A CHARACTERIZATION OF CERTAIN IRREDUCIBLE SYMPLECTIC 4-FOLDS

YASUNARI NAGAI

ABSTRACT. We give a characterization of irreducible symplectic fourfolds which are given as Hilbert scheme of points on a K3 surface.

1. INTRODUCTION

In the theory of the moduli problem of K3 surfaces, Kummer surfaces played a very important role. It is easy to characterize Kummer surfaces.

Proposition 1.1. (See [2], Chap. VIII §6) Let S be a K3 surface. If S contains 16 disjoint $\mathbb{P}^1$'s $C_1, \ldots, C_{16}$ and $D = \sum C_i$ is 2-divisible in Pic(S) then S is isomorphic to a Kummer surface.

The density of Kummer surfaces in the moduli space and this characterization enable us to derive the Global Torelli Theorem for arbitrary K3 surfaces from that for Kummer surfaces.

A higher dimensional analogue of a K3 surface is an irreducible symplectic manifold.

Definition 1.2. A compact Kähler manifold $X$ of dimension $2n$ is said to be irreducible symplectic if the following conditions are satisfied.

(i) $X$ admits a symplectic form, i.e. there exists a $d$-closed holomorphic 2-form $\sigma$ such that $\sigma^{\wedge n}$ is nowhere vanishing.

(ii) $h^0(X, \Omega^2_X) = 1$, i.e. any non-zero holomorphic 2-form is the symplectic form up to constant.

(iii) $X$ is simply connected.

An irreducible symplectic manifold is also called hyper-Kähler in the literature (see [3, 4]).

It seems that the moduli behaviour of irreducible symplectic manifolds is similar to that of K3 surfaces. Although Namikawa recently found a counterexample to the Global Torelli Problem in higher dimensions [5], one still believes that some kind of Global Torelli Theorem should hold, but even a convincing conjectural version of it is missing for the time being.

With a view towards the Global Torelli Problem for irreducible symplectic manifolds, it is important to ask for some “typical” objects in the moduli spaces in question and to give their characterization. This question in general seems to be quite hard. It is natural to restrict ourselves to a special case as our first step.

Date: Revised 21/09/2002.

Mathematics Subject Classification (2000): Primary 14J32, Secondary 32Q20.
The Hilbert scheme of points on a K3 surface is an example of irreducible symplectic manifold which is important and seems to be rather easy to handle, for it has a very explicit description, in particular in dimension four.

Example 1.3. (cf. [3]) Let $S$ be a smooth surface, $\text{Hilb}^n(S)$ the Hilbert scheme of 0-dimensional sub-schemes of length $n$ and $\text{Sym}^n(S) = S^n/\mathbb{G}_n$ the $n$-th symmetric product of $S$. Beauville [3] showed that the natural morphism (Hilbert-Chow morphism)

$$F : \text{Hilb}^n(S) \to \text{Sym}^n(S)$$

is a crepant birational morphism and that if $S$ is a K3 surface, the Hilbert scheme $\text{Hilb}^n(S)$ is an irreducible symplectic manifold of dimension $2n$. In the case $n = 2$, the description of $F$ is quite easy. The singular locus $\Sigma$ of $\text{Sym}^2(S)$ is isomorphic to $S$ and $\text{Sym}^2(S)$ is locally of the form $\mathbb{C}^2 \times (A_1$ surface singularity) along $\Sigma$. It is easy to show that $F$ is simply the blowing-up of $\text{Sym}^2(S)$ along $\Sigma$. Considering the action of $\mathbb{G}_2$, we have the following diagram

$$\begin{array}{ccc}
\text{Bl}_\Delta(S \times S) & \xrightarrow{\tilde{F}} & S \times S \\
\downarrow & & \downarrow \\
\text{Hilb}^2(S) & \xrightarrow{F} & \text{Sym}^2(S)
\end{array}$$

(1)

where $\Delta$ is the diagonal of $S \times S$.

