Scattering for 3D Cubic Focusing NLS on the Domain Outside a Convex Obstacle Revisited

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Abstract In this article, we consider the focusing cubic nonlinear Schrödinger equation (NLS) in the exterior domain outside of a convex obstacle in \(\mathbb{R}^3\) with Dirichlet boundary conditions. We revisit the scattering result below ground state in Killip–Visan–Zhang [The focusing cubic NLS on exterior domains in three dimensions. Appl. Math. Res. Express. AMRX, 1, 146–180 (2016)] by utilizing the method of Dodson and Murphy [A new proof of scattering below the ground state for the 3d radial focusing cubic NLS. Proc. Amer. Math. Soc., 145, 4859–4867 (2017)] and the dispersive estimate in Ivanovici and Lebeau [Dispersion for the wave and the Schrödinger equations outside strictly convex obstacles and counterexamples. Comp. Rend. Math., 355, 774–779 (2017)], which avoids using the concentration compactness. We conquer the difficulty of the boundary in the focusing case by establishing a local smoothing effect of the boundary. Based on this effect and the interaction Morawetz estimates, we prove that the solution decays at a large time interval, which meets the scattering criterion.

Keywords Schrödinger equation, exterior domain, global well-posedness, scattering criterion

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1 Introduction
Consider the Cauchy problem of the nonlinear Schrödinger equation with Dirichlet boundary condition

\[
\begin{align*}
    i\partial_t u + \Delta u &= -|u|^2 u =: F(u), \quad (t, x) \in \mathbb{R} \times \Omega, \\
    u(0, x) &= \phi(x), \\
    u(t, x) &= 0, \quad x \in \partial\Omega,
\end{align*}
\]

(1.1)

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where $\Omega$ is the exterior of a smooth, compact, strictly convex obstacle $\Omega^c \subset \mathbb{R}^3$ with smooth boundary $\partial \Omega$, and $\Delta$ is the Dirichlet Laplacian operator. It is easy to find that the solution $u$ to equation (1.1) with sufficient smooth conditions possesses the energy conservation

$$E_\Omega(u(t)) := \int_{\Omega} \left[ \frac{1}{2} |\nabla u(t, x)|^2 - \frac{1}{4} |u(t, x)|^4 \right] dx = E_\Omega(u_0)$$

and the mass conservation

$$M_\Omega(u(t)) := \int_{\Omega} |u(t, x)|^2 dx = M_\Omega(u_0).$$

When $\Omega = \mathbb{R}^3$, the Cauchy problem

$$\begin{cases}
    i\partial_t u + \Delta u + |u|^2 u = 0, \\
    u(0, x) = u_0(x),
\end{cases}$$

(1.4)

is scale invariant. More precisely, the class of solutions to (1.4) is left invariant by the scaling

$$u(t, x) \mapsto \lambda u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$ 

Moreover, one can also check that the only homogeneous $L^2_x$-based Sobolev space that is left invariant under (1.5) is $\dot{H}^s_x(\mathbb{R}^3)$. Hence, we say that the Cauchy problem (1.1) is $\dot{H}^s$-critical.

We will consider the well-posedness and long time behavior of the Cauchy problem (1.1) with initial data in the energy spaces. To do it, we first recall the classical Sobolev spaces on the domain $\Omega$.

**Definition 1.1** For integer $k \geq 1$ and $1 \leq p \leq \infty$, we denote $H^{k,p}_0(\Omega)$ as the closure of $C_c^\infty(\Omega)$ under the norm

$$\|u\|_{H^{k,p}_0(\Omega)} := \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p(\Omega)}.$$ 

If $p = 2$, we also write $H^k_0(\Omega) = H^{k,2}_0(\Omega)$ for simplicity.

In fact, $-\Delta$ is an unbounded and positive semi-definite symmetric operator on $C_c^\infty(\Omega)$. We define the corresponding quadratic form by

$$Q(u, v) = \int_{\Omega} \nabla u(x) \nabla \bar{v}(x) dx, \quad \text{for } u, v \in C_c^\infty(\Omega).$$

The extension of the form $Q$ is unique and well defined on $H^1_0(\Omega)$. Then the Friedrichs extension of $Q$ gives the Dirichlet Laplacian on $\Omega$, $-\Delta_\Omega$, which is a self-adjoint operator and with form domain $Q(-\Delta_\Omega) = D(\sqrt{-\Delta_\Omega})$. By the spectral theorem, we are able to denote the spectral measure $E(\lambda)$ and the operators by

$$\varphi(\sqrt{-\Delta_\Omega}) = \int_{[0, \infty)} \varphi(\lambda) dE(\lambda).$$

Thus, the linear operator $e^{it\Delta_\Omega}$ associated to the free Schrödinger equation on $\Omega$ is well defined and unitary on $L^2(\Omega)$. And we can define the Sobolev spaces based on the operator $\Delta_\Omega$.

**Definition 1.2** For $s \geq 0$ and $1 < p < \infty$, let $\dot{H}^{s,p}_D(\Omega)$ and $\dot{H}^{s,p}_D(\Omega)$ denote the completions of $C_c^\infty(\Omega)$ under the norms

$$\|f\|_{\dot{H}^{s,p}_D(\Omega)} := \|(\Delta_\Omega)^{\frac{s}{2}} f\|_{L^p(\Omega)} \quad \text{and} \quad \|f\|_{\dot{H}^{s,p}_D(\Omega)} := \|(1 - \Delta_\Omega)^{\frac{s}{2}} f\|_{L^p(\Omega)}.$$ 

When $p = 2$ we also write $\dot{H}^s_D(\Omega)$ and $\dot{H}^s_D(\Omega)$ for $\dot{H}^{s,2}_D(\Omega)$ and $\dot{H}^{s,2}_D(\Omega)$, respectively.
These two definitions are equivalent under certain conditions, see Proposition 2.1 below.

For the Euclidean space $\mathbb{R}^d$, the linear operator $e^{it\Delta}$ obeys the dispersive estimates and the Strichartz estimates. Owing to this, the local well-posedness theory of the solutions to equation (1.4) with the general power type nonlinearities $F(u) = |u|^{p-1}u$ is standard. For the defocusing energy subcritical $(F(u) = -|u|^{p-1}u, 1 + \frac{4}{d} < p < 1 + \frac{4}{d-2})$ cases, the solutions with initial data in $H^1(\mathbb{R}^d)$ are globally well-posed and scatter, see Cazenave [3] and Killip–Visan [16] and references therein.

