Algebraic solution for the vector potential in the Dirac equation

H S Booth*, G Legg, P D Jarvis

School of Mathematics and Physics, University of Tasmania
GPO Box 252-21, Hobart Tas 7001, Australia

Abstract

The Dirac equation for an electron in an external electromagnetic field can be regarded as a singular set of linear equations for the vector potential. Radford’s method of algebraically solving for the vector potential is reviewed, with attention to the additional constraints arising from non-maximality of the rank. The extension of the method to general spacetimes is illustrated by examples in diverse dimensions with both c- and a-number wavefunctions.

*Centre for Mathematics and its Applications, Australian National University. hbooth@wintermute.anu.edu.au
1 Introduction

The Maxwell-Dirac equations are the coupled nonlinear partial differential equations which describe a classical electron interacting with an electromagnetic field. They are also the equations from which quantum electrodynamics is derived. Since the mathematical foundations of the latter remain unclear, the Maxwell-Dirac equations continue to be of interest [1, 2, 3]. Recently Radford [4] handled the Maxwell-Dirac equations firstly by solving the Dirac equation for the electromagnetic potential in terms of the wavefunction and its derivatives, and then substituting this solution in the Maxwell equations. This approach subsequently led to some physically interesting results [5, 6, 7] (for a review see [8]).

Despite the viability and potential importance of Radford’s algebraic solution, at least for the treatment of the equations of classical electrodynamics for electrons and photons, it appears that the method has not before appeared in this context. One analysis which reached negative conclusions about the approach, and which may have engendered the lack of attention to it in the literature, is that of Eliezer [9]. In that paper, it was noted that the determinant of the matrix of coefficients of the vector potential $A_\mu$ in the Dirac equation actually vanishes, and that therefore a unique algebraic inversion was not possible. The aim of the present paper is to reconcile [4] and [9], and to emphasise the legitimacy of the algebraic Ansatz, despite the negative conclusions of [9], by a careful analysis of the nature of the Dirac equation regarded as a linear system [10] for $A_\mu$. The main result is that the Dirac equation is indeed invertible if a real solution for the vector potential is required, and moreover that the treatment entrains an additional set of polynomial constraints on the wavefunction and its partial derivatives which must be carried forward in any further analysis. In §2 below, the abstract formalism is developed, and (for the 4 dimensional case) it is shown how the explicit manipulations, which rely on the structure of the Dirac algebra to derive the solution for the vector potential and the additional constraints, conform to the general setting (it is also pointed out that the solution can be regarded as including the mass, or more generally a Lorentz scalar potential, as a fifth unknown). This is done both in Lorentz-covariant Dirac spinor notation, and in van der Waerden 2-spinor notation. In §3, the case of arbitrary (flat) spacetimes with signature $(t,s)$ is taken up. Known results on the structure of the Dirac algebra (formally, the Clifford algebra $C(t,s)$) in these cases are used to give an enumeration of constraints which are quadratic in the wavefunction and derivatives (in addition to current and partial axial current conservation, which hold in all cases). The four-dimensional results are recovered, and generalised to the case of $a$-number as well as $c$-number wavefunctions. A major outcome is a tabulation (table 2) of such constraints as to fermion wavefunction statistics and metric signature in diverse dimensions. Concluding remarks and prospects for further development of the work are given in §4 below.
2 The 4 dimensional Dirac equation

8 × 4 real system

The Dirac equation for a fermion of charge $q$ described by the spinor wavefunction $\psi$ in the presence of an external electromagnetic potential may be written

$$ q \gamma^\mu \psi A_\mu = (i \gamma^\mu \partial_\mu \psi - m \psi). \quad (1) $$

Following Eliezer[9], we write this as a matrix equation for $A_\mu$;

$$ M^\mu_\alpha A_\mu = Z_\alpha \quad (2) $$

where $M^\mu_\alpha \equiv \gamma^\mu_\alpha \beta \psi_\beta$ for $M \in M_4(\mathbb{C})$, $M : \mathbb{C}^4 \rightarrow \mathbb{C}^4$ and $A, Z \in \mathbb{C}^4$.

In [9] it was noted that $M$ had rank 3 and determinant zero, with a rank 1 right null space, and therefore could not be inverted to obtain a unique solution for the potential. Yet Radford[4] did just this, albeit in the bispinor representation. That work exploited the fact that $A_\mu$ is real[2], which was not used in [9]. The point is that despite the zero determinant, (2) can be inverted if we know that $A_\mu$ is real, and providing that the intersection of the right null space of $M$ with $\mathbb{R}^4$ (as a subspace of $\mathbb{C}^4$) is trivial. Even though the columns of $M$ are not linearly independent as a vector space over $\mathbb{C}$, they are in general linearly independent as a vector space over $\mathbb{R}$.

We may break (2) into real and imaginary parts, yielding a system of 8 real equations in 4 real unknowns, schematically $\mathcal{M}A = \mathcal{Z}$, where

$$ \mathcal{M} = \begin{pmatrix} \frac{1}{2}(M + M^*) \\ \frac{1}{2i}(M - M^*) \end{pmatrix}, \quad \mathcal{Z} = \begin{pmatrix} \frac{1}{2}(Z + Z^*) \\ \frac{1}{2i}(Z - Z^*) \end{pmatrix}. \quad (3) $$

$\mathcal{M}$ is not square; no determinant is defined; yet there are other tests for linear independence and invertibility[10]. To invert a system of $m$ equations in $n$ unknowns, with $m > n$, of the form (3), then we seek an $n \times m$ matrix $G$ such that $G\mathcal{M} = 1$; the unit $n \times n$ matrix, and then $A = G\mathcal{Z}$. If such a solution exists, the rank of $\mathcal{M}$ is $n$ (= 4).

