Bayes Factors and Posterior Estimation: Two Sides of the Very Same Coin

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ABSTRACT

Recently, several researchers have claimed that conclusions obtained from a Bayes factor (or the posterior odds) may contradict those obtained from Bayesian posterior estimation. In this article, we wish to point out that no such “contradiction” exists if one is willing to consistently define one’s priors and posteriors. The key for congruence is that the (implied) prior model odds used for testing are the same as those used for estimation. Our recommendation is simple: If one reports a Bayes factor comparing two models, then one should also report posterior estimates which appropriately acknowledge the uncertainty with regards to which of the two models is correct.

1. Introduction

Recently, several researchers have claimed that conclusions obtained from a Bayes factor (or the posterior odds) may contradict those obtained from Bayesian posterior estimation. For example, Rouder, Haaf, and Vandekerckhove (2018) discuss what they see as “two popular Bayesian approaches that may seem incompatible inasmuch as they provide different answers to what appears to be the same question.” The two approaches in question are referred to as the “estimation approach” and the “Bayes factor approach.” Wagenmakers et al. (2022) also discuss these two approaches and ask why they result in a “paradoxical state of affairs.” Kelter (2022) lament how “Bayesian interval estimates and hypothesis tests can yield contradictory conclusions” and Tendeiro and Kiers (2019) examine an apparent “mismatch between results from tests and credible intervals.” Wagenmakers et al. (2020) go so far as to suggest that, since credible intervals and the Bayes factor are at odds, “the practice of rejecting [the null] whenever a 95% interval does not include the null value […] is not principled and may be misleading in practice.”

In this article, we wish to point out that no such “conflict,” “paradox,” or “mismatch” exists if one is willing to consistently define one’s priors and posteriors. Specifically, we show that if the same (implied) prior model odds are specified, the Bayes factor approach and the estimation approach are in fact, entirely congruent.

Let \( \theta \) be the parameter of interest and suppose there are two different models, Model 0 (\( M_0 \)) and Model 1 (\( M_1 \)), which are a priori probable with probabilities \( \Pr(M_0) \) and \( \Pr(M_1) \), respectively, such that \( \Pr(M_0) + \Pr(M_1) = 1 \). For each of these models, there is a distinct prior distribution for \( \theta \). For model \( i \), let \( \pi_i(\theta) \) be the prior density for \( \theta \), and let \( \pi_i(\theta|\text{data}) \) be the corresponding posterior density such that:

\[
\pi_i(\theta|\text{data}) = \frac{\pi_i(\theta)\Pr(\text{data}|\theta)}{\Pr(\text{data}|M_i)},
\]

for \( i = 0, 1 \), where \( \Pr(\text{data}|\theta) \) is the model distribution of the data given \( \theta \), and:

\[
\Pr(\text{data}|M_i) = \int \Pr(\text{data}|\theta)\pi_i(\theta)d\theta.
\]

Based on the posterior, one may calculate point estimates for \( \theta \), such as the posterior mean or the posterior median, and credible intervals.

One may also be interested in calculating the Bayes factor, \( BF_{10} \), which is defined as the ratio of the posterior odds to the prior odds, and can also be defined as the ratio of the marginal likelihoods of the observed data for the two models:

\[
BF_{10} = \frac{\Pr(M_1|\text{data})\Pr(M_1)}{\Pr(M_0|\text{data})\Pr(M_0)}
\]

In order to determine which of the two models is more likely to be the true data-generating mechanism, one may consider the posterior model probabilities, \( \Pr(M_0|\text{data}) \) and \( \Pr(M_1|\text{data}) \):

\[
\Pr(M_0|\text{data}) = \frac{\Pr(\text{data}|M_0)\Pr(M_0)}{\Pr(\text{data}|M_0)\Pr(M_0) + \Pr(\text{data}|M_1)\Pr(M_1)}
= \frac{\Pr(M_0)}{\Pr(M_0) + BF_{10} \times \Pr(M_1)},
\]

\[
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$$\Pr(M_1|\text{data}) = \frac{\Pr(\text{data}|M_1)\Pr(M_1)}{\Pr(\text{data}|M_0)\Pr(M_0) + \Pr(\text{data}|M_1)\Pr(M_1)} = \frac{\Pr(M_1)}{\Pr(M_0) + \BF_{10} \times \Pr(M_1)}, \tag{3}$$

as well as the posterior evidence for Model 1 versus Model 0:

$$\PO_{10} = \frac{\Pr(M_1|\text{data})}{\Pr(M_0|\text{data})} = \frac{\Pr(M_1)}{\Pr(M_0)} \times \BF_{10}. \tag{4}$$

The Bayes factor, $\BF_{10}$, can be calculated without defining prior model odds (i.e., without specifying $\Pr(M_1)$ and $\Pr(M_0)$). However, in order to determine which model is more likely to be the “true model,” the Bayes factor must be combined with the prior model odds to obtain the posterior odds. Despite this fact, the practice of explicitly specifying prior model odds is “often ignored” by researchers (Tendeiro and Kiers 2019) who simply quote the Bayes factor as “the weight of evidence from the data” in favor of one model relative to another (O’Hagan and Forster 2004). Lavine and Schervish (1999) explain in detail why “such informal use of Bayes factors suffers a certain logical flaw” that can result in incoherent decisions.

Should a researcher make decisions about which model they believe is most likely to be true citing only the Bayes factor, one must work backward in order to determine their “implied prior model odds.” For instance, if someone believes that $M_1$ is more likely to be the true model whenever $\BF_{10} > 1$, and believes that $M_0$ is more likely to be the true model whenever $\BF_{10} < 1$, then it follows that such a person has assumed prior model odds of 1:1. Someone more skeptical of $M_1$ might only believe that $M_1$ is most likely whenever $\BF_{10} > 9$, and that $M_0$ is most likely whenever $\BF_{10} < 9$. Such beliefs would correspond to implied prior model odds of 1:9 (i.e., they would necessarily imply that $\Pr(M_0) = 0.9$ and $\Pr(M_1) = 0.1$). Note that researchers will typically only be willing to conclude with some certainty that $M_0$ is the true model if $\BF_{10}$ falls below some threshold (e.g., if $\BF_{10} < 1/3$) and only be willing to conclude with some certainty that $M_1$ is the true model if $\BF_{10}$ falls above some threshold (e.g., if $\BF_{10} > 3$). These thresholds are chosen based on both the (implied) prior model odds and on the (implied) relative costs of making a false positive conclusion or false negative conclusion versus remaining indecisive; see Lavine and Schervish (1999).