Cauchy theory for the NLS equation in the exterior of the (non-trapping) obstacle, with initial data in $H^1_D$ was initiated in Burq–Gérard–Tzvetkov [2], see also Planchon–Vega [20]. We also refer to Ivanovici [9], Ivanovici–Planchon [11] and Landoulsi [19] for the local well-posedness at the critical regularity. In general domains, we do not have the dispersive estimate and the Strichartz estimates for $e^{it\Delta}$. For the case of exterior domain of a convex obstacle, Ivanovici [9] proved the Strichartz estimates except endpoint case by using the Melrose and Taylor parametrix and she also proved the scattering theory for the energy subcritical NLS on exterior domain of smooth convex obstacle in 3D. Ivanovici and Lebeau [10] proved the dispersive estimates holding only in the 3D case. For more scattering results of defocusing subcritical NLS in the general exterior domains, we refer to Planchon–Vega [20], Ivanovici–Planchon [11], and Blair–Smith–Sogge [1].

In this paper, we consider scattering theory of the solutions to focusing equation (1.1), which is mass supercritical and energy subcritical under the ground state. In fact, the nonlinear elliptic equation

$$-\Delta \varphi + \varphi = |\varphi|^2 \varphi, \quad (1.6)$$

has infinite number of solutions in $H^1(\mathbb{R}^3)$. Then for any solution $\varphi \in H^1(\mathbb{R}^3)$ to (1.6), $e^{it} \varphi$ is a global and non-scattering solution to the Cauchy problem (1.4). Furthermore, there exists a minimal mass solution and we often denote it as $Q$ and call it the ground state, which is positive, radial, exponentially decaying, see Cazenave [3] and Tao [23]. Holmer–Roudenko [8] proved the global well-posedness and scattering theory for radial solutions to equation (1.4) such the following conditions in $\mathbb{R}^3$:

$$(A) \quad E_{\mathbb{R}^3}(u_0) M_{\mathbb{R}^3}(u_0) < E_{\mathbb{R}^3}(Q) M_{\mathbb{R}^3}(Q),$$

$$(B) \quad \|\nabla u_0\|_{L^2(\mathbb{R}^3)} \|u_0\|_{L^2(\mathbb{R}^3)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)} \|Q\|_{L^2(\mathbb{R}^3)}.$$  

Duyckaerts–Holmer–Roudenko [7] removed the radial assumption. Killip–Visan–Zhang [17] proved the following results for exterior domains of convex obstacles in $\mathbb{R}^3$:

**Theorem 1.3** Let $\Omega$ be the exterior of a convex obstacle in $\mathbb{R}^3$. If the initial data $u_0 \in H^1_D(\Omega)$ satisfies

$$(1.7) \quad E_{\Omega}(u_0) M_{\Omega}(u_0) < E_{\mathbb{R}^3}(Q) M_{\mathbb{R}^3}(Q),$$

$$(1.8) \quad \|\nabla u_0\|_{L^2(\Omega)} \|u_0\|_{L^2(\Omega)} < \|\nabla Q\|_{L^2(\mathbb{R}^3)} \|Q\|_{L^2(\mathbb{R}^3)},$$

then, the corresponding solution to the Cauchy problem (1.1) with initial $u_0$ is globally well-posed and scatters.

The proofs of above theorem in [7, 17] utilized the concentration compactness arguments basing on the profile decomposition introduced by Kenig–Merle [13, 14], which have become
powerful and effective methods for many dispersive equations and many other equations.

In this article, we revisit Theorem 1.3, by employing an idea of Dodson–Murphy [5, 6], which provide new proofs in the Euclidean case avoiding uses of concentration compactness.

**Outline of Proof for Theorem 1.3** By the Strichartz estimates and the equivalence of various Sobolev norm definitions, we have the local well-posedness of (1.1) in \( H^1_D(\Omega) \). From the coercivity property (Lemma 2.10 below) under the ground state, we know the solution \( u \) is globally well-posed and of bounded \( H^1_D(\Omega) \) norm. Utilizing the dispersive estimates, we prove that the scattering criterion given by Dodson–Murphy [6] also holds in our case, that is: if for any large time window, there exists a large subinterval such that a space-time norm of of \( u \) is small in it, then \( u \) must scatter.

To end the proof, the main difficulties are how to overcome the effect from boundary \( \partial \Omega \) and the lack of the Galilean invariance. Combining with the concavity of \( \partial \Omega \) and the coercivity property, the Morawetz estimate yields a weaker local smoothing effect on the boundary. On the other hand, as in [7], for the Euclidean case, by the Galilean invariance, one can assume the critical solution \( u_c \) has zero conserved momentum, which yields the spatial translation parameter \( x(t) = o(t) \) (as \( t \to \infty \)). This fact is essential to the preclusion of the critical solution by making use of the Morawetz estimates centered at origin. For our case, the momentum is obviously bounded since \( u \in L^\infty_t H^1_x \). Based on this fact, one could just expect \( |x(t)| \lesssim |t| \). However, the interaction Morawetz identity is defined as an average of the Morawetz action that is centered at any point in \( \mathbb{R}^3 \). Fortunately, since \( u \in L^\infty_t H^1_x \), we are able to prove the smallness \( L^3_{t,x} \)-norm in a large subinterval of any large time interval without employing the Galilean transformation.

Finally, this and a standard continuity argument imply the solution that satisfies the conditions of the scattering criterion.

**Remark 1.4** Our proof is based on the the dispersive estimates of [10], which does not hold true in higher dimensions. Nevertheless, in these cases, it is hopeful that one may prove the corresponding results via establishing weaker dispersive estimates (see for example, the third author [25]).

**Remark 1.5** We remark that the interaction Morawetz estimates also reflect that the solution decays in big ball around any point. In fact, for any fixed \( R > 0 \), we have

\[
\lim_{t \to \infty} \inf_{x(t) \in \mathbb{R}^3} \sup_{x(t) \in B_R} \|u(t, \cdot)\|_{L^3_{x,t}(\Omega \cap B_R)} = 0,
\]

where \( B_R \) is the ball center at origin with radius \( R \). This suffices the scattering criterion for non-radial NLS (1.1) in Tao [21] when \( \Omega = \mathbb{R}^3 \).

**Remark 1.6** In fact, as in [6, 22], one can check that our proof would imply

\[
\|u\|_{L^4_{t,x}(\mathbb{R} \times \Omega)} \lesssim \exp\{ \exp\{ A(E(u_0), M(u_0)) \} \},
\]

where \( A \) is a rational polynomial of \( E(u_0), M(u_0) \) and \( E(Q), M(Q) \). The double-exponential growth is derived from the local smoothing effect of boundary and the interaction Morawetz estimates.

