OPTIMAL SENSORS PLACEMENT FOR DAMAGE DETECTION OF BEAM STRUCTURES

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Abstract. This paper is dedicated to the identifiability of vibrating beam structures with uncertain damages. The probability of damage occurrence is computed assuming a Gaussian distributed random damage parameter. Then, we propose a technique for selecting an optimized solution of sensors placement based on the comparison among the probability of damage occurrence and the probability to detect the damage, where the latter is evaluated exploiting the closed-form asymptotic solution provided by a perturbation approach. This comparison must be intended as an investigation on the minimum number of sensors beyond which monitoring accuracy (estimated by an error function measuring the differences among the two probabilities) increases less than a ‘small’ predetermined threshold.

The capabilities and efficiency of the technique are shown through a parametric analysis on a sample case study: a simply supported beam with a random parameter ruling the evolution of a non-localized damage. The relevant results are presented and discussed, showing which conditions (sensors network) properly characterizes the beam dynamics.
1 INTRODUCTION

Safety assessment of existing structures is a crucial task of many engineering fields. Within this framework, dynamic approaches - mainly concerned with output-only techniques - have been proved useful to locate and quantify structural damages. When attention is paid to “early warning” systems, able to activate specific protocols, a robust identifiability of dynamics features of the structure plays a key role. Despite this increasing interest, structural health monitoring is often addressed in engineering practice relying on simplified rules, where sensors network is established a priori.

Among the approaches dedicated to sensors placement, we may cite the one proposed in [1], based on improved genetic algorithms, and the work [2] where optimal placement for multi-setup modal analysis is considered. Several methods for optimal sensors placement were studied and compared for a suspended bridge in [3]. Conditions for the invertibility of linear system models are discussed in [4]. In the cited papers attention is paid to experimental modal analysis, whereas there are few works on identifiability of damage processes of a structure; the recent work [5] focused on searching the weakest part of the structure in a random context.

This paper deals with the identifiability of vibrating beam structures with uncertain damages. The dynamic behavior of one-dimensional elements has been extensively investigated in the literature, showing how damages, even if microcracks like, clearly affect displacement fields [6] and propagating waves [7, 8]. Here, non-localized damages are considered, which are in general more difficult to identify than narrow damages [9]. Damages are often ‘small’ changes of the healthy system and therefore a perturbation approach can properly describe the effect of the damage, both for standard [10] and state-space dynamic analysis [11].

In [12], a perturbation method has been proposed by some of the authors to derive the asymptotic eigensolution of a vibrating beam with uncertain damage. Then, the statistics of the random damage parameter have been obtained by means of an objective function minimizing the difference among analytical and experimental fractiles of the eigenvalues. The results showed how the second order terms of the perturbation approach significantly increase the identifiability of stochastic damages. Starting from this model, we here propose a technique for obtaining an optimized solution of sensors placement. The proposed approach aims at exploiting the closed-form asymptotic solution of the inverse problem to compare more combinations of number and placement of the sensors. These comparisons rely on an error function measuring the deviation from the analytical probability of damage occurrence.

Due to the asymptotic nature of the research pattern, the performance of the proposed placement optimization technique has a low computational effort. A simply supported beam with a random parameter ruling the evolution of a non-localized damage centred at the midspan is assumed as case study. The results are presented and discussed, showing the capability and the efficiency of the proposed technique in selecting optimal sensors networks.
2 PROPOSED METHOD

2.1 Beam formulation

When shear effect is assumed negligible (according to the Euler-Bernoulli theory), free flexural undamped vibrations of one-dimensional continua are governed by the relationship:

\[
\frac{\partial^2}{\partial x^2} \left( B(x, \varepsilon) \frac{\partial^2 v}{\partial x^2} \right) + \mu(x, \varepsilon) \frac{\partial^2 v}{\partial t^2} = 0 \quad x \in [0, L]
\]

where \( B = EI \) is the flexural stiffness (being \( E \) and \( I \) the Young’s modulus of the material and the moment of inertia of the cross-section, respectively), \( \rho = \rho_v A \) is the linear mass density (being \( \rho_v \) and \( A \) the volume mass density of the material and the area of the cross-section, respectively), \( x \) is the abscissa along the beam, of length \( L \), \( v \) is the transversal displacement, and \( \varepsilon \) is a random variable (see Fig. 1).

