Splittings and robustness for the Heine-Borel theorem*

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Abstract. The Heine-Borel theorem for uncountable coverings has recently emerged as an interesting and central principle in higher-order Reverse Mathematics and computability theory, formulated as follows: \( HBU \) is the Heine-Borel theorem for uncountable coverings given as \( \bigcup_{x \in [0,1]} [x - \Psi(x), x + \Psi(x)] \) for arbitrary \( \Psi : [0,1] \rightarrow \mathbb{R}^+ \), i.e. the original formulation going back to Cousin (1895) and Lindelöf (1903). In this paper, we show that \( HBU \) is equivalent to its restriction to functions continuous almost everywhere, an elegant robustness result. We also obtain a nice splitting \( HBU \leftrightarrow [WHBU^+ + HBC_0 + WKL] \) where \( WHBU^+ \) is a strengthening of Vitali’s covering theorem and where \( HBC_0 \) is the Heine-Borel theorem for countable collections (and not sequences) of basic open intervals, as formulated by Borel himself in 1898.

Keywords: Higher-order Reverse Mathematics · Heine-Borel theorem · Vitali covering theorem · splitting · robustness.

1 Introduction and preliminaries

We sketch our aim and motivation within the Reverse Mathematics program (Section 1.1) and introduce some essential axioms and definitions (Section 1.2).

1.1 Aim and motivation

Reverse Mathematics (RM hereafter) is a program in the foundations of mathematics initiated by Friedman \([5,6]\) and developed extensively by Simpson and others \([25,26]\); an introduction to RM for the ‘mathematician in the street’ may be found in \([27]\). We assume basic familiarity with RM, including Kohlenbach’s higher-order \( RM \) introduced in \([10]\). Recent developments in higher-order \( RM \), including our own, are published in \([13,19]\).

Now, a splitting \( A \leftrightarrow [B + C] \) is a relatively rare phenomenon in second-order \( RM \) where a natural theorem \( A \) can be \textit{split} into two \textit{independent} natural parts \( B \) and \( C \). Splittings are quite common in higher-order \( RM \), as studied in some detail in \([23]\). An unanswered question here is whether the higher-order generalisations of the Big Five of \( RM \) (and related principles) have natural splittings.

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In this paper, we study the Vitali and Heine-Borel covering theorems for uncountable coverings with an eye on splittings. In particular, our starting point is HBU, defined in Section 1.2, which is the Heine-Borel theorem for uncountable coverings $\bigcup_{x \in [0,1]} I^\Psi_x$ for arbitrary third-order $\Psi : [0,1] \to \mathbb{R}^+$ and $I^\Psi_x \equiv (x - \Psi(x), x + \Psi(x))$. This kind of coverings was already studied by Cousin in 1895 ([3]) and Lindelöf in 1903 ([12]). In Section 2.2 we obtain an elegant splitting involving HBU, namely as follows:

$$\text{HBU} \leftrightarrow [\text{WHBU}^+ + \text{HBC}_0 + \text{WKL}], \quad (1.1)$$

where WHBU$^+$ is a strengthening of the Vitali covering theorem and where HBC$_0$ is the Heine-Borel theorem for countable collections (and not sequences) of open intervals, as formulated by Borel himself in [1]. In Section 2.1 we prove HBU $\leftrightarrow$ HBU$^\omega$, where the latter is HBU restricted to functions $\Psi : [0,1] \to \mathbb{R}^+$ continuous almost everywhere on the unit interval. By contrast, the same restriction for the Vitali covering theorem results in a theorem equivalent to weak weak König's lemma WWKL. The results in Section 2.1 were obtained following the study of splittings involving ‘continuity almost everywhere’. The proof of Theorem 2.2 (in a stronger system) was suggested to us by Dag Normann. In general, this paper constitutes a spin-off from our joint project with Dag Normann on the Reverse Mathematics and computability theory of the uncountable (see [13, 17, 19]).

Finally, the foundational and historical significance of our results is as follows.

Remark 1.1 First of all, as shown in [13, 15, 16], the third-order statements HBU and WHBU cannot be proved $\mathrm{Z}^\omega_2$, a conservative extension of $\mathrm{Z}_2$ based on third-order comprehension functionals. A sceptic of third-order objects could ‘downplay’ this independence result by pointing to the outermost quantifier of HBU and WHBU and declare that the strength of these principles is simply due to the quantification over all third-order functions. This point is moot in light of HBU $\leftrightarrow$ HBU$^\omega$ proved in Theorem 2.2, and the central role of ‘continuity almost everywhere’ in e.g. the study of the Riemann integral and measure theory.

Secondly, our first attempt at obtaining a splitting for HBU was to decompose the latter as HBU$^\omega + \text{WHBU}$, where WHBU allows one to reduce an arbitrary covering to a covering generated by a function that is continuous almost everywhere. Alas, this kind of splitting does not yield independent conjuncts, which is why we resort to stronger notions like countability, namely in Section 2.2.

Thirdly, the splitting in (1.1) has some historical interest as well: Borel himself formulates the Heine-Borel theorem in [1] using countable collections of intervals rather than sequences of intervals (as in second-order RM). In fact, Borel’s proof of the Heine-Borel theorem in [1] p. 42 starts with: Let us enumerate our intervals, one after the other, according to whatever law, but determined. He then proceeds with the usual ‘interval-halving’ proof, similar to Cousin in [3].

