Bell's inequality for \(n\) spin-\(s\) particles

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The Mermin-Klyshko inequality for \(n\) spin-\(\frac{1}{2}\) particles and two dichotomic observables is generalized to \(n\) spin-\(s\) particles and two maximal observables. It is shown that some multiparty multilevel Greenberger-Horne-Zeilinger states [A. Cabello, Phys. Rev. A 63, 022104 (2001)] maximally violate this inequality for any \(s\). For a fixed \(n\), the magnitude of the violation is constant for any \(s\), which provides a simple demonstration and generalizes the conclusion reached by Gisin and Peres for two spin-\(s\) particles in the singlet state [Phys. Lett. A 162, 15 (1992)]. For a fixed \(s\), the violation grows exponentially with \(n\), which provides a generalization to any \(s\) of Mermin’s conclusion for \(n\) spin-\(\frac{1}{2}\) particles [Phys. Rev. Lett. 65, 1838 (1990)].

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I. INTRODUCTION

Einstein, Podolsky, and Rosen (EPR) [1] believed that the results of experiments on a local subsystem of a composite physical system can be predicted with certainty from the results of local experiments in other regions would be determined by the local properties of the subsystem. However, the violation of Bell’s inequality by quantum mechanics [2] meant a spectacular departure from EPR’s point of view. According to quantum mechanics, the results of local experiments cannot be described in terms of classical local properties.

On the other hand, it was commonly accepted that classical properties would emerge for large quantum systems. The adjective “large” usually means either systems composed of many particles or systems with a high number of internal degrees of freedom. Early violations of Bell’s inequalities [3, 4] involved pairs of spin-\(\frac{1}{2}\) particles in the singlet state [4]. However, the EPR argument is also applicable to pairs of spin-\(s\) particles in the singlet state or to systems of \(n\) spin-\(\frac{1}{2}\) particles in Greenberger, Horne, and Zeilinger (GHZ) states [5]. Violations of Bell’s inequalities for the two spin-\(s\) singlet state have been extensively discussed [6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and have stimulated some recent experiments for \(s\) particles in Greenberger-Horne-Zeilinger states [16].

Violations of Bell’s inequalities for the two spin-\(s\) singlet state have been extensively discussed [6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and have stimulated some recent experiments for \(s = 1\) [10, 11, 12, 13, 14, 15]. On the other hand, violations of Bell’s inequalities for \(n\) spin-\(\frac{1}{2}\) particles have attracted much attention [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. However, a study of Bell’s inequalities for systems of \(n\) spin-s particles and the limit of both \(n \to \infty\) and \(s \to \infty\) was still missing.

In order to place our discussion in a suitable context, we shall review some of the earlier violations of Bell’s inequalities for two spin-\(s\) particles and for \(n\) spin-\(\frac{1}{2}\) particles.

Mermin [6] showed that a pair of spin-\(s\) particles in the singlet state violates a particular Bell’s inequality involving four local spin component observables \(\hat{S}_1 \cdot \hat{a}\), \(\hat{S}_1 \cdot \hat{b}\), \(\hat{S}_2 \cdot \hat{b}\), and \(\hat{S}_2 \cdot \hat{c}\). He found that the range of settings for which the violation occurs vanishes as \(1/s\) when \(s \to \infty\). Subsequently, however, Mermin and Schwarz [7] found evidence that this vanishing might be peculiar to the chosen inequality (see also [12, 13]).

Ögren [10] studied the original Bell’s inequality [3] for three different ways of defining dichotomic observables from \(\hat{S}_1 \cdot \hat{a}\), \(\hat{S}_1 \cdot \hat{b}\), \(\hat{S}_2 \cdot \hat{b}\), and \(\hat{S}_2 \cdot \hat{c}\). He found that the range of settings for which the singlet state of two spin-\(s\) particles violates Bell’s inequality is of the same magnitude, at least for small \(s\), and larger than those obtained in Ref. [6].