Curiously, some researchers adopt different (implied) prior model odds for testing and for estimation. For instance, based on the idea that there is a fundamental distinction between testing (“is the effect, $\theta$, present or absent?”) and estimation (“how big is the effect, $\theta$, assuming it is present?”) (Wagenmakers et al. 2018), researchers often use a Bayes factor comparing $M_1$ (“effect is present”) to a point null $M_0$ (“effect is absent”) for testing (with implied) prior model odds of 1:1, but then assume $M_1$ (“effect is present”) is the true model for estimation (with implied prior model odds of 1:0). As we shall see in the next section with a simple example, this curious practice of adopting different (implied) prior model odds for testing and for estimation is the root cause of the so-called “paradoxical state of affairs.”

2. Flipping a Possibly Biased Coin

As a concrete example, consider observing $X$ “heads” out of $N$ coin flips from a possibly biased coin. This is the same example as considered by Wagenmakers et al. (2022); see also Puga, Krzywinski, and Altman (2015a,b). Curiously, while the biased coin example has long been part of statistical folklore, such a coin does not actually exist in the physical world; see Gelman and Nolan (2002).

We assume that the observed coin flip data are the result of a Binomial distribution where the $\theta$ parameter corresponds to the probability of obtaining a “heads” such that:

$$\Pr(\theta) = f_{\text{Binom}}(X, N|\theta),$$

where $f_{\text{Binom}}()$ is the Binomial probability mass function.

We define two different priors, one for $M_0$, and another for $M_1$. For $M_0$, a “point null” prior states that the coin is fair (i.e., “heads” and “tails” are equally likely), such that $\theta = 0.5$. For $M_1$, the prior states that the probability of a “heads” could be anywhere between 0 and 1, with equal likelihood, as described by a Beta distribution: $\theta \sim \text{Beta}(\alpha, \beta)$, where $\alpha = \beta = 1$. The Beta(1,1) distribution is equivalent to a Uniform(0,1) distribution. See panels A and B in Figure 1.

The prior density functions are therefore defined as follows:

$$\pi_0(\theta) = \delta_{0.5}(\theta),$$

and

$$\pi_1(\theta) = \begin{cases} 1, & \text{if } \theta \in [0,1] \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where $\delta_{0.5}()$ is the Dirac delta function at 0.5 which can be informally thought of as a probability density function which is zero everywhere except at 0.5, where it is infinite.

Suppose we observe $X = 60$ “heads” out of $N = 100$ coin flips. Then we can easily calculate the following from Equation (1):

$$\Pr(\text{data}|M_0) = f_{\text{Binom}}(60, 100, 0.5) = 0.0108,$$

and

$$\Pr(\text{data}|M_1) = \int_{t=0}^{t=1} f_{\text{Binom}}(60, 100, t)dt = 0.0099.$$ 

These functions are plotted in panels D and E of Figure 1 with the grey vertical dashed line corresponding to observed data of $X = 60$. The ratio of these two numbers is equal to the Bayes factor:

$$\BF_{10} = \frac{\Pr(\text{data}|M_1)}{\Pr(\text{data}|M_0)} = \frac{0.0099}{0.0108} = 0.913. \tag{6}$$

In general, the Bayes factor for this scenario can be computed as

$$\BF_{10} = (N + 1)\left(\frac{N}{X}\right)^\alpha \beta^X (1 - \theta_0)^{N-X},$$

where $\theta_0 = 0.5$. Now suppose the prior probability of each of the two models is equal, such that $\Pr(M_0) = \Pr(M_1) = 0.5$. Then, following Equations (2)–(4), we obtain:

$$\Pr(M_0|\text{data}) = \frac{0.5}{0.5 + 0.913 \times 0.5} = 0.523.$$
Figure 1. Flipping a possibly biased coin ($X = 60$ and $N = 100$). The first row shows prior density functions (with the grey numbers next to the arrows corresponding to point mass of the "spike"); middle row shows probability model for the data (with the gray vertical dashed lines corresponding to observed data of $X = 60$); and the lower row shows posterior density functions (with the grey numbers next to the arrows corresponding to point mass of the "spike"). Left column corresponds to $M_0$, middle column corresponds to $M_1$, and right column corresponds to the "mixed/averaged" model.

$$\Pr(M_1|\text{data}) = 0.913 \times 0.5 + 0.913 \times 0.5 = 0.477,$$

and

$$\text{PO10} = 0.5 \times 0.913 = 0.913,$$

which indicates modest support for $M_0$ relative to $M_1$.

Each of the two models has a corresponding posterior distribution plotted in panels G and H of Figure 1. If one assumes that $M_1$ is the correct model, then the posterior distribution of $\theta$ can be derived analytically (since the Binomial and Beta are conjugate distributions) as

$$\pi_1(\theta|\text{data}) = f_{\text{Beta}}(\theta, X + \alpha, N - X + \beta) = f_{\text{Beta}}(\theta, 60 + 1, 100 - 60 + 1) = f_{\text{Beta}}(\theta, 61, 41),$$

where $f_{\text{Beta}}()$ is the Beta probability density function. The posterior mean, $\hat{\theta}_1$, posterior median, $\tilde{\theta}_1$, and a 95% equal-tailed credible interval, 95%CrI($\theta_1$), can then be calculated as $\hat{\theta}_1 = 61/(61 + 41) = 0.598$, $\tilde{\theta}_1 = 0.599$, and 95%CrI($\theta_1$) = [0.502, 0.691]. This credible interval, notably, does not include 0.5.
If, alternatively, one assumes that \( M_0 \) is the correct model, then, consequently, the posterior distribution is \( \pi_0(\theta|data) = \delta_{0.5}(\theta) \). The posterior mean, median, and 95% equal-tailed credible interval will be: \( \delta_0 = 0.500, \bar{\delta}_0 = 0.500, \) and 95% CrI(\( \theta_0 \)) = [0.500, 0.500]. This credible interval only includes 0.500, indicating total certainty with regards to the true value of \( \theta \). As such, it is a very conservative 95% credible interval in the sense that 100% (instead of exactly 95%) of the posterior weight is within the interval.

