Isgur–Wise function in a relativistic model of constituent quarks

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November 6, 2000

Abstract

The integral representation for Isgur – Wise function (IWF) is obtained in the framework of instant–form relativistic Hamiltonian dynamics for mesons with one heavy quark. The upper and lower limits are calculated for the slope parameter of IWF $\rho^2$ by the model independent way. IWF is calculated for different wave functions of quarks in the meson. The difference between the limits of $\rho^2$ equals 1/3. The constraint on the slope parameter is in a good agreement with experiments. The weak dependence of IWF on the choice of wave functions is found.

PACS numbers: 12.39.Ki, 12.60.Rc, 13.20.He

Key words: constituent quark model, semileptonic decays, Isgur–Wise function

The quark composite systems with one heavy quark have been the focus of attention for the last several years. In the infinite heavy quark mass limit the additional spin-flavour symmetries appear and simplify the calculation of matrix elements of electroweak transitions \[.\] Therefore all mesons form factors of the electroweak transitions are determined by a single universal Isgur – Wise function (IWF) $\xi_{IW}(w)$ \[.\] ($w = (v \cdot v')$, $v'_\mu$ and $v_\mu$ are 4–velocities of initial and final meson).

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The calculation of this function is nonperturbative in principle. Several approaches exist to this problem, for example, the calculations on lattices [3], QCD sum rules [2, 4, 5, 6, 7, 8, 9], quasipotential approach [10, 11], and different forms of relativistic Hamiltonian dynamics (RHD): light-front dynamics [12, 13], point [14] and instant forms [15, 16].

The slope parameter $\rho^2 = -\xi'_{IW}(w = 1)$ is of special interest, while studying the properties of IWF. This interest, in particular, is explained by recent measurements of this value [17, 18, 19]. By these experiments, one can discriminate different models or, additionally, constraint parameters of the models. The results of calculation of the slope parameter in different approaches differ greatly [2, 3, 4, 6, 10, 12, 13, 15, 16].

In this work, IWF is calculated for mesons which contain one heavy quark. For slope parameter $\rho^2$ the strong restriction is obtained. This result depends weakly on interaction model for quarks. The interval of restriction on the slope parameter equals 1/3.

The calculations of IWF are performed for different phenomenological wave functions of quarks in meson. It is shown that IWF depends weakly enough on the choice of the model wave functions (about 10% at $w \leq 2$).

For calculation in our work we use the instant form of RHD in the version developed by the authors [20, 21] (RHD is discussed in greater detail in Ref. [22] and references therein). Our approach is closely related to that in Refs. [13, 16]. However, our approach contains some original features, e.g., the method of construction of current transition matrix elements [21].

In our work, IWF is calculated from the semileptonic decay of pseudoscalar meson form factors [21]. The hadron part of invariant amplitude of the semileptonic decay of pseudoscalar meson can be expressed through two form factors $F_{\pm}$:

$$
\langle \vec{p}_c | J^\mu | \vec{p}_c' \rangle = P_\mu^- F_-(t) + P_\mu^+ F_+(t),
$$

where $\vec{p}_c, \vec{p}_c'$ are 3-momenta of mesons, $P_\mu^- = (p_c - p_c')^\mu$, $P_\mu^+ = (p_c + p_c')^\mu$, $t = (p_c - p_c')^2$.

In our approach, the form factors $F_-(t), F_+(t)$ have the following integral representation in the impulse approximation [21]:

$$
F_\pm(t) = \int_{m_Q + m_q}^{\infty} d\sqrt{s} d\sqrt{s'} \varphi_c(k(s)) G_{\pm}^{(0)}(s, t, s') \varphi_{c'}(k(s')), \tag{2}
$$

where $G_{\pm}^{(0)}(s, Q^2, s')$ are the so-called free two-partical form factors, $\varphi_{c'}, \varphi_c$ are phenomenological wave functions of initial and final mesons in the sense of RHD. They are normalized with relativistic density states:

$$
\varphi(k) = \sqrt{\sqrt{s}(1 - (m_Q^2 - m_q^2)^2/s^2)} u(k) k, \quad \int k^2 u^2(k) dk = 1. \tag{3}
$$
Where \( k(s) = \sqrt{(s^2 - 2s(m_Q^2 + m_q^2) + (m_Q^2 - m_q^2)^2)/4} \), \( m_Q \), \( m_q \) are masses of heavy and light quarks respectively, \( u(k) \) is the non-relativistic wave function.

