The ODE Method for Asymptotic Statistics in Stochastic Approximation and Reinforcement Learning

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Abstract
The paper concerns the stochastic approximation recursion,
\[ θ_{n+1} = θ_n + α_{n+1} f(θ_n, Φ_{n+1}) , \quad n ≥ 0 , \]
where the estimates \( θ_n ∈ \mathbb{R}^d \), and \( Φ := \{ Φ_n \} \) is a stochastic process on a general state space, satisfying a conditional Markov property that allows for parameter-dependent noise. In addition to standard Lipschitz assumptions and conditions on the vanishing step-size sequence, it is assumed that the associated mean flow \( \frac{d}{dt} θ_t = \mathcal{F} (θ_t) \), is globally asymptotically stable, with stationary point denoted \( θ^* \). The main results are established under additional conditions on the mean flow and a version of the Donsker-Varadhan Lyapunov drift condition known as (DV3) for \( Φ \):
(i) An appropriate Lyapunov function is constructed that implies convergence of the estimates in \( L_4 \).
(ii) A functional central limit theorem (CLT) is established, as well as the usual one-dimensional CLT for the normalized error. Moment bounds combined with the CLT imply convergence of the normalized covariance \( \mathbb{E}[z_n z_n^\top] \) to the asymptotic covariance \( Σ_θ \) in the CLT, where \( z_n := (θ_n - θ^*) / √{α_n} \).
(iii) The CLT holds for the normalized version \( z_{PR}^n \) of the averaged parameters \( θ_{PR}^n \), subject to standard assumptions on the step-size. Moreover, the normalized covariance of both \( θ_{PR}^n \) and \( z_{PR}^n \) converge to \( Σ_{PR}^θ \), the minimal covariance of Polyak and Ruppert.
(iv) An example is given where \( f \) and \( \mathcal{F} \) are linear in \( θ \), and \( Φ \) is a geometrically ergodic Markov chain but does not satisfy (DV3). While the algorithm is convergent, the second moment of \( θ_n \) is unbounded and in fact diverges.

This arXiv version 3 represents a major extension of the results in prior versions.
The main results now allow for parameter-dependent noise, as is often the case in applications to reinforcement learning.

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1 Introduction

The stochastic approximation (SA) method of Robbins and Monro [51] was designed to solve the $d$-dimensional root-finding problem $\overline{f}(\theta^*) = 0$, where $\overline{f}: \mathbb{R}^d \to \mathbb{R}^d$ is defined as the expectation $\overline{f}(\theta) := \mathbb{E}[f(\theta, \Phi)]$, $\theta \in \mathbb{R}^d$, with $f : \mathbb{R}^d \times X$ and $\Phi$ a random variable with values in $X$. Interest in SA has grown over the past decade with increasing interest in reinforcement learning (RL) and other “stochastic algorithms” [57, 37, 1].

Algorithm design begins with recognition that $\theta^*$ is a stationary point of the mean flow,

$$\frac{d}{dt} \theta_t = \overline{f}(\theta_t).$$

A standard assumption is that this ODE is globally asymptotically stable, so that in particular solutions converge to $\theta^*$ from each initial condition $\theta_0 \in \mathbb{R}^d$. In applications, the first step in the “ODE method” is to construct the function $f$ that determines $\overline{f}$ so that stability and other desirable properties are satisfied.

The mean flow can be approximated using an Euler scheme. Subject to standard Lipschitz conditions, it is recognized that the Euler approximation is robust to “measurement noise”, which motivates the SA recursion,

$$\theta_{n+1} = \theta_n + \alpha_{n+1} f(\theta_n, \Phi_{n+1}),$$

where $\{\alpha_n\}$ is the non-negative step-size sequence, and the distribution of $\Phi_n$ converges to that of $\Phi$ as $n \to \infty$. Writing $\Delta_{n+1} := f(\theta_n, \Phi_{n+1}) - \overline{f}(\theta_n)$, the interpretation as a noisy Euler approximation is made explicit:

$$\theta_{n+1} = \theta_n + \alpha_{n+1} \overline{f}(\theta_n) + \Delta_{n+1}, \quad n \geq 0.$$

Analysis of the SA recursion traditionally proceeds by comparison with solutions of the mean flow (1) [7, 3]. This requires a time-scaling: Take, $\tau_0 = 0$ and define,

$$\tau_{k+1} = \tau_k + \alpha_{k+1}, \quad k \geq 0.$$

Two continuous-time processes are then compared:

(i) **Interpolated parameter process:**

$$\Theta_t = \theta_k \text{ when } t = \tau_k, \text{ for each } k \geq 0,$$

and defined for all $t$ through piecewise linear interpolation.

(ii) **Re-started ODE:** For each $n \geq 0$, let $\{\hat{\theta}_t^{(n)} : t \geq \tau_n\}$ denote the solution to (1), initialized according to the current parameter estimate:

$$\frac{d}{dt} \hat{\theta}_t^{(n)} = \overline{f}(\hat{\theta}_t^{(n)}), \quad t \geq \tau_n, \quad \hat{\theta}_{\tau_n}^{(n)} = \theta_n.$$

Iteration of (3) gives, for any $0 < n < K$,

$$\Theta_{\tau_K} = \theta_n + \sum_{i=n}^{K-1} \alpha_{i+1} \overline{f}(\theta_i) + \sum_{i=n}^{K-1} \alpha_{i+1} \Delta_{i+1} = \Theta_{\tau_n} + \int_{\tau_n}^{\tau_K} \overline{f}(\Theta_\tau) \, d\tau + \mathcal{E}_K^{(n)},$$

where $\mathcal{E}_K^{(n)}$ is the sum of cumulative disturbances and the error resulting from the Riemann-Stieltjes approximation of the integral. The disturbance term $\mathcal{E}_K^{(n)}$ will vanish with $n$ uniformly in $K$ subject to conditions on $\{\Delta_i\}$ and the step-size. This is, by definition, the ODE approximation of $\{\theta_n\}$ [7, 3].

A closer inspection of the theory shows that two ingredients are required to establish convergence of SA: The first is stability of the mean flow (1), as previously noted. The second requirement is
that the parameter sequence \( \{ \theta_n \} \) obtained from the SA recursion is bounded with probability one. Rates of convergence through the central limit theorem (CLT) and the law of the iterated logarithm require further assumptions [3, 33, 30, 7, 22, 42].

There are two well-known approaches to establishing boundedness of the parameter sequence based on properties of the mean-flow. The relationship between these two approaches is discussed in [37, Ch. 4] and in [60, 59].

**Lyapunov criterion:** The existence of a smooth function \( V : \mathbb{R}^d \rightarrow \mathbb{R}_+ \) that has a Lipschitz gradient \( \nabla V \), along with \( \varepsilon > 0 \) satisfying

\[
\bar{f}(\theta)^\top \nabla V (\theta) \leq -\varepsilon \| \theta - \theta^* \|^2 \quad \text{for all } \theta.
\]

See [28] for examples of this approach and further history.

**ODE@\( \infty \):** Suppose that the following limit exists for each \( \theta \):

\[
\bar{f}_\infty (\theta) := \lim_{r \to \infty} r^{-1} \bar{f}(r\theta).
\]

This defines the vector field for the ODE@\( \infty \) of [8, 7]. The so-called (see, e.g., [4, 49]) “Borkar-Meyn theorem” states that: If this ODE is globally asymptotically stable, and the process \( \Phi = \{ \Phi_n \} \) such that the sequence \( \{ \Delta_n \} \) appearing in (3) is a martingale difference sequence, then the sequence \( \{ \theta_n \} \) is bounded with probability one. Relaxations of the assumptions of [8] are given in [4, 49]. Reference [50] presents an extension of [8] in which \( \Phi = \{ \Phi_n \} \) is parameter dependent. The setting is adversarial: It is assumed that the ODE@\( \infty \) is stable under the worst-case noise sequence.

The advantage of the ODE@\( \infty \) over Lyapunov techniques is that \( \bar{f}_\infty \) is often very simple compared to the vector field \( \bar{f} \), and its stationary point is always the origin. Consider for example stochastic gradient descent for the Rastrigin loss function \( \Gamma \), whose gradient is given by \( \nabla \Gamma (x) \vert_i = [2x_i + b \sin(2\pi x_i)] \), with \( b > 0 \). In this case \( \bar{f} = -\nabla \Gamma \) results in \( \bar{f}_\infty = -2x \).

Of the many applications of the results in [8], the majority are in the context of RL. However, two essential assumptions in this prior work appear to be often overlooked and may indeed be violated, especially in RL applications:

- The martingale difference sequence assumption in [8] holds only in very special cases, such as tabular Q-learning; the martingale difference property is exploited in [58] to obtain error bounds for this special case.
- It is assumed in [8] that \( \Phi \) is a Markov chain, which rules out parameter-dependent exploration, such as \( \varepsilon \)-greedy policies.

One of the main aims of this paper is to rectify these problems. We provide much more realistic assumptions on the “noise” process \( \Phi \), which are valid in a broad range of applications and lead to far stronger conclusions than anticipated in prior research. Specifically, we consider a family of Markov processes \( \Phi^\theta \) and assume that the evolution of each \( \Phi^\theta \) is specified by a transition kernel \( P^\theta \) from family \( \{ P^\theta : \theta \in \mathbb{R}^d \} \); these define a conditional Markov property as in [45, 50]. For example, such a parameter dependent model may take the form,

\[
\Phi_{n+1} = g(\Phi_n, \theta_n, W_{n+1}),
\]

in which \( W = \{ W_n \} \) is an independent and identically distributed (i.i.d.) sequence. Such models are required, e.g., in analysis of Q-learning, of off-policy TD-learning with parameter-dependent exploration, and of actor-critic methods [57, 37].

It is assumed that each transition kernel \( P^\theta \) satisfies (DV3)—a slightly weaker version of the well-known Donsker-Varadhan Lyapunov drift condition [13, 14]—which is used in [31, 32] to establish exact large deviations asymptotics for the partial sums of functions of a Markov chain; see Section 2.2. Details and connections with other drift conditions are discussed in the next section.

The main results we obtain are the following. We hope to make the theory more accessible by conveying the main ideas in the body of the paper, and leaving tedious calculations to the Appendix.
Contributions

- Subject to Lipschitz bounds on \( f \) and \( \overline{f} \), a Lyapunov function is constructed for the joint process \( \{ \theta_n, \Phi_n \} \). It satisfies a drift condition that implies boundedness of \( \{ \theta_n \} \) almost surely and in \( L_4 \). (Thms. 1 and 2.)

- Conditions are provided that ensure almost sure convergence for the parameter sequence, and also an associated CLT: \( z_n \xrightarrow{d} N(0, \Sigma_\alpha) \), where the convergence is in distribution, with \( z_n := \hat{\theta}_n / \sqrt{\alpha_n} \), \( \hat{\theta}_n := \theta_n - \theta^* \), and with the covariance \( \Sigma_\alpha \) explicitly identified. A corresponding functional CLT (FCLT) is also established (Thms. 3 and 4).

Convergence and the FCLT follow quickly from the strong bounds in Thm. 2 combined with techniques in the existing literature [7]. The following results are not in the current literature in any form, except in a few very special cases discussed below.

- Conditions are provided to ensure the sequence \( \{ z_n \} \) is bounded in \( L_4 \), and convergence of the normalized covariance is established in Thm. 4:

\[
\lim_{n \to \infty} \frac{1}{\alpha_n} \mathbb{E} [\hat{\theta}_n \hat{\theta}_n^T] = \lim_{n \to \infty} \mathbb{E} [z_n z_n^T] = \Sigma_\alpha. \tag{10}
\]

These results have significant implications for estimates obtained using the averaging technique of Polyak and Ruppert:

\[
\theta_{n}^{PR} = \frac{1}{n} \sum_{k=1}^{n} \theta_k, \quad n \geq 1. \tag{11}
\]

- **Thm. 5**: Suppose that the step-size \( \alpha_{n+1} = 1/(n+1)^\rho \) is used in (2), with \( \frac{1}{2} < \rho < 1 \). Under the same conditions as above, the CLT holds for the normalized sequence \( z_{n}^{PR} := \sqrt{n} \hat{\theta}_{n}^{PR} \), with \( \bar{\theta}_{n}^{PR} = \theta_{n}^{PR} - \theta^* \). Moreover, the rate of convergence is optimal, in that,

\[
\lim_{n \to \infty} n \mathbb{E} [\bar{\theta}_{n}^{PR} (\bar{\theta}_{n}^{PR})^T] = \lim_{n \to \infty} \mathbb{E} [z_{n}^{PR} (z_{n}^{PR})^T] = \Sigma_{PR}^{\alpha}, \tag{12}
\]

where \( \Sigma_{PR}^{\alpha} \) is minimal in a matricial sense [53, 46]. The proof (in Appendix A.6) also establishes the refinement,

\[
\mathbb{E} [\| A^* \bar{\theta}_{n}^{PR} + \frac{1}{n} H_n^* \|_2^2] \leq b_{17} \alpha_n^2, \quad n \geq 1, \tag{13}
\]

in which the matrix \( A^* \) is defined in (42) below, \( \{ H_k^* : k \geq 1 \} \) is a martingale, and the constant \( b_{17} \) is independent of \( \alpha \), but depends on \( \rho \) and the initial condition \( (\theta_0, \Phi_0) \). This bound easily implies (12) based on the definition of \( \Sigma_{PR}^{\alpha} \) in Thm. 5.

The discussion surrounding Prop. 3 indicates that condition (DV3) cannot be dropped. On the other hand, (DV3) is not overly restrictive, e.g., it holds for linear state space models with disturbance whose marginal has Gaussian tails, and the continuous-time version holds for the Langevin diffusion under mild conditions [32].

However, (DV3) fails for Markov chains on Euclidean space that are skip-free, i.e., when the increments \( \Phi_{n+1} - \Phi_n, n \geq 0 \), are deterministically bounded. For example, the M/M/1 queue in discrete time is geometrically ergodic under the standard load assumption [39, Ch. 3], but it does not satisfy (DV3). This motivates the further contribution:

- An example is given in which the assumptions of Thm. 1 hold; a scalar model in which \( \overline{f}(\theta) = -\theta \). Consequently, \( \lim_{n \to \infty} \theta_n = \theta^* = 0 \) with probability one from each initial condition. The driving noise is in some sense ideal: The (parameter-independent) Markov chain \( \Phi \) is
constructed so that it is reversible and geometrically ergodic. It also satisfies the L-mixing conditions imposed in [23]. However, (DV3) does not hold. Therefore, not only is there is no available theory to obtain moment bounds but indeed it is shown in Prop. 4 that the second moment diverges: \( \lim_{n \to \infty} E[\|\theta_n\|^2] = \infty \).

Prior work There are few results on convergence of moments, such as (10) or (12), in the SA literature. Most closely related to this paper is [23] which contains the error bound \( E[\|\tilde{\theta}_n\|^q]^{1/q} = O(1/\sqrt{n}) \) for every \( q \geq 1 \), but this conclusion is for a version of SA that requires resetting: A compact region \( \Theta^0 \subset \mathbb{R}^d \) is specified, along with a state \( \theta^0 \in \Theta^0 \), and the algorithm sets \( \theta_{n+1} = \theta^0 \) if \( \theta_n \notin \Theta^0 \). It is assumed that \( \theta^* \) lies in the interior of \( \Theta^0 \). The major statistical assumptions include Condition 1.1: \( \{f(\theta, \Phi_{n+1})\} \) and \( \{\partial_\theta f(\theta, \Phi_{n+1})\} \) are L-mixing, and Condition 1.2: For some \( \varepsilon_0 > 0 \), \( \sup_n E[\exp(\varepsilon_0 f(\theta, \Phi_{n+1}))] \) is uniformly bounded for \( \theta \) in compact sets. It is not clear how these conclusions can be extended to the present setting, or if finer results such as (12) are even achievable.

The optimal asymptotic variance in the CLT for SA and techniques to obtain the optimum for scalar recursions were introduced by Chung [9], soon after the introduction of SA. See also [54, 19] for early work, and surveys in [62, 45, 3, 33, 7]. The averaging technique came in the independent work of Ruppert [53] and Polyak and Juditsky [46, 47]; see [43] for an elegant technical summary, and [15] for the best results for linear SA. The reader is referred to [3, 33, 7] for more history on the substantial literature on asymptotic statistics for SA.

The roots of the ODE@\( \infty \) can be traced back to the fluid model techniques for stability of networks [10, 11, 39], which were extended to skip-free Markov chains in [20]. See [7, 37] for further history since [8].

Much of the recent literature on convergence rates for SA is motivated by applications to RL, and seeks finite-\( n \) error bounds rather than asymptotic results of the form considered here. The article [56] treats temporal difference (TD) learning (a particular RL algorithm) with Markovian noise, which justifies the linear SA recursion of the form \( f(\theta, \Phi) = A(\Phi)\theta + b(\Phi) \). The primary focus is on the constant step-size setting, with parallel results for vanishing step-size sequences. The uniform bounds assumed in [56] will hold under uniform ergodicity and boundedness of \( A \) and \( b \) as functions of \( \Phi \). Improved bounds are obtained in [16], but subject to i.i.d. noise; this paper also contains a broad survey of results in this domain. While not precisely finite-\( n \) bounds as in [56], they are asymptotically tight since they are refinements of the CLT.

Finite-\( n \) bounds are obtained in [43, 2] for stochastic gradient descent, with martingale difference noise. This statistical assumption is relaxed in [17, 44] for linear SA recursions; the assumptions of [17] include (DV3), along with conditions on the functions \( A \) and \( b \) that are related to the bounds imposed here. Most of the effort is devoted to algorithms with constant step-size combined with averaging as in (11).

In [26] a functional law of the iterated logarithm is obtained for linear SA recursions with constant step-size, and subject to an L-mixing condition similar to [23].

2 Background and Assumptions

This section contains some of the main assumptions that will remain in effect for most of our result, and an outline of important background material.

2.1 SA model

The stochastic process \( \Phi = \{\Phi_n\} \) evolves on a Polish state space \( X \), equipped with its Borel sigma-algebra \( B(X) \). Its dynamics are defined by a family of transition kernels \( \{P_\theta : \theta \in \mathbb{R}^d\} \). In the special
We next lay out notation for a single Markov chain \( \Phi \). The mean flow vector field is then defined as the expectation, where \( s \) and \( g, h \) are measurable functions of \( x \) for each \( n \in \mathbb{N} \). The one-step minorization condition under (16):

\[
R(x, A) \geq s_n(x) \nu(A), \quad \text{for } A \in \mathcal{B}(X) \text{ and all } x \in X.
\]

In analysis it is often more convenient to work with the resolvent \( \Phi + \nu \) with transition kernel \( P_\nu \), for \( \theta \in \mathbb{R}^d \) fixed. For the realization (9), this evolves according to the recursion, \( \Phi_{n+1} = g(\Phi_n, \theta, W_{n+1}) \), \( n \geq 0 \).

It is assumed that each \( \Phi \) is geometrically ergodic with unique probability invariant measure \( \pi_\theta \). The mean flow vector field is then defined as the expectation,

\[
\bar{f}(\theta) = \int f(\theta, x) \pi_\theta(dx).
\]

For any \( \mathcal{B}(X) \)-measurable function \( g : X \to \mathbb{R}^m \) and any measure \( \mu \) on \( (X, \mathcal{B}(X)) \), we use the compact notation \( \mu(g) := \int g(x) \mu(dx) \), whenever the integral is well defined.

### 2.2 Markov chains

We next lay out notation for a single Markov chain \( \Phi \) on \( X \) with transition kernel \( P \). As above, it is assumed that \( \Phi \) is geometrically ergodic with unique probability invariant measure \( \pi \).

A pair \( (C, \nu) \) are called small if \( C \in \mathcal{B}(X) \) and \( \nu \) is a probability measure on \( (X, \mathcal{B}(X)) \), such that, for some \( \varepsilon > 0 \) and \( n_0 \geq 1 \) the minorization condition holds:

\[
P_{n_0}(x, A) \geq \varepsilon \nu(A), \quad \text{for } A \in \mathcal{B}(X) \text{ and } x \in C.
\]

If the particular \( \nu \) is not of interest, then we simply say that the set \( C \) is small. It is sometimes convenient to replace the set \( C \) by a function \( s : X \to \mathbb{R}_+ \). Then we say that \( s \) is small if,

\[
P_{n}(x, A) \geq \varepsilon s(x) \nu(A), \quad \text{for } A \in \mathcal{B}(X) \text{ and all } x \in X.
\]

In analysis it is often more convenient to work with the resolvent \( R := \sum_{n=0}^{\infty} 2^{-n} P_n \), which satisfies a one-step minorization condition under (16):

\[
R(x, A) \geq s_+(x) \nu(A), \quad \text{for } A \in \mathcal{B}(X) \text{ and all } x \in X,
\]

where \( s_+(x) = 2^{-n_0-1} \varepsilon \int R(x, dy) s(y) \).

The Markov chain is called aperiodic if there exists a probability measure \( \nu \) and \( \varepsilon > 0 \) such that, for each \( x \in X \) and \( A \in \mathcal{B}(X) \), there is \( n(x, A) \geq 1 \) such that:

\[
P_{n}(x, A) \geq \varepsilon \nu(A), \quad \text{for all } n \geq n(x, A).
\]

For any measurable function \( g : X \to [1, \infty) \), the Banach space \( L^g_\infty \) is defined to be the set of measurable functions \( \phi : X \to \mathbb{R} \) satisfying:

\[
\|\phi\|_g := \sup_{x} \frac{1}{g(x)} |\phi(x)| < \infty.
\]

For a pair of functions \( g, h : X \to [1, \infty) \), and any linear operator \( \widehat{P} : L^g_\infty \to L^h_\infty \), the induced operator norm is denoted:

\[
\|\widehat{P}\|_{g,h} := \sup\left\{ \frac{\|\widehat{P}\phi\|_h}{\|\phi\|_g} : \phi \in L^g_\infty, \|\phi\|_g \neq 0 \right\}.
\]
We write $\|\tilde{P}\|_g$ instead of $\|\tilde{P}\|_{g,h}$ when $g = h$. In particular, we view of transition kernels $Q$ as linear operators acting on functions $g$ on $X$ via $Qg(x) = \int g(y)P(x, dy)$, $x \in X$.

Suppose that $\pi'(g) := \int g\pi < \infty$. Then the rank-one operator $1 \otimes \pi$ has finite induced operator norm, $\|1 \otimes \pi\|_{g,h} < \infty$, for any choice of $h$, where,

$$[1 \otimes \pi] \phi(x) = \pi(\phi), \quad \text{for } x \in X \text{ and } \phi \in L^g_\infty.$$ 

Consider the centered semigroup $\{\tilde{P}^n = P^n - 1 \otimes \pi : n \geq 0\}$. The chain $\Phi$ is called $g$-uniformly ergodic if $\pi'(g)$ is finite, and $\|\tilde{P}^n\|_g \to 0$ as $n \to \infty$. Necessarily the rate of convergence is geometric, and it is known that seemingly weaker definitions of geometric ergodicity are equivalent to the existence of $g$ for which $g$-uniform ergodicity holds [41]. We say $\Phi$ is geometrically ergodic if it is $g$-uniformly ergodic for some $g$.

The Lyapunov drift criterion (V4) holds with respect to the Lyapunov function $v : X \to [1, \infty)$, if:

For a small function $s$, and constants $\delta > 0, b < \infty$,

$$E[v(\Phi_{k+1}) - v(\Phi_k) | \Phi_k = x] \leq -\delta v(x) + bs(x), \quad x \in X. \quad (V4)$$

Under (V4) and aperiodicity, the Markov chain is $v$-uniformly ergodic [41, Ch. 15.2.2].

We establish almost sure (a.s.) boundedness and convergence of $\{\theta_n\}$ in (2) subject to (V4) for the family of transition kernels $\{P_\theta : \theta \in \mathbb{R}^d\}$, and suitable conditions on the step-size sequence and on $\tilde{f}$. A stronger drift condition is imposed to establish moment bounds:

For functions $V : X \to \mathbb{R}_+$, $W : X \to [1, \infty)$, a small function $s : X \to [0, 1]$, and $b > 0$:

$$E[\exp(V(\Phi_{k+1})) | \Phi_k = x] \leq \exp(V(x) - W(x) + bs(x)), \quad x \in X. \quad (DV3)$$

The bound (DV3) implies (V4) with $v = e^V$; see [32] or [41, Ch. 20.1].

