Neutral minima in two-Higgs doublet models

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February, 2007

Abstract. We study the neutral minima of two-Higgs doublet models, showing that these potentials can have at least two such minima with different depths. We analyse the phenomenology of these minima for the several types of two-Higgs doublet potentials, where CP is explicitly broken, spontaneously broken or preserved. We discover that it is possible to have a neutral minimum in these potentials where the masses of the known particles have their standard values, with another deeper minimum where those same particles acquire different masses.

1 Introduction

The standard model (SM) of electroweak interactions is a remarkably successful theory, but its scalar sector is yet untested. Numerous theories with a larger scalar content have been proposed over the years in the attempt to increase the predictive power of the model and offer explanations for problems such as baryogenesis, CP violation, the hierarchy question and others. In models with more than one scalar, the possibility of the scalar potential having more than one minimum arises, and some of those minima can break the \(SU(3)\) \(\times U(1)\) gauge symmetry of the SM. Thus one is left with the possibility of imposing charge and/or colour breaking bounds on the parameters of the theory: excluding those combinations of parameters for which the potential's deeper minimum breaks charge and/or colour conservation. This appealing idea was first considered in the framework of supersymmetric theories \cite{1} and applied extensively to such models \cite{2}. Recently, it has also been applied to the Zee model \cite{3} and to the two-Higgs doublet model (2HDM) \cite{4,5}. The results obtained in this last reference were generalized in ref. \cite{6} to models with an arbitrary number of Higgs doublets \cite{7}.

The results of \cite{4} may be summarised as follows: in the 2HDM, whenever a minimum that preserves charge and CP (we dub these “normal” minima) exists, that minimum is certainly deeper than any charge breaking (CB) stationary point. Further, the global minimum of the potential is a normal one, and the CB stationary point ends up being a saddle point. A similar conclusion holds for the spontaneous breaking of the CP symmetry: when a normal minimum exists, it is certainly deeper than any possible CP stationary point, and the global minimum is a
normal one. However, unlike the CB case, the question of whether the CP stationary point can be a minimum was left unanswered in ref. [4]. There is then the possibility that the potential may have a CP minimum, with a normal one lying below it.

The importance of these results is to ensure the stability of the normal minimum against spontaneous charge or CP breaking: if one finds a normal minimum in the 2HDM, one may rest assured that there is no deeper CB or CP minimum for which the system may eventually evolve via tunneling. However, the work of ref. [4] did not answer the following question: how many normal minima does the 2HDM have? Are they all acceptable minima, with phenomenology according to the experimental data? And if there are several normal minima, which is the deepest one? Several of these questions were addressed for the 2HDM in recent papers [8], but in this paper we will focus on other aspects not treated in those works. A recent work [9] studied the minimum structure of the next-to-minimal Supersymmetric Standard Model.

In the present work we will take a close look at the normal minimum structure of several types of 2HDM potentials. We will show that there is indeed the possibility that several normal minima coexist in the 2HDM, and if they are not degenerate, they will give rise to different phenomenologies. This paper is structured as follows: in section 2 we will introduce our formalism, write down the most general 2HDM potential and analyse the several types of theories one obtains by imposing symmetries on the model. In section 3 we will discuss the vacuum structure of the model and review the results of [4] about the differences in depths of the scalar potential at two of these possible vacua. In section 4 we will analyse normal minima in potentials where CP is broken - explicitly or spontaneously - and, in section 5, in potentials where CP is always left unbroken. We conclude, in section 6, with a general overview.

\section{The 2HDM potentials}

Let us consider two $SU(2)$ doublets, with hypercharge $Y = 1$, given by

$$
\Phi_1 = \left( \varphi_1 + i\varphi_2 \right) , \quad \Phi_2 = \left( \varphi_3 + i\varphi_4 \right) .
$$

where all the $\varphi_i$ are real functions. Their numbering may seem odd, but it simplifies the writing of the scalar mass matrices. With these two fields one can build four $SU(2) \times U(1)$ real quadratic invariants,

$$
x_1 \equiv |\Phi_1|^2 = \varphi_1^2 + \varphi_2^2 + \varphi_5^2 + \varphi_7^2
$$

$$
x_2 \equiv |\Phi_2|^2 = \varphi_3^2 + \varphi_4^2 + \varphi_6^2 + \varphi_8^2
$$

$$
x_3 \equiv Re(\Phi_1^\dagger\Phi_2) = \varphi_1\varphi_3 + \varphi_2\varphi_4 + \varphi_5\varphi_6 + \varphi_7\varphi_8
$$

$$
x_4 \equiv Im(\Phi_1^\dagger\Phi_2) = \varphi_1\varphi_4 - \varphi_2\varphi_3 + \varphi_5\varphi_8 - \varphi_6\varphi_7 .
$$

The most general 2HDM potential (for an overview, see for instance [7, 11]) is therefore a polynomial on the $x$'s, with all possible linear and quadratic terms in these variables. That is,

$$
V = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 + b_{11} x_1^2 + b_{22} x_2^2 + b_{33} x_3^2 + b_{44} x_4^2 + b_{12} x_1 x_2 + b_{13} x_1 x_3 + b_{14} x_1 x_4 + b_{23} x_2 x_3 + b_{24} x_2 x_4 + b_{34} x_3 x_4 ,
$$

where the coefficients $a_i$ and $b_{ij}$ are all real, the former having dimensions of mass squared and the latter being dimensionless. Under a CP transformation of the form $\Phi_i \rightarrow \Phi_i^*$, $x_1$, $x_2$ and $x_3$ remain the same but $x_4$ switches signal. Thus, the terms of the potential which are linear in $x_4$ break CP explicitly. The most general explicit CP-breaking potential has therefore 14
real parameters. Through an appropriate choice of basis for \{\Phi_1, \Phi_2\} it is possible to reduce that number to 11 \cite{12}. It is easy to see how to reduce the number of parameters to 12. The quadratic part of (3) may be written as

\[ a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = \begin{pmatrix} \Phi_1 & \Phi_2 \end{pmatrix} \begin{pmatrix} \frac{a_1}{2} & \frac{a_3 - i a_4}{2} \\ \frac{a_3 + i a_4}{2} & a_2 \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}. \]

(4)

Through an unitary transformation on the fields, it is possible to diagonalize this \(2 \times 2\) matrix. Written in terms of the new fields, the quadratic terms are now just two, whereas the quartic terms linear in \(a\) terms linear in \(x\) can be expressed in terms of the two others, which leaves us with 11 independent real parameters.

For instance, if one asks that the potential is invariant under a global \(U(1)\) symmetry (\(\Phi_1 \rightarrow e^{i\alpha} \Phi_1, \Phi_2 \rightarrow \Phi_2\)), then the terms linear in \(x_3\) and \(x_4\) vanish and the potential becomes

\[ V_A = a_1 x_1 + a_2 x_2 + b_{11} x_1^2 + b_{22} x_2^2 + b_{33} x_3^2 + b_{44} x_4^2 + b_{12} x_1 x_2. \]

(5)

We call this the potential A. It is very important to have \(b_{33} \neq b_{44}\) in this model, otherwise one has a massless axion\(^1\). Alternatively, with a \(U(1)\) symmetry (\(\Phi_1 \rightarrow e^{i\alpha} \Phi_1, \Phi_2 \rightarrow \Phi_2\)), the model is also greatly simplified and one obtains the potential B,

\[ V_B = a_1 x_1 + a_2 x_2 + a_3 x_3 + b_{11} x_1^2 + b_{22} x_2^2 + b_{33} (x_3^2 + x_4^2) + b_{12} x_1 x_2, \]

(6)

where the symmetry imposes \(b_{33} = b_{44}\). This potential is the analogous of the supersymmetric one. Notice that the \(a_3\) term softly breaks the \(U(1)\) symmetry. It was left there to prevent the appearance of a massless axion in the theory. Unlike the most general case, here one cannot “rotate away” the \(a_3\) term without introducing further parameters in the quartic terms of \(V_B\). In order to ensure that a minimum exists away from the origin, a trivial calculation shows that we must have \(a_3^2 > 4a_1 a_2\). As was shown in ref. \cite{13}, the imposition of these symmetries makes it impossible for spontaneous CP breaking to occur in models A or B. Another property of these symmetries is that they are exactly those that are necessary to prevent flavour changing neutral currents (FCNC) in the theory, a phenomenon that plagues both the potential \(3\) and its restriction to the case of CP not being explicitly broken.

