Loewy series of parabolically induced $G_1T$-Verma modules

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Abstract

Assuming the Lusztig conjecture on the irreducible characters for reductive algebraic groups in positive characteristic $p$, which is now a theorem for large $p$, we show that the modules for their Frobenius kernels induced from the simple modules of $p$-regular highest weights for their parabolic subgroups are rigid and determine their Loewy series.

Let $G$ be a reductive algebraic group over an algebraically closed field $k$ of positive characteristic $p$, $P$ a parabolic subgroup of $G$, $T$ a maximal torus of $P$, and $G_1$ (resp. $P_1$) the Frobenius kernel of $G$ (resp. $P$). In this paper we study the structure of $G_1T$-modules induced from the simple $P_1T$-modules of $p$-regular highest weights. Thus our study goes parallel to parabolically induced Verma modules in characteristic 0. In case $P$ is a Borel subgroup of $G$, assuming Lusztig’s conjecture for the irreducible characters for $G_1T$, which is now a theorem for large $p$ thanks to [AJS], [KL], [L94], [KT], or more recently to [F], H. H. Andersen and the second author of the present paper showed that the induced modules are rigid and determined their Loewy series [AK]. We now show that the parabolically induced modules are also rigid and describe their Loewy series.

To go into more details, let $B$ be a Borel subgroup of $P$ containing $T$, $\Lambda$ the character group of $B$, $R \subset \Lambda$ the root system of $G$ relative to $T$, and $R^+$ the positive system of $R$ such that the roots of $B$ are $-R^+$. We let $R^s$ denote the set of simple roots, and $I$ a subset of $R^s$ such that the root subgroups $U_\alpha$ of $G$ associated to $\alpha \in I$ generate $P$ together with $B$. Denote by $\hat{\nabla}_P$ the induction functor from the category of $P_1T$-modules to the category of $G_1T$-modules, and let $\hat{L}^P(\lambda)$ denote the simple $P_1T$-module of highest weight $\lambda \in \Lambda$. Our object of study is $\hat{\nabla}_P(\hat{L}^P(\lambda))$. After stating some generalities in $\S\S 1$ and 2, we specialize into the case where $\lambda$ is $p$-regular, i.e., if $\alpha^\vee$ is the coroot of each root $\alpha$ and if $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$, the case when $p \langle \lambda + \rho, \alpha^\vee \rangle$ for all roots $\alpha$. If $M$ is a finite dimensional $G_1T$-module, we call the sum of its simple submodules the socle of $M$ and denote it by $\text{soc}M = \text{soc}^1M$. If $\pi : M \to M/\text{soc}M$ is the quotient, we let $\text{soc}^2M = \pi^{-1}\text{soc}(M/\text{soc}M)$.

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and repeat to construct a filtration $0 < \text{soc}M < \text{soc}^2M < \cdots < M$, called the socle series of $M$. Dually, we call the intersection of all its maximal submodules the radical of $M$ and denote it by $\text{rad}M = \text{rad}^1M$. Letting $\text{rad}^iM = \text{rad}(\text{rad}^{i-1}M)$ for $i > 1$, one obtains a filtration $M > \text{rad}M > \text{rad}^2M > \cdots > 0$, called the radical series of $M$. It is known that the minimal $i$ such that $\text{soc}^iM = M$ and the minimal $j$ such that $\text{rad}^jM = 0$ coincide, called the Loewy length of $M$ and denoted $\ell\ell(M)$. By definition each $\text{soc}_iM = \text{soc}^iM/\text{soc}^{i-1}M$, called the $i$-th socle layer of $M$, and $\text{rad}_iM = \text{rad}^iM/\text{rad}^{i+1}M$, called the $i$-th radical layer of $M$, are semisimple. Any filtration $0 < M^1 < M^2 < \cdots < M$ with each subquotient semisimple has the length at least $\ell\ell(M)$. If the length of the filtration such a filtration $M^\bullet$ is $\ell\ell(M)$, then $\text{soc}^iM \geq M^i \geq \text{rad}^{\ell\ell(M)-i+1}M$ for each $i$. We say $M$ is rigid iff the socle series and the radical series of $M$ coincide. In §3 we employ graded representation theory from [AJS] to show that the induction functor $\hat{\text{End}}$ of a projective $p\mathbb{Z}R$-generator of the block. Assuming Lusztig’s conjecture for the irreducible characters of $G,T$, Andersen, Jantzen and Soergel [AJS] showed that the algebra $E$ for a $p$-regular block is $(p\mathbb{Z}R \times \mathbb{Z})$-graded and is Koszul with respect to its $\mathbb{Z}$-gradation. We show in §4 that the rigidity of $\hat{\text{End}}(\hat{L}^p(\lambda))$ for $p$-regular $\lambda$ follows from a result in [BCS]. Unlike the case $P = B$ the number of $G,T$-composition factors of $\hat{\text{End}}(\hat{L}^p(\lambda))$ varies depending on the highest weight $\lambda$. Nonetheless, we show also in §4 that the Loewy length of $\hat{\text{End}}(\hat{L}^p(\lambda))$ is uniformly $\ell(w_0 w_I) + 1$ with $w_0$ (resp. $w_I$) the longest element of the Weyl group $W$ (resp. $W_I$) of $G$ (resp. $P$). In §5 we determine the Loewy series of $\hat{\text{End}}(\hat{L}^p(\lambda))$.

Given a category $C$ and its objects $X$ and $Y$, $C(X,Y)$ will denote the set of morphisms in $C$ from $X$ to $Y$.

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1° Some generalities

Let $G$ be a reductive algebraic group over an algebraically field $k$ of positive characteristic $p$, $B$ a Borel subgroup of $G$, $T$ a maximal torus of $B$, $\Lambda$ the character group of $B$, $R \subset \Lambda$ the root system of $G$ relative to $T$, and $R^+$ the positive system of $R$ such that the roots of $B$ are $-R^+$. We let $R^s \subset R^+$ denote the set of simple roots, and $\Lambda^+ \subset \Lambda$ the set of dominant weights of $\Lambda$. For each $\alpha \in R$ we let $\alpha^\vee$ denote the coroot of $\alpha$. Let $W$ be the Weyl group of $G$ generated by the reflections $s_\alpha$, $\alpha \in R$, and $\ell$ the length function on $W$ with respect to the simple reflections. Let $w_0$ be the longest element of $W$.

For each $\alpha \in R$ let $U_\alpha$ denote the root subgroup of $G$ associated to $\alpha$. Let $I \subseteq R^s$ and $P = P_I = \langle B, U_\alpha \mid \alpha \in I \rangle$ the standard parabolic subgroup of $G$ associated to $I$, and let $L_I$ denote its standard Levi subgroup. Let $R_I \subseteq R$ denote the root system of $L_I$ with its induced positive system $R_I^+$. Put $\Lambda_P = \{ \lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle = 0 \ \forall \alpha \in I \}$ and $\Lambda_I^+ = \{ \lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in I \}$. Let $W_I$ be the Weyl group of $P$ and $w_I$ its longest element. Put $w_I = w_0 w_I$. Let $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ and $\rho_P = \frac{1}{2} \sum_{\alpha \in R^+ \setminus R_I^+} \alpha \in \Lambda \otimes \mathbb{Z} \mathbb{Q}$. For simplicity we will assume $G$ is semisimple and simply connected. Let $W_p = W \ltimes p\mathbb{Z}R$,
$W_{I,P} = W_I \ltimes p\mathbb{Z}R_I$, and $\rho_I = \frac{1}{2} \sum_{\alpha \in R_I^+} \alpha = \rho - \rho_P$. For $x \in W_p$, we will write $x \cdot \lambda$ for $x(\lambda + \rho) - \rho$. In case $x \in W_{I,P}$, $x \cdot \lambda = x(\lambda + \rho_I) - \rho_I$. We will also let $-(x \cdot \lambda) - 2\rho = -x(\lambda + \rho) - \rho$.

(1.1) Let $\alpha_0$ be the highest short root of $R$ and let $h = \langle \rho, \alpha_0^\vee \rangle + 1$ the Coxeter number of $G$.

**Lemma:** $2\rho_P = w_I\rho + \rho = w_0(w^I \cdot 0) \in \Lambda_P \cap \Lambda^+$ with $\langle 2\rho_P, \alpha^\vee \rangle \in [2, h] \ \forall \alpha \in R^* \setminus I$.

**Proof:** One has

$$w_0(w^I \cdot 0) = w_0(w_0w_I\rho - \rho) = w_I\rho + \rho = \rho + w_I\frac{1}{2} \left( \sum_{\beta \in R^+ \setminus R_I^+} \beta + \sum_{\beta \in R_I^+} \beta \right)$$

$$= \frac{1}{2} \left( \sum_{\beta \in R^+ \setminus R_I^+} \beta + \sum_{\beta \in R_I^+} \beta \right) + \frac{1}{2} \left( \sum_{\beta \in R^+ \setminus R_I^+} \beta - \sum_{\beta \in R_I^+} \beta \right) = \sum_{\beta \in R^+ \setminus R_I^+} \beta = 2\rho_P.$$

If $\alpha \in I$, $\langle 2\rho_P, \alpha^\vee \rangle = \langle w_I\rho + \rho, \alpha^\vee \rangle = \langle \rho, w_I\alpha^\vee \rangle + 1 = 0$, and hence $2\rho_P \in \Lambda_P$. If $\alpha \in R^* \setminus I$, $\langle 2\rho_P, \alpha^\vee \rangle = \langle w_I\rho + \rho, \alpha^\vee \rangle = \langle \rho, w_I\alpha^\vee \rangle + 1 \leq \langle \rho, \alpha_0^\vee \rangle + 1 = h$.

(1.2) If $H \leq K$ are closed subgroups of $G$, we let $\text{ind}_H^K$ denote the induction functor from the category $H\text{Mod}$ of rational $H$-modules to the category $K\text{Mod}$ of rational $K$-modules: if $M \in H\text{Mod}$, $\text{ind}_H^K M = \{ f \in \text{Sch}_K(K, M) \mid f(kh) = h^{-1}f(k) \ \forall k \in K \forall h \in H \}$. We let $\text{Dist}(H)$ (resp. $\text{Dist}(K)$) denote the algebra of distributions on $H$ (resp. $K$) and let $\text{coind}_H^K = \text{Dist}(K) \otimes_{\text{Dist}(H)} \text{dist}(H)$ denote the coinduction functor from $\text{Dist}(H)\text{Mod}$ to $\text{Dist}(K)\text{Mod}$. For a finite dimensional $H$-module $M$ we will mean by $M^*$ the $k$-linear dual of $M$. By $\otimes$ we will always mean $\otimes_k$ unless otherwise specified. Let $H_1$ denote the Frobenius kernel of $H$.

If $M$ is a $P$-module, $\text{coind}_{P_1}^{G_1} M$ extends to a $G_1P$-module with $P$ acting on $\text{Dist}(G_1)$ and $\text{Dist}(P_1)$ by the adjoint action and as given on $M$, in which case we will write $\text{coind}_{P_1}^{G_1} M$ for $\text{coind}_{P_1}^{G_1} M$ [\text{\cite{I.8.20}}]. Let $\text{Ru}(P)$ denote the unipotent radical of $P$.

**Proposition (cf. \text{\cite{II.3.5}}):** Let $M \in P\text{Mod}$.

(i) There is an isomorphism of $G_1P$-modules $\text{ind}_{P_1}^{G_1} M \simeq \text{coind}_{P_1}^{G_1} M \otimes (2(1 - p)\rho_P)$.

(ii) If $M$ is finite dimensional, there is an isomorphism of $G_1P$-modules

$$(\text{ind}_{P_1}^{G_1} M)^* \simeq \text{ind}_{P_1}^{G_1} (M^* \otimes (2(p - 1)\rho_P)).$$

**Proof:** Recall from [\text{\cite{I.8.20}}] an isomorphism of $G_1P$-modules

(1) $\text{coind}_{P_1}^{G_1} M \simeq \text{ind}_{P_1}^{G_1} (M \otimes \chi|_P(\chi)^{-1})$,

(2) $\text{ind}_{P_1}^{G_1} M)^* \simeq \text{ind}_{P_1}^{G_1} (M^* \otimes \chi|_P(\chi)^{-1})$ if dim $M < \infty$,

where $\chi$ (resp. $\chi'$) is a 1-dimensional representation of $G$ (resp. $P$) through which $G$ (resp. $P$) acts on $\text{Dist}(G_1)^{G_1} = \{ \mu \in \text{Dist}(G_1) | \rho_\ell(x)\mu = \mu \ \forall x \in G_1 \}$ (resp. $\text{Dist}(P_1)^{P_1} = \{ \mu \in \text{Dist}(P_1) | \rho_\ell(x)\mu = \mu \ \forall x \in P_1 \}$).
\{ \mu \in \text{Dist}(P_1) | \rho_1(x)\mu = \mu \ \forall x \in P_1 \} \right\}$, where $\rho_1$ denotes the left regular action. As $\chi$ is trivial by [1] II.3.4/1.9.7, (1) and (2) read, resp.,

(3) \quad \text{coind}_{P_1}^{G_1 \mathcal{T}} M \simeq \text{ind}_{P_1}^{G_1 \mathcal{T}} (M \otimes (\chi')^{-1}) ,

(4) \quad (\text{ind}_{P_1}^{G_1 \mathcal{T}} M)^* \simeq \text{ind}_{P_1}^{G_1 \mathcal{T}} (M^* \otimes (\chi')^{-1}) .

Recall from [1] I.9.7 that $\chi'$ is given by $g \mapsto \det(\text{Ad}(g))^{p-1}$, $g \in P$. In particular, $\chi'$ factors through $P/\text{Ru}(P)$, and is trivial on the derived subgroup of $L_I$. To compute $\chi'$, therefore, we have only to consider the adjoint representation of $T$ on $\text{Lie}(P) = \text{Lie}(T) \oplus \bigoplus_{\beta \in R^+} \text{Lie}(U_{-\beta}) \oplus \bigoplus_{\alpha \in R_I^+} \text{Lie}(U_{\alpha})$. Thus for each $t \in T$

$$ \det(\text{Ad}(t)) = \left( \sum_{\beta \in R^+} -\beta + \sum_{\alpha \in R_I^+} \alpha \right) (t) = \left( -\sum_{\beta \in R^+} \beta - \sum_{\alpha \in R_I^+} \alpha \right) (t) = -\sum_{\beta \in R^+ \setminus R_I^+} \beta (t) .$$

It follows that $\chi' = (p-1)(-2\rho_P)$, and hence the assertions.

(1.3) Likewise, write $\text{coind}_{P_1 T}^{G_1 \mathcal{T}} M$ for the $G_1 T$-module $\text{coind}_{P_1}^{G_1 \mathcal{T}} M$ in case $M$ is a $P_1 T$-module.

**Proposition:** Let $M \in P_1 T \text{Mod}$.

(i) There is an isomorphism of $G_1 T$-modules $\text{ind}_{P_1 T}^{G_1 \mathcal{T}} M \simeq \text{coind}_{P_1 T}^{G_1 \mathcal{T}} (M \otimes 2(1-p)\rho_P)$.

(ii) If $M$ is finite dimensional, there is an isomorphism of $G_1 T$-modules

$$ (\text{ind}_{P_1 T}^{G_1 \mathcal{T}} M)^* \simeq \text{ind}_{P_1 T}^{G_1 \mathcal{T}} (M^* \otimes 2(p-1)\rho_P) .$$

(1.4) If $L$ is a simple $P$-module, the $P$-action on $L$ factors through $P/\text{Ru}(P)$, affording a simple $L_I$-module of highest weight belonging to $\Lambda_I^+$. For each $\lambda \in \Lambda^+$ (resp. $\lambda \in \Lambda_I^+$), we let $L(\lambda)$ (resp. $L^P(\lambda)$) denote the simple $G$- (resp. $P$-) module of highest weight $\lambda$. Likewise for simple $P_1 T$-modules. For each $\lambda \in \Lambda$ we let $\hat{L}(\lambda)$ (resp. $\hat{L}_P^P(\lambda)$) denote the simple $G_1 T$- (resp. $P_1 T$-) module of highest weight $\lambda$. Let $\Lambda_p = \{ \lambda \in \Lambda | \langle \lambda, \alpha^\vee \rangle \in [0, p] \ \forall \alpha \in R^+ \}$. Each $\lambda \in \Lambda$ admits a decomposition $\lambda = \lambda^0 + p\lambda^1$ with $\lambda^0 \in \Lambda_p$ and $\lambda^1 \in \Lambda$. Thus $\hat{L}_P^P(\lambda) \simeq L^P(\lambda^0) \otimes p\lambda^1$; if $\lambda^0 = \lambda_I^0 + \lambda_I^1$ with $\lambda_I^1 \in \Lambda_p$, then $\hat{L}_P^P(\lambda) \simeq L^P(\lambda_I^0) \otimes (\lambda_I^1 + p\lambda^1) \simeq L^P(\lambda^0) \otimes p\lambda^1$. In particular,

(1) \quad \{ \text{ind}_{P_1 T}^{G_1 \mathcal{T}} (\hat{L}_P^P(\lambda)) \}^* \simeq \text{ind}_{P_1 T}^{G_1 \mathcal{T}} (\hat{L}_P^P((-w_I) \bullet \lambda))) \otimes p(2\rho_P + w_I\lambda^1 - \lambda^1) \quad \text{with } 2\rho_P + w_I\lambda^1 - \lambda^1 \in \mathbb{Z}R.$$

If $H$ is a closed subgroup of $G$ and if $M$ is an $H$-module, we let $\text{soc}_H M$ (resp. $\text{rad}_H M$) denote the socle (resp. the radical) of $M$, and put $\text{hd}_H M = M/(\text{rad}_H M)$.