Figure 1: Beam model.

In what follows we assume:

1. \( \rho(x, \varepsilon) = \rho; \)
2. \( B(x, \varepsilon) = B_0 + \varepsilon B_1(x), \ |\varepsilon| \ll 1; \)
3. \( \varepsilon \sim \mathcal{N}(\mu_\varepsilon, \sigma_\varepsilon); \)

that is: 1. changes of the mass are assumed negligible (a fairly common hypothesis when one deals with damage identification); 2. \( \varepsilon \) is a ‘small’ perturbation, linearly scaling the stiffness variation \( B_1(x) \) of an initial (uniform) bending stiffness \( B_0; \) 3. the random parameter \( \varepsilon \) is assumed Gaussian distributed with mean value \( \mu_\varepsilon \) and standard deviation \( \sigma_\varepsilon \). Although these positions reduce the entirety of the problem, they provide a useful simplification, allowing for a proper description of stochastic reductions (damage) and increases (reinforcement) of the beam stiffness. Specifically, in our applications we pose a negative function \( B_1(x) \) (which establishes where the damage is located), so that the random parameter \( \varepsilon \) can be regarded as a random damage, where \( \varepsilon = 0 \) implies an healthy beam, whereas a positive value of the parameter indicates a damage state (the higher the parameter, the greater the damage).

With these assumptions, Eq. (1) leads to:

\[
\frac{\partial^2}{\partial x^2} \left( (B_0 + \varepsilon B_1(x)) \frac{\partial^2 v}{\partial x^2} \right) + \rho \frac{\partial^2 v}{\partial t^2} = 0 \quad x \in [0, L] \quad \varepsilon \sim \mathcal{N}(\mu_\varepsilon, \sigma_\varepsilon) \quad |\varepsilon| \ll 1
\]

and expanding all the terms in:

\[
\frac{\partial^2}{\partial x^2} \left( (B_0 + \varepsilon B_1(x)) \frac{\partial^2 v}{\partial x^2} \right) + \rho \frac{\partial^2 v}{\partial t^2} = 0
\]
one gets:
\[
\varepsilon \frac{\partial^2 B_1(x)}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + 2\varepsilon \frac{\partial B_1(x)}{\partial x} \frac{\partial^2 v}{\partial x^2} + (B_0 + \varepsilon B_1(x)) \frac{\partial^4 v}{\partial x^4} + \rho \frac{\partial^2 v}{\partial t^2} = 0
\]
(4)

Then, a separation of variables (modal projection):
\[
v(x, t) = \sum_{i=1}^{\infty} \phi_i(x) q_i(t, \varepsilon) \quad q_i(t, \varepsilon) = C_i \sin (\omega_i(\varepsilon)t + \theta_i) \quad \omega_i(\varepsilon) = \lambda_i^2(\varepsilon) \sqrt{\frac{B_0}{\rho}}
\]
(5)
gives the governing equation (from now on: \(\bullet\)' = \(\partial(\bullet)/\partial x\), \(\bullet\) = \(\partial(\bullet)/\partial t\)):
\[
\varepsilon B''_i(x) \phi''_i(x, \varepsilon) + 2\varepsilon B'_i(x) \phi''_i(x, \varepsilon) + (B_0 + \varepsilon B_1(x)) \phi'''_i(x, \varepsilon) = -\frac{\dot{q}_i(t, \varepsilon)}{q_i(t, \varepsilon)} = \omega_i^2(\varepsilon)
\]
(6)
where \(\phi_i(x, \varepsilon)\) are the mode shapes (eigenvectors) and \(f_i = \omega_i/(2\pi)\) the relevant frequencies (related to the eigenvalues \(\lambda_i\)).

