Einstein-Schrödinger theory using Newman-Penrose tetrad formalism

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Abstract. The Einstein-Schrödinger theory is modified to include a large cosmological constant caused by zero-point fluctuations. This “extrinsic” cosmological constant which multiplies the symmetric metric is assumed to be nearly cancelled by Schrödinger’s “bare” cosmological constant which multiplies the nonsymmetric fundamental tensor, such that the total cosmological constant is consistent with measurement. This modified Einstein-Schrödinger theory is expressed in Newman-Penrose form, and tetrad methods are used to confirm that it closely approximates ordinary general relativity and electromagnetism.

A solution for the connections in terms of the fundamental tensor is derived in the tetrad frame. The tetrad form of an exact electric monopole solution is shown to approximate the Reissner-Nordström solution and to be of Petrov type-D.

PACS numbers: 04.40.Nr, 98.80.Es, 04.20.Jb, 12.10.-g

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1. Introduction

The Einstein-Schrödinger theory without a cosmological constant was originally proposed by Einstein and Straus in 1946\cite{1, 2, 3, 4, 5}. Schrödinger made an important contribution to the theory by generalizing it to include a cosmological constant, and by showing that the theory can be derived from a very simple Lagrangian density if this cosmological constant is assumed to be non-zero\cite{6, 7, 8, 9}. This more general theory is usually called Schrödinger’s Affine Field Theory or the Einstein-Schrödinger Theory. This theory is a generalization of ordinary general relativity which allows a non-symmetric fundamental tensor and connection. Einstein and Schrödinger suspected that the antisymmetric part of the fundamental tensor might contain the electromagnetic field, but despite much effort this has never been demonstrated.

Recently we have shown\cite{10, 11} that a well motivated modification of the Einstein-Schrödinger theory does indeed closely approximate ordinary general relativity and electromagnetism, the modification being the addition of a cosmological constant caused by zero-point fluctuations. It is reasonable to assume that the Einstein-Schrödinger theory must eventually be quantized to accurately predict reality, and this cosmological constant can be viewed as a kind of zeroth order quantization effect\cite{12, 13, 14}. This “extrinsic” cosmological constant which multiplies the symmetric metric is assumed to be nearly cancelled by Schrödinger’s “bare” cosmological constant which multiplies the nonsymmetric fundamental tensor, resulting in a total cosmological constant which is consistent with measurement. The fact that these two cosmological constants multiply different fields has the effect of creating a Lorentz force, and also
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fixes some other problems with the original theory. The fine-tuning of cosmological constants is less objectionable when one considers that it is similar to renormalization methods which are commonplace in quantum field theory. For example, to account for self energy, “bare” particle masses become large (infinite if the cutoff wavenumber goes to infinity), but the total “physical” mass remains small. In a similar manner in the present theory, Schrödinger’s “bare” cosmological constant becomes large, but the total “physical” cosmological constant remains small. This can be viewed as a kind of energy density renormalization of the original Einstein-Schrödinger theory to account for zero-point fluctuations, and with this quantization effect included, the theory closely approximates ordinary general relativity and electromagnetism.

As in [10, 11] and in papers by several other authors [18, 19, 20, 16], the metric is defined as,

\[ g^{\sigma\tau} = \frac{\sqrt{-N}}{\sqrt{-g}} N^{-1(\sigma\tau)}, \quad \sqrt{-g} = \left[ -\det(\sqrt{-g}g^{\mu\nu}) \right]^{1/(n-2)} = \left[ -\det(\sqrt{-N}N^{-1(\mu\nu)}) \right]^{1/(n-2)}. \] (1)

Here \( N_{\mu\nu} \) is the fundamental tensor, \( N = \det(N_{\mu\nu}), g = \det(g_{\mu\nu}), \) “\( n \)” is the dimension, and \( N^{-1(\sigma\tau)} \) is the inverse of \( N_{\sigma\tau} \) so that \( N^{-1(\sigma\tau)}N_{\sigma\tau} = \delta_{\mu}^{\nu}. \) When \( N_{\mu\nu} \) is symmetric, the equation gives \( g_{\mu\nu} = N_{\mu\nu} \) as desired. In this paper, raising and lowering of indices is always done using \( \bigotimes \), and covariant derivative “\( ; \)” is done using the Christoffel connection \( \Gamma^\alpha_{\beta\gamma} \) formed from \( \bigotimes \).

The field equations of the Einstein-Schrödinger theory, including an extrinsic cosmological term [10], energy-momentum tensor [19], and charge currents [15, 16] are,

\[ \begin{align*}
\alpha R(\sigma\mu) + \Lambda_b N(\sigma\mu) + \Lambda_e g_{\sigma\mu} &= \frac{8\pi G}{c^4} \left( T_{\sigma\mu} - \frac{1}{(n-2)} g_{\sigma\mu} T^\alpha_{\alpha} \right),
\end{align*} \] (2)

\[ \begin{align*}
\alpha R(\sigma\mu,\nu) + \Lambda_b N(\sigma\mu,\nu) &= 0,
\end{align*} \] (3)

\[ \begin{align*}
N_{\sigma\mu,\beta} - \sigma^\alpha_{\beta\gamma} N_{\alpha\mu} - \sigma^\alpha_{\beta\mu} N_{\alpha\sigma} &= -\frac{8\pi}{c(n-1)} \frac{\sqrt{-g}}{\sqrt{-N}} \left( N_{\sigma[\alpha} N_{\beta]\mu] + \frac{1}{(n-2)} N_{[\alpha\beta]} N_{\mu\sigma} \right) j^\alpha, \quad (4)
\end{align*} \]

\[ \begin{align*}
\alpha \Gamma^\alpha_{\beta\gamma} &= \sigma^\alpha_{\beta\gamma}.
\end{align*} \] (5)

Here \( \alpha R_{\sigma\mu} = R_{\sigma\mu}(\sigma \Gamma) \) is the Ricci tensor, \( \sigma^\alpha_{\beta\gamma} \) is the non-symmetric connection, \( T_{\sigma\mu} \) is an energy-momentum tensor with no electromagnetic component, and \( j^\alpha \) is a charge current. Like Schrödinger, we include a bare cosmological constant \( \Lambda_b \) because this allows a very simple derivation of the theory [3, 10]. The total \( \Lambda \) is then

\[ \Lambda = \Lambda_b + \Lambda_e. \] (6)

From a theorem of tensor calculus [17], [3] implies that \( \alpha R(\sigma\mu) + \Lambda_b N(\sigma\mu) \) is a curl, so (6) can be written in the completely equivalent form

\[ \begin{align*}
\alpha R(\sigma\mu) + \alpha \Gamma^\alpha_{\beta\gamma} + 2\Lambda_b A_{\sigma(\mu]} + \Lambda_e N_{\sigma(\mu]} &= 0,
\end{align*} \] (7)

and (8) can be combined together in the completely equivalent form,

\[ \begin{align*}
\alpha R_{\sigma\mu} + \alpha \Gamma^\alpha_{\beta\gamma} + 2\Lambda_b A_{\sigma(\mu]} + \Lambda_b N_{\sigma\mu} + \Lambda_e g_{\sigma\mu} &= \frac{8\pi G}{c^4} \left( T_{\sigma\mu} - \frac{1}{(n-2)} g_{\sigma\mu} T^\alpha_{\alpha} \right). \quad (8)
\end{align*} \]

The tensor term \( \alpha \Gamma^\alpha_{\beta\gamma} \) is needed to retain Hermitian symmetry [1, 15, 16, 11], and \( \alpha R_{\sigma\mu} + \alpha \Gamma^\alpha_{\beta\gamma} \) is sometimes called the Hermitianized Ricci tensor. Contracting (11) with \( N^{-1(\mu\nu)} \) shows that this extra term vanishes when charge currents are absent,

\[ \begin{align*}
\alpha \Gamma^\alpha_{\beta\gamma} &= -\frac{8\pi}{c(n-1)(n-2)} \frac{\sqrt{-g}}{\sqrt{-N}} N_{\beta\gamma} = \frac{1}{2} N^{-1(\mu\nu)} N_{\sigma\mu,\beta} = \frac{1}{2N} \frac{\partial N}{\partial N_{\sigma\mu}} N_{\sigma\mu,\beta} = \frac{(\sqrt{-N})_{,\beta}}{\sqrt{-N}} \quad (9)
\end{align*} \]

\[ \begin{align*}
\alpha \Gamma^\alpha_{\beta\gamma} &= \frac{8\pi}{c(n-1)(n-2)} \frac{\sqrt{-g}}{\sqrt{-N}} N_{\beta\gamma} = (\ln \sqrt{-N})_{,\beta} = 0. \quad (10)
\end{align*} \]
Multiplying (2) by \(-N^{-\nu\rho}N^{-\nu\tau}\) and using (9) gives

\[
N^{-\nu\rho,\beta} + \Gamma^\tau_{\nu\beta} N^{-\nu\rho} + \Gamma^\rho_{\beta\nu} N^{-\nu\tau} = \frac{8\pi}{c(n-1)} \frac{\sqrt{-g}}{\sqrt{-N}} \left( j^\rho\delta^\tau_{\beta} + \frac{1}{(n-2)} \delta^\rho N_{[\alpha\beta]} N^{-\nu\tau} \right),
\]

(11)

Taking the antisymmetric part of (12) and contracting gives Ampere’s law

\[
f^\tau_{\rho,\tau} = \left( \frac{\sqrt{-g} f^\tau_{\rho\tau}}{\sqrt{-g}} \right) = \left( \frac{\sqrt{-N} N^{-\nu\rho}}{\sqrt{-g}} \right) = \frac{4\pi}{c} j^\rho,
\]

(13)

and the continuity equation

\[
j^\nu_{\rho,\rho} = (c/4\pi) f^\nu_{\rho,\tau} \rightarrow 0,
\]

(14)

where we define

\[
f^\rho_{\nu,\nu} = -\frac{\sqrt{-N}}{\sqrt{-g}} N^{-\nu\rho}.
\]

(15)

The Einstein equations are obtained by combining (2) with its contraction,

\[
\frac{8\pi G}{c^4} T^\tau_{\sigma\mu} = \delta^\tau_{\sigma\mu} + \Lambda_b \left( N_{[\sigma\mu]} - \frac{1}{2} \delta_{\sigma\mu} N_{\rho} \right) + \Lambda_c \left( 1 - \frac{n}{2} \right) g_{\sigma\mu},
\]

(16)

where we define

\[
\delta^\tau_{\sigma\mu} = \delta^\tau_{\sigma\mu} - \frac{1}{2} g_{\sigma\mu} \delta^\tau_{\rho}.
\]

(17)

Using (13) and (15), a generalized contracted Bianchi identity with charge currents can be derived for this theory [10, 11],

\[
\delta^\tau_{\sigma\nu} = \frac{3}{2} f^\nu_{\rho\sigma} R^\sigma_{[\nu\rho,\sigma]} + \frac{4\pi}{c} j^\nu (\delta^\sigma_{[\rho\sigma]} + \delta^\sigma_{[\nu\sigma]}).
\]

(18)

Using only the definitions (1, 15), another useful identity is derived in [11],

\[
\left( N_{[\sigma\nu]} - \frac{1}{2} \delta_{\sigma\nu} N_{\rho} \right)_{\mu} - \frac{3}{2} f^\nu_{\rho\mu} N_{[\nu\rho,\sigma]} = f^\nu_{\rho\mu} N_{[\mu\rho]}.
\]

(19)

Using [11, 3, 19, 21, 22], the divergence of the Einstein equations (16) gives the ordinary Lorentz force equation of general relativity and electromagnetism [11]

\[
\frac{8\pi G}{c^4} T^\nu_{\sigma\nu} = \frac{4\pi}{c} j^\nu (\delta^\rho_{[\nu\sigma]} + \delta^\rho_{[\nu\sigma]}) + \Lambda_b \frac{4\pi}{c} j^\nu N_{[\nu\sigma]} = \frac{8\pi}{c} \delta^\nu_{[\beta]} A_{[\sigma\nu]}.
\]

(20)

Assuming \( T_{\beta\nu} = \mu c^2 u_{\beta} u_{\nu} \), with \( u^\alpha = dx^\alpha / ds = \frac{1}{c} \mu c^2 m \) and \( (\mu u^\nu)_{\nu} = 0 \) from [14],

\[
\frac{c^2}{G} \Lambda_b j^\nu A_{[\sigma\nu]} = T^\nu_{\sigma\nu} = (\mu c^2 u_{\nu} u_{\sigma})_{\nu} = \mu c^2 u_{\nu} u_{\sigma,\nu} = \frac{\mu}{m} \left( m c \frac{d x_{\sigma}}{d s} \right) \cdot c \frac{d x_{\nu}}{d s}.
\]

(21)

Here the conversion to cgs units is,

\[
f_{\sigma\mu} = \sqrt{-\frac{2G}{c^4\Lambda_b}} f_{\sigma\mu}^{(cgs)}, \quad A_{\sigma} = \sqrt{-\frac{2G}{c^4\Lambda_b}} A_{\sigma}^{(cgs)}, \quad j_{\sigma} = \sqrt{-\frac{2G}{c^4\Lambda_b}} j_{\sigma}^{(cgs)}, \quad Q = \sqrt{-\frac{2G}{c^4\Lambda_b}} Q^{(cgs)}.
\]

(22)

Eq. (11) describes an implicit algebraic dependence of \( \delta^\rho_{[\sigma\nu]} \) on \( N_{[\sigma\mu]} \) and \( N_{[\beta\sigma]} \).