Free two-particle form factors \( G_{\pm}^{(0)} \) have been calculated in Ref. [21]:

\[
G_{\pm}^{(0)}(s, t, s') = g(s, t, s') \left\{ [-f_3(t)\lambda(s, t, s')(s - s' - t) - \right.
\]
\[-2f_1(t) s' [t (s + s' - t) - \eta_1 (s - s' - t) + \eta_2 (s - s' + t)]\cos \alpha +
\]
\[+f_6(t) M_2 \xi(s, t, s') s' t \sin \alpha \},
\]
\[
g(s, t, s') = \frac{\sqrt{s(s + s' - t)}\Theta(s, t, s')}{4\sqrt{s'}[\lambda(s, t, s')]^{3/2} \sqrt{\lambda(s', M_1^2, M_2^2)\lambda(s, M_3^2, M_4^2)}}.
\]

\[\eta_1 = M_2^2 - M_1^2, \quad \eta_2 = M_2^2 - M_3^2, \quad \eta_3 = M_2^2 - M_4^2 \]
\[\Theta(s, t, s') = \theta(s - s_1) - \theta(s - s_2), \quad \lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + bc + ac),
\]
\[\xi(s, t, s') = \left\{-M_2^2\lambda(s, t, s') - ss't - \eta_2(t + \eta_3) - s\eta_3^2 - s'\eta_1^2 +
\]
\[+s'\eta_1(s - s' + t) + \eta_1 \eta_2(s + s' - t)\}^{1/2}.
\]

\( \theta \) is Heaviside step function, \( s_1 \) and \( s_2 \) are variables that determine the kinematically admissible range of \( s \) and \( s' \) which are given by
\[
s_{1,2} = s' + t - \frac{1}{2M_2^2}(s' + \eta_2)(t + \eta_3) \mp \frac{1}{2M_2^2}\sqrt{\lambda(s', M_1^2, M_2^2)\lambda(t, M_3^2, M_4^2)}.
\]
\[\alpha = \omega_1 + \omega_2 + \omega_3, \quad \omega_i \] are parameters of Wigner rotation:

\[
\omega_1 = \arctan \frac{\xi(s, t, s')}{M_1(s + s' - t + 2\sqrt{ss'}) + \sqrt{ss'((s + \sqrt{s'}) - \eta_2\sqrt{s} - \eta_1\sqrt{s'})},
\]
\[
\omega_2 = \arctan \frac{\xi(s, t, s')}{M_3(s + s' - t + 2\sqrt{ss'}) + \sqrt{s(s + s' + \sqrt{ss'}) + \eta_2\sqrt{s} + \eta_1\sqrt{s}'}},
\]
\[
\omega_3 = \arctan \frac{\xi(s, t, s')}{M_3s' + M_2(s - t) + \sqrt{s'[(M_3 + M_2)^2 - t] + \eta_2(M_2 + M_3)}}.
\]

\( M_1 \) is the light quark mass, \( M_{2,3} \) are the heavy quark masses. Quark form factors in the approximation of pointlike quarks have a form [21]:

\[
f_1(t) = (M_2 + M_3) f(t), \quad f_3(t) = -(M_2 - M_3) f(t),
\]
\[ f_6(t) = -\frac{4M_2}{M_3^2} f(t), \quad f(t) = \left[(M_3 + M_2)^2 - t\right]^{-1/2}. \]

The matrix element of the semileptonic decay of pseudoscalar meson can be determined in terms of form factors \( h_{\pm}(w) \) (see, for example, [12]):

\[
\langle \vec{p}_c|J^\mu|\vec{p}_c'\rangle = \sqrt{M_cM_{c'}} \left[ h_+(w)(v + v')^\mu + h_-(w)(v - v')^\mu \right].
\] (5)

Here \( M_c, M_{c'} \) are masses of mesons in initial and final states.

