A General Rheology for Dense Fluid-Particle Mixtures

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Abstract

Granular-fluid mixtures widely exist in nature, and with the increase of solid fractions, their rheology behaviors are still elusive in different conditions. The challenge in the prediction of their dynamics is due to the unclear combined effect of fluid-solid and solid-solid interactions on the macroscopic behaviors. In this contribution, we generalize and unify the frictional rheologies of both dense suspensions and dry granular media using a dimensionless number based on the ratio between physically interpretable length scales, which covers a wide range of flow regimes and unifies the existing theories for dry and immersed granular flows. These results support the power-law scaling behaviors near jamming reported in recent investigations. The findings establish a general constitutive framework for visco-inertial granular flows and are important
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for a better understanding of granular-fluid mixtures in both natural and engineering situations, and also provide a reliable method to understand the multi-phase, multi-scale flows under complex force fields.

Keywords: Granular materials, Rheology, Immersed state, Length scale

1 Introduction

Granular flows are ubiquitous in natural phenomena, such as landslides, debris flows, and rock falls [1–3], and they can exhibit different flow behaviors akin to solids, fluids, or gases [4–7]. Complex environmental conditions, and highly dissipative interactions make it difficult to obtain a unified constitutive law for their flow characteristics. On one hand, following pioneering works [8–10] on dry granular flows in steady state conditions, it has been determined that the apparent friction coefficient $\mu$ and the solid fraction $\phi$ can be considered as a function of the inertial number $I$. The inertial number $I = \dot{\gamma}d/\sqrt{P/\rho_s}$ is defined as a ratio of the microscopic time scale ($\sqrt{d^2/\rho_s/P}$) to the macroscopic deformation time scale ($1/\dot{\gamma}$) [7, 9], where $\dot{\gamma}$ is the shear rate, $d$ is the average particle diameter, $P$ is the pressure applied to the granular sample, and $\rho_s$ is the particle density. Lacaze et al. [11] verified this theory through transient granular column collapse experiments, showing a successful application of the $\mu(I)$ theory to granular flows.

On the other hand, when granular materials are fully submerged, Cassar et al. [12] proposed the viscous number, $I_v = \eta_f \dot{\gamma}/P$, where $\eta_f$ is the fluid dynamic viscosity. Boyer et al. [13] generalized the $\mu(I_v)$ constitutive law for suspensions. Trulsson et al. [14] further investigated the rheology of submerged granular flows in the visco-inertial regime, where both contact forces and hydrodynamic forces play important roles, and proposed a combined dimensionless number $K = \lambda I^2 + I_v$ for successfully describing the submerged granular flows in different flow regimes (by varying the viscosity of the interstitial fluid), but this work based on the assumption that the effect of fluid compared to the contact force is a constant represented by $\lambda$ in different conditions, and $\lambda$ is obtained by fitting. More effort is needed to establish a universal constitutive law suitable for the complex granular flow where both particle interactions and hydrodynamic forces are non-negligible. In this case, a proper constitutive relationship is still lacking. In this work, we aim to tackle this problem using both theoretical derivations and numerical simulations.

2 Results

Theoretical derivation of dimensionless length scale. As discussed in the pioneering works [7, 12], the rheology of the granular flow is represented by the microscopic particle movement time scale $t_f$ divided by the macroscopic rearrangement time scale $T = 1/\dot{\gamma}$. $T$ is the time for one layer of particles to
be sheared over one particle diameter, $d$, with respect to the other, and $t_f$ can be seen as the time for a particle to be compressed by the confining pressure $P$ to travel over a particle diameter. Following the work of Cassar et al. [12], the characteristic equilibrium of a single particle inside an immersed granular assembly is given by

$$
\frac{(\pi/6) \rho_s d^3}{\rho_s d^3 \frac{du}{dt}} = \frac{(\pi/4) P d^2 - F_d},
$$

where $F_d$ is the hydrodynamic force in submerged conditions and $u$ is the particle velocity. In granular flows, the Reynolds number is usually very low, hence the hydrodynamic force can be assumed as the Stokes force $F_d = 3\pi \eta_f u d$. During the settling process of a single particle, the particle velocity increases until the hydrodynamic force is equal to the driving force, and the particle reaches the terminal velocity $u_f = P d/(12 \eta_f)$. In previous works [10, 13], in the inertial regime (or dry granular flows), the drag force is neglected, which assumes that particles travel with a constant acceleration $a_c = 3P/(2 \rho_s d)$, with a deduced settling time $t_f = t_{dry} = \sqrt{4 \rho_s d^2/(3P)}$ and a time scale ratio $t_f/T = (2/\sqrt{3})I$.

