The averaging method on slow-fast phase spaces with symmetry

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Abstract. In the context of the perturbation theory, we study a normalization problem for a wide class of Hamiltonian systems of “quasiadiabatic” type on slow-fast $G$-spaces. We show that a normalization transformation can be defined as an isomorphism between the original Poisson bracket and its $G$-average.

1. Introduction

In the study of Hamiltonian models with slowly or rapidly varying Hamiltonians, in the framework of the averaging method and the theory of adiabatic approximation [12], [14], [15], [16], we deal with phase spaces whose symplectic (Poisson) structures depend on a perturbation parameter in a singular way. As a consequence, this leads to a non-standard perturbative setting in the sense that the corresponding unperturbed dynamics is non-Hamiltonian. On the other hand, the adiabatic theory is closely related with some interesting geometric structures such as Hannay-Berry connections [8], [9], [19] and (weak) coupling symplectic and Poisson structures [1], [3], [7].

In this paper, we are interested in a “quasiadiabatic” situation and study a wide class of Hamiltonian systems with slow-fast variables with the following feature: the leading term $H_0$ of the Hamiltonian depends on the slow variables and the fast variables appear only in the perturbation $H_1$. The particular case when $H_0 \equiv 0$ corresponds to the adiabatic situation. More precisely, our problem is formulated as follows. Let $(S, \{\cdot,\cdot\}_S)$ and $(P, \{\cdot,\cdot\}_P)$ be symplectic and Poisson manifolds, respectively. Let $G$ be a compact connected Lie group and $\mathfrak{g}$ its Lie algebra. Viewing the direct product $M = S \times P$ as the total space of the trivial Poisson fiber bundle $\pi : M \to S$ (the canonical projection), we assume that there is a family of Hamiltonian $G$-actions on $M$ with a fiberwise momentum map $J : M \to \mathfrak{g}$ in the sense of [8]. On the other hand, equipping the total space $M$ with $\varepsilon$-dependent product Poisson structure $\{\cdot,\cdot\}_M = \{\cdot,\cdot\}_S + \frac{1}{\varepsilon}\{\cdot,\cdot\}_P$ ($\varepsilon$ is a small parameter), we consider a perturbed Hamiltonian system on $(M, \{\cdot,\cdot\}_M)$ of the form $H = H_0 \circ \pi + \varepsilon H_1$, where $H_0 \in C^\infty(S)$ and $H_1 \in C^\infty(M)$. Under a natural symmetry hypothesis for the unperturbed system, we study the following normalization problem: find a near identity transformation $T_\varepsilon$ on $M$ which brings the original model into a system which is $\varepsilon^2$ close to a $G$-symmetric Hamiltonian system. One of the main motivations of this problem is to understand a geometric meaning of the first normalization step in the proof of the classical adiabatic theorem [12], [14], [15] which involves action-angle variables and generating functions. On the other hand, such a normalization question is closely related to the semiclassical quantization on slow-fast
phase spaces [21]. Our key observation [6] is that a normalization transformation can be defined as an isomorphism between the original symplectic (Poisson) structure and its $G$-average. In particular, this result gives a free coordinate normalization in the adiabatic case. The main difficulty occurs when the fiberwise Poisson structure $\{,\}_P$ is degenerate. In this case, the "naive" averaging does not work because the averaging procedure does not respect the Jacobi identity for Poisson tensors which is essentially a nonlinear equation. Here, we use an approach developed in [5] which is a combination of the coupling Poisson method [1], [3] with the theory of Hannay-Berry connections [8]. The paper is organized as follows. In Section 2 we examine the normalization problem for a relatively simple class of slow-fast Hamiltonian systems with two degree of freedom [2]. In Sections 3 and 4, we generalize these results to the case where the fiberwise Poisson structure $\{,\}_P$ is of arbitrary type.

2. Hamiltonian Models with Rapidly Varying Perturbations

On the standard phase space $(\mathbb{R}^4, dp_1 \wedge dq_1 + dP_2 \wedge dQ_2)$, let us consider a Hamiltonian system of the form [2]

$$H = H_0(p_1, q_1) + \varepsilon H_1(p_1, q_1, P_2, Q_2),$$

where $\varepsilon \ll 1$ is a small parameter and $\kappa \in [0, 1]$ is a constant. After rescaling $p_2 = \frac{P_2}{\varepsilon^\kappa}, q_2 = \frac{Q_2}{\varepsilon^{1-\kappa}}$, we get the $\varepsilon$-dependent symplectic form

$$\Omega = dp_1 \wedge dq_1 + \varepsilon dp_2 \wedge dq_2.$$ (2.1)

and the Hamiltonian

$$H = H_0(p_1, q_1) + \varepsilon H_1(p_1, q_1, p_2, q_2)$$ (2.2)

depending regularly on $\varepsilon$. According to the appearance of the small parameter $\varepsilon$ in the corresponding Poisson bracket

