An adaptive $C^0$IPG method for the Helmholtz transmission eigenvalue problem

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Abstract

The interior penalty methods using $C^0$ Lagrange elements ($C^0$IPG) developed in the last decade for the fourth order problems are an interesting topic in academia at present. In this paper, we discuss the adaptive fashion of $C^0$IPG method for the Helmholtz transmission eigenvalue problem. We give the a posteriori error indicators for primal and dual eigenfunctions, and prove their reliability and efficiency. We also give the a posteriori error indicator for eigenvalues and design a $C^0$IPG adaptive algorithm. Numerical experiments show that this algorithm is efficient and can get the optimal convergence rate.

Key words: transmission eigenvalues, interior penalty Galerkin method, Lagrange elements, a posteriori error estimates, adaptive algorithm.

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1 Introduction

The transmission eigenvalues can be used to obtain estimates for the material properties of the scattering object [1][2][3], and have theoretical importance in the uniqueness and reconstruction in inverse scattering theory [4]. In recent years, the computation of transmission eigenvalues has attracted the attention of many researchers. The first numerical treatment of the transmission eigenvalue problem appeared in [5] where three finite element methods, including the Argyris, continuous and mixed methods, are proposed for

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the Helmholtz transmission eigenvalues, and has been further developed by
6,7,8,9,10,11,12,13,14,15,16,17,18,19 et al.. C⁰ interior penalty Galerkin (C⁰IPG) method, developed in the last decade
[20,21], is a new class of Galerkin methods for fourth order problems. The
researches for C⁰IPG methods have been an interesting topic in academia at
present. There exist many researches for fourth order elliptic equations (see
[20,21,22,23,24]) and for eigenvalue problems (see [8,25,26,27,28,29]) by
C⁰IPG methods.

The a posteriori error estimates and adaptive finite element methods are al-
ways the main streams of scientific and engineering computing. The idea of the
a posteriori error estimates was first introduced by Babuska and Rheinboldt
[30] in 1978. Up to now, many excellent works have been summarized in the
books such as [31,32,33]. And a posteriori error estimates of residual type of
C⁰IPG method of fourth order elliptic equations also have been summarized
in [21].

Inspired by the works mentioned above, in this paper, based on the weak
formulation proposed in [16,17], we propose a new C⁰IPG discrete scheme
(see (2.25)) and discuss the a posteriori error estimates and adaptive algo-
rithm of C⁰IPG method for the Helmholtz transmission eigenvalue problem.
We give the a posteriori error indicators for primal and dual eigenfunctions
and eigenvalues. We prove that the indicators for both primal and dual eigen-
functions are reliable and efficient, and analyze the reliability of the indicator
for eigenvalues. Based on the given indicators, we design an adaptive algo-
rithm. Numerical experiments show that this algorithm is efficient and can
get the optimal convergence rate. Compared with adaptive C¹ conforming fi-
nite element algorithm in [9], the adaptive C⁰IPG algorithm is simpler to be
constructed and implemented numerically.

In this paper, regarding the basic theory of finite element methods, we refer
to [33,34,35,36,37].

Throughout this paper, the letter C (with or without subscripts) denotes a
positive constant independent of mesh size h, which may not be the same
constant in different places. For simplicity, we use the symbol a ≤ b to mean
that a ≤ Cb and the symbol a ≈ b to mean a ≲ b ≲ a.

2 A C⁰IPG discrete scheme

Consider the Helmholtz transmission eigenvalue problem: Find k ∈ C, w, σ ∈
L²(Ω), w − σ ∈ H²(Ω) such that
\[ \Delta w + k^2 nw = 0, \quad in \ \Omega, \quad (2.1) \]
\[ \Delta \sigma + k^2 \sigma = 0, \quad in \ \Omega, \quad (2.2) \]
\[ w - \sigma = 0, \quad on \ \partial \Omega, \quad (2.3) \]
\[ \frac{\partial w}{\partial \gamma} - \frac{\partial \sigma}{\partial \gamma} = 0, \quad on \ \partial \Omega, \quad (2.4) \]

where \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) is a bounded simply connected inhomogeneous medium, \( \gamma \) is the unit outward normal to \( \partial \Omega \) and the index of refraction \( n = n(x) \) is positive.

Let \( W^{s,p}(\Omega) \) denote the usual Sobolev space with norm \( \| \cdot \|_{s,p} \), \( H^s(\Omega) = W^{s,2}(\Omega) \), and \( \| \cdot \|_{s,2} = \| \cdot \|_s \), \( H^0(\Omega) = L^2(\Omega) \) with the inner product \( (u,v)_0 = \int_\Omega u v dx \). Denote \( H^2_0(\Omega) = \{ v \in H^2(\Omega) : v|_{\partial \Omega} = \frac{\partial v}{\partial \gamma}|_{\partial \Omega} = 0 \} \). Let \( H^{-1}(\Omega) \) be the “negative space” with norm \( \| \cdot \|_{-1} \).

Define Hilbert space \( H = H^2_0(\Omega) \times L^2(\Omega) \) with norm \( \| (v,z) \|_H = \| v \|_2 + \| z \|_0 \), and define \( H^1 = H^1_0(\Omega) \times H^{-1}(\Omega) \) with norm \( \| (v,z) \|_{H^1} = \| v \|_1 + \| z \|_{-1} \).

Since \( L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \) compactly and \( H^2(\Omega) \hookrightarrow H^1(\Omega) \) compactly, \( H \hookrightarrow H^1 \) compactly.

In this paper, we suppose that \( n \in W^{1,\infty}(\Omega) \) satisfying the following condition

\[ 1 + \delta \leq n(x) \text{ in } \Omega, \]

for some constant \( \delta > 0 \). And the argument is the same if \( 0 < n(x) \leq 1 - \varrho \) in \( \Omega \) \((\varrho > 0)\) holds.

From \([38,39]\) we know that the problem \((2.1)-(2.4)\) can be written as the following equivalent weak formulation: Find \( k \in \mathbb{C}, u \in H^2_0(\Omega) \) such that

\[ (\frac{1}{n-1} \Delta u, \Delta v)_0 = k^2 (\nabla u, \nabla (\frac{n}{n-1}v))_0 + k^2 (\nabla (\frac{1}{n-1}u), \nabla v)_0 - k^4 (\frac{n}{n-1}u, v)_0, \quad \forall v \in H^2_0(\Omega). \]

Introduce an auxiliary variable \( \omega = k^2 u \), and let \( \lambda = k^2 \), then we arrive at a linear weak formulation (see \([16,17]\)): Find \( \lambda \in \mathbb{C}, (u, \omega) \in H \setminus \{0\} \) such that

\[ A((u, \omega), (v, z)) = \lambda B((u, \omega), (v, z)), \quad \forall (v, z) \in H, \quad (2.5) \]

where

\[ A((u, \omega), (v, z)) = ((\frac{1}{n-1} - \mu) \Delta u, \Delta v)_0 + \mu \int_\Omega D^2 u : D^2 \bar{v} dx + (\omega, z), \quad (2.6) \]

with constant \( \mu > 0 \), \( \frac{1}{n-1} - \mu \geq 0 \), and
\[
B((u, \omega), (v, z)) = \left( \nabla \left( \frac{1}{n-1} u \right), \nabla v \right)_0 + \left( \nabla u, \nabla \left( \frac{n}{n-1} v \right) \right)_0 - (\omega, \frac{n}{n-1} v)_0 + (u, z)_0.
\]

It is obvious that \( A(\cdot, \cdot) \) is a selfadjoint, continuous sesquilinear form on \( \mathbf{H} \times \mathbf{H} \),

\[
A((v, z), (v, z)) \gtrsim \| (v, z) \|^2_{\mathbf{H}},
\]

and for any given \((f, g) \in \mathbf{H}^1\), \( B((f, g), (v, z)) \) is a continuous linear form on \( \mathbf{H} \),

\[
|B((f, g), (v, z))| \lesssim \| (f, g) \|_{\mathbf{H}^1} \| (v, z) \|_{\mathbf{H}^1}, \quad \forall (v, z) \in \mathbf{H}^1.
\]

We use \( A(\cdot, \cdot) \) and \( \| \cdot \|_A = A(\cdot, \cdot)^{\frac{1}{2}} \) as an inner product and norm on \( \mathbf{H} \), respectively.

The source problem associated with (2.5) is as follows: Find \((\psi, \varphi) \in \mathbf{H}\) such that

\[
A((\psi, \varphi), (v, z)) = B((f, g), (v, z)), \quad \forall (v, z) \in \mathbf{H}.
\]

From Lax-Milgram theorem we know that (2.9) has one and only one solution. Therefore, we define the corresponding solution operator \( T : \mathbf{H}^1 \to \mathbf{H} \) by

\[
A(T(f, g), (v, z)) = B((f, g), (v, z)), \quad \forall (v, z) \in \mathbf{H}.
\]

Then (2.5) has the equivalent operator form:

\[
T(u, \omega) = \frac{1}{\lambda}(u, \omega).
\]

From (2.10) we have

\[
\| T(f, g) \|_{\mathbf{H}} \lesssim \| (f, g) \|_{\mathbf{H}^1}, \quad \forall (f, g) \in \mathbf{H}^1.
\]

Thus we know that \( T : \mathbf{H} \to \mathbf{H} \) is compact, and \( T : \mathbf{H}^1 \to \mathbf{H}^1 \) is compact.

Consider the dual problem of (2.5): Find \( \lambda^* \in \mathbb{C}, (u^*, \omega^*) \in \mathbf{H} \setminus \{0\} \) such that

\[
A((v, z), (u^*, \omega^*)) = \overline{\nabla} B((v, z), (u^*, \omega^*)), \quad \forall (v, z) \in \mathbf{H}.
\]

The source problem associated with (2.13) is as follows: Find \((\psi^*, \varphi^*) \in \mathbf{H}\) such that
Define the corresponding solution operator \( T^* : H^1 \rightarrow H \) by

\[
A((v, z), T^*(f, g)) = B((v, z), (f, g)), \quad \forall (v, z) \in H.
\]

Then (2.13) has the equivalent operator form:

\[
T^*(u^*, \omega^*) = \lambda^{*-1} (u^*, \omega^*).
\]

From (2.10) and (2.15) we know that \( T^* \) is the adjoint operator of \( T \) in the sense of inner product \( A(\cdot, \cdot) \). So the primal and dual eigenvalues are connected via \( \lambda = \overline{\lambda} \) (see [16]).

Denote

\[
\mathbb{S} = \left( \frac{2d}{1 + d}, 2 \right].
\]

We need the following regularity assumption:

\( R(\Omega) \). For any \( \xi \in H^{-1}(\Omega) \), there exists \( \psi \in W^{3,p_0}(\Omega) \) satisfying

\[
\Delta \left( \frac{1}{n - 1} \Delta \psi \right) = \xi, \quad \text{in } \Omega, \quad \psi = \frac{\partial \psi}{\partial \gamma} = 0 \quad \text{on } \partial \Omega,
\]

and

\[
\| \psi \|_{3,p_0} \leq C_\Omega \| \xi \|_{-1},
\]

where \( p_0 \in \mathbb{S}, \ C_\Omega \) denotes the prior constant dependent on the \( n(x) \) and \( \Omega \) but independent of the right-hand side \( \xi \) of the equation.

Let \( \pi_h \) be a shape-regular mesh, for any element \( \kappa \in \pi_h \), let \( h_\kappa \) denote diameter of \( \kappa \), \( h = \max_{\kappa \in \pi_h} h_\kappa \). And let

\[
S^h = \{ v \in C(\overline{\Omega}) \cap H^1_0(\Omega) : v|_\kappa \in P_m, \forall \kappa \in \pi_h \},
\]

where \( P_m \) is the set of all polynomials in \( d \) variables of degree \( \leq m (m \geq 2) \).

Let \( H^h = S^h \times S^h \). Then \( H^h \subset H^1 \) but \( H^h \not\subset H \).