**Proposition 1.** Suppose that the Markov chain $\Phi$ is aperiodic. Then:

(i) Under (V4), there exists $g \in (0,1)$ and $b_v < \infty$ such that for each $g \in L^\infty_v$,

$$|E[g(\Phi_n) | \Phi_0 = x] - \pi'(g)| \leq b_v\|g\|_v v(x)g^n, \quad n \geq 0, x \in X. \quad (19a)$$

(ii) If (DV3) holds then the conclusion in (i) holds with $v = e^V$. Consequently, for each $g \in L^V_\infty$ and $\kappa > 0$, we can find $b_{\kappa,g} < \infty$ such that

$$E[|g(\Phi_n)|^\kappa | \Phi_0 = x] \leq b_{\kappa,g}v(x), \quad n \geq 0, x \in X. \quad (19b)$$

**Proof.** The geometric convergence (19a) follows from [41, Th. 16.1.4], and (19b) follows from $v$-uniform ergodicity and the fact that $|||g|^\kappa||_v < \infty$ under our assumptions [31, 32]. \hfill \Box

### 2.3 Disturbance decomposition and ODE solidarity

The decomposition of Métivier and Priouret [36] provides a representation for $\{\Delta_{n+1} := \tilde{f}(\theta_n, \Phi_{n+1}) : n \geq 0\}$, where $\tilde{f}(\theta, x) := f(\theta, x) - \tilde{f}(\theta)$ for $\theta \in \mathbb{R}^d$ and $x \in X$, with $f$ defined in (2) and $\tilde{f}$ defined in (15). Let $\tilde{f} : \mathbb{R}^d \times X \to \mathbb{R}^d$ solve Poisson’s equation:

$$E[\tilde{f}(\theta, \Phi_{n+1})] - \tilde{f}(\theta, \Phi_n^\theta) | \Phi_n^\theta = x] = -\tilde{f}(\theta, x), \quad \theta \in \mathbb{R}^d, x \in X. \quad (20)$$
Provided a measurable solution exists, we obtain a disturbance decomposition based on the three sequences:

\[ \zeta_{n+1} := \hat{f}(\theta_n, \Phi_{n+1}) - \mathbb{E}[\hat{f}(\theta_n, \Phi_{n+1}) | \mathcal{F}_n], \quad T_n := \psi(\theta_n, \Phi_n), \]

\[ \Upsilon_n := \frac{1}{\alpha_n} \left[ \psi(\theta_n, \Phi_n) - \psi(\theta_{n-1}, \Phi_n) \right], \quad \text{with} \quad \psi(\theta, x) := f(\theta, x) - \hat{f}(\theta, x) \]

(21)

Recall the sequence \( \{\tau_K\} \) defined in (4).

**Lemma 1.** We have,

\[ \Delta_n = \zeta_n - T_n + T_n - 1 - \alpha_n \Upsilon_n, \quad \text{and}, \]

\[ \sum_{i=n+1}^{K} \alpha_i \Delta_i = M_{\tau_K} - M_{\tau_n} + \mathcal{E}_{n,K}, \quad 0 \leq n < K, \]

with \( M_{\tau_K} := \sum_{k=1}^{K} \alpha_k \zeta_k, \quad 1 \leq K < \infty, \quad M_{\tau_0} = 0, \)

\[ \mathcal{E}_{n,K} := - \sum_{i=n+1}^{K} \alpha_i^2 \Upsilon_i + \alpha_n T_n - \alpha_{K+1} T_K - \sum_{i=n+1}^{K} \left[ \alpha_i - \alpha_{i+1} \right] T_i. \]

Moreover, \( \{\zeta_k\} \) is a martingale difference sequence and \( \{M_{\tau_K}\} \) is a martingale.

**Proof.** The representation for \( \{\Delta_{n+1}\} \) follows from (20), and (22) is then obtained through summation by parts. \( \square \)

The disturbance decomposition in [36] subsequent papers is applied in the simpler setting in which \( \Phi \) is Markovian [3, 33, 7]. Obtaining useful bounds on \( \{\Upsilon_{n+1}\} \) is more challenging in our present, more general setting. The value of Lemma 1 is clear only after we establish that \( \{\mathcal{E}_{n,K}\} \) has a very small contribution to estimation error as compared to the martingale \( \{M_{\tau_K}\} \); much of the work in the Appendix is devoted to quantifying this claim.

The indexing of the martingale \( \{M_{\tau_K}\} \) by \( \tau_K \) (defined in (4)) is to facilitate ODE approximations. These are defined by comparison of two continuous-time processes: \( \Theta = \{\Theta_t\} \), defined in (5), and the solution \( \theta^{(n)} = \{\theta^{(n)}_t\} \) to the re-started ODE (6). The two processes are compared over time blocks of length approximately \( T \), with \( T > 0 \) chosen sufficiently large.

Let \( T_0 = 0 \) and \( T_{n+1} = \min\{\tau_k : \tau_k \geq T_n + T\} \). The \( n \)-th time block is \([T_n, T_{n+1}]\) and \( T \leq T_{n+1} - T_n \leq T + \bar{\alpha} \) by construction, where \( \bar{\alpha} := \sup_k \alpha_k \) is assumed bounded by 1 in Assumption (A1) below. The following notation is required throughout:

\[ m_0 := 0, \quad \text{and} \quad m_n \text{ denotes the integer satisfying } \tau_{m_n} = T_n, \quad \text{for each } n \geq 1. \]

(23)

**Lemma 1** is one step in establishing the ODE approximation,

\[ \lim_{n \to \infty} \sup_{T_n \leq t \leq T_{n+1}} \|\Theta_t - \theta^{(n)}_t\| = 0, \quad \text{a.s.} \]

(24)

This, combined with global asymptotic stability of the mean flow (1), quickly leads to convergence of \( \{\theta_n\} \), provided this sequence is bounded. Boundedness is established through examination of a scaled vector field defined next.
2.4 ODE@∞

The ODE@∞ technique for establishing stability of SA is based on the scaled vector field, denoted $\bar{f}_c(\theta):=\overline{f}(c\theta)/c$, for any $c \geq 1$, and requires the existence of a continuous limit: $\bar{f}_\infty(\theta):=\lim_{c \to \infty} \bar{f}_c(\theta)$ for any $\theta \in \mathbb{R}^d$. The ODE@∞ is then denoted:

$$\frac{d}{dt} \theta_t = \bar{f}_\infty(\theta_t).$$

We always have $\bar{f}_\infty(0) = 0$, and asymptotic stability of this equilibrium is equivalent to global asymptotic stability [8, 7].

An ODE approximation is obtained for the scaled parameter estimates: On denoting $c_n := \max\{1, \|\theta_{m_n}\|\}$, and $\hat{\theta}_k = \theta_k/c_n$ for each $n$ and $m_n \leq k < m_{n+1}$, we compare two processes in continuous time on the interval $[T_n, T_{n+1})$:

$$\hat{\Theta}_t = \frac{1}{c_n} \Theta_t, \quad T_n \leq t < T_{n+1},$$

$$\frac{d}{dt} \hat{\Theta}_t = \bar{f}_c(\hat{\Theta}_t), \quad \text{with initial condition } \hat{\theta}_{T_n} = \hat{\Theta}_{T_n} = \frac{1}{c_n} \theta_{m_n}.$$  (26a)

The scaled iterates $\{\hat{\theta}_k\}$ satisfy, for $m_n \leq k < m_{n+1}$,

$$\hat{\theta}_{k+1} = \hat{\theta}_k + \alpha_{k+1} f_{c_n}(\hat{\theta}_k, \Phi_{k+1}) = \hat{\theta}_k + \alpha_{k+1} [\bar{f}_c(\hat{\theta}_k) + \Delta_{k+1}],$$

where $f_{c_n}(\theta_k, \Phi_{k+1}) := f(c_n \theta_k, \Phi_{k+1})/c_n$ and $\Delta_{k+1} := \Delta_{k+1}/c_n$. The representation (27) motivates an ODE analysis similar to what is performed to establish (24). The desired approximation is established in Prop. 10:

$$\lim_{n \to \infty} \sup_{T_n \leq t < T_{n+1}} ||\hat{\Theta}_t - \hat{\theta}_t|| = 0, \quad \text{a.s.}$$

The short proof of Prop. 9 shows that (28) implies a.s. boundedness of $\{\theta_k\}$.

3 Main Results

Since we will be working with a family of Markov chains $\{\Phi^\theta\}$, where each $\Phi^\theta$ has transition kernel $P_\theta$ in some family $\{P_\theta : \theta \in \mathbb{R}^d\}$, we require a family of drift conditions. For simplicity we assume a common Lyapunov function, and a common small set condition in the form (17). In particular, from now on, when we say that (DV3) holds we mean the following:

For functions $V : X \to \mathbb{R}_+$, $W : X \to [1, \infty)$, $s : X \to [0, 1]$, and $b > 0$,

$$E\left[ \exp(V(\Phi^\theta_{k+1})) \mid \Phi_k^\theta = x \right] \leq \exp(V(x) - W(x) + bs(x)),$$

for all $x \in X, \theta \in \mathbb{R}^d$. Moreover, $s(x) > 0$ for all $x$, and there is a probability measure $\nu$ on $\mathcal{B}(X)$ such that,

$$R_\theta(x, A) \geq s(x) \nu(A), \quad \text{for } A \in \mathcal{B}(X) \text{ and all } x \in X, \theta \in \mathbb{R}^d.$$  (29)

The uniform version of (V4) is analogous.

We require a Lipschitz continuity assumption for the family of transition kernels: For a constant $b_d$, and any $\theta, \theta' \in \mathbb{R}^d$,

$$\|P_\theta - P_{\theta'}\|_H \leq \frac{b_d}{1 + \|\theta\| + \|\theta'\|} \|\theta - \theta'\|,$$

where the choice of the function of $H : X \to [1, \infty)$ depends on the context; recall (18) for notation.
In applications to RL, this Lipschitz condition can be obtained by design. In particular, in the stability analysis of Q-learning found in [38], exploration is designed so that $P_{t\theta} = P_{\theta}$ for $\|\theta\| \geq M$ and all $r \geq 1$, where $M \gg 1$ is a design choice. Local Lipschitz continuity of $\{P_\theta\}$ in $L^H_\infty$ then implies the global bound (30).

Our main results depend on Assumptions (A1)–(A3) below. Assumption (A1) is standard in the SA literature, and (A3) is the starting point for analysis based on the ODE at $\infty$. The special assumptions on $\Phi$ in (A2) are less common; especially the introduction of (DV3).

(A1) The non-negative gain sequence $\{\alpha_n\}$ satisfies:

$$0 \leq \alpha_n \leq 1, \text{ for all } n, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} \alpha_n^2 < \infty. \quad (31)$$

Moreover, with $\gamma_n := \alpha_{n+1}^{-1} - \alpha_n^{-1}$, $\gamma = \lim_{n \to \infty} \gamma_n$ exists and is finite.

(A2) There exists a measurable function $L : \mathbb{X} \to \mathbb{R}_+$ such that for each $x \in \mathbb{X}$,

$$\|f(0,x)\| \leq L(x),$$
$$\|f(\theta, x) - f(\theta', x)\| \leq L(x)\|\theta - \theta'\|, \quad \theta, \theta' \in \mathbb{R}^d, \quad (32)$$

with $L : \mathbb{X} \to \mathbb{R}_+$. The remaining parts of (A2) depend on the drift condition imposed:

- Under (V4), it is assumed that $L^8 \in L^p_\infty$ and that (30) holds with $H = v^\delta$, for any $1/4 \leq \delta \leq 1$.
- Under (DV3), it is assumed that $\delta_L := \|L\|_W < 1$ (so that $L \in L^W_\infty$) and that (30) holds with $H = 1 + V^p$ for $p = 1, 2$. In addition, for each $r > 0$,

$$S_W(r) := \{x : W(x) \leq r\} \quad \text{is either small or empty for any } P_\theta. \quad (33a)$$
$$b_V(r) := \sup\{V(x) : x \in S_W(r)\} < \infty. \quad (33b)$$

(A3) The scaled vector field $\mathcal{F}_\infty(\theta)$ exists: $\mathcal{F}_c(\theta) \to \mathcal{F}_\infty(\theta)$ as $c \to \infty$, for each $\theta \in \mathbb{R}^d$. Moreover, the ODE (25) is globally asymptotically stable.

Given $c \geq 1$, let $\phi_c(t, \theta)$ denote the solution to,

$$\frac{d}{dt} \theta_t = \mathcal{F}_c(\theta_t), \quad \theta_0 = \theta, \quad (34)$$

and define $\phi_\infty(t, \theta)$ accordingly, so that it solves (25) with initial condition $\theta_0 = \theta$. It is known that (A3) implies global exponential asymptotic stability of the ODE at $\infty$:

**Proposition 2.** Under (A3) there exists $T_r > 0$ such that $\|\phi_\infty(t, \theta)\| \leq \frac{1}{2}\|\theta\|$ for $t \geq T_r$ and every $\theta \in \mathbb{R}^d$. Moreover:

(i) There exist positive constants $b$ and $\delta$ such that, for any $\theta \in \mathbb{R}^d$ and $t \geq 0$,

$$\|\phi_\infty(t, \theta)\| \leq b\|\theta\|e^{-\delta t}.$$

(ii) There exists $c_0 > 0$ and $\frac{1}{2} < \varrho_t < 1$ such that whenever $\|\theta\| \leq 1$,

$$\|\phi_c(t, \theta)\| \leq \varrho_t, \quad \text{for all } t \in [T_r, T_r + 1], \quad c \geq c_0.$$

Part (i) is [8, Lem. 2.6] and (ii) is [7, Cor. 4.1].
3.1 Convergence and moment bounds

Thm. 1 is anticipated from well-established theory.

**Theorem 1.** Suppose that the mean flow (1) is globally asymptotically stable. If, in addition, (V4) and (A1)–(A3) hold, then the parameter sequence \( \{\theta_k\} \) converges a.s. to the invariant set of \( \frac{d}{dt} \theta_t = f(\theta_t) \).

The main challenge in the proof is establishing boundedness of the parameter sequence. Once this is established, convergence of \( \{\theta_k\} \) to the invariant set of the ODE follows from standard arguments [7, Th. 2.1], subject to uniform bounds on the terms in Lemma 1. The proof of boundedness is established in Appendix A.3, based on the ODE approximation (28). The inequalities obtained in Appendix A.3 will also be used to obtain moment bounds.

The subscript ‘\( r \)’ in \( T_r \) stands for “relaxation time” for the ODE@\( \infty \). Moment bounds require either a bound on \( T_r \), or a stronger bound on \( L \). Recall \( \delta_L := ||L||_W \) was defined in (A2). We say that \( L = o(W) \) if,

\[
\lim_{r \to \infty} \sup_{x \in X} \frac{|L(x)|}{r} = 0.
\]

That is, \( \lim_{r \to \infty} ||L||_{W_r} = 0 \), with \( W_r(x) := \max\{r,W(x)\} \) for \( x \in X \). These are summarized in the following two-part assumption.

(A4a) \( T_r \) in Prop. 2 can be chosen so that \( T_r < 1/(4\delta_L) \)  \hspace{1cm} (A4b) \( L = o(W) \).

**Theorem 2.** Suppose that (DV3) and (A1)–(A3) hold. Assume in addition that (A4) holds in form (a) or (b).

Then, \( \sup_{k \geq 0} E \left[ (||\theta_k|| + 1)^4 \exp(V(\Phi_{k+1})) \right] < \infty \).

The bound \( T_r < 1/(4\delta_L) \) assumed in Assumption (A4a) appears difficult to validate. It is included because it is what is essential in the proof of Thm. 2. Assumption (A4b) implies (A4a) provided we modify the function \( W \) used in (DV3).

**Lemma 2.** Suppose that (DV3) holds, subject to the bounds (33). Then:

(i) (DV3) holds and the bounds (33) continue to hold for the pair \((V,W_r)\) for any \( r \geq 1 \), with \( W_r(x) := \max\{r,W(x)\} \).

(ii) If in addition (A4b) holds, then (A4a) also holds with \( W \) replaced by \( W_{r_0} \) for \( r_0 \geq 1 \) sufficiently large.

The proof of Lemma 2 follows from the definitions.

The bound (DV3) is used in multiplicative ergodic theory for Markov chains [31, 32]. Motivation in the present paper is similar, and made clear from consideration of the linear scalar recursion, with \( f(\theta, \Phi) = a(\Phi)\theta \), so that for \( m,n \geq 0 \),

\[
|\theta_{n+m}| = |\theta_m \prod_{k=m+1}^{n+m} (1 + \alpha_k a(\Phi_k))| \leq |\theta_m| \exp \left( \sum_{k=m+1}^{n+m} \alpha_k a(\Phi_k) \right). \tag{35}
\]

This suggests that strong assumptions are required to obtain bounds on \( L_p \) moments of the parameter sequence even in this simplest of examples. The following is a crucial step in the proof of Thm. 2. Its proof is contained in Appendix A.1.

**Proposition 3.** The following holds under (DV3): For any initial conditions \( \theta_0, \Phi_0, \) any \( n_0,n \), and any non-negative sequence \( \{\delta_k : 1 \leq k \leq n - 1\} \) satisfying \( \sum \delta_k \leq 1 \),

\[
E \left[ \exp \left( V(\Phi_{n_0+n}) + \sum_{k=n_0}^{n_0+n-1} \delta_k W(\Phi_k) \right) \bigg| F_{n_0} \right] \leq b_v^2 e^{b_v \Phi_{n_0}} \text{ a.s.}, \tag{36}
\]

where \( b_v > 0 \) is as in Prop. 1, and \( b > 0 \) is the constant in (DV3).
3.2 Asymptotic statistics

We now turn to rates of convergence, and for this it is assumed that the ODE (1) is globally asymptotically stable with unique equilibrium denoted \( \theta^* \in \mathbb{R}^d \). The accent ‘tilde’ is used to represent error: We write \( \Theta^{(n)}(t_k) := \Theta(t_k) - \theta^{(n)}(t_k) \) and \( \tilde{\theta}_k := \Theta_k - \theta^* \). Two normalized error sequences are considered:

\[
    z_k := \frac{1}{\sqrt{\alpha_k}} \tilde{\theta}_k, \quad Z^{(n)}_k := \frac{1}{\sqrt{\alpha_k}} \Theta^{(n)}(t_k), \quad k \geq n.
\]  

The domain of the latter is extended to all \( \tau \geq \tau_n \) by piecewise linear interpolation. It is convenient to move the origin of the time axis as follows:

\[
    Z^{(n)}_\tau := Z^{(n)}_{\tau_n + \tau}, \quad \tau \geq 0.
\]

The two fundamental approximations of interest here are:

**Central Limit Theorem** (CLT): For any bounded continuous function \( g: \mathbb{R}^d \to \mathbb{R} \),

\[
    \lim_{k \to \infty} \mathbb{E}[g(z_k)] = \mathbb{E}[g(X)], \quad \text{where } X \sim N(0, \Sigma_\theta),
\]

for an appropriate covariance matrix \( \Sigma_\theta \).

**Functional Central Limit Theorem** (FCLT): The sequence of stochastic processes \( \{Z^{(n)}_\tau: n \geq 1\} \) converges in distribution to the solution of the Ornstein-Uhlenbeck equation,

\[
    dX_t = FX_t dt + D dB_t, \quad X_0 = 0,
\]

where \( B = \{B_t\} \) a standard \( d \)-dimensional Brownian motion, and \( F, D \) are \( d \times d \) matrices.

Consider the family of martingale difference sequences, parametrized by \( \theta \),

\[
    \zeta_n(\theta) := \hat{f}(\theta, \Phi^\theta_n) - \mathbb{E}[\hat{f}(\theta, \Phi^\theta_n) | F_{n-1}], \quad \Phi^\theta \text{ is Markovian with tr. kernel } P_\theta.
\]

The steady-state covariance matrices are denoted:

\[
    \Sigma^*_\zeta(\theta) := \mathbb{E}_\theta[\zeta_n(\theta) \zeta_n(\theta)^\top], \quad \Sigma^* = \Sigma^*_\zeta(\theta^*). \tag{42}
\]

The CLT is obtained under the assumption that \( A^* \) is Hurwitz, where \( A^* := A(\theta^*) \) with \( A = A(\theta) \) given in (A5a) below. This assumption combined with global asymptotic stability of the mean flow implies that it is globally exponentially asymptotically stable [35, Prop. A.11]: for some \( b_\theta < \infty \), and \( \varrho_\theta > 0 \),

\[
    \| \phi(t; \theta) - \theta^* \| \leq b_\theta \| \theta - \theta^* \| e^{-\varrho_\theta t}, \quad \theta \in \mathbb{R}^d, \quad t \geq 0 \tag{43}
\]

This bound is required in consideration of bias, starting with the approximation \( \mathbb{E}[\theta_k - \theta^*] \approx \phi(t_k; \theta_n) - \theta^* \) for large \( n \) and all \( k \geq n \). For the CLT to hold, the bias must decay faster than \( 1/\sqrt{\alpha_k} \). Assumption (A5b) is introduced to ensure this.

(A5a) \( \overline{f}: \mathbb{R}^d \to \mathbb{R}^d \) is continuously differentiable in \( \theta \), and the Jacobian matrix \( A = \partial \overline{f} \) is uniformly bounded and uniformly Lipschitz continuous.

(A5b) The step-size is \( \alpha_n = 1/n^\rho \) with \( \frac{1}{2} < \rho \leq 1 \), and (43) holds with \( \varrho_\theta > 0 \). It is furthermore assumed that \( \varrho_\theta > 1/2 \) in the special case \( \rho = 1 \).

**Theorem 3** (FCLT). Suppose (DV3), (A1)–(A4) and (A5a) hold. Assume moreover that the mean flow (1) is globally asymptotically stable, and that \( \frac{1}{2} I + A^* \) is Hurwitz, where \( A^* := A(\theta^*) \) with \( A = A(\theta) \) given in (A5a) and \( \gamma \) defined in (A1). Then, the FCLT holds: \( \{Z^{(n)}_\tau: \tau \geq 0\} \) converges in distribution to the solution of (40), with \( F = \frac{1}{2} I + A^* \), \( D \) any solution to \( \Sigma^* = DD^\top \), and \( \Sigma_\theta > 0 \) is the unique solution to the Lyapunov equation:

\[
    \left[ \frac{1}{2} \gamma I + A^* \right] \Sigma_\theta + \Sigma_\theta \left[ \frac{1}{2} \gamma I + A^* \right]^\top + \Sigma^* = 0. \tag{44}
\]
We write \( h = o(\omega) \) if \( \lim_{t \to \infty} \sup_{\theta \in \mathbb{R}^d} \frac{|h(\theta)|}{\max\{r, \omega(\theta)\}} = 0 \), for functions \( h, \omega : \mathbb{R} \to \mathbb{R}_+ \). We can now establish a strong version of the CLT which, in addition to weak convergence, includes convergence of the normalized covariance.

**Theorem 4 (CLT).** Suppose that (A5b) holds, along with the assumptions of Thm. 3. Then the CLT (39) holds, with asymptotic covariance given in (44). Moreover, the set of functions for which (39) holds includes any continuous function \( g : \mathbb{R}^d \to \mathbb{R} \) satisfying \( \|g\| = o(\omega) \) with \( \omega(\theta) = \|\theta\|^4 \) for \( \theta \in \mathbb{R}^d \). In particular, the following limit holds:

\[
\lim_{n \to \infty} \frac{1}{\alpha_n} \mathbb{E}[\hat{\theta}_n^T \hat{\theta}_n] = \Sigma_\theta.
\]

Thm. 4 suggests that the step-size \( \alpha_{n+1} = \alpha_0/(n+1) \) is required to obtain the optimal convergence rate of \( O(1/n) \) for the mean-square error (subject to the constraint that \( \frac{\alpha}{\omega} I + A^* \) is Hurwitz). In fact, it is far better to use a larger step-size sequence and employ averaging via (11). This is made clear in Thm. 5.

