1. Introduction

One of the non-perturbative alternatives commonly pursued in quantum field theories with coupling constants generically of order unity or larger, is the semiclassical approximation. In this approximation, a conveniently chosen subset of the entire space of degrees of freedom is quantized in the classical background provided by the remaining degrees of freedom. An oft-quoted example of this approach is the formulation of quantum theories of ‘matter’ fields in classical gravitational backgrounds: one starts in this approach with the assumption that the quantum fluctuations of the gravitational background may be ignored. Attempts to go beyond the confines of this approximation would no doubt entail a fully consistent quantum theory of gravitation which is still not available. Fortunately there exist kinematical domains in several situations of physical interest where such an approximation turns out to be an exact one, in that, fluctuations of the non-quantized degrees of freedom essentially decouple from the physics. In other words, within the kinematical window, the entire dynamics is captured by the semiclassical approximation, and no further correction is ever needed. Thus, one can account for all quantum gravitational effects that may occur within this kinematical regime, without having (or needing) a full theory of quantum gravity. Even if we were to be able, some day, to formulate such a theory, the physical predictions of this theory, when restricted to the appropriate kinematical domain must match the results of the semiclassical approach. In what follows, we wish to dwell on some instances of this fortuitous occurrence. We mention in passing that the ‘solvable’ (classical)
part of the theory still has nontrivial information about interactions in contrast to standard perturbation theory which is generally around free field theory.

The kinematical arena for what follows is the large centre-of-mass (cm) energy \( \sqrt{s} \), fixed, small momentum transfer \( \sqrt{t} \) regime; we shall be interested in the limiting situation \( s/t \to \infty \). Clearly this corresponds to almost forward scattering at tiny scattering angles and large impact parameters. In this kinematical regime there is a decoupling of local field degrees of freedom from the dynamics. The residual degrees of freedom form a much simpler lower dimensional field theory which appears like a boundary field theory. Furthermore, the interplay between gravitational and electromagnetic interactions become especially interesting in this kinematical regime when one of the particles is magnetically charged. In this case the fine structure constant of electromagnetism \( \alpha \) does not evolve with \( s \), but increases with increasing \( t \). So if \( t \) is held fixed then \( \alpha \) does not run at all. Thus, in the kinematical region of interest, one expects gravitational interactions to dominate over electromagnetism. With monopole-charge scattering, though, this is not the case as we show below.

In this latter situation, with both gravity and electromagnetism mediated by their respective shock wave fronts, the possibility of an ‘interference’ of these shock fronts must be taken into account. If these shock waves do not mix, then earlier results on the amplitudes for Planckian scattering of point charged particles (including ours) shall stand vindicated. On the other hand, possible mixing of these shocks will lead to new physics like a neutral test particle experiencing effects due to the charge of a boosted charged black hole. Our analysis, albeit heuristic and hence tentative, appears to indicate that shock waves do not mix in the case of Einstein gravity, but they do for the dilatonic extension currently in vogue, inspired by string theory.

The survey is organized as follows. In the second section we review earlier literature\(^1,2\) on pure electric charge-charge scattering within the “shock-wave” picture. Scaling arguments leading to a truncation of the Maxwell action and eventually to the shock wave picture will be summarized for completeness. Then we introduce magnetic monopoles in the theory, and proceed to generalize the foregoing formalism to calculate the scattering amplitude. Particular attention will be paid to subtleties arising from problems like the Dirac string singularity. In the third section gravity will be introduced and the interactions involving both electromagnetism and gravity will be studied. Next we discuss charge-charge and charge-monopole scattering at Planckian energies. The relative contributions of electromagnetic and gravitational scattering in the two cases will be contrasted in detail. We will also comment on the behavior of singularities, namely the poles in the scattering amplitude and how they differ from one process to another. In the next section we address the issue of mixing of shock waves. The Reissner-Nordstrom metric which
describes the static gravitational field due to a point charge, is boosted to luminal velocities to determine whether the resulting gravitational shock wave depends on the charge. We present arguments to the effect that the charge-dependent piece of the boosted metric can be removed by a ‘global’ diffeomorphism and hence does not contribute to the shock wave. Quite a contrasting situation is then shown to exist for a dilatonic black hole, where the shock wave geometry depends explicitly on the charge, and hence characterizes a mixing of the two species. Physical consequences of this mixing are discussed. We end with a few concluding remarks on future outlook and applications of this approach.

2. Electromagnetic Scattering at High Energies

2.1 Effective Theory at High Energies

Suppose there are two spinless charged particles moving at very high velocities, such that the center of mass energy $\sqrt{s}$ is very high. At these energies and very low squared momentum transfer $t$, the scattering is almost exclusively in the forward direction, with a very large impact parameter. This small ratio between the scales of momenta in the transverse and longitudinal directions enables us to associate two widely different length scales with the longitudinal and transverse directions, Thus, we scale the null coordinates $x^\pm$ such that $x^\alpha \rightarrow \lambda x^\alpha$ and $x^i \rightarrow x^i$, where $\alpha$ runs over the light cone indices $+, -, \ldots$, while $i$ signifies the transverse coordinates $x, y$. Note that $\lambda$ is not a new parameter in the theory; it merely signifies the large kinematical ratio under consideration. Under this scaling the $A_\mu$ s transform as $A_\alpha \rightarrow \lambda^{-1} A_\alpha$. The transverse $A_i$ s remain unchanged. The transformed action now has the form

$$S = -\frac{1}{4} \int d^4x \left( \lambda^{-2} F_{\alpha\beta} F^{\alpha\beta} + 2 F_{\alpha i} F^{\alpha i} + \lambda^2 F_{ij} F^{ij} \right) \quad (1).$$

The parameter $\lambda$ may now be chosen to depend on $s$:

$$\lambda = \frac{k}{\sqrt{s}} \rightarrow 0,$$

where $k$ is a finite constant having dimensions of energy. Then the limit $s \rightarrow \infty$ becomes equivalent to the limit $\lambda \rightarrow 0$. Thus in this kinematical regime, the transverse part of the action with $F_{ij}$ can be ignored and what we have left is an effective action of the form

$$S = -\frac{1}{4} \int d^4x \left( \lambda^{-2} F_{\alpha\beta} F^{\alpha\beta} + 2 F_{\alpha i} F^{\alpha i} \right).$$
Notice that in the partition function the fluctuations of the term $F_{\alpha\beta} F^{\alpha\beta}$ are suppressed in the imaginary exponent (due to the smallness of $\lambda$) and the configuration with the dominant contribution is $F_{\alpha\beta} = 0$ i.e. $F^{+-} = E_z = 0$. This shows that the electric field is localized in the transverse plane. Similarly, if we write the original action in the dual formalism, with the $F_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu}$, then $\tilde{F}^{+-} = B_z = 0$. This brings us to the shock wave picture: fields due to processes characterized by longitudinal momenta that are overwhelmingly larger than transverse momenta are essentially confined to the plane (called the ‘shock front’) perpendicular to the direction of motion of the source particles. Thus, if we were to describe the gauge field interaction in terms of currents $j^\mu$, then only the light cone components $j^\pm$ would be physically relevant. Furthermore, if these currents were to be associated with charges moving almost luminality, then

$$j_\pm = j_\pm (x^\pm, \vec{r}_\perp) \ j^i (x) = 0$$

This allows us to define two functions $k^+$ and $k^-$, where

$$j_+ = \partial_- k^- (x^-, \vec{r}_\perp)$$
$$j_- = \partial_+ k^+ (x^+, \vec{r}_\perp)$$

(2)

