Quotient maps of 2,3-uniform tilings of the plane on the torus

Marbarisha M. Kharkongor¹, Debashis Bhowmik², Dipendu Maity¹

¹Department of Sciences and Mathematics, Indian Institute of Information Technology Guwahati, Bongora, Assam-781 015, India.

{marbarisha.kharkongor, dipendu}@iiitg.ac.in/{marbarisha.kharkongor, dipendumaity}@gmail.com.

²Department of Mathematics, Indian Institute of Technology Patna, Patna 801 106, India.
debashisiitg@gmail.com.

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Abstract

A 2-uniform tiling is an edge-to-edge tiling by regular polygons having 2 distinct transitivity classes of vertices. There are 20 distinct 2-uniform tilings (these are of 14 different types) on the plane, and since the plane is the universal cover of the torus, it is natural to explore maps on the torus that correspond to the 2-uniform tilings. In this article, we discuss that if a map is the quotient of a plane’s 2-uniform lattice then what would be the bounds of the number of vertex orbits.

A 3-uniform tiling is an edge-to-edge tiling by regular polygons having 3 distinct transitivity classes of vertices. There are 61 distinct 3-uniform tilings on the plane. In this article, we discuss that if a map is the quotient of a plane’s 3-uniform lattice then what would be the bounds of the number of vertex orbits.

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1 Introduction

A map is a connected 2-dimensional cell complex on a surface. Equivalently, it is a cellular embedding of a connected graph on a surface. For a map K, let V(K) be the vertex set of K and u ∈ V(K). The faces containing u form a cycle (called the face-cycle at u) C_u in the dual graph of K. That is, C_u is of the form (F_{1,1} · · · F_{1,n_1}) · · · (F_{k,1} · · · F_{k,n_k})- F_{1,1}, where F_{i,ℓ} is a p_i-gon for 1 ≤ ℓ ≤ n_i, 1 ≤ i ≤ k, p_r ≠ p_{r+1} for 1 ≤ r ≤ k − 1 and p_k ≠ p_1. In this case, the vertex u is said to be of type [p_1^{n_1}, . . . , p_k^{n_k}] (addition in the suffix is modulo k). A map K is said to be k-semiequivelar (or in short, semiequivelar) if V(K) = V_1 ⊔ V_2 ⊔ · · · ⊔ V_k such that the type of the vertices of V_i (1 ≤ i ≤ k) is same and the types of the vertices in V_i and V_j (i ̸= j, 1 ≤ i, j ≤ k) are different. In this article, we use semiequivelar to mean k-semiequivelar for some k. A semiequivelar map is said to be an equivelar map if it consists of single type of faces with all the vertices are of same type.

A 2-uniform tiling is a tiling of the plane having 2 distinct transitivity classes of vertices. A vertex-transitive map (or tiling) is a map (or tiling) on a closed surface (or on the plane) on which the automorphism group acts transitively on the set of vertices. A 2-uniform tiling or map will have vertices that we could label X, and others that we could label Y. Each X vertex can be mapped onto every other X vertex, but cannot be mapped to any Y vertex.
A semiregular tiling of $\mathbb{R}^2$ is also known as Archimedean, or homogeneous, or uniform tiling. In [6], Grünbaum and Shephard showed that there are exactly eleven types of Archimedean tilings on the plane. These types are $[3^6]$, $[3^4, 6^1]$, $[3^3, 4^2]$, $[3^2, 4^1, 3^1, 4^1]$, $[3^1, 6^1, 3^1, 6^1]$, $[3^1, 4^1, 6^1, 4^1]$, $[3^1, 12^2]$, $[4^1, 6^1, 12^1]$, $[4^1, 6^1, 12^1]$, $[4^1, 6^1, 12^1]$. Clearly, a semiregular tiling on $\mathbb{R}^2$ gives a semiequivelar map on $\mathbb{R}^2$. But, there are semiequivelar maps on the plane which are not (not isomorphic to) an Archimedean tiling. In fact, there exists $[p^q]$ equivelar maps on $\mathbb{R}^2$ whenever $1/p + 1/q < 1/2$ (see in [1], [5]). Thus, we have

**Proposition 1.1.** There are infinitely many types of equivelar maps on the plane $\mathbb{R}^2$.

We know that the plane is the universal cover of the torus. Since there are infinitely many equivelar maps on the plane, it is natural to ask what are the other types of semiequivelar maps that exist on the torus. Here we have the following result.

**Proposition 1.2.** [2, 3] Let $X$ be a semiequivelar map on a surface $M$. If $M$ is the torus then the type of $X$ is $[3^6]$, $[6^3]$, $[4^4]$, $[3^1, 6^1]$, $[3^3, 4^2]$, $[3^2, 4^1, 3^1, 4^1]$, $[3^1, 6^1, 3^1, 6^1]$, $[3^1, 4^1, 6^1, 4^1]$, $[3^1, 12^2]$, $[4^1, 8^2]$ or $[4^1, 6^1, 12^1]$.

We know that all the Archimedean tilings are vertex-transitive. But, it is not true on the torus. Here, we know the following.

**Proposition 1.3.** [2, 3] Let $X$ be an equivelar map on the torus. If the type of $X$ is $[3^6]$, $[4^4]$, $[6^3]$ or $[3^3, 4^2]$ then $X$ is vertex-transitive. If the type is $[3^2, 4^1, 3^1, 4^1]$, $[3^1, 6^1, 3^1, 6^1]$, $[3^1, 4^1, 6^1, 4^1]$, $[3^1, 12^2]$, $[4^1, 8^2]$, $[3^4, 6^1]$ or $[4^1, 6^1, 12^1]$, then there exists a semiequivelar toroidal map which is not vertex-transitive.

The 2-uniform tilings of the plane $\mathbb{R}^2$ are the generalization of vertex-transitive tilings on the plane. We know from [6, 7, 3] that there are twenty 2-uniform tiling of types

$[3^6; 3^3, 4^2], [3^6; 3^2, 4^1, 3^1, 4^1], [3^4, 6^1; 3^2, 6^2], [3^3, 4^2; 3^1, 6^1, 4^1], [3^3, 4^2; 3^1, 3^1, 4^1],$

$[3^6; 3^2, 4^1, 12^1], [3^1, 4^1, 6^1, 4^1, 12^1], [3^2, 4^1, 3^1, 4^1, 12^1], [3^2, 6^1, 3^1, 6^1],$

$[3^1, 3^1, 6^1, 12^2], [3^1, 4^2, 6^1, 3^1, 4^1, 6^1, 4^1], [3^1, 4^2, 6^1, 3^1, 6^1, 12^2], [3^1, 4^2, 6^1, 3^1, 4^1, 6^1, 4^1], [3^1, 4^2, 6^1, 3^1, 6^1, 12^2], [3^1, 4^2, 6^1, 3^1, 4^1, 6^1, 4^1], [3^1, 4^2, 6^1, 3^1, 6^1, 12^2], [3^1, 4^2, 6^1, 3^1, 4^1, 6^1, 4^1], [3^1, 4^2, 6^1, 3^1, 6^1, 12^2], [3^1, 4^2, 6^1, 3^1, 4^1, 6^1, 4^1], [3^1, 4^2, 6^1, 3^1, 6^1, 12^2], [3^1, 4^2, 6^1, 3^1, 4^1, 6^1, 4^1], [3^1, 4^2, 6^1, 3^1, 6^1, 12^2], [3^1, 4^2, 6^1, 3^1, 4^1, 6^1, 4^1], [3^1, 4^2, 6^1, 3^1, 6^1, 12^2], [3^1, 4^2, 6^1, 3^1, 4^1, 6^1, 4^1], [3^1, 4^2, 6^1, 3^1, 6^1, 12^2]$. on the plane (see in Section 3). We know that the plane is the universal cover of the torus. We also know that the maps of the above fourteen types also exist on the torus. Thus, it is natural to ask that if $X$ is a map on the torus and $\eta: Y \to X$ is a covering map where $Y$ is an 2-uniform tiling on the plane, then what would be the number of orbits of vertices of $X$. Here, we know the following.

**Proposition 1.4.** [5] Let $Z$ be a semiequivelar map on the torus and $\eta: K_5 \to Z$ be a covering map ($K_5$ given in Section 3). Let the vertices of $Z$ form $m$ Aut($Z$)-orbits. Then, $m \leq 4$.

In this article, we prove the following.

**Theorem 1.5.** Let $X$ be a semiequivelar map on the torus and $\eta: Y \to X$ be a covering map. Let the vertices of $X$ form $m$ Aut($X$)-orbits. Let $K_i$ for $1 \leq i \leq 20$ (in Section 3) denote the 2-uniform tilings on the plane.

1. If $Y = K_1, K_9, K_{11}, K_{14}$, or $K_{17}$ then $m \leq 6$.
2. If $Y = K_2, K_7, K_{10}$, or $K_{16}$ then $m \leq 4$.
3. If $Y = K_3, K_4, K_8, K_{12}, K_{13}$, or $K_{15}$ then $m = 2$.
4. If $Y = K_{18}$ or $K_{19}$ then $m \leq 3$.
5. If $Y = K_6$ then $m \leq 7$.
6. If $Y = K_{20}$ then $m \leq 9$. 


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A 3-uniform tiling is an edge-to-edge tiling by regular polygons having 3 distinct transitivity classes of vertices. A vertex-transitive map is a map on a closed surface on which the automorphism group acts transitively on the set of vertices. A 3-uniform tiling or map will have vertices that we could label $X_1, X_2, X_3$. Each $X_i$ vertex can be mapped onto every other $X_j$ vertex, but cannot be mapped to any $X_j$ for $j \neq i$ vertices.

The $k$-uniform tilings of the plane $\mathbb{R}^2$ are the generalization of vertex-transitive tilings on the plane. We know that the plane is the universal cover of the torus.

If $k = 3$, we know from [6, 7, 4] that there are 61 3-uniform tiling on the plane. In this article, we prove the following. If $X$ is a map on the torus and $\eta: Y \rightarrow X$ is a covering map where $Y$ is an 3-uniform tiling on the plane, then what would be the number of orbits of vertices of $X$. Precisely, we have have the following result.

**Theorem 1.6.** Let $X$ be a semiequivelar map on the torus and $\eta: Y \rightarrow X$ be a covering map. Let the vertices of $X$ form $m \ \text{Aut}(X)$-orbits. Let $K_i$ for $1 \leq i \leq 61$ (in Section 4) denote the 3-uniform tilings on the plane. If $X = K_1, K_2, \ldots, \text{or}, K_{61}$ then $m \leq 15$.

## 2 Proof of Theorems 1.5

Let $K_i$ (given in Section 3) be of type $A$, where

$$A \in \{[3^6; 3^4, 6^1], [3^6; 3^3, 4^2], [3^6, 3^2, 4^1, 4^1], [3^6, 3^2, 4^1], [3^6, 3^2, 4^1, 12^1], [3^6, 3^2, 6^2],$$

$$[3^4, 12^1], [3^5, 6^1], [3^4, 6^1], [3^5, 4^2, 3^1], [3^5, 4^2], [3^4, 12^1, 3^1], [3^4, 6^1],$$

$$[3^4, 12^1, 6^1], [3^4, 12^1, 4^1], [3^5, 4^2, 3^1, 4^1], [3^5, 6^1, 3^1, 4^1], [3^5, 6^1, 4^1, 12^1],$$

$$[3^5, 6^1, 4^1, 12^1, 3^1, 12^2], [3^5, 6^1, 4^1, 12^1, 4^1], [3^5, 6^1, 4^1, 12^1, 6^1], [3^5, 6^1, 4^1, 12^1, 4^1, 6^1, 12^2]\}.$$  

Grünebaum and G. C. Shephard [9, 7] and Krötenheerdt [4] have discussed the existence and uniqueness of the 2-uniform tilings $K_i, i = 1, 2, \ldots, 20$ of the plane. Thus, we have the following.

**Proposition 2.1.** The 2-uniform maps $K_i \ (1 \leq i \leq 20)$ are unique up to isomorphism.

**Remark 2.2.** Let $S = \{0, 1, 2, \ldots, m - 1\}$. Let $S + n := \{0 + n, 1 + n, 2 + n, \ldots, m + n - 1\}$. Define $S_m = \{S + nk \ : \ k \in \mathbb{Z} \land 2 \mid (k - 1)\}$. Let $K_m$ be a map on the plane defined as follows. Let the vertex set of $K_m$ be $\{(i, j) \in \mathbb{Z} \times \mathbb{Z} \}$ on $\mathbb{R}^2$. The points $(i, j)$ and $(i_1, j_1)$ are connected by an edge if $(i, j) \sim (i_1, j_1).$ Thus, the edges of $K_m$ :

1. $(i, j) \sim (i_1, j_1)$ if $|i - i_1| = 1$ and $j = j_1$ or $i = i_1$ and $|j - j_1| = 1$,
2. $(i, j) \sim (i + 1, j + 1), (t, j) \sim (t + 1, j - 1)$ if $i \in S_m, j$ is even and $t \in \mathbb{Z} \setminus S_m$.

Thus, we get an object $K_m$ for each $m \geq 2$. Clearly, $K_x \not\cong K_y$ for $x \neq y$ since the number of continuous quadrilaterals are different. Therefore, there are infinitely many 2-semiequivelar tilings of the plane.

**Proof of Theorem 1.5.** Let $X_1$ be a semiequivelar map on the torus that is the quotient of the plane’s 2-uniform lattice $K_1$. Let the vertices of $X_1$ form $m_1 \ \text{Aut}(X_1)$-orbits. Let $K_1$ be as in Section 3. Let $V_1 = V(K_1)$ be the vertex set of $K_1$. Let $H_1$ be the group of all the translations of $K_1$. So, $H_1 \leq \text{Aut}(K_1)$.

Since $X_1$ is a semiequivelar map on the torus that is the quotient of the plane’s 2-uniform lattice $K_1$ (as by Proposition 2.1, $K_1$ is unique), so, we can assume, there is a polyhedral covering map $\eta_1 : K_1 \rightarrow X_1$ where $X_1 = K_1 / \Gamma_1$ for some fixed element (vertex, edge or face)
free subgroup $\Gamma_1 \leq \text{Aut}(K_1)$. Hence $\Gamma_1$ consists of translations and glide reflections. Since $X_1 = K_1/\Gamma_1$ is orientable, $\Gamma_1$ does not contain any glide reflection. Thus $\Gamma_1 \leq H_1$.

We take the middle point of the line segment joining vertices $a_0$ and $a_3$ as the origin $(0,0)$ of $K_1$. Let $\alpha_1 := a_9 - a_0$, $\beta_1 := a_{14} - a_0$ and $\gamma_1 := a_{20} - a_0$ in $K_1$. Then

$$H_1 := \langle x \mapsto x + \alpha_1, x \mapsto x + \beta_1, x \mapsto x + \gamma_1 \rangle.$$ 

Under the action of $H_1$, vertices of $K_1$ form twelve orbits. Consider the subgroup $G_1$ of $\text{Aut}(K_1)$ generated by $H_1$ and the map (the half rotation) $x \mapsto -x$. So,

$$G_1 = \{ \alpha : x \mapsto \varepsilon x + ma_1 + nb_1 + r\gamma_1 : \varepsilon = \pm 1, m,n,r \in \mathbb{Z} \} \cong H_1 \rtimes \mathbb{Z}_2.$$ 

Clearly, under the action of $G_1$, vertices of $K_1$ form six orbits. The orbits are

$$O_1 := \langle a_0 \rangle, O_2 := \langle a_1 \rangle, O_3 := \langle a_2 \rangle, O_4 := \langle b_{18} \rangle, O_5 := \langle b_{20} \rangle, O_6 := \langle b_{28} \rangle.$$ 

**Claim 1.** If $K \leq H_1$ then $K \leq G_1$. The alphabet $K$ has already been used in the statement of [3].

Let $g \in G_1$ and $k \in K$. Then $g(x) = \varepsilon x + ma + nb + rc$ and $k(x) = x + pa + qb + \ell c$ for some $m,n,r,p,q,\ell \in \mathbb{Z}$ and $\varepsilon \in \{1,-1\}$. Therefore,

$$\begin{align*}
(g \circ k \circ g^{-1})(x) &= (g \circ k)(\varepsilon(x - ma - nb - rc)) \\
&= g(\varepsilon(x - ma - nb - rc)) + pa + qb + \ell c \\
&= x - ma - nb - rc + \varepsilon(pa + qb + \ell c) + ma + nb + rc \\
&= x + \varepsilon(pa + qb + \ell c) \\
&= k^\varepsilon(x).
\end{align*}$$ 

Thus, $g \circ k \circ g^{-1} = k^\varepsilon \in K$. This completes the claim.

By Claim 1, $\Gamma_1$ is a normal subgroup of $G_1$. Therefore, $G_1/\Gamma_1$ acts on $X_1 = K_1/\Gamma_1$. Since $O_1 := \langle a_0 \rangle, O_2 := \langle a_1 \rangle, O_3 := \langle a_2 \rangle, O_4 := \langle b_{18} \rangle, O_5 := \langle b_{20} \rangle, O_6 := \langle b_{28} \rangle$ are the $G_1$-orbits, it follows that $\eta_1(O_j)$ for $j = 1, 2, \ldots, 6$ are the $(G_1/\Gamma_1)$-orbits. Since the vertex set of $K_1$ is $\sqcup_{j=1}^6 \eta_j(O_j)$ and $G_1/\Gamma_1 \leq \text{Aut}(X_1)$, it follows that the number of $\text{Aut}(X_1)$-orbits of vertices is $\leq 6$.

Let $X_9 = K_9/\Gamma_9$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_9 \leq \text{Aut}(K_9)$. Let the vertices of $X_9$ form $m_9 \text{ Aut}(X_9)$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_5$ as the origin $(0,0)$ of $K_9$ (see Section [3]). Let $\alpha_9 := a_1 - a_0$, $\beta_9 := a_2 - a_0$ and $\gamma_9 := a_3 - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_9 := \langle x \mapsto x + \alpha_9, x \mapsto x + \beta_9, x \mapsto x + \gamma_9 \rangle$ and

$$G_9 = \{ \alpha : x \mapsto \varepsilon x + m_9\alpha_9 + n_9\beta_9 + r_9\gamma_9 : \varepsilon = \pm 1, m_9, n_9, r_9 \in \mathbb{Z} \} \cong H_9 \rtimes \mathbb{Z}_2.$$ 

By the same arguments as above and in Claim 1, $\Gamma_9 \leq G_9$ and the number of $G_9/\Gamma_9$-orbits of vertices of $X_9$ is six. Therefore, $G_9/\Gamma_9$ acts on $X_9 = K_9/\Gamma_9$. Since $O_1 := \langle a_0 \rangle, O_2 := \langle b_0 \rangle, O_3 := \langle b_{18} \rangle, O_4 := \langle b_{33} \rangle, O_5 := \langle a_8 \rangle, O_6 := \langle b_{34} \rangle$ are the $G_9$-orbits, it follows that $O_1 := \langle a_0 \rangle, O_2 := \langle b_0 \rangle, O_3 := \langle b_{18} \rangle, O_4 := \langle b_{33} \rangle, O_5 := \langle a_8 \rangle, O_6 := \langle b_{34} \rangle$ are the $(G_9/\Gamma_9)$-orbits. Since the vertex set of $X_9$ is $\sqcup_{j=1}^6 \eta_9(O_j)$ and $G_9/\Gamma_9 \leq \text{Aut}(X_9)$, it follows that the number of $\text{Aut}(X_9)$-orbits of vertices is $\leq 6$. 