**Remark 1.7** Our arguments can be used to prove the similar results for general focusing energy subcritical cases \( (F(u) = -|u|^{p-1}u, \frac{7}{3} < p < 5) \), which has been considered in Yang [24].
This article is organized as follows: in Section 2, we recall some basic facts on the domain. Section 3 is devoted to proving the scattering under the assumption of smallness of $L^5_{t,x}$-norm of the solution. In Section 4, we verify the scattering criterion.

We conclude the introduction by giving some notations which will be used throughout this paper. We always use $X \lesssim Y$ to denote $X \leq CY$ for some constant $C > 0$. $X \sim Y$ stands for $X \lesssim Y$ and $Y \lesssim X$. Similarly, $X \lesssim_u Y$ indicates there exists a constant $C := C(u)$ depending on $u$ such that $X \leq C(u)Y$. The symbol $\nabla$ refers to the spatial derivation. For $M = \mathbb{R}^3$ or a domain in $\mathbb{R}^3$, we use $L^r(M)$ to denote the Banach space of functions $f : M \to \mathbb{C}$ whose norm

$$\|f\|_{L^r(M)} = \left(\int_M |f(x)|^r \, dx \right)^{\frac{1}{r}}$$

is finite, with the usual modifications when $r = \infty$. For a time slab $I$, we use $L^r_tL^s_x(I \times M)$ to denote the space-time norm

$$\|f\|_{L^r_tL^s_x(I \times M)} = \left(\int_I \|f(t,x)\|_{L^s_x(M)}^r \, dt \right)^{\frac{1}{r}}$$

with the usual modifications when $q$ or $r$ is infinite.

2 Basic Tools and the Local Theory

In this section we give some basic harmonic tools and the local well-posedness theory for the Cauchy problem (1.1). In this section, we assume that $\Omega$ is the complement of a compact convex body $\Omega^c \subset \mathbb{R}^3$ with smooth boundary.

First, we recall the following proposition.

Proposition 2.1 (Equivalence of the Sobolev norms [18]) Let $1 < p < \infty$. If $0 \leq s < \min\{1, \frac{3}{p}, \frac{3}{q}\}$, then

$$\|(-\Delta)^{\frac{s}{2}} f\|_{L^p(\mathbb{R}^3)} \lesssim \|(-\Delta)^{\frac{s}{2}} f\|_{L^p(\Omega)}$$

for all $f \in C_c^\infty(\Omega)$. (2.1)

Using this proposition, we have

Corollary 2.2 (Fractional product rule [18]) For all $f, g \in C_c^\infty(\Omega)$, we have

$$\|(-\Delta)^{\frac{s}{2}} (fg)\|_{L^p(\Omega)} \lesssim \|(-\Delta)^{\frac{s}{2}} f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)} + \|f\|_{L^p(\Omega)} \|(-\Delta)^{\frac{s}{2}} g\|_{L^q(\Omega)}$$

with the exponents satisfying $1 < p, p_1, q_2 \leq \infty$, $1 < p_2, q_1 \leq \infty$,

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$$

and $0 < s < \min\{1, \frac{1}{p_1} + \frac{1}{p_2}, \frac{3}{q_1} + \frac{3}{q_2}\}$.

Corollary 2.3 (Fractional chain rule [18]) Suppose $G \in C^1(\mathbb{C})$, $s \in (0, 1]$, and $1 < p, p_1, p_2 < \infty$ are such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $0 < s < \min\{1, \frac{1}{p_1} + \frac{1}{p_2}, \frac{3}{p_2} + \frac{3}{p_2}\}$. Then

$$\|(-\Delta)^{\frac{s}{2}} G(f)\|_{L^p(\Omega)} \lesssim \|G'(f)\|_{L^{p_1}(\Omega)} \|(-\Delta)^{\frac{s}{2}} f\|_{L^{p_2}}$$

We need the chain rule for fractional derivatives on $\mathbb{R}^d$, which will be useful for the local theory.

Proposition 2.4 (Chain rule for fractional derivatives [15]) If $F \in C^2$, with $F(0) = 0$, $F'(0) = 0$, and $|F''(a + b)| \leq C\{|F''(a)| + |F''(b)|\}$, and $|F'(a + b)| \leq C\{|F'(a)| + |F'(b)|\}$, we have, for
\[ 0 < \alpha < 1, \]
\[ \| \Lambda^\alpha F(u) \|_{L^p_t(L^q_x)} \leq C \| F'(u) \|_{L^{p_1}(L^{q_1})} \| \Lambda^\alpha u \|_{L^{p_2}(L^{q_2})}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \]
and
\[ \| \Lambda^\alpha [F(u) - F(v)] \|_{L^p_t(L^q_x)} \]
\[ \leq C \| [F'(u)w_{L^{p_1}(L^{q_1})} + [F'(v)w_{L^{p_1}(L^{q_1})}] \| \Lambda^\alpha (u - v) \|_{L^{p_2}(L^{q_2})} \]
\[ + C \| F'(u)w_{L^{p_1}(L^{q_1})} + [F'(v)w_{L^{p_1}(L^{q_1})}] \| \Lambda^\alpha u \|_{L^{p_2}(L^{q_2})} + \| \Lambda^\alpha v \|_{L^{p_2}(L^{q_2})} \| (u - v) \|_{L^{p_1}(L^{q_1})}, \]
where \( \Lambda = (-\Delta_{\mathbb{R}^3})^{\frac{1}{2}} \).

Next, we recall the dispersive estimates.

