NULL CONTROLLABILITY OF A DEGENERATE PARABOLIC EQUATION WITH DELAY

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Abstract. We are concerned about the null controllability of a linear degenerate parabolic equation with one delay parameter on the line (0, 1), where the control force is exerted on a subdomain of (0, 1) or on the boundary. For that we show how Carleman estimate can be used to establish such results. The second novelty, we discuss the problem of boundary control for parabolic degenerate equations with delay.

1. Introduction

Consider the following linear degenerate parabolic equation with delay

\[
\begin{align*}
&y_t = (a(x)y_x)_x + b(t)y + c(t)y(t-h) + \mathbb{1}_\omega u(t) \quad \text{on } Q \\
&C_y = 0 \quad \text{on } \Sigma \\
y(0) = y_0 \quad \text{in } (0,1) \\
y = \Theta \quad \text{in } (0,1-h)
\end{align*}
\]

where \( T > 0, Q = (0,T) \times (0,1), \Sigma = (0,T) \times \{0,1\} \), the delay term \( h > 0, (0,1-h) = (-h,0) \times (0,1), \mathbb{1}_\omega \) is the characteristic function of an open set \( \omega \subset (0,1), b,c \in L^\infty(Q), y_0 \in L^2(0,1), \Theta \in L^2((0,1)-h) \) and \( u \in L^2(Q) \). The function \( a \) is a diffusion coefficient which degenerates at 0 (i.e., \( a(0) = 0 \)) and we shall admit two types of degeneracy for \( a \), namely weak and strong degeneracy. Indeed, \( a \) can be either weak degenerate (WD), i.e.,

\[
\text{(WD)} \quad \left\{ \begin{array}{l}
(i) \ a \in C((0,1]) \cup C^1((0,1)), a > 0 \ \text{in } (0,1], \ a(0) = 0, \\
(ii) \ \exists K \in [0,1) \text{ such that } xa'(x) \leq Ka(x), \ \forall x \in [0,1],
\end{array} \right.
\]

or strong degenerate (SD), i.e.,

\[
\text{(SD)} \quad \left\{ \begin{array}{l}
(i) \ a \in C^1((0,1]), a > 0 \ \text{in } (0,1], \ a(0) = 0, \\
(ii) \ \exists K \in [1,2] \text{ such that } xa'(x) \leq Ka(x) \ \forall x \in [0,1], \\
(iii) \ \exists K \in (1,K], x \mapsto \frac{a(x)}{x^\theta} \text{ is nondecreasing near } 0, \ \text{if } K > 1, \\
\end{array} \right.
\]

The boundary condition \( C_y = 0 \) is either \( y(t,0) = y(t,1) = 0 \) in the weak degenerate case (WD) or \( y(t,1) = (ay_x)(t,0) = 0 \) in the strong degenerate case (SD).

Approximate controllability of infinite-dimensional retarded linear systems has been studied in [10, 11, 7]. Recently, Ammar-Khodja et al. gave in [3] the first null controllability result for retarded non degenerate parabolic equations with a localized in space control function. In the present paper we use the same technique as [3] to establish null controllability result for retarded degenerate parabolic equations with a localized in space control function.

We give also a particular interest to degenerate parabolic problems with delay under a boundary control. Indeed, when the boundary control is exerted at the bound \( x = 1 \), we show that our

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problem can be transformed into a parabolic degenerate problem with one delay parameter on a larger domain $(0, 2)$, with a control interval located in $(1, 2)$.

In the sequel, if $O$ is an open subset of $(0, 1)$ and $r \in \mathbb{R} \setminus \{0\}$, we set

$$O_r = \begin{cases} (0, r) \times O & \text{if } r > 0 \\ (r, 0) \times O & \text{if } r < 0. \end{cases}$$

This paper is concerned with the $L^2$ null controllability for system (1.1) which we now recall.

**Definition 1.1.** System (1.1) is said to be null controllable at time $T$. If for any $(y_0, \theta) \in L^2(0, 1) \times L^2(-h, 0, L^2(0, 1))$ there exists a control $u \in L^2(Q)$ such that the associated solution $y$ to (1.1) satisfies $y(T) = 0$ in $(0, 1)$.

