Accelerated Target Updates for Q-learning

Bowen Weng, Huaqing Xiong and Wei Zhang

Abstract—This paper studies accelerations in Q-learning algorithms. We propose an accelerated target update scheme by incorporating the historical iterates of Q functions. The idea is conceptually inspired by the momentum-based accelerated methods in the optimization theory. Conditions under which the proposed accelerated algorithms converge are established. The algorithms are validated using commonly adopted testing problems in reinforcement learning, including the FrozenLake grid world game, two discrete-time LQR problems from the Deepmind Control Suite, and the Atari 2600 games. Simulation results show that the proposed accelerated algorithms can improve the convergence performance compared with the vanilla Q-learning algorithm.

I. INTRODUCTION

Reinforcement learning (RL) aims to study how an agent learns a policy through interacting with its environment to minimize the accumulative loss for a task. RL has received dramatically growing attention and success in various tasks, such as playing go [26], Atari [19], bipedal walking [8] and studying control systems [16], [17], to name a few. This paper focuses on the Q-learning algorithm which is a model-free RL algorithm to find an estimate of the optimal action value function.

Ever since the first proposal of the Q-learning algorithm in 1989 [33], the method has been studied extensively in finite state-action space. When the size of the state-action space is relatively small, the Q-function can be explicitly represented as a tabular function leading to a convenient proof of convergence [14].

When the state-action space is continuous or considerably large, Q-learning usually requires function approximations. Lewis et. al. studied the Q-learning problem for linear control systems [1] and extended the method to the continuous time domain [29], [31]. They consider value iteration with appropriate sampling and customized Q-function structure. The step of target update was later introduced in the Deep Q-Network (DQN) learning [19] with the Q-function being parameterized as a deep neural network. DQN illustrates great success in playing video games [22] that significantly exceeds human-level of performance. This also leads to various improved algorithms for Q-learning [30] and for general RL [25], [26].

Besides the exploration on the improved Q-learning algorithms with better performance in applications, another line of research lies in the convergence analysis of variants of the Q-learning algorithms [4], [5], [7], [9], [11]. Given that the training speed largely determines how an algorithm can contribute to the real application, accelerating the convergence is always of great interest. Optimization theory has provided effective schemes of acceleration with theoretic guarantees. One of the most popular schemes is based on the so-called momentum idea by involving more historical information into the update. Momentum-based algorithms, including Heavy-ball (HB) [23], Nesterov’s accelerated gradient (NAG) [21], have been proved to be able to accelerate the convergence when loss functions are strongly convex [12], [21]. Under a general setting without strong convexity, its acceleration has not been well understood theoretically. Nevertheless, its convergence can still be guaranteed [34] and numerical results show great success [3], [10], [15]. This also motivates our proposal of a new momentum-based accelerated scheme in Q-learning.

Our contribution in this paper are twofold. First, we propose a new accelerated Q-learning scheme which can be applied to both finite and continuous state-action space. Conditions under which the accelerated algorithms converge are derived under some mild assumptions. Our scheme is motivated by the momentum idea from the optimization theory, and may potentially inspire more works to use elegant optimization algorithms in RL. Second, we numerically verify the proposed algorithms in various challenging tasks, including a grid world game, two linear quadratic regulation problems from the Deepmind Control Suite [28], and the Atari 2600 video games. The simulation studies show improved convergence performance than the vanilla Q-learning. Notice that our validation differs from many existing RL studies in the control literature which are only validated using hand crafted examples (mostly linear system).

The rest of the paper is organized as follows. Section 2 introduces the background of Q-learning in finite state-action space and proposes an accelerated scheme following with convergence guarantee. In section 3, we extend our accelerated scheme to the case where state-action space is continuous or considerably large and provide stability result for the linear parametric approximation value function. Section 4 presents numerical results for both finite and continuous cases and explore the potential promise of our algorithms in more complicated applications.

II. ACCELERATED Q-LEARNING WITH TABULAR Q-FUNCTIONS

In this section, we consider the finite state-action space case for which the Q-function can be explicitly represented as a tabular function. We first introduce the basic background
of Q-learning. Then we propose an accelerated scheme for the Q-learning algorithm and show its convergence in finite state-action space.

