Coherence, Path-Predictability and I-Concurrence: A Triality

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It is well known that fringe contrast is not a good quantifier of the wave nature of a quanton in multipath interference. A new interference visibility, based on the Hilbert-Schmidt coherence is introduced. It is demonstrated that this visibility is a good quantifier of wave nature, and can be experimentally measured. A generalized path predictability is introduced, which reduces to the predictability of Greenberger and Yasin, for the case of two paths. In a multipath, which-way interference experiment, the new visibility, the predictability and the I-concurrence (quantifying the entanglement between the quanton and the path-detector), are shown to follow a tight triality relation. It quantifies the essential role that entanglement plays in multipath quantum complementarity, for the first time.

I. INTRODUCTION

We have come a long way since Neils Bohr proposed his principle of complementarity in 1928 [1], but instead of fading away, the topic has increasingly become more interesting and involved. Wootters and Zurek were the first ones to quantitatively analyze the phenomenon of complementarity or wave-particle duality, as it is more commonly called [2]. Greenberger and Yasin analyzed the problem with the assumption that unequal beams in a two-path interferometer allows for predicting, to a certain degree, which of the two paths the quanton followed. They established a duality relation [3]

\begin{equation}
P^2 + V^2 \leq 1,
\end{equation}

which involved a path predictability \( P \) and an interference visibility, which is just the fringe contrast \( V = (I_{\text{max}} - I_{\text{min}})/(I_{\text{max}} + I_{\text{min}}) \), \( I_{\text{max}}, I_{\text{min}} \) being the maximum and minimum intensities in a region on the screen. Here, equal beams would imply no path information. It was further refined by Jaeger, Shimony and Vaidman [4]. This approach, however, does not take into account the fact that even when the two beams are equal, one may introduce a probe to find out which of the two paths the quanton followed. Such a scenario was considered by Englert [5], where he studied a two-path interferometer in the presence of a path-detector. He arrived at a duality relation

\begin{equation}
D^2 + V^2 \leq 1,
\end{equation}

involving a path distinguishability \( D \) and the fringe contrast \( V \). After a lull of nearly two decades, the issue of wave-particle duality saw a resurgence when the concept of complementarity was quantitatively extended to multipath interference [6–8]. The multipath complementarity relations used a coherence introduced in the context of quantum information theory [9], instead of conventional interference visibility. This coherence measure has been shown to be a good quantifier of the wave nature of a quanton [10]. The duality relation, based on path-predictability, was also generalized to the multipath case using coherence [11]. However, the two kinds of duality relations, one based on predictability, and the other on distinguishability, remained separated.

Jakob and Bergou [12] were the first to point out that, in two-path interference, the predictability and distinguishability were connected through entanglement, and the duality relation (2) can be written as a triality relation between predictability, visibility and concurrence [13], which is a measure of entanglement. This was followed by a similar realization in classical optics [14]. This idea was later investigated in a multipath quantum setting, and a multipath predictability, coherence [9] and entanglement were shown to follow a triality relation [15]. While that analysis used a robust measure of coherence, the quantifier of entanglement could not be shown to be a proper entanglement measure. Although the analysis...
brought out the role of entanglement in multipath complementarity, it could not be rigorously quantified. There have been other interesting works trying to find an operational entanglement measure from complementarity in the general case of multipartite and multi-dimensional systems [16].

In the following, we investigate the issue of complementarity in multipath interference and show that a robust entanglement measure, namely I-concurrence [17], plays an essential role in it.

II. PRELIMINARIES

A. Visibility measure

Two decades back it was experimentally demonstrated that in multipath interference, conventional visibility, defined as fringe contrast, can increase under path-selective decoherence or selective path information [18]. Subsequent debate concluded that conventional visibility does not quantify the wave nature of a quanton in multipath interference, and cannot be used to analyze the notion of Bohr’s complementarity [19–21]. In the experiment of Mei and Weitz [18], in a multipath interference, one particular path was given an additional 180° phase flip. A selective decoherence was applied on the particular path, by scattering photons off the quanton. Counterintuitively, this resulted in an increase in the fringe contrast of the interference pattern.