1.2 Preliminaries

We introduce some axioms and definitions from (higher-order) RM needed below. We refer to [10] §2 or [13] §2 for the definition of Kohlebach’s base theory $\mathrm{RCA}_0^\omega$. 
and basic definitions like the real numbers $\mathbb{R}$ in $\text{RCA}_0^\omega$. For completeness, some definitions are included in the technical appendix, namely Section A.

Some axioms of higher-order arithmetic First of all, the functional $\varphi$ in $(\exists^2)$ is clearly discontinuous at $f = 11$;… in fact, $(\exists^2)$ is equivalent to the existence of $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = 1$ if $x > 0$, and 0 otherwise ([10] $\S$3).

$$(\exists^2 \leq 1)(\forall f^1)[(\exists n)(f(n) = 0) \leftrightarrow \varphi(f) = 0].$$

(32)

Related to $(\exists^2)$, the functional $\mu^2$ in $(\mu^2)$ is also called Feferman’s $\mu$ ([10]).

$$(\exists^2)(\forall f^1)[(\exists n)(f(n) = 0) \to [f(\mu(f)) = 0 \land (\forall i < \mu(f))(f(i) \neq 0)] \land [(\forall n)(f(n) \neq 0) \to \mu(f) = 0]].$$

(32)

Intuitively, $\mu^2$ is the least-number-operator, i.e. $\mu(f)$ provides the least $n \in \mathbb{N}$ such that $f(n) = 0$, if such there is. We have $(\exists^2) \leftrightarrow (\mu^2)$ over $\text{RCA}_0^\omega$ and $\text{ACA}_0^\omega \equiv \text{RCA}_0^\omega + (\exists^2)$ proves the same second-order sentences as $\text{ACA}_0$ by [9] Theorem 2.5.

Secondly, the Heine-Borel theorem states the existence of a finite sub-covering for an open covering of certain spaces. Now, a functional $\Psi : \mathbb{R} \to \mathbb{R}^+$ gives rise to the canonical cover $\cup_{x \in I} I^\Psi_x$ for $I \equiv [0,1]$, where $I^\Psi_x$ is the open interval $(x - \Psi(x), x + \Psi(x))$. Hence, the uncountable covering $\cup_{x \in I} I^\Psi_x$ has a finite sub-covering by the Heine-Borel theorem; in symbols:

Principle 1.2 (HBU) $(\forall \Psi : \mathbb{R} \to \mathbb{R}^+)(\exists y_0, \ldots, y_k \in I)(\exists x \in I)(x \in \cup_{i \leq k} I^\Psi_{y_i}).$

Cousin and Lindelöf formulate their covering theorems using canonical covers in [3][12]. This restriction does not make much of a difference, as studied in [24].

Thirdly, let WHBU be the following weakening of HBU:

Principle 1.3 (WHBU) For any $\Psi : \mathbb{R} \to \mathbb{R}^+$ and $\varepsilon > 0$, there are pairwise distinct $y_0, \ldots, y_k \in I$ with $1 - \varepsilon < \sum_{i < k} |J^\Psi_{y_i}|$, where $J^\Psi_{y_{i+1}} := I^\Psi_{y_{i+1}} \setminus (\cup_{j \leq i} I^\Psi_{y_j})$.

As discussed at length in [14], WHBU expresses the essence of the Vitali covering theorem for uncountable coverings; Vitali already considered the latter in [31]. Basic properties of the gauge integral ([28]) are equivalent to HBU while WHBU is equivalent to basic properties of the Lebesgue integral (without RM-codes: [14]). By [13][14][15], $Z^2_2$ proves HBU and WHBU, but $Z^2_2$ cannot. The exact definition of $Z^2_2$ and $Z^2_2$ is in the aforementioned references and Section A.2. What is relevant here is that $Z^2_2$ and $Z^2_2$ are conservative extensions of $Z_2$ by [9] Cor. 2.6], i.e. the former prove the same second-order sentences as the latter.

We note that HBU (resp. WHBU) is the higher-order counterpart of WKL (resp. WWKL), i.e. weak König’s lemma (resp. weak weak König’s lemma) from RM as the ECF-translation ([10][29]) maps HBU (resp. WHBU) to WKL (resp. WWKL), i.e. these are (intuitively) weak principles. We refer to [10] $\S$2 or Remark A.1 for a discussion of the relation between ECF and $\text{RCA}_0^\omega$.

Finally, the aforementioned results suggest that (higher-order) comprehension as in $Z^2_2$ is not the right way of measuring the strength of HBU. As a better alternative, we have introduced the following axiom in [22].
Principle 1.4 (BOOT) \((∀Y^2)(∃Y ⊆ R)(∀n^0)[n ∈ X ↔ (∃f^1)(Y(f, n) = 0)]\).

By [22, §3], BOOT is equivalent to convergence theorems for nets, we have the implication \(\text{BOOT} \rightarrow \text{HBU}\), and \(\text{RCA}_0^\omega + \text{BOOT}\) has the same first-order strength as \(\text{ACA}_0\). Moreover, BOOT is a natural fragment of Feferman’s projection axiom (Proj1) from [1]. Thus, BOOT is a natural axiom that provides a better ‘scale’ for measuring the strength of HBU and its ilk, as discussed in [17,22].