Another possible symmetry to impose on the potential is that it remains invariant under an interchange between both fields (\(\Phi_1 \leftrightarrow \Phi_2\)). Then the potential becomes \cite{14}

\[ V_C = a_1 (x_1 + x_2) + a_3 x_3 + b_{11} (x_1^2 + x_2^2) + b_{33} x_3^2 + b_{44} x_4^2 + b_{12} x_1 x_2 + b_{13} (x_1 + x_2) x_3. \]

(7)

Again, no basis changes can reduce the number of parameters in this model.

\(^1\)In fact, without this restriction, the potential has the \(U(1)\) symmetry of model B without its soft breaking term.
The potential $V_C$ was first studied by Branco and Rebelo in [14], introducing an extra term of the form $a_2 (x_1 - x_2)$ to softly break the symmetry imposed on the $\Phi$ fields. The consequence is that the resulting potential may have spontaneous CP breaking, without FCNC. Similarly, adding to $V_A$ a term of the form $a_3 x_3$, one breaks softly the $Z_2$ symmetry that characterizes that potential. Once again, this gives rise to the possibility of spontaneous CP breaking. In fact, in ref. [12] it was shown that the potentials $A$ and $C$ are equivalent - it is possible to obtain the one from the other with a basis transformation.

Interestingly, no soft breaking of the $U(1)$ symmetry of the potential $V_B$ gives rise to spontaneous CP breaking. The potential written in eq. (6) has already a soft breaking term - the $a_3$ term - and the only remaining term quadratic in the fields one could add to the potential would be $a_4 x_4$ - and that term explicitly breaks the CP symmetry.

Let us summarise what we have discussed about the 2HDM potentials in terms of the number of independent real parameters that they have:

- Potentials where CP is explicitly broken, which have 11 real parameters. FCNCs occur.
- Potentials with 9 real parameters where explicit CP conservation has been imposed, but for which one might have spontaneous CP breaking. Again, FCNCs occur.
- Potentials with 7 real parameters where, besides explicit CP conservation, one has also imposed a discrete symmetry ($Z_2$ or permutation of $\Phi_1$ and $\Phi_2$). No spontaneous symmetry breaking occurs, FCNCs are excluded. With an additional term that breaks softly the discrete symmetry, spontaneous CP might arise.
- Potentials with 6 real parameters, with explicit CP conservation and invariant under a global $U(1)$ symmetry. Again there is no FCNC but these models have a massless axion. If one adds to the potential a term that softly breaks the global $U(1)$, that axion acquires a mass but there is no possibility of spontaneous CP breaking.

There is an interesting possible classification of these potentials, which we will present in appendix [B].

3 The stationary points of the 2HDM

The vacuum structure of the 2HDM is very simple [10], with three different types of vev configuration. For a charge and CP conserving vacuum, the fields $\Phi_1$ and $\Phi_2$ have vevs of the form

$$\Phi_1 \rightarrow \left( \begin{array}{c} 0 \\ v_1 \end{array} \right), \quad \Phi_2 \rightarrow \left( \begin{array}{c} 0 \\ v_2 \end{array} \right)$$

(8)

where $v_1$ and $v_2$ are real numbers. For a vacuum that breaks charge conservation, we have

$$\Phi_1 \rightarrow \left( \begin{array}{c} 0 \\ v'_1 \end{array} \right), \quad \Phi_2 \rightarrow \left( \begin{array}{c} \alpha \\ v'_2 \end{array} \right).$$

(9)

Again, all vevs in this equation are real. Finally, a CP breaking vacuum occurs when the fields acquire vevs of the form

$$\Phi_1 \rightarrow \left( \begin{array}{c} 0 \\ v''_1 + i\delta \end{array} \right), \quad \Phi_2 \rightarrow \left( \begin{array}{c} 0 \\ v''_2 \end{array} \right).$$

(10)
Vacua with \( \alpha \) and \( \delta \) simultaneously non-zero are not considered because the minimisation conditions of the potential forbid them.\(^2\)

The structure of the potential of eq. (3) is such that it may have several stationary points, and they may be of different natures. In ref. [4] we obtained a remarkable result that relates the difference in the depths of the potential in each of the three types of possible stationary points. Namely, the difference between the value of the potential at a CB stationary point, \( V_{CB} \), and the value of potential at a normal stationary point, \( V_N \), is given by

\[
V_{CB} - V_N = \frac{M_{H^\pm}^2}{2v^2} \left[ (v_1' v_2 - v_2' v_1)^2 + \alpha^2 v_1^2 \right],
\]

where \( v^2 = v_1^2 + v_2^2 \) and \( M_{H^\pm}^2 \) is the value of the squared mass of the charged Higgs scalar, evaluated at the normal stationary point. What this equation tells us is that when the normal stationary point is a minimum,\(^3\) then, all of the eigenvalues of the squared scalar mass matrices being positive, we will have \( M_{H^\pm}^2 > 0 \) and the deeper stationary point will be the normal minimum. Furthermore, in ref. [4] we were able to prove that the CB stationary point is unique, and that eq. (11), when \( M_{H^\pm}^2 > 0 \), also implies that it is, necessarily, a saddle point. The stability of the normal minimum against tunneling to a deeper charge breaking stationary point is thus ensured in the 2HDM.

A similar result holds for the comparison between the CP and normal stationary points. In the case where it makes sense to define a CP symmetry - that is, in potentials where it is not explicitly broken - the difference between the value of the potential in the stationary points with vevs given by eqs. (8) and (10) is given by [4]

\[
V_{CP} - V_N = \frac{M_A^2}{2v^2} \left[ (v_1'' v_2 - v_2'' v_1)^2 + \delta^2 v_2^2 \right].
\]

\( M_A^2 \) is now the value of the squared pseudoscalar mass at the normal stationary point. Again, if this squared mass is positive, for instance if the normal stationary point is a minimum, then the normal minimum is indeed deeper than the CP breaking one. Again, the CP stationary point being unique, the stability of the normal minimum against tunneling is guaranteed. However, unlike the CB case, we were not able to prove, in ref. [4], that the CP stationary point is necessarily a saddle point when the normal stationary point is a minimum. Thus, even though we have proved that the global minimum is a normal one, we cannot discard the possibility that above it a CP breaking minimum exists.

