EXISTENCE AND UNIQUENESS OF TRI-TRONQUÉÉ
SOLUTIONS OF THE SECOND PAINLEVÉ HIERARCHY

N. JOSHI AND M. MAZZOCCO

Abstract. The first five classical Painlevé equations are known to have solutions described by divergent asymptotic power series near infinity. Here we prove that such solutions also exist for the infinite hierarchy of equations associated with the second Painlevé equation. Moreover we prove that these are unique in certain sectors near infinity.

1. Introduction

The second Painlevé hierarchy is an infinite sequence of nonlinear ordinary differential equations containing

\begin{equation}
\text{P}^{(n)}_{\text{II}} : \quad V'' = 2V^3 + xV + \alpha, \quad V = V(x), \quad \alpha \text{ const},
\end{equation}

as its simplest equation. This hierarchy is a symmetry reduction of the mKdV hierarchy \cite{1,2,7}, and for this reason, it is believed that its elements all possess the Painlevé Property, i.e. all the movable singularities of all solutions are poles \cite{13}. We conjecture that, in fact, all solutions of every equation in the hierarchy are meromorphic in the complex \(x\)-plane.

Another remarkable property of the equations belonging to the second Painlevé hierarchy, denoted by \(\text{P}^{(n)}_{\text{II}}\), is their irreducibility. That is, for generic values of the parameter \(\alpha\) in each equation of the hierarchy, there is no transformation, within a certain class described in \cite{14}, that maps any of these equations to a linear equation or to a lower order nonlinear ordinary differential equation. This result has been proved only for the case of \(\text{P}^{(1)}_{\text{II}}\) (see \cite{16}), but we conjecture that it is the case for all the other equations in the hierarchy. In fact, we show evidence that no \(n\)-th order member of the hierarchy can be reduced to a lower-order member of the hierarchy.

The main aim of this paper is the asymptotic study of solutions of each \(n\)-th equation \(\text{P}^{(n)}_{\text{II}}\) in the second Painlevé hierarchy. Since \(\infty\) is a non-Fuchsian singularity for each equation \(\text{P}^{(n)}_{\text{II}}\), one expects that the generic solution of \(\text{P}^{(n)}_{\text{II}}\) possesses asymptotic behaviours that are given by (meromorphic combinations of) hyperelliptic functions (see \cite{12}). We concentrate here, however, on solutions described by divergent power series as \(x \to \infty\). In particular, we focus on solutions that possess no poles outside a circle of sufficiently large radius in certain sectors of the \(x\)-plane. We show that there exist unique true solutions with such behaviour in certain sectors of the \(x\)-plane. These have not been proved before for \(n \geq 2\).

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To be more precise, consider $P_{II}$. The asymptotic study of its solutions, in the limit $x \to \infty$, began with the work of Boutroux [3]. The sectors of validity of its asymptotic behaviours are described by six rays: $\arg(x) = j\pi/3, \ j = 0, \ldots, 5$. All solutions are meromorphic in the complex $x$-plane and the general solution is asymptotic to Jacobian elliptic functions whose modulus varies with $\arg(x)$ (see [10]).

Locally, the poles of the solution of $P_{II}$ are aligned with the lattice of periodicity of its asymptotic behaviour. However, as $\arg(x)$ changes within each sector of validity, this lattice slowly changes. Since the elliptic function has poles in each period parallelogram, so does the solution of $P_{II}$. In other words, the general solution has an infinite number of poles within each sector.

However, there also exist two types of one-parameter families of solutions, called tronquée by Boutroux, that possess no poles whatsoever in an annular sector

$$\Omega_j = \left\{ x \in \mathbb{C} \mid |x| > |x_0|, \ |\arg(x) - j\pi/3| < \pi/3 \right\}, \ j = 0, \ldots, 5,$$

for some $x_0$. For a given $j$, as $|x| \to \infty$, $x \in \Omega_j$, such a tronquée solution either has asymptotic behaviour

$$y(x) = \left( \frac{-x}{2} \right)^{1/2} \left( 1 + O(x^{-\frac{2}{3}(1-\epsilon)}) \right),$$

for an appropriate choice of branch of the square root, or

$$y(x) = \left( \frac{-\alpha}{x} \right) \left( 1 + O(x^{-\frac{2}{3}(1-\epsilon)}) \right),$$

for some $\epsilon > 0$. Boutroux characterised each sector $\Omega_j$ by its ray of symmetry $\arg(x) = j\pi/3$. He showed that there also exist tri-tronquée solutions that are asymptotically pole-free along three successive such rays. There are six such solutions. In this paper, we prove the existence and uniqueness of the analogous tri-tronquée solutions for all differential equations of the second Painlevé hierarchy.

The hierarchy we consider arises as a symmetry reduction of the KdV (Korteweg-de Vries) hierarchy which is defined by

$$\partial_t U + \partial_x L_n \{ U \} = 0, \ n \geq 1$$

where

$$\partial_\xi L_{n+1} \{ U \} = (\partial_\xi \xi + 4U \partial_\xi + 2U_\xi)L_n \{ U \}$$

$$L_1 \{ U \} = U$$

The fact that at each step equation (5a) can be integrated to obtain the differential operator $L_{n+1}$ was proved in [11].

After the transformation $U = W_\xi - W^2$, and reduction $W(\xi) = V(x)/((2n + 1)t_{2n+1})^{1/(2n+1)}, x = \xi/((2n + 1)t_{2n+1})^{1/(2n+1)}$ (see [3] for details), we get the $P_{II}$ hierarchy

$$P_{II}^{(n)}: \left( \frac{d}{dx} + 2V \right) L_n \{ V_x - V^2 \} = xV + \alpha_n, \ n \geq 1$$

where $\alpha_n$ are constants and $L_n$ is the operator defined by Equation (5a) with $\xi$ replaced by $x$. We give a more explicit form of (6) in Proposition 2.1 below.
n = 1, Equation (6) is $P_{II}$, whereas for $n = 2$ (noting that $L_2\{U\} = U_{\xi\xi} + 3U^2$) it is

\begin{equation}
V_{4x} - 10V^2V_{2x} - 10VV_x^2 + 6V^5 = xV + \alpha_2,
\end{equation}

where we use the notation

\begin{equation}
V_{mx} := \frac{d^mV}{dx^m}.
\end{equation}

Below we will also use $V_m \equiv V_{mx}$.