Note that the multiplication of a row of $G$ with a column of $\mathcal{M}$ is actually a Euclidean real inner product. If the columns of $\mathcal{M}$ are understood to be spinors over an 8 dimensional real basis, we can accept the same interpretation for the rows of $G$. The existence of an 8 dimensional real basis thus supplies us with a definition of a real inner product between spinors, for which we will use the notation $(\phi \bullet \psi)$. It is easy to verify that $(\phi \bullet \psi)$ is actually equal to the real part of the standard complex inner product:

$$ (\phi \bullet \chi) = \Re <\phi, \chi> = \frac{1}{2} (\phi^\dagger \chi + \psi^\dagger \chi). \quad (4) $$

---

1 In this section standard Cartesian coordinates $x^\mu$, $\mu = 0, 1, 2, 3$ for four dimensional Minkowski space with $(1, 3)$ metric $(\eta_{\mu\nu})_{\mu,\nu=0,1,2,3} = \text{diag}(+,-,-,-)$ are introduced. Affixes for Dirac spinors are introduced as $\psi_\alpha$, $\alpha = 1, 2, 3, 4$, while the Dirac matrices (generators of the Clifford algebra $\mathbb{C}(1, 3)$ in the standard basis) satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$; for conventions see 11]. In §3 below, the notation is generalised to dimension $d = t + s$ with metric signature $(t, s)$.

2 The most telling use of the reality of $A_\mu$ is implicit in the bispinor representation, where half the equations are conjugated, taking $A_\mu$ to be real. This results in systems of equations where the matrix operating on $A_\mu$ can have non-zero determinant 6]. This transformation cannot be performed with a matrix transformation on $\mathbb{C}^4$, and does not preserve the determinant.
The system of $m$ equations in $n$ unknowns entails[10] that the right hand side of the equation should satisfy $m - n$ ($8 - 4 = 4$) additional consistency conditions, arising from the fact that $\mathcal{Z}$ must fall in the column space of $\mathcal{M}$. To find these consistency conditions, we seek a further $m - n$ linearly independent spinor rows $\chi$ that have zero real inner product with the columns of $\mathcal{M}$ ($\chi$ span the left null space of $\mathcal{M}$). The consistency conditions may then be written $(\chi \bullet Z) = 0$. $\mathcal{G}$ is not unique, for any linear combination of the rows $\chi$ can be added without changing its effect on $\mathcal{M}$.

It is not necessary to work explicity in 8 real components: regardless of which basis we use, the columns of $\mathcal{M}$ and the rows of $\mathcal{G}$ are just spinors in a vector space over $\mathbb{R}$, and the matrix multiplication is just the calculation of inner products using (4). All that we require for the inversion is to find spinors $\phi_{\nu}$ where $(\phi_{\nu} \bullet \gamma^\mu \psi) \propto \delta_{\nu}^\mu$. For the consistency conditions, we require 4 linearly independent spinors $\chi$ such that $(\chi \bullet \gamma^\mu \psi) = 0$. As will now be shown, the structure of the Dirac algebra indeed admits such rows, $\phi_{\nu}$ and $\chi$.

**Inversion**

Let $\phi_{\nu} = \gamma^0 \gamma_{\nu} \psi$. Then

$$(\phi_{\nu} \bullet \gamma^\mu \psi) = \frac{1}{2} (\psi^\dagger \gamma^\dagger_{\nu} \gamma^0 \gamma^\mu \psi + \psi^\dagger \gamma^\mu \gamma^0 \gamma_{\nu} \psi).$$

We use the hermiticity of $\gamma^0$ and $\gamma^0 \gamma^\mu$:

$$\gamma^\mu \gamma^0 = \gamma^\mu \gamma^0 = (\gamma^0 \gamma^\mu)^\dagger = \gamma^0 \gamma^\mu,$$

and likewise for $\gamma_{\nu}$. Then

$$(\phi_{\nu} \bullet \gamma^\mu \psi) = \frac{1}{2} (\psi^\dagger \gamma^0 (\gamma_{\nu} \gamma^\mu + \gamma^\mu \gamma_{\nu}) \psi) = \delta_{\nu}^\mu \overline{\psi} \psi$$

where the Dirac conjugate $\overline{\psi}$ is defined in the usual way[3] as $\overline{\psi} = \psi^\dagger \gamma^0$.

Applying the real inner product with the same rows to the right-hand side of the Dirac equation (1), gives explicitly

$$\frac{1}{2} (i \psi^\dagger \gamma^0 \gamma^\nu \gamma^\mu \partial_\mu \psi - i \partial_\mu \psi^\dagger \gamma^0 \gamma^\mu \gamma^\nu \psi - 2m \psi^\dagger \gamma^0 \gamma^\nu \psi).$$

The last term is identified with the current $j^\nu \equiv \overline{\psi} \gamma^\nu \psi$, so we can write the solution for the vector potential

$$A_\mu = \frac{1}{2q} \frac{i (\psi \gamma_\mu \overline{\psi} - \overline{\psi} \overline{\gamma_\mu \psi}) - 2m j^\mu}{\psi \psi}.$$  

---

3 General properties and nomenclature for the Dirac algebra in generic dimensions and spacetime signatures (including four dimensional Minkowski space) are given in §3 below.
Consistency Conditions

It is possible to show by the use of (4) two separate sufficient conditions for spinors \( \chi \) to have zero real inner product with \( \gamma^\mu \psi \):

1. \( \chi = \Gamma \psi \), where \( \Gamma \) is a matrix in the Dirac algebra such that \( \Gamma^\dagger \gamma^\mu \) is antihermitean; Alternatively,

2. \( \chi = \Upsilon \psi^* \) where \( \Upsilon \) is a matrix in the Dirac algebra such that \( \Upsilon^\dagger \gamma^\mu \) is antisymmetric.

