Slavnov-Taylor identities for noncommutative $\text{QED}_4$

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Abstract

In this work we present an analysis of the one-loop Slavnov-Taylor identities in noncommutative $\text{QED}_4$. The vectorial fermion-photon and the triple photon vertex functions were studied, with the conclusion that no anomalies arise.
I. INTRODUCTION

Quantum field theories defined in a noncommutative space have been under intense scrutiny in the last years \[1, 2\]. The outcome of these investigations have unveiled various unusual and intriguing aspects which are consequences of their inherent nonlocality. Among these properties, the most peculiar one is the transmutation of part of the ultraviolet divergences into infrared ones, a property that has been called infrared/ultraviolet mixing \[3\]. From a technical viewpoint, the mixing is due to the separation of the contributions of Feynman diagrams in parts nonplanar, which are ultraviolet finite but may present an infrared singularity, and planar, which may have only ultraviolet divergences. Aside the potentially dangerous character of the infrared divergences, the mere separation of the amplitudes in planar and nonplanar parts may obstruct the ultraviolet renormalization of noncommutative theories.

In the commutative setting, it is well known that Slavnov-Taylor (ST) identities \[4\] play a fundamental role in the renormalization of non-Abelian gauge theories \[4, 5\]. It is therefore essential to verify to what extension these identities are affected by the noncommutativity of the underlying space. In this work we will present a detailed analysis of the one-loop ST identities in noncommutative QED\(_4\). As we will explicitly verify, there are no anomalies and the usual renormalization procedure is not basically modified.

We would like to point out some relevant studies on the subject. For the pure noncommutative \(U(N)\) Yang-Mills model, the compatibility of dimensional renormalization with the ST identities have been verified in \[6\] up to one-loop order. Reference \[7\] contains an explicit on-shell verification of the one-loop ST identity for the trilinear fermion-photon vertex. In the tree approximation, the identities have been verified in various scattering processes in \[8\]. They were also used in \[9\] to investigate the dependence of the two point function of the gauge field on the gauge parameter. To prove the absence of radiative corrections to the Chern-Simons coefficient, the axial gauge identities were used and explicitly verified in a one-loop calculation \[10\].

This work is organized as follows. In section II we introduce our basic notation and the Feynman rules for noncommutative QED\(_4\). Section III provides a formal derivation for the ST identities. In particular, using these relations the longitudinal part of photon propagator is fixed and the identities for the vectorial fermion-photon and triple photon vertex functions are presented. In Section IV these identities are subjected to a detailed analysis taking in consideration the counterterms needed to control the ultraviolet behavior. Section V contains some final comments and a discussion of our results.
II. NONCOMMUTATIVE QED\textsubscript{4}

Classically, the noncommutative QED\textsubscript{4} is described by the action

\[ S_{INV} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} \ast F^{\mu\nu} + \bar{\psi} \ast (i\not{D} - m)\psi \right], \tag{1} \]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]_* \), with \([A_\mu, A_\nu]_* = A_\mu \ast A_\nu - A_\nu \ast A_\mu\), is the field strength, \( D_\mu \psi = \partial_\mu - ieA_\mu \ast \psi \) is a gauge covariant derivative and the star (Moyal) product is defined by

\[ \phi_1(x) \ast \phi_2(x) \equiv e^{\frac{i}{2} \theta_{\mu\nu}(\partial_\mu \phi_1(x) \partial_\nu \phi_2(y))} \Big|_{y=x}, \tag{2} \]

where \( \theta_{\mu\nu} \) is a real antisymmetric matrix and \( \xi \) is a parameter which sets the strength of the noncommutativity.

The above action is invariant under the gauge transformations

\[ \delta A_\mu = \frac{1}{e} D_\mu \Lambda \equiv \frac{1}{e}(\partial_\mu \Lambda - i e[A_\mu, \Lambda]_*), \]
\[ \delta \psi = i \Lambda \ast \psi \quad (\delta \bar{\psi} = -i \bar{\psi} \ast \Lambda), \tag{3} \]

To complete the quantum version of the model, we need to add to (1) a gauge fixing, \( S_{GF} \), and the corresponding Faddeev-Popov, \( S_{FP} \), actions. For the general class of Lorentz gauges in which we will work

\[ S_{GF} + S_{FP} = \int d^4x \left[ -\frac{1}{2\alpha}(\partial_\mu A_\mu)_*^2 + \partial_\mu \bar{C} \ast (\partial_\mu C - i e[A_\mu, C]_*) \right], \tag{4} \]

where \( \alpha \) is the gauge fixing parameter. As it happens in the commutative gauge theories, the total action \( S = S_{INV} + S_{GF} + S_{FP} \) is not invariant under gauge transformations anymore but instead has a BRST symmetry such that

\[ \delta A_\mu = -\frac{1}{e}(\partial_\mu C - i e[A_\mu, C]_*)\lambda, \]
\[ \delta \psi = -iC\lambda \ast \psi \quad (\delta \bar{\psi} = i\bar{\psi} \ast C\lambda), \]
\[ \delta C = iC \ast C\lambda, \]
\[ \delta \bar{C} = -\frac{1}{\alpha e}(\partial_\mu A_\mu)^2, \tag{5} \]

where \( \lambda \) is a constant Grassmannian parameter. At a formal level, the invariance of the action under these transformations imply in relations between the Green functions as it will be shortly verified. For an explicit calculation, we will need the Feynman rules for the model which are fixed
as follows. First, the free propagators are the same as in the commutative version of the model, i.e.,

\[ p = \frac{i}{p - m}, \]  
(6)

\[ \mu \ldots \nu = -\frac{i}{p^2} \left[ g^{\mu\nu} - (1 - \alpha) \frac{p_\mu p_\nu}{p^2} \right], \]  
(7)

\[ p = \frac{i}{p^2}, \]  
(8)

for the fermion, photon and ghost field propagators, respectively. Introducing the notation \( p \wedge k \equiv \frac{1}{2} \xi \theta^{\mu\nu} p_\mu k_\nu \), we determine the vertices as being

\[ = -ie\gamma^\mu e^{ip \wedge k}, \]  
(9)

\[ = 2e \sin(p \wedge q) \gamma^{\mu\alpha}(p, q, k), \]  
(10)

\[ = -4ie^2 \left[ (g^{\mu\beta} g^{\alpha\nu} - g^{\mu\nu} g^{\alpha\beta}) \sin(p \wedge r) \sin(q \wedge k) \right. \]
\[ + \left. (g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta}) \sin(q \wedge p) \sin(r \wedge k) \right. \]
\[ + \left. (g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\alpha\nu}) \sin(p \wedge k) \sin(r \wedge q) \right], \]  
(11)

\[ = 2ek^\mu \sin(p \wedge k), \]  
(12)

where \( \gamma^{\mu\alpha}(p, q, k) = (p - q)^\alpha g^{\mu\nu} + (q - k)^\mu g^{\nu\alpha} + (k - p)^\nu g^{\alpha\mu} \).
III. SLAVNOV-TAYLOR IDENTITIES FOR THE GENERATING FUNCTIONALS: FORMAL ASPECTS

Following the standard procedure adopted in commutative gauge theories, we start by considering the generating functional for the Green functions of the basic fields and their BRST variations,

\[ Z[J, \eta, \bar{\eta}, \zeta, \bar{\zeta}; K, v, \omega, \bar{\omega}] = \int D\mu D\psi D\bar{\psi} DC D\bar{C} e^{i(S + S_{\text{source}})}, \quad (13) \]

where \( S \) was given in the previous section and

\[ S_{\text{source}} = \int d^4x \left( J_\mu \ast A_\mu + \bar{\eta} \ast \psi + \psi \ast \eta + \bar{\zeta} \ast C + C \ast \zeta + K_\mu \ast \frac{1}{\epsilon} (\partial^\mu C - i\epsilon [A_\mu, C_*]) + iv \ast C \ast C + i\bar{\omega} \ast C \ast \bar{\psi} + i\bar{\psi} \ast C \ast \omega \right). \quad (14) \]