Like in the non degenerate case [3], for a solution $y$ to (1.1), the property $y(t_0) = 0$ in $(0, 1)$ for some $t_0 > 0$ and $u \in L^2((0, t_0) \times (0, 1))$ does not imply that $y(t) = 0$ for $t > t_0$ even if we choose $u \equiv 0$ for $t > t_0$. Of course, this is due to the presence of the delay term in the equation. Indeed, let us introduce the function

$$z(t, s, x) = y(t + s, x), \quad (t, s, x) \in (0, T) \times (0, 1)_- h.$$  

The equation (1.1) can then be written as follow

$$\begin{cases} y_t = (a(x)y_x)_x + b(t)y + c(t)z_{|z=0} + I_\omega u(t) & \text{on } Q \\ z_t = \frac{\partial z}{\partial x} & \text{on } (0, T) \times (0, 1)_- h \\ Cy = 0 & \text{on } \Sigma \\ y(0, \cdot) = y_0 & \text{in } (0, 1) \\ z|_{z=0} = \theta & \text{in } (0, 1)_- h. \end{cases}$$  

(1.4)

Our first main result in this paper is the following.

**Theorem 1.2.** Let $T > 0$. Assume that $b, c \in L^\infty(Q)$ and

$$\lim_{t \to T^-} (T - t)^4 \ln \|c(t)\|_{(0,1)} = -\infty. \quad (1.5)$$  

Then, for any $(y_0, \theta) \in M_2 = L^2(0, 1) \times L^2((0, 1)_- h)$, there exists $u \in L^2((0, T) \times \omega)$ such that the associated solution of (1.1) satisfies $y(T) = 0$ in $(0, 1)$. Moreover, the control $u$ can be chosen such that

$$\|u\|_{L^2((0, T) \times \omega)} \leq C_T \|y_0, \theta\|_{M_2} \quad (1.6)$$  

for a positive constant $C_T$ depending only on $T$ and $\omega$.

In the nondegenerate case, the exponent of $(T - t)$ in the condition (1.5) is 1, whereas in the degenerate case the right exponent is 4. This fact is due to the corresponding weighted functions used in Carleman estimates of degenerate parabolic equations. To establish Theorem 1.2, we need to prove an observability inequality of the following adjoint problem associated to (1.1).

$$\begin{cases} -W_t = (a(x)W_x)_x + b(t)W + (I_{[0,T]}cW)(t + h) & \text{on } Q \\ CW = 0 & \text{on } \Sigma \\ W(T, \cdot) = W_0 & \text{in } (0, 1). \end{cases}$$  

(1.7)

To this end, we use Carleman estimate established in [4].

This paper is organized as follows. In section 2, we briefly recall the result concerning the well-posedness of problem (1.1). The proof of Theorem 1.2 is given in section 4. It relies on a so-called observability inequality, which we state in section 3, for the solutions of the adjoint problem (1.7) associated to linear system (1.1). This result uses the global Carleman estimates [8, 2] that we recall in section 4. And finally Section 5 of this paper is devoted to the case of boundary control.

All along the article, we use generic constants for the estimates, whose values may change from line to line.
2. Well-posedness

Likewise in [3] from results in Artola [5], we have the following wellposedness result.

**Proposition 2.1.** If \((y_0, \Theta, u) \in L^2(0,1) \times L^2((0,1) - h) \times L^2(Q)\) then there exists a unique solution \(y \in L^2((-h,T) \times (0,1))\) of (1.1) such that

\[
y \in L^2(0,T;H^1) \cap C([0,T];L^2(0,1)), \quad yt \in L^2(0,T;H^{-1})
\]

and there exists \(C_T > 0\) which does not depend on \((y_0, \Theta, u)\) such that

\[
\sup_{t \in [0,T]} \|y(t)\|^2_{L^2(0,1)} + \int_0^T (\|\sqrt{a}y_x\|^2_{L^2} + \|y_t\|_{H^{-1}}^2) dt \leq C_T \left( \|y_0\|^2_{L^2(0,1)} + \|\Theta\|^2_{L^2((0,1) - h)} + \|u\|^2_{L^2(Q)} \right)
\]

for a constant \(C_T > 0\).

Furthermore, if \((y_0, \Theta, u) \in H^1(0,1) \times L^2((0,1) - h) \times L^2(Q)\), this solution satisfies

\[
y \in L^2(0,T;H^1) \cap C([0,T];H^1(0,1)), \quad yt \in L^2(0,T;L^2(0,1))
\]

and there exists \(C_T > 0\) which does not depend on \((y_0, \Theta, u)\) such that

\[
\sup_{t \in [0,T]} \|y(t)\|^2_{H^1} + \int_0^T (\|y_t\|^2_{L^2} + \|ay_x\|^2_{L^2}) dt \leq C_T \left( \|y_0\|^2_{H^1(0,1)} + \|\Theta\|^2_{L^2((0,1) - h)} + \|u\|^2_{L^2(Q)} \right).
\]

3. Observability inequality

This section is devoted to characterize the null controllability of the linear system (1.1).