The form factors in Eq. (5) are connected with the form factors in Eqs. (1), (2) by the following formula:

\[
h_{\pm}(w) = \frac{1}{2\sqrt{M_cM_{c'}}} \left[ F_+(t) (M_c \pm M_{c'}) + F_-(t) (M_c \mp M_{c'}) \right].
\] (6)

Scalar product of 4–velocities is determined through the square of momentum transfer:

\[
w = \left( M_c^2 + M_{c'}^2 - t \right)/(2M_cM_{c'}). \]

The IWF can be obtained from expressions (2), (6) by finding the limit \( M_2 \sim M_3 \sim m_Q \to \infty, M_1 = m_q \):

\[
\lim_{m_Q \to \infty} h_+(w) = \xi_{IW}(w), \quad \lim_{m_Q \to \infty} h_-(w) = 0.
\] (7)

In our calculations, we have assumed that the wave functions in Eq. (2) become independent on the flavour of heavy quark and coincide in initial and final states in limit (7).

The result of calculation of IWF can be presented in following form:

\[
\xi_{IW}(w) = \frac{1}{\sqrt{2(w+1)}} \int_0^\infty \int_{-1}^1 k^2 dk dz \frac{\sqrt{k^2 + m_q^2}}{\sqrt{k^2 + m_q^2}} \cos \omega(u(k)u(k')), \quad (8)
\]

\[
k' = \left[ (\sqrt{w^2 - 1}kz + w\sqrt{k^2 + m_q^2})^2 - m_q^2 \right]^{1/2},
\]

\[
\omega = \arctan \frac{k\sqrt{(w^2 - 1)(1 - z^2)}}{(w + 1)m_q + \sqrt{k^2 + m_q^2 + \sqrt{k^2 + m_q^2}}},
\]

The condition \( \xi_{IW}(1) = 1 \) is a consequence of the normalization (3).

In analogy with the works [14, 15, 16], the slope parameter of IWF at \( w = 1 \) can be represented as the sum of three terms:

\[
\rho^2 = \rho_{space}^2 + \rho_{WR}^2 + \rho_{quark}^2.
\] (9)

\[
\rho_{space}^2 = \frac{1}{3} \int_0^\infty k^2 \left(k^2 + m_q^2\right) \left(\frac{du}{dk}\right)^2 dk + \frac{1}{6} \int_0^\infty \frac{k^4 u^2(k)}{k^2 + m_q^2} dk - \frac{1}{2},
\] (10)
\[ \rho_{WR}^2 = \frac{1}{6} \int_0^\infty \frac{k^4 u^2(k)}{(\sqrt{k^2 + m_q^2} + m_q)^2} \, dk, \quad \rho_{\text{quark}}^2 = \frac{1}{4}. \]

The terms in Eq. (9) describe the contributions of form of wave function (with the consideration of relativistic normalization (8)), Wigner rotation, and quark current, respectively. The contribution \( \rho_{WR}^2 \) and the second term in \( \rho_{\text{space}}^2 \) have relativistic nature. Let us remark that our value of \( \rho_{\text{quark}}^2 \) coincides with that from [15].

The integrals of the wave function can be transformed into space coordinate representation:

\[ \int_0^\infty k^2 \left( \frac{d u}{d k} \right)^2 \, dk = \int_0^\infty r^4 \psi_0^2(r) \, dr = \langle r^2 \rangle, \]
\[ \int_0^\infty k^4 \left( \frac{d u}{d k} \right)^2 \, dk = \int_0^\infty r^4 \left( \frac{d \psi_0}{d r} \right)^2 \, dr, \quad \int_0^\infty r^2 \psi_0^2(r) \, dr = 1. \] (11)

Here \( \langle r^2 \rangle \) is the mean square radius in the ground state with wave function \( \psi_0(r) \).

Notice that all integrals in Eqs. (9), (10) are positive. From equations (9), (10), (11) the restriction on possible values of \( \rho^2 \) can be established. To do so, let us consider the relativistic terms in Eqs. (9), (10) \( (\rho_{WR}^2 \text{ and the second term in } \rho_{\text{space}}^2) \).

