Water Wave Modeling Using Complete Solution of Laplace Equation
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Abstract—Analytical solution of Laplace equation using variable separation method, consists of two velocity potentials. However, only one component has been used. This research used both velocity potential equation components. With the potential equation, water wave surface equation and the related wave constants were formulated using kinematic free surface boundary condition and surface momentum equation. The characteristic of water wave surface that was produced was observed, both in deep water and shallow water.

Keywords—Complete Solution of Laplace Equation, Water wave surface equation.

I. INTRODUCTION
The completion of Laplace equation using variable separation method in Dean (1991), produces two potential velocities, i.e. cosine and sine components. However, the application only exists in the cosine component in formulating various characters of water wave. This research does not discuss the use of the second component in Dean (1991), rather it studies the characteristic of water wave surface if the two velocity potential components are used simultaneously, i.e. water wave surface equation is formulated using a complete velocity potential. The formulation begins by formulating the final form of the two velocity potential components. Then, in each velocity potential, water wave surface and wave amplitude equations were formulated. Using wave amplitude equation and surface momentum equation, equations for wave number \(k\) and wave constant \(G\) were formulated.

It is obtained that the two velocity potential components have similar wave number \(k\), wave constant \(G\), but with different water wave surface equations. As the final water wave surface is the superposition or the sum of the two water wave surface equations.

The characteristic of the water wave surface equation consists of maximum wave amplitude in a wave period, wave length, correlation between wave amplitude and wave height and other produced water wave surface profile was studied.

II. TOTAL VELOCITY POTENTIAL EQUATION
The form of Laplace equation solution (Dean (1991)), after periodic boundary condition was performed against time \(t\) is

\[
\varphi(x, z, t) = A \cos kx (Ce^{kz} + De^{-kz}) \sin \sigma t + B \sin kx (Ce^{kz} + De^{-kz}) \sin \sigma t \\
\]

\(\sigma = \frac{2\pi}{T}\) is angular frequency, whereas \(T\) is wave period. Constant \(k\) was obtained by performing lateral periodic boundary condition and wave number \(k = \frac{2\pi}{L}\) was obtained where \(L\) is wave length. Therefore, \(k\) in (1) there is only one value of wave number \(k\) and one value of wavelength \(L\). Based on the linear characteristic of Laplace equation, then (1) can be written as,

\[
\varphi(x, z, t) = \Phi_A(x, z, t) + \Phi_B(x, z, t) \quad (2)
\]

Where,

\[
\Phi_A(x, z, t) = A \cos kx (Ce^{kz} + De^{-kz}) \sin \sigma t \\
\Phi_B(x, z, t) = B \sin kx (Ce^{kz} + De^{-kz}) \sin \sigma t
\]

(3) and (4), the values of constants \(A, B, C\) and \(D\) should be determined.

Equation (3) was performed at flat bottom (Dean (1991))

\[
\varphi_A(x, z, t) = G_A \cosh(k + z) \cos kx \sin \sigma t
\]

was obtained. Similar procedure was performed in (4),

\[
\varphi_B(x, z, t) = G_B \cosh(k + z) \sin kx \sin \sigma t
\]

was obtained.
As has been mentioned, the two velocity potentials have similar wave number $k$. There should have been one wave constant, i.e. $G = G_A = G_B$, but to ensure a proof will be done in the following chapters. In the previous researches Hutahaean (2019 a, b) formulated equations for wave number $k$ and wave constant $G$ using kinematic free surface boundary condition (KFSBC) and surface momentum equation. So is the case with this research, KFSBC equation and surface momentum equation will be used to formulate equations for wave number $k$ and wave constant $G$. At the same time, this research is a improvement on the procedure of KFSBC integration against time $t$, in Hutahaean (2019a,b).

