Permutation Reconstruction from MinMax-Betweenness Constraints

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Abstract
In this paper, we investigate the reconstruction of permutations on \(\{1, 2, \ldots, n\}\) from betweenness constraints involving the minimum and the maximum element located between \(t\) and \(t + 1\), for all \(t = 1, 2, \ldots, n - 1\). We propose two variants of the problem (directed and undirected), and focus first on the directed version, for which we draw up general features and design a polynomial algorithm in a particular case. Then, we investigate necessary and sufficient conditions for the uniqueness of the reconstruction in both directed and undirected versions, using a parameter \(k\) whose variation controls the stringency of the betweenness constraints. We finally point out open problems.

Keywords: betweenness, permutation, algorithm, genome, common intervals

1 Introduction
The BETWEENNESS problem is motivated by physical mapping in molecular biology and the design of circuits [2]. In this problem, we are given the set \([n] := \{1, 2, \ldots, n\}\), for some positive integer \(n\), and a set of \(m\) betweenness constraints \((m > 0)\), each represented as a triple \(x \xrightarrow{\rightarrow} y\) with \(x, a, y \in [n]\) and signifying that \(a\) is required to be between \(x\) and \(y\). The goal is to find a permutation on \([n]\) satisfying a maximum number of betweenness constraints. In [2], it is shown that the BETWEENNESS problem is NP-complete even in the particular case where all the constraints have to be satisfied.

In this paper we are interested in a problem related to the BETWEENNESS problem, which also finds its motivations in molecular biology. Given \(K\) \((K \geq 2)\) permutations on the same set \([n]\), representing \(K\) genomes given by the sequences of their genes, a common interval of these permutations is a subset of \([n]\) whose elements are consecutive (i.e. they form an interval) on each of the \(K\) permutations. Common intervals thus represent regions of the genomes which have identical gene content, but possibly different gene order. Computing common intervals or specific subclasses of them in linear time (up to the number of output intervals) has been done by case-by-case approaches until recently, when we proposed [3] a common linear framework, whose basis is the notion of MinMax-profile. The MinMax-profile of a permutation \(P\) forgets the order of the elements in a permutation, and keeps only essential betweenness information, defined as, for each \(t \in [n - 1]\), the minimum and maximum value in the interval delimited by the elements \(t\) (included) and \(t + 1\) (included) on \(P\) (with no restriction on the relative positions of \(t\) and \(t + 1\) on \(P\)). When \(K\) permutations are available, their MinMax-profile is defined similarly, by considering for every \(t \in [n - 1]\) the global minimum and the global maximum of the \(K\) intervals delimited by \(t\) and \(t + 1\) on the \(K\) permutations. We show in [3] that, assuming the permutations have been renumbered such that one of them is the identity permutation, the MinMax-profile of \(K\) permutations is all we need to find common intervals, as well as all the specific subclasses of common intervals defined in the literature, in linear time (up to the number of output intervals).

Hence, the MinMax-profile is a simplified representation of a (set of) permutation(s), which is sufficient to efficiently solve a number of problems related to finding common intervals in permutations. Moreover, it may be computed in linear time [3]. However, it can be easily seen that distinct (sets of) permutations may have the same MinMax-profile, implying that the MinMax-profile captures a part, but not all, of the information in the (set of) permutation(s).

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In this paper, we study the reconstruction of a permutation from a given MinMax-profile, and discuss possible generalizations.

2 Definitions and Problems

In the remaining of the paper, permutations are defined on \([n]\) and are increased with elements 0 and \(n+1\), added respectively at the beginning and the end of each permutation (and assumed to be fixed). This is due to the need to make the distinction between a permutation and its reverse order permutation.

**Definition 1.** [3] The MinMax-profile of a permutation \(P\) on \([n] \cup \{0, n+1\}\) is the set of MinMax-constraints

\[
\text{MinMax}(P) = \{t_{\text{min}_t, \text{max}_t} + 1 \mid 0 \leq t \leq n\}
\]

where \(\text{min}_t\) (\(\text{max}_t\), respectively) is the minimum (maximum respectively) element in the interval delimited on \(P\) by the element \(t\) (included) and the element \(t+1\) (included).

Note that the relative positions on \(P\) (i.e. which one is on the left of the other) of \(t, t+1\) on the one hand, and of \(\text{min}_t, \text{max}_t\) on the other hand are not indicated by a MinMax-profile. In the case where the relative positions of \(t\) and \(t+1\) are known for all \(t\), we use the term of directed MinMax-profile and the notations \(t_{\text{min}_t, \text{max}_t} + 1\) when \(t\) is on the left of \(t+1\), respectively \(t_{\text{min}_t, \text{max}_t} + 1\) when \(t+1\) is on the left of \(t\).

**Example 1.** Let \(P = (0 \ 6 \ 4 \ 7 \ 2 \ 9 \ 1 \ 8 \ 5 \ 3 \ 10)\) a permutation on \([9] \cup \{0, 10\}\). Then its MinMax-profile is (note that the MinMax-constraints sharing an element are concatenated):

\[
0 \ [0.9] 1 \ [1.9] 2 \ [1.9] 3 \ [1.9] 4 \ [1.9] 5 \ [1.9] 6 \ [4.7] 7 \ [1.9] 8 \ [1.9] 9 \ [1.10] 10
\]

whereas its directed MinMax-profile is:

\[
0 \ [0.9] 1 \ [1.9] 2 \ [1.9] 3 \ [1.9] 4 \ [1.9] 5 \ [1.9] 6 \ [4.7] 7 \ [1.9] 8 \ [1.9] 9 \ [1.10] 10
\]

Notice that the MinMax-profile and the directed MinMax-profile of any permutation obtained by arbitrarily permuting the elements \([3, 5, 8]\) are the same, showing that a (directed or not) MinMax-profile may correspond to several distinct permutations.

The MinMax-profile of a set \(\mathcal{P}\) of permutations is defined similarly [3], by requiring that \(\text{min}_t\) and \(\text{max}_t\) be defined over the union of the intervals delimited by \(t\) (included) and \(t+1\) (included) on the \(K\) permutations in \(\mathcal{P}\). This definition is given here for the sake of completeness, but is little used in the paper.

We distinguish between the MinMax-profile of a (set of) permutation(s) and a MinMax-profile:

**Definition 2.** A MinMax-profile on \([n] \cup \{0, n+1\}\) is a set of MinMax-constraints

\[
F = \{t_{\text{min}_t, \text{max}_t} + 1 \mid 0 \leq t \leq n\}
\]

with \(0 \leq m_t \leq t < t+1 \leq M_t \leq n+1\).

Again, a MinMax-profile is directed when for all \(t, 0 \leq t \leq n\), the relative position of \(t\) with respect to \(t+1\) is given. A MinMax-profile may be the MinMax-profile of some permutation, or of a set of permutations, but may also be the profile of no (set of) permutation(s). We limit this study to one permutation, and therefore formulate the following problem:
MinMax-Betweenness

Input: A positive integer $n$, a MinMax-profile $F$ on $[n] \cup \{0, n + 1\}$.

Question: Is there a permutation $P$ on $[n] \cup \{0, n + 1\}$ whose MinMax-profile is $F$?

The MinMax-Betweenness problem is obviously related to the Betweenness problem, since looking for a permutation $P$ with MinMax-constraints defined by $F$ means satisfying a number of betweenness constraints. Some differences exist however, as $F$ also defines non-betweenness constraints.

More precisely, each MinMax-constraint $t \xleftarrow{m_t, M_t} t + 1$ from $F$ may be expressed using the betweenness constraints (abbreviated B-constraints):

$$t \xleftarrow{m_t} t + 1, t \xrightarrow{M_t} t + 1$$

along with the non-betweenness constraints (abbreviated NB-constraints):

$$-(t \xleftarrow{j} t + 1), j = 0, 1, \ldots, m_t - 1, M_t + 1, \ldots, n + 1.$$}

Remark 1. It is worth noticing here that in a (directed or not) MinMax-profile which corresponds to at least one permutation on $[n] \cup \{0, n + 1\}$, the value $0$ ($n + 1$ respectively) should only occur in one precise MinMax-constraint, namely the one involving $0$ and $1$ ($n$ and $n + 1$ respectively). Otherwise, $0$ ($n + 1$ respectively) cannot be the leftmost (rightmost, respectively) value in the permutation. In the remainder of the paper, it is assumed that this condition has been verified before further investigations, and assume therefore that $0$ and $n + 1$ are respectively located in places $0$ and $n + 1$.