The constant factor $2/\sqrt{3} \approx 1.15$ is usually ignored. In the viscous regime, we assume that the particle travels with the maximum velocity $u_f = P d/(12 \eta_f)$ for the characteristic length $d$, the settling time is $t_f = t_{sub} = 12 \eta_f / P$, and the time scale ratio $t_f/T = 12 I_v$. However, when the particles flow in the fluid where the inertial force is comparable to the hydrodynamic force, also known as the viscous-inertial regime, we can describe their rheology by neither the inertial number $I$ nor the viscous number $I_v$ individually.

We solve Eq. 1 and obtain the settling velocity of particles in a fluid, $u(t) = u_f(1 - e^{-at})$, where $a = 18 \eta_f / (\rho_s d^2)$. The distance of a sphere settling in fluid
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Fig. 2 (a) Correlation between the apparent frictional coefficient \( \mu \) and the solid fraction \( \phi \). The inset plots the normalized friction \( \mu/\mu_c \) and normalized solid fraction \( \phi/\phi_c \). (b) We plot \( \phi/\phi_c \) as a function of the time ratio \( G \). With the increasing of \( ST \), \( \phi_c - \phi \propto G^\alpha \), where \( \alpha \) decreases from 1 to 0.5.

is \( l = \int_0^T u(t)dt = u_f T - \frac{u_f}{a}(1 - e^{-at}) \). Hence, the microscopic time \( t_f \) of a particle traveling for \( d \) is \( t_f = \frac{12\eta_f}{P} + \frac{\rho_s d^2}{18\eta_f} \left[ 1 + W\left( -e^{-216\eta_f^2/(\rho_s d^2 P)} - 1 \right) \right] \). We can obtain two characteristic dimensionless numbers. The ratio between the microscopic time scale \( t_f \) and macroscopic rearrangement time scale \( T \) is

\[
G = \frac{t_f}{T} = 12I_v + \frac{I^2}{18I_v} \left[ 1 + W\left( -e^{-\frac{216I_v^2}{18I_v}} \right) \right],
\]

(2)

where \( W(\cdot) \) is the Lambert-W function. The derivation details is in Appendix.A.

We then define a microscopic characteristic length scale, \( l \), as the traveling distance of a particle over the rearrangement time, \( T \), that

\[
l = \int_0^T u(t)dt = u_f T - \frac{u_f}{a}(1 - e^{-at}) = \frac{Pd}{12\eta_f \dot{\gamma}} - \frac{P\rho_s d^3}{216\eta_f^2} \left( 1 - e^{-\frac{18\eta_f}{\rho_s d^2 \dot{\gamma}}} \right). \]

(3)

We calculate the dimensionless length scale, \( G \), as the ratio between macroscopic deformation length, \( L = d \), and \( l \) so that

\[
G = \frac{L}{l} = \frac{216I_v^2}{18I_v - I^2 \left( 1 - e^{-18I_v/I^2} \right)}. \]

(4)

**Physical interpretation of \( I \), \( I_v \), and \( K \).** As shown in Table.1, \( G \) and \( G \) are consistent with the previous works in different conditions, and \( ST = I/I_v \) is used to measure the different proportion effect between the fluid and contact effects in different flow regimes. Within two extreme conditions, where granular
materials change from a free-fall regime to a viscous regime ($ST$ varies from $\infty$ to 0), $G$ naturally transforms from $I$ to $I_v$, which implies that, if we relate the solid fraction to $G$ as $\phi_c - \phi \propto G^\alpha$, the exponential parameter, $\alpha$, needs to vary from 1.0 to 0.5 as the regime of a granular system changes. However, $G$ exhibits a more convenient transition from $I^2$ to $I_v$ so that $\lim_{ST \to \infty} G^{0.5} = (4I^2/3)^{0.5}$ and $\lim_{ST \to 0} G^{0.5} = (12I_v)^{0.5}$, which recovers the relationship of $\phi_c - \phi \propto G^\beta$, where $\beta = 0.5$ and will be further verified and discussed based on both our simulation results and experimental data acquired from previous studies.

$G$ and $G$ are obtained from the microscopic momentum equation, they both naturally contain the mixed effect of fluid-solid and solid-solid interactions. We can deduce the visco-inertial number $K = I_v + \lambda I^2$ with a varying $\lambda(St) = (1 - e^{-18/ST})^{18 - ST(1 - e^{-18/ST})}$ from the dimensionless length scale $G$, where $St = I^2/I_v$, the derivation detail is in the Appendix.B.

