On the Supremum of Random Dirichlet Polynomials

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Abstract: We study the supremum of some random Dirichlet polynomials
\[ D_N(t) = \sum_{n=2}^{N} \varepsilon_n d_n n^{-\sigma - it}, \]
where \((\varepsilon_n)\) is a sequence of independent Rademacher random variables, the weights \((d_n)\) are multiplicative and \(0 \leq \sigma < 1/2\). The particular attention is given to the polynomials \(\sum_{n \in \mathcal{E}, \varepsilon_n n^{-\sigma - it}}\), \(\mathcal{E} = \{2 \leq n \leq N : P^+(n) \leq p_r\}\), \(P^+(n)\) being the largest prime divisor of \(n\). We obtain sharp upper and lower bounds for supremum expectation that extend the optimal estimate of Halász-Queffélec

\[ \mathbb{E} \sup_{t \in \mathbb{R}} \left| \sum_{n=2}^{N} \varepsilon_n n^{-\sigma - it} \right| \approx \frac{N^{1-\sigma}}{\log N}. \]

Our approach in proving these results is entirely based on methods of stochastic processes, in particular the metric entropy method.

1. Introduction and main results

Let \(\{d_n, n \geq 1\}\) be a sequence of real numbers. Let \(s = \sigma + it\) denote a complex number. The study of the supremum of the Dirichlet polynomials

\[ P(s) = \sum_{n=2}^{N} d_n n^{-s} \]

over lines \(\{s = \sigma + it, t \in \mathbb{R}\}\) is naturally related to that of corresponding Dirichlet series, via the abscissa of uniform convergence

\[ \sigma_u = \inf \left\{ \sigma : \sum_{n=2}^{\infty} d_n n^{-\sigma - it} \text{ converges uniformly over } t \in \mathbb{R} \right\}, \]

through the relation

\[ \frac{\log \sup_{t \in \mathbb{R}} \left| \sum_{n=2}^{N} d_n n^{-it} \right|}{\log N} = \limsup_{N \to \infty} \sigma_u. \]

One can refer to Bohr [B], Bohnenblust and Hille [BH], Helson [H], Hardy and Riesz [HR], Queffélec [Q3] for this background and related results. This of course, basically justifies the investigation of the supremum of Dirichlet polynomials (see for instance Konyagin and Queffélec [KQ]).

The following classical reduction step enables to replace the Dirichlet polynomial by some relevant trigonometric polynomial. In order to recall this reduction, we introduce the necessary

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notation. Let $2 = p_1 < p_2 < \ldots$ be the sequence of all primes. If $n = \prod_{j=1}^{\tau} p_j^{a_j(n)}$, we write $\omega(n) = \{a_j(n), 1 \leq j \leq \tau\}$. Let $\pi(N)$ denote, as usual, the number of prime numbers that are less or equal to $N$. Finally, let $T = [0,1)= \mathbb{R}/\mathbb{Z}$ be the torus. Let us fix $N$, put $\mu = \pi(N)$, and define, for $\tilde{z} = (z_1, \ldots, z_\mu) \in T^\mu$,

$$Q(\tilde{z}) = \sum_{n=2}^{N} d_n n^{-\sigma} \varepsilon^{2i\pi \omega(n) \cdot \tilde{z}},$$

The famous H. Bohr’s observation ([Q1-3]), states that

$$\sup_{t \in \mathbb{R}} |P(\sigma + it)| = \sup_{\tilde{z} \in T^\mu} |Q(\tilde{z})| .$$

(1.1)

This indeed follows straightforwardly from Kronecker’s Theorem (see [HW], Theorem 442, p.382).

A parallel study is also developed for random Dirichlet polynomials and random Dirichlet series in the papers of Halász [Ha1-2], Queffélec [Q1-3], Bayart, Konyagin and Queffélec [BKQ], Kahane [K] and of Yu [Y1-3],[STY], Hedenmalm and Saksman [HS]. Such investigations concerning random Dirichlet series (as well as random power series) go back to earlier works of Hartman [Har], Clarke [C], Dvoretzky and Erdös [DE1-2], Dvoretzky and Chojnacki [DC].

Let $\varepsilon = \{\varepsilon_i, i \geq 1\}$ be (here and throughout the whole paper) a sequence of independent Rademacher random variables ($\mathbb{P}\{\varepsilon_i = \pm 1\} = 1/2$) defined on a basic probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Consider the random Dirichlet polynomials

$$D(s) = \sum_{n=2}^{N} \varepsilon_n d_n n^{-\sigma - it} .$$

(1.2)

When $d_n \equiv 1$, some results about the suprema are known. If $\sigma = 0$, then for some absolute constant $C$, and all integers $N \geq 2$

$$C^{-1} N \frac{1}{\log N} \leq \mathbb{E} \sup_{t \in \mathbb{R}} \left| \sum_{n=2}^{N} \varepsilon_n n^{-it} \right| \leq C \frac{N}{\log N} .$$

(1.3)

This has been proved by Halász (see [Q2-3]). In [Q2-3] (see also [Q1] for a first result), Queffélec extended Halász’s result to the range of values $0 \leq \sigma < 1/2$; and provided a probabilistic proof of the original one, using Bernstein’s inequality for polynomials, properties of complex Gaussian processes and the sieve method introduced by Halász. He obtained that for some constant $C_\sigma$ depending on $\sigma$ only, and all integers $N \geq 2$

$$C_\sigma^{-1} \frac{N^{1-\sigma}}{\log N} \leq \mathbb{E} \sup_{t \in \mathbb{R}} \left| \sum_{n=2}^{N} \varepsilon_n n^{-\sigma - it} \right| \leq C_\sigma \frac{N^{1-\sigma}}{\log N} .$$

(1.4)

This in fact admits a stronger form

$$C_\sigma^{-1} \leq \mathbb{E} \sup_{N \geq 2} \sup_{t \in \mathbb{R}} \left| \frac{\sum_{n=2}^{N} \varepsilon_n n^{-\sigma - it}}{N^{1-\sigma} \log N} \right| \leq C_\sigma .$$

(1.4')
A proof is given at the end of Section 4. We shall hereafter use and simplify Queffélec’s probabilistic argument, notably reducing the proof of the upper bound part to the study of suitable real Gaussian processes (which can be easily reduced to a single one). Further, we will not use Bernstein’s inequality, unlike in both previous proofs. A simple metric entropy argument is indeed sufficient, making the proof entirely based upon stochastic processes methods.

By developing this approach, we will also study the case when the $d_n$’s are not constant and random Dirichlet polynomials are supported by other sets than intervals of integers $[2, N]$. At this regard, we consider the following natural extension. For any integer $n > 1$, let $P^+(n)$ denote the largest prime divisor of $n$. Let $1 \leq M < N$ be two positive integers and define

$$S(N, M) = \{2 \leq n \leq N : P^+(n) \leq M\}.$$

Since $S(N, N) = [2, N]$, these sets naturally generalize the notion of interval of integers. By using the standard notation

$$\Psi(N, M) := \#(S(N, M)),$$

$u = (\log N)/\log M$, we have ([T], Theorem 6 p.405)

$$\Psi^*(N, M) := \frac{\Psi(N, M)}{N} = \rho(u) + O\left(\frac{1}{\log y}\right),$$

uniformly for $x \geq y \geq 2$, where $\rho(u)$ is the Dickman function, namely the unique continuous function on $[0, \infty]$, having a derivative on $[0, \infty]$, and such that

$$\left\{\begin{array}{ll}
\rho(v) & = 1, \\
v\rho'(v) + \rho(v - 1) & = 0,
\end{array}\right. \quad (0 \leq v \leq 1),
\quad (v > 1).$$

It is known that $\rho(u)$ is a decreasing positive function and that $\log \rho(u) \sim -u \log u$, as $u \to \infty$. In other words, $\rho$ decreases as fast as the inverse of Gamma function. By setting $M = N^\varepsilon$ in (1.5) we see that $\Psi(N, N^\varepsilon) \sim N \rho(\varepsilon^{-1})$ for any fixed $0 < \varepsilon < 1$.

In view of (1.5), we sometimes refer to $\Psi^*$ as to Dickman-type function.

Fix some positive integer $\tau \leq \pi(N)$, and recall that $p_1 < p_2 < \ldots$ is the sequence of primes. Put

$$\mathcal{E}_\tau = \mathcal{E}_\tau(N) = \{2 \leq n \leq N : P^+(n) \leq p_\tau\}.$$ 

Note that for $\mu = \pi(N)$ we have $\mathcal{E}_\mu = \{2, \ldots, N\}$.

The $\mathcal{E}_\tau$-based Dirichlet polynomials were already considered in [Q3]. One motivation for considering them, related to Rudin-Shapiro problem, will be explained later.

We begin with a result that contains both above mentioned estimates (1.3) and (1.4).