The solution for \( \delta^\rho_{[\beta\sigma]} N_{[\mu\nu]} \) yields the Christoffel connection for the symmetric case,

\[
\Gamma^\alpha_{\sigma\mu} = \frac{1}{2} g^{\alpha\nu} (g_{\mu\nu,\sigma} + g_{\nu\sigma,\mu} - g_{\sigma\mu,\nu}).
\]

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There is also a fairly simple solution for the special case $f^\sigma_\mu f^\mu_\sigma = \det(f^\mu_\nu) = 0$. The solution for $^\sigma \Gamma_\sigma^\rho_\beta (N_\mu \nu)$ is much more complicated in the general non-symmetric case, and this has been a big obstacle in working with the theory. Here we will derive a solution in the Newman-Penrose tetrad frame which applies for all cases except $f^\sigma_\mu f^\mu_\sigma = \det(f^\mu_\nu) = 0$. It is given as an addition $Y^\tau_\nu_\beta$ to the Christoffel connection,

$$^\sigma \Gamma^\tau_\nu_\beta = \Gamma^\tau_\nu_\beta + Y^\tau_\nu_\beta.$$  \hspace{1cm} (24)

Extracting $Y^\tau_\nu_\beta$ from the Ricci tensor gives,

$$^\sigma R^\sigma_\mu + ^\sigma \Gamma^\alpha_\alpha_\sigma^\beta - \bar{\Gamma}^\alpha_\alpha_\sigma^\beta = \bar{\Gamma}^\alpha_\alpha_\sigma^\beta \bar{Y}^\alpha_\nu_\beta + \bar{\Gamma}^\alpha_\nu_\beta \bar{Y}^\alpha_\nu_\beta.$$  \hspace{1cm} (25)

Here $R^\sigma_\mu = R^\sigma_\mu (\Gamma)$ is the ordinary Ricci tensor. Substituting (25) into (8) or into (2,3,7,16), and working in the tetrad frame, one then has an explicit version of the field equations. It must be mentioned that the solution for $^\sigma \Gamma^\alpha_\sigma^\beta_\gamma (N_\mu \nu)$ in the Newman-Penrose tetrad frame has in a sense already been done in [22], prior to the development of the full Newman-Penrose formalism. However, [22] uses a different metric definition and does not include charge currents. This reference also uses completely non-standard conventions and notation, and does not simplify results.

This paper is organized as follows. In §2 the Newman-Penrose formalism is applied to the non-symmetric fields of the Einstein-Schrödinger theory. In §3 the field equations are expressed in tetrad form. In §4 a solution is derived for $^\sigma \Gamma^\alpha_\sigma^\beta_\gamma (N_\mu \nu)$ in the tetrad frame. In §5 the theory is shown to approximate ordinary general relativity and electromagnetism when a cosmological constant from zero-point fluctuations is assumed. In §6 the tetrad form of an exact electric monopole solution is shown to approximate the Reissner-Nordström solution and to be of Petrov type-D.

2. Application of the Newman-Penrose Formalism

In the remainder of this paper we will assume $n=4$. Let us write (11) as

$$W^\sigma_\mu = \frac{\sqrt{-N}}{\sqrt{-g}} N^{-\mu_\sigma} = g^\sigma_\mu + f^\sigma_\mu.$$  \hspace{1cm} (26)

It is proven in [22] and in Appendix D that if we do not have $f^\sigma_\mu f^\mu_\sigma = \det(f^\mu_\nu) = 0$, then $W^\sigma_\mu$ can be decomposed such that,

$$W^\sigma_\mu = W^{ab} \epsilon^a_\sigma \epsilon^b_\mu,$$  \hspace{1cm} (27)

$$W^{ab} = \begin{pmatrix} 0 & (1+\bar{u}) & 0 & 0 \\ (1-\bar{u}) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1+\bar{i}u) \\ 0 & 0 & -(1-\bar{i}u) & 0 \end{pmatrix},$$  \hspace{1cm} (28)

$$g_{ab} = g^{ab} = W^{(ab)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$  \hspace{1cm} (29)

$$-f_{ab} = f^{ab} = W^{[ab]} = \begin{pmatrix} 0 & \bar{u} & 0 & 0 \\ -\bar{u} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{i}u \\ 0 & \bar{i}u & 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (30)

Here tetrad indices are indicated with Latin letters. As with the usual Newman-Penrose formalism, the tetrads $e^a_\sigma$ and inverse tetrads $e^\alpha_\mu$ both consist of two real
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vectors and two complex conjugate vectors, and are related via raising and lowering of indices with $g^{ab}$ and $g_{\sigma\mu}$,

\begin{align}
  v_\sigma &= e_1^\sigma, \quad u_\sigma = e_2^\sigma, \quad m_\sigma = e_3^\sigma, \quad m^*_\sigma = e_4^\sigma, \\
  l_\sigma &= e_2^\sigma, \quad n_\sigma = e_1^\sigma, \quad m_\sigma = -e_4^\sigma, \quad m^*_\sigma = -e_3^\sigma, \\
  \delta^{\sigma}_{\mu} &= e_\alpha^\sigma e^{\alpha\mu}, \quad \delta^a_b = e_b^\sigma e^a_\sigma, \\
  e &= \det(e^\alpha_\nu) = \varepsilon^{\alpha\beta\sigma\mu} l_\alpha n_\beta m_\sigma m^*_\mu, \\
  e^* &= -e.
\end{align}

One difference from the usual Newman-Penrose formalism is that gauge freedom is restricted so that only type III tetrad transformations can be used. If $W^{\sigma\mu}$ is real, the scalars “$\hat{u}$” (u grave) and “$\check{u}$” (u check) are real and are given by\cite{22}

\begin{align}
  \hat{u} &= \sqrt{\omega - \ell/4}, \\
  \check{u} &= \sqrt{\omega + \ell/4}, \\
  \omega &= (\ell/4)^2 - f/g, \\
  f &= \det(f_{\mu\nu}), \quad g = \det(g_{\mu\nu}), \\
  f/g &= -\check{u}^2 \hat{u}^2, \\
  \ell &= f^\sigma_{\mu} f^\mu_{\sigma} = 2(\hat{u}^2 - \check{u}^2).
\end{align}

If $W^{\sigma\mu}$ is instead Hermitian, things are unchanged except that “$\hat{u}$” and “$\check{u}$” are imaginary instead of real.

From\cite{20 28}, the fundamental tensor of the Einstein-Schrödinger theory is,

\begin{align}
  N^{-\sigma\nu}_{\abc} &= \sqrt{-g_0} \begin{pmatrix}
    0 & (1-\hat{u}) & 0 & 0 \\
    0 & 0 & 0 & -(1-i\check{u}) \\
    (1+\hat{u})\hat{c}/\check{c} & 0 & 0 & 0 \\
    0 & 0 & 0 & -(1+i\check{u})\hat{c}/\check{c}
  \end{pmatrix}, \\
  N_{\bc} &= \begin{pmatrix}
    0 & (1-\hat{u})\hat{c}/\check{c} & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & (1+i\check{u})\hat{c}/\check{c} \\
    0 & 0 & 0 & -(1+i\check{u})\hat{c}/\check{c}
  \end{pmatrix},
\end{align}

where

\begin{align}
  \hat{c} &= \frac{1}{\sqrt{1+\hat{u}^2}} = \sqrt{1-s^2}, \\
  \hat{u} &= \hat{s}/\hat{c}, \\
  \check{c} &= \frac{1}{\sqrt{1-\hat{u}^2}} = \sqrt{1+s^2}, \\
  \check{u} &= \check{s}/\check{c}, \\
  \sqrt{-N_0} &= \sqrt{-\det(N_{\abc})} = \frac{i}{\hat{c}\check{c}}. \\
  \sqrt{-g_0} &= \sqrt{-\det(g_{\abc})} = i, \\
  \sqrt{-N} &= \sqrt{-N_0} e = \frac{ie}{\hat{c}\check{c}}, \\
  \sqrt{-g} &= \sqrt{-g_0} e = ie.
\end{align}

Note the correspondence of $\hat{s}, \hat{c}, \hat{u}$ and $\check{s}, \check{c}, \check{u}$ to circular and hyperbolic trigonometry functions.
Covariant derivative is done in the usual fashion,

\[ T^a_{b;c} = e^a_{\sigma} e^{\mu}_{\tau} T^\mu_{\mu:;c} e^\tau_{c;\tau} = T^a_{b;c} + \gamma^a_{\alpha d} T^d_{b} - \gamma^d_{b c} T^a_{d}. \]

(52)

For the spin coefficients we will follow the conventions of Chandrasekhar,\[24\]

\[ \gamma_{abc} = \frac{1}{2} (\lambda_{abc} + \lambda_{cab} - \lambda_{bca}) = e_a^{\mu} e_b^{\mu;\alpha} e_c^{\alpha}, \]

(53)

\[ \gamma_{abc} = -\gamma_{bac}, \quad \gamma_{\alpha bc} = 0, \]

(54)

\[ \lambda_{abc} = (e_{b\sigma;\mu} - e_{b\mu;\sigma}) e_a^{\sigma} e_c^{\mu} e_a^{\mu;\mu} = \lambda_{abc} - \lambda_{cba}, \]

(55)

\[ \lambda_{abc} = -\lambda_{cba}, \]

(56)

\[ \rho = \gamma_{314} , \quad \mu = \gamma_{243} , \quad \tau = \gamma_{312} , \quad \pi = \gamma_{241} , \]

(57)

\[ \kappa = \gamma_{311} , \quad \sigma = \gamma_{313} , \quad \lambda = \gamma_{244} , \quad \nu = \gamma_{242} , \]

(58)

\[ \epsilon = (\gamma_{211} + \gamma_{341})/2 , \quad \gamma = (\gamma_{212} + \gamma_{342})/2 , \]

(59)

\[ \alpha = (\gamma_{214} + \gamma_{344})/2 , \quad \beta = (\gamma_{213} + \gamma_{343})/2 . \]

(60)

With these coefficients and with other tetrad quantities, complex conjugation causes the exchange 3→4, 4→3. As usual we may also define directional derivative operators,

\[ D = e^\alpha_1 \frac{\partial}{\partial x^\alpha} , \quad \Delta = e^\alpha_2 \frac{\partial}{\partial x^\alpha} , \quad \delta = e^\alpha_3 \frac{\partial}{\partial x^\alpha} , \quad \delta^* = e^\alpha_4 \frac{\partial}{\partial x^\alpha}. \]

(61)

3. The Field Equations in Newman-Penrose Form

Substituting (52) into (8) gives the field equations,

\[ 8\pi G \left( T_{bd} - \frac{1}{2} g_{bd} T^a_a \right) = R_{bd} + 2 \Lambda_b A_{[b|d]} + \Lambda_b N_{bd} + \Lambda_c g_{bd} + \nabla^c R_{bd} \left( T^a_a \right) - \Lambda_c \nabla^c g_{bd} \]

\[ - \nabla^a \Phi_{b|d} - \nabla^b \Phi_{a|d} + \nabla^c \Phi_{d|a} - \nabla^d \Phi_{c|a} - \nabla^c \Phi_{d|a} - \nabla^d \Phi_{c|a}. \]

(62)

Taking the symmetric and antisymmetric parts of this and rearranging gives,

\[ R_{bd} = \frac{8\pi G}{c^4} \left( T_{bd} - \frac{1}{2} g_{bd} T^a_a \right) - \Lambda_b N_{bd} - \Lambda_c g_{bd} \]

\[ - \nabla^a \Phi_{b|d} - \nabla^b \Phi_{a|d} + \nabla^c \Phi_{d|a} - \nabla^d \Phi_{c|a} - \nabla^c \Phi_{d|a} - \nabla^d \Phi_{c|a}. \]

\[ \Lambda_b N_{bd} = 2 \Lambda_b A_{[b|d]} - \nabla^c \Phi_{d|a} + \nabla^d \Phi_{c|a}. \]

(63)

(64)

The usual Ricci identities will be valid if we define \( \Phi_{ab} \) values in terms of the right-hand side of (63),

\[ \Phi_{10} = -R_{11}/2, \quad \Phi_{22} = -R_{22}/2, \quad \Phi_{02} = -R_{33}/2, \quad \Phi_{20} = -R_{44}/2, \]