In this paper, we show that each equation $P^{(n)}_{II}$ of the hierarchy possesses solutions that are free of poles in sectors of angular width $2n\pi/(2n+1)$ as $|x| \to \infty$ and describe their asymptotic behaviour. Furthermore, there exist unique solutions that are pole-free in sectors of angular width $4n\pi/(2n+1)$. We call such solutions \textit{tri-tronquée} solutions. In the second order case these solutions occur as important solutions of transition phenomena for PDEs such as the modified Korteweg de Vries equation (see [8]) and for ODEs with slowly changing parameters (see [9]). We expect that the tri-tronquée solutions of $P^{(n)}_{II}$ will occur in higher order PDEs with transition phenomena.

Our main result is

**Theorem 1.1.** For each integer $n \geq 1$, there exists $x_0$, $|x_0| > 1$, and sectors

\begin{align*}
S_n &= \{ x \in \mathbb{C} \mid |x| > |x_0|, |\arg(x - x_0)| < n\pi/(2n+1) \} \\
\Sigma_n &= \{ x \in \mathbb{C} \mid |x| > |x_0|, |\arg(x - x_0)| < 2n\pi/(2n+1) \}
\end{align*}

in which the following results hold.

1. Equation (6) has formal solutions

\begin{align}
V_{f,\infty} &= \left(\frac{(-1)^n x}{2c_n}\right)^{\frac{1}{2n}} \sum_{k=0}^{\infty} a_k^{(n)} \left(\frac{2n}{2n+1} x^{\frac{2n+1}{2}}\right)^{-k}, \quad a_0^{(n)} = 1 \\
V_{f,0} &= \left(\frac{-2\alpha_n}{(2n+1)x}\right) \sum_{k=0}^{\infty} b_k^{(n)} \left(\frac{2n}{2n+1} x^{\frac{2n+1}{2}}\right)^{-k}, \quad b_0^{(n)} = 1
\end{align}

where

\begin{equation}
c_n := \frac{2^{2n-1}\Gamma(n+1/2)}{\Gamma(n+1)\Gamma(1/2)}
\end{equation}

and the coefficients $a_k^{(n)}$, $b_k^{(n)}$, $k \geq 1$, are given by substitution.

2. In each sector $S_n$ there exist true solutions $V_{\infty}$ and $V_0$ of Equation (6) with asymptotic behaviour

\begin{align*}
V_{\infty} &\sim x \to \infty V_{f,\infty} \\
V_0 &\sim x \to \infty V_{f,0}
\end{align*}

3. The true solutions $V_{\infty}$ and $V_0$ of Equation (6) are unique in proper sub-sectors of $\Sigma_n$ containing respectively $2n$ among the following half-lines

\begin{align}
\arg(x) &= \frac{2j + 1}{2n+1} \pi, \quad j = 0, \ldots, 2n, \\
or
\arg(x) &= \pi + \frac{2j + 1}{2n+1} \pi, \quad j = 0, \ldots, 2n.
\end{align}
The theorem above is proved in Section 2.

Remark 1.2. We mention that although the elements of the P\textsubscript{II} hierarchy are integrable (see [3]), we make no use of this fact in this paper. To our knowledge there is no general result linking integrability to existence of tri-tronquée-type solutions (for example in the case of Painlevé VI equation such solutions have not yet been found).

2. Proof of Theorem [13]

We start the proof by deriving a more explicit expression for the P\textsubscript{II} hierarchy.

Proposition 2.1. The differential equations P\textsuperscript{(n)}\textsubscript{II} have the form

\begin{equation}
V_{2n} = P_{2n-1}(V_0, V_1, \ldots, V_{2n-2}) + xV_0 + \alpha_n + \beta_n V_0^{2n+1},
\end{equation}

where $V_n := \frac{d^nV}{dz^n}$. $P_{2n-1}$ is a polynomial in $V_0, V_1, \ldots, V_{2n-2}$ of degree $2n - 1$, of the form

\begin{equation}
P_{2n-1} = \sum_{\langle k \rangle = 2n+1, k_0 \leq 2n-1} b_{k_0, \ldots, k_{2n-2}} V_0^{k_0} V_1^{k_1} \cdots V_{2n-2}^{k_{2n-2}}.
\end{equation}

Here $k$ is a multi-index $k = (k_0, \ldots, k_{2n-2})$, with norm

\begin{equation}
\langle k \rangle := \sum_{p=0}^{2n-2} (p+1)k_p,
\end{equation}

$b_{k_0, \ldots, k_{2n-2}}$ are constants (some of which may be zero), and

\begin{equation}
\beta_n = (-1)^{n+1}2^{2n} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + 1)\Gamma(\frac{1}{2})}.
\end{equation}

Proof. The result (13) can be derived from (13) by proving that

\begin{equation}
L_n(V_1 - V_0^2) = V_{2n-1} + \tilde{\beta}_n V_0^{2n}
+ \sum_{\langle k \rangle = 2n, k_0 \leq 2n-2} a_{k_0, \ldots, k_{2n-2}} V_0^{k_0} V_1^{k_1} \cdots V_{2n-2}^{k_{2n-2}}
\end{equation}

where $\tilde{\beta}_n = \frac{-\beta_n}{2}$, and $a_{k_0, \ldots, k_{2n-2}}$ denote constants. Let us derive (12) from (14) first and then prove (14) by induction.