Let $X_{11} = K_{11}/\Gamma_{11}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{11} \leq \text{Aut}(K_{11})$. Let the vertices of $X_{11}$ form $m_{11}\text{ Aut}(X_{11})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_3$ as the origin $(0,0)$ of $K_{11}$. Let $\alpha_{11} := a_{15} - a_0$, $\beta_{11} := a_0 - a_0$ and $\gamma_{11} := a_{37} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{11} := \langle x \mapsto x + \alpha_{11}, x \mapsto x + \beta_{11}, x \mapsto x + \gamma_{11} \rangle$

$$G_{11} = \{ \alpha : x \mapsto \varepsilon x + m_{11}\alpha_{11} + n_{11}\beta_{11} + r_{11}\gamma_{11} : \varepsilon = \pm 1, m_{11}, n_{11}, r_{11} \in \mathbb{Z} \} \cong H_{11} \times \mathbb{Z}_2.$$  

By the same arguments as above and in Claim 1, $\Gamma_{11} \leq G_{11}$ and the number of $G_{11}/\Gamma_{11}$-orbits of vertices of $X_{11}$ is six. Therefore, $G_{11}/\Gamma_{11}$ acts on $X_{11} = K_{11}/\Gamma_{11}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle b_0 \rangle$, $O_5 = \langle b_2 \rangle$, $O_6 = \langle b_{46} \rangle$ are the $G_{11}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle b_0 \rangle$, $O_5 = \langle b_2 \rangle$, $O_6 = \langle b_{46} \rangle$ are the $(G_{11}/\Gamma_{11})$-orbits. Since the vertex set of $X_{11}$ is $\sqcup_{j=1}^{6}\eta_{11}(O_j)$ and $G_{11}/\Gamma_{11} \leq \text{Aut}(X_{11})$, it follows that the number of $\text{Aut}(X_{11})$-orbits of vertices is $\leq 6$.

Let $X_{14} = K_{14}/\Gamma_{14}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{14} \leq \text{Aut}(K_{14})$. Let the vertices of $X_{14}$ form $m_{14}\text{ Aut}(X_{14})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_3$ as the origin $(0,0)$ of $K_{14}$. Let $\alpha_{14} := a_9 - a_3$, $\beta_{14} := a_{30} - a_3$ and $\gamma_{14} := a_{34} - a_3 \in \mathbb{R}^2$. Similarly as above, define $H_{14} := \langle x \mapsto x + \alpha_{14}, x \mapsto x + \beta_{14}, x \mapsto x + \gamma_{14} \rangle$

$$G_{14} = \{ \alpha : x \mapsto \varepsilon x + m_{14}\alpha_{14} + n_{14}\beta_{14} + r_{14}\gamma_{14} : \varepsilon = \pm 1, m_{14}, n_{14}, r_{14} \in \mathbb{Z} \} \cong H_{14} \times \mathbb{Z}_2.$$  

By the same arguments as above and in Claim 1, $\Gamma_{14} \leq G_{14}$ and the number of $G_{14}/\Gamma_{14}$-orbits of vertices of $X_{14}$ is six. Therefore, $G_{14}/\Gamma_{14}$ acts on $X_{14} = K_{14}/\Gamma_{14}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle b_0 \rangle$, $O_5 = \langle b_2 \rangle$, $O_6 = \langle b_{4} \rangle$ are the $G_{14}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle b_0 \rangle$, $O_5 = \langle b_2 \rangle$, $O_6 = \langle b_{4} \rangle$ are the $(G_{14}/\Gamma_{14})$-orbits. Since the vertex set of $X_{14}$ is $\sqcup_{j=1}^{6}\eta_{14}(O_j)$ and $G_{14}/\Gamma_{14} \leq \text{Aut}(X_{14})$, it follows that the number of $\text{Aut}(X_{14})$-orbits of vertices is $\leq 6$.

Let $X_{17} = K_{17}/\Gamma_{17}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{17} \leq \text{Aut}(K_{17})$. Let the vertices of $X_{17}$ form $m_{17}\text{ Aut}(X_{17})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_3$ as the origin $(0,0)$ of $K_{17}$. Let $\alpha_{17} := a_{20} - a_4$, $\beta_{17} := a_{7} - a_4$ and $\gamma_{17} := a_{16} - a_4 \in \mathbb{R}^2$. Similarly as above, define $H_{17} := \langle x \mapsto x + \alpha_{17}, x \mapsto x + \beta_{17}, x \mapsto x + \gamma_{17} \rangle$

$$G_{17} = \{ \alpha : x \mapsto \varepsilon x + m_{17}\alpha_{17} + n_{17}\beta_{17} + r_{17}\gamma_{17} : \varepsilon = \pm 1, m_{17}, n_{17}, r_{17} \in \mathbb{Z} \} \cong H_{17} \times \mathbb{Z}_2.$$  

By the same arguments as above and in Claim 1, $\Gamma_{17} \leq G_{17}$ and the number of $G_{17}/\Gamma_{17}$-orbits of vertices of $X_{17}$ is nine. Therefore, $G_{17}/\Gamma_{17}$ acts on $X_{17} = K_{17}/\Gamma_{17}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle b_0 \rangle$, $O_5 = \langle b_1 \rangle$, $O_6 = \langle b_{2} \rangle$ are the $G_{17}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle b_0 \rangle$, $O_5 = \langle b_1 \rangle$, $O_6 = \langle b_{2} \rangle$ are the $(G_{17}/\Gamma_{17})$-orbits. Since the vertex set of $X_{17}$ is $\sqcup_{j=1}^{6}\eta_{17}(O_j)$ and $G_{17}/\Gamma_{17} \leq \text{Aut}(X_{17})$, it follows that the number of $\text{Aut}(X_{17})$-orbits of vertices is $\leq 6$. Thus, it completes the part (1).

Let $X_2 = K_2/\Gamma_2$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_2 \leq \text{Aut}(K_2)$. Let the vertices of $X_2$ form $m_2\text{ Aut}(X_2)$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_3$ as the origin $(0,0)$ of $K_2$. Let $\alpha_2 := a_6 - a_0$, $\beta_2 := a_{18} - a_0$ and $\gamma_2 := a_{24} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_2 := \langle x \mapsto x + \alpha_2, x \mapsto x + \beta_2, x \mapsto x + \gamma_2 \rangle$

$$G_2 = \{ \alpha : x \mapsto \varepsilon x + m_2\alpha_2 + n_2\beta_2 + r_2\gamma_2 : \varepsilon = \pm 1, m_2, n_2, r_2 \in \mathbb{Z} \} \cong H_2 \times \mathbb{Z}_2.$$
By the same arguments as above and in Claim 1, \( \Gamma_2 \leq G_2 \) and the number of \( G_2/\Gamma_2 \)-orbits of vertices of \( X_2 \) is four. Therefore, \( G_2/\Gamma_2 \) acts on \( X_2 = K_2/\Gamma_2 \). Since

\[
O_1 = \langle a_0 \rangle, \ O_2 = \langle a_1 \rangle, \ O_3 = \langle a_2 \rangle, \ O_4 = \langle b_0 \rangle
\]

are the \( G_2 \)-orbits, it follows that \( \eta_2(O_j) \) for \( j = 1, 2, 3, 4 \) are the \( (G_2/\Gamma_2) \)-orbits. Since the vertex set of \( Y \) is \( \bigcup_{j=1}^{4} \eta^j \) and \( G_2/\Gamma_2 \leq \text{Aut}(X_2) \), it follows that the number of \( \text{Aut}(X_2) \)-orbits of vertices is \( \leq 4 \).

Let \( X_7 = K_7/\Gamma_7 \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_7 \leq \text{Aut}(K_7) \). Let the vertices of \( X_7 \) form \( m_7 \text{ Aut}(X_7) \)-orbits. We take the middle point of the line segment joining vertices \( b_0 \) and \( b_1 \) as the origin \((0,0)\) of \( K_7 \). Let \( \alpha_7 := b_{22} - b_0, \ \beta_7 := b_{28} - b_0 \) and \( \gamma_7 := b_{41} - b_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_7 := \langle x \mapsto x + \alpha_7, x \mapsto x + \beta_7, x \mapsto x + \gamma_7 \rangle \) and

\[
G_7 = \{ \alpha: x \mapsto \varepsilon x + m_7 \alpha_7 + n_7 \beta_7 + r \gamma_7 : \varepsilon = \pm 1, m_7, n_7, r_7 \in \mathbb{Z} \} \cong H_7 \times \mathbb{Z}_2.
\]

By the same arguments as above and in Claim 1, \( \Gamma_7 \leq G_7 \) and the number of \( G_7/\Gamma_7 \)-orbits of vertices of \( X_7 \) is four. Therefore, \( G_7/\Gamma_7 \) acts on \( X_7 = K_7/\Gamma_7 \). Since \( O_1 = \langle a_0 \rangle, \ O_2 = \langle b_0 \rangle, \ O_3 = \langle b_5 \rangle, \ O_4 = \langle b_6 \rangle \) are the \( G_7 \)-orbits, it follows that \( O_1 = \langle a_0 \rangle, \ O_2 = \langle b_0 \rangle, \ O_3 = \langle b_5 \rangle, \ O_4 = \langle b_6 \rangle \) are the \( (G_7/\Gamma_7) \)-orbits. Since the vertex set of \( X_7 \) is \( \bigcup_{j=1}^{4} \eta^j \) and \( G_7/\Gamma_7 \leq \text{Aut}(X_7) \), it follows that the number of \( \text{Aut}(X_7) \)-orbits of vertices is \( \leq 4 \).

Let \( X_{10} = K_{10}/\Gamma_{10} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{10} \leq \text{Aut}(K_{10}) \). Let the vertices of \( X_{10} \) form \( m_{10} \text{ Aut}(X_{10}) \)-orbits. We take the middle point of the line segment joining vertices \( b_0 \) and \( b_1 \) as the origin \((0,0)\) of \( K_{10} \). Let \( \alpha_{10} := b_6 - b_0, \ \beta_{10} := b_{23} - b_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_{10} := \langle x \mapsto x + \alpha_{10}, x \mapsto x + \beta_{10} \rangle \) and

\[
G_{10} = \{ \alpha: x \mapsto \varepsilon x + m_{10} \alpha_{10} + n_{10} \beta_{10} : \varepsilon = \pm 1, m_{10}, n_{10} \in \mathbb{Z} \} \cong H_{10} \times \mathbb{Z}_2.
\]

By the same arguments as above and in Claim 1, \( \Gamma_{10} \leq G_{10} \) and the number of \( G_{10}/\Gamma_{10} \)-orbits of vertices of \( X_{10} \) is three. Therefore, \( G_{10}/\Gamma_{10} \) acts on \( X_{10} = K_{10}/\Gamma_{10} \). Since \( O_1 = \langle a_0 \rangle, \ O_2 = \langle a_1 \rangle, \ O_3 = \langle b_0 \rangle, \ O_4 = \langle b_0 \rangle \) are the \( G_{10} \)-orbits, it follows that \( \eta_0(O_1), \eta_0(O_2), \eta_0(O_3), \eta_0(O_4) \) are the \( (G_{10}/\Gamma_{10}) \)-orbits. Since the vertex set of \( X_{10} \) is \( \eta_0(O_1) \cup \eta_0(O_2) \cup \eta_0(O_3) \cup \eta_0(O_4) \) and \( G_{10}/\Gamma_{10} \leq \text{Aut}(X_{10}) \), it follows that the number of \( \text{Aut}(X_{10}) \)-orbits of vertices is \( \leq 4 \).

Let \( X_{16} = K_{16}/\Gamma_{16} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{16} \leq \text{Aut}(K_{16}) \). Let the vertices of \( X_{16} \) form \( m_{16} \text{ Aut}(X_{16}) \)-orbits. We take the middle point of the line segment joining vertices \( a_0 \) and \( a_4 \) as the origin \((0,0)\) of \( K_{16} \). Let \( \alpha_{16} := a_4 - a_1 \) and \( \beta_{16} := a_8 - a_1 \in \mathbb{R}^2 \). Similarly as above, define \( H_{16} := \langle x \mapsto x + \alpha_{16}, x \mapsto x + \beta_{16} \rangle \) and

\[
G_{16} = \{ \alpha: x \mapsto \varepsilon x + m_{16} \alpha_{16} + n_{16} \beta_{16} : \varepsilon = \pm 1, m_{16}, n_{16} \in \mathbb{Z} \} \cong H_{16} \times \mathbb{Z}_2.
\]

By the same arguments as above and in Claim 1, \( \Gamma_{16} \leq G_{16} \) and the number of \( G_{16}/\Gamma_{16} \)-orbits of vertices of \( X_{16} \) is three. Therefore, \( G_{16}/\Gamma_{16} \) acts on \( X_{16} = K_{16}/\Gamma_{16} \). Since \( O_1 = \langle a_1 \rangle, \ O_2 = \langle b_1 \rangle, \ O_3 = \langle a_3 \rangle, \ O_4 = \langle b_2 \rangle \) are the \( G_{16} \)-orbits, it follows that \( \eta_0(O_1), \eta_0(O_2), \eta_0(O_3), \eta_0(O_4) \) are the \( (G_{16}/\Gamma_{16}) \)-orbits. Since the vertex set of \( X_{16} \) is \( \eta_0(O_1) \cup \eta_0(O_2) \cup \eta_0(O_3) \cup \eta_0(O_4) \) and \( G_{16}/\Gamma_{16} \leq \text{Aut}(X_{16}) \), it follows that the number of \( \text{Aut}(X_{16}) \)-orbits of vertices is \( \leq 2 \) and hence, \( m_{16} \leq 4 \). Thus, it completes the part (2).

Let \( X_3 = K_3/\Gamma_3 \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_3 \leq \text{Aut}(K_3) \). Let the vertices of \( X_3 \) form \( m_3 \text{ Aut}(X_3) \)-orbits.
We take the middle point of the line segment joining vertices \( a_0 \) and \( a_1 \) as the origin \((0,0)\) of \( K_3 \). Let \( \alpha_3 := a_1 - a_0 \) and \( \beta_3 := a_{33} - a_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_3 := \langle x \mapsto x + \alpha_3, x \mapsto x + \beta_3 \rangle \) and
\[
G_3 = \{ \alpha : x \mapsto \varepsilon x + m_3 \alpha_3 + n_3 \beta_3 : \varepsilon = \pm 1, m_3, n_3 \in \mathbb{Z} \} \cong H_3 \rtimes \mathbb{Z}_2.
\]
By the same arguments as above and in Claim 1, \( \Gamma_3 \leq G_3 \) and the number of \( G_3/\Gamma_3 \)-orbits of vertices of \( X_3 \) is two. Therefore, \( G_3/\Gamma_3 \) acts on \( X_3 = K_3/\Gamma_3 \). Since \( O_1 = \langle a_1 \rangle, O_2 = \langle b_1 \rangle \) are the \( G_3 \)-orbits, it follows that \( \eta_3(O_1), \eta_3(O_2) \) are the \( (G_3/\Gamma_3) \)-orbits. Since the vertex set of \( X_3 \) is \( \eta_3(O_1) \cup \eta_3(O_2) \) and \( G_3/\Gamma_3 \leq \text{Aut}(X_3) \), it follows that the number of \( \text{Aut}(X_3) \)-orbits of vertices is \( \leq 2 \) and hence, \( m_3 = 2 \).

Let \( X_4 = K_4/\Gamma_4 \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_4 \leq \text{Aut}(K_4) \). Let the vertices of \( X_4 \) form \( m_4 \text{ Aut}(X_4) \)-orbits. We take the middle point of the line segment joining vertices \( a_0 \) and \( a_1 \) as the origin \((0,0)\) of \( K_4 \). Let \( \alpha_4 := a_1 - a_0 \) and \( \beta_4 := a_{43} - a_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_4 := \langle x \mapsto x + \alpha_4, x \mapsto x + \beta_4 \rangle \) and
\[
G_4 = \{ \alpha : x \mapsto \varepsilon x + m_4 \alpha_4 + n_4 \beta_4 : \varepsilon = \pm 1, m_4, n_4 \in \mathbb{Z} \} \cong H_4 \rtimes \mathbb{Z}_2.
\]
By the same arguments as above and in Claim 1, \( \Gamma_4 \leq G_4 \) and the number of \( G_4/\Gamma_4 \)-orbits of vertices of \( X_4 \) is two. Therefore, \( G_4/\Gamma_4 \) acts on \( X_4 = K_4/\Gamma_4 \). Since \( O_1 = \langle a_1 \rangle, O_2 = \langle b_1 \rangle \) are the \( G_4 \)-orbits, it follows that \( \eta_4(O_1), \eta_4(O_2) \) are the \( (G_4/\Gamma_4) \)-orbits. Since the vertex set of \( X_4 \) is \( \eta_4(O_1) \cup \eta_4(O_2) \) and \( G_4/\Gamma_4 \leq \text{Aut}(X_4) \), it follows that the number of \( \text{Aut}(X_4) \)-orbits of vertices is \( \leq 2 \) and hence, \( m_4 = 2 \).

Let \( X_8 = K_8/\Gamma_8 \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_8 \leq \text{Aut}(K_8) \). Let the vertices of \( X_8 \) form \( m_8 \text{ Aut}(X_8) \)-orbits. We take the middle point of the line segment joining vertices \( a_0 \) and \( a_1 \) as the origin \((0,0)\) of \( K_8 \). Let \( \alpha_8 := a_3 - a_0 \) and \( \beta_8 := a_4 - a_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_8 := \langle x \mapsto x + \alpha_8, x \mapsto x + \beta_8 \rangle \) and
\[
G_8 = \{ \alpha : x \mapsto \varepsilon x + m_8 \alpha_8 + n_8 \beta_8 : \varepsilon = \pm 1, m_8, n_8 \in \mathbb{Z} \} \cong H_8 \rtimes \mathbb{Z}_2.
\]
By the same arguments as above and in Claim 1, \( \Gamma_8 \leq G_8 \) and the number of \( G_8/\Gamma_8 \)-orbits of vertices of \( X_8 \) is two. Therefore, \( G_8/\Gamma_8 \) acts on \( X_8 = K_8/\Gamma_8 \). Since \( O_1 = \langle a_0 \rangle, O_2 = \langle b_0 \rangle \) are the \( G_8 \)-orbits, it follows that \( \eta_8(O_1), \eta_8(O_2) \) are the \( (G_8/\Gamma_8) \)-orbits. Since the vertex set of \( X_8 \) is \( \eta_8(O_1) \cup \eta_8(O_2) \) and \( G_8/\Gamma_8 \leq \text{Aut}(X_8) \), it follows that the number of \( \text{Aut}(X_8) \)-orbits of vertices is \( \leq 2 \) and hence, \( m_8 = 2 \).

Let \( X_{12} = K_{12}/\Gamma_{12} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{12} \leq \text{Aut}(K_{12}) \). Let the vertices of \( X_{12} \) form \( m_{12} \text{ Aut}(X_{12}) \)-orbits. We take the middle point of the line segment joining vertices \( a_0 \) and \( a_1 \) as the origin \((0,0)\) of \( K_{12} \). Let \( \alpha_12 := a_1 - a_0 \) and \( \beta_12 := a_{12} - a_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_{12} := \langle x \mapsto x + \alpha_{12}, x \mapsto x + \beta_{12} \rangle \) and
\[
G_{12} = \{ \alpha : x \mapsto \varepsilon x + m_{12} \alpha_{12} + n_{12} \beta_{12} : \varepsilon = \pm 1, m_{12}, n_{12} \in \mathbb{Z} \} \cong H_{12} \rtimes \mathbb{Z}_2.
\]
By the same arguments as above and in Claim 1, \( \Gamma_{12} \leq G_{12} \) and the number of \( G_{12}/\Gamma_{12} \)-orbits of vertices of \( X_{12} \) is two. Therefore, \( G_{12}/\Gamma_{12} \) acts on \( X_{12} = K_{12}/\Gamma_{12} \). Since \( O_1 = \langle a_0 \rangle, O_2 = \langle b_0 \rangle \) are the \( G_{12} \)-orbits, it follows that \( \eta_{12}(O_1), \eta_{12}(O_2) \) are the \( (G_{12}/\Gamma_{12}) \)-orbits. Since the vertex set of \( X_{12} \) is \( \eta_{12}(O_1) \cup \eta_{12}(O_2) \) and \( G_{12}/\Gamma_{12} \leq \text{Aut}(X_{12}) \), it follows that the number of \( \text{Aut}(X_{12}) \)-orbits of vertices is \( \leq 2 \) and hence, \( m_{12} = 2 \).
Let \( X_{13} = K_{13}/\Gamma_{13} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{13} \leq \text{Aut}(K_{13}) \). Let the vertices of \( X_{13} \) form \( m_{13} \) \( \text{Aut}(X_{13}) \)-orbits with \( m_{13} \geq 2 \). We take the middle point of the line segment joining vertices \( a_0 \) and \( a_1 \) as the origin \((0,0)\) of \( K_{13} \). Let \( \alpha_{13} := a_1 - a_0 \) and \( \beta_{13} := a_19 - a_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_{13} := \langle x \mapsto x + \alpha_{13}, x \mapsto x + \beta_{13} \rangle \) and
\[
G_{13} = \{ \alpha : x \mapsto \varepsilon x + m_{13}\alpha_{13} + n_{13}\beta_{13} : \varepsilon = \pm 1, m_{13}, n_{13} \in \mathbb{Z} \} \cong H_{13} \times \mathbb{Z}_2.
\]
By the same arguments as above and in Claim 1, \( \Gamma_{13} \leq \text{Aut}(K_{13}) \) and the number of \( G_{13}/\Gamma_{13} \)-orbits of vertices of \( X_{13} \) is two. Therefore, \( G_{13}/\Gamma_{13} \) acts on \( X_{13} = K_{13}/\Gamma_{13} \). Since \( O_1 = \langle a_0 \rangle, O_2 = \langle b_0 \rangle \) are the \( G_{13} \)-orbits, it follows that \( \eta_{13}(O_1), \eta_{13}(O_2) \) are the \( (G_{13}/\Gamma_{13}) \)-orbits. Since the vertex set of \( X_{13} \) is \( \eta_{13}(O_1) \sqcup \eta_{13}(O_2) \) and \( G_{13}/\Gamma_{13} \leq \text{Aut}(X_{13}) \), it follows that the number of \( \text{Aut}(X_{13}) \)-orbits of vertices is \( \leq 2 \) and hence, \( m_{13} = 2 \).