Some basic definitions We introduce the higher-order definitions of ‘open’ and ‘countable’ set, as can be found in e.g. [15,17,19].

First of all, open sets are represented in second-order RM as countable unions of basic open sets ([26, II.5.6]), and we refer to such sets as ‘RM-open’. By [26, II.7.1], one can effectively convert between RM-open sets and (RM-codes for) continuous characteristic functions. Thus, a natural extension of the notion of ‘open set’ is to allow arbitrary (possibly discontinuous) characteristic functions, as is done in e.g. [15,17,21], which motivates the following definition.

Definition 1.5 [Sets in \(\text{RCA}_0^\omega\)] We let \(Y : R \rightarrow R\) represent subsets of \(R\) as follows: we write ‘\(x \in Y\)’ for ‘\(Y(x) >_R 0\)’ and call a set \(Y \subseteq R\) ‘open’ if for every \(x \in Y\), there is an open ball \(B(x, r) \subseteq Y\) with \(r^0 > 0\). A set \(Y\) is called ‘closed’ if the complement, denoted \(Y^c = \{x \in R : x \notin Y\}\), is open.

For open \(Y\) as in Definition 1.5, the formula ‘\(x \in Y\)’ has the same complexity (modulo higher types) as for RM-open sets, while given \((∃^2)\) it is equivalent to a ‘proper’ characteristic function, only taking values ‘0’ and ‘1’. Hereafter, an ‘(open) set’ refers to Definition 1.5. ‘RM-open set’ refers to the definition from second-order RM, as in e.g. [26, II.5.6].

Secondly, the definition of ‘countable set’ (Kunen; [11]) is as follows in \(\text{RCA}_0^\omega\).

Definition 1.6 [Countable subset of \(\mathbb{R}\)] A set \(A \subseteq R\) is countable if there exists \(Y : R \rightarrow N\) such that \((∀x, y ∈ A)(Y(x) =_R 0 Y(y) → x =_R y)\). If \(Y : R \rightarrow N\) is also surjective, i.e. \((∀n ∈ N)(∃x ∈ A)(Y(x) = n)\), we call \(A\) strongly countable.

Hereafter, ‘(strongly) countable’ refers to Definition 1.6 unless stated otherwise. We note that ‘countable’ is defined in second-order RM using sequences ([26 V.4.2]), a notion we shall call ‘enumerable’.

Thirdly, we have explored the connection between HBU, generalisations of HBU, and fragments of the neighbourhood function principle NFP from [30] in [22,24]. In each case, nice equivalences were obtained assuming \(A_0\) as follows.

Principle 1.7 (\(A_0\)) For \(Y^2\) and \(A(σ) ≡ (∃g ∈ 2^N)(Y(g, σ) = 0)\), we have
\[
(∀f ∈ N^N)(∃n ∈ N)A(\overline{f}n) \rightarrow (∃G^2)(∀f ∈ N^N)A(\overline{f}G(f)),
\]
where \(\overline{f}n\) is the finite sequence \((f(0), f(1), \ldots, f(n-1))\).

As discussed in [22,24], the axiom \(A_0\) is a fragment of NFP and can be viewed as a generalisation of QF-AC^{1,0}, included in \(\text{RCA}_0^\omega\). As an alternative to \(A_0\), one could add ‘extra data’ or moduli to the theorems to be studied.
2 Main results

In Section 2.1, we show that $\text{HBU}$ is equivalent to $\text{HBU}_\infty$, i.e. the restriction to functions continuous almost everywhere, while the same restriction applied to $\text{WHBU}$ results in a theorem equivalent to $\text{WWKL}$ (see [26, X.1] for the latter). In Section 2.2 we establish the splitting (1.1) involving $\text{HBU}$.

2.1 Ontological parsimony and the Heine-Borel theorem

We introduce $\text{HBU}_\infty$, the restriction of $\text{HBU}$ from Section 1.2 to functions continuous almost everywhere, and establish $\text{HBU} \leftrightarrow \text{HBU}_\infty$ over RCA. The same restriction for $\text{WHBU}$ turns out to be equivalent to weak weak König’s lemma ($\text{WWKL}$; see [26, X.1]), well-known from second-order RM.

We first need the following definition, where we note that the usual\(^1\) definition of ‘measure zero’ is used in RM.

**Definition 2.1** [Continuity almost everywhere] We say that $\Psi : [0, 1] \to \mathbb{R}$ is continuous almost everywhere if it is continuous outside of an RM-closed set $E \subset [0, 1]$ which has measure zero.

Let $\text{HBU}_\infty$ be $\text{HBU}$ restricted to functions continuous almost everywhere as in the previous definition. The proof of the following theorem (in a stronger system) was suggested by Dag Normann, for which we are grateful.

**Theorem 2.2** The system $\text{RCA}_0^\omega$ proves $\text{HBU} \leftrightarrow \text{HBU}_\infty$.

**Proof.** First of all, as noted in Section 1.2 $(\exists^2)$ is equivalent to the existence of a discontinuous $\mathbb{R} \to \mathbb{R}$-function, namely by [10, Prop. 3.12]. Thus, in case $\neg(\exists^2)$, all functions on $\mathbb{R}$ are continuous. In this case, we trivially obtain $\text{HBU} \leftrightarrow \text{HBU}_\infty$.