The $n$-th equation $P_{2n-1}$ (see Equation (13)) has the form

\begin{equation}
\left(\frac{\partial}{\partial x} + 2V_0\right)\left(V_{2n-1} + \tilde{\beta}_n V_0^{2n} + \sum_{\langle k \rangle = 2n, k_0 \leq 2n-2} a_{k_0, \ldots, k_{2n-2}} V_0^{k_0} V_1^{k_1} \cdots V_{2n-2}^{k_{2n-2}}\right) =
\end{equation}

\begin{equation}
= xV_0 + \alpha_n
\end{equation}
that is

\[
V_{2n} + 2n \tilde{\beta}_n V_0^{2n-1} V_1 + \\
+ \sum_{(k) = 2n} a_{k_0, \ldots, k_{2n-2}} \left( k_0 V_0^{k_0-1} V_1^{k_1+1} \ldots V_{2n-2}^{k_{2n-2}+1} + k_1 V_0^{k_0} V_1^{k_1-1} V_2^{k_2+1} \ldots V_{2n-2}^{k_{2n-2}+} + k_{2n-2} V_0^{k_0} V_1^{k_1} \ldots V_{2n-2}^{k_{2n-2}-1} V_{2n-1} \right) + \\
+ 2V_0 V_{2n-1} + 2 \tilde{\beta}_n V_0^{2n+1} + \\
+ 2 \sum_{(k) = 2n} a_{k_0, \ldots, k_{2n-2}} V_0^{k_0+1} V_1^{k_1} \ldots V_{2n-2}^{k_{2n-2}+} = \\
= x V_0 + \alpha_n \\
\Rightarrow \\
V_{2n} + 2 \tilde{\beta}_n V_0^{2n+1} - \sum_{(k) = 2n+1} b_{k_0, \ldots, k_{2n-1}} V_0^{k_0} V_1^{k_1} \ldots V_{2n-1}^{k_{2n-1}+} = \\
= x V_0 + \alpha_n
\]

for some suitable constants \( b_{k_0, \ldots, k_{2n-1}} \). For \( \beta_n = -2 \tilde{\beta}_n \) we obtain \( 14 \) as desired.

We now want to prove \( 14 \) by induction. It is trivially true for \( n = 1 \) with \( \beta_1 = 2 \). Let us assume it true for \( n \) and prove it for \( n + 1 \). Observe that

\[
L_{n+1} = \frac{\partial^2 L_n}{\partial x^2} + 2(V_1 - V_2) L_n + 2 \int (V_1 - V_2) \frac{\partial L_n}{\partial x} \, dx,
\]

where \((V_1 - V_2) \frac{\partial L_n}{\partial x} \, dx\) is always an exact form (see \( 13 \)). We obtain

\[
L_{n+1} = V_{2n+1} + 2n(2n-1) \tilde{\beta}_n V_0^{2n-2} V_1^2 + 2n \tilde{\beta}_n V_0^{2n-1} V_1 + \\
+ \sum_{(k) = 2n} a_{k_0, \ldots, k_{2n-2}} \left( k_0 (k_0-1) V_0^{k_0-2} V_1^{k_1+2} \ldots V_{2n-2}^{k_{2n-2}+2} + \\
+ k_0 (k_0+1) V_0^{k_0} V_1^{k_1+1} \ldots V_{2n-2}^{k_{2n-2}+1} + \\
+ k_0 (k_0-2) V_0^{k_0-2} V_1^{k_1+1} \ldots V_{2n-2}^{k_{2n-2}+1} V_{2n-1} + \\
+ k_0 (k_0+2) V_0^{k_0+1} V_1^{k_1+2} \ldots V_{2n-2}^{k_{2n-2}+2} V_{2n-1} + \\
+ k_0 (k_0-3) V_0^{k_0-2} V_1^{k_1+1} \ldots V_{2n-2}^{k_{2n-2}+1} V_{2n-1} + \\
+ k_0 (k_0+3) V_0^{k_0+1} V_1^{k_1+2} \ldots V_{2n-2}^{k_{2n-2}+2} V_{2n-1} + \\
+ 2V_1 \sum_{(k) = 2n} a_{k_0, \ldots, k_{2n-2}} V_0^{k_0} V_1^{k_1} \ldots V_{2n-2}^{k_{2n-2}+2} V_{2n-1} - \\
- 2 \tilde{\beta}_0 V_0^{2n+2} - 2 V_1 \sum_{(k) = 2n} a_{k_0, \ldots, k_{2n-2}} V_0^{k_0} V_1^{k_1} \ldots V_{2n-2}^{k_{2n-2}+2} + \\
+ 2 \int (V_1 - V_2) \left[ V_{2n} + 2n \tilde{\beta}_n V_0^{2n-1} V_1 + \\
+ \sum_{(k) = 2n} a_{k_0, \ldots, k_{2n-2}} \left( k_0 V_0^{k_0-1} V_1^{k_1+1} \ldots V_{2n-2}^{k_{2n-2}+1} + \\
+ k_1 V_0^{k_0} V_1^{k_1-1} V_2^{k_2+1} \ldots V_{2n-2}^{k_{2n-2}+1} + \\
+ k_{2n-2} V_0^{k_0} V_1^{k_1} \ldots V_{2n-2}^{k_{2n-2}-1} V_{2n-1} \right) \right] \, dx.
\]
Thus

$$L_{n+1} = V_{2(n+1)-1} - 2\tilde{\beta}_n V_0^{2(n+1)} - \frac{4n}{2n+1} \tilde{\beta}_n V_0^{2(n+1)} + \sum_{(k)=2n}^{k_0 \leq 2(n+1)-2} \tilde{a}_{k_0, \ldots, k_{2(n+1)-2}} V_0^{k_{2(n+1)-2}} V_{k_0} V_{k_1} \cdots V_{2(n+1)-2}$$

that has the required form for \( \tilde{\beta}_{n+1} = -2\tilde{\beta}_n \frac{2n+1}{n+1} \), i.e. \( \tilde{\beta}_{n+1} = -2\tilde{\beta}_n \frac{2n+1}{n+1} \). From \( \beta_1 = 2 \) we obtain the right value of \( \beta_n \).

