Solving the Hamilton-Jacobi equation for gravitationally interacting electromagnetic and scalar fields

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Abstract

The spatial gradient expansion of the generating functional was recently developed by Parry, Salopek, and Stewart to solve the Hamiltonian constraint in Einstein-Hamilton-Jacobi theory for gravitationally interacting dust and scalar fields. This expansion is used here to derive an order-by-order solution of the Hamiltonian constraint for gravitationally interacting electromagnetic and scalar fields. A conformal transformation and functional integral are used to derive the generating functional up to the terms fourth order in spatial gradients. The perturbations of a flat Friedmann-Robertson-Walker cosmology with a scalar field, up to second order in spatial gradients, are given. The application of this formalism is demonstrated in the specific example of an exponential potential.

1. Introduction

Hamilton-Jacobi (HJ) theory has many applications in the perturbative and non-perturbative analysis of dynamical systems in classical mechanics. Peres developed an Einstein-Hamilton-

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Jacobi (EHJ) formulation of general relativity in which a generating functional has to satisfy the momentum and Hamiltonian constraints of general relativity \[1\]. In the framework of quantum cosmology, it was known that the momentum constraints require any wave functional to be diffeomorphism invariant \[3\]. In a WKB approximation, such a requirement translates into the diffeomorphism invariance of the generating functional. The Hamiltonian constraint is a non-linear functional partial differential equation that governs the time evolution of the generating functional.

Based on Peres’ formalism, Parry, Salopek, and Stewart used a series expansion of the generating functional in spatial gradients of the fields to derive an order-by-order approximate solution of the Hamiltonian constraint for general relativity with matter fields (see Ref. \[4\], from now on referred to as PSS). Such a generating functional is diffeomorphism invariant in each order of the expansion. Salopek and Bond used this formalism to show how non-linear effects of the metric and scalar fields may be included in stochastic inflationary models. The main advantage of this analysis is that the lapse function and shift vectors do not appear in the EHJ equations. Therefore, one obtains a coordinate-free approach to cosmological perturbations. In the above models matter fields consist of self-interacting scalar and dust fields.

In this article the above formalism is extended to minimally coupled gravitationally interacting scalar and electromagnetic fields. Such minimally coupled electromagnetic fields give rise to conformally invariant field equations. Hence, the electromagnetic field energy density is proportional to \(1/a^4\) where \(a\) is the scale factor. Consequently, the electromagnetic field is diluted away during the De Sitter expansion phase of the inflationary cosmologies. To break the conformal invariance, a direct coupling of gravity to electromagnetism \[5\] or corrections due to the quantum conformal anomaly have been considered \[6\]. A coordinate-free approach to the perturbative analysis of cosmological models with electromagnetic fields could eventually lead to a better understanding of the primordial magnetic fields. The generating functional up to the third order in the spatial gradient expansion is given in section \(2\). Following PSS, section \(3\) is a demonstration of how a recursion relation and a functional integral in superspace can be used to derive the higher order terms in the spatial gradient expansion from the previous terms. Section \(4\) is an exhibition of the gauge fixing and the solution of the field equations.
2. ADM reduction and the EHJ equations

The action for minimally coupled gravitationally interacting neutral scalar and electromagnetic fields can be written as

\[ I = \int \sqrt{g} \left[ \frac{1}{2} R - \frac{1}{2} \phi_{,\mu} \phi^{,\mu} - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] d^4 x, \mu = (0, 1, 2, 3), \]  

(1)

where \( F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \) is the electromagnetic field strength and \( V(\phi) \) is the scalar field potential. ADM reduction of the above action is achieved by defining the 3-metric \( \gamma_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu \) and the vector potential \( A = A_|| + A_\perp \), such that \( (A_||)_\mu = \gamma^{\nu} A_{\nu,\mu}, (A_\perp)_\mu = -(n^\nu A_\nu)n_\mu \), where \( n_\mu \) is the unit vector field normal to space-like hypersurfaces of simultaneity parametrized by \( t \). In the basis \( (\partial_t, \partial_i), i = 1, 2, 3 \), such that \( n_\mu (\partial_i) = 0 \) and \( n^\mu = (\partial_t, -N^i \partial_i)/N \) (no sum), the following relations hold: \( (A_||)_i = \gamma^{\nu} A_{\nu,i}, A_0 = N A_\perp + N^i A_i \), and

\[ ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \]  