In short, if we define a vector $k$ such that

$$k(x) = k^+ (x^+, \vec{r}_\perp) - k^- (x^-, \vec{r}_\perp)$$

then

$$j^\alpha = \epsilon^{\alpha\beta} \partial_\beta k.$$

The flatness condition $F^{+-} = 0$ above admits a solution in terms of the light cone components of the gauge potential $A_\pm = \partial_\pm \Omega$. If, further, we impose the Landau gauge $\partial_\mu A^\mu = 0$, $\Omega$ obeys d’Alembert’s equation

$$\partial_+ \partial_- \Omega = 0$$

which implies

$$\Omega = \Omega^+ (x^+, \vec{r}_\perp) + \Omega^- (x^-, \vec{r}_\perp).$$

It is then easy to show that the electromagnetic Lagrange density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu$$

can be written as

$$\mathcal{L} = -\frac{1}{2} \partial_- \Omega^- \tilde{\nabla}^2 \partial_+ \Omega^+ - \frac{1}{2} \partial_+ \Omega^+ \tilde{\nabla}^2 \partial_- \Omega^- - \partial_+ k^+ \partial_- \Omega^- - \partial_- k^- \partial_+ \Omega^+$$
which reduces to a total derivative in the light cone coordinates
\[ L = -\partial_-(\frac{1}{2}\Omega^- \bar{\nabla}^2 \partial_+ \Omega^+ + \partial_+ k^+ \Omega^-) - \partial_+(\frac{1}{2}\Omega^+ \bar{\nabla}^2 \partial_- \Omega^- + \partial_- k^- \Omega^+). \]

This shows that the action \( S = \int d^4x L \) is a surface term defined on the boundary of null plane:
\[ S = \oint d\tau \int d^2r_\perp \left( \frac{1}{2}\Omega^- \bar{\nabla}^2 \dot{\Omega}^+ - \frac{1}{2}\Omega^+ \bar{\nabla}^2 \dot{\Omega}^- + k^+ \Omega^- - k^- \Omega^+ \right). \quad (3) \]

Here all the quantities are evaluated on the boundary contour parametrized by the affine parameter \( \tau \). An overdot denotes \( \partial/\partial \tau \). Thus \( \Omega \)s are the only surviving dynamical degrees of freedom in the problem. This simplification of the action has its origin in the kinematics of the situation. One can show that the shock front discussed earlier emerges as the solution of the classical equations of motion for the above action.

### 2.2 Charge-charge scattering

The foregoing analysis allows us to compute exactly the \( S \)-matrix for the scattering of two highly energetic particles assumed to carry electric charge. Making use of Lorentz covariance of the theory, we will do the calculations in a special inertial frame in which one of the charges moves with velocity close to luminal, while the other is moving relatively slowly. The shock wave front due to the former extends over the entire transverse plane. Thus, the target particle, assumed to be moving in a direction opposite to that of the source, encounters this shock wave and its wave function acquires an Aharonov-Bohm type phase factor. The overlap between the wave functions of the target particle before and after its encounter with the shock front leads to the scattering amplitude.

In the limit \( \beta \to 1 \), the potential of the ultrarelativistic ‘source’ particle is found to be a pure gauge almost everywhere except on the shock plane where it has a discontinuity,
\[ \tilde{A}^0 = \tilde{A}^z = -2e' \ln(\mu r_\perp) \delta(x^-), \]
\[ \tilde{A}^i = 0, \quad i = 1, 2. \quad (4) \]

Here, \( \mu \) is a dimensional parameter inserted to make the logarithm in (4) dimensionless. The potential \( \tilde{A}^\mu \) is singular on the shock plane \( (x^- = 0) \) and is gauge equivalent to the potential \( \tilde{A}'^\mu \) where \( \tilde{A}'^\mu = \tilde{A}^\mu + \partial^\mu \Lambda \), \( \Lambda \) being a Lorentz scalar. Choosing \( \Lambda \) to be \( -2e' \theta(x^-) \ln \mu r_\perp \), we get
\[ \tilde{A}'^0 = \tilde{A}'^3 = 0, \quad \tilde{A}'_\perp = -2e' \theta(x^-) \bar{\nabla} \ln \mu r_\perp \quad (5) \]
We see that the gauged transformed vector potential is a pure gauge everywhere except on the hyperplane \( x^- = 0 \) which is also the shock plane. Thus as one expects, the fields are non-vanishing only on this plane and are given by

\[
E^i = \frac{2e' r^i_\perp}{r^2_\perp} \delta(x^-), \quad E^z = 0
\]

\[
B^i = -\frac{2e' \epsilon_{ij} r^j_\perp}{r^2_\perp} \delta(x^-), \quad B^z = 0.
\]

(6)

These singular field configurations cause an instantaneous interaction with the (slower) target particle. We now proceed to calculate the results of such interaction\(^3\).

Consider the wave function of the slow particle; for early times \( t < z \) the particle is free and its wavefunction is just a plane wave given by,

\[
\psi_<(x^\pm, \vec{r}_\perp) = \psi_0 = \exp[ipx] \text{ for } x^- < 0.
\]

with momentum eigenvalue \( p^\mu \). Immediately after the shock front passes by, its interaction with the gauge potential enters via the ‘minimal coupling prescription’ by which we replace all the \( \partial^\mu \)'s by \( \partial^\mu - ieA^\mu \). The corresponding wavefunction acquires a multiplicative phase factor \( \exp \left( ie \int dx^\mu A_\mu \right) \). Thus from equation (5), for \( x^- > 0 \), the modified wavefunction is

\[
\psi_>(x^\pm, \vec{r}_\perp) = \exp \left[ -ie e' \ln(\mu^2 r^2_\perp) \right] \psi_0 \text{ for } x^- > 0
\]

(7)

The wavefunction \( \psi_> \) can now be expanded in terms of the complete set of momentum eigenfunctions (plane waves) with suitable coefficients in the following form

\[
\psi_> = \int dk_+ d^2k_\perp A(k_+, \vec{k}_\perp) \exp[i\vec{k}_\perp \cdot \vec{r}_\perp - ik_+ x^- - ik_- x^+]
\]

(8)

with the on shell condition \( k_+ = (k^2_\perp + m^2)/k_- \). Here,

\[
A(k_+, k_\perp) = \frac{\delta(k_+ - p_+)}{(2\pi)^2} \int d^2r_\perp \exp i \left( -2ee' \ln(\mu r_\perp) + \vec{q} \cdot \vec{r}_\perp \right),
\]

where \( \vec{q} \equiv \vec{p}_\perp - \vec{k}_\perp \) is the transverse momentum transfer, \( k \) and \( p \) being the final and initial momenta respectively. The integration over the transverse \( x - y \) plane can be performed exactly\(^3\) yielding the amplitude

\[
f(s,t) = \frac{k_+}{4\pi k_0} \delta(k_+ - p_+) \frac{\Gamma(1 - ie e')}{\Gamma(ie e')} \left( \frac{4}{-t} \right)^{1-ie e'}.
\]

(9)
where we have put in the canonical kinematical factors. \( t \equiv -q^2 \) is the transverse momentum transfer. With this amplitude, one can easily show that the scattering cross section is

\[
\frac{d^2\sigma}{dk_\perp^2} \sim \frac{(ee')^2}{t^2},
\]

where we have used a property of the gamma function, namely \( |\Gamma(a + ib)| = |\Gamma(a - ib)| \), \( a \) and \( b \) being real.

It has been shown\(^2\) that this scattering amplitude is identical to the amplitude obtained in the Eikonal approximation where virtual momenta of exchanged quanta are ignored in comparison to external momenta, leading to a resummation of a class of Feynman graphs. Since, generically \( ee' = O\left(\frac{1}{137}\right)\), this approximation will receive usual perturbative radiative corrections. The second order pole singularity in the cross section as \( t \to 0 \) is, of course, typical of processes where massless quanta are exchanged.