In the non-relativistic limit these positive terms disappear in Eq. (10) and we obtain the lower bound for the slope parameter:

\[ \rho^2 \geq \frac{1}{3} \int_0^\infty r^4 \left( \frac{d \psi_0}{d r} \right)^2 \, dr + \frac{1}{3} m_q^2 \langle r^2 \rangle - \frac{1}{4}. \] (12)

In the ultrarelativistic limit these relativistic terms in Eq. (9) take the maximum values and we obtain the upper bound for \( \rho^2 \):

\[ \rho^2 \leq \frac{1}{3} \int_0^\infty r^4 \left( \frac{d \psi_0}{d r} \right)^2 \, dr + \frac{1}{3} m_q^2 \langle r^2 \rangle + \frac{1}{12}. \] (13)

Let us remark that the difference between these maximum and minimum of the slope parameter equals 1/3.

Now we shall use the results of works [23]. In these works a recurrence formula was derived:

\[ \langle r^{2l} \rangle = \frac{1}{2 \mu} \frac{l(2l + 1) \langle r^{2(l-1)} \rangle}{E_1(l) - E_0}. \] (14)

Here \( l \) is the orbital angular momentum, \( \langle r^{2l} \rangle \) multipole moments in ground state of two-body system, \( E_0 \) ground state energy, \( E_1(l) \) energy of the lowest level of multipolarity \( l \), \( \mu \) reduced mass, in the limit (7), \( \mu = m_q \). Recurrent formula (14) is satisfied within 1% accuracy as equality for confining potential.
For estimation of mean square radius in (12) and (13) by means of Eq. (14), we have used two measurements of energy levels of D - meson (1\s_0 and 1\p_1) [24]. For further estimation, let us transform integrals in expressions (12) and (13):

\[
\frac{1}{3} \int_0^\infty r^4 \left( \frac{d\psi_0}{dr} \right)^2 dr = 1 + \frac{1}{3} \int_0^\infty \psi_0(r) \left[ (\hat{r}\hat{p})^2 - i (\hat{r}\hat{p}) \right] \psi_0(r) r^2 dr .
\] (15)

Here \(\hat{p}\) is the operator of canonical 3-momentum of system.

For model independent estimation of integral in right part of Eq. (15), we have used quasiclassical approximation. In ground state, this approximation gives \((\hat{r}\hat{p}) = 0\) because the mean value of \(r(t)\) sweeps a circle in the quasiclassical approximation. Therefore, the left hand side of Eq. (15) is approximately equal to unit.

For estimation of quantum corrections of this result, it is possible to calculate exactly the integral in the left part of Eq. (15) in some models. Let us perform the calculation for the model with linear confinement and Coulomb behavior at small distances [25]:

\[
u(r) = N_T e^{-\alpha r^{3/2} - \beta r} , \quad \alpha = \frac{2}{3} \sqrt{2} \mu a , \quad \beta = \mu b .
\] (16)

In Eq.(16), \(a\) and \(b\) are the parameters of linear and Coulomb parts of potential, respectively. We use the value of \(b = 4/3\alpha_s\). In doing so, we have used the value \(\alpha_s = 0.52\) at a scale of heavy meson masses. The parameter of linear part of potential \(a = 0.0816\) GeV\(^2\) was obtained by fitting of the mean square radius in ground state of \(D\)-meson which was calculated from Eq. (14). In this way the quantum correction \(\simeq 0.07\) was obtained.

As follows from the general conditions of application of the quasiclassical approximation, the less steep potential \((\sim r^\alpha , \alpha < 1\) or \(\sim \ln r\)) must result in smaller quantum corrections. Hence, for such class of models, the following restriction is valid to within \(\simeq 0.07\) accuracy:

\[
\frac{3}{4} \leq \rho^2 - \frac{1}{3} m_q^2 \langle r^2 \rangle \leq \frac{13}{12} .
\] (17)

The light quark mass \(m_q\) is now the only free parameter in our estimations. In our relativistic calculation in Refs. [20], [21] we used the value \(m_q = 0.25\) GeV. This calculation are in reasonably good agreement with experiments on electroweak properties of light mesons. In this case, Eq. (14) gives \(\langle r^2 \rangle = 0.422\) fm\(^2\). Hence, inequalities (17) are reduced to the following limitation:

\[
0.98 \leq \rho^2 \leq 1.31 \quad (18)
\]