A. Q-learning

For a finite state space \( \mathcal{X} \), action space \( \mathcal{U} \), and noise space \( \mathcal{W} \) with a nonnegative bounded cost function \( R : \mathcal{X} \times \mathcal{U} \times \mathcal{W} \rightarrow [0, R_{\text{max}}] \), we consider a discrete-time optimal sequential decision problem as follows.

\[
\text{minimize } \pi \quad J_\pi(x_0) = \mathbb{E}_{w} \left\{ \sum_{t=0}^{\infty} \gamma^t R(x_t, \pi(x_t), w_t) \right\},
\]
subject to \( x_{t+1} = f(x_t, u_t, w_t) \).

That is, starting at a state \( x_0 \in \mathcal{X} \) with the time-invariant dynamical system \( f \), one seeks to minimize the expected accumulated loss with a discount factor \( \gamma \in (0, 1) \) over feasible (stationary) policy \( \pi : \mathcal{X} \rightarrow \mathcal{U} \) with \( u_t = \pi(x_t) \).

We define \( J^*(x) := J_{\pi^*}(x) \) as the optimal value function when applying the optimal policy \( \pi^* \). Then we can define the optimal Q-function as

\[
Q^*(x, u) = \mathbb{E}_{w} \left\{ R(x, u, w) + \gamma J^*(f(x, u, w)) \right\}. \tag{1}
\]

In other words, the agent experiences an immediate reward at state \( x \), and follows the optimal policy \( \pi^* \) thereafter. We have

\[
J^*(x) = \min_{u \in U(x)} Q^*(x, u), \forall x \in \mathcal{X}, \tag{2}
\]
where \( U(x) \) is the admissible set of actions at state \( x \). Correspondingly, the optimal policy can also be obtained from the Q-function:

\[
\pi^*(x) = \arg\min_{u \in U(x)} Q^*(x, u). \tag{3}
\]

We can see that the optimal policy can be obtained from the optimal Q-function \( Q^*(x, u) \) without the knowledge of the system dynamics \( f \). This “model-free” property is one of the major advantages of Q-learning. Hereafter, we focus on how to obtain \( Q^*(x, u) \).

When the state-action space is finite, Q-function can be explicitly represented as a tabular function as

\[
Q_{i+1}(x, u) = (1 - \tau)Q_i(x, u) + \tau \mathbb{E}_{w} \{ R(x, u, w) \} + \gamma \min_{u' \in U(x')} Q_i(x', u'), \tag{4}
\]
where \( x' = f(x, u, w) \) and \( \tau \in (0, 1) \). It is called synchronous Q-learning when all the state-action pairs are updated simultaneously, and asynchronous when it updates only one data entry each step. In this paper, we focus on the synchronous algorithm.

We define an operator \( T \) as

\[
(TQ)(x, u) = \mathbb{E}_{w} \{ R(x, u, w) + \gamma \min_{u' \in U(x')} Q(x', u') \}, \tag{5}
\]
where \( x' = f(x, u, w) \).

Remark 2.1: Note that \((TQ)(x, u)\) has already incorporated the noise contribution through the expectation operation. In the remaining, we will simplify the notations by omitting the parameter \((x, u)\) when no ambiguity arises, e.g. \( Q := Q(x, u) \), \( TQ := TQ(x, u) \).

The vanilla Q-learning \((4)\) can be viewed as the iterative update of Q-function given a fixed target \( TQ_i \), and is called target update as

\[
Q_{i+1} = (1 - \tau)Q_i + \tau TQ_i, \tau \in (0, 1). \tag{6}
\]

Let \( \| \cdot \| = \| \cdot \|_\infty \), i.e., the infinity norm or sup-norm. In [7], the authors showed that the contraction property holds for the operator \( T \) in sup-norm with the discount factor \( \gamma \) as follows.

**Theorem 2.2 ([7]):** Given the operator \( T \) defined as \((5)\) with a fixed point \( Q^* \), i.e. \( TQ^* = Q^* \), for any bounded \( Q \) and \( Q' \),

\[
\| TQ - TQ' \| \leq \gamma \| Q - Q' \|. \tag{7}
\]

Based on this theorem, the convergence of the sequence \( \{Q_i\} \) generated by \((6)\) immediately follows.

\[
\| Q_{i+1} - Q^* \| \leq (1 - \tau) \| Q_i - Q^* \| + \tau \gamma \| TQ_i - TQ^* \| \leq (1 - \tau + \tau \gamma) \| Q_i - Q^* \|. \tag{8}
\]

We have seen the convergence of the vanilla Q-learning is immediate when using the contraction property \((7)\). From now on, we focus on how to improve the convergence performance based on the same contraction property by introducing a carefully designed accelerated scheme.