**Proposition 3.1.** Let \(T > 0\), system (1.1) is \(L^2\) null controllable at time \(T\) if and only if there exists a constant \(C_T > 0\) such that for any \(W_0 \in L^2(0,1)\), the solution of the backward linear system (1.7) satisfies the estimate

\[
\int_0^1 W^2(0) dx + \int_{-h}^0 \int_0^1 |(cW \mathds{1}_{[0, \min(h,T)]})(s + h)|^2 dxds \leq C_T \int_{\omega T} W^2 dx dt.
\]

As proved by Ammar-Khodja et al. [3] in the nondegenerate case, this result is a consequence of the two lemmas in the sequel. Indeed, we denote by \(y_u\) the solution of (1.1) which obtained for \(y_0 = 0, \Theta = 0\) and arbitrary \(u \in L^2(Q)\), and let \(y^H\) be the solution of (1.1) associated with \(u = 0\) and arbitrary initial data \((y_0, \Theta) \in M_2 = L^2(0,1) \times L^2((0,1) - h))\). For \(T > 0\), let us also introduce the following solution operators

\[
S_T : \quad M_2 \quad \rightarrow \quad L^2(0,1) \\
(y_0, \Theta) \quad \rightarrow \quad S_T(y_0, \Theta) = y^H(T),
\]

and

\[
L_T : \quad L^2(Q) \quad \rightarrow \quad L^2(0,1) \\
u \quad \rightarrow \quad L_T u = y^u(T).
\]

From Proposition 2.1, we infer that \(S_T \in L(M_2, L^2(0,1))\) and \(L_T \in L(L^2(Q), L^2(0,1))\). If \(y\) is the solution of (1.1) associated with \((y_0, \Theta, u)\), we have

\[
y(T) = S_T(y_0, \Theta) + L_T u.
\]

Therefore, with these notations, the \(L^2\) null controllability property at time \(T > 0\) is equivalent to the following problem:

for all \((y_0, \Theta) \in M_2\), find \(u \in L^2(Q)\) such that \(L_T u = -S_T(y_0, \Theta)\) \hspace{1cm} (3.2)

The last problem has a solution if and only if

\[
R(S_T) \subset R(L_T),
\]

where \(R(L)\) denote the range of the operator \(L\).

Again, we recall the following well-known result due to Zabczyk (See [12, Theorem 2.2, p. 208]),
Lemma 3.2. Let $X, Y, Z$ be three Hilbert spaces, $X^*, Y^*, Z^*$ their dual spaces and $F \in \mathcal{L}(X, Z)$, $G \in \mathcal{L}(Y, Z)$. Assume that $Y$ is separable. Then $R(F) \subset R(G)$ if and only if there exists a constant $C > 0$ such that

$$\|F^*z\|_{X^*} \leq C\|G^*z\|_{Y^*}, \quad z \in Z^*,$$

where $F^*$ and $G^*$ are the adjoint operators.

Now assume that $X = M_2$, $Y = L^2(Q)$, $Z = L^2(0, 1)$, $F = S_T$ and $G = L_T$. Then, the inclusion (3.3) is equivalent to

$$\|S_T^*y_0\|^2_{M_2} \leq C_T\|L_T^*y_0\|^2_{L^2(Q)}, \quad y_0 \in L^2(0, 1),$$

where $C_T > 0$ does not depend on $y_0$.

Lemma 3.3. Let $W_0 \in L^2(0, 1)$ and $W$ be the associated solution of (1.7). Then

$$S_T^*W_0 = \left(W(0), (cW\mathbb{1}_{[0,\min\{h,T\}]})(\cdot + h)\right), \quad L_T^*W_0 = \mathbb{1}_\omega W. \quad (3.4)$$

Proof. Let $y$ be the solution of (1.1) associated with $(y_0, \Theta, u) \in M_2 \times L^2(Q)$ and $W$ be the solution of (1.7) associated with $W_0$. Multiplying the equation of (1.1) by $W$ and integrating over $Q$ yields the equality

$$\int_0^T \int_0^1 y_1 W = \int_0^T \int_0^1 (a(x)y_x)_x W + \int_0^T \int_0^1 b(t)yW + \int_0^T \int_0^1 c(t)Wy(t - h) + \int_0^T \int_0^1 W\mathbb{1}_\omega u(t). \quad (3.5)$$

Otherwise, by integrating by parts we have

$$\int_0^T \int_0^1 y_1 Wdxdt = \int_0^T \int_0^1 (y(T)W_0 - y_0W(0))dx - \int_0^T \int_0^1 Wdydt. \quad (3.6)$$