$$\{p_1, q_1\} = 1,$$

$$\{p_2, q_2\} = \frac{1}{\varepsilon},$$

we can separate the space coordinates into slow $(p_1, q_1)$ and fast $(p_2, q_2)$ variables. Viewing the equations of motion of (2.1), (2.2)

$$\dot{p}_1 = -\frac{\partial H_0}{\partial q_1} - \varepsilon \frac{\partial H_1}{\partial q_1}, \quad \dot{q}_1 = \frac{\partial H_0}{\partial p_1} + \varepsilon \frac{\partial H_1}{\partial p},$$ (2.3)

$$\dot{p}_2 = -\frac{\partial H_1}{\partial q_2}, \quad \dot{q}_2 = \frac{\partial H_1}{\partial p_2}$$ (2.4)

as a perturbed dynamical system, we observe that for $\varepsilon = 0$ the unperturbed system

$$\dot{p}_1 = -\frac{\partial H_0}{\partial q_1}, \quad \dot{q}_1 = \frac{\partial H_0}{\partial p_1},$$ (2.5)

$$\dot{p}_2 = -\frac{\partial H_1}{\partial q_2}, \quad \dot{q}_2 = \frac{\partial H_1}{\partial p_2}$$ (2.6)

is not Hamiltonian relative to $\Omega$. Therefore, we deal with non-standard perturbative Hamiltonian setting.

Remark 2.1 In particular when $H_0 \equiv 0$. we arrive at the adiabatic situation. In this case, in the theory of adiabatic approximation [14], [15], system (2.1), (2.2) is called a slow-fast Hamiltonian system. The corresponding unperturbed system is a family of Hamiltonian systems with one degree of freedom parameterized by the slow variables.
As usual, we need some good properties of the unperturbed system. We make the following assumption (the symmetry hypothesis): the unperturbed system (2.5), (2.6) admits a first integral \( J = J(p_1, q_1, p_2, q_2) \) such that the flow \( \text{Fl}_{J}^{t} \) of the vector field

\[
V_{J} := \frac{\partial J}{\partial p_2} \frac{\partial}{\partial q_2} - \frac{\partial J}{\partial q_2} \frac{\partial}{\partial p_2}
\]

is 2\( \pi \)-periodic, \( \text{Fl}_{J}^{t} \) is an infinitesimal generator of the \( S^1 \)-action on \( \mathbb{R}^4 = \mathbb{R}^2_{p_1, q_1} \times \mathbb{R}^2_{p_2, q_2} \). The circle \( S^1 = \mathbb{R}/2\pi \mathbb{Z} \) acts along the slices \( \{p_1, q_1\} \times \mathbb{R}^2_{p_2, q_2} \) in a Hamiltonian fashion, with momentum map \( J_{p_1, q_1} \in \mathbb{R} \), parametrically depending on the slow variables. Remark that we have the following freedom in the choice of \( J \) one can replace \( J \) by \( J \rightarrow J + f \circ \pi \), where \( f : \mathbb{R}^2_{p_1, q_1} \rightarrow \mathbb{R} \) is a first integral of Hamiltonian system (2.5) with one-degree of freedom and \( \pi : \mathbb{R}^2_{p_1, q_1} \times \mathbb{R}^2_{p_2, q_2} \rightarrow \mathbb{R}^2_{p_1, q_1} \) is the canonical projection.

We are interested in the following normalization question [6]. Fix an \( S^1 \)-invariant open domain \( N \) in \( \mathbb{R}^4 \) with compact closure. By a near identity transformation we mean a smooth family of mappings \( T_{\varepsilon} : N \rightarrow \mathbb{R}^4, \varepsilon \in (-\delta, \delta) \) such that \( T_0 = \text{id} \) and \( T_{\varepsilon} \) is a diffeomorphism onto its image. We say that a near identity transformation \( T_{\varepsilon} \) is a normalization transformation of first order, if the pull-back of our original model (2.1), (2.2) via \( T_{\varepsilon} \) is a Hamiltonian system of the form

\[
(N, \Omega^{\text{inv}}, H_0 + \varepsilon H_1^{\text{inv}} + O(\varepsilon^2)), \quad (2.7)
\]

where \( \Omega^{\text{inv}} \) and \( H_1^{\text{inv}} \) are some \( S^1 \)-invariant symplectic form and function on \( N \), respectively. If such a normalization exists, then we get the approximate model \( (N, \Omega^{\text{inv}}, H_0 + \varepsilon H_1^{\text{inv}}) \) which is a \( S^1 \)-symmetric Hamiltonian system. Consider the \( S^1 \)-average of the \( \varepsilon \)-dependent symplectic form \( \Omega \) (2.1):

\[
\langle \Omega \rangle := \frac{1}{2\pi} \int_0^{2\pi} (\text{Fl}_{J}^{t})^{*} \Omega dt.
\]

The \( S^1 \)-average of an arbitrary form on \( N \) is defined in the same way. Since the exterior differential commutes with the averaging operator, the 2-form \( \langle \Omega \rangle \) is closed but it is not necessarily nondegenerate for all \( \varepsilon \neq 0 \).