We present below, in Section 3, our analysis of the Directed MinMax-Betweenness problem, proposing a first algorithmic approach and pointing out the main difficulties for reaching a complete polynomial solution. In Section 4, we identify a polynomial particular case for the directed version. In Section 5 we propose to generalize MinMax-profiles to $k$-profiles, by introducing a parameter $k$ which allows to progressively increase the amount of information contained in a $k$-profile, up to a value $k_0$ which allows to identify each permutation by its $k_0$-profile. Section 6 is the conclusion.

3 Seeking an algorithm for Directed MinMax-Betweenness

3.1 A naïve approach

Let $F$ be a directed MinMax-profile on $[n] \cup \{0, n + 1\}$. The most intuitive idea for solving Directed MinMax-Betweenness is to build a simple directed graph $G$ (i.e. with no loops or multiple arcs) whose vertex set $V(G)$ is $[n] \cup \{0, n + 1\}$ and whose arcs $(x, y)$ indicate the precedence relationships between the elements on each permutation corresponding to the given $k$-profile (i.e. $x$ is on the left of $y$). If a permutation exists, $G$ must be a directed acyclic graph (or DAG). The MinMax-constraints from $F$ directly define arcs using: 1) the relative order between $t$ and $t + 1$, for each $t \in [n]$ (the corresponding arcs of $G$ are called R-arcs), and 2) the B-constraints (resulting into B-arcs).

Further arcs may be dynamically obtained by repeatedly invoking: 3) the transitivity of the precedence relationship (resulting into T-arcs), and 4) the NB-constraints (resulting into NB-arcs).
Algorithm 1 The Build-Easy-Arcs algorithm

Input: A directed MinMax-profile $F$ over $[n] \cup \{0, n + 1\}$.
Output: Either the answer “No” (meaning no permutation exists), or the pair $(G, \text{SilNB})$ where $G$ is the DAG containing all deducible $R-$, $N-$, $T-$ and $NB-$arcs, and $\text{SilNB}$ is the set of silent $NB$-constraints.

(Note: Arcs are added only if they do not create loops, nor multiple arcs with common source and target.)

1: $G \leftarrow ([n] \cup \{0, n + 1\}, \emptyset)$
2: for each $t \in [n] \cup \{0\}$ do
3: if $t \in \{m_{l}, M_{l}\}$, $t + 1 \in F$ then $tl \leftarrow t$; $tr \leftarrow t + 1$ else $tl \leftarrow t + 1$; $tr \leftarrow t$ end if
4: add the $R$-arc ($tl$, $tr$) to $G$
5: add the $B$-arcs ($tl$, $m_{t}$), ($m_{t}$, $tr$), ($tl$, $M_{t}$), ($M_{t}$, $tr$) to $G$ // according to (4)
6: end for
7: $\text{SilNB} \leftarrow$ the set of all $NB$-constraints $\lnot(t \xleftrightarrow{a} t + 1)$ deduced from $F$ // according to (2)
8: $G \leftarrow \text{Build-Closure}(G, \text{SilNB})$
9: remove from $\text{SilNB}$ all $NB$-constraints $\lnot(t \xleftrightarrow{a} t + 1)$ for which a setting is already found
10: if $G$ is not a DAG then
11: output “No”
12: else
13: output $(G, \text{SilNB})$
14: end if

Algorithm 2 The Build-Closure algorithm

Input: A simple directed graph $G$ with vertex set $[n] \cup \{0, n + 1\}$, a set $NBc$ of $NB$-constraints on $[n] \cup \{0, n + 1\}$.
Output: The $NB$-transitive closure of $G$ using the $NB$-constraints in $NBc$.

(Note: Arcs are added only if they do not create loops, nor multiple arcs with common source and target.)

1: while a $T$-arc or an $NB$-arc $(x, y)$ may be added do
2: add $(x, y)$ to $G$
3: end while
4: output($G$)

Algorithm 1 shows these steps. After the construction of the $R$- and $B$-arcs (steps 2-6), either transitivity or $NB$-constraints may be arbitrarily invoked to add supplementary arcs as long as possible, performing what we call the $NB$-transitive closure of $G$. This is done by the Build-Closure algorithm (Algorithm 2), called in step 8 of Algorithm 1. It is clear that in step 1 of the Build-Closure algorithm a $T$-arc $(x, y)$ may be added iff there is a vertex $c$ such that $(x, c)$ and $(c, y)$ are arcs, but $(x, y)$ is not an arc. The condition for adding the $NB$-arc $(x, y)$ is slightly more complex, as $(x, y)$ may be added iff

- either an $NB$-constraint $\lnot(y \xleftrightarrow{a} z)$ with $z \in \{y - 1, y + 1\}$ exists in $NBc$, and $(x, z)$ is an arc,
- or an $NB$-constraint $\lnot(x \xleftrightarrow{b} z)$ with $z \in \{x - 1, x + 1\}$ exists in $NBc$, and $(z, y)$ is an arc.

Clearly, this naive approach for $MinMax$-betweenness attempts to exploit all the $MinMax$-constraints. Unfortunately, for some $NB$-constraints $\lnot(t \xleftrightarrow{a} t + 1)$ Algorithm 1 may provide no setting (i.e. neither the arcs $(t, a)$ and $(t + 1, a)$, nor the arcs $(a, t)$ and $(a, t + 1)$ are present in $G$), as shown below. These constraints are called silent $NB$-constraints, and are returned by the algorithm together with $G$, if $G$ is a DAG (step 13).

Example 2. Let $F$ be defined on $[9] \cup \{0, 10\}$ by the following $MinMax$-constraints:

$$
\begin{array}{ccccccc}
0 & {\overset{[0,9]}{\rightarrow}} & 1 & {\overset{[1.9]}{\rightarrow}} & 2 & {\overset{[1.9]}{\rightarrow}} & 3 & {\overset{[1.9]}{\rightarrow}} & 4 & {\overset{[1.9]}{\rightarrow}} & 5 & {\overset{[4.7]}{\rightarrow}} & 6 & {\overset{[1.9]}{\rightarrow}} & 7 & {\overset{[1.9]}{\rightarrow}} & 8 & {\overset{[1.9]}{\rightarrow}} & 9 & {\overset{[1.10]}{\rightarrow}} & 10
\end{array}
$$

Figure shows the $R-$, $B-$ and $T-$ arcs used by Algorithm 1 to build the directed graph deduced from $F$. Vertices 0 and 10 are left apart in this figure, since the constraints they are involved in allow only to
Figure 1: Directed acyclic graph $G$ obtained from Algorithm 1 using the directed $\text{MinMax}$-profile $0 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 5 \leftrightarrow 6 \leftrightarrow 7 \leftrightarrow 8 \leftrightarrow 9 \leftrightarrow 10$. For simplicity reasons, vertices 0 and 10 and the arcs incident to at least one of them are omitted. The pairs of vertices $\{2, 4\}$, $\{2, 6\}$ and $\{2, 7\}$ are not arcs, they show the silent NB-constraint $\neg (6 \leftrightarrow 7)$ and the pair $\{2, 4\}$, that are related by coherent arc directions.

Our problem is now this one:

(P) Given $G$ and a set of silent NB-constraints, decide whether a setting is possible for each silent NB-constraint such that the graph resulting by transitive closure is a DAG.

Unfortunately, the following result shows the difficulty of the problem:

Claim 1. (1) Problem (P) is NP-complete even when the silent NB-constraints involve disjoint triples of vertices.

Notice however that the graph $G$ we obtain at the end of Algorithm 1 may have particular features (that we have not identified) making that we are dealing with a particular case of problem (P). Claim 1 shows therefore that our problem is potentially difficult, but does not prove its hardness.

Remark 2. From an algorithmic point of view, we may notice that with the output of Algorithm 1 we may easily find a parameterized algorithm for $\text{MinMax}$-$\text{BETWEENNESS}$. Given $G$ and $\text{SilNB}$, we have $O(2^{\lvert \text{SilNB} \rvert})$ possible settings to test, thus resulting into an FPT algorithm with parameter $s$ given by the number of silent NB-constraints.
3.2 Further analysis of arc propagation

With the aim of forcing the setting of some appropriately chosen silent NB-constraint, let us now analyze the impact of adding an arbitrary arc \((a_1, b_1)\) to \(G\), where \(a_1\) and \(b_1\) are non-adjacent vertices from \(G\). Denote \(G + (a_1, b_1)\) the graph obtained from \(G\) by adding the arc \((a_1, b_1)\), and let \(G^3\) be the NB-transitive closure of \((G + (a_1, b_1), \text{SilNB})\), i.e. the directed graph obtained by performing Build-Closure\((G + (a_1, b_1), \text{SilNB})\).