(2)

Then one follows with the procedure outlined in Ref. [7] to derive the Lagrangian \( L \) for gravitationally interacting electromagnetic and scalar fields. With the momenta \( \pi^{ij} = \delta L / \delta \dot{\gamma}_{ij}, \pi^\phi = \delta L / \delta \dot{\phi} \), \( E^i = \delta L / \delta A_i \), where \( \dot{\cdot} := d/dt \), after a Legendre transformation the Hamiltonian is

\[ \text{Hamiltonian} = \int (N^\mu \mathcal{H}_\mu + A_0 \mathcal{G}) d^3 x, \]  

(3)

where

\[
\begin{align*}
\mathcal{H}_0 & = \gamma^{-1/2} \pi^{ij} \pi^{kl} (2 \gamma^{il} \gamma_{jk} - \gamma_{ij} \gamma_{kl}) + \gamma^{1/2} [V(\phi) - R/2 + F^{il} F_{il}/4 + \phi^i \phi_{,i}/2] \\
& + \gamma^{-1/2} [E^i E_i + (\pi^\phi)^2]/2 = 0, \text{ Hamiltonian constraint,} \\
\mathcal{H}_i & = -2 \pi^i_{,ij} + F_{it} E^t + \pi^\phi \phi_{,i} = 0, \text{ momentum constraint,} \\
\mathcal{G} & = -E^i |_i = 0, \text{ Gauss law constraint.}
\end{align*}
\]  

(4)

\( |_i \) is the 3-covariant derivative (for covariant derivatives of tensor densities and sign conventions see [8]). Utilizing Hamilton’s equations, the evolution equations for the fields are as
follows:

\[
\dot{\phi} = N\gamma^{-1/2}\pi^\phi + N^i\phi_i, \quad (5)
\]

\[
\dot{\gamma}_{ij} = N2\gamma^{-1/2}\pi^{kl}(2\gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl}) + 2N_{(ij)}, \quad (6)
\]

\[
\dot{A}_i = N\gamma^{-1/2}E_i + N^jF_{ji} + A_{0,i}. \quad (7)
\]

The evolution equations for the momenta are considerably more complicated. They are given by

\[
\dot{\pi}^\phi = -N\gamma^{-1/2}\frac{dV}{d\phi} - \frac{1}{2}\gamma^{1/2} \left( N_{,m}\phi^m + N\phi^l|_l \right) - \left( N^m\pi^\phi \right)_m, \quad (8)
\]

\[
\dot{\pi}^{ij} = N\gamma^{-1/2} \left\{ \gamma^{ij} \left[ \pi^{mn}\pi_{mn} - \frac{1}{2}(\pi)^2 \right] - 4\pi^{im}\pi^j_m + 2\pi^{ij}\pi \right. \\
+ \frac{1}{4}\gamma^{ij}E^i_mE_l - \frac{1}{2}E^i_jE^j + \frac{1}{2}\gamma^{ij}(\pi^\phi)^2 \left. \right\} - \frac{1}{2}N\gamma^{1/2} \left\{ R^{ij} - \frac{1}{2}\gamma^{ij}R \right.
\]

\[
- F^i_kF^{jk} + \frac{1}{4}\gamma^{ij}F_{lm}F^{lm} + \frac{1}{2}\gamma^{ij}\phi_i\phi^j - \phi^i\phi^j + \gamma^{ij}V(\phi) \right\}
\]

\[
+ \frac{1}{2}\gamma^{1/2} \left( N_{|ij} - N_{|il}\gamma^{ij} \right) + \left( N^m\pi^{ij} \right)_n - 2\pi^{n(i}N^{j)}_n, \quad (9)
\]

\[
\dot{E}^i = \gamma^{1/2} \left( N_{,m}E^mi + N^{Fmi}_m \right) + \left( N^mE^i - N^{i}E^m \right)_m. \quad (10)
\]

Instead of solving the evolution equations for the fields and momenta, one can try to solve the EHJ equations. The EHJ equations are derived by the substitutions

\[
\pi^{ij} = \frac{\delta S}{\delta\gamma^{ij}}, \quad \pi^\phi = \frac{\delta S}{\delta\phi}, \quad E^i = \frac{\delta S}{\delta A_i}, \quad (11)
\]

in \( H_\mu \) and \( G \). \( S = S[\gamma_{ij}, \phi, A_i] \) is the generating functional (Hamilton’s principal function) \[\text{[4]}\]. The Hamiltonian constraint is a hyperbolic functional partial differential equation for \( S \). After solving the EHJ equations, \([5]-[7]\) and \([\text{[3]}]\) yield the full set of the evolution equations.