We use the normalized Hilbert-Schmidt coherence to define a generalized visibility for a quanton passing through a multipath interferometer (see FIG. 1)

\[ V^2 = \frac{n}{n-1} \sum_{i \neq j} |\rho_{ij}|^2 = \mathcal{C}_{HS}, \quad (3) \]

where \( \mathcal{C}_{HS} \) is the normalised Hilbert-Schmidt coherence. The system is assumed to have a \( n \)-dimensional Hilbert space, which arises out of the quanton passing through \( n \) paths. The states corresponding to the quanton following each path are mutually orthogonal, and can be used to construct a Hilbert space basis. The density matrix element \( \rho_{ij} \) is taken between two such states. This visibility satisfies all the criteria proposed by Dür for a valid measure of interference and wave properties [22]. It is worth noting that, for the two slit interference case, the generalized visibility reduces to \( V = 2|\rho_{12}| \), which can be easily shown to be equal to the usual visibility measure \( V = (I_{\max} - I_{\min})/(I_{\max} + I_{\min}) \).

It might be useful to mention at this point that in classical optics, the interference visibility in a two-slit interference is equal to the degree of mutual coherence between the two slits, for equal intensity in the two beams. Thus for a two-slit experiment, \( 2|\rho_{12}| \) can be identified with mutual coherence function \( \gamma_{12} \), at least for equal intensity in the two beams. The classical optical equivalent of the Hilbert-Schmidt coherence then would be proportional to the square of mutual coherence function between two slits, summed over all slit pairs, \( \sum_{i \neq j} |\gamma_{ij}|^2 \).

Consider a quanton passing through \( n \)-path interferometer, which could have a multi-slit or some kind of beam splitter, as the first element. The state corresponding to the \( i \)-th path is represented by the pure state \( |\psi_i\rangle \). In general the density operator of the quanton may be mixed, and will have the general form

\[ \rho = \sum_{j=1}^{n} \sum_{k=1}^{n} \rho_{jk} |\psi_j\rangle \langle \psi_k|. \quad (4) \]

The diagonal elements represent the fractional populations of the various beams. We allow the possibility that the phase \( \theta_k \) of the \( i \)th beam may be shifted by \( \theta_k \), in order to analyze the experimental scenario of Mei and Weitz [18]. On coming out, the state of the quanton is

\[ \rho = \sum_{j=1}^{n} \sum_{k=1}^{n} \rho_{jk} |\psi_j\rangle \langle \psi_k| e^{i(\theta_j - \theta_k)}, \quad (5) \]

where \( \rho_{jk} \) are assumed to be real without loss of generality, as any phases can be absorbed in the exponential factor. In the multi-slit interference experiment, after coming out, the beams spread and overlap and are registered at different positions on the screen. In another kind of multipath interference, the beams may be split and recombined, as happens in a Mach-Zehnder interferometer. In general, we may assume that each beam is split into new channels, whose states may be represented by \( |\xi_i\rangle \). The simplest case would be where all the original beams have equal overlap with a particular output channel, say \( |\xi_i\rangle \). This translates to an assumption that \( \langle \xi_i | \psi_k \rangle = A_i \) is same for all \( k \). The probability \( I \) of finding the quanton in the \( i \)-th output channel (not the \( i \)-th path) is then given by \( I = \langle \xi_i | \rho |\xi_i\rangle \). For the present case, this probability is given by

\[ I = |A_i|^2 \left[ \sum_{j=1}^{n} \rho_{jj} + \sum_{j \neq k} \rho_{jk} e^{i(\theta_j - \theta_k)} \right] \]

\[ = |A_i|^2 \left[ 1 + \sum_{j \neq k} \rho_{jk} e^{i(\theta_j - \theta_k)} \right] \]

\[ = |A_i|^2 \left[ 1 + \sum_{j \neq k} |\rho_{jk}| \cos(\theta_j - \theta_k) \right]. \quad (6) \]

If one were to simulate Mei and Weitz experiment in this situation, one would first change the phase of the \( i \)-th path as \( \theta_i \to \theta_i + \pi \). It is to be noted that our visibility \( V \), given by (3), is unaffected by this phase change, whereas the fringe contrast does get affected.