III. THE FORMULATION OF WAVE NUMBER $k$ AND WAVE CONSTANT $G_A$ USING $\Phi_A$.

3.1. Water wave surface equation

The first step in formulating equation for wave number $k$ and wave constant $G$ is the formulation of water wave surface equation to obtain wave amplitude equation. The formulation was performed using KFSBC. KFSBC equation using weighted total acceleration is (Hutahaean 2019 a,b,c),

$$ \frac{\partial \eta}{\partial t} = w - u \eta \frac{\partial u}{\partial x} \quad \ldots \ldots (7) $$

Where $w$ is weighting coefficient with the value of 2.87-3.14 (Hutahaean 2019 c), $\eta = \eta(x, t)$ is water wave surface elevation against still water level ($z = 0$). $u_{p}$ is water particle velocity at horizontal-x direction at the water surface ($z = \eta$), whereas $w_{p}$ is the water particle velocity at vertical z direction at the surface water. Using (5), equations of particles velocity at horizontal and vertical directions were obtained, i.e.

$$ w(x, z, t) = -\frac{\partial \Phi_A}{\partial x} = G_A k \cosh h + z \sin k x \sin \alpha \ldots \ldots (8) $$

$$ u(x, z, t) = -\frac{\partial \Phi_A}{\partial z} = -G_A k \sinh h + z \cos k x \sin \alpha \ldots \ldots (9) $$

(8) and (9) were performed at $z = \eta$ and substituted to (7),

$$ \frac{\partial \eta}{\partial t} = -G_A k \sinh h + \eta \cos k x \sin \alpha \ldots \ldots (10) $$

Water wave surface equation was obtained by integrating (10) against time $t$. It’s visible that (10) is a non-linear function against time $t$ which is difficult to complete its integration. However, there are two arguments to make it simple, where the two arguments produce similar conclusion.

The first argument is that the velocity potential equation was obtained using variable separation method, i.e. velocity potential $\Phi$ that is regarded to have a form of $\Phi(x, z, t) = X(x) Z(z) T(t)$, where $X(x)$ is only a function of $x$, $Z(z)$ is only a function of $z$ and $T(t)$ is only a function of time $t$. In this case $Z(z) = \cosh h + z$. In relation with this, $\eta$ on the right side of the equation, both $\sinh h (h + \eta)$ or in $\cosh (h + \eta) \frac{\partial \eta}{\partial x}$ are not the function of time $t$, even though $\eta = \eta(x, t)$. Hence (10) can be written as,

$$ \frac{\partial \eta}{\partial t} = -G_A k \left( \cosh (h + \eta) \sinh (h + \eta) \frac{\partial \eta}{\partial x} \right) \sin \alpha \ldots \ldots (11) $$

The second argument is that for a periodical function against time $t$, the element

$$ \frac{\partial \eta}{\partial t} = -G_A k \left( \cosh (h + \eta) + 2 \sinh (h + \eta) \frac{\partial \eta}{\partial x} \right) \sin \alpha \ldots \ldots (12) $$

At the characteristics point, where in this research the characteristic point is a point where $\cos k x = \sin k x = \cos \alpha t = \sin \alpha t$, (12) can be written as

$$ \eta(x, t) = \frac{G_A k}{\gamma \alpha} \left( \cosh (h + \eta) + \sinh (h + \eta) \frac{\partial \eta}{\partial x} \right) \cos \alpha t \ldots \ldots (13) $$

The form $\cos k x$ was selected because the first term of the elements in the parentheses is more dominant than the second element because of the presence of $\frac{\partial \eta}{\partial x}$ element in the second term. It is defined a wave amplitude equation,

$$ A = \frac{G_A k}{\gamma \alpha} \left( \sinh (h + \eta) + \cosh (h + \eta) \frac{\partial \eta}{\partial x} \right) \ldots \ldots (14) $$

Using (14) the water wave surface equation becomes

$$ \eta(x, t) = A \cos k x \cos \alpha t $$

Water wave surface equation was obtained at the velocity potential component $\Phi_A$, i.e.
\[ \eta_{\alpha}(x, t) = A \cos kx \cos \sigma t \quad \cdots \cdots (15) \]
\[ \frac{d\eta_{\alpha}}{dx} = -k \sin kx \cos \sigma t \]

From (14),
\[ A_{\eta,1\alpha} = \frac{gA_k}{2y^2} \sinh (h + \eta_{\alpha}) \quad \cdots \cdots (16) \]
\[ A_{\eta,2\alpha} = \frac{gA_k}{y^2} \sinh (h + \eta_{\alpha}) \frac{\partial A}{\partial x} \quad \cdots \cdots (17) \]
\[ \eta_{\alpha}(x, t) = \frac{gA_k}{y^2} (A_{\eta,1\alpha} \cos kx + A_{\eta,2\alpha} \sin kx) \cos \sigma t \quad \cdots \cdots (18) \]

So, water wave surface equation consists of 4 (four) equations, i.e. (15), (16), (17) and (18), where wave amplitude in (15) is as input or known number.