**Theorem 1.1.**

a) **Upper bound.** Let $0 \leq \sigma < 1/2$. Then there exists a constant $C_\sigma$ such that for any integer $N \geq 2$ it is true that

$$E \sup_{t \in \mathbb{R}} \left| \sum_{n \in \mathcal{E}_\tau} \varepsilon_n n^{-\sigma-it} \right| \leq \begin{cases} 
C_\sigma N^{1/2-\sigma} \tau^{1/2} / (\log N)^{1/2} & \text{if } N^{1/2} \leq \tau \leq N, \\
C_\sigma N^{3/4-\sigma} / (\log N)^{1/2} & \text{if } N^{1/2} / \log N \leq \tau \leq N^{1/2}, \\
C_\sigma N^{1/2-\sigma} \tau^{1/2} / (\log N)^{1/2} & \text{if } 1 \leq \tau \leq N^{1/2} / \log N.
\end{cases}$$

b) **Lower bound.** Let $0 \leq \sigma < 1/2$. Then there exists a constant $C_\sigma$ such that for every $N \geq 2$,

$$E \sup_{t \in \mathbb{R}} \left| \sum_{n \in \mathcal{E}_\tau} \varepsilon_n n^{-\sigma-it} \right| \geq \begin{cases} 
C_\sigma N^{1/2-\sigma} \tau^{1/2} / (\log \tau)^{1/2} & \Psi^*\left(\frac{N}{p_\tau}, p_\tau/2\right)^{1/2},
\end{cases}$$

\[\]
Sharpness of the result. It is instructive to compare the lower and upper bounds obtained in Theorem 1.1.

Consider three cases, as in the upper bound of this theorem:

Case I: \( N^{1/2} \leq \tau \leq N \).
Here the Dickman function vanishes from the lower bound and we have \( \log \tau \sim \log N \). It follows from the theorem

\[
C_1(\sigma) \frac{N^{1/2-\sigma \tau^{1/2}}}{(\log N)^{1/2}} \leq E \sup_{t \in \mathbb{R}} \left| \sum_{n \in \mathcal{E}_\tau} \varepsilon_n n^{-\sigma-\nu} \right| \leq C_2(\sigma) \frac{N^{1/2-\sigma \tau^{1/2}}}{(\log N)^{1/2}}.
\]

Thus our bounds are optimal.

Case II: \( \frac{N^{1/2}}{\log N} \leq \tau \leq N^{1/2} \).
Again the Dickman function vanishes from the lower bound and we have \( \log \tau \sim \log N \). Thus

\[
C_1(\sigma) \frac{N^{1/2-\sigma \tau^{1/2}}}{(\log N)^{1/2}} \leq E \sup_{t \in \mathbb{R}} \left| \sum_{n \in \mathcal{E}_\tau} \varepsilon_n n^{-\sigma-\nu} \right| \leq C_2(\sigma) \frac{N^{3/4-\sigma}}{(\log N)^{1/2}}.
\]

The ratio of the right and the left hand side satisfies

\[
1 \leq \frac{N^{1/4}}{\tau^{1/2}} \leq (\log N)^{1/2}.
\]

Thus a logarithmic gap appears.

Case III: \( 1 \leq \tau \leq \frac{N^{1/2}}{\log N} \).
Assume first that \( \tau \geq N^\varepsilon \) for some fixed \( \varepsilon > 0 \), necessarily with \( \varepsilon < 1/2 \). Then the Dickman function produces in the lower bound just an extra constant depending on \( \varepsilon \). We have

\[
C_1(\sigma, \varepsilon) \frac{N^{1/2-\sigma \tau^{1/2}}}{(\log \tau)^{1/2}} \leq E \sup_{t \in \mathbb{R}} \left| \sum_{n \in \mathcal{E}_\tau} \varepsilon_n n^{-\sigma-\nu} \right| \leq C_2(\sigma) \frac{N^{1/2-\sigma \tau^{1/2}}}{(\log \tau)^{1/2}}.
\]

The gap is still of the logarithmic order:

\[
1 \leq (\log \tau)^{1/2} \leq (\log N)^{1/2}.
\]

One should notice that an upper estimate \( C N^{1/2-\sigma (\tau \log \log N)^{1/2}} \) slightly weaker than our bound in Case III was obtained in [Q3].

It is also worth of mentioning that our approach to the lower bounds is very different from that in the preceding works [Q3], [KQ] based on deterministic estimates valid for any polynomial, see e.g. lower bound in (1.6) below. It would be interesting to check whether the optimisation of parameters in deterministic estimates enables to this approach to compete with our lower bound on the whole range of \( \tau \).

Unfortunately, if \( \tau \) is relatively small, namely \( \log \tau \ll \log N \), the gap between the upper and the lower bounds in Theorem 1.1 becomes rather significant due to the small factor \( \Psi^* \) in the lower bound. Our next result, although being not optimal, shows that the presence of \( \Psi^* \) is really crucial.
Theorem 1.2. Let $0 \leq \sigma < 1/2$. Then there exists a constant $C_\sigma$ such that for any integer $N \geq 2$ and $\tau > \exp\{(\log \log N)^2\}$ it is true that

$$N^{1/2-\sigma} \Psi^*\left(\frac{N}{p_\tau}, p_\tau/2\right)^{1/2} \leq \mathbb{E} \sup_{t \in \mathbb{R}} \left| \sum_{n \in E_\tau} \varepsilon_n n^{-\sigma-\text{int}} \right| \leq C_\sigma N^{1/2-\sigma} \tau^{1/2} \Psi^*\left(\frac{N}{p_\tau}, p_\tau/2\right)^{1/2}.$$

Estimates of $\ell_1$-type. The reader familiar with evaluation of Rademacher processes may wonder whether the brutal $\ell_1$-estimates

$$\mathbb{E} \sup_{t \in \mathbb{R}} \left| \sum_{n \in E_\tau} \varepsilon_n n^{-\sigma-\text{int}} \right| \leq \sum_{n \in E_\tau} n^{-\sigma} := L(N, \tau)$$

are useful at least in some zone of parameters. In our context the answer is negative. Actually, one can show that

$$L(N, \tau) \geq c N^{1-\sigma} \Psi^*(N, p_\tau) \sim c N^{1-\sigma} \rho \left(\frac{\log N}{\log p_\tau}\right).$$

This is too much for good upper bounds, as one can see from two following examples. The first one handles large $\tau$ and the second one deals with small $\tau$.

1) Let $\varphi_\tau \sim N^h$ with $1/2 < h \leq 1$. Then we see that

$$L(N, \tau) \geq c(h) N^{1-\sigma}$$

while the upper bound from Theorem 1.1 yields a better estimate

$$\mathbb{E} \sup_{t \in \mathbb{R}} \left| \sum_{n \in E_\tau} \varepsilon_n n^{-\sigma-\text{int}} \right| \leq C_\sigma N^{1/2-\sigma} \tau^{1/2} \Psi^*\left(\frac{N}{p_\tau}, p_\tau/2\right)^{1/2} \approx C_\sigma N \left(1 + \frac{h}{2}\right)^{1/2} \lesssim C_\sigma N \log N.$$

The gap between the two upper bounds is at least logarithmic for $h = 1$ and polynomial for $h < 1$.

2) Let $\varphi_\tau \sim \exp\{\log \log N\}^A$ with $A \geq 2$. Then we see that

$$L(N, \tau) \geq c N^{1-\sigma} \rho \left(\frac{\log N}{\log \log N}\right)^A \geq c N^{1-\sigma} \exp\left(-c \frac{\log N}{(\log \log N)^{A-1}}\right)$$

while the upper bound from Theorem 1.2 yields a better estimate

$$\mathbb{E} \sup_{t \in \mathbb{R}} \left| \sum_{n \in E_\tau} \varepsilon_n n^{-\sigma-\text{int}} \right| \leq C_\sigma N^{1/2-\sigma} \exp\left(-c \frac{\log N}{(\log \log N)^{A-1}}\right).$$

The gap between the two upper bounds is polynomial. One observes that $\ell_1$-estimate becomes even worse when $\tau$ decreases and approaches the critical zone.

Rudin-Shapiro polynomials. The upper bound in Theorem 1.2 is known to be related with Rudin-Shapiro problem for Dirichlet polynomials. Let us recall first the classical setting. For any trigonometric polynomial we have

$$\sum_{n=0}^{N-1} |a_n| \leq \sup_{t \in \mathbb{R}} \left| \sum_{n=0}^{N-1} a_n e^{int} \right| \leq \sum_{n=0}^{N-1} |a_n|. \quad (1.6)$$
The arguments for getting the lower bound are the inequality between the sup-norm and $L_2$-norm, the orthogonality of $(e^{int})_n$ and Hölder inequality.

Rudin and Shapiro constructed a fairly simple sequence $a_n \in \{-1, +1\}$ such that the right order of the lower bound is attained:

$$\sup_{t \in \mathbb{R}} \left| \sum_{n=0}^{N-1} a_n e^{int} \right| \leq (2 + \sqrt{2}) \sqrt{N + 1} \sim (2 + \sqrt{2}) \frac{\sum_{n=0}^{N-1} |a_n|}{\sqrt{N}}.$$ 

Consider now the Dirichlet polynomials instead of the trigonometric ones. It is known from [KQ] and [Q3] that for any $(a_n)$

$$\sup_{t \in \mathbb{R}} \left| \sum_{n=0}^{N-1} a_n n^t \right| \geq \alpha_1 \frac{\sum_{n=0}^{N-1} |a_n|}{\sqrt{N}} \exp\{\beta_1 \sqrt{\log N \log \log N}\}. \quad (1.7)$$

and for some $(a_n)$

$$\sup_{t \in \mathbb{R}} \left| \sum_{n=0}^{N-1} a_n n^t \right| \leq \alpha_2 \frac{\sum_{n=0}^{N-1} |a_n|}{\sqrt{N}} \exp\{\beta_2 \sqrt{\log N \log \log N}\}, \quad (1.7)$$

with some universal constants $\alpha_{1,2}, \beta_{1,2}$.

