SEMI-INVARIANT SUBMANIFOLDS IN METRIC GEOMETRY OF AFFINORS

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Abstract. We introduce a generalization of structured manifolds as the most general Riemannian metric $g$ associated to an affinor (tensor field of $(1,1)$-type) $F$ and initiate a study of their semi-invariant submanifolds. These submanifolds are generalizations of CR-submanifolds of almost complex geometry and semi-invariant submanifolds of several interesting geometries (almost product, almost contact and others). We characterize the integrability of both invariant and anti-invariant distribution; the special case when $F$ is covariant constant with respect to $g$ gives major simplifications in computations.

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Introduction

The geometry of manifolds endowed with geometrical structures has been intensively studied and several important results have been published, see Yano-Kon [14]. The more important classes of such manifolds are formed by almost complex, almost product, almost contact, almost paracontact manifolds for which the cited book offers a good introduction. The geometry of submanifolds in these manifolds is very rich and interesting, as well, see for example the classical [7] or the more recent survey [8]. CR-submanifolds introduced by Bejancu in [2] (for almost complex geometry) respectively [5] (for almost contact geometry) had a great impact on the developing of the theory of submanifolds in these ambient manifolds; a proof of this fact is given by the books [4] and [13].

In the present paper we first introduce the concept of $(g,F,\mu)$-manifold which contains as particular cases all the above types of structures. Then we study semi-invariant submanifolds of a $(g,F,\mu)$-manifold, which are extensions of CR-submanifolds to this general class of manifolds. We find necessary and sufficient conditions for the integrability of both distributions on a semi-invariant submanifold, see Theorems 3.1 and 3.3. Particularly, we prove that some semi-invariant submanifolds carry a natural foliation, in Theorem 4.4 and we obtain characterizations of totally geodesic foliations on semi-invariant submanifolds in Theorems 4.8 and 4.10. For a particular value of the real parameter $\mu$ we can connect our study with the almost symplectic geometry and this fact opens some possible further applications in physical sciences having as an example the relationship between CR-structures and Relativity pointed out in the last Chapter of [4].

This work is dedicated to Professor Aurel Bejancu on the occasion of his 65th birthday. His ideas represent a starting point for several important studies as the present Bibliography partially shows.

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1. Metric geometry of affinors and submanifolds

Let $M$ be an $m$-dimensional manifold for which we denote by $C^\infty(M)$ the algebra of smooth functions on $M$ and by $\Gamma(TM)$ the $C^\infty(M)$-module of smooth sections of the tangent bundle $TM$ of $M$; let $X, Y, Z, ...$ denote such vector fields. We use the same notation $\Gamma(V)$ for any other vector bundle $V$ over $M$. Let also $T^1_1M$ be the $C^\infty(M)$-module of smooth sections of the tangent bundle $TM$ of $M$; let also $T^1_1M$ be the $C^\infty(M)$-module of $\Gamma(TM \otimes T^*M)$ i.e. the real space of tensor fields of $(1,1)$-type on $M$. Let consider a fixed $F \in T^1_1M$ usually called affinor ([9]) or vector 1-form; the remarkable affinor of every manifold is the Kronecker tensor field $I = (\delta_j^i)$.

Fix $\mu \in \{-1, +1\}$. Let now $g$ be a Riemannian metric on $M$.

**Definition 1.1** $M$ is a $(g, F, \mu)$-manifold if :

$$g(FX, Y) + \mu g(X, FY) = 0.$$ (1.1)

The geometry of the data $(M, g, F, \mu)$ is called affinor-metric geometry. If in particular, $F_x$ is nondegenerate at any point $x \in M$ then we say that $M$ is a nondegenerate $(g, F, \mu)$-manifold; otherwise, $M$ is called degenerate $(g, F, \mu)$-manifold.