This coin flip scenario is a good example for illustrating the so-called “incompatibility” discussed by Rouder, Haaf, and Vandekerckhove (2018) and others: The BF_{10} = 0.913 indicates that the data are more likely to occur under the null, while 95% CrI(\( \theta_1 \)) = [0.502, 0.691] excludes 0.500. However, there is no reason to expect that the BF_{10} and 95% CrI(\( \theta_1 \)) will be congruent. If someone decides that \( M_0 \) (“the coin is fair”) is more likely because BF_{10} < 1, necessarily their implied prior model odds must be 1:1. However, if they also claim that Pr(\( \theta \in 95\text{% CrI}(\theta_1)|data \)) = 0.95, then they must also believe that Pr(\( M_1 \) = 1 and \( M_0 \) = 0 which is clearly incompatible with the implications made for testing. To be clear, testing and estimation will only be congruent if the (implied) prior model odds adopted for both are the same. In the next section, we review two different ways this can be achieved.

### 3. Two Equivalent Approaches

If someone assumes that \( M_1 \) is the correct model (and therefore reports 95% CrI(\( \theta_1 \))), then there is really no sense for them to compute a Bayes factor (or the posterior odds), since the (implied) prior odds are definitive with respect to which of the two models is “correct.” van Ravenzwaaij and Wagenmakers (2021) explain as follows: If one assumes that \( M_1 \) is the correct model, then the null model is “deemed false from the outset, and hence no amount of data can either support or undercut it.”

Alternatively, if someone is uncertain about which of the two models is correct, estimates and credible intervals for \( \theta \) should take this uncertainty into account. This can be achieved by either (a) Bayesian model averaging (BMA), or by (b) defining a single “mixture” prior. Both approaches, as we will demonstrate, will deliver the very same results.

Under the BMA approach, one averages the posterior distributions under each of the two models, \( \pi_0(\theta|data) \) and \( \pi_1(\theta|data) \), weighted by their relative posterior model probabilities, to obtain an appropriate “averaged” posterior distribution:

\[
\pi_{\text{BMA}}(\theta|data) = \pi_0(\theta|data) \text{Pr}(M_0|data) + \pi_1(\theta|data) \text{Pr}(M_1|data).
\]  

(9)

Alternatively, under the single “mixture” prior approach, one defines a single model with a single “mixture” prior, \( \pi_{\text{mix}}(\theta) \), defined as a weighted combination of \( \pi_0(\theta) \) and \( \pi_1(\theta) \) such that:

\[
\pi_{\text{mix}}(\theta) = \pi_0(\theta) \text{Pr}(M_0) + \pi_1(\theta) \text{Pr}(M_1).
\]

In this case, the corresponding posterior, \( \pi_{\text{mix}}(\theta|data) \), can also be written as a weighted combination such that:

\[
\pi_{\text{mix}}(\theta|data) \propto (\text{Pr}(M_0)\pi_0(\theta) + \pi_1(\theta)) \text{Pr}(data|\theta).
\]  

(10)

The \( \pi_{\text{mix}}(\theta|data) \) posterior and the \( \pi_{\text{BMA}}(\theta|data) \) posterior are in fact identical since:

\[
\pi_{\text{mix}}(\theta|data) \propto \text{Pr}(M_0)\pi_0(\theta) \text{Pr}(data|\theta)
\]

\[
+ \text{Pr}(M_1)\pi_1(\theta) \text{Pr}(data|\theta)
\]

\[
\propto m_0 \text{Pr}(M_0) + m_1 \text{Pr}(M_1)
\]

\[
\propto \text{Pr}(M_0)\pi_0(\theta|data) + \text{Pr}(M_1)\pi_1(\theta|data)
\]

\[
\propto \pi_{\text{BMA}}(\theta|data),
\]

(11)

where \( m_i = \text{Pr}(M_i|data), \) for \( i = 0, 1 \). Thus, because densities are normalized to integrate to 1, we have that \( \pi_{\text{mix}}(\theta|data) = \pi_{\text{BMA}}(\theta|data) \). As such, point estimates and credible intervals obtained with the BMA approach will be identical to those obtained with the single “mixture” prior approach.

This equality is not always acknowledged in the literature. For example, in describing a mixture model for Bayesian meta-analysis, Röver, Wandel, and Friede (2019) note that the two approaches are equal (“the mixture prior effectively results in a model-averaging approach”), while Gronau et al. (2019) and Bartóš et al. (2021) in proposing a similar approach for Bayesian meta-analysis, do not point out that the BMA approach is exactly equivalent to the single “mixture” prior approach.

Going forward we write simply \( \pi(\theta|data) \) instead of using the \( \pi_{\text{mix}}(\theta|data) \) or \( \pi_{\text{BMA}}(\theta|data) \) notation, and use \( \pi(\theta) \) to denote the “mixed”/“averaged” prior instead of \( \pi_{\text{mix}}(\theta) \) or \( \pi_{\text{BMA}}(\theta) \).

In our example of the possibly biased coin, with the BMA approach, we obtain, from Equation (9):

\[
\pi(\theta|data) = 0.523 \times \delta_{0.5}(\theta) + 0.477 \times f_{\text{beta}}(\theta, 61, 41).
\]

Obtaining MCMC draws from this posterior is simple: With 1 million draws from \( \pi_0(\theta|data) \) and another 1 million draws from \( \pi_1(\theta|data) \), one can obtain 1 million draws from \( \pi(\theta|data) \) by combining together approximately 523 thousand draws from \( \pi_0(\theta|data) \) with 477 thousand draws from \( \pi_1(\theta|data) \). See panel I of Figure 1 where the \( \pi(\theta|data) \) function is plotted. From these combined draws, we can then compute the posterior mean, \( \bar{\theta} \), the posterior median, \( \bar{\theta} \), and the 95% equal-tailed credible interval, 95% CrI(\( \theta \)), as \( \bar{\theta} = 0.547, \bar{\theta} = 0.500, \) and 95% CrI(\( \theta \)) = [0.500, 0.676]. Note that the 95% equal-tailed credible interval includes 0.5. Also, note that due to the discontinuity in the posterior, this is a conservative credible interval and is not actually equal-tailed: The [0.500, 0.676] interval includes 96.39% of the posterior weight, since \( \text{Pr}(\theta < 0.500|data) = 0.011 \) and \( \text{Pr}(\theta > 0.676|data) = 0.025 \).

Estimation based on \( \pi(\theta|data) \) will be entirely congruent with testing based on the posterior model odds, \( \text{PO}_{10}, \) (or the Bayes factor, \( \text{BF}_{10} \)) since both estimation and testing are done using the very same (implied) prior model odds. One could
also calculate all of these numbers analytically. For the posterior mean, we calculate:

\[
\hat{\theta} = \Pr(M_0|\text{data}) \hat{\theta}_0 + \Pr(M_1|\text{data}) \hat{\theta}_1 \\
= 0.523 \times 0.500 + 0.477 \times 0.598 \\
= 0.547. 
\]

For the 95% equal-tailed credible interval, we calculate

\[
95\% \text{CrI}(\theta) = \left[ Q_{\theta|\text{data}}(0.025), Q_{\theta|\text{data}}(0.975) \right], 
\]

where \( Q_z(q) \) is the \( q \)th quantile of \( Z \), and \( \theta_0 = 0.5 \). Finally, for the posterior median, we calculate \( \hat{\theta} = Q_{\theta|\text{data}}(0.5) = 0.500 \).