Let \( X_{15} = K_{15}/\Gamma_{15} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{15} \leq \text{Aut}(K_{15}) \). Let the vertices of \( X_{15} \) form \( m_{15} \) \( \text{Aut}(X_{15}) \)-orbits with \( m_{15} \geq 2 \). We take the middle point of the line segment joining vertices \( a_0 \) and \( a_19 \) as the origin \((0,0)\) of \( K_{15} \). Let \( \alpha_{15} := a_2 - a_0 \) and \( \beta_{15} := a_{10} - a_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_{15} := \langle x \mapsto x + \alpha_{15}, x \mapsto x + \beta_{15} \rangle \) and
\[
G_{15} = \{ \alpha : x \mapsto \varepsilon x + m_{15}\alpha_{15} + n_{15}\beta_{15} : \varepsilon = \pm 1, m_{15}, n_{15} \in \mathbb{Z} \} \cong H_{15} \times \mathbb{Z}_2.
\]
By the same arguments as above and in Claim 1, \( \Gamma_{15} \leq \text{Aut}(K_{15}) \) and the number of \( G_{15}/\Gamma_{15} \)-orbits of vertices of \( X_{15} \) is two. Therefore, \( G_{15}/\Gamma_{15} \) acts on \( X_{15} = K_{15}/\Gamma_{15} \). Since \( O_1 = \langle a_0 \rangle, O_2 = \langle b_0 \rangle \) are the \( G_{15} \)-orbits, it follows that \( \eta_{15}(O_1), \eta_{15}(O_2) \) are the \( (G_{15}/\Gamma_{15}) \)-orbits. Since the vertex set of \( X_{15} \) is \( \eta_{15}(O_1) \sqcup \eta_{15}(O_2) \) and \( G_{15}/\Gamma_{15} \leq \text{Aut}(X_{15}) \), it follows that the number of \( \text{Aut}(X_{15}) \)-orbits of vertices is \( \leq 2 \) and hence, \( m_{15} = 2 \). Thus, it completes the part (3).

Let \( X_{18} = K_{18}/\Gamma_{18} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{18} \leq \text{Aut}(K_{18}) \). Let the vertices of \( X_{18} \) form \( m_{18} \) \( \text{Aut}(X_{18}) \)-orbits with \( m_{18} \geq 2 \). We take the middle point of the line segment joining vertices \( a_0 \) and \( a_28 \) as the origin \((0,0)\) of \( K_{18} \). Let \( \alpha_{18} := a_2 - a_0 \) and \( \beta_{18} := a_{20} - a_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_{18} := \langle x \mapsto x + \alpha_{18}, x \mapsto x + \beta_{18} \rangle \) and
\[
G_{18} = \{ \alpha : x \mapsto \varepsilon x + m_{18}\alpha_{18} + n_{18}\beta_{18} : \varepsilon = \pm 1, m_{18}, n_{18} \in \mathbb{Z} \} \cong H_{18} \times \mathbb{Z}_2.
\]
By the same arguments as above and in Claim 1, \( \Gamma_{18} \leq \text{Aut}(K_{18}) \) and the number of \( G_{18}/\Gamma_{18} \)-orbits of vertices of \( X_{18} \) is three. Therefore, \( G_{18}/\Gamma_{18} \) acts on \( X_{18} = K_{18}/\Gamma_{18} \). Since \( O_1 = \langle a_0 \rangle, O_2 = \langle a_1 \rangle, O_3 = \langle b_1 \rangle \) are the \( G_{18} \)-orbits, it follows that \( \eta_{18}(O_1), \eta_{18}(O_2), \eta_{18}(O_3) \) are the \( (G_{18}/\Gamma_{18}) \)-orbits. Since the vertex set of \( X_{18} \) is \( \eta_{18}(O_1) \sqcup \eta_{18}(O_2) \sqcup \eta_{18}(O_3) \) and \( G_{18}/\Gamma_{18} \leq \text{Aut}(X_{18}) \), it follows that the number of \( \text{Aut}(X_{18}) \)-orbits of vertices is \( \leq 3 \).

Let \( X_{19} = K_{19}/\Gamma_{19} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{19} \leq \text{Aut}(K_{19}) \). Let the vertices of \( X_{19} \) form \( m_{19} \) \( \text{Aut}(X_{19}) \)-orbits with \( m_{19} \geq 2 \). We take the middle point of the line segment joining vertices \( b_0 \) and \( b_2 \) as the origin \((0,0)\) of \( K_{19} \). Let \( \alpha_{19} := b_0 - b_2 \) and \( \beta_{19} := b_7 - b_2 \in \mathbb{R}^2 \). Similarly as above, define \( H_{19} := \langle x \mapsto x + \alpha_{19}, x \mapsto x + \beta_{19} \rangle \) and
\[
G_{19} = \{ \alpha : x \mapsto \varepsilon x + m_{19}\alpha_{19} + n_{19}\beta_{19} : \varepsilon = \pm 1, m_{19}, n_{19} \in \mathbb{Z} \} \cong H_{19} \times \mathbb{Z}_2.
\]
By the same arguments as above and in Claim 1, \( \Gamma_{19} \leq \text{Aut}(K_{19}) \) and the number of \( G_{19}/\Gamma_{19} \)-orbits of vertices of \( X_{19} \) is three. Therefore, \( G_{19}/\Gamma_{19} \) acts on \( X_{19} = K_{19}/\Gamma_{19} \). Since \( O_1 = \langle b_0 \rangle, \)
Let \(X_6 = K_6/\Gamma_6\) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \(\Gamma_6 \leq \text{Aut}(K_6)\). Let the vertices of \(X_6\) form \(m_6\) \(\text{Aut}(X_6)\)-orbits. We take the middle point of the line segment joining vertices \(a_6\) and \(a_6\) as the origin \((0,0)\) of \(K_6\). Let \(\alpha_6 := a_{23} - a_3, \beta_6 := a_{16} - a_6\) and \(\gamma_6 := a_{39} - a_7 \in \mathbb{R}^2\). Similarly as above, define \(H_6 := \langle x \mapsto x + \alpha_6, x \mapsto x + \beta_6, x \mapsto x + \gamma_6\rangle\) and
\[
G_6 = \{ \alpha : x \mapsto \varepsilon x + m_6\alpha_6 + n_6\beta_6 + r_6\gamma_6 : \varepsilon = \pm 1, m_6, n_6, r_6 \in \mathbb{Z} \} \cong H_6 \times \mathbb{Z}_2.
\]
By the same arguments as above and in Claim 1, \(\Gamma_6 \leq G_6\) and the number of \(G_6/\Gamma_6\)-orbits of vertices of \(X_6\) is seven. Therefore, \(G_6/\Gamma_6\) acts on \(X_6 = K_6/\Gamma_6\). Since \(O_j = \langle a_j \rangle, j = 0, 1, \ldots , 5\) \(O_6 = \langle b_1 \rangle\) are the \(G_6\)-orbits, it follows that \(\theta_6(O_j), j = 0, 1, \ldots , 6\) are the \((G_6/\Gamma_6)\)-orbits. Since the vertex set of \(X_6\) is \(\bigcup_{j=0}^{6} \theta_6(O_j)\) and \(G_6/\Gamma_6 \leq \text{Aut}(X_6)\), it follows that the number of \(\text{Aut}(X_6)\)-orbits of vertices is \(\leq 7\). Thus, it completes the part (5).

Let \(X_{20} = K_{20}/\Gamma_{20}\) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \(\Gamma_{20} \leq \text{Aut}(K_{20})\). Let the vertices of \(X_{20}\) form \(m_{20}\) \(\text{Aut}(X_{20})\)-orbits. We take the middle point of the line segment joining vertices \(a_0\) and \(a_6\) as the origin \((0,0)\) of \(K_{20}\). Let \(\alpha_{20} := a_{24} - a_7, \beta_{20} := a_{14} - a_5\) and \(\gamma_{20} := a_20 - a_4 \in \mathbb{R}^2\). Similarly as above, define \(H_{20} := \langle x \mapsto x + \alpha_{20}, x \mapsto x + \beta_{20}, x \mapsto x + \gamma_{20}\rangle\) and
\[
G_{20} = \{ \alpha : x \mapsto \varepsilon x + m_{20}\alpha_{20} + n_{20}\beta_{20} + r_{20}\gamma_{20} : \varepsilon = \pm 1, m_{20}, n_{20}, r_{20} \in \mathbb{Z} \} \cong H_{20} \times \mathbb{Z}_2.
\]
By the same arguments as above and in Claim 1, \(\Gamma_{20} \leq G_{20}\) and the number of \(G_{20}/\Gamma_{20}\)-orbits of vertices of \(X_{20}\) is nine. Therefore, \(G_{20}/\Gamma_{20}\) acts on \(X_{20} = K_{20}/\Gamma_{20}\). Since \(O_1 = \langle a_1 \rangle, O_2 = \langle a_2 \rangle, O_3 = \langle a_3 \rangle, O_4 = \langle a_4 \rangle, O_5 = \langle a_5 \rangle, O_6 = \langle a_6 \rangle, O_7 = \langle b_3 \rangle, O_8 = \langle b_0 \rangle, O_9 = \langle b_{11} \rangle\) are the \(G_{20}\)-orbits, it follows that \(O_1 = \langle a_1 \rangle, O_2 = \langle a_2 \rangle, O_3 = \langle a_3 \rangle, O_4 = \langle a_4 \rangle, O_5 = \langle a_5 \rangle, O_6 = \langle a_6 \rangle, O_7 = \langle b_3 \rangle, O_8 = \langle b_0 \rangle, O_9 = \langle b_{11} \rangle\) are the \((G_{20}/\Gamma_{20})\)-orbits. Since the vertex set of \(X_{20}\) is \(\bigcup_{j=1}^{9} \theta_{20}(O_j)\) and \(G_{20}/\Gamma_{20} \leq \text{Aut}(X_{20})\), it follows that the number of \(\text{Aut}(X_{20})\)-orbits of vertices is \(\leq 9\). Thus, it completes the part (6).
3  2-uniform tilings of the plane
4 3-uniform tilings of the plane
5 Proof of Theorem 1.6

Proposition 5.1. The maps $K_i$ ($1 \leq i \leq 61$) (in Section 4) are unique up to isomorphism.

Proof of Theorem 1.6. Let $X_1$ be a semiequivelar map on the torus that is the quotient of the plane’s 3-uniform lattice $K_1$. Let the vertices of $X_1$ form $m_1 \text{Aut}(X_1)$-orbits. Let $K_1$ be as in Section 4. Let $V_1 = V(K_1)$ be the vertex set of $K_1$. Let $H_1$ be the group of all the translations of $K_1$. So, $H_1 \leq \text{Aut}(K_1)$.

Since $X_1$ is a semiequivelar map on the torus that is the quotient of the plane’s 3-uniform lattice $K_1$ (as by Proposition 5.1, $K_1$ is unique), so, we can assume, there is a polyhedral covering map $\eta_1 : K_1 \rightarrow X_1$ where $X_1 = K_1/\Gamma_1$ for some fixed element (vertex, edge or face) free subgroup $\Gamma_1 \leq \text{Aut}(K_1)$. Hence $\Gamma_1$ consists of translations and glide reflections. Since $X_1 = K_1/\Gamma_1$ is orientable, $\Gamma_1$ does not contain any glide reflection. Thus $\Gamma_1 \leq H_1$.

We take the middle point of the line segment joining vertices $a_0$ and $a_1$ as the origin $(0,0)$ of $K_1$. Let $\alpha_1 := a_2 - a_0$ and $\beta_1 := a_7 - a_0$ in $K_1$. Then

$$H_1 := \langle x \mapsto x + \alpha_1, x \mapsto x + \beta_1 \rangle.$$ 

Under the action of $H_1$, vertices of $K_1$ form five orbits. Consider the subgroup $G_1$ of $\text{Aut}(K_1)$ generated by $H_1$ and the map (the half rotation) $x \mapsto -x$. So,

$$G_1 = \{ \alpha : x \mapsto \varepsilon x + m\alpha_1 + n\beta_1 : \varepsilon = \pm 1, m, n \in \mathbb{Z} \} \cong H_1 \rtimes \mathbb{Z}_2.$$

Clearly, under the action of $G_1$, vertices of $K_1$ form three orbits. The orbits are

$$O_1 := \langle a_0 \rangle, O_2 := \langle b_0 \rangle, O_3 := \langle c_0 \rangle.$$

Claim 1. If $S \leq H_1$ then $S \leq G_1$.

Let $g \in G_1$ and $s \in S$. Then $g(x) = \varepsilon x + ma + nb + rc$ and $s(x) = x + pa + qb + \ell c$ for some $m, n, r, p, q, \ell \in \mathbb{Z}$ and $\varepsilon \in \{1, -1\}$. Therefore,

$$(g \circ s \circ g^{-1})(x) = (g \circ s)(\varepsilon(x - ma - nb - rc))$$

$$= g(\varepsilon(x - ma - nb - rc) + pa + qb + \ell c)$$

$$= x - ma - nb - rc + \varepsilon(pa + qb + \ell c) + ma + nb + rc$$

$$= x + \varepsilon(pa + qb + \ell c)$$

$$= s^\varepsilon(x).$$

Thus, $g \circ j \circ g^{-1} = s^\varepsilon \in S$. This completes the claim.

By Claim 1, $\Gamma_1$ is a normal subgroup of $G_1$. Therefore, $G_1/\Gamma_1$ acts on $X_1 = K_1/\Gamma_1$. Since

$$O_1 := \langle a_0 \rangle, O_2 := \langle b_0 \rangle, O_3 := \langle c_0 \rangle$$

are the $G_1$-orbits, it follows that $\eta_1(O_j)$ for $j = 1, 2, \ldots, 6$ are the $(G_1/\Gamma_1)$-orbits. Since the vertex set of $K_1$ is $\sqcup_{j=1}^6 \eta_1(O_j)$ and $G_1/\Gamma_1 \leq \text{Aut}(X_1)$, it follows that the number of $\text{Aut}(X_1)$-orbits of vertices is 3.

Let $X_2 = K_2/\Gamma_2$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_2 \leq \text{Aut}(K_2)$. Let the vertices of $X_2$ form $m_2 \text{Aut}(X_2)$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_1$ as the origin $(0,0)$ of $K_2$ (see Section 4). Let $\alpha_2 := a_4 - a_0$ and $\beta_2 := a_2 - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_2 := \langle x \mapsto x + \alpha_2, x \mapsto x + \beta_2 \rangle$ and

$$G_2 = \{ \alpha : x \mapsto \varepsilon x + m_2\alpha_2 + n_2\beta_2 : \varepsilon = \pm 1, m_2, n_2 \in \mathbb{Z} \} \cong H_2 \times \mathbb{Z}_2.$$
By the same arguments as above and in Claim 1, $G_2 \le G_2$ and the number of $G_2/\Gamma_2$-orbits of vertices of $X_2$ is three. Therefore, $G_2/\Gamma_2$ acts on $X_2 = K_2/\Gamma_2$. Since $O_1 = (a_0)$, $O_2 = (b_0)$, $O_3 = (c_0)$ are the $G_2$-orbits, it follows that $O_1 = (a_0)$, $O_2 = (b_0)$, $O_3 = (c_0)$ are the $(G_2/\Gamma_2)$-orbits. Since the vertex set of $X_2$ is $\bigcup_{j=1}^3 \eta_2(O_j)$ and $G_2/\Gamma_2 \le \text{Aut}(X_2)$, it follows that the number of $\text{Aut}(X_2)$-orbits of vertices is 3.

Let $X_3 = K_3/\Gamma_3$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_3 \le \text{Aut}(K_3)$. Let the vertices of $X_3$ form $m_3 \text{ Aut}(X_3)$-orbits. We take the middle point of the line segment joining vertices $b_1$ and $c_1$ as the origin $(0, 0)$ of $K_3$ (see Section 4). Let $\alpha_3 := c_2 - c_1$ and $\beta_3 := c_5 - c_1 \in \mathbb{R}^2$. Similarly as above, define $H_3 := \langle x \mapsto x + \alpha_3, x \mapsto x + \beta_3, x \mapsto x + \gamma_3 \rangle$ and

$$G_3 = \lbrace \alpha : x \mapsto \varepsilon x + m_3 \alpha_3 + n_3 \beta_3 : \varepsilon = \pm 1, m_3, n_3 \in \mathbb{Z} \rbrace \cong H_3 \times \mathbb{Z}_2.$$ 

By the same arguments as above and in Claim 1, $\Gamma_3 \le G_3$ and the number of $G_3/\Gamma_3$-orbits of vertices of $X_3$ is eight. Therefore, $G_3/\Gamma_3$ acts on $X_3 = K_3/\Gamma_3$. Since $O_1 = (a_0)$, $O_2 = (a_5)$, $O_3 = (b_0)$, $O_4 = (b_1)$, $O_5 = (b_5)$, $O_6 = (c_0)$, $O_7 = (c_1)$, $O_8 = (c_2)$ are the $G_2$-orbits, it follows that $O_1 = (a_0)$, $O_2 = (a_5)$, $O_3 = (b_0)$, $O_4 = (b_1)$, $O_5 = (b_5)$, $O_6 = (c_0)$, $O_7 = (c_1)$, $O_8 = (c_2)$ are the $(G_3/\Gamma_3)$-orbits. Since the vertex set of $X_3$ is $\bigcup_{j=1}^8 \eta_3(O_j)$ and $G_3/\Gamma_3 \le \text{Aut}(X_3)$, it follows that the number of $\text{Aut}(X_3)$-orbits of vertices is $\le 8$.

Let $X_4 = K_4/\Gamma_4$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_4 \le \text{Aut}(K_4)$. Let the vertices of $X_4$ form $m_4 \text{ Aut}(X_4)$-orbits. We take the middle point of the line segment joining vertices $b_3$ and $b_1$ as the origin $(0, 0)$ of $K_4$ (see Section 4). Let $\alpha_4 := b_4 - b_3$ and $\beta_4 := b_1 - b_3 \in \mathbb{R}^2$. Similarly as above, define $H_4 := \langle x \mapsto x + \alpha_4, x \mapsto x + \beta_4 \rangle$ and

$$G_4 = \lbrace \alpha : x \mapsto \varepsilon x + m_4 \alpha_4 + n_4 \beta_4 : \varepsilon = \pm 1, m_4, n_4 \in \mathbb{Z} \rbrace \cong H_4 \times \mathbb{Z}_2.$$ 

By the same arguments as above and in Claim 1, $\Gamma_4 \le G_4$ and the number of $G_4/\Gamma_4$-orbits of vertices of $X_4$ is three. Therefore, $G_4/\Gamma_4$ acts on $X_4 = K_4/\Gamma_4$. Since $O_1 = (a_0)$, $O_2 = (b_3)$, $O_3 = (c_0)$ are the $G_4$-orbits, it follows that $O_1 = (a_0)$, $O_2 = (b_3)$, $O_3 = (c_0)$ are the $(G_4/\Gamma_4)$-orbits. Since the vertex set of $X_4$ is $\bigcup_{j=1}^3 \eta_4(O_j)$ and $G_4/\Gamma_4 \le \text{Aut}(X_4)$, it follows that the number of $\text{Aut}(X_4)$-orbits of vertices is 3.