**Proposition 2.2.** For each \( n \geq 1 \), the change of variables

$$V(x) = f(x) u(z), \quad z = g(x)$$

where

$$f(x) = x^{1/(2n)}, \quad g(x) = \frac{2n}{2n+1} x^{(2n+1)/2n}$$

maps \( P_{11}^{(n)} \) to

$$\frac{d^{2n}}{dx^{2n}} u = -\sum_{l=0}^{2n-1} a_{2n,l} x^{-2n-2} \frac{d^l}{dx^l} u(z) + \frac{2n}{2n+1} u(z) + \beta_n u^{2n+1} + \frac{2n}{2n+1} a_{2n,l} + z^{-2n-1} \sum_{m_0, \ldots, m_{2n-2} = 0} a_{m_0, \ldots, m_{2n-2}} (z) \prod_{l=0}^{2n-2} \left( \frac{d^l}{dx^l} u(z) \right),$$

where \( a_{m_0, \ldots, m_{2n-2}}(z) \) are polynomials of degree \( \leq 2n \) and \( a_{m,0} \) are some constant coefficients and the sum \( \sum_{m_0, \ldots, m_{2n-2} = 0} a_{m_0, \ldots, m_{2n-2}}(z) \) is zero for \( n = 1 \). Furthermore, equation (14) admits formal series expansions

$$(17) \quad u_{f,\infty} = \left( \frac{(-1)^n}{2c_n} \right)^{\frac{n}{2}} \sum_{k=0}^{\infty} a_k^{(n)} z^{-k}, \quad a_0^{(n)} = 1$$

$$(18) \quad u_{f,0} = \left( \frac{-\alpha_n}{2n z} \right)^n \sum_{k=0}^{\infty} b_k^{(n)} z^{-k}, \quad b_0^{(n)} = 1$$

**Remark 2.3.** For \( P_{11} \), Equation (14) leads to

$$f_0 g_1^2 \frac{d^2}{dz^2} u + (2f_1 g_1 + f_0 g_2) \frac{d}{dz} u + f_2 u = 2f_0^3 u^3 + x f_0 u + \alpha_1,$$

where we are using the notation above, i.e. \( f_m = \frac{d^m f}{dx^m} \) and \( g_m = \frac{d^m g}{dx^m} \). A maximal dominant balance is given by

$$f_0 g_1^2 = f_0^3 = x f_0$$

$$\Rightarrow f_0 = x^{1/2}, \quad g_1 = x^{1/2}.$$

The result:

$$V(x) = x^{1/2} u(z), \quad z = 2x^3/3$$

leads to

$$\frac{d^2}{dz^2} u = 2u^3 + u + \frac{1}{z} \left( \frac{2}{3} \alpha_1 - \frac{d}{dx} u \right) + \frac{u}{2z^2}$$

The new variables \( u, z \) in (14) were first given by Boutroux [3].
Proof. Let us first prove the following formula:

\begin{equation}
\frac{d^p}{dx^p} V(x) = \sum_{l=0}^{p} \hat{a}_{p,l} x^{l+1-\frac{2n}{2n+1}(p+1)} \frac{d^l}{dz^l} u(z),
\end{equation}

for some constants \( \hat{a}_{p,l} \), \( \hat{a}_{p,p} = \left( \frac{2n+1}{2n} \right)^{\frac{n+1}{n+1}} \).  

First let us show by induction that

\begin{equation}
V_p = \sum_{l=0}^{p} U_{p,l}(g_1, g_2, \ldots, g_p) \frac{d^l}{dz} u(z)
\end{equation}

where for \( l = 1, \ldots, p \),

\[ U_{p,l}(g_1, g_2, \ldots, g_p) = \sum_{n_1+2n_2+\cdots+pn_p=p+1}^{n_1+2n_2+\cdots+pn_p=1}^{n_1+2n_2+\cdots+pn_p=1} c_{n_1,n_2,\ldots,n_p} g_1^{n_1} \cdots g_p^{n_p}. \]

for some constant coefficients \( c_{n_1,n_2,\ldots,n_p} \), and

\[ U_{p,0} = f_p = g_{p+1}, \quad U_{p,p} = f_0(g_1)^p = (g_1)^p+1. \]

The formula (22) is obvious for \( p = 0 \), with \( U_{0,0} = f_0 = g_1 \). Suppose (22) is valid for some \( p \geq 1 \) let us prove it for \( p + 1 \).

\[ V_{p+1} = \frac{d}{dx} V_p = \frac{d}{dx} \sum_{l=0}^{p} U_{p,l}(g_1, g_2, \ldots, g_p) \frac{d^l}{dz^l} u(z) = \sum_{l=0}^{p} \left( \frac{d}{dx} (U_{p,l}) \frac{d^l}{dz^l} u(z) + U_{p,l}(g_1) \frac{d}{dz} \frac{d^l}{dz^l} u(z) \right) \]

so that we obtain (22) for \( p + 1 \) with

\[ U_{p+1,l} = \frac{d}{dx} U_{p,l} + g_l U_{p,l-1}, \quad \text{for } l = 1, \ldots, p \]

and

\[ U_{p+1,0} = \frac{d}{dx} U_{p,0}, \quad U_{p+1,p+1} = g_1 U_{p,p}. \]

Let us express the polynomials \( U_{p,l} \) as functions of \( z \):

\[ g_1^{n_1} \cdots g_p^{n_p} \propto z^{(\sum_{l=1}^{p} (\frac{2n}{2n+1}) l) + 1} (n_1+\cdots+pn_p) \propto z^{l(p+1)} \]

so that we obtain formula (21), for \( l = 1, \ldots, p - 1 \) and

\[ U_{p,0} \propto z^{1-\frac{2n}{2n+1}(p+1)}, \quad U_{p,p} = \left( \frac{2n+1}{2n} \right)^{\frac{n+1}{n+1}}. \]

This shows formula (21).