3.2. Equation for \( k \) and \( G_A \).

The next step is formulating equations for wave number \( k \) and wave constant \( G_A \). The equation to calculate the two parameters can be obtained using (13) and surface momentum equation. (13) is differentiated against horizontal-x axis.
\[ \frac{d\eta}{dx} = -\frac{G_A k^2}{y^2} \]
\[ \left( \sinh (h + \eta) + \cosh (h + \eta) \frac{\partial \eta}{\partial x} \right) \sin kx \cos \sigma t \quad \cdots \cdots (19) \]

Bearing in mind that there are two variables that need to be calculated, then two equations are needed. As the second equation is surface momentum equation, where convective velocity is ignored.
\[ y \frac{d\eta}{dt} = -\frac{g\eta}{y^2} \quad \cdots \cdots (20) \]

Substitute \( \frac{d\eta}{dt} \), where \( \frac{d\eta}{dt} \) was obtained from (8), and substitute \( \frac{\partial \eta}{\partial x} \) with (19),
\[ y \sigma G_A k \cosh (h + \eta) \sin kx \cos \sigma t = \frac{gG_A k^2}{y^2} \left( \sinh (h + \eta) + \cosh (h + \eta) \frac{d\eta}{dx} \right) \sin kx \cos \sigma t \]

The equation is divided with \( G_A \cosh (h + \eta) \sin kx \cos \sigma t \) for \( \sin kx \cos \sigma t \) which is not the same with zero and remembering that in deep water \( \tanh (h + \eta) = 1 \)
\[ y^2 \sigma^2 = gk \left( 1 - \frac{\eta}{\partial x} \right) \]

Substitute \( \frac{\eta}{\partial x} \) with (19),
\[ y^2 \sigma^2 = gk \left( 1 - \frac{gA k^2}{y^2} \left( \sinh (h + \eta) + \cosh (h + \eta) \frac{\partial \eta}{\partial x} \right) \right) \]

Keeping in mind (14),
\[ y^2 \sigma^2 = gk (1 - k \sin kx \cos \sigma t) \]

The left side of the equation is constant number, therefore the right side should be constant, maximum value of \( \sin kx \cos \sigma t = 1 \) is used
\[ y^2 \sigma^2 = gk (1 - kA) \quad \cdots \cdots (21) \]

This equation is an equation to calculate wave number \( k \) in the deep water. This equation has a maximum wave amplitude value and at the same time is a critical wave steepness for a wave period, i.e. in a large wave amplitude, \( (1 - kA) = 0 \) can occur, or
\[ A_{\max} = \frac{k}{2\sigma} \quad \cdots \cdots (22) \]

The calculation of (22) can be done if wavelength \( L \) is already known. In the case that wavelength is not known, the equation for wave amplitude maximum can be obtained by bearing in mind that (23) is a quadratic equation for wave number \( k \), with a real root if the determinant is greater than zero. Wave amplitude maximum was achieved at determinant valued equal to zero, \( d = g^2 - 4gA \sigma^2 = 0 \), so obtained
\[ A_{\max} = \frac{9g}{4g^2 \sigma^2} \quad \cdots \cdots (23) \]

By equating \( A_{\max} \) with (22) and (23), critical wavelength in a wave period was obtained, i.e.
\[ L_{\min} = \frac{9g}{2g^2 \sigma^2} \quad \cdots \cdots (24) \]

(22), (23) and (24) only apply for just one component, in this case is \( \Phi_A \).

As has been stated that from the two velocity potentials \( \Phi_A \) and \( \Phi_B \), there is only one value of wave number \( k \), therefore it can be estimated that by using \( \Phi_B \) the form of wave number equation that is similar with (21) will be obtained.