**Lemma 2.2** The \( S^1 \)-average \( \langle \Omega \rangle \) has the representation

\[
\langle \Omega \rangle = \Omega - \varepsilon d\Theta,
\]

where the 1-form \( \Theta = \theta_1 dp_1 + \theta_2 dq_1 \) is given by

\[
\theta = \frac{1}{2\pi} \int_0^{2\pi} (t - \pi)(\text{Fl}_{J}^{t})^{*} d_1 J dt. \quad (2.9)
\]

and has zero \( S^1 \)-average,

\[
< \Theta > = 0. \quad (2.10)
\]

Here \( d_1 J = \frac{\partial J}{\partial p_1} dp_1 + \frac{\partial J}{\partial q_1} dq_1 \). Moreover, there exists \( \delta > 0 \) such that

\[
\Omega_{\lambda} = (1 - \lambda)\langle \Omega \rangle + \lambda \Omega \quad (2.11)
\]

is a symplectic form on \( N \) for all \( \varepsilon \in (0, \delta) \) and \( \lambda \in [0, 1] \).
Representations (2.8), (2.9) are derived from the following property of closed forms with respect to a given $S^1$-action: every closed form splits into a $S^1$-invariant form and an exact one (see, for example [18]). The proof of the second statement of the lemma follows from the following argument. Consider the vector fields

\begin{align*}
Y^\lambda_1 &= \frac{\partial}{\partial p_1} + (1 - \lambda)(\frac{\partial \theta_1}{\partial p_2} \frac{\partial}{\partial q_2} - \frac{\partial \theta_1}{\partial q_2} \frac{\partial}{\partial p_2}), \\
Y^\lambda_2 &= \frac{\partial}{\partial q_1} + (1 - \lambda)(\frac{\partial \theta_2}{\partial p_2} \frac{\partial}{\partial q_2} - \frac{\partial \theta_2}{\partial q_2} \frac{\partial}{\partial p_2}).
\end{align*}

(2.12) (2.13)

Then, by a straightforward computation one can show that the determinant of the matrix of $\Omega^\lambda$ with respect to the basis $Y_1, Y_2, \frac{\partial}{\partial p_2}, \frac{\partial}{\partial q_2}$ equals $\varepsilon^2 (1 - \varepsilon \Delta^\lambda)^2$, where

\[ \Delta^\lambda = (1 - \lambda)(\frac{\partial \theta_2}{\partial p_1} \frac{\partial}{\partial q_1} - \frac{\partial \theta_1}{\partial q_1} \frac{\partial}{\partial p_1} + (1 - \lambda)(\frac{\partial \theta_1}{\partial p_2} \frac{\partial}{\partial q_2} - \frac{\partial \theta_2}{\partial q_2} \frac{\partial}{\partial p_2})] \]

Since $\bar{N}$ is compact, there exists $\delta > 0$ such that $1 - \varepsilon \Delta^\lambda \neq 0$ on $\bar{N}$ for all $\varepsilon \in (0, \delta)$ and $\lambda \in [0, 1]$.

Now, we formulate our main result.

**Theorem 2.3** Under the symmetry hypothesis above, for small enough $\varepsilon$, there exists a normalization transformation $T_\varepsilon : N \rightarrow \mathbb{R}^4$ which is defined as a symplectomorphism between the original symplectic structure $\Omega$ and its $S^1$-average $\langle \Omega \rangle$

\[ \langle \Omega \rangle = T_\varepsilon^* \Omega. \]

(2.14)

The transformation $T_\varepsilon$ brings system (2.1), (2.2) into the form (2.7) with $\Omega^{\text{inv}} = \langle \Omega \rangle$ and

\[ H_1^{\text{inv}} = \langle H_1 \rangle, \]

(2.15)

where $\langle H_1 \rangle$ is the $S^1$-average of $H_1$.

To construct a normalization map $T_\varepsilon$, one can use a parameter-dependent version of the Moser homotopy method [7]. For every $\varepsilon \in (0, \delta)$, consider the curve of symplectic forms $\Omega^\lambda$ (2.11) on $N$ joining $\langle \Omega \rangle$ with $\Omega$. Then, there exists a unique $\lambda$-dependent vector field $Z^\lambda$ on $N$ satisfying the equation $\int_{Z^\lambda} \Omega^\lambda = -\varepsilon d\theta$. In terms of vector fields in (2.12), (2.13), it can be expressed as follows

\[ Z^\lambda = \frac{\varepsilon}{1 - \varepsilon \Delta^\lambda} [-\theta_2 Y^\lambda_1 + \theta_1 Y^\lambda_2]. \]

(2.16)

Finally, shrinking again $\delta$, we can arrange that the flow $F^\lambda_{Z^\lambda}$ of the time-dependent vector field $Z^\lambda$ is defined on $N$ for all $\lambda \in [0, 1]$ and $\varepsilon \in (0, \delta)$. The desired near identity transformation in (2.14) is given by $T_\varepsilon = F^\lambda_{Z^\lambda} \mid_{\lambda=1}$.

