Boolean 2-designs and the embedding of a 2-design in a group

Andrea Caggegi
Dipartimento di Metodi e Modelli Matematici
Viale delle Scienze Ed. 8, I-90128 Palermo (Italy)
caggegi@unipa.it

Alfonso Di Bartolo
Dipartimento di Matematica e Applicazioni
Via Archirafi 34, I-90123 Palermo (Italy)
alfonso@math.unipa.it

Giovanni Falcone
Dipartimento di Metodi e Modelli Matematici
Viale delle Scienze Ed. 8, I-90128 Palermo (Italy)
gfalcone@unipa.it

Abstract
This paper is the extended abstract of a talk given at the conference Combinatorics 2008, Costermano (Italy) 22-28 June 2008. We try to embed a $t$-design $\mathcal{D}$ in a finite commutative group in such a way that the sum of the $k$ points of a block is zero. We can compute the number of blocks of the boolean 2-design having all the non zero vectors of $\mathbb{Z}_2^n$ as the set of points and the $k$-subsets of elements the sum of which is zero as blocks.

1. Preliminaries.

We recall that a $t-(v, k, r_t)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a finite set $\mathcal{P}$ with $|\mathcal{P}| = v$, the elements of which are called points, together with a family $\mathcal{B}$ of subsets of $\mathcal{P}$, called blocks, such that any block contains exactly $k$ points and $t$ points are contained in exactly $r_t$ blocks. For any $s < t$, any $t-(v, k, r_t)$ design $\mathcal{D}$ is an $s-(v, k, r_s)$ design with

$$r_s = r_t \frac{(v-s)(v-s-1)\cdots(v-t+1)}{(k-s)(k-s-1)\cdots(k-t+1)}.$$

The number $r_1$ of blocks containing any given point is consequently a constant. It is usual to denote $r_1$ simply by $r$ and $r_2$ by $\lambda$. If we denote by $b = |\mathcal{B}|$ the number of blocks, a necessary condition for the existence of a $t-(v, k, r_t)$ design is $vr = bk$.

Labelling the points of $\mathcal{P}$ with $P_1, \ldots, P_v$ and the blocks of $\mathcal{B}$ with $b_1, \ldots, b_b$, the incidence matrix $A = (a_{ij})$ of $\mathcal{D}$ is defined putting

$$a_{ij} = \begin{cases} 1 & \text{if } P_i \in b_j \\ 0 & \text{if } P_i \notin b_j \end{cases}.$$
It holds $A^TA = (r-\lambda)I + \lambda J$, where $J$ is the $v \times v$ matrix the entries of which are all equal 1. It follows that $|A^TA| = rk(r-\lambda)^{(v-1)}$. A 2-design where $v = b$, or equivalently $r = k$, is called symmetric and its incidence matrix is such that $AA^T = A^TA$.

The complementary design of $\mathcal{D} = (\mathcal{P}, B)$ is the $t-(v, v-k, \tilde{r}_t)$ design $\tilde{\mathcal{D}} = (\mathcal{P}, \tilde{B})$ where $\tilde{B}$ is the set of $(v-k)$-tuples of $\mathcal{P}$ which are the complement $\mathcal{P}\setminus b$ of a block $b \in B$. Consequently, we have $\tilde{r}_t = (b-2r+\lambda) \frac{(k-2)\cdots(k-t+1)}{(v-2)\cdots(v-t+1)}$.

The supplementary design of $\mathcal{D} = (\mathcal{P}, B)$ is the $t-(v, k, r_t)$ design $\hat{\mathcal{D}} = (\mathcal{P}, \hat{B})$ where $\hat{B}$ is the set of unordered $k$-tuples of distinct points of $\mathcal{P}$ which are not blocks of $B$.

1. Remark: The $t-(v, k, r_t)$ design $\mathcal{D}$ is the complementary design of $\tilde{\mathcal{D}}$ as well as the supplementary design of $\hat{\mathcal{D}}$. The complementary design of the supplementary design of $\mathcal{D}$ is a $t-(v, v-k, \tilde{r}_t)$ design which is equal to the supplementary design of the complementary design of $\mathcal{D}$.$\square$

The derived design of $\mathcal{D} = (\mathcal{P}, B)$ at the point $P$ is the design $\text{Der}_P \mathcal{D} = (\mathcal{P}\setminus P, \text{Der}_P B)$ where $\text{Der}_P B = \{b \setminus P : P \in b \in B\}$.