Since $-W_t = (a(x)W_x)_x + b(t)W + (cW\mathbb{1}_{[0,T]})(t + h)$. Thus (3.5) becomes

$$\int_0^1 (y(T)W_0 - y_0W(0))dx + \int_0^T \int_0^1 y(cW\mathbb{1}_{[0,T]})(t + h) = \int_0^T \int_0^1 c(t)Wy(t - h) + \int_0^T \int_0^1 W\mathbb{1}_\omega u(t).$$

If $T > h$ :

$$\int_0^T c(t)Wy(t - h) = \int_0^h c(t)Wy(t - h) + \int_h^T c(t)Wy(t - h) = \int_{-h}^0 \Theta(t)(cW\mathbb{1}_{[0,t]})(t + h) + \int_0^T y(cW\mathbb{1}_{[0,T]})(t + h).$$

If $T < h$ :

$$\int_0^T c(t)Wy(t - h) = \int_{-h}^0 \Theta(t)(cW\mathbb{1}_{[0,T]})(t + h).$$

We can summarize these two cases writing

$$\int_0^T c(t)Wy(t - h) = \int_{-h}^0 \Theta(t)(cW\mathbb{1}_{[0,\min\{h,T\}]}) + \int_0^T y(cW\mathbb{1}_{[0,T]})(t + h). \quad (3.7)$$

Thus, we deduce from (3.7) that

$$\langle S_T^*(y_0, \Theta) + L_T^*u, W_0 \rangle_{L^2(0,1)} = \left\langle (y_0, \Theta), (W(0), (cW\mathbb{1}_{[0,\min\{h,T\}]})(\cdot + h)) \right\rangle_{M_2} + \int_0^T \int_0^1 W\mathbb{1}_\omega u(t). \quad (3.8)$$

Taking successively $(y_0, \Theta) = (0, 0)$ and $u = 0$ in this last identity leads to (3.4). \qed
4. Null controllability

In this section we give the proof of the main result. Meanwhile let us recall and establish the following results. Indeed taking into account Carleman estimates established in [1, 8, 2], and consider the following equation

\[\begin{cases}
z'(t) + Mz + bz = f & \text{in } ((\varsigma, \varsigma + l) \times (0, 1)) \\
Cz = 0 & \text{in } ((\varsigma, \varsigma + l) \times \{0\}) \\
z(\varsigma + l) = z_0 & \text{in } (0, 1),
\end{cases}\]

with \(f \in L^2((\varsigma, \varsigma + l) \times (0, 1))\), \(b \in L^\infty((\varsigma, \varsigma + l) \times (0, 1))\), \(z_0 \in L^2(0, 1)\) and \(l, \varsigma\) are real numbers such that \(l > 0\).

By using the interval \((\varsigma, \varsigma + l)\) instead of the interval \((0, T)\) in the Propositions 3.4 and 3.5 [8], the weighed functions become as follow

\[
\begin{align*}
\theta(t) &= \frac{1}{(t - \varsigma)^4 + (l - t)^4} \\
\Phi(t, x) &= \Phi(t) \Psi(x) \\
\Psi(x) &= \left(e^{\sigma(x)} - e^{2\rho \|\sigma\|_\infty}\right)
\end{align*}
\]

with \(\sigma\) is a function in \(C^2([0, 1])\) satisfying \(\sigma(x) > 0\) in \((0, 1)\), \(\sigma(0) = \sigma(1) = 0\) and \(\sigma_x \neq 0\) in \([0, 1] \setminus \tilde{\omega}\) for some open \(\tilde{\omega} \subset \Omega\), \(c_0 = \int_0^1 \frac{x}{a(x)} dx\) and \(d > 4c_0\). Thus, applying the previous propositions (with \(\tau = 0\)) to (4.1), we get the following result.

**Theorem 4.1.** Let \(\omega \subset (0, 1)\) be a non empty subset. Then there exist two positive constants \(C\) and \(s_0\) such that for every \((z_0, f) \in L^2(0, 1) \times L^2((\varsigma, \varsigma + l) \times (0, 1))\) the solution \(z\) of (4.1) satisfies

\[
\int_\varsigma^c (s \theta a(x) z^2_x + (s \theta)^3 z^2_x e^{2s \varphi}) dx dt 
\]

\[
\leq C \left( \int_\varsigma^c f^2 e^{2s \varphi} dx dt + \int_\varsigma^c \int_\omega (s \theta)^3 z^2 e^{2s \varphi} dx dt \right)
\]

for all \(s > s_0\).

Therefore, we get the following lemma as a consequence.