**Table 1** Generalization the previous rheological laws with $G$ and $G$

| Flow regimes | Rheological laws | $G = t_f/T$ | $G = \mathcal{L}/l$ |
|--------------|-----------------|-------------|---------------------|
| Inertial     | $\phi_c - \phi \propto I[7, 9, 10]$ | $\lim_{ST \to \infty} G = 2I/\sqrt{3}$ | $\lim_{ST \to \infty} G = 4I^2/3$ |
| Viscous      | $\phi_c - \phi \propto I_v^{0.5}[12, 13]$ | $\lim_{ST \to 0} G = 12I_v$ | $G/12 = I_v + \lambda(st)I^2$ |
| Visco-inertial | $\phi_c - \phi \propto K^{0.5}[14]$ | - | $G = \phi_c - \phi \propto G^\alpha$ |
| General law  | -               | -           | $\phi_c - \phi \propto G^{0.5}$ |

Here, we only present the changing of solid fraction with the dimensionless numbers, because the apparent frictional coefficient usually shows the same scaling law with the solid fraction.

The derivation details of $G$ and $G$ transform into different form with different $ST$ are in the Appendix.A.

**Numerical simulation setup.** We use DEM with frictional interactions modeled by a Hookean contact law with energy dissipation [15] to implement a series of three-dimensional shearing simulations in dry and submerged conditions. The setup is shown in Fig.1(a). We establish periodic boundaries in both x - and y - directions, and set up two rough plates vertical to the z - direction to apply the shear velocity in the x - direction. The bottom plate is fixed in the z-direction, while the top plate exerts a constant pressure to the granular system. The system contains 2800 spherical particles, their radii are uniformly distributed in the range of 0.07 – 0.1 cm, their density is 1.18 g/cm$^3$, the normal stiffness is $5 \times 10^7$ dyn/cm, and the tangential stiffness is half of the normal one. The restitution coefficient and the frictional coefficient are 0.2 and 0.3, respectively. The shear rate and the confining pressure are in the range of 0.15 - 50 s$^{-1}$ and 80 - 300 Pa. For the submerged condition, the interstitial fluid viscosities vary from 0.001 to 10 dyn-s/cm$^2$. The particles are subjected to three types of forces: contact force, drag force, and lubrication forces, which will be discussed in the Method Section.

**Solid fraction and apparent friction.** As shown in Fig.1(b),(c),(d), the granular system reaches the steady-state when the solid fraction, $\phi$, and the apparent friction coefficient, $\mu$, become stable. Meanwhile, we obtain a linear
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Fig. 3  Simulation data of dry and submerged conditions with different viscosities. We plot \( \phi/\phi_c \) as a function of (a) \( L/l = 4I^2/3 \), (b) \( L/l = 12I_v \), (c) \( K = I_v + \lambda I^2 \), \( \lambda = 0.1 \), (d) \( L/l = G \), with solid line given by Eq.5.

Fig. 4  Simulation data of dry and submerged conditions with different viscosities. We plot \( \mu/\mu_c \) as a function of (a) \( L/l = 4I^2/3 \), (b) \( L/l = 12I_v \), (c) \( K = I_v + \lambda I^2 \), \( \lambda = 0.1 \), (d) \( L/l = G \), with solid line given by Eq.6.

velocity profile along the z - direction. We plot both our simulation results and the experimental data obtained from Ref.[16] in Fig.2(a). Although the implementation of the experiment and simulation is not a one-to-one match, the relationships between the solid fraction and the apparent friction show the same trend. The normalized solid fraction \( \phi/\phi_c \) and normalized apparent frictional coefficient \( \mu/\mu_c \) shows a single relation as shown inside Fig.2(a).

As investigated in previous works, \( \phi \) follows a power-law relationship, \( (\phi_c - \phi) \propto \left( t_f/T \right)^{\alpha} \), where \( \alpha \) is a scaling parameter. In dry granular systems, \( t_f/T = t_{dry}/T = 2I/\sqrt{3} \) and \( \alpha = 1.0 \) [7, 9]. As shown in Fig.3(a), as decreasing the viscosity of the fluid, until the fluid effect is nearly nil, \( (\phi_c - \phi) \propto I \) as shown in Fig.3(a). In a submerged condition, when the fluid effect is dominant, \( t_f/T = t_{sub}/T = 12I_v \) and \( \alpha = 0.5 \), as shown in Fig.3(b), with the increase of the viscosity of fluid above a threshold (here \( \eta_f = 1 \text{ g}/(\text{cm}\cdot\text{s}) \)), data collapse to \( (\phi_c - \phi) \propto \left( I_v \right)^{0.5} \), which is also obtained in Refs.[12, 13]. However, when the fluid viscosity varies in a wide range, the fluid effect and contact effect are both significant, \( \alpha \) varies with the viscosity of fluid [14]. Although, \( G = t_f/T \) naturally contains the varying effects of fluid and grain contact in different conditions, and smoothly transforms from \( I = t_{dry}/T \) to \( I_v = t_{sub}/T \), the exponents of \( \alpha \) for \( (\phi_c - \phi) \propto \left( t_f/T \right)^{\alpha} \) based rheological law from dry to very viscous regime is still hold an unclear decay from 1 to 0.5 that leads to \( (\phi_c - \phi) \propto G \) being not universal as shown in Fig.2(b). On the contrary, as we have predicted in the previous section, \( G \) obtains a good collapse of all data, which shows that the dimensionless length scale \( G \) can well describe the change of both inertial and hydraulic effects during flow regime transitions. Thus, we
Fig. 5  Evolution of $\lambda$ of $K = I_v + \lambda I^2$ in terms of $St$.