Therefore, the lower bound for Dirichlet polynomials is necessarily worse than in the classical case. Notice also that the construction of example (1.7) in [Q3] is a probabilistic one; no explicit example of Rudin-Shapiro type is known for Dirichlet polynomials. It turns out that Theorem 1.2 generates a new family of random polynomials satisfying (1.7).

Indeed, take any $\sigma \in [0, \frac{1}{2})$ and choose $\tau$ in the optimal way. Namely, let

$$\log \tau \sim \left(\frac{\log N}{2}\right)^{1/2} (\log \log N)^{1/2}.$$ 

Set $a_n = \varepsilon_n n^{-\sigma} 1_{\{n \in E_\tau\}}$. It is easy to see that

$$\sum_{n=0}^{N} |a_n| = \sum_{n \in E_\tau} n^{-\sigma} \sqrt{N} \geq c N^{1/2-\sigma} \Psi_*(N, p_\tau),$$

while by Theorem 1.2 we have the bound for the average of the left hand side in (1.7):

$$\mathbb{E} \sup_{t \in \mathbb{R}} \left| \sum_{n=0}^{N-1} a_n n^t \right| \leq C_\sigma N^{1/2-\sigma} \tau^{1/2} \Psi^* \left(\frac{N}{p_\tau^2}, p_\tau\right)^{1/2}$$

$$= C_\sigma N^{1/2-\sigma} \exp\left\{\frac{1}{2} \left(\frac{\log N}{2}\right)^{1/2} (\log \log N)^{1/2} + \frac{1}{2} \log \Psi^* \left(\frac{N}{p_\tau^2}, p_\tau\right)\right\}.$$ 

Since by properties of Dickman function,

$$\log \Psi^* \left(\frac{N}{p_\tau^2}, p_\tau\right) \sim \log \rho \left(\frac{\log (N/p_\tau)}{\log p_\tau}\right) \sim - \frac{\log (N/p_\tau)}{\log p_\tau} \log \frac{\log (N/p_\tau)}{\log p_\tau}$$

$$\sim - \frac{\log N}{\log \tau} \log \frac{\log N}{\log \tau} \sim - (2 \log N)^{1/2} \left(\frac{\log \log N}{2}\right)^{1/2} = - \left(\frac{\log N}{2}\right)^{1/2} (\log \log N)^{1/2}$$
and by the same arguments
\[ \log \Psi^* (N, p_\tau) \sim -\left( \frac{\log N}{2} \right)^{1/2} \left( \log \log N \right)^{1/2}, \]
we finally obtain
\[ \mathbb{E} \sup_{t \in \mathbb{R}} \left| \sum_{n=0}^{N-1} a_n n^t \right| \leq C \sigma \sum_{n=0}^{N} |a_n| \exp \left\{ -\left( \frac{\log N}{2} \right)^{1/2} \left( \log \log N \right)^{1/2} \right\}, \]
as required in (1.7).

A particular case of this example with \( \sigma = 0 \) was considered in [Q3]. Our calculation yields a slightly better constant in the exponent. The question about the best possible constant raised in [KQ] seems still to be open.

2. Proof of the upper bound in Theorem 1.1.

The principle of the proof of the upper bound is as follows. Once operated the reduction to the study of a random polynomial \( Q \) on the multidimensional torus by using (1.1), the proof then consists of two different steps based on a decomposition \( Q = Q_1 + Q_2 \). The study of the supremum of the polynomial \( Q_1 \) is made by using the metric entropy method.

The investigation of the supremum of the polynomial \( Q_2 \) is undertaken by using first the contraction principle, reducing the study to the one of a complex valued Gaussian process. The latter task is carried out by means of Slepian’s Comparison Lemma, and by a careful study of the \( L^2 \)-metric induced by this process.

Now, we turn to the rigorous proof of the upper bound and introduce some notation.

We can represent \( E_\tau \) as the union of disjoint sets
\[ E_j = \{ 2 \leq n \leq N : P^+(n) = p_j \}, \quad j = 1, \ldots, \tau. \]

For \( \tilde{z} \in T^\tau \) we put
\[ Q(\tilde{z}) = \sum_{j=1}^{\tau} \sum_{n \in E_j} \varepsilon_n n^{-\sigma} e^{2i\pi (a(n), \tilde{z})}. \]
By (1.1) we have
\[ \sup_{t \in \mathbb{R}} \left| \sum_{j=1}^{\tau} \sum_{n \in E_j} \varepsilon_n n^{-\sigma} e^{2i\pi (a(n), \tilde{z})} \right| = \sup_{\tilde{z} \in T^\tau} |Q(\tilde{z})|. \]

Let \( 1 \leq \nu < \tau \) be fixed. Write \( Q = Q_1 + Q_2 \) where
\[ Q_1(\tilde{z}) = \sum_{P^+(n) \leq p_\nu} \varepsilon_n n^{-\sigma} e^{2i\pi (a(n), \tilde{z})}, \quad Q_2(\tilde{z}) = \sum_{p_\nu < P^+(n) \leq p_\tau} \varepsilon_n n^{-\sigma} e^{2i\pi (a(n), \tilde{z})}. \]

First, evaluate the supremum of \( Q_2 \). Introduce the following random process
\[ X^\varepsilon(\gamma) = \sum_{\nu < j \leq \tau} \alpha_j \sum_{n \in E_j} \varepsilon_n n^{-\sigma} e^{2i\pi (a(n), \tilde{z})}, \quad \gamma \in \Gamma, \]
where \( \gamma = (\alpha_j)_{\nu < j \leq \tau}, (\beta_m)_{1 \leq m \leq N/2} \) and \( \Gamma = \{ \gamma : |\alpha_j| \vee |\beta_m| \leq 1, \nu < j \leq \tau, 1 \leq m \leq N/2 \} \).

Writing

\[
Q_2(\bar{z}) = \sum_{\nu < j \leq \tau} e^{2i\pi \nu j} \sum_{n \in E_j} \varepsilon_n n^{-\sigma} e^{2i\pi \{ \sum_{k \neq j} a_k(n) z_k + [a_j(n) - 1] z_j \}}
\]

and considering separately the imaginary and real parts of \( e^{2i\pi a_j(n) z_j} \) and \( e^{2i\pi \sum_{k \neq j} a_k(n) z_k} \), easily shows that

\[
Q_2(\bar{z}) = X^e(\gamma_1(\bar{z})) + i X^e(\gamma_2(\bar{z})) = X^e(\gamma_3(\bar{z})) + X^e(\gamma_4(\bar{z}))
\]

where

\[
\gamma_1(\bar{z}) = (\{ \cos(2\pi \nu j) \}_{\nu < j \leq \tau}, (\cos(2\pi \sum_k a_k(m) z_k))_{1 \leq m \leq N/2})
\]

\[
\gamma_2(\bar{z}) = (\{ \sin(2\pi \nu j) \}_{\nu < j \leq \tau}, (\cos(2\pi \sum_k a_k(m) z_k))_{1 \leq m \leq N/2})
\]

etc. Therefore, we obtain

\[
\sup_{\bar{z} \in \mathcal{T}_\tau} |Q_2(\bar{z})| \leq 4 \sup_{\gamma \in \Gamma} |X^e(\gamma)|.
\]

By the contraction principle ([K] p.16-17)

\[
\mathbb{E} \sup_{\bar{z} \in \mathcal{T}_\tau} |Q_2(\bar{z})| \leq 4 \sqrt{\frac{\pi}{2}} \mathbb{E} \sup_{\gamma \in \Gamma} |X^e(\gamma)|,
\]

where \( \{X(\gamma), \gamma \in \Gamma\} \) is the same process as \( X^e(\gamma) \) except that the Rademacher random variables \( \varepsilon_n \) are replaced by independent \( N(0,1) \) random variables \( \mu_n \):

\[
X(\gamma) = \sum_{\nu < j \leq \tau} \alpha_j \sum_{n \in E_j} \mu_n n^{-\sigma} \beta_m \frac{\mu_j}{p_j}.
\]

The problem now reduces to estimating the supremum of the real valued Gaussian process \( X \). Towards this aim, we examine the \( L^2 \)-norm of its increments:

\[
\|X - X'\|^2 = \sum_{\nu < j \leq \tau} \sum_{n \in E_j} n^{-2\sigma} [\alpha_j \beta_m \frac{\mu_j}{p_j} - \alpha'_j \beta'_m \frac{\mu'_j}{p'_j}]^2 \leq 2 \sum_{\nu < j \leq \tau} \sum_{n \in E_j} n^{-2\sigma} \left[ (\alpha_j - \alpha'_j)^2 + (\beta_m \frac{\mu_j}{p_j} - \beta'_m \frac{\mu'_j}{p'_j})^2 \right],
\]

where we have used the identity \( \alpha_j \beta_m \frac{\mu_j}{p_j} - \alpha'_j \beta'_m \frac{\mu'_j}{p'_j} = (\alpha_j - \alpha'_j) \beta_m \frac{\mu_j}{p_j} + (\beta_m \frac{\mu_j}{p_j} - \beta'_m \frac{\mu'_j}{p'_j}) \alpha'_j \).