The final case of interest occurs in the potential with explicit CP breaking. In that situation, both the vevs of eq. (8) and those of eq. (10) have the same physical meaning: both stationary points break the same symmetry and the existence of a relative phase between the vevs of each field does not distinguish them. For this potential, we refer to the vevs of (8) as the “\( N_1 \) stationary point”, and those of (10) as the “\( N_2 \) stationary point”. It may be shown [12] that it is possible to pass from a vev structure of the type \( N_2 \) to vevs of the \( N_1 \) type by a specific basis transformation - the complex phase of eq. (10) is absorbed in the parameters of the potential, and the final vevs of both \( \Phi_1 \) and \( \Phi_2 \) are real. However, this does not mean that the \( N_1 \) and \( N_2 \) stationary points are the same - simply that there is a field basis for which \( N_2 \) may be written with real vevs. In that new basis, however, the \( N_1 \) vevs would have acquired a relative complex phase. In ref. [4] we found an interesting relationship between the difference in the depths of

\(^2\)Except for a very special case in the explicit CP breaking potential. Even in that case, though, via a basis change, that vacuum may be reduced to one with \( \alpha \neq 0 \) and \( \delta = 0 \).

\(^3\)Even more generally, when it is a stationary point for which \( M_{H^\pm}^2 > 0 \).
the potential at the $N_1$, $N_2$ stationary points. Namely,

$$V_{N_2} - V_{N_1} = \frac{1}{2} \left[ \left( \frac{M_1^2}{v^2} \right)_{N_1} - \left( \frac{M_1^2}{v^2} \right)_{N_2} \right] \left[ (v''_1 v_2 - v''_2 v_1)^2 + \delta^2 v_2^2 \right] .$$  \tag{13}

In this equation we have $(v')^2_{N_1} = v_1^2 + v_2^2$ and $(v')^2_{N_2} = v''_1 + v''_2 + \delta^2$, and $(M_1^2)_{N_1,2}$ are the squared charged scalar masses at each of the $N_1$, $N_2$ stationary points. A similar expression was found in ref. [8]. Equation (13) tells us that the deepest stationary point will be the one with the largest ratio between the square of the charged Higgs mass and $v^2$.

Let us now consider the stationarity equations that give rise to the different stationary points that we have been discussing. We have mentioned that the CB and CP stationary points are unique since they are given by linear equations on the vevs. However, this is not true for the normal stationary point. Let us begin with the most general 2HDM potential written in a basis where $a_3 = a_4 = 0$. We make $\phi_5 = v_1$ and $\phi_6 = v_2$ in eq. (2) and minimise (3) by solving the equations $\partial V / \partial \phi_i = 0$, $i = 1 \ldots 8$. Most of these equations are trivially satisfied. The non-trivial ones are

$$\frac{\partial V}{\partial \phi_5} = 2 a_1 v_1 + 4 \ b_{11} \ v_1^3 + 2 (b_{12} + b_{33}) v_1 v_2^2 + 3 b_{13} v_1 v_2 + b_{23} v_2^3 = 0$$

$$\frac{\partial V}{\partial \phi_6} = 2 a_2 v_2 + 4 \ b_{22} \ v_2^3 + 2 (b_{12} + b_{33}) v_1^2 v_2 + b_{13} v_1^3 + 3 b_{23} v_1 v_2^2 = 0 ,$$ \tag{14}

and

$$\frac{\partial V}{\partial \phi_7} = v_1 \ (b_{14} v_1^2 + b_{24} v_2^2 + b_{34} v_1 v_2) = 0$$

$$\frac{\partial V}{\partial \phi_8} = - v_2 \ (b_{14} v_1^2 + b_{24} v_2^2 + b_{34} v_1 v_2) = 0 .$$ \tag{15}

Notice that one cannot have solutions of the form $\{v_1 = 0, v_2 \neq 0\}$ or $\{v_1 \neq 0, v_2 = 0\}$, unless some parameters of the potential are set to zero ($b_{23}$, $b_{24}$ and $b_{13}$, $b_{14}$ respectively). Since there is no symmetry forcing those parameters to be zero, they have to be present in the potential. We now define the usual polar coordinates $v_1 = v \cos \beta$ and $v_2 = v \sin \beta$. A trivial solution of these equations is clearly $v = 0$. Excluding that case, the stationarity conditions become

$$v^2 = \frac{1}{\cos^2 \beta} \ b_{23} \tan^3 \beta + 2 (b_{12} + b_{33}) \tan^2 \beta + 3 b_{13} \tan \beta + 4 b_{11}$$ \tag{16}

and

$$- a_2 b_{23} \tan^4 \beta + [4 a_1 b_{22} - 2 a_2 (b_{12} + b_{33})] \tan^3 \beta$$

$$+ 3 (a_1 b_{23} - a_2 b_{13}) \tan^2 \beta + [2 a_1 (b_{12} + b_{33}) - 4 a_2 b_{11}] \tan \beta + a_1 b_{13} = 0$$ \tag{17}

and both equations (16) reduce to

$$b_{24} \tan^2 \beta + b_{34} \tan \beta + b_{14} = 0 .$$ \tag{18}

Eq. (16) tells us that, other than its sign, the value of $v$ is determined unequivocally by $\tan \beta$. Eq. (17) is a quartic equation on $\tan \beta$, having at most four possible real solutions. These two equations describe therefore eight possible solutions $\{v_1, v_2\}$, due to the ambiguity on the sign of $v$. The 2HDM potential (3) is however invariant under the transformation $\Phi_1 \rightarrow -\Phi_1$ and $\Phi_2 \rightarrow -\Phi_2$, so that these eight solutions correspond to only four different physical scenarios. Adding the trivial solution $v_1 = v_2 = 0$, we have a total of nine solutions. However, we must contend with eq. (18) as well, which is a quadratic equation on $\tan \beta$. Then, there are at most
two different values of $\tan \beta$ which satisfy all equations. This means that we have a maximum of five stationary points.

For potentials with explicit CP conservation, equations (15) are trivially satisfied, since $b_{14} = b_{24} = b_{34} = 0$. Therefore, equation (18) does not exist and the potential could have a total of nine stationary points.

At this point we ask: can we have more than one normal minimum, with different depths? The answer is yes, and to see this we make use of Morse’s inequalities [15]: for a given real function of two variables, let $m_0$, $m_1$ and $m_2$ be the number of its minima, saddle points and maxima, respectively. For a polynomial function in $v_1$ and $v_2$, bounded from below, such as the one we are dealing with, Morse’s inequalities state that:

- $m_0 \geq 1$;
- $m_1 \geq m_0 - 1$;
- $m_0 - m_1 + m_2 = 1$.

We know that the 2HDM potential has $m_0 + m_1 + m_2 = 2n + 1$ stationary solutions, $n = 0, \ldots, 4$: at most $2n$ real roots of eqs. (16), (17) and (18) plus the trivial solution $v_1 = v_2 = 0$. Hence we find that $m_0 + m_2 = n + 1$. Let us analyse the several possibilities for the number of minima $m_0$, depending on the number of solutions $n$. Simply counting all the different combinations of extrema leads us to:

- $n = 0$: we have necessarily $m_0 = 1$, the minimum is unique but is located at the origin, $v_1 = v_2 = 0$, which means that there is no $SU(2)_W \times U(1)_Y$ symmetry breaking. This case is excluded on physical grounds.
- $n = 1$: we find two possibilities:
  - $m_0 = 1$, which is the previous case.
  - $m_0 = 2$, which means two degenerate minima away from the origin, related to one another by a change of sign of the vevs. This situation corresponds to an acceptable symmetry breaking and it means that there are no normal minima with different depths. This would be the “standard” situation.
- $n = 2$: there are three possibilities:
  - $m_0 = 1$ or 2 are like the previous cases.
  - $m_0 = 3$, one uninteresting minimum at the origin, and two degenerate ones away from it. This situation would also be the “standard” one, as there would be no normal minima with different depths.
- $n = 3$: there are now three qualitatively different cases:
  - $m_0 = 1, 2, 3$, like above.
  - $m_0 = 4$, this case corresponds to two pairs of degenerate minima away from the origin. Nothing forces these two pairs of minima to have the same depth. We might therefore have one normal minimum deeper than another.
- $n = 4$: we have:
  - $m_0 = 1, 2, 3, 4$, like above.
\(- m_0 = 5\) is similar to the \(m_0 = 4\) case examined above, with an extra minimum present at the origin.