3. The spatial gradient expansion and the order-by-order solution of the EHJ equations

The momentum constraint implies that the generating functional is diffeomorphism invariant \[\text{[1]-[3]}\]. One such-diffeomorphism invariant quantity is \( S = \int f[\phi, \gamma_{ij}, A_i]\gamma^{1/2}d^3x \). More generally, a diffeomorphism invariant \( S \) can be a multiple integral of some multi-point functions.

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The contribution of such highly non-local terms could be important, for example, if the spatial inhomogeneities are correlated. However, at least in the lowest orders, the contribution of such terms to the generating functional are expected to be insignificant in the generic case. Likewise, the Gauss law constraint implies that $S$ is gauge-invariant, e.g. $S = S[F_{ij}]$. Other gauge-invariant quantities like $\oint A_l dx^l$ could also be included in $S$. However, if the space-like surfaces of simultaneity are simply connected, one can write all such quantities in terms of $F_{ij}$ using Stokes theorem. Non-simply connected three-manifolds are not considered here.

The Hamiltonian constraint determines the time evolution of the fields. Following PSS, an order-by-order solution of the Hamiltonian constraint is achieved by the expansion of the generating functional in spatial gradients:

$$S = \sum_{n=0}^{\infty} \lambda^n S^{(n)}. \quad (12)$$

where $\lambda^n$ denotes the number of spatial gradients in $S^{(n)}$. It turns out that the scalar field is indispensable in this model and dominates the dynamics of the space-time. Moreover, in the limit $R \to 0$, in the absence of electromagnetism, the zeroth order solution is exact. Therefore, merely based on dimensional grounds, this heuristically suggests the expansion parameter should obey $\lambda \propto \frac{\bar{R}}{V(\phi)}$. In this relation, $\bar{R}$ is an appropriate combination of the curvature invariants of dimension $L^{-2}$ such that in the flat space limit where the three-curvature is vanishing $\bar{R} = 0$, and $V(\phi)$ represents the potential energy density of the scalar field. The convergence of the above series is an unsolved problem [11].

An order-by-order solution of the EHJ equation is achieved by substituting (12) in the first equation in (4) and the subsequent expansion of $\mathcal{H}_0$ in spatial gradients:

$$\mathcal{H}_0 = \sum_{n=0}^{\infty} \lambda^n \mathcal{H}^{(n)} \quad (13)$$

and requiring the EHJ equation to vanish at each order. In the above equation

$$\mathcal{H}^{(0)} = \gamma^{-1/2} \frac{\delta S^{(0)}}{\delta \gamma_{ij}} \frac{\delta S^{(0)}}{\delta \gamma_{kl}} (2 \gamma_{il} \gamma_{jk} - \gamma_{ij} \gamma_{kl}) + \gamma^{1/2} V(\phi) + \gamma^{-1/2} \frac{\delta S^{(0)}}{\delta \phi}^2 / 2 = 0 \quad (14)$$

One can easily obtain the first few terms in (12) by an ansatz. The zeroth order term

$$S^{(0)} = -2 \int \gamma^{1/2} H(\phi) d^3 x, \quad (15)$$
called the long-wavelength approximation (LWA), is the same as in PSS for some arbitrary function \( H(\phi) \). Inserting \( S^{(0)} \) in (14) yields
\[
-3H^2 + V(\phi) + 2\left(\frac{dH}{d\phi}\right)^2 = 0.
\]

(16)