The invariance of the functional integral (13) under the field-coordinate transformation (5) and the nilpotency of that variations imply the ST identity

\[ \int d^4x \left( J_\mu \frac{\delta W}{\delta K_\mu} - \bar{\zeta} \frac{\delta W}{\delta v} - \frac{1}{\alpha \epsilon} \partial_\mu \frac{\delta W}{\delta J_\mu} \ast \zeta - \bar{\eta} \frac{\delta W}{\delta \bar{\omega}} + \frac{\delta W}{\delta \omega} \ast \eta \right) = 0, \quad (15) \]

where \( W = -i \ln Z \) is the generating functional for the connected Green functions. Furthermore, by subjecting the functional integral to an arbitrary variable change \( \delta \bar{C} \), we may derive that

\[ \zeta = e\partial_\mu \frac{\delta W}{\delta K_\mu}. \quad (16) \]

As usual, the generating functional \( \Gamma \) of proper (one-particle-irreducible) vertex functions is obtained by a Legendre transformation

\[ W[J, \eta, \bar{\eta}, \zeta, \bar{\zeta}; K, v, \omega, \bar{\omega}] = \Gamma[A_{cl}, \psi_{cl}, \bar{\psi}_{cl}, C_{cl}, \bar{C}_{cl}; K, v, \omega, \bar{\omega}] + \int d^4x \left( J_\mu \ast A_{cl}^\mu + \bar{\eta} \ast \psi_{cl} + \bar{\psi}_{cl} \ast \eta + \bar{\zeta} \ast C_{cl} + \bar{C}_{cl} \ast \zeta \right), \quad (17) \]

where we have introduced the classical fields

\[ A_{cl}^\mu = \frac{\delta W}{\delta J_\mu}, \quad \psi_{cl} = \frac{\delta W}{\delta \eta}, \quad \bar{\psi}_{cl} = -\frac{\delta W}{\delta \bar{\eta}}, \quad C_{cl} = \frac{\delta W}{\delta \zeta}, \quad \bar{C}_{cl} = -\frac{\delta W}{\delta \bar{\zeta}}. \quad (18) \]

From these definitions, it follows that

\[ \frac{\delta \Gamma}{\delta A_{cl}^\mu} = -J_\mu, \quad \frac{\delta \Gamma}{\delta \psi_{cl}} = \bar{\eta}, \quad \frac{\delta \Gamma}{\delta \bar{\psi}_{cl}} = -\eta, \quad \frac{\delta \Gamma}{\delta C_{cl}} = \bar{\zeta}, \quad \frac{\delta \Gamma}{\delta \bar{C}_{cl}} = -\zeta. \quad (19) \]

In terms of \( \Gamma \) the identities (15) and (16) become

\[ \int d^4x \left( \frac{\delta \Gamma}{\delta A_{cl}^\mu} \ast \frac{\delta \Gamma}{\delta K_\mu} + \frac{\delta \Gamma}{\delta C_{cl}} \ast \frac{\delta \Gamma}{\delta v} - \frac{1}{\alpha \epsilon} (\partial_\mu A_{cl}^\mu) \ast \frac{\delta \Gamma}{\delta C_{cl}} + \frac{\delta \Gamma}{\delta \psi_{cl}} \ast \frac{\delta \Gamma}{\delta \bar{\omega}} + \frac{\delta \Gamma}{\delta \omega} \ast \frac{\delta \Gamma}{\delta \psi_{cl}} \right) = 0 \quad (20) \]

5
and

\[ i \frac{\delta \Gamma}{\delta C_{cl}} = -e \partial_{\mu} \frac{\delta \Gamma}{\delta K_{\mu}} . \]  

(21)

The identity (20) can be simplified by redefining \( \Gamma \):

\[ \Gamma \to \Gamma - \frac{1}{2\alpha} \int d^4x (\partial_{\mu} A_{cl}^{\mu})^2 \]  

(22)

so that we obtain

\[ \int d^4x \left[ \frac{\delta \Gamma}{\delta A_{cl}^{\mu}} \frac{\delta \Gamma}{\delta K_{\mu}} + \frac{\delta \Gamma}{\delta \bar{\psi}} \frac{\delta \Gamma}{\delta \psi} + \frac{\delta \Gamma}{\delta \bar{\omega}} \frac{\delta \Gamma}{\delta \psi} \right] = 0 \]  

(23)

and

\[ i \frac{\delta \Gamma}{\delta C_{cl}} + e \partial_{\mu} \frac{\delta \Gamma}{\delta K_{\mu}} = 0. \]  

(24)

Let us now consider some specific applications of the above identities.

A. The photon propagator

As a first application of the identities derived in the previous section, we will now prove that the longitudinal part of the photon propagator is not modified by radiative corrections. To this end, we twice differentiate the generating functional of the connected Green functions (15) with respect to \( \zeta(y) \) and \( J^\nu(z) \) and set all sources equal to zero, which gives

\[ - \frac{1}{\alpha e} \partial_{\nu} \frac{\delta^2 W}{\delta J^\nu(z) J^\mu(y)} + \frac{\delta^2 W}{\delta \zeta(y) \delta K^\nu(z)} = 0, \]  

(25)

where we have introduced the notation \( O| \) to imply that the object \( O \) at the left of the vertical bar has to be calculated with all sources equal to zero. But, from Eq. (16) it follows that

\[ \partial_{\nu} \frac{\delta^2 W}{\delta \zeta(y) \delta K^\nu(z)} = \frac{1}{e} \delta(z - y) \]  

(26)

so that the photon propagator \( D_{\mu\nu}(z - y) = -i \frac{\delta^2 W}{\delta J^\nu(z) \delta J^\mu(y)} \) must satisfy

\[ \partial_{\mu} \partial_{\nu} D_{\mu\nu}(z - y) = -i \alpha \delta(z - y), \]  

(27)

which in momentum space becomes

\[ q^\mu q^\nu D_{\mu\nu}(q) = -i \alpha. \]  

(28)

Now, compatibility with this constraint requires the propagator to have the general form

\[ D_{\mu\nu}(q) = \left( g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) D_T(q^2) + \frac{q^\mu q^\nu}{q^2} D_\theta(q^2) - \frac{i \alpha}{q^2} \frac{q^\mu q^\nu}{q^2}. \]  

(29)
Notice that, because of the charge conjugation properties \(9, 11\), terms of the type
\[
\tilde{q}^\mu q^\nu + \tilde{q}^\nu q^\mu \quad \frac{\partial^2}{q^2}
\]
are not allowed in the decomposition (29). Thus the longitudinal part of the propagator is the same as in the free approximation. Notice also that
\[
q^\mu D_{\mu\nu}(q) = -i\frac{\alpha q^\nu}{q^2},
\]
which will be useful in the next section when we will analyze the ST identity for the vectorial vertex function.