This trivial analysis shows us that, if there are more than two solutions for \(\tan \beta\), then the 2HDM may have more than one normal minimum away from the origin at different depths. However, no more than two such minima can exist.

Why is this interesting? We already know that in the 2HDM, when a normal minimum exists, then the *global* minimum of the theory is normal [4]. However, another interesting possibility might arise: the minimisation equations may give us a normal minimum \(N_1\) with vevs \(\{v_1, v_2\}\) and another normal minimum \(N_2\) with vevs \(\{\hat{v}_1, \hat{v}_2\}\). Which is the deepest? Suppose that we find \(N_1\) for which the vevs are such that the SM phenomenology is satisfied, namely, \(v_1^2 + v_2^2 = (246 \text{ GeV})^2\) and the \(W\) and \(Z\) boson masses are according to their experimental values; but \(N_2\) is deeper, for which the sum of the squared vevs \(\hat{v}_1^2 + \hat{v}_2^2\) has a completely different value, contrary to experimental data. This situation is clearly undesirable since it can lead to tunneling between an acceptable minimum and one with gauge boson and fermion masses different from their measured values. A trivial calculation (see appendix A) shows us that the difference in depth of the potential at \(N_1\) and \(N_2\) is given by

\[
V_{N_2} - V_{N_1} = \frac{1}{2} \left[ \left( \frac{M_H^2}{v^2} \right)_{N_1} - \left( \frac{M_H^2}{\hat{v}^2} \right)_{N_2} \right] (v_1 \hat{v}_2 - v_2 \hat{v}_1)^2,
\]

a result very similar to eq. (13). Again, it is the squared mass of the charged scalars divided by the respective \(v^2\) which “controls” the difference in depths of the potential at the stationary points. As we will shortly show, it is possible, depending on the value of the parameters of the potential, to have such “coexisting” minima in situations of interest for particle physics phenomenology.

4 Neutral minima in potentials with CP breaking

As mentioned before, one of the most interesting features of the 2HDM is that it allows for breaking of the CP symmetry. We will now analyse the possibility of coexistence of neutral minima in this potential, for the two cases of CP breaking: explicit and spontaneous.

4.1 Potential with explicit CP breaking

As we saw earlier, for the 2HDM potential with explicit CP breaking the value of \(\tan \beta\) is determined by a set of two equations, (17) and (18), one quartic and another quadratic. This means that we may have, at most, two different values of \(\tan \beta\). We are therefore in the \(n = 2\) case discussed above. As was shown, this means that the normal minimum exists and is unique.

This conclusion, however, was derived for stationary points with real vevs, of the form (8). We know that in this potential CP is not defined and as such stationary points of the form of eq. (10), in which the vevs have a relative complex phase, have the same physical relevance as stationary points with real vevs. Equation (13) gives us the difference in depth of the 2HDM potential with explicit CP breaking at two neutral stationary points, \(N_1\) (with real vevs) and \(N_2\) (with vevs with a relative complex phase). This result was first obtained in ref. [4], but one question was left open in that work: is it possible to find values of the potential such that one can find two minima that verify eq. (13)? We will perform a numerical study to prove that such a situation is indeed possible.
The stationarity conditions for a $N_1$ stationary point with vevs such as those of eq. (8) were shown in eq. (14). Similarly, for a $N_2$ stationary point, with vevs of the form (10), the stationarity conditions are

$$2a_1 v''_1 + 4b_{11} v''_1 (v''_1 + \delta^2) + 2(b_{12} + b_{33}) v''_1 v''_2 + b_{13} (3v''_1 + \delta^2) v''_2 + b_{23} v''_3$$

$$- 2b_{14} v''_1 v'_2 \delta - b_{34} v''_2 \delta = 0$$

$$2a_2 v''_2 + 4b_{22} v''_2 + 2b_{12} v''_1 (v''_1 + \delta^2) + 2b_{33} v''_1 v''_2 + b_{13} v'_1 (v''_1 + \delta^2) + 3b_{23} v''_1 v''_2 - b_{14} (v''_1 + \delta^2) \delta - 3b_{24} v''_2 \delta - 2b_{34} v''_2 \delta + 2b_{44} v''_2 \delta = 0$$

$$2a_1 \delta + 4b_{11} \delta (v''_1 + \delta^2) + 2b_{12} \delta v''_2 + 2b_{13} v''_1 v''_2 \delta - b_{14} (v''_1 + 3\delta^2) v''_2$$

$$- b_{24} v''_3 - b_{34} v''_1 v''_2 + 2b_{44} v''_2 \delta = 0.$$  \hspace{1cm} (20)

Because we are looking for simultaneous $N_1$ and $N_2$ minima, these equations have to be solved together with the stationarity conditions (14) and (15). There are two ways to do this: (a), generate a set of $\{a_i, b_{jk}\}$ parameters and use both $N_1$ and $N_2$ stationarity conditions to determine the 5 vevs; or (b) generate all but 6 of the parameters $\{a_i, b_{jk}\}$ and also the 5 vevs and use the stationarity conditions to determine the remaining 6 potential parameters. We chose option (b) for two reasons. Firstly, we want one of the minima to describe the “real” world. As such, we know what the sum of the squared vevs should be ($(246 \text{ GeV})^2$) at that minimum. It is easier to use this information by inputing the vevs to begin with. Secondly, the stationarity conditions (14), (20) are cubic in the vevs, and therefore very difficult to solve analytically. Numerical methods need to be used, which are difficult to control and time-consuming. Determining the potential parameters requires nothing more elaborate than solving a set of linear equations, once the vevs have been specified. Our method to solve the stationarity equations, then, consists in: (1) generating values for the vevs, $\{v_1, v_2\}$ and $\{v''_1, v''_2, \delta\}$, and all but 4 of the $b_{ij}$; (2) using eqs. (14) to determine the value of the parameters $\{a_1, a_2\}$; (3) choose the parameters $\{b_{14}, b_{24}, b_{34}, b_{44}\}$ so that they satisfy the remaining stationarity equations, namely:

$$\begin{bmatrix}
2 v''_1 v''_2 \delta \\
\delta (v''_1 + \delta^2) \\
v''_2 (v''_1 + 3\delta^2) \\
v''_2 \\
v''_2 \\
v''_2 \\
v''_2 \\
0
\end{bmatrix}
\begin{bmatrix}
2 v''_1 v''_2 \\
\delta (v''_1 + \delta^2) \\
v''_1 v''_2 \\
v''_1 v''_2 \\
v''_1 v''_2 \\
v''_1 v''_2 \\
v''_1 v''_2 \\
0
\end{bmatrix}
\begin{bmatrix}
2 v''_1 v''_2 \\
\delta (v''_1 + \delta^2) \\
v''_1 v''_2 \\
v''_1 v''_2 \\
v''_1 v''_2 \\
v''_1 v''_2 \\
v''_1 v''_2 \\
0
\end{bmatrix}
\begin{bmatrix}
b_{14} \\
b_{24} \\
b_{34} \\
b_{44}
\end{bmatrix} = \begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4
\end{bmatrix},$$

with

$$S_1 = 2a_1 v'' + 4b_{11} v'' (v'' + \delta^2) + 2(b_{12} + b_{33}) v'' v'' + b_{13} (3v'' + \delta) v'' + b_{23} v''^3$$

$$S_2 = 2a_2 v'' + 4b_{22} v'' + 2b_{12} v'' (v'' + \delta^2) + 2b_{33} v'' v'' + b_{13} v'' (v'' + \delta^2) + 3b_{23} v'' v'' - b_{14} (v'' + \delta^2) \delta - 3b_{24} v'' \delta - 2b_{34} v'' \delta + 2b_{44} v'' \delta = 0$$

$$S_3 = 2a_1 \delta + 4b_{11} \delta (v'' + \delta^2) + 2b_{12} \delta v'' + 2b_{13} v'' v'' \delta - b_{14} (v'' + 3\delta^2) v''$$

$$S_4 = 0.$$  \hspace{1cm} (22)