We give the following result as an analogy of Proposition [1.4].

Theorem 1.4. Let $X$ be a projective irreducible symplectic fourfold. Assume that there exists a birational morphism $f : X \to Y$ which contracts an irreducible divisor $E$ to a surface $S \subset Y$ such that

(i) The general fibre of $f|_E : E \to S$ is isomorphic to $\mathbb{P}^1$,
(ii) $E$ is 2-divisible in $\text{Pic}(X)$,
(iii) $E^3 = 192$.

Then, $S$ is a K3 surface and $X$ is isomorphic to the Hilbert scheme $\text{Hilb}^2(S)$ of $S$.

Remark 1.4.1. If $X$ is deformation equivalent to some $\text{Hilb}^2(T)$ for a K3 surface $T$, the condition (iii) can be replaced by

(iii’) $q_X(E) = -8$.

where $q_X$ is the Beauville-Bogomolov form on $H^2(X, \mathbb{Z})$ (see [3, 6], see also Remark 2.7.1).

Remark 1.4.2. It would be natural to pose the following question: If $E$ is an irreducible divisor on $X$ with $q_X(E) < 0$, then there exist an irreducible symplectic fourfold $X'$ birational to $X$ and a birational morphism $f : X' \to Y'$ which contracts the strict transform of $E$ on $X'$? Clearly, the answer will be affirmative if every flop of symplectic 4-fold is a Mukai flop as conjectured, for the termination of flops for terminal fourfolds is already known.
The next natural problem to consider is the density of the birational (bimeromorphic) models of Hilbert schemes made from Kummer surfaces in the connected component of the moduli space containing an irreducible symplectic fourfold which is birational to the Hilbert scheme of some K3 surface. But even this seems to be rather hard question.

The rough idea to prove the theorem is to trace backward Beauville’s proof of Example 1.3. It uses more or less elementary and standard techniques. It contains several ingredients. One is a numerical computation using Holomorphic Lefschetz theorem of Atiyah-Singer. Another is the decomposition theorem of Kähler manifolds with trivial first Chern class. The result of Wierzba [7] on divisorial contractions of symplectic manifolds is also used in an essential way. The remaining part consists of geometric arguments based on the geometry of K3 surfaces.

Notation. Through this paper we work with the following notation. Let $X$, $Y$, $f$, $E$ and $S$ be as in the theorem above. Theorems 1.4(ii) and 1.5 in [7] imply that $E$ is a $\mathbb{P}^1$ bundle and $S$, which is the singular locus of $Y$, is a smooth surface with $K_S \sim 0$. Furthermore they infer that $\tilde{Y}$ is analytically locally isomorphic to $\mathbb{C}^2 \times (A_1$ surface singularity) at each point of $S$. Put $D = \frac{1}{2}E$ and take a double covering $p : \tilde{X} \to X$ defined by $\mathcal{O}(D)$. Then $p$ is ramified at $E \subset \tilde{X}$ and $p(\tilde{E}) = E$. $\tilde{X}$ is smooth since $E$ is smooth and we have the following diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
p \downarrow & & \downarrow q \\
X & \xrightarrow{f} & Y
\end{array}
$$

where $\tilde{f}$ and $q$ is the Stein factorization of $f \circ p$. In fact $\tilde{Y}$ is a smooth fourfold, $\tilde{f}$ is the blowing-up of $\tilde{Y}$ along a smooth centre $\tilde{S}$ with the exceptional divisor $\tilde{E}$, and $q_{\tilde{S}} : \tilde{S} \sim \to S$ is an isomorphism. Note that $K_{\tilde{Y}} \sim 0$, for $f$ is crepant and $q$ has no ramification divisor.

Remark 1.4.3. The projectivity assumption in Theorem 1.4 is made to apply Wierzba’s result in our argument. If Wierzba’s description on divisorial contraction of symplectic manifolds is valid for non-projective ones, the projectivity assumption would not be necessary.