Several definitions are needed before going further. Given an NB-constraint \(\neg(a \leftrightarrow t + 1)\), the vertex \(a\) of \(G\) is called the top of the NB-constraint, whereas the pair \(\{t, t + 1\}\) is called the basis of the NB-constraint. An arc \((x, y)\) is new if it is an arc of \(G^3\) but not of \(G\), and is old if it is an arc of \(G\). New arcs are obtained using Build-Closure\((G + (a_1, b_1), \text{SilNB})\) according to a certain linear order, resulting from the arbitrary choices made in step 1. This order is denoted \(\alpha\), such that \((x_1, y_1)\alpha(x_2, y_2)\) means that \((x_1, y_1)\) is created by Build-Closure before \((x_2, y_2)\). Then, the following claim is simple:

**Claim 2.** For each new arc \((v, w)\), there exists a series of new arcs \(U := (v_1, w_1), (v_2, w_2), \ldots, (v_z, w_z)\) such that \((v_1, w_1) = (a_1, b_1), (v_z, w_z) = (v, w), (v_i, w_i)\alpha(v_{i+1}, w_{i+1})\) for all \(i\) with \(1 \leq i \leq z - 1\) and each arc \((v_{i+1}, w_{i+1})\), \(1 \leq i \leq z - 1\), is obtained from the preceding one \((v_i, w_i)\) using one of the following cases:

1. \(w_{i+1} = v_i\) and \((v_{i+1}, w_i)\) is either an old arc, or a new arc such that \((v_{i+1}, w_i)\alpha(v_{i+1}, w_{i+1})\); in this case \((v_{i+1}, w_{i+1})\) is a new \(T\)-arc.

2. \(w_i = v_{i+1}\) and \((v_i, v_{i+1})\) is the basis of an NB-constraint of \(\text{SilNB}\) with top \(v_i\); in this case \((v_i, v_{i+1})\) is a new \(N\)-arc.

3. \(v_i = v_{i+1}\) and \((v_i, v_{i+1})\) is either an old arc, or a new arc such that \((w_i, v_i)\alpha(v_{i+1}, w_{i+1})\); in this case \((v_i, v_{i+1})\) is a new \(T\)-arc.

4. \(v_i = v_{i+1}\) and \((v_i, v_{i+1})\) is the basis of an NB-constraint from \(\text{SilNB}\) with top \(v_i\); in this case \((v_i, v_{i+1})\) is a new \(N\)-arc.

**Proof.** In order to obtain \((v_{i+1}, w_{i+1})\), we need to apply either the transitivity (step 2 in Algorithm 2 for a \(T\)-arc, which gives cases 1 and 3), or an NB-constraint from \(\text{SilNB}\) (again step 2 in Algorithm 2 but for an \(N\)-arc, which gives cases 2 and 4).

The sequence \(U\) is called a setting sequence for \((v, w)\), whereas the index \(i\) of an arc \((v_i, w_i)\) is called its range in \(U\). From now on, the case in Claim 2 used to deduce one arc from the preceding one in a setting sequence is indicated between the two arcs.

**Example 3.** For the example in Figure 1 if \(a_1 = 2\) and \(b_1 = 7\), then \(U := (2, 7)\odot(2, 6)\odot(2, 4)\) is a setting sequence for \((2, 4)\) using case 4 followed by case 3 in Claim 2 to go from one arc to the next one.

Now, let \(a_1, a_2, \ldots, a_s\) (respectively \(b_1, b_2, \ldots, b_t\)) be the subsequence of \(v_1, \ldots, v_z\) (respectively of \(w_1, \ldots, w_z\)) obtained by replacing consecutive copies of the same vertex with only one copy of that vertex. Equivalently, if \((a_i, b_j)\) is an arc of \(U\), then the next arc is either \((a_{i+1}, \alpha b_j)(\) cases 1 and 2 in Claim 2) or \((a_i, b_{j+1})(\) cases 3 and 4 in Claim 2). Of course, we have \(a_1 = v_1, a_s = v_z = v, b_1 = w_1\) and \(b_t = w_z = w\).

**Example 4.** Consider \(P = \{0 \ 7 \ 4 \ 10 \ 2 \ 1 \ 12 \ 8 \ 3 \ 9 \ 5 \ 11 \ 6 \ 13\}\), and let \(F\) be the MinMax-profile of \(P\). For \(F\), apply Algorithm 1 to obtain the graph \(G\) and the set \(\text{SilNB}\). Then \(G\) - that the reader is invited to build it himself - is partitioned into three sets, respectively made of: the vertices preceding the pair \{1, 12\}, the pair \{1, 12\} (in this order, and with no intermediate vertex), and the vertices following the pair \{1, 12\}. The set \(\text{SilNB}\) is \(\{8 \odot 11, 9, 5 \odot 3 \odot 6\}\), and thus involves only vertices in the third set, which induces in \(G\) the subgraph \(G'\) with vertex set \(\{3, 5, 6, 8, 9, 11\}\) and arcs
Remark 3. Notice that we could possibly have \( a_i = a_l \), for distinct \( i, l \in \{1, 2, \ldots, s\} \), i.e., they correspond to the same vertex of \( G \). If two arcs with the same endpoint are set in distant steps of the setting process represented by \( U \). We could also possibly have \( a_i = b_j \) for some \( i, j \) if, for instance, \( a_1, \ldots, a_h \) (with \( h > i \)) are distinct, \( b_1, \ldots, b_{j-1} \) are distinct, \( (a_h, b_{j-1}) \) is a new arc and \( (b_{j-1}, a_i) \) is an old arc (making that the vertex \( b_j \) is equal to \( a_i \), and thus by transitivity - or case 3 in Claim\(^2\) - one sets \( (a_h, a_i) \)).

Remark 4. Also note that for every pair of arcs \( (a_i, b_j) \) and \( (a_p, b_q) \) from \( U \), we have either \( i \leq p \) and \( j \leq q \) (when \( (a_i, b_j) \alpha (a_p, b_q) \), or \( i \geq p \) and \( j \geq q \) (when \( (a_p, b_q) \alpha (a_i, b_j) \)). It is therefore understood, here and in the subsequent of the paper, that in case \( a_i = a_l \) for some \( i \neq l \), we make a clear difference between the arcs \( (a_i, b_j) \) of \( U \) and the arcs \( (a_i, b_j) \) of \( U \). These arcs are incident with the same vertex of \( G \) but this vertex is called \( a_i \) in the first case, and \( a_l \) in the second one.

Example 4 (cont’d). We have \( a_2 = a_4 = 5 \) and \( b_1 = b_4 = 9 \), but when we refer to the new arcs containing \( a_2 \) we only refer to the arc \( (5, 3) \) and when we refer to the new arcs containing \( a_4 \) we only refer to the arc \( (5, 9) \). Similarly, when we refer to the new arcs incident with \( b_1 \) we refer only to the arc \( (11, 9) \) whereas when we refer to those incident with \( b_4 \) we mean the arcs \( (6, 9) \) and \( (5, 9) \).

In order to represent arc propagation, we need to look closer to the partial subgraph \( H_U \) of \( G^1 \) given by the set of distinct vertices used in the setting sequence \( U \), the arcs in \( U \) and the arcs used to deduce each arc of \( U \) from the previous one, using Claim\(^2\). The graph \( H_U \) is defined as:

\[
V(H_U) = \{a_i \in V(G) \mid 1 \leq i \leq s\} \cup \{b_j \in V(G) \mid 1 \leq j \leq r\}
\]

\[
E(H_U) = U \cup \{(a_{i+1}, a_i) \mid \exists b_j : (a_{i+1}, b_j) \text{ is deduced from } (a_i, b_j) \text{ in } U \text{ using case 1}\}
\cup \{(a_i, a_{i+1}) \mid \exists b_j : (a_i, b_j) \text{ is deduced from } (a_{i+1}, b_j) \text{ in } U \text{ using case 2}\}
\cup \{(b_{j+1}, b_j) \mid \exists a_i : (a_i, b_{j+1}) \text{ is deduced from } (a_i, b_j) \text{ in } U \text{ using case 3}\}
\cup \{(b_j, b_{j+1}) \mid \exists a_i : (a_i, b_{j+1}) \text{ is deduced from } (a_i, b_j) \text{ in } U \text{ using case 4}\}
\]

The graph \( H_U \) is called the setting path associated with \( U \) (or, alternatively, a setting path for \( (a_s, b_r) \)). Notice that case 2 (respectively case 4) in Claim\(^2\) may be included in case 1 (respectively case 3).
when the basis is the arc $(a_{i+1}, a_i)$ (arc $(b_i, b_{i+1})$ respectively). The definition of $H_U$ keeps as case 2 (respectively case 4) only the configuration not included in case 1 (respectively case 3). See Figure 2b).