### B. The vectorial vertex function

The ST identity for the vectorial fermion-photon vertex, the proper part of \(\langle 0|T(\psi\bar{\psi}A_\mu)|0\rangle\), can be derived by turning off all the sources after differentiating the functional equation (15) with respect to the sources \(\eta(y), \bar{\eta}(x), \text{and} \zeta(z)\). The result is
\[
\frac{1}{\alpha e} \frac{\partial^\mu}{\partial \bar{\eta}(x)\bar{\eta}(y)\delta J^\mu(z)} = \frac{\delta^3 W}{\delta \zeta(z)\delta \eta(x)\delta \omega(y)} - \frac{\delta^3 W}{\delta \zeta(z)\delta \eta(y)\delta \omega(x)},
\]
or, equivalently,
\[
\frac{1}{\alpha e} \frac{\partial^\mu}{\partial \bar{\eta}(x)\bar{\eta}(y)\delta J^\mu(z)} = i \langle 0|T(C(z)\psi(x)\bar{C}(y)\bar{\psi}(y)\delta \omega(x))|0\rangle
\]
\[
- ie \langle 0|T(C(z)\psi(x)\bar{C}(y)\bar{\psi}(y)\delta \omega(y))|0\rangle
\]
i.e.,
\[
\frac{1}{\alpha e} \frac{\partial^\mu}{\partial \bar{\eta}(x)\bar{\eta}(y)\delta J^\mu(z)} = ie^{i\partial_x(\bar{C}(z)\bar{\psi}(y)C(y))}|_{x=x}
\]
\[
- ie^{i\partial_y(\bar{C}(z)\bar{\psi}(y)C(y))}|_{y=y}
\]
where \(\partial_x \wedge \partial_x = \frac{1}{2}\tau^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\). Notice that as consequence of this identity
\[
\delta \langle 0|T(\psi(x)\bar{\psi}(y)\bar{C}(z))|0\rangle = 0.
\]
We may translate the above equations into identities for the proper, one-particle irreducible, vertex functions. These functions are given by
\[
\langle 0|T(\psi(x)\bar{\psi}(y)A_\mu(z))|0\rangle
\]
\[
= - \int d^4x' d^4y' d^4z' S_F(x - x')\Gamma^\nu(x', y', z') S_F(y' - y)D_{\mu\nu}(z - z'),
\]
\[ i e^{i \partial_x \wedge \partial_y} \langle 0 | T(\psi(\hat{x}) \bar{\psi}(y) C(x) \bar{C}(z)) | 0 \rangle |_{\hat{x} = x} \]
\[ = \int d^4 y' d^4 z' H_1(x, y', z') S_F(y' - y) \Delta(z' - z), \]
\[ i e^{i \partial_x \wedge \partial_y} \langle 0 | T(\psi(x) \bar{\psi}(y) C(\bar{y}) \bar{C}(z)) | 0 \rangle |_{\bar{y} = y} \]
\[ = \int d^4 x' d^4 z' S_F(x - x') H_2(x', y, z') \Delta(z' - z), \]

where \( S_F(x - x') \) and \( \Delta(z' - z) \) are the fermion and ghost fields propagators, respectively, \( \Gamma^\nu(x', y', z') \) is the vectorial proper vertex,

\[ \Gamma^\nu(x', y', z') = \frac{\delta^3 \Gamma}{\delta \psi_{cl}(x') \delta \bar{\psi}_{cl}(y') \delta A_{cl\nu}(z')}, \]

\[ H_1(x, y', z') = i \int d^4 u d^4 v e^{i \partial_x \wedge \partial_y} S_F(x - u) \Delta(\hat{x} - v) \Gamma(u, y', v, z') |_{\hat{x} = x} \]

and

\[ H_2(x', y, z') = i \int d^4 u d^4 v \Gamma(x', u, v, z') e^{i \partial_x \wedge \partial_y} S_F(u - y) \Delta(\hat{y} - v) |_{\bar{y} = y}, \]

in which

\[ \Gamma(u, y', v, z') = \frac{\delta^4 \Gamma}{\delta \psi_{cl}(u) \delta \bar{\psi}_{cl}(y') \delta C_{cl}(v) \delta \bar{C}_{cl}(z')} \]

is the fermion-ghost four-vertex.

In momentum space, Eq. (36) reads

\[ \Gamma^\nu(k, p, q) q^\mu D_{\mu\nu}(-q) = -i e \left[ S_F^{-1}(k) H_1(k, p, q) \Delta(q) - H_2(k, p, q) S_F^{-1}(p) \Delta(q) \right], \]

where

\[ H_1(k, p, q) = i \int \frac{d^4 k'}{(2\pi)^4} e^{i k' \wedge k} S_F(k') \Delta(k - k') \Gamma(k', p, k - k', q) \]

and

\[ H_2(k, p, q) = i \int \frac{d^4 p'}{(2\pi)^4} e^{-i p' \wedge p} \Gamma(k, p', p' - p, q) S_F(p') \Delta(p' - p), \]

with \( k = p + q \).

Similarly, we may determine \( \Delta(q) \) from the Dyson-Schwinger equation,

\[ \Delta(q) = \Delta_{(0)}(q) - \Delta_{(0)}(q) \Sigma_C(q) \Delta(q), \]

where \( \Delta_{(0)}(q) = \frac{i}{q^2} \) and \( \Sigma_C(q) \) denotes the proper self-energy operator of the ghost field. Therefore, it is easy to verify that

\[ \Delta(q) = \frac{i}{q^2[1 + b(q^2)]}, \]
in which the self-energy has been expressed as \( i\Sigma_C(q) = q^2b(q^2) \).

Thus, with the expressions (31) and (46), we can rewrite the identity (42) as follows

\[
q_\mu \Gamma^\mu(k, p, q)[1 + b(q^2)] = ie[S_F^{-1}(k)H_1(k, p, q) - H_2(k, p, q)S_F^{-1}(p)]. 
\]  

(47)

By considering that energy-momentum conservation holds at the vertices \( \Gamma^\mu(k, p, q) \) and \( H(k, p, q) \), we can write

\[
\Gamma^\mu(k, p, q) = i\epsilon(2\pi)^4\delta^4(k - p - q)\tilde{\Gamma}^\mu(p, p + q)
\]

(48)

and

\[
H(k, p, q) = (2\pi)^4\delta^4(k - p - q)\tilde{H}(p, p + q).
\]

(49)

With this representation we may obtain from Eq. (47) that

\[
q_\mu \tilde{\Gamma}^\mu(p, p + q)[1 + b(q^2)] = S_F^{-1}(p + q)\tilde{H}_1(p, p + q) - \tilde{H}_2(p, p + q)S_F^{-1}(p).
\]

(50)

\[\text{C. The triple photon vertex}\]

To obtain ST identity for the triple photon vertex, the proper part of \( \langle 0|T(A_\mu A_\nu A_\lambda)|0\rangle \), we differentiate the functional equation (15) with respect to \( \zeta(x) \), \( J^\nu(y) \) and \( J^\lambda(z) \) and turn off all the sources. The result is

\[
\frac{1}{\alpha e} \delta_x^\mu \delta_j^\nu \delta_j^\lambda \left| \frac{\delta^3W}{\delta J^\mu(x)\delta J^\nu(y)\delta J^\lambda(z)} \right| = \frac{\delta^3W}{\delta \zeta(x)\delta K^\nu(y)\delta J^\lambda(z)} + \frac{\delta^3W}{\delta \zeta(x)\delta J^\nu(y)\delta K^\lambda(z)},
\]

(51)

or in terms of the Green functions,

\[
-\frac{1}{\alpha} \delta_x^\mu \langle 0|T(\tilde{A}_\mu(x)A_\nu(y)A_\lambda(z))|0\rangle = \langle 0|T(\tilde{C}(x)D_\nu^{AD}(y)C(y)A_\lambda(z))|0\rangle
\]

\[
+ \langle 0|T(\tilde{C}(x)A_\nu(y)D_\lambda^{AD}(z)C(z))|0\rangle,
\]