Let $X_5 = K_5/\Gamma_5$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_5 \le \text{Aut}(K_5)$. Let the vertices of $X_5$ form $m_5 \text{ Aut}(X_5)$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $c_1$ as the origin $(0, 0)$ of $K_5$ (see Section 4). Let $\alpha_5 := a_0 - a_0$ and $\beta_5 := a_32 - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_5 := \langle x \mapsto x + \alpha_5, x \mapsto x + \beta_5 \rangle$ and

$$G_5 = \lbrace \alpha : x \mapsto \varepsilon x + m_5 \alpha_5 + n_5 \beta_5 : \varepsilon = \pm 1, m_5, n_5 \in \mathbb{Z} \rbrace \cong H_5 \times \mathbb{Z}_2.$$ 

By the same arguments as above and in Claim 1, $\Gamma_5 \le G_5$ and the number of $G_5/\Gamma_5$-orbits of vertices of $X_5$ is twelve. Therefore, $G_5/\Gamma_5$ acts on $X_5 = K_5/\Gamma_5$. Since $O_1 = (a_0)$, $O_2 = (a_1)$, $O_3 = (a_2)$, $O_4 = (a_3)$, $O_5 = (a_3)$, $O_6 = (c_0)$, $O_7 = (b_0)$, $O_8 = (c_0)$, $O_9 = (c_1)$, $O_{10} = (c_2)$, $O_{11} = (c_3)$, $O_{12} = (c_4)$ are the $G_5$-orbits, it follows that $O_1 = (a_0)$, $O_2 = (a_1)$, $O_3 = (a_2)$, $O_4 = (a_3)$, $O_5 = (a_3)$, $O_6 = (a_5)$, $O_7 = (b_0)$, $O_8 = (c_0)$, $O_9 = (c_1)$, $O_{10} = (c_2)$, $O_{11} = (c_3)$, $O_{12} = (c_4)$ are the $(G_5/\Gamma_5)$-orbits. Since the vertex set of $X_5$ is $\bigcup_{j=1}^{12} \eta_5(O_j)$ and $G_5/\Gamma_5 \le \text{Aut}(X_5)$, it follows that the number of $\text{Aut}(X_5)$-orbits of vertices is $\le 12$.

Let $X_6 = K_6/\Gamma_6$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_6 \le \text{Aut}(K_6)$. Let the vertices of $X_6$ form $m_6 \text{ Aut}(X_6)$-orbits.
We take the middle point of the line segment joining vertices $a_0$ and $a_1$ as the origin $(0,0)$ of $K_6$ (see Section 4). Let $\alpha_6 := a_2 - a_0$ and $\beta_6 := a_{22} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_6 := \langle x \mapsto x + \alpha_6, x \mapsto x + \beta_6 \rangle$ and

$$G_6 = \{ \alpha : x \mapsto \varepsilon x + m_6 \alpha + n_6 \beta : \varepsilon = \pm 1, m_6, n_6 \in \mathbb{Z} \} \cong H_6 \times \mathbb{Z}_2.$$  

By the same arguments as above and in Claim 1, $\Gamma_6 \leq G_6$ and the number of $G_6/\Gamma_6$-orbits of vertices of $X_6$ is seven. Therefore, $G_6/\Gamma_6$ acts on $X_6 = K_6/\Gamma_6$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_0 \rangle$, $O_3 = \langle a_8 \rangle$, $O_4 = \langle b_0 \rangle$, $O_5 = \langle c_0 \rangle$, $O_6 = \langle c_1 \rangle$, $O_7 = \langle c_{10} \rangle$ are the $G_6$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_0 \rangle$, $O_3 = \langle a_8 \rangle$, $O_4 = \langle b_0 \rangle$, $O_5 = \langle c_0 \rangle$, $O_6 = \langle c_1 \rangle$, $O_7 = \langle c_{10} \rangle$ are the $(G_6/\Gamma_6)$-orbits. Since the vertex set of $X_6$ is $\cup_{j=1}^{10} \eta_6(O_j)$ and $G_6/\Gamma_6 \leq \text{Aut}(X_6)$, it follows that the number of $\text{Aut}(X_6)$-orbits of vertices is $\leq 7$.

Let $X_7 = K_7/\Gamma_7$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_7 \leq \text{Aut}(K_7)$. Let the vertices of $X_7$ form $m_7 \text{Aut}(X_7)$-orbits. We take the point $a_0$ as the origin $(0,0)$ of $K_7$ (see Section 4). Let $\alpha_7 := a_1 - a_0$, $\beta_7 := a_2 - a_0$ and $\gamma_7 := a_3 - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_7 := \langle x \mapsto x + \alpha_7, x \mapsto x + \beta_7, x \mapsto x + \gamma_7 \rangle$ and

$$G_7 = \{ \alpha : x \mapsto \varepsilon x + m_7 \alpha + n_7 \beta + r_7 \gamma : \varepsilon = \pm 1, m_7, n_7, r_7 \in \mathbb{Z} \} \cong H_7 \times \mathbb{Z}_2.$$  

By the same arguments as above and in Claim 1, $\Gamma_7 \leq G_7$ and the number of $G_7/\Gamma_7$-orbits of vertices of $X_7$ is seven. Therefore, $G_7/\Gamma_7$ acts on $X_7 = K_7/\Gamma_7$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle b_0 \rangle$, $O_3 = \langle b_{19} \rangle$, $O_4 = \langle b_{17} \rangle$, $O_5 = \langle c_0 \rangle$, $O_6 = \langle c_1 \rangle$, $O_7 = \langle c_2 \rangle$ are the $G_7$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle b_0 \rangle$, $O_3 = \langle b_{19} \rangle$, $O_4 = \langle b_{17} \rangle$, $O_5 = \langle c_0 \rangle$, $O_6 = \langle c_1 \rangle$, $O_7 = \langle c_2 \rangle$ are the $(G_7/\Gamma_7)$-orbits. Since the vertex set of $X_7$ is $\cup_{j=1}^{10} \eta_7(O_j)$ and $G_7/\Gamma_7 \leq \text{Aut}(X_7)$, it follows that the number of $\text{Aut}(X_7)$-orbits of vertices is $\leq 7$.

Let $X_8 = K_8/\Gamma_8$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_8 \leq \text{Aut}(K_8)$. Let the vertices of $X_8$ form $m_8 \text{Aut}(X_8)$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_1$ as the origin $(0,0)$ of $K_8$ (see Section 4). Let $\alpha_8 := a_{28} - a_0$ and $\beta_8 := a_{46} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_8 := \langle x \mapsto x + \alpha_8, x \mapsto x + \beta_8 \rangle$ and

$$G_8 = \{ \alpha : x \mapsto \varepsilon x + m_8 \alpha + n_8 \beta : \varepsilon = \pm 1, m_8, n_8 \in \mathbb{Z} \} \cong H_8 \times \mathbb{Z}_2.$$  

By the same arguments as above and in Claim 1, $\Gamma_8 \leq G_8$ and the number of $G_8/\Gamma_8$-orbits of vertices of $X_8$ is ten. Therefore, $G_8/\Gamma_8$ acts on $X_8 = K_8/\Gamma_8$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle a_3 \rangle$, $O_5 = \langle a_4 \rangle$, $O_6 = \langle a_5 \rangle$, $O_7 = \langle b_0 \rangle$, $O_8 = \langle c_0 \rangle$, $O_9 = \langle c_1 \rangle$, $O_{10} = \langle c_2 \rangle$ are the $G_8$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle a_3 \rangle$, $O_5 = \langle a_4 \rangle$, $O_6 = \langle a_5 \rangle$, $O_7 = \langle b_0 \rangle$, $O_8 = \langle c_0 \rangle$, $O_9 = \langle c_1 \rangle$, $O_{10} = \langle c_2 \rangle$ are the $(G_8/\Gamma_8)$-orbits. Since the vertex set of $X_8$ is $\cup_{j=1}^{10} \eta_8(O_j)$ and $G_8/\Gamma_8 \leq \text{Aut}(X_8)$, it follows that the number of $\text{Aut}(X_8)$-orbits of vertices is $\leq 10$.

Let $X_9 = K_9/\Gamma_9$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_9 \leq \text{Aut}(K_9)$. Let the vertices of $X_9$ form $m_9 \text{Aut}(X_9)$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_1$ as the origin $(0,0)$ of $K_9$ (see Section 4). Let $\alpha_9 := a_8 - a_0$, $\beta_9 := a_{15} - a_0$ and $\gamma_9 := a_{18} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_9 := \langle x \mapsto x + \alpha_9, x \mapsto x + \beta_9, x \mapsto x + \gamma_9 \rangle$ and

$$G_9 = \{ \alpha : x \mapsto \varepsilon x + m_9 \alpha + n_9 \beta + r_9 \gamma : \varepsilon = \pm 1, m_9, n_9, r_9 \in \mathbb{Z} \} \cong H_9 \times \mathbb{Z}_2.$$  

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By the same arguments as above and in Claim 1, \( \Gamma_9 \leq G_9 \) and the number of \( G_9/\Gamma_9 \)-orbits of vertices of \( X_9 \) is ten. Therefore, \( G_9/\Gamma_9 \) acts on \( X_9 = K_9/\Gamma_9 \). Since \( O_1 = \langle a_0 \rangle, O_2 = \langle a_1 \rangle, O_3 = \langle a_2 \rangle, O_4 = \langle b_0 \rangle, O_5 = \langle c_0 \rangle, O_6 = \langle c_1 \rangle, O_7 = \langle c_2 \rangle, O_8 = \langle c_3 \rangle, O_9 = \langle c_4 \rangle, O_{10} = \langle c_5 \rangle \) are the \( G_9 \)-orbits, it follows that \( O_1 = \langle a_0 \rangle, O_2 = \langle a_1 \rangle, O_3 = \langle a_2 \rangle, O_4 = \langle b_0 \rangle, O_5 = \langle c_0 \rangle, O_6 = \langle c_1 \rangle, O_7 = \langle c_2 \rangle, O_8 = \langle c_3 \rangle, O_9 = \langle c_4 \rangle, O_{10} = \langle c_5 \rangle \) are the \( G_9/\Gamma_9 \)-orbits. Since the vertex set of \( X_9 \) is \( \sqcup_{j=1}^{10} \eta_j(O_j) \) and \( G_9/\Gamma_9 \leq \text{Aut}(X_9) \), it follows that the number of \( \text{Aut}(X_9) \)-orbits of vertices is \( \leq 10 \).

Let \( X_{10} = K_{10}/\Gamma_{10} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{10} \leq \text{Aut}(K_{10}) \). Let the vertices of \( X_{10} \) form \( m_{10} \text{ Aut}(X_{10}) \)-orbits. We take the middle point of the line segment joining vertices \( a_0 \) and \( a_{12} \) as the origin \((0,0)\) of \( K_{10} \) (see Section 4). Let \( \alpha_{10} := a_1 - a_0 \) and \( \beta_{10} := a_{22} - a_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_{10} := \langle x \mapsto x + \alpha_{10}, x \mapsto x + \beta_{10} \rangle \) and

\[
G_{10} = \{ \alpha : x \mapsto \varepsilon x + m_{10}\alpha_{10} + n_{10}\beta_{10} : \varepsilon = \pm 1, m_{10}, n_{10} \in \mathbb{Z} \} \cong H_{10} \times \mathbb{Z}_2.
\]

By the same arguments as above and in Claim 1, \( \Gamma_{10} \leq G_{10} \) and the number of \( G_{10}/\Gamma_{10} \)-orbits of vertices of \( X_{10} \) is three. Therefore, \( G_{10}/\Gamma_{10} \) acts on \( X_{10} = K_{10}/\Gamma_{10} \). Since \( O_1 = \langle a_0 \rangle, O_2 = \langle b_0 \rangle, O_3 = \langle c_0 \rangle \) are the \( G_{10} \)-orbits, it follows that \( O_1 = \langle a_0 \rangle, O_2 = \langle b_0 \rangle, O_3 = \langle c_0 \rangle \) are the \( (G_{10}/\Gamma_{10}) \)-orbits. Since the vertex set of \( X_{10} \) is \( \sqcup_{j=1}^{3} \eta_j(O_j) \) and \( G_{10}/\Gamma_{10} \leq \text{Aut}(X_{10}) \), it follows that the number of \( \text{Aut}(X_{10}) \)-orbits of vertices is 3.

Let \( X_{11} = K_{11}/\Gamma_{11} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{11} \leq \text{Aut}(K_{11}) \). Let the vertices of \( X_{11} \) form \( m_{11} \text{ Aut}(X_{11}) \)-orbits. We take the middle point of the line segment joining vertices \( a_0 \) and \( a_{13} \) as the origin \((0,0)\) of \( K_{11} \) (see Section 4). Let \( \alpha_{11} := a_1 - a_0 \) and \( \beta_{11} := a_{22} - a_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_{11} := \langle x \mapsto x + \alpha_{11}, x \mapsto x + \beta_{11} \rangle \) and

\[
G_{11} = \{ \alpha : x \mapsto \varepsilon x + m_{11}\alpha_{11} + n_{11}\beta_{11} : \varepsilon = \pm 1, m_{11}, n_{11} \in \mathbb{Z} \} \cong H_{11} \times \mathbb{Z}_2.
\]

By the same arguments as above and in Claim 1, \( \Gamma_{11} \leq G_{11} \) and the number of \( G_{11}/\Gamma_{11} \)-orbits of vertices of \( X_{11} \) is three. Therefore, \( G_{11}/\Gamma_{11} \) acts on \( X_{11} = K_{11}/\Gamma_{11} \). Since \( O_1 = \langle a_0 \rangle, O_2 = \langle b_0 \rangle, O_3 = \langle c_0 \rangle \) are the \( G_{11} \)-orbits, it follows that \( O_1 = \langle a_0 \rangle, O_2 = \langle b_0 \rangle, O_3 = \langle c_0 \rangle \) are the \( (G_{11}/\Gamma_{11}) \)-orbits. Since the vertex set of \( X_{11} \) is \( \sqcup_{j=1}^{3} \eta_j(O_j) \) and \( G_{11}/\Gamma_{11} \leq \text{Aut}(X_{11}) \), it follows that the number of \( \text{Aut}(X_{11}) \)-orbits of vertices is 3.

Let \( X_{12} = K_{12}/\Gamma_{12} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{12} \leq \text{Aut}(K_{12}) \). Let the vertices of \( X_{12} \) form \( m_{12} \text{ Aut}(X_{12}) \)-orbits. We take the middle point of the line segment joining vertices \( a_0 \) and \( a_{11} \) as the origin \((0,0)\) of \( K_{12} \) (see Section 4). Let \( \alpha_{12} := a_1 - a_0 \) and \( \beta_{12} := a_{19} - a_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_{12} := \langle x \mapsto x + \alpha_{12}, x \mapsto x + \beta_{12} \rangle \) and

\[
G_{12} = \{ \alpha : x \mapsto \varepsilon x + m_{12}\alpha_{12} + n_{12}\beta_{12} : \varepsilon = \pm 1, m_{12}, n_{12} \in \mathbb{Z} \} \cong H_{12} \times \mathbb{Z}_2.
\]

By the same arguments as above and in Claim 1, \( \Gamma_{12} \leq G_{12} \) and the number of \( G_{12}/\Gamma_{12} \)-orbits of vertices of \( X_{12} \) is three. Therefore, \( G_{12}/\Gamma_{12} \) acts on \( X_{12} = K_{12}/\Gamma_{12} \). Since \( O_1 = \langle a_0 \rangle, O_2 = \langle b_2 \rangle, O_3 = \langle c_0 \rangle \) are the \( G_{12} \)-orbits, it follows that \( O_1 = \langle a_0 \rangle, O_2 = \langle b_2 \rangle, O_3 = \langle c_0 \rangle \) are the \( (G_{12}/\Gamma_{12}) \)-orbits. Since the vertex set of \( X_{12} \) is \( \sqcup_{j=1}^{3} \eta_j(O_j) \) and \( G_{12}/\Gamma_{12} \leq \text{Aut}(X_{12}) \), it follows that the number of \( \text{Aut}(X_{12}) \)-orbits of vertices is 3.

Let \( X_{13} = K_{13}/\Gamma_{13} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{13} \leq \text{Aut}(K_{13}) \). Let the vertices of \( X_{13} \) form \( m_{13} \text{ Aut}(X_{13}) \)-orbits. We take the middle point of the line segment joining vertices \( a_0 \) and \( a_{14} \) as the origin.
Let \( \alpha_{13} := a_1 - a_0 \) and \( \beta_{13} := a_{24} - a_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_{13} := \langle x \mapsto x + \alpha_{13}, x \mapsto x + \beta_{13} \rangle \) and
\[
G_{13} = \{ \alpha : x \mapsto \varepsilon x + m_{13}\alpha_{13} + n_{13}\beta_{13} : \varepsilon = \pm 1, m_{13}, n_{13} \in \mathbb{Z} \} \cong H_{13} \times \mathbb{Z}_2.
\]
By the same arguments as above and in Claim 1, \( \Gamma_{13} \leq G_{13} \) and the number of \( G_{13}/\Gamma_{13} \)-orbits of vertices of \( X_{13} \) is three. Therefore, \( G_{13}/\Gamma_{13} \) acts on \( X_{13} = K_{13}/\Gamma_{13} \). Since \( O_1 = \langle a_0 \rangle \), \( O_2 = \langle b_3 \rangle \), \( O_3 = \langle c_2 \rangle \) are the \( G_{13} \)-orbits, it follows that \( O_1 = \langle a_0 \rangle \), \( O_2 = \langle b_3 \rangle \), \( O_3 = \langle c_2 \rangle \) are the \( (G_{13}/\Gamma_{13}) \)-orbits. Since the vertex set of \( X_{13} \) is \( \bigcup_{j=1}^{13} \eta_{13}(O_j) \) and \( G_{13}/\Gamma_{13} \leq \text{Aut}(X_{13}) \), it follows that the number of \( \text{Aut}(X_{13}) \)-orbits of vertices is 3.

Let \( X_{15} = K_{15}/\Gamma_{15} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{15} \leq \text{Aut}(K_{15}) \). Let the vertices of \( X_{15} \) form \( m_{15} \text{ Aut}(X_{15}) \)-orbits. We take the middle point of the line segment joining vertices \( c_0 \) and \( c_3 \) as the origin \((0,0)\) of \( K_{15} \) (see Section 4). Let \( \alpha_{15} := a_1 - a_0 \), \( \beta_{15} := a_2 - a_0 \) and \( \gamma_{15} := a_3 - a_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_{15} := \langle x \mapsto x + \alpha_{15}, x \mapsto x + \beta_{15}, x \mapsto x + \gamma_{15} \rangle \) and
\[
G_{15} = \{ \alpha : x \mapsto \varepsilon x + m_{15}\alpha_{15} + n_{15}\beta_{15} + r_{15}\gamma_{15} : \varepsilon = \pm 1, m_{15}, n_{15}, r_{15} \in \mathbb{Z} \} \cong H_{15} \times \mathbb{Z}_2.
\]
By the same arguments as above and in Claim 1, \( \Gamma_{15} \leq G_{15} \) and the number of \( G_{15}/\Gamma_{15} \)-orbits of vertices of \( X_{15} \) is ten. Therefore, \( G_{15}/\Gamma_{15} \) acts on \( X_{15} = K_{15}/\Gamma_{15} \). Since \( O_1 = \langle a_0 \rangle \), \( O_2 = \langle b_3 \rangle \), \( O_3 = \langle c_2 \rangle \) are the \( G_{15} \)-orbits, it follows that \( O_1 = \langle a_0 \rangle \), \( O_2 = \langle b_3 \rangle \), \( O_3 = \langle c_2 \rangle \) are the \( (G_{15}/\Gamma_{15}) \)-orbits. Since the vertex set of \( X_{15} \) is \( \bigcup_{j=1}^{16} \eta_{15}(O_j) \) and \( G_{15}/\Gamma_{15} \leq \text{Aut}(X_{15}) \), it follows that the number of \( \text{Aut}(X_{15}) \)-orbits of vertices is \( \leq 10 \).