We chose random values for the potential’s parameters, such that the $b_{ij}$ couplings were of the same order - we considered $b$ parameters in the range $10^{-3} \leq b_{ij} \leq 10$. To be certain that the solutions of the stationarity conditions correspond to minima, we calculated the eigenvalues of the squared scalar mass matrices and verified that, except for the three zeros corresponding to the Goldstone bosons, the remaining ones are positive. The mass matrix expressions for any 2HDM potential may be found in ref. [4].
Let us see, for instance, if it is possible that the $N_1$ and $N_2$ minima have the same value for the squared vevs, but different scalar masses. In other words, can the 2HDM accommodate two minima which predict the same $W$, $Z$ and fermion masses, but different scalar spectra? The answer is, yes. To see this, we solve eqs. (22) by inputing values for the vevs such that $v_1^2 + v_2^2 = v_1''^2 + v_2''^2 + \delta^2 = (246 \text{ GeV})^2$. We also input the several $b_{ij}$ parameters and scan the parameter space, accepting only sets of parameter values for which both $N_1$ and $N_2$ are minima. To make these minima of some physical interest, we also demanded that all the scalar masses be larger than 100 GeV, but inferior to 1 TeV. The results we found are illustrated in fig. (1), where we plot the difference of potential depths at $N_1$ and $N_2$ against the difference in the charged scalar masses at both minima. One immediately observes that the $N_1$ and $N_2$ minima are equally likely to be the deepest minima. We also remark that it is not difficult to find combinations of $\{a, b\}$ parameters and vevs for which these minima coexist: for 100000 sets of parameters for which $N_1$ is a minimum obeying the criteria described above, about 10% have a $N_2$ minimum “alongside”. Hence, for the same set of parameters of the potential we can have two different minima, both of them predicting the same values for the gauge boson masses and fermions, but with different spectra of scalar particles. Tunneling from $N_1$ to $N_2$ would therefore only change the values of Higgs’ masses.
4.2 Potentials with spontaneous CP breaking

For the 2HDM where CP is not explicitly broken, all of the couplings $b_{4i}$ are set to zero, except for $b_{44}$, so that the stationarity condition of eq. (15) is trivially satisfied. Then the number of possible normal minima goes up - we have a total of nine possible stationary points from which, as explained above, we may have a maximum of two non-degenerate minima away from the origin. For these potentials, though, the stationary points of the type $N_2$ now correspond to a spontaneous breaking of the CP symmetry.

For illustrative purposes, let us see if the following situation might occur: a CP breaking minimum with $v = 246$ GeV above a normal one. At this normal minimum, the value of $v$ is not fixed a priori. To achieve this end we once again solve the stationarity conditions of the 2HDM potential by generating random values for most of the parameters and vevs and thus obtaining linear equations on the remaining unknowns. Solving the linear set of equations thereof resulting produces a complete set of parameters for the potential, for which we then proceed to investigate whether both stationary points - the normal and the CP one - are minima, by analysing the eigenvalues of the corresponding squared scalar mass matrices.

![Figure 2: Masses of the neutral Higgs bosons at a CP breaking minimum with $v = 246$ GeV versus the expected $W$ mass at the deeper normal minimum. For each combination of potential parameters, the lightest Higgs mass is represented by crosses, the second lightest mass by stars, the heaviest one by circles.

The results we found are shown in fig. (2), where we plot the masses of the neutral Higgs

\footnote{Since, we repeat, according to the results of ref. [4] the reverse is not possible.}
bosons at the CP minimum, versus the $W$ boson mass expected at the normal minimum. For visualisation purposes, the vertical axis was limited to values above $10^{-5}$ GeV, which removed from view several points corresponding to very light Higgs masses. The normal minimum is found to have depth similar to that of the CP one, but the resulting $W$ masses are in general very different. An interesting feature of the CP violating minimum is also shown in this plot: the neutral Higgs masses are extremely low, much lower than the most recent experimental bounds, with the exception of the heaviest neutral mass, represented by circles in fig. (2), which may reach hundreds of GeV. The second heaviest neutral Higgs (stars in fig. (2)) is also very light. For the lightest neutral Higgs, the maximum value we obtained for its mass was of the order of 0.1 GeV. One must also emphasize that these sets of parameters where CP minima coexist with normal ones are extremely difficult to find. Out of a total of about 10 million normal minima, we only find CP ones with a sum of squared vevs equal to 246 GeV for some 160 of those. Our conclusion, then, is that though coexisting normal and CP minima are numerically possible, they are extremely unlikely, at least in the range of parameters we chose. When they are found, the CP minimum seems to be characterized by having a very low mass for the lightest Higgs boson. This would seem to exclude the strange possibility that we are “living” in a CP minimum with standard $W$ masses, with a normal minimum lying underneath it. Notice, however, that this does not exclude, at all, the possibility that the potential has parameters such that the global minimum of the theory is CP violating. The conclusion of [4] only imply that in this case no normal minimum exists.

We have also considered the situation where the normal minimum has $v = 246$ GeV, with a CP minimum above it, with vevs determined by the stationarity conditions. In cosmological terms this would raise the possibility of a universe “resting” for some time in such a CP breaking minimum immediately after the Big Bang, before tunneling the present normal minimum. Again we found that this situation only occurs for a very small subset of all the parameter space that was scanned. The results obtained are very similar to those shown in fig. (2). The $W$ masses found at the CP minimum and the neutral Higgs masses at the normal minimum tend to be smaller than 100 GeV. Nevertheless, we have found several cases where the Higgs masses are compatible with present experimental bounds.

5 Normal minima in potentials where CP is conserved

In the previous sections we looked at the simultaneous existence of two neutral minima, one with real vevs, the other with complex ones. As we showed in section 3, however, it is possible to have several normal minima with real vevs $\{v_1, v_2\}$, not all of them having the same depths. This is already a possibility for the potentials with soft CP breaking, analysed in the previous section, but the case we treated there was the most interesting situation of a CP-breaking minimum coexisting with a normal one. However, for potentials where CP breaking is impossible the existence of several normal minima for the same potential is of interest. As was explained in section 2 there are two different potentials where spontaneous CP breaking is not possible, depending on the symmetries one has imposed on the 2HDM model. As we will now show, the normal vacua structure is very different for these two potentials.