have

$$\frac{\phi(G)}{\phi_c} = \frac{1}{1 + b\sqrt{G}},$$

(5)

where $b = 0.2$ is a constant suitable for both the present system and the experimental data from Ref.[16], a different $b = 0.3$ for 2-D condition collapses the simulation data in Ref.[17]. The 2-D collapse data is shown in the Supplementary files. The dimensionless numbers, $I$, and $I_v$ can only describe the behavior in certain regimes, but not in the full spectrum of flow behaviors. As for the dimensionless number $K = I_v + \lambda I^2$, $\lambda$ is different among different works. For 2-D configurations, Trulsson et al.[14] suggests $\lambda = 0.635$. However, in the work of Ref.[17], $I_m = \sqrt{\alpha_i I_v + \alpha_i I^2} = (1/\alpha_v)\sqrt{I_v + (\alpha_i/\alpha_v)I^2}$ is used to describe the rheological law of submerged granular flows, $\alpha_i = 1.0$, and $\alpha_v = 2.0$, it could be seen a $K$ number with $\lambda = \alpha_i/\alpha_v = 0.5$. For 3-D conditions, $\lambda = 0.1$ in Ref.[16], but in Ref.[18] $\lambda = 0.635$ when $K < 0.01$, and $\lambda = 0.3$ for the rest. According to the changing of the length scale in different conditions, the parameter $\lambda(St)$ of $K$ varies in the range of 0.056-0.11 in different flow regimes as shown in Fig.5, hence with a $\lambda = 0.1$ obtained in Ref.[16], $K$ also shows a good collapse of all the experimental and simulation data in 3-D, however $G$ is superior due to the lack of a free fitting parameter ($\lambda$).

In Fig. 4(a), we plot the relationship between $\mu$ and $I$, where we find that the $\mu - I$ rheology is sufficient to describe the constitutive relationship of systems in inertial regimes [dry, $\eta_f = 0.001 \text{ g/(cm-s)}$, $\eta_f = 0.01 \text{ g/(cm-s)}$]. However, as we increase the fluid viscosity, to reach the viscous regime, the $\mu - I$ relationship of systems with $\eta_f = 0.1 \text{ g/(cm-s)}$, $\eta_f = 1 \text{ g/(cm-s)}$ and $\eta_f = 10 \text{ g/(cm-s)}$ deviates from the others. Plotting the relationship between $\mu$ and $I_v$ in Fig. 4(b) shows that the data have a good collapse for systems with $\eta_f = 0.1 - 10 \text{ g/(cm-s)}$, but cannot capture the behavior of systems with $\eta_f = 0.01$ and $\eta_f = 0.001$ and the dry samples. Systems in different regimes result in distinct rheological behaviors, which further indicates that universal rheology is needed to describe granular flows among free-fall, inertial, and viscous regimes. The rheology of dense suspensions in different flow regimes can be described by the dimensionless number $\mathcal{G}$ as shown in Fig.4(d), while $\mu$ for both dry and submerged conditions with a broad range of different viscosities could be generalized as a function of the dimensionless length scale $\mathcal{G} = \mathcal{L}/l$,
with the following equation

$$\mu(G)/\mu_c = 1 + G_0 \sqrt{G},$$  \hspace{1cm} (6)

where $\mu_c$ is the minimum apparent friction coefficient, $G_0 = 1.25$ is a constant obtained by fitting. Due to $\lambda$ varying in a short range 0.056-0.11, even $K$ with a constant fitting parameter $\lambda = 0.1$ also shows good collapses. As the investigation above, we find that the square root of dimensionless length scale shows a universal linear relation with the solid fraction and apparent frictional coefficient. Due to the relation between the time scale ratio and the dimensionless length scale could be expressed as $L/l = (t_f/T)^\kappa$. In a dry condition, the macroscopic deformation $L = d\dot{\gamma}t$, and the microscopic particle displacement $l = 1/2a_c t^2$, and $\kappa = 2$. However, in a very viscous regime, $l = u_f t$ and $\kappa = 1$. Between these two extreme regime, $l = u_f t - u_f/a(1 - e^{-at})$, $\kappa$ decays from 2 to 1. Hence, we can obtain the following relations

