Topologically quantized current in quasiperiodic Thouless pumps

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Thouless pumps are topologically nontrivial states of matter with quantized charge transport, which can be realized in atomic gases loaded into an optical lattice. This topological state is analogous to the quantum Hall state. However, contrarily to the exact and extremely precise and robust quantization of the Hall conductance, the pumped charge is strictly quantized only when the pumping time is a multiple of a characteristic time scale, i.e., the pumping cycle duration. Here, we show instead that the pumped current becomes exactly quantized, independently from the pumping time, if the system is led into a quasiperiodic, incommensurate regime. In this quasiperiodic and topologically nontrivial state, the Bloch bands and the Berry curvature become flat, the pumped charge becomes linear in time, while the current becomes steady, topologically quantized, and proportional to the Chern number. The quantization of the current is exact up to exponentially small corrections. This has to be contrasted with the case of the commensurate (nonquasiperiodic) regime, where the current is not constant, and the pumped charge is quantized only at integer multiples of the pumping cycle.

The hallmark of topological states of matter is the exact quantization of a physical observable in terms of a conserved quantity, the topological invariant\cite{1, 2}. A paradigmatic example is that of the quantum Hall conductance, which is quantized as integer (or fractional) multiples of $e^2/h$ with a precision which exceeds one part in a billion\cite{1, 3, 4}. Moreover, this quantization is robust against perturbations, i.e., it persists in the presence of disorder, defects, impurities, or imperfections of the experimental sample. This led to extremely precise definition of the electrical resistance standard and experimental determination of the fine-structure constant\cite{5}.

A topologically equivalent state of matter is the Thouless pump\cite{2, 6–14}, which can be engineered, for instance, with ultracold atoms\cite{15–18} in a superlattice created by the superposition of two optical lattices with different wavelengths\cite{19–22}. When the superlattice is adiabatically and periodically varied in time, the charge pumped through the atomic cloud is quantized in terms of the topological invariant, i.e., the Chern number of the Bloch bands\cite{2}. However, the charge is quantized only when the duration of the pumping process is an integer multiple $nT$ of a characteristic time $T$, which correspond to a full adiabatic cycle, and deviations from the quantized value are linear in time $\propto t - nT$. In this sense, the quantization of the pumped charge is not exact: This constitutes a fundamental hindrance to the realization of metrological standards.

Here, we will show that the exact quantization of the pumped current can be indeed realized by a Thouless pump in the quasiperiodic regime. In optical lattices, quasiperiodicity\cite{23–28} is realized using a superposition of two standing waves with incommensurate wavelengths, i.e., their ratio $\alpha$ is an irrational number. In this regime, the translational symmetry is completely broken, the familiar concept of Brillouin zone becomes ill-defined, and the usual definition of the Chern number as integral of the Berry curvature breaks down. In order to illustrate the exact quantization of the current in a realistic experimental setup, we will derive an effective tight-binding model describing an atomic gas in a bichromatic potential\cite{22}, which coincides with a generalized Harper-Hofstadter Hamiltonian\cite{29–34} with an extra spatial-dependent tunneling term. Secondly, we will operatively define the Chern number in the quasiperiodic regime, by taking the limit of an ensemble of periodic and topologically-equivalent (homotopic) states which progressively approximate the quasiperiodic system. In this limit, the Bloch bands and the Berry curvature become asymptotically flat. Finally, we describe the experimental fingerprint of the quasiperiodic nontrivial topological state, which can be observed in the charge transport and in the adiabatic evolution of the centre of mass of the atomic gas. Whereas in the commensurate (nonquasiperiodic) case, the current is not constant and the pumped charge is quantized only at exact multiples of the pumping cycle, we find that the quasiperiodic nontrivial state is characterized by a steady and topologically quantized pumping current, independently from the duration of the pumping process. The exact quantization of the current is a direct consequence of quasiperiodicity, and may contribute to the definition of a more accurate and direct definition of current standards\cite{35}.