**Lemma 2.5** (Dispersive estimate [10])
\[ \| e^{it\Delta_\alpha} f \|_{L^p_t(L^q_x)} \lesssim t^{-\frac{\alpha}{2}} \| f \|_{L^1_t(L^q_x)}. \quad (2.2) \]

Combining this with the endpoint Strichartz estimate of Keel–Tao, we have the following Strichartz estimates:

**Proposition 2.6** (Strichartz estimates [9, 12]) Let \( q, \tilde{q} \geq 2, \) and \( 2 \leq r, \tilde{r} \leq \infty \) satisfying
\[ \frac{3}{2} - \frac{1}{2} \leq \frac{2}{q} + \frac{2}{\tilde{q}} = \frac{2}{r} + \frac{2}{\tilde{r}}. \]
Then, the solution \( u \) to \((\partial_t + \Delta)u = F\) on an interval \( I \ni 0 \) satisfies
\[ \| u \|_{L^q_tL^r_x(I \times \Omega)} \lesssim \| u_0 \|_{L^2(\Omega)} + \| F \|_{L^q'_tL^r'_x(I \times \Omega)}. \quad (2.3) \]

We define the \( S(I) \) and \( W(I) \) norm for an interval \( I \) by
\[ \| u \|_{S(I)} = \| u \|_{L^q_tL^r_x(I \times \Omega)} \quad \text{and} \quad \| u \|_{W(I)} = \| u \|_{L^q'_tL^{r'}_x(I \times \Omega)}. \quad (2.4) \]

Thus, by Strichartz, Corollary 2.3, Proposition 2.4, we have

**Theorem 2.7** (Local well-posedness [4, 13]) Assume that \( u_0 \in \dot{H}^1_x(\Omega), 0 \in I, \) and \( \| u_0 \|_{\dot{H}^1_x(\Omega)} \leq A. \) Then there exists \( \delta = \delta(A) \) such that if \( \| e^{it\Delta_\alpha} u_0 \|_{S(I)} \leq \delta, \) there exists a unique solution \( u \) to (1.1) in \( I \times \Omega, \) with \( u \in C(I; \dot{H}^1_x(\Omega)) \) such that
\[ \| (-\Delta_\alpha)^{\frac{1}{2}} u \|_{W(I)} + \sup_{t \in I} \| u(t) \|_{\dot{H}^1_x(\Omega)} \leq C A, \quad \| u \|_{S(I)} \leq 2 \delta. \quad (2.5) \]

Moreover, if \( u_{0,k} \to u_0 \) in \( \dot{H}^1_x(\Omega), \) we obtain the corresponding solutions \( u_k \to u \) in \( C(I; \dot{H}^1_x(\Omega)). \)

**Remark 2.8** From standard arguments, we have if \( u \) is a global solution and such that
\[ \| u \|_{S(\mathbb{R})} < \infty, \]
then \( u \) scatters in both directions.

We need the following refined Gagliardo–Nirenberg inequality, which follows from the sharp Gagliardo–Nirenberg inequality and the Pohozaev identities of the ground state.

**Lemma 2.9** (Refined Gagliardo–Nirenberg inequality, [6]) For \( f \in H^1(\mathbb{R}^3) \) and any \( \xi \in \mathbb{R}^3, \)
\[ \| f \|^4_{L^4(\mathbb{R}^3)} \leq \frac{4}{3} \left( \frac{\| f \|^2_{L^2(\mathbb{R}^3)} \| f \|^2_{\dot{H}^1(\mathbb{R}^3)}}{\| f \|^2_{L^2(\mathbb{R}^3)} \| f \|^2_{\dot{H}^1(\mathbb{R}^3)}} \right) \inf_{\xi \in \mathbb{R}^3} \| e^{i \xi \cdot \cdot} f \|^2_{\dot{H}^1(\mathbb{R}^3)}. \quad (2.6) \]

Before the end of this section, we recall the coercivity property for functions under the ground state \( Q \) (i.e., satisfying the conditions (A) and (B)). We denote \( M_{\mathbb{R}^3} \) and \( E_{\mathbb{R}^3} \) as the mass and energy on \( \mathbb{R}^3 \) respectively.
Lemma 2.10 (Coercivity) Let \( u_0 \in H^1_D(\Omega) \) satisfy \( E_\Omega(u_0)M_\Omega(u_0) < (1 - \delta)E_{\mathbb{R}^3}(Q)M_{\mathbb{R}^3}(Q) \).

If \( \|u_0\|_{L^2(\Omega)} \|u_0\|_{H^1(\Omega)} \leq \|Q\|_{L^2(\mathbb{R}^3)} \|Q\|_{H^1(\mathbb{R}^3)} \), then there exists \( \delta' = \delta'(\delta) > 0 \) so that

\[
\|u(t)\|_{L^2(\Omega)} \|u(t)\|_{H^1(\Omega)} \leq (1 - \delta') \|Q\|_{L^2(\mathbb{R}^3)} \|Q\|_{H^1(\mathbb{R}^3)}
\]

holds for all \( t \in I \), where \( u : I \times \Omega \to C \) is the maximal lifespan solution. In particular, \( I = R \) and \( u \) is uniformly bounded in \( H^1(\Omega) \).

Moreover, for any function \( f \in H^1_D(\Omega) \) such that (2.7), there exists \( \rho = \rho(\delta') > 0 \) such that

\[
\|f^2\|_{H^1(\Omega)} - \frac{3}{4}\|f^4\|_{L^1(\Omega)} \geq \rho(\|f^2\|_{H^1(\Omega)} + \|f^4\|_{L^1(\Omega)}).
\]

Proof. The proof follows from Proposition 2.1 above, Lemma 2.3 and Lemma 2.4 in [5]. \( \square \)

Remark 2.11 Suppose \( u_0 \in H^1_D(\Omega) \) satisfies (1.7) and (1.8). Then by the above lemma, the maximal-lifespan solution \( u \) to (1.1) with initial data \( u_0 \) obeys

\[
\|u(t)\|_{L^2(\Omega)} \|u(t)\|_{H^1(\Omega)} \leq (1 - \delta') \|Q\|_{L^2(\mathbb{R}^3)} \|Q\|_{H^1(\mathbb{R}^3)}
\]

for all \( t \) in the lifespan of \( u \). In particular, \( u \) remains bounded in \( H^1_D(\Omega) \) and hence is global.

3 Scattering Criterion

In this section, we prove a scattering criterion for solutions of the Cauchy problem (1.1).

Proposition 3.1 Suppose that \( u \) is a global solution to (1.1), satisfying

\[
\|u\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3 \times \Omega)} \leq E.
\]

There exist \( \epsilon = \epsilon(E, \Omega) > 0 \) and \( T_0 = T_0(\epsilon, E, \Omega) > 0 \) satisfying that if for any \( a \in \mathbb{R} \) there exists \( T \in \mathbb{R} \) such that \( [T - \epsilon^{-5}, T] \subset (a, a + T_0) \) and

\[
\|u\|_{L^5_{t,x}([T - \epsilon^{-5}, T] \times \Omega)} \leq \epsilon,
\]

then \( u \) scatters forward in time.

Proof. Using the fact

\[
u(t) = e^{i(t-T)\Delta} + i \int_T^t e^{i(t-s)\Delta} |u|^2 u(s) ds
\]

By the Strichartz estimates and continuity method, there exists \( \eta = \eta(E, \Omega) \) such that if for any \( T > 0 \),

\[
\|e^{i(t-T)\Delta} u(T)\|_{L^5_{t,x}((T, \infty) \times \Omega)} \leq \eta,
\]

then \( u \) scatters forward.