(52)

where \( D_\nu^{AD}(y) \) denotes the covariant derivative in the adjoint representation, \( D_\nu^{AD}(y)C(y) = \partial_\nu^{AD}(y)C(y) - i\epsilon[A_\nu(y), C(y)] \). Thus, we can rewrite the above expression as

\[
-\frac{1}{\alpha} \delta_x^\mu \langle 0|T(\tilde{A}_\mu(x)A_\nu(y)A_\lambda(z))|0\rangle = \partial_{j\nu}\langle 0|T(\tilde{C}(x)C(y)A_\lambda(z))|0\rangle
\]

\[
+ 2e \sin(\partial_y \cdot \partial_{\hat{y}})\langle 0|T(\tilde{C}(x)A_\nu(y)C(\hat{y})A_\lambda(z))|0\rangle + \partial_{j\lambda}\langle 0|T(\tilde{C}(x)A_\nu(y)C(z))|0\rangle
\]

\[
+ 2e \sin(\partial_z \cdot \partial_{\hat{z}})\langle 0|T(\tilde{C}(x)A_\nu(y)A_\lambda(z)C(\hat{z}))|0\rangle,
\]

(53)

where, after the application of the differential operators, we must identify \( \hat{y} \) and \( \hat{z} \) respectively with \( y \) and \( z \).
These Green functions have the following one-particle irreducible decomposition:

\begin{align}
(0|T(A_\mu(x)A_\nu(y)A_\lambda(z))|0) &= \int d^4x'd^4y'd^4z' D_{\mu\nu'}(x-x')D_{\nu\nu'}(y-y')D_{\lambda\lambda'}(z-z')\Gamma^{\nu'\lambda'}(x', y', z'), \\
\partial_{y'}(0|T(\bar{C}(x)C(y)A_\lambda(z))|0) &= i \int d^4x'd^4y'd^4z'\partial(y-y')D_{\lambda\lambda'}(z-z')G^\rho(x', y', z'), \\
2e\sin(\partial_y \partial_{\bar{y}})(0|T(C(x)A_\nu(y)C(\bar{y})A_\lambda(z))|0) &= -i \int d^4x'd^4y'd^4z'\Delta(x-x')G_{\nu}^{\lambda'}(x', y, z')D_{\lambda\lambda'}(z'-z), \\
\partial_{z\lambda}(0|T(\bar{C}(x)A_\nu(y)C(z))|0) &= i \int d^4x'd^4y'd^4z'\Delta(x-x')D_{\nu\rho}(y-y')\partial_{z\lambda}D_{\lambda\lambda'}(z-z')G^\rho(x', z', y'), \\
2e\sin(\partial_z \partial_{\bar{z}})(0|T(C(x)A_\nu(y)A_\lambda(z)C(\bar{z})|0) &= -i \int d^4x'd^4y'd^4z'\Delta(x-x')G_{\lambda}^{\nu'}(x', z, y')D_{\nu\nu'}(y'-y),
\end{align}

where \( \Gamma^{\nu'\lambda'}(x', y', z') \) and \( G^\rho(x', z', y') \) are the triple gauge and the ghost-gauge vertices respectively,

\begin{align}
\Gamma^{\nu'\lambda'}(x', y', z') &= \frac{\delta^3\Gamma}{\delta A_{cl\mu'}(x')\delta A_{cl\nu'}(y')\delta A_{cl\lambda'}(z')}, \\
G^\rho(x', z', y') &= \frac{\delta^3\Gamma}{\delta C_{cl}(x')\delta C_{cl}(y')\delta A_{cl\rho}(z')}.
\end{align}

Also

\begin{equation}
G_{\nu}^{\lambda'}(x', y, z') = -2e\int d^4u\, d^4v\, \sin(\partial_y \partial_{\bar{y}})\Delta(y-u)D_{\nu'}(y - v)\Gamma^{\nu\lambda'}(x', u, v, z'),
\end{equation}

in which

\begin{equation}
\Gamma^{\nu\lambda'}(x', u, v, z') = \frac{\delta^4\Gamma}{\delta C_{cl}(x')\delta C_{cl}(u)\delta A_{cl\nu'}(v)\delta A_{cl\lambda'}(z')}.
\end{equation}

In momentum space the Eq. (53) reads

\begin{equation}
p^\mu\Gamma_{\mu\nu\lambda}(p, q, k)[1 + b(p^2)] = G_{\lambda\nu'}(p, q, k)\{(q^2g_{\nu'\nu} - q'^2q_{\nu'})[1 + \Pi_T(q^2)] + \Pi_\theta(q^2)\tilde{q}'\tilde{q}_{\nu'}\} + G_{\nu\lambda'}(p, k, q)\{(k^2g_{\lambda'\lambda} - k'^2k_{\lambda'})[1 + \Pi_T(k^2)] + \Pi_\theta(k^2)\tilde{k}'\tilde{k}_{\lambda'}\}
\end{equation}

with

\begin{equation}
G_{\lambda\nu'}(p, q, k) = -2e\int \frac{d^3q'}{(2\pi)^3}\sin(q' \wedge q)i\Delta(q')iD_{\nu\alpha}(q - q')\Gamma_{\alpha\lambda}(p, q', q - q', k),
\end{equation}

10
where we have used the ghost propagator and the inverse of the photon propagator
\[
\left. iD_{\mu\nu}^{-1}(q) = -i \left\{ (q^2g_{\mu\nu} - q_\mu q_\nu) [1 + \Pi_T(q^2)] - \frac{q_\mu q_\nu}{\alpha} - \Pi_{\theta}(q^2)q_\mu q_\nu \right\}, \quad (65)
\]
which satisfy
\[
q^\mu D_{\mu\nu}^{-1}(q) = -\frac{1}{\alpha} q^2 q_\nu. \quad (66)
\]

Using the energy-momentum conservation at the vertices \(\Gamma_{\mu\nu\lambda}(p, q, k)\) and \(G_{\nu'\lambda}(p, q, k)\), so that
\[
\Gamma_{\mu\nu\lambda}(p, q, k) = \frac{2\pi}{4} \delta^4(p + q + k) \tilde{\Gamma}_{\mu\nu\lambda}(p, q, -p - q) \quad (67)
\]
and
\[
G_{\nu'\lambda}(p, q, k) = \frac{2\pi}{4} \delta^4(p + q + k) \tilde{G}_{\nu'\lambda}(p, q, -p - q), \quad (68)
\]
we get
\[
p^\mu \tilde{\Gamma}_{\mu\nu\lambda}(p, q, k)[1 + b(p^2)] = \tilde{G}_{\nu'\lambda}(p, q, k)\{(q^2g_{\nu'\lambda} - q_{\nu'} q_\lambda)[1 + \Pi_T(q^2)] + \Pi_{\theta}(q^2)q_{\nu'} q_\lambda\} \quad (69)
\]
\[
+ \tilde{G}_{\nu'\lambda}(p, q, k)\{[k^2g_{\nu'\lambda} - k_{\nu'} k_\lambda][1 + \Pi_T(k^2)] + \Pi_{\theta}(k^2)k_{\nu'} k_\lambda\}.
\]

IV. SLAVNOV-TAYLOR IDENTITIES AT ONE-LOOP: EXPLICIT CALCULATIONS

A. The vectorial vertex function

The ST identities derived previously are valid only in a formal way since the radiative corrections contain ultraviolet divergences. To eliminate these divergences counterterms must be introduced so that the action for noncommutative QED\(_4\) becomes
\[
S = \int d^4x \left[ -\frac{Z_3}{4} G_{\mu\nu} G^{\mu\nu} - i\frac{e Z_1}{2} [A_\mu, A_\nu] + \frac{e^2 Z_4}{4} [A_\mu, A_\nu] [A^\mu, A^\nu] + Z_2 \bar{\psi} i \slashed{D} \psi - (m + \delta m) \bar{\psi} \psi + e Z_1 F_{\mu\nu} \star A_\psi - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + \tilde{Z}_3 \partial_\mu \tilde{C} \partial^\mu C - i\epsilon \tilde{Z}_1 \partial_\mu \tilde{C} \star [A^\mu, C] \right], \quad (70)
\]
where \(G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\).