When \( \theta_0 \) is on the boundary of the 95% credible interval, the interval will necessarily be conservative (and not equal-tailed) in the sense that either \( \Pr(\theta < Q_{\theta|\text{data}}(0.025)|\text{data}) > 0.025 \) or \( \Pr(\theta > Q_{\theta|\text{data}}(0.975)|\text{data}) > 0.025 \) and therefore the interval will include more than 95% of the posterior weight. See Campbell and Gustafson (2022) for a discussion on this point.

With the single “mixture” prior approach, from Equation (10), we obtain:

\[
\pi(\theta|\text{data}) \propto 0.5 \times f_{\text{Binom}}(X, N, \theta) \times \left( \delta_{0.5}(\theta) + 1_{(0.01)}(\theta) \right), 
\]

where \( 1_{(0.01)}(\theta) \) is an indicator function equal to 1 if \( \theta \in (0, 1) \) and equal to 0 otherwise, as in Equation (5). One might recognize this as a version of the well-known “spike-and-slab” model (see van den Bergh et al. (2021)) and Monte Carlo sampling directly from this posterior can be challenging when using popular MCMC software such as JAGS and Stan. An easy workaround is to introduce a latent parameter, \( \omega \), such that the “mixed prior” is defined in a hierarchical way as follows:

\[
\pi(\theta|\omega) = (1 - \omega)\pi_0(\theta) + \omega\pi_1(\theta), \\
\omega \sim \text{Bernoulli}(\Pr(M_1)).
\]

This hierarchical strategy is often referred to as the “product space method”; see Carlin and Chib (1995) and more recently Lodewyckx et al. (2011). See also the discussion about testing as mixture estimation in Robert (2016) and the discussion about unification via the spike-and-slab model in Rouder, Haaf, and Vandekerckhove (2018).

Employing the “product space method” we obtain (using JAGS) 1 million draws from \( \pi(\theta|\text{data}) \) and can calculate the posterior mean, \( \hat{\theta} \), the posterior median, \( \hat{\theta} \), and the 95% equal-tailed credible interval, 95%CrI(\theta), as \( \hat{\theta} = 0.547 \), \( \hat{\theta} = 0.500 \), and 95%CrI(\theta) = [0.500, 0.676]. These numbers are identical to those obtained using the BMA approach.

The “product space method” is also advantageous since \( \hat{\omega} \), the posterior mean of \( \omega \), is the posterior probability of \( M_1 \). As such, calculation of the posterior odds and of the Bayes factor is straightforward:

\[
\frac{\hat{\omega}}{1 - \hat{\omega}}
\]

and:

\[
BF_{10} = \left( \frac{\hat{\omega}}{1 - \hat{\omega}} \right) / \left( \Pr(M_1) / \Pr(M_0) \right).
\]

For the possibly biased coin example, we obtain \( \hat{\omega} = 0.477 \), \( PO_{10} = 0.913 \), and \( BF_{10} = 0.913 \). These numbers are equal the values calculated analytically in Equations (6) and (7).

When the probability of \( M_0 \) given the data is nonnegligible, ignoring the \( M_0 \) model “paints an overly optimistic picture of what values \( \theta \) is likely to have” (Wagenmakers et al. 2022). As such, there can be a substantial difference between the estimates obtained from the posterior under \( M_1 \) and estimates from the “mixture” or “averaged” posterior. In our example of the possibly biased coin, there is modest evidence in favor of \( M_0 \) with \( \Pr(M_0|\text{data}) = 0.523 \), and as such there is a notable difference between \( \hat{\theta}_1 = 0.509 \) and \( \hat{\theta} = 0.547 \), and between 95%CrI(\theta)_1 = [0.502, 0.691] and 95%CrI(\theta) = [0.500, 0.676].

4. Observing a Single Coin Flip

Wagenmakers et al. (2020) consider the coin flip example but with only a single flip (i.e., with \( N = 1 \)) which lands tails (i.e., with \( X = 0 \)). In this simple, yet surprisingly interesting scenario, we have, from Equation (1):

\[
\Pr(\text{data}|M_0) = f_{\text{Binom}}(0, 1, 0.5) = 0.500,
\]

and

\[
\Pr(\text{data}|M_1) = \int_{t=0}^{t=1} f_{\text{Binom}}(0, 1, t) dt = 0.500.
\]

The ratio of these two numbers is equal to the Bayes factor:

\[
BF_{10} = \frac{\Pr(\text{data}|M_1)}{\Pr(\text{data}|M_0)} = \frac{0.500}{0.500} = 1,
\]

indicating that the single coin flip is “perfectly uninformative” (Wagenmakers et al. 2020) with regards to determining whether the data support \( M_1 \) over \( M_0 \) or vice-versa.

Wagenmakers et al. (2020) do not explicitly define any prior model odds for this example, but do consider \( \pi_0(\theta) \), a Beta(0.5,0.5) prior on \( \theta \), for estimation. Using this prior, a 95% credible interval for \( \theta \) is calculated as 95%CrI(\theta)_1 = [0.23, 1.00], and a 66% credible interval for \( \theta \) is calculated as 66%CrI(\theta)_1 = [0.70, 1.00]. Wagenmakers et al. (2020) then explain their results as follows: “It appears paradoxical that data can be perfectly
uninformative for comparing $[M_0$ to $M_1]$, and at the same
time provide reason to believe that $\theta > 0.5$ rather than $\theta < 0.5$.

As was the case with $N = 100$ coin flips, the “paradox” here
is simply due to the fact that different implied prior model odds
are being used for testing and for estimation. If I have “reason to
believe” that $Pr(\theta \in [0.70, 1.00]|data) = 0.66$, then necessarily, I
could not have believed, prior to flipping the coin, that either $M_0$
or $M_1$ were the true data generating mechanism. This is because a
belief that $Pr(\theta \in [0.70, 1.00]|data) = 0.66$ necessarily implies a
 corresponding a priori belief in the $\pi_1(\theta)$ prior.