As an example of (1), take \( \chi = i \gamma^0 \psi \). Both the left hand side as well as the mass term of (1) vanish (as \( i \gamma^0 \) is itself antihermitean), leaving

\[
0 = -\psi^\dagger \gamma^0 \gamma^\mu \partial^\mu \psi - \partial^\mu \psi^\dagger (\gamma^0 \gamma^\mu)^\dagger \psi.
\]

This is the normal current conservation equation,

\[
\partial \cdot j \equiv \partial^\mu j_\mu = \partial^\mu \psi^\dagger \gamma^0 \gamma^\mu \psi = 0. \tag{6}
\]

Also satisfying condition (1), take \( \chi = i \gamma^0 \gamma_5 \psi \), using the antihermiticity of \( i \gamma^0 \gamma_5 \gamma^\mu \). The hermiticity of \( i \gamma_5 \gamma^0 \) now ensures that the mass term survives, and manipulations on the right hand side of (1) lead in a similar way to the equation for partial conservation of axial current \( j_5^\nu \equiv \psi \gamma_5 \gamma^\nu \psi \) as the second consistency condition:

\[
\partial \cdot j_5 + 2im \overline{\psi} \gamma_5 \psi = 0. \tag{7}
\]

As an example of the sufficient condition (2), take \( \chi = \gamma_5 \psi^* \) and \( \chi = i \gamma_5 \psi^* \) respectively, where \( \psi^* \) is the charge conjugation matrix. We evaluate these inner products with \( \gamma^\mu \psi \) using the hermitean conjugate \((\gamma_5 \psi^*)^\dagger = \psi^\dagger (\gamma_5 \psi)^\dagger = -\psi^\dagger C \gamma_5 \). These inner products are the real and imaginary parts of \(-\psi^\dagger C \gamma_5 \gamma^\mu \psi \) respectively, which is zero by the antisymmetry of \( C \gamma_5 \gamma^\mu \). Applying the same row operation(s) to the right hand side gives \( 0 = \psi^\dagger C \gamma_5 (i \gamma^\mu \partial^\mu \psi - m \psi) \) or

\[
\psi^\dagger C \gamma_5 \partial \psi = 0 \tag{8}
\]

(after using the antisymmetry of \( C \gamma_5 \) itself to eliminate the mass term) yielding one complex condition, or two real conditions on the spinor. This is the result previously reported by Eliezer [9] (who attributed to Dirac the antisymmetry argument using \( C \gamma_5 = \alpha_x \alpha_z \) in the standard representation). The consistency conditions (1), (7), and (8) are equivalent to Radford’s [4] ‘reality’ conditions.

Alternative inversion

As mentioned above, the choice of nonsingular matrix inverting \( M \), and consequently the form of the final expression for \( A \), is not unique. As an alternative choose \( \phi_\mu = i \gamma^0 \gamma_5 \gamma^\mu \psi \). We then find by a similar working to (4), using the anticommuting property of \( \gamma_5 \) with \( \gamma^\mu \), that

\[
(i \gamma^0 \gamma_5 \gamma_\nu \psi \cdot \gamma^\mu \psi) = -\delta^\mu_\nu \overline{\psi} \gamma_5 \psi.
\]
Applying the inner product with the same rows to the right hand side of the Dirac equation (1), in this case the mass term vanishes, yielding an alternative solution for the vector potential:

$$A_\mu = \frac{i}{2q} \overline{\psi} \gamma_5 \gamma_\mu \partial_\mu \psi - \overline{\psi} \gamma_5 \gamma_\mu \frac{\partial}{\gamma_\mu \psi}.$$  \hspace{1cm} (9)

That (5) and (9) are indeed equivalent, and equivalent to [4], follows from the use of Fierz identities together with the auxiliary constraints (see §3 below).

**8 × 5 real system**

The inversion (5) does not contain any mass term. However, note that the pseudoscalar consistency condition (7) can be written

$$m = \frac{i}{2} \overline{\psi} \gamma_5 \partial_\mu \psi + \overline{\psi} \gamma_5 \partial_\mu \psi.$$ \hspace{1cm} (10)

The similarity between (9) and (10) suggests that the original system could have been considered as 8 real equations in 5 unknowns, $qA_0, \ldots , qA_3$, and $m$ (or more generally a Lorentz scalar potential). In this system, (9) and (10) provide an inversion, while (8) and the real and imaginary parts of (8) provide the 3 consistency conditions.