One representation of the covariance matrix \( \Sigma_{\theta}^{PR} \) appearing in (12) is based on the stationary version of the Markov chain with transition kernel \( P_{th} \), denoted \( \{\Phi_k^* : k \in \mathbb{Z}\} \). Let \( \Sigma_\Delta^* \) denote the asymptotic covariance of \( \{\Delta_k^* = f(\theta^*, \Phi^*_k) : k \in \mathbb{Z}\} \), and \( G := -(A^*)^{-1} \) the stochastic Newton-Raphson gain of Ruppert [52]. Thm. 5 that follows justifies the representation,

\[
\Sigma_{\theta}^{PR} = G \Sigma_\Delta^* G^T.
\]

The standard definition of asymptotic covariance gives:

\[
\Sigma_\Delta^* = \sum_{k=-\infty}^{\infty} \mathbb{E}[f(\theta^*, \Phi^*_0) f(\theta^*, \Phi^*_k)^T] = \Sigma_\zeta^*.
\]

We also have \( \Sigma_\Delta^* = \Sigma_\zeta^* \) (defined in (42))—a consequence of the noise decomposition (21).

It is well known that \( \Sigma_{\theta}^{PR} \) is the optimal achievable covariance for SA when \( \{\Delta_{n+1}\} \) is a martingale difference sequence, and this is achievable using the averaging technique of Polyak and Ruppert (11) (recall the history at outlined at the close of Section 1). The following extends this result to the far more general setting of this paper.

**Theorem 5 (Optimizing asymptotic covariance).** Suppose the assumptions of Thm. 4 hold. Then, (12) holds for the estimates (11) with \( \Sigma_{\theta}^{PR} \) given in (45).

Thm. 5 improves upon Thm. 4 in two respects: The mean square error convergence is accelerated to \( O(1/n) \) rather than \( O(\alpha_n) \), and the asymptotic covariance is optimal – that is, minimal. Its proof follows from Prop. 17, in which the coupling bound (13) is established.

### 3.3 Counterexample

Consider the M/M/1 queue with uniformization, that is, a reflected random walk \( Q = \{Q_n\} \) on \( X = \{0, 1, 2, \ldots\} \),

\[
Q_{n+1} = \max(0, Q_n + D_{n+1}), \quad n \geq 0,
\]

in which \( D \) is i.i.d on \( \{-1, 1\} \) with \( \alpha = P\{D_k = 1\}, \mu = 1 - \alpha = P\{D_k = -1\} \). This is a reversible, geometrically ergodic Markov chain when \( \rho = \alpha/\mu < 1 \), with geometric invariant measure \( \pi \) and with \( \eta := E_{\pi}[Q_n] = \rho/(1 - \rho) \).

However, the drift inequality (DV3) fails to hold for any unbounded function \( W \) [32].
Consider the scalar stochastic approximation recursion,
\[
\theta_{n+1} = \theta_n + \frac{1}{n+1} \{(Q_{n+1} - \eta - 1)\theta_n + W_{n+1}\}, \quad \theta_0 \in \mathbb{R},
\]  
(48)
where \(W = \{W_n\}\) is i.i.d. and independent of \(Q\), with Gaussian marginal \(N(0, 1)\). The mean vector field associated with (48) is linear,
\[
\overline{f}(\theta) := \mathbb{E}_\pi [(Q_{n+1} - \eta - 1)\theta + W_{n+1}] = -\theta,
\]
so that Thm. 1 implies convergence. Moreover, this example satisfies all of the assumptions imposed in [23], so that \(L_p\) bounds hold subject to the resetting step imposed there.

To obtain moment bounds for (48) without resetting, it is helpful to compare solutions: For two initial conditions \(\theta^1_0, \theta^2_0\) we obtain a bound similar to (35):
\[
|\theta^1_n - \theta^2_n| \leq |\theta^1_0 - \theta^2_0| \exp \left( \sum_{k=1}^{n} \alpha_k (Q_k - \eta - 1) \right).
\]
We find that the right-hand side cannot admit useful bounds for all \(\rho < 1\).

**Proposition 4.** The following hold for the SA recursion (48):

(i) For all \(\rho < 1\), we have \(\lim_{n \to \infty} \theta_n = \theta^* = 0\) a.s.

(ii) But if \(\rho > 1/2\), then \(\lim_{n \to \infty} \mathbb{E}[\theta^2_n] = \infty\).

The proof of Prop. 4 is rooted in large deviations theory for reflected random walks. A standard object of study is the scaled process \(q_n^\rho = \frac{1}{n} Q_{\lfloor nt \rfloor}\). As \(n \to \infty\) this converges to zero for each \(t\) with probability one. The large deviations question of interest is the probability of a large excursion over a finite time-window as illustrated in Fig. 1. There is elegant theory for estimating this probability, e.g., [21], which leads to the proof of Prop. 4.

![Figure 1: Constraint region for scaled queue length process.](image)

**Prop. 4 does not** say that the CLT fails, rather, it says the limiting covariance cannot be used to inform the convergence analysis. Given the form (48) and the fact that \(\{\theta_n\}\) converges to zero, we might establish a CLT by comparison with the recursion,
\[
\theta^\rho_{n+1} = \theta^\rho_n + \frac{1}{n+1} [\overline{f}(\theta^\rho_n) + W_{n+1}] = \theta^\rho_n + \frac{1}{n+1} [-\theta^\rho_n + W_{n+1}],
\]
for which the CLT does hold with asymptotic variance equal to one.

All of assumptions for the CLT imposed in [45] are satisfied for this example, except for one: Assumption C.6 would imply the following: For each \(r \geq 1\) there exists \(b_{p,r}\) such that for each initial condition \(\theta_0\) and \(Q_0 = q_0\),
\[
\mathbb{E}_{\theta_0, q_0} [\mathbb{I}\{\|\theta_k\| \leq r : 1 \leq k \leq n\} Q^p_n] \leq b_{p,r} [1 + q^{p+1}_0].
\]
This is unlikely, since \(\theta_k \to 0\) almost surely, and theory implies the weaker bound when the indicator function is removed: \(\mathbb{E}_{q_0} [Q^p_n] \leq b_p [1 + q^{p+1}_0]\); see [41, Prop. 14.4.1].
Fig. 2 shows histograms of the normalized error $\sqrt{n}\theta_n$ for two values of load. In each case the approximating density is $N(0, 1)$. What is missing from these plots is outliers that were removed before plotting the histograms. For load $\rho = \alpha/\mu = 3/7$ the outliers were few and not large. For $\rho = 6/7$, nearly 1/3 of the samples were labeled as outliers, and in this case the outliers were massive: Values exceeding $10^{20}$ were observed in about 1/5 of runs.

4 Conclusions

This paper provides simple criteria for convergence and asymptotic statistics for stochastic approximation and RL. There are of course many important avenues for future research:

(i) We are currently investigating the value of resetting as defined in [23]. In particular, with resetting, can versions of Thms. 4 and 5 be obtained under (V4) and appropriate assumptions on $f$?

(ii) Extensions to two time-scale SA are of interest in many domains. Such generalizations will be made possible based on the present results, combined with ideas from [29, 27, 61].

(iii) Is there any hope for finer asymptotic statistics, such as an Edgeworth expansion or finite-$n$ bounds of Berry-Esséen type?

(iv) Many practitioners in RL opt for constant step-size algorithms. When $\alpha_n \equiv \alpha_0$ is fixed but sufficiently small, it is possible to obtain the drift condition (82) under the assumptions of Thm. 2 (with (A1) abandoned). In [34] it is shown that this implies that the fourth moment of $\tilde{\theta}_n$ is uniformly bounded. Open questions here include:

- Can we establish a topological formulation of geometric ergodicity for $\{(\theta_n, \Phi_n) : n \geq 0\}$ to justify steady-state bias and variance formulae? Ideas in the prior work [25] might be combined with recent advances on topological ergodicity for Markov chains—see [12, 48] and their references.

- Once we have established some form of ergodicity, does it follow that $\{z_k\}$ converges in distribution as $n \to \infty$? Is the limit approximately Gaussian for small $\alpha_0$? Can we obtain convergence of moments as in Thm. 4?

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A Technical Proofs

The appendix begins with bounds on sums of functions of the Markov chains $\Phi^\theta$. Appendix A.1 is devoted to bounds under (V4) leading to a proof of Prop. 3, and finer bounds are applied in Appendix A.2 to obtain bounds on solutions to Poisson’s equation (20). The remainder of the appendix is organized in correspondence with the presentation of the main results: Appendix A.3 establishes boundedness of parameter estimates under the assumptions of Thm. 1, and hence convergence. Appendix A.4 contains parallel theory for establishing $L^p$ bounds on the parameter estimates and, based on this, Appendix A.5 contains proofs of the main results of the paper concerning asymptotic statistics.

A.1 Markov chain bounds

This subsection is organized into two parts. We first investigate implications of (V4), and then establish a key lemma under a weaker drift condition.

Consequences of (V4)  Recall the constant $b_v$ was introduced in Prop. 1 in the ergodic theorem (19a), which is equivalently expressed
\[ ||P^n - 1 \otimes \pi||_v \leq b_v \varrho^n. \]
In this Appendix we consider the family of transition kernels $\{P^\theta : \theta \in \mathbb{R}^d\}$. It follows from the main result of [40] that the following hold under (V4) or (DV3), for constants $b_v, \varrho < 1$ independent of $\theta$:
\[ ||e^{P^n}||_v \leq b_v \varrho^n \quad \text{and} \quad ||P^n\theta||_v \leq b_v, \quad n \geq 0, \quad (49) \]
with $v := e^V$ under (DV3), and $e^{P^n} := P^n - 1 \otimes \pi^\theta$.

We can generalize the second bound in (49) to the stochastic process $\Phi$:

**Lemma 3.** Under (DV3) we have the bound $\mathbb{E}[v(\Phi_n) | F_k] \leq b_v v(\Phi_k)$ a.s., for a possibly larger constant $b_v$, for any initial conditions $\theta_0, \Phi_0$, and any $n > k \geq 0$.

**Proof.** Subject to (DV3), denote $S = \{x : \exp(W(x) - bs(x)) \leq 2\}$. Then, for any $\theta_0, \Phi_0$,
\[
\mathbb{E}[v(\Phi_{m+1}) | F_m] \leq \exp\left(V(\Phi_m) - W(\Phi_m) + bs(\Phi_m)\right) \\
\quad \leq \mathbb{I}_S(\Phi_m) \frac{1}{2} v(\Phi_m) + \mathbb{I}_S(\Phi_m) \left[ \sup_{x \in S} \exp\left(V(x) - W(x) + bs(x)\right) \right] \\
\quad \leq \frac{1}{2} v(\Phi_m) + b_S, \quad m \geq 0,
\]
where $b_S < \infty$ can be found due to (33b). The desired result then follows, with $b_v = 2b_S$, by induction and the smoothing property of conditional expectation.

These uniform bounds lead to the proof of Prop. 3:

**Proof of Prop. 3.** The proof amounts to obtaining a bound on the nonlinear program, $\max\{\Gamma(\delta) : \delta \in \mathbb{R}^n_+, \sum \delta_k \leq 1\}$, with $\Gamma$ the function of $\delta$ on the left-hand side of eq. (36). To simplify discussion we take $n_0 = 0$, so that the objective becomes,
\[ \Gamma(\delta) := \mathbb{E}\left[\exp\left(V(\Phi_n) + \sum_{k=0}^{n-1} \delta_k W(\Phi_k)\right)\right], \]
with $\theta_0 \in \mathbb{R}^d$ and $\Phi_0 = x \in X$ arbitrary.

The function $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}_+ \cup \{\infty\}$ is strictly convex, which means the maximum is attained at an extreme point: The optimizer $\delta^*$ is a vector with all entries but one equal to zero. Consequently, $\Gamma(\delta^*) = \max_{0 \leq k < n} \mathbb{E}[\exp(V(\Phi_n) + W(\Phi_k))]$.
To complete the proof of (36) we obtain the bound,
\[
E\left[ \exp\{V(\Phi_n) + W(\Phi_k)\} \right] \leq b_n^2 e^{b} v(x), \quad 0 \leq k \leq n - 1, \quad \Phi_0 = x \in X. \tag{50}
\]
On applying Lemma 3 we have for each \( k \),
\[
E[\exp(V(\Phi_n) + W(\Phi_k)) | \mathcal{F}_{k+1}] \leq b_n v(\Phi_{k+1}) \exp(W(\Phi_k)),
\]
Hence by the smoothing property of conditional expectation,
\[
E[\exp(V(\Phi_n) + W(\Phi_k))] \leq b_n E[\exp(V(\Phi_{k+1}) + W(\Phi_k))] \leq b_n e^{b} E[\exp(V(\Phi_k))],
\]
where the second inequality follows from (DV3) and the uniform bound \( s(x) \leq 1 \). This and a second application of (49) completes the proof of (50).

\[\square\]

**Condition (V3) and its consequences.** Condition (V3) is a relaxation of (V4): For functions \( G, H : X \to [1, \infty] \), a small function \( s : X \to (0, 1] \), and constant \( b < \infty \),
\[
E[H(\Phi^\theta_{k+1}) - H(\Phi^\theta_k) | \Phi^\theta_k = x] \leq -G(x) + bs(x), \quad x \in X. \tag{51}
\]
Or in operator-theoretic notation, \( P_\theta H \leq H - G + bs \). Consequences of this drift criterion is a focus of [41, Ch. 14]. Condition (DV3) implies (V3) with \( V = H \) and \( G = W \).

In later results, in particular Prop. 6, we construct a solution to (51) in which the small function differs from the function \( s \) appearing in (29). The next result justifies replacing \( s \) by an indicator function of a sublevel set of the function \( H \).

**Lemma 4.** Suppose that (51) and (29) hold. Then the set \( C_m = \{ x : H(x) \leq m \} \) is either empty or small for each \( m \geq 1 \): There exists \( \varepsilon_m > 0 \) independent of \( \theta \) such that \( R_\theta(x, A) \geq \varepsilon_m \nu(A) \) for each \( A \in \mathcal{B}(X) \) and \( x \in C_m \).

**Proof.** The fact that \( C_m \) is small follows from [41, Th. 14.2.3]. The proof of the lemma involves obtaining \( \varepsilon_m \), depending only on \( \{H, b, s, \nu\} \).

It is assumed in (29) that \( s \) is everywhere positive. Consequently, there is \( \delta > 0 \) such that the set \( S = \{ x : s(x) \geq \delta \} \) satisfies \( \nu(S) > 0 \). We next obtain a bound on the \emph{first return time} to the set \( S \), denoted \( \tau_S = \min\{k \geq 1 : \Phi^\theta_k \in S\} \). We obtain from [41, Th. 14.2.2] and the assumption that \( G \geq 1 \),
\[
E[\tau_S] \leq V(x) + b \mathbb{E}\left[ \sum_{k=0}^{\tau_S} s(\Phi^\theta_k) \right] \leq V(x) + b \frac{1}{\nu(S)} \sum_{k=0}^{\tau_S} R_\theta(\Phi^\theta_k, S), \quad \text{for all } x = \Phi^\theta_0,
\]
where the second inequality follows from (29). The following bound on the sum on the right-hand side is justified in the proof of [41, Th. 14.2.3]:
\[
E\left[ \sum_{k=0}^{\tau_S} R_\theta(\Phi^\theta_k, S) \right] = \sum_{n=0}^{\infty} 2^{-n-1} E\left[ \sum_{k=0}^{\tau_S} \mathbb{1}\{\Phi^\theta_{k+n} \in S\} - \sum_{n=0}^{\infty} 2^{-n-1} n = 1.
\]
From the definition of \( C_m \) we conclude that \( E[\tau_S] \leq B_m := m + b/\nu(S) \) for all \( x \in C_m \).

Markov’s inequality then gives \( \mathbb{P}\{\tau_S \leq n\} \geq 1 - B_m/n \) for any \( n \) and any \( x \in C_m \). Hence with \( n_m \geq 1 \) chosen so that \( B_m/n_m \leq 1/2 \), we have \( \sum_{k=1}^{n_m} P^\theta_k(x, S) = \mathbb{P}\{\tau_S \leq n_m\} \geq 1/2 \) for \( x \in C_m \).

We have \( P^\theta_k R_\theta \leq 2^{-k-1} R_\theta \) for any \( k \), giving for \( x \in C_m \) and \( A \in \mathcal{B}(X) \),
\[
\left[ \sum_{k=1}^{n_m} 2^{-k-1} \right] R_\theta(x, A) \geq \left( \sum_{k=1}^{n_m} P^\theta_k R_\theta \right)(x, A) \geq \frac{1}{2} \min_{y \in S} R_\theta(y, A) \geq \frac{1}{2} \delta \nu(A).
\]
This gives the desired inequality with \( \varepsilon_m = \frac{1}{2} \delta \left[ \sum_{k=1}^{n_m} 2^{-k-1} \right]^{-1} \). \[\square\]
A.2 Poisson’s equation and consequences

This subsection is devoted to bounds on the solution to Poisson’s equation \( \hat{f} \) appearing in (20), which is the basis of the noise decomposition (21).

We first require a representation in terms of the fundamental kernel [41], defined as the inverse \( Z_\theta := [I - \hat{P}_\theta]^{-1} \), with \( \hat{P}_\theta \) defined below (49). We will see that the inverse exists on an appropriate domain under the assumptions of our main results. Throughout the Appendix we consider exclusively the solution to (20) defined by:

\[
\hat{f}(\theta, x) := \int Z_\theta(x, dy)f(\theta, y), \quad \theta \in \mathbb{R}^d, \ x \in X.
\] (52)

When adopting operator theoretic notation we denote \( \hat{f}_\theta(x) = \hat{f}(\theta, x) \) and \( f_\theta(x) = f(\theta, x) \) for \( x \in X \), giving \( \hat{f}_\theta = Z_\theta f_\theta \).

Subject to geometric ergodicity, the fundamental kernel may be expressed as the sum:

\[
Z_\theta = \sum_{k=0}^{\infty} \hat{P}_\theta^k = I + \sum_{k=1}^{\infty} [P_\theta^k - I \otimes \pi_\theta].
\] (53)

Under (V3) we then have \( \|\hat{f}_\theta\| \leq c_0 v(x) \) for all \( x \), with \( c_0 < \infty \) for each \( \theta \). The following provides more useful bounds.

**Proposition 5.** Suppose that (51) and (29) hold. Then the fundamental kernels admit the uniform bound \( \|Z_\theta\|_{G,H} \leq b_2 := b_v = b_1 + b_0 [1 + b_0 b_v] \), with \( b_v := 1 + 2b \) and \( b_0 := b + \|s/H\|_\infty^{-1} \).

**Proof.** Under the assumptions of the proposition the Markov chain is positive Harris recurrent with \( \pi_\theta(G) < \infty \) and \( \|Z_\theta\|_{G,H} < \infty \) [24, Th. 2.3]. The remaining work is to construct the universal constant \( b_2 \).

The minorization condition is expressed \( R_\theta \geq s \otimes \nu \), which justifies the notation \( U_\theta = \sum_{n=0}^{\infty} (R_\theta - s \otimes \nu)^n \). It is well known that \( \mu_\theta = \nu U_\theta \) is an invariant measure for both \( R_\theta \) and \( P_\theta \), that \( \mu_\theta(s) = 1 \), and that \( U_\theta s = 1 \) everywhere (see [31, 32] for history).

Based on this theory we obtain a representation for the fundamental kernel as follows: First, \( Z_{\theta,r} := U_\theta[I - 1 \otimes \pi_\theta] \) is a version of the fundamental kernel for \( R_\theta \), so that \( R_\theta Z_{\theta,r} = Z_{\theta,r} - [I - 1 \otimes \pi_\theta] \).

The identity \( R_\theta[P_\theta - I] = R_\theta - I \) then implies that \( Z_{\theta,0} := R_\theta Z_{\theta,0} \) solves \( P Z_{\theta,0} = Z_{\theta,0} - [I - 1 \otimes \pi_\theta] \).

We add a rank one operator to obtain:

\[
Z_\theta := 1 \otimes \pi_\theta + Z_{\theta,0} = 1 \otimes \pi_\theta + U_\theta R_\theta[I - 1 \otimes \pi_\theta].
\]

This solves \( P Z_\theta = Z_\theta - [I - 1 \otimes \pi_\theta] \), and also \( \pi_\theta(Z_\theta g) = \pi_\theta(g) \) for any \( g \in L^2_G \), which is consistent with (53).

The value of \( b_2 \) will be obtained as an upper bound on the right-hand side of the inequality,

\[
\|Z_\theta\|_{G,H} \leq \|1 \otimes \pi_\theta\|_{G,H} + \|U_\theta R_\theta\|_{G,H}[1 + \|1 \otimes \pi_\theta\|_G]. \] (54)

The first step is to express (V3) in terms of the resolvent. Writing (51) as \( |P - I| H \leq -G + bs \) we apply \( R_\theta \) to each side and use \( R_\theta[P - I] = R_\theta - I \) to obtain a version of (V3) for the resolvent kernel, \( |R_\theta - I| H = R_\theta[P - I] H \leq -R_\theta G + b R_\theta s \), and hence \( R_\theta G \leq b R_\theta s + [I - (R_\theta - s \otimes \nu)] H \).

Following standard arguments (e.g., [31, Lem. 3.2]) we conclude that:

\[
U_\theta R_\theta G \leq H + b U_\theta R_\theta s = H + b U_\theta R_\theta s. \] (55)
From the definition of $U_\theta$ we have $U_\theta R_\theta = U_\theta (R_\theta - s \otimes \nu) + [U_\theta s] \otimes \nu = U_\theta - I + [U_\theta s] \otimes \nu$. Recalling $\mu_\theta(s) = 1$ and $U_\theta s \equiv 1$ gives $U_\theta R_\theta s \leq U_\theta s + \nu(s) U_\theta s \leq 2$, and from (55),

$$\|U_\theta R_\theta\|_{G,H} \leq \sup_x \frac{1}{H(x)} [H(x) + bU_\theta R_\theta s(x)] \leq b_\theta := 1 + 2b.$$

Next, consider $\|1 \otimes \pi_\theta\|_{G,H} = \sup_x \pi_\theta(G)/H(x) \leq \pi_\theta(G)$. We have,

$$\pi_\theta(G) = \frac{\mu_\theta(G)}{\mu_\theta(X)} \leq \mu_\theta(G) = \mu_\theta(R_\theta G) = \nu(U_\theta R_\theta G) \leq \nu(H) \|U_\theta R_\theta\|_{G,H} \leq \nu(H)b_\theta,$$

where the first inequality follows from $\mu_\theta = \nu U_\theta \geq \nu$, so that $\mu_\theta(X) \geq 1$, and the identity that follows is a consequence of invariance of $\mu_\theta$.

To bound $\nu(H)$ we combine the pair of bounds $R_\theta H(x) \geq s(x)\nu(H)$ and $R_\theta H \leq H + bs(x)$, to obtain $\nu(H) \leq b + H(x)/s(x)$. The ratio admits the bound $H(x)/s(x) \leq \|s/H\|_{\infty}^{-1}$ for all $x$, from which we obtain $\nu(H) \leq b_\nu := b + \|s/H\|_{\infty}^{-1}$, and hence $\|1 \otimes \pi_\theta\|_{G,H} \leq b_\nu b_\nu$. This and (54) completes the proof.

\[\square\]

Lipschitz bounds on $\hat{f}_\theta$ are obtained by invoking the resolvent equation,

$$Z_\theta - Z_{\theta'} = Z_\theta [\hat{P}_\theta - \hat{P}_{\theta'}] Z_{\theta'}, \quad \theta, \theta' \in \mathbb{R}^d, \tag{56}$$

and its corollary $\pi_\theta - \pi_{\theta'} = \pi_\theta [\hat{P}_\theta - \hat{P}_{\theta'}] Z_{\theta'}$. This is one ingredient in perturbation theory for Markov chains [55], which is at the heart of the actor-critic method of RL—see commentary in [37]. Lemma 5 provides conditions under which (56) is justified and an essential bound.