In a typical experiment\cite{19, 20} a topologically nontrivial Thouless pump is realized by an ultracold Fermi gas loaded into a dynamically controlled bichromatic lattice, which is a superposition of a stationary lattice (short lattice) and a dynamical sliding lattice (long lattice) in the form

$$V(x, \phi) = V_S \cos^2 \left( \frac{\pi x}{d_S} \right) + V_L \cos^2 \left( \frac{\pi x}{d_L} + \frac{\phi}{2} \right),$$  \hspace{1cm} (1)

where $V_{S,L}$ and $d_{S,L}$ are the depths and the wavelengths respectively for the long and short lattices, and $\phi$ is the phase difference between the two lattices, which we assume to vary linearly in time $\phi = 2\pi t/T$ with a period $T$. The commensuration $\alpha = d_S/d_L$ between the two lattices can be in principle controlled by tilting the long lattice beam with respect to the short lattice\cite{14}. We assume a deep lattice regime $V_S > E_r$ (here $E_r = \hbar^2/(8M d_S^2)$ is the recoil energy of the short lattice\cite{16}. If $V_S > V_L$ the continuum Hamilto-
with \( q \) the long lattice as a perturbation\[9\]. This gives rise to localized states at the minima of the short lattice and treat-even close in the limit case site potential \( H \) Hamiltonian corresponding to a generalized Harper equation \( \alpha \) commensurate case, i.e., \( \alpha = p/q \) integer coprimes, α
\[E_k = \frac{n^2}{L^2} + k x, \varphi \] for \( n = 1 \)
\[\left[ \frac{1}{2} V + K \sin (\pi \alpha) \cos (k + \pi \alpha) \right] c_{k+1}^\dagger c_{k+2} + \text{h. c.}, \quad (3)\]
where the momentum can be restricted in the first Brillouin zone \( k \in [0, 2\pi/q] \). Figure 1 shows the energy spectra of the tight-binding model calculated for different values of \( K \). These spectra are a deformed version of the Hofstadter butterfly\[30\]: Indeed, whereas the Hofstadter butterfly \( K = 0 \) is symmetric with respect to the transformations \( \alpha \rightarrow 1 - \alpha \) and \( E \rightarrow -E \) (which correspond to \( k \rightarrow k + \pi \)), the spatially-dependent tunneling term breaks these symmetries. These asymmetries are inherited by the properties of the continuous potential in Eq. (1) (cf. Ref. [22]). For small \( K \), one can assume that the intraband gaps remain open for \( K \rightarrow 0 \) and are thus homeomorphic to the gaps of the Hofstadter butterfly. This mandates that the intraband gaps of the model are topologically nontrivial and characterized by a nonzero Chern number \( C \) which satisfies the diophantine equation \( -pC \equiv j \mod q \) (analogously to the Hofstadter butterfly \( K = 0 \)). We notice that, unlike the case of the Hofstadter butterfly \( K = 0 \), the energy spectra around \( E = 0 \) are gapped for \( \alpha = p/q \) with \( q \) even. These central gaps correspond to a nontrivial Chern number satisfying the diophantine equation \( -pC \equiv (q/2) \mod q \).

In the commensurate case, where \( \alpha \) is a rational number \( \alpha \in \mathbb{Q} \), assuming homogeneously populated bands below the Fermi level \( E_F \), and at zero temperature \( T = 0 \), the total charge pumped during an adiabatic evolution \( \phi \rightarrow \phi + 2\pi \) is quantized and equal to the Chern number \( C \) of the filled Bloch bands\[2\] \( Q = C = (1/2\pi) \int_0^{2\pi} \delta_{\phi} \cdot \partial_{\phi} W d\phi \). Here \( \Omega = \sum \Theta(E_F - E_i) \Omega_i \) is the total Berry curvature at the Fermi level \( E_F \), with \( \Theta(E) \) the Heaviside step function and \( \Omega_i = \oint_{\delta_{\phi} u_i} [\partial_{\phi} u_i] \) the Berry curvature of the \( i \)th band, defined in terms of the Bloch wavefunctions \( |\psi_i(k, x)\rangle = e^{i\phi} |u_i(k, x)\rangle \). On the other hand, the current \( I = \partial_t Q = (1/T) \int_0^{2\pi/q} d\phi \Omega \) is not quantized and it is not constant during the pumping process, and oscillates around an average value \( \langle I \rangle = 2\pi \Theta(\Omega)/(qT) \) with a maximum variation \( \delta I \leq 2\pi \delta \Omega/(qT) \) where \( \delta \Omega = \max \Omega - \min \Omega \).