(65)

\[ \Phi_{11} = -(R_{12} + R_{34})/4, \quad \Lambda = R/24 = (R_{12} - R_{34})/12. \]

(66)

(67)

From (60) (22), Ampère's law \[103\] becomes,

\[ 4\pi c j^c = f^{bc}_{-b} + \gamma^b_{ab} f^{ac}_{-a} + \gamma^c_{ab} f^{ba}_{-b}, \]

(68)

\[ 4\pi c j^2 = f^{12}_{-1} + \gamma^3_{12} f^{12} + \gamma^4_{12} f^{12} + \gamma^2_{34} f^{34} + \gamma^2_{43} f^{34} \]

\[ = D\dot{u} - \rho^* \dot{u} - \rho \dot{u} - \rho \dot{u} + \rho^* \dot{u}, \]

\[ = D\dot{u} - \rho \dot{w} - \rho^* \dot{w}, \]

(69)

\[ 4\pi c j^1 = f^{21}_{-2} + \gamma^3_{21} f^{21} + \gamma^4_{21} f^{21} + \gamma^1_{34} f^{34} + \gamma^3_{43} f^{34} \]

(70)

(71)

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\[ \Delta \bar{u} - \mu \bar{u} - \mu \bar{u} + \mu \bar{u} - \mu \bar{u} \]
\[ \Delta \bar{u} - \mu \bar{w} - \mu \bar{w}, \]
\[ \frac{4\pi}{c} j^4 = \gamma^4_{31} j^{34} + \gamma^4_{32} j^{34} + \gamma^4_{12} f^{21} + \gamma^4_{21} f^{12} \]
\[ = -i \delta \dot{u} - \pi \dot{u} + \pi \dot{u} + \pi \dot{u} \]
\[ = -i \delta \dot{u} + \tau \dot{u} + \tau \dot{u}, \]

where
\[ w = \dot{u} + i \dot{u} \]

The connection equations are easier to work with in contravariant form than in covariant form. Multiplying \( \mathbf{1} \) by \( \sqrt{-N} / \sqrt{-g} \) and using \( \mathbf{2} \) gives

\[ 0 = O_{bd} = \sqrt{-N} \left( N^{-cd} b + \gamma_{ab} N^{ac} + \gamma_{bc} N^{ad} + \gamma_{ab} N^{ad} + \gamma_{bc} N^{ad} \right) \]
\[ + \frac{8 \pi}{3c} \left( j^{[4]} \delta_b^c + \frac{1}{2} j^a N_{[ab]} c \dot{c} (1 + \dot{u}) \right), \]

\[ 0 = O_{11} = \gamma_{1b}^{1} (1 - \dot{u}) \]
\[ 0 = O_{22} = \gamma_{1b}^{2} (1 + \dot{u}) \]
\[ 0 = O_{33} = -\gamma_{2b}^{2} (1 - \dot{u}) - \gamma_{3b}^{2} (1 + \dot{u}), \]

\[ 0 = O_{34}^{b} = \pm i \dot{u} - (1 + i \dot{u}) \left( \frac{\sqrt{-N}}{\sqrt{-g}} b + \gamma_{ab} N^{ac} + \gamma_{bc} N^{ad} \right) \]
\[ + \frac{8 \pi}{3c} \left( \pm j^{[4]} \delta_b^c + \frac{1}{2} j^a N_{[ab]} c \dot{c} (1 + \dot{u}) \right), \]

\[ 0 = O_{24}^{b} = \gamma_{3b} (-1 + \dot{u}) + \gamma_{2b} (1 + \dot{u}) \]
\[ = \pm \gamma_{3b} w + \gamma_{2b} (1 + \dot{u}) \pm \gamma_{2b} (1 - \dot{u}) \pm \gamma_{3b} (1 - \dot{u}) \pm \gamma_{3b} (1 + \dot{u}), \]

\[ 0 = O_{3b}^{13} = \gamma_{2b} (1 + \dot{u}) + \gamma_{4b} (1 + \dot{u}) \]
\[ = \pm \gamma_{2b} w + \gamma_{4b} (1 + \dot{u}) + \gamma_{4b} (1 - \dot{u}) \]

To save space in the equations above we are using the notation,

\[ -O_{bd} = +O_{bd} = O_{bd}^{cd}, -Y_{cd}^{b} = +Y_{cd}^{b} = Y_{bc}^{d}. \]
4. An Exact Solution for the Connection Addition in the Tetrad Frame

The connection equations can be solved by forming linear combinations of them where all of the $\Upsilon^a_{bc}$ terms cancel except for the desired one. The required linear combinations are listed in Appendix A and the calculations are done in Appendix B. The result, split into symmetric and antisymmetric components is below:

\[
\Upsilon^2_{(12)} = \frac{e^2}{\bar{e}^2} \left( \Delta \bar{u} - \frac{4\pi}{3c} \bar{u} j^2 \right),
\]

\[
\Upsilon^1_{(12)} = \frac{e^2}{\bar{e}^2} \left( \Delta \bar{u} + \frac{4\pi}{3c} \bar{u} j^1 \right),
\]

\[
\Upsilon^4_{(34)} = -\frac{e^2}{\bar{e}^2} \left( \bar{u} \delta \bar{u} + \frac{4\pi}{3c} \bar{u} j^4 \right),
\]

\[
\Upsilon^1_{(11)} = \bar{u} \Delta \bar{u} c^2 - \bar{u} \Delta \bar{u} c^2 + \frac{4\pi}{3c} \bar{u} j^2,
\]

\[
\Upsilon^2_{(22)} = \bar{u} \Delta \bar{u} c^2 - \bar{u} \Delta \bar{u} c^2 - \frac{4\pi}{3c} \bar{u} j^1,
\]

\[
\Upsilon^3_{(33)} = \bar{u} \delta \bar{u} c^2 - \bar{u} \delta \bar{u} c^2 - \frac{4\pi}{3c} \bar{u} j^4,
\]

\[
\Upsilon^2_{(11)} = \Upsilon^1_{(22)} = \Upsilon^3_{(44)} = 0,
\]

\[
\Upsilon^2_{(23)} = \frac{i\bar{u}}{2} (\Delta \bar{u} c^2 - i \delta \bar{u} c^2) - \frac{2\pi}{3c} \bar{u} j^4,
\]

\[
\Upsilon^1_{(13)} = -\frac{i\bar{u}}{2} (\Delta \bar{u} c^2 + i \delta \bar{u} c^2) - \frac{2\pi}{3c} \bar{u} j^4,
\]

\[
\Upsilon^3_{(13)} = -\frac{\bar{u}}{2} (\bar{D} \bar{u} c^2 + i \bar{D} \bar{u} c^2) + \frac{2\pi}{3c} \bar{u} j^2,
\]

\[
\Upsilon^3_{(23)} = -\frac{\bar{u}}{2} (\Delta \bar{u} c^2 - i \Delta \bar{u} c^2) - \frac{2\pi}{3c} \bar{u} j^1,
\]

\[
\Upsilon^4_{(12)} = -\frac{\bar{u}}{2} (\Delta \bar{u} c^2 - i \Delta \bar{u} c^2) - \frac{2\pi}{3c} \bar{u} j^1,
\]

\[
\Upsilon^4_{(34)} = -\frac{i\bar{u}}{2} (\bar{D} \bar{u} c^2 + \rho w - \rho^* w^*),
\]

\[
\Upsilon^3_{(43)} = -\frac{i\bar{u}}{2} (\bar{D} \bar{u} c^2 + \mu w - \mu^* w^*),
\]

\[
\Upsilon^2_{(13)} = \frac{\kappa w}{\bar{z}}, \quad \Upsilon^1_{(24)} = -\frac{\nu w}{\bar{z}},
\]

\[
\Upsilon^3_{(13)} = \frac{\sigma w}{\bar{z}}, \quad \Upsilon^4_{(24)} = -\frac{\lambda w}{\bar{z}},
\]

\[
\Upsilon^4_{(11)} = \frac{\kappa w}{\bar{z}}, \quad \Upsilon^3_{(22)} = -\frac{\nu w}{\bar{z}},
\]

\[
\Upsilon^3_{(33)} = \frac{\sigma w}{\bar{z}}, \quad \Upsilon^1_{(44)} = -\frac{\lambda w}{\bar{z}},
\]

\[
\Upsilon^2_{(12)} = -\frac{\bar{u} \Delta \bar{u}}{\bar{e}^2} + \frac{4\pi}{3c} \bar{u} j^2,
\]

\[
\Upsilon^1_{(12)} = -\bar{u} \Delta \bar{u} - \frac{4\pi}{3c} \bar{u} j^1,
\]
\[ \Upsilon_{[34]} = -\frac{i\sigma^2}{3} \dot{u} - 4\pi^2 \frac{\alpha^2}{3c} j^4, \] (112)

\[ \Upsilon_{[23]} = \frac{1}{2} (\delta \dot{u} c^2 - i \delta \dot{u} i^2) - \frac{2\pi^2}{3c} j^4, \] (113)

\[ \Upsilon_{[13]} = -\frac{1}{2} (\delta \dot{u} c^2 + i \delta \dot{u} i^2) - \frac{2\pi^2}{3c} j^4, \] (114)

\[ \Upsilon_{[3]} = \frac{1}{2} (\Delta \dot{u} c^2 + i \Delta \dot{u} i^2) - \frac{2\pi^2}{3c} j^4, \] (115)

\[ \Upsilon_{[3]} = -\frac{1}{2} (\Delta \dot{u} c^2 - i \Delta \dot{u} i^2) - \frac{2\pi^2}{3c} j^4, \] (116)

\[ \Upsilon_{[12]} = \frac{1}{2} \left( \delta \dot{u} \frac{c^2}{c^2} + \tau w - \pi^* w^* \right), \] (117)

\[ \Upsilon_{[34]} = \frac{1}{2} \left( i \Delta \dot{u} \frac{c^2}{c^2} + \rho w - \rho^* w^* \right), \] (118)

\[ \Upsilon_{[4]} = -\frac{1}{2} \left( i \Delta \dot{u} \frac{c^2}{c^2} - \mu w + \mu^* w^* \right), \] (119)

\[ \Upsilon_{[13]} = -\frac{\kappa w}{z}, \quad \Upsilon_{[24]} = \frac{-\nu w}{z}, \] (120)

\[ \Upsilon_{[13]} = \frac{\sigma w}{z}, \quad \Upsilon_{[24]} = \frac{\lambda w}{z}, \] (121)

where

\[ \dot{z} = [(1 + i\dot{u})^2(1 + i\dot{u}) + (1 \mp i\dot{u})^2(1 \mp i\dot{u})]/2 = 1 + 2i\dot{u} - \dot{u}^2, \] (122)

\[ \dot{\bar{z}} = [(1 \mp \dot{u})^2(1 \mp i\dot{u}) + (1 \pm i\dot{u})^2(1 \mp i\dot{u})]/2 = 1 + 2i\dot{u} + \dot{u}^2. \] (123)

As an error check, it is easy to verify that these results agree with (9) and (5).

\[ \Upsilon_{(ab)} = \dot{u} \dot{\bar{u}} c^2 - \dot{u} \dot{u} c^2 + \frac{8\pi}{3c} \left( \dot{u} c^2 \delta_b^a j^4 \right) - i\dot{u} \dot{u} c^2 \delta^a_b j^4 \] (124)

\[ = -\left( \frac{\sqrt{-g_0}}{\sqrt{-g_0}} \right)_b \left( \frac{\sqrt{-g_0}}{\sqrt{-g_0}} \right)_b + \frac{4\pi}{3c} \sqrt{-g_0} j^a N_{[ab]}, \] (125)

\[ \Upsilon_{[ba]} = 0. \] (126)

5. Approximation of Classical General Relativity and Electromagnetism

In [10, 11] it is shown that this theory closely approximates ordinary general relativity and electromagnetism. Here we will confirm this by deriving some of the results in [10, 11] with tetrad methods.

We will make much use of the small-skew approximation [11],

\[ |f_{\alpha \beta}| \ll 1. \] (127)

It is a widely accepted technique, and is used heavily in research on this topic. We will refer to equations as being accurate to order \( f^1 \) or \( f^2 \) etc., meaning that higher order terms such as \( f^\alpha \beta f^\beta \alpha f^\alpha \mu \) are being ignored. To a limited extent we will also use the approximation of small rates of change and small spatial curvatures,

\[ |f^\alpha \beta \mu / f^\alpha \beta| \ll \sqrt{\Lambda_b}, \] (128)

\[ |C_{\sigma \mu \alpha \rho}| \ll \Lambda_b, \] (129)
where $C_{\sigma \mu \rho}$ is the Weyl tensor. The symbols $||$ mean the largest measurable component for some standard spherical or cartesian coordinate system. If an equation has a tensor term which can be neglected in one coordinate system, it can be neglected in any coordinate system, so it is only necessary to prove it in one coordinate system. We will see that the approximations above are satisfied to such an extreme degree that it is simply not necessary to define them more rigorously.