Let \( X_{16} = K_{16}/\Gamma_{16} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{16} \leq \text{Aut}(K_{16}) \). Let the vertices of \( X_{16} \) form \( m_{16} \text{ Aut}(X_{16}) \)-orbits. We take the middle point of the line segment joining vertices \( b_0 \) and \( b_3 \) as the origin \((0,0)\) of \( K_{16} \) (see Section 4). Let \( \alpha_{16} := a_1 - a_0 \), \( \beta_{16} := a_2 - a_0 \) and \( \gamma_{16} := a_3 - a_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_{16} := \langle x \mapsto x + \alpha_{16}, x \mapsto x + \beta_{16}, x \mapsto x + \gamma_{16} \rangle \) and
\[
G_{16} = \{ \alpha : x \mapsto \varepsilon x + m_{16}\alpha_{16} + n_{16}\beta_{16} + r_{16}\gamma_{16} : \varepsilon = \pm 1, m_{16}, n_{16}, r_{16} \in \mathbb{Z} \} \cong H_{16} \times \mathbb{Z}_2.
\]
By the same arguments as above and in Claim 1, \( \Gamma_{16} \leq G_{16} \) and the number of \( G_{16}/\Gamma_{16} \)-orbits of vertices of \( X_{16} \) is eleven. Therefore, \( G_{16}/\Gamma_{16} \) acts on \( X_{16} = K_{16}/\Gamma_{16} \). Since \( O_1 = \langle a_0 \rangle \), \( O_2 = \langle b_0 \rangle \), \( O_3 = \langle b_1 \rangle \), \( O_4 = \langle b_2 \rangle \), \( O_5 = \langle b_3 \rangle \), \( O_6 = \langle c_0 \rangle \), \( O_7 = \langle c_1 \rangle \), \( O_8 = \langle c_2 \rangle \), \( O_9 = \langle c_3 \rangle \), \( O_{10} = \langle c_4 \rangle \), \( O_{11} = \langle c_5 \rangle \) are the \( G_{16} \)-orbits, it follows that \( O_1 = \langle a_0 \rangle \), \( O_2 = \langle b_0 \rangle \), \( O_3 = \langle b_1 \rangle \), \( O_4 = \langle b_2 \rangle \), \( O_5 = \langle b_3 \rangle \), \( O_6 = \langle c_0 \rangle \), \( O_7 = \langle c_1 \rangle \), \( O_8 = \langle c_2 \rangle \), \( O_9 = \langle c_3 \rangle \), \( O_{10} = \langle c_4 \rangle \), \( O_{11} = \langle c_5 \rangle \) are the \( (G_{16}/\Gamma_{16}) \)-orbits. Since the vertex set of \( X_{16} \) is \( \bigcup_{j=1}^{16} \eta_{16}(O_j) \) and \( G_{16}/\Gamma_{16} \leq \text{Aut}(X_{16}) \), it follows that the number of \( \text{Aut}(X_{16}) \)-orbits of vertices is \( \leq 11 \).

Let \( X_{17} = K_{17}/\Gamma_{17} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{17} \leq \text{Aut}(K_{17}) \). Let the vertices of \( X_{17} \) form \( m_{17} \text{ Aut}(X_{17}) \)-orbits. We take the middle point of the line segment joining vertices \( a_0 \) and \( a_3 \) as the origin \((0,0)\) of \( K_{17} \) (see Section 4). Let \( \alpha_{17} := a_6 - a_0 \), \( \beta_{17} := a_{15} - a_0 \) and \( \gamma_{17} := a_{21} - a_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_{17} := \langle x \mapsto x + \alpha_{17}, x \mapsto x + \beta_{17}, x \mapsto x + \gamma_{17} \rangle \) and
\[
G_{17} = \{ \alpha : x \mapsto \varepsilon x + m_{17}\alpha_{17} + n_{17}\beta_{17} + r_{17}\gamma_{17} : \varepsilon = \pm 1, m_{17}, n_{17}, r_{17} \in \mathbb{Z} \} \cong H_{17} \times \mathbb{Z}_2.
\]
By the same arguments as above and in Claim 1, \( \Gamma_{17} \leq G_{17} \) and the number of \( G_{17}/\Gamma_{17} \)-orbits of vertices of \( X_{17} \) is ten. Therefore, \( G_{17}/\Gamma_{17} \) acts on \( X_{17} = K_{17}/\Gamma_{17} \). Since \( O_1 = \langle a_0 \rangle \),
\[ O_2 = \langle a_1 \rangle, O_3 = \langle a_2 \rangle, O_4 = \langle b_0 \rangle, O_5 = \langle c_0 \rangle, O_6 = \langle c_1 \rangle, O_7 = \langle c_{15} \rangle, O_8 = \langle c_{16} \rangle, O_9 = \langle c_{17} \rangle, O_{10} = \langle c_{18} \rangle \text{ are the } G_{17}\text{-orbits, it follows that } O_1 = \langle a_0 \rangle, O_2 = \langle a_1 \rangle, O_3 = \langle a_2 \rangle, O_4 = \langle b_0 \rangle, O_5 = \langle c_0 \rangle, O_6 = \langle c_1 \rangle, O_7 = \langle c_{15} \rangle, O_8 = \langle c_{16} \rangle, O_9 = \langle c_{17} \rangle, O_{10} = \langle c_{18} \rangle \text{ are the } (G_{17}/\Gamma_{17})\text{-orbits.} \]

Since the vertex set of \( X_{17} \) is \( \sqcup_{j=1}^{10} \eta_j(\Omega_j) \) and \( G_{17}/\Gamma_{17} \leq \text{Aut}(X_{17}) \), it follows that the number of \( \text{Aut}(X_{17})\text{-orbits of vertices is } \leq 10. \]

Let \( X_{18} = K_{18}/\Gamma_{18} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{18} \leq \text{Aut}(K_{18}) \). Let the vertices of \( X_{18} \) form \( m_{18} \text{ Aut}(X_{18})\)-orbits. We take the middle point of the line segment joining vertices \( a_0 \) and \( c_2 \) as the origin \((0,0)\) of \( K_{18} \) (see Section [4]). Let \( \alpha_{18} := a_{11} - a_0, \beta_{18} := a_{18} - a_0 \) and \( \gamma_{18} := a_{31} - a_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_{18} := \langle x \mapsto x + \alpha_{18}, x \mapsto x + \beta_{18}, x \mapsto x + \gamma_{18} \rangle \) and

\[ G_{18} = \{ \alpha : x \mapsto \varepsilon x + m_{18}\alpha_{18} + n_{18}\beta_{18} + r_{18}\gamma_{18} : \varepsilon = \pm 1, m_{18}, n_{18}, r_{18} \in \mathbb{Z} \} \cong H_{18} \times \mathbb{Z}_2 \]

By the same arguments as above and in Claim 1, \( \Gamma_{18} \leq G_{18} \) and the number of \( G_{18}/\Gamma_{18}\text{-orbits of vertices of } X_{18} \text{ is } \leq 13. \]

Let \( X_{19} = K_{19}/\Gamma_{19} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{19} \leq \text{Aut}(K_{19}) \). Let the vertices of \( X_{18} \) form \( m_{19} \text{ Aut}(X_{19})\)-orbits. We take the point \( b_0 \) as the origin \((0,0)\) of \( K_{19} \) (see Section [4]). Let \( \alpha_{19} := b_1 - b_0, \beta_{19} := b_2 - b_0 \) and \( \gamma_{19} := b_3 - b_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_{19} := \langle x \mapsto x + \alpha_{19}, x \mapsto x + \beta_{19}, x \mapsto x + \gamma_{19} \rangle \) and

\[ G_{19} = \{ \alpha : x \mapsto \varepsilon x + m_{19}\alpha_{19} + n_{19}\beta_{19} + r_{19}\gamma_{19} : \varepsilon = \pm 1, m_{19}, n_{19}, r_{19} \in \mathbb{Z} \} \cong H_{19} \times \mathbb{Z}_2 \]

By the same arguments as above and in Claim 1, \( \Gamma_{19} \leq G_{19} \) and the number of \( G_{19}/\Gamma_{19}\text{-orbits of vertices of } X_{19} \text{ is } \leq 5. \]

Let \( X_{20} = K_{20}/\Gamma_{20} \) be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup \( \Gamma_{20} \leq \text{Aut}(K_{20}) \). Let the vertices of \( X_{20} \) form \( m_{20} \text{ Aut}(X_{20})\)-orbits. We take the middle point of the line segment joining vertices \( a_0 \) and \( a_3 \) as the origin \((0,0)\) of \( K_{20} \) (see Section [4]). Let \( \alpha_{20} := a_6 - a_0, \beta_{20} := a_{12} - a_0 \) and \( \gamma_{20} := a_{18} - a_0 \in \mathbb{R}^2 \). Similarly as above, define \( H_{20} := \langle x \mapsto x + \alpha_{20}, x \mapsto x + \beta_{20}, x \mapsto x + \gamma_{20} \rangle \) and

\[ G_{20} = \{ \alpha : x \mapsto \varepsilon x + m_{20}\alpha_{20} + n_{20}\beta_{20} + r_{20}\gamma_{20} : \varepsilon = \pm 1, m_{20}, n_{20}, r_{20} \in \mathbb{Z} \} \cong H_{20} \times \mathbb{Z}_2 \]

By the same arguments as above and in Claim 1, \( \Gamma_{20} \leq \text{Aut}(K_{20}) \) and the number of \( G_{20}/\Gamma_{20}\text{-orbits of vertices of } X_{20} \text{ is } \leq 9. \) Therefore, \( G_{20}/\Gamma_{20} \text{ acts on } X_{20} = K_{20}/\Gamma_{20}. \) Since \( O_1 = \langle a_0 \rangle, O_2 = \langle a_1 \rangle, O_3 = \langle a_2 \rangle, O_4 = \langle b_0 \rangle, O_5 = \langle b_1 \rangle, O_6 = \langle b_8 \rangle, O_7 = \langle b_3 \rangle, O_8 = \langle c_0 \rangle, O_9 = \langle c_1 \rangle, O_{10} = \langle c_{15} \rangle, O_{11} = \langle c_{16} \rangle, O_{12} = \langle c_{17} \rangle, O_{13} = \langle c_{18} \rangle \text{ are the } (G_{20}/\Gamma_{20})\text{-orbits.} \]

Since the vertex set of \( X_{20} \) is \( \sqcup_{j=1}^{9} \eta_j(\Omega_j) \) and \( G_{20}/\Gamma_{20} \leq \text{Aut}(X_{20}) \), it follows that the number of \( \text{Aut}(X_{20})\text{-orbits of vertices is } \leq 9. \)
Let $X_{21} = K_{21}/\Gamma_{21}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{21} \leq \text{Aut}(K_{21})$. Let the vertices of $X_{21}$ form $m_{21}$ Aut($X_{21}$)-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_1$ as the origin $(0, 0)$ of $K_{21}$ (see Section 4). Let $\alpha_{21} := a_2 - a_0$ and $\beta_{21} := a_8 - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{21} := \langle x \mapsto x + \alpha_{21}, x \mapsto x + \beta_{21} \rangle$ and

$$G_{21} = \{\alpha : x \mapsto \varepsilon x + m_{21}\alpha_{21} + n_{21}\beta_{21} : \varepsilon = \pm 1, m_{21}, n_{21} \in \mathbb{Z} \} \cong H_{21} \rtimes \mathbb{Z}_2.$$  

By the same arguments as above and in Claim 1, $\Gamma_{21} \leq G_{21}$ and the number of $G_{21}/\Gamma_{21}$-orbits of vertices of $X_{21}$ is four. Therefore, $G_{21}/\Gamma_{21}$ acts on $X_{21} = K_{21}/\Gamma_{21}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle c_0 \rangle$, $O_3 = \langle c_1 \rangle$, $O_4 = \langle b_0 \rangle$ are the $G_{21}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle c_0 \rangle$, $O_3 = \langle c_1 \rangle$, $O_4 = \langle b_0 \rangle$ are the $(G_{21}/\Gamma_{21})$-orbits. Since the vertex set of $X_{21}$ is $\sqcup_{j=1}^{4}\eta_{21}(O_j)$ and $G_{21}/\Gamma_{21} \leq \text{Aut}(X_{21})$, it follows that the number of $\text{Aut}(X_{21})$-orbits of vertices is $\leq 4$.

Let $X_{22} = K_{22}/\Gamma_{22}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{22} \leq \text{Aut}(K_{22})$. Let the vertices of $X_{22}$ form $m_{22}$ Aut($X_{22}$)-orbits. We take the middle point of the line segment joining vertices $a_0$ and $b_1$ as the origin $(0, 0)$ of $K_{22}$ (see Section 4). Let $\alpha_{22} := a_1 - a_0$ and $\beta_{22} := a_2 - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{22} := \langle x \mapsto x + \alpha_{22}, x \mapsto x + \beta_{22} \rangle$ and

$$G_{22} = \{\alpha : x \mapsto \varepsilon x + m_{22}\alpha_{22} + n_{22}\beta_{22} : \varepsilon = \pm 1, m_{22}, n_{22} \in \mathbb{Z} \} \cong H_{22} \rtimes \mathbb{Z}_2.$$  

By the same arguments as above and in Claim 1, $\Gamma_{22} \leq G_{22}$ and the number of $G_{22}/\Gamma_{22}$-orbits of vertices of $X_{22}$ is six. Therefore, $G_{22}/\Gamma_{22}$ acts on $X_{22} = K_{22}/\Gamma_{22}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle b_0 \rangle$, $O_3 = \langle b_1 \rangle$, $O_4 = \langle c_{10} \rangle$, $O_5 = \langle c_8 \rangle$, $O_6 = \langle a_{10} \rangle$ are the $G_{22}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle b_0 \rangle$, $O_3 = \langle b_1 \rangle$, $O_4 = \langle c_{10} \rangle$, $O_5 = \langle c_8 \rangle$, $O_6 = \langle a_{10} \rangle$ are the $(G_{22}/\Gamma_{22})$-orbits. Since the vertex set of $X_{22}$ is $\sqcup_{j=1}^{6}\eta_{22}(O_j)$ and $G_{22}/\Gamma_{22} \leq \text{Aut}(X_{22})$, it follows that the number of $\text{Aut}(X_{22})$-orbits of vertices is $\leq 6$.

Let $X_{23} = K_{23}/\Gamma_{23}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{23} \leq \text{Aut}(K_{23})$. Let the vertices of $X_{23}$ form $m_{23}$ Aut($X_{23}$)-orbits. We take the point $b_0$ as the origin $(0, 0)$ of $K_{23}$ (see Section 4). Let $\alpha_{23} := b_1 - b_0$ and $\beta_{23} := b_2 - b_0 \in \mathbb{R}^2$. Similarly as above, define $H_{23} := \langle x \mapsto x + \alpha_{23}, x \mapsto x + \beta_{23} \rangle$ and

$$G_{23} = \{\alpha : x \mapsto \varepsilon x + m_{23}\alpha_{23} + n_{23}\beta_{23} : \varepsilon = \pm 1, m_{23}, n_{23} \in \mathbb{Z} \} \cong H_{23} \rtimes \mathbb{Z}_2.$$  

By the same arguments as above and in Claim 1, $\Gamma_{23} \leq G_{23}$ and the number of $G_{23}/\Gamma_{23}$-orbits of vertices of $X_{23}$ is five. Therefore, $G_{23}/\Gamma_{23}$ acts on $X_{23} = K_{23}/\Gamma_{23}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle b_0 \rangle$, $O_4 = \langle c_{01} \rangle$, $O_5 = \langle c_1 \rangle$ are the $G_{23}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle b_0 \rangle$, $O_4 = \langle c_{01} \rangle$, $O_5 = \langle c_1 \rangle$ are the $(G_{23}/\Gamma_{23})$-orbits. Since the vertex set of $X_{23}$ is $\sqcup_{j=1}^{5}\eta_{23}(O_j)$ and $G_{23}/\Gamma_{23} \leq \text{Aut}(X_{23})$, it follows that the number of $\text{Aut}(X_{23})$-orbits of vertices is $\leq 5$.

Let $X_{24} = K_{24}/\Gamma_{24}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{24} \leq \text{Aut}(K_{24})$. Let the vertices of $X_{24}$ form $m_{24}$ Aut($X_{24}$)-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_6$ as the origin $(0, 0)$ of $K_{24}$ (see Section 4). Let $\alpha_{24} := a_{12} - a_0$, $\beta_{24} := a_{30} - a_0$ and $\gamma_{24} := a_{43} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{24} := \langle x \mapsto x + \alpha_{24}, x \mapsto x + \beta_{24}, x \mapsto x + \gamma_{24} \rangle$ and

$$G_{24} = \{\alpha : x \mapsto \varepsilon x + m_{24}\alpha_{24} + n_{24}\beta_{24} + r_{24}\gamma_{24} : \varepsilon = \pm 1, m_{24}, n_{24}, r_{24} \in \mathbb{Z} \} \cong H_{24} \rtimes \mathbb{Z}_2.$$  

By the same arguments as above and in Claim 1, $\Gamma_{24} \leq G_{24}$ and the number of $G_{24}/\Gamma_{24}$-orbits of vertices of $X_{24}$ is fifteen. Therefore, $G_{24}/\Gamma_{24}$ acts on $X_{24} = K_{24}/\Gamma_{24}$. Since $O_1 = \langle a_0 \rangle$,
orbits. We take the middle point of the line segment joining vertices
edge or face) free subgroup $\Gamma$ of vertices is
it follows that
is
is
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are the $(G_{24}/\Gamma_{24})$-orbits. Since the vertex set of $X_{24}$ is $\bigcup_{j=1}^{15} \eta_{24}(O_j)$ and $G_{24}/\Gamma_{24} \leq \text{Aut}(X_{24})$, it follows that the number of $\text{Aut}(X_{24})$-orbits of vertices is $\leq 15$.

Let $X_{25} = K_{25}/\Gamma_{25}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{25} \leq \text{Aut}(K_{25})$. Let the vertices of $X_{25}$ form $m_{25}$ $\text{Aut}(X_{25})$-orbits. We take the middle point of the line segment joining vertices $b_0$ and $b_1$ as the origin $(0,0)$ of $K_{25}$ (see Section 4). Let $\alpha_{25} := b_2 - b_0$ and $\beta_{25} := b_5 - b_0 \in \mathbb{R}^2$. Similarly as above, define $H_{25} := \langle x \mapsto x + \alpha_{25}, x \mapsto x + \beta_{25} \rangle$

$G_{25} = \{ \alpha : x \mapsto \varepsilon x + m_{25}\alpha_{25} + n_{25}\beta_{25} : \varepsilon = \pm 1, m_{25}, n_{25} \in \mathbb{Z} \} \cong H_{25} \rtimes \mathbb{Z}_2.$

By the same arguments as above and in Claim 1, $\Gamma_{26} \leq G_{26}$ and the number of $G_{26}/\Gamma_{26}$-orbits of vertices of $X_{26}$ is four. Therefore, $G_{26}/\Gamma_{26}$ acts on $X_{26} = K_{26}/\Gamma_{26}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle b_0 \rangle$, $O_4 = \langle c_0 \rangle$, $O_5 = \langle c_1 \rangle$ are the $G_{26}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle b_0 \rangle$, $O_4 = \langle c_0 \rangle$, $O_5 = \langle c_1 \rangle$ are the $(G_{26}/\Gamma_{26})$-orbits. Since the vertex set of $X_{26}$ is $\bigcup_{j=1}^{4} \eta_{26}(O_j)$ and $G_{26}/\Gamma_{26} \leq \text{Aut}(X_{26})$, it follows that the number of $\text{Aut}(X_{26})$-orbits of vertices is $\leq 4$.

Let $X_{27} = K_{27}/\Gamma_{27}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{27} \leq \text{Aut}(K_{27})$. Let the vertices of $X_{27}$ form $m_{27}$ $\text{Aut}(X_{27})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_2$ as the origin $(0,0)$ of $K_{27}$ (see Section 4). Let $\alpha_{27} := a_5 - a_0$ and $\beta_{27} := a_{15} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{27} := \langle x \mapsto x + \alpha_{27}, x \mapsto x + \beta_{27} \rangle$

$G_{27} = \{ \alpha : x \mapsto \varepsilon x + m_{27}\alpha_{27} + n_{27}\beta_{27} : \varepsilon = \pm 1, m_{27}, n_{27} \in \mathbb{Z} \} \cong H_{27} \rtimes \mathbb{Z}_2.$

By the same arguments as above and in Claim 1, $\Gamma_{28} \leq G_{28}$ and the number of $G_{28}/\Gamma_{28}$-orbits of vertices of $X_{28}$ is five. Therefore, $G_{28}/\Gamma_{28}$ acts on $X_{28} = K_{28}/\Gamma_{28}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle b_0 \rangle$, $O_4 = \langle c_0 \rangle$, $O_5 = \langle c_1 \rangle$ are the $G_{28}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle b_0 \rangle$, $O_4 = \langle c_0 \rangle$, $O_5 = \langle c_1 \rangle$ are the $(G_{28}/\Gamma_{28})$-orbits. Since the vertex set of $X_{28}$ is $\bigcup_{j=1}^{5} \eta_{28}(O_j)$ and $G_{28}/\Gamma_{28} \leq \text{Aut}(X_{28})$, it follows that the number of $\text{Aut}(X_{28})$-orbits of vertices is $\leq 5$.