Let \( p \in \mathbb{S} \), from the trace theorem with scaling we have the following trace inequality:

\[
\int_{\partial \kappa} |w|^2 ds \lesssim h_\kappa^{d-2d/p-1} \| w \|_{0,p,\kappa}^2 + h_\kappa^{2+d-2d/p-1} \| w \|_{1,p,\kappa}^2, \quad \forall \kappa \in \pi_h.
\]
Let $E$ denote the set of all $(d - 1)$-faces in $\pi_h$ ($d = 2, 3$). We decompose $E = E^i \cup E^b$ where $E^i$ and $E^b$ refer to interior faces and faces on the boundary $\partial \Omega$, respectively. For each $\ell \in E^i$, we choose an arbitrary unit normal vector $\gamma_\ell$ and denote the two triangles sharing this face by $\kappa^-$ and $\kappa^+$, where $\gamma_\ell$ points from $\kappa^-$ to $\kappa^+$. We set the jump and average on $\ell$ as

$$[[\partial v / \partial \gamma_\ell]] = \nabla(v|_{\kappa^+}) \cdot \gamma_\ell - \nabla(v|_{\kappa^-}) \cdot \gamma_\ell,$$

(2.19)

$$\{((1 - \mu)\Delta v)\} = \frac{1}{2}( (1 - \mu)\Delta v|_{\kappa^-} + (1 - \mu)\Delta v|_{\kappa^+}),$$

(2.20)

$$\{\partial^2 v / \partial \gamma_\ell^2\} = \frac{1}{2}(\partial^2 v / \partial \gamma_\ell^2|_{\kappa^-} + \partial^2 v / \partial \gamma_\ell^2|_{\kappa^+})$$

(2.21)

with $\partial^2 v / \partial \gamma_\ell^2 = \gamma_\ell \cdot (D^2 v)\gamma_\ell$.

For any $\ell \in E^b$ which is a face of $\kappa$, we take $\gamma_\ell$ to be the unit normal vector pointing towards the outside of $\Omega$ and set

$$[[\partial v / \partial \gamma_\ell]] = -\gamma_\ell \cdot \nabla(v|_{\kappa}),$$

(2.22)

$$\{((1 - \mu)\Delta v)\} = (1 - \mu)\Delta v|_{\kappa}, \quad \{\partial^2 v / \partial \gamma_\ell^2\} = \partial^2 v / \partial \gamma_\ell^2|_{\kappa}.$$  

(2.23)

Define piecewise Sobolev space

$$W^{3,p}(\Omega, \pi_h) = \{ v \in C(\bar{\Omega}) \cap H^1_0(\Omega) : v|_{\kappa} \in W^{3,p}(\kappa) \forall \kappa \in \pi_h \} \quad (p \in \mathbb{S}).$$

Referring [8,21,25], we define

$$A_h((u, \omega), (v, \omega)) = \sum_{\kappa \in \pi_h} \int_{\kappa} (1 - \mu)\Delta u \Delta \pi dx + \mu \int_{\kappa} D^2 u : D^2 \pi dx$$

$$+ \sum_{\ell \in E^i} \int_{\ell} \{((1 - \mu)\Delta u)\}[\partial \pi / \partial \gamma_\ell] + \{((1 - \mu)\Delta v)\}[\partial \pi / \partial \gamma_\ell] ds$$

$$+ \mu \sum_{\ell \in E^i} \int_{\ell} \{\partial^2 u / \partial \gamma_\ell^2\}[\partial \pi / \partial \gamma_\ell] + \{\partial^2 v / \partial \gamma_\ell^2\}[\partial \pi / \partial \gamma_\ell] ds$$

$$+ \sigma \sum_{\ell \in E^b} \int_{\ell} \{\partial u / \partial \gamma_\ell\}[\partial \pi / \partial \gamma_\ell] ds + \sum_{\kappa \in \pi_h} \int_{\kappa} \omega \pi dx,$$

(2.24)

where $\sigma > 1$ is the penalty parameter, and $\hat{\ell} = h_\ell$ is the diameter of $\ell$.

We give the following $C^0$IPG discrete scheme of (2.5): Find $\lambda_h \in C, (u_h, \omega_h) \in H_h \setminus \{0\}$ such that

$$...$$
\[ A_h((u_h, \omega_h), (v, z)) = \lambda_h B((u_h, \omega_h), (v, z)), \quad \forall (v, z) \in H_h. \quad (2.25) \]

We define the mesh-dependent norms \( \| \cdot \|_h \) and \( \| \| \|_h \) on \( W^{3,p}(\Omega, \pi_h) \times L^2(\Omega) \) as

\[
\|(u, \omega)\|_h^2 = \sum_{\kappa \in \pi_h} \|u\|_{2,\kappa}^2 + \sigma \sum_{\ell \in \mathcal{E}} \|\left[\frac{\partial u}{\partial \gamma_\ell}\right]\|_{0,\ell}^2 + \sum_{\kappa \in \pi_h} \|\omega\|_{0,\kappa}^2, \quad (2.26)
\]

\[
\|\|u, \omega\||_h^2 = \|\|u\|_h^2 + \frac{1}{\sigma} \sum_{\ell \in \mathcal{E}} \|\{\Delta u\}\|_{0,\ell}^2 + \frac{1}{\sigma} \sum_{\ell \in \mathcal{E}} \|\{\frac{\partial^2 u}{\partial \gamma_\ell^2}\}\|_{0,\ell}^2. \quad (2.27)
\]

By the trace inequality \( (2.18) \) with \( p = 2 \) and the inverse estimates we have

\[
\|\Delta v\|_{0,\ell} \lesssim \ell^{-\frac{1}{2}} \|v\|_{2,\kappa}, \quad \|\{\frac{\partial^2 u}{\partial \gamma_\ell^2}\}\|_{0,\ell} \lesssim \ell^{-\frac{1}{2}} \|v\|_{2,\kappa}, \quad \forall v \in S^h. \quad (2.28)
\]

So on \( H_h \) the two norms \( \| \cdot \|_h \) and \( \| \| \|_h \) are equivalent.

For any \( (u, \omega), (v, z) \in W^{3,p}(\Omega, \pi_h) \times L^2(\Omega) \), by the Schwartz inequality we can deduce

\[
\begin{align*}
|\lambda_h B((u_h, \omega_h), (v, z))| &\lesssim \sum_{\kappa \in \pi_h} \|\Delta u\|_{0,\kappa} \|\Delta v\|_{0,\kappa} + \sum_{\kappa \in \pi_h} \|u\|_{2,\kappa} \|\pi\|_{2,\kappa} \\
&+ \sum_{\ell \in \mathcal{E}} \left( \sqrt{\frac{\ell}{\sigma}} \|\{\Delta u\}\|_{0,\ell} \sqrt{\frac{\sigma}{\ell}} \|\{\frac{\partial \pi}{\partial \gamma_\ell}\}\|_{0,\ell} + \sqrt{\frac{\ell}{\sigma}} \|\{\Delta v\}\|_{0,\ell} \sqrt{\frac{\sigma}{\ell}} \|\{\frac{\partial \pi}{\partial \gamma_\ell}\}\|_{0,\ell} \right) \\
&+ \sum_{\ell \in \mathcal{E}} \left( \sqrt{\frac{\ell}{\sigma}} \|\{\frac{\partial^2 u}{\partial \gamma_\ell^2}\}\|_{0,\ell} \sqrt{\frac{\sigma}{\ell}} \|\{\frac{\partial \pi}{\partial \gamma_\ell}\}\|_{0,\ell} + \sqrt{\frac{\ell}{\sigma}} \|\{\frac{\partial^2 v}{\partial \gamma_\ell^2}\}\|_{0,\ell} \sqrt{\frac{\sigma}{\ell}} \|\{\frac{\partial \pi}{\partial \gamma_\ell}\}\|_{0,\ell} \right) \\
&+ \sigma \sum_{\ell \in \mathcal{E}} \frac{1}{\sqrt{\ell}} \|\frac{\partial u}{\partial \gamma_\ell}\|_{0,\ell} \frac{1}{\sqrt{\ell}} \|\frac{\partial \pi}{\partial \gamma_\ell}\|_{0,\ell} + \sum_{\kappa \in \pi_h} \|\omega\|_{0,\kappa} \|\pi\|_{0,\kappa} \\
&\lesssim \|\|u, \omega\||_h \|\|v, z\||_h. \quad (2.29)
\end{align*}
\]

And for any \( (u_h, \omega_h), (v, z) \in H_h \), we have

\[
|\lambda_h B((u_h, \omega_h), (v, z))| \leq C \|\|u, \omega\||_h \|\|v, z\||_h. \quad (2.30)
\]

And referring \[8,23\], when \( \sigma \) is large enough, by \( (2.28) \) and the Young inequality we deduce
\[
A_h((u_h, \omega_h), (u_h, \omega_h)) \geq C_1 \sum_{k \in \Pi_h} \left( \|\Delta u_h\|_0^2 + \|u_h\|_2^2 \right) \\
- \sqrt{C_1} \left( \sum_{k \in \Pi_h} \|\Delta u_h\|_{0,k}^2 \right)^{\frac{1}{2}} \frac{C_1}{\sqrt{C_1}} \left( \sum_{\ell \in E} \frac{1}{\ell} \|\left[\partial u_h / \partial \gamma \right]\|_{0,\ell}^2 \right)^{\frac{1}{2}} \\
- \sqrt{C_1} \left( \sum_{k \in \Pi_h} \|u_h\|_{2,k}^2 \right)^{\frac{1}{2}} \frac{C_1}{\sqrt{C_1}} \left( \sum_{\ell \in E} \frac{1}{\ell} \|\left[\partial u_h / \partial \gamma \right]\|_{0,\ell}^2 \right)^{\frac{1}{2}} \\
+ \sigma \sum_{\ell \in E} \frac{1}{\ell} \left[\partial u_h / \partial \gamma \right]_0^2 + \sum_{k \in \Pi_h} \|\omega_h\|_{0,k}^2 \\
\geq C_1 \left( \sum_{k \in \Pi_h} \|\Delta u_h\|_{0,k}^2 \right) + \left( \frac{\sigma}{2} - \frac{C_2}{2C_1} \right) \sum_{\ell \in E} \frac{1}{\ell} \left[\partial u_h / \partial \gamma \right]_{0,\ell}^2 \\
+ \frac{C_1}{2} \sum_{k \in \Pi_h} \|u_h\|_{2,k}^2 + \left( \frac{\sigma}{2} - \frac{C_2}{2C_1} \right) \sum_{\ell \in E} \frac{1}{\ell} \left[\partial u_h / \partial \gamma \right]_{0,\ell}^2 + \sum_{k \in \Pi_h} \|\omega_h\|_{0,k}^2 \\
\geq \|(u_h, \omega_h)\|_{H_h}^2, \quad \forall (u_h, \omega_h) \in H_h.
\]

Consider the $C^0$IPG discrete scheme of (2.9): Find $(\psi_h, \varphi_h) \in H_h$ such that

\[
A_h((\psi_h, \varphi_h), (v, z)) = B((f, g), (v, z)), \quad \forall (v, z) \in H_h.
\]

We introduce the corresponding solution operator: $T_h : H^1 \to H_h$:

\[
A_h(T_h, (f, g), (v, z)) = B((f, g), (v, z)), \quad \forall (v, z) \in H_h.
\]

Then (2.25) has the operator form:

\[
T_h(u_h, \omega_h) = \frac{1}{\lambda_h}(u_h, \omega_h).
\]

The $C^0$IPG discrete scheme of (2.13) is given by: Find $\lambda^*_h \in \mathbb{C}, (u^*_h, \omega^*_h) \in H_h \setminus \{0\}$ such that

\[
A_h((v, z), (u^*_h, \omega^*_h)) = \overline{\lambda_h} B((v, z), (u^*_h, \omega^*_h)), \quad \forall (v, z) \in H_h.
\]

Define the solution operator $T_h^* : H^1 \to H_h$ satisfying

\[
A_h((v, z), T_h^*(f, g)) = B((v, z), (f, g)), \quad \forall (v, z) \in H_h.
\]

Thus (2.35) has the following equivalent operator form:

\[
T_h^*(u_h^*, \omega_h^*) = \lambda_h^{-1}(u_h^*, \omega_h^*).
\]
It can be proved that $T_h^*$ is the adjoint operator of $T_h$ in the sense of inner product $A_h(\cdot, \cdot)$. In fact, $\forall (u, \omega), (v, z) \in H_h$, from (2.33) and (2.36) we have

$$A_h(T_h(u, \omega), (v, z)) = B((u, \omega), (v, z)) = A_h((u, \omega), T_h^*(v, z)).$$

Hence, the primal and dual eigenvalues are connected via $\lambda_h = \bar{\lambda}_h$.