**Lemma 4.2.** Let \(T > 0\) and \(T_h = \max(0, T - h)\) and assume \(b, c \in L^\infty((0, T) \times (0, 1))\). Then, there exist positive constants \(C, s_0\) such that for any \(W_0 \in L^2(0, 1)\), the associated solution \(\varphi\) of (1.7) satisfies

\[
\int_{T_h}^T e^{-2s M \theta} \int_0^1 W^2 dx dt \leq C s^3 \int_{T_h}^T \int_\omega W^2 dx dt
\]

for all \(s > s_0\), where \(\theta(t) = \frac{1}{(t - T_h)^4 + (T - t)^4}\) and \(M = \max_{x \in (0, 1)} |\psi(x)|\) (the function \(\psi\) is defined in (4.2)).

**Proof.** On \((T_h, T) \times (0, 1)\), the solution \(W\) of (1.7) satisfies

\[
\begin{cases}
W' + MW = -bW & \text{in } (T_h, T) \times (0, 1) \\
CW = 0 & \text{in } (T_h, T) \times \{0, 1\}.
\end{cases}
\]

Applying (4.3) with \(\varsigma = T_h\), \(l = T - T_h\), we get

\[
\int_{T_h}^T \int_0^1 \left(s \theta a(x) W^2_x + s^3 \theta^3 \frac{x^2}{a(x)} W^2_x \right) e^{2s \varphi} dx dt \leq C \left( \int_{T_h}^T \int_\omega s^3 \theta^3 W^2 e^{2s \varphi} dx dt \right)
\]

for all \(s > s_0\), where \(\varphi\) and \(\Phi\) as in (4.2). On the other hand, we have

\[
\int_0^1 W^2 dx dt = \int_0^1 \left( \frac{1}{2} \frac{1}{2} \frac{1}{2} W^2 \right) \frac{3}{4} \left( \frac{x^2}{a} W^2 \right) \frac{1}{4} dx dt
\]
Thus, we infer the estimate \( \lim E \) Differentiating \( W \) satisfies equation \( x \mapsto \frac{x^2}{a} \) is nondecreasing on \((0, 1)\). Hence, by Hardy-Poincaré inequality [4, Proposition 2.1], one has
\[
\int_0^1 \frac{a^4}{x^7} W^2 dx = \int_0^1 \frac{p(x)}{x^2} W^2 dx \leq C \int_0^1 \frac{a(x)}{x^2} W^2 dx \leq C \int_0^1 a(x)W^2 dx, \tag{4.5}
\]
and so
\[
\int_{T_0}^T \int_0^1 W^2 dx dt \leq C \int_{T_0}^T \int_0^1 \left( aW^2 + (s\theta)^3 \frac{x^2}{a} W^2 \right) dx dt. \tag{4.6}
\]
Since \( M = \max_{x \in (0, 1)} |\psi(x)| \) and \( \theta \geq \theta(T + T) \), we get from the previous inequality
\[
\int_{T_0}^T e^{-2sM\theta} \int_0^1 W^2 dx dt \leq C \int_{T_0}^T e^{-2sM\theta} \int_0^1 \left( aW^2 + (s\theta)^3 \frac{x^2}{a} W^2 \right) dx dt.
\]

Seeing that \( \lim_{t \to (T_0 - T)} \theta^3(t)e^{2sT} = 0 \), there exists a positive constant \( C > 0 \) such that
\[
\int_{T_0}^T \int_0^1 s^3 \theta^3 W^2 e^{2sT} dx dt \leq Cs^3 \int_{T_0}^T W^2 dx dt.
\]
Thus, we infer the estimate \( (4.4) \). \( \square \)

The following monotonicity argument is of great utility to establish observability estimate, the proof is similar to that one given in [3].

**Lemma 4.3.** Let
\[
K = 5 + 2\|b\|_\infty + \|c\|_\infty. \tag{4.7}
\]
Then, for any \( W \) satisfying equation \( (1.7) \), the function
\[
E(t) = e^{Kt} \left( \int_0^1 W^2(t) dx + \int_0^1 \int_t^{t+\min\{h,T\}} (cW^2_{|0,T})^2(s) ds dx \right), \quad t \in [0, T] \tag{4.8}
\]
is non decreasing. 