2. Geometry of $\tilde{Y}$

In this section, we prove the following proposition.

**Proposition 2.1.** Under the assumption of Theorem 1.4 and the notation above, $\tilde{Y}$ is isomorphic to a product of two K3 surfaces.

Our strategy is to apply the following famous decomposition theorem to $\tilde{Y}$.

**Theorem 2.2** (cf. [3]). Let $Z$ be a compact Kähler manifold with $K_Z \sim 0$. Then there exists a finite étale covering $\tilde{Z} \to Z$ such that $\tilde{Z}$ is isomorphic to a product of varieties of following types

(i) complex torus,
(ii) Calabi-Yau manifold i.e. compact Kähler manifold $W$ for which $K_W \sim 0$, $h^i(W, \mathcal{O}_W) = 0$ for $0 < i < \dim W$, and $\pi_1(W) = \{e\}$.

(iii) irreducible symplectic manifold.

Thanks to this powerful theorem, Proposition 2.1 is reduced to the following

**Proposition 2.3.** (i) $\pi_1(\tilde{Y}) = \{e\}$. (ii) $h^0(\tilde{Y}, \Omega^2_{\tilde{Y}}) = 2$.

**Proof of Proposition 2.3 ⇒ Theorem 2.1.** Applying Theorem 2.2 under the condition (i) of Proposition 2.3, $\tilde{Y}$ itself decomposes into a product of Calabi-Yau manifolds and irreducible symplectic manifolds. Since $\tilde{Y}$ is of dimension 4, a product of two K3 surfaces, a Calabi-Yau fourfold or a compact irreducible symplectic fourfold is possible. But (ii) of Proposition 2.3 asserts that the last two cases do not happen. Q.E.D.

To compute these quantities from the condition (iii) of Theorem 1.4, we use Holomorphic Lefschetz formula of Atiyah-Singer [1].

**Definition 2.4.** Let $M$ be a compact complex manifold, and $g$ an automorphism of finite order of $M$. The **holomorphic Lefschetz number** $L_{\text{hol}}(g)$ is defined by

$$L_{\text{hol}}(g) = \sum (-1)^p \text{trace}(g^* : H^p(M, \mathcal{O}_M)).$$

**Theorem 2.5** (Holomorphic Lefschetz formula, [1]). As in the notation of the definition above. Assume further that the fixed point set $M^g = \{x \in M \mid g(x) = x\}$ is smooth. Then the formula

$$L_{\text{hol}}(g) = \int_{M^g} \prod \theta \mathcal{U}(N_{M^g/M}(\theta)) \cdot \text{td}(M^g)$$

holds, where $\text{td}(M^g)$ denotes the Todd class of $M^g$, $N_{M^g/M}$ the normal bundle, $N_{M^g/M}(\theta) \subset N_{M^g/M}$ the eigen-sub-bundle of $(g_{N_{M^g/M}})^*$ with the eigenvalue $e^{i\theta}$, and

$$\mathcal{U}(x_1, x_2, \ldots) = \left\{ \prod \left( \frac{1 - e^{-x_j - i\theta}}{1 - e^{-i\theta}} \right) \right\}^{-1}.$$

The general formula itself is very complicated, but in our case the formula becomes easy to handle.

**Lemma 2.6.** Notation as in §1. Let $g$ be an involution of $\tilde{Y}$ induced by $q : \tilde{Y} \to Y$. Then $L_{\text{hol}}(g) \in \mathbb{Z}$ and we have

$$L_{\text{hol}}(g) = \frac{1}{48} (c_2(T_S) + 3c_2(N_{S/\tilde{Y}})).$$

In particular if $\tilde{S}$ is a K3 surface,

$$L_{\text{hol}}(g) = \frac{1}{2} + \frac{1}{16} c_2(N_{S/\tilde{Y}})$$

and if $S$ is an abelian surface,

$$L_{\text{hol}}(g) = \frac{1}{16} c_2(N_{S/\tilde{Y}}).$$
Proof. Since \( g \) is an involution, eigenvalues of \( g^* \) on each cohomology group must be \( \pm 1 \), therefore trace \( g^* \in \mathbb{Z} \), in particular \( L_{\text{hol}}(g) \in \mathbb{Z} \).