Due to translational invariance, the Hamiltonian in Eq. (3) is periodic in the momentum \( k \rightarrow k + 2\pi/q \). However, it is not periodic in the phase since \( H(\phi + 2\pi q) \neq H(\phi) \). By direct substitution into Eqs. (1) and (2), one can show that a phase shift \( \phi \rightarrow \phi + 2\pi q \) is equivalent to a translation \( n \rightarrow n + c \) where \( c \) satisfies the diophantine equation \( p c \equiv m \mod q \). Thus, the Hamiltonian is “unitarily” periodic\[12, 13\] in the phase \( \phi \) up to lattice translations, in the sense that it is periodic up to unitary transformations (translations) \( H(\phi + 2\pi q) = T^c H(\phi) T^c \), where \( T \) translates the lattice by one site. This is a general property of the continuous model in Eq. (1) as well as the discrete tight-binding model in

Figure 1. Energy spectra of the tight-binding Hamiltonian in Eq. (3) calculated for \( V = 2J \) and \( K = J \) (a) and \( J/2 \) (b) respectively. The spectra are asymmetric with respect to the transformations \( E \rightarrow -E \) and \( \alpha \rightarrow 1 - \alpha \), and they are gapped around \( E = 0 \) for \( \alpha = p/q \) with \( q \) even. The central gap at \( \alpha = 1/2 \) has Chern number \( C = -1 \). The symmetries are restored and the central gaps at \( \alpha = p/q \) with \( q \) even close in the limit case \( K \rightarrow 0 \) (not shown).
as a consequence, the energies and Berry curvatures are periodic in the phase \( \phi \to \phi + 2\pi/q \). By considering the periodicities in the momentum and phase, one can conclude that the energies and Berry curvatures of Bloch bands satisfy

\[
E_n(k + n2\pi/q, \phi + m2\pi/q) = E_n(k, \phi),
\]
\[
\Omega_n(k + n2\pi/q, \phi + m2\pi/q) = \Omega_n(k, \phi).
\]

Hence, the energy bands and the Berry curvatures become flat in the limit of large denominators \( q \). Consequently, the pumped charge at well-defined fractions of the pumping cycle \( \Delta \phi = 2\pi m/q \) is quantized as fractions of the Chern number\(^{12,13}\) \( Q = mC/q = (1/2\pi) \int_0^{\phi + 2\pi m/q} \frac{d\phi}{q} \int_0^{2\pi/q} dk \Omega \).

We will now consider the incommensurate and quasiperiodic case, that is, when \( \alpha \) is an irrational number \( \alpha \in \mathbb{R} - \mathbb{Q} \). In order to do so, we will approach quasi-periodicity by taking the limit of the commensurate case with \( \alpha_n = p_n/q_n \to \alpha \), with \( p_n/q_n \) a succession of rational approximations of the number \( \alpha \), obtained in terms of continued fraction representation. Every irrational number \( \alpha \) can be written uniquely as an infinite continued fraction\(^{37}\) as \( \alpha = [a_0; a_1, a_2, \ldots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} \) with \( a_i \) integers. The successive approximations obtained by truncating the continued fraction representation \( \alpha_n = [a_0; a_1, a_2, \ldots, a_n] \) are rational numbers, and converge to \( \alpha \). We thus consider the ensemble of Hamiltonians \( H^{(\alpha_n)} \) describing the commensurate systems with commensurations \( \alpha_n = p_n/q_n = [a_0; a_1, a_2, \ldots, a_n] \).

We assume that the insulating gap at the Fermi level remains open, such that the Hamiltonians \( H^{(\alpha_n)} \) are homotopic and thus topologically equivalent. Since the denominator \( q_n \) increases for \( n \to \infty \), the Brillouin zone \( [0, 2\pi/q_n] \) shrinks at each successive approximations and therefore becomes ill-defined in the quasiperiodic limit. Thus, the usual definition of the Chern number as an integral of the Berry curvature in the Brillouin zone needs to be redefined. However, it is clear that in the quasiperiodic limit \( (q_n \to \infty) \), following Eq. (4), the energy bands and the Berry curvatures become constant, i.e., they no longer depend on the momentum and phase. Hence, the limit of the Berry integral for \( n \to \infty \) converges, and we can define the Chern number in the quasiperiodic limit as

\[
C_\alpha = \lim_{\alpha_n \to \alpha} \frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi}{q_n} \int_0^{2\pi/q_n} dk \Omega^{(\alpha_n)} = \lim_{\alpha_n \to \alpha} \frac{2\pi}{q_n} \int_0^{2\pi} \frac{d\phi}{q_n} \int_0^{2\pi/q_n} dk \Omega^{(\alpha_n)} = \frac{\Delta \phi}{2\pi} C_\alpha.
\]