In this paper, we suppose that $\{\lambda_j\}$ and $\{\lambda_{j,h}\}$ are enumerations of the eigenvalues of (2.25) and (2.25) respectively according to the same sort rule, each repeated as many times as its multiplicity, and $\lambda = \lambda_i$ is the $i$th eigenvalue with the algebraic multiplicity $q$ and the ascent $\alpha$, $\lambda_i = \lambda_i+1 = \cdots, \lambda_{i+q-1}$, and $\lambda_h = \lambda_{i,h}$. When $\|T_h - T\|_{H^1} \to 0$, $q$ eigenvalues $\lambda_{i,h}, \cdots, \lambda_{i+q-1,h}$ of (2.25) will converge to $\lambda$.

Let $E$ be the spectral projection associated with $T$ and $\lambda$, then $\text{ran}(E) = \text{null}((\lambda^{-1} - T)^q)$ is the space of generalized eigenfunctions associated with $\lambda$ and $T$, where $\text{ran}$ denotes the range and $\text{null}$ denotes the null space. Let $E_h$ be the spectral projection associated with $T_h$ and the eigenvalues $\lambda_{i,h}, \cdots, \lambda_{i+q-1,h}$, then $\text{ran}(E_h)$ is the space spanned by all generalized eigenfunctions corresponding to all eigenvalues $\lambda_{i,h}, \cdots, \lambda_{i+q-1,h}$. In view of the adjoint problem (2.13) and (2.35), the definitions of $E^*$, $\text{ran}(E^*)$, $E_h^*$ and $\text{ran}(E_h^*)$ are analogous to $E$, $\text{ran}(E)$, $E_h$ and $\text{ran}(E_h)$ (see [24]).

The error estimate of the $C^0$IPG method for eigenvalue problems is based on the error estimate of the $C^0$IPG method for the corresponding source problems. Next using argument as in [25] we well prove the a priori error estimates for the source problem (2.9).

From Lemma 3.1 in [25] we known that (2.9) admits a unique solution $(\psi, \varphi) \in (W^{3,p_0}(\Omega) \cap H_0^2(\Omega)) \times H_0^1(\Omega)$ and

$$\|(\psi, \varphi)\|_{W^{3,p_0,2}} \leq C_R \|(f, g)\|_{H^1}, \quad \forall (f, g) \in H^1,$$

where $p_0 \in \mathbb{S}$, $C_R$ denotes the prior constant.

Denote $A((u, \omega), (v, z)) \equiv a(u, v) + (\omega, z)_0$, $A_h((u, \omega), (v, z)) \equiv a_h(u, v) + (\omega, z)_0$, $\|(u, \omega)\|_h \equiv \|u\|_h^2 + \|\omega\|_0^2$, $\|(u, \omega)\|_h \equiv \|u\|_h^2 + \|\omega\|_0^2$, $B'(f, v) = \int_\Omega \nabla f \cdot \nabla \varphi dx$. Define the auxiliary operator $K : H_0^1(\Omega) \to H_0^2(\Omega)$ by

$$a(Kf, v) = B'(f, v), \quad \forall v \in H_0^2(\Omega).$$

Then for any $f \in H_0^1(\Omega)$ it is valid that $Kf \in W^{3,p_0}(\Omega)$ and

$$\|Kf\|_{3,p_0} \lesssim \|f\|_1.$$}

Referring (3.7)-(3.9) in [25] we can deduce
\[ A_h((\psi, \varphi), (v, z)) = B((f, g), (v, z)), \quad \forall (v, z) \in H_h, \quad (2.41) \]
\[ a_h(K f, v) = B'(f, v), \quad \forall v \in S^h. \quad (2.42) \]

From (2.41) and (2.32) we get
\[ A_h((\psi, \varphi) - (\psi_h, \varphi_h), (v, z)) = 0, \quad \forall (v, z) \in H_h. \quad (2.43) \]

Define the operator
\[ I_h(\psi, \varphi) = (I^1_h \psi, I^2_h \varphi), \]
where \( I^1_h : H^1_0(\Omega) \cap C^0(\overline{\Omega}) \to S^h \) is the Lagrange nodal interpolation operator and \( I^2_h : L^2(\Omega) \to S^h \) is defined by
\[ (\varphi - I^2_h \varphi, z)_0 = 0, \quad \forall z \in S^h. \]

From Lemma 3.3 in [25], for any \((\psi, \varphi) \in W^{m+1,p}(\Omega) \times W^{m-1,2}(\Omega)\), the following estimates hold:
\[ \|
\begin{pmatrix} \psi, \varphi \end{pmatrix} - I_h(\psi, \varphi) \|_h \leq C h^{m-1+(\frac{1}{2} - \frac{1}{p})d} (\|\psi\|_{m+1,p,\Omega} + \|\varphi\|_{m-1,\Omega}), \quad (2.44) \]
\[ \|
\begin{pmatrix} \psi, \varphi \end{pmatrix} - I_h(\psi, \varphi) \|_{H^1} \leq C h^{m+(\frac{1}{2} - \frac{1}{p})d} (\|\psi\|_{m+1,p,\Omega} + \|\varphi\|_{m-1,\Omega}). \quad (2.45) \]

From a Poincaré-Friedrichs inequality [41] we get
\[ \| (v, z) \|_{H^1} = \| v \|_1 + \| z \|_{-1} \lesssim \| v \|_h + \| z \|_0 \]
\[ \lesssim \| (v, z) \|_h, \quad \forall (v, z) \in H_h. \quad (2.46) \]

Let \((v, z) = T_h(f, g)\) in (2.33), and we get
\[ \| T_h(f, g) \|_h \leq C \| (f, g) \|_{H^1}, \quad \forall (f, g) \in H^1. \quad (2.47) \]

**Lemma 2.1.** Let \((\psi, \varphi)\) and \((\psi^*, \varphi^*)\) be the solution of (2.9) and (2.14), respectively, and let \((\psi_h, \varphi_h)\) and \((\psi^*_h, \varphi^*_h)\) be the \(C^0\)IPG approximation solution of (2.9) and (2.14), respectively. Assume that \((\psi, \varphi), (\psi^*, \varphi^*) \in W^{m+1,p}(\Omega) \times H^{m-1}(\Omega) (p \in \mathbb{S})\), then
\[ \|
\begin{pmatrix} \psi, \varphi \end{pmatrix} - (\psi_h, \varphi_h) \|_h \lesssim h^{m-1+(\frac{1}{2} - \frac{1}{p})d} (\|\psi\|_{m+1,p} + \|\varphi\|_{m-1}), \quad (2.48) \]
\[ \|
\begin{pmatrix} \psi^*, \varphi^* \end{pmatrix} - (\psi^*_h, \varphi^*_h) \|_h \lesssim h^{m-1+(\frac{1}{2} - \frac{1}{p})d} (\|\psi^*\|_{m+1,p} + \|\varphi^*\|_{m-1}), \quad (2.49) \]

furthermore assume \(R(\Omega)\) holds, then
\[ \| (\psi, \varphi) - (\psi_h, \varphi_h) \|_{H^1} \lesssim h^{m+\left(1 - \frac{1}{p} - \frac{1}{p_0}\right)d} (\|\psi\|_{m+1,p} + \|\varphi\|_{m-1}), \]  
\[ \| (\psi^*, \varphi^*) - (\psi^*_h, \varphi^*_h) \|_{H^1} \lesssim h^{m+\left(1 - \frac{1}{p} - \frac{1}{p_0}\right)d} (\|\psi^*\|_{m+1,p} + \|\varphi^*\|_{m-1}). \]  

**Proof.** From (2.31), (2.43), (2.29) and (2.44), we deduce

\[ \| I_h(\psi, \varphi) - (\psi_h, \varphi_h) \|_2 \leq A_h(I_h(\psi, \varphi) - (\psi_h, \varphi_h)), \]

\[ = A_h(I_h(\psi, \varphi) - (\psi, \varphi), I_h(\psi, \varphi) - (\psi_h, \varphi_h)) \]

\[ \lesssim h^{m-1+d\left(\frac{1}{2} - \frac{1}{p}\right)} (\|\psi\|_{m+1,p} + \|\varphi\|_{m-1}), \]

thus we get

\[ \| (\psi, \varphi) - (\psi_h, \varphi_h) \|_h \leq \| (\psi, \varphi) - I_h(\psi, \varphi) \|_h + \| I_h(\psi, \varphi) - (\psi_h, \varphi_h) \|_h \]

\[ \lesssim h^{m-1+d\left(\frac{1}{2} - \frac{1}{p}\right)} (\|\psi\|_{m+1,p} + \|\varphi\|_{m-1}), \]

which is the desired result (2.48). By the same argument we can prove (2.49). Denote \( e = \psi - \psi_h \). From (2.39), (2.42), (2.43) with \( z = 0 \), (2.29), (2.48), (2.44) and (2.40), we deduce

\[ |B'(e, e)| = |a_h(Ke, e)| = |a_h(e, Ke - I^1_h Ke)| \]

\[ \lesssim \|e\|_h \|Ke - I^1_h Ke\|_h \]

\[ \lesssim h^{m-1+d\left(\frac{1}{2} - \frac{1}{p}\right)} \|\psi\|_{m+1,p} h^{1+d\left(\frac{1}{2} - \frac{1}{p_0}\right)} \|Ke\|_{3,p_0} \]

\[ \lesssim h^{m+\left(1 - \frac{1}{p} - \frac{1}{p_0}\right)d} \|\psi\|_{m+1,p} \|e\|_1, \]

i.e.,

\[ \|e\|_1 \lesssim h^{m+\left(1 - \frac{1}{p} - \frac{1}{p_0}\right)d} \|\psi\|_{m+1,p}. \]  

From (2.9) and (2.32) we have \( \varphi = f \in H^1_0(\Omega) \) and

\[ (\varphi - \varphi_h, z)_0 = 0, \quad \forall z \in S^h. \]

So

\[ \|\varphi - \varphi_h\|_{m-1} \lesssim h^m \|\varphi\|_{m-1}. \]  

From (2.52) and (2.53) we get the desired result (2.50). By the same argument we can prove (2.51). The proof is completed. \( \square \)

Based on Lemma 2.1, using argument as Theorem 3.3 and Theorem 3.4 in [25] we can prove the following a priori error estimates for the eigenvalue problem.
Theorem 2.1. Assume that $R(\Omega)$ holds and $n \in W^{1,\infty}(\Omega) \cap H^2(\Omega)$, then

$$|\left(\frac{1}{q}\sum_{j=i}^{i+q-1} \lambda_{j,h}^{-1}\right)^{-1} - \lambda| \lesssim \|T - T_h\|_{\text{ran}(E)}\|_{H^1};$$  \hspace{1cm} (2.54)

assume $\text{ran}(E) \subset W^{m+1,p}(\Omega) \times H^{m-1}(\Omega)$ ($p \in \mathbb{S}$), then

$$\|T - T_h\|_{\text{ran}(E)}\|_{H^1} \lesssim h^{m+(1-\frac{1}{p} - \frac{1}{p_0})d};$$ \hspace{1cm} (2.55)

furthermore, assume that $(u_h, \omega_h)$ is an eigenfunction corresponding to $\lambda_h$ and $\|(u_h, \omega_h)\|_h = 1$, then there exists eigenfunction $(u, \omega)$ corresponding to $\lambda$ such that

$$\|(u_h, \omega_h) - (u, \omega)\|_{H^1} \lesssim h^{m+(1-\frac{1}{p} - \frac{1}{p_0})d};$$ \hspace{1cm} (2.56)

$$\|\|(u_h, \omega_h) - (u, \omega)\|_h \lesssim h^{m-1+(1-\frac{1}{2} - \frac{1}{p})d}. \hspace{1cm} (2.57)$$

In addition, when the eigenvalue $\lambda$ is non-defective, for $(u^*, \omega^*) \in \text{ran}(E^*)$ with $\|(u^*, \omega^*)\|_h = 1$, there exists $(u_h^*, \omega_h^*) \in \text{ran}(E_h^*)$ such that

$$\|(u^*, \omega^*) - (u_h^*, \omega_h^*)\|_{H^1} \lesssim h^{m+(1-\frac{1}{p} - \frac{1}{p_0})d};$$ \hspace{1cm} (2.58)

$$\|\|(u^*, \omega^*) - (u_h^*, \omega_h^*)\|_h \lesssim h^{m-1+(1-\frac{1}{2} - \frac{1}{p})d};$$ \hspace{1cm} (2.59)

for $(u_h^*, \omega_h^*) \in \text{ran}(E_h^*)$ with $\|(u_h^*, \omega_h^*)\|_h = 1$, there exists $(u^*, \omega^*) \in \text{ran}(E^*)$ such that

$$\|(u_h^*, \omega_h^*) - (u^*, \omega^*)\|_{H^1} \lesssim h^{m+(1-\frac{1}{p} - \frac{1}{p_0})d};$$ \hspace{1cm} (2.60)

$$\|\|(u_h^*, \omega_h^*) - (u^*, \omega^*)\|_h \lesssim h^{m-1+(1-\frac{1}{2} - \frac{1}{p})d};$$ \hspace{1cm} (2.61)

$$|\lambda_h - \lambda| \lesssim h^{2m-2+2(\frac{1}{2} - \frac{1}{p})d}. \hspace{1cm} (2.62)$$