As an equation for \( G_A \), surface momentum equation (22) and water wave surface equation (15) were used and were performed at characteristic point.
\[ G_A = \frac{gA}{y \sigma \cosh (h + \frac{k}{2})} \quad \cdots \cdots (25) \]

IV. THE FORMULATION OF WAVE NUMBER \( k \) AND WAVE CONSTANT \( G_B \) USING \( \Phi_B \).

4.1. Water wave surface equation.

Particle velocity equations to horizontal and vertical directions were formulated using \( \Phi_B \), in (6).
\[ u(x, z, t) = -\frac{\partial \Phi_B}{\partial x} = -G_k \cosh (h + z) \cos kx \sin \sigma t \cdots (26) \]
\[ w(x, z, t) = -\frac{\partial \Phi_B}{\partial z} \]
\[ = -G_b \sinh h (h + z) \sin k z \sin \sigma \theta \ldots(27) \]

Substitute (26) and (27) that was performed at \( z = \eta \) to (7),

\[ \frac{d \eta}{dt} = -G_b \sinh h (h + \eta) \sin k z \sin \sigma \theta \]

+ \( G_b k \cosh h (h + \eta) \cos k z \sin \theta \)

As has been performed in previous section, the right side of equation (28) can be written as,

\[ \frac{d \eta}{dt} = -G_b k \left( \sinh h (h + \eta) \sin k x \right) - \cosh h (h + \eta) \cos k x \cos \sigma \theta \]

(29)

Then, it was integrated against time \( t \).

\[ \eta(x, t) = \]

\[ \frac{G_b k}{\gamma \sigma} \left( \sinh h (h + \eta) \sin k x \right) - \cosh h (h + \eta) \cos k x \cos \sigma \theta \]

(30)

At the characteristic point, (30) can be written as

\[ \eta(x, t) = \]

\[ \frac{G_b k}{\gamma \sigma} \left( \sinh h (h + \eta) - \cosh h (h + \eta) \frac{d \eta}{dx} \right) \sin k z \cos k x \cos \sigma \theta \]

(31)

Selected to use \( \sinh k x \) because the term in the parentheses on the right side is more dominant than the second term where there is \( \frac{d \eta}{dx} \). Defined a wave amplitude equation,

\[ A = \frac{G_b k}{\gamma \sigma} \left( \sinh h (h + \eta) - \cosh h (h + \eta) \frac{d \eta}{dx} \right) \sin k z \cos k x \cos \sigma \theta \]

(32)

Water wave surface equation becomes

\[ \eta(x, t) = A \sin k z \cos k x \cos \sigma \theta \]

(33)

From (32) water wave surface equation was obtained from velocity potential component \( \Phi_b \) is

\[ \eta_{wb}(x, t) = A \sin k z \cos k x \cos \sigma \theta \]

\[ \frac{d \eta_{wb}}{dx} = k A \cos k z \cos \sigma \theta \]

(34)

\[ A_{x,1b} = \frac{G_b k}{\gamma \sigma} \sinh h (h + \eta_{0a}) \]

(35)

\[ A_{x,2b} = \frac{G_b k}{\gamma \sigma} \sinh h (h + \eta_{0a}) \frac{d \eta_{0a}}{dx} \]

Thus, water wave surface equation that was obtained with \( \Phi_b \) also consists of 4 (four) equations, i.e. (33), (34), (35) and (36), where wave amplitude in (33) is as an input or known number.

4.2. Equation for \( k \) and \( G_b \).

(31) was differentiated against horizontal-x axis,

\[ \frac{d \eta}{dx} = \frac{G_b k^2}{\gamma \sigma} \left( \sinh h (h + \eta) - \cosh h (h + \eta) \frac{d \eta}{dx} \right) \cos k x \cos \sigma \theta \]

(37)

Next, surface momentum equation was performed where convective velocity was ignored,

\[ \frac{d \eta}{dx} = -g \eta \]