**Proposition:** For each $\lambda \in \Lambda$

$$ \text{soc}_{G_1 T} \text{ind}_{P_1 T}^{G_1 \mathcal{T}} (\hat{L}_P^P(\lambda)) = \hat{L}(\lambda) ,$$

$$ \text{hd}_{G_1 T} \text{ind}_{P_1 T}^{G_1 \mathcal{T}} (\hat{L}_P^P(\lambda)) = \hat{L}(-w_I\lambda^0 - p\lambda^1 + 2(p-1)\rho_P^*)$$

$$ = \hat{L}(w_I \bullet \lambda) \otimes p(\lambda^1 - 2\rho_P - w_I\lambda^1 + w_0((-w_I) \bullet \lambda)^1 - ((-w_I) \bullet \lambda)^1 ).$$
\textbf{Proof:} For each \( \lambda \in \Lambda \) we have \( \soc_{PT}(\ind^{PT}_{B_{1T}} \lambda) \simeq \soc_{PT}(\ind^{(P/R\nu(P))_{1T}}\lambda) = \hat{L}^P(\lambda) \). Then
\[
\ind^{G_{1T}}_{PT} \hat{L}^P(\lambda) \leq \ind^{G_{1T}}_{PT} \ind^{PT}_{B_{1T}}(\lambda) \simeq \ind^{G_{1T}}_{B_{1T}} \lambda.
\]

It follows that \( \soc_{G_{1T}}(\ind^{G_{1T}}_{PT} \hat{L}^P(\lambda)) = \{ \ind^{G_{1T}}_{PT} \hat{L}^P(\lambda) \} \cap \soc_{G_{1T}}(\ind^{G_{1T}}_{B_{1T}} \lambda) = \hat{L}(\lambda) \). Then
\[
\hd_{G_{1T}}(\ind^{G_{1T}}_{PT} \hat{L}^P(\lambda)) \simeq \{ \soc_{G_{1T}}(\ind^{G_{1T}}_{PT} \hat{L}^P(\lambda))^* \}^* \\
\quad \simeq \{ \soc_{G_{1T}}(\ind^{G_{1T}}_{B_{1T}} (\hat{L}^P(\lambda))^* \otimes 2(p-1)\rho_P) \}^* \text{ by (1.2.ii)}. 
\]

Now \( \hat{L}^P(\lambda)^* = (L^P(\lambda^0) \otimes p\lambda^1)^* = L^P(\lambda^0)^* \otimes -p\lambda^1 = L^P(-w_I\lambda^0) \otimes -p\lambda^1 = \hat{L}^P(-w_I\lambda^0 - p\lambda^1) \). Also \( \forall \nu \in \Lambda \)
\[
\hat{L}^P(\nu) \otimes 2(p-1)\rho_P \leq (\ind^{PT}_{B_{1T}} (\nu) \otimes 2(p-1)\rho_P) \text{ by the tensor identity,}
\]
and hence
\[
\hat{L}^P(\nu) \otimes 2(p-1)\rho_P \leq \soc_{PT}(\ind^{PT}_{B_{1T}} (\nu) \otimes 2(p-1)\rho_P) = \hat{L}^P(\nu \otimes 2(p-1)\rho_P).
\]

It follows that
\[
\hd_{G_{1T}}(\ind^{G_{1T}}_{PT} \hat{L}^P(\lambda)) \simeq \{ \soc_{G_{1T}}(\ind^{G_{1T}}_{PT} \hat{L}^P(-w_I\lambda^0 - p\lambda^1 + 2(p-1)\rho_P)) \}^* \\
\quad = \hat{L}(-w_I\lambda^0 - p\lambda^1 + 2(p-1)\rho_P)^*. 
\]

Finally,
\[
-w_I\lambda^0 - p\lambda^1 + 2(p-1)\rho_P = -w_I\lambda^0 - p\lambda^1 + (p-1)(w_I\rho + \rho) \text{ by (1.1)} \\
= -w_I(\lambda^0 + \rho) - \rho + p(w_I\rho + \rho - \lambda^1) = (-w_I) \bullet \lambda + p(w_I\lambda^1 + w_I\rho + \rho - \lambda^1) \\
= (-w_I) \bullet \lambda + p(w_I\lambda^1 + 2\rho_P - \lambda^1) \text{ by (1.1) again.}
\]

Thus
\[
\hat{L}(-w_I\lambda^0 - p\lambda^1 + 2(p-1)\rho_P)^* = \{ \hat{L}((-w_I) \bullet \lambda + p(w_I\lambda^1 + 2\rho_P - \lambda^1)) \}^* \\
= \hat{L}(-w_0((-w_I) \bullet \lambda)) \otimes p\{-w_I\lambda^1 - 2\rho_P + \lambda^1 + w_0((-w_I) \bullet \lambda)^1 - ((-w_I) \bullet \lambda)^1 \}
\]
with \( -w_0((-w_I) \bullet \lambda) = -w_0(-w_I(\lambda + \rho) - \rho) = w_0w_I(\lambda + \rho) - \rho = w_0w_I \bullet \lambda = w^t \bullet \lambda. \)
\[
(1.5) \text{ Corollary: Let } \lambda \in \Lambda.
\]

(i) \( \hd_{PT} \hat{L}(\lambda) = \hat{L}^P(\lambda) \text{ while } \soc_{PT} \hat{L}(\lambda) = \hat{L}^P(w_Iw_0\lambda^0 + p\lambda^1) \).

(ii) If \( \lambda \in \Lambda^+ \), \( \hd_P L(\lambda) = L^P(\lambda) \text{ while } \soc_P L(\lambda) = L^P(w_Iw_0\lambda) \).

\textbf{Proof:} (i) For each \( \nu \in \Lambda \)
\[
P_T \Mod(\hat{L}(\lambda), \hat{L}^P(\nu)) \simeq G_T \Mod(\hat{L}(\lambda), \ind^{G_{1T}}_{PT} \hat{L}^P(\nu)) = \delta_{\lambda \nu \emptyset} \text{ by (1.4)}. 
\]

It follows that \( \hd_{PT} \hat{L}(\lambda) = \hat{L}^P(\lambda) \). Then
\[
\soc_{PT} \hat{L}(\lambda) \simeq \{ \hd_{PT} (\hat{L}(\lambda))^* \}^* = \{ \hd_{PT} (\hat{L}(-w_0\lambda^0 - p\lambda^1))^* \}^* = \hat{L}^P(-w_0\lambda^0 - p\lambda^1)^* \\
= \{ L^P(-w_0\lambda^0) \otimes -p\lambda^1 \}^* \simeq L^P(w_Iw_0\lambda^0) \otimes p\lambda^1 = \hat{L}^P(w_Iw_0\lambda^0 + p\lambda^1). 
\]
(ii) For each $\mu \in \Lambda_f^+$

$$P\text{Mod}(L(\lambda), L^P(\mu)) \cong G\text{Mod}(L(\lambda), \text{ind}_{G}^E L^P(\mu)) \leq G\text{Mod}(L(\lambda), \text{ind}_{G}^E \text{ind}_{B}^P(\mu))$$

$$\cong G\text{Mod}(L(\lambda), \text{ind}_{G}^E(\mu)) = \delta_{\mu,k}.$$  

It follows that $\text{hd}_{P} L(\lambda) = L^P(\lambda)$. Then

$$\text{soc}_{P} L(\lambda) \cong \{ \text{hd}_{P}(L(\lambda)^*)\}^* = \{ \text{hd}_{P} L(-w_{0}\lambda)\}^* = L^P(-w_{0}\lambda)^* = L^P(w_{0}w_{0}\lambda).$$

(1.6) Let $H$ be a closed subgroup of $G$ and $\phi$ an automorphism of $H$. If $M$ is an $H$-module, by $^\phi M$ we will mean an $H$-module of ambient $k$-linear space $M$ with the $H$-action twisted by $\phi^{-1}$ [I.2.15/3.5]: $\forall h \in H, \forall m \in M$, the action of $h$ on $m$ in $^\phi M$ is given by $\phi^{-1}(h)m$. In particular, under the conjugate action of $W$ on $T$, $\forall w \in W$ and $\forall \lambda \in \Lambda$,

$$w_{\lambda} = w \lambda.$$

If $K$ is a closed subgroup of $H$ and $V$ is a $K$-module, there is an isomorphism of $^uH$-modules [I.3.5.4]

$$\text{ind}_{K}^H(V) \cong \text{ind}_{K}^uH(w_{V}).$$

Throughout the rest of the paper we will abbreviate $\text{ind}_{P_{1}T}^{G_{1}T}$ (resp. $\text{ind}_{B_{1}T}^{P_{1}T}$) as $\hat{\nabla}_{P}$ (resp. $\hat{\nabla}_{P}^{w}$). More generally, for $w \in W$ let $^{w}P = wPw^{-1}$ and put $\hat{\nabla}_{^{w}P} = \text{ind}_{^{w}P}^{G_{1}T}$, $\hat{\nabla}_{^{w}P}^{w} = \text{ind}_{^{w}P}^{B_{1}T}$. Let also $\hat{\nabla}_{w} = \text{ind}_{^{w}P}^{G_{1}T}$; we will abbreviate $\hat{\nabla}_{e}$ as $\hat{\nabla}$. For each $\lambda \in \Lambda$ and $w \in W$ we will write $\lambda(w)$ for $\lambda + (p - 1)(w \cdot 0)$ after [AJS]. Then

$$\text{ind}_{K}^H(V) \cong \text{ind}_{K}^uH(w_{V}).$$

(3) \quad $^{u}\hat{\nabla}_{P}(\hat{L}^{P}(\lambda)) \cong \hat{\nabla}_{w_{P}}^{w}(^{u}\hat{L}^{P}(\lambda))$ by (2)

$$\leq \hat{\nabla}_{w_{P}}^{w}(^{u}\hat{\nabla}^{P}(\lambda))$$

$$\cong \hat{\nabla}_{w_{P}}^{w}(\hat{\nabla}_{B}^{w}(\lambda))$$ again

$$\hat{\nabla}_{w}(\lambda)$$ by (1)

$$= \hat{\nabla}_{w}(w \cdot \lambda - w \cdot 0) \cong \hat{\nabla}_{w}(w \cdot \lambda + (p - 1)(w \cdot 0)) \otimes -p(w \cdot 0)$$

$$= \hat{\nabla}_{w}((w \cdot \lambda)(w)) \otimes -p(w \cdot 0).$$

(1.7) Put $\hat{\Delta} = \text{coind}_{B_{1}T}^{G_{1}T}$. Let $\tau$ be the Chevalley antiinvolution of $G$ such that $\tau|_{T} = \text{id}_{T}$ [I.1.16], and hence $\tau(U_{\alpha}) = U_{-\alpha}$ for each $\alpha \in R$. If $H$ is a subgroup of $G$ and if $M$ is a finite dimensional $H$-module, let $M^\tau$ be the $\tau(H)$-module with the ambient space $M^*$ and the $\tau(H)$-action twisted by $\tau$: $\forall x \in \tau(H), \forall f \in M^*, \forall m \in M$, $(xf)(m) = f(\tau(x)m)$. Recall from [I.9.3.5] that there is a functorial isomorphism $(\tau^{*}) \circ \hat{\nabla} \cong \hat{\Delta} \circ (\tau^{*})$ on the category of finite dimensional $B_{1}T$-modules. More generally, put $B^{+} = \tau B$, $P^{+} = \tau P = \langle B^{+}, U_{-\alpha}|_{\alpha \in I} \rangle$ and let $\Delta_{P} = \text{coind}_{P_{1}T}^{G_{1}T}$. If $M$ is a finite dimensional $P_{1}T$-module, there is an isomorphism of $G_{1}T$-modules

$$\hat{\nabla}_{P}(M) \cong \hat{\Delta}(M^\tau).$$
Let $U_1^+(w_1) = \prod_{\beta \in R^+ \setminus R_I} U_{\beta,1}$ be the Frobenius kernel of the unipotent radical of $P^+$. If $V$ is a $G_1T$-module, let $V_{U_1^+(w_1)} = \{ v \in V | xv = v \forall x \in U_1^+(w_1) \}$. If $M$ is a $B_1T$-module, as $G_1 = U_1^+(w_1)P_1$, $\hat{\nabla}(M)^{U_1^+(w_1)} = \{ \text{Sch}_k(G_1T,M)^{B_1T} \}^{U_1^+(w_1)} = \text{Sch}_k(P_1T,M)^{B_1T}$ maintains a structure of $P_1T$-module such that

$$\hat{\nabla}(M)^{U_1^+(w_1)} = \hat{\nabla}P(M).$$

Recall also that each $\hat{\nabla}(\lambda)$, $\lambda \in \Lambda$, is projective/injective as $B_1^+T$-module [J, II.9.5]. As $U_1^+(w_1)$ is a normal subgroup of $B_1^+$, $U_1^+(w_1)$ is exact in $B_1^+$ [J, I.6.5.2], and hence $\hat{\nabla}(\lambda)$ remains injective/projective as $U_1^+(w_1)$-module.

2° Translation functors

For $\lambda, \mu \in \Lambda$ let $T_{I,\lambda}^\mu$ denote the translation functor on the $G_1T$-modules. If $M$ is a $L_{I,1}T$-module, we say $M$ belongs to $\lambda$ iff all the $L_{I,1}T$-composition factors of $M$ are highest weights belonging to $W_{I,\mu} \cdot \lambda$. We let $T_{I,\lambda}^\mu$ denote the translation functor on the $L_{I,1}T$-modules.

For each $\alpha \in R$ and $n \in \mathbb{Z}$ let $H_{\alpha,n} = \{ v \in \Lambda \otimes \mathbb{R} | \langle v + \rho, \alpha^\vee \rangle = pn \}$. We call a connected component of $\Lambda \otimes \mathbb{R} \setminus \cup_{\alpha \in R, n \in \mathbb{Z}} H_{\alpha,n}$ an alcove. If $F \subseteq \Lambda \otimes \mathbb{R}$, $\Gamma$ will denote the closure of $F$ in $\Lambda \otimes \mathbb{R}$. We say $\lambda \in \Lambda$ is $p$-regular iff $\lambda$ lies in an alcove. If $x \in W_p$ and $A$ is an alcove, we will write $xA$ to mean $x \cdot A$.

(2.1) Lemma: Let $\eta \in \Lambda$ and $E$ a simple $G$-module of extremal weight $\eta$. If $w\eta \in \Lambda^+$, $w \in W_I$, and if $w \alpha + \alpha$ is not a weight of $E$.

Proof: Let $x \in W$ with $x \eta \in \Lambda^+$, and put $\nu = x\eta$, $\nu' = w\eta$. Let $J = \{ \beta \in I | \langle \nu', \beta^\vee \rangle = 0 \}$, $W_J = \{ s_\beta | \beta \in J \}$, $W^J = \{ y \in W | y\beta > 0 \ \forall \beta \in J \}$, and write $xw^{-1} = y_1y_2$ with $y_1 \in W^J$, $y_2 \in W_J$. Just suppose $w\eta + \alpha$ is a weight of $E$. Then $\nu + y_1\alpha = y_1(\nu' + \alpha)$ would also be a weight of $E$. As $\nu$ is the highest weight of $E$, $y_1\alpha < 0$, and hence $\alpha \notin J$. Then $0 < \langle \nu', \alpha^\vee \rangle = (y_1\nu', y_1\alpha^\vee) = (\nu, y_1\alpha^\vee)$, and hence $y_1\alpha > 0$, absurd.

(2.2) Proposition: Let $\lambda, \mu \in \Lambda$ with $\mu$ lying in the closure of the facet $\lambda$ belongs to with respect to $W_p$. Regarding an $L_{I,1}T$-module as a $P_1T$-module through the quotient $P \rightarrow P/Ru(P)$, there is a functorial isomorphism of $G_1T$-modules on the category of $L_{I,1}T$-modules

$$T_{I,\lambda}^\mu \hat{\nabla}(?) \simeq \hat{\nabla}(T_{I,\lambda}^\mu(?)).$$

Proof: Let $M$ be an $L_{I,1}T$-module belonging to the $\lambda$-block, and $E$ a simple $G$-module of extremal weight $\mu - \lambda$. Let $pr_\mu$ (resp. $pr_{I,\mu}$) be the projection to the $\mu$-block of $G_1T$- (resp. $L_{I,1}T$-) modules. Thus $T_{I,\lambda}^\mu \hat{\nabla}(M) = pr_\mu(E \otimes \hat{\nabla}(M))$. If $w(\mu - \lambda) \in \Lambda^+$ with $w \in W_I$ and $v \in E \setminus 0$ is of weight $w(\mu - \lambda)$, then $\text{Dist}(L_I)v$ is by (2.1) an $L_I$-module of highest weight $w(\mu - \lambda)$. If we put $E' = \text{Dist}(L_I)v$, $T_{I,\lambda}^\mu M = pr_{I,\mu}(E' \otimes M)$ [J, Remark II.7.6.1]. Thus

$$T_{I,\lambda}^\mu \hat{\nabla}(M) = pr_\mu(E \otimes \hat{\nabla}(M)) \simeq pr_\mu(\hat{\nabla}(E \otimes M))$$

$$\geq pr_\mu(\hat{\nabla}(E' \otimes M)) \geq \hat{\nabla}(pr_{I,\mu}(E' \otimes M)) = \hat{\nabla}(T_{I,\lambda}^\mu(M)).$$
As it becomes an isomorphism for $M = \hat{\nabla}^P(x \bullet \lambda)$ and $x \in W_{I,p}$, the isomorphism for general $M$ follows using the five lemma.

(2.3) **Corollary:** Let $\lambda, \mu \in \Lambda$. Assume that $\mu$ lies in the closure of the facet $\lambda$ belongs to with respect to $W_p$. Let $F_I$ be the facette $\lambda$ belongs to with respect to $W_{I,p}$ and let $\hat{F}_I$ be its upper closure with respect to $W_{I,p}$. Then

$$T^\mu_\lambda \hat{\nabla}_P(\hat{L}^P(\lambda)) \simeq \begin{cases} \hat{\nabla}_P(\hat{L}^P(\mu)) & \text{if } \mu \in \hat{F}_I, \\ 0 & \text{else}, \end{cases}$$

in the first case of which one has a commutative diagram of $G_1T$-modules

$$\begin{array}{ccc} T^\mu_\lambda \hat{\nabla}(\lambda) & \sim & \hat{\nabla}(\mu) \\ \downarrow & & \downarrow \\ T^\mu_\lambda \hat{\nabla}_P(\hat{L}^P(\lambda)) & \sim & \hat{\nabla}_P(\hat{L}^P(\mu)). \end{array}$$

(2.4) For $\alpha \in R$ and $n \in \mathbb{Z}$ let $s_{\alpha,n}$ denote the reflection in the wall $H_{\alpha,n}$.