The "\( \alpha \)" component part is easily controlled as follows,

\[
\sum_{\nu < j \leq \tau} \sum_{n \in E_j} n^{-2\sigma} (\alpha_j - \alpha'_j)^2 \leq \sum_{\nu < j \leq \tau} (\alpha_j - \alpha'_j)^2 \frac{p_j^{-2\sigma}}{m \leq N/p_j} \sum_{m \leq N/p_j} m^{-2\sigma} \leq C_\sigma \sum_{\nu < j \leq \tau} (\alpha_j - \alpha'_j)^2 \left( \frac{N^{1-2\sigma}}{p_j} \right).
\]

(2.1)

For the "\( \beta \)" component part, we have

\[
\sum_{\nu < j \leq \tau} \sum_{n \in E_j} \frac{(\beta_m \frac{\mu_j}{p_j} - \beta'_m \frac{\mu'_j}{p'_j})^2}{n^{2\sigma}} \leq \sum_{m \leq N/p_j} (\beta_m - \beta'_m)^2 \left( \sum_{\nu < j \leq \tau} \frac{1}{mp_j^{2\sigma}} \right) \left( \sum_{m \leq N/p_j} \frac{1}{mp_j^{2\sigma}} \right) = K_m^2 (\beta_m - \beta'_m)^2.
\]

(2.2)
Now we evaluate the coefficients $K_m$. Consider two cases.

1) $m \leq N/p_{\tau}$. Then $mp_j \leq mp_{\tau} \leq N$ for all $j \leq \tau$ and, by using the standard estimate (see [HW], Theorem 8, p.10)

$$p_j \sim j \log j$$

we have

$$K_m^2 = \sum_{\nu<j \leq \tau} (mp_j)^{-2\sigma} \leq m^{-2\sigma} \sum_{j \leq \tau} p_j^{-2\sigma} \leq C m^{-2\sigma} \sum_{j \leq \tau} (j \log j)^{-2\sigma} \leq C m^{-2\sigma} \tau^{-2\sigma} \log \tau$$

Thus

$$\sum_{m \leq N/p_{\tau}} K_m \leq C_{\sigma} \frac{\tau^{1/2}}{p_{\tau}^2} \sum_{m \leq N/p_{\tau}} m^{-\sigma} \leq C_{\sigma} \left( \frac{N}{p_{\tau}} \right)^{1-\sigma} \frac{\tau^{1/2}}{p_{\tau}^2} \leq C_{\sigma} \frac{N^{1-\sigma} \tau^{1/2}}{p_{\tau} \log \tau}.$$

2) $N/p_{\nu} \geq m > N/p_{\tau}$. Then take a unique $k \in (\nu, \tau]$ such that $N/p_k < m \leq N/p_{k-1}$. We have

$$K_m^2 = \sum_{\nu<j \leq k-1} (mp_j)^{-2\sigma} \leq m^{-2\sigma} \sum_{j \leq k-1} p_j^{-2\sigma} \leq C_{\sigma} m^{-2\sigma} \sum_{j \leq k-1} (j \log j)^{-2\sigma} \leq C_{\sigma} m^{-2\sigma} \frac{k^{-2\sigma}}{(\log k)^2\sigma} \leq C_{\sigma} m^{-2\sigma} \frac{k}{p_k^{2\sigma}} \leq C_{\sigma} m^{-2\sigma} \frac{k}{(N/m)^{2\sigma}} \leq C_{\sigma} \frac{k}{N^{2\sigma}}.$$

Since $k \log k \leq C p_k \leq C \frac{N}{m}$, we have

$$k \leq C \frac{N}{m} (\log(m/N))^{-1}.$$

We arrive at $K_m \leq C_{\sigma} N^{-\sigma} \left( \frac{N}{m} \right)^{1/2} (\log(N/m))^{-1/2}$. It follows that

$$\sum_{m \leq N/p_{\nu}} K_m \leq C_{\sigma} N^{-\sigma} \sum_{m \leq N/p_{\nu}} \left( \frac{N}{m} \right)^{1/2} (\log(m/N))^{-1/2} \leq C_{\sigma} N^{1-\sigma} \int_{0}^{1/p_{\nu}} u^{-1/2} (\log(1/u))^{-1/2} du \leq C_{\sigma} N^{1-\sigma} p_{\nu}^{-1/2} (\log p_{\nu})^{-1/2} \leq C_{\sigma} N^{1-\sigma} \nu^{1/2} \log \nu.$$

Now define a second Gaussian process by putting for all $\gamma \in \Gamma$

$$Y(\gamma) = \sum_{\nu<j \leq \tau} \left( \frac{N^{1-2\sigma}}{p_j} \right)^{1/2} \alpha_j \xi'_j + \sum_{m \leq N/p_{\nu}} K_m \beta_m \xi''_m := Y'_\gamma + Y''_\gamma,$$
where \( \xi'_i, \xi''_j \) are independent \( \mathcal{N}(0,1) \) random variables. It follows from (2.1) and (2.2) that for some suitable constant \( C_\sigma \), one has the comparison relations: for all \( \gamma, \gamma' \in \Gamma \),

\[
\|X_\gamma - X_{\gamma'}\|_2 \leq C_\sigma \|Y_\gamma - Y_{\gamma'}\|_2.
\]

By virtue of the Slepian comparison lemma (see [L], Theorem 4 p.190), since \( X_0 = Y_0 = 0 \), we have

\[
E \sup_{\gamma \in \Gamma} |X_\gamma| \leq 2E \sup_{\gamma \in \Gamma} X_\gamma \leq 2C_\sigma E \sup_{\gamma \in \Gamma} Y_\gamma \leq 2C_\sigma E \sup_{\gamma \in \Gamma} |Y_\gamma|.
\]

It remains to evaluate the supremum of \( Y \). First of all,

\[
E \sup_{\gamma \in \Gamma} |Y'(\gamma)| \leq N^{\frac{1}{2} - \sigma} \sum_{\nu < j \leq \tau} p_j^{-1/2}
\]

By (2.3), we have

\[
\sum_{\nu < j \leq \tau} p_j^{-1/2} \leq \sum_{i < j \leq \tau} p_j^{-1/2} \leq \frac{C_\tau^{1/2}}{(\log \tau)^{1/2}},
\]

thus

\[
E \sup_{\gamma \in \Gamma} |Y'(\gamma)| \leq C N^{\frac{1}{2} - \sigma} \frac{\tau^{1/2}}{(\log \tau)^{1/2}}. \tag{2.4}
\]

To control the supremum of \( Y'' \), we use our estimates for the sums of \( K_m \) and write that

\[
E \sup_{\gamma \in \Gamma} |Y''(\gamma)| \leq \sum_{m \leq N/p_\nu} K_m 
\]

\[
\leq C_\sigma \left( \frac{N^{1-\sigma}}{\nu^{1/2} \log \nu} + \frac{N^{1-\sigma}}{\tau^{1/2} \log \tau} \right) \leq C_\sigma N^{1-\sigma} \frac{1}{\nu^{1/2} \log \nu}. \tag{2.5}
\]

Now, we turn to the supremum of \( Q_1 \). Towards this aim, introduce the auxiliary Gaussian process

\[
Y(\underline{z}) = \sum_{P^+(n) \leq p_\nu} n^{-\sigma} \{ \vartheta_n \cos 2\pi \langle \underline{a}(n), \underline{z} \rangle + \vartheta'_n \sin 2\pi \langle \underline{a}(n), \underline{z} \rangle \}, \quad \underline{z} \in \mathbb{T}^\nu,
\]

where \( \vartheta, \vartheta'_j \) are independent \( \mathcal{N}(0,1) \) random variables. By symmetrization (see e.g. Lemma 2.3 p. 269 in [PSW]),

\[
E \sup_{\underline{z} \in \mathbb{T}^\nu} |Q_1(\underline{z})| \leq \sqrt{8\pi} E \sup_{\underline{z} \in \mathbb{T}^\nu} |Y(\underline{z})|,
\]