### 2.3 Charge - monopole scattering\(^4\)

Now that we have calculated the amplitude of the scattering of two charges, one can inquire as to what changes, if any, will take place if we replace one of the charges by a Dirac magnetic monopole. This question is worth pursuing for various reasons. First of all, the (albeit imagined) existence of monopoles will imply that the Maxwell equations assume a more symmetric form, due to the property of duality of field strengths and electric and magnetic charges. Within quantum mechanics, as Dirac has shown, monopoles offer a unique explanation of the quantized nature of electric charge. But as is well-known, introduction of monopoles in the theory brings in other problems such as singularities in the vector potential. It will be interesting to see how one can deal with them in the present formalism and investigate the range of validity of the shock wave picture in this context. One should also keep in mind the fact that a satisfactory local quantum field theory for monopoles is still lacking. Further, given Dirac’s quantization condition, monopole electrodynamics cannot be understood in perturbative terms around some non-interacting situation. Thus, as advertized earlier, the shock wave picture may be one of the few important probes available for such processes.

Recall, however, it is not possible to choose a single non-singular potential to describe the field of the monopole everywhere. We need at least two such potentials, each being well behaved in some region and being related by a local gauge transformation in the overlapping region. In spherical polar coordinates, these potentials
can be chosen as

\[
\vec{A}^I = \frac{g}{r \sin \theta} (1 - \cos \theta) \hat{\phi} , \quad 0 \leq \theta < \pi \\
\vec{A}^{II} = -\frac{g}{r \sin \theta} (1 + \cos \theta) \hat{\phi} . \quad 0 < \theta \leq \pi .
\]

(10)

The Dirac strings associated with the two potentials are along the semi infinite lines \( \theta = \pi \) and 0 respectively, i.e. along the negative and positive halves of the \( z \) axis. \( \vec{A}^I \) and \( \vec{A}^{II} \) become singular along these two lines respectively. It may be noted that here we have made the gauge choice \( A^0 = 0 \), and have chosen an orientation of our coordinates such that only the \( x \) and \( y \) components survive. In the region \(-\pi < \phi < \pi \), where either of \( \vec{A}^I \) or \( \vec{A}^{II} \) may be used, they are related by a gauge transformation with the gauge parameter \( 2g\hat{\phi} \). It can be readily verified that

\[
\nabla \times \vec{A}^I = \nabla \times \vec{A}^{II} = \frac{g}{r^2} \hat{r}.
\]

Here the curls are taken in the respective regions of validity of the potentials. In the following calculations for convenience we shall work with \( \vec{A}^I \) only, but all subsequent results will be independent of this particular choice.

As in the last section, we give the monopole a Lorentz boost of magnitude \( \beta \) along the positive \( z \) axis. It can be shown that if equations (10) are rewritten in cartesian coordinates, then \( \vec{A}^I \) transforms to

\[
\beta A^I_i = \frac{-g\epsilon_{ij}r^j}{r^2} \left[ 1 - \frac{z - \beta t}{R_{\beta}} \right] \hat{\phi}.
\]

(11)

Before proceeding further, let us examine the behavior of the Dirac strings under Lorentz boosts. For this purpose it is convenient to write equation (11) in the following form.

\[
\beta \vec{A}^I = \frac{g}{r} \left[ 1 - \frac{z - \beta t}{R_{\beta}} \right] \hat{\phi}
\]

(12)

On the \( z \)-axis (\( \theta = 0 \) or \( \pi \)), we have \( r \rightarrow 0 \) implying that \( R_{\beta} \rightarrow |z - \beta t| \). Thus the above equation reduces to

\[
\beta \vec{A}^I = \frac{g}{r} \left[ 1 - \text{sgn}(z - \beta t) \right] \hat{\phi}.
\]

Thus for \( z > \beta t \), i.e. in front of the boosted monopole the vector potential vanishes, while it becomes singular behind it (\( z < \beta t \)). It is as if the monopole drags the Dirac string along with it and as in the static case, the semi-infinite line of singularity originates from it. Similarly, by looking at the boosted potential \( \vec{A}^{II} \), it can be easily
verified that for this, the string is always in front of the monopole and ‘pushed’ by it as it moves. These results also hold in the limit $\beta \to 1$, i.e. for the potential,

$$\vec{A}_0^I \equiv \lim_{\beta \to 1} \beta \vec{A}_i^I = \frac{2g}{r_\perp} \theta(x^-) \hat{\phi}. \quad (13)$$

The corresponding electromagnetic fields are

$$B^i = \frac{2gr_i}{r_\perp^2} \delta(x^-), \quad B^z = 0$$

$$E^i = \frac{2ge_{ij}r_j}{r_\perp^2} \delta(x^-), \quad E^z = 0. \quad (14)$$

Unlike the fields of a charge in motion, here the magnetic field is radial, whereas the electric field is circular on the shock plane. Here also $\vec{A}_0^I$ is a pure gauge everywhere except on the null plane $x^- = 0$. It may be noted that the above $\vec{E}$ and $\vec{B}$ fields can be obtained by making the following transformations in (14): $e' \to g$, $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$. This is a consequence of the duality symmetry in Maxwell’s equations incorporating monopoles.

To compute the scattering amplitude, we first rewrite $\vec{A}_0^I$ in (12) as a total derivative in the following form

$$\vec{A}_0^I = 2g \theta(x^-) \nabla \phi. \quad (15)$$

We note in passing that the gauge potentials for a luminally boosted electric charge (13) and monopole (15), both given as total derivatives on the transverse plane, form the real and imaginary parts respectively of the gradient of the holomorphic function $lnz$ where $z \equiv r_\perp e^{i\phi}$, where $\phi$ is now the azimuthal angle on the transverse plane.

For $t < z$, i.e. before the arrival of the monopole with its shock front, the wave function of the charge $e$ is once again the plane wave

$$\psi_<(x^\pm, \vec{r}_\perp) = \psi_0 \text{ for } x^- < 0. \quad (16)$$

After encountering the shock wave, it is modified by the gauge potential dependent phase factor. The final form of the wave function is

$$\psi_>(x^\pm, \vec{r}_\perp) = \exp[i2eg\phi] \psi_0' \text{ for } x^- > 0$$

by virtue of the potential (15) with the usual requirement of continuity. At this point we make the additional assumption of Dirac quantization namely, for an interacting
monopole-charge system, the magnitudes of their electric and magnetic charge must be constrained by the relation

\[ eg = \frac{n}{2}, \quad n = 0, \pm 1, \pm 2, \ldots\]

Thus we get

\[ \psi_\gamma = e^{in_\phi} \psi_0. \]

This sort of phase factor in the small angle scattering of a monopole and a charge was first found by Goldhaber\(^5\) using more standard techniques. Expanding \(\psi_\gamma\) in plane waves as before we get an integral expression for the scattering amplitude as follows

\[ A(k_+, k_-) = \frac{\delta(k_+ - p_+)}{(2\pi)^2} \int d^2r_\perp \exp i (n_\phi + \vec{q} \cdot \vec{r}_\perp). \quad (17) \]

Once again \(\vec{q} \equiv p_\perp - k_\perp\) is the momentum transfer and as before we have the dispersion relation \(k_+ = (k_{\perp}^2 + m^2)/k_-\). By conveniently choosing the orientation of the transverse axes as in the previous section, the angular integration gives \((1/q^2) \int_0^\infty d\rho \ \rho J_n(\rho)\), where \(J_n(\rho)\) is the Bessel function of order \(n\). This integral is also standard and the result is

\[ \left(\frac{1}{-t}\right) \frac{2\Gamma(1 + \frac{n}{2})}{\Gamma(\frac{n}{2})}, \quad (18) \]