3 The a posteriori error analysis of $C^0$IPG discrete scheme for the source problem (2.9)

In 2012, Brenner \cite{21} proposed and analyzed the a posteriori error estimates of $C^0$IPG methods for biharmonic equation. Based on \cite{21}, in this section we discuss a posteriori error estimates of $C^0$IPG discrete scheme (2.32) for the source problem (2.9).
Denote

\[ F = F(f, g) = -\Delta \left( \frac{1}{n-1} f \right) - \frac{n}{n-1} \Delta f - \frac{n}{n-1} g, \quad \text{in } \kappa, \]

where \( f, g \in W^{3,p}(\Omega, \pi_h) \), and denote

\[
\eta_\kappa(F, \psi_h) = h_\kappa^2 \| F - \Delta \left( \frac{1}{n-1} \Delta \psi_h \right) \|_{0, \kappa}, \quad \forall \kappa \in \pi_h,
\]

\[
\eta_{\ell,1}(\psi_h) = \frac{\sigma}{\ell^2} \left\| \left[ \frac{\partial \psi_h}{\partial \gamma_\ell} \right] \right\|_{0, \ell}, \quad \forall \ell \in \mathcal{E},
\]

\[
\eta_{\ell,2}(\psi_h) = \mu \ell^2 \left\| \left[ \frac{\partial^2 \psi_h}{\partial \gamma_\ell^2} \right] \right\|_{0, \ell}, \quad \forall \ell \in \mathcal{E}^i,
\]

\[
\eta_{\ell,3}(\psi_h) = \ell^2 \left\| \left[ \frac{\partial \left( \frac{1}{n-1} \Delta \psi_h \right)}{\partial \gamma_\ell} \right] \right\|_{0, \ell}, \quad \forall \ell \in \mathcal{E}^i,
\]

\[
\eta_{\ell,4}(\psi_h) = \ell^2 \left\| \left[ \left( \frac{1}{n-1} - \mu \right) \Delta \psi_h \right] \right\|_{0, \ell}, \quad \forall \ell \in \mathcal{E}^i.
\]

Then the residual-based error indicator \( \eta_h \) is defined by

\[
\eta_h^2(F, \psi_h; \varphi_h, \kappa) = \eta_\kappa^2(F, \psi_h) + \sum_{\ell \in \mathcal{E} \cap \partial \kappa} \eta_{\ell,1}^2(\psi_h)
+ \frac{1}{2} \sum_{\ell \in \mathcal{E} \cap \partial \kappa} \left\{ \eta_{\ell,1}^2(\psi_h) + \eta_{\ell,2}^2(\psi_h) + \eta_{\ell,3}^2(\psi_h) + \eta_{\ell,4}^2(\psi_h) \right\}
+ \sum_{\ell \in \mathcal{E} \cap \partial \kappa} \left\| \frac{n+1}{n-1} \|_{0, \ell} h_{\ell}^2 \eta_{\ell,1}(f) + \| f - \varphi_h \|_{0, \kappa}^2 + h^4 \sum_{\ell \in \mathcal{E} \cap \partial \kappa} \eta_{\ell,1}^2(\varphi_h) \right\}, \quad (3.6)
\]

\[
\eta_h^2(F, \psi_h, \varphi_h, \Omega) = \sum_{\kappa \in \pi_h} \eta_h^2(F, \psi_h, \varphi_h, \kappa). \quad (3.7)
\]

Let \( P_j(\Omega, \pi_h) \) be the space of piecewise polynomial functions of degree \( \leq j \) and \( \bar{g} \in P_j(\Omega, \pi_h) \) denote the \( L^2 \) orthogonal projection of \( g \). And denote

\[
\tilde{F} = -\Delta \left( \frac{1}{n-1} f \right) - \frac{n}{n-1} \Delta f - \frac{n}{n-1} g,
\]

\[
\tilde{\eta}_\kappa(F, \psi_h) = h_\kappa^2 \| \tilde{F} - \Delta \left( \frac{1}{n-1} \Delta \psi_h \right) \|_{0, \kappa}, \quad \forall \kappa \in \pi_h,
\]

\[
\tilde{\eta}_{\ell,3}(\psi_h) = \ell^2 \left\| \left[ \frac{\partial \left( \frac{1}{n-1} \Delta \psi_h \right)}{\partial \gamma_\ell} \right] \right\|_{0, \ell}, \quad \forall \ell \in \mathcal{E}^i,
\]

\[
\tilde{\eta}_{\ell,4}(\psi_h) = \ell^2 \left\| \left[ \left( \frac{1}{n-1} - \mu \right) \Delta \psi_h \right] \right\|_{0, \ell}, \quad \forall \ell \in \mathcal{E}^i.
\]

\[\text{13}\]
The data oscillations are defined by

\[
Osc_j(F) = (\sum_{\kappa \in \pi_h} h_\kappa^4 \| F - \hat{F} \|_{0,\kappa}^2)^{\frac{1}{2}}, \tag{3.12}
\]

\[
Osc_j(\eta_{\ell,3}) = (\sum_{\kappa \in \pi_h} \sum_{\ell \in E \cap \partial \kappa} (\eta_{\ell,3}(\psi_h) - \tilde{\eta}_{\ell,3}(\psi_h))^2)^{\frac{1}{2}}, \tag{3.13}
\]

\[
Osc_j(\eta_{\ell,4}) = (\sum_{\kappa \in \pi_h} \sum_{\ell \in E \cap \partial \kappa} (\eta_{\ell,4}(\psi_h) - \tilde{\eta}_{\ell,4}(\psi_h))^2)^{\frac{1}{2}}. \tag{3.14}
\]

**Theorem 3.1.** Let \((\psi, \varphi)\) and \((\psi_h, \varphi_h)\) be the solution of (2.29) and (2.32), respectively. Assume that \(R(\Omega)\) holds and \(n \in W^{1,\infty}(\Omega) \cap H^2(\Omega)\), then

\[
\| (\psi, \varphi) - (\psi_h, \varphi_h) \|_h \lesssim \eta_h(F, \psi_h, \varphi_h, \Omega). \tag{3.15}
\]

**Proof.** Brenner introduced the enriching operator \(E_h : S^h \to H^2(\Omega)\) and proved (see (4.4) in [21])

\[
\sum_{\kappa \in \pi_h} (h_\kappa^{-4} \| v - E_h \psi \|_{0,\kappa}^2 + h_\kappa^{-2} | v - E_h \psi |_{1,\kappa}^2 + | v - E_h \psi |_{2,\kappa}^2)
\]

\[
\lesssim \sum_{\kappa \in \pi_h} \frac{1}{\ell} \| \| [\partial \psi / \partial \gamma_{\ell}] \|_{0,\ell}^2, \quad \forall v \in S^h. \tag{3.16}
\]

Denote \(E_h(u_h, \omega_h) = (E_h u_h, E_h \omega_h)\).

Due to (2.26) we need to bound \(\sigma \sum_{\ell \in E} \frac{1}{\ell} \| \| [\partial \psi / \partial \gamma_{\ell}] \|_{0,\ell}^2 \) and \(\sum_{\kappa \in \pi_h} \| \psi - \psi_h \|_{2,\kappa}^2 + \sum_{\kappa \in \pi_h} \| \varphi - \varphi_h \|_{2,\kappa}^2\).

Since \(\sigma > 1\), from (3.12) we get

\[
\sigma \sum_{\ell \in E} \frac{1}{\ell} \| \| [\partial \psi / \partial \gamma_{\ell}] \|_{0,\ell}^2 = \sigma \sum_{\ell \in E} \frac{1}{\ell} \| \| [\partial \psi_h / \partial \gamma_{\ell}] \|_{0,\ell}^2 \leq \sum_{\ell \in E} \eta_{\ell,1}^2. \tag{3.17}
\]

From (3.16) and (3.12) we have

\[
\sum_{\kappa \in \pi_h} \| \psi - \psi_h \|_{2,\kappa}^2 + \sum_{\kappa \in \pi_h} \| \varphi - \varphi_h \|_{2,\kappa}^2
\]

\[
\leq 2 \sum_{\kappa \in \pi_h} (\| \psi - E_h \psi_h \|_{2,\kappa}^2 + | \psi_h - E_h \psi_h |_{2,\kappa}^2)
\]

\[
+ 2 \sum_{\kappa \in \pi_h} (\| \varphi - E_h \varphi_h \|_{2,\kappa}^2 + | \varphi_h - E_h \varphi_h |_{2,\kappa}^2)
\]

\[
\lesssim \| (\psi, \varphi) - E_h(\psi_h, \varphi_h) \|_H^2 + \sum_{\ell \in E} \eta_{\ell,1}^2(\psi_h) + h^4 \sum_{\ell \in E} \eta_{\ell,1}^2(\varphi_h). \tag{3.18}
\]
By duality we have

\[
\| (\psi, \varphi) - E_h(\psi_h, \varphi_h) \|_H \approx \sup_{(v, z) \in H \setminus \{0\}} \frac{A((\psi, \varphi) - E_h(\psi_h, \varphi_h), (v, z))}{\| (v, z) \|_H}.
\] (3.19)

Denote

\[
A_\kappa((u, \omega), (v, z)) = \int_\kappa \left( \frac{1}{n-1} - \mu \right) \Delta u, \Delta \bar{v} dx + \mu \int_\kappa D^2 u : D^2 \bar{v} dx + \int_\kappa \omega \bar{z} dx.
\] (3.20)

From (2.6), (3.20), (2.9) and (2.32) we get

\[
A((\psi, \varphi) - E_h(\psi_h, \varphi_h), (v, z)) = \sum_{\kappa \in \pi_h} A_\kappa((\psi_h, \varphi_h) - E_h(\psi_h, \varphi_h), (v, z)) - \sum_{\kappa \in \pi_h} A_\kappa((\psi_h, \varphi_h), (v, z) - I_h(v, z))
\]

\[
+ A((\psi, \varphi), (v, z)) - \sum_{\kappa \in \pi_h} A_\kappa((\psi_h, \varphi_h), I_h(v, z))
\]

\[
= \sum_{\kappa \in \pi_h} A_\kappa((\psi_h, \varphi_h) - E_h(\psi_h, \varphi_h), (v, z))
\]

\[
- \sum_{\kappa \in \pi_h} A_\kappa((\psi_h, \varphi_h), (v, z) - I_h(v, z))
\]

\[
+ A_h((\psi, \varphi), I_h(v, z)) - \sum_{\kappa \in \pi_h} A_\kappa((\psi_h, \varphi_h), I_h(v, z))
\]

\[
+ B((f, g), (v, z) - I_h(v, z)) \equiv I_1 - I_2 + I_3 - I_4 + I_5.
\] (3.21)