(38)

\[ \eta_{0a} \]

was obtained from (26) whereas \( \frac{d \eta}{dx} \) from (37) and an equation was obtained,

\[ -\gamma \sigma G_b k \cosh h (h + \eta) \cos k x \cos \sigma \theta = -\frac{G_b k^2}{\gamma \sigma} \left( \sinh h (h + \eta) - \cosh h (h + \eta) \frac{d \eta}{dx} \right) \cos k x \cos \sigma \theta \]

The two terms of the equation were divided with \( -G_b k \cosh h (h + \eta) \cos k x \cos \sigma \theta \), and keeping in mind that in deep water \( \tanh h (h + \eta) = 1 \),

\[ \gamma \sigma \frac{d \eta}{dx} = g \left( 1 - \frac{d \eta}{dx} \right) \]

Substitute (37)

\[ \gamma \sigma \frac{d \eta}{dx} = g \left( 1 - \frac{d \eta}{dx} \right) \sinh h (h + \eta) \cos k z \cos \sigma \theta \]

(39)

Bearing in mind (32) and by taking \( \cos k x \cos \sigma \theta = 1 \),

\[ \gamma \sigma \frac{d \eta}{dx} = g k \left( 1 - \frac{d \eta}{dx} \right) \sinh h (h + \eta) \cos k z \cos \sigma \theta \]

(40)

Equation for wave number \( k \), i.e. (39) by using \( \Phi_b \) is the same as wave number equation formulated using \( \Phi_A \), i.e. (21), so it is proven that at potential velocity that is the superposition of two velocity potentials, each component has similar wave number.

Furthermore, the equation for wave constant \( G_b \) was obtained using surface momentum equation performed at characteristic point, obtained,

\[ G_b = \frac{g A}{\gamma \sigma \cosh (h + \eta/2)} \]

(40)

Compare to (25), the two velocity potentials have similar wave constant \( G \) i.e. \( G_A = G_B = G \).
V. SUMMARY

The description in Chapter III and Chapter IV shows that both velocity potentials have similar wave number equation and wave constant, so that both have similar wave number $k$ and wave constant $G$. Similarly,

1. Wave number equation from (21) and (39)

$$\gamma^2 \sigma^2 = gk - gAk^2$$

2. Wave constant $G$ equation from (25) and (40),

$$G = \frac{gA}{\gamma \sigma \cos k (h + \frac{d}{2})}$$

2. Water wave surface equation has a slightly different form.

a. Water wave surface equation of $\Phi_A$

$$\eta_A(x, t) = A \cos k \xi \cos \sigma t \quad \ldots \ldots (15)$$

$$\frac{d\eta_A}{dx} = -kA \sin k \xi \cos \sigma t$$

$$A_{R,1A} = \frac{gk}{\gamma \sigma} \sinh k(h + \eta_0) \quad \ldots \ldots (16)$$

$$A_{R,2A} = \frac{gk}{\gamma \sigma} \sinh k(h + \eta_0) \frac{d\eta_0}{dx} \quad \ldots \ldots (17)$$

$$\eta_A(x, t) = \frac{gk}{\gamma \sigma} \left(A_{R,1A} \cos k \xi + A_{R,2A} \sin k \xi \right) \cos \sigma t \ldots \ldots (18)$$

b. Water wave surface equation of $\Phi_B$

$$\eta_B(x, t) = A \sin k \xi \cos \sigma t \quad \ldots \ldots (33)$$

$$\frac{d\eta_B}{dx} = kA \cos k \xi \cos \sigma t$$

$$A_{R,1B} = \frac{gk}{\gamma \sigma} \sinh k(h + \eta_0) \quad \ldots \ldots (34)$$

$$A_{R,2B} = \frac{gk}{\gamma \sigma} \sinh k(h + \eta_0) \frac{d\eta_0}{dx} \quad \ldots \ldots (35)$$

$$\eta_B(x, t) = \frac{gk}{\gamma \sigma} \left(A_{R,1B} \sin k \xi - A_{R,2B} \cos k \xi \right) \cos \sigma t \ldots \ldots (36)$$

c. Total water wave surface equation is

$$\eta = \eta_A + \eta_B$$

VI. RESULT OF MODEL

6.1. The calculation of deep water depth

Deep water depth was obtained using the criteria

$$\tanh h_0 k_0 \left(h_0 + \frac{d_0}{2h_0}\right) = 1.$$ For a wave amplitude that is much smaller than deep water depth $h_0$, $k_0 h_0 \left(1 + \frac{d_0}{2h_0}\right) = k_0 h_0 = constant$ applies. $\tanh h_0 k_0 = 1$ can be obtained at $k_0 h_0 = a_0 \pi$ where $\tanh (a_0 \pi) = 1$. Thus, $k_0 h_0 = a_0 \pi$ and $h_0 = \frac{\pi}{k_0} \ldots \ldots (41)$