Let us consider worst-case values of (127,128,129). We will assume that $\Lambda_c$ is caused by zero-point fluctuations with a cutoff wave-number $27, 26, 25, 28, 29$

\[ k_c = C_c/l_P, \quad C_c \sim 1, \]  

where $l_P = \sqrt{\hbar G/c^3} = 1.6 \times 10^{-33}\text{cm}$ is the Planck length. Then assuming all of the known fundamental particles we have [12, 10].

\[ \Lambda_b \approx -\Lambda_c \sim C_z k_c^4 l_P^2 \sim C_c C_z/l_P^2 \sim 10^{66}\text{cm}^{-2}, \quad C_z \sim 60/2\pi? \]  

(131)

For a charged particle with $f^1_0 = Q/r^2$, applying the conversion to cgs units [22] shows that $|f^1_0| = 1$ would occur at the radius

\[ r_c = \sqrt{|Q|} = \sqrt{\frac{e^2}{2G/c^4 \Lambda_b}} = \sqrt{l_P \sqrt{\frac{2e^2}{c^4 \hbar \Lambda_b}}} = \frac{l_P}{C_c} \left( \frac{2\alpha}{C_z} \right)^{1/4} \sim 10^{-33}\text{cm} \]  

(132)

where $\alpha = e^2/hc \approx 1/137$ is the fine structure constant. For atomic radii near the Bohr radius ($a_0 = \hbar^2/m_e e^2 = 5.3 \times 10^{-9}\text{cm}$) we have,

\[ |f^1_0| \sim r_e^2/a_0^2 \sim 10^{-50}, \]  

(133)

\[ |f^1_0|/\sqrt{\Lambda_b} f^1_0 \sim 2/\sqrt{\Lambda_b} a_0 \sim 10^{-24}. \]  

(134)

For the smallest radii probed by high energy particle physics experiments ($10^{-17}\text{cm}$),

\[ |f^1_0| \sim r_e^2/(10^{-17})^2 \sim 10^{-32}, \]  

(135)

\[ |f^1_0|/\sqrt{\Lambda_b} f^1_0 \sim 2/\sqrt{\Lambda_b} 10^{-17} \sim 10^{-16}. \]  

(136)

The fields at this radius are larger than near any macroscopic charged object, and are also larger than the strongest plane-wave fields. We must also consider rates of change for the highest energy gamma rays ($10^{20}\text{eV}$) where

\[ |f^1_0|/\sqrt{\Lambda_b} f^1_0 \sim E/hc \sqrt{\Lambda_b} \sim 10^{-8}. \]  

(137)

The largest observable values of the Weyl tensor might be expected to occur near the Schwarzschild radius, $r_s = 2Gm/c^2$, of black holes, where it takes on values around $r_s/r^3$. However, since the lightest black holes have the smallest Schwarzschild radius, they will create the largest value of $r_s/r^3 = 1/r^3_s$. The lightest black hole that we can expect to observe would be of about one solar mass, where

\[ \frac{C_{1010}}{\Lambda_b} \sim \frac{1}{\Lambda_b r^2_s} = \Lambda_b \left( \frac{c^2}{2Gm_{\odot}} \right)^2 \sim 10^{-77}. \]  

(138)

Clearly the approximations [127,128,129] are extremely accurate. This is particularly true for the small-skew approximation because terms with higher powers of $f^\sigma_{\mu}$ are usually two powers higher than leading order terms, so that from (132) they will be $< 10^{-64}$ of the leading order terms.
The tetrad formalism allows the small-skew approximation to be stated somewhat more rigorously as \(|\hat{u}| \ll 1, |\hat{u}| \ll 1\). From \([139,140]\), a charged particle will have \(\hat{u} \approx Q/r^2, \hat{u} = 0\). From \([141,146,143,20,30]\) we have, to second order in \(\hat{u}\) and \(\hat{u}\),

\[
\begin{align*}
\hat{c}/\hat{c} & \approx 1 + \hat{u}^2/2 + \hat{u}^2/2 = 1 + \hat{u}^2 - \ell/4, \\
-\hat{c}/\hat{c} & \approx -1 + \hat{u}^2/2 + \hat{u}^2/2 = -1 + \hat{u}^2 + \ell/4,
\end{align*}
\]

(139)

(140)

\[
N_{(ab)} = \begin{pmatrix}
0 & \hat{c}/\hat{c} & 0 & 0 \\
\hat{c}/\hat{c} & 0 & 0 & 0 \\
0 & 0 & \hat{c}/\hat{c} & 0 \\
0 & 0 & 0 & -\hat{c}/\hat{c}
\end{pmatrix} \approx g_{ab} + f_a^\gamma f_{cb} - \frac{1}{4} g_{ab} \ell,
\]

(141)

\[
N_{[ab]} = \begin{pmatrix}
0 & -\hat{u}\hat{c}/\hat{c} & 0 & 0 \\
\hat{u}\hat{c}/\hat{c} & 0 & 0 & 0 \\
0 & 0 & 0 & i\hat{u}\hat{c}/\hat{c} \\
0 & 0 & -i\hat{u}\hat{c}/\hat{c} & 0
\end{pmatrix} \approx f_{ab}.
\]

(142)

These \(n = 4\) results match the order \(f^2\) approximations derived in \([10]\),

\[
N_{(\sigma\mu)} \approx g_{\sigma\mu} + f_{\sigma}^\nu f_{\nu\mu} - \frac{1}{2(n-2)} g_{\sigma\mu} \ell,
\]

(143)

\[
N_{[\sigma\mu]} \approx f_{\sigma\mu}.
\]

(144)

The next higher order terms of \([139,140]\) will be two orders higher in \(\hat{u}\) and \(\hat{u}\) than the leading order terms. This confirms that the next higher order terms in \([143,144]\) will be two orders higher in \(f^\mu_\nu\) than the leading order terms, and from \([135]\) these terms must be \(< 10^{-64}\) of the leading order terms. Also, while we are mostly ignoring the special case \(f^\sigma_\mu f^\mu_\sigma = \det(f^\mu_\nu) = 0\) in this paper, it is easy to show from Appendix D that for this case, \([143,144]\) are exact instead of approximate.

In Appendix C it is shown that to second order in \(\hat{u}\) and \(\hat{u}\), the exact \(n = 4\) solution for \(\Upsilon^\sigma_{bc}\) in \([4]\) matches the order \(f^2\) approximation derived in \([10]\),

\[
\begin{align*}
\Upsilon^\alpha_{(\sigma\mu)} & \approx f^\tau_{(}\sigma f\mu)^{\alpha}\tau + f^{\alpha\tau} f_{(\tau(\sigma;\mu) + \frac{1}{4(n-2)} (\ell,^\alpha g_{\sigma\mu} - 2 \ell,_{(\sigma} \delta^\alpha_{\mu)})} \\
& + \frac{4\pi}{c(n-2)} \ell^p (f^\alpha_\mu g_{\sigma\mu} + \frac{1}{2(n-1)} f_{p(\sigma} \delta^\alpha_{\mu)}), \\
\Upsilon^\alpha_{[\sigma\mu]} & \approx \frac{1}{2} (f_{\sigma\mu} + f_{\sigma}^{\mu;\alpha} - f^\alpha_{\sigma;\mu}) + \frac{8\pi}{c(n-1)} f_{[\alpha} \delta^\alpha_{\mu]}, \\
\Upsilon^\alpha_{\alpha\sigma} & \approx \frac{-1}{2(n-2)} \ell,_{\alpha} + \frac{8\pi}{c(n-1)(n-2)} f_{\alpha\sigma}.
\end{align*}
\]

(145)

(146)

(147)

The tetrad version of \([145,146]\) in Appendix C differs from the exact solution in \([4]\) only by the factors \(\hat{c},\hat{c},\hat{z},\hat{z}\), and from \([139,140,122,123]\) these factors will induce terms which are two orders higher in \(\hat{u}\) and \(\hat{u}\) than the leading order terms. This confirms that the next higher order terms in \([145,147]\) will be two orders higher in \(f^\mu_\nu\) than the leading order terms, and from \([135]\) these terms must be \(< 10^{-64}\) of the leading order terms.

Substituting \([143]\) into \([16]\) and using \([11]\) gives the order \(f^2\) Einstein equations,

\[
\begin{align*}
\varepsilon G_{\sigma\mu} & \approx \frac{8\pi G}{c^2} T_{\sigma\mu} - \Lambda_\ell (f_{\sigma}^\nu f_{\nu\mu} - \frac{1}{4} g_{\sigma\mu} f^{\nu\rho} f_{\nu\rho}) + \Lambda \left( \frac{n}{2} - 1 \right) g_{\sigma\mu}.
\end{align*}
\]

(148)

With the conversion to cgs units \([22]\), the second term in \([148]\) is the ordinary electromagnetic energy-momentum tensor. By substituting \([145,146,147]\) into \([25,17]\),
one can derive \[^{10}{11}\] an order \( f^2 \) approximation of \( \mathcal{G}_{\sigma \mu} \) in terms of the ordinary Einstein tensor \( \mathcal{G}_{\sigma \mu} = R_{\sigma \mu} - (1/2)g_{\sigma \mu} R \). However, without actually doing the calculation, it is easy to see that \( \mathcal{G}_{\sigma \mu} \) and \( G_{\sigma \mu} \) can differ only by second order terms such as \( f^\tau (\sigma f_{\mu})^\alpha_{\tau \alpha} \) and \( f^\nu (\sigma f_{\mu})^\alpha_{\mu \nu} \). This result applies with or without charge currents since \( 4 \pi f^\rho / c = f^\rho_{\rho \tau} \) from \[^{13}\]. Therefore from \[^{13},^{14}\], such additional terms must be \( < 10^{-16} \) of the ordinary electromagnetic term.

Combining \[^{7},^{14},^{25},^{140}\] gives, to order \( f^2 \),

\[
2 \Lambda_b A_{[\sigma,\mu]} + \Lambda_b f_{\sigma \mu} \approx - \left( \mathcal{G}_{[\sigma,\mu]} + \mathcal{G}^\alpha_{\alpha [\sigma,\mu]} \right) \approx - \mathcal{G}^\alpha_{[\sigma,\mu];\alpha} \quad (149)
\]

\[
\approx - \frac{1}{2} \left( f_{\sigma \mu;}^\alpha + f_{\mu \nu}^\alpha - f_{\sigma (\mu ;\alpha)}^\alpha - \frac{8 \pi}{c(n-1)} j_{[\sigma,\mu]} \right) \quad (150)
\]

\[
\approx - \frac{3}{2} f_{[\sigma,\mu];\alpha}^\alpha + 2 f_{[\sigma,\mu];\alpha}^\alpha - \frac{8 \pi}{c(n-1)} j_{[\sigma,\mu]},
\quad (151)
\]

\[
\approx - \frac{3}{2} f_{[\sigma,\mu];\alpha}^\alpha + 4 f_{[\sigma,\mu];\alpha}^\alpha + \frac{8 \pi}{c(n-1)} j_{[\sigma,\mu]} - \frac{8 \pi}{c(n-1)} j_{[\sigma,\mu]},
\quad (152)
\]

\[
f_{\sigma \mu} \approx 2 A_{[\mu,\sigma]} + \partial_{[\tau,\alpha]} \varepsilon_{\mu \tau} \quad (153)
\]

where

\[
f_{[\sigma,\mu];\alpha} = - \frac{2 \Lambda_b}{3} \partial_{\tau} \varepsilon_{\sigma \mu \alpha}, \quad \partial_{\tau} = \frac{1}{4 \Lambda_b} f_{[\mu,\sigma];\alpha} + \frac{4 \pi}{c} j_{[\sigma,\mu];\alpha}.
\quad (154)
\]

From \[^{13},^{14}\], the last term of \[^{15}\] can only contribute \( < 10^{-32} \) of \( f_{\mu \nu} \). From \[^{13},^{14},^{15},^{15}\], the \( 4 f_{[\alpha,\mu];\alpha} / \Lambda_b \) term can only contribute \( < 10^{-77} \) of \( f_{\sigma \mu} \) because of the antisymmetrized second derivative. Ignoring these terms and taking the divergence of \[^{15}\] using \[^{13}\], the \( \partial_{[\tau,\alpha]} \varepsilon_{\mu \tau} \) term drops out and we get a close approximation to Maxwell’s equations,

\[
\mathcal{F}_{\sigma \mu;}^\rho = 2 A_{[\mu,\sigma]}^\rho \approx \frac{4 \pi}{c} j_{\mu},
\quad (155)
\]

\[
\mathcal{F}_{[\mu,\nu]} = 2 A_{[\mu,\sigma]} = 0 \quad \text{(from the definition \( \mathcal{F}_{\sigma \mu} = 2 A_{[\mu,\sigma]} \)).}
\quad (156)
\]