Here we note an important difference with the previously calculated charge-charge amplitude. There the arguments of the gamma functions were complex, whereas in this case they are real. In fact, the amplitude in this case is simply

\[ f(s, t) = \frac{k_+}{2\pi k_0} \delta(k_+ - p_+) \left(\frac{n}{-t}\right), \quad (19) \]

where we have incorporated the canonical kinematical factors. Such factorization makes the expression for the amplitude simple. We observe that it is proportional to the monopole strength \(n\). It follows that the scattering cross section becomes

\[ \frac{d^2\sigma}{dk_\perp^2} \sim \frac{n^2}{t^2} \quad (20) \]

It may be mentioned that we would have obtained the same result if we had used the second of the gauge potentials in (13) and performed the Lorentz boost etc. One way to see this is by noting that the potentials, boosted to \(\beta \approx 1\), are both gauge equivalent to a gauge potential \(A'_{\mu}\) given by

\[ \vec{A}'_\perp = 0 = A'_+; \quad A'_- = 2g\phi\delta(x^-) \text{ everywhere}. \quad (21) \]
The apparent disappearance of the Dirac string singularity in this gauge is a red herring; the gauge transformation has flipped the Dirac string onto the shock plane, thus preventing it from being manifest. More importantly, the gauge potential, though globally defined functionally, is not single-valued, being a monotonic function of a periodic angular variable. Thus, the singularity has been traded in for non-single-valuedness. Of course, the theory of fields which are not single-valued functions is in no way easier to formulate than that for singular fields. It is interesting to note further that for boost velocities that are subluminal, one cannot obtain a globally defined potential $A'^\mu$ in any gauge.

We would like to make a few more remarks at this point. First of all, if we choose another Lorentz frame in which the electric charge is lightlike while the monopole is moving slowly, we would get identical results from the dual formalism wherein one introduces a gauge potential $A^M_\mu$ such that the dual field strength $\tilde{F}_{\mu\nu} \equiv \partial_{[\mu}A^M_{\nu]}$. If this gauge potential is used to define electric and magnetic fields, the standard field tensor $F_{\mu\nu}$ must satisfy a Bianchi identity of the form $\partial_\mu F^{\mu\nu} = 0$ which would then imply that the gauge potential due to a point charge must have a Dirac string singularity. Further, the monopole will behave identically to the point charge of the usual formalism, so that our method above is readily adapted to produce identical consequences. Second, one can also treat the scattering of two Dirac monopoles in the same kinematical limit exactly as in subsection 2.1, using this dual formalism. This would yield a result identical to the one for the electric charge case, with $e$ and $e'$ being replaced by $g$ and $g'$, the monopole charges. Finally, having dealt with particles carrying either electric or magnetic charge, it is straightforward to extend our calculations when one of them is a dyon, that is, it has both electric and magnetic charge. The electromagnetic fields on the shock front of the boosted dyon will be the superposition of the fields produced by a fast charge and a monopole. Also, depending upon the nature of the charge on the other particle (electric or magnetic), one must employ the usual or the dual formalism.

2.4 Dyon - dyon scattering

With the above observations we are in a position to address the problem of dyon-dyon scattering in this formalism. Consider two dyons $(e_1, g_1)$ and $(e_2, g_2)$, where the ordered pair denotes its electric and magnetic charge contents respectively. Let us assume that the first one is ultra relativistic. By means of an electromagnetic duality transformation we can ‘rotate’ the dyon $(e_1, g_1)$ by an angle $\theta$, so that the new values of electric and magnetic charges become $e'$ and $g'$. The same duality transformation rotates the second dyon to $(e, g)$. Since physical observables do not depend on the parameter $\theta$, we can make use of this symmetry and choose it to be
such that
\[ \tan \theta = \frac{g_2}{e_2}. \]  
(22)

This implies that the first dyon transforms to
\[ \begin{align*}
    e' &= \frac{e_1 e_2 + g_1 g_2}{\sqrt{e_2^2 + g_2^2}} \\
    g' &= \frac{-e_1 g_2 + g_1 e_2}{\sqrt{e_2^2 + g_2^2}}
\end{align*} \]  
(23)

while for the second dyon
\[ \begin{align*}
    e &= \sqrt{e_2^2 + g_2^2} \\
    g &= 0.
\end{align*} \]  
(24)

This shows that the slow test dyon has been rotated to a pure electric charge. Then from the results derived previously, the total phase shift in its wavefunction after being hit by the shock wave of the dyon \((e', g')\) is \( [e'e' \ln \mu^2 r_{\perp}^2 + 2eg'\phi] \). Having found this, we can express this in terms of the parameters of the two dyons we started with. The result is \([e_1 e_2 + g_1 g_2] \ln \mu^2 r_{\perp}^2 - 2(e_1 g_2 - g_1 e_2) \phi \]. The calculation of the scattering amplitude now becomes straightforward. It may be noted that the quantities \((e_1 e_2 + g_1 g_2)\) and \((e_1 g_2 - g_1 e_2)\) are the only combinations of the electric charges \(e_1, e_2\) and the magnetic charges \(g_1, g_2\) that are invariant under duality rotations. Thus it is remarkable that the total phase shift and hence the scattering amplitude depends only on these combinations. Alternatively, we could also have made the choice \(\tan \theta = -e_2/g_2\), in which case \(e\) would become zero and the second dyon transforms into a monopole. Obviously these different choices are merely for convenience and the scattering amplitude does not depend on it. Thus dyon-dyon scattering can always be reduced to dyon-charge or dyon-monopole scattering. Also note that by a duality rotation the usual Dirac quantization condition gets transformed into the generalized expression
\[ e_1 g_2 - e_2 g_1 = \frac{n}{2} \]  
(25)

This implies that the second term in the phase shift becomes \(n\phi\) as in the charge-monopole scattering case.

Finally, we can ask the question as to what happens if we consider a massive vector field e.g. that described by the Proca Lagrangian
\[ \mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{\mu^2}{2} A_\mu A^\mu \]  
(26)
The solution in the static limit for $A^\mu$ in the Lorentz gauge is given by

$$A^0 = \frac{e'}{r} \exp (-\mu r), \quad A^i = 0,$$  \quad (27)

where $e'$ is a point charge at rest. Formally we can apply a Lorentz boost to this potential and try to take the limit $\beta \to 1$. The result is

$$\beta A^\mu = \eta^\mu \frac{e' \exp (-\mu R_\beta / \sqrt{1 - \beta^2})}{R_\beta}$$  \quad (28)

which vanishes identically when we take the limit $\beta \to 1$. Thus no shock wave emerges in this case and there are no $\delta$-function electromagnetic fields on the null plane $x^- = 0$. This observation can also be understood as follows. In the formulation of the boundary field theory in section 2.1 it was shown that the gauge parameter $\Omega (\Omega^+, \Omega^-)$ was the only dynamical degree of freedom in the theory and the corresponding equations of motion yielded the shock wave picture. On the other hand, the Lagrangian of the massive vector field does not have the required gauge invariant structure to admit of such a parameter. This accounts for the absence of the shock wave.

3. Electromagnetic vs Gravitational Scattering at Planckian Energies

3.1 Spacetime around a massless particle

At Planckian cm energies the Einstein action also undergoes a truncation akin to the electromagnetic situation. The spacetime geometry that emerges for a particle boosted to velocities close to luminal, is expected to emerge from the coupling of the above truncated action to a suitably constrained matter energy-momentum tensor. This has been done in ref.[1]. Identical answers can however be obtained by a process of boosting the static (Schwarzschild) metric due to a point particle, adopted in ref.[3]; we sketch this approach below. Essentially this boosting means the mapping of a solution of Einstein equation with a massless particle, to Minkowski space but with one of the null coordinates shifted non-trivially, now without any massless particle present$.^9$. It is argued below how this can be interpreted as a gravitational shock wave.