**2-spinor analysis**

Radford[4] and Booth and Radford[5] used van der Waerden notation in order to derive a complex form of the vector potential subject to additional reality conditions. Here the 2 spinor version is reached via the Weyl representation of the Dirac algebra (see for example [11]), wherein

$$\psi_\alpha = \begin{pmatrix} u_a \\ \bar{u}^a \end{pmatrix}, \quad \psi^c_\alpha = \begin{pmatrix} v_a \\ \bar{v}^a \end{pmatrix}, \quad \bar{\psi}^\alpha = - \begin{pmatrix} \bar{v}^a \\ u_a \end{pmatrix}.$$  

A generic matrix $\Gamma$ in the Dirac algebra has matrix elements

$$\Gamma_{\alpha}^{\beta} = \begin{pmatrix} \Gamma^{a}_{b} & \Gamma^{\dot{a}b} \\ \Gamma^{\dot{a}b} & \Gamma^{\dot{a}b} \end{pmatrix};$$

in particular,

$$\gamma_{\mu}^{\alpha} {}_{\beta} = - \begin{pmatrix} 0 & \bar{\sigma}^\mu_{ab} \\ \sigma_{\mu ab} & 0 \end{pmatrix},$$

where

$$(\sigma^\mu)_{0 \leq \mu \leq 3} = (\sigma^0, \sigma), \quad (\bar{\sigma}^\mu)_{0 \leq \mu \leq 3} = (\sigma^0, -\sigma).$$ \hspace{1cm} (11)

The Pauli matrices are

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
The definitions (11) are consistent with
\[
C_{\alpha\beta} = - \begin{pmatrix} \varepsilon_{ab} & 0 \\ 0 & \varepsilon_{\dot{a}\dot{b}} \end{pmatrix}, \quad C^{\alpha\beta} = - \begin{pmatrix} \varepsilon^{ab} & 0 \\ 0 & \varepsilon^{\dot{a}\dot{b}} \end{pmatrix}
\]

together with \( \varepsilon = i\sigma^2 \), that is, component-wise,
\[
\varepsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon^{ab} = -\varepsilon_{ab}, \quad \varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{\dot{a}\dot{b}}.
\]

Starting then from
\[
q\gamma^\mu A_\mu \psi = (i\gamma^\mu \partial_\mu - m)\psi
\]
and transcribing to 2-spinor form, the Dirac equation reads directly
\[
q\bar{A}_{\alpha\beta} \bar{v}^\beta = i\bar{\sigma}_{\alpha\beta} \bar{v}^\beta + mu, \quad qA^{\alpha\beta} u_\beta = i\sigma^{\alpha\beta} u_\beta + m\bar{v}^\alpha
\]
where\(^5\)
\[
\bar{A}_{\alpha\beta} \equiv -\bar{\sigma}^{\mu\alpha}_{\beta} A_\mu, \quad A^{\alpha\beta} = \sigma^{\mu}_{\alpha\beta} A_\mu.
\]

Finally taking complex conjugates,
\[
U : \quad qA^{d\bar{c}} \bar{v}_d = -i\sigma^{d\bar{c}} \bar{v}_d + mu^c,
\]
\[
V : \quad qA^{\bar{a}\bar{b}} u_\bar{b} = i\sigma^{\bar{a}\bar{b}} u_\bar{b} + m\bar{v}^\bar{a},
\]
\[
\bar{U} : \quad qA^{\bar{c}\bar{d}} v_\bar{d} = -i\sigma^{\bar{c}\bar{d}} v_\bar{d} + m\bar{v}^{\bar{c}},
\]
\[
V : \quad qA^{ba} \bar{u}_b = -i\sigma^{ba} \bar{u}_b + mv^a.
\]

Thus by taking combinations of the form \( \alpha(\bar{V}^b u^b - \bar{U}^{\bar{a}} u^\bar{a}) \), \( \beta(U^b \bar{v}^\bar{a} - V^{\bar{a}} \bar{v}^\bar{a}) \) and using \( u_d v^b - v_d u^b = \delta^b_d (u_v v^c) \) the vector potential can be isolated, with general 'solution'
\[
A^{\bar{c}\bar{d}} = -\frac{i}{q} \left[ \alpha(u^d \sigma^{d\bar{c}} v_c + u^d \sigma^{d\bar{c}} v_c) - \beta(v^d \sigma^{d\bar{c}} \bar{v}_c - \bar{v}^d \sigma^{d\bar{c}} \bar{v}_c) - 2im(\alpha u^d \bar{v}^\bar{c} + \beta v^d \bar{v}^\bar{c}) \right]. \quad (12)
\]

with arbitrary parameters \( \alpha, \beta \). As emphasised by Radford\(^4\), all these forms are equivalent, subject to the hermiticity conditions satisfied by the potential itself. The latter can be imposed via the two-spinor projections of \( A^{\alpha\beta} \), namely
\[
(A^{\alpha\beta} \bar{u}_\alpha u_\beta)^* = (A^{\dot{a}\dot{b}} \bar{u}_\dot{a} u_\dot{b}),
\]
\[
(A^{\alpha\beta} \bar{v}_\bar{a} v_\bar{b})^* = (A^{\dot{a}\dot{b}} \bar{v}_\dot{a} v_\dot{b}),
\]
\[
(A^{\alpha\beta} \bar{u}_\dot{a} v_\bar{b})^* = (A^{\dot{a}\dot{b}} \bar{u}_\dot{a} v_\dot{b}). \quad (13)
\]

Substitution of these conditions into a suitable form of (12), for example with \( \alpha = 1, \beta = 0 \), leads to
\[
\partial_\mu (v^a \sigma^{a\mu}_{\alpha\beta} \bar{v}^\beta + \bar{u}_\alpha \sigma^{\alpha\beta}_{\mu a} u_\beta) = 0,
\]
\[
\partial_\mu (v^a \sigma^{a\mu}_{\alpha\beta} \bar{v}^\beta + \bar{u}^\beta \sigma^{\alpha\beta}_{\mu a} u_\beta) = 2im(v^a u_a - \bar{u}_\bar{a} \bar{v}^\bar{a}),
\]
\[
u^a \sigma^{a\mu}_{\alpha\beta} \partial_\mu \bar{v}^\beta = \bar{u}_\alpha \sigma^{\alpha\beta}_{\mu a} \partial_\mu u_\beta. \quad (14)
\]