**Proposition:** Let $\lambda, \mu \in \Lambda$ with $\lambda$ lying in an alcove $A$ and $\mu \in \overline{A}$. Assume $\{x \in W_p|x \bullet \mu = \mu\} = \{e, s_{\alpha,n}\}$ for some $\alpha \in R^+_I$ and $n \in \mathbb{Z}$. If $M$ is an $L_{I,1}T$-module belonging to $\mu$, there is an isomorphism of $G_1T$-modules

$$T^\lambda_\mu \hat{\nabla}_P(M) \simeq \hat{\nabla}_P(T^\lambda_{I,\mu}M),$$

regarding $M$ and $T^\lambda_{I,\mu}M$ as $P_1T$-modules via the quotient $P \to P/Ru(P)$.

**Proof:** Arguing as in (2.2) yields $T^\lambda_\mu \hat{\nabla}_P(M) \simeq \hat{\nabla}_P(T^\lambda_{I,\mu}M)$. On the other hand, if $M = \hat{\nabla}^P(x \bullet \mu)$ for some $x \in W_{I,p}$,

$$\text{ch } T^\lambda_\mu \hat{\nabla}_P(\hat{\nabla}^P(x \bullet \mu)) = \text{ch } T^\lambda_\mu \hat{\nabla}(x \bullet \mu) = \hat{\nabla}(x \bullet \lambda) + \hat{\nabla}(xs_{\alpha,n} \bullet \lambda)$$

while

$$\text{ch } \hat{\nabla}_P(T^\lambda_{I,\mu} \hat{\nabla}^P(x \bullet \mu)) = \hat{\nabla}_P(\hat{\nabla}^P(x \bullet \mu)) + \hat{\nabla}_P(\hat{\nabla}^P(xs_{\alpha,n} \bullet \lambda)) \quad \text{as } s_{\alpha,n} \in W_{I,p}$$

$$= \hat{\nabla}(x \bullet \lambda) + \hat{\nabla}(xs_{\alpha,n} \bullet \lambda).$$

By additivity the character equality holds for general $M$, and hence the assertion.

(2.5) **Corollary:** Let $\lambda, \mu \in \Lambda$ and keep the assumptions on $\lambda$ and $\mu$ from (2.4).

(i) $T^\lambda_\mu \hat{\nabla}_P(\hat{L}^P(\mu))$ admits a $G_1T$-filtration whose subquotients are $\hat{\nabla}_P(\hat{L}^P(x \bullet \lambda))$, $x \in W_{I,p}$, with multiplicity $m_x \in \mathbb{N}$ such that $\text{ch } T^\lambda_{I,\mu} \hat{L}^P(\mu) = \sum_{x \in W_{I,p}} m_x \hat{\nabla}_P(\hat{L}^P(x \bullet \lambda))$.

(ii) If $\lambda < s_{\alpha,n} \bullet \lambda$, then $\text{soc}_{G_1T} T^\lambda_{I,\mu} \hat{\nabla}_P(\hat{L}^P(\mu)) = \hat{L}(\lambda)$.

**Proof:** For (i) argue as in (2.2). As $\hat{\nabla}_P(\hat{L}^P(\mu)) \leq \hat{\nabla}(\mu)$, (ii) follows from the fact that $\text{soc}_{G_1T} T^\lambda_{I,\mu} \hat{\nabla}(\mu) = \hat{L}(\lambda)$. 

8
3° Grading the induction functor

In this section we employ graded representation theory from \textcite{AJS} to show that our induction functor $\hat{\nabla}_p$ can be graded on $p$-regular blocks. To facilitate reference to \textcite{AJS}, we will adapt to their notations except for $k = k$, $\Lambda = X$, and $\check{L} = L_k$.

Let $S_k$ be the symmetric algebra on $ZR \otimes \mathbb{k}$ over $\mathbb{k}$ and $\hat{S}_k$ its completion along the maximal ideal $m$ generated by $R$. We will denote each $x \in R$ in $S_k$ by $h_a$ after \textcite{AJS} 14.3. Fix a $p$-regular weight $\lambda^+$ belonging to the bottom dominant alcove, and put $\Omega = W_p \cdot \lambda^+$, $Y = pZR$. For all the unexplained notations we refer to \textcite{AJS}.

(3.1) Let us first recall \textcite{AJS} §18 to suit our objectives. The category of finite dimensional $G_1 T$-modules belonging to the block $\Omega$ may be identified with $C_k(\Omega)$ from \textcite{AJS}. For each $\lambda \in \Omega$ let $Q_k(\lambda)$ be the projective cover and the injective hull of $\check{L}(\lambda)$ in $C_k(\Omega)$. If $Q = \bigoplus_{w \in W} Q_k(w \cdot \lambda^+)$, $Q$ is a projective $L$-generator of $C_k(\Omega)$ \textcite{AJS} E.3. Thus, if $E_{\Omega,k} = C_k(\Omega)^{(Q, ?)}_{\Omega}(Q, ?)$ is an isomorphism of $Y$-graded $k$-algebra. Letting $\tilde{Q} = C_k(\Omega)^{(Q, ?)}_{\Omega}(Q, ?)$ gives an equivalence of categories from $C_k(\Omega)$ to $E_{\Omega,k} \text{modgr}_Y$ with quasi-inverse $v = Q \otimes E_{\Omega,k}$. \textcite{AJS} E.4.

Let $C(\Omega, \hat{S}_k)$ denote the deformation category over $\hat{S}_k$ of $C_k(\Omega)$. If $C(\Omega, \hat{S}_k)$ is its full subcategory consisting of the objects that are free over $\hat{S}_k$, there is a fully faithfull functor $C(\Omega, \hat{S}_k)$ to the combinatorial category $K(\Omega, \hat{S}_k)$ \textcite{AJS} 9.4. Each $Q_k(\lambda)$ lifts to a projective object $Q_{\hat{S}_k}(\lambda)$ of $C(\Omega, \hat{S}_k)$, and $\mathcal{V}_\Omega Q_{\hat{S}_k}(\lambda)$ admits a graded $S_k$-form $Q(\lambda)$ in the graded combinatorial category $C(\Omega, \hat{S}_k)$. If $P = \bigoplus_{w \in W} Q(w \cdot \lambda^+)$ and if $E_\Omega = C_k(\Omega)^{(P, ?)}_{\Omega}(P, ?)$, then $E_\Omega$ is a $(Y \times \mathbb{Z})$-graded $S_k$-algebra of finite type and there is an isomorphism of $Y$-graded $k$-algebras $E_\Omega \otimes \mathbb{k} \simeq E_{\Omega,k}$. Thus $E_{\Omega,k}$ comes equipped with a structure of finite dimensional $(Y \times \mathbb{Z})$-graded $k$-algebra. We denote by $\tilde{C}_k(\Omega)$ the category of finite dimensional $(Y \times \mathbb{Z})$-graded $E_{\Omega,k}$-modules after \textcite{AJS} 18.18 and let $\bar{v}$ denote the functor from $\tilde{C}_k(\Omega)$ to $C_k(\Omega)$ composite of the forgetful functor from $\tilde{C}_k(\Omega)$ to $E_{\Omega,k} \text{modgr}_Y$ and $v$ \textcite{AJS} 18.19. Each $Q_k(\lambda)$, $Z_\lambda(w(\lambda))$, and $\check{L}(\lambda)$, $\lambda \in \Omega$, $\lambda \in \Omega$, $w \in W$, admits a graded object $\check{Q}_k(\lambda)$, $\check{Z}_\lambda(w(\lambda))$, and $\check{L}(\lambda)$ in $\tilde{C}_k(\Omega)$, respectively, such that $\bar{v}Q_k(\lambda) \simeq Q_k(\lambda)$, $\bar{v}Z_\lambda(w(\lambda)) \simeq Z_\lambda(w(\lambda))$, and $\bar{v}\check{Q}_k(\lambda) \simeq L_k(\lambda)$ in $C_k(\Omega)$ \textcite{AJS} 18.8 and 18.10.

(3.2) Fix $\lambda_\nu \in \Lambda_\nu^+ \cap W_p \cdot \lambda^+$ with $\langle \lambda_\nu^+ + \rho, \alpha^\vee \rangle < p \forall \alpha \in R_\nu^+$. Let $\Omega_\nu = W_{I_p} \cdot \lambda_\nu^+$ and let $C(\nu)$ denote the category of finite dimensional $L_{I_1 T}$-modules belonging to the block $\Omega_\nu$. Put $Y_\nu = pZ\mathbb{I}$. For each $\nu \in \Lambda$ let $Q_{\lambda_\nu}(\nu)$ be the projective cover of $\check{L}(\nu)$ as $L_{I_1 T}$-module. If $Q_l = \bigoplus_{w \in W_{I_p}} Q_{I_1}(w \cdot \lambda_\nu^+)$, it is a projective $L_{I_1}$-generator of $C_k(\Lambda_\nu)$. Let $E_{\lambda_\nu} = C_k(\Lambda_\nu)^{(Q_{I_1}, ?)}_{\lambda_\nu}(Q_{I_1}, ?)$ is equipped with a structure of finite dimensional $(Y_I \times \mathbb{Z})$-graded $k$-algebra. Let $\tilde{C}_k(\Omega)$ denote the category of $(Y_I \times \mathbb{Z})$-graded $E_{\lambda_\nu} \text{modgr}_Y$-modules, and construct $\check{\nabla}_{I_1}(\lambda), \check{L}_{I_1}(\lambda) \in \tilde{C}(\Omega_I), \lambda \in \Omega_I$, just like $\check{\nabla}(\lambda), \check{L}(\lambda)$ for $G$.

Unless otherwise specified we will regard an $L_{I_1 T}$-module as a $P_1 T$-module via inflation along the quotient $P \rightarrow P/Ru(P) \simeq L_I$.
Lemma: There is a functorial isomorphism from the category of $Y_I$-graded $E_{\Omega_I,k}$-modules of finite type to $C_k(\Omega)$

$$Q \otimes_{E_{\Omega_I,k}} C_k(\Omega)^2(Q, \tilde{\nabla}_P(Q)) \otimes_{E_{\Omega_I,k}} \tilde{\nabla}_P(Q) \simeq \tilde{\nabla}_P(QI \otimes_{E_{\Omega_I,k}} I).$$

Proof: Let $\tilde{M}$ be a $Y_I$-graded $E_{\Omega_I,k}$-module of finite type. As $Q \otimes_{E_{\Omega_I,k}}$ and $C_k(\Omega)^2(Q, ?)$ are quasi-inverse to each other, $Q \otimes_{E_{\Omega_I,k}} C_k(\Omega)^2(Q, \tilde{\nabla}_P(Q)) \otimes_{E_{\Omega_I,k}} \tilde{\nabla}_P(Q) \simeq \tilde{\nabla}_P(QI) \otimes_{E_{\Omega_I,k}} \tilde{M}$, which is isomorphic to $\tilde{\nabla}_P(QI \otimes_{E_{\Omega_I,k}} \tilde{M})$ if $\tilde{M}$ is isomorphic to $E_{\Omega_I,k}$. In general, apply the five lemma to a natural homomorphism of $G_1T$-modules $\tilde{\nabla}_P(QI) \otimes_{E_{\Omega_I,k}} \tilde{M} \rightarrow \tilde{\nabla}_P(QI \otimes_{E_{\Omega_I,k}} \tilde{M}).$

(3.3) We will show that the lemma above refines to a commutative diagram

(1)

in such a way that for each $\lambda \in \Omega_I$

$$C_k(\Omega)^2(Q, \tilde{\nabla}_P(QI)) \otimes_{E_{\Omega_I,k}} \tilde{\nabla}_I, k(\lambda) \simeq \tilde{\nabla}_k(\lambda) \langle \delta(\lambda) - \delta_I(\lambda) \rangle,$$

where $\delta$ (resp. $\delta_I$) is the length function on $\Omega$ (resp. $\Omega_I$) [AJS, 17.1].

To justify the commutative diagram, we have only to show that $C_k(\Omega)^2(Q, \tilde{\nabla}_P(QI))$ is equipped with a structure of $(Y \times Z)$-graded left $E_{\Omega_I,k}$ and $(Y_I \times Z)$-graded right $E_{\Omega_I,k}$-bimodule. For that we first deform the functor $\tilde{\nabla}_P$. Put $S_I,k = S_k(ZI \otimes_Z k)$ to be the symmetric algebra over $k$ on $ZI \otimes_Z k$. We will write $A_G$ (resp. $A_I$) for $S_{\frac{1}{\alpha}}$ (resp. the completion of $S_I,k$ with respect to the maximal ideal generated by $ZI \otimes_Z k$). For each $\beta \in R^+$ let $A_G^{\beta} = A_G[\frac{1}{h_\alpha} \mid \alpha \in R^+ \setminus \{\beta\}]$, $A_I^{\beta} = A_G[\frac{1}{h_\alpha} \mid \alpha \in R^+]$, and for $\beta \in R_I^+$ put $A_I^{\beta} = A_I[\frac{1}{h_\alpha} \mid \alpha \in R_I^+ \setminus \{\beta\}]$, $A_I^{\beta} = A_I[\frac{1}{h_\alpha} \mid \alpha \in R_I^+]$. We will regard $A_G$ as an $A_I$-algebra via inclusion $R_I \hookrightarrow R$; in case $R$ has two lengths, if a component $I'$ of $I$ consists only of long roots, we take $h_\alpha = d_\alpha H_\alpha$ for each $\alpha \in R_{I'}$ with $d_\alpha$ for $R$ instead of $h_\alpha = H_\alpha$. Though this deviates from the convention in [AJS, 14.4/p. 11], it causes no difference to our application. Thus $A_I^{\lambda}$ is an $A_I^{\lambda}$-algebra, and for $\beta \in R_I^+$

$$A_G^{\beta} \simeq \begin{cases} A_I^{\beta \otimes A_I} A_G^{\beta} & \text{if } \beta \in R_I^+ \\ A_I^{\lambda \otimes A_I} A_G^{\lambda} & \text{else} \end{cases}$$

For a $W_{I,I'}$-orbit $\Gamma_I$ in $\Lambda$ define $C(\Gamma_I, A_I), C(\Gamma_I, A_G^{\beta})$ for $\beta \in R_I^+$, $C(\Gamma_I, A_G)$, $C(\Gamma_I, A_G^{\beta})$ for $\beta \in R^+$, $\mathcal{F}(\Gamma_I, A_I), \mathcal{K}(\Gamma_I, A_I), \mathcal{K}(\Gamma_I, S_{I,k})$ and $\mathcal{C}(\Gamma_I)$ for $L_I$ just as for $G$; precisely these are defined first for the semisimple part of $L_I$ and then extended to $L_I$ in a natural way. For
\( \nu \in \Lambda \) define likewise \( Z_{I,A_I}(\nu), Z^{\rho}_{I,A_I}(\nu) = Z_{I,A_I}(\nu) \) for \( \beta \in R_I^+ \), and \( Z^{\vartheta}_{I,A_I}(\nu) = Z_{I,A_I}(\nu) \) as well as \( Z_{I,A_G}(\nu), Z_{I,A_G}(\nu) \) for \( \beta \in R^+ \), and \( Z_{I,A_G}(\nu) \).