$$\phi_c - \phi \propto \sqrt{\frac{L}{l}} \propto \sqrt{\left(\frac{t_f}{T}\right)^\kappa},$$  \hspace{1cm} (7)

and this is the reason why $\alpha = \kappa/2$ decays from 1 to 0.5 in the $(t_f/T)^\alpha$-based rheologies[7, 9, 10, 12, 13]. In the work of Ref.[19], the dynamical processes of the granular flow are strongly correlated to the size of structural scales, and the jamming transitions are related to the emergence of large scale structures[20]. A characterized length scale related to time ratios is deduced to govern the rheology of granular flows[21]. In the present work, we generalizes these $(t_f/T)^\alpha$- and characterized length scale based works and provide a universal rheological law for granular flow in the whole flow regimes.

3 Discussion

Granular materials are omnipresent in nature and are crucial to human beings. On one hand, granular mixed with fluid transport on the earth can generate templates for human settlement. On the other hand, the unpredictable dynamic behaviors such as debris flows and submarine landslides cause lots of human lives and cause numerous loss of properties. We propose a general constitutive relationship that is suitable for granular flows in different conditions. The dimensionless length scale $G$ can accurately describe the granular material flow in both subaqueous and subaerial conditions, where the effect of fluid and grain forces change under different confining pressures, fluid viscosities, and macroscopic deformations. $\mu - G$ rheology naturally transforms into $\mu - I$, when hydrodynamic effects are negligible, and it converges to $\mu - I_v$ when those effects are significant, which could be used to formulate constitutive models for large-scale prediction at larger scales than the ones we have explored in this study.
Further, in the present work, we show that the characteristic time ratio and characteristic length scale between the microscopic and macroscopic scale, are the reliable and useful parameters to investigate a multi-phase, multi-scale system. The microscopic mechanism of a material completely reflects the macroscopic behaviors. The characteristic length scale is naturally connected with the changing of dimensional parameters such as solid fraction, which is more convenient to be used to extract the macroscopic features dependent on the solid fraction. This method could be used to investigate granular materials under more complicated force fields.

4 Methods

The numerical simulation is implemented by the Discrete Element Method[15]. The particles are subjected to the contact forces, lubrication forces, and drag forces, and their displacements are calculated by

\[ m_i \frac{d^2 r_i}{dt^2} = \sum_k f_{ik}^n \mathbf{n}_k + \sum_j f_{ij}^{lubr,n} + \sum_j f_{ij}^{lubr,t} + f_i^{drag} \]  

(8)

\[ I_i \frac{d \mathbf{w}_i}{dt} = \sum_k f_{ik}^t \mathbf{t}_k \times \mathbf{c}_k + \sum_j f_{ij}^{lubr,t} \times \mathbf{c}_{ij} \]  

(9)

where \( m_i, \mathbf{r}_i, I_i, \mathbf{w}_i \) are the mass, position, inertia matrix, and rotation vector of particle \( i \), respectively. \( f_{ik}^n, f_{ik}^t \) are the normal contact force, and tangential contact force subjected by particle \( k \). \( \mathbf{n}_k \) is the normal unit vector pointing from the center of particle \( i \) to particle \( k \), \( \mathbf{t}_k \) is the tangential unit vector pointing in the opposite direction of the relative tangential displacement between the contact particle \( i \) and \( k \). \( \mathbf{c}_k \) is the vector from the center of particle \( i \) to the contact point with particle \( k \). \( f_{ij}^{lubr,n} \) and \( f_{ij}^{lubr,t} \) are the lubrication forces in the normal and tangential direction between particle \( i \) and \( j \). \( \mathbf{c}_{ij} \) is the vector from the center of particle \( i \) to the point where the surface of particle \( i \) interacting with the joining vector between the center of particle \( i \) and particle \( j \). \( f_i^{drag} \) is the drag force for particles given by

\[ f_i^{drag} = 3 \pi \eta \mathbf{d} \left[ \mathbf{u}^f(z_i) - \mathbf{u}^p_i \right] , \]  