Let $X_{29} = K_{29}/\Gamma_{29}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{29} \leq \text{Aut}(K_{29})$. Let the vertices of $X_{29}$ form $m_{29}$ $\text{Aut}(X_{29})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $c_1$ as the origin

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Let $\alpha_{29} := a_2 - a_0$ and $\beta_{29} := a_{13} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{29} := \langle x \mapsto x + \alpha_{29}, x \mapsto x + \beta_{29} \rangle$ and

$$G_{29} = \{ \alpha : x \mapsto \varepsilon x + m_{29} \alpha_{29} + n_{29} \beta_{29} : \varepsilon = \pm 1, m_{29}, n_{29} \in \mathbb{Z} \} \cong H_{29} \rtimes \mathbb{Z}_2.$$  

By the same arguments as above and in Claim 1, $\Gamma_{29} \leq G_{29}$ and the number of $G_{29}/\Gamma_{29}$-orbits of vertices of $X_{29}$ is nine. Therefore, $G_{29}/\Gamma_{29}$ acts on $X_{29} = K_{29}/\Gamma_{29}$. Since $O_1 = (a_0), O_2 = (a_1), O_3 = (b_1), O_4 = (b_7), O_5 = (b_8), O_6 = (c_0), O_7 = (c_1), O_8 = (c_{30}), O_9 = (c_{37})$ are the $G_{29}$-orbits, it follows that $O_1 = (a_0), O_2 = (a_1), O_3 = (b_1), O_4 = (b_7), O_5 = (b_8), O_6 = (c_0), O_7 = (c_1), O_8 = (c_{30}), O_9 = (c_{37})$ are the $(G_{29}/\Gamma_{29})$-orbits. Since the vertex set of $X_{29}$ is $\bigsqcup_{j=1}^9 \eta_{29}(O_j)$ and $G_{29}/\Gamma_{29} \leq \text{Aut}(X_{29})$, it follows that the number of $\text{Aut}(X_{29})$-orbits of vertices is $\leq 9$.

Let $X_{30} = K_{30}/\Gamma_{30}$ be a semi-equivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{30} \leq \text{Aut}(K_{30})$. Let the vertices of $X_{30}$ form $m_{30}$ $\text{Aut}(X_{30})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $b_1$ as the origin $(0,0)$ of $K_{30}$ (see Section 4). Let $\alpha_{30} := a_2 - a_0$ and $\beta_{30} := a_{38} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{30} := \langle x \mapsto x + \alpha_{30}, x \mapsto x + \beta_{30} \rangle$ and

$$G_{30} = \{ \alpha : x \mapsto \varepsilon x + m_{30} \alpha_{30} + n_{30} \beta_{30} : \varepsilon = \pm 1, m_{30}, n_{30} \in \mathbb{Z} \} \cong H_{30} \rtimes \mathbb{Z}_2.$$ 

By the same arguments as above and in Claim 1, $\Gamma_{30} \leq G_{30}$ and the number of $G_{30}/\Gamma_{30}$-orbits of vertices of $X_{30}$ is eight. Therefore, $G_{30}/\Gamma_{30}$ acts on $X_{30} = K_{30}/\Gamma_{30}$. Since $O_1 = (a_0), O_2 = (a_1), O_3 = (b_1), O_4 = (b_7), O_5 = (b_8), O_6 = (c_0), O_7 = (c_1), O_8 = (c_{14}), O_9 = (c_{14}), O_9 = (c_{14})$ are the $G_{30}$-orbits, it follows that $O_1 = (a_0), O_2 = (a_1), O_3 = (b_1), O_4 = (b_7), O_5 = (b_8), O_6 = (c_0), O_7 = (c_1), O_8 = (c_{14}), O_9 = (c_{14})$ are the $(G_{30}/\Gamma_{30})$-orbits. Since the vertex set of $X_{30}$ is $\bigsqcup_{j=1}^8 \eta_{30}(O_j)$ and $G_{30}/\Gamma_{30} \leq \text{Aut}(X_{30})$, it follows that the number of $\text{Aut}(X_{30})$-orbits of vertices is $\leq 8$.

Let $X_{31} = K_{31}/\Gamma_{31}$ be a semi-equivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{31} \leq \text{Aut}(K_{31})$. Let the vertices of $X_{31}$ form $m_{31}$ $\text{Aut}(X_{31})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $b_1$ as the origin $(0,0)$ of $K_{31}$ (see Section 4). Let $\alpha_{31} := a_1 - a_0$ and $\beta_{31} := a_2 - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{31} := \langle x \mapsto x + \alpha_{31}, x \mapsto x + \beta_{31} \rangle$ and

$$G_{31} = \{ \alpha : x \mapsto \varepsilon x + m_{31} \alpha_{31} + n_{31} \beta_{31} : \varepsilon = \pm 1, m_{31}, n_{31} \in \mathbb{Z} \} \cong H_{31} \rtimes \mathbb{Z}_2.$$ 

By the same arguments as above and in Claim 1, $\Gamma_{31} \leq G_{31}$ and the number of $G_{31}/\Gamma_{31}$-orbits of vertices of $X_{31}$ is six. Therefore, $G_{31}/\Gamma_{31}$ acts on $X_{31} = K_{31}/\Gamma_{31}$. Since $O_1 = (a_0), O_2 = (a_1), O_3 = (b_1), O_4 = (b_{14}), O_5 = (c_{14}), O_6 = (c_{14})$ are the $G_{31}$-orbits, it follows that $O_1 = (a_0), O_2 = (a_1), O_3 = (b_1), O_4 = (b_{14}), O_5 = (c_{14}), O_6 = (c_{14})$ are the $(G_{31}/\Gamma_{31})$-orbits. Since the vertex set of $X_{31}$ is $\bigsqcup_{j=1}^8 \eta_{31}(O_j)$ and $G_{31}/\Gamma_{31} \leq \text{Aut}(X_{31})$, it follows that the number of $\text{Aut}(X_{31})$-orbits of vertices is $\leq 6$.

Let $X_{32} = K_{32}/\Gamma_{32}$ be a semi-equivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{32} \leq \text{Aut}(K_{32})$. Let the vertices of $X_{32}$ form $m_{32}$ $\text{Aut}(X_{32})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_3$ as the origin $(0,0)$ of $K_{32}$ (see Section 4). Let $\alpha_{32} := a_6 - a_0, \beta_{32} := a_{20} - a_0$ and $\gamma_{32} := a_{27} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{32} := \langle x \mapsto x + \alpha_{32}, x \mapsto x + \beta_{32}, x \mapsto x + \gamma_{32} \rangle$ and

$$G_{32} = \{ \alpha : x \mapsto \varepsilon x + m_{32} \alpha_{32} + n_{32} \beta_{32} + r_{32} \gamma_{32} : \varepsilon = \pm 1, m_{32}, n_{32}, r_{32} \in \mathbb{Z} \} \cong H_{32} \rtimes \mathbb{Z}_2.$$ 

By the same arguments as above and in Claim 1, $\Gamma_{32} \leq G_{32}$ and the number of $G_{32}/\Gamma_{32}$-orbits of vertices of $X_{32}$ is twelve. Therefore, $G_{32}/\Gamma_{32}$ acts on $X_{32} = K_{32}/\Gamma_{32}$. Since $O_1 = (a_0)$,
orbits. We take the middle point of the line segment joining vertices

Let $X_{33} = K_{33}/\Gamma_{33}$ be a semi-equivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{33} \leq \text{Aut}(K_{33})$. Let the vertices of $X_{33}$ form $m_{33} \text{Aut}(X_{33})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_2$ as the origin $(0,0)$ of $K_{33}$ (see Section 4). Let $\alpha_{33} := a_6 - a_0$, $\beta_{33} := a_{12} - a_0$ and $\gamma_{33} := a_{18} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{33} := \langle x \mapsto x + \alpha_{33}, x \mapsto x + \beta_{33}, x \mapsto x + \gamma_{33} \rangle$ and

$$G_{33} = \{ \alpha : x \mapsto \varepsilon x + m_{33}\alpha_{33} + n_{33}\beta_{33} + r_{33}\gamma_{33} : \varepsilon = \pm 1, m_{33}, n_{33}, r_{33} \in \mathbb{Z} \} \cong H_{33} \rtimes \mathbb{Z}_2$$

By the same arguments as above and in Claim 1, $\Gamma_{33} \leq G_{33}$ and the number of $G_{33}/\Gamma_{33}$-orbits of vertices of $X_{33}$ is twelve. Therefore, $G_{33}/\Gamma_{33}$ acts on $X_{33} = K_{33}/\Gamma_{33}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$ $O_4 = \langle b_0 \rangle$, $O_5 = \langle c_0 \rangle$, $O_6 = \langle c_1 \rangle$, $O_7 = \langle c_2 \rangle$, $O_8 = \langle c_3 \rangle$, $O_9 = \langle c_4 \rangle$, $O_{10} = \langle c_5 \rangle$, $O_{11} = \langle c_6 \rangle$, $O_{12} = \langle c_7 \rangle$ are the $G_{33}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$ $O_4 = \langle b_0 \rangle$, $O_5 = \langle c_0 \rangle$, $O_6 = \langle c_1 \rangle$, $O_7 = \langle c_2 \rangle$, $O_8 = \langle c_3 \rangle$, $O_9 = \langle c_4 \rangle$, $O_{10} = \langle c_5 \rangle$, $O_{11} = \langle c_6 \rangle$, $O_{12} = \langle c_7 \rangle$ are the $G_{33}/\Gamma_{33}$-orbits. Since the vertex set of $X_{33}$ is $\sqcup_{j=1}^{12} \eta_{33}(O_j)$ and $G_{33}/\Gamma_{33} \leq \text{Aut}(X_{33})$, it follows that the number of $\text{Aut}(X_{33})$-orbits of vertices is $\leq 12$.

Let $X_{35} = K_{35}/\Gamma_{35}$ be a semi-equivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{35} \leq \text{Aut}(K_{35})$. Let the vertices of $X_{35}$ form $m_{35} \text{Aut}(X_{35})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_2$ as the origin $(0,0)$ of $K_{35}$ (see Section 4). Let $\alpha_{35} := a_{12} - a_0$ and $\beta_{35} := a_{43} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{35} := \langle x \mapsto x + \alpha_{35}, x \mapsto x + \beta_{35} \rangle$ and

$$G_{35} = \{ \alpha : x \mapsto \varepsilon x + m_{35}\alpha_{35} + n_{35}\beta_{35} \in \mathbb{Z} : \varepsilon = \pm 1, m_{35}, n_{35} \in \mathbb{Z} \} \cong H_{35} \rtimes \mathbb{Z}_2$$

By the same arguments as above and in Claim 1, $\Gamma_{35} \leq G_{35}$ and the number of $G_{35}/\Gamma_{35}$-orbits of vertices of $X_{35}$ is four. Therefore, $G_{35}/\Gamma_{35}$ acts on $X_{35} = K_{35}/\Gamma_{35}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle c_1 \rangle$ $O_4 = \langle b_0 \rangle$, $O_5 = \langle b_1 \rangle$ are the $G_{35}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle c_1 \rangle$ $O_4 = \langle b_0 \rangle$, $O_5 = \langle b_1 \rangle$ are the $G_{35}/\Gamma_{35}$-orbits. Since the vertex set of $X_{35}$ is $\sqcup_{j=1}^{4} \eta_{35}(O_j)$ and $G_{35}/\Gamma_{35} \leq \text{Aut}(X_{35})$, it follows that the number of $\text{Aut}(X_{35})$-orbits of vertices is $\leq 4$.

Let $X_{36} = K_{36}/\Gamma_{36}$ be a semi-equivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{36} \leq \text{Aut}(K_{36})$. Let the vertices of $X_{36}$ form $m_{36} \text{Aut}(X_{36})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_2$ as the origin $(0,0)$ of $K_{36}$ (see Section 4). Let $\alpha_{36} := a_{11} - a_0$ and $\beta_{36} := a_{35} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{36} := \langle x \mapsto x + \alpha_{36}, x \mapsto x + \beta_{36} \rangle$ and

$$G_{36} = \{ \alpha : x \mapsto \varepsilon x + m_{36}\alpha_{36} + n_{36}\beta_{36} \in \mathbb{Z} : \varepsilon = \pm 1, m_{36}, n_{36} \in \mathbb{Z} \} \cong H_{36} \rtimes \mathbb{Z}_2$$

By the same arguments as above and in Claim 1, $\Gamma_{36} \leq G_{36}$ and the number of $G_{36}/\Gamma_{36}$-orbits of vertices of $X_{36}$ is five. Therefore, $G_{36}/\Gamma_{36}$ acts on $X_{36} = K_{36}/\Gamma_{36}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle c_1 \rangle$ $O_4 = \langle b_0 \rangle$, $O_5 = \langle b_1 \rangle$ are the $G_{36}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle c_1 \rangle$ $O_4 = \langle b_0 \rangle$, $O_5 = \langle b_1 \rangle$ are the $G_{36}/\Gamma_{36}$-orbits. Since the vertex set of $X_{36}$ is $\sqcup_{j=1}^{5} \eta_{36}(O_j)$ and $G_{36}/\Gamma_{36} \leq \text{Aut}(X_{36})$, it follows that the number of $\text{Aut}(X_{36})$-orbits of vertices is $\leq 5$.

Let $X_{37} = K_{37}/\Gamma_{37}$ be a semi-equivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{37} \leq \text{Aut}(K_{37})$. Let the vertices of $X_{37}$ form $m_{37} \text{Aut}(X_{37})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_2$ as the origin.
(0,0) of $K_{37}$ (see Section 4). Let $\alpha_{37} := a_{12} - a_0$ and $\beta_{37} := a_{42} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{37} := \langle x \mapsto x + \alpha_{37}, x \mapsto x + \beta_{37} \rangle$ and

$$G_{37} = \{ \alpha : x \mapsto \varepsilon x + m_{37}\alpha_{37} + n_{37}\beta_{37} : \varepsilon = \pm 1, m_{37}, n_{37} \in \mathbb{Z} \} \cong H_{37} \rtimes \mathbb{Z}_2.$$ 

By the same arguments as above and in Claim 1, $\Gamma_{37} \leq G_{37}$ and the number of $G_{37}/\Gamma_{37}$-orbits of vertices of $X_{37}$ is five. Therefore, $G_{37}/\Gamma_{37}$ acts on $X_{37} = K_{37}/\Gamma_{37}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle c_1 \rangle$, $O_4 = \langle b_0 \rangle$, $O_5 = \langle b_1 \rangle$ are the $G_{37}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle c_1 \rangle$, $O_4 = \langle b_0 \rangle$, $O_5 = \langle b_1 \rangle$ are the $(G_{37}/\Gamma_{37})$-orbits. Since the vertex set of $X_{37}$ is $\sqcup_{j=1}^5 \eta_{37}(O_j)$ and $G_{37}/\Gamma_{37} \leq \text{Aut}(X_{37})$, it follows that the number of $\text{Aut}(X_{37})$-orbits of vertices is $\leq 5$.

Let $X_{38} = K_{38}/\Gamma_{38}$ be a semi-equivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{38} \leq \text{Aut}(K_{38})$. Let the vertices of $X_{38}$ form $m_{38}$ $\text{Aut}(X_{38})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_2$ as the origin $(0,0)$ of $K_{38}$ (see Section 4). Let $\alpha_{38} := a_{10} - a_0$ and $\beta_{38} := a_{35} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{38} := \langle x \mapsto x + \alpha_{38}, x \mapsto x + \beta_{38} \rangle$ and

$$G_{38} = \{ \alpha : x \mapsto \varepsilon x + m_{38}\alpha_{38} + n_{38}\beta_{38} : \varepsilon = \pm 1, m_{38}, n_{38} \in \mathbb{Z} \} \cong H_{38} \rtimes \mathbb{Z}_2.$$ 

By the same arguments as above and in Claim 1, $\Gamma_{38} \leq G_{38}$ and the number of $G_{38}/\Gamma_{38}$-orbits of vertices of $X_{38}$ is five. Therefore, $G_{38}/\Gamma_{38}$ acts on $X_{38} = K_{38}/\Gamma_{38}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle b_0 \rangle$, $O_3 = \langle b_23 \rangle$, $O_4 = \langle a_1 \rangle$, $O_5 = \langle c_1 \rangle$ are the $G_{38}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle b_0 \rangle$, $O_3 = \langle b_23 \rangle$, $O_4 = \langle a_1 \rangle$, $O_5 = \langle c_1 \rangle$ are the $(G_{38}/\Gamma_{38})$-orbits. Since the vertex set of $X_{38}$ is $\sqcup_{j=1}^5 \eta_{38}(O_j)$ and $G_{38}/\Gamma_{38} \leq \text{Aut}(X_{38})$, it follows that the number of $\text{Aut}(X_{38})$-orbits of vertices is $\leq 5$.

Let $X_{39} = K_{39}/\Gamma_{39}$ be a semi-equivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{39} \leq \text{Aut}(K_{39})$. Let the vertices of $X_{39}$ form $m_{39}$ $\text{Aut}(X_{39})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_6$ as the origin $(0,0)$ of $K_{39}$ (see Section 4). Let $\alpha_{39} := a_{12} - a_0$, $\beta_{39} := a_{36} - a_0$ and $\gamma_{39} := a_{52} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{39} := \langle x \mapsto x + \alpha_{39}, x \mapsto x + \beta_{39}, x \mapsto x + \gamma_{39} \rangle$ and

$$G_{39} = \{ \alpha : x \mapsto \varepsilon x + m_{39}\alpha_{39} + n_{39}\beta_{39} + r_{39}\gamma_{39} : \varepsilon = \pm 1, m_{39}, n_{39}, r_{39} \in \mathbb{Z} \} \cong H_{39} \rtimes \mathbb{Z}_2.$$ 

By the same arguments as above and in Claim 1, $\Gamma_{39} \leq G_{39}$ and the number of $G_{39}/\Gamma_{39}$-orbits of vertices of $X_{39}$ is fifteen. Therefore, $G_{39}/\Gamma_{39}$ acts on $X_{39} = K_{39}/\Gamma_{39}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle a_3 \rangle$, $O_5 = \langle a_4 \rangle$, $O_6 = \langle a_5 \rangle$, $O_7 = \langle c_0 \rangle$, $O_8 = \langle b_0 \rangle$, $O_9 = \langle b_1 \rangle$, $O_{10} = \langle c_4 \rangle$, $O_{11} = \langle c_5 \rangle$, $O_{12} = \langle b_5 \rangle$, $O_{13} = \langle c_8 \rangle$, $O_{14} = \langle c_11 \rangle$, $O_{15} = \langle c_12 \rangle$ are the $G_{32}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle a_3 \rangle$, $O_5 = \langle a_4 \rangle$, $O_6 = \langle a_5 \rangle$, $O_7 = \langle c_0 \rangle$, $O_8 = \langle b_0 \rangle$, $O_9 = \langle b_1 \rangle$, $O_{10} = \langle c_4 \rangle$, $O_{11} = \langle c_5 \rangle$, $O_{12} = \langle b_5 \rangle$, $O_{13} = \langle c_8 \rangle$, $O_{14} = \langle c_11 \rangle$, $O_{15} = \langle c_12 \rangle$ are the $(G_{39}/\Gamma_{39})$-orbits. Since the vertex set of $X_{39}$ is $\sqcup_{j=1}^{15} \eta_{39}(O_j)$ and $G_{39}/\Gamma_{39} \leq \text{Aut}(X_{39})$, it follows that the number of $\text{Aut}(X_{39})$-orbits of vertices is $\leq 15$.