Next, let us verify (2.15). By using (2.16) and the standard time expansion for flows, we deduce the relation $H \circ T_\varepsilon = H_0 + \varepsilon (H_1 - \Theta_{H_0}) + O(\varepsilon^2)$, where

\[ \Theta_{H_0} = -\frac{\partial H_0}{\partial q_1} \theta_1 + \frac{\partial H_0}{\partial p_1} \theta_2 \]

It follows from (2.10) that

\[ \langle \Theta_{H_0} \rangle = 0. \]

(2.17)

Let $V$ be the vector field of unperturbed system (2.5), (2.6). By the symmetry hypothesis, $V$ is $S^1$-invariant and hence

\[ \langle V \rangle = V. \]

(2.18)
One the other hand, one can show that
\[
\langle V \rangle = \frac{\partial H_0}{\partial p_1} \frac{\partial}{\partial q_1} - \frac{\partial H_0}{\partial q_1} \frac{\partial}{\partial p_1} + \frac{\partial F}{\partial p_2} \frac{\partial}{\partial q_2} - \frac{\partial F}{\partial q_2} \frac{\partial}{\partial p_2},
\]
where \( F = \langle H_1 \rangle + \Theta H_0 \). This representation follows from the fact \([5],[6]\) : the vector fields \( Y^0_1 \) and \( Y^0_2 \) are \( S^1 \)-invariant, \([Y^0_1, V] = [Y^0_2, V] = 0\). Comparing (2.18) with (2.19) implies that
\[
\langle H_1 \rangle + \Theta H_0 = H_1 - f \circ \pi,
\]
for a certain \( f \in C^\infty(\pi(N)) \). Finally, averaging both side of this equality and using (2.17), we get
\[
f \circ \pi = -\langle \Theta H_0 \rangle = 0.
\]

Now, let us see under which conditions we have a Hamiltonian \( S^1 \)-space.

**Proposition 2.4** The \( S^1 \)-action is Hamiltonian relative the symplectic form \( \langle \Omega \rangle \) if and only if \( J \) is a first integral of the vector fields \( Y^0_1 \) and \( Y^0_2 \),
\[
\frac{\partial J}{\partial p_1} + \frac{\partial \theta_1}{\partial p_2} \frac{\partial J}{\partial q_2} - \frac{\partial \theta_1}{\partial q_2} \frac{\partial J}{\partial p_2} = 0,
\]
\[
\frac{\partial J}{\partial p_1} + \frac{\partial \theta_2}{\partial p_2} \frac{\partial J}{\partial q_2} - \frac{\partial \theta_2}{\partial q_2} \frac{\partial J}{\partial p_2} = 0.
\]

The result follows from (2.8), (2.9) and the following representation for the Poisson tensor \( \Pi \) associated to \( \langle \Omega \rangle \):
\[
\Pi = \frac{1}{1 - \varepsilon \Delta_0} Y^0_1 \wedge Y^0_2 + \frac{1}{\varepsilon} \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_2}.
\]

Remark that condition (2.21), (2.22) is equivalent to the following \( \langle d_1 J \rangle = 0 \). One can try to satisfy conditions (2.21), (2.22) by using the freedom in the choice of \( J \). For example, this is the case, when the \( S^1 \)-action is free \([2]\). For more details, see \([8],[9]\) and Section 3.

**Remark 2.5** In the adiabatic case, \( H_0 \equiv 0 \), the transformation \( T_\varepsilon \) in Theorem 2.3 can be viewed as a free coordinate normalization in the classical adiabatic theorem \([14],[15]\) (the usual method uses action-angle variables and generating functions). The symmetry hypothesis is provided by the compactness of the level sets of \( H_1 \) for frozen values of \( p_1, q_1 \). In a regular domain, the cyclic integral \( J \) is defined as the standard action and satisfies conditions (2.21), (2.22).

To complete this section, we notice that in the case when \( H_1 \) is a quadratic function in the fast variables \( p_2, q_2 \) and the domain \( \pi(N) \) is foliated by periodic trajectories of 1-dimensional Hamiltonian system (2.5), the question on the existence of a cyclic integral is reduced to the stability question for a family of time-periodic linear Hamiltonian systems on the plane \([2]\).