By the Duhamel formula, we have

\[
e^{i(t-T)\Delta} u(T) = e^{i\xi^\alpha u_0} + i \int_0^T e^{i(t-s)\Delta} (|u|^2 u)(s) ds.
\]

First, by the Strichartz estimates, there exists \( T_1 > 0 \) such that, if \( T > T_1 \),

\[
\|e^{i\xi^\alpha u_0}\|_{L^5_{t,x}((T, \infty) \times \Omega)} < \frac{1}{\xi}\eta.
\]

Take \( a = T_1, \epsilon = \eta^2, T \) as in the assumption (3.2) and make a decomposition

\[
[0, T] = [0, T - \epsilon^{-5}] \cup [T - \epsilon^{-5}, T] := I_1 \cup I_2.
\]
Then by the Strichartz estimates,
\[ \|u\|_{L_t^5 H^{\frac{30}{11}}_x((T-\epsilon^{-5}, T] \times \Omega)} \leq C\|u_0\|_{H^1} + C\|u\|_{L_t^\infty L_{x}^{5,1}(I_2 \times \Omega)}^2 \|u\|_{L_t^5 H^{\frac{30}{11}}_x((T-\epsilon^{-5}, T] \times \Omega)}. \]

Using (3.2) and the continuity method, we have
\[ \|u\|_{L_t^5 H^{\frac{30}{11}}_x((T-\epsilon^{-5}, T] \times \Omega)} \lesssim 1. \]

Thus, we have
\[ \left\| \int_{I_2} e^{i(t-s)\Delta_n} (|u|^2 u) (s) ds \right\|_{L_t^5 L_x^{\frac{30}{11}}(T, \infty) \times \Omega)} \lesssim \|u\|_{L_t^5 L_x^{5,1}(I_2 \times \Omega)}^2 \|u\|_{L_t^5 H^{\frac{30}{11}}_x((T-\epsilon^{-5}, T] \times \Omega)} \lesssim \epsilon^2. \quad (3.6) \]

Next, we consider the corresponding contribution of \( I_1 \). By the Duhamel formula and the Strichartz estimates, we have
\[ \left\| \int_{I_1} e^{i(t-s)\Delta_n} (|u|^2 u)(s) ds \right\|_{L_t^5 L_x^{\frac{30}{11}}(T, \infty) \times \Omega)} = \left\| e^{i(t-(T-T_{\epsilon^{-5}})\Delta_n} u(T-T_{\epsilon^{-5}}) + e^{i(t-T_{\epsilon^{-5}})\Delta_n} u_0 \right\|_{L_t^5 L_x^{\frac{30}{11}}(T, \infty) \times \Omega)} \lesssim 1. \]

On the other hand, employing the dispersive estimates and the Sobolev embedding, we have
\[ \left\| \int_{I_1} e^{i(t-s)\Delta_n} (|u|^2 u)(s) ds \right\|_{L_t^5 L_x^{\frac{30}{11}}(T, \infty) \times \Omega)} \lesssim \left\| \int_{I_1} \frac{1}{(t-s)^\frac{3}{2}} ds \right\|_{L_t^5(T, \infty)} \|u\|_{L_t^\infty H^{\frac{15}{2}}_x(\mathbb{R} \times \Omega)}^3 \lesssim \epsilon^\frac{3}{2}. \]

Thus, by interpolation, we have
\[ \left\| \int_{I_1} e^{i(t-s)\Delta_n} (|u|^2 u)(s) ds \right\|_{L_t^5 L_x^{\frac{30}{11}}(T, \infty) \times \Omega)} \lesssim \epsilon^\frac{15}{32}, \]
which together with (3.5) and (3.6) implies (3.3). Therefore, we complete the proof.

### 4 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. First we prove a local smoothing effect property on the Boundary \( \partial \Omega \) by utilizing a Morawetz-type estimate. Then we prove the interaction Morawetz estimates for the solution in the Theorem 1.3. Finally, we prove Theorem 1.3 by showing the solution satisfies the conditions of the scattering criterion in the previous section.

Let \( \chi_R(x) \) be a smooth function on \( \mathbb{R}^3 \) and such that \( \chi_R(x) = 1 \) when \( |x| \leq \frac{R}{2} \) and \( \chi_R(x) = 0 \) when \( |x| \geq \frac{3R}{2} \). We need the following coercivity property, which follows similar proof of Lemma 3.2 in [5].

**Lemma 4.1 (Coercivity on balls)** There exists \( R = R(\delta, M(u), Q) > 0 \) sufficiently large such that for any point \( z \in \mathbb{R}^3 \),
\[ \sup_{t \in \mathbb{R}} \| \chi_R(\cdot - z) u(t) \|_{L_t^2(\Omega)} \| \chi_R(\cdot - z) u(t) \|_{H^1_x(\Omega)} < (1 - \delta) \| Q \|_{L_t^2(\mathbb{R}^3)} \| Q \|_{H^1_x(\mathbb{R}^3)} \quad (4.1) \]
ununiformly in \( t \).

In particular, by Lemma 2.10, there exists \( \delta' = \delta'(\delta) > 0 \) so that
\[ \| \chi_R(\cdot - z) u(t) \|_{H^1_x(\Omega)}^2 - \frac{3}{4} \| \chi_R(\cdot - z) u(t) \|_{L_t^4(\Omega)}^4 \geq \delta' \| \chi_R(\cdot - z) u(t) \|_{H^1_x(\Omega)}^2 \quad (4.2) \]
uniformly for \( t \in \mathbb{R} \).
First note that

$$\|\chi_R(\cdot - z)u(t)\|_{L^2_x(\Omega)} \leq \|u(t)\|_{L^2_x(\Omega)}$$

uniformly for $t \in \mathbb{R}$. Thus, it suffices to show the $\dot{H}^1_x(\Omega)$ term. Using the integration by parts and Dirichlet boundary, we get

$$\int_\Omega |\nabla (\chi_R(\cdot - z)u)|^2 dx = \int_\Omega \chi_R(\cdot - z)u \Delta (\chi_R(\cdot - z)u) dx$$

$$= \int_\Omega \chi_R(\cdot - z)^2 |\nabla u|^2 - \chi_R(\cdot - z)\Delta (\chi_R(\cdot - z)) |u|^2 dx$$

Then the result follows.