We have

\[
I_2 = \sum_{\kappa \in \pi_h} A_\kappa((\psi_h, \varphi_h), (v, z) - I_h(v, z))
\]

\[
= \sum_{\kappa \in \pi_h} \int_\kappa \left( \frac{1}{n-1} - \mu \right) \Delta \psi_h \Delta (v - T^1_h v) dx + \mu \int_\kappa D^2 \psi_h : D^2 (v - T^1_h v) dx
\]

\[
+ \sum_{\kappa \in \pi_h} \int_\kappa \varphi_h (z - T^2_h z) dx \equiv J_1 + J_2 + J_3,
\] (3.22)

and by the Green’s formula we have
\[
J_1 = \sum_{\kappa \in \pi_h} \int_{\kappa} \nabla \left( \frac{1}{n - 1} - \mu \right) \Delta \psi_h \nabla (v - I^1_h v) dx \\
+ \sum_{\kappa \in \pi_h} \int_{\partial \kappa} \left( \frac{1}{n - 1} - \mu \right) \Delta \psi_h \frac{\partial(v - I^1_h v)}{\partial \gamma} ds \\
= \sum_{\kappa \in \pi_h} \int_{\kappa} \Delta \left( \frac{1}{n - 1} - \mu \right) \Delta \psi_h (v - I^1_h v) dx \\
- \sum_{\kappa \in \pi_h} \int_{\partial \kappa} \nabla \left( \frac{1}{n - 1} - \mu \right) \Delta \psi_h (v - I^1_h v) \cdot \gamma ds \\
+ \sum_{\kappa \in \pi_h} \int_{\partial \kappa} \left( \frac{1}{n - 1} - \mu \right) \Delta \psi_h \frac{\partial(v - I^1_h v)}{\partial \gamma} ds \\
= \sum_{\kappa \in \pi_h} \int_{\kappa} \Delta \left( \frac{1}{n - 1} - \mu \right) \Delta \psi_h (v - I^1_h v) dx \\
+ \sum_{\ell \in \mathcal{E}} \int_{\ell} \left[ \nabla \left( \frac{1}{n - 1} - \mu \right) \Delta \psi_h \right] (v - I^1_h v) ds \\
- \sum_{\ell \in \mathcal{E}} \int_{\ell} \left\{ \left( \frac{1}{n - 1} - \mu \right) \Delta \psi_h \right\} \left[ \frac{\partial(v - I^1_h v)}{\partial \gamma_{\ell}} \right] ds \\
- \sum_{\ell \in \mathcal{E}} \int_{\ell} \left\{ \left( \frac{1}{n - 1} - \mu \right) \Delta \psi_h \right\} \left\{ \frac{\partial(v - I^1_h v)}{\partial \gamma_{\ell}} \right\} ds, \quad (3.23)
\]

and by the Green’s formula (see also (7.10) in [21]) we have

\[
J_2 = \mu \left\{ \sum_{\kappa \in \pi_h} \int_{\kappa} (\Delta^2 \psi_h) (v - I^1_h v) dx + \sum_{\ell \in \mathcal{E}} \int_{\ell} \left[ \frac{\partial(\Delta \psi_h)}{\partial \gamma_{\ell}} \right] (v - I^1_h v) ds \\
+ \sum_{\ell \in \mathcal{E}} \int_{\ell} \left\{ \frac{\partial^2 \psi_h}{\partial \gamma_{\ell}^2} \right\} \left[ \frac{\partial I^1_h v}{\partial \gamma_{\ell}} \right] ds - \sum_{\ell \in \mathcal{E}} \int_{\ell} \left\{ \frac{\partial^2 \psi_h}{\partial \gamma_{\ell}^2} \right\} \left\{ \frac{\partial(v - I^1_h v)}{\partial \gamma_{\ell}} \right\} ds \\
- \sum_{\ell \in \mathcal{E}} \int_{\ell} \left\{ \frac{\partial^2 \psi_h}{\partial \gamma_{\ell} \partial t_{\ell}} \right\} \left( \frac{\partial(v - I^1_h v)}{\partial t_{\ell}} \right) ds \right\}, \quad (3.24)
\]

By (2.24) we get
\[ I_3 - I_4 = \sum_{\ell \in E} \int \left\{ \left( \frac{1}{n-1} - \mu \right) \Delta \psi_h \right\} \left[ \frac{\partial I_1^h v}{\partial \gamma_{\ell}} \right] \]
\[ + \left\{ \left( \frac{1}{n-1} - \mu \right) \Delta I_1^h v \right\} \left[ \frac{\partial \psi_h}{\partial \gamma_{\ell}} \right] ds \]
\[ + \mu \sum_{\ell \in E} \int \left\{ \left( \frac{\partial^2 \psi_h}{\partial \gamma_{\ell}^2} \right) \left[ \frac{\partial I_1^h v}{\partial \gamma_{\ell}} \right] \right\} + \left\{ \frac{\partial^2 I_1^h v}{\partial \gamma_{\ell}^2} \right\} \left[ \frac{\partial \psi_h}{\partial \gamma_{\ell}} \right] ds \]
\[ + \sigma \sum_{\ell \in E} \frac{1}{\ell} \int \left[ \frac{\partial \psi_h}{\partial \gamma_{\ell}} \right] \left[ \frac{\partial I_1^h v}{\partial \gamma_{\ell}} \right] ds , \quad (3.25) \]

from the Green’s formula we get

\[ B((f, g), (v, z)) = (\nabla \left( \frac{1}{n-1} f \right), \nabla v)_0 + (\nabla f, \nabla (\frac{n}{n-1} v))_0 - (g, \frac{n}{n-1} v)_0 + (f, z)_0 \]
\[ = - \sum_{\kappa} \int_{\kappa} \nabla \left( \frac{1}{n-1} f \right) \tau d\sigma - \sum_{\kappa} \int_{\kappa} \frac{n}{n-1} \Delta f \tau d\sigma - \left( \frac{n}{n-1} g \right) \tau |_{\partial \kappa} + (f, z)_0 \]
\[ + \sum_{\kappa} \int_{\partial \kappa} \frac{\partial \left( \frac{1}{n-1} f \right)}{\partial \gamma} |_{\partial \kappa} \tau d\sigma \]
\[ \equiv \sum_{\kappa} \int_{\kappa} F \tau d\sigma + (f, z)_0 + \sum_{\ell \in E} \int_{\ell} \left[ \frac{\partial \left( \frac{1}{n-1} f \right)}{\partial \gamma} \right] \tau d\sigma \]
\[ + \sum_{\ell \in E} \int_{\ell} \left[ \frac{n}{n-1} \frac{\partial f}{\partial \gamma} \right] \tau d\sigma , \quad (3.26) \]

thus

\[ I_5 = B((f, g), (v, z)) - I_h(v, z) = \sum_{\kappa} \int_{\kappa} F(v - I_1^h v) d\sigma \]
\[ + (f, z - I_h^2 z)_0 + \sum_{\ell \in E} \int_{\ell} \left[ \frac{n + 1}{n-1} \frac{\partial f}{\partial \gamma} \right] (v - I_1^h v) d\sigma. \quad (3.27) \]

Substituting (3.22), (3.25) and (3.27) into (3.21), we obtain
\[
A((\psi, \varphi) - \mathbf{E}_h(\psi_h, \varphi_h), (v, z))
= I_1 + \sum_{\kappa \in \Pi_h} \int (F - \Delta(\frac{1}{n-1} \Delta \psi_h))(v - I_h^1 v) + (f - \varphi_h)(z - I_h^2 z) \, dx
\]

\[
- \sum_{\ell \in E} \int [\nabla [(\frac{1}{n-1} - \mu) \Delta \psi_h] \cdot \gamma](v - I_h^1 v) \, ds
+ \sum_{\ell \in E} \int \{[(\frac{1}{n-1} - \mu) \Delta \psi_h] \{[\partial(v - I_h^1 v) \, \partial \gamma_{\ell} \}} ds
\]

\[
- \mu \sum_{\ell \in E} \int [[\partial^{2} \psi_{h} \frac{\partial}{\partial \gamma_{\ell}}][v - I_{h}^{1} v] ds - \mu \sum_{\ell \in E} \int \{\{\partial^{2} \psi_{h} \frac{\partial}{\partial \gamma_{\ell}}] \} ds
\]

\[
+ \mu \sum_{\ell \in E} \int \{[[\partial^{2} \psi_{h} \frac{\partial}{\partial \gamma_{\ell}}] \} \} \right] ds + \mu \sum_{\ell \in E} \int \{[[\partial^{2} \psi_{h} \frac{\partial}{\partial \gamma_{\ell}}] \} \} \right] ds
\]

\[
+ \sum_{\ell \in E} \int \{((\frac{1}{n-1} - \mu) \Delta I_{h}^{1} v) [[\partial \psi_{h} \frac{\partial}{\partial \gamma_{\ell}}] ds + \mu \sum_{\ell \in E} \int \{[[\partial^{2} \psi_{h} \frac{\partial}{\partial \gamma_{\ell}}] \} \} \right] ds
\]

\[
+ \mu \sum_{\ell \in E} \int \{[[\partial^{2} \psi_{h} \frac{\partial}{\partial \gamma_{\ell}}] \} \} \right] ds + \sigma \sum_{\ell \in E} \int [[\partial \psi_{h} \frac{\partial}{\partial \gamma_{\ell}}] ds
\]

\[
\equiv I_1 + G_2 + G_3 + \cdots + G_{15}.
\]  

By (3.20), the Schwarz inequality, (3.16) and (3.2) we get

\[
|I_1| = | \sum_{\kappa \in \Pi_h} A_{\kappa}((\psi_h, \varphi_h) - \mathbf{E}_h(\psi_h, \varphi_h), (v, z)) |
\]

\[
\lesssim \sum_{\kappa \in \Pi_h} ( \| \frac{1}{n-1} \|_{0, \infty, \kappa} |\psi_h - E_h \psi_h|_{2, \kappa} + \| \varphi_h - E_h \varphi_h \|_{0, \kappa} ) \| (v, z) \|_{\mathbf{H}}
\]

\[
\lesssim \sum_{\ell \in E} \int ( \| \frac{1}{n-1} \|_{0, \infty, \kappa} \eta_{\ell, 1} (\psi_h)^2 + \eta_{\ell, 1} (\varphi_h)^2 ) \| (v, z) \|_{\mathbf{H}}.
\]  

by (3.1) we get
\[ |G_2| \lesssim \left( \sum_{\kappa \in \pi_h} h_\kappa^4 \| F - \Delta \left( \frac{1}{n-1} \Delta \varphi_h \right) \|_{0, \kappa}^2 \right)^{\frac{1}{2}} \left( \sum_{\kappa \in \pi_h} h_\kappa^{-4} \| v - I_h^1 v \|_{0, \kappa}^2 \right)^{\frac{1}{2}} \]

\[ + \| f - \varphi_h \|_0 \| z - I_h^2 z \|_0 \lesssim \left( \sum_{\kappa \in \pi_h} \eta_\kappa^2 \| v \|_2 \right) + \| f - \varphi_h \|_0 \| z \|_0, \]

by (3.4) we get

\[ |G_3 + G_6| \lesssim \left( \sum_{\ell \in E} \hat{\ell}^3 \frac{\| \nabla \left( \frac{1}{n-1} \Delta \varphi_h \right) \cdot \gamma \|_{0, \ell}^2}{\| \nabla \psi_h \|_{0, \ell}^2} \right)^{\frac{1}{2}} \left( \sum_{\ell \in E} \hat{\ell}^{-3} \| v - I_h^1 v \|_{0, \ell}^2 \right)^{\frac{1}{2}} \]

\[ \lesssim \left( \sum_{\ell \in E} \eta_{\ell,3}^2 \right)^{\frac{1}{2}} \| v \|_2, \]

we see

\[ G_4 + G_{10} = 0, \quad G_7 + G_{12} = 0, \]

by (3.5) we get

\[ |G_5| \lesssim \left( \sum_{\ell \in E} h_\ell \left[ \left( \frac{1}{n-1} - \mu \right) \Delta \varphi_h \right]^2 \right)^{\frac{1}{2}} \| v \|_2 \lesssim \left( \sum_{\ell \in E} \eta_{\ell,4}^2 \right)^{\frac{1}{2}} \| v \|_2, \]

by (3.3) we get

\[ |G_8| \lesssim \left( \sum_{\ell \in E} \eta_{\ell,2} (\psi_h)^2 \right)^{\frac{1}{2}} \| v \|_2, \]

by (3.2), the trace theorem with scaling and a standard inverse estimate, we deduce

\[ |G_9| \lesssim \mu \left( \sum_{\ell \in E} \hat{\ell} \left[ \left( \frac{\partial^2 \psi_h}{\partial \gamma \partial \ell} \right)^2 \right]_{0, \ell} \right)^{\frac{1}{2}} \left( \sum_{\ell \in E} \hat{\ell}^{-1} \left( \frac{\partial (v - I_h^1 v)}{\partial \ell} \right)^2 \right)^{\frac{1}{2}} \]

\[ \lesssim \mu \left( \sum_{\ell \in E} \eta_{\ell,1} (\psi_h)^2 \right)^{\frac{1}{2}} \| v \|_2, \]

by (3.2) we get
\[ |G_{11}| \lesssim \frac{1}{\sigma} \left( \sum_{\ell \in E} \frac{1}{n-1} - \mu \right) \eta_{1,0}(\psi_h^2) \frac{1}{2} |v|_2, \]
\[ |G_{13}| \lesssim \frac{1}{\sigma} \left( \sum_{\ell \in E} \eta_{1,0}(\psi_h^2) \right) \frac{1}{2} |v|_2, \]
\[ |G_{14}| \lesssim \left( \sum_{\ell \in E} \eta_{1,0}(\psi_h^2) \right) \frac{1}{2} |v|_2, \]
\[ |G_{15}| \lesssim \left( \sum_{\ell \in E} \frac{n+1}{n-1} \eta_{1,0}(\psi_h^2) \right) \frac{1}{2} |v|_2. \]

Substituting these estimates into (3.28), we obtain
\[ A((\psi, \varphi) - E_h(\psi_h, \varphi_h), (v, z)) \lesssim \eta_h(F, \psi_h, \varphi_h, \Omega) \|(v, z)\|_H. \tag{3.30} \]

Combining (3.17)-(3.19) and (3.30) we obtain (3.15). \(\square\)

Using the argument in Theorem 8 in [21], we can prove the following theorem.

