Numerical investigation of non-Gaussianities in the phase and modulus of density Fourier modes

Qin Jian¹,², Pan Jun⁴, Yu Yu¹,²*, Zhang Pengjie¹,²,³†

¹Department of Astronomy, School of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai, 200240, China
²Key Laboratory for Particle Astrophysics and Cosmology (MOE)/Shanghai Key Laboratory for Particle Physics and Cosmology, China
³Tsung-Dao Lee Institute, Shanghai Jiao Tong University, Shanghai, 200240, China
⁴National Astronomical Observatories, Chinese Academy of Sciences, Beijing 100101, China

ABSTRACT

We numerically investigate non-Gaussianities in the late-time cosmological density field in Fourier space. We explore various statistics, including the two-point and three-point probability distribution function (PDF) of phase and modulus, and two & three-point correlation function of of phase and modulus. We detect significant non-Gaussianity for certain configurations. We compare the simulation results with the theoretical expansion series of Matsubara (2007). We find that the $O(V^{-1/2})$ order term alone is sufficiently accurate to describe all the measured non-Gaussianities in not only the PDFs, but also the correlations. We also numerically find that the phase-modulus cross-correlation contributes $\sim 50\%$ to the bispectrum, further verifying the accuracy of the $O(V^{-1/2})$ order prediction. This work demonstrates that non-Gaussianity of the cosmic density field is simpler in Fourier space, and may facilitate the data analysis in the era of precision cosmology.

Key words: cosmology: large scale structure – cosmology: dark matter

1 INTRODUCTION

The structure originates from the initial density fluctuations which are thought to be (or at least nearly) Gaussian, under the inflation scenarios (Guth 1981; Sato 1981; Linde 1982). The statistical properties of random Gaussian fields are completely characterized by the two-point correlation function (or its Fourier counterpart, the power spectrum), which are very convenient and popular tool to characterize the large-scale structure (Peebles 1980). On the other hand, due to the gravitational evolution of the Universe, the late-time matter density fields are highly non-Gaussian (Bernardeau et al. 2002a; Mellier 1999; Bartelmann & Schneider 2001; Munshi et al. 2008; Kilbinger 2015). As a result, cosmological information goes beyond the two-point statistics (Rimes & Hamilton 2005; Rimes & Hamilton 2006; Neyrinck & Szapudi 2007; Neyrinck et al. 2006). To effectively characterize and extract information form the late-time density fields, Cosmologists turn to non-Gaussian statistics such as the $n$-point correlation functions (Bernardeau et al. 2002b; Takada & Jain 2003; Sembolini et al. 2011; Fu et al. 2014), the polyspectra (e.g., Sefusati et al. 2006), Minkowski functionals (Mecke et al. 1994; Hikage et al. 2003a; Shirasaki & Yoshida 2014; Kratochvil et al. 2012; Matsubara et al. 2020; Matsubara & Kuriki 2020), counts of clusters, peaks, and voids (Jain & Van Waerbeke 2000; Mariani et al. 2009; Kratochvil et al. 2010; Liu et al. 2015a,b; Qin et al. 2018; Pisani et al. 2019), and statistics based on ideas with neural networks (e.g., Gupta et al. 2018; Ribli et al. 2019a,b; Cheng et al. 2020; Cheng & Ménard 2021). Applications of these non-Gaussian statistics to the large-scale structure of the Universe are quite popular (e.g., Hikage et al. 2003b, 2006; Liu et al. 2020; Fan et al. 2010; Yang et al. 2011; Liu & Haiman 2016; Liu et al. 2016; Kacprzak et al. 2016; Shan et al. 2017; Giocoli et al. 2018; Hall & Mead 2018; Martinet et al. 2018; Coulton et al. 2019; Munshi et al. 2019).