Since $\text{RCA}_0^\omega$ is a classical system, we have the law of excluded middle as in $\neg(\exists^2) \vee (\exists^2)$. As we have provided a proof in the first case $\neg(\exists^2)$, it suffices to provide a proof assuming $(\exists^2)$, and the law of excluded middle finishes the proof.

Hence, for the rest of the proof, we may assume $(\exists^2)$.

Secondly, the Cantor middle third set $\mathcal{C} \subset [0, 1]$ is available in $\text{RCA}_0$ by (the proof of) [26, IV.1.2] as an RM-closed set, as well as the well-known recursive homeomorphism from Cantor space $2^\mathbb{N}$ to $\mathcal{C}$ defined as $H : 2^\mathbb{N} \to [0, 1]$ and $H(f) := \sum_{n=0}^{\infty} \frac{2f(n)}{3^{n+1}}$. Note that given $\exists^2$, we can decide whether $x \in \mathcal{C}$ or not.

Thirdly, we prove $\text{HBU}_\infty \to \text{HBU}_\varepsilon$, where the latter is $\text{HBU}$ for $2^\mathbb{N}$ as follows:

$$(\forall G^2)(\exists f_0, \ldots, f_k \in 2^{\mathbb{N}})(\forall g \in 2^{\mathbb{N}})(\exists i \leq k)(g \in \overline{\bigcup_{i \in \sigma} G(f_i)})(\text{HBU}_\varepsilon)$$

and where $|\sigma|$ is the open neighbourhood in $2^{\mathbb{N}}$ of sequences starting with the finite binary sequence $\sigma$. The equivalence $\text{HBU} \leftrightarrow \text{HBU}_\varepsilon$ may be found in [13,16].

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\(^1\) A set $A \subset \mathbb{R}$ is **measure zero** if for any $\varepsilon > 0$ there is a sequence of basic open intervals $(I_n)_{n \in \mathbb{N}}$ such that $\bigcup_{n \in \mathbb{N}} I_n$ covers $A$ and has total length below $\varepsilon$. 

Now assume \(\text{HBC}_{\omega}\) and fix \(G^2\) and define \(\Psi : [0, 1] \to \mathbb{R}^+\) using (\(3^2\)) as:

\[
\Psi(x) := \begin{cases} 
\frac{d(x,C)}{2^{n(x,C)}} & x \notin C \\
1 & \text{otherwise}
\end{cases},
\]

where \(I(x)\) is the unique \(f \in 2^{\mathbb{N}}\) such that \(H(f) = x\) in case \(x \in C\), and \(00\ldots\) otherwise. Note that the distance function \(d(x,C)\) exists given \(\text{ACA}_0\) by [7, Theorem 1.2]. Clearly, \(3^2\) allows us to define this function as a third-order object that is continuous on \([0,1] \setminus C\). Since \(C\) has measure zero (and is RM-closed), apply \(\text{HBU}_{\omega}\) to \(\bigcup_{x \in [0,1]} I^x\). Let \(y_0, \ldots, y_k\) be such that \(\cup_{i \leq k} I^y_i\) covers \([0,1]\). By the definition of \(\Psi\) in (2.1), if \(x \in [0,1] \setminus C\), then \(C \cap I^y_x = \emptyset\). Hence, let \(z_0, \ldots, z_m\) be those \(y_i \in C\) for \(i \leq k\) and note that \(\cup_{j \leq m} I^y_j\) covers \(C\). Clearly, \(I(z_0), \ldots, I(z_m)\) yields a finite sub-cover of \(\bigcup_{f \in 2^\mathbb{N}} \overline{G(f)}\), and \(\text{HBU}_{\omega}\) follows. \(\square\)

We could of course formulate \(\text{HBU}_{\omega}\) with the higher-order notion of ‘closed set’ from \([15]\), and the equivalence from the theorem would still go through. The proof of the theorem also immediately yields the following.

**Corollary 2.3 (\(\text{ACA}_0^\omega\))** \(\text{HBU}\) is equivalent to the Heine-Borel theorem for canonical coverings \(\bigcup_{x \in E} I^x\), where \(E \subset [0,1]\) is RM-closed and has measure zero.

As expected, Theorem 2.2 generalises to principles that imply \(\text{HBU}\) over \(\text{RCA}_0^\omega\) (see \([17]\,\text{Figure 1}\) for an overview) and that boast a third-order functional to which the ‘continuous almost everywhere’ restriction can be naturally applied. An example is the following corollary involving \(\text{BOOT}\).

**Corollary 2.4** The system \(\text{RCA}_0^\omega\) proves \(\text{BOOT} \leftrightarrow \text{BOOT}_{\omega}\), where the latter is

\[(\exists X \subset \mathbb{N})(\forall n^0)(n \in X \leftrightarrow (\exists x \in [0,1])(Y(x,n) = 0)),\]

where \(\lambda x.Y(x,n)\) is continuous almost everywhere on \([0,1]\) for any fixed \(n \in \mathbb{N}\).