For someone who believes in the $\pi_1(\theta)$ prior, the above Bayes
factor, $BF_{10}$, regardless of it’s value, is meaningless since both
$Pr(M_0) = 0$ and $Pr(M_1) = 0$. The prior model odds will be ill-
defined (i.e., $PO_{10} = 0$) and therefore posterior model odds
cannot be determined regardless of the observed data. To be
clear, the Bayes factor is only meaningful to someone if they are
willing to consider that both models, $M_0$ and $M_1$, are a priori,
plausible.

Notably, if one assumes prior model odds of 1:1 (such that
$Pr(M_0) = Pr(M_1) = 0.5$), estimation based on the
“mixture”/“averaged” posterior will be entirely congruent with the
“perfectly uninformative” $BF_{10} = 1$. Indeed, we obtain
95%CrI($\theta$) = [0.025, 0.776] and a conservative
66%CrI($\theta$) = [0.188, 0.500], which suggests that even a single
flip is informative with regards to inferring $\theta$. However, we also obtain:

$$Pr(M_0|data) = \frac{0.5}{0.5 + 1 \times 0.5} = 0.500,$$

and

$$Pr(M_1|data) = \frac{1 \times 0.5}{0.5 + 1 \times 0.5} = 0.500,$$

which suggests that, while we have gained some information
about $\theta$, $M_0$ and $M_1$ remain equally likely to be the true model.
When framed in this manner, there is nothing paradoxical:
Conclusions about $\theta$ and conclusions about $M_0$ versus $M_1$
are entirely congruent since both are based on the same prior model
odds. Figure 2 plots the “mixture”/“averaged” posterior where we see that, notably, exactly 50% of posterior mass for $\theta$ is at
exactly $\theta = 0.500$. Looking at the area under the curve, one can
also determine that $Pr(\theta \neq 0.5) = Pr(\theta < 0.5) + Pr(\theta >
0.5) = 0.375 + 0.125 = 0.5$.

Using the “mixture”/“averaged” posterior for estimation
(with the same prior model odds that are used for testing) will
ensure that there is no “incompatibility” or “mismatch” between
testing models and estimating $\theta$. We emphasize that this applies
universally, regardless of how $M_0$ and $M_1$ are defined. Notably,
some researchers have claimed that the “incompatibility”
problem only occurs when considering point-null hypotheses
(e.g., Tendeiro and Kiers (2019) write that: “The problem is
directly related to the use of the point null model”). In the next
section, we consider an example that does not involve a point
null.

### 5. With an Interval Null Model

Instead of questioning whether or not the coin is exactly fair,
suppose one wishes to determine whether or not the coin is fair
within some negligible margin. Indeed, it could be argued that
there is no such thing as a perfectly fair coin (Diaconis, Holmes,
and Montgomery 2007) and that bias is only consequential if it
is of a certain nonnegligible magnitude.

In other words, one might argue that $\theta$ will never be exactly
equal to 0.5, and therefore consider “nonoverlapping hypotheses”
(Morey and Rouder 2011) with priors for two competing
models, $M_2$ and $M_3$, defined by partitioning a Uniform(0,1)
density function as follows:

$$\pi_2(\theta) = \begin{cases} 10, & \text{if } \theta \in \Theta_0 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\pi_3(\theta) = \begin{cases} 1/0.9, & \text{if } \theta \in \Theta_1 \\ 0, & \text{otherwise.} \end{cases}$$

where $\Theta_0 = [0.45, 0.55]$ and $\Theta_1 = (0.45) \cup (0.55, 1)$;
see panels A and B of Figure 3. The corresponding posterior
distributions (see panels G and H of Figure 3) are:

$$\pi_2(\theta|data) \propto f_{\text{Beta}}(\theta, X + 1, N - X + 1) \times 10 \times 1_{[\Theta_0]}(\theta),$$

and

$$\pi_3(\theta|data) \propto f_{\text{Beta}}(\theta, X + 1, N - X + 1) \times 1.11 \times 1_{[\Theta_1]}(\theta).$$

Having observed $X = 60$ “heads” out of $N = 100$ coin flips,
we calculate from Equation (1) (see panels D and E of Figure 3):

$$Pr(data|M_2) = \int_{t=0.45}^{t=0.55} 10 \times f_{\text{Binom}}(60, 100, t)dt = 0.016,$$

$$Pr(data|M_3) = \frac{1}{0.9} \left[ \int_{t=0.45}^{t=1} f_{\text{Binom}}(60, 100, t)dt + \int_{t=0.55}^{t=1} f_{\text{Binom}}(60, 100, t)dt \right] = 0.009.$$

and

$$BF_{32} = \frac{Pr(data|M_3)}{Pr(data|M_2)} = \frac{0.009}{0.016} = 0.584.$$

For estimation, suppose one assumes a Uniform(0,1) prior
(i.e., the $\pi_1(\theta)$ prior as defined in Equation (5)), so as to give
equal prior weight to all values between 0 and 1. From the corre-
sponding $\pi_1(\theta|data)$ posterior (see Equation (8)), one obtains
a posterior mean of $\hat{\theta}_1 = 0.598$, a posterior median of $\hat{\theta}_1 = 0.599$, and 95%CrI($\theta_1$) = [0.502, 0.691]. One can also calculate
$Pr(\theta \in \Theta_1|data) = 0.840$.

This is therefore another example of the so-called “mis-
match” between Bayesian testing and posterior estimation: The
$BF_{32} = 0.584$ appears to support the null, while 95%CrI($\theta_1$) =
[0.502, 0.691] excludes 0.500. As was the case in the previous
examples with the point-null hypothesis, the “mismatch” here is
due to different implied prior model odds being used for testing
and for estimation.
Figure 2. Observing a single coin flip \((X = 0\) and \(N = 1\)). The first row shows prior density functions (with the gray numbers next to the arrows corresponding to point mass of the “spike”); middle row shows probability model for the data (with the gray vertical dashed lines corresponding to observed data of \(X = 60\)); and the lower row shows posterior density functions (with the gray numbers next to the arrows corresponding to point mass of the “spike”). Left column corresponds to \(M_0\), middle column corresponds to \(M_1\), and right column corresponds to the “mixed/averaged” model.