When considering the n-point correlation function or the polyspectra, the number of configurations explodes with the number of points, which makes the measurement difficult. Methods to avoid the high complexity of the n-point functions are proposed. One is the local transformation, which can significantly reduce the non-Gaussianity in the field, and enhance the cosmological information in the two-point statistics (e.g., Coles & Jones 1991; Neyrinck et al. 2009; Neyrinck 2011; McCullagh et al. 2016; Joachimi et al. 2011; Yu et al. 2011; Simpson et al. 2011; Carron & Szapudi 2013; Giblin et al. 2018). However, recently Qin et al. (2020) revealed that the copula is clearly non-Gaussian in the differential form. Since copula is invariant under local transformation, the above finding implies that local transformation can not reduce non-Gaussianity perfectly. On the other hand, in the Fourier space, the Line Correlation Function (LCF) relies on pure phase information to simplify the evaluation of non-Gaussianities (Obreschkow et al. 2013; Wolsteholme et al. 2015; Eggemeier & Smith 2017). The LCF is constructed from the phase correlations of the density field and has been applied, for instance, to constrain the growth rate of structure (Byun et al. 2020), to probe redshift-space distortions (Franco et al. 2019; Eggemeier et al. 2015), and to improve the cosmological constraints (Ali et al. 2018; Byun et al. 2017).

Theory of the Fourier mode probability distribution has been constructed by Matsubara (2007). Based on the generating function and
2 Qiu et al.

the cumulant expansion theorem, Matsubara (2007) obtained the general expression of the N-point Fourier mode distribution function, and the Edgeworth expansion series. Hikage et al. (2004) found that the leading order non-Gaussian term is already sufficient to describe the 3-point phase probability distribution function (PDF) in N-body simulations. The LCF statistics is also based on the theory of Matsubara (2007). Recently, Li et al. (2021) investigated the growth of the Fourier mode moduli, providing an alternative approach on the Fourier mode study.

Given the importance of the Matsubara (2007) theory in understanding the Fourier mode statistics, we carry out more comprehensive numerical investigations. The Fourier mode statistics is expanded into series of $O(V^{-1/2})$. So the key task is to quantify the relative difference of these terms. We measure both the 2-point and 3-point phase PDFs. We also derive and compare the N-point correlation functions of phases and moduli. The major finding is that the expansion to $O(V^{-1/2})$ is sufficient for all the investigated statistics. This agrees with the previous finding of Hikage et al. (2004) on the three-point phase PDF. Surveys such as BOSS, eBOSS, DESI and PFS have volumes of $\sim 10^{9} h^{-3} \text{Gpc}^3$, much larger than the simulation volumes that we investigate. Therefore a major conclusion is that the $O(V^{-1/2})$ order non-Gaussianity expression of Matsubara (2007) is sufficiently accurate for these surveys. Another interesting finding is that the phase-modulus cross-correlation is responsible for $\sim 50\%$ in the measured bispectrum.

This paper is organized as follows. In §2 we review the analytical formulas of the probability distribution functions of the Fourier modes, and we present the analytical predictions of the two and three point correlation functions of the phases and modulus derived from the PDFs. And we introduce our method to disrupt the phase-modulus correlation and investigate the influence on polyspectra. In §3, we describe the simulation data, present the results measured from simulations, and compare to the analytical formulas. We summarize the results in §4.

2 THE DISTRIBUTION FUNCTION OF FOURIER MODES

Starting from the the cumulant expansion theorem of the Fourier coefficients, Matsubara (2007) derived the general expression of N-point probability distribution function of Fourier modes $\delta(k) = |\delta(k)| e^{i \phi(k)}$. We summarize the major results in Matsubara (2007), and then derive the major statistics that we will investigate in this paper.

2.1 Preliminaries of Fourier mode distribution

The nonlinear evolution of the universes drives the n-point PDF $P_n$ of Fourier modes of $\delta(k_j)$ ($j = 1, \cdots, n$) to deviate from the Gaussian PDF $P_{G,n}$. Following Matsubara (2007), the arguments of PDF are taken as $A_1, \theta_1, \cdots, A_n, \theta_n$. Here $A_i \equiv A_{k_i}$ is the normalized Fourier modulus ($A_k = |\delta(k)|/\sqrt{P(k)}$) and $\theta_i \equiv \theta_{k_i}$ is the phase. For brevity, we often ignore the arguments in $P_n$ and $P_{G,n}$. The Gaussian PDF

$$P_{G,n} = \prod_{i=1}^{n} P_i, \quad P_i = \frac{1}{\sqrt{2\pi \sigma_i^2}} e^{-A_i^2/2\sigma_i^2}\pi^{-1}.$$ (1)