If one wants to know which path the quanton followed, one needs to have the quanton paths entangled with a path-detector. The combined state of the quanton and the path-detector, after the two have interacted, will be
of the form
\[
\rho' = \sum_{j=1}^{n} \sum_{k=1}^{n} \rho_{jk} |\psi_j \rangle \langle \psi_k | e^{i(\theta_j - \theta_k)} \otimes |d_j \rangle \langle d_k |,
\]
where \{ |d_i \rangle \} are the states of the path detector, assumed to be normalized, but not necessarily mutually orthogonal. The reduced density operator of the quanton is obtained by tracing over the path-detector states:
\[
\rho_r' = \sum_{j=1}^{n} \sum_{k=1}^{n} \rho_{jk} |\psi_j \rangle \langle \psi_k | e^{i(\theta_j - \theta_k)} \langle d_k | d_j \rangle.
\]
In order to reproduce the Mei and Weitz experimental situation, we assume that the path-detector can only tell if the quanton passed through path \( \ell \) or not, it cannot discriminate between any other paths. This is achieved by having all the path detector states being identical, (say) equal to \(|d_1\rangle\), and only \(|d_i\rangle\) being different from \(|d_1\rangle\). Now if one were to calculate the visibility, as defined by (3), for this state of the quanton, it will turn out to be
\[
\mathcal{V}^2 = \frac{1}{n-1} \sum_{i \neq j} |\rho_{ij}|^2 \langle d_i | d_j \rangle^2.
\]
Since \(|\langle d_i | d_j \rangle| = 1\) for all \(i, j\) except those involving \(\ell\), one can easily see that
\[
\mathcal{V}^2 \neq \frac{1}{n-1} \sum_{i \leq j} |\rho_{ij}|^2
\]
or
\[
\mathcal{V}^2 \leq \mathcal{V}^2,
\]
which means that the visibility will never increase (equality is achieved for the trivial case when visibility is zero i.e., for diagonal quanton states) if one tries to find out if the quanton passed through path \(\ell\) or not. Thus the new visibility works satisfactorily even for Mei and Weitz experiment, where conventional visibility fails.

The next question we address is, how to measure this visibility in an interference experiment. The probability given by (6), can also be interpreted as the intensity at visibility in an interference experiment. The probability experiment, where conventional visibility fails. The next question we address is, how to measure this visibility in an interference experiment. The probability by having all the path detector states being identical, (say) equal to \(|d_1\rangle\), and only \(|d_i\rangle\) being different from \(|d_1\rangle\). Now if one were to calculate the visibility, as defined by (3), for this state of the quanton, it will turn out to be
\[
\mathcal{V}^2 = \frac{1}{n-1} \sum_{i \neq j} |\rho_{ij}|^2 \langle d_i | d_j \rangle^2.
\]

Now one can repeat the experiment with another pair of paths, and so on, until one has gone over all path pairs. If all paths are equally probable, \(\rho_{ii} = \rho_{jj} = \frac{1}{n}\), \(V_{ij} = n|\rho_{ij}|\). The mean square of the visibilities, of all path pairs, is given by
\[
\frac{1}{n(n-1)} \sum_{\text{pairs}} V_{ij}^2 = \frac{n}{n-1} \sum_{i \neq j} |\rho_{ij}|^2 = \mathcal{V}^2.
\]
This is a very interesting result, which shows that the multipath visibility \(\mathcal{V}\) is the root mean square of the fringe contrasts of interference from all path pairs. So, another way \(\mathcal{V}\) can be measured is by opening only a pair of paths at a time, and measuring conventional visibility, and then taking root mean square over all path pairs. Such a procedure is very much feasible, and selectively opening path pairs has already been experimentally demonstrated [26]. If the paths are not equally probable, each two-path visibility has to be multiplied with \(\frac{1}{n} (\rho_{ii} + \rho_{jj})\) to normalize the weights before taking the average. It can be shown that \(\frac{n}{n(n-1)} \sum_{\text{pairs}} (\rho_{ii} + \rho_{jj})^2 V_{ij}^2 = \mathcal{V}^2\).