Claim 2 and the definition of $H_U$ allow us to have a basis for future analysis, but also show us that the choice of one arc $(a_1, b_1)$ has effects that are difficult to measure accurately. The NP-completeness of the problem $(P)$ (see Claim 1) comes from this complex arc propagation, which makes that different setting sequences with the same initial arc may lead to conflicts, i.e. to circuits.

**Example 5.** In Figure 3 we present a configuration (which is a subgraph of $G$) showing that not each possible setting is a correct setting, since imposing the existence of one arc $(a_1, b_1)$ may induce circuits in the graph $G^1$. In this configuration, setting the arc $(a_1, b_1)$ implies the additional arcs $(a_3, b_2)$ and $(a'_3, b'_2)$, and thus the construction of a circuit. A $MinMax$-profile inducing such a configuration in the associated DAG $G$ is the following one (where $a_1 = 18, b_1 = 12, a_2 = 22, a_3 = 21, a'_2 = 16, a'_3 = 15, b_2 = 25, b'_2 = 8$):

\[
\begin{align*}
0 &\rightarrow [0,27] & 1 &\rightarrow [1,29] & 2 &\rightarrow [1,29] & 3 &\rightarrow [1,29] & 4 &\rightarrow [1,29] & 5 &\rightarrow [1,29] \\
10 &\rightarrow [1,29] & 11 &\rightarrow [1,29] & 12 &\rightarrow [8,25] & 13 &\rightarrow [1,29] & 14 &\rightarrow [1,29] & 15 &\rightarrow [1,29] \\
18 &\rightarrow [1,29] & 19 &\rightarrow [1,29] & 20 &\rightarrow [1,29] & 21 &\rightarrow [5,22] & 22 &\rightarrow [1,29] & 23 &\rightarrow [1,29] \\
24 &\rightarrow [15,25] & 25 &\rightarrow [1,29] & 26 &\rightarrow [1,29] \\
\end{align*}
\]

In this example, the $MinMax$-constraints in bold define the arcs needed to obtain the configuration in Figure 3 and some additional arcs. The elements involved in these $MinMax$-constraints are, in every permutation with this $MinMax$-profile, on the left of 1 and 29 (the minimum and maximum elements), which are neighbors and in this order on each permutation. The remaining of the elements are on the right of 1 and 29, and are intended to complete the set $\{1, 2, \ldots, 30\}$ without any participation to the configuration.

In order to find polynomial particular cases, we need to be able to control the form of the setting paths, and this is what we do in the subsequent. To this end, notice that:

**Remark 5.** The vertices 0 and $n + 1$ belong to no setting path. Indeed, according to Remark 1 it is assumed that they are definitely located at places 0 and $n + 1$ respectively, and thus their relative positions with respect to any other element are known. No arc incident to any of them may thus be added, as would be the case if they belonged to some setting path.
4 Polynomial case for Directed MinMax-Betweenness

Say that a MinMax-profile \( F \) on \([n] \cup \{0, n + 1\}\) is linear if the inclusion between sets defines a linear order on the intervals \([m_t..M_t] \), \(1 \leq t \leq n - 1\), where the notation \((a..b)\) denotes the set of integers \(x\) with \(a \leq x \leq b\). We show in this section that the problem Directed MinMax-Betweenness is polynomial for linear MinMax-profiles.

Given \( c \in [n] \), let \( NB(c) = \{t \mid 1 \leq t \leq n - 1, c < m_t \text{ or } M_t < c\} \). In other words, \( NB(c) \) is the set of values \( t \) such that \( \{t, t + 1\} \) is the basis of an NB-constraint with top \( c\).

**Claim 3.** Let \( F \) be a linear profile on \([n] \cup \{0, n + 1\}\). Then the inclusion between sets defines a linear order denoted \( \prec \) on the sets \( NB(c), 1 \leq c \leq n\).

**Proof.** By contradiction, assume that \( c_1 \) and \( c_2 \) exist such that \( NB(c_1) \setminus NB(c_2) \) contains \( t_1 \) and \( NB(c_2) \setminus NB(c_1) \) contains \( t_2 \). Then \( t_1 \neq t_2\).

In the case where \( c_1 < m_{t_1} \) and \( c_2 < m_{t_2} \), assume w.l.o.g. that \( c_1 < c_2 \). Then \( c_1 < m_{t_2} \) and thus \( t_2 \in NB(c_1) \), a contradiction. The case where \( c_1 > M_{t_1} \) and \( c_2 > M_{t_2} \) is similar.

In the case where \( c_1 < m_{t_1} \) and \( c_2 > M_{t_2} \), recall that by hypothesis \( F \) is linear, and thus either \([m_{t_1}..M_{t_1}] \subseteq [m_{t_2}..M_{t_2}] \) or vice-versa. If \([m_{t_1}..M_{t_1}] \subseteq [m_{t_2}..M_{t_2}] \), then \( m_{t_1} < M_{t_1} \leq M_{t_2} \) and with \( c_2 > M_{t_2} \) we deduce that \( t_1 \in NB(c_2) \), a contradiction. If \([m_{t_2}..M_{t_2}] \subseteq [m_{t_1}..M_{t_1}] \), then \( m_{t_2} \leq M_{t_2} < M_{t_1} \) and with \( c_1 < m_{t_1} \) we deduce that \( t_2 \in NB(c_1) \), a contradiction. \( \blacksquare \)

Now, assume Algorithm [**1**] has been applied for \( F \), and let \((G, SilNB)\) be its output, assuming \( G \) is a DAG. To finish the algorithm for \( F \), we apply Algorithm [**3**]. The following claim is easy but very useful.

**Claim 4.** The vertex \( b_1 \) chosen in Algorithm [**3**] has the following properties:

(a) \( NB(b_1) \) is maximum with respect to the linear order \( \prec \) on the set \( \{NB(c) \mid 1 \leq c \leq n \text{ and } c \text{ is the top of at least one constraint from } SilNB\} \).

(b) \( b_1 \) does not belong to a basis, but is a top for all the basis defining constraints from \( SilNB\).

**Proof.** The first affirmation is clear by the choice of \( b_1 \) in step 3 of the algorithm and Claim [**3**]. The second affirmation is deduced by contradiction. If \( b_1 \) belonged to a basis \( \{b_1, b_1 + 1\} \) or \( \{b_1 - 1, b_1\} \) with top \( c \), then we would have \( NB(c) \not\subseteq NB(b_1) \) since the basis \( \{b_1, b_1 + 1\} \) or \( \{b_1 - 1, b_1\} \) cannot have top \( b_1 \) (the vertices of a basis are by definition distinct from its top). The second part of affirmation (b) results directly from affirmation (a). \( \blacksquare \)

In the next claims, we show the correctness of our algorithm. To this end, each arc \((a_i, b_j)\) of \( U \) (and thus of \( H_U \)) is called a local new arc with respect to \( U \), in order to make the difference with the arcs from \( G^1 \) which are new but do not belong to \( U \), termed non-local new arcs. Similarly, a vertex \( a_i \) of \( H_U \) is a local top if there exists \( b_q \) such that \( \neg(b_q \xrightarrow{a} b_{q+1}) \) is an NB-constraint used by \( U \), i.e. one of the arcs \((a_i, b_q)\) and \((a_i, b_{q+1})\) is deduced from the other in \( H_U \), using case 4. The pair \( \{b_q, b_{q+1}\} \) is in this case a local basis. The symmetric definitions hold for a vertex \( b_j \) (instead of \( a_i \)). Note that a local basis has a unique local top, by Remark [**4**].

For any vertex \( a_i \), we also denote \( first_U(i) \) the minimum \( u \) with \( 1 \leq u \leq r \) such that \((a_i, b_u)\) belongs to \( U \).

**Claim 5.** Let \( F \) be a directed linear profile on \([n] \cup \{0, n + 1\}\) and let \( a_1 \) and \( b_1 \) be chosen as in Algorithm [**3**] Then the following affirmations hold:

(a) Let \((v, w)\) be a new arc of \( G^1 \) and let \( U \) be setting sequence for \((v, w)\) with arc sources \( a_1, a_2, \ldots, a_s \) and arc targets \( b_1, b_2, \ldots, b_s \). Then there is no old arc \((b_i, a_i)\) in \( G^1 \), with \( 1 < i \leq s \).