We apply Theorem 2.5 under \( M = \widetilde{Y} \) and \( M^g = S \). Since \( g \) produces the two dimensional locus of \( A_1 \) singularities \( S \), we have

\[
(g_{N_{\widetilde{S}/\widetilde{Y}}})^* = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Therefore, \( \det(1 - (g_{N_{\widetilde{S}/\widetilde{Y}}})^*) = 4 \) and the only possible \( \theta \) is \( \theta = \pi \), i.e. \( N_{\widetilde{S}/\widetilde{Y}}(\pi) = N_{\widetilde{S}/\widetilde{Y}} \). Since the rank of this bundle is 2, we have only to consider \( \mathcal{U}^\pi \) in 2 variables. By definition

\[
\mathcal{U}^\pi(x_1, x_2) = \frac{2}{1 + e^{-x_1}} \cdot \frac{2}{1 + e^{-x_2}}.
\]

This implies

\[
\mathcal{U}^\pi(N_{\widetilde{S}/\widetilde{Y}}(\pi)) = 1 - \frac{1}{2} c_1(N_{\widetilde{S}/\widetilde{Y}}) + \frac{1}{4} c_2(N_{\widetilde{S}/\widetilde{Y}}).
\]

Note that \( \text{td}(\widetilde{S}) = 1 + \frac{1}{12} c_2(T_S) \) because \( K_{\widetilde{S}} \sim 0 \). Combining these, we get the formula. For the last assertion, note that \( S \) is either a K3 surface or a complex 2-torus by the Enriques-Kodaira’s classification. Q.E.D.

For the computation of the second Chern class \( c_2(N_{\widetilde{S}/\widetilde{Y}}) \), we prepare an easy lemma.

Lemma 2.7. \( \widetilde{E}^4 = c_2(N_{\widetilde{S}/\widetilde{Y}}) \).

Proof. Note that \( \widetilde{h} \) is a blowing-up of the smooth variety \( \widetilde{Y} \) along the smooth centre \( \widetilde{S} \). Therefore, we can regard \( \widetilde{h} = \widetilde{f}\mid_{\widetilde{E}} \) as \( \widetilde{h} : \widetilde{E} = \mathbb{P}(N_{\widetilde{S}/\widetilde{X}}) \to S \). We have a line bundle \( L \) on \( \widetilde{E} \) such that there is an exact sequence of bundle maps

\[
0 \longrightarrow L \longrightarrow \widetilde{h}^* N_{\widetilde{S}/\widetilde{X}} \longrightarrow O_{\widetilde{E}}(1) \longrightarrow 0.
\]

By naturality, we have \( c_2(\widetilde{h}^* N_{\widetilde{S}/\widetilde{X}}) = c_2(N_{\widetilde{S}/\widetilde{X}}) \cdot [F] \in H^4(\widetilde{E}, \mathbb{Z}) \), where \( F \) is a fibre of \( \widetilde{h} \). This implies \( c_2(N_{\widetilde{S}/\widetilde{X}}) = c_2(\widetilde{h}^* N_{\widetilde{S}/\widetilde{X}}) \cdot O_{\widetilde{E}}(1) = L \cdot O_{\widetilde{E}}(1)^2 \). On the other hand, one has trivially \( \widetilde{h}^* c_1(N_{\widetilde{S}/\widetilde{X}}) = c_1(\widetilde{h}^* N_{\widetilde{S}/\widetilde{X}}) = c_1(O_{\widetilde{E}}(1)) + c_1(L) \).