### B. Predictability measure

A generalized predictability can be defined as,
\[
\mathcal{P}^2 = \sum_{i=1}^{n} \rho_{ii}^2 - \frac{1}{n-1} \sum_{i \neq j} \rho_{ii}\rho_{jj} = 1 - \frac{n-1}{n} \sum_{i \neq j} \rho_{ii}\rho_{jj}
\]
where \(\rho\) is the density matrix of the quanton and \(n\) is the dimension of the density matrix (number of paths). As is evident from the above expression \(\mathcal{P}\) varies from zero to one, therefore normalized. Although defined in a very different manner, it turns out that (16) is identical to the predictability defined by Dürr [22]. To show that (16) is a good measure of generalized predictability, we show that it follows all of Dürr’s criteria.

(i) \(\mathcal{P}\) is a polynomial function of \(\rho_{ii}\), therefore a continuous function involving only the diagonal elements.

(ii) When the quanton’s path is known for sure, i.e., \(\rho_{ii} = 1\) and \(\rho_{jj} = 0\) \(\forall j \neq i\) (only \(i^{th}\) slit is open), predictability reaches its global maximum i.e., \(\mathcal{P} = 1\).

(iii) When quanton’s probability of going through all paths are equally likely, i.e., \(\rho_{ii} = \frac{1}{n} \forall i\), generalized
predictability is zero, since \( P^2 = 1 - \frac{2n}{n-1} \sum_{i,j \neq j} \frac{1}{\rho_{ii} + \rho_{jj}} \).

(iv) A change of probabilities (diagonal elements \( \rho_{ii} \)) towards equalization leads to decrease in the generalized predictability. To show that, consider \( \rho_{11} < \rho_{22} \) and a small parameter \( 0 < \varepsilon < \rho_{22} - \rho_{11} \). A shift in probabilities, \( \rho_{11} = \rho_{11} + \varepsilon \) and \( \rho_{22} = \rho_{22} - \varepsilon \), with other elements unchanged yields the following,

\[
P^2 - \hat{P}^2 = \frac{2n\varepsilon}{n-1} (\rho_{22} - \rho_{11} - \varepsilon).
\] (17)

Since, \( \varepsilon > 0 \) and \( \varepsilon < \rho_{22} - \rho_{11} \), one obtains \( P > \hat{P} \), therefore, a decrease in predictability.

Therefore, generalized predictability as defined in (16) satisfies Dürr’s criteria. Furthermore, for two beam interferometer case (\( n = 2 \)), it reduces to,

\[
P = |\rho_{11} - \rho_{22}|
\] (18)

which matches with the predictability proposed by Greenberger and Yasin [3].

Interestingly, this multipath predictability can also be written as root-mean-square of the two-path predictabilities from all path pairs:

\[
P^2 = \frac{1}{n-1} \sum_{\text{pairs}} (\rho_{ii} + \rho_{jj})^2 P_{ij}^2,
\] (19)

where \( P_{ij} \) is the path predictability for the \( i \)’th and \( j \)’th path, \( a \ la \) Greenberger and Yasin. The fact that our \( n \)-path predictability can also be expressed in terms of the conventional two-slit predictability for path pairs, has not been recognized earlier, although the predictabilities proposed by Greenberger and Yasin [3] and Dürr [22] have been around for a long time.

C. Path distinguishability

A multipath distinguishability has been proposed earlier as [7]

\[
D^2 = 1 - \left( \frac{1}{n-1} \sum_{i \neq j} \sqrt{\rho_{ii} \rho_{jj}} |\langle d_i | d_j \rangle|^2 \right)^2,
\] (20)

which can be related to unambiguous quantum state discrimination. We define the path-distinguishability \( D \) in a different way:

\[
D^2 = 1 - \frac{n}{n-1} \sum_{i \neq j} \rho_{ii} \rho_{jj} |\langle d_i | d_j \rangle|^2.
\] (21)

In the spirit of complementarity, our proposed distinguishability measure satisfies the following basic criteria for being a reliable path quantifier [22]:

(i) When the detector states \( \{ |d_i\rangle \} \) form an orthonormal basis, i.e., \( \langle d_i | d_j \rangle = \delta_{ij} \), the path distinguishability \( D \) is maximum (\( D = 1 \)), since in this case all the paths can be unambiguously discriminated.