Of course formula (21) gives

\[ V_{2n} = \sum_{l=0}^{2n} \bar{a}_{2n,l} x^{l+1-\frac{2n}{2n+1}(p+1)} \frac{d^l}{dz^l} u(z) = \frac{2n+1}{2n} \frac{d^{2n}}{dz^{2n}} u + \sum_{l=0}^{2n-1} \bar{a}_{2n,l} x^{l+1-\frac{2n}{2n+1}(p+1)} \frac{d^l}{dz^l} u. \]

Let us now compute the polynomial \( P_{2n-1} \) as a function of \( z, u \) and its derivatives. Recall formula (13):

\[ P_{2n-1} = \sum_{(k)\geq 2n+1 \atop k_0 \leq 2n-1} b_{k_0, \ldots, k_{2n-2}} V_0^{k_0} V_1^{k_1} \cdots V_{2n-2}^{k_{2n-2}}. \]
By (21), we have

\[ V_p^{(k_p)} = \sum_{s_0^p + \ldots + s_p^p = k_p} \prod_{l=0}^p \hat{a}_{s_0^p,\ldots,s_p^p} \left( z^{(l+1)} - z^{(p+1)} \frac{d^l}{dz^l} u(z) \right) s_l^p = z^{k_p - \frac{2n}{(p+1)}(p+1)} \sum_{s_0^p + \ldots + s_p^p = k_p} \prod_{l=0}^p \left( z^{l} \frac{d^l}{dz^l} u(z) \right) s_l^p = z^{k_p - \frac{2n}{(p+1)}(p+1)} \sum_{s_0^p + \ldots + s_p^p = k_p} \prod_{l=0}^p \left( \frac{d^l}{dz^l} u(z) \right) s_l^p. \]

However, for \( p \neq 0 \), \( \sum_{l=0}^p l s_l^p < p \sum_{l=0}^p s_l^p = pk_p \), for \( p \neq 1 \) and \( \sum_{l=0}^1 l s_l^1 \leq k_1 \), we have that

\[ V_p^{(k_p)} = z^{k_p - \frac{2n}{(p+1)}(p+1)} \sum_{s_0^p + \ldots + s_p^p = k_p} h_{s_0^p,\ldots,s_p^p}(z) \prod_{l=0}^p \left( \frac{d^l}{dz^l} u(z) \right) s_l^p, \]

where \( h_{s_0^p,\ldots,s_p^p}(z) \) are polynomials in \( z \) of degree \( < pk_p \) for \( p \neq 1 \) and of degree \( \leq k_1 \) for \( p = 1 \). Now to compute \( P_{2n-1} \) as a function of \( z \) let us first observe that

\[ \prod_{p=0}^{2n-2} z^{k_p - \frac{2n}{(p+1)}(p+1)} = z^{k_0 + \ldots + k_{2n-2} - 2n}. \]

Thus \( V_{k_1}^{(k_1)} \cdots V_{2n-2}^{(k_{2n-2})} \) has the form

\[ \prod_{p=0}^{2n-2} \sum_{s_0^p + \ldots + s_p^p = k_p} h_{s_0^p,\ldots,s_p^p}(z) \prod_{l=0}^p \left( \frac{d^l}{dz^l} u(z) \right) s_l^p, \]

where the \( l \)-th derivative of \( u, \frac{d^l}{dz^l} u(z) \), appears for every \( p \) as \( \left( \frac{d^l}{dz^l} u(z) \right) s_l^p \). Assuming that \( s_l^p = 0 \) for all \( l > p \) we can write

\[ \prod_{p=0}^{2n-2} \sum_{s_0^p + \ldots + s_p^p = k_p} h_{s_0^p,\ldots,s_p^p}(z) \prod_{l=0}^{2n-2} \left( \frac{d^l}{dz^l} u(z) \right) s_l^p, \]

Let us collect together all the derivatives of \( u \) of the same order, we obtain that the \( l \)-th derivative of \( u \) appears with the power \( m_l = \sum_{p=0}^{2n-2} s_l^p \). Thus \( \sum_{l=0}^{2n-2} m_l^l = \sum_{p=0}^{2n-2} \sum_{p=0}^{2n-2} s_l^p = \sum_{p=0}^{2n-2} k_p \). Now the sum \( \sum_{p=0}^{2n-2} k_p \) is maximal for \( k_0 = 2n-1 \), \( k_1 = 1 \) and \( k_p = 0 \) for all \( p \neq 0,1 \) because \( \sum_{p=0}^{2n-2} (p+1)k_p = 2n+1 \) and \( k_0 \leq 2n-1 \).

As a consequence we have \( 0 \leq m_l \leq \sum_{p=0}^{2n-2} k_p \leq 2n \). We then obtain

\[ V_{k_1}^{(k_1)} \cdots V_{2n-2}^{(k_{2n-2})} = \sum_{m_0,\ldots,m_{2n-2}=0}^{2n} \hat{h}_{m_0,\ldots,m_{2n-2}}(z) \prod_{l=0}^{2n-2} \left( \frac{d^l}{dz^l} u(z) \right) m_l, \]

for some polynomials \( \hat{h}_{m_0,\ldots,m_{2n-2}}(z) \) of degree \( < \sum_{p=0}^{2n-2} pk_p \). In fact \( V_{k_1}^{(k_1)} \) contributes with a polynomial of degree \( k_1 \) and for \( p \neq 1 \) each \( V_p^{(k_p)} \) contributes with a polynomial of degree \( < pk_p \). Since \( \sum_{p=0}^{2n-2} (p+1)k_p = 2n+1 \), for \( k_1 \neq 0 \) (2k_1 is even) at least some other \( k_p \) must be non-zero giving total degree \( < \sum_{p=0}^{2n-2} pk_p \).