Lastly, we recall that a Steiner $k$-tuple system is a $t-(v, k, r_t)$ design with $t = k-1$ and $r_t = 1$. Among the Steiner quadruple systems, i.e. $3-(v, 4, 1)$ designs, we find the boolean quadruple system of order $2^n$, which is defined, for $n \geq 3$, as the $3-(2^n, 4, 1)$ design obtained putting $\mathcal{B}$ to be the set of all quadruples of distinct vectors of $\mathcal{P} = \mathbb{Z}_2^n$ the sum of which is zero.

2. Embedding in a group.

We start by asking a natural question: what designs are subsets of a finite commutative group, so that the sum of the elements in a block is a constant? If this constant is zero, there is a unique way to define such a group. Let $\mathfrak{G}$ be the free commutative group generated by the $v$ points of $\mathcal{P}$ and let $\mathfrak{R}$ be the subgroup of $\mathfrak{G}$ generated by the $b$ elements of the form

$$\sum_{X \in b_j} X \quad (j = 1, \cdots, b),$$

where $b_j$ is a block of $\mathcal{B}$. The subgroup $\mathfrak{R}$ is clearly generated by the rows of the incidence matrix of $\mathcal{D}$. Finally, define the group $\mathfrak{G}_\mathcal{D} = \mathfrak{G}/\mathfrak{R}$ and consider the map $\mathcal{P} \longrightarrow \mathfrak{G}_\mathcal{D}$, $X \mapsto x = X + \mathfrak{R}$.

2. Example: Let $\mathcal{D}$ be the $2-(9, 3, 1)$ Steiner triple system. After reducing the rows of the incidence matrix by elementary integer linear
combinations

Thus $D$ is isomorphic to the point-line design of the affine plane $\mathbb{Z}_3^3$. Here, after reducing the rows of the incidence matrix

$$A = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}, \quad \sim
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

we find that $\mathcal{G}_D = (\mathbb{Z}_3)^3$, and the (images of the) points of $D$ are:

$$P_1 = (0, 2, 2) \quad P_2 = (1, 1, 2) \quad P_3 = (2, 0, 2)$$
$$P_4 = (1, 2, 1) \quad P_5 = (2, 1, 1) \quad P_6 = (0, 0, 1)$$
$$P_7 = (2, 2, 0) \quad P_8 = (0, 1, 0) \quad P_9 = (1, 0, 0).$$

Thus $D = \{(x_1, x_2, x_3) \in (\mathbb{Z}_3)^3 : x_1 + x_2 + x_3 = 1\}$. This gives evidence of the fact that $D$ is isomorphic to the point-line design of the affine plane of order 3. Now we consider the $2 - (9, 6, 5)$ design of pairs of parallel lines in the affine plane of order 3. Here, after reducing the rows of the incidence matrix

$$A = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0
\end{pmatrix}, \quad \sim
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 \\
0 & 1 & 0 & 0 & 0 & -2 & 0 & 2 & -1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & -2 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 1 & 2 & 0 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

we find that $\mathcal{G}_D = \mathbb{Z}_2 \oplus (\mathbb{Z}_3)^3$, and the points of $D$ are:

$$P_1 = (1; 0, 2, 2) \quad P_2 = (1; 1, 1, 2) \quad P_3 = (1; 2, 0, 2)$$
$$P_4 = (1; 1, 2, 1) \quad P_5 = (1; 2, 1, 1) \quad P_6 = (1; 0, 0, 1)$$
$$P_7 = (1; 2, 2, 0) \quad P_8 = (1; 0, 1, 0) \quad P_9 = (1; 1, 0, 0)$$

We remark that it is unexpected that the coordinates of the points are precisely the same as before, apart from the first one. Moreover, even if the first coordinate is constant, it seems to play an important role, since it distinguishes blocks from the first case: note that $(P_1 + P_2 + P_3)$ and $(P_4 + P_5 + P_6)$ here are not zero, whereas $(P_1 + P_2 + P_3) + (P_4 + P_5 + P_6) = (0; 0, 0, 0).$ 

□
In the following proposition, which has clearly a connection with the computation of the $p$-rank of the incidence matrix in [4] and [3], shows that the exponent of $G_D$ divides $k(r - \lambda)$.