**Lemma 5.** Suppose that for functions measurable functions $H_i : X \to [1, \infty)$, $i = 1, 2, 3$, we have

$$\|P_\theta\|_{H_i} + \|Z_\theta\|_{H_{i, H_{i+1}}} < \infty, \text{ and } \pi_\theta(H_{i+1}) < \infty \text{ for each } i = 1, 2 \text{ and each } \theta.$$

Then, $\|P_\theta - P_{\theta'}\|_{H_i} \leq \|P_\theta - P_{\theta'}\|_{H_1} + \pi_\theta(H_2)\|P_\theta - P_{\theta'}\|_{H_2} \|Z_{\theta'}\|_{H_{i, H_2}}$ for each $\theta, \theta' \in \mathbb{R}^d$.

When (49) holds we have $\|Z_\theta\|_{\infty} \leq b_\nu/(1 - \theta)$. Consequently, if (A2) also holds then $\hat{f}_\theta \in L^\nu_c$, and from (56) we obtain a useful formula for differences: For $\theta, \theta' \in \mathbb{R}^d$,

$$\hat{f}_\theta - \hat{f}_{\theta'} = Z_\theta [f_\theta - f_{\theta'}] + Z_\theta [\hat{P}_\theta - \hat{P}_{\theta'}] \hat{f}_{\theta'} \tag{57}$$

Far better bounds are obtained in Prop. 7 whose proof is based on the following:

**Proposition 6.** (i) Under (V4), $\sup_{\pi} \|Z_\pi\|_{\nu,x} < \infty$ for each $\epsilon \in (0, 1]$.

(ii) Under (DV3), $\sup_{\pi} \|Z_\pi\|_{G_p, V_p} < \infty$ with $G_p = 1 + W V^{p-1}$, $V_p = 1 + V^p$, for each $p \geq 1$.

**Proof.** Part (i) follows from Jensen’s inequality: If (V4) holds, then the same drift condition holds for $\nu^\epsilon$ using $P_\theta^\epsilon \leq (P_\theta^\nu)^\epsilon$; see [24, Th. 2.3] or [41, Lem. 15.2.9].

The proof of part (ii) also begins with Jensen’s inequality. The function $G(x) = \log(x)^p$ is concave on the interval $[a_p, \infty]$ with $a_p = \exp(p - 1)$. Consequently, under (DV3),

$$\mathbb{E}[V(\Phi_{k+1}^\theta) + W(\Phi_k^\theta) - bs(\Phi_k^\theta) + k_p^p] \mid \mathcal{F}_k] \quad \text{ (where } k_p = a_p + b)$$

$$= \mathbb{E}[G(\exp(V(\Phi_{k+1}^\theta) + W(\Phi_k^\theta) - bs(\Phi_k^\theta) + k_p^p)) \mid \mathcal{F}_k]$$

$$\leq G(\exp(V(\Phi_k^\theta) + k_p^p)) = [V(\Phi_k^\theta) + k_p^p]^p.$$

From this we obtain a version of the drift condition (51):

$$\mathbb{E}[V_p(\Phi_{k+1}^\theta) \mid \mathcal{F}_k] \leq V_p(\Phi_k^\theta) - \delta_p G_p(\Phi_k^\theta) + b_p \|S(\Phi_k^\theta),$$

where $\delta_p > 0$, $b_p < \infty$, and $S = \{x : V_p(x) \leq m\}$ for some $m$. The proof is completed on combining Prop. 5 and Lemma 4.

\[\square\]

**Proposition 7.** Suppose that (A2) holds. We then obtain a Lipschitz bound on $\hat{f}$ defined in (52) under one of the following conditions:
(i) If $(V4)$ holds and $L^4 \in L^v_{\infty}$, then there exists a constant $b_f < \infty$ such that:

\[
\| \hat{f}(\theta, x) \| \leq b_f v(x)^{\frac{1}{4}} [1 + \| \theta \|],
\]  
\[
\| \hat{f}(\theta, x) - \hat{f}(\theta', x) \| \leq b_f v(x)^{\frac{1}{4}} \| \theta - \theta' \|, \quad \theta, \theta' \in \mathbb{R}^d, x \in X.
\]

(ii) If $(DV3)$ holds and $L \in L^w_{\infty}$, then there exists a constant $b_f < \infty$ such that:

\[
\| \hat{f}(\theta, x) \| \leq b_f (1 + V(x)) [1 + \| \theta \|],
\]  
\[
\| \hat{f}(\theta, x) - \hat{f}(\theta', x) \| \leq b_f (1 + V(x)) \| \theta - \theta' \|, \quad \theta, \theta' \in \mathbb{R}^d, x \in X.
\]

Under either (i) or (ii), the function $\hat{f}$ solves Poisson’s equation in the form (20).

**Proof.** We adopt the alternate notation $\hat{f}_\theta(x) = \hat{f}(\theta, x)$ and $f_\theta(x) = f(\theta, x)$.

We begin with (i). It follows from (32) and the assumption $L^4 \in L^v_{\infty}$ that,

\[
\| f_\theta(x) \| \leq L^4 [1 + \| \theta \|] \leq b_f(\theta) v(x)^{\frac{1}{4}}, \quad \text{where } b_f(\theta) = \| L \|_{v^{\frac{1}{4}}} [1 + \| \theta \|].
\]

That is, $\| f_\theta \|_{v^{1/4}} \leq b_f(\theta)$, and we then obtain (58a):

\[
\frac{1}{v(x)^{\frac{1}{4}}} \| \hat{f}_\theta(x) \| \leq \| Z_\theta f_\theta \|_{v^{1/4}} \leq b_f(\theta) \| Z_\theta \|_{v^{1/4}}, \quad \text{for each } \theta \in \mathbb{R}^d, x \in X.
\]

The proof of (58b) uses (57):

\[
\| \hat{f}_\theta - \hat{f}_\theta' \|_{v^{1/4}} \leq \| Z_\theta f_\theta - f_\theta' \|_{v^{1/4}} + \| Z_\theta \|_{v^{1/4}} \| \hat{P}_\theta - \hat{P}_\theta' \|_{v^{1/4}} \| \hat{f}_\theta' \|_{v^{1/4}}
\]

\[
\leq \| Z_\theta \|_{v^{1/4}} \| L \|_{v^{1/4}} \| \theta - \theta' \| + \| Z_\theta \|_{v^{1/4}} \| \hat{P}_\theta - \hat{P}_\theta' \|_{v^{1/4}} \| \hat{f}_\theta' \|_{v^{1/4}},
\]

where the second inequality follows from (32) in (A2). Applying (58a) and Lemma 5 with $H_i = v^{\frac{1}{4}}$ for each $i$ we bound the second term:

\[
\| \hat{P}_\theta - \hat{P}_\theta' \|_{v^{1/4}} \| f_\theta' \|_{v^{1/4}} \leq \left( \frac{B_d}{1 + \| \theta \| + \| \theta' \|} \right) b_f [1 + \| \theta \|] \leq b_f B_d \| \theta - \theta' \|,
\]

where $B_d = b_d \sup_{\| \theta \|, \| \theta' \|} [1 + \pi_\theta(V_1) \| Z_\theta \|_{V_1, V_2}]$.

Part (ii) requires repeated applications of Prop. 6 (ii), which tells us that $Z_\theta$ is a bounded linear operator from $L^G_{\infty}$ to $L^W_{\infty}$ for each $p$, with $G_p = 1 + W V_p$ and $V_p = 1 + V_p$. Recalling (18), this is expressed $\| Z_\theta \|_{G_p, V_p} < \infty$ for each $p \geq 1$. Applying this result with $p = 1$ gives $\| \hat{f}_\theta \|_{V_1} \leq \| Z_\theta \|_{G_1, V_1} \| f_\theta \|_{G_1}$. This bound combined with (32) implies (59a).

The proof of (59b) uses (57) as in the proof of (i): We have, with $p = 2$,

\[
\| \hat{f}_\theta - \hat{f}_\theta' \|_{V_2} \leq \| Z_\theta \|_{G_2, V_2} \| f_\theta - f_\theta' \|_{G_2} + \| Z_\theta \|_{G_2, V_2} \| \hat{P}_\theta - \hat{P}_\theta' \|_{V_1, G_2} \| Z_\theta \|_{G_1, V_1} \| f_\theta' \|_{G_1}.
\]

Each term is bounded via Prop. 6 (ii) as in the proof of (58b), except for a slightly different application of Lemma 5:

\[
\| \hat{P}_\theta - \hat{P}_\theta' \|_{V_1, G_2} \leq \| \hat{P}_\theta - \hat{P}_\theta' \|_{V_1} \leq \frac{B_d}{1 + \| \theta \| + \| \theta' \|} \| \theta - \theta' \|,
\]

where $B_d = b_d \sup_{\| \theta \|, \| \theta' \|} [1 + \pi_\theta(V_1) \| Z_\theta \|_{V_1, V_2}] \leq b_d \sup_{\| \theta \|, \| \theta' \|} [1 + \pi_\theta(V_1) \| Z_\theta \|_{G_2, V_2}]$. 

\[\square\]
The following is a crucial corollary to Prop. 7.

Proposition 8. Under (A2) we obtain bounds on the terms in the decomposition of $\Delta_{n+1}$ in Lemma 1. Under (V4),

\[ \|z_{n+1}\| \leq b_{\bar{f}} ([E[v^{1/2}(\Phi_{n+1}) \mid F_n] + v^{1/2}(\Phi_{n+1})] [1 + \|\theta_n\|] ) \]

\[ \|T_{n+1}\| \leq b_{\bar{f}} v^{1/2}(\Phi_{n+1}) [1 + \|\theta_n\|] \] \hspace{1cm} (60a)

\[ \|Y_{n+1}\| \leq b_{\bar{f}} L v^{1/2}(\Phi_{n+1}) [1 + \|\theta_n\|] . \]

Under (DV3),

\[ \|z_{n+1}\| \leq b_{\bar{f}} [2 + E[V(\Phi_{n+1}) \mid F_{n+1}] + V(\Phi_{n+1})] [1 + \|\theta_n\|] \]

\[ \|T_{n+1}\| \leq b_{\bar{f}} [1 + V(\Phi_{n+1})][1 + \|\theta_n\|] \] \hspace{1cm} (60b)

\[ \|Y_{n+1}\| \leq b_{\bar{f}} [1 + V(\Phi_{n+1})] L(\Phi_{n+1}) [1 + \|\theta_n\|] . \]

A.3 Almost-Sure Bounds

As discussed immediately after Thm. 1, boundedness of the parameter sequence $\{\theta_k\}$ is sufficient to obtain a.s. convergence under (A1) and (A2). Lemma 6 is one ingredient in establishing convergence; the bounds are easily established under (A1).

Lemma 6. Under (A1), the following hold: (i) $\lim_{n \to \infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$, (ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ (iii) $\lim_{n \to \infty} \sum_{m_n+1}^{m_n+1-1} |\sqrt{\alpha_{k+1}} - \sqrt{\alpha_{k+2}}| = 0$.

Recall from Section 2.4 that we introduce a “hat” on the parameter or its candidate ODE approximation to denote the scaled process. In particular, $\hat{\theta}_k = \theta_k/\epsilon_n$ for $m_n \leq k < m_{n+1}$, with $\epsilon_n := \max\{1, \|\theta_{m_n}\|\}$. In Prop. 9 we show that the ODE approximation (28) for $\{\theta_k\}$ implies boundedness of $\{\theta_k\}$. The remainder of the section is devoted to verifying the (28) under the assumptions of Thm. 1.

Proposition 9. If (28) holds for each initial condition, then there is a deterministic constant $b_0 < \infty$ such that $\lim_{k \to \infty} \sup_{k \to \infty} \|\theta_k\| \leq b_0$, for each $(\theta_0, \Phi_0)$.

Proof. Assuming that $T_{n+1} - T_n \geq T_r$ for each $n$, and recalling the definition (26b),

\[ \frac{1}{\max\{1, \|\theta_{m_n}\|\}} \|\theta_{m_n+1}\| = \|\hat{\theta}_{T_{n+1}}\| \leq \|\hat{\theta}_{T_{n+1}}\| + \|\hat{\theta}_{T_{n+1}} - \hat{\theta}_{T_{n+1}}\| \leq \frac{1}{2} + o(1), \]

in which $\hat{\theta}_{T_{n+1}} := \lim_{t \to T_{n+1}} \hat{\theta}_t$, from which it follows that $\lim_{n \to \infty} \|\theta_{m_n}\| \leq 2$ a.s. Applying the same arguments to $\|\theta_k\|/\|\theta_{m_n}\|$, \n
\[ \frac{1}{\max\{1, \|\theta_{m_n}\|\}} \max_{m_n \leq k < m_{n+1}} \|\theta_k\| \leq \sup_{T_n \leq T_{n+1}} \|\hat{\theta}_t\| + \sup_{T_n \leq t < T_{n+1}} \|\hat{\theta}_t - \hat{\theta}_t\| . \]

This combined with (28) establishes the desired bound with $b_0 = 2 \sup_{0 \leq t \leq T_{n+1}} \sup_{T_n \leq t < T_{n+1}} \|\hat{\theta}_t\|$, where the inner supremum is over all solutions to the ODE@\infty satisfying $\|\hat{\theta}_0\| \leq 1$. \hfill \Box

Given Prop. 9, the remaining work required in establishing boundedness of the parameter estimates rests on establishing solidarity with the ODE@\infty:
Proposition 10. If (V4) and (A1)–(A2) hold, then (28) follows:
\[
\lim_{n \to \infty} \sup_{T_n \leq t < T_{n+1}} \|\hat{\Theta}_t - \hat{\Phi}_t\| = 0, \quad \text{and} \quad \sup_{k \geq 0} \|\hat{\Theta}_k\| < \infty \quad \text{a.s.}
\]

The proof of the proposition takes up the remainder of this subsection. While many arguments follow closely [8] and [7, Th. 4.1], the Markovian setting introduces additional complexity. Key inequalities established here are also required for moment bounds.

We begin with an application of the fundamental kernel and Prop. 6.

Proposition 11. Suppose that the assumptions of Thm. 1 hold, so that in particular (V4) holds along with the bound (30) using \( v \) and \( L^8 \in L^\infty_\phi \). Consider any function \( g: \mathcal{X} \to \mathbb{R} \) satisfying \( |g|^{8/3} \in L^\infty_\phi \), and denote \( \tilde{g}_\theta = g - \pi_\theta(g) \). Then, the partial sums converge: For each initial condition \( \Phi_0, \theta_0 \), there is a square-integrable random variable \( S_\infty^\theta \) such that,
\[
\lim_{n \to \infty} \sum_{i=1}^n \alpha_i \tilde{g}_\theta(\Phi_i) = S_\infty^\theta \quad \text{a.s.}
\]

Proof. The proof is similar to [3, Part II, Sec. 1.4.6, Prop. 7], though more complex because the noise is not exogenous. In the exogenous case we only require \( g^2 \in L^\infty_\phi \), and \( \tilde{g}_\theta \) is independent of \( \theta \).

Denote \( \tilde{g}_\theta := Z_\theta g \). Prop. 6 tells us that \( |\tilde{g}_\theta(x)| \leq b_g x^{3/8} \) for a fixed constant \( b_g \) and all \( \theta, x \). We have \( \mathbb{E}[\tilde{g}_\theta(\Phi_{i+1}) | F_i] = \tilde{g}_\theta(\Phi_i) - \tilde{g}_\theta(\Phi_i) \) for each \( i \), from which we obtain \( \tilde{g}_\theta(\Phi_i) = \zeta_i^\theta - \{T_i^\theta - T_{i-1}^\theta\} + \alpha_i R_i^\theta \), where,
\[
\zeta_i^\theta = \tilde{g}_\theta(\Phi_{i+1}) - \mathbb{E}[\tilde{g}_\theta(\Phi_{i+1}) | F_i], \quad T_i^\theta = \tilde{g}_\theta(\Phi_{i+1}), \quad R_i^\theta = \frac{1}{\alpha_i} [\tilde{g}_\theta(\Phi_i) - \tilde{g}_\theta(\Phi_{i-1})].
\]

Hence the partial sums of interest can be expressed,
\[
S_n^\theta := \sum_{i=1}^n \alpha_i \tilde{g}_\theta(\Phi_i) = M_n^\theta - \sum_{i=1}^n \alpha_i \{T_i^\theta - T_{i-1}^\theta\} + \sum_{i=1}^n \alpha_i^2 R_i^\theta,
\]
where \( M_n^\theta = \sum_{i=1}^n \alpha_i \zeta_i^\theta \) is a martingale that is square integrable:
\[
\mathbb{E}[(M_n^\theta)^2] = \sum_{i=1}^n \alpha_i^2 \mathbb{E}[(\zeta_i^\theta)^2] \leq b_g^2 (\sup_{n \geq 0} \mathbb{E}[v(\Phi_n)]) \sum_{i=1}^\infty \alpha_i^2.
\]

Therefore, it is convergent to a square-integrable random variable, denoted \( M_\infty^\theta \).

Convergence of the second term in (61) is established using summation by parts:
\[
\sum_{i=1}^n \alpha_i \{T_i^\theta - T_{i-1}^\theta\} = \alpha_n T_n^\theta - \alpha_1 T_0^\theta - \sum_{i=1}^{n-1} T_i^\theta \{\alpha_{i+1} - \alpha_i\}.
\]

From this we obtain a candidate expression for the limit:
\[
S_\infty^\theta = M_\infty^\theta + R_\infty^\theta + R_\infty^\theta + \alpha_1 T_0^\theta, \quad \text{with} \quad R_\infty^\theta = \sum_{i=1}^\infty T_i^\theta \{\alpha_{i+1} - \alpha_i\}, \quad R_\infty^\theta = \sum_{i=1}^\infty \alpha_i^2 R_i^\theta.
\]

Justification requires \( \lim_{n \to \infty} \alpha_n T_n^\theta = 0 \) a.s., and the existence of the two infinite sums in (62) as square integrable random variables.

The first limit is obtained as follows:
\[
\mathbb{E} \left[ \sum_{n=1}^\infty \{\alpha_n T_n^\theta\}^2 \right] \leq \left( \sup_{n \geq 0} \mathbb{E}[\{T_n^\theta\}^2] \right) \sum_{n=1}^\infty \alpha_n^2 < \infty.
\]
This implies that \( \alpha_n T_n^2 \to 0 \) as \( n \to \infty \) a.s. and in \( L_2 \). Similarly, applying the triangle inequality in \( L_2 \),

\[
\| R_{\infty}^2 \|_{L_2} \leq \sum_{i=1}^{\infty} \| T_i^2 \|_{L_2} |\alpha_{i+1} - \alpha_i| \leq \sqrt{\sup_{n \geq 0} \mathbb{E}[\{T_n^2\}]^2} \sum_{i=1}^{\infty} |\alpha_{i+1} - \alpha_i| < \infty.
\]

To show that \( R_{\infty}^2 \) exists as a random variable in \( L_2 \) it is sufficient to obtain a uniform \( L_2 \) bound on \( \{ R_i^2 \} \). In view of (56) we have for any \( x \) and \( i \), and with \( H = v^{3/8} \),

\[
\frac{1}{H(x)} |\theta_i(x) - \hat{\theta}_{i-1}(x)| \leq b'_g Z_{\theta} \| H \|_H \hat{\theta} - \hat{P}_\theta \|_H, \quad \theta = \theta_i, \ \theta' = \theta_{i-1}.
\]

This, combined with (30) implies that there is a constant \( b'_g \) such that,

\[
| R_i^2 | \leq b'_g \left[ \frac{1}{\alpha_i} H(\Phi_i) \| \theta_i - \theta_{i-1} \| \right]^2 \leq b'_g H^2(\Phi_i)^2 \leq b'_g v^{1/4}(\Phi_i) v^{3/4}(\Phi_i) \leq b'_g v(\Phi_i).
\]

Applying Lemma 3 it follows that \( \sup_i \mathbb{E}[| R_i^2 |] < \infty. \)

This is now used in the following proof establishing boundedness of \( \{ \hat{\theta}_k \} \).

**Lemma 7.** Under (V4) and (A1)–(A2), we have,

\[
1 + \| \hat{\theta}_{k+1} \| \leq \exp(\alpha_{k+1} L(\Phi_k)) (1 + \| \hat{\theta}_k \|), \quad k \geq 0, \quad (63a)
\]

\[
\limsup_{k \to \infty} \| \hat{\theta}_k \| \leq 2 \exp(\bar{\pi}(L) T), \quad \text{where} \quad \bar{\pi}(L) = \sup_{\theta} \pi_{\theta}(L) < \infty. \quad (63b)
\]

**Proof.** We have \( \| \hat{\theta}_{k+1} \| \leq \| \hat{\theta}_k \| + \alpha_{k+1} \| f_{c_n}(\hat{\theta}_k, \Phi_{k+1}) \| \) for each \( m_n \leq k < m_{n+1} \). The Lipschitz bound for \( f \) is inherited by \( f_{c_n} \), giving \( \| f_{c_n}(\hat{\theta}_k, \Phi_{k+1}) \| \leq L(\Phi_{k+1})(1 + \| \hat{\theta}_k \|) \). Applying the bound \( 1 + z \leq e^z \) with \( z = \alpha_{k+1} L(\Phi_{k+1}) \) gives (63a).

On iterating this bound we obtain, for each \( m_n < k \leq m_{n+1} \),

\[
1 + \| \hat{\theta}_k \| \leq (1 + \| \hat{\theta}_{m_n} \|) \exp \left( \sum_{i=m_{n+1}}^{k} \alpha_i L(\Phi_i) \right) \leq 2 \exp \left( \sum_{i=m_{n+1}}^{k} \alpha_i L(\Phi_i) \right),
\]

which implies, with with \( g_{\theta} = L - \pi_{\theta}(L) \),

\[
\max_{m_n \leq k \leq m_{n+1}} \| \hat{\theta}_k \| \leq 2 \exp \left( \sum_{i=m_{n+1}}^{m_{n+1}} \alpha_i L(\Phi_i) \right) \leq 2 \exp(\bar{\pi}(L)[T_{n+1} - T_n]) \exp \left( \sum_{i=m_{n+1}}^{m_{n+1}} \alpha_i g_{\theta}(\Phi_i) \right),
\]

where we used \( \| \hat{\theta}_{m_n} \| \leq 1 \) in the first inequality. The bound (63b) follows from Prop. 11. \( \Box \)

The simple proof of the following error bounds is omitted.

**Lemma 8.** Under (A1) there is a constant \( b_0 \) such that for each \( n \geq 1 \), \( m_n < k \leq m_{n+1} \) and \( t \in [\tau_{k-1}, \tau_k] \),

\[
\| \Theta - \hat{\theta}^{(n)} \| \leq \max \{ \| \Theta_{\tau_k} - \hat{\theta}^{(n)} \|, \| \Theta_{\tau_{k-1}} - \hat{\theta}^{(n)} \| \} + b_0 \max \{ \| \Theta_{\tau_k}^{(n)} \|, \| \Theta_{\tau_{k-1}}^{(n)} \| \} \alpha_k
\]

\[
\| \hat{\Theta} - \hat{\theta} \| \leq \max \{ \| \hat{\Theta}_{\tau_k} - \hat{\theta}^{(n)} \|, \| \hat{\Theta}_{\tau_{k-1}} - \hat{\theta}^{(n)} \| \} + b_0 \max \{ \| \hat{\theta}^{(n)} \|, \| \hat{\theta}^{(n)} \| \} \alpha_k.
\]

27
Bounds on $\|\hat{\Theta}_t - \hat{\Delta}_t\|$ for $t \in \{\tau_k : k \geq 0\}$ require the representation of $\Delta_{k+1}$ in (21). Denote the scaled variables by:

$$
\Delta_{k+1} := \dot{\zeta}_{k+1} - \dot{T}_{k+1} + \dot{T}_k + \alpha_{k+1} \dot{Y}_{k+1}
$$

$$
:= [\dot{\zeta}_{k+1} - \dot{T}_{k+1} + \dot{T}_k - \alpha_{k+1} \dot{Y}_{k+1}] / c_n, \quad m_n \leq k < m_{n+1}.
$$

(64a)

On scaling the bounds in Prop. 8 we obtain,

$$
\|\hat{\zeta}_{k+1}\| \leq b_f [E[v^{\frac{3}{2}}(\Phi_{k+1}) | \mathcal{F}_{k+1}] + v^{\frac{3}{2}}(\Phi_{k+2})] [1 + \|\hat{\theta}_k\|]
$$

(64b)

$$
\|\dot{T}_{k+1}\| \leq b_f v^{\frac{3}{2}}(\Phi_{k+1}) [1 + \|\hat{\theta}_{k+1}\|]
$$

(64c)

$$
\|\dot{Y}_{k+1}\| \leq b_f \|L\| v^{\frac{3}{2}}(\Phi_{k+1}) [1 + \|\hat{\theta}_k\|].
$$

(64d)

**Proposition 12.** Under (V4) and (A1)–(A2):

(i) The martingale \(\left\{ \sum_{k=1}^{n} \alpha_k \hat{\zeta}_k : n \geq 1 \right\}\) converges a.s. as \(n \to \infty\).