**Example 2.6** On the phase space \( (\mathbb{R}^4, \Omega) \), consider the Hamiltonian of the form
\[
H = \frac{1}{2} p_1^2 + \frac{1}{4} q_1^4 + \varepsilon \left( p_2^2 + \frac{\delta}{2} q_2^2 \right).
\]
According to \([13]\), for \( \varepsilon \ll 1 \) and every fixed \( \delta \in (0,1) \), this Hamiltonian system is non-integrable but the corresponding unperturbed system is strongly stable and hence admits a cyclic first integral \( J \), which is also quadratic in \( p_2, q_2 \).
3. General Slow-Fast Phases Spaces with Symmetry

Here we generalize the results of the previous Section to a wide class of slow-fast phase spaces with symmetry. Let $M = S \times P$ be the product of a manifold $S$ and a Poisson manifold $P$ equipped with Poisson bracket of general type

$$\{x^\alpha, x^\beta\}_P = \Lambda^{\alpha\beta}(x).$$

In particular, we admit that the rank of the Poisson tensor $\Lambda = (\Lambda^{\alpha\beta}(x))$ is not necessarily constant. We will think of $M$ as the total space of the trivial Poisson fibration $\pi: S \times P \to S$ over the base $S$ with typical fiber $(P, \{\cdot, \cdot\}_P)$.

Let $G$ be a connected compact Lie group and $\mathfrak{g}$ its Lie algebra. We assume that there is a left action $\Phi = \{\Phi_g\}$ of $G$ on $M$, $G \ni g \mapsto \Phi_g \in \text{Diff}(M)$, such that for all $\xi \in S$ and $g \in G$ the following conditions hold:

- the fiber $\{\xi\} \times P$ over $\xi$ is invariant under $\Phi_g$;
- the restriction of the diffeomorphism $\Phi_g$ to $\{\xi\} \times P$ is Poisson;
- there exists a fiberwise momentum map, that is, a smooth mapping $J: M \to \mathfrak{g}^*$ such that

$$\frac{d}{dt} |_{t=0} \Phi_{\exp(ta)}^{-1} \frac{\partial}{\partial x^\gamma} = -\Lambda^{\gamma\beta}(x) \frac{\partial J_a}{\partial x^\beta} \frac{\partial}{\partial x^\gamma} $$

(3.1)

for every $a \in \mathfrak{g}$. Here, $J_a: M \to \mathbb{R}$ is given by $J_a(\xi, x) = (\xi(\xi, x), a)$ and throughout the article, the summation over repeated indices will be understood.

In other words, condition (3.1) means that the restriction of the $G$-action to each fiber is Hamiltonian with momentum map $J |_{\{\xi\} \times P}$. Therefore, according to the terminology introduced in [8], we have a family of Hamiltonian $G$-actions on the (trivial) Poisson bundle. A $C^\infty$ function $k = k(\xi, x)$ on $M$ is said to be a fiberwise Casimir function if $\Lambda^{\gamma\beta}(x) \frac{\partial K}{\partial x^\beta} = 0$. It is clear that every fiberwise Casimir function is $G$-invariant. Denoting the subspace of all fiberwise Casimir functions by $\text{Casim}(M, \Lambda)$, we observe that the freedom in the choice of the fiberwise momentum map $J$ in (3.1) is given by transformation

$$J_a \mapsto J_a + K_a$$

(3.2)

for any linear mapping $K: \mathfrak{g} \to \text{Casim}(M, \Lambda)$.

Using the action of the compact Lie group $G$, one can define the $G$-average of an arbitrary tensor field $T$ on $M$ by the standard formula

$$\langle T \rangle = \int_G \Phi_g^* T \, dg$$

which gives a $G$-invariant tensor field $\langle T \rangle$ on $M$ of the same type. Here $dg$ is the normalized Haar measure on $G$.

Now, we assume additionally, that the base $S$ is a symplectic manifold $(S, \{\cdot, \cdot\}_S)$ carrying a nondegenerate Poisson bracket

$$\{\xi^i, \xi^j\}_S = \omega^{ij}(\xi) \quad (\det(\omega^{ij}) \neq 0),$$

Rescaling the bracket on $P$ by a factor $\frac{1}{\varepsilon}$ ($\varepsilon > 0$), we equip the total space $M = S \times P$ with the $\varepsilon$-dependent product Poisson bracket

$$\{\xi^i, \xi^j\}_M = \omega^{ij}(\xi),$$

(3.3)
\[
\{\xi^i, x^\alpha\}_M = 0, \tag{3.4}
\]
\[
\{x^\alpha, x^\beta\}_M = \frac{1}{\varepsilon} \Lambda^{\alpha\beta}(x). \tag{3.5}
\]

In the context of the perturbation theory for Hamiltonian systems on the phase space \((M, \{\cdot, \cdot\}_M)\) (see, Section 4), the small parameter \(\varepsilon\) will play the role of a perturbation parameter. In this case, the variables \(\xi \in S\) and \(x \in P\) are said to be slow and fast, respectively. Fixing a fiberwise momentum map \(\mathbb{J}\), we will call the setup \((M = S \times P, \{\cdot, \cdot\}_M, G, \Phi, \mathbb{J})\) a slow-fast \(G\)-space. This is not a canonical (Poisson) \(G\)-space, since the \(G\)-action does not preserve the Poisson bracket \(\{\cdot, \cdot\}_M\), in general. Indeed, it easy to see that the bracket \(\{\cdot, \cdot\}_M\) is invariant under the \(G\)-action only in the case when the family of the Hamiltonian \(G\)-actions is trivial (independent of the slow variables). The following result says that one can correct this “defect” of our phase space by slightly deforming the Poisson bracket \(\{\cdot, \cdot\}_M\).