Next, we make some preparation for the Morawetz estimates. Let $n(x)$ be the outer normal vector at $x \in \partial \Omega$ and define the outer derivative by

$$\partial_n f = \nabla f \cdot n.$$ 

Denote $dS$ to be the induced measure on $\partial \Omega$.

Let $\eta > 0$ small, $\chi(x) = 1$ for $|x| \leq 1 - \eta$ and $\chi = 0$ for $x \geq 1$. Let $R > 1$ large, and define

$$\phi(x) = \frac{1}{\omega_3 R^3} \int_{\mathbb{R}^3} \chi^2 \left( \frac{x - s}{R} \right) \chi^4 \left( \frac{s}{R} \right) ds,$$

and

$$\phi_1(x) = \frac{1}{\omega_3 R^3} \int_{\mathbb{R}^3} \chi^2 \left( \frac{x - s}{R} \right) \chi^4 \left( \frac{s}{R} \right) ds,$$

where $\omega_3$ is the volume of unit ball in $\mathbb{R}^3$. Then we have

$$|\phi - \phi_1| \lesssim \eta.$$

Let

$$\psi(x) = \frac{1}{|x|} \int_0^{|x|} \phi(r) dr,$$

which satisfies

$$|\psi(x)| \leq \min \left\{ 1, \frac{R}{|x|} \right\} \quad \text{and} \quad \partial_k \psi(x) = \frac{x_k}{|x|^2} [\psi(x) - \phi(x)].$$

One can also deduce that

$$\partial_k [\psi(x)x_k] = 3\phi(x) + 2(\psi - \phi)(x), \quad (4.3)$$

where the repeated indices are summed.

4.1 Local Smoothing Effect

We define the Morawetz action by

$$M(t) = 2 \text{Im} \int_\Omega \psi(x) x [\bar{u} \nabla u] dx. \quad (4.4)$$

Then, $|M(t)| \lesssim R$.

**Proposition 4.2** For large $T_0 > 1$ and any time interval $I = [a, a + T_0] \subset \mathbb{R}$, we have

$$\frac{1}{T_0} \int_I \int_{\partial \Omega} |\partial_n u|^2(t, x) dS(x) dt \lesssim \frac{1}{(\log T_0)^2}. \quad (4.5)$$
Proof From the identity

\[ 2\partial_t \text{Im}(\bar{u}u_k) = \partial_k |u|^4 + \partial_k \Delta |u|^2 - 4\partial_j \text{Re}(\bar{u}_j u_k), \]  

(4.6), and integration by parts, we have

\[
\partial_t M(t) = 4 \int_\Omega \left[ \phi(x)|\nabla u|^2(t, x) - \frac{3}{4} \phi_1(x)|u|^4(t, x) \right] dx
\]

(4.7), where \( \phi \). By the Coercivity property Lemma 4.1, there exists \( R > 0 \), such that the first term of (4.10) is nonnegative for \( R > R_1 \). And the nonnegativity for second term of (4.8) follows from the facts \( \phi - \phi_1 \) and

\[ |\psi(x)| + |\nabla \phi| + |\nabla \psi| \lesssim |\psi - \phi| \left( 1 + \frac{1}{|x|} \right) + |\nabla \phi| \leq \frac{1}{\eta R} + \min \left\{ \frac{|x|}{\eta R}, \frac{R}{|x|} \right\} + \min \left\{ \frac{1}{\eta R}, \frac{R}{|x|^2} \right\}, \]  

(4.11)

we have

\[ \frac{1}{J} \int_{R_0}^{e^\eta R_0} \left( |\phi - \phi_1| + |\psi - \phi| + |\nabla \phi| + |\nabla \psi| \right) \frac{dR}{R} \lesssim \eta + \frac{1}{J \eta} + \frac{1}{R_0 \eta J}. \]  

(4.12)

Thus, we can deduce that

\[-\frac{1}{T_0} \int_I \frac{1}{J} \int_{R_0}^{e^\eta R_0} \int_{\partial \Omega} \psi(x)x \cdot n(x)|\partial_n u|^2(t, x) dx dR dt \lesssim \eta + \frac{1}{J \eta} + \frac{1}{R_0 \eta J} + \frac{R_0 e^{\eta J}}{T_0 J} + \frac{1}{\eta^2 J R_0^2}. \]  

(4.13)

Since the boundary \( \partial \Omega \) of \( \Omega \) is concave and compact, we have \( -\psi(x)x \cdot n(x) = -x \cdot n(x) \gtrsim 1 \) for \( x \in \partial \Omega \), which yields

\[ \frac{1}{T_0} \int_I \int_{\Omega} |\partial_n u|^2(t, x) dx dt \lesssim \frac{1}{J \eta R_0} + \frac{1}{J \eta} + \eta + \frac{R_0 e^{\eta J}}{T_0 J} + \frac{1}{\eta^2 J R_0^2}. \]  

(4.14)

Then the conclusion follows by taking \( \eta = R_0^{-1} = J^{-\frac{1}{2}} = (\log T_0)^{-\frac{1}{2}} \).

4.2 Interaction Morawetz Estimates

We define the interaction Morawetz quantity

\[ M_R(t) = 2 \int_{\Omega \times \Omega} |u|^2(t, y)|\psi(x - y)(x - y)| \text{Im}[\bar{u}\nabla u](t, x) dx dy. \]  

(4.15)

One can easily find that for any \( R > 0 \) and \( t \in \mathbb{R} \),

\[ |M_R(t)| \lesssim \text{RE}_0^2. \]
Theorem 4.3 (Interaction Morawetz Estimates) For arbitrary small \( \varepsilon > 0 \), there exists \( T_0, R_0 > 0 \) large and \( \eta > 0 \) small enough satisfying that: for any interval \( I = [a, a + T_0] \), there exists \( \xi = \xi(s,t,R) \in \mathbb{R}^3 \) such that

\[
\frac{1}{T_0} \int_{R_0}^{T_0} \frac{1}{R^3} \int_{R^3} \int_{\Omega \times \Omega} \left| \chi \left( \frac{-s}{R} \right) u \right|^2 (t,y) \left| \nabla \left( \chi \left( \frac{-s}{R} \right) u \xi \right) \right|^2 (t,x) dxdydsdt \frac{dR}{R} \leq \varepsilon,
\]

(4.16)

where \( u^\xi(t,x) = e^{i\xi t} u(t,x) \).