**Theorem 3.2.** Under the condition of Theorem 3.1, we have

\[ \eta_h(F, \psi_h, \varphi_h, \Omega) \lesssim \|(\psi, \varphi) - (\psi_h, \varphi_h)\|_h \]
\[ + \text{Osc}_m(F) + \text{Osc}_m(\eta_{l,3}(\psi_h)) + \text{Osc}_m(\eta_{l,4}(\psi_h)). \tag{3.31} \]

4 The a posteriori error analysis of \(C^0\text{IPG}\) discrete scheme for the eigenvalue problem (2.5)

Now, we analyze the a posteriori error of the \(C^0\text{IPG}\) eigenpair \((\lambda_h, u_h, \omega_h)\).

Consider the source problem (2.9) associated with (2.5) with \((f, g) = \lambda_h(u_h, \omega_h)\).

Then its generalized solution \((\psi, \varphi) = \lambda_h T_h(u_h, \omega_h)\) and the \(C^0\text{IPG}\) approximation \((\psi_h, \varphi_h) = \lambda_h T_h(u_h, \omega_h) = (u_h, \omega_h)\). Let \(v = 0\) in (2.25), we get \(\omega_h = \lambda_h u_h\).

Thus, in (3.6), we have

\[ \left( \sum_{\ell \in E} \frac{n+1}{n-1} h_{\ell}^4 \eta_{\ell,1}(f)^2 \right) \frac{1}{2} = \left( \sum_{\ell \in E} \frac{n+1}{n-1} h_{\ell}^4 \eta_{\ell,1}(\lambda_h u_h)^2 \right) \frac{1}{2} \approx \left( \sum_{\ell \in E} h_{\ell}^4 \eta_{\ell,1}(u_h)^2 \right) \frac{1}{2}, \]
\[ h^4 \sum_{\ell \in E \cap \partial \kappa} \eta_{\ell,1}^2(\varphi_h) = h^4 \sum_{\ell \in E \cap \partial \kappa} \eta_{\ell,1}^2(\lambda_h u_h) \approx \sum_{\ell \in E \cap \partial \kappa} h^4 \eta_{\ell,1}^2(u_h), \]
\[ \|f - \varphi_h\|_{0,\kappa}^2 = \|\lambda_h u_h - \omega_h\|_{0,\kappa}^2 = 0. \]

Hence, from (3.6), (3.7), (3.15) and (3.31) we obtain
Theorem 4.1. Referring Lemma 4.1 in [16] we can deduce the following theorem.

Proof. See Lemma 3.5 in [25].

Lemma 4.1. The following lemma is a generalization of the Lemma 9.1 in [34].

\begin{align}
\eta_\kappa^2(F, u_\kappa, \omega_\kappa, \kappa) &= \eta_\kappa^2(F, u_\kappa) + \sum_{\ell \in E \cap \partial \kappa} \eta^2_{\ell,1}(u_\kappa) + \frac{1}{2} \sum_{\ell \in E \cap \partial \kappa} \{\eta^2_{\ell,1}(u_\kappa)
+ \eta^2_{\ell,2}(u_\kappa) + \eta^2_{\ell,3}(u_\kappa) + \eta^2_{\ell,4}(u_\kappa)\} + O\left(\sum_{\ell \in E \cap \partial \kappa} h^4 \eta^2_{\ell,1}(u_\kappa)\right),
\end{align}

\begin{align}
\eta_\kappa^2(F, u_\kappa, \omega_\kappa, \Omega) &= \sum_{\kappa \in \pi_h} \eta^2_\kappa(F, u_\kappa, \omega_\kappa, \kappa),
\end{align}

\begin{align}
&\|\lambda_h T(u_\kappa, \omega_\kappa) - \lambda_h T(u_\kappa, \omega_\kappa)\|_h \lesssim \eta_\kappa(F, u_\kappa, \omega_\kappa, \Omega),
\end{align}

\begin{align}
&\eta_\kappa(F, u_\kappa, \omega_\kappa, \Omega) \lesssim \|\lambda_h T(u_\kappa, \omega_\kappa) - \lambda_h T(u_\kappa, \omega_\kappa)\|_h
+ Osc_m(F) + Osc_m(\eta_{\ell,3}(u_\kappa)) + Osc_m(\eta_{\ell,4}(u_\kappa)).
\end{align}

where \(f = \lambda_h u_\kappa, g = \lambda_h \omega_\kappa\) in \(F\).

It is noted that \(O\left(\sum_{\ell \in E \cap \partial \kappa} h^4 \eta^2_{\ell,1}(u_\kappa)\right)\) is higher order small than \(\sum_{\ell \in E \cap \partial \kappa} \eta^2_{\ell,1}(u_\kappa) + \frac{1}{2} \sum_{\ell \in E \cap \partial \kappa} \eta^2_{\ell,1}(u_\kappa)\), so it can be neglected in actual numerical computation.

The following lemma is a generalization of the Lemma 9.1 in [34].

Lemma 4.1. Let \((\lambda, u, \omega)\) and \((\lambda^*, u^*, \omega^*)\) be the eigenpair of (2.5) and (2.13), respectively. Then for any \((v, z), (v^*, z^*)\) \(\in H_h\), when \(B((v, z), (v^*, z^*)) \neq 0\) it is valid that

\begin{align}
\frac{A_h((v, z), (v^*, z^*))}{B((v, z), (v^*, z^*))} - \lambda = \frac{A_h((u, \omega) - (v, z), (u^*, \omega^*) - (v^*, z^*))}{B((v, z), (v^*, z^*))}
- \lambda \frac{B((u, \omega) - (v, z), (u^*, \omega^*) - (v^*, z^*))}{B((v, z), (v^*, z^*))}.
\end{align}

Proof. See Lemma 3.5 in [25]. \(\Box\)

Referring Lemma 4.1 in [16] we can deduce the following theorem.

Theorem 4.1. Assume that \(\lambda\) and \(\lambda_h\) are the \(ith\) eigenvalues of (2.5) and (2.25), respectively, \((u_\kappa, \omega_\kappa)\) is a eigenfunction corresponding to \(\lambda_h\) with \(\|u_\kappa(\omega_\kappa)\|_h = 1\), the ascent \(\alpha\) of \(\lambda\) is equal to 1, and assume that \(R(\Omega)\) holds and \(n \in W^{1,\infty}(\Omega) \cap H^2(\Omega)\). Let \((\bar{u}_\kappa, \bar{\omega}_\kappa)\) be the orthogonal projection of \((u_\kappa, \omega_\kappa)\) to \(\text{ran}(E_h^\ast)\) in the sense of inner product \(A_h(\cdot, \cdot)\), and

\begin{align}
(u_\kappa^\ast, \omega_\kappa^\ast) = \frac{(\bar{u}_\kappa, \bar{\omega}_\kappa)}{\|\bar{u}_\kappa, \bar{\omega}_\kappa\|_h}.
\end{align}

Then there exist \((u, \omega) \in \text{ran}(E)\) and \((u^\ast, \omega^\ast) \in \text{ran}(E^\ast)\) such that \((u_\kappa, \omega_\kappa) - (u, \omega)\) and \((u_\kappa^\ast, \omega_\kappa^\ast) - (u^\ast, \omega^\ast)\) satisfy (2.56)-(2.57) and (2.60)-(2.61) respectively, and
\[ |\lambda_h - \lambda| \lesssim \|(u_h, \omega_h) - (u, \omega)\|_h \|(u_h^*, \omega_h^*) - (u^*, \omega^*)\|_h \]
\[ + \|(u_h, \omega_h) - (u, \omega)\|_{H^1} \|(u_h^*, \omega_h^*) - (u^*, \omega^*)\|_{H^1}. \]  

(4.5)

**Proof.** From \( \alpha = 1 \), we know \( \text{ran}(E^*) \) is the space of eigenfunctions associated with \( \lambda^* \). Choose \( (u, \omega) \in \text{ran}(E) \) such that \((2.56)-(2.57)\) hold. Define

\[ f((v, z)) = A(E(v, z), (u, \omega)), \quad \forall (v, z) \in H. \]

Since for all \((v, z) \in H\) one has

\[ |f((v, z))| = |A(E(v, z), (u, \omega))| \leq \|E(v, z)\|_A \|(u, \omega)\|_A \]
\[ \lesssim \sqrt{\lambda}\|E(v, z)\|_{H^1} \lesssim \|E\|_{H^1}\|(v, z)\|_A, \]

\( f \) is a linear and bounded functional on \( H \) and \( \|f\|_A \lesssim \|E\|_{H^1} \). Using the Riesz Theorem, we know there exists \((u^*, \omega^*) \in H\) satisfying \( \|(u^*, \omega^*)\|_A = \|f\|_A \) and

\[ A((v, z), (u^*, \omega^*)) = A(E(v, z), (u, \omega)). \]  

(4.6)

For any \((v, z) \in H\), notice \( E(I - E)(v, z) = 0 \), then

\[ A((v, z), (\lambda^* - T^*)(u^*, \omega^*)) = A((\lambda - T)(v, z), (u^*, \omega^*)) \]
\[ = A((\lambda - T)E(v, z), (u^*, \omega^*)) + A((\lambda - T)(I - E)(v, z), (u^*, \omega^*)) = 0, \]
i.e., \((\lambda^* - T^*)(u^*, \omega^*) = 0\), hence \((u^*, \omega^*) \in \text{ran}(E^*)\). By \((4.6)\) we have

\[ \lambda B((u, \omega), (u^*, \omega^*)) = A((u, \omega), (u^*, \omega^*)) = A(E(u, \omega), (u, \omega)) \]
\[ = A((u, \omega), (u, \omega)) \approx A_h((u_h, \omega_h), (u_h, \omega_h)) \approx 1. \]  

(4.7)

Then, there exits \((\bar{u}_h^*, \bar{\omega}_h^*) \in \text{ran}(E_h^*)\) such that \((\bar{u}_h^*, \bar{\omega}_h^*) - (u^*, \omega^*)\) satisfies \((2.58)\), and from \((2.56), (2.58),\) and \((4.7)\), when \( h \) is small enough, there is a positive constant \( C_0 \) independent of \( h \) such that

\[ |B((u_h, \omega_h), (\bar{u}_h^*, \bar{\omega}_h^*))| \geq C_0. \]

Since \((\bar{u}_h, \bar{\omega}_h)\) is the orthogonal projection of \((u_h, \omega_h)\) to \( \text{ran}(E_h^*)\) in the sense of inner product \( A_h(\cdot, \cdot), \)
\[ |B((u_h, \omega_h), (u_h^*, \omega_h^*))| = \frac{1}{\lambda_h} A_h((u_h, \omega_h), (u_h^*, \omega_h^*)) \]
\[ \geq \frac{1}{\lambda_h} A_h((u_h, \omega_h), \frac{(u_h^*, \omega_h^*)}{\|u_h^*, \omega_h^*\|_h}) \]
\[ \geq \frac{1}{\|u_h^*, \omega_h^*\|_h} |B((u_h, \omega_h), (u_h^*, \omega_h^*))| \geq C_0. \]

In (4.33), chose \((v, z) = (u_h, \omega_h)\) and \((v^*, z^*) = (u_h^*, \omega_h^*)\), and chose \((u^*, \omega^*)\) such that \((u_h^*, \omega_h^*) - (u^*, \omega^*)\) satisfies (2.60)-(2.61), noting that
\[ \lambda_h = A((u_h, \omega_h), (u_h^*, \omega_h^*)) / B((u_h, \omega_h), (u_h^*, \omega_h^*)), \]
we obtain (4.34). \(\square\)

**Remark 4.1.** When \(\lambda\) is a simple eigenvalue, ran\((E_h^*\)) is a one-dimensional space spanned by the eigenfunction \((u_h^*, \omega_h^*)\) of (2.35) with the mesh size \(h\). When the multiplicity \(q > 1\) of \(\lambda\), in actual computation we can use the two sided Arnoldi algorithm to compute both left and right eigenfunctions of (2.25) at the same time, and obtain \((u_h, \omega_h)\) and \((u_h^*, \omega_h^*)\).