Since \(\varepsilon\) is a small parameter, a suitable solution of Eq. (5) can be achieved through a perturbation approach. Moreover, the dynamic system under investigation is non-nilpotent (non-defective), therefore the eigensolution is a perturbation series of integer powers of the small parameter (a Taylor series):
\[
\begin{align*}
\lambda_i(\varepsilon) &= \lambda_{0i} + \varepsilon \lambda_{1i} + \varepsilon^2 \lambda_{2i} + \ldots \\
\phi_i(x, \varepsilon) &= \phi_{0i}(x) + \varepsilon \phi_{1i}(x) + \varepsilon^2 \phi_{2i}(x) + \ldots
\end{align*}
\]
(7)

Defining the operator:
\[
L[\bullet] = -B_0 \lambda^2_0(\bullet) + B_0(\bullet)''''
\]
(8)
and balancing the same powers in \(\varepsilon\), Eq. (6) and Eq. (7) can be written as a series of equations:
\[
\begin{align*}
\varepsilon^0 : L[\phi_0(x)] &= 0 \\
\varepsilon^1 : L[\phi_1(x)] &= 4B_0 \lambda^3_0 \lambda_1 \phi_0(x) - B''_0(x) \phi''_0(x) - 2B'_0(x) \phi''_0(x) - B_1(x) \phi'''_0(x) \\
\varepsilon^2 : L[\phi_2(x)] &= 6B_0 \lambda^3_0 \lambda^2_1 \phi_0(x) + 4B_0 \lambda^3_0 \lambda_2 \phi_0(x) + 4B_0 \lambda^3_0 \lambda_1 \phi_1(x) + \\
&\quad -2B''_0(x) \phi''_0(x) + 2B'_0(x) \phi''_0(x) + B_1(x) \phi'''_0(x)
\end{align*}
\]
(9)
where, for the sake of brevity, the dependence on \(i\) has been omitted.

The solution of the so-called generating equation Eq. (9) is the well-known one:
\[
\phi_{0i}(x) = C_1 \cos(\lambda_{0i}x) + C_2 \sin(\lambda_{0i}x) + C_3 \cosh(\lambda_{0i}x) + C_4 \sinh(\lambda_{0i}x)
\]
(10)
where the eigenvalues \(\lambda_{0i}\) and three of the four constants \(C_j\), \(j = 1, \ldots, 4\) (the fourth one is a generic constant scaling the mode) are evaluated imposing the four boundary conditions.

Once the couples \((\lambda_{0i}; \phi_{0i}(x))\) are known, Eq. (9) gives \((\lambda_{1i}; \phi_{1i}(x))\), Eq. (9) gives \((\lambda_{2i}; \phi_{2i}(x))\) and so on. After some simplifications, the following expressions are obtained:
\[
\begin{align*}
\lambda_{1i} &= \frac{\int_0^L \phi_{0i}(x) B''_i(x) \phi''_0(x) dx + 2 \int_0^L \phi_{0i}(x) B'_i(x) \phi''_0(x) dx + \lambda^4_0 \int_0^L B_1(x) \phi''_0(x) dx}{4B_0 \lambda^3_0 \int_0^L \phi''_0(x) dx} \\
\lambda_{2i} &= \frac{-3 \lambda^2_{1i} \int_0^L \phi''_0(x) dx - 2 \lambda_{0i} \lambda_{1i} \int_0^L \phi_{0i}(x) \phi_{1i}(x) dx}{2 \lambda_{0i} \int_0^L \phi''_0(x) dx} \\
&\quad + \frac{\int_0^L \phi_{0i}(x) B''_1(x) \phi''_0(x) dx + 2 \int_0^L \phi_{0i}(x) B'_1(x) \phi''_0(x) dx + \int_0^L \phi_{0i}(x) B_1(x) \phi''_0(x) dx}{4B_0 \lambda^3_0 \int_0^L \phi''_0(x) dx}
\end{align*}
\]
(11)
for the first and second order terms of the eigenvalues, and:

\[
\begin{align*}
\phi_{1i}(x) &= \alpha_{ii}\phi_{0i}(x) + \sum_{k=1, k \neq i}^{\infty} \alpha_{ik}\phi_{0k}(x) \\
\alpha_{ik} &= \int_0^L \phi_{0k}(x)B_{1i}''(x)\phi_{0i}'(x)dx + 2\int_0^L \phi_{0k}(x)B_{1i}'(x)\phi_{0i}''(x)dx + \lambda_{0i}^4 \int_0^L \phi_{0k}(x)B_1(x)\phi_{0i}(x)dx \\
\phi_{2i}(x) &= \beta_{ii}\phi_{0i}(x) + \sum_{k=1, k \neq i}^{\infty} \beta_{ik}\phi_{0k}(x) \\
\beta_{ik} &= -4B_0\lambda_{0i}^3\lambda_{1i}\int_0^L \phi_{0k}(x)\phi_{1i}(x)dx + \frac{B_0(\lambda_{0i}^4 - \lambda_{0k}^4)\int_0^L \phi_{0k}^2(x)dx}{B_0(\lambda_{0i}^4 - \lambda_{0k}^4)\int_0^L \phi_{0k}^2(x)dx} + \\
&\quad + \int_0^L \phi_{0k}(x)B_{1i}''(x)\phi_{0i}'(x)dx + 2\int_0^L \phi_{0k}(x)B_{1i}'(x)\phi_{0i}''(x)dx + \int_0^L \phi_{0k}(x)B_1(x)\phi_{1i}(x)dx \\
&\quad \frac{B_0(\lambda_{0i}^4 - \lambda_{0k}^4)\int_0^L \phi_{0k}^2(x)dx}{B_0(\lambda_{0i}^4 - \lambda_{0k}^4)\int_0^L \phi_{0k}^2(x)dx}
\end{align*}
\]

for the first and second order terms of the eigenvectors.

### 2.2 Optimal sensors placement

For a given beam, once assigned the stiffness \(B(x, \varepsilon) = B_0 + \varepsilon B_1(x)\), the linear mass density \(\rho(x, \varepsilon) = \rho\) and the length \(L\), Eqs. (10–12) furnish a closed-form solution of the eigenvalues and eigenvectors as:

\[
\begin{align*}
\lambda_i(\varepsilon) &= \lambda_{0i} + \varepsilon \lambda_{1i} + \varepsilon^2 \lambda_{2i} + \ldots \\
\phi_i(x, \varepsilon) &= \phi_{0i}(x) + \varepsilon \phi_{1i}(x) + \varepsilon^2 \phi_{2i}(x) + \ldots
\end{align*}
\]

(13)

here adopted to estimate the identifiability of damages in the vibrating uncertain beam.

The damage state of the beam is governed by the random parameter \(\varepsilon\), therefore the damage probability \(P_d\) can be evaluated as:

\[
P_d = P(\varepsilon > 0) = \int_0^\infty f(\varepsilon) d\varepsilon
\]

(14)

being \(f(\varepsilon)\) the probability density function of \(\varepsilon\). In detail, since the random parameter \(\varepsilon\) has been assumed gaussian distributed, the probability of damage \(P_d\) depends only on the ratio among the standard deviation \(\sigma_\varepsilon\) and the mean value \(\mu_\varepsilon\), that is, \(P_d\) is a function of the so-called coefficient of variation \(CV_\varepsilon = \sigma_\varepsilon / \mu_\varepsilon\). Figure 2 shows the variation of the probability of damage \(P_d\) when the coefficient of variation \(CV_\varepsilon\) is increased from 0 to 1.