**Proof.** In case \(\neg(\exists\mathbb{Z})\), all functions on \(\mathbb{R}\) are continuous by \([10]\,\text{Prop. 3.12}\); in this case, the equivalence is trivial. In case \(\exists\mathbb{Z}\), the forward direction is immediate, modulo coding real numbers given \(\exists\mathbb{Z}\). For the reverse direction, fix \(Y^2\) and note that we may restrict the quantifier (\(\exists f^1\)) in \(\text{BOOT}\) to \(2^\mathbb{N}\) without loss of generality. Indeed, \(\mu^2\) allows us to represent \(f^1\) via its graph, a subset of \(\mathbb{N}^2\), which can be coded as a binary sequence. Now define

\[
Z(x,n) := \begin{cases} 
0 & x \in C \land Y(I(x),n) = 0 \\
1 & \text{otherwise}
\end{cases},
\]

where \(C\) and \(I\) are as in the theorem. Note that \(\lambda x.Z(x,n)\) is continuous outside of \(C\). By \(\text{BOOT}_{\omega}\), there is \(X \subset \mathbb{N}\) such that for all \(n \in \mathbb{N}\), we have:

\[
n \in X \leftrightarrow (\exists x \in [0,1])(Z(x,n) = 0) \leftrightarrow (\exists f \in 2^\mathbb{N})(Y(f,n) = 0),\]

where the last equivalence is by the definition of \(Z\) in (2.2). \(\square\)
Splittings and robustness for the Heine-Borel theorem

Next, we show that the Vitali covering theorem as in $\text{WHBU}$ behaves quite differently from the Heine-Borel theorem as in $\text{HBU}$. Recall that the Heine-Borel theorem applies to open coverings of compact sets, while the Vitali covering theorem applies to Vitali coverings of any set $E$ of finite (Lebesgue) measure. The former provides a finite sub-covering while the latter provides a sequence that covers $E$ up to a set of measure zero. As argued in [14], $\text{WHBU}$ is the combinatorial essence of Vitali’s covering theorem.

Now, let $\text{WHBU}_a$ be $\text{WHBU}$ restricted to functions continuous almost everywhere, as in Definition 2.1; recall that $Z^2_2\omega$ cannot prove $\text{WHBU}_a$.

**Theorem 2.5** The system $\text{RCA}_0^{\omega} + \text{WKL}$ proves $\text{WHBU}_a$.

**Proof.** Let $\psi : [0, 1] \to \mathbb{R}^+$ be continuous on $[0, 1] \setminus E$ with $E \subset [0, 1]$ of measure zero and RM-closed. Fix $\varepsilon > 0$ and let $\bigcup_{n \in \mathbb{N}} I_n$ be a union of basic open intervals covering $E$ and with measure at most $\varepsilon/2$. Then $[0, 1]$ is covered by:

$$\bigcup_{q \in \mathbb{Q} \setminus E} B(q, \psi(q)) \bigcup \bigcup_{n \in \mathbb{N}} I_n.$$ (2.3)

Indeed, that the covering in (2.3) covers $E$ is trivial, while $[0, 1] \setminus E$ is (RM)-open. Hence, $x_0 \in [0, 1] \setminus E$ implies that $B(x_0, r) \subset [0, 1] \setminus E$ for $r > 0$ small enough and for $q \in \mathbb{Q} \cap [0, 1]$ close enough to $x_0$, we have $x_0 \in B(q, \psi(q))$. By [26, IV.1], WKL is equivalent to the countable Heine-Borel theorem. Hence, there are $q_0, \ldots, q_k \in \mathbb{Q} \setminus E$ and $n_0 \in \mathbb{N}$ such that the finite union $\bigcup_{i=1}^k B(q_i, \psi(q_i)) \bigcup \bigcup_{j=0}^{n_0} I_j$ covers $[0, 1]$. Since the measure of $\bigcup_{j=0}^{n_0} I_j$ is at most $\varepsilon/2$, the measure of $\bigcup_{i=1}^k B(q_i, \psi(q_i))$ is at least $1 - \varepsilon/2$, as required by $\text{WHBU}_a$. \hfill \Box

**Corollary 2.6** The system $\text{RCA}_0^{\omega}$ proves $\text{WWKL} \leftrightarrow \text{WHBU}_a$.

**Proof.** The reverse implication is immediate in light of the RM of $\text{WWKL}$ in [26, X.1], which involves the Vitali covering theorem for countable coverings (given by a sequence). For the forward implication, convert the cover from (2.3) to a Vitali cover and use [26, X.1.13]. \hfill \Box

Finally, recall Remark [1.1] discussing the foundational significance of the above.

### 2.2 Splittings for the Heine-Borel theorem

We establish a splitting for $\text{HBU}$ as in Theorem [4.9] based on known principles formulated with countable sets as in Definition [1.6]. As will become clear, there is also some historical interest in this study.

First of all, the following principle $\text{HBC}_0$ is studied in [17, §3], while the (historical and foundational) significance of this principle is discussed in Remark [1.1].

The aforementioned system $Z^2_2\omega$ cannot prove $\text{HBC}_0$.

2 An open covering $V$ is a Vitali covering of $E$ if any point of $E$ can be covered by some open in $V$ with arbitrary small (Lebesgue) measure.
Principle 2.7 (HBC0) For countable \( A \subseteq \mathbb{R}^2 \) with \((\forall x \in [0,1])(\exists(a,b) \in A)(x \in (a,b)), \) there are \((a_0,b_0), \ldots, (a_k,b_k) \in A\) with \((\forall x \in [0,1])(\exists i \leq k)(x \in (a_i,b_i)).\)

Secondly, the second-order Vitali covering theorem has a number of equivalent formulations (see [26, X.1]), including the statement a countable covering of \([0, 1]\) has a sub-collection with measure zero complement. Intuitively speaking, the following principle WHBU\(^+\) strengthens ‘measure zero’ to ‘countable’. Alternatively, WHBU\(^+\) can be viewed as a weakening of the Lindelöf lemma, introduced in [12] and studied in higher-order RM in [13, 10].