\(^5\)Hermiticity, and raising and lowering of indices are entailed in the relations \( \bar{\sigma}^{\mu}_{\alpha\beta} = \varepsilon_{ab} \varepsilon_{\alpha\dot{b}} \sigma^{\mu}_{ab} \), \( (\sigma^{\mu}_{\alpha\beta})^* = \sigma^{\mu}_{\beta\alpha} \).
Referring (12) to a fixed tetrad basis, $\bar{u}_a u_b$, $\bar{v}_a v_b$, $\bar{u}_a v_b$ and $\bar{v}_a u_b$ via appropriate contractions and using (14) then shows directly that all forms are equivalent to the manifestly hermitean version with $\alpha = \beta = 1$. As expected from the general analysis in the previous subsection, (12) with $\alpha = \beta = 1$ agrees with the previous result (5) expressed in the Weyl basis, and (14) are equivalent to current and partial axial current conservation and the additional complex pseudoscalar identity (8) satisfied by the Dirac wavefunction.

3 Higher dimensional extensions

In the previous section it was shown that the 4 dimensional Dirac equation can be regarded as a singular set of real linear equations for the vector potential $A_\mu$. Gaussian elimination in this $8 \times 4$ real system then leads to a solution for $A_\mu$ and also implies a set of four additional linearly independent constraints, linear in $\partial_\mu \psi$, which can be identified in this case with bilinear identities proposed by Radford[4] and Booth and Radford[5] using van der Waerden 2-spinor notation. Either approach ultimately derives from the structure of the Dirac algebra and the symmetry properties of the $\gamma$ matrices. The same analysis is extended here in Dirac notation to higher dimensional cases and different metric signatures (in flat spacetime). Also, an important distinction to make is that between c-number and a-number (or Grassmann-valued) Dirac spinors. For the latter, a Gaussian elimination argument requires a formal treatment of linear algebra over Grassmann-extended ground fields[12]; for present purposes it suffices to assume that the count of solutions and constraints goes through.

The explicit construction of the Dirac algebra in higher dimensions has been reviewed by Tanii[13]; see also [14, 15, 16]. The Dirac matrices satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu},$$

(15)

where the spacetime metric signature is taken as $(t,s)$ for even dimensions $t+s = d$, indices being labelled $\mu, \nu = 0, 0', 0''$, \ldots; 1, 2, \ldots, $s$, with chirality determined by the equivalent of $\gamma_5$, $\hat{\gamma} \equiv \gamma_0 \gamma_0' \cdots \gamma_1 \gamma_2 \cdots \gamma_d$.

There are three involutive automorphisms associated respectively with complex conjugation, transposition and hermitean conjugation under which the $\gamma$ matrices acting on complex spinors $\psi_\alpha$ undergo similarity transformations:

$$\gamma_\mu^\beta_\alpha = \delta_A A_{\alpha\alpha'} \gamma_\mu^{\alpha'}_{\beta'} (A^{-1})^{\beta\beta}$$

(16)

$$\gamma_\mu^\beta_\alpha = \delta_B B_{\alpha\alpha'} \gamma_\mu^{\alpha'}_{\beta'} (B^{-1})^{\beta\beta}$$

(17)

$$\gamma_\mu^\beta_\alpha = \delta_C C_{\alpha\alpha'} \gamma_\mu^{\alpha'}_{\beta'} (C^{-1})^{\beta\beta}$$

(18)

where $\delta_A, \delta_B, \delta_C$ are sign factors depending on the spacetime:

$$\delta_A = \delta_B \delta_C;$$

- Take the sum and difference of (14a) and (14b)

7 The notation $\gamma_\mu^{(a)}, \gamma_\mu^{(b)}, \gamma_\mu^{(c)}$ is used for the expressions on the right hand sides of (18) (without the sign factors), extended to arbitrary elements of the Dirac algebra $\Gamma$ (see below). Note that in the previous section the index conventions for the $A$ and $C$ matrices differ from that used here in the general case.
Table 1: The values of $\epsilon_B$ as a function of $\delta_B = \pm 1$ and $s - t \pmod{8}$ (from \[13\]; $\times$ indicates that no representation with the specified signs exists).

| $s - t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---------|---|---|---|---|---|---|---|---|
| $\epsilon_+$ | $+1$ | $\times$ | $-1$ | $-1$ | $\times$ | $+1$ | $+1$ |
| $\epsilon_-$ | $+1$ | $+1$ | $\times$ | $-1$ | $-1$ | $\times$ |

$$\delta_B = \pm 1; \quad \delta_C = \delta_B(-1)^{t+1}.$$  

$A$, $B$ and $C$ are related through the definitions of Dirac and charge conjugation for spinors,

$$\psi_{\alpha} = B_{\alpha}^{\beta} \psi_{\beta}, \quad \overline{\psi}^{\alpha} = \psi^{* \beta}(A^{-1})^{\beta \alpha}, \quad \psi^{\alpha} = C_{\alpha \beta} \overline{\psi}^{\beta},$$

so that

$$B_{\alpha}^{\beta} = C_{\alpha \beta} (A^{-1})^{\beta \beta'}.$$  

The fundamental identity $B^* B = \epsilon_B I$ determines the existence of Majorana spinors for metrics in which $\epsilon_B = +1$, and $\delta_B = -1$. Dimensions for which this is possible can be read from table 1 which gives the values of $\epsilon_b = \sqrt{2} \cos[\frac{1}{8} \pi (s - t - \delta_B)]$. Finally, the implementation of parity in the Dirac algebra is determined by the symmetry of $C$, $C^T = \epsilon_C C, \epsilon_C = (\delta_B)^t (-1)^{t(t-1)} \epsilon_B$.