Peres [14] and Gisin and Peres [15] found dichotomic operators such that two spin-\(s\) particles in the singlet state violate the Clauser-Horne-Shimony-Holt [3] (CHSH) inequality and that the magnitude of the violation (that is, the ratio of the quantum correlation to the maximal classical one) tends to a constant [14] or is constant [15] for any \(s\).

An experimental violation of Bell’s inequalities for an optical analog of the singlet state of two spin-1 particles has been recently reported in Ref. [16].

On the other hand, Mermin [18] has shown that the correlations found by \(n\) spacelike separated observers who share \(n\) spin-\(\frac{1}{2}\) particles in a GHZ state maximally violate a Bell’s inequality involving two local spin component observables per particle by a factor that increases exponentially with \(n\). Mermin’s inequality for \(n\) spin-\(\frac{1}{2}\) particles distinguishes between the \(n\) even and odd cases. Ardehali [19] derived a similar inequality that leads to a higher violation for even \(n\). Finally, Belinsky and Klyshko [22] proposed an elegant single inequality that leads to a maximum violation for arbitrary \(n\). This inequality is mostly referred to as the Mermin-Klyshko inequality.

The structure of this paper is as follows: In Sec. II we introduce a generalization for any spin of the Mermin-Klyshko inequality using two maximal observables (i.e., represented by nondegenerated operators) per particle. In Sec. III we show that maximally entangled states of two spin-\(s\) particles and some multiparticle multilevel GHZ states defined in Ref. [23] maximally violate the inequality presented in Sec. II.
In Sec. IV we present the conclusions of our research: On one hand, we reach Gisin and Peres’s conclusion in Ref. [15], namely that for two particles in a maximally entangled state the ratio of the quantum correlation to the maximal classical one is constant as $s$ grows. Moreover, we extend Gisin and Peres’s conclusion to systems of three or more particles. On the other hand, we generalize to any $s$ Mermin’s conclusion in Ref. [18] that the ratio of the quantum correlation to the maximal classical one grows exponentially with the number of particles. In addition, the inequality presented in Sec. II would allow us to translate the proofs of Bell’s theorem without inequalities for multiparticle multilevel GHZ states introduced in Ref. [28] into feasible experimental tests.

II. THE MERMIN-KLYSHKO INEQUALITY FOR $N$ SPIN-$S$ PARTICLES

Let us consider a system with $n \geq 2$ distant spin-$s$ particles, $1, \ldots, n$ shared by $n$ distant observers which perform spacelike local experiments, chosen between $A_j^{(s)}$ and $B_j^{(s)}$, on his/her particle $j$. Let us choose units in which $\hbar = 1$ and let $A_j^{(s)}$ and $B_j^{(s)}$ be physical observables on particle $j$ taking values $-s, -s+1, \ldots, s$.

The correlation $A_1^{(s)} \ldots A_n^{(s)} = A_1^{(s)} \ldots A_n^{(s)}$ of $A_1^{(s)}, \ldots, A_n^{(s)}$ is defined as

$$A_1^{(s)} \ldots A_n^{(s)} = \sum_{m_1, \ldots, m_n = -s} m_1 \ldots m_n P(A_1^{(s)} = m_1, \ldots, A_n^{(s)} = m_n),$$

where $P(A_1^{(s)} = m_1, \ldots, A_n^{(s)} = m_n)$ is the joint probability of obtaining $A_1^{(s)} = m_1, \ldots, A_n^{(s)} = m_n$ when $A_1^{(s)}, \ldots, A_n^{(s)}$ are measured.

Let us consider the linear combination of $2^{2f(n/2)}$ correlations, where $f(x)$ is the greatest integer less than or equal to $x$, defined recursively by

$$M_n^{(s)} = M_{n-1}^{(s)} (A_n^{(s)} + B_n^{(s)}) + K_{n-1}^{(s)} (A_n^{(s)} - B_n^{(s)}),$$

letting $M_1^{(s)} = A_1^{(s)}$, and $K_1^{(s)}$ being the same as $M_n^{(s)}$ but exchanging the $A$’s for $B$’s.

