A Mixed Graph Achieving a Moore-like Bound

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Abstract

Mixed graphs have both directed and undirected edges. A mixed cage is a regular mixed graph of given girth with minimum possible order. In this paper we construct a mixed cage of order 30 that achieves the mixed graph analogue of the Moore bound for degree 3, out-degree 1, and girth 6.

1 Notation and Terminology

A mixed graph is a graph with both directed and undirected edges. We refer to directed edges as arcs and undirected edges as edges. The degree of a vertex $v$ in a mixed graph $G$ is the number of edges incident with $v$, whereas the in-degree and out-degree are the numbers of arcs incident to and from $v$. $G$ is regular if all three degrees are constant as $v$ ranges over $V(G)$.

A cycle in a mixed graph is a sequence of vertices $v_0, v_1, \cdots, v_k$ such that there are no repeated vertices except that $v_0 = v_k$, each pair of consecutive vertices $(v_i, v_{i+1})$ is either an edge or an arc, and there are no repeated edges or arcs. The girth of a mixed graph is the length of a shortest cycle. Note that this definition considers the possibility of 1-cycles (loops) and 2-cycles.

Mixed graphs have been studied in the context of the degree/diameter problem. We follow the notation used in that literature and denote the degree and outdegree of a regular mixed graph by $r$ and $z$, respectively.
An \((r, z, g)\)-graph is a regular mixed graph with degree \(r\), out-degree \(z\), and girth \(g\). An \((r, z, g)\)-cage is an \((r, z, g)\)-graph of minimum possible order. We denote this minimum order by \(f(r, z, g)\).

Recall [2] that the Moore bound for an \(r\)-regular graph of diameter \(d\) is given by:

\[
n(r, d) \geq \frac{r(r - 1)^d - 2}{r - 2}
\]  

(1)

In [1] Araujo-Pardo, Hernández-Cruz, and Montellano-Ballesteros consider the problem of finding mixed cages. They focus on the case \(z = 1\) and determine a lower bound for \(f(r, 1, g)\) based on the Moore bound. Their idea is to attach undirected Moore trees to each vertex of a directed path of length \(g - 1\), choosing trees whose depth is as large as possible while still guaranteeing that all tree vertices are distinct.

So let \(v_0, v_1, \cdots, v_{g-1}\) be the vertices of a directed path of length \(g - 1\). Using edges, attach a Moore tree of depth \(i\) to both \(v_i\) and \(v_{g-1-i}\) for \(0 \leq i \leq \lfloor g/2 \rfloor\). Note: if \(i = g-i-1\) we attach only one tree. The base path contains arcs and the Moore trees contain edges. This gives the following bound.

**Theorem 1 (The AHM Bound [1])**

\[
f(r, 1, g) \geq \sum_{i=0}^{g-1} n(r, \min(i, g-i-1))
\]

**Theorem 2**

\[
f(3, 1, 6) = 30
\]

**Proof.** The AHM bound for \(r = 3\), \(z = 1\), and \(g = 6\) is 30. The graph \(G\) shown in the figure has order 30 and has the require parameters.

The graph can be describe algebraically as follows. Let

\[
V(G) = \{v(i, j) \mid 0 \leq i < 3, \ 0 \leq j < 10\}
\]

be the vertex set. For \(0 \leq i < 3\), let

\[
V_i = \{v(i, j) \mid 0 \leq j < 10\}.
\]
Each $V_i$ induces a directed 10-cycle. These three cycles are shown in figure. Let $V_0$ be the set of vertices in the lower partition of the figure and let $V_1$ and $V_2$ be the sets of vertices on the outer and inner (resp.) cycles in the upper part of the figure. Imagine the vertices labeled such that vertex $v_{i,0}$ is the rightmost vertex on each of the three directed 10-cycles and that the other vertices are labeled in counter-clockwise order. The arcs and edges are given as follows, where second indices are computed modulo 10:

(a) arc($v(0, j), v(0, j+1)$)
(b) edge($v(0, j), v(1, j)$)
(c) edge($v(0, j), v(2, j+5)$)
(d) edge($v(1, j), v(2, j+2)$)
(e) edge($v(1, j), v(2, j-2)$)

In the figure, edges of type (b) and (c) are not drawn but indicated by color: vertices in $V_0$ are adjacent to those vertices in $V_1$ and $V_2$ that have the matching color.

**Note:** Recently another graph achieving the AHM bound was found by the author. This graph is for the case $(6, 1, 6)$. The graph has order 90, which is the AHM bound, and can be constructed as a lift of the complement of the line graph of $K_6$. The adjacency matrix for the graph can be found at the following location.

[https://cs.indstate.edu/ge/MixedCages/g90.txt](https://cs.indstate.edu/ge/MixedCages/g90.txt)

**References**

[1] G. Araujo-Pardo, C. Hernández-Cruz, and J. J. Montellano-Ballesteros, Mixed Cages. *Graphs and Combinatorics*, 35 (2019), 989–999.

[2] M. Miller and J. Sirán, Moore Graphs and Beyond: A Survey of the Degree/Diameter Problem, *Electronic Journal of Combinatorics* DS14, May 16, 2013.
Figure 1: The unique smallest \((3, 1, 6)\)-graph of order 30. Vertices in the lower figure are adjacent to vertices in the upper figure that have the matching color. The automorphism group has two generators: a rotation of \(\pi/5\) (in both figures) and an involution that transposes the inner and outer cycles in the upper figure. These generators commute, so the group is \(\mathbb{Z}_2 \times \mathbb{Z}_{10}\).