Combining these we get \( c_2(N_{\widetilde{S}/\widetilde{X}}) = \widetilde{h}^* c_1(N_{\widetilde{S}/\widetilde{X}}) \cdot c_1(O_{\widetilde{E}}(1))^2 + \widetilde{E}^4 \), for \( O_{\widetilde{E}}(1) = (-\widetilde{E})_{\widetilde{E}} \). But we have \( c_1(N_{\widetilde{S}/\widetilde{X}}) = c_1(T_{\widetilde{S}/\widetilde{Y}}) - c_1(T_S) = 0 \) and therefore \( c_2(N_{\widetilde{S}/\widetilde{X}}) = \widetilde{E}^4 \). Q.E.D.

Remark 2.7.1. We show the converse of Theorem 2.4 using this lemma. Let \( X = \text{Hilb}^2(S) \) for some K3 surface \( S \). Noting that (I) fits into the diagram (II), (i,ii) of the Theorem 2.4 are evident. In the notation of (I) we have \( N_{\Delta/\Delta \times S} \cong T_S \cong \Omega_1^S \), for \( S \) is K3. Therefore, for the exceptional divisor \( \widetilde{E} \) of \( F \), we get...
\[ \tilde{E}^4 = c_2(N_{\Delta/S \times S}) = c_2(\Omega_S^1) = 24. \] By the projection formula and ramification, we have

\[ (3) \quad \tilde{E}^4 = \left( \frac{1}{2} p^* E \right)^4 = \frac{1}{16} (p^* E)^4 = \frac{1}{8} E^4, \]

so that \( E^4 = 8 \cdot 24 = 192. \)

**Corollary 2.8.** Under the assumption and notation as in \( \S 1 \), \( \tilde{S} \) is a K3 surface.

**Proof.** Assume the contrary, *i.e.* assume \( \tilde{S} \) be an abelian surface. Note that \( \tilde{E}^4 = 24 \) by (3). Lemmas 2.6 and 2.7 imply \( L_{hol}(g) = \frac{24}{16} \notin \mathbb{Z} \), which is a contradiction. Q.E.D.

Now is the time to prove Proposition 2.3.

**Proof of Proposition 2.3.** (i) Let \( \hat{E}, \tilde{E} \) be tubular neighbourhoods of \( E, \tilde{E} \) and set \( X^o = X \setminus E, \tilde{X}^o = X \setminus \tilde{E}. \) Then we have

\[ X = X^o \cup \hat{E}, \quad X^o \cap \hat{E} = \hat{E} \setminus E, \]
\[ \tilde{X} = \tilde{X}^o \cup \tilde{E}, \quad \tilde{X}^o \cap \tilde{E} = \tilde{E} \setminus \tilde{E}. \]

Since the K3 surface \( S \cong \tilde{S} \) is simply connected and \( E \rightarrow S \) and \( \tilde{E} \rightarrow \tilde{S} \) are \( \mathbb{P}^1 \)-bundles, the homotopy exact sequence infers

\[ \pi_1(E) = \pi_1(\tilde{E}) = \{ e \}, \quad \pi_1(\hat{E}) = \pi_1(\tilde{E}) = \{ e \}. \]

Note that \( \pi_1(X) = \{ e \} \) because \( X \) is irreducible symplectic. By Van Kampen’s theorem

\[ \begin{array}{ccc}
\pi_1(X) & \cong & \{ e \} \\
\downarrow & & \downarrow \\
\pi_1(X^o) & \cong & \{ e \} \\
\phi & \cong & \pi_1(\hat{E}) \\
\phi & \cong & \pi_1(\tilde{E}) \cong \mathbb{Z} \\
\end{array} \]

we know \( \phi \) is surjective. Consider the following commutative diagram

\[ \begin{array}{ccc}
\pi_1(\tilde{X}^o) & \cong & \pi_1(\tilde{E} \setminus \tilde{E}) \cong \mathbb{Z} \\
\downarrow p_* & & \downarrow p_* \\
\pi_1(X^o) & \cong & \pi_1(\hat{E} \setminus \hat{E}) \cong \mathbb{Z} \\
\phi & \cong & \phi \\
\end{array} \]

