde}. $$

After the calculation of every term and simplification, we have

$$ l_a l_b = -\frac{1}{6} \eta_{ab} l^2 \ + \frac{1}{6} W^B_{\ A \ cde} W^B_{\ A \ cde}. $$

Then, if we contract $a$ and $b$ with $\eta_{ab}$, we have that $W^B_{\ A \ cde} W^B_{\ A \ cde} = 0$, and then, $l^2 = -l^2 = 0$, so $l_a$ is a null vector.
\[ l_{a}l_{b} = \frac{1}{6} W_{B \quad acd}^{A} W_{A \quad b \quad c d}^{B}, \quad l^2 = 0. \]  
(A.38)

We can do the same analysis with the product \( lW \). This time, after a quite long calculation, we get
\[ l_{a}W_{B \quad a \quad b \quad c d}^{A} = \frac{1}{3} W_{C \quad a e [d]}^{B} W_{C \quad a e |c |}^{A} = 0. \]  
(A.39)

If we now contract the indices \( a \) and \( b \) we get the following expression
\[ l_{a}W_{B \quad acd}^{A} = i_{3} W_{C \quad a e [d]}^{B} W_{C \quad a e |c |}^{A} = 0. \]  
(A.40)

The last identity can be checked by using the anti-self-duality of \( W \). Therefore, \( W \) is transverse to \( l \). Finally, we have to compute the product of two \( W \)'s. After simplification, the painful calculation yields the following result:
\[ W_{B \quad abc}^{A} W_{C \quad e f g}^{D} = 1^{4} W_{D \quad abc}^{B} W_{C \quad e f g}^{B} + \frac{3}{4} \delta_{A \quad D}^{B} \delta_{C \quad B}^{D} \left( 6 l_{[e} \delta_{f]}^{a} \delta_{g]}^{b} l_{a}^{c} + \frac{4}{3} l_{e}^{a} \delta_{b}^{c} l_{f}^{a} \right) \]
\[ + \frac{3}{4} \left( W_{D \quad abc}^{a} W_{C \quad d e f}^{a} - W_{D \quad abc}^{a} W_{C \quad d e f}^{a} \right). \]  
(A.41)

This is a very complex and rich identity and it can deliver a lot of information. For example, if we contract two pairs of indices, this expression simplifies to
\[ W_{B \quad abc}^{A} W_{C \quad e f g}^{D} = (2 \delta_{A \quad D}^{B} \delta_{C \quad B}^{D} - \delta_{A \quad B}^{D} \delta_{C \quad D}^{B}) l_{a} l_{b}. \]  
(A.42)

If we only contract two indices, the result is
\[ W_{B \quad abc}^{A} W_{C \quad e f g}^{D} = \frac{1}{2} \left( 5 \delta_{A \quad D}^{B} \delta_{C \quad D}^{B} - 4 \delta_{B \quad D}^{A} \delta_{C \quad D}^{B} \right) l_{a} l_{b} + \frac{1}{2} l_{a} l_{b} + W_{D \quad h e c}^{B} W_{C \quad a l / f}^{h}. \]  
(A.43)

In order to make further progress, we need to introduce another null vector, \( n_{a} \), such that
\[ n^2 = 0, \quad l^2 n_{a} = 1. \]  
(A.44)

Then, we introduce a 2-form, \( \mathcal{A}^{A} _{B \quad ab} \), defined as
\[ \mathcal{A}^{A} _{B \quad ab} \equiv n^{a} W_{B \quad abc}. \]  
(A.45)

By using the anti-self-duality of \( W \), it can be shown that there is a converse relation
\[ W_{B \quad abc} = 3 \mathcal{A}^{A} _{B \quad [ab] c}. \]  
(A.46)
By construction, $\mathcal{J}^A_B$ is transverse to $n$ and $l$:

$$n^a \mathcal{J}^A_B ab = l^a \mathcal{J}^A_B ab = 0.$$  \hspace{1cm} (A.47)

Therefore, this object has a four-dimensional character and it lives in the four dimensions transverse to $n$ and $l$. In this space, $\mathcal{J}^A_B$ is self-dual

$$\tilde{\varepsilon}^{abcd} \mathcal{J}^A_B ab = \frac{1}{2} \varepsilon^{abcd} \mathcal{J}^A_B ab = \mathcal{J}^A_B ab.$$  \hspace{1cm} (A.48)

where

$$\varepsilon^{abcd} \equiv \varepsilon^{abcdef} l_a n_f,$$  \hspace{1cm} (A.49)

is the Levi-Civita symbol in the the 4-dimensional space transverse to the two null directions $l, n$.

Our next goal is to find the product $\mathcal{J}^A_B \cdot \mathcal{J}^C_D$. With this aim, we contract equation (A.43) with $n^a n^c$, obtaining

$$\mathcal{J}^A_B a c \mathcal{J}^C_D c b = -\frac{1}{16} \left( \delta^A_D (\delta^B_C) + 3 \delta^A_C (\delta^B_D) \right) \tilde{\delta}^a_b + \frac{i}{2} \left( -\delta^A_D \delta^C_D a b + \delta^C_D \delta^A_D a b \right),$$  \hspace{1cm} (A.50)

where

$$\tilde{\delta}^a_b \equiv \delta^a_b - l^a n_b - n^a l_b.$$  \hspace{1cm} (A.51)

is the identity of the 4-dimensional space transverse to the two null directions $l, n$ and can be used as the projector onto this space.