Consider now the Dirac equation for $\psi$ and conjugate forms:

$$qA \psi = (i\partial - m)\psi, \quad qA \psi^c = (-i\partial - \delta_B m)\psi^c, \quad \overline{qA} = \overline{\psi}(-i\partial - \delta_A m), \quad \overline{qA} = \overline{\psi}^c(i\partial - \delta_C m)$$

(19)

where the sign factors follow from the explicit definitions in (18). Write (19) as $\Psi$, $\overline{\Psi}$, $\Psi^c$ and $\overline{\Psi}^c$ respectively. Then, as in the four dimensional case, the solution for the potential follows by taking combinations $\overline{\psi} \gamma_{\mu} \Psi + \overline{\Psi} \gamma_{\mu} \psi$ which force the isolation of $A_{\mu}$ through an anticommutator of $\gamma$ matrices:

$$A_{\mu} = \frac{1}{2q} \frac{i(\overline{\psi} \gamma_{\mu} \overline{\psi} - \overline{\psi} \gamma_{\mu} \psi)}{\overline{\psi} \psi} - 2s_A \overline{m} \overline{\psi} \gamma_{\mu} \psi.$$  

(20)

where $s_A = \frac{1}{2}(1 + \delta_A) \equiv \frac{1}{2}(1 - (-1)^t)$ (compare (3)).

The identification of additional identities satisfied by the Dirac wavefunction given (19), (20) amounts to determining the structure of a certain ideal in the free algebra of rational expressions in $\psi$, $\psi^*$ and partial derivatives $\partial^\mu \psi$, $\partial^\mu \psi^*$, for appropriately smooth wavefunctions. Such expressions do not necessarily form linear representations of the
spacetime Lorentz symmetry group \( SO(t, s) \), and the distinguished role played by the constraints following from Gaussian elimination is not clear. Here, the quadratic case is analysed, by analogy with the four-dimensional case, and in relation to the counting suggested by dimensional considerations. Polynomials in \( \psi, \bar{\psi} \) and derivatives do decompose with respect to the Lorentz algebra. Moreover, there is a natural bi-grading by degree: in the quadratic case, this is simply by fermion number \( F \). Thus the cases \( \bar{\psi} \cdot \psi \), with \( F = 0 \), and \( \psi \cdot \bar{\psi} \), or equivalently \( \bar{\psi}^c \cdot \psi \), with \( F = 2 \), can be considered (as the latter will necessarily be complex, constraints in \( \bar{\psi} \cdot \psi \) do not require separate treatment). A case by case analysis follows.

\( F = \pm 2 \):

As is well known, products of two spinors admit a decomposition into antisymmetric tensor representations of the Lorentz group[17]. Noting that

\[
\bar{\psi}^c \Gamma \psi \equiv \epsilon_B (C^{-1} \Gamma)^{\alpha\beta} \psi_\alpha \psi_\beta,
\]

it is clear that an explicit decomposition is provided by the linear basis for the Dirac algebra comprising the antisymmetric \( p \)-fold products of \( \gamma \) matrices[13, 18] \( \{ \gamma_\mu, \gamma_{\mu\nu}, \ldots, \gamma_{\mu_1\mu_2...\mu_p}, \ldots \} \), \( p = 1, \ldots, d \). Furthermore, for appropriate \( p \), depending on the spacetime signature, dimension and statistics of the spinor wavefunctions, (21) is identically zero, leading after contraction with \( A_\mu \) to a sequence of differential identities in the Dirac wavefunction, after implementation of (19, 20). If the statistics of the Dirac wavefunction is specified as \( \psi_\alpha \psi_\beta = f \psi^\beta \psi^\alpha \), then using

\[
\bar{\psi}^c \Gamma \psi = \epsilon_C f \bar{\psi}^c \Gamma^{(c)} \psi,
\]

and

\[
(\gamma_{\mu_1\mu_2...\mu_p})^{(c)} = \delta^{p+1}_C \gamma_{\mu_p...\mu_2\mu_1\mu} = \delta^{p+1}_C (-1)^{\frac{1}{2}(p+1)p} \gamma_{\mu_1\mu_2...\mu_p},
\]

we require

\[
\epsilon_C f \delta^{p+1}_C (-1)^{\frac{1}{2}(p+1)p} = -1.
\]

Note also[18]

\[
\gamma_\mu \gamma_{\mu_1\mu_2...\mu_p} = (-1)^p \gamma_{\mu_1\mu_2...\mu_p} \gamma_\mu + 2 p \eta_{[\mu_1} \gamma_{\mu_2...\mu_p]} = -(-1)^p \gamma_{\mu_1\mu_2...\mu_p} \gamma_\mu + \frac{2}{(p+1)!} \gamma_{\mu_1\mu_2...\mu_p}.
\]