If the mass \(m_q\) is varied within the limits in which the modern relativistic calculations are performed in different works (0.20 \(\leq m_q \leq 0.33\) GeV) then the region of possible \(\rho^2\) increases:

\[
0.93 \leq \rho^2 \leq 1.38 \quad (19)
\]
The derived restrictions are examined in our work for the following different model wave functions: the harmonic-oscillator potential model (see, for example, [26])

$$u(k) = N_{HO} \exp \left(-k^2/2b^2\right),$$

(20)

the power law wave function (see, for example, [27])

$$u(k) = N_{PL} (k^2/b^2 + 1)^{-n}, \quad n = 2, 3,$$

(21)

and already mentioned wave function with linear confinement and a Coulomb behavior at small distances which is described by Eq. (16). The wave function parameters were determined by fitting of mean square radius of the $D-$ meson which was calculated by formula (14). The fitting gives the following values: in the model corresponding to Eq. (20) we obtain $b = 0.372$ GeV, in the model (21) we find $b = 0.526$ GeV for $n = 2$ and $b = 0.731$ GeV for $n = 3$. With the presented wave functions the following values for slope parameter in Eq. (1) were obtained: $\rho^2 = 1.32$ in the model (20), $\rho^2 = 1.13$ in the model (21) for $n = 2$ and $\rho^2 = 1.20$ for $n = 3$, and $\rho^2 = 1.20$ in the model (16).

The result in model of harmonic-oscillator satisfies inequalities (18) within the accuracy of our calculations. Let us remind in this connection that the restrictions (17) were obtained for the model with potential increasing slower than linearly. It is possible that the restrictions (17) are valid for wider class of models. The calculations with exact formulae (9), (10) for models (16), (21) satisfy the restriction (18).

Let us compare our results (18), (19) with recent experimental values of $\rho^2$. Result of CLEO Collaboration [17], [19] ($\rho^2 = 1.67\pm0.11\text{ (stat)}\pm0.22\text{ (syst)}$) satisfies our “soft” result (19) and is slightly outside the region (18). The average LEP result [18], [19] ($\rho^2 = 1.13\pm0.08\text{ (stat)}\pm0.16\text{ (syst)}$) satisfies the ”hard” restriction (18).

Let us consider results for $\rho^2$ in other approaches. The two restrictions on slope parameter $\rho^2$ were obtained in QCD sum rules: $\rho^2 \geq 0.25$ [4], $\rho^2 \leq 1.7$ [3]. Quasipotential approach gives $\rho^2 \geq 0.81$ [11]. Another restriction was obtained in the framework of RHD [15] $\rho^2 \geq 3/4$. As we see, our restriction (18) does not contradict these results but reinforces them. However, our result, as well as that of work [13] and experiments [17], [18], is in poor correlation with the result of Ref. [5] obtained in the framework of QCD sum rules $\rho^2 \leq 0.75 \pm 0.15$.

Now let us consider the calculations in the framework of instant form of RHD. In work [16], the calculations are performed for different interaction models of quarks in meson and results are within the range from 0.97 to 1.28, that is very close to (18). The result of quasipotential approach [10] coincides with that of ours, $\rho^2 = 1.02$. 

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The results of different calculations of $\rho^2$ in the framework of QCD sum rules differ considerably: $1.00 \pm 0.02$, $0.84 \pm 0.02$, $0.66 \pm 0.05$, $0.70 \pm 0.25$. First and fourth of these values are near our range.

The lattice calculations are in good correlation with our result, but we should point the large errors of these calculations: $\rho^2 = 0.9^{+0.2+0.4} -0.3-0.2$.

Now let us discuss briefly the calculation of IWF in our approach. In fig.1 the results of calculation for IWF with the described model wave functions are shown. The calculation was carried out with the parameters of this work at $m_q=0.25$ GeV. As can be seen from the fig.1, one of the most interesting features of our calculations is the weak dependence of IWF on the choice of model wave functions at $w \leq 2$. Hence, the properties of IWF are mainly determined by the relativistic kinematics of light quark or so-called ”light” degrees of freedom.