(ii) When all the detector states are parallel, i.e., state of quanton and the detector is separable, and probability through all the slits are equal, i.e., \( \rho_{ii} = 1/n \forall i \), then the path distinguishability \( D \) is minimum (\( D = 0 \)).

(iii) Whenever the detector states \( |d_i\rangle \) and \( |d_j\rangle \) are made more orthogonal i.e., with an increment in \( |\langle d_i | d_j \rangle| \), the distinguishability increases.

(iv) Distinguishability and visibility both can not increase simultaneously under any operation, which is evident through the fact \( D^2 + V^2 = 1 \).

For the case of two paths, (21) reduces to the distinguishability defined by Englert [5]. Next we address the issue of measuring this distinguishability. Taking cue from the visibility, we note that if one were to close all paths, except paths \( i, j \), the path distinguishability, based on minimum error state discrimination, is given by

\[
D_{ij} = \sqrt{1 - \frac{4\rho_{ii} \rho_{jj}}{(\rho_{ii} + \rho_{jj})^2}} |\langle d_i | d_j \rangle|^2,
\] (22)

which is essentially Englert’s distinguishability, generalized for unequal path probabilities by using the Helstrom bound [27]. As done in the case of visibility, one can repeat the interference experiment by selectively opening different pairs of slits, while measuring the distinguishability. When all the path are equally probable, the mean square of the distinguishabilities, from all pairs of paths, is given by

\[
\frac{2}{n(n-1)} \sum_{\text{pairs}} D_{ij}^2 = 1 - \frac{n}{n-1} \sum_{i \neq j} \rho_{ii} \rho_{jj} |\langle d_i | d_j \rangle|^2 = D^2,
\] (23)

which is the same as (21). Thus the distinguishability too can be measured experimentally, as the root mean square of conventional distinguishabilities, from all path pairs. When the paths are not equally probable, the distinguishability can be measured through the conventional two-slit distinguishability from path pairs \( D_{ij} \) and the path probabilities \( \rho_{ii} \) as follows

\[
D^2 = \frac{n}{2(n-1)} \sum_{\text{pairs}} (\rho_{ii} + \rho_{jj})^2 D_{ij}^2
\]

+ \frac{1}{2(n-1)} \sum_{\text{pairs}} (\rho_{ii} + \rho_{jj})^2.
\] (24)

The last two terms in the above relation may be interpreted as a measure of the inequality in the probability of various path pairs, by rewriting them as \( 1 - \frac{n}{n(n-1)} \sum_{\text{pairs}} (\rho_{ii} + \rho_{jj})^2 \). For equally probable paths \( \rho_{ii} + \rho_{jj} = 2/n \) for all \( i, j \). One can check that for equally probable paths, only the first term survives and reduces to (23). For \( n = 2 \), there is only one path pair, and there is no question of inequality between different pairs. In that case, the last two terms will be zero even if the two paths are unequally probable.
III. COMPLEMENTARITY

A. Role of entanglement

Consider a quanton passing through a $n$-path interferometer, such that its quantum state is given by the following pure state,

$$|\psi\rangle = \sum_{i=1}^{n} c_i |\psi_i\rangle,$$

(25)

where, $c_i \in \mathbb{C}$. The density operator for the quanton can be written as $\rho = \sum_{i,j} c_i c_j^* |\psi_i\rangle \langle \psi_j|$. Using (3) and (16), one obtains a $n$-path duality relation as follows. The sum of the squares of the predictability and visibility is given by

$$\mathcal{P}^2 + \mathcal{V}^2 = 1 - \frac{n}{n-1} \sum_{i \neq j} \rho_{ii} \rho_{jj} - |\rho_{ij}|^2.$$

(26)