In particular,

$$M_2^{(s)} = A_1^{(s)} A_2^{(s)} + A_1^{(s)} B_2^{(s)} + B_1^{(s)} A_2^{(s)} - B_1^{(s)} B_2^{(s)}$$

and

$$M_3^{(s)} = 2 \left( A_1^{(s)} B_2^{(s)} B_3^{(s)} + B_1^{(s)} A_2^{(s)} B_3^{(s)} + B_1^{(s)} B_2^{(s)} A_3^{(s)} - A_1^{(s)} A_2^{(s)} A_3^{(s)} \right).$$

In any theory in which local variables of particle $j$ determine the results of local observables $A_j^{(s)}$ and $B_j^{(s)}$, the absolute value of $M_n^{(s)}$ is bound as follows:

$$|M_n^{(s)}| \leq 2^{n-1}s^n.$$ (5)

This is the generalization to spin-$s$ of the Mermin-Klyshko inequality. If $A_j$ and $B_j$ are observables taking values $-1$ or $1$ (i.e., for $s = 1$), or for $s = \frac{1}{2}$ and choosing units in which $2\hbar = 1$, then we obtain the Mermin-Klyshko inequality [22]. If, in addition, $n$ is odd and greater than 3, then (up to a factor $2^{f((n-1)/2)}$) we obtain Mermin’s inequality [18]. If $n = 2$ we obtain the CHSH inequality [3].

The bounds in inequality (5) can be easily derived as follows: In any local-realistic theory, for any individual system, observables $A_j$ and $B_j$ have predefined values $a_j$ and $b_j$, respectively. Each of these values is constrained to lie between $-s$ and $s$. Since $M_n^{(s)}$ is linear in each local observable (fixing the value of the other 2$n-1$ local observables), $M_n^{(s)}$ will take its extremal values when local observables take their extremal values, $-s$ or $s$. The various combinations of $a_j = \pm s$ and $b_j = \pm s$ always give $\pm 2^{n-1}s^n$, Q.E.D.

III. VIOLATIONS OF THE GENERALIZED MERMIN-KLYSHKO INEQUALITY

For a $n$ spin-$s$ particle system in a quantum pure state $|\psi\rangle$, the quantum correlation of $A_1, \ldots, A_n$ is defined as $\langle \psi | A_1^{(s)} \otimes \cdots \otimes A_n^{(s)} | \psi \rangle$, where $A_1^{(s)}, \ldots, A_n^{(s)}$ are the self-adjoint operators that represent the local observables $A_1^{(s)}, \ldots, A_n^{(s)}$.

Let us consider the following local operators on particle $j$:

$$\hat{A}_j^{(s)} = \begin{pmatrix} s & -s+1 & \cdots & s-1 \\ s-1 & \cdots & -s+1 & s \\ \vdots & \ddots & \ddots & \ddots \\ s & \cdots & s-1 & s \end{pmatrix},$$ (6)

$$\hat{B}_j^{(s)} = \begin{pmatrix} s & -s+1 & \cdots & s-1 \\ s-1 & \cdots & -s+1 & s \\ \vdots & \ddots & \ddots & \ddots \\ s & \cdots & s-1 & s \end{pmatrix}.$$ (7)

$\hat{A}_j^{(s)}$ and $\hat{B}_j^{(s)}$ are diagonal $(2s+1) \times (2s+1)$ matrices, with nondegenerated eigenvalues $-s, -s+1, \ldots, s-1, s$.

In addition, let us recursively define the following operator on the composite system consisting on $n \geq 2$ spin-$s$ particles:

$$\hat{M}_n^{(s)} = \hat{M}_{n-1}^{(s)} \otimes (\hat{A}_1^{(s)} + \hat{B}_1^{(s)}) + \hat{K}_{n-1}^{(s)} \otimes (\hat{A}_1^{(s)} - \hat{B}_1^{(s)}),$$