Principle 2.8 (WHBU\(^+\)) For \( \Psi : [0, 1] \to \mathbb{R}^+ \), there is a sequence \((y_n)_{n \in \mathbb{N}}\) in \([0, 1]\) such that \([0, 1] \setminus \bigcup_{n \in \mathbb{N}} I^\Psi_{y_n}\) is countable.

Note that WHBU\(^+\) + HBC0 yields a conservative\(^3\) extension of RCA\(_0^\omega\), i.e. the former cannot imply HBU without the presence of WKL. Other independence results are provided by Theorem 2.10.

We have the following theorem, where \( A_0 \) was introduced in Section 1.2. This axiom can be avoided by enriching\(^4\) the antecedent of HBC0.

Theorem 2.9 The system RCA\(_0^\omega\) + \( A_0 \) proves

\[
(\text{WHBU}^+ + \text{HBC}_0 + \text{WKL}) \leftrightarrow \text{HBU},
\]

where the axiom \( A_0 \) is only needed for HBU \( \rightarrow \) HBC0.

Proof. First of all, in case \( \neg(3^2) \), all functions on \( \mathbb{R} \) are continuous, rendering \( \text{WHBU}^+ + \text{HBC}_0 \) trivial while HBU reduces to WKL. Hence, for the rest of the proof, we may assume \((3^2)\), by the law of excluded middle as in \((3^2) \lor \neg(3^2)\).

For the reverse implication, assume \( A_0 + \text{HBU} \) and let \( A \) be as in HBC0. The functional \( 3^2 \) can uniformly convert real numbers to a binary representation. Hence (2.5) is equivalent to a formula as in the antecedent of \( A_0 \):

\[
(\forall x \in [0, 1])(\exists n \in \mathbb{N}) [(\exists (a,b) \in A)(a < [x](n + 1) - \frac{1}{2^n} \land [x](n + 1) + \frac{1}{2^n} < b)] \tag{2.5}
\]

where \([x](n)\) is the \( n \)-th approximation of the real \( x \), given as a fast-converging Cauchy sequence. Apply \( A_0 \) to (2.5) to obtain \( G : [0, 1] \to \mathbb{N} \) such that \( G(x) = n \) as in (2.3). Apply HBU to \( \bigcup_{x \in [0,1]} I^\Psi_x \) for \( \Psi(x) := \frac{1}{2^G(x)} \). The finite sub-cover \( y_0, \ldots, y_k \in [0, 1] \) provided by HBU gives rise to \((a_i, b_i) \in A\) containing \( I^\Psi_y \) for \( i \leq k \) by the definition of \( G \). Moreover, HBU implies WKL as the latter is equivalent to the ‘countable’ Heine-Borel theorem as in [26, IV.1]. Clearly, the empty set is countable by Definition 1.6 and HBU \( \rightarrow \) WHBU\(^+\) is therefore trivial.

\(^3\) The system RCA\(_0^\omega\) + \( \neg(3^2) \) is an \( L_2 \)-conservative extension of RCA\(_0^\omega\) and the former readily proves WHBU\(^+\) + HBC0. By contrast HBU \( \rightarrow \) WKL over RCA\(_0^\omega\).

\(^4\) In particular, one would add a function \( G : [0, 1] \to \mathbb{R}^2 \) to the antecedent of HBC0 such that \( G(x) \in A \) and \( x \in (G(x)(1), G(x)(2)) \) for \( x \in [0, 1] \). In this way, the covering is given by \( \bigcup_{x \in [0,1]} (G(x)(1), G(x)(2)) \).
For the forward implication, fix $\Psi : [0, 1] \to \mathbb{R}^+$ and let $(y_n)_{n \in \mathbb{N}}$ be as in $\text{WHBU}^+$. Define ‘$x \in B$’ as $x \in [0, 1] \setminus \bigcup_{n \in \mathbb{N}} I_{y_n}^\Psi$ and note that when $B$ is empty, the theorem follows as $\text{WKL}$ implies the second-order Heine-Borel theorem ([26], IV.1)). Now assume $B \neq \emptyset$ and define $A$ as the set of $(a, b)$ such that either $(a, b) = I_{y_n}^\Psi$ for $x \in B$, or $(a, b) = I_{y_n}^\Psi$ for some $n \in \mathbb{N}$. Note that in the first case, $(a, b) \in A$ if and only if $\frac{a+b}{2} \in B$, i.e. defining $A$ does not require quantifying over $\mathbb{R}$. Moreover, $A$ is countable because $B$ is: if $Y$ is injective on $B$, then $W$ defined as follows is injective on $A$:

$$W((a, b)) := \begin{cases} 2Y(\frac{a+b}{2}) & a+b \in B \\ H((a, b)) & \text{otherwise} \end{cases},$$

where $H((a, b))$ is the least $n \in \mathbb{N}$ such that $(a, b) = I_{y_n}^\Psi$, if such there is, and zero otherwise. The intervals in the set $A$ cover $[0, 1]$ as in the antecedent of $\text{HBC}_0^0$, and the latter now implies $\text{HBU}$. 