(ii) \(\lim_{n \to \infty} \max_{m_n \leq k \leq m_{n+1}} \alpha_k \|\dot{T}_{k+1}\| = 0\)

(iii) \(\lim_{n \to \infty} \sum_{k=m_n}^{m_{n+1}-1} |\alpha_{k+1} - \alpha_{k+2}| \|\dot{Y}_{k+1}\| = 0\) a.s.

(iv) \(\lim_{n \to \infty} \sum_{k=m_n}^{m_{n+1}-1} \alpha_k^2 \|\dot{Y}_{k+1}\| = 0\) a.s.

**Proof.** Lemma 7 tells us that \(b_\theta := \sup_{k \geq 1} \|\hat{\theta}_k\| < \infty\) a.s. Given (64b),

$$
E[\|\dot{\zeta}_{k+1}\|^2 | \mathcal{F}_k] \leq 4b_f^2 E[v^{\frac{3}{2}}(\Phi_{k+1}) | \mathcal{F}_k](1 + \|\hat{\theta}_k\|)^2
$$

$$
\leq 4b_f^2 (1 + b_\theta)^2 E[v^{\frac{3}{2}}(\Phi_{k+1}) | \mathcal{F}_k], \quad \text{for each } k \geq 1,
$$

$$
\sum_{k=1}^{\infty} \alpha_k^2 E[\|\dot{\zeta}_{k+1}\|^2 | \mathcal{F}_k] \leq 4b_f^2 (1 + b_\theta)^2 \sum_{k=1}^{\infty} \alpha_k^2 E[v^{\frac{3}{2}}(\Phi_{k+1}) | \mathcal{F}_k], \quad \text{a.s.}
$$

The right-hand side is finite a.s. since by Lemma 3 and Jensen’s inequality,

$$
E \left[ \sum_{k=1}^{\infty} \alpha_k^2 E[v^{\frac{3}{2}}(\Phi_{k+1}) | \mathcal{F}_k] \right] \leq b_f^2 v^{\frac{3}{2}}(x) \sum_{k=1}^{\infty} \alpha_k^2 < \infty, \quad x = \Phi_0 \in X
$$

Part (i) then follows by martingale convergence.

For (ii), it follows from (64c) that for each \(k \geq 1\),

$$
\max_{m_n < k \leq m_{n+1}} \alpha_k \|\dot{T}_{k+1}\| \leq b_f (1 + b_\theta) \max_{m_n < k \leq m_{n+1}} \alpha_k v^{\frac{3}{2}}(\Phi_k).
$$

On denoting \(\tilde{\theta}_k = v^{\frac{3}{2}} - \pi_\theta(v^{\frac{3}{2}})\), the series \(\{\sum_{k=0}^{n} \alpha_k \tilde{\theta}_k(\Phi_k)\}\) is convergent, by Prop. 11, and hence a Cauchy sequence. Consequently,

$$
\lim_{n \to \infty} \max_{m_n < k \leq m_{n+1}} \alpha_k v^{\frac{3}{2}}(\Phi_k) = \lim_{n \to \infty} \max_{m_n < k \leq m_{n+1}} |\alpha_k \tilde{\theta}_k(\Phi_k)| = 0.
$$

Part (iii) follows from Lemma 6 (ii) and Prop. 11, and arguments similar to the proof of (ii). Part (iv) follows from the fact that \(\|\alpha_k \dot{Y}_{k+1}\|\) is vanishing as \(k \to \infty\): It is square summable in \(L_2\) and almost surely du to Lemma 7 and (60b).
The identity (22) combined with (27) gives for $k > m_n$,

$$\hat{\theta}_k = \hat{\theta}_{m_n} + \sum_{i=m_n}^{k-1} \alpha_{i+1} \mathcal{J}_{c_n}(\hat{\theta}_i) + \tilde{\varepsilon}_k,$$

with $\tilde{\varepsilon}_k := \sum_{i=m_n+1}^k [\alpha_i \hat{\xi}_i - \alpha_i^2 \hat{T}_i - [\alpha_i - \alpha_{i+1}] \hat{T}_i] + \alpha_{m_n+1} \hat{T}_{m_n} - \alpha_k \hat{T}_k$.

(65)

Lemma 9 is based on (65) and the representation:

$$\hat{\vartheta}_{\tau_{k+1}} = \hat{\theta}_{m_n} + \int_{T_n}^{T_{k+1}} \mathcal{J}_{c_n}(\hat{\theta}_t) \, dt, \quad k \geq m_n.$$  (66a)

Lemma 9. For $m_n \leq k < m_{n+1}$,

$$\hat{\theta}_{k+1} - \hat{\vartheta}_{\tau_{k+1}} = \sum_{i=m_n}^k \alpha_{i+1} \mathcal{J}_{c_n}(\hat{\theta}_i) - \int_{T_n}^{T_{k+1}} \mathcal{J}_{c_n}(\hat{\theta}_t) \, dt + \tilde{\varepsilon}_{k+1}$$

$$= \sum_{i=m_n}^k \alpha_{i+1} \{ \mathcal{J}_{c_n}(\hat{\theta}_i) - \mathcal{J}_{c_n}(\hat{\vartheta}_{\tau_{i+1}}) \} + \tilde{\varepsilon}_{k+1} + \tilde{\varepsilon}_{k+1}^D,$$  (66b)

where $\{\tilde{\varepsilon}_k : m_n < k \leq m_{n+1}\}$ is given in (65), and $\tilde{\varepsilon}_{k+1}^D$ is the discretization error resulting from the Riemann-Stieltjes approximation of the integral (66b).  (66c)

Proof. The identities (66b, 66c) follow from (65) and (66a). \hfill \Box

We next obtain bounds on the error sequences:

$$b^D(n) := \max_{m_n < k \leq m_{n+1}} \|\tilde{\varepsilon}_k^D\|, \quad b^N(n) := \max_{m_n < k \leq m_{n+1}} \|\tilde{\varepsilon}_k\|$$  (67a)

from which we obtain the desired ODE approximation for the scaled parameter sequence.

Lemma 10. Under (V4) and (A1)–(A2), the limit (28) holds, along with the bounds,

$$\max_{m_n \leq j \leq m_{n+1}} \|\hat{\theta}_j - \hat{\vartheta}_{\tau_j}\| \leq b(n) \exp(\tilde{L}T).$$  (67b)

where $\tilde{L}$ is the Lipschitz constant for $\bar{f}$, and $b(n) = b^D(n) + b^N(n)$ is a vanishing sequence:

$$b^D(n) \leq b_{10} \sum_{i=m_n+1}^{m_{n+1}} \alpha_i^2, \quad \text{with } b_{10} \text{ a deterministic constant};$$  (67c)

$$b^N(n) \leq \max_{m_n \leq j < m_{n+1}} \left\{ \| \sum_{i=m_n+1}^k \alpha_i \hat{\xi}_i \| + \alpha_{i+1} \|\hat{T}_i\| \right\} + \alpha_{m_n+1} \|\hat{T}_{m_n}\|$$

$$+ \sum_{i=m_n+1}^k \left\{ \alpha_i^2 \|\hat{T}_i\| + |\alpha_i - \alpha_{i+1}| \|\hat{T}_i\| \right\}$$  (67d)
Proof. The bound (67c) follows from uniform Lipschitz continuity of \( \mathcal{F}_{cn} \), and (67d) follows from the definition of \( \hat{\mathcal{E}}_k \) in (65). The conclusion that \( b(n) = b^D(n) + b^V(n) \) is a vanishing sequence follows from (67c) and Prop. 12.

Lipschitz continuity of \( \mathcal{F}_{cn} \) also gives,
\[
\sum_{i=m_n}^k \alpha_{i+1}(\mathcal{F}_{cn}(\hat{\theta}_i) - \mathcal{F}_{cn}(\hat{\theta}_{\tau_i})) \leq \bar{L} \sum_{i=m_n}^k \alpha_{i+1}\|\hat{\theta}_i - \hat{\theta}_{\tau_i}\|,
\]
where \( \bar{L} \) is the common Lipschitz constant for any of the scaled functions \( \{\mathcal{F}_c : c > 0\} \). Consequently, from (66c) and the triangle inequality,
\[
\max_{m_n \leq j \leq k+1} \|\hat{\theta}_j - \hat{\theta}_{\tau_j}\| \leq \bar{L} \sum_{i=m_n}^k \alpha_{i+1} \max_{m_n \leq j \leq i} \|\hat{\theta}_j - \hat{\theta}_{\tau_j}\| + b(n). \tag{68}
\]
The discrete Gronwall inequality and (68) implies (67b).

Finally, (68) together with Lemma 8 completes the proof that (28) holds. \( \square \)

### A.4 \( L_p \) Bounds

The proof of Thm. 2 is similar to the proof of Thm. 1, but more challenging. Starting with the scaled iterates \( \{\hat{\theta}_k\} \), we first show in Prop. 13 that the conditional fourth moment of \( \hat{\theta}_k \) is uniformly bounded in \( k \). Prop. 15 establishes an ODE approximation of \( \{\hat{\theta}_k\} \) in the \( L_4 \) sense, based on the almost sure ODE approximation Prop. 10. The ODE approximation of \( \{\hat{\theta}_k\} \) combined with Assumption (A3) leads to a recursive “contraction inequality” in Lemma 13 which then quickly leads to the proof of Thm. 2.

**Details of Proof** Recall that \( T > 0 \) specifies the length of time blocks used in the ODE method described in Section 2.3. We fix \( T = T_r \) throughout this section with \( T_r \) defined in (A4). All of the bounds obtained in this Appendix are for “large \( n \)”:

Let \( n_g \geq 1 \) denote a fixed integer satisfying,
\[
\tilde{\alpha} := \sup\{\alpha_n : n \geq n_g\} \leq \frac{3}{4}\left(\frac{1}{4 \delta_L} - T_r\right), \tag{69a}
\]
where \( \delta_L := \|L\|_W \). For \( n \geq n_g \) we obtain under (A4),
\[
4 \delta_L \sum_{k=m_n}^{m_{n+1}-1} \alpha_{k+1} \leq 4 \delta_L (T_r + \tilde{\alpha}) \leq 4 \delta_L \left(T_r + \frac{3}{4}\left(\frac{1}{4 \delta_L} - T_r\right)\right) = \frac{3}{4} + \frac{1}{4}T_r \cdot 4 \delta_L < 1. \tag{69b}
\]

Recall the definition \( \hat{\theta}_k := \theta_k/c_n \) for \( m_n \leq k \leq m_{n+1} \), where \( c_n = \max\{1, \|\theta_{m_n}\|\} \).

**Proposition 13.** Under (DV3) and (A1)-(A4), the following bounds hold for each \( n \geq n_g \) and \( m_n \leq k < m_{n+1} \):

\begin{itemize}
  \item[(i)] \( \mathbb{E}[\exp(V(\Phi_{k+1})) (1 + \|\hat{\theta}_k\|)^4 \mid \mathcal{F}_{m_n+1}] \leq 16b^2_n e^b \exp(V(\Phi_{m_n+1})) \);
  \item[(ii)] \( (\|\theta_k\| + 1)^4 \leq 16 \exp(4 \delta_L S_{m_n,m_{n+1}-1}^W)(\|\theta_{m_n}\| + 1)^4 \), where,
\end{itemize}
\[
S_{\ell,k}^W := \sum_{j=\ell}^k \alpha_{j+1} W(\Phi_{j+1}), \quad 0 \leq \ell \leq k. \tag{70}
\]
Proof. The bound $L \leq \delta L W$ combined with (63a) gives,

$$1 + \|\hat{\theta}_{k+1}\| \leq \exp(\delta L \alpha_{k+1} W(\Phi_{k+1}))(1 + \|\hat{\theta}_k\|).$$

Iterating this inequality, we obtain for each $m_n \leq k < m_{n+1}$,

$$1 + \|\hat{\theta}_{k+1}\| \leq \exp(\delta L S_{m_{n},k} W)(1 + \|\hat{\theta}_{m_n}\|)$$

$$\Rightarrow c_n + \|\theta_{k+1}\| \leq \exp(\delta L S_{m_{n},k} W)(c_n + \|\theta_{m_n}\|).$$

(71)

Then (ii) follows from the facts that $1 \leq c_n \leq 1 + \|\theta_{m_n}\|$.

Since $1 + \|\hat{\theta}_{m_n}\| \leq 2$ by construction, (71) also gives,

$$\exp(\{V(\Phi_{k+1}) + 4\delta L S_{m_{n},k} W\}) \leq 16\exp(\{V(\Phi_{k+1}) + 4\delta L S_{m_{n},k} W\}|_{\mathcal{F}_{m_n+1}}),$$

where the final inequality follows from Prop. 3 and (69). This establishes (i). □

For any random vector $X$, we denote $\|X\|^{(n)}_4 := E[\|X\|^4 | \mathcal{F}_{m_n+1}]^{\frac{1}{4}}$, and obtain bounds for various choices of $X$.

**Proposition 14.** Under (DV3) and (A1)–(A4), there exists a constant $b_{14} < \infty$ such that for all $n \geq n_g$ and $m_n \leq k < m_{n+1},$

$$\|\hat{\zeta}_{k+1}\|_4^{(n)} \leq b_{14} \exp(\frac{1}{4} V(\Phi_{m_{n+1}}))$$

$$\|\bar{\zeta}_{k+1}\|_4^{(n)} \leq b_{14} \exp(\frac{1}{4} V(\Phi_{m_{n+1}}))$$

$$\|\hat{\bar{\zeta}}_{k+1}\|_4^{(n)} \leq b_{14} \exp(\frac{1}{4} V(\Phi_{m_{n+1}})).$$

(72a, 72b, 72c)

**Proof.** Bounds identical to (60b) hold for the scaled noise components:

$$\|\hat{\zeta}_{k+1}\| \leq b_f^{(2 + E[V(\Phi_{k+1})] | \mathcal{F}_k] + V(\Phi_{k+1})}[1 + \|\hat{\theta}_k\|]$$

$$\|\bar{\zeta}_{k+1}\| \leq b_f^{(1 + V(\Phi_{k+1})][1 + \|\hat{\theta}_k\|]}$$

$$\|\hat{\bar{\zeta}}_{k+1}\| \leq b_f^{[1 + V(\Phi_{k+1})] L(\Phi_{k+1})[1 + \|\hat{\theta}_k\|].$$

Consider first the martingale difference term $\hat{\zeta}_{k+1}$:

$$\|\hat{\zeta}_{k+1}\|^4 \leq b_f^{4(2 + E[V(\Phi_{k+1})] | \mathcal{F}_k] + V(\Phi_{k+1}))^4(1 + \|\hat{\theta}_k\|)^4.$$

Taking conditional expectations gives,

$$E[\|\hat{\zeta}_{k+1}\|^4 | \mathcal{F}_{m_n+1}]$$

$$\leq b_f^{4} E[(2 + E[V(\Phi_{k+1})] | \mathcal{F}_k] + V(\Phi_{k+1}))^4(1 + \|\hat{\theta}_k\|)^4 | \mathcal{F}_{m_n+1}]$$

$$= b_f^{4} E[(1 + \|\hat{\theta}_k\|)^4 E[(2 + E[V(\Phi_{k+1})] | \mathcal{F}_k] + V(\Phi_{k+1}))^4 | \mathcal{F}_k] | \mathcal{F}_{m_n+1}]$$

We next apply the bound the bound $V^{4} \leq b_5 e V$ for a finite constant $b_5$, along with $(2 + E[V(\Phi_{k+1})] | \mathcal{F}_k] + V(\Phi_{k+1}))^4 \leq 32 + 16E[V^4(\Phi_{k+1}) | \mathcal{F}_k] + 16V^4(\Phi_{k+1})$, to conclude,

$$E[(2 + E[V(\Phi_{k+1})] | \mathcal{F}_k] + V(\Phi_{k+1}))^4 | \mathcal{F}_k] \leq 32 + 32E[V^4(\Phi_{k+1}) | \mathcal{F}_k]$$

$$\leq 32[1 + b_5 e]E[exp(V(\Phi_{k+1})) | \mathcal{F}_k]$$

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Therefore, \( E[\| \hat{z}_{k+1} \|^4 \mid F_{m_n+1}] \leq 32[1 + b_4^2]E[\exp(\Phi_{k+1}) (1 + \| \hat{\theta}_k \|)^4 \mid F_{m_n+1}] \). The bound (72a) then follows from Prop. 13 (i).

The telescoping term admits the bound \( \| \hat{F}_{j+1} \|^4 \leq b_j^4[1 + V(\Phi_{j+1})]^4[1 + \| \hat{\theta}_j \|]^4 \) for any \( m_n \leq j < m_{n+1} \). The fact \( \|(1 + V)^4\|_v < \infty \) gives (72b).

Turning to the final term, we have for some \( b_0 < \infty \),

\[
E[\| \tilde{Y}_{k+1} \|^4 \mid F_{m_n+1}] \leq b_j^4\sum_{m \leq k \leq n} E[(1 + \| \hat{\theta}_k \|)^4 \{ [1 + V(\Phi_{k+1})] L(\Phi_{k+1}) \}^4 \mid F_{m_n+1}]
\]

where the existence of \( b_0 < \infty \) in the second inequality follows from the fact that \( \| L^p \|_v < \infty \) and \( \|(1 + V)^p\|_v < \infty \) for any \( p \geq 1 \) under (A2). Consequently,

\[
E[\| \tilde{Y}_{k+1} \|^4 \mid F_{m_n+1}] \leq b_j^4 b_0 E[(1 + \| \hat{\theta}_k \|)^4 \exp(V(\Phi_{k+1})) \mid F_{m_n+1}].
\]

The desired bound in (72c) follows from another application of Prop. 13 (i).

**Lemma 11.** Let \( \{X_k\} \) be a martingale difference sequence satisfying \( E[\|X_k\|^4] < \infty \) for each \( k \). There exists a constant \( b_{11} < \infty \) such that for any non-negative scalar sequence \( \{\delta_k\} \) and \( m > 0, n > m \),

\[
E[\max_{m \leq k \leq n} \| \sum_{i=m}^k \delta_i X_i \|^4] \leq b_{11} \left( \sum_{i=m}^n \delta_i^2 \right) \max_{m \leq k \leq n} E[\|X_k\|^4].
\]

**Proof.** Burkholder’s inequality [3, Lem. 6, Ch. 3, Part II] implies the desired bound: For a constant \( b_{11} < \infty \),

\[
E[\max_{m \leq k \leq n} \| \sum_{i=m}^k \delta_i X_i \|^4] \leq b_{11} E\left[ \left( \sum_{i=m}^n \delta_i^2 \right)^2 \left( \sum_{i=m}^n \|X_i\|^4 \right) \right] \leq b_{11} \left( \sum_{i=m}^n \delta_i^2 \right)^2 \max_{m \leq k \leq n} E[\|X_k\|^4],
\]

where the second inequality follows from the Cauchy-Schwarz inequality.

The bound (68) continues to hold:

\[
\max_{m_n \leq j \leq k+1} \| \hat{\theta}_j - \hat{\theta}_{\tau_j} \| \leq \bar{L} \sum_{i=m_n}^k \alpha_{i+1} \max_{m_n \leq j \leq i} \| \hat{\theta}_j - \hat{\theta}_{\tau_j} \| + b^p(n) + b^N(n), \quad (73)
\]

The first error term is bounded by a deterministic vanishing sequence (recall (67c)), and \( \{b^N(n)\} \) defined in (67a) is bounded in the following.

**Lemma 12.** For a deterministic vanishing sequence \( \{E^{12}(n) : n \geq 1\} \) we have:

\[
\| b^N(n) \|_4^{(n)} \leq E^{12}(n) \exp(V(\Phi_{m_n+1})).
\]
Proof. Applying Lemma 11 to the sum of martingale difference terms in (67d),

\[ E \left[ \max_{m_n < j < k} \left\| \sum_{i=m_n+1}^{j} \alpha_i \tilde{z}_i \right\|^4 \middle| \mathcal{F}_{m_n+1} \right] \leq b_1 \left( \sum_{i=m_n+1}^{m_{n+1}} \alpha_i^2 \right) \leq b_1 \left( \frac{1}{2} \right)^4 (V(\Phi_{m_n+1}))) \left( \sum_{i=m_n+1}^{m_{n+1}} \alpha_i^2 \right)^2, \]

where the second inequality follows from (72a) in Prop. 14.

Consider next the supremum involving \( \{ \tilde{T}_n \} \):

\[ E \left[ \max_{m_n \leq i < k} \alpha_i^4 \left\| \tilde{T}_i \right\|^4 \middle| \mathcal{F}_{m_n+1} \right] \leq \sum_{i=m_n}^{m_{n+1}-1} \alpha_i^4 E[\left\| \tilde{T}_i \right\|^4 \middle| \mathcal{F}_{m_n+1}] \leq [b_1]^4 \exp(V(\Phi_{m_n+1})) \sum_{i=m_n}^{m_{n+1}-1} \alpha_i^4. \]

By the triangle inequality and (72b) of Prop. 14,

\[ \left\| \sum_{i=m_n+1}^{k} |\alpha_i - \alpha_{i+1}| \left\| \tilde{T}_i \right\| \right\|_4^{(n)} \leq \sum_{i=m_n+1}^{k} |\alpha_i - \alpha_{i+1}| \left\| \tilde{T}_i \right\|_4 \leq [b_1]^4 \exp(1/4 V(\Phi_{m_n+1})) \sum_{i=m_n+1}^{m_{n+1}} |\alpha_i - \alpha_{i+1}|. \]

Lemma 6 (ii) asserts that the sum vanishes as \( n \to \infty \).

Similarly, it follows from (72c) of Prop. 14 that for \( k < m_{n+1} \),

\[ \left\| \sum_{i=m_n+1}^{k} \alpha_i^2 \left\| \tilde{T}_i \right\| \right\|_4^{(n)} \leq [b_1]^4 \exp(1/4 V(\Phi_{m_n+1})) \sum_{i=m_n+1}^{m_{n+1}} \alpha_i^2. \]

Combining these bounds completes the proof. \( \square \)

The next result extends Prop. 10 to an \( L_4 \) bound.

**Proposition 15.** Under (DV3) and (A1)–(A4), there is a vanishing deterministic sequence \( \{ E_n^{15} : n \geq 1 \} \) such that:

\[ E \left[ \sup_{t \in [T_n, T_{n+1}]} \left\| \hat{\Theta}_t - \hat{\theta}_t \right\|^4 \middle| \mathcal{F}_{m_n+1} \right] \leq E_n^{15} \exp(1/4 V(\Phi_{m_n+1})). \]  

(74)

**Proof.** Returning to (73), recall that \( b^P(n) \) is a vanishing deterministic sequence. Applying Lemma 12, we can find a deterministic vanishing sequence \( \{ b(n) \} \) such that,

\[ E \left[ \max_{m_n \leq j \leq k+1} \left\| \hat{\theta}_j - \hat{\theta}_j \right\|^4 \middle| \mathcal{F}_{m_n+1} \right] \leq \hat{L} \sum_{i=m_n}^{k} \alpha_{i+1} E \left[ \max_{m_n \leq j \leq i} \left\| \hat{\theta}_j - \hat{\theta}_j \right\|^4 \middle| \mathcal{F}_{m_n+1} \right] + b(n) \exp(1/4 V(\Phi_{m_n+1})), \]

for all \( m_n \leq k < m_{n+1} \). By the discrete Gronwall’s inequality,

\[ E \left[ \max_{m_n \leq j \leq m_{n+1}} \left\| \hat{\theta}_j - \hat{\theta}_j \right\|^4 \middle| \mathcal{F}_{m_n+1} \right] \leq b(n) \exp(1/4 V(\Phi_{m_n+1})) \exp(\hat{L}[T + 1]). \]

This combined with Lemma 8 completes the proof. \( \square \)

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Lemma 13. Under (DV3) and (A1)–(A4), we can find a constant $b_{13} < \infty$ such that for all $n \geq n_g$,

$$
E[\|\theta_{m+1}\|^4 \mid F_{m+1}] \leq \left( \varrho_r + \mathcal{E}^{15}_n \exp\left(\frac{1}{4} V(\Phi_{m+1})\right)\right)(\|\theta_m\| + 1) + b_{13},
$$

where $0 < \varrho_r < 1$ is defined in Prop. 2.