(10)

where \( \mathbf{u}^f \) and \( \mathbf{u}^p \) are the velocities of the fluid and the particle. Further, we assume that the fluid velocity is linearly distributed in the z-direction inspired by the work of Ref.[14]. It needs to be mentioned that the drag force in the present work is based on the condition of a low Reynolds number in the dense granular flow. The lubrication force [22] between particle \( i \) and \( j \) in the normal
and tangential directions are

\[ f_{ij}^{lubr,n} = 6\pi \eta_f R_{\text{eff}}^2 v_{rel}^n \frac{\delta_g}{\delta_g}, \quad \text{and} \]
\[ f_{ij}^{lubr,t} = 6\pi \eta_f R_{\text{eff}}^2 v_{rel}^t \left[ \frac{8}{15} \ln \frac{R_{\text{eff}}}{\delta_g} + 0.9588 \right], \tag{11} \]

where \( R_{\text{eff}} = \left[ 2(d_i^{-1} + d_j^{-1}) \right]^{-1} \), \( d_i \) and \( d_j \) are the diameters of particles \( i \) and \( j \) and \( \delta_g \) is the distance between the surfaces of particles \( i \) and \( j \). The cut-off length of \( \delta_g \) is \( R_{\text{eff}} \). \( v_{rel}^n \) and \( v_{rel}^t \) are the relative velocities between the nearest points of two particles in the normal and tangential direction. We calculate the stress, \( \sigma_{ij} \), of the granular assembly excluding the grain near the plate through the contact pairs [23]. The normal pressure and shear stress are given by, \( P = -\sigma_{ii}/3, \tau = \sqrt{\tau_{ij}^n \tau_{ij}^t}/2 \), where \( \tau_{ij} = P\delta_{ij} + \sigma_{ij} \).

\textbf{Data availability.} All relevant data are available upon request from the authors.

\textbf{Appendix A Time ratio and length scale derivation}

We solve Eq.1 and obtain the settling velocity of particles in a fluid, \( u(t) = u_f(1 - e^{-at}) \), where \( a = 18\eta_f/\rho_s d^2 \). The traveling distance of a particle is \( d = \int_0^{t_f} u(t)dt = u_f t_f - \frac{u_f}{a}(1 - e^{-at_f}) \), which is equal to

\[ a \left( t_f - \frac{1}{a} - \frac{d}{u_f} \right) e^a \left( t_f - \frac{1}{a} - \frac{d}{u_f} \right) = e^{-\frac{ad}{u_f}}. \tag{A1} \]

Eq.A1 has a form of Lambert W-function, \( X e^X = Y \), where \( X \) is the independent variable, \( Y \) is the dependent variable, and \( X \) can be calculated using the Lambert W-function \( W \) as \( X = W(Y) \) [24]. Hence, the solution for \( t_f \) is expressed as

\[ t_f = \frac{d}{u_f} + \frac{1}{a} + \frac{1}{a} W \left( -e^{-\frac{ad}{u_f} - 1} \right) \]
\[ = \frac{12\eta_f}{P} + \frac{\rho_s d^2}{18\eta_f} \left[ 1 + W \left( -e^{-216\eta_f^2/(\rho_s d^2 P)} - 1 \right) \right]. \tag{A2} \]
The ratio between the microscopic time scale $t_f$ and macroscopic rearrangement time scale $T$ is

$$G = \frac{t_f}{T} = 12I_v + \frac{I_v^2}{18I_v} \left[ 1 + W \left( -e^{-\frac{216I_v^2}{ST^2}} - 1 \right) \right]$$

$$= 12I_v \left[ 1 + \frac{ST^2}{216} + \frac{ST^2}{216} W \left( -e^{-\frac{216}{ST^2}} - 1 \right) \right]$$

$$= \frac{2I}{\sqrt{3}} \left[ \frac{6\sqrt{3}}{ST} + \frac{ST\sqrt{3}}{36} + \frac{ST\sqrt{3}}{36} W \left( -e^{-\frac{216}{ST^2}} - 1 \right) \right]. \quad (A3)$$

When the inertial force is dominant, the hydrodynamic force is nearly nil, and one can obtain

$$\lim_{ST \to \infty} G = \frac{2I}{\sqrt{3}} \left[ \frac{6\sqrt{3}}{ST} + \frac{ST\sqrt{3}}{36} + \frac{ST\sqrt{3}}{36} W \left( -e^{-\frac{216}{ST^2}} - 1 \right) \right]$$

$$= \frac{2I}{\sqrt{3}} \left[ 1 + W \left( -e^{-\frac{216}{ST^2}} - 1 \right) \right]. \quad (A4)$$