so that we are again led to evaluating the supremum of a real valued Gaussian process. For \( \underline{z}, \underline{z}' \in \mathbb{T}^\nu \) put \( \|Y(\underline{z}) - Y(\underline{z}')\|_2 := d(\underline{z}, \underline{z}') \),
and observe that
\[
d(z, z')^2 = 4 \sum_{n \in P^+(n) \leq p_\nu} \frac{1}{n^{2\sigma}} \sin^2(\pi (q(n, z - z')) \leq 4\pi^2 \sum_{n \in P^+(n) \leq p_\nu} \frac{1}{n^{2\sigma}} |(q(n), z - z')|^2
\]
\[
\leq 4\pi^2 \sum_{n \in P^+(n) \leq p_\nu} n^{-2\sigma} \left[ \sum_{j=1}^\nu |a_j(n)|z_j - z'_j| \right]^2
\]
\[
= 4\pi^2 \sum_{n \in P^+(n) \leq p_\nu} \sum_{j_1, j_2=1}^\nu a_{j_1}(n)a_{j_2}(n) |z_{j_1} - z'_{j_1}| |z_{j_2} - z'_{j_2}| n^{-2\sigma}
\]
\[
= 4\pi^2 \sum_{j_1, j_2=1}^\nu a_{j_1}(n)a_{j_2}(n) |z_{j_1} - z'_{j_1}| |z_{j_2} - z'_{j_2}| n^{-2\sigma}
\]
\[
\leq 4\pi^2 \sum_{j_1, j_2=1}^\nu |z_{j_1} - z'_{j_1}| |z_{j_2} - z'_{j_2}| \sum_{b_1, b_2=1}^\infty b_1b_2 \sum_{n \leq N, a_{j_1}(n)=b_1, a_{j_2}(n)=b_2} n^{-2\sigma}
\]
\[
\leq 4\pi^2 \sum_{j_1, j_2=1}^\nu |z_{j_1} - z'_{j_1}| |z_{j_2} - z'_{j_2}| \sum_{b_1, b_2=1}^\infty b_1b_2 p_{j_1}^{-2b_1}\sigma p_{j_2}^{-2b_2}\sigma \sum_{k \leq N p_{j_1}^{-b_1} p_{j_2}^{-b_2}} k^{-2\sigma}
\]
\[
\leq C_\sigma N^{1-2\sigma} \sum_{j_1, j_2=1}^\nu |z_{j_1} - z'_{j_1}| |z_{j_2} - z'_{j_2}| \sum_{b_1, b_2=1}^\infty b_1b_2 p_{j_1}^{-2b_1}\sigma p_{j_2}^{-2b_2}\sigma [p_{j_1}^{-b_1} p_{j_2}^{-b_2}]^{-1-2\sigma}
\]
\[
= C_\sigma N^{1-2\sigma} \sum_{j_1, j_2=1}^\nu |z_{j_1} - z'_{j_1}| |z_{j_2} - z'_{j_2}| \sum_{b_1, b_2=1}^\infty b_1b_2 p_{j_1}^{-b_1} p_{j_2}^{-b_2}
\]
\[
= C_\sigma N^{1-2\sigma} \left\{ \sum_{j=1}^\nu |z_j - z'_j| \sum_{b=1}^\infty b p_j^{-b} \right\}^2.
\]

Thus,
\[
d(z, z') \leq C_\sigma N^{1/2-\sigma} \left\{ \sum_{j=1}^\nu |z_j - z'_j| \sum_{b=1}^\infty b p_j^{-b} \right\}.
\]

**Remark.** In the middle of the long calculation, we did not use the fact that the variable \(k\) satisfies \(P^+(k) \leq p_\nu\). Actually, this observation permits to introduce an extra factor related to the Dickman function, something like \(\rho(\log N/\log \nu)\). This is helpful once \(\nu\) is very small with respect to \(N\) (see the upper bound in Theorem 1.2).

Now we explore the entropy properties of the metric space \((T^\nu, d)\). Towards this aim, take \(\varepsilon \in (0, 1)\) and cover \(T^\nu\) by rectangular cells so that if \(z\) and \(z'\) belong to the same cell we have
\[
|z_j - z'_j| \leq \begin{cases} \frac{\varepsilon}{\log \log \nu}, & 1 \leq j \leq \nu^{1/2}, \\ \frac{\varepsilon}{\nu^{1/2}}, & \nu^{1/2} < j \leq \nu. \end{cases}
\]

Thus, every cell is a product of two cubes of different size and dimension. The necessary number of cells \(M(\varepsilon)\) is bounded as follows,
\[
M(\varepsilon) \leq \left( \frac{\log \log \nu}{\varepsilon} \right)^{\nu^{1/2}} \varepsilon^{-(\nu - \nu^{1/2})} = (1/\varepsilon)^\nu(\log \log \nu)^{\nu^{1/2}}.
\]
Let us now evaluate the distance $d(z, z')$ for $z, z'$ satisfying (2.8). By (2.7) we have

$$d(z, z') \leq C\sigma N^{1/2-\sigma} \{d_1 + d_2 + d_3\},$$

where

$$d_1 = \sum_{j=1}^{\nu} |z_j - z_j'| \sum_{b=2}^{\infty} b p_j^{-b},$$

$$d_2 = \sum_{\nu^{1/2} < j \leq \nu} |z_j - z_j'| p_j^{-1},$$

$$d_3 = \sum_{j \leq \nu^{1/2}} |z_j - z_j'| p_j^{-1}.$$

For any $j \geq 1$ we have

$$\sum_{b=2}^{\infty} b p_j^{-b} = \sum_{b=2}^{\infty} b \left(\frac{2}{p_j}\right)^{b-2} \leq \left(\frac{2}{p_j}\right)^2 \sum_{b=2}^{\infty} b 2^{-b} = C p_j^{-2}. \tag{2.9}$$

Hence,

$$d_1 \leq \left(\sum_{j=1}^{\nu} C p_j^{-2}\right) \max_{j \leq \nu} |z_j - z_j'| \leq C\varepsilon.$$

Similarly,

$$d_2 \leq \left(\sum_{\nu^{1/2} < j \leq \nu} p_j^{-1}\right) \max_{\nu^{1/2} < j \leq \nu} |z_j - z_j'| \leq C \left(\sum_{\nu^{1/2} < j \leq \nu} (j \log j)^{-1}\right) \varepsilon \leq C \int_{\nu^{1/2}}^{\nu} \frac{du}{u \log u} \varepsilon = C(\log \log \nu - \log(\log \nu/2)) \varepsilon = C(\log 2) \varepsilon.$$

Finally,

$$d_3 \leq \left(\sum_{j=1}^{\nu} p_j^{-1}\right) \max_{j \leq \nu^{1/2}} |z_j - z_j'| \leq C \left(\sum_{j=1}^{\nu} (j \log j)^{-1}\right) \frac{\varepsilon}{\log \log \nu} \leq C \varepsilon.$$

By summing up three estimates, we have $d(z, z') \leq C\sigma N^{1/2-\sigma} \varepsilon$ which enables the evaluation of the metric entropy.

Let $N(T^\nu, d, u)$ be the minimal number of balls of radius $u$ that cover the space $(T^\nu, d)$. We have

$$\log N(T^\nu, d, C\sigma N^{1/2-\sigma} \varepsilon) \leq \log M(\varepsilon) \leq \nu |\log \varepsilon| + \nu^{1/2} \cdot \log \log \nu.$$  

Observe also that

$$\|Y(\bar{z})\|_2 \leq C\sigma N^{1/2-\sigma}, \quad \bar{z} \in T^\nu. \tag{2.10}$$
Hence, $D := \text{diam}(T^\nu, d) \leq C_{\sigma} N^{\frac{1}{2} - \sigma}$, and by the classical Dudley’s entropy theorem (see [L], Theorem 1 p.179), for any fixed $z \in T^\nu$

$$E \sup_{z' \in T^\nu} |\Upsilon(z') - \Upsilon(z)| \leq C_{\sigma} \int_0^D \left[ \log N(T^\nu, d, u) \right]^{1/2} du \leq C_{\sigma} \int_0^{C_{\sigma} N^{1/2 - \sigma}} \left[ \log N(T^\nu, d, u) \right]^{1/2} du$$

$$= C_{\sigma} N^{1/2 - \sigma} \int_0^1 \left[ \log N(T^\nu, d, C_{\sigma} N^{1/2 - \sigma} \varepsilon) \right]^{1/2} d\varepsilon$$

$$\leq C_{\sigma} N^{1/2 - \sigma} \int_0^1 \left[ \nu |\log \varepsilon| + \log \log \nu \cdot \nu^{1/2} \right]^{1/2} d\varepsilon$$

$$\leq C_{\sigma} N^{1/2 - \sigma} \nu^{1/2}.$$ 

Using again (2.10), we have

$$E \sup_{z' \in T^\nu} |\Upsilon(z')| \leq C_{\sigma} N^{1/2 - \sigma} \nu^{1/2}. \quad (2.11)$$

The final stage of the proof provides the optimal choice of the parameter $\nu$ balancing the quantities (2.4), (2.5), and (2.11). As suggests the Theorem’s claim, we consider three cases.

**Case 1.** $N^{1/2} \leq \tau \leq N$. Obviously, this case contains the results of Halász and Queffélec. In this case we choose

$$\nu = \frac{\tau}{\log N}$$

thus balancing (2.4) and (2.11). We obtain from both terms the bound $C_{\sigma} \frac{N^{1/2 - \sigma} \tau^{1/2}}{(\log N)^{1/2}}$ while the term (2.5) is negligible. The correctness condition $\nu \leq \tau$ is obvious.

**Case 2.** $N^{1/2} (\log N)^{-1} \leq \tau \leq N^{1/2}$. In this case we choose

$$\nu = N^{1/2} (\log N)^{-1}$$

thus balancing (2.5) and (2.11). We obtain from both terms the bound $C_{\sigma} \frac{N^{3/4 - \sigma}}{(\log N)^{1/2}}$ while the term (2.4) is negligible. The correctness condition $\nu \leq \tau$ is obvious for the range under consideration.

**Case 3.** $1 \leq \tau \leq N^{1/2} (\log N)^{-1}$. Here we just set $\nu = \tau$. It means that we do not need the splitting of the polynomial in two parts. Formally, the quantities (2.4) and (2.5) are not necessary and we obtain the bound $C_{\sigma} N^{1/2 - \sigma} \tau^{1/2}$ directly from (2.11).

The upper bound is proved completely.