The relation (1.1) says that the $g$-adjoint of $F$ is $F^* = -\mu F$. In literature there is an abundance of examples of $(g, F, \mu)$-manifolds. Some of the main examples are presented here:

**Examples 1.2**

1. An almost Hermitian manifold ([4, p. 11]) $(M, g, J)$ is a nondegenerate $(g, F, \mu = +1)$-manifold; the nondegeneration is a consequence of $J^2 = -I$.

2. An almost parahermitian manifold ([1]) $(M, g, P)$ is a nondegenerate $(g, F, \mu = +1)$-manifold while an almost Riemannian product manifold is a nondegenerate $(g, F, \mu = -1)$-manifold; the nondegeneration is a consequence of $P^2 = I$.

3. An almost contact metric manifold ([4, p. 15]) $(M, g, \varphi, \xi, \eta)$ is a $(g, F, \mu = +1)$-manifold; as $\varphi(\xi) = 0$, $M$ is degenerate.

4. An almost paracontact manifold ([12]) $(M, g, \varphi, \xi, \eta)$ is a $(g, F, \mu = +1)$-manifold. As in the previous example we have $\varphi(\xi) = 0$ and therefore $M$ is degenerate.

5. The general case of a nondegenerate $(g, F, \mu = +1)$-manifold is called structured manifold in [11].

Recall that a real $2m$-dimensional manifold $M$ is called an almost symplectic manifold if it is endowed with a nondegenerate 2-form $\Omega \in \Lambda^2(M)$. We derive the following characterization:

**Proposition 1.3** Let $M$ be a $(g, F, \mu = +1)$-manifold. Then $M$ is nondegenerate if and only if $\Omega$ defined by:

$$\Omega(X, Y) = g(FX, Y)$$ (1.2)

is an almost symplectic structure. In this case $m$ is even.

**Proof** $\Omega$ is skew-symmetric from $\mu = +1$. A straightforward computation yields that $\Omega$ is nondegenerate if and only if $M$ is a nondegenerate $(g, F, \mu = +1)$-manifold. $\square$

**Example 1.4** For Example 1.2.1 $\Omega$ is exactly the fundamental or Kähler 2-form and then inspired by this fact we introduce:

**Definition 1.5** For a nondegenerate $(g, F, \mu = +1)$-manifold $\Omega$ is call the fundamental 2-form.
In the last part of this section let us recall briefly the geometry of Riemannian submanifolds. Consider an $n$-dimensional submanifold $N$ of $M$. Then the main objects induced by the Levi-Civita connection $\nabla$ of $(M, g)$ on $N$ are involved in the well known Gauss-Weingarten equations:
\[
\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X V = -A_V X + \nabla^\perp_X V,
\]
for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(T^\perp N)$. Here $\nabla$ is the Levi-Civita connection on $N$, $h$ is the second fundamental form of $N$, $A_V$ is the Weingarten operator with respect to the normal section $V$ and $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp N$ of $N$. Let us point out that $h$ and $A_V$ are related by:
\[
g(h(X, Y), V) = g(A_V X, Y).
\]
If $h$ vanishes identically on $N$ then $N$ is called totally geodesic.

2. Submanifolds in affinor-metric geometry

Next, we consider a submanifold $N$ of a $(g, F, \mu)$-manifold $M$. Then $g$ induces a Riemannian metric on $N$ which we denote by the same symbol $g$. Then, following the definition given by Bejancu [2] for CR-submanifolds we introduce a special class of submanifolds of $M$ as follows:

**Definition 2.1** $N$ is a semi-invariant submanifold of $M$ if there exists a distribution $D$ on $N$ satisfying the conditions:
(i) $D$ is $F$-invariant:
\[
F(D_x) \subset D_x, \quad \forall x \in N.
\]
(ii) The complementary orthogonal distribution $D^\perp$ to $D$ in $TN$ is $F$-anti-invariant, that is:
\[
F(D^\perp_x) \subset T^\perp_x N, \quad \forall x \in N.
\]
(iii) $F^2(D^\perp)$ is a distribution on $N$.