Since \( a_{m_0,\ldots,m_{2n-2}}(z) := z^{k_0 + \ldots + k_{2n-2}} \hat{h}_{m_0,\ldots,m_{2n-2}}(z) \) are polynomials of degree
the equation as

\[ \sum_{m_0, \ldots, m_{2n-2}} a_{m_0, \ldots, m_{2n-2}}(Y) \prod_{l=0}^{2n-2} \left( \frac{d^l}{dz^l} u(z) \right)^{m_l}, \]

where \( a_{m_0, \ldots, m_{2n-2}} \) have the desired properties. This concludes the proof of (14).

From (14) it is straightforward to compute the leading terms of (17) and (18) for the solutions \( u(z) \) in the case when \( d^k u/dz^k \ll u, \) for \( |z| > 1. \) Moreover, rewriting the equation as

\[
\frac{2n}{2n+1} u(z) + \frac{2n}{2n+1} \frac{\partial^n}{\partial z^n} u(z) - \frac{2n}{2n+1} \sum_{l=0}^{2n-1} \alpha_{2n,l} z^{l-2n} \left( \frac{d^l}{dz^l} u(z) \right) + \left( \frac{2n}{2n+1} \right) \prod_{l=0}^{2n-2} \left( \frac{d^l}{dz^l} u(z) \right)^{m_l},
\]

and iterating on partial sums of (17) or (18), we are led to formal solutions of the desired form.

To prove the third statement we need the following

**Theorem 2.4.** Consider the linear system

\[
\varepsilon^l \frac{d^l}{dz^l} Y = A(z, \varepsilon) Y,
\]

where \( Y \) is a column vector of \( m \) components, \( z \in \mathbb{C}, \) \( h \) is a positive integer and \( A(z, \varepsilon) \) is a \( m \times m \) matrix function admitting asymptotic expansion of the form

\[
A(z, \varepsilon) \sim \sum_{0}^{\infty} A_{r}(z) \varepsilon^r, \quad \varepsilon \to 0,
\]

that is uniformly valid for \( \varepsilon \in \Sigma \cap \{\varepsilon | 0 < \varepsilon \leq \varepsilon_0\}, \) \( \Sigma \) being some sector of the \( \varepsilon \)-plane, with coefficients \( A_{r}(z) \) holomorphic in a ball, \( z \in B(z_0, \alpha). \) Denote the eigenvalues
of $A_0(z)$ by $\mu_1(z), \ldots, \mu_m(z)$. If $\mu_1(z_0), \ldots, \mu_n(z_0)$ are pairwise distinct, then the system (23) admits a fundamental solution $Y(z)$ of the form

$$Y(z) = \hat{Y}(z, \varepsilon) \exp(Q(z, \varepsilon)),$$

where $\hat{Y}(z, \varepsilon)$ admits asymptotic expansion of the form

$$\hat{Y}(z, \varepsilon) \sim \sum_{\tau=0}^{\infty} \hat{Y}_\tau(z) \varepsilon^\tau, \quad \varepsilon \to 0,$$

that is uniformly valid for $\varepsilon \in \hat{\Sigma} \cap \{ \varepsilon \mid 0 < \varepsilon \leq \varepsilon_0 \}$, $\varepsilon_0 < \varepsilon$, $\hat{\Sigma}$ being a small enough sub-sector of $\Sigma$ centered around any arbitrary ray $\arg(\varepsilon) = \alpha$, for any $\alpha$ that is not an odd multiple of $\pi$, with coefficients $\hat{Y}_\tau(z)$ holomorphic in $z \in B(z_0, \hat{a}) \subset B(z_0, a)$, $\hat{a} > a$, and with $\det \hat{Y}_0(z) \neq 0$. The matrix $Q(z, \varepsilon)$ is diagonal of the form

$$Q(z, \varepsilon) = \sum_{\tau=1}^{h} \hat{Q}_\tau(z) \varepsilon^{-\tau},$$

where $\hat{Q}_h(z) = \text{diag} \left( \int_{z_0}^{z} \mu_1(t) \, dt, \ldots, \int_{z_0}^{z} \mu_m(t) \, dt \right)$.

Proof. It can be found in [17, 14].

Let us consider two adjacent sectors $\hat{S}_n$ and $\hat{S}_n$ and two solutions, $\hat{V}(x)$ and $\hat{V}(x)$ of (12) having the same asymptotic behaviour (3) or (4) in the given sectors $\hat{\Sigma}_n$ and $\hat{\Sigma}_n$ respectively. For every fixed $j = 0, \ldots, 2n$, we take, in the case of asymptotic behavior (3), the sectors

$$\hat{S}_n = \left\{ x \in \mathbb{C} \mid |x| > |x_0|, \frac{(2j-1)\pi}{2n+1} + \varepsilon < |\arg(x)| < \frac{(2j+1)\pi}{2n+1} + \varepsilon \right\},$$

and in the case of asymptotic behavior (4) we take the sectors

$$\hat{S}_n = \left\{ x \in \mathbb{C} \mid |x| > |x_0|, \frac{(2j+1)\pi}{2n+1} - \varepsilon < |\arg(x)| < \frac{(2j+3)\pi}{2n+1} - \varepsilon \right\},$$

for $\varepsilon > 0$ small enough. In both cases these sectors intersect in a small sector $\mathcal{S}_e$ of opening $< 2\varepsilon$ containing in the first case the line

$$\arg(x) = \frac{2j+1}{2n+1} \pi,$$

and, in the second case, the line

$$\arg(x) = \pi + \frac{2j+1}{2n+1} \pi.$$

We want to show that in this small sector $\mathcal{S}_e$ the two solutions $\hat{V}(x)$ and $\hat{V}(x)$ coincide. If this is the case, we can then analytically extend $\hat{V}(x)$ to the sector $\Sigma_n = \hat{S}_n \cup \hat{S}_n$, this extension is unique and the third statement of our Theorem 1.1 is proved.