**Proof.** At first, let us consider a smooth data \( W_0 \in H^1_s(0, 1) \) and set
\[
m = \min\{h, T\}, \quad \psi = cW^2_{|0,T}
\]
and
\[
E_1 = \int_0^1 W^2(t) dx + \int_0^1 \int_t^{t+m} \psi^2(t) dx.
\]
Differentiating \( E_1 \) with respect to \( t \) gives
\[
E'_1(t) = \left( 2 \int_0^1 W W'(t) dx + \int_0^1 \psi^2(t + m) - \psi^2(t) dx \right). \tag{4.9}
\]
Thus, using \( (1.7) \) we get
\[
2 \int_0^1 W W'(t) dx = 2 \int_0^1 W (a(x)W_x + b(t)W + (cW^2_{|0,T})(t + h)) dx
\]
\[= 2 \int_0^1 a(x)W_x^2 - b(t)W^2 - W(cW \mathbb{1}_{[0,T]})(t + h)dx\]
\[= 2 \int_0^1 a(x)W_x^2 - b(t)W^2 - W\psi(t + m)dx. \quad (4.10)\]

The last equality comes from the fact that, either \( h \geq T \) and then \( \psi(t + h) = (cW \mathbb{1}_{[0,T]})(t + h) = 0 \) on \((0, T)\), or \( h < T \) and then \( m = h \). From (4.9)-(4.10) and Young’s inequality

\[KE_1(t) + E'_1(t) = K \left( \int_0^1 W^2(t)dx + \int_0^1 t^{+m} \psi^2(t)dx \right) + 2 \int_0^1 a(x)W_x^2 - b(t)W^2 - W\psi(t + m)\]
\[\quad + \int_0^1 \psi^2(t + m) - \psi^2(t)\]
\[\geq (K - 2||b||_\infty - ||c||_\infty - 4) \int_0^1 W^2(t)dx / 4 \int_0^1 \psi^2(t + m).\]

Since \( E'(t) = e^{Kt}(KE_1(t) + E'_1(t)) \) and \( K - 2||b||_\infty - ||c||_\infty - 4 = 1 \), we see that \( E' > 0 \) on \((0, T)\) and using then a density argument, we get the result for any \( W_0 \in L^2(0, 1) \). \( \square \)

The following intermediate estimate is also of great interest.

**Lemma 4.4.** Under the hypotheses of Lemma 4.2 and Lemma 4.3, assume moreover that \( c \) satisfies (1.5). Then for any \( T > 0 \), there exists a constant \( C_T = C_T(\omega, ||b||_\infty, ||c||_\infty, T, h) > 0 \) such that any solution of (1.7) satisfies

\[\int_0^1 W^2(0)dx + \int_{-h}^0 \int_{(0,1)} ||(cW \mathbb{1}_{[0, min(h,T)]})(s + h)||^2 dxds \leq C \int_{\omega_T} W^2 dxdt. \quad (4.11)\]

**Proof.** From Lemma 4.2, for \( \nu = \frac{T - T_h}{4} \), we have

\[\int_{T_h + \nu}^T e^{-2sM\theta} \int_0^1 W^2 dx dt \leq Cs^3 \int_{T_h + \nu}^T \int_{\omega_T} W^2 dxdt.\]

Using the energy \( E \) defined in (4.8), we can write

\[\int_{T_h + \nu}^T e^{-2sM\theta - Kt} E(t) dt = \int_{T_h + \nu}^T e^{-2sM\theta - Kt} \left( \int_0^1 W^2(t)dx + \int_0^1 t^{+min(h,T)} (cW \mathbb{1}_{[0,T]})(s) dxds \right)\]
\[= \int_{T_h + \nu}^T e^{-2sM\theta} \int_0^1 W^2(t)dx + \int_{T_h + \nu}^T e^{-2sM\theta} \int_0^1 t^{+min(h,T)} (cW \mathbb{1}_{[0,T]})(s) dxds\]
\[\leq Cs^3 \int_{T_h + \nu}^T \int_{\omega_T} W^2 dxdt + \int_{T_h + \nu}^T e^{-2sM\theta} \int_0^1 t^{+min(h,T)} \left( cW \mathbb{1}_{[0,T]} \right) dxds. \quad (4.12)\]

The hypothesis (1.5) is equivalent to the following: for any \( r > 0 \), there is \( \delta > 0 \) such that

\[||c(t)||_{L^\infty((0,1) \setminus [\overline{\rho}, \overline{\sigma})}, \ t \in (T - \delta, T)\]

Thus, we have

\[||c(t)||_{L^\infty((0,1) \setminus [\overline{\rho}, \overline{\sigma})} \leq (1 + e^{\frac{T}{r}}) ||c||_\infty e^{-\frac{r}{r-M}}, \ t \in (T_h, T)\]