Since \( \tilde{X}^o \rightarrow X^o \) is étale of degree 2, \( \pi_1(\tilde{X}^o) \xrightarrow{p} \pi_1(X^o) \) is injective but not surjective. It follows that \( \tilde{\phi} \) is also surjective, because \( \pi_1(\tilde{E} \setminus \tilde{E}) \rightarrow \pi_1(\hat{E} \setminus E) \) is of index 2. Again using Van Kampen’s theorem, we get \( \pi_1(\tilde{X}) = \{ e \} \), therefore \( \pi_1(Y) = \{ e \} \), for a birational map \( \tilde{f} \) does not change the fundamental group.
(ii) The condition $h^0(X, \Omega^2_X) = 1$ implies $h^0(Y, \Omega^2_Y) = 1$. Therefore,
\[ L_{hol}(g) = 1 + \{1 - (h^0(\bar{Y}, \Omega^2_{\bar{Y}}) - 1)\} + 1 = 4 - h^0(\bar{Y}, \Omega^2_{\bar{Y}}). \]
On the other hand, by Lemmas 2.6, 2.7, and Corollary 2.8 we see
\[ L_{hol}(g) = \frac{1}{2} + \frac{24}{16} = 2. \]
Combining these, we get $h^0(\bar{Y}, \Omega^2_{\bar{Y}}) = 2$. Q.E.D.

3. CONCLUSION OF THE PROOF OF THEOREM 1.4

In this section, we complete the proof of Theorem 1.4.

In the last section, we have shown that $\bar{Y} \cong T_1 \times T_2$, where, $T_1, T_2$ are K3 surfaces. To prove Theorem 1.4, we investigate the action of $g$ on $\bar{Y}$.

**Proposition 3.1.** $T_1 \cong T_2 \cong \tilde{S}$. We may assume $\tilde{S}$ is the diagonal of $\bar{Y} \cong \tilde{S} \times \tilde{S}$.

**Proof.** Let $p_i : \bar{Y} \to T_i$ ($i = 1, 2$) be the projections. To prove the proposition, it is enough to show
\[ \dim p_1(\tilde{S}) = \dim p_2(\tilde{S}) = 2, \]
for these imply $p_i|\tilde{S} =: \phi_i : \tilde{S} \to T_i$ ($i = 1, 2$) is generically finite, but $K_{\tilde{S}} \sim 0$ and $K_{T_i} \sim 0$ imply $\phi_i$ has neither exceptional divisor nor ramification divisor, i.e. $\phi_i$ is isomorphism. In the following, we show (4) via case by case consideration.

If $\dim p_1(\tilde{S}) = 0$, i.e. $p_1(\tilde{S}) = \{t\} \subset T_1$, we get $\tilde{S} = p_1^{-1}(t)$, therefore $N_{\tilde{S}/\bar{Y}} = 0$ so that $c_2(N_{\tilde{S}/\bar{Y}}) = 0$. This contradicts to Lemma 2.7. By the same argument for $p_2$ we have dim $p_1(\tilde{S}) \geq 1$ ($i = 1, 2$).

**Claim.** If $\dim p_1(\tilde{S}) = 1$, then $\dim p_2(\tilde{S}) = 2$. Moreover for $Z = p_1^{-1}(p_1(\tilde{S}))$, we have $g(Z) = Z$.