As we have learned from the central limit theorem, non-Gaussian corrections to $P_n$ decrease with the number of independent Fourier modes. Therefore they have explicit dependence on the volume $V$ in measuring the Fourier mode. Matsubara (2007) derived the full expression (its Eq. 48) in series of $V^{-1/2}, V^{-1/3}, \cdots$:

$$\frac{P_n}{P_{G,n}} - 1 = \sum_{i=1}^{n} C_{i/2}.$$ (2)

$C_{i/m}$ is the non-Gaussian correction of $V^{-i/m}$ dependence. So in general case, the leading order non-Gaussian correction is

$$C_{-1/2} = V^{-1/2}$$

$$\times \sum_{k_1, k_2, k_3} A_1 A_2 A_3 \cos(\theta_1 + \theta_2 - \theta_3) p^{(3)}(k_1, k_2, -k_3).$$ (3)

The symbol “uhs” indicates the “upper half sphere”: $k_z \geq 0$ of the $k$-space. The bispectrum is defined through

$$\langle \delta(k_1) \delta(k_2) \delta(k_3) \rangle = V^{-1/2} \frac{K_{k_1+k_2+k_3}}{\sqrt{2}} p^{(3)}(k_1, k_2, k_3),$$

$$p^{(3)}(k_1, k_2, k_3) = \frac{\delta^{uhs}_{k_1+k_2+k_3}}{\sqrt{P(k_1)P(k_2)P(k_3)}}.$$ (4)

Notice that the normalized bispectrum $p^{(3)}(k_1, k_2, k_3)$ does not depend on $V$, so the non-Gaussian correction to $P$ decreases with increasing volume, at the rate of $V^{-1/2}$.

The next-to-leading order non-Gaussian term $C_{-1}$ has a $V^{-1}$ dependence. So it is sub-dominant to $C_{-1/2}$ when $C_{-1/2} \neq 0$. However, for some configurations $C_{-1/2} = 0$ (e.g. $P_3$ with $k_1 + k_2 + k_3 \neq 0$) and $C_{-1}$ will be the dominant contribution of non-Gaussianity. $C_{-1}$ depends on both the bispectrum and trispectrum and the full expression is given by Eq. 58 of Matsubara (2007).

2.2 Derived statistics

The above results allow us to derive various statistics, such as the phase distribution and phase correlation function, modulus distribution and modulus correlation function, and also the phase-modulus cross-correlation.

2.2.1 Phase distribution and correlation function

Using Eq. 3, we can obtain the phase PDF $P(\theta_1, \cdots, \theta_n)$.

$$\frac{P(\theta_1, \cdots, \theta_n)}{P_{G}(\theta_1, \cdots, \theta_n)} = 1 + \frac{3}{4\sqrt{V}} \sum_{uhs} \cos(\theta_1 + \theta_2 - \theta_3)$$

$$\times p^{(3)}(k_1, k_2, -k_3)/O(V^{-1})$$

(5)

Here $P_{G}(\theta_1, \cdots, \theta_n) = (2\pi)^{-n}$. Therefore in general the phase distribution show non-Gaussianity at the order of $V^{-1/2}$. We can then derive the n-point phase correlation functions. For example, the 2-point phase correlation

$$\langle \theta_k \theta_{2k} \rangle = (\theta_k \theta_{2k})_G + \frac{\sqrt{\pi}}{4\sqrt{V}} p^{(3)}(k, k, -2k)$$

$$= \pi^3 + \frac{\sqrt{\pi}}{4\sqrt{V}} p^{(3)}(k, k, -2k)/O(V^{-1}).$$ (6)