letting $\hat{M}_1^{(s)} = \hat{A}_1^{(s)}$, and $\hat{K}_1^{(s)}$ being the same as $\hat{M}_n^{(s)}$ but exchanging the $\hat{A}$’s for $\hat{B}$’s.
As can be easily checked, $\hat{M}_n^{(s)}$ is a linear combination of $2^n(n/2)$ operators of the type $\hat{A}_1^{(s)} \otimes \cdots \otimes \hat{A}_n^{(s)}$ (all of them commuting if $n$ is odd, but not if it is even). The greatest eigenvalue of $\hat{M}_n^{(s)}$ is $2^{(n-1)/2}s^n$, which is non-degenerated. Let us consider the corresponding eigenstate $\left| \mu_n^{(s)} \right>$, characterized by the equation

$$\hat{M}_n^{(s)} \left| \mu_n^{(s)} \right> = 2^{(n-1)/2}s^n \left| \mu_n^{(s)} \right> \quad (9)$$

For $n = 2$, $\left| \mu_n^{(s)} \right>$ is a maximally entangled state of two spin-$s$ particles. For $n \geq 3$, $\left| \mu_n^{(s)} \right>$ is a generalized GHZ state, as defined in Ref. [28], and allows us to develop an EPR-like argument for observables $A_j$ and $B_j$. For $n$ odd (even), $\left| \mu_n^{(s)} \right>$ and all the operators (a subset of mutually commuting operators) of the type $\hat{A}_1^{(s)} \otimes \cdots \otimes \hat{A}_n^{(s)}$ included in $\hat{M}_n^{(s)}$ allow us to develop a GHZ-like proof without inequalities of Bell’s theorem [28] (see [28] for the details).

In this paper, however, we are interested in violations of inequality [3]. For that purpose, let us take a look at the prediction of quantum mechanics for the state $\left| \mu_n^{(s)} \right>$ for the combination of correlations appearing in inequality [3]. Observable $M_n^{(s)}$ is represented in quantum mechanics by the self-adjoint operator $\hat{M}_n^{(s)}$. Therefore, as can be immediately seen in Eq. [3], according to quantum mechanics the expected value for $M_n^{(s)}$ in the state $\left| \mu_n^{(s)} \right>$ is given by

$$\left< \mu_n^{(s)} \left| \hat{M}_n^{(s)} \right| \mu_n^{(s)} \right> = 2^{(n-1)/2}s^n \quad (10)$$

This value violates inequality [3]. Indeed, it can be proved that this is the maximum allowed violation of inequality [3]. The proof is simple for $n$ odd. Then, $M_n^{(s)}$ is a linear combination with coefficients $\pm 2^{(n-1)/2}$ of $2^n-1$ operators of the type $\hat{A}_1^{(s)} \otimes \cdots \otimes \hat{A}_n^{(s)}$, and each of these correlations is bound by $\pm s^n$. Therefore, for $n$ odd, the maximum value that $M_n^{(s)}$ can reach is, by definition, $2^{(n-1)/2}s^n$, Q.E.D.

If $n$ is even the proof is more difficult (for $n = 2$ and $s = 1$, or for $s = 2$ and choosing units in which $2\hbar = 1$, proofs can be found in Refs. [30, 31]).

IV. CONCLUSIONS

The ratio between the quantum correlation given by Eq. (10) and the maximal classical one, which appears in Eq. (4), is

$$\max_{M_n^{(s)}} = 2^{(n-1)/2} \forall s \quad (11)$$

That is, for a fixed $n \geq 2$ the contradiction between quantum mechanics and local realism is constant as the spin $s$ increases. For $n = 2$ the same conclusion was reached by Gisin and Peres in Ref. [13]. Therefore, our analysis is in agreement with Gisin and Peres’ and generalizes it to systems of $n \geq 2$ particles.

On the other hand, ratio (11) shows that for a fixed $s$, the correlations found by distant observers violate the classical bound by a factor that increases exponentially with the number $n$ of particles. For $s = 1$ the same conclusion was reached by Mermin in Ref. [18]. Thus our analysis generalizes Mermin’s to systems of spin $s \geq \frac{1}{2}$.