The principles $\text{WHBU}^+$ and $\text{HBC}_0$ are ‘quite’ independent by the following theorem, assuming the systems therein are consistent.

**Theorem 2.10** The system $\text{Z}_2^+ + \text{QF-AC}^{0,1} + \text{WHBU}^+$ cannot prove $\text{HBC}_0$. The system $\text{RCA}_0^\omega + \text{HBC}_0 + \text{WHBU}^+$ cannot prove $\text{WKL}_0$.

**Proof.** For the first part, suppose $\text{Z}_2^+ + \text{QF-AC}^{0,1} + \text{WHBU}^+$ does prove $\text{HBC}_0$. The latter implies $\text{NIN}$ as follows by [17, Cor. 3.2]:

$$(\forall Y : [0, 1] \to \mathbb{N})(\exists x, y \in [0, 1])(Y(x) = Y(y) \land x \neq_R y). \quad \text{(NIN)}$$

Clearly, $\neg\text{NIN}$ implies $\text{WHBU}^+$, and we obtain that $\text{Z}_2^+ + \text{QF-AC}^{0,1} + \neg\text{NIN}$ proves a contradiction, namely $\text{WHBU}^+$ and its negation. Hence, $\text{Z}_2^+ + \text{QF-AC}^{0,1}$ proves $\text{NIN}$, a contradiction by [17, Theorem 3.1], and the first part follows.

For the second part, the $\text{ECF}$-translation (see Remark A.1) converts $\text{HBC}_0 + \text{WHBU}^+$ into a triviality.

Finally, we discuss similar results as follows. Of course, the proof of Theorem 2.9 goes through *mutatis mutandis* for $\text{WHBU}^+ + \text{HBC}_0$ formulated using strongly countable sets. Moreover, (2.0) can be proved in the same way as (2.4), assuming additional countable choice as in $\text{QF-AC}^{0,1}$:

$$\text{WHBU} \leftrightarrow [\text{WHBU}^+ + \text{WHBC}_0 + \text{WWKL}], \quad (2.6)$$

where $\text{WHBC}_0$ is $\text{HBC}_0$ with the conclusion weakened to the existence of a sequence $(a_n, b_n)_{n \in \mathbb{N}}$ of intervals in $A$ with measure at least one. Also, if we generalize $\text{HBU}$ to coverings of any separably closed set in $[0, 1]$, the resulting version of (2.4) involves $\text{ACA}_0$ rather than $\text{WKL}_0$ in light of [8, Theorem 2].
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A Reverse Mathematics: second- and higher-order

A.1 Reverse Mathematics

Reverse Mathematics (RM hereafter) is a program in the foundations of mathematics initiated around 1975 by Friedman ([5, 6]) and developed extensively by Simpson ([26]). The aim of RM is to identify the minimal axioms needed to prove theorems of ordinary, i.e. non-set theoretical, mathematics. We refer to [27] for a basic introduction to RM and to [25, 26] for an overview of RM. The details of Kohlenbach’s higher-order RM may be found in [10], including the base theory $\text{RCA}_0^\omega$. The latter is connected to $\text{RCA}_0$ by the ECF-translation as follows.

Remark A.1 (The ECF-interpretation) The (rather) technical definition of ECF may be found in [29, p. 138, § 2.6]. Intuitively, the ECF-interpretation $[A]_{\text{ECF}}$ of a formula $A \in L_\omega$ is just $A$ with all variables of type two and higher replaced by type one variables ranging over so-called ‘associates’ or ‘RM-codes’; the latter are (countable) representations of continuous functionals. The ECF-interpretation connects $\text{RCA}_0^\omega$ and $\text{RCA}_0$ (see [10, Prop. 3.1]) in that if $\text{RCA}_0^\omega$ proves $A$, then $\text{RCA}_0$ proves $[A]_{\text{ECF}}$, again ‘up to language’, as $\text{RCA}_0$ is formulated using sets, and $[A]_{\text{ECF}}$ is formulated using types, i.e. using type zero and one objects.

In light of the widespread use of codes in RM and the common practise of identifying codes with the objects being coded, it is no exaggeration to refer to ECF as the canonical embedding of higher-order into second-order arithmetic.

We now introduce the usual notations for common mathematical notions.

Definition A.2 (Real numbers and related notions in $\text{RCA}_0^\omega$)

a. Natural numbers correspond to type zero objects, and we use ‘$n^0$’ and ‘$n \in \mathbb{N}$’ interchangeably. Rational numbers are defined as signed quotients of natural numbers, and ‘$q \in \mathbb{Q}$’ and ‘$<_{\mathbb{Q}}$’ have their usual meaning.

b. Real numbers are coded by fast-converging Cauchy sequences $q(\cdot) : \mathbb{N} \to \mathbb{Q}$, i.e. such that $(\forall n^0, i^0)(|q_n - q_{n+1}| <_{\mathbb{Q}} \frac{1}{2^i})$. We use Kohlenbach’s ‘hat function’ from [10, p. 289] to guarantee that every $q^1$ defines a real number.
c. We write ‘\(x \in \mathbb{R}\)’ to express that \(x^1 := (q_{t,1})\) represents a real as in the previous item and write \([x](k) := q_k\) for the \(k\)-th approximation of \(x\).