**Proof of the claim.** Assume $\dim p_1(\tilde{S}) = \dim p_2(\tilde{S}) = 1$. Let $C_i = p_i(\tilde{S})$ ($i = 1, 2$). Then we have
\[ \tilde{S} \subset p_1^{-1}(C_1) \cap p_2^{-1}(C_2) = C_1 \times C_2. \]
Since $\tilde{S}$ is irreducible, so are $C_1$ and $C_2$. But this implies that $\tilde{S}$ is actually the product of these two curves, which is absurd. Therefore we may assume $\dim p_2(\tilde{S}) = 2$ and $\varphi_2 : \tilde{S} \to T_2$ is an isomorphism. The second assertion of the claim is equivalent to $p_1(g(Z)) = C_1$. Assume the contrary, i.e. $p_1(g(Z)) = T_1$. Then, we have the following diagram
Clearly \( \kappa(Z) = -\infty \), where \( \kappa(Z) \) the Kodaira dimension of \( Z \). The sub-additivity property of Kodaira dimension (cf. [4]) implies that irreducible components of any fibre of \( \psi_1 \) are rational curves. On the other hand, connected components of the general fibre of \( \varphi_1 : \tilde{S} \rightarrow C \) are elliptic curves so that \( \psi_1 \) contains an elliptic curve as its fibre, a contradiction. (End of the proof of the claim) Q.E.D.

To prove the proposition, what we have to do is to get a contradiction assuming \( \dim p_1(\tilde{S}) = 1 \) and \( g(Z) = Z \), thanks to the claim. Consider the diagram

\[
\begin{array}{c}
Z = C_1 \times T_2 \xrightarrow{p_1} C_1 \\
\tilde{S} \xrightarrow{\varphi_1} C_1 \\
\end{array}
\]

Let \( U_t = p_1^{-1}(t) \ (t \in C_1) \) and consider the normalizations

\[
\begin{array}{c}
\tilde{S} \xrightarrow{\varphi_1} \tilde{S} \xrightarrow{\rho_1} C_1 \\
\end{array}
\]

The involution \( g \) on \( \tilde{S} \) ascends to \( \tilde{S} \). Since \( \tilde{S} \) is the anti-canonical map of \( Z \), \( g \) descends to \( \tilde{S} \). This implies \( \dim p_1(g(U_t)) = 0 \) therefore \( p_1(g(U_t)) = \{ t \} \), since \( U_t \cap \tilde{S} = g(U_t) \cap \tilde{S} \neq \emptyset \). Then we have a family of automorphisms \( \{ \psi_t : U_t \rightarrow U_t \} \). There is a commutative diagram of isomorphisms

\[
\begin{array}{c}
U_t \xrightarrow{p_1} C_1 \xrightarrow{\rho_1} \tilde{S} \\
U_t \xrightarrow{p_2} \tilde{S} \xrightarrow{\rho_1} \tilde{S} \\
\end{array}
\]

Since a K3 surface has no infinitesimal automorphism, \( \rho_t \) is independent of \( t \). But \( \psi_t \) fixes the points on \( D_t = U_t \cap \tilde{S} \) and \( \tilde{S} = \bigcup_{t \in C_1} D_t \), \( \rho_t \) induces the identity on \( \tilde{S} \) and also on \( U_t \). This contradicts \( g \neq \text{id} \). Q.E.D.

Finally the following proposition completes our proof of the Main Theorem, in view of Example [1,3].

**Proposition 3.2.** Notation as above. The action of \( g \) on \( \tilde{Y} = \tilde{S} \times \tilde{S} \) is the permutation of two components.

**Proof.** Consider the diagram

\[
\begin{array}{c}
\tilde{S} \xrightarrow{\text{diag.}} \tilde{Y} = \tilde{S} \times \tilde{S} \xrightarrow{p_2} \tilde{S} \\
\tilde{S} \xrightarrow{p_1} \tilde{S} \\
\end{array}
\]

Let \( T_t = p_1^{-1}(t) \ (t \in \tilde{S}) \) and \( \Gamma_i = \{ t \in \tilde{S} \mid \dim p_i \circ g(T_t) = 1 \} \). Note that \( \Gamma_i \) is a locally closed set in Zariski topology.