Suppose by contradiction that the two solutions, $\hat{V}(x)$ and $\hat{V}(x)$ of (12) having the same asymptotic behaviour (3) or (4) differ in the given sector $\mathcal{S}_e$. Then their difference $W(x) := \hat{V}(x) - \hat{V}(x)$ and all its derivatives vanish asymptotically as $x \to \infty$ in $\mathcal{S}_e$. Suppose that $W(x) \neq 0$ for some value $x$. 
Let us consider the differential equation satisfied by \( W(x) \)

\[
W_{2n} = P_{2n-1}(\tilde{V}_0, \ldots, \tilde{V}_{2n-2}) - P_{2n-1}(\tilde{V}_0, \ldots, \tilde{V}_{2n-2}) + xW_0 + \beta_n(\tilde{V}_0^{2n+1} - \tilde{V}_0^{2n+1}),
\]

where \( \tilde{V}_m = \frac{d^m\tilde{V}}{dx^m}, \tilde{V}_m = \frac{d^m\tilde{V}}{dx^m} \) and \( W_m = \frac{d^mW}{dx^m} \) as above. Now

\[
\tilde{V}_0^{2n+1} - \tilde{V}_0^{2n+1} = \sum_{k=0}^{2n} \tilde{V}_0^{2n-k} \tilde{V}_0^k W_0
\]

and

\[
P_{2n-1}(\tilde{V}_0, \ldots, \tilde{V}_{2n-2}) - P_{2n-1}(\tilde{V}_0, \ldots, \tilde{V}_{2n-2}) = \sum_{\{k\}=2n+1}^{\infty} b_{k_0, \ldots, k_{2n-2}} \left( \tilde{V}_0^{k_0} \tilde{V}_1^{k_1} \ldots \tilde{V}_{2n-2}^{k_{2n-2}} - \tilde{V}_0^{k_0} \tilde{V}_1^{k_1} \ldots \tilde{V}_{2n-2}^{k_{2n-2}} \right)
= Q_{2n-1}W_0 + Q_{2n-2}W_1 + \cdots + Q_1W_{2n-2}
\]

where

\[
Q_{2n-p-1} = b_{p_0, \ldots, p_{2n-2}}^{(p)} \tilde{V}_0^{k_0} \tilde{V}_1^{k_1} \ldots \tilde{V}_{p-1}^{k_{p-1}} \left( \sum_{l=0}^{k_p-1} \tilde{V}_p^{k_p-1-l} \tilde{V}_p^l \right) \tilde{V}_{p+1}^{k_{p+1}} \ldots \tilde{V}_{2n-2}^{k_{2n-2}},
\]

for some constants \( b_{p_0, \ldots, p_{2n-2}}^{(p)} \). So we have

\[
W_{2n} = Q_{2n-1}W_0 + Q_{2n-2}W_1 + \cdots + Q_1W_{2n-2} + xW_0 + \beta_n \sum_{k=0}^{2n} \tilde{V}_0^{2n-k} \tilde{V}_0^k W_0.
\]

We want to study the asymptotic behaviour of the non-zero solutions \( W(x) \) of this linear differential equation as \( x \to \infty \) in the sector \( \Sigma_n \).

We first deal with the case \( \mathbb{R} \). Let us perform a variable rescaling \( \varepsilon^{2n}x = z \), such that \( z \in \mathbb{B}(z_0, a) \) for some \( a > 0 \) and \( \varepsilon \to 0 \), in the sector \( \Sigma_n \). From \( \mathbb{R} \) we obtain

\[
\tilde{V} \left( \frac{z}{\varepsilon^{2n}} \right), \; \tilde{V} \left( \frac{z}{\varepsilon^{2n}} \right) \sim -1 a_0 z^{\frac{1}{2}} + O(\varepsilon^{2n})
\]

so that the polynomials \( Q_{2n-p-1} \) are all rescaled as \( Q_{2n-p-1} \to \varepsilon^{2n+2}Q_{2n-p-1} \), for some polynomials \( q_{2n-p-1}(z) \). In fact \( V_p^{kp} \sim \varepsilon^{k_p(2np-1)} \frac{d^{p+1}}{dz^{p+1}} \) and since

\[
k_0(-1) + k_1(2n-1) + \cdots + k_{p-1}(2n(p-1) - 1) + (k_p-1)(2np-1) + k_{p+1}(2n(p+1) - 1) + \cdots + k_{2n-2}(2n(2n-2) - 1) = \sum_{l=0}^{2n-2} k_l(2nl - 1) - (2np - 1) = 2n \left( \sum_{l=0}^{2n-2} k_l - p \right) - \sum_{l=0}^{2n-2} k_l + 1 \geq 2n \left( \sum_{l=0}^{2n-2} k_l - p \right) - 2n + 2 \geq 2n - 2,
\]

because \( \sum_{l=0}^{2n-2} k_l \geq \sum_{l=0}^{2n-2} k_l + 1 \), we have

\[
\tilde{V}_0^{k_0} \tilde{V}_1^{k_1} \ldots \tilde{V}_{p-1}^{k_{p-1}} \left( \sum_{l=0}^{k_p-1} \tilde{V}_p^{k_p-1-l} \tilde{V}_p^l \right) \tilde{V}_{p+1}^{k_{p+1}} \ldots \tilde{V}_{2n-2}^{k_{2n-2}} \equiv O\left( \varepsilon^{2(n+1)} \right).
\]

Analogously the polynomial \( \sum_{k=0}^{2n} \tilde{V}_0^{2n-k} \tilde{V}_0^k \) can be expanded as \( (2n + 1) \frac{d^{2n}}{dz^{2n}} + O(\varepsilon) \). The differential equation then becomes:

\[
\varepsilon^{4n^2} \frac{d^{2n}}{dz^{2n}} W = \varepsilon^{2n+2} \sum_{k=0}^{2n-2} q_{2n-k-1}(z) \frac{d^{k}}{dz^{k}} W + \frac{z}{\varepsilon^{2n}} W - \frac{2}{\varepsilon^{2n}} z W + O(\varepsilon) W.
\]
Such equation can be put into system form by
\[ Y_1 = W(z), \quad Y_{k+1} = \varepsilon^{2n+1} \frac{d}{dz} Y_k, \quad k = 1, \ldots, 2n - 1. \]