Some particular classes of semi-invariant submanifolds are defined as follows. Let $p$ and $q$ be the ranks of the distributions $D$ and $D^\perp$ respectively. If $q = 0$, that is $D^\perp = \{0\}$, we say that $N$ is an $F$-invariant submanifold of $M$. If $p = 0$, that is $D = \{0\}$, we call $N$ an $F$-anti-invariant submanifold of $M$.

If $pq \neq 0$ then $N$ is called a proper semi-invariant submanifold. Now, we denote by $\tilde{D}$ the complementary orthogonal vector bundle to $F(D^\perp)$ in $T^\perp N$. If $\tilde{D} = \{0\}$ then we say that $N$ is a normal $F$-semi-invariant submanifold.

Thus, $N$ is an $F$-invariant, respectively $F$-anti-invariant, if and only if:
\[
F(TN) \subset TN \quad (\text{resp. } F(TN) \subset T^\perp N).
\]

$N$ is normal $F$-semi-invariant if and only if:
\[
F(D^\perp) = T^\perp N.
\]

**Examples 2.2**

1) For Example 1.2.1 we obtain the classical concept of CR-submanifold of Bejancu [4 p. 20]; the condition iii) is satisfied from $J^2 = -I$.
2) For Example 1.2.2 we obtain the notion of semi-invariant submanifold; for the almost parahermitian case see [1] while for the second case see [3]. The condition iii) is satisfied again from $P^2 = -I$.
3) For Example 1.2.3 we obtain the notion of semi-submanifold [4 p. 100] with $\xi \in T^\perp N$. This last condition implies $TN \subset \ker \eta$ and since $\varphi|_{\ker \eta}$ is an almost complex structure we
2) If is a normal semi-invariant submanifold then \( T(v) \) take \( X \) and: 

\[
F \quad \text{that is} \quad M \quad \text{the above distributions satisfy}
\]

1) (vi) Take \( X \) on \( N \); we need only to show (2.1) we obtain for \( X = PX + QX \in \Gamma(TN) \): 

\[
FX = \varphi X + \omega X
\]

where we put: 

\[
\varphi = F \circ P, \quad \omega = F \circ Q.
\]

Thus \( \varphi \) is a tensor field of \((1,1)\)-type on \( N \) while \( \omega \) is a \( F(D^\perp) \)-valued vector 1-form on \( N \). Thus we derive:

**Proposition 2.5** Let \( N \) be a semi-invariant submanifold of a \((g,F,\mu)\)-manifold \( M \). Then:

(iv) \( N \) is a \((g,\varphi,\mu)\)-manifold.

(v) \( F^2(D^\perp) \) is a vector subbundle of \( D^\perp \).

(vi) The vector bundle \( \tilde{D} \) is \( F \)-invariant i.e. for all \( x \in N \) we have: \( F(\tilde{D}_x) \subset \tilde{D}_x \).

**Proof** (iv) By definition, \( g \) is a Riemannian metric on \( N \) and \( \varphi \) is a tensor field of \((1,1)\)-type on \( N \); we need only to show (2.1). By using (1.1) for \( F \) we obtain for \( X,Y \in \Gamma(TN) \):

\[
g(\varphi X, Y) = g(FPX, Y) = g(FPX, PY) = -\mu g(PX, FPY) = -\mu g(X, \varphi Y).
\]

(v) Take \( X \in \Gamma(D) \) and \( Y \in \Gamma(D^\perp) \) in (2.1): \( g(X, F^2Y) = -\mu g(FX, FY) = 0 \) since \( FX \in \Gamma(D) \) and \( FY \in \Gamma(T^\perp N) \). Hence \( F^2(D^\perp) \) is orthogonal to \( D \) and by condition (iii) we deduce that \( F^2(D^\perp) \) is a vector subbundle of \( D^\perp \).