B. Accelerated Q-learning

In optimization and deep learning, momentum-based schemes, including Heavy-ball (HB) [23] and Nesterov’s accelerated gradient (NAG) [21], have been widely used to accelerate the convergence of gradient based algorithms. Their stability and convergence behaviors have been well studied for a wide class of convex and nonconvex problems. Inspired by the momentum idea, we propose a class of accelerated Q-learning (AQL) algorithm for the tabular Q-function learned in the synchronized manner where all the state-action pairs are updated simultaneously. The update of AQL performs as follows.

\[
P_{i+1} = Q_i + \tau (TQ_i - Q_i),
\]

\[
Q_{i+1} = P_{i+1} + \beta_1 (P_{i+1} - P_i) + \beta_2 (Q_i - Q_{i-1}), \tag{9}
\]
with \( \beta_1 \in (0, 1), \beta_2 \in (0, 1) \).

Notice that when we take \( \beta_1 = 0 \), then the update \((9)\) only involves one-step historical information \( Q_{i-1} \). This applies the same idea as HB by taking \( Q_i - Q_{i-1} \) as the momentum term and is referred as HB-AQL below. When taking \( \beta_1 \neq 0 \), we also involve \( TQ_{i-1} \) into the update, which is motivated by the idea of NAG and thus denoted as NesAQL. We additionally comment that the speedy Q-learning proposed in [4] only contains \( TQ_{i-1} \) in the update without using the historical information \( Q_{i-1} \), and thus can be somehow covered by our algorithm. In fact, by tuning \( \beta_1 \) and \( \beta_2 \) separately, we can view \((9)\) as a general AQL algorithm, and thus better performance can be expected. Given a general AQL with parameters \((\tau, \beta_1, \beta_2)\), we now
establish conditions under which AQL converges to the optimal Q function.

**Theorem 2.3:** Fix $\gamma \in (0, 1)$. Let $X$ and $U$ be finite state and action space. Q-function $Q(x, u)$ is a tabular function defined over $X \times U$. For $\tau \in (0, 1)$, $\beta_1 \in (0, 1)$, $\beta_2 \in (0, 1)$ satisfying

$$1 - \tau + \tau \gamma < \frac{1 - 2\beta_2}{1 + 2\beta_1},$$

(10)

the sequence $\{Q_i\}$ generated by (9) satisfies

$$\lim_{\tau \to 1} \|Q_i - Q^*\| = 0.$$

(11)

**Proof:** Observe that the update of $P_{i+1}$ is the same as (6) and thus we can still use (9) to bound the following sup-norm

$$\|P_{i+1} - Q^*\| \leq (1 - \tau + \gamma) \|Q_i - Q^*\|.$$  

(12)

Then we have

$$\|Q_{i+1} - Q^*\| = \|[(1 + \beta_1)(P_{i+1} - Q^*) - \beta_1(P_i - Q^*)
+ \beta_2(Q_i - Q^*) - \beta_2(Q_{i-1} - Q^*)]\|
\leq[(1 + \beta_1)(1 - \tau + \gamma) + \beta_2] \|Q_i - Q^*\|
+ [\beta_1(1 - \tau + \gamma) + \beta_2] \|Q_{i-1} - Q^*\|.$$  

(13)

Denote $z_i = \|Q_i - Q^*\|$ and the above inequality can be expressed as an element-wise vector inequality as

$$\begin{bmatrix} z_{i+1} \\ z_i \end{bmatrix} \leq \begin{bmatrix} a_1 & a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_i \\ z_{i-1} \end{bmatrix},$$  

(14)

where $a_1 = (1 + \beta_1)(1 - \tau + \gamma) + \beta_2, a_2 = \beta_1(1 - \tau + \gamma) + \beta_2$.