The phase correlation function of 3 Fourier modes

$$\langle \rho_1^{m_1} \rho_2^{m_2} \rho_3^{m_3} \rangle = \frac{\langle \rho_1^{m_1} \rho_2^{m_2} \rho_3^{m_3} \rangle_G}{O(1)} + \frac{\langle \rho_1^{m_1} \rho_2^{m_2} \rho_3^{m_3} \rangle}{\langle \rho_1^{m_1} \rho_2^{m_2} \rho_3^{m_3} \rangle_G}$$

(7)

$$= \frac{(2\pi)^{m_1+m_2+m_3}}{(m_1+1)(m_2+1)(m_3+1)} + \left( \frac{\langle \rho_1^{m_1} \rho_2^{m_2} \rho_3^{m_3} \rangle}{NG} \right).$$
Here $\theta_{12} \equiv \theta_{k_1 + k_2}$. The non-Gaussian contribution $(\theta_1^m \theta_2^m \theta_{12}^m)_{\text{NG}}$ is generally nonzero. For instance,

$$\langle \theta_1 \theta_2 \theta_{12} \rangle = \frac{4\pi^4}{3} + \frac{\pi^{3/2}}{2\sqrt{V}} p^{(3)}(k_1, k_2, -k_1 - k_2) + O(V^{-1}).$$

The correlation function $\langle e^{i(\theta_1 + \theta_2 - \theta_{12})} \rangle$ combines these polynomials and has

$$\langle e^{i(\theta_1 + \theta_2 - \theta_{12})} \rangle = \frac{\pi^{3/2}}{8\sqrt{V}} p^{(3)}(k_1, k_2, -k_1 - k_2) + O(V^{-1}).$$

### 2.2.2 Modulus distribution and correlation function

In contrast, the modulus distribution is much more Gaussian, since from Eq. 3 we have

$$\mathcal{P}(A_1, \cdots, A_n) = 1 + O(V^{-1}).$$

(10)

So for mild non-Gaussian field, we expect the modulus distribution can be well approximated by the N-point independent Rayleigh distribution,

$$\mathcal{P}_G(A_1, \cdots, A_n) = \prod_i \mathcal{P}(A_i).$$

(11)

These barely correlated modulus combined with random phases (we call it the randomization of phases) yield Gaussian fields.

We can also derive the N-point correlation functions of the Fourier coefficients from the distribution, such as the 2-point moduli correlation

$$\langle A_k^m A_{2k}^n \rangle = \langle A_k^m A_{2k}^n \rangle + O(V^{-1}) = I(m)I(n) + O(V^{-1}).$$

(12)

Here

$$I(n) = \int_0^\infty A^n \cdot 2 N^{-\Delta^2} \, dA = \Gamma(1 + \frac{N}{2}).$$

At the order of $V^{-1}$ and above, there are non-vanishing non-Gaussian corrections to $\mathcal{P}(A)$, and the modulus correlation functions. These corrections are evaluated in §3.

### 2.2.3 Phase-modulus cross correlation

Nevertheless, the nearly Gaussian modulus distribution does not indicate that the modulus contains no non-Gaussian information. The point is that the moduli are correlated with phases. It is interesting to ask how much this modulus-phase correlation contribute to the non-Gaussianity. This can be quantified by the following operation. We generate a new field with phase correlations retained, but modulus-phase correlations removed by replacing the original moduli with random variables generated by the Rayleigh distribution. Then from equation (5) the joint distribution of the Fourier modes of the new field has

$$\mathcal{P}^\prime_n = 1 + \sum_{k_1, k_2, k_3} \frac{\pi^{3/2}}{4\sqrt{V}} \cos(\theta_1 + \theta_2 - \theta_3) p^{(3)}(k_1, k_2, -k_3).$$

(13)

Integration over $\mathcal{P}^\prime_n$ yields the bispectrum of the new field

$$p^{(3)}(k_1, k_2, -k_3) = \left(\frac{\pi}{4}\right)^3 \times p^{(3)}(k_1, k_2, -k_3).$$

(14)

We can see that there is a factor of $(\pi/4)^3 \approx 0.483$ decrease of the bispectrum if we decorrelate the phase-modulus. This empirically reflects the contribution to non-Gaussianity of the phase-modulus correlation.