Electromagnetism has no dynamical degrees of freedom at this order. The LWA is very important in structure formation after inflation. Therefore, it is unlikely that electromagnetic fields play a significant role in structure formation in this model. For \( n > 0 \)
\[
\mathcal{H}(n) = \gamma^{1/2}(2\gamma_{ij}\gamma_{jk} - \gamma_{ij}\gamma_{kl}) \left( 2\frac{\delta S^{(0)}}{\delta \gamma_{ij}} \frac{\delta S^{(n)}}{\delta \gamma_{kl}} + \sum_{p=1}^{n-1} \frac{\delta S^{(p)}}{\delta \gamma_{ij}} \frac{\delta S^{(n-p)}}{\delta \gamma_{kl}} \right) + \frac{1}{2} \gamma^{-1/2} \gamma_{ij} \sum_{p=1}^{n-1} \frac{\delta S^{(p)}}{\delta A_i} \frac{\delta S^{(n-p)}}{\delta A_j} + \gamma^{-1/2} \frac{\delta S^{(0)}}{\delta \phi} \frac{\delta S^{(n)}}{\delta \phi} + \frac{1}{2} \gamma^{-1/2} \sum_{p=1}^{n-1} \frac{\delta S^{(p)}}{\delta \phi} \frac{\delta S^{(n-p)}}{\delta \phi} + \nu^{(n)},
\]
\[
\nu^{(2)} = \gamma^{1/2}(-R + \phi \phi, i + F_{ij}F^{ij}/2)/2,
\]
\[
\nu^{(n)} = 0, \text{ for } n \neq 2.
\]

(17)

At each order, \( \mathcal{H}(n) = 0 \) is a linear hyperbolic functional differential equation in the unknown functional \( S^{(n)} \).

\( S^{(2)} \) is the next non-vanishing term given by
\[
S^{(2)} = \int \gamma^{1/2}(J(\phi)R + K(\phi)\phi, i \phi^i + L(\phi)F_{ij}F^{ij})d^3x.
\]

(18)

The above terms are not the only terms quadratic in spatial gradients in \( S^{(2)} \). However, the remaining terms are either equal to the above terms modulo surface integrals or vanish identically. For example, \( \int \gamma^{1/2}F^{ij}\epsilon_{ijk}\phi^k d^3x = -\int \gamma^{1/2}(F^{ij}k\epsilon_{ijk}\phi) d^3x = 0 \) due to Maxwell equations \( F_{[ij]} = 0 \). It is easy to verify that \( S^{(2)} \) satisfies the momentum and Gauss law constraints. One notices that \( \frac{\delta S^{(2)}}{\delta A_i} \) is already quadratic in spatial derivatives and does not appear in the second order EHJ equation \( \mathcal{H}^{(2)} = 0 \). After inserting \( S^{(2)} \) in the second order EHJ equation and grouping together the coefficients of \( R, \phi |^l |, \phi^i \phi, i, \) and \( F_{ik} F^{ik} \) one
respectively has

\[ s_1 := HJ - 2 \frac{dH}{d\phi} \frac{dJ}{d\phi} - \frac{1}{2} = 0, \quad s_3 := HK - 4J \frac{d^2J}{d\phi^2} + 2 \frac{dH}{d\phi} \frac{dK}{d\phi} + \frac{1}{2} = 0, \]

\[ s_2 := -H \frac{dJ}{d\phi} + K \frac{dH}{d\phi} = 0, \quad s_4 := HL + 2 \frac{dH}{d\phi} \frac{dL}{d\phi} - \frac{1}{4} = 0. \]  

(19)

At first sight, (19) seems to be an over-determined system for three unknown functions, \( J, K, \) and \( L \). However, by solving \( s_2 = 0 \) for \( K \), one can show that \( s_3 \) is not independent and obeys the relation

\[ s_3 = 2 \frac{ds_2}{d\phi} - s_1 + \frac{ds_1}{d\phi} H \left( \frac{dH}{d\phi} \right)^{-1}. \]

In the spatial gradient expansion of the generating functional \( S \) for the scalar fields in the absence of electromagnetism, there is no contribution from odd order terms. Electromagnetism makes non-trivial contributions to odd orders. The only non-vanishing term in \( S^{(3)} \) is

\[ S^{(3)} = \int \gamma^{1/2} M(\phi) F^{ij} \epsilon_{ikl} F^{kl} d^3x. \]  

(20)