(b) All arcs \((b_1, x)\) of \( G^1 \) are old.
Algorithm 3 The Linear-Profile algorithm

Input: The output \((G, \text{SilNB})\) of Algorithm 1 for a directed linear \(\text{MinMax}\)-profile \(F\) on \([n] \cup \{0, n+1\}\).
Output: A permutation \(F\) with the \(\text{MinMax}\)-profile \(F\).

1: while \(\text{SilNB} \neq \emptyset\) do
2: \(C \leftarrow \{c \in [n] \mid c\text{ is the top of at least one constraint from SilNB}\}\)
3: Choose \(b_1 \in C\) s.t. \(|NB(b_1)| = \max\{|NB(c)| \mid c \in C\}\)
4: Choose \(a_1\) such that \(\neg(a_1 \xrightarrow{b_1} a_1 + 1) \in \text{SilNB}\).
5: \(G \leftarrow \text{Build-Closure}(G + (a_1, b_1), \text{SilNB})\)
6: end while
7: \(P \leftarrow \text{topologically sort } G\)
8: Output\((P)\)

Proof. To prove (a), we assume by contradiction that the affirmation is false, and choose \((v, w), U\) and \((b_1, a_i)\) such that the arc \((a_i, b_{\text{first}_U(i)})\) is the smallest with respect to the order \(\alpha\). Several cases occur.

i) If \(\{a_{i-1}, a_i\}\) is a local basis (case 2 in the definition of \(H_U\)), then \(b_1\) is also a top of it (by Claim 4(b) and thus from the old arc \((b_1, a_i)\) we deduce the existence of the old arc \((b_1, a_{i-1})\) (computed by the call of Build-Closure in step 8 of Algorithm 1). But then the choice of \((b_1, a_i)\) is contradicted, since \((a_{i-1}, b_{\text{first}_U(i-1)})\alpha(a_i, b_{\text{first}_U(i)})\).

ii) If \((a_i, a_{i-1})\) is an old arc (case 1 in the definition of \(H_U\), with an old arc), then \((b_1, a_{i-1})\) is also an old arc, computed by the call of Build-Closure in step 8 of Algorithm 1. As before, the choice of \((b_1, a_i)\) is contradicted.

iii) If \((a_i, a_{i-1})\) is a new local arc (case 1 in the definition of \(H_U\), with a local new arc), then this arc belongs to \(U\) and was built before \((a_i, b_{\text{first}_U(i)})\) since it must be built before its use. Then there exist \(p, q\) with \(1 \leq p < i - 1\) and \(1 \leq q \leq \text{first}_U(i)\) such that \((a_p, b_q)\) and \((a_i, a_{i-1})\) are the same arc, but with different notations due to its multiple use in \(H\) (see Remark 3). In particular, \(a_p\) and \(a_i\) are the same vertex of \(H_U\), and thus \((b_1, a_p)\) is an old arc of \(H_U\), with \(p < i\). Once again, the choice of \((b_1, a_i)\) is contradicted, since \((a_p, b_{\text{first}_U(p)})\alpha(a_i, b_{\text{first}_U(i)})\).

iv) Finally, if \((a_i, a_{i-1})\) is a non-local new arc (case 1 in the definition of \(H_U\), with a non-local new arc), then it was built before \((a_i, b_{\text{first}_U(i)})\). Consequently, there exists a setting sequence \(T\) for \((a_i, a_{i-1})\) with arc sources \(c_1 = a_1, c_2, \ldots, c_g = a_i\) and arc targets \(d_1 = b_1, d_2, \ldots, d_h = a_{i-1}\). In this setting sequence, we have that \((b_1, c_g)\) is an old arc, and \((c_g, d_h) = (a_i, a_{i-1})\). Then, \((c_g, d_{\text{first}_T(g)})\alpha(a_i, a_{i-1})\alpha(a_i, b_{\text{first}_U(i)})\), contradicting again the choice of \(U\) and \((b_1, a_i)\).

To prove (b), assume by contradiction that some arcs \((b_1, x)\) are created by Build-Closure\((G + \{a_1, b_1\}, \text{SilNB})\), and let \((b_1, x_1)\) be the smallest of them according to the order \(\alpha\). Then in a setting sequence \(U\) for \((b_1, x_1)\) with arc sources \(a_1, a_2, \ldots, a_s\) and arc targets \(b_1, b_2, \ldots, b_r\), we have \((b_1, x_1) = (a_p, b_q)\) for some \(p, q\) with \(1 < p \leq s\) and \(1 < q \leq r\). Then the pair \(\{a_{p-1}, a_p\}\) is not a basis since \(a_p = b_1\) and by Claim 4(b), \(b_1\) belongs to no basis. Then, \((a_p, a_{p-1})\) is an arc. This arc cannot be old, since then recalling that \(a_p = b_1\) we have that \((b_1, a_{p-1})\) is an old arc thus contradicting affirmation (a). Then \((a_p, a_{p-1})\) must be a new arc. Now, we have by case 1 in Claim 2 that \((a_p, a_{p-1})\alpha(a_p, b_{\text{first}_U(p)})\alpha(a_p, b_q)\). Since \((a_p, a_{p-1}) = (b_1, x_1)\) and \((a_p, b_q) = (b_1, x_1)\) we deduce that \((b_1, a_{p-1})\alpha(b_1, x_1),\) thus contradicting the choice of \((b_1, x_1)\). Say that a setting sequence \(U\) for \((v, u)\) with arc sources \(a_1, a_2, \ldots, a_s\) and arc targets \(b_1, b_2, \ldots, b_r\) is canonical if \(H\) has the following properties:

\((a)\) \(b_1\) and (if it exists) \(b_2\) are distinct from \(a_i\), \(1 \leq i \leq s\), and \((b_1, b_2)\) is an old arc.
(b) \( r \leq 2 \).

(c) \((a_i, b_1) \in U\), for all \( i \) with \( 1 \leq i \leq s \).

**Claim 6.** Let \( F \) be a directed linear profile on \([n] \cup \{0, n + 1\}\) and let \( a_1 \) and \( b_1 \) be chosen as in Algorithm 3. Let \((v, w)\) be a new arc of \( G^1 \). Then, for each setting sequence \( U \) for \((v, w)\) with arc sources \( a_1, a_2, \ldots, a_s \) and arc targets \( b_1, b_2, \ldots, b_r \), there is a canonical setting sequence \( U^0 \) for \((v, w)\) with arc sources \( a_1, a_2, \ldots, a_s \) and arc targets \( b_1, b_2 \), and (whenever \( b_1 \neq b_r \)) \( b'_{r-1} = b_r \).

**Proof.** The proof is by induction on the range \( k \) of \((v, w)\) (or, equivalently, of \((a_s, b_r)\)) in a setting sequence \( U \) for \((v, w)\). Recall that the arc with range 1 is \((a_1, b_1)\).

In the case \( k = 2 \), we have either \( r = 1 \) (when cases 1 or 2 in Claim 2 are used to obtain the second arc), or \( s = 1 \) (when cases 3 or 4 are used). When \( r = 1 \) we are already done. When \( s = 1 \), by Claim 5(b) we know that \((b_1, b_2)\) is an old arc, and we are done.

In the general case, assume by inductive hypothesis that the claim holds for all arcs with range less than \( k \) in some setting sequence, and that the range of \((v, w)\) (or, equivalently, of \((a_s, b_r)\)) in \( U \) is \( k \). We have two cases.