$a_0$ is a determined constant number, for example in CERC (1984) $a_0 = 1$ was used, so that $\frac{h_0}{a_0} = 0.5$ was obtained. In this research, in addition to the criteria of $\tanh (a_0 \pi) = 1$, the value $a_0$ was also determined based on other reviews.

Hutahaean (2019b) obtained that the larger the $a_0$, the larger the breaker depth and breaker height will be, so $a_0$ can be determined indiscriminately.

For a large wave amplitude, $\tanh \left(h_0 + \frac{d_0}{2h_0}\right) = 1$, where $k_0 \left(h_0 + \frac{d_0}{2h_0}\right) = a_0 \pi$, revision on $a_0$ was done against $a_0$,

$$a_0 = k_0 \left(h_0 + \frac{d_0}{2h_0}\right) \quad \ldots \ldots (42)$$

Therefore the values of wave frequency $\sigma$ and $A_0$ parameter were absorbed in the value of $a_0$. For the following calculation, $\alpha = \alpha_A$ was used. The value of $\alpha_0$ cannot be used too large, e.g. 2.25, where $tanh (2.25\pi) = 1$, but it should taking into consideration the characteristic of breaking that was produced, with the best value of $\alpha_0 = 1.6 - 1.9$. Hutahaean (2019b) obtained that with $\alpha_0 = 1.65$, breaker depth that is in accordance with CERC (1984) was obtained.

**Table 1.** Wave characteristic in several wave periods

| $T$ (sec.) | $A_{max}$ (m) | $L_0$ (m) | $h_0$ (m) | $\frac{h_0}{L_0}$ |
|-----------|---------------|-----------|-----------|-------------------|
| 6         | 0.53          | 6.9       | 6.04      | 0.88              |
| 7         | 0.72          | 9.39      | 8.22      | 0.88              |
| 8         | 0.95          | 12.26     | 10.73     | 0.88              |
| 9         | 1.2           | 15.52     | 13.58     | 0.88              |
| 10        | 1.48          | 19.16     | 16.77     | 0.88              |
| 11        | 1.79          | 23.19     | 20.29     | 0.88              |
| 12        | 2.13          | 27.6      | 24.15     | 0.88              |
| 13        | 2.5           | 32.39     | 28.34     | 0.88              |
| 14        | 2.89          | 37.56     | 32.87     | 0.88              |
| 15        | 3.32          | 43.12     | 37.73     | 0.88              |

Table (1) shows the result of the calculation of wave characteristic for several wave periods which includes deep water wave amplitude maximum $A_{max}$, deep water wave length $L_0$ and deep water depth $h_0$. The wave amplitude looks small but it will produce a large wave height, where the relation of wave height that is twice wave amplitude cannot be used. The calculation was done using the value of $\alpha_0 = 1.75$ the values of $\gamma = 2.05$, where this value was obtained with the procedure in Hutahaean (2019 c.d), whereas as wave amplitude maximum, (23) was used.
6.2. Water wave surface profile

The model was performed using wave period 8 sec., wave amplitude 0.95 m, \( \gamma = 2.05 \) and \( \alpha_0 = 1.75 \). The result of the model can be seen in Fig.1.a., Fig.1.b. and Fig.1.c.