Since for the pure state given by (25) $\rho_{ij} = c_i c_j^*$ resulting in $\rho_{ii} \rho_{jj} = |\rho_{ij}|^2$, the duality relation saturates for the pure state case. When a quanton is given by an ensemble $\{|\psi^\alpha\rangle, |\psi^{\alpha'}\rangle\}$, where $\alpha \geq 0$ with $\sum_{\alpha} p^{\alpha} = 1$, and $|\psi^\alpha\rangle = \sum_i c_i^{\alpha} |\psi_i\rangle$. The resulting density matrix is mixed with the elements given by $\rho_{ij} = \sum_{\alpha} p^{\alpha} c_i^{\alpha} c_j^{\alpha*}$:

$$\rho_{ii} \rho_{jj} - |\rho_{ij}|^2 = \sum_{\alpha} \left| c_i^{\alpha}\right|^2 \sum_{\beta} \left| c_j^{\beta}\right|^2 - \left| \sum_{\alpha} c_i^{\alpha} c_j^{\alpha*} \right|^2 \geq 0.$$

(27)

Here, the last inequality follows from the Cauchy-Schwarz inequality. Therefore, in general we have the following $n$-path duality relation,

$$\mathcal{P}^2 + \mathcal{V}^2 \leq 1,$$

(28)

which saturates for pure $\rho$. This result was already derived by Dürr [22], but now sets the stage for investigating the case where there is a path-detector in place, which can perform a which-path or which-way detection.

Consider an interferometer with an added path-detecting device, such that the quanton going through the $n$ paths is described by the state

$$|\Psi\rangle = \sum_{i=1}^{n} c_i |\psi_i\rangle |d_i\rangle,$$

(29)

where, $|\psi_i\rangle$ is the state of the quanton corresponding to its passing through the $i^{th}$ path, and $|d_i\rangle$ is the state of the path-detector, corresponding to that possibility. The states $\{|d_i\rangle\}$ are assumed to be normalized, but not necessarily orthogonal.

The reduced density operator of the quanton, after tracing over the path-detector states is,

$$\rho_r = \sum_{i,j} c_i c_j^* \langle d_j |d_i \rangle |\psi_i\rangle \langle \psi_j|.$$

(30)

In terms of the density operator for the quanton in the absence of the path-detector, one can write (30) as,

$$\rho_r = \sum_{i,j} \rho_{ij} \langle d_j |d_i \rangle |\psi_i\rangle \langle \psi_j|.$$

(31)

For this reduced density operator, one can write (26) as,

$$\mathcal{P}^2 + \mathcal{V}^2 = 1 - \frac{n}{n-1} \sum_{i \neq j} \rho_{ii} \rho_{jj} - |\rho_{ij}|^2$$

$$= 1 - \frac{n}{n-1} \sum_{i \neq j} \rho_{ri} \rho_{rj} + \frac{n}{n-1} \sum_{i \neq j} |\rho_{rij}|^2.$$

(32)

One can define a normalized entanglement measure $\mathcal{E}$ such that,

$$\mathcal{E}^2 = \frac{n}{n-1} \sum_{i \neq j} \left( \rho_{ri} \rho_{rj} - |\rho_{rij}|^2 \right) = \frac{n}{2(n-1)} E^2$$

(33)

where $E$ is the generalized concurrence [28], and coincides with I-concurrence [17], when the bipartite state under consideration is a pure state. Eqn. (32) can then be written as,

$$\mathcal{P}^2 + \mathcal{V}^2 = 1 - \mathcal{E}^2,$$

(34)

or as a triality relation,

$$\mathcal{P}^2 + \mathcal{V}^2 + \mathcal{E}^2 = 1.$$

(35)

Therefore, the generalized predictability, Hilbert-Schmidt coherence, and I-concurrence obey a tight triality relation. It is interesting to note that the path-distinguishability measure, as defined in the previous section, is related to the generalized predictability through I-concurrence as follows,

$$\mathcal{D}^2 = \mathcal{P}^2 + \mathcal{E}^2.$$

(36)

This above relation shows that the two different kinds of path knowledge, predictability and distinguishability, are quantitatively connected through a measure of entanglement, the I-concurrence. It very naturally generalizes Englert’s duality relation (2) to multipath interference:

$$\mathcal{D}^2 + \mathcal{V}^2 = 1.$$

(37)

Needless to say, Englert’s duality relation is recovered for $n = 2$. Since the above relation is an equality, it is obvious that this multipath distinguishability does not suffer from the shortcoming of Dürr’s multipath distinguishability, pointed out by Bimonte and Musto [29].