**Claim.** \( \dim \Gamma_i \leq 1 \quad (i = 1, 2) \).
Proof of the claim. Assume the contrary, i.e. assume that \( \Gamma \subset \tilde{S} \) contains an Zariski open set. For any \( t \in \tilde{S} \),

\[
g(T_t) \xrightarrow{p_1} C_t \subset \tilde{S}
\]

with \( C_t \) is a rational curve containing \( t \). Since \( t \) sweeps over an open set of \( \tilde{S} \), \( \{C_t\} \) is a covering family of rational curves on \( \tilde{S} \). This is impossible because \( \tilde{S} \) is a K3 surface. (End of the proof of the claim) Q.E.D.

Therefore, there exists a Zariski open set \( V \subset \tilde{S} \) such that

\[
(\dim p_1 \circ g(T_t), \dim p_2 \circ (T_t)) = (0, 2), (2, 2) \text{ or } (2, 0)
\]

for any \( t \in V \).

In the first case, we have \( g(T_t) = T_t \) and \( g \) induces \( j_t \in \text{Aut}(T_t) \cong \text{Aut}(\tilde{S}) \).

Since \( \tilde{S} \) has no infinitesimal automorphism, \( j_t \) is constant with respect to \( t \in \tilde{S} \).

But \( j_t(t) = t \) for any \( t \in V \) implies \( j_t = \text{id} \), contradiction.

In the second case, \( g(T_t) \) is the graph of an automorphism \( f_t \in \text{Aut}(\tilde{S}) \).

Since \( g(T_t) \cap g(T_{t'}) = \emptyset \) for \( t \neq t' \), we have \( f_t \neq f_{t'} \). This contradicts the discreteness of \( \text{Aut}(\tilde{S}) \).

Therefore only the last case can happen. This implies \( g(p_1^{-1}(t)) = p_2^{-1}(t) \), for \( T_t \cap \tilde{S} \neq \emptyset \). Exchanging the roles of \( p_1 \) and \( p_2 \), we also have \( g(p_2^{-1}(t)) = p_1^{-1}(t) \).

Finally we get

\[
\begin{array}{ccc}
\{(s,t)\} & \xrightarrow{g} & p_1^{-1}(s) \cap p_2^{-1}(t) \\
\downarrow & & \downarrow \\
p_2^{-1}(s) \cap p_1^{-1}(t) & = & \{(t,s)\}
\end{array}
\]

for \( s, t \in V \). This shows \( g \) is a permutation of \( p_1 \) and \( p_2 \). Q.E.D.

Acknowledgement. The author would like to express his profound gratitude to Prof. Yujiro Kawamata, his supervisor, for comments and continuous encouragement. He also thanks Tetsushi Ito and Shunsuke Takagi for stimulating discussions, Prof. Keiji Oguiso for his comments and indicating an improvement of the proof of Proposition [3.1]. He is also indebted to the referee for his careful reading of the manuscript and many valuable comments and suggestions.

References

[1] Atiyah, M.F., Singer, I., The index of elliptic operators III, Ann. Math. (2) 87, 546-604 (1968)
[2] Barth, W., Peters, C., Van de Ven, A., Compact Complex Surfaces, Springer-Verlag (1984)
[3] Beauville, A., Variétés Kähleriennes dont la première classe de Chern est nulle, J. Diff. Geom. 18, 775-782 (1983)
[4] Kawamata, Y., Minimal models and the Kodaira dimension of algebraic fiber spaces, J. Reine Angew. Math. 363, 1-46 (1985)
[5] Namikawa, Y., Counter-example to global Torelli problem for irreducible symplectic manifolds, preprint, e-print arXiv:math.AG/0110114 to appear in Math. Ann.
[6] Huybrechts, D., Compact hyperkähler manifolds: basic results, Invent. math. 135, No.1, 63-113 (1999)
[7] Wierzba, J., Contractions of Symplectic Varieties, preprint, e-print arXiv:math.AG/9910136 to appear in J. Alg. Geom.
Yasunari Nagai, Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan

E-mail address: nagai@ms.u-tokyo.ac.jp