5.1 Normal vacua in the potential $V_A$

The stationarity conditions for the potential $V_A$ may be obtained from the most general ones written in eqs. (14) and (15) if one makes $b_{13} = b_{23} = 0$. Since for this potential $b_{14} = b_{24} = 0$.
\( b_{34} = 0 \), eq. (15) is trivially satisfied and because \( b_{13} = b_{23} = 0 \), eqs. (14) become
\[
\frac{\partial V}{\partial \phi_5} = 2a_1 v_1 + 4b_{11}v_1^3 + 2(b_{12} + b_{33})v_1v_2^2 = 0
\]
\[
\frac{\partial V}{\partial \phi_6} = 2a_2 v_2 + 4b_{22}v_2^3 + 2(b_{12} + b_{33})v_1^2v_2 = 0 .
\]
(23)

We can identify the trivial solution, \( v_1 = v_2 = 0 \), and three non-trivial ones:

- **Solution I** (if \( a_2 < 0 \)):
  \[
  v_1 = 0 , \quad v_2^2 = -\frac{a_2}{2b_{22}} .
  \]
  At this stationary point the value of the potential is
  \[
  V_I = -\frac{a_2^2}{4b_{22}} .
  \]
(24)

- **Solution II** (if \( a_1 < 0 \)):
  \[
  v_1^2 = -\frac{a_1}{2b_{11}} , \quad v_2 = 0 .
  \]
  At this stationary point the value of the potential is
  \[
  V_{II} = -\frac{a_1^2}{4b_{11}} .
  \]
(27)

- **Solution III**:
  \[
  v_1^2 = \frac{2b_{22}a_1 - (b_{12} + b_{33})a_2}{(b_{12} + b_{33})^2 - 4b_{11}b_{22}} , \quad v_2^2 = \frac{2b_{11}a_2 - (b_{12} + b_{33})a_1}{(b_{12} + b_{33})^2 - 4b_{11}b_{22}} .
  \]
  At this stationary point the value of the potential is
  \[
  V_{III} = \frac{b_{22}a_1^2 + b_{11}a_2^2 - (b_{12} + b_{33})a_1a_2}{4b_{11}b_{22} - (b_{12} + b_{33})^2} .
  \]
(29)

The solutions I and II correspond to models very similar to the SM although, for certain realizations of the 2HDM, these solutions may correspond to cases where either the up or down quarks are massless. The solution III is the most interesting one, and we will analyse it in detail. To do so, let us first consider the relations one can establish between the \( \{ a, b \} \) parameters and some of the physical parameters of the model at a stationary point III: the neutral CP-even scalar masses \( M_h \) and \( M_H \), the angle \( \beta \) with \( \tan \beta = v_2/v_1 \) and the rotation angle \( \alpha \) that diagonalises the matrix of the squared masses of the CP-even scalars. These relations are found in ref. [16], and are given by

\[
a_1 = -\frac{1}{4 \cos \beta} \left[ \cos \alpha \cos(\beta - \alpha) M_H^2 - \sin \alpha \sin(\beta - \alpha) M_h^2 \right]
\]
\[
a_2 = -\frac{1}{4 \sin \beta} \left[ \sin \alpha \cos(\beta - \alpha) M_H^2 + \cos \alpha \sin(\beta - \alpha) M_h^2 \right]
\]
\[
b_{11} = \frac{1}{4v^2 \cos^2 \beta} \left( \cos^2 \alpha M_H^2 + \sin^2 \alpha M_h^2 \right)
\]
\[
b_{22} = \frac{1}{4v^2 \sin^2 \beta} \left( \sin^2 \alpha M_H^2 + \cos^2 \alpha M_h^2 \right) .
\]
(30)

\(^5\)Requiring that the potential be bounded from below implies that \( b_{11} > 0 \) and \( b_{22} > 0 \).
There are more relations, but these four will be all that we will need for our purposes. Let us assume that we have a set of parameters of the potential for which the solution III defined above exists and is a minimum, and for which the solutions I and II exist as well. The question we now ask ourselves is, under these circumstances, is it possible that either I or II are also minima, and deeper than III? Let us start by comparing the value of the potential at the stationary point I and the minimum III. Making use of the fact that at any stationary point the value of the potential is given by \((a_1 v^2_1 + a_2 v^2_2)/2\) and using eqs. (30), we may rewrite \(V_I\) and \(V_{III}\) as

\[
V_I = -\frac{v^2}{4} \left[ \frac{\sin \alpha \cos(\beta - \alpha) M_H^2 + \cos \alpha \sin(\beta - \alpha) M_h^2}{\sin^2 \alpha M_H^2 + \cos^2 \alpha M_h^2} \right]^2
\]

\[
V_{III} = -\frac{v^2}{4} \left[ \cos^2(\beta - \alpha) M_H^2 + \sin^2(\beta - \alpha) M_h^2 \right].
\]

Notice that we are rewriting the values of the potential in terms of the physical parameters (Higgs masses, etc) of the minimum III. Can we have \(V_I < V_{III}\)? According to the expressions above that occurs if

\[
\left[\frac{\sin \alpha \cos(\beta - \alpha) M_H^2 + \cos \alpha \sin(\beta - \alpha) M_h^2}{\sin^2 \alpha M_H^2 + \cos^2 \alpha M_h^2}\right]^2 > \cos^2(\beta - \alpha) M_H^2 + \sin^2(\beta - \alpha) M_h^2
\]

\[
\left[\sin \alpha \cos(\beta - \alpha) M_H^2 + \cos \alpha \sin(\beta - \alpha) M_h^2\right]^2 > \left[\cos^2(\beta - \alpha) M_H^2 + \sin^2(\beta - \alpha) M_h^2\right] \left(\sin^2 \alpha M_H^2 + \cos^2 \alpha M_h^2\right),
\]

where the last step was only possible because we are working under the assumption that the solution III is a minimum - therefore, \(M_h^2 > 0\) and \(M_H^2 > 0\), and the inequality in eqs. (32) is not changed. Developing this inequality leads to a straightforward conclusion:

\[
V_I < V_{III} \Rightarrow \cos(2 \beta) < -1.
\]

A similar impossibility is found if one investigates the case \(V_{II} < V_{III}\) when III is a minimum. Which means that, if the solution III is a minimum, then it is certainly the global minimum of the theory. Then, the 2HDM potential cannot tunnel from a phenomenologically acceptable minimum III, where all quarks are massive, to a deeper solution I or II, where either the up or down quarks could be massless.

As a curiosity, we may also have a situation where the deepest minimum is either of the form I or II. In that case, there are two observations to make: (a) the solution III is necessarily not a minimum; and (b), there is the possibility that both solutions I and II are simultaneously minima. According to eqs. (25) and (27), the specific values of the parameters will determine which of the two solutions corresponds to the deepest minimum.

### 5.2 Normal vacua in the potential \(V_B\)

We remind the readers that for the potential \(V_B\) we cannot choose a field basis so that \(a_3\) and \(\{b_{13}, b_{23}\}\) are simultaneously zero. The stationarity conditions for normal vevs in the potential \(V_B\) are very similar to those of eq. (14), namely

\[
\frac{\partial V}{\partial \varphi_5} = 2 a_1 v_1 + a_3 v_2 + 4 b_{11} v_1^3 + 2 (b_{12} + b_{33}) v_1 v_2^2 = 0
\]

\[
\frac{\partial V}{\partial \varphi_6} = 2 a_2 v_2 + a_3 v_1 + 4 b_{22} v_2^3 + 2 (b_{12} + b_{33}) v_1^2 v_2 = 0 .
\]
The presence of the $a_3$ terms makes it impossible to solve analytically these equations. We can however follow a similar strategy to the one we used to treat the most general 2HDM potential: with polar coordinates $v$ and $\beta$, we obtain an equation for $v^2$ in terms of $\beta$,

$$v^2 = -\frac{1}{\cos^2 \beta} \frac{2a_1 + a_3 \tan \beta}{2 (b_{12} + b_{33}) \tan^2 \beta + 4 b_{11}},$$

(35)

and a quartic equation for $\tan \beta$,

$$a_3 b_{22} \tan^4 \beta + [2 a_1 b_{22} - a_2 (b_{12} + b_{33})] \tan^3 \beta + \{a_1 (b_{12} + b_{33}) - 2 a_2 b_{11}\} \tan \beta - a_3 b_{11} = 0.$$  

(36)

This is an equation very similar to the one we studied in section 3 so the conclusions we reached there are still valid: this potential can have, at most, two pairs of non-degenerate minima away from the origin.