**Theorem 1** For spacetime dimension \( d = 4k \), a single Lorentz scalar or pseudoscalar bilinear complex differential constraint of the form

\[
\bar{\psi}^c A \psi = 0 \quad \text{or} \quad \bar{\psi}^c \gamma A \psi = 0
\]

exists, whenever \( \epsilon_C \delta_C f = \mp 1 \), respectively. For spacetimes of dimension \( d = 4k + 2 \), both scalar and pseudoscalar complex constraints (23) hold if \( \epsilon_C \delta_C f = -1 \); otherwise neither holds. In general a sequence of Fierz type identities

\[
\delta_p \bar{\psi}^c \gamma_{\mu_1\mu_2...\mu_p} A \psi = p A_{\mu_1} \bar{\psi}^c \gamma_{\mu_2...\mu_p} \psi,
\]

holds, where \( \delta_p = \frac{1}{2}(1 - \epsilon_C \delta^{p+1}_C f(-1)^{\frac{1}{2}(p+1)}) \). Note that (23), (20) are regarded as differential conditions on \( \psi \) via the substitutions (19), (20).
Proof
Clearly (24), together with the interchange sign factors implies (24) after contraction with $A_{\mu}$. After substituting (20) and rationalising, the identity is therefore quartic, and generically of Fierz type. The only bilinear cases occur when $p = 0$ (in which case the condition $\epsilon_C \delta_C f = -1$ refers to the symmetry of $\overline{\psi} A \psi$), or for $p = d$ (for which $\gamma_{\mu}$ and $\tilde{\gamma}$ anticommute, and the charge conjugation of $\tilde{\gamma}$ determines whether the pseudoscalar identity holds).

\[ \square \]

$\mathbb{F} = 0$ : Theorem 2 The real type Fierz type identities

\[ \overline{\psi} A_{\mu_1\mu_2...\mu_p} \psi = (-1)^p \overline{\psi} \gamma_{\mu_1\mu_2...\mu_p} A \psi + 2p A_{\mu_1} \overline{\psi} \gamma_{\mu_2...\mu_p} \psi \]  

(27)

hold, where the appropriate parts of (19) are to be used on the left and right-hand sides, respectively. Current conservation and partial conservation of axial current hold for all $d$:

\[ \partial^\mu \overline{\psi} \gamma_\mu \psi = 0, \]
\[ \partial^\mu \overline{\psi} \tilde{\gamma} \gamma_\mu \psi + 2i m s A (\overline{\psi} \tilde{\gamma} \psi) = 0. \]  

(28)

Proof
(27) follows directly by contraction of (24a) with $A_{\mu}$. (28) uses (19) together with anticommutativity of $\gamma_{\mu}$ and $\tilde{\gamma}$ (See (24b)).

\[ \square \]

Explicit enumeration of these results is complicated because of the multitude of subcases involved. In table 2 the counting and nature of the identities is illustrated in diverse dimensions, and for given fermion statistics and metric signatures for the Minkowski, conformal and Euclidean spacetimes. The explicit expressions for the sign factors in the representations of the Dirac algebra available imply the expressions given for $\delta_C \epsilon_C$, namely $\epsilon_B, \delta_B \epsilon_B$ and $(-1)^{t+1/2} \delta_B \epsilon_B$ for $t = 1, t = 2$ and $t = 3$ respectively. The entries within each metric signature class then indicate for which type of fermion wavefunction statistics ($c$-number, $f = +1$, or $a$-number, $f = -1$ respectively) the indicated scalar $S$ or pseudoscalar $P$ complex identity exists. Where $\epsilon_B = +1$ exists, bracketed entries indicate the choice $\delta_B = -1$ or $\delta_B = +1$, consistent with the availability either of Majorana ($M$) or pseudoMajorana ($pM$) spinors respectively, in that dimension and spacetime signature.

In table 2, the entry for $c$-number wavefunctions in four dimensional Minkowski space corresponding to a single complex bilinear pseudoscalar identity is the original Dirac equation case. For $a$-number fermions, there exists the corresponding scalar equivalent (reflecting the properties of $\gamma_{\mu}$ and $\gamma_{\mu} \gamma_5$ in the Dirac algebra). Taking into account the two real constraints (current and partial axial current conservation), there is in either case a total of four real conditions, in accord with the count needed to accompany the four dimensional vector potential in the $8 \times 4$ linear system (see §2).
The counting of constraints in $d$ dimensions must be examined carefully in relation to Gaussian elimination in the corresponding $2^d + 1 \times d$ real linear system. For example in two-dimensional Minkowski space there are no complex ($\mathbb{F} = \pm 2$) bilinear constraints for $c$-number wavefunctions, and thus only two real ($\mathbb{F} = 0$) bilinear constraints, whereas for $a$-number wavefunctions there are two real plus two complex identities. The apparent under- or over-determination of the system (at the quadratic level) must be reconciled with the remaining Fierz-type identities (of quartic type). Two dimensions is a special case because of the abelian nature of the Lorentz group, but in other cases, the Lorentz decomposition of the remaining higher order constraints also bears on the counting. For example, in $d = 6$ ($16 \times 6$ real system) there are again either no or two complex bilinear constraints (see table 2). In the latter case, a further $16 - 6 - 2 - 4 = 4$ independent real conditions are needed. These may represent 4 additional scalar conditions at higher order, or a real 6-vector which satisfies two additional conditions equivalent to two of the scalar and pseudoscalar conditions. Similar considerations apply to the $d = 8$ and $d = 10$ cases.