Proof By the identities (4.6) and

\[
\partial_t |u|^2 = -2\partial_n \text{Im}(\bar{u}u_k),
\]

(4.17)

we have

\[
\partial_t M_R(t) = \int_{\Omega \times \Omega} |u|^2(t,y)\psi(x-y)(x-y)\nabla |u|^4(t,x) dxdy
\]

\[
+ \int_{\Omega \times \Omega} |u|^2(t,y)\psi(x-y)(x-y)\nabla \Delta |u|^2(t,x) dxdy
\]

\[
- 4 \int_{\Omega \times \Omega} |u|^2(t,y)\psi(x-y)(x_k-y_k) \text{Re} (\partial_j (\bar{u}u_k)(t,x)) dxdy
\]

\[
- 4 \int_{\Omega \times \Omega} \partial_j \text{Im}(\bar{u}u_j)(t,y)\psi(x-y)(x-y)_k \text{Im}(\bar{u}u_k)(t,x) dxdy.
\]

(4.18)

By integration by parts and the Dirichlet boundary condition of \( u \), we have

\[
(4.18) = - \int_{\Omega \times \Omega} |u|^2(t,y)[3\phi(x-y) + 2(\psi - \phi)(x-y)]|u|^4(t,x) dxdy
\]

\[
- 3 \int_{\Omega \times \Omega} |u|^2(t,y)\phi_1(x-y)|u|^4(t,x) dxdy
\]

\[
- 2 \int_{\Omega \times \Omega} |u|^2(t,y)(\psi - \phi)(x-y)|u|^4(t,x) dxdy
\]

\[
- 3 \int_{\Omega \times \Omega} |u|^2(t,y)(\psi - \phi_1)(x-y)|u|^4(t,x) dxdy.
\]

(4.22)

Here, we view (4.23) and (4.24) as error terms from the definitions the cutoff functions.

\[
(4.19) = \int_{\Omega \times \Omega} |u|^2(t,y)\nabla_x \left[ 3\phi(x-y) + 2(\psi - \phi)(x-y) \right] \nabla \left[ |u|^2(t,x) \right] dxdy
\]

\[
+ 2 \int_{\Omega} \int_{\partial \Omega} |u|^2(t,y)\psi(x-y)(x-y)\tilde{n}_x|\partial_n|u|^2(t,x) dS(x)dy.
\]

(4.25)

As above, we also regard (4.25) as an error term. We will apply the local smoothing effect to the estimation of (4.26),

\[
(4.20) = 4 \int_{\Omega \times \Omega} |u|^2(t,y)\phi(x-y)|\nabla u|^2(t,x) dxdy
\]

\[
+ 4 \int_{\Omega \times \Omega} |u|^2(t,y)P_{ij}(x-y)(\psi - \phi)(x-y) \text{Re} (\bar{u}u_k) dxdy
\]

\[
- 4 \int_{\Omega} \int_{\partial \Omega} |u|^2(t,y)\psi(x-y)(x-y)_k \text{Re} (\partial_n \bar{u}u_k)(t,x) dS(x)dy.
\]

(4.27)
\( (4.21) = -4 \int \int_{\Omega \times \Omega} \partial_{y_j} \text{Im}(\bar{u}u_j)(t,y)\psi(x-y)(x-y)\chi(t,x) dxdy \)

\[ = -4 \int \int_{\Omega \times \Omega} \phi(x-y) \text{Im}(\bar{u}\nabla u)(t,y) \text{Im}(\bar{u}\nabla u)(t,x) dxdy \\ -4 \int \int_{\Omega \times \Omega} \text{Im}(\bar{u}\nabla u_j)(t,y) P_{jk}(x-y) [\psi(x-y) - \phi(x-y)] \text{Im}(\bar{u}\nabla u_k)(t,x) dxdy, \]

where \( P_{ij}(x) = \delta_{ij} - \frac{x_i x_j}{|x|^2} \).

From the fact that \( \psi - \phi \geq 0 \) and Cauchy–Schwarz, we have

\[ (4.28) + (4.31) \]

\[ = 4 \int \int_{\Omega \times \Omega} |u|^2(t,y) |\nabla_y u(t,x)|^2 (|\psi - \phi|)(x-y) dxdy \]

\[ -4 \int \int_{\Omega \times \Omega} \text{Im}[\bar{u} \nabla_x u](t,y) \text{Im}[\bar{u} \nabla_y u](t,x) (|\psi - \phi|)(x-y) dxdy \geq 0, \]

where \( \nabla_z \) is the angular derivation centered at \( z \in \mathbb{R}^3 \). By the compactness and convexity of \( \partial\Omega \), we have

\[ |(4.26) + (4.29)| \]

\[ = \left| 2 \int_{\Omega} \int_{\partial \Omega} |u|^2(t,y) \psi(x-y)(x-y)n(x) |\partial_n u|^2(t,x) dxdy \right| \]

\[ \lesssim R \int_{\Omega} \int_{\partial \Omega} |u|^2(t,y) |\partial_n u|^2(t,x) dxdy. \]

By a direct computation, one has

\[ \frac{\omega_3 R^3}{4} [(4.27) + (4.30)] \]

\[ = \int_{\mathbb{R}^3} \int_{\Omega \times \Omega} \chi^2 \left( \frac{x-s}{R} \right) \chi^2 \left( \frac{y-s}{R} \right) \]

\[ \left[ |u|^2(t,y) |\nabla u|^2(t,x) - \text{Im}(\bar{u}\nabla u)(t,y) \text{Im}(\bar{u}\nabla u)(t,x) \right] dxdyds \]

\[ = \int_{\mathbb{R}^3} \int_{\Omega \times \Omega} \chi^2 \left( \frac{x-s}{R} \right) \chi^2 \left( \frac{y-s}{R} \right) |u|^2(t,y) |\nabla u|^2(t,x) dxdyds, \]

and

\[ \xi(t,s,R) = -\frac{\int_{\Omega} \chi^2 \left( \frac{x-s}{R} \right) \text{Im}(\bar{u}\nabla u)(t,x) dx}{\int_{\Omega} \chi^2 \left( \frac{x-s}{R} \right) |u|^2(t,x) dx} \]

or \( \xi = 0 \) if \( \int_{\Omega} \chi^2 \left( \frac{x-s}{R} \right) |u|^2(t,x) dx = 0 \).