Notice that $z_i \geq 0, \forall i$, thus the inequality preserves when we can take 2-norm for both sides of (13).

$$\begin{bmatrix} z_{i+1} \\ z_i \end{bmatrix}_2 \leq \begin{bmatrix} a_1 & a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_i \\ z_{i-1} \end{bmatrix}_2.$$  

(15)

Denote $T = \begin{bmatrix} a_1 & a_2 \\ 1 & 0 \end{bmatrix}$, and $\rho(T) = \max_i |\lambda_i(T)|$ where $\lambda_i(T)$ is the $i$th eigenvalue of the matrix $T$. Note that $\rho(T) < 1$ can imply $\lim_{i \to \infty} \begin{bmatrix} z_{i+1} \\ z_i \end{bmatrix}_2 = 0$, and thus imply that $\lim_{i \to \infty} z_i = \lim_{i \to \infty} \|Q_i - Q^*\| = 0.$

It remains to find out the choice of parameter algorithms $(\tau, \beta_1, \beta_2)$ such that $\rho(T) = \frac{a_2 + \sqrt{a_1^2 + 4a_2}}{2a_2} < 1$. By further simplification we find that the parameters satisfying (10) can guarantee the bound and thus complete the proof.

We have proved the convergence of AQL in finite state-action space. We will also show it can numerically outperform the vanilla Q-learning algorithm (6) in section IV-B. In the next section, we will extend this acceleration scheme to a more general and significant case in which we deal with continuous state and action spaces.

### III. ACCELERATED Q-LEARNING WITH Q-FUNCTION APPROXIMATION

In this section, we extend our accelerated scheme to the general continuous state-action space, and provide theoretic convergence guarantee under the linear approximation architecture of Q-function. We also point out that our accelerated algorithm can be practically applied to arbitrary parametric approximation architectures including deep neural networks.

We consider a bounded continuous state space $X \subset \mathbb{R}^N$ and action space $U \subset \mathbb{R}^M$ with a nonnegative bounded cost function $R : X \times U \times W \to [0, R_{\text{max}}]$ where $W$ is the noise. In this case, it is often impossible or extremely difficult to write Q-function as an explicit tabular function, and thus the update rule of (6) is no longer applicable.

To handle this problem, we consider a parametric function $\hat{Q}(x, u; \theta)$ as an approximation of Q-function. The parameter vector $\theta$ is of finite and relatively lower dimension and thus easier to implement. The approximation architectures can be rich through different choices of the function class [7]. Among them, we will mostly focus on the linear architecture, that is,

$$\hat{Q}(x, u; \theta) = \Phi(x, u)\theta \hat{Q} = \sum_{j=1}^{M} \Phi_j(x, u)\theta_j \hat{Q},$$  

(16)

where $\theta \in \mathbb{R}^s$, and $\Phi : X \times U \to \mathbb{R}^{(M+N) \times s}$ is matrix function of size $(M + N) \times s$, where the columns $\Phi_j$ represent nonlinear kernel (feature) functions. Sutton and Barto [27] and Bertsekas [6] have stressed the importance of this specific architecture. A well-known application of the linear approximation architecture related to control is learning linear quadratic regulation (LQR) without knowing the model, which will be explained in detail following with numerical results in section IV-C. Hereafter, we aim to find out the optimal estimate of $Q^*$ from the linear approximation function space.

First we consider an expected target loss defined as:

$$L(\theta_i) = \mathbb{E}[TQ_{i-1}(x, u; \theta_{i-1}) - \Pi(T\hat{Q}_i)(x, u; \theta_i)]^2,$$  

(17)

where

$$TQ_{i-1} := TQ_{i-1}(x, u; \theta_{i-1}) = R(x, u, w) + \gamma \min_{u' \in U(\omega') \cup \omega'} \hat{Q}_{i-1}(x', u'; \theta_{i-1})$$

with $x' = f(x, u, w)$. We further denote $\Pi(T\hat{Q}_i) := \Pi(T\hat{Q}_i)(x, u; \theta_i)$ as the approximation or projection of $T\hat{Q}_i$ in the chosen linear function space at iteration $i$. To be specific, given $\theta_{i-1}$ and a state-action pair $(x, u)$ at iteration $i$, we want to minimize (16) over $\theta_i$ to find out $\Pi(T\hat{Q}_i)$, that is

$$\rho(\Pi(T\hat{Q}_i)) = \arg\min_{\theta_i \in \mathbb{R}^s} L(\theta_i).$$  

(18)

One can obtain $\rho(\Pi(T\hat{Q}_i))$ by solving a least square problem. When state dimension is high or the dynamics is complex and noisy, one other option is to perform gradient-based optimization algorithm, e.g. SGD, through differentiating the loss function with respect to the function parameters $\theta_i$. With
sufficient time steps run by the optimization algorithm, we propose the following assumption.