To sum up, the modulus distribution of Fourier modes is nearly Gaussian and largely independent to each other. The leading order non-Gaussian correction is of the order $V^{-1}$. In contrast, the phase distribution is significantly more non-Gaussian. The leading order non-Gaussian correction, for general cases, is of the order $V^{-1/2}$ and is proportional to the bispectrum. In this work, we will numerically test these results for the cosmological density fields, including:

- The PDF of the Fourier modulus.
- The correlation functions of Fourier moduli. Direct comparison of the PDF may not be clear in terms of accuracy, especially for two or more point cases. The comparison of correlation functions can show differences more clearly in these cases.
- The PDF of the phase closure (e.g., $\theta_1 + \theta_2 - \theta_{12}$). To first order, the joint PDFs of phases are in the form of phase closures, which we compare to the distribution measured from simulated density fields.
- The phase correlation functions, which are statistics complementary to the phase closure distributions.

### 3 RESULTS

#### 3.1 Simulation data

The simulations we use for the test were run with 3072 particles in a box with volume $V = (600 \, \text{Mpc}/h)^3$, and a flat cosmology specified by $\Omega_m = 0.268$, $\Omega_{\Lambda} = 0.732$, $H_0 = 71 \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1}$, $\sigma_8 = 0.83$, $n_s = 0.968$. The details of the simulations and the CosmicGrowth

Figure 1. The one-point distribution function of normalized Fourier modulus of the density fields at three redshifts (from top to bottom panels). Three cases of wave number $k \sim 0.2 \, \text{Mpc}/h$, $0.5 \, \text{Mpc}/h$ and $1.0 \, \text{Mpc}/h$ (averaged with bin width $0.1 \, \text{Mpc}/h$) are shown from left to right panels, respectively. The solid lines show the Rayleigh distribution and the dotted lines show the non-Gaussian prediction Eq. (15). The three colors represent the results from the three realizations of the simulation. The non-Gaussian prediction is not distinguishable from the Rayleigh distribution.
simulation series are described in Jing et al. (2007) & Jing (2019). The density fields are sampled at redshifts \( z = 0, 1, 5 \) with the grid size \( 1 \text{Mpc}/h \). The mean number of particles per grid is 134.2, so we can safely neglect the effect of shot noise. The simulations have three different realizations, allowing us to better handle the cosmic variance.

### 3.2 Fourier modulus

Fig. 1 shows the one-point PDF of the normalized Fourier modulus, for \( z = 0, 1, 5 \) and three \( k \) bins at \([0.1, 0.3], [0.4, 0.6]\) and \([0.9, 1.0]\) (in units of \( h/\text{Mpc} \)). It is very close to Gaussian. The results indicates the central limit theory, i.e., the simulation box is large such that the Fourier transform sums over a large number of independent modes. This behavior has also been predicted by Fan & Bardeen (1995). The leading order non-Gaussian correction is

\[
\frac{P(A)}{P_G(A)} - 1 = \frac{1}{V} \left( \frac{1}{4} A^4 - A^2 + \frac{1}{2} \right) P^{(4)}(k, -k, -k, -k). 
\]  

(15)

We use the trispectrum measured from the same simulations to calculate this correction and plot it in Fig. 1. We find that it is totally negligible.

Fig. 2 shows the 2-point moduli PDF \( P(A_1, A_2) \) of \( k_1 \) and \( k_2 = 2k_1 \). Again the PDF is very close to Gaussian. The leading order non-Gaussian correction, \( P_2/P_{G,2} - 1 \) is

\[
\frac{1}{2V} \times \left\{ -2A_1^2A_2^2 - \frac{1}{2}A_1^4 + 2A_1^2 + A_2^2 - 1 \right\} P^{(3)}(k_1, k_1, -k_2) + \frac{1}{V} \times \left\{ A_1^2A_2^2 - A_1^2 - A_2^2 + 1 \right\} P^{(4)}(k_1, k_2, -k_1, -k_2).
\]

(16)

Figure 2 also shows the prediction including the above corrections. Again, their impacts are negligible and the 2-point PDF of moduli is well described by the Gaussian Fourier mode distribution.