As established by Liao, Midya, and Berg (2021), the implied prior model odds that correspond to estimation with the Uniform(0,1) prior (i.e., under the \(\pi_1(\theta|\text{data})\) posterior) can be calculated as

\[
\text{IPMO}_1 = \frac{\Pr_1(\theta \in \Theta_1)}{\Pr_1(\theta \in \Theta_0)} = \frac{9}{1},
\]

where

\[
\Pr_1(\theta \in \Theta_1) = \int_{\Theta_1} \pi_1(\theta) d\theta = \int_{t=0}^{t=0.45} 1dt + \int_{t=0.55}^{t=1} 1dt = 0.90.
\]

and

\[
\Pr_1(\theta \in \Theta_0) = \int_{\Theta_0} \pi_1(\theta) d\theta = \int_{t=0.55}^{t=0.45} 1dt = 0.10.
\]

Notably, by multiplying these implied prior model odds by the Bayes factor, one obtains posterior odds that favor \(M_3\) and which are entirely congruent with the \(\pi_1(\theta|\text{data})\) posterior:

\[
\text{IPMO}_1 \times \text{BF}_{32} = 9 \times 0.584 = 5.255
\]

\[
= \frac{\Pr_1(\theta \in \Theta_1|\text{data})}{\Pr_1(\theta \in \Theta_0|\text{data})} = \frac{0.840}{0.160}.
\]

Alternatively, if one assumes equal prior model odds (i.e., assumes a priori that \(\Pr(M_2) = \Pr(M_3) = 0.5\)), then the posterior odds favor \(M_3\):

\[
\text{PO}_{32} = \frac{\Pr(M_3)}{\Pr(M_2)} \times \text{BF}_{32} = \frac{0.5 \times 0.584}{0.5} = 0.584.
\]
Estimation that is congruent with $PO_{32} = 0.584$ can be done by invoking the “mixed/averaged” prior:
\[
\pi(\theta) = \Pr(M_2)\pi_2(\theta) + \Pr(M_3)\pi_3(\theta),
\]
and the corresponding “mixed/averaged” posterior:
\[
\pi(\theta|\text{data}) \propto \Pr(M_2|\text{data})\pi_2(\theta|\text{data}) + \Pr(M_3|\text{data})\pi_3(\theta|\text{data})
\]
\[
\propto 0.631 \times f_{\text{Beta}}(\theta, X + 1, N - X + 1) \times 1_{[\theta_0]}(\theta) + 0.369 \times f_{\text{Beta}}(\theta, X + 1, N - X + 1) \times 1_{[\theta_1]}(\theta)
\]
From the $\pi(\theta|\text{data})$ posterior, one obtains a posterior mean of $\hat{\theta} = 0.557$, a posterior median of $\tilde{\theta} = 0.543$, and $95\%\text{CrI}(\theta) = [0.479, 0.673]$.

To reiterate, testing and estimation will only be congruent if the (implied) prior model odds adopted for both are the same. In this example, when estimation is based on a Uniform$(0,1)$ prior for $\theta$, the implied prior model odds are $9:1$, whereas when testing decisions about which model is most likely to be true are based on whether or not $BF_{32} > 1$, the implied prior model odds are $1:1$. As such, it is no surprise that the two approaches provide different answers.

6. A Difference in Personality Types?

As an example from applied research, consider Heck et al. (2022) who present the analysis of data concerning the difference between type D personality and nontype D personality for two outcome variables: Body Mass Index (BMI) and Negative
Affectivity Score (NAS). See summary data in Table 1 and note that these data were originally analyzed by Lin et al. (2019).

Heck et al. (2022) consider two competing models, a null model, $M_0$, which corresponds to the lack of an association between personality type and outcome, and an alternative model, $M_1$, which suggests the existence of an association. Let $\theta$ be the parameter of interest, the standardized effect size. The difference in personality types is associated with an increase of $\theta$ standard deviations in the outcome variable and a Normal model for the data is defined as

$$
\Pr(\text{data} | \theta, \sigma^2) = \prod_{i=1}^{n} f_{\text{Normal}}(Y_i, \mu + X_i \times \frac{\sigma \theta}{2}, \sigma^2),
$$

where $f_{\text{Normal}}()$ is the Normal probability density function, $Y_i$ is the continuous outcome variable for the $i$th observation (i.e., $Y_i$ is the $i$th subject's BMI score in the first analysis, and the $i$th subject's NAS score in the second analysis), and $X_i$ corresponds to the $i$th subject's personality type such that $X_i = -1$ indicates a non-type D personality and $X_i = 1$ indicates a type D personality.

We have three parameters, $\mu$, $\theta$, and $\sigma^2$ for which we must define priors and we define these priors for both $M_0$ and $M_1$ as

$$
\pi_0(\mu) = 1, \quad \pi_0(\sigma^2) = \frac{1}{\sigma^2}, \quad \pi_0(\theta) = \delta_0(\theta),
$$

and

$$
\pi_1(\mu) = 1, \quad \pi_1(\sigma^2) = \frac{1}{\sigma^2}, \quad \pi_1(\theta) = f_{\text{Cauchy}}(\theta, 0.707),
$$

where, for both $M_0$ and $M_1$, noninformative Jeffreys’ priors are defined for $\sigma^2$, and for $\mu$; and $f_{\text{Cauchy}}(x, s) = ((\pi s^2)(1 + (x/s)^2))^{-1}$ is the Cauchy probability density function evaluated at $x$, with scale parameter $s$. Specifying noninformative Jeffreys’ priors is mathematically convenient for calculating the Bayes factor. However, these improper priors cannot be accurately specified in popular MCMC software such as WinBugs, JAGS, and greta; see Faulkenberry (2018).

For the BMI outcome, Heck et al. (2022) report a Bayes factor of $BF_{10} = 1/4.2$ and sample from the posterior $\pi_1(\theta | \text{data})$ (thereby implicitly assuming $M_1$ is correct) to obtain a posterior median of $\hat{\theta}_1 = 0.05$, and 95% CrI($\theta_1$) = [-0.35, 0.45]. The posterior mean (not reported) is $\tilde{\theta}_1 = -2.36$. The evidence in favor of $M_1$ is overwhelming, $\Pr(M_1 | \text{data}) > 0.99$, and therefore estimates from the “mixture”/“averaged” posterior (obtained using either the BMA approach or the “mixture” prior approach) are nearly identical: $\hat{\theta} = -2.36, \tilde{\theta} = -2.36$, and 95% CrI($\theta$) = [-2.85, -1.87].

7. Flipping One Million Coins

To better explain why we believe researchers should report estimates from the “mixture”/“averaged” posterior instead of estimates from $\pi_1(\theta | \text{data})$ or other posteriors, we return one final time to the coin flip example, but this time with $N = 10$ flips. Let us explicitly assume that $M_0$ and $M_1$ are a priori equally likely: $\Pr(M_0) = \Pr(M_1) = 0.5$.