**Assumption 3.1:** Given $\theta_{i-1}$ and a state-action pair $(x, u)$ at iteration $i$, $T\hat{Q}_i$ can be sufficiently learned with an error uniformly bounded by $\epsilon$ via minimizing the loss $\ell$, i.e.,

$$\|\Pi(T\hat{Q}_i) - T\hat{Q}_i\| < \epsilon, \forall i.$$  \hspace{1cm} (18)

This assumption has been proven valid for various classes of functions including convex [35] and some non-convex approximation architectures such as shallow neural network [2], [18].

So far we have obtained the parameter of $\Pi(T\hat{Q}_i)$, i.e. $\theta_{i}^{\Pi(T\hat{Q}_i)}$ satisfying Assumption 3.1 via minimizing (16) at iteration $i$. We then perform the target update over parameter space as

$$\theta_{i+1}^{\hat{Q}} = (1-\tau)\theta_{i}^{\hat{Q}} + \tau \theta_{i}^{\Pi(T\hat{Q}_i)},$$  \hspace{1cm} (19)

with $\tau \in (0, 1)$ and $\theta_{i}^{\hat{Q}}$ being the parameter of the function $\hat{Q}(x, u; \theta_i)$. Similar to AQL, we propose the accelerated scheme of (19), named parametric AQL (PAQAL), as follows.

$$\Delta_{i+1} = \theta_{i+1}^{\hat{Q}} - \theta_{i}^{\hat{Q}},$$

$$\theta_{i+1}^{\hat{Q}} = \theta_{i}^{\hat{Q}} + \beta_1 (\Delta_{i+1} - \Delta_i) + \beta_2 (\theta_{i}^{\hat{Q}} - \theta_{i-1}^{\hat{Q}}),$$  \hspace{1cm} (20)

where $\beta_1 \in (0, 1)$, $\beta_2 \in (0, 1)$ are the hyper-parameters of PAQAL. Similar to the tabular case, we will refer to (20) as HBPAQL when $\beta_1 = 0$, and NesPAQL when $\beta_1 \neq 0$.

Notice that under the linear approximation architecture, we can transfer the update in the parameter space as (20) into the value space. Specifically, we multiply $\hat{Q}(x, u)$ from the left to all the terms in (20) and obtain

$$\hat{\hat{Q}}_{i+1} = \hat{Q}_i + \tau (\Pi(T\hat{Q}_i) - \hat{Q}_i),$$

$$\hat{Q}_{i+1} = \hat{\hat{Q}}_{i+1} + \beta_1 (\hat{\hat{Q}}_{i+1} - \hat{\hat{Q}}_i) + \beta_2 (\hat{Q}_i - \hat{\hat{Q}}_{i-1}).$$  \hspace{1cm} (21)

Observe that (21) is similar to the AQL algorithm [9]. The only difference is that we have $\Pi(T\hat{Q}_i)$ instead of $T\hat{Q}_i$. Generally the contraction property [7] cannot be directly applied to $\Pi(T\hat{Q}_i)$ [7], and thus the technique to prove convergence for tabular AQL is no longer applicable. With assumptions made before, however, we can still guarantee the stability with careful arguments which are concluded as the following theorem.

**Theorem 3.2:** Fix $\gamma \in (0, 1)$. We consider the linear approximation architecture as (15) and Assumption 3.1 holds. For $\tau \in (0, 1)$, $\beta_1 \in (0, 1)$, $\beta_2 \in (0, 1)$ satisfying

$$1 - \tau + \tau \gamma < \frac{1 - 2\beta_2}{1 + 2\beta_1},$$  \hspace{1cm} (22)

the sequence $\{\hat{Q}_i\}$ generated by (20) satisfies

$$\lim_{i \to \infty} \|\hat{Q}_i - \hat{Q}^*\| \leq \frac{(1 + 2\beta_1)}{1 - \tau} \epsilon,$$  \hspace{1cm} (23)

where $c = \frac{a_1 + \sqrt{a_1^2 + 4a_2^2}}{2}$ with $a_1 = (1 + \beta_1)(1 - \tau + \gamma \tau) + \beta_2$ and $a_2 = \beta_1 (1 - \tau + \gamma \tau) + \beta_2$.