We begin by considering the one-loop contributions to the vectorial vertex function. Writing the \(\tilde{\Gamma}_{\mu}(p, p + q)\), \(S^{-1}(p + q)\) and \(\tilde{H}(p, p + q)\) expansions as
\[
\tilde{\Gamma}_{\mu}(p, p + q) = Z_1 F_{\mu\nu} \epsilon^{\nu\lambda\rho\sigma} + \Lambda_{\mu}(p, p + q) \epsilon^{\nu\lambda\rho\sigma}, \quad (71)
\]
\[
\tilde{H}_i(p, p + q) = [Z_i + B_i(p, p + q) \epsilon^{\nu\lambda\rho\sigma}] \quad \text{for } i = 1, 2 \quad (72)
\]
and

\[ S_F^{-1}(p) = [Z_2 \not{p} - m - \delta m - \Sigma(p)], \quad (73) \]

where \( \Lambda^\mu(p, p + q), \Sigma(p) \) and \( B(p, p + q) \) are the one-loop contributions. Thus, the ST identity for the vectorial vertex \([50]\), in the tree approximation, becomes

\[ Z_1 F \tilde{Z}_3 g e^{ip \wedge q} = [Z_2 Z_5(p + q) - (m + \delta m)Z_5]e^{ip \wedge q} - [Z_2 Z_6(p) - (m + \delta m)Z_6]e^{ip \wedge q}, \quad (74) \]

so that the validity of the ST identity requires that

\[ Z_5 = Z_6 \quad \text{and} \quad \tilde{Z}_3/Z_5 = Z_2/Z_1 F. \quad (75) \]

For the one-loop approximation, we have

\[ q_\mu \Lambda^\mu_a(p, p + q) + q_\mu \Lambda^\mu_b(p, p + q) + g b(q^2) = \Sigma(p) - \Sigma(p + q) + (p + q - m)B_1(p, p + q) \]
\[ - B_2(p, p + q)(p - m). \quad (76) \]

The diagrams representing these contributions are given by: (from now on, we restrict ourselves to the Feynman gauge, \( \alpha = 1 \))

\[ -i e \Lambda^\mu_a(p, p + q)e^{ip \wedge q} \quad (77) \]

\[ = \int \frac{d^4 l}{(2\pi)^4} \frac{-ig_{\alpha\beta}}{l^2} (-ie\gamma^\alpha)iS_0(p + q - l)(-ie\gamma^\beta)iS_0(p - l)(-ie\gamma^\beta)e^{-2il\wedge q}e^{ip \wedge q}, \]

with

\[ iS_0(p) = \frac{i}{p - m}. \quad (78) \]

This contribution is entirely nonplanar and using \( g = (q + p - l - m) - (p - l - m) \) can be shown to satisfy

\[ q_\mu \Lambda^\mu_a(p, p + q) = \Sigma_{np}(p) - \Sigma_{np}(p + q), \quad (79) \]

where the nonplanar fermion self-energy is

\[ -i\Sigma_{np}(p) = \int \frac{d^4 l}{(2\pi)^4} \frac{-ig_{\alpha\beta}}{l^2} (-ie\gamma^\alpha)iS_0(p - l)(-ie\gamma^\beta)e^{-2il\wedge q}. \quad (80) \]
\[ 2. \quad \int \frac{d^4l}{(2\pi)^4} \frac{-ie\alpha\beta}{(p + q - l)^2} \frac{-ie\gamma\rho}{(p - l)^2} (-ie\gamma^\alpha)iS_0(l)(-ie\gamma^\lambda) \times (2e)_{\gamma^\mu\rho}(q, -p - q + l, p - l) \frac{1}{2i} \left( 1 - e^{2ilq}e^{-2ipq} \right) e^{ipq}, \]

whose planar part logarithmically diverges. In fact its pole part (PP) is given by

\[ \text{PP}[-ie\Lambda_\mu^\mu(p, p + q)] = \frac{3}{16\pi^2} \frac{1}{\epsilon} \gamma^\mu \]

so that, in the minimal dimensional regularization scheme,

\[ Z_{1F} = 1 - \frac{3}{16\pi^2} \frac{1}{\epsilon}, \]

which agrees with the result of previous calculation [12]. Contracting \( q_\mu \) in the expression (81), we get

\[ q_\mu \Lambda_\mu^\mu(p, p + q) = \int \frac{d^4l}{(2\pi)^4} \frac{-ie\alpha\beta}{(p + q - l)^2} \frac{-ie\gamma\rho}{(p - l)^2} (-ie\gamma^\alpha)iS_0(l)(-ie\gamma^\lambda) \left( 1 - e^{2ilq}e^{-2ipq} \right) \times \left\{ (p + q - l)^\rho q^\beta + (p - l)^\beta q^\rho - [(p + q - l) \cdot q + (p - l) \cdot q] q^\beta \rho \right\}, \]

which may be further simplified using

\[ (p + q - l)^\rho q^\beta + (p - l)^\beta q^\rho = (p + q - l)^\rho (p + q - l)^\beta - (p - l)^\rho (p - l)^\beta, \]

and

\[ (p + q - l) \cdot q + (p - l) \cdot q = (p + q - l)^2 - (p - l)^2, \]

to yield

\[ q_\mu \Lambda_\mu^\mu(p, p + q) = \Sigma(p) - \Sigma(p + q) - \Sigma_{np}(p) + \Sigma_{np}(p + q) \]  
\[ + \int \frac{d^4l}{(2\pi)^4} \frac{-i}{(p + q - l)^2} \frac{i}{(p - l)^2} (ie)(q - l)iS_0(l)(ie)(q - l) \left( 1 - e^{2ilq}e^{-2ipq} \right) \]
\[ - \int \frac{d^4l}{(2\pi)^4} \frac{-i}{(p + q - l)^2} \frac{i}{(p - l)^2} (ie)(q - l)iS_0(l)(ie)(q - l) \left( 1 - e^{2ilq}e^{-2ipq} \right), \]

where \( \Sigma \) is similar to \( \Sigma_{np} \) of Eq. (80), but without the phase factor.

For the last term in left-hand side of Eq. (76), we get

\[ \int \frac{d^4l}{(2\pi)^4} \frac{-i}{(p + q - l)^2} \frac{i}{(p - l)^2} (ie)(q - l)iS_0(l)(ie)(q - l) \left( 1 - e^{2ilq}e^{-2ipq} \right), \]
we find yielding 

Since

Then, Let us now consider the one-loop contributions to the right-hand side of Eq. (76). We have

Hence, for the left-hand side of Eq. (76), we find

Since

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Hence, for the left-hand side of Eq. (76), we find

Let us now consider the one-loop contributions to the right-hand side of Eq. (76). We have

yielding

where

Z_5 = 1 - \frac{e^2}{16\pi^2} \frac{1}{e}. (93)
Notice that in the Landau gauge $Z_5 = 1$ as it happens in ordinary commutative QCD \[5\]. Now,

$$(\phi + q - m) B_1(p, p + q) = \int \frac{d^4l}{(2\pi)^4} \frac{i}{(p + q - l)^2} \frac{i}{(p - l)^2} (ie)(\phi + q - m) iS_0(l) (ie)(\phi + q - l) \times \left( 1 - e^{2il\wedge q} e^{-2ip\wedge q} \right)$$

and making the substitution $(\phi + q - m) \rightarrow (\phi + q - l) + (l - m)$, we obtain

$$(\phi + q - m) B_1(p, p + q) = \int \frac{d^4l}{(2\pi)^4} \frac{i}{(p + q - l)^2} \frac{i}{(p - l)^2} (ie)(\phi + q - l) iS_0(l) (ie)(\phi + q - l) \times \left( 1 - e^{2il\wedge q} e^{-2ip\wedge q} \right) + i \int \frac{d^4l}{(2\pi)^4} \frac{i}{(p + q - l)^2} \frac{i}{(p - l)^2} (ie)(\phi + q - l) (ie) \times \left( 1 - e^{2il\wedge q} e^{-2ip\wedge q} \right).$$