All other third order terms like \( F^{ij} \phi, F^{mn} F^{lp} \epsilon_{mnp}, F^{ij} \epsilon_{ij} \) either vanish or are total divergences. To solve the third-order EHJ equation \( \mathcal{H}^{(3)} = 0 \) one has to use the relation \( \Gamma_{ji} = \gamma^{-1/2}(\gamma^{1/2})_{ji} \). The solution yields

\[ MH + \frac{dM}{d\phi} \frac{dH}{d\phi} = 0. \]  

(21)

\( S^{(3)} \) non-trivially satisfies the momentum and Gauss law constraints. To show this, one frequently uses the identity \( V s \epsilon_{ij} = V_i \epsilon_{sjk} + V_j \epsilon_{isk} + V_k \epsilon_{ij}s \) for any vector field \( V_i \), achieved from

\[ \frac{\delta}{\delta \gamma_{rs}} (\epsilon^{ijk}) = \frac{\delta}{\delta \gamma_{rs}} \left( \gamma^{si} \gamma^{jm} \gamma^{kn} \epsilon_{ilmn} \right) \] and multiplication of both sides by \( V_r \). Equations (19), (19), and (21) form a set of differential equations, easily solvable for most relevant potentials. Nevertheless, the full set of differential equations becomes increasingly complicated at higher orders. As in PSS one can use the expression for \( S^{(0)} \) and the conformal transformation

\[ f_{ij} = \gamma_{ij} \Omega^{-2}(u), \quad u := \int (-2 \frac{dH}{d\phi})^{-1} d\phi, \quad \frac{d\Omega}{du} = H \Omega, \]  

(22)

to solve the EHJ equations (17). The EHJ equations transforms into

\[ \frac{\delta S^{(n)}}{\delta u(x)} |_{f_{ij}, A_i} = -\tilde{\mathcal{R}}^{(n)} \]  

(23)
where
\[
\hat{\mathcal{R}}^{(n)} = f^{-1/2} \Omega^{-3} (2 f u f_{j k} - f_{i j} f_{k l}) \sum_{p=1}^{n-1} \frac{\delta S^{(p)}}{\delta f_{i j}} \frac{\delta S^{(n-p)}}{\delta f_{k l}}
\]
\[
+ \frac{f^{-1/2}}{8 \Omega^3} \left( \frac{dH}{d\psi} \right)^{-2} \sum_{p=1}^{n-1} \left( \frac{\delta S^{(p)}}{\delta u} - 2 \frac{\delta S^{(p)}}{\delta f_{i m}} f_{i m} H \right) \frac{\delta S^{(n-p)}}{\delta u}
\]
\[
- 2 \frac{\delta S^{(n-p)}}{\delta f_{p q}} f_{p q} H + \frac{f^{-1/2}}{2 \Omega} f_{i j} \sum_{p=1}^{n-1} \frac{\delta S^{(p)}}{\delta A_i} \frac{\delta S^{(n-1)}}{\delta A_j} + \bar{\nu}^{(n)}
\]
\[
\bar{\nu}^{(2)} = f^{1/2} \left\{ \frac{\Omega}{2} \hat{R} + \frac{d}{d u} (H \Omega) u_i u_j f^{i j} - \frac{1}{4 \Omega} F_{i k} F_{i m} f^{i l} f^{k m} \right\}, \quad \bar{\nu}^{(n)} = 0, \text{ for } n \neq 2.
\]
\(\hat{R}\) is the conformal curvature and all indices are raised and lowered with the conformal metric \(f_{i j}\).

The proof of the integrability of (23) for gravitationally interacting electromagnetic and scalar fields proceeds similar to what was outlined in [12]. By using the expression for \(\bar{\nu}^{(2)}\) from (24), a functional integration of (23) gives rise to the following expression for \(S^{(2)}\):
\[
S^{(2)} = \int f^{1/2} \left( j(u) \hat{R} + k(u) u_i u^i + 2 \nu(u) \right) d^3 x,
\]  
where
\[
j(u) := \int_0^u \frac{\Omega(u')}{2} d u' + D_1, \quad \nu(u) := \int_0^u \frac{d u'}{4 \Omega(u')} + D', \quad k(u) := H \Omega.
\]  