In this way we obtain a system of the form (28) with \( h = 2n + 1 \) and leading matrix
\[
A_0(z) = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
f(z) & 0 & \ldots & \ldots & 0
\end{pmatrix},
\]
with \( f(z) := (1 - 2n)z \). The eigenvalues of \( A_0 \) are \( \mu_k(z) = (2n - 1)^\frac{i\pi}{n} z^\frac{i\pi}{n} \exp(\frac{i\pi}{n}) \), \( k = 1, \ldots, 2n \). Applying Theorem 2.4, we conclude that there exists a fundamental solution \( Y(z) \) of our system of the form (28) with asymptotic behaviour
\[
Y(z) \sim \sum_{0}^{\infty} \tilde{Y}_r(z) \varepsilon^r \exp \left( \frac{1}{\varepsilon^h} \text{diag}(\nu_1(z), \ldots, \nu_{2n}(z)) + O(\varepsilon^{1-h}) \right)
\]
where \( \nu_k(z) = 2n(2n-1)^\frac{i\pi}{n} \exp(\frac{i\pi}{n}) z^\frac{i\pi}{n} \) and \( \det(\tilde{Y}_0(z)) \neq 0 \). Such a fundamental solution \( Y(z) \) exists for every sub-sector of \( S_\varepsilon \) centered around the line (10). In fact such lines are never odd multiples of \( \frac{\pi}{n} \). This shows that in any sub-sector of \( S_\varepsilon \) centered around the line (10) each solution \( W(x) \) of (28) has asymptotic behaviour with leading term
\[
W(x) \sim W_0 \text{diag} \left( \exp \left( \frac{2n(2n-1)^\frac{i\pi}{n}}{1 + 2n} \exp(\frac{i\pi}{n}) z^\frac{i\pi}{n} \right), k = 1, \ldots, 2n \right)
\]
where \( W_0 \) is a nonsingular constant matrix. This leads to a contradiction because the asymptotic behaviour of each solution \( W(x) \) of (28) or of at least one of its derivatives is oscillatory along those lines, and does not vanish asymptotically as assumed at the beginning.

We now deal with the case (31). Let us perform a variable rescaling \( \tau x = z \), such that \( z \in B(z_0, \alpha) \) and \( \tau \to 0 \), in the sector \( \Sigma_\varepsilon \). Then \( \tilde{V} \) and \( \tilde{V} \) are accordingly rescaled to \( \tau \tilde{v} \) and \( \tau \tilde{v} \) respectively. As a consequence
\[
Q_{2n-p-1} \frac{d^n}{dz^n} W_p = \tau^{k_0 + \cdots + k_{p-1} + (p+1)(k_{p-1} + (p+1)k_{p-1} + \cdots + (2n-1)k_{2n-2} - 2p+1) \frac{d^n W}{dz^n}}
\]
for some polynomial \( q_{2n-p-1} \). The differential equation then becomes:
\[
\tau^{2n} \frac{d^{2n}}{dz^{2n}} W = Q_{2n} W + \tau Q_{2n-1} \frac{d}{dz} W + \cdots + \tau^{2n-1} Q_{1} \frac{d^{2n-1}}{dz^{2n-1}} W + W + \beta_0 \tau^{2n}(\tilde{v}^{2n} \cdots + \tilde{v}^{2n}) W.
\]

By the rescaling
\[
Y_1(z) := W(z), \quad Y_{k+1} := \varepsilon^k \frac{d}{dz} Y_k(z), \quad k = 1, \ldots, 2n - 1,
\]
where \( \varepsilon = \tau^{\frac{i\pi}{n}} \) and \( h = 2n + 1 \) we obtain again a system of the form (28) with leading matrix \( A_0(z) \) of the same form (31) with \( f(z) = z \). The eigenvalues of \( A_0(z) \) are \( \mu_k(z) = z^{\frac{i\pi}{n} \exp(\frac{i\pi}{n}) \exp(\frac{i\pi}{n}), k = 1, \ldots, 2n \). Applying Theorem 2.4,
we conclude that there exists a fundamental solution $Y(z)$ of our system of the form (23) with asymptotic behaviour

$$Y(z) \sim \sum_{r=0}^{\infty} \hat{Y}_r(z) \varepsilon^r \exp \left( \frac{1}{\varepsilon} \text{diag} (\nu_1(z), \ldots, \nu_{2n}(z)) \right)$$

where $\nu_k(z) = \frac{2n}{1+2n} \exp \left( \frac{i\pi}{2n} \right) \exp \left( \frac{i\pi k}{n} \frac{1+2n}{4} \right)$. Such solution $Y(z)$ exists for every sub-sector of $\mathcal{S}_\epsilon$ centered around the line (11). In fact such lines are never odd multiples of $\frac{\pi}{2}$. This shows that in any sub-sector of $\mathcal{S}_\epsilon$ centered around the line (11) each solution $W(x)$ of (28) has asymptotic behaviour with leading term

$$W(x) \sim W_0 \text{diag} \left( \exp \left( \frac{2n}{1+2n} \right) \exp \left( \frac{i\pi}{2n} \right) \exp \left( \frac{i\pi k}{n} \frac{1+2n}{4} \right) \right) k = 1, \ldots, 2n$$

where $W_0$ is a non-singular constant matrix. This leads to a contradiction because the asymptotic behaviour of each solution $W(x)$ of (28) or of at least one of its derivatives is oscillatory along those lines, and not vanishing asymptotically as assumed at the beginning.

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School of Mathematics and Statistics F07, University of Sydney, NSW2006 Sydney, Australia
E-mail address: nalini@maths.usyd.edu.au

Mathematical Institute, University of Oxford, 24-29 St Giles, Oxford OX1 3LB, UK
and DPMMS, Wilberforce Road, Cambridge CB3 0WB, UK.
E-mail address: M.Mazzocco@dpmms.cam.ac.uk