3. **Proposition:** For any $x = X + R \in G_D$ we have $k(r - \lambda)x = 0$. If $\mathcal{D}$ has a partition in blocks, then $(r - \lambda)x = 0$.

**Proof.** Summing the points of the $r$ blocks through any given point $X$ we find

$$(r - \lambda)x + \lambda \cdot \sum_{y = Y + R} y = 0,$$

hence $(r - \lambda)x = 0$, if $\mathcal{D}$ has a partition in blocks. Otherwise, let $X_1, \cdots, X_k$ be the points of a given block, then

$$0 = (r - \lambda) \sum_{x_i = X_1 + R} x_i = -k \cdot \lambda \cdot \sum_{y = Y + R} y.$$ 

As $k \cdot \lambda \cdot \sum_{y = Y + R} y = 0$, the assertion follows. \hfill \Box

We remark that the injectivity of the map $\mathcal{P} \rightarrow G_D$, $X \mapsto x = X + R$, is not always guaranteed. In particular we find that, if $v \equiv 1 \pmod{12}$, any Steiner triple system $\mathcal{D}$ of cardinality $v$ is not embeddable in $G_D$. In fact, as a consequence of [3], we have that the $b \times v$ incidence matrix of $\mathcal{D}$ has rank $v$ over any field of characteristic $p \neq 3$ and it has rank $v - 1$ over a field of characteristic $p = 3$. This forces $G_D$ to be a cyclic group of a $3$-power order, but by the above Proposition the exponent of $G_D$ must be a divisor of 3. Since $\mathcal{D}$ has more than three points, we can see that $\mathcal{P}$ is not embeddable in $G_D$.

The following proposition, though trivial, has a fine corollary.

4. **Proposition:** The map is injective if and only if for any $v \in \mathbb{Z}^b$ and for any permutation matrix $H$, we have

$$vA \neq (1, -1, 0, \cdots, 0)H.$$ 

\hfill \Box

5. **Corollary:** (Irrationality condition for the injectivity): If the map is not injective, then there exists a vector $v \in \mathbb{Z}^b$ such that

$$\langle v, v \rangle = vAATvT = 2.$$ 

**Proof.** The assertion follows from the fact that, for any permutation matrix $H$, we have $HH^T = I$. \hfill \Box
6. Corollary: A symmetric $2-(v,k,\lambda)$ design $D$ is embeddable in $G_D$ (unless $D$ is a triangle).

Proof. It is known that a 2-design is symmetric if and only if $AA^T = A^TA = (k-\lambda)I + \lambda J$. Hence, for any non-zero integer $b$-tuple $v = (x_1, \ldots, x_b)$, we have

$$\langle v, v \rangle = vAA^Tv = (k-\lambda)\sum_{i=1}^{b} x_i^2 + \lambda \left( \sum_{i=1}^{b} x_i \right)^2 > 2.$$

As pointed out in Proposition 3, it is necessary that the exponent of $G_D$ is a divisor of $k(r-\lambda)$. Now we show that for the existence of a $2-(v,k,\lambda)$ design having a $p$-group as the set of points it is sufficient that $kx = 0$.

7. Proposition: Let $\mathcal{P}$ be the Galois field with $q = p^t$ elements, $p \geq 2$ a prime. For any $k = mp$, with $2 < k < q$, let $\mathcal{B}$ be the family of unordered $k$-tuples of distinct elements of $\mathcal{P}$ the sum of which is zero. Then $D = (\mathcal{P}, \mathcal{B})$ is a $2-(q,k,\lambda)$ design.