Recall from (1.7) the parabolic subgroup \( P^+ = \langle B^+, U_{-\alpha} | \alpha \in I \rangle \). Let \( \Gamma \) be the \( W_p \)-orbit in \( \Lambda \) containing \( \Gamma_I \). Regarding an object of \( \mathcal{F} \mathcal{C}(\Gamma_I, A_I) \) as a \( \text{Dist}(P_I^+) \)-module by the quotient \( P^+ \to P^+/\text{Ru}(P^+) \), define a functor \( \nabla_{P,A_I} : \mathcal{F} \mathcal{C}(\Gamma_I, A_I) \to \mathcal{F} \mathcal{C}(\Gamma, A_G) \) via

\[
M \mapsto (\text{Dist}(G_1) \otimes_{\text{Dist}(P_1^+)} M^r)^r \otimes_{A_I} A_G \simeq \{ \text{Dist}(G_1) \otimes_{\text{Dist}(P_1^+)} (M \otimes_{A_I} M)^r \}^r,
\]

which reduces to \( \nabla_P \) by reduction to \( \mathbb{k} \). For each \( \nu \in \Gamma_I \) one has

\[
(3) \quad \nabla_{P,A_I}(Z_{I,A_I}(\nu))^r \simeq Z_{A_G}(\nu)^r.
\]

(3.4) Let \( U_1(w_I) = \prod_{\beta \in R^+ \setminus R_I} U_{-\beta,1} \) be the Frobenius kernel of the unipotent radical of \( P \) and \( \text{Dist}^+(U_1(w_I)) \) the augmentation ideal of \( \text{Dist}(U_1(w_I)) \). Let \( \Gamma \) be an arbitrary \( W_p \)-orbit. For each \( M \in \mathcal{C}(\Gamma, A_G) \) put

\[
M_n = M/\text{Dist}^+(U_1(w_I)) M \simeq \{ \text{Dist}(U_1(w_I))/\text{Dist}^+(U_1(w_I)) \} \otimes_{\text{Dist}(U_1(w_I))} M
\]

the module of \( \text{Dist}^+(U_1(w_I)) \)-coinvariants of \( M \). If \( M = Z_{A_G}(\nu), \nu \in \Lambda \), taking the \( \tau \)-dual of (1.7) yields an isomorphism in \( \mathcal{C}(W_{I,p} \bullet \nu, A_G) \)

\[
(1) \quad Z_{A_G}(\nu)_n \simeq Z_{I,A_I}(\nu) \otimes_{A_I} A_G \simeq Z_{I,A_G}(\nu).
\]

Let \( \beta \in R_I^+, \nu \in \Gamma \) with \( \beta \uparrow \nu > \nu \), and put \( \Gamma_I = W_{I,p} \bullet \nu \). One has from [AJS] 8.6, as \( d_\beta \in \mathbb{k}^\times \) by the standing hypothesis on \( p \) [AJS] 14.4, as

\[
(2) \quad \text{Ext}^1_{\mathcal{C}(\Gamma,A_G)}(Z^{\rho}_{A_G}(\nu), Z^{\rho}_{A_G}(\beta \uparrow \nu)) \simeq A_G h^1_\beta / A_G \simeq (A_I^2 h^{-1}_\beta / A_I^2) \otimes_{A_I} A_G^2
\]

\[
\simeq \text{Ext}^1_{\mathcal{C}(\Gamma', A_G)}(Z^{\rho}_{I,A_I}(\nu), Z^{\rho}_{I,A_I}(\beta \uparrow \nu)) \otimes_{A_I} A_G^2
\]

\[
\simeq \text{Ext}^1_{\mathcal{C}(\Gamma', A_G)}(Z_{I,A_G}^\beta(\nu), Z_{I,A_G}^\beta(\beta \uparrow \nu)) \quad \text{by [AJS] 3.2}.
\]

**Lemma:** Assume \( \beta \uparrow \nu > \nu \). If \( 0 \to Z^{\rho}_{A_G}(\beta \uparrow \nu) \to M \to Z^{\rho}_{A_G}(\nu) \to 0 \) is exact in \( \mathcal{C}(\Gamma,A_G^2) \), applying \( ?_n \) to the sequence yields an exact sequence \( 0 \to Z^{\rho}_{I,A_G^2}(\beta \uparrow \nu) \to M_n \to Z^{\rho}_{I,A_G^2}(\nu) \to 0 \) with \( M \) projective in \( \mathcal{C}(\Gamma,A_G^2) \) iff \( M_n \) projective in \( \mathcal{C}(\Gamma_I,A_G^2) \). Conversely, applying \( \text{Dist}(G_1) \otimes_{\text{Dist}(P_1^+)} ?_n \) to the latter sequence recovers the former. Likewise, if \( 0 \to Z^{\rho}_{I,A_I}(\beta \uparrow \nu) \to M' \to Z^{\rho}_{I,A_I}(\nu) \to 0 \) is an exact sequence in \( \mathcal{C}(\Gamma_I,A_G^2) \) with \( M' \) projective, then applying \( \text{Dist}(G_1) \otimes_{\text{Dist}(P_1^+)} ?_n \otimes_{A_I} A_G^2 \) yields an exact sequence \( 0 \to Z^{\rho}_{A_G}(\beta \uparrow \nu) \to \text{Dist}(G_1) \otimes_{\text{Dist}(P_1^+)} M' \otimes_{A_I} A_G^2 \to Z^{\rho}_{A_G}(\nu) \to 0 \) with \( \text{Dist}(G_1) \otimes_{\text{Dist}(P_1^+)} M' \otimes_{A_I} A_G^2 \) projective in \( \mathcal{C}(\Gamma,A_G^2) \).

**Proof:** Assume the sequence \( 0 \to Z^{\rho}_{A_G}(\beta \uparrow \nu) \to M \to Z^{\rho}_{A_G}(\nu) \to 0 \) is exact. As \( ?_n \simeq \{ \text{Dist}(U_1(w_I))/\text{Dist}^+(U_1(w_I)) \} \otimes_{\text{Dist}(U_1(w_I))} \) and as \( Z^{\rho}_{A_G}(\nu) \simeq \text{Dist}(U_1) \simeq \text{Dist}(U_1(w_I)) \) \( \otimes_{\mathbb{k}} \) \( \text{Dist}((B \cap L_{I,1})) \) is free over \( \text{Dist}(U_1(w_I)) \), \( 0 \to Z^{\rho}_{A_G}(\beta \uparrow \nu)_n \to M_n \to Z^{\rho}_{A_G}(\nu)_n \to 0 \) remains exact with \( Z^{\rho}_{A_G}(\beta \uparrow \nu)_n \simeq Z_{I,A_G}(\beta \uparrow \nu) \) and \( Z^{\rho}_{A_G}(\nu)_n \simeq Z_{I,A_G}(\nu) \).
Recall from [AJS, 12.4] how each $M$ is constructed. Let $w_\beta \in W_I$ with $w_\beta^{-1} \beta \in I$. Let $v_\nu^\beta \in Z^\beta_{AG}(\nu)$ of weight $\nu(w_\beta)$ corresponding to the standard generator $1 \otimes 1$ of $Z^\beta_{AG}(\nu(w_\beta))$ under the isomorphism $Z^\beta_{AG}(\nu) = Z^\beta_{AG}(\nu) \simeq Z^\beta_{AG}(\nu(w_\beta))$, and define $v_\nu^\beta_{\beta_\nu} \in Z^\beta_{AG}(\beta \uparrow \nu)$ likewise. Write $\langle \nu + \rho, \beta' \rangle \equiv p - n \mod p$ with $n \in [0, p]$, and put $z_\nu = E_{-\beta} v_\nu^\beta b + v_\nu^\beta \in Z_K(\beta \uparrow \nu) \oplus Z_K(\nu)$ for each $b \in A^0_G h^{-1}_\beta$ with $K = \text{Frac}(A_G)$, so $z_\nu$ is of weight $\nu(w_\beta)$. Then $M^\nu_{\beta_\nu}(b) = \text{Dist}(G_1) v_\nu^\beta_{\beta_\nu} A^\beta_G + \text{Dist}(G_1) z_\nu A^\beta_G$ living in $Z_K(\beta \uparrow \nu) \oplus Z_K(\nu)$, and the sequence reads $v_\nu^\beta_{\beta_\nu}$ mapping to itself while $z_\nu \mapsto v_\nu^\beta_{\beta_\nu}$. Now

$$\text{Dist}(G_1) \simeq \text{Dist}(w_\beta U_1) \otimes \text{Dist}(w_\beta B^+_1)$$

$$\simeq \text{Dist}(w_\beta U_1(w_I)) \otimes \text{Dist}(w_\beta(B \cap L_I)_1) \otimes \text{Dist}(w_\beta B^+_1)$$

$$\simeq \text{Dist}(U_1(w_I)) \otimes \text{Dist}(w_\beta(B \cap L_I)_1) \otimes \text{Dist}(w_\beta B^+_1)$$

as $w_\beta U(w_I) = w_\beta \prod_{\alpha \in R^+ \setminus R_I} U^{-\alpha} = \prod_{\alpha \in R^+ \setminus R_I} U^{-\alpha} = U(w_I)$. Thus $(\text{Dist}(G_1)) v_\nu^\beta_{\beta_\nu})_n \simeq \text{Dist}(w_\beta(B \cap L_I)_1) v_\nu^\beta_{\beta_\nu}$, $(\text{Dist}(G_1)) z_\nu)_n \simeq \text{Dist}(w_\beta(B \cap L_I)_1) z_\nu$, and hence $M^\nu_{\beta_\nu}(b)_n = \text{Dist}(L_I, 1) v_\nu^\beta_{\beta_\nu} A^\beta_G + \text{Dist}(L_I, 1) z_\nu A^\beta_G$. It follows from [AJS, 8.7] that $M^\nu_{\beta_\nu}(b)$ is projective in $\mathcal{C}(\Omega, A^\beta_G)$ iff $A^\beta_G b = A^\beta_G h^{-1}_\beta / A^\beta_G$ iff $M^\nu_{\beta_\nu}(b)_n$ is projective in $\mathcal{C}(\Omega_I, A^\beta_G)$. Likewise the last assertion follows from (1).

(3.5) We now transfer from $\mathcal{F}(\Omega, A_G)$ (resp. $\mathcal{F}(\Omega_I, A_I)$) to the combinatorial category $\mathcal{K}(\Omega, A_G)$ (resp. $\mathcal{K}(\Omega_I, A_I)$) via the faithfully flat functor $\mathcal{V}_\Omega$ (resp. $\mathcal{V}_{\Omega_I}$). Define a functor $\mathcal{I} : \mathcal{K}(\Omega_I, A_I) \to \mathcal{K}(\Omega, A_G)$ as follows: for each $\mathcal{M} \in \mathcal{K}(\Omega_I, A_I)$ and $\lambda \in \Omega$ set

$$(\mathcal{I}, \mathcal{M})(\lambda) = \begin{cases} \mathcal{M}(\lambda) \otimes_{A_I} A^\beta_G & \text{if } \lambda \in \Omega_I \\ 0 & \text{else}, \end{cases}$$

and for each $\beta \in R^+$ set

$$(\mathcal{I}, \mathcal{M})(\lambda, \beta) = \begin{cases} \mathcal{M}(\lambda, \beta) \otimes_{A_I} A^\beta_G & \text{if } \lambda \in \Omega_I \text{ and } \beta \in R_I^+ \\ \mathcal{M}(\lambda) \otimes_{A_I} A^\beta_G & \text{if } \lambda \in \Omega_I \text{ and } \beta \notin R_I^+ \\ \mathcal{M}(\beta \uparrow \lambda) \otimes_{A_I} A^\beta_G & \text{if } \beta \uparrow \lambda \in \Omega_I \text{ and } \beta \notin R_I^+ \text{ else.} \end{cases}$$

Let $\mathcal{D}(\Omega, A_G)$ (resp. $\mathcal{D}(\Omega_I, A_I)$) be the full subcategory of $\mathcal{C}(\Omega, A_G)$ (resp. $\mathcal{C}(\Omega_I, A_I)$) consisting of objects admitting a $Z_{AG}$- (resp. $Z_{I,A_I}$-) filtration. We want to show a functorial isomorphism $\mathcal{V}_\Omega \circ \widehat{\mathcal{V}}_{P,A_I} \simeq \mathcal{I} \circ \mathcal{V}_{\Omega_I}$ from $\mathcal{D}(\Omega_I, A_I)$ to $\mathcal{D}(\Omega, A_G)$.

Recall that $\mathcal{V}_\Omega$ and $\mathcal{V}_{\Omega_I}$ are defined with specific choice of extensions according to Theorem of Good Choices [AJS, 13.4].

**Lemma:** Let $\lambda \in \Omega_I$ and $\beta \in R_I^+$ with $\beta \uparrow \lambda > \lambda$. Let $e^\beta(\lambda) \in \text{Ext}^1_{\mathcal{C}(\Omega, A^\beta_G)}(Z^\beta_{AG}(\lambda), Z^\beta_{AG}(\beta \uparrow \lambda))$ and $e^\beta(\lambda) \in \text{Ext}^1_{\mathcal{C}(\Omega_I, A^\beta_G)}(Z^\beta_{I,A_I}(\lambda), Z^\beta_{I,A_I}(\beta \uparrow \lambda))$ chosen according to Theorem of Good Choices. Let $Y^\beta_{AG}(\lambda) \in \mathcal{C}(\Omega, A^\beta_G)$ (resp. $Y^\beta_{I,A_I}(\lambda) \in \mathcal{C}(\Omega_I, A^\beta_G)$) be the module representing $e^\beta(\lambda)$ (resp. $e^\beta(\lambda)$). Then $Y^\beta_{I,A_I}(\lambda) \otimes_{A_I} A^\beta_G = Y^\beta_{AG}(\lambda)$ and $\text{Dist}(G_1) \otimes_{\text{Dist}(P^+_1)} Y^\beta_{I,A_I}(\lambda) \otimes_{A_I} A^\beta_G = Y^\beta_{AG}(\lambda)$. 

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Proof: Write \( \lambda = w_1 \cdot \lambda^+ + p \gamma_1 = w_2 \cdot \lambda^+_I + p \gamma_2 \) with \( w_1 \in W, w_2 \in W_I, \gamma_1 \in ZR, \gamma_2 \in ZR_I \). By [AJS] 13.25 we may assume \( \lambda^+_I = w_2^{-1} w_1 \cdot \lambda^+ \). Then for each \( \alpha \in R^+_I \)

\[
(\text{w}_2^{-1} \text{w}_1)^{-1} \alpha = w_1^{-1} w_2 \alpha > 0,
\]
and hence

\[
w_1^{-1} \alpha > 0 \quad \text{iff} \quad w_2^{-1} \alpha > 0.
\]

Recall from [AJS] 13.2.5 that \( e^\beta(\lambda) = b^\beta(\lambda) e^\beta_0(\lambda) \), and thus \( e^\beta_I(\lambda) = b^\beta_I(\lambda) e^\beta_{I,0}(\lambda) \) likewise with the RHS’s specified as follows. By [AJS] 12.12.5

\[
b^\beta(\lambda) = \begin{cases} e^\beta_{w_1 \cdot \lambda^+, -\rho} d(w_1 \cdot \lambda^+, -\rho, s_\beta) \kappa(\beta) & \text{if } w_1^{-1} \beta > 0 \\ e^\beta_{w_1 \cdot \lambda^+, -\rho} d(w_1 \cdot \lambda^+, -\rho, s_\beta) h_\beta & \text{else}
\end{cases}
\]

with \( \kappa(\beta) = \prod_{\alpha \in R^+_I} h^\beta_{\alpha, \alpha^\vee} \), where \( w_\beta \in W_I \) such that \( w_1^{-1} \beta \in I \), and thus

\[
b^\beta_I(\lambda) = \begin{cases} e^\beta_{I, w_2 \cdot \lambda^+_I, -\rho_I} d_I(w_2 \cdot \lambda^+_I, -\rho_I, s_\beta) \kappa_I(\beta) & \text{if } w_2^{-1} \beta > 0 \\ e^\beta_{I, w_2 \cdot \lambda^+_I, -\rho_I} d_I(w_2 \cdot \lambda^+_I, -\rho_I, s_\beta) h_\beta & \text{else}
\end{cases}
\]

with \( \kappa_I(\beta) = \prod_{\alpha \in R^+_I} h^{-\beta, \alpha^\vee}_\alpha \). By (2) the two cases in (3) and (4) agree, and \( \kappa(\beta) = \kappa_I(\beta) \).

By [AJS] 12.12.5

\[
\varepsilon^\beta_{w_1 \cdot \lambda^+, -\rho} = \prod_{\alpha \in R^+_I} (-1)^{-\rho - w_1 \cdot \lambda^+, \alpha^\vee} \hat{\alpha}^{-\rho - w_1 \cdot \lambda^+} \prod_{\alpha \in R^+_I \setminus \beta} (-1)^{-\rho - w_1 \cdot \lambda^+, \alpha^\vee}
\]

with \( \hat{\alpha}(\nu) = \begin{cases} 1 & \text{if } \langle \nu, \alpha^\vee \rangle > 0 \\ 0 & \text{else}
\end{cases} \) for each \( \nu \in \Lambda \) [AJS] A.1.1., and thus

\[
\varepsilon^\beta_{I, w_2 \cdot \lambda^+_I, -\rho_I} = \prod_{\alpha \in R^+_I \setminus \beta} (-1)^{-\rho_I - w_2 \cdot \lambda^+_I, \alpha^\vee} \hat{\alpha}^{-\rho_I - w_2 \cdot \lambda^+_I} \prod_{\alpha \in R^+_I \setminus \beta} (-1)^{-\rho_I - w_2 \cdot \lambda^+_I, \alpha^\vee}.
\]

One has \( -\rho - w_1 \cdot \lambda^+ = -\rho - w_1 \cdot (w_2^{-1} w_1)^{-1} \cdot \lambda^+_I = -\rho - \rho_I - w_2 \cdot \lambda^+_I \). If \( \alpha \in R^+ \) with \( s_\beta \alpha < 0 \), \( \alpha \in R^+_I \), and hence \( -\rho - w_1 \cdot \lambda^+, \alpha^\vee = -\rho - \rho_I - w_2 \cdot \lambda^+_I, \alpha^\vee = -\rho_I - w_2 \cdot \lambda^+_I, \alpha^\vee \) and \( \hat{\alpha}^{-\rho - w_1 \cdot \lambda^+} = \hat{\alpha}^{-\rho_I - w_2 \cdot \lambda^+_I} \). Thus \( \varepsilon^\beta_{w_1 \cdot \lambda^+, -\rho} = \varepsilon^\beta_{I, w_2 \cdot \lambda^+_I, -\rho_I} \). By [AJS] 13.2.2

\[
d(w_1 \cdot \lambda^+, -\rho, s_\beta) = \prod_{\alpha \in R^+_I \setminus \beta \alpha < 0} \frac{[k_\alpha; w_1 \cdot \lambda^+ + \rho]}{h_\alpha}
\]

\[
d_I(w_2 \cdot \lambda^+_I, -\rho_I, s_\beta) = \prod_{\alpha \in R^+_I \setminus \beta \alpha < 0} \frac{[k_\alpha; w_2 \cdot \lambda^+_I + \rho_I]}{h_\alpha}
\]
By (2) again the products run over the same subset of $R^+$. By [AJS] 13.1.4
\[ [k_\alpha; w_1 \cdot \lambda^+ + \rho] = (H_\alpha + \langle w_1 \cdot \lambda^+ + \rho, \alpha^\vee \rangle) H_\alpha^{-1} h_\alpha \]
with $\langle w_1 \cdot \lambda^+ + \rho, \alpha^\vee \rangle = \langle w_1 (w_2^{-1} w_1)^{-1} \cdot \lambda^+ + \rho, \alpha^\vee \rangle = \langle w_2 \cdot \lambda^+_I + \rho, \alpha^\vee \rangle$.
and hence $d(w_1 \cdot \lambda^+ + \rho, s_\beta) = d_I(\rho, s_\beta)$ and $b^\beta(\lambda) = b^\beta_I(\lambda)$.