**Proof:** Under the linear approximation architecture, we have seen that (20) can be equivalently represented as (21). Further with Assumption 3.1 and the contraction property of the operator $T$ as [7], we obtain

$$\|\hat{\hat{Q}}_{i+1} - \hat{Q}^*\| = \|[(1-\tau)(\hat{Q}_i - \hat{Q}^*) + \tau (\Pi(T\hat{Q}_i) - \hat{Q}^*)] + \tau (\Pi(T\hat{Q}_i) - T\hat{Q}_i) + \tau (T\hat{Q}_i - \hat{Q}^*)\|$$

$$\leq (1 - \tau)\|\hat{Q}_i - \hat{Q}^*\| + \tau \|T\hat{Q}_i - \hat{Q}^*\|$$

$$\leq (1 - \tau + \tau \gamma)\|\hat{Q}_i - \hat{Q}^*\| + \tau \epsilon.$$  \hspace{1cm} (24)

Then we have

$$\|\hat{Q}_{i+1} - \hat{Q}^*\| = \|\hat{\hat{Q}}_{i+1} - \hat{Q}^*\| - \beta_1 (\hat{\hat{Q}}_{i+1} - \hat{Q}^*) - \beta_2 (\hat{Q}_i - \hat{Q}^*)$$

$$\leq [(1 - \tau)(\hat{Q}_i - \hat{Q}^*) + \tau (\Pi(T\hat{Q}_i) - \hat{Q}^*)]$$

$$\leq [(1 - \beta_1)(1 - \tau + \gamma \tau) + \beta_2] \|\hat{Q}_i - \hat{Q}^*\| + (1 + \beta_1)\tau \epsilon$$

$$+ \beta_1 (1 - \tau + \gamma \tau) + \beta_2 \|\hat{Q}_i - \hat{Q}^*\| + \beta_1 \tau \epsilon.$$

Let $z_i = \|\hat{Q}_i - \hat{Q}^*\|$. Then the above inequality can be equivalently represented by

$$z_{i+1} + c_1 z_i \leq c_2 (z_i + c_1 z_{i-1}) + (1 + 2\beta_1)\epsilon.$$  \hspace{1cm} (25)

where $c_1 = \frac{a_1 + \sqrt{a_1^2 + 4a_2^2}}{2}$, $c_2 = \frac{a_1 + \sqrt{a_1^2 + 4a_2^2}}{2}$ with $a_1 = (1 + \beta_1)(1 - \tau + \gamma \tau) + \beta_2$, $a_2 = \beta_1 (1 - \tau + \gamma \tau) + \beta_2$.

When the condition (22) is satisfied, one can check that $c_1 > 0$ and $c_2 < 1$ and by induction we further have

$$z_{i+1} + c_1 z_i \leq c_2 (z_i + c_1 z_{i-1}) + (1 + 2\beta_1)\epsilon$$

$$\leq c_2^2 (z_{i-1} + c_1 z_{i-2}) + \tau (1 + 2\beta_1) (1 + c_2)\epsilon$$

$$\cdots$$

$$\leq c_2^i (z_{i-1} + c_1 z_{i-2}) + \tau (1 + 2\beta_1) \epsilon \sum_{k=0}^{i-1} c_2^k.$$  \hspace{1cm} (26)

Let $i \to \infty$ and we can complete the proof by

$$\lim_{i \to \infty} \|\hat{Q}_i - \hat{Q}^*\| < \lim_{i \to \infty} (z_{i+1} + c_1 z_i) < \frac{(1 + 2\beta_1)}{1 - c_2} \epsilon.$$  \hspace{1cm} (27)

The above convergence results of the proposed PAQAL are proved under the linear approximation architecture. We emphasize that, however, the proposed PAQAL have a wider range of application with the choice of richer approximation architectures. When Q-function is parameterized as a deep neural network with $\theta_{i}^{\hat{Q}}$ being the network weights, for example, [19] is referred as target update. Combining [19] with SGD learning $T\hat{Q}$, we have the exact form of deep Q-network (DQN) learning [19]. Such neural network approximation architecture can be combined with the update rule of the proposed PAQAL [20]. Although formal convergence results cannot be guaranteed, faster convergence are often observed (see Section IV-D).