Fig.1.a. shows that \( \eta_A \) and \( \eta_B \) (\( \eta_A \) water wave profile of \( \Phi_A \), \( \eta_B \) water wave profile of \( \Phi_B \)) have cnoidal profile, and both have similar profile size, i.e. wave crest elevation \( \eta_{\text{max}} = 1.62 \) m, whereas wave trough elevation \( \eta_{\text{min}} = -0.66 \) m, therefore wave profile is asymmetric where \( \eta_{\text{max}} \) is not the same as \( |\eta_{\text{min}}| \). Wave height \( H = 1.62 + 0.66 = 2.28 \) m, Wilson parameter value (1963), \( \frac{\eta_{\text{max}}}{H} = 0.711 \), with this parameter value the wave profile belongs to cnoidal wave profile (Table 2).

### Table 2: Wave type according to Wilson criteria (1963)

| Wave Type       | \( \frac{\eta_{\text{max}}}{H} \) |
|-----------------|-----------------------------------|
| Airy waves      | \(< 0.505 \)                      |
| Stoke’s waves   | \(< 0.635 \)                      |
| Cnoidal waves   | \( 0.635 < \frac{\eta_{\text{max}}}{H} < 1 \) |
| Solitary waves  | \( = 1 \)                        |

Furthermore, the calculation of water wave surface characteristic was performed in several wave periods with wave amplitude maximum, i.e. in equation (37), \( A_{\text{max}} = \frac{g}{4\gamma^2\sigma^2} \). The result of the calculation is presented in Table (3) and Table (4).

### Table 3: Water wave surface characteristics at wave amplitude maximum

| \( T \) (sec.) | \( \eta_{\text{min}} \) (m) | \( \eta_{\text{max}} \) (m) | \( H \) (m) | \( \frac{\eta_{\text{max}}}{H} \) |
|----------------|-----------------------------|-----------------------------|-------------|-------------------------------|
| 8              | -1.31                       | 1.25                        | 2.561       | 0.487                         |

Fig.1.b. Wave profile \( \eta = \eta_A + \eta_B \) in a wave period of 8 sec., \( A = 0.95 \) m

Fig.1.c. Wave profile in a wave period of 8 sec., \( A = 0.95 \) m

One thing that should be noticed is that there is a concavity in in the wave crest. A wave with a sharp wave crest can hardly be seen in a wave in the deep water, it always looks flat. Wave crest in Fig. 1b can be stated as flat, which is quite in accordance with the one in the nature.
In the maximum wave amplitude, the Wilson parameter value \( \frac{H_{\text{max}}}{H} = 0.487 \) for all wave period shows that the wave belongs to to Airy wave, where \( H_{\text{max}} \) is quite close with \( |\eta_{\text{min}}| \). Furthermore in Table (4), the value of \( \frac{H}{A} \) = 2.71 shows that an approach that the value of wave height \( H \) is twice the value of wave amplitude \( A \) cannot be determined or performed. Wave steepness \( \frac{H}{L} = 0.209 \), this condition exceeds the criteria of critical wave steepness from Michell (1893) with the value of \( \frac{H}{L} = 0.142 \).

Table 4: The value of \( \frac{H}{A} \) and wave steepness \( \frac{H}{L} \) in wave amplitude maximum.

| \( T \hspace{0.5cm} \text{(sec.)} \) | \( A \hspace{0.5cm} \text{(m)} \) | \( L \hspace{0.5cm} \text{(m)} \) | \( \frac{H}{A} \) | \( \frac{H}{L} \) |
|----------------|----------------|----------------|--------------|--------------|
| 8              | 0.945          | 12.265         | 2.71         | 0.209        |
| 9              | 1.196          | 15.522         | 2.71         | 0.209        |
| 10             | 1.477          | 19.163         | 2.71         | 0.209        |
| 11             | 1.787          | 23.188         | 2.71         | 0.209        |
| 12             | 2.127          | 27.595         | 2.71         | 0.209        |
| 13             | 2.496          | 32.386         | 2.71         | 0.209        |
| 14             | 2.894          | 37.56          | 2.71         | 0.209        |
| 15             | 3.323          | 43.118         | 2.71         | 0.209        |

Note: wave height can be seen in Table (3)