Combining these estimates above, we have

\[ \frac{1}{R^3} \int_{\mathbb{R}^3} \int_{\Omega \times \Omega} \chi \left( \frac{x-s}{R} \right) u^2(t,y) \left[ \nabla \left( \frac{x-s}{R} \right) u^\xi(t,x) \right]^2 dxdyds \]

\[ \lesssim \frac{1}{\eta^2 R^3} + \partial_t M(t) + \int_{\Omega} \int_{\partial \Omega} |u|^2(t,y) |x \cdot n(x)| |\partial_n u|^2(t,x) dxdy \]

\[ + \int_{\Omega \times \Omega} |u|^2(t,y) u^4((\psi - \phi)(x-y) + (\phi - \phi_1)(x-y)) dxdy \]

\[ + \int_{\Omega \times \Omega} |u|^2(t,y) u^n u(t,x) |\nabla (\psi + \phi)(x-y)| dxdy. \]
By Lemma 4.1 and (4.11), for sufficiently large $R > 0$, we have
\[ \frac{1}{J T_0} \int_{R_0}^{R e^T} \int_{I} \frac{1}{R^3} \int_{\mathbb{R}^3} \int_{\Omega \times \Omega} \left| \chi \left( \frac{s - y}{R} \right) u^2 (t, y) \right|^{\frac{3}{2}} \left( \frac{\nabla \left( \frac{s - y}{R} \right)}{R} \right) \left( t, x \right) \, dx \, dy \, ds \, dt \frac{dR}{R} \]
\[ \lesssim \frac{1}{J T_0} \left( R_0 e^T + \frac{\log T_0}{2} \right) \frac{1}{R_0 J} \frac{1}{J^2} + \eta, \]
which implies the conclusion (4.16) by taking $\eta = J^{-\frac{1}{2}} = R_0^{-1} = \varepsilon$ and $\log T_0 = e^{\varepsilon^2}$.

4.3 Proof of Theorem 1.3

By the interaction Morawetz estimates and the Sobolev embedding, there exists $T_0 > 0$ and $R \in [R_0, e^T R_0]$ such that for any interval $I = [a, a + T_0]$
\[ \frac{1}{T_0} \int_{I} \frac{1}{R^3} \int_{\mathbb{R}^3} \left| \chi \left( \frac{s - y}{R} \right) u(t) \right|^{\frac{3}{2}} \left( \frac{\nabla \left( \frac{s - y}{R} \right)}{R} \right) \left( t, x \right) \, dx \, dy \, dt \lesssim \varepsilon. \]

Thus there exists $\theta \in [0, 1]^3$ such that
\[ \frac{1}{T_0} \int_{I} \sum_{z \in \mathbb{Z}^3} \left| \chi \left( \frac{s - R}{R} (z + \theta) \right) u(t) \right|^{\frac{3}{2}} \left( \frac{\nabla \left( \frac{s - R}{R} (z + \theta) \right)}{R} \right) \left( t, x \right) \, dx \, dy \, dt \lesssim \varepsilon. \]

Therefore, there exists a subinterval $I_0 = [b - \varepsilon^{-\frac{1}{4}}, b] \subset I$ such that
\[ \int_{I_0} \sum_{z \in \mathbb{Z}^3} \left| \chi \left( \frac{s - R}{R} (z + \theta) \right) u(t) \right|^{\frac{3}{2}} \left( \frac{\nabla \left( \frac{s - R}{R} (z + \theta) \right)}{R} \right) \left( t, x \right) \, dx \, dy \, dt \lesssim \varepsilon^\frac{1}{2}. \]

This together with the Gagliardo–Nirenberg inequality
\[ \|f\|_{L^3} \lesssim \|f\|_{L^2}^2 \|\nabla f\|_{L^2}^2 \]
implies that
\[ \int_{I_0} \sum_{z \in \mathbb{Z}^3} \left| \chi \left( \frac{s - R}{R} (z + \theta) \right) u(t) \right|^{\frac{3}{2}} \left( \frac{\nabla \left( \frac{s - R}{R} (z + \theta) \right)}{R} \right) \left( t, x \right) \, dx \, dy \, dt \lesssim \varepsilon^\frac{1}{2}. \] (4.34)

On the other hand, by Hölder’s inequality and Sobolev embedding, we have
\[ \sum_{z \in \mathbb{Z}^3} \left| \chi \left( \frac{s - R}{R} (z + \theta) \right) u(t) \right|_{L^2(\Omega)} \left| \chi \left( \frac{s - R}{R} (z + \theta) \right) u(t) \right|_{L^8(\Omega)} \lesssim 1, \]
which yields
\[ \sum_{z \in \mathbb{Z}^3} \left| \chi \left( \frac{s - R}{R} (z + \theta) \right) u(t) \right|_{L^2(\Omega)}^2 \lesssim 1. \] (4.35)

Now, we have, by Cauchy–Schwarz, (4.34), (4.35) and $|I_0| = \varepsilon^{-1/4}$,
\[ \|u\|^3_{L^3(I_0 \times \Omega)} \leq \int_{I_0} \sum_{z \in \mathbb{Z}^3} \left| \chi \left( \frac{s - R}{R} (z + \theta) \right) u(t) \right|^3_{L^3(\Omega)} \, dt \]
\[ \leq \int_{I_0} \left( \sum_{z \in \mathbb{Z}^3} \left| \chi \left( \frac{s - R}{R} (z + \theta) \right) u(t) \right|^4_{L^3(\Omega)} \right)^{\frac{1}{2}} \left( \sum_{z \in \mathbb{Z}^3} \left| \chi \left( \frac{s - R}{R} (z + \theta) \right) u(t) \right|^2_{L^3(\Omega)} \right)^{\frac{1}{2}} \, dt \]
\[
\left(\int_{I_0} \left| \sum_{z \in \mathbb{Z}^3} \left| \chi \left( \frac{\cdot - R(z + \theta)}{R} \right) u(t) \right|^4 dt \right|^{1/4} \right)^{1/2} \left(\int_{I_0} \sum_{z \in \mathbb{Z}^3} \left| \chi \left( \frac{\cdot - R(z + \theta)}{R} \right) u(t) \right|^2 dt \right)^{1/2} \leq \varepsilon^{1/4}.
\]

By interpolation, we have
\[
\|u\|_{L^6(\mathbb{R} \times \Omega)} \leq \|u\|_{L^6(\mathbb{R} \times \Omega)}^{1/4} \|u\|_{L^6(\mathbb{R} \times \Omega)}^{7/4} \leq \varepsilon^{1/24 + 7/48} \leq \varepsilon^{7/16},
\]
where we have used the fact that
\[
\|u\|_{L^6(\mathbb{R} \times \Omega)} \lesssim \langle |I| \rangle^{1/3},
\]
which is a direct consequence of the Strichartz estimates, the Sobolev inequality and a standard continuity argument. Then, by the scattering criterion in Proposition 3.1, the conclusion follows.

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