To verify the previous statement, we performed a scan of the parameter space of the $V_B$ potential searching for such minima. As before, our procedure was to randomly generate a partial set of parameters for the potential and two pairs of vevs $\{v_1, v_2\}$, such that $v_1^2 + v_2^2 = (246 \text{GeV})^2$ for the $N_1$ minimum, and a second pair $\{\hat{v}_1, \hat{v}_2\}$, such that $\hat{v}_1^2 + \hat{v}_2^2$ has a value between $(1 \text{ GeV})^2$ and $(1000 \text{ GeV})^2$ for $N_2$. We then use the stationarity conditions (35) to determine, solving a set of linear equations, the parameters $\{a_1, a_2, a_3, b_{33}\}$. So that minima of the type $N_1$ have some phenomenological relevance we excluded the combinations of parameters which produced scalar masses too high (above 1000 GeV) or too low (below about 100 GeV). We found many such $N_1$ solutions, and for a small subset of those the minima $N_1$ and $N_2$ coexist. In fig. 3 we show the plot of the mass of the lightest neutral Higgs boson mass at the minimum $N_1$ - the standard one, for which $M_W = 80.5$ GeV - versus the mass expected for the W boson at the normal minimum $N_2$, when the deepest minimum found is $N_2$. For many other points, we found that $N_1$ was the deepest minimum. The difference in depths of both minima can be large. The remarkable thing that this figure demonstrates is that we may be “living” in a perfectly reasonable $N_1$ minimum, where the known particle masses are those that have been measured; but “below” lies a deeper $N_2$ minimum, with exactly the same gauge symmetries but very different particle mass spectra. In those circumstances, then, tunneling to the deeper minimum - and to a universe with very different particles - is, in principle, possible. We remark that there doesn’t seem to be anything particular - any special combination of parameters of range of values of particle masses, for instance - characterizing either of the two regimes, deeper $N_1$ or deeper $N_2$. We have lightest Higgs masses ranging from $\sim 100$ GeV to $\sim 800$ GeV for the $N_1$ minimum in either regime.

There are two fulcrum observations to make respecting these conclusions. Firstly, the percentage of parameter space where both minima were found to coexist is extremely small: out of a generated 15 million $N_1$ minima, a $N_2$ minimum existed in only about 37000, and of those in only about 1/3 of the cases was $N_2$ deeper than $N_1$. Therefore, this feature of the 2HDM potential is a rare occurrence, one such case for every thousand trials, but it exists nevertheless. Secondly, even if the $N_2$ minimum exists and is below $N_1$, it is not clear whether that means the $N_1$ minimum is unacceptable or even “dangerous”. If the tunneling time from $N_1$ to $N_2$ is vastly superior to the current age of the universe, then $N_1$ would be an acceptable vacuum of the theory. The calculation of tunneling probabilities in models with more than one scalar is a very complex undertaking [17], and outside the scope of the present work.
6 Conclusions

We have performed a thorough analysis of the structure of neutral minima in 2HDM potentials. We have shown that it is possible to have coexisting neutral minima in such potentials. From a careful study of the stationarity conditions we were able to establish the maximum number of possible minima that might exist in those conditions. For the 2HDM potential with explicit CP breaking, we concluded that there might only be one normal minimum with real vevs. However, for this potential, there is no physical distinction between stationary points with real vevs or with vevs which have a relative complex phase. We were then able to scan the parameter space of the potential and discover many combinations of parameters for which these two types of stationary points are indeed minima, with different depths. We chose a particularly curious case in which both of those minima predict identical values for the masses of the known particles (gauge bosons and fermions) but have completely different scalar spectra.

In 2HDM potentials with spontaneous CP breaking we found that CP breaking minima and normal minima can coexist side by side in the potential, but the combinations of parameters corresponding to this situation are extremely rare. Further, requiring that the CP minimum describe the current known particle masses would imply a spectrum of scalars with very low masses, thus seemingly ruled out by experiments.

For potentials where CP breaking is not possible, we can still have normal minima coexisting side by side. For the class of potentials $V_A$, which have a $Z_2$ symmetry, we found that it is possible
to demonstrate, analytically, that if there is a minimum where both fields $\Phi$ have non-zero vevs, that minimum is unique and certainly the global minimum of the theory. However, for another class of potentials - $V_B$, which have a $U(1)$ symmetry - we may have combinations of potential parameters for which two normal minima exist, with different vevs and different scalar spectra. In particular, we showed that it is possible to have the least deep of those minima with $v = 246$ GeV, and the deeper one to have a completely different value for the squared sum of the vevs. This raises the possibility of tunneling between both minima. In fact, the results shown in this paper raise several interesting questions regarding cosmology: is it possible that tunneling between two normal minima, such as were found for the potential $V_B$, occurs in less than the age of the universe? Could we have a cosmological evolution described by a 2HDM potential with spontaneous CP breaking in which the universe first “rests” in a CP breaking minimum, before tunneling or sliding down to the normal minimum that it currently occupies? And what would be the consequences of such an evolution in what regards questions of baryogenesis and matter-antimatter asymmetry? The study of tunneling in models with several scalars is complex, so these questions lie beyond the scope of this present work.

Acknowledgments: We thank L. Sanchez for many useful discussions. This work is supported by Fundação para a Ciência e Tecnologia under contract POCI/FIS/59741/2004. P.M.F. is supported by FCT under contract SFRH/BPD/5575/2001. R.S. is supported by FCT under contract SFRH/BPD/23427/2005.

A Difference between the depth of the 2HDM potential at two normal minima

As was mentioned in sec. 3 we have the possibility, in 2HDM potentials, of having multiple stationary points with vevs of the “normal” type of eq. (8). Let us then consider two such stationary points, one with vevs $\{v_1, v_2\}$, which we will call $N_1$, and another, $N_2$, with vevs $\{\hat{v}_1, \hat{v}_2\}$. Let us also introduce the notation used in [4], and define a vector $X = [x_1 \ x_2 \ x_3 \ x_4]$, containing the values of the $x$ variables of eq. (2) at each of the stationary points. We also define the quantities $V'_i = \partial V/\partial x_i$. For instance, $V_1 = a_1 + 2b_{11} x_1 + b_{12} x_2 + b_{13} x_3 + b_{14} x_4$. The vector $V'$ is thus defined as $V' = [V'_1 \ V'_2 \ V'_3 \ V'_4]$, evaluated at each of the stationary points. The stationarity conditions of the 2HDM potential (eqs. (14) to (18) for the most general potential, or the equivalent ones for the more restricted potentials) imply that, at the $N_1$ stationary point we have

$$X_1 = \begin{bmatrix} v_1^2 \\ v_2^2 \\ v_1 v_2 \\ 0 \end{bmatrix}, \quad V' = \begin{pmatrix} -V'_3 \\ -2v_1 v_2 \end{pmatrix}_{N_1} \begin{pmatrix} v_1^2 \\ v_2^2 \\ 0 \end{pmatrix}, \quad (37)$$

with analogous definitions at the $N_2$ stationary point for the vectors $X_2$ and $\hat{V}'$, with the obvious replacements $v_i \rightarrow \hat{v}_i$. The quantity $-V'_3/2v_1 v_2$, as was shown in ref. [4], is related to the value of the squared charged scalar mass. Namely, we have

$$\left(\frac{V'_3}{2v_1 v_2}\right)_{N_1} = \left(\frac{M^2_{H^\pm}}{v_1^2 + v_2^2}\right)_{N_1} = \left(\frac{M^2_{H^\pm}}{v^2}\right)_{N_1}, \quad (38)$$

with an analogous result for $N_2$. Finally, let us use two more definitions introduced in ref. [4],