The \( \partial_{[\tau,\alpha]} \varepsilon_{\mu \tau} \) term of \[^{15}\] can also be neglected from \[^{13},^{14}\]. However, it is interesting to consider the case where \[^{12}\] is not satisfied, but where there are no extreme spatial curvatures \[^{12}\] so that \( 4 f_{[\alpha,\mu];\alpha} / \Lambda_b \) is still negligible. Then, taking the curl of \[^{15}\], the \( 2 A_{[\mu,\sigma]} \) and \( j_{[\sigma,\mu]} \) terms drop out and we see that the additional vector field \( \partial_{\rho} \) obeys a form of the Proca equation,

\[
\Lambda_b \partial_{\rho} \approx - \partial_{[\rho,\nu]}^\nu.
\quad (157)
\]

This equation suggests the possibility of a \( \partial_{\rho} \) particle, and this is discussed in detail in §6-§7 of \[^{10}\]. There it is shown that if a \( \partial_{\rho} \) particle does result from \[^{15}\], it would apparently have a negative energy. However, the other odd feature of this particle is that from \[^{15},^{130},^{131}\], its minimum frequency \( \omega = \sqrt{2 \Lambda_b c} = \sqrt{2 C_z c} \) would exceed the zero-point cutoff frequency \( c k_c = c C_z / l_P \), and we assume this would prevent the particle from existing. Whether the cutoff of zero-point fluctuations is caused by a discreteness of spacetime near the Planck length \[^{20},^{23},^{23},^{29}\] or by some other effect, we simply assume that this cutoff would also apply to real fundamental particles \[^{27}\]. Comparing the two frequency expressions, we see that this argument only applies if

\[
C_z > 1 / \sqrt{2 C_z},
\quad (158)
\]

where \( C_c \) and \( C_z \) are defined by \[^{130},^{131}\]. Since the prediction of a negative energy particle would probably be inconsistent with reality, this theory should be approached cautiously when considering it with values of \( C_c \) and \( C_z \) which do not satisfy \[^{158}\].
6. An Electric Monopole Solution

Here we assume $T_{\mu\nu} = 0$, $j^\rho = 0$, which is the Einstein-Schrödinger equivalent of electro-vac general relativity. The Newman-Penrose tetrads of the electric monopole solution derived in [10] are similar to those of the Reissner-Nordström solution [24], except for the $\tilde{c}$ factors,

$$e_{1\alpha} = l_\alpha = (1, -1/a\tilde{c}, 0, 0), \quad e_{1\alpha} = l_\alpha = (1/a\tilde{c}, 1, 0, 0), \quad (159)$$

$$e_{2\alpha} = n_\alpha = \frac{1}{2}(a\tilde{c}, 1, 0, 0), \quad e_{2\alpha} = n_\alpha = \frac{1}{2}(1, -a\tilde{c}, 0, 0), \quad (160)$$

$$e_{3\alpha} = m_\alpha = -r\sqrt{\tilde{c}/2} (0, 0, 1, i \sin \theta), \quad e_{3\alpha} = m_\alpha = \frac{1}{r\sqrt{2\tilde{c}}} (0, 0, 1, i \csc \theta), \quad (161)$$

where from [10] and [131-132],

\[ \dot{u} = 0 \quad \dot{s} = 0 \quad \dot{\tilde{c}} = 1, \quad (162) \]

\[ \ddot{u} = \frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{Q}{\tilde{c} r^2}, \quad (163) \]

\[ \ddot{s} = \frac{Q}{r^2}, \quad (164) \]

\[ \tilde{c} = \sqrt{1 + \dot{\tilde{c}}^2} = \sqrt{1 + \sqrt{1 + \frac{Q^2}{r^4}}}, \quad (165) \]

\[ a = 1 - \frac{2m}{r} - \frac{\Lambda_b r^2}{3} - \frac{\Lambda_c V}{r} = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3} - (\Lambda_b - \Lambda) \left( \frac{Q^2}{2r^2} - \frac{Q^4}{40r^6} \right), \quad (166) \]

\[ V = \int \sqrt{r^4 + Q^2} \, dr = \frac{1}{3} \left[ r\sqrt{r^4 + Q^2} - Q^3/2 \left( 2 \arctan \left( \frac{Q}{r} \right) \pi/4 \right) \right] \]

\[ = \int r^2 \left[ 1 + \frac{Q^2}{2r^2} - \frac{Q^4}{8r^6} \right] \, dr = \frac{r^3}{3} - \frac{Q^2}{2r} + \frac{Q^4}{40r^5} \cdots, \quad (167) \]

\[ \Lambda_b \approx -\Lambda_c \approx \pm 10^{66} \text{cm}^{-2}, \quad \Lambda \approx 10^{-56} \text{cm}^{-2}, \quad \Lambda/\Lambda_b \approx 10^{-122}, \quad (168) \]

\[ Q = e \sqrt{-2G/c^4 \Lambda_b} \approx \sqrt{\pm 1} \times 10^{-66} \text{cm}^2. \quad (170) \]

In (168), the term $-\Lambda_b Q^2/2r^2 = e^2 G/c^4 r^2$ matches a term appearing in the Reissner-Nordström solution, and the remaining $Q$ terms are negligible for ordinary radii.

The nonzero tetrad derivatives are,

\[ e_{11,1} = -\left( \frac{1}{a\tilde{c}} \right)' = \frac{(ac)'}{2}, \quad e_{20,1} = -r\sqrt{\tilde{c}/2} i \cos \theta, \quad (171) \]

\[ e_{32,1} = -\sqrt{\tilde{c}/2} - \frac{\tilde{c}''}{2\sqrt{2\tilde{c}}} = -\frac{\tilde{c}^2 + \tilde{s}^2}{2\sqrt{2\tilde{c}}} = -\frac{1}{\sqrt{2\tilde{c}}}, \quad e_{33,1} = e_{32,1} i \sin \theta. \quad (172) \]

From these and (155), the $\lambda_{abc}$ coefficients are

\[ \lambda_{a1b} = e_{11,1}(e_a e_b e_c), \quad \lambda_{221} = e_{20,1}(e_2 e_1 e_0) = \frac{(ac)'}{2}, \quad (173) \]

\[ \lambda_{123} = e_{20,1}(e_1 e_3 e_0) = e_{20,1}(e_1 e_3 e_0) = 0, \quad (174) \]

\[ \lambda_{223} = e_{20,1}(e_2 e_3 e_0) = 0, \quad (175) \]

\[ \lambda_{324} = e_{20,1}(e_3 e_4 e_0) = 0, \quad (176) \]
\[\lambda_{132} = e_{30,1}(c_0^0c_1^1 - c_1^0c_2^1) = 0,\]  
\(\lambda_{233} = -e_{32,1}c_1^1c_2^1 - e_{33,1}c_1^2 - e_{33,1}c_1^3 = 0,\]  
\(\lambda_{243} = -e_{42,1}c_1^2c_2^3 - e_{43,1}c_2^1c_3^3 = -2\left(\frac{-1}{\sqrt{2\hat{c}}}\right)\left(\frac{-a\hat{c}}{2}\right)\frac{1}{r\sqrt{2\hat{c}}} = -\frac{a}{2r\hat{c}},\]  
\(\lambda_{441} = e_{42,1}c_1^2c_1^1 + e_{43,1}c_1^3c_1^1 = 0,\]  
\(\lambda_{431} = e_{32,1}c_1^2c_1^1 + e_{33,1}c_1^3c_1^1 = 2\left(\frac{-1}{\sqrt{2\hat{c}}}\right)\frac{1}{r\sqrt{2\hat{c}}} = -\frac{1}{r\hat{c}^2},\]  
\(\lambda_{334} = e_{33,2}c_1^3c_1^2 - e_3^2c_1^3 = 2(-r\sqrt{\hat{c}/2}i\cos \theta)\left(\frac{i\csc \theta}{r\sqrt{2\hat{c}}}\right)\frac{1}{r\sqrt{2\hat{c}}} = \frac{\cot \theta}{r\sqrt{2\hat{c}}}.\]  

From [24], the spin coefficients are similar to those of the Reissner-Nordström solution [24], except for the \(\hat{c}\) factors,

\[\rho = \gamma_{314} = \lambda_{431} = -\frac{1}{r\hat{c}^2},\]  
\[\mu = \gamma_{243} = \lambda_{243} = -\frac{a}{2r\hat{c}},\]  
\[\beta = \frac{1}{2}(\gamma_{213} + \gamma_{343}) = \frac{1}{2}\lambda_{334} = \frac{\cot \theta}{2r\sqrt{2\hat{c}}},\]  
\[\alpha = \frac{1}{2}(\gamma_{214} + \gamma_{344}) = \frac{1}{2}\lambda_{344} = -\frac{\cot \theta}{2r\sqrt{2\hat{c}}},\]  
\[\gamma = \frac{1}{2}(\gamma_{212} + \gamma_{342}) = \frac{1}{2}\lambda_{221} = \frac{(a\hat{c})'}{4},\]  
\[\epsilon = \frac{1}{2}(\gamma_{211} + \gamma_{341}) = 0,\]  
\[\tau = \gamma_{312} = 0,\]  
\[\pi = \gamma_{241} = 0,\]  
\[\kappa = \gamma_{311} = 0,\]  
\[\sigma = \gamma_{313} = 0,\]  
\[\lambda = \gamma_{244} = 0,\]  
\[\nu = \gamma_{242} = 0.\]  

The type-D classification of this solution is evident because \(\kappa = \sigma = \lambda = \nu = \epsilon = 0\) and from the Weyl tensor components calculated with MAPLE,

\[\Psi_2 = -\frac{1}{c}\left(1 + \frac{2Q^2}{r^4}\right)\left(m + \frac{\Lambda c}{2r^5} + \frac{\Lambda c}{6r^4}\right) + \frac{\Lambda c Q^2}{6r^4} + \frac{Q^2}{2\hat{c}r^6},\]  
\[\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0.\]  

The electromagnetic vector potential from [10] and [169, 170] completes the solution,

\[A_\alpha = (A_0/a\hat{c}, A_0/2, 0, 0),\]  
\[A_0 = \frac{Q}{r}\left(1 - \frac{4\Lambda}{3\Lambda_b}\right) + \frac{Qm}{\Lambda_b r^4} + \left(1 - \frac{\Lambda}{\Lambda_b}\right)\left(\frac{2Q^3}{5r^5} - \frac{Q^5}{30r^9} \ldots\right).\]  

Here all terms except \(Q/r\) are negligible for ordinary radii, assuming [169, 170].
From (26,28,159,161,164-166), the tetrad solution matches the solution in [10].

\[
W^{\sigma\mu} = e^\sigma_a W^{ab} e_b^\mu
\]

(200)

\[
= e^\sigma_a \begin{pmatrix}
0 & 1+\tilde{u} & 0 & 0 \\
1-\tilde{u} & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{ac} & 1 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{-a\tilde{c}}{2} & 0 \\
0 & 0 & 0 & \frac{1}{r\sqrt{2c}} \\
0 & 0 & 0 & \frac{r\sqrt{2c}}{1}
\end{pmatrix}
\]

(201)

\[
= \begin{pmatrix}
\frac{1}{ac} & \frac{1}{2} & 0 & 0 \\
1-\frac{a\tilde{c}}{2} & 0 & 0 & 0 \\
0 & 0 & \frac{r\sqrt{2c}}{1} & \frac{1}{r\sqrt{2c}} \\
0 & 0 & -\frac{r\sqrt{2c}}{1} & \frac{1}{r\sqrt{2c}}
\end{pmatrix}
\begin{pmatrix}
(1+\tilde{u}) & (1+\tilde{u}a\tilde{c}) & 0 & 0 \\
(1-a\tilde{c}) & -a\tilde{c} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{r\sqrt{2c}} \\
0 & 0 & \frac{1}{r\sqrt{2c}} & \frac{1}{r\sqrt{2c}}
\end{pmatrix}
\]

(202)

\[
= \frac{1}{c} \begin{pmatrix}
1/a & -\tilde{s} & 0 & 0 \\
\tilde{s} & -a\tilde{c}^2 & 0 & 0 \\
0 & 0 & -1/r^2 & 0 \\
0 & 0 & 0 & -1/r^2\sin^2\theta
\end{pmatrix}
\]

(203)

\[
g^{\sigma\mu} = W^{(\sigma\mu)} = \frac{1}{c} \begin{pmatrix}
1/a & 0 & 0 & 0 \\
0 & -a\tilde{c}^2 & 0 & 0 \\
0 & 0 & -1/r^2 & 0 \\
0 & 0 & 0 & -1/r^2\sin^2\theta
\end{pmatrix}
\]

(204)

\[
g_{\sigma\mu} = \bar{c} \begin{pmatrix}
a & 0 & 0 & 0 \\
0 & -1/a\tilde{c}^2 & 0 & 0 \\
0 & 0 & -r^2 & 0 \\
0 & 0 & 0 & -r^2\sin^2\theta
\end{pmatrix}
\]