On the other hand, the probability to detect the damage resorting to a statistical dynamic analysis is related to the reduction of the eigenvalues:

\[
P_{ddi} = P(\lambda_i(\varepsilon) < \lambda_{0i}) = P(\lambda_i(\varepsilon) - \lambda_{0i} < 0) \quad i = 1, \ldots, n_\lambda
\]

(15)

being \(n_\lambda\) the number of the first (measured) eigenvalues (function of the sampling frequency of the sensors).

Neglecting the terms greater than the second order, the perturbative solution Eq. (13) gives a closed-form solution for the reduction of the eigenvalues:

\[
\lambda_i(\varepsilon) - \lambda_{0i} = \varepsilon \lambda_{1i} + \left(\frac{\varepsilon}{\sigma_\varepsilon}\right)^2 \sigma_\varepsilon^2 \lambda_{2i}
\]

(16)
where, since \( \varepsilon \) has been assumed a gaussian parameter, \( \varepsilon \sim \mathcal{N}(\mu_\varepsilon, \sigma_\varepsilon) \), the term \((\varepsilon/\sigma_\varepsilon)^2\) has a noncentral chi-squared distribution \( \chi^2(k_\chi, \lambda_\chi) \), with a unitary degree of freedom \( k_\chi \) and non-centrality parameter \( \lambda_\chi \) equals to \((\mu_\varepsilon/\sigma_\varepsilon)^2\).

The probability density function of the term \( \lambda_i(\varepsilon) - \lambda_{0i} \) is therefore a finite mixture of two known continuous distributions:

\[
f(\lambda_i(\varepsilon) - \lambda_{0i}) = w_{1i} f(\varepsilon) + w_{2i} g(\varepsilon)
\]

with:

\[
\begin{align*}
  f(\varepsilon) & : \varepsilon \sim \mathcal{N}(\mu_\varepsilon, \sigma_\varepsilon) \\
  g(\varepsilon) & : \varepsilon \sim \chi^2(1, (\mu_\varepsilon/\sigma_\varepsilon)^2)
\end{align*}
\]

and being \( w_{1i}, w_{2i} \) the relevant weights, which must satisfy the two conditions: \( w_{1i}, w_{2i} \geq 0 \) and \( w_{1i} + w_{2i} = 1 \). Comparing Eq. (16) with Eq. (17), a suitable choice for the two weights is:

\[
w_{1i} = \frac{|\lambda_{1i}|}{|\lambda_{1i}| + \sigma_\varepsilon^2|\lambda_{2i}|} \quad w_{2i} = \frac{\sigma_\varepsilon^2|\lambda_{2i}|}{|\lambda_{1i}| + \sigma_\varepsilon^2|\lambda_{2i}|}
\]

The probability to detect the damage \( P_{d\text{di}} = P(\lambda_i(\varepsilon) < \lambda_{0i}) \) is therefore:

\[
P_{d\text{di}} = \int_{\Omega_\varepsilon} w_{1i} f(\varepsilon) + w_{2i} g(\varepsilon) d\varepsilon \quad i = 1, \ldots, n_\lambda
\]

where Eqs. (18–19) give the integrand (the mixture distribution), whereas the domain \( \Omega_\varepsilon \) is a function of \( \lambda_{1i} \) and \( \lambda_{2i} \) (see Eq. (16)).

Generally, the two derivatives \( \lambda_{1i} \) and \( \lambda_{2i} \) are both negative (see Fig. 8 of the case study); for the sake of brevity, the probability \( P_{d\text{di}} \) is explicitly evaluated only in this case. When \( \lambda_{1i} \) and \( \lambda_{2i} \) are negative, the domain is \( \Omega_\varepsilon = (-\infty, -\lambda_{1i}/\lambda_{2i}) \cup (0, \infty) \) and the integral in Eq. (20) provides the probability to detect the damage as:

\[
P_{d\text{di}} = \gamma_{1i} w_{1i} \frac{1}{2} \left( 1 + \text{Erf} \left[ \frac{1}{\sqrt{2CV_\varepsilon}} \right] + \text{Erfc} \left[ \frac{\lambda_{1i} + \lambda_{2i}/\mu_\varepsilon}{\sqrt{2CV_\varepsilon} \lambda_{2i}/\mu_\varepsilon} \right] \right) + \gamma_{2i} w_{2i} \quad i = 1, \ldots, n_\lambda
\]
being Erf the error function and Erfc the complementary error function, and where $\gamma_{1i}$ and $\gamma_{2i}$ are penalty indexes affecting the first and the second order term, respectively. These two indexes play a key role in finding the solution and several definitions can be adopted. In this work, penalty is related to the discretization error of experimental mode shapes, i.e., $\gamma_{1i}$ and $\gamma_{2i}$ are evaluated comparing the continuous (analytical) mode shapes and the discretized (measurable by the network) ones. Specifically, mode shapes are compared using the absolute values of the areas between the mode shape and the beam axis as:

$$
\gamma_{1i} = 1 - \frac{|\tilde{A}_{1i} - A_{1i}|}{A_{1i}} \quad \gamma_{2i} = 1 - \frac{|\tilde{A}_{2i} - A_{2i}|}{A_{2i}}
$$

(22)

Figure 3: Areas under the graph of $|\phi_{1i}(x)|$ for different values of the number $n_s$ of sensors.

The discrepancy between the probability of damage occurrence, Eq. 14, and the probability to detect the damage, Eq. 21, is therefore a (scalar) measure of the sensor network capabilities in damage detection, and can be profitably implemented for selection purposes. In detail, for a given beam and a given network, we assume as overall measure of the capability in damage detection the following average difference:

$$
\Delta P = \sum_{i=1}^{n_\lambda} \frac{1}{n_\lambda} \Delta P_i \quad \Delta P_i = P_d - P_{ddi}
$$

(24)
which is a function of: the number \( n_\lambda \) of measured eigenvalues, the positions \( x_s \) of sensors along the beam axis and the number \( n_s \) of sensors, \( \Delta P = \Delta P(n_\lambda, n_s, x_s) \). Assuming that an optimal sensors placement is the one requiring less sensors, the best sensors network for a given number \( n_\lambda \) of measured eigenvalues can be defined as:

\[
\min n_s : \Delta P \leq \hat{\Delta} P
\]  

(25)

being \( \hat{\Delta} P \) a chosen threshold value for the difference among the probability of damage and the probability of detecting the damage.

3 CASE STUDY

We consider a simply supported (hinged-hinged) beam. The following properties are assumed for the undamaged scenario: bending stiffness \( B_0 = 162 \cdot 10^6 \text{Nm}^2 \), linear mass density \( \rho = 450 \text{kg/m} \) and length \( L = 6 \text{~m} \). Values are consistent with a concrete beam of Young’s modulus \( E = 30 \cdot 10^9 \text{~Pa} \), volume mass density \( \rho_V = 2500 \text{~kg/m}^3 \) and rectangular cross-section of base 0.30 m and height 0.60 m, Fig. 4.

![Figure 4: Case study.](image)

Damage is introduced using for \( B_1(x) \):

\[
B_1(x) = -B_0 e^{-\frac{(x-\mu_b)^2}{2\sigma_b^2}} \Rightarrow B(x,\varepsilon) = B_0 \left( 1 - \varepsilon e^{-\frac{(x-\mu_b)^2}{2\sigma_b^2}} \right)
\]  

(26)

where \( \mu_b \) characterizes the position of the damage and \( \sigma_b \) the relevant spatial amplitude. Here the damage is assumed centered at the midspan (\( \mu_b = L/2 \)) and scattered around it (\( \sigma_b = L/30 \)), see again Fig. 4. For the gaussian random parameter \( \varepsilon \), we set the mean value \( \mu_\varepsilon = 0.20 \) and the standard deviation \( \sigma_\varepsilon = 0.04 \).