Once again we choose to carry out the analysis in a Lorentz frame in which the velocity of one particle is very much greater than that of the other. We know that the space time around a point particle is spherically symmetric and is described by what is known as the Schwarzschild metric. If we assume the mass $m$ of the particle
to be small, then it is given in the Minkowski coordinates \((T,x,y,Z)\) by,

\[
\frac{ds^2}{dT^2} = -\left(1 - \frac{2Gm}{R}\right)dT^2 + \left(1 + \frac{2Gm}{R}\right)(dx^2 + dy^2 + dZ^2).
\]  

(29)

where \(R = \sqrt{x^2 + y^2 + Z^2}\) and \(m \ll R/G\). If the above coordinate system is moving with a relative velocity \(\beta\) with respect to coordinates \((t,x,y,z)\) then the two are related by a Lorentz transformation of the form

\[
T = t \cosh \theta - z \sinh \theta,
\]

\[
Z = -t \sinh \theta + z \cosh \theta,
\]

(30)

\(\theta\) is called the rapidity which is related to the boost velocity by the relation

\[
\tanh \theta = \beta.
\]

Now to take the limit \(\beta \to 1\) or alternatively \(\theta \to \infty\), we also set

\[
m = 2p_0 e^{-\theta},
\]

where the rest energy of the particle is \(2p_0 > 0\). This parametrization is consistent with the fact that the mass of the particle must exponentially vanish as its velocity approaches that of light. When we substitute equation (30) in equation (29), we have the metric due to a particle moving at the speed of light in the \(x^+\)-direction (i.e along \(x^- = 0\)). In terms of the lightcone and the transverse coordinates this metric becomes

\[
\frac{ds^2}{dx^2} = \left(1 + \frac{2Gm}{R}\right)[-dx^- dx^+ + dx^2 + dy^2] + \frac{4Gm}{R} \left[\frac{p_0}{m} dx^- + \frac{m}{4p_0} dx^+\right]^2,
\]

with

\[
R^2 = x^2 + y^2 + \left(\frac{p_0}{m} x^- - \frac{m}{4p_0} x^+\right)^2.
\]

(31)

Using this and neglecting terms of order \(m\) or above, we get the limiting form of the metric

\[
\lim_{m \to 0} ds^2 = -dx^- \left(dx^+ - 4Gp_0 \frac{dx^-}{|x^-|}\right) + dx^2 + dy^2,
\]

where the limit is evaluated at \(x^- \neq 0\) and \((x^+, x, y)\) fixed. Defining a new set of coordinates through the relation

\[
dx^+ = dx^+ - \frac{4Gp_0}{|x^-|} dx^-,
\]

\[
dx^- = dx^-,
\]

\[
dx^i = dx^i,
\]

(32)
we observe that the above metric is just a flat Minkowski metric
\[ ds^2 = -dx^- dx'^+ + dx'^2 + dy'^2. \] (33)

The crucial point to note here is that the metric suffers a discontinuity at \( x^- = 0 \) through the term \( |x^-|^{-1} \). Now, taking the leading order terms in equation (33), it can be shown that \( dx^-/x^- = dR/R \), which gives
\[ dx'^+ = dx^+ - \theta(x^-) \frac{4Gp_0 dR}{R}. \]

A solution of the above equation near the null plane ( \( |x^-| \to 0 \) ) is,
\[ x'^+ = x^+ + 2Gp_0 \theta(x^-) \ln \left( \mu^2 r_\perp^2 \right). \] (34)

Note that the coordinates \( x^- \) and \( x^i \) remain unchanged. This step function at the null plane \( x^- = 0 \) is the gravitational equivalent of the electromagnetic shock-wave. There we had a similar discontinuity in the gauge potential \( A^\mu \). Here we have two flat regions of space-time corresponding to \( t < z \) and \( t > z \) which are glued together at the null plane \( t = z \) (or \( x^- = 0 \)). However there is a shift of coordinates at this plane given by equation (32). It is as if a two dimensional flat space-time on the \( t - z \) plane is cut along the line \( t = z \) and pasted back again after being shifted along this line by the amount given above.

Now that we have found the metric around a lightlike particle, in principle we should be able to predict the behavior of another (slower) test particle encountering it. Since the sole effect of the gravitational shock wave is the cutting and pasting of the Minkowski space along the null direction \( x^- = 0 \) after a shift of the \( x^+ \) coordinate, it is easy to see that the test particle wave function will acquire a phase factor upon passing through this shock front. One more remark is in order at this point. The logarithmic singularity in the expression for the shift in the coordinate \( x^+ \) in equation (32) causes an infinite time delay of all interactions via virtual particle exchanges. This shows that it is the shock wave interactions which dominate over all standard field theoretic effects like particle creation via brehmstrahlung etc. However, as we shall show later, the gravitational shock wave may not dominate in all situations where other interactions mediated also by shock wavefronts exist.

3.2 Gravitational scattering

To begin with we will assume the particles to be neutral and as before, also spinless. We look at the behavior of the wavefunction of a slow test particle in the
background metric of the lightlike particle carrying with it a ‘gravitational’ shock wave. Before the arrival of the shock wave \((x^- < 0)\), the test particle is in a flat space time as derived in the last section. Thus, as before, its quantum mechanical wave function is a plane wave of the form

\[
\psi_< (x^\pm, \vec{r}_\perp) = e^{i p x}
\]

with definite momentum \(p^\mu\). This can be written in terms of the lightcone and transverse coordinates as

\[
\psi_<(x^\pm, r_\perp) = \exp i \left[ p_\perp x_\perp - p_+ x^- - p_- x^+ \right]
\]

On encountering the shock wave, it is transported to another flat space time defined by \(x^- > 0\) which is related to the previous one by a shift in the \(x^+\) coordinates. From the explicit expression for this shift in equation (32) we see that the wavefunction immediately gets modified into

\[
\psi_>(x^\pm, \vec{r}_\perp) = \exp i \left[ p_\perp x_\perp - p_- (x^+ + 2Gp_0 \ln r_\perp^2) \right],
\]

which is also a plane wave but in the new coordinates. We have put \(\mu = 1\) in equation (32) and evaluated the above at \(x^- = 0^+\). Noting that the factor \(2Gp_- p_0\) can be written as \(Gs\), the phase shift in the final wave function is \(-Gs \ln r_\perp^2\). But this is just the electromagnetic phase shift that we got in the last section in the case of charge-charge scattering with \(Gs\) replacing the earlier coupling \(ee'\). This implies that the scattering amplitude will also be the same as the previous case with this replacement. Consequently we have for the gravitational scattering of the two particles,

\[
f(s, t) = \frac{k_+}{4\pi k_0} \delta(k_+ - p_+) \frac{\Gamma(1 - iGs)}{\Gamma(iGs)} \left( \frac{4}{-t} \right)^{1-iGs}.
\]  

(35)

The corresponding differential cross section is

\[
\frac{d^2\sigma}{d\vec{k}_\perp^2} \sim \frac{G^2 s^2}{t^2}
\]

Despite the striking similarity with electromagnetism, there is an important difference here. The coupling is now proportional to \(s\), the square of the center of mass energy. The above cross section seems to increase without limit with increase of \(s\), thus violating unitarity. To understand this, we must note that at super-Planckian energies one expects gravitational collapse and inelastic processes to take place. Hence the above expression fails to be a faithful representation of the actual
scattering and one has to invoke a full theory of quantum gravity at such extreme energies\(^3\). Similar arguments hold good for all the other cross sections found in this paper.