In the next section, we will provide numerical results to see more complicated and promising applications of our accelerated scheme.
IV. SIMULATION RESULTS

In this section, we numerically evaluate the performance of the proposed AQL and PAQL algorithms in various challenging problems including the FrozenLake grid world game, two discrete-time LQR problems from the Deepmind Control Suite [28] and Atari 2600 games. For the three aforementioned problem domains, the Q-function is defined as a tabular function, a parametric function with linear approximation and a deep convolutional neural network, respectively. Before proceeding to the simulation results, we first clarify the appropriate sampling strategy with respect to the nature of different Q-learning problems.

A. Appropriate sampling

The Q-learning algorithm essentially ensembles the value iteration method. Without knowledge of underlying transition probabilities or system dynamics, certain sampling strategy is required. Given the trade-off between finite samples and the desired approximation performance, an appropriate sampling strategy is playing a crucial role in the practice of Q-learning.

For learning with tabular Q-functions, we repeatedly measure results on the same entry of state-action pair to approximate the expectation. Ideally, this requires abusing the simulation by setting the system to arbitrary state, followed with an action being uniformly sampled from the admissible action space. Practically, we do sampling in an episodic manner, with one episode being a process that starts at an initial state and terminates after finite number of steps or when certain condition is met. The sampled data set is constructed with multiple episodes such that state-action pairs are visited infinitely often. This also aligns with the assumptions for theoretical convergence of Q-learning [14].

Uniform sampling is practically difficult when state space is continuous or considerably large. The collected episodes for each iterate, namely the experience, are discarded after each step of target learning. This slows down the learning speed with extra sampling time and potentially breaks the i.i.d. assumption for SGD. We adopt $\epsilon$-greedy [20] and prioritized experience replay [24], both of which are common techniques in RL with parametric approximation. $\epsilon$-greedy balances the exploitation of current policy $\pi_i$ and exploration in the action space. By introducing a parameter $\epsilon$ that decays as training proceeds, the probability of using the policy $\pi_i$ to sample data increases accordingly. Prioritized experience replay establish a “memory” of samples that contains multiple episodes from recent rounds of iterates. The probability for each data entry to be sampled in training is based on the magnitude of its temporal difference error. Therefore, important transitions possess stochastic prioritization during the training stage.

With the theoretical study usually assuming certain ideal form of sampling, the importance of an appropriate sampling method is often underestimated. When dealing with more complex problems with high dimensions and intricate dynamics, as we will illustrate later, careful design of sampling strategies is necessary in order to achieve desired performance.

B. FrozenLake

FrozenLake is a classic baseline problem for Q-learning. An agent controls the movement of a character in a grid world. Some tiles of the grid are walkable, and others lead to the agent falling into the water. Additionally, the movement direction of the agent is uncertain and only partially depends on the chosen direction. The agent is rewarded for finding a feasible path to a goal tile. The environment for FrozenLake is a $4 \times 4$ grid world. We consider two sub-tasks, the FrozenLake and the FrozenLake8x8 with a bigger grid world (Fig. 1(a)). In both Frozenlake tasks, “S” is the safe starting point, “F” is the safe frozen surface, “H” stands for the hole that terminates the game, and “G” is the target state that comes with an immediate reward of 1. This forms a problem with state space size of $16 \times 64$ for FrozenLake8x8, action space size of 4 and reward space $R \in \{0, 1\}$.

As discussed in section III, Q-function for such case can be explicitly defined as a tabular function. For evaluation purpose, we have access to the true transition probability, and can find the ground truth optimal Q-function $Q^*$ using dynamic programming. We then test the original Q-learning algorithm (6) and the proposed AQL (9) algorithm on the two tasks. Detailed comparisons are illustrated in Fig. 2. We set $\tau = 0.1$, $\beta_1 = 0.1$, $\beta_2 = 0.1$ for corresponding algorithms. For all the listed algorithms, a higher $\tau$ could achieve faster convergence, but also causes the oscillation during the later iterates. This is due to the over-estimations of Q values and can be further resolved by double Q-learning [13]. The proposed AQL algorithm is also compatible with double Q-learning but further comparisons are out of the scope of this paper.

