Gouy phase and matter waves

I G da Paz, M C Nemes and J G Peixoto de Faria

1 Departamento de Física, Instituto de Ciências Exatas, Universidade Federal de Minas Gerais, C.P.702, CEP 30161-970, Belo Horizonte, MG, Brazil
2 Departamento Acadêmico de Ciências Básicas, Centro Federal de Educação Tecnológica de Minas Gerais, Av Amazonas 7675, CEP 30510-000, Belo Horizonte, MG, Brazil

E-mail: irismar@fisica.ufmg.br

Abstract. We propose an experiment involving atoms and a standing electromagnetic wave in a cavity as a possibility to determine the analogous of Gouy’s phase in optics for matter waves. The advantages and shortcomings of the proposed experiment are analyzed indicating its feasibility at the edge of present day technology.

1. Introduction

In 1890 Gouy showed that a focused electromagnetic beam will acquire an additional axial $\pi$ phase shift with respect to a plane wave as it evolves through its focus [1–3]. This phase shift is now called the Gouy phase shift or phase anomaly and is best known in the case of an obstructed paraxial gaussian beam [4]. Due to the formal similarity between the paraxial equation in wave optics and the Schrödinger equation for matter waves a question which arises naturally is if a similar phase anomaly may occur in the region around the focus of an atomic beam.

In order to answer this question, in this contribution we present the evolution of an atomic beam described by a gaussian wave packet interacting dispersively with a cavity field. We analyze the feasibility of the experiment in terms of present limitations in the necessary devices such as lenses and interferometers.

2. The model

The model we use is the following [5–7]: consider two-level atoms moving along the $Oz$ direction and that they penetrate a region where a stationary electromagnetic field is maintained. The region is the interval $z = -L$ until $z = 0$. The atomic linear moment in this direction is such that the de Broglie wavelength associated is much smaller than the wavelength of the electromagnetic field. We assume that the atom moves classically along direction $Oz$ and the atomic transition of interest is detuned with the mode of the electromagnetic field (dispersive interaction). The Hamiltonian for this model is given by

$$H = \frac{\hat{p}_x^2}{2M} + g(\hat{x})\hat{a}^\dagger\hat{a}$$

(1)

where $M$ is the atom mass, $\hat{p}_x$ and $\hat{x}$ are the linear momentum and position along the direction $Ox$, $\hat{a}^\dagger$ and $\hat{a}$ are the creation and destruction operators of a photon of the electromagnetic mode, respectively. The coupling between atom and field is given by de function $g(x) = \alpha \varepsilon_0^2(x)$ where $\alpha$ is the atomic linear susceptibility, $\alpha = \frac{\phi^2}{\hbar \Delta}$, where $\phi^2$ is the square of the dipole moment and $\Delta$ is the detuning. $\varepsilon_0$
corresponds to the electric field amplitude in vacuum. The effective interaction time is \( t_L = \frac{L}{v_z} \), where \( v_z \) is the longitudinal velocity of the atoms.

The dynamics of the closed system is governed by the Schrödinger equation

\[
i\hbar \frac{d}{dt} |\Psi\rangle = \hat{H}_{AF} |\Psi\rangle.
\]

(2)

At \( t = 0 \) the state of the system is given by a direct product of the state corresponding to the transversal component of the atom and a field state, \( |\Psi_{cm}\rangle \otimes |\Psi_F\rangle \). The field state can be expanded in the eigenstates of the number operator \( \hat{a}^\dagger \hat{a} \):

\[
|\Psi_F\rangle = \sum_n \omega_n |n\rangle, \quad \sum_n |\omega_n|^2 = 1.
\]

(3)

When atom and field interact the atomic and field states get entangled. We can then write

\[
|\Psi(t)\rangle = \sum_n \omega_n \int_{-\infty}^{+\infty} dx \, \psi_n(x, t) |x\rangle \otimes |n\rangle,
\]

where

\[
i\hbar \frac{\partial}{\partial t} \psi_n(x, t) = \left\{-\frac{\hbar}{2M} \nabla^2 + g(x)n \right\} \psi_n(x, t),
\]

(5)

or, if one defines

\[
|\Psi_n(t)\rangle = \int_{-\infty}^{+\infty} dx \, \psi_n(x, t) |x\rangle,
\]

the Eq. (5) takes the form

\[
i\hbar \frac{d}{dt} |\Psi_n(t)\rangle = \left[ \frac{\hat{p}_x^2}{2M} + g(\hat{x})n \right] |\Psi_n(t)\rangle.
\]

(7)