Let $X_{40} = K_{40}/\Gamma_{40}$ be a semi-equivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{40} \leq \text{Aut}(K_{40})$. Let the vertices of $X_{40}$ form $m_{40}$ $\text{Aut}(X_{40})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_3$ as the origin $(0,0)$ of $K_{40}$ (see Section 4). Let $\alpha_{40} := a_{17} - a_0$ and $\beta_{40} := a_{15} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{40} := \langle x \mapsto x + \alpha_{40}, x \mapsto x + \beta_{40} \rangle$ and

$$G_{40} = \{ \alpha : x \mapsto \varepsilon x + m_{40}\alpha_{40} + n_{40}\beta_{40} : \varepsilon = \pm 1, m_{40}, n_{40} \in \mathbb{Z} \} \cong H_{40} \rtimes \mathbb{Z}_2.$$
By the same arguments as above and in Claim 1, $\Gamma_{40} \leq G_{40}$ and the number of $G_{40}/\Gamma_{40}$-orbits of vertices of $X_{40}$ is seven. Therefore, $G_{40}/\Gamma_{40}$ acts on $X_{40} = K_{40}/\Gamma_{40}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle a_{b_1} \rangle$, $O_5 = \langle a_{b_2} \rangle$, $O_6 = \langle a_{b_3} \rangle$, $O_7 = \langle a_{b_4} \rangle$ are the $G_{40}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle a_{b_1} \rangle$, $O_5 = \langle a_{b_2} \rangle$, $O_6 = \langle a_{b_3} \rangle$, $O_7 = \langle a_{b_4} \rangle$ are the $(G_{40}/\Gamma_{40})$-orbits. Since the vertex set of $X_{40}$ is $\bigcup_{j=1}^{7} \eta_{40}(O_j)$ and $G_{40}/\Gamma_{40} \leq \text{Aut}(X_{40})$, it follows that the number of $\text{Aut}(X_{40})$-orbits of vertices is $\leq 7$.

Let $X_{41} = K_{41}/\Gamma_{41}$ be a semiequivalent map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{41} \leq \text{Aut}(K_{41})$. Let the vertices of $X_{41}$ form $m_{41} \text{ Aut}(X_{41})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_3$ as the origin $(0,0)$ of $K_{41}$ (see Section 4). Let $\alpha_{41} := a_{11} - a_0$, $\beta_{41} := a_{15} - a_0$ and $\gamma_{41} := a_{21} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{41} := \langle x \mapsto x + \alpha_{41}, x \mapsto x + \beta_{41}, x \mapsto x + \gamma_{41} \rangle$ and

$$G_{41} = \{ \alpha : x \mapsto \varepsilon x + m_{41}\alpha_{41} + n_{41}\beta_{41} + r_{41}\gamma_{41} : \varepsilon = \pm 1, m_{41}, n_{41}, r_{41} \in \mathbb{Z} \} \cong H_{41} \times \mathbb{Z}_2$$

By the same arguments as above and in Claim 1, $\Gamma_{41} \leq G_{41}$ and the number of $G_{41}/\Gamma_{41}$-orbits of vertices of $X_{41}$ is nine. Therefore, $G_{41}/\Gamma_{41}$ acts on $X_{41} = K_{41}/\Gamma_{41}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle b_{18} \rangle$, $O_5 = \langle b_{0} \rangle$, $O_6 = \langle b_{1} \rangle$, $O_7 = \langle b_{2} \rangle$, $O_8 = \langle b_{19} \rangle$, $O_9 = \langle b_{24} \rangle$, $O_0 = \langle b_{20} \rangle$ are the $G_{41}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle b_{18} \rangle$, $O_5 = \langle b_{0} \rangle$, $O_6 = \langle b_{1} \rangle$, $O_7 = \langle b_{2} \rangle$, $O_8 = \langle b_{19} \rangle$, $O_9 = \langle b_{24} \rangle$ are the $(G_{41}/\Gamma_{41})$-orbits. Since the vertex set of $X_{41}$ is $\bigcup_{j=1}^{9} \eta_{41}(O_j)$ and $G_{41}/\Gamma_{41} \leq \text{Aut}(X_{41})$, it follows that the number of $\text{Aut}(X_{41})$-orbits of vertices is $\leq 9$.

Let $X_{42} = K_{42}/\Gamma_{42}$ be a semiequivalent map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{42} \leq \text{Aut}(K_{42})$. Let the vertices of $X_{42}$ form $m_{42} \text{ Aut}(X_{42})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_3$ as the origin $(0,0)$ of $K_{42}$ (see Section 4). Let $\alpha_{42} := a_{11} - a_0$, $\beta_{42} := a_{15} - a_0$ and $\gamma_{42} := a_{21} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{42} := \langle x \mapsto x + \alpha_{42}, x \mapsto x + \beta_{42}, x \mapsto x + \gamma_{42} \rangle$ and

$$G_{42} = \{ \alpha : x \mapsto \varepsilon x + m_{42}\alpha_{42} + n_{42}\beta_{42} + r_{42}\gamma_{42} : \varepsilon = \pm 1, m_{42}, n_{42}, r_{42} \in \mathbb{Z} \} \cong H_{42} \times \mathbb{Z}_2$$

By the same arguments as above and in Claim 1, $\Gamma_{42} \leq G_{42}$ and the number of $G_{42}/\Gamma_{42}$-orbits of vertices of $X_{42}$ is nine. Therefore, $G_{42}/\Gamma_{42}$ acts on $X_{42} = K_{42}/\Gamma_{42}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle b_{24} \rangle$, $O_5 = \langle b_{25} \rangle$, $O_6 = \langle b_{14} \rangle$, $O_7 = \langle b_{5} \rangle$, $O_8 = \langle b_{17} \rangle$, $O_9 = \langle b_{24} \rangle$, $O_0 = \langle b_{20} \rangle$ are the $G_{42}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle b_{24} \rangle$, $O_5 = \langle b_{25} \rangle$, $O_6 = \langle b_{14} \rangle$, $O_7 = \langle b_{5} \rangle$, $O_8 = \langle b_{17} \rangle$, $O_9 = \langle b_{24} \rangle$ are the $(G_{42}/\Gamma_{42})$-orbits. Since the vertex set of $X_{42}$ is $\bigcup_{j=1}^{9} \eta_{42}(O_j)$ and $G_{42}/\Gamma_{42} \leq \text{Aut}(X_{42})$, it follows that the number of $\text{Aut}(X_{42})$-orbits of vertices is $\leq 9$.

Let $X_{43} = K_{43}/\Gamma_{43}$ be a semiequivalent map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{43} \leq \text{Aut}(K_{43})$. Let the vertices of $X_{43}$ form $m_{43} \text{ Aut}(X_{43})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_3$ as the origin $(0,0)$ of $K_{43}$ (see Section 4). Let $\alpha_{43} := a_{7} - a_0$ and $\beta_{43} := a_{13} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{43} := \langle x \mapsto x + \alpha_{43}, x \mapsto x + \beta_{43} \rangle$ and

$$G_{43} = \{ \alpha : x \mapsto \varepsilon x + m_{43}\alpha_{43} + n_{43}\beta_{43} : \varepsilon = \pm 1, m_{43}, n_{43} \in \mathbb{Z} \} \cong H_{43} \times \mathbb{Z}_2$$

By the same arguments as above and in Claim 1, $\Gamma_{43} \leq G_{43}$ and the number of $G_{43}/\Gamma_{43}$-orbits of vertices of $X_{43}$ is seven. Therefore, $G_{43}/\Gamma_{43}$ acts on $X_{43} = K_{43}/\Gamma_{43}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle a_{24} \rangle$, $O_5 = \langle a_{25} \rangle$, $O_6 = \langle a_{26} \rangle$, $O_7 = \langle b_0 \rangle$ are the $G_{43}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle a_{24} \rangle$, $O_5 = \langle a_{25} \rangle$, $O_6 = \langle a_{26} \rangle$, $O_7 = \langle b_0 \rangle$ are the $(G_{43}/\Gamma_{43})$-orbits. Since the vertex set of $X_{43}$ is $\bigcup_{j=1}^{7} \eta_{43}(O_j)$ and $G_{43}/\Gamma_{43} \leq \text{Aut}(X_{43})$, it follows that the number of $\text{Aut}(X_{43})$-orbits of vertices is $\leq 7$. 

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Let $X_{44} = K_{44}/\Gamma_{44}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{44} \leq \text{Aut}(K_{44})$. Let the vertices of $X_{44}$ form $m_{44}$ Aut($X_{44}$)-orbits. We take the middle point of the line segment joining vertices $0$ and $b_{18}$ as the origin $(0, 0)$ of $K_{44}$ (see Section 4). Let $a_{44} := b_1 - b_0$ and $\beta_{44} := b_{30} - b_0 \in \mathbb{R}^2$. Similarly as above, define $H_{44} := \langle x \mapsto x + \alpha_{44}, x \mapsto x + \beta_{44} \rangle$ and

$$G_{44} = \{ \alpha : x \mapsto \varepsilon x + m_{44}\alpha_{44} + n_{44}\beta_{44} : \varepsilon = \pm 1, m_{44}, n_{44} \in \mathbb{Z} \} \cong H_{44} \rtimes \mathbb{Z}_2.$$ 

By the same arguments as above and in Claim 1, $\Gamma_{44} \leq G_{44}$ and the number of $G_{44}/\Gamma_{44}$-orbits of vertices of $X_{44}$ is three. Therefore, $G_{44}/\Gamma_{44}$ acts on $X_{44} = K_{44}/\Gamma_{44}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_{18} \rangle$, $O_3 = \langle b_0 \rangle$ are the $G_{44}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_{18} \rangle$, $O_3 = \langle b_0 \rangle$ are the $(G_{44}/\Gamma_{44})$-orbits. Since the vertex set of $X_{44}$ is $\sqcup_{j=1}^{3} \eta_{44}(O_j)$ and $G_{44}/\Gamma_{44} \leq \text{Aut}(X_{44})$, it follows that the number of Aut($X_{44}$)-orbits of vertices is 3.

Let $X_{45} = K_{45}/\Gamma_{45}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{45} \leq \text{Aut}(K_{45})$. Let the vertices of $X_{45}$ form $m_{45}$ Aut($X_{45}$)-orbits. We take the middle point of the line segment joining vertices $b_0$ and $b_{11}$ as the origin $(0, 0)$ of $K_{45}$ (see Section 4). Let $a_{45} := b_1 - b_0$ and $\beta_{45} := b_{28} - b_0 \in \mathbb{R}^2$. Similarly as above, define $H_{45} := \langle x \mapsto x + \alpha_{45}, x \mapsto x + \beta_{45} \rangle$ and

$$G_{45} = \{ \alpha : x \mapsto \varepsilon x + m_{45}\alpha_{45} + n_{45}\beta_{45} : \varepsilon = \pm 1, m_{45}, n_{45} \in \mathbb{Z} \} \cong H_{45} \rtimes \mathbb{Z}_2.$$ 

By the same arguments as above and in Claim 1, $\Gamma_{45} \leq G_{45}$ and the number of $G_{45}/\Gamma_{45}$-orbits of vertices of $X_{45}$ is four. Therefore, $G_{45}/\Gamma_{45}$ acts on $X_{45} = K_{45}/\Gamma_{45}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_{11} \rangle$, $O_3 = \langle b_{21} \rangle$, $O_4 = \langle b_0 \rangle$ are the $G_{45}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_{11} \rangle$, $O_3 = \langle b_{21} \rangle$, $O_4 = \langle b_0 \rangle$ are the $(G_{45}/\Gamma_{45})$-orbits. Since the vertex set of $X_{45}$ is $\sqcup_{j=1}^{4} \eta_{45}(O_j)$ and $G_{45}/\Gamma_{45} \leq \text{Aut}(X_{45})$, it follows that the number of Aut($X_{45}$)-orbits of vertices is $\leq 4$.

Let $X_{46} = K_{46}/\Gamma_{46}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{46} \leq \text{Aut}(K_{46})$. Let the vertices of $X_{46}$ form $m_{46}$ Aut($X_{46}$)-orbits. We take the middle point of the line segment joining vertices $b_0$ and $b_{14}$ as the origin $(0, 0)$ of $K_{46}$ (see Section 4). Let $a_{46} := b_1 - b_0$ and $\beta_{46} := b_{85} - b_0 \in \mathbb{R}^2$. Similarly as above, define $H_{46} := \langle x \mapsto x + \alpha_{46}, x \mapsto x + \beta_{46} \rangle$ and

$$G_{46} = \{ \alpha : x \mapsto \varepsilon x + m_{46}\alpha_{46} + n_{46}\beta_{46} : \varepsilon = \pm 1, m_{46}, n_{46} \in \mathbb{Z} \} \cong H_{46} \rtimes \mathbb{Z}_2.$$ 

By the same arguments as above and in Claim 1, $\Gamma_{46} \leq G_{46}$ and the number of $G_{46}/\Gamma_{46}$-orbits of vertices of $X_{46}$ is four. Therefore, $G_{46}/\Gamma_{46}$ acts on $X_{46} = K_{46}/\Gamma_{46}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle b_0 \rangle$, $O_3 = \langle b_{60} \rangle$, $O_4 = \langle b_{75} \rangle$ are the $G_{46}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle b_0 \rangle$, $O_3 = \langle b_{60} \rangle$, $O_4 = \langle b_{75} \rangle$ are the $(G_{46}/\Gamma_{46})$-orbits. Since the vertex set of $X_{46}$ is $\sqcup_{j=1}^{4} \eta_{46}(O_j)$ and $G_{46}/\Gamma_{46} \leq \text{Aut}(X_{46})$, it follows that the number of Aut($X_{46}$)-orbits of vertices is $\leq 4$.

Let $X_{47} = K_{47}/\Gamma_{47}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{47} \leq \text{Aut}(K_{47})$. Let the vertices of $X_{47}$ form $m_{47}$ Aut($X_{47}$)-orbits. We take the middle point of the line segment joining vertices $b_0$ and $b_{16}$ as the origin $(0, 0)$ of $K_{47}$ (see Section 4). Let $a_{47} := b_1 - b_0$ and $\beta_{47} := b_{51} - b_0 \in \mathbb{R}^2$. Similarly as above, define $H_{47} := \langle x \mapsto x + \alpha_{47}, x \mapsto x + \beta_{47} \rangle$ and

$$G_{47} = \{ \alpha : x \mapsto \varepsilon x + m_{47}\alpha_{47} + n_{47}\beta_{47} : \varepsilon = \pm 1, m_{47}, n_{47} \in \mathbb{Z} \} \cong H_{47} \rtimes \mathbb{Z}_2.$$ 

By the same arguments as above and in Claim 1, $\Gamma_{47} \leq G_{47}$ and the number of $G_{47}/\Gamma_{47}$-orbits of vertices of $X_{47}$ is five. Therefore, $G_{47}/\Gamma_{47}$ acts on $X_{47} = K_{47}/\Gamma_{47}$. Since $O_1 = \langle b_0 \rangle$,
orbits. We take the middle point of the line segment joining vertices $a_1$ of vertices is $≤$ edge or face free subgroup $Γ_{X}O$.

By the same arguments as above and in Claim 1, $Γ_{X}O$ acts on $X_{48} = K_{48}/Γ_{48}$ is seven. Therefore, $G_{48}/Γ_{48}$ acts on $X_{48} = K_{48}/Γ_{48}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle b_1 \rangle$, $O_4 = \langle a_2 \rangle$, $O_5 = \langle a_0 \rangle$, $O_6 = \langle a_1 \rangle$, $O_7 = \langle b_0 \rangle$ are the $G_{48}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle b_1 \rangle$, $O_4 = \langle a_2 \rangle$, $O_5 = \langle a_0 \rangle$, $O_6 = \langle a_1 \rangle$, $O_7 = \langle b_0 \rangle$ are the $(G_{48}/Γ_{48})$-orbits. Since the vertex set of $X_{48}$ is $\sqcup_{j=1}^{7}η_{48}(O_j)$ and $G_{48}/Γ_{48} ≤ \text{Aut}(X_{48})$, it follows that the number of $\text{Aut}(X_{48})$-orbits of vertices is $≤ 7$.

Let $X_{49} = K_{49}/Γ_{49}$ be a semi-equivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $Γ_{49}$ is three. Therefore, $G_{49}/Γ_{49}$ acts on $X_{49} = K_{49}/Γ_{49}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle b_0 \rangle$ are the $G_{49}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle b_0 \rangle$ are the $(G_{49}/Γ_{49})$-orbits. Since the vertex set of $X_{49}$ is $\sqcup_{j=1}^{3}η_{49}(O_j)$ and $G_{49}/Γ_{49} ≤ \text{Aut}(X_{49})$, it follows that the number of $\text{Aut}(X_{49})$-orbits of vertices is $3$.

Let $X_{50} = K_{50}/Γ_{50}$ be a semi-equivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $Γ_{50}$ is three. Therefore, $G_{50}/Γ_{50}$ acts on $X_{50} = K_{50}/Γ_{50}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle b_0 \rangle$ are the $G_{50}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle b_0 \rangle$ are the $(G_{50}/Γ_{50})$-orbits. Since the vertex set of $X_{50}$ is $\sqcup_{j=1}^{3}η_{50}(O_j)$ and $G_{50}/Γ_{50} ≤ \text{Aut}(X_{50})$, it follows that the number of $\text{Aut}(X_{50})$-orbits of vertices is $3$.

Let $X_{51} = K_{51}/Γ_{51}$ be a semi-equivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $Γ_{51}$ is three. Therefore, $G_{51}/Γ_{51}$ acts on $X_{51} = K_{51}/Γ_{51}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle b_0 \rangle$ are the $G_{51}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle b_0 \rangle$ are the $(G_{51}/Γ_{51})$-orbits. Since the vertex set of $X_{51}$ is $\sqcup_{j=1}^{3}η_{51}(O_j)$ and $G_{51}/Γ_{51} ≤ \text{Aut}(X_{51})$, it follows that the number of $\text{Aut}(X_{51})$-orbits of vertices is $3$.
Let $\alpha_{51} := a_{14} - a_0$ and $\beta_{51} := a_7 - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{51} := \langle x \mapsto x + \alpha_{51}, x \mapsto x + \beta_{51} \rangle$ and

$$G_{51} = \{ \alpha : x \mapsto \varepsilon x + m_{51}\alpha_{51} + n_{51}\beta_{51} : \varepsilon = \pm 1, m_{51}, n_{51} \in \mathbb{Z} \} \cong H_{51} \rtimes \mathbb{Z}_2.$$

By the same arguments as above and in Claim 1, $\Gamma_{51} \leq G_{51}$ and the number of $G_{51}/\Gamma_{51}$-orbits of vertices of $X_{51}$ is three. Therefore, $G_{51}/\Gamma_{51}$ acts on $X_{51} = K_{51}/\Gamma_{51}$. Since $O_1 = \langle a_1 \rangle$, $O_2 = \langle a_2 \rangle$, $O_3 = \langle b_2 \rangle$, $O_4 = \langle b_2 \rangle$, $O_5 = \langle a_3 \rangle$, $O_6 = \langle a_0 \rangle$, $O_7 = \langle b_0 \rangle$, $O_8 = \langle b_1 \rangle$, $O_9 = \langle b_2 \rangle$, $O_{10} = \langle a_8 \rangle$, $O_{11} = \langle a_{10} \rangle$, $O_{12} = \langle b_3 \rangle$, $O_{13} = \langle b_4 \rangle$, $O_{14} = \langle a_{11} \rangle$, $O_{15} = \langle a_9 \rangle$ are the $G_{50}$-orbits, it follows that $O_1 = \langle a_1 \rangle$, $O_2 = \langle a_2 \rangle$, $O_3 = \langle b_2 \rangle$, $O_4 = \langle b_2 \rangle$, $O_5 = \langle a_3 \rangle$, $O_6 = \langle a_0 \rangle$, $O_7 = \langle b_0 \rangle$, $O_8 = \langle b_1 \rangle$, $O_9 = \langle b_2 \rangle$, $O_{10} = \langle a_8 \rangle$, $O_{11} = \langle a_{10} \rangle$, $O_{12} = \langle b_3 \rangle$, $O_{13} = \langle b_4 \rangle$, $O_{14} = \langle a_{11} \rangle$, $O_{15} = \langle a_9 \rangle$ are the $(G_{51}/\Gamma_{51})$-orbits. Since the vertex set of $X_{51}$ is $\sqcup_{j=1}^{15} \eta_{51}(O_j)$ and $G_{51}/\Gamma_{51} \leq \text{Aut}(X_{51})$, it follows that the number of $\text{Aut}(X_{51})$-orbits of vertices is $\leq 15$.