Similarly,

$$B_2(p, p + q) \Rightarrow B_2(p, p + q) e^{ip\wedge q} = \int \frac{d^4l}{(2\pi)^4} \frac{i}{(p + q - l)^2} \frac{i}{(p - l)^2} \frac{i}{(p - l)^2} (ie)(\phi - I) iS_0(l) (ie)(\phi - I) \times \left( 1 - e^{2il\wedge q} e^{-2ip\wedge q} \right) e^{ip\wedge q}.$$}

Therefore,

$$B_2(p, p + q)(\phi - m) = \int \frac{d^4l}{(2\pi)^4} \frac{i}{(p + q - l)^2} \frac{i}{(p - l)^2} (ie)(\phi - l) iS_0(l) (ie)(\phi - l) \times \left( 1 - e^{2il\wedge q} e^{-2ip\wedge q} \right) + i \int \frac{d^4l}{(2\pi)^4} \frac{i}{(p + q - l)^2} \frac{i}{(p - l)^2} (ie)(\phi - l) (ie) \times \left( 1 - e^{2il\wedge q} e^{-2ip\wedge q} \right),$$

where we also have done the replacement $(\phi - m) \rightarrow (\phi - l) + (l - m)$. Finally, summing the above results,

$$(\phi + q - m) B_1(p, p + q) - B_2(p, p + q)(\phi - m) = + \int \frac{d^4l}{(2\pi)^4} \frac{i}{(p + q - l)^2} \frac{i}{(p - l)^2} (ie)(\phi + q - l) iS_0(l) (ie)(\phi + q - l) \times \left( 1 - e^{2il\wedge q} e^{-2ip\wedge q} \right)$$
Thus, the left-hand side of Eq. (76) is identical to the right-hand side as we can see from Eq. (91). Therefore, the ST identity for the vectorial vertex is satisfied at one-loop.

B. The triple photon vertex

Writing the expansions for $\tilde{\Gamma}^{\mu\nu\lambda}(p, q, k)$ and $\tilde{G}^{\lambda\nu}(p, q, k)$, defined in (59) and (61), as

$$\tilde{\Gamma}^{\mu\nu\lambda}(p, q, k) = 2e Z_1 \sin(p \wedge q) \gamma^{\mu\nu\lambda}(p, q, k) + 2e \sin(p \wedge q) \Lambda^{\mu\nu\lambda}(p, q, k),$$

$$\tilde{G}^{\lambda\nu}(p, q, k) = -2e \tilde{Z}_1 \sin(p \wedge q) g^{\lambda\nu} + 2e \sin(p \wedge q) B^{\lambda\nu}(p, q, k),$$

we obtain the ST identity (69) for the triple photon vertex, in the tree approximation,

$$Z_1 \tilde{Z}_3 \left[(k^2 g^{\lambda\nu} - k^{\lambda} k^{\nu}) - (q^2 g^{\nu\rho} - q^{\nu} q^{\rho})\right] = \tilde{Z}_1 Z_3 \left[(k^2 g^{\lambda\nu} - k^{\lambda} k^{\nu}) - (q^2 g^{\nu\rho} - q^{\nu} q^{\rho})\right],$$

which requires that

$$\tilde{Z}_3 / \tilde{Z}_1 = Z_3 / Z_1.$$

On the other hand, the one-loop approximation is given by

$$p^\mu \gamma^{\mu\nu\lambda}(p, q, k) b(p^2) + p^\mu \Lambda^{\mu\nu\lambda}(p, q, k) = \Pi^{\nu\lambda}(q) - \Pi^{\lambda\nu}(k)$$

$$+ B^{\lambda\nu}(p, q, k)(q^2 g^{\nu\rho} - q^{\nu} q^{\rho}) + B^{\nu\lambda}(p, k, q)(k^2 g^{\lambda\rho} - k^{\lambda} k^{\rho}),$$

where we have introduced the photon self-energy $\Pi^{\nu\nu}(q) \equiv \Pi_T(q^2)(q^2 g^{\nu\rho} - q^{\nu} q^{\rho}) + \Pi_\theta(q^2) q^{\nu} q^{\nu}$.

The contributions with a fermion loop in the left and right-hand sides of Eq. (103) are directly identified when we consider the diagrams:

$$= -\int \frac{d^4 l}{(2\pi)^4} \text{tr}(-ie\gamma^\nu) i S_0(q + l)(-ie\gamma^\mu) i S_0(p + q + l)$$
\[ x(-ie\gamma^\lambda)iS_0(l)e^{-ip\wedge q}, \]

These diagrams are different only in the circulation of the momentum integration. Since \( C\gamma^\mu C^{-1} = -\gamma^{T\mu} \) and \( CS_0(l)C^{-1} = S_0^T(-l) \), we can rewrite the above expression as

\[
2e\sin(p \wedge q)\Lambda_{a_2}^{\mu\nu\lambda}(p, q, k) = 2e\sin(p \wedge q)\Lambda_{a_2}^{\mu\nu\lambda}(p, q, k)
\]

Thus, by summing the two diagrams, we obtain

\[
2e\sin(p \wedge q)\Lambda_{a_2}^{\mu\nu\lambda}(p, q, k) = 2e\sin(p \wedge q)\left[\Lambda_{a_1}^{\mu\nu\lambda}(p, q, k) + \Lambda_{a_2}^{\mu\nu\lambda}(p, q, k)\right]
\]

i.e.,

\[
\Lambda_{a}^{\mu\nu\lambda}(p, q, k) = ie^2 \int \frac{d^4l}{(2\pi)^4} \text{tr} \gamma^\nu S_0(q + l)\gamma^\mu S_0(p + q + l)\gamma^\lambda S_0(l)\sin(p \wedge q),
\]

Therefore, using also \((\bar{p} + q + l - m) - (\bar{q} + l - m)\), we get

\[
p_{\mu}\Lambda_{a}^{\mu\nu\lambda}(p, q, k) = \Pi_{a}^{\nu\lambda}(q) - \Pi_{a}^{\nu\lambda}(k),
\]

where

\[
i\Pi_{a}^{\mu}(q) = -\int \frac{d^4l}{(2\pi)^4} \text{tr}(-ie\gamma^\nu)iS_0(q + l)(-ie\gamma^\lambda)iS_0(l)
\]

is the photon self-energy, with a fermion loop.