The above PDF may not be most suitable to demonstrate the non-Gaussian corrections. So we compare the moduli correlation functions between two Fourier modes, which is a compressed version of the two-point moduli PDF. We only investigate the cases of \( k_2 = 2k_1 \) and \( z = 0 \). But we investigate different orders in \( A (\langle A_m^mA_n^n \rangle) \), with \((m, n) = (1, 1), (1, 2), (2, 2)\). For all cases, the simulation results agree well with the Gaussian prediction (Eq. 12).

The leading order non-Gaussian term, \( O(V^{-1}) \), is

\[
\frac{1}{2V} \times \left\{ -2I(m + 2)I(n + 2) - \frac{1}{2}I(m + 4) + 2I(m + 2) + I(n + 2) - 1 \right\} P^{(3)}(k, k, -2k) + \frac{1}{V} \times \left\{ I(m + 2)I(n + 2) - I(m + 2) - I(n + 2) + 1 \right\} P^{(4)}(k, 2k, -k, -2k).
\]

(17)

Figure 3 shows that the non-Gaussian correction is tiny. To quan-
Figure 4. The reduced two-point distribution (data points) of the Fourier phases of the density fields for $k_2 = 2k_1$ at three redshifts (from top to bottom panels). Cases for $|k_1| = 0.23 \text{Mpc}/h$, $0.53 \text{Mpc}/h$ and $1.02 \text{Mpc}/h$ are shown from left to right panels, respectively. $|k_1|$ is averaged from bins of width $0.1 \text{Mpc}/h$. The three colors represent the three realizations of the simulation. The solid lines represent analytical prediction to the $O(V^{-1/2})$ order and dotted lines to the $O(V^{-1})$ order. The dotted lines are not distinguishable from the solid lines.

Figure 5. Same as Fig 4 but for configuration $k_2 = 3k_1$ and the solid lines represent analytical prediction to the $O(V^{-1})$ order. To the $O(V^{-1/2})$ order a uniform distribution is expected.

The situation is different for other configurations, such as $k_1 = 3k_2$. In such case, the $O(V^{-1})$ vanishes, and the leading order non-Gaussian correction is

$$O(V^{-1}) = \frac{\pi}{8V} \cos(3\theta_k - \theta_{3k}) p^{(4)} (k,k,-3k)$$

(20)

Fig. 5 shows that in such case, the non-Gaussian correction in both simulation and theory are small, and can be neglected within the simulation statistical uncertainties. For other configurations ($k_2 \neq 2k_1$ and $k_2 \neq 3k_1$), non-Gaussian corrections only show in the order of $O(V^{-3/2})$ or above. Therefore for these configurations, phases can be treated as uncorrelated.

Then we compare the correlation functions. For $k_2 = 2k_1$,

$$\langle \theta_k \theta_{2k} \rangle - \pi^2 = \frac{\sqrt{\pi}}{4V} \left[ p^{(3)} (k,k,-2k) \right]$$

(21)

with,

$$O(V^{-1}) = \frac{1}{16V} \left[ p^{(3)} (k,k,-2k) \right]^2.$$  

(22)

But for $k_2 = 3k_1$, the leading order correction is $O(V^{-1})$,

$$\langle \theta_k \theta_{3k} \rangle - \pi^2 = \frac{\pi}{24V} p^{(4)} (k,k,k,3k) + \cdots$$

(23)

For this reason, we do not detect the impact of non-Gaussianity in the case of $k_2 = 3k_1$ (Fig. 6). In contrast, the detection of non-Gaussianity in the configuration $k_2 = 2k_1$ is more significant at most $k$ ranges. The deviation measured from the simulation (Fig. 6) agrees excellently with the theoretical prediction (Eq. 21, $O(V^{-1/2})$). The possible exception is at small $k$. But given the relatively less Fourier modes, the statistical fluctuations are larger and the deviation from Gaussian is statistically insignificant.