In this limit, the Chern number is simply proportional to the Berry curvature, which diverges asymptotically as \( \Omega^{(\alpha_n)} \sim qC/2\pi \). Moreover, due to the flat Berry curvature, we obtain that the pumped charge pumped during adiabatic transformations, for any initial and final values of the phase \( \phi \to \phi + \Delta \phi \) becomes

\[
Q_\alpha = \lim_{\alpha_n \to \alpha} \frac{1}{2\pi} \int_0^{\phi + \Delta \phi} \frac{d\phi}{q_n} \int_0^{2\pi/q_n} dk \Omega^{(\alpha_n)} = \frac{\Delta \phi}{2\pi} C_\alpha,
\]

whereas the instantaneous charge current becomes

\[
I_\alpha = \lim_{\alpha_n \to \alpha} \frac{1}{2\pi} \int_0^{2\pi/q_n} dk \Omega^{(\alpha_n)} = \frac{C}{T}.
\]

In the quasiperiodic limit, the Berry curvature becomes flat, the pumped charge is linear in the phase difference \( \Delta \phi \), whereas the current \( I = \partial_t Q \) becomes constant and proportional to the Chern number, defined in Eq. (5).

The quantization of the pumped current is the distinctive fingerprint of the quasiperiodic topological state. In order to illustrate how to verify this experimentally, we now consider the original continuous Hamiltonian which describes the ultracold atomic gas in a bichromatic potential. The atomic gas is typically confined by a shallow harmonic potential \( V_T(x/d_S)^2 \). The continuous system is thus described by the time-dependent Schrödinger equation

\[
\hat{h} \partial_t \Psi(x,t) = \left[ -\frac{\hbar^2 \partial_x^2}{2M} + V(x,t) + \frac{V_T}{d_S^2} x^2 \right] \Psi(x,t).
\]
The pumped current $I = \partial_t Q$ can be linked to a simple physical observable, i.e., the center of mass of the atomic cloud. The variation of the center of the cloud $\langle x(t) \rangle = (1/N) \int_{-\infty}^{\infty} |x| \Psi(x,t)|^2$ is proportional to the pumped charge[10, 12], that is $Q = \rho \langle [x(t + \Delta t)] - \langle x(t) \rangle \rangle$ where $\rho = j/(qds)$ is the number of atoms $j$ per unit cell $qds$. The length $L$ of the cloud can be controlled by the trapping potential $V_T$. For a given choice of the commensuration $\alpha = p/q$, the length of the superlattice unit cell is equal to $qds$. Therefore, assuming the number of filled bands to be $j$, the total length of a cloud of $N$ atoms is given by $N/j$ unit cells. Hence, the number of atoms $N$ must be an integer multiple of the filling factor $j$, and the trapping potential $V_T$ must be tuned such that the length of the cloud satisfies

$$N \equiv j \frac{L}{qds}. \quad (9)$$

This condition guarantees a uniform density of the atomic cloud, equal to $j$ atoms per unit cell $qds$. Moreover, in order to minimize thermal and nonadiabatic effects, one should consider a filling factor $j = p$ which correspond to the large central gap of Fig. 1 with Chern number $C = -1$.

Figures 2(a) and (b) show respectively the pumped current $I$ and the current $I = \partial_t Q$ pumped through the atomic cloud obtained by calculating the center of the cloud of the continuous system described in Eq. (8) in the adiabatic limit, for the central gap with Chern number $C = -1$. Different curves correspond to successive rational approximations of the commensuration $\alpha = 1/\Phi^2 \in \mathbb{R} - \mathbb{Q}$, where $\Phi$ is the golden ratio. We tune the trapping potential such that the length of the cloud $L$ satisfies Eq. (9). We notice that the presence of the harmonic trap necessarily breaks the periodicity in the phase $\phi$. Figures 2(c) and (d) show respectively the charge and current calculated by numerical integrating Eqs. (6) and (7) for the effective tight-binding Hamiltonian in Eq. (3). The charge and the current calculated for the continuous lattice (in the presence of the harmonic trap) and for the effective tight-binding model show the same asymptotic behavior approaching the quasiperiodic limit. The pumped charge is quantized as integer fractions of the Chern number $\rho/m_q C$ for well-defined fractions of the pumping period $\Delta \phi = 2\pi m/q$. For increasing denominators $q$, the pumped charge approximates a linear dependence $Q = \Delta \phi C \alpha/2\pi$, whereas the current approaches its quantized value $I_{\alpha} = C\alpha/T$ for $\alpha_n \rightarrow \alpha$.