Finally, we contract everything with Pauli matrices, in order to express the equation in terms of the real components

$$\mathcal{J}^x_{a b} \equiv (\sigma^x)^B_A \mathcal{J}^A_B a b.$$  \hspace{1cm} (A.52)

The result is

$$\mathcal{J}^x_{a b} \mathcal{J}^y_{c b} = -\delta^{xy} \tilde{\delta}^a_b + \varepsilon^{xyz} \tilde{\mathcal{J}}^y_{a b},$$  \hspace{1cm} (A.53)

or, hiding the spacetime indices,

$$\mathcal{J}^x \cdot \mathcal{J}^y = -\delta^{xy} + \varepsilon^{xyz} \mathcal{J}^z.$$  \hspace{1cm} (A.54)

Hence, the objects $\mathcal{J}^x$ satisfy the algebra of quaternions.

\textbf{Appendix B. Connection and curvature components}

Let us consider the metric of the supersymmetric field configurations of ungauged $\mathcal{N} = (1, 0)$, $d = 6$ supergravity:

$$ds^2 = 2f(du + \beta)(dv + H du + \omega) - f^{-1} \gamma_{ab} dx^a dx^b.$$  \hspace{1cm} (B.1)

The components of the inverse metric are given by

$$g^{uu} = -\beta^2 f, \quad g^{w} = f^{-1} + f \beta^2 (\beta_m H - \omega_m), \quad g^{mn} = f \beta^m,$$

$$g^{w_0} = -HF^2 / \beta^2 - 2HF^{-1} + 2 \beta^m \omega_m H - f \omega^2, \quad g^{w_0} = f (\omega_m - \beta^m H),$$

$$g^{mn} = -f \gamma_{mn}.$$  \hspace{1cm} (B.2)

\footnote{Lowering one index it would give the induced metric on that space.}
where the indices \( m \) and \( n \) in \( \omega_{mn}, \beta^m \) have been raised with \( \gamma^{mn} \). We introduce a Vielbein:

\[
e^+ = f(du + \beta^1), \quad e^- = dv + Hdu + \omega, \quad e^m = f^{-1/2}v^m,
\]

where \( v^m \) is a Vielbein of the metric \( \gamma_{mn} \). The metric then is

\[
ds^2 = e^+ \otimes e^- + e^- \otimes e^+ - \delta_{mn} e^m \otimes e^n
\]

and the inverse Vielbein is

\[
e_+ = f^{-1}(\partial_u - H\partial_v), \quad e_- = \partial_v, \quad e_m = f^{1/2}v_m - f^{1/2}\beta_m\partial_u - f^{1/2}(\omega_m - \beta_m)\partial_v.
\]

The spin connection \( \omega_{ab} = \omega_{cab}e^c \) is defined through Cartan’s first structure equation

\[
de e^a = \omega^a_b \wedge e^b,
\]

and, for the above metric and Vielbein basis it has the following non-vanishing components

\[
\begin{align*}
\omega_{++m} &= f^{-1/2}(\omega_m - v_mH), \\
\omega_{+-m} &= \omega_{-+m} = \omega_{m+} = \frac{1}{2}f^{-1/2}\left[\frac{1}{2}d\omega_m + f\dot{\beta}_m - v_m\beta\right], \\
\omega_{++m} &= f^{-1}v_{[m]} + \frac{1}{2}f(d\omega)_{mn} - f\dot{v}_mH\beta_n + f\omega_m\beta_n, \\
\omega_{--m} &= -\omega_{m-} = \frac{1}{2}f^2(d\beta)_{mn} + f^2\dot{\beta}_{[m}\beta_n, \\
\omega_{m+n} &= \frac{1}{2}f^{-1/2}(\omega_m - v_mH) + f^{-1/2}(\dot{d}\omega)_{mn} - f\dot{v}_mH\beta_n + f\omega_m\beta_n + f^{1/2}[\dot{\beta}_{[m}\beta_n - \dot{\beta}_{m]\beta_n], \\
\omega_{m+n} &= f^{1/2}[\dot{v}_m\omega_n - \dot{v}_n\omega_m] + f^{1/2}[\dot{v}_m\omega_n - \dot{v}_n\omega_m].
\end{align*}
\]

In these expressions, \( \dot{v}_m \equiv dv\dot{v}_m, \dot{\omega}_m = v_m\partial_u\dot{\omega}_m \) and \( (d\omega)_{mn} = 2v_m\dot{\omega}_n \) and \( \omega_{m+n} \) is the spin connection associated to the Vielbein \( v^m \).

The curvature 2-form is defined by

\[
R^a_b = \frac{1}{2}R^a_{bcde} \wedge e^d = d\omega^a_b - \omega^a_c \wedge \omega^c_b.
\]

In this work we only need to compute a component of the Ricci tensor, namely \( R_{++} = R^a_{++a} = R^m_{++m} \). We get the following result:

\[
R_{++} = -\tilde{\nabla}^2 H + \tilde{\nabla}^m \dot{\omega}_m - \beta_m(\tilde{\omega}^m - \partial^m H)
- (\tilde{\omega}^m - \partial^m H)(2\dot{\beta}_m + 2\tilde{\beta}^m_{[m]} + \tilde{\beta}^m_{[m]} - \dot{v}_{[m]}\beta_n) + \frac{1}{4}f^2 G^2 + 5f^{-4}J^2 - 2f^{-3}J^2 + \partial_h(f^{-2}v_m^m) + f^{-2}\dot{v}_{(mn)}\dot{v}_{(mn)}
\]

\[32\]
where $\tilde{\nabla}$ stands for the covariant derivative with respect to the connection $\tilde{\omega}_{mnr}$, and

$$G \equiv \tilde{d}\omega - \tilde{d}H \wedge \beta + \dot{\omega} \wedge \beta = D\omega - \tilde{d}H \wedge \beta. \quad (B.10)$$

**ORCID iDs**

Pablo A Cano  
https://orcid.org/0000-0001-8932-7197  
Tomás Ortín  
https://orcid.org/0000-0001-6810-1279

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