The complementary functionals \(D_1, D'\) are constants of integration. In the next order \(\frac{\delta S^{(3)}}{\delta u} = 0\), therefore \(S^{(3)} = \int f^{1/2} F_{i j} \varepsilon_{i k l} F^{k l} d^3 x\) is the most general form of \(S^{(3)}\) in which ; and \(\varepsilon_{i j k}\) are the covariant derivative and Levi-Civita tensor associated with \(f_{i j}\) respectively. Conformal transformation of this expression gives rise to
\[
S^{(3)} = \int f^{1/2} \Omega^2 F_{i j} \varepsilon_{i k l} F^{k l} d^3 x.
\]  

Functional integration of (23) in the next order gives rise to
\[
\begin{align*}
S^{(4)} &= \int d^3 x f^{1/2} \left\{ - \ell(u) \hat{R}^{ij} \hat{R}_{ij} - (3 \ell(u)/8 + m(u)) \hat{R}^2 - n(u) (\hat{R}_{ij} - f^{ij} \hat{R}/2) u_i u_j \\
+ & \ r(u) u_i^2 u_j + s(u) (F_{i m} F^{i m})^2 + t(u) u_i u_j F_{i m} F^{i m} + v(u) u_i u_j F_{i m} F^{i m} \\
+ & \ w(u) u_i u_j F_{i m} F^{i m} + x(u) F_{i m} F^{i m} + y(u) F_{i m} F^{i m} + z(u) \hat{R} F_{i m} F^{i m} \\
+ & \ a(u) \hat{R}_{i k l} F^{i k m} F^{l m} \right\}.
\end{align*}
\]
where $\ell(u), m(u), n(u), r(u)$ are given in PSS. In the above expression $s(u), \cdots, a(u)$ are defined as

$$s'(u) = \Omega^{-3}[-11l^2/4 + (\frac{dH}{d\phi})^{-2}(\Omega^{-1}/4 + Hl)^2/8],$$

$$a'(u) = 8j\Omega^{-3},$$

$$z'(u) = \Omega^{-3}\left(-\frac{5}{2}j + \frac{1}{4}(\frac{dH}{d\phi})^{-2}(1/8 + Hl\Omega/2 - Hj\Omega^{-1}/4 - H^2jl\right),$$

$$a'(u) = 8j\Omega^{-3},$$

$$v'(u) = \Omega^{-2}(-4lH + 3\Omega^{-1}/2),$$

$$w'(u) = 4l\Omega^{-1}(2lH + \Omega^{-1}),$$

$$t'(u) = \Omega^{-2}(Hl - \Omega^{-3}/3),$$

in which $' := d/du$.

4. The evolution equations of the fields

Once the EHJ equations are solved, the evolution equations for the fields are obtained from (5-7) and (11). A judicious choice of gauge greatly simplifies the field equations. In the almost synchronous gauge ($N^i = 0$) if $u$ is the time parameter, from (5) it follows that in the LWA (i.e $S = S(0)$) the lapse obeys $N^{(1)} = 1$. The superscript $(n)$ means that the right hand side of the equation contains terms up to $(n - 1)$th order in spatial gradients.

The choice of $u$ as the time parameter is valid as long as the geometry is sufficiently close to that of the homogeneous models (for a relevant discussion see Ref. [13]). Then it is useful to replace (3) with the equivalent evolution equation

$$\dot{f}_{ij} = 2N\Omega^{-3}f^{-1/2} \frac{\delta S}{\delta f_{kl}} (2f_{jk}f_{il} - f_{ij}f_{kl}) - 2Hf_{ij}$$

for the conformal metric $f_{ij}$ which is related to $\gamma_{ij}$ via

$$\gamma_{ij} = \exp\left\{-\int (\frac{dH}{d\phi})^{-1}Hd\phi\right\} f_{ij}.$$  

In the LWA $\dot{f}_{ij}^{(1)} = 0$. $f_{ij}^{(1)}$ is the seed metric that contains no dynamical degrees of freedom. The first non-trivial evolution equation for $\dot{A}_i$ in the temporal gauge $A_0 = 0$, is obtained
from the conformal transformation of (7):

\[ \dot{A}_i^{(3)} = N f^{-1/2} \Omega^{-1} f^i u \frac{\delta S^{(2)}}{\delta A_l} = -4 \Omega^{-1} F^{m \mid i ; m}. \]  