Whence, choosing \( s \) sufficiently large such that \( e^{-2sM\theta} < 1 \) on \((0, T) \times (0, 1)\), we have

\[I \leq \int_{T_h + \nu}^T e^{-2sM\theta} dt \int_{T_h + \nu}^T \left( \int_{(0,1) \setminus [\overline{\rho}, \overline{\sigma})} (cW)^2(s)dx + \int_{\omega_T} (cW)^2(s)dx \right) ds\]
\[\leq 3C_0 \nu \left( \int_{T_h + \nu}^T e^{-2sM\theta} \int_0^1 W^2 dxdt + ||c||_\infty^2 \int_{\omega_T} W^2 dxdt \right), \quad (4.13)\]
with $C_0 = (1 + e^{\frac{\pi}{4}}\|c\|_\infty)^2$, since $\nu = (T - T_h)/4$. Now, for $t \in [T_h + \nu, T]$ and $r > 0$ such that
\[ r \geq \frac{sM}{(\nu)^4}, \] (4.14)
we have
\[ e^{-2sM\theta} = e^{-(\nu - T_h)\|c\|_\infty^2} \geq e^{-2sM\theta_{T_h}} \geq e^{-(\nu - T_h)\|c\|_\infty^2}. \]
Thus, going back to (4.13), taking into account (4.4) in Lemma 4.2, we infer
\[ I \leq 3C_0\nu \left( \int_{T_h + \nu}^{T} e^{-(\nu - T_h)\|c\|_\infty^2} \int_{\omega_T} W^2 \, dx \, dt \right)^2 \]
\[ \leq 3C_0\nu \left( \int_{T_h + \nu}^{T} e^{-2sM\theta_{T_h}} \int_{\omega_T} W^2 \, dx \, dt \right)^2 \]
\[ \leq 3C_0\nu \left( C_{s^3} \int_{T_h + \nu}^{T} \int_{\omega_T} W^2 \, dx \, dt + \|c\|_\infty^2 \int_{\omega_T} W^2 \, dx \right) \]
\[ \leq 3C_0\nu \left( C_{s^3} + \|c\|_\infty^2 \right) \int_{\omega_T} W^2 \, dx. \] (4.15)
With this last inequality, (4.12) becomes
\[ \int_{T_h + \nu}^{T} e^{-2sM\theta_{T_h} - Kt} E(t) \, dt \leq C\left( 1 + 3C_0\nu(C_{s^3} + \|c\|_\infty^2) \right) \int_{\omega_T} W^2 \, dx. \] (4.16)

Now, from Lemma 4.3, we get from this last estimate
\[ \left( \int_{T_h + \nu}^{T - \nu} e^{-2sM\theta_{T_h} - Kt} \, dt \right) E(0) \leq \left( \int_{T_h + \nu}^{T} e^{-2sM\theta_{T_h} - Kt} \, dt \right) E(0) \]
\[ \leq \int_{T_h + \nu}^{T} e^{-2sM\theta_{T_h} - Kt} E(t) \, dt \]
\[ \leq C\left( 1 + 3C_0\nu(C_{s^3} + \|c\|_\infty^2) \right) \int_{\omega_T} W^2 \, dx. \]
Since $\theta(T_h - \nu) = \theta(T + \nu) = \frac{A^8}{3(T - T_h)^4}$ and $\theta(t) \leq \theta(T - \nu)$, $t \in [T_h + \nu, T - \nu]$, we deduce
\[ \int_{T_h + \nu}^{T - \nu} e^{-2sM\theta_{T_h} - Kt} \, dt \geq \frac{1}{2}(T - T_h) e^{-2sM\theta(T - \nu) - K(T - \nu)} \]
\[ \geq \frac{1}{2}(T - T_h) e^{-\frac{2^7M}{s(T - T_h)^4} - KT}. \]
Thus, choosing $s$ large enough in (4.4), one has
\[ E(0) \leq C\left( 1 + 3C_0\nu(C_{s^3} + \|c\|_\infty^2) \right) \left( \frac{2C}{C_{s^3}} \right) \int_{\omega_T} W^2 \, dx. \] (4.17)
Therefore, to conclude the proof, observe that
\[ \int_{-\frac{1}{2}}^{1} |(cW[I_{[0,-\min(h,T)]}](s + h)|^2 \, dx \, ds \leq \int_{0}^{1} \int_{-\frac{1}{2}}^{1} |(cW)(s + h)|^2 \, dx \, ds \]
\[ \leq \int_{0}^{1} \int_{0}^{\min(h,T)} |(cW)(s)|^2 \, dx \, ds. \]
Since
\[ E(0) = \int_{0}^{1} W^2(0) \, dx + \int_{0}^{1} \int_{0}^{\min(h,T)} (cW[I_{[0,T]}])^2(s) \, ds \, dx \]

it follows that
\[ \int_{0}^{1} W^2(0) \, dx + \int_{-\frac{1}{2}}^{1} |(cW[I_{[0,-\min(h,T)]}](s + h)|^2 \, dx \, ds \]
\[
\begin{align*}
&\leq 2C \left( 1 + 3C_{0}C_{0}(C_{0}^{2} + \|c\|_{\infty}^{2}) \right) e^{\frac{2^{17}h_{n}K_{s}}{\nu(T - T_{h})}} + K T \int_{\omega T}^{T} W^{2} dx dt.
\end{align*}
\]