(vi) Take \( X \in \Gamma(TN) \), \( Y \in \Gamma(D^\perp) \) and \( V \in \Gamma(\tilde{D}) \). Then we obtain:

\[
g(FV, X) = -\mu g(V, FX) = -\mu g(V, \varphi X + \omega X) = 0
\]

and:

\[
g(FV, FY) = -\mu g(V, F^2Y) = 0
\]

since \( \varphi X \in \Gamma(D) \), \( \omega X \in \Gamma(FD^\perp) \) and \( F^2Y \in \Gamma(D^\perp) \). Thus \( F\tilde{D} \) is orthogonal to \( TN \oplus FD^\perp \), that is \( F\tilde{D} \) is a vector subbundle of \( \tilde{D} \). This completes the proof of the proposition. \( \square \)

In the non-degenerated case we have equalities for the above inclusions:

**Corollary 2.6** Let \( N \) be a semi-invariant submanifold of a nondegenerate \((g,F,\mu)\)-manifold \( M \). Then:

1) the above distributions satisfy:

\[
F(D) = D, \quad F^2(D^\perp) = D^\perp, \quad F(\tilde{D}) = \tilde{D}.
\]

2) if \( \mu = +1 \) then \( D^\perp \) and \( F(D^\perp) \) are Lagrangian distribution on \((TM, \Omega)\). In particular if \( N \) is a normal semi-invariant submanifold then \( T^\perp N \) is a Lagrangian submanifold in \((TM, \Omega)\).
Proof We need to prove only 2).
2.1) Let $X, Y \in \Gamma(D^\perp)$; then $\Omega(X, Y) = g(FX, Y) = 0$ since $FX \in \Gamma(T^\perp N)$ while $Y \in \Gamma(T N)$.
2.2) Let $X, Y \in \Gamma(F(D^\perp))$; then $\Omega(X, Y) = g(FX, Y) = 0$ since $FX \in \Gamma(T N)$ while $Y \in \Gamma(T^\perp N)$.

The second part of the above Corollary is extremely important since it relates the geometry of semi-invariant submanifolds with the almost symplectic geometry, a topic very studied from the point of view of applications in Analytical Mechanics.

3. Integrability of distributions on a semi-invariant submanifold

Let $N$ be a semi-invariant submanifold of a $(g, F, \mu)$-manifold $M$. Then we recall that the Nijenhuis tensor field of $F$ is defined as follows ([4, p.11]):
\[
N_F(X, Y) = [FX, FY] + F^2[X, Y] - F[FX, Y] - F[X, FY],
\] (3.1)
for any $X, Y \in \Gamma(TM)$. In a similar way, the Nijenhuis tensor field of $\varphi$ on $N$ is given by:
\[
N_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y],
\] (3.2)
for any $X, Y \in \Gamma(T N)$. We recall that a tensor field of $(1, 1)$-type defines an integrable structure on a manifold if and only if its Nijenhuis tensor field vanishes identically on the manifold. Now we obtain necessary and sufficient conditions for the integrability of $D$ and $D^\perp$ in terms of Nijenhuis tensor fields of $F$ and $\varphi$.

**Theorem 3.1** Let $N$ be a semi-invariant submanifold of a $(g, F, \mu)$-manifold $M$. Then the following assertions are equivalent:
1) $D$ is an integrable distribution.
2) The Nijenhuis tensor field of $\varphi$ satisfies:
\[
Q \circ N_\varphi = 0, \quad \forall X, Y \in \Gamma(D).
\] (3.3)
3) The Nijenhuis tensor fields of $F$ and $\varphi$ satisfy the equality: $N_F = N_\varphi$ on $D$.