3.3 Fourier phases

For the phase PDF, the one-point PDF is always flat (Matsumara 2007). So we start with the 2-point PDF to examine the non-Gaussian corrections.

3.3.1 Two-point

Fig. 4 shows the phase PDF of $k_1 = 2k_2$. Firstly, it shows statistically significant deviation from the Gaussian case (flat PDF). The deviation is well described by the leading order non-Gaussian correction ($O(V^{-1/2})$),

$$O(V^{-1/2}) = \frac{\sqrt{\pi}}{2V} \cos(2\theta_k - \theta_{2k}) p^{(3)} (k,k,-2k).$$

(18)

We also calculate the second order correction ($O(V^{-1})$)

$$O(V^{-1}) = \frac{1}{2V} \cos \left( 2(2\theta_k - \theta_{2k}) \right) \left[ p^{(3)} (k,k,-2k) \right]^2.$$  

(19)

Fig. 4 shows that the $O(V^{-1})$ is negligible comparing to the leading order $O(V^{-1/2})$ term.

The situation is different for other configurations, such as $k_1 = 3k_2$. In such case, the $O(V^{-1/2})$ vanishes, and the leading order correction
corrections are

\[ O(V^{-1/2}) = \frac{\pi^{3/2}}{4\sqrt{V}} \cos(\theta_1 + \theta_2 - \theta_3) p(3) \langle k_1, k_2, -k_3 \rangle, \quad (24) \]

and the next-to-leading order correction is

\[ O(V^{-1}) = \frac{\pi}{V} \cos(2(\theta_1 + \theta_2 - \theta_3)) \left[p(3) \langle k_1, k_2, -k_3 \rangle \right]^2. \quad (25) \]

For any configurations, non-Gaussian corrections are of the order \( O(V^{-3/2}) \) or higher, and are negligible.

Figures 7 & 8 shows the reduced three-point distribution of the Fourier phases for configuration \( k_1 + k_2 = k_3 \) and \( 2k_1 + k_2 = k_3 \) respectively. Both cases are measured from the configuration that \( |k_1| \) and \( |k_2| \) lie in range \([0.5, 0.6] h^{-1}\text{Mpc}/h\) and open angle \( \alpha \) between \( k_1, k_2 \) lie in range \([10^\circ, 21^\circ], [59^\circ, 61^\circ]\) and \([119^\circ, 121^\circ]\). For \( k_1 + k_2 = k_3 \), the non-Gaussian correction is significant (Fig. 7). Due to the large sample size of \( k \) modes, the detection of non-Gaussianity is improved comparing to the case of two-point distribution. The measurement agrees well with the leading order correction in theory \( O(V^{-1/2}) \). The next-to-leading order correction \( O(V^{-1}) \) is negligible.

For \( 2k_1 + k_2 = k_3 \), we detect no significant deviation from the Gaussian distribution. This is also expected in the theory (Eq. 25).

For \( k_1 + k_2 = k_3 \) the prediction is determined by a factor of \( p(4) \) (Eq. 26) which are marked out in each panels of Fig 8.

We then expect the analytical formulas to describe the 3-point phase correlation functions, to a better accuracy than the 2-point cases. Taking configuration \( k_1 + k_2 = k_3 \) as an example, from Eq. 24 we have

\[ \langle \theta_1 \theta_2 \theta_3 \rangle = \frac{4\pi^4}{3} + \frac{\pi^{3/2}}{2\sqrt{V}} p(3) \langle k_1, k_2, -k_1 - k_2 \rangle + O(V^{-1}). \quad (27) \]

\[ O(V^{-1}) = \frac{1}{8V} \left[p(3) \langle k_1, k_2, -k_1 - k_2 \rangle \right]^2. \quad (28) \]
shows the 3-pt phase correlation functions at redshift 1. The reduced bispectrum is excellently. To test the influence of disrupting the modulus-phase correlation on Gaussianity, which agrees with the leading order theory prediction estimate the measurement errorbars. Again we detect significant non-Gaussianity, which agrees with the leading order theory prediction excellently.