Fig. 1: (a) FrozenLake8x8 Environment, (b) LQR with 6 Masses and 2 Linear Actuators, (c) Boxing from Atari 2600 Games

Fig. 2: Comparing Various Methods for FrozenLake Games
C. Linear Quadratic Regulation

The problem of infinite-horizon discrete-time linear quadratic regulation (LQR) considers a linear system

\[ x_{t+1} = Ax_t + Bu_t, \tag{25} \]

with cost function in the quadratic form as

\[ J = \sum_{t=0}^{\infty} (x_t^T Q x_t + u_t^T R u_t + 2x_t^T N u_t). \tag{26} \]

Let the positive definite \( P \) be the unique solution to the discrete-time algebraic Riccati equation (DARE)

\[ P = A^T PA - (A^T PB + N)(R + B^T PB)^{-1}(B^T PA + N^T) + Q. \tag{27} \]

We have the optimal control as \( u^*_t = -K^* x_t \) with

\[ K^* = (R + B^T PB)^{-1}(N^T + B^T PA). \tag{28} \]

Following the procedure established in section III we parameterize a quadratic Q-function with a matrix parameter \( H \) in the form of

\[ Q(x, u; H) = \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}. \tag{29} \]

The linear policy corresponding to (29) satisfies \( u = -K x, K = 2H_{uu}^{-1}H_{ux} \). We evaluate the performance of various Q-learning methods at each iterate \( \tau \) with the Euclidean norm \( \|K^* - K^*\|_2 \).

In this section, the linear system is constructed as a coupled mass damper system with \( n \) masses, serially connected through linear joints (see Fig. [I](b)) with \( m \) joints being actuated. The system has the state dimension of \( 2n \) with position state \( x_p \) and velocity state \( x_v \). The action dimension is \( m \). The reward is quadratic with respect to the position and controls, i.e. \( R = \frac{1}{2} x_p^T Q x_p + \eta_2 u^T Ru \) with control cost coefficient \( \eta = 0.1 \). The system is a default RL benchmark from the Deepmind control suite [28]. These tasks are more challenging than the typical LQR problems being considered in the control literature. We consider two sub-tasks, the "LQR_2.1" with \( n = 2, m = 1 \) and the "LQR_6.2" with \( n = 6, m = 2 \) which take 4269 and 11840 iterates respectively to converge to \( K^* \) through DARE.

We compare the performance of proposed PAQL algorithms with Q-learning in Fig [3] and Fig. [4]. For both tasks, we let \( \tau = 0.9, \beta_1 = 0.2, \beta_2 = 0.2 \) for corresponding algorithms. The learning process of DARE is also included. Direct comparison regarding the training time with DARE is not fair given that DARE requires system dynamics but Q-learning methods are model-free. In our illustration, we exclude the sampling time and consider the number of value iterations required to achieve certain level of desired performance (Table [I]).

D. Atari 2600 games

We further evaluate the performance of PAQL with two Atari 2600 games. It is a challenging RL benchmark task that takes high-dimensional high-frequency video sequence (\( \text{dim}(X) = 84 \times 84 \times 4 \)) as state and real video game control keys as action. The performance for each algorithm is justified empirically by the average return of 100 trails of episodes. The Q-function is parameterized as a deep convolutional neural network. Various improved techniques for DQN [30], [32] are disabled. Hyper-parameters are set as \( \tau = 0.9, \beta_1 = 0.2, \beta_2 = 0.2 \). The algorithm is implemented based on the open.ai baseline, which is a set of high-quality implementations of RL algorithms. The original DQN implementation and its variants are roughly on par with scores in published papers, which mostly exceeds expert level of human play. Results are illustrated in Fig. [5].

The Q-function is structured with millions of parameters. The sampling and target learning with SGD both consume a significant amount of time and computational power. On a dual-GPU machine with the PAQL algorithm, the training for the game Pong takes 0.6 million samples in 15 minutes. For the Boxing game to achieve the illustrated results, PAQL takes 2 million samples in 40 minutes. On the contrast, DQN...
would require at least 4 million samples to acquire similar performance.

V. CONCLUSION

We propose AQL and PAQL, a set of accelerated Q-learning algorithms, which are provably stable in finite and continuous state-action space with mild assumptions, respectively. We empirically evaluate the algorithms with various challenging tasks and verify that the proposed algorithms can accelerate the convergence of the vanilla Q-learning algorithm.

There are several opportunities for the future extension of this work. We are motivated, for example, to explore more complicated adaptive accelerated schemes to improve the convergence performance. For the continuous state-action space, it is also important to study the stability of the target update when using a general nonlinear approximation architecture such as neural network, which will be of our future interest.

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