Due to $\lim_{ST \to +\infty} W\left( -e^{-\frac{216}{ST^2}} - 1 \right) = W(-1/e) = -1$, we apply the Taylor expansion to $W(x)$ as $x \to -1/e$ so that $W(x) = -1 + \sqrt{2(1 + ex)} - O(1 + ex)$ [25]. Thus,

$$\lim_{ST \to \infty} G = \frac{2I}{\sqrt{3}} \left[ \frac{6\sqrt{3}}{36} \right] \left[ \sqrt{2(1 - e^{-\frac{216}{ST^2}})} - O \left( 1 - e^{-\frac{216}{ST^2}} \right) \right]$$

$$\approx \frac{2I}{\sqrt{3}} \left[ \frac{6\sqrt{3}}{36} \right] \left[ \sqrt{\frac{216}{ST^2}} - O \left( \frac{216}{ST^2} \right) \right]$$

$$= \frac{2I}{\sqrt{3}} \left[ 1 + O \left( \frac{6\sqrt{3}}{ST} \right) \right] = \frac{t_{dry}}{T}. \quad (A5)$$

For the dimensionless length scale

$$\lim_{I_v \gg I_v} G = \frac{216I_v^2}{18I_v - I_v^2(1 - e^{-18I_v/I_v^2})}$$

$$\approx \frac{216I_v^2}{18I_v - I_v^2(18I_v/I_v^2 - 162I_v^2/I_v^4 + O(I_v^3/I_v^6))}$$

$$= \frac{216I_v^2}{162I_v^2/I_v^2 + O(I_v^3/I_v^4)}$$

$$= 4I^2/3. \quad (A6)$$
When the hydrodynamic force is dominant, one can obtain
\[
\lim_{ST \to +0} G = 12I_v \left[ 1 + \frac{ST^2}{216} + \frac{ST^2}{216} W \left( -e^{-\frac{216}{ST^2}} \right) \right]
\]
\[
= 12I_v = \frac{t_{sub}}{T},
\]
where \( W(0) = 0 \). The dimensionless length scale is
\[
\lim_{I \ll I_v} G = \frac{216I_v^2}{18I_v - I^2(1 - e^{-18I_v/I^2})} \approx \frac{216I_v^2}{18I_v - I^2} = 12I_v.
\]

As granular materials change from a free fall regime to a viscous regime, the dimensionless number \( G \) naturally transforms from an inertial number to a viscous number.

**Appendix B  Relationships between \( G \) and \( K \)**

First, we assume that \( G \propto K \), then, \( G = \beta K = \beta(I_v + \lambda I^2) \).

\[
\frac{216I_v^2}{18I_v - I^2(1 - e^{-18I_v/I^2})} \beta I_v = \beta \lambda I^2
\]
\[
216I_v^2 - 18\beta I_v^2 + \beta I_v J^2(1 - e^{-18I_v/I^2})
\]
\[
18I_v - I^2(1 - e^{-18I_v/I^2}) = \beta \lambda I^2,
\]
we select \( \beta = 12 \)

\[
\lambda = \frac{(1 - e^{-18I_v/I^2})}{18 - I^2/I_v(1 - e^{-18I_v/I^2})}.
\]

represent \( St = I^2/I_v \), we obtain \( \lambda(St) = \frac{(1-e^{-18/ST})}{18-St(1-e^{-18/ST})} \). Hence, \( G = 12(I_v + \lambda(St)I^2) \).

**Appendix C  Critical apparent friction and solid fraction**

The critical solid fraction and apparent friction in different conditions are list in Table.C1.
Table C1  Critical apparent friction and solid fraction

| η (g/(cm·s)) | Dry | 0.001 | 0.01 | 0.2 | 1 | 10 | Dry | Water |
|--------------|-----|-------|------|-----|---|----|-----|-------|
| φc           | 0.605| 0.605 | 0.605| 0.605| 0.609| 0.609| 0.596| 0.586 |
| µc           | 0.363| 0.363 | 0.363| 0.35 | 0.35 | 0.34 | 0.386| 0.41  |

1 Experimental data from Ref.[16]

References

[1] Hutter, K., Koch, T., Pluüiss, C. & Savage, S. B. The dynamics of avalanches of granular materials from initiation to runout. part ii. experiments. Acta Mechanica 109 (1), 127–165 (1995).

[2] Tegzes, P., Vicsek, T. & Schiffer, P. Avalanche dynamics in wet granular materials. Physical Review Letters 89 (9), 094301 (2002).

[3] Yang, G. C., Jing, L., Kwok, C. Y. & Sobral, Y. D. Pore-scale simulation of immersed granular collapse: Implications to submarine landslides. Journal of Geophysical Research: Earth Surface 125 (1) (2020).

[4] Jaeger, H. M., Nagel, S. R. & Behringer, R. P. Granular solids, liquids, and gases. Reviews of modern physics 68 (4), 1259 (1996).