3.4 Impact of phase-modulus cross-correlation

To test the influence of disrupting the modulus-phase correlation on the polyspectra, we make new realizations from the simulation results by randomizing the modulus or phases. That is, for the Fourier coefficients $f(k) \equiv A_k e^{i\theta_k}$, to randomize phases we replace the phase $\theta_k$ for each pixel with an independent random phase $\theta'_k$, while keeping the modulus fixed. The first new field $f'(k) = A_k e^{i\theta'_k}$ is then Gaussian Random Field (GRF) as the modulus are independently Rayleigh distributed (Eq. 10). To realize the randomization of the modulus, we keep the phases fixed. And we replace $A_k$ of each pixel with an independent random modulus $A'_k$ sampled from the Rayleigh distribution (Eq. 11). The second new field $f''(k) = A'_k e^{i\theta_k}$ then retain the phase information with phase-modulus correlation removed. We then measure the bispectrum from the two new fields and compare to the original field.

The analytical prediction of the influence of the bispectrum for disrupting phase-modulus correlation is equation (14), that is, a factor of $\left(\frac{x}{\hat{x}}\right)^3 \sim 0.5$ decrease in the bispectrum. On the other hand, the phase randomized field is Gaussian (to first order), vanished bispectrums are then predicted for this field. In Fig. 10 the reduced bispectrum from the density field at $z = 0$ and that from the randomized fields are plotted. The round and triangular data points show the results of the phase randomized field and modulus randomized field, respectively. The bispectrums are measured from the configuration that $|k_1|$ and $|k_2|$ lie in range $[0.5, 0.6] h^{-1}\text{Mpc}/h$ and open angle $\alpha$ (between $k_1, k_2$) lie in ranges with bin width $2^\circ$. The x-axis are the average open angles within each $\alpha$ bins. The dashed line (zero value line) and the square bars represent the prediction for the randomized fields and the star points show the result of the original field.

The results agree with the predictions for both fields, indicating again the validity of the perturbation theory and the first order approximations. The decrease of the bispectrum of the modulus randomized field implies the contribution of the Fourier modulus to the non-Gaussianity, in way of phase-modulus coupling.

4 SUMMARY

In this work, we analyzed the two-point and three-point probability distributions of modulus and phase of Fourier modes through N-body simulations, and made comparison to the theory prediction. We further measured the phase and modulus correlation functions, and derived the theoretical prediction and made the comparisons. We also investigated the phase-modulus cross-correlation. We found that it contributes $\sim 50\%$ to the measured bispectrum.

We found that the agreement between the simulation data and $O(V^{-1/2})$ order approximation is generally good, especially for the three point cases due to a larger sample size. These results are con-
sistent with the numerical investigation of three-point phase PDF (Hikage et al. 2004). We also calculate the $O(V^{-1})$ order correction, and found that they are always negligible. The analysis of (Hikage et al. 2004) used simulation volumes $V \leq (300h^{-1}Mpc)^3$. Our simulation volume is $V = (600h^{-1}Mpc)^3$. Most galaxy and weak lensing surveys have larger volume, at least of the order $(1000h^{-1}Mpc)^3$. This means that for the related Fourier mode analysis of these surveys, the $O(V^{-1/2})$ term is the only non-Gaussian correction that we need to include. The same methodology can be extended to the 2D field such as the weak lensing convergence field and the SZ effect. We expect weaker non-Gaussianity (e.g., Joachimi et al. 2011; Munshi et al. 2014; Yu et al. 2011, 2012, 2016; Chen et al. 2020; Zhang & Sheth 2007), and therefore the $O(V^{-1/2})$ term (or equivalently the $O(f_{sky}^{-1/2})$ term) will be sufficient for the 2D Fourier mode (or harmonic mode) PDF.

This work further demonstrates that, thanks to the central limit theorem, non-Gaussianities in Fourier space are weaker and simpler. Therefore data analysis in Fourier space has specific advantage, and may be worthy of further investigation.

DATA AVAILABILITY

All data included in this study are available upon request by contact with the corresponding author.

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