**Theorem 3.1.** Let \(N \subseteq M = S \times P\) be \(G\)-invariant open subset with compact closure. For all sufficiently small \(\varepsilon\), there exists a near identity transformation
\[
T_\varepsilon : N \rightarrow M, \ T_0 = \text{id}
\]
which brings the original Poisson bracket (3.3)-(3.5) into a \(G\)-invariant Poisson bracket \(\{\cdot, \cdot\}_N^{\text{inv}}\) on \(N\),
\[
\{f_1 \circ \Phi_g, f_2 \circ \Phi_g\}_N^{\text{inv}} = \{f_1, f_2\}_N^{\text{inv}} \circ \Phi_g.
\]

The proof of this statement is based on the same idea as the proof of Theorem 2.3. First, we construct a \(G\)-invariant Poisson bracket \(\{\cdot, \cdot\}_N^{\text{inv}}\) and then we define the near identity transformation \(T_\varepsilon\) as a Poisson isomorphism between \(\{\cdot, \cdot\}_M\) and \(\{\cdot, \cdot\}_N^{\text{inv}}\). But, in the contrast to the symplectic case we have the following difficulty: if the Poisson structure \(\{\cdot, \cdot\}_p\) is degenerate, then the \(G\)-average of the Poisson tensor of \(\{\cdot, \cdot\}_M\) is a \(G\)-invariant bivector field which does not satisfy the Jacobi identity, in general.

According to [5], an \(G\)-invariant Poisson bracket \(\{\cdot, \cdot\}_N^{\text{inv}}\) on \(N\) can be constructed by combining the coupling Poisson method [3] and the theory of Hannay-Berry connections [8]. In coordinates, this bracket is given by the formulas [5]:
\[
\{\xi^i, \xi^j\}_N^{\text{inv}} = -\mathcal{F}^{ij}(\xi, x), \tag{3.6}
\]
\[
\{\xi^i, x^\sigma\}_N^{\text{inv}} = \mathcal{F}^{is}(\xi, x)\Lambda^{s\nu}(x) \frac{\partial \theta_s(\xi, x)}{\partial x^\nu}, \tag{3.7}
\]
\[
\{x^\alpha, x^\beta\}_N^{\text{inv}} = \frac{1}{\varepsilon} \Lambda^{\alpha\beta}(x) \tag{3.8}
+ \Lambda^{\alpha\nu}(x) \frac{\partial \theta_\alpha(\xi, x)}{\partial x^\nu} \mathcal{F}^{ij}(\xi, x) \frac{\partial \theta_j(\xi, x)}{\partial x^\nu} \Lambda^{\mu\beta}(x).
\]

Here, (local) smooth functions \(\theta_i\) and \(\mathcal{F}_{ij}\) on \(S\) are defined by
\[
\theta_i = -\int^1_G \left(\int_0^1 \Phi_{\exp(\alpha)}^{\mathbb{J}} \frac{\partial \mathbb{J}_\alpha}{\partial \xi_i} dt\right) dg + k_i \quad (g = \exp a), \tag{3.9}
\]
and \(\mathcal{F}^{is}(\xi, x)\mathcal{F}_{sj}(\xi, x) = \delta^i_j\), where
\[
\mathcal{F}_{ij}(\xi, x) = \omega_{ij}(\xi) - \varepsilon \frac{\partial \theta_i}{\partial \xi_j} - \frac{\partial \theta_j}{\partial \xi_i} + \{\theta_i, \theta_j\}_P. \tag{3.10}
\]
Moreover, \( k_i \in \text{Casim}(N,A) \) are fiberwise Casimir functions determined the conditions \( \langle \theta_i \rangle = 0 \).
Remark that the right hand side of (3.9) is well-defined because of the property: every connected compact Lie group of exponential type (for details, see [8]). The local functions \( \theta_i \) and \( F_{ij} \) on \( S \) give the global horizontal 1-form \( \theta = \theta_j(x) \, d\xi^i \) and 2-form \( F = \frac{1}{2} F_{ij}(x) \, d\xi^i \wedge d\xi^j \) on the trivial Poisson bundle \( M = S \times P \) which are related to the Hannay-Berry connection and its curvature [5], [8]. Since the closure \( \bar{N} \) is compact, there exists \( \delta > 0 \) such that \( \text{det}(F_{ij}(\xi, x)) \neq 0 \) and hence the \( G \)-invariant Poisson bracket \( \{ \cdot, \cdot \}^\text{inv}_N \) is defined on \( N \) for all \( \varepsilon \in (0, \delta) \). We have the following properties: the invariant Poisson bracket is a \( \varepsilon \)-deformation of the original one,
\[
\{ \cdot, \cdot \}^\text{inv}_N = \{ \cdot, \cdot \}_M + O(\varepsilon)
\]
and the characteristic distributions of the Poisson structures \( \{ \cdot, \cdot \}^\text{inv}_N \) and \( \{ \cdot, \cdot \}_M \) coincide on \( N \).