Proof. The proof is similar to Prop. 9. We begin with Prop. 2, giving:

$$
\|\phi_c(t, \theta_0)\| \leq \varrho_r < 1, \text{ for all } t \in [T_r, T_r + 1], \ c \geq c_0.
$$

By the triangle inequality, for each $n \geq n_g$ and $c_{m_n} = \max\{1, \|\theta_{m_n}\|\}$,

$$
\frac{1}{c_{m_n}} E[\|\theta_{m+1}\|^4 \mid F_{m+1}] = E[\|\hat{\theta}_{T_n+1}\|^4 \mid F_{m+1}] \frac{1}{4}
$$

$$
\leq E[\|\hat{\theta}_{T_n+1} - \hat{\theta}_{T_n+1}\|^4 \mid F_{m+1}] \frac{1}{4} + E[\|\hat{\theta}_{T_n+1}\|^4 I\{\|\theta_{m_n}\| \geq c_0\} \mid F_{m+1}] \frac{1}{4}.
$$

As in the proof of Prop. 9, $\|\hat{\theta}_{T_n+1}\|^4 I\{\|\theta_{m_n}\| < c_0\} \leq \sup_{0 \leq t \leq T+1} \sup_{\delta} \|\hat{\theta}_t\|$, where the inner supremum is over all solutions to the ODE $\dot{\theta} = \varrho \theta$ satisfying $\|\hat{\theta}_0\| \leq c_0$.

Prop. 15 and (76) give, respectively,

$$
E[\|\hat{\theta}_{T_n+1} - \hat{\theta}_{T_n+1}\|^4 \mid F_{m+1}] \frac{1}{4} \leq \mathcal{E}^{15}_n \exp\left(\frac{1}{4} V(\Phi_{m+1})\right),
$$

$$
E[\|\hat{\theta}_{T_n+1}\|^4 I\{\|\theta_{m_n}\| \geq c_0\} \mid F_{m+1}] \frac{1}{4} (\|\theta_{m_n}\| + 1) \leq \varrho_r (\|\theta_{m_n}\| + 1).
$$

Combining these bounds gives (75). \qed

The ODE approximation in (74) combined with the asymptotic stability of the origin for $\frac{d}{dt}\theta_t = \tilde{f}_\infty(\theta_t)$ (see (A3)) leads to the following “contraction bound”:

Lemma 14. Under (DV3) and (A1)–(A4), there exists constants $0 < \varrho^{14} < 1$, $b_{14} < \infty$, and a deterministic and vanishing sequence $\{\mathcal{E}^{14}_n : n \geq n_g\}$ such that for all $n \geq n_g$,

$$
E\left[\|\theta_{m+1}\| + 1\right]^4 \mid F_{m+1} \leq \left(\varrho^{14} + \mathcal{E}^{14}_n V(\Phi_{m+1})\right)(\|\theta_m\| + 1)^4 + b_{14}.
$$

The following simple bound is used twice in the proof:

Lemma 15. $b_{c,\varepsilon} := \sup_{x} \{(x + c)^4 - (1 + \varepsilon)x^4\} < \infty$, for any $c, \varepsilon > 0$.

Proof of Lemma 14. By Lemma 15, for any $\varepsilon > 0$, we have,

$$
E\left[\|\theta_{m+1}\| + 1\right]^4 \mid F_{m+1} \leq (1 + \varepsilon)E\left[\|\theta_{m+1}\|^4 \mid F_{m+1}\right] + b_{1,\varepsilon}.
$$

It follows from (75) that,

$$
E[\|\theta_{m+1}\|^4 \mid F_{m+1}] \leq \left\{ \left( \varrho_r + \mathcal{E}^{15}_n \exp\left(\frac{1}{4} V(\Phi_{m+1})\right)\right)(\|\theta_m\| + 1) + b_{13} \right\}^4.
$$

By Lemma 15 once more, for any $\varepsilon_0 > 0$,

$$
\left\{ \left( \varrho_r + \mathcal{E}^{15}_n \exp\left(\frac{1}{4} V(\Phi_{m+1})\right)\right)(\|\theta_m\| + 1) + b_{13} \right\}^4
$$

$$
\leq (1 + \varepsilon_0) \left( \varrho_r + \mathcal{E}^{15}_n \exp\left(\frac{1}{4} V(\Phi_{m+1})\right)\right)^4 (\|\theta_m\| + 1)^4 + b_1.
$$

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The desired contraction is as follows: For constants 

$$E$$

and hence $$L$$

that for all $$n$$

Most of the work involves establishing the bound (82).

Proof of Thm. 2. It will follow that the mean of $$\beta L$$

It is assumed throughout that $$0 < \delta < 1$$ and

Applying (DV3) and (33b), the following simple bound holds,

$$L^b_{m+1} = (\|\theta_{m+1}\| + 1)^4 E[\exp(V(\Phi_{m+1}) + \epsilon^0 W(\Phi_{m+1})) | \mathcal{F}_{m+1}] - 1. \tag{81}$$

The desired contraction is as follows: For constants $$\eta_0 < 1$$ and $$n_0 \geq 1,$$

It will follow that the mean of $$L^b_m$$ is uniformly bounded, and then the proof of Thm. 2 is completed by invoking Prop. 13 (ii) and Prop. 3.

Proof of Thm. 2. Most of the work involves establishing the bound (82).

Applying (DV3) and (33b), the following simple bound holds,

$$L^b_{m+1} \leq \exp(V(\Phi_{m+1}) + b) L^a_m \ a.s., \tag{83}$$

and hence $$L^b_{m+1} \leq \exp(b V(r) + b) L^a_{m+1}$$ when $$W(\Phi_{m+1}) \leq r,$$ for any $$n$$ and $$r.$$

The bound (77) combined with the definitions gives:

$$E[L^a_{m+1} | \mathcal{F}_{m+1}] \leq q^{14} L^a_m + \epsilon^a L^a_{m+1} \|\theta_{m+1}\| + 1 + 1)^4 E[V(\Phi_{m+1}) | \mathcal{F}_{m+1}] + b_{14} \tag{84}$$

By definition and the smoothing property of conditional expectation,

$$E[L^b_{m+1} | \mathcal{F}_{m+1}] = E[(1 + \|\theta_{m+1}\|)^4 \exp(V(\Phi_{m+1}) + \epsilon^0 W(\Phi_{m+1})) | \mathcal{F}_{m+1}].$$
Using Prop. 13 (ii) implies that $E[L_{m_n+1}^b | F_{m_n}]$ is bounded above by:

$$16(\|\theta_{m_n}\| + 1)^4 E[\exp(V(\Phi_{m_n+1}) + \varepsilon^\circ W(\Phi_{m_n+1}) + 4\delta \sum_{k=m_n+1}^{m_{n+1}} \alpha_k W(\Phi_k)) | F_{m_n}].$$

Recall that by construction $\varepsilon^\circ + 4\delta \sum_{k=m_n+1}^{m_{n+1}} \alpha_k \leq 1$ for each $n \geq n_g$ from (81). Prop. 3 then implies,

$$E[L_{m_n+1}^b | F_{m_n}] \leq 16(\|\theta_{m_n}\| + 1)^4 b_c^2 e^b E[\exp(V(\Phi_{m_n+1})) | F_{m_n}]$$

$$= 16b_c^2 e^b \exp(-\varepsilon^\circ W(\Phi_{m_n})) L_{m_n}^b$$

$$E[L_{m_n+1} | F_{m_n}] \leq \varrho^4 L_{m_n}^a + (\varepsilon_n^a + 16b^2 e^b) \exp(-\varepsilon^\circ W(\Phi_{m_n})) L_{m_n}^b + b_{14},$$

where the second bound follows from the first and (84).

To obtain the desired bound (82) we consider two cases. If the coefficient of $L_{m_n}^b$ satisfies,

$$(\varepsilon_n^a + 16b^2 e^b) \exp(-\varepsilon^\circ W(\Phi_{m_n})) \leq \varrho^4,$$

then we obtain the desired bound with $\varrho_0 = \varrho^4$ and $b_0 = b_{14}$. In the contrary case, we have,

$$\exp(-\varepsilon^\circ W(\Phi_{m_n})) < (\varepsilon_n^a + 16b^2 e^b) / \varrho^4 < (\varepsilon_n^a + 16b^2 e^b) / \varrho^4.$$  

We apply (83), which implies that we can find a constant $b_V^*$ that is independent of $\beta$, such that $L_{m_n}^b \leq b_V^* L_{m_n}^a$, giving (note that $\exp(-\varepsilon^\circ W(\Phi_{m_n})) < 1$):

$$E[L_{m_n+1} | F_{m_n}] \leq \varrho^4 L_{m_n}^a + (\varepsilon_n^a + \beta 16b^2 e^b) b_V^* L_{m_n}^a + b_{14}.$$  

Choose $0 < \beta < 1$ so that $16b^2 e^b b_V^* < 1 - \varrho^4$, and define,

$$\varrho_0 = \varrho^4 + \max_{n \geq n_0} \{(\varepsilon_n^a + 16b^2 e^b) b_V^* \}.$$  

We can choose $n_0 \geq 1$ large enough so that $\varrho_0 < 1$. Choosing $b_L = b_0$ gives (82):

$$E[L_{m_n+1} | F_{m_n}] \leq \varrho_0 L_{m_n}^a + b_0 \leq \varrho_0 L_{m_n} + b_0, \quad n \geq n_0,$$

which gives $\sup_{n \geq 0} E[L_{m_n}] < \infty$. Prop. 13 (ii) combined with Prop. 3 finishes the proof. □

### A.5 Asymptotic statistics

Recall that the FCLT is concerned with the continuous-time process defined in (38), and the CLT concerns the discrete-time stochastic process defined in (37). Most of the work here concerns the FCLT, with the CLT obtained as a corollary.

For the FCLT it is convenient to introduce the notation:

$$y_k^{(n)} := \tilde{\Theta}_k^{(n)} = \theta_k - \bar{\delta}_k^{(n)}, \quad z_k^{(n)} := Z_k^{(n)} = \frac{y_k^{(n)}}{\sqrt{\alpha_k}}, \quad k \geq n.$$  

Analysis is based on familiar comparisons: For $k \geq n$,

$$\delta_{k+1}^{(n)} = \delta_k^{(n)} + \alpha_{k+1}[\tilde{\delta}_{k+1}^{(n)} - \mathcal{E}_{k}^D]$$

$$\theta_{k+1} = \theta_k + \alpha_{k+1}[\tilde{\delta}_{k+1}^{(n)} + \Delta_{k+1}],$$

where $\mathcal{E}_{k}^D$ denotes the error in replacing the integral $\int^{\tau_{k+1}}_{\tau_k} \tilde{\mathcal{F}}(\delta_{t}^{(n)}) \, dt$ by $\alpha_{k+1}\tilde{\mathcal{F}}(\delta_{t_k}^{(n)})$.

We require the following corollary to Thm. 2:

**Corollary 1.** Suppose the conditions of Thm. 2 hold. Then, there exists a finite constant $b_1$ such that for all $n \geq n_g$,

$$\|\zeta_{n+1}\| \leq b_1, \quad \|\mathcal{F}_n\| \leq b_1, \quad \|\Upsilon_{n+1}\| \leq b_1.$$  

$$\|\zeta_{n+1}\| \leq b_1, \quad \|\mathcal{F}_n\| \leq b_1, \quad \|\Upsilon_{n+1}\| \leq b_1.$$  

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Overview of proofs of the CLT and FCLT Define $w_n = \min\{k : \tau_k \geq \tau_n + T\}$. If $n = n_0$ for some integer $n_0$, then $w_n = m_{n+1}$ follows from (23). We obtain an update for the scaled error sequence $\{Z_{n}^{(n)} : n \leq k \leq w_n\}$ in order to establish tightness of the family of stochastic processes: $\{Z_{n}^{(n)} : t \in [0, T]\}_{n=1}^{\infty}$ defined in (38) and also identify the limit.

The proof of Lemma 16 is found in Appendix A.5.

**Lemma 16.** Under (A2)–(A5a) we have the approximation,
\[
y_{k+1}^{(n)} = y_{k+1}^{(n)} + \alpha_{k+1} [A_{k}^{(n)} y_{k}^{(n)} + \mathcal{E}_{k}^{T} + \mathcal{E}_{k}^{D} + \Delta_{k+1}], \quad k \geq n, \tag{88}
\]
where $\overline{A}_{k}^{(n)} = A(\overline{\theta}_{\tau_k}^{(n)})$, and the error terms are interpreted and bounded as follows:

(i) $\mathcal{E}_{k}^{T}$ is the error in the Taylor expansion:
\[
\mathcal{E}_{k}^{T} := \overline{f}(\theta_{k}) - \overline{f}(\overline{\theta}_{\tau_k}^{(n)}) - \overline{A}_{k}^{(n)}[\theta_{k} - \overline{\theta}_{\tau_k}^{(n)}], \quad \|\mathcal{E}_{k}^{T}\| = O(\|y_{k}^{(n)}\|^2 \wedge \|y_{k}^{(n)}\|) \tag{89}
\]

(ii) $\|\mathcal{E}_{k}^{D}\| = O(\alpha_k \|\theta_k\|)$ (defined below; cf. (86)).

(iii) $\Delta_{k+1} = \zeta_{k+1} - T_{k+1} + T_{k} - \alpha_{k+1} Y_{k+1}$, in which $\{\zeta_k\}$ is a martingale difference sequence. The remaining terms $T_{k} = \overline{f}(\theta_k, \Phi_{k+1})$, and $-\alpha_{k+1} Y_{k+1} = \overline{f}(\theta_{k+1}, \Phi_{k+2}) - \overline{f}(\theta_k, \Phi_{k+2})$ satisfy, with $b_\gamma$ a finite constant,
\[
\|T_{k}\| \leq b_\gamma (1 + V(\Phi_{k+2})) \|\theta_{k+1} - \theta_k\|, \quad \|\alpha_{k+1} Y_{k+1}\| \leq b_\gamma (1 + V(\Phi_{k+2}) \|\theta_{k+1} - \theta_k\|.
\]

The following companion to Lemma 6 is required. The proof is omitted.

Recall that $\gamma_k$ is defined in (A1) and $n_g$ is defined in (69a).

**Lemma 17.** The following bounds hold for each $k \geq 1$,
\[
\sqrt{\frac{\alpha_k}{\alpha_{k+1}}} = 1 + \frac{\gamma_k}{2} \alpha_k + O(\alpha_k^2), \quad |\alpha_k - \alpha_{k+1}| = \gamma_k \alpha_k^2 + O(\alpha_k^3), \tag{90a}
\]
Moreover, with $b_{17} = \varepsilon^{T/2}/T$ and every $n$ satisfying $n \geq n_g$,
\[
\prod_{i=l}^{k-1} (1 + \gamma_i \alpha_i)^{\frac{1}{2}} \leq \exp\left(\frac{1}{2} \sum_{i=n}^{w_n} \gamma_i \alpha_i\right) \tag{90b}
\]
\[
\prod_{i=l}^{k-1} \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_{i+1}}} \leq 1 + b_{17} \tau_k - \tau_l, \quad n \leq l < k \leq w_n. \tag{90c}
\]
Dividing each side of (88) by $\sqrt{\alpha_{k+1}}$ gives,
\[
z_{k+1}^{(n)} = \sqrt{\frac{\alpha_k}{\alpha_{k+1}}} z_k^{(n)} + \alpha_{k+1} \left[\sqrt{\frac{\alpha_k}{\alpha_{k+1}}} A_k^{(n)} z_{k}^{(n)}\right] + \sqrt{\alpha_k + 1} \mathcal{E}_{k}^{T} + \mathcal{E}_{k}^{D} + \Delta_{k+1},
\]
This combined with Lemma 17 provides the approximation:

**Lemma 18.** For $k \geq n$,
\[
z_{k+1}^{(n)} = z_k^{(n)} + \alpha_{k+1} \left[\sqrt{\frac{\alpha_k}{\alpha_{k+1}}} + \frac{\gamma_k}{2} I + A_k^{(n)}\right] z_{k}^{(n)} + \sqrt{\alpha_{k+1}} \zeta_{k+1} + \sqrt{\alpha_{k+1}} \mathcal{E}_{k}^{11a}, \tag{91a}
\]
where $\mathcal{E}_{k}^{11a} := -T_{k+1} + T_{k} + \mathcal{E}_{k}^{T} + \mathcal{E}_{k}^{D} - \alpha_{k+1} Y_{k+1} + \sqrt{\alpha_{k+1}} \mathcal{E}_{k}^{\alpha}$, \tag{91b}
with $\mathcal{E}_{k}^{\alpha}$, satisfying $\|\mathcal{E}_{k}^{\alpha}\| = O(\alpha_k \|z_{k}^{(n)}\|$, is the error due to the approximation of $\frac{\alpha_k}{\alpha_{k+1}}$:
\[
\mathcal{E}_{k}^{\alpha} = \frac{1}{\alpha_{k+1}} \left[\sqrt{\frac{\alpha_k}{\alpha_{k+1}}} - \left\{1 + \frac{\gamma_k}{2} \alpha_k\right\}\right] z_{k}^{(n)} + \left(\frac{\alpha_k}{\alpha_{k+1}} - 1\right) A_k^{(n)} z_{k}^{(n)}.
\]

The following summarizes the bounds obtained in this subsection. It will be seen that these bounds combined with standard arguments complete the proof of Thm. 3.

**Proposition 16.** The following hold under the assumptions of Thm. 3, for any $T > 0$: There exists a constant $b_{16} < \infty$ such that:

\[
\mathbb{E}[\|z^{(n)}_k - z^{(n)}_l\|^2] \leq b_{16}|\tau_k - \tau_l|^2, \\
\mathbb{E}[\|z^{(n)}_k\|^2] \leq b_{16},
\]

for all $n \geq n_g$ and $n \leq l < k \leq w_n$.

We next explain why (A5b) combined with the conclusions of Thm. 3 and Prop. 16 imply the CLT. Let $\{m_n\}$ be any increasing sequence satisfying the assumptions in the convergence theory: $\tau_{m_{n+1}} \geq \tau_{m_n} + T$ for each $n$, and $\lim_{n \to \infty} [\tau_{m_{n+1}} - \tau_{m_n}] = T$. The FCLT implies the following limit for any continuous and bounded function $g: \mathbb{R}^d \to \mathbb{R}$:

\[
\lim_{n \to \infty} \mathbb{E}[g(z^{(m_n)}_{m_{n+1}})] = \mathbb{E}[g(X_T)];
\]
here $X_T$ is Gaussian: The solution to (40) with initial condition $X_0 = 0$. Prop. 16 implies that we can go beyond bounded functions. The following uniform bound holds,

\[
\max_{m_n \leq k \leq m_{n+1}} \|z^{(m_n)}_k\|_4 \leq b_z,
\]

where $b_z$ grows exponentially with $T$, but independent of $n$. It follows that (92) holds for any function satisfying $g = o(\omega)$, with $\omega$ the quartic function introduced in Thm. 4. The extension to unbounded functions is via uniform integrability.

The challenge then is to replace $z^{(m_n)}_{m_{n+1}}$ by $z_n := (\theta_n - \theta^*)/\sqrt{\alpha_n}$ in this limit, which amounts to bounding bias:

\[
z^{(m_n)}_{m_{n+1}} = \frac{1}{\sqrt{\alpha_{m_{n+1}}}} \{\theta_{m_{n+1}} - \theta^{(m_n)}_{m_{n+1}}\} = z_{m_{n+1}} + \frac{1}{\sqrt{\alpha_{m_{n+1}}}} \{\theta^* - \theta^{(m_n)}_{m_{n+1}}\},
\]

It is here that we require the stability condition (43), which provides the bound,

\[
\|\theta^* - \theta^{(m_n)}_{m_{n+1}}\| = \|\theta^* - \phi(\tau_{m_{n+1}} - \tau_{m_n}; \theta_{m_n})\| \leq \sqrt{\alpha_{m_n}} b_e \omega^{-\alpha} e^{-\omega T} \|z_{m_n}\|,
\]

and hence from the prior identity,

\[
\|z_{m_{n+1}}\| \leq \|z^{(m_n)}_{m_{n+1}}\| + \xi_{m_n}(T) \|z_{m_n}\| \\
\|z_{m_{n+1}} - z^{(m_n)}_{m_{n+1}}\| \leq \xi_{m_n}(T) \|z_{m_n}\|,
\]

where $\xi_{m_n}(T) = \sqrt{\frac{\alpha_{m_n}}{\alpha_{m_{n+1}}}} b_e \omega^{-\alpha} e^{-\omega T}$.

It will be seen that $\xi_{m_n}(T)$ is vanishing with $T$, uniformly in $n$.

It is convenient here to flip the window backwards in time: For each $k \geq w_1$ denote $w_k^- = \max\{n : w_n \leq k\}$, and let,

\[
\xi_k^-(T) = \left(\sqrt{\frac{\alpha_{w_k^-}}{\alpha_k}}\right) b_e \omega^{-\alpha} e^{-\omega T}.
\]

Lemma 23 tells us that $\{\xi_k^-(T)\}$ vanishes with $T$, uniformly in $k$, from which we obtain the following:

**Lemma 19.** Suppose that the conclusions of Thm. 3 and Prop. 16 hold. If in addition (A5b) is satisfied, then:

(i) $\lim_{k \to \infty} \mathbb{E}[g(z^{-}_k)] = \mathbb{E}[g(X_T)]$ for any continuous function satisfying $g = o(\omega)$. 

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The error $L_2$ norm $\|z_k\|_2$ is uniformly bounded in $k$, and moreover,
\[
\limsup_{k \to \infty} \|z_k - z_k^{(w_k)}\|_2^4 \leq \mathcal{E}_T^{19} \to 0, \quad \text{as } T \to \infty.
\]

Proof of Thm. 4. The proof is presented here with the understanding that Thm. 3, Prop. 16 and Lemma 19 have been established. Full details of the proofs are found below.

We first establish (39) for bounded and continuous functions, and for this it is sufficient to restrict to characteristic functions. That is, it is sufficient to establish the family of limits,
\[
\lim_{k \to \infty} \mathbb{E}[\phi_v(z_k)] = \mathbb{E}[\phi_v(X)], \quad v \in \mathbb{R}^d,
\]
where $\phi_v(z) = \exp(jv^T z)$ for $z \in \mathbb{R}^d$ and $j = \sqrt{-1}$ [6, Th. 29.4]. Letting $L_{\phi_v}$ denote a Lipschitz constant for $\phi_v$, Lemma 19 (ii) and Jensen’s inequality give,
\[
\limsup_{k \to \infty} |\mathbb{E}[\phi_v(z_k)] - \mathbb{E}[\phi_v(z_k^{(w_k)})]| \leq L_{\phi_v} \limsup_{k \to \infty} \mathbb{E}[\|z_k - z_k^{(w_k)}\|] \leq L_{\phi_v} \{\mathcal{E}_T^{19}\}^{1/4}.
\]
Combining this with Lemma 19 (i) gives,
\[
\limsup_{k \to \infty} |\mathbb{E}[\phi_v(z_k)] - \mathbb{E}[\phi_v(X)]| \leq L_{\phi_v} \{\mathcal{E}_T^{19}\}^{1/4} + |\mathbb{E}[\phi_v(X_T)] - \mathbb{E}[\phi_v(X)]|.
\]
The right-hand side vanishes as $T \to \infty$ for any $v$, from which we conclude that (39) holds for bounded and continuous functions.