### 4 Conclusions

In this work the linearity of the Dirac equation has been exploited as a vehicle to obtain an algebraic solution for the vector potential, and previous discussions in the literature ([4] and [9]) have been reconciled. In addition to its role in the problem of obtaining (classical) solutions of the full nonlinear Maxwell-Dirac equations, the algebraic method has potential application to the nonabelian case, and to Duffin-Kemmer algebras rather than Clifford algebras. Modifications to electrodynamics such as the Born-Infeld theory, and indeed the nonrelativistic limit of the Dirac equation itself, may also be amenable to further study by the method.

Table 2: Enumeration of the type of complex bilinear identity (scalar, $S$ or pseudoscalar, $P$) admitted by the Dirac wavefunction of the indicated statistics, for various dimensions $d$ and metrics of Minkowski $(1,d-1)$, conformal $(2,d-2)$ and Euclidean $(d,0)$ signature (see text for details).

| $d$ | $(1,d-1)$ | $(2,d-2)$ | $(d,0)$ |
|-----|-----------|-----------|---------|
|     | $\varepsilon_B$ $S$ $P$ | $\delta_B\varepsilon_B$ $S$ $P$ | $\pm\delta_B\varepsilon_B$ $S$ $P$ |
| 2   | $+1$ $(a)$ $(a)$ $(M)$ | $-$ $-$ $-$ $-$ | $-1$ $(c)$ $(c)$ $(M)$ |
| 4   | $\mp 1$ $(a)$ $(c)$ $(M)$ | $\pm 1$ $(c)$ $(a)$ $(M)$ | $\pm 1$ $a$ $c$ $a$ $-$ |
| 6   | $-1$ $c$ $c$ $-$ | $-1$ $(c)$ $(c)$ $(M)$ | $+1$ $(a)$ $(a)$ $(pM)$ |
| 8   | $\pm 1$ $(a)$ $(c)$ $(pM)$ | $\mp 1$ $c$ $a$ $c$ $-$ | $\mp 1$ $(a)$ $(c)$ $(M)$ |
| 10  | $+1$ $(a)$ $(a)$ $(M)$ | $+1$ $(a)$ $(a)$ $(pM)$ | $-1$ $(c)$ $(c)$ $(M)$ |
Acknowledgements

The authors are grateful to Prof Angas Hurst for discussions, and for providing copies of his own notes, and to Prof Christie Eliezer for informing one of us (HS B) of his paper and original correspondence with Dirac. We also thank Profs Tony Bracken and Bob Delbourgo for interest in the work and discussions of various aspects. Finally PDJ thanks Wim Tholen for inspiration.

References

[1] M. Esteban, V. Georgiev, E. Séré, Stationary solutions of the Maxwell-Dirac and the Klein-Gordon-Dirac equations, Calc. Var. 4, 265-281, (1996).

[2] M. Flato, J.C.H. Simon, E. Taflin, Asymptotic Completeness, Global Existence and the Infrared Problem for the Maxwell-Dirac Equations, Memoirs of the American Mathematical Society, 127 (606) (2) (May 1997).

[3] V. Georgiev, Small amplitude solutions of the Maxwell-Dirac equations, Indiana Univ. Math. J. 40 (3), 845-883 (1991).

[4] C.J. Radford, Localised Solutions of the Dirac-Maxwell Equations, J. Math. Phys. 37 (9), 4418-4433 (1996).

[5] H.S. Booth, C.J.Radford, The Dirac-Maxwell equations with cylindrical symmetry, J. Math. Phys. 38 (3), 1257-1268 (1997).

[6] C.J. Radford, H.S. Booth Magnetic Monopoles, electric neutrality and the static Maxwell-Dirac equations, J.Phys.A:Math.Gen. 32, 5807-5822 (1999).

[7] H.S. Booth, The Static Maxwell-Dirac Equations PhD Thesis, University of New England (1998).

[8] H.S. Booth, Nonlinear electron solutions and their characteristics at infinity, The ANZIAM Journal (formerly J.Aust.M.S (B)) (to appear).

[9] C.J. Eliezer, A Consistency Condition for Electron Wave Functions, Camb. Philos. Soc. Trans. 54 (2), 247-250 (1958).

[10] G. Strang, Linear Algebra and its Applications, Academic Press, New York, (1976).

[11] C. Itzykson and J.B. Zuber, Quantum Field Theory, McGraw Hill, New York (1980).

[12] B.S. deWitt, Supermanifolds, Cambridge University Press, Cambridge (1985).

[13] Yoshiaki Tanii, Physics Department, Faculty of Science, Saitama University, Introduction to supergravities in diverse dimensions, from YITP workshop on Supersymmetry, 27-30 March 1996, STUPP-98-146, hep-th/9802138 (February, 1998).
[14] P.S. Howe, G. Sierra, P.K. Townsend *Supersymmetry in six dimensions*, Nuclear Phys B221, 331-348 (1983).

[15] Y. Choquet-Bruhat, C. deWitt-Morette, *Analysis, Manifolds and Physics II: 92 Applications*, North Holland, Amsterdam, (1989).

[16] R. Coquereaux, *Modulo 8 periodicity of real Clifford algebras and particle physics*, Phys Lett 115B (5), 389-395 (1982).

[17] G.R.E. Black, R.C. King and B.G. Wybourne, *Kronecker products for compact semisimple Lie groups*, J Phys A16 (3), 1555-1589 (1983).

[18] D.A. Akyeampong and R. Delbourgo, *Dimensional regularization, abnormal amplitudes and anomalies*, Nuovo Cim 23A, 578-93 (1974).