Next, we will use the harmonic approximation for \( g(\hat{x}) \) which is a fine approximation provided the dispersion of the wavepacket in the transverse direction \( d \) is much smaller than the wavelength of the electromagnetic field mode \( \lambda \) [7]. Taking the main terms of the Taylor expansion of the function \( g(x) \),

\[
g(x) \approx g_0 - \frac{g_1^2}{g_2} + \frac{1}{2} g_2 (x - x_f),
\]

(8)

we get

\[
i\hbar \frac{d}{dt} |\Psi_n(t)\rangle = \left[ \frac{\hat{p}_x^2}{2M} + \frac{1}{2} M \Omega_n^2 (\hat{x} - x_f)^2 \right] |\Psi_n(t)\rangle
\]

\[
= \hat{H}_n |\Psi_n(t)\rangle,
\]

(9)

where \( x_f = -g_1/g_2 \) and \( \Omega_n^2 = n g_2 / M \). In order to obtain focalization of the atomic beam it is crucial that the initial state be compressed in momentum since only then this initial momentum compression is transferred dynamically to the \( x \) coordinate and a focus can be obtained [6, 8].

**Focalization of the atomic beam**

Let us assume, as an initial atomic state, the compressed vacuum state

\[
\langle x | \psi_n(t=0) \rangle = \psi_n(x, t=0) = \left( \frac{1}{d \sqrt{\pi}} \right)^{1/2} \exp \left( -\frac{x^2}{2d^2} \right)
\]

(10)

where \( d \) is the initial width of the packet and \( d > b_n = \sqrt{\hbar} / (M \Omega_n) \). We known that the most general gaussian state will remain gaussian through quadratic hamitonians. Therefore the time evolution can be
given by the time evolution of the parameters needed to specify the state. The most general gaussian state can be
written as [9]

$$\Psi(x) = \left(\frac{u}{\pi}\right)^{1/4} \exp\left(-i\frac{x\bar{\rho}}{2\hbar} + i\Phi\right) \exp\left[-\frac{(x - \bar{x})^2}{2} + i\frac{\bar{p}x}{\hbar}\right],$$  \hspace{1cm} (11)

where the parameters $\bar{x}$, $\bar{\rho}$, $u$ and $v$ are time dependent functions, and $\Phi$ is a real phase known as the Gouy’s phase.

Now inserting this function in the Schrödinger equation we get the time evolution for the parameters $\bar{x}$, $\bar{\rho}$, $K = u + iv$ and $\Phi$. We get

$$\bar{x}(t < t_L) = -x_f \cos \Omega_n t,$$

(12)

$$\bar{\rho}(t < t_L) = M \Omega_n x_f \sin \Omega_n t,$$

(13)

and

$$K(t < t_L) = \left(\cos \Omega_n t + \frac{i b_n^2}{d^2} \sin \Omega_n t\right)^{-1} \left(\frac{1}{d^2} \cos \Omega_n t + i \frac{1}{b_n} \sin \Omega_n t\right),$$

(14)

for the initial conditions $\bar{x}_0 = -x_f$, $\bar{\rho}_0 = 0$, $u = d^{-2}$ and $v = 0$. Moreover, at this stage

$$\Phi \sim \Omega_n t.$$  \hspace{1cm} (15)

Also, from Eq. (14) we obtain

$$u(t < t_L) = \left[d^2 \left(\cos^2 \Omega_n t + \frac{b_n^4}{d^2} \sin^2 \Omega_n t\right)\right]^{-1}.$$  \hspace{1cm} (16)

Now $u^{-1}$ is the width of the gaussian wavepacket squared.

When the atomic beam leaves the region of the electromagnetic field, the atomic state evolves freely. The equations of motion can be obtained analogously and we get for $t > t_L$

$$\bar{x}(t > t_L) = -x_f \cos \phi_n + \Omega(t - t_L)x_f \sin \phi_n,$$

(17)

$$\bar{\rho}(t > t_L) = M \Omega_n x_f \sin \phi_n,$$

(18)

$$K(t > t_L) = \frac{b_n^2}{d^2} \cos \phi_n + i \sin \phi_n$$

(19)

and

$$d^2 u(t > t_L) = \left[\left(\cos \phi_n - \frac{t - t_L}{\tau} \sin \phi_n\right)^2 + \frac{b_n^4}{d^2} \left(\sin \phi_n + \frac{t - t_L}{\tau} \cos \phi_n\right)^2\right]^{-1},$$

(20)

where $\phi_n = \Omega_n t_L$ and $\tau = M b_n^2 / \hbar$.