Consider the following thought experiment. By drawing from the “mixture”/“averaged” prior, $\pi(\theta)$, one obtains a coin for which $\theta = \theta^{[1]}$. Flipping this coin $N = 10$ times generates a total of $X = 4$ “heads” and the posterior mean, under $M_1$, is

$$
\hat{\theta}_1 = \frac{(X + \alpha)}{(N + \alpha + \beta)} = \frac{(4 + 1)}{(10 + 1 + 1)} = \frac{5}{12} = 0.417.
$$

The equal-tailed 95% credible interval is 95% CrI($\theta_1$) = [0.167, 0.692]. From Equation (12), the posterior mean, taking into account the model uncertainty, is

$$
\hat{\theta} = \Pr(M_0 | \text{data})\hat{\theta}_0 + \Pr(M_1 | \text{data})\hat{\theta}_1 = 0.693 \times 0.5 + 0.307 \times 0.417 = 0.474,
$$

and the equal-tailed 95% credible interval is 95% CrI($\theta$) = [0.227, 0.616].

Now suppose we obtain a second coin from the “mixed”/“averaged” prior for which $\theta = \theta^{[2]}$. Flipping this second coin $N = 10$ times generates a total of $X = 8$ “heads.” Then the posterior mean, under $M_1$, is $\hat{\theta}_1 = \frac{8}{12} = 0.750$, and the equal-tailed 95% credible interval is 95% CrI($\theta_1$) = [0.482, 0.940]. The posterior mean, taking into account the model uncertainty, is $\hat{\theta} = 0.669$, and the equal-tailed 95% credible interval is

indicating considerable evidence in favor of the null model.

Under $M_0$, we have $\theta = 0$. Therefore, we can obtain a Monte Carlo sample from the “mixture”/“averaged” posterior, $\pi(\theta | \text{data})$, by simply combining 192 thousand draws for $\theta$ from $\pi_1(\theta | \text{data})$ with 808 thousand zeros. From this “combined” sample we calculate $\hat{\theta} = 0.00$, $\tilde{\theta} = 0.01$, and 95% CrI($\theta$) = [-0.18, 0.28]. One can also obtain these estimates using Equation (13) with $\theta_0 = 0$. These numbers are notably different than those based on the $\pi_1(\theta | \text{data})$ posterior reported by Heck et al. (2022).

For the NAS outcome, Heck et al. (2022) report a Bayes factor of $BF_{10} = 10^{20}$ and sample from the posterior $\pi_1(\theta | \text{data})$ (implicitly assuming $M_1$ is correct) to obtain a posterior median of $\hat{\theta}_1 = -2.36$, and 95% CrI($\theta_1$) = [-2.85, -1.87]. The posterior mean (not reported) is $\tilde{\theta}_1 = -2.36$. The evidence in favor of $M_1$ is overwhelming, $\Pr(M_1 | \text{data}) > 0.99$, and therefore estimates from the “mixture”/“averaged” posterior (obtained using either the BMA approach or the “mixture” prior approach) are nearly identical: $\hat{\theta} = -2.36, \tilde{\theta} = -2.36$, and 95% CrI($\theta$) = [-2.85, -1.87].

Table 1. The sample mean, $\bar{X}$, and sample standard deviation, SD, for each of the two groups (type D personality and non-type D personality) and each of the two outcome variables (body mass index (BMI) and negative affectivity score (NAS)).

| Outcome | Non-Type D ($n = 193$) | Type D ($n = 23$) |
|---------|------------------------|------------------|
| BMI     | $\bar{X} = 26.0$ (SD=4.9) | $\bar{X} = 25.7$ (SD=4.4) |
| NAS     | $\bar{X} = 5.5$ (SD=4.2) | $\bar{X} = 15.4$ (SD=3.5) |

NOTE: The data were originally analyzed by Lin et al. (2019).
Table 2. One million coins: For the jth coin we observe $X^{(j)}$ “heads” out of $N = 10$ flips, we calculate the posterior estimates.

| j | $\theta^{(j)}$ | $X^{(j)}$ | N | $\hat{\theta}_1$ | $\hat{\theta}$ | $95\%$CrI($\theta_1$) | $95\%$CrI($\theta$) |
|---|---|---|---|---|---|---|---|
| 1 | 0.50 | 4 | 10 | 0.417 | 0.474 | [0.167,0.692] | [0.227,0.616] |
| 2 | 0.77 | 8 | 10 | 0.750 | 0.669 | [0.177,0.750] | [0.550,0.930] |
| 3 | 0.91 | 9 | 10 | 0.250 | 0.331 | [0.067,0.877] | [0.070,0.500] |
| 4 | 0.73 | 8 | 10 | 0.750 | 0.669 | [0.177,0.750] | [0.550,0.930] |
| 5 | 0.87 | 9 | 10 | 0.833 | 0.801 | [0.587,0.977] | [0.500,0.976] |
| 6 | 0.50 | 10 | 10 | 0.667 | 0.573 | [0.067,0.877] | [0.445,0.860] |
| 7 | 0.50 | 5 | 10 | 0.500 | 0.500 | [0.032,0.688] | [0.032,0.688] |
| 8 | 0.31 | 10 | 10 | 0.167 | 0.199 | [0.024,0.500] | [0.024,0.500] |
| ... | ... | ... | ... | ... | ... | ... | ... |
| 1,000,000 | 0.60 | 10 | 10 | 0.583 | 0.526 | [0.383,0.773] | [0.383,0.773] |

NOTE: Whenever 0.50 is on the boundary of the 95%CrI($\theta_1$) interval, the interval will be conservative. Specifically, for $X = 1$ and $X = 9$, 96.97% of the posterior weight will be within the interval; for $X = 2$ and $X = 8$, 95.29% of the posterior weight will be within the interval.