Another important point to note is the structure of poles in the scattering amplitude (35). It seems that there is a ‘bound state’ spectrum at

\[ G_s = -iN, \ N = 1, 2, 3, \ldots. \]

It has been remarked in ref.\([6]\) that the \(t\)-dependence of the residues of the poles can be expressed as polynomials in \(t\) with degree \(N - 1\). Thus, the largest spins of the bound states are \(N - 1\). This is similar to the Regge behavior of hadronic resonances, albeit with an imaginary slope. It remains to be seen whether these poles are ‘physical’ in the sense they correspond to resonant states or as argued in ref.\([1]\) are just artifacts of our kinematical approximations. Nevertheless, we will show in the subsequent sections that the introduction of electromagnetism does have an effect on their location in the complex \(s\) plane.\(^1\)

### 3.4 Charge-charge versus gravitational scattering

After having considered the pure gravitational scattering, we introduce electromagnetic interactions. Once again we choose a frame such that the ‘source’ charge \(e'\) has an electromagnetic shock wave associated with it. The electric and magnetic fields on the shock front are those found in the previous section, given in equation (5). We assume at this point that the resultant effect of the combined shock wave (gravitational and electromagnetic) on the test particle is to produce a phase shift in its wave function which is the sum of the individual phase shifts. This tacitly presumes the independence of the gravitational and electromagnetic shock waves\(^2\) which is by no means self-evident and warrants justification. In a later section, we shall present a first attempt at such a justification\(^14\). Since both the phase shifts are proportional to \(\ln \mu^2 r^2_+\), the net effect is succinctly captured by the shift \(G_s \rightarrow G_s + ee'\), with the final form of the wave function after it crosses the null plane \(x^- = 0\) being

\[
\psi_>(x^\pm, x_\perp) = \exp \left[ -i \left( ee' + Gs \right) \ln \mu^2 r^2_+ + ipx \right].
\]

Consequently, the scattering amplitude becomes

\[
f(s, t) = \frac{k_+}{4\pi k_0} \frac{\delta(k_+ - p_+)}{\Gamma(1 - iee' - iG_s)} \frac{\Gamma(1 - iG_s)}{\Gamma(i ee' + iG_s)} \left( \frac{4}{-t} \right)^{1 - iee' - iG_s}.
\] (36)

\(^1\)For attempts in three and four dimensions towards improving the ‘gravity eikonal’ see [10-13]
This gives the cross section,
\[
\frac{d^2\sigma}{dk^2} \sim \frac{1}{t^2} (ee' + Gs)^2 .
\] (37)

To compare the relative magnitudes of the two terms, we recall that the electromagnetic coupling constant \( ee' \) evolves only with \( t \) through radiative corrections and not with \( s \). Thus in the kinematical regime that we are considering, it remains fixed at its low energy value. For example, if the particles carry one electronic charge each, then \( ee' \sim 1/137 \). On the other hand, at Planck scales, the second term in the cross section is of order unity. This shows that gravity is the principal contributor in the scattering process and electromagnetic effects can be treated as small perturbations. Likewise, the poles of the scattering amplitude (36) are shifted by \( O(\alpha) \) corrections to the pure gravity poles. Observe that these poles appear only when gravitational interactions are taken into account, because it is only in this case that the interaction is a (monotonically increasing) function of energy.

3.5 Charge-monopole versus gravitational scattering

Motivated by the conclusions of the last section, we now proceed to investigate whether they undergo any modifications when we assume one of the particles to carry a magnetic charge. In other words, will gravity still dominate over electromagnetic interactions at Planckian energies? With the replacement of the electric charge \( e' \) of the fast moving particle by a magnetic charge \( g \), the fields on the electromagnetic shock front are given by equation (15). As before, when it crosses the charge \( e \), we add the gravitational and electromagnetic phase shifts in its wavefunction. While the former is still \( -Gs \ln r^2 \), the latter, as seen from equation (2.39), is now \( in\phi \). Thus, charge-monopole electromagnetic effects cannot be incorporated by a shift of \( Gs \), in contrast to the charge-charge case. Thus the wavefunction assumes the form
\[
\psi_>(x^\pm, x_\perp) = \exp [i (n\phi - Gs \ln \mu^2 r^2_\perp + ipx)]
\]

Due to the azimuthal dependence, the calculation of the overlap with momentum eigenstates has to be done ab initio. Clearly, the relevant integral for the evaluation of \( f(s, t) \) is
\[
\int d^2 r_\perp \exp[i(n\phi - Gs \ln \mu^2 r^2_\perp + \vec{q}.\vec{r}_\perp)].
\]

Once again, the integral over \( \phi \) is readily done, and the above reduces to
\[
\frac{1}{q^2} \int^\infty_0 d\rho \rho^{1-2Gs} J_n(\rho) .
\]
Here \( J_n(\rho) \) is the Bessel function of order \( n \). The above integral is again a standard one\(^4\) and finally we get the amplitude

\[
 f(s, t) = \frac{k_+}{4\pi k_0} \delta(k_+ - p_+) \left( \frac{n}{2} - iGs \right) \frac{\Gamma\left(\frac{n}{2} - iGs\right)}{\Gamma\left(\frac{n}{2} + iGs\right)} \left( \frac{4}{-t} \right)^{1-iGs} \tag{38}
\]

and hence the cross section

\[
 \frac{d^2\sigma}{dk^2_\perp} \sim \frac{1}{t^2} \left( \frac{n^2}{4} + G^2 s^2 \right) \tag{39}
\]

Since \( n \) is at least of order unity, it is clear from the above expression, that for \( \sqrt{s} \approx M_{\text{pl}} \), both the terms are of the same order of magnitude. This means that unlike charge-charge scattering, even at Planck scale gravity is no longer the dominant shock wave interaction. Electromagnetism with monopoles becomes equally important. This dramatic difference from the charge-charge case is a consequence of the Dirac quantization condition, which restricts the values of \( e \) and \( g \) from being arbitrarily small. In fact, the above may be considered to be a rephrasal of the strong coupling aspects of the monopole sector in electromagnetism and of the gravitational interactions at Planck scale. As already mentioned earlier, gravitational effects would indeed tend to dominate for \( Gs >> 1 \) if the Dirac quantum number \( n \) is held fixed. But it is far from clear if, in this circumstance, the simple-minded semiclassical analysis performed above will go through without modification. Indeed, as explained in ref.[3, super-Planckian energies will most probably entail real black hole collisions with the ensuing technical complications.

Returning once more to the analytic structure of \( f(s, t) \), we see that now they occur at

\[
 Gs = -i \left( N + \frac{n}{2} \right),
\]

that is a shift in \( s \) by half-odd integral values. Once again, the spectrum of these ‘bound states’ is no longer a perturbation on the spectrum in the pure gravity situation. More interestingly, the shift observed above due primarily to the monopoles strongly suggest another possibility: the Saha phenomenon\(^8\). Recall that, this implies that any charge-monopole pair composed of spinless particles will, as a consequence of Dirac quantization, possess half-odd integral quantized (field) angular momentum. If we blithely regard the integer \( N \), which also occurs in the spectrum of bound states in pure gravitational scattering, as the \( \text{spin} \) of the states, then it is enticing to consider the shift by one-half the Dirac quantum number \( n \) in the charge-monopole case to be the extra spin due to the field angular momentum that the system would pick up in accord with Saha’s predictions. Further, if one speculatively associates the Regge-like behavior observed in purely gravitational scattering
with the spectrum of some string theory (albeit with imaginary slope parameter), then the spectrum with charge-monopole electromagnetic scattering can as well be speculated to correspond to some supersymmetric string theory. In any event the role of electric-magnetic duality, were we to actually discern any such string structures, can hardly be over-emphasized.