So, in the present work, the Isgur–Wise function is calculated in the framework of the instant form of relativistic Hamiltonian dynamics. A strong model-independent restriction is obtained for the slope parameter of Isgur–Wise function. This restriction reinforces the restrictions that were given in other approaches. We have also shown that Isgur–Wise function depends weakly on the choice of the model wave functions at $w \leq 2$.

We are grateful to A.V.Gorokhov for numerous useful discussions. This work was supported in part by the program ”Russian Universities – Basic Researches” (grant N 02.01.28).
References

[1] N. Isgur and M.B. Wise, Phys. Lett. B232 (1989). p.113.; B237 (1990) p.527.
[2] M. Neubert, Phys. Rev. D45 (1992) p.2451.; D47 (1993) p.4063.
[3] K.C. Bowler et al. UKQCD Collab., Phys.Rev. D52 (1995) 5067.
[4] J.D. Bjorken, Preprint of Stanford Univ., SLAC-PUB-5278 (1990).
[5] M.B. Voloshin, Phys. Rev. D46 (1992) p.3062.
[6] E. Bagan , P. Ball and P. Gosdzinsky, Phys. Lett. B301 (1993) p.101.
[7] B. Blok, M. Shifman, Phys. Rev. D47 (1993) p.2949.
[8] M. Narison, Phys.Lett. B325 (1994) p.197.
[9] E. de Rafael and J. Taron, Phys. Rev. D50 (1994) p.373.
[10] R.N. Faustov and V.O. Galkin, Z. Phys. C66 (1995) p.119.
[11] D. Ebert, R.N. Faustov, V.O. Galkin, Phys.Rev.D, 61 (2000) 014016.
[12] H. Cheng, C. Cheung and C. Hwang, Phys. Rev. D55 (1996) p.1559.
[13] S. Simula, Phys.Lett. B373 (1996) p.193.
[14] B.D. Keister, e–print arxiv hep-ph/9703310
[15] A. Le Yaouanc, L. Oliver, O. Pène and J.-C. Raynal, Phys. Lett. B365 (1995) p.319; Phys. Lett. B386 (1996) p.304.
[16] V. Morénas, A. Le Yaouanc, L. Oliver, O. Pène and J.-C. Raynal, Phys. Lett. B386 (1996) p.315; Phys. Rev. D56 (1997) p.5668.
[17] J.P.Alexander et al.(CLEO Collab.) Preprint CLEO CONF 00–03, 2000.
[18] D. Abbaneo et al. (LEP Collaborations) Preprint CERN–EP–2000–016, 2000.
[19] M.Battaglia, arXiv:hep-ex/0009020
[20] E.V. Balandina, A.F. Krutov and V.E. Troitsky, Theor.Math.Fiz. 103 (1995)p.41; J.Phys. G22 (1996) p.1585.
[21] E.V. Balandina, A.F. Krutov, V.E. Troitsky and O.I. Shro, Phys. At. Nuc. 63 (2000) p.244.
[22] B.D. Keister and W.N. Polyzou, Advances in Nucl. Phys. 20 (1991) p.225.
[23] R.J. Lombard and J. Mareš, Phys. Rev. D59 (1999) 076005; Phys. Lett. B472 (2000) p.150.
[24] D.E. Groom et al., Europ. Phys. J. C15 (2000) p.1.
[25] H. Tezuka, J. Phys.A: Math.Gen. 24 (1991) p.5267.
[26] P.L. Chung, F. Coester and W.N. Polyzou, Phys. Lett. B205 (1988) p.545.
[27] F. Schlumpf, Phys. Rev. D50 (1994) p.6895.
Figure captions

Fig. 1. Results of the calculations of Isgur–Wise function with the different model wave functions. Dotted line: the power law wave function of Eq. (21) with $n = 2$; dashed line: the same model with $n = 3$; dot–dashed line: the wave function in the model with linear confinement and Coulomb behavior at small distances of Eq. (16); solid line: harmonic–oscillator wave function of Eq. (20). Wave function parameters were determined by fitting of mean square radius of $D$–meson calculated with formula (14).
Fig. 1.