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \quad B = \begin{bmatrix} 2b_{11} & b_{12} & b_{13} & b_{14} \\ b_{12} & 2b_{22} & b_{23} & b_{24} \\ b_{13} & b_{23} & 2b_{33} & b_{34} \\ b_{14} & b_{24} & b_{34} & 2b_{44} \end{bmatrix}.$$
With this vector and matrix of parameters, we may write the value of the potential at each of the stationary points in a very concise manner. As was shown in [4] the value of the potential at \( N_1 \) is given by
\[
V_{N_1} = \frac{1}{2} A^T X_1 = -\frac{1}{2} X_1^T B X_1
\] (40)
and the vector \( V' \) at \( N_1 \) has a very simple expression: \( V' = A + B X_1 \). Likewise, the value of the potential at \( N_2 \) will be given by \( V_{N_2} = \frac{1}{2} A^T X_2 \) and \( V' \), at this stationary point, is given by \( \hat{V}' = A + B X_2 \).

With all necessary definitions introduced we may now demonstrate eq. (19). From the definitions of \( X \) and \( V' \) and their values at the stationary points given by eq. (37), it is simple to see that we have
\[
X_1^T \hat{V}' = X_1^T (A + B X_2) = X_1^T A + X_1^T B X_2 = 2V_{N_1} + X_1^T B X_2
\]
\[
X_2^T V' = X_2^T (A + B X_1) = X_2^T A + X_2^T B X_1 = 2V_{N_2} + X_2^T B X_1
\] (41)
where the last equality follows from eq. (40), and its analogue for the \( N_2 \) stationary point. Notice that, because the matrix \( B \) is symmetric, the two terms \( X_1^T B X_2 \) and \( X_2^T B X_1 \) are identical. So that, subtracting these two equations, we obtain
\[
V_{N_2} - V_{N_1} = \frac{1}{2} \left( X_2^T V' - X_1^T \hat{V}' \right) .
\] (42)

Now, using equations (37) and (38) we can write
\[
X_2^T V' = \begin{bmatrix} \hat{v}_2^2 & \hat{v}_2^2 & \hat{v}_1 \hat{v}_2 & 0 \end{bmatrix} \begin{pmatrix} \frac{M_{H^+}^2}{v^2} \\ \frac{v_2}{v_1} \\ -2v_1v_2 \\ 0 \end{pmatrix} = \left( \frac{M_{H^+}^2}{v^2} \right)_{N_1} (v_1 \hat{v}_2 - v_2 \hat{v}_1)^2 \] (43)
and an equation entirely analogous to this one for \( X_1^T \hat{V}' \). From these two equations one obtains the result expressed in equation (19).

B Classification of the several 2HDM potentials

As explained in section 2 there are many different types of 2HDM potentials, depending on whether CP is or is not conserved, and on the types of symmetries that one imposes on the models. There is however a simple way of grouping those several potentials in specific categories, characterized by a single number, which we call the “index” of the potential. With the definitions of the real vectors \( X \) and \( A \) and the matrix \( B \) in the previous appendix, it is trivial to see that the most general 2HDM potential is written as
\[
V = A^T X + \frac{1}{2} X^T B X .
\] (44)

Now, \( B \) being a real and symmetric matrix, it can be diagonalised by a given orthogonal transformation \( O \), such that
\[
O B O^T = \begin{bmatrix} \hat{b}_1 & 0 & 0 & 0 \\ 0 & \hat{b}_2 & 0 & 0 \\ 0 & 0 & \hat{b}_3 & 0 \\ 0 & 0 & 0 & \hat{b}_4 \end{bmatrix} .
\] (45)
Accordingly, the vectors $X$ and $A$ are transformed by the matrix $O$,

$$X \rightarrow \hat{X} = OX, \quad A \rightarrow \hat{A} = OA,$$

so that the potential is now written as

$$V = \hat{a}_i \hat{x}_i + \hat{b}_i \hat{x}_i^2$$

with a sum on the index $i$ assumed. Because $B$ is a $4 \times 4$ matrix and $A$ and $X$ vectors with four elements, we conclude that through the transformation $O$ we can write the potential in terms of only eight quantities $\hat{a}$ and $\hat{b}$ - we say that this potential has index eight. Notice that $O$ is not a basis transformation, and that we have not reduced the number of independent parameters of the potential - were we to study the stationary points of $V$, we would need the original 14 real parameters of the potential (or 11, with a suitable basis transformation) to do so. Equation (47) is nothing more than a simpler way of writing the potential.

For the potential with explicit CP conservation, the matrix $B$ has zeros on its fourth row and column, except the diagonal element, $b_{44}$. Also, $a_4 = 0$, which means that the last entry of the vector $A$ is zero. In this case, the transformation $O$ that diagonalises $B$ has zeros in its fourth row and column, except the $(4, 4)$ element, which is equal to 1. Then, the rotated vector $\hat{A} = OA$ still has a zero in its fourth entry. The rotated matrix $\hat{B}$ has $a$ priori four independent eigenvalues, so this potential ends up being written in terms of seven parameters - three $\hat{a}_i$ and four $\hat{b}_i$. The potential with explicit CP conservation has therefore index seven.

What about the potentials $V_A, V_B, V_C$, for which extra symmetries have been imposed? Well, for $V_A$ and $V_B$ the matrix $B$ has further zeros, since $b_{13} = b_{33} = 0$. The matrix $O$ is therefore block diagonal, with a $2 \times 2$ matrix in its first two rows and columns and the identity in the third and fourth positions. The transformation $O$ therefore does not affect the values of $a_3$, $a_4$, $b_{33}$ and $b_{44}$. The transformed elements therefore satisfy the conditions $\hat{a}_3 = a_3$, $\hat{a}_4 = 0$ ($a_4$ is already zero for these potentials, since they explicitly preserve CP), $\hat{b}_3 = 2b_{33}$ and $\hat{b}_4 = 2b_{44}$.

The potential $V_A$ (eq. (5)) has $a_3 = 0$ and $b_{33} \neq b_{44}$. Therefore, after the transformation $O$, it will have index six - two $\hat{a}_i$ and four $\hat{b}_i$ parameters. If one includes a soft breaking term (an $a_3$ term) then there is an extra parameter - the softly broken $V_A$ has index seven. For the potential $V_B$, because $b_{33} = b_{44}$, two of the eigenvalues of $B$ are equal. Therefore, there are three different $\hat{a}_i$ parameters and three $\hat{b}_i$ ones - $V_B$ has index six. Notice that, had we allowed a massless axion in this model (or in $V_A$), the index number would have been reduced by one. Finally, for the potential $V_C$ of eq. (7), the diagonalisation of the matrix $B$ and corresponding rotation of the vector $A$ leads to the conclusion that this model has index six, and that the softly broken $V_C$ has index seven. The study of the stationary points of the potential through the diagonalization of an analogue of matrix $B$ was done in ref. [8]. An interesting observation is that all the potentials where CP can be broken (explicitly or spontaneously) have index larger or equal to seven.

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