**Lemma 4.2.** Assume that the ascent \(\alpha = 1\) of \(\lambda, (u_h, \omega_h)\) is an eigenfunction corresponding to \(\lambda_h\) and \(\|u_h, \omega_h\|_h = 1\), then there exists eigenfunction \((u, \omega)\) corresponding to \(\lambda\) such that
\[ |\lambda_h - \lambda| + \|u_h, \omega_h\| = (u, \omega)\| H^1 \leq \|T - T_h\|_h (u_h, \omega_h)\| H^1. \quad (4.8) \]

**Proof.** Using the argument as in proposition 5.3 in [40] we can deduce
\[ \|u_h, \omega_h\| = (u, \omega)\| H^1 \leq \|T - T_h\|_h (u_h, \omega_h)\| H^1. \quad (4.9) \]
Simple calculation shows
\[ B((T - T_h)(u_h, \omega_h), (u^*, \omega^*)) = B(T(u_h, \omega_h), (u^*, \omega^*)) - B(T_h(u_h, \omega_h), (u^*, \omega^*)) \]
\[ = \lambda^{-1} A(T(u_h, \omega_h), (u^*, \omega^*)) - B(T_h(u_h, \omega_h), (u^*, \omega^*)) \]
\[ = \lambda^{-1} B((u_h, \omega_h), (u^*, \omega^*)) - \lambda_h^{-1} B((u_h, \omega_h), (u^*, \omega^*)) \]
\[ = (\lambda^{-1} - \lambda_h^{-1}) B((u_h, \omega_h), (u^*, \omega^*)), \]
where \((u^*, \omega^*)\) satisfies Theorem 4.1.
Then the above equality implies
\[ |\lambda_h - \lambda| \leq \|T - T_h\|_h (u_h, \omega_h)\| H^1. \quad (4.10) \]
Combining (4.9) and (4.10) we get (4.8). \(\square\)
Referring [42] et al., we give the relationship between the \(C^0\)IPG eigenvalue
Lemma 4.3. Let \((\lambda_h, (u_h, \omega_h))\) be the \(i\)th eigenpair of (2.25) with \(\|(u_h, \omega_h)\|_h = 1\), \(\lambda\) be the \(i\)th eigenvalue of (2.5), then there exists an eigenfunction \((u, \omega)\) corresponding to \(\lambda\), such that

\[
\|(u_h, \omega_h) - (u, \omega)\|_h = \lambda_h \|T(u_h, \omega_h) - T_h(u_h, \omega_h)\|_h + R_1,
\]

where \( |R_1| \lesssim \|(T - T_h)(u_h, \omega_h)\|_{H^1} \).

**Proof.** From (2.11), (2.12) and (4.8) we have

\[
\|(u, \omega) - \lambda_h T(u_h, \omega_h)\|_h = \|\lambda T(u, \omega) - \lambda_h T(u_h, \omega_h)\|_h
\]

\[
\lesssim \|\lambda(u, \omega) - \lambda_h (u_h, \omega_h)\|_{H^1} \lesssim \|(T - T_h)(u_h, \omega_h)\|_{H^1}.
\]

Denote

\[
\|(u_h, \omega_h) - (u, \omega)\|_h = \lambda_h \|(T - T_h)(u_h, \omega_h)\|_h + R_1.
\]

From the triangle inequality and (4.12) we deduce

\[
| R_1 | = | \|(u_h, \omega_h) - (u, \omega)\|_h - \lambda_h \|(T - T_h)(u_h, \omega_h)\|_h |
\]

\[
= | \|(u_h, \omega_h) - (u, \omega)\|_h - \|\lambda_h T(u_h, \omega_h) - (u, \omega)\|_h |
\]

\[
\leq \|(u, \omega) - \lambda_h T(u_h, \omega_h)\|_h \lesssim \|(T - T_h)(u_h, \omega_h)\|_{H^1}.
\]

Due to (4.13) and (4.14), (4.11) is obtained. \(\square\)

**Theorem 4.2.** Let \((\lambda_h, (u_h, \omega_h))\) be the \(i\)th eigenpair of (2.25) with \(\|(u_h, \omega_h)\|_h = 1\), \(\lambda\) be the \(i\)th eigenvalue of (2.5). Assume that \(R(\Omega)\) holds and \(n \in W^{1,\infty}(\Omega) \cap H^2(\Omega)\), then there exists an eigenfunction \((u, \omega)\) corresponding to \(\lambda\), such that

\[
\|(u_h, \omega_h) - (u, \omega)\|_h \lesssim \eta_h(F, u_h, \omega_h, \Omega),
\]

\[
\eta_h(F, u_h, \omega_h, \Omega) \lesssim \|(u_h, \omega_h) - (u, \omega)\|_h + \text{Osc}_m(F) + \text{Osc}_m(\eta_{\ell,3}(u_h)) + \text{Osc}_m(\eta_{\ell,4}(u_h)).
\]

**Proof.** Combining (4.11) with (4.1) we get (4.15). Combining (4.11) with (4.2) and neglecting the higher order small quantity \(R_1\) we get (4.16). \(\square\)

For the dual problem (2.13), denote

\[
F^* = F^*(f, g) = \frac{-1}{n - 1} \Delta f - \Delta \left(\frac{n}{n - 1} f\right) - \frac{n}{n - 1} g.
\]

Using the same argument as in Theorem 4.2 we can prove the following theorem.
\textbf{Theorem 4.3.} Let \((\lambda^*_h, (u^*_h, \omega^*_h))\) be the \(i\)th eigenpair of (2.35) with \(\|(u^*_h, \omega^*_h)\|_h = 1\), \(\lambda^*\) be the \(i\)th eigenvalue of (2.13). Assume that \(R(\Omega)\) holds and \(n \in W^{1,\infty}(\Omega) \cap H^2(\Omega)\), then there exists an eigenfunction \((u^*, \omega^*)\) corresponding to \(\lambda^*\), such that

\[
\|(u^*_h, \omega^*_h) - (u^*, \omega^*)\|_h \lesssim \eta_h(F^*, u^*_h, \omega^*_h, \Omega),
\]

where \(f = \lambda^*_h u^*_h, g = \lambda^*_h \omega^*_h\) in \(F^*\).

\textbf{Theorem 4.4.} Under the condition of Theorem 4.1, the following estimate holds

\[
|\lambda_h - \lambda| \lesssim \eta_h^2(F, u_h, \omega_h, \Omega) + \eta_h^2(F^*, u^*_h, \omega^*_h, \Omega) + R_2,
\]

where

\[
R_2 = \sum_{\kappa \in \pi_h} h_{\kappa}^{2\alpha} \|(u, \omega) - I_h(u, \omega)\|_{H^{2,\alpha}(\kappa)}^2 + \sum_{\kappa \in \pi_h} h_{\kappa}^{2\alpha} \|(u^*, \omega^*) - I_h(u^*, \omega^*)\|_{H^{2,\alpha}(\kappa)}^2.
\]

\textbf{Proof.} Thanks to Poincaré-Friedrichs inequalities in [41], we have \(\|(u_h, \omega_h) - (u, \omega)\|_{H^1} \lesssim \|(u_h, \omega_h) - (u, \omega)\|_h\) and \(\|(u^*_h, \omega^*_h) - (u^*, \omega^*)\|_{H^1} \lesssim \|(u^*_h, \omega^*_h) - (u^*, \omega^*)\|_h\). Thus from (4.5) we get

\[
|\lambda_h - \lambda| \lesssim \|(u_h, \omega_h) - (u, \omega)\|_h \|(u^*_h, \omega^*_h) - (u^*, \omega^*)\|_h
\]

Due to (2.29), the triangle inequality, (2.30), (2.31) and the interpolation estimate, we deduce

\[
\|(u_h, \omega_h) - (u, \omega)\|_h^2 \\
\leq \|(u_h, \omega_h) - I_h(u, \omega)\|_h^2 + \|(u, \omega) - I_h(u, \omega)\|_h^2 \\
\lesssim \|(u_h, \omega_h) - I_h(u, \omega)\|_h + \|(u, \omega) - I_h(u, \omega)\|_h \\
\lesssim \|(u, \omega) - I_h(u, \omega)\|_h \\
\lesssim \eta_h^2(F, u_h, \omega_h, \Omega) + \sum_{\kappa \in \pi_h} h_{\kappa}^{2\alpha} \|(u, \omega) - I_h(u, \omega)\|_{H^{2,\alpha}(\kappa)}^2.
\]

Similarly, we can get

\[
\|(u^*_h, \omega^*_h) - (u^*, \omega^*)\|_h^2 \\
\lesssim \eta_h^2(F, u^*_h, \omega^*_h, \Omega^*) + \sum_{\kappa \in \pi_h} h_{\kappa}^{2\alpha} \|(u^*, \omega^*) - I_h(u^*, \omega^*)\|_{H^{2,\alpha}(\kappa)}^2.
\]

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Submitting the above two estimate into (4.20), we get (4.19).  

\textbf{Remark 4.2.} From Theorems 4.2 and 4.3, we know the indicator \( \eta^2_h(F, u_h, \omega_h, \Omega) + \eta^2_h(F^\ast, u^\ast_h, \omega^\ast_h, \Omega) \) of the eigenfunction error \( \| (u_h, \omega_h) - (u, \omega) \|_h^2 + \| (u^\ast_h, \omega^\ast_h) - (u^\ast, \omega^\ast) \|_h^2 \) is reliable and efficient up to data oscillation, so Algorithm 1 can generate a good graded mesh, which makes approximation eigenfunctions can get the optimal convergent rate \( h^m \) in \( \| \cdot \|_h \). And thus we are able to expect to get \( R_2 \lesssim h^{2(m-1)} \), thereby from (4.19) have \( |\lambda_h - \lambda| \lesssim h^{2(m-1)} \). Therefore, we think that \( \eta^2_h(F, u_h, \omega_h, \Omega) + \eta^2_h(F^\ast, u^\ast_h, \omega^\ast_h, \Omega) \) can be viewed as the indicator of \( \lambda_h \). The numerical experiments in Section 5 show this indicator of \( \lambda_h \) is reliable and efficient. And \( \lambda_h \) can achieve the optimal convergent rate.

5 Adaptive algorithms and Numerical Experiment

Using the a posteriori error estimates and consulting the existing standard algorithms (see, e.g., [9,42]), we present the following algorithm.

\textbf{Algorithm 1}

Choose the parameter \( \sigma, \mu, 0 < \theta < 1 \).

\textbf{Step 1.} Set \( l = 0 \) and pick any initial mesh \( \pi_{h_l} \) with the mesh size \( h_l \).

\textbf{Step 2.} Solve \((2.25)\) on \( \pi_{h_l} \) for discrete solution \((\lambda_{h_l}, (u_{h_l}, \omega_{h_l}))\) with \( \| (u_{h_l}, \omega_{h_l}) \|_h = 1 \) and find \((u_{h_l}^\ast, \omega_{h_l}^\ast) \in ran(E_{k_l}^\ast) \) by (4.14) (also see Remark 4.1).

\textbf{Step 3.} Compute the local indicators \( \eta^2_h(F, u_{h_l}, \omega_{h_l}, \kappa) + \eta^2_h(F^\ast, u_{h_l}^\ast, \omega_{h_l}^\ast, \kappa) \).

\textbf{Step 4.} Construct the \( \pi_{h_{l+1}} \) by \textbf{Marking strategy E}.

\textbf{Step 5.} Refine \( \pi_{h_{l+1}} \) to get a new mesh \( \pi_{h_{l+1}} \) by procedure \textbf{Refine}.

\textbf{Step 6.} Set \( l = l + 1 \) and goto Step 2.