Imposing the boundary conditions, Eq. (10) leads to \( \lambda_{0i} = i\pi/L \), for the eigenvalues, and \( \phi_{0i} = \sin(\lambda_{0i}x) \), for the (unitary scaled) mode shapes; then, the perturbation approach, Eqs. (11–12), furnishes the first and the second order eigensolutions. The first four couples \( (\lambda_{0i}; \phi_{0i}(x)) \) are shown in Fig. 5 whereas the first order \( (\lambda_{1i}; \phi_{1i}(x)) \) and second order
(\(\lambda_{2i}; \phi_{2i}(x)\)) terms are in Fig. 6 and Fig. 7 respectively. Accordingly to the damage position (centered at the midspan), the damage does not break the symmetry, but the symmetric mode shapes (the first and the third ones) are more sensitive to the damage than the antisymmetric modes (the second and the fourth ones).

As we expected (see the paragraph 2.2), \(\lambda_{1i}\) and \(\lambda_{2i}\) are both negative; the trends of the first four eigenvalues \(\lambda_i\) (normalized respect the unperturbed ones, \(\lambda_{0i}\)) are in Fig. 8, which shows the improvement due to the second order terms and the low sensitivity of the antisymmetric modes.

The coefficient of variation \(CV_\varepsilon = \sigma_\varepsilon / \mu_\varepsilon\) is 0.04/0.20 = 0.20, hence the probability \(P_d\) of damage (Eq. (14)) is close to 100% (see Fig. 2). The probability \(P_{dd}\) to detect the damage, provided by Eq. (21), is here evaluated considering four measured eigenvalues \((n_\lambda = 4)\) and assuming the network to be composed of equally spaced sensors. With this assumptions, the ruling parameter for the optimal choice of the network is only the number of sensors, \(n_s\).

Figure 9 shows the values obtained for the penalty indexes \(\gamma_{1i}\) and \(\gamma_{2i}\) when only one sensor, located at the midspan, is adopted: due to the symmetry of the problem, a single sensor at the centerline fails to detect the anti-symmetric vibration modes (the second and the fourth ones) and, therefore, the relative penalty indexes are practically equal to 0. Moreover, a direct comparison among the normalized first (top left) and second (bottom left) order modes shows that, although very similar, the eigenvectors \(\phi_{1i}(x)\) are not coincident with the \(\phi_{2i}(x)\).

Figure 10 shows the values of the average difference \(\Delta P\) (Eq. (24)) when the number of sensors, \(n_s\), is increased from 1 to 10. The percentage decreases rapidly and, if the threshold value \(\tilde{\Delta}P\) is set at 10% (see Eq. (25)), we obtain as optimal sensors placement the one considering eight sensors (framed in red in Fig. 10).
Figure 6: First order eigensolution.

Figure 7: Second order eigensolution.
Figure 8: Paths of the normalized eigenvalues vs the damage parameter; first order (dashed lines) and second order (continuous lines) solutions.

Figure 9: Penalty indexes for one single sensor, located at the midspan.
4 CONCLUSIONS

The paper focuses on the identifiability of damages in transversely vibrating uncertain 1-D continua. A perturbation technique has been used to obtain the asymptotic eigensolution up to the second order, where the smallness parameter acts as Gaussian distributed damage parameter. The network capabilities in the damage detection was then evaluated comparing the damage probability with the probability of detecting the damage with a given sensors network, where the latter exploits the closed-form (perturbative) eigensolution together with a penalty rule. The proposed method has been implemented to compare several sensors scenarios of a simply supported beam undergoing a non-localized damage centered at the midspan.

Apart from these first results, some more refinements are under investigations to improve the effectiveness of the proposed method, namely, a generalization of the procedure by considering the most plausible damages (here the optimization refers to a given, expected, damaged scenario) and by introducing a penalty index accounting for the sensitivity of the sensors (i.e., the lower frequency shift detectable by the sensors). Furthermore, the definition used to measure the accuracy of the sensor networks, based on an average difference (Eq. 24), could be ‘weaken’ or ‘strengthen’ (looking at a lower or greater difference); the relevant effects on the optimal sensors placements are being analyzed.

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