### 3. Proof of the lower bound in Theorem 1.1.

Let $\underline{d} = \{d_n, n \geq 1\}$ be a sequence of reals. Recall that by (1.1) we have

$$\sup_{t \in \mathbb{R}} \left| \sum_{j=1}^\tau \sum_{n \in E_j} d_n \varepsilon_n n^{-\sigma - it} \right| = \sup_{z \in T^\tau} |Q(z)|,$$

where

$$Q(z) = \sum_{j=1}^\tau \sum_{n \in E_j} d_n \varepsilon_n n^{-\sigma} e^{2\pi i \langle a(n), z \rangle}.$$
Consider the subset \( Z \) of \( T^\tau \) defined by

\[
Z = \{ z = \{ z_j, 1 \leq j \leq \tau \} : z_j = 0, \text{ if } j \leq \tau / 2, \text{ and } z_j \in \{0, 1/2\}, \text{ if } j \in (\tau / 2, \tau] \}.
\]

Observe that the imaginary part of \( Q \) vanishes on \( Z \), since for any \( z \in Z \) and any \( n \) it is true that

\[
e^{2\pi (a(n), z)} = \cos(2\pi (a(n), z)) = (-1)^{2(a(n), z)}.
\]

Hence, \( Q \) takes the following simple form on \( Z \)

\[
Q(z) = \sum_{\tau / 2 < j \leq \tau} \sum_{n \in E_j} d_n \varepsilon_n n^{-\sigma} (-1)^{2(a(n), z)}.
\]

This is no longer a trigonometric polynomial, but simply a finite rank Rademacher process.

For \( j \in (\tau / 2, \tau] \) define

\[
L_j = \{ n = p_j \hat{n} : \hat{n} \leq \frac{N}{p_j} \text{ and } P^+(\hat{n}) \leq p_{\tau / 2} \}.
\]

Since

\[
E_j \supset L_j, \quad j = 1, \ldots, \tau,
\]

the sets \( L_j \) are pairwise disjoint.

Put for \( z \in Z \),

\[
Q'(z) = \sum_{\tau / 2 < j \leq \tau} \sum_{n \in L_j} d_n \varepsilon_n n^{-\sigma} (-1)^{2(a(n), z)}.
\]

We now recall a useful fact.

**Lemma 3.1.** Let \( X = \{X_z, z \in Z\} \) and \( Y = \{Y_z, z \in Z\} \) be two finite sets of random variables defined on a common probability space. We assume that \( X \) and \( Y \) are independent and that the random variables \( Y_z \) are all centered. Then

\[
E \sup_{z \in Z} |X_z + Y_z| \geq E \sup_{z \in Z} |X_z|.
\]

**Proof.** Let \( \mathcal{F} \) be the \( \sigma \)-field generated by \( Y \). Then

\[
E \sup_{z \in Z} |X_z + Y_z| = E \left( \sup_{z \in Z} |X_z + Y_z| \left| \mathcal{F} \right| \right) \\
\geq E \left( \sup_{z \in Z} |E (X_z + Y_z) \left| \mathcal{F} \right| \right) \\
= E \left( \sup_{z \in Z} |X_z + E Y_z| \right) \\
= E \sup_{z \in Z} |X_z|.
\]

\[\blacksquare\]
Clearly, since \( \{Q(z) - Q'(z), z \in \mathbb{Z}\} \) and \( \{Q'(z), z \in \mathbb{Z}\} \) are independent,
\[
E \sup_{z \in \mathbb{Z}} |Q(z)| \geq E \sup_{z \in \mathbb{Z}} |Q'(z)|.
\]

We now proceed to a direct evaluation of \( Q'(z) \) by proving

**Proposition 3.2.** There exists a universal constant \( c \) such that for any system of coefficients \((d_n)\)
\[
c \sum_{\tau/2 < j \leq \tau} |\sum_{n \in \mathcal{L}_j} d_n^2|^{1/2} \leq E \sup_{z \in \mathbb{Z}} |Q'(z)| \leq \sum_{\tau/2 < j \leq \tau} |\sum_{n \in \mathcal{L}_j} d_n^2|^{1/2}.
\]

**Proof.** For any \( n \in \mathcal{L}_j \), we have \( 2\langle a(n), z \rangle = 2z_j \), so that
\[
\sum_{n \in \mathcal{L}_j} d_n \varepsilon_n (-1)^{2\langle a(n), z \rangle} = (-1)^{2z_j} \sum_{n \in \mathcal{L}_j} d_n \varepsilon_n(\omega).
\]
Thus
\[
Q'(z) = \sum_{\tau/2 < j \leq \tau} (-1)^{2z_j} \sum_{n \in \mathcal{L}_j} d_n \varepsilon_n(\omega).
\]
Let \( \omega \in \Omega \). We can select \( z_j = z_j(\omega) = 0 \) or \( 1/2 \), \( \tau/2 < j \leq \tau \), according to the sign \( + \) or \( - \) of the sum \( \sum_{n \in \mathcal{L}_j} d_n \varepsilon_n(\omega)n^{-\sigma} \). This implies that
\[
\sup_{z \in \mathbb{Z}} |Q'(z)| = \sum_{\tau/2 < j \leq \tau} |\sum_{n \in \mathcal{L}_j} d_n \varepsilon_n|.
\]
Consequently, by the Khintchine inequalities for Rademacher sums [KS]
\[
E \sup_{z \in \mathbb{Z}} |Q'(z)| \geq \sum_{\tau/2 < j \leq \tau} E \left| \sum_{n \in \mathcal{L}_j} d_n \varepsilon_n \right| \geq c \sum_{\tau/2 < j \leq \tau} \left( E \left| \sum_{n \in \mathcal{L}_j} d_n \varepsilon_n n^{-\sigma} \right|^2 \right)^{1/2}
\]
\[
= c \sum_{\tau/2 < j \leq \tau} \left( \sum_{n \in \mathcal{L}_j} d_n^2 \right)^{1/2}.
\]
The upper bound immediately follows from the Cauchy-Schwarz inequality. 

**Corollary 3.3.** If \((d_n)\) is a multiplicative system, we have
\[
E \sup_{z \in \mathbb{Z}} |Q'(z)| \geq c \sum_{\tau/2 < j \leq \tau} d_{p_j} \left( \sum_{n \leq N/p_j} d_n^2 \right)^{1/2}
\]

Now we can finish the proof of Theorem 1.1.

**Proof of the lower bound in Theorem 1.1.** If \( d_n \equiv n^{-\sigma} \), we get from the above corollary
\[
E \sup_{z \in \mathbb{T}} \left| \sum_{j=1}^{\tau} \sum_{n \in \mathcal{E}_j} \varepsilon_n n^{-\sigma} e^{2i\pi \langle a(n)z \rangle} \right| \geq E \sup_{z \in \mathbb{Z}} |Q'(z)|
\]
\[
\geq \frac{C}{N^{\sigma}} \sum_{\tau/2 < j \leq \tau} \# \{ m \leq N/p_j : P^+(m) \leq p_j/2 \}^{1/2}
\]
\[
= \frac{C}{N^{\sigma}} \sum_{\tau/2 < j \leq \tau} \Psi\left( \frac{N}{p_j}, \frac{p_j}{2} \right)^{1/2}.
\]
Since
\[ \Psi\left(\frac{N}{p_j}, \frac{p_{\tau/2}}{2}\right) \geq \Psi\left(\frac{N}{p_{\tau/2}}\right) \]
\[ = \frac{N}{p_{\tau/2}} \Psi^*(\frac{N}{p_{\tau/2}}) \]
\[ \geq \frac{c N}{\tau \log \tau} \Psi^*(\frac{N}{p_{\tau/2}}), \]
we obtain
\[ \mathbb{E} \sup_{z \in T} \left| \sum_{j=1}^{\tau} \sum_{n \in E_j} \varepsilon_n n^{-\sigma} e^{2\pi i g(n) z} \right| \geq \frac{c N}{N^\sigma} \frac{\tau}{2} \left[ \frac{c N}{\tau \log \tau} \Psi^*(\frac{N}{p_{\tau/2}}) \right]^{1/2} \]
\[ = c N^{1/2-\sigma} \left( \frac{\tau}{\log \tau} \right)^{1/2} \Psi^*(\frac{N}{p_{\tau/2}})^{1/2}, \]
as asserted.

4. Proof of Theorem 1.2.
We need to prove the upper bound, since the lower bound was obtained in Theorem 1.1. Moreover, we are only going to show how the calculations concerning the upper bound of Theorem 1.1 should be corrected in order to get an extra Dickman-type factor.

Step 1. Some remarks on semi-asymptotic formula for Dickman function.
We discuss the so called semi-asymptotic formula (see [BT])
\[ \Psi(ax, y) = a^{\alpha(x, y)} \Psi(x, y) (1 + O(1/\bar{u})) \quad (4.1) \]
where \( \bar{u} = \min\{\log x, y\} / \log y \) and
\[ \alpha(x, y) = \frac{\log(1 + y/\log x)}{\log y} = 1 - \frac{\log \log x}{\log y} + \frac{\log(1 + \log x/\log y)}{\log y} \]
\[ = 1 - \frac{\log \log x}{\log y} + O\left(\frac{\log x}{y \log y}\right). \]

Since in our zone \( y > \log x \), we have
\[ O\left(\frac{\log x}{y}\right) = O(1) = o(\log \log x). \]
Therefore \( \alpha \leq 1 \) for \( x \) large enough. We also see that \( \alpha \to 1 \) when \( x \to \infty \), hence \( \alpha \geq 2/3 \) for all \( x \) large enough. We will use in the sequel that \( 2/3 \leq \alpha \leq 1 \).