Table 3. Simulation study results: Average values obtained from the subset set of “coins” for which $X = x$ “heads”.

| x | # of coins | $x/N$ | $\hat{\theta}_1$ | $\hat{\theta}$ | $E(\theta^{(j)}|X^{(j)} = x)$ |
|---|---|---|---|---|---|
| 0 | 45,864 | 0.0 | 0.083 | 0.083 | 0.087 |
| 1 | 50,180 | 0.1 | 0.167 | 0.199 | 0.198 |
| 2 | 67,281 | 0.2 | 0.250 | 0.331 | 0.331 |
| 3 | 104,309 | 0.3 | 0.333 | 0.427 | 0.427 |
| 4 | 148,022 | 0.4 | 0.417 | 0.474 | 0.474 |
| 5 | 168,651 | 0.5 | 0.500 | 0.500 | 0.500 |
| 6 | 148,502 | 0.6 | 0.583 | 0.526 | 0.526 |
| 7 | 103,961 | 0.7 | 0.667 | 0.573 | 0.573 |
| 8 | 67,353 | 0.8 | 0.750 | 0.669 | 0.670 |
| 9 | 49,939 | 0.9 | 0.833 | 0.801 | 0.801 |
| 10 | 45,938 | 1.0 | 0.917 | 0.912 | 0.913 |

NOTE: For example, among the subset of 104,309 coins that produced $X = 3$, the average of $\theta^{(1)}$ values is 0.427, a numerical approximation to $E(\theta^{(1)}|X^{(1)} = 3)$. Whereas the posterior mean estimate (taking into account the model uncertainty) whenever $X^{(1)} = 3$ is $\hat{\theta} = 0.427$.

95%CrI($\theta$) = [0.500, 0.930]. Note that due to the discontinuity in the posterior, this is a conservative 95% credible interval that is not actually equal-tailed: The [0.500, 0.930] interval includes 95.29% of the posterior weight, since $Pr(\theta < 0.500|\text{data}) = 0.022$ and $Pr(\theta > 0.930|\text{data}) = 0.025$.

We implemented this process by simulation again and again obtaining 1 million coins from the “mixed”/“averaged” prior (i.e., generating 1 million values $\theta^{(1)}, \ldots, \theta^{(1000000)}$ from $\pi(\theta)$), and flipping each of these coins 10 times; see Table 2. In total, we obtained a total of 148,022 coins for which $X = 4$; see Table 3.

Because this is a simulation, it is possible to reveal the underlying true values of $\theta^{(1)}, \ldots, \theta^{(1000000)}$. For the subset of coins for which $X = 4$, the average of these values is 0.474. For lack of better notation we write $E(\theta^{(j)}|X^{(j)} = 4) = 0.474$. Notably, this equals $\hat{\theta}$ when $X = 4$ (see Equation (15)), but does not equal $\hat{\theta}_1$ when $X = 4$ (see Equation (14)). Nor does it equal $\frac{1}{10}$. In this sense, both $\hat{\theta}_1$ and $\frac{1}{10}$ overestimate $\theta$, when $X = 4$. Table 3 shows these comparisons for $X = 0, \ldots, 10$.

Furthermore, amongst the 148,022 simulated $\theta^{(j)}$ values for which $X = 4$, exactly 140,459 are within the interval of 95%CrI($\theta_1$)=[0.227,0.616]. Notably, this proportion is 140,459/148,022 = 0.95. In comparison, the proportion of the 148,022 values within the interval of 95%CrI($\theta$)=[0.167,0.692] is 0.98. Again, for lack of better notation, we write $Pr(\theta^{(j)} \in 95\%\text{CrI}(\theta_1)|X^{(j)} = 4) = 0.98$, and $Pr(\theta^{(j)} \in 95\%\text{CrI}(\theta)|X^{(j)} = 4) = 0.95$. In this sense, the 95%CrI($\theta_1$) interval is too wide, whereas the 95%CrI($\theta$) is appropriately wide. Table 4 shows these comparisons for $X = 0, \ldots, 10$.

The behavior exhibited in Tables 3 and 4 is completely general mathematically. Generically, let $(\theta^{(j)}, X^{(j)})$ be a random draw as is used to construct Table 2, row by row, for example, $\theta^{(j)}$ is drawn from $\pi(\theta)$, and then $X^{(j)}$ is drawn from “the model” with $\theta$ set equal to $\theta^{(j)}$. Thus, the average of $\theta^{(j)}$ when $X = x$ (the last column of Table 3) is simply (a Monte Carlo approximation to) $E(\theta^{(j)}|X^{(j)} = x)$, for each $x$. But since $(\theta^{(j)}, X^{(j)})$ is nothing more than a joint draw from the amalgamation of the prior and statistical model, $E(\theta^{(j)}|X^{(j)} = x)$ and $\hat{\theta}$ when $X = x$, are one and the same. By definition then, if we repeatedly sample parameter and data pairs this way, the average of the parameter values among draws yielding a specific data value is the posterior mean of the parameter given that data value. The same calibration idea applies to credible intervals so that $Pr(\theta^{(j)} \in 95\%\text{CrI}(\theta)|X^{(j)} = 4)$ = 0.95. In this sense, the 95%CrI($\theta_1$) interval is too wide, whereas the 95%CrI($\theta$) is appropriately wide. Table 4 shows these comparisons for $X = 0, \ldots, 10$.

This strongly supports the argument that when considering model uncertainty via a Bayes factor, $\hat{\theta}$ is the appropriate estimate to report and 95%CrI($\theta$) is the appropriate credible interval. While these sorts of calibration properties of Bayesian estimators with respect to repeated sampling of data under different parameter values feature sporadically in the literature (e.g., Rubin and Schenker (1986)), they do not seem to be widely known. They do however, underpin the scheme of Cook, Gelman, and Rubin (2006) to validate Markov chain Monte Carlo approximations to posterior quantities and they are also discussed more recently in Gustafson and Greenland (2009). Importantly, we note that this sense of calibration arises only when the prior truly corresponds to the data-generating process.

8. Conclusion

In conclusion, there is no mismatch between the Bayes factor (or the posterior odds) and Bayesian posterior estimation with an
appropriately defined prior and posterior. Our recommendation is therefore very simple. If one reports a Bayes factor comparing $M_0$ and $M_1$, then one should also report posterior estimates based on the “mixed/averaged” posterior, $\pi(\theta|data)$, with prior model odds appropriately specified. Researchers should refrain from reporting posterior estimates based on $\pi_1(\theta|data)$ (which implicitly assumes $M_1$ is correct) or estimates based on other posteriors for which the Bayes factor is meaningless. We see no reason why disregarding $M_0$ “for the purpose of parameter estimation” (Wagenmakers and Gronau 2020) is advisable.

On a final note, in order to keep things as accessible as possible, all of the examples considered in this article were univariate inference problems involving only two models. Our conclusions and recommendations, however, also apply to more complex problems. On this point, we refer interested readers to Rossell and Telesca (2017), who consider strategies for how to use the same prior for both estimation and model selection in high-dimensional settings to achieve coherence between estimation and testing, and to Lavine and Schervish (1999) and Oelrich et al. (2020), who discuss Bayes factor based model selection in scenarios involving more than two models.

**Supplementary Materials**

R code to replicate all figures and results is provided in the supplementary materials.

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