From now on, differently for the previous calculations, the contributions to the ST identity turns out to be very involved and a complete verification is unfeasible. In this situation we restrict ourselves in to verify the matching of the divergent parts of the two sides of Eq. (103).
The diagram for the ghost self-energy has already been considered in (87) and its PP is given by

\[
\text{PP}[p\mu\gamma^{\mu\nu\lambda}\gamma\mu\nu(p, q) b(p^2)] = \frac{e^2}{16\pi^2} \frac{1}{\epsilon} [(k^2 g^{\lambda\nu} - k^\lambda k^\nu) - (q^2 g^{\nu\lambda} - q^\nu q^\lambda)].
\] (111)

In the sequel we consider the diagrams \(\Lambda^{\mu\nu\lambda}\) of the left-hand side of (103), with ghost loop,

\[
v = 2e \sin(p \land q) \Lambda^{\mu\nu\lambda}_{b1}(p, q, k)
\] (112)

\[
= -i^3 (2e)^3 \int \frac{d^4 l}{(2\pi)^4} \frac{l^\lambda (p + l)^\mu (p + q + l)^\nu}{l^2 (p + l)^2 (p + q + l)^2} \times \sin(l \land p) \sin(l \land p + l \land q) \sin(l \land q + p \land q),
\]

and photon loop,

\[
v = 2e \sin(p \land q) \Lambda^{\mu\nu\lambda}_{b2}(p, q, k)
\] (113)

\[
= -i^3 (2e)^3 \int \frac{d^4 l}{(2\pi)^4} \frac{(p - q - l)^\lambda (-p - l)^\nu (-l)^\mu}{l^2 (p + l)^2 (p + q + l)^2} \times \sin(l \land p) \sin(-l \land p - l \land q) \sin(l \land q + p \land q),
\]

that have the same phase factors and, therefore, can be calculated analogously. Their PP contributions are

\[
\text{PP}[p\mu\Lambda^{\mu\nu\lambda}_{b1, b2,c}(p, q, k)] = -\frac{19e^2}{96\pi^2} \frac{1}{\epsilon} [(k^2 g^{\lambda\nu} - k^\lambda k^\nu) - (q^2 g^{\nu\lambda} - q^\nu q^\lambda)].
\] (115)
The remain diagrams for $\Lambda^{\mu \nu \lambda}$, are given by

\[ p+1 = 2e \sin(p \wedge q) \Lambda_{\mu \nu \lambda}^{d1}(p, q, k) \] (116)

\[
\frac{1}{2}(-i)^2(2e)(-4ie^2) \int \frac{d^4l}{(2\pi)^4} \frac{(l - p)^\lambda g^{\mu \nu} + (g_\alpha^\nu + 2)(2l + p)^\mu g^{\nu \lambda} + (l + 2p)^\nu g^{\mu \lambda}}{l^2(p + l)^2} \\
\times \sin(l \wedge p) \sin(-l \wedge p - l \wedge q) \sin(l \wedge q + p \wedge q)
\]

\[
+ \frac{1}{2}(-i)^2(2e)(-4ie^2) \int \frac{d^4l}{(2\pi)^4} \frac{(l + 2p)^\lambda g^{\mu \nu} + (g_\alpha^\nu - 2)(2l + p)^\mu g^{\nu \lambda} + (l - p)^\nu g^{\mu \lambda}}{l^2(p + l)^2} \\
\times \sin(l \wedge p) \sin(l \wedge q) \sin(-l \wedge p - l \wedge q - p \wedge q)
\]

\[
+ \frac{1}{2}(-i)^2(2e)(-4ie^2) \int \frac{d^4l}{(2\pi)^4} \frac{3p^\lambda g^{\mu \nu} - 3p^\nu g^{\mu \lambda}}{l^2(p + l)^2} \sin^2(l \wedge p) \sin(p \wedge q),
\]

\[ q+1 = 2e \sin(p \wedge q) \Lambda_{\nu \mu \lambda}^{d2}(p, q, k) \] (117)

\[
\frac{1}{2}(-i)^2(2e)(-4ie^2) \int \frac{d^4l}{(2\pi)^4} \frac{(l - q)^\lambda g^{\mu \nu} + (l + 2q)^\mu g^{\nu \lambda} + (g_\alpha^\nu - 2)(2l + q)^\nu g^{\mu \lambda}}{l^2(q + l)^2} \\
\times \sin(l \wedge q) \sin(-l \wedge p - l \wedge q) \sin(l \wedge p - p \wedge q)
\]

\[
+ \frac{1}{2}(-i)^2(2e)(-4ie^2) \int \frac{d^4l}{(2\pi)^4} \frac{(l + 2q)^\lambda g^{\mu \nu} + (l - q)^\mu g^{\nu \lambda} + (g_\alpha^\nu - 2)(2l + q)^\nu g^{\mu \lambda}}{l^2(q + l)^2} \\
\times \sin(l \wedge p) \sin(l \wedge q) \sin(-l \wedge p - l \wedge q + p \wedge q)
\]

\[
+ \frac{1}{2}(-i)^2(2e)(-4ie^2) \int \frac{d^4l}{(2\pi)^4} \frac{3q^\mu g^{\nu \lambda} - 3q^\nu g^{\mu \lambda}}{l^2(q + l)^2} \sin^2(l \wedge q) \sin(p \wedge q), \text{ and}
\]

\[ k-1 = 2e \sin(p \wedge q) \Lambda_{\mu \nu \lambda}^{d3}(p, q, k) \] (118)

\[
\frac{1}{2}(-i)^2(2e)(-4ie^2) \int \frac{d^4l}{(2\pi)^4} \frac{-(g_\alpha^\nu - 2)(2l - k)^\lambda g^{\mu \nu} + (l + k)^\mu g^{\nu \lambda} - (l - 2k)^\nu g^{\mu \lambda}}{l^2(p + q + l)^2} \\
\times \sin(l \wedge p) \sin(-l \wedge p - l \wedge q) \sin(-l \wedge q - p \wedge q)
\]
\[+ \frac{1}{2}(-i)^2(2e)(4i^2e^2) \int \frac{d^4l}{(2\pi)^4} \frac{-(g^\alpha_\alpha - 2)(2l - k)^\lambda g^{\mu\nu} - (l - 2k)^\mu g^{\nu\lambda} - (l + k)^\nu g^{\mu\lambda}}{l^2(p + q + l)^2} \times \sin(l \wedge q) \sin(-l \wedge p - l \wedge q) \sin(-l \wedge p + p \wedge q)\]

\[+ \frac{1}{2}(-i)^2(2e)(-4ie^2) \int \frac{d^4l}{(2\pi)^4} \frac{3k^\nu g^{\mu\lambda} - 3k^\mu g^{\lambda\nu}}{l^2(p + q + l)^2} \sin^2(l \wedge p + l \wedge q) \sin(p \wedge q),\]

where their PP contributions take the form

\[\text{PP}[\mu \lambda_{d1,d2,d3}(p, q, k)] = \frac{9e^2}{32\pi^2} \frac{1}{\epsilon} [(k^2 g^{\lambda\nu} - k^\lambda k^\nu) - (q^2 g^{\nu\lambda} - q^\nu q^\lambda)].\] (119)

Therefore, the sum of all PP contributions of the left-hand side of Eq. (103) becomes

\[\text{PP}[\mu \lambda(p, q, k)b(p^2) + \Lambda^\mu_{b_1,b_2,c,d1,d2,d3}(p, q, k)] = \frac{7e^2}{48\pi^2} \frac{1}{\epsilon} [(k^2 g^{\lambda\nu} - k^\lambda k^\nu) - (q^2 g^{\nu\lambda} - q^\nu q^\lambda)].\] (120)

Let us now look to the right-hand side of the identity (103). The diagrams are

\[v \quad q \quad l \quad q+1 \quad v = i\Pi_b^{\nu\lambda}(q)\] (121)

\[v \quad q \quad l \quad q+1 \quad v = i\Pi_c^{\nu\lambda}(q)\] (122)

\[v \quad q \quad l \quad q+1 \quad v = i\Pi_d^{\nu\lambda}(q)\] (123)

\[= \frac{1}{2}(-i)^2(2e)^2 \int \frac{d^4l}{(2\pi)^4} \frac{\gamma^{\alpha\beta}(l, q, -l - q)\gamma^{\beta\nu\alpha}(l + q, -q, -l)}{l^2(q + l)^2} \sin^2(l \wedge q),\]
so that

\[ \text{PP}[\Pi_{b,c,d}^\nu(q) - \Pi_{b,c,d}^\nu(k)] = \frac{5e^2}{24\pi^2} \left[ (k^2g^{\lambda\nu} - k^\lambda k^\nu) - (q^2g^{\nu\lambda} - q^\nu q^\lambda) \right]. \]  \hspace{1cm} (124)