Prop. 16 and Lemma 19 imply that $\{g(z_k)\}$ is uniformly integrable when $g = o(\omega)$, which justifies extension of (39) to unbounded and continuous functions satisfying this bound.

Tightness and the FCLT We begin with a proof of tightness of the distributions of the two families of stochastic processes $\{Z^{(n)}_{t\omega}, t \in [0,T]\}_{n=1}^\infty$ and $\{\theta^{(n)}_t, t \in [0,T]\}_{n=1}^\infty$, along with $L_p$ bounds required for refinements of the FCLT.

Using a well-known criterion for tightness of probability measure over $C([0,T];\mathbb{R}^d)$ in [5, Th. 12.3], we establish a form of uniform continuity: There exists a constant $b_{16} < \infty$ such that for all $n \geq n_g$ and $n \leq l < k \leq w_n$,
\[
\mathbb{E}[\|\theta^{(n)}_{\tau_k} - \theta^{(n)}_{\tau_l}\|_2^4] \leq b_{16} |\tau_k - \tau_l|^2, \quad (95a)
\]
\[
\mathbb{E}[\|Z^{(n)}_{\tau_k\omega} - Z^{(n)}_{\tau_l\omega}\|_2^4] \leq b_{16} |\tau_k - \tau_l|^2. \quad (95b)
\]
The bound (95a) follows directly from Thm. 2. The bound (95b) is the major part of Prop. 16 that is the main outcome of this subsection, whose proof requires the proof of Lemma 16 and another simple lemma.

Proof of Lemma 16. Part (ii) follows from Lipschitz continuity of $\overline{f}$, and part (iii) follows from Lemma 1 and (59) of Prop. 7.

For part (i) we begin with the definition (89), written in the form,
\[
\mathcal{E}_k^T = R(\theta^{(n)}_{\tau_k}, \theta_k), \quad R(\theta^o, \theta) = \overline{f}(\theta) - \overline{f}(\theta^o) - A(\theta^o)(\theta - \theta^o) \quad \text{for } \theta, \theta^o \in \mathbb{R}^d,
\]
and with $A(\theta^o) = \partial \overline{f}(\theta^o)$. The bound $\|R(\theta^o, \theta)\| = O(\|\theta^o - \theta\|^2)$ follows from (A5a). Also, $R(\theta^o, \theta)$ is Lipschitz continuous in $\theta$ with Lipschitz constant $2\overline{L}$, giving,
\[
\|R(\theta^o, \theta)\| \leq b_0 \min(\|\theta^o - \theta\|, \|\theta^o - \theta\|^2).
\]
The desired bound in (i) then follows from the definition $\mathcal{E}_k^T = R(\theta^{(n)}_{\tau_k}, \theta_k)$ and the definition $y^{(n)}_k = \theta_k - \theta^{(n)}_{\tau_k}$. 
\]

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Recalling the notation $\overline{A}_k^{(n)} = A(\overline{\alpha}_k^{(n)})$ introduced before (88), consider the collection of $d \times d$ matrices \( \{P_{l,k}^A : 0 \leq l \leq k\} \) defined by induction: for each $l$,

\[
P_{l,l}^A = I, \quad \text{and} \quad P_{l,k}^A = [I + \alpha_k \overline{A}_{k-1}^{(n)}]P_{l,k-1}^A, \quad k > l
\]

We also consider the scaled matrices \( \{P_{l,k} = \sqrt{\alpha_l / \alpha_k} P_{l,k}^A : 0 \leq l \leq k\} \). These are written informally as

\[
P_{l,k}^A := \prod_{i=l}^{k-1} [I + \alpha_{i+1} \overline{A}_i^{(n)}], \quad 0 \leq l < k, \quad P_{l,l}^A := I,
\]

\[
P_{l,k} := \prod_{i=l}^{k-1} [I + \alpha_{i+1} \overline{A}_i^{(n)}] \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_{i+1}}}, \quad 0 \leq l < k, \quad P_{l,l} := I.
\]

With $n > 0$ and $l \geq n$ fixed, iterating (88) for $k > l$, gives,

\[
y_k^{(n)} = P_{l,k} y_l^{(n)} + \sum_{j=l+1}^{k} \alpha_j P_{j,k} \left[ E_{j-1}^T + E_{j-1}^D + \Delta_j \right].
\]

We consider next a similar recursion for \( \{z_k^{(n)}\} \) obtained by iterating (91a).

**Lemma 20.** For each $n$, and $n \leq l < k \leq w_n$,

\[
z_k^{(n)} = P_{l,k} z_l^{(n)} + \sum_{j=l+1}^{k} \sqrt{\alpha_j} P_{j,k} \left[ E_{j-1}^T + E_{j-1}^D + \Delta_j \right].
\]

Moreover, there exists a deterministic constant $b_\tau < \infty$ such that for all $n > 0$,

(i) $\max_{n \leq l < k \leq w_n} \{\|P_{l,k}^A\| + \|P_{l,k}\|\} \leq b_\tau$, \quad (ii) $\|P_{l,k} - I\| \leq b_\tau (\tau_k - \tau_l)$.

**Proof.** Dividing both sides of (97) by $\sqrt{\alpha_k}$ gives (98) since $P_{j,k} = \sqrt{\alpha_l / \alpha_k} P_{j,k}^A$.

Since $\|A(\theta)\| \leq L$ for all $\theta \in \mathbb{R}^d$, we apply Lemma 17 to obtain the pair of bounds:

\[
\|P_{j,k}^A\| \leq \prod_{i=j+1}^{k} (1 + \alpha_i L) \leq \exp(\bar{L}T),
\]

\[
\prod_{i=l}^{k-1} \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_{i+1}}} \leq 1 + b[\tau_k - \tau_l] \leq 1 + bT, \quad n \leq l < k \leq w_n.
\]

Therefore, $P_{j,k}^A, P_{j,k}$ are uniformly bounded for all $n \leq j < k < w_n$, and all $n$. This establishes (i), and writing

\[
P_{l,k}^A - I = [I + \alpha_k \overline{A}_{k-1}^{(n)}] [P_{l,k-1}^A - I] + \alpha_k \overline{A}_{k-1}^{(n)}\]

brings us one step towards (ii): For a fixed constant $b_\gamma$,

\[
\|P_{l,k}^A - I\| \leq b_\gamma (\tau_k - \tau_l), \quad k > l.
\]

The remainder of the proof of (ii) uses the identity,

\[
P_{l,k} = [P_{l,k} - P_{l,k}^A] + P_{l,k}^A = \prod_{i=l}^{k-1} \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_{i+1}}} - 1] P_{l,k} + P_{l,k}^A.
\]
Applying (99b) and (99c) establishes (ii):

\[
\|P_{l,k} - I\| \leq \left\| \prod_{i=l}^{k-1} \frac{\sqrt{\alpha_i}}{\sqrt{\alpha_{i+1}}} - 1 \right\| \|P_{l,k}^A\| + \|I - P_{l,k}^A\|
\]

\[
\leq b[\tau_k - \tau_l] \exp(\lambda T) + \|I - P_{l,k}^A\|
\]

\[
\leq b_p[\tau_k - \tau_l], \quad \text{with } b_p := b \exp(\lambda T) + b_y
\]

Next we establish a bound for each disturbance term on the right-hand side of (98). The bound (i) in Lemma 21 is far from tight. Once we obtain \(L_4\) bounds on \(\{z_k^{(n)} : n \leq k \leq w_n\}\), we can expect that the left-hand side of (i) will vanish as \(n \to \infty\). Our immediate goal is only to establish \(L_4\) bounds.

**Lemma 21.** The following hold under the assumptions of Thm. 3: There exist a constant \(b_{21} < \infty\) and a vanishing sequence \(\{\varepsilon_n^{21} : n \geq 1\}\) such that for all \(n > 0\) and \(l < k \leq w_n\):

\[(i) \| \sum_{j=l+1}^{k} \sqrt{\alpha_j} P_{j,k} \varepsilon_j^{T-1} \|_4 \leq b_{21} \sum_{j=l}^{k-1} \alpha_j \|z_j^{(n)}\|_4.\]

\[(ii) \| \sum_{j=l+1}^{k} \sqrt{\alpha_j} P_{j,k} [\varepsilon_j^{D-1} - \alpha_j Y_j] \|_4 \leq \varepsilon_n^{21}(\tau_k - \tau_l)^{\frac{1}{2}}.\]

\[(iii) \| \sum_{j=l+1}^{k} \sqrt{\alpha_j} P_{j,k} [T_j - T_{j-1}] \|_4 \leq \min\{\varepsilon_n^{21}, b_{21}(\tau_k - \tau_l)^{\frac{1}{2}}\}.\]

\[(iv) \| \sum_{j=l+1}^{k} \sqrt{\alpha_j} P_{j,k} \zeta_j \|_4 \leq b_{21}(\tau_k - \tau_l)^{\frac{1}{2}}.\]

The following bounds are required in (i)–(iii): For a finite constant \(b_{100}\):

\[
\|\varepsilon_j^{T}\|_4 \leq b_{100} \sqrt{\alpha_j} \|z_j^{(n)}\|_4 \quad \|\varepsilon_j^{D}\|_4 \leq \alpha_j b_{100},
\]

\[
\|Y_j\|_4 \leq b_{100}, \quad \|T_j\|_4 \leq b_{100}.
\]

The first bound follows from \(\varepsilon_j^T = O(||y_j^{(n)}||^2 \wedge ||y_j^{(n)}||) \leq O(\sqrt{\alpha_j} ||z_j^{(n)}||)\). The remaining bounds follow from Lemma 16 combined with Thm. 2 and Corollary 1.

**Proof of Lemma 21.** Applying Lemma 20 and (100),

\[
\| \sum_{j=l+1}^{k} \sqrt{\alpha_j} P_{j,k} \varepsilon_j^{T-1} \| \leq b_{100} b_p \sum_{j=l+1}^{k} \sqrt{\alpha_j} \varepsilon_{j-1}^{T-1} \|z_{j-1}^{(n)}\|.
\]

The bound in (i) then follows. The proof of (ii) is similar.

Applying summation by parts to the objective in (iii) gives,

\[
\sum_{j=l+1}^{k} \sqrt{\alpha_j} P_{j,k} [T_j - T_{j-1}]
\]

\[
= \sqrt{\alpha_k} T_k - \sqrt{\alpha_{l+1}} P_{l+1,k} T_l + \sum_{j=l+1}^{k} \sqrt{\alpha_j} \alpha_{j-1} \left\{ \gamma_{j-1} I + A(\theta_j^{(n)}) \right\} P_{j,k} T_j,
\]

\[
\leq \sqrt{\alpha_k} T_k - \sqrt{\alpha_{l+1}} P_{l+1,k} T_l + \sum_{j=l+1}^{k} \sqrt{\alpha_j} \alpha_{j-1} \left\{ \gamma_{j-1} I + A(\theta_j^{(n)}) \right\} P_{j,k} T_j,
\]

\[
= b_{21}(\tau_k - \tau_l)^{\frac{1}{2}}.
\]
which leads to the bound,

$$\left\| \sum_{j=l+1}^{k} \sqrt{\alpha_j} P_{j,k} [T_j - T_{j-1}] \right\|$$

$$\leq \sqrt{\alpha_k} \|T_k\| + \sqrt{\alpha_{l+1}} \|P_{l+1,k} T_l\| + \sum_{j=l+1}^{k} \sqrt{\alpha_j \alpha_{j-1}} \left\| \{ \gamma_j I + A(\theta^{(n)}_{\tau_j}) \} P_{j,k} T_j \right\|.$$ 

Using (100) and the inequality \((a + b + c)^4 \leq 16\{a^4 + b^4 + c^4\}\), we may increase the constant \(b_{100}\) to obtain,

$$E \left[ \left\| \sum_{j=l+1}^{k} \sqrt{\alpha_j} P_{j,k} [T_j - T_{j-1}] \right\|^4 \right]$$

$$\leq b_{100} \alpha_k^2 E \|T_k\|^4 + b_{100} \alpha_{l+1}^2 E \|T_l\|^4 + b_{100} E \left[ \left( \sum_{j=l+1}^{k} \sqrt{\alpha_{j+1} \alpha_j} \|T_j\| \right)^4 \right].$$ 

The first two terms on the right-hand side are bounded as follows:

$$\alpha_k^2 E \|T_k\|^4 + \alpha_{l+1}^2 E \|T_l\|^4 \leq b_{100}^4 \sum_{j=l+1}^{k} \alpha_j^2 \leq b_{100}^4 \left( \sum_{j=l+1}^{k} \alpha_{j+1} \right)^2.$$ 

The right-hand side vanishes with \(n\), as it is bounded by a constant times \((\tau_k - \tau_l)^2\). Next,

$$E \left[ \left( \sum_{j=l+1}^{k} \sqrt{\alpha_{j+1} \alpha_j} \|T_j\| \right)^4 \right] = \left( \sum_{j=l+1}^{k} \alpha_j \right)^4 E \left[ \left( \frac{1}{\sum_{j=l+1}^{k} \alpha_j} \sum_{j=l+1}^{k} \sqrt{\alpha_{j+1} \alpha_j} \|T_j\| \right)^4 \right]$$

$$\leq \left( \sum_{j=l+1}^{k} \alpha_j \right)^3 \sum_{j=l+1}^{k} \alpha_{j+1} \alpha_j E \|T_j\|^4$$

$$\leq T^3 b_{100}^4 \sum_{j=l+1}^{k} \alpha_{j+1} \alpha_j \leq (\tau_k - \tau_l)^2 E_n,$$ 

where the first inequality follows by Jensen’s inequality, and the final inequality holds with \(E_n := T^3 b_{100}^4, \max \{ \alpha_{j+1}^2 / \alpha_j : j \geq n \}\), which vanishes as \(n \to \infty\). Combining these bounds gives (iii).

For each \(n, l \geq n\) define \(\Gamma_{l,l}^{(n)} = 0\) and \(\Gamma_{l,k}^{(n)} := \sum_{j=l+1}^{k} \sqrt{\alpha_j} P_{j,k} \xi_j\) for \(k > l\). We have the recursive representation,

$$\Gamma_{l,k}^{(n)} = \sqrt{\alpha_{k-1} \alpha_k} [I + \alpha_k A^{(n)}_{k-1}] \Gamma_{l,k-1}^{(n)} + \sqrt{\alpha_k} \xi_k$$

$$= \Gamma_{l,k-1}^{(n)} + \left[ (\sqrt{\alpha_{k-1} \alpha_k} - 1) I + \sqrt{\alpha_{k-1} \alpha_k} A^{(n)}_{k-1} \right] \Gamma_{l,k-1}^{(n)} + \sqrt{\alpha_k} \xi_k,$$

and summing each side then gives,

$$\Gamma_{l,k}^{(n)} = \sum_{j=l+1}^{k} \left[ (\sqrt{\alpha_{j-1} \alpha_j} - 1) I + \sqrt{\alpha_{j-1} \alpha_j} A(\theta^{(n)}_{\tau_j}) \right] \Gamma_{l,j}^{(n)} + \sum_{j=l+1}^{k} \sqrt{\alpha_j} \xi_j.$$ 

Applying Lemma 17 once more we obtain, with a possibly larger constant \(b_{100} < \infty\),

$$\| (\sqrt{\alpha_{j-1} \alpha_j} - 1) I + \sqrt{\alpha_j \alpha_{j+1}} A(\theta^{(n)}_{\tau_j}) \| \leq b_{100} \alpha_j, \quad j > 0.$$ 

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Then, by the triangle inequality,
\[ \|\Gamma_{l,k}^{(n)}\|_4 \leq b_{100} \sum_{j=l+1}^{k} \alpha_j \|\Gamma_{l,j}^{(n)}\|_4 + b_{l,k}, \quad b_{l,k} := \| \sum_{j=l+1}^{k} \sqrt{\alpha_j} \zeta_j \|_4. \]
Denote \( \bar{b}_{l,k} = \max_{l<j \leq k} b_{l,j} \). The discrete Gronwall’s inequality then gives,
\[ \| \Gamma_{l,k}^{(n)}\|_4 \leq \exp(b_{100} \sum_{j=l+1}^{k} \alpha_j \bar{b}_{l,k}) \leq \exp(b_{100} T) \bar{b}_{l,k}, \quad n \leq l < k \leq w_n, \]
and we obtain via Lemma 11,
\[ \bar{b}_{l,k}^4 \leq E \left[ \max_{l \leq j \leq k} \left\{ \sum_{i=l}^{j} \sqrt{\alpha_i} \zeta_i \right\}^4 \right] \leq b_{11} \max\left\{ \frac{\| \zeta_j \|_4^4}{\| \alpha_j \|_4^4} \right\} \leq b_{11} \max\left\{ \frac{\| \zeta_j \|_4^4}{\| \alpha_j \|_4^4} \right\} (\tau_k - \tau_l)^2. \]
This and Corollary 1 establishes \( \bar{b}_{l,k} \leq b_r b_{1/4} (\tau_k - \tau_l)^{\frac{1}{2}}, \) and concludes the proof of (iv). \( \square \)

**Proof of Prop. 16.** Applying the triangle inequality to (98) in Lemma 20,
\[ \| z^{(n)}_k \|_4 \leq \| \sum_{j=n+1}^{k} \sqrt{\alpha_j} P_{j,k} \mathcal{E}^T_{j-1} \|_4 + \| \sum_{j=n+1}^{k} \sqrt{\alpha_j} P_{j,k} \zeta_j \|_4 \]
\[ + \| \sum_{j=n+1}^{k} \sqrt{\alpha_j} P_{j,k} [\mathcal{E}^D_{j-1} - \alpha_j \mathcal{Y}_j - \mathcal{T}_{j-1} + \mathcal{T}_j] \|_4, \quad n \leq k \leq w_n. \]
Applying Lemma 21 to the above inequality implies that for \( n \) sufficiently large,
\[ \| z^{(n)}_k \|_4 \leq b_{21} \sum_{j=n}^{k-1} \alpha_j \| z^{(n)}_j \|_4 + 3 b_{21} [\tau_k - \tau_n]^{\frac{1}{2}} \leq b_{21} \sum_{j=n}^{k} \alpha_j \| z^{(n)}_j \|_4 + 3 b_{21} T^{\frac{1}{2}}. \]
The discrete Gronwall’s inequality gives:
\[ \sup_n \sup_{n \leq k \leq w_n} \| z^{(n)}_k \|_4 < \infty. \] \( (101) \)
An application of the triangle inequality to (98), for arbitrary \( n \leq l < k \leq w_n \), gives,
\[ \| z^{(n)}_k - z^{(n)}_l \|_4 \leq \| P_{l,k} - I \| \| z^{(n)}_l \|_4 + \| \sum_{j=l+1}^{k} \sqrt{\alpha_j} P_{j,k} \mathcal{E}^T_{j-1} \|_4 + \| \sum_{j=l+1}^{k} \sqrt{\alpha_j} P_{j,k} \zeta_j \|_4 \]
\[ + \| \sum_{j=l+1}^{k} \sqrt{\alpha_j} P_{j,k} [\mathcal{E}^D_{j-1} - \alpha_j \mathcal{Y}_j - \mathcal{T}_{j-1} + \mathcal{T}_j] \|_4. \]
The proof is completed on combining this with (101), Lemma 20 (ii) and Lemma 21. \( \square \)

Having established tightness of the sequence of stochastic processes \( \{ Z^{(n)}_n \} : n \geq 1 \), the next step in the proof of the FCLT is to characterize any sub-sequential limit. The following variant of Lemma 21 will be used in this step.

**Lemma 22.** The following hold under the assumptions of Thm. 3: There exists a vanishing sequence \( \{ \mathcal{E}^2_n : n \geq 1 \} \) such that for all \( n > 0 \) and \( n \leq l < k \leq w_n \),
(i) \( E \left[ \max_{l \leq \ell < k} \left\| \sum_{j=l}^{\ell} \sqrt{\alpha_j + 1} \mathcal{E}_j^T \right\|^2 \right] \leq \mathcal{E}_n^{22} \).

(ii) \( E \left[ \max_{l \leq \ell < k} \left\| \sum_{j=l}^{\ell} \sqrt{\alpha_j + 1} [\mathcal{E}_j^P - \alpha_j \mathcal{Y}_{j+1}] \right\|^2 \right] \leq \mathcal{E}_n^{22} \).

(iii) \( E \left[ \max_{l \leq \ell < k} \left\| \sum_{j=l}^{\ell} \sqrt{\alpha_j + 1} [\mathcal{T}_{j+1} - \mathcal{T}_j] \right\|^2 \right] \leq \mathcal{E}_n^{22} \).

Proof. The proofs of (ii) and (iii) are identical to the corresponding bounds in Lemma 21. Part (i) requires the tighter bound \( \mathcal{E}_j^T = O(\|y_j^{(n)}\|^2 \wedge \|y_j^{(n)}\|) \), so that,

\[
\|\mathcal{E}_j^T\|^2 \leq b_0^2 \min\{\alpha_j^2 \|z_j^{(n)}\|^4, \alpha_j \|z_j^{(n)}\|^2\},
\]

for some constant \( b_0 \), and also,

\[
E[\|\mathcal{E}_j^T\|^2] \leq b_0^2 \min\{\alpha_j^2 E[\|z_j^{(n)}\|^4], \alpha_j E[\|z_j^{(n)}\|^2]\}.
\]

Applying the triangle inequality in \( L_2 \) gives,

\[
E \left[ \max_{l \leq \ell < k} \left\| \sum_{j=l}^{\ell} \sqrt{\alpha_j + 1} \mathcal{E}_j^T \right\|^2 \right]^\frac{1}{2} \leq \sum_{j=n}^{w_n-1} \sqrt{\alpha_j + 1} E \left[ \|\mathcal{E}_j^T\|^2 \right]^\frac{1}{2}
\]

\[
\leq b_0 \sum_{j=n}^{w_n-1} \sqrt{\alpha_j + 1} \min\{\alpha_j \|z_j^{(n)}\|^2, \sqrt{\alpha_j} \|z_j^{(n)}\| \}
\]

Choose \( n_0 \geq 1 \) so that \( \alpha_j^2 E[\|z_j^{(n)}\|^4] \leq \alpha_j E[\|z_j^{(n)}\|^2] \) for \( j \geq n_0 \). The preceding then gives, for all \( n \geq n_0, l \geq n \) and \( l \leq k \leq w_n \),

\[
E \left[ \max_{l \leq \ell < k} \left\| \sum_{j=l}^{\ell} \sqrt{\alpha_j + 1} \mathcal{E}_j^T \right\|^2 \right]^\frac{1}{2} \leq b_0 \sum_{j=n}^{w_n-1} \sqrt{\alpha_j + 1} \alpha_j \|z_j^{(n)}\|^2 \leq b_0 T \left( \max_{j \geq n} \sqrt{\alpha_j + 1} \|z_j^{(n)}\|^2 \right).
\]

The right-hand side vanishes as \( n \to \infty \). \( \Box \)

We next place the problem in the setting of [18, Ch. 7], which requires a particular decomposition. As motivation, first write (40) (with \( F = \frac{1}{2} I + A^* \)) as,

\[
X_\tau = Y_\tau + M_\tau, \quad 0 \leq \tau \leq T, \quad \text{where} \quad Y_\tau = \int_0^\tau \left[ \frac{1}{2} I + A^* \right] X_t \, dt,
\]

and \( \{M_\tau\} \) is Brownian motion. The required decomposition of \( Z_{\tau|\tau_-}^{(n)} \) (defined in (38)) is,

\[
Z_{\tau|\tau_-}^{(n)} = Y_{\tau|\tau_-}^{(n)} + M_{\tau|\tau_-}^{(n)}, \quad 0 \leq \tau \leq T, \quad n \geq 1,
\]

in which \( \{M_{\tau|\tau_-}^{(n)}\} \) is a martingale for each \( n \), and the first term approximates \( \int_0^\tau \left[ \frac{1}{2} I + A^* \right] Z_{\tau|\tau_-}^{(n)} \, dt \). The representation (91a) suggests the choice of martingale,

\[
M_{\tau|\tau_-}^{(n)} = \sum_{j=n+1}^k \sqrt{\alpha_j} \zeta_j, \quad \tau_k \leq \tau < \tau_{k+1},
\]

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valid for $\tau \geq \tau_{n+2}$, with $M_{\tau_{n+2}} := 0$ for $\tau < \tau_{n+2}$. Its covariance is denoted $\Sigma^{(n)} := E[M_{\tau_{n+2}}(M_{\tau_{n+2}})^T]$. The representation (91) combined with Lemma 22 then gives us (102) with $Y^{(n)}_{\tau_{n+2}} := Z^{(n)}_{\tau_{n+2}} - M_{\tau_{n+2}}$, and a useful bound:

$$Y^{(n)}_{\tau_{n+2}} = \int_0^\tau \left[ \frac{\tau}{2} I + A(\hat{\delta}^{(n)}_{\tau_{n+2}}) \right] Z^{(n)}_{\tau_{n+2}} dt + \mathcal{E}^Z_{\tau_{n+2}}(n), \quad \tau_k \leq \tau < \tau_{k+1},$$

in which

$$\lim_{n \to \infty} E\left[ \sup_{0 \leq \tau \leq T} ||\mathcal{E}^Z_{\tau_{n+2}}(n)||^2 \right] = 0.$$

(103)

**Proof of Thm. 3.** We apply Theorem 4.1 of [18, Ch. 7] to obtain the FCLT. It is sufficient to establish the following bounds:

$$0 = \lim_{n \to \infty} E\left[ \sup_{0 \leq \tau \leq T} ||Z^{(n)}_{\tau_{n+2}} - Z^{(n)}_{\tau_{n+2}}||^2 \right]$$

(104a)

$$0 = \lim_{n \to \infty} E\left[ \sup_{0 \leq \tau \leq T} ||Y^{(n)}_{\tau_{n+2}} - Y^{(n)}_{\tau_{n+2}}||^2 \right]$$

(104b)

$$0 = \lim_{n \to \infty} E\left[ \sup_{0 \leq \tau \leq T} ||\Sigma^{(n)}_{\tau_{n+2}} - \Sigma^{(n)}_{\tau_{n+2}}|| \right]$$

(104c)

$$0 = \lim_{n \to \infty} P\left\{ \sup_{0 \leq \tau \leq T} ||Y^{(n)}_{\tau_{n+2}} - \int_0^\tau \left[ \frac{\tau}{2} I + A^* \right] Z^{(n)}_{\tau_{n+2}} dt \right|| \geq \varepsilon \right\}, \quad \varepsilon > 0$$

(104d)

$$\tau \Sigma_{MD} = \lim_{n \to \infty} \Sigma^{(n)}_{\tau_{n+2}}, \quad 0 \leq \tau \leq T.$$  

(104e)

These five equations correspond to equations (4.3)–(4.7) of [18, Ch. 7].