(32)

Since \( d/du \) and \( \delta \) commute, the evolution equation for \( F_{ij} \) is easily derived from the above equations to be

\[ \dot{F}_{ij}^{(3)} = 8 \Omega^{-1} l(u) F_{[ij]m}^{\mid m}, \]

(33)

with the solution

\[ F_{ij}^{(3)} = \exp \left\{ - \left( 8 \int du \ \Omega^{-1} l(u) \right) \delta_i^k \delta_j^l \nabla_k \nabla^m \right\} F_{ml}. \]

(34)

In the above equation and what follows, \( F_{ij} \) is an arbitrary antisymmetric tensor field, \( \nabla \) refers to the covariant derivative with respect to the seed metric and the indices are raised with the seed metric. The exponential of the matrix differential operator is defined as:

\[ \exp \{ \cdots \} := \left\{ \delta^m_i \delta^l_j - \left( 8 \int \Omega^{-1} l(u) du \right) \delta_i^k \delta_j^l \nabla_k \nabla^m \right\} F_{ml}. \]

(35)

Once the evolution equations for the fields are solved, the evolution equations for the momenta are easily derived from (11). In particular, the electric field obeys the equation

\[ E_{i}^{(3)} = \delta S^{(2)} \] 

(37)

In higher orders (3) shows that \( N^{(n)} \neq 1 \). For example,

\[ N^{(3)} = 1 - \Omega^{-3} \left[ \tilde{R}^{(1)} \left( \frac{dj}{du} - jH \right) + F_{kl} F_{mn} f^{(1)km} f^{(1)ln} \left( lH + \frac{dl}{du} \right) \right] \left( -2 \frac{dH}{d\phi} \right)^{-2} \]

(38)

where \( \tilde{R}^{(1)} \) is the three-curvature associated with the seed metric \( f^{(1)}_{ij} \). Obviously, the higher order evolution equations are more complicated. The third order evolution of the conformal metric is derived from (30) and (38) to obey

\[ \dot{f}_{ij}^{(3)} = \Omega^{-3} \left\{ \tilde{R}^{(1)} f_{ij}^{(1)} \left[ \frac{H}{2} \left( \frac{dH}{d\phi} \right)^{-2} (jH - \frac{dj}{du}) + j \right] - 4 j \tilde{R}_{ij}^{(1)} \right. \]

\[ + F_{kl} F_{mn} f^{(1)km} f^{(1)ln} f_{ij}^{(1)} \left[ -\frac{H}{2} \left( \frac{dH}{d\phi} \right)^{-2} (lH + \frac{dl}{du}) + 3l \right] - 8 l F_{in} F_{jm} f^{(1)mn} \right\}. \]

(39)
As a demonstration of an application of the formalism developed so far, one could compute $f^{(3)}_{ij}$ for an arbitrary magnetic field $F_{kl}$ and seed metric $f^{(1)}_{ij}$ with a scalar potential $V = V_0 \exp \left\{ -\sqrt{\frac{2}{p}} \phi \right\}$. The general solution of (16) for $p \neq 1/3$ is given in [10]. The general parametric solution of (16) with $H$ and $\phi$ as functions of an independent variable $v$ is,

$$H = \left[ \frac{V_0}{3} \right]^{\frac{1}{2}} \exp \left\{ -\phi \sqrt{\frac{3}{2}} \cosh v \right\}, \quad (40)$$

where $\phi_m$ is the integration constant. A special solution of (16) for $p \neq 1/3$ is

$$H = \left[ \frac{V_0}{3 - 1/p} \right]^{\frac{1}{2}} \exp \left\{ \frac{\phi}{\sqrt{2p}} \right\}, \quad (42)$$

that corresponds to the Halliwell attractor [2]. By using (22), (39), (42) and with a choice of time parameter such that $\lim_{u \to 0} \phi = -\infty$, the contribution of all the terms second order in spatial gradients to the evolution of the conformal metric at this order is

$$f^{(3)}_{ij} = \frac{-2c^{-2}}{(p + 1)} \left[ \frac{V_0}{p(3p - 1)} \right]^{-p} u^{-2p+1} \tilde{R}^{(1)}_{ij}$$