Finally, applying the homotopy method for Poisson structures [1], [4] one can show that, for small enough \( \varepsilon \), a Poisson isomorphism \( \tau_\varepsilon \) between Poisson brackets \( \{ \cdot, \cdot \}^\text{inv}_N \) and \( \{ \cdot, \cdot \}_M \) is given as the time-1 flow of a time-dependent vector field on \( N \).

The next point is to think of the phase space \( (N, \{ \cdot, \cdot \}^\text{inv}_N, G, \Phi) \) as a Hamiltonian \( G \)-space. As we mentioned above, the freedom in the choice of a fiberwise momentum map \( J \) is given by (3.2). Different choices of \( J \) lead to different (but isomorphic) \( G \)-invariant Poisson structures.

**Theorem 3.2** Let \( \{ \cdot, \cdot \}^\text{inv}_N \) be the \( G \)-invariant Poisson bracket (3.6)-(3.8) associated to a fiberwise momentum map \( J \). Then, the \( G \)-action on \( N \) is Hamiltonian relative to \( \{ \cdot, \cdot \}^\text{inv}_N \) if and only if
\[
(d_S J) = 0 \quad \text{on} \ N,
\]
where \( d_S \) is the partial exterior differential on \( M = S \times P \) along \( S \). In this case, the corresponding momentum map is \( \varepsilon J \). Moreover, the Hamiltonian \( G \)-action on \( N \) admits a \( G \)-equivariant momentum map \( \bar{J} : N \to \mathfrak{g}^* \) given by
\[
\bar{J}_a = \varepsilon \int_G J_{\text{Ad}^{-1}_g a} \circ \Phi_g dg.
\]

By a straightforward computation one can check that the condition for infinitesimal generators (3.1) of the \( G \)-action to be Hamiltonian relative to \( \{ \cdot, \cdot \}^\text{inv}_N \) just coincides with (3.11). Formula (3.12) for an \( G \)-equivariant momentum map is due to [17] (see, also [11]). Recall that the equivariance of \( J \) relative to the co-adjoint action \( \text{Ad}_g^* \) of \( G \) on \( \mathfrak{g}^* \) means that \( J \circ \Phi_g = \text{Ad}_g^* \circ J \). In particular, this means that \( \bar{J} : N \to \mathfrak{g}^* \) is a Poisson map, where the co-algebra \( \mathfrak{g}^* \) is equipped with "minus" Lie-Poisson bracket. (for more details, see [11])

**Remark 3.3** Hypothesis (3.11), called the adiabatic condition, was introduced in [8], [9] in the context of the theory of Hannay-Berry connections on symplectic and Poisson fiber bundles. The existence of a fiberwise momentum map satisfying (3.11) is related to vanishing of a certain cohomology class associated to the \( G \)-action.

**Example 3.4** Let us take \( S = T^*\mathbb{R}^3_0 \) equipped with the usual symplectic structure \( dp \wedge dq \) and \( P = \text{so}^*(3) \) carrying the cyclic Poisson bracket \( \{ x_a, x_b \} = -\varepsilon_{a\beta\gamma} x_\gamma \). On the space \( M = T^*\mathbb{R}^3_0 \times \text{so}^*(3) \), let us consider the action of the circle \( S^1 = \mathbb{R}/2\pi\mathbb{Z} \) given by the infinitesimal generator \( V = (x \times \frac{\partial}{\partial q^i}) \cdot \frac{\partial}{\partial \xi^i} \). This \( S^1 \)-action corresponds to the rotations in the \( x \)-space around the axis \( \frac{\partial}{\partial q^i} \).
One can check that all hypotheses above are satisfied and we get a family of Hamiltonian \( S^1 \)-actions with fiberwise momentum map \( J = -\left( \frac{\partial}{\partial q^i}, x \right) \). In this case, \( J \) satisfies condition (3.11) and formulas (3.6)-(3.8) lead to the following \( S^1 \)-invariant Poisson brackets on \( M \):
\[
\{ p_i, p_j \} = -\frac{\varepsilon}{q^i} \varepsilon_{ijk} q^k x_a.
\]
\[
\{p_i, q^j\} = \delta^j_i, \quad \{q^i, q^j\} = 0,
\]
\[
\{p_i, x_\alpha\} = -\frac{1}{|q|^2} \epsilon_{\alpha\beta\gamma} \epsilon_{\beta ij} q^j x_\gamma,
\]
\[
\{q^i, x_\alpha\} = 0,
\]
\[
\{x_\alpha, x_\beta\} = -\frac{1}{\varepsilon} \epsilon_{\alpha\beta\gamma} x_\gamma.
\]

Such a kind of Poisson structures appears in the Hamiltonian formulation of Wong’s equations [10] describing a particle motion in a Wu-Yang monopole.