Observe that (104a) is vacuous since $Z^{(n)}_{\tau_{n+2}}$ is continuous for each $n$. Proofs of the remaining limits are established in order:

(104b): We have $Y^{(n)}_{\tau_{n+2}} - Y^{(n)}_{\tau_{n+2}} = \sqrt{\alpha_{j-1}} \zeta_j$ if $\tau = \tau_j$ for some $j$, and zero otherwise. Consequently, for each $n$,

$$E\left[ \sup_{0 \leq \tau \leq T} ||Y^{(n)}_{\tau_{n+2}} - Y^{(n)}_{\tau_{n+2}}||^2 \right] \leq E\left[ \max_{n \leq j \leq w_n} \alpha_{j-1} ||\zeta_j||^2 \right] \leq \sqrt{E\left[ \max_{n \leq j \leq w_n} \alpha_{j-1}^2 ||\zeta_j||^4 \right]}.$$

The right-hand side is bounded as follows:

$$E\left[ \max_{n \leq j \leq w_n} \alpha_{j-1}^2 ||\zeta_j||^4 \right] \leq \sum_{j=n}^{\infty} E\left[ \alpha_{j-1}^2 ||\zeta_{j-1}||^4 \right] \leq \left( \max_{j \geq n} E\left[ ||\zeta_{j-1}||^4 \right] \right) \sum_{j=n}^{\infty} \alpha_{j-1}^2.$$

The fourth moment is uniformly bounded by Corollary 1, and Assumption (A1) then implies that the right-hand side vanishes as $n \to \infty$.

(104c): $\Sigma^{(n)}_{\tau_{n+2}} - \Sigma^{(n)}_{\tau_{n+2}} = \alpha_{j-1} E[\zeta_{j-1}^T \zeta_{j-1}]$ if $\tau = \tau_j$ for some $j$, and zero otherwise. Applying Corollary 1 once more gives, $\lim_{n \to \infty} \sup_{0 \leq \tau \leq T} ||\Sigma^{(n)}_{\tau_{n+2}} - \Sigma^{(n)}_{\tau_{n+2}}|| = 0$.

(104d): Applying the definitions,

$$Y^{(n)}_{\tau_{n+2}} - \int_0^\tau \left[ \frac{\tau}{2} I + A^* \right] Z^{(n)}_{\tau_{n+2}} dt = \mathcal{E}^Z_{\tau_{n+2}}(n) + \int_0^\tau \left[ A(\hat{\delta}^{(n)}_{\tau_{n+2}}) - A^* \right] Z^{(n)}_{\tau_{n+2}} dt,$$

and hence for each $n$ and $0 \leq \tau \leq T$,

$$||Y^{(n)}_{\tau_{n+2}} - \int_0^\tau \left[ \frac{\tau}{2} I + A^* \right] Z^{(n)}_{\tau_{n+2}} dt|| \leq ||\mathcal{E}^Z_{\tau_{n+2}}(n)|| + \int_0^T \left[ ||A(\hat{\delta}^{(n)}_{\tau_{n+2}}) - A^*|| Z^{(n)}_{\tau_{n+2}} || dt. $$

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The limit in (103) tells us that \( \sup_{0 \leq t \leq T} \| \mathcal{E}_t^Z(n) \| \) converges to zero as \( n \to \infty \), where the convergence is in \( L_2 \). We next show that the second term vanishes in \( L_1 \).

Thanks to Assumption (A5a) we can apply Lipschitz continuity of \( A \) to obtain the following bound, for some fixed constant \( L_A < \infty \),

\[
E \left[ \int_0^T \| A(\bar{\theta}_{t-}^{(n)}) - A^* \| Z_{t-}^{(n)} \| dt \right] \leq L_A \int_0^T E[\| \bar{\theta}_{t-}^{(n)} - \theta^* \| \| Z_{t-}^{(n)} \|] \ dt \\
\leq L_A \sqrt{\int_0^T E[\| \bar{\theta}_{t-}^{(n)} - \theta^* \|^2] \ dt \int_0^T E[\| Z_{t-}^{(n)} \|^2] \ dt}.
\]

Thm. 1 combined with Thm. 2 implies that the first integral converges to zero. Corollary 1 tells us that \( E[\| Z_{t-}^{(n)} \|^2] \) is bounded in \( n \) and \( t \geq 0 \).

The limit (104e) is immediate from the definitions. \( \square \)

**Technical results for the CLT** Recall that the proof of Thm. 4 rests on Thm. 3, Prop. 16 and Lemma 19. It remains to prove the lemma, which requires the following bounds whose proof is omitted.

**Lemma 23.** The following bound holds under (A5b): For a constant \( b \) independent of \( k \) or \( T \):

\[
\alpha_{w_k^-}/\alpha_k \leq \begin{cases} 
b \exp(T), & \rho = 1 \\
b(1 + T)^{\rho/(1-\rho)}, & \rho < 1. 
\end{cases}
\]

Consequently, with \( \{\xi_k^-(T) : k \geq w_1\} \) defined in (94), \( \lim_{T \to \infty} \sup_{k \geq w_1} \xi_k^-(T) = 0 \).

**Proof of Lemma 19.** Part (i) is immediate from Thm. 3 and Prop. 16. For (ii) we first establish boundedness of \( \| z_k \|_4 \), which is based on (93) written in “backwards form”:

\[
\| z_k \| \leq \| z_k^{(w_k^-)} \| + \xi_k^-(T)\| z_{w_k^-} \| \quad \text{and} \quad \| z_k - z_k^{(w_k^-)} \| \leq \xi_k^-(T)\| z_{w_k^-} \|. \tag{105}
\]

For given \( n \geq 1 \), suppose that the integer \( k \) satisfies \( m_n < k \leq m_{n+1} \), with \( \{m_n\} \) used in (92). We then have \( m_{n-1} < w_k^- \leq m_n \), and from the first bound in (105),

\[
b_{n+1}^z := \max_{m_n < k \leq m_{n+1}} \| z_k \|_4 \leq \max_{m_{n-1} < j \leq m_n} \| z_j^{(w_j^-)} \|_4 + \bar{\xi}^-(T) \max_{m_{n-1} < j \leq m_n} \| z_j \|_4 \\
\leq B^z + \bar{\xi}^-(T)b_n^z,
\]

where \( B^z \) is a finite constant bounding each of \( \| z_j^{(w_j^-)} \|_4 \), and \( \bar{\xi}^-(T) = \sup_{k \geq w_1} \xi_k^-(T) \). Lemma 23 tells us that we can choose \( T \) so that \( \bar{\xi}^-(T) < 1 \), which implies that \( \{\| z_k \|_4 : k \geq 1\} \) is bounded. The second bound in (105) gives the final conclusion:

\[
\limsup_{k \to \infty} \| z_k - z_k^{(w_k^-)} \|_4 \leq E_T^{19} := \bar{\xi}^-(T) \limsup_{k \to \infty} \| z_k \|_4.
\]

\( \square \)
A.6 Optimizing the Rate of Convergence

The covariance matrix (46) appears in consideration of the pair of martingales,

\[ H_K := \sum_{k=1}^{K} \zeta_k, \quad H_K^* := \sum_{k=1}^{K} \zeta_k^*, \quad 1 \leq K < \infty, \]

(106)
in which \( \zeta_{n+1}^* := \hat{f}(\theta^*, \Phi_{n+1}^*) - \mathbb{E}[\hat{f}(\theta^*, \Phi_{n+1}^*) | \mathcal{F}_n] \) for any \( n \geq 0 \) (in analogy with (21)). We have:

\[ \Sigma^*_\Delta = \lim_{n \to \infty} \frac{1}{n} \text{Cov} (H_n) = \lim_{n \to \infty} \frac{1}{n} \text{Cov} (H_n^*). \]

The following result implies Thm. 5:

**Proposition 17.** Under the assumptions of Thm. 4 there is a constant \( b_{17} \), depending on the initial condition \((\theta_0, \Phi_0)\), such that (13) holds.

**Proof.** We begin with a version of (88):

\[ \tilde{\theta}_{k+1} = \tilde{\theta}_k + \alpha_{k+1} [A\tilde{\theta}_k + \mathcal{E}T_k + \Delta_{k+1}], \quad k \geq 0, \]

where \( \mathcal{E}T_k = \tilde{f}(\theta_k) - A\tilde{\theta}_k \), which satisfies \( \|\mathcal{E}T_k\| \leq L_A \|\tilde{\theta}_k\|^2 \), with \( L_A \) the Lipschitz constant for \( A(\theta) \).

Dividing each side by \( \alpha_{k+1} \) and summing from \( k = 1 \) to \( n \) gives,

\[ \frac{1}{\alpha_{n+1}} \tilde{\theta}_{n+1} - \frac{1}{\alpha_1} \tilde{\theta}_1 = A^* \sum_{k=1}^{n} \tilde{\theta}_k + \sum_{k=1}^{n} \mathcal{E}T_k + \Delta_{k+1} + \gamma_k \tilde{\theta}_k, \]

with \( \{\gamma_k\} \) defined in (A1), which for the choice of step-size assumed here gives \( \gamma_k \leq \rho k^{\rho-1} \). Rearranging terms and dividing by \( n \),

\[ A^* \tilde{\theta}_{n+1}^R + n^{-1} H_{n+2}^* = n^{-1} H_{n+1}^* + \frac{1}{n\alpha_{n+1}} \tilde{\theta}_{n+1} - \frac{1}{n\alpha_1} \tilde{\theta}_1 - \frac{1}{n} \sum_{k=1}^{n} \|\mathcal{E}T_k + \Delta_{k+1} + \gamma_k \tilde{\theta}_k\|, \]

where \( H_{n+1}^* \) is defined in (106). Applying Lemma 1 gives:

\[ \|A^* \tilde{\theta}_{n+1}^R + n^{-1} H_{n+1}^*\| \leq \frac{1}{n} \sum_{k=1}^{n} [L_A \|\tilde{\theta}_k\|^2 + \alpha_{k+1} \|\mathcal{T}_{n+1}\|] \]

(107a)

\[ + \frac{1}{n} \|\tilde{\theta}_{n+1} - \tilde{\theta}_1\| + \frac{1}{n} \sum_{k=1}^{n} \gamma_k \|\tilde{\theta}_k\| \]

(107b)

\[ + \frac{1}{n} \|H_{n+1} - H_{n+1}^*\| \]

(107c)

\[ + \frac{1}{n} \|\mathcal{T}_{n+1} - \mathcal{T}_1\| + \frac{1}{n} \|H_{n+1}^* - H_n^*\| \]

(107d)

On taking \( L_2 \) norms we obtain,

\[ \mathbb{E}[\|A^* \tilde{\theta}_{n+1}^R + n^{-1} H_{n+1}^*\|^{2}]^{1/2} \leq \varepsilon_n^a + \varepsilon_n^b + \varepsilon_n^c + \varepsilon_n^d, \]

in which \( \varepsilon_n^a \) is the \( L_2 \) norm of the right-hand side of (107a), \( \varepsilon_n^b \) is the \( L_2 \) norm of the right-hand side of (107b), \( \varepsilon_n^c \) is the \( L_2 \) norm of the right-hand side of (107c), and \( \varepsilon_n^d \) is the \( L_2 \) norm of the right-hand side of (107d). Repeated application of the bound \( \mathbb{E}[\|\tilde{\theta}_n\|^r] \leq O(\alpha_n^{r/2}) \) for \( 1 \leq r \leq 4 \), gives \( \varepsilon_n^a = O(\alpha_n) \), \( \varepsilon_n^b = O(n^{-1+\rho/2}) \), \( \varepsilon_n^c = O(n^{-(1+\rho)/2}) \) and \( \varepsilon_n^d = O(n^{-1}) \).

The error term \( \varepsilon_n^a \) dominates, which completes the proof. \( \square \)
A.7 Counterexample

The proof of Prop. 4 begins with a representation of (48):

\[
\theta_n = \theta_0 \prod_{k=1}^{n} [1 + \alpha_k (Q_k - \eta - 1)] + \sum_{k=1}^{n} \alpha_k W_k \prod_{l=k+1}^{n} [1 + \alpha_l (Q_l - \eta - 1)].
\] (108)

The two elementary lemmas that follow are used to obtain lower bounds. The first is obtained based on comparison of the sum with the integral \(\int_{n/2+1}^{n+1} \frac{1}{x} \, dx\).

**Lemma 24.** For \(n \geq 1\), \(\sum_{k=n/2+1}^{n} \frac{1}{k} \geq \log 2 - 1/n\).

Denote \(\sigma = \log(1+2\delta)/(2\delta)\) with \(\delta = \mu - \alpha\). A first and second order Taylor series approximations of the logarithm leads to the following:

**Lemma 25.** If \(\delta > 0\) then \(\sigma \geq 1 - \delta\) and \(\exp(\sigma x) \leq 1 + x\) for \(x \in [0, 2\delta]\). Moreover, \(\sigma \geq \delta/(4\alpha \log 2)\) if \(\alpha > 1/3\).

The two lemmas provide bounds on (108), subject to constraints on the sample paths of the queue. The next step in the proof is to present a useful constraint, and bound the probability that it is satisfied. For this we turn to the functional LDP for a reflected random walk.

Define for any \(n \geq 1\) the piecewise constant function of time \(q^n_t = \frac{1}{n} Q_{\lceil nt \rceil}\). For each \(\varepsilon \in [0, 1/2]\) we define a constraint on the scaled process as follows: Let \(\mathcal{X}\) denote the set of all càdlàg functions \(\phi : [0, 1] \to \mathbb{R}_+\), and let \(\mathcal{R}_\varepsilon \subset \mathcal{X}\) consist of those functions satisfying the following strict bounds:

\[
\delta \varepsilon + \min\{\delta t, \delta(1-t)\} \leq q_t \leq 2\delta t \quad \text{for all} \ t \in [\varepsilon, 1] \text{ when } q \in \mathcal{R}_\varepsilon.
\] (109)

If \(q^n \in \mathcal{R}_\varepsilon\) this implies a bound on the term within the product in (108):

**Lemma 26.** For given \(\varepsilon \in (0, 1/2)\), suppose that \(q^n \in \mathcal{R}_\varepsilon\). Then, there is \(n_\varepsilon\) such that the following bounds hold for each \(n \geq n_\varepsilon\), and each \(\ell\) satisfying \(\varepsilon n \leq \ell \leq n\):

\[
\min\{\delta, \delta(n\alpha_\ell - 1)\} \leq \alpha_\ell (Q_\ell - \eta - 1) \leq 2\delta.
\] (110)

**Proof.** Fix \(\ell \geq \varepsilon n\) and denote \(t = \ell/n\). Under the assumption that \(\alpha_\ell = 1/\ell\),

\[
\frac{1}{t} q^n_t = \frac{1}{t} \frac{1}{n} Q(\ell) = \alpha_\ell Q(\ell).
\]

The upper bound is then immediate: If \(q^n \in \mathcal{R}_\varepsilon\) then,

\[
\alpha_\ell (Q_\ell - \eta - 1) \leq \alpha_\ell Q(\ell) = \frac{1}{t} q^n_t \leq 2\delta.
\]

The lower bound proceeds similarly, where we let \(n_\varepsilon = (\eta + 1)/(|\varepsilon^2\delta|)\). For \(\ell \geq \varepsilon n\),

\[
\alpha_\ell (Q_\ell - \eta - 1) = \frac{1}{t} q^n_t - \frac{1}{\ell} (\eta + 1) \geq \frac{1}{t} q^n_t - \frac{1}{\varepsilon n} (\eta + 1)
\]

\[
\geq \frac{1}{t} \left( \delta \varepsilon + \min\{\delta t, \delta(1-t)\} \right) - \frac{1}{\varepsilon n} (\eta + 1)
\]

\[
\geq \frac{1}{t} \min\{\delta t, \delta(1-t)\}, \quad \text{when } n \geq n_\varepsilon,
\]

where the final inequality uses the bound \(\frac{1}{t} \delta \varepsilon \geq \frac{1}{\varepsilon n} (\eta + 1)\) for \(n \geq n_\varepsilon\) and \(t \leq 1\). The proof is completed on substituting \(1/t = n/\ell\). \(\square\)
We now have motivation to bound the probability that $q^n \in \mathcal{R}_\varepsilon$.

The log-moment generating function for the distribution of $D_{k+1}$ appearing in (47) is $\Lambda(\vartheta) = \log(\alpha e^\vartheta + \mu e^{-\vartheta})$. Its convex dual is finite only for $|v| \leq 1$, with

$$I(v) = \frac{1}{2}(1 + v) \log\left(\frac{1 + v}{2\alpha}\right) + \frac{1}{2}(1 - v) \log\left(\frac{1 - v}{2\mu}\right), \quad -1 \leq v \leq 1. \quad (111)$$

**Lemma 27.** For each $\varepsilon \in (0, 1/2)$ the following holds:

$$\lim_{n \to \infty} \frac{1}{n} \log(P\{q^n \in \mathcal{R}_\varepsilon\}) = -\{\varepsilon I(1+\varepsilon) + (\frac{1}{2} - \varepsilon) I(\delta)\} \geq -\frac{1}{2} \delta^2 + O(\varepsilon^2). \quad (112)$$

*Proof.* The functional LDP for the sequence $\{q^n\} \subset \mathcal{X}$ is obtained from the LDP for $\{D_k : k \geq 1\}$ via the contraction principle [21], which for (112) gives

$$\lim_{n \to \infty} \frac{1}{n} \log(P\{q^n \in \mathcal{R}_\varepsilon\}) = -\inf_{\phi \in \mathcal{R}_\varepsilon} \int_0^1 I(\frac{d}{dt} \phi_t) dt.$$ 

Convexity of $I$ implies that the optimizer $\phi^*$ is piecewise linear as illustrated in Fig. 1, with slope $\delta(1+\varepsilon)$ on the interval $[0,\varepsilon]$, $\delta$ on the interval $(\varepsilon, \frac{1}{2})$, and $-\delta$ on the remainder of the interval. The proof of the limit in (112) is completed on recognizing that $I(-\delta) = 0$.

The inequality in (112) is established in two steps. First, $\varepsilon I(1+\varepsilon) = \varepsilon I(\delta) + O(\varepsilon^2)$. The proof is completed on substituting $v = \delta$ in (111) to obtain $I(\delta) = \delta \log(1 + \delta/\alpha) \leq \delta^2/\alpha$. \hfill $\square$

*Proof of Prop. 4.* The almost sure convergence in (i) follows from Thm. 1.

For (ii), let $\varepsilon \in (0, \frac{1}{2})$ be fixed and denote $n_0 = \lceil \varepsilon n \rceil$ (the least integer greater than or equal to $\varepsilon n$). Given $\{W_n\}$ is i.i.d. with zero mean and unit variance, we have for each $n \geq n_0$,

$$E[\theta_n^2] \geq E\left[\frac{\alpha_n^2}{\alpha_{n-1}} W_{n-1}^2 \prod_{k=n_0}^n [1 + \alpha_k(Q_k - \eta - 1)^2]\right] = \alpha_{n_0}^2 E[P_n^Q],$$

$$P_n^Q := \prod_{k=n_0}^n [1 + \alpha_k(Q_k - (\eta + 1))^2].$$

The remainder of the proof consists of establishing the following bound:

$$\log E[P_n^Q] \geq \log E[I\{q^n \in \mathcal{R}_\varepsilon\} P_n^Q] \geq \log P\{q^n \in \mathcal{R}_\varepsilon\} + [2n(\log 2 - \varepsilon)]\delta \sigma. \quad (113)$$

Since $\varepsilon \in (0, \frac{1}{2})$ can be arbitrarily close to zero, (112) combined with Lemma 25 leads to the conclusion that the right-hand side of (113) is unbounded as $n \to \infty$.

**Lemma 26** gives $\alpha_0 [Q_k - (\eta + 1)] \in [0, 2\delta]$ for $n_0 \leq k \leq n$ if $q^n \in \mathcal{R}_\varepsilon$, and so by Lemma 25,

$$P_n^Q \geq \exp\left(\sigma \left[\sum_{k=n_0}^n 2\alpha_k(Q_k - (\eta + 1))\right]\right), \quad q^n \in \mathcal{R}_\varepsilon.$$

A second application of **Lemma 26** provides bounds on each term in the sum:

$$\alpha_k[Q_k - (\eta + 1)] \geq \begin{cases} \delta, & n_0 \leq k \leq n/2, \\ \delta[n\alpha_k - 1], & k > n/2, \end{cases} \quad \text{whenever } q^n \in \mathcal{R}_\varepsilon.$$
Putting these bounds together gives,

\[
P_n^Q \geq \exp\left( \sigma \sum_{k=n_0}^{n/2} 2\alpha_k [Q_k - (\eta + 1)] + \sigma \sum_{k=n/2+1}^{n} 2\alpha_k [Q_k - (\eta + 1)] \right)
\]

\[
\geq \exp\left( (n - 2n_0 + 2)\delta\sigma \right) \exp\left( -\delta\sigma n + 2\sigma\delta n \sum_{k=n/2+1}^{n} \alpha_k \right) \geq \exp\left( [2n \log 2 - 2n\varepsilon] \delta\sigma \right),
\]

where the last inequality follows from \( n \sum_{k=n/2+1}^{n} \alpha_k \geq n \log 2 - 1 \) (see Lemma 24), and the substitution \( n_0 = n\varepsilon \). This establishes (113) and completes the proof. \( \square \)