(205)

\[
f^{\sigma\mu} = W^{[\sigma\mu]} = \frac{1}{c} \begin{pmatrix}
0 & -\tilde{s} & 0 & 0 \\
\tilde{s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(206)

\[
f_{\sigma\mu} = W_{[\sigma\mu]} = \frac{1}{c} \begin{pmatrix}
0 & \tilde{s} & 0 & 0 \\
-\tilde{s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(207)

\[
N_{\sigma\mu} = \frac{\sqrt{-N}}{\sqrt{-g}} W^{\sigma\mu T} = \begin{pmatrix}
\tilde{a}\tilde{c}^2 & \tilde{s} & 0 & 0 \\
0 & 1/a & 0 & 0 \\
0 & 0 & -r^2 & 0 \\
0 & 0 & 0 & -r^2\sin^2\theta
\end{pmatrix}
\]

(208)

Also, from (200,145,51,84) we have,

\[
1/e = det(e^\sigma_\nu) = i\csc\theta/\tilde{c}r^2,
\]

(209)

\[
e = det(e^\sigma_\nu) = -i\tilde{c}r^2\sin\theta,
\]

(210)

\[
\sqrt{-N} = \sqrt{-N_c} e = ie/\tilde{c}e = -r^2\sin\theta,
\]

(211)

\[
\sqrt{-g} = \sqrt{-g_c} e = ie = \tilde{c}r^2\sin\theta.
\]

(212)

This solution reduces to the Papapetrou type I solution [30] for \(\Lambda_e = 0\), \(\Lambda_b = \Lambda\), and to the Schwarzschild solution for \(Q = 0\). Note from (170,172,165) that for an elementary charge, \(\tilde{e}^2\) diverges from one near \(r_e = \sqrt{|Q|} \sim 10^{-33}\) cm, and it goes to zero there if \(Q\) is imaginary. This \(r_e\) surface may lie inside or outside of the Schwarzschild radius depending on the charge/mass ratio.
7. Conclusions

The Einstein-Schrödinger theory is modified to include a large cosmological constant caused by zero-point fluctuations. This extrinsic cosmological constant which multiplies the symmetric metric is assumed to be nearly cancelled by Schrödinger’s bare cosmological constant which multiplies the nonsymmetric fundamental tensor, resulting in a total cosmological constant which is consistent with measurement. Sections §1 and §5 demonstrate that this theory closely approximates ordinary general relativity and electromagnetism, confirming the results in [10, 11] using different methods. This is corroborated by the close approximation of the electric monopole solution to the Reissner-Nordström solution, and by its Petrov type-D classification.

The Einstein-Schrödinger theory is very workable in Newman-Penrose form. The presented solution of the connection equations is fairly simple and is applicable for all cases except $f^\nu_{\mu\rho} = \det(f^\nu_{\mu}) = 0$. It is very compatible with symbolic algebra programs such as REDUCE or MAPLE. Given the amenability of Newman-Penrose methods to type-D solutions, this paper might be useful in finding a charged rotating solution to the Einstein-Schrödinger theory, either in its original or modified form.

Appendix A. Linear Combinations to Solve for the Connection Addition

In the following linear combinations of equations (80-90), the right-hand-side $\Upsilon_{bc}^a$ terms cancel,

\[ \pm \Upsilon_{12}^2 = \pm \Upsilon_{12}^2 + \frac{1}{2} \left( \pm O_{1}^{12} - O_{2}^{22} - \pm O_{1}^{21} \frac{1+\bar{u}}{1+\bar{u}} \right), \]  

(A.1)

\[ \pm \Upsilon_{12}^1 = \pm \Upsilon_{12}^1 + \frac{1}{2} \left( \pm O_{1}^{12} - O_{1}^{11} - \pm O_{2}^{21} \frac{1+\bar{u}}{1+\bar{u}} \right), \]  

(A.2)

\[ \pm \Upsilon_{34}^4 = \pm \Upsilon_{34}^4 + \frac{1}{2} \left( \pm O_{3}^{44} + O_{4}^{44} + \pm O_{3}^{43} \frac{1+i\bar{u}}{1+i\bar{u}} \right), \]  

(A.3)

\[ \Upsilon_{11}^1 = \Upsilon_{11}^1 + \frac{1}{2} (-O_{1}^{21} - O_{1}^{12} + O_{2}^{22}), \]  

(A.4)

\[ \Upsilon_{22}^2 = \Upsilon_{22}^2 + \frac{1}{2} (-O_{2}^{12} - O_{2}^{21} + O_{1}^{11}), \]  

(A.5)

\[ \Upsilon_{33}^3 = \Upsilon_{33}^3 + \frac{1}{2} ( O_{3}^{43} + O_{3}^{34} - O_{4}^{44}), \]  

(A.6)

\[ \Upsilon_{11}^2 = \Upsilon_{11}^2 - \frac{1}{2} O_{1}^{22}, \]  

(A.7)

\[ \Upsilon_{22}^3 = \Upsilon_{22}^3 - \frac{1}{2} O_{2}^{21}, \]  

(A.8)

\[ \Upsilon_{44}^3 = \Upsilon_{44}^3 + \frac{1}{2} O_{4}^{43}, \]  

(A.9)

\[ \pm \Upsilon_{23}^2 = \pm \Upsilon_{23}^2 + \frac{1}{2} \left( \pm O_{2}^{24} - \pm O_{1}^{14} - \pm O_{4}^{12} \frac{1+i\bar{u}}{1+\bar{u}} \right), \]  

(A.10)

\[ \pm \Upsilon_{13}^1 = \pm \Upsilon_{13}^1 + \frac{1}{2} \left( -\pm O_{2}^{22} + \pm O_{1}^{14} - \pm O_{3}^{21} \frac{1+\bar{u}}{1+\bar{u}} \right), \]  

(A.11)

\[ \pm \Upsilon_{13}^3 = \pm \Upsilon_{13}^3 + \frac{1}{2} \left( \pm O_{1}^{12} - \pm O_{3}^{23} + \pm O_{4}^{34} \frac{1+\bar{u}}{1+i\bar{u}} \right), \]  

(A.12)
Appendix B. Calculation of the Exact Connection Addition

\[
\pm \Upsilon_{33} = \pm \Upsilon_{33} + \frac{1}{2} \left( \pm \Upsilon_{11}^2 - \pm \Upsilon_{13}^1 + \pm \Upsilon_{24}^2 \right), \tag{A.13}
\]

\[
\pm \Upsilon_{12} = \pm \Upsilon_{12} + \frac{1}{2} \left( \pm \Upsilon_{11}^2 - \pm \Upsilon_{13}^1 + \pm \Upsilon_{24}^2 \right) \tag{A.14}
\]

\[
\pm \Upsilon_{34} = \pm \Upsilon_{34} + \frac{1}{2} \left( \pm \Upsilon_{11}^2 - \pm \Upsilon_{13}^1 + \pm \Upsilon_{24}^2 \right) \tag{A.15}
\]

\[
\pm \Upsilon_{43} = \pm \Upsilon_{43} + \frac{1}{2} \left( \pm \Upsilon_{11}^2 - \pm \Upsilon_{13}^1 + \pm \Upsilon_{24}^2 \right) \tag{A.16}
\]

\[
\pm \Upsilon_{13} = \pm \Upsilon_{13} + \frac{1}{2} \left( \pm \Upsilon_{11}^2 - \pm \Upsilon_{13}^1 + \pm \Upsilon_{24}^2 \right) \tag{A.17}
\]

\[
\pm \Upsilon_{24} = \pm \Upsilon_{24} + \frac{1}{2} \left( \pm \Upsilon_{11}^2 - \pm \Upsilon_{13}^1 + \pm \Upsilon_{24}^2 \right) \tag{A.18}
\]

\[
\pm \Upsilon_{43} = \pm \Upsilon_{43} + \frac{1}{2} \left( \pm \Upsilon_{11}^2 - \pm \Upsilon_{13}^1 + \pm \Upsilon_{24}^2 \right) \tag{A.19}
\]

\[
\pm \Upsilon_{33} = \pm \Upsilon_{33} + \frac{1}{2} \left( \pm \Upsilon_{11}^2 - \pm \Upsilon_{13}^1 + \pm \Upsilon_{24}^2 \right) \tag{A.20}
\]

\[
Y_{11} = Y_{11} + \frac{1}{2} \left( \pm \Upsilon_{11}^2 - \pm \Upsilon_{13}^1 + \pm \Upsilon_{24}^2 \right) \tag{A.21}
\]

\[
Y_{22} = Y_{22} + \frac{1}{2} \left( \pm \Upsilon_{11}^2 - \pm \Upsilon_{13}^1 + \pm \Upsilon_{24}^2 \right) \tag{A.22}
\]

\[
Y_{33} = Y_{33} + \frac{1}{2} \left( \pm \Upsilon_{11}^2 - \pm \Upsilon_{13}^1 + \pm \Upsilon_{24}^2 \right) \tag{A.23}
\]

\[
Y_{44} = Y_{44} + \frac{1}{2} \left( \pm \Upsilon_{11}^2 - \pm \Upsilon_{13}^1 + \pm \Upsilon_{24}^2 \right) \tag{A.24}
\]

Appendix B. Calculation of the Exact Connection Addition

Performing the linear combinations in Appendix A and using (A.11) and the notation (B.12) gives

\[
\pm \Upsilon_{12} = \mp \frac{D \bar{u}}{(1 \pm \bar{u})} + \frac{4\pi}{3c(1 \pm \bar{u})} j^2, \tag{B.1}
\]

\[
\pm \Upsilon_{12} = \mp \frac{\Delta \bar{u}}{(1 \pm \bar{u})} + \frac{4\pi}{3c(1 \pm \bar{u})} j^1, \tag{B.2}
\]

\[
\pm \Upsilon_{34} = \mp \frac{i \delta \bar{u}}{(1 \pm \bar{u})} + \frac{4\pi}{3c(1 \pm \bar{u})} j^4, \tag{B.3}
\]

\[
\Upsilon_{11} = \frac{D\sqrt{-N_o}}{\sqrt{-N_o}} + \frac{4\pi \bar{u} c^2}{3c} j^2 = \bar{u}D\bar{u}c^2 - \bar{u}D\bar{u}c^2 + \frac{4\pi \bar{u} c^2}{3c} j^2, \tag{B.4}
\]

\[
\Upsilon_{22} = \frac{\Delta \sqrt{-N_o}}{\sqrt{-N_o}} - \frac{4\pi \bar{u} c^2}{3c} j^1 = \bar{u}\Delta\bar{u}c^2 - \bar{u}\Delta\bar{u}c^2 - \frac{4\pi \bar{u} c^2}{3c} j^1, \tag{B.5}
\]

\[
\Upsilon_{33} = \frac{\delta \sqrt{-N_o}}{\sqrt{-N_o}} - \frac{4\pi i \bar{u} \bar{c} c^2}{3c} j^4 = \bar{u}\delta\bar{u}c^2 - \bar{u}\delta\bar{u}c^2 - \frac{4\pi i \bar{u} \bar{c} c^2}{3c} j^4, \tag{B.6}
\]

\[
\Upsilon_{11} = \Upsilon_{22} = \Upsilon_{44} = 0, \tag{B.7}
\]

\[
\Upsilon_{23} = \pm \frac{1}{2}(\tau w + \pi^* w^*) - \frac{1}{2}(\mp \delta \bar{u} c^2 - \bar{u} \delta \bar{u} c^2)(1 \pm \bar{u}) \pm \frac{2\pi(2 \mp 3i\bar{u})}{3c(1 \pm i\bar{u})} j^4 \tag{B.8}
\]
Einstein-Schrödinger theory using Newman-Penrose tetrad formalism

\[ = \pm \frac{1}{2} (i\rho \delta^2 c^2 (1 \mp i \bar{u}) - \delta \bar{u}^2 \mp \bar{u} \delta \bar{u}^2 (1 \pm i \bar{u}) = \frac{2\pi}{3c(1 \mp i \bar{u})} j^4 \]  
(B.9)

\[ = \pm \frac{1}{2} (\delta \bar{u}^2 - i \delta \bar{u}^2) (1 \pm i \bar{u}) \mp \frac{2\pi}{3c(1 \mp i \bar{u})} j^4, \]  
(B.10)

\[ \pm \Upsilon_{13}^{1} = \pm \frac{1}{2} (\tau w + \pi^* w^*) - \frac{1}{2} (\rho \delta \bar{u}^2 - \bar{u} \delta \bar{u}^2) (1 \mp i \bar{u}) = \frac{2\pi (2 \pm 3 \bar{u})}{3c(1 \pm i \bar{u})} j^4 \]  
(B.11)

\[ = \pm \frac{1}{2} (i \rho \delta \bar{u}^2 + \delta \bar{u}^2 \mp \bar{u} \delta \bar{u}^2 (1 \pm i \bar{u}) = \frac{2\pi}{3c(1 \pm i \bar{u})} j^4 \]  
(B.12)