Step 2. Main estimate and the adjustment of the previous proof.
We still use the notation \( \Psi^*(x, y) = x^{-1} \) but skip \( y \) everywhere since \( y = p_\nu \). In other words, we denote \( \Psi(x) := \Psi(x, p_\nu) \) and \( \Psi^*(x) := \Psi^*(x, p_\nu) \).

Let \( b^* = 1 \) for \( b = 1 \) and \( b^* = 2b/3 \) for \( b = 2, 3, \ldots \). We will prove that for all \( b_1, b_2 \geq 1, j_1, j_2 \leq \nu \)
\[ \Psi\left(\frac{N}{p_{j_1} p_{j_2}}\right) \leq C \frac{N}{p_{b_1 p_{j_1} b_2 p_{j_2}}} \Psi^*\left(\frac{N}{p_{b_2}}\right). \quad (4.2) \]
Once (4.2) is proved, the calculation from (2.6) is updated as follows. Let denote \( D_j = |z_j - z'_j| \). Then

\[
d(z, z')^2 \leq C \sum_{j_1, j_2 \leq \nu} D_{j_1} D_{j_2} \sum_{b_1, b_2 = 1}^{\infty} b_1 b_2 p_{j_1}^{-2b_1} p_{j_2}^{-2b_2} \Psi \left( \frac{N}{p_{j_1}^{b_1} p_{j_2}^{b_2}} \right) \left( \frac{N}{p_{j_1}^{b_1} p_{j_2}^{b_2}} \right)^{-2\sigma}
\]

\[
= C N^{-2\sigma} \sum_{j_1, j_2 \leq \nu} D_{j_1} D_{j_2} \sum_{b_1, b_2 = 1}^{\infty} b_1 b_2 \Psi \left( \frac{N}{p_{j_1}^{b_1} p_{j_2}^{b_2}} \right)
\]

\[
\leq C N^{1-2\sigma} \Psi^* \left( \frac{N}{p^*_\nu} \right) \sum_{j_1, j_2 \leq \nu} D_{j_1} D_{j_2} \sum_{b_1, b_2 = 1}^{\infty} b_1 b_2 p_{j_1}^{-b_1} p_{j_2}^{-b_2} \sigma
\]

\[
= C N^{1-2\sigma} \Psi^* \left( \frac{N}{p^*_\nu} \right) \left( \sum_{j \leq \nu} D_j \sum_{b = 1}^{\infty} b p_j^{-b} \right)^2.
\]

Now everything continues as in the proof of Theorem 1.1 but with an extra factor \( \Psi^* \left( \frac{N}{p^*_\nu} \right) \).

The minor change (corresponding to (2.9) is that

\[
\sum_{b=2}^{\infty} b p_j^{-b} = \sum_{b=2}^{\infty} \left( \frac{2}{p_j} \right)^{b} 2^{-b} \leq \left( \frac{2}{p_j} \right)^{4/3} \sum_{b=2}^{\infty} b 2^{-b} = C \frac{1}{p_j^{4/3}},
\]

hence still

\[
d_1 \leq \sum_{j=1}^{\nu} \frac{C}{p_j^{4/3}} \max_j D_j \leq C \varepsilon.
\]

**Step 3. The proof of inequality (4.2).**

We consider three cases

1. \( b_1, b_2 \geq 2 \). By applying (4.1) with \( x = \frac{N}{p_{j_1}^{b_1} p_{j_2}^{b_2}} \) and \( a = p_{j_1}^{b_1} p_{j_2}^{b_2} \), we get

\[
\Psi(N) = \left( \frac{p_{j_1}^{b_1} p_{j_2}^{b_2}}{p_{j_1}^{b_1} p_{j_2}^{b_2}} \right)^{\alpha} \Psi \left( \frac{N}{p_{j_1}^{b_1} p_{j_2}^{b_2}} \right) (1 + O(1/\bar{u})).
\]

Once \( \bar{u} \) is large enough and \( \alpha \geq 2/3 \) we have

\[
\Psi \left( \frac{N}{p_{j_1}^{b_1} p_{j_2}^{b_2}} \right) \leq C \Psi(N) \left( p_{j_1}^{b_1} p_{j_2}^{b_2} \right)^{-2/3}.
\]

Similarly, we pass from \( \Psi(N) \) to \( \Psi \left( \frac{N}{p^*_\nu} \right) \). By using \( \alpha \leq 1 \), we have

\[
\Psi(N) = [ p_j^{2\alpha} \Psi \left( \frac{N}{p^*_\nu} \right) (1 + O(1/\bar{u})) \leq C p_j^{2} \Psi \left( \frac{N}{p^*_\nu} \right) = CN \Psi^* \left( \frac{N}{p^*_\nu} \right).
\]

By combining two estimates we get

\[
\Psi \left( \frac{N}{p_{j_1}^{b_1} p_{j_2}^{b_2}} \right) \leq C \left( p_{j_1}^{b_1} p_{j_2}^{b_2} \right)^{-2/3} N \Psi^* \left( \frac{N}{p^*_\nu} \right).
\]
as required.

2. $b_1 = b_2 = 1$. By applying (4.1) with $x = \frac{N}{p_{j_1}p_{j_2}}$ and $a = \frac{p_{j_2}^\sigma}{p_{j_1}p_{j_2}}$, we get, using $\alpha \leq 1$,

$$
\Psi \left( \frac{N}{p_{j_1}p_{j_2}} \right) \leq C \left( \frac{p_{j_2}^\sigma}{p_{j_1}p_{j_2}} \right)^\alpha \Psi \left( \frac{N}{p_{j_1}p_{j_2}} \right) \leq C \frac{p_{j_2}^\sigma}{p_{j_1}p_{j_2}} \Psi \left( \frac{N}{p_{j_1}p_{j_2}} \right) = C \left( \frac{N}{p_{j_1}p_{j_2}} \right)^\alpha \Psi^* \left( \frac{N}{p_{j_1}p_{j_2}} \right),
$$

as required.

3. $b_1 = 1, b_2 \geq 2$. By applying (4.1) with $x = \frac{N}{p_{j_1}p_{j_2}}$ and $a = p_{j_2}^{b_2}$, we get, using $\alpha \geq 2/3$,

$$
\Psi \left( \frac{N}{p_{j_1}p_{j_2}} \right) = \left( p_{j_2}^{b_2} \right)^\alpha \Psi \left( \frac{N}{p_{j_1}p_{j_2}} \right) (1 + O(1/\bar{u})),
$$

hence

$$
\Psi \left( \frac{N}{p_{j_1}p_{j_2}} \right) \leq C p_{j_2}^{-2b_2/3} \Psi \left( \frac{N}{p_{j_1}} \right) = C p_{j_2}^{-2} \Psi \left( \frac{N}{p_{j_1}} \right).
$$

Yet, letting $p_{j_2} = 1$ in (4.3), we have

$$
\Psi \left( \frac{N}{p_{j_1}} \right) \leq C \frac{N}{p_{j_1}} \Psi^* \left( \frac{N}{p_{j_1}} \right),
$$

and we are done with case 3. Therefore, the proof of (4.2) is complete.

We finish the section by giving a proof of (1.4'). Only the upper bound needs a proof. Fix some large integer $M$. Let $\{g_n, n \geq 1\}$ be a sequence of independent $\mathcal{N}(0, 1)$ distributed random variables. By contraction principle, there is an absolute constant $C$ such that

$$
\mathbb{E} \sup_{N \leq M} \sup_{t \in \mathbb{R}} \frac{\sum_{n=2}^{N} g_n n^{-\sigma - it}}{N^{1-\sigma}(\log N)^{-1}} \leq C \mathbb{E} \sup_{N \leq M} \sup_{t \in \mathbb{R}} \frac{\sum_{n=2}^{N} g_n n^{-\sigma - it}}{N^{1-\sigma}(\log N)^{-1}}.
$$

We now need the following inequality (see [W1] p.451) which is a simple consequence of Borell-Sudakov-Tsirelson inequality: if $G_1, \ldots, G_N$ are Gaussian random vectors with values in a separable Banach space $(B, \| \cdot \|)$, then

$$
\mathbb{E} \sup_{1 \leq k \leq N} \| G_k \| \leq C \left\{ \sup_{1 \leq k \leq N} \mathbb{E} \| G_k \| + \mathbb{E} \sup_{1 \leq k \leq N} \sigma_k |\zeta_k| \right\}
$$

where $\sigma_k = \sup_{f \in B^*, \| f \| \leq 1} (\mathbb{E} \langle f, G_k \rangle^2)^{1/2}$, $k = 1, \ldots, N$, $\{\zeta_k, 1 \leq k \leq N\}$ is a sequence of independent $\mathcal{N}(0, 1)$ distributed random variables, and $C$ is a universal constant.