Finally, let us see the diagrams \( B_{\lambda\nu}^\lambda(p, q, k) \) and \( B_{\nu\lambda}^\nu(p, k, q) \), given by

\[ = 2e \sin(p \wedge q) B_{\alpha\beta\gamma}^\lambda(p, q, k) \]  \hspace{1cm} (125)

\[ = i^2(-i)(2e)^3 \int \frac{d^4l}{(2\pi)^4 \mid l^2(p + l)^2(p + q + l)^2} \times \sin(l \wedge p) \sin(-l \wedge p - l \wedge q) \sin(-l \wedge q - p \wedge q), \]

\[ = i(-i)^2(2e)^3 \int \frac{d^4l}{(2\pi)^4 \mid l^2(p + l)^2(p + q + l)^2} \times \sin(l \wedge p) \sin(-l \wedge p - l \wedge q) \sin(l \wedge q + p \wedge q), \]  \hspace{1cm} (126)

\[ = 2e \sin(p \wedge q) B_{\alpha\lambda\gamma}^\nu(p, k, q) \]  \hspace{1cm} (127)

\[ = i^2(-i)(2e)^3 \int \frac{d^4l}{(2\pi)^4 \mid l^2(p + l)^2(p + q + l)^2} \times \sin(l \wedge p) \sin(l \wedge p + l \wedge q) \sin(l \wedge q + p \wedge q), \]  \hspace{1cm} (128)
\[ i(-i)^{2} (2\epsilon)^{3} \int \frac{d^{4}l}{(2\pi)^{4}} \frac{(-l)_{\alpha}^{\mu} \gamma_{\lambda}(p + l, q, -p - q - l)}{l^{2}(p + l)^{2}(p + q + l)^{2}} \times \sin(l \wedge p) \sin(-l \wedge p - l \wedge q) \sin(l \wedge q + p \wedge q), \]

where their PP contributions are

\[ \text{PP}[B^{\lambda}_{\nu}(p, q, k)(q^{2} g^{\nu\nu} - q^{\nu} q^{\nu}) + B^{\nu}_{\lambda}(p, k, q)(k^{2} g^{\lambda\lambda} - k^{\lambda} k^{\lambda})] = -\frac{e^{2}}{16\pi^{2}} \frac{1}{\epsilon} \left[ (k^{2} g^{\lambda\nu} - k^{\lambda} k^{\nu}) - (q^{2} g^{\nu\lambda} - q^{\nu} q^{\lambda}) \right]. \]  

(129)

From this, we see that the renormalization constant \( \tilde{Z}_{1} \) must be

\[ \tilde{Z}_{1} = 1 - \frac{e^{2}}{16\pi^{2}} \frac{1}{\epsilon}, \]

(130)

so that \( \tilde{Z}_{1} = Z_{5} \) and thus we obtain the relations

\[ Z_{2}/Z_{1F} = Z_{3}/\tilde{Z}_{1} = Z_{3}/Z_{1}. \]  

(131)

Therefore, the sum of all PP contributions of the right-hand side of Eq. (103) becomes

\[ \text{PP}[\Pi_{b,c,d}^{\nu\lambda}(q) - \Pi_{b,c,d}^{\lambda\nu}(k) + B^{\lambda}_{\nu}(p, q, k)(q^{2} g^{\nu\nu} - q^{\nu} q^{\nu}) + B^{\nu}_{\lambda}(p, k, q)(k^{2} g^{\lambda\lambda} - k^{\lambda} k^{\lambda})] = \frac{7e^{2}}{4\pi^{2}} \frac{1}{\epsilon} \left[ (k^{2} g^{\lambda\nu} - k^{\lambda} k^{\nu}) - (q^{2} g^{\nu\lambda} - q^{\nu} q^{\lambda}) \right], \]

(132)

which is the same result as for the left-hand side, Eq. (120).

Besides these ultraviolet divergent parts, arising from the planar parts of the diagrams, we have also infrared singular parts (SP) coming from the nonplanar parts of the same diagrams, at \( p, q, k = 0 \). Explicit calculations, combining denominators with Feynman parameters and using nonplanar integrals, give us the SP for the diagrams \( \Lambda^{\mu\nu\lambda} \) on the left-hand side of Eq. (103):

\[ \text{SP}[2\epsilon \sin(p \wedge q)\Lambda_{b1,b2,c1,d1,d2,d3}^{\mu\nu\lambda}(p, q, k)] = \frac{4e^{3}}{\pi^{2}} \sin(p \wedge q)_{p \wedge q} \left( \frac{\bar{p}^{\mu}\bar{p}^{\nu}\bar{p}^{\lambda}}{\xi \bar{p}^{4}} + \frac{\bar{q}^{\mu}\bar{q}^{\nu}\bar{q}^{\lambda}}{\xi \bar{q}^{4}} + \frac{\bar{k}^{\mu}\bar{k}^{\nu}\bar{k}^{\lambda}}{\xi k^{4}} \right), \]

(133)

where we are not taking into account the logarithmic singularities. Contracting \( q_{\mu} \) in the above expression, we obtain

\[ \text{SP}[q_{\mu}\Lambda_{b1,b2,c1,d1,d2,d3}^{\mu\nu\lambda}(p, q, k)] = \frac{2e^{3}}{\pi^{2}} \left( \frac{\bar{q}^{\nu}\bar{q}^{\lambda}}{\xi^{2} \bar{q}^{4}} - \frac{\bar{k}^{\nu}\bar{k}^{\lambda}}{\xi^{2} k^{4}} \right), \]

(134)

which is exactly the same SP for the photon self-energy diagrams on the right-hand side of Eq. (103),

\[ \text{SP}[\Pi_{b1,b2,c1,d1,d2,d3}^{\mu\nu\lambda}(q) - \Pi_{b1,b2,c1,d1,d2,d3}^{\lambda\nu\mu}(k)] = \frac{2e^{3}}{\pi^{2}} \left( \frac{\bar{q}^{\nu}\bar{q}^{\lambda}}{\xi^{2} \bar{q}^{4}} - \frac{\bar{k}^{\nu}\bar{k}^{\lambda}}{\xi^{2} k^{4}} \right). \]

(135)

The other diagrams of the ST identity (103) contribute only with logarithmic SP. These singularities are not problematic as they are integrable.
V. FINAL COMMENTS

In this work, for some specific Green functions, we have analyzed the ST identities in the context of noncommutative QED\textsubscript{4}. Special attention was given to the vectorial fermion-photon and triple photon vertex functions, explicitly verifying that no anomalies arise. The validity of these identities imply that, in spite of the presence of dangerous infrared singularities, the ultraviolet structure is not essentially modified by the noncommutativity. In fact, although the individual pole parts have been changed and new divergences appeared, the counterterms are related as they should in a non-Abelian situation. This however does not preclude the occurrence of dangerous infrared singularities which, in higher orders, jeopardizes the perturbative series. To extend our results to higher orders, our study must therefore be supplemented by some mechanism to control the mentioned singularities. One possibility is to consider the effect of supersymmetry; as known supersymmetric theories have a better ultraviolet behavior and consequently they may be free from dangerous infrared/ultraviolet mixing. This is what happens in susy noncommutative QED\textsubscript{4}.

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