We compare next $e_0^\beta(\lambda)$ and $e_0^{\rho,\beta}(\lambda)$. Take $\omega \in \Lambda$ in the upper closure of the facet $\lambda$ belongs to with respect to $\langle s_\beta, r \rangle$. By [AJS] 12.3.1
\[ \epsilon_0^\beta = \epsilon_{1,0}^\beta = \epsilon_0^\beta \]
By definition [AJS] 12.5 again
\[ \epsilon_0^\beta = \prod_{\alpha < 0, \omega \alpha > 0} (-1)^{\langle \omega - \lambda, \alpha^\vee \rangle} \prod_{\alpha < 0, \omega \alpha > 0} (-1)^{\langle \omega - \lambda, \alpha^\vee \rangle} = \epsilon_{I, \omega}^\beta. \]
By definition [AJS] A.7.1 and A.2.1
\[ d(\omega, \lambda, s_\beta) = \prod_{\alpha < 0, \omega \alpha > 0} d(\omega, \lambda, \alpha) = \prod_{\alpha < 0, \omega \alpha > 0} \left( \frac{H_\alpha + \langle \omega + \rho, \alpha^\vee \rangle}{H_\alpha + \langle \lambda + \rho, \alpha^\vee \rangle} \right) = d_I(\omega, \lambda, s_\beta). \]
It follows in (5) that $\epsilon_{I, \omega}^\beta d(\omega, \lambda, s_\beta) h_\beta^{-1} = \epsilon_{I, \omega}^\beta d_I(\omega, \lambda, s_\beta) h_\beta^{-1}$. By [AJS] 12.12.1
\[ t_0^\beta[\omega, \lambda] = t[\omega, \lambda, a_{\omega, \lambda}], \quad t_{I,0}^\beta[\omega, \lambda] = t_I[\omega, \lambda, a_{I, \omega}]. \]
with
\[ a_{\omega, \lambda} = a_{\lambda, \omega} \epsilon_{\omega, \lambda} = \epsilon_{\omega, \lambda} \] by [AJS] A.12
\[ a_{I, \omega, \lambda} = \epsilon_{I, \omega, \lambda} \] by (6)
\[ a_{I, \omega, \lambda} = a_{I, \omega, \lambda} \epsilon_{I, \omega, \lambda} = a_{I, \omega, \lambda}. \]
We have $t[\omega, \lambda, a_{\omega, \lambda}] = t[\omega, \lambda, e, e]$ by [AJS] 12.8.2 with $e \in E_{\omega - \lambda} \setminus 0$, $E$ a simple $G$-module of extremal weight $\omega - \lambda$ [AJS] 11.1, and with $\tilde{e} = a_{\lambda, \omega}(-1)^{n} E_{-\beta_i} e \in E_{\omega - \lambda} \setminus 0$.
\[ \langle \lambda + \rho, \beta^\vee \rangle \equiv p - n \mod p, \ n \in [0, p], \] by [AJS] 12.3.1. Recall from [AJS] 12.6 the definition of $t[\omega, \lambda, e, e] : \text{Ext}^1_{C(\Omega)}(Z_{A_G}^\beta(\lambda), Z_{A_G}^\beta(\beta \uparrow \lambda)) \to H_\beta^{-1} A_G^\beta/A_G^\beta = h_\beta^{-1} A_G^\beta/A_G^\beta$.

Let $\xi \in \text{Ext}^1_{C(\Omega)}(Z_{A_G}^\beta(\lambda), Z_{A_G}^\beta(\beta \uparrow \lambda))$ represented by a short exact sequence
\[ 0 \to Z_{A_G}^\beta(\beta \uparrow \lambda) \xrightarrow{i} M \xrightarrow{j} Z_{A_G}^\beta(\lambda) \to 0. \]
As $H_\beta \xi = 0$, there is $j' \in C(\Omega, A_G^\beta)(Z_{A_G}^\beta(\lambda), M)$ with $j \circ j' = H_\beta \text{id}_{Z_{A_G}^\beta(\lambda)}$. Apply the translation functor $T_{\chi}^\omega$ to (8) to obtain a split exact sequence
\[ 0 \to T_{\chi}^\omega Z_{A_G}^\beta(\beta \uparrow \lambda) \xrightarrow{T_{\chi}^\omega i} T_{\chi}^\omega M \xrightarrow{T_{\chi}^\omega j} T_{\chi}^\omega Z_{A_G}^\beta(\lambda) \to 0. \]
Let \( i' \in C(A^β_{\omega,e}(\lambda), T_\lambda Z^β_{\lambda-A}(\lambda)) \) with \( i' \circ T_\lambda^β j = id_{T_\lambda Z^β_{\lambda-A}(\lambda)} \). Recall from [AJS] 11.2.1 isomorphisms \( f_e : Z^β_{\lambda-A}(\omega) \to T_\lambda Z^β_{\lambda-A}(\lambda) = \text{pr}(E \otimes Z^β_{\lambda-A}(\lambda)) \) via \( 1 \otimes 1 \mapsto \text{pr}(e \otimes 1 \otimes 1) \) and \( f_e : Z^β_{\lambda-A}(\omega) \to T_\lambda Z^β_{\lambda-A}(\beta \uparrow \lambda) = \text{pr}(E \otimes Z^β_{\lambda-A}(\beta \uparrow \lambda)) \) via \( 1 \otimes 1 \mapsto \text{pr}(\bar{e} \otimes 1 \otimes 1) \). If \( a \in A^β_{\omega,e} \), then \( t[\omega, e, \bar{e}] = aH^{-1}_{\beta} + A^β_{\omega,e} \). Now recall the \( L_\lambda \)-submodule \( E' \) of \( E \) from (2.2) and choose \( e_l = e \in E' \) and \( \bar{e}_l = \bar{e} \in E' \) to define \( t[\omega, e, \bar{e}] : Ext^1_{C(\Omega, A^β_{\lambda,\lambda})}(Z^β_{\lambda,\lambda}(\lambda), Z^β_{\lambda,\lambda}(\beta \uparrow \lambda)) \to H^{-1}_{\lambda} A^β_{\lambda}/A^β_{\lambda} \) likewise.

As we have natural isomorphisms from (2.2) or rather from its \( \tau \)-dual

\[
\begin{align*}
\text{Dist}(G_1) &\otimes_{\text{Dist}(P^+)} Z^β_{\lambda,\lambda}(\lambda) \otimes_{A^β_{\lambda}} A^β_{\lambda} \simeq Z^β_{\lambda-A}(\lambda), \\
\text{Dist}(G_1) &\otimes_{\text{Dist}(P^+)} Z^β_{\lambda,\lambda}(\beta \uparrow \lambda) \otimes_{A^β_{\lambda}} A^β_{\lambda} \simeq Z^β_{\lambda-A}(\beta \uparrow \lambda), \\
\text{Dist}(G_1) &\otimes_{\text{Dist}(P^+)} T_\lambda Z^β_{\lambda,\lambda}(\lambda) \otimes_{A^β_{\lambda}} A^β_{\lambda} \simeq T_\lambda Z^β_{\lambda-A}(\lambda), \\
\text{Dist}(G_1) &\otimes_{\text{Dist}(P^+)} T_\lambda Z^β_{\lambda,\lambda}(\beta \uparrow \lambda) \otimes_{A^β_{\lambda}} A^β_{\lambda} \simeq T_\lambda Z^β_{\lambda-A}(\beta \uparrow \lambda),
\end{align*}
\]

the commutative diagram

\[
\begin{align*}
\text{Ext}^1_{C(\Omega, A^β_{\lambda,\lambda})}(Z^β_{\lambda,\lambda}(\lambda), Z^β_{\lambda,\lambda}(\beta \uparrow \lambda)) &\xrightarrow{\text{Dist}(G_1) \otimes_{\text{Dist}(P^+)} Z^β_{\lambda,\lambda}(\lambda)} H^{-1}_{\lambda} A^β_{\lambda}/A^β_{\lambda} \\
\text{Ext}^1_{C(\Omega, A^β_{\lambda,\lambda})}(Z^β_{\lambda,\lambda}(\lambda), Z^β_{\lambda-A}(\lambda)) &\xrightarrow{\text{Dist}(G_1) \otimes_{\text{Dist}(P^+)} Z^β_{\lambda-A}(\lambda)} H^{-1}_{\lambda} A^β_{\lambda}/A^β_{\lambda}
\end{align*}
\]

follows. More precisely, if \( Y^β_{\lambda}(\lambda) \) (resp. \( Y^β_{\lambda}(\lambda) \)) is the module representing \( e^β(\lambda) \) (resp. \( e^β(\lambda) \)), then \( \text{Dist}(G_1) \otimes_{\text{Dist}(P^+)} Y^β_{\lambda,\lambda}(\lambda) \otimes_{A^β_{\lambda}} A^β_{\lambda} = Y^β_{\lambda-A}(\lambda) \) with \( Y^β_{\lambda,\lambda}(\lambda) \otimes_{A^β_{\lambda}} A^β_{\lambda} = Y^β_{\lambda-A}(\lambda) \) by (3.4).

(3.6) We are now ready to show

**Theorem:** There is a functorial isomorphism from \( D(\Omega, A^β_{\lambda}) \) to \( D(\Omega, A^β_{\lambda}) \)

\[
\forall \Omega \circ \nabla_{\lambda,\lambda} \simeq \forall \Omega_{\lambda,\lambda}.
\]

**Proof:** For each \( X \in D(\Omega, A^β_{\lambda}) \) and \( M \in D(\Omega, A^β_{\lambda}) \)

\[
C(\Omega, A^β_{\lambda})(X, \nabla_{\lambda,A^β_{\lambda}}(M)) \simeq C(\Omega, A^β_{\lambda})(X, (\text{Dist}(G_1) \otimes_{\text{Dist}(P^+)} M^T) \otimes_{A^β_{\lambda}} A^β_{\lambda})
\]

\[
\simeq C(\Omega, A^β_{\lambda})(X, \{\text{Dist}(G_1) \otimes_{\text{Dist}(P^+)} (M \otimes_{A^β_{\lambda}} A^β_{\lambda})\}^T)
\]

\[
\simeq C(\Omega, A^β_{\lambda})(\text{Dist}(G_1) \otimes_{\text{Dist}(P^+)} (M \otimes_{A^β_{\lambda}} A^β_{\lambda})^T, X^T) \quad \text{by [AJS] 4.5.5}
\]

\[
\simeq C_L(\lambda)((M \otimes_{A^β_{\lambda}} A^β_{\lambda})^T, \text{Ann}_{X^T}(\text{Dist}^+(U^+_1(w)))
\]

with

\[
\text{Ann}_{X^T}(\text{Dist}^+(U^+_1(w))) = \{f \in \text{Mod}_{A}(X, A^β_{\lambda}) \mid 0 = xf = f(\tau(x)) \forall x \in \text{Dist}^+(U^+_1(w)))
\]

\[
\simeq \{X/\text{Dist}^+(U^+_1(w)))X \}^T = (X^n)^T.
\]
Thus

\[ C(\Omega, A_G)(X, \hat{\nabla}_{P,A_I}(M)) \simeq C(\Omega, A_G)((M \otimes_{A_I} A_G)^\tau, (X_n)^\tau) \]
\[ \simeq C(\Omega, A_G)(X_n, M \otimes_{A_I} A_G). \]

It follows for each \( \lambda \in \Omega \) that

\[
(\mathcal{V}_\Omega \circ \hat{\nabla}_{P,A_I})(M)(\lambda) = C(\Omega, A_G^\theta)(Z^\theta(\lambda), \hat{\nabla}_{P,A_I}(M)^\theta)
\]
\[ \simeq C(\Omega I, A_G^\theta)(Z^\theta_I(\lambda), M^\theta \otimes_{A_I} A_G^\theta)
\]
\[ \simeq C(\Omega I, A_G^\theta)(Z^\theta_I(\lambda), M^\theta) \otimes_{A_I} A_G^\theta \text{ by [AJS 3.2]}
\]
\[ = \begin{cases} 
\mathcal{V}_{\Omega I}(M)(\lambda) \otimes_{A_I} A_G^\theta & \text{if } \lambda \in \Omega I \\
0 & \text{else}
\end{cases}
\]
\[ = (I \circ \mathcal{V}_{\Omega I})(M)(\lambda). \]

Assume \( \lambda \in \Omega I \). If \( \beta \in R_+^I \),

\[
(\mathcal{V}_\Omega \circ \hat{\nabla}_{P,A_I})(M)(\lambda, \beta) = C(\Omega, A_G^\beta)(Y_{\hat{A}_G}^\beta(\lambda), \hat{\nabla}_{P,A_I}(M)^\beta)
\]
\[ \simeq C(\Omega I, A_G^\beta)(Y_{I,A_I}^\beta(\lambda), M^\beta \otimes_{A_I} A_G^\beta) \text{ likewise by (3.5)}
\]
\[ \simeq \mathcal{V}_{\Omega I}(M)(\lambda, \beta) \otimes_{A_I} A_G^\beta = (I \circ \mathcal{V}_{\Omega I})(M)(\lambda, \beta). \]

If \( \beta \in R_+^I \setminus R_+^I \),

\[
(\mathcal{V}_\Omega \circ \hat{\nabla}_{P,A_I})(M)(\lambda, \beta) = C(\Omega, A_G^\beta)(Y_{\hat{A}_G}^\beta(\lambda), \hat{\nabla}_{P,A_I}(M)^\beta)
\]
\[ \simeq C(\Omega I, A_G^\beta)(M^\beta \otimes_{A_I} A_G^\beta)^\tau, (Y_{A_G}^\beta(\lambda)^\tau)
\]
\[ \simeq C(\Omega I, A_G^\beta)((M^\beta \otimes_{A_I} A_G^\beta)^\tau, (Z_{I,A_I}^\theta(\lambda) \otimes_{A_I} A_G^\theta)^\tau) \text{ as } A_G^\beta \simeq A_I^\theta \otimes_{A_I} A_G^\beta
\]
\[ \text{in this case, and hence } Y_{\hat{A}_G}^\beta(\lambda)^\tau \simeq (Z_{I,A_I}^\theta(\lambda) \oplus Z_{I,A_I}^\theta(\beta \uparrow \lambda)) \otimes_{A_I} A_G^\beta
\]
\[ \simeq C(\Omega I, A_I^\beta)(Z_{I,A_I}^\theta(\lambda), M^\beta) \otimes_{A_I} A_G^\beta
\]
\[ = \mathcal{V}_{\Omega I}(M)(\lambda) \otimes_{A_I} A_G^\beta = (I \circ \mathcal{V}_{\Omega I})(M)(\lambda, \beta). \]

If \( \lambda \in \Omega \setminus \Omega_I \) and if \( \beta \in R_+ \setminus R_+^I \) with \( \beta \uparrow \lambda \in \Omega_I \), we have likewise

\[ (\mathcal{V}_\Omega \circ \hat{\nabla}_{P,A_I})(M)(\lambda, \beta) \simeq \mathcal{V}_{\Omega I}(M)(\beta \uparrow \lambda) \otimes_{A_I} A_G^\beta = (I \circ \mathcal{V}_{\Omega I})(M)(\lambda, \beta). \]

If \( \lambda \in \Omega \setminus \Omega_I \) and if \( \beta \uparrow \lambda \notin \Omega_I \),

\[ (\mathcal{V}_\Omega \circ \hat{\nabla}_{P,A_I})(M)(\lambda, \beta) = 0 = (I \circ \mathcal{V}_{\Omega I})(M)(\lambda, \beta). \]

(3.7) Define finally a functor \( \tilde{\mathcal{I}} : \tilde{\mathcal{K}}(\Omega_I, S_{I,k}) \to \tilde{\mathcal{K}}(\Omega, S_k) \) just like \( \mathcal{I} \) as follows: for each \( \mathcal{M} \in \mathcal{K}(\Omega_I, A_I) \) and \( \lambda \in \Omega \) set

\[ (\tilde{\mathcal{I}}(\mathcal{M}))(\lambda) = \begin{cases} 
\mathcal{M}(\lambda) \otimes_{S_{I,k}} S_k^0 & \text{if } \lambda \in \Omega I \\
0 & \text{else,}
\end{cases} \]
and for each $\beta \in R^+$ set
\[
(\tilde{I}(M))(\lambda, \beta) = \begin{cases} 
M(\lambda, \beta) \otimes S_{I,k} S^\beta_k & \text{if } \lambda \in \Omega_I \text{ and } \beta \in R_I^+ \\
M(\lambda) \otimes S_{I,k} S^\beta_k & \text{if } \lambda \in \Omega_I \text{ and } \beta \notin R_I^+ \\
M(\beta \uparrow \lambda) \otimes S_{I,k} S^\beta_k & \text{if } \beta \uparrow \lambda \in \Omega_I \text{ and } \beta \notin R_I^+ \\
0 & \text{else.}
\end{cases}
\]

From [AJS 14.10] one has, in particular, for each $\lambda \in \Omega_I$
\[
(1) \quad \tilde{I}(Z_{I,\lambda}^{w_I}) \simeq Z_\lambda^{w_0}.
\]

Let $Q_{I,A_I} = \bigoplus_{w \in W_I} Q_{I,A_I}(w \cdot \lambda^\uparrow) \in C(\Omega_I, A_I)$ with $Q_{I,A_I}(w \cdot \lambda^\uparrow)$ the lift of the projective cover of $\hat{L}^I(w \cdot \lambda^\uparrow)$ for $L_I$ over $A_I$. Let $P_I$ be a graded $S_{I,k}$-form of $V_{\Omega_I}(Q_{I,A_I})$.

**Lemma:** One has an isomorphism in $K(\Omega, A_G)$
\[
\tilde{I}(P_I) \otimes_S k \simeq \mathcal{I}(V_{\Omega_I}(Q_{I,A_I})) \simeq V_{\Omega_I} \circ \hat{\nabla}_{P,A_I}(Q_{I,A_I}).
\]

**Proof:** The first isomorphism follows from the definition that $P_I \otimes_{S_{I,k}} A_I \simeq V_{\Omega_I}(Q_{I,A_I})$, and the second from (3.6).