We have shown that a proper entanglement measure, the I-concurrence, is an integral part of complementarity in multipath interference. For $n = 2$, (35) reduces to

$$P^2 + V^2 + \mathcal{E}^2_2 = 1,$$

(38)

where is $P$ is the predictability of Greenberger and Yasin.
For unequally probable paths, the above expression modifies to \( E^2 = \frac{2}{n(n-1)} \sum_{\text{pairs}} E_{ij}^2 \), where \( E_{ij} \) is the concurrence of entanglement between the quanton and the path-detector when all except path \( i, j \) are blocked. This is true if all the paths are equally probable. For unequally probable paths, the above expression modifies to \( E^2 = \frac{n}{2(n-1)} \sum_{\text{pairs}}(\rho_{ii} + \rho_{jj})^2 E_{ij}^2 \). This is an interesting connection between concurrence and I-concurrence which becomes meaningful in the kind of experiment we have been discussing, where pairs of paths are opened at a time.

### IV. PERSPECTIVE

As mentioned before, a triality relation has been obtained earlier in the context of multi-slit quantum interference [15]. Let us look at the results obtained here in that perspective. The relation obtained earlier involved the coherence measure introduced in the context of quantum information theory [9]. That coherence has been shown to be a proper measure of coherence, in the sense that it does not increase under any incoherent operation. On the other hand, the Hilbert-Schmidt coherence used by us, although very commonly used, has been shown to not be a robust coherence monotone, as it may increase under some incoherent operations. Nevertheless, it has been recently highlighted that in addition to having a simpler structure, the Hilbert-Schmidt coherence has information theoretic significance, in particular, through its manifestation as quantum uncertainty [23]. Furthermore, the shortcoming of monotonicity is absent in the genuine quantum coherence framework [24]. Therefore, Hilbert-Schmidt coherence may still be good enough to quan-
tify visibility in multipath interference. From a practical consideration too, the coherence measure of Baumgratz et al. has a simpler connection to multislit interference [10], which has also been experimentally demonstrated [25, 26].

The predictability used in both the works has been shown to be a proper measure of path predictability. Both can be easily measured in a multipath interference experiment, as it only involves measuring the intensities at each slit, or in each path.

The measure of entanglement is an aspect in which the earlier work falls short. The quantifier used there has not yet been shown to be a proper entanglement measure. On the other hand, the quantifier used for entanglement in the present work is I-concurrence, a well-known entanglement monotone. This advantage allows the possibility of experimentally inferring the degree of entanglement between the quanton and an ancilla system.

Thus the two formulations connecting predictability, entanglement, and interference, are equivalent, but both have their strengths and weaknesses. For two-path interference, they become identical.

V. CONCLUSION

In conclusion, we have introduced an interference visibility based on the Hilbert-Schmidt coherence, and shown that it constitutes a good measure of wave nature, and does not suffer from the shortcomings of the conventional fringe contrast, which emerges in certain experiments [18]. Interestingly, this visibility can be obtained as a root mean square of the fringe contrasts from all pairs of paths, if the experiment is done by opening only a pair of paths at a time. Selectively opening pairs of slit in a multi-slit interference experiment has already been realized [26].

A multipath predictability has been introduced, which is a natural generalization of the two-path predictability of Greenberger and Yasin [3]. Interestingly, this multipath predictability can also be measured using the two-path predictabilities of pairs of paths. A multipath distinguishability has also been introduced which, again, can be expressed in terms of root mean square of the two-path distinguishability (introduced by Englert) of all pairs of paths. The predictability and visibility introduced here, together with the I-concurrence (related to entanglement between the quanton and a path-detector), are shown to follow a tight triality relation. This brings out the essential role of entanglement in the phenomenon of complementarity, in the general context of multipath interference. As an interesting offshoot, the normalized I-concurrence turns out to be the root mean square of concurrences relating to all path pairs.

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