\[ = \pm \frac{1}{2} (\delta \bar{u}^2 + i \delta \bar{u}^2) (1 \pm i \bar{u}) \mp \frac{2\pi}{3c(1 \pm i \bar{u})} j^4, \]  
(B.13)

\[ \pm \Upsilon_{13}^{3} = \pm \frac{1}{2} (\rho w + \rho^* w^*) + \frac{1}{2} (\rho \delta \bar{u}^2 - \bar{u} \delta \bar{u}^2) (1 \mp i \bar{u}) = \frac{2\pi (2 \pm 3 \bar{u})}{3c(1 \pm i \bar{u})} j^2 \]  
(B.14)

\[ = \pm \frac{1}{2} (\rho \delta \bar{u}^2 (1 \mp \bar{u}) + i \delta \bar{u}^2 \mp \bar{u} \delta \bar{u}^2 (1 \pm \bar{u}) = \frac{2\pi}{3c(1 \mp \bar{u})} j^2 \]  
(B.15)

\[ = \pm \frac{1}{2} (\delta \bar{u}^2 + i \delta \bar{u}^2) (1 \pm \bar{u}) \mp \frac{2\pi}{3c(1 \pm \bar{u})} j^2, \]  
(B.16)

\[ \pm \Upsilon_{23}^{3} = \pm \frac{1}{2} (\mu w + \mu^* w^*) + \frac{1}{2} (\mu \delta \bar{u}^2 - \bar{u} \delta \bar{u}^2) (1 \mp \bar{u}) = \frac{2\pi (2 \pm 3 \bar{u})}{3c(1 \pm \bar{u})} j^2 \]  
(B.17)

\[ = \pm \frac{1}{2} (-\Delta \bar{u}^2 (1 \mp \bar{u}) + i \Delta \bar{u}^2 \mp \bar{u} \Delta \bar{u}^2) (1 \pm \bar{u}) = \frac{2\pi}{3c(1 \mp \bar{u})} j^2 \]  
(B.18)

\[ = \pm \frac{1}{2} (-\Delta \bar{u}^2 - i \Delta \bar{u}^2) (1 \pm \bar{u}) \mp \frac{2\pi}{3c(1 \mp \bar{u})} j^2, \]  
(B.19)

\[ \pm \Upsilon_{12}^{1} = \frac{1}{2(1 \pm \bar{u})} \left( \pm \rho w (1 \pm i \bar{u}) = \pi^* w^* (1 \mp i \bar{u}) \pm \delta \bar{u}^2 \frac{e^2}{c^2} + \bar{u} \delta \bar{u} - \frac{4\pi}{c} i \bar{u} j \right) \]  
(B.20)

\[ = \frac{\pm 1}{2(1 \pm \bar{u})} \left( \delta \bar{u}^2 \frac{e^2}{c^2} + \tau w - \pi^* w^* \right), \]  
(B.21)

\[ \pm \Upsilon_{34}^{2} = \frac{1}{2(1 \pm i \bar{u})} \left( \pm \rho w (1 \mp \bar{u}) = \rho^* w^* (1 \mp \bar{u}) \mp \rho \bar{u}^2 \frac{e^2}{c^2} - \bar{u} \rho \bar{u} + \frac{4\pi}{c} \bar{u} j \right), \]  
(B.22)

\[ = \frac{\pm 1}{2(1 \pm i \bar{u})} \left( i \rho \bar{u}^2 \frac{e^2}{c^2} + \rho w - \rho^* w^* \right), \]  
(B.23)

\[ \pm \Upsilon_{43}^{1} = \frac{1}{2(1 \mp i \bar{u})} \left( \pm \mu w (1 \mp \bar{u}) = \mu^* w^* (1 \mp \bar{u}) \mp \mu \bar{u}^2 \frac{e^2}{c^2} - \bar{u} \mu \bar{u} - \frac{4\pi}{c} \mu j \right) \]  
(B.24)

\[ = \frac{\pm 1}{2(1 \mp i \bar{u})} \left( i \mu \bar{u}^2 \frac{e^2}{c^2} - \mu w + \mu^* w^* \right), \]  
(B.25)

\[ \pm \Upsilon_{13}^{2} = \frac{\kappa w(\bar{u} \mp 1)}{\pm}, \quad \pm \Upsilon_{24}^{1} = \frac{-\nu w(\bar{u} \mp 1)}{\pm}, \]  
(B.26)

\[ \pm \Upsilon_{13}^{4} = \frac{\sigma w^2(\bar{u} \mp 1)}{\pm}, \quad \pm \Upsilon_{24}^{3} = \frac{-\lambda w(\bar{u} \mp 1)}{\pm}, \]  
(B.27)

\[ \Upsilon_{11}^{2} = \frac{\kappa w^2}{\pm}, \quad \Upsilon_{22}^{3} = \frac{-\nu w^2}{\pm}, \]  
(B.28)

\[ \Upsilon_{33}^{2} = \frac{-\sigma w^2}{\pm}, \quad \Upsilon_{44}^{1} = \frac{-\lambda w^2}{\pm}. \]  
(B.29)
Appendix C. Check of the Approximate Connection Tetrad Addition Formula

Here we will show that the order $f^2$ approximation of $\Upsilon^\alpha_{\rho\sigma}$ in $\Upsilon_{(12)}$ matches the exact solution in $\Upsilon_{(41)}$ for $\hat{c} = \hat{c} = \hat{z} = \hat{z} = 1$, which amounts to a second order approximation in $\hat{u}$ and $\hat{u}$. Much use is made of $g_{ab}$ and $f_{ab}$ from $\Upsilon_{(29)}$, $\gamma_{cab} = -\gamma_{abc}$ from $\Upsilon_{(52)}$, $\ell_{e}/4 = \hat{u}_{a} - \hat{u}_{a}$ from $\Upsilon_{(41)}$, and the field equations $\Upsilon_{(41)}$. To save space, only one component of each type will be shown.

In tetrad form $\Upsilon_{(14)}$ becomes,

\[
\Upsilon_{c(de)} \approx \frac{1}{2} (f^{a}_{d}(f_{ec,a} - \gamma_{ca} f_{be} - \gamma_{eb} f_{dc}) + f^{b}_{e}(f_{dc,a} - \gamma_{da} f_{be} - \gamma_{eb} f_{dc})
\]
\[
+ f^{a}_{c}(f_{bd,e} - \gamma_{ae} f_{bd} - \gamma_{bd} f_{ec}) + f^{b}_{c}(f_{ed,a} - \gamma_{ad} f_{be} - \gamma_{bd} f_{ec})
\]
\[
+ \frac{1}{8}(\ell_{c}g_{de} - \ell_{d}g_{ec} - \ell_{e}g_{dc})
\]
\[
+ \frac{2\pi}{c} \left(j^{a}_{f}(f_{ca}g_{de} + \frac{1}{3}j^{a}_{f}g_{ed} + \frac{1}{3}j^{a}_{f}g_{de})\right),
\]
\[\text{(C.1)}\]

\[
\Upsilon_{1(12)} \approx \frac{1}{2} (f^{a}_{1}(f_{21,a} - \gamma_{a2} f_{1b} - \gamma_{b2} f_{1a}) + f^{b}_{1}(f_{a1,a} - \gamma_{a2} f_{1b} - \gamma_{b1} f_{1a})
\]
\[
+ f^{a}_{1}(f_{a2,2} - \gamma_{a2} f_{1b} - \gamma_{b2} f_{1a}) + f^{b}_{1}(f_{a1,2} - \gamma_{a1} f_{1b} - \gamma_{b2} f_{1a})
\]
\[
+ \frac{1}{8}(\ell_{1}g_{12} - \ell_{1}g_{21} - \ell_{2}g_{11})
\]
\[
+ \frac{2\pi}{c} \left(j^{2}_{f}(f_{12}g_{12} + \frac{1}{3}j^{2}_{f}g_{12} + \frac{1}{3}j^{2}_{f}g_{12})\right)
\]
\[\text{(C.2)}\]

\[
= \hat{u}D\hat{u} - \frac{4\pi}{3c} \hat{u}j^{2},
\]
\[\text{(C.3)}\]

\[
\Upsilon_{2(11)} \approx f^{a}_{1}(f_{12,a} - \gamma_{a2} f_{1b} - \gamma_{b2} f_{1a}) + f^{b}_{2}(f_{a1,1} - \gamma_{a1} f_{1b} - \gamma_{b1} f_{1a})
\]
\[
+ \frac{1}{8}(\ell_{1}g_{11} - \ell_{1}g_{12} - \ell_{1}g_{21})
\]
\[
+ \frac{2\pi}{c} \left(j^{1}_{f}(f_{21}g_{11} + \frac{1}{3}j^{2}_{f}g_{12} + \frac{1}{3}j^{2}_{f}g_{12})\right)
\]
\[\text{(C.4)}\]

\[
= \hat{u}D\hat{u} - \frac{4\pi}{3c} \hat{u}j^{2},
\]
\[\text{(C.5)}\]

\[
\Upsilon_{1(11)} \approx f^{a}_{1}(f_{11,a} - \gamma_{a2} f_{1b} - \gamma_{b1} f_{1a}) + f^{b}_{1}(f_{a1,1} - \gamma_{a1} f_{1b} - \gamma_{b1} f_{1a})
\]
\[
+ \frac{1}{8}(\ell_{1}g_{11} - \ell_{1}g_{11} - \ell_{1}g_{11})
\]
\[
+ \frac{2\pi}{c} \left(j^{2}_{f}(f_{12}g_{11} + \frac{1}{3}j^{2}_{f}g_{12} + \frac{1}{3}j^{2}_{f}g_{12})\right)
\]
\[\text{(C.6)}\]

\[
= 0,
\]
\[\text{(C.7)}\]

\[
\Upsilon_{1(12)} \approx \frac{1}{2} (f^{a}_{2}(f_{31,a} - \gamma_{a3} f_{1b} - \gamma_{b3} f_{1a}) + f^{b}_{3}(f_{21,a} - \gamma_{a3} f_{1b} - \gamma_{b3} f_{1a})
\]
\[
+ f^{a}_{3}(f_{a1,3} - \gamma_{a3} f_{1b} - \gamma_{b3} f_{1a}) + f^{b}_{3}(f_{a3,2} - \gamma_{a3} f_{1b} - \gamma_{b2} f_{1a})
\]
\[
+ \frac{1}{8}(\ell_{1}g_{31} - \ell_{1}g_{31} - \ell_{3}g_{21})
\]
\[
+ \frac{2\pi}{c} \left(j^{3}_{f}(f_{12}g_{31} + \frac{1}{3}j^{3}_{f}g_{32} + \frac{1}{3}j^{3}_{f}g_{32})\right)
\]
\[\text{(C.8)}\]

\[
= \frac{1}{2}(i\hat{u}D\hat{u} + \hat{u}D\hat{u}) - \frac{1}{2}(\hat{u}D\hat{u} - \hat{u}D\hat{u}) - \frac{2\pi}{3c} \hat{u}j^{4}
\]
\[\text{(C.9)}\]
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\[ \Upsilon_{3(12)} \approx \frac{1}{2} \left( f^a_1(f_{33,a} - \gamma_{2a}^b f_{b3} - \gamma_{3a}^b f_{b3} + f^a_2(f_{13,a} - \gamma_{2a}^b f_{b1} - \gamma_{3a}^b f_{b1}) + f^a_3(f_{a1,2} - \gamma_{a2}^b f_{b1} - \gamma_{a1}^b f_{b2} + f^a_4(f_{a1,1} - \gamma_{a1}^b f_{b1} - \gamma_{a2}^b f_{b1})) \right) \\
+ \frac{1}{8} \left( \ell_{2g12} - \ell_{1g23} - \ell_{2g13} \right) \\
+ \frac{2\pi}{c} \left( j^4 f_{34g12} + \frac{1}{3} j^2 f_{21g23} + \frac{1}{3} j^4 f_{12g13} \right) \] (C.11)

\[ \Upsilon_{1(13)} \approx \frac{1}{2} \left( f^a_1(f_{31,a} - \gamma_{2a}^b f_{b1} - \gamma_{1a}^b f_{b3} + f^a_3(f_{11,a} + \gamma_{1a}^b f_{b1} - \gamma_{2a}^b f_{b3})) + f^a_2(f_{a1,3} - \gamma_{a3}^b f_{b1} - \gamma_{a2}^b f_{b3} + f^a_1(f_{a3,1} - \gamma_{a1}^b f_{b3} - \gamma_{a2}^b f_{b3})) \right) \\
+ \frac{1}{8} \left( \ell_{1g13} - \ell_{1g31} - \ell_{3g11} \right) \\
+ \frac{2\pi}{c} \left( j^2 f_{12g13} + \frac{1}{3} j^2 f_{21g31} + \frac{1}{3} j^2 f_{13g11} \right) \] (C.18)

\[ \Upsilon_{3(11)} \approx \frac{1}{2} \left( f^a_1(f_{13,a} - \gamma_{1a}^b f_{b3} - \gamma_{3a}^b f_{b3} + f^a_2(f_{a1,1} - \gamma_{a1}^b f_{b1} - \gamma_{a1}^b f_{b3} - \gamma_{a3}^b f_{b3})) + f^a_3(f_{a1,3} - \gamma_{a3}^b f_{b1} - \gamma_{a3}^b f_{b3} + f^a_1(f_{a3,1} - \gamma_{a1}^b f_{b3} - \gamma_{a3}^b f_{b3})) \right) \\
+ \frac{1}{8} \left( \ell_{2g11} - \ell_{1g13} - \ell_{1g31} \right) \\
+ \frac{2\pi}{c} \left( j^2 f_{34g11} + \frac{1}{3} j^2 f_{21g31} + \frac{1}{3} j^2 f_{21g13} \right) \] (C.21)

\[ \Upsilon_{1(13)} \approx \frac{1}{2} \left( f_{12,1} - \gamma_{11}^b f_{b2} - \gamma_{12}^b f_{b1} + f_{12,2} - \gamma_{12}^b f_{b2} - \gamma_{21}^b f_{b1} - f_{11,1} + \gamma_{12}^b f_{b1} + \gamma_{11}^b f_{b2} \right) \\
+ \frac{4\pi}{3c} \left( j^4 g_{ee} - j^4 g_{de} \right), \] (C.24)