**Proof** Firstly, we note that $D$ is integrable if and only if:
\[
Q([X, Y]) = 0, \quad \forall X, Y \in \Gamma(D).
\] (3.4)
Since the last three terms in the right side of (3.2) lie in $\Gamma(D)$ we deduce that:
\[
Q \circ N_\varphi(X, Y) = Q([FX, FY]), \quad \forall X, Y \in \Gamma(D).
\] (3.5)
As $M$ is nondegenerate we deduce that $\varphi$ is an automorphism on $\Gamma(D)$. Thus the equivalence of 1) and 2) follows directly. Next, we obtain for any $X, Y \in \Gamma(D)$:
\[
N_F(X, Y) = N_\varphi(X, Y) + F\omega([X, Y]) - \omega([\varphi X, Y]) - \omega([X, \varphi Y]).
\] (3.6)
If $D$ is integrable then the last three terms of (3.6) vanishes and this yields 3). Conversely, suppose that $N_F = N_\varphi$ on $D$; then:
\[
F\omega([X, Y]) = \omega([\varphi X, Y] + [X, \varphi Y]).
\] (3.7)
Obviously the right-hand-side of the previous equation is in $\Gamma(F(D^\perp)) \subset \Gamma(T^{\text{bot}} N)$. On the other hand, the left-hand-side is in $\Gamma(F^2D^\perp) \subset \Gamma(T N)$; we conclude that both sides in (3.7) must vanish.

Finally, from: $F^2Q([X, Y]) = 0$ and $F^2$ automorphism of $\Gamma(TM)$ we deduce 1).
Remark 3.2 For Example 1.2.1 the equivalence of 1) and 2) is exactly the Theorem 2.2. of [4, p. 25] while the equivalence of 1) and 3) is the Theorem 2.1. of [4, p. 25].

Now, we consider $X, Y \in \Gamma(D^\perp)$. Then taking into account that $\varphi X = \varphi Y = 0$ we get:

$$N_\varphi(X, Y) = F^2 P[X, Y]$$

and this enables us to state the following:

Theorem 3.3 Let $N$ be a semi-invariant submanifold of a nondegenerate $(g, F, \mu)-$manifold. Then $D^\perp$ is integrable if and only if the Nijenhuis tensor field of $\varphi$ vanishes identically on $D^\perp$.

Remark 3.4 For Example 1.2.1 the above result is the Theorem 2.3. of [4, p. 26].

4. A natural foliation on a semi-invariant submanifold

Let $\nabla$ be the Levi-Civita connection on $M$ with respect to the Riemannian metric $g$. Then $F$ is a parallel tensor field on $M$ if:

$$\nabla F = 0.$$  \hfill (4.1)

Examples 4.1

1) For Example 1.2.1 we have the notion of Kähler manifold.
2) For Example 1.2.2, in the first part we have the concept of para-Kähler manifold while for the second part the notion of locally Riemannian product manifold.
3) For Example 1.2.3 we get the notion of cosymplectic manifold.

In the present section we study the geometry of semi-invariant submanifolds of $(g, F, \mu)$-manifolds with parallel tensor field $F$. First, we prove the following:

Proposition 4.2 Let $N$ be a semi-invariant submanifold of a nondegenerate $(g, F, \mu)$-manifold with parallel tensor field $F$. Then for all $X, Y \in \Gamma(D^\perp)$:

$$A_{F X} Y - A_{F Y} X = \varphi([X, Y]).$$ \hfill (4.2)

Proof By using the Weingarten equation and the parallelism condition we get:

$$A_{F X} Y = \nabla^\perp_F X - \nabla_Y FX = \nabla^\perp_F X - F(\nabla_X Y).$$ \hfill (4.2)

Writing a similar equation by interchanging $X$ and $Y$ and then subtracting we obtain:

$$A_{F X} Y - A_{F Y} X = \nabla^\perp_F X - \nabla^\perp_X F Y + F([X, Y]),$$ \hfill (4.3)

since $\nabla$ is a torsion-free linear connection. Thus (4.2) is obtained by equalizing the tangent parts to $N$ in the above equation. \hfill \Box

Example 4.3 The relation (4.2) becomes for Example 1.2.1 the equation (2.2) of [4, p. 43].