\textbf{Marking Strategy E}

Given parameter \( 0 < \theta < 1 \):

\textbf{Step 1.} Construct a minimal subset \( \tilde{\pi}_{h_l} \) of \( \pi_{h_l} \) by selecting some elements in \( \pi_{h_l} \) such that

\[
\sum_{\kappa \in \mathcal{E}_{h_l}} (\eta^2_{h_l}(F, u_{h_l}, \omega_{h_l}, \kappa) + \eta^2_{h_l}(F^\ast, u_{h_l}^\ast, \omega_{h_l}^\ast, \kappa)) \geq \theta(\eta^2_{h_l}(F, u_{h_l}, \omega_{h_l}, \Omega) + \eta^2_{h_l}(F^\ast, u_{h_l}^\ast, \omega_{h_l}^\ast, \Omega)).
\]

\textbf{Step 2.} Mark all the elements in \( \tilde{\pi}_{h_l} \).

We compute the transmission eigenvalues on the unit square domain with a slit \([0, 1]^2 \setminus [0.5, 1]\) and the L-shaped domain \([-1, 1]^2 \setminus [0, 1] \times [-1, 0] \) using Algorithm 1 with \( m = 2, 3 \). All the initial meshes are made up of congruent triangles. And the mesh sizes take \( h_0 = \frac{\sqrt{2}}{32} \) and \( h_0 = \frac{\sqrt{2}}{16} \) for the domain with a slit and the L-shaped domain, respectively. \( \theta = 0.25 \) and \( \theta = 0.5 \) for \( m = 2 \) and \( m = 3 \), respectively. Our programs uses MATLAB2012a and the iFEM
package (see [43]) on a HP-Z230 workstation (CPU 3.6GHZ and RAM 32GB). We use the sparse solver eigs to solve (2.25) and (2.35) for eigenvalues. Before showing the results, some symbols need to be explained:

\[ k_j = \sqrt{\lambda_j}; \]

\( \lambda_{j,h} \): the \( j \)th eigenvalue derived from the \( l \)th iteration using Algorithm 1,

\[ k_{j,h} = \sqrt{\lambda_{j,h}}; \]

DOF: the number of degrees of freedom.

The accurate eigenvalues for the problems on the two above domains are unknown. For the domain with a slit, we take \( k_1 \approx 2.80677803, k_2 \approx 2.98066000 \) for \( n = 16 \), and take \( k_1 \approx 4.14438323, k_7 \approx 5.57000885 - 1.31142340i \) for \( n = 8 + x - y \). For the L-shaped domain, we take \( k_1 \approx 1.47609911, k_2 \approx 1.56972499 \) for \( n = 16 \), and take \( k_1 \approx 2.30212024, k_5 \approx 2.92423162 - 0.56458999i \) for \( n = 8 + x - y \). All of them are obtained by Algorithm 1. And we think them relatively accurate. By computation we also know that the first ten smallest eigenvalues are all simple.

We present some adaptive refined mesh in Figure 1, and the curves of the error of the numerical eigenvalues in Figures 2 \( \sim \) 5.

From Figure 1, we can see that the singularities of the eigenfunctions for the two domain mainly center on the corner points.

From Figures 2 \( \sim \) 5, we see that the curves of the indicator are parallel to the curves of the error of \( \lambda_{j,h} \), which shows the posteriori error estimators are reliable and efficient for all the cases; we also see that the accuracy of the numerical eigenvalues on adaptive meshes, better than that on uniform meshes, can get the optimal convergence order \( O(\text{DOF}^{-m+1}) \), \( m = 2, 3 \).

However, from Figures 2 \( \sim \) 5, we also see that there exists fluctuation in the results on adaptive meshes when \( \text{DOF} \) is large enough. This is probably the consequence of the performance of linear algebra routine on this problem. To treat such problems to get higher accurate approximation much more careful design of the routine is needed.

References

[1] F. Cakoni, M. Cayoren, D. Colton, Transmission eigenvalues and the nondestructive testing of dielectrics, Inverse Problems, 24, 065016 (2008).

[2] F. Cakoni, D. Gintides, H. Haddar, The existence of an infinite discrete set of transmission eigenvalues, SIAM J. Math. Anal., 42, 237-255 (2010).

[3] J. Sun, Estimation of transmission eigenvalues and the index of refraction from Cauchy data, Inverse Problems, 27, 015009 (2011).

[4] D. Colton, R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, 2nd ed. Vol. 93 in Applied Mathematical Sciences, Springer, New York (1998).
Fig. 1. Adaptive meshes for the smallest eigenvalue on the domain with a slit with $DOF = 28688, n = 16, m = 3$ (left up), on the L-shaped domain with $DOF = 29972, n = 16, m = 3$ (right up), on the domain with a slit with $DOF = 29564, n = 8 + x - y, m = 3$ (left down) and on the L-shaped domain with $DOF = 32954, n = 8 + x - y, m = 3$ (right down).

Fig. 2. The convergence rates of eigenvalues for the domain with a slit (left) and for the L-shaped domain (right) when $n = 16, m = 2, \sigma = 30, \mu = \frac{1}{10}$.
Fig. 3. The convergence rates of eigenvalues for the domain with a slit (left) and for the L-shaped domain (right) when $n = 16, m = 3, \sigma = 30, \mu = \frac{1}{15}$.

Fig. 4. The convergence rates of eigenvalues for the domain with a slit (left) and for the L-shaped domain (right) when $n = 8 + x - y, m = 2, \sigma = 20, \mu = \frac{1}{5}$.

Fig. 5. The convergence rates of eigenvalues for the domain with a slit (left) and for the L-shaped domain (right) when $n = 8 + x - y, m = 3, \sigma = 20, \mu = \frac{1}{5}$.
[5] D. Colton, P. Monk, J. Sun, Analytical and computational methods for transmission eigenvalues, Inverse Problems, 26, 045011 (2010).

[6] J. An, J. Shen, A spectral-element method for transmission eigenvalue problems, J. Sci. Comput., 57, 670-688 (2013).

[7] F. Cakoni, P. Monk, J. Sun, Error analysis for the finite element approximation of transmission eigenvalues, Comput. Meth. Appl. Math., 14, 419-427 (2014).

[8] H. Geng, X. Ji, J. Sun, L. Xu, C0IP methods for the transmission eigenvalue problem, J. Sci. Comput., 68, 326-338 (2016).

[9] J. Han, Y. Yang, An adaptive finite element method for the transmission eigenvalue problem, J. Sci. Comput., 69, 326-338 (2016).

[10] X. Ji, J. Sun, T. Turner, Algorithm 922: a mixed finite element method for Helmholtz transmission eigenvalues, ACM Transaction on Math. Soft., 38, 29:1-8 (2012).

[11] X. Ji, J. Sun, H. Xie, A multigrid method for Helmholtz transmission eigenvalue problems, J. Sci. Comput., 60, 276-294 (2014).

[12] P. Monk, J. Sun, Finite element methods of Maxwell transmission eigenvalues, SIAM J. Sci. Comput., 34, B247–264 (2012).

[13] J. Sun, Iterative methods for transmission eigenvalues, SIAM J. Numer. Anal., 49, 1860-1874 (2011).

[14] J. Sun, L. Xu, Computation of Maxwells transmission eigenvalues and its applications in inverse medium problems, Inverse Problems, 29, 104013 (18pp) (2013).

[15] Y. Yang, H. Bi, H. Li, J. Han, Mixed method for the Helmholtz transmission eigenvalues, SIAM J. Sci. Comput., 38, A1383-A1403 (2016).

[16] Y. Yang, J. Han, H. Bi, Error estimates and a two grid scheme for approximating transmission eigenvalues, arXiv: 1506.06486 V2 [math. NA] 2 Mar 2016.

[17] Y. Yang, J. Han, H. Bi, Non-conforming finite element methods for transmission eigenvalue problem, Comput. Methods Appl. Mech. Engrg., 307, 144-163 (2016).

[18] J. Han, Y. Yang, An $H^m$–conforming spectral element method on multidimensional domain and its application to transmission eigenvalues, Sci. China Math., 60, 1529-1542 (2017).

[19] A. Kleefeld,A numerical method to compute electromagnetic interior transmission eigenvalues, Inverse Problems, 29(10), 104012 (2013).
[20] G. Engel, K. Garikipati, T. Hughes, M. Larson, L. Mazzei, R. Taylor, Continuous/discontinuous finite element approximations of fourth order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity, Comput. Methods Appl. Mech. Engrg., 191, 3669–3750 (2002).

[21] S.C. Brenner, C\(^0\) interior penalty methods, In Frontiers in Numerical Analysis-Durham 2010, Lecture Notes in Computational Science and Engineering 85, 79-147, Springer-Verlag (2012).

[22] S.C. Brenner, L. Sung, C\(^0\) interior penalty methods for fourth order elliptic boundary value problems on polygonal domains, J. Sci. Comput. 22/23, 83-118 (2005).

[23] T. Gudi, A new error analysis for discontinuous finite element methods for the linear elliptic problems, Math. Comp., 79, 2169-2189 (2010).

[24] X. Ji, J. Sun, Y. Yang, Optimal penalty parameter for C\(^0\)IPDG, Appl. Math. Lett., 37, 112-117 (2014).

[25] Y. Yang, H. Bi, H. Li, J. Han, A C\(^0\)IPG method and its error estimates for the Helmholtz transmission eigenvalue problem, J. Comput. Appl. Math., 326, 71-86 (2017).

[26] G.N. Wells, N.T. Dung, A C\(^0\) discontinuous Galerkin formulation for Kirchhoff plates, Comput. Methods Appl. Mech. Engrg., 196, 3370-3380 (2007).

[27] S.C. Brenner, P. Monk, J. Sun, C\(^0\)IPG method for biharmonic eigenvalue problems, R.M. Kirby et al. (eds.), Spectral and High Order Methods for Partial Differential Equations, ICOSAHOM 2014, Lecture Notes in Computational Science and Engineering 106. Springer International Publishing, Switzerland(2015).

[28] S.C. Brenner et al., Adaptive C\(^0\) interior penalty method for biharmonic eigenvalue problems, In Numerical Solution of PDE Eigenvalue Problems, Oberwolfach Rep. 10(4), pp. 3265-3267 (2013).

[29] H. Li, Y. Yang, C\(^0\)IPG adaptive algorithms for biharmonic eigenvalue problem, Numer. Algor., DOI 10.1007/s11075-017-0388-8 (2017).

[30] I. Babuska, W.C. Rheinboldt, Error estimates for adaptive finite element computations, SIAM J. Numer. Anal., 15, 736-754 (1978).

[31] M. Ainsworth, J.T. Oden, A Posterior Error Estimation in Finite Element Analysis, Wiley-Inter science, New York (2011).

[32] R. Verfürth, A Posteriori Error Estimation Techniques, Oxford University Press, USA (2013).

[33] Z. Shi, M. Wang, Finite Element Methods, Scientific Publishers, Beijing (2013).
[34] I. Babuska, J.E. Osborn, Eigenvalue Problems, in: P.G. Ciarlet, J.L. Lions, (Ed.), Finite Element Methods (Part 1), Handbook of Numerical Analysis, vol.2, Elsevier Science Publishers, North-Holland, 640-787 (1991).

[35] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, 2nd ed.. Springer-Verlag, New york (2002).

[36] P.G. Ciarlet, Basic Error Estimates for Elliptic Problems, in: P.G. Ciarlet, J.L. Lions, (Ed.), Finite Element Methods (Part1), Handbook of Numerical Analysis, vol.2, Elsevier Science Publishers, North-Holland, 21-343 (1991).

[37] J.T. Oden, J.N. Reddy, An Introduction to the Mathematical Theory of Finite Elements, Courier Dover Publications, New York (2012).

[38] F. Cakoni, H. Haddar, On the existence of transmission eigenvalues in an inhomogeneous medium, Appl. Anal., 88, 475-493 (2009).

[39] B.P. Rynne, B.D. Sleeman, The interior transmission problem and inverse scattering from inhomogeneous media, SIAM J. Math. Anal., 22, 1755-1762 (1991).

[40] F. Chatelin, Spectral Approximations of Linear Operators, Academic Press, New York (1983).

[41] S.C. Brenner, K. Wang, J. Zhao, Poincaré-Friedrichs inequalities for piecewise $H^2$ functions, Numer. Funct. Anal. Optim., 25, 463-478 (2004).

[42] X. Dai, J. Xu., A. Zhou, Convergence and optimal complexity of adaptive finite element eigenvalue computations, Numer. Math. 110, 313-355 (2008).

[43] L. Chen, iFEM: An integrated finite element method package in MATLAB, Technical Report, University of California at Irvine (2009).