4. Shock Wave Mixing

We now turn to the issue of mixing of the two species of shock waves, viz., gravitational and electromagnetic for the Planckian scattering of charged particles. The gravitational shock wave relevant to two particle scattering in Minkowski space has been obtained in one of two ways: either by demanding that the Minkowskian geometry with a lightlike particle present is the same as an empty Minkowski space with coordinates shifted along the geodesic of the particle, or by a process of ‘boosting’ the metric of a massive particle to luminal velocities when its mass exponentially decays to zero. Since there are well known (electrically and magnetically) charged black hole solutions of the Einstein equation, boosting such solutions to the velocity of light would of course produce both gravitational and electromagnetic shock waves. It is then important to determine whether these two species of shock waves actually mix. The problem may be stated succinctly as follows: the calculation of the amplitude in the shock wave picture entails computing the phase factor that the shock wave induces on the wave function of the target particle. When both particles carry charge, phase factors are induced by electromagnetism and gravity independently of the other. It has been assumed in the literature that, with both shock waves present, the net phase factor is simply the sum of the individual phase factors. In other words, the gravitational and electromagnetic shock waves are assumed to travel collinearly without interaction, even though they are extremely localized singular field configurations. We seek a possible justification of this supposition in what follows.

4.1 Decoupling in the Reissner-Nordström Case

The gravitational field due to a stationary point particle of mass $M$ and electric charge $Q$ is given by the standard Reissner-Nordström metric

$$ds^2 = (1 - \frac{2GM}{r} + \frac{GQ^2}{r^2})dt^2 - (1 - \frac{2GM}{r} + \frac{GQ^2}{r^2})^{-1}dr^2 - r^2d\Omega^2,$$

where $G$ is Newton’s constant. The question we address here is: if the particle is Lorentz-boosted to a velocity $\beta \sim 1$, what will be the nature of the gravitational field as observed in the ‘stationary’ frame? Let us assume, for simplicity and without
loss of generality, that the particle is boosted in the $+z$ direction, so that $z, t$ are related to the transformed coordinates $Z, T$ according to

\begin{align*}
T &= t \cosh \rho + z \sinh \rho, \\
Z &= t \sinh \rho + z \cosh \rho.
\end{align*}

(41)

The parameter $\rho$ is called rapidity: $\beta = \tanh \rho$. The null coordinates are defined as usual as $x^\pm = t \pm z$. The boosting involves parametrizing the mass of the black hole as $M = 2pe^{-\rho}$ where $p$ is the momentum of the boosted particle, lying almost entirely in the longitudinal direction. $p$ is usually kept fixed at a large value in the boosting process, and the limit of the boosted metric is evaluated as $\rho \to \infty$. In this limit, the Reissner-Nordström metric assumes the form

\begin{equation}
\begin{aligned}
ds^2 &\to dx^-\{dx^+ - dx^- \left[ 2Gp \left/ |x^-| \right. - \frac{GQ^2}{(x^-)^2} \right]\} - d\bar{x}_+^2.
\end{aligned}
\end{equation}

(42)

The boosted metric does indeed seem to depend explicitly on the charge $Q$. But, notice that this dependence is confined to a part of the metric that can be removed by a diffeomorphism, albeit one that is singular at the origin. However, the part that goes as $1/|x^-|$ cannot be removed by any diffeomorphism; this latter, of course, is precisely the part that is associated with the gravitational shock wave\footnote{We note here that our approach and results for the Reissner-Nordström case differ somewhat from those of ref. [15] where, in fact, the limiting procedure employed appears to yield vanishing electromagnetic shock waves in the luminal limit.}. Further, not only is its coefficient independent of $Q$, it is identical to the coefficient of the $1/|x^-|$ term in the boosted Schwarzschild metric\footnote{We note here that our approach and results for the Reissner-Nordström case differ somewhat from those of ref. [15] where, in fact, the limiting procedure employed appears to yield vanishing electromagnetic shock waves in the luminal limit.}. All memory of the charge of the parent black hole solution is obliterated upon boosting, insofar as the gravitational shock wave is concerned. Consequently, the mutual transparency of the two shock waves follows immediately.\footnote{We note here that our approach and results for the Reissner-Nordström case differ somewhat from those of ref. [15] where, in fact, the limiting procedure employed appears to yield vanishing electromagnetic shock waves in the luminal limit.} Hence the net phase shift of the wave function of a test particle moving in the two shock waves is simply the sum of the phase factors induced individually by each shock wave. This, in the case of scattering of two electric charges, simply amounts to the replacement $Gs \to Gs + ee'$ as mentioned in refs. [2,3]. A similar decoupling of electromagnetic and gravitational shock waves can be seen by boosting a magnetically charged Reissner-Nordström solution, which justifies once again the determination of the net phase factor in the test charge wave function as the sum of the individual phase factors in Planckian charge-monopole scattering\footnote{We note here that our approach and results for the Reissner-Nordström case differ somewhat from those of ref. [15] where, in fact, the limiting procedure employed appears to yield vanishing electromagnetic shock waves in the luminal limit.}.

The foregoing analysis is completely general, and requires no assumption on the strength of the charge $Q$, except perhaps that it be finite. However, an analysis of the singularities of the the metric in (40) indicates that, if one is to abide by the
dictates of Cosmic Censorship, the charge $Q$ must obey $Q \leq M$. In the extremal limit, the boosting procedure adopted above forces $Q$ to decay exponentially to zero as the rapidity runs to infinity. The electromagnetic shock wave, by withering away in this limit, then trivially decouples from the gravitational one. This was first pointed out in ref.[15].

4.2 Non-decoupling in Dilaton Gravity

The charged black hole solution of four dimensional dilaton gravity, obtained as a part of an effective low energy theory from the heterotic string compactified on some compact six-fold, is given, following [16,17], as

$$ds^2 = (1 - \frac{\alpha}{Mr})^{-1} \left[ (1 - \frac{2GM}{r})dt^2 - (1 - \frac{2GM}{r})^{-1}dr^2 - (1 - \frac{\alpha}{Mr})^2 d\Omega^2 \right],$$

(43)

where, $\alpha \equiv Q^2 e^{-2\phi_0}$, with $\phi_0$ being the asymptotic value of the dilaton field $\phi$. The metric reduces to the Schwarzschild metric when $\alpha = 0$, and, not surprisingly, shares the coordinate singularity at $r = 2GM$ which becomes the event horizon for the curvature singularity at $r = 0$. In addition, there is the ‘singularity’ at $r = \frac{\alpha}{M}$ which is not necessarily a coordinate singularity. We shall return to this point later.

We now apply the boosting procedure elaborated in the last section to this metric. The mass of the black hole is parametrized as $M = 2pe^{-\rho}$ and the Lorentz-transformed metric is evaluated in the limit as the rapidity $\rho \to \infty$ for fixed large $p$. The result can be expressed as the Minkowski metric in terms of shifted coordinate differentials,

$$ds^2 \to d\tilde{x}^+ d\tilde{x}^- - (d\tilde{x}_\perp)^2,$$

where,

$$d\tilde{x}^+ = dx^+ - \left( \frac{4Gp}{|x^-|} \right) dx^-$$

$$d\tilde{x}^- = dx^- \left( \frac{1 - \frac{\alpha}{2p|x^-|}}{1 - \frac{\alpha}{p|x^-|}} \right)$$

$$d\tilde{x}_\perp = d\tilde{x}_\perp.$$

(44)

Several features emerge immediately from these equations; of these, the most striking is the explicit dependence on the charge $\alpha$ of terms that will surely contribute to the gravitational shock wave because of their non-differentiable functional form. No less important is the fact that, in this case, the coordinate $x^-$ which, in the Schwarzschild (and Reissner-Nordström) case(s), defined the null surface ($x^- = 0$)

---

3This metric is the so-called string metric[16]. What follows is equally valid for the Einstein metric.
along which the two Minkowski spaces were to be glued, is now itself subject to transformation by such a discontinuous function, again explicitly depending on $\alpha$. Before examining these aspects in detail, we note in passing that the results reduce to those in the Schwarzschild case in the limit $\alpha = 0$, as indeed is expected.