(3.8) Let $E_{\Omega_I} = \hat{K}(\Omega_I, S_{I,k})^\otimes (P_I, P_I)^{\otimes 2}$, which is a $(Y_I \times \mathbb{Z})$-graded $S_{I,k}$-algebra of finite type, responsible for the structure of $(Y_I \times \mathbb{Z})$-graded $\mathbb{K}$-algebra on $E_{\Omega_I,k} \simeq E_{\Omega_I} \otimes_{S_{I,k}} \mathbb{K}$. Now set $J = \hat{K}(\Omega, S_{k})^\otimes (P, \tilde{I}(P_I))$, which comes equipped with a structure of $(Y \times \mathbb{Z})$-graded left $E_{\Omega}$ and $(Y_I \times \mathbb{Z})$-graded right $E_{\Omega_I}$-bimodule. If $J_k = J \otimes_{S_k} \mathbb{K}$, it is thus a $(Y \times \mathbb{Z})$-graded left $E_{\Omega,k}$ and $(Y_I \times \mathbb{Z})$-graded right $E_{\Omega_I,k}$-bimodule.

**Corollary:** The parabolic induction functor $\hat{\nabla}_p$ is $\mathbb{Z}$-graded by the bimodule $J_k$ in such a way
\[
\begin{array}{ccc}
\tilde{C}_k(\Omega_I) & \xrightarrow{J_k \otimes E_{\Omega_I,k}} & \tilde{C}_k(\Omega) \\
\vee & \downarrow & \vee \\
L_{I,1} T \text{Mod} & \circ & G_1 T \text{Mod} \\
\downarrow & \circ & \downarrow \\
P_1 T \text{Mod} & \xrightarrow{\hat{\nabla}_p} & G_1 T \text{Mod}
\end{array}
\]

that for each $\lambda \in \Omega_I$ there is an isomorphism in $\tilde{C}_k(\Omega)$
\[
J_k \otimes E_{\Omega_I,k} \tilde{\nabla}_{I,k}(\lambda) \simeq \tilde{\nabla}_k(\lambda)(\delta(\lambda) - \delta_I(\lambda)).
\]

**Proof:** The commutativity of the diagram follows from (3.2) by the isomorphism of left
\(E_{\Omega,k}\) and right \(E_{\Omega,k}\)-bimodules

\[
J_k \simeq J \otimes_{S_k} A_G \otimes_{A_G} \mathbb{k}
\]

\[
\simeq \mathcal{K}(\Omega, A_G)^{(\mathcal{V}_\Omega(Q_{A_G}), \hat{\mathcal{V}}_\Omega(P_I) \otimes_{S_k} A_G)) \otimes_{A_G} \mathbb{k} \quad \text{by [AJS] 18.9.3}
\]

\[
\simeq \mathcal{K}(\Omega, A_G)^{(\mathcal{V}_\Omega(Q_{A_G}), \mathcal{V}_\Omega \circ \hat{\mathcal{V}}_{P,A_I}(Q_{I,A_I}))} \otimes_{A_G} \mathbb{k} \quad \text{by (3.7)}
\]

\[
\simeq C(\Omega, A_G)^{(Q_{A_G}, \hat{\mathcal{V}}_{P,A_I}(Q_{I,A_I}))} \otimes_{A_G} \mathbb{k} \quad \text{by [AJS] 18.9.5/6}
\]

\[
\simeq C_k(\Omega)^{(Q, \hat{\mathcal{V}}_P(Q_I))} \text{ from (3.3.3)}.
\]

Also,

\[
J_k \otimes_{E_{\Omega,k}} \hat{\mathcal{V}}_{I,k}(\lambda)
\]

\[
\simeq (J \otimes_{E_{\Omega,k}} \hat{\mathcal{V}}_{I,S_{I,k}}(\lambda)) \otimes_{S_{I,k}} \mathbb{k} = \{K(\Omega, S_k)^{(P, \hat{K}_\Omega(P_I))} \otimes_{E_{\Omega,I}} \hat{\mathcal{V}}_{I,S_{I,k}}(\lambda)\} \otimes_{S_{I,k}} \mathbb{k}
\]

\[
= \{K(\Omega, S_k)^{(P, \hat{K}_\Omega(\hat{Z}_{I_{I,k}}(\lambda)^{-}\delta_l(\lambda)))} \otimes_{S_{I,k}} \mathbb{k} \quad \text{by the five lemma}
\]

\[
= K(\Omega, S_k)^{(\mathcal{V}_\mathcal{S}(\mathcal{S}_{\mathcal{S}}(\lambda)) \otimes_{S_{I,k}} \mathbb{k} \quad \text{by (3.7.1)}
\]

\[
= \hat{\mathcal{V}}_{S_{I,k}}(\lambda)(\delta(\lambda) - \delta_l(\lambda)) \otimes_{S_{I,k}} \mathbb{k} = \hat{\mathcal{V}}_{S_{I,k}}(\lambda)(\delta(\lambda) - \delta_l(\lambda)).
\]

4° Rigidity

Keep the notations of §3. We will show that all \(\hat{\mathcal{V}}_P(\hat{\mathcal{L}}^P(\lambda))\) for \(p\)-regular \(\lambda \in \Lambda\) are \(\mathbb{Z}\)-graded. Assuming Lusztig’s conjecture on the irreducible character formulae for \(G_1 T\)-modules [LN0]/[Kat], [AJS] has shown that the endomorphism algebra of a projective \(Y\)-generator for the block of \(\lambda\) is Koszul. We show that the rigidity of \(\hat{\mathcal{V}}_P(\hat{\mathcal{L}}^P(\lambda))\) follows from a result of [BGS]. The Lusztig conjecture is now a theorem for large \(p\) thanks to [AJS], [KL], [KT], [L94] and more recently [E]. We will also find the Loewy length of \(\hat{\mathcal{V}}_P(\hat{\mathcal{L}}^P(\lambda))\) for a \(p\)-regular \(\lambda \in \Lambda\) to be uniformly \(\ell(w^{l}) + 1\).

Thus fix a \(p\)-regular weight \(\lambda\) and put \(\Omega = W_p \bullet \lambda\). For \(M \in C_k(\Omega)\), we let \([M : \hat{\mathcal{L}}(\mu)]\), \(\mu \in \Omega\), denote the multiplicity of simple \(\hat{\mathcal{L}}(\mu)\) among the \(C_k(\Omega)\)-composition factors of \(M\).

(4.1) Let us first recall the construction of \(\hat{L}_k(\lambda)\), which is a slight simplification over the one in [AJS] 18.12. As \(\hat{Z}_k(\lambda)\) has a simple head in \(E_{\Omega,k}\operatorname{modgr}_Y\), by the categorical equivalence \(v\), the radical \(\operatorname{rad}_E_{\Omega,k}\operatorname{modgr}_Y \hat{Z}_k(\lambda)\) of \(\hat{Z}_k(\lambda)\) in the category \(E_{\Omega,k}\operatorname{modgr}_Y\) is maximal. But \(\operatorname{rad}_E_{\Omega,k}\operatorname{modgr}_Y \hat{Z}_k(\lambda)\) belongs to \(C_k(\Omega)\) by [AJS] E.11, and hence coincides with the radical \(\hat{c}(\Omega)\operatorname{rad}_{\hat{c}(\Omega)} \hat{Z}_k(\lambda)\) in the category of \(C_k(\Omega)\). We set \(\hat{L}_k(\lambda) = \hat{Z}_k(\lambda)/\operatorname{rad}_{\hat{c}(\Omega)} \hat{Z}_k(\lambda)\). Then \(\hat{v}\hat{L}_k(\lambda) = Q \otimes_{E_{\Omega,k}} \hat{L}_k(\lambda) \simeq \{Q \otimes_{E_{\Omega,k}} \hat{Z}_k(\lambda)\}/\{Q \otimes_{E_{\Omega,k}} \operatorname{rad}_{\hat{c}(\Omega)} \operatorname{modgr}_Y \hat{Z}_k(\lambda)\} \simeq Z_k(\lambda)/\operatorname{rad}_{\hat{c}(\Omega)} Z_k(\lambda) \sim \hat{L}(\lambda)\).

In turn, \(\hat{L}_k(\lambda) \simeq H_{\Omega,k}\hat{L}(\lambda)\) in \(E_{\Omega,k}\operatorname{modgr}_Y\) while \(H_{\Omega,k}\hat{L}(\lambda) = C_k(\Omega)^{(Q, \hat{L}(\lambda))} \simeq C_k(\Omega)((Q_k(\lambda), \hat{L}(\lambda))\) as \(Q_k(\lambda)\) is the projective cover of \(\hat{L}(\lambda)\), and hence \(\hat{L}_k(\lambda)\) is of dimension 1.

By the equivalence \(v\) the \(\hat{L}_k(\lambda)\), \(\lambda \in \Omega\), exhaust the simple objects of \(E_{\Omega,k}\operatorname{modgr}_Y\). If
$\tilde{L}$ is a simple object of $\tilde{C}_k(\Omega)$, then
\[
0 \neq E_{\Omega,k}^{\text{modgr}}_Y(\tilde{L}, \tilde{L}_k(\lambda)) \quad \text{for some } \lambda \in \Omega
\]
\[
= \bigoplus_{i \in \mathbb{Z}} E_{\Omega,k}^{\text{modgr}}_Y(\tilde{L}, \tilde{L}_k(\lambda)) = \bigoplus_{i \in \mathbb{Z}} \tilde{C}_k(\Omega)(\tilde{L}, \tilde{L}_k(\lambda)(-i)),
\]
and hence $\tilde{C}_k(\Omega)(\tilde{L}, \tilde{L}_k(\lambda)(i)) \neq 0$ for some $i$. Then $\tilde{L} \simeq \tilde{L}_k(\lambda)(i)$ in $\tilde{C}_k(\Omega)$ by their simplicity. Such $\lambda$ and $i$ are unique by [AJS] 18.8. Thus we have obtained the first 2 parts of

**Proposition:** (i) Each $\tilde{L}_k(\lambda)$, $\lambda \in \Omega$, is 1-dimensional.

(ii) Each simple object of $\tilde{C}_k(\Omega)$ is isomorphic to some $\tilde{L}(\lambda)(i)$ for unique $\lambda \in \Omega$ and $i \in \mathbb{Z}$. Any simple object of $E_{\Omega,k}^{\text{modgr}}_Y$ is isomorphic to some $\tilde{L}(\lambda)$ for unique $\lambda \in \Omega$.

(iii) If $M \in \tilde{C}_k(\Omega)$, the radical (resp. socle) series of $M$ in $E_{\Omega,k}^{\text{modgr}}_Y$ and in $\tilde{C}_k(\Omega)$ coincide.

**Proof:** (iii) We show first that each radical layer $\text{rad}^{i}_{\text{modgr}}_Y \tilde{M}/\text{rad}^{i+1}_{\text{modgr}}_Y \tilde{M}$ remains semisimple in $\tilde{C}_k(\Omega)$. As it inherits the structure of $\tilde{C}_k(\Omega)$ from $M$ by [AJS] E.11, we may assume $M$ is semisimple in $E_{\Omega,k}^{\text{modgr}}_Y$. If $L$ is a simple component of $M$ in $E_{\Omega,k}^{\text{modgr}}_Y$, as $L$ is 1-dimensional by (i), each $(E_{\Omega,k})_{Y \times \{i\}}$, $i \neq 0$ annihilates $L$ while each element of $(E_{\Omega,k})_{Y \times \{0\}}$ is acting by a scalar, and hence $M$ is semisimple also in $\tilde{C}_k(\Omega)$; each $\mathbb{Z}$-homogeneous component $M_i$ of $M$ must be $E_{\Omega,k}$-stable. On the other hand, each $\text{rad}^{i}_{\tilde{C}_k(\Omega)} \tilde{M}/\text{rad}^{i+1}_{\tilde{C}_k(\Omega)} \tilde{M}$ is semisimple in $E_{\Omega,k}^{\text{modgr}}_Y$ as each simple component is 1-dimensional by (i) again. It now follows that the radical series of $M$ in $E_{\Omega,k}^{\text{modgr}}_Y$ and $\tilde{C}_k(\Omega)$ coincide.

The socle version of [AJS] E.11 holds, and hence also the assertion about the socle series of $M$.

(4.2) Assume now Lusztig’s conjecture on the irreducible characters of $G_1T$-modules. Then $E_{\Omega,k}$ is Koszul with respect to its $\mathbb{Z}$-gradation thanks to [AJS] 18.17. In particular, $E_{\Omega,k}$ is positively graded: $E_{\Omega,k} = \bigoplus_{i \in \mathbb{N}} (E_{\Omega,k})_i$ with $(E_{\Omega,k})_0 = \bigcap_{w \in W} \mathbb{Z}[\pi_w]$, and is generated by $(E_{\Omega,k})_1$ over $k$ by [RGS] Props. 2.1.3 and 2.3.1, where $\pi_w : \bigcap_{w \in W} Q_k(x \cdot \lambda^+) \rightarrow Q_k(w \cdot \lambda^+)$ is the projection. Let $E_{\Omega,k}^{\text{modgr}}_Z$ denote the category of finite dimensional $\mathbb{Z}$-graded $E_{\Omega,k}$-modules.

**Proposition:** Assume the Lusztig conjecture.

(i) Each $\tilde{L}_k(\lambda)$, $\lambda \in \Omega$, is homogeneous of degree 0 with respect to the $\mathbb{Z}$-grading. In particular, each $\tilde{L}_k(w \cdot \lambda^+)$, $w \in W$, is isomorphic to $\mathbb{Z}[\pi_w]$ in $E_{\Omega,k}^{\text{modgr}}_Z$.

(ii) Each simple object of $E_{\Omega,k}^{\text{modgr}}_Z$ is isomorphic to some $\tilde{L}_k(w \cdot \lambda^+)(i)$ for unique $w \in W$ and $i \in \mathbb{Z}$.

(iii) If $M \in \tilde{C}_k(\Omega)$, the radical (resp. socle) series of $M$ in $E_{\Omega,k}^{\text{modgr}}_Z$ and in $\tilde{C}_k(\Omega)$ coincide.
Proof: (i) Recall from (4.1) that the \( \mathbb{Z} \)-grading on \( E_{\Omega,k} \) arises from that of \( E_\Omega \). Thus \( k\pi_w = k(\pi_w \otimes 1) \) if \( \pi_w : \prod_{x \in W} Q(x \cdot \lambda^+) \to Q(w \cdot \lambda^+) \) is the projection. But

\[
\tilde{K}(\Omega, S_k)^2(\bigoplus_{x \in W} Q(x \cdot \lambda^+), Q(w \cdot \lambda^+))
\]

\[
= \bigoplus_{x \in W} \bigoplus_{\gamma \in Y} \bigoplus_{i \in \mathbb{Z}} K(\Omega, S_k)(Q(x \cdot \lambda^+)[\gamma], Q(w \cdot \lambda^+))_i \quad \text{by definition [AJS] E.1, E.3}
\]

\[
= \bigoplus_{x \in W} \bigoplus_{\gamma \in Y} K(\Omega, S_k)(Q(x \cdot \lambda^+)[\gamma], Q(w \cdot \lambda^+)) \quad \text{by [AJS] E.1}
\]

\[
\simeq \bigoplus_{x \in W} \bigoplus_{\gamma \in Y} K(\Omega, S_k)(Q(x \cdot \lambda^+ + \gamma), Q(w \cdot \lambda^+)) \quad \text{by [AJS] 17.6/18.5}
\]

with \( K(\Omega, S_k)(Q(x \cdot \lambda^+ + \gamma), Q(w \cdot \lambda^+)) = \bigoplus_{i > 0} K(\Omega, S_k)(Q(x \cdot \lambda^+ + \gamma), Q(w \cdot \lambda^+))_i \) unless \( x \cdot \lambda^+ + \gamma = w \cdot \lambda^+ \) while \( K(\Omega, S_k)(Q(w \cdot \lambda^+), Q(w \cdot \lambda^+))_0 = S_k \text{id}_{Q(w \cdot \lambda^+)} \) [AJS 17.9]. On the other hand,

\[
\tilde{Z}_k(w \cdot \lambda^+ + \gamma) = \tilde{K}(\Omega, S_k)^2(\bigoplus_{x \in W} Q(x \cdot \lambda^+), Z_{w \cdot \lambda^+ + \gamma}(\delta(w \cdot \lambda^+ + \gamma))) \otimes S_k k
\]

by definition [AJS 18.10.1 and 18.12]

\[
\simeq \bigoplus_{x \in W} \bigoplus_{\gamma \in Y} K(\Omega, S_k)(Q(x \cdot \lambda^+ + \nu), Z_{w \cdot \lambda^+ + \gamma}(\delta(w \cdot \lambda^+ + \gamma))) \otimes S_k k \quad \text{as above.}
\]

Each \( K(\Omega, S_k)(Q(x \cdot \lambda^+ + \nu), Z_{w \cdot \lambda^+ + \gamma}(\delta(w \cdot \lambda^+ + \gamma))) \) is a direct summand of \( \tilde{K}(\Omega, S_k)(Q(x \cdot \lambda^+ + \nu), Q(w \cdot \lambda^+ + \gamma)) \) by [AJS 15.10 and 17.6/18.9], and hence \( K(\Omega, S_k)(Q(x \cdot \lambda^+ + \nu), Z_{w \cdot \lambda^+ + \gamma}(\delta(w \cdot \lambda^+ + \gamma))) = K(\Omega, S_k)(Q(x \cdot \lambda^+ + \nu), Z_{w \cdot \lambda^+ + \gamma}(\delta(w \cdot \lambda^+ + \gamma))) > 0 \) unless \( x \cdot \lambda^+ + \nu = w \cdot \lambda^+ + \gamma \), i.e., \( x = w \) and \( \nu = \gamma \), by [AJS 17.9] again while

\[
K(\Omega, S_k)(Q(w \cdot \lambda^+ + \gamma), Z_{w \cdot \lambda^+ + \gamma}(\delta(w \cdot \lambda^+ + \gamma)))_0
\]

\[
\simeq K(\Omega, S_k)(Z_{w \cdot \lambda^+ + \gamma}(2R^+ - \delta(w \cdot \lambda^+ + \gamma)), Z_{w \cdot \lambda^+ + \gamma}(\delta(w \cdot \lambda^+ + \gamma)))_0
\]

by [AJS 15.10 and 17.6.2]

\[
\simeq (S_k)_0 \quad \text{by [AJS] 15.10.2}.
\]

Thus the epi \( \tilde{Z}_k(w \cdot \lambda^+ + \gamma)/\tilde{Z}_k(w \cdot \lambda^+ + \gamma) > 0 \to \tilde{L}_k(w \cdot \lambda^+ + \gamma) \) is an isomorphism of \( E_{\Omega,k} \text{mod}_{\mathbb{Z}} \) by dimension, and hence \( \tilde{L}_k(w \cdot \lambda^+ + \gamma) \) is of degree 0. In particular, \( \tilde{L}_k(w \cdot \lambda^+) \simeq k(\tilde{\pi}_w \otimes 1) \).