The focus

The focus will be located in the atomic beam region where the width of the wavepacket is minimal. In other words, when $u(t > t_L)$ be a maximum there will be the focus. This will happen when the function

$$D(t) = \left(\cos \phi_n - \frac{t - t_L}{\tau} \sin \phi_n\right)^2 + \frac{b_n^4}{d^2} \left(\sin \phi_n + \frac{t - t_L}{\tau} \cos \phi_n\right)^2$$  \hspace{1cm} (21)

attains its minimum value. The time for which its derivative vanishes is given by

$$t_f = \frac{z_f + L}{v_z} = t_L + \tau \left(1 - \frac{b_n^4}{d^2}\right)^2 \sin \phi_n \cos \phi_n$$

(22)
therefore the focus is located at

\[ z_f = v_z \tau \frac{1 - b_n^4}{d^4} \tan \phi_n + \tan^2 \phi_n. \] (23)

**Phase Anomaly**

We now, considering the equation for \( u(t > t_L) \) Eq. (20), we get for the phase \( \Phi \)

\[
\Phi(t > t_L) = -\frac{b_n^2}{d^2} \left( \sin^2 \phi_n + b_n^2 \cos^2 \phi_n \right) \\
\times \arctan \left\{ \frac{d^2 \left[ \frac{t-t_L}{\tau} \left( \tan^2 \phi_n + b_n^4 \right) - \left( 1 - b_n^4 \right) \tan \phi_n \right]}{b_n^2 \left( 1 + 2 \tan^2 \phi_n \right)^{1/2} \tan \phi_n} \right\}. \] (24)

At the focus \( \Phi = 0 \), as expected. Therefore the atomic wavefunction suffers a \( \pi/2 \) variation around \( z_f \).

The fact that this variation is only \( \pi/2 \), in contrast with the value of \( \pi \) for the light, as described in the introduction of this work, is due to the fact that the quantum lens focuses the atomic beam in the \( Ox \) direction, keeping the \( Oy \) direction unperturbed (i.e., the electromagnetic field acts as a cylindrical lens).

3. A possible Experiment with Cesium atoms

The setup showed in Fig. 1 is similar to the one used to measure the Gouy phase shift in gaussian light beam. Here, a stationary field yielded in the cavity works out as a quantum lens. The Mach-Zhender interferometer is built using integrated atom optics: the paths are atomic waveguides produced by focusing a detuned laser light with microfabricated cylindrical lenses [10] or by magnetic fields produced by microwires carrying electrical currents [11]. We compute the values of the experimental parameters of interest considering a beam of cold cesium atoms.

We choose the detuning \( |\Delta| \sim 2 \pi \times 10^{11} \) Hz, the coupling strength \( g \sim 2 \pi \times 10^7 \) Hz, and we consider the mean photon number in the field mode as \( \bar{n} \sim 100 \) and the corresponding atomic transition the \( D_2 \) line (\( \lambda \sim 852 \) nm). In this case, \( \frac{\omega_{ab}}{|\Delta|} = 10^{-3} \), \( |\Delta| \omega_{ab} \sim 0.3 \times 10^{-3} \) (\( \omega_{ab} \) is the atomic transition frequency), \( b \sim 5.16 \times 10^{-8} \) m, \( \Omega_n = 1.80 \times 10^5 \) rad/s. We must have \( b \ll \lambda \) to get a valid harmonic approximation. So we get \( d \sim 80 \) nm as the initial transversal width of the atomic wavepacket. There are two parameters remaining that need to be fit: \( L \) (stationary field waist) and \( v_z \) (longitudinal atomic velocity). Taking \( t_L \sim 1 \) \( \mu s \), the atomic velocity can be calculated imposing that \( z_f^{(n)} \sim 10^{-4} \) m. So we get \( v_z \sim 30 \) m/s, which implies that the stationary field waist must be approximately \( L \sim 30 \) \( \mu m \).

**References**

[1] Gouy C R 1890 Acad. Sci. Paris 110 1251.
[2] Gouy C R 1891 Ann. Chim. Phip. Ser. 6 24.
[3] Siegman A E 1986 Lasers (Calif.: Mill Valley, University Science) p 682.
[4] Saleh B E A and Teich M 1991 Fundamentals of Photonics. (New York: John Wiley Sons) p 87.
[5] Averbukh I Sh, Akulin V M and Schleich W P 1994 Phys. Rev. Lett. 72 437.
[6] Rohwedder B and Orszag M 1996 Phys. Rev. A 54 5076.
[7] Schleich W P 2001 Quantum Optics in Phase Space (Berlin: Wiley-VCH) p 583.
[8] da Paz I G, Nemes M C and de Faria J G P to be published.
[9] Bialynicki-Birula, I 1998 Acta Phys. Polonica B 29 3569.
[10] Birkl G, Buchkremer F B J, Dumke R and Estmer W 2001 Opt. Comun. 191 67.
[11] Mabuchi H, Ye J and Kimble H J 1999 Appl. Phys. B 68 1095.