The conclusion follows by taking equalities in (4.14) and (4.4), replacing then \( K \) (see (4.7)) and \( s \) by their values, which completes the proof. \( \square \)

**Proof of Theorem 1.2.** From Proposition 3.1, the system (1.1) is null controllable if and only if every solution \( W \) of its adjoint system (1.7) satisfies the estimate (3.1). Assume that \( c \) satisfies (1.5) and thanks to Lemma 4.4 we get the estimate (3.1). This completes the proof. \( \square \)

### 5. Boundary control

Now, let us consider the following boundary controlled degenerate delay equation

\[
\begin{align*}
\begin{cases}
y_{t} = (a(x)y_{x})_{x} + b(t)y + c(t)y(t - h) & \text{on } Q \\
y(t, 0) = 0, \text{ for } (\text{WD}) & \text{on } (0, T) \\
(a'y_{x})(t, 0) = 0, \text{ for } (\text{SD}) & \text{on } (0, T) \\
y(t, 1) = h(t) & t \in (0, T) \\
y(0, \cdot) = y_{0} & \text{in } (0, 1) \\
y = \Theta & \text{in } (0, 1 - h),
\end{cases}
\end{align*}
\]

where the control is acting at the point \( x = 1 \), in which the diffusion coefficient \( a \) do not vanish. We have the following result.

**Theorem 5.1.** Assume that Hypothesis (1.5) is satisfied. Then for any \((y_{0}, \theta) \in M_{2} = L^{2}(0, 1) \times L^{2}((-h, 0) \times (0, 1)), \) there exists a control \( h \in L^{2}((0, T)) \) such that the associated solution of (5.1) satisfies \( y(T) = 0 \) in \((0, 1), \)

**Proof.** Since the control is acting on \( x = 1 \), we use the same technique as in [4] consisting to transform the boundary control problem (5.1) into the following distributed control problem

\[
\begin{align*}
\begin{cases}
\ddot{y} = (\ddot{a}(x)\ddot{y}_{x})_{x} + \ddot{b}(t)\ddot{y} + \ddot{c}(t)\ddot{y}(t - h) + \textbf{1}_{\omega}u(t) & \text{on } (0, T) \times (0, 2) \\
\ddot{y}(t, 0) = 0, \text{ for } (\text{WD}) & \text{on } (0, T) \\
(\ddot{a}y_{x})(t, 0) = 0, \text{ for } (\text{SD}) & \text{on } (0, T) \\
\ddot{y}(t, 2) = 0 & \text{on } (0, T) \\
\ddot{y}(0, \cdot) = \ddot{y}_{0} & \text{in } (0, 2) \\
\ddot{y} = \ddot{\Theta} & \text{in } (0, 2 - h)
\end{cases}
\end{align*}
\]

where \( \omega \subset (1, 2), \) and

\[
\ddot{a}(x) = \begin{cases}
a(x), x \in [0, 1] \\
a(1), x \in (1, 2]
\end{cases}, \quad \ddot{b}(t, x) = \begin{cases}
b(t, x), x \in [0, 1] \\
0, \quad x \in (1, 2]
\end{cases}, \quad \ddot{c}(t, x) = \begin{cases}
c(t, x), x \in [0, 1] \\
0, \quad x \in (1, 2],
\end{cases}
\]

\[
\ddot{y}_{0}(x) = \begin{cases}
y_{0}(x), x \in [0, 1] \\
0, \quad x \in (1, 2]
\end{cases}, \quad \ddot{\Theta}(t, x) = \begin{cases}
\Theta(t, x), x \in [0, 1 - h] \\
0, \quad x \in (1, 2 - h].
\end{cases}
\]

It is not difficult to see that the assumption (1.5) implies \( \lim_{t \to T^{-}}(T - t)^{4} \ln \|\ddot{c}(t)\|_{(0, 2) \setminus \omega} = -\infty. \)

Therefore, we apply Theorem 1.2 to the system (5.2). The right boundary control for (5.1) is then defined by \( \ddot{h}(t) = \ddot{y}(t, 1). \) \( \square \)
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