(ii) Let \( L \) be a simple object of \( E_{\Omega,k} \text{mod}_{\mathbb{Z}} \). As \( (E_{\Omega,k})_0 L = 0 \), \( L \) is a \( (E_{\Omega,k})_0 \)-module. Then \( L \) is by its simplicity isomorphic to some \( k\pi_w(i), w \in W, i \in \mathbb{Z} \).

(iii) now follows from (ii) just like (4.1.iii), applying [AJS] E.11] to the pair \( (Y \times \mathbb{Z}, Z) \) in place of \( (Y \times \mathbb{Z}, Y) \).

(4.3) We are now to obtain from [BGS Prop. 2.4.1] the rigidity of \( \check{\nabla}_P(\check{L}) \), as well as \( \nabla(\lambda) \) and \( \check{Q}(\lambda) = Q_k(\lambda) \) for each \( \lambda \in \Omega \) demonstrated first in [AK] by a different method using Vogon’s version of the Lusztig conjecture.

Lemma: Assume the Lusztig conjecture on the irreducible characters of \( G_1 T \)-modules.
Let $M \in \mathcal{C}_\kappa(\Omega)$. If $M$ has a simple socle and a simple head as an object of $E_{\Omega, k}\text{modgr}_Y$, then $M$ is rigid in $E_{\Omega, k}\text{modgr}_Y$.

**Proof:** By the hypothesis $M$ has a simple socle and a simple head in $(E_{\Omega, k}\text{modgr}_Y)$ by (4.1) and (4.2). If $\text{hd}_{E_{\Omega, k}\text{modgr}_Y} M$ (resp. $\text{soc}_{E_{\Omega, k}\text{modgr}_Y} M$) is concentrated in degree $j$ (resp. $k$), from [BGS, Prop. 2.4.1]

\[
\text{rad}_{E_{\Omega, k}\text{modgr}_Y}^i M = M_{\geq i+j} \quad \text{and} \quad \text{soc}_{E_{\Omega, k}\text{modgr}_Y}^i M = M_{\geq k-i+1} \quad \forall i.
\]

Thus $M_{\geq k-i+1} = \text{soc}_{E_{\Omega, k}\text{modgr}_Y}^i M \geq \text{rad}_{E_{\Omega, k}\text{modgr}_Y}^{\ell(\ell(M))} M = M_{\geq \ell(\ell(M)) - i + j}$, and hence $k - i + 1 \leq \ell(\ell(M)) - i + j$. As the equality holds for $i = 0$, $k + 1 = \ell(\ell(M)) + j$. Then $\forall i$, $k - i + 1 \leq \ell(\ell(M)) - i + j = \ell(\ell(M)) - i + k + 1 - \ell(\ell(M)) = k - i + 1$, and hence

\[
\text{soc}_{E_{\Omega, k}\text{modgr}_Y}^i M = \text{soc}_{E_{\Omega, k}\text{modgr}_Y}^i M = M_{\geq \ell(\ell(M)) + j - i} = \text{rad}_{E_{\Omega, k}\text{modgr}_Y}^{\ell(\ell(M)) - i} M = \text{rad}_{E_{\Omega, k}\text{modgr}_Y}^{\ell(\ell(M)) - i} M.
\]

(4.4) Recalling from (1.4) that each $\hat{\nabla}_P(\hat{L}^P(\lambda))$ has a simple socle and a simple head yields

**Theorem:** Assume the Lusztig conjecture. Each $\hat{\nabla}_P(\hat{L}^P(\lambda))$ for $p$-regular $\lambda$ is rigid.

(4.5) To determine eventually the Loewy series of $\hat{\nabla}_P(\hat{L}^P(\lambda))$, we have to compute its Loewy length. As $\ell(\ell(\hat{\nabla}_P(\hat{L}^P(\lambda)))) = \ell(\ell(w^t \hat{\nabla}_P(\hat{L}^P(\lambda))))$, we will compute $\ell(\ell(w^t \hat{\nabla}_P(\hat{L}^P(\lambda))))$.

**Lemma:** $\text{hd}_{G_1 T}(w^t \hat{\nabla}_P(\hat{L}^P(\lambda))) = \hat{L}(w^t \lambda) \otimes -p(w^t \rho \cdot 0)$.

**Proof:** We may assume $\lambda^\rho = 0$. By (1.4)

\[
\text{hd}_{G_1 T}(w^t \hat{\nabla}_P(\hat{L}^P(\lambda))) = w^t \text{hd}_{G_1 T}(\hat{\nabla}_P(\hat{L}^P(\lambda)))
\]

\[
= w^t \{ \hat{L}(w^t \lambda) \otimes p(-2\rho_P + w_0((-w_1) \lambda)^1 - ((-w_1) \lambda)^1) \}
\]

\[
= w^t \{ L((w^t \lambda)^0) \otimes p((w^t \lambda)^0 - 2\rho_P + w_0((-w_1) \lambda)^1 - ((-w_1) \lambda)^1) \}
\]

\[
= L((w^t \lambda)^0) \otimes p((w^t \lambda)^0 - 2\rho_P + w_0((-w_1) \lambda)^1 - ((-w_1) \lambda)^1)
\]

while $\hat{L}(w^t \lambda) \otimes -p(w^t \rho \cdot 0) = L((w^t \lambda)^0) \otimes p((w^t \lambda)^1 - (w^t \cdot 0))$. Thus we are to show

(1) \quad $(w^t \lambda)^1 - (w^t \cdot 0) = w^t \{ (w^t \lambda)^1 - 2\rho_P + w_0((-w_1) \lambda)^1 - ((-w_1) \lambda)^1 \}$.

Write $w_1 \lambda = \mu^0 + p\mu^1$ with $\mu^0 \in \Lambda_P$ and $\mu^1 \in \Lambda$. Thus $\mu^0$ is $p$-regular. As $w^t \lambda = w_0 \cdot (\mu^0 + p\mu^1) = w_0 \cdot \mu^0 + p\mu_0$, $(w^t \lambda)^1 = w_0\mu_1 - \rho$. Likewise, as $(-w_1) \lambda = (-1) \cdot (\mu^0 + p\mu^1) = (-1) \cdot \mu^0 - p\mu_1$, $((-w_1) \lambda)^1 = -\mu^1 - \rho$. It follows that the LHS of (1) is equal by (1.1) to

\[
w_0\mu_1 - \rho - w_0w_1 \cdot 0 = w_0\mu_1 - \rho - w_0w_1\rho - \rho = w_0\mu_1 - \rho - w_02\rho_P = w_0(\mu_1 + \rho - 2\rho_P)
\]

while the RHS of (1) is equal to

\[
w^t \{ w_0\mu_1 - \rho - 2\rho_P + w_0(\mu_1 - \rho) - (\mu^1 - \rho) \} = w^t(-2\rho_P + \mu^1 + \rho)
\]

\[
= w_0(w_1\mu^1 + w_1\rho - 2\rho_P) \quad \text{as } w_12\rho_P = 2\rho_P.
\]
Thus we are left to verify that $\mu^1 + \rho = w_I(\mu^1 + \rho)$, for which we have only to check $\langle \mu^1 + \rho, \alpha^\vee \rangle = 0 \forall \alpha \in I$. But

\[
\langle p + p(\mu^1 + \rho, \alpha), \rho \rangle \geq \langle \mu^0 + p\mu^1 + pp + \rho, \alpha^\vee \rangle = \langle \mu^1 \cdot \lambda + p\rho + \rho, \alpha^\vee \rangle = \langle \lambda + \rho, w_I\alpha^\vee \rangle + p
\]

and hence $\langle \mu^1 + \rho, \alpha^\vee \rangle = 0$, as desired.

(4.6) Recall from (1.6.3) that $w^I \hat{\nabla}_P(\hat{L}^P(\lambda)) \leq \hat{\nabla}_{w^I}(\langle w^I \cdot \lambda \rangle) \otimes -p(w^I \cdot 0)$. Recall also from [AK] an intertwining homomorphism $\phi_w \in G_1 T \text{Mod}(\hat{\nabla}_w((w \cdot \lambda)(w)), \hat{\nabla}(w \cdot \lambda)) \setminus 0$ for each $w \in W$, which is unique up to $\mathbb{K}^\times$. As $1 = \hat{\nabla}(w^I \cdot \lambda) = \hat{\nabla}_{w^I}(\langle w^I \cdot \lambda \rangle) = \hat{L}(w^I \cdot \lambda)$ by [AK, 1.2.3], one obtains from (4.5) a commutative diagram of $G_1 T$-modules

\[
\begin{array}{ccc}
\hat{\nabla}_{w^I}(\langle w^I \cdot \lambda \rangle) & \otimes -p(w^I \cdot 0) & \hat{L}(w^I \cdot \lambda) \otimes -p(w^I \cdot 0) \\
\text{w}^I \hat{\nabla}_P(\hat{L}^P(\lambda)) & \rightarrow & \hat{L}(w^I \cdot \lambda) \otimes -p(w^I \cdot 0),
\end{array}
\]

As $\phi_{w^I}(\text{soc}_{\ell(w^I)} \hat{\nabla}_{w^I}(\langle w^I \cdot \lambda \rangle)) = 0$ [AK], we must have

\[
\ell(\hat{\nabla}_{w^I}(\hat{L}^P(\lambda))) \geq \ell(w^I) + 1.
\]

On the other hand, there is another intertwining homomorphism $\phi'_{w^I} \in G_1 T \text{Mod}(\hat{\nabla}_{w^0}(\langle w^I \cdot \\
\lambda \rangle(w^0)), \hat{\nabla}_{w^I}(\langle w^I \cdot \lambda \rangle)) \setminus 0$. As

\[
\begin{align*}
\text{hd}_{G_1 T} \hat{\nabla}_{w^0}(\langle w^I \cdot \lambda \rangle(w^0)) & \otimes -p(w^I \cdot 0) \\
& = \text{hd}_{G_1 T} \hat{\Delta}(w^I \cdot \lambda) \otimes -p(w^I \cdot 0) \text{ by [AK] 1.2} \\
& = \hat{L}(w^I \cdot \lambda) \otimes -p(w^I \cdot 0) = \text{hd}_{G_1 T}(\hat{\nabla}_{w^I}(\hat{L}^P(\lambda))),
\end{align*}
\]

one obtains as in (1) another commutative diagram

\[
\begin{array}{ccc}
\hat{\nabla}_{w^0}(\langle w^I \cdot \lambda \rangle(w^0)) & \otimes -p(w^I \cdot 0) & \\
\phi'_{w^I} \otimes -p(w^I \cdot 0) & \hat{\nabla}_{w^I}(\langle w^I \cdot \lambda \rangle(w^I)) & \otimes -p(w^I \cdot 0),
\end{array}
\]

with $\phi'_{w^I}(\text{soc}_{\ell(w^0) - \ell(w^I)} \hat{\nabla}_{w^0}(\langle w^I \cdot \lambda \rangle(w^0))) = 0$. Assuming the Lusztig conjecture we have $\ell(\hat{\nabla}_{w^0}(\langle w^I \cdot \lambda \rangle(w^0))) = \ell(w^0) + 1$. It follows that

\[
\ell(\hat{\nabla}_{w^I}(\hat{L}^P(\lambda))) \leq \ell(w^0) + 1 - \{\ell(w^0) - \ell(w^I)\} = \ell(w^I) + 1.
\]

Thus, together with (2), we have obtained
Theorem: Assume the Lusztig conjecture. For any $p$-regular $\lambda \in \Lambda$
\[ \ell(\hat{\nabla}_P(\hat{L}^P(\lambda))) = \ell(w^I) + 1. \]

(4.7) Remark: This is a generalization of [KY] 1.4 and [K09], where we found for $G$ of
rank at most 2 or in case $G = GL_{n+1}(k)$ with $P$ a maximal parabolic such that $G/P \simeq P^n$
for any $n \in \mathbb{N}$ that $\ell(\hat{\nabla}_P(\lambda)) = \ell(w^I) + 1$ for $p$-regular $\lambda \in \Lambda_P$. In fact, for $G/P \simeq P^n$ we
computed $\ell(\hat{\nabla}_P(\lambda))$ for any $\lambda \in \Lambda_P$ in [K09] 2.3 dispensing with the Lusztig conjecture.

(4.8) Recall that $\tilde{Z}_k(\lambda)/\tilde{Z}_k(\lambda)_{>0} \simeq \tilde{L}_k(\lambda) \simeq \text{hd}_{E_{\Omega,k}\text{modgr}_Z} \tilde{Z}_k(\lambda)$ for each $\lambda \in \Omega$. It follows
that the $\mathbb{Z}$-gradation on $\tilde{Z}_k(\lambda)$ is such that for each $j \in \mathbb{N}$
\[ \tilde{Z}_k(\lambda)_{\geq j} = \text{rad}_{E_{\Omega,k}\text{modgr}_Z}^j \tilde{Z}_k(\lambda) = \text{rad}_{E_{\Omega,k}\text{modgr}_Y}^j \tilde{Z}_k(\lambda) \]
\[ = \text{soc}_{E_{\Omega,k}\text{modgr}_Z}^{|R^+|+1-j} \tilde{Z}_k(\lambda) = \text{soc}_{E_{\Omega,k}\text{modgr}_Z}^{|R^+|+1-j} \tilde{Z}_k(\lambda), \]
and hence
\[ \text{soc}_{E_{\Omega,k}(\Omega)}^{|R^+|+1-j} Z_k(\lambda) = \tilde{v}(\tilde{Z}_k(\lambda)_{\geq j}) = \text{rad}_{E_{\Omega,k}(\Omega)}^j Z_k(\lambda). \]

More generally,

Proposition: Assume the Lusztig conjecture. The $\mathbb{Z}$-gradation on each $\tilde{Z}_k^w(\lambda)$, $\lambda \in \Omega$, $w \in W$, is such that for each $i \in \mathbb{N}$
\[ \text{rad}_{E_{\Omega,k}(\Omega)}^i Z_k^w(\lambda(w)) = \text{rad}_{G_{1T}}^i \tilde{\nabla}_{ww_0}(\lambda(ww_0)) = \tilde{v}(\tilde{Z}_k^w(\lambda)_{\geq -\ell(w)+i}) \]
\[ = \text{soc}_{E_{\Omega,k}(\Omega)}^{|R^+|+1-i} Z_k^w(\lambda(w)) = \text{soc}_{G_{1T}}^{|R^+|+1-i} \tilde{\nabla}_{ww_0}(\lambda(ww_0)). \]

Thus $\forall \mu \in \Omega$,
\[ [\text{rad}_{E_{\Omega,k}(\Omega)}^i Z_k^w(\lambda(w)) : \hat{L}(\mu)] = [\tilde{Z}_k^w(\lambda) : \tilde{L}_k(\mu)(-\ell(w)+i)] \]
\[ = [\text{soc}_{E_{\Omega,k}(\Omega)}^{|R^+|+1-i} Z_k^w(\lambda(w)) : \hat{L}(\mu)], \]

where the middle term is the multiplicity of simple $\tilde{L}_k(\mu)(-\ell(w)+i)$ in $\tilde{Z}_k^w(\lambda)$ considered as objects of $E_{\Omega,k}\text{modgr}_Z$.

Proof: One has from [AJS] 15.3.2
\[ k = C_k(\Omega)(Z_k^w(\lambda(w)), Z_k(\lambda)) = \tilde{C}_k(\Omega)(Z_k^w(\lambda), \tilde{Z}_k(\lambda)(-2\ell(w))). \]

Let $j \in \mathbb{Z}$ minimal such that $\tilde{Z}_k^w(\lambda)_{\geq j} \neq 0$, so $\tilde{Z}_k^w(\lambda)_{\geq j}/\tilde{Z}_k^w(\lambda)_{> j} = \text{hd}_{E_{\Omega,k}\text{modgr}_Z} \tilde{Z}_k^w(\lambda) = \text{hd}_{E_{\Omega,k}\text{modgr}_Y} \tilde{Z}_k^w(\lambda) = H_{\Omega,k}(\text{rad}_{C_k(\Omega)} Z_k^w(\lambda(w))),$ which is sent to
\[ \tilde{Z}_k(\lambda)_{\geq j+2\ell(w)}/\tilde{Z}_k(\lambda)_{> j+2\ell(w)} = (\tilde{Z}_k(\lambda)(-2\ell(w))\rangle_{\geq j}/(\tilde{Z}_k(\lambda)(-2\ell(w))\rangle_{> j} \]
\[ = H_{\Omega,k}(\text{rad}_{C_k(\Omega)}(\ell(w)) Z_k(\lambda)) \]
\[ = \tilde{Z}_k(\lambda)_{\geq \ell(w)}/\tilde{Z}_k(\lambda)_{> \ell(w)} \] by above.

Thus $j = -\ell(w)$. As $\ell(\tilde{Z}_k^w(\lambda(w))) = |R^+| + 1$, the assertion follows.
Untwisting $w_I$ of (4.6.3) reads

\[
Z_k^{w_I w_0}(\lambda \langle w_I w_0 \rangle)
\]

Thus one obtains a commutative diagram in $E_{\Omega, k} \text{modgr}_Y$.

Recall that $C_k(\Omega)(Z_k^{w_I w_0}(\lambda \langle w_I w_0 \rangle), Z_k^{w_0}(\lambda \langle w_0 \rangle))$ is 1-dimensional. On the other hand, each $Z_k^{w}(\lambda \langle w \rangle), w \in W$, admits a graded object $\tilde{Z}_k^{w}(\lambda) \in C_k(\Omega)$ such that $v\tilde{Z}_k^{w}(\lambda) \simeq Z_k^{w}(\lambda \langle w \rangle)$.