First of all, the charge $\alpha$ may be chosen to be small by taking a large value of $\phi_0$, so that, with a large value of $p$, one can binomially expand the denominators in the rhs of the first two equations in (44); this yields, for points away from $x^- = 0$,

$$
\begin{align*}
\hat{d}x^+ &= dx^+ - \left[ \frac{4Gp}{|x^-|} + \frac{4\alpha}{(x^-)^2} \right] dx^- + O(\alpha^2/p) \\
\hat{d}x^- &= dx^- + \frac{\alpha}{2p|x^-|} dx^- + O(\alpha^2/p^2).
\end{align*}
$$

(45)

We now observe that, as far as the shift in $dx^+$ is concerned, the part that will contribute to the gravitational shock wave is in fact, independent of $\alpha$, and, furthermore, is identical to the result in the Schwarzschild and hence the Reissner-Nordström case. As for the latter solution, the $\alpha$-dependent part may be rendered innocuous by a smooth diffeomorphism.

This is however not the case for the shift in $dx^-$, which is explicitly $\alpha$-dependent. Clearly, the gravitational shock wave now possesses a more complicated geometrical structure than in the earlier examples. The geometry can no longer be expressed as two Minkowski spaces glued after a shift along the null surface $x^- = 0$, for now there is a discontinuity in the $x^-$ coordinate at that very point, in contrast to the previous cases where it was continuous. This discontinuity has a rather serious implication: unlike in the earlier situation wherein the coordinate $x^-$ could well serve as the affine parameter characterizing the null geodesic of a test particle crossing the gravitational shock wave (cf. [3]), a null geodesic is actually incomplete in this situation. To see this in more detail, consider the geodesic equations of a very light particle moving in the Lorentz-boosted metric (42),

$$
\begin{align*}
\dot{T} &= \left( \frac{1 - \frac{\alpha}{Mr}}{1 - \frac{2GM}{r}} \right) E \\
r^2\dot{\phi} &= \left( 1 - \frac{2GM}{r} \right) L \\
\dot{r}^2 &= \left( 1 - \frac{\alpha}{Mr} \right)^2 \left[ E^2 - \frac{L^2}{r^2} \left( \frac{1 - \frac{2GM}{r}}{1 - \frac{\alpha}{Mr}} \right) \right].
\end{align*}
$$

(46)

Unlike the geodesic equations for a boosted Schwarzschild metric, which can be solved perturbatively in a power series in the mass $M$ (or alternatively in the parameter $e^{-\rho}$ (where $\rho$ is the rapidity))\textsuperscript{9}, these equations do not admit any perturbative
solution because of the singularity at $r = \alpha/M$. Taking recourse to singular perturbation theory does not evade the problem; the definition of a continuous affine parameter is not possible in this case. It follows that the singularity in question must be a curvature singularity. Although for the Reissner-Nordström case also, for generic value of the charge, the singularity at $r = 0$ is no longer hidden by the event horizon, there are no other singularities away from this point. In the present instance, the singularity at $r = 0$ is actually protected by the Schwarzschild horizon. One might consider imposing an extremal condition on the charge $\alpha$ (vid. [16]): $\alpha = 2M^2$ to mitigate the circumstances. However, this limit is not interesting for our purpose, for the same reason that the extremal Reissner-Nordström is not – the charge decays exponentially to zero with the rapidity going off to infinity.

The non-decoupling of gravitational and electromagnetic effects that we see here can be made more articulate if one proceeds to actually calculate the phase shift of the wave function of a test particle encountering the gravitational shock wave, notwithstanding the pathologies delineated above. The equations (44) above for the differentials are consistent with the following finite shifts, obtained by generalizing results of ref.[9],

\begin{align}
    x^+ &= x^+ - 2Gp \ln \mu^2 r^2 \\
    x^- &= x^- + \frac{\alpha}{2p} \ln \mu^2 r^2 \\
    \vec{x}^+ &= \vec{x}^-. \tag{47}
\end{align}

With these, following [3] we can easily calculate the net phase shift of the wave function of a test particle due purely to gravitational effects:

$$
\Phi_{total} = (Gs + \frac{k^2}{2s}) \ln \mu^2 r^2. \tag{48}
$$

Here, $k_\perp$ is the transverse momentum of the test particle. Thus, even if the test particle is electromagnetically neutral, its wave function undergoes a phase shift that depends on the charge of the black hole boosted to produce the gravitational shock wave. This is a novel phenomenon, in our opinion, although, strictly speaking, in the kinematical regime under consideration, the magnitude of the effect is small. Nevertheless, the mixing of the electromagnetic and gravitational shock waves, in this case is quite obvious.

The scattering amplitude for a test particle encountering such a gravitational shock wave can be calculated following ref.[3]. Modulo standard kinematical factors and irrelevant constants, the answer is

$$
f(s, t) \sim \frac{1}{t} \frac{\Gamma \left(1 - i(Gs + \frac{k^2}{2s})\right)}{\Gamma \left(i(Gs + \frac{k^2}{2s})\right)}. \tag{49}
$$
Since the calculation is performed in a coordinate frame in which the test particle is assumed to be moving slowly, the amplitude does not appear manifestly Lorentz-invariant, although there is nowhere any violation of Lorentz invariance. The only likely outcome of such a calculation will be the replacement of the quantity \( k_\perp^2 \) by the squared momentum transfer \( t \) up to some numerical coefficient of \( \mathcal{O}(1) \). As a consequence, the poles in (49) would undergo a shift of \( \mathcal{O}(\alpha G t/N^2) \) from their integer-valued (given by \( N \)) positions on the imaginary axis found in the Schwarzschild case\(^3\). This shift is quite different from similar shifts when electromagnetic effects are included based upon a decoupling assumption\(^1\,^\cdots\,^3\). The non-decoupling is manifest from the coefficient \( \alpha G \) in this case. Also, the electromagnetic shifts are always constant independent of \( t \), in contrast to what we find here.

5. Conclusions

The predominance of shock waves as instantaneous mediators of electromagnetic and gravitational interactions at Planckian centre-of-mass energies and low fixed momentum transfers has thus been established even for situations where a reduced local field theory description is not immediately available. Charge-monopole interactions show significant departures from ordinary charge-charge interactions at Planckian energies to the extent that they begin to compete with gravitational interactions at these energies.

The decoupling of electromagnetic and gravitational shock waves have now been established for the case of general relativity, justifying thereby earlier results incorporating both fields for electrically and magnetically charged particles scattering at Planckian centre-of-mass energies. The shifts in the poles due to electromagnetic effects stand vindicated. Admittedly, in the general relativity case the poles appear to be artifacts of the large impact parameter approximation\(^1\). However, as will be reported elsewhere in the near future\(^13\), the scattering amplitude does exhibit poles even beyond the eikonal approximation. The nature of the shift due to the dilaton coupling tends to reinforce the speculation that string theory may actually provide a way to compute corrections to this approximation as a power series in \( t \).

The results may also have implications for black holes. The effect of infalling particles collapsing gravitationally onto a black hole has been analyzed\(^9\) to produce a shift of the classical event horizon. If we also subscribe to the view\(^18\) that this shift essentially involves generalizing the flat space gravitational shock wave to a curved background, then a particle whose fields are obtained by boosting fields of a dilaton black hole would cause extra shifts of the horizon of a Schwarzschild black hole. In addition, with electric and magnetic charges present, novel contributions are to be expected for any S-matrix proposal for dilatonic black holes.
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