4. Perturbed Hamiltonian Dynamics

Suppose that we start with a perturbed Hamiltonian system on the slow-fast \( G \)-space \((M = S \times P, \{\cdot, \cdot\}_M, G, \Phi, \mathbb{J})\) of the form

\[
H(\xi, x, \varepsilon) = H_0(\xi) + \varepsilon H_1(\xi, x)
\]

for some \( H_0 \in C^\infty(S) \) and \( H_1 \in C^\infty(P) \). Since, the leading term \( H_0 \) of the Hamiltonian is independent of the fast variables \( x \), the corresponding equations of motion have the regular dependence in the perturbation parameter \( \varepsilon \) at \( \varepsilon = 0 \) and are written in the bracket form as follows

\[
\frac{d\xi^i}{dt} = \{H_0, \xi^i\}_S + \varepsilon \{H_1, \xi^i\}_S,
\]

\[
\frac{dx^\alpha}{dt} = \{H_1, x^\alpha\}_P.
\]

But, because of the singularity of the Poisson bracket \( \{\cdot, \cdot\}_M \) at \( \varepsilon = 0 \), the unperturbed system

\[
\frac{d\xi^i}{dt} = \{H_0, \xi^i\}_S,
\]

\[
\frac{dx^\alpha}{dt} = \{H_1, x^\alpha\}_P.
\]

is no longer Hamiltonian relative to \( \{\cdot, \cdot\}_M \).

**Theorem 4.1** Assume that on the \( G \)-invariant open domain \( N \subseteq M \) with compact closure, the momentum map \( \mathbb{J} \) is a vector-valued integral of motion of the unperturbed system (4.1),(4.2). Then, for small enough \( \varepsilon \), there exists a near identity transformation \( T_\varepsilon : N \rightarrow M \) which takes the original perturbed Hamiltonian model

\[
(M, \{\cdot, \cdot\}_M, H = H_0 + \varepsilon H_1)
\]

into the Hamiltonian system

\[
(N, \{\cdot, \cdot\}_N^{\text{inv}}, H \circ T_\varepsilon = \bar{H} + O(\varepsilon^2))
\]

where \( \{\cdot, \cdot\}_N^{\text{inv}} \) is the \( G \)-invariant Poisson bracket (3.6)-(3.8) associated to the fiberwise momentum map \( \mathbb{J} \) and

\[
\bar{H} = H_0 + \varepsilon \langle H_1 \rangle.
\]
The proof of this statement is based on Theorem 3.1. By using the homotopy method, the normalization transformation $T_\varepsilon$ is constructed as the Poisson isomorphism between Poisson structures $\{\cdot, \cdot\}_N^{\text{inv}}$ and $\{\cdot, \cdot\}_M$ and it is uniquely determined by the fiberwise momentum map $\mathbb{J}$. On the other hand, the relationship between $H$ and $\mathbb{J}$ is given by the symmetry hypothesis for the unperturbed system. Putting these arguments together, one can show that the pull-back of $H$ by $T_\varepsilon$ is exactly of the form (4.3), (see also [6]).

Therefore, Theorem 4.1 says that after transformation $T_\varepsilon$ the original perturbed system becomes $\varepsilon^2$-close to the $G$-symmetric Hamiltonian system $(N, \{\cdot, \cdot\}_N^{\text{inv}}, H)$. As a direct consequence of Theorem 3.2 and Theorem 4.1, we derive the following result.

**Corollary 4.2** If in addition to hypothesis of Theorem 4.1, the momentum map $\mathbb{J}$ satisfies the adiabatic condition (3.11), then the approximate model $(N, \{\cdot, \cdot\}_N^{\text{inv}}, H)$ is a Hamiltonian system with $G$-symmetry, in particular,

$$\{\bar{H}, \bar{\mathbb{J}}_{a}\}_N^{\text{inv}} = 0,$$

$$\{\bar{\mathbb{J}}_{a}, \bar{\mathbb{J}}_{b}\}_N^{\text{inv}} = -\bar{\mathbb{J}}_{[a,b]}$$

for any $a, b \in g$. Here $\bar{\mathbb{J}}$ is the equivariant momentum map given by (3.12).

Some typical examples of Hamiltonian models to which the above results can be applied, are related to the particle motion with spin in a magnetic field [6] and the rigid body motion [20].

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