Proof. As the sum of the elements in the ground field is zero, we have that $\mathcal{B} \neq \emptyset$. Let now $P_1, P_2 \in \mathcal{B}$, let $Q_1, Q_2 \in \mathcal{P}$ and let $\rho(X) = AX + T$, with $A, T \in \mathcal{P}$ and $A \neq 0$, be the affinity of the affine line defined on $\mathcal{P}$, such that $\rho(P_i) = Q_i$. Then

$$\sum_{Q \in \rho(b)} Q = \sum_{P \in \mathcal{B}} \rho(P) = \sum_{P \in \mathcal{B}} (AP + T) = kT.$$

As $k \equiv 0 \ (p)$ we find that $\rho(b)$ is in $\mathcal{B}$, hence the number of unordered $k$-tuples of $\mathcal{B}$ containing $Q_1, Q_2$ is equal to the number of unordered $k$-tuples of $\mathcal{B}$ containing $P_1, P_2$. □

8. Remark: A remarkable case is when $k = p$ and $\mathcal{P}$ is an $n$-dimensional affine space over a Galois field with $p^m$ elements, $m > 1$. □

3. Boolean designs.

In this section we consider the case $p = 2$.

9. Proposition: Let $\mathcal{P}$ be a $n$-dimensional affine space over the Galois field with 2 elements. For any $k = 2m$, with $2 < k < 2^n$, let $\mathcal{B}$ be the family of unordered $k$-tuples of distinct elements of $\mathcal{P}$ the sum of which is zero. Then $D = (\mathcal{P}, \mathcal{B})$ is a $3-(2^n,k,\lambda_3)$ design.
Proof. Since a line in \( P \) cannot contain three points and the group of affinity is transitive on the triangles, we can move any three distinct points of \( P \) onto any three distinct points of \( P \) with an affinity. The proof follows as in Proposition 7.

10. Proposition: Let \( P \) be the set of non–zero vectors of \( \mathbb{Z}_2^n \) and, for any \( k = 2, 3, \ldots, 2^n - 2 \), let \( B_k \) be the family of unordered \( k \)-tuples of distinct elements of \( P \) the sum of which is zero. Then \( D_k = (P, B_k) \) is a \( 2 - (v = 2^n - 1, k, \lambda_k) \) design such that \( (k + 1)b_{k+1} + (v - k + 1)b_{k-1} = \binom{v}{k} \). Consequently we have \( b_k = \binom{v}{k} \alpha_{\lfloor k/2 \rfloor} \) where

\[
\alpha_h = \frac{1}{v-2h} \left( 1 - \sum_{i=0}^{h-2} (-1)^i \prod_{j=0}^{i} \frac{1+2(h-j)!!}{v-2(h-j-1)!!} \right) \\
= \frac{1}{v-2h} \sum_{i=0}^{h-1} (-1)^i \frac{(2h+1)!!}{(2(h-i)+1)!!} \cdot \frac{(v-2h)!!}{(v-2(h-i-1))!!}.
\]

11. Remark: According to [5] (see also sequences A010085-89 in [6]), the numbers \( b_k \) of blocks of \( D_k \) are equal to the weights of the huge \( (2^n - 1, 2^n - n - 1, 3) \)-Hamming code \( C \), whereas the numbers \( \tilde{b}_k \) of blocks of \( \tilde{D}_k \) are the numbers of weights \( k \) vectors of a \( C \) which belong to weight 1 cosets of \( C \).

12. Remark: The supplementary design \( \tilde{D}_k \) of the above one is the \( 2 - (2^n - 1, k, \tilde{r}_k) \) design defined, for any \( 1 < k < 2^n - 1 \), on the set \( P \) of non–zero vectors of \( \mathbb{Z}_2^n \) by the family \( B \) of unordered \( k \)-tuples of distinct elements of \( P \) the sum of which is different from zero.

13. Remark: For \( k = 4 \) in Proposition 9 we get the boolean system of order \( 2^n \). We remark that in the affine space \( \mathbb{Z}_2^n \) a necessary and sufficient condition for four distinct points to lie in an affine plane is that their sum is zero, that is the boolean quadruple system of order \( 2^n \) is the classical design of two-dimensional subspaces of an affine space over \( \mathbb{Z}_2 \). Consider now the design where \( k = 8 \). We remark that, in the affine spaces \( \mathbb{Z}_2^4 \) and \( \mathbb{Z}_2^5 \), a necessary and sufficient condition for eight distinct points \( P_1, \ldots, P_8 \) to lie in a two disjoint planes is that their sum is zero. To see this, we can assume that any four of \( \mathcal{I} = \{P_1, \ldots, P_4\} \) do not lie in a plane. Taking the sum of any three points of \( \mathcal{I} \), we get \( \binom{2^n}{3} \) further points, not in \( \mathcal{I} \), which are mutually distinct, otherwise two of the eight points of \( \mathcal{I} \) are equal. For \( n = 4, 5 \) we get then a contradiction. Hence the design we get for \( n = 4, 5 \) and \( k = 8 \) is the classical design of disjoint
pairs of two-dimensional subspaces of an affine space over \( \mathbb{Z}_2 \). Things change for \( \mathbb{Z}_2^6 \), because the following 8-ple, the sum of which is zero,