Applying this inequality gives

$$
\mathbb{E} \sup_{N \leq M} \sup_{t \in \mathbb{R}} \frac{\sum_{n=2}^{N} g_n n^{-\sigma - it}}{N^{1-\sigma}(\log N)^{-1}} \leq C \mathbb{E} \sup_{N \leq M} \sup_{t \in \mathbb{R}} \frac{\sum_{n=2}^{N} g_n n^{-\sigma - it}}{N^{1-\sigma}(\log N)^{-1}} + C \mathbb{E} \sup_{N \leq M} |\zeta_N\sigma_N|
$$

$$
\leq C_\sigma + C \mathbb{E} \sup_{N \leq M} |\zeta_N\sigma_N|,
$$

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Proposition 3.4. We have obtained

\[ \sigma_N \leq C \sup_{t \in \mathbb{R}} \left\| \sum_{n=2}^{N} g_n n^{-\sigma-it} \right\|_2 \leq C \left( \sum_{n=2}^{N} n^{-2\sigma} \right)^{1/2} \leq C_{\sigma} N^{1/2-\sigma} \frac{1}{N^{1-\sigma} (\log N)^{-1}} = C_{\sigma} \log N \cdot N^{1/2}. \]

It is an obvious fact that \( E \sup_{N \leq M} \left| \sum_{n=2}^{N} g_n n^{-\sigma-it} \right| \) is bounded uniformly in \( M \) by some absolute constant. So that, there exists a constant \( C_{\sigma} \) depending on \( \sigma \) only, such that for any \( M \)

\[ E \sup_{N \leq M} \sup_{t \in \mathbb{R}} \left| \sum_{n=2}^{N} g_n n^{-\sigma-it} \right| \leq C_{\sigma}. \]

The claimed result follows immediately.

Note to conclude that the same argument applies to our upper bounds results with minor modifications (by introducing suitable blocks).

5. Other results.

In this section we test our technique on some other sets of coefficients.

Let \( \{d_n, n \geq 1\} \) be a sequence of multiplicative weights: \( d_{nm} = d_n d_m \) whenever \( n, m \) are coprimes. Denote

\[ B_m = \sum_{2 \leq n \leq m} d_n^2. \quad (5.1) \]

By choosing \( \tau = \mu : = \pi(N) \) in the lower bound of Proposition 3.2, we get

\[ E \sup_{\bar{z} \in \mathcal{T}^s} \left| \sum_{n=2}^{N} d_n \varepsilon_n n^{-\sigma} e^{2i\pi (a(n)) \bar{z}} \right| \geq E \sup_{\bar{z} \in \mathcal{Z}} |Q''(\bar{z})| \geq C N^{-\sigma} \sum_{\mu/2 < j \leq \mu} d_{p_j} \left( \sum_{\bar{n} \leq N/p_j} d_{\bar{n}}^2 \right)^{1/2}. \]

Note that for large \( N \) in the case \( \tau = \mu \) the sets \( \mathcal{L}_j \) reduce to \( \{ n = p_j \bar{n} : \bar{n} \leq N/p_j \} \). Indeed, if \( \bar{n} \leq N/p_j \) and if there is an \( s \geq \mu/2 \) such that \( p_s | \bar{n} \), then this implies that

\[ N \geq p_j p_s \geq p_{\mu/2}^2 \sim (\mu \log \mu)^2/4 \sim N^2/4, \]

which is impossible for large \( N \). Thus necessarily \( P^+(\bar{n}) \leq p_{\mu/2} \). Thereby,

\[ E \sup_{\bar{z} \in \mathcal{T}^s} \left| \sum_{n=2}^{N} d_n \varepsilon_n n^{-\sigma} e^{2i\pi (a(n)) \bar{z}} \right| \geq C N^{-\sigma} \sum_{\mu/2 < j \leq \mu} d_{p_j} \left( \sum_{\bar{n} \leq N/p_j} d_{\bar{n}}^2 \right)^{1/2} \]

\[ = C N^{-\sigma} \sum_{\mu/2 < j \leq \mu} d_{p_j} B_{N/p_j}^{1/2}. \]

We have obtained

Proposition 3.4. There exists a universal constant \( C, N_0 \) such that for any \( 0 \leq \sigma < 1/2 \), any integer \( N \geq N_0 \) and any multiplicative sequence of weights \( (d_n) \)

\[ E \sup_{t \in \mathbb{R}} \left| \sum_{n=2}^{N} \varepsilon_n d_n n^{-\sigma-it} \right| \geq C N^{-\sigma} \sum_{\mu/2 < j \leq \mu} d_{p_j} B_{N/p_j}^{1/2}, \]
where \( B_m \) is defined in (5.1).

Apply this to the case \( d_n = d(n) \), where \( d(n) = \# \{ d : d \mid n \} \) is the divisor function. Although these weights are very irregular, their sums behave regularly, in particular,

\[
\sum_{n=1}^{N} d^2(n) \sim \left( \frac{N}{\pi^2} \right) \log^3 N.
\]

as \( N \) tends to infinity. The last estimate immediately provides \( B_m \sim (m/\pi^2) \log^3 m \), hence (noticing that \( d_p = 2 \) and \( \mu \sim N/\log N \))

\[
\sum_{\mu/2 < j \leq \mu} d_{p_j} B_{N/p_j}^{1/2} \sim \sum_{\mu/2 < j \leq \mu} (2N/p_j \pi^2)^{1/2} \log^{3/2} N \approx \frac{2N^{1/2}}{\pi} \sum_{\mu/2 < j \leq \mu} \frac{1}{p_j} \log^{3/2} \frac{N}{p_j} \approx \frac{2N^{1/2}}{\pi} \sum_{\mu/2 < j \leq \mu} \frac{1}{(j \log j)^{1/2}} \approx \frac{N^{1/2}}{(\log \mu)^{1/2}} \sim \frac{N}{\log N}.
\]

Now, let \( \{ P_k, k \in K \} \) be a finite set of mutually coprime numbers. Consider the set of integers

\[
E = \{ n : n = \prod_{k \in K} P_k^{\alpha_k}, \quad \alpha_k \in \{0, 1\} \}
\]

and the associated Dirichlet polynomial

\[
D_E(t) = \sum_{n \in E} \varepsilon_n n^{-\sigma - it} = \sum_{n=2}^{N} \varepsilon_n \chi_E(n)n^{-\sigma - it},
\]

where \( N = \prod_{k \in K} P_k \). We prove the following.

**Proposition 3.5.** There exists a universal constant \( C \) such that, for any \( \sigma \geq 0 \) and any \( \{ P_k, k \in K \} \)

\[
E \sup_{t \in \mathbb{R}} |D_E(t)| \geq C \prod_{k \in K} \left( 1 + P_k^{-2\sigma} \right)^{1/2} \sup_{G \subseteq K} \frac{\sum_{j \in G} P_j^{-\sigma}}{\prod_{k \in G} \left( 1 + P_k^{-2\sigma} \right)^{1/2}}.
\]

**Proof.** By (1.1) we have

\[
\sup_{t \in \mathbb{R}} |D_E(t)| = \sup_{\zeta \in \mathbb{T}^\mu} |Q(\zeta)|,
\]

where \( \mu = |K| \) and

\[
Q(\zeta) = \sum_{n=2}^{N} \chi_E(n)\varepsilon_n n^{-\sigma} e^{2\pi i (a(n), \zeta)}.
\]

Let \( A \subset K \) and \( B = K \setminus A \). We assume that both \( A \) and \( B \) are nonempty sets. Define for \( j \in B \),

\[
B_j = \{ n \in E : \alpha_k = 0 \text{ if } k \in B, k \neq j, \alpha_j = 1 \}
\]
and $Z \subset \mathbb{T}^r$ by

$$Z = \left\{ \hat{z} = \{ z_k, 1 \leq k \leq 2r \} : z_k = 0, \text{ if } k \in A, \text{ and } z_k \in \{0, 1/2\} \text{ if } k \in B \right\}.$$ 

For $j \in B$, $n \in B_j$ and $z \in Z$, we have $2\langle a(n), \hat{z} \rangle = 2 \sum_{k \in K} \alpha_k z_k = 2z_j = \pm 1$, so that similarly to our previous lower bound

$$\sup_{\hat{z} \in Z} |Q(\hat{z})| \geq \sum_{j \in B} \left| \sum_{n \in B_j} \varepsilon_n n^{-\sigma} \right|,$$

almost surely. Hence

$$\mathbb{E} \sup_{\hat{z} \in Z} |Q(\hat{z})| \geq C \sum_{j \in B} \left( \mathbb{E} \left| \sum_{n \in B_j} \varepsilon_n n^{-2\sigma} \right|^2 \right)^{1/2} = C \sum_{j \in B} P_j^{-\sigma} \left( \prod_{(\alpha_k) \in \{0,1\}^A} P_k^{-2\sigma \alpha_k} \right)^{1/2} \prod_{k \in A} \left( 1 + P_k^{-2\sigma} \right)^{1/2} \left\{ \sum_{j \in B} P_j^{-\sigma} \right\}.$$

Therefore

$$\mathbb{E} \sup_{t \in \mathbb{R}} |D_E(t)| \geq C \sup_{A \subseteq K, A \neq K} \prod_{k \in A} \left( 1 + P_k^{-2\sigma} \right)^{1/2} \left\{ \sum_{j \in A^c} P_j^{-\sigma} \right\} \sup_{A \subseteq K, A \neq K} \prod_{k \in A^c} \left( 1 + P_k^{-2\sigma} \right)^{1/2} \left\{ \sum_{j \in A^c} P_j^{-\sigma} \right\}.$$

\begin{flushright}
\Box
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