[5] Roux, J.-N. & Combe, G. Quasistatic rheology and the origins of strain. Comptes Rendus Physique 3 (2), 131–140 (2002).

[6] Goldhirsch, I. Rapid granular flows. Annual review of fluid mechanics 35 (1), 267–293 (2003).

[7] GDR & MiDi. On dense granular flows. The European Physical Journal E 14, 341–365 (2004).

[8] Iordanoff, I. & Khonsari, M. Granular lubrication: toward an understanding of the transition between kinetic and quasi-fluid regime. J. Trib. 126 (1), 137–145 (2004).

[9] Da Cruz, F., Emam, S., Prochnow, M., Roux, J.-N. & Chevoir, F. Rheophysics of dense granular materials: Discrete simulation of plane shear flows. Physical Review E 72 (2), 021309 (2005).

[10] Jop, P., Forterre, Y. & Pouliquen, O. A constitutive law for dense granular flows. Nature 441 (7094), 727–730 (2006).

[11] Lacaze, L. & Kerswell, R. R. Axisymmetric granular collapse: a transient 3d flow test of viscoplasticity. Physical Review Letters 102 (10), 108305 (2009).
A General Rheology for Dense Fluid-Particle Mixtures

[12] Cassar, C., Nicolas, M. & Pouliquen, O. Submarine granular flows down inclined planes. *Physics of fluids* **17** (10), 103301 (2005).

[13] Boyer, F., Guazzelli, É. & Pouliquen, O. Unifying suspension and granular rheology. *Physical Review Letters* **107** (18), 188301 (2011).

[14] Trulsson, M., Andreotti, B. & Claudin, P. Transition from the viscous to inertial regime in dense suspensions. *Physical Review Letters* **109** (11), 118305 (2012).

[15] Cundall, P. A. & Strack, O. D. A discrete numerical model for granular assemblies. *Geotechnique* **29** (1), 47–65 (1979).

[16] Tapia, F., Ichihara, M., Pouliquen, O. & Guazzelli, E. Viscous to inertial transition in dense granular suspension. *Phys. Rev. Lett.* **129**, 078001 (2022). URL https://link.aps.org/doi/10.1103/PhysRevLett.129.078001. https://doi.org/10.1103/PhysRevLett.129.078001.

[17] Amarsid, L. *et al.* Viscoinertial regime of immersed granular flows. *Physical Review E* **96** (1), 012901 (2017).

[18] Ness, C. & Sun, J. Flow regime transitions in dense non-brownian suspensions: Rheology, microstructural characterization, and constitutive modeling. *Physical Review E* **91** (1), 012201 (2015).

[19] Bonnoit, C., Lanuza, J., Lindner, A. & Clement, E. Mesoscopic length scale controls the rheology of dense suspensions. *Physical review letters* **105** (10), 108302 (2010).

[20] Bocquet, L., Colin, A. & Ajdari, A. Kinetic theory of plastic flow in soft glassy materials. *Physical review letters* **103** (3), 036001 (2009).

[21] DeGiuli, E., Düring, G., Lerner, E. & Wyart, M. Unified theory of inertial granular flows and non-brownian suspensions. *Physical Review E* **91** (6), 062206 (2015).

[22] Goldman, A. J., Cox, R. G. & Brenner, H. Slow viscous motion of a sphere parallel to a plane wall—i motion through a quiescent fluid. *Chemical engineering science* **22** (4), 637–651 (1967).

[23] Goldhirsch, I. & Goldenberg, C. On the microscopic foundations of elasticity. *The European Physical Journal E* **9** (3), 245–251 (2002).

[24] Scott, T. C., Mann, R. & Martinez Ii, R. E. General relativity and quantum mechanics: towards a generalization of the lambert w function a generalization of the lambert w function. *Applicable Algebra in Engineering, Communication and Computing* **17** (1), 41–47 (2006).
[25] Veberič, D. Lambert w function for applications in physics. *Computer Physics Communications* **183** (12), 2622–2628 (2012).

**Acknowledgments.** This work is supported by the National Natural Science Foundation of China (NSFC grant NO. 12172305). We thank Westlake High-Performace Computing Center for computational resources and related assistance. The simulations were based on the MECHSYS open source library (http://mechsys.nongnu.org).

**Author contributions.** Z.G. and T.M. designed the study; S.A.G.T developed the simulation method; Z.G. performed the numerical simulation, the theoretical analysis and analyzed the data; H.E.H, T.M, and S.A.G.T edited the manuscript. All authors read critically and participate to the writing of the manuscript.

**Competing interests.** The authors declare no competing interests.