Therefore, the approach presented in this paper unifies and generalizes some previous results, in particular, those in Refs. [14, 15, 22], and unifies the conclusions reached in Refs. [14, 15, 18]: Neither a large spin nor a large number of particles nor a large number of large spin particles guarantee classical behavior.

In addition, this approach allows us to translate the proofs without inequalities of Bell’s theorem for multiparty multilevel GHZ states introduced in Ref. [23] into Bell’s inequalities that can be tested in real experiments.

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[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[2] J.S. Bell, Physics (Long Island City, N.Y.) 1, 195 (1964).
[3] J.F. Clauser, M.A. Horne, A. Shimony, and R.A. Holt, Phys. Rev. Lett. 23, 880 (1969).
[4] D. Bohm, Quantum Theory (Prentice-Hall, Englewood Cliffs, New Jersey, 1951).
[5] D.M. Greenberger, M.A. Horne, and A. Zeilinger, in Bell’s Theorem, Quantum Theory, and Conceptions of the Universe, edited by M. Kafatos (Kluwer, Dordrecht, 1989), p. 69.
[6] N.D. Mermin, Phys. Rev. D 22, 356 (1980).
[7] N.D. Mermin and G.M. Schwarz, Found. Phys. 12, 101 (1982).
[8] A. Garg and N.D. Mermin, Phys. Rev. Lett. 49, 901 (1982); Phys. Rev. Lett. 49, 1294 (1982).
[9] A. Garg and N.D. Mermin, Phys. Rev. D 27, 339 (1983).
[10] M. Ögren, Phys. Rev. D 27, 1766 (1983).
[11] S.L. Braunstein and C.M. Caves, Phys. Rev. Lett. 61, 662 (1988).
[12] A.L. Sanz and J.L. Sánchez Gómez, An. Fis., Ser. A 86, 77 (1990).
[13] M. Ardehali, Phys. Rev. D 44, 3336 (1991).
[14] A. Peres, Phys. Rev. A 46, 4413 (1992).
[15] N. Gisin and A. Peres, Phys. Lett. A 162, 15 (1992).
[16] A. Lamas-Linares, J.C. Howell, and D. Bouwmeester, Nature (London) 412, 887 (2001).
[17] J.C. Howell, A. Lamas-Linares, and D. Bouwmeester, Phys. Rev. Lett. 88, 030401 (2002).
[18] N.D. Mermin, Phys. Rev. Lett. 65, 1838 (1990).
[19] S.M. Roy and V. Singh, Phys. Rev. Lett. 67, 2761 (1991).
[20] R.K. Clifton, M.L.G. Redhead, and J.N. Butterfield, Found. Phys. 21, 149 (1991).
[21] M. Ardehali, Phys. Rev. A 46, 5375 (1992).
[22] A.V. Belinsky and D.N. Klyshko, Usp. Fiz. Nauk 163, 1 (1993) [Phys. Usp. 36, 653 (1993)].
[23] S.L. Braunstein and A. Mann, Phys. Rev. A 47, R2427 (1993).
[24] M. Żukowski and D. Kaszlikowski, Phys. Rev. A 56, R1682 (1997).
[25] N. Gisin and H. Bechmann-Pasquinucci, Phys. Lett. A 246, 1 (1998).
[26] R.F. Werner and M.M. Wolf, Phys. Rev. A 61, 062102 (2000).
[27] R.F. Werner and M.M. Wolf, Phys. Rev. A 64, 032112 (2001).
[28] A. Cabello, Phys. Rev. A 63, 022104 (2001).
[29] The operators of the type $\hat{A}_1^{(s)} \otimes \cdots \otimes \hat{A}_n^{(s)}$ appearing in Mermin’s original inequality for $n$ spin-$\frac{1}{2}$ particles in Ref. [18] are mutually compatible both for $n$ odd and even, and can be used to develop a GHZ-like proof of Bell’s theorem without inequalities in both cases.
[30] B.S. Cirel’son, Lett. Math. Phys. 4, 93 (1980).
[31] L.J. Landau, Phys. Lett. A 120, 54 (1987).