**Case A.** The arc preceding \((a_s, b_r)\) in \( U \) is \((a_s, b_{r-1})\). By inductive hypothesis, for \((a_s, b_{r-1})\) there is a canonical setting sequence \( U^0 := \{(a_1, b_1), (a_2, b_1), \ldots, (a_s, b_1)\} \) and (if \( b_{r-1} \neq b_1 \)) \((a_s, b'_2)\), meaning that \( b'_{r-1} = b_{r-1} \) when \( b'_2 \) exists, and \( b_1 = b_{r-1} \) when \( b'_2 \) does not exist. We have two (sub)cases:

**A.1.** When \( b'_2 \) exists, we have that \( U' := U^0 \cup \{(a_s, b_r)\} \) (this is concatenation) is a setting sequence for \((a_s, b_r)\), in which \((a_s, b_r)\) is obtained from \((a_s, b'_2)\) using the same case of Claim 2 as used in \( U \). Notice that the case 3 with a new arc \((b'_2, b_r)\) cannot appear, since then in any setting sequence \( U' \) for \((b'_2, b_r)\) with arc sources \( c_x \) and arc targets \( d_y \), we have that \((b'_2, b_r) = (c_i, d_j)\) for some \( i \) and \( j \), implying that \((b_1, c_i)\) is an old arc (as \( c_i = b'_2\), a contradiction with Claim 5(a)). Then only case 3 with an old arc, and case 4 may occur. Both cases imply that \((b_1, b_r)\) is an old arc, as follows. In case 3 with an old arc \((b'_2, b_r)\), the transitivity using the old arc \((b'_2, b_r)\) implies indeed the construction of \((b_1, b_r)\) in step 8 of Algorithm 1. If \( b'_2, b_r \) is a local basis (i.e. case 4 is used), we deduce that \( b_1 \) is a top for it, by Claim 4(b). Now, since \((b_1, b'_2)\) is an old arc by inductive hypothesis, we deduce that \((b_1, b_r)\) is also an old arc obtained from the NB-constraint with top \( b_1 \) and basis \( \{b'_2, b_r\} \). Thus \( b_1, b_r \) is an old arc in all cases. Then \( U^2 = \{(a_1, b_1), \ldots, (a_s, b_1)\} \) is a setting sequence for \((a_s, b_r)\), which is canonical if we ensure that \( b_r \) is distinct from all \( a_i \), \( 1 \leq i \leq s \). This is guaranteed by Claim 5(a).

**A.2.** When \( b'_2 \) does not exist, we have that \( U' := U^0 \cup \{(a_s, b_r)\} \) is a canonical setting sequence for \((a_s, b_r)\). Indeed, as \( b_1 = b_{r-1} \) we know that \((a_s, b_r)\) is obtained from \((a_s, b_1)\) using case 3 or 4 in Claim 2. Moreover, by Claim 4(b), \( b_1 \) belongs to no basis, thus \((b_1, b'_2)\) is an old or new arc. But the latter possibility is forbidden by Claim 5(b).

**Case B.** The arc preceding \((a_s, b_r)\) in \( U \) is \((a_{s-1}, b_r)\). By inductive hypothesis, for \((a_{s-1}, b_r)\) there is a canonical setting sequence \( U^0 := \{(a_1, b_1), (a_2, b_1), \ldots, (a_{s-1}, b_1)\} \) and (if \( b_r \neq b_1 \)) \((a_{s-1}, b'_2)\), meaning that \( b'_2 = b_r \) when \( b'_2 \) exists, and \( b_1 = b_r \) when \( b'_2 \) does not exist. We have two (sub)cases:

**B.1.** When \( b'_2 \) exists, we show that the sequence \( U^1 := \{(a_1, b_1), \ldots, (a_{s-1}, b_1), (a_s, b_1), (a_s, b'_2)\} \) is the sought canonical sequence. Clearly, \((a_{s-1}, b_1)\) is obtained from \((a_1, b_1)\) using the setting sequence \( U^0 \) from which \((a_{s-1}, b_1)\) is useless in this case. Also, \((a_s, b'_2)\) is obtained from \((a_s, b_1)\) and \((b_1, b'_2)\) by transitivity (case 3 in Claim 2). It remains to show that \((a_s, b_1)\) is deduced from \((a_{s-1}, b_1)\) and \((a_{s-1}, a_s)\) in \( U \). \((a_{s-1}, a_s)\) is used to deduce \((a_s, b_1)\) from \((a_{s-1}, b_1)\), using either case 1 or case 2 in Claim 2. If case 1 is used, then \((a_s, a_{s-1})\) is an arc (new or old), and it allows to deduce \((a_s, b_1)\) from \((a_{s-1}, b_1)\) using the transitivity. If case 2 is used, then \((a_{s-1}, a_s)\) is a local basis, thus \( b_1 \) is a top of it. The resulting NB-constraint allows to deduce \((a_s, b_1)\) from \((a_{s-1}, b_1)\) in this case too.
B.2 When \( b'_r \) does not exist, we have that \( b_r = b_1 \) and \( U^1 := U^0(a_s, b_1) \) is a canonical setting sequence for \((a_s, b_r)\).

**Claim 7.** Let \( F \) be a directed linear profile on \([n] \cup \{0, n + 1\}\). Then the NB-transitive closure \( G^1 \) obtained in step 5 of Algorithm 3 when \( b_1 \) (respectively \( a_1 \)) are chosen as in step 3 (respectively step 4) has no circuit.

**Proof.** Assume a circuit \( d_1, d_2, \ldots, d_c, c \geq 2, \) is created in \( G^1 \). Because of the transitive closure, a shortest such circuit has length 2. Let then \( d_1, d_2 \) form a 2-circuit and assume that (at least) \((d_1, d_2)\) is a new arc. Then, according to Claim 6, there exists a canonical setting path with vertices \( a_1, \ldots, a_s(= d_1) \) and \( b_1, \ldots, b_r(= d_2) \) \((r \in \{1, 2\})\). Consequently \((d_2, d_1)\) cannot be an old arc, since then in \( G \) either we have directly that \((b_1, d_1)\) is an old arc (when \( r = 1 \) and thus \( d_2 = d_1 \)) or the transitivity guarantees the same conclusion when \( r = 2 \). But this yields a contradiction with Claim 5(a).

We deduce that \((d_2, d_1)\) is a new arc, implying again the existence of a canonical setting path with vertices \( a'_1, \ldots, a'_r(= d_2) \) and \( b'_1, \ldots, b'_r(= d_1) \) \((r' \in \{1, 2\})\). But \( b'_1 = b_1 \) and \( a'_1 = a_1 \). Consequently we have either that \( b_1 = d_1 \) (when \( r' = 1 \)) or that \((b_1, d_1)\) is an old arc (when \( r' = 2 \)). In the former case we have a contradiction with affirmation (a) in the definition of a canonical setting path since \( d_1 = a_s = b_1 \). In the latter case, we have again a contradiction with Claim 5(a). \( \square \)

We are now ready to prove the main theorem:

**Theorem 1.** **Directed MinMax-Betweenness is polynomial for linear MinMax-profiles.**

**Proof.** Given a linear MinMax-profile \( F \), we first apply Algorithm 1 and, if it returns a pair \((G, SilNB)\), we apply Algorithm 3. To show the correctness of the algorithm, we show the answer is "No" iff there is no permutation whose MinMax-profile is \( F \).

If the answer is "No", then Algorithm 1 returns that \( G \) is not a DAG, which occurs iff some MinMax-constraints from \( F \) cannot be simultaneously satisfied. Thus, there is no permutation with MinMax-profile \( F \).

Now, assume there is no permutation whose MinMax-profile is \( F \), and suppose by contradiction that the algorithm returns a permutation \( P \). We show that \( P \) satisfies all the MinMax-constraints in \( F \), yielding a contradiction with the hypothesis. The permutation \( P \) is output at the end of Algorithm 3 showing that Algorithm 1 finishes with a DAG \( G \). Then, in Algorithm 2, every execution of the while loop in steps 1-6 satisfies at least one silent NB-constraint and, by Claim 7, creates no circuit. Therefore, the pair \((G, SilNB)\) obtained at the end of each execution consists again in a DAG \( G \) with \( R-, B-, T-\) and NB-arcs, and a set \( SilNB \) with smaller size than the previous one. Thus the while loop ends when \( SilNB = \emptyset \) and yields a DAG \( G \) that satisfies all the constraints imposed by the MinMax-profile \( F \). Any topological order of \( G \), including \( P \), is thus a permutation with MinMax-profile \( F \). The hypothesis that there is no permutation with MinMax-profile \( F \) is thus contradicted.

The execution time of the algorithm is clearly dominated by the \(|SilNB|\) computations of the NB-transitivity closure in step 5 of Algorithm 3. Now, the number of NB-constraints in \( SilNB \) is in \( O(n^2) \) (we have at most one NB-constraint \( \neg(t \leftrightarrow t + 1) \) for each \( t \) and each \( a \) and the NB-transitivity closure is clearly performed in polynomial time, thus the resulting algorithm is polynomial. \( \square \)

## 5 Generalizations

In this section, we generalize the definition of MinMax-profiles so as to allow them to carry different amounts of information, depending on an integer parameter \( k \).