Let $X_{52} = K_{52}/\Gamma_{52}$ be a semi-equivaler map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{52} \leq \text{Aut}(K_{52})$. Let the vertices of $X_{52}$ form $m_{52} \text{ Aut}(X_{52})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $b_0$ as the origin $(0,0)$ of $K_{52}$ (see Section 4). Let $\alpha_{52} := a_3 - a_0$ and $\beta_{52} := a_5 - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{52} := \langle x \mapsto x + \alpha_{52}, x \mapsto x + \beta_{52} \rangle$ and

$$G_{52} = \{ \alpha : x \mapsto \varepsilon x + m_{52}\alpha_{52} + n_{52}\beta_{52} : \varepsilon = \pm 1, m_{52}, n_{52} \in \mathbb{Z} \} \cong H_{52} \rtimes \mathbb{Z}_2.$$

By the same arguments as above and in Claim 1, $\Gamma_{52} \leq G_{52}$ and the number of $G_{52}/\Gamma_{52}$-orbits of vertices of $X_{52}$ is six. Therefore, $G_{52}/\Gamma_{52}$ acts on $X_{52} = K_{52}/\Gamma_{52}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle b_1 \rangle$, $O_3 = \langle b_0 \rangle$, $O_4 = \langle b_1 \rangle$, $O_5 = \langle a_3 \rangle$, $O_6 = \langle b_2 \rangle$, $O_7 = \langle b_4 \rangle$, $O_8 = \langle a_{10} \rangle$, $O_9 = \langle a_{11} \rangle$, $O_{10} = \langle a_9 \rangle$ are the $G_{52}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle b_1 \rangle$, $O_3 = \langle b_0 \rangle$, $O_4 = \langle b_1 \rangle$, $O_5 = \langle a_3 \rangle$, $O_6 = \langle b_2 \rangle$ are the $(G_{52}/\Gamma_{52})$-orbits. Since the vertex set of $X_{52}$ is $\sqcup_{j=1}^{6} \eta_{52}(O_j)$ and $G_{52}/\Gamma_{52} \leq \text{Aut}(X_{52})$, it follows that the number of $\text{Aut}(X_{52})$-orbits of vertices is $\leq 6$.

Let $X_{53} = K_{53}/\Gamma_{53}$ be a semi-equivaler map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{53} \leq \text{Aut}(K_{53})$. Let the vertices of $X_{53}$ form $m_{53} \text{ Aut}(X_{53})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $b_1$ as the origin $(0,0)$ of $K_{53}$ (see Section 4). Let $\alpha_{53} := a_1 - a_0$ and $\beta_{53} := a_{42} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{53} := \langle x \mapsto x + \alpha_{53}, x \mapsto x + \beta_{53} \rangle$ and

$$G_{53} = \{ \alpha : x \mapsto \varepsilon x + m_{53}\alpha_{53} + n_{53}\beta_{53} : \varepsilon = \pm 1, m_{53}, n_{53} \in \mathbb{Z} \} \cong H_{53} \rtimes \mathbb{Z}_2.$$

By the same arguments as above and in Claim 1, $\Gamma_{53} \leq G_{53}$ and the number of $G_{53}/\Gamma_{53}$-orbits of vertices of $X_{53}$ is five. Therefore, $G_{53}/\Gamma_{53}$ acts on $X_{53} = K_{53}/\Gamma_{53}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle b_0 \rangle$, $O_3 = \langle a_{12} \rangle$, $O_4 = \langle a_3 \rangle$, $O_5 = \langle a_{32} \rangle$ are the $G_{53}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle b_0 \rangle$, $O_3 = \langle a_{12} \rangle$, $O_4 = \langle a_3 \rangle$, $O_5 = \langle a_{32} \rangle$ are the $(G_{53}/\Gamma_{53})$-orbits. Since the vertex set of $X_{53}$ is $\sqcup_{j=1}^{5} \eta_{53}(O_j)$ and $G_{53}/\Gamma_{53} \leq \text{Aut}(X_{53})$, it follows that the number of $\text{Aut}(X_{53})$-orbits of vertices is $\leq 5$.

Let $X_{54} = K_{54}/\Gamma_{54}$ be a semi-equivaler map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{54} \leq \text{Aut}(K_{54})$. Let the vertices of $X_{54}$ form $m_{54} \text{ Aut}(X_{54})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_{14}$ as the origin $(0,0)$ of $K_{54}$ (see Section 4). Let $\alpha_{54} := a_1 - a_0$ and $\beta_{54} := a_{81} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{54} := \langle x \mapsto x + \alpha_{54}, x \mapsto x + \beta_{54} \rangle$ and

$$G_{54} = \{ \alpha : x \mapsto \varepsilon x + m_{54}\alpha_{54} + n_{54}\beta_{54} : \varepsilon = \pm 1, m_{54}, n_{54} \in \mathbb{Z} \} \cong H_{54} \rtimes \mathbb{Z}_2.$$
By the same arguments as above and in Claim 1, $\Gamma_{54} \leq G_{54}$ and the number of $G_{54}/\Gamma_{54}$-orbits of vertices of $X_{54}$ is three. Therefore, $G_{54}/\Gamma_{54}$ acts on $X_{54} = K_{54}/\Gamma_{54}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_{51} \rangle$, $O_3 = \langle b_0 \rangle$, are the $G_{54}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_{51} \rangle$, $O_3 = \langle b_0 \rangle$ are the $(G_{54}/\Gamma_{54})$-orbits. Since the vertex set of $X_{54}$ is $\sqcup_{j=1}^{54} O_j$ and $G_{54}/\Gamma_{54} \leq \text{Aut}(X_{54})$, it follows that the number of $\text{Aut}(X_{54})$-orbits of vertices is 3.

Let $X_{55} = K_{55}/\Gamma_{55}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{55} \leq \text{Aut}(K_{55})$. Let the vertices of $X_{55}$ form $m_{55}$ $\text{Aut}(X_{55})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $b_1$ as the origin $(0, 0)$ of $K_{55}$ (see Section 4). Let $\alpha_{55} := a_1 - a_0$ and $\beta_{55} := a_{51} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{55} := \langle x \mapsto x + \alpha_{55}, x \mapsto x + \beta_{55} \rangle$ and

$$G_{55} = \{ \alpha : x \mapsto \varepsilon x + m_{55}\alpha_{55} + n_{55}\beta_{55} : \varepsilon = \pm 1, m_{55}, n_{55} \in \mathbb{Z} \} \cong H_{55} \times \mathbb{Z}_2.$$

By the same arguments as above and in Claim 1, $\Gamma_{55} \leq G_{55}$ and the number of $G_{55}/\Gamma_{55}$-orbits of vertices of $X_{55}$ is five. Therefore, $G_{55}/\Gamma_{55}$ acts on $X_{55} = K_{55}/\Gamma_{55}$. Since $O_1 = \langle b_0 \rangle$, $O_2 = \langle a_0 \rangle$, $O_3 = \langle a_{12} \rangle$, $O_4 = \langle a_{23} \rangle$, $O_5 = \langle a_{32} \rangle$ are the $G_{55}$-orbits, it follows that $O_1 = \langle b_0 \rangle$, $O_2 = \langle a_0 \rangle$, $O_3 = \langle a_{12} \rangle$, $O_4 = \langle a_{23} \rangle$, $O_5 = \langle a_{32} \rangle$ are the $(G_{55}/\Gamma_{55})$-orbits. Since the vertex set of $X_{55}$ is $\sqcup_{j=1}^{55} O_j$ and $G_{55}/\Gamma_{55} \leq \text{Aut}(X_{55})$, it follows that the number of $\text{Aut}(X_{55})$-orbits of vertices is 5.

Let $X_{56} = K_{56}/\Gamma_{56}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{56} \leq \text{Aut}(K_{56})$. Let the vertices of $X_{56}$ form $m_{56}$ $\text{Aut}(X_{56})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $b_1$ as the origin $(0, 0)$ of $K_{56}$ (see Section 4). Let $\alpha_{56} := a_1 - a_0$ and $\beta_{56} := a_{45} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{56} := \langle x \mapsto x + \alpha_{56}, x \mapsto x + \beta_{56} \rangle$ and

$$G_{56} = \{ \alpha : x \mapsto \varepsilon x + m_{56}\alpha_{56} + n_{56}\beta_{56} : \varepsilon = \pm 1, m_{56}, n_{56} \in \mathbb{Z} \} \cong H_{56} \times \mathbb{Z}_2.$$

By the same arguments as above and in Claim 1, $\Gamma_{56} \leq G_{56}$ and the number of $G_{56}/\Gamma_{56}$-orbits of vertices of $X_{56}$ is six. Therefore, $G_{56}/\Gamma_{56}$ acts on $X_{56} = K_{56}/\Gamma_{56}$. Since $O_1 = \langle b_0 \rangle$, $O_2 = \langle a_0 \rangle$, $O_3 = \langle a_{12} \rangle$, $O_4 = \langle a_{25} \rangle$, $O_5 = \langle a_{32} \rangle$, $O_6 = \langle a_{33} \rangle$ are the $G_{56}$-orbits, it follows that $O_1 = \langle b_0 \rangle$, $O_2 = \langle a_0 \rangle$, $O_3 = \langle a_{12} \rangle$, $O_4 = \langle a_{25} \rangle$, $O_5 = \langle a_{32} \rangle$, $O_6 = \langle a_{33} \rangle$ are the $(G_{56}/\Gamma_{56})$-orbits. Since the vertex set of $X_{56}$ is $\sqcup_{j=1}^{56} O_j$ and $G_{56}/\Gamma_{56} \leq \text{Aut}(X_{56})$, it follows that the number of $\text{Aut}(X_{56})$-orbits of vertices is 6.

Let $X_{57} = K_{57}/\Gamma_{57}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{57} \leq \text{Aut}(K_{57})$. Let the vertices of $X_{57}$ form $m_{57}$ $\text{Aut}(X_{57})$-orbits. We take the middle point of the line segment joining vertices $b_0$ and $b_1$ as the origin $(0, 0)$ of $K_{57}$ (see Section 4). Let $\alpha_{57} := b_1 - b_0$ and $\beta_{57} := b_{20} - b_0 \in \mathbb{R}^2$. Similarly as above, define $H_{57} := \langle x \mapsto x + \alpha_{57}, x \mapsto x + \beta_{57} \rangle$ and

$$G_{57} = \{ \alpha : x \mapsto \varepsilon x + m_{57}\alpha_{57} + n_{57}\beta_{57} : \varepsilon = \pm 1, m_{57}, n_{57} \in \mathbb{Z} \} \cong H_{57} \times \mathbb{Z}_2.$$

By the same arguments as above and in Claim 1, $\Gamma_{57} \leq G_{57}$ and the number of $G_{57}/\Gamma_{57}$-orbits of vertices of $X_{57}$ is six. Therefore, $G_{57}/\Gamma_{57}$ acts on $X_{57} = K_{57}/\Gamma_{57}$. Since $O_1 = \langle a_1 \rangle$, $O_2 = \langle a_0 \rangle$, $O_3 = \langle b_0 \rangle$, $O_4 = \langle b_1 \rangle$, $O_5 = \langle b_8 \rangle$, $O_6 = \langle b_{15} \rangle$ are the $G_{57}$-orbits, it follows that $O_1 = \langle a_1 \rangle$, $O_2 = \langle a_0 \rangle$, $O_3 = \langle b_0 \rangle$, $O_4 = \langle b_1 \rangle$, $O_5 = \langle b_8 \rangle$, $O_6 = \langle b_{15} \rangle$ are the $(G_{57}/\Gamma_{57})$-orbits. Since the vertex set of $X_{56}$ is $\sqcup_{j=1}^{57} O_j$ and $G_{57}/\Gamma_{57} \leq \text{Aut}(X_{57})$, it follows that the number of $\text{Aut}(X_{57})$-orbits of vertices is 6.

Let $X_{58} = K_{58}/\Gamma_{58}$ be a semiequivelar map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{58} \leq \text{Aut}(K_{58})$. Let the vertices of $X_{58}$ form $m_{58}$ $\text{Aut}(X_{58})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_3$ as the origin.
of vertices of 

By the same arguments as above and in Claim 1, $\Gamma_{58} \leq G_{58}$ and the number of $G_{58}/\Gamma_{58}$-orbits of vertices of $X_{58}$ is twelve. Therefore, $G_{58}/\Gamma_{58}$ acts on $X_{58} = K_{58}/\Gamma_{58}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle b_0 \rangle$, $O_5 = \langle b_6 \rangle$, $O_6 = \langle b_3 \rangle$, $O_7 = \langle b_4 \rangle$, $O_8 = \langle b_8 \rangle$, $O_9 = \langle a_8 \rangle$, $O_{10} = \langle a_{10} \rangle$ are the $G_{58}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle b_0 \rangle$, $O_5 = \langle a_6 \rangle$, $O_6 = \langle b_3 \rangle$, $O_7 = \langle b_4 \rangle$, $O_8 = \langle a_8 \rangle$, $O_9 = \langle b_7 \rangle$, $O_{10} = \langle b_8 \rangle$, $O_{11} = \langle b_9 \rangle$, $O_{12} = \langle a_{10} \rangle$ are the $(G_{58}/\Gamma_{58})$-orbits. Since the vertex set of $X_{58}$ is $\bigcup_{j=1}^{12}\eta_{58}(O_j)$ and $G_{58}/\Gamma_{58} \leq \text{Aut}(X_{58})$, it follows that the number of $\text{Aut}(X_{58})$-orbits of vertices is $\leq 12$.

Let $X_{59} = K_{59}/\Gamma_{59}$ be a semi-equivaleral map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{59} \leq \text{Aut}(K_{59})$. Let the vertices of $X_{59}$ form $m_{59} \text{ Aut}(X_{59})$-orbits. We take the middle point of the line segment joining vertices $b_0$ and $b_1$ as the origin $(0,0)$ of $K_{59}$ (see Section 4). Let $a_{59} = b_1 - b_0$ and $\beta_{59} = b_{18} - b_0 \in \mathbb{R}^2$. Similarly as above, define $H_{59} = \langle x \mapsto x + a_{59}, x \mapsto x + \beta_{59} \rangle$ and

$$G_{59} = \{\alpha : x \mapsto \varepsilon x + m_{59}a_{59} + n_{59}\beta_{59} : \varepsilon = \pm 1, m_{59}, n_{59} \in \mathbb{Z} \} \cong H_{59} \times \mathbb{Z}_2.$$ By the same arguments as above and in Claim 1, $\Gamma_{59} \leq G_{59}$ and the number of $G_{59}/\Gamma_{59}$-orbits of vertices of $X_{59}$ is six. Therefore, $G_{59}/\Gamma_{59}$ acts on $X_{59} = K_{59}/\Gamma_{59}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle b_1 \rangle$, $O_5 = \langle a_6 \rangle$, $O_6 = \langle b_3 \rangle$, $O_7 = \langle b_4 \rangle$, $O_8 = \langle a_8 \rangle$, $O_9 = \langle b_7 \rangle$, $O_{10} = \langle a_{10} \rangle$ are the $G_{59}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle b_1 \rangle$, $O_5 = \langle a_6 \rangle$, $O_6 = \langle b_3 \rangle$, $O_7 = \langle b_4 \rangle$, $O_8 = \langle a_8 \rangle$, $O_9 = \langle b_7 \rangle$, $O_{10} = \langle a_{10} \rangle$ are the $(G_{59}/\Gamma_{59})$-orbits. Since the vertex set of $X_{59}$ is $\bigcup_{j=1}^{6}\eta_{59}(O_j)$ and $G_{59}/\Gamma_{59} \leq \text{Aut}(X_{59})$, it follows that the number of $\text{Aut}(X_{59})$-orbits of vertices is $\leq 6$.

Let $X_{60} = K_{60}/\Gamma_{60}$ be a semi-equivaleral map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{60} \leq \text{Aut}(K_{60})$. Let the vertices of $X_{60}$ form $m_{60} \text{ Aut}(X_{60})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_3$ as the origin $(0,0)$ of $K_{60}$ (see Section 4). Let $a_{60} = a_3 - a_0$ and $\beta_{60} = a_{54} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{60} = \langle x \mapsto x + a_{60}, x \mapsto x + \beta_{60} \rangle$ and

$$G_{60} = \{\alpha : x \mapsto \varepsilon x + m_{60}a_{60} + n_{60}\beta_{60} : \varepsilon = \pm 1, m_{60}, n_{60} \in \mathbb{Z} \} \cong H_{60} \times \mathbb{Z}_2.$$ By the same arguments as above and in Claim 1, $\Gamma_{60} \leq G_{60}$ and the number of $G_{60}/\Gamma_{60}$-orbits of vertices of $X_{60}$ is eight. Therefore, $G_{60}/\Gamma_{60}$ acts on $X_{60} = K_{60}/\Gamma_{60}$. Since $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle b_1 \rangle$, $O_5 = \langle b_0 \rangle$, $O_6 = \langle a_{14} \rangle$, $O_7 = \langle a_{15} \rangle$ are the $G_{60}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle b_1 \rangle$, $O_5 = \langle b_0 \rangle$, $O_6 = \langle a_{14} \rangle$, $O_7 = \langle b_{15} \rangle$, $O_8 = \langle b_{14} \rangle$, $O_9 = \langle a_{15} \rangle$ are the $(G_{60}/\Gamma_{60})$-orbits. Since the vertex set of $X_{60}$ is $\bigcup_{j=1}^{8}\eta_{60}(O_j)$ and $G_{60}/\Gamma_{60} \leq \text{Aut}(X_{60})$, it follows that the number of $\text{Aut}(X_{60})$-orbits of vertices is $\leq 8$.

Let $X_{61} = K_{61}/\Gamma_{61}$ be a semi-equivaleral map on the torus for some fixed element (vertex, edge or face) free subgroup $\Gamma_{61} \leq \text{Aut}(K_{61})$. Let the vertices of $X_{61}$ form $m_{61} \text{ Aut}(X_{61})$-orbits. We take the middle point of the line segment joining vertices $a_0$ and $a_3$ as the origin $(0,0)$ of $K_{61}$ (see Section 4). Let $a_{61} = a_3 - a_0$ and $\beta_{61} = a_{49} - a_0 \in \mathbb{R}^2$. Similarly as above, define $H_{61} = \langle x \mapsto x + a_{61}, x \mapsto x + \beta_{61} \rangle$ and

$$G_{61} = \{\alpha : x \mapsto \varepsilon x + m_{61}a_{61} + n_{61}\beta_{61} : \varepsilon = \pm 1, m_{61}, n_{61} \in \mathbb{Z} \} \cong H_{61} \times \mathbb{Z}_2.$$ By the same arguments as above and in Claim 1, $\Gamma_{61} \leq G_{61}$ and the number of $G_{61}/\Gamma_{61}$-orbits of vertices of $X_{61}$ is eight. Therefore, $G_{61}/\Gamma_{61}$ acts on $X_{61} = K_{61}/\Gamma_{61}$. Since $O_1 = \langle a_0 \rangle,$
$O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle a_{15} \rangle$, $O_5 = \langle b_0 \rangle$, $O_6 = \langle b_1 \rangle$, $O_6 = \langle b_{14} \rangle$, $O_6 = \langle b_{15} \rangle$ are the $G_{61}$-orbits, it follows that $O_1 = \langle a_0 \rangle$, $O_2 = \langle a_1 \rangle$, $O_3 = \langle a_2 \rangle$, $O_4 = \langle a_{15} \rangle$, $O_5 = \langle b_0 \rangle$, $O_6 = \langle b_1 \rangle$, $O_6 = \langle b_{14} \rangle$, $O_6 = \langle b_{15} \rangle$ are the $(G_{61}/\Gamma_{61})$-orbits. Since the vertex set of $X_{61}$ is $\biguplus_{j=1}^{8} \eta_{61}(O_j)$ and $G_{61}/\Gamma_{61} \leq \text{Aut}(X_{61})$, it follows that the number of $\text{Aut}(X_{61})$-orbits of vertices is $\leq 8$. □

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7 Conflicts of Interest Statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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