6.5. Water wave surface profile at breaker point
To obtain water wave surface profile at breaker point, the values of \( \Phi, k \) and \( A \) are needed at breaker point. To obtain the value of the three wave parameters, shoaling and breaking analysis was performed. The shoaling and breaking model used in this research looks similar to the one in Hutahaean (2019 b), the model that was not discussed here. Bearing in mind that the two wave potentials have similar equations for wave amplitude, wave constant and wave number, then the shoaling and breaking model will also be similar to the model in Hutahaean (2019 b) that was formulated using \( \Phi_A \).

a. Water wave surface profile
As an example of water wave surface profile at breaker point, a wave with wave period \( T = 8 \) sec., wave amplitude \( A = 0.95 \) m and bottom slope \( \frac{\partial h}{\partial x} = -0.005 \) was used. Water wave surface profile at the breaker point is presented in Fig. 2a. and Fig. 2b.

\[
\eta = \eta_A + \eta_B
\]

At the breaker point, wave profile is asymmetrical, where \( \eta_{\text{max}} = 1.90 \) m, \( \eta_{\text{min}} = -1.059 \) m, where wave height \( H = 2.958 \) m, whereas Wilson parameter \( \frac{H_{\text{max}}}{H} = 0.642 \). With this parameter value, the type of the wave is cnoidal wave type. There are two phenomena that should be paid attention to, first, the occurrence of wave setup where in the deep water the wave profile is symmetrical, whereas in the shallow water the wave trough part is lifted. The next phenomenon is the separation of a wave from the two velocity potentials that were used, where there are two wave
crest. The presence of two adjacent waves also found in the coastal water. A more vivid example is tsunami wave on the coast or land, consist of two large main wave crests. In the profile of the breaking wave, it is also visible that there is a wave trough in front of wave crest. This also occurs in tsunami, where prior to the coming of the peak of the tsunami, the coastal water recedes first.

b. Adjustment with breaker height index equation
The result of breaker height model was calibrated against the average value of 5 (five) breaker height indexes. The adjustment was performed by multiplying wave constant $G$ with 0.336. Whereas breaker depth $h_b$ was adjusted with breaker depth from SPM (1984), by changing the values of $\alpha_0$, where $\alpha_0 = 1.76$ resulted in a breaker height that fit with breaker depth from SPM (1984). The breaker height index equations used as comparators are breaker height index (BHI) equations from Komar and Gaughan (1972), Larson, M. and Kraus, N.C. (1989), Smith and Kraus (1990), Gourlay (1992) and Rattana Pitikonand Shibayama (2000), with the comparison result is presented in Table (5).

Table 5: Comparison of breaker height model with BHI

| $T$ (sec.) | $H_b$ (m) (model) | $H_b$ (m) (BHI) | $h_b$ (m) (model) | $h_b$ (m) (SPM) |
|-----------|-------------------|-----------------|------------------|-----------------|
| 8         | 2.602             | 2.955           | 3.673            | 3.697           |
| 9         | 3.293             | 3.741           | 4.65             | 4.678           |
| 10        | 4.065             | 4.617           | 5.741            | 5.776           |
| 11        | 4.919             | 5.586           | 6.948            | 6.989           |
| 12        | 5.854             | 6.644           | 8.269            | 8.317           |
| 13        | 6.87              | 7.797           | 9.705            | 9.761           |
| 14        | 7.968             | 9.041           | 11.256           | 11.321          |
| 15        | 9.146             | 10.378          | 12.921           | 12.996          |

Note :BHI is the average value of breaker height from 5 (five) breaker height index equations.

As has been stated that the adjustment of breaker height was performed by multiplying wave constant $G$ at breaker point with a coefficient of 0.477.

VII. CONCLUSION
Both components of velocity potential equation as the solution of Laplace equation have similar wave number and wave constant, so that both can be performed as a unity to model water wave mechanics.

Water wave surface equation from each velocity potential component has different form, where the total of water wave surface equation is the sum of the two water wave surface equations. However, as has been stated that both have similar wave amplitude value and equation, wave number and wave constant $G$. Both produced similar wave profile. Therefore, both water wave surface equation are actually identical. Wave separation in the shallow water, also occurs in the nature, shows that the two velocity potentials should have been used. In addition, water wave surface resultant have different wave height with each component of water wave surface. This also strengthens that the two velocity potential components of Laplace equation should have been used all simultaneously.

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