Now, we can state the following main result:

Theorem 4.4 Let $N$ be a semi-invariant submanifold of a nondegenerate $(g, F, \mu = +1)$-manifold with parallel tensor field $F$. Then the $F$-anti-invariant distribution $D^\perp$ is integrable.

Proof For any $X, Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(D)$ we have:

$$g(A_{F X} Y, Z) = -g(F \nabla_Y X, Z) = +\mu g(\nabla_Y X, F Z) = -\mu g(X, \nabla_Y F Z) =$$

$$= \mu^2 g(F X, \nabla_Y Z) = \mu^2 g(F X, [Y, Z] + \nabla_Z Y) = \mu^2 g(F X, \nabla_Z Y).$$ \hfill (4.4)
Also, we have:

\[ g(A_{FY}X, Z) = \mu^2 g(FY, \tilde{\nabla} Z X) = -\mu^2 g(F\tilde{\nabla} Z Y, X) = \mu^3 g(\tilde{\nabla} Z Y, FX). \]  
\( (4.5) \)

Comparing (4.4) and (4.5) we deduce that for \( \mu = 1 \):

\[ g(A_{FX} Y - A_{FY} X, Z) = 0 \]

which means that \( A_{FX} Y - A_{FY} X \in \Gamma(D^\perp) \). On the other hand, from (4.2) we conclude that:

\[ A_{FX} Y - A_{FY} X \in \Gamma(D) \]

and thus we have that:

\[ A_{FX} Y - A_{FY} X = 0. \]  
\( (4.6) \)

Finally, returning to (4.2) and taking into account that \( F \) is nondegenerate we deduce that:

\[ P[X, Y] = 0, \]

that is, \( D^\perp \) is integrable. □

**Remark 4.5** For Example 1.2.1 the above result is part (i) of Theorem 1.1. of [4, p. 39].

Regarding the integrability of \( D \) we prove the following:

**Theorem 4.6** Let \( N \) be a semi-invariant submanifold of a nondegenerate \( (g, F, \mu) \)-manifold \( M \) with parallel tensor field \( F \). Then the \( F \)-invariant distribution \( D \) is integrable if and only if the second fundamental form \( h \) of \( N \) satisfies for any \( X, Y \in \Gamma(D) \) and \( Z \in \Gamma(D^\perp) \):

\[ g(h(X, \varphi Y) - h(Y, \varphi X), F Z) = 0. \]  
\( (4.7) \)

**Proof** By using the Gauss equation we deduce that:

\[ \nabla X \varphi Y + h(X, \varphi Y) = \varphi(\nabla X Y) + \omega(\nabla X Y) + F h(X, Y). \]  
\( (4.8) \)

Write a similar equation by interchanging \( X \) and \( Y \), and then subtracting we obtain:

\[ \nabla X \varphi Y - \nabla Y \varphi X + h(X, \varphi Y) - h(Y, \varphi X) = \varphi([X, Y]) + \omega([X, Y]) \]  
\( (4.9) \)

since \( h \) is symmetric and \( \nabla \) is a torsion-free linear connection. Equalize the normal parts in the above equation and obtain:

\[ h(X, \varphi Y) - h(Y, \varphi X) = \omega([X, Y]). \]  
\( (4.10) \)

Now, suppose that \( D \) is integrable; then (4.7) is immediately. Conversely, if (4.6) is satisfied, then with (4.10) we deduce that:

\[ 0 = -\mu g(Q[X, Y], F^2 Z). \]

Since \( F \) is nondegenerate we infer that \( F^2 \) is an automorphism of \( \Gamma(D^\perp) \) and hence \( D \) is integrable. □

**Remark 4.7** In particular, if \( F \) is an almost complex structure on \( M \) then we obtain the results of Bejancu [4] and Blair-Chen [6] respectively, for CR-submanifolds.