It follows that

\[
E_{\Omega, k} \text{modgr}_Y(H_{\Omega, k}Z_k^{w_I w_0}(\lambda \langle w_I w_0 \rangle), H_{\Omega, k}Z_k^{w_0}(\lambda \langle w_0 \rangle))
\]

Recall that $C_k(\Omega)(Z_k^{w_I w_0}(\lambda \langle w_I w_0 \rangle), Z_k^{w_0}(\lambda \langle w_0 \rangle))$ is a single $j \in \mathbb{Z}$ by dimension; in fact, $j = 0$ by [AJS 15.3.2]. Then, taking $\eta \in \tilde{C}_k(\Omega)(\tilde{Z}_k^{w_I w_0}(\lambda), \tilde{Z}_k^{w_0}(\lambda)) \setminus 0$, $\text{im}(\eta) \in \tilde{C}_k(\Omega)$ with $v(\text{im}(\eta)) = \nabla^i_P(\hat{L}_P(\lambda))$. This gives another proof that $\nabla^i_P(\hat{L}_P(\lambda))$ is $\mathbb{Z}$-graded, and hence is rigid.

**Corollary:** Assume the Lusztig conjecture. The $\mathbb{Z}$-gradation on $\text{im}(\eta)$ is such that for each $i \in \mathbb{N}$

\[
v((\text{im}(\eta))_{\geq -i}) = \text{rad}^{i-1}_{G_{i+1}}\nabla^i_P(\hat{L}_P(\lambda)) = \text{soc}^{i+1}_{G_{i+1}}\nabla^i_P(\hat{L}_P(\lambda)).
\]

**Proof:** As

\[
\text{soc}_{G_{i+1}}\nabla^i_P(\hat{L}_P(\lambda)) = \text{soc}_{G_{i+1}} Z_k^{w_0}(\lambda \langle w_0 \rangle) \quad \text{by (1)}
\]

\[
= \bar{v}(\tilde{Z}_k^{w_0}(\lambda \langle w_0 \rangle))_0 \quad \text{by (4.8)},
\]

\[
\bar{v}(\text{im}(\eta)_0) = \text{soc}_{G_{i+1}}\nabla^i_P(\hat{L}_P(\lambda)), \text{ and hence the assertion.}
\]

$^5$ The Loewy series
Let us write $\hat{\nabla}_P(\hat{L}^P(\lambda))$. We continue to assume the Lusztig conjecture.

(5.1) Let us first recall from [AK] or from [AJS, 18.19] a formula for the socle series of $\nabla(\lambda)$:

$$Q_{\mu, \lambda} = \sum_j q^{\frac{d(\mu, \lambda) - j}{2}} [\text{soc}_{j+1} \hat{\nabla}(\lambda) : \hat{L}(\mu)],$$

where $d(\mu, \lambda) = d(A, C)$ is the distance from alcove $A$ containing $\mu$ to alcove $C$ containing $\lambda$ and $Q_{\mu, \lambda} = Q_{A,C}$ is a periodic inverse Kazhdan-Lusztig polynomial defined in [L80].

On the other hand, the Lusztig conjecture for the $L_{I,T}$-modules asserts

$$\text{ch} \hat{L}^P(\lambda) = \sum_{\mu \in W_{I,T} \cdot \lambda} (-1)^{d_I(\mu, \lambda)} \hat{P}_{\mu, \lambda}(1) \text{ch} \hat{\nabla}^T(\mu),$$

where $d_I(\mu, \lambda)$ is the distance from alcove $A$ containing $\mu$ to alcove $C$ containing $\lambda$ with respect to $W_{I,T}$ and $\hat{P}_{\mu, \lambda} = \hat{P}_{A,C}$ is Kato’s periodic Kazhdan-Lusztig polynomial for $W_{I,T}$ [Kat]. We will prove a formula

$$\sum_j q^{\frac{d(\mu, \lambda) - j}{2}} [\text{soc}_{j+1} \hat{\nabla}_P(\hat{L}^P(\lambda)) : \hat{L}(\mu)] = \sum_{\nu \in W_{I,T} \cdot \lambda} Q_{\mu, \nu}(-1)^{d_I(\nu, \lambda)} \hat{P}_{\nu, \lambda}.$$

The formula reduces to (1) in case $I = \emptyset$, i.e., when $P$ is a Borel subgroup. It also holds for $P = G$ by the inversion formula $\sum_{\nu} Q_{\mu, \nu} (-1)^{d_I(\nu, \lambda)} \hat{P}_{\nu, \lambda} = \delta_{\mu, \lambda}$ [L80, 11.10]/[Kat, p. 129].

(5.2) Let us write $\hat{\nabla}_{k}(\lambda) = \hat{Z}_{k}^{\omega_{\mu}}(\lambda)$ for each $\lambda \in \Omega$. Then the formula (5.1.1) reads by (4.8) with $q^{\frac{t^2}{2}} = t$

$$Q_{\mu, \lambda}(t^2) = \sum_j t^{d(\mu, \lambda) - j} [\hat{\nabla}_{k}(\lambda) : \hat{L}_{k}(\mu)(-j)].$$

If we write $Q_{\lambda, \nu}(t) = \sum_j Q_{\lambda, \nu} t^j$ with $Q_{\lambda, \nu} \in \mathbb{Z}$, the formula (1) reads in the Grothendieck group of $E_{I,k}$

$$[\hat{\nabla}_{k}(\lambda)] = \sum_{j \in \mathbb{Z}} \sum_{\mu \in \Omega} Q_{d(\mu, \lambda) - j}^{\lambda} [\hat{L}_{k}(\mu)(-j)],$$

inverting which reads, if we write $\hat{P}_{\mu, \lambda}(t) = \sum_{j \in \mathbb{Z}} \hat{P}_{\mu, \lambda} t^j$

$$[\hat{L}_{k}(\lambda)] = \sum_{j \in \mathbb{Z}} \sum_{\mu \in \Omega} (-1)^{d(\mu, \lambda)} \hat{P}_{\mu, \lambda} (-1)^{d_I(\nu, \lambda)} \hat{P}_{\nu, \lambda} \forall \mu \in \Lambda.$$

**Theorem:** Assume the Lusztig conjecture. If $\lambda$ is a $p$-regular weight, the Loewy series of $\nabla_P(\hat{L}^P(\lambda))$ is given by

$$\sum_{j \in \mathbb{N}} q^{\frac{d(\mu, \lambda) - j}{2}} [\text{soc}_{j+1} \hat{\nabla}_P(\hat{L}^P(\lambda)) : \hat{L}(\mu)] = \sum_{\nu \in W_{I,T} \cdot \lambda} Q_{\mu, \nu}(-1)^{d_I(\nu, \lambda)} \hat{P}_{\nu, \lambda} \forall \mu \in \Lambda.$$
Proof: Put $\tilde{\nabla}_P = J_k \otimes \Omega_1$ from (3.8) for simplicity. In the Grothendieck group of $\tilde{\mathcal{C}}_k(\Omega_I)$ the formula (3) reads $[\tilde{L}_{I,k}(\lambda)] = \sum_{j \in \mathbb{Z}, \mu \in \Omega} (-1)^{d_I(\mu, \lambda)} \hat{P}_{\mu, \lambda, j + d_I(\mu, \lambda)} [\tilde{\nabla}_P(\mu)](j)$. Put $n_\lambda = \delta(\lambda) - \delta_I(\lambda)$, so $\tilde{\nabla}_P(\tilde{\nabla}_k(\lambda)) \simeq \tilde{\nabla}_k(\lambda)(n_\lambda)$ by (3.8). As $\tilde{\nabla}_P$ is exact, so is $\tilde{\nabla}_P$ by (3.8) also. Then

\[
[\tilde{\nabla}_P(\tilde{L}_{I,k}(\lambda))] = \sum_{\mu, j} (-1)^{d_I(\mu, \lambda)} \hat{P}_{\mu, \lambda, j + d_I(\mu, \lambda)} [\tilde{\nabla}_P(\tilde{L}_{I,k}(\mu))(j)]
\]

\[
= \sum_{\mu, j} (-1)^{d_I(\mu, \lambda)} \hat{P}_{\mu, \lambda, j + d_I(\mu, \lambda)} [\tilde{\nabla}_k(\mu)(n_\mu + j)]
\]

\[
= \sum_{\mu, j} (-1)^{d_I(\mu, \lambda)} \hat{P}_{\mu, \lambda, j + d_I(\mu, \lambda)} \sum_{k, \nu} Q_{d(\nu, \mu), k - \delta(\mu, \nu)}^\nu [\tilde{L}_k(\nu)(n_\mu + j - k)]
\]

\[
= \sum_{\mu, j, \nu, \nu} (-1)^{d_I(\mu, \lambda)} \hat{P}_{\mu, \lambda, j + d_I(\mu, \lambda)} Q_{d(\nu, \mu), k - \delta(\mu, \nu)}^\nu [\tilde{L}_k(\nu)(n_\mu + j - k)]
\]

Recall now $\text{im}(\eta)$ from (4.9). As $\tilde{L}_{I,k}(\lambda) \leq \tilde{\nabla}_k(\lambda)$, $\tilde{\nabla}_P(\tilde{L}_{I,k}(\lambda)) \leq \tilde{\nabla}_P(\tilde{\nabla}_k(\lambda)) \simeq \tilde{\nabla}_k(\lambda)(n_\lambda)$. As $\text{im}(\eta) \leq \tilde{\nabla}_k(\lambda)$, it follows that $\tilde{\nabla}_P(\tilde{L}_{I,k}(\lambda)) \simeq \text{im}(\eta)(n_\lambda)$. Thus

\[
[soc_{i+1} \nabla_P(\tilde{L}^P(\lambda)) : \tilde{L}(\nu)] = [\text{im}(\eta) : \tilde{L}_k(\nu)(-i)] \quad \text{by (4.9)}
\]

\[
= [\tilde{\nabla}_P(\tilde{L}_{I,k}(\lambda))(-n_\lambda) : \tilde{L}_k(\nu)(i)] = [\tilde{\nabla}_P(\tilde{L}_{I,k}(\lambda)) : \tilde{L}_k(\nu)(n_\lambda - i)]
\]

\[
= \sum_{\mu, j} (-1)^{d_I(\mu, \lambda)} \hat{P}_{\mu, \lambda, j + d_I(\mu, \lambda)} Q_{d(\nu, \mu), \delta(\mu) - \delta_I(\mu) - i + j}^\nu
\]

\[
= \sum_{\mu, j} (-1)^{d_I(\mu, \lambda)} \hat{P}_{\mu, \lambda, j + d_I(\mu, \lambda)} Q_{d(\nu, \mu), -i - j}^\nu
\]

\[
= \sum_{\mu, j} (-1)^{d_I(\mu, \lambda)} \hat{P}_{\mu, \lambda, j + d_I(\mu, \lambda)} Q_{d(\nu, \mu), -i - j}^\nu
\]

and hence

\[
\sum_i t^{d(\nu, \lambda) - i} [soc_{i+1} \nabla_P(\tilde{L}^P(\lambda)) : \tilde{L}(\nu)] = \sum_i t^{d(\nu, \lambda) - i} \sum_{\mu, j} (-1)^{d_I(\mu, \lambda)} \hat{P}_{\mu, \lambda, j + d_I(\mu, \lambda)} Q_{d(\nu, \mu), -i - j}^\nu
\]

\[
= \sum_i \sum_{\mu, j} (-1)^{d_I(\mu, \lambda)} \hat{P}_{\mu, \lambda, j + d_I(\mu, \lambda)} Q_{d(\nu, \mu), -i - j}^\nu
\]

\[
= \sum_j \sum_{\mu, j} (-1)^{d_I(\mu, \lambda)} \hat{P}_{\mu, \lambda, j + d_I(\mu, \lambda)} Q_{d(\nu, \mu), -i - j}^\nu
\]

\[
= \sum_{\mu} (-1)^{d_I(\mu, \lambda)} \hat{P}_{\mu, \lambda} (t^2) Q_{d(\nu, \mu)}^\nu (t^2)
\]

as desired.

(5.3) Given a simple $G_1 T$-module, the formula (5.1.3) is not necessarily accessible to locate a simple factor in the Loewy layers of $\nabla_P(\tilde{L}^P(\lambda))$. The following are particularly important factors in the study of the Frobenius direct image of the structure sheaf of $G/P$ [HKR], [KY]. Let $W^1 = \{ w \in W \mid \ell(ww') = \ell(w) + \ell(w') \ \forall w' \in W_I \}$, which forms a complete set of representatives of $W/W_I$. 

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**Proposition:** Assume the Lusztig conjecture. Let $\lambda \in \Lambda$ be p-regular. If $w \in W^I$, $L((w \cdot \lambda)^0) \otimes p(w^{-1} \cdot (w \cdot \lambda)^1)$ appears in the $((\ell(w) + 1))$st socle layer of $\nabla_P(\tilde{L}^P(\lambda))$.

**Proof:** In the commutative diagram (4.6.1) put $\phi = \phi_w$. Write $w^I = s_{i_1}s_{i_2} \ldots s_{i_m}$ in a reduced expression with $m = \ell(w^I)$, and put $y_r = s_{i_1}s_{i_2} \ldots s_{i_r}$ for $r \leq m$. Then $y_r^{-1}w^I = s_{i_{r+1}} \ldots s_{i_m} \in W^I$. Recall from [AK] that $\phi_w : \nabla_{w^I}((w^I \cdot \lambda)(w^I)) \to \nabla(w^I \cdot \lambda)$ is the composite

$$
\nabla_{w^I}((w^I \cdot \lambda)(w^I)) = \nabla_{s_{i_1} \ldots s_{i_m}}((w^I \cdot \lambda)(s_{i_1} \ldots s_{i_m})) \xrightarrow{\phi_m} \nabla_{s_{i_1} \ldots s_{i_{m-1}}}((w^I \cdot \lambda)(s_{i_1} \ldots s_{i_{m-1}})) \xrightarrow{\phi_{m-1}} \ldots \xrightarrow{\phi_2} \nabla_{s_{i_1}}((w^I \cdot \lambda)(s_{i_1})) \xrightarrow{\phi_1} \nabla(w^I \cdot \lambda).
$$

Put $L = \text{soc}(\nabla_{y_r}((w^I \cdot \lambda)(y_r)) \otimes -p(w^I \cdot 0))$ and $\phi_r = \{(\phi_{r+1} \circ \cdots \circ \phi_m) \otimes -p(w^I \cdot 0)\}_{w \in \nabla_P(\tilde{L}^P(\lambda))}$. As $\phi_m \neq 0$ and as each $\phi_i$ annihilates the socle of its domain, we must have $\ell(\text{soc}(\phi_r)) = \ell(\nabla_P(L^P(\lambda)) - (m - r) + 1$ by (4.6). Then $L = \text{soc}(\nabla_{y_r}) = \text{rad}_r(\text{soc}(\phi_r))$, which is a quotient of $\text{rad}_r(w^I \nabla_P(L^P(\lambda)))$. Thus $L$ lies in $\text{rad}_r(w^I \nabla_P(L^P(\lambda)))$. It follows from the rigidity of $\nabla_P(L^P(\lambda))$ that $(w^I)^{-1}L$ appears in its socle layer of level $\ell(\nabla_P(L^P(\lambda)) - r = \ell(w^I) + 1 - r = m + 1 - r = \ell(y_r^{-1}w^I) + 1$. Recall now from [AK] 1.2.4] that

$$
L = \tilde{L}((y_r^{-1} \cdot (w^I \cdot \lambda)^0) + p(y_r \cdot (y_r^{-1} \cdot (w^I \cdot \lambda)^1)) \otimes -p(w^I \cdot 0) = \tilde{L}((y_r^{-1}w^I \cdot \lambda)^0 + p(y_r \cdot (y_r^{-1}w^I \cdot \lambda)^1)) \otimes -p(w^I \cdot 0).
$$

Thus

$$(w^I)^{-1}L = L((y_r^{-1}w^I \cdot \lambda)^0) \otimes p((w^I)^{-1}y_r \cdot (y_r^{-1}w^I \cdot \lambda)^1) \otimes -p(w^I \cdot 0) = L((y_r^{-1}w^I \cdot \lambda)^0) \otimes p((w_r^{-1}w^I \cdot \lambda)^0) \otimes p((y_r^{-1}w^I)^{-1} \cdot (y_r^{-1}w^I \cdot \lambda)^1) = L((y_r^{-1}w^I \cdot \lambda)^0) \otimes p((y_r^{-1}w^I)^{-1} \cdot (y_r^{-1}w^I \cdot \lambda)^1).
$$

Finally, we check that any $w \in W^I$ may be realized as $y_r^{-1}w^I$ as above. Let $w \in W^I$. As $\ell(ww_I) = \ell(w) + \ell(w_I)$, one can write $w_0 = s_{j_1} \ldots s_{j_r}w_I$ with $r = \ell(w_0) - \ell(w) - \ell(w_I)$. Then $w^I = w_0w_I = s_{j_1} \ldots s_{j_r}w$ with $\ell(w^I) = r + \ell(w)$. Thus, putting $y_r = s_{j_1} \ldots s_{j_r}$, yields $w = y_r^{-1}w_I$, as desired.

**Remark:** This is a generalization of [KY1 1.5], [K09] and [K12 3.5]. In case $\lambda = 0$ we constructed for $G$ of rank at most 2 [KY] or in case $G = GL_{n+1}(k)$ and $P$ is maximal parabolic such that $G/P \simeq \mathbb{P}^n$ for any $n \in N$ [K09] a Karoubi complete strongly exceptional sequence $\{E_w \mid w \in W^I\}$ for the bounded derived category of coherent $\mathcal{O}_{G_C/P_C}$-modules out of $G_1\text{Mod}(L((w \cdot 0)^0), \text{soc}_{\ell(w)+1}(\nabla_P(0)))$, where $G_C$ and $P_C$ are the groups over the complex number field corresponding to $G$ and $P$, respectively. Our (5.3) assures at least that $G_1\text{Mod}(L((w \cdot 0)^0), \text{soc}_{\ell(w)+1}(\nabla_P(0)) \neq 0$ in general for large $p$.  

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