In tetrad form (14) becomes,

\[ \Upsilon_{c[de]} \approx \frac{1}{2} \left( f_{d,e,c} - \gamma_{de}^b f_{be} + \gamma_{ce}^b f_{bd} + f_{ce,d} - \gamma_{cd}^b f_{be} - \gamma_{cd}^b f_{bd} + f_{cd,e} + \gamma_{ce}^b f_{bd} + \gamma_{de}^b f_{be} \right) \\
+ \frac{4\pi}{3c} \left( j^4 g_{ee} - j^4 g_{de} \right), \] (C.24)
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\[ + \frac{4\pi}{3c} (j_1g_{21} - j_2g_{11}) \]  \hspace{1cm} (C.25)
\[ = - D\dot{u} + \frac{4\pi}{3c} f^2 , \]  \hspace{1cm} (C.26)
\[ \mathcal{T}_{1[23]} \approx \frac{1}{2} (f_{23,1} - \gamma_{21}^b f_{63} - \gamma_{31}^b f_{2b} + f_{13,2} - \gamma_{12}^b f_{63} - \gamma_{32}^b f_{1b} - f_{23,3} + \gamma_{13}^b f_{62} + \gamma_{23}^b f_{1b}) \]
\[ + \frac{4\pi}{3c} (j_2g_{31} - j_3g_{21}) \]  \hspace{1cm} (C.27)
\[ = \frac{1}{2} (\delta \dot{u} - \gamma_{231} i \dot{u} - \gamma_{231} \dot{u} - \gamma_{321} i \dot{u} + \gamma_{132} \dot{u}) - \frac{4\pi}{3c} j_3 \]  \hspace{1cm} (C.28)
\[ = \frac{1}{2} (\delta \dot{u} - \pi^* w^* - \tau w) - \frac{4\pi}{3c} j_3 \]  \hspace{1cm} (C.29)
\[ = \frac{1}{2} (\delta \dot{u} - i \delta \dot{u}) - \frac{2\pi}{3c} j^4 , \]  \hspace{1cm} (C.30)
\[ \mathcal{T}_{3[12]} \approx \frac{1}{2} (f_{12,3} - \gamma_{13}^b f_{62} - \gamma_{12}^b f_{1b} + f_{32,1} - \gamma_{31}^b f_{62} - \gamma_{21}^b f_{3b} - f_{31,2} + \gamma_{32}^b f_{61} + \gamma_{12}^b f_{1b}) \]
\[ + \frac{4\pi}{3c} (j_3g_{21} - j_2g_{31}) \]  \hspace{1cm} (C.31)
\[ = \frac{1}{2} (-\delta \dot{u} + \gamma_{321} \dot{u} + \gamma_{321} i \dot{u} + \gamma_{132} \dot{u} - \gamma_{312} i \dot{u}) \]  \hspace{1cm} (C.32)
\[ = - \frac{1}{2} (\delta \dot{u} + \tau w - \pi^* w^*) , \]  \hspace{1cm} (C.33)
\[ \mathcal{T}_{1[13]} \approx \frac{1}{2} (f_{13,1} - \gamma_{13}^b f_{63} - \gamma_{12}^b f_{1b} + f_{31,1} - \gamma_{31}^b f_{63} - \gamma_{11}^b f_{1b} - f_{31,3} + \gamma_{31}^b f_{61} + \gamma_{11}^b f_{1b}) \]
\[ + \frac{4\pi}{3c} (j_1g_{31} - j_3g_{11}) \]  \hspace{1cm} (C.34)
\[ = - \gamma_{311} i \dot{u} + \gamma_{311} \dot{u} \]  \hspace{1cm} (C.35)
\[ = - \kappa w . \]  \hspace{1cm} (C.36)

Appendix D. The Non-symmetric Matrix Decomposition Theorem

**Theorem:** Assume \( W^{\sigma \mu} \) is a real tensor, \( f^{\sigma \mu} = W^{\sigma [\sigma \mu]} \), and \( g^{\sigma \mu} = W^{(\sigma \mu)} \) is an invertible metric which can be put into Newman-Penrose tetrad form

\[ g_{\alpha \beta} = g^{\alpha \beta} = g^{\alpha \beta} e^\alpha_e e^\beta_e = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} , \]  \hspace{1cm} (D.1)

\[ l^\sigma = e_1^\sigma , \quad n^\sigma = e_2^\sigma , \quad m^\sigma = e_3^\sigma , \quad m^* = e_4^\sigma \]  \hspace{1cm} (D.2)
\[ l_\sigma = e^2_\sigma , \quad n_\sigma = e^1_\sigma , \quad m_\sigma = -e^4_\sigma , \quad m^*_\sigma = -e^3_\sigma \]  \hspace{1cm} (D.3)
\[ \delta^\mu_\mu = e_\sigma^a e^\alpha_e , \quad \delta^b = e_\sigma^a e^\alpha_e , \]  \hspace{1cm} (D.4)

where \( l_\sigma \) and \( n_\sigma \) are real, \( m_\sigma \) and \( m^*_\sigma \) are complex conjugates. Then tetrads may be chosen such that

\[ W^{\alpha \beta} = W^{\alpha \beta} e^\alpha_e e^\beta_e = \begin{pmatrix} 0 & (1+\dot{u}) & 0 & 0 \\ (1-\dot{u}) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1+i\dot{u}) \\ 0 & 0 & -(1-i\dot{u}) & 0 \end{pmatrix} , \]  \hspace{1cm} (D.5)
where \( \dot{u} \) and \( \ddot{u} \) are real, except if \( f^\sigma_{\mu} f^{\mu}_{\nu} = \text{det}(f^{\mu}_{\nu}) = 0 \), in which case tetrads may be chosen such that

\[
W^{ab} = W^{\alpha^\beta} e^a_{\alpha} e^b_{\beta} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & -\dot{u} & -\ddot{u} \\
0 & \dot{u} & 0 & -1 \\
0 & \ddot{u} & -1 & 0
\end{pmatrix}, \quad (D.6)
\]

where \( \dot{u} \) is real.

**Proof:** From [24] p.51, \( f_{ab} \) can be parameterized by the three complex scalars

\[
\phi_0 = f_{13}, \quad \phi_1 = (f_{12} + f_{43})/2, \quad \phi_2 = f_{42}. \quad (D.7)
\]

From [24] p.53-54, there are a series of tetrad transformations which do not alter \( D.1 \).

Type I:

\[
l_\sigma \to l_\sigma, \quad m_\sigma \to m_\sigma + al_\sigma, \quad m_\sigma^* \to m_\sigma^* + a*l_\sigma, \quad n_\sigma \to n_\sigma + a^*m_\sigma + am_\sigma^* + aa^*l_\sigma, \quad (D.8)
\]

\[
\phi_0 \to \phi_0, \quad \phi_1 \to \phi_1 + a^*\phi_0, \quad \phi_2 \to \phi_2 + 2a^*\phi_1 + (a^*)^2\phi_0. \quad (D.9)
\]

Type II:

\[
n_\sigma \to n_\sigma, \quad m_\sigma \to m_\sigma + bn_\sigma, \quad m_\sigma^* \to m_\sigma^* + b^*n_\sigma, \quad l_\sigma \to l_\sigma + b^*m_\sigma + bm_\sigma^* + bb^*n_\sigma, \quad (D.10)
\]

\[
\phi_0 \to \phi_0, \quad \phi_1 \to \phi_1 + b\phi_2, \quad \phi_0 \to \phi_0 + 2b\phi_1 + b^2\phi_2. \quad (D.11)
\]

Type III:

\[
n_\sigma \to n_\sigma A, \quad l_\sigma \to l_\sigma/A, \quad m_\sigma \to m_\sigma e^{i\theta}, \quad m_\sigma^* \to m_\sigma^* e^{-i\theta}, \quad (D.12)
\]

\[
\phi_0 \to \phi_0 e^{i\theta}/A, \quad \phi_1 \to \phi_1, \quad \phi_2 \to \phi_0 e^{-i\theta}A. \quad (D.13)
\]

Using type I and II transformations, we can always make either \( \phi_2 = 0 \) or \( \phi_0 = 0 \) by solving a quadratic equation and performing a tetrad transformation with

\[
a^* = \frac{-2\phi_1 \pm \sqrt{(2\phi_1)^2 - 4\phi_2\phi_0}}{2\phi_0} \quad \text{or} \quad b = \frac{-2\phi_1 \pm \sqrt{(2\phi_1)^2 - 4\phi_2\phi_0}}{2\phi_2}. \quad (D.14)
\]

Note that a type I transformation does not alter \( \phi_0 \) and a type II transformation does not alter \( \phi_2 \). Therefore, if \( \phi_1 \neq 0 \) at this point, we can make \( \phi_0 = \phi_2 = 0 \) by doing a second transformation of the opposite type to the first one with

\[
b = -\frac{\phi_0}{2\phi_1} \quad \text{or} \quad a^* = -\frac{\phi_2}{2\phi_1}. \quad (D.15)
\]

Then with \( \ddot{u} = -2Re(\phi_1) \), \( \dot{u} = -2Im(\phi_1) \), we have from (D.7) (D.1),

\[
f_{ab} = \begin{pmatrix} 0 & \ddot{u} & 0 & 0 \\ \ddot{u} & 0 & 0 & 0 \\ 0 & 0 & 0 & i\ddot{u} \\ 0 & 0 & -i\ddot{u} & 0 \end{pmatrix}, \quad f^{a}_{b} = \begin{pmatrix} \ddot{u} & 0 & 0 & 0 \\ 0 & -\ddot{u} & 0 & 0 \\ 0 & 0 & i\ddot{u} & 0 \\ 0 & 0 & -i\ddot{u} & 0 \end{pmatrix}, \quad f^{ab} = \begin{pmatrix} 0 & \ddot{u} & 0 & 0 \\ 0 & 0 & 0 & \ddot{u} \\ 0 & 0 & -\ddot{u} & 0 \\ 0 & 0 & 0 & \ddot{u} \end{pmatrix}. \quad (D.16)
\]

This is the first case (D.5). The procedure above fails if \( \phi_1 = 0 \) in (D.15) in which case there is only one nonzero scalar, either \( \phi_0 \) or \( \phi_2 \). If the nonzero scalar is \( \phi_0 \), it can be changed to \( \phi_0 \) by doing type II transformation with \( b = 1 \) followed by a type I transformation with \( a^* = -1 \). Furthermore, we can make \( \phi_0 \) real by doing a type III transformation. Then with \( \ddot{u} = \phi_0 \) we have from (D.7) (D.1),

\[
f_{ab} = \begin{pmatrix} 0 & 0 & \ddot{u} & \ddot{u} \\ 0 & 0 & 0 & 0 \\ -\ddot{u} & 0 & 0 & 0 \\ -\ddot{u} & 0 & 0 & 0 \end{pmatrix}, \quad f^{a}_{b} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \ddot{u} & 0 & 0 \\ 0 & 0 & \ddot{u} & 0 \\ 0 & 0 & 0 & \ddot{u} \end{pmatrix}, \quad f^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\ddot{u} & -\ddot{u} \\ 0 & 0 & \ddot{u} & \ddot{u} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (D.17)
\]

This is the second case (D.6). Since \( f^\sigma_{\mu} f^{\mu}_{\nu} = f^a_{\mu} f^b_{\nu} \) and \( \text{det}(f^{\mu}_{\nu}) = \text{det}(f^{a}_{b}) \), we see from (D.16) (D.17) that this second case occurs if and only if \( f^\sigma_{\mu} f^{\mu}_{\sigma} = \text{det}(f^{\mu}_{\nu}) = 0 \). This proves the theorem.
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