d. Two reals \(x, y\) represented by \(q_{t,1}\) and \(r_{t,1}\) are equal, denoted \(x =_\mathbb{R} y\), if

\[(\forall n^0)((|g_n - r_n| \leq 2^{-n+1})).\]

Inequality ‘\(<_\mathbb{R}\)’ is defined similarly. We sometimes omit the subscript ‘\(\mathbb{R}\)’ if it is clear from context.

e. Functions \(F : \mathbb{R} \to \mathbb{R}\) are represented by \(\Phi^{1 \to 1}\) mapping equal reals to equal reals, i.e. extensionality as in \((\forall x, y \in \mathbb{R})(x =_\mathbb{R} y \implies \Phi(x) =_\mathbb{R} \Phi(y))\).

f. Binary sequences are denoted ‘\(f, g \in C^\omega\)’ or ‘\(f, g \in 2^\mathbb{N}\)’. Elements of Baire space are given by \(f^1, g^1\), but also denoted ‘\(f, g \in \mathbb{N}^\mathbb{N}\)’.

**Notation A.3 (Finite sequences)** The type for ‘finite sequences of objects of type \(\rho\)’ is denoted \(\rho^*\), which we shall only use for \(\rho = 0, 1\). Since the usual coding of pairs of numbers goes through in \(\text{RCA}_0\), we shall not always distinguish between 0 and 0*. Similarly, we assume a fixed coding for finite sequences of type 1 and shall make use of the type ‘\(1^*\)’. In general, we do not always distinguish between ‘\(s^\rho\)’ and ‘\(\langle s^\rho \rangle\)’, where the former is ‘the object \(s\) of type \(\rho\)’, and the latter is ‘the sequence of type \(\rho^*\) with only element \(s^\rho\)’. The empty sequence for the type \(\rho^*\) is denoted by ‘\(\langle \rangle \)’, usually with the typing omitted. Furthermore, we denote by ‘\(|s| = n\)’ the length of the finite sequence \(s^\rho = \langle s^\rho_0, s^\rho_1, \ldots, s^\rho_{n-1} \rangle\), where \(|\langle \rangle| = 0\), i.e. the empty sequence has length zero. For sequences \(s^\rho, t^\rho\), we denote by ‘\(s * t\)’ the concatenation of \(s\) and \(t\), i.e. \((s * t)(i) = s(i)\) for \(i < |s|\) and \((s * t)(j) = t(|s| - j)\) for \(|s| \leq j < |s| + |t|\). For a sequence \(s^\rho\), we define \(\pi N := \langle s(0), s(1), \ldots, s(N - 1) \rangle\) for \(N^0 < \langle s \rangle\). For a sequence \(\alpha^{0 \to \rho}\), we also write \(\pi \alpha = \langle \alpha(0), \alpha(1), \ldots, \alpha(N - 1) \rangle\) for any \(N^0\). Finally, \((\forall q^0 \in Q^\rho).A(q)\) abbreviates \((\forall q^0 < |Q|).A(Q(i))\), which is (equivalent to) quantifier-free if \(A\) is.

**A.2 Further systems**

We define some standard higher-order systems that constitute the counterpart of e.g. \(\Pi^1_2\)-\text{CA}_0 and \(\mathbb{Z}_2\). First of all, the Suslin functional \(S^2\) is defined in [10] as:

\[
(\exists S^2 \leq_2 1)(\forall f^1)(\exists g^0)(\forall n^0)(f(\pi n) = 0) \iff S(f) = 0.
\]

(S2)

The system \(\Pi^1_2\)-\text{CA}_0 \equiv \text{RCA}_0^\omega + (S^2)\) proves the same \(\Pi^1_2\)-sentences as \(\Pi^1_2\)-\text{CA}_0 by [20] Theorem 2.2. By definition, the Suslin functional \(S^2\) can decide whether a \(\Sigma^1_1\)-formula as in the left-hand side of (S2) is true or false. We similarly define the functional \(S_k^2\) which decides the truth or falsity of \(\Sigma^1_k\)-formulas from \(L_2\); we also define the system \(\Pi^1_k\)-\text{CA}_0 as \(\text{RCA}_0^\omega + (S_k^2)\), where \((S_k^2)\) expresses that \(S_k^2\) exists. We note that the operators \(\nu_n\) from [2] p. 129] are essentially \(S_n^2\) strengthened to return a witness (if existant) to the \(\Sigma^1_n\)-formula at hand.

Secondly, second-order arithmetic \(\mathbb{Z}_2\) readily follows from \(\bigcup k \Pi^1_k\)-\text{CA}_0, or from:

\[
(\exists E^3 \leq_3 1)(\forall Y^2)(\exists f^1)(Y(f) = 0) \iff E(Y) = 0,
\]

(33)

and we therefore define \(\mathbb{Z}_2^2 \equiv \text{RCA}_0^\omega + (33)\) and \(\mathbb{Z}_2^\omega \equiv \bigcup k \Pi^1_k\)-\text{CA}_0, which are conservative over \(\mathbb{Z}_2\) by [9] Cor. 2.6. Despite this close connection, \(\mathbb{Z}_2^\omega\) and \(\mathbb{Z}_2^2\) can behave quite differently, as discussed in e.g. [13] §2.2. The functional from (33) is also called ‘\(3^3\)’, and we use the same convention for other functionals.