Hence, the pumped current $I_{\alpha}$ in the quasiperiodic limit is quantized and equal to the Chern number (in elementary units). This result has been obtained by only using the symmetry of the systems, in particular the unitary periodicity in the phase $\phi$. We now determine the asymptotic behavior of the current as one approaches the quasiperiodic limit. For $K = 0$, Eq. (3) reduces to the Harper-Hofstadter Hamiltonian: In this case, it has been shown numerically and perturbatively[38] that the Berry curvature takes the form $\Omega^{(q)} \approx F + Ge^{-\pi/\xi}[\cos(qk) + \cos(q\phi)]$. It is reasonable to extrapolate this result also to the case $K \neq 0$. From Eq. (5), one has that $F = qC/2\pi$, whereas $G \propto q^2$ (see Ref. [38]).

Hence, the flattening of the Berry curvature is exponential, and for large $q$ one has asymptotically $\delta \Omega^{(q)} \approx q^2 e^{-q/\xi}$ where $g > 0$ is a constant. Thus, the current approaches its quantized value exponentially as

$$\delta I = |I - I_{\alpha}| \lesssim \frac{2\pi g q}{T} \exp \left( -\frac{q}{\xi} \right) \approx \frac{2\pi q}{T\sqrt{D}|\alpha - \alpha_n|} \exp \left( -\frac{1}{\xi \sqrt{D}|\alpha - \alpha_n|} \right). \quad (10)$$

where $|\alpha - \alpha_n| \sim 1/Dq^2$ with $D < \sqrt{5}$, due to the Dirichlet’s approximation theorem and Hurwitz’s theorem[37]. Thus, Eq. (10) describes the scaling behavior of the current in the quasiperiodic limit, in terms of the difference $|\alpha - \alpha_n|$ between the irrational commensuration $\alpha$ and its successive rational approximations $\alpha_n = p_n/q_n$.

In Fig. 2(e) we verify numerically the scaling behavior of the current, by calculating the variations $\delta I$ using Eq. (7) for the effective tight-binding Hamiltonian in Eq. (3). The current approaches its quantized value $I_{\alpha} = C\alpha/T$ in the quasiperiodic limit $\alpha_n \rightarrow \alpha$ exponentially as $\propto \exp(-q/\xi)$. By linear fitting the exponential decays, we determine numerically the scaling length $\xi$, as a function of $J$ and $K$. The numerical analysis suggests a functional dependence of the form $\xi \propto J \exp(-\beta K$. The current is quantized and equal to the Chern number (in natural units), and approaches its quantized value exponentially fast in the quasiperiodic limit.

In summary, we have shown how a quasiperiodic and topologically nontrivial Thouless pump can be realized by an atomic gas confined in a bichromatic and quasiperiodic optical lattice, which is a superposition of two harmonic potentials with incommensurate wavelengths. This system is characterized by a topological invariant defined as the limit of the Chern numbers of an ensemble of topologically-equivalent and periodic Hamiltonians. The distinctive fingerprint of this quasiperiodic and topologically nontrivial state is the exact quantization of the current in units of the Chern number, which is a consequence of the flattening of the Bloch bands and of the Berry curvature in the quasiperiodic limit. This exact quantization is measurable in a typical experimental setting of ultracold atomic gases in optical lattices, and may open new perspectives for a more accurate and direct definition of current standards.

The work of P. M. is supported by the Japan Science and Technology Agency (JST) of the Ministry of Education, Culture, Sports, Science and Technology (MEXT), JST CREST Grant. No. JPMJCR19T2 and by the (MEXT)-Supported Program for the Strategic Research Foundation at Private Universities “Topological Science” (Grant No. S1511006). The work of M. N. is partially supported by the Japan Society for the Promotion of Science (JSPS) Grant-in-Aid for Scientific Research (KAKENHI) Grant Numbers 16H03984 and 18H01217 and by a Grant-in-Aid for Scientific Research on Innovative Areas “Topological Materials Science” (KAKENHI Grant No. 15H05855) from MEXT of Japan.
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