Now, for \( \mu = 1 \) we denote by \( F^\perp \) the natural foliation defined by the \( F \)-anti-invariant distribution \( D^\perp \) and call it the \( F \)-anti-invariant foliation on \( N \). We recall that \( F^\perp \) is called a **totally geodesic foliation** if each leaf of \( F^\perp \) is totally geodesic immersed in \( N \). Thus \( F^\perp \) is totally geodesic if and only if the Levi-Civita connection \( \nabla \) of \( N \) satisfies for all \( Y, Z \in \Gamma(D^\perp) \):

\[ \nabla_Y Z \in \Gamma(D^\perp). \]  
\( (4.11) \)
Theorem 4.8 Let $N$ be a semi-invariant submanifold of a nondegenerate $(g, F, \mu)$-manifold $M$ with parallel tensor field $F$. Then the following assertions are equivalent:

(i) The $F$-anti-invariant foliation is totally geodesic.

(ii) The second fundamental form $h$ of $N$ satisfies for all $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$:

$$h(X,Y) \in \Gamma(\tilde{D}).$$

(4.12)

(iii) $D^\perp$ is $A_V$-invariant for any $V \in \Gamma(FD^\perp)$ that is we have for all $Y \in \Gamma(D^\perp)$:

$$A_Y Y \in \Gamma(D^\perp).$$

(4.13)

Proof We have for any $X \in \Gamma(D)$ and $Y, Z \in \Gamma(D^\perp)$:

$$g(\nabla_Y Z, F X) = g(\tilde{\nabla}_Y Z, F X) = -\mu g(\tilde{\nabla}_Y F Z, X) =$$

$$= \mu g(A_{F Z} Y, X) = \mu g(h(X,Y), F Z).$$

(4.14)

Now, suppose that $\mathcal{F}^\perp$ is totally geodesic; then the first term of (4.14) vanishes. Hence the last term in (4.14) vanishes which implies ii). Conversely, suppose (4.12) is satisfied. Then from (4.14) we deduce (4.11) since $F$ is an automorphism of $\Gamma(D)$. This proves the equivalence of (i) and (ii). The equivalence of (ii) and (iii) is straightforward. $\square$

Remark 4.9 For Example 1.2.1 the equivalence of (i) and (ii) is the Theorem 1.3. of [4, p. 41].

Finally, we can prove the following:

Theorem 4.10 Let $N$ be a semi-invariant submanifold of a nondegenerate $(g, F, \mu)$-manifold with parallel tensor field $F$. Then the $F$-invariant distribution $D$ is integrable and the foliation $\mathcal{F}$ defined by $D$ is totally geodesic if and only if the second fundamental form $h$ of $N$ satisfies for all $X, Y \in \Gamma(D)$:

$$h(X,Y) \in \Gamma(\tilde{D}).$$

(4.15)

Proof $D$ is integrable and $\mathcal{F}$ is totally geodesic if and only if for all $X, U \in \Gamma(D)$:

$$\nabla_X U \in \Gamma(D).$$

(4.16)

This is equivalent to:

$$g(\tilde{\nabla}_X U, Z) = 0,$$

(4.17)

for all $Z \in \Gamma(D^\perp)$. As $F$ is an automorphism of $\Gamma(D)$ we can write the above equality as follows:

$$g(\tilde{\nabla}_X F Y, Z) = 0,$$

(4.18)

for all $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$, which is equivalent to:

$$g(\tilde{\nabla}_X Y, F Z) = 0.$$ 

(4.19)

By using the Gauss equation, the last relation is equivalent to:

$$g(h(X, Y), F Z) = 0,$$

(4.20)

which completes the proof of the theorem. $\square$

Remark 4.11 For Example 1.2.1 the above result is Theorem 1.2. of [4, p. 40].
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