\[
(0,0,0,0,0,0) \quad (1,0,0,0,0,0) \quad (0,1,0,0,0,0) \quad (0,0,1,0,0,0) \\
(0,0,0,1,0,0) \quad (0,0,0,0,1,0) \quad (0,0,0,0,0,1) \quad (1,1,1,1,1,1),
\]
is not the disjoint union of two affine subplanes.

14. **Definition:** Let \( D_k = (\mathcal{P}, \mathcal{B}_k) \) be the \( 2 - (2^n - 1, k; \lambda_k) \) design where \( \mathcal{P} \) is the set of non-zero vectors of \( \mathbb{Z}_2^n \) and \( \mathcal{B}_k \) is the family of unordered \( k \)-tuples of distinct elements of \( \mathcal{P} \) the sum of which is zero. We say that the block \( b \in \mathcal{B}_k \) is reducible if it is the union of two disjoint blocks \( b_1 \in \mathcal{B}_{k_1}, b_2 \in \mathcal{B}_{k_2} \), where \( k_1 + k_2 = k \).

15. **Proposition:** Let \( \{e_i : i = 1, 2, \ldots, n\} \) be the canonical basis of \( \mathbb{Z}_2^n \) and let \( c_k = \{e_1, \ldots, e_{k-1}, e_1 + e_2 + \cdots + e_{k-1}\} \). Then any block \( b \in \mathcal{B}_k \) is irreducible if and only if \( b \) is contained in the orbit of \( c_k \) under \( GL(n, 2) \).

**Proof.** Let \( b = \{P_1, \ldots, P_{k-1}, P_k = \sum_{j=1}^{k-1} P_j\} \) be an irreducible block. The claim follows if we show that \( \{P_1, \ldots, P_{k-1}\} \) are linearly independent. By contradiction, we may assume without loss of generality that \( P_{k-1} = \sum_{j=1}^{k-2} \alpha_j P_j \), with \( \alpha_j = 0, 1 \), but not all zero. If any \( \alpha_j = 1 \), then \( P_k = 0 \), a contradiction. Thus some \( \alpha_j = 0 \). But this forces \( b \) to be reducible, a contradiction.

16. **Corollary:** The number of irreducible blocks \( b \in \mathcal{B}_k \) is \( \prod_{i=1}^{k} (2^n - 2^{i-1}) \).

17. **Remark:** The family \( \mathcal{B} \) of unordered \( k \)-tuples of linearly dependent vectors of \( \mathcal{P} \) is such that \( b \in \mathcal{B} \) if and only if \( b \) contains a \( h \)-tuple of non-zero vectors the sum of which is zero, for some \( h \leq k \). This shows that \( D = (\mathcal{P}, \mathcal{B}) \) is a 2-design, the supplementary of which is the 2-design defined by the family \( \tilde{\mathcal{B}} \) of unordered \( k \)-tuples of linearly independent vectors of \( \mathcal{P} \).

**References**

[1] T. Beth, D. Jungnickel, H. Lenz, *Design theory*, 2nd ed. Cambridge University Press (1999).

[2] C. J. Colbourn, J. H. Dinitz, *The CRC Handbook of Combinatorial Designs*, CRC Press, Boca Raton (1996).

[3] J. Doyen, X. Hubaut, M. Vandensavel, Ranks of incidence matrices of Steiner triple systems, Math. Z. 163, pp. 251259 (1978)

[4] N. Hamada, On the \( p \)-rank of the incidence matrix of a balanced or partially balanced incomplete block design and its applications
to error correcting codes, Hiroshima Math. J. Volume 3, Number 1, pp. 153-226 (1973)

[5] J. R. Schatz, On the Weight Distributions of Cosets of a Linear Code, Amer. Math. Monthly, Vol. 87, No. 7, pp. 548-551 (1980)

[6] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at www.research.att.com/njas/sequences/ (1996-2008)