$$+ \frac{c^{-2}}{(1 - p)} \left[ \frac{V_0}{p(3p - 1)} \right]^{-p} u^{-4p+1} \left( -\frac{1}{4} F_{mn} F^{mn} f^{(1)}_{ij} + 2 F_i^n F_j^n \right), \quad p \neq 1 \quad (43)$$

where $c$ is the integration constant and the seed metric is absorbed to raise indices. Integration of the above equation yields

$$f^{(3)}_{ij} = \frac{c^{-4}}{(1 - p)(1 - 2p)} \left[ \frac{V_0}{p(3p - 1)} \right]^{-2p} u^{-4p+2} \left( -\frac{1}{4} F_{mn} F^{mn} f^{(1)}_{ij} + F_i^n F_j^n \right)$$

$$+ \frac{c^{-2}}{p^2 - 1} \left[ \frac{V_0}{p(3p - 1)} \right]^{-p} u^{-2p+2} \tilde{R}_{ij} + f^{(1)}_{ij}. \quad (44)$$

After taking (31) and (43) into account, the three-metric is given by

$$\gamma^{(3)}_{ij} = \frac{c^{-2}}{(1 - p)(1 - 2p)} \left[ \frac{V_0}{p(3p - 1)} \right]^{-p} u^{-2p+2} \left( -\frac{1}{4} F_{mn} F^{mn} f^{(1)}_{ij} + F_i^n F_j^n \right)$$

$$+ \frac{1}{p^2 - 1} u^2 \tilde{R}^{(1)}_{ij} + c^2 \left[ \frac{V_0}{p(3p - 1)} \right]^{p} u^p f^{(1)}_{ij}. \quad (45)$$
Two points regarding the above equation are in order. First, as a perturbation of a flat Friedmann-Robertson-Walker (FRW) model, the growth of inhomogeneities from terms second order in spatial gradients is encoded in $\gamma^{(3)}_{ij}$. One notices that with the choice of $u$ as the time parameter, such a growth resulting from the curvature is insensitive to the exact form of the exponential potential. In other words, deviations from a flat FRW cosmology resulting from a magnetic field evolve quite differently from that of the inhomogeneities due to the space-time geometry. In particular, one notices that if $p > 1$, the contribution of a primordial magnetic field to spatial inhomogeneities decays rapidly.

Secondly, the reader is cautioned against a mini-superspace perturbation of a flat FRW model, namely, a perturbation within a mini-superspace cosmology. For example, for a vanishing magnetic field in a Bianchi I cosmology where the conformal three-curvature is vanishing (i.e. $\tilde{R}_{ij} = 0$), one would get the obviously incorrect result $\gamma^{(n)}_{ij}(t) = \gamma^{(1)}_{ij}(t)$ for $n \neq 1$ (because $S^{(n)} = 0$ for $n \neq 0$). The exact evolution equation for such a model could be derived from (8-10) and (5-7) which could be solved numerically for generic initial conditions. This problem also exists in the dust model treated in PSS which has been quite successful in deriving the contribution of short wavelength fields to structure formation and the anisotropies of microwave background radiation.

5. Conclusion

The spatial gradient expansion of the generating functional was developed by PSS to solve the Hamiltonian constraint in EHJ formulation of general relativity for gravitationally interacting dust and scalar fields. The spatial gradient expansion could be consistently applied to solve the Hamiltonian constraint for gravitationally interacting electromagnetic and scalar fields. At each order, the EHJ equation is a linear functional partial differential equation in the unknown functional $S^{(n)}$ which after a conformal transformation could be integrated to yield $S^{(n)}$. $S^{(2)}$ and $S^{(3)}$ were calculated in detail and $S^{(4)}$ was given. Such an order-by-order solution of the EHJ equation gives rise to order-by-order corrections to the fields evolving in a flat FRW model. The corrections are due the presence of spatial inhomogeneities and magnetic field. Not surprisingly, such corrections start with terms second order in spatial gradients. The formalism was applied to the specific example of a scalar field with potential
\[ V = V_0 \exp \left\{ -\sqrt{2} \phi \right\} \]. Contributions of all the terms second order in spatial gradients to the metric were derived.

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