**Definition 3.** Let \( k \) be a positive integer with \( 1 \leq k \leq n + 1 \). The \( k \)-profile of a permutation \( P \) on \([n] \cup \{0, n + 1\}\) is the set of \( k \)-constraints

\[
M_k(P) = \bigcup_{i=1}^{k} \{t \in [\min_{t+1, \max_{t+1}}] \mid t + i \leq n + 1 \}
\]
where \( \min_{t,i} \) \( \max_{t,i} \) (\( \max_{t,i} \) \( \max_{t,i} \) respectively) is the minimum (maximum respectively) value in the interval delimited on \( P \) by the element \( t \) (included) and the element \( t + i \) (included).

Note that \( \text{MinMax} \) as defined in Section\(^2 \) are the \( I \)-profiles. A \( k \)-profile is thus a \( \text{MinMax} \)-profile augmented with longer-range information of the same nature as the \( \text{MinMax} \)-profile itself, for pairs \( \{t, t + i\} \) with \( i \) at most equal to \( k \). Then all the definitions related to \( \text{MinMax} \)-profiles generalize to \( k \)-profiles in an obvious way, allowing us to state the following variant of the \( \text{MinMax} \)-Betweenness Problem:

\[ \text{(Directed or not) } k-\text{MinMax Betweenness} \]

**Input:** A positive integer \( n \), a (directed or not) \( k \)-profile \( F_k \) on \([n] \cup \{0, n + 1\} \).

**Question:** Is there a permutation \( P \) on \([n] \cup \{0, n + 1\} \) whose \( k \)-profile is \( F_k \)?

Similarly to the \( \text{MinMax} \)-Betweenness problem, the \( k-\text{MinMax Betweenness} \) problem provides a \( k \)-profile, which imposes \( B \)-constraints and \( \text{NB} \)-constraints for the permutations associated with it, if any. The existence of at least one permutation raises the question of its uniqueness, allowing to know whether the permutation is characterized by its \( k \)-profile or not. More formally, we state the two following problems:

\[ \text{(Directed or not) } \text{MinMax-Reconstruction} \]

**Input:** A positive integer \( n \).

**Requires:** Find the minimum value of \( k \) such that (directed or not) \( k-\text{MinMax Betweenness} \) has at most one solution, for each possible \( k \)-profile \( F_k \) on \([n] \cup \{0, n + 1\} \).

\[ \text{(Directed or not) } \text{Unique k-MinMax Betweenness} \]

**Input:** A positive integer \( n \), a (directed or not) \( k \)-profile \( F_k \) on \([n] \cup \{0, n + 1\} \).

**Requires:** Decide whether \( F_k \) is the \( k \)-profile of a unique permutation on \([n] \cup \{0, n + 1\} \), or not. In the positive case, find the unique permutation associated with \( F_k \).

Problems \( k-\text{MinMax Betweenness} \) and \( \text{Unique k-MinMax Betweenness} \) are clearly related, but do not allow easy deductions in one sense or the other. For instance, even if we have a solution for the \( \text{Directed MinMax-Betweenness} \) in the case of a linear profile (see Section\(^4 \)), we know nothing about the uniqueness of the permutation \( P \) the algorithm outputs (when such a permutation exists).

In the subsequent, we solve the \( \text{MinMax-Reconstruction} \) problem in the undirected case, and give a lower bound for the directed case. We assume \( \text{wlog} \) that the \( k \)-profiles we use are compatible with the assumption that \( 0 \) and \( n + 1 \) are respectively the leftmost and the rightmost element in the permutations we are dealing with. Then we prove the following result:

**Theorem 2.** The minimum \( k \) in (directed or not) \( \text{MinMax-Reconstruction} \) satisfies:

\[ (a) \quad k = \max\{1, n - 3\} \text{ in } \text{MinMax-Reconstruction}. \]

\[ (b) \quad k \geq \left\lceil \frac{n-3}{2} \right\rceil \text{ in } \text{Directed MinMax-Reconstruction}, \text{ for } n \geq 4. \text{ For } n = 1, 2, 3, \text{ we have } k = 1. \]

The proof is based on the following claim.

**Claim 8.** Let \( n \) be a positive integer, and \( F_k \) be a (directed or not) \( k \)-profile on \([n] \cup \{0, n + 1\} \) \( 1 \leq k \leq n + 1 \). Then:

1. In all the permutations whose \( k \)-profile is \( F_k \) (if any), the elements \( 1 \) and \( n \) have precisely the same positions, denoted \( q_1 \) and \( q_n \).
2. If \( l = \min\{q_1, q_n\} \) and \( r = \max\{q_1, q_n\} \), then the sets \( X, Y \) and \( Z \) of elements situated respectively between the positions 0 and \( l \) (for \( X \)), \( l \) and \( r \) (for \( Y \)), \( r \) and \( n+1 \) (for \( Z \)) are the same over all the permutations with \( k \)-profile \( F_k \) (if any).

**Proof.** Assuming at least one permutation corresponding to \( F_k \) exists, let \( P \) be such a permutation. Denote \( q_1 \) the position of 1 on \( P \) and successively consider the B-constraints

\[
\begin{align*}
    n &\gets m_{n,n+1}^{-1} n + 1, n - 1 \gets m_{n-1,n}^{-1} n, \ldots, 2 \gets m_{2,3}^{-1} 3.
\end{align*}
\]

The first B-constraint places \( n \) on the left of 1 iff \( m_{n,n+1} = 1 \), the second one places \( n-1 \) on the opposite side of 1 with respect to \( n \) iff \( m_{n-1,n} = 1 \) and so on. Each element in \( \{n, n-1, \ldots, 2\} \) is deterministically placed on the left or on the right of 1 depending only on those B-constraints. As a consequence, 1 is at the same place \( q_1 \) in all permutations corresponding to \( F_k \).

A similar reasoning may be done with the element \( n \) and the B-constraints:

\[
0 \gets M_0^{-1} 1, 1 \gets M_1^{-2} 2, \ldots, n-2 \gets M_{n-2,n-1}^{-1} n-1.
\]

We similarly deduce that \( n \) is at the same place \( q_n \) in all permutations corresponding to \( F_k \), and the sets of elements situated respectively on its left and right are the same in all permutations.

Putting together the previous deductions, whatever the order of \( q_1 \) and \( q_n \), we have that - on the one hand - \( X \cup Y \) and \( Z \) are identical in all permutations, and - on the other hand - \( X \) and \( Y \cup Z \) are identical in all permutations. The conclusion follows. \( \blacksquare \)

**Proof of Theorem**. We now prove affirmations (a) and (b).

**Proof of affirmation (a).** For \( n = 1 \), it is trivial. When \( n \in \{1, 2, 3\} \) it is easy to prove, using Claim \( \text{8} \) that the 1-profile guarantees the uniqueness of the associated permutation. When \( n \geq 4 \), assume by contradiction that \( k < n - 3 \) and let \( P \) be a permutation on \( [n] \cup \{0, n+1\} \) whose elements in positions 1 to 4 are \( p_1 = k + 2, p_2 = k + 3, p_3 = 1 \) and \( p_4 = n \). Let \( F_k \) be the \( k \)-profile of \( P \). According to Claim \( \text{8} \) the elements 1 and \( n \) are situated respectively at positions 3 and 4 in all permutations associated with \( F_k \), and positions 1 and 2 are occupied (whatever the order) by the elements \( k+2 \) and \( k+3 \). Now, in \( F_k \) the \( k \)-constraints involving one of the elements \( k+2 \) and \( k+3 \) and another element following 1 on its right are useless for fixing the places of \( k+2 \) and \( k+3 \) since these constraints have the minimum and maximum element 1 and \( n \). The only possibly useful \( k \)-constraints are those involving \( 0, 1, k+2 \) and \( k+3 \), but these integers have pairwise difference larger than \( k \) except for \( k+2 \) and \( k+3 \). Now, \( k+2 \) and \( k+3 \) are involved in the 1-constraint \( k+2 [k+2,k+3] k+3 \), which does not fix them on the places 1 and 2 of the permutation. Thus, there are at least two permutations with \( k \)-profile \( F_k \), a contradiction. We thus have \( k \geq n - 3 \).

We now show that if \( k = n - 3 \), then there is at most one permutation on \( [n] \) whose \( k \)-profile is \( F_k \). This is shown by induction on \( n \).

When \( n = 4 \) and \( k = 1 \), Claim \( \text{8} \) guarantees that, if at least one permutation with the given 1-profile exists, then 1 and 4 have fixed places, and 2 (respectively 3) is located in the same set among \( X, Y, Z \) in all suitable permutations. If 2 and 3 are in different sets, then the uniqueness is guaranteed. Otherwise, either 2 and 3 are in a set delimited by the position of 1, and then the constraint \( 1 \overset{[m_{1,2},M_{1,2}]}{\Rightarrow} 2 \) allows to deduce whether 3 separates 1 and 2 or not (thus fixing the positions of 2 and 3), or they are in a set delimited by \( n (= 4) \), and then the constraint \( 3 \overset{[m_{3,4},M_{3,4}]}{\Rightarrow} 4 \) allows to deduce whether 2 separates 3 and 4 or not. In all cases, all the elements are located at fixed places, thus the permutation associated with the 1-profile is unique.

Assume now, by inductive hypothesis, that for all \( n' < n \), a \( (n'-3) \)-profile either has no associated permutation, or has exactly one. Let now \( F_{n-3} \) be a \( (n-3) \)-profile for permutations on \( [n] \cup \{0, n+1\} \), and let \( q_1, q_n, l, r, X, Y, Z \) be defined according to Claim \( \text{8} \) assuming at least one permutation exists. Denote \( P \) any of these permutations, extended with 0 and \( n+1 \). Let \( W = X \cup \{0, n\} \), if \( q_n < q_1 \), and \( W = X \cup Y \cup \{0, 1, n\} \), otherwise. We show that:
Indeed, if in the initial $P_n$ (such that by the cardinality of $W_n$ one hand, $P$ are respectively the minimum and maximum element in $P[0..n]$). Now, renumber the elements of $P[0..n]$ from 0 to $n' + 1$ according to their increasing values, where $n' < n$ and $n' + 1$ is at position $q_n$. Then the resulting permutation is a permutation $P'$ on $[n']$ augmented with 0 and $n' + 1$.

Denote $F'_{n'}$ the $(n' - 3)$-profile of this permutation, and let us show that $P'$ is unique. For $n' \geq 4$, we show that when $F_{n-3}$ is known, $F'_{n-3}$ is also known, and then apply inductive hypothesis to deduce that $F'_{n-3}$ (and thus $F_{n-3}$) fixes the places of the elements in $P[0..n]$. Whereas for $n' = 2, 3$ we show that there are enough 1-constraints deduced from $F_{n-3}$ to guarantee the uniqueness of $P'$. The case $n' = 1$ is trivial.

Let $h \in [m_{h,h+i},m_{h,h+i+1}]$ be a constraint on $P'$, which belongs to $F'_{n-3}$ if $n' \geq 4$ (i.e. $1 \leq i \leq n' - 3$) and to $F_1$ if $n' = 2, 3$. Let $b(h), b(m_{h,h+i}), b(M_{h,h+i})$ and $b(h + i)$ be respectively the labels of $h, m_{h,h+i}, M_{h,h+i}, h + i$ before renumbering. Then the difference between the labels of $h$ and $h + i$ in the initial $P$ satisfies:

$$b(h + i) - b(h) \leq (h + i) - h + (n - n' - 1).$$

Indeed, if $x$ elements of $P$ are between $n$ and $n + 1$, then the total number of elements in $P$ is, on the one hand, $n + 2$ (the cardinality of $[n] \cup \{0, n + 1\}$) and, on the other hand, $1 + n' + 1 + x + 1$ (given by the cardinality of $W$, by $x$ and by the element $n + 1$). Then $x = n - n' - 1$, and it represents the maximum number of elements that can miss between $b(h + i)$ and $b(h)$, additionally to the values separating them in $P'$, i.e. $(h + i) - h$. But then from equation (4) we deduce:

$$b(h + i) - b(h) \leq i + n - n' - 1.$$  

Case $n' \geq 4$. From equation (5) we deduce with $i \leq n' - 3$ that $b(h + i) - b(h) \leq n' - 3 + n - n' - 1 \leq n - 4$, meaning that $b(h)[b(m_{h,h+i}),b(M_{h,h+i})]b(h+i)$ is a constraint from $F_{n-3}$, yielding the constraint $h[m_{h,h+i},M_{h,h+i}]h+i$ of $F'_{n-3}$ after renumbering. Of course, this affirmation is true since the renumbering keeps the order between the elements, and thus the (renumbered) minimum and maximum value of each given interval. Thus $F'_{n-3}$ is deducible from $F_{n-3}$ and, by inductive hypothesis, the permutation $P'$ is uniquely determined by $F'_{n-3}$.

Case $n' = 2$. Then, as assumed above, $i = 1$ and thus equation (5) implies $b(h + 1) - b(h) \leq 1 + n - 2 - 1 = n - 2$ which is larger than $n - 3$. This shows that all the 1-constraints on $P'$ with $b(h + 1) - b(h) \leq n - 3$ are deducible from constraints in $F_{n-3}$ but the 1-constraints on $P'$ with $b(h+1)-b(h) = n-2$ are not. These latter 1-constraints are obtained when $b(h+1)-b(h) = 1+n-2-1$ (according to equation (5)), that is, when $b(h+1)-b(h) = n-2$. To achieve this with 0, $n$ and the other two elements $e, f$ in $W$ (w.l.o.g. assume $e \leq f$) we must have either $e = n - 2$, and thus $f = n - 1$, (such that $e - 0 = n - 2$), or $e = 1$ and $f = n - 1$ (such that $f - e = n - 2$), or $e = 1$ and $f = 2$ (such that $n - f = n - 2$). In all cases, exactly one 1-constraint is missing (i.e. not resulting from $F_{n-3}$) but the uniqueness of the permutation is still guaranteed, since the two other 1-constraints are sufficient to fix the elements in a 4-permutation (including the endpoints 0 and 4).

Case $n' = 3$. Using (5) and the information that $i = 1$, we deduce that $b(h + 1) - b(h) \leq 1 + n - 3 - 1 = n - 3$, and thus all the 1-constraints are available for $P'$. As the theorem is true for permutations on 3 elements, then we are done.

Affirmation (3) is proved. Similarly, we show that the places of the elements situated on each permutation $P$ between the element 1 (in position $q_1$) and the element $n + 1$ are fixed. Thus, all the elements of each permutation $P$ are in fixed places, and there is only one permutation $P$ with the $(n-3)$-profile $F'_{n-3}$.

Proof of affirmation (b). Similarly to the undirected case, in the directed case assume by contradiction that $k < \lceil \frac{n-3}{2} \rceil$ and build $P$ as in the undirected case, but with $p_2 = 2k + 3$ instead of $p_2 = k + 3$ (thus
6 Conclusions and Perspectives

In this paper, we investigated some problems related to the construction of a permutation from a MinMax-profile or, more generally, from some k-profile, with 1 ≤ k ≤ n. For the first of these problems, the MinMax-BETWEENNESS problem, we noticed the main difficulties of the directed version and gave a polynomial particular case.

The undirected version is even more difficult, due to differences with respect to the directed version that we present hereafter. First, as the relative position of t and t + 1 (i.e., the arc of G between t and t + 1) is not directly given by the MinMax-profile, the B-constraints cannot be directly exploited as in steps 3-5 of Algorithm[1]. The construction of those two types of arcs, the R-arcs and the B-arcs, must therefore be integrated into the Build-Closure algorithm, where the B-constraints must be considered as well as the NB-constraints when seeking new arcs to be added to G. It may be noticed that, similarly to the case of the NB-constraints, any of the B-constraints t \rightarrow t + 1, t \leftarrow t + 1 has two possible settings, resulting either in the set of new arcs \( \text{Arcs}^+ = \{(t, t + 1), (t, M_t), (M_t, t + 1)\} \), or in the set of new arcs \( \text{Arcs}^- = \{(t + 1, t), (m_t, t), (M_t, t), (t + 1, M_t)\} \). When one arc is set, then the four other arcs are set accordingly. When no arc is set, the B-constraints are silent. The algorithm obtained from Algorithm[1] by performing the indicated changes thus outputs either the answer No, or G and two sets SilNB and SilB of silent NB- and silent B-constraints respectively. We thus arrive at the second main difference between the directed and undirected case. Any setting sequence must allow to deduce new arcs also using the B-constraints, thus adding cases to those already in Claim[2] and yielding the study of the arc propagation in G even more complicated that in the directed case.

For both versions, and also for the more general k-MinMax BETWEENNESS problem, the algorithmic difficulty of the problem is an open problem. The same holds for the DIRECTED MinMax-RECONSTRUCTION problem. Also, being able to recognize a k-profile allowing to reconstruct exactly one permutation, i.e., solving (Directed or not) UNIQUE k-MinMax BETWEENNESS, would allow to identify a subclass of permutations perfectly represented by their k-profile.

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